Conditional Speed of Branching Brownian Motion, Skeleton Decomposition and Application to Random Obstacles

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Abstract

We study a branching Brownian motion $Z$ in $\mathbb{R}^d$, among obstacles scattered according to a Poisson random measure with a radially decaying intensity. Obstacles are balls with constant radius and each one works as a trap for the whole motion when hit by a particle. Considering a general offspring distribution, we derive the decay rate of the annealed probability that none of the particles of $Z$ hits a trap, asymptotically in time $t$. This proves to be a rich problem motivating the proof of a more general result about the speed of branching Brownian motion conditioned on non-extinction. We provide an appropriate skeleton decomposition for the underlying Galton-Watson process when supercritical and show through a non-trivial comparison that the doomed particles do not contribute to the asymptotic decay rate.

Keywords: Branching Brownian motion, Poissonian traps, Random environment, Hard obstacles, Rightmost particle

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1. Introduction

Branching Brownian motion (BBM) among random obstacles functioning as traps has been studied recently in \cite{6,7,9,12,16}. We study a BBM that evolves in $\mathbb{R}^d$, where a radially decaying field of Poissonian traps are...
present. We are mainly interested in the trap-avoiding probability of the system up to time \( t \), where the underlying Galton-Watson process (GWP) has a general offspring distribution. Investigation of this problem leads us to finding the speed of BBM conditioned on non-extinction, which is of independent interest.

Let \( Z = (Z(t))_{t \geq 0} \) be a \( d \)-dimensional BBM with branching rate \( \beta > 0 \) and offspring distribution \( \lambda \). The process starts with a single particle at the origin, which performs a Brownian motion in \( \mathbb{R}^d \) for a random time which is distributed exponentially with constant parameter \( \beta \). Then, the particle dies and simultaneously gives birth to a random number of particles distributed according to the offspring distribution \( \lambda \), which is a probability measure on \( \mathbb{N} = \{0, 1, 2, \ldots\} \). Similarly, each offspring particle repeats the same procedure independently of all others, starting from the position of her parent. In this way, one obtains a measure-valued Markov process \( Z = (Z(t))_{t \geq 0} \), where \( Z_t \) can be identified as a particle configuration for given \( t \geq 0 \). By assumption, \( Z(0) = \delta_0 \). The total mass process \( |Z| = (|Z(t)|)_{t \geq 0} \) is a continuous time branching process with rate \( \beta \), and the number of particles in generation \( n \) of \( |Z| \) is a GWP \( (N(n))_{n \in \mathbb{N}} \) with offspring distribution \( \lambda \). The initial particle present at \( t = 0 \) constitutes the 0th generation, the offspring of the initial particle constitute the 1st generation, and so forth. We denote the extinction time of the process \( |Z| \) by \( \tau \), which is formally defined as \( \tau = \inf \{ t \geq 0 : |Z(t)| = 0 \} \), where we use the convention that \( \inf \emptyset = \infty \). We then denote the event of extinction of the process \( |Z| \) by \( \mathcal{E} \), and formally write \( \mathcal{E} = \{ \tau < \infty \} \). We use the term non-extinction for the event \( \mathcal{E}^c \). Let \( P \) be the probability corresponding to the process \( Z \), and \( E \) the corresponding expectation.

The branching Brownian motion is assumed to live in a random environment consisting of Poissonian traps. Let \( \mathcal{B}(\mathbb{R}^d) \) be the \( d \)-dimensional Borel sets, and let \( \Pi \) denote the Poisson random measure on \( \mathcal{B}(\mathbb{R}^d) \), with a spatially dependent locally finite mean measure \( \nu \) such that \( d\nu/dx \) exists and is continuous on \( \mathbb{R}^d \) and

\[
\frac{d\nu}{dx} \sim \frac{l}{|x|^{d-1}}, \quad |x| \to \infty, \quad l > 0,
\]

where \( dx \) is for the Lebesgue measure. A random trap configuration \( K \) with radius \( a \) on \( \mathbb{R}^d \) is defined as

\[
K = \bigcup_{x_i \in \text{supp}(\Pi)} \bar{B}(x_i, a),
\]

where \( \bar{B}(x_i, a) \) is the closed ball of radius \( a > 0 \) centered at \( x_i \). Let \( \mathbb{P} \) be
the probability for the Poisson random measure, and $E$ the corresponding expectation.

For $A \in \mathcal{B}(\mathbb{R}^d)$ and $t \geq 0$, notice that $|\text{supp}(Z(t)) \cap A|$ is the number of particles located in $A$ at time $t$. For $t \geq 0$, let

$$R(t) = \bigcup_{s \in [0, t]} \text{supp}(Z(s))$$

be the range of $Z$ up to time $t$. Now let $T$ be the first time that $Z$ hits a trap,

$$T = \inf \{t \geq 0 : |\text{supp}(Z(t)) \cap K| > 0\} = \inf \{t \geq 0 : R(t) \cap K \neq \emptyset\}.$$ 

Then, the event of trap-avoiding up to time $t$ of $Z$ among the Poissonian traps is given by $\{T > t\}$.

We aim to analyze $(E \times P)(T > t)$, the annealed (averaged) trap-avoiding probability of the system up to time $t$, for a BBM with a general offspring distribution $\lambda$. In [7], the asymptotic decay of the annealed probability has been found as $t \to \infty$ in the strictly dyadic branching case, that is, $\lambda(2) = 1$. On the way to its generalization, we find out that the problem proves to be rich enough to require several stand-alone results that we also contribute in this paper. The most important of these is on the speed of BBM conditioned on non-extinction as given in Theorem 1. When the GWP is supercritical, the particles are grouped into those with infinite or finite line of descent, so-called “skeleton” and “doomed” particles, respectively. In other words, a skeleton decomposition is performed to analyze the problem. We show by a comparison argument that the doomed particles do not contribute to the conditional speed. Here, the speed of BBM refers to the growth rate of the radius of the minimal ball containing the range of the BBM. It has first been studied by McKean [14, 15], later by Bramson [1, 2] and Chauvin and Rouault [5], and is still subject to active research.

In Theorem 2, we prove the asymptotic decay of the annealed trap-avoiding probability as a large deviation result. It is an important and non-trivial application of Theorem 1. The problem of trap-avoiding asymptotics for BBM among Poissonian traps has been previously studied by Engländer [6] for the case $d \geq 2$ and where the trap intensity was uniform. Then, in search for extending the result to the case $d = 1$, Engländer and den Hollander [7] considered the more interesting case where the trap intensity was radially decaying as given in [1]. Both of these works were for a strictly dyadic underlying GWP, and later Öz and Çağlar [16] studied the uniform Poisson intensity case for GWP with a general offspring distribution. In
the present work, we consider a radially decaying trap intensity as in (1),
and a general offspring distribution for the underlying GWP. We show that,
conditioned on non-extinction, the doomed particles of the decomposition
of the supercritical GWP do not contribute at all to the asymptotic decay
rate of the trap-avoiding probability. In comparison with the strictly dyadic
case, the mean and another functional of the offspring distribution appear
as parameters in the rate expression (see (5)). We refer the reader to [8] for
an interesting survey article on the topic of BBM among Poissonian traps,
and to [9, 12] for various related problems.

Theorem 3 is a technical result identifying the rate function of Theorem 2
through tedious analysis.

The organization of the paper is as follows. In Section 2, we give the
statements of the three theorems described above as our main results. Sec-
section 3 includes the essential lemmas together with their proofs. In sections
4, 5 and 6, we prove theorems 1, 2 and 3, respectively.

2. Main Results

To formulate our main results, we introduce further notation. Let $f$ be
the probability generating function (p.g.f.) of the offspring distribution, and
$\mu$ be the mean number of offspring,

$$f(s) = \sum_{j=0}^{\infty} \lambda(j)s^j,$$

$$\mu = \sum_{j=0}^{\infty} j \lambda(j),$$

and define

$$m = \mu - 1.$$

Throughout this work, we assume that $\mu < \infty$ and without loss of generality
that $\lambda(1) = 0$ (see the proof of Lemma 4). Now let $q = P(\mathcal{E})$ be the
probability of extinction for the underlying GWP, and let

$$\alpha = 1 - f'(q).$$

Note that $\lambda(0) = 0$ implies that $q = 0$ since two or more offspring is produced
each time a particle branches. When $\lambda(0) = 0$, it then follow that $\alpha = 1$ as
$f'(0) = \lambda(1)$, and $\lambda(1) = 0$ by assumption. Also, $q = 1$ when $\mu \leq 1$, and
$q \in (0, 1)$ when $\lambda(0) > 0$ and $\mu > 1$. 
For $r, b \geq 0$, define
\[ g_d(r, b) = \int_{B_r(0)} \frac{dx}{|x + be|^{d-1}}, \]
where $e = (1, 0, \ldots, 0)$ is the unit vector in the direction of the first coordinate. Finally, let
\[ l_{cr}^* = l_{cr}^*(m, \beta, d) = \frac{1}{s_d} \sqrt{\frac{\beta}{2m}}, \]
where $s_d$ is the surface area of the $d$-dimensional unit ball ($s_1 = 2$, $s_2 = 2\pi$, $s_3 = 4\pi$, etc.).

Finally, recall from (2) that $\cup_{s \in [0, t]} Z(s)$ denotes the range of $Z$ up to time $t$, and define
\[ M(t) = \inf \{ r > 0 : R(t) \subseteq B_r(0) \} \quad \text{for} \quad d \geq 1, \]
to be the radius of the minimal ball containing $R(t)$. Now we state our main results.

**Theorem 1** (Conditional speed of BBM). *Suppose that the underlying GWP is supercritical, i.e., $m > 0$. Then, for $d \geq 1$, conditioned on non-extinction, $M(t)$ converges to $\sqrt{2\beta m}$ in $P$-probability as $t \to \infty$.*

Regarding the speed of BBM, we refer the reader to [14, 15] for the case $d = 1$, and where $\lambda(0) = 0$. We note that almost sure speed results exist too (see e.g. [11]), but for our purposes, convergence in probability suffices. For $d \geq 2$ and $\lambda(0) = 0$, we derive the speed in Proposition 1 in Section 4. Theorem 1, which is also proved in Section 4, covers the general case where $\lambda(0)$ can be nonzero. In this result, we see that conditioned on non-extinction of the underlying GWP, the speed remains as $\sqrt{2\beta m}$, which can be explained as follows. When the supercritical GWP is decomposed into skeleton and doomed particles (see Lemma 1), the doomed particles do not contribute to the speed, and the skeleton particles alone form another GWP with the same mean as the original one.

**Theorem 2** (Variational formula). *Fix $d, f, \beta, a$. Define
\[ I(l, f, \beta, d) = \min_{\eta \in [0,1], c \in [0, \sqrt{2} \beta m]} \left\{ \beta \alpha \eta + \frac{\eta^2}{2\eta} + lg_d(\sqrt{2\beta m}(1 - \eta), c) \right\}. \]  
(For $\eta = 0$ put $c = 0$ and $g_d(\sqrt{2\beta m}, 0) = s_d \sqrt{2\beta m}.$)
1. If \( \lambda(0) = 0 \), then
\[
\lim_{t \to \infty} \frac{1}{t} \log(\mathbb{E} \times P)(T > t) = -I(l, f, \beta, d).
\]

2. If \( \lambda(0) > 0 \) and \( m > 0 \), then
\[
\lim_{t \to \infty} \frac{1}{t} \log(\mathbb{E} \times P)(T > t | \mathcal{E}^c) = -I(l, f, \beta, d).
\]

3. If \( \lambda(0) > 0 \) and \( m \leq 0 \), then
\[
\lim_{t \to \infty} (\mathbb{E} \times P)(T > t) = (\mathbb{E} \times P)(T > \tau) > 0,
\]
where \( \tau = \inf \{ t \geq 0 : |Z(t)| = 0 \} \) is the extinction time and \( \mathcal{E} \) is the event of extinction for the underlying GWP.

In comparison with the variational result given in \([7, \text{Theorem 1.1}]\) for dyadic branching, Theorem 2 contributes two extra factors, both naturally depending only on \( f \), in the expression for the rate function (5). The extra factor \( \alpha \) in the first term appears when \( \lambda(0) > 0 \) (if \( \lambda(0) = 0 \), then \( \alpha = 1 \)). It is a consequence of the modified p.g.f. for the offspring distribution of the skeleton particles, which are part of the decomposition of the supercritical GWP conditioned on non-extinction. The extra factor \( m \) inside \( g_d \) appears, because the speed of BBM is now no longer \( \sqrt{2} \beta \) but \( \sqrt{2} \beta m \). Since \( \alpha = 1 = m \) for strictly dyadic branching, the result in \([7]\) is recovered by (5) as a special case.

The next result is purely analytic, and solves the variational problem for the rate function \( I(l, f, \beta, d) \). We generalize the corresponding result in \([7]\).

**Theorem 3 (Crossover).** Fix \( f, \beta, a \).

1. For \( d \geq 1 \) and all \( l \neq l_{cr} \), the variational problem has a unique pair of minimizers, denoted by \( \eta^* = \eta^*(l, f, \beta, d) \) and \( c^* = c^*(l, f, \beta, d) \).

2. For \( d = 1 \),
\[
\begin{align*}
l \leq l_{cr} : \quad & I(l, f, \beta, d) = \beta \frac{l}{l_{cr}}, \\
l > l_{cr} : \quad & I(l, f, \beta, d) = \beta \alpha,
\end{align*}
\]
and
\[
\begin{align*}
l < l_{cr} : \quad & \eta^* = 0, \quad c^* = 0, \\
l > l_{cr} : \quad & \eta^* = 1, \quad c^* = 0,
\end{align*}
\]
where \( l_{cr} = \alpha t_{cr}^* = \frac{\alpha}{2} \sqrt{\frac{3}{2m}} \).

3. For \( d \geq 2 \),

\[
\begin{align*}
I(l, f, \beta, d) &= \beta l, \\
I(l, f, \beta, d) &= \beta(\alpha \wedge \frac{l}{t_{cr}^*}),
\end{align*}
\]

and

\[
\begin{align*}
l < l_{cr} & : \quad \eta^* = 0, \quad c^* = 0, \\
l > l_{cr} & : \quad 0 < \eta^* < 1, \quad 0 < c^* < \sqrt{2\beta},
\end{align*}
\]

where \( l_{cr} = \gamma_d l_{cr}^* \), with

\[
\gamma_d = \frac{-1 + \sqrt{1 + 4m\alpha M_d^2}}{2mM_d^2} \in (0, \alpha),
\]

where

\[
M_d = \frac{1}{2s_d m} \max_{R \in (0, \infty)} [g_d(R, 0) - g_d(R, 1)].
\]

Furthermore, \( c^* > \sqrt{2\beta m(1 - \eta^*)} \) when \( l > l_{cr} \).

Comparing this theorem to the corresponding one in [7], we see that in this case the extra factors \( \alpha \) and \( m \) appear ubiquitously in the formulas. Again, we note that the formulas above reduce to the corresponding ones in [7] upon setting \( \alpha = 1, m = 1 \).

3. Preparations

In this section we prove seven preparatory lemmas. Lemma 1 is needed directly in the proof of Theorem 2. Lemma 2 and Lemma 3 are needed in the proof of Proposition 1 and Theorem 1, which are central in the proof of Theorem 2. Lemma 4 is needed in the proof of both Theorem 1 and Theorem 2. Lemma 5 and Lemma 6 are needed to prove Theorem 1. Lemma 7 is used in the proof of Proposition 1.

**Lemma 1.** Let \( r > 0 \) and \( b > 0 \). Then,

\[
\lim_{t \to \infty} \frac{1}{lt} \nu(B_{rt}(bte)) = g_d(r, b).
\]
Proof. By substituting \( y = x/t \), we see that
\[
g_d(rt, bt) = \int_{B_{rt}(0)} \frac{dx}{|x + bte|^{d-1}} = t \int_{B_r(0)} \frac{dy}{|y + be|^{d-1}} = t g_d(r, b). \tag{12}
\]
Recall that integrals of asymptotically equivalent positive continuous functions are asymptotically equivalent if one of the integrals diverges. Apply this result to the radial integral. By assumption, \( d\nu/dx \) is continuous on \( \mathbb{R}^d \). As the \( d\)-dimensional volume element has radial component \( |x|^{d-1}dx \), the integrands of both radial integrals below are continuous on \( \mathbb{R}^+ \). Hence, we have
\[
\nu(B_{rt}(bte)) = \int_{B_{rt}(bte)} \frac{d\nu}{dx} dx \sim \int_{B_{rt}(bte)} \frac{1}{|x|^{d-1}} dx = l g_d(rt, bt). \tag{13}
\]
Combining (12) and (13) yields \( \lim_{t \to \infty} \frac{\nu(B_{rt}(bte))}{t g_d(r, b)} = 1 \), which implies the result. \( \square \)

**Lemma 2** (Overproduction). Let \( \delta > 0 \). Then for any \( t \)
\[
P(|Z(t)| > e^{m\beta t + \delta}) \leq e^{-\delta t}.
\]
Proof. Use the fact that \( E|Z(t)| = e^{m\beta t} \) (see [10]) and then apply Markov inequality. \( \square \)

**Lemma 3** (Comparison). Suppose that \( \lambda(0) = 0 \). Let \( B \subset \mathbb{R}^d \) be open or closed. Let \( x \in B \). Let \( P_x \) be the law of Brownian motion, denoted by \( W \), starting from \( x \), and let \( P_{\delta x} \) be the law of BBM starting from \( \delta x \). Define the first exit times from \( B \)
\[
\psi_B = \inf \{ t \geq 0 : W(t) \in B^c \}
\]
\[
\hat{\psi}_B = \inf \{ t \geq 0 : |\text{supp}(Z(t)) \cap B^c| \geq 1 \}.
\]
Then for any \( k \in \mathbb{N} \) and \( t \geq 0 \)
\[
P_{\delta x} (\hat{\psi}_B > t ||Z(t)| \leq k) \geq [P_x (\psi_B > t)]^k.
\]
Proof. See the corresponding proof in [2], which is written for strictly dyadic branching, and note that the proof can easily be extended to the case of general \( \lambda \)-GWP with \( \lambda(0) = 0 \), since the population does not decrease for such processes, that is, if for any \( k \in \mathbb{N}_+ \), \(|Z(t)| \leq k\), then \(|Z(s)| \leq k\) for all \( s \in [0, t] \). \( \square \)
Lemma 4 (Decomposition of the supercritical GWP). Let $N = (N(n))_{n \in \mathbb{N}}$ be a supercritical GWP with offspring distribution $\lambda$, where $\lambda(0) > 0$. Let $f$ be the corresponding p.g.f. Let $q$ be the probability of extinction for $N$, and define $\bar{q} = 1 - q$. Let $N^*(n)$ be the number of particles in generation $n$ that have an infinite line of descent.

1. The law of $N$ given extinction is the same as that of a GWP with p.g.f.
   
   \[
   \tilde{f}(s) := \frac{f(qs)}{q}.
   \]

2. The law of $N^* = (N^*(n))_{n \in \mathbb{N}}$ given non-extinction is the same as that of a GWP with p.g.f.

   \[
   f^*(s) := \frac{[f(q + \bar{q}s) - q]/\bar{q}}{q}.
   \]

3. The law of $N$ given non-extinction is the same as that of a process $\bar{N}$ generated as follows: Let $N^*$ be the process consisting of the particles of infinite line of descent. To each particle of $N^*$ having $n_0$ children, add $n_1$ more children, that each start an independent GWP with p.g.f. $\tilde{f}$, where $n_1$ has the p.g.f.

   \[
   f_{n_1}(s) := \frac{(D_{n_0}f)(qs)}{(D_{n_0}f)(q)}.
   \]

   where $D$ is the differentiation operator, and all $n_1$ and all GWPs added are mutually independent given $N^*$. The resultant process is $\bar{N}$.

4. Let $X = (X(t))_{t \in \mathbb{R}^+}$ be a continuous time branching process with rate $\beta$ such that the number of particles in generation $n$ of $X$ is $N(n)$ for $n \in \mathbb{N}$. Let $f$ be the p.g.f. for the offspring distribution of $N$, and let $X^*(t)$ be the number of particles with infinite line of descent at time $t$. Then, the process $X^* = (X^*(t))_{t \in \mathbb{R}^+}$ given non-extinction is equal in law to a continuous time branching process with rate $\beta(1 - f'(q))$ and offspring distribution, whose p.g.f. is

   \[
   \bar{f} := \frac{[f^*(s) - f'(q)s](1 - f'(q))}{1 - f'(q)}.
   \]

Remark. The reader should note that in (15), positive mass on 1 is allowed. For example, if $\lambda(0) = 1/3$, $\lambda(2) = 2/3$, $f(s) = 1/3 + (2/3)s^2$, then $q = 1/2 = \bar{q}$, and $f'(q) = 2/3$, while $f^*(s) = (2/3)s + (1/3)s^2$. This distribution still has the same mean as originally (i.e., 4/3), but here we put 2/3 weight on 1.

What (17) says is that, equivalently (without putting mass on 1), it has rate $\beta/3$ and p.g.f. $\bar{f} = s^2$. 
Proof. See [13, Proposition 4.10] for a proof of parts 1, 2 and 3, and [3, Section 1.12] for further details on the decomposition of supercritical GWPs. For a proof of part 4, observe that a continuous time branching process with rate $\beta$ and offspring distribution $\lambda$ is equal in law to a continuous time branching process with rate $\beta(1 - \lambda(1))$ and offspring distribution $\bar{\lambda}$, where $\bar{\lambda}(1) = 0$ and $\bar{\lambda}(j) = \lambda(j)/(1 - \lambda(1))$ for $j \neq 1$. To complete the proof, see the proof of part 3 of [16, Theorem 1].

Lemma 5. If $s \in [0, 1]$, then $f^*(s) \leq \tilde{f}(s)$.

Proof. By (14) and (15), we need to show that

\[ \frac{f(q + qs) - q}{\bar{q}} \leq f(qs)/q. \]  

(18)

The left-hand side of (18) is equal to $[f(s + q(1 - s)) - q]/(1 - q)$. Since $f$ is convex, by Jensen’s inequality, it is enough to show that

\[ \frac{s + (1 - s)f(q) - q}{1 - q} \leq f(qs)/q. \]

Since $f$ is greater than or equal to the identity on $[0, q]$, $s \in [0, 1]$ implies that $f(qs) \geq qs$. We now have, using $f(q) = q$, that

\[ \frac{s + (1 - s)f(q) - q}{1 - q} = \frac{s + (1 - s)q - q}{1 - q} = \frac{f(qs)}{q}. \]

Lemma 6. Let $Z_1 = (Z_1(t))_{t \geq 0}$ and $Z_2 = (Z_2(t))_{t \geq 0}$ be two BBMs with the same constant branching rate $\beta > 0$, and p.g.f.’s $\phi_1$ and $\phi_2$ for the offspring distributions, respectively. Let $P_{s,x}^1$ and $P_{s,x}^2$ be the corresponding probabilities for processes that start at time $s$ with a single particle at position $x \in \mathbb{R}^d$. Assume that $\phi_1 \leq \phi_2$ on $[0, 1]$. Then, for any Borel set $B$, and any fixed time $t \geq s$,

\[ P_{s,x}^1(R_1(t) \subset B) \leq P_{s,x}^2(R_2(t) \subset B), \]  

(19)

where $R_j(t)$ is the range of process $Z_j$ up to time $t$ for $j = 1, 2$.

Proof. Clearly, if $x \notin B$, then $P_{s,x}^1(R_1(t) \subset B) = 0 = P_{s,x}^2(R_2(t) \subset B)$, so the inequality in (19) is trivially satisfied. Now, assume that $x \in B$. Define

\[ u^j(s, x, t) := P_{s,x}^j(R_j(t) \subset B), \quad s \leq t, \quad \text{for} \quad j = 1, 2. \]

Let $\tau^j$, $j = 1, 2$, be the first branching time after $s$, for a single particle located at $x \in \mathbb{R}^d$ at time $s$. Then, $\tau^j$ is exponentially distributed with
Define $F$ have Note that if $w$ negative, and if $π$ where
within the time interval $[s,r]$ that starts at time $s$ and position $x ∈ ℝ^d$ remains in $B$ until time $t$, while $E^{j}(r)$ denotes the expectation conditioned on the event that $Y$ remains in $B$ within the time interval $[s,r]$. Thus, $φ_1 ≤ φ_2$ implies that

$$(u^1 − u^2)(s,x,t) ≤ \int_s^t \pi(r − s, x, B) \times$$

$$E^{j}(r) \left[ φ_1 \left( u^1(r,Y,r,t) \right) − φ_1 \left( u^2(r,Y,r,t) \right) \right] Q(dr). \quad (20)$$

Set $s = 0$ and use that $Q$ has an explicitly known bounded density $g = g(t)$ to arrive at

$$(u^1 − u^2)(0,x,t) ≤ \int_0^t \pi(r) g(r) E^{(r)} \left[ φ_1 \left( u^1(r,Y,r,t) \right) − φ_1 \left( u^2(r,Y,r,t) \right) \right] dr. \quad (21)$$

Letting $v^j(x,t) := u^j(0,x,t)$, $j = 1, 2$, and making the substitution $r → t−r$,

$$(v^1−v^2)(x,t) ≤ \int_0^t \pi(t−r) g(t−r) E^{(t−r)}_0 \left[ φ_1 \left( v^1(Y_{t−r},r) \right) − φ_1 \left( v^2(Y_{t−r},r) \right) \right] dr. \quad \text{ (21)}$$

Now let $w := (v^1 − v^2) ∨ 0$, where $∨$ denotes maximum. Recall that $φ_1$ is monotone nondecreasing and convex on $[0,1]$, having a bounded derivative. Hence $φ_1(v^1) − φ_1(v^2) ≥ 0$ holds if and only if $v^1 − v^2 ≥ 0$, in which case $φ_1(v^1) − φ_1(v^2) ≤ C(v^1 − v^2)$ for some $C > 0$. Therefore,

$$w(x,t) ≤ C \int_0^t \pi(t−r) g(t−r) E^{(t−r)}_0 w(Y_{t−r},r) dr. \quad \text{ (22)}$$

Note that if $w(x,t) = 0$, then (22) holds since the right-hand side is nonnegative, and if $w(x,t) ≥ 0$, then (22) holds by the definition of $w$ and by (21). Define $F(r) := E^{(t−r)}_0 w(Y_{t−r},r)$, and note that $F(t) = w(x,t)$. We have

$$F(t) ≤ C \int_0^t \pi(t−r) g(t−r) F(r) dr,$$
and, by Gronwall’s inequality, we conclude that $F(t) \leq 0$. Hence, $w(x, t) \leq 0$. But $w(x, t) \geq 0$ by definition, therefore $w(x, t) = 0$, that is $u^1 - v^2 \geq 0$, which means that $u^1 \leq u^2$, as claimed.

Lemma 7 (Brownian hitting times in $\mathbb{R}^d$). Let $B$ be a Brownian motion in $\mathbb{R}^d$ starting at the origin, with corresponding probability $P_0$, and let $k > 0$. Then, as $t \to \infty$,

$$P_0 \left( \sup_{0 \leq s \leq t} |B_s| \geq kt \right) = \exp \left\{ - \frac{k^2 t}{2} (1 + o(1)) \right\}. \quad (23)$$

Furthermore, for $d = 1$, even the following stronger statement is true. Let $m_t = \inf_{0 \leq s \leq t} B_s$ and $M_t = \sup_{0 \leq s \leq t} B_s$. (The process $M_t - m_t$ is the range process of $B$.) Then, as $t \to \infty$,

$$P_0 (M_t - m_t \geq kt) = \exp \left\{ - \frac{k^2 t}{2} (1 + o(1)) \right\}. \quad (24)$$

Proof. For $d = 1$, recall that [10, equation 7.3.3] for any $a > 0$,

$$P_0 \left( \sup_{0 \leq s \leq t} |B_s| \geq a \right) = \sqrt{\frac{2}{\pi}} \int_{a/\sqrt{t}}^{\infty} \exp(-y^2/2)dy.$$

Setting $a = kt$ above, we see via l’Hôpital’s rule that

$$P_0 \left( \sup_{0 \leq s \leq t} |B_s| \geq kt \right) = \exp \left\{ - \frac{k^2 t}{2} (1 + o(1)) \right\}.$$

We now extend this result to $d \geq 2$. Note that the lower bound holds trivially since the projection of a $d$-dimensional Brownian motion onto the first coordinate axis is a one-dimensional Brownian motion. In order to prove the upper bound, we first prove (24).

Let $d = 1$. Define

$$\theta_c = \inf \{ s \geq 0 | M_s - m_s = c \}.$$

According to p.199 in [4] and references therein, the Laplace transform of $\theta_c$ satisfies

$$E \exp \left( - \frac{\lambda^2}{2} \theta_c \right) = \frac{2}{1 + \cosh(\lambda c)}, \quad \lambda > 0.$$

Hence, by exponential Markov inequality,

$$P_0 (\theta_c \leq t) \leq \exp \left( \frac{\lambda^2}{2} t \right) E \exp \left( - \frac{\lambda^2}{2} \theta_c \right) = \exp \left( \frac{\lambda^2}{2} t \right) \frac{2}{1 + \cosh(\lambda c)}.$$
Taking $c = kt$, one obtains
\[ P_0(M_t - m_t \geq kt) = P_0(\theta_{kt} \leq t) \leq \exp \left( \frac{\lambda^2}{2} t \right) \frac{2}{1 + \cosh(\lambda kt)}. \]

Optimizing, we see that the estimate is the sharpest when $\lambda = k$, in which case, we arrive at the desired upper bound. Since $P_0 \left( \sup_{0 \leq s \leq t} |B_s| \geq kt \right) \leq P_0(M_t - m_t \geq kt)$, the lower bound coincides with the upper bound, which implies the result.

For $d \geq 2$, let us consider, more generally, the Brownian motion starting from $x$, denoted by $B^x$, $x \in \mathbb{R}^d$, and let $r = |x|$. We will use the well known fact [17, Example 8.4.1] that if $R = |B|$ and $r > 0$, then the process $R$, called the $d$-dimensional Bessel process is the strong solution of the one-dimensional stochastic differential equation on $(0, \infty)$:
\[ R_t = r + \int_0^t \frac{d - 1}{2R_s} ds + W_t, \]
where $W$ is the standard Brownian motion.

Using the strong Markov property of Brownian motion, applied at the first hitting time of the $\rho$-sphere, it is clear that in order to verify the lemma, it suffices to prove it when $B$ is replaced by $B^x$, and $|x| = \rho > 0$. This follows, since for every fixed $\rho$ such that $0 < \rho < kt$, we have
\[ P_0 \left( \sup_{0 \leq s \leq t} |B_s| \geq kt \right) \leq P_0 \left( \sup_{0 \leq s \leq t+\tau} |B_s| \geq kt \right) = P_\rho \left( \sup_{0 \leq s \leq t} R_s \geq kt \right), \tag{25} \]
where $\tau$ is the first hitting time of $B$ on the $\rho$-sphere, and $\{P_\tau; r > 0\}$ are the probabilities for $R$ (or for $W$).

Now consider $B^x$ and define the sequence of stopping times $0 = \tau_0 < \sigma_0 < \tau_1 < \sigma_1, \ldots$ with respect to the filtration generated by $R$ (and thus, also with respect to the one generated by $W$) as follows:
\[ \tau_0 := 0; \quad \sigma_0 := \inf \{ s > 0 | R_s = \rho/2 \}, \]
and for $i \geq 1$,
\[ \tau_{i+1} := \inf \{ s > \sigma_i | R_s = \rho \}; \quad \sigma_{i+1} := \inf \{ s > \tau_{i+1} | R_s = \rho/2 \}. \]
Note that for $i \geq 0$ and $s \in [\tau_i, \sigma_i]$,
\[ R_s = \rho + \int_{\tau_i}^s \frac{d - 1}{2R_z} dz + W_s - W_{\tau_i} \leq \rho + \rho^{-1}(d - 1)\Delta_i + W_s - W_{\tau_i} \tag{26} \]
where $\Delta_i^s := s - \tau_i$.

Since $R_s \leq \rho$ for $\sigma_i \leq s \leq \tau_i + 1$ and $i \geq 0$, it is also clear that for $t > \rho / k$, the relation $\sup_{0 \leq s \leq t} R_s \geq kt$ is tantamount to

$$\sup_{i \geq 0} \sup_{\tau_i \wedge t < s \leq \sigma_i \wedge t} R_s \geq kt.$$ 

Putting this together with (26), it follows that $P_\rho(\sup_{0 \leq s \leq t} R_s \geq kt)$ can be upper estimated by

$$P_\rho(\exists i \geq 0, \exists \tau_i \wedge t < s \leq \sigma_i \wedge t : W_s - W_{\tau_i} \geq kt - \rho^{-1}(d - 1)\Delta_i^s - \rho).$$

This can be further upper estimated by

$$P_\rho\left(M_t - m_t \geq \left[k - \frac{d - 1}{\rho} - \delta\right]t\right) = P_0\left(M_t - m_t \geq \left[k - \frac{d - 1}{\rho} - \delta\right]t\right),$$

for any $\delta > 0$, as long as $t \geq \rho / \delta$, where the last equality follows from the translation invariance of Brownian motion. To complete the proof, fix $\rho$, $\delta > 0$, let $t \to \infty$, and use the already proven relation (24), along with (25). Finally let $\delta \to 0$ and $\rho \to \infty$. \hfill $\square$

4. Speed of BBM

In this section, we first state a well-known convergence in probability result regarding the speed of BBM, which holds when the dimension is one, and when each particle is certain to give at least one offspring, i.e., $\lambda(0) = 0$. We then extend this result to any dimension, and finally to the case where $\lambda(0) > 0$. The results of this section are central in proving Theorem 2.

Recall from (4) that $M(t) = \inf \{r > 0 : R(t) \subseteq B_r(0)\}$ for $d \geq 1$ denotes the radius of the minimal ball containing the range of the process up to time $t$, where $R(t)$ stands for the range up to time $t$.

**Proposition 1 (Speed of BBM).** Suppose that $\lambda(0) = 0$. For $d \geq 1$, $\frac{M(t)}{t}$ converges to $\sqrt{2}\beta m$ in $P$-probability as $t \to \infty$.

**Remark.** It is easy to see that the quantity $\beta m$ is invariant under changing the setting from the ‘canonical one’ (that is, putting zero mass on 1) to a non-canonical one (that is, assigning positive mass to 1).

**Proof.** For $d = 1$, the reader is referred to [14, 15] and [6]. Now let $d \geq 2$. Since the projection of $Z$ onto the 1st coordinate axis is a one-dimensional
BBM with branching rate \( \beta \), the lower estimate for convergence in probability follows from the result for \( d = 1 \) and the inequality

\[
P \left( \frac{M(t)}{t} > \sqrt{2\beta m - \varepsilon} \right) \geq P^* \left( \frac{M(t)}{t} > \sqrt{2\beta m} - \varepsilon \right)
\]

\( \forall \varepsilon > 0 \) and \( \forall t \), where \( P^* \) denotes the law of the one-dimensional projection of \( Z \). To prove the upper estimate for convergence in probability, let \( \varepsilon > 0 \) and let \( B = B_{(\sqrt{2\beta m} + \varepsilon)}(0) \). Pick \( \delta > 0 \) such that

\[
\frac{1}{2}(\sqrt{2\beta m + \varepsilon})^2 > \beta m + \delta. \tag{27}
\]

Recall that \( \psi_B \) and \( \hat{\psi}_B \) are the first exit times from \( B \) for a single Brownian particle starting at the origin and for a BBM, respectively. See that these events are identical:

\[
\{ \frac{M(t)}{t} > \sqrt{2\beta m + \varepsilon} \} = \{ \hat{\psi}_B \leq t \}.
\]

Estimate

\[
P \left( \frac{M(t)}{t} > \sqrt{2\beta m + \varepsilon} \right) \leq P(|Z(t)| > |e^{(\beta m + \delta)t}|) + P(\hat{\psi}_B \leq t | |Z(t)| \leq |e^{(\beta m + \delta)t}|). \tag{28}
\]

By Lemma 2, the first term on the right-hand side tends to zero exponentially fast. Now consider the second term. Recall that \( P_x \) is the law of Brownian motion starting from \( x \), and \( P_{\delta x} \) is the law of BBM starting from \( \delta x \). By Lemma 3 we have the estimate

\[
P_{\delta x}(\hat{\psi}_B > t | |Z(t)| \leq |e^{(\beta m + \delta)t}|) \geq [P_0(\psi_B > t)]|e^{(\beta m + \delta)t}|.
\]

By Lemma 7

\[
[P_0(\psi_B > t)]|e^{(\beta m + \delta)t}| = \left[ 1 - \exp \left( -\frac{(\sqrt{2\beta m + \varepsilon})^2}{2t} \right) \exp(o(t)) \right]|e^{(\beta m + \delta)t}|. \tag{29}
\]

By (27), using the binomial expansion, right-hand side of (29) tends to 1 exponentially fast as \( t \to \infty \), so that (28) yields the desired upper estimate

\[
P \left( \frac{M(t)}{t} > \sqrt{2\beta m + \varepsilon} \right) \to 0,
\]

which completes the proof. \( \square \)
4.1. Proof of Theorem 4.1

We emphasize that in this proof, the branching rate for all types of particles for all processes considered is constant and is equal to \( \beta \), and all probabilities written should be understood as conditioned on the non-extinction of the underlying GWP.

First, decompose the supercritical GWP into two sets of particles: particles with infinite line of descent, which, following the terminology in [18], we call skeleton particles; and particles with finite line of descent, which we call the doomed particles. By Lemma 4, each skeleton particle gives rise to a GWP consisting only of skeleton particles with p.g.f.

\[
f^*(s) := \left[ f(q + \bar{q}s) - q \right] / \bar{q},
\]

where \( q \) is the probability of extinction for the underlying GWP and \( \bar{q} = 1 - q \). Also, by Lemma 4, each doomed particle starts her own single-type GWP with p.g.f.

\[
\tilde{f}(s) := f(qs) / q.
\]

Observe that \((f^*)'(1) = f'(1) = \mu\). Let \( \kappa = \tilde{f}'(1) = f'(q) \), and define \( k = \kappa - 1 \). It is clear that each skeleton particle produces at least one skeleton particle, and can produce doomed particles as well, whereas a doomed particle only produces doomed particles.

Now we translate this decomposition into the language of multi-type GWPs, see [3, Chapter 5]. We define the skeleton particles to be of type 1, and the doomed particles to be of type 2, so that the underlying GWP is a two-type GWP, and we have the following decomposition for the BBM:

\[
(Z(t))_{t \geq 0} = (Z^1(t), Z^2(t))_{t \geq 0},
\]

where \( Z^1 \) is the process consisting of the skeleton particles, and \( Z^2 \) is the one consisting of the doomed particles. Note that \( Z^1 \) by itself is a BBM, however, \( Z^2 \) is not. Let \(|Z^j(t)|, j = 1, 2\) be the number of particles of type \( j \) existing at time \( t \). Observe that conditioning \( Z \) on non-extinction is equivalent to the initial condition \((|Z^1(0)|, |Z^2(0)|) = (1, 0)\), which says that the process starts with one skeleton particle, hence never dies out. For \( d \geq 1 \), let us define the range of \( Z^1 \) as \( R^1 \), and the range of \( Z^2 \) as \( R^2 \). Similarly, define

\[
M^1(t) := \inf \{ r > 0 : R^1(t) \subseteq B_r(0) \},
\]

\[
M^2(t) := \inf \{ r > 0 : R^2(t) \subseteq B_r(0) \}.
\]
Now since $M(t) = \max \{M^1(t), M^2(t)\}$, $f^*(0) = 0$, $(f^*)'(1) - 1 = m$, and $\beta m$ in the speed is invariant under how one describes the branching (see the remark after Proposition 1), Proposition 1 immediately gives the desired lower bound: \( \forall \varepsilon > 0, \)

\[
P\left( \frac{M(t)}{t} > \sqrt{2\beta m - \varepsilon} \right) \to 1 \quad \text{as} \quad t \to \infty.
\]

In order to obtain the desired upper bound, we find the mean matrix for the two-type continuous time branching process $|Z| = (|Z^1|, |Z^2|)$. Let

\[
f^{(i)}(s, t) = \sum_{i, j \geq 0} p^{(i)}(i, j) s^i t^j
\]

be the p.g.f.’s that determine the distribution of the offspring produced by particles of type 1 and type 2, respectively, where $p^{(k)}(i, j)$, $k = 1, 2$ is the probability that a particle of type $k$ gives $i$ offspring of type 1 and $j$ offspring of type 2. It follows that

\[
f^*(s) = \sum_{i=0}^{\infty} \left( \sum_{j=0}^{\infty} p^{(1)}(i, j) \right) s^i,
\]

and

\[
\tilde{f}(t) = \sum_{j=0}^{\infty} \left( \sum_{i=0}^{\infty} p^{(2)}(i, j) \right) t^j,
\]

where $p^{(2)}(i, j) = 0$ for $i = 1, 2, \ldots$ since a doomed particle never produces a skeleton particle. In order to find the mean matrix, we first calculate the expected number of offspring of type 2 for a particle of type 1, denoted by $E[N(2) | N(0) = (1, 0)]$, where the discrete-time process $(N(n))_{n \in \mathbb{N}}$ represents the number of particles in generation $n$ of $|Z|$. Let $L$ denote the random variable for the number of offspring for a particle in the original GWP, $L^*$ of which are skeleton particles. By [13, Proposition 4.10], the joint p.g.f. of $L - L^*$ and $L^*$ is

\[
E \left[ s^{L-L^*} t^{L^*} \right] = f(qs + \tilde{q}t).
\]
It follows that

$$E[s^L - L^* t^L | L^* \geq 1] = \sum_{j=0, k=1}^{\infty} P(L - L^* = j, L^* = k)s^j t^k$$

$$\quad \quad \quad = \frac{f(qs + \bar{q}t) - \sum_{j=0}^{\infty} P(L = j, L^* = 0)s^j}{1 - q}$$

$$\quad \quad \quad = \frac{f(qs + \bar{q}t) - \sum_{j=0}^{\infty} P(L = j | L^* = 0)P(L^* = 0)s^j}{1 - q}$$

$$\quad \quad \quad = \frac{f(qs + \bar{q}t) - f(qs)}{1 - q},$$

where we have used (31) in the fourth equality. Setting $s = t$ in (32) yields the p.g.f. for the total number of offspring for a skeleton particle:

$$E[s^L | L^* \geq 1] = \frac{f(s) - f(qs)}{\bar{q}}. \quad (33)$$

Finally, differentiating (33) with respect to $s$, setting $s = 1$ and then subtracting $\mu$ gives the desired expectation

$$E[N(2)|N(0) = (1, 0)] = (\mu - \kappa)\frac{\bar{q}}{\bar{q}}. \quad (34)$$

It follows that the mean matrix [3, Section 5.7] for this two-type continuous time branching process is given by:

$$A(t) := \left( \begin{array}{cc} e^{\beta tm} & \frac{q}{\bar{q}}[e^{\beta tm} - e^{\beta tk}] \\ 0 & e^{\beta tk} \end{array} \right),$$

where $A_{ij}(t)$ is the expected population of type $j$ at time $t$ given that the process starts with a single particle of type $i$.

The strategy now is to compare the two-type BBM $Z$ at hand with a modified two-type BBM, say $\tilde{Z}$, prove the result for $\tilde{Z}$, and finally conclude that the result holds for $Z$ by comparison. Let $\tilde{Z} = (\tilde{Z}^1, \tilde{Z}^2)$ be a two-type BBM, $\tilde{P}$ be the corresponding probability, and $\tilde{f}^{(1)}(s), \tilde{f}^{(2)}(s)$ be the p.g.f.’s for the offspring of particles of type 1 and type 2 respectively, such that the p.g.f.’s for every $0 \leq s \leq 1$ and every $0 \leq t \leq 1$ satisfy

$$\tilde{f}^{(1)}(s, t) = f^{(1)}(s, t) \quad \quad \quad (35)$$

$$\tilde{f}^{(2)}(s, t) = \sum_{j=0}^{\infty} \left( \sum_{i=0}^{\infty} \tilde{p}^{(1)}(j, i) \right) t^j, \quad \quad (36)$$
which says that a particle of type 1 of $\bar{Z}$ and that of $Z$ behave exactly alike, whereas a particle of type 2 of $\bar{Z}$ produces no offspring of type 1 similar to $Z$, but produces offspring of her own type just as a particle of type 1 of $Z$ produces offspring of type 1, with mean $\mu > 1$. The mean matrix for this modified process is

$$A(t) := \begin{pmatrix} e^{\beta tm} & \frac{g(m-k)}{\beta t}e^{\beta tm} \\ 0 & e^{\beta tm} \end{pmatrix}.$$ 

Define similarly as before the range of $Z^1$ as $R^1$, and the range of $Z^2$ as $R^2$. Further define

$$M^1_t := \inf \left\{ r > 0 : R^1(t) \subseteq B_r(0) \right\},$$

$$M^2_t := \inf \left\{ r > 0 : R^2(t) \subseteq B_r(0) \right\}.$$ 

By (35) and (36), and since $(f^*')'(1) - 1 = m$ and $f^*(0) = 0$, Proposition 1 implies that for every $\varepsilon > 0$

$$P \left( \frac{M^1_t}{t} > \sqrt{2\beta m + \varepsilon} \right) = P \left( \frac{M^1_t}{t} > \sqrt{2\beta m + \varepsilon} \right) \to 0 \quad \text{as} \quad t \to \infty.$$ 

Furthermore, since particles of type 1 of $\bar{Z}$ behave exactly the same way as those of $Z$, and particles of type 2 of $\bar{Z}$ produce offspring according to $f^*$ while particles of type 2 of $Z$ produce offspring according to $\tilde{f}$, by Lemma 5 and Lemma 6, to complete the proof it suffices to show that

$$P \left( \frac{M^2_t}{t} > \sqrt{2\beta m + \varepsilon} \right) \to 0 \quad \text{as} \quad t \to \infty \quad \text{for any} \quad \varepsilon > 0.$$ 

We proceed as in the proof of Proposition 1. Let $B = B_{(\sqrt{2\beta m + \varepsilon})t}(0)$, and $\delta > 0$ satisfy (27) as before. Define $\psi_B = \inf \left\{ t \geq 0 : |\text{supp}(Z^2(t)) \cap B^c| \geq 1 \right\}$, and estimate as before,

$$P \left( \frac{M^2_t}{t} > \sqrt{2\beta m + \varepsilon} \right) \leq P \left( |Z^2(t)| > |\psi_B| \right) + P \left( \psi_B \leq t \right) \leq P \left( |Z^2(t)| \leq |\psi_B| \right).$$ 

By the fact that $E[|Z^2(t)| | Z(0) = (1,0)] = \frac{q}{\beta t}(m-k)\beta t e^{\beta tm}$ and the Markov inequality, the first term on the right-hand side tends to zero. The second term on the right-hand side tends to zero by an extension of Lemma 3. Note that since each particle of type 2 of $\bar{Z}$ produces at least one particle of its own type (i.e., $\tilde{f}^2(s,0) = 0$), we may extend Lemma 3 to apply to particles of type 2 of $\bar{Z}$. This completes the proof of Theorem 1.
5. Proof of Theorem 2

Part 3 of the theorem follows since $\tau < \infty$ almost surely for a non-supercritical process. For a detailed proof of part 3, see [16, Theorem 1]. Since $\lambda(0) = 0$ gives $\alpha = 1$ and $P(E^c) = 1$, conditioning the process on $E^c$ is the same as not conditioning it when $\lambda(0) = 0$. Hence, we may extend part 2 to cover for the case $\lambda(0) = 0$, and prove the first two parts of the theorem together. Note that the key ingredients in the proof are results concerning the decomposition of supercritical GW-processes and the speed of BBM, namely Lemma 4 and Theorem 1.

5.1. Proof of the lower bound

Let $m > 0$. Fix $\beta, d, f$ and $\eta, c$. The scenario below yields the desired lower bound for the annealed survival probability conditioned on non-extinction, i.e., $(\mathbb{E} \times P)(T > t|E^c)$.

1. Recall that the first particle must be of infinite line of descent once the process is conditioned on non-extinction. Suppress the branching so that there is precisely one skeleton particle in the system up to time $\eta t$. This is equivalent to requiring that whenever a skeleton particle branches within the period $[0, \eta t]$, it gives precisely one offspring of its own kind. Therefore, by the fourth part of Lemma 4, this event has probability $\exp[-\beta \alpha \eta t]$.

2. For $d \geq 2$, empty the two-sided cylinder (“tube”) $T_t$, defined by

$$T_t = \left\{ x = (x_1, \ldots, x_d) \in \mathbb{R}^d : |x_1| \leq kt + h(t), \sqrt{x_2^2 + \ldots + x_d^2} \leq r(t) + h(t) \right\},$$

where $k > c$, and $t \mapsto r(t)$ and $t \mapsto h(t)$ are non-negative mappings that are picked such that $\lim_{t \to \infty} r(t)/t = 0$, $\lim_{t \to \infty} r(t) = \infty$ and $\lim_{t \to \infty} h(t)/t = 0$, $\lim_{t \to \infty} h(t) = \infty$. By the upper bounds above on $r(t)$ and $h(t)$, Lemma 1 gives that the probability to empty $T_t$ is $\exp[o(t)]$ for $d \geq 2$. For $d = 1$, empty the line $T^1_t := \{ x_1 \in \mathbb{R} : |x_1| \leq 2h(t) \}$, which again costs a probability of $\exp[o(t)]$.

3. For $d \geq 2$, move the single skeleton particle (see the remark after Lemma 4) during the period $[0, \eta t]$ so that it is at a specific site at distance $ct + o(t)$ from the origin at time $\eta t$ (assume without loss of generality that the specific site has first coordinate $ct + o(t)$ and all
other coordinates zero), and confine it to the smaller tube $\hat{T}_t$, defined by

$$\hat{T}_t = \left\{ x = (x_1, \ldots, x_d) \in \mathbb{R}^d : |x_1| \leq kt, \sqrt{x_2^2 + \ldots + x_d^2} \leq r(t) \right\},$$

up to time $\eta t$.

Let $P_0$ be the $d$-dimensional Wiener measure, $W^1$ be the projection of the $d$-dimensional Brownian motion onto the first coordinate axis, and decompose the Brownian motion into an independent sum $W = W^1 + W^{d-1}$, where $W^{d-1}$ is hence implicitly defined. Define the events

$$A_t = \left\{ ct \leq |W^1_{\eta t}| \leq ct + o(t) \right\},$$
$$B_t = \left\{ |W^1_s| \leq kt \quad \forall \quad 0 \leq s \leq \eta t \right\},$$
$$C_t = \left\{ |W^{d-1}_s| \leq r(t) \quad \forall \quad 0 \leq s \leq \eta t \right\}.$$

Note that the event $A_t \cap B_t \cap C_t$ is exactly the desired scenario for this part. The formula for the transition density for $d$-dimensional Brownian motion gives

$$P_0(A_t) = \exp\left[ -\frac{c^2 t}{2\eta} + o(t) \right],$$
$$P_0(B_t) = \exp\left[ -\frac{k^2 t}{2\eta} + o(t) \right],$$

so it follows that

$$\exp\left[ -\frac{c^2 t}{2\eta} + o(t) \right] = P_0(A_t) \geq P_0(A_t \cap B_t) \geq P_0(A_t) - P_0(B_t^c)$$
$$= \exp\left[ -\frac{c^2 t}{2\eta} + o(t) \right] - \exp\left[ -\frac{k^2 t}{2\eta} + o(t) \right]$$
$$= \exp\left[ -\frac{c^2 t}{2\eta} + o(t) \right],$$

which yields

$$P_0(A_t \cap B_t) = \exp\left[ -\frac{c^2 t}{2\eta} + o(t) \right]. \quad (37)$$

Since $r(t) \to \infty$, we have $P_0(C_t) = \exp[o(t)]$. Combining this with (37) and using the independence of $W^1$ and $W^{d-1}$, we find

$$P_0(A_t \cap B_t \cap C_t) = \exp\left[ -\frac{c^2 t}{2\eta} + o(t) \right].$$

For $d = 1$, note that the function in (5) is minimized when $c = 0$ (see the proof of Theorem 3). Confine the single skeleton particle to the one-dimensional tube $T^1_t := \{x_1 \in \mathbb{R} : |x_1| \leq h(t) \}$ up to time $\eta t$.

Since the length of the tube tends to infinity as $t$ tends to infinity, the probability of this event is $\exp[o(t)]$.
4. Confine all the doomed particles that are created within the period 
\([0, \eta t]\) to the larger tubes \(T_t\) and \(T^1_t\) for \(d \geq 2\) and \(d = 1\), respectively, 
up to time \(t\). Since the number of occurrences of branching up to time 
\(t\) along a single ancestral line is a Poisson process with mean \(\beta t\), and a 
skeleton particle on average produces \((\mu - \kappa)q/(1-q)\) doomed particles 
every time it branches (see the proof of Theorem 1), it follows that at 
most \(\lfloor \beta \eta t (\mu - \kappa)q/(1-q) \rfloor \) doomed particles are produced along the single 
skeletal line up to time \(\eta t\) with probability \(\exp\left[ o(\eta t) \right] \). Let the radii 
of the subtrees initiated by these \(\lfloor \beta \eta t (\mu - \kappa)q/(1-q) \rfloor =: n(t)\) doomed 
directed particles be \(\rho_1, \rho_2, \ldots, \rho_n(t)\). If \(K(t) := \max \{ \rho_1, \rho_2, \ldots, \rho_n(t) \} \), then 
\[ P(K(t) < h(t)) = P(\rho_1 < h(t) )^{n(t)}. \] (38)

Now, since a doomed subtree is almost surely finite, \(\lim_{t \to \infty} h(t) = \infty\) implies that 
\(P(\rho_1 < h(t)) \to 1\), which in turn implies that the probability in (38) is \(\exp\left[ o(\eta t) \right]\). We have already confined the single 
skeleton particle to the smaller tube \(\hat{T}_t\) for \(d \geq 2\) (\(\hat{T}^1_t\) for \(d = 1\)). Since 
each doomed particle that is produced along the single skeletal line 
within the period \([0, \eta t]\) must be produced within the smaller tube 
\(\hat{T}_t(\hat{T}^1_t)\), and the dimensions of the larger tube \(T_t(T^1_t)\) all exceed the 
corresponding dimensions of the smaller tube by \(h(t)\), we conclude 
that all of the doomed particles that are created within the period 
\([0, \eta t]\) stay inside the larger tube with probability \(\exp\left[ o(\eta t) \right]\).

5. Empty a \(\sqrt{2\beta m(1 - \eta)}t\)-ball around the position of the single skeleton 
particle at time \(\eta t\). By Lemma \[\[\text{this event has probability} \]
\exp \left[ -lg_d(\sqrt{2\beta m(1 - \eta)}t, c) + o(t) \right]. \]

6. Require the branching system initiated by the single skeleton particle 
present at time \(\eta t\) to stay inside the \(\sqrt{2\beta m(1 - \eta)}t\)-ball during the remaining 
time \((1 - \eta)t\). When \(m > 0\), by Proposition \[\[\text{and Theorem} \]
the probability of this event conditioned on non-extinction is \(\exp[ o(\eta t)]\).

Since the motion, branching and trap formation mechanisms are independent of each other, minimizing the cost of all these events over the parameters \(\eta\) and \(c\) gives us with the desired lower estimate for \((\mathbb{E} \times P)(T > t\mathcal{E}^c)\). 
Note that the survival scenario described above is composed of three large

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1 By radius we mean the radius of the smallest ball centered at the root of the subtree, containing the (finite) range of the subtree.
deviation events: suppressing the branching for a linear time, moving the Brownian particle to a linear distance, and emptying a ball with linear radius. The remaining three components, namely parts two, four and six are not large deviation events and do not give exponential costs.

5.2. Proof of the upper bound

Fix $\beta, d, f$. Let $0 < \varepsilon < 1$. Recall that $|Z(t)|$ is the number of particles in the system at time $t$, $|Z^\ast(t)|$ of which are skeleton particles. For $t > 1$, define

$$\eta_t = \sup \left\{ \eta \in [0, 1] : |Z^\ast(\eta_t)| \leq \lfloor t^{d+\varepsilon} \rfloor \right\}.$$  

(39)

Before we proceed, we prove the following lemma:

**Lemma 8.**

$$P \left( \frac{i}{n} \leq \eta_t | \mathcal{E}^c \right) = \exp \left[ -\beta \alpha \frac{i}{n} t + o(t) \right].$$

**Proof.** Observe that $\left\{ \frac{i}{n} \leq \eta_t \right\} = \left\{ |Z^\ast(i/n t)| \leq \lfloor t^{d+\varepsilon} \rfloor \right\}$. By part 4 of Lemma 4, we know that the process $|Z^\ast|$ conditioned on non-extinction of $|Z|$ is equal in law to a continuous time branching process with rate $\beta(1 - f'(q)) = \beta \alpha$ and offspring distribution $\bar{\lambda}$, where $\bar{\lambda}(0) = \bar{\lambda}(1) = 0$. On the event $\mathcal{E}^c$, since the first particle is of infinite line of descent and $t > 1$, the inequality $P \left( \frac{i}{n} \leq \eta_t | \mathcal{E}^c \right) \geq \exp \left[ -\beta \alpha \frac{i}{n} t + o(t) \right]$ follows trivially.

Now consider a strictly dyadic branching Brownian motion $\tilde{Z}$. We have $P(|\tilde{Z}(t)| = k) = e^{-\beta t} (1 - e^{-\beta t})^{k-1}$ for every $k \in \mathbb{N}$ and for every $t \geq 0$, see [10]. It follows that $P(|\tilde{Z}(t)| \geq k) = (1 - e^{-\beta t})^k$. Using the binomial expansion $(1 - x)^n = 1 - nx + \binom{n}{2} x^2 - \ldots + (-1)^n x^n$, we obtain

$$P(|\tilde{Z}(t)| \leq \lfloor t^{d+\varepsilon} \rfloor) = 1 - (1 - e^{-\beta t})^{\lfloor t^{d+\varepsilon} \rfloor}$$

$$= e^{-\beta t} \left( t^{d+\varepsilon} \right) - \left( \frac{t^{d+\varepsilon}}{2} \right) e^{-\beta t} + \ldots + (-1)^{\lfloor t^{d+\varepsilon} \rfloor - 1} e^{-\beta t (t^{d+\varepsilon} - 1)}$$

$$\leq e^{-\beta t \lfloor t^{d+\varepsilon} \rfloor} = \exp[-\beta t + o(t)].$$

The result follows by comparison with the strictly dyadic case, since $\bar{\lambda}(0) = \bar{\lambda}(1) = 0$.  

Henceforth, all probabilities written should be understood as conditioned
on $E^c$. Now, for every $n \in \mathbb{N}$,

$$(E \times P)(T > t)$$

$$= \sum_{i=0}^{n-1} (E \times P) \left( \{ T > t \} \cap \left\{ \frac{i}{n} \leq \eta_t < \frac{i+1}{n} \right\} \right) + (E \times P) \left( \{ T > t \} \cap \{ \eta_t = 1 \} \right)$$

$$\leq \sum_{i=0}^{n-1} \exp \left[ -\beta \alpha \frac{i}{n} t + o(t) \right] (E \times P) \left( \{ T > t \} \cap \{ \eta_t = 1 \} \right) + \exp[-\beta t + o(t)],$$

(40)

where we use Lemma 8, and introduce the conditional probabilities

$$P_t^{(i,n)}(\cdot) = P \left( \cdot \left| \frac{i}{n} \leq \eta_t < \frac{i+1}{n} \right\} \right), \quad i = 0, 1, \ldots, n - 1.$$

Recall that $\{ \eta_t < (i + 1)/n \} = \{|Z^*(t(i + 1)/n)| > |t^{d+\varepsilon}|\}$. Let $A_t^{(i,n)}$, $i = 0, 1, \ldots, n - 1$ to be the event that among the $|Z^*(t(i + 1)/n)|$ skeleton particles alive at time $t(i + 1)/n$, there are $\leq |t^{d+\varepsilon}|$ particles such that the ball with radius

$$\rho_t^{(i,n)} := (1 - \varepsilon) \sqrt{2 \beta m (1 - \frac{i+1}{n}) t}$$

(41)

around the particle is non-empty (i.e., contains a point from $\omega$). Estimate

$$(E \times P_t^{(i,n)})(T > t) \leq (E \times P_t^{(i,n)})(A_t^{(i,n)})$$

$$+ (E \times P_t^{(i,n)})(T > t \mid [A_t^{(i,n)}]^c).$$

(42)

On the event $[A_t^{(i,n)}]^c$ there are $> |t^{d+\varepsilon}|$ balls containing a trap, and by Proposition 1 and (41), the BBM emanating from the center of each ball exists this ball in the remaining time $(1 - (i + 1)/n)t$ with a probability tending to 1 as $t \to \infty$. By radial symmetry, the BBM hits a trap inside this ball with a probability $C_1/|\rho_t^{(i,n)}|^{d-1}$ when exiting, where $C_1 > 0$ is a constant depending on the geometry. Hence, the second term on the right-hand side of (42) is bounded above by

$$\left[ 1 - \frac{C_1}{|\rho_t^{(i,n)}|^{d-1}} \right] |t^{d+\varepsilon}| \leq \exp[-C_2 t^{1+d}]$$

uniformly in all parameters, where $C_2 > 0$ is another constant. Hence, the second term on the right-hand side of (42) is superexponentially small (SES).
Now consider the first term on the right-hand side of (42). Since $g_d(r, bt) = t g_d(r, b)$ for $r, b > 0$, the cost of clearing a ball with radius linear in $t$ is exponentially small in $t$ as $t \to \infty$. Then, for large $t$, we have

\[
(\mathbb{E} \times P_t^{(i,n)})(A_t^{(i,n)} | |Z^*(t(i+1)/n)| = [t^{d+\epsilon}]) \\
\geq (\mathbb{E} \times P_t^{(i,n)})(A_t^{(i,n)} | |Z^*(t(i+1)/n)| = [t^{d+\epsilon}] + (j + 1))
\]

for every $j \in \{1, 2, \ldots\}$. It follows that

\[
(\mathbb{E} \times P_t^{(i,n)})(A_t^{(i,n)} | |Z^*(t(i+1)/n)| = [t^{d+\epsilon}]) + 1).
\]

We continue to estimate from above as follows. Each skeleton particle alive at time $t(i+1)/n$ is at a random point, whose spatial distribution is identical to that of $W(t(i+1)/n)$, where $W$ denotes the standard Brownian motion. Let $x_0$ be the center of the empty ball at time $t(i+1)/n$ that is closest to the origin. Let $0 \leq c < \infty, \delta > 0$, and define $B_j, j = 1, \ldots, [t^{d+\epsilon}] + 1$ to be the event that $j$th skeleton particle at time $t(i+1)/n$ is at a distance $ct \leq r < (c + \delta)t$ from the origin and that the $\rho_t^{(i,n)}$-ball around it is empty. Then,

\[
(\mathbb{E} \times P_t^{(i,n)})(A_t^{(i,n)} \cap \{ct \leq |x_0| < (c + \delta)t\} | |Z^*(t(i+1)/n)| = [t^{d+\epsilon}]) + 1) \\
\leq (\mathbb{E} \times P_t^{(i,n)})(B_1 \cup \ldots \cup B_{[t^{d+\epsilon}] + 1}) \\
\leq ([t^{d+\epsilon}] + 1)(\mathbb{E} \times P_t^{(i,n)})(B_1) \\
= ([t^{d+\epsilon}] + 1) \exp \left[-\frac{c^2}{2(i+1)/n}t + o(t)\right] \\
\times \exp \left[-lg_d \left(1 - \epsilon \right) \sqrt{2\beta(m-1)(1 - \frac{i+1}{n}, c) t + O(\delta)t + o(t)}\right],
\]

where we have used the independence of BBM and trap mechanisms, and Lemma $\square$ in passing to the last equality. Indeed, on the event $A_t^{(i,n)}$ conditioned on $|Z^*(t(i+1)/n)| = [t^{d+\epsilon}] + 1$, there is at least 1 skeleton particle
with an empty ball around it. From \cite{42}, \cite{43} and \cite{11}, we obtain

\[(\mathbb{E} \times P_t^{(i,n)})(T > t)\]
\[\leq \sum_{j=0}^{n-1} \left( \mathbb{E} \times P_t^{(i,n)} \right) \left( \left\{ \frac{j}{n} \sqrt{2 \beta t} \leq |x_0| < \frac{j+1}{n} \sqrt{2 \beta t} \right\} \cap A_t^{(i,n)} \right) \]
\[+ (\mathbb{E} \times P_t^{(i,n)}) \left( \left\{ |x_0| \geq \sqrt{2 \beta t} \right\} \cap A_t^{(i,n)} \right) + \text{SES} \]
\[\leq \sum_{j=0}^{n-1} \left( \lfloor t^{d+\varepsilon} \rfloor + 1 \right) \exp \left[ -\frac{\beta j^2/n^2}{(i+1)/n} t + o(t) \right] \]
\[\times \exp \left[ -l_{gd} \left( (1 - \varepsilon) \sqrt{2 \beta m(1 - \frac{i+1}{n})}, \frac{j}{n} \sqrt{2 \beta} \right) t + O(1/n)t + o(t) \right] \]
\[+ \exp[-\beta t + o(t)] + \text{SES}. \quad (45)\]

Here, the SES comes from the second term on the right-hand side of \cite{42}. Also, in passing to the last inequality, we have used that the probability of the event \{\{|x_0| \geq \sqrt{2 \beta t}\}\} is bounded above by the probability that a single Brownian particle is at a distance \geq \sqrt{2 \beta t} at time \(t\), which is \exp[-\beta t + o(t)].

Substituting (45) into (40), and optimizing over \(i, j \in \{0, 1, \ldots, n-1\}\) gives

\[\limsup_{t \to \infty} \frac{1}{t} \log(\mathbb{E} \times P)(T > t) \leq -\min_{i, j \in \{0, 1, \ldots, n-1\}} \left\{ \frac{\beta \alpha_i}{n} + \frac{\beta j^2/n^2}{(i+1)/n} + l_{gd} \left( (1 - \varepsilon) \sqrt{2 \beta m(1 - \frac{i+1}{n})}, \frac{j}{n} \sqrt{2 \beta} \right) \right\}. \]

Now let \(\eta = i/n\), \(c = j/n\sqrt{2 \beta}\), let \(n \to \infty\), use the continuity of the functional form from which the minimum is taken, and finally let \(\varepsilon \to 0\) to obtain the desired upper bound,

\[\limsup_{t \to \infty} \frac{1}{t} \log(\mathbb{E} \times P)(T > t) \leq -I(l, f, \beta, d). \]

Note that \(c > \sqrt{2 \beta}\) cannot minimize

\[\left\{ \frac{\beta \alpha \eta + c^2}{2 \eta} + l_{gd}(\sqrt{2 \beta m(1 - \eta)}, c) \right\} \quad (46)\]

since this makes (46) greater than \(\beta\), which can be improved by setting \(c = 0\) and \(\eta = 1\).
6. Proof of Theorem 3

Let
\[ G_d(\eta, c) = \beta \alpha \eta + \frac{c^2}{2\eta} + \log g_d(\sqrt{2\beta/m}(1 - \eta), c) \] (47)
so that
\[ I(l, f, \beta, d) = \min_{\eta \in [0, 1], c \in [0, \sqrt{2\beta}]} G_d(\eta, c). \] (48)

To see the existence of the minimizers \( \eta^*, c^* \) of (48), note that \( G_d \) is lower semicontinuous on the compact subset \([0, 1] \times [0, \sqrt{2\beta}]\) of \( \mathbb{R}^2 \) since
\[ \lim_{(\eta, c) \to (0, 0)} \beta \alpha \eta + \log g_d(\sqrt{2\beta/m}(1 - \eta), c) = \log \sqrt{2\beta m} \]
and \( \frac{c^2}{2\eta} \geq 0. \)

We refer the reader to [7] for the proof of their uniqueness when \( l \neq l_{cr} \).

Consider \( d = 1 \). Since \( g_1(r, b) = 2r \), the minimum over \( c \) in (48) is taken at \( c^* = 0 \), so that (48) reduces to
\[ I(l, f, \beta, d) = \min_{\eta \in [0, 1]} \{ \beta \alpha \eta + 2l \sqrt{2\beta/m}(1 - \eta) \} \]
\[ = \min_{\eta \in [0, 1]} \{ \eta(\beta \alpha - 2l \sqrt{2\beta/m}) + 2l \sqrt{2\beta/m} \}. \]

This proves (6) and (7), and identifies \( l_{cr} \) as the solution to \( \beta \alpha = 2l \sqrt{2\beta/m} \).

Now consider \( d \geq 2 \). We have
\[ G_d(\eta, c) - G_d(0, 0) = \beta \alpha \eta + \frac{c^2}{2\eta} + l[g_d(\sqrt{2\beta/m}(1 - \eta), c) - g_d(\sqrt{2\beta/m}, 0)], \] (49)
where the last term on the right-hand side is less than or equal to zero, with equality if and only if \( (\eta, c) = (0, 0) \) (this follows from the definition of \( g_d \)). Suppose that \( (\eta, c) = (0, 0) \) is a minimizer when \( l = l_0 \). Then, by uniqueness of minimizers, the right-hand side of (49) is strictly positive for all \( (\eta, c) \neq (0, 0) \) when \( l = l_0 \). As the last term in the right-hand side of (49) is strictly negative for all \( (\eta, c) \neq (0, 0) \), it follows that for all \( l < l_0 \), the right-hand side of (49) is zero when \( (\eta, c) = (0, 0) \) and strictly positive otherwise. Therefore, there must exist \( l_{cr} \in (0, \infty) \) such that
1. \( (\eta, c) = (0, 0) \) is the minimizer when \( l < l_{cr} \),
2. \( (\eta, c) = (0, 0) \) is not the minimizer when \( l > l_{cr} \).
Henceforth, we will take this as the definition of $l_{cr}$ for $d \geq 2$. This proves the first parts of (8) and (9). Note that $l_{cr} \in (0, \infty)$, because the last term on the right-hand side of (49) decreases without bound as $l \to \infty$ and tends to zero as $l \to 0$ for fixed $(\eta, c) \neq (0, 0)$.

Now let $d \geq 2$ and $l > l_{cr}$. We know that $(\eta, c) = (0, 0)$ is not a minimizer. The combination $\eta^* = 0$, $c^* > 0$ is not possible because $G(0, c) = \infty$ for all $c > 0$. The combination $\eta^* > 0$, $c^* = 0$ is ruled out as well because $G(\eta, 0)$ takes its minimum either at $\eta = 0$ or $\eta = 1$, so $\eta^* > 0$ would imply that $\eta^* = 1$. However, $\eta^* = 1$ can be excluded via [7, Lemma 4.1]. Also, $c^* = \sqrt{2\beta}$ is not possible since this yields $G_d > \beta$, whereas $(\eta, c) = (1, 0)$ is not a minimizer yet yields $G_d \leq \beta$ since $\alpha \in (0, 1]$. Hence, we conclude that the minimizers for $d \geq 2$ and $l > l_{cr}$ appear in the interior of $[0, 1] \times [0, \sqrt{2\beta}]$. This proves the second parts of (8) and (9).

In the rest of the proof, we find $l_{cr}$ for $d \geq 2$. For $R \geq 0$, let

$$g_d(R) = \int_{B_R(0)} \frac{dx}{|x + e|^d}. \tag{50}$$

Then, we may write (47) as

$$G_d(\eta, c) = \beta \alpha \eta + \frac{c^2}{2\eta} + l \cdot g_d \left( \frac{\sqrt{2\beta m} (1 - \eta)}{c} \right). \tag{51}$$

The stationary points are the solutions of the equations $\frac{\partial G_d}{\partial \eta} = 0$ and $\frac{\partial G_d}{\partial c} = 0$, which yield

$$0 = \beta \alpha - \beta mv^2 - l \sqrt{2\beta m} g_d'(u)$$

$$0 = \sqrt{2\beta m}v + l[g_d(u) - ug_d'(u)] \tag{52}$$

upon setting

$$u = \frac{\sqrt{2\beta m} (1 - \eta)}{c}, \quad v = \frac{c}{\sqrt{2\beta m}}. \tag{53}$$

Eliminating $g_d'$ in (52) gives

$$\frac{l}{\sqrt{2\beta m}} g_d(u) = -v + \frac{1}{2m} u(\alpha - mv^2). \tag{54}$$

It follows by (51), (53) and (54) that for $d \geq 2$ and $l > l_{cr}$,

$$I(l, f, \beta, d) = G_d(u^*, v^*) = \beta (\alpha - mv^*). \tag{55}$$

Since the minimizer $(u^*, v^*)$ must satisfy (51) (note that there may be more than one pair $(u, v)$ that satisfies (51)), we conclude by (55) that $v^*$ is the maximal value of $v > 0$ on the curve in the $(u, v)$-plane given by (54).

Recall the following lemma from [7], where a proof is given as well.
Lemma 9. \( R \mapsto g'_d(R) \) is strictly increasing on \((0, 1)\), infinite at 1, and strictly decreasing on \((1, \infty)\).

Using (3), we may write (54) as
\[
g_d(u) \ll 2 s_d m l^*_cr = -v + \frac{1}{2m} u(\alpha - mv^2).
\] (56)

Since \( g_d(u) \sim s_d u \) as \( u \to \infty \) by (50), we obtain from Lemma 9 that the left-hand side of (56) is strictly convex on \((0, 1)\), has infinite slope at 1, is strictly concave on \((1, \infty)\), and has limiting derivative \( l/(2ml^*_cr) \) (see Figure 1).

Figure 1. Qualitative plot of: (1) \( u \mapsto \frac{g_d(u)l}{2 s_d m l^*_cr} \), (2) \( u \mapsto -v + \frac{1}{2m} u(\alpha - mv^2) \).
The dotted line is \( u \mapsto \frac{1}{2m} l^*_cr \).

Firstly, see that \( l_{cr} \leq \alpha l^*_cr \). Indeed, assume the contrary. Then, there exists \( \bar{l} \in (\alpha l^*_cr, l_{cr}) \), which implies that \((\eta, c) = (0, 0)\) is the unique minimizer for \( G_d \) when \( l = \bar{l} \). However, \( G_d(0, 0) = \beta \bar{l}/l^*_cr > \beta \alpha = G_d(1, 0) \), which contradicts \((\eta, c) = (0, 0)\) being the minimizer. This implies that \( l_{cr} \leq \alpha l^*_cr \).

We claim that \( l_{cr} \) can be identified by the following formula
\[
l_0 := \sup \left\{ l > 0 : \frac{l}{l_{cr}*} \frac{1}{2 s_d m} g_d(u) > -\sqrt{\frac{1}{m} \left( \alpha - \frac{l}{l_{cr}*} \right) + \frac{1}{2m} \frac{l}{l_{cr}*} u} \quad \forall u \in (0, \infty) \right\}.
\] (57)

In order to prove our claim, i.e., \( l_0 = l_{cr} \), we need to show that \((\eta, c) = (0, 0)\) is the minimizer for \( G_d \) when \( l < l_0 \), and \((\eta, c) = (0, 0)\) is not the
minimizer when \( l > l_0 \). Equivalently, as \( G_d(0,0) = \beta l/l_{cr}^* \), we need to show via \((55)\) that for \( l > l_0 \) there exists a \( v > \sqrt{1/m(\alpha - \beta l/l_{cr}^*)} \) such that \( v \) satisfies \((56)\) for some \( u > 0 \), and that for \( l < l_0 \) there is no such \( v \). As we know that \( l_{cr} > 0 \) and \((\eta, c) = (0, 0)\) is the unique minimizer when \( l < l_{cr} \), there must exist an \( l_1 > 0 \) such that the inequality in \((57)\) holds for all \( u \in (0, \infty) \) when \( l = l_1 \), hence \( 0 < l_1 < l_0 \). Now take any \( 0 < l_2 < l_1 \), and see that the inequality in \((57)\) holds for all \( u \in (0, \infty) \) also for \( l = l_2 \). This implies that if \( l < l_0 \), then the curve on the left-hand side of \((56)\) and the line on the right-hand side do not touch when \( v > \sqrt{1/m(\alpha - \beta l/l_{cr}^*)} \). As both terms on the right-hand side of \((56)\) decrease as \( v \) increases and the left-hand side of \((56)\) does not depend on \( v \), the curve on the left-hand side of \((56)\) and the line on the right-hand side do not touch for \( v > \sqrt{1/m(\alpha - \beta l/l_{cr}^*)} \) as well when \( l < l_0 \).

Now consider \( l > l_0 \). By definition of \( l_0 \), there exists \( u \in (0, \infty) \), say \( \tilde{u} \), such that

\[
\frac{l_0}{l_{cr}^*} \frac{1}{2s_d m} g_d(u) = -\sqrt{\frac{1}{m} \left( \alpha - \frac{l_0}{l_{cr}^*} \right)} \frac{1}{2} \frac{l_0}{l_{cr}^*} u.
\]

Note that since \( g_d(u) < s_d u \) for all \( u \in (0, \infty) \), \((58)\) implies that we have the strict inequality \( l_0 < \alpha l_{cr}^* \). Now take any \( l_1 \in (l_0, \alpha l_{cr}^*) \), and observe that at \( u = \tilde{u} \), the right-hand side of \((58)\) becomes greater than the left-hand side if we replace \( l_0 \) by \( l_1 \). Also, since \( g_d(u) \sim s_d u \), the left-hand side of \((58)\) must be greater than the right-hand side for large \( u \) if we replace \( l_0 \) by \( l_1 \). As the functions on both sides of \((58)\) are continuous with respect to \( u \), it follows by the intermediate value theorem that there exists \( u \in (0, \infty) \), say \( \tilde{u} \), such that \((58)\) holds, with \( l_0 \) replaced by \( l_1 \). This in turn implies that \( G_d(u, v) = G_d(\tilde{u}, \sqrt{1/m(\alpha - l_1/l_{cr}^*)}) = \beta l_1/l_{cr}^* \), but since the minimizer is unique, this shows that \((\eta, c) = (0, 0)\) is not the minimizer for \( l_0 < l_1 < l_{cr}^* \). As we have shown that \((\eta, c) = (0, 0)\) is not the minimizer when \( l \in (l_0, l_{cr}^*) \), we conclude by definition of \( l_{cr} \) that \((\eta, c) = (0, 0)\) is not the minimizer for \( G_d \) for all \( l > l_0 \). Hence, \( l_0 = l_{cr} \).

Now put

\[
g_d(u) = s_d u - g_d(u) \quad M_d = \frac{1}{2s_d m} \max_{u \in (0, \infty)} \hat{g}_d.
\]

Then, \((57)\) reads

\[
l_{cr} = \sup \left\{ l > 0 : \sqrt{\frac{1}{m} \left( \alpha - \frac{l}{l_{cr}^*} \right)} > \frac{1}{l_{cr}^*} M_d \right\}.
\]

\[
(59)
\]
This implies that
\[
\sqrt{\frac{1}{m} \left( \frac{\alpha - l_{cr}}{l_{cr}^*} \right)} = \frac{l_{cr}}{l_{cr}^*} M_d,
\]
which completes the proof of (10) and (11).

We finally show that \( u^* \in (0, 1) \) when \( l > l_{cr} \). Note that for a fixed \( v \), if the line on the right-hand side of (56) cuts the curve on the left-hand side at more than one point, then by (55), this \( v \) cannot be \( v^* \) since the minimizer is unique. Hence, we are looking for a \( v \) value such that (56) is satisfied for exactly one \( u \) value, and this pair is \( (u^*, v^*) \). This implies that when \( v = v^* \), the line on the right-hand side of (56) is tangent to the curve on the left-hand side at \( u = u^* \). Since \( v^* > \sqrt{1/m(\alpha - l/l_{cr}^*)} \), the slope of the line on the right-hand side of (56) is < \( 1/(2m)(l/l_{cr}^*) \). Then, since the left-hand side of (56) has infinite slope at \( u = 1 \), is concave on \((1, \infty)\), and decreases asymptotically to \( 1/(2m)(l/l_{cr}^*) \), it follows that \( u^* \in (0, 1) \). By (53), we conclude that \( c^* > \sqrt{2/3m}(1 - \eta^*) \). This completes the proof of Theorem 3.

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