Primordial Black Holes from the QCD Transition?

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(March 21, 2022)

Abstract

Can a violent process like sudden reheating after supercooling at the onset of a first-order QCD transition improve the possibility of primordial black hole formation? Underdensities reheat earlier than overdensities, there is a short period of huge pressure differences, hence fluid acceleration. Density perturbations on scales far below the Hubble radius $\lambda \ll R_H$ get an amplification which grows quadratically in wavenumber, the amplifications at the horizon scale are small. Primordial black hole formation cannot be sufficiently amplified by the QCD transition unless the initial spectrum is fine tuned.

98.80.Cq, 12.38.Mh, 95.35.+d

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Typeset using REVTEX
I. INTRODUCTION AND CONCLUSION

The transition from a quark-gluon plasma to a hadron gas in the early universe took place at a temperature $T_\star \sim 150$ MeV. Recent results of lattice QCD indicate a first-order phase transition for the physical values of the $u$, $d$, $s$ quark masses [1]. The mass inside the Hubble horizon at the QCD transition is $\sim 1 M_\odot$. Crawford and Schramm [2] first proposed that black hole formation at the QCD transition could account for dark matter today. Recently Jedamzik [3] proposed to identify such primordial black holes (PBH) with the MACHO’s (massive compact halo objects) observed by microlensing [4]. He pointed out that the formation of PBH’s should be particularly efficient during the QCD transition due to a significant decrease in the effective sound speed during the transition. Schmid, Schwarz and Widerin [5,6] showed that during a first-order QCD transition the sound velocity $c_s = (\partial p/\partial \rho)^{1/2}$ vanishes (for wavelengths much larger than the bubble separation), there are no pressure gradients, no restoring forces, and preexisting cosmological density perturbations go into free fall. They computed the amplification of linear cosmological perturbations due to the QCD transition. The amplification is very large for scales $\lambda$ far below the horizon, amplification $\propto k^n$ with $n = 1$ in the bag model, $n = 3/4$ in a fit to lattice QCD data. There is no amplification for superhorizon modes in accordance with a general theorem. Near the horizon the first peak (above the incoming spectrum of cosmological perturbations) has a height (amplification factor) of only 1.5 for the fit to the lattice QCD data and 2.0 for the bag model. The position of the first peak is at $M_{RAD} \approx 0.15 M_\odot$, which corresponds to $\lambda/R_H \approx 0.6$ at the end of the first order QCD transition. It was concluded that primordial black hole formation is unlikely, because the amplifications of perturbations due to the QCD transition are large only far below the Hubble scale.

In this paper we ask, whether a violent process like sudden reheating after supercooling at the beginning of the QCD transition could enhance the amplification of preexisting density perturbations and improve the possibility of black hole formation at the QCD transition.

The amount of supercooling $(T_\star - T_{sc})/T_\star \equiv \eta$ is proportional to the dimensionless ratio $(\sigma^{3/2}/(l T_\star^{1/2}))$, where $\sigma$ = surface tension, $l$ = latent heat, see e.g. [7]. Data are available only for quenched lattice QCD (gluons only, no quarks) [8]: $l/T_\star^4 \approx 1.40(9)$ and a very small surface tension $\sigma/T_\star^3 \approx 0.0155(16)$. This gives a very small supercooling of $\eta \approx 10^{-3}$ [7]. In view of the uncertainties in the lattice QCD determinations and the absence of data for $l$ and $\sigma$ from QCD including three physical quarks we consider the possibility of a larger surface tension and/or a smaller latent heat giving more supercooling. For illustrative purposes in our figures we used $\eta = 1/10$.

The amplification of perturbations due to sudden reheating at the beginning of a first order QCD transition occurs because reheating happens at a certain supercooling temperature $T_{sc}$. Hence underdensities reheat earlier than overdensities. (For subhorizon scales Newtonian time is applicable.) During this short period of time, there are spatial variations of pressure of $O(\eta)$ leading to huge pressure gradients (compared to the small preexisting cosmological pressure gradients) and huge fluid accelerations, hence preexisting overdensities get a very violent but very short compressional impulse. The fluid velocity effectively jumps proportional to the wave number $k$. For scales far below the Hubble radius this jump becomes the dominant contribution to the fluid velocity throughout the transition. In the bag model [9], which we shall consider here for simplicity, the resulting amplification of
density perturbations for scales far below the horizon grows as $k^2$, i.e. with one additional enhancement factor $\propto k$ due to sudden reheating. For horizon and superhorizon scales general relativity must be used.

The general relativistic time-evolution requires the choice of a foliation of space-time, and for linear perturbations this defines a gauge. The hypersurface of sudden reheating is a slice of constant energy density, and we use the corresponding gauge, the uniform density (UD) gauge, to evolve through sudden reheating. In this gauge the relevant variables are continuous at sudden reheating. To evolve before and after reheating the UD gauge is not suitable, because it is singular in the subhorizon limit, instead we use the uniform expansion (UE) gauge, which is nonsingular, and which is defined by requiring that the fundamental observers, who are at rest on a slice, have a uniform Hubble expansion. In the UE gauge the relevant variables are discontinuous at sudden reheating, and these discontinuities are obtained in a very simple way by using the gauge transformation between the UD and the UE gauges. In the superhorizon limit the state variables in UE gauge do not jump. Moreover we prove the theorem that phase transitions do not affect at all the laws of evolution for $\delta \equiv \delta \rho / \rho$ of the growing mode (i.e. growing in relative importance) in UE gauge in the limit $k_{\text{phys}}^2 \ll \{H^2, \dot{H}\}$,

$$
(\delta_{\text{UE}})_{\text{growing mode}} = \text{const} \left( \frac{k_{\text{phys}}}{H} \right)^2.
$$

(1)

This translates into the ‘conservation law’ $(\varphi_{\text{UE}})_{\text{growing mode}} = \text{const}$, where $4 \triangle \varphi_{\text{UE}} \equiv -(3)^{-1} R(\Sigma_{\text{UE}})$, by using the energy constraint. Because of this theorem the superhorizon modes during the QCD transition remain unaffected in the sense of Eq. (1). Near the horizon the first amplification peak gives a factor 2.2 for a supercooling of 10% in the bag model, the additional enhancement due to sudden reheating at the horizon scale is about 10%.

The enhancement of black hole formation at the QCD transition is discussed in the last section. For standard models of structure formation without a large tilt, the amplitudes are not big enough to produce a cosmologically relevant amount of black holes [10]. A tilted spectrum could be fine tuned to produce black holes at the QCD scale. But this tilted spectrum would need a break just below the QCD scale in order not to overproduce smaller black holes, a second fine-tuning. We conclude that the QCD transition enhances the probability of black hole formation, but for an observable amount of black holes today the preexisting spectrum would have to be fine tuned around the QCD scale, and the major effect would not be due to the QCD transition.

II. SUPERCOOLING AND SUDDEN REHEATING

The quark-gluon plasma, photons and leptons are tightly coupled via strong, electromagnetic and weak interactions, $\Gamma / H \gg 1$, and make up a single perfect (i.e. dissipationless) radiation fluid at scales $\lambda > 10^{-7} R_H$, see [7]. The pressure for perfect fluids with negligible chemical potential only depends on temperature. In the bag model [9], which we will consider for simplicity, the quark-gluon plasma (QGP) is described by $p_{\text{bag}}^{\text{QGP}} = p_{\text{ideal}}^{\text{QGP}} - B$, i.e.
\[ p_{\text{QGP}} = \frac{\pi^2}{90} s_{\text{QGP}}^* T^4 - B, \]

where \( g^* \) is the effective number of relativistic helicity states, and \( B \) is the bag constant. Above the phase transition temperature we include \( u, d \) quarks and gluons in the quark-gluon plasma and \( \gamma, e, \mu, \nu \)’s in the photon-lepton fluid. Below the phase transition we model the radiation fluid as a hadron gas (HG) of massless pions, tightly coupled to the photons and leptons.

In the cosmological expansion the temperature in the bag model drops \( \propto 1/a \), see Fig. \[.] To derive this result we use \( s = dp/dT \), which comes from the Maxwell relation for the free energy, hence \( s_{\text{QGP}} = s_{\text{ideal}} \propto T^3 \). We further use that the entropy in a comoving volume is conserved, \( s \propto 1/a^3 \), and finally obtain \( T \propto 1/a \). The energy density is given by \( \rho = Ts - p \) from the first law and homogeneity, hence \( \rho_{\text{QGP}} = \rho_{\text{ideal}} + B \). It follows that the sound velocity \( (c_s^2)_{\text{QGP}} = (\partial p/\partial T)_{\text{QGP}} = 1/3 \) and that \( p/\rho < 1/3 \). The bag constant is determined by the transition temperature \( T_s \) and \( \Delta g^* \equiv g_{\text{QGP}}^* - g_{\text{HG}}^* \) via \( p_{\text{QGP}}(T_s) = p_{\text{HG}}(T_s) \), \[ B = (\pi^2/90)\Delta g^* T_s^4 \]. Similarly the latent heat is given via \( l \equiv \Delta s, l = (2\pi^2/45)\Delta g^* T_s^4 \).

For supercooling, \( T < T_s \), the free energy density \( f = -p(T) \) for the QGP is higher than for the HG, and therefore the QGP is not the equilibrium state. However the fluid supercools in the metastable quark-gluon phase until the formation and growth of hadronic bubbles starts reheating the universe at \( T_{sc} < T_s \). We give a short overview of bubble nucleation and sudden reheating, a more detailed discussion of bubble nucleation can be found e.g. in \[\]. Assuming homogeneous bubble nucleation (no ‘dirt’) the probability to form a bubble is proportional to \( \exp(\Delta S) \), where \( \Delta S \) is the change in entropy by creating a bubble. \( \Delta S \) is determined by the work required to form a (spherical) bubble, \( -T_s \Delta S = \Delta F = (p_{\text{QGP}} - p_{\text{HG}})^\frac{4\pi}{3} R^3 + \sigma 4\pi R^2 \), where \( \sigma \) is the surface tension of a hadronic bubble. Only for large enough bubbles \( (R \geq R_{\text{crit}}) \) free energy is gained. The probability to form a hadronic bubble with critical radius per unit volume and unit time is given by

\[ I(T) = I_0 \exp \left( -\frac{\Delta F_{\text{crit}}}{T} \right), \]

with \( \Delta F_{\text{crit}} = 16\pi \sigma^3/[3(p_{\text{HG}} - p_{\text{QGP}}^2)] \). For small supercooling, \( \eta \equiv 1 - T/T_s \ll 1 \), we obtain \( (p_{\text{QGP}} - p_{\text{HG}}) \approx (T_s - T)\Delta (dp/dT) = -\eta T_s \Delta s = -\eta l \), where \( l \) is the latent heat. Therefore the probability to form a critical bubble per unit volume and unit time can be written as

\[ I \approx T_s^4 \exp \left( -A/\eta^2 \right), \]

with \( A \equiv 16\pi \sigma^3/(3l^2 T_s) \). A numerical prefactor to \( T_s^4 \) would be irrelevant for our purposes.

The surface tension \( \sigma \) and the latent heat \( l \) are the crucial parameters for \( T_{sc} \). Data are available only for quenched lattice QCD (gluons only, no quarks) \[\]: \( l/T_s^4 \approx 1.40(9) \) and a very small surface tension \( \sigma/T_s^3 \approx 0.0155(16) \). There are no values for unquenched QCD available yet. Using the results from quenched lattice QCD we find \( A = 2.9 \times 10^{-5} \).

The bubbles grow most probably by weak deflagration \[\] \[\], and the latent heat released from the bubbles is distributed into the surrounding QGP by acoustic waves and by neutrinos, whose mean free path is \( 10^{-7} R_{\text{H}} \), of the order of the bubble separation. This reheats the QGP to \( T_s \) and prohibits further bubble formation. The supercooling temperature fraction \( \eta \) can be estimated by the schematic case of one single bubble nucleated per Hubble volume and per Hubble time,
\[
\eta \approx \left[ \frac{A}{4 \ln(T_\star / H_\star)} \right]^{1/2} \approx 4 \times 10^{-4}.
\]

(5)

This rough estimate gives a final supercooling fraction which is only \( \approx 20\% \) too low compared to the more realistic case of one bubble nucleated per cm\(^3\) per \(10^{-6}\) of a Hubble time, see [7]. The time needed for the supercooling is given by \( \Delta t_{\text{sc}} / t_H = \eta / (3c_s^2) = \mathcal{O}(10^{-3}) \). In the figures we took \( \eta = 10^{-1} \) for illustrative purposes. At the maximal supercooling the universe suddenly reheats up to the QCD transition temperature \( T_\star \). The sudden reheating implies a jump in pressure \( [p] \equiv p_* - p_{\text{sc}} \).

\[
\frac{[p]}{\rho + p} = \eta,
\]

(6)

up to linear order in \( \eta \). This jump in pressure depends on the maximal supercooling and can be a huge effect. In the case of the cosmological QCD transition it is \( \approx 100 \) times larger than preexisting COBE normalized perturbations. On the other hand, the jump in entropy density is small since it has to be quadratic in \( \eta \) due to the second law, \( [s] / s = \frac{3}{2} \eta^2 \).

The reheating is treated as a sudden process happening at the maximal supercooling, see Fig. [1]. This is justified since the reheating time is \( \approx 10^{-6}t_H \ll \Delta t_{\text{sc}} \), which follows from the following consideration. Bubbles present at a given time have been nucleated typically during the preceding time interval \( \Delta t_{\text{nucl}} \equiv I / (dI / dt) \). Using the relation between time and supercooling \( \eta, \, d\eta / dt = 3c_s^2 / t_H \), we find

\[
\Delta t_{\text{nucl}} / t_H = \eta^2 / (6Ac_s^2) = \mathcal{O}(10^{-6}).
\]

(7)

The corresponding bubble nucleation distance is a few cm [14,15].

During the reversible part of the first order phase transition thermal equilibrium between the quark-gluon plasma and the hadron gas is maintained. We showed in [6] that pressure gradients and the isentropic sound speed (for wavelengths \( \lambda \) much larger than the bubble separation), \( c_s = (\partial p / \partial T)_{s}^{1/2} \), must be zero during a first-order phase transition of a fluid with negligible chemical potential (i.e. no relevant conserved quantum number). The sound speed must be zero, because for such a fluid the pressure can only depend on the temperature, \( p(T) \), and because the transition temperature \( T_\star \) has a given value, it cannot depend on any parameter, hence \( p(T_\star) = p_* \) is a given constant, and \( c_s = 0 \). During the entire QCD transition the sound speed stays zero and suddenly rises back to the radiation value \( c_s = 1 / \sqrt{3} \) after the transition is completed. In contrast pressure varies continuously and goes below the ideal radiation fluid value \( p = \rho / 3 \), but stays positive.

III. AMPLIFICATION OF SUBHORIZON PERTURBATIONS

The evolution of linear cosmological perturbations is first analyzed on subhorizon scales, where Newtonian concepts for space and time (for an expanding radiation fluid) are applicable. The perturbations on subhorizon scales get an amplification factor which is quadratic in wave number \( k \) for \( k \gg H \).

The maximal supercooling is reached when bubble nucleation becomes efficient and the radiation fluid suddenly reheats, i.e. it happens on a hypersurface of constant temperature,
\[ T = T_{sc}, \text{ hence uniform energy density. On this hypersurface of locally sudden reheating, } \Sigma_{RH}, \text{ pressure jumps uniformly, see Fig. 2. } \]

At a given Newtonian time, overdensities have higher temperature than underdensities and therefore reheat later. This time delay (lapse of time) of the actual reheating of a fluid element on \( \Sigma_{RH} \) from the average Newtonian time of reheating, \( t_{RH} \), is denoted by \( \Delta t(x) \) and shown in Fig. 2 for an overdensity of one perturbation mode \( k \). The time delay \( \Delta t(x) \) follows from the condition of uniform energy density on the surface of reheating, \( \rho(x, t + \Delta t(x)) = \text{uniform}, \]

\[ \epsilon(x)|^{(-)} + \Delta t(x) \frac{d\rho}{dt}|^{(-)} = 0. \tag{8} \]

\( \epsilon(x) \equiv \delta \rho \) denotes the energy density perturbation at a given time, \( (-) \) means fixed time immediately before the short period during which the various fluid elements reheat. Inserting the continuity equation for the FRW background \( d\rho/dt = -3H(\rho + p) \) at the time of maximal supercooling gives

\[ \Delta t(x) = \frac{1}{3H} \left. \epsilon(x) \right|^{(-)} . \tag{9} \]

One can apply the same argument for the perturbations and the background just after reheating. The time delay of \( \Sigma_{RH} \) is the same whether evaluated before or after the hypersurface, and this condition gives the discontinuity equation for \( \epsilon \),

\[ \left[ \frac{\epsilon}{\rho + p} \right]^{(+)} - \left[ \frac{\epsilon}{\rho + p} \right]^{(-)} = 0. \tag{10} \]

\( \rho \) is continuous at the actual hypersurface \( \Sigma_{RH} \), but the effects of the fluid expansion is different in the two media. This generates an effective discontinuity in \( \epsilon \).

We now integrate the evolution equations for the relativistic fluid over the very short time period of order \( \Delta t \) when reheating happens at some \( x \). We split \( \rho \) and \( p \) into a homogeneous term and an inhomogeneous term, \( \rho(x, t) = \rho(t) + \epsilon(x, t), p(x, t) = p(t) + \pi(x, t) \). In distinction to usual perturbation theory the pressure inhomogeneities compared to the background energy density \( \rho \) are of order \( \Delta T/T \). The pressure inhomogeneities are the dominant terms, the driving terms, all remaining terms can be neglected during the short time interval when reheating occurs at various \( x \). In this approximation the continuity equation (energy conservation) and the Euler equation (momentum equation) read,

\[ \partial_t \epsilon(x, t) = -3H \pi(x, t) \tag{11a} \]
\[ \partial_t S(x, t) = -\nabla \pi(x, t), \tag{11b} \]

where \( S \) is the momentum density and \( S = (\rho + p)v \) for a radiation fluid.

The pressure inhomogeneity \( \pi(x, t) \) is a sequence of step functions with step size \([p]\), while \( \nabla \pi(x, t) \) is a sequence of Dirac delta functions. It is necessary to distinguish the unperturbed pressure \( p(t) \), which makes a jump \([p]\) at \( t = t_{RH} \), from the average pressure \( \bar{p}(t) \), which does not make a jump at \( t = t_{RH} \). It is very convenient to define the pressure
perturbations as \( \pi(x, t) \equiv p(x, t) - p(t) \), therefore \( \pi(x, t) \neq p(x, t) - \bar{p}(t) \). The dominant pressure inhomogeneity is

\[
\pi(x, t) = [p] \{\theta(t - t_{\text{RH}} - \Delta t(x)) - \theta(t - t_{\text{RH}})\}, \tag{12}
\]

with \([p] \equiv p^+(t) - p^-(t)\). Integrating the continuity equation Eq. (11a) over the short period of huge pressure differences gives the jump condition for \( \epsilon \),

\[
[\epsilon(x)] = [p] 3H \Delta t(x) = [p] \frac{\epsilon(x)}{\rho + p}^{(-)}, \tag{13}
\]

in agreement with Eq. (10).

Integrating the Euler equation Eq. (11b) over the short period of huge pressure differences, gives the jump condition for the momentum density \( S \),

\[
[S(x)] = [p] \int dt \delta(t - t_{\text{RH}} - \Delta t(x)) \nabla(\Delta t)
= [p] \frac{1}{3H} \nabla \frac{\epsilon(x)}{\rho + p}^{(-)}. \tag{14a}
\]

The discontinuity in \( S \) is a local discontinuity at the actual hypersurface of reheating (produced by the \( \delta \) functions of Eq. (11b) and Eq. (14a)). It is a sudden impulse imparted on each fluid element.

In the evolution equations Eqs. (11a), (11b) different Fourier modes are coupled during the short period of reheating. Therefore the standard procedure to evolve each \( k \) mode separately is not applicable at reheating. However the resulting jump conditions obtained by working in \( x \) space, Eqs. (13), (14b), are valid for each \( k \) mode separately.

On subhorizon scales the discontinuity in \( \epsilon \) is independent of \( k \) (for a preexisting Harrison-Zel’dovich spectrum), the discontinuity in \( S \) is proportional to \( k_{\text{phys}} \) and is the dominant effect in the subhorizon limit.

The evolution of density perturbations before and after sudden reheating can be computed in a Jeans analysis generalized to a relativistic fluid for each Fourier mode separately, see [6]. The preexisting subhorizon perturbations before sudden reheating are acoustic oscillations, see Fig. 3. In the bag model the sound speed \( c_s = 1/\sqrt{3} \) and the amplitudes \( A_{\text{in}} \) for \( \epsilon \) and \( |\sqrt{3}S| \) are equal until sudden reheating. At \( t_{\text{RH}} \), \( \epsilon \) and \( S \) jump according to Eqs. (13), (14b). After sudden reheating, during the reversible part of the QCD transition, the sound speed \( c_s \equiv 0 \), i.e. the restoring force in the acoustic oscillations vanishes as first shown in [5,6], and density perturbations go into free fall. Since the QCD transition lasts less than a Hubble time, gravity is negligible for subhorizon scales and density perturbations have approximately constant velocity, \( \epsilon(t) = \epsilon^+(t) - (t - t_{\text{RH}})\nabla S^+(t) \). This gives an amplification which is linear in \( k_{\text{phys}} \) and proportional to \( S^+(t) \). After the QCD transition is completed, one has acoustic oscillations again and the total amplification factor for \( k \gg H \) is

\[
\frac{A_{\text{out}}}{A_{\text{in}}} = \frac{k}{k_1} \frac{k}{k_2} \cos \varphi_{\text{in}} - \sin \varphi_{\text{in}}. \tag{15}
\]

\( \varphi_{\text{in}} \) is the phase of the incoming acoustic oscillation at \( t_{\text{RH}} \). The scale \( k_1 \equiv \sqrt{3}/\Delta t_{\text{trans}} \approx R_H^{-1} \) depends on the duration of the reversible QCD transition, \( \Delta t_{\text{trans}} \). This contribution is due
to the free fall and starts just below the Hubble scale, see Fig. 4. $k_2$ depends on the amount of supercooling, $k_2 \equiv H_{RH}^2\sqrt{3}/\eta$. The amplification factor is quadratic in $k$ for $(\lambda/R_H) < \eta$.

Jump conditions like Eqs. (13), (14b) must be used whenever there is a jump in the background quantities (as opposed to a jump only in $c_s^2$ as in the case without supercooling and sudden reheating). Only the background pressure is allowed to jump not the energy density, according to the continuity equation. Due to the second law of thermodynamics a fluid can only heat instantaneously, sudden cooling is impossible.

**IV. GENERAL RELATIVISTIC ANALYSIS**

A slicing of space-time (with space-like hypersurfaces on which time is defined to be constant) is needed to formulate time-evolution equations in general relativity. We restrict ourselves to time-orthogonal foliations. For linear cosmological perturbations a choice of a foliation already fixes a gauge. For sudden reheating there is one special slice, the hypersurface of reheating $\Sigma_{RH}$, which is a surface of constant energy density. It is a fixed-time slice in uniform density (UD) gauge, see [16].

We use the uniform density gauge, the gauge adapted to our problem, to evolve through sudden reheating. We now show that in UD gauge the relevant variables are continuous (do not jump). The hypersurface of sudden reheating, $\Sigma_{RH}$, has definite extrinsic curvature $K_{ij}$ and intrinsic curvature $(3)R_{ij}$ at each point on $\Sigma_{RH}$, hence these quantities are continuous at reheating. The momentum density $S$ in UD gauge also stays continuous at reheating due to the momentum constraint of general relativity, $D_j K^j_i - D_i K^j_j = 8\pi G S_i$, see e.g. [17]. The energy density is uniform in space and continuous in time in UD gauge.

The UD gauge is not suitable for evolving before and after reheating, because the UD gauge is a singular gauge in the subhorizon limit, $H/k_{phys} \to 0$, see below. Therefore we shall solve the dynamics before and after sudden reheating using the uniform expansion gauge (UE), which is nonsingular both in the superhorizon and subhorizon limits and in which the physical dynamics will turn out to be most transparent (in contrast to the UD gauge). However in the UE gauge the relevant variables are discontinuous (jump) at sudden reheating. The uniform (Hubble) expansion gauge is defined by requiring that the fundamental observers, who are at rest on the slice, $\overline{u}(\text{obs}) = \overline{u}(\Sigma)$, have uniform Hubble expansion, i.e. the perturbation of the mean extrinsic curvature of $\Sigma$, $\delta \left[ \text{tr} K^i_j(\Sigma) \right] \equiv \kappa$, is zero, $\kappa_{UE} \equiv 0$.

The geometrical properties of a hypersurface $\Sigma$ in the longitudinal sector are given by the following variables, Bardeen 1989 [16]: The perturbation of the trace of the extrinsic curvature $\delta \left[ \text{tr} K^i_j(\Sigma) \right] \equiv \kappa$, the traceless part of the extrinsic curvature (shear $\sigma^i_j$ of normals), which is determined by $\chi$ through $\sigma^i_j(\Sigma) = -\left( \partial^i \partial_j - \frac{1}{3} \delta^i_j \triangle \right) \chi$, and $(3)R(\Sigma) = -4 \triangle \varphi$. The lapse of time between the perturbed hypersurfaces is $(1 + \alpha)$. The connection between the geometric variables and the gauge potentials (in the longitudinal sector with time-orthogonal coordinates),

$$ds^2 = -(1 + 2\alpha) \, dt^2 + a(t)^2 [\delta_{ij}(1 + 2\varphi) + 2\partial_i \partial_j \gamma] \, dx^i dx^j, \quad (16)$$

is as follows: $\chi = a^2 \dot{\varphi}$, $\kappa = -3(\dot{\varphi} - H \alpha) - \triangle \chi$.

The state of the fluid (before or after sudden reheating) on a hypersurface of fixed time (in some gauge) is given by the following variables: The energy density perturbation $\delta \rho \equiv \epsilon$ and
the momentum density $S$, which can be written as $S = \nabla \psi$ in the longitudinal sector (sector of scalar perturbations). $p(\rho)$ is given by the bag model before and after the irreversible reheating. There are no anisotropic stresses for our perfect fluid (QCD, $\gamma$, leptons). The total set of variables in the longitudinal sector is $(\epsilon, \psi), (\kappa, \chi, \varphi), \alpha$.

For the dynamical equations in UE gauge we choose to take the evolution equations of the fluid, $\nabla_\mu T^{\mu\nu} = 0$, which are the continuity equation and (in the longitudinal sector) the 3-divergence of the Euler equation,

$$
\partial_t \epsilon = -3H(\epsilon + \pi) - \Delta \psi - 3H(\rho + p)\alpha \tag{17a}
$$

$$
\partial_t \psi = -3H\psi - (\rho + p)\alpha, \tag{17b}
$$

with $\pi \equiv \delta p$. The system of dynamical equations is closed by Einstein’s $R^0_0$-equation,

$$
(\Delta + 3\dot{H})\alpha = 4\pi G(\epsilon + 3\pi), \tag{17c}
$$

together with the equation of state. This set of equations has exactly the same structure as the Jeans equations: Two first-order time-evolution equations for the fluid state variables $(\epsilon, \psi)_{UE}$, whose initial values can be chosen free of constraints, supplemented by an elliptic equation for $\alpha_{UE}$, which plays the role of the gravitational potential. The lapse $\alpha$ is the only geometrical variable needed to solve the dynamical evolution in UE gauge. The other geometric perturbations, i.e. the shear $\chi$ and the intrinsic curvature $\varphi$, do not appear in our closed set of evolution equations, Eqs. (17a-17c). If desired, $\varphi$ and $\chi$ can be computed from the state variables via the energy and momentum constraints, $\Delta \varphi = -4\pi G\epsilon, \Delta \chi = -12\pi G\psi$ on any UE slice.

To make the gauge transformation from the uniform expansion gauge (independent variables $\epsilon, \psi$) to the uniform density gauge (independent variables $\psi, \kappa$) we need the gauge transformation formulae for $(\epsilon, \psi, \kappa) \{16\}$

$$
\tilde{\epsilon} - \epsilon = -3H(\rho + p)\Delta t \tag{18a}
$$

$$
\tilde{\psi} - \psi = -(\rho + p)\Delta t \tag{18b}
$$

$$
\tilde{\kappa} - \kappa = -(3\dot{H} - k^2_{phys})\Delta t, \tag{18c}
$$

where the variables with tilde correspond to the new gauge and $\Delta t$ is the lapse of time from the old to the new hypersurface, see Fig. 2.

We now derive the discontinuity conditions at sudden reheating for the uniform expansion gauge. We use the state variables $(\epsilon, \psi)_{UE}$ just before reheating and use Eq. (18a) with $\tilde{\epsilon}_{UE} \equiv 0$ to obtain

$$
\Delta t(-) = \frac{\epsilon_{UE}(-)}{3H(\rho + p(-))}, \tag{19}
$$

where $(-)$ means $t_{UE}$ immediately before reheating. The momentum potential $\psi_{UE}$ and the extrinsic curvature $\kappa_{UE}$ follow from Eqs. (18b), (18c),

$$
\psi_{UE} = \psi_{UE}(-) - \frac{\epsilon_{UE}(-)}{3H}, \tag{20a}
$$

$$
\kappa_{UE} = -(3\dot{H}(-) - k^2_{phys})\frac{\epsilon_{UE}(-)}{3H(\rho + p(-))}. \tag{20b}
$$
ψ_{UD} and κ_{UD} stay continuous at sudden reheating. We make the analogous gauge transformation (from UE to UD) after sudden reheating. From the κ_{UD} equation we get the discontinuity condition for ǫ,

\[
\left[ \frac{3\dot{H} - k_{\text{phys}}^2}{\rho + p} \epsilon_{\text{UE}} \right] = 0, \tag{21}
\]

where the jump in the background quantity \( \dot{H} = -4\pi G(\rho + p) \) is the given input, and the jump in ǫ is the output. In the subhorizon limit \( [\epsilon/(\rho + p)] \to 0 \) and \( \Delta t^(-) = \Delta t^{(+)} \), i.e. we recover the results of the analysis using Newtonian geometry, Eq. (13). In the superhorizon limit ǫ does not jump, \( [\epsilon] \to 0 \), but \( \Delta t^(-) \neq \Delta t^{(+)} \). This means that the coordinate regions \( t_{\text{UE}} < t_{\text{RH}} \) and \( t_{\text{UE}} > t_{\text{RH}} \) cover certain regions of space-time doubly and other regions not at all. Applying the ψ_{UD} equation before and after reheating we obtain

\[
[\psi_{\text{UE}}] = \frac{1}{3H} [\epsilon_{\text{UE}}]. \tag{22}
\]

The resulting modification of the perturbation amplitude due to the QCD phase transition (transfer function) is shown in Fig. 4.

On superhorizon scales \((\epsilon, \psi)_{\text{UE}}\) do not jump. An even stronger statement can be made for \( \delta_{\text{UE}} \equiv \epsilon_{\text{UE}}/\rho \): The evolution of the growing mode (i.e. growing in relative importance) in \( \delta_{\text{UE}} \) is given by

\[
(\delta_{\text{UE}})_{\text{growing mode}} = \text{const} \left( \frac{k_{\text{phys}}}{H} \right)^2 \tag{23}
\]

for scales \( k_{\text{phys}}^2 \ll \{H^2, |\dot{H}|\} \) and \( p/\rho = w < 1 \). Written in this form the growing mode of the state variable \( \delta_{\text{UE}} \) is manifestly unaffected during phase transitions or any other changes in the equation of state. The proof goes as follows. We take the \( k_{\text{phys}}^2 \ll |\dot{H}| \) limit of the \( R_{\theta\theta} \) equation, \(-3(\rho + p)\alpha = (\epsilon + 3\pi)\), and we insert this into the continuity equation, Eq. (17a), and into the Euler equation, Eq. (17b),

\[
\partial_t \epsilon + 2H \epsilon = k_{\text{phys}}^2 \psi \tag{24a}
\]
\[
\partial_t \psi + 3H \psi = \frac{1}{3} \epsilon. \tag{24b}
\]

We note that the pressure gradients have dropped out. For the mode which will turn out to be the growing mode the two terms on the left-hand side of the Euler equation do not cancel, and \( \epsilon \) is relevant in the Euler equation, hence \( \psi = \mathcal{O}(\epsilon/H) \). Therefore the \( \psi \) term is negligible in the continuity equation for the growing mode if \( k_{\text{phys}}^2 \ll H^2 \), \( \partial_t (\ln \epsilon) = -2\partial_t (\ln a) \). Hence \( \epsilon = \text{const} a^{-2} \) and using the Friedmann equation \( \delta_{\text{UE}} \equiv \epsilon_{\text{UE}}/\rho = \text{const} \left( k_{\text{phys}}^2/H^2 \right) \). For the decaying mode the two terms on the left-hand side of the continuity equation do not cancel, and \( \psi \) is relevant in the continuity equation, hence \( \epsilon = \mathcal{O} \left( k_{\text{phys}}^2 \psi /H \right) \). Therefore the \( \epsilon \) term is negligible in the Euler equation for the decaying mode, \( \partial_t (\ln \psi) = -3\partial_t (\ln a) \). Hence \( \psi = \text{const} a^{-3} \), and the continuity equation gives \( \epsilon \propto a^{-2} \int a^{-3} dt \). With \( a \propto t^{(1+w)} \) it follows that this is indeed the decaying mode for \( w \equiv p/\rho < 1 \). The law, Eq. (24), can be translated (using the energy constraint) into the ‘conservation law’ \((\varphi_{\text{UE}})_{\text{growing mode}} = \text{const.}\)
This is consistent with the 'conservation law' \[ \zeta \text{growing mode} = \text{const} \text{ for } k^2_{\text{phys}} \ll H^2, \]

where \( \zeta \equiv \varphi + \frac{\epsilon}{3(\rho_p + p)} \).

Deruelle and Mukhanov \[19\] have analyzed jump conditions (matching conditions) in zero shear (ZS) gauge, also called longitudinal gauge, where the shear of the normals to the equal time hypersurfaces is zero by definition. They focussed on superhorizon physics. The matching conditions are much more complicated in zero shear gauge than in uniform expansion gauge; no superhorizon discontinuities in UE state variables, complicated discontinuity conditions in ZS in Ref. \[19\]. This is connected to the fact that the zero shear gauge is singular in the superhorizon limit (Bardeen 1988 \[16\]), and one particular aspect of this is that \( \Delta_t^{\text{ZS,UD}}/\Delta t^{\text{UE,UD}} \propto (H/k_{\text{phys}})^2 \to \infty \) in the superhorizon limit.

Finally we show that the uniform density gauge is singular in the subhorizon limit. For a Harrison-Zel’dovich spectrum the subhorizon density contrast \( \delta_{\text{UE}} \equiv \epsilon/\rho = O(10^{-4}) \) is independent of \( k \), the state variables \( (\epsilon, \psi)_{\text{UE}} \) agree with \( (\epsilon, \psi) \) in a Jeans analysis. From Eq. (19) and the subhorizon limit of Eq. (20b) we obtain \( \kappa_{\text{UD}}/H = O(k^2_{\text{phys}} \delta_{\text{UE}}/H^2) \to \infty \).

\[ \text{V. BLACK HOLE FORMATION AT THE QCD TRANSITION?} \]

Black holes form in a radiation dominated universe if the density contrast of a top hat perturbation inside the Hubble radius is in the range \( 1/3 \leq \delta_H \leq 1 \) \[20\]. For an observable amount of 1 \( M_\odot \) black holes today, i.e. \( \Omega_{\text{BH}}^{(0)} = O(1) \), the fraction \( \beta \) of energy density converted to black holes at the QCD transition must be \( O(a_{\text{QCD}}/a_{\text{equality}}) \approx 10^{-8} \). For a gaussian distribution of density fluctuations the fraction \( \beta \) of \( \rho_{\text{BH}} \) at the time of formation is given by,

\[ \beta \equiv \frac{\rho_{\text{BH}}}{\rho_{\text{tot}}} \approx \frac{1}{\sqrt{2\pi} \delta_{\text{rms}}} \int_{1/3}^1 \exp(-\frac{\delta^2}{2\delta_{\text{rms}}^2})d\delta \]

Without any enhancement from the QCD transition this requires \( \delta_{\text{rms}} \approx 0.06 \) \[21\]. The sudden reheating at the onset of a first-order QCD transition leads to huge amplifications of density perturbations on scales far below the Hubble horizon. But at the horizon scale the QCD transition gives enhancement factors of 2.0 and 2.2 for the bag model without resp. with sudden reheating for a supercooling of 10\%, Fig. 4, an additional enhancement of 10\%. For lattice QCD without sudden reheating the enhancement factor is 1.5 \[6\] in our linear perturbation treatment. This indicates a corresponding reduction in the required preexisting perturbation spectrum at the solar mass scale. Cardall and Fuller \[22\] used a qualitative argument of Carr and Hawking \[23\] and the bag model and also obtained a factor 2 reduction in the required preexisting perturbation spectrum. These QCD factors of 1.5 or ~ 2 are so modest that a preexisting Harrison-Zel’dovich spectrum with COBE normalization is very far from giving a cosmologically relevant amount of black holes \[10\]. One would have to put in a fine-tuned tilt \( (n - 1) \approx 0.36 \) to get the desired amount of black holes. However, this tilted spectrum would overproduce primordial black holes on scales which are only a factor 50 below the Hubble radius at the transition. Therefore a break in the preexisting spectrum below the QCD scale would be required, a second fine tuning.
We conclude that the QCD transition with or without sudden reheating enhances the probability of black hole formation, but the preexisting spectrum needs to be fine tuned around the QCD scale, and the major effect would not be due to the QCD transition.

ACKNOWLEDGMENTS

We thank Dominik Schwarz for useful discussions. P. W. thanks the Swiss National Science Foundation for financial support.
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FIG. 1. The evolution of temperature $T$ as a function of the scale factor $a$ in the bag model. For illustrative purposes the supercooling is $\Delta T/T_\star = 10^{-1}$.

FIG. 2. A space-time diagram at the time of sudden reheating: The supercooled (2) and the reheated (4) epoch are separated by the hypersurface of reheating $\Sigma_{\text{RH}}$ (3). On $\Sigma_{\text{RH}}$ temperature, hence pressure, jumps uniformly. $\Delta t(x)$ denotes the lapse of time compared to the average time, $t_{\text{RH}}$, for a certain fluid element to reach the hypersurface of reheating. For subhorizon wavelengths $t_{\text{RH}}$ is the average Newtonian time for reheating, for horizon and superhorizon wavelengths (general relativistic case), $t = t_{\text{RH}}$ is a hypersurface of constant time in a given gauge, e.g. in the uniform expansion gauge.
FIG. 3. The evolution in conformal time $\eta$ of the density contrast ($\delta_{\text{RAD}} \equiv \epsilon/\rho$) and the velocity ($\dot{\psi}_{\text{RAD}} \equiv ik \cdot \mathbf{S}/(k\rho) = (\rho + p)v/\rho$) for the radiation fluid in uniform expansion (Hubble) gauge. At sudden reheating, the fluid velocity jumps. During the reversible part of the QCD transition, marked by the 2 vertical lines, the velocity stays approximately constant and the density contrast grows linearly. The incoming amplitude is normalized to 1.

FIG. 4. The modifications of the radiation fluid amplitude $A_{\text{out}}/A_{\text{in}}$ (transfer function) due to the QCD transition in the bag model with a supercooling of $\Delta T/T^* = 10^{-1}$. On the horizontal axis the wavenumber $k$ is represented by the RAD mass contained in a sphere of radius $\pi/k$. $M_H$ is the mass inside the horizon at the QCD transition.