Quantum Principal Fiber Bundles: Topological Aspects *

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Abstract

We introduce the notion of locally trivial quantum principal bundles. The base space and total space are compact quantum spaces (unital C*-algebras), the structure group is a compact matrix quantum group. We prove that a quantum bundle admits sections if and only if it is trivial. Using a quantum version of Čech cocycles, we obtain a reconstruction theorem for quantum principal bundles. The classification of bundles over a given quantum space as a base space is reduced to the corresponding problem, but with an ordinary classical group playing the role of structure group. Some explicit examples are considered.

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1 Introduction

Quantum groups [12, 11, 4] are by now a well-established notion, and one that attracts considerable interest, both as a subject of pure mathematics and as a tool in theoretical physics. One of the approaches to quantum groups is to view them within the wider context of non-commutative geometry [2], as quantum spaces endowed with a particular additional structure. More specifically, we adopt here the point of view, developed by Woronowicz and collaborators [12, 10, 8, 13], where a (generally non-commutative) $C^*$-algebra is interpreted as a generalization of the (commutative) $C^*$-algebra of continuous functions on a locally compact topological space. In this sense, the theory of $C^*$-algebras may be considered as an extension of the theory of a certain category of classical topological spaces (point sets)\(^1\). Such a framework provides a convenient starting point for further development of non-commutative geometry: classical geometrical notions, such as differential structures, differential forms, metrics, connections, etc. are to be re-defined in a way applicable to non-commutative $C^*$-algebras.

Within this context, it is natural to seek non-commutative (or ‘quantum’) extensions of classical geometrical constructions involving Lie groups, now generalized to matrix quantum groups [12]. For instance, quantum homogeneous spaces and quotient spaces of quantum groups were studied by Podles [10, 9, 8]. The aim of the present paper is to lay the groundwork for a theory of locally trivial quantum principal fiber bundles. Such a construction is of obvious intrinsic interest, as it generalizes a very important and natural object of classical geometry. On the physical side, it is well known that principal bundles provide the natural geometrical setting for classical Yang-Mills theory. One may expect that quantum principal bundles should play a corresponding role for a non-commutative Yang-Mills theory, attempts to formulate which are currently under way.

In the present work, we follow the general approach to quantum spaces sketched above: the total bundle space and base space are replaced by (compact) quantum spaces, represented by unital $C^*$-algebras, and the structure group is replaced by a (compact matrix) quantum group. The classical definitions are extended by translating them into dual form, involving the algebras of continuous functions on the corresponding spaces, and giving up commu-

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\(^1\)For an example of work along these lines, see [3].
tativity. This is not entirely straightforward: the quantum definitions and statements must make no reference to points of the spaces involved (quantum spaces are in no sense point sets). An example of the difficulties one encounters is given by the concept of free action of a group. This does not appear to admit a satisfactory generalization to quantum spaces. For instance, the definition proposed in [5, 1] is formulated in terms of a mapping which is not a $C^*$-algebra morphism. Our choice is to impose a quantum version of local triviality, a stronger requirement.

The scope of the present paper is restricted to formulating the basic definitions and extracting their most immediate consequences, specifically, those of a ‘topological’ nature. The study of differential geometric structures on quantum principal bundles within the framework of our proposed theory is left to future publications. Moreover, it should be pointed out that our work can be extended in a number of directions. For instance, one may work with open (instead of closed) coverings of the base space of the bundle, as is more conventional in classical geometry; this is a rather technical point, involving the more intricate theory of noncompact quantum spaces [13]. A more substantial point concerns our choice of the quantum extension of the notion of Cartesian product; though certainly not a unique choice, it is the one most frequently considered in similar work [14].

In section 2 we review the basic definitions and facts concerning quantum group theory as developed by Woronowicz. Section 3 introduces trivial quantum principal bundles, their sections and automorphisms. Bundle automorphisms are defined in such a way that their set forms a group in the ordinary sense. In the commutative case, this group coincides with the usual group of gauge transformations of a trivial principal bundle. In addition, the classical one to one correspondence between sections and trivializations also extends to our theory. Section 4 begins with the general definition of locally trivial quantum principal bundles. Our definition is somewhat more restrictive than the one proposed in [1]. This is because we were careful to preserve in full the correspondence with the classical theory: if all ‘function’ algebras are taken to be commutative, all the objects of our theory reduce to their classical counterparts. Such an approach allows us to prove a number of powerful results; the first (Th. 2) states that existence of a section is equivalent to global triviality of the bundle. Next, we present an outline of Čech cohomology theory for quantum principal bundles, which is then employed to give a reconstruction theorem (Th. 3), stating that a quantum principal bundle
may be recovered from the corresponding quantum Čech cocycle. The next of our main results (Th. [4]) is that two bundles with isomorphic sub-bundles are themselves isomorphic. Theorem [3] states that every bundle has what we call a ‘classical sub-bundle’, i.e. a sub-bundle with the structure group being an ordinary (classical) group. Taken together with Theorem [3], the classification of quantum principal bundles is reduced to that of bundles over the same base space, with the structure group being the classical subgroup of the corresponding quantum group. This extends considerably a result of [7], where only bundles over commutative base spaces were considered.

Following a brief digression on associated bundles in section 5, in section 6 we present some explicit examples: we describe all $SU_q(2)$ principal bundles over the quantum unit disk of Klimek and Lesniewski [6], and over the Podleś spheres [8]. An appendix is devoted to a brief summary of the principal concepts and operations of the theory of quantum spaces.

## 2 Quantum groups

All $C^*$-algebras which will be considered in this paper are understood to be separable $C^*$-algebras with unit; correspondingly, all algebra homomorphisms are unital $C^*$-algebra homomorphisms. Homomorphisms (linear, multiplicative, $*$-preserving mappings) of $C^*$-algebras into the complex number field will be termed functionals for the sake of brevity. We use the notation $m_A : A \otimes A \to A$ (where $A$ is a $C^*$-algebra) for the linear mapping (not an algebra homomorphism in general) defined by $m_A(a \otimes b) = ab$. For a review of the basic notions of the theory of compact quantum spaces, as they are applied in the present paper, we refer the reader to the appendix.

The definition below follows [12].

**Definition 1** $G = (A, u)$ is called a (compact matrix) quantum group if $A \neq \{0\}$ is a $C^*$-algebra, $u = (u_{ij})$ is a $N \times N$ matrix with entries in $A$, and

1. $A$ is the smallest $C^*$ algebra containing all matrix elements of $u$,

2. There exists a homomorphism $\Delta : A \to A \otimes A$ such that

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj},$$

(1)
3. \( u \) and \( u^T \) are invertible.

In the case when \( A \) is a commutative algebra, \( (A, u) \) will be called a classical (matrix) group. We will often employ the notation \( A = C(G) \).

The above definition implies the following

**Proposition 1**

1. \( (\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta \) (co-associativity);

2. There exists a unique functional \( \varepsilon \) on \( A \), the counit, with the properties

\[
\varepsilon(u_{ij}) = \delta_{ij}\tag{2}
\]

\[
(\varepsilon \otimes \text{id})\Delta = (\text{id} \otimes \varepsilon)\Delta = \text{id}.\tag{3}
\]

3. Let us denote by \( \mathcal{A} \) the dense subalgebra in \( A \) generated by \( u_{ij} \). There exists a (unique) anti-homomorphism (i.e. linear and anti-multiplicative mapping) \( S : \mathcal{A} \to \mathcal{A} \) called the antipode, such that

\[
S(u_{ij}) = (u^{-1})_{ij}
\]

\[
S(S(a^*)^*) = a
\]

for all \( a \in \mathcal{A} \).

A representation of the quantum group \( G \) is an invertible \( M \times M \) matrix \( v \) with entries in \( A \) such that \( \Delta(v_{mn}) = \sum_p v_{mp} \otimes v_{pn} \). In particular, \( u \) is a representation of \( G \). For any representation \( v \) of \( G \), \( \varepsilon(v_{mn}) = \delta_{mn} \) and \( S(v_{mn}) = (v^{-1})_{mn} \).

Two representations are called equivalent if the two corresponding matrices are related by a similarity transformation given by a matrix with entries in \( C \). A representation is reducible if its matrix is (up to similarity) block-diagonal. Otherwise, a representation is called irreducible. For more details, see [12].

**Definition 2** A subgroup of the quantum group \( G = (A, u) \) is the triple \( (G, H, \theta_{HG}) \), where \( H = (B, v) \) is a quantum group, \( \dim v = \dim u \), and \( \theta_{HG} : A \to B \) is a \( C^* \) homomorphism such that \( \theta_{HG}(u_{ij}) = v_{ij} \).
Note that $\theta_{HG}$ is necessarily a $C^*$ epimorphism.

We now introduce a special subgroup, which exists for any quantum group $G$, and which will play an important role in the sequel. Moreover, this classical subgroup corresponds to a group in the usual sense.

For a quantum group $G$ consider the set of functionals on $C(G)$. This set $G/\!\!/\!\!C^*$ is equipped with the natural structure of a group: the group multiplication is given by 

$$\phi \cdot \psi = (\phi \otimes \psi)\Delta$$

for $\phi, \psi \in G/\!\!/\!\!C^*$; the neutral element is the functional $\varepsilon$, the co-unit. We thus see that $G/\!\!/\!\!C^*$ is a semigroup with unit; but, since it is also a compact topological space, $G/\!\!/\!\!C^*$ is a group $\mathbb{R}$. In fact, it is a compact subgroup of $GL(N, \mathbb{C})$ with $N = \dim u$: its elements are given by $\phi(u_{ij})$ for $\phi \in G/\!\!/\!\!C^*$, and group multiplication is equivalent to matrix multiplication.

Consider now $C(G/\!\!/\!\!C^*)$, the algebra of continuous functions on the group $G/\!\!/\!\!C^*$. One can define a natural homomorphism $\rho : C(G) \to C(G/\!\!/\!\!C^*)$ by the formula

$$[\rho(f)](\phi) = \phi(f).$$

The triple $(G, G/\!\!/\!\!C^*, \rho)$, where $G/\!\!/\!\!C^*$ as a quantum group is $G/\!\!/\!\!C^* = (C(G/\!\!/\!\!C^*), \rho(u_{ij}))$, is what we call the classical subgroup of $G$.

Observe that the kernel of $\rho$ coincides with the commutator of $C(G)$, i.e. the smallest closed *-ideal in $C(G)$ which contains the commutator of any two elements of $C(G)$.

**Definition 3** Let $C(X)$ be a $C^*$ algebra with unit, and $G$ be a quantum group. We say that a $C^*$ homomorphism $\Gamma : C(X) \to C(X) \otimes C(G)$ is an action of $G$ on the quantum space $X$ if:

1. $(\Gamma \otimes \text{id})\Gamma = (\text{id} \otimes \Delta)\Gamma$,

2. The closure of the span of $(I \otimes y)\Gamma x, x \in C(X), y \in C(G)$ is equal to $C(X) \otimes C(G)$.

The above definition is taken from [10]. For completeness, we also quote the following theorem [10] (see also [9]):

**Theorem 1** Let $\Gamma$ be an action of the quantum group $G$ on the quantum space $X$. Then $C(X)$ may be decomposed as the (closure of the) direct sum of
invariant subspaces corresponding to the inequivalent irreducible representations of \( G \), with the multiplicity (possibly infinite) of any given representation being uniquely determined.

3 Trivial quantum principal bundles

In the present section we collect the basic facts concerning trivial quantum principal bundles. This will serve to introduce the concepts and methods which will be applied in the sequel to the general case of (locally trivial) quantum principal bundles.

**Definition 4** Let \( C(P), C(X) \) be \( C^* \) algebras, \( G = (C(G), u) \) a quantum group, \( \Gamma : C(P) \to C(P) \otimes C(G) \) an action of \( G \), and \( \pi : C(X) \to C(P) \) an injective homomorphism, such that

\[
\Gamma \pi (f) = \pi (f) \otimes I
\]

for all \( f \in C(X) \).

\((C(P), C(X), G, \pi, \Gamma)\) will be called a trivial quantum principal bundle if there exists a \( (C^*) \) isomorphism \( \Phi : C(P) \to C(X) \otimes C(G) \) such that

\[
\Phi \pi (f) = f \otimes I
\]

for all \( f \in C(X) \), and

\[
(\Phi \otimes id) \Gamma = (id \otimes \Delta) \Phi.
\]

\( \Phi \) will be called a trivialization of the bundle.

**Definition 5** Let \((C(P), C(X), G, \pi, \Gamma)\) and \((C(P'), C(X), G, \pi', \Gamma')\) be trivial quantum principal bundles.

A \( C^* \) isomorphism \( \Xi : C(P) \to C(P') \) will be called an isomorphism of trivial quantum principal bundles if:

\[
\Xi \pi = \pi'
\]

and

\[
(\Xi \otimes id) \Gamma = \Gamma' \Xi.
\]
It therefore follows that a trivial quantum principal bundle is isomorphic as a bundle to \((C(X) \otimes C(G), C(X), G, \text{id} \otimes I, \text{id} \otimes \Delta)\), and the freedom to choose a trivialization corresponds to automorphisms of this (product) bundle.

The following lemma provides a more explicit description of the automorphisms of a trivial quantum principal bundle. It is the main technical tool which will be employed in this paper.

**Lemma 1** Let \(\Psi\) be an automorphism of the trivial principal bundle \((C(X) \otimes C(G), C(X), G, \text{id} \otimes I, \text{id} \otimes \Delta)\). The homomorphism \(\tau_\Psi : C(G) \to C(X)\), uniquely determined from \(\Psi\) through the formula

\[
\tau_\Psi = (\text{id} \otimes \varepsilon) \Psi(I \otimes \text{id})
\]

(9)
takes values in the center of \(C(X)\). Conversely, given any \(\tau\) with the above property, the formula

\[
\Psi_\tau = (m_{C(X)} \otimes \text{id})(\text{id} \otimes \tau \otimes \text{id})(\text{id} \otimes \Delta)
\]

(10)
uniquely defines \(\Psi_\tau\), which is a bundle automorphism.

**Proof:** The algebra \(C(X) \otimes C(G)\) is generated by two mutually commuting subalgebras, \(C(X) \otimes I\) and \(I \otimes C(G)\). The latter is in turn generated by the elements \(I \otimes u_{ij}\). Thus the automorphism \(\Psi\) is uniquely determined by the formulas

\[
\Psi(f \otimes I) = f \otimes I,
\]

\[
\Psi(I \otimes u_{ij}) = \Psi_{ij} \in C(X) \otimes C(G).
\]

Since \(\Psi\) is an isomorphism,

\[
(f \otimes I)\Psi_{ij} = \Psi_{ij}(f \otimes I).
\]

(11)
The condition that \(\Psi\) commutes with \(\text{id} \otimes \Delta\) leads to

\[
(\Psi \otimes \text{id})(\text{id} \otimes \Delta)I \otimes u_{ij} = (\text{id} \otimes \Delta)\Psi(I \otimes u_{ij}),
\]
i.e.

\[
(\Psi \otimes \text{id})I \otimes u_{ik} \otimes u_{kj} = (\text{id} \otimes \Delta)\Psi_{ij},
\]

\[
\sum_k \Psi_{ik} \otimes u_{kj} = (\text{id} \otimes \Delta)\Psi_{ij}.
\]
Applying to both sides $\text{id} \otimes \varepsilon \otimes \text{id}$ we obtain

$$[(\text{id} \otimes \varepsilon) \Psi_{ik}] \otimes u_{kj} = \Psi_{ij}$$

by eq. 3.

But $(\text{id} \otimes \varepsilon) \Psi_{ik} = \tau(u_{ik})$, therefore

$$\Psi_{ij} = \tau(u_{ik}) \otimes u_{kj}.$$ 

Thus in virtue of eq. 3:

$$(\tau(u_{ik})f - f\tau(u_{ik})) \otimes u_{kj} = 0.$$ 

Now, the matrix $u$ is invertible, hence $I \otimes u_{kj}$ are also elements of an invertible matrix. It follows that

$$\tau(u_{ik})f - f\tau(u_{ik}) = 0$$

for all $f \in C(X)$.

This proves the first claim of the lemma.

Secondly, given $\tau : C(G) \to Z(C(X))$, $\Psi_\tau$ is a homomorphism uniquely specified by the given formula. The presence of the diagonal mapping $m : C(X) \otimes C(X) \to C(X)$ causes no problems due to the values of $\tau$ being central.

It remains to be shown that $\Psi$ is an isomorphism, i.e. the inverse $\Psi^{-1}$ exists. But it is easily verified that $\Psi^{-1}$ is obtained in the analogous way from $\tau S$. □

A simple consequence of the above lemma is the following:

**Corollary:** Since $\tau$ takes values in the center of $C(X)$, $\tau$ vanishes on the kernel of the projection onto the classical subgroup $\rho : C(G) \to C(G/)$.

Thus every such $\tau$ is in one-to-one correspondence with a homomorphism $\tau/ : C(G/) \to Z(C(X))$, such that $\tau = \tau/ \circ \pi/$. 

**Proposition 2** The set of homomorphisms $\tau : C(G) \to Z(C(X))$ forms a group isomorphic to the group of automorphisms of the trivial principal
bundle \((C(X) \otimes C(G), C(X), G, \text{id} \otimes I, \text{id} \otimes \Delta)\). The group structure is given by
\[
\tau_1 \cdot \tau_2 = m_{C(X)}(\tau_1 \otimes \tau_2) \Delta \\
\tau^{-1} = \tau S \\
e = I_{C(X)} \varepsilon
\]
giving the multiplication, inversion, and unit element, respectively.

### 3.1 Sections of a trivial quantum principal bundle

**Definition 6** For a trivial principal bundle \((C(P), C(X), G, \pi, \Gamma)\), let \(s : C(P) \to C(X)\) be a homomorphism such that \(s \pi = \text{id}\). We will call \(s\) a section of the bundle.

It turns out that, similarly as in the classical case, every section of a trivial quantum principal bundle determines a trivialization, and every pair of sections determines a bundle automorphism. This is the subject of the following lemma.

**Lemma 2**

a) There exists a canonical one-to-one correspondence between the set of sections of \((C(P), C(X), G, \pi, \Gamma)\) and the set of trivializations.

b) Every pair of sections \(s_1, s_2\) determines a unique bundle automorphism \(\Xi_{12} : C(P) \to C(P)\) such that \(s_1 \Xi_{12} = s_2\).

**Proof:** First let us observe that any trivialization \(\Phi\) may be used to determine a section, via
\[
s_\Phi = (\text{id} \otimes \varepsilon)\Phi.
\]
Observe now that given a section \(s' : C(X) \otimes C(G) \to C(X)\) of the product bundle, an automorphism of the product bundle is obtained by
\[
\Psi_{s'} = (s' \otimes \text{id})(\text{id} \otimes \Delta).
\]
That this is indeed an automorphism is verified by observing that \(\tau_{s'} : C(G) \to C(X)\), given by
\[
\tau_{s'} = s'(I \otimes \text{id}),
\]
satisfies the assumptions of lemma [1] and by the procedure of that lemma corresponds precisely to \(\Psi_{s'}\).
Next, observe that composing the section $s'$ with an automorphism $\Xi : C(X) \otimes C(G) \rightarrow C(X) \otimes C(G)$ gives

$$\Psi_{s'}\Xi = \Psi_{s'}\Xi.'$$

Taking an arbitrary trivialization $\Phi : C(P) \rightarrow C(X) \otimes C(G)$, we define $\Phi_s : C(P) \rightarrow C(X) \otimes C(G)$ by $\Phi_s = \Psi_{s'}\Phi$, where $s' = s\Phi^{-1}$. It is easily verified that $\Phi_s$ is independent of the choice of $\Phi$. Moreover, for any trivialization $\Phi'$,

$$\Phi_{s\Phi'} = \Phi'.$$

This establishes point a).

Now, take a pair of sections $s_1, s_2$. As a simple consequence of the above, we can write

$$\Xi_{12} = \Phi_{s_1}^{-1}\Phi_{s_2} : C(P) \rightarrow C(P),$$

which fulfills point b), completing the proof. □

4 Quantum principal bundles: general definition and basic properties

The present section begins with a general definition of (locally trivial) quantum principal bundles. This definition is basically a transcription, into the dual language of function algebras (now not necessarily commutative) of the usual classical definition, except for one feature: local triviality is prescribed by using a finite covering of the base space by closed (not open) sets. It is of course possible to use open coverings, but at the cost of complications related to the more difficult theory of noncompact quantum spaces (see e.g. [IR]). As the present approach is sufficient for the purposes addressed in this paper, we choose to leave this extension to future work.

Definition 7 Let $C(P), C(X)$ be $C^*$ algebras, $G = (C(G), u)$ a quantum group, $\pi : C(X) \rightarrow C(P)$ an injective homomorphism, $\Gamma : C(P) \rightarrow C(P) \otimes C(G)$ an action of $G$.

We will call $(C(P), C(X), G, \pi, \Gamma)$ a quantum principal bundle if:

a) $\Gamma\pi = \pi \otimes I$
b) there exists a finite covering \((C(U_i), \kappa_i)_{i \in I}\) of \(C(X)\) and a family \((\hat{\kappa}_i)_{i \in I}\) of surjective homomorphisms \(\hat{\kappa}_i : C(P) \to C(U_i) \otimes C(G)\) forming a covering of \(C(P)\) and obeying the following equations:

\[
\hat{\kappa}_i \pi = (\text{id} \otimes I) \kappa_i,
\]

\[
(\hat{\kappa}_i \otimes \text{id}) \Gamma = (\text{id} \otimes \Delta) \hat{\kappa}_i.
\]

Obviously, a trivial quantum principal bundle is a special case of the present definition: take \(\kappa = \text{id}_{C(X)}, \hat{\kappa} = \Phi\) — any trivialization. The definitions of bundle isomorphism and section can be trivially extended to the general case.

**Theorem 2** A quantum principal bundle admits a section iff it is trivial.

**Proof:** By definition, a trivial bundle admits a trivialization, and it was proven above that any trivialization determines a section.

To prove the converse: take the mapping

\[
\Phi_s : C(P) \to C(X) \otimes C(G)
\]

given by \(\Phi_s = (s \otimes \text{id}) \Gamma\). We claim that \(\Phi_s\) is a trivialization for any section \(s\).

It is easily verified that \(\Phi_s \pi = \text{id} \otimes I\) and \((\Phi_s \otimes \text{id}) \Gamma = (\text{id} \otimes \Delta) \Phi_s\). It remains to be shown that \(\Phi_s\) is bijective.

First, observe that given a section \(s\) and a family of local trivializations \(\hat{\kappa}_i\) over a covering \(\kappa_i\), the section \(s\) descends to a family of local sections \(s_i : C(U_i) \otimes C(G) \to C(U_i)\), fulfilling \(\kappa_i s = s_i \hat{\kappa}_i\) and \(s_i (\text{id} \otimes I) = \text{id}\). Indeed, since \(\hat{\kappa}_i\) are surjective, given (for fixed \(i\))

\[
C(U_i) \otimes C(G) \ni f_i = \hat{\kappa}_i(f)
\]

the element \(f \in C(P)\) is determined up to an element of \(\ker \hat{\kappa}_i\). For \(s_i\) to be well defined, we must show that \(\kappa_i s(\ker \hat{\kappa}_i) = \{0\}\).

Take \(h \in \ker \hat{\kappa}_i\); \(h\) may be uniquely decomposed into

\[
h = \pi s(h) + (h - \pi s(h)),
\]

where the second term is in \(\ker s\), in virtue of \(s \pi = \text{id}\). One has therefore

\[
\hat{\kappa}_i(h) = 0 = \hat{\kappa}_i \pi s(h) = (\text{id} \otimes I) \kappa_i s(h),
\]
hence $\kappa_is(h) = 0$, as required.

Next, the family of sections $s_i$ determines a family of automorphisms

$$\Phi_{s_i} : C(U_i) \otimes C(G) \to C(U_i) \otimes C(G),$$

$$\Phi_{s_i} = (s_i \otimes \text{id})(\text{id} \otimes \Delta).$$

One easily verifies that

$$\Phi_{s_i}\hat{\kappa}_i = (\kappa_i \otimes \text{id})\Phi_s. \tag{12}$$

Consider now the direct sum algebra $\bigoplus_i C(U_i) \otimes C(G)$, the direct sum of local automorphisms $\bigoplus_i \Phi_{s_i}$, and the morphisms

$$\hat{\mathcal{K}} : C(P) \to \bigoplus_i C(U_i) \otimes C(G),$$

$$\mathcal{K} \otimes \text{id} : C(X) \otimes C(G) \to \bigoplus_i C(U_i) \otimes C(G),$$

defined by

$$\hat{\mathcal{K}} = \bigoplus_i \hat{\kappa}_i,$$

$$\mathcal{K} \otimes \text{id} = \bigoplus_i (\kappa_i \otimes \text{id}).$$

Since $\bigcap_i \ker \kappa_i = \{0\}$ and $\bigcap_i \ker \hat{\kappa}_i = \{0\}$, $\hat{\mathcal{K}}$ and $\mathcal{K} \otimes \text{id}$ are injective. As a consequence of eq. 12, one has

$$(\bigoplus_i \Phi_{s_i})\hat{\mathcal{K}} = (\mathcal{K} \otimes \text{id})\Phi_s.$$

As stated above, $\mathcal{K} \otimes \text{id}$ is injective; moreover, $\bigoplus_i \Phi_{s_i}$ is bijective. Hence $\Phi_s$ is injective. But since $\hat{\mathcal{K}}$ is also injective, and

$$(\bigoplus_i \Phi_{s_i})^{-1}(\mathcal{K} \otimes \text{id})\Phi_s = \hat{\mathcal{K}},$$

it follows that $\Phi_s$ is bijective, completing the proof. □
4.1 Quantum Čech cocycles and bundle reconstruction

Now we introduce the concept of “cocycle on a quantum space with values in a quantum group”. This will play an important role in subsequent developments, especially in the following reconstruction theorem and in our main theorem. In fact — it may be loosely said — our definition makes the cocycles take values in the classical subgroup of the quantum group. That this should be the case could be foreseen on the basis of lemma 1.

**Definition 8** Let \( C(X) \) be a quantum space with covering \( \{ \kappa_i \} \), \( \kappa_i : C(X) \to C(U_i) \). Since \( \kappa_i \) are surjective, \( C(U_i) \approx C(X)/\ker \kappa_i \). We shall denote \( C(U_{ij}) \equiv C(X)/\{ \ker \kappa_i + \ker \kappa_j \} \), where \( \{ \ker \kappa_i + \ker \kappa_j \} \) is the smallest two-sided closed \(*\)-ideal containing \( \ker \kappa_i \) and \( \ker \kappa_j \). Analogously, \( C(U_{ijk}) \equiv C(X)/\{ \ker \kappa_i + \ker \kappa_j + \ker \kappa_k \} \). We shall also need the natural projections \( \Pi^{ij}_k : C(U_{ij}) \to C(U_{ijk}) \).

A collection of homomorphisms \( \tau_{ij} : C(G) \to Z(C(U_{ij})) \) will be called a \( C(G) \)-valued cocycle on \( C(X) \) associated with the covering \( \{ \kappa_i \} \) if the following conditions hold:

1. \( \tau_{ii} = I_{C(U_i)} \varepsilon \)
2. \( \tau_{ji} = \tau_{ij} \circ S \)
3. Defining the product
   \[
   \tau_{ij} * \tau_{jk} : C(G) \to C(U_{ijk})
   \]
   by
   \[
   \tau_{ij} * \tau_{jk} = m_{C(U_{ijk})}(\Pi^{ij}_k \tau_{ij} \otimes \Pi^{jk}_i \tau_{jk}) \Delta,
   \]
   require
   \[
   \tau_{ij} * \tau_{jk} = \Pi^{ik}_j \tau_{ik}.
   \]

**Definition 9** Two cocycles \( \{ \tau_{ij} \} \), \( \{ \tau'_{ij} \} \) associated with the same covering \( \{ \kappa_i \} \) of \( C(X) \) will be called equivalent if, for some family of homomorphisms \( \{ \sigma_i \} \), \( \sigma_i : C(G) \to Z(C(U_i)) \):

\[
\tau'_{ij} = m_{C(U_{ij})}(m_{C(U_i)} \otimes \text{id})(\Pi^i_j \otimes \text{id} \otimes \Pi^j_i)(\sigma_i \otimes \tau_{ij} \otimes \sigma_j S)(\text{id} \otimes \Delta) \Delta,
\]
where \( \Pi^i_j : C(U_i) \to C(U_{ij}) \) is the natural homomorphism onto the quotient.
Note: The above definition is correct, in spite of the occurrence of the quantum group antipode $S$ and the diagonal mapping $m_{C(U_i)}$. This is due to the fact that both $\sigma_i$ and $\tau_{ij}$ are required to be valued in the centers of the corresponding algebras. In particular, this implies that they vanish on the kernel of the classical projection of $C(G)$.

**Proposition 3** Any quantum principal bundle \((C(P), C(X), G, \pi, \Gamma)\) together with a set of local trivializations \((C(U_i), \kappa_i, \hat{\kappa}_i)\) determines a cocycle \(\{\tau_{ij}\}\) on \(C(X)\) valued in \(C(G)\), associated with the covering \(\{\kappa_i\}\).

**Proof:** First, observe that if \(\ker(\Pi_i^j \otimes \text{id})\hat{\kappa}_i = \ker(\Pi_i^j \otimes \text{id})\hat{\kappa}_j\), one can define \(\Phi_{ij}: C(U_{ij}) \otimes C(G) \to C(U_{ij}) \otimes C(G)\) by the formula

\[
\Phi_{ij} = [(\Pi_j^i \otimes \text{id})\hat{\kappa}_i][(\Pi_j^i \otimes \text{id})\hat{\kappa}_j]^{-1}
\]

and \(\Phi_{ij}\) thus defined will be a trivial principal bundle automorphism. Moreover, \(\Phi_{ij}\Phi_{ji} = \text{id}\) and

\[
[(\Pi_k^j \otimes \text{id})\Phi_{ij}][(\Pi_k^j \otimes \text{id})\Phi_{jk}] = (\Pi_k^j \otimes \text{id})\Phi_{ik}.
\]

By lemma 3, the corresponding \(\{\tau_{ij}\}\) fulfill the conditions for being a \(C(G)\)-valued cocycle on \(C(X)\) associated with \((\kappa_i, C(U_i))\). \(\square\)

The initial assumption follows from the subsequent lemma:

**Lemma 3** A \(G\)-invariant two-sided ideal \(i \subset C(U) \otimes C(G)\) is uniquely determined by \(i \cap \pi(C(U)) = i \cap [C(U) \otimes I]\), and is of the form \(j \otimes C(G)\), where \(j\) is a two-sided ideal in \(C(U)\).

**Proof:** According to theorem 1,

\[
i = \bigoplus_{\alpha \in \hat{G}} W_{\alpha},
\]

where \(\hat{G}\) is the set of irreducible inequivalent representations of \(G\), and \(W_{\alpha}\) are the corresponding invariant subspaces. Since the action \(\Gamma : i \to i \otimes C(G)\) is the restriction of \(\text{id} \otimes \Delta\), any set of elements \(f_j \in W_{\alpha}\) such that \(\Gamma f_j =
\[ \sum_k f_k \otimes u^{(a)}_{\kappa_j} \] must be of the form \( f_j = \sum_i h_i \otimes u^{(a)}_{\kappa_j} \), \( h_i \in C(U) \). Since \( i \) is an ideal in \( C(U) \otimes C(G) \),

\[
i \ni \sum_j (\sum_i h_i \otimes u^{(a)}_{\kappa_j})(I \otimes S^{(a)}(u^{(a)})) = h_k \otimes I,
\]

proving the claim. \( \square \)

Another simple consequence of the above lemma is the following corollary; essentially, it means that a \( G \)-invariant subset of the bundle space \( P \) is determined by its projection onto the base space \( X \).

**Corollary:** Consider \( \ker \hat{\kappa}_i \) — an ideal in \( C(P) \). For any \( j \), \( \hat{\kappa}_j(\ker \hat{\kappa}_i) \) is a \( G \)-right invariant ideal in the product bundle \( C(U_j) \otimes C(G) \), and is therefore determined by its invariant elements, which belong to \( \kappa_j(\ker \kappa_i) \otimes I \). An ideal \( i' \) in \( C(P) \) with the property that \( \hat{\kappa}_j(i') = \kappa_j(\ker \kappa_i) \otimes C(G) \) for all \( j \) is the one generated by \( \pi(\ker \kappa_i) \). But since \( \bigcap_j \ker \kappa_j = \{0\} \), this ideal is unique.

By the same token, \( \ker(\Pi_j^i \otimes id)\hat{\kappa}_i \) is generated by \( \pi(\ker \Pi_j^i \kappa_i) \).

However, \( \ker(\Pi_j^i \kappa_i) = \{\ker \kappa_i + \ker \kappa_j\} = \ker(\Pi_j^i \kappa_j) \). It thus follows that

\[
\ker(\Pi_j^i \otimes id)\hat{\kappa}_i = \ker(\Pi_j^i \otimes id)\hat{\kappa}_j.
\]

Now, let \( \{\hat{\kappa}_i\} \) and \( \{\hat{\kappa}_i'\} \) be two sets of local trivializations of a given bundle \( (C(P), C(X), G, \pi, \Gamma) \) associated with the same covering \( \{\kappa_i\} \) of the base space \( C(X) \). We state without proof the following

**Proposition 4** \( \{\hat{\kappa}_i\} \) and \( \{\hat{\kappa}_i'\} \) determine equivalent cocycles \( \{\tau_{ij}\} \) and \( \{\tau_{ij}'\} \), respectively.

It is moreover clear that two isomorphic principal bundles over the same base space, supplied with local trivializations over the same covering of the base space, also determine equivalent cocycles.

Now we proceed to consider the inverse problem: \( i.e. \) given an equivalence class of cocycles, we will reconstruct the corresponding quantum principal bundle, up to bundle isomorphism.
Theorem 3 Let \( \{ \tau_{ij} \} \) be a \( G \)-valued cocycle on the quantum space \( C(X) \), associated with the covering \(( C(U_i), \kappa_i )\). There exists a unique (up to isomorphism) quantum principal bundle \(( C(P), C(X), G, \pi, \Gamma )\) provided with a set of local trivializations \( \{ \hat{\kappa}_i \} \), such that \( \{ \tau_{ij} \} \) is the corresponding cocycle.

For any cocycle \( \{ \tau_{ij}' \} \) equivalent to \( \{ \tau_{ij} \} \), the corresponding bundle is the same (up to isomorphism) and the corresponding \( \{ \hat{\kappa}_i' \} \) are such that \( \hat{\kappa}_i' = \hat{\kappa}_i \Phi_i \), where \( \{ \Phi_i \} \) are automorphisms of the trivial bundles \( C(U_i) \otimes C(G) \).

Proof: To construct a representative \( C(P) \) of the isomorphism class of bundles fulfilling the claim of the theorem, we apply the procedure of connected sum of quantum spaces described in the Appendix. We form the connected sum of trivial bundles \( C(U_i) \otimes C(G) \); the overlaps between the components of the sum are given by \( C(U_{ij}) \otimes C(G) \), and the isomorphisms \( \Phi_{ij} \) between the overlaps are constructed from elements of the cocycle \( \{ \tau_{ij} \} \) following lemma \( 1 \). The bundle structure on the disjoint sum \( \bigoplus_i ( C(U_i) \otimes C(G) ) \) is given by its trivial bundle structure, i.e. \( \pi = \text{id} \otimes I \) and \( \Gamma = \text{id} \otimes \Delta \). It is easily verified that the connected sum is a \( G \)-invariant subalgebra of the disjoint sum, and that it contains the image of \( \pi \) restricted to \( C(X) \) (understood as a connected sum of \( C(U_i) \)). Finally, \( \{ \hat{\kappa}_i \} \) are given by the canonical projections onto the components of the connected sum. It is also clear that the cocycle determined by \( C(P) \) and \( \{ \hat{\kappa}_i \} \) is again \( \{ \tau_{ij} \} \).

A cocycle \( \{ \tau_{ij}' \} \) which is equivalent to \( \{ \tau_{ij} \} \) determines isomorphisms \( \Phi_{ij}' : C(U_{ij}) \otimes C(G) \to C(U_{ij}) \otimes C(G) \), which are given by \( \Phi_{ij}' = \Phi_i \Phi_{ij} \Phi_j^{-1} \), where \( \Phi_i \) are the projections to \( C(U_i) \) of a family of automorphisms of \( C(U_i) \otimes C(G) \). Together, the automorphisms \( \Phi_i \) determine an automorphism of the disjoint sum \( \bigoplus_i ( C(U_i) \otimes C(G) ) \), under which \( C(P) \) is taken to \( C(P') \). These two bundles are therefore isomorphic. \( \square \)

4.2 The classical sub-bundle
In this subsection we will show that all the data of a quantum principal bundle over \( C(X) \) with structure group \( G \) is actually contained in its ‘classical sub-bundle’, a bundle over \( C(X) \) with structure group \( G/\), the classical subgroup of \( G \).

Definition 10 Let the quantum group \( H = ( C(H), v ) \) be a subgroup of the quantum group \( G = ( C(G), u ) \), with \( \rho : C(G) \to C(H) \) — the subgroup
surjection. We will call \((C(Q), C(X), H, \pi_Q, \Gamma_Q)\) a sub-bundle of the principal bundle \((C(P), C(X), G, \pi_P, \Gamma_P)\) with ‘co-embedding’ \(\eta : C(P) \to C(Q)\), if \(\eta\) is a surjective homomorphism, and:

\[
\eta \pi_P = \pi_Q, \\
(\eta \otimes \rho) \Gamma_P = \Gamma_Q \eta.
\]

**Proposition 5** Let \((C(Q), C(X), H, \pi_Q, \Gamma_Q)\) be a sub-bundle of the principal bundle \((C(P), C(X), G, \pi_P, \Gamma_P)\). Given a set of local trivializations \(\{\hat{\lambda}_i\}\) of \(C(Q)\), the formula

\[
\hat{\kappa}_i = [((\text{id} \otimes \varepsilon_H) \hat{\lambda}_i \eta) \otimes \text{id}] \Gamma_P
\]

provides a set of local trivializations of the bundle \(C(P)\), over the same covering of \(C(X)\).

**Proof:** The algebraic properties required for \(\{\hat{\kappa}_i\}\) are easily verified. The proof that they are surjective proceeds by techniques analogous to those employed in Theorem 2. \(\square\)

**Theorem 4** Let \(C(P)\) and \(C(P')\) be two quantum principal bundles with the same base space \(C(X)\) and structure group \(G\). If \(C(P)\) and \(C(P')\) both have a sub-bundle \(C(Q)\) over \(C(X)\), with structure group \(H\) being a sub-group of \(G\) under the same co-embedding \(\rho : C(G) \to C(H)\), they are isomorphic.

**Proof:** Using Proposition 5 we construct local trivializations \(\{\hat{\kappa}_i\}, \{\hat{\kappa}'_i\}\) of \(C(P)\) and \(C(P')\), respectively, in both cases over the same covering of \(C(X)\). By Proposition 3 these data provide two \(G\)-valued cocycles on \(C(X)\) associated with this covering. Obviously, these cocycles are identical: they are obtained by composing the cocycle determined by \(C(Q)\) with the co-embedding \(\rho\). By Theorem 3, the bundles \(C(P)\) and \(C(P')\) are therefore isomorphic. \(\square\)

**Theorem 5** Let \((C(P), C(X), G, \pi, \Gamma)\) be a quantum principal bundle, \(\rho : C(G) \to C(G/)\) the homomorphism onto the classical subgroup.

There exists a unique sub-bundle \((C(P/), C(X), G/, \pi/, \Gamma/)\) with co-embedding \(\eta : C(P) \to C(P/)\), which we will call the classical sub-bundle.
Proof: Take an arbitrary set of local trivializations of \(C(P)\), \(\{\hat{\kappa}_i\}\), and define \(\ker \eta = \bigcap_i \ker (\text{id} \otimes \rho) \hat{\kappa}_i\). \(C(P)\) is defined to be \(C(P)/\ker \eta\), with \(\eta\) the natural projection onto the quotient.

Indeed, observe that for every \(i\), \(\ker (\text{id} \otimes \rho) \hat{\kappa}_i\) is \(G/\)-invariant, since \(\hat{\kappa}_i\) is \(G\)-covariant and \(\text{id} \otimes \rho\) is \(G/\)-covariant. Hence \((\text{id} \otimes \rho) \Gamma\) projects to a \(G/\)-action \(\Gamma/\) on \(C(P)/\). The atlas \(\{\hat{\kappa}_i\}\) on \(C(P)\) defines an atlas \(\{\hat{\kappa}_i/\}\) on \(C(P)/\) by the formula

\[
(\text{id} \otimes \rho) \hat{\kappa}_i = \hat{\kappa}_i/ \circ \eta. \tag{13}
\]

To complete the proof we now proceed to show that:

1. \(\pi(C(X)) \cap \ker \eta = \{0\}\), thus \(\pi/ = \eta \pi\) is injective, as required;

2. The subalgebra of \(G/\)-invariant elements in \(C(P)/\) is equal to \(\pi/(C(X))\).

For the first point, observe that, by the formula relating \(\hat{\kappa}_i\) and \(\kappa_i\) (definition 7, b),

\[
(\text{id} \otimes \rho) \hat{\kappa}_i \pi x = \kappa_i(x) \otimes I_{C(G/)}
\]

for all \(i\) and for all \(x \in C(X)\). However, \(\bigcap_i \ker \kappa_i = \{0\}\), proving the first claim above.

To prove the second claim: let \(f \in C(P)/\) be \(G/\)-invariant, \(i.e.\) \(\Gamma/f = f \otimes I\). Consider now \(\eta^{-1} f \subset C(P)\). For any \(f' \in \eta^{-1} f\),

\[ g = (\text{id} \otimes \mu_{G/})(\text{id} \otimes \rho) \Gamma f' \in \eta^{-1} f, \]

where \(\mu_{G/}\) is the Haar measure on \(G/\), is \(G/\)-invariant, since \(\eta^{-1} f\) is \(G/\)-invariant and closed in \(C(P)\). Thus for any \(f\) we can take \(f' \in \eta^{-1} f\) to be \(G/\)-invariant.

From \(f'\) we obtain a collection \(\{f_i\}, f_i \in C(U_i)\), by taking \(f_i = (\text{id} \otimes \varepsilon) \hat{\kappa}_i f'\). It remains to be shown that there exists an \(h \in C(X)\) such that for all \(i\), \(f_i = \kappa_i h\). If the latter holds, then obviously \(f = \eta \pi h\): since \(f'\) is \(G/\)-invariant, \(\hat{\kappa}_i f' \in C(U_i) \otimes C(G)\) is also \(G/\)-invariant. As a consequence,

\[
(\text{id} \otimes \rho) \hat{\kappa}_i f' = [(\text{id} \otimes \varepsilon) \hat{\kappa}_i f'] \otimes I,
\]

since for any \(G/\)-invariant \(x \in C(G)\), \(\rho(x) = I \varepsilon(x)\).

To show that such an \(h \in C(X)\) exists, it is sufficient to prove that for any pair \(i, j\),

\[
\Pi^i_j f_i = \Pi^j_i f_j.
\]

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To show this, we re-express the LHS by using the local automorphisms $\Phi_{ij}$ introduced in the proof of Proposition 3 and their expression in terms of the cocycle $\{\tau_{ij}\}$, leading to

$$\Pi_{ij}f_i = m_{C(U_{ij})}(\Pi_i^j \otimes \tau_{ji})\hat{\kappa}_j f'.$$

Now, since $\ker \rho \subset \ker \tau_{ji}$, one may write $\tau_{ji} = \tau_{ji}' \rho$. Making use of the $G/\text{-invariance of } f'$, we obtain

$$m_{C(U_{ij})}(\Pi_i^j \otimes \tau_{ji})\hat{\kappa}_j f' = m_{C(U_{ij})}(\Pi_i^j \otimes \tau_{ji}'(I_{C(G)}\varepsilon))\hat{\kappa}_j f' = \Pi_i^j (\text{id} \otimes \varepsilon)\hat{\kappa}_j f' = \Pi_i^j f_j.$$

This proves the claim. □

**Theorem 6** Let $(C(Q), C(X), H, \pi_Q, \Gamma_Q)$ be a quantum principal bundle such that its structure group $H$ is the classical subgroup of a certain quantum group $G$: $C(H) = C(G/).$ Then there exists a unique (up to isomorphism) principal bundle $(C(P), C(X), G, \pi_P, \Gamma_P)$ such that $C(Q)$ is its classical sub-bundle.

**Proof:** The bundle $(C(P)$ is obtained via the Reconstruction Theorem (Th. 3). It suffices to observe that the cocycle uniquely determined by $C(Q)$ and a set of its local trivializations $(\hat{\kappa}_i^Q, C(U_{ij}))$ extends uniquely to a $G$-valued cocycle (over the same covering of $C(X)$):

$$\tau_{ij}^P : C(G) \to C(U_{ij}),$$

$$\tau_{ij}^P = \tau_{ij}^Q \rho,$$

where $\rho$ is the canonical epimorphism $\rho : C(G) \to C(H)$. It is obvious that the bundle reconstructed from $\{\tau_{ij}^P\}$ fulfills the claim of the theorem. □

**Remark:** The bundle $C(P)$ may also be reconstructed as

$$C(P) = \{ f \in C(Q) \otimes C(G) : (\Gamma_Q \otimes \text{id} - \text{id} \otimes (\rho \otimes \text{id}) \Delta) f = 0 \},$$

with $\Gamma_P = \text{id} \otimes \Delta |_{C(P)}$, $\eta : C(P) \to C(Q)$ given by $\eta = \text{id} \otimes \varepsilon |_{C(P)}$, etc. We leave the proof as an exercise to the reader.
5 Associated bundles

In the classical situation, given a principal bundle $P$ with structure group $G$, and a vector space $V$ which carries a representation of $G$, it is standard to define a vector bundle associated to $P$ as a suitable quotient space of $P \times V$. In this brief section we limit ourselves to giving a quantum analog of this definition, and stating and proving a simple proposition: a bundle associated to a trivial principal bundle is itself trivial.

**Definition 11** Let $V$ be a finite-dimensional linear space carrying a representation $T$ of the quantum group $G$. A bundle associated to the principal bundle $(C(P), C(X), G, \pi, \Gamma)$, corresponding to the representation $T$, is the subspace $F$ of $C(P) \otimes V$ determined by

$$F = \{ \alpha \in C(P) \otimes V : (\Gamma \otimes \text{id} - \text{id} \otimes T)\alpha = 0 \}.$$  

$F$ is naturally endowed with the structure of a left module over $C(X)$: for $a \in C(X), \alpha \in F$, $a \cdot \alpha = \pi(a)\alpha$.

**Proposition 6** Let $C(P)$ be a trivial principal bundle. Then any associated bundle $F$ is trivial, i.e. $F \approx C(X) \otimes V$.

*Proof:* $C(P)$, being trivial, may be identified (up to isomorphism) with $C(X) \otimes C(G)$. Thus $F$ is a subspace of $C(X) \otimes C(G) \otimes V$. Note that $\text{id} \otimes T : C(X) \otimes V \to C(X) \otimes C(G) \otimes V$ is an injective mapping into $F \subset C(X) \otimes C(G) \otimes V$. We will show that the image of $\text{id} \otimes T$ is actually equal to $F$. Indeed, for $\alpha \in F$:

$$(\text{id} \otimes T)(\text{id} \otimes \varepsilon \otimes \text{id})\alpha = (\text{id} \otimes \varepsilon \otimes \text{id} \otimes \text{id})(\text{id} \otimes \text{id} \otimes T)\alpha = (\text{id} \otimes \varepsilon \otimes \text{id} \otimes \text{id})(\text{id} \otimes \Delta \otimes \text{id})\alpha = \alpha,$$

proving the claim. □
6 Examples

In the present section we will give a few elementary examples of quantum principal fiber bundles over non-commutative base spaces, with the matrix quantum group $SU_q(2)$ as structure group. We begin by recalling the definition of the quantum group $SU_q(2)$ [12]:

**Definition 12** Let $q \in [-1, 1] \setminus \{0\}$. The quantum group $SU_q(2) = (A, u)$, where $A$ is the universal $C^*$ algebra generated by two elements $\alpha, \gamma$ satisfying the relations

\[
\begin{align*}
\alpha^*\alpha + \gamma^*\gamma &= I, \\
\alpha\alpha^* + q^2\gamma^*\gamma &= I, \\
\gamma\gamma^* &= \gamma^*\gamma, \\
\alpha\gamma &= q\gamma\alpha, \\
\alpha\gamma^* &= q\gamma^*\alpha,
\end{align*}
\]

and

\[
\begin{align*}
u &= \left( \begin{array}{c}
\alpha, \\
-q\gamma^* \\
\gamma, \\
\alpha^*
\end{array} \right). \quad (15)
\end{align*}
\]

For $q = 1$ this reduces to the classical $SU(2)$ group. For $q \neq 1$, the classical subgroup $SU_q(2)/$ is given by $SU_q(2)/ = (C(S^1), v)$, where $S^1 = \{e^{i\phi} : \phi \in \mathbb{R}\}$ and

\[
\begin{align*}
v &= \left( \begin{array}{c}
\zeta, \\
0 \\
0, \\
\zeta
\end{array} \right),
\end{align*}
\]

with $\zeta \in C(S^1)$, $\zeta(e^{i\phi}) = e^{i\phi}$. The co-embedding $\rho : SU_q(2) \to C(S^1)$ is given by $\rho(\alpha) = \zeta$, $\rho(\gamma) = 0$.

For our first example, we will consider $SU_q(2)$ principal bundles over the quantum disk $[6]$. The quantum disk is defined in the following:

**Definition 13** Let $\mu \in (0, 1)$. The quantum disk $C(D_\mu)$ is the universal $C^*$ algebra generated by the element $z$ satisfying the relation

\[
zz^* - z^*z = \mu(I - zz^*)(I - z^*z). \quad (16)
\]

In [6] it was shown that the algebra $C(D_\mu)$ has the following inequivalent irreducible $*$-representations:

1. A family of one-dimensional representations (functionals) defined by $z \mapsto e^{i\phi}$, with $\phi \in \mathbb{R}$. We see that this family of functionals is parametrized by elements of $S^1$: one may say that $S^1$ forms the classical boundary of the quantum space $D_\mu$. 

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2. An infinite-dimensional representation \( t \) defined as follows: let \( \mathcal{H} \) be the Hilbert space spanned by an orthonormal basis \( \{e_n\}_{n \in \mathbb{N}} \). Then
\[
t(z)e_n = \begin{cases} 0, & n = 0 \\ \frac{n\mu}{1+n\mu}e_{n-1}, & n \geq 1; \end{cases}
\tag{17}
\]
\[
t(z^*)e_n = \sqrt{\frac{(n+1)\mu}{1+(n+1)\mu}}e_{n+1}.
\tag{18}
\]
Furthermore (see Th. IV.7 of [3]), the algebra \( C(D_\mu) \) is isomorphic to \( C^*(S) \), the unital \( C^* \) algebra generated by the operator \( S \) on \( \mathcal{H} \), defined by \( Se_n = e_{n+1} \) for all \( n \in \mathbb{N} \). This is true independently of the value of \( \mu \in (0,1) \).

In particular, it follows that the representation \( t \) is faithful. The problem of classifying \( SU_q(2) \) principal bundles over \( D_\mu \) is solved by the following lemma:

**Lemma 4** A \( C^* \) algebra \( C(X) \) which admits a faithful irreducible representation does not admit any nontrivial covering; that is, given any covering \( \{\kappa_i\}, \{C(U_i)\}_{i=1}^n \) of \( C(X) \), for some \( i \), \( \ker \kappa_i = \{0\} \).

**Proof:** Let us assume, to the contrary, that for all \( i \), \( \ker \kappa_i = \{0\} \). Since \( \bigcap_i \ker \kappa_i = \{0\} \), then for any collection \( \{f_i\}_{i=1}^n \) such that \( f_i \in \ker \kappa_i \), we have
\[
f_n f_{n-1} \cdots f_2 f_1 = 0.
\]
Since we have a faithful representation, then (identifying elements of \( C(X) \) with their images under this representation) for any \( f_i \in \ker \kappa_1 \) there exists an \( x_1 \in \mathcal{H} \) such that \( f_1 x_1 \neq 0 \). Note that, since \( \ker \kappa_1 \) is an ideal in \( C(X) \), and the representation is irreducible, the image of \( x_1 \) under the action of \( \ker \kappa_1 \) must form at least a dense subset in \( \mathcal{H} \) — being a subspace of \( \mathcal{H} \) invariant under the action of \( C(X) \). Now, for any \( f_2 \in \ker \kappa_2 \), there must exist, in \( \ker \kappa_1 x_1 \), a vector \( x_2 \) such that \( f_2 x_2 = f_2 f_1 x_1 \neq 0 \). Applying this argument repeatedly, we obtain
\[
f_n f_{n-1} \cdots f_2 f_1 \neq 0,
\]
contradicting the assumption. \( \square \)

From the above lemma we conclude that \( D_\mu \) does not admit nontrivial coverings. As a simple consequence, we can now state:
**Proposition 7** All quantum principal fiber bundles over the quantum disk are trivial.

As another example, we now consider a ‘quantum sphere’, obtained by gluing together two copies of the quantum disk $D_\mu$. This procedure is an instance of the general construction of connected sum of quantum spaces (see Appendix), and does not differ essentially from gluing together two ordinary (classical) disks to form a classical sphere $S^2$.

**Definition 14** By quantum two-dimensional sphere $C(S^2_\mu)$ we mean the subalgebra of the direct sum $C(D_\mu) \oplus (D_\mu)$ determined by the condition

$$C(S^2_\mu) = \{ f_1 \oplus f_2 \in C(D_\mu) \oplus C(D_\mu) : \psi(f_1) = \psi(f_2) \text{ for all functionals } \psi \}.$$ 

Equivalently, since (see above) the set of functionals on $C(D_\mu)$ may be identified with $S^1$, we can introduce the classical projection $\rho : C(D_\mu) \to C(S^1)$ by

$$\rho(f)(\psi) = \psi(f);$$

then the condition above reads $\rho(f_1) = \rho(f_2)$.

The above construction provides automatically a non-trivial two-element covering of $S^2_\mu$, given by

$$\kappa_{1,2} : C(S^2_\mu) \to C(D_\mu), \quad \kappa_{1,2}(f_1 \oplus f_2) = f_{1,2}.$$ 

It is easily seen that $\text{ker } \kappa_1 \cap \text{ker } \kappa_2 = \{0\}$, thus we indeed have a covering. Observe now that

$$C(S^2_\mu) \supset \text{ker } \kappa_1 = \{0 \oplus f_2 : \rho(f_2) = 0\},$$

and similarly for $\text{ker } \kappa_2$; thus $\{\text{ker } \kappa_1 + \text{ker } \kappa_2\}$ consists of elements of the form $f_1 \oplus f_2$ such that $\rho(f_1) = \rho(f_2) = 0$. Therefore,

$$C(S^2_\mu)/\{\text{ker } \kappa_1 + \text{ker } \kappa_2\} = C(S^2_\mu)/\ker(\rho \oplus \rho) = C(S^1).$$

One may thus say that the two quantum disks intersect along an ‘equator’, which is an ordinary circle $S^1$.

\textsuperscript{2}The quantum sphere here defined in fact coincides, as a $C^*$-algebra, with those introduced by Podle\'s in\textsuperscript{8}.
According to theorem 3, principal bundles over the base space $C(S^2_\mu)$ may be reconstructed from a cocycle $\tau : C(G) \to C(S^1)$. Following theorem 3, $\tau$ is determined uniquely by a cocycle $\tau/ : C(G/) \to C(S^1)$, valued in the classical subgroup of $C(G)$. It follows that the classification of the $G$ quantum principal bundles over $C(S^2_\mu)$ obtained as above is equivalent to that of classical $G/$ principal bundles over $S^2$. In particular, $SU_q(2)$ bundles over $C(S^2_\mu)$ are classified by the integers (for $q \neq 1$).

In fact, the above classification of bundles over $C(S^2_\mu)$ is exhaustive: That any principal bundle with base space $C(S^2_\mu)$ admits a set of trivializations over the covering we have been using is a simple consequence of corollary 4.1 and proposition 7.

For the next example 3, consider the quantum space obtained from $D_\mu$ by identifying all points of the boundary $S^1$, i.e. the subalgebra of $C(D_\mu)$ consisting of elements $f$ such that $\rho(f) = cI \in C(S^1)$, $c \in C$. As shown by the analysis of 4, this algebra has only one nontrivial ideal, given by $\rho(f) = 0$; therefore, it does not admit any nontrivial covering. As a consequence, all principal bundles with this base space are trivial.

On the other hand, the above algebra is obtained in 8 as a quantum quotient space of $SU_q(2)$ by an action of its classical subgroup $U(1)$. The quantum group $SU_q(2)$ thus displays part of the features of a $U(1)$-principal bundle over the quotient (for $q = 1$ this is the Hopf fibration of $S^3 \approx SU(2)$ over $S^2$). It is not, however, a principal bundle according to our definition (for $q \neq 1$), as it is not locally trivial.

A slight extension of the above examples may be obtained by removing one or more non-intersecting disks from a classical sphere $S^2$ and gluing quantum disks onto the $S^1$ boundaries. The reader will easily find that the classification of $G$-principal quantum bundles over the base spaces thus obtained reduces to that of $G/$-principal classical bundles over $S^2$. The corresponding remains true with $S^2$ replaced by any compact two-dimensional surface.

It remains a challenge to find interesting examples where quantum spaces are glued together in an ‘essentially non-commutative’ way.

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3This was also introduced in 8 as an example of a quantum sphere.
A Appendix

Compact quantum spaces are a generalization of the notion of compact topological spaces. It turns out that many basic notions of the theory of compact topological spaces admit noncommutative extensions. In this appendix, we review such extensions for those notions which find essential application in the present paper: cartesian product, closed subset, intersection and union of closed subsets, disjoint and connected union of quantum spaces, covering of a quantum space by closed subsets, and classical subset of a quantum space. All our definitions are straightforward dualizations of the corresponding definitions for point sets; we choose to summarize them here for the sake of completeness and clarity.

We recall that within the approach adopted in the present paper, a compact quantum space (a. k. a. noncommutative topological space) is represented by a (separable) unital $C^*$-algebra, in general not commutative. In the commutative case, this $C^*$-algebra may be identified with the algebra $C(X)$ of continuous complex functions on an ordinary compact topological space $X$; we denote by $C(X)$ the algebra corresponding to the quantum space $X$ also in the noncommutative case. Mappings between quantum spaces correspond to unital $C^*$-homomorphisms; in particular, homeomorphisms are represented by algebra isomorphisms.

Fiber bundles are a generalization of cartesian products of topological spaces. For our purposes, a suitable extension of the notion of cartesian product to quantum spaces is the following: consider the $C^*$-algebras $C(X_1)$ and $C(X_2)$. By a theorem due to Gelfand, Naimark and Segal (see [3]), every (separable) $C^*$-algebra has a faithful continuous representation in $B(H)$, the algebra of bounded linear operators on a (separable) Hilbert space $H$. Moreover, this representation is isometric (norm-preserving). Thus $C(X_1)$ and $C(X_2)$ may be identified with certain closed subalgebras of $B(H)$. Their algebraic tensor product is contained in $B(H) \otimes B(H) \subset B(H \otimes H)$. Completing the image of $C(X_1) \otimes C(X_2)$ with respect to the norm in $B(H \otimes H)$, we obtain a separable $C^*$-algebra, which we identify as the tensor product of $C(X_1)$ with $C(X_2)$. It turns out that the resulting algebra does not depend on the choice of (faithful) representations of $C(X_1)$ and $C(X_2)$. The quantum space corresponding to this algebra will be understood as the cartesian product of the quantum spaces $X_1$ and $X_2$. For commutative algebras $C(X_1)$ and $C(X_2)$, the above construction yields the algebra of continuous functions
on the (topological) cartesian product $X_1 \times X_2$.

By a closed subset $Y$ of the quantum space $X$ we mean the quantum space represented by a $C^*$-algebra $C(Y)$ obtained as a quotient algebra of $C(X)$ by a (closed, two-sided) $*$-ideal $i$. The natural projection homomorphism of $C(X)$ onto the quotient plays the role of the canonical embedding of $Y$ in $X$.

Given two closed subsets $Y_1, Y_2$ of the quantum space $X$, we can define their intersection $Y_1 \cap Y_2 \subset X$ as the quantum space corresponding to the quotient of $C(X)$ by the minimal ideal containing both $i_1$ and $i_2$, the ideals involved in constructing $Y_1$ and $Y_2$. Obviously this ideal $i_{12}$ is unique and consists of elements of the form $f_1 + f_2$, $f_1 \in i_1$, $f_2 \in i_2$. This allows us to treat $Y_1 \cap Y_2$ also as a subset of $Y_1$ and $Y_2$.

Similarly, the union $Y_1 \cup Y_2 \subset X$ is represented by the quotient of $C(X)$ by $i_1 \cap i_2$, and clearly contains $Y_1$, $Y_2$ and $Y_1 \cap Y_2$ as closed subspaces. The reader will easily find that the above definitions extend straightforwardly to arbitrary finite families of closed subsets of a quantum space, and obey the usual identities of set calculus.

Now we introduce a few notions enabling elementary surgery operations on quantum spaces. The disjoint union of quantum spaces, $X_1 \cup X_2$, is represented by the direct sum algebra $C(X_1) \oplus C(X_2)$, and obviously contains $X_1$ and $X_2$ as closed subspaces, with $X_1 \cap X_2 = \emptyset$ (the empty set is represented by the trivial algebra $\{0\}$). Given some additional data, one may furthermore form connected sums of quantum spaces: take $Y_1 \subset X_1$ and $Y_2 \subset X_2$, closed subsets of the respective quantum spaces; provided $Y_1$ and $Y_2$ are isomorphic, a choice of this isomorphism allows us to identify them, giving a connected union of $X_1$ with $X_2$. The precise definition is as follows: let $C(Y_1) = C(X_1)/i_1$ and $C(Y_2) = C(X_2)/i_2$, and let $\Phi_{12} : C(Y_1) \to C(Y_2)$ be a given $C^*$ algebra isomorphism. The connected union of $X_1$ and $X_2$ corresponding to these data is defined to be the quantum space represented by the $C^*$ subalgebra of $C(X_1) \oplus C(X_2)$ consisting of elements $(f_1, f_2)$ obeying the condition

$$\Phi_{12}\Pi_1 f_1 = \Pi_2 f_2,$$

where $\Pi_{1,2}$ denote the respective embedding homomorphisms. The above construction generalizes in a straightforward way to finite families of quantum spaces, provided the corresponding isomorphisms $\Phi_{ij}$ obey suitable consistency conditions, in exact analogy with the case of ordinary topological
spaces. In the case when \( X_1 \) and \( X_2 \) are themselves given as closed subsets of another quantum space, then with the natural choice for the required data, their connected union coincides with their union as subsets.

A (finite) closed covering of a quantum space \( X \) is a finite family of closed subsets \( U_i \) of \( X \) such that \( X = \bigcup_i U_i \). In more detail, a covering of \( X \) is given by a finite family of algebras \( C(U_i) \) and surjective homomorphisms \( \kappa_i : C(X) \to C(U_i) \), such that \( \bigcap_i \ker \kappa_i = \{0\} \). It must be noted here that existence of a nontrivial (finite closed) covering is a restrictive condition on the quantum space \( X \); by nontrivial covering we mean that \( \ker \kappa_i \neq \{0\} \) for each \( i \).

As the final notion, we now introduce the classical subset \( X/ \) of the quantum space \( X \). This is defined as the quantum space represented by the quotient of \( C(X) \) by its commutator ideal, i.e. the smallest closed \( * \)-ideal containing all elements of the form \( fg - gf \), for all \( f, g \in C(X) \). We see that \( C(X/ \) thus obtained is a commutative \( C^* \) algebra; thus, by the Gelfand-Naimark theorem, it may be identified with the algebra of continuous functions on the set of its (linear and multiplicative) functionals. This set carries the natural structure of a compact topological space, and we identify it with \( X/ \).

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