Computing the dimension of ideals in group algebras, with an application to coding theory

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Abstract
We study the problem of computing the dimension of a left/right ideal in a group algebra $F[G]$ of a finite group $G$ over a field $F$ by relating the dimension to the rank of an appropriate matrix, originating from a regular right/left representation of $G$. In particular, the dimension of a principal ideal is equal to the rank of the matrix representing a generator. From this observation, we establish a bound and an efficient algorithm for the computation of the dimension of an ideal in a group ring. Since group codes are ideals in finite group rings, our algorithm allows to efficiently compute their dimension.

Keywords: Group algebra, ideal, group code, representation, rank, characteristic polynomial.

Mathematics Subject Classification (2010): 20C05, 16D25, 16S34

1 Introduction and preliminaries

Let $G = \{g_1, g_2, \ldots, g_n\}$ be a finite multiplicative group of order $n = |G|$, with neutral element $g_1 = 1$. Let $F$ be a field of characteristic $p$. Finite fields of order $q = p^m$ are denoted as $F_q$. The group algebra $F[G]$ of $G$ over $F$ consists of the formal sums

$$\sum_{j=1}^{n} \alpha_i g_i, \quad \alpha_i \in F, \quad g_i \in G,$$

\[\text{(1)}\]

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where the $\alpha_i's$ are called coefficients. The sum in $F[G]$ is defined coefficientwise, that is the coefficients of the same $g_j$ are added according to the addition in $F$. The product is performed by applying the distributive law, and the group elements are multiplied according to the rule in $G$. The group algebra $F[G]$ is a vector space of dimension $n$ over the field $F$, and has the structure of an associative ring with identity. It is commutative if and only if $G$ is commutative. It turns out that the structure of the ideals depends on the group $G$ and the field characteristic. If the field characteristic does not divide the group order, or is $0$, the group ring is semisimple by Maschke’s Theorem (see [8]). In addition, every ideal is principal and generated by an idempotent, [3]. If the field characteristic divides the group order, then the group ring is not semisimple in general [10].

A group code of length $n$ is a linear code which is the image of an ideal $I \subseteq F[G]$ via an isomorphism $\phi : F[G] \to F^n$. In other words, an ideal of $F[G]$ is a group code in $F^n$, for a given choice of a basis of $F[G]$. Denote by $C = \phi(I)$ the group code corresponding to the ideal $I$, then $C$ is an $[n,k]$-code, where $k = \dim F I$ (see [12] for more details). In this paper, we study the problem of how to efficiently compute the dimension of $I$. Our approach uses some elementary tools from representation theory.

Every representation $D : G \to GL_m(K)$ of $G$ over an extension field $K$ of $F$ induces a representation of the group algebra $F[G]$, which we denote again by $D$

$$
\sum_{j=1}^n \alpha_j g_j \mapsto \sum_{j=1}^n \alpha_j D(g_j).
$$

Here $M_m(K)$ is the ring of $m \times m$ matrices with entries in $K$. In particular a regular representation of $G$ induces a representation of $F[G]$ over $F$.

The main results of the paper are contained in Section 2. In Proposition 1 we relate the dimension of a proper left/right ideal $I \subseteq F[G]$ to the rank of a matrix constructed using a regular right/left representation of $G$. In Proposition 4 and Theorem 5 we discuss how to compute an idempotent generator of a given left/right ideal and show that the dimension of the left/right ideal that it generates can be obtained from the characteristic polynomial of the matrix from Proposition 1. In Corollary 7 we reduce the computation of the dimension of a principal left/right ideal to the computation of a characteristic polynomial, even in the case when the ideal is not generated by an idempotent. In Section 3 we discuss the applications to coding theory and present some examples.
2 Ideal dimension

Computing the dimension of a left/right ideal in \(\mathbb{F}[G]\) has both theoretical relevance and practical applications. The solution that we propose is based on the representation of \(\mathbb{F}[G]\) induced by a regular representation of \(G\). In the next proposition, we relate the dimension of a left/right ideal to the rank of a matrix built using the regular right/left representation of \(G\). In the rest of the section, we discuss how to efficiently compute this rank. In particular, we relate it to the characteristic polynomial of a suitable matrix.

Proposition 1. Let \(G = \{g_1, \ldots, g_n\}\) be a group, \(\mathbb{F}\) be a field, and let \(A = \mathbb{F}[G]\). Let \(f_1, \ldots, f_t \in A, f_i = \sum_{j=1}^n \alpha_{ij}g_j\) with \(\alpha_{ij} \in \mathbb{F}\). Let \(I = A f_1 + \ldots + A f_t\) and \(J = f_1 A + \ldots + f_t A\). Let \(\rho(f_1), \ldots, \rho(f_t)\) be the matrices corresponding to \(f_1, \ldots, f_t\) in the representation of \(A\) induced by the regular right representation \(\rho\) of \(G\). Let \(\lambda(f_1), \ldots, \lambda(f_t)\) be the matrices corresponding to \(f_1, \ldots, f_t\) in the representation of \(A\) induced by the regular left representation \(\lambda\) of \(G\). Define block matrices

\[
\rho(I) = \begin{bmatrix} \rho(f_1) \\ \vdots \\ \rho(f_t) \end{bmatrix} \quad \text{and} \quad \lambda(J) = \begin{bmatrix} \lambda(f_1) \\ \vdots \\ \lambda(f_t) \end{bmatrix}.
\]

Then

\[
\dim_{\mathbb{F}} I = \text{rank} \rho(I) \quad \text{and} \quad \dim_{\mathbb{F}} J = \text{rank} \lambda(J).
\]

Proof. For each \(f = \sum_{i=1}^n a_i g_i \in A\), define \(f^* = \sum_{i=1}^n a_i g_i^{-1}\). We have \(\rho(f^*) = (m_{ij}) \in M_n(\mathbb{F})\), where

\[
g_i f = m_{i1} g_1 + \cdots + m_{in} g_n.
\]

In other words, the entries in the i-th row of \(\rho(f_k^*)\) are the coefficients of \(g_i f_k\) with respect to the \(\mathbb{F}\)-basis \(g_1, \ldots, g_n\) of \(A\). The elements \(g_i f_k\) for \(i = 1, \ldots, n\) and \(k = 1, \ldots, t\) generate \(I\) as \(\mathbb{F}\)-vector space, hence

\[
\dim_{\mathbb{F}} I = \text{rank} \begin{bmatrix} \rho(f_1^*) \\ \vdots \\ \rho(f_t^*) \end{bmatrix}.
\]

The permutation of \(G\) that exchanges \(g_i\) and \(g_i^{-1}\) for each \(i\) induces a permutation of the columns...
of $\rho(f^*)$ for all $f \in A$, which sends $\rho(f^*)$ to $\rho(f)$. Hence

$$\dim_F I = \text{rank} \begin{bmatrix} \rho(f_1^*) \\ \vdots \\ \rho(f_t^*) \end{bmatrix} = \text{rank} \begin{bmatrix} \rho(f_1) \\ \vdots \\ \rho(f_t) \end{bmatrix} = \text{rank} \rho(I).$$

Similarly, for the right ideal $J = f_1A + \cdots + f_tA$ we consider the regular left representation $\lambda$ of $G$. The entries in the $i$-th row of $\lambda(f_k)$ are the coefficients of $f_kg_i$ with respect to the $F$-basis $g_1, \ldots, g_n$. Since the elements $f_kg_i$ generate $J$ as an $F$-vector space, the rank of $\lambda(J)$ equals the dimension of $J$. □

The computation of characteristic polynomials is straightforward, and can be used to compute or bound the rank of a matrix. In this context, Proposition 1 has interesting consequences. We start by establishing a preliminary result. This result is essentially contained in Theorem 24.2 of [3], but we present it here in the form that we will need.

**Lemma 2.** Let $A$ be a semisimple ring. Let $I \subset A$ be a proper left (resp., right) ideal. Then $I = Ae$ (resp., $I = eA$), where $e \in A$ is an idempotent and $a = ae$ (resp. $a = ea$) for all $a \in I$. Further, any idempotent generator of $I$ is of the form $e = 1 - \epsilon$, where $\epsilon$ is an idempotent and $\epsilon e = e\epsilon = 0$.

If in addition $I$ is a two-sided ideal, then $I = (e)$ for some $e \in A$ idempotent. Moreover $A = I \oplus J$ where $J = 0 :_A I$ is the annihilator of $I$, and $J = (\epsilon)$.

**Proof.** We give the proof for the case of left ideals. The proof for right ideals is analogous. Since $A$ is semisimple, $A = I \oplus J$, where $J \subset A$ is a left ideal. Write $1 = e + \epsilon$, where $e \in I$ and $\epsilon \in J$.

Multiplying the identity on the left by $e$, and using the fact that $J$ is a left ideal, we obtain that

$$ee = e - e^2 \in I \cap J = 0.$$

Therefore $e = e^2 \in I$ is an idempotent, and the same holds for $\epsilon \in J$.

Clearly $Ae \subseteq I$. In order to show the reverse inclusion, observe that for any $a \in I$ we have $ae = a - ae \in I \cap J = 0$, therefore $a = ae \in Ae$. It follows that $I = Ae$ and $a = ae$ for all $a \in I$. This also shows that $J = Ae$ and $ae = a$ for all $a \in J$.

If in addition $I$ is a two-sided ideal, then $A = I \oplus J$ where $J$ is also a two-sided ideal. Hence $I = (e)$ and $J = (\epsilon)$ with $ee = e\epsilon = 0$. Therefore $IJ = JI = 0$ and

$$J \subseteq 0 :_A I = \{a \in A \mid aI = Ia = 0\}.$$
Conversely, let $a \in 0 :_A I$. Then $a = ae + ae = ae \in J$. Therefore $J = 0 :_A I$, as claimed.

**Remarks 3.**

1. Since in general a semisimple ring has elements which are not idempotents, there exist ideals which are generated by an element which is not an idempotent. However, the lemma implies that they have another generator, which is an idempotent.

   E.g., let $A = \mathbb{F}_5[S_3]$ and let $f = 1 + (12) \in A$. Then $f^2 = 2f \neq f$, however $Af = A(3f)$ and $e = 3f$ is idempotent.

2. An idempotent matrix $M \in M_n(\mathbb{F})$ has minimal polynomial $z, z - 1$ or $z^2 - z$. Hence its characteristic polynomial has the form $z^k(z - 1)^{n-k}$ for some $0 \leq k \leq n$.

   In the previous example, following the notation of Proposition 1 we have

   $\rho(f) = \begin{bmatrix}
   1 & 1 & 0 & 0 & 0 & 0 \\
   1 & 1 & 0 & 0 & 0 & 0 \\
   0 & 0 & 1 & 1 & 0 & 0 \\
   0 & 0 & 1 & 1 & 0 & 0 \\
   0 & 0 & 0 & 1 & 1 \\
   0 & 0 & 0 & 1 & 1 
\end{bmatrix}$.

   The matrix $\rho(f)$ is not idempotent, and has characteristic polynomial $z^3(z - 2)^3$. Notice that $2 \in \mathbb{F}_5^*$ has order 4, and $\rho(f)^4 = 3\rho(f)$ has characteristic polynomial $z^3(z - 1)^3$.

Remark 3.1 raises the question of how to compute an idempotent generator of a given ideal $I$. E.g. let $I = Af$ be a left ideal with a given generator $f \in A$. Then $e - 1 \in \{a \in A \mid fa = 0\}$, but the right annihilator of $f$ is not always easy to compute. However, the problem has a simple solution if we consider a representation of $A$, since it reduces to a simple linear algebra problem.

In the next proposition we discuss the case of left ideals. Right ideals can be treated similarly.

**Proposition 4.** Let $D$ be a representation of $G = \{g_1, \ldots, g_n\}$ such that $D(g_1), \ldots, D(g_n)$ are linearly independent. Then $D$ induces an injective representation $D$ of $A = \mathbb{F}[G]$, and the right annihilators of $f \in A$ corresponds via $D$ to the solutions over $\mathbb{F}$ of the linear system

$$D(f) \left( \sum_{i=1}^{n} x_i D(g_i) \right) = 0$$
in the variables \(x_1\ldots x_n\). In other words, solutions \((a_1,\ldots,a_n)\in\mathbb{F}^n\) of the system corresponds to right annihilators \(a = a_1g_1 + \ldots + a_ng_n\) of \(f\).

**Proof.** A representation \(D: G \to GL_m(\mathbb{F})\) induces a representation \(D: A \to \mathcal{M}_m(\mathbb{F})\). \(D\) is injective since \(D(g_1),\ldots,D(g_n)\) are linearly independent. Hence \(D\) is an isomorphism between \(A\) and its image, in particular \(fa = 0\) iff \(D(fa) = D(f)D(a) = 0\). Let \(a = a_1g_1 + \ldots + a_ng_n\), then \(D(a) = a_1D(g_1) + \ldots + a_nD(g_n)\) and \(D(f)D(a) = 0\) iff \((a_1,\ldots,a_n)\in\mathbb{F}^n\) is a solution of the linear system

\[
D(f) \left( \sum_{i=1}^{n} x_iD(g_i) \right) = 0.
\]

\[\]

We can now state the first consequence of Proposition 1. Notice that if \(p \nmid n = |G|\), then \(\mathbb{F}[G]\) is semisimple by Maschke’s Theorem (see [8]). In particular, by Lemma 2 every left/right ideal of \(A\) has an idempotent generator, which one can compute using Proposition 4. In that case, the next result allows us to reduce the computation of the dimension of a left/right ideal to the computation of a characteristic polynomial.

**Theorem 5.** Let \(G = \{g_1,\ldots,g_n\}\) be a group, \(\mathbb{F}\) be a field, and let \(A = \mathbb{F}[G]\). Let \(f = \alpha_1g_1+\ldots+\alpha_ng_n \in A\) and let \(I = Af\) be a proper left ideal (resp., \(I = fA\) be a proper right ideal) of \(A\). Let \(F = \rho(f)\) be the matrix associated to \(f\) in the regular right representation \(\rho\) of \(G\) (resp., let \(F = \lambda(f)\) be the matrix associated to \(f\) in the regular left representation \(\lambda\) of \(G\)). Let \(z^kg(z)\) be the characteristic polynomial of \(F\), where \(z \nmid g(z)\).

Then

\[n - k \leq \dim_{\mathbb{F}} I \leq n - 1.\]

If in addition \(f\) is idempotent, then \(\dim_{\mathbb{F}} I = n - k\).

**Proof.** It follows from Proposition 1 that

\[\dim_{\mathbb{F}} I = \text{rank}(F) \geq n - k.\]

Since \(I\) is proper, \(\dim_{\mathbb{F}} I \leq n - 1\).

If in addition \(f\) is idempotent, then \(F\) is an idempotent matrix, hence

\[\ker F = \ker F^m \text{ for any } m \geq 1.\]
Therefore, the algebraic and geometric multiplicities of 0 as an eigenvalue of $F$ coincide, and

$$\dim F I = \text{rank}(F) = n - k,$$

where the first equality again follows from Proposition 1.

In the case that $p \mid n$, there may be ideals of $F[G]$ which have no idempotent generator. If $f$ is not idempotent, it is possible that $\text{rank}(F) > n - k$. Next we give some examples where $A$ is commutative and $I = (f)$ has $\dim F I = \text{rank}(F) > n - k$. In particular, we provide examples where $n - k = 0$ and $\dim F I = \ell$ for any $\ell$ which divides $n/p$, $p = \text{char} F$.

**Example 6.** We follow the notation of Theorem 5. Let $G$ be a cyclic group of order $n = \ell m$ and let $p$ be a prime, $p \mid m$. Let $H$ be a subgroup of $G$ of order $p$, and let

$$f = \sum_{h \in H} h \in F[G]$$

where $F$ is a field. For a suitable reordering of the elements of $G$, the matrix $F \in M_n(F)$ associated to $f$ in a regular representation of $G$ is the block matrix

$$F = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 0 \\
0 & 0 & \cdots & 0 & 1
\end{bmatrix} \ell \text{ blocks}$$

where 0 and 1 denote the matrix of size $m \times m$ filled with 0 and 1 respectively. If $I = (f) \subset A$, then

$$\dim F I = \text{rank}(F) = \ell$$

by Proposition 1. It is easy to check that the characteristic polynomial of $F$ is $z^{n-\ell} (z-m)^\ell$. Hence $k = n-\ell$ if $\text{char} F \neq p$, and $k = n$ if $\text{char} F = p$. In particular, if $\text{char} F = p \mid n$, then

$$\dim F I = \ell > n - k = 0.$$
Corollary 7. Let $f \in A = \mathbb{F}[G]$ and let $I = Af$ (resp., $I = fA$). Let $F$ be the matrix associated to $f$ in the regular right representation (resp., in the regular left representation) of $G$. Let

$$M = F \quad \text{if } G \text{ is commutative},$$

and let

$$M = \begin{bmatrix} 0 & F \\ F^t & 0 \end{bmatrix} \quad \text{if } G \text{ is not commutative.}$$

Let $x$ be a transcendental element over $\mathbb{F}$ and let $X$ be the diagonal matrix with eigenvalues $1, x, \ldots, x^{m-1}$, where $m$ is the size of the matrix $M$. Let $z^k g(z, x)$ be the characteristic polynomial of the matrix $XM$, where $z \nmid g(z, x)$. Then

$$\dim_{\mathbb{F}} I = \begin{cases} \deg z^k g(z, x) & \text{if } G \text{ is commutative,} \\ \deg z^k g(z, x)/2 & \text{if } G \text{ is not commutative.} \end{cases}$$

Remark 8. If in Corollary 7 we let $x$ be a random element in $\mathbb{F}$ (or a suitable algebraic extension field), we obtain a faster randomized algorithm to compute the dimension of $I$. Such an approach works well in practice, and we can increase the probability of obtaining the correct result by repeating the computation for different random values of $x$.

3 Application to coding theory and examples

A group code is an $\mathbb{F}_q$-linear code which is image of an ideal $I \subseteq \mathbb{F}_q[G]$ via the isomorphism $\phi : \mathbb{F}[G] \to \mathbb{F}^n$ which sends $g_i \in G$ to the $i$th element of the canonical basis of $\mathbb{F}^n$. The length of $\phi(I)$ is equal to $n$, the cardinality of $G$, and the dimension of the code is the dimension of $I$ over $\mathbb{F}_q$. The method that we propose therefore allows us to efficiently compute the dimension of group codes. See [2, 6] for an overview of group code properties, and [4] for an efficient encoding, and corresponding syndrome decoding.

Our method will be illustrated by two simple examples. We observe that the right regular representation of a group $G = \{g_1, \ldots, g_n\}$ can be easily obtained from a modified Cayley table: The element in position $(i, j)$ in the table is $g_i^{-1}g_j$. Letting $g_1$ be the neutral element, the entries on the diagonal are all equal to $g_1$. For a given $f = \sum_{i=1}^n x_i g_i \in A = \mathbb{F}[G]$, the matrix $\rho(f)$ associated to $f$ in the right regular representation $\rho$ of $A$ is obtained by substituting $g_i$ by $x_i$ in the Cayley table for $i = 1, \ldots, n$. 

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Example 9. Consider the Klein group $K_4 = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{e, \alpha, \beta, \alpha \beta\}$, where $e$ is the neutral element, $\alpha^2 = \beta^2 = 1$, and $\alpha \beta = \beta \alpha$. Let $a, b, c, d \in \mathbb{F}$, then the matrix associated to $f = ae + b\alpha + c\beta + d\alpha \beta \in \mathbb{F}[K_4]$ in the right regular representation $\rho$ of $\mathbb{F}[K_4]$ is

$$\rho(f) = \begin{bmatrix}
    a & b & c & d \\
    b & a & d & c \\
    c & d & a & b \\
    d & c & b & a
\end{bmatrix}.$$ 

If $\text{char}(\mathbb{F}) \neq 2$, then $\rho(f)$ is diagonalizable

$$\rho(f) \sim \begin{bmatrix}
    a + b + c + d & 0 & 0 & 0 \\
    0 & a - b - c + d & 0 & 0 \\
    0 & 0 & a + b - c - d & 0 \\
    0 & 0 & 0 & a - b + c - d
\end{bmatrix}.$$ 

In particular, the characteristic polynomial of $\rho(f)$ is

$$\phi_f(z) = (z - a - b - c - d)(z - a + b + c - d)(z - a - b + c + d)(z - a + b - c + d) = z^k g(z)$$

with $g(0) \neq 0$, and

$$4 - k = \deg g(z) = \text{rank} \, \rho(f) = \text{dim}_\mathbb{F}(f).$$

Notice that every $f \in \mathbb{F}[K_4]$ is idempotent. In this case $\mathbb{F}[K_4]$ is semisimple, and the diagonal form of $\rho(f)$ corresponds to the decomposition of $\mathbb{F}[K_4]$ as a direct sum of ideals

$$\mathbb{F}[K_4] = (e + \alpha + \beta + \alpha \beta) \oplus (e - \alpha - \beta + \alpha \beta) \oplus (e + \alpha - \beta - \alpha \beta) \oplus (e - \alpha + \beta - \alpha \beta),$$

where each of the ideals appearing as direct summands is minimal and idempotent.

If $\text{char}(\mathbb{F}) = 2$, then $\rho(f)$ can be brought into the form

$$\rho(f) \sim \begin{bmatrix}
    a + b + c + d & 0 & 0 & 0 \\
    b & a + b + c + d & 0 & 0 \\
    c & c + d & a + d & b + c \\
    d & c + d & b + c & a + d
\end{bmatrix}.$$ 

The characteristic polynomial of $\rho(f)$ is

$$\phi_f(z) = (z + a + b + c + d)^4.$$
If \( f \) is not invertible, then \( a + b + c + d = 0 \), in particular \( \phi_f(z) = z^4 \). Hence \( \operatorname{rank} \rho(f) \in \{0, 1, 2\} \). In particular, \( \operatorname{rank} \rho(f) = 0 \) if and only if \( f = 0 \). Moreover \( \operatorname{rank} \rho(f) = 1 \) if and only if \( a = b = c = d \neq 0 \).

In this case we have \( (f) = (e+\alpha+\beta+\alpha\beta) \) and \( \dim_\mathbb{F}(f) = 1 \). Finally, if \( a, b, c, d \) are not all equal and either \( b \neq 0 \) or \( b \neq c \) or \( c \neq d \), then \( \operatorname{rank} \rho(f) = 2 \). In this case \( f \in \{e + \alpha, e + \beta, e + \alpha\beta\} \) and \( \dim_\mathbb{F}(f) = 2 \).

Notice that if \( \operatorname{char}(\mathbb{F}) = 2 \), then \( \mathbb{F}[K_4] \) is not semisimple and every ideal equals its annihilator.

**Example 10.** Consider the symmetric group \( S_3 = \{1, (12), (13), (23), (123), (132)\} \).

Each element \( g \in \mathbb{F}[S_3] \) is of the form

\[
g = a1 + b(12) + c(13) + d(23) + e(123) + f(132)
\]

where \( a, b, c, d, e, f, \in \mathbb{F} \), and is represented by the matrix

\[
\rho(g) = \begin{bmatrix}
a & b & c & d & e & f \\
b & a & e & f & c & d \\
c & f & a & e & d & b \\
d & e & f & a & b & c \\
f & c & d & b & a & e \\
e & d & b & c & f & a \\
\end{bmatrix}
\]

where \( \rho \) denotes the right regular representation of \( G \). If \( \operatorname{char}(\mathbb{F}) \neq 2, 3 \), then \( \rho(f) \) can be brought in the following form

\[
\rho(g) \sim \begin{bmatrix}
a + b + c + d + e + f & 0 & 0 & 0 & 0 & 0 \\
0 & a + b + c + d + e + f & 0 & 0 & 0 & 0 \\
0 & 0 & a - f & e - f & -c + d & b - c \\
0 & 0 & -e + f & a - e & b - d & c - d \\
0 & 0 & -c + d & b - c & a - f & e - f \\
0 & 0 & b - d & c - d & -e + f & a - e \\
\end{bmatrix}
\]

Therefore \( A = \mathbb{F}[S_3] = I_1 \oplus I_2 \oplus I_3 = I_1 \oplus I_2 \oplus J_1 \oplus J_2 \), where \( I_1, I_2, I_3 \) are minimal two-sided ideals

\[
I_1 = A(1 + (12) + (13) + (23) + (123) + (132)), \quad I_2 = A(1 - (12) - (13) - (23) + (123) + (132)), \\
I_3 = A(1 - (123)) + A((12) - (23)) + A((13) - (23)) + A((123) - (132)) = A(2 \cdot 1 - (123) - (132)).
\]
We have \( \dim_F I_1 = \dim_F I_2 = 1 \) and \( \dim_F I_3 = 4 \). Moreover \( I_3 \) can be decomposed as the direct sum of two left ideals: \( I_3 = J_1 \oplus J_2 \) with

\[
J_1 = A(1 + (12) - (23) - (123)), \quad J_2 = A(1 - (12) + (23) - (132))
\]

and \( \dim_F J_1 = \dim_F J_2 = 2 \). Using (3) it is easy to check that in each case the dimension of the ideal is as predicted by Theorem 5. It is also easy to check that \( I_1, I_2, J_1, J_2 \) are minimal left ideals, hence any nonzero \( I = A f \subseteq A \) is the sum or one or more among them. In particular \( \dim_F I \) is the sum of the dimensions of the corresponding ideals, again as predicted by Theorem 5.

If \( \text{char}(F) = 2 \), then

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & a + e + f & a + b + c + d + e + f & 0 & 0 & 0 \\
0 & a + b + c + d + e + f & a + e & e + f & b + c & c + d \\
0 & 0 & e + f & a + f & b + d & b + c \\
0 & 0 & b + c & c + d & a + e & e + f \\
0 & 0 & b + d & b + c & e + f & a + f \\
\end{bmatrix}
\]

\( \rho(g) \sim \)

Denoting by \( I_1, I_2, I_3, J_1, J_2 \) the same ideals as before, we have \( I_1 = I_2 \) and an easy computation involving the above matrix yields

\( \dim_F I_1 = 1, \quad \dim_F I_3 = 4, \quad \dim_F J_1 = \dim_F J_2 = 2. \)

Notice that if \( \text{char}(F) = 2 \), then \( F[S_3] \) is no longer semisimple.

If \( \text{char}(F) = 3 \), then \( I_1, I_2 \subseteq I_3 \) where

\[
I_3 = A(1 - (123)) + A((12) - (23)) + A((13) - (23)) + A((123) - (132)) = A(1 + (123) + (132))
\]

and

\( \dim_F I_1 = \dim_F I_2 = 1, \quad \dim_F I_3 = 2. \)

Again \( F[S_3] \) is not semisimple.

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