Entanglement-breaking channels in infinite dimensions

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Abstract

We give a representation for entanglement-breaking channels in separable Hilbert space that generalizes the “Kraus decomposition with rank one operators” and use it to describe the complementary channels. We also give necessary and sufficient condition of entanglement-breaking for a general quantum Gaussian channel. Application of this condition to one-mode channels provides several new cases where the additivity conjecture holds in full generality.

1 Introduction

In the paper [1] we gave a general integral representation for separable states in the tensor product of infinite dimensional Hilbert spaces and proved the structure theorem for the quantum communication channels that are entanglement-breaking, which generalizes the finite-dimensional result of Horodecki, Shor and Ruskai [2].

In what follows $\mathcal{H}$ denotes separable Hilbert space; $\mathfrak{S}(\mathcal{H})$ – the Banach space of trace-class operators in $\mathcal{H}$, and $\mathfrak{S}(\mathcal{H})$ – the convex subset of all density operators $\rho$ in $\mathcal{H}$. We shall also call them states for brevity, having in mind that a density operator $\rho$ uniquely determines a normal state on the algebra $\mathfrak{B}(\mathcal{H})$ of all bounded operators in $\mathcal{H}$ (see e. g. [3]). Equipped with the trace-norm distance, $\mathfrak{S}(\mathcal{H})$ is a complete separable metric space.

A state $\rho \in \mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is called separable if it belongs to the convex closure of the set of all product states in $\mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. It is shown in [1] that separable states are precisely those which admit a representation

$$\rho = \int_{\mathcal{X}} \rho_1(x) \otimes \rho_2(x) \pi(dx), \quad (1)$$

where $\pi(dx)$ be a Borel probability measure and $\rho_j(x), j = 1, 2$, are Borel $\mathfrak{S}(\mathcal{H}_j)$-valued functions on some complete separable metric space $\mathcal{X}$.

A channel with input space $\mathcal{H}_A$ and output space $\mathcal{H}_B$ is bounded linear completely positive trace-preserving map $\Phi : \mathfrak{T}(\mathcal{H}_A) \to \mathfrak{T}(\mathcal{H}_B)$. Channel $\Phi$ is called entanglement-breaking if for arbitrary Hilbert space $\mathcal{H}_R$ and arbitrary
state $\rho \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_R)$ the state $(\Phi \otimes \text{Id}_R)(\rho) \in \mathcal{S}(\mathcal{H}_B \otimes \mathcal{H}_R)$, where $\text{Id}_R$ is the identity map in $\mathcal{L}(\mathcal{H}_R)$, is separable.

It is shown in [1] that the channel $\Phi$ is entanglement-breaking if and only if there is a complete separable metric space $\mathcal{X}$, a Borel $\mathcal{S}(\mathcal{H}_B)$-valued function $\rho_B(x)$ and an observable $M$ in $\mathcal{H}_A$ with the outcome set $\mathcal{X}$ given by probability operator-valued measure (POVM) $M(dx)$ (which is a measure on $\mathcal{X}$ taking values in the positive cone of $\mathcal{B}(\mathcal{H}_A)$), with $M(\mathcal{X}) = I$) such that

$$\Phi(\rho) = \int_{\mathcal{X}} \rho_B(x) m_{\rho}(dx),$$

(2)

where $m_{\rho}(S) = \text{Tr} \rho M(S)$ for all Borel subsets $S \in \mathcal{B}(\mathcal{X})$. This gives a continual version of the class of channels introduced in [4] and can be regarded as a generalization of a result in [2] to infinite dimensions.

In finite dimensions entanglement-breaking channels form a large class in which the famous additivity conjecture for the classical capacity holds as shown by Shor [5]. Generalization of this property to infinite dimensions is by no means straightforward. First, we define generalized ensemble as arbitrary probability distribution on the state space $\mathcal{S}(\mathcal{H}_A)$ [6], [7]. The average state of the ensemble is given by the barycenter $\bar{\rho}_\pi = \int \rho \pi(\rho)$.

Let $A$ be an arbitrary subset of $\mathcal{S}(\mathcal{H}_A)$, then the $A$-constrained $\chi-$capacity of the channel $\Phi$ is defined as

$$C^\chi(\Phi, A) = \sup_{\pi: \bar{\rho}_\pi \in A} \int \mathcal{H}(\Phi[\rho] \mid \Phi[\bar{\rho}_\pi]) \pi(d\rho),$$

where $\mathcal{H}(\rho; \sigma)$ is the quantum relative entropy. In case the output entropy is finite on $A$ this amounts to

$$C^\chi(\Phi, A) = \sup_{\sigma \in A} \left[ \mathcal{H}(\Phi[\sigma]) - \hat{H}_\Phi(\sigma) \right],$$

(3)

where

$$\hat{H}_\Phi(\sigma) = \inf_{\pi: \rho_\pi = \sigma} \int \mathcal{H}(\Phi[\rho]) \pi(d\rho)$$

(4)

is the convex closure of the output entropy $\mathcal{H}(\Phi[\rho])$. When $A = \mathcal{S}(\mathcal{H}_A)$ and $\pi$ runs through ordinary ensembles given by probability distributions with finite supports, this reduces to the familiar definition of (unconstrained) $\chi-$ capacity $C^\chi(\Phi)$.

Then, as shown in [8], for an entanglement-breaking channel $\Phi_1$ and arbitrary channel $\Phi_2$ the additivity conjecture holds in its strongest form

$$C^\chi(\Phi_1 \otimes \Phi_2, A_1 \otimes A_2) = C^\chi(\Phi_1, A_1) + C^\chi(\Phi_2, A_2),$$

(5)

where

$$A_1 \otimes A_2 = \{ \rho \in \mathcal{S}(\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}) : \text{Tr}_2 \rho \in A_1, \text{Tr}_1 \rho \in A_2 \}.$$

Moreover, the convex closure is superadditive

$$\hat{H}_{\Phi_1 \otimes \Phi_2}(\sigma_{12}) \geq \hat{H}_{\Phi_1}(\sigma_1) + \hat{H}_{\Phi_2}(\sigma_2)$$

(6)
for any state $\sigma_{12}$.

In applications the constraints of the form

$$A(H, E) = \{ \rho \in \mathcal{S}(\mathcal{H}_A) : \text{Tr}\rho H \leq E \} , \quad (7)$$

where $H$ is positive selfadjoint operator (typically the energy operator, see [6]) are of the major interest. In this case one shows as in [9], that (5) implies

$$C_\chi(\Phi^\otimes n, A(H \otimes \ldots \otimes I + \ldots I \otimes \ldots \otimes H, nE)) = nC_\chi(\Phi, A(H, E))$$

and hence

$$C(\Phi, H, E) = C_\chi(\Phi, A(H, E)) = \sup_{\pi : \text{Tr}\bar{\rho} \leq E} \int H(\Phi[\rho]; \Phi[\bar{\rho}_x]) \pi(d\rho) \quad (8)$$

for any entanglement-breaking channel $\Phi$, where $C(\Phi, H, E)$ is the classical capacity of the channel $\Phi$ with the input energy constraint as defined in [6].

The paper has two self-consistent parts. In part I (Sec. 2, 3) we give another representation for entanglement-breaking channels in separable Hilbert space, that generalizes the “Kraus decomposition with rank one operators”. We also find complementary channels and remark that coherent information for anti-degradable channel is always non-positive.

Part II (Sec. 4-6) is devoted to Gaussian entanglement-breaking channels. We give necessary and sufficient condition of entanglement-breaking for a general quantum Gaussian channel. Application of this condition to one-mode channels provides several new cases where the additivity conjecture holds in the full generality.

## 2 A representation for entanglement-breaking channels

Here we further specify the formula (2), by employing the representation for POVM from [10]. Basing on this specification, we give explicit description of the Stinespring isometry for the channel $\Phi$ and of the complementary channel.

**Lemma 1.** (Radon-Nikodym theorem for POVM) For every POVM $M$ on $X$, there exist a positive $\sigma$-finite measure $\mu$ on $X$, a dense domain $D \subset \mathcal{H}$, and a countable family of functions $x \rightarrow a_k(x)$ such that for almost all $x$, $a_k(x)$ are linear functionals on $D$, satisfying

$$\int_X \sum_k |\langle a_k(x)|\psi \rangle|^2 \mu(dx) = \|\psi\|^2 ; \quad \psi \in D, \quad (9)$$

and

$$\langle \psi| M(S) |\psi \rangle = \int_S \sum_k |\langle a_k(x)|\psi \rangle|^2 \mu(dx); \quad \psi \in D. \quad (10)$$
Note that \( a_k(x) \) are in general unbounded functionals defined only on \( D \), nevertheless we find it convenient to continue use of the “bra-ket” notation for such functionals.

**Proof:** We follow the proof of Radon-Nikodym theorem for instruments from [10]. By Naimark’s theorem [3], there exist a Hilbert space \( K \), and sharp observable \( E \) given by spectral measure \( \{ E(S); S \in B(\mathcal{X}) \} \) in \( K \), and an isometry \( W \) from \( \mathcal{H} \) to \( K \) such that

\[
M(S) = W^* E(S) W, \quad S \in B(\mathcal{X}).
\]

(11)

According to von Neumann’s spectral theorem, \( K \) can be decomposed into the direct integral of Hilbert spaces

\[
K = \int_{\mathcal{X}} \oplus \mathcal{H}(x) \mu(dx)
\]

(12)

with respect to some positive \( \sigma \)-finite measure \( \mu \), diagonalizing the spectral measure \( E \):

\[
E(S)\phi = \int_S \oplus \phi(x) \mu(dx),
\]

(13)

where \( \phi(x) \in \mathcal{H}(x) \) are the components of the vector \( \phi \in K \) in the decomposition (12). Let us fix a measurable field of orthonormal bases \( \{ e_{k,x} \} \) in the direct integral (12) and denote \( \phi_k(x) = \langle e_{k,x} | \phi(x) \rangle \), where the inner product is in \( \mathcal{H}(x) \) (note that \( \phi_k(x) \) are defined \( \mu \)-almost everywhere).

Let \( \psi \in \mathcal{H} \), then the decomposition of the vector \( W\psi \in K \) reads

\[
W\psi = \int_{\mathcal{X}} \oplus \sum_k (W\psi)_k e_{k,x} \mu(dx),
\]

where

\[
\int_{\mathcal{X}} \sum_k |(W\psi)_k(x)|^2 \mu(dx) = ||\psi||^2; \quad \psi \in \mathcal{H},
\]

(14)

since \( W \) is isometric. Since \( W \) is linear, we have for \( \mu \)-almost all \( x \)

\[
(W\left( \sum_j \lambda_j \psi_j \right))(x) = \sum_j \lambda_j (W\psi_j)_k(x),
\]

(15)

where \( \{ \psi_j \} \subset \mathcal{H} \) is a fixed system of vectors, and \( \lambda_j \) are complex numbers, only finite number of which are non-zero.

Let us now fix an orthonormal basis \( \{ \psi_j \} \subset \mathcal{H} \) and let \( D = \text{lin}\{ \psi_j \} \) be its linear span. We define linear functionals \( a_k(x) \) on \( D \) by the relation

\[
\langle a_k(x) | \sum_j \lambda_j \psi_j \rangle = \sum_j \lambda_j (W\psi_j)_k(x),
\]

where \( (W\psi_j)_k^0 \) is a fixed representative of the equivalence class \( (W\psi_j)_k \). Then by (15), for any fixed \( \psi \in D \) there is a subset \( \mathcal{X}_\psi \subset \mathcal{X} \), such that \( \mu(\mathcal{X} \setminus \mathcal{X}_\psi) = 0 \) and

\[
\langle a_k(x) | \psi \rangle = (W\psi)_k(x), \quad \text{for all } k, \text{ all } x \in \mathcal{X}_\psi.
\]

(16)
Combining (13) and (16), we obtain (10). The normalization condition (9) follows from (14) and (16). □

**Theorem 1.** For any entanglement-breaking channel $\Phi$, there exist a complete separable metric space $\mathcal{Y}$, a positive $\sigma$-finite measure $\nu$ on $\mathcal{Y}$, a dense domain $D \subset \mathcal{H}_A$, a measurable function $y \to a(y)$, defined for almost all $y$, such that $a(y)$ are linear functionals on $D$, satisfying

$$\int_{\mathcal{Y}} |\langle a(y) | \psi \rangle|^2 \nu(dy) = \| \psi \|^2; \quad \psi \in D,$$

(17)

and a measurable family of unit vectors $y \to b(y)$ in $\mathcal{H}_B$ such that

$$\Phi[\rho] = \int_{\mathcal{Y}} |\langle b(y) | \psi \rangle|^2 \nu(dy) \text{ for } \rho = |\psi\rangle\langle \psi | \text{ with } \psi \in D.$$

(18)

This is infinite-dimensional analog of the “Kraus decomposition with rank one operators” from [2] with the difference that it is in general continual and the Kraus operators $V(y) = |b(y)\rangle\langle a(y)|$ are unbounded and only densely defined. As shown in [1], there are entanglement-breaking channels for which the usual Kraus decomposition with bounded rank one operators does not exist.

**Proof.** By making the spectral decomposition of the density operators $\rho_B(x) = \sum \lambda_l(x) |b_l(x)\rangle\langle b_l(x)|$ in (2) and using lemma 1, we find

$$\Phi[\rho] = \int_{\mathcal{Y}} \sum_{k,l} \lambda_l(x) |b_l(x)\rangle\langle b_l(x)| |\langle a_k(x) | \psi \rangle|^2 \mu(dx); \quad \psi \in D.$$

(19)

Let $\mathcal{Y}$ be the space of triples $y = (x, k, l)$, with naturally defined metric and the countably finite measure defined by

$$\nu(S \times k \times l) = \int_S \lambda_l(x) \mu(dx).$$

Define $a(y) = a_k(x)$ and $b(y) = b_l(x)$, then (17) follows from (9) and (18) – from (19). □

### 3 Remarks on complementary channels

In general, the Stinespring representation

$$\Phi[\rho] = \text{Tr}_E V \rho V^*$$

holds in the infinite dimensional case for arbitrary channel, where $\mathcal{H}_E$ is environment space and $V : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$ is an isometry. The complementary channel is defined as

$$\tilde{\Phi}[\rho] = \text{Tr}_B V \rho V^*.$$
If there exists a channel $T : \mathcal{F}(\mathcal{H}_B) \rightarrow \mathcal{F}(\mathcal{H}_E)$ such that $\tilde{\Phi} = T \circ \Phi$, then $\Phi$ is degradable \cite{11}, and if $\Phi = T' \circ \tilde{\Phi}$ for some channel $T' : \mathcal{F}(\mathcal{H}_E) \rightarrow \mathcal{F}(\mathcal{H}_B)$, then $\Phi$ is anti-degradable \cite{12}.

**Proposition 1.** If $\Phi$ is degradable (anti-degradable) channel then $I_c(\rho, \Phi) \geq 0$ (resp. $I_c(\rho, \Phi) \leq 0$) for arbitrary density operator $\rho$ such that $H(\Phi[\rho]) < \infty$, $H(\tilde{\Phi}[\rho]) < \infty$.

**Proof.** As noticed in \cite{13}, there is a formula which relates the coherent information and the $\chi -$ quantity giving the upper bound for the classical information. Namely, for arbitrary pure-state decomposition $\rho = \sum_j \pi_j \rho_j$

$$I_c(\rho, \Phi) = \chi_B - \chi_E,$$

where

$$\chi_B = H(\Phi[\rho]) - \sum_j \pi_j H(\Phi[\rho_j])$$

and similarly for $\chi_E$. But $\chi_B = \sum_j \pi_j \left[ H(\Phi[\rho_j]; \Phi[\rho]) - H(\Phi[\rho_j]; \Phi[\rho]) \right]$, whence

$$I_c(\rho, \Phi) = \sum_j \pi_j \left[ H(\Phi[\rho_j]; \Phi[\rho]) - H(\tilde{\Phi}[\rho_j]; \tilde{\Phi}[\rho]) \right].$$

The assertion then follows from the monotonicity of the relative entropy and the definition of (anti-)degradable channel. $\square$

As observed in \cite{12}, anti-degradable channels have zero quantum capacity $Q(\Phi)$. Proposition 1 provides a short proof of this statement. Let $\Phi$ be anti-degradable, then such is $\Phi \otimes n$, hence $I_c(\rho, \Phi \otimes n) \leq 0$. Then the coding theorem for the quantum capacity implies

$$Q(\Phi) = \lim_{n \to \infty} n^{-1} \sup_{\rho} I_c(\rho, \Phi \otimes n) = 0.$$

For the entanglement-breaking channel \cite{18} we introduce the environment space $\mathcal{H}_E = L^2(\nu)$, and define the operator $V : \mathcal{D} \rightarrow \mathcal{H}_B \otimes \mathcal{H}_E$ by the formula

$$(V \psi)(y) = |b(y)\rangle \langle a(y)\psi|.$$

Then $V$ is isometric by \cite{17} and hence uniquely extends to the isometry $\mathcal{H}_A \rightarrow \mathcal{H}_B \otimes \mathcal{H}_E$. It is the Sinespring isometry for channel $\Phi$ namely

$$\text{Tr}_E V \rho V^* = \Phi[\rho], \quad \text{for } \rho = |\psi\rangle \langle \psi| \text{ with } \psi \in \mathcal{D}$$

by \cite{18}. On the other hand, $\text{Tr}_B V \rho V^*$ is the integral operator in $\mathcal{H}_E = L^2(\nu)$ defined by the kernel

$$\sigma_\rho(y_2, y_1) = \langle b(y_2)|b(y_1)\rangle \langle a(y_1)|\psi\rangle \overline{\langle a(y_2)|\psi\rangle},$$

which completely describes the complementary channel $\tilde{\Phi}$.

**Example.** Let $\mathcal{H}_A = L^2(\nu)$, $\mathcal{D} = C(\nu') \cap L^2(\nu)$, then the relation

$$\Phi[\rho] = \int |b(y)\rangle \langle b(y)||\psi(y)|^2 \nu(dy); \quad \text{for } \rho = |\psi\rangle \langle \psi| \text{ with } \psi \in \mathcal{D}$$
defines entanglement-breaking channel. This extends to
\[ \Phi[\rho] = \int_Y |b(y)\rangle\langle b(y)| \rho(y, y) dy, \]
where \( \rho(y_2, y_1) \) is the kernel of the integral operator \( \rho \) in \( L^2(\nu) \) (for which the diagonal value \( \rho(y, y) \) is unambiguously defined \[14\]). The output of the complementary channel is the integral operator in \( \mathcal{H}_E = L^2(\nu) \) with the kernel
\[ \sigma\rho(y_2, y_1) = \langle b(y_2)|b(y_1)\rangle \rho(y_2, y_1). \]

In finite dimensions every entanglement-breaking channel is anti-degradable\[1\]. In infinite dimensions, use representation \[15\] and define the entanglement-breaking channel \( T' : \mathcal{H}_E \rightarrow \mathcal{H}_B \) by the formula
\[ T'[\sigma] = \int_Y |b(y)\rangle\langle b(y)| \sigma(y, y) \nu(dy), \]
then \( \Phi[\rho] = T'\Phi[\rho] \) for \( \rho = |\psi\rangle\langle \psi| \) with \( \psi \in \mathcal{D} \), hence the assertion follows.

4 Gaussian observables

In what follows we shall consider real vector space \( Z \) equipped with different bilinear forms \( \alpha, \Delta, \ldots \). For concreteness and convenience of notation we shall consider \( Z \) as the space of column vectors with components in \( \mathbb{R} \). Then the forms are given by matrices which we denote by the same letter, e. g. \( \alpha(w, z) = w^T \alpha z \), where \( T \) denotes transposition. If \( \alpha \) is an inner product on \( Z \), then \( (Z, \alpha) \) is an Euclidean space, while if \( \Delta \) is a nondegenerated skew-symmetric form then \( (Z, \Delta) \) is a symplectic space. We call symplectic space \( (Z, \Delta) \) standard if the commutation matrix, corresponding to the symplectic form \( \Delta(z, z') \), is
\[ \Delta = \text{diag} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \]
Let \( 2n = \text{dim} Z \), and let \( d^{2n}z \) denote element of symplectic volume in \( (Z, \Delta) \). Symplectic Fourier transform and its converse are given by
\[ \hat{f}(w) = \int e^{i\Delta(w, z)} f(z) d^{2n}z; \quad f(z) = (2\pi)^{-2n} \int e^{-i\Delta(w, z)} \hat{f}(w) d^{2n}w. \]

Quantization on a symplectic space \( (Z, \Delta) \) is given by (irreducible) Weyl system in a Hilbert space \( \mathcal{H} \), which is a family of unitary operators \( \{W(z), z \in Z\} \) satisfying the canonical commutation relations, one of the equivalent forms of which is
\[ W(z)^* W(w) W(z) = \exp(i\Delta(w, z)) W(w). \]  
By Stone’s theorem, \( W(z) = \exp(iRz) \), where \( R \) is the row vector of selfadjoint operators in \( \mathcal{H} \) satisfying the Heisenberg commutation relations, see \[3\], Ch. V, for detail.

\[1\] We are indebted to M.-B. Ruskai for this observation \[15\].
Assume we have two symplectic spaces $Z_A, Z_B$ with the corresponding Weyl systems in Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$. Let $M$ an observable in $\mathcal{H}_A$ with the outcome set $Z_B$, given by positive operator-valued measure $M(d^{2n}z)$. In what follows we skip the index $B$ so that $Z = Z_B$ etc. It is completely determined by the operator characteristic function

$$\phi_M(w) = \int e^{i\Delta(w,z)}M(d^{2n}z).$$

A function $\phi(w)$ is characteristic function of an observable if and only if it satisfies the following conditions:

1. $\phi(0) = I$;
2. $\phi(w)$ is weakly continuous at $w = 0$;
3. for any choice of a finite subset $\{w_j\} \subset Z_B$ the block matrix with operator entries $\phi(w_j - w_k)$ is nonnegative definite.

In the case $\|\phi_M(w)\|$ is integrable on $Z_B$, measure $M(d^{2n}z)$ has density $p_M(z)M(d^{2n}z)$, which is a.e. defined on $Z_B$, taking values in the cone of bounded positive operators in $\mathcal{H}_A$, such that $\int p_M(z)d^{2n}z = I$. Moreover,

$$p_M(z) = (2\pi)^{-2n} \int e^{-i\Delta(w,z)}\phi_M(w)d^{2n}w. \quad (21)$$

Observable $M$ will be called Gaussian (canonical in [16], [3]) if its operator characteristic function has the form

$$\phi_M(w) = W_A(Kw) \exp \left( -\frac{1}{2}\mu(w,w) \right) = \exp \left( iR_A Kw - \frac{1}{2}\mu(w,w) \right), \quad (22)$$

where $K: Z_B \rightarrow Z_A$ is a linear operator and $\mu$ is a bilinear form on $Z_B$. A necessary and sufficient condition for (22) to define an observable is the matrix inequality

$$\mu \geq \frac{i}{2} K^T \Delta_A K. \quad (23)$$

Indeed, (22) apparently satisfies conditions 1,2 and (23) is equivalent to the condition 3 since for an operator function given by (22)

$$\phi(w_j - w_k) = W(Kw_k)^*W(Kw_j) \exp \left[ -\frac{i}{2} \Delta(Kw_j, Kw_k) - \frac{1}{2}\mu(w_j - w_k, w_j - w_k) \right]$$

$$= C_j^*C_j \exp \left[ \mu(w_j, w_k) - \frac{i}{2} \Delta(Kw_j, Kw_k) \right],$$

where $C_j = W(Kw_j) \exp \left[ -\frac{i}{2}\mu(w_j, w_j) \right]$, and nonnegative definiteness of matrices with scalar entries $\mu(w_j, w_k) - \frac{1}{2}\Delta(Kw_j, Kw_k)$, with arbitrary choice of finite subset $\{w_j\}$, is equivalent to that for $\exp \left[ \mu(w_j, w_k) - \frac{i}{2}\Delta(Kw_j, Kw_k) \right]$ (see [3], proof of Theorem 5.1, Ch. V).
Observable $M$ is sharp if and only if $\mu = 0$, in which case it is the spectral measure of commuting selfadjoint operators $R_A K$. In a sense opposite is the following case:

**Proposition 2.** Let $\mu$ be nondegenerate and $K$ invertible, then observable $M$ has operator density given by

$$p_M(z) = W_A(K'^{-1} z)\sigma_A W_A(K'^{-1} z)^* \frac{1}{(2\pi)^n |\det K|},$$

where $\sigma_A$ is Gaussian state with zero mean and correlation function $\mu(K^{-1} z, K^{-1} z')$ and $K' = \Delta^{-1} K^T \Delta$ is symplectic transpose.

**Proof.** Since $\mu$ is nondegenerate, $\|\phi_M(w)\| = \exp \left(-\frac{1}{2} \mu(w, w)\right)$ is integrable, and applying (21) we get

$$p_M(z) = (2\pi)^{-2n} \int e^{-i\Delta(w,z)} W_A(Kw) \exp \left(-\frac{1}{2} \mu(w, w)\right) d^2n w$$

$$= (2\pi)^{-2n} \int e^{i\Delta(u, K'^{-1} z)} W_A(-u) \exp \left(-\frac{1}{2} \mu(K^{-1} u, K^{-1} u)\right) \frac{d^2n u}{|\det K|}$$

$$= (2\pi)^{-2n} \int W_A(K'^{-1} z) W_A(-u) W_A(K'^{-1} z)^* \exp \left(-\frac{1}{2} \mu(K^{-1} u, K^{-1} u)\right) \frac{d^2n u}{|\det K|}$$

$$= \frac{1}{(2\pi)^n |\det K|} W_A(K'^{-1} z) \left[ \int W_A(-u) \exp \left(-\frac{1}{2} \mu(K^{-1} u, K^{-1} u)\right) \frac{d^2n u}{(2\pi)^n} \right] W_A(K'^{-1} z)^*.$$

Here we used change of variable $u = -Kw$ in the second line, the relation (20) in the third line and in the fourth line the term in squared brackets is the Weyl transform of the characteristic function of the state $\sigma_A$. □

Let us describe an explicit construction of Naimark’s dilation of observable $M$ in the spirit of [2], Prop. 5.1, Ch. II.

**Proposition 3.** Assume the condition (23) holds, then there exist Bosonic system $\mathcal{H}_C$ with canonical observables $R_C$ such that $\mathcal{H}_B \subseteq \mathcal{H}_A \otimes \mathcal{H}_C$ and Gaussian state $\rho_C \in \mathcal{S}(\mathcal{H}_C)$ for which

$$M(S) = \text{Tr}_C (I_A \otimes \rho_C) E_{AC}(S), \quad S \subseteq Z_B,$$

(24)

where $E_{AC}$ is a sharp observable in $\mathcal{H}_A \otimes \mathcal{H}_C$ given by the joint spectral measure of commuting selfadjoint operators

$$X_B = \Delta_B^{-1} (R_A K + R_C K_C)^T,$$

(25)

where $K_C : Z_B \to Z_C$ is operator such that

$$K_C^T \Delta C K_C = -K^T \Delta_A K.$$

(26)

**Proof.** The condition (20) means that $K_C^T \Delta C K_C + K^T \Delta_A K = 0$, that is commutativity of operators (25). By adapting the proof of Prop. 8.1 from Ch. VI of [2], we obtain a symplectic space $(Z_C, \Delta_C)$, operator $K_C : Z_B \to Z_C$ and
an inner product in $Z_C$, given by symmetric matrix $\alpha_C \geq \frac{i}{2} \Delta_C$ such that \eqref{eq:26} holds along with

$$K_C^T \alpha_C K_C = \mu.$$ 

Then the characteristic function of $E_{AC}$ is

$$\phi_{E_{AC}}(w_B) = \int \exp[i \Delta(w_B, z_B)] E_{AC} \left( d^n z_B \right) = \exp \left( i w_B^T \Delta_B X_B = \exp \left( R_A K + R_C K_C \right) w_B \right) = W_A(Kw_B) W_C(K_Cw_B),$$

whence

$$\text{Tr}_C \left( I_A \otimes \rho_C \right) \phi_{E_{AC}}(w_B) = W_A(Kw_B) \exp \left( - \frac{1}{2} \alpha_C(K_Cw_B, K_Cw_B) \right) = \phi_M(w_B),$$

and \eqref{eq:24} follows. \hfill $\square$

5 Gaussian entanglement-breaking channels

Recall that characteristic function of a state $\rho$ is given by

$$\varphi(z) = \text{Tr} \rho W(z).$$

Channel $\Phi : \mathcal{H}_A \rightarrow \mathcal{H}_B$ transforming states according to the rule

$$\varphi_B(z_B) = \varphi_A(Kz_B) f(z_B), \quad \text{(27)}$$

where $K$ is a linear map between output and input symplectic spaces $(Z_B, \Delta_B)$, $(Z_A, \Delta_A)$ and $f$ is a complex function, is called linear Bosonic. Necessary and sufficient condition on $f$ is nonnegative definiteness of matrices with scalar entries

$$f(w_j - w_k) \exp \left[ - \frac{i}{2} \Delta(w_j, w_k) + \frac{i}{2} \Delta(Kw_j, Kw_k) \right],$$

with arbitrary choice of finite subset $\{w_j \} \subset Z_B$ \cite{17, 18}.

If, additionally, $f$ is a Gaussian characteristic function, the channel is Gaussian. Thus for Gaussian channel, transformation of states is described as

$$\varphi_B(z_B) = \varphi_A(Kz_B) \exp \left[ im(z_B) - \frac{1}{2} \alpha(z_B, z_B) \right]. \quad \text{(28)}$$

The triple $(K, m, \alpha)$ is called parameters of the Gaussian channel. Without loss of generality we assume $m \equiv 0$. Necessary and sufficient condition on the parameters of Gaussian channel is

$$\alpha \geq \frac{i}{2} \left[ \Delta_B - K^T \Delta_A K \right]. \quad \text{(29)}$$
This follows from the condition for the general linear Bosonic channel applied to Gaussian $f$ similarly to the proof of (23). Importance of this condition in the matrix form was emphasized in [19].

**Theorem 2.** Let $\Phi$ be quantum Gaussian channel with parameters $(K,0,\alpha)$. It is entanglement-breaking if and only if $\alpha$ admits decomposition

$$\alpha = \nu + \mu, \quad \text{where} \quad \nu \geq \frac{i}{2} \Delta_B, \quad \mu \geq \frac{i}{2} K^T \Delta_A K.$$  \hfill (30)

In this case $\Phi$ has the representation

$$\Phi[\rho] = \int_{Z_B} W(z)\sigma_B W(z)^* m_\rho (d^2n z),$$  \hfill (31)

where $\sigma_B$ is Gaussian state with parameters $(0,\nu)$, and $m_\rho (S) = \text{Tr} \rho M_A(S)$, $S \subseteq Z_B$, is the probability distribution of the Gaussian observable $M_A$ with characteristic function [22].

**Proof.** First, assume $\alpha$ admits the decomposition and consider the channel defined by (31); we have to show that

$$\Phi^*[W_B(w)] = W_A(Kw) \exp \left[ -\frac{1}{2} \alpha(w,w) \right].$$  \hfill (32)

Indeed, for arbitrary $\rho$

$$\text{Tr}_B \Phi^*[W_B(w)] = \text{Tr}_B \Phi[\rho] W_B(w) = \int_{Z_B} \text{Tr}_B W_B(z)\sigma_B W_B(z)^* W_B(w) m_\rho (d^2n z)$$

$$= \int_{Z_B} \text{Tr}_B (W_B(z)^* W_B(w) W_B(z)) m_\rho (d^2n z)$$

$$= \text{Tr}_B W_B(w) \int_{Z_B} \exp [i \Delta(w,z)] m_\rho (d^2n z)$$

$$= \exp \left[ -\frac{1}{2} \nu(w,w) \right] \text{Tr}_B \rho M_A(w)$$

$$= \text{Tr}_B W_A(Kw) \exp \left[ -\frac{1}{2} \nu(w,w) - \frac{1}{2} \mu(w,w) \right],$$

whence (32) follows.

Conversely, let $\Phi$ be Gaussian and entanglement-breaking. We will use Gaussian version of the proof from [1], generalizing the Choi-Jamiołkowski correspondence to infinite-dimensional channels. Fix a Gaussian state $\rho_A$ in $\mathcal{S}(\mathcal{H}_A)$ of full rank and let $\{ |e_i \rangle \}_{i=1}^{+\infty}$ be the basis of eigenvectors of $\rho_A$ with the corresponding (positive) eigenvalues $\{ \lambda_i \}_{i=1}^{+\infty}$. Consider the unit vector

$$|\Omega \rangle = \sum_{i=1}^{+\infty} \sqrt{\lambda_i} |e_i \rangle \otimes |e_i \rangle$$

in the space $\mathcal{H}_A \otimes \mathcal{H}_A$, then $|\Omega \rangle \langle \Omega |$ is Gaussian purification of $\rho_A$. Since $\Phi$ is entanglement-breaking, the Gaussian state

$$\rho_{AB} = (\text{Id}_A \otimes \Phi) [ |\Omega \rangle \langle \Omega |]$$  \hfill (34)

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in $\mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is separable. As follows from the proof of Proposition 1 in [20], this implies representation

$$\rho_{AB} = \int \int_{Z_A Z_B} W_A(z_A)\sigma_A W_A(z_A)^* \otimes W_B(z_B)\sigma_B W_B(z_B)^* P(d^{2m}z_A d^{2n}z_B),$$

where $\sigma_A, \sigma_B$ are Gaussian states and $P$ is a Gaussian probability distribution. One then shows as in [1] that the relation

$$M_A(S) = \rho_A^{-1/2} \left[ \int \int_{Z_A} W_A(z_A)\sigma_A W_A(z_A)^* \otimes W_B(z_B)\sigma_B W_B(z_B)^* P(d^{2m}z_A d^{2n}z_B) \right] \rho_A^{-1/2},$$

where bar means complex conjugation in the basis of eigenvectors of $\rho_A$, defines an observable on Borel subsets $S \subseteq Z_B$, and the representation (31) holds for the channel $\Phi$ with these $M_A$ and $\sigma_B$. Let us denote $\nu$ the correlation function of the state $\sigma_B$; without loss of generality we can assume its mean is zero. It remains to show that $M_A$ is Gaussian observable with the characteristic function (22) where $\mu = \alpha - \nu$. But from (32)

$$\Phi^*[W_B(w)] = \exp \left[ -\frac{1}{2} \nu(w, w) \right] \phi_{M_A}(w)$$

for any channel $\Phi$ with the representation (31), whence taking into account (32), we indeed get (22) with $\mu = \alpha - \nu$. □

A necessary condition for the decomposability (30) and hence for the channel to be entanglement breaking is

$$\alpha \geq \frac{i}{2} (\Delta_B + K^T \Delta_A K).$$

In general this condition implies that for any input Gaussian state of the channel $\text{Id}_A \otimes \Phi$, the output has positive partial transpose. Indeed, this channel transforms the correlation matrix of the input state according to the rule

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \rightarrow \begin{bmatrix} I & 0 \\ 0 & K^T \end{bmatrix} \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & K \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \alpha \end{bmatrix} \equiv \alpha_{AB}.$$

The right hand side representing the correlation matrix of the output state satisfies

$$\alpha_{AB} \geq \frac{i}{2} \begin{bmatrix} I & 0 \\ 0 & K^T \end{bmatrix} \begin{bmatrix} \Delta_A & 0 \\ 0 & \Delta_A \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & K \end{bmatrix} + \frac{i}{2} \begin{bmatrix} 0 & 0 \\ 0 & \pm \Delta_B - K^T \Delta_A K \alpha \end{bmatrix}$$

$$= \frac{i}{2} \begin{bmatrix} \Delta_A & 0 \\ 0 & \pm \Delta_B \end{bmatrix},$$

where in the estimate of the second term we used (33) with its transpose. However, this is necessary and sufficient for the output state to have positive partial
transpose [20]. As shown in [20], there are nonseparable Gaussian states with positive partial transpose, therefore the condition (35) is in general weaker than (30).

The condition of the theorem is automatically fulfilled in the special case

\[ K^T \Delta_A K = 0. \]

In this case operators \( R_A K \) commute hence \( M_A \) sharp observable given by is their joint spectral measure and the probability distribution \( m_\rho(d^n z) \) can be arbitrarily sharply peaked around any point \( z \) by appropriate choice of the state \( \rho \). Hence in this case it is natural to identify \( \Phi \) as c-q (classical-quantum) channel determined by the family of states \( z \rightarrow W(z)\sigma_B W(z)^* \).

6 The case of one mode

Let us apply theorem 2 to the case of one Bosonic mode \( A = B \), where

\[ \Delta_A(z, z') = \Delta_B(z, z') = \Delta(z, z') = x'y - xy'. \]

As shown in [22], by choosing appropriate canonical unitary transformations \( U_1, U_2 \), any one mode Gaussian channel with parameters \((K, 0, \alpha)\) can be transformed via

\[ \Phi'[\rho] = U_2 \Phi^* [U_1 \rho U_1^*] U_2^* \]

to one of the following normal forms, where \( N \geq 0, k \) is real number:

\[ A) \quad K [x, y] = k [x, 0]; \quad \alpha(z, z) = (N + \frac{1}{2}) (x^2 + y^2); \]
\[ B_1) \quad K [x, y] = [x, y]; \quad \alpha(z, z) = \frac{1}{2} y^2; \]
\[ B_2) \quad K [x, y] = [x, y]; \quad \alpha(z, z) = N (x^2 + y^2); \]
\[ C) \quad K [x, y] = k [x, y]; \quad k > 0, k \neq 1; \quad \alpha(z, z) = (N + \frac{|1 - k^2|}{2}) (x^2 + y^2); \]
\[ D) \quad K [x, y] = k [x, -y]; \quad k > 0; \quad \alpha(z, z) = (N + \frac{(1 + k^2)}{2}) (x^2 + y^2). \]

The case \( B_2) \) representing channel with additive classical noise, and \( C) \) representing attenuator/amplifier, are of major interest in applications [15], [23].

We have only to find the form \( K^T \Delta_A K \) and check the decomposability (30) in each of these cases. We rely upon the simple fact that

\[ (N + \frac{1}{2}) I \geq \frac{i}{2} \Delta \]

if and only if \( N \geq 0.\)

\[ A) \quad K^T \Delta K = 0, \text{ hence } \Phi \text{ is c-q (in fact essentially classical) channel; } \]
\[ B) \quad K^T \Delta K = \Delta, \text{ hence the necessary condition (35) requires } \alpha \geq i\Delta. \text{ This is never fulfilled in the case } B_1) \text{ due to degeneracy of } \alpha. \text{ Thus the channel is } \]
not entanglement breaking (in fact it has infinite quantum capacity as shown in [22]). On the other hand, in the case $B_2$ the condition (30) is fulfilled with $\nu = \mu = \alpha/2$ if and only if $N \geq 1$, hence $\Phi$ is entanglement breaking in this case:

- $C)$ $K^T \Delta K = k^2 \Delta$. It is clear that in this case the decomposability condition holds if and only if $\alpha \geq \frac{1}{2}(1 + k^2)\Delta$, which is equivalent to $N + \frac{|1-k^2|}{2} \geq \frac{(1+k^2)}{2}$ or

$$N \geq \min \left(1, k^2 \right).$$

This gives the condition for the entanglement breaking (which also formally includes the case $B_2$);

- $D)$ $K^T \Delta K = -k^2 \Delta$. Again the decomposability condition holds if and only if $\alpha \geq \frac{1}{2}(1 + k^2)\Delta$, which always holds, hence the channel is entanglement breaking for all $N \geq 0$.

Thus the additivity property (5) holds for one-mode Gaussian channels of the form $A), D)$ with arbitrary parameters, and $B_2), C)$ with parameters satisfying (30). To compare this with previous results, the only case where additivity of $C_\chi(\Phi, E, c)$ with the special energy constraint ($E = a^\dagger a$) was established, is $C)$ with $N = 0, k < 1$ (pure loss channel) [24], which does not intersect with our result. The actual computation of $C_\chi(\Phi, E, c)$ is in general an open problem: there is a natural conjecture that the $\chi-$capacity of quantum Gaussian channel with quadratic energy constraint is attained on a Gaussian ensemble of pure Gaussian states, but so far this was only established for c-q channels [4] and the pure loss channel [24]. If the conjecture is true, then in the cases $B_2), C), D)$ the optimal ensemble is the complex Gaussian distribution $P(d^2 z)$ with zero mean and variance $c$ on the coherent states $W_A(z)\rho_0 W_A(z)^*$ where $\rho_0$ is the vacuum state. Hence $C(\Phi, E, c) = g(k^2 c + N_0) - g(N_0)$, where

$$N_0 = \begin{cases} (k^2 - 1)_+ + N, & \text{case } B_2), C); \\ k^2 + N, & \text{case } D). \end{cases}$$

is the mean number of quanta in the output corresponding to the vacuum input state and $g(x) = (x + 1) \log(x + 1) - x \log x$.

In general entanglement-breaking channels have zero quantum capacity $Q(\Phi) = 0$, cf. Sec. 3 In this connection it is notable that the domain (30) coincides with zero quantum capacity domain obtained in [20] from completely different argument. However in any case this is superseded by the broader domain found in [23] from degradability analysis.

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