Research Note

Near-minimum-fuel Strategy for Continuous Single-axis Maneuvers of a Rigid Body*

Donghoon KIM1 and Henzhe LEEGHIM2

1Department of Aerospace Engineering, Mississippi State University, Mississippi State, MS 39762, USA
2Department of Aerospace Engineering, Chosun University, Gwangju 61452, Republic of Korea

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1. Introduction

Optimal rigid body maneuvers have been investigated by many researchers.1,2) Unlike the minimum-energy maneuver, minimum-time and fuel maneuvers require instantaneous switching of controls (bang-bang and bang-off-bang, respectively), which cannot be generated practically. In addition, switch-times are usually very sensitive to modeling errors.3) Several studies have been conducted to resolve the issue of minimum-time maneuvers.4,5)

In this study, the authors suggest a near-minimum-fuel control law for single-axis, rest-to-rest maneuvers to avoid discontinuities in control profiles that are not attractive for structures having flexible parts; especially because their high-frequency behavior can potentially excite undesirable flexible body responses. Note that the mass of fuel usage is neglected in that the moments of inertia are not varied during rotational maneuvers, which are conducted by motorized actuators. Numerical simulations are conducted to highlight the performance of the strategy suggested.

2. Dynamics and Kinematics for Single-axis Maneuvers

The rotational dynamics equation of a rigid body is given as6)

\[ \dot{\omega}(t) = \mathbf{J}^{-1} \mathbf{u}(t), \]

where, \( I \in \mathbb{R}^{3 \times 3} \) is the positive definite inertia tensor for the spacecraft, \( \omega(t) \in \mathbb{R}^3 \) is the angular velocity vector of the spacecraft, \( \times \) is the cross product symbol, and \( \mathbf{u}(t) \in \mathbb{R}^3 \) is the control torque vector.

The governing kinematics differential equation for the Euler angles (3-2-1 set) is given as7)

\[ \dot{\theta}(t) = \frac{\omega(t)}{s\theta_2(t)} \begin{bmatrix} s\theta_3(t) & c\theta_3(t) & 0 \\ s\theta_2(t)c\theta_3(t) & -s\theta_2(t)s\theta_3(t) & 0 \\ -c\theta_2(t)s\theta_3(t) & c\theta_2(t)c\theta_3(t) & s\theta_2(t) \end{bmatrix}, \]

where, \( s \) and \( c \) denote sine and cosine functions, respectively.

For the special case of a single-axis maneuver, the rotational dynamic and kinematic equations in Eqs. (1) and (2) are simply expressed as

\[ \dot{\theta}(t) = \omega(t), \quad \dot{\omega}(t) = \frac{\mathbf{u}(t)}{J}, \]

where, \( \mathbf{u}(t) \) is the maneuvering time and the smooth function, \( \mathbf{a}(\delta, T, t) \) is the state vector and inertia element of \( I \) for describing the rotating axis, respectively. The scalar control input, \( \mathbf{a}(t) \), is constrained by the maximum control input, \( \mathbf{u}_{max} \), as

\[ |\mathbf{u}(t)| \leq \mathbf{u}_{max}. \]

3. Near-minimum-fuel Solution for Continuous Control Profiles

Note that strict minimum-fuel maneuvers can be conducted using bang-off-bang control.8) From Eq. (3), the second-order differential equation is given as

\[ \dot{\mathbf{J}}\dot{\theta}(t) = \mathbf{u}(t) = \mathbf{u}_{max}\mathbf{a}(\delta, T, t), \]

where, \( \mathbf{a}(\delta, T, t) = \mathbf{a}(t) \) (\( \delta \) and \( T \) are the constants), is defined as9)

\[ \mathbf{a}(t) = b + ct + dt^2 + et^3, \]

where, \( b, c, d, \) and \( e \) are the constant coefficients. Note that the coefficients are found for each time interval by imposing the four boundary conditions listed in Table 1 on Eq. (6) and its time derivative equation:

\[ \dot{a}(t) = c + 2dt + 3et^2. \]

| Interval | Boundary condition |
|----------|--------------------|
| \([a, b]\) | \(a(0) = a'(0) = 0, a(\delta) = -1, a'(\delta) = 0\) |
| \([b, c]\) | \(a(\delta) = -1, a'(\delta) = 0, a(t_1) = -1, a'(t_1) = 0\) |
| \([c, d]\) | \(a(t_1) = -1, a'(t_1) = 0, a(t_2) = 0, a'(t_2) = 0\) |
| \([d, e]\) | \(a(t_2) = 0, a'(t_2) = 0, a(t_3) = 0, a'(t_3) = 0 \) |
| \([e, t_0]\) | \(a(t_3) = 0, a'(t_3) = 0, a(t_4) = 1, a'(t_4) = 0\) |
| \([t_0, T]\) | \(a(t_4) = 1, a'(t_4) = 0, a(T) = 0, a'(T) = 0\) |
To provide a smooth transition between \(-u_{\text{max}}\) and \(u_{\text{max}}\), \(a(t)\) needs to be determined for each time interval. Assuming that the initial angular velocity and time are zeros, the cubic polynomial equation is found as

\[
a(t) = \begin{cases} 
\frac{t^2}{\delta^3} \left( 3 - \frac{2t}{\delta} \right), & \text{if } t \in [0, \delta]; \\
\frac{t^2}{\delta^3} \left( 3 - \frac{2(t-t_1)}{\delta} \right), & \text{if } t \in [\delta, t_1]; \\
-1, & \text{if } t \in [t_1, t_2]; \\
-1 + \frac{(t-t_1)^2}{\delta^2} \left[ 3 - \frac{2(t-t_1)}{\delta} \right], & \text{if } t \in [t_2, t_3]; \\
\frac{(t-t_3)^2}{\delta^2} \left[ 3 - \frac{2(t-t_3)}{\delta} \right], & \text{if } t \in [t_3, t_4]; \\
1, & \text{if } t \in [t_4, t_5]; \\
1 - \frac{(t-t_5)^2}{\delta^2} \left[ 3 - \frac{2(t-t_5)}{\delta} \right], & \text{if } t \in [t_5, T]; 
\end{cases}
\]

(8)

where,

\[
\delta = \beta \kappa, \quad \kappa = T - \sqrt{T^2 - 8(p_0 - p_T)}, \quad p_0 = \frac{J \theta(t_0)}{u_{\text{max}}}, \quad p_T = \frac{J \theta(T)}{u_{\text{max}}}, \quad t_1 = \frac{T}{2} - \frac{\delta + \zeta}{2}, \quad t_2 = \frac{T}{2} + \frac{\delta - \zeta}{2}, \quad t_3 = \frac{T}{2} - \frac{\delta - \zeta}{2}, \quad t_4 = \frac{T}{2} + \frac{\delta + \zeta}{2}, \quad t_5 = T - \delta, \quad \zeta = \sqrt{(T - \delta)^2 - 4(p_0 - p_T)}.
\]

Note that the term \(8(p_0 - p_T)\) should be less than or equal to \(T^2\), and \(\delta\) should be less than or equal to \(\kappa/4\). An unpleasant sharp control profile, however, is given when \(\delta = \kappa/4\). Obviously, the unit for all parameters is seconds except for the constant smoothing parameter \(\beta\), \(p_0\), and \(p_T\), the unit of which is seconds-squared. See the Appendix for the derivation procedure.

In this paper, the sharp control profile case is not considered as a continuous solution. Thus, \(\beta\) has to lie between 0 and 1/4 in order to avoid the sharp control profile case.

Using Eqs. (5) and (8), the angular acceleration, \(\theta(t)\), is calculated. By integrating the angular acceleration found, the angular velocity and angle are calculated as

\[
\omega(t) = u_{\text{max}} v(t) / J, \quad \theta(t) = u_{\text{max}} p(t) / J,
\]

(9)

where,

\[
v(t) = \begin{cases} 
v(t_0) - \frac{t^3}{\delta^3} \left( 1 - \frac{t}{2\delta} \right), & \text{if } t \in [0, \delta]; \\
v(\delta) - \frac{(t - \delta)^3}{\delta^3}, & \text{if } t \in [\delta, t_1]; \\
v(t_1) - \frac{(t - t_1)^3}{\delta^3} \left( 1 - \frac{t - t_1}{2\delta} \right), & \text{if } t \in [t_1, t_2]; \\
v(t_2), & \text{if } t \in [t_2, t_3]; \\
v(t_3) + \frac{(t - t_3)^3}{\delta^3} \left( 1 - \frac{t - t_3}{2\delta} \right), & \text{if } t \in [t_3, t_4]; \\
v(t_4) + t - t_4, & \text{if } t \in [t_4, t_5]; \\
v(t_5) + t - t_5 - \frac{(t - t_5)^3}{\delta^3} \left( 1 - \frac{t - t_5}{2\delta} \right), & \text{if } t \in [t_5, T]; 
\end{cases}
\]

(10)

\[
p(t) = \begin{cases} 
p(t_0) - \frac{t^3}{\delta^3} \left( \frac{1}{4} - \frac{t}{10\delta} \right), & \text{if } t \in [0, \delta]; \\
p(\delta) - \frac{v(\delta)(t - \delta) - \frac{1}{2}(t - \delta)^2}{\delta}, & \text{if } t \in [\delta, t_1]; \\
p(t_1) + \frac{v(t_1)(t - t_1) - \frac{(t - t_1)^3}{2}}{\delta}, & \text{if } t \in [t_1, t_2]; \\
p(t_2) + \frac{v(t_2)(t - t_2)}{\delta}, & \text{if } t \in [t_2, t_3]; \\
p(t_3) + \frac{v(t_3)(t - t_3)}{\delta}, & \text{if } t \in [t_3, t_4]; \\
p(t_4) + \frac{v(t_4)(t - t_4) + \frac{(t - t_4)^2}{2}}{\delta}, & \text{if } t \in [t_4, t_5]; \\
p(t_5) + \frac{v(t_5)(t - t_5) + \frac{(t - t_5)^2}{2}}{\delta}, & \text{if } t \in [t_5, T]; 
\end{cases}
\]

(11)

From Eq. (5), the smoothing control is calculated as

\[
u(t) = u_{\text{max}} a(t).
\]

(12)

### 4. Numerical Simulation

To highlight the technique proposed, a rest-to-rest maneuver case is considered and numerical simulations are performed using the parameters listed in Table 2. Arbitrary parameter values are selected for validating the performance of the method currently suggested, but real data will be used for three-dimensional control of rigid bodies; especially, for cases where only two-axes control torque is available in
the future. Note that several sequential rotations around the remaining control axes provide a suboptimal control solution for the underactuated rigid body.10) Smoothing parameter $\beta$ is assumed to be $1/9$, and the switch-times are calculated as

$$
(\delta, t_1, t_2, t_3, t_4, t_5)
$$

$$
= (0.7463, 1.4245, 2.1708, 7.8292, 8.5755, 9.2537).
$$

The simulation results are shown in Fig. 1. Note that bang-off-bang indicates minimum fuel results, and smoothing represents near-minimum results. The states satisfy the boundary conditions prescribed and a smooth continuous control profile is obtained, but a lower singular control time is required. By decreasing the value of $\beta$, the singular control time for the smooth continuous control profile approaches the singular control time for the bang-off-bang control profile.

5. Conclusions

A methodology is suggested to generate smooth continuous control profiles for the fuel-minimization problem related to the single-axis maneuvers of rigid bodies. The formulations are derived and numerical results are demonstrated for a single-axis, rest-to-rest maneuver case. These formulations can be generalized and extended for solving problems of higher dimensionality, especially for underactuated system control.

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**Appendix**

For \( t \in [t_0, \delta] \), coefficients \( a \) and \( b \) are determined by introducing the first two boundary conditions in Table 1 into Eqs. (6) and (7)

\[
a(0) = b = 0, \quad a'(0) = c = 0. \tag{13}
\]

Inserting Eq. (13) and introducing the remaining boundary conditions in Table 1 into Eqs. (6) and (7) leads to

\[
a(\delta) = \delta^2 (d + e\delta) = -1, \tag{14}
\]
\[
a'(\delta) = \delta(2d + 3e\delta) = 0. \tag{15}
\]

From Eq. (15), the coefficient \( d \) is expressed by

\[
d = -\frac{3}{2e\delta}, \quad \delta \neq 0. \tag{16}
\]

Inserting Eq. (16) into Eq. (14) leads to

\[
e = \frac{2}{\delta^3}. \tag{17}
\]

and \( d \) is found by inserting Eq. (17) into Eq. (16)

\[
d = -\frac{3}{\delta^2}. \tag{18}
\]

Substituting Eqs. (13), (17), (18) into Eq. (6) leads to

\[
a(t) = -\frac{t^2}{\delta^2} \left( 3 - \frac{2t}{\delta} \right). \tag{19}
\]

Integrating Eq. (19) leads to

\[
v(t) = v(t_0) - \frac{t^3}{\delta^3} \left( 1 - \frac{t}{2\delta} \right), \tag{20}
\]

and integrating Eq. (20) leads to

\[
p(t) = p(t_0) - \frac{t^4}{\delta^4} \left( \frac{1}{4} - \frac{t}{10\delta} \right), \tag{21}
\]

where, the initial values \( v(t_0) = J\omega(t_0)/u_{\text{max}} \) and \( p(t_0) = J\theta(t_0)/u_{\text{max}} \) are found from Eq. (9). Similarly, \( a(t) \), \( v(t) \), and \( p(t) \) for each time interval can be found.

Matthew Phillip Cartmell

*Associate Editor*