DILOGARITHM IDENTITIES, q-DIFFERENCE EQUATIONS
AND THE VIRASORO ALGEBRA

EDWARD FRENKEL AND ANDRÁS SZENES

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1. Introduction

In this paper we give a new proof of the identity

\[ \sum_{j=1}^{n-1} L(\delta_j) = \frac{\pi^2}{6} \frac{2n-2}{2n+1}, \]

where \( L(z) \) is the Rogers dilogarithm function:

\[ L(z) = -\int_0^z \log(1-w) \, d \log w + \frac{1}{2} \log z \log(1-z), \quad 0 \leq z \leq 1, \]

and \( \delta_j = \frac{\sin^2\left(\frac{\pi}{2n+1}\right)}{\sin^2\left(\frac{\pi j + 1}{2n+1}\right)} \).

This equality was proved by Richmond and Szekeres from the asymptotic analysis of the Gordon identities [1] and also by Kirillov using analytic methods [2]. There is a wide class of identities of this type, which emerged in recent works on two-dimensional quantum field theories and statistical mechanics. They appear in the calculation of the critical behavior of integrable models, using the so-called Thermodynamic Bethe Ansatz equation ([3, 4, 5, 6, 7, 8]). This is an elegant, but mathematically non-rigorous method.

In this paper, we present a new approach to the identities, which relies on the concepts of quantum field theory, and is rigorous at the same time. We hope this will lead to a better understanding of the mathematical structures behind general integrable two-dimensional quantum field theories.

The idea goes back to Feigin [9]: There is a certain increasing filtration on each irreducible minimal representation of the Virasoro algebra (or another conformal algebra) by finite-dimensional subspaces, whose characters are connected by a q-difference equation. This equation can be used to obtain an expression for the asymptotics of the character of this representation in terms of values of the dilogarithm. On the

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E-mail addresses: frenkel@math.harvard.edu, szenes@math.mit.edu
other hand, it is known that this asymptotics is determined by the (effective) central charge \( c_{\text{eff}} \) (cf. [10]), and this gives us an identity.

To illustrate that the values of the dilogarithm should appear in the asymptotics of the solutions to \( q \)-difference equations, consider the equation

\[
f(qx) = (1 - aq)f(x), \quad 0 < q < 1.
\]

Then

\[
\lim_{n \to \infty} f(q^n) = f(1)^{\infty} \prod_{n=1}^{\infty} (1 - aq^n).
\]

Let us calculate the asymptotics of this solution as \( q \to 1 \). We have

\[
\log q \log f(q^n) = \log q (\log f(1) + \sum_{n=1}^{\infty} \log(1 - aq^n)).
\]

This expression can be interpreted as a Riemann sum. Thus in the limit \( q \to 1 \) it gives

\[
- \int_0^1 \log(1 - ax) \, d \log x = L(a) - \frac{1}{2} \log a \log(1 - a).
\]

We will show that this approach can be carried out for the irreducible representations of the \((2, 2n + 1)\) models of the Virasoro algebra. In this case \( c_{\text{eff}} = (2n - 2)/(2n + 1) \), and the corresponding identity is (1).

Let us remark that according to Goncharov [11], the dilogarithm identities of this type define elements of torsion in \( K_3(\mathbb{R}) \).

2. (2, 2n + 1) minimal models of the Virasoro algebra

The Virasoro algebra is the Lie algebra generated by elements \( \{L_i, i \in \mathbb{Z}\} \) and \( C \), with the relations

\[
[L_i, L_j] = (i - j)L_{i+j} + \frac{1}{12}(i^3 - i)\delta_{i,-j}C \quad \text{and} \quad [L_i, C] = 0.
\]

Fix an integer \( n > 1 \). Let \( 1_n \) be the one-dimensional representation of the subalgebra generated by \( \{C, L_i, i \geq -1\} \), on which the elements \( L_i \) act by 0, and \( C \) acts by multiplication by \( 1 - 3(2n - 1)^2/(2n + 1) \). Denote by \( \tilde{V}_n \) the induced module of the Virasoro algebra. It is \( \mathbb{Z} \)-graded with respect to the action of \( L_0 \), \( \deg(L_{-i}) = i \). By the Poincaré-Birkhoff-Witt theorem this module has a \( \mathbb{Z} \)-graded basis consisting of the monomials \( \{L_{-m_1} \cdots L_{-m_p} v \mid m_1 \geq m_2 \geq \cdots \geq m_p > 1\} \), where \( v \) is the generator of \( 1_n \). The module \( \tilde{V}_n \) has a unique vector \( u \) of degree \( 2n + 1 \), such that \( L_i u = 0 \) for \( i > 0 \). The quotient \( V_n \) of the module \( \tilde{V}_n \) by the submodule generated by \( u \) inherits the \( \mathbb{Z} \)-grading, and is irreducible [12, 13]. It is called the vacuum representation of the \((2, 2n + 1)\) minimal model of the Virasoro algebra [14].

Let \( C_n = \{(m_1, \ldots, m_p) \mid m_1 \geq \cdots \geq m_p > 1, m_i \geq m_{i+n-1} + 2\} \).
Proposition 1 ([15]). The image of the set
\[ \{ L^{-m_1} \cdots L^{-m_p} v | (m_1, \ldots, m_p) \in C_n \} \]
under the homomorphism \( V_n \to V_n \) gives a linear basis of \( V_n \).

Remark 1. While the argument in [15] relied on the Gordon identities, this statement can be proved directly, using only the representation theory of the Virasoro algebra (see [16]).

For every integer \( N > 0 \), introduce the subspaces \( W^r_N, 0 \leq r < n \), of the module \( V_n \), which are linearly spanned by the vectors
\[ \{ L^{-m_1} \cdots L^{-m_p} v | (m_1, \ldots, m_p) \in C_n, m_1 \leq N, m_{r+1} \leq N - 1 \}. \]

For any \( \mathbb{Z} \)-graded vector space \( V = \bigoplus_{m=0}^{\infty} V(m) \) with \( \dim V(m) < \infty \), let \( \text{ch} V = \sum_{m=0}^{\infty} \dim V(m) q^m \), which is called the character of \( V \). Denote by \( w^r_N \) the character of \( W^r_N \), and introduce \( w_N = (w^0_N, \ldots, w^{n-1}_N) \) and \( w_0 = (0, \ldots, 1) \).

Lemma 2. We have the following recursion relation:
\[ w_N = M_n(q)^N w_{N-1}, \]
where
\[ M_n(x) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & x & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & x^{n-2} & \cdots & x & 1 \\ x^{n-1} & x^{n-2} & \cdots & x & 1 \end{pmatrix}. \]

This gives us the following formula for the character of \( V_n \):
\[ \text{ch} V_n = w^0_0 \prod_{N=1}^{\infty} M_n(q^N) w_0 = w^0_0 \cdots M_n(q^N) \cdots M_n(q^2) M_n(q) w_0. \]

Thus \( \text{ch} V_n \) is equal to \( w^0_0 \lim_{q \to \infty} f(q^n) \), where \( f(x) \) is the solution to the \( q \)-difference equation \( f(qx) = M_n(x) f(x) \) with the initial condition \( f(1) = w_0 \).

The character of \( V_n \) is a function in \( q \) for \( 0 < q < 1 \). We will study its asymptotics as \( q \to 1 \).

The following result was proved by Kac and Wakimoto, and follows from the mod-}

ular properties of \( \text{ch} V_n \).

Proposition 3 ([10]).
\[ - \lim_{q \to 1} \log q \log \text{ch} V_n = \frac{\pi^2}{6} \frac{2n - 2}{2n + 1}. \]

Remark 2. The number \( \frac{2n - 2}{2n + 1} \) is called the effective central charge of the \( (2, 2n + 1) \) minimal model.

In the next section, we will use (3) to derive another expression for this asymptotics.
3. ASYMPTOTICS OF THE INFINITE PRODUCT

The following lemma relates the asymptotics of \((3)\) to the asymptotics of the infinite product of the highest eigenvalues of \(M_n(q^N)\).

**Lemma 4.** Denote by \(d_n(x)\) the eigenvalue of \(M_n(x)\) of maximal absolute value.

(a) For any \(x, 0 < x \leq 1\), the matrix \(M_n(x)\) has \(n\) different real eigenvalues and \(d_n(x) > 0\).

(b) The asymptotic behavior of \(\text{ch} V_n\) is the same as that of the infinite product of the \(d_n(q^N)\), more precisely

\[
- \lim_{q \to 1} \log q \log \text{ch} V_n = - \lim_{q \to 1} \log q \log \prod_{N=1}^{\infty} d_n(q^N).
\]

Similarly to the calculation of \((2)\), the right hand side of \((4)\) can be written as the following integral:

\[
\int_0^1 \log d_n(x) \, d \log x.
\]

The rest of this section is devoted to the calculation of this integral.

Denote by \(Q_n(\lambda, x)\) the characteristic polynomial of \(M_n(x)\). Our computation is based on an explicit rational parametrization of the curve \(Q_n(\lambda, x) = 0\).

**Remark 3.** On this curve, \(\lambda\) and \(x\) define two algebraic functions. Our integral \((5)\) is the integral of \(\log \lambda \, d \log x\) along a path on the curve. We would like to stress that it is the rationality of this curve, that allows us to express this integral in terms of values of the dilogarithm function.

Introduce the \(n \times n\) matrix

\[
L_n(x) = \begin{pmatrix}
1 + x & -1 & 0 & \ldots & 0 & 0 & 0 \\
-x & 1 + x & -1 & \ldots & 0 & 0 & 0 \\
0 & -x & 1 + x & \ldots & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \ldots & -x & 1 + x & -1 \\
0 & 0 & 0 & \ldots & 0 & -x & x
\end{pmatrix},
\]

and observe that \(M_n(x)^{-2} = x^{-n}L_n(x)\). Then the roots of the characteristic polynomial \(P_n(\mu, x) = \det(L_n(x) - \mu)\) are connected to the roots of \(Q_n(\lambda, x)\) via the formula \(\lambda^2 = x^n/\mu\). On the other hand, we have the following recursion for \(P_n(x)\):

\[P_n(\mu, x) = (1 + x - \mu)P_{n-1}(\mu, x) - xP_{n-2}(\mu, x),\]

with initial conditions \(P_0(\mu, x) = 1\) and \(P_1(\mu, x) = x - \mu\).
Introduce \( t = x/(x - \mu + 1)^2 \) and \( u = 1/(x - \mu + 1) \). In these new variables our recursion can be written as
\[
u^{-1} \frac{P_{n-1}}{P_n} = \frac{1}{1 - t \left(u^{-1} \frac{P_{n-2}}{P_{n-1}}\right)}.
\]
This can be solved by the continuous fraction
\[
u^{-1} \frac{P_{n-1}}{P_n} = \frac{1}{t \quad \frac{1}{1 - t \quad \frac{1}{t \quad \ddots \quad \frac{1}{t \quad \frac{1}{1 - u}}}}}
\]
As a consequence the equation \( P_n = 0 \) is equivalent setting the denominator of this continuous fraction to 0. This leads to the following parametrization for \( u \):
\[
u = A_n(t), \quad \text{where} \quad A_n(t) = 1 - \frac{t}{t \quad \frac{1}{t \quad \ddots \quad \frac{1}{t \quad \frac{1}{1 - t}}}}
\]
Naturally, \( A_n(t) \) satisfies the recursion \( A_n(t) = 1 - t/A_{n-1}(t) \) with the initial condition \( A_1(t) = 1 \). A quick calculation shows that as functions of the parameter \( t \) our original variables \( \lambda \) and \( x \) take the form
\[
x_n(t) = \frac{t}{A_n^2(t)} \quad \text{and} \quad \lambda_n(t) = \frac{A_1(t)A_2(t) \ldots A_{n-1}(t)}{A_n^{-1}(t)}.
\]

**Lemma 5.** For any \( \alpha \), \( 0 < \alpha \leq 1 \), there are \( n \) real positive solutions \( t_1(\alpha) < \cdots < t_n(\alpha) \) to the equation \( x_n(t) = \alpha \). The numbers \( \lambda_n(t_1(\alpha)), \ldots, \lambda_n(t_n(\alpha)) \) are the eigenvalues of the matrix \( M_n(\alpha) \), and
\[
d_n(\alpha) = \lambda_n(t_1(\alpha)).
\]
The lemma implies that our integral (5) can be written as
\[
\int_{t=0}^{t=t_1(1)} \log \lambda_n(t) \, d \log x_n(t).
\]
(6)
Introduce the rational functions \( f_i(t) = 1 - A_{i+1}(t)/A_i(t) \). The following key formula can be proved by induction:
\[
f_i(t) = \frac{t^i}{(1 - f_1(t))^{2(i-1)} \cdots (1 - f_{i-1}(t))^2}.
\]
(7)
In terms of the these functions our variables can be expressed as follows:

\[ x_n = \frac{t}{(1 - f_1)^2(1 - f_2)^2 \cdots (1 - f_{n-1})^2} \]

\[ \lambda_n = \frac{1}{(1 - f_1)(1 - f_2)^2 \cdots (1 - f_{n-1})^{n-1}} \]

Now we can calculate the integral (6). We begin with a more general integral

\[ \int_{0}^{\gamma} \log \lambda_n(t) \, d \log x_n(t), \]

where \( \gamma > 0 \) is such that \( \lambda_n(t) \) and \( x_n(t) \) have no poles on the interval \([0, \gamma]\). Substituting the formulas above, we obtain

\[ -\int_{0}^{\gamma} \sum_{j=1}^{n-1} j \log(1 - f_j(t)) \, d \left( \log t - \sum_{i=1}^{n-1} 2 \log(1 - f_i(t)) \right) = \]

\[ -\int_{0}^{\gamma} \sum_{j=1}^{n-1} \log(1 - f_j(t)) \, d \log \left( \frac{t^j}{\prod_{i=1}^{j-1} (1 - f_i(t))^{2(j-i)}} \right) + \]

\[ \int_{0}^{\gamma} \sum_{i,j=1}^{n-1} 2 \min(i, j) \log(1 - f_j(t)) \, d \log(1 - f_i(t)). \]

Applying formula (7) to the first summand, and partial integration to the second, our integral takes the form

\[ -\int_{0}^{\gamma} \sum_{j=1}^{n-1} \log(1 - f_j(t)) \, d \log f_j(t) + \sum_{i,j=1}^{n-1} \min(i, j) \log(1 - f_j(\gamma)) \log(1 - f_i(\gamma)). \]

Now by the definition of the Rogers dilogarithm, this can be written as

\[ \sum_{j=1}^{n-1} L(f_j(\gamma)) - \frac{1}{2} \sum_{j=1}^{n-1} \log f_j(\gamma) \log(1 - f_j(\gamma)) + \]

\[ \sum_{i,j}^{n-1} \min(i, j) \log(1 - f_j(\gamma)) \log(1 - f_i(\gamma)). \]

To complete our proof, we need to describe some properties of the numbers \( \delta_i \) of (1).

**Lemma 6.** Fix an integer \( n > 1 \). The numbers \( \delta_i, 1 \leq i \leq n-1 \), satisfy the following properties:

(a) \( f_i(\delta_1) = \delta_i \)

(b) \( \delta_i = \prod_{j=1}^{i-1} (1 - \delta_j)^{2 \min(i, j)} \)

(c) \( \delta_1 = A_n(\delta_1)^2, \) thus \( x_n(\delta_1) = 1 \). Moreover, \( t_1(1) = \delta_1 \).
Combining the calculation above with property (c), we see that the integral (6) equals to the expression (8), with $\gamma = \delta_1$. Using property (a) this is simply

$$\sum_{j=1}^{n-1} L(\delta_j) - \frac{1}{2} \sum_{j=1}^{n-1} \log \delta_j \log(1 - \delta_j) + \sum_{i,j} \min(i, j) \log(1 - \delta_i) \log(1 - \delta_j).$$

Finally, according to property (b), the last two terms cancel. Hence we obtain the following

**Proposition 7.**

$$\int_0^1 \log d_n(x) \, d \log x = \sum_{j=1}^{n-1} L(\delta_j).$$

This result, together with Lemma 4 and Proposition 3, proves identity (1).

**Remark 4.** We are certain that this method is applicable to other models of conformal field theory.

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**References**

1. B.Richmond, G.Szekeres, *Some formulas related to dilogarithms, the zeta function and the Andrews-Gordon identities*, J. Australian Math. Soc. (Series A) 31 (1981) 362-373
2. A.N.Kirillov, *On identities for the Rogers dilogarithm function related to simple Lie algebras*, Zap. Nauch. Sem. LOMI, 164 (1987) 121-133
3. A.N.Kirillov, N.Reshetikhin, *Exact solution of the XXZ Heisenberg model of spin S*, Zap. Nauch. Sem. LOMI, 160 (1987) 211-222
4. V.Bazhanov, N.Reshetikhin, *Restricted solid-on-solid models connected with simply laced algebras and conformal field theory*, J. Phys. A 23 (1990) 1477-1492
5. A.Kuniba, T.Nakanishi, *Spectra in conformal field theory and the Rogers dilogarithm*, Preprint SMS-042-92, June 1992, to appear in Mod. Phys. Lett. A
6. W.Nahm, A.Recknagel, M.Terhoeven, *Dilogarithm identities in conformal field theory*, Preprint BONN-HE-92-35, November 1992
7. Al.Zamolodchikov, *Thermodynamic Bethe Ansatz in relativistic models: scaling 3-state Potts and Lee-Yang models*, Nucl. Phys. B342 (1990) 695-720
8. T.Klassen, E.Meltzer, *Purely elastic scattering theories and their ultraviolet limit*, Nucl. Phys. B338 (1990) 485-528
9. B.Feigin, private communication
10. V.Kac, M.Wakimoto, *Modular invariant representations of infinite-dimensional Lie algebras and superalgebras*, Proc. Natl. Acad. Sci. USA, 85 (1988) 4956-4960
11. A.Goncharov, private communication
12. V.Kac, *Contravariant form for infinite-dimensional Lie algebras and superalgebras*, Lect. Notes in Phys. 94 (1979) 441-445
13. B. Feigin, D. Fuchs, *Representations of the Virasoro algebra*, in “Representations of infinite-dimensional Lie groups and Lie algebras”, eds. A.M. Vershik and D.P. Zhelobenko, Gordon and Breach, 1990, 465-554

14. A. Belavin, A. Polyakov, A. Zamolodchikov, *Infinite conformal symmetry in two-dimensional quantum field theory*, Nucl. Phys. B241 (1984) 333-380

15. B. Feigin, T. Nakanishi, H. Ooguri, *The annihilating ideals of minimal models*, Int. J. Mod. Phys. Suppl. 1A (1992) 217-238

16. B. Feigin, E. Frenkel, in preparation.