THE RANK OF THE 2ND GAUSSIAN MAP
FOR GENERAL CURVES

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Abstract. We prove that, for the general curve of genus $g$, the 2nd Gaussian map $\mu$ is injective if $g \leq 17$ and surjective if $g \geq 18$. The proof relies on the study of the limit of $\mu$ when the general curve of genus $g$ degenerates to a general stable binary curve, i.e. the union of two rational curves meeting at $g + 1$ points.

Introduction

Let $X$ be a smooth, projective curve of genus $g$ and let $\mathcal{L}$ be a line bundle on $X$. Consider the product $X \times X$, with the projections $p_1, p_2$ to the factors, and the natural morphism $p$ to the symmetric product $X(2)$. One has $p_*(p_1^*\mathcal{L} \otimes p_2^*\mathcal{L}) = \mathcal{L}^+ \oplus \mathcal{L}^-$, where $\mathcal{L}^+$ are the invariant and anti-invariant line bundles with respect to the involution $(x, y) \mapsto (y, x)$. One has $H^0(\mathcal{L}^+) \cong \text{Sym}^2 H^0(\mathcal{L})$ and $H^0(\mathcal{L}^-) \cong \wedge^2 H^0(\mathcal{L})$. Restriction to the diagonal of $X(2)$ gives rise to two maps

$$\mu_{\mathcal{L},1} : \text{Sym}^2 H^0(\mathcal{L}) \to H^0(\mathcal{L}^\odot 2), \quad \omega_{\mathcal{L},1} : \wedge^2 H^0(\mathcal{L}) \to H^0((\mathcal{L}^\odot 2) \otimes K_X),$$

where $K_X$ is the canonical bundle of $X$. Both maps have a well known geometric meaning. The former is given by considering the map $\phi_{\mathcal{L}} : X \to \mathbb{P}^r := \mathbb{P}(H^0(\mathcal{L}))^*$ defined by the complete linear series determined by $\mathcal{L}$ and by pulling back to $X$ forms of degree two in $\mathbb{P}^r$. The latter is given by considering the composition $\gamma$ of $\phi_{\mathcal{L}}$ with the Gauss map $X$ to the Grassmannian of lines $\mathbb{G}(1, r)$ and by pulling back to $X$ via $\gamma$ forms of degree one in $\mathbb{P}^r$. The maps $\mu_{\mathcal{L},1}$ and $\omega_{\mathcal{L},1}$ are the first instances of two hierarchies of maps $\mu_{\mathcal{L},k}$ and $\omega_{\mathcal{L},k}$, defined for all positive integers $k$, and called by some authors higher Gaussian maps of $X$. They are inductively defined by iterated restrictions to the diagonal of $X(2)$. Precisely for all $k \geq 2$ one has

$$\mu_{\mathcal{L},k} : \ker(\mu_{\mathcal{L},k-1}) \to H^0((\mathcal{L}^\odot 2 \otimes K_X^\otimes 2(k-1))), \quad \omega_{\mathcal{L},k} : \ker(\omega_{\mathcal{L},k-1}) \to H^0((\mathcal{L}^\odot 2 \otimes K_X^\otimes 2(k-1))).$$

These maps are particularly interesting when $\mathcal{L} \cong K_X$, in which case we will simply denote them as $\mu_k$ and $\omega_k$. They are both defined at a general point of the moduli space of curves $\mathcal{M}_g$ and it is natural to guess that they have some modular meaning. Indeed, $\mu_1$ is the codifferential, at the point corresponding to $X$, of the Torelli map $\tau : \mathcal{M}_g \to \mathcal{A}_g$, and Noether’s theorem says it is surjective if and only if $X$ is non-hyperelliptic.

The map $\omega_1$ is called the Wahl map, and it is related to important deformation and extendability properties of the canonical image of the curve (cf. [BM], [W]). Because of this, it has been studied by various authors, too many to be quoted here. One the most interesting results concerning it is perhaps a theorem first proved by Ciliberto, Harris and Miranda in [CHM], to the effect that $\omega_1$ is surjective, as expected, for a general curve of genus $g = 10$ and $g \geq 12$. Moreover, this map is injective, as expected, for a general curve of genus $g \leq 8$, cf. [CM2]. Unexpectedly, the Wahl map is not of maximal rank for a general curve of genus $g = 9, 11$.

In general, all maps $\mu_k$ and $\omega_k$ are supposed to be meaningful in the geometry of curves, especially of curves with general moduli. Here we will look in particular at the map $\mu_2 : \mathcal{I}_2(K_X) \to H^0(X, K_X^\otimes 4)$, where $\mathcal{I}_2(K_X)$ is the vector space of forms of degree two vanishing on the canonical model of $X$. From now on we will simply denote this map by $\mu$, and we will call it the 2nd Gaussian map of $X$. This map was first considered by Green-Griffiths in [G] and its importance resides in the fact that it is related to the 2nd fundamental form of the moduli space of curves $\mathcal{M}_g$, embedded in $\mathcal{A}_g$ via the Torelli map, cf. [CPT], [CF1], [CF2].

Despite the unexpected behaviour of the Wahl map for genus $g = 9, 11$, a reasonable working hypothesis is that the 2nd Gaussian map $\mu$ should be of maximal rank for a general curve of any genus $g$. A dimension...
count shows that this is equivalent to say that $\mu$ should be injective for a general curve of genus $g \leq 17$ and surjective if $g \geq 18$. So far, the best result in this direction has been proved by Colombo and Frediani in [CF3], where, by studying hyperplane sections of high genera of K3 surfaces, they show that $\mu$ is surjective for a general curve of genus $g > 152$. For other interesting results concerning $\mu$, see also [CF2, CFP].

In this paper, we prove the maximal rank property for every genus:

**Theorem 1.** The 2nd Gaussian map $\mu: I_2(K_X) \to H^0(X, K_X^{\otimes 4})$ for $X$ a general curve of any genus $g$ has maximal rank, namely it is injective for $g \leq 17$ and surjective for $g \geq 18$.

As shown in [CPT], the map $\mu$ has a lifting $\rho: I_2(K_X) \to \text{Sym}^2(H^0(K_X^{\otimes 2}))$, which is the datum of the second fundamental form of the Torelli embedding at the point corresponding to $X$ in the non-hyperelliptic case. As proved in [CF2], Corollary 3.4, $\rho$ is injective for all non-hyperelliptic curves $X$. Our result shows that if $X$ is general, then the image of $\rho$ is transversal to the kernel of the multiplication map $\text{Sym}^2(H^0(K_X^{\otimes 2})) \to H^0(K_X^{\otimes 4})$.

The proof of Theorem 1 is by degeneration to a reducible nodal curve for which the limit of $\mu$, described in [17] has maximal rank. The theorem then follows by upper semicontinuity. We do not use graph curves here, i.e. the curves exploited in [CHM], because for them the limit of $\mu$ is more difficult to understand. We used instead a general binary curve, i.e. a stable curve of genus $g$ consisting of two rational components meeting at $g + 1$ points, which are general on both components. For such a curve $C$ we explicitly write down the ideal $I_2(K_C)$ in [12]. In [13] we describe the 2nd Gaussian map for $C$ modulo torsion, and then, in [2] we deal with the torsion part. By direct computations performed with Maple (the script is presented and commented in the Appendix), we verified the injectivity for a general binary curve of genus $g \leq 17$ and the surjectivity for $g = 18$. Finally, in [14] we proceed by induction on $g$ to complete the argument for $g \geq 19$.

The behaviour of $\mu$, and its connection with the curvature of $M_g$ in $A_g$, indicates possible relations of the surjectivity of $\mu$ with the Kodaira dimension of $M_g$ being non-negative. This, we think, would be a great subject for future research. Also interesting is the study of the Gaussian maps $\mu_k, w_k$ for higher values of $k$. The maps $\mu_k$ are related to higher fundamental forms of the Torelli immersion of $M_g$ in $A_g$ at a non-hyperelliptic point. Are these maps also of maximal rank for a general curve?

In this paper we work over the complex field and we will use standard notation in algebraic geometry. In particular, if $X$ is a Gorenstein curve, $\Omega_X^1$ will denote its sheaf of Kähler differentials and $K_X$ will denote its dualizing sheaf or canonical bundle, or a canonical divisor. In general, we will indifferently use sheaf, bundle or divisor notation. We will often write $H^i(L)$ instead of $H^i(X, L)$ for cohomology spaces.

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1. **The 2nd Gaussian map for a stable curve**

Let $X$ be a stable curve of genus $g$. We will denote by $I_2(K_X)$ the vector space of forms of degree 2 vanishing on the canonical model of $X$. If $X$ is smooth, the 2nd Gaussian map $\mu: I_2(K_X) \to H^0(X, K_X^{\otimes 4})$ is locally defined as follows.

Fix a basis $\{\omega_i\}$ of $H^0(K_X)$, and write it in a local coordinate $z$ as $\omega_i = f_i(z) \, dz$. Let $Q \in I_2(K_X)$, with $Q = \sum_{i,j} s_{ij}\omega_i \otimes \omega_j$, the matrix $(s_{ij})$ being symmetric. Since $\sum_{i,j}s_{ij} f_i f_j = 0$, one has $\sum_{i,j}s_{ij} f'_i f'_j = 0$. The local expression of $\mu(Q)$ is then (cf., e.g., [CF2])

$$
\mu(Q) = \sum_{i,j} s_{ij} f'_i f'_j \, (dz)^4 = - \sum_{i,j} s_{ij} f'_i f'_j \, (dz)^4.
$$

(1)

If $X$ is nodal, one can similarly define the 2nd Gaussian map $\mu: I_2(K_X) \to H^0(X, \text{Sym}^2(\Omega_X^1) \otimes K_X^{\otimes 2})$ which is locally defined in a similar way as in [11]. Precisely, let $\{\omega_i\}$ be a basis of $H^0(K_X)$. In local coordinates, we can write $\omega_i = f_i \, \xi$, where $f_i$ is a regular function and $\xi$ is a local generator of the canonical bundle $K_X$. Then $\mu$ is locally defined by

$$
\mu(Q) = - \sum_{i,j} s_{ij} \, df_i \, df_j \, \xi^{\otimes 2}.
$$

(2)
Given a flat degeneration over a disc of a general curve to a stable curve $X$, the 2nd Gaussian map for $X$ is the flat limit of the 2nd Gaussian map for the general curve.

It is useful to describe in some detail the space $H^0(X, \text{Sym}^2(\Omega^1_X) \otimes K^\otimes 2)$. First remark that $\text{Sym}^2(\Omega^1_X)$ has torsion $T$ supported at the nodes of $X$. So we have a short exact sequence

$$0 \to T \to \text{Sym}^2(\Omega^1_X) \to \mathcal{F}_X \to 0,$$

where $\mathcal{F}_X$ is a non-locally free, rank 1, torsion free sheaf on $X$.

**Lemma 2.** (a) For every node $p$ of $X$, $T_p$ is a 3-dimensional vector space; if the local equation of $X$ around $p$ is $xy = 0$, then $T_p$ is spanned by $dx dy, x dx dy$ and $y dx dy$.

(b) If $X_i$ are the irreducible components of the normalization $\pi: \tilde{X} \to X$ of $X$, one has

$$\mathcal{F}_X \cong \bigoplus_i \pi_* K_X^\otimes 2.$$

**Proof.** Since $y dx = -x dy$, a local section of $\text{Sym}^2(\Omega^1_{\tilde{X}})$ around a node $xy = 0$ can be uniquely written as $f(x) (dx)^2 + g(x, y) dx dy + h(y) (dy)^2$, where $g(x, y)$ is linear. Then (a) is a local computation and (b) follows from (a).

As a consequence, since $K_{X|\tilde{X}_i} = K_{X_i}(D_i)$ where $D_i$ be the divisor of nodes on $X_i$, one has

$$H^0(X, \text{Sym}^2(\Omega^1_X) \otimes K_X^\otimes 2) \cong T \oplus \bigoplus_i H^0(X_i, K_{X_i}^\otimes 4(2D_i)).$$

where $T \cong \mathbb{C}^3\delta$, with $\delta$ the number of nodes of $X$.

### 2. Canonical binary curves

Let $[x_1, \ldots, x_g]$ be homogenous coordinates in $\mathbb{P}^{g-1}$, $g \geq 3$. Let $p_h = [0, \ldots, 0, 1, 0, \ldots, 0]$, with 1 at the $h$-th place, $1 \leq h \leq g$, be the coordinate points and $u = [1, 1, \ldots, 1]$ the unit point. Take $C_1, C_2$ two distinct rational normal curves in $\mathbb{P}^{g-1}$ passing through $p_h$, $1 \leq h \leq g$, and $u$. Then $C_1, C_2$ intersect transversally at these $g + 1$ points and have no further intersection.

We may and will assume that $C_k, k = 1, 2$, is the closure of the image of the map $f_k$ given by

$$t \mapsto f_k(t) = \left[\frac{1}{t - \alpha_{k,1}}, \frac{1}{t - \alpha_{k,2}}, \ldots, \frac{1}{t - \alpha_{k,g}}\right],$$

where $\alpha_{k,i} \in \mathbb{C}$, $k = 1, 2$, $i = 1, \ldots, g$. In particular, $f_k(\alpha_{k,h}) = p_h, h = 1, \ldots, g$, and $f_k(\infty) = u$. For our purposes, the $\alpha_{k,i}$’s will be general in $\mathbb{C}$. Actually, we will often consider them as indeterminates on $\mathbb{C}$.

The curve $C = C_1 \cup C_2$ is the limit of a general canonical curve $X \subset \mathbb{P}^{g-1}$ of genus $g$, and $C$ is canonical too, i.e. $\mathcal{O}_C(1) \cong K_C$. The curve $C$ is usually called a canonical binary curve.

**Proposition 3.** A canonical binary curve $C = C_1 \cup C_2$ is projectively normal.

**Proof.** The assertion is trivial for $g = 3$, which is the minimum allowed value of $g$. So we may assume $g \geq 4$. By Theorem 1.2 in [S], it suffices to show that there are $g - 2$ smooth points of $C$ spanning a $\mathbb{P}^{g-3}$ which meets $C$ scheme-theoretically at these $g - 2$ points only. Choose $g - 2$ general points on $C_1$ and let $\Lambda \cong \mathbb{P}^{g-3}$ be their span. This meets transversally $C_1$ at these points. We claim that $\Lambda$ does not meet $C_2$. Otherwise choose $g - 4$ general points on $C_1$ and project $C$ down to $\mathbb{P}^3$ from their span. The image of $C_1$ is a rational normal cubic $\Gamma_1$, whereas $C_2$ projects birationally (cf. [CC]) to a non-degenerate rational curve $\Gamma_2$ of degree larger than 3, thus $\Gamma_1$ and $\Gamma_2$ are distinct. Moreover the general secant line to $\Gamma_1$ would meet $\Gamma_2$, which is impossible by the trisecant lemma (see the focal proof in [ChC]).

**Remark 4.** The only information that we will need from the above proposition is that $C$ is quadratically normal, which is equivalent to

$$\dim(\mathcal{I}_2(K_C)) = \begin{pmatrix} g - 2 \\ 2 \end{pmatrix}.$$
We are now interested in explicitly describing the vector space $I_2(K_C)$ of degree two forms vanishing on $C$, i.e. the domain of the map $\mu$ for $C$. The analysis we are going to make will provide another proof that the general binary curve $C$ is quadratically normal.

For $k = 1, 2$, set

$$A_k(t) = \prod_{i=1}^{g} (t - \alpha_{k,i}).$$

For each $h = 0, \ldots, g$, the coefficients $c_{k,h}$ of $t^{g-h}$ in $A_k(t)$ are, up to sign, the elementary symmetric functions

$$c_{k,0} = 1, \quad c_{k,h} = (-1)^h \sum_{1 \leq i_1 < i_2 < \cdots < i_h \leq g} \alpha_{k,i_1} \alpha_{k,i_2} \cdots \alpha_{k,i_h}.$$  

Note that the index $h$ is the degree of $c_{k,h}$ as a polynomial in the $\alpha_{k,i}$'s.

Fix $k \in \{1, 2\}$. Since $C_k$ passes through the coordinate points, the equation of a quadric $Q \subset \mathbb{P}^{g-1}$ containing $C_k$ has the form

$$\sum_{1 \leq i < j \leq g} s_{ij}x_i x_j = 0,$$

with the conditions

$$P_k(t) := \sum_{1 \leq i < j \leq g} \frac{A_k(t)}{(t - \alpha_{k,i})(t - \alpha_{k,j})} s_{ij} = \sum_{n=0}^{g-2} P_{k,n} t^n \equiv 0,$$

where $P_k(t)$ is a polynomial in $t$ of degree $g-2$ whose coefficients are linear polynomials $P_{k,n}(s_{ij})$ in the $s_{ij}$'s, $n = 0, \ldots, g-2$. By expanding the product $A_k(t)$ one sees that the coefficients $p_{k,h;i,j}$ of $s_{ij}$ in $P_{k,g-2-h}$, $h = 0, \ldots, g-2$, are

$$p_{k,0;i,j} = 1, \quad p_{k,1;i,j} = -\sum_{i_i \neq i,j} \alpha_{k,i_i}, \quad p_{k,h;i,j} = (-1)^h \sum_{i_1 < i_2 < \cdots < i_h \atop a_i \neq i,j} \alpha_{k,i_1} \alpha_{k,i_2} \cdots \alpha_{k,i_h}, \quad 2 \leq h \leq g-2,$$  

namely the elementary symmetric functions, removing the $i$ and $j$ terms, up to sign. Again the index $h$ coincides with the degree of $p_{k,h;i,j}$ as a homogeneous polynomial in the $\alpha_{k,i}$'s.

Consider also the polynomials

$$Q_{k,n}(s_{ij}) := \sum_{1 \leq i < j \leq g} \left( \sum_{m=0}^{g-2-n} \alpha_{k,i}^m \alpha_{k,j}^{g-2-n-m} \right) s_{ij}, \quad n = 0, \ldots, g-2,$$

and let $q_{k,h;i,j} = \sum_{m=0}^{h} \alpha_{k,i}^m \alpha_{k,j}^{g-2-n-m}$ be the coefficient of $s_{ij}$ in $Q_{k,g-2-h}$, $h = 0, \ldots, g-2$. Also in this case the index $h$ coincides with the degree of $q_{k,h;i,j}$ as a homogeneous polynomial in the $\alpha_{k,i}$'s.

**Remark 5.** The coefficient $q_{k,h;i,j}$ of $s_{ij}$ in $Q_{k,g-2-h}$ can be recursively computed by

$$q_{k,0;i,j} = 1, \quad q_{k,1;i,j} = \alpha_{k,i} + \alpha_{k,j}, \quad q_{k,h;i,j} = q_{k,1;i,j} q_{k,h-1;i,j} - \alpha_{k,i} \alpha_{k,j} q_{k,h-2;i,j}, \quad 2 \leq h \leq g-2.$$  

Note that all the monomials $\alpha_{k,i}^m \alpha_{k,j}^{h-m}$, $m = 0, \ldots, h$, in particular $\alpha_{k,i}^h$ and $\alpha_{k,j}^{g-2-h-1}$, appear in $q_{k,h;i,j}$ with coefficient 1. Note also the recursive formula

$$q_{k,h;i,j} = \alpha_{j} q_{k,h-1;i,j} + \alpha_{j}^h, \quad 1 \leq h \leq g-2.$$

We will need the following lemma:

**Lemma 6.** Fix $k \in \{1, 2\}$. For each $n = 0, \ldots, g-2$, one has

$$P_{k,n} = \sum_{m=0}^{g-2-n} c_{k,m} Q_{k,n+m}.$$  

In particular, the linear system

$$P_{k,n}(s_{ij}) = 0, \quad n = 0, \ldots, g-2,$$

in the $s_{ij}$'s is equivalent to the linear system

$$Q_{k,n}(s_{ij}) = 0, \quad n = 0, \ldots, g-2.$$  

Proof. One has $P_{k,g-2} = Q_{k,g-2}$ and $P_{k,g-3} = Q_{k,g-3} + c_{k,1}Q_{k,g-2}$. Next we proceed by induction: formula (9) is equivalent to

$$p_{k,h;i,j} = \sum_{l=0}^{h} c_{k,l} q_{k,h-l;i,j}, \quad \text{for } h = 0, \ldots, g - 2. \quad (12)$$

For $h = 0, 1$, (12) clearly holds. Since the index $k$ is fixed, we omit it. For $2 \leq h \leq g - 2$, one has

$$p_{h;i,j} - c_{h} q_{0;i,j} = (\alpha_i + \alpha_j) p_{h-1;i,j} - \alpha_i \alpha_j p_{h-2;i,j} \quad \text{(by induction)}$$

$$= c_{h-1} q_{1;i,j} + \sum_{l=0}^{h-2} c_l (q_{h-1-l;i,j} q_{1;i,j} - \alpha_i \alpha_j q_{h-2-l;i,j}) = \sum_{l=0}^{h-1} c_l q_{h-l;i,j},$$

which proves (12) and therefore (9). Since $c_{k,0} = 1$, the base change matrix between the $Q_{k,n}$’s and the $P_{k,n}$’s is unipotent triangular, hence it is invertible. The equivalence between (10) and (11) follows. \qed

Next we can give the announced description of $\mathcal{I}_2(K_C)$.

**Proposition 7.** Let $g \geq 3$. For a general choice of $\alpha_{k,i}$, $1 \leq k \leq 2$, $1 \leq i \leq g$, one has that

(a) the linear system (11) has maximal rank $g - 1$;

(b) the linear system

$$Q_{1,0}(s_{ij}) = \cdots = Q_{1,g-2}(s_{ij}) = Q_{2,0}(s_{ij}) = \cdots = Q_{2,g-3}(s_{ij}) = 0, \quad (13)$$

has maximal rank $2g - 3$.

**Proof.** (a) Since the index $k$ is fixed, we drop it here. Let us consider the matrix

$$U := U(\alpha_1, \ldots, \alpha_g) = (q_{h;i,j})_{0 \leq h \leq g-2, 1 \leq i < j \leq g}$$

of size $(g - 1) \times \binom{g}{2}$, where the pairs $(i, j)$ are lexicographically ordered. We have to prove that there is a minor of $U$ of order $g - 1$ which is not identically zero. We show this for the minor $D := D(\alpha_1, \ldots, \alpha_g)$ determined by the first $g - 1$ columns, indexed by $(1, i)$ with $2 \leq i \leq g$. This is true if $g = 3$, so we proceed by induction on $g$. Look at $D$ as a polynomial in $\alpha_g$: it has degree $g - 2$ and the coefficient of $\alpha_g^{g-2}$ is $D(\alpha_1, \ldots, \alpha_{g-1})$ (cf. Remark 4), which is non-zero by induction. This proves the assertion.

Equivalently, by subtracting from each row the previous one multiplied by $\alpha_1$ and using (8) (cf. Remark 5), one sees that $D$ is the Vandermonde determinant $V(\alpha_2, \ldots, \alpha_g) = \prod_{2 \leq i < j \leq g}(\alpha_j - \alpha_i)$ of $\alpha_2, \ldots, \alpha_g$.

(b) We use the same idea of the proof of (a). Form a matrix $Z := Z(\alpha_{k,i})_{1 \leq k \leq 2, 1 \leq i < j \leq g}$ of size $(2g - 3) \times \binom{g}{2}$ by concatenating vertically $U$ (for $k = 1$) and the matrix

$$W := W(\alpha_2, \ldots, \alpha_g) = (q_{h,k;i,j})_{1 \leq h \leq g-2, 1 \leq i < j \leq g}.$$ 

It suffices to prove that the minor $M := M(\alpha_{k,i})_{1 \leq k \leq 2, 1 \leq i < j \leq g}$ of $Z$ determined by the first $2g - 3$ columns, indexed by $(1, i), (2, j)$ with $2 \leq i \leq g$ and $3 \leq j \leq g$, is not identically zero as a polynomial in the $\alpha_{k,i}$’s. This is clearly true for $g = 3$, so we proceed by induction on $g$. Look at $M$ as a polynomial in $\alpha_1, g$ and $\alpha_2$: one sees that the monomial $\alpha_1^{g-2} \alpha_2^{g-3}$ appears in $M$ with the coefficient $(\alpha_2 - \alpha_1)^2 M(\alpha_{k,i})_{1 \leq k \leq 2, 1 \leq i < j \leq g-1}$, which is non-zero by induction, proving the assertion.

Equivalently, looking at $M$ as a polynomial in $\alpha_1, \alpha_2$, one sees that the coefficient of the monomial $\alpha_1^{g-2}$ is the product of the two Vandermonde determinants $V(\alpha_2, \ldots, \alpha_g) V(\alpha_1, \ldots, \alpha_{g-1})$. \qed

**Remark 8.** The solutions of the linear system (11), as well as those of (10), give us the quadrics containing the rational normal curve $C_k$, whereas the solutions of (13) give us the quadrics in $\mathcal{I}_2(K_C)$ for the binary curve $C = C_1 \cup C_2$.

3. Binary curves: the 2nd Gaussian map modulo torsion

Let $C = C_1 \cup C_2$ be a general binary curve. In this section we will consider the composition $\nu$ of the 2nd Gaussian map for $C$ with the projection to the non-torsion part of $H^0(C, \text{Sym}^2(\Omega_C) \otimes K_C^*)$ (cf. formula 8 in [1]). Specifically, for $k = 1, 2$, we will look at the map

$$\nu_k: \mathcal{I}_2(K_C) \to H^0(C_k, K_{C_k}^{\otimes 4}(2D_k))$$

where $D_k$ is a divisor of degree $g + 1$ on $C_k$, therefore $\nu = \nu_1 \oplus \nu_2$ and

$$H^0(C_k, K_{C_k}^{\otimes 4}(2D_k)) \cong H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(2g - 6)).$$
The map $\nu_k$ can be explicitly written down, by taking into account \(\ref{eq:1f} \) and the description of the ideal $I_2(K_C)$ (see \(\ref{eq:2f} \)). Precisely, let $Q \in I_2(K_C)$ be of the form \(\ref{eq:7} \) where the $s_{ij}$'s are solutions of \(\ref{eq:9} \). Then
\[
\nu_k(Q) = \sum_{1 \leq i \neq j \leq g} \frac{1}{(t - \alpha_{k,i})^2(t - \alpha_{k,j})^2} s_{ij}(dt)^4 \in H^0(C_k, K_C^\otimes 4(2D_k)).
\]
To look at this as a section of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2g - 6))$, we multiply by $A_k^2(t)$. Hence
\[
\nu_k(Q) = \sum_{1 \leq i < j \leq g} \frac{A_k^2(t)}{(t - \alpha_{k,i})^2(t - \alpha_{k,j})^2} s_{ij} =: R_k(t)
\]
is a polynomial in $t$ whose apparent degree is $2g - 4$, but its coefficient of degree $2g - 4$ is $P_{k,g-2}$ and the one of degree $2g - 5$ is proportional to $P_{k,g-3}$, hence they vanish and $R_k(t)$ has actual degree $2g - 6$.

Using this explicit description \(\ref{eq:14} \) of $\nu$, we asked Maple to compute its rank for low values of $g$ (see the Appendix for Maple scripts). The result is the following:

\begin{prop}
The map $\nu$ has maximal rank for $g \leq 18$, namely $\nu$ is injective for $g \leq 10$ and it is surjective for $11 \leq g \leq 18$.
\end{prop}

\begin{cor}
The 2nd Gaussian map $\mu$ is injective for the general curve of genus $g \leq 10$.
\end{cor}

\section*{4. Binary curves: the torsion}

Let $C = C_1 \cup C_2$ be a general binary curve as in \(\ref{eq:4} \). In \(\ref{eq:4} \) we may replace $f_k$, $1 \leq k \leq 2$, with
\[ A_k(t)f_k(t) = [\phi_{k,1}(t), \ldots, \phi_{k,g}(t)], \quad \phi_{k,i}(t) = \frac{A_k(t)}{(t - \alpha_{k,i})}. \]

Now we consider the restriction $\tau$ of the 2nd Gaussian map for $C$ to $\ker(\nu)$, which lands in the torsion part $T$ of $H^0(C, \text{Sym}^2(\Omega_C^1) \otimes K_C^\otimes 2)$, cf. formula \(\ref{eq:12} \). By taking into account Lemma \(\ref{lem:2} \) (a), a direct computation shows that the composition of $\tau$ with the projection on the torsion part $T_{p_h}$ at the coordinate point $p_h$ is as follows: if $Q \in \ker(\nu)$ is of the form \(\ref{eq:1f} \), then $Q$ is mapped to
\[
dx dy \sum_{i \neq j} s_{ij}\phi_{i,1}^*(\alpha_{1,i})\phi_{2,j}^*(\alpha_{2,h}) + 2x dx dy \sum_{i \neq j} s_{ij}\phi_{i,1}^*(\alpha_{1,h})\phi_{2,j}^*(\alpha_{2,h}) + 2y dx dy \sum_{i \neq j} s_{ij}\phi_{i,1}^*(\alpha_{1,h})\phi_{2,j}^*(\alpha_{2,h}).
\]
where $s_{ji} = s_{ij}$ and $x, y$ are local coordinates around $p_h$ such that $C_1: y = 0$ and $C_2: x = 0$. The description of the torsion at the unitary point $u$ is similar. Replace $f_k$ by the parametrization $\frac{1}{2} f_k(\frac{1}{2})$. Again a direct computation shows that the composition of $\tau$ with the projection on $T_u$ is
\[
Q \mapsto dx dy \sum_{i \neq j} s_{ij} \alpha_{1,i} \alpha_{2,j} + 2x dx dy \sum_{i \neq j} s_{ij} \alpha_{1,i}^2 \alpha_{2,j} + 2y dx dy \sum_{i \neq j} s_{ij} \alpha_{1,i} \alpha_{2,j}^2
\]
where $s_{ji} = s_{ij}$ and $x, y$ are local coordinates around $u$ such that $C_1: y = 0$ and $C_2: x = 0$.

Consider the following commutative diagram with exact rows
\[
\begin{array}{cccccc}
0 & \longrightarrow & T & \longrightarrow & H^0(C, \text{Sym}^2(\Omega_C^1) \otimes K_C^\otimes 2) & \longrightarrow & H^0(C_1, K_{C_1}^\otimes 2(2)) \oplus H^0(C_2, K_{C_2}^\otimes 2(2)) \cong H^0(\mathcal{F}_C) \\
& \tau \downarrow & & & & & \nu \\
0 & \longrightarrow & \ker(\nu) & \longrightarrow & I_2(K_C) &
\end{array}
\]

We asked Maple to compute the rank of the map $\tau$ for $11 \leq g \leq 18$ (see the script in the Appendix). Taking into account diagram \(\ref{eq:13} \), the result is the following:

\begin{prop}
Let $C$ be a general binary curve of genus $g$. The maps $\tau$ and $\mu$ have maximal rank for $g \leq 18$, namely they are injective for $g \leq 17$ and surjective for $g = 18$.
\end{prop}

\begin{cor}
The map $\mu$ is injective for the general curve of genus $g \leq 17$, and surjective for $g = 18$.
\end{cor}

\section*{5. The induction step}

In this section we prove the main result of this paper, namely the surjectivity of the 2nd Gaussian map $\mu$ for the general curve of genus $g \geq 18$.

Let $C \subset \mathbb{P}^{g-1}$ be a nodal canonical curve and let $p \in C$ be a node. Let $\tilde{C} \to C$ be the partial normalization of $C$ at $p$, and let $p_1, p_2 \in \tilde{C}$ be the points over $p$. Note that the projection from $p$ maps
$C$ to the canonical model of $\tilde{C}$ in $\mathbb{P}^g-2$ and we will assume that this induces an isomorphism of $\tilde{C}$ to its canonical model. Consider the following diagrams

$$
0 \longrightarrow H^0(F_{\tilde{C}}) \longrightarrow H^0(F_C) \longrightarrow \mathcal{O}_{2p_1} \oplus \mathcal{O}_{2p_2} \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow \tilde{T} \longrightarrow T \longrightarrow T_p
$$

where $\bar{T}$ is the torsion subsheaf of $\text{Sym}^2(\Omega^1_C)$, $\nu, \tau$ are the maps of diagram (18) for the curve $C$ and $\tilde{\nu}, \tilde{\tau}$ are the corresponding ones for $\tilde{C}$. Diagrams (19) are commutative and the horizontal sequences are exact, hence the next lemma is clear:

**Lemma 13.** If $\tilde{\nu}$ and $\chi$ are surjective, then $\nu$ is also surjective. $\square$

We apply this to prove:

**Theorem 14.** If $C = C_1 \cup C_2$ is a general binary curve of genus $g \geq 18$, then $\mu$ is surjective for $C$.

**Proof.** The case $g = 18$ has been done by a direct computation, cf. Proposition [11] We then proceed by induction on $g$: the commutativity of the diagram (18) and the previous lemma show that it is enough to prove the surjectivity of $\chi$ and $\tau_p$, where $p$ is a node of $C$. We will do this for $p = u$ the unitary point.

In this situation, the map $\nu$ is the one $\nu_1 \oplus \nu_2$ considered in [34]. Therefore $\chi = \chi_1 \oplus \chi_2$, where $\chi_k$ is the composition of $\nu_k$ with the restriction to $\mathcal{O}_{2p_1}$, $k = 1, 2$. In local coordinates, $\chi_k(Q)$ is the pair formed by the constant term and the coefficient of the degree-one term of the Taylor expansion around $p$ of the polynomial $\nu_k(Q)$. In [33] we computed $\nu_k$ using a local coordinate $t$ on $C_k$. In this coordinate, the point $p = [1, \ldots, 1]$ corresponds to $t = \infty$. Therefore, if $Q \in \mathcal{I}_2(K_C)$ is of the form (9), with the $s_{ij}$’s satisfying (13), then $\chi_k(Q)$ is the pair of coefficients of the highest degrees $2g - 6$ and $2g - 7$ of the polynomial $\nu_k(Q)$, i.e. of the polynomial $R_k(t)$ given in (14). We denote by $R_{k;2g-6}$ and $R_{k;2g-7}$ these coefficients, which are linear polynomials in the $s_{ij}$’s. We will now compute them.

Fix the index $k$ and omit it. By expanding $A^2$ in (14), one sees that the coefficient of $s_{ij}$ in $R_{2g-6}$ is

$$4p_{2i,j} + \sum_{i,j \neq i,j}^g \alpha^2_i = 4p_{2i,j} + n_2 - (\alpha^2_i + \alpha^2_j),$$

where $n_2 = \sum_{m=1}^g \alpha^2_m$ is independent of $i, j$, and $p_{2i,j}$ is the coefficient of $s_{ij}$ in $R_{2g-4}$, cf. [7]. By (10), this means that

$$R_{2g-6} = 4p_{g-4} + n_2 p_{g-2} - \sum_{i<j}^g (\alpha^2_i + \alpha^2_j)s_{ij} = -\sum_{i<j}^g (\alpha^2_i + \alpha^2_j)s_{ij}.$$

Similarly, one sees that the coefficient of $s_{ij}$ in $R_{2g-7}$ is twice

$$-4p_{3i,j} - \sum_{i,j \neq i,j \text{ both} \neq i,j} \alpha^2_i \alpha_{i+1} = -4p_{3i,j} + n_3 + n_2 q_{1i,j} - c_1(\alpha^2_i + \alpha^2_j) - (\alpha^2_i + \alpha^2_j) - q_{3i,j},$$

where $n_3 = -\sum_{m=1}^g \alpha^3_m$ is independent of $i, j$. Therefore, taking into account account (10) and (11), one has

$$R_{2g-7} = -2c_1 R_{2g-6} - 2 \sum_{i<j} (\alpha^2_i + \alpha^2_j)s_{ij}.$$
Recall that ker(ν) is defined in $Z_2(K_C)$ by the vanishing of the polynomials $R_k(t)$, $k = 1, 2$, whose coefficients of degree at most $2g - 8$ are polynomials in the $\alpha_{k,i}$'s of degree at least $4$. By the description of the torsion at the unitary point given in (17), we need to show the rank maximality of the matrix

$$Y' = \mathcal{N}((\alpha_{k,i})_{1 \leq k \leq 2, 1 \leq i < j \leq g})$$

of size $(2g + 4) \times \left(\binom{g}{2}\right)$ obtained by concatenating vertically the above matrix $Y$ and the matrix of size $3 \times \left(\binom{g}{2}\right)$ whose rows are $(\alpha_{1,i}\alpha_{2,j} + \alpha_{1,j}\alpha_{2,i})_{1 \leq i < j \leq g}$, $(\alpha_{1,i}^2\alpha_{2,j} + \alpha_{1,j}^2\alpha_{2,i})_{1 \leq i < j \leq g}$, and $(\alpha_{1,i}\alpha_{2,j}^2 + \alpha_{1,j}\alpha_{2,i}^2)_{1 \leq i < j \leq g}$. We claim that the minor $N' = \mathcal{N}((\alpha_{k,i})_{1 \leq k \leq 2, 1 \leq i < j \leq g})$ of $Y'$ determined by the first $2g + 4$ columns, indexed by $(1, i)$, $(2, j)$, $(3, \ell)$, with $2 \leq i \leq g$, $3 \leq j \leq g$, $4 \leq \ell \leq 10$ is non-zero for $g \geq 10$. We verified the case $g = 10$ with Maple (see the script in the Appendix) and the induction is the same as before because the monomial $\alpha_{1,i}^2\alpha_{2,j}^2$ appears in $N'$ again with coefficient $(\alpha_{2,2} - \alpha_{2,1})N'(\alpha_{k,i})_{1 \leq k \leq 2, 1 \leq i < j \leq g}$. This concludes the proof that $\tau_p$ is surjective, hence the proof of the theorem. □

**Corollary 15.** The 2nd Gaussian map $\mu$ is surjective for the general curve of genus $g \geq 18$. □

**Appendix:** Maple scripts for computations

We list here the Maple script we run. We will explain it afterwards: for this purpose, we added line numbers at each five lines.

```maple
alpha[1]:=[3,12,21,29,37,43,46,54,62,65,72,81,85,89,94,97,105]:
alpha[2]:=[6,18,24,36,39,42,45,52,60,63,71,80,84,86,91,96,104,108]:
for g from 4 to 18 do
    listsij:=[seq(seq(s[i,j],j=i+1..g),i=1..g)]:
    for k from 1 to 2 do
        A[k]:=mul(t-alpha[k][i],i=1..g):
        R[k]:=add(add(s[i,j]*(A[k]^2)/((t-alpha[k][i])^2*(t-alpha[k][j])^2),
            j=i+1..g),i=1..g):
    end do:
    Z:=linalg[genmatrix](linalg[genmatrix](EqsKerNu,listsij)),'r0') mod 109:
    printf("For g=%2d, one has dim I2(K)=%3d, ",g,nops(listsij)-r0):
    EqsKerNu:=[seq(seq(primpart(coeff(R[k],t,n)),n=0..2*g-6),k=1..2)]:
    K:=Gausselim(linalg[stackmatrix](Zref,'r0')) mod 109:
    printf("dim Ker(nu)=%2d, corank(nu)=%d, ",nops(listsij)-r1,4*g-10-r1+r0):
    for k from 1 to 2 do
        phi1[k,i]:=diff(A[k]/(t-alpha[k][i]),t):
        phi2[k,i]:=diff(phi1[k,i],t):
        for h from 1 to g do
            phi1e[k,i,h]:=eval(phi1[k,i],t=alpha[k][h]):
            phi2e[k,i,h]:=eval(phi2[k,i],t=alpha[k][h]):
        end do:
        phi1e[k,i]+phi1e[2,i,h]+phi1e[2,j,h]:
        phi2e[k,i]+phi2e[2,i,h]+phi2e[2,j,h]:
    end do:
end do:
```

We have used the Maple scripts for computations listed above. We will explain them afterwards.
In lines 1–2, we define the $\alpha_{k,i}$’s which will be used. We chose them randomly. In line 3, we start the main loop which runs for $4 \leq g \leq 18$. In line 4, we collect the unknowns $\{x_{i,j}\}_{1 \leq i \leq j \leq g}$ in the list listsi: there are $\binom{g}{2}$ of them. In lines 6–8 we define the polynomials $A_k(t)$ and $R_k(t)$, cf. [13] and [14].

In lines 10–13, we define the matrix $Z$ associated to the linear system [13], whose solutions give us the quadrics in $I_2(K_2)$, cf. the proof of Proposition [7]. In line 14, Maple computes the rank $r_0$ of $Z$ via Gaussian elimination, by calculating modulo 109 to speed up computations. The resulting matrix in row echelon form is called $Z_{\text{ref}}$. As expected by Proposition [7] (b), Maple finds $r_0 = 2g - 3$ for each $g = 4, \ldots, 18$. In line 15, Maple prints out the genus $g$ and $\dim(I_2(K_2)) = \binom{g}{2} - r_0 = (g-2)^2$. 

In line 16, we collect in EqsKerNu the list of equations which determine $\ker(\nu)$, cf. the definition [14] of $\nu$ in [8]. In lines 17–18, Maple computes the rank $r_1$ of the linear system $\text{EqsKerNu} \cap \ker(Z_{\text{ref}})$, again via Gaussian elimination modulo 109, and the resulting row echelon matrix is called $K$. Maple finds that $r_1 = \binom{g}{2}$ for $4 \leq g \leq 10$ and that $r_1 = 69 - 13$ for $11 \leq g \leq 18$. Therefore the rank of $\nu$ is $r_1 - r_0 = (g-2)^2$ for $4 \leq g \leq 10$, and $= 4g - 10$ for $11 \leq g \leq 18$. This proves Proposition [9].

In line 19, Maple prints out the dimension of $\ker(\nu)$ and the corank of $\nu$, that is $4g - 10 - r_1 + r_0$. In lines 20–25, we define the 1st derivative $\phi_1$ and the 2nd one $\phi_2$ of the $\phi_{k,i}$’s, cf. [15]. We then define their evaluations $\phi_{\text{1e}}, \phi_{\text{2e}}$ at the coordinate point $p_h$. Using them, in lines 26–33 we compute the torsion at $p_h$, $h = 1, \ldots, g$, cf. [16], and, in lines 34–39, the torsion at the unit point $u$, cf. [17].

In lines 40–41, we collect in EqsKerTau the equations which determine $\ker(\tau)$ and Maple computes the rank $r_2$ of $\text{EqsKerTau} \cap \ker(K)$, via Gaussian elimination modulo 109 as before. Maple finds that $r_2 = \binom{g}{2}$ for $4 \leq g \leq 17$ and that $r_2 = 152$ for $g = 18$. Therefore the rank of $\tau$ is $r_2 - r_1 = (g^2 - 13g + 26)/2$ for $11 \leq g \leq 17$, and is $57$ for $g = 18$. This proves Proposition [11].

In line 42, Maple prints out the the dimension of $\ker(\tau)$ and the corank of $\tau$, that is $3g + 3 - r_2 + r_1$. Finally, in lines 43–59, Maple computes the minors $N$ (when $g = 7$) and $N'$ (when $g = 10$), needed in the proof of Theorem [14] and it prints out that $N$ mod $5 = 4$ and $N'$ mod $23 = 16$.

References

[ACGH] A. Arbarello, M. Cornalba, P. Griffiths, J. Harris, Geometry of Algebraic Curves, vol. I, Grundlehrer der math. Wissenschaft 267. Springer Verlag, 1985.

[BM] A. Beauville and J.-Y. Mérindol, Sections hyperplanes des surfaces K3, Duke Math. J., 55 (4) (1987), 873–878.

[CC] C. Ciliberto, C. Ciliberto, A few remarks on the lifting problem, Proceedings, Conf. of Algebraic Geometry, Paris, 1992, Astérisque 218 (1993), 95–109.

[ChC] L. Chiantini, C. Ciliberto, A few remarks on the lifting problem, Proceedings, Conf. of Algebraic Geometry, Paris, 1992, Astérisque 218 (1993), 95–109.

[CHM] C. Ciliberto, J. Harris, R. Miranda, On the surjectivity of the Wahl map, Duke Math. J. 57 (1988), 829–858.

[CM1] C. Ciliberto, R. Miranda, Gaussian maps for certain families of canonical curves, Complex projective geometry (Trieste, 1989/Bergen, 1989), London Math. Soc. Lecture Note 179, Cambridge Univ. Press, 1992, 106–127.

[CM2] C. Ciliberto, R. Miranda, Gaussian maps for canonical curves of low genus, Duke Math. J. 61 (1990), 417–443.
[CF1] E. Colombo, P. Frediani, Some results on the second Gaussian map for curves, arXiv:0805.3422, to appear on Michigan J. Math.

[CF2] E. Colombo, P. Frediani, Siegel metric and curvature of the moduli space of curves, arXiv:0805.3425, to appear on Trans. A.M.S.

[CF3] E. Colombo, P. Frediani, On the second Gaussian map for curves on a K3 surface, preprint, arXiv:0905.2330.

[CFP] E. Colombo, P. Frediani, G. Pareschi, Hyperplane sections of abelian surfaces, preprint, arXiv:0903.2781.

[CPT] E. Colombo, G.P. Pirola, A. Tortora, Hodge-Gaussian Maps, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 30 (2001), 125–146.

[G] M.L. Green, Infinitesimal methods in Hodge theory, in Algebraic Cycles and Hodge Theory, Torino 1993, Lecture Notes in Mathematics 1594, Springer, 1994, 1–92.

[S] F.O. Schreyer, A standard basis approach to syzygies of canonical curves, J. reine angew. Math. 421 (1991), 83–123.

[W] J. Wahl, The Jacobian algebra of a graded Gorenstein singularity, Duke Math. J. 55 (4) (1987), 843–871.

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