Canonical metrics on Hartogs domains

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Abstract

An \( n \)-dimensional Hartogs domain \( D_F \) with strongly pseudoconvex boundary can be equipped with a natural Kähler metric \( g_F \). This paper contains two results. In the first one we prove that if \( g_F \) is an extremal Kähler metric then \( (D_F, g_F) \) is holomorphically isometric to an open subset of the \( n \)-dimensional complex hyperbolic space. In the second one we prove the same assertion under the assumption that there exists a real holomorphic vector field \( X \) on \( D_F \) such that \( (g_F, X) \) is a Kähler–Ricci soliton.

Keywords: Kähler metrics; Hartogs domain; extremal metrics; Kähler–Ricci solitons; generalized scalar curvatures; canonical metrics.

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1 Introduction and statements of the main results

The study of the existence and uniqueness of a preferred Kähler metric on a given complex manifold \( M \) is a very interesting and important area of research, both from the mathematical and from the physical point of view. Many definitions of canonical metrics (Einstein, constant scalar curvature, extremal, Kähler–Ricci solitons and so on) have been given both in the compact and in the noncompact case (see e.g. \([2], [15] \) and \([24] \)). In the noncompact case many important questions are still open. For example Yau raised the question on the classification of Bergman Einstein metrics on strongly pseudoconvex domains and S.–Y. Cheng conjectured that if the Bergman metric on a strongly pseudoconvex domain is Einstein, then the domain is biholomorphic to the ball (see \([13] \)).

In this paper we are interested in extremal Kähler metrics and Kähler–Ricci solitons on a particular class of strongly pseudoconvex domains, the so called Hartogs domains (see the next section for their definition and main properties).

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Our main results are the following theorems.

**Theorem 1.1** Let \((D_F, g_F)\) be an \(n\)-dimensional strongly pseudoconvex Hartogs domain. Assume that \(g_F\) is an extremal Kähler metric. Then \((D_F, g_F)\) is holomorphically isometric to an open subset of the \(n\)-dimensional complex hyperbolic space.

**Theorem 1.2** Let \((D_F, g_F)\) be an \(n\)-dimensional strongly pseudoconvex Hartogs domain and let \(X\) be a real holomorphic vector field on \(D_F\) such that \((g_F, X)\) is a Kähler–Ricci soliton. Then \(g_F\) is Kähler–Einstein. Consequently, \((D_F, g_F)\) is holomorphically isometric to an open subset of the \(n\)-dimensional complex hyperbolic space.

Notice that (compare with Cheng’s conjecture above) the assumptions on the metric \(g_F\) in Theorem 1.1 and Theorem 1.2 are weaker than Einstein’s condition. To this regard it is worth pointing out that when \(g_F\) equals the Bergman metric on \(D_F\), then \((D_F, g_F)\) is holomorphically isometric to an open subset of the complex hyperbolic space (see Theorem 1.3 in [10] for a proof).

The paper is organized as follows. In the next section, after recalling the definition of Hartogs domains, we analyze their pseudoconvexity, and we prove a lemma regarding their generalized scalar curvatures. Sections 3 and 4 are dedicated to the proofs of Theorem 1.1 and Theorem 1.2 respectively.

## 2 Strongly pseudoconvex Hartogs domains

Let \(x_0 \in \mathbb{R}^+ \cup \{+\infty\}\) and let \(F : [0, x_0) \rightarrow (0, +\infty)\) be a decreasing continuous function, smooth on \((0, x_0)\). The Hartogs domain \(D_F \subset \mathbb{C}^n\) associated to the function \(F\) is defined by

\[
D_F = \{(z_0, z_1, \ldots, z_{n-1}) \in \mathbb{C}^n \mid |z_0|^2 < x_0, \ |z_1|^2 + \cdots + |z_{n-1}|^2 < F(|z_0|^2)\}.
\]

One can prove that the assumption of strongly pseudoconvexity of \(D_F\) is equivalent (see Proposition 2.1 below) to the fact that the natural \((1, 1)\)-form on \(D_F\) given by

\[
\omega_F = \frac{i}{2} \partial \bar{\partial} \log \frac{1}{F(|z_0|^2) - |z_1|^2 - \cdots - |z_{n-1}|^2}
\]

(1)
is a Kähler form on $D_F$. The Kähler metric $g_F$ associated to the Kähler form $\omega_F$ is the metric we will be dealing with in the present paper. Observe that for $F(x) = 1 - x, 0 \leq x < 1$, $D_F$ equals the $n$-dimensional complex hyperbolic space $\mathbb{C}H^n$ and $g_F$ is the hyperbolic metric, i.e. $g_F = g_{hyp}$. In the 2-dimensional case this metric has been considered in [11] and [21] in the framework of quantization of Kähler manifolds. In [20], the first author studied the Kähler immersions of $(D_F, g_F)$ into finite or infinite dimensional complex space forms, [9] is concerned with the existence of global symplectic coordinates on $(D_F, \omega_F)$ and [10] deals with the Riemannian geometry of $(D_F, g_F)$ (in particular in this paper one can find necessary and sufficient conditions in terms of $F$ for the completeness of the metric $g_F$).

**Proposition 2.1** Let $D_F$ be a Hartogs domain in $\mathbb{C}^n$. Then the following conditions are equivalent:

(i) the $(1,1)$-form $\omega_F$ given by (1) is a Kähler form;

(ii) the function $-\frac{x F'(x)}{F(x)}$ is strictly increasing, namely $-(\frac{x F'(x)}{F(x)})' > 0$ for every $x \in [0, x_0)$;

(iii) the boundary of $D_F$ is strongly pseudoconvex at all $z = (z_0, z_1, \ldots, z_{n-1})$ with $|z_0|^2 < x_0$;

(iv) $D_F$ is strongly pseudoconvex.

**Proof:** (i) $\Leftrightarrow$ (ii) Set

$$A = F(|z_0|^2) - |z_1|^2 - \cdots - |z_{n-1}|^2.$$  \hspace{1cm} (2)

Then $\omega_F$ is a Kähler form if and only if the real-valued function $\Phi = -\log A$ is strictly plurisubharmonic, i.e. the matrix $g_{\alpha \bar{\beta}} = (\frac{\partial^2 \Phi}{\partial z_\alpha \partial \bar{z}_\beta})$, $\alpha, \beta = 0, \ldots, n - 1$ is positive definite, where

$$\omega_F = \frac{i}{2} \sum_{\alpha, \beta=0}^{n-1} g_{\alpha \beta} dz_\alpha \wedge d\bar{z}_\beta.$$  \hspace{1cm} (3)

A straightforward computation gives

$$\frac{\partial^2 \Phi}{\partial z_0 \partial \bar{z}_0} = \frac{F'(|z_0|^2)|z_0|^2 - (F''(|z_0|^2)|z_0|^2 + F'(|z_0|^2))A}{A^2},$$

$$\frac{\partial^2 \Phi}{\partial z_0 \partial \bar{z}_\beta} = \frac{F'(|z_0|^2)z_0 \bar{z}_\beta}{A^2}, \quad \beta = 1, \ldots, n - 1.$$
and

\[ \frac{\partial^2 \Phi}{\partial z_\alpha \partial \bar{z}_\beta} = \delta_{\alpha \beta} A + \bar{z}_\alpha z_\beta, \quad \alpha, \beta = 1, \ldots, n - 1. \]

Then, by setting

\[ C = F''(|z_0|^2)|z_0|^2 - (F'''(|z_0|^2)|z_0|^2 + F'(|z_0|^2))A, \tag{4} \]

one sees that the matrix \( h = (g_{\alpha \beta}) = (\frac{\partial^2 \Phi}{\partial z_\alpha \partial \bar{z}_\beta})_{\alpha, \beta = 0, \ldots, n - 1} \) is given by:

\[
\begin{pmatrix}
C & -F'z_0 z_1 & \ldots & -F'z_0 z_n & -F'z_0 z_{n-1} \\
-F'z_0 \bar{z}_1 & A + |z_1|^2 & \ldots & \bar{z}_1 z_n & \bar{z}_1 z_{n-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-F'z_0 \bar{z}_n & z_n \bar{z}_1 & \ldots & A + |z_n|^2 & \bar{z}_n z_{n-1} \\
-F'z_0 \bar{z}_{n-1} & z_{n-1} \bar{z}_1 & \ldots & z_{n-1} \bar{z}_n & A + |z_{n-1}|^2
\end{pmatrix} \tag{5}
\]

First notice that the \((n - 1) \times (n - 1)\) matrix obtained by deleting the first row and the first column of \( h \) is positive definite. Indeed it is not hard to see that, for all \( 1 \leq \alpha \leq n - 1 \),

\[
\det \begin{pmatrix}
A + |z_\alpha|^2 & \bar{z}_\alpha z_{\alpha+1} & \ldots & \bar{z}_\alpha z_{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
\bar{z}_{n-1} z_\alpha & \bar{z}_{n-1} z_{\alpha+1} & \ldots & A + |z_{n-1}|^2
\end{pmatrix} = A^{n-\alpha} + A^{n-\alpha-1}(|z_\alpha|^2 + \cdots + |z_{n-1}|^2) > 0. \tag{6}
\]

On the other hand, by the Laplace expansion along the first row, we get

\[
\det(h) = \frac{C}{A^{2n}} [A^{n-1} + A^{n-2}(|z_1|^2 + \cdots + |z_{n-1}|^2)] + \frac{F'z_0 z_1}{A^{2n}} \det \begin{pmatrix}
-F'z_0 \bar{z}_1 & \bar{z}_1 z_1 & \ldots & \bar{z}_1 z_{n-1} \\
-F'z_0 \bar{z}_2 & A + |z_2|^2 & \ldots & \bar{z}_2 z_{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
-F'z_0 \bar{z}_{n-1} & z_{n-1} \bar{z}_1 & \ldots & A + |z_{n-1}|^2
\end{pmatrix} + \cdots + \frac{(-1)^{n} F'z_0 \bar{z}_{n-1}}{A^{2n}} \det \begin{pmatrix}
-F'z_0 \bar{z}_1 & A + |z_1|^2 & \ldots & z_{n-2} \bar{z}_1 \\
-F'z_0 \bar{z}_2 & z_1 \bar{z}_2 & \ldots & z_{n-2} \bar{z}_2 \\
\vdots & \vdots & \ddots & \vdots \\
-F'z_0 \bar{z}_{n-1} & z_{n-1} \bar{z}_1 & \ldots & z_{n-2} \bar{z}_{n-1}
\end{pmatrix}.
\]
\[
\begin{align*}
&= \frac{C}{A^{2n}} [A^{n-1} + A^{n-2}(|z_1|^2 + \cdots + |z_{n-1}|^2)] + \\
&\quad + \frac{F'2|z_0|^2|z_1|^2}{A^{2n}} \det \left( \begin{array}{cccc}
-1 & z_2 & \ldots & z_{n-1} \\
-\bar{z}_2 & A + |z_2|^2 & \ldots & z_{n-1}\bar{z}_2 \\
\vdots & \vdots & \ddots & \vdots \\
-\bar{z}_{n-1} & z_2\bar{z}_{n-1} & \ldots & A + |z_{n-1}|^2
\end{array} \right) + \cdots + \\
&\quad + (-1)^n \frac{F'^2|z_0|^2|z_{n-1}|^2}{A^{2n}} \det \left( \begin{array}{cccc}
-\bar{z}_1 & A + |z_1|^2 & \ldots & z_{n-2}\bar{z}_1 \\
-\bar{z}_2 & z_1\bar{z}_2 & \ldots & z_{n-2}\bar{z}_2 \\
\vdots & \vdots & \ddots & \vdots \\
-1 & z_1 & \ldots & z_{n-2}
\end{array} \right) = \\
&\quad + \frac{1}{A^{n+2}} [CA + (C - F'^2|z_0|^2)(|z_1|^2 + \cdots + |z_{n-1}|^2)].
\end{align*}
\]

By substituting (2) and (4) into this last equality one gets

\[
\det(h) = -\frac{F^2}{A^{n+1}} \left( \frac{xF'}{F} \right)' \bigg|_{x=|z_0|^2}.
\]

(7)

Hence, by (6) and (7), the matrix \(\frac{\partial^2 \Phi}{\partial z_0 \partial \bar{z}_\beta}\) is positive definite if and only if \(\left( \frac{xF'}{F} \right)' < 0\).

Before proving equivalence \((ii) \iff (iii)\) we briefly recall some facts on complex domains (see e.g. [16]). Let \(\Omega \subseteq \mathbb{C}^n\) be any complex domain of \(\mathbb{C}^n\) with smooth boundary \(\partial \Omega\), and let \(z \in \partial \Omega\). Assume that there exists a smooth function \(\rho : \mathbb{C}^n \to \mathbb{R}\) (called defining function for \(\Omega\) at \(z\)) satisfying the following: for some neighbourhood \(U\) of \(z\), \(\rho < 0\) on \(U \cap \Omega\), \(\rho > 0\) on \(U \setminus \Omega\) and \(\rho = 0\) on \(U \cap \partial \Omega\); \(\text{grad} \ \rho \neq 0\) on \(\partial \Omega\). In this case \(\partial \Omega\) is said to be strongly pseudoconvex at \(z\) if the Levi form

\[
L(\rho, z)(X) = \sum_{\alpha, \beta = 0}^{n-1} \frac{\partial^2 \rho}{\partial z_\alpha \partial \bar{z}_\beta}(z) X_\alpha \bar{X}_\beta
\]

is positive definite on

\[
S_\rho = \{(X_0, \ldots, X_{n-1}) \in \mathbb{C}^n \mid \sum_{\alpha=0}^{n-1} \frac{\partial \rho}{\partial z_\alpha}(z) X_\alpha = 0\}
\]
(it is easily seen that this definition does not depend on the particular defining function \( \rho \)).

\((ii) \iff (iii)\) Let now \( \Omega = D_F \) and let us fix \( z = (z_0, z_1, \ldots, z_{n-1}) \in \partial D_F \) with \(|z_0|^2 < x_0\). Then, \(|z_1|^2 + \cdots + |z_{n-1}|^2 = F(|z_0|^2)\). In this case

\[
\rho(z_0, z_1, \ldots, z_{n-1}) = |z_1|^2 + \cdots + |z_{n-1}|^2 - F(|z_0|^2)
\]

is a (globally) defining function for \( D_F \) at \( z \), the Levi form for \( D_F \) reads as

\[
L(\rho, z)(X) = |X_1|^2 + \cdots + |X_{n-1}|^2 - (F' + F''|z_0|^2)|X_0|^2 \quad (8)
\]

and

\[
S_{\rho} = \{(X_0, X_1, \ldots, X_{n-1}) \in \mathbb{C}^n \mid -F'\bar{z}_0X_0 + \bar{z}_1X_1 + \cdots + \bar{z}_{n-1}X_{n-1} = 0\}. \quad (9)
\]

We distinguish two cases: \( z_0 = 0 \) and \( z_0 \neq 0 \). At \( z_0 = 0 \) the Levi form reads as

\[
L(\rho, z)(X) = |X_1|^2 + \cdots + |X_{n-1}|^2 - F'(0)|X_0|^2
\]

which is strictly positive for any non-zero vector \( (X_0, X_1, \ldots, X_{n-1}) \) (not necessarily in \( S_{\rho} \)) because \( F \) is assumed to be decreasing.

If \( z_0 \neq 0 \) by (9) we obtain

\[
X_0 = \frac{\bar{z}_1X_1 + \cdots + \bar{z}_{n-1}X_{n-1}}{-\bar{z}_0} \quad \text{which, substituted in (8), gives:}
\]

\[
L(X, z) = |X_1|^2 + \cdots + |X_{n-1}|^2 - \frac{F' + F''|z_0|^2}{F'|z_0|^2} |\bar{z}_1X_1 + \cdots + \bar{z}_{n-1}X_{n-1}|^2. \quad (10)
\]

Therefore we are reduced to show that:

\((xF'/F')' < 0 \) for \( x \in (0, x_0) \) if and only if \( L(X, z) \) is strictly positive for every \( (X_1, \ldots, X_{n-1}) \neq (0, \ldots, 0) \) and every \( (z_0, z_1, \ldots, z_{n-1}) \in \partial D_F \), \( 0 < |z_0|^2 < x_0 \).

If \((xF'/F')' < 0 \) then

\[
(F' + xF'')F < xF'^2 \quad \text{and, since } F(|z_0|^2) = |z_1|^2 + \cdots + |z_{n-1}|^2,
\]

we get:

\[
L(X, z) > |X_1|^2 + \cdots + |X_{n-1}|^2 - \frac{1}{F'(|z_0|^2)} |\bar{z}_1X_1 + \cdots + \bar{z}_{n-1}X_{n-1}|^2 = \frac{(|X_1|^2 + \cdots + |X_{n-1}|^2)(|z_1|^2 + \cdots + |z_{n-1}|^2) - |\bar{z}_1X_1 + \cdots + \bar{z}_{n-1}X_{n-1}|^2}{|z_1|^2 + \cdots + |z_{n-1}|^2}
\]

and the conclusion follows by the Cauchy-Schwarz inequality.
Conversely, assume that $L(X, z)$ is strictly positive for every $(X_1, \ldots, X_{n-1}) \neq (0, \ldots, 0)$ and each $z = (z_0, z_1, \ldots, z_{n-1})$ such that $F(|z_0|^2) = |z_1|^2 + \cdots + |z_{n-1}|^2$. By inserting $(X_1, \ldots, X_{n-1}) = (z_1, \ldots, z_{n-1})$ in (10) we get

$$L(z, z) = F(|z_0|^2) \left( 1 - \frac{F' + F'' |z_0|^2}{F' |z_0|^2} F(|z_0|^2) \right) > 0$$

which implies $(xF'/F)' < 0$.

Finally, the proof of the equivalence (ii)$\iff$(iv) is completely analogous to that given in [11] (Proposition 3.4 and Proposition 3.6) for the 2-dimensional case, to which the reader is referred.

Remark 2.2 Notice that the previous proposition is a generalization of Proposition 3.6 in [11] proved there for the 2-dimensional case.

Recall (see e.g. [18]) that the Ricci curvature and the scalar curvature of a Kähler metric $g$ on an $n$-dimensional complex manifold $(M, g)$ are given respectively by

$$\text{Ric}_{\alpha\bar{\beta}} = -\frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta} (\log \det (h)), \quad \alpha, \beta = 0, \ldots, n-1$$

and

$$\text{scal}_g = \sum_{\alpha, \beta = 0}^{n-1} g^{\beta \bar{\alpha}} \text{Ric}_{\alpha \bar{\beta}},$$

where $g^{\beta \bar{\alpha}}$ are the entries of the inverse of $(g_{\alpha \bar{\beta}})$, namely $\sum_{\alpha = 0}^{n-1} g^{\beta \bar{\alpha}} g_{\alpha \bar{\gamma}} = \delta_{\beta \gamma}$.

When $(M, g) = (D_F, g_F)$, using (5) it is not hard to check the validity of the following equalities.

$$g^{00} = \frac{A}{B} F,$$

$$g^{\beta 0} = \frac{A}{B} F' z_\alpha \bar{z}_\beta, \quad \beta = 1, \ldots, n-1,$$

$$g^{\beta \bar{\alpha}} = \frac{A}{B} (F' + F'' |z_0|^2) z_\alpha \bar{z}_\beta, \quad \alpha \neq \beta, \ \alpha, \beta = 1, \ldots, n-1,$$

$$g^{\bar{\beta} \bar{\alpha}} = \frac{A}{B} [B + (F' + F'' |z_0|^2) |z_\beta|^2], \quad \beta = 1, \ldots, n-1,$$
where
\[
B = B(|z_0|^2) = F'^2|z_0|^2 - F'(F' + F'')|z_0|^2.
\]
Now, set
\[
L(x) = \frac{d}{dx}[\frac{d}{dx}\log(x F'^2 - F(F' + F'' x))].
\]
A straightforward computation using (7) and (11) gives:

\[
\begin{align*}
\text{Ric}_{\overline{0}0} &= -L(|z_0|^2) - (n + 1)g_{\overline{0}0}, \quad (17) \\
\text{Ric}_{\alpha\overline{\beta}} &= -(n + 1)g_{\alpha\overline{\beta}}, \quad \alpha > 0. \quad (18)
\end{align*}
\]

Then, by (12), the scalar curvature of the metric \(g_F\) equals

\[
\text{scal}_{g_F} = -L(|z_0|^2)g^{0\overline{0}} - (n + 1) \sum_{\alpha,\beta=0}^{n-1} g^{\beta\overline{\beta}} g_{\alpha\overline{\beta}} = -L(|z_0|^2)g^{0\overline{0}} - n(n + 1),
\]

which by (13) reads as

\[
\text{scal}_{g_F} = -\frac{A B}{F} - n(n + 1). \quad (19)
\]

We conclude this section with Lemma 2.3 below which will be used in the proof of our results. This lemma is a generalization of a result proved by the first author for 2-dimensional Hartogs domains (see Theorem 4.8 in [21]). We first recall the definition of generalized scalar curvatures. Given a Kähler metric \(g\) on an \(n\)-dimensional complex manifold \(M\), its generalized scalar curvatures are the \(n\) smooth functions \(\rho_0, \ldots, \rho_{n-1}\) on \(M\) satisfying the following equation:

\[
\frac{\det(g_{\alpha\overline{\beta}} + t\text{Ric}_{\alpha\overline{\beta}})}{\det(g_{\alpha\overline{\beta}})} = 1 + \sum_{k=0}^{n-1} \rho_k t^{k+1}, \quad (20)
\]

where \(g_{\alpha\overline{\beta}}\) are the entries of the metric in local coordinates. Observe that for \(k = 0\) we recover the value of the scalar curvature, namely

\[
\rho_0 = \text{scal}_g.
\]

The introduction and the study of these curvatures (in the compact case) are due to K. Ogiue [23] to whom the reader is referred for further results.
In particular, in a joint paper with B.Y. Chen [1], he studies the constancy of one of the generalized scalar curvatures. Their main result is that, under suitable cohomological conditions, the constancy of one of the $\rho_k$'s, $k = 0, \ldots, n - 1$, implies that the metric $g$ is Einstein.

**Lemma 2.3** Let $(D_F, g_F)$ be an $n$-dimensional Hartogs domain. Assume that one of its generalized scalar curvatures is constant. Then $(D_F, g_F)$ is holomorphically isometric to an open subset of the $n$-dimensional hyperbolic space.

**Proof:** By (17), (18) we get

$$\frac{\det(g_{\alpha\beta} + tRic_{\alpha\beta})}{\det(g_{\alpha\beta})} = (1 - (n + 1)t)^n - tL(1 - (n + 1)t)^{n-1} \frac{AF}{B}.$$ 

So the generalized curvatures of $(D_F, g_F)$ are given by

$$\rho_k = (n+1)^k (-1)^{k+1} \binom{n-1}{k} \left[ \frac{n(n+1)}{k+1} + \frac{AF}{B} \right], \quad k = 0, \ldots, n-1. \quad (22)$$

Notice that, for $k = 0$, we get $\rho_0 = -\frac{AF}{B} - n(n+1) = \text{scal}_{g_F}$, (compare with (19)) in accordance with (21).

Thus, $\rho_k$ is constant for some (equivalently, for any) $k = 0, \ldots, n - 1$ if and only if $\frac{AF}{B}$ is constant. Since $A = F(|z_0|^2) - |z_1|^2 - \cdots - |z_{n-1}|^2$ depends on $z_1, \ldots, z_{n-1}$ while $\frac{LF}{B}$ depends only on $z_0$, this implies that $L = 0$, i.e.

$$\frac{d}{dx} \left[ x \frac{d}{dx} \log(xF'^2 - F(F' + F''x)) \right]_{x = |z_0|^2} \equiv 0.$$

Now, we continue as in the proof of Theorem 4.8 in [21] and conclude that $F(x) = c_1 - c_2 x$, $x = |z_0|^2$, with $c_1, c_2 > 0$, which implies that $D_F$ is holomorphically isometric to an open subset of the complex hyperbolic space $\mathbb{CH}^n$ via the map

$$\phi : D_F \to \mathbb{CH}^n, \quad (z_0, z_1, \ldots, z_{n-1}) \mapsto \left( \frac{z_0}{\sqrt{c_1/c_2}}, \frac{z_1}{\sqrt{c_1}}, \ldots, \frac{z_{n-1}}{\sqrt{c_1}} \right).$$

$\square$
3 Proof of Theorem 1.1

Extremal metrics were introduced and christened by Calabi [4] in the compact case as the solution for the variational problem in a Kähler class defined by the square integral of the scalar curvature. Therefore they are a generalization of constant scalar curvature metrics. Calabi himself constructs nontrivial (namely with nonconstant scalar curvature) metrics on some compact manifolds. Only recently extremal Kähler metrics were rediscovered by several mathematicians due to their link with the stability of complex vector bundles (see e.g. [3], [8], [14], [19] and [22]). Obviously extremal metrics cannot be defined in the noncompact case as the solutions of a variational problem involving some integral on the manifold. Nevertheless they can be alternatively defined (also in the noncompact case) as those metrics such that the (1,0)-part of the Hamiltonian vector field associated to the scalar curvature is holomorphic. Therefore, in local coordinates an extremal metric must satisfy the following system of PDE’s (see [4]):

\[
\frac{\partial}{\partial \bar{z}_\gamma} \left( \sum_{\beta=0}^{n-1} g^{\bar{\beta} \alpha} \frac{\partial \text{scal}_g}{\partial z_\beta} \right) = 0,
\]

for every \( \alpha, \gamma = 0, \ldots, n-1 \). Notice that in the noncompact case, the existence and uniqueness of such metrics are far from being understood. For example, only recently in [6] (see also [7]), it has been shown the existence of a nontrivial extremal and complete Kähler metric in a complex one-dimensional manifold.

Proof of Theorem 1.1 In order to use equations (23) for \((D_F, g_F)\) we write the entries \(g^{\bar{\beta} \alpha}\) by separating the terms depending only on \(z_0\) from the other terms. More precisely, (13), (14), (15) and (16) can be written as follows.

\[
g^{00} = P_{00} + Q_{00}(|z_1|^2 + \cdots + |z_{n-1}|^2),
\]

\[
g^{0\alpha} = \bar{z}_0 z_\alpha [P_{0\alpha} + Q_{0\alpha}(|z_1|^2 + \cdots + |z_{n-1}|^2)], \quad \alpha = 1, \ldots, n-1,
\]

\[
g^{\alpha \bar{\alpha}} = F + P_{\alpha \bar{\alpha}} |z_\alpha|^2 - (1 + Q_{\alpha \bar{\alpha}} |z_\alpha|^2) \sum_{k \neq \alpha} |z_k|^2 - R_{\alpha \bar{\alpha}} |z_\alpha|^4, \quad \alpha = 1, \ldots, n-1,
\]
\[ g^{\bar{\alpha}\alpha} = \bar{z}_\beta z_\alpha [P_{ab} + Q_{ab}(|z_1|^2 + \cdots + |z_{n-1}|^2)], \quad \alpha \neq \beta, \quad \alpha, \beta = 1, \ldots, n-1, \]

where

\[
\begin{align*}
    P_{00} &= \frac{F^2}{B}, \quad Q_{00} = -\frac{F}{B}, \\
    P_{0a} &= \frac{F'F}{B}, \quad Q_{0a} = -\frac{F'}{B}, \\
    P_{aa} &= \frac{F(F' + F''|z_0|^2)}{B} - 1, \quad Q_{aa} = R_{aa} = \frac{F' + F''|z_0|^2}{B}, \\
    P_{ab} &= \frac{F(F' + F''|z_0|^2)}{B}, \quad Q_{ab} = -\frac{F' + F''|z_0|^2}{B}
\end{align*}
\]

are all functions depending only on \(|z_0|^2|\).

We also have (cfr. (19))

\[
\text{scal}_{g_F} = -n(n+1) + G(F - |z_1|^2 - \cdots - |z_{n-1}|^2) \tag{24}
\]

where

\[
G = G(|z_0|^2) = -\frac{L(|z_0|^2)F(|z_0|^2)}{B(|z_0|^2)}.
\]

Assume that \(g_F\) is an extremal metric, namely equation (23) is satisfied.

We are going to show that \(\text{scal}_{g_F}\) is constant and hence by Lemma 2.3 \((D_F, g_F)\) is holomorphically isometric to an open subset of \((CH^n, g_{hyp})\). In order to do that, fix \(i \geq 1\) and let us consider equation (23) when \(g = g_F\) for \(\alpha = 0, \gamma = i\).

We have

\[
\frac{\partial \text{scal}_{g_F}}{\partial \bar{z}_0} = G' z_0 (F - |z_1|^2 - \cdots - |z_{n-1}|^2) + z_0 GF'
\]

\[
\frac{\partial \text{scal}_{g_F}}{\partial z_i} = -G z_i.
\]

So, equation (23) gives
\[ \frac{\partial}{\partial \bar{z}_i} \left\{ \left[ P_{00} + Q_{00} \sum_{k=1}^{n-1} |z_k|^2 \right] \left[ G' \bar{z}_0 (F - \sum_{k=1}^{n-1} |z_k|^2) + z_0 GF' \right] - 
\]

\[ -z_0 G \left[ P_{0a} + Q_{0a} \sum_{k=1}^{n-1} |z_k|^2 \right] \sum_{k=1}^{n-1} |z_k|^2 \right) = 0, \]

namely

\[ Q_{00} \bar{z}_i \left[ G' \bar{z}_0 (F - \sum_{k=1}^{n-1} |z_k|^2) + z_0 GF' \right] - G' \bar{z}_i \left[ P_{00} + Q_{00} \sum_{k=1}^{n-1} |z_k|^2 \right] - 
\]

\[ -z_0 G Q_{0a} \bar{z}_i \sum_{k=1}^{n-1} |z_k|^2 - z_0 \bar{z}_i G \left[ P_{0a} + Q_{0a} \sum_{k=1}^{n-1} |z_k|^2 \right] = 0. \]

Deriving again with respect to \( \bar{z}_i \), we get

\[ -2Q_{00} G' \bar{z}_0 z_i^2 - 2G Q_{0a} z_0 z_i^2 = 0. \]

Let us assume \( z_0 \bar{z}_i \neq 0 \). This implies \( Q_{00} G' + G Q_{0a} = 0 \), i.e. \( G F' + F G' = 0 \) or, equivalently, \( G = \frac{c}{F} \) for some constant \( c \in \mathbb{R} \). The proof of Theorem 1.1 will be completed by showing that \( c = 0 \). In fact, in this case \( G = 0 \) on the open and dense subset of \( D_F \) consisting of those points such that \( z_0 \bar{z}_i \neq 0 \) and therefore, by (24), \( \text{scal}_{g_F} \) is constant on \( D_F \). In order to prove that \( c = 0 \), let us now consider equation (23) for \( \alpha = i, \gamma = i \).
This implies

\[-G'\left|z_0\right|^2 z_i^2 \left(\frac{P_{0a} + Q_{0a}}{z_i} \sum_{k=1}^{n-1} |z_k|^2\right) + \bar{z}_0 z_i^2 Q_{0a} \left[G' z_0 (F - \sum_{k=1}^{n-1} |z_k|^2) + GF' z_0\right] -
\]

\[-P_{aa} G z_i^2 + G z_i^2 Q_{aa} \sum_{k \neq 0,i} |z_k|^2 + 2G z_i^2 \bar{z}_i R_{aa} - G z_i^2 Q_{ab} \sum_{k \neq 0,i} |z_k|^2.\]

If we divide by \(z_i^2\) (we are assuming \(z_i \neq 0\)) and derive again the above expression with respect to \(\bar{z}_i\) we get

\[-G'\left|z_0\right|^2 Q_{0a} + GR_{aa} = 0.\]

By the definitions made at page 11 this is equivalent to

\[\frac{G' F'\left|z_0\right|^2 + G(F' + F''|z_0|^2)}{B} = 0,\]

i.e. \((G F' x)' = 0, x = \left|z_0\right|^2\). Substituting \(G = \frac{c}{P}\) in this equality we get \(c(F' x) = 0\). Since \((F' x) < 0\) (by (ii) in Proposition 2.1) \(c\) is forced to be zero, and this concludes the proof. \(\square\)

### 4 Proof of Theorem 1.2

A Kähler–Ricci soliton on a complex manifold \(M\) is a pair \((g, X)\) consisting of a Kähler metric \(g\) and a real holomorphic vector field \(X\) on \(M\) such that

\[\text{Ric}_g = \lambda g + L_X g, \tag{25}\]

for some \(\lambda \in \mathbb{R}\), where \(L_X g\) is the Lie derivative of \(g\) along \(X\), i.e.

\[(L_X g)(Y, Z) = X(g(Y, Z)) - g([X, Y], Z) - g(Y, [X, Z]), \tag{26}\]

for \(Y, Z\) vector fields on \(M\). A real holomorphic vector field \(X\) is the real part of a holomorphic vector field, namely, in local complex coordinates \((z_0, \ldots, z_{n-1})\) on an open subset \(U \subset M\),

\[X = \sum_{k=0}^{n-1} \left(f_k \frac{\partial}{\partial z_k} + \bar{f}_k \frac{\partial}{\partial \bar{z}_k}\right) \tag{27},\]
for some holomorphic functions \( f_k, k = 0, \ldots, n - 1 \) on \( U \).

We refer the reader to [5, 25, 26] for the existence and uniqueness of Kähler–Ricci solitons on compact manifolds and to [12] for the noncompact case. Kähler–Ricci solitons generalize Kähler–Einstein metrics. Indeed any Kähler–Einstein metric \( g \) on a complex manifold \( M \) gives rise to a trivial Kähler–Ricci soliton by choosing \( X = 0 \) or \( X \) Killing with respect to \( g \). Obviously if the automorphism group of \( M \) is discrete then a Kähler–Ricci soliton \((g, X)\) is nothing but a Kähler–Einstein metric \( g \).

Our Theorem 1.2 asserts that a Kähler–Ricci soliton \((g_F, X)\) on a pseudoconvex Hartogs domain \( D_F \) is necessarily trivial. Notice that the automorphism group of \( D_F \) is not discrete (see also [17]).

**Proof of Theorem 1.2** Let \((g_F, X)\) be a Kähler–Ricci soliton. By applying both sides of (25) to the pair \((\frac{\partial}{\partial z_0}, \frac{\partial}{\partial \bar{z}_0})\) and taking into account (17) one gets:

\[
-L(|z_0|^2) = \gamma g_{00} + \sum_{k=0}^{n-1} \left( f_k \frac{\partial g_{0\bar{0}}}{\partial z_k} + \bar{f}_k \frac{\partial g_{0\bar{0}}}{\partial \bar{z}_k} \right) + \sum_{k=0}^{n-1} \left( \frac{\partial f_k}{\partial z_0} g_{k\bar{0}} + \frac{\partial \bar{f}_k}{\partial \bar{z}_0} g_{0k} \right) 
\]

(28)

where

\[
\gamma = \lambda + (n + 1).
\]

(29)

By (5), we have

\[
\tilde{C} = \sum_{k=0}^{n-1} C_k (f_k \bar{z}_k + \bar{f}_k z_k) + C(\phi_0 + \bar{\phi}_0) - F' \sum_{k=1}^{n-1} \left( z_0 \frac{\partial f_k}{\partial z_0} + \bar{z}_0 \frac{\partial \bar{f}_k}{\partial \bar{z}_0} \right)
\]

(30)

where we have set \( \tilde{C} = -A^2 L - \gamma C \), \( C_k = \frac{A^2 \partial \phi_0}{\partial x_k} \) \((x_k = |z_k|^2)\) and \( \phi_0 = \frac{\partial f_0}{\partial z_0} \)
(A and \( C \) are given by (2) and (4) respectively).

Now, by applying the operator \( \frac{\partial^4}{\partial z_i \partial \bar{z}_i \partial z_i \partial \bar{z}_i} \) \((i = 1, \ldots, n - 1)\) to both sides of this equation we get

\[
-4L = \sum_{k=0}^{n-1} \frac{\partial^4 C_k}{\partial z_i \partial \bar{z}_i \partial z_i \partial \bar{z}_i} (f_k \bar{z}_k + \bar{f}_k z_k) + 2 \sum_{k=0}^{n-1} \frac{\partial^3 C_k}{\partial z_i \partial \bar{z}_i \partial z_i} \left( f_k \delta_{ik} + \frac{\partial \bar{f}_k}{\partial \bar{z}_i} \right) + \\
+ 2 \sum_{k=0}^{n-1} \frac{\partial^3 C_k}{\partial z_i \partial \bar{z}_i \partial z_i} \left( f_k \delta_{ik} + \frac{\partial f_k}{\partial z_i} \right) + 4 \sum_{k=0}^{n-1} \frac{\partial^2 C_k}{\partial z_i \partial \bar{z}_i} \left( \frac{\partial f_k}{\partial z_i} \delta_{ik} + \frac{\partial \bar{f}_k}{\partial \bar{z}_i} \delta_{ik} \right)
\]

(31)
\[
+ \sum_{k=0}^{n-1} \frac{\partial^2 C_k}{\partial z_i^2} \frac{\partial^2 \bar{f}_k}{\partial \bar{z}_i^2} z_k + \sum_{k=0}^{n-1} \frac{\partial^2 C_k}{\partial z_i^2} \frac{\partial^2 \bar{f}_k}{\partial \bar{z}_i^2} \bar{z}_k + 2 \sum_{k=0}^{n-1} \frac{\partial C_k}{\partial z_i} \frac{\partial^2 \bar{f}_k}{\partial \bar{z}_i^2} \delta_{ik} + \\
+ 2 \sum_{k=0}^{n-1} \frac{\partial C_k}{\partial z_i} \frac{\partial^2 \bar{f}_k}{\partial \bar{z}_i^2} \delta_{ik} + \frac{\partial^4 C}{\partial z_i^2 \partial \bar{z}_i^2} (\phi_0 + \bar{\phi}_0) + 2 \frac{\partial^4 C}{\partial z_i^2 \partial \bar{z}_i} \frac{\partial \bar{\phi}_0}{\partial \bar{z}_i} + \\
+ 2 \frac{\partial^3 C}{\partial z_i \partial \bar{z}_i^2} \frac{\partial \phi_0}{\partial z_i} + \frac{\partial^2 C}{\partial z_i^2} \frac{\partial \bar{\phi}_0}{\partial z_i} + \frac{\partial^2 C}{\partial \bar{z}_i^2} \frac{\partial \bar{\phi}_0}{\partial \bar{z}_i}.
\]

Since \(C\) and \(C_k\) are rotation invariant, by evaluating the previous expression at \(z_1 = \cdots = z_{n-1} = 0\) and taking into account that

\[
\frac{\partial^4 C_0}{\partial z_i^2 \partial \bar{z}_i^2} \big|_{\{z_1 = \cdots = z_{n-1} = 0\}} = -8x \frac{F'^3}{F^3},
\]

\[
\frac{\partial^2 C_i}{\partial z_i \partial \bar{z}_i} \big|_{\{z_1 = \cdots = z_{n-1} = 0\}} = 2x \frac{F'^2}{F^2},
\]

\[
\frac{\partial^4 C}{\partial z_i^2 \partial \bar{z}_i^2} \big|_{\{z_1 = \cdots = z_{n-1} = 0\}} = 0,
\]

we have

\[
L = 2x \frac{F'^3}{F^3} (f_0 \bar{z}_0 + \bar{f}_0 \bar{z}_0) - 2x \frac{F'^2}{F^2} (\phi_i + \bar{\phi}_i),
\]

(31)

where \(\phi_i = \frac{\partial f_i}{\partial z_i}\).

Now, let \(i = 1, \ldots, n-1\). By applying both sides of (25) to the pair \(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}\) one gets

\[
-\gamma \ g_{ii} = \sum_{k=0}^{n-1} \left( f_k \frac{\partial g_{ii}}{\partial z_k} + f_k \frac{\partial g_{ii}}{\partial \bar{z}_k} \right) + \sum_{k=0}^{n-1} \left( \frac{\partial f_k}{\partial z_j} g_{ki} + \frac{\partial f_k}{\partial \bar{z}_j} g_{ki} \right)
\]

(32)

where \(\gamma\) is given by (29). By (5) and (18) this means

\[
-\gamma \ A + |z_i|^2 = \sum_{k=1}^{n-1} A^{-3}[2|z_i|^2 + A(1 + \delta_{ik})](f_k \bar{z}_k + \bar{f}_k z_k) - \\
- A^{-3} F'(2|z_i|^2 + A)(f_0 \bar{z}_0 + \bar{f}_0 z_0) - \frac{F'}{A^2} \left( \frac{\partial f_0}{\partial \bar{z}_i} z_0 \bar{z}_i + \frac{\partial f_0}{\partial z_i} \bar{z}_0 z_i \right) + \\
+ \frac{1}{A^2} \sum_{k=1}^{n-1} \left[ \frac{\partial f_k}{\partial z_i} (\bar{z}_k z_i + \delta_{ki} A) + \frac{\partial f_k}{\partial \bar{z}_i} (z_k \bar{z}_i + \delta_{ki} A) \right].
\]

(33)
If we evaluate both sides of this equation at \( z_1 = \cdots = z_{n-1} = 0 \) we get

\[-\gamma F = -F'(f_0\bar{z}_0 + \bar{f}_0z_0) + F(\phi_i + \bar{\phi}_i). \tag{34}\]

Moreover, by multiplying equation (33) by \( A^2 \), by applying the operator \( \frac{\partial^2}{\partial z_i \partial \bar{z}_i} \) to both sides and evaluating at \( z_1 = \cdots = z_{n-1} = 0 \) one gets

\[0 = -\frac{F'}{F}(f_0\bar{z}_0 + \bar{f}_0z_0) + (\phi_i + \bar{\phi}_i). \tag{35}\]

Finally, by comparing (31) with (35), one gets \( L = 0 \) and hence, by the proof of Lemma 2.3, \((D_F, g_F)\) is holomorphically isometric to an open subset of \((\mathbb{C}H^n, g_{hyp})\) and we are done. (Notice that equations (34) and (35) yield \( \gamma = 0 \) and by (25) with \( g_F = g_{hyp} \) one gets that \( X \) is a Killing vector field with respect to \( g_{hyp} \), as expected). \( \square \)

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