A Hybrid Quantum Secret Sharing Scheme based on Mutually Unbiased Bases

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Abstract. With the advantages of both classical and quantum secret sharing, many practical hybrid quantum secret sharing have been proposed. In this paper, we propose a hybrid quantum secret sharing scheme based on mutually unbiased bases and monotone span program. First, the dealer sends the shares in the linear secret sharing to the participants in the authorization set via a secure channel. Then, the dealer and participants perform unitary transformation on a $d$-dimensional quantum state sequentially, and the dealer publishes the measurement result confidentially to the participants in the authorization set to recover the secret. The verifiability of the scheme is guaranteed by the Hash function. Next, the correctness and security of the scheme are proved and our scheme is secure against the general eavesdropper attacks. Finally, a specific example is employed to further clarify the flexibility of the scheme and the detailed comparison of similar quantum secret sharing schemes also shows the superiority of our proposed scheme.

Keywords: Quantum secret sharing · Mutually unbiased bases · Verifiability · Access structure.

1 Introduction

As a combination of cryptography and quantum mechanics, quantum cryptography plays an important role in cryptography. Compared with classical cryptography on the basis of computational complexity, quantum cryptography based on the laws of quantum physics can achieve unconditional security. Many branches of quantum cryptography have been developed, such as quantum key distribution (QKD)[1,2], quantum key agreement (QKA)[3-5], quantum secure direct communication (QSDC)[6,7], quantum teleportation[8,9], quantum signature[10,11], quantum authentication[12-14], quantum secret sharing (QSS)[15-30] and so on.

Quantum secret sharing (QSS) is an important research field in quantum cryptography, which means that the dealer divides a secret into several shadows and sends them to multiple participants. Only the participants in authorized sets can recover the secret, and the participants in unauthorized sets can not recover the secret. Since Hillery et al. [15] proposed the first quantum secret

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sharing scheme by using GHZ state in 1999, a growing number of QSS schemes [16-30] have been proposed. For example, Williams et al. [22] described and experimentally demonstrated a three-party quantum secret sharing protocol using polarization-entangled photon pairs. Tsai et al. [23] used the entanglement property of W-state to propose the first three-party SQSS protocol. Song et al. [24] demonstrated a \((t, n)\) threshold \(d\)-level quantum secret sharing scheme. A verifiable \((t, n)\) threshold quantum secret sharing scheme was proposed using the \(d\)-dimensional Bell state and the Lagrange interpolation by Yang et al. in Ref. [25]. Hao et al. [26] put forward a secret sharing scheme using the mutually unbiased bases on the \(p^2\)-dimensional quantum system. Bai et al. [27] proposed the concept of decomposition of quantum access structure to design a quantum secret sharing scheme. In Ref. [28], Liu et al. study the local distinguishability of the 15 kinds of seven-qudit quantum entangled states and then proposed a \((k, n)\) threshold quantum secret sharing scheme. A new improving quantum secret sharing scheme was proposed by Xu et al. [29], in which more quantum access structures can be realized by the scheme than the one proposed by Nascimento et al. [30].

Although many schemes have been proposed, the verifiability and the flexibility of the schemes are also important issues worth of consideration. In this paper, we propose a hybrid and verifiable quantum secret sharing scheme based on mutually unbiased bases and the monotone span program, which focuses on transmitting a \(d\)-dimensional quantum state among the dealer Alice and participants and the application of the linear secret sharing. Each participant in a authorization set can perform a unitary transformation on the received particle and send it to the next one until the last one sends it to Alice. They can recover the secret by the linear secret sharing and the measurement value sent by Alice. Verifiability ensures that the secret recovered in each authorization set is the original one, and it also ensures that once a dishonest participant appears, he will be found. Compared with the threshold scheme, the quantum secret sharing scheme based on the access structure realizes the different influences of participants in the process of recovering secrets, thereby achieving the flexibility of the scheme.

By comparison, our scheme shows all the advantages of the previous QSS and the unique advantages, such as,

1. It uses a qudit state instead of a qubit state.
2. The participants can check the authenticity of the recovered secret.
3. It needs fewer quantum resources and quantum operations.
4. It has general access structure.
5. It reduces the communication costs and computation complexity.

This paper is organized as follows. In section 2, we illustrate the preliminary knowledge related to the proposed scheme. The new proposed scheme is introduced in section 3. Section 4 give a proof of the correctness, verifiability and security of the proposed scheme. In section 5, we give an example to further illustrate our proposed scheme. Finally, the comparison and conclusion is given in section 6 and section 7.
2 Preliminaries

In this section, we introduce the preliminary knowledge of our scheme.

2.1 Access structure

Definition 1 Let \( \mathcal{P} = \{P_1, P_2, \ldots, P_n\} \) be a set of participants, an access structure \( \Gamma \subseteq 2^\mathcal{P} \) is a family of authorized sets of participants.

Definition 2 If \( \Gamma \) is the access structure on \( \mathcal{P} \), then any set in \( \Gamma \) is called the authorization subset on \( \mathcal{P} \), which is called the authorization set for short. If \( A \in \Gamma, A \subseteq B \subseteq \mathcal{P} \), then \( B \in \Gamma \). The family of the unauthorized sets is called an adversary structure, that is to say, \( \Gamma^c := \Delta \).

Example 1 Let \( \mathcal{P} = \{P_1, P_2, P_3, P_4\}, \Gamma = \{A_1, A_2, A_3\} \), where \( A_1 = \{P_1, P_2, P_3\}, A_2 = \{P_1, P_2, P_4\}, A_3 = \{P_1, P_2, P_3, P_4\} \). So

\[
\Delta = \begin{cases}
\{0, \{P_1\}, \{P_2\}, \{P_3\}, \{P_1, P_2\}, \{P_1, P_3\}, \{P_1, P_4\}, \{P_2, P_3\}, \{P_2, P_4\}, \{P_3, P_4\}, \{P_1, P_2, P_3\}\}, \\
\{\{P_2, P_3\}, \{P_2, P_4\}, \{P_3, P_4\}, \{P_1, P_3, P_4\}\}.
\end{cases}
\]

2.2 Monotone span program

MSP was introduced in Ref. [31] by Karchmer and Wigderson as a model of computation to design the linear secret sharing scheme.

Definition 3 \( \mathcal{M}(\mathcal{F}, M, \psi, \xi) \) is a monotone span program(MSP), where \( M \) is a \( k \times l \) matrix over a finite field \( \mathcal{F} \), \( \psi : \{1, 2, \ldots, k\} \to \mathcal{P} \) is a surjective labeling map, \( \xi = (1, 0, \ldots, 0)^T \in \mathcal{F}^l \) is defined as the target vector. For any \( A \subseteq \mathcal{P} = \{P_1, P_2, \ldots, P_n\} \), there is a corresponding eigenvector \( \delta_A = \langle \delta_1, \delta_2, \ldots, \delta_n \rangle \in \{0, 1\}^n \) if and only if \( P_i \in A, \delta_i = 1 \). The Boolean function \( f : \{0, 1\}^n \to \{0, 1\}, f(\delta_A) = 1 \) represents the corresponding \( \varepsilon \) rows of \( M \), where \( \psi(\varepsilon) \in A, \varepsilon \in \{1, 2, \ldots, k\} \).

Definition 4 A monotone span program (MSP) is called a MSP for access structure \( \Gamma \), if it can be satisfied that \( \forall A \in \Gamma, \exists \lambda_A \in \mathcal{F}^k \Rightarrow M^T_A \lambda_A = \xi \), and \( \forall A \in \Delta, \exists h = (1, h_2, \ldots, h_4) \in \mathcal{F}^l \Rightarrow M_A h = 0 \in \mathcal{F}^m \).

Example 2 \( \mathcal{M}(\mathcal{F}, M, \psi, \xi) \) is an MSP of access structure \( \Gamma \) as shown in example 1, where \( \mathcal{F} = \mathbb{Z}_5, \psi(i) = Bob_i, i \in \{1, 2, 3, 4\}, \xi = (1, 0, 0, 0)^T, M = \begin{pmatrix} 1 & 0 & 3 & 4 \\ 0 & 0 & 2 & 1 \\ 3 & 4 & 1 & 0 \\ 1 & 2 & 4 & 0 \end{pmatrix} \).

Therefore, \( \lambda_{A_1} = (1, 1, 0)^T, \lambda_{A_2} = (1, 1, 0)^T, \lambda_{A_3} = (1, 1, 3, 4)^T \).

2.3 Linear secret sharing

Monotone span program is utilized to design the linear secret sharing scheme, which is aimed that the dealer Alice shares a secret \( s \) among \( k \) shareholders \( Bob_1, Bob_2, \ldots, Bob_k \) according to the MSP for access structure \( \Gamma \). It includes the following two phases as follows.
Distribution phase
Alice prepares a random vector $\rho = (s, \rho_2, \ldots, \rho_l)^T \in \mathbb{C}^l$ and computes $s = M\rho = (s_1, \ldots, s_k)^T$. Then, she sends $s_i$ to $\psi(i)$ via a secure channel.

Reconstruction phase
Let $s_A$ be indicated the vector for the authorized set $A$. The participants in $A$ restore the secrets cooperatively as follows.

$$s^T_A\lambda_A = (M_A\rho)^T\lambda_A = \rho^T (M_A^T\lambda_A) = \rho^T\xi = s. \quad (1)$$

2.4 Necessary quantum properties

**Definition 5** Mutually unbiased base is defined that two sets of standard orthogonal bases $A_1 = \{|\varphi_1\rangle, |\varphi_2\rangle, \ldots, |\varphi_d\rangle\}$ and $A_2 = \{|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_d\rangle\}$, which defined over a $d$-dimensional complex space $C^d$ in Ref.[32,33], if the following relationship is satisfied

$$|\langle \varphi_i | \psi_j \rangle| = \frac{1}{\sqrt{d}}. \quad (2)$$

If any two of the set of standard orthogonal bases $\{A_1, A_2, \ldots, A_m\}$ in space are unbiased, then this set is called an unbiased bases set. Besides, it can be found $d + 1$ mutually unbiased bases if $d$ is an odd prime number.

**Definition 6** The computation base is expressed as $\{|k\rangle | k \in D \}$, and the remaining groups can be expressed as:

$$|v_{l}^{(j)}\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} w^{k(l+jk)} |k\rangle, \quad (3)$$

where $|v_{l}^{(j)}\rangle$ represents the $l$-th vector in the $j$-th bases, $w = e^{2\pi i/l}, j \in D, D = \{0, 1, \cdots, d - 1\}$. These mutually unbiased bases satisfy the following conditions:

$$|\langle v_{l}^{(j)} | v_{l}^{(j')} \rangle| = \frac{1}{\sqrt{d}}, j \neq j'. \quad (4)$$

**Definition 7** In Ref.[34], the two unitary transformations $X_d$ and $Y_d$ that we need to use in this paper can be expressed as:

$$X_d = \sum_{m=0}^{d-1} w^m |m\rangle \langle m|, Y_d = \sum_{m=0}^{d-1} w^{m^2} |m\rangle \langle m|. \quad (5)$$

Implementing (5) on $|v_{l}^{(j)}\rangle$ in turn, we can obtain:

$$X_d^x Y_d^y |v_{l}^{(j)}\rangle = |v_{l+x}^{(j+y)}\rangle. \quad (6)$$

For the convenience of expression, $X_d^x Y_d^y$ is denoted as $U_{x,y}$, that is,

$$U_{x,y} |v_{l}^{(j)}\rangle = |v_{l+x}^{(j+y)}\rangle. \quad (7)$$
3 Proposed scheme

In this section, we construct a verifiable quantum secret sharing scheme that includes a dealer Alice and \( n \) shareholders \( Bob_1, Bob_2, \cdots, Bob_n \). The access structure \( \Gamma \) can be expressed as \( \Gamma = \{ A_1, A_2, \cdots, A_r \} \), where \( A_i (i = 1, 2, \cdots, r) \) is a authorization set. For the convenience of description, the authorization set is recorded as \( A_i = \{ Bob_1^{(i)}, Bob_2^{(i)}, \cdots, Bob_m^{(i)} \} \), \( (1 \leq m \leq n) \). Without losing generality, it is assumed that the participants in the authorization set \( A_i = \{ Bob_1^{(i)}, Bob_2^{(i)}, \cdots, Bob_m^{(i)} \} \) want to recover the secret \( s \). The specific steps of the scheme are as follows.

3.1 Distribution phase

Alice implements the following steps.

3.1.1 Select a random vector \( \rho = (S_1, \rho_2, \rho_3, \cdots, \rho_l)^T \) according to authorization set \( A_i \).

3.1.2 Calculate \( s = M_n \times \rho = (s_1, s_2, \cdots, s_n)^T \), \( i = 1, 2, \cdots, r \) and send \( s_i \) to \( \psi(j) = Bob_j (j = 1, 2, \cdots, n) \) through the quantum secure channel.

3.1.3 Compute and publish \( H_1 = h(S_i), H_2 = h(s) \), where \( h() \) is a public Hash function.

3.1.4 Prepare a quantum state \( |\phi \rangle = |\phi_0^0 \rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j \rangle \) and perform a unitary operation \( U_0^{(i)} \) to get the quantum state \( |\phi_0^{(i)} \rangle = U_0^{(i)} |\phi_0^0 \rangle = 1_{\psi_0}^{(i)} \), where \( \phi_0^{(i)} = s \) is the secret, \( q_0^{(i)} \) is a secret value known only to Alice. Then, she sends the quantum state \( |\phi_0^{(i)} \rangle \) performed by the unitary operation to the first participant \( Bob_1^{(i)} \) in the authorization set \( A_i \).

3.2 Reconstruction phase

Participants in \( A_i = \{ Bob_1^{(i)}, Bob_2^{(i)}, \cdots, Bob_m^{(i)} \} \), \( (1 \leq m \leq n) \) can recover the secret by the following steps.

3.2.1 After receiving the quantum state \( |\phi_0 \rangle \), the first participant \( Bob_1^{(i)} \) performs unitary operation \( U_{p_1}^{(i)} \) on it and gets the quantum state \( |\phi_1^{(i)} \rangle = U_{p_1}^{(i)} |\phi_0 \rangle = 1_{\psi_1}^{(i)} \).

3.2.2 The other participants \( Bob_j^{(i)} (j = 2, 3, \cdots, m) \) in the authorization set \( A_i \) perform the same operation as in step 3.2.1, which means that after receiving
In order to reconstruct the secret, each participant in authorization set $A_i$ when all the participants act and transmit, the final quantum state is

$$|\phi\rangle_m = \prod_{k=0}^{m} U_{p_k(i), a_k(i)} |\varphi_0^i\rangle = |\varphi_{m}^i\rangle.$$  (8)

3.2.3 When Alice receives the final quantum state $|\phi\rangle_m$, she can know that it satisfies the following condition on account of $q_0(i), q_1(i), \ldots, q_m(i)$,

$$q_0(i) + q_1(i) + \cdots + q_m(i) = q_i.$$  (9)

She selects the measurement bases $M_{q_i} = \{ |\varphi_j(q_i)\rangle \mid j \in D \}$ to measure it, and then infers the following condition should be established in the authorization set $A_i$

$$p_0(i) + p_1(i) + \cdots + p_m(i) = p_0(i) + S_i = r_i.$$  (10)

If it is established, Alice checks whether $H_1$ of the participants are equal to the published one. If so, the measurement results $r_i$ will be sent to all participants in the authorization set $A_i$ through the secure channel and then it move to the next step. If not, the scheme is terminated.

3.2.4 In order to reconstruct the secret, each participant in authorization set $A_i$ can recover the secret by calculating $s = p_0 = r_i - \sum_{i=1}^{m} p_i = r_i - S_i$.

4 Correctness, verifiability and security

In this section, the provability of the correctness, verifiability and security of our scheme is given.

4.1 Correctness

**Theorem 1** If a $d$-dimensional quantum state in mutually unbiased bases is

$$|v_l^j\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} u^{k(l+jk)} |k\rangle,$$

and a unitary operation $U_{x,y} = X_d^x Y_d^y$ is performed
on it, then it will become another state $|v^{(j+y)}_{l+x}\rangle$, that is, $U_{x,y} |v^{(j)}_{l+x}\rangle = |v^{(j+y)}_{l+x}\rangle$.

**Proof** When implementing $Y_d^y X_d^x$ on $|v^{(j)}_{l+x}\rangle$ in turn, we can obtain,

$$
X_d^x Y_d^y |v^{(j)}_{l+x}\rangle = X_d^x \left( \sum_{m=0}^{d-1} w^{ym^2} |m\rangle \langle m| \right) \left( \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} w^{k(l+j+k)} |k\rangle \right) = \frac{1}{\sqrt{d}} \sum_{m=0}^{d-1} w^{ym} |m\rangle \langle m| \sum_{k=0}^{d-1} w^{k(l+j+y+k)} |k\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} w^{k[l+j+y+k]} |k\rangle = |v^{(j+y)}_{l+x}\rangle. 
$$

This completes the proof.

**Lemma 1** In the secret sharing scheme, according to Theorem 1, the initial state selected by Alice is $|\phi\rangle = |\phi_0^0\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle$, and the unitary operation $U_{p^{(i)}_k,q^{(i)}_k}$, $k = 0, 1, \cdots, m$ is performed on the states sequentially by Alice and all the participants in the authorization set $A_i$, then the final state is $|\phi^{(i)}_m\rangle = \left( \prod_{u=0}^{m} U_{p_u,q_u} \right) |\phi\rangle$, that is, $|\phi^{(i)}_m\rangle = \prod_{k=0}^{m} U_{p^{(i)}_k,q^{(i)}_k} |\phi_0^{(i)}\rangle = \left( \sum_{k=0}^{m} q^{(i)}_k \right) |\phi_0^{(i)}\rangle$. When Alice announces the measurement result $r_i$ via the quantum secure channel to the participants in $A_i$, they can restore the secret $s = p_0 = r_i - \sum_{k=1}^{m} p^{(i)}_k = r_i - S_i$.

### 4.2 Verifiability

On one hand, before Alice sends the measurement result, she can check $H_1$ to ensure that the secret value recovered by linear secret sharing is correct, which provides a prerequisite for participants to recover the correct secret. On the other hand, each participant can check

$$
H_2 = h(s), \tag{12}
$$

to ensure that the recovered secret is the original one.

### 4.3 Security

We analyze the security of our scheme against the general attacks here.

**Entangle and measure attack** We assume that eavesdropper Eve intercepts the particles sent among Alice and the participants and then uses a unitary operation $U_E$ to entangle an ancillary state $|E\rangle$ on the transmitted particle. In order to steal secret information by measuring the ancillary state, Eve act the unitary operator $U_E$ on $|E\rangle$ and the transmitted particle. To simplify the
description, we consider the bases corresponding to \( j = 0 \), namely, \(|v_l^{(0)}\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} w^{kl} |k\rangle\), so

\[
U_E |k\rangle |E\rangle = \sum_{h=0}^{d-1} a_{kh} |h\rangle |e_{kh}\rangle, \tag{13}
\]

\[
U_E |v_l^{(0)}\rangle |E\rangle = U_E \left( \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} w^{kl} |k\rangle \right) |E\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} \sum_{h=0}^{d-1} a_{kh} |h\rangle |e_{kh}\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} \sum_{h=0}^{d-1} a_{kh} |v^{(0)}_m\rangle |e_{kh}\rangle, \tag{14}
\]

where \( w = e^{\frac{2\pi i}{d}} \), \(|E\rangle\) is the initial state of the auxiliary space, \(|e_{kh}\rangle\) are pure ancillary states determined uniquely by the unitary operation \( U_E \), so

\[
\sum_{h=0}^{d-1} |a_{kh}|^2 = 1, k \in \{0, 1, \cdots, d-1\}. \tag{15}
\]

For the sake of avoiding the rising error rate, Eve has to set \( a_{kh} = 0 \), \( k, h \in \{0, 1, \cdots, d-1\}, k \neq h \). Therefore, (11) and (12) can be simplified to

\[
U_E |k\rangle |E\rangle = a_{kk} |e_{kk}\rangle, \tag{16}
\]

\[
U_E |v_l^{(0)}\rangle |E\rangle = \frac{1}{d} \sum_{k=0}^{d-1} \sum_{m=0}^{d-1} w^{k(l-m)} a_{kk} |v^{(0)}_m\rangle |e_{kk}\rangle. \tag{17}
\]

Similarly, to avoid the eavesdropping check, Eve has to set

\[
\sum_{k=0}^{d-1} w^{k(l-m)} a_{kk} |e_{kk}\rangle = 0, \tag{18}
\]

where \( m \in \{0, 1, \cdots, d-1\}, m \neq l \). For any \( l \in \{0, 1, \cdots, d-1\} \), we can obtain \( d \) equations

\[
a_{00} |e_{00}\rangle = a_{11} |e_{11}\rangle = \cdots = a_{d-1,d-1} |e_{d-1,d-1}\rangle. \tag{19}
\]

So, whatever quantum state Eve uses, he can only get the same information from the auxiliary particles. Similar analysis can be used for the other quantum states \(|v_l^{(j)}\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} w^{k(l+jk)} |k\rangle\), so the entanglement measurement attack is invalid in our scheme.
**Intercept and resend attack** The eavesdropper Eve intercepts the transmitted particles among Alice and the participants and resends some forged particles. For a simple description, we suppose that the eavesdropper Eve intercepts the quantum state \( |\phi_k\rangle \) sent by Bob\(_k\) to Bob\(_{k+1}\). However, he does not know any information about the measurement bases and only chooses the correct measurement bases with the probability of \( \frac{1}{d} \) to get measure outcome

\[
p_0 + \sum_{i=1}^{k} p_i.
\]

(20)

Even if the result is measured with the probability of \( \frac{1}{d} \), the secret information cannot be obtained because \( p_i, i \in \{k+1, \cdots, m\} \) is unknown. If Alice shares \( n \) secret information, the probability that eavesdropper succeed will be \( \left( \frac{1}{d} \right)^n \). With the increase of the number of \( n \), there will be \( \lim_{n \to \infty} \left( \frac{1}{d} \right)^n = 0 \). The other is that Eve intercepts the \( s_i \) sent by Alice to the participants, but the \( s_i \) does not carry any information of the secret. In short, Eve cannot obtain the secret in intercept-and-resend attack.

**Forgery attack** If Alice shares a fake \( s_i \) to Bob\(_i\), the secret \( s \) will not be restored by the participants in \( A_i \). If one or some of the participants perform the false unitary operation, they will be found by Alice because the measurement result will be inconsistent with Alice’s expectation. What’s more, even if some dishonest participants performed the fake unitary transformation and Alice successfully measured the expected result, there is no use for this attack. Because the recovered secret \( s' \) with \( H_2' = h(s') \neq H_2 = h(s) \) guaranteed. So, the forgery attack is useless.

**Collusion attack** If the participants in \( B_i, B_i \subseteq A_i \) collude to restore the secret, they must obtain the \( s_k(i) \) and \( \lambda_k(i) \) of each participant in the authorization set \( A_i \) to recover \( S_i \). When \( B_i \subseteq A_i \), they cannot get the other’s secret share information, so this attack is unsuccessful.

### 5 Example

Here, we explain our scheme more clearly by giving an example.

**Example 3** According to the MSP and the access structure \( \Gamma \) in the example 2, assuming Alice wants to share secret \( s = 3 \in \mathbb{Z}_5 \) among the four participants Bob\(_1\), Bob\(_2\), Bob\(_3\), Bob\(_4\), she prepares a random vector \( \rho = (4, 1, 0, 2)^T \) firstly and then computes \( s = M\rho = (s_1, s_2, s_3, s_4)^T = (2, 2, 1, 1)^T \). Next, she sends \( s_i \) to Bob\(_i\), \( (i = 1, 2, 3, 4) \) via a secure channel and publishes \( H_1 = h(4) \), \( H_2 = h(3) \). Without losing generality, we assume that the participants in \( A_1 \) want to restore the secret. The dealer Alice prepares a state \( |\phi \rangle = |\phi_0\rangle = \frac{1}{\sqrt{4}} \sum_{i=0}^{4} |i\rangle \) and performs
In this case, Alice selects $M = 3$ to obtain $|\phi\rangle_0 = U_{3,2} |\phi_0^0\rangle = |\phi_3^2\rangle$, where $p_0 = s = 3$ is the secret, $q_0 = 2 \in \mathbb{Z}_5$ is a randomly selected secret value only known by Alice. Next she sends the quantum state $|\phi\rangle_0 = |\phi_3^2\rangle$ to Bob$_1$. After receiving $|\phi\rangle_0 = |\phi_3^2\rangle$, Bob$_1$ performs the unitary operation $U_{\lambda_{s1},\lambda_1} = U_{2,1}$ to get $|\phi\rangle_1 = |\phi_0^3\rangle$ and sends it to Bob$_2$. When receiving $|\phi\rangle_1 = |\phi_0^3\rangle$, Bob$_2$ performs the unitary operation $U_{\lambda_{s2},\lambda_2} = U_{2,1}$ to get $|\phi\rangle_2 = |\phi_2^3\rangle$ and sends it to Bob$_3$. After receiving $|\phi\rangle_2 = |\phi_2^3\rangle$, Bob$_3$ performs $U_{\lambda_{s3},\lambda_3} = U_{0,0}$ to get $|\phi\rangle_3 = |\phi_2^2\rangle$ and sends it to Alice. For the authorization set $A_1$, when all participants act and transmit particle, the final quantum state is

$$|\varphi\rangle_{\text{final}} = \left( \prod_{i=0}^{3} U_{p_i,q_i} \right) |\varphi_0^0\rangle = \left| \varphi_3^0 \right\rangle = |\varphi_2^4\rangle. \quad (21)$$

In this case, Alice selects $M_4 = \{ |\varphi_j^{(4)}\rangle \mid j \in \{0,1,2,3,4\} \}$ to measure $|\varphi\rangle_{\text{final}} = |\varphi_2^4\rangle$ and records the measurement result $r_1$. Afterwards, Alice checks whether $r_1 = 2$ and $H_1 = h(4)$ are true. If not, the scheme is terminated. If they are established, the measurement result $r_1$ is sent to each participant in $A_1$ through a quantum secure channel. After the participant receives it, the secret $s$ can be recovered as

$$s = p_0^{(1)} = r_1 - p_1^{(1)} - p_2^{(1)} - p_3^{(1)} = r_1 - \lambda_1^{(1)} s_1^{(1)} - \lambda_2^{(1)} s_2^{(1)} - \lambda_3^{(1)} s_3^{(1)}. \quad (22)$$

That is $s = 2 - (2 + 2 + 0) = 3$. Last but not least, they can check $H_1$ to make certain of the authenticity of the secret.

### 6 Comparison

In this section, we give a comparison among our scheme and other similar $d$-dimensional QSS schemes\cite{24,35,36} in terms of basic properties, computational complexity and communication costs. The schemes in Ref.\cite{24,36} are the threshold QSS, however the scheme in Ref.\cite{35} and ours are the general access structure QSS. The general access structure makes the level and influence of the participants different, making the scheme more flexible. They all use the Hash function to make the verifiability of the $d$-dimensional QSS scheme. The scheme proposed by Song et al.\cite{24} shared a classical secret by utilizing polynomials according to the Lagrange interpolation formula. The transformation of the particles includes some operations such as $d$-level CNOT, QTF, Inverse QTF, and generalized Pauli operator. However, the general access structure QSS is far more flexible and practical than the threshold one. In Ref.\cite{35}, Mashadi proposed a hybrid secret sharing based on the quantum Fourier transform and monotone span program, in which the participants recover the secret by means of measuring the entangled state. The number of unitary operators is not much different in the premise, while the number of required quantum states and the number of measurement operations are greatly reduced, which consumes less quantum resources.
and the scheme is more practical. Qin et al.\cite{36} put forward a verifiable \((t,n)\) threshold QSS using \(d\)-dimensional Bell state and they realize the authentication of quantum state transmission by adding some decoy particles. According to the Lagrange interpolation and the unitary operation, they can recover the secret with measuring the final Bell state. The Specific comparison of basic property among Ref.\cite{24,35,36} and ours is given in Table 1. The comparison of the computational complexity and communication costs of the general access structure QSS\cite{35} and the new is given in Table 2.

| Property          | Song\cite{24}       | Mashhadi\cite{35} | Qin\cite{36} | New          |
|-------------------|---------------------|-------------------|--------------|--------------|
| Model             | \((t,n)\) threshold| General           | \((t,n)\) threshold| General     |
| Verification      | Hash function       | Hash function     | Hash function| Hash function|
| Secret            | Classic             | Classic           | Classic      | Classic      |
| Dimension         | \(d\)               | \(d\)             | \(d\)        | \(d\)        |
| Method            | LI                  | MSP, LC           | LI           | MSP, MUB, LC |
| NQO               | \(QFT,QFT^{-1}\)   | \(QFT, Pauli\)   | UO           | UT           |

| Property                               | Mashhadi\cite{35} | Ours |
|----------------------------------------|--------------------|------|
| Number of message particles            | \(m - 1\)         | 1    |
| Unitary operation                      | \(m\)              | \(m + 1\) |
| \(QTF\)                                | 1                   | –    |
| Measure operation                      | \(m\)              | 1    |
| Hash function                          | 2                   | 2    |

Remark 1. LI: Lagrange interpolation, MSP: Monotone span program, MUB: Mutually unbiased bases, LC: Linear computation, NQO: necessary quantum operation, QTF: Quantum Fourier Transform, \(QFT^{-1}\): Inverse Quantum Fourier Transform, UO: Unitary operation, UT: Unitary transformation.

## 7 Conclusions

The verifiable quantum secret sharing scheme based on the access structure is very useful in practice. In this paper, we construct a verifiable quantum secret
sharing scheme based on the property of the mutually unbiased base and the monotone span program. The dealer and participants in the authorization set can restore secret through the transformation and transmission of a $d$-dimensional quantum state as well as linear secret sharing. In addition, the correctness, verifiability and security analysis of the scheme have been proved. Finally, a specific example and a comparison are given to further clarify the advantages and practicality of our scheme.

For the future work, the verifiability of the scheme is analyzed from the view that the recovered secret is consistent with the original one. However, the issue of mutual authentication among the participants in the authorization set is still worth studying.

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