Localization length exponent in two models of quantum Hall plateau transitions

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(Dated: April 26, 2018)

Motivated by the recent numerical studies on the Chalker-Coddington network model that found a larger-than-expected critical exponent of the localization length characterizing the integer quantum Hall plateau transitions, we revisited the exponent calculation in the continuum model and in the lattice model, both projected to the lowest Landau level or subband. After including the corrections of an irrelevant length scale, we obtain \( \nu = 2.47 \pm 0.03 \), which is larger but still consistent with the earlier results in the two models, unlike what was found recently in the network model. The scaling of the total number of conducting states, as determined by the Chern number calculation, is accompanied by the irrelevant length scale exponent \( \gamma = 4.3 \) in the lattice model, indicating that the topology number calculation is not severely affected by irrelevant perturbations.

I. INTRODUCTION

In the presence of a strong perpendicular magnetic field, the Hall conductance of two-dimensional electron gases at low temperatures is quantized to integral multiples of \( e^2/h \). Tuning either magnetic field or Fermi energy \( E_F \), one can drive an integer quantum Hall transition from one plateau to another. The transition is controlled by a correlation length \( \xi \), which diverges as

\[
\xi(E) \sim |E - E_c|^{-\nu},
\]

as the Fermi energy crosses a critical energy \( E_c \) (or the magnetic field \( H \) crosses a critical value \( H_c \), in which case \( H \) replaces \( E \) in the above formula). The critical exponent \( \nu \) is believed to be universal and has been studied extensively both theoretically and experimentally.

Prior to 2009, for non-interacting electrons, the consensus was \( \nu \approx 2.38 \pm 0.05 \). This value was also supported by experiments on samples with short-range scatterers even though electron-electron interactions are present in experiments, and may be relevant.

Quite unexpectedly, when Slevin and Ohtsuki redid the finite-size scaling study of the Chalker-Coddington model including corrections due to an irrelevant operator, they found \( \nu = 2.593 \) [2.587, 2.598], significantly larger than the previously reported values. They found the irrelevant length scale exponent \( \gamma \approx 0.17 \), which gave rise to large corrections, and altered the exponent \( \nu \) significantly. Several calculations followed. Obuse et al. reported \( \nu = 2.55 \pm 0.01 \) and a larger irrelevant length exponent \( \gamma = 1.29 \). Amado et al. obtained \( \nu = 2.616 \pm 0.014 \) but found logarithmic irrelevant length scale correction. Other calculations by Dahlhaus et al., Fulga et al., and Slevin and Ohtsuki in the CCN model also confirmed \( \nu = 2.56 \sim 2.6 \). Obuse, Gruzberg, and Evers perform a stability analysis of the finite-size scaling of the CCN model and reported \( \nu = 2.62 \pm 0.06 \), with an irrelevant length scale correction exponent \( \gamma \geq 0.4 \), which, as the authors pointed out, is considerably larger than most recently reported values. All these works show that with irrelevant length scale correction included, the localization length exponent \( \nu = 2.55 \sim 2.62 \). However, the smallness of the leading irrelevant length exponent indicates that great care is needed in numerical studies using finite-size scaling.

Despite the flurry of new and seemingly consistent results, the correct value of the localization length exponent remains under debate. Gruzberg et al. questioned the regular lattice setup of the CCN and considered a general network with geometric disorder. Numerical simulations of this new model found \( \nu = 2.374 \pm 0.018 \) in agreement with the earlier results. Bondesan et al. developed an effective Gaussian free field approach of the Chalker-Coddington network model of the integer quantum Hall plateau transition. Even though the theory confirmed that the spectrum of multifractal dimensions at the transition is parabolic, the authors warned that numerical calculations may suffer from existence of an irrelevant perturbation which is close to marginal (or even marginal).

Consequently, numerical calculations of \( \nu \) in alternative models are desirable to help resolve the disagreements, especially ones that are purely two-dimensional, and do not have to rely on the crossover to one-dimension to analyze the data. Earlier, Huo and Bhatt obtained \( \nu = 2.4 \pm 0.1 \) employed a two-dimensional approach using Chern number calculations in the continuum Landau-level model. The same result was also found in the disordered Hofstadter model. No attempt, however, has been made for the two models including corrections due to irrelevant length scales, as done in the CCN model.

In this paper, we revisit the continuum Landau-level model and the Hofstadter lattice model with short-range impurities. After projecting the disorder potential into the lowest Landau level (subband), we calculate the Chern number for all eigenstates to identify the conducting states and perform finite-size scaling of their total number. Combining results in the two models, we obtain a slightly larger exponent \( \nu = 2.47 \pm 0.03 \) than earlier re-
sults $\nu = 2.4 \pm 0.1$, but the two are still consistent, given the larger error bars of the previous work. In addition, we find that the corrections to scaling are very small, (e.g., the irrelevant length exponent is large, $y = 4.3 \pm 0.2$ in the lattice model), which can explain the relative accuracy of the earlier results on smaller sizes using the Chern number approach, unlike transfer matrix methods. We also perform finite-size scaling of the width of the density of conducting states, where the results appear to be somewhat less reliable. We attribute this to the fluctuations in the tails of the density of conducting states in finite-sized samples, which renders calculations of higher moments less reliable. Thus, the zeroth moment (the total number of conducting states) is more reliable than the width of density of conducting states, which is the second moment.

The paper is organized as follows. In Sec. II we introduce the two models we study. We briefly review the Chern number calculation method and discuss our scheme for analyzing our data. The next section describes our main results. We first analyze the finite-size scaling of the total number of conducting states in the lattice model and in the continuum model in Sec. IIIA. We then present the finite-size scaling of the width of the density of conducting states in the two models in Sec. IIIB. We discuss the potential errors in the Chern number calculations in Sec. IIIC. Finally, in Sec. IV we give a summary of our results and discuss our conclusions in light of other work.

II. MODELS AND METHOD

In this paper we present two microscopic models for quantum Hall plateau transitions. The first model describes electrons hopping on a tight-binding lattice with uniform magnetic flux and on-site disorder. The second model describes electrons in continuum with a short-range impurity potential projected into the lowest Landau level. In this section we review the basics of the two models. We also describe the Chern number calculation method used to extract the localization length critical exponents, and the statistical measures used to categorize the data.

A. The lattice model

We consider the two-dimensional tight-binding model

$$\mathcal{H} = - \sum_{\langle ij \rangle} \left( t e^{i\theta_{ij}} c_i^\dagger c_j + \text{h.c.} \right) + \sum_i c_i^\dagger c_i,$$

where $\theta_{ij} = \frac{\pi}{2} \int_{l_i} \mathbf{A} \cdot dl$ in the presence of a perpendicular magnetic field $B$. We choose the Landau gauge $\mathbf{A} = (0, Bx, 0)$. The magnetic flux $\phi$ per unit cell is

$$\frac{\phi}{\phi_0} = \frac{Ba^2}{\hbar c/e} = \frac{1}{2\pi} \sum \theta_{ij},$$

where $\phi_0 = \hbar c/e$ is the flux quantum, and the disorder potential $\epsilon_i$ are independent variables with identical uniform distribution on $[-W, W]$. As the flux $\phi$ per unit cell varies, the clean model has a self-similar energy spectrum, known as the Hofstadter butterfly. We choose the flux $\phi$ per unit cell as $\phi_0/3$. The clean Hamiltonian is translationally invariant and can be diagonalized to three subbands,

$$H_0(\mathbf{k})|\phi_n(\mathbf{k})\rangle = E_n(\mathbf{k})|\phi_n(\mathbf{k})\rangle$$

where the subband index $n = 0, 1, 2$. The three subbands carry Chern number $1, -2$, and $1$, and each contains $N_\phi = L_1L_2/3$ states for an $L_1 \times L_2$ lattice. To study a single-step plateau transition, we truncate the Hilbert space by projecting the disorder potentials into the lowest subband,

$$V(r) = \sum_{\mathbf{k},\mathbf{k}'} |\phi_0(\mathbf{k})\rangle \langle \phi_0(\mathbf{k})| V(r) |\phi_0(\mathbf{k}')\rangle \langle \phi_0(\mathbf{k}')|. \quad (5)$$

The ratio $t/W$ can be set to 0 after the projection, which fixes the critical energy at $E_c = 0$.

B. The continuum model

We consider the two-dimensional electron in presence of a magnetic field

$$H = \frac{1}{2m}(\mathbf{P} + \frac{e}{c} \mathbf{A})^2 + V(r) = \frac{\Pi^2}{2} + V(r) \quad (6)$$

where the symmetric gauge $\mathbf{A} = \frac{1}{2} B(y, -x)$ is used, and $V(r)$ is the random potential. The generator of infinitesimal magnetic translation is

$$\kappa \equiv (\kappa_1, \kappa_2) = \mathbf{P} - \frac{e}{c} \mathbf{A} = \Pi(-B), \quad (7)$$

with $[\kappa_1, \kappa_2] = -i$, where we set the magnetic length $l_B = 1$ and $\hbar = 1$ for convenience. For a finite $L_1 \times L_2$ system, the translation operator $t(L_j \hat{e}_j) = \exp(ik_jL_j)$ satisfies the magnetic algebra

$$t(\mathbf{a})t(\mathbf{b}) = \exp[i(\mathbf{a} \times \mathbf{b}) \cdot \hat{z}] t(\mathbf{b})t(\mathbf{a}). \quad (8)$$

When the number of flux quanta in an $L_1 \times L_2$ sample $N_\phi = L_1L_2/(2\pi)$ is an integer, $t(L_1 \hat{e}_1)$ and $t(L_2 \hat{e}_2)$ commute with each other. By defining two primitive translations, $t_j = t(L_j/N_\phi \hat{e}_j) = \exp(ik_jL_j/N_\phi)$ for $j = 1, 2$, we are able to construct a Landau-like stripe basis $|\phi_m\rangle$ in the lowest Landau level by requiring

$$\begin{cases}
    t_1|\phi_m\rangle = |\phi_{m+1}\rangle, \\
    t_2|\phi_m\rangle = \exp(-i2\pi m/N_\phi)|\phi_m\rangle.
\end{cases} \quad (9)$$
The random potential is then projected into the lowest Landau level
\[
V(r) = \sum_{jk} |\phi_j\rangle\langle\phi_j| \sum_q V_q e^{i q r} |\phi_k\rangle\langle\phi_k|
\]  
(10)
where \( \kappa = (\kappa_1, \kappa_2) = (2\pi n / L_1, 2\pi n / L_2) \) and \( m, n \) are integers. For simplicity, we consider square samples \( L_1 = L_2 = L \) with generalized boundary condition
\[
t(\vec{L}_j|\psi_j) = e^{i \theta_j|\psi_j}, \quad j = 1, 2
\]  
(11)
with \( \theta_j \in [0, 2\pi] \), the Hamiltonian matrix is calculated as
\[
H_{jk}(\theta_1, \theta_2) = \sum_{mn} V_{mn} e^{-i \pi (m^2 + n^2)/2 N_o} e^{-i \pi mn / N_o}
\]
\[
\times e^{i [m\theta_2 - n\theta_1] / N_o} e^{-i \pi m j / N_o} |\phi_j \rangle \langle \phi_k - n|.
\]  
(12)
For particle-hole symmetric potential, the critical energy is also expected at \( E_c = 0 \).

C. The Chern number calculation

In either case, we can impose generalized boundary conditions. The boundary condition averaged Hall conductance for a state \( \psi_m \) is a topological invariant
\[
\langle \sigma_{xy}(m) \rangle = C(m) e^2 / h,
\]  
(13)
where the first Chern number
\[
C(m) = \frac{1}{2\pi i} \int_0^{2\pi} d\theta_x d\theta_y \left[ \langle \frac{\partial \psi_m}{\partial \theta_x} \frac{\partial \psi_m}{\partial \theta_y} \rangle - h.c. \right].
\]  
(14)
For efficient numerical calculation, we follow the method of Fukui et al.\textsuperscript{26} We divide the \( 2\pi \times 2\pi \) boundary condition space into a grid of \( L_g \times L_g \) sites
\[
\vec{\theta} = (\theta_x, \theta_y) = \frac{2\pi}{L_g} (l_x, l_y), \quad l_x, l_y = 0, \ldots, L_g - 1,
\]  
(15)
so primitive vectors \( \hat{\mu} \) are vectors in the directions of \( \mu = x \) and \( y \) with length \( 2\pi/L_g \). For the \( m \)th eigenstate, we define a \( U(1) \) link variable
\[
U^m_{\mu}(\vec{\theta}) = \langle \phi_m(\vec{\theta})|\phi_m(\vec{\theta} + \hat{\mu}) \rangle / N^m_{\mu}(\vec{\theta}),
\]  
(16)
where \( N^m_{\mu}(\vec{\theta}) = |\langle \phi_m(\vec{\theta})|\phi_m(\vec{\theta} + \hat{\mu}) \rangle| \) is the normalization factor. The Berry phase in a unit cell at \( \vec{\theta} \) is then calculated by a gauge invariant formula
\[
\gamma_m(\vec{\theta}) = \frac{1}{i} \ln U^m_{\mu}(\vec{\theta}) U^m_{\mu}(\vec{\theta} + \hat{x}) U^m_{\mu}(\vec{\theta} + \hat{y})^{-1} U^m_{\mu}(\vec{\theta})^{-1},
\]  
which is restricted to \((-\pi, \pi]\). The Chern number \( C(m) \) is obtained by summing the phase over all lattice sites
\[
C(m) = \frac{1}{2\pi} \sum_{\vec{\theta}} \gamma_m(\vec{\theta}).
\]  
(17)
Due to the introduction of the locally gauge-invariant formulation of Chern number, the efficiency of the method has been improved greatly even with a coarsely discretized boundary condition space. As pointed by Arovas et al.\textsuperscript{23} the sensitivity of the nodes of the wave function to the smooth change in boundary conditions can be used to distinguish the localized and extended states. Consequently, we follow Huo and Bhatt\textsuperscript{20} to identify states with nonzero Chern numbers as conducting states.

The total number of conducting states per sample is defined as
\[
N_c = \int_{-\infty}^{\infty} \rho_c(E) dE,
\]  
(18)
where \( \rho_c(E) \) is the density of conducting states. In the vicinity of the critical energy \( E_c \), the localization length diverges as \( \xi(E) \sim |E - E_c|^{-\nu} \). For a finite size system with linear size \( L \sim \sqrt{N_o} \), one expects that the states with \( \xi(E) > L \) are conducting. The number of such states scale as \( N_c \sim L^2 \rho(E_c)|E_m - E_c| \sim N_o^{1-1/2\nu} \), to the lowest order, where \( E_m \) is determined by \( \xi(E_m) \sim L \). One can also define the width of \( \rho_c(E) \) as the square root of its second moment
\[
\Delta E = \left[ \int_{-\infty}^{\infty} (E - E_c)^2 \rho_c(E) dE / \int_{-\infty}^{\infty} \rho_c(E) dE \right]^{1/2}.
\]  
(19)
Roughly speaking, the width is set by \( L \sim \xi(E) \sim \Delta E^{-\nu} \), so it is expected to follow \( \Delta E \sim L^{-1/\nu} \sim N_o^{-1/(2\nu)} \). The lowest-order scaling of \( N_c \) and \( \Delta E \) may need irrelevant length corrections.

D. Goodness of fit

In this paper we compare various scaling hypotheses with and without irrelevant length corrections. After obtaining the best-fit values of parameters, we need to decide which hypothesis is the best description of the data. Our result of the critical exponent is then obtained from the best-fit with the best hypothesis. In this subsection, we discuss the acceptability of the scaling hypothesis.

Generally speaking, we fit \( N \) data points \((x_n, y_n)\) with \( n = 0, \ldots, N - 1 \) with measurement errors \( \sigma_n \) to a model that has \( M \) adjustable parameters \( a_0, \ldots, a_{M - 1} \). The model has a functional relationship between the measured variables and parameters \( y(x) = y(x; a_0, \ldots, a_{M - 1}) \), minimizing the parameters \( a_0, \ldots, a_{M - 1} \) is equivalent to minimizing the chi-square
\[
\chi^2 = \sum_{n=0}^{N-1} \left[ \frac{y_n - y(x_n; a_0, \ldots, a_{M - 1})}{\sigma_n} \right]^2.
\]  
(20)
A typical value of \( \chi^2 \) for a moderately good fit is \( \chi^2 \approx \text{dof} \), where \( \text{dof} = N - M \) is the degrees of freedom, or, equivalently, the reduced chi-square \( \chi^2 / \text{dof} \approx 1 \). Relatively smaller value of \( \chi^2 \) indicates a better fit. To
give a quantitative measure for the goodness-of-fit of the model, one may use the cumulative probability function $Q(\chi^2|\text{dof})$, which is the probability that the observed chi-square will exceed a particular value $\chi^2$ by chance even for a correct model:

$$Q(\chi^2|\text{dof}) = 1 - P\left(\frac{\text{dof}}{2}, \frac{\chi^2}{2}\right),$$

(21)

where $P(a, x)$ is the incomplete Gamma function. Normally, a larger value of $Q$ is taken as an indicator of a better fit. If $Q$ is too small, the reason could be either the model is unlikely to be true and can be rejected, or the size of the measurement errors $\sigma_n$ are larger than stated. If $Q$ is close to 1, it could be caused by overestimation of the measurement errors.

## III. RESULTS

### A. The number of conducting states

We begin with the study of square lattices with $L = 6$-81. The largest system contains almost 30 times more states than the largest did twenty years ago. The number of disorder realizations ranges from $10^5$ for $L = 6$ to $10^2$ for $L = 81$. In the Chern number calculation we choose a grid size such that the number of grid points along in each dimension is $L_g = 30$ for $L \leq 30$ and $L_g \approx 2\sqrt{3L}$ for $L = 36$-81. Figure 1 shows the density of states $\rho(E)$ and the density of conducting states $\rho_c(E)$ for several system sizes.

The total density of states $\rho(E)$ remains the same bell shape for all system sizes, while the density of conducting states $\rho_c(E)$ shrinks as the system size increases, implying that in the thermodynamic limit only states with $E = 0$ are extended. To prove this, we follow the same procedure in earlier literature to study the scaling behavior of the total number of conducting states per sample $N_c$ and the width $\Delta E$ of the density of conducting states $\rho_c(E)$, which are the zeroth and (the normalized) second moment of the density of nonzero Chern states.

We begin by fitting the number of conducting states to

$$N_c = aN_\phi^{1-1/(2\nu)}.$$

(22)

A chi-square fitting generates $\nu = 2.49 \pm 0.01$, as shown by the blue dashed line in Fig. 2. The reduced chi-square $\tilde{\chi}^2 = 0.732$ and the goodness of fit $Q = 0.599$. The straightforward power-law fit for all system sizes results in a reasonable reduced chi-square $\tilde{\chi}^2$. Deviation is visually noticeable only for $L = 6$ in Fig. 2. This can be improved if we include the irrelevant length scale corrections.

The leading correction can be accounted for by assuming

$$N_c = a(1 + bN_\phi^{-y/2})N_\phi^{1-1/(2\nu)},$$

(23)

where $y$ is the leading irrelevant length exponent. A chi-square fitting now generates $\nu = 2.479 \pm 0.005$ and $y = 4.3 \pm 0.2$, as illustrated in Fig. 2. The reduced chi-square $\tilde{\chi}^2 = 0.631$ and the goodness of fit $Q = 0.804$. One expects this form is valid only for $|bN_\phi^{-y/2}| \ll 1$. Indeed, the correction term $bN_\phi^{-y/2}$ varies from $-0.06$ for $L = 6$ to $-10^{-7}$ for $L = 81$. With or without the correction, the critical exponent $\nu$ remains unchanged within error bars. This is supported by the similar $\tilde{\chi}^2$ or $Q$ for the two fits. Strictly speaking, the fit with the irrelevant length exponent is better, as would be expected given the larger number of parameters. However, it is debatable whether the improvement is enough to warrant increasing the number of parameters. Further, it is worth pointing out that the irrelevant length exponent is large, $y = 4.3$. This is strong evidence that the irrelevant (or marginal) perturbation at the criticality has a weak effect on the Chern number calculation and the scaling of the number of conducting states.

For the continuum model, we study system size from $N_\phi = 32$ to 2048, with disorder realizations from $16384$ for $N_\phi = 32$ to 128 for $N_\phi = 2048$. In the Chern number calculation we use grid size $L_g = 50$ for all system sizes.

![Figure 1](image1.png)

**FIG. 1.** (a) Disorder averaged total density of states $\rho(E)$ and (b) density of conducting states $\rho_c(E)$ in the lattice model for system sizes $L = 9, 18, 36, \text{and} 72$.

![Figure 2](image2.png)

**FIG. 2.** Log-log plot of the number of conducting states per sample $N_c$ versus the total number of states per sample $N_\phi$ in the lattice model. The blue dashed line is the power-law fit without corrections and the solid red line shows the fit with irrelevant length corrections.

sizes. Fig. 3 fits the number of conducting states as $N_c = a N_\phi^{1-1/(2\nu)}$. For all available system sizes, we find $\nu = 2.45 \pm 0.02$ with $\bar{\chi}^2 = 1.27$ and $Q = 0.25$. We note that even in the absence of the irrelevant length corrections, the exponent is in good agreement with the value obtained above in the lattice model. We also fit data only in systems with $N_\phi \geq 256$, the result is $\nu = 2.41 \pm 0.07$ with $\bar{\chi}^2 = 1.12$ and $Q = 0.35$. The exponent agrees with the previous one for all systems within the error bar, which is now larger due to the fewer system sizes.

The agreement in $\nu$ between all systems and larger system only in Fig. 3 already suggests that the existence of the irrelevant length corrections in the continuum model is difficult to demonstrate. Attempt to fit all data with the corrections leads to large error bars in $\nu = 2.49 \pm 0.16$ and $y = 1.7 \pm 7.4$, which indicates that a single leading irrelevant length correction is not a good hypothesis. This is also supported by a larger $\chi^2 = 1.47$ and a smaller $Q = 0.17$. In general, the fits in the continuum model are not as good as those in the lattice model, judging from relatively large $\chi^2$ and small $Q$. It is possible that the existence of a dangerous (or marginal) irrelevant operator may cause slower disappearing corrections to scaling. We attempt to fit all data in the continuum model to a power law with logarithmic corrections

$$N_c = a (1 + b/\ln N_\phi) N_\phi^{1-1/(2\nu)}.$$  \hfill (24)

As illustrated in Fig. 3 it generates $\nu = 2.51 \pm 0.08$ with $\bar{\chi}^2 = 1.30$ and $Q = 0.24$. The goodness of fit is almost the same as in the absence of the corrections, and the large error bar in the exponent $\nu$ prevents us from concluding on the existence of a dangerous irrelevant operator within the range of the system sizes we study.

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B. The width of the density of conducting states

While the critical exponent $\nu$ is expected to be universal in different models of the plateau transitions with different numerical methods, the same does not appear to be true of the irrelevant length exponent. To show this, we consider the width $\Delta E$ of the density of conducting states. With the corrections of the leading irrelevant length, one expects $\Delta E = c (1 + d N_\phi^{-y/2}) N_\phi^{-1/(2\nu)}$. A chi-square fitting gives $\nu = 2.40 \pm 0.02$ and $y = 1.2 \pm 1$ with the reduced chi-square $\bar{\chi}^2 = 1.032$ and the goodness of fit $Q = 0.415$, as shown in Fig. 4. The correction term $d N_\phi^{-y/2}$ varies from $-0.12$ for $L = 6$ to $-0.005$ for $L = 81$. The correction is twice as large for small system sizes as that in the scaling of $N_c$, which indicates that we may need to consider additional corrections.

The comparison of the scalings of $N_c$ and $\Delta E$ shows that even with the same set of data, namely $\rho_c(E)$, one obtains exponent $\nu$ with a slight difference. Judging from the smaller size of the corrections and, in particular, the larger value of $y$, we conclude that the scaling of $N_c$ is more reliable. This is consistent with empirical evidence from simulations of other disordered systems such as spin glasses that lower order moments of distributions can be calculated more reliably (they also seem to converge faster to equilibrium than high order moments). An additional evidence is that we can perform a power-law fitting of $\Delta E$ without the irrelevant length corrections, $\Delta E \sim c N_\phi^{-1/(2\nu)}$, for system sizes $L \geq 30$ (i.e., $N_\phi \geq 300$), as shown by the blue dashed line in Fig. 4. The result is $\nu = 2.48 \pm 0.02$ with $\bar{\chi}^2 = 1.444$ and $Q = 0.205$. Therefore, the value obtained from larger systems is consistent with that obtained from the scaling of $N_c$. 

FIG. 3. Log-log plot of the number of conducting states $N_c$ vs. $N_\phi$ in the continuum model. The blue dashed line is the power-law fit without any corrections and the red solid line shows the fit with logarithmic corrections. Data points are taken from $N_\phi = 16$ to $N_\phi = 2048$.

FIG. 4. Log-log plot of the width $\Delta E$ of the density of conducting states vs. $N_\phi$ in the lattice model. The error bars are much smaller than the symbols. The red solid line shows the power-law fit with leading irrelevant length corrections for all data points. The blue dashed line is the power-law fit without any corrections for systems with $L \geq 30$ (or $N_\phi \geq 300$).
Therefore, the scaling of \( \Delta E \) in the two models are not. The observation confirms that the finite-size fluctuations are more significant in the tails of the density of conducting states. If we fit all data to the form \( \Delta E = cN_{\phi}^{-1/(2\nu)} \), we obtain \( \nu = 2.31 \pm 0.01 \) with the reduced chi-square \( \chi^2 = 0.58 \) and the goodness of fit \( Q = 0.81 \). When we only fit data with \( N_{\phi} \geq 256 \), we obtain \( \nu = 2.34 \pm 0.05 \) with \( \chi^2 = 0.91 \) and \( Q = 0.47 \). Within the error bars, we cannot draw a definite conclusion on the trend of \( \nu \) in the thermodynamic limit. Like the scaling of \( N_c \), including either power-law corrections or logarithmic corrections shows no significant improvement.

As we discussed at the end of Sec. III A, the localization length critical exponent \( \nu \) obtained from the scaling of \( N_c \) is consistent in the lattice model and the continuum model. However, the values of \( \nu \) obtained from the scaling of \( \Delta E \) in the two models are not. The observations confirm that the finite-size fluctuations are more significant in the tails of the density of conducting states. In other words, the values obtained from the scaling of \( \Delta E \) are less reliable, unless we can approach much larger system sizes. We also studied the width of the density of conducting states for different percentiles and found that the fluctuations are too large to allow accurate analysis.

C. Errors in the Chern Number calculation

Because Chern number, or the dimensionless Hall conductance, is a topological invariant, the main errors in the Chern number approach come from the discretization of the boundary condition space into a grid. The accuracy of Chern numbers thus depends on the grid size \( L_g \). But increasing \( L_g \) will slow down the Chern number calculation. To estimate the optimal \( L_g \), e.g., in the lattice model, we consider uniformly distributed \( \gamma_m \) in Eq. (17) in the boundary condition space. For a state \( m \) with Chern number \( C(m) \), we expect

\[
\bar{\gamma}_m = \frac{2\pi C(m)}{L_g^2},
\]

for each \( \bar{\theta} \). Sign errors of \( \gamma_m \) occur due to the restriction of the phase to \( (-\pi, \pi) \) when \( \gamma_m \) is comparable to \( \pi \), i.e.,

\[
\frac{2\pi C(m)}{L_g^2} \sim \epsilon \pi
\]

where \( \epsilon \) can be understood as the error rate of the state. This gives a rough estimate of the error rate per state with \( C(m) = O(1) \) as \( \epsilon \sim 2/L_g^{-2} \).

We implement a simple consistency check for the Chern number calculation: the total Chern number of each sample should be one. When we obtain a sample that fails the total Chern number check, we reject all states in the sample. This happens when at least one of the states in the sample has an incorrect Chern number. For small enough \( \epsilon \), the rejection rate \( r \) for a sample with \( L^2/3 \) states is

\[
r \approx \epsilon \left( L^2/3 \right) \sim \frac{2}{3} \left( \frac{L}{L_g} \right)^2
\]

Therefore, in the lattice model we choose \( L_g \approx \sqrt{2}/3L \) for \( L \geq 36 \) and \( L_g = 30 \) for \( L \leq 30 \), in order to keep the rejection rate low.

We also run numerical tests to estimate the error rate of the Chern number calculation for individual states and the consequent errors in \( N_c \) and \( \Delta E \), which we use to extract \( \nu \). In the tests we study the computational errors of lattice systems with linear size \( L = 6-21 \), each with \( N_s = 10,000 \) random potential realizations. Our calculation finds that the rejection rate \( r \) vanishes as \( L_g^{-\eta} \) with \( \eta \) varying between 2.5-3.3, which decreases even faster than our simple expectation. When we fix \( L_g = 30 \), the rejection rate \( r \) is no more than \( 5 \times 10^{-3} \) and increases with size as \( L^{1.6} \), which is close to \( L^2 \) as we expect in Eq. (27). The better-than-expected observations are likely due to the fact that the majority of the Chern numbers is zero and their percentage increases with \( L \). With the numerical results, we can assume that Chern numbers calculated with \( L_g = 60 \) are sufficiently accurate and can be used
as a reference. We then calculate Chern numbers with $L_y = 30$. By comparing them, we estimate the relative error in $N_c$ is around $10^{-3}$ and the relative error in $\Delta E$ is even smaller. Compared to the sample-to-sample fluctuations, we find the errors due to the Chern number calculation are at least one order of magnitude smaller and can be neglected.

IV. SUMMARY AND DISCUSSION

In summary, we revisit two models for the integer quantum Hall plateau transition: the disordered Hofstadter lattice model and the continuum Landau-level model. We perform high-precision Chern number calculation for system size up to $N_x = 2187$ in the lattice model and 2048 in the continuum model. We use multiple measures to rule out the errors in the Chern number calculation caused by the discretization of the integration grid. Using nonzero Chern number as criterion, we calculate the density of conducting states $\rho_c(E)$ for varies system sizes.

By fitting the total number of conducting states $N_c$ to a power law, we obtain the localization length critical exponent $\nu = 2.49 \pm 0.01$ in the lattice model and $\nu = 2.45 \pm 0.02$ in the continuum model. In the lattice model we obtain $\nu = 2.479 \pm 0.005$ with better goodness of fit after we include the leading irrelevant length corrections. However, the bare power-law fit cannot be improved in the continuum model by the inclusion of either the leading irrelevant length corrections or logarithmic corrections caused by a dangerous (marginal) irrelevant perturbation. Based on these results, we conclude $\nu = 2.47 \pm 0.03$ if the two models are in the same universality class. The conservative error bar is chosen to be the sum of the error bars in the two models. The larger exponent is found in much larger systems than in the earlier studies. In fact the largest system in the present study has 16 times the flux quanta of that in the earlier study for the continuum model and almost 30 times the flux quanta for the previous lattice model. However, given the fact that earlier studies had large error bars of 0.1, the new result is still consistent with the earlier estimates.

In both models the irrelevant length corrections for $N_c$ are small. Even in the lattice model where the corrections are apparently larger, the leading irrelevant length exponent is found to be $y = 4.3 \pm 0.2$, which is substantially larger than the value found in the Chalker-Coddington network model, suggesting that the topological number based calculation is not sensitive to the inclusion of irrelevant perturbations. This also explains why earlier Chern number calculations revealed the long-standing result of $\nu = 2.4 \pm 0.1$ in systems with no more than 128 magnetic flux quanta.

We also attempt to determine $\nu$ from the scaling of the width of the density of conducting states $\rho_c(E)$, which is related to the second moment of $\rho_c$. The value is found to be appreciably smaller than that obtained by the scaling of $N_c$ and is model-dependent: $2.40 \pm 0.02$ for the lattice model and $2.31 \pm 0.01$ for the continuum model. Nevertheless, fitting the data from larger systems only in the lattice model gives $\nu = 2.48 \pm 0.02$, which is then consistent with the value from the scaling of $N_c$. These results emphasize the presence of finite-size fluctuations in the Chern number calculation. However, these fluctuations mainly affect the tails of $\rho_c(E)$, so the analysis of its high moments needs special care.

Finally, the localization length critical exponent $\nu = 2.47 \pm 0.03$ in the present study differs from the value (varying from 2.55 to 2.62) found in recent studies of the Chalker-Coddington network model. Recently, Gruzberg et al. argued that the network model may not capture all types of disorder that are relevant at the integer quantum Hall plateau transition. However, the authors found in the numerical simulation of a geometrically disordered network model $\nu = 2.374 \pm 0.018$, which is significantly smaller than that found in the present study for the lattice model and the continuum model. A resolution of the discrepancies between the network model and the lattice and continuum models remains an interesting problem for the future.

V. ACKNOWLEDGEMENTS

The authors thank Ferdinand Evers, Tomi Ohtsuki and Keith Slevin for helpful discussions. This work was supported by the 973 Program under Project No. 2012CB927404, by the NSFC under Grant No. 11174246, and by DOE grant DE-SC0002140 (RNB).

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