The derivative of the Minkowski function

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Abstract. We prove new results on the derivative of the Minkowski question mark function.

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§ 1. Introduction

1.1. The Minkowski function $?(x)$. The function $?(x)$ (a strictly increasing one-to-one map of the closed interval $[0, 1]$ onto itself whose derivative vanishes almost everywhere) was first considered by Minkowski [1] in 1911. Later Salem [2] gave an equivalent definition of $?(x)$, which was thereafter referred to as the Minkowski function, as follows. If

$$x = [x_1, x_2, \ldots, x_t, \ldots] = \frac{1}{x_1 + \frac{1}{x_2 + \cdots + \frac{1}{x_t + \cdots}}}$$

(1.1)

is an ordinary continued fraction expansion (finite or infinite) of a number $x \in [0, 1]$ with positive integer terms $x_1, x_2, \ldots, x_t, \ldots$, then

$$?(x) = \frac{1}{2x_1-1} - \frac{1}{2x_1+x_2-1} + \cdots + \frac{(-1)^{n+1}}{2x_1+x_2+\cdots+x_n-1} + \cdots.$$  

(1.2)

The Minkowski function possesses a number of very interesting properties, which can be found, for example, in [2]–[5]. In particular the derivative $?(x)$ can be equal only to 0 or $+\infty$ at every $x$ where it exists. The story that follows is about the choice between these two values.

1.2. The history of the question. Given an irrational number $x$ as in (1.1), we consider the sum $S_x(t)$ of the terms with subscripts from 1 to $t$ in the continued fraction:

$$S_x(t) = x_1 + x_2 + \cdots + x_t.$$
A theorem on the connection between the value of the derivative $?'(x)$ and the limiting behaviour of the ratio $S_x(t)/t$ was proved in [5]. In particular, the equality $?'(x) = +\infty$ was shown to be related to whether the inequality $S_x(t)/t < \kappa_1$ holds for all sufficiently large $t$, where

$$\kappa_1 = \frac{2\log(1 + \sqrt{5})}{\log 2} - 2 = 1.3884 \ldots .$$  \hspace{1cm} (1.3)

Some theorems generalizing and improving the results in [5] on the derivative of the Minkowski function were proved in [6]–[9]. In particular, it was asked in [7] whether the limiting behaviour of $S_x(t)$ and $\kappa_1 t$ can be compared in one way or another in the case when $?'(x) = 0$ instead of $?'(x) = +\infty$. To state the corresponding theorem in [7], which was improved in [8], we define the real number

$$\kappa_4 = \sqrt{\frac{2\log(1 + \sqrt{5}) - 3\log 2}{\log 2}} = 0.7486 \ldots .$$  \hspace{1cm} (1.4)

**Theorem A** ([7], [8]). (i) Let $x$ be an irrational number in $(0, 1)$ such that the derivative $?'(x)$ exists and $?'(x) = 0$. Let $\psi = \psi(t)$ be a function assuming only positive real values and tending to $+\infty$ in a monotone and sufficiently slow manner that we asymptotically have

$$\psi(t) = o\left(\frac{t}{\log t}\right).$$

Then, for all sufficiently large $t$ (depending on $x$ and $\psi$), we have

$$\max_{u \leq t} (S_x(u) - \kappa_1 u) \geq \kappa_4 \sqrt{t \log t} \left(1 - \frac{1}{\psi(t)}\right).$$

(ii) There is an irrational number $x \in (0, 1)$ such that $?'(x) = 0$ and the following inequality holds for all sufficiently large $t$:

$$S_x(t) - \kappa_1 t \leq 2\sqrt{2} \cdot \kappa_4 \sqrt{t \log t} \left(1 + 2^{5\log \log t}\right).$$

A similar question was considered in [8] in the case when the terms are bounded. Namely, for every $n \geq 5$, let $E_n$ be the set of all irrational numbers $x$ in the interval $(0, 1)$ such that all their terms $x_j$ are bounded above by $n$:

$$x_j \leq n \quad \forall j \in \mathbb{N}.$$  

In this case we write $\kappa_1^{(n)}$ for the constant analogous to $\kappa_1$. Its value is

$$\kappa_1^{(n)} = \frac{(n + 1) \log \Phi - \log \mu_n}{(n - 1) \log \sqrt{2} - \log \mu_n + 2 \log \Phi}, \quad n \geq 5,$$  \hspace{1cm} (1.5)

where $\Phi = (1 + \sqrt{5})/2$ and $\mu_n = (n + 2 + \sqrt{n^2 + 4n})/2$.  


Remark 1.1. An easy calculation using (1.3), (1.4) and (1.5) shows that for every $n \geq 5$ there is a real number $\Theta = \Theta_n$ with $|\Theta| \leq 1$ and

$$
\kappa_1^{(n)} = \kappa_1 + \frac{2(\kappa_4)^2 \log n}{n} + \Theta \frac{10 \log^2 n}{n^2}.
$$

(1.6)

The following theorem is analogous to Theorem A.

**Theorem B** ([8], [9]). (i) Suppose that $x \in E_n$, where $n \geq 5$, the derivative $?'(x)$ exists and $?'(x) = 0$. Then, for all sufficiently large $t$ (depending on the value of $n$), at least one of the following bounds holds:

$$
\max_{u \leq t} (S_x(u) - \kappa_1^{(n)} u) > \frac{1}{13} \sqrt{t}
$$

if $n \geq 42$, or

$$
\max_{u \leq t} (S_x(u) - \kappa_1^{(n)} u) > \frac{2}{7n^{2.5}} \sqrt{t}
$$

otherwise.

(ii) For every given $n \geq 5$ there is a number $x \in E_n$ such that $?'(x) = 0$ and the following inequality holds for all sufficiently large $t$:

$$
\max_{u \leq t} (S_x(u) - \kappa_1^{(n)} u) \leq S_x(t) - \kappa_1^{(n)} t \leq \left(2^{2/3} n^{2/3} + 21 n^{1/3}\right) \sqrt{t}.
$$

(1.8)

In a similar vein, the following theorem enables us to compare the limiting behaviour of $S_x(t)$ and $\kappa_2 t$ in the case when $?'(x) = +\infty$. Here the constants $\kappa_2$ and $\lambda_n$ (for $n \in \mathbb{N}$) have the following values:

$$
\kappa_2 = \frac{4 \log \lambda_5 - 5 \log \lambda_4}{\log \lambda_5 - \log \lambda_4 - \log \sqrt{2}} = 4.401 \ldots,
$$

(1.9)

$$
\lambda_n = 0.5(n + \sqrt{n^2 + 4}).
$$

(1.10)

**Theorem C** ([7]). (i) Let $x$ be an irrational number in $(0, 1)$ such that the derivative $?'(x)$ exists and $?'(x) = +\infty$. Then, for all sufficiently large $t$,

$$
\max_{u \leq t} (\kappa_2 u - S_x(u)) \geq 10^{-8} \sqrt{t}.
$$

(1.11)

(ii) There is an irrational number $x \in (0, 1)$ such that $?'(x) = +\infty$ and, for all sufficiently large $t$,

$$
\max_{u \leq t} (\kappa_2 u - S_x(u)) \leq \kappa_2 t - S_x(t) \leq 200 \sqrt{t}.
$$

(1.12)

Remark 1.2. The bounds in (1.7) and (1.8) are of the same order with respect to $t$ (namely, $O(\sqrt{t})$). However, their dependence on $n$ is different, even in the sense of monotonicity: the coefficient of $\sqrt{t}$ is an absolute constant in (1.7) and an increasing function of $n$ in (1.8). Making these coefficients closer to each other is an interesting open problem. Another unsolved problem is the non-coincidence of the constant factors of $\sqrt{t}$ in (1.11) and (1.12) (they are equal to $10^{-8}$ and 200). In the present paper we strengthen the inequalities (1.7), (1.8), (1.11) and (1.12). In particular, we replace the coefficient of $\sqrt{t}$ in (1.8) by a constant independent of $n$. 
1.3. Our main results. We introduce the following constants: \( \mu_n' = (n + 2 - \sqrt{n^2 + 4n})/2 \) is the conjugate of \( \mu_n \),

\[
c^{(n)} = \left( \frac{\Phi^2}{\mu_n} \right)^{1/(n-1)} \sqrt{2}, \quad \gamma_n = \frac{2\Phi + \mu_n' - 2\Phi \mu_n' + n\Phi^2}{\sqrt{5n^2 + 20n}}. \tag{1.13}
\]

Note that \( c^{(n)} > 1.09 \) for \( n \geq 5 \) and \( c^{(n)} \) tends to \( \sqrt{2} \) as \( n \) increases. Moreover, \( \gamma_n > 1.06 \) for \( n \geq 5 \) and \( \gamma_n \) tends to \( \Phi^2/\sqrt{5} \) as \( n \) increases. We also need the following quantities: \( \lambda = \sqrt{2\lambda_4/\lambda_5} \approx 1.1537043 \),

\[
\gamma = \frac{2 + \sqrt{5}}{\sqrt{20}} \cdot \frac{5 + \sqrt{29}}{2\sqrt{29}} \left( 1 + [\overline{4,\overline{5}}] \right)^2 \approx 0.9982728, \tag{1.14}
\]

where the bar means the infinitely repeated period of a continued fraction. Note that \( \log \gamma < 0 \). The main purpose of our investigation is to prove the following two theorems.

**Theorem 1.1.** (i) Suppose that \( n \geq 5 \), \( x \in E_n \), the derivative \( ?'(x) \) exists and \( ?'(x) = 0 \). Then, for every sufficiently small positive \( \varepsilon \), the following inequality holds for all sufficiently large \( t \) (depending on \( x \) and \( \varepsilon \)):

\[
\max_{u \leq t} (S_x(u) - \kappa_1^{(n)} u) \geq \sqrt{\frac{4(\kappa_1^{(n)} - 1)(n + 1 - 2\kappa_1^{(n)})\log(\gamma_n - \varepsilon)}{3(n - 1)\log c^{(n)}}} \sqrt{t}. \tag{1.15}
\]

In particular, the following inequality holds for all sufficiently large \( t \) under the same conditions provided that \( n \) is sufficiently large:

\[
\max_{u \leq t} (S_x(u) - \kappa_1^{(n)} u) > 0.4852 \sqrt{t}. \tag{1.16}
\]

(ii) For every \( n \geq 5 \) there is an \( x \in E_n \) such that \( ?'(x) = 0 \) and, for all sufficiently large \( t \), at least one of the following inequalities holds:

\[
\max_{u \leq t} (S_x(u) - \kappa_1^{(n)} u) \leq 4.78 \sqrt{t} \tag{1.17}
\]

if \( n \) is sufficiently large, and

\[
\max_{u \leq t} (S_x(u) - \kappa_1^{(n)} u) \leq 7 \sqrt{t} \tag{1.18}
\]

otherwise.

**Theorem 1.2.** (i) Let \( x \) be an irrational number of the form (1.1) such that the derivative \( ?'(x) \) exists and \( ?'(x) = +\infty \). Let \( \varepsilon < 10^{-20} \) be an arbitrarily small positive number. Then, for every sufficiently large \( t \), we have

\[
\max_{u \leq t} (\kappa_2 u - S_x(u)) \geq \frac{2}{\sqrt{3}} \sqrt{\frac{-(5 - \kappa_2)(\kappa_2 - 4)\log(\gamma + \varepsilon)}{\log \lambda}} \sqrt{t} > 0.06222 \sqrt{t}. \tag{1.19}
\]

(ii) There is an irrational \( x \in (0, 1) \) such that \( ?'(x) = +\infty \) and, for all sufficiently large \( t \), we have

\[
\max_{u \leq t} (\kappa_2 u - S_x(u)) \leq 0.26489 \sqrt{t}. \tag{1.20}
\]
Remark 1.3. Using the notation (1.10), we can rewrite the number $\gamma$ in (1.14) in the form
\[ \gamma = \frac{2\lambda_4 \lambda_5 - 4\lambda_5 - 5\lambda_4 + 22}{\sqrt{580}}. \]

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§ 2. The main notation and starting lemmas

Let $A_t = (a_1, a_2, \ldots, a_t)$ be an arbitrary sequence of $t$ positive integers, where $t \geq 0$ is an integer. We write $[A_t]$ for the finite continued fraction
\[ [A_t] = [a_1, a_2, \ldots, a_t] = \frac{1}{a_1 + \frac{1}{a_2 + \cdots + \frac{1}{a_t}}} \]  

(2.1)

If $t > 0$, we write $\widetilde{A}_t$, $(A_t)^-$ and $(A_t)_-$ for the finite sequences $(a_t, a_{t-1}, \ldots, a_1)$, $(a_1, a_2, \ldots, a_{t-1})$ and $(a_2, a_3, \ldots, a_t)$ respectively. (If $t = 0$, then $A_t$ and $\widetilde{A}_t$ are empty.) With reference to the value of the continued fraction (2.1), the numbers
\[ q_0 = 1, \quad q_1 = a_1, \quad q_2 = \langle A_2 \rangle = a_2a_1 + 1, \quad \ldots, \quad q_t = \langle A_t \rangle = a_tq_{t-1} + q_{t-2}, \]

are called the denominators of the convergents (see [10]) or the continuants of the finite sequences $\emptyset$, $A_1, A_2, \ldots, A_t$ (see [11]). In particular, for the empty sequence $\emptyset = A_0$, we put by definition $\langle \emptyset \rangle = 1$ and $\left[ \langle \emptyset \rangle \right] = \langle \emptyset^- \rangle = \langle \emptyset_- \rangle = 0$. The equalities $[A_t] = \langle (A_t)^- \rangle/\langle A_t \rangle$ and $[\widetilde{A}_t] = \langle (A_t)^- \rangle/\langle A_t \rangle$ still hold when $t = 0$. An example of using this notation is the known formula (see [11])
\[ \langle A \rangle \langle B \rangle \leq \langle A, B \rangle = \langle A \rangle \langle B \rangle + \langle A^- \rangle \langle B^- \rangle = \langle A \rangle \langle B \rangle (1 + [\widetilde{A}][B]) \leq 2\langle A \rangle \langle B \rangle \]  

(2.2)

for finite sequences $A$ and $B$. Note that applying (2.2) twice, we arrive at the relation
\[ a \langle A \rangle \langle B \rangle \leq \langle A, a, B \rangle = \langle A \rangle \langle B \rangle (a + [\widetilde{A} + [B])] \leq \langle A \rangle \langle B \rangle (a + 2). \]  

(2.3)

In particular, an application of (2.3) enables us to obtain the bound
\[ \frac{a}{a + 1} \leq \frac{\langle A, a, B \rangle}{\langle A, a + 1, B \rangle} \leq \frac{a + 2}{a + 3}. \]  

(2.4)

In what follows we always assume that every irrational number $x \in (0,1)$ under consideration is associated with an infinite sequence of positive integers $(x_1, x_2, \ldots, x_t, \ldots)$ by the formula (1.1). For every $t \in \mathbb{N}$ we denote the finite subsequence $(x_1, x_2, \ldots, x_t)$ by $X_t$. We also write $a$, $b$ and so on for the irrational numbers in the interval $(0,1)$ whose infinite continued fraction expansions are of the form $a = [a_1, a_2, \ldots]$, $b = [b_1, b_2, \ldots]$ and so on. By capital letters with
positive integer subscripts we always denote the initial intervals of these expansions; for example, \( A_s = (a_1, a_2, \ldots, a_s) \) and \( B_t = (b_1, b_2, \ldots, b_t) \), where the subscripts of \( A_s \) and \( B_t \) are equal to their number of elements. Conversely, given any finite sequences \( A_s, B_t, \ldots \), we have in mind the existence of irrational numbers \( a = [a_1, a_2, \ldots] \), \( b = [b_1, b_2, \ldots] \), \ldots in \((0, 1)\) such that the initial intervals of their continued fraction expansions are \( A_s, B_t, \ldots \) respectively.

Given a positive integer \( \nu \leq t \), we denote the sum of the first \( \nu \) elements of \( A_t, B_t, \ldots \) by \( S_\nu(A), S_\nu(B), \ldots \). When \( n \geq 5 \), we also write \( \varphi_n(a, \nu), \varphi_n(b, \nu), \ldots \) for the differences \( \varphi_n(a, \nu) = S_\nu(a) - \kappa_1^{(n)} \nu, \varphi_n(b, \nu) = S_\nu(b) - \kappa_2^{(n)} \nu, \ldots \) and so on.

Finally, we define \( \varphi_{a,2}(\nu), \varphi_{b,2}(\nu), \ldots \) by the formulae \( \varphi_{a,2}(\nu) = \kappa_2 \nu - S_\nu(a), \varphi_{b,2}(\nu) = \kappa_2 \nu - S_\nu(b), \ldots \). The following lemma is an obvious corollary of this notation and the inequality \( 4 < \kappa_2 < 5 \).

**Lemma 2.1.** For an arbitrary finite sequence \( A_t \) and any \( \nu \in \{2, 3, \ldots, t\} \), the inequality \( \varphi_{a,2}(\nu) > \varphi_{a,2}(\nu - 1) \) holds if and only if \( a_\nu < 4 \). Equivalently, \( \varphi_{a,2}(\nu) < \varphi_{a,2}(\nu - 1) \) if and only if \( a_\nu > 5 \).

The present paper is a continuation of [7]–[9] and shares their background ideas. In particular, our investigation of the values of the derivative of the Minkowski function is based on the following lemmas, whose first analogues were proved in [6].

**Lemma 2.2** ([8]). Suppose that \( \mathbf{x} \in (0, 1) \) is irrational and

\[
\lim_{t \to +\infty} \frac{\langle X_{t-1} \rangle}{\sqrt{2} S_x(t)} = +\infty. \tag{2.5}
\]

Then the derivative \(?'(\mathbf{x})\) exists and is equal to infinity.

Conversely, if the derivative \(?'(\mathbf{x})\) at an irrational point \( \mathbf{x} \in (0, 1) \) exists and is equal to infinity, then

\[
\lim_{t \to +\infty} \frac{\langle X_t \rangle}{\sqrt{2} S_x(t)} = +\infty. \tag{2.6}
\]

**Lemma 2.3** ([8]). For every \( n \geq 5 \) and any \( \mathbf{x} \in \mathbf{E}_n \), the derivative \(?'(\mathbf{x})\) exists if and only if one of the following relations holds:

\[
\frac{\langle X_t \rangle}{\sqrt{2} S_x(t)} \to 0 \quad \text{or} \quad \frac{\langle X_t \rangle}{\sqrt{2} S_x(t)} \to +\infty \quad \text{as} \quad t \to +\infty. \tag{2.7}
\]

Moreover, \(?'(\mathbf{x})\) is equal to zero (resp. infinity) if the first (resp. second) condition in (2.7) holds.

### § 3. Two-sided estimates of periodic continuants

In what follows we need estimates for the continuants

\[
\langle 1, 1, \ldots, 1 \rangle, \quad \langle n, n, \ldots, n \rangle, \quad \langle 1, n, 1, n, \ldots, 1, n \rangle.
\]

To obtain them, we recall Binet's formula

\[
\langle 1, 1, \ldots, 1 \rangle = \frac{1}{\sqrt{5}} \phi^{l+1} + \frac{1}{\sqrt{5}} (-1)^{l+1} \phi^{l+1}, \tag{3.1}
\]

\( l \) digits

\( l \) pairs

\( k \) numbers
and its generalizations
\[
\langle 1, n, 1, n, \ldots, 1, n \rangle_{\text{pairs}} = \frac{\sqrt{n^2 + 4n + n}}{2\sqrt{n^2 + 4n}} (\mu_n)^l + \frac{\sqrt{n^2 + 4n - n}}{2\sqrt{n^2 + 4n}} (\mu'_n)^l,
\]
(3.2)
\[
\langle n, n, \ldots, n \rangle_{\text{numbers}} = \frac{n + \sqrt{n^2 + 4}}{2\sqrt{n^2 + 4}} \left( \frac{n + \sqrt{n^2 + 4}}{2} \right)^l + \frac{\sqrt{n^2 + 4} - n}{2\sqrt{n^2 + 4}} \left( \frac{n - \sqrt{n^2 + 4}}{2} \right)^l,
\]
(3.3)
which hold for all positive integers \( l \) and \( n \). The second summands in (3.1)–(3.3) tend exponentially to zero. Discarding these remainder terms, we can write (3.1)–(3.3) in the form of two-sided inequalities with only principal terms:
\[
\Phi^{l-1} \leq \Phi \frac{\sqrt{5} \Phi^l (1 - 2^{-l})}{\sqrt{5}} \leq \langle 1, 1, \ldots, 1 \rangle_{\text{digits}} \leq \Phi \frac{\sqrt{5} \Phi^l (1 + 2^{-l})}{\sqrt{5}} \leq \Phi^l,
\]
(3.4)
\[
\frac{\sqrt{n^2 + 4n + n}}{2\sqrt{n^2 + 4n}} (\mu_n (1 - 2^{-l})) \leq \langle 1, n, 1, n, \ldots, 1, n \rangle_{\text{pairs}} \leq \frac{\sqrt{n^2 + 4n + n}}{2\sqrt{n^2 + 4n}} (\mu_n (1 + 2^{-l}))
\]
(3.5)
\[
\frac{\lambda_n}{\sqrt{n^2 + 4}} (\lambda_n - n^{-l})^l \leq \langle n, n, \ldots, n \rangle_{\text{numbers}} \leq \frac{\lambda_n}{\sqrt{n^2 + 4}} (\lambda_n + n^{-l})^l.
\]
(3.6)
The following simplified version of (3.6) can be deduced from it:
\[
(\lambda_n)^{l-1} < \langle n, n, \ldots, n \rangle_{\text{numbers}} < (\lambda_n)^l.
\]
(3.7)

§ 4. Inequalities with pulverized continuants

Consider a continuant \( \beta = \langle X, y, Z \rangle \), where \( X \) and \( Z \) are finite sequences of numbers and \( y \) is a positive integer. Put
\[
\beta_y = [\overleftarrow{X}] + [Z].
\]
(4.1)
We stress that the quantity (4.1) depends not on the value of \( y \), but on its location in the continuant \( \beta \). It is assumed that \( n \geq 5 \).

**Definition 4.1.** Let \( m \) be an integer such that
\[
1 < m < n,
\]
and let \( A^{(0)}, A^{(1)}, \ldots, A^{(m)} \) be finite sequences of numbers in \( \{1, 2, \ldots, n\} \) (possibly repeating) consisting of at least 9 elements. We consider numbers \( z_1, z_2, \ldots, z_m \) with
\[
z_1 = z_2 = \cdots = z_m = m,
\]
define a finite sequence
\[
A = (A^{(0)}, z_1, A^{(1)}, z_2, A^{(2)}, \ldots, z_{m-1}, A^{(m-1)}, z_m, A^{(m)})
\]
(4.2)
and put $\beta = \langle A \rangle$. Let $\beta_{z_1}, \beta_{z_2}, \ldots, \beta_{z_m}$ be the numbers defined in (4.1) for the continuant $\beta$. Suppose that the expressions

$$
\beta_{\text{max}} = \max\{\beta_{z_1}, \beta_{z_2}, \ldots, \beta_{z_m}\}, \quad \beta_{\text{min}} = \min\{\beta_{z_1}, \beta_{z_2}, \ldots, \beta_{z_m}\}
$$

satisfy the inequality

$$
\beta_{\text{max}} - \beta_{\text{min}} \leq 0.1. \tag{4.4}
$$

Then the passage from one continuant to another by the rule

$$
\langle A \rangle \mapsto \langle A^{(0)}, 1, A^{(1)}, m + 1, A^{(2)}, m + 1, A^{(3)}, \ldots, m + 1, A^{(m-1)}, m + 1, A^{(m)} \rangle \tag{4.5}
$$

is referred to as a pulverization of the sequence $A$ in (4.2).

**Lemma 4.1.** Suppose that $B_t$ is obtained from $A_t$ by pulverization. Then, for every positive integer $\nu \leq t$, we have

$$
\langle A_\nu \rangle \geq \langle B_\nu \rangle, \tag{4.6}
$$

$$
0 \leq S_a(\nu) - S_b(\nu) < n. \tag{4.7}
$$

**Proof.** The upper bound in (4.7) for every $\nu$ follows immediately since $z_1 < n$.

We now prove (4.6). Let $l$ and $p$ be such that $a_l = z_1$, $a_p = z_m$ and $l < p$. We first assume that $\nu < l$. Then (4.6) holds with equality: its left-hand side and right-hand side coincide.

It remains to prove (4.6) for $\nu \geq l$. To do this, we represent the pulverization as a process of $m$ steps by the formulae $(z_1 \mapsto 1)$ at the first step, $(z_2 \mapsto z_2 + 1)$ at the second step, and so on until $(z_m \mapsto z_m + 1)$ at the $m$th step, as in (4.5). To estimate the possible changes of $\beta_{z_1}, \beta_{z_2}, \ldots, \beta_{z_m}$ in the course of this process, we take three facts into account. First, a real number is determined by the convergent with denominator $Q$ of its continued fraction with accuracy $\pm Q^{-2}$. Second, a lower bound for the denominator $Q$ of a continued fraction of length $l$ is given by the first inequality in (3.4). Third, there are at least 9 elements between $z_j$ and $z_{j+1}$ in the sequence $A_t$. Since the quantity $\beta_{z_k}$ ($k = 1, 2, \ldots, m$) is the sum of two continued fractions, its total change in the course of the process does not exceed

$$
2(\Phi^{-18} + \Phi^{-36} + \cdots) < 0.001. \tag{4.8}
$$

To estimate the change of the continuants in (4.5) at various steps of the process, we use the inequality (2.4) and make the following remarks. First, when increasing a term equal to $z_k = m$ by 1, we increase the continuant by a factor of at most

$$
\frac{\langle A, m + 1, B \rangle}{\langle A, m, B \rangle} = \frac{m + 1 + \left\lceil A \right\rceil + B}{m + \left\lceil A \right\rceil + B} \leq \exp\left(\frac{1}{m + \left\lceil A \right\rceil + B}\right)
$$

and, continuing this chain of inequalities and using (4.8), we have

$$
\frac{\langle A, m + 1, B \rangle}{\langle A, m, B \rangle} \leq \exp\left(\frac{1}{m + \beta_{z_k} - 0.001}\right). \tag{4.9}
$$
Second, when decreasing a term equal to \( z_1 = m \) by \( (z_1 - 1) \), we decrease the continuant by a factor of at least

\[
\langle A, z_1, B \rangle = \frac{m + [\hat{A}] + [B]}{1 + [\hat{A}] + [B]} \geq \frac{m + \beta_{z_1} + 0.001}{1 + \beta_{z_1} + 0.001}
\]  

(4.10)

because \( f(y) = (m + y)/(1 + y) \) is a decreasing function of \( y = [\hat{A}] + [B] \) for every fixed \( m \).

Dividing the bound (4.10) by the product of the bounds (4.9) over \( k = 2, 3, \ldots, m \), we obtain the inequality

\[
\langle A, \nu \rangle \langle B, \nu \rangle \geq \frac{m + \beta_{z_1} + 0.001}{1 + \beta_{z_1} + 0.001} \exp\left(\sum_{k=2}^{m} \frac{1}{m + \beta_{z_k} - 0.001}\right)^{-1}.
\]  

(4.11)

Each term of the sum in (4.11) is a decreasing function of \( \beta_{z_k} \). Moreover, \( \beta_{z_k} \geq \beta_{z_1} - 0.1 \) for every \( k = 2, 3, \ldots, m \) in view of (4.4). Therefore, continuing the chain of inequalities in (4.11), we have

\[
\langle A, \nu \rangle \langle B, \nu \rangle \geq \frac{m + \beta_{z_1} + 0.001}{1 + \beta_{z_1} + 0.001} \exp\left(\frac{m - 1}{m + \beta_{z_1} - 0.101}\right).
\]  

(4.12)

The right-hand side of (4.12) is a decreasing function of \( \beta_{z_1} \) for a fixed \( m \) (this can be verified by calculating its derivative). Hence it attains its minimum value at the maximum value of \( \beta_{z_1} = 2 \):

\[
\langle A, \nu \rangle \langle B, \nu \rangle \geq \frac{m + 2 + 0.001}{3 + 0.001} \exp\left(\frac{1 - m}{m + 2 - 0.101}\right) = \frac{m + 2.001}{3.001} \exp\left(\frac{1 - m}{m + 1.899}\right). 
\]  

(4.13)

The result of the inequality (4.13) is an increasing function of \( m \), which can again be verified by taking its derivative with respect to \( m \) (and thus temporarily regarding \( m \geq 2 \) as a continuous quantity). Hence it attains its minimum value at \( m = 2 \). Therefore we see from (4.13) that

\[
\langle A, \nu \rangle \langle B, \nu \rangle \geq \frac{2 + 2.001}{3.001} \exp\left(\frac{1 - 2}{2 + 1.899}\right) = \frac{4.001}{3.001} \exp\left(\frac{-1}{3.899}\right) > 1.03 > 1.
\]  

(4.14)

This proves (4.6) in view of (4.14). \( \square \)

In the following lemma, the sequence \( B_t = (b_1, b_2, \ldots, b_t) \) is obtained from \( X_t \) by a deterministic algorithm.

**Lemma 4.2.** For every \( x \in E_n \) (where \( n \geq 5 \)) and every integer \( t > 0 \) there is a finite sequence \( B_t \) consisting only of 1s and \( n \)s such that the following inequalities hold for all positive integers \( \nu \leq t \):

\[
\langle X, \nu \rangle \geq \langle B, \nu \rangle, \tag{4.15}
\]

\[
0 \leq S_X (\nu) - S_B (\nu) \leq 400n(n + n^2) \leq n^7. \tag{4.16}
\]
Proof. We subject the sequence $X_t$ to a number of transformations (mostly pulverizations) resulting in a sequence $B_t$ with the desired properties. To prove (4.16), we shall derive analogues of (4.6) and (4.7) for several pulverizations (in contrast to a single pulverization in Lemma 4.1).

Let $X' = (x_{i_1}, x_{i_2}, \ldots, x_{i_k})$ be the sequence of all elements of $X_t$ whose values lie in the interval $[2, n - 1]$. We represent $X'$ as the union of ten subsequences $X^{(\mu)} = (x_{j_1}^{(\mu)}, x_{j_2}^{(\mu)}, \ldots, x_{j_p}^{(\mu)}), \mu = 0, 1, 2, \ldots, 9$, each of which consists of the elements of $X'$ whose label is congruent to $\mu$ modulo 10. Choose one of these $\mu$.

Writing $\beta = \langle X_t \rangle$, we define $\beta_x$ by the formula (4.1) for every $x$ in $X^{(\mu)}$. Put $\beta_x = \beta_x^{(0)}$. This notation is needed because the algorithm to be applied to $X^{(\mu)}$ will change the values of $\beta_x$ compared to their initial values $\beta_x^{(0)}$. We recall that the quantities $\beta_{\max}$ and $\beta_{\min}$ were defined in (4.3).

For every $\gamma = 1, 2, \ldots, 39, 40$ we endow the interval $[(\gamma - 1)/20, \gamma/20)$ with the label $\gamma$. All these intervals together form a partition of $[0, 2)$. Hence every quantity $\beta_x^{(0)}$ belongs to one of them. We represent the sequence $X^{(\mu)}$ as the union of forty subsequences $X^{(\mu, \gamma)} = (x_{q_1}^{(\mu, \gamma)}, x_{q_2}^{(\mu, \gamma)}, \ldots, x_{q_p}^{(\mu, \gamma)}), \gamma = 1, 2, \ldots, 40$, in such a way that, for every element $x$ of each subsequence, the corresponding quantity $\beta_x$ belongs to the interval labelled by $\gamma$ (all the numbers $q_1, q_2, \ldots, q_{p'}$ are still congruent to $\mu$ modulo 10). Choose one of these $\gamma$. Note that for any elements $a$ and $b$ of $X^{(\mu, \gamma)}$ we have $|\beta_a - \beta_b| \leq 0.05$ by construction.

We represent the sequence $X^{(\mu, \gamma)}$ as the union of fewer than $n$ subsequences $X^{(\mu, \gamma, m)} = (x_{s_1}^{(\mu, \gamma, m)}, x_{s_2}^{(\mu, \gamma, m)}, \ldots, x_{s_p'\gamma, m})$, $m = 2, 3, \ldots, n - 1$, consisting of equal elements of numerical value $m$ (all the numbers $s_1, s_2, \ldots, s_{p'}$ are still congruent to $\mu$ modulo 10 and we still have $|\beta_a - \beta_b| \leq 0.05$ for any elements $a$ and $b$ of the sequence $X^{(\mu, \gamma, m)}$). Choose one of these $m$ (to begin with, the smallest one, say, $m = 2$) and assume that the elements of $X^{(\mu, \gamma, m)}$ are enumerated in ascending order:

$$1 \leq s_1 < s_2 < \cdots < s_{p'} \leq t.$$  

We also fix some values of $\mu$ and $\gamma$ and begin to iterate the following procedure. If the current sequence $X^{(\mu, \gamma, m)}$ still contains at least $n$ elements, then we choose any $n$ of them:

$$z_1 = x_{s_1}^{(\mu, \gamma, m)}, \quad z_2 = x_{s_2}^{(\mu, \gamma, m)}, \quad \cdots, \quad z_n = x_{s_n}^{(\mu, \gamma, m)}.$$  

Assume that the indices $s_1, s_2, \ldots, s_n$ are least possible. Take the first $m$ elements (4.17) and pulverize the sequence $X_t$. (Denote the resulting sequence again by $X_t$ and take the next step of the algorithm, and so on.) By Lemma 4.1, this step does not increase the continuant $\langle X_{t'} \rangle$. By the argument in the proof of Lemma 4.1, the following inequality holds after any number of steps:

$$\beta_{\max} - \beta_{\min} \leq |\beta_a - \beta_a^{(0)}| + |\beta_a^{(0)} - \beta_b^{(0)}| + |\beta_b^{(0)} - \beta_b| < 0.001 + 0.05 + 0.001 < 0.1,$$

where $\beta_a = \beta_{\max}$ and $\beta_b = \beta_{\min}$ for some $a$ and $b$, so that the pulverization is well defined.

The inequality (4.15) for the result of this multi-step algorithm also follows from Lemma 4.1, which says that (4.6) holds for each individual step.
We now prove (4.7). To do this, we recall that a single pulverization of $X_t$ replaces the numbers $z_1, z_2, \ldots, z_m$, which are equal to $m$, by the quantities

$$1, \ z_2 + 1, \ z_3 + 1, \ \ldots, \ z_{m-1} + 1, \ z_m + 1$$

(4.18)

respectively, none of which is equal to $m$. Hence the elements (4.18) do not belong to the set $X^{(\mu, \gamma, m)}$ and, therefore, do not participate in the subsequent steps at this stage of the algorithm.

Thus, in the notation (4.17), $s_1$ at every step (beginning with the second) is not smaller than $s_{m+1}$ at the previous step. Therefore, independently of the value of $\nu$, the interval $[1, \nu]$ can partially contain the set of indices $s_1, s_2, \ldots, s_m$ at only one step. At the other steps, this interval contains either all these indices, or none of them. The steps corresponding to the last two of these three cases do not influence the validity of the bound (4.7) since they do not change the sum $S_x(\nu)$. In the first case, the bound (4.7) holds by Lemma 4.1.

Consider the remaining elements of $X^{(\mu, \gamma, m)}$, choose the first $m$ of them, and pulverize the sequence obtained at the previous step, and so on. To get rid of the last (at most $n$) elements of $\{2, 3, \ldots, n - 1\}$ (if $X^{(\mu, \gamma, m)}$ contains any), simply replace each of them by 1. This change decreases all partial sums by less than $n^2$, and the continuants also decrease. There is an appropriate correction in (4.16).

We then pass to the next value of $m$, and so on, until we reach the maximum $m = n - 1$. Hence the total number of values needed of $m$ is less than $n$. We also take into account the fact that the algorithm and the related calculations are to be applied for all $\mu = 0, 1, 2, \ldots, 9$ and $\gamma = 1, 2, \ldots, 40$. Therefore all the changes of partial sums are to be multiplied by $400n$. The resulting finite sequence is denoted by $B_t$. □

§ 5. A preliminary lower bound for $\varphi_x, 2(t)$

We note that $\left(\frac{\lambda_4}{\lambda_5}\right)^5 > 1$.

Lemma 5.1. Let $m$ and $k$ be positive integers with $k = m + 1$ and

$$mt \leq S_a(t) < kt.$$ 

Then we have the bound

$$\langle A_t \rangle \leq 4(\lambda_m)^{r_m}(\lambda_k)^{r_k},$$

(5.1)

where the non-negative integers $r_m$ and $r_k$ satisfy the equalities

$$S_a(t) = mr_m + kr_k, \quad r_m + r_k = t.$$ 

(5.2)

In particular, if

$$4t < S_a(t) < 5t,$$ 

(5.3)

then we have

$$\langle A_t \rangle \leq 4 \left(\frac{(\lambda_4)^5}{(\lambda_5)^4}\right)^{\varphi_{a, 2}(t)/\kappa_2} \sqrt{2} S_a(t).$$ 

(5.4)
Proof. It was shown in Theorem 5 of [12] that

\[
\langle A_t \rangle \leq \begin{cases} 
\langle m, m, \ldots, m \rangle & \text{if } S_a(t) \equiv 0 \pmod{t}, \\
\frac{r_m}{r_m} \text{ numbers} & \langle k, m, m, \ldots, m, k, k, \ldots, k \rangle \text{ otherwise.}
\end{cases}
\]  

To be definite, assume that \( S_a(t) \not\equiv 0 \pmod{t}. \) Then we consider the continuant in the lower row of (5.5). Interchange the first \( k \) and all subsequent \( m. \) Using (2.2), one can show that this operation (henceforth referred to as a ‘reflection’) can change the continuant by a factor of at most two. Hence, again using (2.2), we have

\[
\langle A_t \rangle \leq 2\langle m, m, \ldots, m, k, k, \ldots, k \rangle \leq 4\langle m, m, \ldots, m \rangle \langle k, k, \ldots, k \rangle.
\]

Applying (3.7), we continue the chain of inequalities (5.6) and obtain (5.1):

\[
\langle A_t \rangle \leq \frac{r_m}{r_m} \text{ numbers} \langle k, k, \ldots, k \rangle \leq 4(\lambda_m)^{r_m}(\lambda_k)^{r_k}.
\]  

But if \( S_a(t) \equiv 0 \pmod{t}, \) then we similarly obtain (5.1) from the upper row of (5.5):

\[
\langle A_t \rangle \leq \langle m, m, \ldots, m \rangle \langle k, k, \ldots, k \rangle - 4(\lambda_m)^{r_m}(\lambda_k)^{r_k}.
\]

In particular, under the condition (5.3) we have \( m = 4, k = 5. \) Hence the equalities (5.2) take the form \( t = r_4 + r_5, \) \( S_a(t) = 4r_4 + 5r_5. \) Regarding them as a system of linear equations in \( r_4 \) and \( r_5 \) and solving this system, we obtain

\[
r_4 = 5t - S_a(t), \quad r_5 = S_a(t) - 4t.
\]

Substituting (5.8) into the inequality (5.1) (already proved), we obtain (5.4):

\[
\langle A_t \rangle \leq 4(\lambda_4)^{5t-S_a(t)}(\lambda_5)^{S_a(t)-4t} = 4\left(\frac{\lambda_4}{\lambda_5}\right)^{2t-S_a(t)}/\kappa_2 \sqrt{2} S_a(t).
\]

The equality in (5.9) follows from the notation (1.9). \( \square \)

In what follows we write the inequality (1.19) in Theorem 1.2 in the form

\[
\max_{u \leq t}(\kappa_2 u - S_x(u)) = \max_{u \leq t} \varphi_{x, 2}(u) \geq M_2 \sqrt{t},
\]

where the constant \( M_2 > 0 \) is undetermined at the moment. In view of Theorem C, we can assume that \( 10^{-8} \leq M_2 \leq 200. \) In the argument that follows, we mostly assume that the inequality (5.10) does not hold. We shall prove that this assumption leads to a contradiction for sufficiently small \( M_2. \) The supremum (rounded down) of these \( M_2 \) is the quantity occurring in the statement of Theorem 1.2.

Lemma 5.2. Let \( x \) be an irrational number in \((0, 1)\) such that the derivative \( ?'(x) \) exists and is equal to infinity. Suppose that (5.10) does not hold for some infinite sequence \( T^{(2)} \) consisting of positive integers \( t. \) Then the inequality \( \varphi_{x, 2}(t) \geq 0 \) holds for all sufficiently large elements \( t \in T^{(2)}. \)
Proof. Consider the subsequence of all \( t \) in \( \mathcal{T}^{(2)} \) such that \( S_X(t) \geq 5t \). This inequality implies that \( k \geq m \geq 5 \). Therefore,

\[
\frac{\lambda_k}{\sqrt{2}^k} < \frac{\lambda_m}{\sqrt{2}^m} \leq \frac{\lambda_5}{\sqrt{2}^5} = 0.918 \ldots < 0.92. \tag{5.11}
\]

Put \( A_t = X_t \) and apply (5.7). As \( t \) tends to infinity, so does the maximum of \( r_m \) and \( r_k \). Suppose that the inequality \( S_a(t) \geq 5t \) holds for arbitrarily large \( t \). Then dividing (5.1) by \( \sqrt{2}^{S_a(t)} \), we deduce from (5.11) that

\[
\frac{\langle A_t \rangle}{\sqrt{2}^{S_a(t)}} \leq 4 \left( \frac{\lambda_m}{\sqrt{2}^m} \right)^{r_m} \left( \frac{\lambda_k}{\sqrt{2}^k} \right)^{r_k} < 4 \left( \frac{\lambda_5}{\sqrt{2}^5} \right)^{\max\{r_m,r_k\}} < 4(0.92)^{\max\{r_m,r_k\}} \to 0
\]

as \( t \) tends to infinity. This contradicts the condition \( ?'(x) = +\infty \). Thus, if the derivative \( ?'(x) \) exists and is equal to infinity, then the inequality \( S_X(t) < 5t \) holds for all sufficiently large \( t \).

Since (5.10) does not hold, we have \( S_X(t) > 4t \). This enables us to use (5.4). It follows from Lemma 2.2 that \( \langle X_t \rangle/\sqrt{2}^{S_X(t)} \to +\infty \) as \( t \to \infty \). Therefore we have the chain of inequalities

\[
4 \leq \frac{\langle X_t \rangle}{\sqrt{2}^{S_X(t)}} \leq 4 \left( \frac{\lambda_4}{\lambda_5} \right)^{\varphi_{x,2}(t)/\kappa_2} \tag{5.12}
\]

If, in addition, \( \varphi_{x,2}(t) < 0 \) for infinitely many values of \( t \), then the lower bound in (5.12) becomes greater than the upper bound for these \( t \). This contradiction shows that \( \varphi_{x,2}(t) \geq 0 \) for all sufficiently large \( t \). \( \square \)

§ 6. Unit variation

Let \( A, B, C \) be finite (possibly empty) sequences of positive integers. We consider an integer \( \tau \geq 4 \). Given any \( a \in \mathbb{N} \) with \( a \leq \tau - 1 \), we put \( b = \tau - a \). Thus \( a + b = \tau \). We define a function

\[
F(a) = F_{A,B,C;\tau}(a) = \langle A, a, B, b, C \rangle = \langle A, a, B, \tau - a, C \rangle = \langle \overrightarrow{C}, \tau - a, \overrightarrow{B}, a, \overrightarrow{A} \rangle. \tag{6.1}
\]

The last equality in (6.1) emphasizes the fact that the location of \( a \) and \( b \) in the original sequence may be arbitrary. Each of them may be to the left or to the right of the other.

Definition 6.1. Let \( i \in \{\pm 1\} \) be an integer of modulus 1, and let \( a \) and \( b \) be any positive integers with \( a + i \geq 1 \) and \( b - i \geq 1 \). Then the change of continuants by the rule

\[
\langle A, a, B, b, C \rangle \mapsto \langle A, a + i, B, b - i, C \rangle
\]

is referred to as a unit variation [7].

In other words, a unit variation is the passage from the quantity \( F(a) \) defined in (6.1) to \( F(a - 1) \) or \( F(a + 1) \) provided that \( a - 1 \geq 1 \) or \( b - 1 \geq 1 \) respectively. The following lemma is useful in the study of properties of unit variations.
Lemma 6.1 ([7]). Suppose that the maximal element of the sequence \((A, B, C)\) does not exceed a positive integer \(n\). Assume that \(1 < a \leq b < n\) and at least one of the following properties holds: \(a < b\) or \([B] = [\overline{B}]\). Then

\[
F(a) \geq \left(1 + \frac{1}{16(n + 2)^3}\right) F(a - 1).
\] (6.2)

We recall that every irrational number \(x\) under consideration is associated with an infinite sequence of positive integers \((x_1, x_2, \ldots, x_t, \ldots)\) by the formula (1.1). The subsequence of \(t\) initial elements is denoted by \(X_t\). Let \(x_i\) and \(x_j\) be arbitrary elements of \(X_t\). The pair \((x_i, x_j)\) is said to be bad if \(x_i \geq 4\), \(i < j\) and \(x_i > x_j + 1\). A bad pair \((x_i, x_j)\) is said to be close if, for every \(k\) lying between \(i\) and \(j\), none of the pairs \((x_i, x_k)\) or \((x_k, x_j)\) is bad.

Lemma 6.2. Let \((x_i, x_j)\) be an arbitrary close bad pair. Then \(x_k = x_i - 1\) for all \(k\) lying between \(i\) and \(j\).

Proof. When \(j - i = 1\), the lemma is trivial. Indeed, there are no elements between \(i\) and \(j\). When \(j - i > 1\), we consider an arbitrary \(k\) between \(i\) and \(j\). On the one hand, \(x_k \leq x_i - 1\) since otherwise the pair \((x_k, x_j)\) would be bad. On the other hand, \(x_k \geq x_i - 1\) since otherwise the pair \((x_i, x_k)\) would be bad. Thus \(x_k = x_i - 1\). \(\square\)

Choose an arbitrary close bad pair \((x_i, x_j)\) in the sequence \(X_t\). Consider the unit variation

\[
\langle x_1, \ldots, x_i, \ldots, x_j, \ldots, x_t \rangle \rightarrow \langle x_1, \ldots, x_i - 1, \ldots, x_j + 1, \ldots, x_t \rangle.
\] (6.3)

Lemma 6.3. Let \((x_i, x_j)\) be an arbitrary close bad pair in the continuant \(\langle X_t \rangle\). Then the unit variation (6.3) enlarges the continuant.

Proof. Let \(B\) be the finite sequence lying between \(x_i\) and \(x_j\). Then \([B] = [\overline{B}]\) by Lemma 6.2. Put \(a = x_j + 1\) and \(b = x_i - 1\). Then \(F(a) > F(a - 1)\) by (6.2). In other words, by Lemma 6.1, the unit variation (6.3) enlarges the continuant. \(\square\)

Lemma 6.4. For every continuant \(\langle X_t \rangle\) there is a continuant \(\langle B_t \rangle = \langle b_1, b_2, \ldots, b_t \rangle\) with the following properties.

1) There are no bad pairs in \(\langle B_t \rangle\).

2) For every integer \(\nu \leq t\) we have

\[
\frac{\langle B_\nu \rangle}{\sqrt{2} S_b(\nu)} \geq \frac{\langle X_\nu \rangle}{\sqrt{2} S_x(\nu)}.
\]

In particular,

\[
\sum_{\nu=1}^{t} \frac{\langle B_\nu \rangle}{\sqrt{2} S_b(\nu)} > \sum_{\nu=1}^{t} \frac{\langle X_\nu \rangle}{\sqrt{2} S_x(\nu)}.
\] (6.4)

3) \(\max_{\nu \leq t} \varphi_{b,2}(\nu) = \max_{\nu \leq t} \varphi_{x,2}(\nu)\).

Proof. Choose an arbitrary close bad pair \((x_i, x_j)\) in the continuant \(\langle X_t \rangle\). Consider the unit variation (6.3). Denote the resulting continuant by \(\langle X'_t \rangle = \langle x'_1, x'_2, \ldots, x'_t \rangle\).

We claim that property 2) holds for \(B_t = X'_t\). Indeed, if \(\nu < i\), then the fraction

\[
\frac{\langle X_\nu \rangle}{\sqrt{2} S_x(\nu)}
\] (6.5)
does not change. If \( \nu \geq j \), then \( \langle X'_\nu \rangle > \langle X_\nu \rangle \) by Lemma 6.3 and, since \( S_X'(\nu) = S_X(\nu) \), the fraction (6.5) increases. Now suppose that \( i \leq \nu < j \). Then it follows from (2.4) that \( \langle X'_\nu \rangle / \langle X_\nu \rangle > (a_i - 1)/a_i \geq 3/4 \). On the other hand, \( S_X(\nu) \) decreases by 1. Hence the fraction (6.5) increases by a factor of at least \( 3\sqrt{2}/4 > 1.06 \). Hence some fractions (6.5) do not change while the others increase. Thus the sum of the fractions (6.5) over all \( \nu \) increases at every step of the algorithm. This proves property 2).

We now prove that property 3) also holds after the unit variation (6.3). In this situation, we distinguish two cases: \( x_i \geq 6 \) and \( x_i \leq 5 \).

First assume that \( x_i \geq 6 \). Then Lemma 6.2 yields that \( x_\nu \geq 5 \) for all \( \nu \) in the interval \( i < \nu < j \). However, by Lemma 2.1, the maximum of \( \varphi_{x,2}(\nu) \) is attained at a point \( \nu \) with \( x_\nu \leq 4 \). Hence this maximum is not attained for \( i < \nu < j \). On the other hand, the quantity \( S_X(\nu) \) does not change when \( \nu < i \) or \( \nu \geq j \). It follows that the quantity \( \max_{\nu \leq \nu'} \varphi_{x,2}(\nu) \) does not change under the unit variation (6.3).

Now assume that \( x_i \leq 5 \). As in the first case, Lemma 6.2 yields that \( x_k \leq 4 \) for all \( k \) with \( i < k < j \). Hence, by Lemma 2.1, the maximum of the function \( \varphi_{x,2}(\nu) \) for \( i \leq \nu \leq j \) is attained at \( \nu = j \). But the quantity \( S_X(j) \) does not change under the unit variation (6.3).

To complete the proof of the lemma, we only need to show that the algorithm terminates (note that every replacement (6.3) not only destroys close bad pairs but can create new ones). To do this, we write \( M \) for the largest term of the continuant \( \langle X_i \rangle \) at the current step of the algorithm. It follows from the definition of a bad pair that the unit variation (6.3) does not enlarge \( M \). However, the sum of the fractions in (6.4) increases at every step of the algorithm. Therefore, performing (6.3) successively for all close bad pairs, we shall eventually arrive at a continuant without bad pairs. In other words, property 1) holds. We denote the resulting continuant by \( \langle B_t \rangle \). □

Lemma 6.4 would enable us to make all the terms of the continuant equal to 4 and 5, were it not for two cases when the necessary unit variation of the form (6.3) cannot be performed. First, all the elements of the resulting continuant \( \langle B_t \rangle \) to the right of \( b_i > 5 \) may be greater than or equal to 5. Second, all the elements to the left of \( b_j < 4 \) may be less than or equal to 4. We consider these two cases in the next two lemmas.

**Lemma 6.5.** Let \( x \) be an irrational number as in (0,1) such that the derivative \( ?'(x) \) exists and is equal to infinity. Suppose that (5.10) does not hold for some infinite sequence \( T^{(2)} \) of positive integers \( t \). Then the following properties hold for all sufficiently large elements \( t \in T^{(2)} \), where \( t' = [t - t^{2/3}] \).

1) For every \( k \) with \( 1 \leq k \leq t' \), the arithmetic mean of the elements of the subinterval \( (b_k, b_{k+1}, \ldots, b_t) \) does not exceed 5.

2) \( \max_{\nu \leq t'} \varphi_{b,2}(\nu) \leq \max_{\nu \leq t} \varphi_{b,2}(\nu) = \max_{\nu \leq t} \varphi_{x,2}(\nu) \).

**Proof.** We shall prove that there is an integer \( t' \) in the interval \( t - t^{2/3} < t' \leq t \) with properties 1) and 2).

If all the elements of the continuant \( \langle B_t \rangle \) do not exceed 5, then the properties 1) and 2) hold automatically for \( t' = t \). We now assume the opposite: there are terms greater than 5. Let \( b_s \) be the leftmost of them. We put \( t' = s - 1 \). Then property 1) clearly holds for this choice of \( t' \).
To prove property 2), note that the inequality in this property holds in view of the shrinking of the range of \( \nu \), and the equality holds by the previous lemma. Hence property 2) also holds.

Notice that all elements of the subsequence \( (b_1, b_2, \ldots, b_t) \) greater than or equal to 5. Indeed, assume that \( b_{t+k} \leq 4 \) for some \( k \geq 2 \). Then \( (b_1, b_{t+k}) \) is a left bad pair. But there are no left bad pairs in \( \langle B_t \rangle \) and we arrive at a contradiction.

It remains to verify that \( t - t' < t^{2/3} \). Assume the opposite. Recall that \( \varphi_{\nu,2}(t) = \varphi_{\nu,2}(t) > 0 \) in this case by Lemmas 5.2 and 6.4. Therefore,

\[
0 < \varphi_{\nu,2}(t) = \left( \kappa_2 t' - S_b(t') \right) + \left( \kappa_2 (t - t') - \sum_{n=t'+1}^{t} b_n \right) \geq \left( \kappa_2 t' - S_b(t') \right) - (\kappa_2 - 5)t^{2/3}.
\]

(6.6)

Hence, it follows from the third part of Lemma 6.4 and (6.6) that

\[
\max_{\nu \leq t} \varphi_{\nu,2}(\nu) = \max_{\nu \leq t} \varphi_{\nu,2}(\nu) \geq \kappa_2 t' - S_b(t') > (5 - \kappa_2)t^{2/3} > M_2\sqrt{t}
\]

for sufficiently large \( t \). This contradicts the assumption that (5.10) does not hold. The resulting contradiction proves that \( t - t' < t^{2/3} \). \( \square \)

Note that, by the choice of \( t' \), the continuant \( \langle B_{t'} \rangle \) contains no terms greater than 5. This continuant \( \langle B_{t'} \rangle \) occurs in the statement and proof of the following lemma.

**Lemma 6.6.** Under the hypotheses of Lemma 6.5, there is a continuant \( \langle C_{t'} \rangle = \langle c_1, c_2, \ldots, c_{t'} \rangle \) with the following properties, where \( t' = [t - t^{2/3}] \).

1) All the elements of \( \langle C_{t'} \rangle \) are equal to 4 or 5. Moreover, \( c_1 = 5 \) and \( c_{t'} = 4 \).

2) For every \( \nu \leq t' \) we have

\[
\frac{\langle C_{t'} \rangle}{\sqrt{2} S_{c}(\nu)} \geq \frac{1}{2} \min \left( \frac{\langle B_{t'} \rangle}{\sqrt{2} S_{b}(\nu)}, \frac{1}{5} (1.05)^\nu \right) \geq \frac{1}{10} \min \left( \frac{\langle A_{t'} \rangle}{\sqrt{2} S_{a}(\nu)}, (1.05)^\nu \right).
\]

(6.7)

3) We have the chain of inequalities

\[
\max_{\nu \leq t'} \varphi_{c,2}(\nu) \leq \max_{\nu \leq t'} \varphi_{b,2}(\nu) + 1 \leq \max_{\nu \leq t} \varphi_{x,2}(\nu) + 1.
\]

(6.8)

**Proof.** Let \( \langle B_{t'} \rangle = \langle b_1, b_2, \ldots, b_{t'} \rangle \) be the continuant obtained in the previous lemma. As in its proof, we claim that the arithmetic mean of the elements of the subsequence \( (b_1, b_2, \ldots, b_{t'}) \) is greater than 4 for sufficiently large \( t \) and for \( \nu > t^{2/3} \).

Indeed, let \( b_s \) be the rightmost term such that the arithmetic mean of the elements of the sequence \( (b_1, b_2, \ldots, b_s) \) is less than or equal to 4. If there are no such \( s \), then none of the elements \( b_1, b_2, \ldots, b_s \) is less than 4. Indeed, suppose that \( b_p < 4 \). Since the arithmetic mean of \( (b_1, b_2, \ldots, b_p) \) is greater than 4, there is an element \( b_k \) greater than or equal to 5. Hence there is a left bad pair \( (b_k, b_p) \) in \( \langle B_{t'} \rangle \), contrary to the definition of \( \langle B_{t'} \rangle \). This also shows that all the elements to the right of \( b_s \) are greater than or equal to 4.

Furthermore, note that if we choose an \( s < t^{2/3} \) as described, then the arithmetic mean of the elements \( (b_1, b_2, \ldots, b_s) \) is exactly equal to 4 (because the continuant \( \langle B_{t'} \rangle \) contains no digits greater than 5). It follows that if some of these
elements are less than 4, then there are also elements greater than 4. We write \(b_j\) for the leftmost element such that \(b_j = 5\), and \(b_i\) for the rightmost element such that \(b_i \leq 3\). Note that \(i < j\) (otherwise \(b_i\) and \(b_j\) form a left bad pair). By our choice, \(b_k = 4\) for every \(k\) in the interval \(i < k < j\). Consider the replacement

\[
\langle b_1, \ldots, b_i, \ldots, b_j, \ldots, b_r \rangle \rightarrow \langle b_1, \ldots, b_i + 1, \ldots, b_j - 1, \ldots, b_r \rangle. \quad (6.9)
\]

Denote the resulting continuant by \(\langle B'_r \rangle = \langle b_1', b_2', \ldots, b_r' \rangle\). Iterating unit variations of the form (6.9), we can transform the subsequence \((b_1, b_2, \ldots, b_s)\) into a sequence of 4's only. Hence we obtain a continuant all of whose terms are 4 or 5. We call it \(\langle C_r \rangle\). The first part of property 1) holds for \(\langle C_r \rangle\). In what follows we verify properties 2) and 3), and at the end of the proof of the lemma we return to the second part of property 1).

Property 3) clearly holds since at every step of the form (6.9) we have \(S_{b'}(\nu) \geq S_b(\nu)\) for all \(\nu\) in the interval \(1 \leq \nu \leq t'\). The rest of (6.8) follows from the previous lemma. This proves property 3) even without the additional term +1.

We now verify property 2). First suppose that \(\nu \geq s\). Since, by Lemma 6.3, the reflections (6.9) enlarge the continuant \(\langle B_r \rangle\) and do not change the sum \(S_b(\nu)\), we have \(\langle C_r \rangle/\sqrt{2}S_{c}(\nu) \geq \langle B_r \rangle/\sqrt{2}S_{b}(\nu)\), and thus property 2) is proved even without the coefficient 1/2. But if \(\nu < s\), then the quantity \(\langle C_r \rangle/\sqrt{2}S_{c}(\nu)\) admits the following lower bound in view of (3.7):

\[
\frac{\langle C_r \rangle}{\sqrt{2}S_{c}(\nu)} = \frac{\left(\frac{4, 4, \ldots, 4}{\sqrt{2}^4\nu}\right)^{\nu}}{\frac{1}{5}\left(\frac{\lambda_4}{4}\right)\nu} > \frac{1}{5}(1.05)^{\nu}.
\]

The rest of (6.7) follows from the previous lemma.

We now return to the proof of the second part of property 1). To perform it, we replace \(c_1\) by 5, and \(c_r\) by 4 (if their values were different). These replacements change every quantity \(S_b(\nu), \nu = 1, 2, \ldots, t'\), by at most 1, and every quantity \(\langle C_r \rangle\) by a factor of at most two. This accounts for the corrections 1/2 and +1 in the statements of properties 2) and 3) respectively. □

§ 7. Inequalities with continuants: reflections

In this section we consider three finite sequences: \(A, B\) and \(C\), of which the middle one (that is, \(B\)) and at least one of the extreme ones (\(A\) or \(C\)) is non-empty.

Definition 7.1 ([13]). The change of continuants by the rule

\[
\langle A, B, C \rangle \mapsto \langle A, \overline{B}, C \rangle
\]

is referred to as the reflection of the sequence \((A, B, C)\).

We write \(A = (a_1, a_2, \ldots, a_w)\) and \(C = (c_1, c_2, \ldots, c_r)\). When \(A\) or \(C\) is empty, we put \(a_w = +\infty\) or \(c_1 = +\infty\) respectively. Consider the quantity

\[
\alpha = \alpha(A, B, C) = (a_w - c_1)(|B| - |\overline{B}|).
\]
Lemma 7.1 ([8]). Suppose that $\alpha = \alpha(A, B, C) \neq 0$. Then
\[
\left( \frac{\langle A, B, C \rangle}{\langle A, \bar{B}, C \rangle} \right)^{\text{sign}(\alpha)} \geq 1.
\]

Note that Lemma 7.1 generalizes a similar lemma of Motzkin and Straus [14].

7.1. Decreasing reflections. In the following lemma, the sequence $C_t = (c_1, c_2, \ldots, c_t)$ is obtained by a deterministic algorithm from the sequence $B_t = (b_1, b_2, \ldots, b_t)$ obtained in Lemma 4.2.

Lemma 7.2. Let $\langle B_t \rangle$ be any continuant all of whose terms are 1 or $n$ (where $n \geq 5$). Then there is a continuant $\langle C_t \rangle$ with the following properties:

1) $\langle C_t \rangle$ contains no interval of the form $(1, 1, \ldots, n, n)$, that is, no pair $(n, n)$ to the right of a pair $(1, 1)$;
2) $S_b(t) = S_c(t)$, $\langle C_t \rangle \leq \langle B_t \rangle$;
3) $\langle C_\nu \rangle / \sqrt{2} S_c(\nu) \leq \langle B_\nu \rangle / \sqrt{2} S_b(\nu)$ for every $\nu \leq t$;
4) $\max_{\nu \leq t}(\varphi_{n,c}(\nu)) = \max_{\nu \leq t}(\varphi_{n,b}(\nu))$.

Proof. We subject $\langle B_t \rangle$ to reflections which will be chosen below. The resulting continuant will be denoted by $\langle C_t \rangle$.

Let $\nu = j_1$ be a maximum point of the function $\varphi_{n,b}(\nu)$. It is uniquely determined since the number $\kappa_1^{(n)}$ is irrational. Since $\varphi_{n,b}(u) > \varphi_{n,b}(u - 1)$ when $b_u = n$ and $\varphi_{n,b}(u) < \varphi_{n,b}(u - 1)$ when $b_u = 1$, we see that $b_{j_1} = n$ and $b_{j_1 + 1} = 1$. Moreover, $b_{j_1 + 2} = 1$ since otherwise $\varphi_{n,b}(j_1 + 2) - \varphi_{n,b}(j_1) \geq n + 1 - 2\kappa_1^{(n)} > 0$ contrary to the definition of $j_1$.

A pair $(n, n)$ of neighbouring elements of the continuant $\langle B_t \rangle$ is said to be good if there is a pair $(1, 1)$ to the left of it. Otherwise we call it bad. Choose the leftmost good pair $(n, n)$. If there are several pairs $(1, 1)$ to the left of it, choose the rightmost of them. We define the following reflection using precisely these pairs $(1, 1)$ and $(n, n)$:

\[
B_t = (\ldots, 1, 1, \ldots, n, n, \ldots) \rightarrow (\ldots, 1, n, \ldots, 1, n, \ldots) = B'_t,
\]

where $B'_t$ is the sequence resulting from a single reflection. By our choices, the subsequence $B$ is periodic of the form $(1, n, 1, n, \ldots, 1, n)$. Indeed, $(n, n)$ cannot occur in $B$ since this pair would be good while we have chosen the leftmost good pair. In similar vein, $(1, 1)$ cannot occur in $B$ since we have chosen the rightmost such pair to the left of $(n, n)$. Hence $B$ cannot contain the element labelled by $j_1$ and, therefore, reflections of the form (7.1) do not change the maximum of the function $\varphi_{n,b}(\nu)$. This argument with $C_t = B'_t$ proves property 4).

Moreover, by Lemma 7.1, the continuant decreases under the reflection (7.1). Hence property 2) in the statement of the lemma holds with $C_t = B'_t$. To verify property 3), we consider the influence of reflections on the quantities $\langle B_\nu \rangle$ and $S_b(\nu)$ for various $\nu$. Let the sequence $B$ in the reflection (7.1) be equal to $(b_s, b_{s+1}, \ldots, b_p)$. Hence $b_{s-1} = b_s = 1$, $b_p = b_{p+1} = n$. If $\nu < s$, then neither quantity changes. But if $\nu > p$, then $\langle B_\nu \rangle$ decreases while $S_b(\nu)$ does not change. In both cases, property 3) holds with $C_t = B'_t$. We now assume that $s \leq \nu \leq p$. 

We first consider the case when \( \nu = s \). In this case \( S_b(\nu) \) increases by \( n - 1 \). We obtain an upper bound for the ratio \( \langle B'_\nu \rangle / \langle B_\nu \rangle \) by dividing the numerator and denominator of the fraction by \( \langle b_1, \ldots, b_{s-1} \rangle \):

\[
\frac{\langle B'_\nu \rangle}{\langle B_\nu \rangle} = \frac{\langle b_1, \ldots, b_{s-1}, n, 1, \ldots, n, 1 \rangle}{\langle b_1, \ldots, b_{s-1}, 1, n, 1, \ldots, n, 1 \rangle} = \frac{1 + [b_{s-1}, \ldots, b_1][n, 1, \ldots, n, 1]}{1 + [b_{s-1}, \ldots, b_1][n, 1, \ldots, n, 1]} < 1.
\]

(7.3)

We deduced (7.2) from the relation \( [b_{s-1}, \ldots, b_1] = [1, \ldots, b_1] > 1/2 \) and the fact that \( f(y) = (n + y)/(1 + y) \) is a decreasing function on the interval \( 0 \leq y \leq 1 \). Thus the reflection (7.1) enlarges the numerator of the fraction \( \langle B_\nu \rangle / \sqrt{2} S_b(\nu) \) by a factor of at most \( (2n + 1)/3 \), and enlarges the denominator by a factor of \( \sqrt{2}^{n-1} \). Since \( n \geq 5 \), the whole fraction decreases under reflection.

We now suppose that \( \nu > s \) and \( b_\nu = 1 \). In this case \( S_b(\nu) \) does not change and \( \langle B_\nu \rangle \) decreases. Indeed, using (2.2) and cancelling common factors in the numerator and denominator, we have

\[
\frac{\langle B'_\nu \rangle}{\langle B_\nu \rangle} = \frac{\langle b_1, \ldots, b_{s-1}, n, 1, \ldots, n, 1 \rangle}{\langle b_1, \ldots, b_{s-1}, 1, n, 1, \ldots, n, 1 \rangle} = \frac{1 + [b_{s-1}, \ldots, b_1][n, 1, \ldots, n, 1]}{1 + [b_{s-1}, \ldots, b_1][1, n, \ldots, 1, n]} < 1.
\]

(7.4)

Consider the case when \( \nu > s \) and \( b_\nu = n \). In this case \( S_b(\nu) \) increases by \( n - 1 \) and \( \langle B'_\nu \rangle / \langle B_\nu \rangle \) satisfies the following equality (obtained using (2.2) and cancelling out):

\[
\frac{\langle B'_\nu \rangle}{\langle B_\nu \rangle} = \frac{\langle b_1, \ldots, b_{s-1}, n, 1, \ldots, n, 1 \rangle}{\langle b_1, \ldots, b_{s-1}, 1, n, 1, \ldots, n, 1 \rangle} = \frac{\langle n, 1, \ldots, n \rangle}{\langle 1, n, \ldots, 1 \rangle} \cdot \frac{1 + [b_{s-1}, \ldots, b_1][n, 1, \ldots, n]}{1 + [b_{s-1}, \ldots, b_1][1, n, \ldots, 1]}.
\]

(7.5)

The resulting expression in (7.4) is the product of two fractions, of which the first is equal to \( n \). This follows from the equality

\[
\langle n, 1, n, \ldots, 1, n \rangle = n \langle 1, n, 1, \ldots, n, 1 \rangle,
\]

which can be proved by induction on the odd length of the continuant. A fraction resembling the second factor in (7.4) was estimated in (7.3). Since a similar bound can be used here, (7.4) gives rise to the estimate

\[
\frac{\langle B'_\nu \rangle}{\langle B_\nu \rangle} \leq n \cdot \frac{(2n + 1)(n + 1)}{(3n + 2)n} = \frac{(2n + 1)(n + 1)}{3n + 2}.
\]

Hence,

\[
\frac{\langle B'_\nu \rangle}{\langle B_\nu \rangle} < \frac{(2n + 1)(n + 1)}{(3n + 2)\sqrt{2}^{n-1}} < 1
\]

when \( n \geq 5 \).

We have checked that properties 2), 3) and 4) hold after one step of the algorithm. Hence they hold throughout the whole algorithm. Since the number of good pairs \( (n, n) \) decreases at every step, they will eventually disappear. Writing \( C_t \) for the sequence obtained at this moment, we conclude that it also possesses property 1). □
Thus, using this lemma, we can get rid of all good pairs \((n, n)\). The following lemma enables us to get rid of bad pairs \((n, n)\) too. We need some new notation. Let \(C'_t = (c'_1, c'_2, \ldots, c'_t)\) be any sequence consisting of 1s and ns. When \(\nu \leq t\) we put

\[
S_{c'}(\nu) = c'_1 + c'_2 + \cdots + c'_\nu, \quad \varphi_{c'}(\nu) = S_{c'}(\nu) - \kappa_1^{(n)} \nu.
\]

Let \(r_1(i)\) (resp. \(r_n(i)\)) be the number of 1s (resp. ns) in the sequence \((c_1, \ldots, c_i)\).

**Lemma 7.3.** Let \(\langle C_t \rangle\) be any continuant all of whose terms are 1 or \(n\) (where \(n \geq 5\)) and containing no good pairs \((n, n)\). Suppose that \(r_1(t) > r_n(t)\). Then there is a continuant \(\langle C'_t \rangle\) with the following properties.
1) \(\langle C'_t \rangle\) contains no pairs \((n, n)\).
2) \(S_{c}(t) = S_{c'}(t), \langle C'_t \rangle \leq 2\langle C_t \rangle\).
3) For every \(\nu \leq t\) we have

\[
\frac{\langle C'_\nu \rangle}{\sqrt{2S_{c'}(\nu)}} \leq 2\max \left\{ \frac{\langle C_\nu \rangle}{\sqrt{2S_{c}(\nu)}}, 4n^2(0.8)^\nu \right\}.
\]

4) \(\max_{u \leq t}(\varphi_{n, c'}(u)) \leq \max_{u \leq t}(\varphi_{n, c}(u))\).

**Proof.** Suppose that the sequence \(C_t\) contains a pair of neighbouring elements \((n, n)\), otherwise there is nothing to prove. The sequence \(C_t\) cannot begin with \((1, 1)\), otherwise all the pairs \((n, n)\) in \(C_t\) are good, contrary to hypothesis. If it begins with a single 1, moving this 1 to the end of the sequence changes the continuant \(\langle C_t \rangle\) by a factor of at most 2. This possible doubling of the continuant is taken into account by means of the coefficients 2 in parts 2) and 3). Thus there is no loss of generality in assuming that \(c_1 = n\). In particular, \(r_1(1) < r_n(1)\).

Let \(s\) be the largest number \(i \leq t\) such that \(r_1(i) \leq r_n(i)\). Since the integers \(r_1(i)\) and \(r_n(i)\) change by no more than 1 (and not simultaneously) as we pass to the next value of \(i\), we have \(r_1(s) = r_n(s)\). Hence the sequence \((c_{s+1}, c_{s+2}, \ldots, c_t)\) (if it is non-empty) cannot begin with \(n\). Moreover, the element \(c_{s+2}\) (if \(s \leq t - 2\)) cannot equal to \(n\). It follows that \((c_{s+1}, c_{s+2}) = (1, 1)\). Hence the sequence \((c_{s+1}, c_{s+2}, \ldots, c_t)\) cannot contain two consecutive \(n\) (otherwise the pair \((n, n)\) would be good, contrary to hypothesis).

Consider the interval \((c_1, c_2, \ldots, c_s)\). Since \(r_1(s) = r_n(s)\), the number of pairs \((1, 1)\) in this interval is equal to the number of pairs \((n, n)\). Moreover, every pair \((n, n)\) lies to the left of every pair \((1, 1)\) because, by hypothesis, there are no good pairs \((n, n)\) in \(\langle C_t \rangle\). Consider the closest pairs \((n, n)\) and \((1, 1)\). By what was said above, the elements between them form a periodic sequence \((1, n, 1, n, \ldots, 1, n)\). Consider the reflection defined using precisely these pairs \((1, 1)\) and \((n, n)\):

\[
C_t = (\ldots, n, n, \ldots, 1, 1, \ldots) \rightarrow (\ldots, 1, 1, \ldots, n, n, \ldots). \quad (7.5)
\]

This reflection makes the number of pairs \((n, n)\) smaller by 1. Denote the resulting sequence again by \(C_t\) and consider an analogous reflection for the new closest pairs \((n, n)\) and \((1, 1)\) and so on, until the interval \((c_1, c_2, \ldots, c_s)\) takes the form \((n, 1, n, 1, \ldots, 1, n)\). Let \(\langle C'_t \rangle\) be the continuant resulting from this series of reflections. Then it possesses property 1) of the present lemma.
By Lemma 7.1, the continuant \( \langle C_t \rangle \) decreases under (7.5). Hence property 2) of the present lemma also holds.

We claim that the following bound holds for all \( \nu \) with \( 1 \leq \nu \leq t \):

\[
S_{c'}(\nu) \leq S_c(\nu).
\]  

(7.6)

Indeed, let the sequence \( B \) in the reflection (7.5) be equal to

\[
(n, 1, n, 1, \ldots, n, 1) = (b_s, b_{s+1}, \ldots, b_p).
\]

Regard (7.5) as the replacement of \( B = (1, n, 1, n, \ldots, 1, n) \). Note that the quantity \( S_c(\nu) \) does not increase under this replacement (it remains unchanged for odd \( \nu - s \) and decreases by \( n - 1 \) for even). Hence the inequality (7.6) holds and, therefore, property 4) of the present lemma holds.

We now verify property 3). It suffices to consider the case when \( \nu < s \) (otherwise the reflections do not change \( S_c(\nu) \) and make the continuant \( \langle C_\nu \rangle \) smaller, so that \( \langle C_\nu' \rangle / \sqrt{2} S_{c'}(\nu) \) can be estimated by the first maximum element). Hence there is no loss of generality in assuming that \( \nu < s \). In this case, using the notation \([\alpha]\) for the integer part of a real number \( \alpha \), we obtain the following upper bound for the fraction \( \langle C_\nu' \rangle / \sqrt{2} S_{c'}(\nu) \):

\[
\frac{\langle C_\nu' \rangle}{\sqrt{2} S_{c'}(\nu)} \leq \frac{\lfloor (\nu+1)/2 \rfloor}{\sqrt{2} (n+1)[\nu/2]} \leq 4n^2 \left( \frac{\mu_n}{\sqrt{2}^{n+1}} \right)^{\lfloor \nu/2 \rfloor} \leq 4n^2 \left( \frac{\mu_n}{\sqrt{2}^{n+1}} \right)^{\nu/2} < 4n^2 (0.8)^\nu.
\]

Here we have used (3.5) and the relations \( \mu_n/\sqrt{2}^{n+1} < 0.64 = (0.8)^2 \) for \( n \geq 5 \).

In the next lemma we transform \( C' \) into a finite sequence \( D_t = (d_1, d_2, \ldots, d_t) \).

**Lemma 7.4.** Let \( C'_t \) be an arbitrary finite sequence all of whose elements are equal to 1 or \( n \) (where \( n \geq 5 \)) and no consecutive elements are equal to \( n \). Then there is a sequence \( D_t \) with the following properties.

1) \( D_t \) is of the form

\[
(1, 1, \ldots, 1, n, \ldots, n, 1, \ldots, 1, n, \ldots, n, 1),
\]

(7.7)

where neither \((m_1, m_2, \ldots, m_\sigma)\) nor \((n_1, n_2, \ldots, n_\sigma)\) contains more than one number smaller than \( n^5 \).

2) For every \( \nu \leq t \) we have \( \langle D_\nu \rangle \leq \langle C'_\nu \rangle \).

3) \( S_d(t) = S_{c'}(t) \).

4) For every \( \nu \leq t \) we have \( 0 \leq S_{c'}(\nu) - S_d(\nu) \leq 2(n - 1)n^5 \).

**Proof.** We act on \( C'_t \) by reflections. Our first aim is to make all the numbers \( m_i \) greater than or equal to \( n^5 \). An element \( c'_{i-1} \) equal to 1 is said to be *free* if \( c'_{i-1} = 1 \), and *tied* if \( c'_{i-1} = n \). We divide the sequence \( C'_t \) into intervals \( E^{(i)} \) in the following recursive way, moving from right to left and beginning with the end of the sequence. Let \( p \) be the number of these intervals (it is not known in advance). The end of the
interval \( E^{(p)} \) is the last element \( c'_t \). The beginning of \( E^{(p)} \) is an element \( c'_{e_p} \) with the following properties:

a) \( c'_{e_p} = 1 \) is free and \( c'_{e_p - 1} \) is tied;

b) \( e_p \) is the maximal number among all indices with the previous property such that the number of free elements in the interval \( (c'_{e_p}, \ldots, c_t) \) is greater than \( n^5 \).

Furthermore, the end of \( E^{(p-1)} \) is \( c'_{e_p - 1} \). The beginning of \( E^{(p-1)} \) (and of all subsequent intervals) is defined similarly to \( E^{(p)} \). The last interval may be shortened: the number of free elements in it may be less than \( n^5 \).

Consider an arbitrary interval \( E^{(i)} \), where \( i > 1 \). It is of the form

\[
\frac{1, \ldots, 1, n, 1, \ldots, n, 1, \ldots, 1, 1, \ldots, 1, n, 1, \ldots, n, 1}{v_1 \quad w_1 \quad v_i \quad w_i},
\]  

(7.8)

where \( v_1 + v_2 + \cdots + v_t > n^5 \) but \( v_2 + \cdots + v_t < n^5 \). We perform a reflection, replacing

\[
C'_t = \frac{1, \ldots, 1, n, 1, \ldots, n, 1, \ldots, 1, 1, \ldots, 1, n, 1, \ldots, n, 1}{v_1 \quad w_1 \quad v_{i-1} \quad w_{i-1} \quad v_i \quad w_i}
\]  

by

\[
\frac{1, \ldots, 1, n, 1, \ldots, n, 1, \ldots, 1, 1, \ldots, 1, n, 1, \ldots, n, 1}{v_1 \quad w_1 \quad v_{i-1} \quad w_{i-1} \quad v_i \quad w_i} = (A, \overline{B}, C).
\]  

(7.10)

It follows from Lemma 7.1 that the continuant \( \langle C'_t \rangle \) decreases under this reflection. As a result of the reflection (7.9), (7.10), we obtain an interval

\[
\frac{1, \ldots, 1, n, 1, \ldots, n, 1, \ldots, 1, 1, \ldots, 1, n, 1, \ldots, n, 1}{v_1 \quad w_1 \quad v_{i-1} + v_i \quad w_{i-1} + w_i}.
\]

Making similar transformations, we finally obtain an interval

\[
\frac{1, 1, \ldots, 1, n, 1, n, 1, \ldots, n, 1}{v_1 + v_2 + \cdots + v_n \quad w_1 + w_2 + \cdots + w_n}.
\]  

(7.11)

Denote the resulting sequence again by \( C'_t \). We want to estimate the possible change in the quantity \( S_{e'}(\nu) \). Since all the reflections act inside the interval, we see that \( S_{e'}(\nu) \) does not change if \( \nu \) lies outside this interval. On the other hand, the interval (7.11) can be obtained from (7.8) by transpositions (or interchanges) of elements equal to 1 and \( n \). The number of 1s requiring a transposition with \( n \) is originally equal to \( v_2 + \cdots + v_t \) and decreases after each transposition. Hence, making no more than \( v_2 + \cdots + v_t \) transpositions, we obtain (7.11). Each transposition makes \( S_{e'}(\nu) \) smaller by at most \( n - 1 \). Since \( v_2 + \cdots + v_t < n^5 \), the total decrease of \( S_{e'}(\nu) \) does not exceed \( (n - 1)n^5 \). Since the intervals are disjoint, this holds for transpositions on all intervals simultaneously. Thus all the numbers \( m_i \)
Using Lemma 7.4, we achieve property 2. Moreover, we have described in Lemmas 4.2, 7.2–7.4 respectively. The transformation and substitute the bound (7.13) into the exponent in (7.12).

Furthermore, using Lemma 7.2, we obtain
\[ \frac{\langle C'_\nu \rangle}{\sqrt{2}S_{c'(\nu)}} \leq \sqrt{2}^{n^7} \frac{\langle X_\nu \rangle}{\sqrt{2}S_{x(\nu)}} \quad \text{and} \quad \max_{u \leq t}(\varphi_{n,c}(u)) \leq \max_{u \leq t}(\varphi_{n,x}(u)). \]

Then, using Lemma 7.3, we deduce that
\[ \frac{\langle C'_\nu \rangle}{\sqrt{2}S_{c'(\nu)}} \leq \sqrt{2}^{n^7+2} \max \left\{ \frac{\langle X_\nu \rangle}{\sqrt{2}S_{x(\nu)}}, 4n^2(0.8)^\nu \right\}, \quad \max_{u \leq t}(\varphi_{n,c'}(u)) \leq \max_{u \leq t}(\varphi_{n,x}(u)). \]

Using Lemma 7.4, we achieve property 2. Moreover, we have \( \max_{u \leq t}(\varphi_{n,d}(u)) \leq \max_{u \leq t}(\varphi_{n,x}(u)) \) and
\[ \frac{\langle D_\nu \rangle}{\sqrt{2}S_{d(\nu)}} \leq \sqrt{2}^{n^7+2+2n^6-2n^5} \max \left\{ \frac{\langle X_\nu \rangle}{\sqrt{2}S_{x(\nu)}}, 4n^2(0.8)^\nu \right\}. \] (7.12)

We finally note that
\[ \sqrt{2}^{n^7+2+2n^6-2n^5} \leq \sqrt{2}^{2n^7} = 2^{n^7} \] (7.13)
and substitute the bound (7.13) into the exponent in (7.12). \( \square \)
7.2. Enlarging reflections. The continuant \( \langle C_{\nu} \rangle \) obtained in Lemma 6.6 occurs in the following lemma.

**Lemma 7.6.** Let \( x \) be an irrational number in \((0, 1)\) such that the derivative \( ?'(x) \) exists and is equal to infinity. Suppose that (5.10) does not hold for some infinite sequence \( T^{(2)} \) consisting of positive integers \( t \). Then for all sufficiently large elements \( t \in T^{(2)} \) there is a continuant \( \langle D_{\nu} \rangle = \langle d_1, d_2, \ldots, d_{\nu} \rangle \) with the following properties, where \( t' = [t - t^2/3] \).

1) \( D_{\nu} \) is of the form

\[
\begin{align*}
\left( \frac{5, 5, \ldots, 5, 4, 4, \ldots, 4, \ldots, 5, 5, \ldots, 5, 4, 4, \ldots, 4, \ldots, 5, 5, \ldots, 5, 4, 4, \ldots, 4, \ldots, 5, 5, \ldots, 5, 4, 4, \ldots, 4, \ldots, 5, 5, \ldots, 5, 4, 4, \ldots, 4, \ldots, 5, 5, \ldots, 5, 4, 4, \ldots, 4, \ldots}, \\
\m_1 \quad n_1 \quad m_1 \quad n_1 \quad m_\sigma \quad n_\sigma
\end{align*}
\]

and neither \((m_1, m_2, \ldots, m_\sigma)\) nor \((n_1, n_2, \ldots, n_\sigma)\) contains more than one number less than or equal to 1000.

2) For every \( \nu \leq t' \) we have

\[
\frac{\langle D_{\nu} \rangle}{\sqrt{2} S_{\delta} (\nu)} \geq \frac{1}{210^6} \frac{\langle C_{\nu} \rangle}{\sqrt{2} S_{\epsilon} (\nu)} \geq 0.1 \min \left( \frac{\langle X_{\nu} \rangle}{\sqrt{2} S_{\epsilon} (\nu)}, (1.05)^\nu \right).
\]

3) \( \max_{\nu \leq t'} \varphi_{d, 2}(\nu) \leq \max_{\nu \leq t'} \varphi_{c, 2}(\nu) \leq \max_{\nu \leq t} \varphi_{x, 2}(\nu) + 1. \)

**Proof.** We act on the continuant \( \langle C_{\nu} \rangle \) by reflections. Recall that \( c_1 = 5 \) and \( c_{\nu} = 4 \). We first achieve that the continuant contains no single terms equal to 5, that is, contains no interval of the form \((4, 5, 4)\). Fix the rightmost single 5 and consider the term 5 closest to it on the left. Perform the reflection

\[
\langle c_1, \ldots, 5, 4, 4, \ldots, 4, 5, 4, \ldots, 4, 4, 4, \ldots, 4, 4, \ldots, 4, 4, \ldots, 4, 4, \ldots, 4, 4, \ldots, 4, 4, \ldots, 4, 4, \ldots \rangle \rightarrow \langle c_1, \ldots, 5, 5, 4, \ldots, 4, 4, 4, \ldots, 4, 4, \ldots, 4, 4, \ldots, 4, 4, \ldots, 4, 4, \ldots, 4, 4, \ldots, 4, 4, \ldots \rangle. \tag{7.14}
\]

Then choose the rightmost of the remaining single 5s and apply (7.14) until all single 5s disappear. Write the resulting continuant as \( \langle C_{\nu}^{(1)} \rangle = \langle c_1^1, c_2^1, \ldots, c_{\nu}^1 \rangle \). Put \( S_{c^1}(\nu) = c_1^1 + c_2^1 + \cdots + c_{\nu}^1 \). It follows from Lemma 7.1 that \( \langle C_{\nu}^{(1)} \rangle \geq \langle C_{\nu} \rangle \) for all \( \nu \). We easily see that the reflections do not ‘overlap’ each other, that is, every element participates in only one reflection. Hence, for every \( \nu \) in the interval \( 1 \leq \nu \leq t' \), we have

\[
0 \leq S_{c^1}(\nu) - S_c(\nu) \leq 1. \tag{7.15}
\]

We can similarly get rid of all intervals of the form

\[
\underbrace{(4, 5, 5, \ldots, 5, 4, \ldots, 4, 4, \ldots, 4, 4, \ldots, 4, 4, \ldots, 4, 4, \ldots)}_m \text{ digits}, \tag{7.16}
\]

where \( m \leq 1000 \). Indeed, suppose that the continuant

\[
\langle C_{\nu}^{(m-1)} \rangle = \langle c_1^{m-1}, c_2^{m-1}, \ldots, c_{\nu}^{m-1} \rangle
\]

contains no intervals of the form \( \underbrace{(4, 5, 5, \ldots, 5, 4, \ldots, 4, 4, \ldots, 4, 4, \ldots, 4, 4, \ldots, 4, 4, \ldots, 4, 4, \ldots, 4, 4, \ldots, 4, 4, \ldots, 4, 4, \ldots, 4, 4, \ldots, 4, 4, \ldots)}_k \text{ digits}, \)

where \( k < m \). Consider the rightmost interval of the form (7.16). Find the element 5 closest to it on the left.
Similarly to (7.14), consider the reflection
\[
\begin{array}{c}
A \langle c_1, \ldots, 5, 4, \ldots, 4, 5, 5, \ldots, 5, 4, \ldots, c_{t'} \rangle \\
m \text{ digits}
\end{array}
\rightarrow
\begin{array}{c}
A \rangle \langle c_1, \ldots, 5, 5, \ldots, 5, 4, \ldots, 4, 4, \ldots, c_{t'} \rangle \\
m \text{ digits}
\end{array}
\]

Repeatedly applying it, we can get rid of all intervals of the form (7.16). Since these reflections do not ‘overlap’ each other, the following inequalities hold for every \( \nu \) in the interval \( 1 \leq \nu \leq t' \):
\[
0 \leq S_{c^m}^{\nu} - S_{c^{m-1}}^{\nu} \leq m, \quad \langle C^{(m)}_{\nu} \rangle \geq \langle C^{(m-1)}_{\nu} \rangle. \tag{7.17}
\]

In total, summing the differences in (7.17) over all \( m \) from 1 to 1000, we have
\[
0 \leq S_{c^{1000}}^{\nu} - S_{c}^{\nu} \leq 500500 < 10^6, \quad \langle C^{(1000)}_{\nu} \rangle \geq \langle C^{(1000)}_{\nu} \rangle, \tag{7.18}
\]

that is,
\[
\frac{\langle C^{(1000)}_{\nu} \rangle}{\sqrt{2} S_{c^{1000}}^{\nu}} \geq \frac{1}{\sqrt{2} 10^6} \left( \frac{\langle C^{(1000)}_{\nu} \rangle}{\sqrt{2} S_{c}^{\nu}} \right). \tag{7.19}
\]

We can similarly get rid of all intervals of the form \((5, 4, 4, \ldots, 4, 5)\), where \(m < 1000\), by acting by transformations of the form
\[
\begin{array}{c}
A \langle c_1^{999+m}, \ldots, 5, 4, \ldots, 4, 5, 5, \ldots, 5, 4, \ldots, c_{t'}^{999+m} \rangle \\
m \text{ digits}
\end{array}
\rightarrow
\begin{array}{c}
A \rangle \langle c_1^{999+m}, \ldots, 5, 5, \ldots, 5, 4, \ldots, 4, 4, \ldots, c_{t'}^{999+m} \rangle \\
m \text{ digits}
\end{array}
\]

Here the continuant \( \langle C^{(999+m)} \rangle \) contains no intervals of the form \((5, 4, 4, \ldots, 4, 5)\) with \(1 \leq k < m\). Denote the resulting continuant \( \langle C^{(2000)}_{\nu} \rangle \) by \( \langle D_{t'} \rangle \). Similarly to (7.18), we have
\[
0 \leq S_{c^{2000}}^{\nu} - S_{c^{1000}}^{\nu} \leq 500500 < 10^6, \quad \langle C^{(2000)}_{\nu} \rangle \geq \langle C^{(2000)}_{\nu} \rangle. \tag{7.20}
\]

Therefore, using (6.7) and (7.19), we finally deduce property 2).

Property 3) follows since the partial sums (that is, \( S_{c}(\nu) \) and so on) do not decrease throughout the algorithm in view of the lower bounds in (7.15), (7.18) and (7.20). □
§ 8. Asymptotic growth of minimal and maximal continuants

8.1. Asymptotic growth of minimal continuants. Recall that the constants \( c^{(n)} \) and \( \gamma_n \) were introduced in (1.13). The meaning of \( \gamma_n \) is clarified in the following lemma.

**Lemma 8.1.** The number \( \gamma_n \) satisfies the equation

\[
\gamma_n = (1 + [\bar{T}] \cdot [\bar{1}, \bar{n}]) (1 + [\bar{T}] \cdot [\bar{n}, \bar{T}]) \frac{\Phi}{\sqrt{5}} \cdot \frac{\sqrt{n^2 + 4n + n}}{2\sqrt{n^2 + 4n}},
\]

where the bar means an infinitely repeated period of a continued fraction.

**Proof.** Our plan is to compute the values of the three infinite continued fractions \([\bar{T}], [\bar{1}, \bar{n}]\) and \([\bar{n}, \bar{T}]\), substitute them into (8.1) and simplify the resulting expression to obtain the right-hand side of (1.13).

Performing the first step of the plan, we obtain

\[
[\bar{T}] = \Phi - 1, \quad [\bar{1}, \bar{n}] = 1 - \mu'_n, \quad [\bar{n}, \bar{T}] = \frac{1 - \mu'_n}{n}.
\]

Putting \( \alpha = 1 - \mu'_n \), we consider the product of the two brackets in (8.1):

\[
(1 + [\bar{T}] \cdot [\bar{1}, \bar{n}]) (1 + [\bar{T}] \cdot [\bar{n}, \bar{T}]) = \frac{(1 + \alpha \Phi - \alpha)(n + \alpha \Phi - \alpha)}{n}.
\]

Expanding the last product of brackets in (8.2) and continuing the chain of equalities, we obtain

\[
(1 + [\bar{T}] \cdot [\bar{1}, \bar{n}]) (1 + [\bar{T}] \cdot [\bar{n}, \bar{T}]) = \frac{n + \alpha \Phi - \alpha + n \alpha \Phi + \alpha^2 \Phi^2 - 2 \alpha^2 \Phi - \alpha n + \alpha^2}{n}.
\]

Substituting \( \Phi^2 = \Phi + 1 \) into (8.3), we have

\[
(1 + [\bar{T}] \cdot [\bar{1}, \bar{n}]) (1 + [\bar{T}] \cdot [\bar{n}, \bar{T}]) = \frac{n + \alpha(\Phi - 1 + n \Phi - n) + \alpha^2(2 - \Phi)}{n}.
\]

Using the equality \( \sqrt{n^2 + 4n + n}/2 = \alpha + n \) and (8.4), we obtain the following formula for the number \( \gamma_n \) defined in (8.1):

\[
\gamma_n = (n + \alpha(\Phi - 1 + n \Phi - n) + \alpha^2(2 - \Phi))(\alpha + n) \frac{\Phi}{\sqrt{5}} \cdot \frac{1}{n \sqrt{n^2 + 4n}}.
\]

Expanding the brackets in (8.5), we have

\[
\gamma_n = (\alpha^3(2 - \Phi) + \alpha^2(\Phi - 1 + n) + \alpha(n \Phi + n^2 \varphi - n^2) + n^2) \frac{\Phi}{n \sqrt{5n^2 + 20n}}.
\]

Substitute \( \alpha^2 = n(1 - \alpha) \) into (8.6) and cancel the factor \( n \). This yields that

\[
\gamma_n = ((\alpha - \alpha^2)(2 - \Phi) + (1 - \alpha)(\Phi - 1 + n) + \alpha(\Phi + n \Phi - n) + n) \frac{\Phi}{\sqrt{5n^2 + 20n}}.
\]
Expanding the pairs of small brackets inside the pair of large brackets in (8.7), collecting similar terms, substituting \( \alpha^2 = n(1 - \alpha) \) and simplifying, we obtain

\[
\gamma_n = \left( (3 - \Phi)\alpha + (n + 1)\Phi - 1 \right) \frac{\Phi}{\sqrt{5n^2 + 20n}}. \tag{8.8}
\]

Expanding all the remaining brackets in (8.8) and using the equalities \( \alpha = 1 - \mu' \) and \( \Phi^2 = \Phi + 1 \), we obtain (1.13). □

We write \( \sigma_{n,d}(t) \) (where \( n \geq 5 \)) for the number of maximal (non-continuable) blocks \((1, n, 1, n, \ldots, 1, n)\) in the continuant \( \langle D_t \rangle \) represented in the form (7.7). Note that it suffices to prove Theorem 1.1 with \( \varepsilon = 10^{-20} \) (the proof can easily be generalized to the case of an arbitrary \( \varepsilon > 0 \)).

**Lemma 8.2.** For any sufficiently large \( t \) we have

\[
\frac{(\gamma_n - 10^{-20})\sigma_{n,d}(t)}{(c(n))\varphi_{n,d}(t)} < \frac{\langle D_t \rangle}{\sqrt{2}S_d(t)} < \frac{(\gamma_n + 10^{-20})\sigma_{n,d}(t)}{(c(n))\varphi_{n,d}(t)}. \tag{8.9}
\]

**Proof.** Write \( r_1 \) and \( r_n \) for the number of elements of \( \langle D_t \rangle \) equal to 1 and \( n \) respectively. Regarding the equalities \( r_1 + r_n = t \) and \( r_1 + nr_n = S_d(t) \) as a system of linear equations in \( r_1 \) and \( r_n \) and solving this system, we have

\[
r_1 = \frac{tn - S_d(t)}{n - 1}, \quad r_n = \frac{S_d(t) - t}{n - 1}. \tag{8.10}
\]

Note that the total length of the blocks consisting of repeated pairs \((1, n)\) is equal to \( 2r_n \) while the total length of those consisting of 1s is equal to \( t - 2r_n \). Hence the following two-sided bound for the continuant \( \langle D_t \rangle \) can be obtained from (3.4) and (3.5) using Lemma 8.1 and putting \( \varepsilon = 10^{-20} \):

\[
\Phi^{t-2r_n}(\mu_n)^{r_n}(\gamma_n - \varepsilon)^{\sigma_{n,d}(t)} < \langle D_t \rangle < \Phi^{t-2r_n}(\mu_n)^{r_n}(\gamma_n + \varepsilon)^{\sigma_{n,d}(t)}.
\]

It follows that

\[
\Phi^t \left( \frac{\mu_n}{\Phi^2} \right)^{r_n} \left( \frac{S_d(t)-t}{(n-1)} \right)^{\sigma_{n,d}(t)} < \langle D_t \rangle < \Phi^t \left( \frac{\mu_n}{\Phi^2} \right)^{r_n} \left( \frac{\Phi_{n+1}}{\mu_n} \right)^{t/(n-1)}. \tag{8.11}
\]

In view of (8.10), we have the chain of equalities

\[
\Phi^t \left( \frac{\mu_n}{\Phi^2} \right)^{r_n} = \Phi^t \left( \frac{\mu_n}{\Phi^2} \right)^{(S_d(t)-t)/(n-1)} = \left( \frac{\mu_n}{\Phi^2} \right)^{S_d(t)/(n-1)} \left( \frac{\Phi_{n+1}}{\mu_n} \right)^{t/(n-1)}. \tag{8.12}
\]
Substituting $S_d(t) = \kappa_1^{(n)} t + \varphi_{n,d}(t)$ into (8.12) and using the definition (1.5) of the constant $\kappa_1^{(n)}$, we obtain
\[
\frac{\Phi^t (\mu_n/\Phi^2)^{r_n}}{\sqrt{2} S_d(t)} = \frac{(\mu_n/\Phi^2)^{S_d(t)/(n-1)} (\Phi^{n+1}/\mu_n)^{t/(n-1)}}{\sqrt{2} S_d(t)}
\]
\[
= \exp \left( \frac{S_d(t)}{n-1} (\log \mu_n - 2 \log \Phi) + \frac{t}{n-1} ((n+1) \log \Phi - \log \mu_n) - S_d(t) \log \sqrt{2} \right)
\]
\[
= \exp \left( \frac{\kappa_1^{(n)} t + \varphi_{n,d}(t)}{n-1} (\log \mu_n - 2 \log \Phi - (n-1) \log \sqrt{2})
\right.
\]
\[
\left. + \frac{t}{n-1} ((n+1) \log \Phi - \log \mu_n) \right) = \exp \left( \frac{t}{n-1} (\kappa_1^{(n)} (\log \mu_n - 2 \log \Phi - (n-1) \log \sqrt{2}) + (n+1) \log \Phi - \log \mu_n) + \frac{\varphi_{n,d}(t)}{n-1} (\log \mu_n - 2 \log \Phi - (n-1) \log \sqrt{2}) \right)
\]
\[
= \left( \frac{\mu_n}{\Phi^2 \sqrt{2}^{n-1}} \right)^{\varphi_{n,a}(t)/(n-1)} = \frac{1}{(c(n))^{\varphi_{n,a}(t)}}.
\]
Comparing the beginning and end of this chain of equalities, we have
\[
\frac{\Phi^t (\mu_n/\Phi^2)^{r_n}}{\sqrt{2} S_d(t)} = \frac{1}{(c(n))^{\varphi_{n,a}(t)}}. \tag{8.13}
\]
Substituting (8.13) into (8.11), we arrive at (8.9). □

### 8.2. Asymptotic growth of maximal continuants.

Let $\langle D_{\nu} \rangle$ be the continuant defined in Lemma 7.6, and let $\sigma_{d,2}(\nu)$ be the number of its (maximal) blocks $(4,4,\ldots,4)$ up to the element $d_{\nu}$.

**Lemma 8.3.** Fix an arbitrarily small positive $\varepsilon \leq 10^{-20}$. Then, for all sufficiently large (depending on $\varepsilon$) positive integers $t$, the following inequality holds for all integers $\nu$ in the interval $t^{1/3} \leq \nu \leq t'$:
\[
(\gamma - \varepsilon)^{\sigma_{d,2}(\nu)} \lambda^{\varphi_{d,2}(\nu)} < \frac{\langle D_{\nu} \rangle}{\sqrt{2} S_d(\nu)} < (\gamma + \varepsilon)^{\sigma_{d,2}(\nu)} \lambda^{\varphi_{d,2}(\nu)}. \tag{8.14}
\]

**Proof.** It suffices to prove the lemma with $\varepsilon = 10^{-20}$ (the general proof is analogous). Let $r_4$ (resp. $r_5$) be the number of elements in $\langle D_{\nu} \rangle$ equal to 4 (resp. 5). As in (5.8), we obtain
\[
r_4 = 5\nu - S_d(\nu), \quad r_5 = S_d(\nu) - 4\nu. \tag{8.15}
\]
We now write $\langle d_1, d_2, \ldots, d_{\nu} \rangle$ by the ‘Morse code rule’ [11], always dissecting the continuant at the interface between 4s and 5s (the second equality in (2.2)). Using the inequality (3.6) for $n = 4$ and 5, we see that
\[
(\lambda_4)^{r_4} (\lambda_5)^{r_5} (\gamma - \varepsilon)^{\sigma_{d,2}(\nu)} < \langle D_{\nu} \rangle < (\lambda_4)^{r_4} (\lambda_5)^{r_5} (\gamma + \varepsilon)^{\sigma_{d,2}(\nu)}. \tag{8.16}
\]
Substituting the values (8.15) of \( r_4 \) and \( r_5 \) into (8.16), we obtain
\[
\lambda_4^{5\nu - S_d(\nu)} \lambda_5^{S_d(\nu) - 4\nu} (\gamma - \varepsilon)^\sigma_{d, 2}(\nu) < \langle D_\nu \rangle < \lambda_4^{5\nu - S_d(\nu)} \lambda_5^{S_d(\nu) - 4\nu} (\gamma + \varepsilon)^\sigma_{d, 2}(\nu). \tag{8.17}
\]
Substituting \( S_d(\nu) = \kappa_2 \nu - \varphi_{d, 2}(\nu) \) into (8.17) and using the definition (1.9) of the constant \( \kappa_2 \), we obtain
\[
\frac{\lambda_4^{5\nu - S_d(\nu)} \lambda_5^{S_d(\nu) - 4\nu}}{\sqrt{2} S_d(\nu)} = \exp((5\nu - S_d(\nu)) \log \lambda_4 + (S_d(\nu) - 4\nu) \log \lambda_5 - S_d(\nu) \log \sqrt{2})
\]
\[
= \exp((5 \log \lambda_4 - 4 \log \lambda_5) \nu - (\log \lambda_4 - \log \lambda_5 + \log \sqrt{2})(\kappa_2 \nu - \varphi_{d, 2}(\nu)))
\]
\[
= \exp((\log \lambda_4 - \log \lambda_5 + \log \sqrt{2})\varphi_{d, 2}(\nu)) = \left(\frac{\lambda_4 \sqrt{2}}{\lambda_5}\right)^\varphi_{d, 2}(\nu) = \lambda^{\varphi_{d, 2}(\nu)}.
\]
Comparing the beginning and end of this chain of equalities, we have
\[
\frac{\lambda_4^{5\nu - S_d(\nu)} \lambda_5^{S_d(\nu) - 4\nu}}{\sqrt{2} S_d(\nu)} = \lambda^{\varphi_{d, 2}(\nu)}. \tag{8.18}
\]
Using (8.17) and (8.18), we arrive at the conclusion of the lemma. \( \square \)

\section*{§ 9. Preliminary estimates for \( \varphi_{n,d}(\nu) \)}

For further use, we write the inequality (1.15) in Theorem 1.1 in the form
\[
\max_{u \leq t} (S_x(u) - \kappa_1^{(n)} u) = \max_{u \leq t} \varphi_{n,d}(u) \geq M^{(n)} \sqrt{7}, \tag{9.1}
\]
where the constant \( M^{(n)} > 0 \) (depending only on \( n \)) is undetermined at the moment. By Theorem B, we can assume that \( M^{(n)} \) satisfies the inequality
\[
\frac{2}{7n^{2.5}} \leq M^{(n)} \leq (2n)^{2/3} + 21n^{1/3},
\]
that is, the quantity \( M^{(n)} \) is bounded. In what follows we mostly assume that the inequality (9.1) does not hold. We shall prove that this assumption leads to a contradiction for sufficiently small \( M^{(n)} \). The supremum (rounded down) of these \( M^{(n)} \) is the quantity occurring in the statement of Theorem 1.1.

We recall that, given any \( x \in E_n \), a finite sequence \( D_t \) was obtained in Lemma 7.5 for every positive integer \( t \).

\begin{lemma}
Suppose that \( x = [x_1, x_2, \ldots, x_t, \ldots] \in E_n \) (where \( n \geq 5 \)), \( T^{(n)} \) is an infinite sequence of positive integers \( t \) for each of which (9.1) does not hold, and \( \varepsilon > 0 \) is sufficiently small. Then, for all sufficiently large numbers \( t \) (depending on \( x \) and \( \varepsilon \)) in \( T^{(n)} \), the finite sequence \( D_t \) satisfies the following inequality for every \( \nu = 1, 2, \ldots, t \):
\[
\varphi_{n,d}(\nu) < \varepsilon t. \tag{9.2}
\]
\end{lemma}

\textbf{Proof.} Fix any \( t \) in \( T \). Since (9.1) does not hold, for \( \nu = 1, 2, \ldots, t \) we have
\[
S_x(\nu) - \kappa_1^{(n)} \nu < M^{(n)} \sqrt{t} < \varepsilon t \tag{9.3}
\]
for all sufficiently large \( t \) (this largeness is independent of \( M^{(n)} \) since \( M^{(n)} \) is bounded). By Lemma 7.5, (9.3) yields (9.2) for every \( \nu = 1, 2, \ldots, t \):

\[
\varphi_{n,d}(\nu) \leq \max_{u \leq t}(\varphi_{n,d}(u)) \leq \max_{u \leq t}(\varphi_{n,x}(u)) = \max_{u \leq t}(S_x(u) - \kappa_1^{(n)} u) < \varepsilon t.
\]

\( \square \)

**Lemma 9.2.** Suppose that \( x \in E_n \) (where \( n \geq 5 \)) and \( ?'(x) = 0 \). Take any sufficiently small \( \varepsilon > 0 \). Then, for all sufficiently large numbers \( t \) (depending on \( x \) and \( \varepsilon \)), the following inequalities hold for all \( \nu \) in the interval \( t^{2/3} \leq \nu \leq t \):

\[
\varphi_{n,d}(\nu) \geq 0,
\]

\[
\sigma_{n,d}(\nu) \leq \frac{\log c^{(n)}}{\log(\gamma_n - \varepsilon)} \varphi_{n,d}(\nu).
\]

**Proof.** Using Lemma 2.3, we obtain the existence of a real number \( T > 0 \) such that for any \( \nu > T \) we have

\[
\max\left\{ \frac{\langle A_{\nu} \rangle}{\sqrt{2} S_{a(\nu)}}, \ 4n^2(0.8)^\nu \right\} \leq \frac{1}{2^{n^2}}.
\]

(9.6)

By Lemma 7.5, it follows from (9.6) that, for every \( \nu > T \), the continuant \( \langle D_{\nu} \rangle \) satisfies the bound

\[
\frac{\langle D_{\nu} \rangle}{\sqrt{2} S_{a(\nu)}} \leq 1.
\]

(9.7)

Suppose that \( t^{2/3} > T + 1 \). Then it follows from (8.14) for \( t = \nu \) and from (9.7) that

\[
\frac{(\gamma_n - \varepsilon) \sigma_{n,d}(\nu)}{(c^{(n)}) \varphi_{n,d}(\nu)} \leq 1.
\]

(9.8)

Taking the logarithm of (9.8), we obtain (9.5).

Solving the inequality (9.5) for \( \varphi_{n,d}(\nu) \), we obtain (9.4). \( \square \)

**Theorem 9.1.** Suppose that \( x \in E_n \) (where \( n \geq 5 \)) and \( ?'(x) = 0 \). Let \( T^{(n)} \) be an infinite sequence of positive integers \( t \) for each of which (9.1) does not hold. Take any sufficiently small \( \varepsilon > 0 \). Then, for all sufficiently large numbers \( t \) (depending on \( x \) and \( \varepsilon \)) in \( T^{(n)} \), the following inequalities hold for all \( \nu, \nu_1 \) and \( \nu_2 \) in the interval \( [t^{2/3}, t] \):

\[
|\varphi_{n,d}(\nu_1) - \varphi_{n,d}(\nu_2)| \leq \varepsilon t,
\]

(9.9)

\[
0 \leq \varphi_{n,d}(\nu) = S_d(\nu) - \kappa_1^{(n)} \nu \leq M^{(n)} \sqrt{t},
\]

(9.10)

\[
\sigma_{n,d}(\nu) \leq \frac{\log c^{(n)}}{\log(\gamma_n - \varepsilon)} M^{(n)} \sqrt{t}.
\]

(9.11)

**Proof.** Take \( t \) large so that \( t^{2/3} > T + 1 \), where \( T \) is the number in the proof of Lemma 9.2. Then, by Lemmas 9.1 and 11.1, the inequalities (9.2) and (9.4) hold simultaneously for all \( \nu \) in the interval \( [t^{2/3}, t] \). Therefore,

\[
|\varphi_{n,d}(\nu_1)| \leq \frac{\varepsilon t}{2}, \quad |\varphi_{n,d}(\nu_2)| \leq \frac{\varepsilon t}{2}.
\]

(9.12)

Using the triangle inequality in (9.12), we obtain (9.9).
For every \( t > T \) in the sequence \( T^{(n)} \), the following bound holds for all \( \nu \) in the interval \( 1 \leq \nu \leq t \) by the definition of this sequence:
\[
\varphi_{n, \nu}(\nu) < M^{(n)} \sqrt{t}.
\]  
(9.13)

By Lemma 7.5, for every \( \nu \) in the interval \( t^{2/3} \leq \nu \leq t \), the upper (resp. lower) bound in (9.10) follows from (9.13) (resp. (9.4)).

Finally, (9.11) follows directly from (9.5) and the upper bound in (9.10). \( \square \)

§ 10. Proof of the first part of Theorem 1.1

For every \( t \) in \( T^{(n)} \), let \( D^{(-)}_t \) be the subsequence of \( D_t \) formed by the elements \( d_\nu \) where \( \nu \) is any integer in the interval \( [t^{2/3}, t] \). We represent \( D^{(-)}_t \) as the disjoint union of the subsequences \( D^{(-)}_t \) and \( D^{(+)}_t \) formed by the elements \( d_\nu \) where \( \nu \) is any integer in the intervals \( [t^{2/3}, 2t/3] \) and \( (2t/3, t] \) respectively. Let \( \sigma^n(D^{(+)}_t) \) and \( \sigma^n(D^{(-)}_t) \) (for \( n \geq 5 \)) be the number of (maximal) blocks of the form \( (1, n, 1, n, \ldots, 1, n) \) in the sequences \( D^{(+)}_t \) and \( D^{(-)}_t \) respectively. Then
\[
\sigma_{n, d}(t) - \sigma^{(n)}(D^{(-)}_t) + \sigma^{(n)}(D^{(+)}_t) \leq \sigma_{n, d}(t) + 1.
\]  
(10.1)

The upper bound in (10.1) differs from \( \sigma_{n, d}(t) \) because when we divide the interval \( [t^{2/3}, t] \) into \( [t^{2/3}, 2t/3] \) and \( (2t/3, t] \), the separating point (that is, \( 2t/3 \)) may belong to a block of the form \( (1, n, 1, n, \ldots, 1, n) \), and then it divides this block into two blocks of the same form.

Put
\[
\varphi^{(n)}(D^{(-)}_t) = \varphi_{n, d}\left(\left\lceil \frac{2t}{3} \right\rceil \right) - \varphi_{n, d}\left([t^{2/3}]\right), \quad \varphi^{(n)}(D^{(+)}_t) = \varphi_{n, d}(t) - \varphi_{n, d}\left(\left\lceil \frac{2t}{3} \right\rceil \right).
\]

In a similar vein,
\[
S(D^{(-)}_t) = S_d\left(\left\lceil \frac{2t}{3} \right\rceil \right) - S_d\left([t^{2/3}]\right), \quad S(D^{(+)}_t) = S_d(t) - S_d\left(\left\lceil \frac{2t}{3} \right\rceil \right).
\]

By Theorem 9.1 we have \( \varphi^{(n)}(D^{(-)}_t) > -\varepsilon t \) and
\[
S(D^{(+)}_t) - \kappa_1^{(n)}\left(t - \left\lceil \frac{2t}{3} \right\rceil \right) = \varphi^{(n)}(D^{(+)}_t) > -\varepsilon t.
\]  
(10.2)

Suppose that
\[
\frac{\sigma^{(n)}(D^{(-)}_t) + \Theta_1}{\sigma_{n, d}(t)} = \mu, \quad \frac{\sigma^{(n)}(D^{(+)}_t) + \Theta_2(t^{2/3} + 1)}{\sigma_{n, d}(t)} = 1 - \mu,
\]  
(10.3)

where \( |\Theta_1|, |\Theta_2| \leq 1 \) and \( \mu \in [0, 1] \) is an arbitrary real parameter.

Let \( r_n(D^{(+)}_t) \) be the number of elements of \( D^{(+)}_t \) that are equal to \( n \). Similarly to the second formula in (8.10) and in view of (10.2), we obtain the bound
\[
r_n(D^{(+)}_t) = \frac{S(D^{(+)}_t) - (t - \left\lceil \frac{2t}{3} \right\rceil)}{n - 1} \geq \frac{(\kappa_1^{(n)} - 1 - \varepsilon)t}{3(n - 1)}.
\]  
(10.4)
By Dirichlet’s principle and in view of (10.3) and (10.4), the sequence \( D_t^+ \) for every \( \varepsilon > 0 \) contains an interval of the form \( \{1, n, 1, n, \ldots, 1, n\} \) whose length \( s \) satisfies the estimate
\[
s \geq \frac{2r_n(D_t^+)}{\sigma(n)(D_t^+)} = \frac{2(\kappa_1(n) - 1 - \varepsilon)t}{3(n-1)\mu\sigma_{n,d}(t)}.
\] (10.5)

Write this interval in the form \( (d_{t_0+1}, d_{t_0+2}, \ldots, d_{t_0+s}) \). Since \( t_0 \geq [2t/3] \), we deduce from (9.5) for \( \nu = t_0 \) and from (10.3) that
\[
\varphi_{n,d}(t_0) \geq \frac{\sigma_{n,d}(t_0)(\gamma_n - \varepsilon)}{\log c(n)} \geq \frac{(1 - \mu)\sigma_{n,d}(t)\log(\gamma_n - \varepsilon)}{\log c(n)}.
\] (10.6)

It follows from (10.5) and (10.6) that
\[
M(n)\sqrt{t} \geq \varphi_{n,d}(t_0 + s) = \varphi_{n,d}(t_0) + s\left(\frac{n+1}{2} - \kappa_1(n)\right)
\geq \frac{(1 - \mu)\sigma_{n,d}(t)\log(\gamma_n - \varepsilon)}{\log c(n)} + \frac{(\kappa_1(n) - 1 - \varepsilon)(n+1-2\kappa_1(n))}{3(n-1)\mu\sigma_{n,d}(t)}.\] (10.7)

Continuing (10.7) by means of the inequality \( a + b \geq \sqrt{4ab} \), which holds for all non-negative numbers \( a \) and \( b \), we have
\[
M(n) \geq \sqrt{\frac{4 - 4\mu}{3\mu}} \sqrt{\frac{(\kappa_1(n) - 1)(n+1-2\kappa_1(n))\log(\gamma_n - \varepsilon)}{(n-1)\log c(n)}}.\] (10.8)

To obtain another version of the bound for \( M(n) \), we consider the subsequence \( D^-_t \) and, given any \( \varepsilon > 0 \), use the Dirichlet principle to show that it contains an interval of the form \( \{1, n, 1, n, \ldots, 1, n\} \) whose length \( s' \) satisfies the estimate
\[
s' \geq \frac{2r_n(D^-_t)}{\sigma(D^-_t)} = \frac{2(D^-_t)(2t/3 - 2t/3) - t^{2/3}}{(n-1)(1 - \mu)\sigma_{n,d}(t)} \geq \frac{(\kappa_1(n) - 1 - \varepsilon)4t}{3(n-1)(1 - \mu)\sigma_{n,d}(t)}.\] (10.9)

Write this interval in the form \( (d_{t_0'+1}, d_{t_0'+2}, \ldots, d_{t_0'+s'}) \). Since \( t_0' \geq t^{2/3} \), we deduce from Theorem 9.1 that \( \varphi_{n,d}(t_0') \geq 0 \). This and (10.9) yield the bound
\[
M(n)\sqrt{t} \geq \varphi_{n,d}(t_0') + s'\left(\frac{n+1}{2} - \kappa_1(n)\right) \geq \frac{(\kappa_1(n) - 1 - \varepsilon)(n+1-2\kappa_1(n))2t}{3(n-1)(1 - \mu)\sigma_{n,d}(t)}.\] (10.10)

It follows from (10.10) and (9.11) that
\[
M(n) \geq \frac{2(\kappa_1(n) - 1 - \varepsilon)(n+1-2\kappa_1(n))\log(\gamma_n - \varepsilon)}{3(n-1)(1 - \mu)\log c(n)M(n)}.\] (10.11)

Solving the inequality (10.11) for \( M(n) \), we obtain
\[
M(n) \geq \sqrt{\frac{2}{3(1 - \mu)}} \sqrt{\frac{(\kappa_1(n) - 1)(n+1-2\kappa_1(n))\log(\gamma_n - \varepsilon)}{(n-1)\log c(n)}}.\] (10.12)
The second square roots in the bounds (10.8) and (10.12) coincide. We notice that the minima of the first square roots in (10.8) and (10.12) on the intervals \([0, 1/2]\) and \([1/2, 1]\) also coincide. They are attained at \(\mu = 1/2\) and are equal to \(2/\sqrt{3}\). Substituting this value into (10.8) and (10.12), we arrive at the bound

\[ M^{(n)} \geq \frac{2}{\sqrt{3}} \sqrt{\frac{(\kappa_1^{(n)} - 1)(n + 1 - 2\kappa_1^{(n)}) \log(\gamma_n - \varepsilon)}{(n - 1) \log e^{(n)}}}. \]  

(10.13)

Thus the inequality

\[ \max_{u \leq t}(\varphi_{n, x}(u)) < M^{(n)} \sqrt{t} \]  

(10.14)

can hold only when the number \(M^{(n)}\) satisfies the bound (10.13). For every fixed sufficiently large \(t\) in the sequence \(T^{(n)}\), we take the infimum of all \(M^{(n)}\) satisfying the bound (10.14). For this infimum, the inequality (10.13) is preserved and the inequality (10.14) becomes an equality. This yields (1.16). The first part of Theorem 1.1 is proved.

§ 11. Upper bounds for \(\sigma_{d,2}(\nu)\)

Let \(M_2 > 0\) be the quantity (unknown at the moment) occurring in (5.10), \(t' = [t - t^{2/3}]\) the parameter in Lemma 7.6, and \(D_t\) the finite sequence in the same lemma. We recall that \(\log (\gamma + \varepsilon)\) is negative for all sufficiently small \(\varepsilon > 0\).

Lemma 11.1. Suppose that \(?'(x) = +\infty\). Take any sufficiently small \(\varepsilon > 0\). Then, for all sufficiently large positive integers \(t\) (depending on \(x\) and \(\varepsilon\)), the following inequalities hold for all integers \(\nu\) in the interval \(t^{2/3} \leq \nu \leq t': \varphi_{d,2}(\nu) \geq 0\) and

\[ \sigma_{d,2}(\nu) \leq -\frac{\log \lambda}{\log (\gamma + \varepsilon)} \varphi_{d,2}(\nu). \]  

(11.1)

Proof. Using Lemma 2.2, we obtain a real number \(T > 0\) such that for any \(\nu > T\) we have

\[ \min \left( \frac{\langle X_{\nu} \rangle}{\sqrt{2} S_x(\nu)}, (1.05)^\nu \right) \geq 10 \cdot 2^{10^6}. \]  

(11.2)

By Lemma 7.6, it follows from (11.2) that the following bound for the continuant \(\langle D_{\nu} \rangle\) holds for any \(\nu < T\):

\[ \frac{\langle D_{\nu} \rangle}{\sqrt{2} S_{a(\nu)}} \geq 1. \]  

(11.3)

Choose \(t\) so large that \(t^{2/3} > T + 1\). Then (11.3) holds for \(\nu\) in the interval \(t^{2/3} \leq \nu \leq t'\). It follows from (8.14) and (11.3) that

\[ (\gamma + \varepsilon)^{\sigma_{d,2}(\nu)} \lambda^{\varphi_{d,2}(\nu)} \geq 1. \]  

(11.4)

Taking the logarithm of (11.4), we obtain (11.1).

In particular, it follows from (11.1) that \(\varphi_{d,2}(\nu) \geq 0\) for all sufficiently large values of \(\nu\). \(\square\)
Theorem 11.1. Suppose that $x \in (0, 1)$ is an irrational number and $?'(x) = +\infty$. Let $T_2$ be an infinite sequence of positive integers $t$ for each of which (5.10) does not hold. Take any sufficiently small $\epsilon > 0$. Then, for all sufficiently large values of $t$ (depending on $x$ and $\epsilon$) in $T_2$, the following inequalities hold for any $\nu$, $\nu_1$ and $\nu_2$ in the interval $[t^{2/3}, t - t^{2/3}]$:

$$|\varphi_{d,2}(\nu_1) - \varphi_{d,2}(\nu_2)| \leq \epsilon t,$$

$$0 \leq \varphi_{d,2}(\nu) = \kappa_2 \nu - S_d(\nu) \leq (M_2 + \epsilon) \sqrt{t},$$

$$\sigma_{d,2}(\nu) \leq -\frac{\log \lambda}{\log (\gamma + \epsilon)} M_2 \sqrt{t}.$$  \hspace{1cm} (11.5) (11.6) (11.7)

Proof. For every $t > T$ in the sequence $T^{(2)}$, the following bound holds for all $\nu$ in the interval $1 \leq \nu \leq t$ by the definition of this sequence:

$$\varphi_{x,2}(\nu) < M_2 \sqrt{t}.$$  \hspace{1cm} (11.8)

Using this and Lemmas 7.6 and 11.1 for the sequences $D_\nu$, we see that, for all sufficiently large $t$, the following inequalities hold for all $\nu$ in the interval $[t^{2/3}, t - t^{2/3}]$:

$$0 < \varphi_{d,2}(\nu) \leq \max_{\nu \leq t} \varphi_{d,2}(\nu) \leq \max_{\nu \leq t} \varphi_{x,2}(\nu) + 1 \leq (M_2 + \epsilon) \sqrt{t}$$

(the required largeness is independent of $M_2$ since $M_2$ is bounded). This in particular proves (11.6).

Moreover, (11.6) yields the following inequalities for all $\nu_1$ and $\nu_2$ in this interval:

$$|\varphi_{d}(\nu_1)| \leq \frac{\epsilon t}{2}, \quad |\varphi_{d}(\nu_2)| \leq \frac{\epsilon t}{2}. $$  \hspace{1cm} (11.9)

Using the triangle inequality in (11.8), we obtain (11.5).

Finally, (11.7) follows directly from (11.1) and the upper bound in (11.6). \hspace{1cm} □

§ 12. Proof of the first part of Theorem 1.2

For every $t$ in $T^{(2)}$, let $D^{1}_{\nu}$ be the subsequence of $D_\nu$ consisting of all the elements $d_\nu$ with $\nu$ in the interval $[t^{2/3}, t']$. We represent $D^{1}_{\nu}$ as the disjoint union of the subsequences $D^{+}_{\nu}$ and $D^{-}_{\nu}$ consisting of the elements $d_\nu$ where $\nu$ is any integer in the intervals $[t^{2/3}, 2t/3]$ and $[2t/3, t']$ respectively. Let $\sigma(D^{+}_{\nu})$ (resp. $\sigma(D^{-}_{\nu})$) be the number of (maximal) blocks of the form $(4, 4, \ldots, 4)$ in the sequence $D^{+}_{\nu}$ (resp. $D^{-}_{\nu}$). Then

$$\sigma_{d,2}(t') - t^{2/3} \leq \sigma(D^{+}_{\nu}) + \sigma(D^{-}_{\nu}) \leq \sigma_{d,2}(t') + 1. $$  \hspace{1cm} (12.1)

The left-hand side of (12.1) differs from $\sigma_{d,2}(t')$ because when we divide the interval $[t^{2/3}, t']$ into $[t^{2/3}, 2t/3]$ and $[2t/3, t']$, the separating point (that is, $2t/3$) may belong to a block of the form $(4, 4, \ldots, 4)$, and then it divides it into two blocks of the same form.

We put

$$\varphi(D^{(-)}_{\nu}) = \varphi_{d,2}\left([\frac{2t}{3}]\right) - \varphi_{d,2}(t^{2/3}], \quad \varphi(D^{(+)}) = \varphi_{d,2}(t') - \varphi_{d,2}\left([\frac{2t}{3}]\right).$$
In a similar vein,
\[ S(D_{t'}^-) = S_d\left(\frac{2t}{3}\right) - S_d([t^{2/3}]), \quad S(D_{t'}^+) = S_d(t') - S_d\left(\frac{2t}{3}\right). \]

By Theorem 11.1 we have \( \varphi(D_{t'}^-) > -\varepsilon t \) and
\[ \kappa_2 \left(t' - \frac{2t}{3}\right) - S(D_{t'}^+) = \varphi(D_{t'}^+) > -\varepsilon t. \] (12.2)

Suppose that
\[ \sigma(D_{t'}^+) + \Theta_1 \sigma_d(t') = \mu, \quad \sigma(D_{t'}^-) + \Theta_2(t^{2/3} + 1) \sigma_d(t') = 1 - \mu, \] (12.3)

where \( |\Theta_1|, |\Theta_2| \leq 1 \) and \( \mu \in [0, 1] \) is an arbitrary real parameter.

Let \( r_4(D_{t'}^+) \) be the number of elements in \( D_{t'}^+ \) that are equal to 4. Similarly to the second formula in (5.8) and in view of (12.2), for all sufficiently large \( t \) we have
\[ r_4(D_{t'}^+) = 5 \left(t' - \frac{2t}{3}\right) - S(D_{t'}^+) \geq \frac{(5 - \kappa_2 - \varepsilon)t}{3} \] (12.4)
since the ratio \( t'/t \) tends to 1 as \( t \) increases. By the Dirichlet principle and in view of (12.3) and (12.4), for every \( \varepsilon > 0 \) the sequence \( D_{t'}^+ \) contains an interval of the form \((4, 4, \ldots, 4)\) whose length \( s \) satisfies the bound
\[ s \geq \frac{r_4}{\sigma(D_{t'}^+)} \geq \frac{(5 - \kappa_2 - \varepsilon)t}{3\mu \sigma_d(t')} . \] (12.5)

Write this interval in the form \((d_{t_0+1}, d_{t_0+2}, \ldots, d_{t_0+s})\). Since \( t_0 \geq \lfloor 2t/3 \rfloor \), we deduce from (11.1) for \( \nu = t_0 \) and from (12.3) that
\[ \varphi_{d,2}(t_0) \geq -\frac{\sigma_{d,2}(t_0)\log(\gamma + \varepsilon)}{\log \lambda} \geq -\frac{\sigma(D_{t'}^-)\log(\gamma + \varepsilon)}{\log \lambda} \geq (\mu - 1)\frac{\sigma_{d,2}(t')\log(\gamma + \varepsilon)}{\log \lambda} . \] (12.6)

It follows from (12.5) and (12.6) that
\[ M_2\sqrt{t} \geq \varphi_{d,2}(t_0 + s) = \varphi_{d,2}(t_0) + s(\kappa_2 - 4) \geq -\frac{(1 - \mu)\sigma_{d,2}(t')\log(\gamma + \varepsilon)}{\log \lambda} + \frac{(5 - \kappa_2 - \varepsilon)t(\kappa_2 - 4)}{3\mu \sigma_{d,2}(t')} . \] (12.7)

Continuing (12.7) by means of the inequality \( a + b \geq \sqrt{4ab} \), which holds for all non-negative numbers \( a \) and \( b \), we have
\[ M_2 \geq \sqrt{\frac{4 - 4\mu}{3\mu}} \sqrt{-\frac{(5 - \kappa_2)(\kappa_2 - 4)\log(\gamma + \varepsilon)}{\log \lambda}} . \] (12.8)
To obtain another version of the bound for $M_2$, we consider the subsequence $D_t^-$ and, given any $\varepsilon > 0$, use the Dirichlet principle to show that it contains an interval of the form $(4, 4, \ldots, 4)$ whose length $s'$ satisfies the inequality

$$s' \geq \frac{r_4}{\sigma(D_{t'})} \geq \frac{5(2t/3 - t^{2/3}) - S_d(2t/3)}{(1 - \mu)\sigma_d(t')},$$

(12.9)

Write this interval in the form $(d_{t_0 + 1}, d_{t_0 + 2}, \ldots, d_{t_0 + s'})$. Since $t_0 \geq t^{2/3}$, we deduce from (11.6) with $\nu = t_0$ that $\varphi_{d_2}(t'_0) \geq 0$. Using this and (12.9), we obtain the bound

$$M_2 \geq \varphi_{d_2}(t'_0) + s'(\kappa_2 - 4) \geq \frac{(5 - \kappa_2 - \varepsilon)(\kappa_2 - 4)2t}{3(1 - \mu)\sigma_{d,2}(t')}.$$  

(12.10)

It follows from (11.7) and (12.10) that

$$M_2 \geq \frac{(5 - \kappa_2 - \varepsilon)(\kappa_2 - 4)2t}{3(1 - \mu)\sigma_{d,2}(t')} \geq \frac{2(5 - \kappa_2 - \varepsilon)(\kappa_2 - 4)\log(\gamma + \varepsilon)\sqrt{t}}{3(1 - \mu)M_2\log \lambda}.$$  

(12.11)

Solving the inequality (12.11) for $M_2$, we obtain

$$M_2 \geq \sqrt{\frac{2}{3(1 - \mu)}} \sqrt{\frac{(5 - \kappa_2)(\kappa_2 - 4)\log(\gamma + \varepsilon)}{\log \lambda}}.$$  

(12.12)

The second square roots in the bounds (12.8) and (12.12) coincide. We notice that the minima of the first square roots in (12.8) and (12.12) on the intervals $[0, 1/2]$ and $[1/2, 1]$ also coincide. They are attained at $\mu = 1/2$ and are equal to $2/\sqrt{3}$. Substituting this value into (12.8) and (12.12), we arrive at the bound

$$M_2 \geq \frac{2}{\sqrt{3}} \sqrt{-\frac{(5 - \kappa_2)(\kappa_2 - 4)\log(\gamma + \varepsilon)}{\log \lambda}}.$$  

(12.13)

Thus the inequality

$$\max_{u \leq t}(\varphi_{a,2}(u)) < M_2 \sqrt{t}$$  

(12.14)

can hold only when $M_2$ satisfies the bound (12.13). For every fixed sufficiently large $t$ in the sequence $T^{(2)}$, we take the infimum of all $M_2$ satisfying (12.14). For this infimum, the inequality (12.13) is preserved and the inequality (12.14) becomes an equality. This yields (1.19). The first part of Theorem 1.2 is proved.

§ 13. Constructions of the number $x$

13.1. Proof of the second part of Theorem 1.1. We construct a number $x = [x_1, \ldots, x_t, \ldots]$ satisfying the assertion of the theorem. Introduce the parameters $t_0 = 16n^{12}$ and $t_i = 4^it_0$, where $i = 1, 2, 3, \ldots$. Partition the infinite sequence $(x_1, \ldots, x_t, \ldots)$ into a countable family of intervals. The interval labelled $i$ is given by the subsequence $(x_{t_i/4+1}, \ldots, x_{t_i})$. The initial interval $(x_1, \ldots, a_{t_0/4})$ may be
chosen arbitrarily, for example, to consist only of 1s. This is irrelevant since \( t_0 \) is an absolute constant. Thus the \( i \)th interval consists of \((3/4)t_i\) elements.

The following notions are convenient. First, given any non-negative integers \( p \) and \( j \) with \( t_p/4 + j \) not exceeding \( t_p \), we define the local \( n \)-deviation (for \( n \geq 5 \)) at the point \( j \) as the quantity

\[
\varphi_{n,p}^{\text{loc}}(j) = \sum_{m=1}^{j} a_{t_p/4+m} - \kappa_1^{(n)} j.
\]

(13.1)

Second, for every \( i \geq 0 \), the quantity

\[
\varphi_{n,i}^{\text{full}} = \varphi_{n,i}^{\text{loc}} \left( \frac{3}{4} t_i \right) = \sum_{m=t_i/4+1}^{t_i} a_m - \frac{3}{4} t_i \kappa_1^{(n)}
\]

is called the total \( n \)-deviation of the \( i \)th interval. Third, the quantity

\[
\varphi_{n,p}^{\text{tot}}(j) = \sum_{m=t_0/4+1}^{t_p+j} a_m - \kappa_1^{(n)} \left( t_p + j - \frac{t_0}{4} \right) = \left( \sum_{i=0}^{p-1} \varphi_{n,i}^{\text{full}} \right) + \varphi_{n,p}^{\text{loc}}(j)
\]

is called the accumulated \( n \)-deviation at the point \( j \). To prove (1.17) (resp. (1.18)), it suffices to construct a number \( x \in E_n \) such that \( ?'(x) = 0 \) and, for any integers \( p \) and \( j \) satisfying \( p \geq 0 \) and \( 1 \leq j \leq 3\sqrt{t_p}/4 \), we have

\[
\varphi_{n,p}^{\text{tot}}(j) < 2.39 \sqrt{t_p} \quad (\text{resp. } \varphi_{n,p}^{\text{tot}}(j) < 3.5 \sqrt{t_p}).
\]

(13.2)

Indeed, the inequalities (1.17) and (1.18), in which the coefficient of \( \sqrt{t} \) is twice as larger as in (13.2), are obtained from (13.2) by substituting the estimate \( t_p \leq 4t \), which holds for all \( t \) in the interval \( t_{p-1} < t \leq t_p \), in view of the equality \( t_p = 4t_{p-1} \).

We now construct the \( i \)th interval. It consists of \( 3\sqrt{t_i}/4 = (3/4)\sqrt{4^{i+2}n^{12}} = 3 \cdot 2^i n^6 \) blocks each of length \( \sqrt{t_i} \). Every block is of the following structure:

\[
\left( 1, n, 1, n, \ldots, 1, n, 1, 1, \ldots, 1 \right)_{k \text{ pairs, } \sqrt{t_i}-2k \text{ numbers}}.
\]

(13.3)

Here \( k \) depends on the number \( i \) of the interval and on the number of the block, to be denoted by \( h \). Thus, \( h \leq 3\sqrt{t_i}/4 \). In the following lemma we consider the situation when the number \( j \) in (13.1) satisfies \( j = h\sqrt{t_i} \).

**Lemma 13.1.** For every \( i \geq 0 \) there is a version of the construction of the \( i \)th interval such that

\[
\left| \varphi_{n,i}^{\text{loc}}(h\sqrt{t_i}) - \frac{4}{3} h \right| < n
\]

(13.4)

for all positive integers \( h \) not exceeding \( 3\sqrt{t_i}/4 \). In particular,

\[
\left| \varphi_{n,i}^{\text{full}} - \sqrt{t_i} \right| < n.
\]

(13.5)
The first term on the right-hand side of (13.8) can be estimated using (13.4): is, the point 1+ $(13.10)$ is maximal when the deviation).

Proof. The proof is by induction on $h$. When $h = 1$, we need to estimate the local $n$-deviation over one block of the form (13.3). Letting $k$ vary from 0 to $\sqrt{\tau_i}/2$, we obtain blocks whose $n$-deviation ranges from $-(\kappa_1(n)-1)\sqrt{\tau_i}$ to $((n+1)/2-\kappa_1(n))\sqrt{\tau_i}$. Since changing $k$ by 1 results in a change of the $n$-deviation by $n-1$, we can choose $k$ in such a way that the local $n$-deviation $\varphi_{i,loc}(\sqrt{\tau_i})$ is equal to $4/3$ with discrepancy at most $n-1$ (for one block, the discrepancy is greater than the resulting value of the deviation).

The rest is similar. If $|\varphi_{n,i}^{loc}((h-1)\sqrt{\tau_i})-4/3(h-1)| < n$, we use the same technique to construct the $h$th block in such a way that (13.4) holds. As $h$ grows, the quantity $(4/3)h$ becomes larger than the discrepancy. In particular, when $h = (3/4)\sqrt{\tau_i}$, the achieved inequality (13.4) takes the form (13.5).

Lemma 13.2. The interval constructed in Lemma 13.4 possesses the following property. For every $i \geq 0$ and all $j$ in the interval $0 < j \leq (3/4)t_i$, we have

$$-n - 1 \leq \varphi_{n,i}^{loc}(j) < 2.5\sqrt{\tau_i} + n. \quad (13.6)$$

In particular, for all sufficiently large $n$ we have

$$-n - 1 \leq \varphi_{n,i}^{loc}(j) < 1.39\sqrt{\tau_i} + n. \quad (13.7)$$

Proof. The lower bound in (13.6) follows immediately from the structure of the block (13.3). Indeed, the function $\varphi_{n,i}^{loc}(j)$ attains its minimum on the interval $[0, (3/4)t_i]$ at a point $j$ where $a_{t_i/4+j} = 1$. On the other hand, since $1+n > 2\kappa_1(n)$, the minimum can be attained only at the first element of a block, that is, the point $j$ satisfies the equality $j = h\sqrt{\tau_i} + 1$ for some non-negative integer $h$. Since $|\varphi_{n,i}^{loc}(j-1)-\varphi_{n,i}^{loc}(j)| < 1$, the lower bound in (13.6) follows from the inequality $\varphi_{n,i}^{loc}(j-1) > -n$, which holds in view of (13.4).

To prove the upper bound in (13.6), we represent $j$ in the form $j = h\sqrt{\tau_i} + r$, where $0 \leq r < \sqrt{\tau_i}$. Then

$$\varphi_{n,i}(j) = \varphi_{n,i}(h\sqrt{\tau_i}) + \left( \sum_{m=1}^r a_{t_i/4+h\sqrt{\tau_i}+m} \right) - \kappa_1(n)r. \quad (13.8)$$

The first term on the right-hand side of (13.8) can be estimated using (13.4):

$$\varphi_{n,i}(h\sqrt{\tau_i}) < \frac{4}{3}h + n \leq \frac{4}{3}\frac{3}{4}\sqrt{\tau_i} + n = \sqrt{\tau_i} + n. \quad (13.9)$$

We now estimate the quantity

$$\left( \sum_{m=1}^r a_{t_i/4+h\sqrt{\tau_i}+m} \right) - \kappa_1(n)r. \quad (13.10)$$

Recall that the structure of the block $(a_{t_i/4+h\sqrt{\tau_i}+1}, \ldots, a_{t_i/4+(h+1)\sqrt{\tau_i}})$ (that is, the $h$th block of the $i$th interval) is described in (13.3). It is clear from this structure that the quantity (13.10) is maximal when $r = 2r_n$, where $r_n$ is the number of elements equal to 1 in block

$\text{Proof.}$
this block. Then \( r_1 + r_n \) (resp. \( r_1 + nr_n \)) is the length (resp. the sum of the elements) in the block. In other words, \( r_1 + r_n = \sqrt{t_i} \) and, by the triangle inequality, we have

\[
|r_1 + nr_n - \kappa_1^{(n)} \sqrt{t_i}| \leq \left| \sum_{m=1}^{(h+1)\sqrt{t_i}} a_{t_i/4+m} - \kappa_1^{(n)} (h+1) \sqrt{t_i} - \frac{4}{3} (h+1) \right|
\]

\[
+ \left| \sum_{m=1}^{h\sqrt{t_i}} a_{t_i/4+m} - \kappa_1^{(n)} h \sqrt{t_i} - \frac{4}{3} h \right| + \frac{4}{3} < n + n + \frac{4}{3} < 3n
\]  

(13.11) in view of (13.4). It follows from (13.11) that \( r_1 + nr_n = \kappa_1^{(n)} \sqrt{t} + O(n) \). Hence,

\[
r_n = \frac{\kappa_1^{(n)} - 1}{n - 1} \sqrt{t_i} + O(1).
\]

(13.12)

Therefore,

\[
\left( \sum_{m=1}^{2r_n} a_{t_i/4+k\sqrt{t_i}+m} \right) - 2\kappa_1^{(n)} r_n = 2r_n \frac{n+1}{2} - 2\kappa_1^{(n)} r_n = r_n (n + 1 - 2\kappa_1^{(n)})
\]

\[
= \sqrt{t_i} \frac{\kappa_1^{(n)} - 1}{n - 1} (n + 1 - 2\kappa_1^{(n)}) + O(n) \leq \sqrt{t_i} (\kappa_1^{(n)} - 1) < 1.5 \sqrt{t_i}.
\]  

(13.13)

Substituting the bounds (13.9) and (13.13) into (13.8), we obtain the upper bound in (13.6). The bound (13.7) can be obtained in a similar way. We only note that the quantity \( \sqrt{t_i} (\kappa_1^{(n)} - 1) \) in (13.13) does not exceed 0.39 \( \sqrt{t_i} \) for sufficiently large \( n \) since \( \kappa_1^{(n)} \) tends to its limiting value \( \kappa_1 \) in view of (1.6). \( \square \)

In the following lemma we estimate the accumulated \( n \)-deviation using Lemmas 13.1 and 13.2.

**Lemma 13.3.** Suppose that \( \varepsilon > 0 \) and \( 1 \leq j \leq (3/4) t_p \). Then, for all sufficiently large \( p \), we have

\[
(1 - \varepsilon) \sqrt{t_p} < \varphi_{n,p}^{\text{tot}}(j) < (3.5 + \varepsilon) \sqrt{t_p}.
\]  

(13.14)

In particular, for all sufficiently large \( n \) we have

\[
(1 - \varepsilon) \sqrt{t_p} < \varphi_{n,p}^{\text{tot}}(j) < (2.39 + \varepsilon) \sqrt{t_p}.
\]  

(13.15)

**Proof.** We use the identity

\[
\varphi_{n,p}^{\text{tot}}(j) = \left( \sum_{i=0}^{p-1} \varphi_{n,i}^{\text{full}} \right) + \varphi_{n,p}^{\text{loc}}(j).
\]  

(13.16)

Substitute the bounds (13.5) and (13.6) into (13.16):

\[
\varphi_{n,p}^{\text{tot}}(j) < \left( \sum_{i=0}^{p-1} (\sqrt{t_i} + n) \right) + (2.5 \sqrt{t_p} + n) = \left( \sum_{i=0}^{p-1} \frac{\sqrt{t_p}}{2^i} \right) + pn + 2.5 \sqrt{t_p} + n.
\]

Continuing this chain, we obtain the bound

\[
\varphi_{n,p}^{\text{tot}}(j) < 3.5 \sqrt{t_p} + (p + 1)n.
\]  

(13.17)
Then the upper bound in (13.14) is obtained from (13.17) by comparing the rates of growth of the geometric progression $\sqrt{t_p} = \sqrt{4p^2 n^{12}} = 4 \cdot 2^p n^6$ and the arithmetic progression $(p + 1)n$. The lower bound can be proved in a similar way.

Replacing (13.6) by (13.7) in this argument for sufficiently large $n$, we arrive at (13.15). □

To complete the proof of the theorem, we need to verify the first relation in (2.7). This can be done using Lemma 8.2. We first obtain an upper bound $\sigma_{n,x}(t)$.

Suppose that $t_{p-1} < t < t_p$. Then $\sigma_{n,x}(t) \leq \sigma_{n,x}(t_p)$ by monotonicity. Since $\sigma_{n,x}(t_p)$ is equal to the total number of blocks constructed up to the element $t_p$ and since the number of blocks of the $i$th interval is always equal to $3\sqrt{t}/4$, we obtain the bound

$$\sigma_{n,x}(t) \leq \sigma_{n,x}(t_p) \leq \frac{3}{4} \sum_{i=0}^{p} \sqrt{t_i} = \frac{3}{4} \sum_{i=0}^{p} \frac{\sqrt{t_p}}{2^{i}} < \frac{3}{2} \sqrt{t_p}. \quad (13.18)$$

Substituting the bounds (13.15) and (13.18) into (8.14) and replacing $\varphi_{n,x}(t)$ by $\varphi_{n,p-1}^{\text{tot}}(t_{p-1})$, we obtain the following bound for $X_t = D_t$ with $t_{p-1} < t < t_p$:

$$\left\langle X_t \right\rangle \leq \frac{(\gamma_n + \varepsilon)\sigma_{n,x}(t)}{(c(n))^{n,x}(t)} \leq \frac{(\gamma_n + \varepsilon)(3/2)\sqrt{t_p}}{(c(n))^{1/2}} \leq 2 \left( \frac{(\gamma_n + \varepsilon)(3/2)}{c(n)} \right)^{1/2}. \quad (13.19)$$

Of course, to justify the first inequality in (13.19), we need to verify that the sequence $X_t$ consists of sufficiently long ‘intervals’ of the form $(1,1,\ldots,1)$ and $(1,n,1,n,\ldots,1,n)$. This is verified by comparing (13.3) and (13.12), which yields the following equalities for every $i \leq p$:

$$k = r_n = \frac{\kappa_1(n)}{n - 1} \sqrt{t_i} + O(1),$$

$$\sqrt{t_i} - 2k = \left(1 - 2\frac{\kappa_1(n)}{n - 1}\right) \sqrt{t_i} + O(1) = \frac{n - 2\kappa_1(n)}{n - 1} \sqrt{t_i} + O(1).$$

Hence it follows from the conditions imposed on the numbers $t_i$ that $k \geq n^5$ and $\sqrt{t_i} - 2k \geq n^5$, as required in the hypotheses of Lemma 8.2.

We easily see that $(\gamma_n + \varepsilon)(3/2) < c(n)$ for all $n \geq 5$. Hence the expression on the right-hand side of (13.19) tends to zero. Therefore, $\varphi'(x) = 0$. The second part of Theorem 1.1 is proved.

13.2. Proof of the second part of Theorem 1.2. We seek $x$ in the form $x = [x_1, \ldots, x_t, \ldots]$. The coefficients $x_1, \ldots, x_t, \ldots$ will be chosen in a special way. To do this, we partition the sequence $(x_1, x_2, \ldots, x_t, \ldots)$ into a countable family of intervals. The interval labelled $i$ is of the form $(x_{t_i/4+1}, \ldots, x_{t_i})$, where $t_0 = 10^4$ and $t_i = 4^4t_0$ for positive integers $i$. The initial interval $(x_1, \ldots, x_{t_0/4})$ consists of 4's only. Since $t_0$ is an absolute constant, this does not influence the value of the derivative. Thus, the $i$th interval consists of $(3/4)t_i$ elements. The following notions are convenient. For every $i$ we define the local deviation as the quantity

$$\varphi_{i}^{\text{loc}}(j) = \kappa_2 j - \sum_{m=1}^{j} a_{t_i/4+m}.$$
Here \( t_i/4 + j \) must be less than or equal to \( t_i \). The quantity
\[
\varphi_i^{\text{full}} = \varphi_i^{\text{loc}} \left( \frac{3}{4} t_i \right) = \kappa_2 \frac{3}{4} t_i - \sum_{m=1}^{3/4 t_i} a_{t_i/4+m}
\]
is called the total deviation of the \( i \)th interval. Finally, the quantity
\[
\varphi_i^{\text{tot}}(j) = \kappa_2 \left( t_i + j - \frac{t_0}{4} \right) - \sum_{m=t_0/4+1}^{t_i+j} a_m = \varphi_i^{\text{loc}}(j) + \sum_{m=0}^{i-1} \varphi_m^{\text{full}}
\]
is called the accumulated deviation at the given point \( j \).

We proceed to construct the \( i \)th interval. It consists of \( \sqrt[4]{t_i} \) blocks (this number is easily seen to be an integer) of length \( \sqrt[4]{t_i} \) each. Every block is of the following structure:
\[
(5, 5, \ldots, 5, 4, 4, \ldots, 4) \quad \text{with} \quad k \text{ numbers} \quad 0.1 \sqrt{t_i} - k \text{ numbers}
\]  \quad (13.20)

Here \( k \) depends on the number \( i \) of the interval and on the number \( m \) of the block. Let \( c \) be a positive constant, to be determined at the end of the proof.

**Lemma 13.4.** For every \( i \geq 1 \) there is a version of the construction of the \( i \)th interval such that, for all positive integers \( m \) not exceeding \( 7.5 \sqrt{t_i} \), we have
\[
\left| \varphi_i^{\text{loc}}(0.1 m \sqrt{t_i}) - c m \right| < 1. \quad (13.21)
\]
In particular,
\[
\left| \varphi_i^{\text{full}} - 7.5 c \sqrt{t_i} \right| < 1. \quad (13.22)
\]

**Proof.** We use induction on \( m \). When \( m = 1 \), we need to estimate the local deviation over one block of the form (13.20). Letting \( k \) vary from 0 to \( 0.1 \sqrt{t_i} \), we obtain blocks with deviation from \( 0.1(\kappa_2-5)\sqrt{t_i} \) (minimum) to \( 0.1(\kappa_2-4)\sqrt{t_i} \) (maximum). Since changing \( k \) by 1 results in a change of the deviation by 1, we can choose \( k \) in such a way that the local deviation \( \varphi_i^{\text{loc}}(\sqrt{t_i}) \) is approximately equal to \( c \) and the absolute discrepancy does not exceed 1. The rest is similar. If
\[
\left| \varphi_i^{\text{loc}}(0.1(m-1)\sqrt{t_i}) - c(m-1) \right| < 1,
\]
then we use the same technique to construct the \( m \)th block in such a way that (13.21) holds. \( \square \)

**Lemma 13.5.** The interval constructed in Lemma 13.4 possesses the following property. For any \( \alpha \) in the interval \( 0 < \alpha \leq 0.75 \) such that \( j = \alpha t_i \in \mathbb{N} \), we have
\[
(10c\alpha - 0.1(\kappa_2-4)(5-\kappa_2))\sqrt{t_i} + O(1) < \varphi_i^{\text{loc}}(\alpha t_i) < 10c\alpha\sqrt{t_i} + O(1). \quad (13.23)
\]

**Proof.** Suppose that \( j = \alpha t_i \) is divisible by \( 0.1\sqrt{t_i} \), in other words, that the finite sequence
\[
(a_{t_i/4+1}, a_{t_i/4+2}, \ldots, a_{t_i-1+j})
\]
consists of exactly \( m = 10\alpha \sqrt{t_i} \) blocks. By Lemma 13.4 we have

\[
    cm - 1 < \varphi_i^{\text{loc}}(0.1m\sqrt{t_i}) < cm + 1. 
\]  

(13.24)

Hence, substituting \( m = 10\alpha \sqrt{t_i} \) into (13.24), we obtain

\[
    10\alpha \sqrt{t_i} - 1 < \varphi_i^{\text{loc}}(\alpha t_i) < 10\alpha \sqrt{t_i} + 1. 
\]  

(13.25)

We also note that substituting \( m \) and \( m + 1 \) into (13.24), we can obtain

\[
    \varphi_i^{\text{loc}}(0.1m\sqrt{t_i}) - \varphi_i^{\text{loc}}(0.1(m - 1)\sqrt{t_i}) = O(1) 
\]

or, equivalently,

\[
    \sum_{n=1}^{0.1\sqrt{t}} x_{t_i/4+0.1m\sqrt{t_i}+n} = 0.1\kappa_2 \sqrt{t} + O(1). 
\]  

(13.26)

We now suppose that \( j \) is represented in the form \( j = 0.1m\sqrt{t_i} + r \), where \( 0 < r < 0.1\sqrt{t_i} \). Then

\[
    \varphi_i^{\text{loc}}(j) = \varphi_i^{\text{loc}}(0.1m\sqrt{t_i}) + \kappa_2 r - \sum_{n=1}^{r} x_{t_i/4+0.1m\sqrt{t_i}+n}. 
\]  

(13.27)

In view of (13.25), the first term on the right-hand side of (13.27) can be approximated by \( 10\alpha \sqrt{t_i} \) with absolute discrepancy at most 1. We now estimate the quantity

\[
    \kappa_2 r - \sum_{n=1}^{r} x_{t_i/4+0.1m\sqrt{t_i}+n}. 
\]  

(13.28)

We easily see from the structure of the block \( (x_{t_i/4+0.1m\sqrt{t_i}+1}, \ldots, x_{t_i/4+(m+1)0.1\sqrt{t_i}}) \) that the quantity (13.28) is minimal when \( r = r_5 \), where \( r_5 \) is the number of elements of this block that are equal to 5. Moreover, (13.28) is maximal when \( r = 0.1\sqrt{t_i} \), that is, at an endpoint of the block. Let \( r_4 \) be the number of elements of this block that are equal to 4. We obtain from (13.26) that \( r_4 + r_5 = 0.1\sqrt{t_i} \) and \( 4r_4 + 5r_5 = 0.1\kappa_2 \sqrt{t_i} + O(1) \). Therefore, \( r_5 = 0.1(\kappa_2 - 4)\sqrt{t_i} + O(1) \). Hence,

\[
    \kappa_2 r_5 - \left( \sum_{n=1}^{r_5} x_{t_i/4+0.1m\sqrt{t_i}+n} \right) = (\kappa_2 - 5)r_5 = -0.1(5 - \kappa_2)(\kappa_2 - 4)\sqrt{t_i} + O(1). 
\]  

(13.29)

Combining (13.25), (13.27) and (13.29), we complete the proof of the lemma. \( \square \)

A bound for the accumulated deviation can easily be obtained from Lemmas 13.4 and 13.5.

**Lemma 13.6.** For any \( i \geq 0 \), an arbitrary \( \alpha \) in the interval \( 0 < \alpha \leq 0.75 \) such that \( j = \alpha t_i \in \mathbb{N} \), any \( \varepsilon > 0 \), and all sufficiently large \( t \) we have

\[
    (7.5c + 10\alpha \varepsilon - 0.1(\kappa_2 - 4)(5 - \kappa_2) - \varepsilon) \sqrt{t_i} < \varphi_i^{\text{tot}}(\alpha t_i) < (7.5c + 10\alpha + \varepsilon) \sqrt{t_i}. 
\]  

(13.30)
Proof. We use the identity
\[
\varphi^\text{tot}_i(j) = \varphi^\text{loc}_i(j) + \sum_{m=0}^{i-1} \varphi^\text{full}_i.
\] (13.31)

In view of (13.22) we have
\[
\sum_{m=0}^{i-1} \varphi^\text{full}_i = \sum_{m=0}^{i-1} (7.5c\sqrt{t_m} + O(1)) = 7.5c \sum_{m=1}^{i} \left(\frac{\sqrt{t_i}}{2^m}\right) + O(\log t_i) = (7.5c + \varepsilon)\sqrt{t_i}.
\] (13.32)

Substituting (13.32) and (13.23) into (13.31), we obtain the required assertion. □

We now need to find a lower bound for \(c\) under which the equality (2.5) holds. (Since \(x \in \mathbf{E}_c\) by construction, (2.5) follows from (2.6).) To do this, we derive an upper bound for \(\sigma_{x,2}(t)\), put \(t = t_i/4 + \alpha t_i\), where \(0 < \alpha \leq 0.75\). It is clear that
\[
\sigma_{x,2}\left(\frac{t_i}{4}\right) = 7.5 \sum_{m=0}^{i-1} \sqrt{t_m} = 7.5 \sum_{m=1}^{i} \frac{\sqrt{t_i}}{2^m} < 7.5\sqrt{t_i}.
\] (13.33)

As a result,
\[
\sigma_{x,2}(t) < (7.5 + 10\alpha)\sqrt{t_i} + 1.
\] (13.34)

Substituting (13.30) and (13.33) into (8.14) and replacing \(\varphi^\text{x}_2(t)\) by \(\varphi^\text{tot}_i(\alpha t_i)\), we obtain the following bound for \(X_t = D_t\) with \(t = t_i/4 + \alpha t_i\):
\[
\frac{(X_t)}{\sqrt{2}S(x(t))} > (\gamma - \varepsilon)^{\sigma_{x,2}(t)} \chi^{\varphi_{x,2}(t)}
\]
\[
> (\gamma - \varepsilon)^{(7.5+10\alpha+\varepsilon)\sqrt{t_i}} \chi^{(7.5c+10ca-((\kappa_2-4)(5-\kappa_2))/10-\varepsilon)\sqrt{t_i}}.
\]

Thus it suffices to show that
\[
(\gamma - \varepsilon)^{7.5+10\alpha} \chi^{7.5c+10ca-0.1(\kappa_2-4)(5-\kappa_2)} > 1
\] (13.35)

for \(0 \leq \alpha \leq 0.75\). Rewriting (13.34), we obtain
\[
((\gamma - \varepsilon)^{7.5} \chi^{-0.1(\kappa_2-4)(5-\kappa_2)} \chi^{7.5c}) ((\gamma - \varepsilon) \chi^c)^{10\alpha} > 1.
\] (13.35)

Taking the logarithms of both factors in (13.35), we see that it suffices to verify two inequalities:
\[
7.5c \log \lambda > 0.1(\kappa_2 - 4)(5 - \kappa_2) \log \lambda - 7.5 \log(\gamma - \varepsilon),
\] (13.36)
\[
c > \frac{\log(\gamma - \varepsilon)}{\log \lambda} = 0.01209 \ldots.
\] (13.37)
Solving (13.36) for all sufficiently small $\varepsilon$, we find the condition
\[ c > \frac{0.1(\kappa_2 - 4)(5 - \kappa_2)}{75} - \frac{\log(\gamma - \varepsilon)}{\log \lambda} = 0.015292 \ldots . \] (13.38)

Moreover, (13.37) holds automatically under this condition. It follows that (2.5) holds for $c = 0.015293$. Therefore,
\[ \varphi_{t_i}^{\text{tot}}(\alpha t_i) < (7.5c + 10c\alpha)\sqrt{t_i} = \left(\frac{7.5c}{\sqrt{\alpha}} + 10c\sqrt{\alpha}\right)\sqrt{\alpha t_i}. \] (13.39)

The sum in brackets in the final part of (13.39) is maximal when $\alpha = 0.75$. In this case we obtain (1.20):
\[ \varphi_{t_i}^{\text{tot}}(\alpha t_i) < 10\sqrt{3}c\sqrt{\alpha t_i} < 0.26489\sqrt{\alpha t_i}. \]

The second part of Theorem 1.2 is proved.

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