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A $q$-deformation of true-polyanalytic Bargmann transforms when $q^{-1} > 1$

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Abstract. We combine continuous $q^{-1}$-Hermite Askey polynomials with new 2D orthogonal polynomials introduced by Ismail and Zhang as $q$-analogs for complex Hermite polynomials to construct a new set of coherent states depending on a nonnegative integer parameter $m$. Our construction leads to a new $q$-deformation of the $m$-true-polyanalytic Bargmann transform on the complex plane. In the analytic case $m = 0$, the obtained coherent states transform can be associated with the Arïk-Coon oscillator for $q' = q^{-1} > 1$. These result may be used to introduce a $q$-deformed Ginibre-type point process.

1. Introduction and statement of the results

In [13], Bargmann introduced a transform which maps isometrically the space $L^2(\mathbb{R})$ onto the Fock space $\mathcal{F}(\mathbb{C})$ of entire functions belonging to $\mathcal{F}_1 := L^2(\mathbb{C}, e^{-zz^*} d\lambda(z)/\pi)$ where $d\lambda(z)$ is the Lebesgue measure on $\mathbb{C}$. Since this transform is strongly linked to the Heisenberg group, it can be seen as a windowed Fourier transform [18]. Hence, the important role it plays in signal processing and harmonic analysis on the phase space [16]. It is also possible to interpret the kernel of this transform in terms of coherent states [5] of the quantum harmonic oscillator whose eigenstates are given by Hermite functions

$$\varphi_j(\xi) = \left( \sqrt{\pi} 2^{j} / j! \right)^{1/2} H_j(\xi) e^{-1/2 \xi^2},$$

$H_j(\cdot)$ being the $j$th Hermite polynomial ([22, p. 50]). A coherent state can be defined by a normalized vector $\Psi_z$ in $L^2(\mathbb{R})$, as a special superposition with the form

$$\Psi_z := (e^{zz^*})^{-1/2} \sum_{j \geq 0} \frac{z^j}{\sqrt{j!}} \varphi_j, \quad z \in \mathbb{C}.$$
It turns out that the coefficients
\[ h_j(z) := \frac{z^j}{\sqrt{j!}}, \quad j = 0, 1, 2, \ldots, \]  
form an orthonormal basis of $\mathcal{F}(\mathbb{C})$. If we denote by $\mathcal{B}_0$ the Bargmann transform, the image of an arbitrary function $f \in L^2(\mathbb{R})$ can be written as
\[ \mathcal{B}_0[f](z) := \pi^{-\frac{1}{4}} \int_{\mathbb{R}} e^{-\frac{1}{2}z^2 - \frac{1}{2}i\xi^2 + \sqrt{2}iz\xi} f(\xi) d\xi, \quad z \in \mathbb{C}. \]  
Otherwise, it was proven [9] that $\mathcal{F}(\mathbb{C})$ coincides with the null space
\[ \mathcal{A}_0(\mathbb{C}) := \{ F \in \mathcal{F}, \Delta F = 0 \} \]  
of the second-order differential operator
\[ \Delta := -\frac{\partial^2}{\partial z \partial \bar{z}} + \bar{z} \frac{\partial}{\partial z}. \]  
The latter one, which acts on the Hilbert space, can be unitarily intertwined to appear as the Schrödinger operator for the motion of a charged particle evolving in a constant and uniform magnetic field normal to the plane. The spectrum of $\Delta$ in $\mathcal{F}$ is the set of eigenvalues $m \in \mathbb{Z}_+$, each of which has an infinite multiplicity, usually called Euclidean Landau levels. For $m \in \mathbb{Z}_+$, the associated eigenspace [9]:
\[ \mathcal{A}_m(\mathbb{C}) := \{ F \in \mathcal{F}, \Delta F = mF \} \]  
is also the $m$th-true-polyanalytic space [4, 27] or the generalized Bargmann space [9]. An orthonormal basis for this space is given by the functions
\[ h_j^m(z) := (-1)^{m+n} (m!j!)^{-1/2} (m \wedge j)!! |z|^{m-j} e^{-i(m-j)\arg(z)} I_{m,n}^{j} (z^2), \quad j = 0, 1, \ldots, \]  
$L_n^{(a)}(\cdot)$ being the Laguerre polynomial ([22, p. 47]), $m \wedge j = \min(m, j)$ and $i^2 = -1$. Note that when $m = 0$, $h_j^0(z)$ reduces to $h_j(z)$ in (3). Therefore, we may replace the coefficients $h_j(z)$ by $h_j^m(z)$ to construct a family of coherent states depending on the parameter $m$. This leads to the coherent states transform $\mathcal{B}_m : L^2(\mathbb{R}) \to \mathcal{A}_m(\mathbb{C})$, defined for any $f \in L^2(\mathbb{R})$ by [24]:
\[ \mathcal{B}_m[f](z) = (-1)^m (2^m m! \sqrt{\pi})^{-1/2} \int_{\mathbb{R}} e^{-\frac{1}{2}z^2 - \frac{1}{2}i\xi^2 + \sqrt{2}iz\xi} H_m(\xi - \frac{z + \bar{z}}{\sqrt{2}}) f(\xi) d\xi, \]  
where $H_m(\cdot)$ denotes the Hermite polynomial. This transform, also called $m$-true-polyanalytic Bargmann transform, has found applications in time-frequency analysis [1], discrete quantum dynamics [2] and determinantal point processes [3]. For more details on (9), see [4] and reference therein.

We also observe that the coefficients (8) can be rewritten in terms of 2D complex Hermite polynomials introduced by Itô [20], as $h_j^m(z) = (m!j!)^{-1/2} H_{m,j}(z, \bar{z})$ where
\[ H_{r,s}(z, w) := \sum_{k=0}^{r+s} (-1)^k k! \binom{r}{k} \binom{s}{k} z^{r-k} w^{s-k}, \quad r, s = 0, 1, 2, \ldots. \]  
For the latter ones, Ismail and Zhang have introduced the following $q$-analogues ([19, p. 9]):
\[ H_{r,s}(z, w|q) := \sum_{k=0}^{r+s} \binom{r+s}{k}_q \binom{r}{k}_q \binom{s}{k}_q (-1)^k q^{(r-k)(s-k)} (q; q)_k z^{r-k} w^{s-k}, \quad z, w \in \mathbb{C} \]  
where
\[ \binom{n}{k}_q = \frac{(q; q)_n}{(q; q)_{n-k}} \frac{(q; q)_k}{(q; q)_k}, \quad k \in \mathbb{Z}_+, \quad (a; q)_n = \prod_{i=0}^{n-1} \left( 1 - aq^i \right) \quad \text{and} \quad (a; q)_\infty = \prod_{i=0}^{\infty} \left( 1 - aq^i \right). \]  
The polynomials (11) can also be rewritten in a form similar to (8) as
\[ H_{r,s}(z, w|q) = (-1)^{r+s} (q; q)_{r+s} z^{r-s} w^{s-r} e^{-i(r-s)\arg(z)} I_{r+s}^{(r-s)}(z|w; q) \]  

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in terms of $q$-Laguerre polynomials $L_n^{(\alpha)}(x; q)$ ([22, p. 108]).

Here, we introduce a new $q$-deformation of the transform (9) with the parameter range $q^{-1} > 1$. The kernel of such a transform may be obtained, up to a normalization factor depending on $z$, as the closed form of a generalized coherent state (a special superposition) that we now construct by replacing the coefficients $h_j^{m,q}(z)$ by a slight modification of the polynomials $H_{m,j}(z, \bar{z}; q)$. More precisely, our superposition combines the new coefficients with continuous $q^{-1}$-Hermite Askey functions [8], which are chosen as $q$-analogs of eigenstates of the harmonic oscillator and may also be associated with the Arif-Coon oscillator for $q' = q^{-1} > 1$ [14]. Precisely, by setting $w = \bar{z}$ in (13), we will be concerned with the following new coefficients

$$h_j^{m,q}(z) := \frac{(-1)^{m+j}(q; q)_m q^{j(m-j)} \left \{ \begin{array}{c} m - j \\ m \\ \end{array} \right \}_{q}^{(1 - q)(q; q)_j} \sqrt{1 - q} \arg(z)}{q^{(m-j)^2 + m+j} \sqrt{(q; q)_m (q; q)_j}} L_m^{(m-j)} \left ( q^{-1}; q \right )$$

where $\alpha = (1 - q)z\bar{z}$. Since $\lim_{q \to 1} L_n^{(\alpha)}(1 - q)x; q) = L_n^{(\alpha)}(x)$ it follows, after straightforward calculations, that $\lim_{q \to 1} h_j^{m,q}(z) = h_j^{m}(z)$ which justifies our choice for the functions (14). Next, as $q$-analogs of eigenstates of the Hamiltonian of the harmonic oscillator, we will be dealing with the functions

$$\phi_j^q(\xi) := \sqrt{\omega_q(\xi)} \left \{ \begin{array}{c} q^{j(j+1)} \\ j \\ \end{array} \right \}_{q}^{(1 - q)(q; q)_j} h_j \left ( \sqrt{\frac{1 - q}{2}} \xi, q \right ), \xi \in \mathbb{R}, \quad j = 0, 1, 2, \ldots,$$

where $h_j(x|q)$ are the continuous $q^{-1}$-Hermite Askey polynomials [8] defined by

$$h_j(x|q) = i^{-j} H_j(i x|q^{-1}),$$

$H_j(x|p)$ being the continuous $p$-Hermite polynomial with $p > 1$ ([22, p. 115]) and

$$\omega_q(\xi) = \pi^{-\frac{1}{2}} q^\frac{1}{2} \cosh(\sqrt{\frac{1 - q}{2}} \xi) e^{-\xi^2}.$$  

Furthermore, in (10, p. 5) Atakishiyev showed that the functions (15) satisfy a Ramanujan-type orthogonality relation on the full real line, which translates to

$$\int_{\mathbb{R}} \phi_j^q(\xi) \phi_k^q(\xi) d\xi = \delta_{jk}$$

in terms of the functions $\{\phi_j^q\}$. The latter ones also satisfy $\lim_{q \to 1} \phi_j^q(\xi) = \phi_j(\xi)$ where $\phi_j(\xi)$ are the Hermite functions (1). This justifies our choice in (15).

Now, with the above material, we are able to define "à la Iwata" [15, 21] a new family of generalized coherent states belonging to $L^2(\mathbb{R})$ by setting

$$\Psi_{z,m,q} := (\mathcal{N}_{m,q}(z\bar{z}))^{-\frac{1}{2}} \sum_{j \geq 0} h_j^{m,q}(z) \phi_j^q,$$

where the normalization factor

$$\mathcal{N}_{m,q}(z\bar{z}) = \frac{q^{2m}((q - 1)z\bar{z}; q)_{\infty}(q^{-1}(q - 1)z\bar{z}; q)_m}{((q - 1)z\bar{z}; q)_m},$$

is defined for every $z \in \mathbb{C}$. These states satisfy the resolution of the identity operator on $L^2(\mathbb{R})$ as

$$\int_{\mathbb{R}} |\Psi_{z,m,q}|^2 d\nu_{m,q}(z) = 1_{L^2(\mathbb{R})}.$$  

Here, the Dirac's bra-ket notation $|\Psi_{z,m,q}\rangle$ means the rank-one operator $\phi \mapsto \langle\Psi_{z,m,q}| \Phi \rangle : \Psi_{z,m,q}, \Phi \in L^2(\mathbb{R})$ and $d\nu_{m,q}(z) := \mathcal{N}_{m,q}(z\bar{z}) d\mu_q(z)$ where $d\mu_q(z)$ is one of many orthogonal measures for the polynomials $h_j^{m,q}(z)$ and it is given by ([19, p. 11]):

$$d\mu_q(z) := \frac{q - 1}{q \log q} (E_q(q^{-1}z\bar{z}))^{-1} d\lambda(z)/\pi,$$
where $E_q(x) = ((q - 1)x; q)_\infty$ defines a $q$-exponential function ([17, p. 11]). Moreover, in the limit $q \rightarrow 1$ the measure $d\mu_q$ reduces to the Gaussian measure $e^{-z\bar{z}}d\lambda(z)/\pi$. Eq. (21) may also be understood in the weak sense as

$$\int_\mathbb{C} \langle f, \Psi_{z,m,q} \rangle \langle \Psi_{z,m,q}, g \rangle dv_{m,q}(z) = \langle f, g \rangle, \quad f, g \in L^2(\mathbb{R}).$$  \hspace{1cm} (23)

Furthermore, straightforward calculations give the overlapping function of two coherent states (19). See Subsection 2.1 below for the proof.

**Proposition 1.** For $m \in \mathbb{Z}_+$ and $q^{-1} > 1$, the following assertion holds true

$$\langle \Psi_{z,m,q}, \Psi_{w,m,q} \rangle_{L^2(\mathbb{R})} = q^{\frac{2m}{2}}((q - 1)z\bar{w}; q)_{\infty} 3\phi_2 \left( \frac{q^{-m}q^w z}{q^{-m}q w}, q, (q - 1)z\bar{w} \right)$$  \hspace{1cm} (24)

for every $z, w \in \mathbb{C}$.

Here, the $3\phi_2$ $q$-series is defined by ([17, p. 4]):

$$3\phi_2 \left( \frac{q^{-n}a, b, c, d}{q^{-n}a, c, d}; q \right) x = \sum_{k=0}^{\infty} \frac{(q^{-n}; q)_k(a; q)_k(b; q)_k}{(c; q)_k(d; q)_k} \frac{x^k}{(q; q)_k}.$$  \hspace{1cm} (25)

In particular, for $z = w$ in (24), the condition $\langle \Psi_{z,m,q}, \Psi_{z,m,q} \rangle_{L^2(\mathbb{R})} = 1$ may provide us with the normalization factor (20). Furthermore, (24) gives an explicit expression for the function

$$K_{m,q}(z,w) := \langle \mathcal{N}_{m,q}(z\bar{w}), \mathcal{N}_{m,q}(w\bar{w}) \rangle_{\mathbb{C}^{q}} \langle \Psi_{z,m,q}, \Psi_{w,m,q} \rangle_{L^2(\mathbb{R})}$$  \hspace{1cm} (26)

which satisfies the limit

$$\lim_{q \rightarrow 1} K_{m,q}(z,w) = e^{z\bar{w}}L_m^{(0)}(|z - w|^2).$$  \hspace{1cm} (27)

The proof of (27) is given in Subsection 2.2 below. Hence, one can say that the closure in $\mathcal{H}_q := L^2(\mathbb{C}, d\mu_q)$ of the linear span of $\{\delta_{j,q} \}_{j \geq 0}$ is a Hilbert space whose reproducing kernel is given in (26) and it will be denoted $\mathcal{A}^q_m(\mathbb{C})$. This space can also be viewed as a $q$-analog of the m-th true-polyanalytic space $\mathcal{A}_m(\mathbb{C})$ in (7) whose reproducing kernel was given by $e^{z\bar{w}}L_m^{(0)}(|z - w|^2)$, see [9].

Eq. (23) also means that the coherent states transform $\mathcal{B}^q_m : L^2(\mathbb{R}) \rightarrow \mathcal{A}^q_m(\mathbb{C})$ defined as usual (see [5, p. 27 for the general theory]) by

$$\mathcal{B}^q_m(f)(z) = \langle \mathcal{N}_{m,q}(z\bar{w}) \rangle_{\mathbb{C}^q} \langle \Psi_{z,m,q}, \Psi_{w,m,q} \rangle_{L^2(\mathbb{R})} f(z), \quad z \in \mathbb{C},$$  \hspace{1cm} (28)

is an isometric map for which we establish the following precise result, see Subsection 2.3 below for the proof.

**Theorem 2.** For $m \in \mathbb{Z}_+$ and $q^{-1} > 1$, the transform $\mathcal{B}^q_m$ is explicitly given by

$$\mathcal{B}^q_m(f)(z) = \gamma_{q,m} \int_{\mathbb{R}} \left[ q^{1+m} \sqrt{1 - q^2 \xi e^{-\text{arsinh}(\sqrt{\frac{q^2}{1 - q^2}} \xi)}}, q^{-m} \sqrt{1 - q^2 \xi e^{\text{arsinh}(\sqrt{\frac{q^2}{1 - q^2}} \xi)}}, \xi \right]$$

$$\times \tilde{Q}_m \left[ \sqrt{\frac{1 - q^2}{2}}, \frac{m+1}{q^2} \sqrt{1 - q^2 \xi}, i q^{m+1} \sqrt{1 - q^2 \xi} / q, \omega_q(\xi) f(\xi) d\xi, \right]$$  \hspace{1cm} (29)

where $\gamma_{q,m} = (-1)^m q \frac{1}{\sqrt{(q; q)_m}}$ and $\tilde{Q}_m$ denotes the $q^{-1}$-AL-Salam-Chihara polynomials.

Here, the polynomial $\tilde{Q}_m$ is defined by ([10, p. 6]):

$$\tilde{Q}_n(\sinh \kappa; t, \tau; q) = q^{-\frac{1}{2}}(i t)^n(i t^{-1} e^\kappa, -i t^{-1} e^{-\kappa}; q)_n 3\phi_2 \left( \frac{q^{-n}, q^{1-n} t e^\kappa, 0}{i q^{1-n} t e^\kappa, -i q^{1-n} t e^{-\kappa}} \right)$$  \hspace{1cm} (30)

where $\kappa \in \mathbb{R}$ and $t, \tau \in \mathbb{C}$. The isometry $\mathcal{B}^q_m$ will be called a $q$-deformation of the true-polyanalytic Bargmann transform $\mathcal{B}_m$ when $q^{-1} > 1$. Indeed, when $q \rightarrow 1$ (29) reduces to (9), see Subsection 2.4 below for the proof.
Corollary 3. For \( m = 0 \), the transform (29) reduces to \( \mathcal{B}_0^q : L^2(\mathbb{R}) \to \mathcal{A}_{0}^q(\mathbb{C}) \), defined by

\[
\mathcal{B}_0^q[f](z) = \int_{\mathbb{R}} \left( -\sqrt{q(1-q)} z e^{\text{argsinh}(\sqrt{\frac{q}{1-q}})} \right) \sqrt{q(1-q)} z e^{-\text{argsinh}(\sqrt{\frac{q}{1-q}})} f(\xi) d\xi
\]

for every \( z \in \mathbb{C} \). In particular, when \( q \to 1 \), \( \mathcal{B}_0^q \) goes to the Bargmann transform (4).

Here, \( \mathcal{A}_{0}^q(\mathbb{C}) \) is the completed space of entire functions in \( \mathcal{S}_{(q)} \), for which the elements

\[
h_j^0,q(z) = ([j]_q!)^{-1/2} q^\frac{j(j-1)}{2} z^j,
\]

where \([j]_q! = \frac{(q; q)_j}{(q-1; q)_j}\), constitute an orthonormal basis. Note that by replacing in (31) the parameter \( q \) by its inverse \( q' = q^{-1} \), we recover the well known orthonormal basis \((|j\rangle_q!^{-1/2} z^j)\) of the classical Arik-Coon type space with \( q' = q^{-1} > 1 \) [25].

Remark 4. For \( m = 0 \), we recover in \( L^2 \left( \mathbb{R}, \sqrt{q(1-q)} d\xi \right) \) the state \( \langle \xi | z, 0, q \rangle \equiv \langle \omega_q(\xi) \rangle^{-1/2} \Psi_{z,0,q}(\xi) \) as a coherent state for the Arik-Coon oscillator with the deformation parameter \( q' = q^{-1} > 1 \), which was constructed by Burban ([14, p. 5]).

Remark 5. In [19, p. 4] Ismail and Zhang have also introduced another class of 2D orthogonal \( q \)-polynomials, here denoted by \( \tilde{H}_{m,j}(z, w | q) \), which also generalize the complex Hermite polynomials [20] and are connected to ones in (11) by

\[
\tilde{H}_{m,j}(z, w | q) = q^{m-j} m^j H_{m,j}(z i, w i | q^{-1}).
\]

In our previous joint work with Arjika [7], we have combined the polynomials \( \tilde{H}_{m,j}(z, \bar{z} | q) \) with the continuous \( q \)-Hermite polynomials \( H_{j}(\xi | q) \) and we have obtained a \( q \)-deformed \( m \)-true-polyanalytic Bargmann transform on \( L^2 \left( \left[ -\frac{\sqrt{q}}{\sqrt{1-q}}, \frac{\sqrt{q}}{\sqrt{1-q}} \right], d\xi \right) \) with \( q^{-1} > 1 \).

Remark 6. The expression (26) may also constitute a starting point to construct a \( q \)-deformation for the determinantal point process associated with an \( m \)th Euclidean Landau level or Ginibre-type point process in \( \mathbb{C} \) as discussed by Shirai [26].

2. Proofs

2.1. Proof of Proposition 1

By (18)-(19), the overlapping function of two coherent states is given by

\[
\langle \Psi_{z,m,q}, \Psi_{w,m,q} | L^2(\mathbb{R}) \rangle = \langle \mathcal{N}_{m,q}(z \bar{z}), \mathcal{N}_{m,q}(w \bar{w}) \rangle^{-\frac{1}{2}} \sum_{j=0}^{\infty} \mathfrak{h}^{m,q}_j(z) \mathfrak{h}^{m,q}_j(w)
\]

\[
= \langle \mathcal{N}_{m,q}(z \bar{z}), \mathcal{N}_{m,q}(w \bar{w}) \rangle^{-\frac{1}{2}} S^{(m)}.
\]

Replacing \( \mathfrak{h}^{m,q}_j(z) \) by their expressions in (14), we can write \( S^{(m)} = S^{(m)}_{\infty} + S^{(m)}_{<\infty} \), where

\[
S^{(m)}_{\infty} = \sum_{j=0}^{m-1} \frac{q^{-m-j} (m-j)^{2m+j} (q-1)^{m-j}(z \bar{w})^{m-j}}{(q; q)_m (q; q)_j} L^{(m-j)}_{m-j} (q^{-1} \alpha; q) L^{(m-j)}_{m-j} (q^{-1} \beta; q)
\]

\[
- \sum_{j=0}^{m-1} \frac{q^{-m-j} (m-j)^{2m+j} (q-1)^{m-j}(z \bar{w})^{m-j}}{(q; q)_m (q; q)_j} L^{(j-m)}_{m} (q^{-1} \alpha; q) L^{(j-m)}_{m} (q^{-1} \beta; q),
\]
and

\[ S^{(m)}_{\infty} = \sum_{j \geq 0} \frac{(q, q; q)_{m+1} \lambda^j (z \bar{w})^j}{(q; q)_m(q; q)_j} L_m^{(j-m)}(q^{-1} \alpha; q) L_m^{(j-m)}(q^{-1} \beta; q) \]

\[ = \frac{q^{m^2+2m}}{\lambda^m} \sum_{j \geq 0} \frac{(q)_j \lambda q^{-m}}{(q; q)_j} L_m^{(j-m)}(q^{-1} \alpha; q) L_m^{(j-m)}(q^{-1} \beta; q), \]

where \( \lambda = (1 - q) z \bar{w}, \alpha = (1 - q) z \bar{w}, \) and \( \beta = (1 - q) w \bar{w}. \) Now, we apply the relation ( [23, p. 3]):

\[ L_n^{(-N)}(x; q) = (-1)^N \frac{1}{N} x^N \frac{(q; q)_n}{(q; q)_N} L_n^{(N)}(x; q) \]

for \( N = j - m, \) \( n = j, \) \( x = x \) in a first time and next for \( x = \beta. \) To obtain that \( S^{(m)}_{\infty}(z, w; q) = 0. \)

For the infinite sum, we rewrite the \( q \)-Laguerre polynomial as ( [22, p. 11]):

\[ L_n^{(\gamma)}(x; q) = \frac{1}{(q; q)_n} 2\Phi_1 \left[ \begin{array}{c} q^{-n}, -x \\ 0 \end{array} \bigg| q; q^{n+\gamma+1} \right] \]

with \( n = m, \gamma = j - m, x = q^{-1} \alpha \) for \( L_m^{(j-m)}(q^{-1} \alpha; q) \) and \( x = q^{-1} \beta \) for \( L_m^{(j-m)}(q^{-1} \beta; q). \) This gives

\[ S^{(m)}_{\infty} = \frac{q^{m^2+2m}}{\lambda^m(q; q)_m} S^{(m)}_q(\alpha; \beta) \]

where

\[ S^{(m)}_q(\alpha; \beta) = \sum_{j \geq 0} \frac{(q)_j \lambda q^{-m}}{(q; q)_j} \sum_{k \geq 0} \frac{(q^{-m}, q^{-1} \alpha; q)_k}{(q; q)_k} \sum_{l \geq 0} \frac{(q^{-m}, q^{-1} \beta; q)_l}{(q; q)_l} \sum_{j \geq 0} \frac{(q)_j q^{m+k+1} \lambda^j}{(q; q)_j} \]

Recalling ( [17, p. 3]):

\[ 2\Phi_1 \left( \begin{array}{c} a, b \\ c \end{array} \bigg| q; q \right) = \sum_{k \geq 0} \frac{(a; q)_k(b; q)_k}{(c; q)_k} \frac{x^k}{(q; q)_k}, \]

the r.h.s of (38) becomes

\[ S^{(m)}_q(\alpha; \beta) = \sum_{j \geq 0} \frac{(q)_j \lambda q^{-m}}{(q; q)_j} \sum_{k \geq 0} \frac{(q^{-m}, q^{-1} \alpha; q)_k}{(q; q)_k} \sum_{l \geq 0} \frac{(q^{-m}, q^{-1} \beta; q)_l}{(q; q)_l} \sum_{j \geq 0} \frac{q^{m+k+1} \lambda^j}{(q; q)_j} \]

Now, by applying the \( q \)-binomial theorem ( [17, p. 11]):

\[ \sum_{n \geq 0} \frac{q^{n}_2}{(q; q)_n} a^n = (-a; q)_\infty \]

for \( a = q^{-m+k+1} \lambda, \) the r.h.s of (40) takes the form

\[ S^{(m)}_q(\alpha; \beta) = \sum_{k, l \geq 0} \frac{(q^{-m}, q^{-1} \alpha; q)_k q^k}{(q; q)_k} \frac{(q^{-m}, q^{-1} \beta; q)_l q^l}{(q; q)_l} (-q^{-m+k+1} \lambda; q)_\infty. \]

By making use of the identity ( [22, p. 9]):

\[ (a; q)_\infty = \frac{(a; q)_\infty}{(aq^2; q)_\infty} \]

for the factor \( (-q^{-m+k+1} \lambda; q)_\infty, \) (43) transforms to

\[ S^{(m)}_q(\alpha; \beta) = (-q^{-m} \lambda; q)_\infty \sum_{k \geq 0} \frac{(q^{-m}, q^{-1} \alpha; q)_k q^k}{(q; q)_k} \sum_{l \geq 0} \frac{(q^{-m}, q^{-1} \beta; q)_l q^l}{(q; q)_l} q^l. \]
Next, by the fact that \((q^{-m} \lambda; q)_{l+k} = (q^{-m} \lambda; q)_{l}(q^{k-m} \lambda; q)_{k}\), it follows that
\[
S_{q}^{(m)}(\alpha; \beta) = (q^{-m} \lambda; q)_{\infty} \sum_{k=0}^{\infty} \frac{(q^{-m}, -q^{-1} \alpha; q)_{k}}{(q^{-m} \lambda, q; q)_{k}} q^{k} 2\phi_{1}\left(\frac{q^{-m}, -q^{-1} \beta}{-q^{k-m} \lambda}; q; q\right).
\]

Using the identity ([17, p. 10]):
\[
2\phi_{1}\left(\frac{q^{-n}, b}{c}; q\right) = \frac{(b^{-1}; q)_{n}}{(c; q)_{n}} b^{n}
\]
for \(n = m, b = -q^{-1} \beta\) and \(c = -q^{k-m} \lambda\), leads to
\[
S_{q}^{(m)}(\alpha; \beta) = (q^{-m} \lambda; q)_{\infty} \sum_{k=0}^{\infty} \frac{(q^{-m}, -q^{-1} \alpha; q)_{k}}{(q^{-m} \lambda, q; q)_{k}} q^{k} \frac{(q^{k+1-m} \lambda; q)_{m}}{(q^{-1} \beta)_{m}}.
\]

Applying the identity ([22, p. 9]):
\[
(aq^{n}; q)_{r} = \frac{(aq; q)_{r}(aq^{r}; q)_{n}}{(aq; q)_{n}}
\]
for \(r = m, n = k, a = q^{1-m} z/w\) and \(a = -q^{-m} \lambda\) in a second time, we arrive at
\[
S_{q}^{(m)}(\alpha; \beta) = \frac{(q^{-1} \beta)_{m}(q^{-1} \lambda; q)_{m}}{(q^{-m} \lambda; q)_{m}} 3\phi_{2}\left(\frac{q^{-n}, -q^{-1} \lambda; q}{q^{1-m} \lambda, -\lambda; w}; q; q\right).
\]

Finally, by the finite Heine transformation ([6, p. 2]):
\[
3\phi_{2}\left(\frac{q^{-n}, \xi, \gamma, q^{1-m} \lambda}{\xi, \gamma, q^{1-n} \lambda}; q; q\right) = \frac{(\xi \tau; q)_{n}}{(\tau, q)_{n}} 3\phi_{2}\left(\frac{q^{-n}, \gamma / \sigma, \xi}{\gamma, \xi \tau}; q; \sigma \tau q^{n}\right)
\]
for parameters \(\xi = q^{z} w, \gamma = -q^{-1} \alpha, \gamma = -\lambda\) and \(\tau = w \zeta\), (48) reads
\[
S_{q}^{(m)}(\alpha; \beta) = \frac{(q^{-1} \beta)_{m}(q^{-1} \lambda; q)_{m}}{(q^{-m} \lambda, w \zeta; q)_{m}} 3\phi_{2}\left(\frac{q^{-n}, q^{w \zeta}, q^{z \lambda}}{q^{-\lambda} \lambda}; q; q, -q^{m-1}(1 - q) w \zeta\right).
\]

Summarizing the above calculations and taking into account the previous prefactors, we arrive at the announced result (24).

\[
\square
\]

2.2. Proof of the limit (27)

Recalling that \(E_{q}(x) = ((q^{-1} x); q)_{\infty}\), then we get that
\[
\lim_{q^{-1}} q^{2m}((q^{-1} x)z w; q)_{\infty} = e^{z w}.
\]

By another side, using (25) together with the fact that \((q^{-n}; q)_{k} = 0, \forall k > n\), the series \(3\phi_{2}\) in (24) terminates as
\[
\sigma_{m, q}(z, w) := \sum_{k=0}^{m} \frac{(q^{-m}, q^{w \zeta}, q^{z \lambda}; q)_{k}}{((q^{-1} z w, q; q)_{k}} \left(q^{m-1}(1 - q) w \zeta\right)^{k}.
\]

Thus, from the identity ([22, p. 10]):
\[
\left[\frac{\gamma}{k}\right]_{q} = (-1)^{k} q^{\gamma - \left\lfloor\frac{k}{q}\right\rfloor} (q^{-\gamma}; q)_{k}.
\]
we, successively, have

\[
\lim_{q \to 1} \sigma_{m,q}(z, w) = \sum_{k=0}^{m} \lim_{q \to 1} \left( \frac{(z^{-1})^k}{k!} \right)
\]

By noticing that the sum in (55) is the evaluation of the Laguerre polynomial \( L_{-1}^{(0)} \) at \(|z - w|^2\), the proof of the limit (27) is completed.

2.3. Proof of Theorem 2

To apply (28), we seek for a closed form for the following series

\[
(N_{m,q}(z\bar{z}))^{\frac{1}{2}} \sum_{j=0}^{m} \frac{(-1)^{m-j}}{\sqrt{(q;q)_m \sqrt{1 - q^{m-j}}}}
\]

which may also be written as

\[
(1-q)^{m\omega(\xi)(q;\bar{q})} \left( \sum_{j=0}^{m} \frac{(-1)^{m-j}}{(z^2 q^{m-j})^{\frac{1}{2}}} \right)\qquad (58)
\]

where \( \alpha = (1-q)z\bar{z} \). Next, replacing the \( q \)-Laguerre polynomial by its expression (36), (58) becomes

\[
\eta^{m,q}(\xi, z) = \sum_{j=0}^{m} \frac{(-1)^{m-j}}{(q;q)_j} L_{-1}^{(j-m)}(q^{-1} \alpha; q) h_j \left( \frac{1 - \frac{q}{2} \xi}{q} \right)
\]

By using the generating function of the \( q^{-1} \)-Hermite polynomials (10, p. 6):

\[
\sum_{n=0}^{\infty} \frac{t^n q^{\frac{n}{2}}}{(q;q)_n} h_n(x|q) = (-te^{\theta}, te^{-\theta}; q)_\infty, \quad \sinh \theta = x
\]

for the parameters \( t = q^{-\frac{1}{2} + k} \sqrt{1 - \bar{q}z} \) and

\[
\sinh \theta = \sqrt{1 - \frac{q}{2} \xi},
\]

the r.h.s of (59) takes the form

\[
\eta^{m,q}(\xi, z) = \frac{1}{(q;q)_m} \sum_{k=0}^{m} \frac{(-1)^{m-k}}{(q;q)_k} q^k (-ye^{\theta} q^k, ye^{-\theta} q^k; q)_\infty
\]

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where \( y = q^{-1-m} \sqrt{1-qz} \). By applying (44), it follows that
\[
\eta^{m,q}(\xi, z) = \frac{(-ye^\theta, ye^{-\theta}; q)_\infty}{(q; q)_m} \sum_{k \geq 0} \frac{(q^{-m}, -q^{-1+\alpha}; q)_k}{(-ye^\theta, ye^{-\theta}; q)_k} q^k
\]
(63)
which can also be expressed as
\[
\eta^{m,q}(\xi, z) = \frac{(-ye^\theta, ye^{-\theta}; q)_\infty}{(q; q)_m} \phi_2 \left[ q^{-m}, -q^{-1+\alpha}, 0 \right] (-ye^\theta, ye^{-\theta}; q). \tag{64}
\]
Next, recalling the definition of the \( q^{-1} \)-Al-Salam-Chihara polynomials in (30) for \( \kappa = \theta, t = iq^{m-1}y \) and \( \tau = iq^{m-2} \sqrt{1-qz} \), (64) reads
\[
\eta^{m,q}(\xi, z) = (-1)^m q^{1/2} (-ye^\theta, ye^{-\theta}; q)_\infty \tilde{Q}_m(sinh\theta; i q^{m-1}y, i q^{m-3} \sqrt{1-qz}; q)
\]
(65)
After some simplifications, we arrive at the following form for the series (56)
\[
\sqrt{\omega_q}(\xi)(-q^{-1/2} \sqrt{1-qze^\theta}, q^{1/2} \sqrt{1-qze^{-\theta}}; q)_\infty
\times (-1)^m q^{1/2} \tilde{Q}_m \left( \frac{1-q}{2}, i q^{m-1} \sqrt{1-qz}, i q^{m-3} \sqrt{1-qz}; q \right).
\tag{66}
\]
This ends the proof.

2.4. Proof of the limit in (29)

To compute the limit of the quantity in (66) as \( q \to 1 \), we first observe that
\[
\lim_{q \to 1} \sqrt{\omega_q}(\xi) = \lim_{q \to 1} \left( \frac{1}{\sqrt{q}} \cos(\sqrt{1-q} \xi) e^{-\xi^2/2} \right)^{1/2} = \pi^{-1/4} e^{-\xi^2/2}.
\tag{67}
\]
Next, we denote
\[
G_q(z; \xi) := (-q^{1/2} \sqrt{1-qze^\theta}, q^{1/2} \sqrt{1-qze^{-\theta}}; q)_\infty.
\tag{68}
\]
Then by (12), we successively obtain
\[
\log G_q(z; \xi) = \sum_{k \geq 0} \log \left( 1 - q^{1/2} k \sqrt{1-qze^\theta} + q^{1/2} k \sqrt{1-qze^{-\theta}} - q^{m+2k-1} (1-q) z^2 \right)
\]
\[
= q^{-1/2} \sqrt{1-qz} (e^\theta - e^{-\theta}) \sum_{k \geq 0} q^k - q^{m+1} (1-q) z^2 \sum_{k \geq 0} q^{2k} + o(1-q)
\]
\[
= q^{1/2} z^{2} (e^\theta - e^{-\theta}) \frac{1}{\sqrt{1-q}} - q^{m+1} z^2 \frac{1}{1+q} + o(1-q).
\]
Thus, form (61) the last equality also reads
\[
\log G_q(z; \xi) = q^{1/2} \sqrt{2} z \xi - q^{m+1} z^2 \frac{1}{1+q} + o(1-q).
\tag{69}
\]
Therefore, when \( q \to 1 \), we have \( \lim_{q \to 1} G_q(z; \xi) = e^{\sqrt{2} z \xi - 1/2 z^2} \). To obtain the limit of the polynomial quantity in (66) as \( q \to 1 \), we recall that the \( q^{-1} \)-Al-Salam-Chihara polynomials can be expressed as (\{11, p. 6\}):
\[
\tilde{Q}_n(s; a, b|q) = q^{-1} \sum_{k=0}^n \frac{n}{k} \binom{n}{k} (ia)^{n-k} h_k(s; b|q)
\tag{70}
\]
in terms of the continuous big \( q^{-1} \)-Hermite polynomials. The latter ones satisfy the limit (\{12, p. 4\}):
\[
\lim_{q \to 1} \kappa^{-n} h_n(\kappa s; 2\kappa b|q) = H_n(s + i b),
\]
and from (70) we conclude that
\[
\lim_{q \to 1} \kappa^{-n} \tilde{Q}_n(k s; 2 i a, 2 i b; q) = H_n(s - a - b).
\] (71)

By applying (71) for \( n = m, s = \xi, a = q^{-m-1} z/\sqrt{2}, b = q^{-m-3} \bar{z}/\sqrt{2} \) and \( \kappa = \sqrt{1 - q} \), we establish the following
\[
\lim_{q \to 1} \frac{(-1)^m q^{1/2(2m)}}{(q; q)_m} \tilde{Q}_m \left( \sqrt{1 - q} \xi; i q^{-m-1} \sqrt{1 - q} z, i q^{-m-3} \sqrt{1 - q} \bar{z}; q \right)
= (-1)^m (2^m m!)^{-1/2} H_m \left( \xi - \frac{z + \bar{z}}{\sqrt{2}} \right).
\]

Finally, by grouping the obtained three limits, we arrive at the assertion in (29). \( \square \)

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