ON AN OPTIMAL STOPPING PROBLEM OF AN INSIDER

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Abstract. We consider the optimal problem

\[ \sup_{\tau \in \mathcal{T}_{\varepsilon,T}} \mathbb{E} \left[ \sum_{i=1}^{n} \phi_i^i (\tau - \varepsilon)^+ \right], \]

where \( T > 0 \) is a fixed time horizon, \((\phi_i^t)_{0 \leq t \leq T}\) is progressively measurable with respect to the Brownian filtration, \( \varepsilon^i \in [0,T] \) is a constant, \( i = 1, \ldots, n \), and \( \mathcal{T}_{\varepsilon,T} \) is the set of stopping times that lie between a constant \( \varepsilon \in [0,T] \) and \( T \). We solve this problem by conditioning and then using the theory of reflected backward stochastic differential equations (RBSDEs). As a corollary, we provide a solution to the optimal stopping problem \( \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E} B(\tau - \varepsilon)^+ \) recently posed by Shiryaev at the International Conference on Advanced Stochastic Optimization Problems organized by the Steklov Institute of Mathematics in September 2012. We also provide its asymptotic order as \( \varepsilon \downarrow 0 \).

1. General result

Let \( T > 0 \) and let \( \{B_t, t \in [0,T]\} \) be a Brownian motion defined on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) and let \( \mathcal{F} = \{\mathcal{F}_t, t \in [0,T]\} \) be the natural filtration augmented by the \( \mathbb{P} \)-null sets of \( \mathcal{F} \). Consider the optimal stopping time problem

\[ v = \sup_{\tau \in \mathcal{T}_{\varepsilon,T}} \mathbb{E} \left[ \sum_{i=1}^{n} \phi_i^i (\tau - \varepsilon)^+ \right], \]

where \((\phi_i^t)_{0 \leq t \leq T}\) is continuous and progressively measurable, \( \varepsilon^i \in [0,T], i = 1, \ldots, n \), are given constants, and \( \mathcal{T}_{\varepsilon,T} \) is the set of stopping times that lie between a constant \( \varepsilon \in [0,T] \) and \( T \). Observe that \( \tau - \varepsilon^i \) is not a stopping time with respect to \( \mathcal{F} \) for \( \varepsilon^i > 0 \). This can be thought of a problem of an insider in which she is allowed to peek \( \varepsilon^i \) into the future for each payoff before making her stopping decision. The solution to (1) is described by the following result:

Theorem 1. Assume \( \mathbb{E} \left[ \sup_{0 \leq t \leq T} (\xi_t^2)^+ \right] < \infty \), where \( \xi_t = \sum_{i=1}^{n} \phi_i^i (t - \varepsilon^i)^+ \), \( 0 \leq t \leq T \). Then the value defined in (1) can be calculated using a reflected backward stochastic differential equation (RBSDE). More precisely, \( v = \mathbb{E} Y_{\varepsilon} \), for any \( \varepsilon \in [0,T] \), where \( (Y_t)_{0 \leq t \leq T} \) satisfies the RBSDE

\[ \xi_t \leq Y_t = \xi_T - \int_t^T Z_s dW_s + (K_T - K_t), \quad 0 \leq t \leq T, \]

\[ \int_0^T (Y_t - \xi_t) dK_t = 0, \]

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Moreover, there exists an optimal stopping time \( \hat{\tau} \) described by

\[
\hat{\tau} = \inf\{t \in [\varepsilon, T] : Y_t = \xi_t\}.
\]

**Proof.** For any \( \tau \in \mathcal{T}_{\varepsilon, T} \),

\[
\mathbb{E}\xi_{\tau} = \mathbb{E}[\mathbb{E}[\xi_{\tau} | F_{\varepsilon}]] \leq \mathbb{E}\left[\operatorname{ess sup}_{\sigma \in \mathcal{T}_{\varepsilon, T}} \mathbb{E}[\xi_{\sigma} | F_{\varepsilon}]\right].
\]

Therefore,

\[
(3) \quad v = \sup_{\tau \in \mathcal{T}_{\varepsilon, T}} \mathbb{E}\xi_{\tau} \leq \mathbb{E}\left[\operatorname{ess sup}_{\tau \in \mathcal{T}_{\varepsilon, T}} \mathbb{E}[\xi_{\tau} | F_{\varepsilon}]\right].
\]

By Theorem 5.2 in [4] there exists a unique solution \((Y, Z, K)\) to the RBSDE in (2). Then by Proposition 2.3 (and its proof) in [4] we have

\[
\sup_{\tau \in \mathcal{T}_{\varepsilon, T}} \mathbb{E}\xi_{\tau} \geq \mathbb{E}\xi_{\hat{\tau}} = \mathbb{E}Y_{\hat{\tau}} = \mathbb{E}Y_{\varepsilon} = \mathbb{E}\left[\operatorname{ess sup}_{\tau \in \mathcal{T}_{\varepsilon, T}} \mathbb{E}[\xi_{\tau} | F_{\varepsilon}]\right].
\]

Along with (3) the last inequality completes the proof. \(\square\)

One should note that the optimal stopping problem we are considering is path dependent (i.e. not of Markovian type) and therefore one would not be able to write down a classical free boundary problem corresponding to (1) unless one is willing to consider free boundary problems in the path dependent PDE framework of [3]. This is the main reason we work with RBSDEs.

Now let us consider Shiryaev’s problem:

\[
(4) \quad v^{(\varepsilon)} = \sup_{\tau \in \mathcal{T}_{0, T}} \mathbb{E}B_{(\tau - \varepsilon)^+},
\]

**Remark 1.** Observe that for \( \varepsilon > 0 \), insider’s value defined in (4) is strictly greater than 0 (and hence does strictly better than a stopper which does not posses the insider information):

\[
v^{(\varepsilon)} \geq \mathbb{E}\left[\max_{0 \leq t \leq \varepsilon \wedge (T - \varepsilon)} B_t\right] = \sqrt{\frac{2}{\pi} (\varepsilon \wedge (T - \varepsilon))} > v^{(0)} = 0,
\]

which shows that there is an incentive for waiting. We also have an upper bound

\[
{v^{(\varepsilon)}} \leq \mathbb{E}\left[\max_{0 \leq t \leq T} B_t\right] = \sqrt{\frac{2T}{\pi}}.
\]

In fact when \( \varepsilon \in [T/2, T] \), \( v^{(\varepsilon)} \) can be explicitly determined as

\[
v^{(\varepsilon)} = \mathbb{E}\left[\max_{0 \leq t \leq T - \varepsilon} B_t\right] = \sqrt{\frac{2(T - \varepsilon)}{\pi}}, \quad \varepsilon \in [T/2, T].
\]

and we have a strict lower bound for \( \varepsilon \in [0, T/2] \)

\[
v^{(\varepsilon)} > \mathbb{E}\left[\max_{0 \leq t \leq \varepsilon} B_t\right] = \sqrt{\frac{2\varepsilon}{\pi}}, \quad \varepsilon \in [0, T/2].
\]

For arbitrary values of \( \varepsilon \in [0, T] \) we have the following result as a corollary of Theorem [4]
Corollary 2. The value defined in (4) can be calculated using an RBSDE. More precisely, $v^\varepsilon = Y_0$ almost surely, where $(Y_t)_{0 \leq t \leq T}$ satisfies the RBSDE (2) with $\xi$ defined as $\xi_t = B_{(t-\varepsilon)^+}$, $0 \leq t \leq T$. Moreover, there exists an optimal stopping time $\hat{\tau}$ described by

$$\hat{\tau} = \inf\{t \geq 0 : Y_t = B_{(t-\varepsilon)^+}\} \geq \varepsilon.$$  

Furthermore, the function $\varepsilon \to v^{(\varepsilon)}$, $\varepsilon \in [0, T]$, is a continuous function.

Proof. The continuity of $\varepsilon \to v^{(\varepsilon)}$, $\varepsilon \in [0, T]$ is a direct consequence of the stability of RBSDEs indicated by Proposition 3.6 in [4]. On the other hand, since $v^{(\varepsilon)} = \sup_{\tau \in \mathcal{T}_{0, T}} \mathbb{E} B_{(\tau-\varepsilon)^+}$, $\delta \in [0, \varepsilon]$, from Theorem 1 we can conclude that $v^{(\varepsilon)} = \mathbb{E} Y_\delta$. The latter implies that $(Y_t)_{t \in [0, T]}$ is a martingale from on $t \in [0, \varepsilon]$. It also shows that the stopping time $\tau_\delta$ defined by

$$\tau_\delta = \inf\{t \in [\delta, T] : Y_t = \xi_t\},$$

is optimal for both the problem $\sup_{\tau \in \mathcal{T}_{0, T}} \mathbb{E} B_{(\tau-\varepsilon)^+}$ and $\sup_{\tau \in \mathcal{T}_{0, T}} \mathbb{E} B_{(\tau-\varepsilon)^+}$, $\delta \in [0, \varepsilon]$. In fact, one can observe that for $t \in [0, \varepsilon)$, $Y_t \geq \mathbb{E} Y_t | \mathcal{F}_t > 0 = \xi_t$, a.s., which implies $dK_t = 0$, $0 \leq t \leq \varepsilon$, a.s, (so the martingale property of $(Y_t)$ for $t \in [0, \varepsilon]$ can also be concluded from this point of view). The martingale property on the other hand implies that stopping time defined in (5) is no less than $\varepsilon$ almost surely. \hfill \Box

Optimal stopping problems of the type

$$\sup_{\tau \in \mathcal{T}_{0, T}} \mathbb{E} \left[ \sum_{j=1}^{m} \varphi^{j}_{\tau + \delta^j} \right],$$

where $\varphi^j$’s and progressively measurable $\delta^j > 0$’s are constants, were considered by [1] and [2]. A generalization combining these type of pay-offs with the one in (1) will be left for future work.

2. Asymptotic behavior of $v^{(\varepsilon)}$ as $\varepsilon \downarrow 0$

The following theorem states that the order of $v^{(\varepsilon)}$ defined in (4) is $\sqrt{2\varepsilon \ln(1/\varepsilon)}$ as $\varepsilon \downarrow 0$, which is the same as Levy’s modulus for Brownian motion. Notice that

$$v^{(\varepsilon)} = \sup_{\tau \in \mathcal{T}_{0, T}} \mathbb{E}[B_{\tau - \varepsilon} - B_{\tau}].$$

Theorem 3.

$$\lim_{\varepsilon \downarrow 0} \frac{v^{(\varepsilon)}}{\sqrt{2\varepsilon \ln(1/\varepsilon)}} = 1.$$  

Before the proof of the above theorem, let us first show two lemmas.

Lemma 4.

$$\lim_{\varepsilon \downarrow 0} \frac{v^{(\varepsilon)}}{\sqrt{2\varepsilon \ln(1/\varepsilon)}} \geq 1.$$
Proof. Let \( d \in (0, 1) \) be a constant, and define \( \tau^* \in \mathcal{T}_e, T \)
\[
\tau^* := \inf \{ n \varepsilon : B_{(n-1)e} - B_{n\varepsilon} \geq d\sqrt{2\varepsilon \ln(1/\varepsilon)}, \ n = 1, \ldots, [T/\varepsilon] - 1 \} \cap T.
\]
Then
\[
\sup_{\tau \in \mathcal{T}_e, T} \mathbb{E}[B_{\tau - \varepsilon} - B_{\tau}] \geq \mathbb{E}[B_{\tau^* - \varepsilon} - B_{\tau^*}] \geq d\sqrt{2\varepsilon \ln(1/\varepsilon)} P(\tau^* \leq \varepsilon[T/\varepsilon] - \varepsilon) + \mathbb{E}[(B_{\tau - \varepsilon} - B_T) 1_{\{\tau > \varepsilon[T/\varepsilon] - \varepsilon\}}].
\]
We have that
\[
P(\tau^* \leq \varepsilon[T/\varepsilon] - \varepsilon) = 1 - P\left( B_{(n-1)e} - B_{n\varepsilon} < d\sqrt{2\varepsilon \ln(1/\varepsilon)}, \ n = 1, \ldots, [T/\varepsilon] - 1 \right)
\]
\[
= 1 - \left[ P\left( B_{\varepsilon} - B_0 < d\sqrt{2\varepsilon \ln(1/\varepsilon)} \right) \right]^{[T/\varepsilon] - 1}
\]
\[
= 1 - \left[ 1 - \int_{d\sqrt{2\varepsilon \ln(1/\varepsilon)}}^{\infty} \frac{1}{\sqrt{2\pi \varepsilon}} e^{-\frac{x^2}{2\varepsilon}} dx \right]^{[T/\varepsilon] - 1}
\]
\[
= 1 - (1 - \alpha)^{\frac{1}{\varepsilon}} ([T/\varepsilon] - 1)^\alpha,
\]
(7)
where
\[
\alpha := \int_{d\sqrt{2\varepsilon \ln(1/\varepsilon)}}^{\infty} \frac{1}{\sqrt{2\pi \varepsilon}} e^{-\frac{x^2}{2\varepsilon}} dx = \frac{1}{2d\sqrt{\pi \ln(1/\varepsilon)}} e^{d^2} (1 + o(1)) \to 0,
\]
by, e.g., [5] (9.20) on page 112. Since \( d \in (0, 1) \), \( ([T/\varepsilon] - 1)^\alpha \to \infty \), and thus
\[
P(\tau^* \leq \varepsilon[T/\varepsilon] - \varepsilon) \to 1, \ \varepsilon \searrow 0.
\]
Therefore,
\[
\liminf_{\varepsilon \searrow 0} \frac{\nu(\varepsilon)}{\sqrt{2\varepsilon \ln(1/\varepsilon)}} \geq \liminf_{\varepsilon \searrow 0} [d P(\tau^* \leq \varepsilon[T/\varepsilon] - \varepsilon) = d.
\]
Then (6) follows by letting \( d \nearrow 1 \).

\[ \square \]

Lemma 5. The family
\[
\left\{ \frac{\sup_{\varepsilon \leq t \leq T} |B_{t - \varepsilon} - B_t|}{\sqrt{2\varepsilon \ln(1/\varepsilon)}} : \varepsilon \in \left( 0, \frac{T}{2} \wedge 1 \right) \right\}
\]
is uniformly integrable.

Proof. Since
\[
\frac{\sup_{\varepsilon \leq t \leq T} |B_{t - \varepsilon} - B_t|}{\sqrt{2\varepsilon \ln(1/\varepsilon)}} \leq \frac{2 \max_{1 \leq n \leq [T/\varepsilon] + 1} \sup_{(n-1)\varepsilon \leq t, t' \leq n\varepsilon} |B_t - B_{t'}|}{\sqrt{2\varepsilon \ln(1/\varepsilon)}} \leq \frac{4 \max_{1 \leq n \leq [T/\varepsilon] + 1} \sup_{(n-1)\varepsilon \leq t \leq n\varepsilon} |B_t - B_{(n-1)e}|}{\sqrt{2\varepsilon \ln(1/\varepsilon)}},
\]

it suffices to show that the family
\[
\{ M_\varepsilon := \max_{1 \leq n \leq \lceil T/\varepsilon \rceil + 1} \sup_{n-1 \leq t \leq n \varepsilon} \left| B_t - B_{(n-1)\varepsilon} \right| : \varepsilon \in \left(0, \frac{T}{2}\right) \}
\]
is uniformly integrable. For \( a \geq 0 \),
\[
P(M_\varepsilon \leq a) = \left[ P \left( \sup_{0 \leq t \leq \varepsilon} |B_t| \leq a \sqrt{\varepsilon \ln(1/\varepsilon)} \right) \right]^{\lceil T/\varepsilon \rceil + 1}.
\]
Hence the density of \( M_\varepsilon \), \( f_\varepsilon \), satisfies that for \( a \geq 0 \),
\[
f_\varepsilon(a) \leq (\lceil T/\varepsilon \rceil + 1) \left[ P \left( \sup_{0 \leq t \leq \varepsilon} |B_t| \leq a \sqrt{\varepsilon \ln(1/\varepsilon)} \right) \right]^{\lceil T/\varepsilon \rceil} \sqrt{\frac{8}{\pi}} \sqrt{\frac{\ln(1/\varepsilon)}{\varepsilon}} e^{-\frac{\ln(1/\varepsilon)}{2} a^2} \leq \frac{4T \sqrt{\ln(1/\varepsilon)}}{\varepsilon} e^{-\frac{\ln(1/\varepsilon)}{2} a^2},
\]
where for the first inequality we use, e.g., \[5\] (8.3) on page 96, and the fact that the density of \( \sup_{0 \leq t \leq \varepsilon} |B_t| \) is no greater than twice the density of \( \sup_{0 \leq t \leq \varepsilon} B_t \). Then we have that for \( N > 0 \),
\[
\mathbb{E} \left[ M_\varepsilon 1_{\{M_\varepsilon > N\}} \right] = \int_N^\infty x f_\varepsilon(x) dx \leq \frac{4T \sqrt{\ln(1/\varepsilon)}}{\varepsilon} \int_N^\infty x e^{-\frac{\ln(1/\varepsilon)}{2} x^2} dx = \frac{4T \varepsilon^{N^2 - 1}}{\sqrt{\ln(1/\varepsilon)}} \leq \frac{T}{2^{N^2 - 3} \sqrt{\ln 2}},
\]
i.e.,
\[
\lim_{N \to \infty} \sup_{\varepsilon \in \left(0, \frac{T}{2}\right]} \mathbb{E} \left[ M_\varepsilon 1_{\{M_\varepsilon > N\}} \right] = 0.
\]
\hfill □

Now let us provide the proof of Theorem 3.

**Proof of Theorem 3.**

\[
\limsup_{\varepsilon \searrow 0} \sup_{T \in \mathbb{T}, T} \mathbb{E}[B_{T-\varepsilon} - B_T] \leq \limsup_{\varepsilon \searrow 0} \mathbb{E} \left[ \sup_{\varepsilon \leq t \leq T} |B_{t-\varepsilon} - B_t| \right] \leq \mathbb{E} \left[ \limsup_{\varepsilon \searrow 0} \sup_{\varepsilon \leq t \leq T} |B_{t-\varepsilon} - B_t| \right] \leq 1,
\]
where we apply Lemma \[5\] for the second inequality, and use Levy’s modulus for Brownian motion (see, e.g., \[5\] Theorem 9.25, page 114) for the third inequality. Together with \[6\], the conclusion follows.

\hfill □

Using the above proof, we can actually show the following result.

**Corollary 6.**

\[
\lim_{\varepsilon \searrow 0} \sup_{T \in \mathbb{T}, T} \mathbb{E}[B_{T-\varepsilon} - B_T] = \lim_{\varepsilon \searrow 0} \mathbb{E} \left[ \sup_{\varepsilon \leq t \leq T} |B_{t-\varepsilon} - B_t| \right] = \lim_{\varepsilon \searrow 0} \mathbb{E} \left[ \sup_{\varepsilon \leq t \leq T} |B_{t-\varepsilon} - B_t| \right] = 1.
\]
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