Global well-posedness and exponential decay to the Cauchy problem of nonhomogeneous Navier-Stokes equations with density-dependent viscosity and vacuum in $\mathbb{R}^2$ *

Xin Zhong†

Abstract

We study global well-posedness of strong solutions for the nonhomogeneous Navier-Stokes equations with density-dependent viscosity and initial density allowing vanish in $\mathbb{R}^2$. Applying a logarithmic interpolation inequality and delicate energy estimates, we show the global existence of a unique strong solution provided that $\|\nabla \mu(\rho_0)\|_{L^q}$ is suitably small, which improves the previous result of Huang and Wang [SIAM J. Math. Anal. 46, 1771–1788 (2014)] to the whole space case. Moreover, we also derive exponential decay rates of the solution. In particular, there is no need to require additional initial compatibility condition despite the presence of vacuum.

Key words and phrases. Nonhomogeneous Navier-Stokes equations; global well-posedness; exponential decay; density-dependent viscosity; vacuum.

2020 Mathematics Subject Classification. 76D05; 76D03.

1 Introduction

In the present paper, we are concerned with nonhomogeneous Navier-Stokes equations with density-dependent viscosity in $\mathbb{R}^2$:

\[
\begin{cases}
\rho_t + \text{div}(\rho \mathbf{u}) = 0, \\
(\rho \mathbf{u})_t + \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) - \text{div}(2\mu(\rho)\mathcal{D}(\mathbf{u})) + \nabla P = 0, \\
\text{div} \mathbf{u} = 0,
\end{cases}
\]

(1.1)

with the initial condition

\[(\rho, \rho \mathbf{u})(x, 0) = (\rho_0, \rho_0 \mathbf{u}_0)(x), \quad x \in \mathbb{R}^2, \quad (1.2)\]

and the far field behavior

\[(\rho, \mathbf{u}) \to (\tilde{\rho}, 0), \quad \text{as } |x| \to \infty, \quad t > 0, \quad (1.3)\]

for a positive constant $\tilde{\rho}$. The unknowns $\rho, \mathbf{u}, P$ are the fluid density, velocity, and pressure, respectively. $\mathcal{D}(\mathbf{u})$ denotes the deformation tensor given by

\[\mathcal{D}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T).\]

The viscosity coefficient $\mu(\rho)$ is a function of the density satisfying

\[\mu \in C^1[0, \infty), \quad \mu \geq \underline{\mu} > 0 \quad (1.4)\]

for some positive constant $\underline{\mu}$.

*This research was partially supported by National Natural Science Foundation of China (Nos. 11901474, 12071359) and the Innovation Support Program for Chongqing Overseas Returnees (No. cx2020082).

†School of Mathematics and Statistics, Southwest University, Chongqing 400715, People's Republic of China (xzhong1014@amss.ac.cn).
The study of the system \((1.1)\) with constant viscosity has a long history, whose mathematical study initiated by the Russian school goes back to the seventies of the last century. On one hand, in the absence of vacuum, Danchin and Mucha [7] investigated the existence of smooth solutions to the Cauchy problem for discontinuous initial density in the whole space \(\mathbb{R}^n\). Under suitable smallness assumptions on the initial data in homogeneous Besov spaces, they proved the global existence of solutions. See also some interesting generalizations in [8,25]. For more information on the results with the initial density away from zero can be found in the monograph [3]. On the other hand, the study on the well-posedness of solutions with initial density allowing vacuum states has grown enormously in recent years. Simon [27] first proved the global existence of weak solutions with finite energy. By introducing the compatibility condition

\[
- \mu \Delta u_0 + \nabla P_0 = \sqrt{\rho_0} g \quad \text{for some } (P_0, g) \in \mathcal{H}^1 \times L^2, \tag{1.5}
\]

Choe and Kim [5] investigated the local existence of strong solutions, which was later improved by Li [18] and Liang [19] without using (1.5). Under the condition (1.5), with the help of a Sobolev inequality of logarithmic type, Huang and Wang [13] (see an improved result [29]) showed the global existence of strong solutions with general large data on two-dimensional (2D) bounded domains. Meanwhile, the case of entire space \(\mathbb{R}^2\) was established by Lü et al. [23] via spatial weighted method. For three-dimensional case, Kim [16] obtained that \((1.1)\) has a unique global strong solution provided \(\|\nabla u_0\|_{L^2}\) is sufficiently small. This result was improved by Craig et al. [6] by requiring that \(\|u_0\|_{\mathcal{H}^1}\) is small enough. Very recently, Danchin and Mucha [9] proved existence and uniqueness of solutions under the assumption that the initial velocity is in \(\mathcal{H}^1\) and that the density is only bounded and nonnegative. These solutions are global in the two-dimensional case for general data and in the three dimensional case if the velocity satisfies a suitable scaling-invariant smallness condition. The key tool to solve the problem is the Lagrangian approach. However, whether their methods work for the whole space case is unknown. As mentioned in their paper: “... the generalization to unbounded domains (even the whole space) within our approach being unclear as regards global-in-time results.” (see [9, p. 1355]).

When it comes to the case that the viscosity \(\mu(\rho)\) depends on the density, it is more involved to investigate the global well-posedness of system \((1.1)\) due to the strong coupling between density and velocity, and the methods used for the case of constant viscosity cannot be applied directly. In fact, for the initial density allowing vacuum states, Lions [20, Chapter 2] first derived the global weak solutions, yet the uniqueness and regularities of such weak solutions are big open questions. Later, Desjardins [10] proposed the so-called pseudo-energy method and established global weak solutions with higher regularity for 2D case provided that \(\|\mu(\rho_0) - 1\|_{L^\infty}\) is suitably small. Meanwhile, if the initial density belongs to some Besov spaces with positive index which guarantee that the initial density is at least a continuous function, Abidi and Zhang [1] can show the uniqueness of the solution in the whole plane. Recently, some attention was focused on the well-posedness of strong solutions to \((1.1)\). For the initial density strictly away from vacuum, Abidi and Zhang [2] proved the global well-posedness to the 3D Cauchy problem of \((1.1)\) under the smallness assumptions on both \(\|u_0\|_{L^2}\), \(\|\nabla u_0\|_{L^2}\) and \(\|\mu(\rho_0) - 1\|_{L^\infty}\). On the other hand, when the initial density allows vanish, under the compatibility condition

\[
- \text{div}(2\mu(\rho_0) \mathcal{D}(u_0)) + \nabla P_0 = \sqrt{\rho_0} g \quad \text{for some } (P_0, g) \in \mathcal{H}^1 \times L^2, \tag{1.6}
\]

which is introduced by Cho and Kim [3] in order to obtain the local existence of solutions solutions (see [24] for an improved result), Huang and Wang [15] and Zhang [28] proved the global existence of strong solutions of \((1.1)\) in 3D bounded domains provided the initial velocity is suitably small in some sense. Applying Desjardins’ interpolation inequality, Huang and Wang [14] also obtained the global strong solutions in 2D bounded domains under the condition that \(\|\nabla \mu(\rho_0)\|_{L^2}\) is suitably small. It should be pointed out that the compatibility condition (1.6) is needed in [14,15,28]. Very recently, by time weighted techniques and energy methods, He et al. [12] and Liu [21] established global well-posedness of strong solutions to the 3D Cauchy problem without using the compatibility condition.
Remark 1.3

2D bounded domains. Theorem 1.1 can be regarded as the uniqueness and regularity theory of Lions’s weak solutions [20] in the case and the following exponential decay rate holds for and the smallness assumption on \(\|\rho_0\|_{L^1}, \|\rho_0 - \bar{\rho}\|_{L^2}\), and \(\|\nabla u_0\|_{L^2}\) such that if

\[
\|\nabla \mu(\rho_0)\|_{L^2} \leq \varepsilon_0,
\]

the problem (1.1)–(1.3) has a unique global strong solution \((\rho, u)\) satisfying for \(\tau > 0\) and \(r \in \left(\frac{2q}{q+2}, q\right)\),

\[
\begin{align*}
\rho &\in L^\infty(0, \infty; L^1 \cap H^1 \cap W^{1,q}) \cap C([0, \infty); H^1 \cap W^{1,q}), \\
\nabla u &\in L^\infty(0, \infty; L^2) \cap L^\infty(\tau, \infty; H^1) \cap L^2(\tau, \infty; W^{1,r}), \\
\nabla P &\in L^\infty(\tau, \infty; L^2) \cap L^2(\tau, \infty; L^r), \\
\nabla u &\in C([\tau, \infty); L^2), \quad \rho u \in C([0, \infty); L^2), \\
t\sqrt{\rho} u_t &\in L^\infty(0, \infty; L^2), \quad t\nabla u_t \in L^2(0, \infty; L^2), \\
e^{\frac{\sigma}{2\tau}} \nabla u, e^{\frac{\sigma t}{\tau}} \sqrt{\rho} u_t &\in L^2(0, \infty; L^2),
\end{align*}
\]

where \(\sigma := \frac{\mu^2}{2\|\rho_0\|_{L^\infty}(\|\rho_0\|_{L^2} + \|\rho_0 - \bar{\rho}\|_{L^2})}\). Moreover, there exists a positive constant \(C\) depends only on \(\mu, \bar{\rho}, q, \|\rho_0\|_{L^\infty}, \|\rho_0\|_{L^1}, \|\rho_0 - \bar{\rho}\|_{L^2}\), and \(\|\nabla u_0\|_{L^2}\) such that

\[
\sup_{0 \leq t < \infty} \|\nabla \rho\|_{L^2} \leq C\|\nabla \rho_0\|_{L^2}, \quad \sup_{0 \leq t < \infty} \|\nabla \mu(\rho)\|_{L^2} \leq C\|\nabla \mu(\rho_0)\|_{L^2},
\]

and the following exponential decay rate holds for \(t \geq 1\),

\[
\|u(\cdot, t)\|_{H^2}^2 + \|\nabla P(\cdot, t)\|_{L^2}^2 + \|\sqrt{\rho} u_t\|_{L^2}^2 \leq Ce^{-\sigma t}.
\]

Remark 1.1

The exponential decay rate (1.11) in Theorem 1.1 is new and somewhat surprising, since the known corresponding decay-in-time rates for strong solutions to the system (1.1) with constant viscosity case are algebraic (see [23]). Moreover, as a direct consequence of (1.10), \(\|\nabla \rho\|_{L^2 \cap L^4}\) remains uniformly bounded with respect to time which is new even for the constant viscosity case (see [23, Lemma 3.5]).

Remark 1.2

It should be noticed that our Theorem 1.1 holds for arbitrarily large initial density with vacuum interiorly and any function \(\mu(\rho)\) satisfying (1.3) with \(\|\nabla \mu(\rho_0)\|_{L^2}\) suitably small, which is in sharp contrast to Abidi and Zhang [3] where they need the initial density strictly away from vacuum and the smallness assumption on \(\|\mu(\rho_0) - 1\|_{L^\infty}\). Moreover, there is no need to impose additional compatibility condition (1.6), which is required in [17] for the global existence of strong solutions in 2D bounded domains.

Remark 1.3

It is not hard to show the strong-weak uniqueness theorem [20, Theorem 2.7] still holds for the initial data \((\rho_0, u_0)\) satisfying (1.7) after modifying its proof slightly. Therefore, our Theorem 1.1 can be regarded as the uniqueness and regularity theory of Lions’s weak solutions [20] in the case of \(\mathbb{R}^2\) with \(\nabla \mu(\rho_0)\) suitably small in the \(L^q\)-norm.

Therefore, our Theorem 1.1 is new and somewhat surprising, since the known corresponding decay-in-time rates for strong solutions to the system (1.1) with constant viscosity case are algebraic (see [23]). Moreover, as a direct consequence of (1.10), \(\|\nabla \rho\|_{L^2 \cap L^4}\) remains uniformly bounded with respect to time which is new even for the constant viscosity case (see [23, Lemma 3.5]).
In the particular case when \( \mu(\rho) \) is a positive constant, it is clear that (1.8) holds true. Thus
Theorem 1.2 implies that for any given (large) initial data \((\rho_0, u_0)\) satisfying (1.7), there exists a unique global strong solution to the problem (1.1)–(1.3) with constant viscosity.

**Theorem 1.2**
For constant \( q \in (2, \infty) \), assume that the initial data \((\rho_0 \geq 0, u_0)\) satisfies (1.7), then the problem (1.1)–(1.3) with \( \mu(\rho) = \mu \) (constant) have a unique global strong solution \((\rho \geq 0, u)\) satisfying (1.9) with \( \sigma = 2\|\rho_0\|_{L^\infty}^2 (\|\rho_0\|_{L^2} + \|\rho_0 - \tilde{\rho}\|_{L^2})^2 \). Moreover, there exists a positive constant \( C \) depends only on \( \mu, \tilde{\rho}, q, \|\rho_0\|_{L^\infty}, \|\tilde{\rho}\|_{L^1}, \|\rho_0 - \tilde{\rho}\|_{L^2}, \) and \( \|\nabla u_0\|_{L^2} \) such that
\[
\sup_{0 \leq t < \infty} \|\nabla \rho\|_{L^2 \cap L^q} \leq C \|\nabla \rho_0\|_{L^2 \cap L^q},
\]
and the following exponential decay rate holds for \( t \geq 1 \),
\[
\|u(\cdot, t)\|_{H^2}^2 + \|\nabla \rho(\cdot, t)\|_{L^2}^2 + \|\sqrt{\rho} u\|_{L^2}^2 \leq Ce^{-\sigma t}.
\]

The main obstacle to the proof of Theorem 1.1 lies in the absence of the positive lower bound for the initial density. As mentioned by many authors [12][15][28], the key ingredient here is to get the time-independent bounds on the \( L^1(0, T; L^\infty) \)-norm of \( \nabla u \) and then the \( L^\infty(0, T; L^q) \)-norm of \( \nabla \mu(\rho) \). It should be pointed out that the crucial techniques of proofs in [14][31] cannot be adapted directly to the situation treated here, since their arguments depend crucially on the boundedness of the domains. Meanwhile, technically, it is hard to modify the three-dimensional analysis of [12] to the two-dimensional case with initial density containing vacuum since the analysis of [12] depends crucially on the a priori \( L^0 \)-bound on the velocity, while in two dimensions it is hard to bound the \( L^p(\mathbb{R}^2) \)-norm of \( u \) in terms of \( \|\sqrt{\rho} u\|_{L^2(\mathbb{R}^2)} \) and \( \|\nabla u\|_{L^2(\mathbb{R}^2)} \) for \( p \geq 2 \). Furthermore, we remark that it seems difficult to apply the methods used in [23] due to time-dependence of the higher order estimates of solutions.

To overcome these difficulties mentioned above, some new mathematical techniques and some new useful energy estimates are needed. First of all, due to the absence of vacuum at infinity, we find that the \( L^2 \)-norm of \( u \) can be bounded by \( \|\nabla u\|_{L^2} \) by using Cauchy-Schwarz inequality and Ladyzhenskaya’s inequality (see (3.12)). Then we derive that \( \|\sqrt{\rho} u\|_{L^2}^2 \) decays with the rate of \( e^{-\sigma t} \) for some \( \sigma > 0 \) depending only on \( \mu, \tilde{\rho}, \|\rho_0\|_{L^\infty}, \|\tilde{\rho}\|_{L^1} \) and \( \|\rho_0 - \tilde{\rho}\|_{L^2} \) (see (5.6)). Next, we need to obtain time weighted estimates of \( \|\nabla u\|_{L^2}^2 \). We multiply (1.1) by \( u \) and see that the key point is to control \( \|\sqrt{\rho} u\|_{L^2}^2 \) (see (3.23)) provided that \( \|\nabla \mu(\rho)\|_{L^2} \leq 1 \) on \([0, T]\). To this end, we make use of an interpolation inequality (see Lemma 2.1) and Gronwall’s inequality to obtain the desired, which is crucial in deriving the time-independent estimates on both the \( L^\infty(0, T; L^2) \)-norm of \( t\sqrt{\rho} u \) and the \( L^2(0, T; L^2) \)-norm of \( t\nabla u \) (see (3.33)). Indeed, the time-weighted estimate is a crucial technique in dropping the compatibility condition for the initial data (see [12][31] for example). Finally, using suitable decay-in-time rates and interpolation arguments, we succeed in obtaining the desired uniform bound (with respect to time) on the \( L^1(0, T; L^\infty) \)-norm of \( \nabla u \) (see (3.44)), which in particular implies \( L^\infty(0, T; L^q) \)-norm of the gradient of the viscosity \( \mu(\rho) \) provided \( \|\nabla \mu(\rho_0)\|_{L^4} \leq \varepsilon_0 \) as stated in Theorem 1.1 (see (3.63)).

The rest of this paper is organized as follows. In Section 2 we collect some elementary facts and inequalities that will be used later. Section 3 is devoted to the a priori estimates. Finally, we give the proof of Theorem 1.1 in Section 4.

## 2 Preliminaries

In this section, we will recall some known facts and elementary inequalities that will be used frequently later.

We begin with the local existence and uniqueness of strong solutions whose proof can be found in [32].

**Lemma 2.1** Assume that \((\rho_0, u_0)\) satisfies (1.7), then there exist a small time \( T > 0 \) and a unique strong solution \((\rho, u)\) to the problem (1.1)–(1.3) in \( \mathbb{R}^2 \times (0, T) \).


Next, the following Gagliardo-Nirenberg inequality (see [25]) will be used later.

**Lemma 2.2 (Gagliardo-Nirenberg)** For \( q \in [2, \infty), r \in (2, \infty), \) and \( s \in (1, \infty), \) there exists some generic constant \( C > 0 \) which may depend on \( q, r, \) and \( s \) such that for \( f \in H^1(\mathbb{R}^2) \) and \( g \in L^s(\mathbb{R}^2) \cap D^{1,r}(\mathbb{R}^2), \) we have

\[
\|f\|_{L^q(\mathbb{R}^2)} \leq C\|f\|_{L^2(\mathbb{R}^2)}^{2/q}\|\nabla f\|_{L^2(\mathbb{R}^2)}^{2 - 2/q},
\]

\[
\|g\|_{L^{q}(\mathbb{R}^2)} \leq C\|g\|_{L^s(\mathbb{R}^2)}^{2/q}+ (r/2)\|\nabla g\|_{L^{r}(\mathbb{R}^2)}^{2 - 2/q}.
\]

Next, the following regularity result on the Stokes system will be useful for our later analysis.

**Lemma 2.3** For positive constants \( \underline{\mu}, \bar{\mu}, \) and \( q \in (2, \infty), \) assume that \( \mu(\rho) \) satisfies

\[
\mu \in C^1[0, \infty), \nabla \mu(\rho) \in L^q, \quad 0 < \underline{\mu} \leq \mu(\rho) \leq \bar{\mu} < \infty.
\]

Let \((u, P)\) be the unique weak solution to the Cauchy problem

\[
\begin{aligned}
-\text{div}(2\mu(\rho)\mathcal{D}(u)) + \nabla P &= F, \quad x \in \mathbb{R}^2, \\
\text{div} u &= 0, \quad x \in \mathbb{R}^2, \\
u &\to 0, \quad |x| \to \infty.
\end{aligned}
\]

Then there exists a positive constant \( C \) depending only on \( \underline{\mu}, \bar{\mu}, \) and \( q \) such that

- If \( F \in L^2(\mathbb{R}^2) \cap L^{q/2+2}(\mathbb{R}^2), \) then \( (\nabla^2 u, \nabla P) \in L^2 \times L^2 \) and

\[
\|\nabla^2 u\|_{L^2} + \|\nabla P\|_{L^2} \leq C\|F\|_{L^2} + C\|\nabla \mu(\rho)\|_{L^q}^{q/2}\|\nabla u\|_{L^2} + C\|\nabla \mu(\rho)\|_{L^q}\|F\|_{L^2}^{q/2+2}. (2.5)
\]

- If \( F \in L^r(\mathbb{R}^2) \cap L^{2q/(q-r)+qr}(\mathbb{R}^2) \) for some \( r \in \left(2q/q+2, q\right), \) then \( (\nabla^2 u, \nabla P) \in L^r \times L^r \) and

\[
\|\nabla^2 u\|_{L^r} + \|\nabla P\|_{L^r} \leq C\|F\|_{L^r} + C\|\nabla \mu(\rho)\|_{L^q}^{q/2+1}\|\nabla u\|_{L^2} + C\|\nabla \mu(\rho)\|_{L^q}\|F\|_{L^2}^{2q/(q-r)+qr}. (2.6)
\]

**Proof.** 1. It follows from (2.1) that

\[
P = -(-\Delta)^{-1}\text{div} F - (-\Delta)^{-1}\text{div}(2\mu(\rho)\mathcal{D}(u)), (2.7)
\]

which together with the Sobolev inequality and properties of Riesz transform yields

\[
\|P\|_{L^{2q}/q} \leq \|(-\Delta)^{-1}\text{div} F\|_{L^{2q}/q} + \|(-\Delta)^{-1}\text{div}(2\mu(\rho)\mathcal{D}(u))\|_{L^{2q}/q}
\]

\[
\leq C\|F\|_{L^{2q}/q} + C\|\mu(\rho)\mathcal{D}(u)\|_{L^{2q}/q}^{2q}
\]

\[
\leq C\|F\|_{L^{2q}/q} + C\|\nabla u\|_{L^{2q}/q}^{2q}. (2.8)
\]

We rewrite the system (2.4) as

\[
\begin{aligned}
-\Delta u + \nabla \left(\frac{P}{\mu(\rho)}\right) &= \frac{F}{\mu(\rho)} + 2\mathcal{D}(u)\nabla \mu(\rho) - P\nabla \mu(\rho) \rho, \quad x \in \mathbb{R}^2, \\
\text{div} u &= 0, \quad x \in \mathbb{R}^2, \\
u &\to 0, \quad |x| \to \infty.
\end{aligned}
\]

Applying the standard \( L^2 \)-estimates to the Stokes system (2.9) together with (2.3), (2.8), (2.4), and Young’s inequality yields

\[
\|\nabla^2 u\|_{L^2} + \|\nabla P\|_{L^2} \leq \|\nabla^2 u\|_{L^2} + C\left\|\nabla \left(\frac{P}{\mu(\rho)}\right)\right\|_{L^2} + C\left\|\frac{P\nabla \mu(\rho)}{\mu(\rho)^2}\right\|_{L^2}
\]
Young's inequality yields problem (1.1)–(1.3) to extend the local strong solution. Thus, let which extends the case of two-dimensional torus

Lemma 2.4 Suppose that proceeding, we rewrite another equivalent form of the system (1.1) as the following

\[ 2. Noting that \( \frac{2q}{2(q-r)+q^r} > 1 \) and \( \frac{qr}{2(q-r)} > 1 \) due to \( r \in (\frac{2q}{q-2}, q) \), we then obtain from (2.7), Sobolev’s inequality, and properties of Riesz transform that

Applying the standard \( L^p \)-estimates to the Stokes system (2.9) together with (2.8), (2.7), (2.1), and Young’s inequality yields

\[
\| \nabla^2 u \|_{L^r} + \| \nabla P \|_{L^r} \\
\leq \| \nabla^2 u \|_{L^r} + C \left( \| \nabla \left( \frac{P}{\mu(\rho)} \right) \|_{L^r} + C \left( \| P \nabla \mu(\rho) \|_{\mu(\rho)^{2+\eta}} \right) \right) \\
\leq C \| F \|_{L^r} + C \| 2D(u) \cdot \nabla \mu(\rho) \|_{L^r} + C \| P \nabla \mu(\rho) \|_{L^r} \\
\leq C \| F \|_{L^r} + C \| \nabla \mu(\rho) \|_{L^r} \| \nabla u \|_{\frac{q^r}{2(q-r)+q^r}} + C \| \nabla \mu(\rho) \|_{L^r} \| P \|_{\frac{q^r}{2(q-r)+q^r}} \\
\leq C \| F \|_{L^r} + C \| \nabla \mu(\rho) \|_{L^r} \| \nabla u \|_{\frac{q^r}{2(q-r)+q^r}} + C \| \nabla \mu(\rho) \|_{L^r} \| P \|_{\frac{q^r}{2(q-r)+q^r}} \\
\leq C \| F \|_{L^r} + C \| \nabla \mu(\rho) \|_{L^r} \| \nabla u \|_{\frac{q^r}{2(q-r)+q^r}} + C \| \nabla \mu(\rho) \|_{L^r} \| P \|_{\frac{q^r}{2(q-r)+q^r}},
\]

which gives the desired (2.6).

Finally, the following logarithmic interpolation inequality has been proven in [30] Lemma 2.4, which extends the case of two-dimensional torus \( T^2 \) in [10] Lemma 1 (see also [11]) to the whole space \( \mathbb{R}^2 \).

Lemma 2.4 Suppose that \( 0 \leq \rho \leq \tilde{\rho} \) and \( u \in H^1(\mathbb{R}^2) \), then we have

\[
\| \sqrt{\rho} u \|_{L^2}^2 \leq C(\tilde{\rho})(1 + \| \sqrt{\rho} u \|_{L^2})\| u \|_{H^1} \sqrt{\log (2 + \| u \|_{H^1}^2)}.
\]

(2.11)

3 A priori estimates

In this section, we will establish some necessary a priori bounds for strong solutions \((\rho, u)\) to the problem (1.1)–(1.3) to extend the local strong solution. Thus, let \( T > 0 \) be a fixed time and \((\rho, u)\) be the strong solution to (1.1)–(1.3) on \( \mathbb{R}^2 \times (0, T) \) with initial data \((\rho_0, u_0)\) satisfying (1.7). Before proceeding, we rewrite another equivalent form of the system (1.1) as the following

\[
\begin{align*}
\rho_t + u \cdot \nabla \rho &= 0, \\
\rho u_t + \rho u \cdot \nabla u - \text{div}(2\mu(\rho)D(u)) + \nabla P &= 0, \\
\text{div } u &= 0.
\end{align*}
\]

(3.1)
In what follows, we denote by
\[ \int \cdot dx = \int_{\mathbb{R}^2} \cdot dx. \]
We sometimes use \( C(f) \) to emphasize the dependence on \( f \).
We begin with the following \( L^p \)-norm estimate of the density.

**Lemma 3.1** It holds that
\[
\sup_{0 \leq t \leq T} \| \rho \|_{L^p} \leq \| \rho_0 \|_{L^p} \quad \text{for} \quad 1 \leq p \leq \infty,
\]
(3.2)
\[
\sup_{0 \leq t \leq T} \| \rho - \tilde{\rho} \|_{L^2} \leq \| \rho_0 - \tilde{\rho} \|_{L^2}.
\]
(3.3)

**Proof.** 1. Since \( 3.1_1 \) is a transport equation, we then directly obtain the desired (3.2). Moreover, \( 3.1_1 \) along with \( \rho_0 \geq 0 \) shows
\[ \inf_{\mathbb{R}^2 \times [0,T]} \rho(x,t) \geq 0. \]
2. For positive constant \( \tilde{\rho} \), we infer from \( 3.1_1 \) that \( \rho - \tilde{\rho} \) satisfies a transport equation
\[ (\rho - \tilde{\rho})_t + u \cdot \nabla (\rho - \tilde{\rho}) = 0, \]
which implies immediately the desired (3.3). \( \square \)

**Remark 3.1** Since \( \mu(\rho) \) is a continuously differentiable function, we deduce from (3.2) and (1.4) that
\[ 0 < \underline{\mu} \leq \mu(\rho) \leq \bar{\mu} := \sup_{[0,\| \rho_0 \|_{L^\infty}]} \mu(\rho) < \infty, \]
(3.4)
and
\[ \| \mu'(\rho) \|_{L^\infty([0,T];L^\infty)} < \infty. \]
Next, the following lemma gives the basic energy estimates.

**Lemma 3.2** It holds that
\[
\sup_{0 \leq t \leq T} \| \sqrt{\rho u} \|_{L^2}^2 + \mu \int_0^T \| \nabla u \|_{L^2}^2 dt \leq \| \sqrt{\rho_0 u_0} \|_{L^2}^2,
\]
(3.5)
and
\[
\sup_{0 \leq t \leq T} \left( e^{\sigma t} \| \sqrt{\rho u} \|_{L^2}^2 \right) + \mu \int_0^T e^{\sigma t} \| \nabla u \|_{L^2}^2 dt \leq \| \sqrt{\rho_0 u_0} \|_{L^2}^2,
\]
(3.6)
where \( \sigma := \frac{\mu \tilde{\rho}^2}{2\| \rho_0 \|_{L^\infty} \left( \| \rho_0 \|_{L^2} + \| \rho_0 - \tilde{\rho} \|_{L^2} \right)^2} \).

**Proof.** 1. Multiplying \( 3.1_2 \) by \( u \) and integrating the resulting equation over \( \mathbb{R}^2 \), we derive that
\[
\frac{1}{2} \frac{d}{dt} \| \sqrt{\rho u} \|_{L^2}^2 + 2 \int \mu(\rho) \mathcal{D}(u) \cdot \nabla u dx = 0.
\]
(3.7)
Noting that
\[ 2 \int \mu(\rho) \mathcal{D}(u) \cdot \nabla u dx = \int \mu(\rho) (\partial_i u^j + \partial_j u^i) \partial_i u^j dx \]
\[ = \frac{1}{2} \int \mu(\rho) (\partial_i u^j + \partial_j u^i) (\partial_i u^j + \partial_j u^i) dx \]
\[ = 2 \int \mu(\rho) | \mathcal{D}(u) |^2 dx, \]
and

\[ 2 \int |\mathcal{D}(u)|^2 dx = \frac{1}{2} \int (\partial_i u_j + \partial_j u_i)(\partial_i u_j + \partial_j u_i) dx \]
\[ = \int |\nabla u|^2 dx + \int \partial_i u_j \partial_j u_i dx = \int |\nabla u|^2 dx, \tag{3.8} \]

we thus obtain from (3.10) and (1.4) that

\[ \frac{d}{dt} \|\sqrt{\rho} u\|_{L^2}^2 + 2 \mu \|\nabla u\|_{L^2}^2 \leq 0. \tag{3.9} \]

Integrating the above inequality over (0, T) gives the desired (3.10).

2. We obtain from Hölder’s inequality that

\[ \tilde{\rho} \int_{\mathbb{R}^2} |u|^2 dx = \int_{\mathbb{R}^2} \rho |u|^2 dx + \int_{\mathbb{R}^2} (\tilde{\rho} - \rho) |u|^2 dx \]
\[ \leq \|\rho\|_{L^2} \|u\|_{L^4}^2 + \|\rho - \tilde{\rho}\|_{L^2} \|u\|_{L^4}^2 \]
\[ \leq \sqrt{2} (\|\rho\|_{L^2} + \|\rho - \tilde{\rho}\|_{L^2}) \|u\|_{L^2} \|\nabla u\|_{L^2}, \tag{3.10} \]

where in the last inequality we have used Ladyzhenskaya’s inequality (see [17, Lemma 1, p. 8])

\[ \|u\|_{L^4(\mathbb{R}^2)}^4 \leq 2 \|u\|_{L^2(\mathbb{R}^2)}^2 \|\nabla u\|_{L^2(\mathbb{R}^2)}^2. \tag{3.11} \]

Hence, we infer from (3.10), Cauchy-Schwarz inequality, (3.2), and (3.3) that

\[ \|u\|_{L^2}^2 \leq \frac{2(\|\rho\|_{L^2} + \|\rho - \tilde{\rho}\|_{L^2})^2}{\tilde{\rho}^2} \|\nabla u\|_{L^2}^2. \tag{3.12} \]

This along with (3.2) yields

\[ \|\sqrt{\rho} u\|_{L^2}^2 \leq \|\rho\|_{L^\infty} \|u\|_{L^2}^2 \leq \frac{2(\|\rho\|_{L^\infty} + \|\rho - \tilde{\rho}\|_{L^2})^2}{\tilde{\rho}^2} \|\nabla u\|_{L^2}^2. \tag{3.13} \]

Consequently, letting \( \sigma := \frac{\tilde{\rho}^2}{2\|\rho\|_{L^\infty}(\|\rho\|_{L^2} + \|\rho - \tilde{\rho}\|_{L^2})^2} \), then we derive from (3.9) and (3.13) that

\[ \frac{d}{dt} (e^{\sigma t} \|\sqrt{\rho} u\|_{L^2}^2) + \mu e^{\sigma t} \|\nabla u\|_{L^2}^2 \leq 0. \tag{3.14} \]

Thus, integrating (3.14) over (0, T) yields the desired (3.10).

\[ \square \]

**Lemma 3.3** Let \( q \) be as in Theorem 1.1 and \( \tilde{\mu} \) be as in (3.4), assume that

\[ \sup_{0 \leq t \leq T} \|\nabla \tilde{\mu}(\rho)\|_{L^2} \leq 1, \tag{3.15} \]

then there exists a positive constant \( C \) depending only on \( \mu, \tilde{\mu}, \tilde{\rho}, q, \|\rho_0\|_{L^\infty}, \|\rho_0\|_{L^1}, \|\rho_0 - \tilde{\rho}\|_{L^2}, \) and \( \|\nabla u_0\|_{L^2} \) such that

\[ \sup_{0 \leq t \leq T} \|\nabla u\|_{L^2}^2 + \int_0^T \|\sqrt{\rho} u\|_{L^2}^2 dt \leq C. \tag{3.16} \]

Moreover, for \( \sigma \) as in Lemma 3.3, one has

\[ \sup_{0 \leq t \leq T} (e^{\sigma t} \|\nabla u\|_{L^2}^2) + \int_0^T e^{\sigma t} \|\sqrt{\rho} u\|_{L^2}^2 dt \leq C. \tag{3.17} \]
Proof. 1. Since $\mu(\rho)$ is a continuously differentiable function, we obtain from (3.11) that

$$[\mu(\rho)]_t + u \cdot \nabla \mu(\rho) = 0.$$ \hfill (3.18)

Multiplying (3.12) by $u_t$ and integrating the resulting equation over $\mathbb{R}^2$ imply that

$$2 \int \mu(\rho) \mathcal{D}(u) \cdot \nabla u_t dx + \int \rho |u_t|^2 dx = - \int \rho u \cdot \nabla u \cdot u_t dx.$$ \hfill (3.19)

Similarly to (3.14), we get

$$2 \int \mu(\rho) \mathcal{D}(u) \cdot \nabla u_t dx = 2 \int \mu(\rho) \mathcal{D}(u) \cdot \mathcal{D}(u)_t dx$$

$$= \frac{d}{dt} \int \mu(\rho) |\mathcal{D}(u)|^2 dx - \int |u| |\nabla \mu(\rho)| |\mathcal{D}(u)|^2 dx,$$

which combined with (3.19) and (3.12) yields

$$\frac{d}{dt} \int \mu(\rho) |\mathcal{D}(u)|^2 dx + \int \rho |u_t|^2 dx = - \int \rho u \cdot \nabla u \cdot u_t dx - \int u \cdot \nabla \mu(\rho) |\mathcal{D}(u)|^2 dx.$$ \hfill (3.20)

By Hölder’s and Gagliardo-Nirenberg inequalities, we have

$$\left| - \int \rho u \cdot \nabla u \cdot u_t dx \right| \leq \frac{1}{4} \|\nabla u_t\|_{L^2}^2 + 2 \|\nabla u\|_{L^4}^2 \|\nabla u\|_{L^4}^2$$

$$\leq \frac{1}{4} \|\nabla u_t\|_{L^2}^2 + C \|\nabla u\|_{L^4}^2 \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2}.$$ \hfill (3.21)

By (3.15), Sobolev’s inequality, (3.12), and Gagliardo-Nirenberg inequality, we arrive at

$$\left| - \int u \cdot \nabla \mu(\rho) |\mathcal{D}(u)|^2 dx \right| \leq C \int |u| |\nabla \mu(\rho)||\nabla u|^2 dx$$

$$\leq C \|\nabla \mu(\rho)||L^\infty|u|_{L^\frac{2}{q}}^2 \|\nabla u\|_{L^4}^2$$

$$\leq C \|u\|_{H^1} \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2}$$

$$\leq C \|\nabla u\|_{L^2}^2 \|\nabla^2 u\|_{L^2}.$$ \hfill (3.22)

Substituting (3.21) and (3.22) into (3.20) leads to

$$\frac{d}{dt} \int \mu(\rho) |\mathcal{D}(u)|^2 dx + \|\nabla u_t\|_{L^2}^2 \leq C \left( \|\nabla u\|_{L^4}^2 + \|\nabla u\|_{L^2} \right) \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2}.$$ \hfill (3.23)

2. Recall that $(u, P)$ satisfies the following Stokes system with variable viscosity

$$\begin{cases}
- \text{div}(2\mu(\rho)\mathcal{D}(u)) + \nabla P = - \rho u_t - \rho u \cdot \nabla u, & x \in \mathbb{R}^2, \\
\text{div } u = 0, & x \in \mathbb{R}^2, \\
u(x) \to 0, & |x| \to \infty.
\end{cases}$$

Applying Lemma 2.3 with $F = - \rho u_t - \rho u \cdot \nabla u$, we obtain from (3.22) and (3.15) that

$$\|\nabla^2 u\|_{L^2} + \|\nabla P\|_{L^2}$$

$$\leq C \left( \|\rho u_t\|_{L^2} + \|\rho u \cdot \nabla u\|_{L^2} \right) + C \|\nabla u\|_{L^2} + C \left( \|\rho u_t\|_{L^\frac{q}{q-1}} + \|\rho u \cdot \nabla u\|_{L^\frac{q}{q-1}} \right)$$

$$\leq C \|\nabla u_t\|_{L^2} + C \|\nabla u\|_{L^4} \|\nabla u\|_{L^4} + C \|\nabla u\|_{L^2}$$

$$+ C \|\rho\|_{L^\frac{q}{q-2}} \|\nabla u_t\|_{L^2} + C \|\rho\|_{L^\frac{q}{q-2}} \|\nabla u\|_{L^4} \|\nabla u\|_{L^4}$$

$$\leq C \|\nabla u_t\|_{L^2} + C \|\nabla u\|_{L^4} \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2} + C \|\nabla u\|_{L^2}$$

$$\leq C \|\nabla u_t\|_{L^2} + C \|\nabla u\|_{L^2} \|\nabla u\|_{L^2} + \frac{1}{2} \|\nabla^2 u\|_{L^2} + C \|\nabla u\|_{L^2},$$
and thus
\[ \|\nabla^2 u\|_{L^2} + \|\nabla P\|_{L^2} \leq C\|\sqrt{\rho} u\|_{L^2} + C\|\sqrt{\rho} u\|_{L^2}^2 \|\nabla u\|_{L^2} + C\|\nabla u\|_{L^2}. \]  
(3.24)

Inserting (3.24) into (3.23) and applying Cauchy-Schwarz inequality, we deduce that
\[ B'(t) + \|\sqrt{\rho} u\|_{L^2}^2 \leq C\|\sqrt{\rho} u\|_{L^2}^2 \|\nabla u\|_{L^2}^2 \]  
(3.25)

where
\[ B(t) := \int \mu(\rho)|\nabla(u)|^2 dx \]
satisfies
\[ \frac{\mu}{2}\|\nabla u\|_{L^2}^2 \leq B(t) \leq \frac{\bar{\mu}}{2}\|\nabla u\|_{L^2}^2 \]  
(3.26)
due to (3.4) and (3.8).

3. It follows from (3.25), (2.11), (3.5), and (3.12) that
\[ B'(t) + \|\sqrt{\rho} u\|_{L^2}^2 \leq C\|\sqrt{\rho} u\|_{L^2}^2 \log (2 + \|\nabla u\|_{L^2}^2) + C\|\nabla u\|_{L^2}^4 + C\|\nabla u\|_{L^2}^2. \]  
(3.27)

3. Setting
\[ f(t) := 2 + B(t), \]
then we deduce from (3.27) and (3.26) that
\[ f'(t) \leq C\|\nabla u\|_{L^2}^2 f(t) + C\|\nabla u\|_{L^2}^2 f(t) \log f(t), \]
which yields
\[ (\log f(t))' \leq C\|\nabla u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2 \log(f(t)). \]  
(3.28)

We thus infer from (3.28), Gronwall’s inequality, (3.3), and (3.26) that
\[ \sup_{0 \leq t \leq T} (\log f(t)) \leq C. \]

This together with (3.26) gives
\[ \sup_{0 \leq t \leq T} \|\nabla u\|_{L^2}^2 \leq C. \]  
(3.29)

Integrating (3.27) over (0, T) together with (3.29) and (3.5) leads to
\[ \int_0^T \|\sqrt{\rho} u\|_{L^2}^2 dt \leq C. \]  
(3.30)

This along with (3.29) implies the desired (3.16).

4. Multiplying (3.27) by $e^{\sigma t}$ and applying (3.24), we derive from (3.26) that
\[ \frac{d}{dt}(e^{\sigma t}B(t)) + e^{\sigma t}\|\sqrt{\rho} u\|_{L^2}^2 \leq Ce^{\sigma t}\|\nabla u\|_{L^2}^2 + \sigma e^{\sigma t}B(t) \leq Ce^{\sigma t}\|\nabla u\|_{L^2}^2. \]  
(3.31)

Integrating (3.31) over (0, T) together with (3.6) and (3.26) leads to (3.17). \qed

**Remark 3.2** Under the condition (3.15), it follows from (2.11), (3.5), (3.12), and (3.16) that
\[ \sup_{0 \leq t \leq T} \|\sqrt{\rho} u\|_{L^4}^2 \leq C. \]  
(3.32)
Lemma 3.4 Let the condition (3.15) be in force, then there exists a positive constant $C$ depending only on $\mu, \tilde{\mu}, \rho, q, \|\rho_0\|_{L^\infty}, \|\rho_0\|_{L^1}, \|\rho_0 - \tilde{\rho}\|_{L^2}$, and $\|\nabla u_0\|_{L^2}$ such that for $i \in \{1, 2\}$,
\[
\sup_{0 \leq t \leq T} \left( t^i \|\sqrt{\rho} u_t\|_{L^2}^2 \right) + \int_0^T t^i \|\nabla u_t\|_{L^2}^2 dt \leq C. 
\] (3.33)
Moreover, for $\sigma$ as that in Lemma 3.2 one has
\[
\sup_{\zeta(T) \leq t \leq T} \left( e^{\sigma t} \|\sqrt{\rho} u_t\|_{L^2}^2 \right) + \int_{\zeta(T)}^T e^{\sigma t} \|\nabla u_t\|_{L^2}^2 dt \leq C, 
\] (3.34)
where $\zeta(T) := \min\{1, T\}$.

Proof. 1. Differentiating (3.12) with respect to $t$, we arrive at
\[
\rho u_{tt} + \rho \nabla u_t - \text{div}(2\mu(\rho)\nabla(u_t)) = -\nabla P_t + \rho_t (u_t + u \cdot \nabla u) - \rho u_t \cdot \nabla u + \text{div}(2\mu\nabla(u)).
\] (3.35)
Multiplying (3.35) by $u_t$ and integrating (by parts) over $\mathbb{R}^2$ and using (1.1) yield
\[
\frac{1}{2} \frac{d}{dt} \int \rho |u_t|^2 dx + 2 \int \mu(\rho) \nabla(u_t) \cdot \nabla u_t dx
\]
\[
= \int \text{div}(\rho u) |u_t|^2 dx + \int \text{div}(\rho u) u \cdot \nabla u \cdot u_t dx - \int \rho u_t \cdot \nabla u \cdot u_t dx - \int 2\mu \nabla(u) \cdot \nabla u_t dx
\]
\[
= J_1 + J_2 + J_3 + J_4. 
\] (3.36)
We obtain from Hölder’s inequality and (2.1) that
\[
\tilde{\rho} \int_{\mathbb{R}^2} |u_t|^2 dx = \int_{\mathbb{R}^2} \rho |u_t|^2 dx + \int_{\mathbb{R}^2} (\tilde{\rho} - \rho) |u_t|^2 dx
\]
\[
\leq \|\sqrt{\rho} u_t\|_{L^2}^2 + \|\tilde{\rho} - \rho\|_{L^2} \|\rho u_t\|_{L^4}^4
\]
\[
\leq \|\sqrt{\rho} u_t\|_{L^2}^2 + C \|\rho - \tilde{\rho}\|_{L^2} \|\rho u_t\|_{L^2} \|\nabla u_t\|_{L^2}, 
\] which combined with Cauchy-Schwarz inequality and (3.6) gives
\[
\|u_t\|_{H^1}^2 \leq C \tilde{\rho}, \rho_0 - \tilde{\rho}\|_{L^2} (\|\sqrt{\rho} u_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2). 
\] (3.37)
This along with Hölder’s inequality, Sobolev’s inequality, (3.5), (3.12), (3.16), and (3.18) indicates that
\[
|J_1| = \left| - \int \rho u_t \cdot \nabla |u_t|^2 dx \right|
\]
\[
\leq 2 \|\rho\|_{L^\infty} \|u_t\|_{L^6} \|\sqrt{\rho} u_t\|_{L^3} \|\nabla u_t\|_{L^2}
\]
\[
\leq C \|\rho\|_{L^\infty} \|u_t\|_{H^1} \|\sqrt{\rho} u_t\|_{L^2} \|\nabla u_t\|_{L^2}
\]
\[
\leq C \|\rho\|_{L^\infty} \|\nabla u_t\|_{L^2} \|\sqrt{\rho} u_t\|_{L^2} \|u_t\|_{H^1} \|\nabla u_t\|_{L^2}
\]
\[
\leq C \|\sqrt{\rho} u_t\|_{L^2} \left( \|\sqrt{\rho} u_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 \right) \|\nabla u_t\|_{L^2}
\]
\[
\leq \frac{C}{8} \|\nabla u_t\|_{L^2}^2 + C \|\sqrt{\rho} u_t\|_{L^2}^2;
\] |J_2| = \left| - \int \rho u_t \cdot \nabla (u \cdot \nabla u) dx \right|
\]
\[
\leq \int \rho |u_t|^2 |u_t| + \rho |u|^2 |\nabla u_t| + \rho |u|^2 |\nabla u| |\nabla u_t| dx
\]
\[
\leq \|\rho\|_{L^\infty} \|u\|_{L^6} \|\nabla u\|_{L^2} \|\nabla u\|_{L^6} \|u_t\|_{L^6} + \|\rho\|_{L^\infty} \|u\|_{L^6} \|\nabla u\|_{L^2} \|\nabla u\|_{L^6} \|u_t\|_{L^6}
\]
11
It follows from (3.24) and (3.32) that

\[ \int |J_3| \leq ||\mathbf{u}||_{L^2}^2 + C||\nabla \mathbf{u}||_{L^2}^2 + C||\nabla \mathbf{u}||_{H^1}^2; \]

\[ \int |J_1| \leq C \int ||\mu|\mathcal{D}(\mathbf{u}_t)|||\nabla \mathbf{u}||_{L^2}^2 \, dx \]

we derive that

\[ \frac{d}{dt} ||\nabla \mathbf{u}||_{L^2}^2 + C||\nabla \mathbf{u}||_{H^1}^2 \leq C||\nabla \mathbf{u}||_{L^2}^2 + C||\nabla \mathbf{u}||_{H^1}^2. \]  

(3.38)

2. Multiplying (3.38) by \( t^i \) (\( i \in \{1, 2\} \)) yields

\[ \frac{d}{dt} (t^i ||\nabla \mathbf{u}||_{L^2}^2) + t^i ||\nabla \mathbf{u}||_{L^2}^2 \leq C t^i ||\nabla \mathbf{u}||_{L^2}^2 + C t^i ||\nabla \mathbf{u}||_{H^1}^2 + i t^{i-1} ||\nabla \mathbf{u}||_{L^2}^2. \]  

(3.39)

It follows from (3.24) and (3.32) that

\[ ||\nabla \mathbf{u}||_{H^1}^2 \leq C ||\nabla \mathbf{u}||_{L^2}^2 + C ||\nabla \mathbf{u}||_{H^2}^2, \]

which together with (3.3) and (3.17) implies

\[ \int_0^T e^{\sigma t} ||\nabla \mathbf{u}||_{H^1}^2 \, dt \leq C. \]  

(3.40)

Thus, for \( \sigma \) as in Lemma 3.2 and any nonnegative integer \( k \), we derive from (3.40) and (3.17) that

\[ \int_0^T t^k ||\nabla \mathbf{u}||_{H^1}^2 \, dt \leq \sup_{0 \leq t \leq T} (t^k e^{\sigma t}) \int_0^T e^{\sigma t} ||\nabla \mathbf{u}||_{H^1}^2 \, dt \leq C, \]  

(3.41)

\[ \int_0^T t^k ||\nabla \mathbf{u}||_{L^2}^2 \, dt \leq \sup_{0 \leq t \leq T} (t^k e^{\sigma t}) \int_0^T e^{\sigma t} ||\nabla \mathbf{u}||_{L^2}^2 \, dt \leq C. \]  

(3.42)

Integrating (3.39) over \( (0, T) \) together with (3.41) and (3.42) leads to the desired (3.33).

3. Multiplying (3.38) by \( e^{\sigma t} \) gives

\[ \frac{d}{dt} (e^{\sigma t} ||\nabla \mathbf{u}||_{L^2}^2) + e^{\sigma t} ||\nabla \mathbf{u}||_{L^2}^2 \leq C e^{\sigma t} ||\nabla \mathbf{u}||_{L^2}^2 + C e^{\sigma t} ||\nabla \mathbf{u}||_{H^1}^2 + e^{\sigma t} ||\nabla \mathbf{u}||_{L^2}^2 \]

\[ \leq C e^{\sigma t} ||\nabla \mathbf{u}||_{L^2}^2 + C e^{\sigma t} ||\nabla \mathbf{u}||_{H^1}^2. \]  

(3.43)

Integrating (3.43) over \( [\zeta(T), T] \) together with (3.34) and (3.40) leads to the desired (3.34). \( \square \)
Lemma 3.5 Let the condition \( (3.15) \) be in force, then there exists a positive constant \( C \) depending only on \( \underline{\mu}, \bar{\mu}, \rho, q; ||\rho_0||_{L^\infty}, ||\rho_0 - \bar{\rho}||_{L^2}, \) and \( ||\nabla u_0||_{L^2} \) such that
\[
\int_0^T ||\nabla u||_{L^\infty} dt \leq C. \tag{3.44}
\]

Proof. 1. Choosing \( 2 < q < q \), we then infer from \((2.12), \text{Lemma } 2.3 \tag{3.16}, \text{and } (3.15)\) that
\[
||\nabla u||_{L^\infty} \leq C||\nabla u||_{L^2}^{\frac{r}{2q-2}} ||\nabla^2 u||_{L^r}^{\frac{q-2}{2q-2}} \\
\leq C||\nabla u||_{L^2}^{\frac{r}{2q-2}} \left( ||\rho u||_{L^q} + ||\rho u \cdot \nabla u||_{L^q} + ||\nabla u||_{L^2} + ||\rho u||_{L^{\frac{2q-2}{2q-2}}(q-2)} + ||\rho u \cdot \nabla u||_{L^{\frac{2q-2}{2q-2}}(q-2)} \right)^{\frac{q-2}{2q-2}} \\
\leq C||\rho u||_{L^2}^{\frac{r}{2q-2}} + C||\rho u \cdot \nabla u||_{L^2}^{\frac{r}{2q-2}} + C||\nabla u||_{L^2} + C||\rho u||_{L^{\frac{2q-2}{2q-2}}(q-2)} + C||\rho u \cdot \nabla u||_{L^{\frac{2q-2}{2q-2}}(q-2)} \tag{3.45}
\]

2. By Sobolev’s inequality, \((2.12), \text{and } (3.17)\), we have
\[
||\rho u||_{L^2}^{\frac{r}{2q-2}} \leq ||\rho||_{L^2}^{\frac{r}{2q-2}} ||u||_{L^2}^{\frac{r}{2q-2}} \\
\leq C \left( ||\rho u||_{L^2} + ||\nabla u||_{L^2} \right)^{\frac{r}{2q-2}} \\
\leq C||\rho u||_{L^2}^{\frac{r}{2q-2}} + C||\nabla u||_{L^2}^{\frac{r}{2q-2}}. \tag{3.46}
\]

Thus, we obtain from Hölder’s inequality and \((3.17)\) that for \( \sigma \) as in Lemma \( 3.2 \)
\[
\int_0^T ||\sqrt{\rho u_t}||_{L^2}^{\frac{r}{2q-2}} dt = \int_0^T e^{-\frac{r}{2q-2} \sigma t} \left( e^{\sigma t} ||\sqrt{\rho u_t}||_{L^2}^{\frac{r}{2q-2}} \right)^{\frac{r}{2q-2}} dt \\
\leq \left( \int_0^T e^{-\frac{r}{2q-2} \sigma t} dt \right)^{\frac{3r-4}{4r-4}} \times \left( \int_0^T e^{\sigma t} ||\sqrt{\rho u_t}||_{L^2}^{\frac{r}{2q-2}} dt \right)^{\frac{r}{2q-2}} \\
\leq C. \tag{3.47}
\]

If \( T \in (0, 1] \), we get from Hölder’s inequality, \((3.33)\), and the fact \( 1 - \frac{r}{3r-4} = \frac{2r-4}{3r-4} > 0 \) that
\[
\int_0^T ||\nabla u_t||_{L^2}^{\frac{r}{2q-2}} dt = \int_0^T t^{-\frac{3r-4}{4r-4}} \left( t^{\frac{3r-4}{2q-2}} ||\nabla u_t||_{L^2}^{\frac{r}{2q-2}} \right)^{\frac{r}{2q-2}} dt \\
\leq \left( \int_0^T t^{-\frac{3r-4}{4r-4}} dt \right)^{\frac{3r-4}{4r-4}} \times \left( \int_0^T t^{\frac{3r-4}{2q-2}} ||\nabla u_t||_{L^2}^{\frac{r}{2q-2}} dt \right)^{\frac{r}{2q-2}} \\
\leq C. \tag{3.48}
\]

If \( T > 1 \), one deduces from \((3.18), \text{Hölder’s inequality, and } (3.14)\) that for \( \sigma \) as in Lemma \( 3.2 \)
\[
\int_0^T ||\nabla u_t||_{L^2}^{\frac{r}{2q-2}} dt = \int_0^1 ||\nabla u_t||_{L^2}^{\frac{r}{2q-2}} dt + \int_1^T ||\nabla u_t||_{L^2}^{\frac{r}{2q-2}} dt \\
\leq C + \int_1^T e^{-\frac{r}{2q-2} \sigma t} \left( e^{\sigma t} ||\nabla u_t||_{L^2}^{\frac{r}{2q-2}} \right)^{\frac{r}{2q-2}} dt \\
\leq C + \left( \int_1^T e^{-\frac{r}{2q-2} \sigma t} dt \right)^{\frac{3r-4}{4r-4}} \times \left( \int_1^T e^{\sigma t} ||\nabla u_t||_{L^2}^{\frac{r}{2q-2}} dt \right)^{\frac{r}{2q-2}} \\
\leq C,
\]

which combined with \((5.46)-(5.48)\) yields
\[
\int_0^T ||\rho u||_{L^2}^{\frac{r}{2q-2}} dt \leq C. \tag{3.49}
\]
3. By Hölder’s inequality, Sobolev’s inequality, (3.2), (3.12), and (3.16), we have
\[
\|\rho u \cdot \nabla u\|_{L^2}^{\frac{r}{r-2}} \leq \|\rho\|_{L^\infty}^{\frac{r}{r-2}} \|u\|_{L^2}^{\frac{r}{r-2}} \|\nabla u\|_{L^\infty}^{\frac{r}{r-2}} \leq C \|u\|_{H^1}^{\frac{r}{r-2}} \|\nabla u\|_{H^1}^{\frac{r}{r-2}} \leq C \|\nabla u\|_{H^1}^{\frac{r}{r-2}},
\]
which together with Hölder’s inequality and (3.40) leads to for \(\sigma\) as in Lemma 3.2
\[
\int_0^T \|\rho u \cdot \nabla u\|_{L^2}^{\frac{r}{r-2}} dt \leq C \int_0^T \|\nabla u\|_{H^1}^{\frac{r}{r-2}} dt
\]
\[
= C \int_0^T e^{-\frac{r}{r-2}\sigma t} (\|\nabla u\|_{H^1}^2)^{\frac{r}{r-2}} dt
\]
\[
\leq C \left( \int_0^T e^{-\frac{r}{r-2}\sigma t} dt \right)^{\frac{r}{r-2}} \left( \int_0^T e^{\sigma t} \|\nabla u\|_{H^1}^2 dt \right)^{\frac{r}{r-2}} \leq C. \tag{3.50}
\]

4. For \(\sigma\) as in Lemma 3.2 one deduces from (3.6) that
\[
\int_0^T \|\nabla u\|_{L^2} dt = \int_0^T e^{-\frac{r}{2}\sigma t} e^{\frac{r}{2}\sigma t} \|\nabla u\|_{L^2} dt \leq \frac{1}{2} \int_0^T (e^{-\sigma t} + e^{\sigma t} \|\nabla u\|_{L^2}^2) dt \leq C. \tag{3.51}
\]
Similarly to (3.49) and (3.50), it is not hard to show that
\[
\int_0^T \|\rho u\|_{L^2}^{\frac{r}{2(r-2)}} \|\nabla u\|_{L^2}^{\frac{r}{2(r-2)}} dt \leq C, \quad \int_0^T \|\rho u \cdot \nabla u\|_{L^2}^{\frac{r}{2(r-2)}} dt \leq C. \tag{3.52}
\]
This along with (3.45) and (3.49)–(3.51) gives the desired (3.44).

\[\Box\]

**Lemma 3.6** Let the condition (3.15) be in force, then there exists a positive constant \(C\) depending only on \(\mu, \tilde{\mu}, \tilde{\rho}, q, \|\rho_0\|_{L^\infty}, \|\rho_0\|_{L^1}, \|\rho_0 - \tilde{\rho}\|_{L^2}\), and \(\nabla \rho_0\|_{L^2}\) such that for \(s \in [2, q)\),
\[
\sup_{0 \leq t \leq T} (\|\nabla \rho\|_{L^2 \cap L^s} + \|\rho_t\|_{L^s}) \leq C \|\nabla \rho_0\|_{L^2 \cap L^s}. \tag{3.53}
\]

**Proof.** Taking spatial derivative \(\nabla\) on the transport equation (3.1) leads to
\[
\partial_t \nabla \rho + u \cdot \nabla^2 \rho + \nabla u \cdot \nabla \rho = 0.
\]
Thus standard energy methods yield for any \(p \in [2, q)\),
\[
\frac{d}{dt} \|\nabla \rho\|_{L^p} \leq C(p) \|\nabla u\|_{L^\infty} \|\nabla \rho\|_{L^p},
\]
which combined with Gronwall’s inequality and (3.44) gives
\[
\sup_{0 \leq t \leq T} \|\nabla \rho\|_{L^2 \cap L^s} \leq C \|\nabla \rho_0\|_{L^2 \cap L^s}. \tag{3.54}
\]
For \(2 \leq s < q\), noticing the following fact
\[
\|\rho_t\|_{L^s} = \|u \cdot \nabla \rho\|_{L^s} \leq \|\nabla \rho\|_{L^s} \|u\|_{L^{\frac{qs}{s-q}}} \leq C \|\nabla \rho\|_{L^s} \|u\|_{H^1},
\]
which together with (3.54), (3.12), and (3.16) yields
\[
\sup_{0 \leq t \leq T} \|\rho_t\|_{L^s} \leq C \|\nabla \rho_0\|_{L^2}. \tag{3.55}
\]
The proof of Lemma 3.6 is complete. \[\Box\]
Lemma 3.7 Let the condition (3.15) be in force, then there exists a positive constant $C$ depending only on $\underline{\mu}, \bar{\mu}, q, \|\rho_0\|_{L^\infty}, \|\rho_0\|_{L^1}, \|\rho_0 - \bar{\rho}\|_{L^2}$, and $\|\nabla u_0\|_{L^2}$ such that for $r \in \left(\frac{2q}{q + 2}, q\right),$

\[
\sup_{0 \leq t \leq T} \left[t \left(\|u\|_{H^2}^2 + \|\nabla P\|_{L^2}^2\right) + \int_0^T t \left(\|\nabla u\|_{W^{1,r}}^2 + \|\nabla P\|_{L^r}^2\right) dt \leq C. \tag{3.56}
\]

Moreover, for $\sigma$ as that in Lemma 3.3 and $\zeta(T)$ as in (3.34), one has

\[
\sup_{\zeta(T) \leq t \leq T} \left[e^{\sigma t} \left(\|u\|_{H^2}^2 + \|\nabla P\|_{L^2}^2\right) \right] \leq C. \tag{3.57}
\]

Proof. 1. It follows from (3.24), (3.32), and (3.12) that

\[
\|u\|_{H^2}^2 + \|\nabla P\|_{L^2}^2 \leq C \|\sqrt{\rho} u\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2,
\]

which combined with (3.33) and (3.17) yields

\[
\sup_{0 \leq t \leq T} \left[t \left(\|u\|_{H^2}^2 + \|\nabla P\|_{L^2}^2\right) \right] \leq C. \tag{3.59}
\]

Furthermore, we get from (3.63), (3.32), and (3.17) that

\[
\sup_{\zeta(T) \leq t \leq T} \left[e^{\sigma t} \left(\|u\|_{H^2}^2 + \|\nabla P\|_{L^2}^2\right) \right] \leq C. \tag{3.60}
\]

2. For $r \in \left(\frac{2q}{q + 2}, q\right)$, we infer from Lemma 2.3, (3.2), (3.12), (3.15), (3.16), Sobolev’s inequality, (3.37), and (3.58) that

\[
\|\nabla u\|_{W^{1,r}}^2 + \|\nabla P\|_{L^r}^2 = \|\nabla u\|_{L^r}^2 + \|\nabla^2 u\|_{L^r}^2 + \|\nabla P\|_{L^r}^2 \leq C\|\nabla u\|_{H^1}^2 + C\|\rho u\|_{L^r}^2 + \|\rho u \cdot \nabla u\|_{L^r}^2 + \|\nabla u\|_{L^2}^2 + \|\rho u\|_{L^{2(2q - r)+q^r}} \frac{2q^r}{2q^r} \|\rho \|_{L^{2(2q - r)+q^r}} \|\nabla u\|_{L^2}^2 + C\|\rho\|_{L^{2(2q - r)+q^r}} \|\nabla u\|_{L^2}^2 + C\|\rho\|_{L^{2(2q - r)+q^r}} \|\nabla u\|_{L^2}^2 + C\|\rho\|_{L^{2(2q - r)+q^r}} \|\nabla u\|_{L^2}^2 + C\|\rho\|_{L^{2(2q - r)+q^r}} \|\nabla u\|_{L^2}^2
\]

which together with (3.17) and (3.33) yields

\[
\int_0^T t \left(\|\nabla u\|_{W^{1,r}}^2 + \|\nabla P\|_{L^r}^2\right) dt \leq C. \tag{3.61}
\]

This finishes the proof of Lemma 3.7 \hfill \Box

Lemma 3.8 Let the condition (3.15) be in force, then there exists a positive number $\varepsilon_0$ depending only on $\underline{\mu}, \bar{\mu}, \bar{\rho}, q, \|\rho_0\|_{L^\infty}, \|\rho_0\|_{L^1}, \|\rho_0 - \bar{\rho}\|_{L^2}$, and $\|\nabla u_0\|_{L^2}$ such that

\[
\sup_{0 \leq t \leq T} \|\nabla \mu(\rho)\|_{L^q} \leq \frac{1}{2} \tag{3.62}
\]

provided that

\[
\|\nabla \mu(\rho_0)\|_{L^q} \leq \varepsilon_0. \tag{3.63}
\]
Thus, we get
\[(\nabla \mu(\rho))_t + u \cdot \nabla^2 \mu(\rho) + \nabla u \cdot \nabla \mu(\rho) = 0.\] (3.64)
Multiplying (3.64) by \(q|\nabla \mu(\rho)|^{q-2} \nabla \mu(\rho)\) and integrating the resulting equation over \(\mathbb{R}^2\) give rise to
\[
\frac{d}{dt} \int |\nabla \mu(\rho)|^q dx + q \int u \cdot \nabla^2 \mu(\rho) \cdot |\nabla \mu(\rho)|^{q-2} \nabla \mu(\rho) dx = -q \int \nabla u \cdot \nabla \mu(\rho) \cdot |\nabla \mu(\rho)|^{q-2} \nabla \mu(\rho) dx.
\]
Integration by parts together with \(\text{div} u = 0\) yields
\[
q \int u \cdot \nabla^2 \mu(\rho) \cdot |\nabla \mu(\rho)|^{q-2} \nabla \mu(\rho) dx = \int u \cdot \nabla((\nabla \mu(\rho))^q) dx = - \int |\nabla \mu(\rho)|^q \text{div} u dx = 0.
\]
Thus, we get
\[
\frac{d}{dt} \|\nabla \mu(\rho)\|_L^q \leq q \int \|\nabla u\| \|\nabla \mu(\rho)\|_L^q \|\nabla \mu(\rho)\|_L^q dx \leq q \|\nabla u\|_L^\infty \|\nabla \mu(\rho)\|_L^q.
\]
This implies that
\[
\frac{d}{dt} \|\nabla \mu(\rho)\|_L^q \leq \|\nabla u\|_L^\infty \|\nabla \mu(\rho)\|_L^q.
\]
which combined with Gronwall’s inequality leads to
\[
\sup_{0 \leq t \leq T} \|\nabla \mu(\rho)\|_L^q \leq \|\nabla \mu(\rho_0)\|_L^q e^{L^\infty dt}. \tag{3.64}
\]
This along with (3.44) gives
\[
\sup_{0 \leq t \leq T} \|\nabla \mu(\rho)\|_L^q \leq C_3 \|\nabla \mu(\rho_0)\|_L^q \tag{3.65}
\]
for some constant \(C_3\) depending only on \(\mu, \tilde{\mu}, \tilde{\rho}, q, \|\rho_0\|_L^\infty, \|\rho_0\|_L^2, \|\rho_0 - \tilde{\rho}\|_L^2, \text{ and } \|\nabla u_0\|_L^2\). Hence, setting \(\varepsilon_0 = \frac{1}{2C_3}\), we obtain the desired (3.62) provided the condition (3.63) holds true. \(\square\)

### 4 Proof of Theorem 1.1

Suppose that the initial data \((\rho_0, u_0, \rho_0, \tilde{\rho}_0, q_0, p_0, \tilde{p}_0, q_0)\) satisfies (1.1), according to Lemma 2.1, there exists a \(T_* > 0\) such that the problem (1.1)–(1.3) has a unique local strong solution \((\rho, u)\) on \(\mathbb{R}^2 \times (0, T_*]\). We plan to extend it to a global one. To this end, let \(\varepsilon_0\) be the constant stated in Lemma 3.3 and
\[
\|\nabla \mu(\rho_0)\|_L^q \leq \varepsilon_0. \tag{4.1}
\]
It follows from (4.2) and (4.3) that
\[
\rho \in C([0, T_*]; L^1 \cap H^1 \cap W^{1,q}). \tag{4.2}
\]
Since \(\mu \in C^1[0, \infty)\), we have
\[
\nabla \mu(\rho) = \mu' \nabla \rho \in C([0, T_*]; L^q), \tag{4.3}
\]
which combined with (4.1) yields that there is a \(T_1 \in (0, T_*)\) such that
\[
\sup_{0 \leq t \leq T_1} \|\nabla \mu(\rho)\|_L^q \leq 1.
\]
Setting
\[
T_* := \sup \{T| (\rho, u) \text{ is a strong solution on } \mathbb{R}^2 \times (0, T]\}, \tag{4.4}
\]
and
\[
T_1 := \sup \{T| (\rho, u) \text{ is a strong solution on } \mathbb{R}^2 \times (0, T] \text{ and } \|\nabla \mu(\rho)\|_L^q \leq 1 \}.
\]

16
Then $T_1^* \geq T_1 > 0$. In particular, Lemmas 3.3–3.8 together with continuity arguments imply that (3.15) in fact holds on $(0, T^*)$. Hence, we have

$$T^* = T_1^*$$

(4.5)

provided that (4.1) holds true. Moreover, for any $0 < \tau < T \leq T^*$, we infer from (3.15), (3.16), and (3.56) that for any $q > 2$,

$$\nabla u \in C([\tau, T]; L^2 \cap L^q),$$

(4.6)

where one has used the standard embedding theory

$$L^\infty(\tau, T; H^1) \cap H^1(\tau, T; H^{-1}) \hookrightarrow C([\tau, T]; L^q)$$

for any $q \in [2, \infty)$.

Now, we claim that $T^* = \infty$. Otherwise, if $T^* < \infty$, we deduce from (4.5) that (3.15) holds at $T = T^*$. Then it follows from (4.2), (4.3), and (4.6) that

$$(\rho, u)(x, T^*) = \lim_{t \to T^*} (\rho, u)(x, t)$$

satisfies the initial conditions (1.7) at $t = T^*$. Thus, taking $(\rho, u)(x, T^*)$ as the initial data, Lemma 2.1 implies that one can extend the strong solutions beyond $T^*$. This contradicts the assumption of $T^*$ in (4.4). Moreover, we obtain (1.10) from (3.54) and (3.65), while the desired exponential decay rate (1.12) follows from (3.34) and (3.57). This completes the proof of Theorem 1.1.

\[\square\]

References

[1] H. Abidi and P. Zhang, On the global well-posedness of 2-D inhomogeneous incompressible Navier-Stokes system with variable viscous coefficient, J. Differential Equations, 259 (2015), no. 8, 3755–3802.

[2] H. Abidi and P. Zhang, Global well-posedness of 3-D density-dependent Navier-Stokes system with variable viscosity, Sci. China Math., 58 (2015), no. 6, 1129–1150.

[3] S. A. Antontesv, A. V. Kazhikov, and V. N. Monakhov, Boundary value problems in mechanics of nonhomogeneous fluids, North-Holland, Amsterdam, 1990.

[4] Y. Cho and H. Kim, Unique solvability for the density-dependent Navier-Stokes equations, Nonlinear Anal., 59 (2004), no. 4, 465–489.

[5] H. J. Choe and H. Kim, Strong solutions of the Navier-Stokes equations for nonhomogeneous incompressible fluids, Comm. Partial Differential Equations, 28 (2003), no. 5-6, 1183–1201.

[6] W. Craig, X. Huang, and Y. Wang, Global wellposedness for the 3D inhomogeneous incompressible Navier-Stokes equations, J. Math. Fluid Mech., 15 (2013), no. 4, 747–758.

[7] R. Danchin and P. B. Mucha, A Lagrangian approach for the incompressible Navier-Stokes equations with variable density, Comm. Pure Appl. Math., 65 (2012), no. 10, 1458–1480.

[8] R. Danchin and P. B. Mucha, Incompressible flows with piecewise constant density, Arch. Ration. Mech. Anal., 207 (2013), no. 3, 991–1023.

[9] R. Danchin and P. B. Mucha, The incompressible Navier-Stokes equations in vacuum, Comm. Pure Appl. Math., 72 (2019), no. 7, 1351–1385.

[10] B. Desjardins, Regularity results for two-dimensional flows of multiphase viscous fluids, Arch. Rational Mech. Anal., 137 (1997), no. 2, 135–158.

[11] B. Desjardins, Regularity of weak solutions of the compressible isentropic Navier-Stokes equations, Comm. Partial Differential Equations, 22 (1997), no. 5-6, 977–1008.
[12] C. He, J. Li, and B. Lü, Global well-posedness and exponential stability of 3D Navier-Stokes equations with density-dependent viscosity and vacuum in unbounded domains, Arch. Ration. Mech. Anal., 239 (2021), no. 3, 1809–1835.

[13] X. Huang and Y. Wang, Global strong solution to the 2D nonhomogeneous incompressible MHD system, J. Differential Equations, 254 (2013), 511–527.

[14] X. Huang and Y. Wang, Global strong solution with vacuum to the two dimensional density-dependent Navier-Stokes system, SIAM J. Math. Anal., 46 (2014), no. 3, 1771–1788.

[15] X. Huang and Y. Wang, Global strong solution of 3D inhomogeneous Navier-Stokes equations with density-dependent viscosity, J. Differential Equations, 259 (2015), no. 4, 1606–1627.

[16] H. Kim, A blow-up criterion for the nonhomogeneous incompressible Navier-Stokes equations, SIAM J. Math. Anal., 37 (2006), no. 5, 1417–1434.

[17] O. A. Ladyzhenskaya, *The mathematical theory of viscous incompressible flow*, Science Publishers, New York-London-Paris, 1969.

[18] J. Li, Local existence and uniqueness of strong solutions to the Navier-Stokes equations with nonnegative density, J. Differential Equations, 263 (2017), no. 10, 6512–6536.

[19] Z. Liang, Local strong solution and blow-up criterion for the 2D nonhomogeneous incompressible fluids, J. Differential Equations, 258 (2015), no. 7, 2633–2654.

[20] P. L. Lions, *Mathematical topics in fluid mechanics, vol. I: incompressible models*, Oxford University Press, Oxford, 1996.

[21] Y. Liu, Global well-posedness of the 2D incompressible Navier-Stokes equations with density-dependent viscosity coefficient, Nonlinear Anal. Real World Appl., 56 (2020), 103156.

[22] Y. Liu, Global existence and exponential decay of strong solutions to the Cauchy problem of 3D density-dependent Navier-Stokes equations with vacuum, Discrete Contin. Dyn. Syst. Ser. B, 26 (2021), no. 3, 1291–1303.

[23] B. Lü, X. Shi, and X. Zhong, Global existence and large time asymptotic behavior of strong solutions to the Cauchy problem of 2D density-dependent Navier-Stokes equations with vacuum, Nonlinearity, 31 (2018), no. 6, 2617–2632.

[24] B. Lü and S. Song, On local strong solutions to the three-dimensional nonhomogeneous Navier-Stokes equations with density-dependent viscosity and vacuum, Nonlinear Anal. Real World Appl., 46 (2019), 58–81.

[25] L. Nirenberg, On elliptic partial differential equations, Ann. Scuola Norm. Sup. Pisa, 13 (1959), 115–162.

[26] M. Paicu, P. Zhang, and Z. Zhang, Global unique solvability of inhomogeneous Navier-Stokes equations with bounded density, Comm. Partial Differential Equations, 38 (2013), 1208–1234.

[27] J. Simon, Nonhomogeneous viscous incompressible fluids: existence of velocity, density, and pressure, SIAM J. Math. Anal., 21 (1990), no. 5, 1093–1117.

[28] J. Zhang, Global well-posedness for the incompressible Navier-Stokes equations with density-dependent viscosity coefficient, J. Differential Equations, 259 (2015), no. 5, 1722–1742.

[29] X. Zhong, Global strong solution and exponential decay for nonhomogeneous Navier-Stokes and magnetohydrodynamic equations, Discrete Contin. Dyn. Syst. Ser. B, doi:10.3934/dcdsb.2020246
[30] X. Zhong, Global well-posedness to the 2D Cauchy problem of nonhomogeneous heat conducting magnetohydrodynamic equations with large initial data and vacuum, accepted by Calc. Var. Partial Differential Equations, 2021.

[31] X. Zhong, Global well-posedness and exponential decay of 2D nonhomogeneous Navier-Stokes and magnetohydrodynamic equations with density-dependent viscosity and vacuum, http://arxiv.org/abs/2102.12288

[32] X. Zhong, Local well-posedness to the 2D Cauchy problem of nonhomogeneous Navier-Stokes equations with density-dependent viscosity and vacuum in unbounded domains, submitted for publication.