DENSITY PROFILES OF COLLISIONLESS EQUILIBRIA. II. ANISOTROPIC SPHERICAL SYSTEMS

ERIC I. BARNES
Department of Physics, University of Wisconsin—La Crosse, La Crosse, WI; barnes.eric@uwln.edu

LILY A. R. WILLIAMS
Department of Astronomy, University of Minnesota, Minneapolis, MN; llrw@astro.unm.edu

ARIF BABUL
Department of Physics and Astronomy, University of Victoria, Victoria, BC, Canada; babul@uvic.ca

AND

JULIANNE J. DALCANTON
Department of Astronomy, University of Washington, Seattle, WA; jd@astro washington.edu

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ABSTRACT

It has long been realized that the dark matter halos that form in cosmological N-body simulations are characterized by density profiles \( \rho(r) \) that, when suitably scaled, have similar shapes. In addition, combining the density and velocity dispersion profiles \( \sigma(r) \), both of which have decidedly non–power-law shapes, leads to a quantity \( \rho/\sigma^3 \) that is a power law in radius over 3 orders of magnitude in radius. Halos’ velocity anisotropy profiles \( \beta(r) \) vary from isotropic near their centers to quite radially anisotropic near the virial radius. Finally, there appears to be a nearly linear correlation between \( \beta \) and the logarithmic density slope \( \gamma \) for a wide variety of halos. This work is part of a continuing investigation of the above interrelationships and their origins using analytical and semianalytical techniques. Our findings suggest that the nearly linear \( \beta-\gamma \) relationship is not just another expression of scale-free \( \rho/\sigma^3 \) behavior. We also note that simultaneously reproducing density and anisotropy profiles like those found in simulations requires \( \beta(r) \) and \( \gamma(r) \) to have similar shapes, leading to nearly linear \( \beta-\gamma \) correlations. This work suggests that the \( \beta-\gamma \) and power-law \( \rho/\sigma^3 \) relations have distinct physical origins.

Subject headings: dark matter — galaxies: kinematics and dynamics — galaxies: structure

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1 INTRODUCTION

Cosmological N-body simulations are important tools that allow us to reconstruct how gravitationally bound objects such as galaxies and clusters form and evolve in the universe. The base-level simulations involve Newtonian gravitational interactions between particles that represent masses of dark matter exclusively. These types of simulations suppress the short-range interactions between masses, making the dynamics collisionless; that is, particles move under the influence of a global potential. The consensus view of the results of such simulations is that all bound structures have common density structures determined by a few parameters. The most often discussed density distribution in the literature is the empirical Navarro-Frenk-White (NFW; Navarro et al. 1996, 1997) function. However, many different functions have been invoked to describe the density profiles of dark matter structures resulting from individual simulations (e.g., Sérsic 1968; Moore et al. 1998; Navarro et al. 2004). While there is a considerable amount of literature attempting to explain the universality of N-body density profiles (Syer & White 1998; Lokas 2000; Nušer 2001; Barnes et al. 2005; Lu et al. 2006), no widely accepted explanation exists.

We are looking to gain a fundamental understanding of the physics driving the universality of dark matter halos in collisionless simulations. Such insight may eventually provide a way of reconciling the differences between simulated halos and the inferred properties of halos of observed galaxies, such as the cusp-core controversy (see, e.g., de Blok et al. 2001; Spekkens et al. 2005). However, the results of the present paper cannot be directly applied to observed galaxies and clusters, because the latter are affected by dissipational baryonic processes, which we ignore in the present work; we concentrate exclusively on the effects of gravity in collisionless systems.

In the past few years, two interesting relationships have been uncovered that link various dynamical properties of halos formed in simulations such as those discussed above. Taylor & Navarro (2001) have found that there is a scale-free relationship between a phase-space proxy and radius: \( \rho/\sigma^3 \propto r^{-\alpha} \), where \( \rho \) is density, \( \sigma \) is velocity dispersion, \( r \) is radial distance, and \( \alpha \) is a constant for any given halo. Interestingly, this relationship is not unique to N-body results. Austin et al. (2005) have shown that halos created semianalytically also have power-law \( \rho/\sigma^3 \) behavior and speculate that this feature is a result of violent relaxation.

The other relationship relates the density and velocity anisotropy profiles of N-body halos (Hansen & Moore 2006). Specifically, the curve describing the logarithmic radial derivative of the logarithm of the density has the same basic shape as the radial profile of a halo’s anisotropy. As with the power-law behavior of \( \rho/\sigma^3 \), there is no obvious basis for this relationship, but together they signal that there are unifying physical processes at work in halo formation.

1 Leverhulme Visiting Professor, Universities of Oxford and Durham.
2 Alfred P. Sloan Foundation Fellow.
This paper and its companion (Barnes et al. 2006, hereafter Paper I) take steps toward uncovering what these processes are. In the previous work, we found that the mechanical equilibrium of halos with velocity isotropy is not the cause of power-law \( r^{-3} \) profiles, as at least two well-known equilibrium density profiles (Hernquist [1990] and King [1966]) do not produce \( r^{-3} \) power laws. However, Navarro et al. (2004) and Sérsic density distributions in mechanical equilibrium produce nearly scale-free \( r^{-3} \) profiles. We have also investigated density profiles that result from solving the isotropic Jeans equation with the constraint that \( \rho \sigma^3 \propto r^{-\alpha} \). In this restricted case, density profiles are similar to those presented by Navarro et al. (2004) but are reproduced better by Sérsic models.

We continue and expand this work by again considering the density profiles that result from solving the Jeans equation under the assumption of scale-free \( \rho \sigma^3 \) (motivated by \( N \)-body and semi-analytical results), but now the velocity distributions will not be restricted to be isotropic. The details of solving the “constrained” Jeans equation are given in § 2. The anisotropic Jeans equation admits a wide variety of solutions. We use generic, but sensible, forms for our adopted anisotropy profiles. These profiles and the solutions that result are discussed in detail in §§ 3 and 4. Our main conclusion follows from these; equilibrium density profiles replicate those seen in simulations if the radial density and anisotropy profiles of any given halo have similar shapes. The degree of connectedness between the \( \rho \sigma^3 \) and density-slope–anisotropy relations is considered in § 5. We end with a summary of our findings and our conclusions (see also Fig. 1).

### 2. THE CONSTRAINED JEANS EQUATION

The Jeans equation is a moment of the collisionless Boltzmann equation. Where the Boltzmann equation describes the evolution of the full phase-space distribution function, which is not easily obtained from simulations, let alone observations of real galaxies,
the Jeans equation relates observationally accessible quantities, such as density and velocity dispersion. Mechanical equilibrium for a spherical, collisionless system with radially dependent anisotropy is determined through the Jeans equation,
\[ \frac{d}{dr} \left[ \frac{\rho(r)\sigma^2(r)}{3 - 2\beta(r)} \right] + \frac{2\beta(r)}{3 - 2\beta(r)} \frac{\rho(r)\sigma^2(r)}{r} = -G\rho(r) \frac{M(r)}{r^2} \]  
(1)

(Jeans 1919; Binney & Tremaine 1987), where \( M(r) \) is the mass enclosed at radius \( r \), \( \rho \) is the density, \( \sigma \) is the total velocity dispersion, and \( \beta \) is the anisotropy. The \( 3 - 2\beta(r) \) terms result from our choice to use the total velocity dispersion rather than the \( r \)-component. While we do not directly calculate phase-space distribution functions here, the density and velocity dispersion functions that will be discussed are everywhere positive, implying that the corresponding distribution functions are physically plausible.

One can reduce the number of functions in equation (1) by assuming an “equation of state” that connects the density to the total dispersion. We use the empirically established relation \( \rho / \sigma^3 = \left( \rho_0 / v_0^3 \right) (r / r_0)^{-3} \). We note that while this connection is consistent with the available numerical evidence (Taylor & Navarro 2001; Dehnen & McLaughlin 2005), it is not the only choice. One could equally well choose to utilize the radial dispersion, as in Dehnen & McLaughlin (2005). At this point in time, it is not clear which choice is the most physically relevant.

Imposing this scale-free \( \rho / \sigma^3 \) constraint and changing to the dimensionless variables \( x \equiv r / r_0 \) and \( y \equiv \rho / \rho_0 \), we rewrite equation (1) as
\[ -\frac{x^2}{y} \left\{ \frac{d}{dx} \left[ \frac{y^{5/3}x^{2n/3}}{3 - 2\beta(x)} \right] + \frac{2\beta(x)}{3 - 2\beta(x)} \frac{y^{5/3}x^{2n/3 - 1}}{r} \right\} = BM(x), \]  
(2)

where \( B = Gm_0 v_0^2 \). Differentiating this equation with respect to \( x \) gives us
\[ \frac{d}{dx} \left( -\frac{x^2}{y} \left\{ \frac{d}{dx} \left[ \frac{y^{5/3}x^{2n/3}}{3 - 2\beta(x)} \right] + \frac{2\beta(x)}{3 - 2\beta(x)} \frac{y^{5/3}x^{2n/3 - 1}}{r} \right\} \right) = Cy x^2, \]  
(3)

where \( C = 4\pi \rho_0 v_0^2 / r_0^2 \). This expression is an extension of equation (8) in Paper I. We eliminate the constant \( C \) by solving for \( y \), differentiating with respect to \( x \) again, and grouping like terms. The resulting constrained Jeans equation is
\[ (2\alpha + \gamma - 6) \left[ \frac{2}{3} (\alpha - \gamma) + 1 \right] (2\alpha - 5\gamma) = 15\gamma'' + 3\gamma' (8\alpha - 5\gamma + 4\beta + 12\delta b_1 - 5) - 3\delta b_1 (4\alpha^2 + \gamma^2 - 8\alpha\gamma + 8\alpha + 7\gamma - 15) \]
\[ - 3\delta^2 [6b_2 (\alpha - \gamma + 1)] - 3\delta [b_3 (54\beta + 144\beta^2 + 24\beta^3)] - 3\delta^\prime [6b_4 (\alpha - \gamma + 1) + 9b_1 b_2 \delta] - 3\delta'' (3b_1). \]  
(4)

In this notation, \( \gamma = \gamma(x) = -d \ln y / dx \) in \( x \) is the logarithmic density slope, \( \delta = \delta(x) = d \ln \beta / dx \). Since \( b_1 = 2(3 - 2\beta), b_2 = (3 + 2\beta) / (3 - 2\beta), b_3 = (3 - 2\beta)^{-3}, \) and the primes indicate derivatives with respect to \( \ln x \).

As discussed previously, the parameter \( \alpha \) is a constant, and we chose to investigate the values \( \alpha = 1.875, \alpha = 35/18 = 1.944, \) and \( \alpha = 1.975. \) These values were chosen based on the findings of Austin et al. (2005). For isotropic systems, \( \alpha = 35/18 \) represents a critical point for solutions of the constrained Jeans equation. Solutions with \( \alpha > 35/18 \) have \( \gamma \)-values that asymptotically approach finite values, and \( \alpha = 1.975 \) is our example of such a case. In terms of density, these solutions approach power-law density profiles at large radius. In the other case, \( \gamma \)-values increase indefinitely with radius, implying density profiles that continually steepen farther from the center. Our value of \( \alpha = 1.875 \) (which is also the value found by Taylor & Navarro 2001) represents this class of solutions.

While aesthetically unappealing in the extreme, equation (4) can be straightforwardly solved numerically for \( \gamma \) over a range of \( \ln x \) once \( \alpha \) and \( \beta(x) \) have been specified [different choices of \( \beta(x) \) are considered in §3, 4.1, and 4.2]. Our approach to the solution sets \( \gamma = 2 \) at the scale length \( r_0 \) (\( x = 1 \)). In effect, we define the scale length of a halo to be the radius where the density slope has the isothermal value. We further define \( x_{\text{vir}} = 10, \) by analogy with \( c = 10 \) NFW profiles [note that this does not mean that \( \rho(x < 10) = 200\rho_{\text{cen}} \). The remaining initial condition is the value of \( \gamma' \) at \( r_0 \). We leave this as a free parameter to be determined by minimizing the differences between the solution \( \gamma \)-profile and those of the NFW,
\[ \rho(x) / \rho_0 = x^{-1}(1 + x)^{-2}, \]  
(5)

and Navarro et al. (2004, hereafter N04),
\[ \log \left[ \rho(x) / \rho_0 \right] = (-2/\mu)(x^n - 1), \]  
(6)
density profiles. In particular, for the NFW profile \( \gamma'(1) = 0.5 \) and for the N04 profile \( \gamma'(1) = 2\mu, \) where \( \mu = 0.17 \) (the best-fit value from N04). We have chosen these profiles because of their ability to empirically describe the collisionless halos that form in a wide variety of numerical simulations as well as for their relative simplicity, but any other profile could be analyzed in the same way.

3. CONSTANT-ANISOTROPY SOLUTIONS

The simplest type of anisotropy distribution is a constant nonzero value. In this case, equation (4) simplifies considerably, as \( \delta \) and its derivatives disappear. Our work in Paper I deals with the special case of \( \beta = 0. \) Here we extend to nonzero, but constant, anisotropies \((-0.2 \leq \beta \leq 1.0)\). The relevant constrained Jeans equation is
\[ (2\alpha + \gamma - 6) \left[ \frac{2}{3} (\alpha - \gamma) + 1 \right] (2\alpha - 5\gamma) = 15\gamma'' + 3\gamma' (8\alpha - 5\gamma + 4\beta - 5). \]  
(7)

Solutions to this equation extend and agree with our earlier work. In general, solutions to the constant-anisotropy constrained Jeans equation do not resemble NFW profiles for any tested combination of constant anisotropy value and \( \alpha \)-value. On the other hand, the constrained Jeans equation solutions can approximate N04 profiles, at least for isotropic and slightly tangentially anisotropic velocity dispersion distributions. For the solutions that best approximate N04 profiles, the anisotropy value decreases as the \( \alpha \)-value increases. Figure 2 shows \( \gamma \)-profiles for \( (a) \alpha = 1.875 \) and \( \beta = 0.1, (b) \alpha = 35/18 = 1.944 \) and \( \beta = -0.1, \) and \( \alpha = 1.975 \) and \( \beta = -0.2. \) Even mild radial anisotropy \( (\beta \geq 0.3) \) drives the solutions to have fairly constant \( \gamma \)-profiles, independent of \( \alpha \).

Figure 2d shows the \( \gamma \)-profile for \( \alpha = 1.875 \) and \( \beta = 0.5. \) We note that the special value \( \alpha = 35/18 \) is obtained analytically only for \( \beta = 0 \) (Austin et al. 2005). For nonzero constant \( \beta, \) the special value of \( \alpha \) can be found by setting the \( \gamma' \)-term in equation (7) equal to zero and utilizing the relationship for power-law density
profiles, $\gamma_{\text{pol}} = 6 - 2\alpha$. The result is that $\alpha_{\text{special}} = 35/18 - 2\beta/9$ (see also Dehnen & McLaughlin 2005).

4. SOLUTIONS FOR ANISOTROPY DISTRIBUTIONS

On relaxing the constant-anisotropy requirement, we are immediately faced with a decision. What form should the anisotropy distribution $\beta(x)$ take? We have chosen two flexible functions to investigate; one is a simple power law, while the other is based on the hyperbolic tangent function. For both functions, we choose boundary anisotropy values, and we have parameters that affect the rates of change between these values. The functions are similar in that they both monotonically increase with $\ln x$, the shape of the distribution, and the range of integration is determined by $\ln x_{\text{bound}} = 25$. We fix the scaled virial radius at the value for an NFW halo with a concentration index of 10, $x_{\text{vir}} = 10$. While our solutions cover large ranges of $x$, we are here interested primarily in the several orders of magnitude surrounding the scale length.

The other distribution that we will refer to as the tanh anisotropy profile. The expression for this is

$$\beta(x) = \frac{1}{2} (\beta_{hi} - \beta_{lo}) \left[ 1 + \tanh(s \ln x) \right] + \beta_{lo},$$  

where again $\beta_{lo}$ is the anisotropy as $\ln x \rightarrow -\ln x_{\text{bound}}$, $\beta_{hi}$ is now the anisotropy as $\ln x \rightarrow \ln x_{\text{bound}}$, and $s$ determines the rate of change between these boundary values. Now, tanh $z = (e^z - e^{-z})/(e^z + e^{-z})$ and $1 + \tanh z = 2e^z/(e^z + e^{-z})$. If $x = s \ln x = \ln x^2$, then $1 + \tanh z = 2x^2/(x^2 + x^{-2}) = 2x^2/(x^2 + 1)$. The tanh form is thus a generalization of commonly discussed anisotropy functions: $\beta \propto r^2(r^2 + r_s^2)$ (Osipkov-Merritt; Osipkov 1979; Merritt 1985) for $s = 1$, and $\beta \propto r^2(r^2 + a)$ (e.g., Mamon & Lokas 2005) for $s = \frac{1}{2}$. The tanh form we adopt allows for more flexibility in the anisotropy profile.

Our approach to solving equation (4) with these anisotropy distributions is as follows: First, choose a profile type to fit to, either the NFW or N04 $\gamma$-distribution. Next, choose pairs of $(\beta_{hi}, \beta_{lo})$-values. With the boundary anisotropy values fixed, one is left with one free parameter in each $\beta$-profile. For each set of $(\beta_{hi}, \beta_{lo})$, we find the $s$ or $q$ that minimizes the average least-squares difference between the $\gamma$-profile that results from solving equation (4) and the chosen $\gamma$-distribution over the range from $10^{-3}x_{\text{vir}}$ to $x_{\text{vir}}$, over which $N$-body simulations produce meaningful results. This procedure provides us with the best approximation to either the NFW or N04 profiles given a set of boundary anisotropy values. Note that we do not use the least-squares values for hypothesis testing or for making statistical comparisons between the various anisotropy forms but, rather, to find the form of the anisotropy.
profile that best reproduces NFW/N04 density profiles while solving the Jeans equation.

4.1. Power-Law Anisotropy Solutions

For the power-law anisotropy solutions, we have investigated a parameter space with $0.0 \leq \beta_{10} \leq 0.9$ and $0.1 \leq \beta_{h1} \leq 1.0$. Points where $\beta_{10} = \beta_{h1}$ have not been included, as they are identical to the constant-anisotropy models ($\S$ 3). Points with $\beta_{10} < 0$ and $\beta_{h1} < 0$ have been excluded because of an instability in the fitting routine for these models. We have also investigated the regions of parameter space where $\beta_{10} > \beta_{h1}$, but these fits are uniformly uninteresting, as the solutions do not resemble any sort of physically relevant density profile. The solutions that best match NFW and N04 profiles all have $\beta_{10} = 0$. Solutions that are matched to NFW profiles tend to have smaller best-fit $\beta_{h1}$ values than those matched to N04 profiles. The best-fit solutions are discussed in detail below.

The best-fit $\gamma$-distributions with power-law anisotropy profiles are shown in Figure 3 (left), where the solutions (solid lines) are matched to the NFW profile (dashed lines). Dash-dotted lines illustrate N04 $\gamma$-profiles. Dotted lines (right) show the anisotropy profile from Mamon & Lokas (2005). Figures 3a and 3b show the $\gamma$- and $\beta$-profiles calculated with $\alpha = 1.875$ that best match the appropriate profile; Figures 3c–3d and 3e–3f illustrate the same distributions but for $\alpha = 35/18$ and $\alpha = 1.975$, respectively. Note that the solutions for the power-law anisotropy profile never resemble the NFW profile; that is, the best fits are not good fits.

Figure 4 illustrates the same relations, but the solutions are matched to the N04 profile. The line styles are the same as in Figure 3. The N04 profile is reproduced with much greater accuracy than the NFW profile, but the match becomes substantially worse as $\alpha$ increases. Note that for the solutions that match the N04 profiles, the best-fit anisotropy profile has an isotropic distribution near the center and a highly radial anisotropic character $(0.8 \leq \beta \leq 1.0)$ near the virial radius. This character of anisotropy profiles has been noted previously in $N$-body simulations (e.g., van Albada 1982; Cole & Lacey 1996). It has also been suggested that such behavior is evidence that an orbital stability process is at work in determining the density structure of equilibrium halos (Barnes et al. 2005).

In numerous $N$-body simulations, a nearly linear relationship has been found between the anisotropy and density-slope profiles (Hansen & Moore 2006). It is important to note that no one simulation displays an exactly linear $\beta$-$\gamma$ profile. Over the range of $\gamma$-values our solutions cover (1 $\leq \gamma \leq 3$), individual simulation $\beta$-$\gamma$ curves are nearly linear, indicating that the anisotropy and density-slope distributions have similar shapes. The variations present in Figure 2 of Hansen & Moore (2006) appear to originate from differing slopes and intercepts for different simulations.

In Figures 5 and 6, we present the $\beta$ versus $\gamma$ profiles (solid lines) obtained from the solutions shown in Figures 3 and 4 and compare them with the relations found in $N$-body simulations. The dashed lines represent the mean trend of the simulation results, and the dash-dotted lines show the extent of variations in those results. Of the six curves in Figures 5 and 6, the best qualitative match to the near-linear relationship found in Figure 2 of Hansen & Moore (2006) is given by the N04-like profile with $\alpha = 1.875$, but other values $1.8 \leq \alpha \leq 1.9$ would also give similar results (Fig. 6a). One difference is that our relation is
Fig. 4.—Power-law $\beta(x)$: left, calculated $\gamma$-profiles that best match the N04 $\gamma$-profile for $\alpha = 1.875, 35/18,$ and $1.975$, respectively; right, illustration of the anisotropy distributions used to calculate the solutions. The line styles are the same as in Fig. 3. The solutions match the designated N04 profile reasonably well, in contrast with the results shown in Fig. 3. [See the electronic edition of the Journal for a color version of this figure.]

Fig. 5.—Correlations between power-law velocity anisotropy $\beta$ and logarithmic density slope $-\gamma$ profiles of constrained Jeans equation solutions for $10^{-3} x_{\text{eq}} \leq x \leq x_{\text{eq}}$: (a), (b), and (c) represent the best-fit NFW solutions with $\alpha = 1.875, 35/18,$ and $1.975$, respectively. The dashed line represents the mean trend found in simulation results (Hansen & Moore 2006), and the dash-dotted lines mark the extent of variations from those simulations. While the solutions are quantitatively consistent with the simulations, their shapes (particularly the concavity) are unlike the results of simulations over similar ranges of $\gamma$-values. [See the electronic edition of the Journal for a color version of this figure.]
Fig. 6.—Same as Fig. 5, but for the best-fit N04 solutions. The most linear (and hence most qualitatively consistent with simulations) relationship in (a) is inconsistent with the quantitative results of simulations. As in Fig. 5, the solid solution curves that quantitatively agree with simulations have more concavity than the simulations themselves. [See the electronic edition of the Journal for a color version of this figure.]

Fig. 7.—Same as Fig. 3, but for tanh $\beta(x)$. Unlike the results shown in Fig. 3, NFW profiles are reproduced very well by these solutions. Note that now the $\gamma$- and $\beta$-profiles share the inflection-point behavior at $\ln x = 0$. This similarity of $\gamma$- and $\beta$-profile shapes implies a nearly linear correlation in the $\beta$-$\gamma$ plane (Fig. 9). (See § 4.2 for details.) [See the electronic edition of the Journal for a color version of this figure.]
“steeper”; we reach total radial anisotropy at $\gamma = 3$, while the mean simulation results reach $\beta \approx 0.5$ at $\gamma = 3$. Solving the constrained Jeans equation for N04-like profiles with $\alpha \gtrsim 1.9$ or for NFW-like profiles does not produce nearly linear $\beta$-$\gamma$ relations, but the solutions are at least consistent with the mean trend within the variations seen in simulations.

4.2. Hyperbolic Tangent Anisotropy Solutions

We now turn to the hyperbolic tangent (tanh) anisotropy distribution solutions. The parameter space for the tanh profile is slightly different from that for the power-law anisotropy distribution: $-0.2 \leq \beta_{\text{lo}} \leq 0.9$ and $0.3 \leq \beta_{\text{hi}} \leq 1.0$. Other combinations of $\beta_{\text{lo}}$ and $\beta_{\text{hi}}$ yield unrealistic density profiles. As in the power-law case, the values of $\beta_{\text{lo}}$ for the solutions that best match both NFW and N04 profiles are small, $\beta_{\text{lo}} \leq 0.1$. The $\beta_{\text{hi}}$ values for the NFW-matched solutions are larger than those for the N04-matched solutions. The specifics of the best-fit solutions are given below.

The solutions that best approximate the NFW profile have anisotropy profiles that range from isotropic near the center to mildly radially anisotropic ($\beta \lesssim 0.5$) at the virial radius. The best-fit solutions’ $\gamma$- and $\beta$-profiles are shown in Figure 7. Again, the line styles are the same as in Figure 3. Unlike the power-law $\beta$ case, it is easy to see that these solutions approximate the asymptotic NFW behavior quite well.

Another difference from the power-law case is the behavior of the solutions that resemble N04 profiles. These solutions tend to have fairly constant anisotropy distributions that are nearly isotropic ($\beta \lesssim 0.3$). This is most evident when looking at the best-fit solutions’ $\gamma$- and $\beta$-profiles (Fig. 8). The N04 $\gamma$-profile is well fitted by the solution, but only when the anisotropy profile is nearly constant and near zero. This agrees with the work presented in Paper I, where a $\beta = 0$ profile produced solutions that are good approximations to N04 profiles.

Figures 9 and 10 show the $\beta$ versus $\gamma$ curves for the NFW-like and N04-like solutions, respectively. The curve for the NFW-like solution with $\alpha = 1.875$ (Fig. 9a) is remarkably linear, again indicating the similarity of the $\beta$- and $\gamma$-distributions. In fact, all of the curves in Figure 9 have substantial regions of linearity. This behavior makes them qualitatively similar to the results of Hansen & Moore (2006; marked by the dashed and dash-dotted lines as before). However, the most linear relation (Fig. 9a, with $\alpha = 1.875$) is also the one that disagrees most strongly with the simulation results. The remaining profiles ($\alpha = 1.944$ and $\alpha = 1.975$) actually show a decent quantitative match to the mean simulation trend. Figure 10 shows that the N04-like density profile that is most linear (again $\alpha = 1.944$ and $\alpha = 1.975$) is a poor quantitative match to simulation results. Overall, the N04-like density profiles produce nearly horizontal $\beta$-$\gamma$ profiles, qualitatively dissimilar to the profiles from simulations. However, we note that profiles created with larger ($\gtrsim 1.944$) $\alpha$-values are quantitatively consistent with simulations.

5. RELATING SCALE-FREE $\rho/\sigma^3$ AND $\beta$ VERSUS $\gamma$

As these results have been found by assuming the power-law nature of $\rho/\sigma^3$, one could ask whether or not the $\beta$ versus $\gamma$ behavior is independent of a scale-free $\rho/\sigma^3$. Since there is no known theoretical motivation for either of these relationships, we attack this question with different tactics.

The extended secondary infall model (ESIM; Williams et al. 2004; Austin et al. 2005) is a semianalytical halo formation scenario based on the work of Ryden & Gunn (1987). In ESIM, spherical halos form by accreting shells of material that have decoupled from Hubble expansion. These halos exist at overdense locations, and their formation includes effects from secondary
Fig. 9.—Same as Fig. 5, but for tanh $\beta$. As in Fig. 6, the most linear relation is quantitatively inconsistent with the results of simulations. Higher $\alpha$-values produce solid curves that have larger regions of linearity and are consistent with simulations. Recall that in this case the $\gamma$- and $\beta$-profiles have similar shapes (particularly the inflection point at $\ln x = 0$). [See the electronic edition of the Journal for a color version of this figure.]

Fig. 10.—Same as Fig. 9, but for the best-fit N04 solutions. Overall, the solid curves are more horizontal than the simulation results. However, larger $\alpha$-values allow for quantitative agreement between the solution curves and simulations. [See the electronic edition of the Journal for a color version of this figure.]
perturbations that can be completely specified and controlled. As shells collapse, their energies change in response to the continually changing potential. In fact, the ESIM halo-collapse process is best described as violent relaxation, because direct two-body effects do not take place (there are no shell-shell interactions), while the shells are allowed to exchange energy by interaction with the global potential. In general, ESIM halos have power-law density distributions over the radial ranges in which N-body simulations show decidedly non–power-law behavior, for example, where NFW profiles change from $\rho \propto r^{-1}$ to $\rho \propto r^{-3}$.

Standard ESIM halos have scale-free $\rho/\sigma^3$ (Austin et al. 2005; Barnes et al. 2005), but what do their $\beta$ versus $\gamma$ curve look like? Figure 11a presents the $\beta$- and $\gamma$-values taken from a standard ESIM halo, and the locus bears no resemblance to the linear relation found in N-body simulations. This implies that the $\beta$-$\gamma$ relationship is not just a manifestation of a scale-free $\rho/\sigma^3$. It is certainly possible that the process responsible for creating scale-free $\rho/\sigma^3$ is not fully expressed in standard ESIM simulations, leading to a different $\beta$-$\gamma$ relationship. However, we suggest that the near-linear relation between $\beta$ and $\gamma$ supports an earlier argument that any generic outcome of collapses that are mostly radial is an instability that produces density distributions with nearly isotropic cores and more radially anisotropic envelopes (Barnes et al. 2005). Because of the semianalytical nature of the ESIM formalism, this instability cannot develop in ESIM halos. However, by manipulating the secondary perturbations present in ESIM halo formation, one can induce this type of anisotropy profile. Doing so radically changes the associated density profile from the single power law of the standard halo to an NFW-like form (Barnes et al. 2005). These changes strongly affect the overall shape of the $\beta$ versus $\gamma$ plot. Figure 11b illustrates the $\beta$-$\gamma$ relationship for an NFW-like ESIM halo. While the points certainly do not fall on a line, they more closely approximate the N-body simulation results. The implication is that the physics driving the density profile shape is also important to establishing the linear $\beta$-$\gamma$ relationship.

6. SUMMARY AND CONCLUSIONS

It is well established empirically that $\rho/\sigma^3$ is a power law in radius for virialized, collisionless halos (Taylor & Navarro 2001; Austin et al. 2005; Dehnen & McLaughlin 2005). We have exploited this relationship and transformed the usual Jeans equation into a constrained form, solely in terms of the logarithmic density slope $\gamma$ and the anisotropy $\beta$. Doing this allows us to solve directly for $\gamma$ once an anisotropy distribution has been chosen.

In an earlier paper (Barnes et al. 2006), we discussed the solutions of this constrained Jeans equation when the anisotropy is assumed to be zero everywhere. Here we have extended our earlier work to include constant, but nonzero, anisotropies, as well as anisotropy distributions. In particular, we have looked at two flexible but distinct types of anisotropy distribution: a power law and a hyperbolic tangent. We find that, in general, the solutions of the constrained Jeans equation that most resemble the empirical NFW and N04 density profiles have $\beta_{h_0} \approx 0$ and $\beta_{h_0} \gtrsim 0.5$. Anisotropy must be an increasing function of radius in halos like those produced in N-body simulations.

A central conclusion of our present work is that there is a distinct similarity between the shapes of radial density and anisotropy profiles. Density profiles that best match the NFW form are found when the anisotropy profile is of the tanh form (Fig. 7, right), which includes the Osipkov-Merritt and Mamon-Lokas forms as special cases. N04-like solutions are found with either profile Figs. 4 and 8, right), but with the tanh $\beta$-profile the anisotropy must be roughly constant over the range $10^{-3}r_{\text{vir}} < r < r_{\text{vir}}$ (Fig. 8).

Another aspect of this profile similarity is given by the $\beta$ versus $\gamma$ plots (Figs. 5, 6, 9, and 10). The combination of $\beta$- and $\gamma$-profiles that produce the most linear curves (as suggested by the results of N-body simulations; Hansen & Moore 2006) are, of course, those profiles that resemble each other most closely. The inflection point in the tanh anisotropy distribution also appears in the NFW density-slope distribution (Figs. 7 and 9, right). In other words, when this second-derivative symmetry is lost, as with NFW-like density and power-law anisotropy profiles (Fig. 3, right), so is the nearly linear correlation between $\beta$ and $\gamma$. However, we note that the most linear $\beta$-$\gamma$ relations found here are also those that are the most quantitatively different from the results of simulations, in that the solid lines in Figures 6a and 9a place the maximum uncertainty range. This suggests that the relationship between $\beta$ and $\gamma$ for any single halo is not strictly linear.

Utilizing the results of semianalytic models of halo formation, we argue that the $\beta$-$\gamma$ relationship is not just a manifestation of a scale-free $\rho/\sigma^3$. Rather, a near-linear relation between $\beta$ and $\gamma$ supports an earlier argument regarding the nature of halo density profiles (Barnes et al. 2005). Velocity distributions with nearly isotropic cores and more radially anisotropic envelopes result from radial collapses. This form of anisotropy distribution necessarily creates a density profile that is less steeply rising in the central regions, since there are fewer radial orbits that plunge close to the center and can support a strong cusp. Likewise, more tangential orbits in the outer regions would increase the density, since the average time an orbit spends far from the center would increase. At least in a qualitative sense, this picture supports the observed similarities between profile shapes; more isotropy (smaller $\beta$) implies less steeply varying density (smaller $\gamma$), and vice versa.
The outcome of this work is that there appear to be at least two physical processes at work in the formation of collisionless dark matter halos in N-body simulations. The evidence presented suggests that one process creates the scale-free $\rho/\sigma^3$ distribution and must be a fairly generic collapse process, as the $\rho/\sigma^3$ behavior is seen in the more restrictive ESIM halos. As a consequence, another process links the anisotropy and density profiles in a unique, nearly linear way.

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