The Impact of Taxation on GMWB Contract in a Stochastic Interest Rate Framework

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Abstract

Modeling taxation in GMWB Variable Annuities has been frequently neglected but accounting for it can significantly improve the explanation of the withdrawal dynamics and lead to a better modeling of the financial cost of these insurance products. The importance of including a model of taxation has first been observed by Moenig and Bauer [13] while considering the Black-Scholes model to describe the underlying security. Anyway, GMWB are long term products and thus accounting for stochastic interest rate has a relevant impact on both the financial evaluation and the policy holder behavior, as observed by Gouden`enge et al. [9]. In this paper we investigate the impact of these two elements together on GMWB evaluation. To this aim we develop a numerical framework which allows one to efficiently compute the fair fee value of a contract. Results show that accounting for both taxation and stochastic interest rate has determinant effects on the behavior of the policy holder and on the cost of GMWB contracts.

Keywords: Variable Annuities, risk-neutral valuation with taxes, stochastic interest rate, optimal withdrawal.

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1 Introduction

Variable Annuities are tax deferred investment contracts with insurance coverage. The market for such products has been steadily growing in the past years all around the world. In particular, according to the Insured Retirement Institute [11], in 2017 the Variable Annuity sales in the United States amounted to over $116 billions, which represents almost half of the total annuity sales in 2017. In this paper we focus on a particular type of Variable Annuity, called Guaranteed Minimum Withdrawal Benefit (GMWB) which promises to return the entire initial investment by means of cash withdrawals during the policy life plus a final payment amounting to the remaining account value at the contract maturity. Usually, the policy holder (hereinafter PH) pays the whole premium as a lumpsum and he is entitled to withdraw at each contract anniversary a variable amount, with a minimum guaranteed. Moreover, if the PH death occurs before the contract maturity, then his heirs receive the remaining account value as a lumpsum payout. The premium paid by the PH determines the risky asset account, which changes over time according to a financial index (usually a fund) and it is also reduced by withdrawals. Thanks to the guarantee, the PH can withdraw money from the account value even if it has run out.

In order to abide the GMWB guarantee, insurers employ hedging techniques which rely on the computation of the fair price of the policy in a risk neutral probability framework. In particular, the hedging costs are offset by deducting a proportional fee from the risky asset account. Moreover, the mortality risk is hedged by using the law of large numbers (see Bernard and Kwak [1] and Lin et al. [12] for an explanation of move-based and semi-static hedging of Variable Annuities). Price and Greeks calculation usually relies on numerical computations, which are based on a convenient model of the product, of the financial market, and nonetheless of the behavior of the PH. In fact, since the PH can choose (within certain limits established by the contract) the amount to be withdrawn, he can decisively drive the total payoff of the contract. Anyway, ordinary techniques for pricing American and Bermudan options produce prices which differ significantly from market observations (Moenig and Bauer [13]). A possible explanation to the theoretical-empirical price gap, can be found in a correct modeling for the dynamics of taxation that the customer must face. In order to reduce this gap, Moenig and Bauer [13] propose to model taxation imposed to the PH and to consider a subjective valuation of the contract. Specifically, they show that when accounting for taxation, PH withdraws less frequently than without taxes and employing ordinary pricing techniques, one can obtain prices which are in line with empirical observations. Moreover, Moenig and Zhu [14] observe that the preferential tax treatment has been one of the key factors that have made Variable Annuities such a popular instrument and thus modeling correctly taxation can improve the explanation of the PH choices in performing withdrawals. We stress out that these investigations have been performed assuming the Black-Scholes model.

Another relevant key in GMWB evaluation is modeling interest rates. As observed by Goudenège et al. [9], since GMWB contracts have long maturities that could last almost 25 years, the Black-Scholes model, with its constant interest rate and volatility, seems to be unsuitable for such a long time interval. Several authors have investigated the possibility of evaluating GMWB contracts considering stochastic interest rate. For example, under the assumption of deterministic withdrawal rates, Peng et al. [15] develop an analytic approximation to the fair value of the GMWB under the Vasicek stochastic interest rate framework. Donnelly et al. [5] consider pricing and Greeks calculation through an Alternating Direction Implicit (ADI) method in the advanced Heston-Hull-White model, assuming constant withdrawals. Yang, Dai and Liu [4] develop a tree based model to include both stochastic interest rate and mortality in their evaluation framework.

In this paper we present an investigation about GMWB pricing and PH behavior when both tax treatment and stochastic interest rate are considers. In particular, following Moenig and Bauer [13] and Moenig and Zhu [14], we model taxation of GMWB with a constant marginal income tax rate $\tau$ on all policy earnings...
(i.e., payout minus initial investment, if positive), assessed after each cash flow. As far as the stochastic interest rate is concerned, we consider the Black-Scholes Hull-White (BS HW) model (Hull [10]) which is often employed by both academics and practitioners for its easiness of calibration. We stress out that considering both taxation and stochastic interest rate is a challenging task. The difficulty is mainly due to the computational effort required. In particular, evaluating a GMWB policy in the considered model is a four (plus time) dimensional problem, which means a high computational cost both in terms of time and memory. In order to manage such a computational effort, we employ the Hybrid PDE (HPDE) method (Briani et al. [2]) which represent an efficient numerical approach, already used in other research works concerning Variable Annuities (Goudenège et al. [8, 9]).

To the best of our knowledge, this is the first analysis on GMWB pricing and PH behavior which accounts for both taxation and stochastic interest rate. Our research could be useful both for the qualitative observations obtained and for the numerical solutions adopted.

The reminder of the paper is organized as follows: Section 2 introduces the models for the underlying process, the interest rate and the mortality of the PH. Section 3 describes the GMWB contract and the taxation model. Section 4 presents the pricing assumptions. Section 5 describes the pricing method and the technical measures. Section 6 is devoted to numerical tests on various examples. Finally, Section 7 presents the conclusions.

2 The Model

In order to establish the reference notation in the rest of the paper, let us introduce the BS HW model.

Underlying process and interest rate

The Hull-White model [10] is one of historically most important interest rate models, which is nowadays often used for option pricing purposes. In particular, the important advantage of the Hull-White model is the existence of closed formulas to compute the prices of bonds, caplets and swaptions. In order to fix the notation, we report the dynamics of the BS HW model:

\[
\begin{align*}
    dS_t &= r_t S_t dt + \sigma S_t dZ^S_t \quad S_0 = S_0, \\
    dr_t &= k (\theta_t - r_t) dt + \omega dZ^r_t \quad r_0 = \bar{r}_0,
\end{align*}
\]

(2.1)

where \(Z^S\) and \(Z^r\) are Brownian motions, and \(d \langle Z^S_t, Z^r_t \rangle = \rho dt\).

The process \(r\) is a generalized Ornstein-Uhlenbeck process: here \(\theta_t\) is not constant but it is a deterministic function which is completely determined by the market values of the zero-coupon bonds by calibration (see Brigo and Mercurio [3]): in this case the theoretical prices of the zero-coupon bonds match exactly the market prices. Let \(P^M(0, T)\) denote the market price of the zero-coupon bonds at time 0 for the maturity \(T\). The market instantaneous forward interest rate is then defined by

\[f^M(0, T) = -\frac{\partial \ln P^M(0, T)}{\partial T} .\]

It is well known that the short rate process \(r\) can be written as

\[r_t = \omega U_t + \beta(t) ,\]

where \(U\) is a stochastic process given by

\[dU_t = -k U_t dt + dZ^r_t \quad U_0 = 0,\]
and $\beta(t)$ is a function

$$\beta(t) = f^M(0,t) + \frac{\omega^2}{2k^2} (1 - \exp(-kt))^2.$$  

Then, the BS HW model can be described by the following relations:

$$
\begin{aligned}
    &dS_t = r_t S_t dt + \sigma S_t dZ_t^S, & S_0 = \bar{S}_0, \\
    &dU_t = -k U_t dt + dZ_t, & U_0 = 0, \\
    &r_t = \omega U_t + \beta(t).
\end{aligned}
$$  

(2.2)

A particular case is called flat curve. In this case, we assume $P^M(t,T) = e^{-\bar{r}_0(T-t)}$ and $f^M(0,T) = \bar{r}_0$. Then

$$\beta(t) = r_0 + \frac{\omega^2}{2k^2} (1 - \exp(-kt))^2,$$

and

$$\theta_t = r_0 + \frac{\omega^2}{2k^2} (1 - \exp(-2kt)).$$

### Mortality

The PH can die before the contract expiration. Following Forsyth and Vetzal [6], the effects of the mortality on the contract can be described by means of two functions. Let the mortality function $M : [0,T] \to \mathbb{R}$ be the probability density representing the random death year of the PH. Thus, the fraction of the original owners who die in $[t,t+dt]$ is equal to $M(t) dt$. Moreover, let $R : [0,T] \to \mathbb{R}$ be the survival function, that is the fraction of the original owners who are still alive at time $t$:

$$R(t) = 1 - \int_0^t M(s) ds.$$  

(2.3)

We remark that the time $t$ is measured from the contract inception and thus $R(0) = 1$. For seek of simplicity, we assume $M$ to be constant between contract’s anniversaries.

### 3 The GMWB contract

We consider here a simple version of the GMWB contract which was first investigated by Moenig and Bauer [13].

We consider an $x$-year old individual that purchased a GMWB policy with a finite integer maturity $T$ against the payment of a single premium $P$. There are three variables which determine the state of a policy at time $t$, namely the account value $X_t$, the benefit base $G_t$ and the tax base $H_t$ whose values at time $t = 0$ are given by

$$X_0 = G_0 = H_0 = P.$$  

(3.1)

During the time between the contract anniversaries $t_i = i, i = 1, \ldots, T$, that is for $t \in [t_{i-1}, t_i[$, the variables $G_t$ and $H_t$ do not change, while $X_t$ varies according to the underlying fund changes. Specifically, let us denote $S_t$ the value of the underlying fund, which evolves according to (2.1). Then, for $t \in [t_{i-1}, t_i[$, $X_t$ follows the same dynamics of $S_t$ with the exception that fees are subtracted continuously, that is

$$dX_t = \frac{X_t}{S_t} dS_t - \varphi X_t dt,$$  

(3.2)

4
where $\varphi$ is the (constant) fee rate. At each anniversary time $t_i$, the continuation of the policy is determined according to the client’s survival during the last year of the contract. In particular, if he has passed away then his heirs receive the account value $X_{t_i}$, net of taxation, and the contract ends. The death benefit is given by

$$db_i = X_{t_i} - \tau (X_{t_i} - H_{t_i})_+,$$

(3.3)

where $\tau$ denotes the constant income tax rate and $(x)_+ = \max(x, 0)$.

On the contrary, if the PH has not passed away, then he is entitled to withdraw an amount $w_i$ within some limits. Specifically, let us denote with $X_{t_i}^-$ and $X_{t_i}^+$ the account values just before and after the withdrawal is made (we use the same notation for $G_{t_i}$ and $H_{t_i}$). The withdrawal amount $w_i$ selected by the PH must satisfy the following relation:

$$0 \leq w_i \leq \max\{X_{t_i}^-, \min\{g^W, G_{t_i}^-\}\}.$$

(3.4)

Here $g^W$ is a positive constant value called the annual guaranteed amount, and it is stated in the contract. Usually $g^W = P/T$ and, in this case, the PH is entitled to withdraw at each contract anniversary exactly an amount equal to $g^W$ throughout the duration of the contract. Such a withdrawal strategy is called static, as the PH do not change the amount to be withdrawn. Such a strategy is easy to be implemented, but it is not the optimal one for the PH.

After the withdrawal has been performed, the new account value is given by

$$X_{t_i}^+ = (X_{t_i}^- - w_i)_+,$$

(3.5)

while the new benefit base and tax base are given by

$$G_{t_{i+1}}^- = G_{t_i}^+ = \begin{cases} (G_{t_i}^- - w_i)_+, & \text{if } w_i \leq g^W \\ (\min\{G_{t_i}^- - w_i, X_{t_i}^+\})_+, & \text{if } w_i > g^W \end{cases}$$

(3.6)

and

$$H_{t_{i+1}}^- = H_{t_i}^+ = H_{t_i}^- - \left( w_i - (X_{t_i}^- - H_{t_i}^-)_+ \right)_+.$$  

(3.7)

respectively.

However, the PH does not receive the whole amount withdrawn, because some fees and tax may have to be applied. Specifically, the PH receives the withdrawn amount reduced by the fees due to the insurer for withdrawing more than the guaranteed amount and also reduced by the taxation, that is

$$w_i - fee_i - tax_i$$

(3.8)

being $fee_i$ the cost for withdrawing an amount exceeding $\min\{g^W, G_{t_i}\}$ and $tax_i$ the income taxes associated with the withdrawal. In particular,

$$fee_i = s_i \cdot \left( w_i - \min\{g^W, G_{t_i}\} \right)_+$$

(3.9)

and

$$tax_i = \tau \cdot \min\left( w_i - fee_i, (X_{t_i}^- - H_{t_i}^-)_+ \right).$$

(3.10)

Here, $s_i$ is a non-negative coefficient called surrender charge. In particular, $s_i$ usually decrease with time and it is zero within the term of the contract.

After the last withdrawal has been made at time $T$, the PH receives the remaining account value net of taxes, that is

$$X_T^+ - \tau (X_T^+ - H_T^+)_+$$

(3.11)

and the contract ends.
4 Pricing assumptions

In this Section we present the pricing method. First of all, we present how to evaluate the GMWB contract assuming PH’s subjective point, that is computing the value of the contract as the expected value of future cash flows earned by the PH, choosing at each contract anniversary the optimal withdrawal. This assumption implies that the PH selects the optimal withdrawal in order to maximize the expected value of its assets, net of taxation. We stress out that the amount withdrawn by the PH is optimal for the PH, but it could be different from the optimal amount computed considering the insurer’s point of view, that is the withdrawal that maximizes the expected value of the assets gross of taxation.

In the second part of this Section we present the evaluation of the policy according to the insurer: we compute the value of the contract as the expected value of future cash flows paid by the insurer (gross of taxation), choosing the optimal withdrawal identified by the PH. We stress out that the evaluation according to the PH’s subjective point is a stand-alone problem, while the evaluation according to the insurer’s subjective point requires the computation of the optimal withdrawals for the PH, that is the computation of the value of the policy according to the PH’s subjective point.

4.1 Policyholder’s subjective valuation

Following Moenig and Bauer [13], we consider the PH’s subjective valuation of the contract. This means that we compute the fair value according to the PH, that is the expected value of future perceived cash flows net of taxes. Let \( V(t, r, X, G, H) \) denote the fair value according to the PH of a GMWB contract at time \( t \) being \( r \) the interest rate, \( X \) the account value, \( G \) the guarantee base and \( H \) the tax base respectively. Here we do not make any assumption on the fact that the PH is alive or not at time \( t \), but rather we consider a generic PH which is alive with probability equal to \( R(t) \), thus applying the law of large numbers to hedge the mortality risk.

We describe now how \( V \) changes at a contract anniversary and between two contract anniversaries.

4.1.1 Value function at a contract anniversary

Let \( t_i, i = 1, \ldots, T \) be the \( i \)-th contract anniversary and let \( V(t_i, r_i, X_{t_i}, G_{t_i}, H_{t_i}) \) and \( V(t_i, r_i, X_{t_i}^+, G_{t_i}^+, H_{t_i}^+) \) represent the value of the contract just before and after the death benefit payment and withdrawal are made. Please, observe that there is no need to distinguish the value of the interest rate before and after the withdrawal, so we simply write \( r_{t_i} \). We assume that the PH select the amount \( w_i \) in order to maximize the expected value of his assets, that is

\[
w_i = \arg\max_{w \in I} V(t_i, r_i, X_{t_i}^+(w), G_{t_i}^+(w), H_{t_i}^+(w)) + R(t_i)(w - fee_i(w) - tax_i(w))
+ (R(t_i) - R(t_{i-1}))db_i, \tag{4.1}
\]

where

\[
I = \left[0, \max\{X_{t_i}, \min\{gW, G_{t_i}\}\}\right]. \tag{4.2}
\]

Please, note that in [4.1] we underlined the dependence of many variables on \( w \) by denoting them as a function of \( w \). Thus the following relation holds,

\[
V(t_i, r_i, X_{t_i}, G_{t_i}, H_{t_i}) = V(t_i, r_i, X_{t_i}^+(w_i), G_{t_i}^+(w_i), H_{t_i}^+(w_i)) + R(t_i)(w_i - fee_i(w_i) - tax_i(w_i))
+ (R(t_i) - R(t_{i-1}))db_i. \tag{4.3}
\]
Moreover, at maturity, the terminal condition is given by the final payoff, that is
\[ V(T^+, r_T, X^+_T, G^+_T, H^+_T) = R(T) \left( X^+_T - \tau (X^+_T - H^+_T)_+ \right). \] (4.4)

We observe that at maturity the optimization problem 4.1 can be easily solved as the continuation value after the payment is the final payoff. It is possible to prove that the optimal withdrawal in this case is
\[ w_T = \min \{ g^W, G_T^- \}, \] (4.5)
and
\[ V(t^-, r_T, X^-_T, G^-_T, H^-_T) = R(T) \left( \max \{ X^-_t, \min \{ g^W, G^-_t \} \} - \tau \left( \max \{ X^-_T, \min \{ g^W, G^-_T \} \} - H^-_T \right)_+ \right) 
+ \left( R(T) - R(T - 1) \right) db_i. \] (4.6)

We observe that both interest rate and taxes appear in equation 4.3 and we aim to investigate on how they affect PH choices.

4.1.2 Dynamics of the value function between two anniversaries

During the time between two contract anniversaries, the variables \( G_t \) and \( H_t \) do not change. Changes in the value of the policy are solely due to the passage of time \( t \) and to the changes of the account value \( X_t \) and of the interest rate \( r_t \). The dynamics of \( V(t, r, X, G, H) \) in \([t_i, t_{i+1}]\) is given by the following partial differential equation (PDE)
\[ V_t + \frac{\sigma^2 X^2}{2} V_{XX} + \frac{\omega^2}{2} V_{rr} + (r - \phi) X V_X + \rho \omega X \sigma V_{Xr} + k (\theta_t - r) V_r - r V = 0 \] (4.7)
with the terminal condition given by
\[ V(t_{i+1}, r, X, G, H) = V(t^-_{i+1}, r, X, G, H). \] (4.8)

4.2 Insurer’s subjective valuation

The valuation according to the insurer differs from the valuation according to the PH for two reasons. First of all, the insurer must shell out an amount gross of taxes, while the customer receives the net amount. Secondly, the insurer has no decision-making power and undergoes the PH’s choices regarding the amount to be withdrawn. Similarly to what done in the previous Section, let \( U(t, r, X, G, H) \) denote the fair value according to the insurer at time \( t \). We describe now how \( U \) changes at a contract anniversary and between two contract anniversaries.

4.2.1 Value function at a contract anniversary

Let \( t_i, i = 1, \ldots, T \) be the i-th contract anniversary and let \( U(t^-_i, r_{t_i}, X^-_{t_i}, G^-_{t_i}, H^-_{t_i}) \) and \( U(t^+_i, r_{t_i}, X^+_{t_i}, G^+_{t_i}, H^+_{t_i}) \) represent the value of the contract just before and after the death benefit payment and withdrawal are made. Let \( w_i \) be the withdrawal determined as the solution of 4.1. The following relation holds,
\[ U(t^-_i, r_{t_i}, X^-_{t_i}, G^-_{t_i}, H^-_{t_i}) = U(t^+_i, r_{t_i}, X^+_{t_i}(w_i), G^+_{t_i}(w_i), H^+_{t_i}(w_i)) + R(t_i)(w_i - fee_i(w_i)) \]
\[ + (R(t_i) - R(t_{i-1})) db_i. \] (4.9)
Moreover, at maturity, the terminal condition is given by the final payoff, that is
\[ U \left( T^+, r_T, X_T^+, G_T^+, H_T^+ \right) = R(T) \left( X_T^+ - \tau \left( X_T^+ - H_T^+ \right) \right). \] (4.10)
Moreover, thanks to (4.5), the contract value before the last payment is given by
\[ U \left( T^-, r_T, X_T^-, G_T^-, H_T^- \right) = R(T) \left( \max \left\{ X_T^-, \min \left\{ g^W, G_i^- \right\} \right\} \right) + \left( R(T) - R(T-1) \right) d_{bi}. \] (4.11)

4.2.2 Dynamics of the value function between two anniversaries

The dynamics of \( U \) in \([t_i, t_{i+1}]\) is given by the following partial differential equation (PDE)
\[ u_t + \frac{\sigma^2}{2} u_{XX} + \frac{\omega^2}{2} u_{rr} + (r - \phi) X u_X + \rho \omega X \sigma u_{Xr} + k(t_i - r) u_r - ru = 0 \] (4.12)
with the terminal condition given by
\[ U \left( t_{i+1}, r, X, G, H \right) = U \left( t_{i+1}, r, X, G, H \right). \] (4.13)

5 Pricing method

The initial fair value of the contract according to the PH’s subjective perspective, denoted by \( V(0, r_0, P, P, P) \), can be computed by solving backward the PDE (4.7) in \([t_i, t_{i+1}]\) for \( i = T-1, \ldots, 0 \), considering the terminal condition in (4.8) and applying relations (4.1) and (4.3) to handle the jumps at each contract anniversary.

With a similar approach, the initial fair value of the contract according to the insurer’s perspective, denoted by \( U(0, r_0, P, P, P) \), can be computed by solving backward the PDE (4.12) in \([t_i, t_{i+1}]\) for \( i = T-1, \ldots, 0 \), considering the terminal condition in (4.13) and applying relations (4.1) and (4.9) to handle the jumps at each contract anniversary. We stress out that computing \( U(0, r_0, P, P, P) \) requires the knowledge of the optimal withdrawals, which can be achieved through the parallel computation of \( V(0, r_0, P, P, P) \).

We stress out that both the evaluation problems are five (including the time variable) dimensional problems and this represent a non-trivial challenge which requires an efficient numerical method to be solved.

5.1 General algorithm

Before going into details, we synthesize the by points the algorithm to compute the contract values \( V(0, r_0, P, P, P) \) and \( U(0, r_0, P, P, P) \) for a given value of \( \varphi \). This procedure can be plugged into the secant method to obtain the fair fee value.

1. Compute the terminal values \( V \left( T^-, r_T, X_T^-, G_T^-, H_T^- \right) \) and \( U \left( T^-, r_T, X_T^-, G_T^-, H_T^- \right) \) according to equations (4.6) and (4.11).
2. For all \( t_i = T - 1, \ldots, 1 \)
   a. Compute \( V \left( t_{i+1}, r_{ti}, X_{ti}^+, G_{ti}^+, H_{ti}^+ \right) \) and \( U \left( t_{i+1}, r_{ti}, X_{ti}^+, G_{ti}^+, H_{ti}^+ \right) \) by solving the PDEs in (4.7) and (4.12);
   b. Solve the maximization problem (4.1);
   c. Compute \( V \left( t_i, r_{ti}, X_{ti}^-, G_{ti}^-, H_{ti}^- \right) \) and \( U \left( t_i, r_{ti}, X_{ti}^-, G_{ti}^-, H_{ti}^- \right) \) by applying relations (4.3) and (4.9).
3. Compute \( V(0, r_0, P, P, P) \) and \( U(0, r_0, P, P, P) \) by solving PDEs in (4.7) and in (4.12).

We solve the PDEs (4.7) and (4.12) by means of the Hybrid PDE (HPDE) method (Briani et al. [2]). This numerical method has already been employed successfully to price Variable Annuities products in other frameworks (Goudenège et al. [8, 9]). For seek of concision, we give here some detail about the HPDE method and we refer the interested reader to Briani et al. [2] for more information.

The HPDE method is called an hybrid method because it combines trees and one dimensional PDE techniques to solve a multidimensional PDE.

In order to develop the algorithm, we consider a tri-dimensional grid \( G = G_X \times G_G \times G_H \), whose points are a triplet of values for the account value, the benefit base and the tax base. In particular, the set \( G_X \) is given by \( G_X = \{0\} \cup G_X^{exp} \), where \( G_X^{exp} \) is a mesh of \( N_X \) points uniform in log around the value \( P \); the sets \( G_G \) and \( G_H \) are two mesh of \( N_G \) and \( N_H \) uniformly distributed points from 0 to \( P \). Moreover, in order to implement the hybrid component of the method, we also consider the quadrinomial tree \( T = \{ (\tau_i, r_{i,j}) \}, \text{ for } i = 0, \ldots, N_T, \text{ and } j = 0, \ldots, 3i \} \) introduced in Goudenège et al. [8]. Specifically, \( r_{i,j} \) represents the \( j \)-th possible state for the process \( r \) at time \( \tau_i = i \Delta \tau \), where \( \Delta \tau = 1/N_T \) is the tree time step and \( N_T \) is a positive integer. We stress out that, since \( 1 = N_T \Delta \tau \), the contract anniversaries are included in the time mesh of \( T \).

The Hybrid PDE method for GMWB pricing consists in employing the Hybrid PDE approach between the contract anniversaries and in applying the changes due to the withdrawals performed at each contract anniversary by the PH. Moreover, the algorithm computes the function \( V \) only for those values of \( X, G, H, r, t \) such that \((X, G, H)\) is a point of \( G \) and \((t, r)\) is a point of \( T \) (the values \( V(0, r_0, P, P, P) \) and \( U(0, r_0, P, P, P) \) are therefore computed).

The optimization problem (4.1) is solved by means of a direct comparison of the values of the objective function considering a mesh of values for \( w \). Specifically, we evaluate the objective function at all the elements of the discrete set \( G_w \cap I \), where \( I \) is given by (4.2) and \( G_w = \{ k\bar{w} : k \in \mathbb{N} \} \), being \( \bar{w} \) the smallest positive withdrawal considered. Moreover, the evaluation of the objective function may require to estimate the value of \( V^+ \) in (4.1) for certain points which may not belong to \( G \): these value are approximated by means of trilinear interpolation on \( G \), which is a fast multivariate interpolation method (Gomes et al. [7]).

5.2 Reducing computational time

It is possible to prove from relations (3.1), (3.6) and (3.7) that the following relation holds

\[
G_{i_t} \leq H_{i_t}, \quad \forall i \in \{0, \ldots, T\}.
\] (5.1)

Thanks to the previous observation, it is possible to reduce the calculation of the function \( V \) to solely the points \((X, G, H)\) of \( G \) which satisfy \( G \leq H \). This allows to halve the computational cost. Obviously, the interpolation algorithm must be adapted to take into account that the value of the function \( V \) is not available at all points of \( G \).

Finally, we point out that another good practice to reduce the computational cost consists in calculating and storing all the coefficients of the trilinear interpolation and only then deal with the optimization problem (4.1) through the grid \( G \).

6 Numerical Results

In this Section we report the results of some numerical tests. Tables 1, 2, 3 report the parameters used in the analysis. Moreover, in order to estimate the mortality functions \( M \) and \( R \), we employ the DAV 2004R
mortality Table, assuming the PH to be, at the inception of the contract, a 65 year old German male (see Forsyth and Vetzal [6] for the Table).

### 6.1 Computing the fair price

In this Section we compute the fair price of a GMWB policy according to the PH’s subjective perspective, that is we compute the fee rate $\varphi$ that equates the premium $P$ and the value of the policy at inception $V(0, r_0, P, P, P)$. To this aim, we employ the secant method to approach the fair value of $\varphi$, computing at each step the value of the contract by employing the algorithm described in 5.1. Therefore, the main goal at each secant step is to compute the value $V(0, r_0, P, P, P)$ assuming a given value for $\varphi$. Results are reported in Table 4.

| Description           | Parameter | Value |
|-----------------------|-----------|-------|
| Time step per year    | $N_T$     | 100   |
| Points in $\mathcal{G}_X$ | $N_X$ | 1002  |
| Points in $\mathcal{G}_G$ | $N_G$ | 41    |
| Points in $\mathcal{G}_H$ | $N_H$ | 41    |
| Smallest withdrawal   | $\bar{w}$ | 2.5   |

Table 3: Parameter choices for the numerical methods.
Table 4: Fair fee rate \( \varphi \) (in basis points) according to PH’s subjective valuation, changing the values of \( r_0 \), \( \sigma \) and \( \tau \).

| \( \sigma \) | \( r_0 = 0.00 \) | \( r_0 = 0.03 \) | \( r_0 = 0.05 \) |
|-----------|----------------|----------------|----------------|
| 0.1       | 706.80         | 1491.63        | 98.55          |
| 0.3       | 383.34         | 40.73          | -11.79         |
| \( \tau \) = 0\% | 152.61         | -77.56         | -5.16          |
| \( \tau \) = 20\% | 1203.45        | 173.43         | -77.56         |

6.2 Comparing the PH withdrawals with and without taxation

We compare the withdrawals performed by the PH considering or not taxation. In particular, we consider a GMWB policy given by the parameters in Tables 1, 2 and 3, considering in particular the case with \( r_0 = 0.03 \), \( \sigma = 0.3 \). Moreover, the value of \( \varphi \) is set as the breakeven fee with taxes, that is \( \varphi = 152.61 \) basis points. Optimal withdrawals \( w_i \) at different anniversaries are reported in Figures 6.1, 6.2 and 6.3. The first column represents \( w_i \) as a function of \( X_{t_i}^- \) and \( r_{t_i} \) considering different values for \( G_{t_i} \) and \( H_{t_i} \) and a zero tax rate, while in the second column considering a positive tax rate (specifically \( \tau = 20\% \)). The area in which the color is darker identifies higher withdrawals. Finally, in the third column, the difference between \( w_i \) without taxation and \( w_i \) with taxation. Here, green areas indicate that withdrawals without tax are higher, while red areas indicate that withdrawals with tax are higher.

We can observe that \( w_i \) depends on all the considered parameters. In general, it increases with both \( X_{t_i}^- \) and \( r_{t_i} \) and decreases with \( G_{t_i} \), but these trends are not always verified. The relation of \( w_i \) with \( H_{t_i} \) and \( \tau \) is more complicated. In particular, we observe that if \( X_{t_i}^- \) is low and \( r_{t_i} \) is high then the PH withdraws more when taxation is applied. Conversely, if \( X_{t_i}^- \) is high and \( r_{t_i} \) is low then the PH withdraws more when no taxation is applied.
Figure 6.1: In the first two columns, the contour plot with respect to $X_2$ (x-axis) and $r_2$ (y-axis) of the optimal withdrawal $w_2$ at time $t_i = 2$ without ($\tau = 0$, first column) and including taxation ($\tau = 0.2$, second column). In the third column the difference between the optimal withdrawal without and with taxation reported in the first two columns.
Figure 6.2: Same contour plots as in 6.1 but considering $t_i = 5$.

Figure 6.3: Same contour plots as in 6.1 but considering $t_i = 8$. 

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6.3 Comparing the PH and the insurer’s subjective prices

In order to assess the impact of taxation on the cost of a GMWB policy, we compute the fair price of a contract according to the insurer’s subjective valuation, that is we compute the fee rate $\varphi$ that equates the premium $P$ and the value of the policy at inception $\mathcal{U}(0, r_0, P, P, P)$. To this aim, we employ the secant method to approach the fair value of $\varphi$, computing at each step the value of the contract by employing the algorithm described in 5.1. Results are reported in Table 5.

We observe that if $\tau = 0$ we obtain the same result as in Table (4): this is correct since there are no taxes and thus the valuation of the PH and of the insurer are the same. Moreover, we observe that if $\tau = 20\%$ the fair value of $\varphi$ is smaller (up to 40 basis points with reference to Table 5) since the PH withdrawal strategy differs from the Insurer optimal one. This phenomenon is more evident when the average interest rate is high and the volatility is low. Thus, this reduces the cost of the policy, leading to prices which are closer to the real market ones.

7 Conclusions

In this paper we have investigate the impact of taxation on GMWB valuation when a stochastic interest rate is considered. We modeled taxation following Moenig and Bauer [13] and Moenig and Zhu [14] and we modeled stochastic interest rate through the Black-Scholes Hull-White model. This analysis combines the effects of taxation and of the variable interest rate which, as already shown separately in other research work, can have a significant impact on the withdrawal choices and thus on the hedging costs.

This analysis has been possible thanks to the use of efficient numerical techniques, in particular the Hybrid PDE method (Briani et al. [2]). The numerical results show that the customer’s choices regarding the withdrawals to be made are significantly influenced by the financial parameters and in particular by the value of the interest rate. In particular, we observed that the PH prefers to withdraw larger amounts when the interest rates and the account value are high. From our analysis, the impact of taxation on withdrawals appears to be less clear-cut and it is difficult to draw absolute conclusions whether it discourages withdrawals or not. Beyond this, the principal effect of taxation is reducing the cost of policies: the strategy followed by the PH in making the withdrawals is sub-optimal for the insurer. This is useful to match theoretical prices to those actually observed on the real market.

| $\sigma$ | $r_0 = 0.00$ | $r_0 = 0.03$ | $r_0 = 0.05$ |
|---------|-------------|-------------|-------------|
| $\tau = 0\%$ | 706.80 | 1491.63 | 98.55 | 383.34 | 40.73 | 173.43 |
| $\tau = 20\%$ | 705.33 | 1486.28 | 67.46 | 362.83 | 1.16 | 143.67 |

Table 5: Fair fee rate $\varphi$ (in basis points) according to the insurer’s subjective valuation, changing the values of $r_0$, $\sigma$ and $\tau$. 

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