HORIZON SADDLE CONNECTIONS, QUASI-HOPF SURFACES AND VEECH GROUPS OF DILATION SURFACES

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ABSTRACT. Dilation surfaces are generalizations of translation surfaces where the geometric structure is modelled on the complex plane up to affine maps whose linear part is real. They are the geometric framework to study suspensions of affine interval exchange maps. However, though the $SL(2,\mathbb{R})$-action is ergodic in connected components of strata of translation surfaces, there may be mutually disjoint $SL(2,\mathbb{R})$-invariant open subsets in components of dilation surfaces. A first distinction is between triangulable and non-triangulable dilation surfaces. For non-triangulable surfaces, the action of $SL(2,\mathbb{R})$ is somewhat trivial so the study can be focused on the space of triangulable dilation surfaces.

In this paper, we introduce the notion of horizon saddle connections in order to refine the distinction between triangulable and non-triangulable dilation surfaces. We also introduce the family of quasi-Hopf surfaces that can be triangulable but display the same trivial behavior as non-triangulable surfaces. We prove that existence of horizon saddle connections drastically restricts the Veech group a dilation surface can have.

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1. INTRODUCTION

A dilation structure on a surface is a geometric surface modelled on the complex plane with structural group the set of real affine maps $x \mapsto ax + b$ with $a \in \mathbb{R}_+^*$ and $b \in \mathbb{C}$. In most cases, a dilation structure is defined on the complement of a finite set of singularities, just like translation structures, see [9].

Dilation surfaces have many common features with translation surfaces: notions of straight lines, angles and direction are well-defined. Thus, they have directional foliations whose dynamics is essentially that of affine interval exchange maps. These foliations have been studied in [1]. Just like moduli spaces of translation surfaces, moduli spaces of dilation surfaces decompose in strata that are analytic orbifolds on which there is an action of $SL(2,\mathbb{R})$ that encompasses a renormalization process.

In strata of translation surfaces, most interesting dynamic properties depend on the closure of the $SL(2,\mathbb{R})$-orbit. Besides, the orbit of a generic translation surface is dense in its connected component of the stratum. We look for an analogous framework for dilation surfaces.

However, there are no $SL(2,\mathbb{R})$-invariant dense open sets in strata of dilation surfaces. An
idea exposed in [2] is that we should focus on nice subsets of strata. A first criterium is
to consider dilation surfaces that are triangulable. Indeed, they form a $SL(2, \mathbb{R})$-invariant
open set.
In this paper, we generalize this criterium by introducing the notion of *horizon saddle
connections*. If a dilation surface displays such objects, then the action of $SL(2, \mathbb{R})$ on it
simplifies drastically.

**Definition 1.1.** In a dilation surface, a $k$-horizon saddle connection $\gamma$ is a saddle connec-
tion (geodesic segment relating two singularities) such that no trajectory crosses $\gamma$ strictly
more than $k$ times.

In particular, non-triangulable dilation surfaces admit 1-horizon saddle connections, see
Proposition 3.1 for details.

Inclusion of the moduli space $\mathcal{T}$ of triangulable dilation surfaces in the moduli space $\mathcal{D}$
of dilation surfaces can be generalized into an infinite sequence of inclusions of $SL(2, \mathbb{R})$-
invariant open sets. The example of quasi-Hopf surfaces (see Subsection 3.2 for details)
shows that there are triangulable surfaces with some horizon saddle connections.

**Theorem 1.2.** In the moduli space of dilation surfaces $\mathcal{D}$, for any $k \geq 1$ the set $\mathcal{H}_k$
of surfaces without $k$-horizon saddle connections is a $SL(2, \mathbb{R})$-invariant open set. We have
\[ \cdots \subset \mathcal{H}_k \subset \cdots \subset \mathcal{H}_1 \subset \mathcal{T} \subset \mathcal{D} \]

Theorem 1.2 is proved in Section 3.

Horizon saddle connections constrain the action of $SL(2, \mathbb{R})$. In the following, the Veech
group $V(X)$ of a dilation surface $X$ is the stabilizer of the action of $SL(2, \mathbb{R})$. There is
a remarkable similarity with the three types of Veech groups for translation surfaces with
poles: continuous, cyclic parabolic or finite, see [5, 6] for details.

**Theorem 1.3.** In a dilation surface $X$ such that there are at least three distinct directions
of horizon saddle connections, then the Veech group of $X$ is finite.

**Theorem 1.4.** In a dilation surface $X$ such that there are exactly two distinct directions
of horizon saddle connections, then there are two cases:
(i) $V(X)$ is finite (of order 1, 2 or 4);
(ii) the Veech group of $X$ is conjugated to $\left\{ \begin{pmatrix} a^k & 0 \\ 0 & a^{-k} \end{pmatrix} \mid k \in \mathbb{Z} \right\}$ with $a \in \mathbb{R}_+^*$ or its product
with the subgroup generated by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

In the latter case, the surface is a rational quasi-Hopf surface.

**Theorem 1.5.** In a dilation surface $X$ such that there is exactly one direction of horizon
saddle connections, then there are three cases:
(i) Hopf surfaces: the Veech group of $X$ is conjugated to $\left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{R}^*, b \in \mathbb{R} \right\}$;
(ii) the Veech group of $X$ is cyclic parabolic: conjugated to $\left\{ \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \mid k \in \mathbb{Z} \right\}$ or its product
with $\{ \pm Id \}$.
(iii) the Veech group of $X$ is trivial or $\{ \pm Id \}$.

Theorems 1.3 to 1.5 are proved in Section 4.

The structure of the paper is the following:
- In Section 2, we recall the background about dilation surfaces: moduli space, linear
holonomy, triangulations.
- In Section 3, we introduce the notion of horizon saddle connections and prove the finiteness lemma. We also discuss some examples: dilation tori, surfaces with chambers, quasi-Hopf surfaces, dilation surfaces with finitely many saddle connections.
- In Section 4, we present some preliminary results about Veech groups and prove the main theorems about restriction of the Veech group in presence of horizon saddle connections.

2. Dilation surfaces

2.1. Dilation structure. A dilation structure on a Riemann surface \( X \) is a kind of affine structure. We follow the definitions of [2].

**Definition 2.1.** A dilation surface is a Riemann surface \( X \) with a finite set \( \Lambda \subset X \) of singularities and an atlas of charts on \( X \setminus \Lambda \) with values in \( \C \) and such that:

(i) transition maps are of the form \( x \mapsto ax + b \) with \( a \in \R^*_+ \).

(ii) the affine structure extends around every element of \( \Lambda \) to a Euclidian cone structure of angle a multiple of \( 2\pi \).

Every objet of \( \C \) invariant under the real affine group makes sense in dilation surfaces. In particular, for each angle \( \theta \in S^1 \), we can define a directional foliation oriented by \( \theta \). Portions of leaves are the geodesics of the affine structure. **Saddle connections** are geodesics joining two conical singularities.

One difficulty in the study of dilation surfaces is that there is no notion of distance since two segment with different lengths and the same direction are equivalent up to affine maps. However, the ratio of lengths of two saddle connections sometimes makes sense.

Let \( \alpha \) and \( \beta \) be two saddle connections of a dilation surface \( X \) that intersect (possibly at the ends of the segment) each other. Then, we consider a chart covering a disk \( D \) (possibly with a slit) centered on the intersection point. In \( D \), the length of the two intersecting branches is well-defined. The portion these branches represent in their saddle connection is also well-defined. Therefore, the **local length ratio** of \( \alpha \) and \( \beta \) is well-defined. It only depends on the choice of the intersection point and possibly the order in the couple \( (\alpha, \beta) \) if the intersection point is a singularity with a nontrivial dilation ratio. This construction will help in some crucial results about Veech groups.

2.2. Linear holonomy. In a dilation surface \( X \), for every path \( \gamma \), we can cover \( \gamma \) with charts of the atlas. The transition map between the first chart and the last chart is an affine map. Its linear part is well-defined up to conjugacy. This number is obviously a topological invariant. Therefore, we have a group morphism \( \rho : H_1(X \setminus \Lambda, \Z) \rightarrow \R^*_+ \) we denote by **linear holonomy** of paths.

The local geometry of a singularity is characterized by two topological numbers associated to a simple loop \( \gamma \) around it:

(i) the linear holonomy \( \rho(\gamma) \);

(ii) the topological index \( i(\gamma) \).

The neighborhood of such a singularity is constructed starting from an infinite cone of angle \( i(\gamma)2\pi \) and a ray starting from the origin of the cone. Then, we identify the right part of the ray with the left part of the ray with a homothety ratio of \( \rho(\gamma) \).

The conical singularities satisfy a Gauss-Bonnet formula. It is remarkable that the dilation ratio appears as something like an "imaginary curvature". The following formula has been proved as Proposition 1 in [3].

**Proposition 2.2.** In a surface of dilation of genus \( g \geq 1 \) with conical singularities \( s_1, \ldots, s_n \) of angle \( 2k_i\pi \) and dilation ratio \( \lambda_i \), we have:
\(i\) \(\sum_{i=1}^{n} (k_i - 1) = 2g - 2;\)

\(ii\) \(\sum_{i=1}^{n} \ln(\lambda_i) = 0.\)

In particular, there is no dilation surface in genus zero. It would be necessary to introduce a notion of pole among the singularities of the affine structure.

**2.3. Moduli space and strata.** For any \(g, n \geq 1\), we can define the moduli space \(D_{g,n}\) of dilation surfaces of genus \(g\) with \(n\) singularities up to diffeomorphisms. This space is an analytic orbifold of real dimension \(6(g-1) + 3n\), see [7] for details.

For any sequence of integers \(a = (a_1, \ldots, a_n)\) of integers such that \(\sum_{i=1}^{n} a_i = 2g - 2\) and any sequence of positive real numbers \(\lambda = (\lambda_1, \ldots, \lambda_n)\) such that \(\prod_{i=1}^{n} \lambda_i = 1\), there is a (possibly empty) stratum \(D_{g,n}(a, \lambda)\) of \(D_{g,n}\). These strata are analytic orbifolds of real dimension \(6(g-1) + 2n + 1\).

\(SL(2, \mathbb{R})\) acts on the moduli space of dilation surface by composition with the coordinates maps with values in \(\mathbb{C}\). The action preserves the linear holonomy \(\rho\) so it also preserves strata. The Veech group \(V(X)\) of a dilation surface \(X\) is the stabilizer of this group action. It is a subgroup of \(SL(2, \mathbb{R})\).

**2.4. Cylinders.** For any closed geodesic, the first return of a small segment orthogonal to the geodesic is a map of the form \(x \mapsto \lambda x\) with \(\lambda \in \mathbb{R}^*_+\). We say that the closed geodesic is flat if \(\lambda = 1\). Otherwise, it is hyperbolic.

Flat closed geodesics belong to families that describe flat cylinders (just like in translation surfaces). Hyperbolic closed geodesics also describe cylinders. We refer to them as affine cylinders. They are obtained from an angular portion of an annulus (or a cyclic cover of an annulus) by gluing the two sides on each other. The isomorphism class of an affine cylinder is completely determined by two numbers:

(i) The affine factor \(\lambda\) (dilation ratio along hyperbolic closed geodesics).

(ii) The angle \(\theta\) (determined by the angular portion of the annulus considered).

In translation surfaces, the horocyclic flow (action of unipotent elements of \(SL(2, \mathbb{R})\) preserving the direction of the closed geodesics of the cylinder) modifies the twist of a flat cylinder. There is a similar phenomenon for affine cylinder.

We consider an affine cylinder of dilation factor \(\lambda\) whose boundary is formed by a saddle connection with a marked point in direction \(\alpha\) and another saddle connection with a marked point in direction \(\beta\). Then we consider the Teichmüller flow \(A_t\) that contracts direction \(\alpha\) with a factor \(e^{-t}\) and expands direction \(\beta\) with a factor \(e^t\). The Teichmüller flow on this surface is periodic and the affine cylinder is invariant by any element \(A_t\) of the flow such that \(t = \frac{\ln(\lambda)}{2}\). This will be useful in Subection 3.3 when we will study quasi-Hopf surfaces.

**2.5. Triangulable dilation surfaces.** In the framework of dilation surfaces, a (geometric) triangulation is a topological triangulation where the edges are saddle connections and where every conical singularity is a vertex. Having a triangulation of dilation surface provides a nice parametrization of its neighborhood in the moduli space (by deforming the triangles). However, not every dilation surface admits a triangulation. Indeed, affine cylinders of angle at least \(\pi\) are not triangulable.

A geodesic trajectory entering in an affine cylinder cannot intersect a hyperbolic geodesic that shares the same direction. Therefore, depending on its direction, it will leave the cylinder or accumulates on the first hyperbolic geodesic that shares the same direction. In the angle of the affine cylinder is at least \(\pi\), then every entering trajectory accumulates...
on some hyperbolic geodesic. In particular, there is no saddle connection joining the two boundaries of the cylinder. Thus, there is no triangulation of the affine cylinder.

Veech proved in some unpublished course notes the converse result (see [8]).

**Theorem 2.3 (Veech).** A dilation surface $X$ admits a geometric triangulation if and only if there is no affine cylinders of angle at least $\pi$ in $X$.

Even if the surface is not triangulable, a maximal system of non-intersecting saddle connections cuts out the surface into a union of affine cylinders of angle at least $\pi$ and a triangulable locus. It will be used as a substitute of triangulation.

2.6. **Invariant $\theta$.** Invariant $\theta$ of a dilation surface was introduced in [2] as a first step to study degeneracy of dilation surfaces. It is informally connected with our problem on finding subclasses of dilation surfaces whose behavior is close to that of translation surfaces. The following result has been proved in [3] as Proposition 5 with a slightly different proof.

**Proposition 2.4.** In a dilation surface $X$, for any real number $a > 0$ there are finitely many affine cylinders of angle at least $a$. In particular, the upper bound $\theta(X)$ among cylinders of $X$ is realized by some affine cylinder.

**Proof.** Affine cylinders of angle at least $\pi$ are disjoint from each other because for a point in such a cylinder, there is only one periodic direction. In the other directions, the trajectory accumulates on some hyperbolic geodesic. Therefore, there are finitely many affine cylinders of angle at least $\pi$.

For cylinders of angle smaller than $\pi$, the reasoning is the following. In any direction, there are at most $g + 1$ homotopically distinct closed hyperbolic geodesics. Since the angle is small, there is only one closed geodesic in its homotopy class for a given direction. Consequently, the sum of the angles of these cylinders is bounded by $(g + 1)\pi$. □

What could be the most important current conjecture about dilation has been formulated in [2]. Does every strictly affine surface (in other words, a dilation surface that is not a translation surface) have an affine cylinder? Does $\theta = 0$ characterizes translation surfaces? Otherwise, we would get an interesting $SL(2, \mathbb{R})$-invariant closed set.

Since every dilation surface of genus one admits a decomposition into cylinders, the conjecture is trivially true. The following proposition settles the case of dilation surfaces of $D_{2,1}$.

**Proposition 2.5.** Any dilation surface of genus two with one singularity such that $\theta = 0$ is a translation surface.

**Proof.** Since $\theta = 0$, there is no affine cylinder at all so the dilation surface is triangulable. Following Theorem 2 of [2], any dilation surface of $D_{2,1}$ contains a cylinder. The sketch of the proof is the following. Every saddle connection is closed and separates the conical singularity into two angular sectors of $(\pi, 5\pi)$ or $(3\pi, 3\pi)$. If we have $(\pi, 5\pi)$, then there is a cylinder bounded the angle $\pi$. Each of the 9 saddle connections of the triangulation defines the same angular partition $(3\pi, 3\pi)$. A clever topological argument proves that it is impossible. The proof is really specific to $D_{2,1}$.

We denote by $A$ the (flat) cylinder and consider a geodesic triangulation such that $A$ is a union of triangles of the triangulation (formed by 6 triangles because of Gauss-Bonnet formula).

If $A$ is formed by 6 triangles, then each of its boundary components is formed by three segments identified with three other segments on the other boundary components. This defines three cylinders transverse to $A$. These cylinders are also flat. The surface is clearly a translation surface.
There is no dilation surface such that $A$ is formed by 5 triangles because every side of the last triangle would be in the same direction.

If $A$ is formed by 4 triangles, the last 2 triangles automatically form a flat cylinder $B$. Then, there are the boundary components of $A$ are formed by two segments. One is related to $B$ and the other is identified by a segment of the other boundary component of $A$. These identified sides define a transverse cylinder of $A$ which should also be flat. The surface is thus a translation surface. If $A$ is formed by 3 triangles, then each of the boundary saddle connections of $A$ is connected with a different triangle among the last three. These three triangles should be cyclically connected. They would form another flat cylinder. Such a gluing does not define a surface of $D_{2,1}$.

If $A$ is formed by 2 triangles, then the last 4 triangles cannot have their three sides connected with different triangles. Thus, there is a pair of them that form a cylinder $B$ disjoint from $A$. The surface is formed by a pair of triangles forming a quadrilateral and two flat cylinders $A$ and $B$. The quadrilateral has two pairs of parallel sides so it is a parallelogram. Such a surface is clearly a translation surface. □

3. Horizon saddle connections

A horizon saddle connection is a segment such that for some number $k$, there is no geodesic trajectory crossing it strictly more than $k$ times. In particular, if $k < k'$, any $k$-horizon saddle connection is a $k'$-saddle connection. There is no such horizon saddle connections in translation surfaces. Indeed, the foliation in the generic direction is minimal so every saddle connection is crossed infinitely many times by some trajectory. Therefore, absence of horizon saddle connections indicates that we are not too far from the case of translation surfaces.

The following proposition is the starting point of the generalization of our triangulability condition.

**Proposition 3.1.** The saddle connections that belong to the boundary of an affine cylinder of angle at least $\pi$ are 1-horizon saddle connections.

**Proof.** A trajectory that enters in such an affine cylinder cannot leave it (they accumulate on an hyperbolic geodesic of the cylinder, see Subsection 2.1 for details). Therefore, no saddle connection of the boundary can be crossed two times. □

The key property of horizon saddle connections is that for any $k$, there are finitely many $k$-horizon saddle connections. This property drastically rigidify the action of $SL(2, \mathbb{R})$. In a similar way, in translation surfaces with poles, most the geometry of a surface is encompassed in a polygon called the core of the surface. The fact that there are finitely many saddle connections in the boundary of the core rigidify the action of $SL(2, \mathbb{R})$ on strata of such surfaces, see [6] for details.

**Lemma 3.2.** In a dilation surface, for any $k \geq 1$, there is at most a finite number of $k$-horizon saddle connections.

**Proof.** We choose a maximal geodesic arc system of $X$ (that is a decomposition into disjoint triangles and affine cylinders of angle at least $\pi$). For any $k$, there is a finite number of topological paths that crosses each edge of the system at most $k$ times. There is a unique geodesic representative in every homotopy class and it minimizes the geometric intersection number. Therefore, there cannot be infinitely many distinct $k$-horizon saddle connections. □

**Remark 3.3.** We actually do not know if there is dilation surface with $k$-horizon saddle connection for arbitrary high $k$. We expect that there is a topological bound on the maximal number of the maximal crossing number of a horizon saddle connection.
Just like triangulable dilation surfaces define a \( SL(2, \mathbb{R}) \)-invariant open set, absence of \( k \)-horizon saddle connections defines for every \( k \) an invariant open set.

**Proof of Theorem 1.2.** Let \( X \) be a surface without \( k \)-horizon saddle connections. We choose a geometric triangulation of \( X \) and a neighborhood of \( X \) in the moduli space such that the same geodesic triangulation holds in the neighborhood. For any topological arc that crosses the edges more than \( k \) times (there is a finite number of such arcs), its geodesic representative in \( X \) is formed by several saddle connections each of which is crossed by trajectories with an arbitrary high number of intersections. Such a trajectory still exists in a neighborhood of \( X \). Thus, for each arc, there is neighborhood where this arc cannot be a horizon saddle connection (at least a compact part of the trajectory that crosses more than \( k \) times persists). Since there is a finite number of such arcs, there is a neighborhood of \( X \) where there is no horizon saddle connections. \( \square \)

3.1. **Genus one.** Even in the simplified situation of dilation surfaces of genus one, triangulated surfaces and surfaces free from horizon saddle connections define distinct open subsets of the strata.

In [4], Ghazouani proved that any dilation torus with \( n \) singularities can be decomposed into at most \( n \) flat and affine cylinders (see Proposition 9). The sketch of the proof is the following: Gauss-Bonnet formula implies that every singularity has a conical angle of \( 2\pi \). A easy argument about homeomorphism of the circle for a transverse curve implies existence of a cylinder. This cylinder is bounded by saddle connections joining marked points. Since they have a conical angle of \( 2\pi \), the other side of the saddle connections also belongs to a cylinder.

A dilation surface belongs to the locus of triangulated surfaces \( TD_{1,n} \) in \( D_{1,n} \) if and only if every affine cylinder of the decomposition has an angle strictly smaller than \( \pi \).

In the following proposition, we introduce a significantly stronger criterium.

**Proposition 3.4.** For a dilation surface \( X \) of genus one that admits a decomposition into \( c \) cylinders. Let \( A_1, \ldots, A_c \subset S^1 \) be the closures of the sets of directions of geodesics of the cylinders of the decomposition. Exactly one of the following two proposition holds:

1. \( \bigcup_{i=1}^c A_i = S^1 \) and every boundary saddle connection of the decomposition is a 1-horizon saddle connection. There are no other horizon saddle connections in the surface.
2. \( \bigcup_{i=1}^c A_i \neq S^1 \) and every saddle connection is crossed infinitely many times by some trajectory.

**Proof.** In the first case, there is a closed geodesic in any direction. Therefore, for any trajectory \( \alpha \), there will be a closed geodesic \( \gamma \) (or equivalently a chain of saddle connections separating two cylinders of the decomposition) in the same direction \( \alpha \) cannot cross. For any saddle connection \( \beta \) that is the boundary of two cylinders, it is clear that if \( \alpha \) crosses \( \beta \) two times, then it should cross \( \gamma \) at least one time. Conversely, any saddle connection that do not separate cylinders is crossed infinitely many times by the closed geodesics of the cylinder.

In the second case, we consider a direction \( \theta \) in the complement of \( \bigcup_{i=1}^c A_i \) that is not the direction of some saddle connection either (since the complement is any open set, it is always realizable). In the directional foliation on a torus, a leaf can be closed, accumulate on a closed geodesic or be minimal in some domain bounded by saddle connections. In
the latter case, the minimal leaf is dense in the whole surface and crosses every saddle connections infinitely many times. If there is a closed geodesic in direction \( \theta \), then it defines a cylinder whose closed geodesics intersect every cylinder of the first decomposition. This cylinder can be completed to form another decomposition into cylinders. Closed geodesics of a given decomposition represent the same free homotopy class. The intersection number of the loops of the two decompositions is not trivial. Therefore, every saddle connection in the boundary of a cylinder of the first decomposition is crossed infinitely many times by some trajectory.

In any stratum \( D_{1,n}(\lambda) \), we define \( H_{1,n}(\lambda) \) to be the \( SL(2, \mathbb{R}) \)-invariant open set (see Theorem 1.2) formed by surface without horizon saddle connections. We have the following strict inclusions of invariant open sets:

\[
H_{1,n}(\lambda) \subsetneq DT_{1,n}(\lambda) \subsetneq D_{1,n}(\lambda)
\]

We proved a dichotomy among dilation tori between those with horizon saddle connections (where one cylinder decomposition covers every direction) and horizon-free dilation tori. In the first case, there is only one cylinder decomposition because any other decomposition would contain a closed geodesic that would cross every cylinder of the first decomposition. Horizon saddle connection make this situation impossible. Therefore, the shape of the cylinders of the unique cylinder decomposition provides global coordinates for this locus in the moduli space. \( SL(2, \mathbb{R}) \) acts separately on each cylinder of the decomposition. We should not expect any interesting mixing behaviour.

3.2. Surfaces with chambers. A natural question in the study of moduli spaces of geometric structures is about the connected components. In [2], the authors study the space \( DT_{2,1} \) of triangulable dilation surfaces of genus two with only one conical singularity. This space fail to be connected and there is an exceptional connected component formed by surfaces split into two chambers separated by a closed saddle connection. This cannot happen in the framework of translation surfaces and we can understand this situation using horizon saddle connections.

A chamber is a dilation surface with boundary formed by a pentagon with two pairs of parallel sides glued on each other. The remaining side is the boundary of the chamber. Gluing the boundaries of two chambers provides a dilation surface of genus two with one singularity.

Clearly, the boundary of a chamber is a 1-horizon saddle connection. Since such a chamber cuts out the surface into two connected components, a trajectory that crosses it should cross it again in the reverse way.

Considering only surfaces without horizon saddle connections, we eliminate the exceptional component formed by surfaces with two chambers. Thus, we could expect that strata of dilation surfaces without horizon saddle connections have exactly the same connected components as translation surfaces.

3.3. Hopf and quasi-Hopf surfaces. Hopf surfaces are examples of non-triangulable dilation surfaces. They are the only case of surfaces of genus at least two where the Veech group is not discrete (see Section 4 for details). They are also the only surfaces whose saddle connections all belong to the same direction.

**Definition 3.5.** A Hopf surface is a dilation surface formed by affine cylinders of angle \( k\pi \) where \( k \) is an integer number and such that the saddle connections of the boundary of every affine cylinder lie in the same directions. In particular, every saddle connection of a Hopf surface is a 1-horizon saddle connection.
We introduce a mild generalization of Hopf surfaces.

**Definition 3.6.** A dilation surface $X$ is quasi-Hopf if there is a pair of directions $\alpha$ and $\beta$ such that $X$ is formed by affine cylinders whose saddle connections of the boundary belong to direction $\alpha$ or $\beta$. We distinguish integer affine cylinders (whose angle is an integer multiple of $\pi$) from non-integer affine cylinders.

Some quasi-Hopf surfaces can be conjugated (using the action of $SL(2, \mathbb{R})$) to surfaces whose affine cylinders have an angle of $\frac{\pi}{2}$. They are examples of triangulable surfaces that admits nevertheless horizon saddle connections.

**Proposition 3.7.** In a quasi-Hopf surface, if the union of the directions of the closed geodesics of any two consecutive cylinders in the decomposition is the whole circle of directions (except perhaps $\alpha$ and $\beta$), then every boundary saddle connection of a cylinder in the decomposition is a $1$-horizon saddle connection. There is no other horizon saddle connection.

**Proof.** A horizon saddle connection cannot cross a cylinder because it would be crossed infinitely many times by a closed geodesic. Therefore, they belong to the boundary of the cylinders of the decomposition. Each of them is a $1$-horizon saddle connection because it is the boundary of two cylinders that together admits a closed geodesic in every direction (excepted $\alpha$ and $\beta$). Therefore, any trajectory crossing such a saddle connection remains forever in one of the two cylinders. \qed

The action of the Teichmüller flow (hyperbolic elements of $SL(2, \mathbb{R})$) contracting direction $\alpha$ and expanding direction $\beta$ on a quasi-Hopf surface is interesting. It crucially depends on the commensurability of the dilation ratios of affine cylinders.

**Proposition 3.8.** We consider a quasi-Hopf surface $X$ formed by affine cylinders whose boundary saddle connections belong to exactly two direction $\alpha$ and $\beta$. The Teichmüller orbit of $X$ in the stratum is closed if and only the dilation ratios of non-integer affine cylinders are log-commensurable.

**Proof.** For any quasi-Hopf surface, we consider the set $C$ of non-integer affine cylinders (we also choose an orientation on them in such a way that it goes from direction $\alpha$ to direction $\beta$). For every cylinder $i \in C$, $\lambda_i$ is the linear holonomy along the hyperbolic geodesics of the cylinder. We consider the Teichmüller flow $A^t$ that contracts direction $\alpha$ with a factor $e^{-t}$ and expands direction $\beta$ with a factor $e^t$.

The dilation action preserves any affine cylinder of $C$ but modifies their twist, see Subsection 2.4. Every non-integer affine cylinder of the decomposition is preserved (with its twist) by any element $A_t$ of the flow such that $t = \frac{\ln(\lambda)}{2}$. Besides, there could be additional symmetries such that the subgroup that preserves the cylinder and its twist is generated by $t = \theta_i = \frac{\ln(\lambda)}{2d_i}$ where $d_i$ is an integer. These exponents $(\theta_1, \ldots, \theta_c)$ define a characteristic ratio. The action of $A^t$ on a cylinder $i$ only depends on the class of $t$ in $\mathbb{R}/\theta_i \mathbb{Z}$. Equivalently, we could consider the image of $t$ in $\mathbb{R}/\theta_i \mathbb{Z}$ by the map $t \mapsto (\theta_1 t, \ldots, \theta_c t)$. Clearly, the Teichmüller orbit is closed if and only if exponents $(\theta_i)_{i \in C}$ are commensurable. In other words, exponents $(\ln(\lambda_i))_{i \in C}$ should be commensurable. \qed

The latter condition defines the subclass of rational quasi-Hopf surfaces.

### 3.4. Dilation surfaces with finitely many saddle connections.

An open question raised in [1] is about characterization of dilation surfaces with finitely many saddle connections. A related question asks if for any dilation surfaces, every point belongs to a saddle connection or a closed geodesic. Surfaces with finitely many saddle connections provides easy examples of this phenomenon. It also exemplifies 2-horizon saddle connections, see
Figure 1. A dilation surface where sides $h, i, j, k, l, m, n, o$ are 1-horizon saddle connections (boundaries of affine cylinders of angle at least $\pi$) whereas saddle connection $f$ is 2-horizon. Trajectory $s$ is an example of trajectory cutting twice $f$.

Figure 1. We can generalize this example to get $k$-horizon saddle connections for an arbitrary number $k$.

We provide here a characterization of dilation surfaces with finitely many saddle connections in terms of types of trajectories.

**Theorem 3.9.** For a dilation surface $X$, the following propositions are equivalent:
(i) There are finitely many saddle connections in $X$.
(ii) Every trajectory either is critical or accumulates on a closed geodesic of an affine cylinder of angle at least $\pi$.

We first need a preliminary result.

**Lemma 3.10.** For a dilation surface $X$, if every critical trajectory in a direction $\theta$ is either a saddle connection or enters in an affine cylinder of angle at least $\pi$, then by cutting out saddle connections in direction $\theta$ we get the following invariant components of the foliation:
(i) Flat cylinders (described by closed geodesics);
(ii) Free components formed by trajectories going from an affine cylinder of angle at least $\pi$ to another.

**Proof.** We draw every saddle connection that belongs to direction $\theta$ and complete them to get a geodesic triangulation of the triangulable locus of the dilation surface. There are finitely many critical trajectories in direction $\theta$ and since they eventually leave the triangulable locus (see Theorem 2.3), their intersection with the triangulated locus is compact and they cut out the triangles into finitely many pieces (trapezoids and triangles). We denote by transverse sides the sides of the pieces that do not belong to direction $\theta$. A transverse side of a piece separates exactly two pieces (because otherwise the piece would be also cut out). Therefore, every regular trajectory in direction $\theta$ that passes through a given piece follows the same discrete path among the pieces. Consequently, such a trajectory is either periodic or eventually leaves the triangulable locus by entering into an affine cylinder of angle at least $\pi$. □

Now we are able to prove the main theorem of the subsection.

**Proof of Theorem 3.9.** Proposition (ii) implies in particular that every trajectory starting from a conical singularity either is a saddle connection or accumulates on a closed geodesic of an affine cylinder. If moreover there are infinitely many saddle connections in the surface, then there is an accumulation point $\theta$ in the set of directions of saddle connections in the surface. Proposition (ii) implies that the decomposition of Lemma 3.10 is only formed by free components.

For a given critical trajectory in direction $\theta$, there are two cases. In the first case, it
eventually enters into an affine cylinder of angle at least $\pi$ and this is the same for any cone of directions close enough to $\theta$. In the second case, the critical trajectory is a saddle connection that separates two free components in the decomposition of Lemma 3.10. For any cone of directions close enough to $\theta$, trajectories ends in the cylinder corresponding to the free component they belong to. Therefore, $\theta$ is not an accumulation point of the set of directions of saddle connections. Proposition (i) thus holds.

Then, we prove that Proposition (i) implies Proposition (ii). We first prove that Proposition (i) implies that every trajectory starting from a conical singularity either is a saddle connection or accumulates on a closed geodesic of an affine cylinder. We draw every saddle connection. Since there is a finite number of them, they cut out angular sectors of conical singularities into irreducible sectors. Critical trajectories of an irreducible sector describe a pencil that does not encounter any other conical singularity. Irreducible sectors may belong to the triangulable locus and then their angular magnitude is strictly smaller than $\pi$. Otherwise, they bound a closed geodesic of an affine cylinder and have an angular magnitude equal to $\pi$.

There are incidence relations between irreducible sectors. Indeed, if we follow a saddle connection bounding an irreducible sector $A$, its end is a conical singularity and rounding this conical singularity with an angle of $\pi$ there are two cases. In the first case we get another saddle connection (in the same direction) and we iterate the construction. In the second case we get a critical trajectory that belongs to an irreducible sector $B$ whose angular magnitude is strictly bigger than $A$ (because trajectories of $B$ in the directions covered by $A$ belong to the pencil of $A$). There are finitely many saddle connections and irreducible sectors so we eventually get a cycle of saddle connections that bound a cylinder. Since there are finitely many saddle connections, this cylinder is affine of angle at least $\pi$. Consequently, every critical trajectory of an irreducible sector eventually ends in the cylinder drawn by the process presented above. Lemma 3.10 then implies that in any direction, regular trajectories either are closed geodesics or accumulate in an affine cylinder of angle at least $\pi$. The first case cannot happen because it would implies existence of a cylinder inside the triangulable locus and we would be able to draw infinitely many saddle connections inside it. Thus, Proposition (ii) holds.

□

4. Veech groups

4.1. General results. The first structure theorem for Veech groups of dilation surfaces has been proved in [2].

**Theorem 4.1** (Dichotomy theorem). Let $X$ be a dilation surface of genus $g \geq 2$, then there are two possible cases:

(i) $X$ is a Hopf surface and $V(X)$ is conjugated to the subgroup of upper triangular elements of $SL(2, \mathbb{R})$;

(ii) $V(X)$ is a discrete subgroup of $SL(2, \mathbb{R})$.

The condition on the genus is necessary because there are some affine tori whose Veech group is $SL(2, \mathbb{R})$. They are constructed as the exponential of some flat tori (see Subsection 4.4 of [2] for details).

Parabolic directions in translations surfaces correspond to cylinder decompositions. There is a similar result for dilation surfaces.

**Proposition 4.2.** Let $X$ be a dilation surface such that $V(X)$ contains a parabolic element. Then the decomposition of $X$ into invariant components of the parabolic direction
is formed by affine and flat cylinders whose boundary saddle connections belong to the parabolic direction. Moduli of the flat cylinders should be commensurable.

Proof. We first follow a part of the proof of Proposition 4 in [2] to prove that in parabolic directions, separatrices are saddle connections. If a separatrix has an accumulation point, then any neighborhood of this point is crossed infinitely many times by the separatrix. Since the parabolic element preserve the direction of the separatrix, it acts as the identity on this neighborhood and thus is the neutral element of the Veech group. Therefore, separatrices have no accumulation points.

Cutting along saddle connections in the parabolic direction provides a decomposition of the dilation surface into invariants components. Since conical singularities are cut out into sectors of angle equal to $\pi$, discrete Gauss-Bonnet implies that the components are topological annuli. These components are thus either flat cylinders or affine cylinders the angle of which are integer multiples of $\pi$. Since the flat cylinders are preserved by the parabolic element, their moduli should be commensurable. □

There is no closed geodesics nor saddle connections in hyperbolic directions of translation surfaces. Such phenomena can appear for some specific dilation surfaces.

Lemma 4.3. Let $X$ be a dilation surface such that $V(X)$ contains a hyperbolic element $\phi$. If a saddle connection $\gamma$ lies in a hyperbolic direction, then $\gamma$ belongs to the common boundary of two cylinders.

Proof. Without loss of generality, we can assume $\gamma$ is vertical and that $\phi$ is expanding in the vertical direction and contracting in the horizontal direction. If $\gamma$ is an edge of some triangle, then $\phi$ provides a sequence of triangles bounding $\gamma$ and such that the directions of the other edges approach the vertical direction. Their local length ratio (see Subsection 2.1 for details) is bounded by below by the ratios of their vertical coordinates. Therefore, either there is a conical singularity inside $\gamma$ (which is impossible) or there is another saddle connection that forms an angle of $\pi$ with $\gamma$. We can iterate with procedure to get a chain of saddle connections in the vertical direction. Since there is a finite number of them, these saddle connections are the boundary of a cylinder. If $\gamma$ is not an edge of a triangle, then it is a boundary saddle connection of an affine cylinder of angle at least $\pi$. □

Proposition 4.4. Let $X$ be a dilation surface such that $V(X)$ contains a hyperbolic element $\phi$. If a saddle connection or a closed geodesic belong to a hyperbolic direction, then $X$ is a quasi-Hopf surface.

Proof. If an affine cylinder is preserved by an hyperbolic element of the Veech group, then its boundary saddle connections belong to invariant directions of the element. Besides, there is no flat cylinder whose closed geodesics belong to an hyperbolic direction. In a given hyperbolic direction, there are at most finitely many saddle connections or closed geodesics. Therefore, at least one power of the hyperbolic element preserves the cylinder they belong to. Consequently, we get invariant cylinders whose boundary is connected with other invariant cylinders until all the surface is covered. □

4.2. Veech groups in presence of horizon saddle connections. There is a lot of examples of dilation surfaces with many directions of horizon saddle connections. Following Proposition 3.4, some dilation tori feature cylinder decompositions such that every boundary saddle connections is horizon. We can also glue chambers (see Subsection 3.2) on the boundary of a polygon. If there are at least three distinct directions of horizon saddle connections, then the Veech group of the surface is finite.

Proof of Theorem 1.3. The set of directions of horizon saddle connections is finite and globally preserved by elements of the Veech group. Therefore, for any element $\phi$ of the Veech group, there is an integer $n$ such that $\phi^n$ preserves every direction of horizon saddle
connections. Since they are at least three, then \( \phi^n \) is the identity. Thus, any element of the Veech group is elliptic. Following Theorem 1 in [2], the Veech group of a dilation surface that is not a Hopf surface is discrete. Therefore, the Veech group of \( X \) is conjugated to a finite rotation group.

In the case of dilation surfaces with exactly two directions of horizon saddle connections, we have to take into account the specific situation of quasi-Hopf surfaces we introduced previously.

**Proof of Theorem 1.4.** A parabolic element cannot globally preserve two distinct directions so the Veech group is discrete (see Theorem 1 of [2]) and contains only elliptic and hyperbolic elements. Elliptic elements globally preserve the two distinct directions so they can only preserve each of them or intertwine them. Therefore, elliptic elements form a finite group conjugated to the trivial group or the finite group of rotations of order two or four. If the Veech group contains a hyperbolic element, then this element preserves the two directions of horizon saddle connections. Every hyperbolic element preserves the same pair of directions. Since the surface is not Hopf (otherwise there would be only one direction of horizon saddle connections), the Veech group is discrete so the hyperbolic elements belong to the same cyclic group. Following Proposition 4.4, the surface is then quasi-Hopf. Proposition 3.8 provides the condition a quasi-Hopf surfaces should satisfy to admit a hyperbolic element of the Veech group that preserves the two directions of the boundary saddle connections of the cylinders.

The case of dilation surfaces with only one direction of horizon saddle connections includes that of Hopf surfaces.

**Proof of Theorem 1.5.** The elliptic elements preserve the direction of horizon saddle connections so they belong to the group \( \{ \pm \text{Id} \} \). Every parabolic element preserves this direction. Therefore, either the Veech group is not discrete (the surface is then a Hopf surface and its described by Theorem 4.1) or the parabolic elements all belong to the same cyclic parabolic group. If there is an hyperbolic element in the Veech group of the surface, then it preserves the horizon saddle connections. These saddle connections thus are boundaries of cylinders whose boundary saddle connections belong to one of the two hyperbolic directions. The surface is then quasi-Hopf. If there is only one direction of horizon saddle connections, then the surface is Hopf.

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