THE FOURIER RESTRICTION NORM METHOD FOR THE ZAKHAROV-KUZNETSOV EQUATION

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Abstract. The Cauchy problem for the Zakharov-Kuznetsov equation is shown to be locally well-posed in $H^s(\mathbb{R}^2)$ for all $s > \frac{1}{2}$ by using the Fourier restriction norm method and bilinear refinements of Strichartz type inequalities.

1. Introduction and main results. We consider the initial value problem

$$
\partial_t u + \partial_x^3 u + \partial_x \partial_y u = \partial_x (u^2)
$$

in $(-T,T) \times \mathbb{R}^d$, $u(0,\cdot) = \phi \in H^s(\mathbb{R}^d)$

for the Zakharov-Kuznetsov equation, which is a higher dimensional generalization of the Korteweg-de Vries equation. In three dimensions, this equation has been derived by Zakharov and Kuznetsov [24, equation (6)] to describe unidirectional wave propagation in a magnetized plasma. A derivation of the two-dimensional equation considered here from the basic hydrodynamic equations was performed by Laedke and Spatschek in [13, Appendix B]. A rigorous justification of equation (1) from the Euler-Poisson system for a uniformly magnetized plasma, valid in both considered space dimensions, was given very recently by Lannes, Linares, and Saut in [14]. Various aspects of the Zakharov-Kuznetsov equation and its generalizations have attracted much attention in recent years. Without attempting to be complete we refer to the papers [23, 4, 2, 19, 5, 15, 17, 22, 16, 21] and references therein. In this paper we will focus on the case $d = 2$.

Regular solutions preserve the $L^2(\mathbb{R}^d)$-norm. Furthermore, there is an underlying Hamiltonian structure and conservation of energy, cp. [18] and references therein.

The three-dimensional version of the Cauchy problem (1) was shown to be locally well-posed in $H^s(\mathbb{R}^3)$ for $s > \frac{3}{8}$ by Linares and Saut in [18], where they used the

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refined energy method of Koch an Tzvetkov, see [12]. This result has been pushed
down to $s > 1$ recently by Ribaud and Vento [20], who proved an essentially sharp
maximal function estimate for the linearized equation and combined this with the
local smoothing effect.

Concerning the Cauchy problem on $\mathbb{R}^2$, global well-posedness in the Sobolev
space $H^s(\mathbb{R}^2)$ has been shown by Faminski˘ı in [4]. Following the argument developed
by Kenig, Ponce, and Vega in [10] he proves the local smoothing effect, a maximal
function estimate as well as a Strichartz type inequality for the linear equation to
obtain local well-posedness by the contraction mapping principle. The global result
is then a consequence of the conservation of energy. Linares and Pastor observed
in [15, Theorem 1.6] that Faminski˘ı’s proof can be optimized to obtain local well-
posedness in the larger data spaces $H^s(\mathbb{R}^2)$ with $1 > s > \frac{3}{4}$. To our knowledge this
is the most advanced result concerning the local problem up to date. We remark
that all of the above mentioned results rely on linear estimates to handle nonlinear
interactions of waves. The purpose of this paper is improve the well-posedness
theory by using genuinely bilinear estimates.

Theorem 1.1. Let $s > \frac{1}{2}$. The initial value problem (1) is locally well-posed in
$H^s(\mathbb{R}^2)$.

The scale invariant Sobolev regularity is $s_c = -1$. We expect that the regularity
threshold can be improved further, but we do not pursue this here.

The paper is organized as follows: We first perform a linear transformation
in the space variables $x$ and $y$, see Subsection 2.1 below. Then, in Subsection
2.2, we introduce the $X^{s,b}$- and restriction spaces adapted to the linear part of
the transformed equation and recall the corresponding Strichartz-type estimates.
Subsection 2.3 is devoted to the proof of several bilinear space time estimates for
free solutions, which play a major role in our analysis. In Section 3 we prove our
main bilinear estimate in Theorem 3.1.

2. Preliminaries. We start by fixing notation. Throughout this paper we denote
the first spatial variable by $x$, its dual Fourier variable by $\xi$, and the second spatial
variable by $y$, and its dual Fourier variable by $\eta$. As usual, $\tau$ is the dual variable of
the time $t$. For $s \in \mathbb{R}$ $J^s_x$ and $I^s_y$ denote the Bessel- and Riesz-potential operators of
order $-s$ with respect to both spatial variables. The corresponding one-dimensional
operators will be called $J^s_x$ and $I^s_y$ respectively. Moreover, we use the
operator $\Lambda^b := F^{-1}(\tau - \xi x - \eta y)F$, where for $a \in \mathbb{R}$ we set $\langle a \rangle := (1 + a^2)^{\frac{1}{2}}$.
Projections onto dyadic intervals in Fourier space receive additional subscripts,
e.g. for $k \in \mathbb{Z}$ we define $P_{x,k} = F^{-1}\chi_{(|\xi| \leq 2^k)}F$, where $\chi$ denotes the (sharp)
characteristic function. $P_{x,\Delta k} = P_{x,k+1} - P_{x,k}$, $P_{x,\geq 1} = Id - P_{x,0}$, and similarly for
the $y$- and $\eta$-variables.

2.1. A linear transformation. We perform a linear change of variables (essentially a rotation) in order to symmetrize the equation. A systematic study of such
transformations in connection with dispersive estimates for cubic phase functions
of two variables can be found in [1]. Let $x' = \mu x + \lambda y$ and $y' = \mu x - \lambda y$ and
$u'(x', y') = u(x, y)$. Then,
$$\begin{align*}
\partial_x u(x, y) &= \mu(\partial_{x'} + \partial_{y'})v(x', y') \\
\partial_y u(x, y) &= \lambda(\partial_{x'} - \partial_{y'})v(x', y')
\end{align*}$$
which implies
\[(\partial_y^3 + \partial_x \partial_y^2)u(x, y) = \mu^3 (\partial_x + \partial_y)^3 v(x', y') + \mu \lambda^2 (\partial_x + \partial_y)(\partial_x - \partial_y)^2 v(x', y')\]
\[= (\mu^3 + \mu \lambda^2) (\partial_y^3 + \partial_x \partial_y^2) v(x', y') + (3 \mu^3 - \mu \lambda^2) (\partial_x^2 \partial_y + \partial_x \partial_y^2 + \partial_y^3) v(x', y').\]

Choosing \( \mu = 4^{-\frac{1}{2}} \) and \( \lambda = \sqrt{34} - \frac{1}{4} \) reduces the above to
\[(\partial_y^3 + \partial_x \partial_y^2)u(x, y) = (\partial_x^3 + \partial_y^3) v(x', y')\]
which implies that we may consider the initial value problem
\[
\partial_t v + (\partial_y^3 + \partial_x \partial_y^2) v = 4^{-\frac{1}{2}} (\partial_x + \partial_y) (v^2) \quad \text{in} \ (-T, T) \times \mathbb{R}^2,
\]
\[v(0, \cdot) = \phi \in H^s(\mathbb{R}^2)\] instead of (1) without changing the well-posedness theory. We define the associated unitary group \( U(t) := e^{-t(\partial_y^3 + \partial_x \partial_y^2)} \).

2.2. Function spaces and linear estimates. In analogy with the KdV theory in [3, 11] we use Bourgain’s \( X^{s, b} \) spaces. We refer the reader to the exposition in [7, Section 2] for more details.

**Definition 2.1.** Let \( s, b \in \mathbb{R} \). The space \( X^{s, b} \) is defined as the space of all tempered distributions \( u \) on \( \mathbb{R} \times \mathbb{R}^2 \) such that \( \hat{u} \in L^2_{\loc}(\mathbb{R} \times \mathbb{R}^2) \) and
\[
\|u\|_{s, b} := \|⟨τ - ξ^3 - η^3⟩^b(ξ)^s \hat{u}(τ, ξ, η)\|_{L^2_{τ,ξ,η}} < +\infty.
\]

Furthermore, for \( T > 0 \) we define the restriction space \( X_T^{s, b} \) as the space of all \( u|_{(0,T)\times\mathbb{R}^2} \) for \( u \in X^{s, b} \), with norm
\[
\|u\|_{s, b; T} := \inf\{\|v\|_{s, b} : v \in X^{s, b}, v|_{(0,T)\times\mathbb{R}^2} = u\}.
\]

Finally, we define the set \( X_{loc}^{s, b} \) of all \( u \) satisfying \( u|_{(0,T)\times\mathbb{R}^2} \in X_T^{s, b} \) for all \( T > 0 \).

Let \( \psi \in C^0([-2, 2)) \) be even, \( 0 \leq \psi \leq 1 \) and \( \psi(t) = 1 \) for \( |t| \leq 1 \), and define \( \psi_T(t) := \psi(t/T) \) for \( T > 0 \).

The following result and its proof can be found in [7, Lemma 2.1].

**Lemma 2.2.** Let \( s, b \in \mathbb{R} \). Then,
\[
\|\psi U\phi\|_{s, b} \lesssim \|\phi\|_{H^s}.
\]

Also, for \(-\frac{1}{2} < b' \leq 0 \leq b \leq b' + 1, \ 0 < T \leq 1,\)
\[
\|\psi_T \int_0^t U(t-s)f(s)ds\|_{s, b} \lesssim T^{1-b+b'}\|f\|_{s, b'}.
\]

Next, let us recall two estimates of Strichartz type. [9, Theorem 3.1 ii)] implies the estimate
\[
\|I_{x}^{\frac{3}{2}} I_{y}^{\frac{3}{2}} U\phi\|_{L^p_{t}L^q_{x,y}} \lesssim \|\phi\|_{L^2_{x,y}} \quad \text{if} \quad \frac{2}{p} + \frac{2}{q} = 1, \ p > 2.
\]

Sobolev embeddings imply
\[
\|U\phi\|_{L^p_{t}L^q_{x,y}} \lesssim \|\phi\|_{L^2_{x,y}} \quad \text{if} \quad \frac{3}{p} + \frac{2}{q} = 1, \ p > 3,
\]
which is a special case of estimate (A.11) from [6].
Lemma 2.3. Let $b > \frac{1}{2}$.

\[ \| I_x^2 I_y^2 u \|_{L_t^2 L_x^2} \lesssim \| u \|_{0,b} \quad \text{if } \frac{2}{p} + \frac{2}{q} = 1, \ p > 2, \]  

(9)

and

\[ \| u \|_{L_t^p L_x^2} \lesssim \| u \|_{0,b} \quad \text{if } \frac{3}{p} + \frac{2}{q} = 1, \ p > 3. \]  

(10)

In particular, we obtain

\[ \| u \|_{L_t^4 L_x^2} \lesssim \| u \|_{0,b} \quad \text{if } b > \frac{5}{12} \]  

(11)

by interpolation (10) for $p = 5$ with the trivial bound $\| u \|_{L_t^2 L_x^2} = \| u \|_{0,0}$. A further interpolation with the conservation of the $L^2$-norm gives

\[ \| u \|_{L_t^4 L_x^2} \lesssim \| u \|_{0,b} \quad \text{if } b > \frac{2}{3p} + \frac{1}{q}. \]  

(12)

2.3. Bilinear estimates for free solutions. For a given measurable function $a : \mathbb{R}^4 \to \mathbb{C}$ of at most polynomial growth we define the bilinear operator $A$ with symbol $a$ via

\[ A(f_1, f_2)(\xi, \eta, \nu) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} a(\xi, \xi_2, \eta, \eta_2) \prod_{j=1}^2 \hat{f}_j(\xi_j, \eta_j) d\xi_1 d\eta_1, \]

initially for $f_1, f_2 \in S(\mathbb{R}^2)$. Similar to [8] let $I_{x,-}^s$ be the bilinear operator with symbol $|\xi_1 - \xi_2|^s$, $I_{x,+}^s$ be the bilinear operator with symbol $|\xi_1 + 2\xi_2|^s$, $I_{y,-}^s$ be the bilinear operator with symbol $|\eta_1 - \eta_2|^s$ and $I_{y,+}^s$ be the bilinear operator with symbol $|\eta_1 + 2\eta_2|^s$. Convolution integrals as in the above definition will henceforth be abbreviated by $I_x$, e. g.

\[ \int_\mathbb{R} f(\xi_1)g(\xi_2)d\xi_1 := \int_{\xi_1 + \xi_2 = \xi} f(\xi_1)g(\xi_2)d\xi_1 = \int f(\xi_1)g(\xi_1)d\xi_1 \]

and similarly, if several variables appear.

Proposition 1. Let $b > \frac{1}{2}$. Then,

\[ \| I_x^2 I_y^2 (P_{y,k} u, v) \|_{L_t^2 L_x^2} \lesssim 2^{\frac{1}{2}} \| u \|_{0,b} \| v \|_{0,b} \]  

(13)

\[ \| I_x^2 I_y^2 (u, P_{y,k} v) \|_{L_t^2 L_x^2} \lesssim 2^{\frac{1}{2}} \| u \|_{0,b} \| v \|_{0,b} \]  

(14)

\[ \| P_{y,k} I_x^2 I_y^2 (u, v) \|_{L_t^2 L_x^2} \lesssim 2^{\frac{1}{2}} \| u \|_{0,b} \| v \|_{0,b} \]  

(15)

Proof. By the transfer principle (i.e. the multilinear generalization of [7, Lemma 2.3]) it suffices to prove the estimates for free solutions $u(t) = U(t)u_0$ and $v(t) = U(t)v_0$ with $\| u_0 \|_{L_t^2 L_x^2} = \| v_0 \|_{L_t^2 L_x^2}$. In this case, the Fourier transform of $I_x^2 I_y^2 (u, v)$ in all three variables is given as

\begin{align*}
\mathcal{F} I_x^2 I_y^2 (u, v)(\xi, \eta, \tau) &= c \int_\mathbb{R} |\xi(\xi_1 - \xi_2)|^{-\frac{1}{2}} \delta(\tau - \xi_1^2 - \xi_2^2) \hat{u}_0(\xi_1, \eta_1) \hat{v}_0(\xi_2, \eta_2) d\xi_1 d\eta_1 \\
&= c \int_\mathbb{R} |\xi(\xi_1 - \xi_2)|^{-\frac{1}{2}} (\hat{u}_0(\xi_1^2, \eta_1)^{\hat{v}_0}(\xi_2, \eta_2) + \hat{u}_0(\xi_2^2, \eta_2)^{\hat{v}_0}(\xi_1, \eta_1)) d\eta_1.
\end{align*}
Remark 1. The above proposition has several useful consequences:

Here $\xi^*$ and $\xi^+_3$ are the solutions of $g(\xi^1_1) = 0$, where $g(\xi_1) = \tau - \xi^3_1 - (\xi - \xi^1_1)^3 - \eta^3_1 - \eta^3_2$. Observe that $\xi^* + \xi^+_3 = \xi$, so by symmetry it suffices to consider only the first contribution to the above expression. To see (13) we assume $u = P_{\eta,k}u$ and use Cauchy-Schwarz to obtain the upper bound

$$2^{\frac{k}{2}} \left( \int_\xi |\xi(\xi^*_1 - \xi^*_2)|^{-1} |\hat{u}_0(\xi^*_1, \eta_1)\hat{v}_0(\xi^*_2, \eta_2)|^2 d\eta_1 \right)^{\frac{1}{2}}.$$  

Squaring and integrating with respect to $\tau$ leads to

$$\|F^{\frac{1}{2}} I^\frac{1}{2}_{x,-}(u, v)(\xi, \eta, \cdot)\|^2 L^2_\xi \lesssim 2^k \int_\xi |\xi(\xi^*_1 - \xi^*_2)|^{-1} |\hat{u}_0(\xi^*_1, \eta_1)\hat{v}_0(\xi^*_2, \eta_2)|^2 d\eta_1 d\tau \lesssim 2^k \int_\xi |\hat{u}_0(\xi_1, \eta_1)\hat{v}_0(\xi_2, \eta_2)|^2 d\eta_1 d\xi_1.$$

Now integration with respect to $\xi$ and $\eta$ gives (the square of) (13), the second estimate (14) then obviously holds true by symmetry.

Alternatively we can first take the $L^2_\xi$ - norm and apply Minkowski’s integral inequality to obtain

$$\|F^{\frac{1}{2}} I^\frac{1}{2}_{x,-}(u, v)(\xi, \eta, \cdot)\|_{L^2_\xi} \lesssim \int_\xi |\xi(\xi^*_1 - \xi^*_2)|^{-\frac{1}{2}} (\hat{u}_0(\xi^*_1, \eta_1)\hat{v}_0(\xi^*_2, \eta_2))_{L^2_\xi} d\eta_1 =: I(\xi, \eta).$$

Now the square of the norm inside the integral equals

$$\int_\xi |\hat{u}_0(\xi_1, \eta_1)\hat{v}_0(\xi_2, \eta_2)|^2 d\eta_1,$$

so that by a second application of Minkowski’s inequality

$$\|I(\cdot, \eta)\|_{L^2_\xi} \lesssim \int_\xi |\hat{u}_0(\cdot, \eta_1)|_{L^2_\xi} |\hat{v}_0(\cdot, \eta_2)|_{L^2_\xi} d\eta_1 \lesssim \|u_0\|_{L^2_{\eta, u}} \|v_0\|_{L^2_{\eta, v}},$$

which gives a bound independent of $\eta$. Finally we use $\|P_{\eta,k}F\|_{L^0_\eta} \lesssim 2^k \|F\|_{L^0_\eta}$ to obtain (15).

**Remark 1.** The above proposition has several useful consequences:

1. As the proof shows, we may replace the dyadic intervals symmetric around zero by intervals $I$ of arbitrary position and length $|I|$, if we change the factor $2^{\frac{k}{2}}$ on the right into $|I|^\frac{1}{2}$. In case of (13) the position of $|I|$ may even depend on $\eta$.

2. Summing up the dyadic pieces in (13) - (15) and using multilinear interpolation we obtain for $s_0, s_1, s_2 \geq 0$ with $s_0 + s_1 + s_2 > \frac{3}{2}$ and $b > \frac{1}{2}$ the inequality

$$\|J_g^{s_0} I^\frac{1}{2}_{x,-}(u, v)\|_{L^2_{x,y}} \lesssim \|J_g^{s_1} u\|_{0,b} \|J_g^{s_2} v\|_{0,b}.  \hspace{1cm} (16)$$

3. By symmetry in $x$ and $y$ we see that all the inequalities (13) - (16) are equally valid with $x$ and $y$ interchanged.
3. The key estimate. Now we are prepared to prove the key estimate for the proof of Theorem 1.1.

**Theorem 3.1.** Let \( s > \frac{1}{2} \). Then for any \( b' \leq -\frac{1}{3} \) and for any \( b > \frac{1}{2} \) the estimate
\[
\| (\partial_x + \partial_y)(u_1 u_2) \|_{s,b'} \lesssim \| u_1 \|_{s,b} \| u_2 \|_{s,b}
\] (17)
holds true for all \( u_1, u_2 \in X^{s,b} \).

**Proof.** Throughout this proof let \( * \) denote the convolution constraint
\[
(\tau, \xi, \eta) = (\tau_1, \xi_1, \eta_1) + (\tau_2, \xi_2, \eta_2)
\]
Under the above constraint it is obvious that
\[
\langle (\xi, \eta) \rangle^s \lesssim \langle (\xi_1, \eta_1) \rangle^s + \langle (\xi_2, \eta_2) \rangle^s \lesssim \langle (\xi_1, \eta_1) \rangle^s \langle (\xi_2, \eta_2) \rangle^s
\]
holds true, which implies that it suffices to prove the claim in the case \( 1 \leq s > \frac{3}{2} \).

Let \( \sigma_0 := \tau - \xi^3 - \eta^3, \sigma_j := \tau - \xi_j^3 - \eta_j^3 \), and \( f_j(\tau, \xi, \eta) := \hat{u}_j(\tau, \xi_j, \eta_j) |\sigma_j|^b \) for \( j = 1, 2 \). The claim is equivalent to the following weighted \( L^2 \) convolution estimate:
\[
\| M(f_1, f_2) \|_{L^2} \lesssim \prod_{j=1}^2 \| f_j \|_{L^2}
\] (18)
where
\[
M(f_1, f_2)(\tau, \xi, \eta) := \frac{\langle (\xi + \eta) \rangle^s \langle (\xi, \eta) \rangle^s}{\langle \sigma_j \rangle^{b-\eta}} \int \prod_{j=1}^2 f_j(\tau_j, \xi_j, \eta_j) \langle \sigma_j \rangle^{b-\eta} d\tau_j d\xi_j d\eta_j.
\]
To show (18) we may assume by symmetry that \( |\eta| \leq |\xi| \). Then we split the domain of integration into three regions, which induces the following decomposition:
\[
\| M(f_1, f_2) \|_{L^2} = R_1 + R_2 + R_3 + R_4.
\]
**Contribution R1:** This corresponds to the region \( |\xi| \lesssim |\xi_1 - \xi_2| \), so that \( |\xi| \lesssim |\xi_1^2 - |\xi_1 - \xi_2|^{\frac{1}{2}} \). Assuming in addition that \( |(\xi_1, \eta_1)| \geq |(\xi_2, \eta_2)| \), which can be done without loss of generality, we obtain the bound
\[
R_1 \lesssim \| I_x^{\frac{1}{2}} I_y^{\frac{1}{2}} (J^s u_1, u_2) \|_{L^2_{t,x,y}} \lesssim \| u_1 \|_{s,b} \| u_2 \|_{s,b},
\]
where we have used (16) with \( s_0 = s_1 = 0 \) and \( s_2 = s > \frac{1}{2} \).

**Contribution R2:** This corresponds to the region where \( |\xi_1 - \xi_2| \ll |\xi| \) and \( |\eta_1| \gtrsim |\xi| \). Here, we have \( |\xi| \sim |\xi_1| \sim |\xi_2| \) and obtain
\[
R_2 \lesssim \| (I_x^{\frac{1}{2}} I_y^{\frac{1}{2}} J^s u_1)(I_x^{\frac{1}{2}} u_2) \|_{L^2_{t,x,y}} + \| (I_x^{\frac{1}{2}} I_y^{\frac{1}{2}} u_1)(J^s u_2) \|_{L^2_{t,x,y}} =: R_{2,1} + R_{2,2},
\]
where
\[
R_{2,1} \lesssim \| I_x^{\frac{1}{2}} I_y^{\frac{1}{2}} J^s u_1 \|_{L^p_{t,x,y}} \| J^s u_2 \|_{L^p_{t,x,y}}
\]
whenever \( \frac{1}{2} + \frac{1}{q} = \frac{3}{p} \). Choosing \( \frac{1}{p} = 1 - s \) we can apply the Strichartz-type estimate (9) to bound the first factor by \( \| u_1 \|_{s,b} \), while for the second we use the estimate (12), so that we arrive at
\[
R_{2,1} \lesssim \| u_1 \|_{s,b} \| u_2 \|_{s,b}.
\]
The contribution \( R_{2,2} \) can be dealt with in exactly the same manner.

**Contribution R3:** We consider the region where \( |\xi_1 - \xi_2| \ll |\xi| \) and \( |\eta_2| \gtrsim |\xi| \). Here, the same argument as for \( R_2 \) applies (with \( u_1 \) and \( u_2 \) interchanged).
Contribution $R_4$: Here, we assume $|\xi_1 - \xi_2| \ll |\xi| \sim |\xi_1| \sim |\xi_2|$ and $|\eta_1| \ll |\xi|$ and $|\eta_2| \ll |\xi|$, thus completing the case by case discussion. We observe that under the convolution constraint the resonance identity
\begin{equation}
\sigma_0 - \sigma_1 - \sigma_2 = 3(\xi_1 \xi_2 + \eta_1 \eta_2) \tag{19}
\end{equation}
holds true (This is similar to the low regularity analysis of the KdV equation, where the analogous identity has been observed in [3, formula 7.46], see also [11]. Note that this similarity is due the transformation performed in Subsection 2.1.). In region $R_4$ this identity implies the inequality
\begin{equation}
\langle \sigma_0 \rangle + \langle \sigma_1 \rangle + \langle \sigma_2 \rangle \gtrsim |\xi|^3, \tag{20}
\end{equation}
which naturally leads to the following further division $R_4 = R_{4.0} + R_{4.1} + R_{4.2}$.

Contribution $R_{4.0}$: This corresponds to the subregion where $\langle \sigma_0 \rangle \gtrsim \langle \sigma_1 \rangle, \langle \sigma_2 \rangle$. Using (20) we estimate of (18) by
\[
R_{4.0} \lesssim \|J^s(u_1 u_2)\|_{L^4_{t,x,y}} \lesssim \|J^s u_1\|_{L^4_{t,x,y}} \|J^s u_2\|_{L^4_{t,x,y}} \lesssim \|u_1\|_{s,b} \|u_2\|_{s,b},
\]
in the last step we have used the estimate (11).

Contribution $R_{4.1}$: Here, we consider the subregion where $\langle \sigma_1 \rangle \gtrsim \langle \sigma_0 \rangle, \langle \sigma_2 \rangle$. We recall the operator $\Lambda^b u_1(\tau_1, \xi_1, \eta_1) = \langle \sigma_1 \rangle^b \tilde{u}_1(\tau_1, \xi_1, \eta_1)$. Using (20) as well as $\langle \sigma_1 \rangle \gtrsim \langle \sigma_0 \rangle$ we obtain the upper bound
\[
R_{4.1} \lesssim \|J^s((\Lambda^b u_1)(u_2))\|_{0,-b} \lesssim \|J^s \Lambda^b u_1\|_{L^4_{t,x,y}} \|J^s u_2\|_{L^4_{t,x,y}},
\]
where first the dual version of (11) and then this estimate itself were applied.

Contribution $R_{4.2}$: This corresponds to the subregion where $\langle \sigma_2 \rangle \gtrsim \langle \sigma_0 \rangle, \langle \sigma_1 \rangle$. This can be treated in precisely the same manner as $R_{4.1}$.

For the sake of completeness, we conclude this paper with a sketch of the proof of Theorem 1.1 based on Theorem 3.1. The ideas are well-known, see e.g. [3, 11, 7]. For $\phi \in H^s(\mathbb{R}^2)$ we solve the integral equation associated to (2)
\[
(\epsilon u + \mathcal{I}(u), t) := 4^{-\frac{4}{3}} \int_0^t U(t-s)(\partial_x + \partial_y)u^2(s)ds
\]
in $X_T^{s,b}$ by means of the contraction mapping principle. Indeed, from Lemma 2.2 and Theorem 3.1 it follows that
\[
\|U(t)\phi + \mathcal{I}(u)\|_{s,b;T} \lesssim \|\phi\|_{H^s} + T^\delta \|u\|_{s,b;T} \leq \|\phi\|_{H^s} + T^\delta \|u\|_{s,b;T}^2.
\]
for some $b > \frac{1}{2}$, $b' < -\frac{1}{2}$ and $\delta > 0$, and similarly
\[
\|\mathcal{I}(u) - \mathcal{I}(v)\|_{s,b;T} \lesssim T^\delta (\|u\|_{s,b;T} + \|v\|_{s,b;T}) \|u - v\|_{s,b;T}.
\]
This implies existence of a fixed point $u \in X_T^{s,b} \subseteq C([-T, T], H^s(\mathbb{R}^2))$ for suitably chosen $T > 0$ (depending on $\|\phi\|_{H^s}$). Based on these estimates one can also prove uniqueness of $u \in X_T^{s,b}$ and continuous dependence on the initial data.

Remark 2 (added to the final version). Shortly after this manuscript was submitted and posted at http://www.arxiv.org/abs/1302.2034, Luc Molinet and Didier Pilod independently derived Theorem 1.1, see http://www.arxiv.org/abs/1302.2933.
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