An algorithm for constructing doubly stochastic matrices for the inverse eigenvalue problem

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Abstract
In this note, we present an algorithm that yields many new methods for constructing doubly stochastic and symmetric doubly stochastic matrices for the inverse eigenvalue problem. In addition, we introduce new open problems in this area that lay the ground for future work.

Keywords: doubly stochastic matrices, inverse eigenvalue problem

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1. Introduction

An \( n \times n \) matrix with real entries is said to be nonnegative if all of its entries are nonnegative. An \( n \times n \) matrix \( A \) over a field \( \mathbf{F} \) (\( \mathbf{F} \) is either the real line \( \mathbb{R} \) or the complex plane \( \mathbb{C} \)) having each row and column sum equal to \( r \), is said to be an \( r \)-generalized doubly stochastic matrix. The set of all \( n \times n \) \( r \)-generalized doubly stochastic matrices with entries in \( \mathbf{F} \) is denoted by \( \Omega^r(n, \mathbf{F}) \). A generalized doubly stochastic matrix is an element of \( \Omega(n, \mathbf{F}) \) where
\[
\Omega(n, \mathbf{F}) = \bigcup_{r \in \mathbf{F}} \Omega^r(n, \mathbf{F}).
\]

Of special importance are the nonnegative elements in \( \Omega^1(n, \mathbb{R}) \) which are called the doubly stochastic matrices. The theory of doubly stochastic matrices is particularly endowed with a large collection of applications in other areas of mathematics and also in other disciplines (see for example [1, 2, 3, 6, 7, 19, 25, 28]).

Let the set of all \( n \times n \) doubly stochastic matrices be denoted by \( \Delta_n \) and the set of all \( n \times n \) symmetric elements in \( \Delta_n \) will be denoted by \( \Delta^s_n \). In addition, let \( \mathbf{M}(n, \mathbf{F}) \) be the algebra of all \( n \times n \) matrices with entries in \( \mathbf{F} \) and \( \text{GL}(n, \mathbf{F}) \) be the general linear group over the field \( \mathbf{F} \).Finally, \( I \) is defined as the imaginary unit and for any matrix (or vector) \( A \), its transpose will be denoted by \( A^T \).

The Perron-Frobenius theorem states that if \( A \) is a nonnegative matrix, then it has a real eigenvalue \( r \) (that is, the Perron-Frobenius root) which is greater than or equal to the modulus of each of the other eigenvalues. Also, \( A \) has an eigenvector \( x \) corresponding to \( r \) such that each of its entries are nonnegative. Furthermore, if \( A \) is irreducible then \( r \) is positive and the entries of \( x \) are also positive (see [1, 2, 11, 15, 25]).

For doubly stochastic matrices, \( r = 1 \) and \( x = e_n = (1, 1, ..., 1)^T \).

An intriguing object of study in the area of matrix theory and mathematical physics is that of the spectral properties and inverse eigenvalue problems for special kinds of matrices. The nonnegative inverse eigenvalue problem (NIEP) is the problem of finding necessary and sufficient conditions for an \( n \)-tuples \( (\lambda_1, \ldots, \lambda_n) \)

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(possibly complex) to be the spectrum of an \( n \times n \) nonnegative matrix \( A \). Although this inverse eigenvalue problem has attracted a considerable amount of interest, for \( n > 3 \) it is still unsolved except in restricted cases. Generally, we have two cases that result in three problems.

- When \( \lambda = (\lambda_1, \ldots, \lambda_n) \) is complex, little is known. The case \( n \leq 3 \) was completely solved in [17] and the solution of the \( 4 \times 4 \) trace-zero nonnegative inverse eigenvalue problem was given in [23].
- When the \( n \)-tuples \( (\lambda_1, \ldots, \lambda_n) \) are real, then we have the following two problems:
  1) The real nonnegative inverse eigenvalue problem (RNIEP) asks which sets of \( n \) real numbers occur as the spectrum of an \( n \times n \) nonnegative matrix \( A \).
  2) The symmetric nonnegative inverse eigenvalue problem (SNIEP) asks which sets of \( n \) real numbers can occur as the spectrum of an \( n \times n \) symmetric nonnegative matrix \( A \).

Each problem remains open. Many partial results for the three problems are known see the recent book [10] and the references therein, for the collection of all known results concerning these problems. Various people raised the question whether the (RNIEP) and (SNIEP) are generally equivalent. In the low dimension \( n \leq 4 \), the two problems are actually equivalent (see [9]). However, for \( n > 4 \), the paper [14] showed that the two problems are generally different. In addition, the paper [9] gives a construction for data \( \lambda = (\lambda_1, \ldots, \lambda_5) \) which is a solution for the (RNIEP) and there is no symmetric nonnegative \( 5 \times 5 \) matrix with spectrum \( \lambda \).

Another object of study in this area that has a big interest is the inverse eigenvalue problem for nonnegative matrices with extra properties. For example, we can consider the same problems for doubly stochastic matrices. So that we have the following problems.

**Problem 1.1.** The doubly stochastic inverse eigenvalue problem denoted by (DIEP), is the problem of determining the necessary and sufficient conditions for a complex \( n \)-tuples to be the spectrum of an \( n \times n \) doubly stochastic matrix. Equivalently, this problem can also be characterized as the problem of finding the region \( \Theta_n \) of \( \mathbb{C}^n \) such that any point in \( \Theta_n \) is the spectrum of an \( n \times n \) doubly stochastic matrix.

Now, when the \( n \)-tuples \( (\lambda_1, \ldots, \lambda_n) \) are all real, then we have the following two problems:

**Problem 1.2.** The real doubly stochastic inverse eigenvalue problem (RDIEP) asks which sets of \( n \) real numbers occur as the spectrum of an \( n \times n \) doubly stochastic matrix. This problem is also equivalent to the problem of finding the region \( \Theta_n^r \) of \( \mathbb{R}^n \) such that any point in \( \Theta_n^r \) is the spectrum of an \( n \times n \) doubly stochastic matrix.

**Problem 1.3.** The symmetric doubly stochastic inverse eigenvalue problem (SDIEP) asks which sets of \( n \) real numbers occur as the spectrum of an \( n \times n \) symmetric doubly stochastic matrix. Equivalently, this problem can also be characterized as the problem of finding the region \( \Theta_n^s \) of \( \mathbb{R}^n \) such that any point in \( \Theta_n^s \) is the spectrum of an \( n \times n \) symmetric doubly stochastic matrix.

**Example 1.** The point \( \alpha = (1,-1/2 + I\sqrt{3}/2, -1/2 - I\sqrt{3}/2) \) is in \( \Theta_3 \) as \( \alpha \) is the spectrum of \( A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \). On the other hand, the point \( \lambda = (1,1/2,1/4) \) is in \( \Theta_3^s \) as \( \lambda \) is the spectrum of \( B = \begin{pmatrix} 7/12 & 1/6 & 1/4 \\ 1/2 & 2/3 & 1/4 \\ 1/3 & 1/6 & 1/2 \end{pmatrix} \). Moreover, \( \lambda \) is in \( \Theta_3^s \) as \( \lambda \) is also the spectrum of \( C = \begin{pmatrix} 13/24 & 7/24 & 1/6 \\ 7/24 & 13/24 & 1/6 \\ 1/6 & 1/6 & 2/3 \end{pmatrix} \).

Although doubly stochastic matrices have been studied extensively, the (DIEP) and (RDIEP) have been considered in [23] [24] where all the results obtained are partial. The (SDIEP) was studied in [12] [13] [20] [21] [24], and earlier work can be found in [12] [22] [27] and all the results obtained are also partial. For general \( n \), all three problems remain open. In addition, the first and last of these problems have been completely
solved for $n = 3$ in [22]. Now in a similar manner to the (NIEP), one could raise the question whether the (RDIEP) and (SDIEP) are generally equivalent. For $n = 3$, we shall prove below that the two problems are actually equivalent. While for $n \geq 4$, it remains a very interesting open problem. In this paper, we are only able to obtain partial solutions concerning all 3 problems.

Recall that $X \in \Omega'(n, F)$ if and only if $Xe_n = re_n$ and $e_n^TX = re_n^T$ if and only if $XJ_n = J_nX = rJ_n$, where $J_n$ is the $n \times n$ matrix with each of its entries is equal to $\frac{1}{n}$. Hence for any $X \in \Omega'(n, F)$, $e_n$ is an eigenvector for both $X$ and $X^T$ corresponding to the eigenvalue $r$. In addition, for any $V$ in $\text{GL}(n, F)$ such that the first row of $V$ is equal to a multiple of $e_n^T$, and each of the last $n - 1$ row sums is zero, then clearly, its inverse $V^{-1}$ has its first column equal to a multiple of $e_n$ and each of the last $n - 1$ column sums of $V^{-1}$ is zero. Such $V$ is said to have pattern $S$. Note that any $n \times n$ matrix $V$ which is orthogonal and has its first column $\frac{1}{\sqrt{n}}e_n$ has pattern $S$. The main results in this paper rely on the following observation for which the proof can be found in [20].

**Observation 1.4.** Let $V \in \text{GL}(n, F)$ has pattern $S$ and $X \in \text{M}(n - 1, F)$, then $V^{-1} \begin{pmatrix} r & 0 \\ 0 & X \end{pmatrix} = A$ in $\Omega'(n, F)$. Conversely, for any $A \in \Omega'(n, F)$ and any $V$ that has pattern $S$, there exists $X \in \text{M}(n - 1, F)$ such that $VAV^{-1} = \begin{pmatrix} r & 0 \\ 0 & X \end{pmatrix}$.

In addition, an obvious necessary conditions for all three above problems are

1. $\sum_{i=1}^{n} \lambda_i^k \geq 0$ for all natural number $k$ which just means that the trace of the nonnegative matrix $A^k$ is nonnegative.
2. $0 \leq |\lambda_i| \leq 1$ for $i = 2, \ldots, n$; as the Perron-Frobenius theorem insures.
3. If $\lambda_i$ is a complex eigenvalue of a doubly stochastic matrix $D$ with nonzero imaginary part, then its conjugate $\bar{\lambda}_i$ is also an eigenvalue of $D$.

Finally, we end this section with the following notation which is used throughout this paper. Let $\Lambda$ be the $n \times n$ diagonal matrix with diagonal entries $1, \lambda_2, \ldots, \lambda_n$ with $1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq -1$ and $\text{trace}(\Lambda) = 1 + \lambda_2 + \ldots + \lambda_n \geq 0$.

2. Construction of Matrices for the SDIEP

In this section, we describe a way of obtaining partial solutions for the (SDIEP). As mentioned earlier, this problem is completely solved only for $n = 3$. Indeed the following theorem is proved in [22].

**Theorem 2.1.** [22] There exists a symmetric $3 \times 3$ doubly stochastic matrix with spectrum $1, \lambda, \mu$ if and only if $-1 \leq \lambda \leq 1$, $-1 \leq \mu \leq 1$, $\lambda + 3\mu + 2 \geq 0$ and $3\lambda + \mu + 2 \geq 0$.

For $n = 4$, a conjecture is given in [15] and for $n \geq 4$, only partial solutions are known. Now recall that $\Delta^*_n$ is a convex polytope of dimension $\frac{1}{2}n(n - 1)$, whose vertices were determined in [16] (see also [8]) where it has been proved that if $A$ is a vertex of $\Delta^*_n$, then $A = \frac{1}{2}(P + P^T)$ for some permutation matrix $P$, although not every $\frac{1}{2}(P + P^T)$ is a vertex. In addition, $e_n$ is always an eigenvector of any $n \times n$ doubly stochastic matrix $A$ so that when $A$ is symmetric, then it is written as $A = V_0\Lambda V_0^T$ for some $V_0$ which is orthogonal and has pattern $S$. Therefore for a fixed such $V_0$ one can ask what relation the $\{\lambda_i\}$ should satisfy for $V_0\Lambda V_0^T$ to be symmetric doubly stochastic. Then the region obtained in this way is a convex region of $\mathbb{R}^n$ and the largest possible subregion of $\Theta^*_n$ one could obtain in this fashion, is attained when the columns of the matrix $V_0$ are the common set of eigenvectors of a maximum number of vertices of $\Delta^*_n$ that mutually commute (note that this is the case in [22, 27]). However this only gives partial solutions since the classification of all such $V_0$ seems to be a difficult problem. Note that classifying all such $V_0$ is equivalent
Given an \( e \) (also known in the literature as a Soules matrix):

\[
\begin{pmatrix}
\frac{1}{n} & \sqrt{\frac{n(n-1)}{n}} & \sqrt{\frac{n(n-1)(n-2)}{n}} & \cdots & \sqrt{\frac{n(n-1)(n-2)\cdots(2)}{n}} \\
\frac{1}{n} & \sqrt{\frac{n(n-1)}{n}} & \sqrt{\frac{n(n-1)(n-2)}{n}} & \cdots & 0 \\
\frac{1}{n} & \sqrt{\frac{n(n-1)}{n}} & \sqrt{\frac{n(n-1)(n-2)}{n}} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{n} & \sqrt{\frac{n(n-1)}{n}} & 0 & \cdots & 0 \\
\frac{1}{n} & 0 & \cdots & \cdots & 0 \\
\frac{1}{n} & 0 & \cdots & \cdots & 0 \\
\frac{1}{n} & 0 & \cdots & \cdots & 0
\end{pmatrix}
\]

for which the matrix \( J \) for components in each of them is zero, then giving 1 region than \( \Gamma \) by constructing another matrix \( V \) sequence complementary to \( \beta \) to the problem of finding the collection \( D \) of all vectors in \( \mathbb{R}^n \) which can serve as the set of eigenvectors of an \( n \times n \) symmetric doubly stochastic matrix. Since \( e \) is always an eigenvector which is orthogonal to all other eigenvectors, then the collection \( D \{ e \} \) is contained in the hyperplane of \( \mathbb{R}^n \) which is orthogonal to \( e \). Equivalently, if \( e_n, x_2, \ldots, x_n \) are the orthonormal eigenvectors of a symmetric doubly stochastic matrix \( A \) corresponding to the eigenvalues \( 1, \lambda_2, \ldots, \lambda_n \) respectively, then \( A = \frac{1}{n} e_n e_n^T + \lambda_2 x_2 x_2^T + \cdots + \lambda_n x_n x_n^T = J_n + \lambda_2 x_2 x_2^T + \cdots + \lambda_n x_n x_n^T \). Therefore, if \( x_2, \ldots, x_n \) are \( n - 1 \) vectors in \( \mathbb{R}^n \) such that the sum of the components in each of them is zero, then giving 1 \( \geq \lambda_2 \geq \cdots \geq \lambda_n \geq -1 \), one can look at the conditions for which the matrix \( J_n + \lambda_2 x_2 x_2^T + \cdots + \lambda_n x_n x_n^T \) is nonnegative. Next, we state the earliest result for the (SDIEP) which is found in \[22\].

**Theorem 2.2.** \[22\] If \( 1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq -1 \) and

\[
\frac{1}{n} + \frac{1}{n(n-1)} \lambda_2 + \frac{1}{n(n-1)(n-2)} \lambda_3 + \cdots + \frac{1}{(2)(1)} \lambda_n \geq 0
\]

then there is a symmetric doubly-stochastic matrix \( D \) such that \( D \) has eigenvalues \( 1, \lambda_2, \ldots, \lambda_n \).

G. Soules \[27\] generalized the above result by considering the following orthogonal pattern \( S \) matrix (also known in the literature as a Soules matrix):

\[
V_n = \begin{pmatrix}
\frac{1}{n} & \sqrt{\frac{n(n-1)}{n}} & \sqrt{\frac{n(n-1)(n-2)}{n}} & \cdots & \sqrt{\frac{n(n-1)(n-2)\cdots(2)}{n}} \\
\frac{1}{n} & \sqrt{\frac{n(n-1)}{n}} & \sqrt{\frac{n(n-1)(n-2)}{n}} & \cdots & 0 \\
\frac{1}{n} & \sqrt{\frac{n(n-1)}{n}} & \sqrt{\frac{n(n-1)(n-2)}{n}} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{n} & \sqrt{\frac{n(n-1)}{n}} & 0 & \cdots & 0 \\
\frac{1}{n} & 0 & \cdots & \cdots & 0 \\
\frac{1}{n} & 0 & \cdots & \cdots & 0 \\
\frac{1}{n} & 0 & \cdots & \cdots & 0
\end{pmatrix}
\]

It is easy to check (see \[27\]) that the symmetric matrix \( A = V_n AV_n^T \) has nonnegative off diagonal entries while the \( i \)th diagonal entry of \( A \) is given by the convex sum

\[
a_{ii} = \frac{1}{n} + \sum_{k=1}^{n} \left( \frac{1}{(k-1)k} \right) \lambda_{n-k+1} + \left( \frac{i-1}{i} \right) \lambda_{n-i+2},
\]

for \( i = 1, \ldots, n \). Moreover the \( a_{ii} \) are increasing so the smallest one is \( a_{11} \). So that if \( \Gamma \) is the convex region defined as the set of all \( n \)-tuples \( (\lambda_1, \ldots, \lambda_n) \) satisfying: \( 1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq -1 \) and \( 1 + \lambda_2 + \cdots + \lambda_n \geq 0 \) with \( a_{ii} \geq 0 \) for \( i = 1, \ldots, n \), then each point in \( \Gamma \) is a solution for the (SDIEP). Next, Soules obtains a larger region than \( \Gamma \) by constructing another matrix \( V_\beta \) from the matrix \( V_n \) above as follows:

Given an \( s \)-long strictly increasing sequence \( \beta \) with values in \( \{1, \ldots, n\} \) where \( 0 < s < n \), let \( \bar{\beta} \) be the \( t \)-long sequence complementary to \( \beta \) \((t = n - s)\). Define the \( s \)-vector \( u = \left( \frac{1}{\sqrt{s}}, \ldots, \frac{1}{\sqrt{s}} \right)^T \in \mathbb{R}^s \) and the \( t \)-vector \( \bar{u} = \left( \frac{1}{\sqrt{n-s}}, \ldots, \frac{1}{\sqrt{n-s}} \right)^T \in \mathbb{R}^{n-s} \). Now let \( w = \left( \frac{1}{\sqrt{s}} \right) u \) and \( \bar{w} = -\left( \frac{1}{\sqrt{n-s}} \right) \bar{u} \). Finally, define \( B_u \) as the \( s \times (s - 1) \) matrix obtained from \( V_s \) by deleting the first column and also \( B_\bar{u} \) is defined similarly. Next, for any \( \beta \) define:

\[
V_\beta = \begin{pmatrix}
u & w & B_u & 0_1 \\
\bar{u} & \bar{w} & 0_2 & B_\bar{u}
\end{pmatrix}
\]

where \( 0_1 \) and \( 0_2 \) are respectively the \( s \times (t - 1) \) and \( t \times (s - 1) \) zero matrices. Then it is a routine computation to check (see \[27\]) that \( V_\beta \) is orthogonal and the symmetric matrix \( A(\beta) = V_\beta V_\beta^T \) has nonnegative off
diagonal entries. Now, if we let $V_β^\sigma$ be the matrix obtained from $V_β$ by permuting its columns according to a permutation $σ$ that leaves columns 1 and 2 fixed and retains the relative order of the columns in $B_α$ and $B_β$ i.e.

$$σ_j = j \text{ for } j = 1, 2 \text{ and } σ_j < σ_{j+1} \text{ for } j ≠ s + 1,$$

then again the symmetric matrix $A(σ, β) = V_β^\sigma A(V_β^\sigma)^T$ would have nonnegative off diagonal entries. Moreover, if $α$ is an $(s - 1)$-long strictly increasing sequence with values in $\{3, ..., n\}$ defined by: $α(j - 2) = σ_j$ for $j = 3, ..., s + 1$, and $α$ is the $(t - 1)$-long sequence complementary to $α$ defined by: $α(j - s - 1) = σ_j$ for $j = s + 2, ..., n$, then each diagonal entry of $A(σ, β)$ is either equal to $Σ_α$ or $Σ_β$ where $Σ_α$ denote the following convex sum:

$$Σ_α = \frac{1}{n} + \frac{n - s}{ns} λ_2 + \sum_{k=1}^{s-1} \frac{λα(s-k)}{(k+1)k},$$

and $Σ_β$ is obtained from $Σ_α$ by replacing $α$ with $β$. Finally, let $n = 2m + 2$ for $n$ even and $n = 2m + 1$ for $n$ odd. Now taking $α$ to be the $m$-long sequence $\{3, 5, 7, ..., n - 2, n\}$, then it is easy to see that $Σ_α$ can be obtained from $Σ_β$ by replacing $λ_{n-2k+2}$ with $λ_{n-2k+1}$ for $k = 1, ..., m$. So that $Σ_α$ is the smallest and then we have the following corollary:

**Corollary 2.3.** If $1 ≥ λ_2 ≥ ... ≥ λ_n ≥ -1$ and

$$\frac{1}{n} + \frac{n - m - 1}{n(m + 1)} λ_2 + \sum_{k=1}^{m} \frac{λ_{n-2k+2}}{(k+1)k} ≥ 0$$

holds, where $n = 2m + 2$ for $n$ even and $n = 2m + 1$ for $n$ odd, then there exists an $n × n$ symmetric doubly stochastic matrix $D$ such that $D$ has eigenvalues $1, λ_2, ..., λ_n$.

2.1. Remarks on Soules’s results.

We can improve on the above corollary by choosing a ‘better’ permutation than $α$ above. By that we mean a permutation that allows us to obtain a larger region of $Θ_n^α$, although this happens on the account of not having a unified condition for all cases of $n$. Indeed, we consider the following 4 cases:

- **$n = 4k + 1$**: Taking $α = \{3, 4, 7, 8, ..., n - 2, n - 1\}$ and $β = \{5, 6, 9, 10, ..., n - 4, n - 3, n\}$ then we obtain the region

  \[\frac{1}{n} + \frac{n - m - 1}{n(m + 1)} λ_2 + \frac{λ_3}{m(m + 1)} + \frac{λ_4}{(m - 1)m} + \frac{λ_5}{(m - 2)(m - 1)} + ... + \frac{λ_{n-2}}{m(m - 2)} + \frac{λ_{n-1}}{m(m - 1)} + \frac{λ_n}{m(m - 2)} ≥ 0\]

- **$n = 4k + 3$**: Let $α = \{3, 4, 7, 8, ..., n - 4, n - 3, n\}$ and then $β = \{5, 6, 9, 10, ..., n - 2, n - 1\}$ and we obtain the region

  \[\frac{1}{n} + \frac{n - m - 1}{n(m + 1)} λ_2 + \frac{λ_3}{m(m + 1)} + \frac{λ_4}{(m - 1)m} + \frac{λ_5}{(m - 2)(m - 1)} + ... + \frac{λ_{n-2}}{m(m - 2)} + \frac{λ_{n-3}}{m(m - 1)} + \frac{λ_n}{m(m - 2)} ≥ 0\]

- **$n = 4k + 2$**: Take $α = \{3, 6, 7, 10, 11, ..., n - 4, n - 3, n\}$ and $β = \{4, 5, 8, 9, ..., n - 2, n - 1\}$ and the region

  \[\frac{1}{n} + \frac{n - m - 1}{n(m + 1)} λ_2 + \frac{λ_3}{m(m + 1)} + \frac{λ_4}{(m - 1)m} + \frac{λ_5}{(m - 2)(m - 1)} + ... + \frac{λ_{n-2}}{m(m - 2)} + \frac{λ_{n-3}}{m(m - 1)} + \frac{λ_n}{m(m - 2)} ≥ 0\]

- **$n = 4k + 2$**: Take $α = \{3, 6, 7, 10, 11, ..., n - 4, n - 3, n\}$ and $β = \{4, 5, 8, 9, ..., n - 2, n - 1\}$ and the region

  \[\frac{1}{n} + \frac{n - m - 1}{n(m + 1)} λ_2 + \frac{λ_3}{m(m + 1)} + \frac{λ_4}{(m - 1)m} + \frac{λ_5}{(m - 2)(m - 1)} + ... + \frac{λ_{n-2}}{m(m - 2)} + \frac{λ_{n-3}}{m(m - 1)} + \frac{λ_n}{m(m - 2)} ≥ 0\]
Let of the following simple observation which is the building block of the algorithm. Before exploring this algorithm, it helps to think the construction described above. In addition, this algorithm can also be used with minor changes to find

Algorithm 1.

- Setp1: For low dimension $k = 2, 3, 4, 5, 6, ...$ we consider all the vertices of $\Delta_k$. For a chosen one of these vertices, we find its eigenvectors (using Maple for example) as orthonormal columns vectors $\frac{1}{\sqrt{k}}e_1, \frac{1}{\sqrt{k}}e_2, ..., \frac{1}{\sqrt{k}}e_k$. 
- Setp2: From the computed eigenvectors in Step1, we form the $k \times k$ orthogonal pattern $S$ matrix $X_k = (\sqrt{\frac{1}{k}}e_1|\sqrt{\frac{1}{k}}e_2|...|\sqrt{\frac{1}{k}}e_k)$ obtained from these eigenvectors, and if we let $\Delta_k$ be the $k \times k$ diagonal matrix with diagonal entries $1, \lambda_2, ..., \lambda_k$ with $1 \geq \lambda_2 \geq ... \geq \lambda_k \geq -1$, and $1 + \lambda_2 + ... + \lambda_k \geq 0$, then we check the conditions such that $X_k \Delta_k X_k^T \geq 0$ for which by this construction, we always have solutions.
- Setp3: Next, for $n > k$ we construct an $n \times n$ matrix $W_n$ by taking the first $n - k + 1$ columns of the Soules matrix $V_n$ and the last $(k - 1)$ columns

$$\begin{pmatrix} x_2 & \cdots & x_k \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

as the $n$ columns of $W_n$. Then find the region for which $W_n \Lambda W_n^T$ is nonnegative.

Setp4: Here we start the improvement process as follows. We construct another matrix $W_\beta$ which will be the analogue of the matrix $V_3$ (defined above) in the following manner: For $k < s < n$, let $u$ be the $s$-vector $(\frac{1}{\sqrt{n}}, ..., \frac{1}{\sqrt{n}})$, $\bar{u}$ be the $(n - s)$-vector $(\frac{1}{\sqrt{n}}, ..., \frac{1}{\sqrt{n}})$, $w = (\sqrt{\frac{1}{n}}u)$ and $\bar{w} = -(\sqrt{\frac{1}{n}}\bar{u})$ be defined as in the previous section. Now define $W_n$ be the $(s - 1) \times s$ matrix obtained from the $s \times s$ matrix $W_s$ by deleting the first column and $B_n$ be the $(n - s - 1) \times (n - s)$ matrix obtained from the Soules matrix $V_{n-s}$ by deleting the first column. Then we distinguish between the two cases:

2.2. New Methods for Constructing Symmetric Doubly Stochastic Matrices

Here we describe an algorithm for practical purposes that yields many new sufficient conditions for the symmetric doubly stochastic inverse eigenvalue problem. In fact, this can thought as a generalization of the construction described above. In addition, this algorithm can also be used with minor changes to find solutions for the ( DIEP) in the complex case (see below). Before exploring this algorithm, it helps to think of the following simple observation which is the building block of the algorithm.

**Observation 2.4.** Let $x$ be an eigenvector of an $n \times n$ doubly stochastic matrix $A$ corresponding to the eigenvalue $\lambda$. If $0_p$ denote the $p \times 1$ zero vector, then $\begin{pmatrix} 0_p \\ x \end{pmatrix}$ and $\begin{pmatrix} x \\ 0_p \end{pmatrix}$ are respectively eigenvectors of the $(n + p) \times (n + p)$ doubly stochastic matrices $I_p \oplus A$ and $A \oplus I_p$ corresponding to $\lambda$.

**Algorithm 1.**

Setp1: For low dimension $k = 2, 3, 4, 5, 6, ...$ we consider all the vertices of $\Delta_k$. For a chosen one of these vertices, we find its eigenvectors (using Maple for example) as orthonormal columns vectors $\frac{1}{\sqrt{k}}e_1, \frac{1}{\sqrt{k}}e_2, ..., \frac{1}{\sqrt{k}}e_k$.

Setp2: From the computed eigenvectors in Step1, we form the $k \times k$ orthogonal pattern $S$ matrix $X_k = (\sqrt{\frac{1}{k}}e_1|\sqrt{\frac{1}{k}}e_2|...|\sqrt{\frac{1}{k}}e_k)$ obtained from these eigenvectors, and if we let $\Delta_k$ be the $k \times k$ diagonal matrix with diagonal entries $1, \lambda_2, ..., \lambda_k$ with $1 \geq \lambda_2 \geq ... \geq \lambda_k \geq -1$, and $1 + \lambda_2 + ... + \lambda_k \geq 0$, then we check the conditions such that $X_k \Delta_k X_k^T \geq 0$ for which by this construction, we always have solutions.

Setp3: Next, for $n > k$ we construct an $n \times n$ matrix $W_n$ by taking the first $n - k + 1$ columns of the Soules matrix $V_n$ and the last $(k - 1)$ columns of $\begin{pmatrix} x_2 & \cdots & x_k \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$ as the $n$ columns of $W_n$. Then find the region for which $W_n \Lambda W_n^T$ is nonnegative.

Setp4: Here we start the improvement process as follows. We construct another matrix $W_\beta$ which will be the analogue of the matrix $V_3$ (defined above) in the following manner: For $k < s < n$, let $u$ be the $s$-vector $(\frac{1}{\sqrt{n}}, ..., \frac{1}{\sqrt{n}})$, $\bar{u}$ be the $(n - s)$-vector $(\frac{1}{\sqrt{n}}, ..., \frac{1}{\sqrt{n}})$, $w = (\sqrt{\frac{1}{n}}u)$ and $\bar{w} = -(\sqrt{\frac{1}{n}}\bar{u})$ be defined as in the previous section. Now define $W_n$ be the $(s - 1) \times s$ matrix obtained from the $s \times s$ matrix $W_s$ by deleting the first column and $B_n$ be the $(n - s - 1) \times (n - s)$ matrix obtained from the Soules matrix $V_{n-s}$ by deleting the first column. Then we distinguish between the two cases:
• Case 1: For \( n = k + 1 \), we take \( W_n = W_{k+1} \) and we then find the region for which \( W_{k+1}AW_{k+1}^T \) is nonnegative.

• Case 2: For \( n > k + 1 \), we let

\[
W_\beta = \begin{pmatrix}
  u & w & 0_1 & W_u \\
  \bar{u} & \bar{w} & B_u & 0_2
\end{pmatrix}
\]

\( 0_1 \) and \( 0_2 \) are zero matrices of suitable orders. Then for any permutation \( \alpha \) of the columns of \( V_\beta \) which preserves the relative order of columns in \( W_u \) and \( B_u \), the conditions for which \( W_\beta AW_\beta^T \) is nonnegative give a convex region \( \Gamma_\alpha \subset \Theta_n^k \) whose all points are solutions for the (SDIEP). Finally, it is worth mentioning here that the union of all \( \Gamma_\alpha \) for all such permutation \( \alpha \), gives a union of convex subsets in \( \Theta_n^k \) whose again all points are solutions for the (SDIEP).

Setp5 Repeat the same process for a different chosen vertex of \( \Delta_k^n \).

Remark 2.5. It should be noted that the above algorithm can also be used by starting with any \( k \times k \) doubly stochastic matrix (i.e. any point of \( \Delta_k^n \)) that is not necessarily a vertex of \( \Delta_k^n \) which is the version of Algorithm1 that we will use for the case of (RDEIP) (see Section 3). However, the advantage of taking a vertex lies in obtaining a larger subset of \( \Theta_n^k \) which in turn by exploiting Step3 and Step4 of Algorithm1 gives a larger region of \( \Theta_n^k \). In addition, the above algorithm can also be used not just for low dimension, but for any dimension \( k \) for which there exists a \( k \times k \) permutation matrix (or any \( k \times k \) doubly stochastic matrix) \( P \) such that the eigenvectors of \( P \) can be computed.

Remark 2.6. With Algorithm 1, the Souls matrix \( V_n \) can be obtained by choosing the vertex \( P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) of \( \Delta_k^n \) in Step1 since \( \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)^T \) and \( \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right)^T \) are the eigenvectors of \( P \).

Example 2. We apply Algorithm 1 on the matrix \( X = \frac{1}{2} \begin{pmatrix}
  1 & 1 & 1 & 1 \\
  1 & -1 & -1 & 1 \\
  1 & -1 & 1 & -1 \\
  1 & 1 & -1 & -1
\end{pmatrix} \), which is the orthogonal pattern \( S \) matrix that diagonalizes all the \( 4 \times 4 \) zero-trace symmetric doubly stochastic matrices (see [21]). Note that the columns of \( X \) are the orthonormal eigenvectors of the following vertices of \( \Delta_4^4 \):

\[
p1 = \begin{pmatrix}
  0 & 1 & 0 & 0 \\
  1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 \\
  0 & 0 & 1 & 0
\end{pmatrix}, \quad p2 = \begin{pmatrix}
  0 & 0 & 0 & 1 \\
  0 & 0 & 1 & 0 \\
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0
\end{pmatrix}, \quad \text{and} \quad p3 = \begin{pmatrix}
  0 & 0 & 0 & 1 \\
  0 & 0 & 1 & 0 \\
  0 & 1 & 0 & 0 \\
  1 & 0 & 0 & 0
\end{pmatrix}
\]

A simple matrix multiplication shows that \( X \Lambda_4 X^T = \)

\[
\frac{1}{2} \begin{pmatrix}
  1 & 1 & 1 & 1 \\
  1 & -1 & -1 & 1 \\
  1 & -1 & 1 & -1 \\
  1 & 1 & -1 & -1
\end{pmatrix} \begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & \lambda_2 & 0 & 0 \\
  0 & 0 & \lambda_3 & 0 \\
  0 & 0 & 0 & \lambda_4
\end{pmatrix} \frac{1}{2} \begin{pmatrix}
  1 & 1 & 1 & 1 \\
  1 & -1 & -1 & 1 \\
  1 & -1 & 1 & -1 \\
  1 & 1 & -1 & -1
\end{pmatrix} = \begin{pmatrix}
  \frac{1}{4} + \frac{\lambda_2}{4} + \frac{\lambda_3}{4} + \frac{\lambda_4}{4} & \frac{1}{4} - \frac{\lambda_2}{4} + \frac{\lambda_3}{4} - \frac{\lambda_4}{4} & \frac{1}{4} - \frac{\lambda_2}{4} - \frac{\lambda_3}{4} + \frac{\lambda_4}{4} & \frac{1}{4} + \frac{\lambda_2}{4} - \frac{\lambda_3}{4} - \frac{\lambda_4}{4} \\
  \frac{1}{4} - \frac{\lambda_2}{4} + \frac{\lambda_3}{4} - \frac{\lambda_4}{4} & \frac{1}{4} + \frac{\lambda_2}{4} + \frac{\lambda_3}{4} + \frac{\lambda_4}{4} & \frac{1}{4} - \frac{\lambda_2}{4} - \frac{\lambda_3}{4} - \frac{\lambda_4}{4} & \frac{1}{4} + \frac{\lambda_2}{4} - \frac{\lambda_3}{4} + \frac{\lambda_4}{4} \\
  \frac{1}{4} - \frac{\lambda_2}{4} - \frac{\lambda_3}{4} + \frac{\lambda_4}{4} & \frac{1}{4} + \frac{\lambda_2}{4} - \frac{\lambda_3}{4} - \frac{\lambda_4}{4} & \frac{1}{4} + \frac{\lambda_2}{4} + \frac{\lambda_3}{4} + \frac{\lambda_4}{4} & \frac{1}{4} - \frac{\lambda_2}{4} + \frac{\lambda_3}{4} - \frac{\lambda_4}{4} \\
  \frac{1}{4} + \frac{\lambda_2}{4} - \frac{\lambda_3}{4} - \frac{\lambda_4}{4} & \frac{1}{4} - \frac{\lambda_2}{4} + \frac{\lambda_3}{4} - \frac{\lambda_4}{4} & \frac{1}{4} + \frac{\lambda_2}{4} + \frac{\lambda_3}{4} - \frac{\lambda_4}{4} & \frac{1}{4} - \frac{\lambda_2}{4} - \frac{\lambda_3}{4} + \frac{\lambda_4}{4}
\end{pmatrix}.
\]
Hence we have the following theorem.

Next, we construct the $n \times n$ orthogonal matrix $W_n$ which in this case, is given by:

$$ W_n = \begin{pmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \cdots & \frac{1}{\sqrt{n-2}} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \cdots & \frac{1}{\sqrt{n-2}} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \cdots & \frac{1}{\sqrt{n-2}} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \cdots & \frac{1}{\sqrt{n-2}} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \cdots & \frac{1}{\sqrt{n-2}} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \cdots & \frac{1}{\sqrt{n-2}} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \cdots & \frac{1}{\sqrt{n-2}} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}. $$

Then a simple matrix multiplication shows the matrices $A = W_n \Lambda W_n^T$ and $B = V_n \Lambda V_n^T$ differ only by the $4 \times 4$ principal submatrices formed from the first 4 rows and the first 4 columns of $A$ and $B$ i.e. $A = \begin{pmatrix} X & C \\ C^T & D \end{pmatrix}$ and $B = \begin{pmatrix} Y & C \\ C^T & D \end{pmatrix}$ where $X$ and $Y$ are $4 \times 4$ matrices. So that by \[27\] all the entries of the symmetric matrix $A = W_n \Lambda W_n^T$ are nonnegative except for the diagonal entries and for $a_{12} = a_{21} = a_{34} = a_{43} = \frac{1}{n} + \frac{\lambda_2}{n(n-1)} + \frac{\lambda_3}{(n-1)(n-2)} + \frac{\lambda_4}{(n-2)(n-3)} + \cdots + \frac{\lambda_{n-3}}{20} - \frac{\lambda_{n-2}}{4} - \frac{\lambda_{n-1}}{4} + \frac{\lambda_n}{4}$. In addition, the first 4 diagonal entries are equal to: $a_{11} = a_{22} = a_{33} = a_{44} = \frac{1}{n} + \frac{\lambda_2}{n(n-1)} + \frac{\lambda_3}{(n-1)(n-2)} + \frac{\lambda_4}{(n-2)(n-3)} + \cdots + \frac{\lambda_{n-3}}{20} + \frac{\lambda_{n-2}}{4} + \frac{\lambda_{n-1}}{4} + \frac{\lambda_n}{4}$.

and the remaining diagonal entries $a_{55}, \ldots, a_{nn}$ are increasing so the smallest one is

$$ a_{55} = \frac{1}{n} + \frac{\lambda_2}{n(n-1)} + \frac{\lambda_3}{(n-1)(n-2)} + \frac{\lambda_4}{(n-2)(n-3)} + \cdots + \frac{\lambda_{n-4}}{30} + \frac{16}{20} \lambda_{n-3}. $$

We can rewrite

$$ a_{55} = \frac{1}{n} + \frac{\lambda_2}{n(n-1)} + \frac{\lambda_3}{(n-1)(n-2)} + \frac{\lambda_4}{(n-2)(n-3)} + \cdots + \frac{\lambda_{n-3}}{4} + \frac{\lambda_{n-3} - \lambda_{n-3}}{4} + \frac{\lambda_{n-3}}{4}. $$

As $1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq -1$, then clearly $a_{11} \leq a_{55}$ and therefore the conditions for which $A = W_n^T \Lambda W_n$ is nonnegative are:

$$ \begin{cases} a_{11} \geq 0 \\ a_{12} \geq 0 \end{cases} \tag{11} $$

Hence we have the following theorem.

**Theorem 2.7.** Let $1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq -1$ and $1 + \lambda_2 + \ldots + \lambda_n \geq 0$. If

$$ \begin{cases} \frac{1}{n} + \frac{\lambda_2}{n(n-1)} + \frac{\lambda_3}{(n-1)(n-2)} + \frac{\lambda_4}{(n-2)(n-3)} + \cdots + \frac{\lambda_{n-3}}{20} + \frac{\lambda_{n-2}}{4} + \frac{\lambda_{n-1}}{4} + \frac{\lambda_n}{4} \geq 0 \\ \frac{1}{n} + \frac{\lambda_2}{n(n-1)} + \frac{\lambda_3}{(n-1)(n-2)} + \frac{\lambda_4}{(n-2)(n-3)} + \cdots + \frac{\lambda_{n-3}}{20} - \frac{\lambda_{n-2}}{4} - \frac{\lambda_{n-1}}{4} + \frac{\lambda_n}{4} \geq 0 \end{cases} $$

then there is an $n \times n$ symmetric doubly stochastic matrix $D$ such that $D$ has eigenvalues $1, \lambda_2, \ldots, \lambda_n$. 8
To improve upon the results of the above theorem, we consider the matrix $W_{\beta} = \begin{pmatrix} u & w & 0_1 & W_u \\ \bar{u} & \bar{w} & B_0 & 0_2 \end{pmatrix}$.

Now let $\alpha$ be the permutation of the columns of $W_{\beta}$ (which of course preserves the relative order of columns in $W_u$ and $B_0$) given by:

- For $n = 2m + 2$ even and $s = m + 1$ and $\alpha = \{4, 6, 8, ..., n-6, n-3, n-1, n\}$ then the new improved conditions become:

$$\begin{align*}
\frac{1}{5} + \frac{n-m-1}{m(m+1)} \lambda_2 + \frac{\lambda_4}{m(m+1)} + \frac{\lambda_6}{m(m+1)} + \frac{\lambda_8}{m(m+1)} + \ldots + \frac{\lambda_{n-6}}{m(m+1)} + \frac{\lambda_{n-3}}{m(m+1)} + \frac{\lambda_{n-1}}{m} + \frac{\lambda_n}{m} & \geq 0 \\
\frac{1}{5} + \frac{n-m-1}{m(m+1)} \lambda_2 + \frac{\lambda_4}{m(m+1)} + \frac{\lambda_6}{m(m+1)} + \frac{\lambda_8}{m(m+1)} + \ldots + \frac{\lambda_{n-6}}{m(m+1)} + \frac{\lambda_{n-3}}{m(m+1)} + \frac{\lambda_{n-1}}{m} + \frac{\lambda_n}{m} & \geq 0
\end{align*}$$

(12)

- While for $n = 2m + 1$ odd and $s = m + 1$, and $\alpha = \{3, 5, 7, ..., n-6, n-3, n-1, n\}$ the new improved conditions become:

$$\begin{align*}
\frac{1}{5} + \frac{n-m-1}{m(m+1)} \lambda_2 + \frac{\lambda_4}{m(m+1)} + \frac{\lambda_6}{m(m+1)} + \frac{\lambda_8}{m(m+1)} + \ldots + \frac{\lambda_{n-6}}{m(m+1)} + \frac{\lambda_{n-3}}{m(m+1)} + \frac{\lambda_{n-1}}{m} + \frac{\lambda_n}{m} & \geq 0 \\
\frac{1}{5} + \frac{n-m-1}{m(m+1)} \lambda_2 + \frac{\lambda_4}{m(m+1)} + \frac{\lambda_6}{m(m+1)} + \frac{\lambda_8}{m(m+1)} + \ldots + \frac{\lambda_{n-6}}{m(m+1)} + \frac{\lambda_{n-3}}{m(m+1)} + \frac{\lambda_{n-1}}{m} + \frac{\lambda_n}{m} & \geq 0
\end{align*}$$

(13)

3. The Real Inverse Eigenvalue Problem For Doubly Stochastic Matrices

First as mentioned earlier, this problem has been considered in [26] where the following theorem has been obtained.

**Theorem 3.1.** [26] Let $\sigma = \{1, \lambda_2, ..., \lambda_n\}$ be a set of real numbers such that

$$1 \geq \lambda_2 \geq ... \geq \lambda_r \geq 0 \geq \lambda_{r+1} \geq ... \geq \lambda_n.$$ 

If $1 \geq \lambda_2 + n\max\{|\lambda_2|, |\lambda_n|\}$, then there exists an $n \times n$ doubly stochastic matrix with spectrum $\sigma$.

In addition, the case $n = 3$ has been used in a proof of a theorem concerning the (SDIEP) in [22] in which the following result holds.

**Theorem 3.2.** Let $X = \begin{pmatrix} p & q & 1-p-q \\ r & s & 1-r-s \\ 1-p-r & 1-q-s & p+q+r+s-1 \end{pmatrix}$ be a doubly stochastic matrix with real eigenvalues 1, $\lambda$, $\mu$. Then $-1 \leq \lambda \leq 1$, $-1 \leq \mu \leq 1$, $\lambda + 3\mu + 2 \geq 0$ and $3\lambda + \mu + 2 \geq 0$.

**Proof.** The first two inequalities are obtained from the Perron-Frobenius theorem. For the other two inequalities, let $x = \lambda + 3\mu + 2$ and $y = 3\lambda + \mu + 2$. We prove that $x$ and $y$ are nonnegative by showing that their sum $x + y$ and their product $xy$ are nonnegative. For, $x + y = 4\lambda + 4\mu + 4 = 4\text{trace}(X) \geq 0$. Now $xy = 3(\lambda + \mu + 1)^2 + 2(\lambda + \mu + 1) + 4\lambda \mu - 1 = 3[\text{trace}(X)]^2 + 2\text{trace}(X) + 4\text{determinant}(X) - 1 = 3(q - r)^2 + 12(p + s)(p + q + r + s - 1) + 12ps$. As the entries of $X$ are nonnegative, therefore $xy \geq 0$, and the proof is complete.

Combining the above theorem with Theorem 3.1, we have the following conclusion.

**Corollary 3.3.** The (RDIEP) and the (SDIEP) are equivalent for the case $n = 3$.

Now returning to the procedures described at the beginning of Section 2 and starting out with a particular $n \times n$ pattern $S$ matrix $V$ which is not orthogonal, we obtain the following theorem that solves the (RDIEP) for some restricted cases.

**Theorem 3.4.** If $1 \geq \lambda_2 \geq ... \geq \lambda_n \geq -1$ and

$$\begin{align*}
1 - (n-1)\lambda_2 + \lambda_3 + ... + \lambda_n & \geq 0 \\
1 + (n-1)\lambda_n & \geq 0
\end{align*}$$

(14)

then there is an $n \times n$ nonsymmetric doubly stochastic matrix $D$ with spectrum $1, \lambda_2, ..., \lambda_n$. 
Proof. Clearly the matrix

\[ V = \begin{pmatrix}
1 & 1 & 1 & \ldots & 1 & 1 \\
1 & -1 & 0 & \ldots & 0 & 0 \\
1 & 0 & -1 & 0 & \ldots & 0 \\
1 & 0 & 0 & -1 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 0 & 0 & 0 & \ldots & 0 \\
1 & 0 & 0 & 0 & \ldots & -1 \\
\end{pmatrix} \]

has pattern \( S \) and its inverse \( V^{-1} \) is given by:

\[ V^{-1} = \begin{pmatrix}
\frac{1}{n} & -\frac{1}{n} & \frac{1}{n} & \ldots & \frac{1}{n} & \frac{1}{n} \\
\frac{1}{n} & \frac{1}{n} & -\frac{1}{n} & \ldots & \frac{1}{n} & \frac{1}{n} \\
\frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \ldots & -\frac{1}{n} & \frac{1}{n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \ldots & \frac{1}{n} & \frac{1}{n} \\
\frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \ldots & \frac{1}{n} & \frac{1}{n} \\
\end{pmatrix} \]

Now the entries of the matrix \( A = (a_{ij}) = VAV^{-1} \) satisfy the following relations:

\[
\begin{align*}
    a_{11} &= \frac{1}{n}(\text{trace}(\Lambda)) \\
    a_{ii} &= \frac{1}{n}(1 + (n - 1)\lambda_i) \quad \text{for } i = 2, \ldots, n \\
    a_{i1} &= \frac{1}{n}(1 + \lambda_2 + \ldots + \lambda_{i-1} - (n - 1)\lambda_i + \lambda_{i+1} + \ldots + \lambda_n) \\
    a_{ij} &= \frac{1}{n}(1 - \lambda_j) \quad \text{for } j > 1 \text{ and } j \neq i
\end{align*}
\]

Note that \( a_{11} \) and \( a_{ij} = \frac{1}{n}(1 - \lambda_j) \) for \( j > 1 \) and \( j \neq i \) are nonnegative since the diagonal entries of \( \Lambda \) are in the decreasing order. In addition, for \( i \neq 1 \), the entries \( a_{11} \) are increasing so the smallest one is \( a_{21} \) and \( a_{ii} \) are decreasing so the smallest is \( a_{nn} \). Therefore the matrix \( A \) is nonnegative if and only if \( a_{21} \geq 0 \) and \( a_{nn} \geq 0 \). Finally \( A \) is doubly stochastic since \( V \) has the pattern \( S \), and the proof is complete.

It is should be noted here that the conditions of the above theorem differ than those of Theorem 3.1 as the point \((1, 1/2, 1/4)\) clearly satisfies the conditions of the above theorem but obviously does not satisfy the conditions of Theorem 3.1 for the case \( n = 3 \). In addition, the conditions of the above theorem are also sufficient for the existence of an \( n \times n \) symmetric doubly stochastic matrix \( D \) with the spectrum \((1, \lambda_2, \ldots, \lambda_n)\).

To see this, it suffices to look at the following result which is Corollary 7 in [12].

**Theorem 3.5.** ([12]) If \( \lambda_2, \ldots, \lambda_n \in [-1/(n-1), 1] \), then there exists an \( n \times n \) symmetric doubly stochastic matrix \( D \) with spectrum \((1, \lambda_2, \ldots, \lambda_n)\).

Here it is worth mentioning that at this stage and with extensive numerical computations, we are not able to find a list of \( n \) (\( \geq 4 \)) real numbers which shows the two problems (RDIEP) and (SDIEP) are different. In conclusion, the question whether (RDIEP) and (SDIEP) are equivalent or not, for \( n \geq 4 \) remains an open problem.

We conclude this section by using the version of Algorithm 1 mentioned in Remark 2.5. More explicitly, we choose a nonsymmetric doubly stochastic matrix with real eigenvalues such as the \( 3 \times 3 \) matrix \( B \) of Example 1. Using Maple, the eigenvectors of \( B \) are given by \((1, 1, 1)^T\), \((1, -2, 1)^T\) and \((1, 1, -2)^T\).

Now the corresponding pattern \( S \) matrix is given by \( P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \). Then its inverse is given by
\[ P^{-1} = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & -1/3 & 0 \\ 1/3 & 0 & -1/3 \end{pmatrix} \] and then \( PA_3P^{-1} = \begin{pmatrix} \frac{1}{1-2\lambda_2+2\lambda_3} & \frac{1-\lambda_2}{1+2\lambda_3} & \frac{1-\lambda_2}{1+2\lambda_3} \\ \frac{1+\lambda_2+2\lambda_3}{1+2\lambda_3} & \frac{1-2\lambda_2+2\lambda_3}{1+2\lambda_3} & \frac{1-\lambda_2}{1+2\lambda_3} \\ \frac{1+\lambda_2+2\lambda_3}{1+2\lambda_3} & \frac{1-\lambda_2}{1+2\lambda_3} & \frac{1+\lambda_2+2\lambda_3}{1+2\lambda_3} \end{pmatrix} \). Then the conditions for which \( PA_3P^{-1} \) is nonnegative is given by Theorem 3.4 for the case \( n = 3 \) with no surprise as \( P \) is a multiple of \( V^{-1} \) which is used in the proof of Theorem 3.4. Following Step 4 of Algorithm 1, we construct the matrix \( W_n \) which is given by:

\[
W_n = \begin{pmatrix}
\frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \ldots & \frac{1}{\sqrt{n-1}} & 1 & 1 \\
\frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \ldots & \frac{1}{\sqrt{n-1}} & -2 & 1 \\
\frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \ldots & \frac{1}{\sqrt{n-1}} & 1 & -2 \\
\frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \ldots & \frac{1}{\sqrt{n-1}} & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \ldots & \frac{1}{\sqrt{n-1}} & 0 & 0 \\
\frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \ldots & \frac{1}{\sqrt{n-1}} & 0 & 0 \\
\end{pmatrix}
\]

A virtually identical description as in the symmetric case shows that \( A = W_n AW_n^{-1} = \begin{pmatrix} X & C \\ GT & D \end{pmatrix} \) and \( B = V_n AV_n^{-1} = \begin{pmatrix} Y \\ GT \\ C \\ D \end{pmatrix} \) where \( X \) and \( Y \) are \( 3 \times 3 \) matrices. Thus a simple check shows that the conditions for which \( A = W_n AW_n^{-1} \) is nonnegative give the following theorem.

**Theorem 3.6.** Let \( 1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq -1 \), such that \( 1 + \lambda_2 + \ldots + \lambda_n \geq 0 \). If

\[
\begin{align*}
\frac{1}{n} + \frac{\lambda_2}{n(n-1)} + \frac{\lambda_3}{(n-1)(n-2)} + \frac{\lambda_4}{(n-2)(n-3)} + \ldots + \frac{\lambda_{n-1}}{n(n-2)} + \frac{\lambda_n}{(n-1)(n-3)} + \ldots + \frac{\lambda_{n-2}}{3(4)} + \frac{2\lambda_{n-1}}{3(4)} + \frac{2\lambda_n}{3} & \geq 0 \\
\frac{1}{n} + \frac{\lambda_2}{n(n-1)} + \frac{\lambda_3}{(n-1)(n-2)} + \frac{\lambda_4}{(n-2)(n-3)} + \ldots + \frac{\lambda_{n-1}}{n(n-2)} + \frac{\lambda_n}{(n-1)(n-3)} + \ldots + \frac{\lambda_{n-2}}{3(4)} + \frac{2\lambda_{n-1}}{3(4)} + \frac{2\lambda_n}{3} & \geq 0
\end{align*}
\]

then \((1, \lambda_2, \ldots, \lambda_n)\) is the spectrum of an \( n \times n \) nonsymmetric doubly stochastic matrix.

Next, we use the improvement process of Step 4 to obtain new results. This can be illustrated by the following example for the case \( n = 6 \).

**Example 3.** we apply the improvement process described in Step 4 of Algorithm 1 to obtain the matrix

\[
W_6 = \begin{pmatrix}
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & 0 & 1 & 1 \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & 0 & -2 & 1 \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 1 & 1 & 0 & 0 \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -2 & 1 & 0 & 0 \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 1 & -2 & 0 & 0 \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -2 & 0 & 0 & 0 \\
\end{pmatrix}
\]

Now an inspection shows that \( W_6 A_6 W_6^{-1} = \begin{pmatrix} B & C \\ C & D \end{pmatrix} \) where

\[
B = \begin{pmatrix}
\frac{1}{6} + \frac{\lambda_3}{6} + \frac{\lambda_4}{6} + \frac{\lambda_5}{6} & \frac{1}{6} + \frac{\lambda_3}{6} - \frac{\lambda_4}{6} - \frac{\lambda_5}{6} \\
\frac{1}{6} + \frac{\lambda_3}{6} - \frac{\lambda_4}{6} + \frac{\lambda_5}{6} & \frac{1}{6} + \frac{\lambda_3}{6} - \frac{\lambda_4}{6} - \frac{\lambda_5}{6} \\
\frac{1}{6} + \frac{\lambda_3}{6} + \frac{\lambda_4}{6} + \frac{\lambda_5}{6} & \frac{1}{6} + \frac{\lambda_3}{6} - \frac{\lambda_4}{6} - \frac{\lambda_5}{6} \\
\frac{1}{6} + \frac{\lambda_3}{6} - \frac{\lambda_4}{6} + \frac{\lambda_5}{6} & \frac{1}{6} + \frac{\lambda_3}{6} - \frac{\lambda_4}{6} - \frac{\lambda_5}{6} \\
\frac{1}{6} + \frac{\lambda_3}{6} + \frac{\lambda_4}{6} + \frac{\lambda_5}{6} & \frac{1}{6} + \frac{\lambda_3}{6} - \frac{\lambda_4}{6} - \frac{\lambda_5}{6} \\
\frac{1}{6} + \frac{\lambda_3}{6} - \frac{\lambda_4}{6} + \frac{\lambda_5}{6} & \frac{1}{6} + \frac{\lambda_3}{6} - \frac{\lambda_4}{6} - \frac{\lambda_5}{6} \\
\end{pmatrix}
\]

\[
C = \begin{pmatrix}
\frac{1}{6} + \frac{\lambda_3}{6} - \frac{\lambda_4}{6} - \frac{\lambda_5}{6} & \frac{1}{6} + \frac{\lambda_3}{6} + \frac{\lambda_4}{6} + \frac{\lambda_5}{6} \\
\frac{1}{6} + \frac{\lambda_3}{6} + \frac{\lambda_4}{6} + \frac{\lambda_5}{6} & \frac{1}{6} + \frac{\lambda_3}{6} - \frac{\lambda_4}{6} - \frac{\lambda_5}{6} \\
\frac{1}{6} + \frac{\lambda_3}{6} + \frac{\lambda_4}{6} + \frac{\lambda_5}{6} & \frac{1}{6} + \frac{\lambda_3}{6} - \frac{\lambda_4}{6} - \frac{\lambda_5}{6} \\
\frac{1}{6} + \frac{\lambda_3}{6} - \frac{\lambda_4}{6} + \frac{\lambda_5}{6} & \frac{1}{6} + \frac{\lambda_3}{6} - \frac{\lambda_4}{6} - \frac{\lambda_5}{6} \\
\frac{1}{6} + \frac{\lambda_3}{6} + \frac{\lambda_4}{6} + \frac{\lambda_5}{6} & \frac{1}{6} + \frac{\lambda_3}{6} - \frac{\lambda_4}{6} - \frac{\lambda_5}{6} \\
\frac{1}{6} + \frac{\lambda_3}{6} - \frac{\lambda_4}{6} + \frac{\lambda_5}{6} & \frac{1}{6} + \frac{\lambda_3}{6} - \frac{\lambda_4}{6} - \frac{\lambda_5}{6} \\
\end{pmatrix}
\]

11
and \( D = \begin{pmatrix} \frac{1}{3} + \frac{2a}{3} + \frac{2a}{3} & \frac{1}{3} - \frac{a}{3} - \frac{2a}{3} & \frac{1}{3} + \frac{2a}{3} - \frac{a}{3} \\ \frac{1}{3} + \frac{2a}{3} + \frac{2a}{3} & \frac{1}{3} - \frac{a}{3} - \frac{2a}{3} & \frac{1}{3} + \frac{2a}{3} - \frac{a}{3} \\ \frac{1}{3} + \frac{2a}{3} + \frac{2a}{3} & \frac{1}{3} - \frac{a}{3} - \frac{2a}{3} & \frac{1}{3} + \frac{2a}{3} - \frac{a}{3} \end{pmatrix} \). Note that for \( 1 \geq \lambda_2 \geq \ldots \geq \lambda_6 \geq -1 \), we have \( \frac{1}{9} + \frac{2a}{3} \geq 0 \). Thus the conditions for which the matrix \( W_{\beta} A_0 W_{\beta}^{-1} \) is nonnegative result in the following conclusion.

**Theorem 3.7.** Let \( 1 \geq \lambda_2 \geq \ldots \geq \lambda_6 \geq -1 \), such that \( 1 + \lambda_2 + \ldots + \lambda_6 \). If

\[
\begin{cases}
\frac{1}{3} + \frac{2a}{3} + \frac{2a}{3} \geq 0 \\
\frac{1}{3} + \frac{2a}{3} + \frac{2a}{3} \geq 0 \\
\frac{1}{3} + \frac{2a}{3} + \frac{2a}{3} \geq 0
\end{cases}
\]

then \((1, \lambda_2, \ldots, \lambda_6)\) is the spectrum of a \( 6 \times 6 \) nonsymmetric doubly stochastic matrix.

4. Constructing doubly stochastic matrices with complex spectrum

We start the section by again mentioning that only the case \( n = 3 \) is completely solved in [22] where the following theorem has been proved.

**Theorem 4.1.** [22] Let \( z \) be a complex number with nonzero imaginary part and \( \bar{z} \) be its complex conjugate. Then \((1, z, \bar{z})\) is the spectrum of a \( 3 \times 3 \) doubly stochastic matrix if and only if \( z \) is in the convex hull of the three cubic roots of unity.

To take advantage of Algorithm 1 for this case, we recall the famous Birkhoff’s theorem that states that \( \Delta_n \) is a convex polytope of dimension \((n - 1)^2\) where its vertices are the \( n \times n \) permutation matrices. Next, we describe how to manipulate Algorithm 1 to obtain solutions for (DIEP) with complex spectrum but with minor changes namely in Step 1 we start out with a nonsymmetirc vertex of \( \Delta_k \). In fact, we can begin Algorithm 1 with any \( k \times k \) doubly stochastic matrix \( X \) with complex spectrum but in this case the orthonormalization process in Step 1 should be dropped if the columns of \( X \) are not orthogonal. We illustrate this idea by considering the following example.

**Example 4.** The vertex \( P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \) of \( \Delta_3 \) has the following three eigenvectors \( \frac{1}{\sqrt{3}} e_3 \), \( x_2 = \frac{1}{\sqrt{3}} (1, w, w)^T \) and \( x_3 = \frac{1}{\sqrt{3}} (1, w^2, w^2)^T \) where \( w = -1/2 + \frac{1}{2} \sqrt{3} \) is the primitive cubic root of unity. Then the \( 3 \times 3 \) complex pattern \( \mathcal{S} \) matrix obtained from these eigenvectors is given by \( X = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & w & w^2 \\ 1 & w^2 & w \end{pmatrix} \) and its inverse is \( X^{-1} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & w & w^2 \\ 1 & w^2 & w \end{pmatrix} \). If we let \( \Pi_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a + Ib & 0 \\ 0 & 0 & a - Ib \end{pmatrix} \) with \( \text{trace}(\Pi_3) = 1 + 2a \geq 0 \) and \( |a + Ib| \leq 1 \), then a simple check shows that

\[
X \Pi_3 X^{-1} = \frac{1}{3} \begin{pmatrix} 1 + 2a & 1 - a - b \sqrt{3} & 1 - a + b \sqrt{3} \\ 1 - a + b \sqrt{3} & 1 + 2a & 1 - a - b \sqrt{3} \\ 1 - a - b \sqrt{3} & 1 - a + b \sqrt{3} & 1 + 2a \end{pmatrix}.
\]

Therefore the conditions for which \( X \Pi_3 X^{-1} \) is nonnegative are simply given by:

\[
\begin{cases}
1 - a + b \sqrt{3} \geq 0 \\
1 - a - b \sqrt{3} \geq 0
\end{cases}
\]
The next step of our algorithm is to form the following $n \times n$ complex pattern $S$ matrix

$$
W_n = \left( \begin{array}{cccc}
\frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \cdots & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \cdots & \frac{1}{\sqrt{2}} & \frac{w^2}{\sqrt{3}} & \frac{w}{\sqrt{3}} \\
\frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \cdots & \frac{1}{\sqrt{2}} & \frac{w^2}{\sqrt{3}} & \frac{w}{\sqrt{3}} \\
\frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \cdots & \frac{1}{\sqrt{2}} & \frac{w^2}{\sqrt{3}} & \frac{w}{\sqrt{3}} \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
\frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \cdots & \frac{1}{\sqrt{2}} & \frac{w^2}{\sqrt{3}} & \frac{w}{\sqrt{3}} \\
\frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \cdots & \frac{1}{\sqrt{2}} & \frac{w^2}{\sqrt{3}} & \frac{w}{\sqrt{3}} \\
\frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \cdots & \frac{1}{\sqrt{2}} & \frac{w^2}{\sqrt{3}} & \frac{w}{\sqrt{3}} \\
\frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \cdots & \frac{1}{\sqrt{2}} & \frac{w^2}{\sqrt{3}} & \frac{w}{\sqrt{3}} \\
\frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \cdots & \frac{1}{\sqrt{2}} & \frac{w^2}{\sqrt{3}} & \frac{w}{\sqrt{3}}
\end{array} \right)
$$

Now if we let $\Pi_n$ be the diagonal matrix $1 \oplus \lambda_2 \oplus \cdots \oplus \lambda_{n-2} \oplus a + Ib \oplus a - Ib$, such that $|a + Ib| \leq 1$ and with $1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-2} \geq -1$ and $\text{trace}(\Pi_n) = 1 + \lambda_2 + \cdots + \lambda_{n-2} + 2a \geq 0$. Then an inspection shows that

$$
W_n^T \Pi_n W_n^{-1} = \left( \begin{array}{ccc}
A & B \\
B^T & C
\end{array} \right)
$$

where $A$ is a $3 \times 3$ circulant matrix whose first row $(a_{11}, a_{12}, a_{13})$ is given by:

$$
\begin{align*}
a_{11} &= \frac{1}{n} + \frac{\lambda_2}{n(n-1)} + \frac{\lambda_3}{(n-1)(n-2)} + \frac{\lambda_4}{(n-2)(n-3)} + \cdots + \frac{\lambda_{n-2}}{(4)(3)} + \frac{2a}{3} + \frac{\lambda_{n-2}}{(4)(3)} - \frac{\lambda_{n-2}}{3} - \frac{\lambda_{n-2}}{3} - \frac{b}{3} \\
a_{12} &= \frac{1}{n} + \frac{\lambda_2}{n(n-1)} + \frac{\lambda_3}{(n-1)(n-2)} + \frac{\lambda_4}{(n-2)(n-3)} + \cdots + \frac{\lambda_{n-2}}{(4)(3)} + \frac{2a}{3} + \frac{\lambda_{n-2}}{(4)(3)} - \frac{\lambda_{n-2}}{3} - \frac{\lambda_{n-2}}{3} - \frac{b}{3} \\
a_{13} &= \frac{1}{n} + \frac{\lambda_2}{n(n-1)} + \frac{\lambda_3}{(n-1)(n-2)} + \frac{\lambda_4}{(n-2)(n-3)} + \cdots + \frac{\lambda_{n-2}}{(4)(3)} + \frac{2a}{3} + \frac{\lambda_{n-2}}{(4)(3)} - \frac{\lambda_{n-2}}{3} - \frac{\lambda_{n-2}}{3} - \frac{b}{3}
\end{align*}
$$

and $B$ is nonnegative and $C$ has nonnegative off-diagonal entries and its diagonal entries are increasing so that the smallest is

$$
c_{11} = \frac{1}{n} + \frac{\lambda_2}{n(n-1)} + \frac{\lambda_3}{(n-1)(n-2)} + \frac{\lambda_4}{(n-2)(n-3)} + \cdots + \frac{\lambda_{n-2}}{(5)(4)} + \frac{3\lambda_{n-2}}{4}.
$$

Thus the conditions for which $W_n^T \Pi_n W_n^{-1}$ is nonnegative gives the following theorem.

**Theorem 4.2.** Let $1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-2} \geq -1$, such that $1 + \lambda_2 + \cdots + \lambda_{n-2} + 2a \geq 0$ and $|a + Ib| \leq 1$. If

$$
\begin{align*}
\frac{1}{n} + \frac{\lambda_2}{n(n-1)} + \frac{\lambda_3}{(n-1)(n-2)} + \frac{\lambda_4}{(n-2)(n-3)} + \cdots + \frac{\lambda_{n-2}}{(4)(3)} + \frac{2a}{3} + \frac{\lambda_{n-2}}{(4)(3)} - \frac{\lambda_{n-2}}{3} - \frac{\lambda_{n-2}}{3} - \frac{b}{3} &\geq 0 \\
\frac{1}{n} + \frac{\lambda_2}{n(n-1)} + \frac{\lambda_3}{(n-1)(n-2)} + \frac{\lambda_4}{(n-2)(n-3)} + \cdots + \frac{\lambda_{n-2}}{(4)(3)} + \frac{2a}{3} + \frac{\lambda_{n-2}}{(4)(3)} - \frac{\lambda_{n-2}}{3} - \frac{\lambda_{n-2}}{3} - \frac{b}{3} &\geq 0 \\
\frac{1}{n} + \frac{\lambda_2}{n(n-1)} + \frac{\lambda_3}{(n-1)(n-2)} + \frac{\lambda_4}{(n-2)(n-3)} + \cdots + \frac{\lambda_{n-2}}{(4)(3)} + \frac{2a}{3} + \frac{\lambda_{n-2}}{(4)(3)} - \frac{\lambda_{n-2}}{3} - \frac{\lambda_{n-2}}{3} - \frac{b}{3} &\geq 0
\end{align*}
$$

then $(1, \lambda_2, \ldots, \lambda_{n-2}, a + Ib, a - Ib)$ is the spectrum of an $n \times n$ doubly stochastic matrix

Note that the above theorem deals with the case of at most two nonreal eigenvalues namely $a + Ib$ and $a - Ib$ in the list $(1, \lambda_2, \ldots, \lambda_{n-2}, a + Ib, a - Ib)$. Now, we can use the improvement process of Step4 to obtain conditions on a list with more than just two complex eigenvalues. This can be illustrated by the following example again for the case $n = 6$.

**Example 5.** we apply the improvement process described in Step4 of our algorithm to obtain the following matrix

$$
W_\beta = \left( \begin{array}{cccccc}
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}
\end{array} \right)
$$
If we let $\alpha$ be the diagonal matrix $1 \oplus f \oplus a + Ib \oplus a - Ib \oplus c + Id \oplus c - Id$, such that $|a + Ib| \leq 1$, $|c + Id| \leq 1$, $-1 \leq f \leq 1$ and $trace(\alpha) = 1 + f + 2a + 2c \geq 0$. Then an inspection shows that the matrix $W_\beta \alpha W^{-1}_\beta$ has the form $W_\beta \alpha W^{-1}_\beta = \begin{pmatrix} B & C \\ C & D \end{pmatrix}$ where

$$B = \begin{pmatrix} \frac{1}{6} + \frac{f}{6} + \frac{2a}{3} & \frac{1}{6} + f - \frac{a}{6} \pm \frac{d\sqrt{3}}{2} \\ \frac{1}{6} + f - \frac{a}{6} \pm \frac{d\sqrt{3}}{2} & \frac{1}{6} + f + \frac{a}{6} - \frac{d\sqrt{3}}{2} \end{pmatrix}$$

$$C = \begin{pmatrix} \frac{1}{6} - \frac{f}{6} - \frac{f}{6} - \frac{f}{6} - \frac{f}{6} - \frac{f}{6} \\ \frac{1}{6} - \frac{f}{6} - \frac{f}{6} - \frac{f}{6} - \frac{f}{6} - \frac{f}{6} \end{pmatrix}$$

and

$$D = \begin{pmatrix} \frac{1}{6} + \frac{f}{6} + \frac{2a}{3} & \frac{1}{6} + f - \frac{a}{6} - \frac{b\sqrt{3}}{3} \\ \frac{1}{6} + f - \frac{a}{6} - \frac{b\sqrt{3}}{3} & \frac{1}{6} + f + \frac{a}{6} + \frac{b\sqrt{3}}{3} \end{pmatrix}$$

Now the conditions for which the matrix $W_\beta \alpha W^{-1}_\beta$ is nonnegative result in the following conclusion.

**Theorem 4.3.** Let $1, f, a + Ib, a - Ib, c + Id, c - Id$ be complex numbers such that $|a + Ib| \leq 1$, $|c + Id| \leq 1$, $-1 \leq f \leq 1$ and $1 + f + 2a + 2c \geq 0$. If

$$\begin{cases} \frac{1}{6} + \frac{f}{6} + \frac{2a}{3} \geq 0 \\ \frac{1}{6} + f - \frac{a}{6} \pm \frac{d\sqrt{3}}{2} \geq 0 \\ \frac{1}{6} + f - \frac{a}{6} - \frac{b\sqrt{3}}{3} \geq 0 \\ \frac{1}{6} + f + \frac{a}{6} + \frac{b\sqrt{3}}{3} \geq 0 \\ \frac{1}{6} + f + \frac{a}{6} + \frac{b\sqrt{3}}{3} \geq 0 \end{cases}$$

then $(1, f, a + Ib, a - Ib, c + Id, c - Id)$ is the spectrum of a $6 \times 6$ doubly stochastic matrix.

Finally, it is worth mentioning here that even applying only the first two steps of our algorithm has an interest of its own since it has the advantage of yielding new conditions for the (DIEP) for the chosen dimension $k$ in Step1. To illustrate this, we include the following example.

**Example 6.** Consider the following $4 \times 4$ nonsymmetric permutation matrix $p = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. Using Maple for example, the complex eigenvectors of $p$ are given by: $x_1 = (1, 1, 1, 1)$, $x_2 = (1, -1, -1, 1)$, $x_3 = (-1, 1, -1, 1)$, and $x_4 = (1, 1, 1, -1)$. Then, from these eigenvectors, we form the complex pattern $S$ matrix

$$X = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix}$$

In addition, we let $\Pi_4$ be the diagonal matrix with diagonal entries as $1, c, a + Ib, a - Ib$, with $trace(\Pi_4) = 1 + c + 2a \geq 0$, $-1 \leq c \leq 1$ and $|a + Ib| \leq 1$. Next, we check the conditions for which $\Pi_4X^{-1}$ is nonnegative as follows. A simple matrix multiplication shows that

$$\Pi_4X^{-1} = \begin{pmatrix} \frac{1}{6} + \frac{f}{6} + \frac{2a}{3} & \frac{1}{6} + f - \frac{a}{6} \pm \frac{d\sqrt{3}}{2} & \frac{1}{6} + f - \frac{a}{6} - \frac{b\sqrt{3}}{3} & \frac{1}{6} + f + \frac{a}{6} + \frac{b\sqrt{3}}{3} \\ \frac{1}{6} + f - \frac{a}{6} \pm \frac{d\sqrt{3}}{2} & \frac{1}{6} + f + \frac{a}{6} - \frac{d\sqrt{3}}{2} & \frac{1}{6} + f + \frac{a}{6} + \frac{b\sqrt{3}}{3} & \frac{1}{6} + f - \frac{a}{6} - \frac{b\sqrt{3}}{3} \\ \frac{1}{6} + f - \frac{a}{6} - \frac{b\sqrt{3}}{3} & \frac{1}{6} + f + \frac{a}{6} - \frac{b\sqrt{3}}{3} & \frac{1}{6} + f + \frac{a}{6} - \frac{b\sqrt{3}}{3} & \frac{1}{6} + f + \frac{a}{6} - \frac{b\sqrt{3}}{3} \\ \frac{1}{6} + f + \frac{a}{6} + \frac{b\sqrt{3}}{3} & \frac{1}{6} + f - \frac{a}{6} + \frac{b\sqrt{3}}{3} & \frac{1}{6} + f - \frac{a}{6} + \frac{b\sqrt{3}}{3} & \frac{1}{6} + f - \frac{a}{6} + \frac{b\sqrt{3}}{3} \end{pmatrix}$$

Note that each diagonal entry of $\Pi_4X^{-1}$ is equal to $\frac{1}{6}trace(\Pi_4)$ and therefore is nonnegative. Hence the conditions for which $\Pi_4X^{-1}$ is nonnegative, are given by:

$$\begin{cases} \frac{1}{6} + f + \frac{2a}{3} \geq 0 \\ \frac{1}{6} + f - \frac{a}{6} \pm \frac{d\sqrt{3}}{2} \geq 0 \\ \frac{1}{6} + f - \frac{a}{6} - \frac{b\sqrt{3}}{3} \geq 0 \\ \frac{1}{6} + f + \frac{a}{6} \pm \frac{b\sqrt{3}}{3} \geq 0 \end{cases}$$

(17)
Thus we have the following theorem.

**Theorem 4.4.** Let $1, c, a+Ib, a-Ib$, be complex numbers with $1+c+2a \geq 0$, $-1 \leq c \leq 1$ and $|a+Ib| \leq 1$. If

$$
\begin{aligned}
1-c-2b &\geq 0 \\
1-c+2b &\geq 0 \\
1+c-2a &\geq 0
\end{aligned}
$$

then $(1, c, a+Ib, a-Ib)$ is the spectrum of a $4 \times 4$ doubly stochastic matrix $D$.

Note that if we wish to have the realizing matrix $D$ has zero trace, then it suffices to add the extra constraint $\text{trace}(\Theta) = 1 + c + 2a = 0$ and then we obtain the following.

**Theorem 4.5.** Let $1, c, a+Ib, a-Ib$, be complex numbers with $1+c+2a = 0$, $-1 \leq c \leq 1$ and $|a+Ib| \leq 1$. If

$$
\begin{aligned}
1-c-2b &\geq 0 \\
1-c+2b &\geq 0 \\
-1 &\leq a \leq 0
\end{aligned}
$$

then $(1, c, a+Ib, a-Ib)$ is the spectrum of a $4 \times 4$ doubly stochastic matrix $D$ with zero trace.

**Conclusion**

We described an algorithm that yields many sufficient conditions for three inverse eigenvalue problems concerning doubly stochastic matrices. Although at this stage we did not offer complete solutions, we leave it for future work to check if it is possible to characterize all the necessary points in Step1 needed for this algorithm to offer complete solutions at least for low dimensions. However, besides that it offers many new partial results, the main importance of this algorithm lies in the fact that it can be used as a checking point in case of a conjecture concerning complete solutions is given for any of these three interesting problems.

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