Dynamics over Signed Networks

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Abstract

A signed network is a network with each link associated with a positive or negative sign. Models for nodes interacting over such signed networks, where two types of interactions are defined along the positive and negative links, respectively, arise from various biological, social, political, and economical systems. As modifications to the conventional DeGroot dynamics for positive links, two basic types of negative interactions along negative links, namely the opposing rule and the repelling rule, have been proposed and extensively studied. This paper reviews some fundamental convergence results that appeared recently in the literature for dynamics over deterministic or random signed networks under a unified algebraic-graphical method. We show that a systematic tool of studying node state evolution over signed networks can be obtained utilizing generalized Perron-Frobenius theory, graph theory, and elementary algebraic recursions.

1 Introduction

In the past decades, the study of network dynamics has attracted tremendous research attentions from a variety of scientific disciplines [10]. Particularly, with its root traced back to 1960s on products of stochastic matrices [49], to 1970s on DeGroot social interactions [12], and to 1980s on distributed optimization [47], consensus algorithms serve as a primary model for social network dynamics as well as being a foundation for some prominent engineering applications of large-scale complex networks [25, 40, 36, 26, 20]. It has become a common understanding that cooperative node dynamics will lead to certain collective network behaviors.

On the other hand, in various biological, social, political, and economical systems, there are often two different, activating or inhibitive, trustful or mistrustful, cooperative or antagonistic, types of node interactions [15, 29, 1]. Using a positive or negative sign to denote the type of a link, the structure of these systems can be modeled as signed graphs. After specifying node dynamical relations along the positive or negative links, the evolution of node states defines signed network dynamics. Consensus algorithms with

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positive and negative links have been recently investigated [2, 17, 18, 19, 27, 31, 32, 42, 50, 23]. There exist two basic types of interactions along the negative links: the opposing negative dynamics [2] where nodes are attracted by the opposite values of the neighbors, and the repelling negative dynamics [17] where nodes tend to be repulsive of the relative position of the states with respect to the neighbors.

1.1 Signed Graphs

Consider a network with \( n \) nodes indexed in the set \( V = \{1, \ldots, n\} \). The structure of the network is represented as an undirected graph \( G = (V, E) \), where an edge (link) \( \{i, j\} \in E \) is an unordered pair of two distinct nodes in the set \( V \). Each edge in \( E \) is associated with a sign, positive or negative, defining \( G \) as a signed graph. The positive and negative edges are collected in the sets \( E^+ \) and \( E^- \), respectively. Then \( G^+ = (V, E^+) \) and \( G^- = (V, E^-) \) are respectively termed positive and negative subgraphs. Throughout the paper and without further specific mention we assume that \( G \) is connected and \( G^- \) contains at least one edge.

For a node \( i \in V \), its positive neighbors are the nodes that share a positive link with \( i \), forming the set \( N^+_i := \{ j : \{i, j\} \in E^+ \} \). Similarly the negative neighbor set of node \( i \) is denoted as \( N^-_i := \{ j : \{i, j\} \in E^- \} \). The set \( N_i = N^+_i \cup N^-_i \) then contains all nodes that interact with node \( i \) in the graph \( G \). We use \( \deg_i = |N_i| \) to denote the degree of node \( i \), i.e., the number of neighbors of node \( i \). Similarly, \( \deg^+_i = |N^+_i| \) and \( \deg^-_i = |N^-_i| \) represent the positive and negative degree of node \( i \), respectively.

1.2 Signed Laplacian

The Laplacian of the positive graph \( G^+ \) is defined as [14]

\[
L_{G^+} := D_{G^+} - A_{G^+}
\]

where \( A_{G^+} \) is the adjacency matrix of the graph \( G^+ \) with \( [A_{G^+}]_{ij} = 1 \) if \( \{i, j\} \in E^+ \) and \( [A_{G^+}]_{ij} = 0 \) otherwise, and \( D_{G^+} = \text{diag}(\deg^+_1, \ldots, \deg^+_n) \) is the degree matrix of the positive subgraph.

The standard definition of Laplacian can be generalized to signed graphs. Let \( D_{G^-} = \text{diag}(\deg^-_1, \ldots, \deg^-_n) \) be the degree matrix of the negative subgraph. Let \( A_{G^-} \) be the \textit{signed} adjacency matrix of the graph \( G^- \), where \( [A_{G^-}]_{ij} = -1 \) if \( \{i, j\} \in E^- \) and \( [A_{G^+}]_{ij} = 0 \) otherwise. Then the matrix

\[
L_{G^-} := D_{G^-} - A_{G^-}
\]

is defined as the signed Laplacian of the negative graph \( G^- \). The signed Laplacian of the signed graph \( G \) is then given by \( L_{G^+} + L_{G^-} \).

Particularly, we can also neglect the sign of edges in \( G^- \) and let \( A^*_{G^-} \) be the adjacency matrix of \( G^- \) with the signs being neglected, i.e., \( [A^*_{G^-}]_{ij} = 1 \) if \( \{i, j\} \in E^- \) and \( [A_{G^+}]_{ij} = 0 \). Then

\[
L^*_{G^-} := D_{G^-} - A^*_{G^-}
\]

is the Laplacian of \( G^- \) neglecting the sign of the links.
1.3 Structural Balance Theory

Introduced in the 1940s [22] and primarily motivated by social-interpersonal and economic networks, a fundamental notion in the study of signed graphs is the so-called structural balance. We recall the following definition (see [10] for a detailed introduction).

**Definition 1** A signed graph $G$ is structurally balanced if there is a partition of the node set into $V = V_1 \cup V_2$ with $V_1$ and $V_2$ being nonempty and mutually disjoint, where any edge between the two node subsets $V_1$ and $V_2$ is negative, and any edge within each $V_i$ is positive.

Known as the Harary’s balance theorem, a signed graph $G$ is structurally balanced if and only if there is no cycle with an odd number of negative edges in $G$ [9]. If $G$ is a complete graph, it turned out that we can verify its structural balance property by simply checking all triangles: $G$ is structurally balanced if and only if among every set of three nodes there are either one or three positive edges [10]. The notion of structural balance can be weakened in the following definition [11].

**Definition 2** A signed graph $G$ is weakly structurally balanced if there is a partition $V = V_1 \cup V_2 \cdots \cup V_m, m \geq 2$ with $V_1, \ldots, V_m$ being nonempty and mutually disjoint, where any edge between different $V_i$’s is negative, and any edge within each $V_i$ is positive.

![Examples of strongly balanced (left), weakly balanced (middle), and unbalanced signed graphs (right).](image)

Figure 1: Examples of strongly balanced (left), weakly balanced (middle), and unbalanced signed graphs (right). Here blue lines represent positive edges; red dashed lines represent negative edges.

It is known that $G$ is weakly structurally balanced if and only if no cycle has exactly one negative edge in $G$ [11]. When $G$ is a complete graph, this condition is equivalent to the fact that there is no set of three nodes among which there is exactly one negative edge [10]. In Figure 1 three basic examples are presented illustrating graph balance.

1.4 Positive/Negative Interactions

Time is slotted at $t = 0, 1, \ldots$. Each node $i$ holds a state $x_i(t) \in \mathbb{R}$ at time $t$ and interacts with its neighbors at each time to revise its state. The interactions rule is specified by the sign of the links. Let
\(\alpha, \beta \geq 0\). We first focus on a particular link \(\{i, j\} \in E\) and specify for the moment the dynamics along this link isolating all other interactions.

- If the sign of \(\{i, j\}\) is positive, each node \(s \in \{i, j\}\) updates its value by
  \[
  x_s(t+1) = x_s(t) + \alpha(x_s(t) - x_s(t)) = (1 - \alpha)x_s(t) + \alpha x_s(t),
  \]
  where \(-s \in \{i, j\} \setminus \{s\}\) with \(\alpha \in (0, 1)\).

- If the sign of \(\{i, j\}\) is negative, each node \(s \in \{i, j\}\) updates its value by either
  - Opposing Rule:
    \[
    x_s(t+1) = x_s(t) + \beta(-x_s(t) - x_s(t)) = (1 - \beta)x_s(t) - \beta x_s(t);
    \]
  - Repelling Rule:
    \[
    x_s(t+1) = x_s(t) - \beta(x_s(t) - x_s(t)) = (1 + \beta)x_s(t) - \beta x_s(t).
    \]

The positive interaction is consistent with DeGroot’s rule of social interactions, which indicates that the opinions of trustful social members are attractive to each other [12]. The opposing rule, introduced in [2], states that a node will be attracted by the opposite of its neighbor’s state if they share a negative link. The repelling rule, introduced in [17], on the other hand states that two nodes sharing a negative link take repulsive interactions rather than attraction. The two parameters \(\alpha\) and \(\beta\) indicate the sign of the underlying link and in the mean time describe the strength of positive and negative links, respectively.

### 1.5 Paper Organization

This paper reviews the existing results on fundamental convergence properties of signed dynamical networks [1, 2, 17, 18, 19, 27, 31, 32, 42, 50, 23, 3]. In the past few years, a variety of signed network models appears in the literature that falls to the categories of opposing and repelling rules defined above regarding the negative node interactions. Various treatments ranging from Lyapunov direct methods [2] to graph lifting [23] and even analysis based on complete observability theory [3] have been used to answer questions concerning with node state consensus or clustering in the asymptotic limit. We form a general signed network model by collecting the node interactions at individual links of the underlying deterministic or random graph. Then an algebraic-graphical method is provided serving as a system-theoretic tool for studying consensus dynamics over signed networks. Combining generalized Perron-Frobenius theory, graph theory, and elementary algebraic recursions, we show that this approach provides simple yet unified proofs to a series of basic convergence results for dynamics over signed networks with both deterministic and random node interactions.

The remainder of the paper is organized as follows. Section 2 presents a series of basic results for dynamics over deterministic networks. Section 3 extends the discussions to random networks with convergence
results established using similar algebraic-graphical analysis but with additional probabilistic ingredient. Finally Section 4 concludes the paper with a few concluding remarks in addition to some discussions on open problems and future directions.

1.6 Notation

Real numbers are in general denoted by lowercase letters \(x, y, a, b, c,\ldots\) and lowercase Greek letters \(\alpha, \beta, \gamma,\ldots\). All vectors are column vectors denoted by bold lowercase letters \(\mathbf{x}, \mathbf{y}, \ldots\). Matrices are denoted with upper case letters such as \(A, B, C,\ldots\). All matrices are real. Given a matrix \(A\), \(A^T\) denotes its transpose and \(A^k\) denotes the \(k\)-th power of \(A\) when it is a square matrix. Likewise the transpose of a vector \(\mathbf{x}\) is denoted by \(\mathbf{x}^T\). The \(ij\)-entry of a matrix \(A\) is denoted by \([A]_{ij}\); the spectrum and spectral radius of a matrix \(A\) is denoted by \(\sigma(A)\) and \(\rho(A)\), respectively; the largest eigenvalue of a symmetric matrix \(A\) is denoted by \(\lambda_{\text{max}}(A)\). The \(n\)-dimensional all-one vector is denoted by \(\mathbf{1}\), and the \(n\)-dimensional unit vector with the \(i\)'th entry being one is \(\mathbf{e}_i\). The node set is always \(V = \{1, \ldots, n\}\), over which a deterministic graph is denoted as \(G\) and a random graph is denoted as \(G\). We use \(W\) to denote a random matrix. Depending on the argument, \(|\cdot|\) stands for the absolute value of a real number or the cardinality of a set. The Euclidean norm of a vector is \(\|\cdot\|\).

2 Deterministic Networks

In this section, we investigate the evolution of the node states with deterministic interactions. The pairwise interactions among the signed links are collected over a deterministic network. We are interested in characterizing the asymptotic limits of the node states and provide some basic convergence theorems. Relevant results in the literature can be seen for instance in \([2, 27, 32, 50, 23]\).

2.1 Fundamental Convergence Results

2.1.1 Opposing Negative Dynamics

With the opposing rule (2) along with the negative links, the update of \(x_i(t)\) reads as

\[
x_i(t+1) = x_i(t) + \alpha \sum_{j \in N_i^+} \left( x_j(t) - x_i(t) \right) - \beta \sum_{j \in N_i^-} \left( x_j(t) + x_i(t) \right)
\]

\[
= \left(1 - \alpha \deg_i^+ - \beta \deg_i^-\right)x_i(t) + \alpha \sum_{j \in N_i^+} x_j(t) - \beta \sum_{j \in N_i^-} x_j(t).
\]

Denote \(\mathbf{x}(t) = (x_1(t)\ldots x_n(t))^T\). We can now rewrite (4) into the following compact form:

\[
\mathbf{x}(t+1) = W_G \mathbf{x}(t) = (I - \alpha L_{G^+} - \beta L_{G^-})\mathbf{x}(t)
\]

where \(L_{G^+}\) and \(L_{G^-}\) are the signed Laplacian of the positive and negative graphs \(G^+\) and \(G^-\), respectively.
Recall that a real matrix (or vector) is called positive (non-negative) if all its entries are positive (non-negative); a stochastic matrix is a nonnegative matrix with row sum equal to one [24]. A key property of the matrix $W_G$ lies in
\[
\sum_{j=1}^{n} |W_G|_{ij} = 1, \ i \in V
\]
which indicates that $W_G$ will become a stochastic matrix if all its entries are put into their absolute values. The following result holds.

**Theorem 1** Assume that $0 < \alpha + \beta < 1/\max_{i \in V} \deg_i$. Then under the opposing rule (2), the following statements hold for any initial value $x(0)$.

(i) If $G$ is structurally balanced subject to partition $V = V_1 \cup V_2$, then $\lim_{t \to \infty} x_i(t) = (\sum_{j \in V_1} x_j(0) - \sum_{j \in V_2} x_j(0))/n, \ i \in V_1$, and $\lim_{t \to \infty} x_i(t) = -((\sum_{j \in V_1} x_j(0) - \sum_{j \in V_2} x_j(0))/n, \ i \in V_2$.

(ii) If $G$ is not structurally balanced, then $\lim_{t \to \infty} x_i(t) = 0, \ i \in V$.

**Proof.** (i) Let $G$ be structurally balanced with partition $V = V_1 \cup V_2$. Consider a gauge transformation given by
\[
z_i(t) = x_i(t), \ i \in V_1; \ z_i(t) = -x_i(t), \ i \in V_2.
\]
The evolution of the $z_i(t)$ follows standard consensus algorithm and the result holds from Theorem 2 in [36].

(ii) Let $0 < \alpha + \beta < 1/\deg_i$ for all $i$. Applying Geršgorin’s Circle Theorem (see, e.g., Theorem 6.1.1 in [24]), it is easy to see that $-1 < \lambda_i(W_G) \leq 1$ for all $\lambda_i \in \sigma(W_G)$. This immediately implies that for any initial value $x(0)$, there exists $y(x(0)) = (y_1(x(0)) \ldots y_n(x(0)))^T$ satisfying $W_G y = y$ such that $\lim_{t \to \infty} x_i(t) = y_i$.

Claim. $|y_1| = \ldots = |y_n|$ for any $x(0)$.

Suppose there are two distinct nodes $i$ and $j$ with $|y_i| \neq |y_j|$. The fact that $W_G y = y$ gives
\[
|y_i| \leq \sum_{j=1}^{n} |W_G|_{ij} \cdot |y_j|, \ i \in V.
\]
This is impossible for a connected graph $G$ noting the fact that $\sum_{j=1}^{n} |W_G|_{ij} = 1, \ i \in V$. This proves the above claim.

Now let $y_* = |y_1| = \ldots = |y_n| \neq 0$ for some $x(0)$. There must be a set $V_*$ (which, of course, may be an empty set) with
\[
y_i = y_*, \ i \in V_*; \ y_i = -y_*, \ i \in V \setminus V_*.
\]
It is straightforward to verify that in order for $W_G y = y$ to hold, all links (if any) in either $V_*$ or $V \setminus V_*$ must be positive, and the links (if any) between $V_*$ and $V \setminus V_*$ must be negative. This is to say, $G$ must be structurally balanced since by our standing assumption $G^{-}$ is nonempty.
We have now completed the proof. □

We remark that the condition \(0 < \alpha + \beta < 1/\max_{i \in V} \deg_i\) in Theorem 1 can be certainly relaxed, e.g., a straightforward one would be \(0 < \alpha \deg_i^+ + \beta \deg_i^- < 1\) for all \(i\). Further relaxations can be obtained making use of the structure of \(L_{G+}^*\) and \(L_{G-}^*\), and the fact that the spectrum of \(W_G\) will be restricted within the unit cycle for sufficiently small \(\alpha\) and \(\beta\).

The essential message of Theorem 1 is that structural balance of \(G\) determines whether one is within the spectrum of \(W_G\). In fact, there holds

\[
\|x(t + 1)\|^2 \leq \lambda_{\max}(W_G^2)\|x(t)\|^2 \leq \|x(t)\|^2
\]

with sufficiently small \(\alpha\) and \(\beta\) guaranteeing \(\lambda_{\max}(W_G^2) \leq 1\). Therefore, the algorithm (4) defines an overall contraction mapping, consistent with the standard consensus algorithms without negative links.

### 2.1.2 Repelling Negative Dynamics

Now consider the repelling rule (3) for negative links. The update of \(x_i(t)\) reads as

\[
x_i(t + 1) = x_i(t) + \alpha \sum_{j \in \mathbb{N}_i^+} (x_j(t) - x_i(t)) - \beta \sum_{j \in \mathbb{N}_i^-} (x_j(t) - x_i(t))
\]

\[
= \left(1 - \alpha \deg_i^+ + \beta \deg_i^-\right)x_i(t) + \alpha \sum_{j \in \mathbb{N}_i^+} x_j(t) - \beta \sum_{j \in \mathbb{N}_i^-} x_j(t).
\]

Recall that the Laplacian of the matrix \(G^-\) neglecting the sign of the links is defined as \(L_{G-}^*\). The algorithm (9) is written into

\[
x(t + 1) = M_G x(t) = (I - \alpha L_{G+}^* + \beta L_{G-}^*) x(t).
\]

From (10), \(M_G 1 = 1\) always holds. We present the following result.

**Theorem 2** Suppose \(G^+\) is connected. Then for any \(0 < \alpha < 1/\max_{i \in V} \deg_i^+\), there exists a critical value \(\beta_s > 0\) for \(\beta\) such that

(i) If \(\beta < \beta_s\), then average consensus is reached along (3) in the sense that \(\lim_{t \to \infty} x_i(t) = \sum_{j=1}^{n} x_i(0)/n\)

for all initial value \(x(0)\);

(ii) If \(\beta > \beta_s\), then \(\lim_{t \to \infty} \|x(t)\| = \infty\) for almost all initial values w.r.t. Lebesgue measure.

**Proof.** Since \(0 < \alpha < 1/\max_{i \in V} \deg_i^+\), all eigenvalues of \(I - \alpha L_{G+}^*\) are nonnegative again invoking the Geršgorin’s Circle Theorem. This in turn leads to the following fact noticing that \(L_{G-}^*\) is positive semi-definite: \(I - \alpha L_{G+}^* + \beta L_{G-}^*\) is positive semi-definite. Define \(J = 11^T/n\) and consider

\[
f(\beta) := \lambda_{\max}\left(I - \alpha L_{G+}^* + \beta L_{G-}^* - J\right).
\]

The Courant-Fischer Theorem (see Theorem 4.2.11 in [24]) implies \(f(\cdot)\) is a continuous and non-decreasing function over \([0, \infty)\). Now that \(G^+\) is connected, we have \(f(0) < 1\) since the second smallest eigenvalue of \(L_{G+}^*\) is positive. Apparently \(f(\infty) > 1\). Therefore, there exists a critical value \(\beta_s > 0\) satisfying \(f(\beta_s) = 1\) such that...
• There holds \( f(\beta) < 1 \) if \( \beta < \beta_* \). In this case, along (10) \( x(t) \) converges to the eigenspace corresponding to the eigenvalue one of \( M_G \), which leads to the average consensus statement in (i).

• There holds \( f(\beta) > 1 \) if \( \beta > \beta_* \). In this case, along (10) \( x(t) \) diverges as long as the initial value \( x(0) \) has a nonzero projection onto the eigenspace corresponding to \( \lambda_{\text{max}}(M_G) \) of \( M_G \). This leads to the almost everywhere divergence statement in (ii).

The proof is now complete. \( \square \)

The condition that \( G^+ \) is a connected graph is crucial for Theorem 2. Once \( G^+ \) becomes disconnected, it is easy to see that one single negative link and an arbitrarily small \( \beta > 0 \) will drive the network state to diverge for almost all initial values.

2.2 Mathematical Reasoning: Eventually Positive Matrices

Theorems 1 and 2 provide elementary but informative characterizations to how negative links influence the network dynamics in the two models:

• With opposing rule, both the positive and negative links contribute to state convergence of the nodes. The overall dynamics has a contraction nature with small \( \alpha \) and \( \beta \). As long as the overall graph \( G \) is connected, the absolute values of node states asymptotically agree; structural balance of the graph merely further determines the existence of nontrivial absolute value agreement in the sense that a bipartite consensus is achieved, i.e., two subgroups of nodes reach consensus, respectively in each of the group, at opposite values.

• With repelling negative dynamics, the negative links produce repulsive interactions with a divergence nature. These negative links are therefore essentially perturbations: the positive links must generate convergence with sufficient speed so that the negative links can be overcome. This requires that the positive graph \( G^+ \) must be connected by itself and results in the critical value of \( \beta \) below which convergence to consensus still holds.

It has been well known that convergence of standard consensus algorithms is closely related to the Perron–Fronenius Theory [36]. Consider a (unsigned) graph \( G \) with (unsigned) Laplacian \( L_G \). A standard consensus algorithm over the graph \( G \), is defined as

\[
x_i(t+1) = x_i(t) + \alpha \sum_{j \in N_i} (x_j(t) - x_i(t)), \quad i \in V
\]

or in vector form,

\[
x(t+1) = S_G x(t),
\]

where \( S_G = I - \alpha L_G \). It turned out that \( S_G \) is a non-negative matrix for \( \alpha < 1/\max_{i \in V} \deg_i \). The Perron–Fronenius Theory is the fundamental reasoning behind the convergence of the algorithm (11) [36]: if and only if \( G \) is connected, there holds

\[
\lim_{t \to \infty} P_G^t = 11^T/n.
\]
In fact, \( \mathbf{1}^T \) and \( \mathbf{1} \) are the left and right eigenvector corresponding to eigenvalue 1 of \( P_G \), known as its Perron–Frobenius eigenvalue.

A matrix \( A \) is called eventually positive if there exists an integer \( k_0 \in \mathbb{N}^+ \) such that \( A^k \) is positive for all \( k \geq k_0 \). If \( G \) is structurally balanced subject to node set partition \( V_1 \) and \( V_2 \), it is easy to see that \( KW_GK^{-1} \) defines a nonnegative stochastic matrix, which is eventually positive if \( 0 < \alpha + \beta < 1 / \max_{i \in V} \deg_i \), where \( K = \text{diag}(k_1, \ldots, k_n) \) with \( k_i = 1, i \in V_1 \) and \( k_i = -1, i \in V_2 \). On the other hand, the matrix \( M_G \) for repelling rule would contain negative values. Letting \( \beta_* \) be the critical value established in Theorem 2

**Proposition 1** Let \( G^+ \) be connected. Then \( M_G \) is eventually positive if \( 0 < \alpha < 1 / \max_{i \in V} \deg_i \) and \( \beta < \beta_* \).

**Proof.** Note that (see Theorem 2.2 in [35]), a matrix \( A \in \mathbb{R}^{n \times n} \) is eventually positive if and only if both \( A \) and \( A^T \) have the strong Perron–Frobenius property: (i) \( \rho(A) \) is a simple positive eigenvalue of \( A \); (ii) the right eigenvector related to \( \rho(A) \) is positive. The statement is immediate by verifying that \( M_G \) has the Perron–Frobenius property under the given conditions, respectively, from the proof of Theorem 2. \( \square \)

### 2.3 Directed Graphs

Directional links in a network can also be associated with signs [48]. We now present generalizations of the previous model and results to signed directed networks. For the ease of presentation, we keep the previous notation and simply adapt them to the directed graph case. Their usage is of course restricted to the current subsection.

Now let the graph \( G = (V, E) \) be a directed graph (digraph), where a link \((i, j) \in E\) is directed starting from \( i \) and pointing to \( j \). A diagraph is termed a signed digraph if each of its links has a positive or negative sign. By revising the definition of positive and negative neighbor sets of node \( i \) to

\[
N_i^+ := \{ j : (j, i) \in E^+ \}; \quad N_i^- := \{ j : (j, i) \in E^- \},
\]

the network dynamics [4] and [9] are then readily defined for the digraph \( G \). The set \( N_i = N_i^+ \cup N_i^- \) continues to represent the overall neighbor set of node \( i \). In this directed graph case we continue to define \( \deg_i^+ = \deg_i^+, \ \deg_i^- = \deg_i^- \), and \( \deg_i = |N_i| \) as the positive, negative, and overall degrees of node \( i \).

The concept of structural balance can be generalized to digraphs by replacing the undirected edges with directional links.

**Definition 3** A signed digraph \( G \) is structurally balanced if there is a partition of the node set into \( V = V_1 \cup V_2 \) with \( V_1 \) and \( V_2 \) being nonempty and disjoint, such that any directional link between \( V_1 \) and \( V_2 \) is negative, and any link whose two end nodes belong to the same \( V_i \) is positive.

The following theorem corresponds to Theorem 1 for signed digraphs.
Consider network dynamics \([4]\) over a digraph \(G\). Assume that \(0 < \alpha + \beta < 1 / \max_{i \in V} \deg_i\). Suppose \(G\) is strongly connected. The following holds for any initial value \(x(0)\).

(i) If \(G\) is structurally balanced subject to partition \(V = V_1 \cup V_2\), then there are \(n\) positive numbers \(w_1, \ldots, w_n\) with \(\sum_{i=1}^{n} w_i = 1\) such that \(\lim_{t \to \infty} x_i(t) = \left( \sum_{j \in V_1} w_j x_j(0) - \sum_{j \in V_2} w_j x_j(0) \right) / n, i \in V_1\) and \(\lim_{t \to \infty} x_i(t) = -\left( \sum_{j \in V_1} w_j x_j(0) - \sum_{j \in V_2} w_j x_j(0) \right) / n, i \in V_2\).

(ii) If \(G\) is not structurally balanced, then \(\lim_{t \to \infty} x_i(t) = 0, i \in V\).

For a digraph \(G\), the adjacency matrix \(A_{G^+}\) of \(G^+\) is given by \([A_{G^+}]_{ij} = 1\) if \((j, i) \in E^+\) and \([A_{G^+}]_{ij} = 0\) otherwise; the signed adjacency matrix \(A_{G^-}\) of \(G^-\) is given by \([A_{G^-}]_{ij} = -1\) if \((j, i) \in E^-\) and \([A_{G^+}]_{ij} = 0\) otherwise. Then \(L_{G^+} := D_{G^+} - A_{G^+}\) and \(L_{G^-} := D_{G^-} - A_{G^-}\) are the signed Laplacian of the directed positive and negative graphs, respectively. The dynamics \(\text{[4]}\) can still be written as into the form of \(\text{[5]}\) with \(W_G = I - \alpha L_{G^+} - \beta L_{G^-}\). Then the \((w_1 \ldots w_n)\) in Theorem 3 is the left eigenvector related to eigenvalue 1 of the matrix \(KW_G\), which therefore depends on \(\alpha\) and \(\beta\).

Again Gershgorin’s Circle Theorem leads to the fact that \(\rho(W_G) \leq 1\). However, the matrix \(W_G\) with a directed graph \(G\) is no longer necessarily symmetric, and therefore it may contain eigenvalues with magnitude 1 with multiplicity even greater than one. We cannot immediately conclude from \(\rho(W_G) \leq 1\) state-convergence of the nodes as shown in the proof of Theorem \(\text{[1]}\) for undirected graphs. We can however bypass this obstacle by imposing a contradiction argument. The proof of Theorem 3 is based on algebraic-graphical analysis, which has been put in the Appendix.

Likewise, the following theorem corresponds to Theorem \(\text{[2]}\) for signed digraphs.

Consider network dynamics \(\text{[7]}\) over a digraph \(G\). Suppose \(G^+\) is strongly connected and fix \(0 < \alpha < 1 / \max_{i \in V} \deg_i^+\). There exists \(\beta_* > 0\) such that for any \(\beta < \beta_*\), there are \(q_1(\beta), \ldots, q_n(\beta) \in \mathbb{R}^+\) with \(\sum_{i=1}^{n} q_i = 1\) satisfying that a consensus is reached at

\[
\lim_{t \to \infty} x_i(t) = \sum_{j=1}^{n} q_j x_j(0), \ i \in V
\]

for all initial value \(x(0)\).

With a digraph \(G\), the standard adjacency matrix \(A_{G^-}^*\) of \(G^-\) is defined by \([A_{G^-}^*]_{ij} = 1\) if \((j, i) \in E^-\) and \([A_{G^-}^*]_{ij} = 0\) otherwise. Then \(L_{G^-}^* = D_{G^-} - A_{G^-}^*\) is the standard Laplacian of \(G^-\). The network dynamics \(\text{[9]}\) can be again represented by \(\text{[10]}\) with \(M_G := I - \alpha L_{G^+} + \beta L_{G^-}^*\). With \(G\) being directed, \(M_G\) is not necessarily symmetric, \(M_G \mathbf{1} = \mathbf{1}\) however continues to hold.

In the statement of Theorem \(\text{[1]}\) for any \(\beta < \beta_*\), \((q_1(\beta) \ldots q_n(\beta))\) is a left eigenvector related to eigenvalue 1 of \(M_G\). The proof of Theorem \(\text{[1]}\) has been put in the Appendix. It is worth emphasizing that the \(\beta_*\) in Theorem \(\text{[1]}\) is merely an upper bound for \(\beta\) under which the network can still reach a consensus in the presence of the negative links, and it is unclear whether such \(\beta_*\) would remain a critical value as the undirected case. The actual value of \(\beta_*\) can be estimated using standard matrix perturbation theory \(\text{[13]}\).
2.4 Weighted Signs, Continuous-time Dynamics, Switching Structures

More sophisticated signed networks can certainly be studied using similar tools and analysis. This sub-section presents a coverage to related results in the literature.

2.4.1 Weighted Signs

The strength of positive and negative links, represented by $\alpha$ and $\beta$, can also be link dependent. This means that for the positive and negative dynamics (1), (2), and (3) along the edge $\{i, j\}$, $\alpha$ and $\beta$ will be replaced by $\alpha_{ij}$ and $\beta_{ij}$, respectively. The results of Theorems 1–4 can be extended to networks with weighted signs straightforwardly [2].

2.4.2 Continuous-time Dynamics

The signed network dynamics considered above clearly have their continuous-time counter part. For the opposing negative dynamics (5), the corresponding node state evolution in continuous time reads as

$$\frac{d}{dt}x(t) = -\left(\alpha L_{G^+} + \beta L_{G^-}\right)x(t).$$

(13)

On the other hand, the continuous-time counter part of the dynamics (10) is

$$\frac{d}{dt}x(t) = -\left(\alpha L_{G^+} - \beta L_{G^-}^*\right)x(t).$$

(14)

Evidently, the asymptotic behavior of (13) and (14) is fully determined by the spectrum of $\alpha L_{G^+} + \beta L_{G^-}$ and $\alpha L_{G^+} - \beta L_{G^-}^*$. They are in fact shifts of the spectrum of $W_G$ and $M_G$, respectively. With continuous-time dynamics, we no longer need to worry about that certain eigenvalues of $\alpha L_{G^+} + \beta L_{G^-}$ and $\alpha L_{G^+} - \beta L_{G^-}^*$ can be outside the unit cycle for large $\alpha$ and $\beta$. Consequently, Theorems 1 and 2 immediately translate to the following statements.

Proposition 2 (i) Along the continuous-time evolution (13), the following hold for any initial value $x(0)$:

- If $G$ is structurally balanced subject to partition $V = V_1 \cup V_2$, then $\lim_{t \to \infty} x_i(t) = \left(\sum_{j \in V_1} x_j(0) - \sum_{j \in V_2} x_j(0)\right)/n$, $i \in V_1$ and $\lim_{t \to \infty} x_i(t) = -\left(\sum_{j \in V_1} x_j(0) - \sum_{j \in V_2} x_j(0)\right)/n$, $i \in V_2$.

- If $G$ is not structurally balanced, then $\lim_{t \to \infty} x_i(t) = 0$, $i \in V$.

(ii) Consider (14) and suppose $G^+$ is connected. Then for any $\alpha > 0$, there exists a critical value $\beta_* > 0$ for $\beta$ such that

- If $\beta < \beta_*$, then an average consensus is reached, i.e., $\lim_{t \to \infty} x_i(t) = \sum_{j=1}^n x_i(0)/n$ for all initial value $x(0)$.

- If $\beta > \beta_*$, then $\lim_{t \to \infty} \|x(t)\| = \infty$ for almost all initial values w.r.t. Lebesgue measure.
The results for opposing negative dynamics can even be extended to nonlinear node interactions \[1, 31\]. As illustrated in (8), under the opposing negative dynamics, both positive and negative links lead to non-expansive network state evolution\[1\]. The mathematical reasoning behind those non-linear generalizations is due to the fact that the non-expansive property can be preserved for suitable nonlinear interaction rules.

### 2.4.3 Switching Network Structures

In the study of standard consensus algorithms, one particular interest was to establish convergence conditions under time-varying network structures \[25, 6, 40, 33\], for which earlier work was dated to 1960s \[49\]. Such analysis can be challenging due to the lacking of a common convergence metric that works for all possible choices of the interaction graphs. Nevertheless, possibilities of generalizing the analysis of time-varying network structures have been shown in the literature \[2, 38, 31, 50, 27, 3\].

Let \( G_t = (V, E_t), t = 0, 1, \ldots \) be a sequence of graphs with each \( G_t \) being a (directed or undirected) signed graph. Then the positive and negative neighbor sets of node \( i \), are determined by connections in \( G_t \) and therefore become time-dependent, denoted \( N^+_i(t) \) and \( N^-_i(t) \), respectively. The network dynamics under the opposing rule (2) are then represented by

\[
x_i(t + 1) = x_i(t) + \alpha \sum_{j \in N^+_i(t)} (x_j(t) - x_i(t)) - \beta \sum_{j \in N^-_i(t)} (x_j(t) + x_i(t)).
\]  

(15)

The following result holds.

**Proposition 3** Suppose there exists a constant \( 0 < \delta < 1 \) such that \( \alpha |N^+_i(t)| + \beta |N^-_i(t)| \leq 1 - \delta \) for all \( i \in V \) and all \( t \geq 0 \).

(i) Let there exist \( T \geq 0 \) such that the graph \( G_{[s, s+T]} := (V, \bigcup_{t=s}^{s+T} E_t) \) is strongly connected for all \( s \geq 0 \). Then along (15), for any initial value \( x(0) \), there exists \( y_s(x(0)) \geq 0 \) such that \( \lim_{t \to \infty} |x_i(t)| = y_s(x(0)) \) for all \( i \in V \).

(ii) Suppose \( G_t \) is undirected for all \( t \geq 0 \). Let the graph \( G_{[s, \infty]} := (V, \bigcup_{t=s}^{\infty} E_t) \) be connected for all \( s \geq 0 \). Then along (15), for any initial value \( x(0) \), there exists \( y_s(x(0)) \geq 0 \) such that \( \lim_{t \to \infty} |x_i(t)| = y_s(x(0)) \) for all \( i \in V \).

**Proof.** The desired conclusions follow from Theorem 2.1 and Theorem 2.2 in \[32\], where the positive and negative weights are link dependent.

The structural balance condition can be generalized to the sequence of graphs \( G_t = (V, E_t) \), under which bipartite consensus result can be similarly established for opposing negative dynamics \[38, 50, 27\]. On the other hand, for repelling negative dynamics, analysis for switching network structures can be extremely challenging since the network state is no longer non-expansive in the presence of one single negative link.

It turned out that in order to preserve convergence to consensus, it is important that at each time step, \[\text{max}_{i \in V} |x_i(t+1)| \leq \text{max}_{i \in V} |x_i(t)|\] as shown in the proof of Theorem 3. Therefore, the network state evolution continues to be non-expansive.

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\[1\] With directed graphs, the statement in general no longer holds under the opposing negative dynamics. However, there still holds that \( \text{max}_{i \in V} |x_i(t+1)| \leq \text{max}_{i \in V} |x_i(t)| \) as shown in the proof of Theorem 3.
the influence of the negative links can be overcome by the positive links. We refer to [3] for such treatment under continuous-time node dynamics.

3 Random Networks

Node interactions happen randomly in many real-world networks, and how consensus can be reached over a random node interaction process have been extensively studied [21, 8, 16, 45, 46, 26, 41]. We proceed to discuss network dynamics over signed random graph processes, where relevant results appeared in [17, 18, 19, 27, 42].

We use the following gossiping model [8] to describe the random node interactions. The undirected, signed graph, \( G = (V, E) \), continues to define the world of the network where interactions take place. Each node initiates interactions at the instants of a rate-one Poisson process, and at each of these instants, picks a node at random to interact with. Under this model, at a given time, at most one node initiates an interaction. This allows us to order interaction events in time and to focus on modeling the node pair selection at interaction times. The node pair selection is then performed as follows.

**Definition 4** Independently at each interaction event \( t \geq 0 \), (i) a node \( i \in V \) is drawn uniformly at random, i.e., with probability \( \frac{1}{n} \); (ii) node \( i \) picks a neighbor \( j \) uniformly with probability \( \frac{1}{\deg i} \) for \( j \in N_i \). In this case, we say that the unordered node pair \( \{i, j\} \) is selected.

The node pair selection process is assumed to be identically distributed and independent over \( t \geq 0 \). Let \((E, \mathcal{F}, \mu)\) be the probability space, where \( \mathcal{F} \) is the discrete \( \sigma \)-algebra on \( E \), and \( \mu \) is the probability measure defined by \( \mu(\{i, j\}) = (1/\deg i + 1/\deg j)/n \) for all \( \{i, j\} \in E \). The node selection process can then be seen as a random event in the product probability space \((\Omega, \mathcal{F}, \mathbb{P})\), where \( \Omega = E^{\mathbb{N}} = \{\omega = (\omega_0, \omega_1, \ldots) : \forall t, \omega_t \in E\}, \mathcal{F} = \mathcal{F}^{\mathbb{N}}, \) and \( \mathbb{P} \) is the product probability measure (uniquely) defined by: for any finite subset \( K \subset \mathbb{N} \), \( \mathbb{P}((\omega_t)_{t \in K}) = \prod_{t \in K} \mu(\omega_t) \) for any \( (\omega_t)_{t \in K} \in E^{|K|} \). For any \( t \in \mathbb{N} \), we define the coordinate mapping \( G_t : \Omega \to E \) by \( G_t(\omega) = \omega_t \), for all \( \omega \in \Omega \). Then formally \( G_t, t = 0, 1, \ldots \) describe the node pair selection process. We denote \( \mathcal{F}_t = \sigma(G_0, \ldots, G_t) \) as the \( \sigma \)-algebra capturing the \( t + 1 \) first interactions of the selection process.

After the pair of nodes \( \{i, j\} \) have been selected at time \( t \), they update their states \( x_i(t) \) and \( x_j(t) \) according to the sign of the link that they share: if the link is positive, they update their states by (1); if the link is negative, they update their states by either (2) or (3). The nodes that are not selected at time \( t \) will keep their states unchanged. In this way, \( x(t), t = 0, 1, \ldots \) specifies a random process over the probability space \((\Omega, \mathcal{F}, \mathbb{P})\), and we are interested in the mean, mean-square, and almost sure convergence of \( x(t) \).

3.1 State Convergence

For opposing and repelling negative dynamics models, we present the following results, respectively, for the mean-square and almost sure convergence of \( x(t) \).
Theorem 5 Let $0 < \alpha, \beta < 1$ and consider opposing rule (2) for dynamics over negative links.

(i) If $G$ is structurally balanced subject to partition $V = V_1 \cup V_2$, then both in the mean-square and almost sure sense there hold

\[ x_i(t) \to \left( \sum_{j \in V_1} x_j(0) - \sum_{j \in V_2} x_j(0) \right) / n, \quad i \in V_1 \tag{16} \]

and

\[ x_i(t) \to -\left( \sum_{j \in V_1} x_j(0) - \sum_{j \in V_2} x_j(0) \right) / n, \quad i \in V_2. \tag{17} \]

(ii) If $G$ is not structurally balanced, then $x_i(t) \to 0$ both in the mean-square and almost sure sense for all $i \in V$.

Theorem 6 Suppose $G^+$ is connected and consider repelling rule (3). For any $0 < \alpha < 1$, there exists $\beta^*(\alpha) > 0$ such that $x_i(t) \to \sum_{j=1}^n x_i(0) / n$ both in mean-square and almost surely for all initial value $x(0)$ if $\beta < \beta^*$.

3.2 Almost Sure Divergence

The following results characterize possible almost sure divergence of $x(t)$ caused by large $\beta$ related to the negative links, respectively, for opposing and repelling models.

Theorem 7 Fix $0 < \alpha < 1$ with $\alpha \neq 1/2$.

(i) Suppose both $G^+$ and $G^-$ are connected. Then under the opposing negative dynamics (2), there exists $\beta_\ast$ such that whenever $\beta > \beta_\ast$, there holds

\[ P\left( \lim_{t \to \infty} \sup_{i \in V} \max_{j \in V} |x_i(t)| = \infty \right) = 1 \tag{18} \]

for almost all initial values w.r.t. Lebesgue measure.

(ii) Under the repelling negative dynamics (3), there exists $\beta_\ast$ such that whenever $\beta > \beta_\ast$, there holds

\[ P\left( \lim_{t \to \infty} \sup_{i,j \in V} |x_i(t) - x_j(t)| = \infty \right) = 1 \tag{19} \]

for almost all initial values w.r.t. Lebesgue measure.

In fact, for both of the two negative dynamics (2) and (3), the node states under random node interactions follow a so-called No-Survivor Property [42], which indicates that every node states (or relative states) will diverge almost surely if the maximum node states (or relative states) diverges almost surely across the entire network. This property is summarized in the following result.

Theorem 8 The following statements hold.
Under the opposing negative dynamics \((\text{2})\), there holds for any \(k \in V\) that
\[
P\left( \limsup_{t \to \infty} |x_k(t)| = \infty \right) = 1. \tag{20}
\]

Under the repelling negative dynamics \((\text{3})\), there holds for any \(k \neq m \in V\) that
\[
P\left( \limsup_{t \to \infty} |x_k(t) - x_m(t)| = \infty \right) = 1. \tag{21}
\]

Theorem 8(i) is a special case of Theorem 3 in [18], where general random graph processes are investigated. Theorem 8(ii) is quoted directly from Theorem 1 in [42]. The two statements are established using a sample-path analysis in lights of the Borel-Cantelli lemma (see, e.g., Theorem 2.3.6 in [13]). The “lim sup” in the above two theorems can be replaced by “lim inf” and the results continue to hold.

### 3.3 Bounded States

Let \(A > 0\) be a constant and define \(\mathcal{P}_A(\cdot)\) by
\[
\mathcal{P}_A(z) = -A, z < -A, \quad \mathcal{P}_A(z) = z, z \in [-A, A], \quad \mathcal{P}_A(z) = A, z > A.
\]
Define the function \(\theta : E \to \mathbb{R}\) so that \(\theta(\{i,j\}) = \alpha\) if \(\{i,j\} \in E^+\) and \(\theta(\{i,j\}) = -\beta\) if \(\{i,j\} \in E^-\). Assume that node \(i\) interacts with node \(j\) at time \(t\). We now consider the following node interaction under the repelling rule:
\[
x_s(t + 1) = \mathcal{P}_A((1 - \theta)x_s(t) + \theta x_{-s}(t)), \quad s \in \{i,j\}. \tag{22}
\]

Now the node dynamics in \((\text{22})\) become nonlinear due to the state constraint. We continue to use \(P\) to denote the overall probability measure capturing the randomness of the updates in the asymmetric constrained model.

The following result shows that with structural balance of \(G\), state clustering is reached almost surely at the two state boundaries.

**Theorem 9** Consider node dynamics \((\text{22})\) and let \(\alpha \in (0, 1/2)\). Assume that \(G\) is a structurally balanced complete graph under the partition \(V = V_1 \cup V_2\). When \(\beta\) is sufficiently large, for almost all initial values \(x(0)\) w.r.t. Lebesgue measure, there exists a binary random variables \(l(x(0))\) taking values in \([-A, A]\) such that:
\[
P\left( \lim_{t \to \infty} x_i(t) = l(x(0)), i \in V_1; \lim_{t \to \infty} x_i(t) = -l(x(0)), i \in V_2 \right) = 1. \tag{23}
\]

The node state clustering results in Theorem 1 and Theorem 9 for opposing rule and repelling rule, respectively, both rely on structural balance of \(G\). It turns out that when \(G\) is a complete graph, weakly structural balance also leads to clustering of node states.

**Theorem 10** Consider node dynamics \((\text{22})\) and let \(\alpha \in (0, 1/2)\). Assume that \(G\) is a weakly structurally balanced complete graph under the partition \(V = V_1 \cup V_2 \cdots \cup V_m\) with \(m \geq 2\). Let \(\alpha \in (0, 1/2)\). When \(\beta\) is sufficiently large, almost sure boundary clustering is achieved in the sense that for almost all initial
value $x(0)$ w.r.t. Lebesgue measure, there are $m$ random variables, $l_1(x(0)), \ldots, l_m(x(0))$, each of which taking values in $\{-A, A\}$, such that:

$$P\left( \lim_{t \to \infty} x_i(t) = l_j(x(0)), \; i \in V_j, \; j = 1, \ldots, m \right) = 1. \tag{24}$$

When the positive graph $G^+$ is connected – therefore there is no structural balance – any node state will touch the two boundaries $-A$ and $A$ an infinite number of times. The result is summarized below.

**Theorem 11** Consider node dynamics (22) and let $\alpha \in (0, 1/2)$. Assume that $G$ is a complete graph and the positive graph $G^+$ is connected. When $\beta$ is sufficiently large, for almost all initial value $x(0)$ w.r.t. Lebesgue measure, there holds for all $i \in V$ that

$$P\left( \liminf_{t \to \infty} x_i(t) = -A, \; \limsup_{t \to \infty} x_i(t) = A \right) = 1. \tag{25}$$

Results of the similar type as Theorems 9, 10 and 11 were established in [42] for a model where asymmetric node updates were also taken into consideration. The assumption that $G$ is a complete graph is merely a technical assumption to simplify the analysis. The proofs of Theorems 9, 10 and 11 are based on stopping time analysis for the process $G_t, t = 0, 1, \ldots$ in lights of the Second Borel-Cantelli Lemma, and they have been put in the Appendix.

### 4 Conclusions

We have surveyed the fundamental results on the convergence properties of dynamics over signed networks. A unified approach was provided in view of generalized Perron-Frobenius theory, graph theory, and elementary algebraic recursions. The results illustrated that dynamical properties of a network depend crucially on the sign structure of the network links, for both deterministic and random node interactions. Many interesting future research directions emerge naturally after the connection between such basic convergence conditions have been clear. First of all, inverse problems such as estimating characteristics of the annotations of links and nodes from observations of various network characteristics at a subset of nodes are of primary interest. Typical questions would include re-construction of node initial values, identification of edge signs, and test of structural balance through a perhaps finite sequence of measurements of the node states [4, 30, 34, 5, 3]. Another interesting direction would be the investigation of controllability issues related to signed networks along the line of research on network controllability [39, 28, 44, 37]. How sign structure of a network system relates to the network controllability or even structural controllability is still an open problem. Finally, it is of interest to look into the scenario when the evolving node states generate feedback to the signs of the network edges. The closed-loop network dynamics will lead to Krause’s type of multi-agent systems where state-dependant interaction structure will inevitably cause high nonlinearity in the state update at the nodes [7].
Appendix

A. Proof of Theorem 3

The statement (i) again follows directly from Theorem 2 in [36] after applying a gauge transformation

\[ z_i(t) = x_i(t), \quad i \in V_1; \quad z_i(t) = -x_i(t), \quad i \in V_2. \]

We now prove the statement (ii) through a contradiction argument. We proceed in steps.

Step 1. Define \( h(t) \) := \( \max_{i \in V} |x_i(t)| \). Observing that (6) continues to hold with a digraph \( G \), we have \( h(t + 1) \leq h(t) \) for all \( t \geq 0 \). Consequently, there is a constant \( h_*(x(0)) > 0 \) such that \( \lim_{t \to \infty} h(t) = h_* \) for any initial value \( x(0) \). We only need to consider the case with \( h_* > 0 \), and by the definition of \( h_* \), for any \( \epsilon > 0 \), there exists \( T(\epsilon) > 0 \) such that

\[ |x_i(t)| \leq h_* + \epsilon, \quad t \geq T. \quad (26) \]

Step 2. Define \( g_i := \liminf_{t \to \infty} |x_i(t)| \). In this step, we show \( g_i = h_* \) for all \( i \in V \). Suppose \( g_{i_0} < h_* \) for some \( i_0 \in V \). By the definition of \( g_i \), for any \( \epsilon > 0 \), there always exists \( t_1 \geq T \) such that

\[ |x_{i_0}(t_1)| \leq g_i + \epsilon. \quad (27) \]

The graph \( G \) is strongly connected. Therefore, the set \( V_1^* := \{ j : i_0 \in N_j \} \) is nonempty. Based on (26), (27) and the fact that \( i_0 \in N_{i_1} \), we then have

\[ |x_{i_1}(t_1 + 1)| = \left| \left( 1 - \alpha |N_{i_1}^+| + \beta |N_{i_1}^-| \right) x_{i_1}(t) + \alpha \sum_{j \in N_{i_1}^+} x_j(t) - \beta \sum_{j \in N_{i_1}^-} x_j(t) \right| \]

\[ \leq \left| 1 - \alpha |N_{i_1}^+| + \beta |N_{i_1}^-| \right| \cdot |x_{i_1}(t)| + \alpha \sum_{j \in N_{i_1}^+} |x_j(t)| + \beta \sum_{j \in N_{i_1}^-} |x_j(t)| \]

\[ \leq \gamma (g_i + \epsilon) + (1 - \gamma) (h_* + \epsilon) \]

\[ = \gamma g_{i_0} + (1 - \gamma) h_* + \epsilon \quad (28) \]

for any \( i_1 \in V_1^* \), where \( \gamma = \min\{\alpha, \beta\} \).

Continuing, we define \( V_2^* := \{ j : \exists i_1 \in V_1^*, \; i_1 \in N_j \} \) as the nodes that have a neighbor in the set \( V_1^* \). Again, the set \( V_2^* \) is nonempty because the graph \( G \) is strongly connected. Repeating the above analysis we have

\[ |x_{i_2}(t_1 + 2)| \leq \gamma^2 g_{i_0} + (1 - \gamma^2) h_* + \epsilon \quad (29) \]

for any \( i_2 \in V_1^* \cup V_2^* \). This process can be further recursively carried out and with \( G \) being strongly connected, there must hold

\[ |x_i(t_1 + n - 1)| \leq \gamma^{n-1} g_{i_0} + (1 - \gamma^{n-1}) h_* + \epsilon, \quad i \in V. \quad (30) \]
Therefore,

\[ h_* \leq \gamma^{n-1} g_{i_0} + (1 - \gamma^{n-1}) h_* + \epsilon, \quad (31) \]

or equivalently,

\[ \gamma^{n-1}(h_* - g_{i_0}) \leq \epsilon. \quad (32) \]

This leads to a contradiction if \( h_* > g_{i_0} \) because \( \epsilon \) in (32) can be arbitrary.

Step 3. Note that, the fact that \( g_i = h_* \) for all \( i \in V \) immediately leads to \( \lim_{t \to \infty} |x_i(t)| = h_* \) for all \( i \in V \) since \( \limsup_{t \to \infty} |x_i(t)| \leq h_* \) by the definition of \( h_* \). It is easy to exclude the case where \( \liminf_{t \to \infty} x_i(t) = -h_* \) and \( \limsup_{t \to \infty} x_i(t) = h_* \) for some \( i \) directly from the dynamics (5). In other words, all node states asymptotically converge. From this point, we can define

\[ V_1 := \{ i \in V : \liminf_{t \to \infty} x_i(t) = h_* \}, \quad V_2 := \{ i \in V : \liminf_{t \to \infty} x_i(t) = -h_* \}. \]

It is then clear that the links between \( V_1 \) and \( V_2 \) can only be negative, and the links inside each subset can only be positive. This proves that the graph \( G \) is structurally balanced.

We have now concluded the proof. \( \square \)

**B. Proof of Theorem 4**

With \( G \) being directed, it still holds that \( M_G 1 = 1 \) since \( M_G = I - \alpha L_G^+ + \beta L_G^- \), where \( L_G^+ 1 = 0 \) and \( \beta L_G^- 1 = 0 \) for digraphs \( G^+ \) and \( G^- \). Therefore, 1 is always an eigenvalue of \( M_G \).

Fix \( \alpha \) with \( 0 < \alpha < 1/\max_{i \in V} \deg_i^+ \). We can define the following two functions:

\[ r(\beta) := \max \{|\lambda_i(M_G)| : \lambda_i(M_G) \in \sigma(M_G) \setminus \{1\}\} \quad (33) \]

as the largest magnitude of the eigenvalues of \( M_G \) which are not equal to one, and

\[ q(\beta) := (q_1(\beta) \ldots q_n(\beta)) \quad (34) \]

with \( q(\beta)M_G = q(\beta) \) and \( \sum_{j=1}^n q_j(\beta) = 1 \).

The following facts stand: (i) \( r(0) < 1 \), and 1 is a simple eigenvalue of \( I - \alpha L_G^+ \) if \( G^+ \) is strongly connected; (ii) \( q(0) \) is a positive row vector. Noticing that both \( r(\cdot) \) and \( q(\cdot) \) are continuous functions, there exists a sufficiently small \( \beta^* \) such that both the two facts hold for \( \beta < \beta^* \), i.e., 1 is a simple eigenvalue of \( M_G \) with \( r(\beta) < 1 \), and \( q(\beta) \) is positive. Therefore, through the Jordan decomposition of \( M_G \), it is easy to see that

\[ \lim_{t \to \infty} M_G^t = 1q(\beta), \]

and this concludes the proof.\(^2\)

\(^2\)In fact, 1 is a simple eigenvalue of \( I - \alpha L_G^+ \) if \( G^+ \) has a directed spanning tree (see, e.g., Proposition 3.8. in [14]).

\(^3\)See the same treatment applied to continuous-time dynamics in Theorem 3.12, [14].
C. Proof of Theorem \[5\]

Let \( e_m = (0 \ldots 1 \ldots 0)^T \) be the \( n \)-dimensional unit vector whose \( m \)’th entry is 1. Under the pair selection process and the opposing rule for negative links, the evolution of the node states can be written as

\[
x(t + 1) = W_t x(t),
\]

where \( W_t, t = 0, 1, \ldots \) is an i.i.d. random matrix process. The distribution of \( W_t \) is given by

\[
P\left( W_t = I - \alpha (e_i - e_j)(e_i - e_j)^T \right) = p_{ij}, \quad \{i, j\} \in E^+
\]

and

\[
P\left( W_t = I - \beta (e_i + e_j)(e_i + e_j)^T \right) = p_{ij}, \quad \{i, j\} \in E^-.
\]

(i) Let \( G \) be structurally balanced subject to partition \( V = V_1 \cup V_2 \). Introduce \( J = 11^T/n, K = \text{diag}(k_1, \ldots, k_n) \) with \( k_i = 1 \) for \( i \in V_1 \) and \( k_i = -1 \) for \( i \in V_2 \), and \( k = (k_1, \ldots, k_n)^T \). Note that, for any realization of \( W_t \), there holds that \( JK W_t = JK \). Thus \( HW_t KJ = JK W_t K = J \), which in turn leads to

\[
(I - J)(K W_t K) = (K W_t K)(I - J)
\]

Consider \( V(t) = \| (I - J)K x(t) \|^2 \). Then

\[
\mathbb{E}\{ V(t + 1) \mid x(t) \} = \mathbb{E}\{ x^T(t) W_t K (I - J) K W_t x(t) \}
\]

\[= \sum_{\{i,j\}\in E^+} p_{ij} (I - 2\alpha(1 - \alpha)(e_i - e_j)(e_i - e_j)^T) + \sum_{\{i,j\}\in E^-} p_{ij} (I - 2\beta(1 - \beta)(e_i - e_j)(e_i - e_j)^T)
\]

\[= I - 2\alpha(1 - \alpha)L_{G^+} - 2\beta(1 - \beta)L_{G^-},
\]

where \( L_{G^+} \) is the weighted Laplacian of \( G^+ \) with \([L_{G^+}]_{ij} = -p_{ij} \) for \( \{i, j\} \in E^+ \), \([L_{G^+}]_{ij} = 0 \) for \( \{i, j\} \in E^+ \) with \( i \neq j \), and \([L_{G^+}]_{ii} = \sum_{j \neq i} p_{ij} \); \( L_{G^-}^* \) is the weighted (but not signed) Laplacian of \( G^- \) with \([L_{G^-}^*]_{ij} = -p_{ij} \) for \( \{i, j\} \in E^- \), \([L_{G^-}^*]_{ij} = 0 \) for \( \{i, j\} \in E^- \) with \( i \neq j \), and \([L_{G^-}^*}]_{ii} = \sum_{j \neq i} p_{ij} \). Noticing that \( \alpha, \beta \in (0, 1) \) implies \( 0 < 2\alpha(1 - \alpha) < 1 \) and \( 0 < 2\beta(1 - \beta) < 1 \), the following facts hold.
F1. $0 \leq \lambda_i(P^*_G) \leq 1$ for all $\lambda_i(P^*_G) \in \sigma(P^*_G)$; $1 \in \sigma(P^*_G)$ with 1 being a corresponding eigenvector.

F2. 1 is an eigenvalue of $\mathbf{1}\mathbf{1}^T$ with multiplicity one and 1 is an associated eigenvector; $\mathbf{1}\mathbf{1}^T$ also has zero as its eigenvalue with multiplicity $n-1$.

F3. $P^*_G$ and $\mathbf{1}\mathbf{1}^T$ commute, i.e., $P^*_G \mathbf{1}\mathbf{1}^T = \mathbf{1}\mathbf{1}^TP^*_G$.

Consequently, all eigenvalues of $P^*_G - \mathbf{1}\mathbf{1}^T/n$ is positive and strictly less than one. We can therefore further conclude that

$$\mathbb{E}\{V(t+1)|x(t)\} \leq \lambda_{\max}(P^*_G - \mathbf{1}\mathbf{1}^T/n)V(t).$$ (41)

This immediately yields that $\mathbb{E}\{V(t)\}$ converges to zero, or equivalently, (16) and (17) hold in the mean-square sense.

Moreover, (41) means that $V(t)$ is a supermartingale, which converges to a limit almost surely by the martingale convergence theorem (Theorem 5.2.9, [13]). Such limit must be zero since $0 < \lambda_{\max}(P^*_G - \mathbf{1}\mathbf{1}^T/n) < 1$ (which implies, $\mathbb{E}\{V(t)\}$ converges to zero exponentially), and this concludes that (16) and (17) hold in the almost sure sense.

(ii) Now we move on to the case where $G$ is not structurally balanced. Consider instead $V_*(t) = \|x(t)\|^2$.

We have

$$\mathbb{E}\{V_*(t+1)|x(t)\} = x^T(t)\mathbb{E}\{W^2_t\}x(t).$$ (42)

Based on (36) and (37), we have

$$P_G := \mathbb{E}\{W^2_t\} = \sum_{\{i,j\} \in E^+} p_{ij}(I - 2\alpha(1 - \alpha)(e_i - e_j)(e_i - e_j)^T) + \sum_{\{i,j\} \in E^-} p_{ij}(I - 2\beta(1 - \beta)(e_i + e_j)(e_i + e_j)^T)

= I - 2\alpha(1 - \alpha)L_{G^+_p} - 2\beta(1 - \beta)L_{G^-_p},$$ (43)

where $L_{G^+_p}$ is the weighted Laplacian of $G^+$ with $[L_{G^+_p}]_{ij} = -p_{ij}$ for $\{i, j\} \in E^+$, $[L_{G^+_p}]_{ij} = 0$ for $\{i, j\} \in E^+$ with $i \neq j$, and $[L_{G^-_p}]_{ij} = \sum_{j \neq i} p_{ij}$; $L_{G^-_p}$ is the weighted and signed Laplacian of $G^-$ with $[L_{G^-_p}]_{ij} = p_{ij}$ for $\{i, j\} \in E^-$, $[L_{G^-_p}]_{ij} = 0$ for $\{i, j\} \in E^-$ with $i \neq j$, and $[L_{G^-_p}]_{ii} = \sum_{j \neq i} p_{ij}$. The main difference between $W_G$ and $P_G$ lies in the weighted edges in $P_G$. Noticing that $\alpha, \beta \in (0, 1)$ implies $0 < 2\alpha(1 - \alpha) < 1$ and $0 < 2\beta(1 - \beta) < 1$, there holds

$$\sum_{j=1}^n |[P_G]_{ij}| = 1.$$ (44)

As discussed previously in Subsection 2.4.1, the lacking of structural balance of $G$ implies that

$$\lambda_{\max}(P_G) < 1$$

as long as $G$ is a connected graph. Consequently, we have

$$\mathbb{E}\{V_*(t+1)|x(t)\} \leq \lambda_{\max}(P_G)V_*(t),$$ (45)
which in turn leads to that $\mathbb{E}\{V_s(t)\}$ tends to zero, and that $V_s(t)$ goes to zero almost surely from the same analysis applied for $V(t)$. Equivalently, we have proved that $x(t)$ converges to zero in mean-square and almost surely.

We have now completed the proof of Theorem 5.

D. Proof of Theorem 6

Let $x_{ave} = \sum_{i \in V} x_i(0)/n$ be the average of the initial beliefs. We introduce $V_p(t) = \sum_{i=1}^n |x_i(t) - x_{ave}|^2 = \|(I - J)x(t)\|^2$. Similar to (39), we have

$$\mathbb{E}\{V_p(t + 1)\mid x(t)\} \leq \lambda_{max}(\mathbb{E}\{W^2(t)\} - J)V(t).$$  \hspace{1cm} (46)

With repelling rule, the distribution of $W_t^p$ is given by

$$P\left(W_t^p = I - \alpha(e_i - e_j)(e_i - e_j)^T\right) = p_{ij}$$  \hspace{1cm} (47)

if $\text{Sgn}\{\{i, j\}\} = +$, and

$$P\left(W_t^p = I + \beta(e_i - e_j)(e_i - e_j)^T\right) = p_{ij}$$  \hspace{1cm} (48)

if $\text{Sgn}\{\{i, j\}\} = -$. As a result, we have

$$\mathbb{E}\{W^2(t)\} = I - 2\alpha(1 - \alpha)L_{G^+} + 2\beta(1 + \beta)L_{G^+}^*.$$  \hspace{1cm} (49)

where $L_{G^+}$ and $L_{G^+}^*$ are defined in (43).

Since $G^+$ is connected, $\lambda_{max}(I - 2\alpha(1 - \alpha)L_{G^+}) < 1$ noticing $0 < \alpha < 1$. Consequently, $\lambda_{max}(\mathbb{E}\{W^2(t)\} - J) < 1$ for all $\beta$ satisfying

$$\beta(1 + \beta) < \frac{\lambda_2(L_{G^+}^*)}{\lambda_{max}(L_{G^+}^*)}\alpha(1 - \alpha),$$  \hspace{1cm} (50)

where $\lambda_2(L_{G^+}^*)$ is the second smallest eigenvalue of $L_{G^+}^*$. Since $g(\beta) = \beta(1 + \beta)$ is nondecreasing, $\lambda_{max}(\mathbb{E}\{W^2(t)\} - J) < 1$ for all $0 \leq \beta < \beta^*$ with

$$\beta^* := \inf_\beta \left\{ \beta(1 + \beta) < \frac{\lambda_2(L_{G^+}^*)}{\lambda_{max}(L_{G^+}^*)}\alpha(1 - \alpha) \right\}.$$  \hspace{1cm} (51)

Applying the same analysis that is used for $V(t)$ and $V_s(t)$, for any $0 \leq \beta < \beta^*$ and from (46), there hold that $\mathbb{E}\{V_p(t)\}$ converges to zero, and that $V_p(t)$ tends to zero almost surely. This completes the proof of Theorem 6.

E. Proof of Theorem 7

(i) Define $h(t) := \max_{i \in V} |x_i(t)|$. The proof is based on the following lemma.

**Lemma 1** Let $\alpha \neq 1/2 \in (0, 1)$ and $\beta \geq 3$. Then $\{h(t + 1) \geq \min\{2\alpha - 1, 1/2\}h(t)\}$ is a sure event.
Proof. We discuss two cases, respectively.

C1. Suppose a pair of nodes \{i, j\} sharing a positive link is selected at time \(t\). If both \(|x_i(t)| < h(t)\) and \(|x_j(t)| < h(t)\) hold, then \(h(t + 1) \geq h(t)\). Therefore, we assume without loss of generality that \(|x_i(t)| = h(t)\). This leads to two scenarios.

(a) Let \(0 < \alpha < 1/2\). Then
\[
|x_i(t + 1)| = |(1 - \alpha)x_i(t) + \alpha x_j(t)| \geq (1 - \alpha)|x_i(t)| - \alpha|x_j(t)| \geq (1 - 2\alpha)h(t). \tag{52}
\]
(b) Let \(1/2 < \alpha < 1\). Then
\[
|x_j(t + 1)| = |(1 - \alpha)x_j(t) + \alpha x_i(t)| \geq \alpha|x_i(t)| - (1 - \alpha)|x_j(t)| \geq (2\alpha - 1)h(t). \tag{53}
\]
We see (52) and (53) lead to \(h(t + 1) \geq |2\alpha - 1|h(t)\).

C2. Suppose a pair of nodes \{i, j\} sharing a negative link is selected at time \(t\). Again we assume without loss of generality that \(|x_i(t)| = h(t)\). We define \(y_i(t) = x_i(t)\) and \(y_j(t) = -x_j(t)\). Then the update of \(y_i(t)\) and \(y_j(t)\) is described by
\[
y_i(t + 1) = y_i(t) + \beta(y_j(t) - y_i(t)) \tag{54}
\]
\[
y_j(t + 1) = y_j(t) + \beta(y_i(t) - y_j(t))
\]
(a) If \(|y_j(t)| \geq 1/2h(t)\), we see obviously from (54) that
\[
h(t + 1) \geq |y_j(t + 1)| \geq h(t)/2 \tag{55}
\]
if \(y_i(t)\) and \(y_j(t)\) have the same sign. Otherwise without loss of generality let \(y_i(t) > 0\) and \(y_j(t) < 0\). Then from (54)
\[
|y_i(t + 1)| = |y_i(t) + \beta(y_j(t) - y_i(t))|
\]
\[
\geq |y_j(t) - y_i(t)| - |y_i(t)|
\]
\[
\geq \frac{3}{2}\beta h(t) - h(t)
\]
\[
\geq h(t)/2 \tag{56}
\]
for \(\beta \geq 1\).
(b) If \(|y_j(t)| < 1/2h(t)\), then there holds for \(\beta \geq 3\) that
\[
|y_i(t + 1)| = |(1 - \beta)y_i(t) + \beta y_j(t)|
\]
\[
\geq (\beta - 1)|y_i(t)| - \beta|y_j(t)|
\]
\[
\geq \frac{1}{2}\beta - 1)h(t)
\]
\[
\geq h(t)/2 \tag{57}
\]
}\)
We see (55), (56), and (57) lead to \( h(t + 1) \geq h(t)/2 \) if \( \beta \geq 3 \).

We have now proved the desired lemma. \( \square \)

With Lemma 1 serves as the same role as the Lemma 5 in [18], the desired conclusion follows from the same argument in view of the strong law of large numbers as the proof of Proposition 1 of [18]. We therefore omit the remaining details.

(ii) The result comes from Theorem 3 in [42]. We therefore refer to the proof therein, which is also based on the strong law of large numbers.

F. Proof of Theorem 9

We quote the following lemma, Lemma 7 in [42]. Note that the proof of Lemma 7 in [42] does not rely on the asymmetric node updates, and therefore the lemma continue to hold for (22).

**Lemma 2** Fix \( \alpha \in (0, 1) \) with \( \alpha \neq 1/2 \). For the dynamics (22) with the random pair selection process, there exists \( \beta^*(\alpha) > 0 \) such that

\[
\mathbb{P}\left( \limsup_{t \to \infty} \max_{i,j \in V} |x_i(t) - x_j(t)| = 2A \right) = 1
\]

for almost all initial beliefs if \( \beta > \beta^* \).

We establish another technical lemma.

**Lemma 3** Fix \( \alpha \in (0, 1/2) \) and \( \beta \geq \max\{2, 1/\alpha\} \). Consider the dynamics (22) with the random pair selection process. Assume that \( G \) is a structurally balanced complete graph under the partition \( V = V_1 \cup V_2 \).

If there are \( i_1 \in V_1, j_1 \in V_2 \) and \( t \geq 0 \) with \( x_{i_1}(t) = -A \) and \( x_{j_1}(t) = A \), then for \( Z = 3(n - 2) \), there exists and a sequence of node pair realizations, \( G_{t+s}(\omega) \) for \( s = 0, 1, \ldots, Z - 1 \) under which there holds

\[
x_{i_1}(t + Z)(\omega) = -A, i \in V_1; x_{i_1}(t + Z)(\omega) = A, i \in V_2.
\]

(58)

**Proof.** We recursively construct such sequence of node pair realizations \( G_{t+s}(\omega) \) for \( s = 0, 1, \ldots, Z - 1 \). Without loss of generality we let \( V_1 \) contains at least two nodes.

Take \( i_2 \neq i_1 \in V_1 \) and let

\[
G_t(\omega) = \{i_1, i_2\}, G_{t+1}(\omega) = \{j_1, i_1\}, G_{t+2}(\omega) = \{j_1, i_2\}.
\]

(59)

Now we investigate the outcome of the above pair selection process. Since \( i_1, i_2 \in V_1 \), they share a positive link whose interaction is defined by (1). Consequently, we conclude from \( x_{i_1}(t) = -A \) and \( \alpha \in (0, 1/2) \) that

\[
x_{i_1}(t + 1)(\omega) \leq 0, x_{i_2}(t + 1)(\omega) \leq (1 - 2\alpha)A.
\]

(60)

Further, since \( \beta \geq 2 \) and \( x_{j_1}(t) = A \), obviously with the chosen \( G_{t+1}(\omega) \) we have \( x_{i_2}(t + 2)(\omega) \leq (1 - 2\alpha)A \) and

\[
x_{i_1}(t + 2)(\omega) = -A, x_{j_1}(t + 2)(\omega) = A.
\]

(61)
Finally, noticing the fact that $\beta \geq 1/\alpha$ there holds
\[ x_{i_1}(t+3)(\omega) = -A, \quad x_{i_2}(t+3)(\omega) = -A, \quad x_{j_1}(t+3)(\omega) = A. \] (62)

Next, we recursively apply the above pair selections for other nodes in $V_1$ and then we get $x_{j_1}(t + 3n_1)(\omega) = A$ and
\[ x_i(t + 3n_1)(\omega) = -A, \quad i \in V_1 \] (63)
with $n_1 = |V_1| - 1$.

Finally, we repeat the same pair selection process for nodes in $V_2$. This will yield
\[ x_i(t + 3(n-2))(\omega) = -A, \quad i \in V_1; \quad x_i(t + 3(n-2))(\omega) = A, \quad i \in V_2. \] (64)
This proved the desired lemma. \qedhere

We now have the necessary tools in hands for the proof of Theorem 9. By Lemma 2, there are two nodes $i^*$ and $j^*$ such that with probability one,
\[ \limsup_{t \to \infty} |x_{i^*}(t) - x_{j^*}(t)| = 2A. \] (65)
We define
\[ T_1 : \inf_{t \geq 0} |x_{i^*}(t) - x_{j^*}(t)| \geq A \]
and then recursively define
\[ T_{m+1} : \inf_{t \geq T_m + Z + 1} |x_{i^*}(t) - x_{j^*}(t)| \geq A \]
for $m = 2, 3, \ldots$. Evidently they form a sequence of stopping times in the random node pair process $G_t, t = 0, 1, \ldots$. From the fact that (65) holds with probability one, $T_m$ is almost surely finite for any $m = 1, 2, \ldots$.

There will be two cases.

C1. Let $i^*$ and $j^*$ belong to different subgroups, say, $i^* \in V_1$ and $j^* \in V_2$. Then by selecting \{i*, j*\} at time $T_m$, we have
\[ x_{i_1}(T_m + 1) = -A, \quad x_{j_1}(T_m + 1) = A, \] (66)
where $i_1$ and $j_1$ are from the set \{i*, j*\} sharing a negative link. Let $i_1 \in V^{i_1}$ and $i_2 \in V^{i_2}$, where each $V^{i_1}$ and $V^{i_2}$ is either $V_1$ or $V_2$. Then Lemma 3 suggests from (66) that
\[ \mathbb{P}(x_i(T_m + Z + 1) = -A, i \in V^{i_1}; \quad x_i(T_m + Z + 1) = A, i \in V^{i_2}) \geq (\min_{(i,j) \in E} p_{ij})^{Z+1}. \] (67)
Note that, since the $T_m$ are stopping times of $G_t, t = 0, 1, \ldots$, by strong Markov property we can invoke the second Borel-Cantelli Lemma (e.g., Theorem 2.3.6 in [13]) to conclude from (67) that almost surely, there is $m_0 \in \mathbb{Z}^+$ such that
\[ x_i(T_{m_0} + Z + 1) = -A, i \in V^{i_1}; \quad x_i(T_{m_0} + Z + 1) = A, i \in V^{i_2}, \]
and therefore
\[ x_i(t) = -A, i \in V_{11}; \quad x_i(t) = A, i \in V_{12} \]
for all \( t \geq T_{m_0} + Z + 1 \) from the structure of the dynamics.

C2. Let \( i_* \) and \( j_* \) belong to the same subgroup, say, \( V_1 \). There must be another node \( k_* \in V_2 \) such that either \( |x_{i_*}(T_m) - x_{k_*}(T_m)| \geq A/2 \) or \( |x_{j_*}(T_m) - x_{k_*}(T_m)| \geq A/2 \). No matter which case it is by selecting the corresponding pair \( \{i_*, k_*\} \) or \( \{j_*, k_*\} \) for time \( T_m \), we obtain two nodes \( i_1 = i_* \) or \( j_* \) and \( j_1 = k_* \) so that
\[ x_{i_1}(T_m + 1) = -A, \quad x_{j_1}(T_m + 1) = A. \]  
(68)

Consequently, this case also ends up with condition (66) and therefore rest of treatment remains the same.

We have now completed the proof of Theorem 10.

G. Proof of Theorem 10

Following Lemma 3 another lemma can be established.

**Lemma 4** Fix \( \alpha \in (0, 1/2) \) and \( \beta \geq \max\{2, 1/\alpha\} \). Consider the dynamics (22) with the random pair selection process. Let \( G \) be a weakly structurally balanced complete graph under the partition \( V = V_1 \cup V_2 \cdots \cup V_m \) for \( m \geq 2 \). If there are \( i_1 \in V_1, j_1 \in V_2 \) and \( t \geq 0 \) with \( x_{i_1}(t) = -A \) and \( x_{j_1}(t) = A \), then for \( Z = 3n - 2m - 2 \), there exists a sequence of node pair realizations, \( G_{t+s}(\omega) \) for \( s = 0, 1, \ldots, Z - 1 \) under which there holds
\[ x_i(t + Z)(\omega) = -A, i \in V_1; \quad x_i(t + Z)(\omega) = A, i \in V_2; \quad x_i(t + Z)(\omega) = T_0 A, i \in V_m, m \geq 3, \]  
(69)
where \( T_0 \) takes value from \( \{1, 0\} \) relying on \( x(t) \).

**Proof.** First of all we apply the node pair selection process in the proof of Lemma 3 and get with \( Z_1 = 3(|V_1| + |V_2| - 2) \) that
\[ x_i(t + Z_1)(\omega) = -A, i \in V_1; \quad x_i(t + Z_1)(\omega) = A, i \in V_2. \]  
(70)

Now take \( k_1 \in V_3 \). Either \( x_{k_1}(t) = x_{k_1}(t + Z_1) < 0 \) or \( x_{k_1}(t) = x_{k_1}(t + Z_1) \geq 0 \) must hold. If \( x_{k_1}(t + Z_1) < 0 \) then letting \( G_{t+Z_1} = \{k_1, j_1\} \) we have \( x_{k_1}(t + Z_1 + 1) = -A, \quad x_{j_1}(t + Z_1 + 1) = A \). Applying the proof of Lemma 3 to \( V_3 \), there is a sequence of node pairs leading to
\[ x_i(t + Z_1 + 3|V_3| - 2) = -A, \quad i \in V_3. \]

Similarly, the other case with \( x_{k_1}(t) = x_{k_1}(t + Z_1) \geq 0 \) leads to
\[ x_i(t + Z_1 + 3|V_3| - 2) = A, \quad i \in V_3. \]
The process can be recursively carried out to the rest of the nodes. The whole process counts \( 3(n - m) + m - 2 = 3n - 2m - 2 \) node pairs. The desired conclusion holds.

The same argument based on stopping times of \( G_t \) and the second Borel-Cantelli Lemma in the proof of Theorem 9 can now be applied to the weakly structural balance case with the help of Lemma 4 and then Theorem 10 holds.

F. Proof of Theorem 11

The proof is based on the following lemma.

**Lemma 5** Fix \( \alpha \in (0, 1/2) \) and \( \beta \geq \max\{2, 1/\alpha\} \). Consider the dynamics (22) with the random pair selection process. Let \( G \) be the complete graph under and \( G^+ \) being connected. If there are \( i_1, j_1 \in \mathbb{V} \) with \( x_{i_1}(t) = -A \) and \( x_{j_1}(t) = A \), then for any \( \epsilon > 0 \), there exist a constant \( Z_0(\epsilon) \) and a sequence of node pair realizations, \( G_{t+s}(\omega) \) for \( s = 0, 1, \ldots, Z - 1 \) under which there holds for some \( i_* \in \mathbb{V} \) that

\[
x_{i_*}(t + Z)(\omega) = -A; \quad x_i(t + Z)(\omega) \geq A - \epsilon, \quad i \neq i_*.
\]

Moreover, the sequence can also be selected so that

\[
x_{i_*}(t + Z)(\omega) = A; \quad x_i(t + Z)(\omega) \leq -A + \epsilon, \quad i \neq i_*.
\]

**Proof.** From our standing assumption the negative graph \( G^- \) contains at least one edge. Let \( k_*, m_* \in \mathbb{V} \) share a negative link. A full proof relies on discussion of all possible sign patterns among the (possibly) four nodes \( i_1, j_1, k_*, m_* \). We just present the analysis for the case with

\[
\{i_1, k_*\} \in E^+, \{i_1, m_*\} \in E^+, \{j_1, k_*\} \in E^+, \{j_1, m_*\} \in E^+.
\]

The other cases are indeed simpler and can be studied via similar techniques, whose details are omitted.

Without loss of generality we let \( x_{m_*}(t) \geq x_{k_*}(t) \). First of all we select \( G_t = \{i_1, k_*\} \) and \( G_{t+1} = \{j_1, m_*\} \). It is then straightforward to verify that

\[
x_{m_*}(t + 2) \geq x_{k_*}(t + 2) + 2\alpha A.
\]

By selecting \( G_{t+2} = \{m_*, k_*\} \) we know from \( \beta \geq 1/\alpha \) that

\[
x_{m_*}(t + 3) = A, \quad x_{k_*}(t + 3) = -A.
\]

Now we let \( i_* = k_* \) and complete the entire node selection process. Define

\[
E_{-i_*} = \{\{i, j\} \in E^+, i, j \neq i_*\}
\]

as all positive links that are not associated with \( i_* \). Note that the positive graph \( G^+ \) is connected, which means that the links in \( E_{-i_*} \) form a connected graph for nodes in \( \mathbb{V} \setminus \{i_*\} \). As a result, noticing the attraction nature of the positive links, it is obvious that we can ensure (71) by alternatively selecting a pair sequence which alternatively choose from \( \{i_*, m_*\} \) and from \( E_{-i_*} \).
If on the other hand (72) is desired then we simply let $i_\star = m_\star$ and apply the same methodology for pair selections. This concludes the proof of the lemma.

In view of Lemma 2 and Lemma 3, the desired theorem is, again, a consequence of the second Borel-Cantelli Lemma.

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