Abel-Prym maps for isotypical components of Jacobians

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Abstract
Let \( C \) be a smooth non-rational projective curve over the complex field \( \mathbb{C} \). If \( A \) is an abelian subvariety of the Jacobian \( J(C) \), we consider the Abel-Prym map \( \varphi_A : C \to A \) defined as the composition of the Abel map of \( C \) with the norm map of \( A \). The goal of this work is to investigate the degree of the map \( \varphi_A \) in the case where \( A \) is one of the components of an isotypical decomposition of \( J(C) \). In this case we obtain a lower bound for \( \text{deg}(\varphi_A) \) and, under some hypotheses, also an upper bound. We then apply the results obtained to compute degrees of Abel-Prym maps in a few examples. In particular, these examples show that both bounds are sharp.

1 Introduction

Given a smooth projective complex curve \( C \) and a fixed point \( q \) in \( C \), the Abel map of \( C \) is the map taking a point \( p \) of \( C \) to the point of the Jacobian \( J(C) \) of \( C \) corresponding to the divisor \( p - q \). It is a very well-known fact that the Abel map is an embedding when \( C \) is non-rational. Now, if \( A \) is an abelian subvariety of \( J(C) \), composing the Abel map of \( C \) with the norm map of \( A \) we get the Abel-Prym map \( \varphi_A : C \to A \). It is easy to see that the image of \( \varphi_A \) is a subcurve of \( A \) generating \( A \). The question of finding bounds for the genus of subcurves of abelian varieties has been considered, for instance, in [1], [8] and [10]. However, not much is known in general about the map \( \varphi_A \).

There are few cases in the literature where the degree of \( \varphi_A \) is calculated. One case that has been studied is when \( A \) is a Prym variety of a covering of smooth projective curves. The first of such cases was that of a double covering \( C \to C' \) that is either étale or ramified at two points. In this case, Mumford showed in [16] that \( \varphi_A \) has degree 2 if \( C \) is hyperelliptic, and degree 1 otherwise. More recently, Lange and Ortega considered the case where the covering is cyclic, in [11], and non-cyclic of degree 3, in [12]. They showed that, under some numerical conditions, the Abel-Prym map of \( A \) has degree 1 or 2. In [4], Brambila-Paz, Gómez-González and Pioli constructed a Prym-Tyurin variety \( A' \)
isogenous to $A$ and, again under some conditions, showed that the Abel-Prym map of $A'$ has degree 1. In [14], Lange and Recillas considered Prym varieties associated to pairs of coverings of smooth projective curves and obtained Abel-Prym maps of degree 1. Finally, in [15] Lange, Recillas and Rojas used correspondences to construct a Prym-Tyurin variety $A$ of exponent 3 having Abel-Prym map of degree 1. Note that in all the cases above, the degree of the Abel-Prym map is one or two. However, as our results show, this is not always the case.

1.1 Main Results

The degree of an Abel-Prym map seems to be rather difficult to understand in full generality. In this work we focus on Abel-Prym maps for Prym-Tyurin components of the isotypical decomposition of a Jacobian variety.

First in Section 3 we define the Abel-Prym map. Using a result of Debarre [8], we obtain in Theorem 3 an upper bound on the degree of a map from a curve to a polarized abelian variety, under some conditions:

**Theorem A** Let $(A, \theta)$ be a polarized abelian variety of dimension $n$ and let $d = \deg(\theta)$. Let $\varphi : C \to A$ be a morphism, where $C$ is a smooth projective complex curve. Assume that $\varphi_*[C] = \frac{k}{(n-1)!} C^{n-1} \theta$ and that $\varphi(C)$ generates $A$. Then

$$\deg(\varphi) \leq \frac{kd}{\sqrt{n}}.$$

Next, in Section 4 we consider a smooth projective complex curve $C$ with an action of a finite group $G$ and the isotypical decomposition $A_1 \times \ldots \times A_n \to J(C)$ of its Jacobian, as introduced in [13]. Now set $\varphi_i = \varphi_{A_i}$ and $C_i = \varphi_i(C)$ so that we have Abel-Prym maps

$$\varphi_i : C \to C_i,$$

for each $i = 1, \ldots, n$. First we show in Proposition 8 that if $\mathrm{dim}(A_i) = 1$ then the degree of $\varphi_i$ is equal to the exponent of $A_i$ in $J(C)$. To deal with the components of higher dimension we let $K_i$ be the kernel of the representation of $G$ associated to the component $A_i$, and consider the quotient curve $\tilde{C}_i = C/K_i$ with quotient morphism

$$\psi_i : C \to \tilde{C}_i.$$

Using the map $\psi_i$, we obtain a lower bound on the degree of the Abel-Prym map $\varphi_i$ and show that, under some conditions, this bound is sharp. More precisely, in Theorems 9 and 11 we show:

**Theorem B** With the above notation, there is a morphism $f_i : \tilde{C}_i \to C_i$ such that $\varphi_i = f_i \circ \psi_i$. In particular

$$\deg(\varphi_i) \geq |K_i|.$$

Moreover, if $A_i$ is a Prym-Tyurin variety of exponent $e(A_i)$ for $C$, we have:
consider the isotypical decomposition of a Jacobian of a smooth curve with an
bounds obtained in Theorems 3 and 9 are sharp. In the last two exam ples we
respectively. We compute the degrees of the Abel-Prym maps and s how that the
denote by \( \hat{e} \) the
isogeny \( \Phi \) of \( A, \theta \) that is, the exponent of the kernel of \( \Phi \), that is, the exponent of the
\( \Psi \cdot \theta \circ \psi \circ \Phi \) of \( i \). By abuse of notation we will also denote by \( N_{A'} \) the unique
A curve \( C \) is a connected, projective and reduced scheme of dimension 1 over
\( g(C) := \dim H^1(C, \mathcal{O}_C) \). For a smooth curve \( C \), we denote by \( J(C) \) the Jacobian variety of \( C \) and by \( \Theta_C \) its theta divisor. Recall that the pair \( (J(C), \Theta_C) \) is a principally polarized abelian variety, that is, \( \deg(\Theta_C) = 1 \).
Moreover, recall that the Abel map \( \alpha: C \to J(C) \) is an embedding.
A principally polarized abelian variety \( (A, \theta) \) is a Prym-Tyurin variety for a smooth curve \( C \) if \( A \) is (isomorphic to) a subvariety of \( J(C) \) and \( i^* \Theta_C \equiv e \theta \), where \( i: A \to J(C) \) is the inclusion map and \( \equiv \) means algebraic equivalence.
Note that in this case \( e(A) = e \).

(ii) If \( \psi^* \) is an embedding then \( A_i \) is a Prym-Tyurin variety of exponent \( \frac{e(A_i)}{|K_i|} \)
for \( \tilde{C}_i \). Moreover, in this case, \( e(A_i) = |K_i| \) if and only if \( A_i = \psi^*(J(\tilde{C}_i)) \).

In Section 3 we apply these results to four cases. In the two initial examples, we consider specific actions of \( \mathbb{Z}_2 \) and \( D_4 \) on a smooth curve of genus 3 and 4, respectively. We compute the degrees of the Abel-Prym maps and show that the bounds obtained in Theorems 3 and 9 are sharp. In the last two examples we consider the isotypical decomposition of a Jacobian of a smooth curve with an action of the dihedral group \( D_p \), for \( p \) an odd prime, and the quaternions group \( Q_8 \), following [2] and [13]. We then use our results to compute or give bounds for the degrees of the Abel-Prym maps in these cases, under some hypotheses on the action.

2 Technical background

In this paper we always work over the field of complex numbers \( \mathbb{C} \). For most of the
following notations, definitions and results concerning abelian varieties we follow [2].

Let \( (A, \theta) \) be a (polarized) abelian variety of dimension \( n \). If \( (d_1, \ldots, d_n) \) is
the type of \( \theta \) then the degree of the polarization \( \theta \) is \( \deg(\theta) = d_1 \cdots d_n \). We
denote by \( (\hat{A}, \hat{\theta}) \) the dual abelian variety of \( A \). Recall that the map \( \Phi_\theta: \hat{A} \to \hat{A} \)
given by \( \Phi_\theta(x) = t_\theta^x \otimes \theta^{-1} \) is an isogeny of degree \( \deg(\Phi_\theta) = (\deg(\theta))^2 \), where
\( t_\theta \) is the translation by \( \theta \) in \( A \). The exponent \( e(\theta) \) of \( \theta \) is the exponent of the
isogeny \( \Phi_\theta \), that is, the exponent of the kernel of \( \Phi_\theta \). Denote by \( \Psi_\theta: A \to A \)
the unique isoegeny such that \( \Phi_\theta \circ \Psi_\theta = e(\theta) id_A \).

Let \( A' \) be an abelian subvariety of \( (A, \theta) \), with canonical embedding \( i: A' \to A \). The exponent of \( A' \) in \( (A, \theta) \) (or simply in \( A \), when there is no possible ambiguity) is \( e(A') = e(i^* \theta) \). Moreover, the norm endomorphism of \( A \) associated to \( A' \) is the composition \( N_{A'} = i \circ \Psi_{i^* \theta} \circ \hat{i} \circ \Phi_\theta \), where \( \hat{i} \) is the dual homomorphism of \( i \). By abuse of notation we will also denote by \( N_{A'} \) the composition

\[
N_{A'} = \Psi_{i^* \theta} \circ \hat{i} \circ \Phi_\theta: A \to A'.
\]
3 Abel-Prym maps

Let $C$ be a smooth non-rational curve and let $(J(C), \Theta_C)$ be its principally polarized Jacobian. If $A$ is an abelian subvariety of $J(C)$, we define the Abel-Prym map of $A$,

$$\varphi : C \to A,$$

as the composition $\varphi = N_A \circ \alpha$ of the Abel map with the norm map of $A$.

**Lemma 1.** Let $C$ be a smooth non-rational curve and let $\varphi : C \to A$ be the Abel-Prym map to an abelian subvariety $A$ of $J(C)$. Then the image $C_A := \varphi(C)$ is a subcurve of $A$ generating $A$. In particular, $g(C_A) \geq \dim(A)$.

**Proof.** The first two assertions follow from the fact that $\alpha(C)$ is isomorphic to $C$ and generates $J(C)$, and the dual $\hat{i}_A$ of the inclusion $i_A : A \hookrightarrow J(C)$ is surjective onto its image.

For the last assertion we consider the normalization map $\nu_A : C'_A \to C_A$ and the composition $f_A : C'_A \to A$ with the inclusion of $C_A$ in $A$. Since the image of $f_A$ generates $A$, then the map $\tilde{f}_A : J(C'_A) \to A$ induced by the universal property is surjective and thus $\dim(A) \leq g(C'_A) \leq g(C_A)$. □

By [9, Prop. II.6.8], the surjective morphism

$$\varphi_A : C \to C_A$$

given by $\varphi_A = \varphi$ is finite and the curve $C_A$ is complete, although possibly singular. It is not easy in this generality to determine when $C_A$ is smooth, or to compute its genus. Under some conditions, we obtain in Theorem 3 an upper bound of the degree of $\varphi_A$. Moreover, in the case where $A$ is is one of the components of an isotypical decomposition of $J(C)$, we will obtain in Theorem 11 a lower bound. First, we need a definition.

The **minimal class** of a polarized abelian variety $(A, \theta)$ of dimension $n$ is defined as

$$z_\theta = \frac{1}{(n - 1)!} \bigwedge^{n-1} \theta.$$

**Lemma 2.** Let $C$ be an irreducible curve on a polarized abelian variety $(A, \theta)$ with $\dim(A) = n$ and $\deg(\theta) = d$. Assume that $C$ generates $A$ and that the class of $C$ in $A$ is $[C] = kz_\theta$. Then

$$k \geq \frac{\sqrt{n}}{d}.$$

**Proof.** By [8, Prop. 4.1], the intersection product of the class $[C]$ with the polarization $\theta$ satisfies $[C] \cdot \theta \geq n \sqrt[4]{d}$. On the other hand, recall that by Riemann-Roch and [2, Prop. 5.2.3] we have $\theta^n = d n!$. Thus this intersection product is $[C] \cdot \theta = kdn$ and the result follows. □
Theorem 3. Let \((A, \theta)\) be a polarized abelian variety of dimension \(n\) and let \(\varphi : C \to A\) be a morphism, where \(C\) is a smooth curve. Assume that \(\varphi_*[C] = k z_0\) and that \(\varphi(C)\) generates \(A\). Then

\[
\deg(\varphi_A) \leq \frac{kd}{\sqrt{d}}
\]

where \(d = \deg(\theta)\).

Proof. Let \(C_A'\) be the normalization of \(C_A = \varphi(C)\) and let \(h : C \to C_A'\) be the induced morphism. Then \(\varphi = g \circ h\), where \(g : C_A' \to A\) is the normalization map of \(C_A\) composed with the inclusion of \(C_A\) in \(A\). Then \(\deg(\varphi_A) = \deg(h)\) and

\[
\varphi_*[C] = g_* (h_*[C]) = g_* (\deg(h)[h(C)]) = \deg(\varphi_A)g_* [C_A'].
\]

Therefore,

\[
g_*[C'_A] = \frac{k}{\deg(\varphi_A)} z_0,
\]

and since \(g_*[C'_A] = [C_A]\), we have by Lemma 2

\[
\frac{k}{\deg(\varphi_A)} \geq \frac{\sqrt{d}}{d}
\]

and the result follows.

Note that if \(A\) is a Prym-Tyurin variety of exponent \(k\) for \(C\) then, by Welker’s criterion and the previous proposition, we have \(\deg(\varphi_A) \leq k\), since \(\deg(\theta) = 1\).

Proposition 4. Let \((A, \theta)\) be a principally polarized abelian variety and let \(\varphi : C \to A\) be a morphism, where \(C\) is a smooth curve. Assume that \(\varphi^*\) is an embedding and \(\varphi_*[C] = k z_0\). Then \(\deg(\varphi_A) = k\) if and only if \(\varphi(C)\) is smooth and \(A = J(\varphi(C))\). In addition, if \(\deg(\varphi_A) = 1\) then \(A = J(C)\).

Proof. Using notation of the proof of Theorem 3 we note that \(\deg(\varphi_A) = k\) if and only if \([C_A] = z_0\). By Matsusaka’s criterion (cf. [2, Remark 12.2.5]), this happens if and only if \(C_A\) is smooth and \(A = J(C_A)\). The last assertion follows from [2, Cor. 12.2.6].

The following result is a direct consequence of [2, Lemma 12.3.1]. For the sake of the completeness, we include it here.

Proposition 5. Let \(f : C \to C'\) be a finite morphism of smooth curves and assume that the pullback \(f^* : J(C') \to J(C)\) is an embedding. Then \(J(C')\) is a Prym-Tyurin variety of exponent \(\deg(f)\) for \(C\).

Proof. We’ll apply Welker’s criterion. Consider the composition

\[
h := \alpha_{C'} \circ f : C \to J(C'),
\]

where \(\alpha_{C'}\) is the Abel map of \(C'\). Then \(h^* = f^* \circ \alpha_{C'}^*\) is an embedding, since \(\alpha_{C'}^*\) is an isomorphism. Moreover, if \(\deg(f) = q\) then \(f_*[C] = q[C']\). This completes the proof, since by Poincaré’s formula [2 Prop. 11.2.1] we have \([C'] = z_{\theta_{C'}}\).
We remark that the hypothesis of $f^*$ being an embedding is fundamental in Proposition 5. Indeed, consider $f : C \to C'$ a non-ramified morphism of degree 2 between a curve $C$ of genus 3 and $C'$ of genus 2. By [2] Theorem 12.3.3, the complementary subvariety $A$ of $f^*(J(C'))$ in $J(C)$ is a (classical) Prym variety, hence $A$ is a Prym-Tyurin variety of exponent 2 for $C$. Therefore $f^*(J(C'))$ is not a Prym-Tyurin variety for $C$, since the polarization induced by $\Theta_C$ in $f^*(J(C'))$ is of type $(1, 2)$, by [2] Cor. 12.1.5. Note that, in this case, the exponent of $f^*(J(C'))$ as an abelian subvariety of $J(C)$ is 2, hence still equal to the degree of $f$.

One might then think that the exponent of $f^*(J(C'))$ as an abelian subvariety of $J(C)$ would always equal to the degree of $f$. However, it is easy to see that this is not the case. Let $f : C \to C'$ be a morphism between a curve $C$ of genus 2 and $C'$ of genus 1. By [2] Thm. 12.3.3(c) and Cor. 11.4.4, then $f$ factors as $f = f_e \circ g$ where $g : C \to C'_g$ and $f_e : C'_g \to C'$ with $C'_g$ of genus 1 and $f_e$ étale. Assume $f_e$ non-constant. Then the exponent of $f^*(J(C'))$ as an abelian subvariety of $J(C)$ is deg($g$), which is strictly smaller than deg($f$).

By [2] Prop. 11.4.3, the pullback map $f^*$ is injective if and only if $f$ does not factor via a cyclic étale covering of degree $\geq 2$. This implies that, in the case of a cyclic étale covering $f : C \to C'$, then $J(C')$ is not a Prym-Tyurin variety for $C$. The next result shows that even if we consider the image $f^*(J(C'))$ of the pullback map, which is already an abelian subvariety of $J(C)$, then we may still not have a Prym-Tyurin variety for $C$.

**Proposition 6.** Let $f : C \to C'$ be a non-constant cyclic étale morphism of smooth curves, where $g(C') \geq 2$ and consider the inclusion map $i : f^*(J(C')) \to J(C)$. If $\deg(f) \neq a \circ (C')$ for some $a \in \mathbb{Z}$, then the induced polarization $i^* \Theta_C$ on $f^*(J(C'))$ is not a multiple of a principal polarization.

**Proof.** For the sake of simplicity, set $\Theta = \Theta_C$, $\Theta' = \Theta_{C'}$, and $g' = g(C')$.

Note that $f^*$ factors as $f^* = i \circ j$, where $j : J(C') \to f^*(J(C'))$ is an isogeny. Set $n = \deg(f)$. By [2] Lemma 12.3.1, we have $(f^*)^* \Theta = n \Theta'$. Hence the type of $(f^*)^* \Theta$ is $(n, \ldots, n)$ and by [2] Theorem 3.6.1, we have

$$\chi((f^*)^* \Theta) = (-1)^d n^g'$$

for some $d \in \mathbb{Z}$. Now, $(f^*)^* = j^* \circ i^*$ and by [2] Cor. 3.6.6 we have

$$\chi((f^*)^* \Theta) = \chi(j^*(i^* \Theta)) = \deg(j) \chi(i^* \Theta).$$

Since $f$ is étale of degree $n$, then the isogeny $j$ also has degree $n$ and we get that

$$\chi(i^* \Theta) = (-1)^d n^{g'-1}. \quad (2)$$

If $i^* \Theta$ is a multiple of a principal polarization on $f^*(J(C'))$, then it is of type $(m, \ldots, m)$ for some $m \in \mathbb{Z}$ and again by [2] Theorem 3.6.1, we have $\chi(i^* \Theta) = (-1)^d m^{g'}$ for some $d' \in \mathbb{Z}$. Thus, (2) gives us

$$m^{g'} = n^{g'-1}.$$
We need to show this can only happen when \( n = a^{g'} \) for some \( a \in \mathbb{Z} \). Let \( n = p_1^{t_1} \cdots p_k^{t_k} \) be the prime decomposition of \( n \). Then
\[
n^{g'-1} = p_1^{r_1(g'-1)} \cdots p_k^{r_k(g'-1)}
\]
and for this to be of the form \( m^{g'} \) we must have
\[
r_i(g' - 1) = s_i g'
\]
for some \( s_i \in \mathbb{Z} \), for every \( i = 1, \ldots, k \). Since \( g' \) and \( g' - 1 \) are coprimes, then \( g' \) divides \( r_i \) and we may write \( r_i = t_i g' \) for some \( t_i \in \mathbb{Z} \) for \( i = 1, \ldots, k \). But then
\[
n = (p_1^{t_1} \cdots p_k^{t_k})^{g'}
\]
and the result is proven. \( \square \)

4 Isotypical decomposition

In this section we focus on Abel-Prym maps for subvarieties of a Jacobian variety arising from the isotypical decomposition introduced in [13]. First we briefly recall this decomposition, for more details see [2, Section 13.6].

Let \( G \) be a finite group acting on a smooth curve \( C \). This action induces an action of \( C \) on the Jacobian variety \( J(C) \) and hence, an algebra homomorphism
\[
\rho : \mathbb{Q}[G] \to \text{End}_{\mathbb{Q}}(J(C)),
\]
where \( \mathbb{Q}[G] \) denotes the rational group algebra of \( G \). Since \( \mathbb{Q}[G] \) is a semi-simple \( \mathbb{Q} \)-algebra of finite dimension, there is an unique decomposition
\[
\mathbb{Q}[G] = Q_1 \times \cdots \times Q_r,
\]
where \( Q_i \) are simple \( \mathbb{Q} \)-algebras. Consider the decomposition of the unit element \( 1 = e_1 + \cdots + e_r \). The elements \( e_i \in Q_i \), seen as elements of \( \mathbb{Q}[G] \), form a set of orthogonal idempotents contained in the center of \( \mathbb{Q}[G] \).

For \( i = 1, \ldots, r \) set \( A_i \subset J(C) \) to be the image of \( \rho(me_i) \), where \( m \) is a positive integer such that \( \rho(me_i) \in \text{End}(J(C)) \). Then by [2, Prop. 13.6.1] \( A_i \) is a \( G \)-stable abelian subvariety of \( J(C) \) with \( \text{Hom}_G(A_i, A_j) = 0 \) for \( i \neq j \). Moreover, the addition map induces an isogeny
\[
A_1 \times \cdots \times A_r \to J(C).
\]
This decomposition is called the isotypical decomposition of \( J(C) \) and we write
\[
J(C) \sim A_1 \times \cdots \times A_r.
\]

The idempotents \( e_i \) can be expressed in terms of representations of \( G \) as
\[
e_i := \frac{\deg \chi_i}{|G|} \sum_{g \in G} \text{tr}_{L_i|Q}(\chi_i(g^{-1}))g,
\]
(3) \{eq:ei\}
where $\chi_i$ is the character of an irreducible representation of $\rho_i: G \to GL(V_i)$ for some complex vector space $V_i$, and $L_i := \mathbb{Q}(\chi_i(g), g \in G)$.

For each $i = 1, ..., n$, let $K_i$ be the kernel of $\rho_i$ and consider the quotient curve $\tilde{C}_i = C/K_i$ with induced morphism

$$\psi_i: C \to \tilde{C}_i.$$  \hspace{1cm} (4)

Note that $\tilde{C}_i$ is smooth, $\psi_i$ has degree $|K_i|$ and, if the action of $G$ on $C$ is known, it is easy to compute the genus of $\tilde{C}_i$.

**Lemma 7.** With the above notation, $A_i$ is an abelian subvariety of $\psi_i^*(J(\tilde{C}_i))$.

**Proof.** By [6, Prop. 5.2] we have $\psi_i^*(J(\tilde{C}_i)) = \text{Im}(p_{K_i})$, where

$$p_{K_i} = \frac{1}{|K_i|} \sum_{k \in K_i} k \in \mathbb{Q}[G].$$

Now, we have $\chi_i(g^{-1}) = \chi_i(kg^{-1}k^{-1}) = \chi_i(g^{-1}k^{-1})$, for all $k \in K_i$ and $g \in G$, where the first equality follows from the property of the character and the second one from the fact that $K_i = \text{ker}(\rho_i)$. So,

$$p_{K_i} e_i = \left( \frac{1}{|K_i|} \sum_{k \in K_i} k \right) \left( \frac{\deg \chi_i}{|G|} \sum_{g \in G} \text{tr}_{L_i|\mathbb{Q}}(\chi_i(g^{-1})) g \right)$$

$$= \frac{\deg \chi_i}{|G||K_i|} \sum_{k \in K_i} \sum_{g \in G} \text{tr}_{L_i|\mathbb{Q}}(\chi_i(kg^{-1}k^{-1})) kg$$

$$= \frac{\deg \chi_i}{|G||K_i|} \sum_{g \in G} \sum_{k \in K_i} \text{tr}_{L_i|\mathbb{Q}}(\chi_i(g^{-1}k^{-1})) kg,$$

and setting $h = kg$, we have

$$p_{K_i} e_i = \frac{\deg \chi_i}{|G||K_i|} \sum_{g \in G} \sum_{h \in K_i} \text{tr}_{L_i|\mathbb{Q}}(\chi_i(h^{-1})) h$$

$$= \frac{|K_i| \deg \chi_i}{|G||K_i|} \sum_{h \in G} \text{tr}_{L_i|\mathbb{Q}}(\chi_i(h^{-1})) h.$$ 

Thus $p_{K_i} e_i = e_i$ and $A_i = \text{Im}(e_i) \subset \text{Im}(p_{K_i}) = \psi_i^*(J(\tilde{C}_i))$. \hfill \Box

With the notation of (1), we set $\varphi_i = \varphi_{A_i}$ and $C_i = C_{A_i}$, so that we have

$$\varphi_i: C \to C_i.$$  \hspace{1cm} (5) \{eq:varphii\}

The next result shows that, when $A_i$ is 1-dimensional, the bound in Theorem 3 is achieved.

**Proposition 8.** With the above notation, if $\dim(A_i) = 1$ then $\deg(\varphi_i) = e(A_i)$. 

Proof. Since \( \dim(A_i) = 1 \) then \( C_i = A_i \) and \( A_i \) is a Prym-Tyurin variety of some exponent \( k = e(A_i) \) for \( C \). Hence, by Proposition 4 we have \( \deg(\varphi_i) = k \).  

To deal with the higher dimensional components of the isotypical decomposition, it is useful to understand the relation between \( \psi_i \) and \( \varphi_i \).

**Theorem 9.** With the above notation, there is a morphism \( f_i: \tilde{C}_i \to C_i \) such that \( \varphi_i = f_i \circ \psi_i \). In particular \( \deg(\varphi_i) \geq |K_i| \).

Proof. First note that given an action of \( G \) on a complex vector space \( V \), we have a representation \( \rho: G \to GL(V) \). The kernel \( K \) of this representation is the intersection of the stabilizers of the elements of \( V \), that is, \( K = \bigcap G_x \), where \( G_x = \{ g \in G \mid g \cdot x = x \} \) for each \( x \in V \). Then we have a morphism \( V \to V/K \) mapping each point \( x \) to its orbit \( O_x = \{ g \cdot x \mid g \in G \} \).

In our case, we have a representation \( \rho_i: G \to GL(V_i) \) with \( K_i = \ker(\rho_i) \). The trivial action of \( K_i \) on \( V_i \) induces a trivial action of \( K_i \) on \( A_i \). Thus, we have compatible actions of \( K_i \) on \( C \) and \( A_i \) and the morphism \( \varphi_i : C \to A_i \) induces a morphism \( \varphi_i : C/K_i \to A_i/K_i \). Therefore, as \( C/K_i = \tilde{C}_i \), \( A_i/K_i \cong A_i \) and \( \ker(\varphi_i) \) becomes \( f_i: \tilde{C}_i \to C_i \).

**Lemma 10.** Let \( A \subset A' \) be abelian subvarieties of the Jacobian of a smooth curve \( C \). Assume that \( (A', \theta') \) and \( (A, \theta) \) are Prym-Tyurin varieties of exponents \( e' \) and \( e \), respectively, for \( C \). Then

\[ e = e' \cdot e_{A'}(A), \]

where \( e_{A'}(A) \) is the exponent of \( A \) as a subvariety of \( A' \).

Moreover, if \( (A', \theta') \) is isomorphic to a principally polarized Jacobian \( (J(C'), \Theta_{C'}) \), then \( A \) is a Prym-Tyurin variety for \( C' \).

Proof. Denote by \( i : A' \hookrightarrow J(C) \), \( j : A \hookrightarrow J(C) \) and \( h : A \to A' \) the inclusion maps such that \( j = i \circ h \). By hypothesis we have

\[ i^* \Theta_C \equiv e' \theta' \quad \text{and} \quad j^* \Theta_C \equiv e \theta. \]

Moreover,

\[ e \theta \equiv j^* \Theta_C \equiv h^*(i^* \Theta_C) \equiv h^*(e' \theta') \equiv e' h^* \theta', \quad (6) \quad \{\text{eq:multexp}\}

which implies that

\[ h^* \theta' \equiv \frac{e}{e'} \theta. \]

Setting \( k = \frac{e}{e'} \), by Lemma 6.2 we have

\[ e_{A'}(A) = e(h^* \theta) = e(k \theta) = ke(\theta) = k. \]

For the last statement, by virtue of (6) it is enough to note that since \( i^* \) and \( j^* \) are embeddings, so is \( h^* \).  

**Theorem 11.** With the above notation, assume \( A_i \) is a Prym-Tyurin variety of exponent \( e(A_i) \) for \( C \). We have:
(i) \( \deg(f_i) \leq \frac{e(A_i)}{|K_i|} \). In particular, if \( e(A_i) = |K_i| \) then \( f_i \) is a normalization and \( \deg(\varphi_i) = |K_i| \).

(ii) If \( \psi_i^* \) is an embedding then \( A_i \) is a Prym-Tyurin variety of exponent \( \frac{e(A_i)}{|K_i|} \) for \( \tilde{C}_i \). Moreover, in this case, \( e(A_i) = |K_i| \) if and only if \( A_i = \psi_i^*(J(\tilde{C}_i)) \).

Proof. (i) Since \( A_i \) is Prym-Tyurin for \( C \), \( \deg(\varphi_i) \leq e(A_i) \), by Theorem 3. By Theorem 9, \( \deg(\varphi_i) = \deg(f_i) \deg(\psi_i) \), and since \( \deg(\psi_i) = |K_i| \), we have \( \deg(f_i) \leq \frac{e(A_i)}{|K_i|} \). Moreover, if \( e(A_i) = |K_i| \) then \( \deg(f_i) \leq 1 \) and thus \( \deg(f_i) = 1 \).

(ii) First note that since \( \psi_i^* \) is an embedding then \( J(\tilde{C}_i) \cong \psi_i^*(J(\tilde{C}_i)) \) and, by Proposition 5, \( J(\tilde{C}_i) \) is a Prym-Tyurin variety of exponent \( |K_i| \) for \( C \). Hence, by Lemma 10, \( A_i \) is a Prym-Tyurin variety of exponent \( \frac{e(A_i)}{|K_i|} \) for \( \tilde{C}_i \). Moreover, if \( e(A_i) = |K_i| \), then \( A_i \) is a Prym-Tyurin variety of exponent 1 for \( \tilde{C}_i \) and hence, by [2, Cor. 12.2.6], we must have \( A_i \cong J(\tilde{C}_i) \), which implies \( A_i = \psi_i^*(J(\tilde{C}_i)) \).

We remark that our results rely strongly on the hypothesis that \( A_i \) is Prym-Tyurin for \( C \). In order to understand Abel-Prym maps for non Prym-Tyurin subvarieties of \( J(C) \), it seems important first to investigate whether the bound in Theorem 3 still holds in the case where \( (\varphi_i)_*[C] \) is not a multiple of the minimal class.

Finally, in [13, Section 2], a further decomposition of the Jacobian is introduced, where each factor \( A_i \) is (non-canonically) decomposed into a product \( B_i \) of isogenous abelian subvarieties of \( A_i \). It would be interesting to study the Abel-Prym map to these \( B_i \) and compare it with the Abel-Prym map to \( A_i \). We plan to address this issue in a future work.

5 Examples

In this section we use the notation of Section 4.

5.1 Action of \( \mathbb{Z}_2 \) over a curve of genus 3

Consider \( \mathbb{Z}_2 = \{0, 1\} \) the cyclic group of order 2 acting on a smooth curve \( C \) of genus 3, such that the induced action on the Jacobian variety of \( C \), is given by

\[
\begin{align*}
\varpi(\alpha_1) &= \alpha_3, & \varpi(\alpha_2) &= \alpha_2 & \varpi(\alpha_3) &= \alpha_1; \\
\varpi(\beta_1) &= \beta_3, & \varpi(\beta_2) &= \beta_2 & \varpi(\beta_3) &= \beta_1;
\end{align*}
\]

where \( \alpha_i, \beta_i \) with \( i = 1, \ldots, 3 \) is the usual sympletic basis of \( J(C) \).

Let us present the isotypical decomposition of \( J(C) \). The character table of \( \mathbb{Z}_2 \), associated the trivial representation \( \rho_0 \) and the sign representation \( \rho_1 \), is given by
By (3) we have
\[ e_0 = \frac{1}{2}(\bar{0} + 1) \quad \text{and} \quad e_1 = \frac{1}{2}(\bar{0} - 1). \]
and the isotypical decomposition is \( J(C) \sim A_0 \times A_1. \)

The abelian variety \( A_0 \) is given by
\[
\langle 2\epsilon_1, 2\epsilon_2, 2\delta_1, 2\delta_2 \rangle_C
\]
with
\[ 2\epsilon_1 = \alpha_1 + \alpha_3, \quad \epsilon_2 = \alpha_2, \quad 2\delta_1 = \beta_1 + \beta_3 \quad \text{and} \quad \delta_2 = \beta_2. \]

Now, \( A_0 \) is a 2-dimensional variety of type \((1,2)\) and hence it is not Prym-Tyurin for \( C \). Since \( \rho_0 \) is the trivial representation, we have \( K_0 = \mathbb{Z}_2 \) and hence \( \deg(\psi_0) = |K_0| = 2 \). Hence, by Theorem [9]

\[ \deg(\psi_0) = 2\deg(f_0) \geq 2. \]

The abelian variety \( A_1 \) has dimension 1 and type \((2)\) and hence, by Proposition [8] \( \deg(\varphi_0) = 2. \)

### 5.2 Action of \( D_4 \) over a curve of genus 4

Consider \( D_4 = \langle a, b \mid a^4 = b^2 = (ab)^2 = 1 \rangle \) the dihedral group of order 8 acting on a smooth curve \( C \) of genus 4 such that the induced action on the Jacobian variety of \( C \), is given by
\[
\begin{align*}
a(\alpha_1) &= \alpha_2, \quad a(\alpha_2) = \alpha_3, \quad a(\alpha_3) = \alpha_4 \quad \text{and} \quad a(\alpha_4) = \alpha_1; \\
a(\beta_1) &= \beta_2, \quad a(\beta_2) = \beta_3, \quad a(\beta_3) = \beta_4 \quad \text{and} \quad a(\beta_4) = \beta_1; \\
b(\alpha_1) &= -\alpha_2, \quad b(\alpha_2) = -\alpha_3, \quad b(\alpha_3) = -\alpha_4 \quad \text{and} \quad b(\alpha_4) = -\alpha_1; \\
b(\beta_1) &= -\beta_2, \quad b(\beta_2) = -\beta_3, \quad b(\beta_3) = -\beta_4 \quad \text{and} \quad b(\beta_4) = -\beta_1.
\end{align*}
\]

where \( \alpha_i, \beta_i \) with \( i = 1, \ldots, 4 \) is the usual sympletic basis of \( J(C) \).

Let us present the isotypical decomposition of \( J(C) \). There are four irreducible representations of degree one and just one of degree two. The character table of \( D_4 \), associated to the irreducible representations \( \rho_0, \ldots, \rho_4 \) (see [7] or [5]) is given by

| \chi \ | 0 | 1 | \{ \alpha^2 \} | \{ \alpha, \alpha^4 \} | \{ b, a^2 b \} | \{ ab, a^4 b \} |
|-----|----|---|-------------|-------------|-------------|-------------|
| \chi_0 | 1 | 1 | 1 | 1 | 1 | 1 |
| \chi_1 | 1 | 1 | 1 | -1 | -1 | -1 |
| \chi_2 | 1 | 1 | -1 | 1 | -1 | 1 |
| \chi_3 | 1 | 1 | -1 | -1 | 1 | 1 |
| \chi_4 | 2 | -2 | 0 | 0 | 0 | 0 |
Again by (3), we have

\[ e_0 = \frac{1}{8} (1 + a + a^2 + a^3 + b + ab + a^2b + a^3b), \]

\[ e_1 = \frac{1}{8} (1 + a + a^2 + a^3 - b - a^2b - a^3b), \]

\[ e_2 = \frac{1}{8} (1 - a + a^2 - a^3 + b - ab + a^2b - a^3b), \]

\[ e_3 = \frac{1}{8} (1 - a + a^2 - a^3 - b + ab - a^2b + a^3b) \]

and

\[ e_4 = \frac{1}{4} (1 - 2a^2). \]

Hence the isotypical decomposition is

\[ J(C) \sim A_0 \times A_1 \times A_2 \times A_3 \times A_4. \]

The varieties \( A_0 \) and \( A_3 \) are both trivial. The abelian varieties \( A_1 \) and \( A_2 \) both have dimension 1 and type (4) and thus, by Proposition 8 we have \( \deg(\varphi_1) = \deg(\varphi_2) = 4 \).

Now, \( A_4 \) is a 2-dimensional variety of type (5 5) given by

\[ \langle 4\epsilon_1, 4\epsilon_2, 4\delta_1, 4\delta_2 \rangle_C \]

\[ \langle 4\epsilon_1, 4\epsilon_2, 4\delta_1, 4\delta_2 \rangle_{\mathbb{Z}} \]

with

\[ 4\epsilon_1 = \alpha_1 - 2\alpha_3, \quad 4\epsilon_2 = \alpha_2 - 2\alpha_4, \quad 4\delta_1 = \beta_1 - 2\beta_3, \quad \text{and} \quad 4\delta_2 = \beta_2 - 2\beta_4. \]

The irreducible representation \( \rho_4 \) has trivial kernel and consequently, \( \deg(\psi_4) = 1 \) and \( C_4 = C \). Moreover, as \( A_4 \) is Prym-Tyurin of exponent \( e(A_4) = 5 \) for \( C \), by [4, Lemma 1.3] \( \deg(\varphi_4) \) divides \( e(A_4) \). Thus \( \deg(\varphi_4) = 1 \) or 5. Assume first that \( \deg(\varphi_4) = 5 \). Then by Proposition [4] \( C_4 \) is smooth and \( A_4 = J(C_4) \). Hence \( g(C_4) = \dim(A_4) = 2 \) and, by Riemann-Hurwitz theorem applied to \( \varphi_4 \), we have

\[ 2g(C) - 2 = \deg(\varphi_4)(2g(C_4) - 2) + R \]

which gives \( R = -4 \), an absurd. Thus, \( \deg(\varphi_4) \neq 5 \) and we must have

\[ \deg(\varphi_4) = 1. \]

In particular, \( \deg(\varphi_4) = |K_4| \) and the bound in Theorem 9 is achieved.
5.3 Action of $D_p$, for $p$ an odd prime

Let $p$ be an odd prime and consider the dihedral group

$$D_p = \langle r, s \mid r^p = s^2 = (rs)^2 = 1 \rangle$$

acting on a smooth curve $C$. Then by [5, Theorem 6.4] we have

$$J(C) \sim A_0 \times A_1 \times A_2,$$

where the $D_p$-action on each factor is given by the trivial action $\rho_0$ on $A_0$, the alternating action $\rho_1$ on $A_1$, and a degree-2 irreducible representation $\rho_2$ on $A_2$.

We assume that for some involution $\kappa$ in $D_p$ the quotient map $C \to C/\langle \kappa \rangle$ is not a cyclic étale covering. Then by [2, Proposition 11.4.3], the pullback map $J(C/\langle \kappa \rangle) \to J(C)$ is an embedding.

Assume $A_0$ non-trivial. Now, since $\rho_0$ is the trivial action, we have $K_0 = D_p$ and $\tilde{C}_0 = C/D_p$. By [5, Theorem 6.4], $A_0$ is isomorphic to $J(\tilde{C}_0)$. Hence by [5, Proposition 5.1] and Proposition 5, $A_0$ is a Prym-Tyurin variety of exponent $2p$ for $C$ and, by Theorem 11, we have

$$\deg(\varphi_0) = 2p.$$

Now assume that $A_1$ is non-trivial. By [5, Section 4], we have $K_1 = \langle r \rangle$ and hence $\tilde{C}_1 = C/\langle r \rangle$ and $|K_1| = p$. Then by Theorem 9

$$\deg(\varphi_1) \geq p.$$

Now consider the commutative diagram

$$\begin{array}{ccc}
J(\tilde{C}_0) & \to & J(C/\langle \kappa \rangle) \\
\downarrow & & \downarrow \\
J(\tilde{C}_1) & \to & J(C).
\end{array}$$

By [5, Proposition 5.1], the map $J(\tilde{C}_0) \to J(C/\langle \kappa \rangle)$ is injective and, by hypothesis $J(C/\langle \kappa \rangle) \to J(C)$ is also injective. Hence $\psi_1^* : J(\tilde{C}_1) \to J(C)$ is injective. Now, by [5, Theorem 6.4], $A_1$ is the pullback to $J(C)$ of the Prym variety $P$ associated to $\tilde{C}_1 \to \tilde{C}_0$, that is, $P$ is the complement of the image of the map $J(\tilde{C}_0) \to J(\tilde{C}_1)$. Since $\tilde{C}_1 \to \tilde{C}_0$ is a degree-2 map, by [2, Theorem 12.3.3], if this map is either unramified or ramified in exactly two points, then $P$ is a Prym-Tyurin variety of exponent 2 for $\tilde{C}_1$. Thus, in this case $A_1$ is a Prym-Tyurin variety of exponent $2p$ for $C$ and, by Theorem 3, we have

$$p \leq \deg(\varphi_1) \leq 2p.$$

Finally, the irreducible representation $\rho_2$ has trivial kernel and $A_2$ is not a Prym-Tyurin variety for $C$, in general. Thus, we know nothing about the degree of the map $\varphi_2$.

We have shown:
Proposition 12. Let \( C \) be a smooth curve admitting an action of the dihedral group \( D_p \), for \( p \) be an odd prime. Assume that for some involution \( \kappa \in D_p \) the quotient map \( C \to C/\langle \kappa \rangle \) is not a cyclic étale covering. If the factors \( A_0 \) and \( A_1 \) of the isotypical decomposition of \( J(C) \) associated to the trivial and the alternating actions, respectively, are non-trivial, then:

- \( \deg(\varphi_0) = 2p; \)
- \( \deg(\varphi_1) \geq p. \) Moreover, if \( \tilde{C}_1 \to \tilde{C}_0 \) is either unramified or ramified in exactly two points, then \( p \leq \deg(\varphi_1) \leq 2p. \)

5.4 Action of the quaternion group \( Q_8 \)

Let \( Q_8 = \{ \pm 1, \pm i, \pm j, \pm ij \mid i^2 = j^2 = (ij)^2 = -1, (-1)^2 = 1 \} \) be the quaternion group acting on a smooth curve \( C \). By [13, Proposition 5.2], the isotypical decomposition of \( J(C) \) with respect to this action is

\[
J(C) \sim A_0 \times A_1 \times A_2 \times A_3 \times A_4,
\]

where the \( Q_8 \)-action on each factor is given by the trivial action on \( A_0 \), and the actions described in [13, Section 4.3] on the other factors.

Assume \( A_0 \) non-trivial. Then \( K_0 = Q_8 \) and \( A_0 \) is isomorphic to \( J(\tilde{C}_0) \), where \( \tilde{C}_0 = C/Q_8 \). By Theorem 9 we have

\[
\deg(\varphi_0) \geq 8.
\]

Moreover, we have equality if \( \psi_0 : C \to \tilde{C}_0 \) does not factor through a cyclic étale cover, by Proposition 5 and Theorem 11.

As for \( A_1, A_2 \) and \( A_3 \), we have by [13, Section 4.3] that \( K_1 = \langle i \rangle, K_2 = \langle j \rangle \) and \( K_3 = \langle ij \rangle \). Thus for \( \ell = 1, 2, 3 \) we have that \( |K_\ell| = 4 \), and \( \psi_0 \) factors as the composition

\[
C \to C/\langle -1 \rangle \to \tilde{C}_\ell \to \tilde{C}_0
\]

where \( \tilde{C}_\ell = C/K_\ell \).

Assume \( A_\ell \) non-trivial. Then by Theorem 9

\[
\deg(\varphi_\ell) \geq 4.
\]

Moreover, by [13, Proposition 5.2] \( A_\ell \) is the pullback to \( J(C) \) of the Prym variety \( P \) associated to the map \( \tilde{C}_\ell \to \tilde{C}_0 \), that is \( P \) is the complement of the image of the map \( J(\tilde{C}_0) \to J(\tilde{C}_\ell) \). By [2, Theorem 12.3.3] if \( \tilde{C}_\ell \to \tilde{C}_0 \) is either unramified or ramified in exactly two points then \( P \) is a Prym-Tyurin variety of exponent 2 for \( \tilde{C}_\ell \) and hence \( A_\ell \) is a Prym-Tyurin variety of exponent 8 for \( C \). In this case, by Theorem 3

\[
4 \leq \deg(\varphi_\ell) \leq 8.
\]

Finally, assume \( A_4 \) non-trivial. Since \( K_4 = \langle 1 \rangle \), we have \( \tilde{C}_4 = C \) and we cannot use Theorem 9 to obtain a lower bound on the degree of \( \varphi_5 \). On
the other hand, by [13 Proposition 5.2] $A_4$ is the Prym variety associated to the degree-2 map $C \to C/\langle -1 \rangle$, that is, the complement of the image of the pullback $J(C/\langle -1 \rangle) \to J(C)$. Thus if this map is either unramified or ramified in exactly two points, then $A_4$ is a Prym-Tyurin variety of exponent 2 for $C$ and by Theorem 3 we have

$$\deg(\varphi_4) \leq 2.$$ 

We have shown:

**Proposition 13.** Let $C$ be a smooth curve admitting an action of the quaternion group $Q_8$. If 

$$J(C) \sim A_0 \times A_1 \times A_2 \times A_3 \times A_4$$

is the isotypical decomposition of the Jacobian of $C$, ordered as in [13 Proposition 5.2], then 

$$\deg(\varphi_0) \geq 8 \quad \text{and} \quad \deg(\varphi_\ell) \geq 4$$

for $\ell = 1, 2, 3$. Moreover, we have

- If $C \to \tilde{C}_0$ does not factor through a cyclic étale cover, then $\deg(\varphi_0) = 8$;
- For $\ell = 1, 2, 3$, if the map $\tilde{C}_\ell \to \tilde{C}_0$ is either unramified or ramified in exactly two points, then $4 \leq \deg(\varphi_\ell) \leq 8$;
- If the map $C \to C/\langle -1 \rangle$ is either unramified or ramified in exactly two points, then $\deg(\varphi_4) \leq 2$.

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