Distributed Continuous-Time Strategy-Updating Rules for Noncooperative Games With Discrete-Time Communication

Xin Cai\textsuperscript{a}, Feng Xiao\textsuperscript{a}, Member, IEEE, Bo Wei\textsuperscript{a}, and Fang Fang\textsuperscript{a}, Senior Member, IEEE

Abstract—In this article, a class of continuous-time noncooperative games in networks of double-integrator agents is explored. The existing methods require that agents communicate with their neighbors in real time. In this article, we propose two discrete-time communication schemes based on the designed continuous-time strategy-updating rule for the efficient use of communication resources. First, the property of the designed continuous-time rule is analyzed to ensure that all agents’ strategies can reach the Nash equilibrium (NE). Then, we propose, respectively, periodic and event-triggered communication schemes for the discrete-time interactions among agents. The rule in the periodic case is implemented synchronously. The rule in the event-triggered case is executed asynchronously without Zeno behaviors. All agents in both cases can reach the NE asymptotically by interacting with neighbors at discrete times. Simulations are performed in networks of Cournot competition to illustrate the effectiveness of the proposed methods.

Index Terms—Double-integrator dynamics, event-triggered communication, Nash equilibrium (NE), noncooperative games, periodic communication.

I. INTRODUCTION

IN THE last decade, there has been a growing interest in adopting game theory to characterize the interactions among decision-making agents in distributed control systems [1]. This is partially because of the wide applications of game theory in smart grids [2], communication networks [3], and sensor networks [4]. In these applications, selfish agents aim to maximize/minimize their own objectives, which may be coupled with the objectives of the others. Noncooperative games provide a powerful tool to analyze how an agent does in response to the actions of other agents. By this method, the coordination objective of multiagent systems can be formulated as the Nash equilibrium (NE) which corresponds to the natural emergent behavior resulting from the interactions of selfish agents [5].

Roughly speaking, the dynamics of noncooperative games can be classified into two categories. One relies on the best-response dynamics, which derive from the most natural “game playing,” i.e., each player changes his strategy to minimize his cost, given strategies of other players [6], [7], [8]. The other relies on the gradient dynamics, which are usually suitable for continuous cost functions. From the perspective of optimization, the strategy of each player is updated along the direction of the fastest descent of its cost function.

Based on the gradient dynamics, the studies of continuous-time noncooperative games in the framework of multiagent systems have recently drawn more and more attention. In this framework, the distributed strategy-updating rules (or NE seeking algorithms) based on consensus protocols designed for games with incomplete information are totally different from the methods proposed in Bayesian games [9], [10], [11]. In such a setting, agents are forced to move to NE with the estimation of other agents’ strategies by communicating with neighbors. The corresponding results are more suitable for networked control systems, such as sensor networks [4], [12], communication networks [3], and smart grids [2], [13]. In consideration of the cyber–physical scenarios where the implementation of distributed algorithms among agents is driven both by inherent dynamics and their own interests, some studies have focused on the games among agents with complex dynamics. Distributed NE seeking algorithms were proposed for second-order linear/nonlinear systems [12], [13], [14]. Besides, (generalized) NE seeking algorithms were proposed for agents with multi-integrator dynamics [3], [15]. A network of general linear systems was steered by a distributed feedback algorithm to NE [16]. For agents with nonlinear dynamics, distributed algorithms were designed to seek NE of games [17], [18].

However, it is a basic feature that the continuous-time communications among agents are assumed in the model setup of continuous-time seeking algorithms proposed in all the above-mentioned work. For example, in mobile sensor networks,
agents (e.g., robots or vehicles) are forced by continuous-time
seeking algorithms [3], [15], which require to be implemented
by the continuous communication among agents. Taking the
communication costs into account, the continuous-time mech-
amism is undoubtedly expensive. Furthermore, due to mobile
sensors with limited energy, it is impossible to apply these
algorithms in practical situations where agents communicate
with each other only at discrete-time instants. To the best
of our knowledge, only a few works have been reported
on discrete-time communication schemes for noncoopera-
tive games. Furthermore, few studies have been reported on
the discrete-time interactions in continuous-time NE seek-
ing algorithms. For continuous-time best-response dynamics
proposed in [6], a self-triggered communication scheme was
designed. In [11] and [19], static event-triggered communica-
tion schemes were proposed for differential games. The
comparison between our proposed method and the existing
work is shown in Table I. These observations motivate us to
study the strategy-updating rules with discrete-time commu-
nication schemes to ensure the convergence of all agents’
strategies. The interaction laws among agents are designed to accurately estimate the strategies of others. During the time interval between any two successive
discrete communication times, outdated information is used.
The longer the communication is disconnected, the further
away the estimations could be driven from the real values. It
may happen that agents’ strategies are far away from the NE
of games. In such a case, how to design discrete-time com-
munication schemes to ensure the convergence of all agents’
strategies to the NE is a critical problem to be solved. Thanks
to various node-based and edge-based event-triggered control
methods proposed in multiagent systems [24], [25], [26], [27],
[28], [29], it is possible to solve this problem.

In this article, the discrete-time communication schemes of a
distributed continuous-time strategy-updating rule for agents
modeled by double-integrator systems is studied. Similar to
the existing work, all agents need to agree to implement
the designed algorithm by exchanging information with their
neighbors. Thus, agents can estimate the strategies of others
through local interactions in a distributed way. A summary of
the main contributions is given below.

1) To remove the requisite of the continuous-time com-
munication in the existing distributed NE seeking
algorithms, a periodic communication scheme is first
proposed to realize periodic and synchronous sampling.
Then, for the purpose of saving communication
resources, we design an asynchronously dynamic event-
triggered communication scheme which depends on an
internal variable with its own dynamics. The scheme is
proven to be Zeno-free. And, it is different from the
self-triggered communication [6] and the static event-
triggering schemes [19], [20]. A self-triggered scheme
predicts the next event time to sample new information
based on available information at the current triggering
time. But in the event-triggering scheme, communication
times are determined only by real-time triggering condi-
tions. Furthermore, for a system, the minimum interevent
interval in a designed dynamic event-triggered scheme
is likely to be larger than that in a static event-triggered
scheme [19], [20].

2) Agents with double-integrator dynamics are studied in
noncooperative games. Different from the updating rule
for the predictive strategy formed by the position and the
velocity proposed in [3], the strategy-updating rule given
in this article utilizes each agent’s own real-time strategy
(i.e., positions) and the estimation of strategies of other
agents for forcing itself to update its strategy along the
direction of the subgradient of its cost function.

3) Continuous cost functions are investigated in this arti-
cle, and the assumption on continuously differentiable
cost functions in the related previous works (see [14],
[16], [23], [30] and the references therein) is relaxed.
The proposed strategy-updating rules synthesize subgra-
dient dynamics and differential inclusions to deal with
continuous cost functions.

The remainder of this article is organized as follows. In
Section II, the problem formulation is given and some related
preliminaries are introduced. In Section III, a continuous-time
strategy-updating rule is proposed. The discrete-time com-
munication schemes are designed in Section IV. Simulation
examples are provided in Section V. Finally, conclusions and
future issues are presented in Section VI.

**Notations:** $\mathbb{R}$ denotes the set of real numbers. $\mathbb{R}_{\geq 0}$ is
the set of non-negative real numbers. The set composed of
non-negative integer numbers is denoted by $\mathbb{Z}_{\geq 0}$. $\mathbb{R}^{n}$ is the
$n$-dimensional real vector space. $\mathbb{R}^{n \times m}$ denotes the set of $n \times m$
real matrices. For a vector $x \in \mathbb{R}^{n}$, its Euclidean norm is $\|x\|_2$. Given
matrix $A \in \mathbb{R}^{n \times n}$, its transpose and the spectral norm are
denoted by $A^T$ and $\|A\|_2$, respectively. For matrices $A$ and $B$, $A \otimes B$
denotes their Kronecker product. $A_k$ and $\lambda_n(A)$ are the second
smallest and the largest eigenvalues of matrix $A$, respectively, and they are
denoted simply by $A_{k}$ and $\lambda_n$. Let $\text{col}(x_1, \ldots, x_n) = [x_1^T, \ldots, x_n^T]^T$. $\text{blk}(A_1, \ldots, A_n)$ is the block
diagonal matrix with diagonal elements $A_1, \ldots, A_n$. $I_n$ and $0_n$
are $n$-dimensional column vectors where all elements are 0
and 1, respectively. $I_n$ denotes the $n \times n$ identity matrix. A
zero matrix is denoted by $0$ with appropriate dimensions. A
set-valued map $\mathcal{F}(x) : \mathbb{R}^{n} \rightrightarrows \mathbb{R}^{n}$ is the map from a vector
$x \in \mathbb{R}^{n}$ to the collection of all subsets of $\mathbb{R}^{n}$.

## II. Problem Formulation and Preliminaries

### A. Problem Formulation

An $N$-person noncooperative game with $N$ agents is consid-
ered here. Let $G = (\mathcal{I}, \Omega, J)$ denote the game, where
agents are indexed in the set \( I = \{1, \ldots, N\} \), \( \Omega = \Omega_1 \times \cdots \times \Omega_N \subset \mathbb{R}^{Nn} \) with agent \( i \)'s strategy set \( \Omega_i \subset \mathbb{R}^n \), \( i \in I \). 

\( J = (J_1, \ldots, J_N) \), \( J_i(x_i, \ldots) : \Omega_i \times \prod_{j \neq i} \Omega_j \rightarrow \mathbb{R} \)

is agent \( i \)'s cost function relying on its own strategy \( x_i \in \Omega_i \) and the other agents’ strategies denoted by vector \( x_{-i} = \text{col}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N) \). Let \( x = \text{col}(x_1, \ldots, x_N) \in \Omega \), which denotes the strategy profile composed of all agents’ strategies. For the game with incomplete information, agents have to estimate strategies of the others by interactions with their neighbors in a network denoted by \( G = (I, \mathcal{E}) \), which may be undirected or directed. For the detailed concepts of graphs, please refer to [31].

We consider that each agent in the network has the inherent dynamics, modeled by a double-integrator system

\[
\begin{align*}
\dot{x}_i &= v_i \\
\dot{v}_i &= u_i.
\end{align*}
\]

In (1), \( x_i \in \mathbb{R}^n \) is agent \( i \)'s strategy, \( v_i \in \mathbb{R}^n \) is the auxiliary state, and \( u_i \in \mathbb{R}^p \) is the control input. Given the other agents’ strategies \( x_{-i} \), agent \( i \) seeks its strategy to minimize its cost in the game, i.e., \( \min_{x_i \in \Omega_i} J_i(x_i, x_{-i}) \). The double-integrator dynamics (1) can represent several types of physical systems, such as autonomous vehicles in traffic scenarios [32], mobile robots in sensor networks [12] and Euler–Lagrange systems [13].

**Assumption 1:** Let \( \Omega_i = \mathbb{R}^n \). Cost \( J_i(x_i, x_{-i}) \) is a continuous function. And, it is convex in \( x_i \) for every fixed \( x_{-i} \) and for all \( i \in I \).

Under Assumption 1, the \( N \)-person noncooperative game formulated in this article admits an NE [33, Th. 4.4]. The cost function \( J_i \) represents a goal or performance metric of agent \( i \), and it may be nonsmooth in many settings. For example, a piecewise linear price function was studied in the Cournot model [34], the performance of compressing sense is measured by \( l_1 \)-norm in [35], and the congestion costs of flow control are assumed to be piecewise smooth in communication networks [36].

In summary, the problems we need to handle include: 1) the proposal of a continuous-time strategy-updating rule for double-integrator agents with continuous cost functions; 2) the design of discrete-time communication schemes to reduce communication loads; and 3) the analysis of the proposed methods that can ensure the asymptotical convergence of strategies to the unique NE of noncooperative game \( G \).

**Remark 1:** Assumption 1 is more general than that in [23]. Under Assumption 1, it is seen that the strategy-updating rule designed in [23] is a special case corresponding to the rule designed in this article. Moreover, the analysis in [23] is only based on ordinary differential equations and the classical Lyapunov stability theory. There is no further study on the periodic communication scheme in the conference version of this article [23]. Although a part of conclusions in this article is similar to that in [23], the conclusions in this article are applicable to the cases with more general cost functions.

### B. Preliminaries

Here, we introduce some necessary notations and lemmas in noncooperative games, convex analysis, and differential inclusions.

**Definition 1** [33, Definition 3.7]: A pure NE of game \( G \) is a strategy profile \( x^* = \text{col}(x_1^*, \ldots, x_N^*) \in \Omega \) satisfying inequality \( J_i(x_i^*, x_{-i}^*) \leq J_i(x_i^*, x_{-i}^*) \) for any \( x_i \in \Omega_i \) and all \( i \in I \).

The NE is a point at which any agent has no willingness to decrease its cost by changing its strategy unilaterally.

**Lemma 1** [33, Corollary 4.2]: Under Assumption 1, game \( G \) admits a pure NE \( x^* \in \Omega \) satisfying

\[
0_n \in \partial J_i(x_i^*, x_{-i}^*) \quad \forall i \in I
\]

where \( \partial J_i(x_i, x_{-i}) \in \mathbb{R}^n \) is the subdifferential of cost function \( J_i \) at the strategy \( x_i \) for fixed \( x_{-i} \).

The condition (2) is a generalization of \( \nabla J_i(x_i^*, x_{-i}^*) = 0_n \) which denotes the gradient of the smooth function \( J_i(x_i, x_{-i}) \) with respect to \( x_i \) and is widely used in [3], [13], [14], [16], and [33]. Let \( F(x) = \prod_{i=1}^N \partial J_i(x_i, x_{-i}) \).

**Assumption 2:** The map \( F(x) : \Omega \Rightarrow \mathbb{R}^{Nn} \) is Lipschitz continuous with Lipschitz constant \( \theta > 0 \) and is strongly monotone; that is, \( (x - x')^T (d - d') \geq w\|x - x'\|^2, \ w > 0 \ \forall x, x' \in \Omega, \ d \in F(x), \ d' \in F(x') \).

**Definition 2** [37, Th. 6.1.2]: If a function \( f : C \rightarrow \mathbb{R} \) is strongly convex on a convex set \( C \subset \mathbb{R}^n \), there exists a constant \( c > 0 \) such that \( (y - y')^T (d - d') \geq c\|y - y'\|^2 \ \forall d \in \partial f(y), \ d' \in \partial f(y') \), where \( \partial f(y) \) is the subdifferential of \( f(\cdot) \) at \( y \).

**Proposition 1** [38, Proposition 9]: Let \( f: \mathbb{R}^n \rightarrow \mathbb{R} \) be a locally Lipschitz and convex function. Then:

1) \( \partial f(y) \subset \mathbb{R}^n \) is nonempty, convex, compact, and all \( d \in \partial f(y) \) satisfy that \( |d| \leq l \) for some \( l > 0 \);

2) \( \partial f(y) \) is upper semi-continuous at \( y \in \mathbb{R}^n \).

A differential inclusion [39] is considered as follows:

\[
\dot{z} \in F(z), \ z(0) = z_0
\]

where \( F: \mathbb{R}^n \Rightarrow \mathbb{R}^n \) is a set-valued map. A solution of (3) is an absolutely continuous curve \( z : [0, T] \rightarrow \mathbb{R}^n \) that satisfies (3) for almost all \( t \in [0, T] \). The set of equilibria of system (3) is \( E = \{ z \in \mathbb{R}^n \mid 0 \in F(z) \} \). If \( F: \mathbb{R}^n \Rightarrow \mathbb{R}^n \) is upper semicontinuous with nonempty, compact, and convex values, there exists a solution to (3) for any initial condition.

**Lemma 2** [38, Th. 4.1]: Let \( V: \mathbb{R}^n \rightarrow \mathbb{R} \) be a smooth function. Define \( S \subset \mathbb{R}^n \) as a strongly positively invariant set under (3). The set-valued Lie derivative of \( V \) with respect to \( F \) at \( z \) is \( \mathcal{L}_F V = \{ \xi \in \mathbb{R}^n \mid (\mathcal{V}(\xi))^T \nu \ \forall \nu \in F(z) \} \). If \( \max \mathcal{L}_F V \leq 0 \) or \( \mathcal{L}_F V = \emptyset \ \forall \xi \in S \), and the evolutions of (3) are bounded, the solutions of (3) starting from \( S \) converge to the largest weakly positively invariant set \( \mathcal{M} \) contained in \( S \cap \{ z \in \mathbb{R}^n \mid 0 \in \mathcal{L}_F V \} \). When \( \mathcal{M} \) is finite, the limit of every solution exists and is an element of \( \mathcal{M} \).

### III. Distributed Continuous-Time Strategy-Updating Rule

For the game with incomplete information, every agent is assumed to estimate the other agents’ strategies and to regulate the estimation by communicating with its neighbors.

Motivated by the augmented pseudo-gradient dynamics designed in [40], we define an estimation vector of agent \( i \) about all agents’ strategies by \( x_i^e = \text{col}(x_1^e, \ldots, x_N^e) \), where \( x_i^e \) is agent \( i \)'s estimation on the strategy of agent \( j \) and \( x_i^e = x_i \). Denote \( x_i^e_{-i} = \text{col}(x_1^e, \ldots, x_{i-1}^e, x_{i+1}^e, \ldots, x_N^e) \). Let \( a \) and \( k \) be...
positive constants to be designed. A strategy-updating rule is proposed as follows:

\[
\begin{align*}
\dot{x}_i &= v_i \\
\dot{v}_i &= -kv_i - \partial_x J_i(x_i, x'_i) - \frac{\alpha}{k} R_i \sum_{j \in \mathcal{I}} a_{ij}(x'_i - x^i) \\
\dot{x}_i' &= -\alpha S_i \sum_{j \in \mathcal{I}} a_{ij}(x^i - x'_i)
\end{align*}
\]  

(4)

where \(a_{ij}\) is the weight on edge \((i, j) \in \mathcal{E}\) of graph \(\mathcal{G}\) and \(\partial_x J_i(x_i, x'_i)\) represents the subdifferential of \(J_i(x_i, x'_i)\) at \(x_i\) with the estimation vector \(x'_i\).

\[
R_i = \begin{bmatrix}
I_{n \times (i-1)n} & I_n & 0_{n \times (N-i)n}
\end{bmatrix}
\]

and

\[
S_i = \begin{bmatrix}
I_{(i-1)n \times (i-1)n} & 0_{(i-1)n \times n} & 0_{(i-1)n \times (N-i)n} \\
0_{(N-i)n \times (i-1)n} & I_{(N-i)n \times n} & 0_{(N-i)n \times (N-i)n} \\
I_{(N-i)n \times (N-i)n} & & \end{bmatrix}
\]

are selection matrices to select the needed elements, that is, \(x_j = R_i x_i\) and \(x'_j = S_i x_i\), \(J_i(x_i, x'_i)\) is agent \(i\)'s cost determined by its own strategy and the estimation of strategies of other agents. The extra correction term \(R_i \sum_{j \in \mathcal{I}} a_{ij}(x'_i - x^i)\) is instrumental in the agreement of agents' estimation vectors. The strategy \(x_i\) is expected to evolve in the direction of any subgradient of cost function \(J_i\) to the NE. Thus, the dynamics of auxiliary state \(v_i\) can be represented as a differential inclusion. Define \(x = \text{col}(x^1, \ldots, x^N)\). A set-valued map \(\mathcal{F}(x)\) is Lipschitz continuous with Lipschitz constant \(\theta > 0\).

Assumption 3: The set-valued map \(\mathcal{F}(x)\) is Lipschitz continuous with Lipschitz constant \(\theta > 0\). Assumption 2 ensures that noncooperative game \(G\) has a unique NE [41, Th. 2]. The assumptions of Lipschitz continuity and monotonicity of involved maps are also used in [40] and [14]. Assumption 3 is an extension of Assumption 2 from the strategy space \(\Omega\) to its augmented space \(\Omega^N\). The assumption on the strong monotonicity of \(\mathcal{F}(x)\) can hold under that each \(J_i(x_i, x'_i)\) is a convex function. The following analysis mainly focuses on undirected graphs, which are assumed to be connected in the following assumption.

Assumption 4: The undirected communication graph \(\mathcal{G}\) is connected.

Let \(x = \text{col}(x^1, \ldots, x_N)\), \(v = \text{col}(v_1, \ldots, v_N)\), \(x = \text{col}(x^1, \ldots, x_N)\), \(\mathcal{R} = \text{blk}[R_1, \ldots, R_N]\), \(\mathcal{S} = \text{blk}[S_1, \ldots, S_N]\), \(\mathcal{S}x = \text{col}(x^1, \ldots, x_N)\), and \(L = L \otimes I_{Nn}\). A compact form of the designed rule (4) for all agents is given by

\[
\begin{align*}
\dot{x}_i &= v_i \\
\dot{v}_i &= -kv_i - \mathcal{F}(x) - \frac{\alpha}{k} \mathcal{R}Lx \\
\dot{S}x &= -\alpha SLx.
\end{align*}
\]  

(5)

Lemma 3: Suppose that Assumptions 1 and 4 hold. \(x^*\) is the NE of game \(G = (\mathcal{I}, \mathcal{O}, J)\) if and only if \((x^*, 0_{Nn}, 1_N \otimes x^*)\) is the equilibrium of dynamical system (5).

The proof is straightforward. We omit it for the limited space.

Theorem 1: Suppose that Assumptions 1–4 hold. Agents, modeled by (1), update their strategies by the strategy-updating rule (4). Then, all agents' strategies can reach the unique NE of game \(G = (\mathcal{I}, \mathcal{O}, J)\) asymptotically, provided that the parameters \(k\) and \(\alpha\) satisfy that \(k > \max\{2\theta/w, \theta + \sqrt{\theta^2 + \alpha\|\mathcal{R}\|}\}, \|\mathcal{R}\|/\lambda_2\}\) and \(\alpha(\lambda_2 - \|\mathcal{R}\|/k) > \theta + k^2\theta/4\), respectively, where \(\theta\) and \(w\) are defined in Assumption 2, \(\lambda_2\) is the second smallest eigenvalue of the Laplacian matrix \(\mathcal{L}\), \(\mathcal{R} = \text{blk}[R_1, \ldots, R_N]\) and \(L = L \otimes I_{Nn}\).

Proof: Define \(\bar{x} = x - x^*, \bar{v} = v - v^*,\) and \(\tilde{x} = x - 1_N \otimes x^*\). The system (5) can be transformed into

\[
\begin{bmatrix}
\dot{\tilde{x}} \\
\dot{\bar{v}} \\
\dot{\bar{x}}
\end{bmatrix} = \mathcal{F}(\tilde{x}, \bar{v}, \bar{x})
\]  

(6)

where

\[
\mathcal{F}(\tilde{x}, \bar{v}, \bar{x}) = \begin{bmatrix}
-k\bar{v} - \mathcal{F}(x) + \mathcal{F}(x^*) - \frac{\alpha}{k} \mathcal{R}L\tilde{x} \\
-\alpha \mathcal{S}L \tilde{x} \\
0
\end{bmatrix}.
\]

(7)

Recall the definition of \(\mathcal{F}(x)\). It follows from Lemma 1 that \(0_{Nn} \in \mathcal{F}(x^*)\).

Consider a Lyapunov candidate function \(V = (1/2)(\|\bar{v}\|^2 + \|\bar{k}\|^2 + \bar{x}^T S^T \bar{S}\tilde{x})\). The set-valued Lie derivative of \(V\) with respect to \(\mathcal{F}\) is given by

\[
\mathcal{L}_x V = \left\{\xi \in \mathbb{R} : \xi = -k\|\bar{v}\|^2 - 2\bar{v}^T (d - d^*) - \frac{2\alpha}{k} \bar{v}^T \mathcal{R}L\tilde{x}
\right\}.
\]

(8)

Let \(\bar{x}^* = (1/N)1_N \otimes 1_N \tilde{x}\) and \(\bar{x}^* = (1/N)1_N \otimes 1_N \tilde{x}\). Then, \(\tilde{x} \in \mathbb{R}^{Nn}\) can be decomposed into two components. One is in the consensus subspace and the other is in the orthogonal complement of the consensus subspace, that is, \(\tilde{x} = \bar{x} + \bar{x}^*\). Since \(\bar{x}^* = (1/N) \otimes x\) for some \(x \in \mathbb{R}^{n}\), it follows that \(\bar{x}^* = 0_{Nn}\); and \((\bar{x}^*)^T \bar{x}^* \leq 2\|\bar{x}^*\|^2\). From the definitions of \(\bar{x}^*\) and \(\bar{x}^*\), \((\bar{x}^*)^T \bar{x}^* = 0\). Thus, \(\|\bar{x}^*\|^2 = \|\bar{x}^*\|^2 + \|\bar{x}^*\|^2\). Define \(d \in \mathcal{F}(1_N \otimes x)\) for some \(x \in \mathcal{O}\). If \(x^i = x = i \in \mathcal{I}, \partial_x J_i(x_i, x'_i) = \partial_x J_i(x_i, x_i)\). It follows from the definitions of \(\mathcal{F}(x)\) and \(\mathcal{F}(x)\) that \(\mathcal{F}(1_N \otimes x) = \mathcal{F}(x)\). Under Assumptions 2 and 3, it follows that:

\[
-2\bar{v}^T (d - d^*) \\ \\
\leq 2\theta \|\bar{v}\| \|\bar{x}\|^2 + 2\theta \|\bar{v}\| \|\bar{x}\|^2 \\
\leq 2\theta \|\bar{v}\|^2 + \theta \|\bar{x}\|^2 + \theta \|\bar{x}\|^2
\]

(7)

and

\[
-k\bar{v}^T (d - d^*) \\ \\
\leq -k\|\bar{x}\|^2 - k\|\bar{x}\|^2 \\
\leq \theta \|\bar{x}\|^2 + \frac{k^2\theta}{4} \|\bar{x}\|^2
\]

(9)

Thus

\[
\xi \leq - (k\bar{x} - 2\theta) \|\bar{x}\|^2 - \left(\frac{k^2\theta}{4} \|\bar{x}\|^2 - \frac{k^2\theta}{4} \|\bar{x}\|^2\right).
\]

Authorized licensed use limited to the terms of the applicable license agreement with IEEE. Restrictions apply.
Since \( \xi \) is arbitrary, it follows that:

\[
\max \mathcal{L}_F V \leq - (k w - \vartheta) \| \dot{x} \|^2 - \left( k - 2 \theta - \frac{\alpha \| R L \|}{k} \right) \| \dot{\nu} \|^2 - \left( \alpha \lambda_2 - \theta - \frac{\alpha \| R L \|}{k} + \frac{\theta k^2 \theta}{4} \right) \| \dot{x}^o \|^2.
\]

If \( k > \max \{ 2 \theta / \nu, \theta + \sqrt{\theta^2 + \alpha \| R L \|}, \| R L \| / \lambda_2 \} \), and \( \alpha (\lambda_2 - \| R L \| / k) > \theta + \frac{k^2 \theta}{4} \), max \( \mathcal{L}_F V < 0 \) with \( \dot{x} \neq 0 \), \( \dot{\nu} \neq 0 \), or \( \dot{x} \neq 0 \nu^o \), max \( \mathcal{L}_F V = 0 \) only if \( \dot{x} = 0 \nu_n \), \( \dot{\nu} = 0 \), and \( \dot{x} = 0 \nu^o_n \), which indicates that all agents' strategies arrive at the NE. Recall that \( V \) is a continuously differentiable, radially unbounded and positive definite function. It follows from system (6) that the origin is the equilibrium point. According to [42, Corollary 4.2 and Th. 4.4], the largest invariant set is given by

\[
M = \left\{ \tilde{x} \in \mathbb{R}^{Nn}, \tilde{\nu} \in \mathbb{R}^{Nn}, \tilde{x} \in \mathbb{R}^{N^2n} | \tilde{x} = 0_{Nn}, \ \tilde{\nu} = 0_{Nn}, \ \tilde{x} = 0_{N^2n} \right\}.
\]

By Lemma 2, any trajectory of (6) starting from an initial condition \((x_0, v_0, \tilde{x}_0)\) converges to the invariant set \( M \). Thus, \((x, v, x)\) converges to the equilibrium \((x^*, 0_{Nn}, 1_N \otimes \dot{x}^*)\) as \( t \rightarrow \infty \). It indicates that all agents' strategies can reach the NE of noncooperative game \( G = (\mathcal{I}, \Omega, J) \).

**Remark 2**: To ensure that each agent estimates the strategies of others accurately, it is necessary to assume the connectivity of the communication graph, which is a global property of the communication graph. In the case that the communication topology is the prior knowledge to agents, parameters \( \alpha \) and \( k \) can be selected to satisfy the conditions in Theorems 1–3. In addition, \( \| R L \| \) only depends on the maximal degree of nodes in the graph, due to the special structure of \( \mathcal{R} \). If the overall communication structure is unknown, the total number of agents is necessary to be known to estimate the algebraic connectivity \( \lambda_2 \) and \( \| R L \| \). The conditions involving the Laplacian matrix of the graph can be relaxed by eigenvalue estimations or adaptive gains, such as the adaptive algorithm proposed in [43].

The above result can be extended to the weight-balanced and strongly connected directed graphs (digraphs). To avoid any confusion, we denote the second smallest eigenvalue of \((1/2)(L + L^T)\) by \( \hat{\lambda}_2 \).

**Corollary 1**: Let \( G \) be a weight-balanced and strongly connected digraph. Under Assumptions 1–3, the strategies of the agents, who have dynamics (1) and update their strategies by the strategy-updating rule (4), can asymptotically reach the unique NE of game \( G = (\mathcal{I}, \Omega, J) \), provided that \( k > \max \{ 2 \theta / \nu, \theta + \sqrt{\theta^2 + \alpha \| R L \|}, \| R L \| / \lambda_2 \} \) and \( \alpha (\hat{\lambda}_2 - \| R L \| / k) > \theta + \frac{\hat{\lambda}_2^2 \theta}{4} \), where \( \theta \) and \( w \) are defined in Assumption 2, \( \hat{\lambda}_2 \) is the second smallest eigenvalue of \((1/2)(L + L^T)\), \( \mathcal{R} = \text{blk} \{ R_1, \ldots, R_N \} \) and \( L = L \otimes I_{Nn} \).

**Proof**: The proof is similar to that for Theorem 1. The difference is the treatment on \((1/2)\tilde{x}^T S \tilde{L} \tilde{x} \) in the Lyapunov function \( V \). For a weight-balanced and strongly connected digraph, the set-valued Lie derivative of \( V \) with respect to \( \mathcal{F} \) is

\[
\mathcal{L}_F V = \left\{ \xi \in \mathbb{R} : \xi = -k \| \dot{\nu} \|^2 - 2\| \dot{x} \|^2 \right\}
\]

The rest analysis is similar to that in the proof of Theorem 1 and is omitted for saving the space.

**IV. DISTRIBUTED STRATEGY-UPDATING RULE WITH DISCRETE-TIME COMMUNICATION**

In this section, discrete-time communication schemes for strategy-updating rule (4) are explored in a reliable and secure communication network. The implementation of strategy-updating rule (4) requires agents to communicate with each other in continuous time, which facilitates the theoretical analysis. Considering the cost and execution mechanism of communication in practical scenarios, we study the strategy-updating rule in discrete-time communication schemes. In this section, communication topologies described by undirected graphs are considered.

Let \( t^i_k | k=1, \ldots, \infty \in \mathbb{R}_+ \), such that \( t^i_k < t^i_{k+1} \), denote the time sequence at which agent \( i \) broadcasts its estimation state \( x^i(t^i_k) \) to its neighbors, for all \( i \in \mathcal{I} \). Before the next time \( t^i_{k+1}, \dot{x}^i(t^i_k) = x^i(t^i_k) \) for \( t \in [t^i_k, t^i_{k+1}) \). Sometimes \( t \) is omitted for simplicity.

For agent \( i \), the strategy-updating rule (4) with discrete-time communication is given by

\[
\dot{x}^i = v_i, \quad \dot{v}_i = -k v_i - \theta_j J_i (x, \dot{x}^i_j) - \alpha \frac{R_i}{k} \sum_{j \in \mathcal{J}} a_{ij} \left( \dot{x}^i - \dot{x}^j \right) - \alpha S_i \sum_{j \in \mathcal{J}} a_{ij} \left( \dot{x}^i - \dot{x}^j \right).
\]

Next, two discrete-time communication schemes are proposed for agents to interact with each other at discrete-time instants. Under these schemes, it is analyzed that all agents' strategies can converge asymptotically to the NE of game \( G = (\mathcal{I}, \Omega, J) \). One is a periodic communication scheme and the other is an event-triggered communication scheme.

**A. Periodic Communication**

In the periodic communication scheme, all agents are equipped with ideal samplers and zero-order holders and communicate with each other synchronously at time period \( \Delta \). In other words, each agent's sensor collects information from neighbors at the sampling time sequence \( t^i_k | k=1, \ldots, \infty \) generated by \( t^i_{k+1} - t^i_k = \Delta \) for all \( i \in \mathcal{I} \). Theorem 2 presents an upper bound on the size of the execution cycle of communications among agents over an undirected graph.

**Theorem 2**: Suppose that Assumptions 1–4 hold. Each agent communicates synchronously with its neighbors over graph \( G \) every \( \Delta \) seconds, starting at zero, where \( \Delta \in (0, \tau) \) and \( \tau \) is the upper bound of communication intervals given by

\[
\tau = \frac{1}{a} \ln \left( 1 + \frac{a \xi}{a + b + b \xi} \right).
\]
In (12), \( \xi^2 = [(\alpha \lambda_2 - \theta)k - \alpha \|RL\| - k^2\theta/4 - \alpha \|L\|/2]/(\alpha \|RL\| + \alpha \|L\| k^2/2) \), \( a = (\theta + 1)/\alpha \), and \( b = \alpha \|S^T SL\| \). Agents, modeled by (1), update their strategies by the strategy-updating rule (11). Then, all agents’ strategies can reach the unique NE of game G asymptotically, provided that parameters \( k \) and \( \alpha \) satisfy that \( k > \max[2\theta/\omega, \theta + \sqrt{\theta^2 + 2\alpha \|RL\| \|L\|/\lambda_2 + \|L\|/\lambda_2}) \) and \( \alpha \lambda_2 - \|RL\|/k - \|L\|/2k > \theta + k^2\theta/4 \), respectively. Here, \( \theta \) and \( w \) are defined in Assumption 2, \( \lambda_2 \) is the second smallest eigenvalue of the Laplacian matrix \( L, R = \text{blk}[R_1, \ldots, R_N] \), and \( L = L \otimes I_{N'} \).

Proof: First, similar to the analysis of Theorem 1, we transfer the equilibrium to the origin. Let \( e_i = \hat{x}(i_j) - x'(t) \forall i \in \mathcal{I} \) and \( e = \text{col}(e_1, \ldots, e_N) \) for \( e = \hat{x} - x \). System (11) can be written in a compact form

\[
\begin{bmatrix}
\dot{\hat{x}}, \dot{\hat{v}}, \dot{S}\hat{x}
\end{bmatrix}^T \in \mathcal{F}_1(\hat{x}, \hat{v}, \hat{x})
\]

where

\[
\mathcal{F}_1(\hat{x}, \hat{v}, \hat{x}) = \begin{bmatrix}
-\dot{\hat{v}} - F(x) + F(x) - \frac{q}{4} |RL| e + \hat{x}
-\alpha \dot{S}L e + \hat{x}
\end{bmatrix}.
\]

Recall the Lyapunov function \( V \) defined in Theorem 1. The set-valued Lie derivative of \( V \) with respect to \( \mathcal{F}_1 \) is given by

\[
\mathcal{L}_{\mathcal{F}_1} V = \{ \xi \in \mathbb{R} : \xi = -k\|\hat{v}\|^2 - 2\hat{v}^T (d - d^*) - \alpha \hat{x}^T L \hat{x}
- \frac{2\alpha}{k} \hat{v}^T RL \hat{x} - \hat{k}^2 (d - d^*) - \alpha \hat{x}^T L e
- \frac{2\alpha}{k} \hat{v}^T RL e, \ x \in F(x), \ d^* \in F(x^*) \}.
\]

Similar to the analysis in Theorem 1, it follows that

\[
\xi \leq -(kw - 2\theta)\|\hat{v}\|^2 - \left(k - 2\theta - \frac{\alpha \|RL\|}{k}\right)\|\hat{v}\|^2
- \left(\alpha \lambda_2 - \theta - \frac{\alpha \|RL\|}{k} - \frac{k^2\theta}{4}\right)\|\hat{x}\|^2
+ \frac{2\alpha}{k} \|RL\| \|\hat{v}\| |e| + \alpha \|L\| \|\hat{x}\| |e|.
\]

By Young’s Inequality, we have that

\[
\frac{2\alpha}{k} \|RL\| \|\hat{v}\| |e| \leq \frac{\alpha \|L\|}{k} (\|\hat{v}\|^2 + \frac{1}{4} \|\hat{x}\|^2).
\]

and

\[
\alpha \|L\| \|\hat{x}\| |e| \leq \frac{\alpha \|R\|}{2k} \|\hat{x}\|^2 + \frac{\alpha k \|L\|}{2} |e|^2.
\]

Substituting (16) and (17) into (15) yields that

\[
\xi \leq -(kw - 2\theta)\|\hat{v}\|^2 - \left(k - 2\theta - \frac{2\alpha \|RL\|}{k}\right)\|\hat{v}\|^2
- \left(\alpha \lambda_2 - \theta - \frac{\alpha \|RL\|}{k} - \frac{k^2\theta}{4}\right)\|\hat{x}\|^2
+ \left(\alpha \|RL\| + \frac{\alpha k \|L\|}{2}\right) |e|^2.
\]

Since \( \xi \) is arbitrary, we have that

\[
\max \mathcal{L}_{\mathcal{F}_1} V \leq -(kw - 2\theta)\|\hat{v}\|^2 - \left(k - 2\theta - \frac{2\alpha \|RL\|}{k}\right)\|\hat{v}\|^2
- \left(\alpha \lambda_2 - \theta - \frac{\alpha \|RL\|}{k} - \frac{k^2\theta}{4}\right)\|\hat{x}\|^2
+ \left(\alpha \|RL\| + \frac{\alpha k \|L\|}{2}\right) |e|^2.
\]

where \( k > \max[2\theta/\omega, \theta + \sqrt{\theta^2 + 2\alpha \|RL\| \|L\|/\lambda_2 + \|L\|/\lambda_2}) \) and \( \alpha \lambda_2 - \|RL\|/k - \|L\|/2k > \theta + k^2\theta/4 \). Let \( \xi^2 = [(\alpha \lambda_2 - \theta)k - \alpha \|RL\| - k^2\theta/4 - \alpha \|L\|/2]/(\alpha \|RL\| + \alpha \|L\| k^2/2) \). If \( |e|^2 < \xi^2 \|\hat{x}\|^2 < \xi^2 \|\hat{x}\|^2, t \in [t_k, t_{k+1}] \), max \( \mathcal{L}_{\mathcal{F}_1} V < 0 \) for all \( t > 0 \). It is clear that each communication time \( t_k \), \( |e| = 0 \). Then, \( e(t) \) grows until next communication time \( t_{k+1} \) when it becomes zero again. The following analysis shows the upper bound of the communication intervals by examining the time period it takes for \( q = |e|/|\hat{x}(t)| \) to evolve from zero to \( \xi \).

\[
\dot{q} = \frac{e^T e}{|e| |\hat{x}|} - \frac{|e| |\hat{x}|}{|\hat{x}|^2} \leq \frac{|e| |\hat{x}|}{|\hat{x}|^2} \leq (1 + q) \frac{|e|}{|\hat{x}|}.
\]

The second inequality follows from the definition of \( q \) and the fact that \( |e| \leq |\hat{x}| \). In addition

\[
\|\hat{x}\| |\hat{v}| + \alpha \|L\| |\hat{x}| |\hat{v}| \leq \alpha \|RL\| |\hat{x}| |\hat{v}| + \alpha \|SL\| |\hat{x}| + \alpha \|SL\| |\hat{x}|.
\]

It follows from the evolution of (11) that \( |\hat{v}|/|\hat{x}| \leq \int_0^t e^{(t-s)F(x)} F(x) + (RL(e + x(s)))/|\alpha |\|SL\| e + x(s)) ds/|t| = 0 \). By the integration mean value theorem, \( |\hat{v}|/|\hat{x}| \leq (1 - e^{-\theta}) \|\hat{x}(s)| + |RL| e + x(s)))/|\alpha |\|SL\| e + x(s)) ds/|t| \) for some fixed \( s \in (0, f) \). It yields that \( |\hat{v}|/|\hat{x}| \leq (\theta + 1)/\alpha \) by the fact that \( \|\hat{x}(s)| < \|\|SL\| e + x(s)) |\|SL\| e + x(s)) \). Using the Comparison Lemma in [42], we have that \( q(t, \psi(t), \alpha) < \psi(t, \phi(t), \alpha) \), where \( \psi(t, \phi(t), \alpha) \) is the solution of \( \dot{\phi} = ((\theta + 1)/\alpha) + 1/\alpha + \alpha \|SL\| (1 + \psi)^2 \), with the initial state \( \phi(0, \phi_0) = \phi_0 \). Then, \( q(t, 0) \leq \psi(t, 0) = (a + b)e^{-\theta}/[(a + b(1 - e^{-\theta}] \), where \( a = (\theta + 1)/\alpha \) and \( b = \alpha \|SL\| \). The time \( \tau \) \( \phi(t, 0) \leq \xi \) is given by \( \tau = (1/\alpha) ln(1 + \alpha \|\hat{x}|/\alpha + b + \xi). \). Then, for \( t_{k+1} - t_k < \tau \), \( |e| < \xi |\hat{x}|(t)|. \) Thus, max \( \mathcal{L}_{\mathcal{F}_1} V = 0 \) if \( \hat{x} = 0 \), \( \hat{v} = 0 \), and \( \hat{x} = 0 \hat{x}(t| \) and max \( \mathcal{L}_{\mathcal{F}_1} V \neq 0 \), otherwise. The largest invariant set is the same as (10). It follows from Lemma 2 that system (13) converges to the origin asymptotically, which indicates that all agents’ strategies can reach the NE of noncooperative game G.

Remark 3: The communication period determined by \( \tau \) in Theorem 2 relies on the communication graph, cost functions of agents, and the designed parameter \( \alpha \). When cost functions are given, and the graph and parameter \( \alpha \) are fixed, \( \tau \) can be determined by (12).
B. Dynamic Event-Triggered Communication

Although the periodic triggering can be realized easily, it may degrade the system performance and use communication resources with low efficiency. In the following, an event-triggered communication scheme is designed to overcome these weaknesses.

A dynamic event-triggered mechanism, which was proposed in [44], is utilized here. Introduce an internal dynamic variable $\eta_i \in \mathbb{R}$ for each agent $i \in \mathcal{I}$, and $\eta_i$ is governed by the following dynamics:

$$
\dot{\eta}_i = -b\eta_i + \frac{1}{2} \sum_{j=1}^{N} a_{ij} \| \tilde{x}_j - \tilde{x}_i \|^2 - (2d_i + \beta_1 + \beta_2) \| \tilde{x}_i - x_i \|^2
$$

where $b > 0$, $\beta_1 = \alpha \| RL \| / k$, $\beta_2 = (\alpha - 2)k\| L \| / 2$, $k > \max \{ 2\theta / \omega, \sqrt{\theta^2 + 2\omega \| RL \|}, \| RL \| / \lambda_2 + \| L \| / (2\lambda_2) \}$, and $\alpha \lambda_2 - \| RL \| / k - \| L \| / (2k) > \lambda_3 + \theta + k^2 \theta / 4 - \| L \| / k$.

Theorem 3: Suppose that Assumptions 1–4 hold. Agent $i$ asynchronously communicates with its neighbors over graph $\mathcal{G}$ at times $\{ t_i^k \}_{k \in \mathbb{Z}_{>0}}$, starting at $t_i^0 = 0$, for all $i \in \mathcal{I}$, according to the following dynamic event-triggering rule:

$$
t_i^{k+1} = \inf \left\{ t \in (t_i^k, \infty) \left| (\beta_1 + \beta_2 + 2d_i) \| \tilde{x}_i - x_i \|^2 \geq \frac{1}{2} \sum_{j=1}^{N} a_{ij} \| \tilde{x}_j - \tilde{x}_i \|^2 + \rho \eta_i \right. \right\}
$$

where $\beta_1, \beta_2, k$, and $\alpha$ are defined in (18), $\rho > 0$, and $\eta_i$ is governed by (18). Agents, modeled by (1), update their strategies by the strategy-updating rule (11). Then, all agents’ strategies can evolve to the unique NE of game $G$ asymptotically.

Proof: Consider the Lyapunov function $V$ defined in Theorem 1, whose set-valued Lie derivative with respect to $\mathcal{F}_1$ is given by (14). According to the analysis in Theorem 2, we have that

$$
\zeta = -k \| \tilde{v} \|^2 - 2\tilde{v}^T (d - d^*) - 2\alpha \tilde{x}^T RL \tilde{x} - k \tilde{x}^T (d - d^*)
\quad - (\alpha - 1) \tilde{x}^T RL \tilde{x} - 2\alpha \tilde{x}^T RL \tilde{x} - (\alpha - 2) \tilde{x}^T L e + s
\quad \forall d \in F(x), \ d^* \in F(x^*)
$$

where $s = -\tilde{x}^T L \tilde{x} - 2\tilde{x}^T L e = -\tilde{x}^T L \tilde{x} + e^T L e$. Similar to the proof of Theorem 2, the analysis is given as follows. From $\tilde{x} = \tilde{x} + \tilde{x}^*$ and $\| \tilde{x} \|^2 = \| \tilde{x} \|^2 + \| \tilde{x}^* \|^2$, it yields that

$$
\zeta \leq -k \| \tilde{v} \|^2 - \left( k - 2\theta - \frac{2\alpha \| RL \|}{k} \right) \| \tilde{v} \|^2
\quad - \left( \alpha - 1 \lambda_2 - \theta - \frac{\alpha \| RL \|}{k} - \frac{k^2 \theta}{4} - \frac{(\alpha - 2) \| L \|}{2k} \right)
\quad \times \| \tilde{x}^* \|^2 + (\beta_1 + \beta_2) \| e \|^2 + s
$$

where $\beta_1$ and $\beta_2$ are defined in (18).

From $L = D - A$ and $D + A \geq 0$ with the degree matrix $D$ and the adjacent matrix $A$ of graph $\mathcal{G}$, it follows that $e^T L e \leq 2e^T(D \otimes I_N) e = 2\sum_{i=1}^{N} d_i \| e_i \|^2$. Therefore, we have that $s = (1/2) \sum_{i=1}^{N} d_i \| e_i \|^2 - \sum_{j=1}^{N} a_{ij} \| \tilde{x}_j - \tilde{x}_i \|^2$. Then

$$
\zeta \leq -(k\omega - 2\theta) \| \tilde{x} \|^2 - \left( k - 2\theta - \frac{2\alpha \| RL \|}{k} \right) \| \tilde{v} \|^2
\quad - \left( \alpha - 1 \lambda_2 - \theta - \frac{\alpha \| RL \|}{k} - \frac{k^2 \theta}{4} - \frac{(\alpha - 2) \| L \|}{2k} \right) \| \tilde{x}^* \|^2
\quad - \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} \| \tilde{x}_j - \tilde{x}_i \|^2
\quad + \sum_{i=1}^{N} (2d_i + \beta_1 + \beta_2) \| e_i \|^2.
$$

Let

$$
\mathcal{F}_2(\tilde{x}, \tilde{v}, \tilde{x}, \eta_i) = \begin{bmatrix}
-\tilde{v} - F(\tilde{x}) + F(\tilde{x}) - \frac{\tilde{x}}{2} RL (\tilde{x} + \tilde{v})
- \alpha SL(e + \tilde{x}) - \frac{b}{2} \sum_{i=1}^{N} a_{ij} \| \tilde{x}_j - \tilde{x}_i \|^2 - (\beta_1 + \beta_2) \| \tilde{x} - x \|^2
\end{bmatrix}
$$

be a set-valued map. Consider a Lyapunov candidate function

$$
V(\tilde{x}, \tilde{v}, \tilde{x}, \eta_i) = \tilde{x}^T \tilde{x} + \tilde{v}^T \tilde{v} + \lambda \| \tilde{x} \|^2 + s
$$

An upper bound of the set-valued Lie derivative of $V$ with respect to $\mathcal{F}_2$ is estimated by

$$
\mathcal{L}_x V_i \leq -\left( k\omega - 2\theta \right) \| \tilde{x} \|^2 - \left( k - 2\theta - \frac{2\alpha \| RL \|}{k} \right) \| \tilde{v} \|^2
\quad - \left( \alpha - 1 \lambda_2 - \theta - \frac{\alpha \| RL \|}{k} - \frac{k^2 \theta}{4} - \frac{(\alpha - 2) \| L \|}{2k} \right) \| \tilde{x}^* \|^2
\quad - \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} \| \tilde{x}_j - \tilde{x}_i \|^2
\quad - b \sum_{i=1}^{N} \eta_i.
$$

For $t \in [t_i^k, t_{i+1}^k)$, substituting triggering condition (19) into dynamics (18) yields that $\dot{\eta}_i \geq -(b + \rho) \eta_i$. Thus, $\eta_i(t) \geq \eta_i(0) e^{-(b + \rho)t}$ for $\eta_i(0) > 0$. Therefore, max $\mathcal{L}_x V_i = 0$ if $\tilde{x} = 0_N \eta_i, \tilde{v} = 0_N \eta_i$, and $\tilde{x}^0 = 0_{N \times n} \eta_i$, and max $\mathcal{L}_x V_i \neq 0$ otherwise. Similar to the analysis in Theorem 2, system (20) can converge asymptotically to the origin.

Then, the Zeno behavior is analyzed by computing a positive lower bound of the interevent times in the event-triggered process. The lower bound is denoted by $t_i^k \in \mathbb{R}_{>0}$, which is the elapsed time of the time that $(\beta_1 + \beta_2 + 2d_i) \| \tilde{x} - x(t) \|^2$ evolves from 0 to $\rho \eta_i$ for all $i \in \mathcal{I}$. Let $\phi = \sqrt{\beta_1 + \beta_2 + 2d_i} \| \tilde{x} - x(t) \| / \sqrt{\rho \eta_i}$. The derivative of $\phi$ with respect to $t$ is given by

$$
\dot{\phi} = -\frac{\sqrt{\beta_1 + \beta_2 + 2d_i} \langle \tilde{x} - x(t), \tilde{x} - x(t) \rangle^T}{\| \tilde{x} - x(t) \| / \sqrt{\rho \eta_i}}
\quad - \frac{\sqrt{\beta_1 + \beta_2 + 2d_i} \| \tilde{x} - x(t) \|}{\rho \eta_i^{3/2}}
\quad \leq \frac{\sqrt{\beta_1 + \beta_2 + 2d_i} \| \tilde{x} - x(t) \|}{\rho \eta_i^{3/2}} + \frac{b + \rho \phi}{2 \rho}.
$$

For $t \in [t_i^k, t_{i+1}^k)$, we have that $\sqrt{\beta_1 + \beta_2 + 2d_i} / \sqrt{\rho \eta_i} < d$ for some positive constant $D$. It yields that $\phi \leq D + (b + \rho) \phi / \rho$. By using the Comparison Lemma [42, Lemma 3.4] and the
fact that \(|\hat{x}_i^k - x_i(t_k)| = 0\) it is further derived that

\[
\psi(t) \leq (2\rho D)/(b + \rho) + \epsilon (2\rho D)(t - t_k) - 1,
\]

\(t \geq t_k\). Then, we have that \(t_k = 2\rho/(b + \rho) \ln((b + \rho)/(2\rho D) + 1)\). There exists a positive low bound of the interevent times, which indicates that the Zeno-behavior is excluded in the designed event-triggered scheme (19).

**Remark 4:** Compared with the studies in [6], [19], and [20], there are mainly two differences, that is, the game models and the communication schemes. In specific, static noncooperative games is studied in this article, while potential games and differential games were studied in [6], [19], and [20]. Moreover, a synchronously periodic sampling scheme and an asynchronously dynamic event-triggered broadcasting scheme are proposed in this article, while a self-triggered communication scheme was designed in [6] and static event-triggered communication schemes were designed in [19] and [20].

**Remark 5:** Although the dynamic event-triggered scheme designed in this article can significantly reduce communication frequency (see Fig. 4 in Section V), the amount of transmitted data may be huge for a large-scale multiagent system. In the case of limited communication bandwidths, the quantized technique may be huge for a large-scale multiagent system. In the case of limited communication bandwidths, the quantized technique should be applied to overcome the above-mentioned weakness. In future work, the combination of event-triggered schemes and quantized methods may be meaningful for distributed NE seeking algorithms applied in practice.

V. SIMULATIONS

Here, an example on networks of Cournot competition is given to illustrate the effectiveness of the designed continuous-time strategy-updating rule (4) and the rule with discrete-time communication (11), respectively.

The competition among distributed energy resources is considered here, where turbine-generator systems can be described by double-integrator agents. The systems communicate with each other on a ring graph [13], [14]. The cost function of agent \(i\) \((i \in \{1, \ldots, 5\})\), is

\[
J_{ii}(x_i, x_{-i}) = \delta_i + \beta_i|x_i - c_i| + \gamma_i x_i^2 - (p - a \sum_{j=1}^5 x_j^2) x_i,
\]

where \(p = 10\), \(a = 0.001\), \([\delta_1, \delta_2, \delta_3, \delta_4, \delta_5]^\top = [5, 8, 6, 9, 7]^\top\), \([\beta_1, \beta_2, \beta_3, \beta_4, \beta_5]^\top = [12, 15, 8, 11, 13]^\top\), \([\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5]^\top = [0.4, 0.5, 0.5, 0.3, 0.3]^\top\), \([c_1, c_2, c_3, c_4, c_5]^\top = [25, 48, 15, 30, 45]^\top\), and \(x(0) = [25, 30, 20, 30, 35]^\top\). The cost functions satisfy Assumptions 1 and 2.

Based on the given cost functions and the communication graph, we obtained that \(w = 0.601\), \(\theta = 1.001\), and \(\lambda_2 = 1.382\). It is further derived that \(||R\|| = 2\) and \(||L|| = 3.618\). According to the conditions in Theorem 1, we select \(k = 4\) and \(\alpha = 5\) for the continuous-time strategy-updating rule (4). And we select \(k = 4\), \(\alpha = 5\), \(b = 0.01\), and \(\rho = 3\) by the conditions in Theorem 3. The evolutions of agents’ strategies updated by (4) and (11) are shown in Fig. 1, where the dash lines depict the strategies updated by the continuous-time communication and the solid lines draw the strategies updated by the event-triggered communication. It is seen that all agents’ strategies converge to the NE of game \(G\) asymptotically. Moreover, the strategy-updating rule based on event-triggered scheme (19) has a similar convergence performance to the continuous-time one. Fig. 2 gives the triggering time sequences of all agents.

For the periodic communication scheme, we select \(k = 4\), \(\alpha = 5\), and \(\Delta = 0.1\) s according to the conditions in Theorem 2. Fig. 3 shows the evolution of all agents’ strategies updated by strategy-updating rule (11) with the periodic communication scheme. It indicates that all agents’ strategies can converge to the NE of game \(G\). In addition, Table II shows the
event times and event intervals of the five agents in the event-triggered scheme. It is seen that the average event interval is greater than the sampling period of the periodic scheme. It is seen from Fig. 4 that the communication frequency in the event-triggered scheme is lower than that in the periodic one. It further indicates that the event-triggered communication scheme can decrease the communication frequency effectively.

VI. CONCLUSION

We have designed a distributed continuous-time strategy-updating rule for double-integrator agents, whose cost functions are continuous and not necessarily continuously differentiable. Our designed rule has been analyzed to ensure the evolution of agents’ strategies to the NE of noncooperative games, if the communication graph is connected and undirected. This property is preserved in strongly connected and weight-balanced communication graphs. Then, discrete-time communication schemes for the implementation of the proposed rule are explored, such as periodic and event-triggered communication schemes. Furthermore, we have established the asymptotical convergence results on the designed strategy-updating rule in periodic and event-triggered communication schemes, and have taken care of the Zeno-behavior of the designed communication schemes. In future work, the networks with time delays, and more complex agents’ dynamics [45], [46] can be considered in the model. And, it may be an interesting issue to study the use of triggered communication schemes in the games with shared constraints.

REFERENCES

[1] L. Pavel and Y. Hong, “Distributed games and Nash equilibrium seeking in multiagent systems over networks: An introduction to the special issue,” IEEE Control Syst. Mag., vol. 42, no. 4, pp. 32–34, Aug. 2022.

[2] Z. Wang et al., “Distributed generalized Nash equilibrium seeking for energy sharing games in prosumers,” IEEE Trans. Power Syst., vol. 36, no. 5, pp. 3973–3986, Sep. 2021.

[3] A. R. Romano and L. Pavel, “Dynamic NE seeking for multi-integrator networked agents with disturbance rejection,” IEEE Trans. Control Netw. Syst., vol. 7, no. 1, pp. 129–139, Mar. 2020.

[4] B. Huang, C. Yang, Z. Meng, F. Chen, and W. Ren, “Distributed nonlinear placement for multi-cluster systems: A time-varying Nash equilibrium-seeking approach,” IEEE Trans. Cybern., vol. 52, no. 11, pp. 11614–11623, Nov. 2022.

[5] G. Hu, Y. Pang, C. Sun, and Y. Hong, “Distributed Nash equilibrium seeking: Continuous-time control-theoretic approaches,” IEEE Control Syst. Mag., vol. 42, no. 4, pp. 68–86, Aug. 2022.

[6] A. Cortés and S. Martinez, “Self-triggered best-response dynamics for continuous games,” IEEE Trans. Autom. Control, vol. 60, no. 4, pp. 1115–1120, Apr. 2015.

[7] B. Swenson, R. Murry, and S. Kar, “On best-response dynamics in potential games,” SIAM J. Control Optim., vol. 56, no. 4, pp. 2734–2767, 2018.

[8] X. Chen, A. Bramstroom, and U. Dieckmann, “Parent-preferred dispersal promotes cooperation in structured populations,” Proc. Roy. Soc. B, Biol. Sci., vol. 286, no. 1895, pp. 1–8, Jan. 2019.

[9] Y. Zheng, J. Ma, and L. Wang, “Consensus of hybrid multi-agent systems,” IEEE Trans. Neural Netw. Learn. Syst., vol. 29, no. 4, pp. 1359–1365, Apr. 2018.

[10] J. Qu, Z. Ji, C. Lin, and H. Yu, “Fast consensus seeking on networks with antagonistic interactions,” Complexity, vol. 2018, Dec. 2018, Art. no. 7831317.

[11] J. Yu and Y. Shi, “Scaled group consensus in multiagent systems with first/second-order continuous dynamics,” IEEE Trans. Cybern., vol. 48, no. 8, pp. 2259–2271, Aug. 2018.

[12] M. Ye, “Distributed Nash equilibrium seeking for games in systems with bounded control inputs,” IEEE Trans. Autom. Control, vol. 66, no. 8, pp. 3833–3839, Aug. 2021.

[13] Z. Deng, “Distributed algorithm design for aggregative games of Euler-Lagrange systems and its application to smart grids,” IEEE Trans. Cybern., vol. 52, no. 8, pp. 3835–3845, Aug. 2022.

[14] X. Cai, F. Xiao, and B. Wei, “Resilient Nash equilibrium seeking in multiagent games under false data injection attacks,” IEEE Trans. Syst., Man, Cybern., vol. 53, no. 1, pp. 275–284, Jan. 2023.

[15] M. Bianchi and S. Grammatico, “Continuous-time fully distributed generalized Nash equilibrium seeking for multi-integrator agents,” Automatica, vol. 129, Jul. 2021, Art. no. 109660.

[16] X. Cai, F. Xiao, B. Wei, M. Yu, and F. Fang, “Nash equilibrium seeking for general linear systems with disturbance rejection,” IEEE Trans. Cybern., early access, Aug. 19, 2022, doi: 10.1109/TCYB.2022.3195361.

[17] Y. Zhang, S. Liang, X. Wang, and H. Ji, “Distributed Nash equilibrium seeking for aggregative games with nonlinear dynamics under external disturbances,” IEEE Trans. Cybern., vol. 50, no. 12, pp. 4876–4885, Dec. 2020.

[18] B. Huang, Y. Zou, and Z. Meng, “Distributed-observer-based Nash equilibrium seeking algorithm for quadratic games with nonlinear dynamics,” IEEE Trans. Syst., Man, Cybern., vol. 51, no. 11, pp. 7260–7268, Nov. 2021.

[19] S. Xue, B. Luo, and D. Liu, “Event-triggered adaptive dynamic programming for zero-sum game of partially unknown continuous-time nonlinear systems,” IEEE Trans. Cybern., vol. 50, no. 9, pp. 3189–3199, Sep. 2020.

[20] Y. Yuan, Y. Wang, and L. Guo, “Event-triggered strategy design for discrete-time nonlinear quadratic games with disturbance compensations: The noncooperative case,” IEEE Trans. Syst., Man, Cybern., vol. 48, no. 11, pp. 1885–1896, Nov. 2018.

[21] C. Shi and G. Yang, “Distributed Nash equilibrium computation in aggregative games: An event-triggered algorithm,” Inf. Sci., vol. 489, pp. 289–302, Jul. 2019.

[22] W. Li and X. Mu, “Event-triggered distributed algorithm for searching general Nash equilibrium with general step-size,” Optim. Control Appl. Methods, vol. 42, no. 2, pp. 526–547, 2021.

[23] X. Cai, F. Xiao, and B. Wei, “A distributed strategy-updating rule with event-triggered communication for noncooperative games,” in Proc. 39th Chin. Control Conf., 2020, pp. 4747–4752.

[24] Y. Sun, Z. Ji, and K. Liu, “Event-based consensus for general linear multiagent systems under switching topologies,” Complexity, vol. 2020, Mar. 2020, Art. no. 5972749.
X. Ge, Q.-L. Han, L. Ding, Y. Wang, and X. Zhang, “Dynamic event-triggered distributed coordination control and its applications: A survey of trends and techniques,” *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 50, no. 9, pp. 3112–3125, Sep. 2020.

S. Liu, B. Niu, G. Zong, X. Zhao, and N. Xu, “Data-driven-based event-triggered optimal control of nonlinear systems with input constraints,” in *Nonlinear Dyn.*, vol. 109, pp. 891–909, May 2022.

Z.-M. Li, X.-H. Chang, and J. H. Park, “Quantized static output feedback fuzzy tracking control for discrete-time nonlinear networked systems with asynchronous event-triggered constraints,” *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 51, no. 6, pp. 3820–3831, Jun. 2021.

M.-Z. Dai, C. Zhang, H. Leung, P. Dong, and B. Li, “Distributed integral-type edge event- and self-triggered synchronization for nonlinear multiagent systems,” *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 52, no. 8, pp. 5259–5270, Aug. 2022.

B. Wei, F. Xiao, F. Fang, and Y. Shi, “Velocity-free event-triggered control for multiple Euler–Lagrange systems with communication time delays,” *IEEE Trans. Autom. Control*, vol. 66, no. 11, pp. 5599–5605, Nov. 2021.

K. Lu, G. Jing, and L. Wang, “Distributed algorithms for searching generalized Nash equilibrium of noncooperative games,” *IEEE Trans. Cybern.*, vol. 49, no. 6, pp. 2362–2371, Jun. 2019.

C. Godsil and G. Royle, *Algebraic Graph Theory*, New York, NY, USA: Springer, 2001.

A. Dreves and M. Gerdts, “A generalized Nash equilibrium approach for optimal control problems of autonomous cars,” *Opt. Control Appl. Methods*, vol. 39, pp. 326–342, Jan/Feb. 2018.

T. Başar and G. Olsder, *Dynamic Noncooperative Game Theory*, 2nd ed. Philadelphia, PA, USA: SIAM, 1999.

K. Lu and Q. Zhu, “Nonsmooth continuous-time distributed algorithms for seeking generalized Nash equilibria of noncooperative games via digraphs,” *IEEE Trans. Cybern.*, vol. 52, no. 7, pp. 6196–6206, Jul. 2022.

S. Jafarpour, V. Cevher, and R. Schapire, “A game theoretic approach to expander-based compressive sensing,” in *Proc. IEEE Int. Symp. Inf. Theory Proc.*, 2011, pp. 464–468.

H. Yin, U. V. Shanbhag, and P. G. Mehta, “Nash equilibrium problems with scaled congestion costs and shared constraints,” *IEEE Trans. Autom. Control*, vol. 56, no. 7, pp. 1702–1708, Jul. 2011.

J.-B. Hiriart-Urruty and C. Lemaréchal, *Fundamentals of Convex Analysis*. Berlin, Germany: Springer, 2001.

J. Cortés, “Discontinuous dynamical systems,” *IEEE Control Syst. Mag.*, vol. 28, no. 3, pp. 36–73, Jun. 2008.

J. Aubin and A. Cellina, *Differential Inclusions*. Berlin, Germany: Springer, 1984.

D. Gajdov and L. Pavel, “A passivity-based approach to Nash equilibrium seeking over networks,” *IEEE Trans. Autom. Control*, vol. 64, no. 3, pp. 1077–1092, Mar. 2019.

J. Rosen, “Existence and uniqueness of equilibrium points for concave n-person games,” *Econometrica*, vol. 33, no. 3, pp. 520–534, Jul. 1965.

H. Khalil, *Nonlinear Systems*, 3rd ed. Upper Saddle River, NJ, USA: Prentice-Hall, 2002.

M. Ye and G. Hu, “Adaptive approaches for fully distributed Nash equilibrium seeking in networked games,” *Automatica*, vol. 129, Jul. 2021, Art. no. 109661.

A. Girard, “Dynamic triggering mechanisms for event-triggered control,” *IEEE Trans. Autom. Control*, vol. 60, no. 7, pp. 1992–1997, Jul. 2015.

Y. Li, Y. Sun, and F. Meng, “New criteria for exponential stability of switched time-varying systems with delays and nonlinear disturbances,” *Nonlinear Anal. Hybrid Syst.*, vol. 26, pp. 284–291, Nov. 2017.

Y. Sun, Y. Tian, and X.-J. Xie, “Stabilization of positive switched linear systems and its application in consensus of multiagent systems,” *IEEE Trans. Autom. Control*, vol. 62, no. 12, pp. 6608–6613, Dec. 2017.

Feng Xiao received the B.S. and M.S. degrees in mathematics from Inner Mongolia University, Hohhot, China, in 2001 and 2004, respectively, and the Ph.D. degree in systems and control from Peking University, Beijing, China, in 2008. He became a Faculty Member with the School of Automation, Beijing Institute of Technology, Beijing, in 2008. From June 2010 to May 2013, he worked as a Postdoctoral Fellow with the Department of Electrical and Computer Engineering, University of Alberta, Edmonton, AB, Canada. From January 2016 to January 2017, he was a Visiting Professor with the Department of Mechanical Engineering, University of Victoria, Victoria, BC, Canada. He was also a Professor with the Harbin Institute of Technology, Harbin, China, and is currently a Professor with the School of Control and Computer Engineering, North China Electric Power University, Beijing, China. His current research interests include group intelligence, control and networked systems, event-triggered control, and hybrid dynamical systems. He is currently the Dean of the School of Control and Computer Engineering, North China Electric Power University, Beijing, China. His research interests include coordination of multiagent systems, event-triggered control, and hybrid dynamical systems.

Bo Wei received the B.S. degree in mathematics from the Hubei University for Nationalities, Enshi, China, in 2011, the M.S. degree in mathematics from China Three Gorges University, Yichang, China, in 2014, and the Ph.D. degree in control science and engineering from the Harbin Institute of Technology, Harbin, China, in 2019. He is currently an Associate Professor with the School of Control and Computer Engineering, North China Electric Power University, Beijing, China. His research interests include coordination of multiagent systems, event-triggered control, and hybrid dynamical systems.

Xin Cai received the B.S. degree in automation from North China Electric Power University, Baoding, China, in 2012, the M.S. degree in control science and engineering from Xinjiang University, Urumqi, China, in 2015, and the Ph.D. degree in control science and engineering from North China Electric Power University, Beijing, China, in 2022. Since 2015, she has been a Lecturer with the School of Electrical Engineering, Xinjiang Power University, Beijing, China. Her research interests are in the fields of optimization and decision-making problems. Xin Cai is currently working as a Control Engineer with the Beijing Electric Power Design Institute, Beijing, China. Xin Cai is a member of the IEEE Control Systems Society (CSS) and the Chinese Institute of Control Engineering (CICE).

Fang Fang (Senior Member, IEEE) received the M.S. degree in control theory and engineering from North China Electric Power University (Baoding Campus), Baoding, China, in 2001, and the Ph.D. degree in thermal power engineering from North China Electric Power University, Beijing, China, in 2005. He is currently a Professor of Control Science and Engineering and the Dean of the School of Control and Computer Engineering, North China Electric Power University. He is leading an Intergovernmental Cooperation Projects of National Key Research and Development Programs Project and the National Natural Science Foundation Project of China. He has authored or coauthored more than 60 high level publications, and headed more than 25 research projects or industrial projects. His current research interests include modeling and control of power generation units, configuration and operation of integrated energy systems, and optimal dispatching of virtual power plants.

Prof. Fang is a Council Member of the Chinese Association of Automation, and the Founding Vice-Chairman of the Chinese Society for Electrical Engineering Technical Committee on Offshore Wind Power and the China Electrotechnical Society Technical Committee on Energy Intelligence.