ABSTRACT

Massive black holes appear to be present in the nuclei of almost all galaxies, but their genesis and evolution are not well understood. As astrophysical black holes are completely characterized by their masses and spins, the observed joint distribution of these quantities contains important clues to their history. We examine the coevolution of mass and spin in binary merger growth scenarios. We find that holes are typically spun down by mergers. Rapid rotation results only if the binary’s larger member already spins quickly and the merger with the smaller hole is consistently near prograde; or, if the binary’s mass ratio approaches unity. If, as some observations have suggested, observed black holes spin rapidly, then this limits the importance of merger scenarios for the growth of black holes.

Subject headings: black hole physics — gravitation — galaxies: active, nuclei — quasars: general

1. Introduction

Black holes span a wide spectrum of masses: the case for stellar mass holes \((M \sim 10 \, M_\odot)\) in the field [e.g., Bailyn et al. (1998)] and supermassive holes \((M \sim 10^6 - 10^9 \, M_\odot)\) in galactic bulges [e.g., Ferrarese (2002); Kormendy & Gebhardt (2001)] is extremely strong; tantalizing evidence suggests middleweight holes \((M \sim 10^2 - 10^4 \, M_\odot)\) as well (Colbert & Mushotzky 1999; Colbert & Ptak 2002; Gebhardt et al. 2000; van der Marel 2001). Stellar mass holes likely form in stellar collapse; the origins of more massive holes remains mysterious. Such holes could form in the collapse of massive gas accumulations; they could grow from smaller holes by accretion; they could grow by capturing stellar mass bodies; and they could grow by repeatedly merging with holes of comparable mass. Any or indeed all of these mechanisms could contribute to the growth of a given hole.
A black hole’s spin may help identify which scenario most strongly impacted its recent history. Since spin likely drives outflows and jets in active galaxies, and since jets are presumed to align with black hole spin (Rees 1978), spin may provide an observational probe of a hole’s recent growth (Merritt 2002). We examine how spin and mass coevolve in mergers. Binaries will form following galaxy mergers (Begelman, Blandford, & Rees 1980), and may harden to the point that gravitational-wave (GW) emission drives its members together. Eventually, they encounter the last stable orbit (LSO), and then plunge and coalesce into a single hole. Our goal is to understand the mass and spin of this remnant hole.

For nearly equal mass holes, this is extremely difficult: we must model the spacetime dynamics of the transition from a binary to a single black hole, accounting for both holes’ spins and the radiated energy and angular momentum. A proper analysis requires mature numerical relativity codes [see, e.g., Lehner (2001)]. The problem is simpler for small mass ratio, \( q \equiv m_2/m_1 \equiv m/M \ll 1 \). This binary is well described as a test particle orbiting a black hole. GW emission shrinks the small hole’s orbit to the LSO, whereupon it plunges into the large hole. Neglecting the final emission of radiation after the plunge, the hole evolves simply: its mass adds the small body’s energy at the LSO, its spin adds the LSO angular momentum.

Because we only need global “conserved” quantities, this description works surprisingly well even for rather large mass ratio. Post-Newtonian analyses (Blanchet 2002; Buonanno & Damour 1999; Damour 2001) show that finite mass ratio typically changes the LSO and its orbital constants by a factor of order \( \eta \equiv mM/(m + M)^2 = q/(1 + q)^2 \leq 0.25 \). The error due to the test particle description is \( \lesssim 0.3 \) for \( q \lesssim 0.5 \). We also may safely neglect the energy and angular momentum radiated in the final merger: although its GW luminosity may be large, its duration will be very short. The mass carried off in this phase, for example, is \( \Delta M \simeq (0.01 - 0.1)Mq^2 \) (Davis, Ruffini, Press, & Price 1971; Sasaki & Nakamura 1982). Neglecting this radiation incurs an error that is less important than other errors built into our approximations, and rapidly becomes negligible for small mass ratio. Likewise, the small hole’s spin can be neglected: since a hole’s spin scales with its mass squared, spin will be less important than the orbital angular momentum, provided we exclude \( q \gtrsim 0.5 \).

We set the speed of light \( c \) and Newton’s constant \( G \) to 1; a useful conversion factor is \( 1 \, M_\odot = 1.5 \text{ km} \). Our binary has masses \( M \) and \( m \); the larger hole has mass \( M \) and spin \( |\mathbf{S}| = aM = amc/G; 0 \leq a \leq M \). Vectors are written in boldface; hatted quantities have been made dimensionless by dividing out powers of mass — e.g., \( \hat{a} = a/M \).
2. Orbital constants and black hole evolution

On reaching the LSO, the smaller hole plunges into the large hole, carrying along its orbital constants — energy $E$, angular momentum parallel to the spin $L_z$, and “Carter constant” $Q$. The Carter constant separates the equations of motion in a Hamilton-Jacobi description of black hole orbits [e.g., Misner, Thorne, and Wheeler (1973) (MTW)]. It is essentially just the “rest” of the orbit’s angular momentum: to very high accuracy (Glampedakis, Hughes, & Kennefick 2002), one can describe the binary as having an angular momentum $L^2 = Q + L^2_z$. We treat $Q$ as $L^2_{\perp}$, angular momentum projected into the equatorial plane (perpendicular to the spin). This treatment resonates with the theory of orbits in axisymmetric potentials: $Q$ is a relativistic analog of the “3rd integral” $I_3$ [cf. Binney & Tremaine (1987)]. Treating $Q$ as $L^2_{\perp}$ is exact for orbits of non-spinning holes; for maximal spin, the error is less than a few percent (Glampedakis, Hughes, & Kennefick 2002).

$E$ includes the rest mass of the orbiting body: bound orbits have $E/m < 1$, unbound orbits $E/m > 1$.

The LSO is a one-parameter set of orbits: each orbit has radius $r$ and constants $(E, L_z, Q)$ determined by the inclination angle $\iota$, defined as

$$\cos \iota = L_z / \sqrt{L_z^2 + Q} \equiv \mu \, .$$

(1)

This angle is very useful: detailed studies (Hughes 2001) show it remains practically constant during inspiral, so a distribution $f(\mu)$ describing an ensemble of binaries at formation likewise describes that ensemble at plunge. To find the constants for circular orbits at plunge, solve

$$R = 0, \quad R' = 0, \quad R'' = 0 \, ;$$

(2)

prime denotes $\partial / \partial r$, and the “potential” $R$ is given by

$$R = \left[ E(r^2 + a^2) - aL_z \right]^2 - \Delta(r) \left[ r^2 + (L_z - aE)^2 + Q \right] \, ,$$

(3)

where $\Delta(r) = r^2 - 2Mr + a^2$. This potential describes the orbit’s radial motion; see MTW, Chap. 33. The LSO is bounded by prograde ($\mu = 1$) and retrograde ($\mu = -1$) equatorial orbits, with constants (Bardeen, Press, & Teukolsky 1972)

$$\dot{r}_{LSO} = r_{LSO}/M$$

$$= 3 + Z_2 \mp \sqrt{(3 - Z_1)(3 + Z_1 + 2Z_2)} \, ,$$

(4)

$$\dot{E}_{LSO} = E_{LSO}/m = \frac{1 - 2v^2 \pm \hat{a}v^3}{\sqrt{1 - 3v^2 \pm \hat{a}v^3}} \, ,$$

(5)

$$\dot{L}_{LSO} = L_{LSO}/m = \pm \dot{r}v \frac{1 \mp 2\hat{a}v^3 + \hat{a}^2v^4}{\sqrt{1 - 3v^2 \pm \hat{a}v^3}} \, .$$

(6)
where \( v \equiv \sqrt{1/r} \), and
\[
Z_1 = 1 + (1 - \hat{a}^2)^{1/3} \left[ (1 + \hat{a})^{1/3} + (1 - \hat{a})^{1/3} \right], \\
Z_2 = \left[ 3\hat{a}^2 + Z_1^2 \right]^{1/2}.
\]
(7)
(8)
The upper sign is for prograde orbits, the lower for retrograde. We now drop the “LSO” subscript, since we always refer to these quantities at the LSO.

Using the bounding cases as initial guesses, it is straightforward to solve Eq. (2) using Newton’s method (Press et al. 1992) for the constants. The results are surprisingly well fit by a simple rule: letting \( \xi \) stand for \( r \), \( E \), or \( L \),
\[
\xi(\mu) \simeq |\xi_{\text{ret}}| + \frac{1}{2}(\mu + 1)(\xi_{\text{pro}} - |\xi_{\text{ret}}|).
\]
(9)
Using Eq. (9) for \( L \), one builds \( L_z(\mu) \approx L(\mu)\mu, L_\perp(\mu) \approx L(\mu)\sqrt{1 - \mu^2} \). These fits are extremely good for small spin, and induce errors of about 5 – 10% for \( \hat{a} \simeq 1 \).

We should also consider eccentricity \( e \). Major mergers are well described by a circular LSO (eccentricity rapidly bleeds away in the formation of a tight binary and by the inspiral), but minor mergers will have significant eccentricity. We find that a black hole growth history barely changes when eccentricity is taken into account: almost identical growth histories are obtained with \( e = 0 \) and with \( e = 1 \). We will confine our discussion to circular orbits.

It is now simple to compute a remnant’s properties. Before merger, the large hole has mass \( M \) and spin \( S = \hat{a}M^2 \) along the \( z \) axis. After merger, the remnant has mass and spin
\[
M' = M[1 + qE(\hat{a}, \mu)] , \\
S'_z = M^2[\hat{a} + qL_z(\hat{a}, \mu)] , \\
S'_\perp = qM^2\hat{L}_\perp(\hat{a}, \mu).
\]
(10)
(11)
(12)
The remnant hole is inclined at an angle \( \Delta \theta \) relative to the original hole, and has spin \( \hat{a}' \):
\[
\Delta \theta = \arccos \left( \frac{S'_z}{\sqrt{S'^2_z + S'^2_\perp}} \right), \\
\hat{a}' = \sqrt{S'^2_z + S'^2_\perp}/M'^2.
\]
(13)
(14)

3. Results

3.1. Single major merger

We now examine the remnant’s properties following a single merger, choosing \( q \) and computing \( (E, L_z, Q) \) as functions of the larger hole’s spin \( \hat{a} \) and the inclination cosine \( \mu \).
We do not yet use the approximation (9), but instead solve Eq. (2) numerically. We then use Eqs. (10) – (14) to describe the remnant.

Two examples, \( q = 0.1 \) and \( q = 0.5 \), are shown in Fig. 1. We show \( \hat{a}' \) as a function of \( \hat{a} \) and \( \mu \). Consider the \( q = 0.1 \) results first. For a broad range of \( \hat{a} \) and \( \mu \), the remnant spins relatively slowly. In many cases, \( \hat{a}' < \hat{a} \): over much of the parameter space, this hole is spun down by the merger. Rapid rotation follows only if the larger hole was already spinning rapidly and plunge occurred at shallow inclination (\( \mu \simeq 1 \)). Nearly nonspinning remnants form near \( \mu \simeq -1, \hat{a} \simeq 0.4 \): the retrograde orbit cancels the hole’s spin. This requires

\[
q \simeq q_{\text{crit}} = \hat{a}/\hat{L}_{\text{ret}}(\hat{a}) \lesssim 0.23 .
\] (15)

When \( q < q_{\text{crit}} \), the spin orientation changes very little by the merger. By contrast, when \( q > q_{\text{crit}} \), the spin is overwhelmed by the plunging body, and the orientation aligns with the plunge angle. Whereas about half of the parameter space for \( q = 0.1 \) leads to remnants with spin \( \hat{a} \lesssim 0.4 \), the half-area contour when \( q = 0.5 \) is at spin \( \hat{a} \simeq 0.8 \).

The slow spin of most remnants spin is simply understood. Angular momentum at the LSO depends strongly on inclination — cf. the fit (9). The magnitude \( |L| \) is small for prograde orbits and large for retrograde orbits. Thus \( |L| \) is smallest when it tends to augment the spin (\( S \) and \( L \) nearly parallel) and is largest when it tends to cancel (\( S \) and \( L \) nearly antiparallel). On average, the hole tends to spin down. This tendency breaks at large \( q \), when \( L \) overwhelms \( S \). The spin of the remnant is then dominated by the orbit at plunge.

### 3.2. Repeated minor mergers

After a hole grows above a certain mass, its subsequent spin evolution may be stochastic. This limit may be described by a Fokker-Planck equation, combining the secular and diffusive changes in the hole’s characteristics. We treat \( \ln M \) and \( \hat{a} \) as independent variables, so that the distribution function (per unit volume \( \hat{a} \) space) \( f(\hat{a}, \ln M) \) evolves via

\[
\frac{\partial f}{\partial \ln M} = -\frac{\partial}{\partial \hat{a}_i} (R_i f) + \frac{1}{2} \frac{\partial^2}{\partial \hat{a}_i \partial \hat{a}_j} (D_{ij} f) ,
\] (16)

where \( \hat{a}_i \) is the \( i \)-th component of \( \hat{a} \),

\[
R_i = \left\langle \frac{\Delta \hat{a}_i}{\Delta \ln M} \right\rangle , \quad D_{ij} = \left\langle \frac{\Delta \hat{a}_i \Delta \hat{a}_j}{\Delta \ln M} \right\rangle ,
\] (17)

and angle brackets mean to average over its distribution [see, e.g., Pathria (1972), Lifshitz & Pitaevski (1980)]. We restrict our attention to a population of black holes that are born
with a specific mass and spin. This solution can be used to integrate over a broad initial population. We also will limit our quantitative analysis to an isotropic distribution of (small) merging holes; generalization to an anisotropic distribution is straightforward though lengthy.

We rewrite Eq. (16) in spherical coordinates attached to the initial spin direction, \( (\hat{a}, \theta, \phi) \). For the isotropic case, symmetry dictates that the only non-zero coefficients are

\[
R \equiv \left< \frac{\Delta \hat{a}}{\Delta \ln M} \right>,
\]

\[
D_\parallel \equiv \left< \frac{(\Delta \hat{a})^2}{\Delta \ln M} \right>, \quad D_\perp \equiv \frac{\hat{a}^2}{2} \left< \frac{(\Delta \theta)^2}{\Delta \ln M} \right>.
\]

Equation (16) then becomes

\[
\frac{\partial f}{\partial \ln M} = \frac{1}{\hat{a}^2} \frac{\partial}{\partial \hat{a}} \left[ \left( \frac{1}{2} \frac{\partial}{\partial \hat{a}} \hat{a}^2 D_\parallel - \hat{a} D_\perp - \hat{a}^2 R \right) f \right] + \frac{1}{2} D_\perp \frac{1}{\hat{a}^2 \sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial f}{\partial \theta}.
\]

Evaluating the coefficients requires that we expand Eqs. (10) – (14) to leading order in \( q \).

For each merger, the changes in \( \hat{a} \) and \( \theta \) satisfy

\[
\delta \hat{a} = q \left[ \hat{L}_z(\hat{a}, \mu) - 2\hat{a} \right] + O(q^2),
\]

\[
\delta \theta = \frac{q}{\hat{a}} \hat{L}(\hat{a}, \mu) \sqrt{1 - \mu^2} + O(q^2).
\]

Combining this with \( \delta \ln M = q \hat{E}(\hat{a}, \mu) \) yields

\[
R = \left< \frac{[\hat{L}_z(\hat{a}, \mu) - 2\hat{a}]}{\hat{E}(\hat{a}, \mu)} \right> \approx -2.4\hat{a};
\]

\[
D_\parallel = \langle q \rangle \left< \frac{[\hat{L}_z(\hat{a}, \mu) - 2\hat{a}]^2}{\hat{E}(\hat{a}, \mu)} \right> \approx \langle q \rangle (4 + 4.9\hat{a}^2);
\]

\[
D_\perp = \frac{\langle q \rangle}{2} \left< \frac{[\hat{L}(\hat{a}, \mu) \sqrt{1 - \mu^2}^2]}{\hat{E}(\hat{a}, \mu)} \right> \approx \langle q \rangle (4 - 0.9\hat{a}^2).
\]

\( \langle q \rangle \) is the typical minor merger mass ratio. In the approximations, we put \( \hat{E} = 1 \) and expand in powers of \( \hat{a} \); this is permissible for \( \hat{a} \leq 0.9 \). The Fokker-Planck approach works well for \( \langle q \rangle \lesssim 0.3 \), and is easily supplemented by direct sums (as in Sec. 3.1) for major mergers.

The Fokker-Planck coefficients can be used to understand semi-quantitatively the evolution of holes that grow through minor mergers. The change in \( \hat{a} \) is dominated by the
resistive term for \( R > (D_\parallel + D_\perp)/\dot{a} \), which is the case for \( \dot{a} \gtrsim 2(q)^{1/2} \). In this case, the fluctuations about the average evolution are small. This average, secular evolution can be found by using the definition of \( R \) and integrating Eq. (23) to find

\[
\dot{a}(t) = \dot{a}(0) \left[ M(0)/M(t) \right]^{2.4}.
\]

If the power were 2 rather than 2.4, this equation would tell us that the hole’s original spin is preserved while the mass grows. The spin dies away somewhat faster because the magnitude of the change for retrograde captures is larger than that of prograde captures.

The orientation evolves diffusively. After growing by \( \Delta M \), the typical misalignment is

\[
\langle \delta \theta \rangle \simeq \sqrt{2D_\perp \Delta \ln M/\dot{a}^2} \simeq 2.7 \sqrt{\frac{\Delta M \langle q \rangle}{M \dot{a}^2}}.
\]

Significant realignment in a single merger occurs only if \( q/\dot{a} \sim 0.3 \), in accord with Eq. (15).

3.3. Rapidly spinning holes

We have argued that a hole is unlikely to acquire rapid spin through mergers. Turn to the converse problem: if we believe that a black hole is spinning with \( \dot{a} \simeq 1 \), how many minor mergers can it have experienced? To be specific, suppose that the hole is born through gravitational collapse or is spun up by accretion to \( \dot{a} \simeq 1 \). If we assume that the hole spun down to its current value by capture from an isotropic distribution of minor mergers, then as \( d\dot{a}/d \ln M \simeq -3 \) for \( \dot{a} \simeq 1 \), the mass acquired must satisfy

\[
\Delta M \lesssim (1 - \dot{a})M/3.
\]

Rapidly spinning holes are extremely unlikely to have suffered a recent merger; their spin must either be original or due to accretion.

4. Discussion

We find that the remnant of a major merger is rarely rapidly rotating: rapid rotation follows only if the larger binary member spun rapidly before merger and the plunge was nearly prograde; or, if the binary’s mass ratio \( q \simeq 1 \). Given the variety of black hole masses seen in galaxies, mergers with \( q \simeq 1 \) should be rare; given the small volume of parameter space leading to rapid rotation, serendipitous configurations leading to a rapid rotation should also be rare. Mounting evidence, mostly from spiral galaxies, suggests that in many cases massive
black holes nonetheless rapidly rotate [e.g., Wilms et al. (2001); Elvis, Risaliti, & Zamorani (2002)]. Our results strongly suggest that this spin cannot come from mergers [e.g. Wilson & Colbert (1995); Kauffmann & Haehnelt (2000)] but, instead, is consistent with the view that black hole mass (and spin) is assembled radiatively [e.g. Small & Blandford (1995); Yu & Tremaine (2002)].

The spin evolution of a hole that grows by repeated minor mergers is neatly described by a Fokker-Planck equation [Eq. (16)], taking a particularly simple form if the mergers arise from an isotropic cluster. In this case, the evolution has a secular component, which is approximately described by a “doctrine of original spin” [\( S = \dot{a}M^2 \) remains roughly constant while \( M \) grows; cf. Eq. (26)], and a diffusive component, with a spectrum of fluctuations governed by the coefficient \( D_\| \) [cf. Eq. (24)]. This is in accord with recent work on models to grow intermediate mass black holes in clusters (Miller 2002).

Finally, we predict the typical angle \( \langle \delta \theta \rangle \) by which a hole’s orientation changes following merger [Eq. (22)]. These results may be of particular observational interest. Jets launched by a black hole’s spin should track its inclination: if the hole is kicked into a new orientation, the jet will “kink”. The angle of the kink should equal the hole’s change in orientation. Applying our predicted dependence for the kink angle on the binary’s parameters may provide some insight into the conditions of the hole’s last merging, untangling a bit of its recent growth history. An abrupt change in inclination [such as discussed in Merritt & Ekers (2002)] requires a comparatively rare major merger.

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Fig. 1.— Spin $\hat{a}'$ of the remnant at mass ratio $q = 0.1$ (left) and $q = 0.5$ (center); right panel is a key. The axes are original spin $\hat{a}$ (vertical) and cosine of plunge inclination $\mu$ (horizontal). When $q = 0.1$, rapid rotation follows only if the original spin was large and the merger was nearly prograde ($\mu \sim 1$). About half of the parameter space yields $\hat{a}' \leq 0.4M$. Higher mass ratio yields more rapidly spinning remnants: when $q = 0.5$, the plunge must be nearly retrograde for $\hat{a}' < 0.6$, and no configuration yields $\hat{a}' < 0.5$. 