1. Definition of the coupling constant from the propagator

The Euclidean two point Green function in momentum space writes in the Landau gauge:

\[ G^{(2)}_{\mu\nu}(p, -p) = G^{(2)}(p^2)\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \]

where \(a, b\) are the color indices.

In any regularization scheme (lattice, dimensional regularization, etc.) with a cut-off \(\Lambda\) (\(d - 4\)) the bare gluon propagator in the Landau gauge is such that

\[ \Gamma(p^2) \equiv \lim_{\Lambda \to \infty} \frac{d \ln [G^{(2)}_{\text{bare}}(p^2, \Lambda)]}{d \ln p^2}, \]

is independent of the regularization scheme.

Lattice calculations provide us with a measurement of the bare propagator, and hence of the observable \(\Gamma\) in eq. (2).

We define the running coupling constant, \(\tilde{\alpha}(p^2)\), through:

\[ \Gamma(p^2) = -\frac{\gamma_0}{4\pi} \tilde{\alpha} - \frac{\gamma_1}{(4\pi)^2} \tilde{\alpha}^2 - \frac{\gamma_2}{(4\pi)^3} \tilde{\alpha}^3 \]

for a given set of real coefficients \(\gamma_0, \gamma_1\) and \(\gamma_2\). Note that in the MOM scheme, the gluon renormalization constant \(Z_3\) is given by \(Z_3(q^2, \Lambda) = q^2 G_{\text{bare}}(q^2, \Lambda)\). Our observable \(\Gamma\) is then the anomalous dimension of the gluon renormalization constant in the MOM scheme. But it is important to realize that this does not imply that we are stuck to the MOM scheme; \(\Gamma\) is defined in a scheme independent way by eq. (2) and can be studied in any general scheme defined by eq. (3). Nevertheless this imposes the value of \(\gamma_0\) to be equal to \(13/2\), the first (universal) term of the anomalous dimension.

Our strategy to study \(\alpha(\mu)\) on the lattice from the propagator is the following:

- Measure on the lattice the bare gluon propagator in Landau gauge, extract \(G^{(2)}_{\text{bare}}(p, a)\) following eq. (3) and compute \(\ln [Z_3(p, a)] = \ln \left[p^2 G_{\text{bare}}(p, a)\right]\).
- Choose \(\gamma_1, \gamma_2\) and integrate the two coupled differential equations:

\[ \frac{d \ln Z_3(q, a)}{d \ln q^2} = -\frac{\gamma_0}{4\pi} \tilde{\alpha} - \frac{\gamma_1}{(4\pi)^2} \tilde{\alpha}^2 - \frac{\gamma_2}{(4\pi)^3} \tilde{\alpha}^3 \]

\[ \frac{\partial \tilde{\alpha}}{\partial \ln q} = -\frac{\beta_0}{2\pi} \tilde{\alpha}^2 - \frac{\beta_1}{(2\pi)^2} \tilde{\alpha}^3 - \frac{\beta_2}{(4\pi)^3} \tilde{\alpha}^4 + O(\tilde{\alpha}^5) \]

\((\tilde{\beta}_2\) is known for any \(\gamma_1\) and \(\gamma_2\))[1]. The solution for \(Z_3(q, a)\) and \(\alpha(q)\) depends on the val-
ues $Z_3(\mu, a)$ and $\bar{\alpha}(\mu)$ given at some initial value $q = \mu$.

- Fit the lattice data with the previous solutions to determine the "best" $Z_3(\mu, a)$ and $\bar{\alpha}(\mu)$. Extract $\Lambda$ (we use three-loop expressions here) and convert to $\Lambda_{\overline{\text{MS}}}$ through $\Lambda_{\overline{\text{MS}}} = \Lambda \exp \left[ \frac{\gamma_1 - \gamma_0}{2\gamma_0\gamma_0} \right]$ where $\gamma_1 \approx 155.3$ is the value of the coefficient $\gamma_1$ in the $\overline{\text{MS}}$ scheme.

2. Hypercubic artifacts

In an hypercubic volume the momenta are the discrete sets of vectors $p_\mu = \frac{2\pi}{a} n_\mu$ where the $n_\mu$ are integer. For a given orbit of the continuum isometry group (characterized by a given value for $n^2$), there are distinct orbits of the hypercubic isometry group. For example $n_\mu = (1,1,1,1)$ and $n_\mu = (2,0,0,0)$.

A usual approach to reduce these hypercubic effects is to use only the "democratic" sets of momenta, i.e. those for which the momentum is equally distributed on the different directions and/or the use of the lattice momenta $\tilde{p}_\mu = \frac{2\pi}{a} \sin \left( \frac{\pi n_\mu}{a} \right)$. We propose another approach, thanks to our large statistics, which allows us to eliminate these hypercubic artifacts. On the lattice, an invariant scalar form factor like $G^{(2)}$ is indeed a function of the 4 invariants: $p^{[n]} = \sum_\mu p_\mu^n$ with $n = 2, 4, 6, 8$. We will neglect the invariants with degree higher than 4 since they vanish at least as $a^4$.

When several orbits exist for one $p^2$, we have found a nice linear behavior $G^{(2)}_{\text{lat}}(p^2, a^2 p^4) = G^{(2)}_{\text{lat}}(p^2, 0) + \frac{2G^{(2)}}{a^2 p^4} \left| a^2 p^{4} = 0 \right.$ $a^2 p^4$. It is then possible to extrapolate to $p^4 = 0$ and we can use $G^{(2)}_{\text{lat}}(p^2) = \lim_{p^4 \to 0} G^{(2)}_{\text{lat}}(p^2, p^4)$ in all our analysis. We have introduced this method in [3].

To reach higher energies and exploit also the cases where only one orbit is available, the method has been extended: from dimensional argument, the slope $\frac{\partial G^{(2)}}{\partial p^4}$ is expected to behave like $1/p^4$ as $L \to \infty$. As shown in [3], we found that the ansatz $\frac{\partial G^{(2)}}{\partial p^4} \left| a^2 p^{4} = 0 \right. = \frac{6}{p^4} (1 + c \exp(-d L p))$ gives a good description at fixed $\beta$ of all our data thus allowing us to extrapolate to $p^4 = 0$.

3. Finite volume effects

In [3], with our lattices at $\beta = 6.2$ and 6.4, we found that the gluon propagator has not yet reach asymptotic scaling at three-loop even at $\mu \simeq 5$ GeV. Higher energies are then required to test scaling. We have now lattices at $\beta = 6.8$, but as we are limited to 244 which is not a too large volume at this value of $\beta$, attention has to be paid to the finite volume effects.

To that end we have generated lattices at $\beta = 6.0$ with several volumes 124, 164, 244 and 324 and tried several functional forms to parametrize the finite volume effects at fixed value of $\beta$. Finally we proposed:

$$G^{(2)}_{\text{lat}}(p^2, L, a) = G^{(2)}_{\text{lat}}(p^2, \infty, a) \times \left( 1 + v_1 \left( \frac{a}{L} \right)^4 + v_2 e^{-v_3 L p} \right)$$

which gives a good parametrization of all the $\beta = 6.0$ data in the UV part. See figure [4].

Once the parameters $v_1, v_2, v_3$ have been determined from the lattices at $\beta = 6.0$, the parametrization (6) can be applied to extrapolate our results on 164 and 244 lattices at $\beta = 6.8$. The good agreement shown by the curves resulting from the extrapolation (figures can be found in [3]) is a crucial test for the validity of such a parametrization for the finite volume effects.

4. Results

To calibrate the lattice at $\beta = 6.2$, we have used the value for the the lattice spacing which has been measured recently with a non-perturbatively improved action (free from O(a) artifacts) [4]. The measured value is: $a^{-1}(\beta = 6.2) = 2.72(11)$ GeV. Other lattices have been calibrated relatively to the one at $\beta = 6.2$ with the results for $a \sqrt{s}$, the string tension in lattice units, published in [3]. We took: $a^{-1}(\beta = 6.0) = 1.94$ GeV, $a^{-1}(\beta = 6.2) = 2.72$ GeV , $a^{-1}(\beta = 6.4) = 3.62$ GeV and $a^{-1}(\beta = 6.8) = 6.03$ GeV.

We have applied the strategy described above to study $\bar{\alpha}$, using the data at $\beta = 6.8$ and found at three-loop using the MOM scheme: $\Lambda_{\overline{\text{MS}}}^{(3\text{loop})} \approx 0.346 \Lambda_{\overline{\text{MS}}}^{(3\text{loop})} = 315 \pm 12$ MeV, where the error is only statistical here.
We can exploit the large scheme space to which we have easily access to define a domain of “good schemes” by imposing some constraints on $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$. The idea is to select schemes for which the successive terms in the expansions in $\tilde{\alpha}$ for the $\beta$-function and $\Gamma$ are not exceedingly large compared to the previous terms (see [2] for the precise definition of the cuts we used). A large scheme dependence in this set of schemes would indicate that we are still far from perturbative scaling. We found that $\tilde{\text{MOM}}$ is a “good scheme” while $\text{MS}$ is not and the result for $\Lambda_{\text{MS}}^{(3\text{loop})}$ ranges from 313 to 323 MeV in the domain of “good schemes”. We consider this dispersion to give an estimate of the effect of higher order terms in the perturbative expansions and the flavorless $\Lambda_{\text{MS}}$ extracted from the propagator up to 9 GeV we quote is then:

$$\Lambda_{\text{MS}}^{(\text{propag.})} = 318 (12) \left( \frac{a^{-1}(\beta = 6.8)}{6.03 \text{ GeV}} \right) \text{MeV} \quad (7)$$

We can compare this result to the one we obtain by direct standard textbook methods from the triple gluon vertex in the $\tilde{\text{MOM}}$ scheme [1]. Data for $\alpha$ are less precise than those for the propagator so we have not yet tried to eliminate the finite volume effects with a method similar to the one used here. Consequently we did not use our $\beta = 6.8$ lattice and stay with our large volume lattices at smaller $\beta$. $\Lambda$ was found to be nearly independent of $\beta$ for $\beta = 6.0, 6.2$ and 6.4 [1]:

$$\Lambda_{\text{MS}}^{(\text{vertex})} = 292 (5) (15) \text{ MeV} \left( \frac{a^{-1}(\beta = 6.8)}{6.03 \text{ GeV}} \right) \quad (8)$$

Our two methods, based on Green functions, give compatible results, both significantly larger than the one obtained over a large energy range with the Schrödinger functional [6].

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Figure 1. Plot (a) shows the free-hypercubic propagators evaluated on $12^4$ (black squares), $16^4$ (black circles), $24^4$ (white circles), $32^4$ (white squares) lattices at $\beta = 6.0$. Plot (b) shows the extrapolation $L \to \infty$ of these free-hypercubic propagators using the parametrization given in Eq. [1].