Canonical Groups for Quantization on the Two-Dimensional Sphere and One-Dimensional Complex Projective Space

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Abstract. Using Isham’s group-theoretic quantization scheme, we construct the canonical groups of the systems on the two-dimensional sphere and one-dimensional complex projective space, which are homeomorphic. In the first case, we take SO(3) as the natural canonical Lie group of rotations of the two-sphere and find all the possible Hamiltonian vector fields, and followed by verifying the commutator and Poisson bracket algebra correspondences with the Lie algebra of the group. In the second case, the same technique is resumed to define the Lie group, in this case SU(2), of CP¹. We show that one can simply use a coordinate transformation from S² to CP¹ to obtain all the Hamiltonian vector fields of CP¹. We explicitly show that the Lie algebra structures of both canonical groups are locally homomorphic. On the other hand, globally their corresponding canonical groups are acting on different geometries, the latter of which is almost complex. Thus the canonical group for CP¹ is the double-covering group of SO(3), namely SU(2). The relevance of the proposed formalism is to understand the idea of CP¹ as a space of where the qubit lives which is known as a Bloch sphere.

1. Introduction

Group-theoretic quantization technique was proposed by Isham [1], [2] is a quantization procedure that is geometrical in nature but with emphasis on the group-theoretical aspects. The idea is to base the quantization primarily on a canonical group, G that describes the symmetries of the phase-space. The canonical group plays the role of a global analogue of the canonical commutation relations in canonical quantization. Due to this idea, an irreducible unitary representation of G, then shows the appropriate quantization of the system. This technique has been used widely by many authors such as [3], [4], [5] etc.

The aim of this paper is to adopt this technique to quantize the simple systems on S² and CP¹, which are both topologically homeomorphic to each other. The cases considered are different from those which Isham and others have discussed in the sense that the phase spaces are not cotangent bundles of some configuration space but they are compact symplectic manifolds. In order to make this paper is more informative, a summary of results for both cases are discussed.
2. Discussions

2.1. *Quantization on the phase space* $S = S^2$

Instead of choosing a canonical group of the phase space, $S = S^2$, we first begin by identifying the set of (globally-defined) canonical observables analogous to the coordinates $q_i$ and $p_j$ in the conventional phase spaces. These observables produce the set of Hamiltonian vector fields on phase space whose integral curves are generated by the one-parameter subgroups of the canonical group. This procedure is then carried out again in the case of $\mathbb{CP}^1$. We choose $G = \text{SO}(3)$ as a candidate of a canonical group of $S^2$. Then, let us consider the phase space, $S = S^2$ is parametrized by the angles $\theta$ and $\phi$. This phase space is endowed with the natural symplectic structure

$$\omega = \sin\theta \, d\theta \wedge d\phi$$  \hspace{1cm} (1)

where $(\theta, \phi)$ is the spherical coordinates. So, one could choose the following set of the globally well-defined functions on $S = S^2$ as

$$u = \sin\theta \cos\phi$$  \hspace{1cm} (2)

$$v = \sin\theta \sin\phi$$  \hspace{1cm} (3)

$$w = \cos\theta$$  \hspace{1cm} (4)

to be the canonical observables. From this set of canonical observables one can obtained the set of corresponding Hamiltonian vector fields $\mathcal{E}_f$ by the relation $\mathcal{E}_f \omega = df$, where $f$ are the observables and $\mathcal{E}_f$ are their Hamiltonian vector fields. From this, they are:

$$\mathcal{E}_u = -\sin\phi \frac{\partial}{\partial \theta} - \cot\theta \cos\phi \frac{\partial}{\partial \phi}$$  \hspace{1cm} (5)

$$\mathcal{E}_v = \cos\phi \frac{\partial}{\partial \theta} - \cot\theta \sin\phi \frac{\partial}{\partial \phi}$$  \hspace{1cm} (6)

$$\mathcal{E}_w = \frac{\partial}{\partial \phi}$$  \hspace{1cm} (7)

After obtaining the set of Hamiltonian vector fields, we could construct their commutator algebra which is as follows

$$[\mathcal{E}_u, \mathcal{E}_v] = -\mathcal{E}_w$$  \hspace{1cm} (8)

$$[\mathcal{E}_v, \mathcal{E}_u] = -\mathcal{E}_w$$  \hspace{1cm} (9)

$$[\mathcal{E}_u, \mathcal{E}_w] = -\mathcal{E}_v$$  \hspace{1cm} (10)

$$[\mathcal{E}_w, \mathcal{E}_v] = \mathcal{E}_u$$  \hspace{1cm} (11)
The Poisson bracket algebra of the observables is

\[
\{u, v\} = w; \quad \{w, u\} = v;
\]

(9)

\[
\{v, w\} = u; \quad \{v, u\} = -w;
\]

\[
\{u, w\} = -v; \quad \{u, u\} = \{v, v\} = \{w, w\} = 0;
\]

\[
\{w, v\} = -u.
\]

The commutator algebra of the Hamiltonian vector fields is thus anti-homomorphic to the Poisson bracket algebra. Their forms suggest that they are further (anti-)homomorphic to the abstract algebra of the SO(3) and thus our choice of the canonical group.

An alternate route is to obtain the action of the group on the phase space, \( S = S^2 \) given by

\[
\mathcal{L}(R, s) = Rs; \quad \forall \ R \in G, s \in S^2.
\]

(10)

We can generate the same Hamiltonian vector fields as before through their integral curves given by the group action. To conclude, SO(3) is considered to be the natural choice of the canonical group on \( S^2 \) and one can proceed to find its irreducible representation for further steps of quantization (this is not discussed here).

2.2 Quantization on the phase space \( S = \mathbb{C}P^1 \)

As we claimed that the homeomorphism of both phase spaces \( \mathbb{C}P^1 \) and \( S^2 \) lead us to assume \( G = SU(2) \) as the natural candidate of the canonical group of \( S = \mathbb{C}P^1 \), due to the homomorphism of the Lie algebras of SO(3) and SU(2).

It is convenient to use the same technique as mentioned before to verify the canonical group of the phase space, \( S = \mathbb{C}P^1 \). In order to simplify the calculations we shall work within the globally well-defined projective complex coordinates \( z \) and its conjugate \( z^* \) on the sphere homeomorphic to \( \mathbb{C}P^1 \) which is given by coordinate transformation,

\[
Z = \tan \frac{\theta}{2} e^{i\phi} \quad \text{and} \quad Z^* = \tan \frac{\theta}{2} e^{-i\phi},
\]

(11)

where \((\theta, \phi)\) are the earlier coordinates of \( S^2 \). One can use the natural symplectic structure on the sphere together with (11) to form the complex symplectic structure on \( \mathbb{C}P^1 \) given by

\[
\omega = \frac{2i dZ \wedge dZ^*}{(1 + |Z|^2)^2}.
\]

(12)

For simplicity the coordinate transformations could also be used to construct the canonical observables from (2) – (4). Hence we have
\[ u' = \frac{Z + Z^*}{1 + |Z|^2}; \quad (13) \]
\[ v' = \frac{-iZ + iZ^*}{1 + |Z|^2}; \quad (14) \]
\[ w' = \frac{1 - |Z|^2}{1 + |Z|^2}, \quad (15) \]

and these observables will further lead to the Hamiltonian vector fields as
\[ \epsilon_u = -\frac{i}{2} (1 - Z^2) \frac{\partial}{\partial Z} + \frac{i}{2} (1 - Z^{*2}) \frac{\partial}{\partial Z^*}; \quad (16) \]
\[ \epsilon_v = \frac{i}{2} (1 + Z^2) \frac{\partial}{\partial Z} + \frac{i}{2} (1 + Z^{*2}) \frac{\partial}{\partial Z^*}; \quad (17) \]
\[ \epsilon_w = iZ \frac{\partial}{\partial Z} - iZ^* \frac{\partial}{\partial Z^*}. \quad (18) \]

From the above vector fields one can get the following commutator algebra as
\[ [\epsilon_u, \epsilon_v] = -\epsilon_w; \quad [\epsilon_u, \epsilon_w] = \epsilon_v; \quad (19) \]
\[ [\epsilon_v, \epsilon_w] = -\epsilon_u; \quad [\epsilon_v, \epsilon_u] = \epsilon_w; \]
\[ [\epsilon_w, \epsilon_u] = -\epsilon_v; \quad [\epsilon_w, \epsilon_v] = [\epsilon_v, \epsilon_u] = [\epsilon_u, \epsilon_w] = 0; \]
\[ [\epsilon_w, \epsilon_v] = \epsilon_u. \]

and the Poisson bracket algebra of the observables as
\[ \{u', v'\} = w'; \quad \{w', u'\} = v'; \quad (20) \]
\[ \{v', w'\} = u'; \quad \{v', u\} = -w'; \]
\[ \{u', w\} = -v'; \quad \{u', u\} = \{v', v\} = \{w', w\} = 0; \]
\[ \{w', v'\} = -u'. \]

The algebraic relations are the same as in (8) and (9). Note that in (13)-(15) and (16)-(18), there is a hidden discrete symmetry under the transformations \( Z \rightarrow Z^*, i \rightarrow -i \). The discrete symmetry here shows that there is a hidden discrete symmetry on the phase-space that keeps the symplectic structure of \( \mathbb{CP}^1 \) invariant. This may have implications in the global structure of the desired canonical group. On the other hand, we can construct the canonical group by exponentiating the vector fields and act upon the space of functions (square-integrable function). This gives a particular representation of the group. It is expected that the canonical group goes to the double cover of \( \text{SO}(3) \) i.e. \( \text{SU}(2) \). Presently details of this necessary lifting are being worked out.
3. Conclusions

In this study, we have shown that the group-theoretic quantization problems of the two-sphere and the one-dimensional complex projective space are related to each other through their algebraic structures. However, at the global level, they give different canonical groups which reflect the different underlying geometry.

Note that the present study only deals with the simplest case of the complex projective space and it is connected to the well-known example of (geometric) quantization on the two-sphere. This motivates us to consider the framework of Isham’s group-theoretic quantization for quantizing complex projective spaces in general. In [12], the complex projective spaces are taken to be the classical phase spaces corresponding to the quantum states in quantum mechanics. It is thus then possible to use this framework to describe multiple qubit or qudit states as arising (geometrically) from a quantization problem and in particular aspects of entanglement.

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