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II: DERIVED LIMITS OVER A POSPACE

SERGEY A. MELIKHOV

Abstract. Čech cohomology $H^n(X)$ of a separable metrizable space $X$ is defined in terms of cohomology of its nerves (or ANR neighborhoods) $P_β$, whereas Steenrod–Sitnikov homology $H_n(X)$ is defined in terms of homology of compact subsets $K_α ⊂ X$.

We show that one can also go vice versa: in a sense, $H^n(X)$ can be reconstructed from $H^*(K_α)$, and if $X$ is finite dimensional, $H_n(X)$ can be reconstructed from $H_∗(P_β)$.

The reconstruction is via a Bousfield–Kan/Araki–Yoshimura type spectral sequence of the form $\lim \limits_p H^q(K_α) \Rightarrow H^{p+q}(X)$, respectively $\lim \limits_p H^q(P_β) \Rightarrow H^{-p}(X)$, except that the derived limits have to be “corrected” so as to take into account a natural topology on the indexing set. The corrected derived limits coincide with the usual ones when the topology is discrete, and in general are applied not to an inverse system but to a “partially ordered sheaf”.

The “correction” of the derived limit functors in turn involves constructing a “correct” (metrizable) topology on the order complex $|P|$ of a partially ordered metrizable space $P$ (such as the hyperspace $K(X)$ of nonempty compact subsets of $X$ with the Hausdorff metric). It turns out that three natural approaches (by using the space of measurable functions, the space of probability measures, or the usual embedding $K(X) → C(X; ℝ)$) all lead to the same topology on $|P|$.

1. INTRODUCTION

Let $X$ be a separable metrizable space, let $K_α$ run over its compact subsets, and let $P_β$ run over the nerves of open covers of $X$ (or over open neighborhoods of $X$ in some ANR). The Steenrod–Sitnikov homology and the Čech cohomology of $X$ are defined by

$$H_n(X) := \colim_α H_n(K_α),$$
$$H^n(X) := \colim_β H^n(P_β),$$

where $H_n(K_α)$ is Steenrod homology (concerning the latter see e.g. [25] or [29]).

It is very natural to try to do everything vice versa, and define some kind of homology of $X$ in terms of $H_n(K_α)$ and some kind of cohomology of $X$ in terms of $H^n(P_β)$. Of course, instead of the direct limits (colim) it would be natural to use inverse limits (lim), as well as their standard “correction terms” — higher derived limits $\lim^p$, $p > 0$.

1.A. Strong (co)homology. This natural path leads rather unambiguously to “strong homology” and “strong cohomology”, which were introduced by Lisica and Mardešić and independently by Miminošvili in the 1980s. Despite doubtless naturality of their
construction, there is a big trouble with these “strong” groups: they cannot be computed in ZFC for the very simplest examples.

**Example 1.1.** Let \( N \) denote the countable discrete space and \( N^+ \) its one-point compactification. The space \( N \times N^+ \) is arguably the very simplest non-compact, non-triangulable space. (Algebraic topology of simplicial complexes and algebraic topology of compacta are completely different stories, which are much, much better understood.) But the \((-1)\)-dimensional strong homology of \( N \times N^+ \) cannot be computed in ZFC \([22]\). Of course, another issue is that one should not really be computing \((-1)\)-dimensional homology at all; but if \( N^+ \) is replaced with the \(n\)-dimensional Hawaiian earring \((R^n \times N)^+\), we have exactly the same problem with the \((n-1)\)-dimensional strong homology \([22]\). Also, the \((n+1)\)-dimensional strong cohomology of the metric quotient \( N \times (R^n \times N)^+ / (N \times \infty) \), which can alternatively be described as the non-compact cluster (=metric wedge) \( \bigvee_{i \in N} (R^n \times N) \), cannot be computed in ZFC for the same reason (see \([32]\; \text{Example 1.3}\)).

This strange issue makes strong homology and cohomology much more interesting for the purposes of Foundations of Mathematics, but at the same time obviously “defective” for the purposes of geometric topology: at least with the current level of depth of human thought, it appears that statements carrying genuine geometric meaning have too low logical complexity to have a chance of being independent of ZFC. Nevertheless, over the years there has appeared a considerable amount of literature about strong homology and strong cohomology (see \([22]\), \([21]\), \([20]\) and references there). One attractive feature of strong homology is its built-in invariance under strong shape. It is an open problem whether Steenrod–Sitnikov homology is an invariant of strong shape. However, as fine shape is now available \([30]\), relevance of this problem is no longer obvious.

1.B. **Topology of the indexing set.** The goal of the present paper is to “do everything vice versa” slightly more carefully, so as to avoid running into set-theoretic troubles. The trouble noted in Example 1.1 is really a very basic trouble with the \( \lim^1 \) functor (see \([32]\; \text{Example 1.2}\)); but it exists only for inverse systems indexed by an uncountable set, or more precisely by a set with no countable cofinal subset. It is therefore natural to look for a “forgotten topology” of the indexing set, and to simply amend the definition of derived limits so as to take this topology into account. This is precisely what we will do in the present paper.

The indexing set we are mostly interested in is the poset \( K(X) \) of all nonempty compact subsets of a separable metrizable space \( X \). (The empty subset is excluded for technical reasons; its exclusion will be harmless.) Given some metric on \( X \), the usual Hausdorff metric (see §4.A) on \( K(X) \) makes it into a separable metric space, whose underlying topology (also known as the Vietoris topology) depends only on that of \( X \). In fact, \( K(X) \) turns out to be a pospace (=partially ordered space): its topology and order agree in the sense that the order relation is a closed set (Lemma 4.1).
As for open covers of $X$ or their nerves, unfortunately they do not seem to carry any natural separable metrizable topology.\footnote{Also, a technical disadvantage is that they do not form a poset; while irreducible open covers do, their topology might be less natural. One can, of course, work with preorders or filtered categories instead of posets, so this is not a serious problem.} While a more elegant solution may exist, one obvious thing to do is to simply embed $X$ in some compact ANR, for example the Hilbert cube $Q$, and look at its open neighborhoods there. Their complements are precisely all compact subsets of $Q \setminus X$, and so the previous construction applies here as well.

1.C. Topological order complex. The usual derived limits are revisited in §2 below, where their usual definition is reformulated in a more geometric form. Namely, the derived limits of an inverse system $D$ indexed by a poset $P$ can be understood as the cohomology groups of a certain sheaf, denoted $\text{holim} D$, over the order complex $\vert P \vert$.

Therefore our next goal is to define an appropriate topology on the order complex of a poset $P$ that is also endowed with a topology.

An obvious approach is to consider the geometric realization of the corresponding simplicial space, also known as the classifying space of the topological category (see §3.A). However, the resulting topology can well be non-metrizable even when the original topology of $P$ is discrete, so it is “wrong” for our purposes.

The next obvious approach is to inject $\vert P \vert$ into some metrizable (perhaps, vector) space naturally “spanned” by $P$ and take the induced topology. But it turns out that there is more than one good way to do so.

The main result of §3 is as follows (see Theorem 3.8).

**Theorem A.** If $P$ is a pospace, the obvious injective maps of the order complex $\vert P \vert$ into the space of measurable functions $[0,1] \to P$ and into the space of probability measures on $P$ induce the same topology on the set $\vert P \vert$.

In §4 we additionally prove the following (see Lemma 4.3(c) and Theorems 4.9, 4.10).

**Theorem B.** The embedding $e$ of $K(X)$ into the space $C(X, \mathbb{R})$ of real-valued functions on $X$, given by $e(A)(x) = d(x,A)$, extends to an injective map of $\vert K(X) \vert$ in $C(X, \mathbb{R})$ such that the induced topology on $\vert K(X) \vert$ does not depend on the choice of the metric $d$ on $X$. Moreover, it coincides with the two topologies of Theorem A.

1.D. Derived limits over a pospace. Let $P$ be a pospace and $\mathcal{F}$ be a posheaf (=partially ordered sheaf) of abelian groups over $P$ (see definition in §5.A). Then there is a certain sheaf $\text{holim} \mathcal{F}$ of abelian groups over the metrizable topological order complex $\vert P \vert$ (see definition in §5.B), and we define $\text{Lim}^p \mathcal{F}$ to be $H^p(\vert P \vert; \text{holim} \mathcal{F})$. When the topology of $P$ is discrete, $\mathcal{F}$ boils down to a $P$-indexed inverse system of abelian groups, and $\text{Lim}^p \mathcal{F}$ coincides with its usual $p$th derived limit.

**Theorem C.** Let $X$ be a separable metrizable space. For each $q \geq 0$ there exists a posheaf $\mathcal{F}^q$ over $K(X)$ such that the stalk $\mathcal{F}^q_{K_\alpha} \simeq H^q(K_\alpha)$ for each $K_\alpha \in K(X)$, and
there exists a spectral sequence of the form  
\[ \text{Lim}^p \mathcal{F}^q = E^p_{2q} \Rightarrow H^{p+q}(X). \]

**Theorem D.** Let \( X \) be a subset of \( S^n \) and let \( U(X) \) be the pospace of its open neighborhoods \( \neq S^n \), ordered by inclusion and topologized by the Hausdorff metric. For each \( q \geq 0 \) there exists a posheaf \( \mathcal{F}_q \) over \( U(X) \) such that the stalk \( (\mathcal{F}_q)_{P_\beta} \simeq H_q(P_\beta) \) for each \( P_\beta \in U(X) \), and there exists a spectral sequence of the form  
\[ \text{Lim}^p \mathcal{F}_q = E^2_{p,q} \Rightarrow H_{q-p}(X). \]

Key ingredients of the proof of Theorem C (see Theorem 5.9) are the Leray spectral sequence of a continuous map, a homotopy equivalence lemma (Theorem 3.4) and a lemma on compatibility of the \( \text{hocolim} \) operator with Leray sheaves (Theorem 5.8(b)). Theorem D (see Theorem 5.10) additionally employs the Sitnikov duality.

All theorems of the present paper are proved in ZFC. Thus it can be argued, in view of Example 1.1, that these results succeed to “expel set theory from algebraic topology”.

2. BACKGROUND: USUAL DERIVED LIMITS REVISITED

Let us now discuss in some detail the usual derived limits \( \text{lim}^p D \) for a diagram \( D \) of abelian groups indexed by a poset \( P \). If \( P \) is viewed as a category\(^2\), then such a diagram \( D \) is simply a functor from \( P \) to the category of abelian groups. It will be more convenient to regard \( D \) as a set of data: groups \( G_p \) for all \( p \in P \) and homomorphisms \( \varphi^p_q : G_p \to G_q \) for all \( p, q \in P \) with \( p \leq q \).

Let us define a new poset \( G = \bigsqcup_{p \in P} G_p \) with \( (p, q) \leq (q, h) \) if and only if \( p \leq q \) and \( \varphi^p_q(h) = h \). The map \( \pi : G \to P \) defined by \( \pi(p, q) = p \) is monotone and so induces a simplicial map \( \mathcal{F} : |G| \to |P| \) between the order complexes\(^3\). The simplicial complex \( |G| \) is known as the *homotopy colimit* \( \text{hocolim} D \), and we will denote the map \( \mathcal{F} \) by \( \text{hocolim} D \).

In fact, \( \text{hocolim} D \) is a cosheaf (in the geometric sense, i.e. a “complete spread” of R. Fox, or equivalently a “display space” of J. Funk, see [46; Appendix B]), and we have \( H_0(|P|; \text{hocolim} D) \simeq \text{colim} D \) and \( H_i(|P|; \text{hocolim} D) = 0 \) for \( i > 0 \).

Here are some details. As a cosheaf of sets, \( \mathcal{F} = \text{hocolim} D \) can be recovered as the cosheafication of its precosheaf\(^4\) of cosections \( F \), defined by \( F(U) = \pi_0(\mathcal{F}^{-1}(U)) \). Namely, the costalks \( \mathcal{F}_x = \mathcal{F}^{-1}(x) \) of \( \mathcal{F} \) are the inverse limits \( \text{lim}^p_U F(U) \) over all open \( U \) containing \( x \), and \( \mathcal{F} \) itself is the projection \( \bigsqcup_{x \in |P|} \mathcal{F}_x \to |P| \), where the disjoint union of sets is endowed with the topology with basis consisting of \( V_{U,s} = \{(x,t) \mid p^U_t(t) = s\} \), where \( p^U_t : \mathcal{F}_x \to F(U) \) is the natural map from the inverse limit, \( U \) runs over open subsets of \( |P| \) and \( s \) runs over \( F(U) \).

\(^2\)With elements of \( P \) as objects, precisely one morphism \( p \to q \) when \( p \leq q \) and no morphisms \( p \to q \) when \( p \nless q \).

\(^3\)The order complex of a poset \( P \) is the simplicial complex with elements of \( P \) as vertices, and with finite chains \( p_1 \leq \cdots \leq p_n \) as simplexes. We endow simplicial complexes with the metric topology.

\(^4\)A *precosheaf* of abelian groups on a topological space \( X \) is a covariant functor from the poset of all open subsets of \( X \) ordered by inclusion to the category of abelian groups.
Let us note that each $\mathcal{F}^{-1}(p \leq q)$ is nothing but the mapping cylinder $\text{cyl}(\varphi^p_q)$. In general, $\mathcal{F}^{-1}(p_1 \leq \cdots \leq p_n)$ is the iterated mapping cylinder $\text{cyl}(\varphi^{p_1}_{p_2}, \ldots, \varphi^{p_{n-1}}_{p_n})$, that is the mapping cylinder of the composition $\text{cyl}(\varphi^{p_1}_{p_2}, \ldots, \varphi^{p_{n-1}}_{p_n}) \xrightarrow{\pi} G_{p_{n-1}} \xrightarrow{\varphi^{p_{n-1}}_{p_n}} G_{p_n}$, where $\pi$ is the natural projection.

The precosheaf $\mathcal{F}$ is of a combinatorial type, known as a "costack" or a "local coefficient system": it is determined by its values on the open stars of simplexes $\text{ost}(\sigma)$, $\sigma \in |\mathcal{P}|$. So the homology of $|\mathcal{P}|$ with coefficients in $\mathcal{F}$ is the homology of the chain complex $C_*(|\mathcal{P}|; F)$ consisting of the groups $C_n = \bigoplus_{\sigma \in |\mathcal{P}|} F_{\sigma}$, where $F_{\sigma} = F(\text{ost}(\sigma))$, with the differential $\partial: C_n \rightarrow C_{n-1}$ defined on each $g_{\sigma} \in F_{\sigma}$ by $\partial(g_{\sigma}) = \sum_{\tau \leq \sigma} [\tau : \sigma] F_{\sigma}(g_\tau)$, where $[\tau : \sigma] = \pm 1$ is the incidence index and $F_{\sigma\tau}: F_{\sigma} \rightarrow F_{\tau}$ is the image of $F$ on the inclusion $\text{ost}(\sigma) \subset \text{ost}(\tau)$. In our case, clearly, $F_{\sigma} = G_{p_1}$ for $\sigma = (p_1 \leq \cdots \leq p_n)$ and $F_{\sigma\tau} = \varphi^{p_1}_{p_2}: G_{p_1} \rightarrow G_{p_2}$ if $\tau = (p_2 \leq \cdots \leq p_n)$, whereas in the $n-1$ other cases $F_{\sigma\tau} = \text{id}: G_{p_1} \rightarrow G_{p_1}$. From this explicit description it is not hard to compute $H_*(|\mathcal{P}|; \mathcal{F})$.

In order to deal with $\lim$ and $\lim^p$ we need to dualize the above. A well-known approach, going back to Bousfield and Kan [7], is by considering the usual homotopy limit, which is built out from the usual mapping cocylinders\(^6\). But it turns out that there is also a different approach, which is arguably more intuitive.

Let $\Gamma$ be the coequalizer of $\bigsqcup_{p \leq q} G_p \times \text{ost}(p \leq q) \xrightarrow{j_1} \bigsqcup_{p \leq q} G_p \times \text{ost}(p)$, that is, the quotient of the latter disjoint union by identifying the images of the former disjoint union under the two inclusions $i_1$ and $j_1$, which arise from the inclusions $i_{pq}: \text{ost}(p \leq q) \subset \text{ost}(p)$ and $j_{pq}: \text{ost}(p \leq q) \subset \text{ost}(q)$. Since $i_1$ and $j_1$ commute with the projection to $|\mathcal{P}|$, we get a map $\mathcal{F}: \Gamma \rightarrow |\mathcal{P}|$, which we will denote by $\text{holim} D$ (breaking with the tradition of understanding holim $D$ as constructed from path spaces).

It is easy to see that $\text{holim} D$ is a sheaf (in the geometric sense, i.e. the "étalé space") — normally a non-Hausdorff one, as is common for sheaves\(^7\), and it turns out, as we will see shortly, that $H^n(|\mathcal{P}|; \text{holim} D) = \lim^n D$.

Let us note that each $\tilde{\mathcal{F}}^{-1}(p \leq q)$ is homeomorphic to $\text{cyl}(\varphi^p_q)$, where $\text{cyl}(X \xrightarrow{f} Y)$ denotes, unconventionally, the pushout of $f \times \text{id}_{\{0,1\}}: X \times \{0,1\} \rightarrow Y \times \{0,1\}$ and the inclusion $X \times \{0,1\} \subset X \times I$. In general, $\tilde{\mathcal{F}}^{-1}(p_1 \leq \cdots \leq p_n)$ is the iterated mapping cocylinder $\text{cyl}(\varphi^{p_1}_{p_2}, \ldots, \varphi^{p_{n-1}}_{p_n})$, that is the mapping cocylinder of the composition $\text{cyl}(\varphi^{p_1}_{p_2}, \ldots, \varphi^{p_{n-1}}_{p_n}) \xrightarrow{\pi} G_{p_{n-1}} \xrightarrow{\varphi^{p_{n-1}}_{p_n}} G_{p_n}$, where $\pi$ is the natural projection.

There is another definition of the sheaf $\text{holim} D$, which may look fancier but is more categorical and will be easier to generalize. Let $P'$ be the poset of nonempty faces of the

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\(^5\)The open star of a simplex $\sigma$ is the union of the interiors of all simplexes containing $\sigma$.

\(^6\)The usual mapping cocylinder of $f: X \rightarrow Y$ is the pullback of $f$ and the evaluation map $Y^I \rightarrow Y$, $p \mapsto p(1)$. Let us note that the usual mapping cylinder of $f$ is the pushout of $f$ and the inclusion $X \rightarrow X \times I$, $x \mapsto (x, 1)$.

\(^7\)Let us note that the coequalizer of the maps $(0, \infty) \times \mathbb{Z} \rightrightarrows \mathbb{R} \times \mathbb{Z}$ given by $(x, n) \mapsto (x, n)$ and $(x, n) \mapsto (x, n + 1)$ is non-Hausdorff, but has a countable base of topology. In contrast, the coequalizer of the maps $[0, \infty) \times \mathbb{Z} \rightrightarrows \mathbb{R} \times \mathbb{Z}$ given by the same formulas is Hausdorff, but does not have a countable base of neighborhoods at 0.
simplicial complex $|P|$; thus $P'$ is the set of all nonempty finite chains of $P$, ordered by inclusion. Sending every chain to its greatest element yields a monotone map $\rho: P' \to P$. Let $\rho^*F: \rho^*G \to P'$ be the pullback of the monotone map $\pi: G \to P$ along $\rho$. Since $\rho^*F$ is monotone, it may be viewed as a continuous map with respect to the Alexandrov topologies\(^9\), and as such it is easily seen to be a sheaf.\(^{10}\) On the other hand, there is a continuous map $g: |P| \to P'$ defined by sending the interior of every simplex to that same simplex regarded as a point of $P'$. The pullback of $\rho^*F$ via $g$ is nothing but the sheaf $\tilde{F} = \text{holim} \; D$. 

The presheaf of sections $\tilde{F}$ of the sheaf $\tilde{F}$ is again determined by its values on the open stars of simplexes, and so is a “stack” or a “local coefficient system”. So the cohomology of $|P|$ with coefficients in $\tilde{F}$ is the cohomology of the cochain complex $C^*(|P|; \tilde{F})$ consisting of the groups $C^n = \prod_{\sigma^n \in |P|} \tilde{F}_\sigma$, where $\tilde{F}_\sigma = \tilde{F}(\text{ost}(\sigma))$, with differential $\delta: C^n \to C^{n+1}$ defined by $(\delta^n)(e^{n+1}) = \sum_{\sigma \subseteq \tau \subseteq +1} [\sigma: \tau] \tilde{F}_{\tau\sigma}(e^n(\sigma))$, where $\tilde{F}_{\tau\sigma}: \tilde{F}_\tau \to \tilde{F}_\sigma$ is the image of $\tilde{F}$ on the inclusion $\text{ost}(\tau) \subseteq \text{ost}(\sigma)$. In our case, clearly, $\tilde{F}_\sigma = G_{p_n}$ for $\sigma = (p_1 \leq \cdots \leq p_n)$ and $\tilde{F}_{\tau\sigma} = \varphi_{p_{n-1}}^{p_{n-1}}: G_{p_{n-1}} \to G_{p_n}$ if $\tau = (p_1 \leq \cdots \leq p_{n-1})$, whereas in the $n-1$ other cases $\tilde{F}_{\tau\sigma} = \text{id}: G_{p_n} \to G_{p_n}$. From this explicit description it immediately follows that $H^n(|P|; \text{holim} \; D) = \text{lim}^p D^n$. 

Now let $\Delta$ be a diagram of simplicial complexes $X_p$ and simplicial maps $f_{p_q}: X_p \to X_q$ indexed by the poset $P$. We may again consider the map $\text{holim} \; \Delta: \text{holim} \; \Delta \to |P|$, which satisfies $(\text{holim} \; \Delta)^{-1}(p_1 \leq \cdots \leq p_n) = \text{cyl}(f_{p_2}^{p_1}, \ldots, f_{p_{n-1}}^{p_{n-2}})$. On the other hand, for each $n$ we have a diagram $D_n$ of abelian groups $H_n(X_p)$ and homomorphisms $(f_{p_q}^p)_*: H_n(X_p) \to H_n(X_q)$ indexed by $P$, and also a diagram $D^n$ of abelian groups $H^n(X_p)$ and homomorphisms $(f_{p_q}^p)^*: H^n(X_q) \to H^n(X_p)$ indexed by the dual poset $P^*$ (i.e. the same set with reversed order). It is not hard to see that the Leray sheaf $\mathcal{H}^n(\text{holim} \; \Delta) \simeq \text{holim} \; D^n$, and the Leray cosheaf $\mathcal{H}_n(\text{holim} \; \Delta) \simeq \text{holoclim} \; D_n$. Consequently, we have a spectral sequence 

$$E_2^{p,q} = \lim^p D^n \simeq H^p(|P|; \text{holim} \; D^n) \Rightarrow H^{p+q}(\text{holim} \; \Delta),$$

which is known as the Bousfield–Kan/Araki–Yoshimura spectral sequence [7], [2], and also a spectral sequence $H_p(|P|; \text{holoclim} \; D_q) \Rightarrow H_{p+q}(\text{holim} \; \Delta)$, which collapses to an isomorphism $\lim D_n \simeq H_n(\text{holim} \; \Delta)$. 

It should be noted that if $(X_p, f_{p_q}^p)$ is a directed system of pointed subcomplexes of a simplicial complex $X$ and their inclusions (directed means that every two are both contained in a third one), then $\text{holim} \; \Delta$ is homotopy equivalent to $X$ [2], [7] (see also

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\(^8\)Thus $\rho^*G$ consists of tuples $(p_1 < \cdots < p_n; g)$, where $g \in G_{p_n}$, and is ordered by $(p_1 < \cdots < p_n; g) \leq (q_1 < \cdots < q_m; h)$ if and only if $(p_1 < \cdots < p_n)$ is a subchain of $(q_1 < \cdots < q_m)$ and $\varphi_{q_m}^{p_n}(g) = h$.

\(^9\)The Alexandrov topology on the set of points of a poset $P$ consists of all subsets $U \subseteq P$ such that if $p \in U$ and $q \geq p$, then also $q \in U$.

\(^{10}\)Indeed, the smallest open neighborhood of an element $(p_1 < \cdots < p_n; g) \in \rho^*G$ consists of all tuples $(q_1 < \cdots < q_m; h) \in \rho^*G$ such that $(p_1 < \cdots < p_n)$ is a subchain of $(q_1 < \cdots < q_m)$ and $h = f_{q_m}^{p_n}(g)$. Since $h$ is uniquely determined by $(q_1 < \cdots < q_m)$, this neighborhood projects homeomorphically onto the smallest open neighborhood of $(p_1 < \cdots < p_n)$ in $P'$, which consists of all chains $(q_1 < \cdots < q_m)$ that have $(p_1 < \cdots < p_n)$ as a subchain.
3. Partially ordered spaces

By a topological poset we mean a topological space $P$ that is also a poset. This is a topological category (in the sense of Segal [42]) with precisely one morphism $p \to q$ whenever $p \leq q$ and no morphisms $p \to q$ otherwise, and with the set of morphisms topologized as a subspace of $P \times P$. (That it is a topological category means that the four structure maps are automatically continuous: the “identity” map from objects to morphisms, the “source” and “target” maps from morphisms to objects, and the “composition” map from morphisms squared to morphisms.) The dual topological poset $P^*$ is the same space with the reversed order.

A pospace (=partially ordered space) is a topological poset such that the order relation $\leq$, viewed as a subset of $P \times P$, is closed in $P \times P$ with respect to the product topology. When $P$ is metrizable, this is equivalent to saying that if $x_n \leq y_n$ for each $n \in \mathbb{N}$ and $x_n \to x$ and $y_n \to y$ as $n \to \infty$, then $x \leq y$.

If $P$ is a topological poset, let $|P|$ denote the set of all formal sums $x = \sum_{i=1}^{n} \lambda_i x_i$, where $n \in \mathbb{N}, x_1 \leq \cdots \leq x_n$ is a nonempty finite chain in $P$, each $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$. We will now consider some natural topologies on $|P|$.

3.A. Wrong construction (weak topology). One natural topology on $|P|$ is well-known. Namely, let $|P|_w$ be the classifying space of $P$ (or, in another terminology, the nerve of $P$), where $P$ is viewed as a topological category [42]. Unfortunately, the topology of $|P|_w$ is “wrong” (for our purposes) in that it generally fails to be metrizable when $P$ is metrizable. In particular, $|P|_w$ is non-metrizable whenever $P$ is discrete (as a space) and the order complex of $P$ (as a poset) is not locally finite. But actually a satisfactory “correction” of the weak topology is already known in this particular case (see Remark 3.1 below).

Let us briefly review the construction of $|P|_w$. By definition, $|P|_w$ is the geometric realization of the simplicial space (=simplicial object in the category of topological spaces, cf. [17]) where every $(n-1)$-simplex is a chain $p_1 \leq \cdots \leq p_n$ of $n$ elements of $P$. Such a chain may be regarded as a monotone map $C: [n] \to P$ (possibly non-injective), where $[n]$ denotes the $n$-element poset $\{1, \ldots, n\}$ with the usual (total) order. All such chains of length $n$ (possibly with repeats) form a subspace $P^{[n]}$ of the $n$-fold Cartesian product $P \times \ldots \times P$ of spaces.

For example, the partially ordered space $[0, 1]$ (the closed unit interval of the real line with its usual total order) yields the “standard skew $n$-simplex” $[0, 1]^{[n]} = \{(x_1, \ldots, x_n) | 0 \leq x_1 \leq \cdots \leq x_n \leq 1\} \subset \mathbb{R}^n$. Let us note that we already have to consider chains with repeats here in order to get the closed simplex.
Every monotone map \( f: [m] \to [n] \) induces a continuous map \( f^*: P[n] \to P[m] \) by \( C \mapsto Cf \). In particular, we have \( f^*: [0,1]^n \to [0,1]^m \). On the other hand, the “standard symmetric \((n-1)\)-simplex” \( \Delta^{n-1} = \{(x_0, \ldots, x_n) \in \mathbb{R}^n \mid x_0, \ldots, x_n \geq 0, x_0 + \cdots + x_n = 1\} \) behaves covariantly: \( f \) induces a map \( f_\times: \mathbb{R}^m \to \mathbb{R}^n \) taking \( \Delta^{m-1} \) into \( \Delta^{n-1} \).

With this notation, \( |P|_w \) is the quotient space of \( \bigsqcup_{n \in \mathbb{N}} P[n] \times \Delta^{n-1} \) by the equivalence relation \((C, f_s(t)) \sim (f^*(C), t)\) for every monotone map \( f: [m] \to [n] \).

Let us note that if we consider only monotone injections so as to avoid degenerate simplexes, this will give a completely unintended topology on \( |P| \), because chains with repeats arise naturally as limits of chains without repeats.

**Remark 3.1.** One can attempt to “correct” the weak topology. Let us fix a metric on \( P \) that is bounded above by 1 and metrize \( \Delta^{n-1} \) by means of the \( l_\infty \) metric on \([0,1]^{[n-1]}\). Let us use the \( l_\infty \) product metric on each summand of \( \bigsqcup_{n \in \mathbb{N}} P[n] \times \Delta^{n-1} \) (note that it will be bounded above by 1) and set the distance between any pair of points in distinct summands equal to 1. This determines a metric, and in particular a metrizable uniform structure, on the disjoint union (beware that it is not the uniformity of the disjoint union, which is generally non-metrizable). Let \( |P|_{uw} \) be the quotient endowed with the topology of the quotient uniformity (not to be confused with the quotient topology).

If \( P \) is discrete (as a space), and is metrized by setting every distance equal to 1, then \( |P|_{uw} \) is metrizable [27]. In the general case, we have three open questions:

1. Is \( |P|_{uw} \) well defined (i.e. independent of the choice of metric on \( P \))?
2. Is \( |P|_{uw} \) metrizable?
3. Is the topology of \( |P|_{uw} \) same as the topology of \( |P|_{HM} \) and \( |P|_{AE} \) (see below)?

3.B. Hartman–Mycielski construction (measurable functions). We will refer to [26; §V] for a detailed discussion of spaces of measurable functions, and we will use some notation introduced there.

Let \( P \) be a topological poset. Given an \( x = \sum_{i=1}^n \lambda_i x_i \in |P| \), where \( x_1 \leq \cdots \leq x_n \), each \( \lambda_i \geq 0 \) and \( \sum_{i=1}^n \lambda_i = 1 \), let us define a step function \( \varphi_x: [0,1] \to P \) by \( \varphi_x(t) = x_k \) if \( t \in I_k \), where \( I_k = [\sum_{i \leq k} \lambda_i, \sum_{i \leq k} \lambda_i) \). Let \( |P|_{HM} \subseteq HM(P) \) be the set of all such functions, endowed with the topology of convergence in measure. If \( P \) is endowed with a metric \( d \), we further endow \( |P|_{HM} \) with the metric \( L_1 \) (which induces the same topology of uniform convergence).

**Example 3.2.** Let \( P = \{0, \ldots, n\} \) with the usual order and with the usual metric (i.e., a subpospace of \( \mathbb{R} \)). Then \( |P|_{HM} \) is isometric to the standard skew \( n \)-simplex \( \Delta = \{(t_1, \ldots, t_n) \in \mathbb{R}^n \mid 0 \leq t_1 \leq \cdots \leq t_n \leq 1\} \) with the \( l_1 \) metric.

Indeed, given step functions \( f, g: [0,1) \to P \) with \( f^{-1}(i) = [t_i, t_{i+1}) \) and \( g^{-1}(i) = [s_i, s_{i+1}) \), where \( 0 = t_0 \leq \cdots \leq t_{n+1} = 1 \) and \( 0 = s_0 \leq \cdots \leq s_{n+1} = 1 \), they correspond to \( T = (t_1, \ldots, t_n) \in \Delta \) and \( S = (s_1, \ldots, s_n) \in \Delta \), with \( l_1(T, S) = \sum_{i=1}^n |t_i - s_i| \).

On the other hand, we have \( f = f_1 + \cdots + f_n \) as elements of the vector space of measurable functions \( I \to \mathbb{R} \), where each \( f_i: I \to P \) is a two-valued step function, \( f_i(t) = 0 \) if \( t < t_i \) and \( f_i(t) = 1 \) if \( t \geq t_i \). Similarly, \( g = g_1 + \cdots + g_n \), where \( g_i(t) = 0 \).
if \( t < s_i \) and \( g_i(t) = 1 \) if \( t \geq s_i \). If \( f_i(t) > g_i(t) \), then \( t_i \leq t < s_i \); and if additionally \( f_j(t) < g_j(t) \), then \( s_j \leq t < t_j \). Hence \( s_j < s_i \) and \( t_i < t_j \), which is a contradiction. Therefore either \( f_i(t) \geq g_i(t) \) for all \( i \), or \( g_i(t) \geq f_i(t) \) for all \( i \). Thus \( L_1(f, g) = \int f(t) - g(t) \, dt \sum_{i=1}^{n_i} L_1(f_i, g_i) = \sum_{i=1}^{n_i} |t_i - s_i| = t_1(T, S) \).

3.C. Resolution. If \( P \) is a poset, let \( A(P) \) be its set of atoms (that is, elements \( p \in P \) such that \( q \not< p \) for each \( q \in P \)). If \( P \) is a topological poset, \( A(P) \) is a topological subposet of \( P \), with discrete order (i.e. no elements are comparable). In particular, \( |A(P)| \) is homeomorphic to \( A(P) \).

Example 3.3. If \( X \) is a metrizable space and \( K(X) \) is the pospace of its nonempty compact subsets (see Lemma 4.1), then \( A(K(X)) = X \).

If \( P \) and \( Q \) are posets, \( P \times Q \) is a poset, where \( (p, q) \leq (p', q') \) iff \( p \leq p' \) and \( q \leq q' \). If \( P \) and \( Q \) are topological posets, \( P \times Q \) is a topological poset. In particular, \( A(P) \times P \) is a topological poset, where \( (a, p) \leq (b, q) \) iff \( a = b \) and \( p \leq q \). It is easy to see that \( |A(P) \times P| \) is homeomorphic to \( A(P) \times |P| \).

Let \( E(P) \) be the topological subposet of \( A(P) \times P \) consisting of all pairs \((a, p)\) such that \( a \leq p \). Thus \( |E(P)| \) is a subspace of \( |A(P) \times P| \), which is identified with the subspace of \( A(P) \times |P| \) consisting of all pairs \((a, \sum_{i=1}^{n_i} \lambda_i x_i)\) such that \( a \leq x_1 \) (and also \( x_1 \leq \cdots \leq x_n \), each \( \lambda_i \geq 0 \) and \( \sum_{i=1}^{n_i} \lambda_i = 1 \)).

Theorem 3.4. \(|E(P)| \) is homotopy equivalent to \( A(P) \).

We will prove and use this theorem only for metrizable \( P \), but the proof straightforwardly extends to the non-metrizable case.

Proof. Let \( \pi: A(P) \times P \to A(P) \) be the projection and let \( \iota: A(P) \to E(P) \subset A(P) \times P \) be defined by \( \iota(a) = (a, a) \). Clearly, \( \pi \iota = \text{id}_{A(P)} \), so in particular, \( \iota \) is an embedding of \( A(P) \) into \( E(P) \subset |E(P)| \). Let us show that \( |E(P)| \) deformation retracts onto the image of \( \iota \). More precisely, we will construct a homotopy \( h_t \) between \( \text{id}_{|E(P)|} \) and the composition \( |E(P)| \subset A(P) \times |P| \xrightarrow{\pi} A(P) \xrightarrow{\iota} E(P) \subset |E(P)| \).

Every pair \((a, \sum_{i=0}^{n_i} \lambda_i x_i) \in |E(P)|\), with \( a \leq x_1 \leq \cdots \leq x_n \), can also be written as \((a, \sum_{i=0}^{n_i} \lambda_i x_i)\), where \( x_0 = a \) and \( \lambda_0 = 0 \). Thus we may assume without loss of generality that \( a = x_1 \). Let \( X = \sum_{i=1}^{n_i} \lambda_i x_i \) and let us define \( h_t(x_1, X) = (x_1, X_t) \), where \( X_t = tx_1 + (1-t)X \). Then \( h_0 = \text{id}_{|E(P)|} \) and \( h_1 = \iota \pi \). It remains to check that \( h_t \) is continuous.

Let \( Y = \sum_{i=1}^{n_i} \mu_i y_i \) and let \( Y_t = ty_1 + (1-t)Y \). Then \( L_1(X, Y) = td(x_1, y_1) + (1-t)L_1(X, Y) \). If we use the \( l_1 \) product metric on the product \( A(P) \times |P| \), then \( d(h_t(x_1, X), h_t(y_1, Y)) = d(x_1, y_1) + L_1(X_t, Y_t) \leq 2d(x_1, y_1) + L_1(X, Y) \). Also, we have \( d(h_t(x_1, X), h_s(x_1, X)) = L_1(X_s, X_t) \leq |s-t| \). It follows that \( h_t \) is continuous.

3.D. Arens–Eels construction (probability measures). A finite measure on a space \( X \) is a function \( \mu: X \to \mathbb{R} \) with support in finitely many points. Thus \( \mu = \sum_{i=1}^{n} \mu_i \delta_{x_i} \), where \( \mu_i = \mu(x_i) \) and \( \delta_x: X \to \mathbb{R} \) is the Dirac measure, defined by \( \delta_x(y) = 1 \) if \( x = y \) and \( 0 \) if \( x \neq y \). Let \( \mu(X) = \sum_{x \in X} \mu(x) \), that is, \( \mu(X) = \sum_{i=1}^{n} \mu_i \) in the previous notation.
If \( \mu(X) = 1 \), then \( \mu \) is called a finite probability measure. Let \( AE(X) \) denote the set of all finite measures on \( X \) and \( AE_V(X) \) its subset consisting of all \( \mu \) with \( \mu(X) = V \). Clearly, \( AE(X) \) is a real vector space and \( AE_V(X) \) is an affine hyperplane in \( AE(X) \) for each \( V \in \mathbb{R} \). Also, \( AE_0(X) \) is vector subspace of \( AE(X) \).

Let us fix a metric \( d \) on \( X \). If \( \nu \in AE_0(X) \), let \( ||\nu|| = \inf \sum_{k=1}^{\infty} |\nu_k|d(x_k, y_k) \), where the infimum is over all representations \( \nu = \sum_{k=1}^{\infty} \nu_k(\delta_{x_k} - \delta_{y_k}) \). Such representations exist since \( \nu(X) = 0 \).

If \( \lambda, \mu \in AE(V) \), let \( \rho(\lambda, \mu) = ||\lambda - \mu|| \). In particular, we get a metric \( \rho \) on the set \( AE_1(X) \) of all finite probability measures, which is called the Kantorovich metric or alternatively the Wasserstein (Vasershtein) metric. If we understand \( \mu \) and \( \nu \) as distributions of masses, \( \rho(\lambda, \mu) \) can be interpreted as the minimal amount of work needed to transport \( \lambda \) into \( \mu \).

Now let \( P \) be a metrizable topological poset. Given a formal sum \( x = \sum_{i=1}^{n} \lambda_i x_i \in |P| \), we have the finite probability measure \( \mu_x := \sum \lambda_i \delta_{x_i} \). Let \( |P|_{AE} \subseteq AE_1(P) \) be the set of all such measures, endowed with the Kantorovich metric \( \rho \).

**Example 3.5.** Let \( P = \{x_1, \ldots, x_n \} \) linearly ordered by \( x_i \leq x_j \) if \( i \leq j \) and with \( d(x_i, x_j) = 1 \) whenever \( i \neq j \). Then \( |P|_{AE} \) is isometric to a homothet of the standard \((n-1)\)-simplex \( \Delta = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \mid \lambda_k \geq 0, \sum_{k=1}^{n} \lambda_k = 1 \} \) with the \( l_1 \) metric.

Indeed, given \( x = \sum_{k=1}^{n} \lambda_k x_k \in |P| \) and \( y = \sum_{k=1}^{n} \mu_k y_k \in |P| \), they correspond to \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \Delta \) and \( \mu = (\mu_1, \ldots, \mu_n) \in \Delta \), with \( l_1(\lambda, \mu) = \sum_{k=1}^{n} |\lambda_k - \mu_k| \).

On the other hand, let \( S^{+} = \{ k \in [n] \mid \lambda_k > \mu_k \} \) and \( S^{-} = \{ k \in [n] \mid \lambda_k < \mu_k \} \). Also let \( N^{+} = \sum_{i \in S^{+}} (\lambda_i - \mu_i) \) and \( N^{-} = \sum_{j \in S^{-}} (\mu_j - \lambda_j) \). Then \( N^{+} - N^{-} = \sum_{k=1}^{n} (\lambda_k - \mu_k) = 1 - 1 = 0 \) and \( N^{+} + N^{-} = \sum_{k=1}^{n} |\lambda_k - \mu_k| = l_1(\lambda, \mu) \). Hence \( N^{+} = l_1(\lambda, \mu)/2 \).

Now we have \( \rho(\mu_x, \mu_y) = \sum_{i \in S^{+}} \sum_{j \in S^{-}} \nu_{ij} d(x_i, x_j) \), where each \( \nu_{ij} > 0 \) and \( \mu_x - \mu_y = \sum_{i \in S^{+}} \sum_{j \in S^{-}} \nu_{ij} (\delta_{x_i} - \delta_{x_j}) \). Then \( \lambda_i - \mu_i = \sum_{j \in S^{+}} \nu_{ij} \) for each \( i \in S^{+} \) and \( \lambda_j - \mu_j = \sum_{i \in S^{+}} \nu_{ij} \) for each \( j \in S^{-} \). Hence \( \rho(\mu_x, \mu_y) = \sum_{i \in S^{+}} \sum_{j \in S^{-}} \nu_{ij} = N^{+} = l_1(\lambda, \mu)/2 \).

**3.E. Comparison.**

**Example 3.6.** The bijection \( \Phi : |P|_{AE} \to |P|_{HM} \), \( \mu_x \mapsto f_x \), is not uniformly continuous in general, even if \( P \) is a pospace with discrete uniform structure. Indeed, let \( P = \{x_i \mid i \in \mathbb{N} \} \) be set of natural numbers with its usual linear order but with elements denoted by \( x_0, x_1, \ldots \) instead of 0, 1, \ldots, and with \( d(x_i, x_j) = 1 \) for every two distinct \( i, j \in \mathbb{N} \). Given an \( n \in \mathbb{N} \), let \( p_n = \frac{1}{n} x_0 + \cdots + \frac{1}{n} x_{n-1} \) and \( q_n = \frac{1}{n} x_1 + \cdots + \frac{1}{n} x_n \). Then \( \rho(\mu_{p_n}, \mu_{q_n}) = \frac{1}{n} d(x_0, x_n) = \frac{1}{n} \). On the other hand, \( L_1(f_{p_n}, f_{q_n}) = \frac{1}{n} d(x_0, x_1) + \cdots + \frac{1}{n} d(x_{n-1}, x_n) = n \frac{1}{n} = 1 \).

**Remark 3.7.** The deformation retraction in Theorem 3.4 is uniformly continuous (in fact, Lipschitz) with respect to the Kantorovich metric. Indeed, in the notation of the proof of Theorem 3.4, \( \rho(X_t, X_s) \leq |s - t| \). (Also, \( \rho(X_t, Y_t) = td(x_1, y_1) + (1 - t)\rho(X, Y) \).)

**Theorem 3.8.** If \( P \) is a metrizable pospace, then \( |P|_{HM} \) and \( |P|_{AE} \) are homeomorphic.

**Proof.** Let us fix some metric on \( P \). Clearly, \( \rho(\mu_x, \mu_y) \leq L_1(f_x, f_y) \).
It remains to show that $\Phi$: $|P|_{AE} \to |P|_{HM}$, $\mu_x \mapsto f_x$, is continuous. We may assume that $P$ has diameter $\leq 1$. Let $\varepsilon > 0$. Let $x = \sum_{i=1}^n \lambda_i x_i$, where $x_1 < \cdots < x_n$, each $\lambda_i \geq 0$ and $\sum_{i=1}^n \lambda_i = 1$. Since $x_1 \not\leq \cdots \not\leq x_n$ and $\not\leq$ is open as a subset of $P \times P$, there exist pairwise disjoint neighborhoods $U_1, \ldots, U_n$ of $x_1, \ldots, x_n$ such that $y_1 \not\leq \cdots \not\leq y_n$ whenever each $y_i \in U_n$. Let $\delta > 0$ be such that $(2n+1)(n+1)\delta < \varepsilon$, each $\lambda_i > \delta$, and each $U_i$ contains the ball of radius $\delta$ about $x_i$. Let $y = \sum_{i=1}^m \mu_i y_i$, where $y_1 < \cdots < y_m$, each $\mu_i \geq 0$ and $\sum_{i=1}^m \mu_i = 1$, and $\mu(x,y) < \delta^2$.

Then there exists a representation $\mu_x - \mu_y = \sum_{j=1}^r \mu_j (\delta_{x_m_j} - \delta_{y_m_j})$ for some $r, m_j$ and $n_j$ such that each $\nu_j \geq 0$ and $\sum_{j=1}^r \nu_j d(x_{m_j}, y_{n_j}) \leq \delta^2$. Let us write $x_{m_j} = x_j$ and $y_{n_j} = y_j$. Thus $\mu_x - \mu_y = \sum_{j=1}^r \nu_j (\delta_{x_j} - \delta_{y_j})$ and $\sum_{j=1}^r \nu_j d(x_j, y_j) \leq \delta^2$. Without loss of generality, $X_1 \leq \cdots \leq X_r$, and if $X_p = X_{p+1}$, then $y_p < Y_{p+1}$. Suppose that $X_p = X_{p+1} = \cdots = X_q$, where either $p = 1$ or $X_{p-1} < X_p$, and either $q = r$ or $X_q < X_{q+1}$.

Let us note that $m_p = m_{p+1} = \cdots = m_q$.

Let $S_{m_p} = \{ j \mid p \leq j \leq q, d(X_j, Y_j) \leq \delta \}$ and $T_{m_p} = \{ j \mid p \leq j \leq q, d(X_j, Y_j) > \delta \}$. Then $\sum_{j \in T_p} \nu_j \leq \sum_{j \in T_p} \nu_j d(x_{m_j}, y_{n_j}) / \delta \leq \sum_{j=1}^r \nu_j d(x_j, y_j) / \delta \leq \delta$. Since $\sum_{j=p}^r \nu_j = \lambda_{m_p} > \delta$, we have $S_{m_p} \neq \emptyset$. Let $k_i = \min S_i$ and $l_i = \max S_i$. Let us note that $m_k = m_i = i$. Since $d(X_{k_i}, Y_{k_i}) \leq \delta$ and $d(X_{k_{i+1}}, Y_{k_{i+1}}) \leq \delta$, we have $Y_{k_i} \in U_{m_k} = U_i$ and $Y_{k_{i+1}} \in Y_{m_{k_{i+1}}} = U_{l_i+1}$. Hence $Y_i \not\leq Y_{k_{i+1}}$ and therefore $Y_i < Y_{k_{i+1}}$ for each $i < n$.

Thus we have $Y_1 < \cdots < Y_1 < Y_2 < \cdots < Y_2 < Y_3 < \cdots$. In other words, if $S = S_1 \cup \cdots \cup S_n$, then $Y_i < Y_j$ whenever $i, j \in S$ and $i < j$. On the other hand, if $T = T_1 \cup \cdots \cup T_n$, then $\sum_{j \in T} \nu_j = \sum_{j=1}^n \sum_{j \in T} \nu_j \leq n \delta$.

Let $I_k = \big( \sum_{j<k} \nu_j, \sum_{j<k} \nu_j \big)$. Let us define a step function $f'_y: [0,1) \to X$ by $f'_y(t) = Y_j$ if $t \in I_j$. Let $I_S = \bigcup_{j \in S} I_j$ and $I_T = \bigcup_{j \in T} I_j$. Then $d(f_x(t), f'_y(t)) \leq \delta$ for each $t \in I_S$ and $\mu(I_T) = \sum_{j \in T} \nu_j \leq n \delta$. Hence $L_1(f_x, f'_y) = \int_I d(f_x(t), f'_y(t)) dt = \int_{I_S} d(f_x(t), f'_y(t)) dt + \int_{I_T} d(f_x(t), f'_y(t)) dt \leq \delta + n \delta = (n+1)\delta$, using that $\mu(I_S) \leq 1$ and $d(f_x(t), f'_y(t)) \leq 1$ for each $t \in I_T$. On the other hand, since $Y_i < Y_j$ whenever $i, j \in S$ and $i < j$, we have $f_y(t) = f'_y(t)$ unless $y$ belongs to the $\mu(I_T)$-neighborhood of the set $\{ \sum_{k<i} \lambda_k \mid k = 0, \ldots, n \}$. Hence $L_1(f_y, f'_y) \leq 2(n+1)\mu(I_T) \leq 2n(n+1)\delta$. Thus $L_1(f_x, f'_y) \leq L_1(f_x, f'_y) + L_1(f'_y, f_y) \leq (2n+1)(n+1)\delta \leq \varepsilon$. \hfill $\square$

4. Simplicial hyperspace

4.1. Hausdorff metric. If $X$ is a metric space, the hyperspace $K(X)$ of its nonempty compact subsets is endowed with the Hausdorff metric

$$d(A, B) = \max \left( \sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b) \right),$$

where $d(a, B) = d(B, a) = \inf_{b \in B} d(a, b)$. Clearly, $X$ isometrically embeds in $K(X)$ via $x \mapsto \{x\}$.

Apart from being a metric space, $K(X)$ is also a poset by inclusion.

Lemma 4.1. $K(X)$ is a pospace.
This follows immediately from (a). Suppose \( A, B \in \mathcal{K}(X) \), \( A \not\supset B \). Thus \( B \) contains a point \( b \notin A \). Let \( \varepsilon = d(A, b)/3 \). If \( B' \) is \( \varepsilon \)-close to \( B \), then \( B' \) contains a point \( b' \) such that \( d(b, b') \leq \varepsilon \). Hence \( d(A, b') \geq 3\varepsilon - \varepsilon > \varepsilon \). Therefore if \( A' \) is \( \varepsilon \)-close to \( A \), then \( b' \notin A' \). Hence \( A' \not\supset B' \).

The following lemma is well-known.

**Lemma 4.2.** (See [3].) Let \( A \) and \( B \) be nonempty subsets of a metric space \( X \). Then

(a) \( \sup_{a \in A} d(a, B) = \sup_{x \in X} d(x, B) - d(x, A) \);
(b) \( d(A, B) = \sup_{x \in X} |d(x, A) - d(x, B)| \).

For the reader’s convenience, we recall the proof for the case where \( A \) and \( B \) are compact (the proof of the general case is only slightly different, but we do not need it).

**Proof.** (a). Since \( d(a, B) = d(a, B) - d(a, A) \), we have the \( \leq \) inequality.

To prove the \( \geq \) inequality, suffices to show that \( d(x, B) - d(x, A) \leq \sup_{a \in A} d(a, B) \) for each \( x \in X \). Since \( A \) is compact, \( d(x, A) = d(x, \alpha) \) for some \( \alpha \in A \), and since \( B \) is compact, \( d(\alpha, B) = d(\alpha, \beta) \) for some \( \beta \in B \). Then \( d(x, B) \leq d(x, \beta) \leq d(x, \alpha) + d(\alpha, \beta) = d(x, A) + d(\alpha, B) \), and hence \( d(x, B) - d(x, A) \leq d(\alpha, B) \leq \sup_{a \in A} d(a, B) \). □

(b). This follows immediately from (a). □

Let \( X \) be a metric space of diameter \( \leq 1 \). By Lemma 4.2(b) \( \mathcal{K}(X) \) admits an isometric embedding \( e \) into the vector space \( C_b(X) \) of bounded continuous functions \( f : X \to \mathbb{R} \) with the norm \( ||f|| = \sup_{x \in X} |f(x)| \), defined by \( e(A)(x) = d(x, A) \) for each nonempty compact \( A \subset X \). In fact, the image of \( e \) lies in the convex subset of \( C_b(X) \) consisting of 1-Lipschitz (in particular, uniformly continuous) functions \( X \to [0, 1] \).

It is well-known that the image of the composition \( X \subset \mathcal{K}(X) \xrightarrow{\sim} C_b(X) \) is a linearly independent set [6].

### 4.B. Simplicial hyperspace of a metric space.

Given a nonempty finite chain in \( \mathcal{K}(X) \), that is, a monotone map \( C : [n] \to \mathcal{K}(X) \) from the totally ordered set \( [n] = \{1, \ldots, n\} \), \( n \geq 1 \), let us write \( C_1 = C(i) \), so that \( C_1 \subset C_2 \subset \cdots \subset C_n \), and let \( |C| \) denote the convex hull of \( e(C([n])) = \{e(C_1), \ldots, e(C_n)\} \) in \( C_b(X) \). The following lemma guarantees that the convex hulls of two injective chains intersect precisely along the convex hull of their maximal common subchain.

**Lemma 4.3.** Let \( A : [n] \to \mathcal{K}(X) \) and \( B : [m] \to \mathcal{K}(X) \) be injective monotone maps.

(a) Let \( f_i = e(A_i) \) and \( g_j = e(B_j) \), and suppose that \( A_n \neq X \) if \( n > 0 \) and \( B_m \neq X \) if \( m > 0 \). If \( \sum_{i=1}^n \lambda_i f_i = \sum_{j=1}^m \mu_j g_j \), where each \( \lambda_i > 0 \) and each \( \mu_j > 0 \), then \( m = n \), each \( f_i = g_i \) and each \( \lambda_i = \mu_i \).

(b) The simplices \( |A| \) and \( |B| \) either coincide or have disjoint interiors.

(c) \( |A| \cap |B| = |C| \), where \( C : [k] \to \mathcal{K}(X) \) is the pullback of \( A \) and \( B \) (that is, \( C([k]) = A([n]) \cap B([m]) \) and \( C \) is injective).

**Proof.** (a). Arguing by induction, we may assume that the assertion is known if \( n \) or \( m \) or both are replaced by smaller numbers. Let \( F = \sum_{i=1}^n \lambda_i f_i \) and \( G = \sum_{j=1}^m \mu_j g_j \). If
Example 4.4. Let $X_n = \{a_1, \ldots, a_n\}$ with $d(a_i, a_j) = 1$ for $i \neq j$. Let $A_i = \{a_1, \ldots, a_i\}$, and let $\Delta_{n-1} = |A|$, the $(n-1)$-simplex spanned by the vectors $e(A_1), \ldots, e(A_n)$ in $C_b(X_n)$. Clearly, $C_b(X_n)$ is nothing but $\mathbb{R}^n$ with the $l_\infty$ norm $\| (x_1, \ldots, x_n) \| = \max(x_1, \ldots, x_n)$, and its points $e(A_1), \ldots, e(A_n)$ are of the form $(0, 1, \ldots, 1), (0, 0, 1, \ldots, 1), \ldots, (0, 0, 0)$. Hence $\Delta_{n-1}$ is the standard skew $(n-1)$-simplex $\{(0, x_2, \ldots, x_n) \mid 0 \leq x_2 \leq \cdots \leq x_n \leq 1\}$ with the $l_\infty$ metric.

(b) Let $X$ be any injective chain. Let $X_0$ be $\operatorname{colim} X$. Then $X_0$ is injective in $\mathcal{I}$. Then $X_0$ is a topological vector space.

(c) This is standard. Trivially $C \subset A \cap |B|$. If $x \in |A| \cap |B|$, then $x$ lies in the interiors of $|A'|$ and $|B'|$ for some injective subchains $A': [n'] \to [n] \xrightarrow{A} K(X)$ and $B': [m'] \to [m] \times K(X)$. Then by (b), $|A'| = |B'|$. Hence $|A'| \subset |C|$ and so $x \in |C|$.

4.C. Examples. The following series of examples, which is not used in the sequel, analyzes the metric on the convex hull in $C_b(X)$ of an individual chain in $K(X)$.

Example 4.4. Let us consider the finite metric space $X_n = \{a_1, \ldots, a_n\}$ with $d(a_i, a_j) = 1$ for $i \neq j$. Let $A_i = \{a_1, \ldots, a_i\}$, and let $\Delta_{n-1} = |A|$, the $(n-1)$-simplex spanned by the vectors $e(A_1), \ldots, e(A_n)$ in $C_b(X_n)$. Clearly, $C_b(X_n)$ is nothing but $\mathbb{R}^n$ with the $l_\infty$ norm $\| (x_1, \ldots, x_n) \| = \max(x_1, \ldots, x_n)$, and its points $e(A_1), \ldots, e(A_n)$ are of the form $(0, 1, \ldots, 1), (0, 0, 1, \ldots, 1), \ldots, (0, 0, 0)$. Hence $\Delta_{n-1}$ is the standard skew $(n-1)$-simplex $\{(0, x_2, \ldots, x_n) \mid 0 \leq x_2 \leq \cdots \leq x_n \leq 1\}$ with the $l_\infty$ metric.

(b) Let $X$ be any metric space with $d(x, y) = 1$ for $x \neq y$ and let $B: [n] \to K(X)$ be any injective chain. Let $X_0$ be as in (a) and let $g: X_0 \to X$ be an embedding such that $g(a_1) \in B_1$ and each $g(a_{i+1}) \in B_{i+1} \setminus B_i$. Then $g$ induces the restriction map $g^*: C_b(X) \to C_b(X_0)$, which clearly restricts to an isometry between $|B|$ and $\Delta_{n-1}$.

Example 4.5. Let $X$ be a metric space consisting of 3 points: $a, b, c$. Let $A_1 = \{a\}$, $A_2 = \{a, b\}$ and $A_3 = \{a, b, c\}$. Let $p = d(a, b)$, $q = d(b, c)$ and $r = d(a, c)$. We have $P := d(A_1, A_2) = p$, $Q := d(A_2, A_3) = \min(q, r)$ and $R := d(A_1, A_3) = \max(p, r)$. Let us note that $P, Q \leq R \leq P + Q$. Then $|A|$ is the 2-simplex spanned by $e(A_1) = (0, p, r)$, $e(A_2) = (0, 0, Q)$ and $e(A_3) = (0, 0, 0)$ in $C_b(X) = \mathbb{R}^3$ with the $l_\infty$ metric. The edges of $|A|$ are of lengths $Q, \max(p, r) = R$ and $\max(p, r - Q) = \max(p, r - q, r - r) = P$ (using that $p + q \geq r$).
Let us note that when \( R > P \), we have \( r = R \) and consequently \( P, Q, R \) determine the vertices \((0, P, R), (0, 0, Q)\) and \((0, 0, 0)\) of \(|A|\). Thus when \( R > P \), the metric on \(|A|\) is determined by the edge lengths (i.e., by its restriction to the set of vertices).

(b) Let \( Y \) be a metric space consisting of 4 points: \( a, b, c_+, c_- \). Let \( B_1 = \{a\}, B_2 = \{a, b\} \) and \( B_3 = \{a, b, c_+, c_-\} \). Let \( p = d(a, b), q_\pm = d(b, c_\pm) \) and \( r_\pm = d(a, c_\pm) \). We have \( P \coloneqq d(B_1, B_2) = p, Q \coloneqq d(B_2, B_3) = \max(Q_+, Q_-), \) where \( Q_\pm = \min(q_\pm, r_\pm) \), and \( R \coloneqq d(B_1, B_3) = \max(p, r_+, r_-) \). Then \(|B|\) is the 2-simplex spanned by \( e(B_1) = (0, p, r_+, r_-), e(B_2) = (0, 0, Q_+, Q_-) \) and \( e(B_3) = (0, 0, 0, 0) \) in \( C_b(Y) = \mathbb{R}^4 \) with the \( l_\infty \) metric. Then the edges of \(|B|\) are of lengths \( \max(Q_+, Q_-) = Q, \max(p, r_+, r_-) = R \) and \( \max(p, r_+ - Q_+, r_- - Q_-) = P \).

Let us note that the metric of \(|B|\) is not determined by the edge lengths even when \( R > P \), and consequently \(|B|\) is generally not isometric to any of the simplexes \(|A|\) described in (a). Indeed, the distance from the vertex of \(|B|\) at the origin to the middle of the opposite side equals \( L \coloneqq \max(p, \frac{r_+ + Q_+}{2}, \frac{r_- + Q_-}{2}) \). Let us assume for simplicity that \( p < r_+ < r_\pm \), so that \( Q_\pm = q_\pm \). Then \( Q = \max(q_+, q_-) \) and \( R = \max(r_+, r_-) \) do not determine \( L = \max\left(\frac{r_+ + q_+}{2}, \frac{r_- + q_-}{2}\right) \). For instance, if \( q_+ = q_- = \varepsilon \) and \( r_+ = r_- + \varepsilon \) for some \( \varepsilon > 0 \), then \( L = (q_+ + r_+ - \varepsilon)/2 = (Q + R - \varepsilon)/2 \), where \( \varepsilon \) can vary.

(c) Let \( Z \) be any metric space and let \( C \colon [n] \to K(Z) \) be any injective chain. Since the \( C_i \) are compact, for each \( i < j \), \( d(C_i, C_j) = d(C_i, x_{ij}) \) for some \( x_{ij} \in C_j \). Let \( Z' \) be the finite subspace of \( Z \) consisting of the \( x_{ij} \) for all \( i < j \). Then the restriction map \( r \colon C_b(Z) \to C_b(Z') \) restricts to an isometry on the vertices of \(|C|\). However, \( r \) need not restrict to an isometry on \(|C|\), because the metric on \( r(|C|) \) generally depends on \( d(x_{ij}, C_k) \), which depend on the choice of the \( x_{ij} \). Indeed, for \( n = 2 \) the metric on \( r(|C|) \subset \{0\} \times \mathbb{R}^3 \subset \mathbb{R}^4 \) is as described in (b), and we have seen that it does depend on the additional parameters.

4.D. Simplicial hyperspace of a metrizable space. If \( X \) is a metric space of diameter \( \leq 1 \), let \( K_\Delta(X) \) denote the union \( \bigcup_{C \in K(X)^{|n|}, n \in \mathbb{N}} |C| \) of the convex hulls in \( C_b(X) \) of all nonempty finite chains in \( K(X) \). When all distances in \( X \) are equal to 1, \( K_\Delta(X) \) is isometric to the geometric realization of \( K(X) \) as a (discrete) poset [27] (see Example 4.4). In general, let us note that although the topology of \( C_b(X) \) does not depend on the metric of \( X \), the subset \( K_\Delta(X) \) of \( C_b(X) \) depends on the embedding \( e \colon K(X) \to C_b(X) \), which in turn depends on the metric of \( X \).

Let \( X_+ = X \sqcup \{p\} \), where \( d(p, x) = 1 \) for each \( x \in X \). Here \( X \) retains the original metric of diameter \( \leq 1 \), so the inclusion \( K(X) \subset K(X_+) \) is an isometry. Let \( K_\Delta^+(X) \) be the union of the convex hulls in \( C_b(X_+) \) of all nonempty finite chains in \( K(X) \subset K(X_+) \).

Lemma 4.6. \( K_\Delta^+(X) \) is isometric to \( K_\Delta(X) \).

Proof. Let us show that the restriction map \( r \colon C_b(X_+) \to C_b(X) \) restricts to an isometry between \( K_\Delta^+(X) \) and \( K_\Delta(X) \). Indeed, for any \( F, G \in K_\Delta(X) \) we have \( F(x) = \sum_{i=1}^n \lambda_i d(x, A_i) \) and \( G(x) = \sum_{j=1}^m \mu_j d(x, B_j) \) for some finite chains \( A \colon [n] \to K(X) \) and \( B \colon [m] \to K(X) \), where each \( \lambda_i > 0 \), each \( \mu_j > 0 \) and \( \sum_{i=1}^n \lambda_i = 1 = \sum_{j=1}^m \mu_j \). Then \( F(p) = 1 = G(p) \), and consequently \( \sup_{x \in X_+} \left| F(x) - G(x) \right| = \sup_{x \in X} \left| F(x) - G(x) \right| \). \( \square \)
Lemma 4.7. For each $n \in \mathbb{N}$, each $\Gamma \in (0, 1]$ and each $\varepsilon > 0$ there exists a $\delta > 0$ such that the following holds.

Let $A: [n] \to K(X) \subset K(X_+)$ and $B: [m] \to K(X) \subset K(X_+)$ be injective monotone maps such that each $d(A_i, A_{i+1}) \geq \Gamma$ and each $d(B_j, B_{j+1}) \geq \Gamma$. Let $f_i = e(A_i)$ and $g_j = e(B_j)$. Let $F = \sum_{i=1}^{n} \lambda_i f_i$ and $G = \sum_{j=1}^{m} \mu_j g_j$, where $1 \geq \lambda_i \geq \Gamma$ and $1 \geq \mu_j \geq \Gamma$ for each $i$ and $j$.

If $\|F - G\| \leq \delta$, then $m = n$, each $d(A_i, B_i) \leq \varepsilon$ and each $|\lambda_i - \mu_i| \leq \varepsilon$.

Lemma 4.7 is not used in the sequel. However, it is a simplified version of Lemma 4.8, whose proof might be easier to read after that of Lemma 4.7.

The proof of Lemma 4.7 is in turn an elaboration on that of Lemma 4.3(a).

Proof. We may assume that $\varepsilon \leq 1$ and $\varepsilon \leq \Gamma$ (by decreasing $\varepsilon$ if needed). Let $\delta = (\varepsilon/2)^{2n}$.

If $n = 0$ but $m > 0$, then $\|G\| \geq \mu_1 \|g_1\| \geq \Gamma$ since $\|g_1\| = 1$. Hence $\|F\| \geq \Gamma - \delta > 0$, which is a contradiction. This establishes the assertion for $n = 0$. Also, if $n > 0$, a similar argument shows that $m > 0$. Arguing by induction, we may assume that the assertion is known if $n$ is replaced by a smaller number.

If $a \in A_1$, then $a$ also lies in each $A_i$, and hence $F(a) = 0$. On the other hand, since each $g_i(a) \geq 0$ and each $\mu_i \geq 0$, we have $\mu_i g_i(a) \leq G(a) \leq F(a) + \delta = \delta$. Hence $d(a, B_i) = g_i(a) \leq \delta/\mu_i \leq \delta/\Gamma$. Similarly, if $b \in B_i$, then $d(A_i, b) \leq \delta/\Gamma$. Hence $d(A_1, B_1) \leq \delta/\Gamma$. Therefore also $\|f_1 - g_1\| \leq \delta/\Gamma$.

For each $x \in A_2$ (or for each $x \in X_+$ if $n = 1$), $\lambda_1 f_1(x) = F(x) \geq G(x) - \delta \geq \mu_1 g_1(x) - \delta \geq \mu_1 f_1(x) - \mu_1 \delta/\Gamma - \delta$. Since $\mu_1 \leq 1$ and $\Gamma \leq 1$, we have $\mu_1 f_1(x) - \mu_1 \delta/\Gamma - \delta \leq 2\delta/\Gamma$. Since $d(A_1, A_2) \geq \Gamma$, there exists an $a \in A_2$ such that $f_1(a) = d(a, A_1) \geq \Gamma$. (If $n = 1$, let $a = p$, the point in $X_+ \setminus X$.) Then $\mu_1 - \lambda_1 \leq 2\delta/\Gamma f_1(a) \leq 2\delta/\Gamma^2$. Similarly, $\lambda_1 - \mu_1 \leq 2\delta/\Gamma^2$, so $|\lambda_1 - \mu_1| \leq 2\delta/\Gamma^2$. Then $||\lambda_1 f_1 - \mu_1 g_1|| \leq ||\lambda_1 (f_1 - g_1)|| + ||(\lambda_1 - \mu_1) g_1|| \leq \delta/\Gamma + 2\delta/\Gamma^2 \leq 3\delta/\Gamma^2$ using that $\lambda_1 \leq 1$, $||g_1|| \leq 1$ and $\Gamma \leq 1$.

We have $\|\sum_{i=2}^{m} \lambda_i f_i - \sum_{j=2}^{m} \mu_j g_j\| = \|F - G + \mu_1 g_1 - \lambda_1 f_1\| \leq \|F - G\| + \|\mu_1 g_1 - \lambda_1 f_1\| \leq \delta + 3\delta/\Gamma^2 \leq 4\delta/\Gamma^2$. We have $4\delta/\Gamma^2 = (\varepsilon/2)^{2n}/\Gamma^2 \leq (\varepsilon/2)^{n-1}$. Then by the induction hypothesis, $m = n$, $d(A_i, B_i) \leq \varepsilon$ for each $i \geq 2$ and $|\lambda_i - \mu_i| \leq \varepsilon$ for each $i \geq 2$. □

Lemma 4.8. For each $n \in \mathbb{N}$, each $\Gamma \in (0, 1]$ and each $\varepsilon > 0$ there exists a $\delta > 0$ such that the following holds.

Let $A: [n] \to K(X) \subset K(X_+)$ and $B: [m] \to K(X) \subset K(X_+)$ be injective monotone maps such that each $d(A_i, A_{i+1}) \geq \Gamma$. Let $f_i = e(A_i)$ and $g_j = e(B_j)$. Let $F = \sum_{i=1}^{n} \lambda_i f_i$ and $G = \sum_{j=1}^{m} \mu_j g_j$, where $1 \geq \lambda_i \geq \Gamma$ for each $i$ and $1 \geq \mu_j \geq \Gamma$ for each $j$.

If $\|F - G\| \leq \delta$, then there exist $1 = l_0 \leq k_1 \leq l_1 \leq k_2 \leq \cdots \leq l_n \leq k_{n+1} = m$ such that $d(A_i, B_j) \leq \varepsilon$ whenever $k_i + 1 \leq j \leq l_i$, each $|\lambda_i - \sum_{j=k_{i-1}+1}^{l_i} \mu_j| \leq \varepsilon$, and each $\sum_{j=l_{i-1}+1}^{l_i} \mu_j \leq \varepsilon$.

Proof. We may assume that $\varepsilon \leq 1$ and $\varepsilon \leq \Gamma$ (by decreasing $\varepsilon$ if needed). Let $\beta = (\varepsilon/6)^{\varphi(n)}$, where $\varphi(0) = 1$ and $\varphi(n) = 4\varphi(n-1) + 1$. Let $\alpha = \beta^2$ and $\delta = \alpha^2$. Thus $\delta = (\varepsilon/6)^{2\varphi(n)}$. 

If $n = 0$, let us consider the point $p$ in $X_+ \setminus X$. Since $F(p) = 0$, we have $\sum_{i=1}^{m} \mu_i = \sum_{i=1}^{m} \mu_i g_i(p) = G(p) \leq \delta = \varepsilon/6$. Arguing by induction, we may assume that the assertion is known if $n$ is replaced by a smaller number.

If $n > 0$, then $||F|| \geq \lambda_1 ||f_1|| \geq \Gamma$ since $||f_1|| = 1$. Hence $||G|| \geq \Gamma - \delta > 0$, and so $m > 0$.

Let $I = \{ i \in [m] \mid \exists a_i \in A_1 \text{ such that } d(a_i, B_i) \geq \alpha \}$. If $I \neq \emptyset$ and $k$ is the greatest element of $I$, then there exists an $a \in A_1$ such that $d(a, B_k) \geq \alpha$. Since $B_1 \subset \cdots \subset B_k$, we also have $d(a, B_i) \geq \alpha$ for all $i \leq k$. Hence $I = \{1, \ldots, k\}$ and the same $a_i = a$ works for each $i \in I$. If $I = \emptyset$, we let $k = 0$.

Let $J = \{ j \in [m] \mid \exists b_j \in B_j \text{ such that } d(A_1, B_j) \geq \beta \}$. If $J \neq \emptyset$ and $l + 1$ is the least element of $J$, then there exists a $b \in B_{l+1}$ such that $d(A_1, b) \geq \beta$. Since $B_{l+1} \subset \cdots \subset B_m$, we also have $b \in B_j$ for all $j \geq l + 1$. Hence $J = \{l + 1, \ldots, m\}$ and the same $b_j = b$ works for each $j \in J$. If $J = \emptyset$, we let $l = m$ and $b = p$ (the point in $X_+ \setminus X$).

Let us note that if $i \geq k + 1$, then $d(x, B_i) < \alpha$ for each $x \in A_1$, so by Lemma 4.2(a), $g_i(x) - f_1(x) = d(x, B_i) - d(x, A_1) \leq \alpha$ for each $x \in X$. Similarly, if $i \leq l$, then $d(x, A_1) < \beta$ for each $x \in B_i$, and so by Lemma 4.2(a), $f_1(x) - g_i(x) = d(x, A_1) - d(x, B_i) \leq \beta$ for each $x \in X$. In particular, if $k + 1 \leq i \leq l$, then $d(A_1, B_i) \leq \max(\alpha, \beta) = \beta < \varepsilon$.

Let $\kappa = \sum_{i=1}^{k} \mu_i$. Since $a \in A_1 \subset \cdots \subset A_n$, we have $F(a) = 0$, so $G(a) \leq \delta$. Since $g_i(a) = d(a, B_i) \geq \alpha$ for each $i \leq k$, and each $\mu_i g_i \geq 0$, we have $\kappa \alpha \leq \sum_{i=1}^{k} \mu_i g_i(a) \leq G(a) \leq \delta$. Hence $\kappa \leq \delta/\alpha = \alpha$.

For each $x \in B_{l+1}$ (or for each $x \in X_+$ if $l = m$) we have $G(x) = \sum_{i=1}^{l} \mu_i g_i(x)$. Since each $\lambda_i f_i \geq 0$, we get

$$\lambda_1 f_1(x) \leq F(x) \leq G(x) + \delta = \sum_{i=1}^{l} \mu_i g_i(x) + \delta \quad (\ast)$$

Since $f_1(b) = d(b, A_1) \geq \beta$, we further get $\sum_{i=1}^{l} \mu_i g_i(b) \geq \lambda_1 \beta - \delta$. On the other hand, since each $g_i(b) \leq 1$, we have $\sum_{i=1}^{k} \mu_i g_i(b) \leq \kappa \leq \alpha$. Since $\alpha + \delta \leq 2\alpha = 2\beta^2 < \Gamma \beta \leq \lambda_1 \beta$, we have $\alpha < \lambda_1 \beta - \delta$, and we conclude that $l > k$ (in particular, $l \geq k$ and $m > 0$).

Let $\nu_1 = \sum_{i=k+1}^{l} \mu_i$. Since $g_i(x) \leq f_1(x) + \alpha$ for each $x \in X$ and each $i \geq k+1$, from $\ast$ we have $\lambda_1 f_1(x) \leq \sum_{i=1}^{l} \mu_i g_i(x) + \delta \leq \kappa + \sum_{i=k+1}^{l} \mu_i (f_1(x) + \alpha) + \delta \leq \alpha + \nu_1 f_1(x) + \nu_1 \alpha + \delta$ for each $x \in B_{l+1}$ (or for each $x \in X_+ \setminus X$ if $l = m$). Since $\nu_1 \leq 1$, we have $\lambda_1 f_1(x) - \nu_1 f_1(x) \leq \alpha + \nu_1 \alpha + \delta \leq 3\alpha$. Since $f_1(b) \geq \beta$, we get $\lambda_1 - \nu_1 \leq 3\alpha/f_1(b) \leq 3\alpha/\beta = 3\beta$.

For each $x \in A_2$ (or for each $x \in X_+$ if $n = 1$) we have $F(x) = \lambda_1 f_1$. Since $g_i(x) \geq f_1(x) - \beta$ for each $i \leq l$, we have $\lambda_1 f_1(x) = F(x) \geq G(x) + \delta \geq \sum_{i=k+1}^{l} \mu_i g_i(x) - \delta \geq \sum_{i=k+1}^{l} \mu_i (f_1(x) - \beta) - \delta = \nu_1 f_1(x) - \nu_1 \beta - \delta$. Since $\nu_1 \leq 1$, we have $\nu_1 f_1(x) - \lambda_1 f_1(x) \leq \nu_1 \beta + \delta \leq 2\beta$. Since $d(A_1, A_2) \geq \Gamma$, there exists an $a' \in A_2$ such that $f_1(a') = d(a', A_1) \geq \Gamma$. (If $n = 1$, let $a' = p$, the point in $X_+ \setminus X$.) Then $\nu_1 - \lambda_1 \leq 2\beta/f_1(a') \leq 2\beta/\Gamma$.

Thus we get $|\lambda_i - \nu_i| \leq 3\beta/\Gamma$. Then $\sum_{i=k+1}^{l} |(\lambda_1 - \nu_1)| \leq \sum_{i=k+1}^{l} |(\lambda_i - \nu_i)| + \sum_{i=k+1}^{l} |(\lambda_i - \nu_i)| \leq 3\beta/\Gamma + \nu_1 \beta \leq 4\beta/\Gamma$ using that $||f_1|| \leq 1$, each $||f_1 - g_i|| \leq \beta$ and $\nu_1 \leq 1$. Also $\sum_{i=1}^{k} |\mu_i g_i| \leq \kappa \leq \alpha \leq \beta/\Gamma$ using that each $||g_i|| \leq 1$. Hence $\sum_{i=1}^{l} |\mu_i g_i| \leq 5\beta/\Gamma$. 

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We have \( \| \sum_{i=2}^{n} \lambda_i f_i - \sum_{j=1+1}^{m} \mu_j g_j \| = \| F - G + \sum_{i=1}^{l} \mu_i g_i - \lambda_1 f_1 \| \leq \| F - G \| + \| \sum_{i=1}^{l} \mu_i g_i - \lambda_1 f_1 \| \leq \delta + 5\beta / \Gamma \leq 6\beta / \Gamma. \) We have \( 6\beta / \Gamma \leq 6(\varepsilon / 6)^{\varphi(n)} / \varepsilon = (\varepsilon / 6)^{4\varphi(n-1)}. \) The assertion now follows from the induction hypothesis.

\[ \square \]

**Theorem 4.9.** The topology of \( K_\Delta(X) \) depends only on the topology of \( X. \)

Unfortunately, the uniform structure of \( K_\Delta(X) \) does not seem to depend only on the uniform structure of \( X \) (at least, the estimates obtained below depend on the dimensions of the two convex hulls).

**Proof.** By Lemma 4.6 it suffices to show that the topology of \( K_\Delta(X) \) depends only on the topology of \( X. \) This makes some difference since \( \| F \| = 1 \) for each \( F \in K_\Delta(X), \) but e.g. \( \| e(X) \| = 0 \) where \( e(X) \in K_\Delta(X). \)

It is well-known that the topology induced on \( K(X) \) by the Hausdorff metric is the Vietoris topology, which is independent of the metric \([33]; \text{Theorem 3.3}]. \) So if \( S \) denotes the underlying set of the metric space \( X = (S, d), \) and \( Y = (S, d') \) for some metric \( d' \) on \( S \) inducing the same topology, then \( \text{id}_S: K(X) \to K(Y) \) is a homeomorphism. Let us extend the composition \( t: e_+ \left( K(X) \right) \xrightarrow{e_+} K(X) \xrightarrow{id_S} K(Y) \xrightarrow{e_+} e_+ \left( K(Y) \right) \) linearly to a map \( T: K_\Delta(X) \to K_\Delta(Y). \) By Lemma 4.3, \( T \) is a bijection. We will show that \( T \) is continuous; by symmetry, \( T^{-1} \) will then also be continuous.

Given an \( x \in K_\Delta^+(X), \) let \( |A| \) be the minimal simplex of \( K_\Delta^+(X) \) containing \( x; \) thus \( A: [n] \to K(X) \) is a nonempty finite chain of nonempty subsets \( A_1 \subset \cdots \subset A_n \subset X \) for some \( n = n(x). \) We have \( x = \sum_{i=1}^{n} \lambda_i f_i, \) where each \( f_i = e(A_i) \) and each \( \lambda_i \geq 0, \) with \( \sum \lambda_i = 1. \) By the minimality, \( A \) is injective, i.e., each \( A_{i+1} \neq A_i, \) and also each \( \lambda_i > 0. \) Then there exists a \( \Gamma \in (0, 1] \) such that each \( d(A_i, A_{i+1}) \geq \Gamma \) and each \( \lambda_i \geq \Gamma. \)

Let \( \varepsilon > 0 \) be given. We need to show that there exists a \( \delta > 0 \) such that if \( y \in K_\Delta^+(X) \) is \( \delta \)-close to \( x, \) then \( T(y) \) is \( \varepsilon \)-close to \( T(x). \) Since \( t \) is continuous, there exists an \( \alpha > 0 \) such that \( t \) sends the \( \alpha \)-neighborhood of each \( f_i \) into the \( \beta \)-neighborhood of \( t(f_i), \) where \( \beta = \varepsilon / (3n + 1). \) We may assume that \( \alpha \leq \beta. \) Let \( \delta = \varepsilon / 4.8 \) be given by Lemma 4.8 for \( \varepsilon / 4.7 = \alpha \) and \( 1.4 / 4.7 = \Gamma. \)

Suppose that \( y \in K_\Delta^+(X) \) is \( \delta \)-close to \( x. \) Let \( |B| \) be the minimal simplex of \( K_\Delta^+(X) \) containing \( y; \) thus \( B: [m] \to K(X) \) is injective, for some \( m = m(y), \) and \( y = \sum_{i=1}^{m} \mu_i g_i, \) where each \( g_i = e(B_i) \) and each \( \mu_i > 0, \) with \( \sum_{i=1}^{m} \mu_i = 1. \) Then by Lemma 4.8 there exist \( 1 = l_0 \leq k_1 \leq l_1 \leq k_2 \leq \cdots \leq k_n = m \) such that \( \| f_j - g_j \| \leq \alpha \) whenever \( k_i + 1 \leq j \leq l_i, \) each \( \lambda_i - \sum_{j=k_i+1}^{l_i} \mu_j \leq \alpha \leq \beta, \) and each \( \sum_{j=l_i+1}^{k_i} \mu_j \leq \alpha \leq \beta. \) We have \( T(x) = \sum_{i=1}^{n} \lambda_i f_i' \) and \( T(y) = \sum_{j=1}^{m} \mu_j g_j', \) where \( f_i' = t(f_i) \) and \( g_j' = t(g_j). \) By the above, \( \| f_i' - g_j' \| \leq \beta \) whenever \( k_i + 1 \leq j \leq l_i. \) For each \( i, \) since \( \| f_i' \| \leq 1 \) and \( \sum_{j=k_i+1}^{l_i} \mu_j \leq 1, \) we have \( \| \lambda_i f_i' - \sum_{j=k_i+1}^{l_i} \mu_j g_j' \| \leq \| (\lambda_i - \sum_{j=k_i+1}^{l_i} \mu_j) f_i' \| + \| \sum_{j=k_i+1}^{l_i} \mu_j (f_i' - g_j') \| \leq \beta + \beta. \) Since each \( \| g_j' \| \leq 1, \) we also have \( \| \sum_{j=l_i+1}^{k_i} \mu_j g_j' \| \leq \beta. \) Hence \( \| T(x) - T(y) \| = \| \sum_{i=1}^{n} \lambda_i f_i' - \sum_{j=1}^{m} \mu_j g_j' \| \leq (3n + 1) \beta = \varepsilon. \)
Proof. By Lemma 4.6, it suffices to show that $K^+_\Lambda(X)$ is homeomorphic to $|K(X)|$.

Given an $x \in K^+_\Lambda(X)$ and an $y \in K^+_\Lambda(X)$, let $|A|$ and $|B|$ be the minimal simplexes of $K^+_\Lambda(X)$ containing $x$ and $y$. Thus $A: [n] \to K(X)$ and $B: [m] \to K(X)$ are nonempty finite chains of nonempty subsets $A_1 \subset \cdots \subset A_n \subset X$ and $B_1 \subset \cdots \subset B_m \subset X$ for some $n$ and $m$, which are injective by the minimality. We have $x = \sum_{i=1}^n \lambda_i f_i$ and $y = \sum_{i=1}^m \mu_i g_i$, where each $f_i = e(A_i)$, each $g_i = e(B_i)$, each $\lambda_i > 0$ and each $\mu_i > 0$, with $\sum \lambda_i = \sum \mu_i = 1$.

We also have the step functions $\varphi_x := \varphi \sum \lambda_i A_i$, $\varphi_y := \varphi \sum \mu_i B_i$ : $I \to K(X)$. Let us prove that the map $\Phi: K^+_\Lambda(X) \to |K(X)|$, defined by $x \mapsto \varphi_x$, is continuous. Let us fix $x$. Then there exists a $\Gamma \in (0,1]$ such that each $d(A_i, A_{i+1}) \geq \Gamma$ and each $\lambda_i \geq \Gamma$. Let $\varepsilon > 0$ and let $\delta = \delta(\varepsilon,\gamma)$ be given by Lemma 4.8. If $y$ is $\delta$-close to $x$, then by Lemma 4.8 there exist $1 = l_0 \leq k_1 \leq l_1 \leq k_2 \leq \cdots \leq l_n \leq k_{n+1} = m$ such that $d(A_i, B_j) \leq \varepsilon$ whenever $k_i + 1 \leq j \leq l_i$, each $|\lambda_i - \sum_{j=k_{i+1}}^{j} \mu_j| \leq \varepsilon$, and each $\sum_{j=l_{i+1}}^{j} \mu_j \leq \varepsilon$.

Using the notation of [26; §19.B], we have $|\varphi_x - \varphi_y| \leq \sum_{i=0}^{n} (\max(p_i, r_i), \max(p_i, q_i)\varepsilon)$, where $p_i = \sum_{j \leq i} \lambda_j$, $q_i = \sum_{j \leq i} \mu_j$ and $r_i = \sum_{j \leq i} \mu_j$. Then $|p_i - r_i| \leq 2\varepsilon$ and $|p_i - q_i| \leq (2i + 1)\varepsilon$ for each $0 \leq i \leq n$. Hence $max(p_i, q_i) - min(p_i, r_i) \leq (4n + 1)\varepsilon$, and consequently $\mu(|\varphi_x - \varphi_y|) \leq (4n + 1)(n + 1)\varepsilon$. Thus $D(\varphi_x, \varphi_y) \leq \varepsilon + \mu(|\varphi_x - \varphi_y|) \leq (4n + 1)(n + 1)\varepsilon$.

It remains to show that $\Phi^{-1}$ is continuous; we will show that it is in fact uniformly continuous. Indeed, by considering a common subdivision of the two triangulations of $[0,1]$ (one with vertices $v_i := \sum_{j \leq i} \lambda_i$ and another with vertices $w_i := \sum_{j \leq i} \mu_i$), we may assume that $m = n$ and each $\lambda_i = \mu_i$. Then $||x - y|| \leq \sum_{i=1}^{n} \lambda_i||f_i - g_i|| = \sum_{i=1}^{n} \lambda_i d(A_i, B_i) = L_1(\varphi_x, \varphi_y)$.

5. Derived limits over a topological poset

5.A. Partially ordered sheaves. A diagram of spaces indexed by a topological poset $P$ is a morphism of topological posets $f : Q \to P$ (i.e., a continuous monotone map) such that if $f(q) \leq p$, then there exists a unique $q' \in f^{-1}(p)$ such that $q \leq q'$.[11] It is not hard to see (cf. [28; §7.6, §5.9]) that this condition is equivalent to saying that $f$ is an order-closed map with order-discrete point-inverses (with respect to the Alexandrov topologies on $P$ and $Q$). Note also that the monotonicity of $f$ is equivalent to its order-continuity. If $f : Q \to P$ is a diagram of spaces, we have the Hatcher maps $f_p^q : f_p \to f_q^r$, where $f_p = f^{-1}(p)$, defined whenever $p \leq q'$, with $f_p^p = id_{f_p}$ (compare [28; §7.6]).

The homotopy colimit of the diagram of spaces $f$ is the natural continuous map hocolim $f : |Q| \to |P|$ (compare [28; §9.8]).

Example 5.1. Let $P$ be a topological poset and $A(P)$ its set of atoms, and let $E(P)$ be the topological subposet of $A(P) \times P$ consisting of all pairs $(a, p)$ such that $a \leq p$ (see details in §3.C). Then the projection $E(P) \subset A(P) \times P \to P$ is a diagram of spaces.

[11] In Logic, monotone maps (between Kripke frames or Heyting algebras) satisfying this property are known as “p-morphisms”. In Category Theory, such maps between posets (viewed as categories) are known as “discrete opfibrations”, and are a special case of “Grothendieck opfibrations”.
Let $\mathcal{C}$ be a concrete category over the category of sets (for example, the category of abelian groups). A $\mathcal{C}$-valued sheaf-diagram over a topological poset $P$ is a $\mathcal{C}$-valued sheaf $F: Q \to P$ that is also a diagram of spaces whose Hatcher maps are $\mathcal{C}$-morphisms. In other words, $F$ is a $\mathcal{C}$-valued sheaf over the space $P$ along with $\mathcal{C}$-morphisms $F_q^p: F_p \to F_q$ for all pairs $p \leq q$ such that each $F_p^q = id_P$ and each $F_q^p F_p^q = F_p^q$.

A $\mathcal{C}$-valued posheaf (=partially ordered sheaf) is a $\mathcal{C}$-valued sheaf-diagram $F: Q \to P$ such that the restriction $F_q^q: \leq q \to \leq q$ of $F \times F: Q \times Q \to P \times P$ is a sheaf ($\mathcal{C}$-valued or equivalently set-valued). Let us note that since $F$ is a sheaf, so is $F \times F$ and consequently also its restriction over any subset of $P \times P$, in particular, over $\leq P$. Thus $F_q^q$ is a posheaf if and only if $\leq q$ is open in $(F \times F)^{-1}(\leq P)$.

**Lemma 5.2.** Let $F: Q \to P$ be a posheaf of sets.

(a) Each $q \in Q$ has a neighborhood $U$ such that $F(U)$ is a neighborhood of $F(q)$ in $P$ and $F$ restricts to a homeomorphism $U \to F(U)$ that is also an isomorphism of posets.

(b) Let $\pi_1, \pi_2: \leq P \subset P \times P \to P$ be the projections onto the two factors. Then the map $f: \pi_1^* Q \to \pi_2^* Q$ given by $F_p^q: F_p \to F_q$, over each pair $(p \leq q') \in \leq P$, is continuous, and hence is a morphism of sheaves $\pi_1^* F \to \pi_2^* F$.

**Proof.** (a). Since $F$ is a sheaf, there exists an open neighborhood $V$ of $q$ in $Q$ such that $F(V)$ is a neighborhood of $F(q)$ in $P$ and $F$ restricts to a homeomorphism $V \to F(V)$. Since $F_q^q$ is also a sheaf, there exists an open neighborhood $W$ of $(q, q)$ in $\leq Q$ such that $F_q^q(W)$ is an open neighborhood of $F_q^q(q, q)$ in $\leq P$ and $F_q^q$ restricts to a homeomorphism $W \to F(W)$. By the definition of product topology, $W$ contains $W_1 \times W_2 \cap \leq Q$, where $W_1$ and $W_2$ are open neighborhoods of $q$ in $Q$. Let $O = V \cap W \cap \leq Q$. Then $F(O)$ is a neighborhood of $F(q)$ in $P$ and $F$ restricts to a homeomorphism $O \to F(O)$; also, $F_q^q(\leq O)$ is an open neighborhood of $F_q^q(q, q)$ in $\leq P$ and $F_q^q$ restricts to a homeomorphism $\leq O \to F_q^q(\leq O)$. By the definition of product topology, $F_q^q(\leq O)$ contains $O_1 \times O_2 \cap \leq P$, where $O_1$ and $O_2$ are open neighborhoods of $F(q)$ in $P$. Let $U = O \cap F^{-1}(O_1 \cap O_2)$ and $\leq U = V \times U \cap \leq U$. Then $F(U)$ is a neighborhood of $F(q)$ in $P$ and $F$ restricts to a homeomorphism $U \to F(U)$; also, $F_q^q(\leq U) = F(U) \times F(U) \cap \leq P$. If $p, p' \in F(U)$ and $p \leq p'$, there exist unique $q, q' \in U$ such that $F(q) = p$ and $F(q') = p'$, and since $F \times F$ restricts to a homeomorphism between $U \times U \cap \leq U$ and $F(U) \times F(U) \cap \leq P$, we must have $q \leq q'$. Thus $F|_U: U \to F(U)$ is also an isomorphism of posets.

(b). Let us note that $\pi_1^* Q$ consists of pairs $(q, p)$, where $q \in Q$, $p \in P$ and $F(q) \leq p$. Similarly, $\pi_2^* Q$ consists of pairs $(p, q)$, where $q \in Q$, $p \in P$ and $p \leq F(q)$. Clearly, $F_q^q: \leq Q \to \leq P$ factors through $\pi_1^* Q$ for each $i$. Since $F_q^q$ is a sheaf and $\pi_1^* F: \pi_1^* Q \to \leq P$ is a sheaf, the resulting map $\rho_i: \leq Q \to \pi_1^* Q$ is also a sheaf. In particular, it is open and continuous. On the other hand, $\rho_i$ is a bijection. Indeed, $\rho_1(q, q') = (q, q')$, where $p' = F(q')$, and we have $q' = F_p^q(q)$, where $p = F(q)$. Thus $\rho_1$ is a homeomorphism, and we may define $f$ to be the composition $\rho_2 \rho_1^{-1}$. 

\[\Box\]
We recall that the Leray sheaf \( \mathcal{H}^n(\pi) \) of a continuous map \( \pi: E \to B \) is the sheafification of the presheaf \( U \mapsto H^n(\pi^{-1}(U)) \).

**Lemma 5.3.** If \( \pi: E \to B \) is a closed map, where \( E \) is metrizable, then \( \mathcal{H}^n(\pi)_b \simeq H^n(\pi^{-1}(b)) \) for each \( b \in B \).

This result is well-known (see [8; Proposition IV.4.2 and Remark 2 to Theorem II.10.6]). We include a self-contained proof for convenience.

**Proof.** (a). Suppose that \( \pi \) is closed, every open neighborhood \( V \) of \( \pi^{-1}(b) \) in \( E \) contains one of the form \( \pi^{-1}(U) \), where \( U \) is an open neighborhood of \( b \) in \( B \); namely, \( U = B \setminus \pi(E \setminus V) \). Hence the group \( \mathcal{H}^n(\pi)_b = \operatorname{colim}_U H^n(\pi^{-1}(U)) \), where \( U \) runs over all open neighborhoods of \( b \) in \( B \), is isomorphic to \( \operatorname{colim}_V H^n(V) \), where \( V \) runs over all open neighborhoods of \( \pi^{-1}(b) \) in \( E \). By Spanier’s tautness theorem (see [43; Theorem 6.6.3]), the latter group is isomorphic to \( H^n(\pi^{-1}(b)) \). \( \square \)

**Theorem 5.4.** Let \( X \) be a metrizable space and let \( E(X) \) be the subspace of \( X \times K(X) \) consisting of all pairs \( (x, K) \) such that \( x \in K \). Let \( \pi \) be the composition of the inclusion \( E(X) \subset X \times K(X) \) and the projection \( X \times K(X) \to K(X) \). Then

(a) \( \pi \) is a closed map, and \( \mathcal{H}^n(\pi)_A \simeq H^n(A) \) for each compact \( A \subset X \);

(b) \( \mathcal{H}^n(\pi) \) is a posheaf with respect to Hatcher maps \( \mathcal{H}^n(\pi)_A^B: \mathcal{H}^n(\pi)_B \to \mathcal{H}^n(\pi)_A \), \( A \subset B \), defined as the restriction (= inclusion induced) maps \( H^n(B) \to H^n(A) \).

Let us note that \( K(X) \) is ordered by reverse inclusion in (b).

**Proof.** (a). Suppose that \( F \subset E(X) \) is a closed subset such that \( \pi(F) \) is not closed. Then there exists a sequence of points \( A_n \in \pi(F) \) converging to a point \( A \notin \pi(F) \). Let us pick any points \( x_n \in A_n \) such that \( (x_n, A_n) \in F \). Suppose that every \( p \in A \) has an open neighborhood \( U_p \) containing only finitely many of the \( x_i \). Since \( A \) is compact, there exist finitely many points \( p_1, \ldots, p_r \in A \) such that \( A \subset U := U_{p_1} \cup \cdots \cup U_{p_r} \). Then \( U \) contains only finitely many of the \( x_i \). On the other hand, since \( A \) is compact and \( U \) is open, there exists a \( q \in \mathbb{N} \) such that \( A_n \subset U \) for all \( n \geq q \). Hence \( U \) contains \( x_n \) for all \( n \geq q \), which is a contradiction. Thus our assumption was wrong, and some \( x \in A \) is a cluster point of the sequence \( x_n \). Then \( x \) is the limit of some subsequence \( x_{n_i} \). Hence \( (x, A) \) is the limit of the sequence \( (x_{n_i}, A_{n_i}) \). Since each \( (x_i, A_i) \in F \) and \( F \) is closed, we get that \( (x, A) \in F \). Hence \( A \in \pi(F) \), which is a contradiction.

Thus \( \pi \) is closed. The second assertion follows from Lemma 5.3 since \( \pi^{-1}(A) \) is homeomorphic to \( A \) for each \( A \in K(X) \). \( \square \)

(b). Let us write \( \mathcal{H}^n(\pi) \) as \( \mathcal{F} : E^n(X) \to K(X) \). Let \( A \) and \( B \) be compact subsets of \( X \) with \( A \subset B \). Let \( \alpha \in \mathcal{F}_A = H^n(A) \) and \( \beta \in \mathcal{F}_B = H^n(B) \) be such that \( \mathcal{F}_A(\beta) = \alpha \), or in other words, \( \beta|_A = \alpha \). Thus \( B \leq A \) in \( K(X) \) and \( \beta \leq \alpha \) in \( E^n(X) \). By Spanier’s tautness theorem (see [43; Theorem 6.6.3]), \( H^n(B) \simeq \operatorname{lim}_V H^n(V) \) over all open neighborhoods \( V \) of \( B \). Hence there exists an open neighborhood \( V \) of \( B \) and a \( \gamma \in H^n(V) \) such that \( \gamma|_B = \beta \). Since the Hausdorff metric induces the Vietoris topology on \( K(X) \), the subset \( W := \{ C \in K(X) \mid C \subset V \} \) of \( K(X) \) is open in \( K(X) \). Actually, \( W \) can be identified
with \( K(V) \), and we have \( A, B \in W \). Also, \( \pi^{-1}(W) \) is an open subset of \( E(X) \) that can be identified with \( E(V) \). Let \( p \) be the composition \( E(V) \subset V \times K(V) \to V \) of the inclusion and the projection, and let \( \sigma = p^* \gamma \in H^n(E(V)) \). For each \( C \in K(V) \), the composition \( C = \pi^{-1}(C) \subset E(V) \xrightarrow{p} V \) coincides with the inclusion \( C \to V \). Hence \( \sigma|_{\pi^{-1}(C)} = \gamma|_C \). On the other hand, by Spanier’s tautness theorem \( \sigma|_{\pi^{-1}(C)} \) coincides with the image of \( \sigma \in H^n(\pi^{-1}(W)) \) in \( \mathcal{F}_C = \text{colim}_U H^n(\pi^{-1}(U)) \), where \( U \) runs over all open neighborhoods of \( C \) in \( K(X) \). Since \( \mathcal{F} \) is the sheafification of the presheaf \( U \mapsto H^n(\pi^{-1}(U)) \) and \( \sigma \in H^n(\pi^{-1}(W)) \), there is a section \( s: W \to E^n(X) \) of \( \mathcal{F} \) over \( W \) given by \( C \mapsto \gamma|_C \in \mathcal{F}_C \).

Let \( s_\leq \) be the restriction of \( s \times s: W \times W \to E^n(X) \times E^n(X) \) to \( \leq W \). Then \( s_\leq(\leq W) \) consists of all tuples \((B', A', \gamma|_{B'}, \gamma|_{A'})\) such that \( A' \subset B' \), and in particular it contains \((B, A, \beta, \alpha)\). But such a tuple \((B', A', \gamma|_{B'}, \gamma|_{A'})\) satisfies \( \gamma|_{A'} = (\gamma|_{B'})|_{A'} \) and hence belongs to \( \leq E^n(X) \). Thus \( s_\leq(\leq W) \subset \leq E^n(X) \). On the other hand, \( \leq W \) is open in \( (\mathcal{F} \times \mathcal{F})^{-1}(\leq E^n(X)) \) since \( s_\leq \) is a section of the sheaf \( \mathcal{F} \times \mathcal{F} |_{(\mathcal{F} \times \mathcal{F})^{-1}(\leq E^n(X))} \) over \( \leq W \). Since \( (B, A, \beta, \alpha) \) could be an arbitrary point of \( \leq E^n(X) \), we conclude that \( \leq E^n(X) \) is open in \( (\mathcal{F} \times \mathcal{F})^{-1}(\leq E^n(X)) \).

5.B. Continuous derived limits. If \( X \) is a topological poset, let \([X]\) denote the topological space with the same underlying set as \( X \) and with \( U \) open in \([X]\) if and only if it is open both in \( X \) and in the Alexandrov topology corresponding to the order on \( X \). If \( f: X \to Y \) is a continuous monotone map between topological posets, it is continuous also as a map \([f]: [X] \to [Y]\).

Let \( P \) be a pospace, and let \([n]\) denote the \( n \)-element poset \( \{1, \ldots, n\} \) with the usual (total) order. A chain of length \( n \) (without repeats) \( p_1 < \cdots < p_n \) in \( P \) may be regarded as an injective monotone map \([n] \to P\). All such chains form a subspace \( P^{[n]} \) of the product \( P^n = P \times \cdots \times P \) of spaces. Let \( \rho_n: P^{[n]} \subset P^n \to P \) be the projection onto the last factor. Let \( P' \) be the topological poset \( \bigcup_{n=1}^\infty P^{[n]} \) of all finite chains in \( P \), ordered by inclusion, with the topology of disjoint union. The map \( \rho: P' \to P \) defined by \( \rho(p_1 < \cdots < p_n) = p_n \) is continuous (since each \( \rho_n \) is continuous) and, clearly, monotone.

Let \( P^\Delta = [P'] \). Thus \( U \) is open in \( P^\Delta \) if and only if \( U \) meets each \( P^{[n]} \) in an open set, and \( c \in U \) implies \( d \in U \) whenever \( c \) is a subchain of \( d \). Let \( \kappa: |P| \to P^\Delta \) be defined by sending the interior of every simplex to that same simplex regarded as a point of \( P' \).

**Lemma 5.5.** If the topology of \( P \) is Hausdorff, then \( \kappa \) is continuous.

**Proof.** Let \( U \) be an open subset of \( P^\Delta \) and let \( x \in U \). Then \( x = (x_1 < \cdots < x_n) \in P^{[n]} \subset P^n \) for some \( n \). By the definition of product topology there exist neighborhoods \( U_1, \ldots, U_n \) of \( x_1, \ldots, x_n \) such that \( (U_1 \times \cdots \times U_n) \cap P^{[n]} \subset U \). Since \( P \) is Hausdorff, we may assume that \( U_1, \ldots, U_n \) are pairwise disjoint. Let \( V_x \) be the set of all nonempty finite chains in \( P \) that contain a subchain \( y_1 < \cdots < y_n \) such that each \( y_i \in U_i \). Since \( (U_1 \times \cdots \times U_n) \cap P^{[n]} \subset U \) and \( U \) is open in \( P^\Delta \), we have \( V_x \subset U \). On the other hand, by [26; Lemma 21.4] (using that \( P \) is \( T_1 \)), \( \kappa^{-1}(V_x) \) is open in \( |P| \). Hence also \( \kappa^{-1}(U) = \bigcup_{x \in U} \kappa^{-1}(V_x) \) is open in \( |P| \). \( \square \)
Let $\mathcal{F} : E \to P$ be a sheaf-diagram. Let us consider the sheaf-diagram $\rho^* \mathcal{F} : \rho^* E \to P'$. Thus $\rho^* E$ consists of all tuples $(p_1 \leq \cdots \leq p_n; g)$, where $(p_1 \leq \cdots \leq p_n) \in P'$ and $g \in \mathcal{F}_{p_n}$, and has the topology of pullback of continuous maps and the order of pullback of monotone maps (namely, of the maps $\mathcal{F} : E \to P$ and $\rho : P' \to P$). Namely, $\rho^* E$ is homeomorphic to $\bigsqcup_{n=1}^{\infty} \rho_n^* E$ and is ordered by $(p_1 \leq \cdots \leq p_n; g) \leq (q_1 \leq \cdots \leq q_m; h)$ if and only if $(p_1 \leq \cdots \leq p_n)$ is a subchain of $(q_1 \leq \cdots \leq q_m)$ and $\mathcal{F}_{q_m}(g) = h$.

Let $E^\Delta_\mathcal{F} = [\rho^* \mathcal{F}]$ and let $\mathcal{F}^\Delta = [\rho^* \mathcal{F}] : E^\Delta_\mathcal{F} \to P^\Delta$. Since $\rho^* \mathcal{F}$ is monotone and continuous, $\mathcal{F}^\Delta$ is continuous.

Also $\mathcal{F}^\Delta$ is an open map. Indeed, since $\rho^* \mathcal{F}$ is a sheaf, it is an open map. Also $\rho^* \mathcal{F}$ is open with respect to the Alexandrov topologies on $\rho^* E$ and $P'$. Indeed, if $U$ is open in the Alexandrov topology on $\rho^* E$, and some $(p_1 \leq \cdots \leq p_n) \in \rho^* \mathcal{F}(U)$ is a subchain of $(q_1 \leq \cdots \leq q_m)$, then $(q_1 \leq \cdots \leq q_m) \in \rho^* \mathcal{F}(U)$ since any $(p_1 \leq \cdots \leq p_n; g) \in U$ gives rise to $(q_1 \leq \cdots \leq q_m; \mathcal{F}_{q_m}(g)) \in U$.

**Lemma 5.6.** If $\mathcal{F}$ is a posheaf, then $\mathcal{F}^\Delta$ is a sheaf.

**Proof.** For a subset $S$ of a poset $Q$ let $|S|$ denote the smallest subset of $Q$ that contains $S$ and is open in the Alexandrov topology; in other words, $|S|$ consists of all $q \in Q$ such that $q \geq p$ for some $p \in Q$.

If $U$ is an open subset of $P^{[n]}$, then $|U|$ is open in $P^\Delta$. Indeed, for each $k \geq 0$, $|U| \cap P^{[n+k]}$ is the union of the preimages of $U$ under the $\binom{n+k}{k}$ projections $P^{[n+k]} \to P^{[n]}$, which are continuous, being the restrictions of the projections $P^{n+k} \to P^n$.

Next, if $V$ is an open subset of $\rho_n^* E$, then $|V|$ is open in $E^\Delta_\mathcal{F}$. Indeed, it suffices to show that $|V| \cap \rho_{n+k}^* E$ is open in $\rho_{n+k}^* E$ for each $k \geq 0$. Let $\pi$ be one of the $\binom{n+k}{k}$ projections $P^{[n+k]} \to P^{[n]}$. Let $f_\pi : \pi^* \rho_n^* E \to \rho_{n+k}^* E$ be given by $f_\pi(c, g) = (c, \mathcal{F}_{\rho_{n+k}(c)}(g))$. Then $\pi^* \rho_n^* \mathcal{F} = (\rho_{n+k}^* \mathcal{F}) \circ f_\pi$ and since $\mathcal{F}$ is a posheaf, it follows from Lemma 5.2(b) that $f_\pi$ is continuous. Thus $f_\pi$ is a homomorphism of sheaves $\pi^* \rho_n^* \mathcal{F} \to \rho_{n+k}^* \mathcal{F}$. It is easy to see that every homomorphism of sheaves is itself a sheaf of sets; and that every sheaf of sets is an open map. In particular, $f_\pi$ is an open map for each projection $\pi$. Clearly, $|V| \cap \rho_{n+k}^* E$ is the union of the sets $f_\pi(V_\pi)$, where $V_\pi$ is the preimage of $V$ under the natural map $\pi^* \rho_n^* E \to \rho_n^* E$. Since the latter is continuous, each $V_\pi$ is open. Thus $|V| \cap \rho_{n+k}^* E$ is open.

Since $\mathcal{F}$ is a sheaf, there exists an open neighborhood $V_{(p, g)}$ of each $(p, g) \in E$ such that $\mathcal{F}$ restricts to a homeomorphism between $V_{(p, g)}$ and an open neighborhood $U_p$ of $p$ in $P$. Since $\mathcal{F}$ is a posheaf, by Lemma 5.2(a) for every $(p', g') \in V_{(p, g)}$ and $(p'', g'') \in V_{(p, g)}$ such that $p' \leq p''$, we have $\mathcal{F}_{p''}(g'') = g''$.

Given an $(p_1 \leq \cdots \leq p_n; g) \in \pi^* E$, let $U$ and $V$ be the preimages of $U_{p_n}$ and $V_{(p_n, g)}$ under $\rho_n : P^{[n]} \to P$ and the natural map $\rho_n^* E \to E$. Since $\rho_n^* E$ is clopen in $\rho^* E$, $V$ is an open neighborhood of $(p_1 \leq \cdots \leq p_n; g)$ in $\pi^* E$ and clearly $\rho^* \mathcal{F}$ restricts to a homeomorphism between $V$ and $U$. By construction, for every $(p_1' \leq \cdots \leq p_n'; g') \in V$ and $(p_1'' \leq \cdots \leq p_n''; g'') \in V$ such that $p_1' \leq p_1''$, we have $\mathcal{F}_{p_1''}(g'') = g''$.

Now $|U|$ consists of all chains $(q_1 \leq \cdots \leq q_m)$ that have a subchain $(p_1' \leq \cdots \leq p_n')$ in $U$; and $|V|$ consists of all tuples $(q_1 \leq \cdots \leq q_m; h)$ such that $(q_1 \leq \cdots \leq q_m)$ has a
subchain \((p'_1 < \cdots < p'_n)\) in \(U\), and \(h = \mathcal{F}^\psi_{q_m}(g')\), where \(g' \in \mathcal{F}_{p_n}\) is the unique element such that \((p'_1 < \cdots < p'_n, g') \in V\). Thus \(h\) is uniquely determined by \((p'_1 < \cdots < p'_n)\).

Given another \((p''_1 < \cdots < p''_n, g'') \in V\) such that \((p''_1 < \cdots < p''_n)\) is a subchain of \((q_1 < \cdots < q_m)\), we have either \(p''_n \leq p'_n\) or \(p'_n \leq p''_n\), since \((q_1 < \cdots < q_m)\) is a chain. By symmetry it suffices to consider the case \(p'_n \leq p''_n\). Then \(\mathcal{F}^{p'_n}(g') = g''\), and consequently \(\mathcal{F}^{q_m}(g') = \mathcal{F}^{p''_n}(g'')\). Thus \(h\) does not depend on the choice of the subchain \((p'_1 < \cdots < p'_n)\) and so is uniquely determined by the chain \((q_1 < \cdots < q_m)\). Hence \(\mathcal{F}\) restricts to a bijection between \([V]\) and \([U]\).

Since \(\mathcal{F}\) is an open continuous map and \([V]\) is open, \(\mathcal{F}\) restricts to an open continuous map between \([V]\) and \([U]\). Since this restriction is also a bijection, it must be a homeomorphism.

We define \(\text{holim} \mathcal{F}\) to be the pullback \(\kappa^* \mathcal{F}^\Delta : \kappa^* E^\Delta_p \to |P|\). In other words,

\[
\text{holim} \mathcal{F} = \kappa^*[\rho^* \mathcal{F}^\Delta] : \kappa^*[\rho^* E] \to |P|.
\]

Lemmas 5.5 and 5.6 imply that \(\text{holim} \mathcal{F}\) is a sheaf.

Given a presheaf \(\mathcal{F}\) of abelian groups over a topological poset \(P\), we define \(\text{Lim}^n \mathcal{F}\), also denoted \(\text{lim}_{p \in \mathcal{P}} \mathcal{F}_p\), to be \(H^n(|P|; \text{holim} \mathcal{F})\) for each \(n = 0, 1, \ldots\). As discussed in §2, when \(P\) has discrete topology, these are the usual derived limits.

5.C. Leray sheaves. Let us call a diagram of spaces \(\pi : E \to P\) an order-isometry if there exist metrics on \(E\) and \(P\), compatible with the topologies, such that \(d(\pi(x), \pi(x')) = d(x, x')\) whenever \(x \leq x'\).

**Lemma 5.7.** Let \(\pi : E \to P\) be a diagram of spaces, with metrizable \(E\) and \(P\). If \(\pi\) is a closed map and an order-isometry, then \(\text{hocolim} \pi : |E| \to |P|\) is a closed map.

**Proof.** Suppose that \(F \subset |E|\) is closed, but \((\text{hocolim} \pi)(F)\) is not. Then there exists a sequence of points \(y_1, y_2, \cdots \in (\text{hocolim} \pi)(F)\) converging to a point \(y \notin (\text{hocolim} \pi)(F)\).

We have \(y = \sum_{i=1}^{n} \lambda_i y_i \in |P|\), where \(y_1 < \cdots < y_n\), each \(y_i \in P\), each \(\lambda_i \geq 0\) and \(\sum_{i=1}^{n} \lambda_i = 1\). Then for each \(k\) we have \(y_k = \sum_{i=1}^{n} \sum_{j \geq i}^{m_{ki}} \lambda_{kij} y_{kij} \in |P|\), where \(y_{k,i,1} < \cdots < y_{k,i,m_{ki}} \leq y_{k,i+1,1}\) for each \(k\) and \(i\), each \(y_{kij} \in P\), and \(\sum_{j=1}^{m_{ki}} \lambda_{kij} = \lambda_i\). Since \(y_{k,i,1} < \cdots < y_{k,i,m_{ki}}\), the formal sum \(\frac{1}{\lambda_i} \sum_{j=1}^{m_{ki}} \lambda_{kij} y_{kij}\) denotes a point \(y_{kij} \in |P|\). Hence we can write formally \(y_k = \sum_{i=1}^{n} \lambda_i y_{ki}\); we will further take this formal equation to encode the more useful observation that the step function \(\varphi_{y_k}\) is the “stacked linear combination” of the step functions \(\varphi_{y_{ki}}\) in the sense that \(\varphi_{y_k}(\lambda_1 + \cdots + \lambda_{i-1} + t \lambda_i) = \varphi_{y_{ki}}(t)\) for \(t \in [0, 1]\).

Therefore \(L_1(\varphi_y, \varphi_{y_k}) = \sum_{i=1}^{n} \lambda_i L_1(\varphi_{y_i}, \varphi_{y_{ki}})\), and consequently for each \(i = 1, \ldots, n\) the sequence \(y_{ki}\) converges to \(y_i\). Here each \(y_i \in P\), but the \(y_{ki}\) need not lie in \(P\); however, each \(y_{kij} \in P\), and for each \(k\) and \(i\) we may choose one of these points \(y'_{ki} := y_{k,i,j_{ki}} \in P\) so that the sequence \(y'_{ki}\) also converges to \(y_i\) for each \(i\) (see [26; Lemma 21.4]). Then, in particular, \(d(y_{ki}, y'_{ki}) \to 0\) as \(k \to \infty\).

Let us pick some points \(x_k \in (\text{hocolim} \pi)^{-1}(y_k) \cap F\). Then for each \(k\) we have \(x_k = \sum_{i=1}^{n} \sum_{j=1}^{m_{ki}} \lambda_{kij} y_{kij} \in |E|\), where \(x_{k,1,i} < \cdots < x_{k,i,m_{ki}} \leq x_{k,i+1,1}\) for each \(k\) and \(i\) and
each $x_{kij} \in (\text{hocolim } \pi)^{-1}(y_{kij})$. (Of course, the $x_{ijk}$ do not necessarily lie in $F$.) Again the formal sum $\frac{1}{\lambda} \sum_{i=1}^{m_k} \lambda x_{kij}$ denotes a point $x_{ki} \in |E|$, and $x_k = \sum_{i=1}^{n_k} \lambda_i x_{kij}$ (which encodes a relation between the step functions). Let $x_{ki} = x_{k,i,j_ki} \in E$; thus $y_{k,i} = \pi(x_{ki})$. Since $\pi$ is a closed map and $\{y_{k,i} \mid k \in \mathbb{N}\}$ is not closed in $P$, $\{x_{k,i} \mid k \in \mathbb{N}\}$ is not closed in $E$. Hence some subsequence $x_{k_i,1}$ converges to a point $x_1 \in E$.

The assertion follows from Lemma 5.3, $\mathcal{H}_{A} \simeq \mathcal{H}^{n}(A)$. On the other hand, using the notation of §5.B with $P = K(X)$, we have $\pi^{-1}(A) \cong A_1$. Hence by (a) and Lemma 5.3, $\mathcal{H}_{A} \simeq \mathcal{H}^{n}(A)$. On the other hand, using the notation of §5.B with $P = K(X)$, we have $\pi^{-1}(A_1) \cong A_1$. Hence by (a) and Lemma 5.3, $\mathcal{H}_{A} \simeq \mathcal{H}^{n}(A)$. On the other hand, using the notation of §5.B with $P = K(X)$, we have $\pi^{-1}(A_1) \cong A_1$. Hence by (a) and Lemma 5.3, $\mathcal{H}_{A} \simeq \mathcal{H}^{n}(A)$. On the other hand, using the notation of §5.B with $P = K(X)$, we have $\pi^{-1}(A_1) \cong A_1$. Hence by (a) and Lemma 5.3, $\mathcal{H}_{A} \simeq \mathcal{H}^{n}(A)$. On the other hand, using the notation of §5.B with $P = K(X)$, we have $\pi^{-1}(A_1) \cong A_1$. Hence by (a) and Lemma 5.3, $\mathcal{H}_{A} \simeq \mathcal{H}^{n}(A)$. On the other hand, using the notation of §5.B with $P = K(X)$, we have $\pi^{-1}(A_1) \cong A_1$. Hence by (a) and Lemma 5.3, $\mathcal{H}_{A} \simeq \mathcal{H}^{n}(A)$. On the other hand, using the notation of §5.B with $P = K(X)$, we have $\pi^{-1}(A_1) \cong A_1$. Hence by (a) and Lemma 5.3, $\mathcal{H}_{A} \simeq \mathcal{H}^{n}(A)$. On the other hand, using the notation of §5.B with $P = K(X)$, we have $\pi^{-1}(A_1) \cong A_1$. Hence by (a) and Lemma 5.3, $\mathcal{H}_{A} \simeq \mathcal{H}^{n}(A)$. On the other hand, using the notation of §5.B with $P = K(X)$, we have $\pi^{-1}(A_1) \cong A_1$. Hence by (a) and Lemma 5.3, $\mathcal{H}_{A} \simeq \mathcal{H}^{n}(A)$. On the other hand, using the notation of §5.B with $P = K(X)$, we have $\pi^{-1}(A_1) \cong A_1$. Hence by (a) and Lemma 5.3, $\mathcal{H}_{A} \simeq \mathcal{H}^{n}(A)$. On the other hand, using the notation of §5.B with $P = K(X)$, we have $\pi^{-1}(A_1) \cong A_1$. Hence by (a) and Lemma 5.3, $\mathcal{H}_{A} \simeq \mathcal{H}^{n}(A)$. On the other hand, using the notation of §5.B with $P = K(X)$, we have $\pi^{-1}(A_1) \cong A_1$. Hence by (a) and Lemma 5.3, $\mathcal{H}_{A} \simeq \mathcal{H}^{n}(A)$. On the other hand, using the notation of §5.B with $P = K(X)$, we have $\pi^{-1}(A_1) \cong A_1$. Hence by (a) and Lemma 5.3, $\mathcal{H}_{A} \simeq \mathcal{H}^{n}(A)$. On the other hand, using the notation of §5.B with $P = K(X)$, we have $\pi^{-1}(A_1) \cong A_1$. Hence by (a) and Lemma 5.3, $\mathcal{H}_{A} \simeq \mathcal{H}^{n}(A)$. On the other hand, using the notation of §5.B with $P = K(X)$, we have $\pi^{-1}(A_1) \cong A_1$. Hence by (a) and Lemma 5.3, $\mathcal{H}_{A} \simeq \mathcal{H}^{n}(A)$. On the other hand, using the notation of §5.B with $P = K(X)$, we have $\pi^{-1}(A_1) \cong A_1$. Hence by (a) and Lemma 5.3, $\mathcal{H}_{A} \simeq \mathcal{H}^{n}(A)$. On the other hand, using the notation of §5.B with $P = K(X)$, we have $\pi^{-1}(A_1) \cong A_1$. Hence by (a) and Lemma 5.3, $\mathcal{H}_{A} \simeq \mathcal{H}^{n}(A)$. On the other hand, using the notation of §5.B with $P = K(X)$, we have $\pi^{-1}(A_1) \cong A_1$. Hence by (a) and Lemma 5.3, $\mathcal{H}_{A} \simeq \mathcal{H}^{n}(A)$. On the other hand, using the notation of §5.B with $P = K(X)$, we have $\pi^{-1}(A_1) \cong A_1$. Hence by (a) and Lemma 5.3, $\mathcal{H}_{A} \simeq \mathcal{H}^{n}(A)$. On the other hand, using the notation of §5.B with $P = K(X)$, we have $\pi^{-1}(A_1) \cong A_1$. Hence by (a) and Lemma 5.3, $\mathcal{H}_{A} \simeq \mathcal{H}^{n}(A)$. On the other hand, using the notation of §5.B with $P = K(X)$, we have $\pi^{-1}(A_1) \cong A_1$. Hence by (a) and Lemma 5.3, $\mathcal{H}_{A} \simeq \mathcal{H}^{n}(A)$. On the other hand, using the notation of §5.B with $P = K(X)$, we have $\pi^{-1}(A_1) \cong A_1$. Hence by (a) and Lemma 5.3, $\mathcal{H}_{A} \simeq \mathcal{H}^{n}(A)$.
of $A$ in $|K(X)|$ and sections $s$, $s'$ of $\mathcal{H}$, $\mathcal{H}'$ over $U$ such that $s(A) = s'(A) = \alpha$ and $s(C) = s'(C) \in \mathcal{H}_C = \mathcal{H}'_C$ for each $C \in U$.

Let $A = \sum_{i=1}^n \lambda_i A_i$ be as above. By Spanier’s tautness theorem (see [43; Theorem 6.6.3]), $H^n(A_1) \simeq \lim_W H^n(V)$ over all open neighborhoods $V$ of $A_1$ in $X$. Hence there exists an open neighborhood $V$ of $A_1$ and a $\gamma \in H^n(V)$ such that $\gamma|_{A_1} = \alpha$. Since the Hausdorff metric induces the Vietoris topology on $K(X)$, the subset $W := \{B \in K(X) | B \subset V\}$ of $K(X)$ is open in $K(X)$. Actually, $W$ can be identified with $K(V)$, and we have $A \in W$. Since $\bar{\rho}$ is continuous, $\bar{\rho}^{-1}(W) = \{(C_1 < \cdots < C_n) \in K(X)^n | C_1 \subset V\}$ is open in $K(X)^n$. Since a chain $C_1 \subset \cdots \subset C_n$ of $K(X)$ satisfies $C_i \subset V$ for some $i$ if and only if it satisfies $C_i \subset V$, in fact $\bar{\rho}^{-1}(W)$ is also Alexandrov open in $K(X)^n$. Hence $[\bar{\rho}^{-1}(W)]$ is an open subset of $|K(X)|$. The open subset $U := \kappa^{-1}[\bar{\rho}^{-1}(W)]$ of $|K(X)|$ consists of all $C = \sum_{i=1}^n \lambda_i C_i$ such that $(C_1 < \cdots < C_n) \in K(X)^n$, $C_i \subset V$, each $\lambda_i > 0$ and $\sum_{i=1}^n \lambda_i = 1$.

By the proof of Theorem 5.4(b), there is a section $s_\mathcal{F}$ of the Leray sheaf $\mathcal{F} := H^n(\pi)$ over $W$ given by $B \mapsto \gamma|B \in \mathcal{F}_B$. Then $s_\mathcal{F} \bar{\rho}$ is a section of $\rho^* \mathcal{F}$ over $\bar{\rho}^{-1}(W)$ given by $(C_1 < \cdots < C_n) \mapsto \gamma|_{C_1} \in \mathcal{F}_{C_1} = (\rho^* \mathcal{F})_{(C_1 < \cdots < C_n)}$. In the alternative language, $s_\mathcal{F} \bar{\rho}$ is a section of $\rho^* \mathcal{F}^*$ over $\rho^{-1}(W^*)$ given by $(C_1 < \cdots < C_n) \mapsto \gamma|_{C_1} \in (\mathcal{F}^*)_C = (\rho^* \mathcal{F}^*)_{(C_1 < \cdots < C_n)}$. Since $s$ and $\rho$ are monotone and continuous (see the proof of Theorem 5.4(b) concerning the monotonicity of $s$), $[s \rho]$ is continuous and hence is a section of $[\rho^* \mathcal{F}^*]$ over $[\rho^{-1}(W^*)]$. But we have $[s \rho] = [s \bar{\rho}]$, $[\rho^* \mathcal{F}^*] = [\rho^* \mathcal{F}]$ and $[\rho^{-1}(W^*)] = [\bar{\rho}^{-1}(W)]$. Therefore $s' := [s \bar{\rho}] \kappa$ is a section of $\mathcal{H}' = \kappa^*[\rho^* \mathcal{F}]$ over $U = \kappa^{-1}[\bar{\rho}^{-1}(W)]$.

Clearly, $s'$ is given by $C \mapsto \gamma|_{C_1} \in \mathcal{F}_C$, where $C$ is as above. In particular, $s'(A) = \gamma|_{A_1} = \alpha$, as desired.

The open subset $(\text{hocolim } \pi)^{-1}(U)$ of $|E(X)| \subset X \times |K(X)|$ consists of all pairs $(x, C)$ where $C = \sum_{i=1}^n \lambda_i C_i \subset U$ and $x \in C_1$. Hence $(\text{hocolim } \pi)^{-1}(U)$ lies in $V \times |K(X)|$. Let $p$ be the composition (hocolim $\pi)^{-1}(U) \subset V \times |K(X)| \to V$ of the inclusion and the projection, and let $\sigma = p^* \gamma \in H^n((\text{hocolim } \pi)^{-1}(U))$. For each $C = \sum_{i=1}^n \lambda_i C_i \subset U$, the composition $C_1 = [\pi|^{-1}(C) \subset (\text{hocolim } \pi)^{-1}(U) \overset{p}{\to} V \text { coincides with the inclusion map } C_1 \to V$. Hence $\sigma|_{(\text{hocolim } \pi)^{-1}(C)} = \gamma|_{C_1}$. On the other hand, by Spanier’s tautness theorem $\sigma|_{(\text{hocolim } \pi)^{-1}(C)}$ coincides with the image of $\sigma \in H^n((\text{hocolim } \pi)^{-1}(U))$ in $\mathcal{H}_C = \text{colim}_O H^n((\text{hocolim } \pi)^{-1}(O))$, where $O$ runs over all open neighborhoods of $C$ in $|K(X)|$. Since $\mathcal{H}$ is the sheafification of $O \mapsto H^n((\text{hocolim } \pi)^{-1}(O))$ and $\sigma \in H^n((\text{hocolim } \pi)^{-1}(U))$, there is a section $s$ of $\mathcal{H}$ over $U$ given by $C \mapsto \gamma|_{C_1} \in \mathcal{F}_C$. In particular, $s(A) = \gamma|_{A_1} = \alpha$, as desired. \hfill $\Box$

5.D. Spectral sequences.

**Theorem 5.9.** Let $X$ be a metrizable space and let $K(X)$ be the pospace of its nonempty compact subsets, topologized by the Hausdorff metric and ordered by reverse inclusion. Then there is a spectral sequence of the form

$$\text{Lim}_K H^q(K_\alpha) \Rightarrow H^{p+q}(X).$$
In more detail, the theorem asserts that for each \( q \geq 0 \) there is a posheaf \( F^q \) over \( K(X) \) such that the stalk \( F^q_{K_\alpha} \simeq H^q(K_\alpha) \) for each \( K_\alpha \in K(X) \), and there is a spectral sequence of the form \( \text{Lim}_p F^q = E^2_{p,q} = H^{p+q}(X) \).

**Proof.** Let \( E(X) \) be the subspace of \( X \times K(X) \) consisting of all pairs \((x, K)\) such that \( x \in K \). Let \( \pi \) be the composition of the inclusion \( E(X) \subset X \times K(X) \) and the projection \( X \times K(X) \to K(X) \).

Then we get a continuous map \( \text{holim} \pi : |E(X)| \to |K(X)| \) (see §5.1A). The Leray spectral sequence of this map runs \( H^p(|K(X)|; \mathcal{H}^q(\text{holim} \pi)) \Rightarrow H^{p+q}(|E(X)|) \) [8].

By Theorem 3.4, \( |E(X)| \) is homotopy equivalent to \( X \).

By Theorem 5.8(b), the Leray sheaf \( \mathcal{H}^q(\text{holim} \pi) \simeq \text{holim} \mathcal{H}^q(\pi) \). By definition, \( \text{Lim}^p \mathcal{H}^q(\pi) = H^p(|K(X)|; \text{holim} \mathcal{H}^q(\pi)) \).

By Theorem 5.4(a), the stalk \( \mathcal{H}^q(\pi)_{K_\alpha} \simeq H^n(K_\alpha) \) for each compact \( K_\alpha \subset X \).

Every finite-dimensional separable metrizable space embeds in some sphere \( S^n \). For subsets of \( S^n \) it is easy to rewrite the previous spectral sequence in terms of homology:

**Theorem 5.10.** Let \( X \) be a subset of \( S^n \) and let \( U(X) \) be the pospace of its open neighborhoods \( \neq S^n \), ordered by inclusion and topologized by the Hausdorff metric. Then there is a spectral sequence of the form

\[
\text{Lim}_p^U H_q(U_\alpha) \Rightarrow H_{q-p}(X).
\]

In more detail, the theorem asserts that for each \( q \geq 0 \) there is a posheaf \( F^q \) over \( K(X) \) such that the stalk \( (F^q)_P \simeq H_q(P_\beta) \) for each \( P_\beta \in U(X) \), and there is a second quadrant homology spectral sequence of the form \( \text{Lim}_p F^q = E^2_{p,q} = H_{q-p}(X) \) (where \( p, q \geq 0 \)).

**Proof.** Clearly, \( U(X) \) is homeomorphic to \( K(S^n \setminus X) \). By the Sitnikov duality (see [29; Theorem 27.1]) we have \( H_i(X) \simeq H^{n-i-1}(S^n \setminus X) \) and \( H_i(U) \simeq H^{n-i-1}(S^n \setminus U) \) for every open neighborhood of \( X \). So the assertion follows from Theorem 5.9. \( \square \)

### 6. Discussion

The results of the present paper suggest that it may be worthwhile to develop an entire theory of derived limits over posets. It must be admitted that the present state of this theory is absolutely unsatisfactory, if not to say downright pathetic.

The following obvious problems still wait to be addressed:

1. Compute \( \text{Lim}_p \) for a few basic examples.
2. How does \( \text{Lim}_p \) behave with respect to cofinal subsets?
3. When does \( \text{Lim}^0 \) coincide with the usual (discretely indexed) inverse limit \( \text{lim} \)?

It is not hard to show that they do coincide in our model setting (that is, for the Leray sheaves \( \mathcal{H}^q(\pi) \) of the map \( \pi : E(X) \to K(X) \)) as long as \( X \) is locally compact. Which is hardly surprising, but even this is not entirely obvious in the absence of answers to the previous question.
On the other hand, there seems to be no reason to expect that $\lim^0 = \lim$ in full generality. Or if they do always coincide, that would actually be pretty bad! If $G_\alpha$ is the Mardešić–Prasolov inverse system (whose $\lim^1$ cannot be computed in ZFC) and $S_\alpha = \text{Hom} \left( \text{Hom}(G_\alpha, S^1), S^1 \right)$, we get a short exact sequence of inverse systems $0 \to G_\alpha \to S_\alpha \to Q_\alpha \to 0$, where $\lim Q_\alpha$ cannot be computed in ZFC.

(4) When does $\text{Lim}^p = \lim^p$?
(5) Obtain vanishing results for $\text{Lim}^p$.
(6) What can be said of $\text{Lim}^p$ over zero-dimensional pospaces?
(7) Understand $\text{Lim}^p$ as derived functors. Deduce a long exact sequence and a formula using an explicit resolution.
(8) Compute $\text{Lim}^p$ for the telescopic chain complexes of [31].
(9) Apply $\text{Lim}^p$ to prove something interesting. In particular, can one establish a version of [32; Theorem 4.9] not involving any assumptions independent from ZFC?

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Disclaimer. I oppose all wars, including those wars that are initiated by governments at the time when they directly or indirectly support my research. The latter type of wars include all wars waged by the Russian state in the last 25 years (in Chechnya, Georgia, Syria and Ukraine) as well as the USA-led invasions of Afghanistan and Iraq.

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Steklov Mathematical Institute of Russian Academy of Sciences, ul. Gubkina 8, Moscow, 119991 Russia

Email address: melikhov@mi-ras.ru