Subhojoy Gupta · Weixu Su

**Dominating surface-group representations into $\text{PSL}_2(\mathbb{C})$ in the relative representation variety**

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**Abstract.** Let $\rho$ be a representation of the fundamental group of a punctured surface into $\text{PSL}_2(\mathbb{C})$ that is not Fuchsian. We prove that there exists a Fuchsian representation that strictly dominates $\rho$ in the simple length spectrum, and preserves the boundary lengths. This extends a result of Gueritaud-Kassel-Wolff to the case of $\text{PSL}_2(\mathbb{C})$-representations. Our proof involves straightening the pleated plane in $\mathbb{H}^3$ determined by the Fock-Goncharov coordinates of a framed representation, and applying strip-deformations.

1. Introduction

Let $S_{g,k}$ be an oriented surface of genus $g \geq 0$ and $k \geq 1$ labelled punctures $p_1, p_2, \ldots, p_k$, with negative Euler characteristic; let $\Pi$ denote its fundamental group.

Given a representation $\rho : \Pi \to \text{PSL}_2(\mathbb{C})$, we define the $\rho$-length of a closed curve $\gamma \in \Pi$ to be the translation length of $\rho(\gamma)$, that is, $l_\rho(\gamma) = \inf_{x \in \mathbb{H}^3} d_{\mathbb{H}^3}(\rho(\gamma) \cdot x, x)$. This is determined by the trace of $\rho(\gamma)$, since $\text{tr}^2(\rho(\gamma)) = 4 \cosh^2(l_\rho(\gamma)/2)$. Note that the translation length is positive if $\rho(\gamma)$ is loxodromic and zero if $\rho(\gamma)$ is parabolic or elliptic. Moreover, if $\rho$ is Fuchsian, that is, can be conjugated to a discrete and faithful representation into $\text{PSL}_2(\mathbb{R})$, then $l_\rho(\gamma)$ coincides with the length of the geodesic representative of $\gamma$ on the hyperbolic surface $\mathbb{H}^2//\Gamma$, where $\Gamma$ is the image of the conjugated representation.

For a fixed $k$-tuple $L = (l_1, l_2, \ldots, l_k) \in \mathbb{R}^k_{\geq 0}$, the relative representation variety for the surface-group $\Pi$ is the space of representations $\text{Hom}(\Pi, L) = \{ \rho : \Pi \to \text{PSL}_2(\mathbb{C}) \mid l_\rho(\gamma_i) = l_i \text{ where } \gamma_i \text{ is the loop around } p_i \}$ where note that each boundary length is fixed, and prescribed by $L$.

A Fuchsian representation $j \in \text{Hom}(\Pi, L)$ is said to dominate a representation $\rho \in \text{Hom}(\Pi, L)$ if

$$\sup_{\gamma} \frac{l_\rho(\gamma)}{l_j(\gamma)} \leq 1$$

(1)

S. Gupta (✉): Department of Mathematics, Indian Institute of Science, Bangalore, India
e-mail: subhojoy@iisc.ac.in

W. Su: School of Mathematics, Sun Yat-sen University, Guangzhou, China
e-mail: suwx9@mail.sysu.edu.cn

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where $\gamma$ varies over all non-peripheral essential simple closed curves on $S_{g,k}$. Moreover, $j$ is said to strictly dominate $\rho$ if the inequality in (1) is strict. Note that these can also described as domination and strict domination in the simple length spectrum – 2 for a discussion about alternative definitions. It follows from the work in [18] that a Fuchsian representation cannot have a strictly dominating Fuchsian representation in the same relative representation variety.

In this note we prove:

**Theorem 1.1.** Let $\mathcal{L} = (l_1, l_2, \ldots, l_k) \in \mathbb{R}^k_{\geq 0}$. For any $\rho \in \text{Hom}(\Pi, \mathcal{L})$ that is not Fuchsian, there exists a Fuchsian representation $j \in \text{Hom}(\Pi, \mathcal{L})$ that strictly dominates $\rho$.

Theorem 1.1 is inspired by the work of Gueritaud-Kassel-Wolff in [8] who used “folded” hyperbolic surfaces to prove a strict domination result in the complete length spectrum, for closed surface-group representations into $\text{PSL}_2(\mathbb{R})$. Around the same time, Deroin-Tholozan ([5]) proved a more general domination result, for representations of a closed surface-group into the isometry group of smooth Riemannian $\text{CAT}(-1)$ spaces, using the theory of harmonic maps. Their result was extended to isometry groups of $\text{CAT}(-1)$ metric spaces in [4], and more recently, to the case of punctured surfaces by Sagman in [16]. In particular, Theorem 1.1 is implicit in [16]; indeed, we rely on his work in the case when $\rho$ is a degenerate and co-axial representation (see Definition 2.1). In the non-degenerate case, however, our proof of Theorem 1.1 avoids harmonic maps, and relies instead on the pleated-surface interpretation of the Fock-Goncharov coordinates of a framed representation into $\text{PSL}_2(\mathbb{C})$, as exploited in [9] and [10].

Our result also improves on Theorem 1.2 of [19]. Indeed, it is known that there is a Bers’ constant $B(g, k, L) > 0$ for any hyperbolic surface with geodesic boundary having lengths above bounded by $L$ (see [13]). It follows directly from Theorem 1.1 that

**Corollary 1.1.** Let $\rho \in \text{Hom}(\Pi, \mathcal{L})$ be a representation such that the length of each peripheral curve is bounded above by $L$. Then there exists a pants decomposition of $S_{g,k}$ such that the $\rho$-lengths of the pants curves are at most $B(g, k, L)$.

We expect that Theorem 1.1 and Corollary 1.1 would be helpful in the study of the relative representation variety. Moreover, we hope that our techniques can be extended to proving analogous results for representations of punctured surface-groups into other complex Lie groups $G$, like $\text{PSL}_n(\mathbb{C})$ for $n > 2$, since we do have Fock-Goncharov coordinates for such representations (see [6]). For any such representation, one can define domination exactly as in (1), where the $\rho$-length of $\gamma$ is defined as the translation-length of $\rho(\gamma)$ in the symmetric space $\text{PSL}_n(\mathbb{C})/\text{PSU}_n$ equipped with a suitable invariant metric. From the work in [6], when the Fock-Goncharov coordinates are real and positive, they determine a Hitchin representation with image in $\text{PSL}_2(\mathbb{C})$. We conjecture:

**Conjecture 1.2.** A generic representation $\rho : \Pi \rightarrow \text{PSL}_n(\mathbb{C})$ has a strictly dominating Hitchin representation $j : \Pi \rightarrow \text{PSL}_n(\mathbb{R})$, in the sense of (1), in the same relative representation variety.
See [3] for a closely related “Metric Domination Conjecture” in the case of a closed surface. There are also some domination conjectures by Lee-Zhang and Tholozan (see [11] and [17]) for Hitchin representations to certain real Lie groups; in a future article we hope to investigate their analogues in the context of punctured-surface-groups and relative representation varieties.

2. Preliminaries

2.1. Framed and non-degenerate representations

Fix a choice of a finite-area hyperbolic metric on $S = S_{g,k}$, such that the punctures are cusps. Passing to the universal cover $\tilde{S} \cong \mathbb{H}^2$, let the Farey set $F_\infty$ be the points in the ideal boundary that are the lifts of the punctures. Note that $F_\infty$ is equipped with an action of the surface-group $\pi_1 = \pi_1(S)$.

In this paper, a framed representation $\hat{\rho}$ is a pair $(\rho, \beta)$ where $\rho \in \text{Hom}(\pi_1, \text{PSL}_2(\mathbb{C}))$ and $\beta : F_\infty \to \mathbb{C}P^1$ is a $\rho$-equivariant map. (Throughout this article, subgroups of $\text{PSL}_2(\mathbb{C})$ are assumed to act on $\mathbb{C}P^1$ by Möbius transformations, and on hyperbolic 3-space $\mathbb{H}^3$ by isometries.) The $\rho$-equivariance implies that if one fixes a fundamental domain $F$ of the action of $\pi_1$ on the universal cover $\tilde{S}$, one obtains for each puncture a choice of a fixed point on $\mathbb{C}P^1$ of the corresponding boundary monodromy.

For the notion of a non-degenerate framed representation, see Definition 2.6 of [10] or Definition 4.3 of [1]. The relevant fact for us is that given a non-degenerate framed representation $\hat{\rho}$, the representation $\rho$ obtained by forgetting the framing is a non-degenerate representation, defined as follows (see Definition 2.4 of [10]):

**Definition 2.1.** A representation $\rho : \pi_1 \to \text{PSL}_2(\mathbb{C})$ is said to be degenerate if either

(a) The image of $\rho$ has a global fixed point on $\mathbb{C}P^1$, and $\rho(\gamma_i)$ is parabolic or identity for each peripheral loop $\gamma_i$, or
(b) The image of $\rho$ preserves a two-point set on $\mathbb{C}P^1$, which is fixed by each $\rho(\gamma_i)$ (where $1 \leq i \leq k$). In this case $\rho$ is said to be co-axial since its image would preserve a geodesic line in $\mathbb{H}^3$.

A representation is then said to be non-degenerate if it is not degenerate.

Note that it follows from this definition that a non-elementary representation is automatically non-degenerate; however there are elementary representations that are non-degenerate – see 2.4 of [10]. Conversely, given a non-degenerate representation $\rho$, one can construct a non-degenerate framed representation $\hat{\rho} = (\rho, \beta)$ where $\beta$ is the framing that assigns to each puncture one of the fixed points of the holonomy/monodromy around it – see Proposition 3.1 of [10].
2.2. Fock-Goncharov coordinates

An ideal triangulation $T$ on $S$ is a collection of geodesic arcs between cusps (the edges), each homotopically non-trivial and non-peripheral, such that the complementary regions are ideal triangles. Given a choice of an ideal triangulation $T$, and a generic framed representation $\hat{\rho}$, one can define non-zero complex numbers to each edge $e \in T$ as follows: consider a lift $\tilde{e}$, and consider the two ideal triangles $\Delta_+^1$ and $\Delta_-^1$ in the lifted ideal triangulation that share the side $\tilde{e}$. The ideal vertices of $\Delta_{\pm}$ determine four points in $F_\infty$, and the image of these under $\beta$ will determine four distinct points in $CP^1$ (here we use the genericity assumption). The Fock-Goncharov coordinate $c(e) \in \mathbb{C}^*$ is then the cross-ratio of these four points.

Let $\hat{\chi}(S)$ be the moduli space of framed representations, namely, the set of framed representations up to the equivalence relation $(\rho, \beta) \sim (A\rho A^{-1}, A \cdot \beta)$ for any $A \in PSL_2(\mathbb{C})$. Then Fock-Goncharov showed that the assignment $[\hat{\rho}] \mapsto \{c(e)\}_{e \in T}$ defines a birational isomorphism $\Phi_T : \hat{\chi}(S) \to (\mathbb{C}^*)^{|T|}$ (see Theorem 1 of [6]).

We shall use the following result that is a consequence of Proposition 3.1 of [10] and Theorem 9.1 of [1]:

**Theorem 2.2.** For a non-degenerate representation $\rho$, there is a choice of a framing $\beta$ and an ideal triangulation $T$ such that the Fock-Goncharov coordinates for the framed representation $\hat{\rho} = (\rho, \beta)$ are well-defined and non-zero.

**Proof.** By Proposition 3.1 of [10], there is a framing $\beta$ such that $\hat{\rho}$ is a non-degenerate framed representation, and then we apply Theorem 9.1 of [1]. Note that in their statement, they assert the existence of a “signed” triangulation $T$, which records the additional data of a sign ($\pm$) at each puncture, which indicates the choice of framing at any puncture with loxodromic monodromy $m$, namely, which of the two fixed-points of $m$. However, we can “correct” our initial choice of framing $\beta$ by swapping these choices, as dictated by these signs. Thus, in our statement we can avoid this additional notion of “signed” triangulations. $\Box$

2.3. Pleated planes in $\mathbb{H}^3$

A pleated plane in $\mathbb{H}^3$ is a map

$$\Psi : \tilde{S} \to \mathbb{H}^3$$

that is totally geodesic on each lift of an ideal triangle on $S$ determined by an ideal triangulation $T$. If $\tilde{e}$ is an edge of the lifted ideal triangulation $\tilde{T}$ on $\tilde{S}$ with two adjacent ideal triangles $\Delta_l$ and $\Delta_r$, then its image is a pleating line of $\Psi$ if $\Psi(\Delta_l \cup \Delta_r)$ is not totally geodesic, i.e. is “bent” at $\Psi(\tilde{e})$. The collection of all pleating lines comprise the pleating locus. Note that the ideal vertices of the ideal triangles in $\tilde{T}$ are precisely the points in the Farey set $F_\infty$.

Given a framed representation $\hat{\rho} = (\rho, \beta)$, one can build a $\rho$-equivariant pleated plane $\Psi$ as follows: send each ideal triangle $\Delta$ of $\tilde{T}$ with ideal vertices $a, b, c$ to the totally-geodesic ideal triangle in $\mathbb{H}^3$ with ideal vertices $\beta(a), \beta(b), \beta(c)$.
If \( \hat{\rho} \) is in addition, non-degenerate, then for any line of the pleating locus (i.e. \( \Psi(\hat{e}) \) for \( \hat{e} \in \hat{T} \)) the two adjacent totally-geodesic ideal triangles \( \Psi(\Delta_{\pm}) \) determine four distinct ideal vertices in \( \mathbb{C}P^1 \). Then the argument of the Fock-Goncharov coordinate \( c(\hat{e}) \in \mathbb{C}^* \) equals the angle between the two totally-geodesic planes containing the two ideal triangles \( \Psi(\Delta_{-}) \) and \( \Psi(\Delta_{+}) \) respectively. The real number \( \ln|c(\hat{e})| \), on the other hand, is the shear-parameter measuring the (signed) distance between points on the common side of \( \Delta_{\pm} \) that are the feet of the perpendiculars from the opposite ideal vertices.

This provides a geometric interpretation of the Fock-Goncharov coordinates.

Note that in the case that all the Fock-Goncharov coordinates are real and positive, the representation \( \rho \) is Fuchsian, and these are the classical shear-coordinates of Teichmüller space (see Theorem 1.7 and section 11 of [6]).

### 2.4. Alternative definitions of domination

We note that our definition of \( j \) dominating \( \rho \) (see (1)) is equivalent to the following two definitions:

(A) The inequality in (1) is true when the supremum is taken over all closed curves in \( \Pi \),

(B) There is a 1-Lipschitz map \( f : \mathbb{H}^2 \rightarrow \mathbb{H}^3 \) that is \( (j, \rho) \)-equivariant, that is, satisfies \( f \circ j(\gamma) = \rho(\gamma) \circ f \) for all \( \gamma \in \Pi \).

Note that (A) can be thought of as domination in the complete length spectrum.

In fact, denote by

\[
K = \sup_{\gamma} \frac{l_{\rho}(\gamma)}{l_j(\gamma)},
\]

where \( \gamma \) varies over all non-peripheral essential simple closed curves (as in (1)), and let \( K' \) denote the supremum where \( \gamma \) varies over all essential closed curves as in (A) above. Clearly, \( K \leq K' \). Then if \( L \) is the minimal Lipschitz constant, over all maps \( f : \mathbb{H}^2 \rightarrow \mathbb{H}^3 \) that are \( (j, \rho) \)-equivariant, as in (B) above, then \( K' \leq L \) (see Lemma 4.5 of [7], or the discussion following Theorem 1.8 in their Introduction).

Thus, \( K \leq L \) and hence \( L < 1 \) implies \( K < 1 \).

If \( L \geq 1 \), then by Theorem 1.3 of [7], there exists a \( (j, \rho) \)-equivariant \( L \)-Lipschitz map \( f \) and a geodesic lamination \( \lambda \) on \( \mathbb{H}^2/j(\Pi) \) that is maximally stretched, i.e., the restriction of \( f \) on \( \lambda \) realizes the Lipschitz constant \( L \). Moreover, the maximal stretch lamination \( \lambda \) is compact and contained in the convex core of \( \mathbb{H}^2/j(\Pi) \).

Suppose that \( L > 1 \), then no geodesic boundary component is in \( \lambda \) since we require their lengths to remain the same. Thus, \( \lambda \) is contained in the interior of the convex core of \( \mathbb{H}^2/j(\Pi) \) and it can be approximated by a sequence of non-peripheral essential simple closed geodesics. This implies that \( K > 1 \). We have shown that (1), (A) and (B) are equivalent.

In the case of a closed surface, when \( \mathbb{H}^2/j(\Pi) \) is a hyperbolic surface without geodesic boundary, if \( L \geq 1 \), we in fact have \( L = K \) by the work in [7] (see
Theorem 1.3, Lemma 4.6 and the proof of Lemma 5.9). Thus, in the closed case, our definition of strict domination, namely, a strict inequality in (1) where the supremum is taken over simple closed curves, is equivalent to the following:

(A’) The inequality in (1) is strict when the supremum is taken over all closed curves in \( \Pi \),

(B’) There is an \( L < 1 \), and an \( L \)-Lipschitz map \( f : \mathbb{H}^2 \to \mathbb{H}^3 \) that is \((j, \rho)\)-equivariant.

However, these alternative definitions of \( j \) strictly dominating \( \rho \) are not all equivalent in the case that \( \mathbb{H}^2/j(\Pi) \) is a hyperbolic surface with at least one geodesic boundary. As mentioned before, we always have \( K \leq K' \leq L \). However, since the representation \( \rho \) doesn’t change the length of the boundary, we always have \( L \geq 1 \). Thus (B’) does not make sense. In fact, if \( K < 1 \), then \( L = 1 \) (otherwise, if \( L > 1 \), then as we have seen above, one can show \( K > 1 \) by approximating the maximal stretch lamination \( \lambda \) by non-peripheral simple closed curves). Note that \( K < 1 \) does not even imply (A’) above, since there could be long non-simple curves most of whose length is near the boundary.

3. Proof of Theorem 1.1

Let \( L = \{l_1, l_2, \ldots, l_k\} \) and suppose \( \rho \in \text{Hom}(\Pi, L) \) is a non-Fuchsian representation. In 3.1 we shall assume that \( \rho \) is non-degenerate; the degenerate case is handled separately in 3.2.

3.1. Case that \( \rho \) is non-degenerate

In this case, by Proposition 3.1 of [10], we know that there exists a framing \( \beta : F_\infty \to \mathbb{C}P^1 \) such that \( \hat{\rho} = (\rho, \beta) \) is a non-degenerate framed representation. We choose an ideal triangulation \( T \) by Theorem 2.2, which determines Fock-Goncharov coordinates \( \{c(e)\}_{e \in T} \in (\mathbb{C}^*)^{\#T} \).

As described in 2, these coordinates determine a \( \rho \)-equivariant pleated plane

\[
\Psi : \hat{S} \to \mathbb{H}^3.
\]

By Proposition 3.3 of [10], the straightening of this pleated plane results in a map \( \overline{\Psi} : \hat{S} \to \mathbb{H}^3 \) whose image lies in the equatorial plane, that is a totally geodesic copy of \( \mathbb{H}^2 \). This is achieved by simply replacing each Fock-Goncharov coordinate by its modulus – see Definition 3.1 of [10]. Moreover, \( \overline{\Psi} \) is the developing map of a hyperbolic surface \( \hat{S} \) homeomorphic to \( S_{g,k} \), with geodesic boundaries or cusps at the punctures. (The cusps arise exactly at the punctures corresponding to zero \( \rho \)-length – see the following lemma). Let \( j_0 : \Pi \to \text{PSL}_2(\mathbb{R}) \) denote the Fuchsian representation corresponding to \( \hat{S} \); the straightened pleated plane \( \overline{\Psi} \) is then \( j_0 \)-equivariant.

The pleating lines for \( \Psi \) determine a measured lamination \( \lambda \) on \( \hat{S} \), comprising a collection \( C \) of disjoint geodesics with weights in \((0, 2\pi)\), namely the bending
angles, such that each geodesic boundary component of $\hat{S}$ has at least one leaf of $\lambda$ spiralling onto it.

Our first observation is:

**Lemma 3.1.** The $j_0$-length of the boundary curve around the $i$-th puncture $p_i$ is equal to $l_i$, for $1 \leq i \leq k$. That is, $j_0 \in \text{Hom}(\Pi, \mathcal{L})$ as well.

**Proof.** Let $s_1, s_2, \ldots, s_k$ be the Fock-Goncharov coordinates for $\hat{\rho}$, associated to the edges of $T$ incident on the $i$-th puncture $p_i$, and let $l_i = \sum_j \ln|s_j|$ be their sum. Then by Lemma 3.2 of [10], we know that the monodromy around the puncture $p_i$ is

(a) Loxodromic if $l_i \neq 0$
(b) Parabolic or identity if $l_i = 0$ and $\sum_j \text{Arg}(s_j) \in 2\pi\mathbb{Z}$, and
(c) Elliptic if $l_i = 0$ but $\sum_j \text{Arg}(s_j) \notin 2\pi\mathbb{Z}$.

Moreover, in each case, the translation length of the monodromy element (i.e. the $\rho$-length of the loop around $p_i$) is precisely $l_i$. By Corollary 3.4 in [10], we then see that for the straightened surface, the monodromy around $p_i$ also has translation length $l_i$. $\Box$

The following geometric lemma shall be used to quantify how the translation length of a non-peripheral loop changes when we straighten:

**Lemma 3.2.** For any $L > 0$, $\alpha \in (0, \pi/2)$ and $\theta \in (-\pi, \pi)$, there is a constant $C > 0$ such that the following holds:

Let $\mathbb{H}^2$ be isometrically embedded as the equatorial plane in $\mathbb{H}^3$, containing a geodesic segment $\ell$ and a bi-infinite geodesic line $\gamma$, such that the two intersect at an angle at least $\alpha$, and $\ell$ has length at least $L$ on either side of $\gamma$. Let $\hat{\ell}$ be the piecewise-geodesic in $\mathbb{H}^3$ obtained when the equatorial plane is pleated along $\gamma$ by a pleating angle at least $\theta$. Then the distance in $\mathbb{H}^3$ between the endpoints of $\hat{\ell}$ is less than $|\ell| - C$.

**Proof.** We denote by $x, y$ be the two endpoints of $\hat{\ell}$, and by $O$ be the intersection point of $\ell$ with $\gamma$. Consider the geodesic triangle with vertices $x, O, y$. The angle at $O$ is some number $\beta > 0$, and it is easy to see that it depends only on the pleating angle, and the intersection angle of $\ell$ with $\gamma$. Let $\beta$ be the pleating angle at least $\theta$.

Claim. There is a constant $\delta = \delta(\alpha, \theta) > 0$ such that $1 + \cos \beta > \delta$.

Proof of claim. It is enough to consider the extreme case that the intersection angle of $\ell$ with $\gamma$ is equal to $\alpha$, and the pleating angle is equal to $\theta$.

Assume that we are working in the unit ball model of $\mathbb{H}^3$, where $O$ is the center of the ball, $\gamma$ is a diameter with endpoints $(1, 0, 0)$ and $(-1, 0, 0)$, and $\ell$ is the entire diameter with endpoints $(\cos \alpha, -\sin \alpha, 0)$ and $(-\cos \alpha, \sin \alpha, 0)$. Then the computation of $\beta$ becomes an elementary Euclidean geometry problem: after we bend the equatorial plane on one side of $\gamma$ by an angle $\theta$ (as shown in Fig. 1), then $\ell$ deforms to $\hat{\ell}$, which is the concatenation of two radial rays from $O$ with endpoints...
Fig. 1. The distance between the endpoints of the geodesic segment \( \ell \) on the equatorial plane decreases by a definite amount, when bent along the geodesic line \( \gamma \) (see Lemma 3.2)

\[(\cos \alpha, -\sin \alpha, 0) \text{ and } (-\cos \alpha, \sin \alpha \cos \theta, -\sin \theta)\] respectively. Since the angle between them at \( O \) equals \( \beta \), we can compute

\[
\cos \beta = (\cos \alpha, -\sin \alpha, 0) \cdot (-\cos \alpha, \sin \alpha \cos \theta, -\sin \theta) = -\cos^2(\alpha) - \sin^2(\alpha) \cos \theta
\]

from which one can deduce that

\[
1 + \cos \beta = 2 \sin^2(\alpha) \sin^2(\frac{\theta}{2})
\]

proving the claim. Note that \( \delta \) depends only on the angles \( \alpha \) and \( \theta \). \( \square \)

Using the law of cosines in hyperbolic trigonometry (see, for example, Chapter 8 of [12]) we have:

\[
\cosh |xy| = \cosh |Ox| \cosh |Oy| - \sinh |Ox| \sinh |Oy| \cos \beta.
\]

Since \( |\ell| = |Ox| + |Oy| \), we obtain

\[
\cosh |\ell| - \cosh |xy| = \sinh |Ox| \sinh |Oy|(1 + \cos \beta) > \sinh |Ox| \sinh |Oy| \delta
\]

Since \( |Ox|, |Oy| \geq L \), and both \( \sinh \) and \( \cosh \) are increasing functions on the positive reals, we are done. \( \square \)

Finally, we shall need the following fact (see Lemma 2.3 of [21]):
Lemma 3.3. (Generalized Collar Lemma) Given a hyperbolic surface $\hat{X}$ of finite type, with finitely many geodesic boundaries and cusps, there exists a $D > 0$ such that any non-peripheral simple closed geodesic $\gamma$ on $\hat{X}$ remains at least distance $D$ away from the geodesic boundary components, and standard horoball-neighborhoods of the cusps.

As a consequence of the distance-decreasing property in Lemma 3.2, we obtain:

Proposition 3.1. The Fuchsian representation $j_0 : \Pi \to \text{PSL}_2(\mathbb{R})$ dominates the representation $\rho$. Moreover, for any simple closed curve $\gamma \in \Pi$ that intersects $\lambda$ on $\hat{S}$, the $j_0$-length of $\gamma$ is strictly greater than its $\rho$-length, such that

$$\sup_{\gamma} \frac{l_\rho(\gamma)}{l_{j_0}(\gamma)} < 1$$

when $\gamma$ varies over all simple closed curves on $S_{g,k}$ that intersect $\lambda$.

Proof. Let $\gamma$ be any simple closed geodesic on $\hat{S}$.

If the developing image of $\tilde{\gamma}$ in the equatorial plane in $\mathbb{H}^3$ does not intersect a pleating line (i.e. a leaf of $\lambda$), then it is not affected by the pleating, and hence the $\rho$-length will be the same as the $j_0$-length.

Else, we can decompose $\gamma$ into a finite union of geodesic arcs $\{\gamma_j\}_{j=1}^N$, such that each $\gamma_j$ has endpoints on $\lambda$, and has its interior disjoint from $\lambda$. Since the ends of leaves of $\lambda$ spiral to the $\partial \hat{S}$ or exit out of cusps, and $\gamma$ is simple, by Lemma 3.3, $\gamma$ does not cross some collar neighborhood of $\partial \hat{S}$ and a horodisk-neighborhoods around the cusps.

Extend $\lambda$ to a maximal ideal triangulation of $\hat{S}$. By the observation above, each $\gamma_j$ does not lie near the cusps of the ideal triangles. Moreover, the intersection of $\gamma$ with each leaf of $\lambda$ (a geodesic side of an ideal triangle) cannot be at an angle close to zero: if it is, $\gamma$ will remain close to the geodesic side of the ideal triangle for a large length, which would force it to lie near the cusp of that ideal triangle, contradicting Lemma 3.3. This implies that any intersection angle is uniformly bounded away from zero, where the uniform bound depends on the constant $D$ introduced in Lemma 3.3.

As a result, the union $\gamma_j \cup \gamma_j'$ of two successive segments satisfies the hypotheses of Lemma 3.2 for some $L, \alpha$ and $\theta$ (which are all independent of the choice of $\gamma$). Note that the length of any $\gamma_j$ is uniformly comparable to $L$, in other words $|\gamma_j| = O(L)$.

The length of $\gamma$ on $\hat{S}$ is equal to

$$l_{j_0}(\gamma) = \sum_j |\gamma_j|.$$

By Lemma 3.2, there exists a $C > 0$ such that

$$l_\rho(\gamma) < \sum_j |\gamma_j| - NC.$$
Thus
\[
\frac{l_\rho(\gamma)}{l_{j_0}(\gamma)} \leq \frac{\sum_j^N |\gamma_j| - NC}{\sum_j^N |\gamma_j|} < 1 - \frac{C}{O(L)}
\]
proving the second statement. 

**Definition 3.4.** *(Filling arcs)* A collection of pairwise-disjoint arcs on \(S_{g,k}\) with endpoints at the punctures, such that each arc is homotopically non-trivial and non-peripheral is *filling* if each complementary component is simply-connected. This happens, for instance, if the collection of arcs determines a maximal ideal triangulation.

In what follows, we shall apply the above definition to the collection of arcs that are the leaves of the geodesic lamination \(\lambda\), whose lifts to the universal cover to \(\tilde{S}\) are the pre-images of the pleating lines for the pleated plane \(\Psi\) (see (2)).

As a consequence of the proof of Proposition 3.1, we then have:

**Corollary 3.2.** If \(\lambda\) is filling, then \(j_0\) strictly dominates \(\rho\).

Thus, in the case when \(\lambda\) is filling, and \(\rho\) is non-degenerate and non-Fuchsian then \(j_0\) is the desired strictly-dominating Fuchsian representation \(j\) in Theorem 1.1.

**Non-filling case**

We shall now deal with the case that \(\lambda\) is *not* filling (we continue with our assumption that \(\rho\) is non-degenerate and non-Fuchsian). Note that the assumption that \(\rho\) is non-Fuchsian implies that the geodesic lamination \(\lambda \neq \emptyset\). From the proof of Lemma 3.1 (see also Corollary 3.4 of [10]), the straightened hyperbolic surface \(\hat{S}\) has a geodesic boundary component for each puncture whose corresponding entry in the tuple \(L\) is positive. Moreover, each puncture that had zero boundary length corresponds to a (finite-volume) cusp in \(\hat{S}\).

By Proposition 3.1, the holonomy \(j_0\) of the hyperbolic surface \(\hat{S}\) dominates \(\rho\); however the \(\rho\)-length and \(j_0\)-length of any simple closed curve that does not intersect \(\lambda\), are equal. In what follows we shall modify \(j_0\) to a strictly dominating representation \(j\). This shall use the operation of (positive) *strip-deformations*, that we shall define below. This had been used in the proof of Lemma 3.4 of [18] — see Definition 1.3 of [2], and the proof of Lemma 4.1 of [8].

In what follows, let \(\Sigma\) be a hyperbolic surface with at least one geodesic boundary component (and possibly some cusps), and let \(\Sigma_c\) be its completion obtained by attaching a funnel to each geodesic boundary component.

**Definition 3.5.** *(Strip-deformation)* Let \(\ell\) be a bi-infinite geodesic line on the complete hyperbolic surface \(\Sigma_c\) such that both ends of \(\ell\) exit out of funnel ends of \(\Sigma_c\) (this could be the same funnel as well). A *(positive)* *strip-deformation with parameter* \(\alpha > 0\) is the operation of cutting along such an infinite geodesic line, say \(\ell\) and...
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Fig. 2. A strip-deformation inserts a hyperbolic strip along a geodesic line embedded in the completion of $\Sigma$

inserting a hyperbolic “strip” of width $\alpha$, which is isometric to the hyperbolic region bounded by two disjoint geodesics at a minimum distance $\alpha > 0$. (See Fig. 2.) The strip is glued without any shear: we first specify a single point of $\ell$, which becomes a pair of points, say $p_-, p_+$, when one cuts along $\ell$. Then the strip is inserted so that $p_\pm$ are identified with the endpoints of the arc of length $\alpha$ which minimizes the distance between the geodesic sides of the strip.

Using positive strip-deformations, one can obtain the following result (see Lemma 4.1 of [8], and [14] for a weaker result):

**Proposition 3.3.** Given a hyperbolic surface $\Sigma$ with non-empty geodesic boundary, there exists a hyperbolic surface $\Sigma'$ homeomorphic to $\Sigma$, such that

$$\sup_{\gamma} \frac{l_\Sigma(\gamma)}{l_{\Sigma'}(\gamma)} < 1$$

where $l_X(\gamma)$ denotes the hyperbolic length of the (geodesic representative of) the curve $\gamma$ on the hyperbolic surface $X$, and $\gamma$ varies over all simple closed curves, including the boundary components.

**Sketch of the proof.** As before, extend $\Sigma$ to a complete hyperbolic surface $\Sigma_c$ by adding funnels to each geodesic boundary component. Choose a collection $\mathcal{G}$ of pairwise-disjoint embedded bi-infinite geodesic lines on $\Sigma_c$ that are filling, and perform a positive strip-deformation (with some positive parameter) on each. The surface $\Sigma'$ is obtained by excising the funnels of the resulting complete hyperbolic surface. The length of a simple closed curve increases by at least the width of the added strip each time it crosses an arc in $\mathcal{G}$. Since any simple closed curve, including the boundary components, intersects $\mathcal{G}$, its length in $\Sigma'$ increases by a definite factor, exactly as in the proof of Proposition 3.1. $\square$

**Remarks.** 1. The same proof, with negative strip-deformations, allows one to shorten lengths of all simple closed curves on a hyperbolic surface with boundary.
For details, see Lemma 4.4 of [8], or [15] (where the operation is called “peeling a strip”).

2. In the construction outlined in the proof of Proposition 3.3, the length of some simple geodesic arc between boundary component(s) of $\Sigma$ would necessarily decrease. This is because by doubling $\Sigma$ across its geodesic boundaries we would obtain a closed surface, and we know that we cannot lengthen all simple closed geodesics on a closed hyperbolic surface (c.f. Proposition 2.1 of [18]). In fact, in the proof of Proposition 3.4 below, we shall see that for sufficiently small strip-deformations on $\Sigma$, the lengths of simple geodesic arcs between boundary component(s) of $\Sigma$ decrease by a uniformly small amount i.e. by a multiplicative factor that is bounded below away from 0.

Recall from the beginning of the section that $\widehat{S}$ is the hyperbolic surface corresponding the “straightened” pleated plane, and $\lambda$ is the “pleating lamination” on it. We shall apply strip-deformations and Proposition 3.3 to various subsurfaces of $\widehat{S}$ in the proof of the following:

**Proposition 3.4.** There is a hyperbolic surface $\widehat{S}_t$ with the underlying topological surface $S_{g,k}$, such that

(A) like $\widehat{S}$, the $i$-th puncture is a cusp if the corresponding entry of $L = (l_1, l_2, \ldots, l_k)$ is zero, and a geodesic boundary component of length $l_i$ otherwise, and

(B) the lengths of simple closed geodesics on $\widehat{S}_t$ are greater than the corresponding geodesics on $\widehat{S}$ in a way that

$$
\sup_{\gamma} \frac{l_\rho(\gamma)}{l_{\widehat{S}}(\gamma)} < 1
$$

where $\gamma$ varies over all non-peripheral simple closed curves on $S_{g,k}$.

**Proof.** In what follows, the surface $\widehat{S}_t$ will be constructed by a suitable deformation of $\widehat{S}$, involving the complementary components of $\lambda$ on $\widehat{S}$ that are not simply-connected. (Recall that such components exists since $\lambda$ is not filling.)

Namely, let $\Sigma^0$ be a connected component of $\widehat{S} \setminus \lambda$ that is not simply-connected. Then the metric completion of $\Sigma^0$ is a “crowned” hyperbolic surface. Recall that a crowned hyperbolic surface is a hyperbolic structure on a punctured surface $S$, such that each puncture corresponds to a “crown end”, bounded by chain of bi-infinite geodesic lines arranged in a cyclic order, such that the positive half-ray of each line and the negative half-ray of the next are asymptotic.

The assumption that $\Sigma^0$ is not simply-connected implies that it is not an ideal polygon. In this case, there is a geodesic representative of the loop around each crown end, and these collection of loops typically bound an embedded hyperbolic surface with geodesic boundary (the convex core of $\Sigma^0$), that we denote by $\Sigma$. (See Fig. 3.) The exceptional case is when $\Sigma^0$ is topologically an annulus, with exactly two crown ends, in which case the convex core $\Sigma$ is a single simple closed geodesic homotopic to the loop around either end.

In this way, we obtain a collection of hyperbolic surfaces with geodesic boundary that we denote by $\Sigma_1, \Sigma_2, \ldots, \Sigma_t$ and possibly some simple closed geodesics
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Fig. 3. The hyperbolic surface \( \hat{S} \) with a subsurface \( \Sigma \) (shown shaded) in the complement of \( \lambda \).

(as in the exceptional case), that we denote by \( \sigma_1, \sigma_2, \ldots, \sigma_m \). Here each \( \Sigma_i \) is connected and embedded in \( \hat{S} \), and the \( \Sigma_i \)'s and \( \sigma_j \)'s are all pairwise disjoint. Consider now the (possibly disconnected) hyperbolic surface

\[
R = \hat{S} \setminus (\Sigma_1 \cup \Sigma_2 \cup \cdots \cup \Sigma_l \cup \sigma_1 \cup \sigma_2 \cup \cdots \cup \sigma_m). \tag{5}
\]

Note that this surface contains all the cusps and/or geodesic boundary components of \( \hat{S} \). We call the geodesic boundary components of \( R \) that arise from the boundary components of the \( \Sigma_i \)'s and the \( \sigma_j \)'s to be the positive boundary components.

We shall now modify \( R \) to a new (topologically identical) hyperbolic surface \( R_t \) by the following operation: choose a collection of pairwise-disjoint non-peripheral geodesic arcs between the positive boundary components of \( R \), such that each positive boundary component has at least one incident arc, and each arc is perpendicular to the positive boundary components at its endpoints. (The shortest arc in a non-trivial homotopy class between boundary components would have the last property.) Let \( R_c \) be the completion of \( R \) obtained by attaching hyperbolic funnels to each of the positive boundary components. These arcs can now be extended to bi-infinite geodesic lines in \( R_c \) that are still pairwise-disjoint (if they intersect, we would obtain a hyperbolic triangle with two right angles, bounded by segments of the two intersecting lines and an arc in a positive boundary component, which is impossible). Hence, we can perform a positive strip deformation with parameter \( t > 0 \) on each of these geodesic lines, to obtain a new hyperbolic surface. Then \( R_t \) is the surface obtained when we remove the funnels from this new surface. By construction, the lengths of all the positive boundary components of \( R \) have increased; we continue to call these the positive boundary components of \( R_t \). Note that the lengths of the other boundary components of \( R \) are not affected in this deformation. The parameter \( t \) in the above discussion will be chosen in the forthcoming discussion (see the Claim below).

First, by applying Proposition 3.3 to each \( \Sigma_i \), we obtain a topologically identical hyperbolic surface \( \Sigma_i' \) such that the lengths of all simple closed geodesics in \( \Sigma_i' \), including the geodesic boundary components, are greater (by a multiplicative factor greater than 1) than the corresponding lengths on \( \Sigma_i \). Moreover, we can choose the
positive parameters on the set of filling arcs in the proof of Proposition 3.3, to make sure that the lengths of the boundary components of each $\Sigma_i'$ match with the lengths of the corresponding positive boundary components of $R_t$. Then we obtain a new hyperbolic surface $\hat{S}_t$ by gluing (a) each $\Sigma_i'$ to $R_t$, and (b) pairs of boundary components of $R_t$ corresponding to the $\sigma_j$s, exactly as dictated by the identification of the boundaries of $\Sigma_i$s and the $\sigma_j$s on $\hat{S}$ (see (5)). There is also a prescribed twist parameter for each such gluing that we now describe: let $\beta$ be the simple closed geodesic on $\hat{S}$ that is either a boundary component of $\Sigma_i$ for some $i$, or $\sigma_j$ for some $j$. Then a choice of a point $p$ on $\beta$ determines points $p_+$ and $p_-$ on the corresponding boundary components of $\Sigma_i$ and $R$ respectively. Recall that the length-increasing deformations defined on $\Sigma_i$ and $R$ are obtained by gluing in hyperbolic strips perpendicular to the boundary on either surface. This will lengthen the boundary components containing $p_+$ and $p_-$ by adding intervals on each, having interiors disjoint from $p_\pm$. In particular, the resulting hyperbolic surfaces $\Sigma_i'$ and $R_t$ still have boundary components with marked points that we can continue denoting by $p_\pm$. In the identification to obtain $\hat{S}_t$, we shall then require that $p_+$ is identified with $p_-$; this uniquely specifies the twist parameter.

By our construction, the surface $\hat{S}_t$ satisfies the requirement (A) in the statement of the Proposition, since the deformations described above do not affect the cusps and geodesic boundary components of $\hat{S}$. We shall now show that we can choose the parameter $t$ so that the requirement (B) of the Proposition is also satisfied.

Let $c$ be a simple closed curve on $\hat{S}$, that is a boundary component of some $\Sigma_i$, or one of the $\sigma_j$s. Note that $c$ has an annular neighborhood in $\hat{S}$, that is bounded by $c$ on one side, and a collection of leaves of $\lambda$ on the other. (The latter is the crown end of the complementary component of $\lambda$ that $c$ is contained in.) This implies that any simple closed geodesic $\gamma$ on $\hat{S}$ that intersects $c$ essentially, must necessarily intersect $\lambda$. Indeed, $\gamma$ must intersect $\lambda$ at least as many times as it intersects $\partial \Sigma_i$. Thus, by Proposition 3.1, we know that

$$\sup_{\gamma} \frac{l_\rho(\gamma)}{l_{\hat{S}}(\gamma)} = \beta < 1$$

when $\gamma$ varies over all simple closed geodesics that are not completely contained in one of the embedded $\Sigma_i$s, and is not one of the $\sigma_j$s.

The key observation now is that the positive strip-deformations resulting in $\hat{S}_t$ changes the fundamental domain for $\hat{S}$ to that of $\hat{S}_t$ continuously, as a function of $t$. As a consequence, we derive the following claim (see Remark (2) after Proposition 3.3):

Claim. For any sufficiently small $t > 0$, we have

$$\sup_{\gamma} \frac{l_\zeta(\gamma)}{l_{\hat{S}}(\gamma)} < \frac{1}{\beta}$$

when $\gamma$ varies over all simple closed curves that are not completely contained in one of the embedded $\Sigma_i$s, and are not one of the $\sigma_j$s. (Here $\beta$ is the constant less than 1 obtained in (6).)

Proof of claim. Any simple closed geodesic $\gamma$ on $\hat{S}$ as above can be decomposed into a finite union of geodesic arcs $\{\gamma_j\}_{j=1}^N$ such that each $\gamma_j$ is either a geodesic
arc between a boundary component of a $\Sigma_i$ (for some $1 \leq i \leq l$) to itself, or a geodesic arc with endpoints on two (or possibly the same) positive boundary components of $R$. In either case, the length of each $\gamma_j$, denoted $|\gamma_j|$, is bounded below by some $L > 0$, that depends on the hyperbolic surface $\hat{S}$ and its subsurfaces $\Sigma_i$s and $R$. Since the fundamental domains for $\hat{S}_t$ and its corresponding subsurfaces in $\mathbb{H}^2$ change continuously as we increase $t$ from 0, the length of each $\gamma_j$ changes continuously. Thus, for any choice of $c > 0$, we can choose $t$ small enough such that this difference of lengths is bounded by $c$ (for each $j$). Then we have

$$l_{\hat{S}_t}(\gamma) > \sum_{j=1}^{N} |\gamma_j| - Nc.$$ and consequently

$$\frac{l_{\hat{S}}(\gamma)}{l_{\hat{S}_t}(\gamma)} \leq \frac{\sum_{j=1}^{N} |\gamma_j|}{\sum_{j=1}^{N} |\gamma_j| - Nc} < \left(1 - \frac{Nc}{\sum_{j=1}^{N} |\gamma_j|}ight)^{-1} < \frac{1}{1 - \frac{c}{L}}.$$ To obtain (7), we choose $c$ such that the right hand side above is equal to $1/\beta$. $\square$

Combining (6) and (7), we see that the inequality (4) in requirement (B) of the Proposition holds when the supremum is taken over all simple closed curves $\gamma$ that are not completely contained in one of the embedded $\Sigma_i$s, and is not one of the $\sigma_j$s.

However, if $\gamma$ is a simple closed curve contained entirely in one of the $\Sigma_i$s, or is one of the $\sigma_j$s, then $l_{\rho}(\gamma) = l_{\hat{S}}(\gamma)$ since $\gamma$ is disjoint from $\lambda$. Also, the length of $\gamma$ on $\hat{S}_t$ is equal to the length of $\gamma$ on the embedded subsurface $\Sigma'_i$. Hence by Proposition 3.3, we obtain

$$\sup_{\gamma} \frac{l_{\rho}(\gamma)}{l_{\hat{S}_t}(\gamma)} < 1$$

when the supremum is taken over all such simple closed curves.

Thus, (4) holds when the supremum is taken over all simple closed curves on $S_{g,k}$, and requirement (B) is satisfied. $\square$

As a consequence, the holonomy $j : \Pi \to \text{PSL}_2(\mathbb{R})$ of the hyperbolic surface $\hat{S}_t$ (obtained in Proposition 3.4) strictly dominates the non-degenerate and non-Fuchsian representation $\rho$ we started with, in the beginning of the section.

3.2. Case that $\rho$ is degenerate

We now handle the remaining case when $\rho$ is a degenerate representation. Recall from Definition 2.1 that there are two possibilities (a) and (b), where the image has exactly one and two global fixed points on $\mathbb{C}P^1$, respectively.

In the case that (a) in Definition 2.1 holds, the monodromy around each puncture of $S_{g,k}$ is parabolic, and the representation $\rho$ lies in the relative character variety $\text{Hom}(\Pi, L)$ where $L = (0, 0, \ldots, 0)$. Then, let $j : \Pi \to \text{PSL}_2(\mathbb{R})$ be any Fuchsian representation such that each of the $k$ punctures is a cusp. Then $j \in \text{Hom}(\Pi, L)$, and strictly dominates $\rho$; indeed, the left hand side of (1) then equals zero.
Finally, in the case that $\rho$ is degenerate and co-axial (i.e. (b) in Definition 2.1 holds), then the image of $\rho$ preserves the geodesic line $\ell$ in $\mathbb{H}^3$, which by a conjugation can be assumed to be to a geodesic line in the equatorial plane $\mathbb{H}^2$. If we identify $\ell$ with $\mathbb{R}$, via an isometry $\Psi$, then each element in the image of $\rho$ acts by a half-translation along $\mathbb{R}$ (i.e has the form $x \mapsto \pm x + c$). Thus, for each $\gamma \in \Pi$, there is a real number $m(\rho(\gamma)) = \Psi(\rho(\gamma) \cdot x) - \Psi(x)$, that is well-defined, i.e. independent of $x$. Moreover, these satisfy: (i) $l_{\rho}(\gamma) = |m(\rho(\gamma))|$ and (ii) $m \circ \rho : \Pi \to \mathbb{R}$ is a homomorphism. Then, as in section 3.1 of [5], this homomorphism to $\mathbb{R}$ can be considered as a defining a virtually-abelian representation $\rho' : \Pi \to \text{PSL}_2(\mathbb{R})$ that preserves a geodesic line $\ell$ in $\mathbb{H}^2$, and acts by translations along it. Note that the translation distance of $\rho'(\gamma)$ along $\mathbb{R}$ is exactly $l_{\rho}(\gamma)$, for each $\gamma \in \Pi$.

Thus, it suffices to find a Fuchsian representation $j$ that strictly dominates $\rho'$ in the sense of (1). To do this, we can apply the techniques of either [8] or [16]. We thank Nathaniel Sagman for the following sketch of the latter approach:

Let $L = (l_1, l_2, \ldots, l_k)$ be the $\rho'$-lengths of the loops around the punctures of $S$. Recall that we had chosen a hyperbolic metric of finite volume on $S = S_{g,k}$ in 2. By Theorem 1.1 of [16], there is a $\rho'$-equivariant map $f : \tilde{S} \to \mathbb{H}^2$ with image $\ell$ such that the Hopf differential $\text{Hopf}(f)$ on $S$ has a pole at the $i$-th puncture of order at most one if $l_i = 0$, and of order two if $l_i > 0$, with a real residue determined by $l_i$. Moreover, by Theorem 1.4 of [16] there is a hyperbolic surface $S'$ of finite volume that is homeomorphic to $S$, with boundary-lengths given by $L$, and a harmonic map $h : S \to S'$ such that $\text{Hopf}(h) = \text{Hopf}(f)$.

By Proposition 3.13 of [16], the energy densities satisfy $e(f) < e(h)$ point-wise, everywhere on $S$. Moreover, as one approaches a puncture of $S$, the ratio $e(f)/e(h) \to 1$, if the monodromy around it is parabolic or hyperbolic (by Theorem 1.1 of [16] and Proposition 3.13 of [20]) and $e(f)/e(h) \to 0$, if the monodromy around the puncture is elliptic (by Proposition 6.1 of [16]). Then, if $j$ is the Fuchsian holonomy of $S'$, the $(j, \rho)$-equivariant map $f \circ h^{-1} : \tilde{h}(\mathbb{H}^2) \to \mathbb{H}^2$ is strictly 1-Lipschitz on any compact subset of $S'$. Since any simple closed geodesic on $S'$ lies in a compact subset of $S'$ by Lemma 3.3, it follows that $j$ strictly dominates $\rho$ in the simple length spectrum, as in (1).

This concludes the proof of Theorem 1.1.
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