\( (T) \)-structures on 2-dimensional \( F \)-manifolds: formal classification

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Abstract: After a brief review on the theory of \((T)\) and \((TE)\)-structures, we determine normal forms for the equivalence classes, under formal isomorphisms, of \((T)\)-structures over 2-dimensional irreducible \( F \)-manifolds.

1 Introduction

The theory of meromorphic connections is a well-established field with importance in many areas of modern mathematics (complex analysis, algebraic geometry, differential geometry, integrable systems etc). An important class of meromorphic connections are the so called \((TE)\)-structures. They are meromorphic connections defined on holomorphic vector bundles over products \( \mathbb{C} \times M \), with pole of Poincaré rank one along the submanifold \( \{0\} \times M \). They also represent the simplest class of meromorphic connections with irregular singularities along \( \{0\} \times M \). The parameter space \( M \) of a \((TE)\)-structure inherits, under a mild additional condition (the so called ‘unfolding condition’) a multiplication \( \circ \) on \( TM \), with nice properties, and a vector field \( E \) which rescales \( \circ \), making \( M \) an \( F \)-manifold with Euler field. The notion of an \( F \)-manifold was introduced for the first time in [6] as a generalization of the notion of a Frobenius manifold [3]. Any Frobenius manifold without metric is an \( F \)-manifold. As shown in [7], there are \( F \)-manifolds which cannot be enriched to a Frobenius manifold. Examples of \( F \)-manifolds arise also in the theory of integrable systems [9, 15] and quantum cohomology [7].

A natural question which arises in this context is to classify the \((TE)\)-structures lying above a given germ of \( F \)-manifolds with Euler field. While a \((TE)\)-structure \( \nabla \) may be seen as a family of meromorphic connections on vector bundles over \( \Delta \) (a small disc around the origin \( 0 \in \mathbb{C} \)), by ‘forgetting’ the derivatives of \( \nabla \) in the parameter space \( M \)-direction (this point of view being crucial in the theory of isomonodromic deformations), we may adopt the alternative view-point and consider \( \nabla \) as a family of flat connections on \( M \) parameterised by \( z \in \mathbb{C}^* \). Such a family has received much attention.
in the theory of meromorphic connections and is referred in the literature as a $(T)$-structure over $M$. Therefore, any $(TE)$-structure underlies a $(T)$-structure but the converse is not always true. The parameter space of a $(T)$-structure inherits the structure of an $F$-manifold (without Euler field), when the unfolding condition is satisfied.

Adopting the second viewpoint, in this paper we make a first step in the classification of $(TE)$-structures which induce a given germ of $F$-manifolds with Euler fields. We consider the simplest case, namely when the $F$-manifold is 2-dimensional (and irreducible) and we determine normal forms for the $(T)$-structures which lie over such an $F$-manifold. The results we prove here will be crucial for future projects, when we shall classify $(TE)$-structures over 2-dimensional (and, possibly bigger dimensional) $F$-manifolds with Euler fields. The 2-dimensional case is considerably simpler, owing to the fact that (unlike higher dimensions) irreducible germs of 2-dimensional $F$-manifolds are classified \[5\]: either they coincide with the germ of the globally nilpotent constant $F$-manifold $N_2$ or they are generically semisimple and belong to a class of germs $I_2(m)$ parameterized by $m \in \mathbb{N}_{\geq 3}$ (see the end of Section 2.2 for the description of these germs). As $F$-manifolds isomorphisms lift to isomorphisms between the spaces of $(T)$- and $(TE)$-structures lying over them, we can (and will) assume, without loss of generality, that our germs of $F$-manifolds are $N_2$ or $I_2(m)$ (with $m \geq 3$). The specific form of these germs will enable us to find the formal normal forms for $(T)$-structures and the classification we were looking for.

Structure of the paper. In Section 2 we recall well-known facts we need on $(T)$, $(TE)$-structures and $F$-manifolds (see e.g. \[4\]). Although our original contribution in this paper refers to $(T)$-structures, we include also basic material on $(TE)$-structure as a motivation and to fix notation and results we shall use in the subsequent stages of our project on classification of $(TE)$-structures. In Sections 3 we determine the formal normal forms for $(T)$-structures over $I_2(m)$ and in Section 4 we study the similar question for $(T)$-structures over $N_2$. A main difference between these two cases lies in the form of formal isomorphisms used in the classification. As opposed to $N_2$ (which has a rich automorphism group), the automorphism group of $I_2(m)$ is finite and formal $(T)$-structures isomorphisms which lift non-trivial automorphisms of $I_2(m)$ do not add much simplification at the classification of $(T)$-structures under formal gauge isomorphisms (formal isomorphisms which act as the identity on $I_2(m)$). For this reason, we content ourselves to determine normal forms for formal gauge isomorphic $(T)$-structures over $I_2(m)$ (see Theorems 15 and 17). As opposed to $I_2(m)$, the $(T)$-structures over $N_2$ will be classified up to formal (not necessarily gauge) isomorphisms.
and the formal automorphisms which lift nontrivial isomorphisms of \( N_2 \) will play a crucial role here. The normal forms in this case are provided by Theorem 22.

2 Preliminary material

We begin by fixing the notation we shall use in this paper.

**Notation 1.** For a complex manifold \( M \), we denote by \( \mathcal{O}_M, \mathcal{T}_M, \Omega^k_M \) the sheaves of holomorphic functions, holomorphic vector fields and holomorphic \( k \)-forms on \( M \) respectively. For a holomorphic vector bundle \( H \), we denote by \( \mathcal{O}(H) \) the sheaf of its holomorphic sections. We denote by \( \Omega^1_{\mathbb{C} \times M}(\log \{0\} \times M) \) the sheaf of meromorphic 1-forms on \( \mathbb{C} \times M \), which are logarithmic along \( \{0\} \times M \). Locally, in a neighborhood of \((0, p)\), where \( p \in M \), any \( \omega \in \Omega^1_{\mathbb{C} \times M}(\log \{0\} \times M) \) is of the form

\[
\omega = \frac{f(z, t)}{z}dz + \sum_i f_i(z, t)dt_i
\]

where \((t_i)\) is a coordinate system around \( p \) and \( f, f_i \) are holomorphic. The ring of holomorphic functions defined on a neighbourhood of \( 0 \in \mathbb{C} \) will be denoted as usual by \( \mathbb{C}\{z\} \). We denote by \( \mathbb{C}\{t, z\} \) the ring of formal power series \( \sum_{n \geq 0} a_n z^n \) where all \( a_n = a_n(t) \) are holomorphic on the same neighbourhood of \( 0 \in \mathbb{C} \).

2.1 \((T)-\) and \((TE)-\)structures

In this section we recall the basic facts on \((T)\) and \((TE)\)-structures.

**Definition 2.** Let \( M \) be a complex manifold and \( H \to \mathbb{C} \times M \) a holomorphic vector bundle.

(a) A \((T)\)-structure over \( M \) is a pair \((H \to \mathbb{C} \times M, \nabla)\) where \( \nabla \) is a map

\[
\nabla : \mathcal{O}(H) \to \frac{1}{z}\mathcal{O}_{\mathbb{C} \times M} \cdot \Omega^1_M \otimes \mathcal{O}(H)
\]

such that for any \( z \in \mathbb{C}^* \) the restriction of \( \nabla \) to \( H|_{\{z\} \times M} \) is a flat connection.

(b) A \((TE)\)-structure over \( M \) is a pair \((H \to \mathbb{C} \times M, \nabla)\) where \( \nabla \) is a flat connection on \( H|_{\mathbb{C} \times M} \) with a pole of Poincaré rank 1 along \( \{0\} \times M \):

\[
\nabla : \mathcal{O}(H) \to \frac{1}{z}\Omega^1_{\mathbb{C} \times M}(\log(\{0\} \times M)) \otimes \mathcal{O}(H).
\]
Any \((TE)\)-structure determines (by forgetting the derivative in the \(z\) direction) a \((T)\)-structure (‘\(E\)’ comes from extension).

Let \((H \to \mathbb{C} \times M, \nabla)\) be a \((TE)\)-structure and \(\Delta \subset \mathbb{C}\) a neighborhood of the origin, \(U \subset M\) a coordinate chart with coordinates \((t_1, \cdots, t_n)\), such that \(H|_\Delta \times U\) is trivial. Using a trivialization \((s_1, \cdots, s_n)\) of \(H|_\Delta \times U\), we write

\[
\nabla(s_i) = \sum_{j=1}^{n} \Omega_{ji} s_j, \quad \Omega = \sum_{i=1}^{n} z^{-1} A_i(z,t) dt_i + z^{-2} B(z,t) dz,
\]

where \(A_i, B\) are holomorphic,

\[
A_i(z,t) = \sum_{k \geq 0} A_i(k) z^k, \quad B(z,t) = \sum_{k \geq 0} B(k) z^k
\]

and \(A_i(k)\) and \(B(k)\) depend only on \(t \in U\). The flatness of the connection \(\nabla\) gives, for any \(i \neq j\),

\[
\begin{align*}
\partial_t A_j - z \partial_z A_i + [A_i, A_j] &= 0, \\
\partial_t B - z^2 \partial_z A_i + z A_i + [A_i, B] &= 0.
\end{align*}
\]

In the case of a \((T)\)-structure, the summand \(z^{-2} B(t, z) dz\) in \(\Omega\) and the equations \((6)\) are dropped. Equations \((5)\) and \((6)\) split according to the powers of \(z\) as follows (with \(A_i(-1) = B(-1) = 0\)):

\[
\partial_t A_i(k - 1) - \partial_z A_i(k - 1) + \sum_{l=0}^{k} [A_i(l), A_j(k - l)] = 0, \quad (7)
\]

\[
\partial_t B(k - 1) - (k - 2) A_i(k - 1) + \sum_{l=0}^{k} [A_i(l), B(k - l)] = 0. \quad (8)
\]

**Definition 3.**

i) An isomorphism \(T : (H, \nabla) \to (\tilde{H}, \tilde{\nabla})\) between two \((T)\)-structures over \(M\) and \(\tilde{M}\) respectively is a holomorphic vector bundle isomorphism \(T : H \to \tilde{H}\) which covers a map of the form \(\text{Id} \times h : \mathbb{C} \times M \to \mathbb{C} \times \tilde{M}\), where \(h : M \to \tilde{M}\) is a biholomorphic transformation, i.e. \(T(H_{(z,p)}) \subset H_{(z,h(p))}\) for any \(p \in M\), and which is compatible with connections:

\[
T(\nabla_{X_p}(s)) = \tilde{\nabla}_{h_s(X_p)}(T(s)), \quad \forall X_p \in T_p M, \quad p \in M, \quad s \in \mathcal{O}(H). \quad (9)
\]

Above, \(T(s) \in \mathcal{O}(\tilde{H})\) is defined by \(T(s)(z, \tilde{p}) := T(s(z, h^{-1}(\tilde{p})))\), for any \(\tilde{p} \in \tilde{M}\).

ii) An isomorphism \(T : (H, \nabla) \to (\tilde{H}, \tilde{\nabla})\) between two \((TE)\)-structures over \(M\) and \(\tilde{M}\) respectively is an isomorphism between their underlying \((T)\)-structures, which satisfies

\[
T(\nabla_{\partial_z}(s)) = \tilde{\nabla}_{\partial_z}(T(s)), \quad \forall p \in M, \quad s \in \mathcal{O}(H). \quad (10)
\]
iii) Two $(T)$ or $(TE)$-structures are isomorphic if there is an isomorphism between them.

Recall that if \( f : N \to \hat{N} \) is a map and \( \pi : \hat{E} \to \hat{N} \) is a bundle over \( \hat{N} \) then \( f^*\hat{E} := \{(e,n) \in \hat{E} \times N, \pi(e) = f(n)\} \) is a bundle over \( N \) with bundle projection \( (e,n) \to n \). Any section \( \hat{s} \in \mathcal{O}(\hat{E}) \) defines a section \( f^*(\hat{s}) \in \mathcal{O}(f^*\hat{E}) \) by \( (f^*(\hat{s}))(n) := (\hat{s}_{f(n)}), n \). If \( f \) is a biholomorphic map, then there is a natural isomorphism \( f^* : \hat{E} \to f^*\hat{E} \) which maps \( \hat{E}_{f(p)} \) onto \( (f^*\hat{E})_p \). Finally, if \( \hat{\nabla} \) is a connection on \( \hat{E} \), then the pull-back connection \( f^*\hat{\nabla} \) is defined by \( (f^*\hat{\nabla})_{X_p}(f^*(\hat{s})) := f^*(\hat{\nabla}_{f_*X_p}(\hat{s})) \), for any \( X_p \in T_pN \), for any \( p \in N \) and \( \hat{s} \in \mathcal{O}(\hat{E}) \). The next lemma can be checked directly. For simplicity, we write \( h^* \) for the pull-back \( (\text{Id} \times h)^* \) (of connections, bundles, etc) and say that \( T \) covers \( h \) (instead of \( \text{Id} \times h \)).

**Lemma 4.** Let \((H, \nabla)\) and \((\hat{H}, \hat{\nabla})\) be two \((T)\)-structures over \( M \) and \( \hat{M} \) respectively. If \( T : (H, \nabla) \to (\hat{H}, \hat{\nabla}) \) is an isomorphism which covers \( h : M \to \hat{M} \), then \( h^* \circ T : (H, \nabla) \to (h^*\hat{H}, h^*\hat{\nabla}) \) is an isomorphism which covers the identity map of \( M \).

Assume that \( T : (H, \nabla) \to (\hat{H}, \hat{\nabla}) \) is a \((T)\) or \((TE)\)-structures isomorphism which covers a biholomorphic map \( h : M \to \hat{M} \). Let \((s_1, \ldots, s_n)\) and \((\hat{s}_1, \ldots, \hat{s}_n)\) be local trivialisations of \( H \) and \( \hat{H} \), over \( \Delta \times U \) and \( \Delta \times \hat{U} \) respectively, where \( U \subset M \) and \( \hat{U} \subset M \) are open subsets and \( \hat{U} = h(U) \). With respect to these trivialisations, \( T|_U \) is given by a holomorphic power series \( T = (T_{ij}) = \sum_{k \geq 0} T(k) z^k \in M(n \times n, \mathcal{O}_{\Delta \times U}) \) with \( T(k) \in M(n \times n, \mathcal{O}_{\Delta \times U}), T(0) \) invertible, such that \( T(s_i) = \sum_{j=1}^n T_{ji} \hat{s}_j \), or, explicitly,

\[
T((s_i)_p) = \sum_{j=1}^n (T_{ji} \circ h)(p)(\hat{s}_j)_{h(p)}, \; \forall p \in U. \tag{11}
\]

Suppose now that \((t_1, \ldots, t_n)\) and \((\hat{t}_1, \ldots, \hat{t}_n)\) are local coordinates of \( M \) and \( \hat{M} \), defined on \( U \) and \( \hat{U} \) respectively. The compatibility \((10)\) and \((11)\) of \( T \) with connections reads

\[
z \partial_i(\hat{T}) + \sum_{j=1}^n (\partial_j h^i)(\hat{A}_j \circ h)\hat{T} - \hat{T} A_i = 0, \; \forall i \tag{12}
\]

\[
z^2 \partial_\bar{z}(\hat{T}) + (\hat{B} \circ h)\hat{T} - \hat{T} B = 0, \tag{13}
\]

where \( \hat{T} := T \circ h \) and \((h^j)\) are the components of the representation of \( h \) in the two charts (the second relation has to be omitted when \( \nabla \) and \( \hat{\nabla} \) are
\( (T)\)-structures. The above equations split according to the powers of \( z \) as

\[
\partial_i \tilde{T}(k-1) + \sum_{l=0}^{k} \left( \sum_{j=1}^{n} (\partial_j h^j)(\tilde{A}_j(l) \circ h)\tilde{T}(k-l) - \tilde{T}(k-l)A_i(l) \right) = 0 \quad (14)
\]

\[
(k-1)\tilde{T}(k-1) + \sum_{l=0}^{k} \left( (\tilde{B}(l) \circ h)\tilde{T}(k-l) - \tilde{T}(k-l)B(l) \right) = 0. \quad (15)
\]

We now discuss a particular class of \((T)\) and \((TE)\)-structures isomorphisms, called gauge isomorphisms. Consider \((H, \nabla)\) a \((T)\) or a \((TE)\)-structure with \(H\) the trivial bundle. Let \((s_1, \cdots, s_n)\) be the standard trivialization of \(H\). Any other trivialization \((\tilde{s}_1, \cdots, \tilde{s}_n)\) of \(H\) is related to \((s_1, \cdots, s_n)\) by an invertible holomorphic matrix \(T\) defined by \(\tilde{s}_i = \sum_{j} T_{ji} s_j\). Suppose that the connection form \(\Omega\) of \(\nabla\) in the basis \((s_1, \cdots, s_n)\) is given by (3), (4) (without the term in \(B\), when \(\nabla\) is a \((T)\)-structure). Then the connection form \(\tilde{\Omega}\) of \(\tilde{\nabla}\) in the new basis \((\tilde{s}_1, \cdots, \tilde{s}_n)\) has the same form with matrices \(\tilde{A}_i\) and \(\tilde{B}\) related to \(A_i\) and \(B\) by

\[
z \partial_i(T) + A_i T - T \tilde{A}_i = 0 \quad (16)
\]

\[
z^2 \partial_i(T) + BT - T \tilde{B} = 0 \quad (17)
\]

or by

\[
\partial_i T(k-1) + \sum_{l=0}^{k} (A_i(l)T(k-l) - T(k-l)\tilde{A}_i(l)) = 0, \quad (18)
\]

\[
(k-1)T(k-1) + \sum_{l=0}^{k} (B(l)T(k-l) - T(k-l)\tilde{B}(l)) = 0. \quad (19)
\]

We say that \(T\) defines a gauge isomorphism between the \((T)\) (or \((TE)\)-structures) with connection forms \(\Omega\) and \(\tilde{\Omega}\) respectively.

**Remark 5.** Obviously, gauge isomorphisms are isomorphisms which preserve the base. When \(h\) is the identity map, relations (14) and (15) reduce to (18) and (19), with the roles of \(A_i, B\) and \(\tilde{A}_i, \tilde{B}\) interchanged. We hope that this difference will not generate confusion. Our conventions regarding gauge isomorphisms coincide with the standard ones used in the literature (see e.g. [14]). For general \((T)\) or \((TE)\)-structures isomorphisms (which do not act on the same bundles or as the identity on the base), the definition (11) of the matrix associated to such an isomorphism seems more natural. The two approaches are anyway equivalent.
A formal isomorphism between two \((T)\) or \((TE)\)-structures \((H, \nabla)\) and \((\tilde{H}, \tilde{\nabla})\) over \((M, 0)\) and \((\tilde{M}, 0)\), which covers a holomorphic map \(h : (M, 0) \to (\tilde{M}, 0)\), is given by a matrix-valued power series \(T = (T_{ij}) = \sum_{k \geq 0} T(k) z^k\), where \(T(k)\) are independent of \(z\) and holomorphic on the same (independent of \(k\)) neighbourhood of the origin \(0 \in \tilde{M}\), and such that, in the above notation, relations (12) and (13) are satisfied (the second in the case of \((TE)\)-structures).

2.2 Unfolding condition, \(F\)-manifolds and \((T)\)-structures

2.2.1 \((T)\)-structures and \(F\)-manifolds

Let \((H, \nabla)\) be a \((T)\)-structure over \(M\). It induces a Higgs field \(C \in \Omega^1(M, \text{End}(K))\) on the restriction \(K := H|_{\{0\} \times M}\), defined by

\[
C_X[a] := [z \nabla_X a], \quad X \in T_M, a \in \mathcal{O}(H),
\]

where \([\;]\) means the restriction to \(\{0\} \times M\) and \(X \in T_M\) is lifted canonically from its domain of definition \(U \subset M\) to \(C \times U\). In the notation from the previous section, \(C\) is given locally by \(\sum_{i=1}^n A_i(0) dt_i\). Relation (7) with \(k = 0\) implies \([A_i(0), A_j(0)] = 0\), i.e. \(C_X C_Y = C_Y C_X\) for any \(X, Y \in T_M\). We write this as \(C \wedge C = 0\).

A \((TE)\)-structure induces additionally an endomorphism \(U \in \text{End}(K)\),

\[
U := [z \nabla_z \partial_a] : \mathcal{O}(K) \to \mathcal{O}(K).
\]

It satisfies \([C, U] = 0\), i.e. \(C_X U = U C_X\) for any \(X \in T_M\).

**Definition 6.** (a) A vector bundle \(K \to M\) with Higgs field \(C\) satisfies the unfolding condition if any point \(p \in M\) has an open neighbourhood \(U\), such that there is \(\zeta \in \mathcal{O}(K|_U)\) with the property that the map \(TU \ni X \to C_X(\zeta) \in K|_U\) is an isomorphism.

(b) A \((T)\)-structure (or a \((TE)\)-structure) satisfies the unfolding condition if the induced data \((K \to M, C)\) satisfy the unfolding condition.

**Definition 7.** A complex manifold \(M\) with a commutative, associative, fiber preserving multiplication \(\circ\) on the holomorphic tangent bundle and unit field \(e \in T_M\) is an \(F\)-manifold if the multiplication satisfies

\[
L_{X \circ Y}(\circ) = X \circ L_Y(\circ) + Y \circ L_X(\circ).
\]

A vector field \(E \in T_M\) is called an Euler field (of weight 1) if

\[
L_E(\circ) = \circ.
\]
The following lemma was proved in Theorem 3.3 of [8]. The proof below is more elegant and shorter.

**Lemma 8.** A $(T)$-structure (respectively, a $(TE)$-structure) $(H \to \mathbb{C} \times M, \nabla)$ with unfolding condition gives rise to an $F$-manifold structure $(\circ, e)$ on $M$ (respectively, to an $F$-manifold structure $(\circ, e, E)$ with Euler field).

**Proof.** The vector bundle $K \to M$ with Higgs field $C$ induced by $\nabla$ defines a multiplication $\circ$ with unit field $e$ on $TM$, by

$$C_{X \circ Y} = C_X C_Y, \ C_e = \text{Id.} \quad (24)$$

When $\nabla$ is a $(TE)$-structure, the induced endomorphism $\mathcal{U}$ of $K$ defines a unique vector field $E \in TM$ with $-C_E = \mathcal{U}$. These statements use the unfolding condition and were proved in Lemma 4.1 of [4]. In order to prove that $(M, \circ, e)$ is an $F$-manifold it is sufficient to find a $(1,0)$-connection $D'$ on the (complex) $C^\infty$-bundle underlying $TM$, with $D'(C) = 0$ (see Lemma 4.2 of [4]). In order to prove that $(M, \circ, e, E)$ is an $F$-manifold with Euler field (when $\nabla$ is a $(TE)$-structure), it is sufficient to prove the existence of a $C^\infty$-endomorphism $\mathcal{Q}$ of $TM$, with $D'(\mathcal{Q}) - [C, \mathcal{Q}] + C = 0$ (see Lemma 4.2 of [4]).

We will define $D'$ and $\mathcal{Q}$ locally, i.e. in a small open subset $U \subset M$. Let $(t_1, \ldots, t_n)$ be a coordinate system on $U$, with coordinate vector fields $\partial_1, \ldots, \partial_n$. Let $\vec{v} := (v_1, \ldots, v_n)$ be a trivialization of $H$ on a neighborhood of $\{0\} \times U$, and let $\Omega$ be the connection form of $\nabla$ in this trivialization. Let $\vec{v}_{\{0\} \times U} := (v_{1}^{(0)}, \ldots, v_{n}^{(0)})$ be the trivialization of $K$ obtained by restricting $\vec{v}$ to $\{0\} \times U$. Define a $(1,0)$-connection $D$ on $K|_U$, by

$$D_{\partial_k}(v_i^{(0)}) = \sum_{j=1}^{n} A_{k}(1)_{ji} v_j^{(0)}. \quad (25)$$

When $\nabla$ is a $(TE)$-structure, define an endomorphism $\mathcal{Q}$ of $K$ by

$$\mathcal{Q}(v_i^{(0)}) := - \sum_{j=1}^{n} B_{j} v_j^{(0)}. \quad (26)$$

From relations (5) and (6),

$$0 = \partial_j A_k(0) - \partial_k A_j(0) + [A_j(1), A_k(0)] + [A_j(0), A_k(1)], \quad (27)$$

$$0 = \partial_j B(0) + A_j(0) + [A_j(1), B(0)] + [A_j(0), B(1)]. \quad (28)$$
From the definitions of $C$ and $\mathcal{U}$,

$$C_{\partial_j}(v_r^{(0)}) = \sum_{k=1}^{n} A_j(0)_{kr} v_k^{(0)}, \quad \mathcal{U}(v_r^{(0)}) = \sum_{k=1}^{n} B(0)_{kr} v_k^{(0)}. \quad (29)$$

Now, a straightforward computation shows that (27) gives $D(C) = 0$ and (28) gives $D(\mathcal{U}) - [C, Q] + C = 0$. More precisely,

$$D(C)_{\partial_j, \partial_i}(v_i^{(0)}) = (D_{\partial_j}(C_{\partial r}) - D_{\partial r}(C_{\partial j}))(v_i^{(0)})$$

$$= D_{\partial_j}(C_{\partial r}(v_i^{(0)})) - C_{\partial r}(D_{\partial_j}(v_i^{(0)})) - D_{\partial r}(C_{\partial j}(v_i^{(0)})) + C_{\partial j}(D_{\partial r}(v_i^{(0)}))$$

$$= \partial_j(A_r(0))_{kr} v_k^{(0)} + (A_r(0))_{ki}(A_j(1))_{sk} v_s^{(0)} - (A_j(1))_{ki}(A_r(0))_{sk} v_s^{(0)}$$

$$- \partial_r(A_j(0))_{ki} v_k^{(0)} - (A_j(0))_{ki}(A_r(1))_{sk} v_s^{(0)} + (A_r(1))_{ki}(A_j(0))_{sk} v_s^{(0)}$$

$$= (\partial_j A_r(0) - \partial_r A_j(0) + [A_r(0), A_j(0)] + [A_j(0), A_r(1)])_{ki} v_k^{(0)} v_s^{(0)}$$

which vanishes from (27) (to simplify notation, we omitted the summation signs). Relation $D(\mathcal{U}) - [C, Q] + C = 0$ can be proved similarly.

Let $h : (\tilde{M}, \tilde{o}, \tilde{e}) \to (M, o, e)$ be an $F$-manifold isomorphism. If $(E, \nabla)$ is a $(T)$-structure over $M$ which induces $(\circ, e)$ then $(h^*E, h^*\nabla)$ is a $(T)$-structure over $\tilde{M}$ which induces $(\tilde{\circ}, \tilde{e})$ and the same holds for $(TE)$-structures. In particular, the spaces of $(T)$ and $(TE)$-structures which lie above isomorphic germs of $F$-manifolds are isomorphic.

### 2.2.2 2-dimensional F-manifolds

There are two types of isomorphism classes of germs of 2-dimensional irreducible $F$-manifolds ([5], Theorem 4.7): $I_2(m)$ with $m \in \mathbb{N}_{\geq 3}$ (generically semisimple) and $\mathcal{N}_2$ (globally nilpotent). As germs of manifolds, $I_2(m)$ and $\mathcal{N}_2$ are $(\mathbb{C}^2, 0)$. In the standard coordinates $(t_1, t_2)$ of $\mathbb{C}^2$, the multiplication of $I_2(m)$ has $\partial_1$ as unit field and $\partial_2 \circ \partial_2 = t_2^{m-2} \partial_2$. Similarly, the multiplication of $\mathcal{N}_2$ has $\partial_1$ as unit field and $\partial_2 \circ \partial_2 = 0$. The next simple lemma describes the automorphism groups of $I_2(m)$ and $\mathcal{N}_2$.

**Lemma 9.** i) The automorphism group of the germ $I_2(m)$ ($m \geq 3$) is cyclic of order $m$, generated by the automorphism

$$(t_1, t_2) \to (t_1, e^{2\pi i/m} t_2).$$

ii) The automorphisms of $\mathcal{N}_2$ are of the form

$$(t_1, t_2) \to (t_1, \lambda(t_2)) \quad (30)$$

where $\lambda \in \mathbb{C}\{t_2\}$, with $\lambda(0) = 0$ and $\dot{\lambda}(0) \neq 0$. 

Our aim in this paper is to describe up to formal isomorphisms all \((T)\)-structures over an arbitrary irreducible germ \((M, 0)\) of 2-dimensional \(F\)-manifolds. From the above comments, we can (and will) assume, without loss of generality, that \((M, 0)\) is either \(I_2(m)\) \((m \geq 3)\) or \(N_2\). Motivated by the above lemma, the \((T)\)-structures lying above \(I_2(m)\) will be classified up to formal gauge isomorphisms. Our classification of \((T)\)-structures over \(N_2\) will be done up to the entire group of formal isomorphisms (not only formal gauge isomorphisms).

2.3 Differential equations

Along this section \(t \in (\mathbb{C}, 0)\) is the standard coordinate.

Lemma 10. Consider the differential equation

\[
\frac{d}{dt}(ah) + a \frac{dh}{dt} = c, \tag{31}
\]

where \(a, c \in \mathbb{C}\{t\}\) are given and the function \(h = h(t)\) is unknown.

i) If \(a(0) \neq 0\), then there is a unique formal solution \(h\) with given \(h(0)\) and this solution is holomorphic.

ii) If \(t = 0\) is a zero of order one for \(a\), then there is a unique formal solution of (31) and this solution is holomorphic.

iii) If \(t = 0\) is a zero of order \(o \geq 2\) for \(a\), then (31) has a formal solution if and only if \(t = 0\) is a zero of order at least \(o - 1\) for \(c\). When it exists, the formal solution is unique and holomorphic.

In all cases, if \(a\) and \(c\) converge on \(\Delta\) (an open disc centred in \(0 \in \mathbb{C}\)), then also the formal solution converges on \(\Delta\).

Proof. As the proof is elementary, we skip the details. The first claim follows from the fundamental theorem of differential equations. For the second and third claims, one checks easily (by taking power series and identifying coefficients) the part concerning the existence of formal solutions. For the convergence, one uses the general result that any formal solution \(u(t)\) of a differential equation of the form

\[
 tu'(t) + A(t)u(t) = b(t), \tag{32}
\]

where \(A : \Delta \to M_n(\mathbb{C})\) and \(b : \Delta \to \mathbb{C}^n\) are holomorphic, is convergent on \(\Delta\). This was proved e.g. in Theorem 5.3 of [16] (see page 22), when \(b = 0\). The case \(b \neq 0\) can be reduced to the case \(b = 0\) in the standard way: if \(u = (u_1, \ldots, u_n)^t\) is a solution of (32) with \(b \neq 0\), one defines \(v := (u_1, \ldots, u_n, 1)^t\) and sees that \(v\) satisfies a differential equation (in dimension \(n + 1\)) of the
same type \((32)\) but with \(b = 0\). One easily shows that in the second and third statements of the lemma, equation \((31)\) reduces to an equation of the form \((32)\) (with \(A\) and \(b\) scalar functions).

For a given function \(f : (C, 0) \to (C, 0)\) and \(n \in \mathbb{Z}_{\geq 1}\) we denote by \(f^n\) the function \(f^n(t) := f(t) \cdots f(t)\) (multiplication \(n\)-times) (not to be confused with the iterated composition \(f \circ \cdots \circ f\)).

**Lemma 11.** Let \(f \in \mathbb{C}\{t\}\) be non-trivial and \(r := \text{ord}_0(f)\). Then there is \(\lambda \in \mathbb{C}\{t\}\), with \(\lambda(0) = 0\) and \(\dot{\lambda}(0) \neq 0\), such that \((\dot{\lambda})^2 \lambda^r = f\). Moreover, any two such functions \(\lambda\) and \(\tilde{\lambda}\) are related by \(\tilde{\lambda}(t) = \lambda_0 \lambda(t)\), where \(\lambda_0 \in \mathbb{C}, \lambda_0^2 = 1\).

**Proof.** As \(r = \text{ord}_0(f)\), we can write \(f(t) = t^r g(t)\) with \(g \in \mathbb{C}\{t\}\) a unit. Similarly, the function \(\lambda\) we are looking for is of the form \(\lambda(t) = tx\), with \(x \in \mathbb{C}\{t\}\) unit. We are looking for \(x\) which satisfies the differential equation

\[(x + t\dot{x})^2 x^r = g.\]  

(33)

As \(g(0) \neq 0\), there is \(k \in \mathbb{C}\{t\}\) unit, such that \(g = k^2\). Similarly, as \(x(0) \neq 0\) we can write \(x = z^2\), for \(z \in \mathbb{C}\{z\}\). Equation \((33)\) is certainly satisfied if \((x + t\dot{x})z^r = k\) or

\[2t(z^{r+2})^r + (r + 2)z^{r+2} = (r + 2)k.\]  

(34)

The differential equation in the unknown function \(y\)

\[2t\dot{y} + (r + 2)y = (r + 2)k\]

has a unique formal solution, which is holomorphic (from the proof of Lemma \([10]\)). As \(k(0) \neq 0\), \(y(0) \neq 0\) and there is \(z \in \mathbb{C}\{t\}\) such that \(z^{r+2} = y\). The function \(z\) satisfies \((34)\) and \(\lambda(t) := tz(t)^2\) is a function we were looking for. The first statement is proved. The second statement follows by taking into account the freedom in the choice of \(z\) and \(k\) in the above argument. \(\Box\)

**Lemma 12.** Let \(C := 4 \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^2 = \frac{2\pi^2}{3}\). For any \(b, l \in \mathbb{Z}_{\geq 2}\) with \(b \geq l\),

\[\sum_{a_i; i \in l, b} (a_1 \cdots a_l)^{-2} \leq C^{l-1} b^{-2},\]  

(35)

where the condition \((*)_{l, b}\) on \((a_1, \cdots, a_l)\) means \(a_i \in \mathbb{Z}_{\geq 1}\) (for any \(1 \leq i \leq l\)) and \(\sum_{i=1}^{l} a_i = b\).
Proof. We prove (35) by induction on \( l \). Let \( l = 2 \). Then

\[
\sum_{a_1,a_2} (a_1a_2)^{-2} = \sum_{a=1}^{b-1} (a^{-1}(b-a)^{-1})^2 = b^{-2} \sum_{a=1}^{b-1} (a^{-1} + (b-a)^{-1})^2 \\
\leq 2b^{-2} \sum_{a=1}^{b-1} (a^{-2} + (b-a)^{-2}) \leq b^{-2}C.
\]

Suppose that (35) holds for any \( l \) and \( m \).

Lemma 13. Let \( \tau \) be a unique formal solution with radius of convergence \( R \) determined by

\[
\sum_{a_1,a_2} (a_1a_2)^{-2} = \sum_{b_1,b_2} \sum_{a_1,...,a_n} (a_1 \cdot a_{n-1} \cdot b_2)^{-2} \\
\leq \sum_{b_1,b_2} (b_2)^{-2} \sum_{a_1,...,a_{n-1}} (a_1 \cdot a_{n-1})^{-2} \\
\leq \sum_{b_1,b_2} (b_2)^{-2} C^{n-2}(b_1)^{-2} \leq C^{n-1}b^{-2},
\]

i.e. (35) holds for \( l = n \) as well.

Before proving the next lemma we remark that for \( f = \sum_{n=0}^{\infty} a_n t^n \in \mathbb{C}\{t\} \) with radius of convergence \( R := \frac{1}{\lim (f_n)_{n=0}} > 0 \), for an arbitrarily small \( \varepsilon > 0 \) a bound \( M > 0 \) with \( |f_n| \leq M(\frac{1}{R} + \varepsilon)^n \) exists.

Lemma 13. Let \( f = \sum_{j=0}^{\infty} f_j t^j \), \( h = \sum_{j=0}^{\infty} h_j t^j \) be two functions from \( \mathbb{C}\{t\} \) and \( m \in \mathbb{Z}_{\geq 1} \). The differential equation

\[
2t\tau'(t) + m\tau(t) - tf(t)\tau^2(t) = h(t)
\]

has a unique formal solution \( \tau = \sum_{j=0}^{\infty} \tau_j t^j \). Moreover, \( \tau \) belongs to \( \mathbb{C}\{t\} \).

Proof. By identifying the coefficients in (36), we obtain that \( \tau_n \) are uniquely determined by

\[
\tau_0 = \frac{h_0}{m}, \quad \tau_n = \frac{1}{m + 2n} \left( \sum_{j+k+p=n-1} f_j \tau_k \tau_p + h_n \right), \quad \forall n \geq 1.
\]

Choose \( M \geq 1 \) and \( r \geq \frac{e^2 M}{m+1} \), where \( C \) is the constant from Lemma 12, such that

\[
|f_j| \leq \frac{r^j M}{(j+1)^2}, \quad |h_j| \leq \frac{r^j M}{(j+1)^2}, \quad \forall j \geq 0
\]
(here we apply the above comments to \( \sum_{j \geq 0} f_j (j+1)^2 z^j \) and \( \sum_{j \geq 0} h_j (j+1)^2 z^j \) which are also convergent). We claim that
\[
|\tau_j| \leq \frac{r^j M^{j+1}}{(j+1)^2}, \forall j \geq 0.
\] (39)

Inequality (39) is satisfied for \( j = 0 \). Suppose that it is satisfied for any \( j \leq n-1 \). From (37),
\[
|\tau_n| \leq \frac{1}{m + 2n} \left( \sum_{j+k+p=n-1} \frac{r^{j+k+p} M^{k+p+3}}{(j+1)^2(k+1)^2(p+1)^2} + \frac{r^n M}{(n+1)^2} \right)
\leq \frac{r^{n-1} M^{n+2}}{m + 2n} \left( \sum_{j+k+p=n-1} (j+1)^{-2}(k+1)^{-2}(p+1)^{-2} + \frac{r^n M}{(n+1)^2} \right)
\leq \frac{r^{n-1} M}{m + 2n} \left( \frac{C^2 M^{n+1}}{(n+2)^2} + \frac{r}{(n+1)^2} \right) = \frac{r^n M^{n+1}}{(n+1)^2} \cdot E,
\]
where in the third line we used Lemma 12 and
\[
E := \frac{(n+1)^2}{r M^n (m + 2n)} \left( \frac{C^2 M^{n+1}}{(n+2)^2} + \frac{r}{(n+1)^2} \right).
\]

where we used \( M \geq 1 \). From \( n \geq 1 \) and \( r \geq \frac{C^2 M}{m+1} \), we obtain that \( E \leq 1 \). Relation (39) follows, and it implies that \( \tau \) converges on \( \Delta(0, \frac{1}{M^r}) \) (the disc of radius \( \frac{1}{M^r} \)).

3 \( (T)\)-structures over \( I_2(m) \)

In this section we find formal normal forms for \( (T)\)-structures over \( I_2(m) \), \( m \geq 3 \). We need to introduce notation.

Notation 14. Along this section \( (t_1, t_2) \) denote the standard coordinates on \( \mathbb{C}^2 \). We shall use the following matrices
\[
C_1 := \mathrm{Id}_2, \quad C_2 := \left( \begin{array}{cc} 0 & t_2^{m-2} \\ 1 & 0 \end{array} \right), \quad D := \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right), \quad E := \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right),
\] (40)

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and the relations between them:

\[(C_2)^2 = t_2^{m-2}C_1, \quad D^2 = C_1, \quad E^2 = 0, \quad (41)\]
\[C_2D = C_2 - 2t_2^{m-2}E = -DC_2, \quad (42)\]
\[C_2E = \frac{1}{2}(C_1 - D), \quad EC_2 = \frac{1}{2}(C_1 + D), \quad (43)\]
\[DE = E = -ED. \quad (44)\]

We shall often use the commutators

\[[C_2, D] = 2(C_2 - 2t_2^{m-2}E), \quad [C_2, E] = -D, \quad [D, E] = 2E. \quad (45)\]

The matrices \(C_1, C_2, D\) and \(E\) form an \(O_{C^2,0}\)-basis of \(M(2 \times 2, O_{C^2,0})\) and

\[\text{Im}([C_2, .]) = O_{C^2,0} \cdot (C_2 - 2t_2^{m-2}E) \oplus O_{C^2,0} \cdot D. \quad (46)\]

For any \(k \geq 0\), let

\[\mathbb{C}[t_2]_{\leq k} := \{f(t_2) \in \mathbb{C}[t_2], \deg f(t_2) \leq k\}.\]

**Theorem 15.**

i) Above the \(F\)-manifold \(I_2(3)\) any \((T)\)-structure is formally gauge isomorphic to the \((T)\)-structure with \(A_1 = C_1\) and \(A_2 = C_2\).

ii) Above the \(F\)-manifold \(I_2(m)\) \((m \geq 4)\), any \((T)\)-structure is formally gauge isomorphic to a \((T)\)-structure of the form

\[A_1 = C_1, \quad A_2 = \begin{pmatrix} 0 & t_2^{m-2} \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & zf \\ 0 & 0 \end{pmatrix} = C_2 + zfE, \quad (48)\]

where \(f \in \mathbb{C}[[z]][t_2]_{\leq m-4}\).

**Proof.** We start with an arbitrary \((T)\)-structure \((H, \nabla)\) above the \(F\)-manifold \(I_2(m)\) with \(m \geq 3\). We choose a trivialization \(\tilde{v} = (v_1, v_2)\) of \(H\) such that the connection matrices \(A_1\) and \(A_2\) of \(\nabla\) in this trivialization satisfy

\[A_1(0) = C_1, \quad A_2(0) = C_2. \quad (49)\]

We will reduce \(\nabla\) to the required normal form in four steps. The first three steps are holomorphic, the fourth step is not holomorphic in general, but it leads to functions in \(\mathbb{C}\{t_2, z\}\) (see notation 1 for this ring).

The **first step** of the normalization is the reduction of \(A_1\) to \(C_1\) and of \(A_2\) to a new matrix \(\bar{A}_2\) with \(A_2(0) = C_2\) and \(\partial_1 \bar{A}_2 = 0\). Consider the system

\[\partial_1 T = -\left(\sum_{k \geq 1} A_1(k) z^{k-1}\right) T, \quad T(z, 0, t_2) = C_1. \quad (50)\]
It has a unique holomorphic solution $T$. We claim that $T$ satisfies
\begin{equation}
T(0) \in \mathcal{O}_{\mathbb{C}^2,0} \cdot C_1 + \mathcal{O}_{\mathbb{C}^2,0} \cdot C_2, \tag{51}
\end{equation}
To prove this claim, we remark that (50) for $z = 0$ gives
\begin{equation}
\partial_1 T(0) = -A_1(1) T(0), \quad T(0)(0,t_2) = C_1. \tag{52}
\end{equation}
On the other hand, relation (7) for $k = 1$ together with (49) gives
\begin{equation}
0 = \partial_1 A_2(0) - \partial_2 A_1(0) + [A_1(0), A_2(1)] + [A_1(1), A_2(0)] = [A_1(1), C_2], \tag{53}
\end{equation}
which implies $A_1(1) = a_1 C_1 + a_2 C_2$ for $a_1, a_2 \in \mathcal{O}_{\mathbb{C}^2,0}$. The differential equation (52) with $A_1(1)$ of this form and ansatz $T(0) = \tau_{01} C_1 + \tau_{02} C_2$, with $\tau_{01}, \tau_{02} \in \mathcal{O}_{\mathbb{C}^2,0}$, and $\tau_{01}(0, t_2) = 1$, $\tau_{02}(0, t_2) = 0$, has a unique solution. We obtain that $T$ satisfies (51), as required. We now change the trivialization $\vec{v}$ by means of $T$. In the new trivialization, $\nabla$ is given by matrices $\tilde{A}_1$ and $\tilde{A}_2$. From (12) for $i = 1$, together with $A_1(0) = C_1$ and (50), we obtain:
\begin{equation}
0 = z \partial_1 T + A_1 T - T \tilde{A}_1 = C_1 T - T \tilde{A}_1 = T(C_1 - \tilde{A}_1) \tag{54}
\end{equation}
which implies $\tilde{A}_1 = C_1$. From (18) for $k = 0$ and $i = 2$,
\begin{equation}
0 = A_2(0) T(0) - T(0) \tilde{A}_2(0) = T(0)(C_2 - \tilde{A}_2(0)), \tag{55}
\end{equation}
where we used (51) and $A_2(0) = C_2$. We obtain $\tilde{A}_2(0) = C_2$. Finally, from (5),
\begin{equation}
0 = z \partial_1 \tilde{A}_2 - z \partial_2 \tilde{A}_1 + [\tilde{A}_1, \tilde{A}_2] = z \partial_1 \tilde{A}_2, \tag{56}
\end{equation}
from which we deduce $\partial_1 \tilde{A}_2 = 0$. The first step is completed.

Owing to the step one, from now on we assume that $A_1 = C_1$, $A_2(0) = C_2$ and $\partial_1 A_2 = 0$.

The second step does not change $A_1 = C_1$ and erases the term $C_1$ in $A_2$. Suppose that
\begin{equation}
A_2 = C_2 + z(a_1 C_1 + a_2 C_2 + a_3 D + a_4 E) \tag{57}
\end{equation}
with $a_1, a_2, a_3, a_4 \in \mathbb{C}\{z, t_2\}$. Let $\tau_1 \in \mathbb{C}\{z, t_2\}$ be the unique solution of
\begin{equation}
\partial_2 \tau_1 = -a_1 \tau_1, \quad \tau_1(z, 0) = 1 \tag{58}
\end{equation}
and $T := \tau_1 C_1$. Relation (12) for $i = 2$ gives
\begin{equation}
0 = z \partial_2 T + A_2 T - T \tilde{A}_2 = (C_2 + z(a_2 C_2 + a_3 D + a_4 E)) - \tilde{A}_2) T. \tag{59}
\end{equation}
Thus
\[ \tilde{A}_2 = C_2 + z(a_2C_2 + a_3D + a_4E), \] (59)
as needed. Remark that the coefficients of \( C_2, D \) and \( E \) in the expressions (57) and (59) of \( A_2 \) and \( \tilde{A}_2 \) are the same.

The third step of the reduction does not change \( A_1 = C_1 \) and brings \( A_2 \) to the form \( C_2 + zfE \) with \( f \in \mathbb{C}\{z, t_2\} \). Suppose

\[ A_2 = C_2 + z(a_2C_2 + a_3D + a_4E) \] (60)
with \( a_2, a_3, a_4 \in \mathbb{C}\{z, t_2\} \). We are searching for a \( T \) and an \( \tilde{A}_2 \) of the form

\[ T = C_1 + z(\tau_3D_3 + \tau_4E) \] (61)
\[ \tilde{A}_2 = C_2 + z(\tilde{a}_1C_1 + \tilde{a}_4E) \] (62)
where \( \tau_3, \tau_4, \tilde{a}_1, \tilde{a}_4 \in \mathbb{C}\{t_2\}[[z]] \), which, together with \( A_2 \), satisfy (12) for \( i = 2 \):

\[
0 = z\partial_z T + A_2 T - T\tilde{A}_2 \\
= z^2\partial_z \tau_3D + z^2\partial_z \tau_4E + z\{C_2, \tau_3D + \tau_4E\} \\
+ (C_2 + z(a_2C_2 + a_3D + a_4E)) - (C_2 + z(\tilde{a}_1C_1 + \tilde{a}_4E)) \\
+ z^2(a_2C_2 + a_3D + a_4E)(\tau_3D + \tau_4E) - z^2(\tau_3D + \tau_4E)(\tilde{a}_1C_1 + \tilde{a}_4E) \\
= z^2\partial_z \tau_3D + z^2\partial_z \tau_4E + z(\tau_3(2C_2 - 4t_2^m-2) - \tau_4D) \\
+ z(a_2C_2 + a_3D + a_4E) - z(\tilde{a}_1C_1 + \tilde{a}_4E) \\
+ z^2(a_2\tau_3(C_2 - 2t_2^m-2) + a_3\tau_3C_1 - a_4\tau_3E) \\
+ a_2\tau_4\left(\frac{1}{2}C_1 - D\right) + a_3\tau_4E) - z^2(\tilde{a}_1\tau_3D + \tilde{a}_4\tau_4E_4 + \tilde{a}_4\tau_3E).
\]

Ordering the terms and dividing once by \( z \), we obtain

\[
0 = C_1(-\tilde{a}_1 + z(a_3\tau_3 + \frac{1}{2}a_2\tau_4)) + C_2(2\tau_3 + a_2 + za_2\tau_3) \\
+ D(-\tau_1 + a_3 + z(\partial_2\tau_3 - \frac{1}{2}a_2\tau_4 - \tilde{a}_1\tau_3)) \\
+ E\left(a_4 - \tilde{a}_4 - 4t_2^m-2\tau_3 + z(\partial_2\tau_4 - 2t_2^m-2a_2\tau_3 -(a_4 + \tilde{a}_4)\tau_3 + a_3\tau_4 - \tilde{a}_1\tau_4)\right).
\]

The coefficient of \( C_2 \) determines \( \tau_3 \) uniquely (\( 2+za_2 \) is a unit in \( \mathbb{C}\{z, t_2\} \)). The coefficient of \( C_1 \) determines \( \tilde{a}_1 \) in terms of \( \tau_4 \). The coefficient of \( D \) determines then \( \tau_4 \). Finally, the coefficient of \( E \) determines \( \tilde{a}_4 \). We proved that \( A_2 \) can be brought to the form (62). Applying the second step to \( A_1 = C_1 \) and \( \tilde{A}_2 \) given by (62), we obtain that \( A_2 \) can be brought (without changing \( A_1 = C_1 \)) into the form \( C_2 + zfE \) (with \( f = \tilde{a}_4 \)), as needed.
The fourth step does not change $A_1 = C_1$ and brings $A_2 = C_2 + za_4E$ (where $a_4 \in \mathbb{C}\{z, t_2\}$) to the normal form $C_2$ if $m = 3$ and $C_2 + z\tilde{a}_4E$, with $\tilde{a}_4 \in \mathbb{C}[z][t_2]_{\leq m-4}$, if $m \geq 4$. We are searching for a $T$ and an $\tilde{A}_2$ of the form

$$T = C_1 + \tau_2C_2 + z(\tau_3D + \tau_4E)$$

(65)

$$\tilde{A}_2 = C_2 + z(\tilde{a}_1C_1 + \tilde{a}_4E),$$

(66)

with $\tilde{a}_1, \tau_2, \tau_3, \tau_4 \in \mathbb{C}\{t_2, z\}$ (see the notation 1 for this ring) and $\tilde{a}_4 \in \mathbb{C}[z][t_2]_{\leq m-4}$, such that, together with $A_2$, they satisfy (12) for $i = 2$:

$$0 = z\partial_2T + A_2T - T\tilde{A}_2$$

$$= z\partial_2\tau_2C_2 + z\tau_2(m-2)t_2^{m-3}E + z^2\partial_2\tau_3D + z^2\partial_2\tau_4E$$

$$+ (C_2 + za_4E) - (C_2 + z(\tilde{a}_1C_1 + \tilde{a}_4E)) + z[C_2, \tau_3D + \tau_4E]$$

$$+ za_4E(\tau_2C_2 + z(\tau_3D + \tau_4E)) - z(\tau_2C_2 + z(\tau_3D + \tau_4E))(\tilde{a}_1C_1 + \tilde{a}_4E)$$

$$= z\partial_2\tau_2C_2 + z\tau_2(m-2)t_2^{m-3}E + z^2\partial_2\tau_3D + z^2\partial_2\tau_4E$$

$$+ z(a_4E - \tilde{a}_1C_1 - \tilde{a}_4E) + z\tau_3(2C_2 - 4t_2^{m-2}E) - z\tau_4D$$

$$+ za_4\tau_2 \frac{1}{2}(C_1 + D) - z^2a_4\tau_3E - z\tilde{a}_4\tau_2C_2 - z^2\tilde{a}_4\tau_3D$$

$$- z^2\tilde{a}_1\tau_4E - z\tilde{a}_4\tau_2\frac{1}{2}(C_1 - D) - z^2\tilde{a}_4\tau_3E.$$  

(67)

Ordering these terms and dividing once by $z$, we obtain

$$0 = C_1(-\tilde{a}_1 + \frac{1}{2}(a_4 - \tilde{a}_4)\tau_2) + C_2(\partial_2\tau_2 + 2\tau_3 - \tilde{a}_1\tau_2)$$

$$+ D(-\tau_4 + \frac{1}{2}(a_4 + \tilde{a}_4)\tau_2 + z(\partial_2\tau_3 - \tilde{a}_1\tau_3))$$

$$+ E \left((m-2)t_2^{m-3}\tau_2 + (a_4 - \tilde{a}_4) - 4t_2^{m-2}\tau_3 + z(\partial_2\tau_4 - (a_4 + \tilde{a}_4)\tau_3 - \tilde{a}_1\tau_4)\right).$$

(68)

We are looking for functions $\tau_2, \tau_3, \tau_4, \tilde{a}_1$ and $\tilde{a}_4$, such that the relations (68) are satisfied. The coefficient of $C_1$ in (68) allows to express $\tilde{a}_1$ in terms of $\tau_2$ and $\tilde{a}_4$:

$$\tilde{a}_1 = F(\tau_2, \tilde{a}_4) := \frac{1}{2}(a_4 - \tilde{a}_4)\tau_2.$$ 

(69)

Using (69), the coefficient of $C_2$ allows to express $\tau_3$ in terms of $\tau_2$ and $\tilde{a}_4$:

$$\tau_3 = G(\tau_2, \tilde{a}_4) := \frac{1}{4}(a_4 - \tilde{a}_4)\tau_2^2 - \frac{1}{2}\partial_2\tau_2.$$ 

(70)

Using (69) and (70), the coefficient of $D$ allows to express $\tau_4$ in terms of $\tau_2$ and $\tilde{a}_4$: $\tau_4 = H(\tau_2, \tilde{a}_4)$, but we shall not use the precise shape of this
expression. Finally, the coefficient of $E$ gives the following equation for $\tau_2$ and $\tilde{a}_4$:

$$t_2^{m-3}((m-2)I + 2t_2\partial_2)(\tau_2) + (a_4 - \tilde{a}_4)(1 - t_2^{m-2}2^2) + zF(\tau_2, \tilde{a}_4) = 0$$  \hspace{1cm} (71)

where

$$F(\tau_2, \tilde{a}_4) := \partial_2H(\tau_2, \tilde{a}_4) - (a_4 + \tilde{a}_4)G(\tau_2, \tilde{a}_4) - F(\tau_2, \tilde{a}_4)\mathcal{H}(\tau_2, \tilde{a}_4).$$

If $m = 3$, an induction argument over the powers of $z$ shows that equation (71) has a unique solution of the form $\tau_2 = \sum_{n \geq 0} \tau_2(n)z^n \in \mathbb{C}[[t_2, z]]$ and $\tilde{a}_4 = 0$. Similarly, if $m \geq 4$, one finds that equation (71) is solved by unique functions $\tau_2 = \sum_{n \geq 0} \tau_2(n)z^n \in \mathbb{C}[[t_2, z]]$ and $\tilde{a}_4 \in \mathbb{C}[[z]][t_2]_{\leq m-4}$. We claim that in both cases, $\tau_2 \in \mathbb{C}\{t_2\}[[z]]$. We give the details only for the case $m \geq 4$ (the case $m = 3$ is similar). From (71), the coefficient $\tau_2(0)$ satisfies

$$t_2^{m-3}((m-2)\tau_2(0) + 2t_2\partial_2\tau_2(0)) + (a_4(0) - \tilde{a}_4(0))(1 - t_2^{m-2}2^2) = 0.$$  \hspace{1cm} (72)

Defining $\tilde{a}_4(0) := [a_4(0)]_{\leq m-4}$ and dividing (72) by $t_2^{m-3}$ we obtain an equation of type (31) for $\tau_2(0)$, namely

$$(m-2)\tau_2(0) + 2t_2\partial_2\tau_2(0) - t_2^{m-2}h(0)\tau_2(0)^2 = -h(0)$$

where $h(0) := t_2^{-(m-3)}(a_4(0) - \tilde{a}_4(0)) \in \mathcal{O}_U$, where $U \subset \mathbb{C}$ is an open disc on which all $a_4(n)$ converge. From Lemma 13, $\tau_2(0)$ is holomorphic on a neighbourhood $U_1 \subset U$ of the origin $0 \in \mathbb{C}$. The higher order coefficients $\tau_2(n)$ (for $n \geq 1$) satisfy

$$t_2^{m-3}((m-2)\tau_2(n) + 2t_2\partial_2\tau_2(n)) - t_2^{m-2}((a_4 - \tilde{a}_4)\tau_2^n(0)) = 0,$$

where $F(n-1) := [F(\tau_2, \tilde{a}_4)](n-1)$ is an expression which depends only on previous $\tau_2(i)$ and $\tilde{a}_4(i)$ ($i \leq n-1$). Let $\tilde{a}_4(n) := [a_4(n) + F(n-1)]_{\leq m-4}$ and $h(n) := t_2^{-(m-3)}(a_4(n) - \tilde{a}_4(n) + F(n-1)) \in \mathcal{O}_{U_1}$. Dividing (73) by $t_2^{m-3}$ we obtain the following equation for the unknown function $\tau_2(n)$:

$$(m-2)\tau_2(n) + 2t_2\partial_2\tau_2(n) - t_2[(a_4 - \tilde{a}_4)\tau_2^n(0)](n) = -h(n)$$

This equation is of type (31), as the coefficient of order $n$ of $(a_4 - \tilde{a}_4)\tau_2^2$ is equal, modulo addition with known functions which involve $\tau_2(i)$ and $\tilde{a}_4(i)$ (with $i \leq n-1$), to $2(a_4(0) - \tilde{a}_4(0))\tau_2(0)\tau_2(n)$. From Lemma 11, $\tau_2(n)$ converges on $U_1$. This is true for any $n \geq 0$. We obtain that $\tau_2$ (as well as $\tau_3$ and $\tau_4$) belong to $\mathbb{C}\{t_2, z\}$ as required. By applying the second step, we eliminate the term $z\tilde{a}_1C_1$ in the expression (66) of $A_2$. This finishes the fourth step and concludes the proof of Theorem 13. \hfill \qed
Remark 16. In the fourth step from the proof of Theorem 15 we used in an essential way Lemmas 10 and 13 on differential equations. The differential equation (36) from Lemma 13 was used to determine \( \tau_2(0) \). The fact that the solutions of this differential equation might have smaller radius of convergence than the coefficients of the equation did not affect our conclusion that \( \tau_2 \in \mathbb{C}\{t_2, z\} \), as the higher order terms \( \tau_2(n) \) (with \( n \geq 1 \)) of \( \tau_2 \) were determined by a simpler differential equation of type (31), and solutions of such differential equations converge on the entire disc where the coefficients do (hence, the radius of convergence of \( \tau_2(n) \) does not tend to zero when \( n \) increases).

Theorem 17. Any two \((T)\)-structures above \( I_2(m) \) \( (m \geq 4) \) of the form (47), (48) are formally gauge non-isomorphic.

Proof. Consider two \((T)\)-structures \( \nabla \) and \( \tilde{\nabla} \) of the form (47), (48), with functions \( f, \tilde{f} \in \mathbb{C}\{z\}[t_2]_{< m-4} \) and assume they are formally gauge isomorphic. Let \( T = \sum_{n \geq 0} T(n) z^n \) be a formal gauge isomorphism between \( \nabla \) and \( \tilde{\nabla} \). In terms of the basis \( \{C_1, C_2, D, E\} \) we write

\[
T(n) = k(n)C_1 + p(n)C_2 + d(n)D + e(n)E
\]

where \( k(n) \), \( p(n) \), \( d(n) \) and \( e(n) \) are independent of \( z \). Relation (18) with \( k = 0 \) and \( i = 1 \) is satisfied. Relation (18) with \( k = 0 \) and \( i = 2 \) implies (using the commutator relations (15)),

\[
2d(0)(C_2 - 2t_2^{m-3}E) - e(0)D = 0,
\]

i.e. \( d(0) = e(0) = 0 \). Relation (18) with \( k = 1 \) and \( i = 1 \) implies (after identifying coefficients of \( C_1, C_2, D \) and \( E \)), that \( k(0) \) and \( p(0) \) are independent of \( t_1 \). Similarly, relation (18) with \( k = 1 \) and \( i = 2 \) gives

\[
\begin{align*}
\partial_2 k(0) + \frac{p(0)}{2}(f(0) - \tilde{f}(0)) &= 0 \\
\partial_2 p(0) + 2d(1) &= 0 \\
p(0)(m - 2)t_2^{m-3} - 4d(1)t_2^{m-2} + k(0)(f(0) - \tilde{f}(0)) &= 0 \\
-e(1) + \frac{p(0)}{2}(f(0) + \tilde{f}(0)) &= 0.
\end{align*}
\]

If, by absurd, \( f(0) \neq \tilde{f}(0) \), then there is \( r \leq m - 4 \) such that the coefficients of order \( r \) in the expansion of \( f(0), \tilde{f}(0) \in \mathbb{C}\{t_2\} \) are different: \( f(0)_r \neq \tilde{f}(0)_r \). But then the coefficient of \( t_2^r \) in the third relation (75) is \( k(0)_0(f(0)_r - \tilde{f}(0)_r) \neq 0 \) (because \( k(0)_0 \neq 0 \), \( T(0) \) being invertible at \( t_2 = 0 \)). We obtain a
contradiction. We deduce that \( f(0) = \tilde{f}(0) \). The third relation (75) combined with \( d(1) = -\frac{1}{2}p(0) \) implies \( p(0) = 0 \) (from Lemma 10). We obtain \( p(0) = d(1) = e(1) = 0 \). The higher order relations (18) with \( k \geq 2 \) show, by an induction argument, that \( f(n) = \tilde{f}(n) \) for any \( n \geq 1 \) and \( p(i) = d(i) = e(i) = 0 \) for any \( i \).

4 \((T)\)-structures over \( \mathcal{N}_2 \)

In this section we classify \((T)\)-structures over \( \mathcal{N}_2 \) up to formal isomorphisms. The first three steps of the proof of Theorem 15 hold also for \((T)\)-structures over \( \mathcal{N}_2 \). The argument is the same, we only need to redefine the matrix \( C_2 \) to have only one non-zero entry, namely the one on the \((2, 1)\)-position, equal to one, and to ignore in relations (63) and (64) the terms which contain \( t_{2}^{m-2} \). Along this section we use this new definition of \( C_2 \) (but the definition of the matrices \( C_1, D \) and \( E \) remains the same). Therefore, without loss of generality we may assume that our \((T)\)-structures have connection form with matrices

\[
A_1 = C_1, \quad A_2 = C_2 + zfE, \tag{76}
\]

where \( f = f(z, t_2) \in \mathbb{C}\{z, t_2\} \). The function \( f \) is said to be associated to \( \nabla \).

**Lemma 18.** Consider a \((T)\)-structure \( \nabla \) with associated function \( f \). If \( f(0) \neq 0 \), then \( r := \text{ord}_0(f(0)) \) is a formal invariant of \( \nabla \). If \( f(0) = 0 \), then any \((T)\)-structure formally isomorphic to \( \nabla \) has this property.

**Proof.** Let \( \tilde{\nabla} \) be a \((T)\)-structure with connection form (with respect to another coordinate system \((\tilde{t}_1, \tilde{t}_2) \) of \( \mathcal{N}_2 \)), given by

\[
\tilde{A}_1 = C_1, \quad \tilde{A}_2 = C_2 + z\tilde{f}E,
\]

where \( \tilde{f} = \tilde{f}(z, \tilde{t}_2) \). Assume that there is a formal isomorphism \( T \) between \( \nabla \) and \( \tilde{\nabla} \), which covers a map \( h : \mathcal{N}_2 \to \mathcal{N}_2 \). As in Section 2.1, we denote by \( \tilde{T} \) the composition \( \tilde{T} := T \circ h \). Let \((t_1, t_2) \to (t_1, \lambda(t_2)) \) be the representation of \( h \) in the two coordinate systems of \( \mathcal{N}_2 \). We write \( \tilde{T} = \sum_{n \geq 0} \tilde{T}(n)z^n \), where

\[
\tilde{T}(n) = \begin{pmatrix} a(n) & e(n) \\ p(n) & b(n) \end{pmatrix}, \tag{77}
\]

where the coefficients \( a(n), e(n), p(n) \) and \( d(n) \) are independent of \( z \). Relation (14) with \( i = 1 \) gives that \( T(n) \) is independent on \( t_1 \), for any \( n \). Relation (14) with \( i = 2 \) and \( k = 0 \) gives

\[
e(0) = 0, \quad b(0) = \lambda a(0). \tag{78}
\]
Relation (11) with $i = 2$ and $k = 1$ gives
\[
e(1) = \partial_2 a(0) + \dot{\lambda}(\tilde{f}(0) \circ \lambda)p(0)
\]
\[
b(1) = \partial_2 p(0) + \dot{\lambda}a(0)
\]
\[
f(0)p(0) = \partial_2 b(0) + \dot{\lambda}e(1)
\]
\[
f(0)a(0) = \dot{\lambda}(\tilde{f}(0) \circ \lambda)b(0).
\] (79)

The last relation (79) and $a(0)b(0)$ non-vanishing at $t_2 = 0$ (because $T(0)$ is invertible at $t_2 = 0$ and $e(0) = 0$), together with $\lambda(0) = 0$, imply our claim. □

The formal classification of $(T)$-structures with $f(0) = 0$ is particularly easy:

**Lemma 19.** Consider a $(T)$-structure $\nabla$ with associated function $f$, such that $f(0) = 0$. Then $\nabla$ is formally isomorphic to the $(T)$-structure $\tilde{\nabla}$ with associated function $\tilde{f} = 0$.

**Proof.** We construct a formal isomorphism $T$ between $\nabla$ and $\tilde{\nabla}$. Suppose that $T$ covers a map $h(t_1, t_2) = (t_1, \lambda(t_2))$. As in the proof of Lemma 18 we define $\tilde{T} = T \circ h = \sum_{n \geq 0} T(n)z^n$ with $T(n)$ of the form (77), independent on $t_1$. We will determine the function $\lambda$ and then the coefficients $a(n), e(n), p(n)$ and $b(n)$ inductively as follows. Recall, from (78), that $e(0) = 0$ and $\dot{\lambda} = \frac{b(0)}{a(0)}$. Relations (12) with $f(0) = \tilde{f}(0) = 0$ give
\[
e(1) = \partial_2 a(0)
\]
\[
b(1) = \partial_2 p(0) + \dot{\lambda}a(1)
\]
\[
\partial_2 b(0) + \dot{\lambda}e(1) = 0.
\] (80)

As $\dot{\lambda} = \frac{b(0)}{a(0)}$, the first and third relation (80) implies that $a(0)b(0)$ is constant. We continue the argument from Lemma 18 and we identify the coefficient of $z^2$ in (11) (applied to $\nabla$, $\tilde{\nabla}$ and $\tilde{T}$). It gives
\[
e(2) = \partial_2 a(1)
\]
\[
f(1)a(0) = \partial_2 e(1)
\]
\[
b(2) = \partial_2 p(1) + \dot{\lambda}a(2)
\]
\[
f(1)p(0) = \partial_2 b(1) + \dot{\lambda}e(2).
\] (81)

Now, the second relation (81) and the first relation (80) imply that
\[
\partial_2^2 a(0) = f(1)a(0).
\] (82)
Let \(a(0)\) be a solution of (82), with \(a(0)|_{t_2=0} \neq 0\), and \(e(1) := \partial_2 a(0)\). With these choices of \(a(0)\) and \(e(1)\), the first relation (80) and the second relation (81) are satisfied. Choose \(b(0)\) such that \(a(0)b(0)\) is constant (different from zero) and \(\lambda \) such that \(\dot{\lambda} = \frac{b(0)}{a(0)}\) and \(\lambda(0) = 0\). Then also the third relation (80) is satisfied. To summarize: identifying the coefficients of \(z^0, z^1\) and \(z^2\) in (14) we determined the functions \(\lambda, a(0), b(0), e(1)\) and we remained with the relations:

\[
\begin{align*}
  b(i) &= \partial_2 p(i - 1) + \dot{\lambda} a(i), \quad i = 1, 2; \\
  e(2) &= \partial_2 a(1), \\
  f(1)p(0) &= \partial_2 b(1) + \dot{\lambda} e(2).
\end{align*}
\]

Suppose that we determined \(a(i)\) (with \(i \leq n - 2\)), \(b(i)\) (with \(i \leq n - 2\)), \(p(i)\) (with \(i \leq n - 3\)) and \(e(i)\) (with \(i \leq n - 1\)). The coefficients of \(z^i\) (with \(i \leq n\)) in (14) give the relations:

\[
\begin{align*}
  e(i) &= \partial_2 a(i - 1), \quad i \leq n \\
  \partial_2 e(i) &= \sum_{r=1}^{i} f(r)a(i - r), \quad i \leq n - 1 \\
  b(i) &= \partial_2 p(i - 1) + \dot{\lambda} a(i), \quad i \leq n \\
  \sum_{r=1}^{i} f(r)p(i - r) &= \partial_2 b(i) + \dot{\lambda} e(i + 1), \quad i \leq n - 1.
\end{align*}
\]  

We now determine \(a(n-1), b(n-1), p(n-2)\) and \(e(n)\) such that the coefficient of \(z^{n+1}\) in (14) vanishes. The coefficient of \(z^{n+1}\) in relation (14) vanishes if and only if

\[
\begin{align*}
  e(n + 1) &= \partial_2 a(n) \\
  \partial_2 e(n) &= \sum_{r=1}^{n} f(r)a(n - r) \\
  b(n + 1) &= \partial_2 p(n) + \dot{\lambda} a(n + 1) \\
  \sum_{r=1}^{n} f(r)p(n - r) &= \partial_2 b(n) + \dot{\lambda} e(n + 1).
\end{align*}
\]  

The second relation (81) and the first relation (83) with \(i = n\) give

\[
\begin{align*}
  \partial_2^2 a(n - 1) &= \sum_{r=1}^{n} f(r)a(n - r).
\end{align*}
\]  

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As $a(i)$ are known for $i \leq n - 2$, (85) can be considered as a differential equation in the unknown function $a(n - 1)$. Choose $a(n - 1)$ to be a solution of this equation. From the first relation (83) with $i = n$, we obtain $e(n)$. Consider now the fourth relation (83) with $i = n - 1$:

$$\sum_{r=1}^{n-1} f(r)p(n-1-r) = \partial_2 b(n-1) + \dot{\lambda}e(n).$$

Replacing in it $b(n-1)$ with its expression given by the third equality (83) with $i = n - 1$, we obtain a differential equation with unknown function $p(n-2)$ (whose coefficients are known):

$$\sum_{r=1}^{n-1} f(r)p(n-1-r) = \partial_2 \left( \partial_2 p(n-2) + \dot{\lambda}a(n-1) \right) + \ddot{\lambda}e(n).$$

Choose $p(n-2)$ to be a solution of this equation. We finally determine $b(n-1)$ from the third relation (83) with $i = n - 1$.

It remains to classify the $(T)$-structures for which $f(0)$ is not identically zero. A first simplification is given in the next lemma.

**Lemma 20.** Let $\nabla$ be a $(T)$-structure over $\mathcal{N}_2$, with associated function $f$ such that $f(0) \neq 0$. Let $r := \text{ord}_0 f(0)$. Then $\nabla$ is formally (even holomorphically) isomorphic to a $(T)$-structure of the same form, with another associated function $\tilde{f}$ which satisfies $\tilde{f}(0) = t^r$.

**Proof.** From Lemma 14, we can find $\lambda \in \mathbb{C}\{t_2\}$, with $\lambda(0) = 0$ and $\dot{\lambda}(0) \neq 0$, such that $(\dot{\lambda})^2 \lambda^r = f(0)$. Consider the isomorphism $T := \text{diag}(1, \dot{\lambda} \circ \lambda^{-1})$ which covers the map $h(t_1, t_2) = (t_1, \lambda(t_2))$. Using relation (14) with $\tilde{T} = T \circ h = \text{diag}(1, \dot{\lambda})$, we obtain that $T$ maps $\nabla$ to a new $(T)$-structure, let us denote it also by $\nabla$, which is given by $A_1 = C_1$ and

$$A_2 = C_2 - \frac{z\dot{\lambda} \circ \lambda^{-1}}{2(\lambda \circ \lambda^{-1})^2}(C_1 - D) + \frac{zf \circ \lambda^{-1}}{(\lambda \circ \lambda^{-1})^2}E.$$

In particular,

$$A_2(1) = -\frac{\ddot{\lambda} \circ \lambda^{-1}}{2(\lambda \circ \lambda^{-1})^2}(C_1 - D) + t^r_2 E$$

(here we used the differential equation satisfied by $\lambda$) and $\nabla$ is of the form (57) from Step 2 of the proof of Theorem 15, with $a_4(0) = t^r_2$. Our claim follows by applying to $\nabla$ the arguments from Step 2 and Step 3 from this proof and by noticing that they leave $a_4(0)$ unchanged: we already remarked
that Step 2 leaves the coefficients of $C_2$, $D$ and $E$ in $A_2$ unchanged (see the end of Step 2, in the proof of Theorem 47). As for Step 3, the coefficient of $E$ in relation (64) (with terms containing $t^{m-2}$ ignored) gives

$$a_4 - \tilde{a}_4 + z(\partial_2\tau_4 - (a_4 + \tilde{a}_4)\tau_3 + a_3\tau_4 - \tilde{a}_1\tau_4) = 0,$$

which implies $a_4(0) = \tilde{a}_4(0)$. □

In the next lemma we simplify further the $(T)$-structures associated function $f$, such that $f(0) = t^2_r$.

**Lemma 21.** Let $\nabla$ be a $(T)$-structure with associated function $f$.

i) The function $f(0)$ is a formal gauge invariant of $\nabla$.

ii) Assume that $f(0) = t^2_r$. If $r = 0$ or $r = 1$, then $\nabla$ is formally gauge isomorphic to a $(T)$-structure with associated function $\tilde{f}(z, t^2) = t^2_r$. If $r \geq 2$ then $\nabla$ is formally gauge isomorphic to a $(T)$-structure with associated function $\tilde{f}(z, t^2) = t^2_r + \sum_{k \geq 1} \tilde{f}(k)z^k \in \mathbb{C}[t^2] [[z]]$, such that $\tilde{f}(k) \in \mathbb{C}[t^2]_{r-2}$, for any $k \geq 1$ (and $\tilde{f}(0) = t^2_r$).

**Proof.** Let $\nabla$ and $\tilde{\nabla}$ be two $(T)$-structures with associated functions $f$ and $\tilde{f}$ respectively. The coefficients $T(n)$ of a formal gauge isomorphism $T = \sum_{n \geq 0} T(n)z^n$ between $\nabla$ and $\tilde{\nabla}$ (if it exists), must satisfy relations (18) with $A_1 = C_1$, $A_2 = C_2 + zfE$ and $\tilde{A}_1 = C_1$, $\tilde{A}_2 = C_2 + z\tilde{f}E$. For any $n \geq 0$, we write $T(n)$ in the form (74). Relation (18) for $k = 0$, $i = 1$ is satisfied and for $k = 0$, $i = 2$ it gives $d(0) = e(0) = 0$. As $T(0)$ is invertible, $k(0)$ is a unit. Relation (18) for $k = 1$, $i = 1$ gives $\partial_1T(0) = 0$, i.e. $k(0)$ and $p(0)$ are independent of $t_1$. For $k = 1$ and $i = 2$ it gives $k(0)$ constant and

$$k(0)(f(0) - \tilde{f}(0)) = 0, \quad d(1) = -\frac{1}{2}\partial_2p(0), \quad e(1) = f(0)p(0). \quad (86)$$

Since $k(0)$ is a unit, we obtain from the first relation in (86) that $f(0)$ is a formal gauge invariant, as needed. Claim i) follows.

We now prove claim ii). In general, relation (18) for $k = n + 1$ ($n \geq 1$)
implies that $k(n), p(n), d(n)$ and $e(n)$ depend only on $t_2$ and

$$\partial_2k(n) + \frac{p(0)}{2}(f(n) - \tilde{f}(n)) + \sum_{l=2}^{n} \frac{p(n + 1 - l)}{2}(f(l - 1) - \tilde{f}(l - 1)) = 0;$$

$$d(n + 1) = -\frac{1}{2}\partial_2p(n);$$

$$e(n + 1) = \partial_2d(n) + \frac{p(0)}{2}(f(n) + \tilde{f}(n)) + \sum_{l=1}^{n} \frac{p(n + 1 - l)}{2}(f(l - 1) + \tilde{f}(l - 1));$$

$$\partial_2e(n) + k(0)(f(n) - \tilde{f}(n)) + \sum_{l=1}^{n} k(n + 1 - l)(f(l - 1) - \tilde{f}(l - 1))$$

$$- \sum_{l=1}^{n} d(n + 1 - l)(f(l - 1) + \tilde{f}(l - 1))) = 0. \quad (87)$$

Assume that $f(0) = t_2^r$. For $n = 1$, the last relation (87) combined with the second and third relations (86) and $f(0) = \tilde{f}(0)$ give

$$\partial_2(t_2^r p(0)) + t_2^r \partial_2 p(0) = k(0)(\tilde{f}(1) - f(1)). \quad (88)$$

From Lemma 10, if $r = 0$ or $r = 1$, equation (88), in the unknown function $p(0)$, has a solution, for any $\tilde{f}(1)$ (in particular, also for $\tilde{f}(1) = 0$). If $r \geq 2$, then (88) has a solution if and only if $\tilde{f}(1) - f(1)$ has a zero of order at least $r - 1$ (again, from Lemma 10). In particular, when $\tilde{f}(1)$ is the polynomial $\tilde{f}(1) := [f(1)]_{r-2}$, it has a solution. We will show that the same phenomena appears at the higher order relations and allows to construct a formal gauge isomorphism between the given $(T)$-structure $\nabla$ and a $(T)$-structure $\tilde{\nabla}$ as in the statement of the lemma. For this, we need to study more carefully relations (87). Since $d(n) = -\frac{1}{2}\partial_2 p(n - 1)$ and $f(0) = \tilde{f}(0)$, the third relation (87) is equivalent to

$$e(n + 1) = -\frac{1}{2}\partial_2^2 p(n - 1) + \frac{p(0)}{2}(f(n) + \tilde{f}(n)) + p(n)f(0)$$

$$+ \sum_{l=2}^{n} \frac{p(n + 1 - l)}{2}(f(l - 1) + \tilde{f}(l - 1)). \quad (89)$$

Consider relation (89) with $n$ replaced by $n - 1$. Using the expression of $e(n)$ given by this relation, $d(n) = -\frac{1}{2}\partial_2 p(n - 1)$ and $f(0) = \tilde{f}(0) = t_2^r$, we write
the last relation (87) in the following form:

\[
\partial_2(t_2^2 p(n-1)) + t_2^2 \partial_2 p(n-1) + k(0)(f(n) - \hat{f}(n))
\]

\[
\partial_2 \left( \frac{1}{2} \partial_2^2 p(n-2) + \frac{p(0)}{2} (f(n-1) + \hat{f}(n-1)) + \sum_{l=2}^{n-1} \frac{p(n-l)}{2} (f(l-1) + \hat{f}(l-1)) \right)
\]

\[
+ \sum_{l=2}^{n} k(n+1-l)(f(l-1) - \hat{f}(l-1))
\]

\[
+ \frac{1}{2} \sum_{l=2}^{n} \partial_2 p(n-l)(f(l-1) + \hat{f}(l-1)) = 0.
\] (90)

Let \( k(0) \in \mathbb{C} \setminus \{0\} \) be arbitrary and \( p(0) \) a solution of (88), with \( \hat{f}(1) = 0 \) when \( r \leq 1 \) and \( \hat{f}(1) = f(1) \leq r - 2 \) when \( r \geq 2 \). Suppose that \( p(r) \) (for any \( r \leq n-2 \)) and \( \hat{f}(r) \) (for \( r \leq n-1 \)) are known. Define \( k(r) \) (1 \( \leq r \leq n-1 \)) (up to constants) such that the first equalities (87) are satisfied and \( d(r), e(r) \) (0 \( \leq r \leq n-1 \)) using the second and third equalities (87). Since we know \( p(r) \) (\( r \leq n-2 \)), \( k(r) \) (\( r \leq n-1 \)) and \( \hat{f}(r) \) (\( r \leq n-1 \)), the last three lines in (90) are known and we may consider (90) as an equation in the unknown function \( p(n-1) \), with suitably chosen function \( \hat{f}(n) \), as follows. From Lemma 10 if \( r = 0 \) or \( r = 1 \) this equation with \( \hat{f}(n) = 0 \) has a solution. When \( r \geq 2 \), there is a unique polynomial \( \hat{f}(n) \) of degree at most \( r-2 \), such that (90) has a solution. Let us choose \( \hat{f}(n) \) in this way and \( p(n-1) \) a solution of (90). Repeating the argument we obtain inductively the required formal isomorphism \( T \).

The formal classification of \((T)\)-structures over \( \mathcal{N}_2 \) is stated as follows.

**Theorem 22.**

i) Any formal \((T)\)-structure over \( \mathcal{N}_2 \) is formally isomorphic to a \((T)\)-structure of the form

\[
A_1 = C_1, \; A_2 = C_2 + zE
\]

\[
A_1 = C_1, \; A_2 = C_2 + zt_2 E
\]

\[
A_1 = C_1, \; A_2 = C_2 + z(t_2' + \sum_{k \geq 1} P_k(t_2) z^k) E, \; (r \in \mathbb{Z}_{\geq 2})
\]

\[
A_1 = C_1, \; A_2 = C_2,
\] (91)

where \( P_k \) are polynomials of degree at most \( r-2 \).

ii) Any two (different) \((T)\)-structures \( \nabla \) and \( \tilde{\nabla} \) from the list (91), at least one of them not belonging to the third line, are formally non-isomorphic. If \( \nabla \) and \( \tilde{\nabla} \) belong to the third line, then they are formally gauge non-isomorphic.
They are formally isomorphic if and only if there exists \( \lambda_0 \in \mathbb{C} \), \( \lambda_0^r = 1 \), such that \( f(k)(t_2) = \lambda_0^{-2} f(k)(\frac{t_2}{\lambda_0}) \) (for any \( k \geq 1 \)) and in this case the formal isomorphism between \( \nabla \) and \( \tilde{\nabla} \) covers the map \( (t_1, t_2) \rightarrow (t_1, \lambda_0 t_2) \).

Proof. All claims, except those which refer to the \((T)\)-structures from the third line, follow from the previous lemmas. The fact that no two different \((T)\)-structures from the third line in the list are formally gauge isomorphic was done implicitly in the argument from Lemma 21 and reduces, again, to Lemma 10. More precisely, if \( \nabla \) and \( \tilde{\nabla} \) are two gauge isomorphic \((T)\)-structures, from the third line from the list, with associated functions \( f \) and \( \tilde{f} \), then \( f(1), \tilde{f}(1) \) are polynomials of degree at most \( r-2 \) and (88) has a solution.

We obtain that \( f(1) = \tilde{f}(1) \) (from Lemma 10) and by induction we conclude that \( f = \tilde{f} \). Assume now that \( \nabla \) and \( \tilde{\nabla} \) are formally isomorphic (but not formally gauge isomorphic). Let \( T \) be a formal isomorphism between them. It covers a map of the form \( h(t_1, t_2) = (t_1, \lambda(t_2)) \), with \( \lambda(0) = 0 \) and \( \dot{\lambda}(0) \neq 0 \). From relation (12), \( A_2(0) = \tilde{A}_2(0) = C_2 \) and \( f(0) = \tilde{f}(0) = t_2^r \), we deduce that \( \lambda \) satisfies \( (\lambda)^2 X = t_2^r \), i.e. \( \lambda(t_2) = \lambda_0 t_2 \), where \( \lambda_0 = 1 \) and \( \lambda_0 \neq 1 \) (from Lemma 11). Consider now the isomorphism \( T_1 \) which covers \( h \) and is determined by the constant matrix \( \tilde{T}_1 := \text{diag}(1, \lambda_0) \). Then \( \nabla^{(1)} := T_1 \cdot \nabla \) is a \((T)\)-structure with \( A_1^{(1)} = C_1, A_2^{(1)} = C_2 + z f^{(1)} E \), where \( f^{(1)}(0) = t_2^r \) and \( f^{(1)}(k)(t_2) = \lambda_0^{-2} f(k)(\frac{t_2}{\lambda_0}) \), for any \( k \geq 1 \). Moreover, \( T \circ T_1^{-1} \) is a formal gauge isomorphism between \( \nabla^{(1)} \) and \( \tilde{\nabla} \). We deduce that \( \nabla^{(1)} \) coincides with \( \nabla \) (from our previous argument) which concludes the proof.

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