Edge waves in an initially stressed visco-elastic plate

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Abstract. The present study analytically investigates the propagation of edge waves in a pre-stressed homogeneous visco-elastic plate. The plate is initially stressed in horizontal direction only. The solution for potential function and displacement components of the edge wave propagation have been obtained separately. Dispersion equation of edge wave has been established in closed form. The horizontal tensile and compressive initial stress parameters give the remarkable effect on the phase velocity of edge wave propagation.

1. Introduction
Viscoelastic materials are considered in the broader context of physical properties. Elastic solids support both shear and hydrostatic stresses and their properties are independent of time or frequency, whereas viscoelastic solids exhibit time and frequency dependence. Isotropic elastic solids are describable by two elastic constants, (viz. shear and bulk modulus). By contrast, viscoelastic materials require a function of time or frequency to describe the behavior. Therefore, a rich set of physical phenomena can occur. For an elastic wave in a body, the velocity of propagation at an edge differs from that at a point far from it. This is due to reflection and refraction of a wave at the edge. The waves propagating at the edge of a plate of finite thickness are referred as Edge wave. The stresses, which exist in an elastic body even though external forces are absent, are termed as initial stresses and the body is said to be initially stressed. Biot [1] mentioned that an initial stress has remarkable effect on the propagation of elastic waves in body. He showed that the propagation of waves in an initially stressed body is fundamentally different from the classical theory of elasticity to a great extent. Ewing et al. [2] discussed that manufacturing process, body forces and unequal rates of heating and cooling, creep and inelastic deformation cause enormous initial stresses in a body. Love [3] predicted that the earth is in a state of high initial stress. For example, due to the atmospheric pressure, gravity, difference in temperature, the large amount of initial stresses may exist inside the earth, so the earth is considered to be an initially stressed. These stresses might exert significant influence on the elastic waves produced by earthquakes, explorations or impacts. Thus, it is imperative to deal with the properties of wave propagation under initial stress. It was Biot [4] who first pointed out that the initial stress influences elastic waves to a great extent. Being motivated by these facts, we have also taken initially stressed plate into account. Also the propagation of edge waves in plates was studied by Dey [5] and Kumar [6]. Dey and De [7] have studied the edge wave propagation in an incompressible anisotropic initially stressed plate of finite thickness. Gupta et al. [8] have deduced the propagation of S-waves in a non-homogeneous anisotropic incompressible and initially stressed medium. Chattopadhyay et al. [9] have studied the propagation of shear waves...
in visco-elastic medium at irregular boundaries. Singh et al.[10] discussed the smooth moving punch in an initially stressed transversely isotropic magneto-elastic medium due to shear wave. To study the effect of viscoelasticity in wave propagation, some attempts have been done earlier but no attempts have been made to study edge wave propagation in a viscoelastic medium. In this paper we investigate the propagation of edge waves in a homogeneous pre-stressed viscoelastic plate of finite thickness. Dispersion equation of edge wave has been obtained in closed form. It has been observed that compressive initial stress and tensile initial stress have their substantial effect on phase velocity of edge wave.

2. Formulation and solution of the problem

To formulate the edge wave propagation in an initially stressed uniformly homogeneous viscoelastic plate of thickness \( H \) and infinite length, we consider \( x \)-axis is along the direction of wave propagation and \( y \)-axis is positive pointing vertically upward. The plate is horizontally initially stressed \( S_{11} \) along the \( x \)-direction respectively.

The dynamical equations of motion for the propagation of edge wave are

\[
(\lambda + 2\mu) \frac{\partial^2 u}{\partial x^2} + \left( \mu + \frac{P}{2} \right) \frac{\partial^2 u}{\partial y^2} + \left( \lambda + \mu - \frac{P}{2} \right) \frac{\partial^2 v}{\partial x \partial y} = \rho \frac{\partial^2 u}{\partial t^2},
\]

and

\[
\left( \mu - \frac{P}{2} \right) \frac{\partial^2 v}{\partial x^2} + \left( \lambda + 2\mu \right) \frac{\partial^2 v}{\partial y^2} + \left( \lambda + \mu + \frac{P}{2} \right) \frac{\partial^2 u}{\partial x \partial y} = \rho \frac{\partial^2 v}{\partial t^2},
\]

where \( \mu = \mu + \mu' \frac{\partial}{\partial t} \), \( \mu \) and \( \mu' \) are stiffness of the material due to elastic and viscoelastic properties. \( P (= -S_{11}) \) is the horizontal initial stress along \( x \)-direction, \( u \) and \( v \) are the displacement components along \( x \) and \( y \) directions respectively. The condition of incompressibility is

\[
e_{xx} + e_{yy} = 0,
\]

where \( e_{xx} = \frac{\partial u}{\partial x} \) and \( e_{yy} = \frac{\partial u}{\partial y} \) are the incremental normal strain tensors. We introduce potential function \( \varphi(x,y,t) \) as follows:

\[
u = -\frac{\partial \varphi(x,y,t)}{\partial y} \quad \text{and} \quad v = \frac{\partial \varphi(x,y,t)}{\partial x},
\]
where $\varphi(x, y, t)$ is a differentiable function.
Using eqns. (1), (2) and (3), we get
\[
(1 - \tilde{\xi}) \frac{\partial^4 \varphi}{\partial x^4} + 2 \frac{\partial^4 \varphi}{\partial x^2 \partial y^2} + (1 + \tilde{\xi}) \frac{\partial^4 \varphi}{\partial y^4} = \frac{1}{\beta^2} \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2},
\]
where $\tilde{\xi} = \frac{P}{2\mu}$ and $\beta^2 = \frac{\mu}{\rho}$.

We have considered the solution of eq. (5) as
\[
\varphi(x, y, t) = \frac{1}{l^2} f(ly) \exp\{i(lx - \alpha t)\},
\]
where $\alpha (= lc)$ and $l$ are frequency of oscillation and wave number.

With the help of eqns. (5) and (6), we obtain the potential function $\varphi(x, y, t)$ as
\[
\varphi(x, y, t) = \frac{1}{l^2} \left\{ c_1 \cosh(\beta_1 ly) + c_2 \cosh(\beta_2 ly) \right\} \sin(lx - \alpha t),
\]
where $c_1$ and $c_2$ are arbitrary constants,
\[
\beta_1^2 = R + \sqrt{R^2 - S^2}, \quad \beta_2^2 = R - \sqrt{R^2 - S^2}, \quad S^2 = \frac{1 - \tilde{\xi}}{1 + \tilde{\xi}} - \frac{\rho \alpha^2}{\mu^2(1 + \tilde{\xi})}, \quad \text{and} \quad R = \frac{1}{1 + \xi} - \frac{\rho \alpha^2}{2\mu^2(1 + \xi)},
\]
with the help of eqns. (4) and (7), we get the displacement components as
\[
u = \frac{1}{l} \left\{ c_1 \beta_1 \sinh(\beta_1 ly) + c_2 \beta_2 \sinh(\beta_2 ly) \right\} \cos(lx - \alpha t),
\]
and
\[
v = \frac{1}{l} \left\{ c_1 \cosh(\beta_1 ly) + c_2 \cosh(\beta_2 ly) \right\} \cos(lx - \alpha t).
\]

2.1. Boundary conditions and dispersion equation
The bounding planes $y = \pm \frac{H}{2}$ are supposed to be free from tractions, hence the boundary conditions are
\[
\begin{align*}
\Delta f_x &= 0, \\
\Delta f_y &= 0,
\end{align*}
\]
at $y = \pm \frac{H}{2},$
where
\[
\Delta f_x = \tilde{\mu} (1 + \tilde{\xi}) \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right),
\]
and
\[
\Delta f_y = \lambda \frac{\partial u}{\partial x} + (\lambda + 2\tilde{\mu}) \frac{\partial v}{\partial y},
\]
are incremental forces per unit initial area at the boundary.

With the help of eqns. (8), (9) and (10) we get the following two equations as
\[
c_1 \left(1 + \beta_1^2\right) \cosh(\Lambda \beta_1) + c_2 \left(1 + \beta_2^2\right) \cosh(\Lambda \beta_2) = 0,
\]
and
\[
c_1 \beta_1 \sinh(\Lambda \beta_1) + c_2 \beta_2 \sinh(\Lambda \beta_2) = 0,
\]
where $\Lambda (= lH/2)$ is the wavelength.

Eliminating arbitrary constants $c_1$ and $c_2$ from eqns. (13) and (14), we get
\[
\beta_2 \left(1 + \beta_2^2\right) \tanh(\Lambda \beta_2) - \beta_1 \left(1 + \beta_2^2\right) \tanh(\Lambda \beta_1) = 0.
\]
This is the dispersion equation of edge wave.
3. Numerical examples and discussion
For the numerical computation of phase velocity of edge wave propagation at the edge of a pre-stressed viscoelastic plate, the following data have been considered [11]:
\[ \mu = 6.77 \times 10^{10} \text{N/m}^2, \rho = 3323 \text{kg/m}^3. \]
Moreover, we consider following data for computation
\[ P/2\mu = -0.2, -0.1, 0.0, 0.2, 0.1; \alpha \mu'/\mu = 0.05. \]

![Figure 2. Variation in dimensionless phase velocity (c/\beta) against dimensionless wave number (1H) for different values of horizontal tensile initial stress when \( \alpha \mu'/\mu = 0.05. \)](image)

![Figure 3. Variation in dimensionless phase velocity (c/\beta) against dimensionless wave number (1H) for different values of horizontal compressive initial stress when \( \alpha \mu'/\mu = 0.05. \)](image)

The dispersion equation for the propagation of edge wave in a visco-elastic plate of finite width under horizontal initial stress is obtained. It relates the dimensionless phase velocity (\( c/\beta \)) with dimensionless wave number (\( lH \)). The variations of dimensionless phase velocity against dimensionless wave number for different values of initial stress parameters have been shown in Figures 2 and 3. Figure 2 gives the effect of horizontal tensile initial stress parameter on the dispersion curve when the values of tensile initial stress parameter are taken -0.2, -0.1, 0.0. However, Figure 3 shows the curves for different values of horizontal compressive initial stress parameter and the values are taken as 0.0, 0.1, 0.2. In both the cases viscosity parameter \( \alpha \mu'/\mu \) kept fixed at 0.05. It is clear from these figures that the dimensionless phase velocity increases with dimensionless wave number when the values of dimensionless horizontal initial stress (tensile and compressive) parameters increase.

4. Conclusion
The propagation of edge wave at the edge of a pre-stressed homogeneous viscoelastic plate has been studied. The dispersion equation for the propagation of edge wave has been obtained in closed form. The dispersion equation relates the phase velocity with wave number. The tensile and compressive initial stress parameters have a significant favoring effect on phase velocity of edge wave. Also it has been observe that the phase velocity increases with wave number when the initial stress parameters increase.
Acknowledgments
The authors convey their sincere thanks to Indian School of Mines, Dhanbad for providing JRF to Ms. Nirmala Kumari and also facilitating us with its best facility.

References
[1] Biot M A 1965 Mechanics of incremental deformations. (New York: John Wiley and Sons Inc)
[2] Ewing W M, Jardetzky W S and Press F 1957 Elastic waves in layered media. (New York: McGraw Hill Book Company Inc)
[3] Love A E H 1944 A treatise on the mathematical theory of elasticity. (New York: Fourth ed. Dover)
[4] Biot M A 1940 The influence of initial stress on elastic waves. J. Appl. Phys. 11(8) 522-530
[5] Das S C and Dey S 1998 Edge wave under initial stress. Appl. Sci. Res. 22 382-389
[6] Kumar S 1959 Edge waves in plates. Int. Symposium on Stress Wave Propagation in Materials. (Pennsylvania State University (Penn.) U.S.A)
[7] Dey S and De P K 2009 Edge wave propagation in an incompressible anisotropic initially stressed plate of finite thickness. Int. J. Comp. Cong. 7(3) 55-60
[8] Gupta S, Kundu S, Verma A K and Verma R 2010 Propagation of S-waves in a non-homogeneous anisotropic incompressible and initially stressed medium. Int. J. Eng., Sci. Technol. 2(2) 31-42
[9] Chattopadhyay A, Gupta S, Sharma V K and Kumari P 2009 Propagation of shear waves in visco-elastic medium at irregular boundaries. Acta Geophys. 58(2) 195-214
[10] Singh A K, Kumari N, Chattopadhyay A and Sahu S A 2015 Smooth moving punch in an initially stressed transversely isotropic magnetoelastic medium due to shear wave. Mech. Adv. Mater. Struct. DOI :10.1080/15376494.2015.1029161
[11] Gubbins D 1990 Seismology and plate tectonics. (London: Cambridge University Press)