Localizatıon of Multi-Dimensional Wigner Distributions

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Abstract

A well known result of P. Flandrin states that a Gaussian uniquely maximizes the integral of the Wigner distribution over every centered disc in the phase plane. While there is no difficulty in generalizing this result to higher-dimensional poly-discs, the generalization to balls is less obvious. In this note we provide such a generalization.

1 Introduction

The Wigner quasi-probability distribution was introduced by Wigner [16] in 1932 in order to study quantum corrections to classical statistical mechanics. Nowadays it lies at the core of the phase-space formulation of quantum mechanics (Weyl correspondence), and has a variety of applications in statistical mechanics, quantum optics, and signal analysis, to name a few. In this note we consider the localization problem of the $n$-particle Wigner distribution in the $2n$-dimensional phase space. We state our results precisely in Theorem 1 below.

Equip the classical phase space $\mathbb{R}^{2n}$ with coordinates $(x,y)$ with $x, y \in \mathbb{R}^n$. The Wigner quasi-probability distribution on $\mathbb{R}^{2n}$, associated with a wave function $\psi \in L^2(\mathbb{R}^n)$ and its complex conjugate $\psi^*$, is defined by

$$W_\psi(x,y) = (2\pi)^{-n} \int_{\mathbb{R}^n} \psi(x+\tau/2)\psi^*(x-\tau/2)e^{-i\tau \cdot y} d\tau$$

The function $W_\psi$ possesses many of the properties of a phase space probability distribution (see e.g., [4]); in particular, it is real. However, $W_\psi$ is not a genuine probability distribution as it can assume negative values.

The localization problem, i.e., estimating the integral of the Wigner distribution over a subregion of the phase space, and the closely related problem of the optimal simultaneous concentration of $\psi$ and its Fourier transform $\hat{\psi}$, have received much attention in the literature both in quantum mechanics, mathematical time-frequency analysis, and signal analysis.

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processing (see e.g. [1, 2, 3, 4, 5, 6, 9, 10, 12, 13, 11], and the references within). Bounds on the $L^p$ norms were found in [7]. More precisely, the problem of interest for us is:

**The Wigner Distribution Localization Problem:** given a measurable set $D \subset \mathbb{R}^{2n}$, find the best possible bounds to the localization function

$$E(D) := \sup_{\psi} \int_D W_\psi \, dx dy,$$

where the supremum is taken over all the functions $\psi \in L^2(\mathbb{R}^n)$ with $\|\psi\|_2 = 1$.

The quantity $E(D)$ is invariant under translations in the phase space, and under the action of the group of linear symplectic transformations (see e.g. [15]). There is no upper bound on $E(D)$; it can be infinite. Indeed, there is a $\psi \in L^2(\mathbb{R}^n)$ such that $\int |W_\psi| \, dx dy = \infty$ [4, sect. 4.6]. An example is $\psi(x) = 1$ if $-\frac{1}{2} < x < \frac{1}{2}$ and $\psi(x) = 0$ otherwise. On the other hand, the $L^p$ norm of $W_\psi$ is bounded [7] for $p \geq 2$ and we can use this information to show that $E(D)$ is bounded by powers of the volume $|D|$. E.g., the $L^\infty$ norm is at most $\pi^{-n}$, so $E(D) \leq \pi^{-n} |D|$.

For certain $D$, however, $E(D)$ is not only finite, it is even less than 1. In [2], Flandrin conjectured this to be true for all convex domains, and he showed that for all centered two-dimensional discs $B^2(r)$ of radius $r$, the standard normalized Gaussian function $\pi^{-1/4} \exp(-x^2/2)$ is the unique maximizer of (1.2). In particular $E(B^2(r)) = 1 - e^{-r^2}$ (see [2], cf. [3]). It follows immediately from the definition of the Wigner distribution that Flandrin’s proof can be easily generalized to higher dimensional poly-discs because the maximization problem then has a simple product structure. A less obvious case is the $2n$-dimensional Euclidean ball $B^{2n}(r)$. The following is the generalization of Flandrin’s result, and our main result:

**Theorem 1.** The standard normalized Gaussian $\pi^{-n/4} \exp(-x^2/2)$ in $L^2(\mathbb{R}^n)$ is the unique maximizer of the Wigner distribution localization problem for any $2n$-dimensional Euclidean ball centered at the origin. In particular,

$$E\left(B^{2n}(r)\right) = \frac{1}{\pi^n} \int_{B^{2n}(r)} e^{-\sum_{i=1}^n (x_i^2 + y_i^2)} \, dx dy = 1 - \frac{\Gamma(n, r^2)}{(n-1)!},$$

where $\Gamma(s, x) = \int_x^\infty t^{s-1} e^{-t} \, dt$ is the upper incomplete gamma function.

**Remarks:** (1.) Owing to the translation covariance of the Wigner distribution, equation (1.3) also applies to a ball of radius $r$ centered anywhere in $\mathbb{R}^{2n}$. It is only necessary to multiply the Gaussian by an appropriate linear form $\exp(a \cdot x)$. Moreover, since the localization function (1.2) is invariant under the action of the group of linear symplectic transformations, Theorem 1 can also be adapted to any image of the Euclidean ball under linear symplectic maps.

(2.) Another generalization is to replace the integral over the ball with the integral over $\mathbb{R}^{2n}$, but with a weight that is a symmetric decreasing function (i.e., a radial and non-increasing function of the radius $\sqrt{x^2 + y^2}$). By the “layer cake representation” [8 sect. 1.13] the standard Gaussian again maximizes uniquely.
2 Proof of Theorem 1

We start with the following preliminaries. Recall that the mixed Wigner distribution of two states \( \psi_1, \psi_2 \in L^2(\mathbb{R}^n) \) is defined by

\[
W_{\psi_1,\psi_2}(x,y) = (2\pi)^{-n} \int_{\mathbb{R}^n} \psi_1(x+\tau/2)\psi_2^*(x-\tau/2)e^{-i\tau y} \, d\tau.
\]

(2.1)

Note that in contrast to (1.1), \( W_{\psi_1,\psi_2} \) is not generally real, but, nevertheless, Hermitian i.e., \( W_{\psi_1,\psi_2} = W_{\psi_2,\psi_1}^* \). Moreover, it is not hard to check that the mixed Wigner distribution is sesquilinear.

Next, let \( \mu = (\mu_1, \ldots, \mu_n) \) be a multi-index of non-negative integers, and let \( x \in \mathbb{R}^n \). The Hermite functions \( H_\mu(x) \) on \( \mathbb{R}^n \) are defined \([14, 15]\) to be the product of the normalized one-dimensional Hermite functions, i.e., \( H_\mu(x) = \prod_{j=1}^n h_{\mu_j}(x_j) \), where

\[
h_k(x) = \pi^{-\frac{1}{4}} (k!)^{-\frac{1}{2}} 2^{-\frac{k}{2}} (-1)^k e^{x^2/2} \frac{d^k}{dx^k} e^{-x^2}.
\]

(2.2)

It is well known that the \( \{H_\mu\} \) form a complete orthonormal system for \( L^2(\mathbb{R}^n) \), and that

\[
\mathbb{H} H_\mu = |\mu| H_\mu,
\]

(2.3)

where \( |\mu| = \sum_{j=1}^n \mu_j \), and \( \mathbb{H} \) is the Schrödinger operator \( \mathbb{H} = -\frac{1}{2} \Delta + \frac{1}{2} |x|^2 - \frac{\mu}{2} \). Here \( \Delta \) denotes the standard \( n \)-dimensional Laplacian. In particular, the sesquilinearity of the Wigner distribution implies that for any \( \psi \in L^2(\mathbb{R}^n) \), one has

\[
W_\psi = \sum_{\mu} \sum_{\nu} \langle \psi, H_\mu \rangle \langle \psi, H_\nu \rangle^* W_{H_\mu, H_\nu}.
\]

(2.4)

The following lemma shows that the integral of the off-diagonal elements of (2.4) over any centered ball \( B^{2n}(r) \) vanishes (cf. \([5]\) Section 2.3).

Lemma 2.1. Let \( \mu, \nu \) be two multi-indices with \( \mu \neq \nu \). Then, for every \( r \geq 0 \), one has

\[
\int_{B^{2n}(r)} W_{H_\mu, H_\nu} \, dx \, dy = 0.
\]

(2.5)

Proof of Lemma 2.1 It is well known (see e.g. \([6]\)) that for the one-dimensional Hermite functions \( \{h_m\} \), one has:

\[
W_{h_j,h_k}(x_1, y_1) = \begin{cases} 
\pi^{-1} (k!/j!)^{1/2} (-1)^k (\sqrt{2} z_1)^j \alpha^{-k} e^{-\|z_1\|^2} L_k^{-j-k}(2|z_1|^2) & \text{if } j \geq k, \\
\pi^{-1} (j!/k!)^{1/2} (-1)^j (\sqrt{2} z_1)^k \alpha^{-j} e^{-\|z_1\|^2} L_j^{-j-k}(2|z_1|^2) & \text{if } k \geq j.
\end{cases}
\]

(2.6)

Here \( z_1 = x_1 + iy_1 \), and \( L_n^\alpha \) are the Laguerre polynomials defined by

\[
L_j^\alpha(x) = \frac{x^{-\alpha} e^x}{j!} \frac{d^j}{dx^j} (e^{-x} x^{j+\alpha}),
\]

(2.7)

for \( j \geq 0 \) and \( \alpha > -1 \). Hence the lemma holds in the 2-dimensional case, i.e., when \( n = 1 \), because the integral of \( z^j \) or \( \overline{z}^j \) over any circle centered at the origin equals zero when...
\[ j \neq 0. \] The higher-dimensional case follows for the same reason from (2.6), the fact that the Wigner distribution function \( W_{H_{\mu},H_{\nu}}(x,y) \) is the product of \( W_{h_{m_j},h_{n_j}}(x_j,y_j) \), and the rotation invariance of the ball \( B^{2n}(r) \).

An immediate corollary of Lemma 2.1, definition (1.2), and equality (2.4) is

**Corollary 2.2.** In the notation above,

\[
\mathcal{E} \left( B^{2n}(r) \right) = \sup_{\mu} \int_{B^{2n}(r)} W_{H_{\mu}} dx dy,
\]

where the supremum is taken over all multi-indices \( \mu = (\mu_1, \ldots, \mu_n) \) of non-negative integers.

The following lemma is the main ingredient in the proof of Theorem 1.

**Lemma 2.3.** For any integer \( \lambda \geq 0 \) and multi-indices \( \mu_1, \mu_2 \) with \( \lambda = |\mu_1| = |\mu_2| \), one has

\[
\int_{B^{2n}(r)} W_{H_{\mu_1}} dx dy = \int_{B^{2n}(r)} W_{H_{\mu_2}} dx dy, \quad \text{for every } r \geq 0.
\]

Postponing the proof of Lemma 2.3, we first conclude the proof of Theorem 1.

**Proof of Theorem 1.** It follows from Corollary 2.2 and Lemma 2.3 above that

\[
\mathcal{E} \left( B^{2n}(r) \right) = \sup_{\lambda} \int_{B^{2n}(r)} W_{H_{\mu_\lambda}} dx dy,
\]

where \( \mu_{\lambda} = (\lambda, 0, \ldots, 0) \), and \( \lambda \) is a non-negative integer. Moreover, from (2.6) and the definition of the Wigner distribution it follow that:

\[
W_{H_{\mu_\lambda}}(x,y) = \frac{(-1)^\lambda}{\pi^n} e^{-\sum_{i=1}^{n}(x_i^2+y_i^2)} L_\lambda(2(x_1^2+y_1^2)),
\]

where \( L_\lambda(z) \) are the \( \alpha = 0 \) Laguerre polynomials (2.7). Setting \( z_j = x_j + iy_j \), we conclude that

\[
\int_{B^{2n}(r)} W_{H_{\mu_\lambda}} dx dy = \int \cdots \int e^{-\sum_{j=2}^{n}|z_j|^2} \left( \int e^{-|z_1|^2} L_\lambda(2|z_1|^2) \right) dz_2 \cdots dz_n.
\]

On the other hand, from Flandrin’s result in the 1-dimensional case [2], it follows that

\[
\int_{B^{2}(\alpha)} W_{h_{\lambda}} dx_1 dy_1 = \int_{|z_1|^2 \leq \alpha^2} (-1)^\lambda e^{-|z_1|^2} L_\lambda(2|z_1|^2) dz_1 \leq \int_{|z_1|^2 \leq \alpha^2} e^{-|z_1|^2} dz_1,
\]

for every radius \( \alpha \geq 0 \). An examination of Flandrin’s proof reveals that the inequality is strict for \( \lambda > 0 \). Hence, for every non-negative integer \( \lambda \) one has

\[
\int_{B^{2n}(r)} W_{H_{\mu_\lambda}} dx dy \leq \pi^{-n} \int \cdots \int e^{-\sum_{j=1}^{n}|z_j|^2} dz_1 \cdots dz_n = 1 - \frac{\Gamma(n, r^2)}{(n-1)!}
\]
with equality only for $\lambda = 0$. The proof of Theorem 1 now follows from (2.11) and (2.14).

**Remark:** The integral in (2.10) is not monotone in $\lambda$ or in $r$ (except for $\lambda = 0$), as might have been thought. See [1, Fig. 2] and [2] for interesting graphs of these integrals as a function of $r$.

For the proof of Lemma 2.3 we shall need the following preliminaries. For a non-negative integer $\lambda$ denote

$$
\mathcal{H}_\lambda = \text{span}\{H_\mu : |\mu| = \lambda\} \subset L^2(\mathbb{R}^n).
$$

It follows from (2.3) above that the space $\mathcal{H}_\lambda$ consists of the eigenfunctions of the rotation invariant Schrödinger operator $\mathbb{H} = -\frac{1}{2} \Delta + \frac{1}{2} |x|^2 - \frac{n}{2}$ with eigenvalue $\lambda$. In particular, it is a finite-dimensional, $O(n)$-invariant subspace of $L^2(\mathbb{R}^n)$ with orthonormal basis $\{H_\mu : |\mu| = \lambda\}$. It follows that for every $R \in O(n)$, and every $\tilde{\mu}$ with $|\tilde{\mu}| = \lambda$, one has:

$$
H_{\tilde{\mu}}(Rx) = \sum_{\nu : |\nu| = \lambda} c_\nu(\tilde{\mu}, R) H_\nu(x),
$$

where the coefficients $c_\nu(\tilde{\mu}, R)$ satisfy $\sum |c_\nu(\tilde{\mu}, R)|^2 = 1$.

We note the following useful fact: In order to identify which coefficients $c_\nu(\tilde{\mu}, R)$ are non-zero, it is only necessary to check the leading powers on the two sides of (2.16). That is, the left side of (2.16) defines a polynomial of degree $\lambda$ in the indeterminates $x_1, \ldots, x_n$. The highest degree terms are the monomials $x_1^{\mu_1} \cdots x_n^{\mu_n}$ with $\sum_{j=1}^n \mu_j = \lambda$, but there are also monomials of degree lower than $\lambda$. In order to show that a given $H_\nu$ appears with a non-zero coefficient in the decomposition (2.16), it is only necessary to show that there is a highest degree monomial $x_1^{\nu_1} \cdots x_n^{\nu_n}$ in the decomposition. It is not necessary to check the lower degree monomials; they will appear automatically because we know that the decomposition contains only Hermite functions of degree $\lambda$ and no others.

**Proof of Lemma 2.3:** Fix a non-negative integer $\lambda$, and $r \geq 0$. We consider the maximum problem

$$
\max_{\mu : |\mu| = \lambda} \int_{B^{2n}(r)} W_{H_\mu} dx dy,
$$

and denote by $\tilde{\mu}$ one of its maximizers.

From the sesquilinearity property of the Wigner distribution and Lemma 2.1, we conclude that for every $R \in O(n)$ one has:

$$
\int_{B^{2n}(r)} W_{H_\nu}(Rx) dx dy = \sum_{\nu} |c_\nu(\tilde{\mu}, R)|^2 \int_{B^{2n}(r)} W_{H_\nu} dx dy.
$$

Since $H_{\tilde{\mu}}$ is a maximizer, this implies that for any $\nu_0$ with $c_{\nu_0}(\tilde{\mu}, R) \neq 0$ one has

$$
\int_{B^{2n}(r)} W_{H_\nu} dx dy = \int_{B^{2n}(r)} W_{H_{\tilde{\mu}}(Rx)} dx dy = \int_{B^{2n}(r)} W_{H_{\nu_0}} dx dy.
$$
as before, namely \( R_{\lambda} \), \( \lambda + 1 = \sum_{j=1}^{n} \mu_j \).

The proof will proceed in two steps. The first is to go from \( \tilde{\mu} \), by a succession of two-dimensional rotations, to \((\lambda, 0, 0, \ldots, 0)\) with \( \lambda = \sum_{j=1}^{n} \tilde{\mu}_j \).

First, we show that there is a rotation \( R' \in O(n) \) with

\[
c_{\tilde{\mu}}(\tilde{\mu}, R') \neq 0, \quad \text{where} \quad \tilde{\mu}' := ((\tilde{\mu}_1 + \tilde{\mu}_2), 0, \tilde{\mu}_3, \ldots, \tilde{\mu}_n).
\]

Thus, \( \tilde{\mu}' \) is also a maximizer. In a similar fashion, we can go from \( \tilde{\mu}' \) to \( \tilde{\mu}'' \), where \( \tilde{\mu}'' := ((\tilde{\mu}_1 + \tilde{\mu}_2 + \tilde{\mu}_3), 0, 0, \tilde{\mu}_4, \ldots, \tilde{\mu}_n) \). Proceeding inductively, we finally arrive at the conclusion that \((\lambda, 0, \ldots, 0)\) is a maximizer.

A rotation \( R' \) that accomplishes the first step to \( \tilde{\mu}' \) is simply \( R' : x_1 \to (x_1 + x_2)/\sqrt{2}, \ x_2 \to (x_1 - x_2)/\sqrt{2}, \ x_j \to x_j \) for \( j > 2 \). The monomial \( x_1^{\tilde{\mu}_1} x_2^{\tilde{\mu}_2} \) becomes \( \frac{1}{2}(x_1 + x_2)^{\tilde{\mu}_1}(x_1 - x_2)^{\tilde{\mu}_2} \) and this obviously contains the monomial \( x_1^{(\tilde{\mu}_1 + \tilde{\mu}_2)} \) with a non-zero coefficient.

The second step is to go in the other direction, from \((\lambda, 0, \ldots, 0)\) to \((\nu_1, \nu_2, \ldots, \nu_n)\) when \( \sum_{j=1}^{n} \nu_j = \lambda \). As before, we do this with a sequence of two-dimensional rotations, the first of which takes us from \((\lambda, 0, \ldots, 0)\) to \((\lambda - \nu_2, \nu_2, 0, \ldots, 0)\). From thence we go to \((\lambda - \nu_2 - \nu_3, \nu_2, \nu_3, 0, \ldots, 0)\), and so forth. This can be accomplished with the same rotation as before, namely \( R' : x_1 \to (x_1 + x_2)/\sqrt{2}, \ x_2 \to (x_1 - x_2)/\sqrt{2}, \ x_j \to x_j \) for \( j > 2 \).

\[\Box\]

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