ON THE DISCRETE LOGARITHMIC MINKOWSKI PROBLEM

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Abstract. A new condition for the existence of solution of the logarithmic Minkowski problem is established. This new condition contains the one established by Zhu [66] and the discrete case established by Böröczky et al. [6] as two important special cases.

1. Introduction

The setting for this paper is $n$-dimensional Euclidean space $\mathbb{R}^n$. A convex body in $\mathbb{R}^n$ is a compact convex set that has non-empty interior. If $K$ is a convex body in $\mathbb{R}^n$, then the surface area measure, $S_K$, of $K$ is a Borel measure on the unit sphere, $S^{n-1}$, defined for a Borel $\omega \subset S^{n-1}$, by

$$S_K(\omega) = \int_{x \in \nu_K^{-1}(\omega)} d\mathcal{H}^{n-1}(x),$$

where $\nu_K : \partial K \rightarrow S^{n-1}$ is the Gauss map of $K$, defined on $\partial K$, the set of points of $\partial K$ that have a unique outer unit normal, and $\mathcal{H}^{n-1}$ is $(n-1)$-dimensional Hausdorff measure.

As one of the cornerstones of the classical Brunn-Minkowski theory, the Minkowski existence theorem can be stated as (see, e.g., [59]): If $\mu$ is not concentrated on a great subsphere of $S^{n-1}$, then $\mu$ is the surface area measure of a convex body if and only if

$$\int_{S^{n-1}} u d\mu = 0.$$ 

The solution is unique up to translation, and even the regularity of the solution is well investigated, see e.g., Lewy [38], Nirenberg [55], Cheng and Yau [12], Pogorelov [58], and Caffarelli [9].

The surface area measure of a convex body has clear geometric significance. Another important measure that is associated with a convex body and that has clear geometric importance is the cone-volume measure. If $K$ is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, then the cone-volume measure, $V_K$, of $K$ is a Borel measure on $S^{n-1}$ defined for each Borel $\omega \subset S^{n-1}$ by

$$V_K(\omega) = \frac{1}{n} \int_{x \in \nu_K^{-1}(\omega)} x \cdot \nu_K(x) d\mathcal{H}^{n-1}(x).$$

For references regarding cone-volume measure see, e.g., [4, 8, 10, 12, 53, 54, 56, 60, 62, 66].

The Minkowski existence theorem deals with the question of prescribing the surface area measure. An important and natural problem is prescribing the cone-volume measure.

Logarithmic Minkowski problem: What are the necessary and sufficient conditions on a finite Borel measure $\mu$ on $S^{n-1}$ so that $\mu$ is the cone-volume measure of a convex body in $\mathbb{R}^n$?

In [43], Lutwak showed that there is an $L_p$ analogue of the surface area measure and posed the associated $L_p$ Minkowski problem which has the classical Minkowski problem and logarithmic

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Minkowski problem as two important cases. If \( p \in \mathbb{R} \) and \( K \) is a convex body in \( \mathbb{R}^n \) that contains the origin in its interior, then the \( L_p \) surface area measure, \( S_p(K, \cdot) \), of \( K \) is a Borel measure on \( S^{n-1} \) defined for a Borel \( \omega \subset S^{n-1} \), by
\[
S_p(K, \omega) = \int_{x \in \nu_{K}^{-1}(\omega)} (x \cdot \nu_K(x))^{1-p} d\mathcal{H}^{n-1}(x).
\]
Obviously, \( S_1(K, \cdot) \) is the classical surface area measure of \( K \) and \( \frac{1}{n} S_0(K, \cdot) \) is the cone-volume measure of \( K \). In recent years, the \( L_p \) surface area measure appeared in, e.g., \([1,4,10,22,23,25,26,30,40,42,55,57,58,59,63,51,56,57,62]\).

Today, the \( L_p \) Minkowski problem is one of the central problems in convex geometric analysis. It can be stated in the following way:

\[ L_p \textbf{ Minkowski problem:} \] For fixed \( p \), what are necessary and sufficient conditions on a finite Borel measure \( \mu \) on \( S^{n-1} \) so that \( \mu \) is the \( L_p \) surface area measure of a convex body in \( \mathbb{R}^n \)?

When \( \mu \) has a density \( f \), with respect to spherical Lebesgue measure, the \( L_p \) Minkowski problem involves establishing existence for the Monge-Ampère type equation:
\[
h^{1-p} \det(h_{ij} + h\delta_{ij}) = f,
\]
where \( h_{ij} \) is the covariant derivative of \( h \) with respect to an orthonormal frame on \( S^{n-1} \) and \( \delta_{ij} \) is the Kronecker delta.

Obviously, the \( L_1 \) Minkowski problem is the classical Minkowski problem. Establishing existence and uniqueness for the solution of the classical Minkowski problem was done by Aleksandrov, and Fenchel and Jessen (see, e.g., \([59]\)). When \( p \neq 1 \), the \( L_p \) Minkowski problem and PDE (1.1) have been studied by, e.g., Lutwak \([43]\), Lutwak and Oliker \([44]\), Guan and Lin \([21]\), Chou and Wang \([14]\), Hug, et al. \([33]\), Böröczky, et al. \([6]\). Additional references regarding the \( L_p \) Minkowski problem and Minkowski-type problems can be found in \([6,11,14,20,24,31,33,36,37,39,43,44,49,52,60,61,67,68]\).

The solutions to the Minkowski problem and the \( L_p \) Minkowski problem connect with some important flows (see, e.g., \([2,3,13,10,34,35]\), and have important applications to Sobolev-type inequalities (see, e.g., \([15,27,29,48,64,65]\).

Most previous work on the \( L_p \) Minkowski problem was limited to the case where \( p > 1 \). The reason that uniqueness of solutions to the \( L_p \) Minkowski problem for \( p > 1 \) can be shown is the availability of mixed volume inequalities established by Lutwak \([43]\). One reason that the \( L_p \) Minkowski problem becomes challenging when \( p < 1 \) is because little is known about the mixed volume inequalities when \( p < 1 \) (see, e.g., \([7]\)). The \( L_0 \) is called the logarithmic case, is clearly the most important case with geometric significance because it is the singular case.

A finite Borel measure \( \mu \) on \( S^{n-1} \) is said to satisfy the \textbf{subspace concentration condition} if, for every subspace \( \xi \) of \( \mathbb{R}^n \), such that \( 0 < \dim \xi < n \),
\[
\mu(\xi \cap S^{n-1}) \leq \frac{\dim \xi}{n} \mu(S^{n-1}),
\]
and if equality holds in (1.2) for some subspace \( \xi \), then there exists a subspace \( \xi' \), that is complementary to \( \xi \) in \( \mathbb{R}^n \), so that also
\[
\mu(\xi' \cap S^{n-1}) = \frac{\dim \xi'}{n} \mu(S^{n-1}).
\]

The measure \( \mu \) on \( S^{n-1} \) is said to satisfy the \textbf{strict subspace concentration inequality} if the inequality in (1.2) is strict for each subspace \( \xi \subset \mathbb{R}^n \), such that \( 0 < \dim \xi < n \).

In \([6]\), Böröczky, et al. gave the following necessary and sufficient conditions for the existence of solutions to the even logarithmic Minkowski problem.
Theorem A. A non-zero finite even Borel measure on the $S^{n-1}$ is the cone-volume measure of an origin-symmetric convex body in $\mathbb{R}^n$ if and only if it satisfies the subspace concentration condition.

Theorem A shows that the subspace concentration condition is a natural condition for all even measures that may arise as the cone-volume measures of origin-symmetric convex bodies. Actually, Böröczky and Henk [5] showed that the cone-volume measure of any convex body whose centroid is the origin satisfies the subspace concentration condition.

However, Zhu [66] proved that any discrete measure on $S^{n-1}$ whose support is in general position is a cone-volume measure. Here, a finite set of vectors in $\mathbb{R}^n$ is in general position if any $n$-element subset is independent.

Theorem B. A discrete measure, $\mu$, on the unit sphere $S^{n-1}$ is the cone-volume measure of a polytope whose outer unit normals are in general position if and only if the support of $\mu$ is in general position and not concentrated on a closed hemisphere of $S^{n-1}$.

A polytope in $\mathbb{R}^n$ is the convex hull of a finite set of points in $\mathbb{R}^n$ provided that it has positive $n$-dimensional volume. The convex hull of a subset of these points is called a facet of the polytope if it lies entirely on the boundary of the polytope and has positive $(n-1)$-dimensional volume. If a polytope $P$ contains the origin in its interior with $N$ facets whose outer unit normals are $u_1, ..., u_N$, and such that if the facet with outer unit normal $u_k$ has $(n-1)$-measure $a_k$ and distance from the origin $h_k$ for all $k \in \{1, ..., N\}$, then

$$V_K(\cdot) = \sum_{k=1}^{N} h_k a_k \delta_{u_k}(\cdot).$$

where $\delta_{u_k}$ denotes the delta measure that is concentrated at the point $u_k$.

Definition. A linear subspace $\xi$ ($1 \leq \dim \xi \leq n-1$) of $\mathbb{R}^n$ is said to be essential with respect to a Borel measure $\mu$ on $S^{n-1}$ if $\xi \cap \text{supp} \mu$ is not concentrated on any closed hemisphere of $\xi \cap S^{n-1}$.

Obviously, if $\xi$ is essential with respect to $\mu$, then $\xi \cap \text{supp} \mu$ contains $1 + \dim \xi$ vectors that span $\xi$, and the origin is a positive linear combination of these vectors.

The main goal of this paper is to provide a common generalization of Theorem B and the sufficiency part of Theorem A, in the case of discrete measures.

Theorem 1.1. Suppose $n \geq 2$, $\mu$ is a discrete measure on $S^{n-1}$ that is not concentrated on any closed hemisphere, and $\mu$ satisfies the subspace concentration condition with respect to any essential linear subspace $\xi$, then $\mu$ is the cone-volume measure of a polytope in $\mathbb{R}^n$.

We would like to note that for the case where $n = 2$, Theorem 1.1 was proved by Stancu [60] by applying the crystalline deformation.

The example in Section 6 shows that a cone-volume measure does not need to satisfy subspace concentration condition with respect to essential linear subspace. New inequalities for cone-volume measures are established in section 6.

2. Preliminaries

In this section, we collect some basic definitions and facts about convex bodies. For general references regarding convex bodies see, e.g., [17, 19, 59, 63].

The vectors of this paper are column vectors. For $x, y \in \mathbb{R}^n$, we will write $x \cdot y$ for the standard inner product of $x$ and $y$, and write $|x|$ for the Euclidean norm of $x$. Suppose $X_1, X_2$ are subspace of
We write \( X_1 \perp X_2 \) if \( x_1 \cdot x_2 = 0 \) for all \( x_1 \in X_1 \) and \( x_2 \in X_2 \). We write \( S^{n-1} = \{ x \in \mathbb{R}^n : |x| = 1 \} \) for the boundary of the Euclidean unit ball \( B^n \) in \( \mathbb{R}^n \), and write \( \omega_n \) for the volume of the unit ball.

Suppose \( C \) is a subset of \( \mathbb{R}^n \), the positive hull, \( \text{pos}(C) \), of \( C \) is the set of all positive combinations of any finitely many elements of \( C \). Let \( \text{lin}(C) \) be the smallest linear subspace of \( \mathbb{R}^n \) containing \( C \). The diameter of \( C \) is defined by

\[
d(C) = \sup \{ |x - y| : x, y \in C \}.
\]

For \( K_1, K_2 \subset \mathbb{R}^n \) and \( c_1, c_2 \geq 0 \), the Minkowski combination, \((c_1 K_1 + c_2 K_2)\), is defined by

\[
c_1 K_1 + c_2 K_2 = \{ c_1 x_1 + c_2 x_2 : x_1 \in K_1, x_2 \in K_2 \}.
\]

The **support function** \( h_K : \mathbb{R}^n \rightarrow \mathbb{R} \) of a compact convex set \( K \) is defined, for \( x \in \mathbb{R}^n \), by

\[
h(K, x) = \max \{ x \cdot y : y \in K \}.
\]

Obviously, for \( c \geq 0 \) and \( x \in \mathbb{R}^n \),

\[
h(cK, x) = h(K, cx) = ch(K, x).
\]

The **convex hull** of two compact sets \( K, L \) in \( \mathbb{R}^n \) is defined by

\[
[K, L] = \{ z : z = \lambda x + (1 - \lambda)y, 0 \leq \lambda \leq 1 \text{ and } x, y \in K \cup L \}.
\]

The **Hausdorff distance** between two convex bodies, \( K \) and \( L \), is defined by

\[
\delta(K, L) = \inf \{ t \geq 0 : K \subset L + tB^n, L \subset K + tB^n \}.
\]

It is known that the Hausdorff distance between two convex bodies, \( K \) and \( L \), is

\[
\delta(K, L) = \max_{u \in S^{n-1}} |h(K, u) - h(L, u)|.
\]

We always consider the space of convex bodies as metric space equipped with the Hausdorff distance. It is known that if a sequence \( \{ K_m \} \) of convex bodies tends to a convex body \( K \) in \( S^{n-1} \), then \( S_{K_m} \) tends weakly to \( S_K \). Therefore \( V_{K_m} \) tends weakly to \( V_K \), as well.

For a convex body \( K \) in \( \mathbb{R}^n \), and \( u \in S^{n-1} \), the support hyperplane \( H(K, u) \) in direction \( u \) is defined by

\[
H(K, u) = \{ x \in \mathbb{R}^n : x \cdot u = h(K, u) \},
\]

the face \( F(K, u) \) in direction \( u \) is defined by

\[
F(K, u) = K \cap H(K, u).
\]

Let \( \mathcal{P} \) be the set of polytopes in \( \mathbb{R}^n \). If the unit vectors \( u_1, ..., u_N \) are not concentrated on a closed hemisphere, let \( \mathcal{P}(u_1, ..., u_N) \) be the subset of \( \mathcal{P} \) such that a polytope \( P \in \mathcal{P}(u_1, ..., u_N) \) if the outer unit normals are a subset of \( \{ u_1, ..., u_N \} \). Let \( \mathcal{P}_N(u_1, ..., u_N) \) be the subset of \( \mathcal{P}(u_1, ..., u_N) \) such that a polytope \( P \in \mathcal{P}_N(u_1, ..., u_N) \) if, \( P \in \mathcal{P}(u_1, ..., u_N) \), and \( P \) has exactly \( N \) facets.

3. **An extreme problem related to the logarithmic Minkowski problem**

Suppose \( \gamma_1, ..., \gamma_N \in \mathbb{R}^+ \), the unit vectors \( u_1, ..., u_N \) are not concentrated on a closed hemisphere. Let

\[
\mu(\cdot) = \sum_{i=1}^{N} \gamma_i \delta_{u_i}(\cdot),
\]

and for \( P \in \mathcal{P}(u_1, ..., u_N) \) define \( \Phi_P : \text{Int} \ (P) \rightarrow \mathbb{R} \) by

\[
\Phi_P(\xi) = \int_{S^{n-1}} \log \ (h(P, u) - \xi \cdot u) \ d\mu(u).
\]
In this section, we study the following extreme problem:

\[
\inf \left\{ \max_{\xi \in \text{Int}(Q)} \Phi_Q(\xi) : Q \in \mathcal{P}(u_1, ..., u_N) \text{ and } V(Q) = |\mu| \right\}.
\]

We will prove that the solution of problem 3.2 solves the corresponding logarithmic Minkowski problem.

For the case where \(u_1, ..., u_N\) are in general position and \(Q \in \mathcal{P}_N(u_1, ..., u_N)\), problem 3.2 was studied in [66]. The results and proofs in this section are similar to [66]. However, for convenience to the readers, we give detailed proofs for these results.

**Lemma 3.1.** Suppose \(\mu\), \(P\) and \(\Phi_P\) are given by (3.4) and (3.1), then there exists a unique point \(\xi(P) \in \text{Int}(P)\) such that

\[
\Phi_P(\xi(P)) = \max_{\xi \in \text{Int}(P)} \Phi_P(\xi).
\]

**Proof.** Let \(0 < \lambda < 1\) and \(\xi_1, \xi_2 \in \text{Int}(P)\). From the concavity of the logarithmic function,

\[
\lambda \Phi_P(\xi_1) + (1 - \lambda) \Phi_P(\xi_2) = \lambda \int_{S^{n-1}} \log(h(P, u) - \xi_1 \cdot u) \, d\mu(u)
\]

\[
+ (1 - \lambda) \int_{S^{n-1}} \log(h(P, u) - \xi_1 \cdot u) \, d\mu(u)
\]

\[
= \sum_{k=1}^{N} \gamma_k \left[ \lambda \log(h(P, u_k) - \xi_2 \cdot u_k) + (1 - \lambda) \log(h(P, u_k) - \xi_2 \cdot u_k) \right]
\]

\[
\leq \sum_{k=1}^{N} \gamma_k \log[h(P, u_k) - (\lambda \xi_1 + (1 - \lambda) \xi_2) \cdot u_k]
\]

\[
= \Phi_P(\lambda \xi_1 + (1 - \lambda) \xi_2),
\]

with equality if and only if \(\xi_1 \cdot u_k = \xi_2 \cdot u_k\) for all \(k = 1, ..., N\). Since the unit vectors \(u_1, ..., u_N\) are not concentrated on a closed hemisphere, \(\mathbb{R}^n = \text{lin}\{u_1, ..., u_N\}\). Thus, \(\xi_1 = \xi_2\). Therefore, \(\Phi_P\) is strictly concave on \(\text{Int}(P)\).

Since \(P \in \mathcal{P}(u_1, ..., u_N)\), for any \(x \in \partial P\), there exists a \(u_{i_0} \in \{u_1, ..., u_N\}\) such that

\[
h(P, u_{i_0}) = x \cdot u_{i_0}.
\]

Thus, \(\Phi_P(\xi) \to -\infty\) whenever \(\xi \in \text{Int}(P)\) and \(\xi \to x\). Therefore, there exists a unique interior point \(\xi(P)\) of \(P\) such that

\[
\Phi_P(\xi(P)) = \max_{\xi \in \text{Int}(P)} \Phi_P(\xi).
\]

\(\square\)

Obviously, for \(\lambda > 0\) and \(P \in \mathcal{P}(u_1, ..., u_N)\),

\[
(3.3)\quad \xi(\lambda P) = \lambda \xi(P),
\]

and if \(P_i \in \mathcal{P}(u_1, ..., u_N)\) and \(P_i\) converges to a polytope \(P\), then \(P \in \mathcal{P}(u_1, ..., u_N)\).

For the case where \(u_1, ..., u_N\) are in general position, the following lemma was proved in [66]. One can easily find that the proofs in [66] also work for the following lemma.

**Lemma 3.2.** Suppose \(\mu\) and \(\Phi_{P_i}\) are given by (3.4) and (3.1), \(P_i \in \mathcal{P}(u_1, ..., u_N)\) and \(P_i\) converges to a polytope \(P\), then \(\lim_{i \to \infty} \xi(P_i) = \xi(P)\) and

\[
\lim_{i \to \infty} \Phi_{P_i}(\xi(P_i)) = \Phi_P(\xi(P)).
\]

The following lemmas will be needed.
Lemma 3.3. Suppose \( \mu, P \) and \( \Phi_P \) are given by (3.0) and (3.1), with \( \xi(P) = o \), then
\[
\sum_{k=1}^{N} \gamma_k \frac{u_k}{h(P, u_k)} = 0.
\]

Proof. Define \( f : \text{Int}(P) \to \mathbb{R}^n \) by
\[
f(x) = \int_{S^{n-1}} \log h(P - x, u) d\mu(u) \\
= \sum_{k=1}^{N} \gamma_k \log(h(P, u_k) - x \cdot u_k).
\]

By conditions,
\[
f(o) = \max_{x \in \text{Int}(P)} f(x).
\]

Thus,
\[
\sum_{k=1}^{N} \gamma_k \frac{u_{k,i}}{h(P, u_k)} = 0,
\]
for all \( i = 1, \ldots, n \), where \( u_k = (u_{k,1}, \ldots, u_{k,n})^T \). Therefore
\[
\sum_{k=1}^{N} \gamma_k \frac{u_k}{h(P, u_k)} = 0.
\]

Lemma 3.4. Suppose the discrete measure \( \mu(\cdot) = \sum_{k=1}^{N} \gamma_k \delta_{u_k}(\cdot) \) is not concentrated on a closed hemisphere, and there exists a \( P \in \mathcal{P}_N(u_1, \ldots, u_N) \) with \( \xi(P) = o \), \( V(P) = |\mu| \) such that
\[
\Phi_P(o) = \inf \left\{ \max_{\xi \in \text{Int}(Q)} \Phi_Q(\xi) : Q \in \mathcal{P}(u_1, \ldots, u_N) \text{ and } V(Q) = |\mu| \right\}.
\]

Then,
\[
V_P = \sum_{k=1}^{N} \gamma_k \delta_{u_k}.
\]

Proof. When noticing Equation (3.3), it is sufficient to establish the lemma under the assumption that \( |\mu| = 1 \).

From the conditions, there exists a polytope \( P \in \mathcal{P}_N(u_1, \ldots, u_N) \) with \( \xi(P) = o \) and \( V(P) = 1 \) such that
\[
\Phi_P(o) = \inf \left\{ \max_{\xi \in \text{Int}(Q)} \Phi_Q(\xi) : Q \in \mathcal{P}(u_1, \ldots, u_N) \text{ and } V(Q) = 1 \right\}.
\]

For \( \delta_1, \ldots, \delta_N \in \mathbb{R} \), choose \( |t| \) small enough so that the polytope
\[
P_t = \bigcap_{i=1}^{N} \{ x : x \cdot u_i \leq h(P, u_i) + t\delta_i \} \in \mathcal{P}_N(u_1, \ldots, u_N),
\]
and
\[
V(P_t) = V(P) + t \left( \sum_{i=1}^{N} \delta_i S_i \right) + o(t),
\]
where \( S_i = |F(P, u_i)| \). Thus,
\[
\lim_{t \to 0} \frac{V(P_t) - V(P)}{t} = \sum_{i=1}^{N} \delta_i S_i.
\]
Let \( \lambda(t) = V(P_t)^{-\frac{1}{n}} \), then \( \lambda(t)P_t \in \mathcal{P}_N(u_1, ..., u_N) \), \( V(\lambda(t)P_t) = 1 \) and
\[
\lambda'(0) = -\frac{1}{n} \sum_{i=1}^{N} \delta_i S_i.
\]
Let \( \xi(t) = \xi(\lambda(t)P_t) \), and
\[
\Phi(t) = \max_{\xi \in \lambda(t)P_t} \int_{S^{n-1}} \log (h(\lambda(t)P_t, u) - \xi \cdot u) \, d\mu(u)
\]
\[
= \sum_{k=1}^{N} \gamma_k \log (\lambda(t)h(P_t, u_k) - \xi(t) \cdot u_k).
\]
From the definition of \( \xi(t) \), Equation (3.5) and the fact that \( \xi(t) \) is an interior point of \( \lambda(t)P_t \), we have
\[
\sum_{k=1}^{N} \gamma_k \frac{u_{k,i}}{\lambda(t)h(P_t, u_k) - \xi(t) \cdot u_k} = 0,
\]
for \( i = 1, ..., n \), where \( u_k = (u_{k,1}, ..., u_{k,n})^T \).

From the fact that \( \xi(P) = 0 \) and Lemma 3.3, we have
\[
\sum_{k=1}^{N} \gamma_k \frac{u_k}{h(P, u_k)} = 0.
\]
Let
\[
F_i(t, \xi_1, ..., \xi_n) = \sum_{k=1}^{N} \gamma_k \frac{u_{k,i}}{\lambda(t)h(P_t, u_k) - (\xi_1 u_{k,1} + ... + \xi_n u_{k,n})}
\]
for \( i = 1, ..., n \). Then,
\[
\left. \frac{\partial F_i}{\partial \xi_j} \right|_{(0, ..., 0)} = \sum_{k=1}^{N} \frac{\gamma_k}{h(P, u_k)^2} u_{k,i} u_{k,j}.
\]
Thus,
\[
\left( \frac{\partial F}{\partial \xi} \right|_{(0, ..., 0)} \right)_{n \times n} = \sum_{k=1}^{N} \frac{\gamma_k}{h(P, u_k)^2} u_k u_k^T,
\]
where \( u_k u_k^T \) is an \( n \times n \) matrix. Since the unit vectors \( u_1, ..., u_N \) are not concentrated on a closed hemisphere, \( \mathbb{R}^n = \text{lin}\{u_1, ..., u_N\} \). Thus, for any \( x \in \mathbb{R}^n \) with \( x \neq 0 \), there exists a \( u_{i_0} \in \{u_1, ..., u_N\} \) such that \( u_{i_0} \cdot x \neq 0 \). Then,
\[
x^T \left( \sum_{k=1}^{N} \frac{\gamma_k}{h(P, u_k)^2} u_k u_k^T \right) x = \sum_{k=1}^{N} \frac{\gamma_k}{h(P, u_k)^2} (x \cdot u_k)^2 \geq \frac{\gamma_{i_0}}{h(P, u_{i_0})^2} (x \cdot u_{i_0})^2 > 0.
\]
Therefore, \( \left( \frac{\partial F}{\partial \xi} \right|_{(0, ..., 0)} \) is positive defined. From this, the fact that \( \xi(0) = 0 \), Equations (3.7), (3.8), and the inverse function theorem,
\[
\xi'(0) = (\xi_1'(0), ..., \xi_n'(0))
\]
exists.

From the fact that $\Phi(0)$ is a minimizer of $\Phi(t)$ (in Equation (3.6)), Equation (3.5), the fact $\sum_{k=1}^{N} \gamma_k = 1$ and Equation (3.8), we have

$$0 = \Phi'(0)$$

$$= \sum_{k=1}^{N} \gamma_k \left( \lambda'(0) h(P, u_k) + \lambda(0) \frac{dh(P, u_k)}{dt} \right)_{t=0} - \xi'(0) \cdot u_k$$

$$= \sum_{k=1}^{N} \gamma_k \left( \sum_{i=1}^{N} \gamma_i S_i \right) h(P, u_k) + \sum_{k=1}^{N} \gamma_k \delta_k - \xi'(0) \cdot \left[ \sum_{k=1}^{N} \frac{\gamma_k h(P, u_k)}{u_k} \right]$$

$$= \sum_{k=1}^{N} \left( \frac{\gamma_k}{h(P, u_k)} - \frac{S_k}{n} \right) \delta_k.$$

Since $\delta_1, ..., \delta_N$ are arbitrary, $\gamma_k = \frac{1}{n} h(P, u_k) S_k$ for $k = 1, ..., N$. □

4. Existence of Solution of the Extreme Problem

In this section, we prove the existence of solution of problem (3.2) for the case where the discrete measure not concentrated on any closed hemisphere of $S^{n-1}$, and satisfies the strict subspace concentration inequality with respect to any essential linear subspace.

Lemma 4.1. Suppose $\mu$ is a discrete probability measure on $S^{n-1}$ that is not concentrated on any closed hemisphere of $S^{n-1}$, and satisfies the strict subspace concentration inequality with respect to any essential linear subspace. If $P_m$ is a sequence of polytopes with $V(P_m) = 1$, $\xi(P_m) = o$, the outer unit normals of $P_m$ is a subset of the support of $\mu$, and $\lim_{m \to \infty} d(P_m) = \infty$, then

$$\int_{S^{n-1}} \log h_{P_m}(u) d\mu(u)$$

is not bounded from above.

Proof. Let $\text{supp}(\mu) = \{u_1, ..., u_N\}$, and $\mu(\{u_i\}) = \gamma_i, i = 1, ..., N$. By taking repeated subsequences, we may assume that

$$h_{P_m}(u_1) \leq ... \leq h_{P_m}(u_N),$$

for all $m \in \mathbb{N}$. Since $\lim_{m \to \infty} d(P_m) = \infty$, we have

$$\lim_{m \to \infty} h_{P_m}(u_1) = 0$$

and

$$\lim_{m \to \infty} h_{P_m}(u_N) = \infty.$$

In particular, we may assume that there exist $q \geq 1$, and

$$1 = \alpha_0 < \alpha_1 < ... < \alpha_q \leq N < N + 1 = \alpha_{q+1}$$

such that if $j = 1, ..., q$, then

$$\lim_{m \to \infty} \frac{h_{P_m}(u_{\alpha_j})}{h_{P_m}(u_{\alpha_{j-1}})} = \infty$$

and if $j = 0, ..., q$, then

$$\lim_{m \to \infty} \sup \frac{h_{P_m}(u_{\alpha_{j+1}-1})}{h_{P_m}(u_{\alpha_j})} < \infty.$$
For \( j = 0, \ldots, q - 1 \), we consider the cone
\[
\Sigma_j = \text{pos}\{u_1, \ldots, u_{\alpha_{j+1}-1}\},
\]
and its negative polar
\[
\Sigma_j^* = \{ v \in \mathbb{R}^n : v \cdot u_i \leq 0 \text{ for all } i = 1, \ldots, \alpha_{j+1} - 1 \}.
\]

We observe that if \( p = 1, \ldots, \alpha_{j+1} - 1 \) and \( v \in \Sigma_j^* \cap S^{n-1} \), then \( \xi(P_m) = o \) and Lemma 3.3 yields that
\[
0 \geq \gamma_p(v \cdot u_p) = -h_{P_m}(u_p) \sum_{i \neq p} \frac{\gamma_i(v \cdot u_i)}{h_{P_m}(u_i)} \geq -h_{P_m}(u_p) \sum_{i \geq \alpha_{j+1}} \frac{\gamma_i(v \cdot u_i)}{h_{P_m}(u_i)},
\]
where Equations (4.2), (4.3) imply that
\[
\sum_{i \geq \alpha_{j+1}} \frac{\gamma_i}{h_{P_m}(u_i)} \geq 0.
\]

We conclude that if \( j = 0, \ldots, q - 1, p = 1, \ldots, \alpha_{j+1} - 1 \) and \( v \in \Sigma_j^* \), then
\[
v \cdot u_p = 0.
\]

Thus, \( \{u_1, \ldots, u_{\alpha_{j+1}-1}\} \) is not concentrated on a closed hemisphere of \( S^{n-1} \cap \text{lin}\{u_1, \ldots, u_{\alpha_{j+1}-1}\} \).

Therefore, for \( j = 0, \ldots, q \),
\[
\text{lin}\{u_1, \ldots, u_{\alpha_{j+1}-1}\} = \text{pos}\{u_1, \ldots, u_{\alpha_{j+1}-1}\}.
\]

Let \( X_j = \text{pos}\{u_1, \ldots, u_{\alpha_{j+1}-1}\}, \) \( d_j = \dim X_j \) for \( j = 0, \ldots, q \), and \( d_{-1} = 0 \). Let \( \tilde{X}_0 = X_0 \), and if \( j = 1, \ldots, q \), then let
\[
\tilde{X}_j = X_{j-1}^\perp \cap X_j.
\]

Obviously, \( \mathbb{R}^n = \sum_{j=1}^q \tilde{X}_j \), \( \tilde{X}_{j_1} \perp \tilde{X}_{j_2} \) for \( j_1 \neq j_2 \), and \( \dim \tilde{X}_j = d_j - d_{j-1} \) for \( j = 0, \ldots, q \).

For \( j = 0, \ldots, q \), it follows from the sublinearity of the support function and (4.3) that there exists \( t_j > 0 \) such that
\[
h_{P_m}(u) \leq t_j h_{P_m}(u_{\alpha_j}) \text{ for } u \in X_j \cap S^{n-1}.
\]

Therefore, if \( t = \max_{j=0, \ldots, q} t_j \), then the \( \dim \tilde{X}_j \)-dimensional volume of \( P_m|\tilde{X}_j \) is at most
\[
(\omega h_{P_m}(u_{\alpha_j}))^{d_j - d_{j-1}},
\]
where \( \omega = t \omega_{d_j - d_{j-1}}^{-1} \) and \( \omega_{d_j - d_{j-1}} \) is the volume of the \( (d_j - d_{j-1}) \)-dimensional unit ball. From Fubini’s formula,
\[
1 = V(P_m) \leq \prod_{j=0}^q (\omega h_{P_m}(u_{\alpha_j}))^{d_j - d_{j-1}}.
\]

From this and the fact that \( \sum_{j=0}^q (d_j - d_{j-1}) = n \), we have
\[
\sum_{j=0}^q \left( \frac{d_j}{n} - \frac{d_{j-1}}{n} \right) \log h_{P_m}(u_{\alpha_j}) \geq - \log \omega.
\]

We rewrite the last inequality as
\[
\log h_{P_m}(u_{\alpha_j}) + \sum_{j=0}^{q-1} \frac{d_j}{n} \log \frac{h_{P_m}(u_{\alpha_j})}{h_{P_m}(u_{\alpha_{j+1}})} \geq \log \omega.
\]
For \( j = 0, ..., q \), we set \( \beta_j = \sum_{i=1}^{\alpha_{j+1}-1} \gamma_i \), and \( \beta_{-1} = 0 \). From the fact that \( X_j = \text{pos}\{u_1, ..., u_{\alpha_{j+1}-1}\} \) and the condition of this lemma, we have, for \( j = 0, ..., q - 1 \),

\[
(4.4) \quad \beta_j < \frac{d_j}{n}.
\]

We conclude that

\[
\sum_{i=1}^{N} \gamma_i \log h_{P_m}(u_i) \geq \sum_{j=0}^{q-1} (\beta_j - \beta_{j-1}) \log h_{P_m}(u_{\alpha_j})
\]

\[
= \log h_{P_m}(u_{\alpha_q}) + \sum_{j=0}^{q-1} \beta_j \log \frac{h_{P_m}(u_{\alpha_j})}{h_{P_m}(u_{\alpha_{j+1}})}
\]

\[
\geq - \log \omega + \sum_{j=0}^{q-1} \left( \beta_j - \frac{d_j}{n} \right) \log \frac{h_{P_m}(u_{\alpha_j})}{h_{P_m}(u_{\alpha_{j+1}})}.
\]

It follows from (4.1), (4.2), (4.4) that if \( j = 0, ..., q - 1 \), then

\[
\lim_{m \to \infty} \left( \beta_j - \frac{d_j}{n} \right) \log \frac{h_{P_m}(u_{\alpha_j})}{h_{P_m}(u_{\alpha_{j+1}})} = \infty.
\]

Therefore

\[
\lim_{m \to \infty} \sum_{i=1}^{N} \gamma_i \log h_{P_m}(u_i) = \infty.
\]

The following lemma will be needed (see, [68], Lemma 3.5).

\textbf{Lemma 4.2.} If \( P \) is a polytope in \( \mathbb{R}^n \) and \( v_0 \in S^{n-1} \) with \( V_{n-1}(F(P,v_0)) = 0 \), then there exists a \( \delta_0 > 0 \) such that for \( 0 \leq \delta < \delta_0 \)

\[
V(P \cap \{ x : x \cdot v_0 \geq h(P,v_0) - \delta \}) = c_n \delta^n + ... + c_2 \delta^2,
\]

where \( c_n, ..., c_2 \) are constants that depend on \( P \) and \( v_0 \).

\textbf{Lemma 4.3.} Suppose the discrete measure \( \mu(\cdot) = \sum_{k=1}^{N} \gamma_k \delta_{u_k}(\cdot) \) is not concentrated on a closed hemisphere. If \( \mu \) satisfies the strict subspace concentration inequality with respect to any essential linear subspace, then there exists a \( P \in \mathcal{P}_N(u_1, ..., u_N) \) such that \( \xi(P) = 0, V(P) = |\mu| \) and

\[
\Phi_P(o) = \inf \left\{ \max_{\xi \in \text{Int}(Q)} \Phi_Q(\xi) : Q \in \mathcal{P}(u_1, ..., u_N) \text{ and } V(Q) = |\mu| \right\},
\]

where \( \Phi_Q(\xi) = \int_{S^{n-1}} \log(h(Q,u) - \xi \cdot u) \, d\mu(u) \).

\textbf{Proof.} It is easily seen that it is sufficient to establish the lemma under the assumption that \( |\mu| = 1 \). Obviously, for \( P, Q \in \mathcal{P}(u_1, ..., u_N) \), if there exists a \( x \in \mathbb{R}^n \) such that \( P = Q + x \), then

\[
\Phi_P(\xi(P)) = \Phi_Q(\xi(Q)).
\]

Thus, we can choose a sequence \( P_i \in \mathcal{P}(u_1, ..., u_N) \) with \( \xi(P_i) = 0 \) and \( V(P_i) = 1 \) such that \( \Phi_{P_i}(o) \) converges to

\[
\inf \left\{ \max_{\xi \in \text{Int}(Q)} \Phi_Q(\xi) : Q \in \mathcal{P}(u_1, ..., u_N) \text{ and } V(Q) = 1 \right\}.
\]

Choose a fixed \( P_0 \in \mathcal{P}(u_1, ..., u_N) \) with \( V(P_0) = 1 \), then

\[
\inf \left\{ \max_{\xi \in \text{Int}(Q)} \Phi_Q(\xi) : Q \in \mathcal{P}(u_1, ..., u_N) \text{ and } V(Q) = 1 \right\} \leq \Phi_{P_0}(\xi(P_0)).
\]
We claim that $P_i$ is bounded. Otherwise, from Lemma 3.1, $\Phi_{P_i}(\xi(P_i))$ is not bounded from above. This contradicts the previous inequality. Therefore, $P_i$ is bounded.

From Lemma 3.2 and the Blaschke selection theorem, there exists a subsequence of $P_i$ that converges to a polytope $P$ such that $P \in \mathcal{P}(u_1, ..., u_N)$, $V(P) = 1$, $\xi(P) = o$ and

$$\Phi_P(o) = \inf \left\{ \max_{\xi \in \text{Int}(Q)} \Phi_Q(\xi) : Q \in \mathcal{P}(u_1, ..., u_N) \text{ and } V(Q) = 1 \right\}.$$  

We next prove that $F(P, u_i)$ are facets for all $i = 1, ..., N$. Otherwise, there exists an $i_0 \in \{1, ..., N\}$ such that

$$F(P, u_{i_0})$$

is not a facet of $P$.

Choose $\delta > 0$ small enough so that the polytope

$$P_\delta = P \cap \{ x : x \cdot u_{i_0} \leq h(P, u_{i_0}) - \delta \} \in \mathcal{P}(u_1, ..., u_N),$$

and (by Lemma 4.2)

$$V(P_\delta) = 1 - (c_n \delta^n + ... + c_2 \delta^2),$$

where $c_n, ..., c_2$ are constants that depend on $P$ and direction $u_{i_0}$.

From Lemma 3.2, for any $i \to 0$ it always true that $\xi(P_{\delta_i}) \to o$. We have,

$$\lim_{\delta \to 0} \xi(P_\delta) = 0.$$

Let $\delta$ be small enough so that $h(P, u_k) > \xi_P(P_\delta) \cdot u_k + \delta$ for all $k \in \{1, ..., N\}$, and let

$$\lambda = V(P_\delta)^{-\frac{1}{n}} = (1 - (c_n \delta^n + ... + c_2 \delta^2))^{-\frac{1}{n}}.$$

From this and Equation (3.3), we have

$$\prod_{k=1}^{N} (h(P_{\delta}, u_k) - \xi(P_{\delta}) \cdot u_k)^{\gamma_k} = \lambda \prod_{k=1}^{N} (h(P_{\delta}, u_k) - \xi(P_{\delta}) \cdot u_k)^{\gamma_k}$$

$$= \lambda \left[ \prod_{k=1}^{N} (h(P, u_k) - \xi(P_\delta) \cdot u_k)^{\gamma_k} \right] \left[ \frac{h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0} - \delta}{h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0}} \right]^{\gamma_{i_0}}$$

$$= \lambda \left[ \prod_{k=1}^{N} (h(P, u_k) - \xi(P_\delta) \cdot u_k)^{\gamma_k} \right] \left[ \frac{(1 - \frac{\delta}{h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0})})^{\gamma_{i_0}}{(1 - (c_n \delta^n + ... + c_2 \delta^2))^{\frac{1}{n}} \right]$$

$$\leq \prod_{k=1}^{N} (h(P, u_k) - \xi(P_\delta) \cdot u_k)^{\gamma_k} \right] \left[ \frac{(1 - \frac{\delta}{d_0})^{\gamma_{i_0}}}{{(1 - (c_n \delta^n + ... + c_2 \delta^2))^{\frac{1}{n}} \right]},$$

where $d_0 = d(P)$ is the diameter of $P$. Thus,

$$\Phi_{\lambda P_\delta} (\xi(\lambda P_\delta)) \leq \Phi_P (\xi(P_\delta)) + B(\delta),$$

where

$$B(\delta) = \gamma_{i_0} \log \left( 1 - \frac{\delta}{d_0} \right) - \frac{1}{n} \log \left( 1 - (c_n \delta^n + ... + c_2 \delta^2) \right).$$

Obviously,

$$B'(\delta) = \gamma_{i_0} \frac{-1/d_0}{1 - \delta/d_0} + \frac{1}{n} \frac{nc_n \delta^{n-1} + ... + 2c_2 \delta}{1 - (c_n \delta^n + ... + c_2 \delta^2)} < 0,$$

when the positive $\delta$ is small enough. From this and the fact that $B_1(0) = 0$,

$$B(\delta) < 0.$$
when the positive $\delta$ is small enough.

From this and Equations (4.6), (4.7), (4.8), there exists a $\delta_0 > 0$ such that $P_{\delta_0} \in \mathcal{P}(u_1, ..., u_N)$ and
\[
\Phi_{\lambda_0 P_{\delta_0}}(\xi(\lambda_0 P_{\delta_0})) \leq \Phi_P(\xi(P_{\delta_0})) \leq \Phi_P(\xi(P)) = \Phi_P(o),
\]
where $\lambda_0 = V(P_{\delta_0})^{-\frac{1}{n}}$. Let $P_0 = \lambda_0 P_{\delta_0} - \xi(\lambda_0 P_{\delta_0})$, then $P_0 \in \mathcal{P}(u_1, ..., u_N)$, $V(P_0) = 1$, $\xi(P_0) = o$ and
\[
\Phi_{P_0}(o) < \Phi_P(o).
\]
This contradicts Equation (4.3). Therefore, $P \in \mathcal{P}_N(u_1, ..., u_N)$. \hfill $\square$

5. Existence of the Solution of the Discrete Logarithmic Minkowski Problem

In this section, we prove the main theorem of this paper.

If $\mu$ is a Borel measure on $S^{n-1}$ and $\xi$ is a proper subspace of $\mathbb{R}^n$, it will be convenient to write $\mu_\xi$ for the restriction of $\mu$ to $S^{n-1} \cap \xi$.

The following lemma will be needed.

**Lemma 5.1.** Suppose $n \geq 2$, $\mu$ is a discrete measure on $S^{n-1}$ that satisfies the subspace concentration condition with respect to any essential linear subspace. If $\xi$ is an essential linear subspace with respect to $\mu$ for which
\[
\mu(\xi \cap S^{n-1}) = \frac{1}{n} \mu(S^{n-1}) \dim \xi,
\]
then $\mu_\xi$ satisfies the subspace concentration condition with respect to any proper linear subspace $\xi' \subset \xi$ such that $\xi' \cap \text{supp}(\mu_\xi)$ is not contained in any closed hemisphere of $\xi' \cap S^{n-1}$.

**Proof.** By conditions,
\[
\mu(\xi \cap S^{n-1}) = \frac{1}{n} \mu(S^{n-1}) \dim \xi.
\]

Suppose there exists a proper subspace $\xi' \subset \xi$ such that $\xi' \cap \text{supp}(\mu_\xi) = (\xi' \cap \text{supp}\mu)$ is not contained in any closed hemisphere of $\xi' \cap S^{n-1}$, then by conditions of the lemma,
\[
\mu_\xi(\xi' \cap (\xi \cap S^{n-1})) = \mu(\xi' \cap (\xi \cap S^{n-1})) = \mu(\xi' \cap S^{n-1}) \leq \frac{1}{n} \mu(S^{n-1}) \dim \xi' = \frac{1}{\dim \xi} \mu_\xi(\xi \cap S^{n-1}) \dim \xi.'
\]

Equality hold if there exists a $\xi'_1$ that complementary to $\xi'$ (in $\mathbb{R}^n$) such that $\mu$ concentrated on $S^{n-1} \cap (\xi' \cup \xi'_1)$. Thus, $\mu_\xi$ concentrated on $(\xi \cap S^{n-1}) \cap (\xi' \cup (\xi'_1 \cap \xi))$. \hfill $\square$

For even measures, the following lemma was proved by Böröczky et al. [6] (Lemma 7.2). One can easily find that their proof works for arbitrary measures, which is the following lemma.

**Lemma 5.2.** Let $\xi$ and $\xi'$ be complementary subspaces in $\mathbb{R}^n$ with $0 < \dim \xi < n$. Suppose $\mu$ is a Borel measure on $S^{n-1}$ that is concentrated on $S^{n-1} \cap (\xi \cup \xi')$, and so that
\[
\mu(\xi \cap S^{n-1}) = \frac{1}{n} \mu(S^{n-1}) \dim \xi.
\]

If $\mu_\xi$ and $\mu_{\xi'}$ are cone-volume measures of convex bodies in the subspaces $\xi$ and $\xi'$, then $\mu$ is the cone-volume measure of a convex body in $\mathbb{R}^n$.

The following lemma will be needed.
Lemma 5.3. Suppose $\mu$ is a Borel measure on $S^{n-1}$, $n \geq 2$, that is not concentrated on any closed hemisphere, and $\mu$ concentrated on two complementary subspaces $\xi$ and $\xi'$ of $\mathbb{R}^n$. Then, $\mu_\xi$ is not concentrated on any closed hemisphere of $\xi \cap S^{n-1}$ and $\mu_{\xi'}$ is not concentrated on any closed hemisphere of $\xi' \cap S^{n-1}$.

Proof. We only need prove that $\mu_\xi$ is not concentrated on any closed hemisphere of $\xi \cap S^{n-1}$.

Suppose $\mu_\xi$ is concentrated on a closed hemisphere, $C$, of $\xi \cap S^{n-1}$. Then, $\mu$ is concentrated on $S^{n-1} \cap \text{pos}\{C \cup \xi'\}$.

However, $S^{n-1} \cap \text{pos}\{C \cup \xi'\}$ is a closed hemisphere of $S^{n-1}$. This contradicts the conditions of the lemma. Therefore, $\mu_\xi$ is not concentrated on any closed hemisphere of $\xi \cap S^{n-1}$. \qed

Theorem 5.4. Suppose $n \geq 1$, $\mu$ is a discrete measure on $S^{n-1}$ that not concentrated on any closed hemisphere, and $\mu$ satisfies the subspace concentration condition with respect to any essential linear subspace, then $\mu$ is the cone-volume measure of a polytope in $\mathbb{R}^n$.

Proof. We first claim that when $\mu$ satisfies the strict subspace concentration inequality with respect to any essential linear subspace, then $\mu$ is the cone-volume measure of a polytope in $\mathbb{R}^n$ (this can be seen from Lemma 5.3 and Lemma 4.3). We prove the theorem by induction.

The theorem is obviously true for $n = 1$. Suppose the theorem is true when the dimensions are no more than $n - 1$. We next prove it is true for dimensions $n$. From the claim at the beginning of the proof, we may suppose that the discrete measure $\mu$ is concentrated on $S^{n-1} \cap (\xi \cup \xi')$ and not concentrated on any closed hemisphere of $S^{n-1}$, where $\xi, \xi'$ are complementary subspaces of $\mathbb{R}^n$ with $0 < \dim \xi < n$. By Lemma 5.3, $\mu_\xi$ is not concentrated on any closed hemisphere of $\xi \cap S^{n-1}$; and $\mu_{\xi'}$ is not concentrated on any closed hemisphere of $\xi' \cap S^{n-1}$. On the other hand, by Lemma 5.1, $\mu_\xi$ satisfies the subspace concentration condition on $\xi \cap S^{n-1}$ with respect to any proper subspace, $\xi_1$, of $\xi$ such that $\xi_1 \cap \mu_\xi$ is not concentrated on a closed hemisphere of $S^{n-1} \cap \xi_1$; and $\mu_{\xi'}$ satisfies the subspace concentration condition on $\xi' \cap S^{n-1}$ with respect to any proper subspace, $\xi'_1$, of $\xi'$ such that $\xi'_1 \cap \mu_{\xi'}$ is not concentrated on a closed hemisphere of $S^{n-1} \cap \xi'_1$. From the induction hypothesis, $\mu_\xi$ is the cone-volume measure of a convex body in $\xi \cap \mathbb{R}^n$, and $\mu_{\xi'}$ is the cone-volume measure of a convex body in $\xi' \cap \mathbb{R}^n$. By Lemma 5.2, $\mu$ is the cone-volume measure of a convex body in $\mathbb{R}^n$. From the condition that $\mu$ is discrete, $\mu$ is the cone-volume measure of a polytope in $\mathbb{R}^n$. \qed

6. New inequalities for cone-volume measures

In this section, we established some inequalities for cone-volume measures.

The following example shows that the cone-volume measure of a convex body does not need to satisfy the subspace concentration condition with respect to essential linear subspace.

Example 6.0. Let $u_1, \ldots, u_n$ be an orthonormal basis of $\mathbb{R}^n$, and let $W = \{x \in u_1^\perp : |x \cdot u_i| \leq 1, \ i = 2, \ldots, n\}$ be an $(n-1)$-dimensional cube. For $r > 0$ and $i = 1, \ldots, n - 1$, $\xi_i = \text{lin}\{u_1, \ldots, u_i\}$ is an essential subspace for the cone-volume measure of the truncated pyramid $P_r = [-ru_1 - rw, u_1 + W]$. If $r > 0$ is small, then $P_r$ approximates $[0, u_1 + W]$, and thus

$$V_{P_r}(\xi_i \cap S^{n-1}) > V_{P_r}([u_1]) = V([0, u_1 + W]) > \frac{1}{n}V(P_r).$$

We next establish some inequalities for the cone-volume measures.

Lemma 6.1. If $K$ is a convex body in $\mathbb{R}^n$, $n \geq 3$, with $0 \in \text{Int}(K)$, then for $u \in S^{n-1}$

$$V_K(\{u\}) + V_K(\{-u\}) + 2(n-1)\sqrt{V_K(\{u\})V_K(\{-u\})} \leq V(K),$$
with equality if and only if $F(K, -u)$ is a translation of $F(K, u)$, $K = [F(K, u), F(K, -u)]$, and $h(K, u) = h(K, -u)$.

In $\mathbb{R}^2$, we have

**Lemma 6.2.** If $K$ is a convex body containing the origin in its interior in $\mathbb{R}^2$, and $u \in S^1$, then

\[
\sqrt{V_K(\{u\})} + \sqrt{V_K(\{-u\})} \leq \sqrt{V(K)},
\]

with equality if and only if $K$ is a trapezoid (or parallelogram) with two sides parallel to $u^\perp$, and $u^\perp$ contains the intersection of the diagonals.

We obtain the following estimate from Lemma 6.1 and Lemma 6.2.

**Corollary 6.3.** If $K$ is a convex body in $\mathbb{R}^n$, $n \geq 2$ with $o \in \text{Int}(K)$ and $u \in S^{n-1}$, then

\[
V_K(\{u\}) \cdot V_K(\{-u\}) \leq \frac{1}{4n^2}(V(K))^2,
\]

with equality if and only if $F(K, -u)$ is a translation of $F(K, u)$, $K = [F(K, u), F(K, -u)]$, and $h(K, u) = h(K, -u)$.

We next prove Lemma 6.1 and Lemma 6.2 together.

**Proof.** We write $| \cdot |$ to denote $(n - 1)$-dimensional measure. Without loss of generality, we can suppose $|F(K, u)| \cdot |F(K, -u)| \neq 0$.

Let $V_K(\{u\}) = \alpha > 0$ and $V_K(\{-u\}) = \beta > 0$, let $h_K(u) = a$ and $h_K(-u) = b$, and for $0 \leq x \leq a + b$ let

\[ K_x = ((a - x)u + u^\perp) \cap K. \]

Since $K$ is a convex body,

\[
\frac{x}{a + b}F(K, -u) + \frac{a + b - x}{a + b}F(K, u) \subset K_x.
\]

From this and the Brunn-Minkowski inequality,

\[
|K_x| \geq \left| \frac{x}{a + b}F(K, -u) + \frac{a + b - x}{a + b}F(K, u) \right|
= \left| \left( \frac{x}{a + b}F(K, -u) + \frac{a + b - x}{a + b}F(K, u) \right)_{u^\perp} \right|
= \left| \frac{x}{a + b}F(K, -u)|_{u^\perp} + \frac{a + b - x}{a + b}F(K, u)|_{u^\perp} \right|
\geq \left( \frac{x}{a + b}|F(K, -u)|_{u^\perp}^{\frac{1}{n-1}} + \frac{a + b - x}{a + b}|F(K, u)|_{u^\perp}^{\frac{1}{n-1}} \right)^{n-1}
= \left( \frac{x}{a + b}|F(K, -u)|_{u^\perp}^{\frac{1}{n-1}} + \frac{a + b - x}{a + b}|F(K, u)|_{u^\perp}^{\frac{1}{n-1}} \right)^{n-1},
\]

with equality if and only if $K_x = \frac{x}{a + b}F(u(K, -u) + \frac{a + b - x}{a + b}F(K, u)$, and $F(K, -u)|_{u^\perp}$ and $F(K, u)|_{u^\perp}$ are homothetic.
Let \( t = \frac{a+b-x}{a+b} \). From (6.3) and Fubini’s formula,
\[
V(K) = \int_{0}^{a+b} |K_x|dx \\
\geq \int_{0}^{a+b} \left( \frac{x}{a+b} |F(K,-u)|^{\frac{1}{n-1}} + \frac{a+b-x}{a+b} |F(K,u)|^{\frac{1}{n-1}} \right)^{n-1} dx \\
= (a+b) \int_{0}^{1} \left( t|F(K,u)|^{\frac{1}{n-1}} + (1-t)|F(K,-u)|^{\frac{1}{n-1}} \right)^{n-1} dt \\
= (a+b) \sum_{i=0}^{n-1} |F(K,u)|^{\frac{i}{n-1}} |F(K,-u)|^{\frac{n-1-i}{n-1}} \int_{0}^{1} t^i(1-t)^{n-1-i} dt \\
= \frac{a+b}{n} \sum_{i=0}^{n-1} |F(K,u)|^{\frac{i}{n-1}} |F(K,-u)|^{\frac{n-1-i}{n-1}}.
\]

(6.4)

Let \( S_1 = |F(K,u)| \) and \( S_2 = |F(K,-u)| \). From (6.4) and the arithmetic-geometric inequality, we have
\[
V(K) = \frac{a+b}{n} \sum_{i=0}^{n-1} S_1^{\frac{i}{n-1}} S_2^{\frac{n-1-i}{n-1}} \\
= \frac{a}{n} S_1 + \frac{b}{n} S_2 + \frac{1}{n} \sum_{i=1}^{n-1} \left( aS_1^{\frac{n-1-i}{n-1}} S_2^{\frac{i}{n-1}} + bS_2^{\frac{n-1-i}{n-1}} S_1^{\frac{i}{n-1}} \right) \\
\geq \alpha + \beta + 2(n-1)\sqrt{\alpha \beta}.
\]

(6.5)

Thus, we get (6.1) and (6.2).

From the equality conditions for (6.3), (6.4) and the arithmetic-geometric inequality, we have, equality holds in (6.5) if and only if \( F(K,u)|_{u^=} \) and \( F(K,-u)|_{u^=} \) are homothetic, \( K = [F(K,u), F(K,-u)] \), and
\[
(6.6) \quad \frac{a}{b} = \left( \frac{S_1}{S_2} \right)^{\frac{2i-n+1}{n-1}},
\]

for all \( 1 \leq i \leq n-1 \).

Therefore, equality holds in (6.2) \( (n=2) \) if and only if \( K \) is a trapezoid (or parallelogram) with two sides parallel to \( u^= \), and \( u^= \) contains the intersection of the diagonals.

When \( n \geq 3 \), Equation (6.6) hold for \( i = 1, ..., n-1 \). Thus, \( \frac{a}{b} = \frac{S_1}{S_2} = 1 \). Therefore, equality holds in (6.1) if and only if \( F(K,-u) \) is a translation of \( F(K,u) \), \( K = [F(K,u), F(K,-u)] \), and \( h_K(u) = h_K(-u) \).

\[\square\]

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