Low-energy sum rules and large-$N_c$ consistency conditions

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Abstract

The large-$N_c$ consistency conditions for axial vector and isovector magnetic couplings of pions to baryons are discussed from the point of view of low-energy current-algebra sum rules (Adler-Weisberger, Cabibbo-Radicati). In particular, we show how the result that ratios of axial vector and isovector magnetic coupling constants get corrections only at the order $1/N_c^2$ follows from the $N_c$-counting of appropriate cross sections. This counting is performed using various approaches at the quark and hadronic level. Other implications of our method are also presented.

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I. INTRODUCTION

Recently there has been a renewed interest in the large-$N_c$ (number of colors) limit of QCD and its implications in the baryon sector. This interest was revived by the derivation of the large-$N_c$ consistency conditions (which we denote by LNCC) by Dashen and Manohar. These conditions concern axial vector coupling constants of pions to baryons, as well as other observables, e.g. isovector magnetic moments. Several recent papers concentrate on these and related issues. The nice and important feature of LNCC is the fact that they reconcile the large-$N_c$ limit of QCD with hadronic physics. They also confirm the special role of the baryon decuplet states, which are crucial to achieve consistency.

The purpose of this paper is to discuss in detail these conditions from the point of view of low-energy current-algebra sum rules. We show that starting from the Adler-Weisberger (AW) and the Cabibbo-Radicati (CR) sum rules one can very straightforwardly obtain the results of Ref. for axial vector couplings and isovector magnetic couplings, respectively. In addition, we get the result that the ratios of isovector magnetic couplings get corrections starting at the order $1/N_c^2$, in analogy to the axial vector case.

The basic elements of our derivation are 1) the special role of the $\Delta$ resonance in pion-nucleon or photon-nucleon scattering in the large-$N_c$ limit (its contribution is extracted from the sum rule and treated separately), and 2) careful $N_c$ counting of the cross sections which remain after the $\Delta$ contribution is extracted (a crucial cancellation of leading-$N_c$ powers occurs).

It was realized a long time ago that the $\Delta$-isobar plays an essential role in the large-$N_c$ physics. In this limit its mass is degenerate with the nucleon, and it is strongly coupled. It was also realized that there is an apparent discrepancy in the $N_c$ counting of the AW sum rule. To resolve this paradox it was proposed to extract the $\Delta$ contribution from the cross section, which reconciled the $N_c$ counting. This procedure immediately led to the first set of LNCC of Refs., which in fact had been known much earlier in context.
of the strong-coupling theory [22] or in the Skyrmion [23]. We show that this analysis may be carried one step further if one carefully examines the \(N_c\) counting of the cross section which remains after extracting the \(\Delta\) contribution (we call it the background contribution). Using various approaches, we show that the leading-\(N_c\) terms in this background contribution cancel, which immediately leads to the second set of LNCC of Refs. [7,8], namely, that the ratios of axial vector coupling constants acquire corrections only of the order \(1/N_c^2\). We point out, that LNCC also imply relations between axial vector couplings of the nucleon to other excited states, such as the Roper resonance, as has been recently suggested by Ref. [24]. A similar analysis for the CR sum rule leads to LNCC for matrix elements of the isovector magnetic moment. In the large-\(N_c\) limit these ratios are the same as for the case of the axial vector couplings [7,8]. Furthermore, they acquire corrections starting at the level \(O(1/N_c^2)\), as in the case of the axial vector couplings. To our knowledge, this result has not been realized before.

Our method can also be used to obtain other large-\(N_c\) consistency results. For instance, the analysis of the Goldberger-Miyazawa-Oehme sum rule [25] discussed in Ref. [26] yields the result that the leading-\(N_c\) part of the isospin-odd pion-nucleon scattering length vanishes, which was found before for the Skyrme model [27–29]. We also point out that the large-\(N_c\) consistency rules lead to the correct \(N_c\) scaling of the pion loop contribution to the isovector charge form factor of the nucleon.

The outline of the paper is as follows: In Sec. II we very briefly review the original derivation of LNCC [7,8]. In Sec. III we turn to the Adler-Weisberger sum rule, and show how these conditions follow if the \(N_c\) counting is done properly for the charge-exchange pion-cross section. The \(\Delta\) contribution is extracted, and the remaining terms are carefully analyzed in Sec. IV. We perform this analysis at the level of quark degrees of freedom (Sec. IVA), as well as at the level of hadronic degrees of freedom (Sec. IVB). In the latter case we consider separately various possible contributions, from the Regge exchange at high energies, through intermediate energy resonance region, to the low-energy threshold region. Interesting implications [24] for coupling of the nucleon to various excited states are
discussed. In Sec. V we derive LNCC for isovector magnetic couplings. Other LNCC results are presented in Sec. VI. Section VII summarizes our basic results and conclusions.

For the reader’s convenience, all $N_c$-counting rules used in this paper are listed in Table I.

II. LARGE-$N_C$ CONSISTENCY CONDITIONS

We begin with a very brief review of the large-$N_C$ consistency conditions as derived in Refs. [7,8]. We remain in the two-flavor sector to avoid some complications when more flavors are present [30]. The first set of relations (from now on called (I)) follows from the examination of the pion-nucleon scattering amplitude, $T_{\pi NN}$, and the observation that the Born diagrams with the nucleon in the intermediate state lead to $N_c^1$ dependence of $T_{\pi NN}$, whereas, according to Witten’s counting [2], it can go at most as $N_c^0$. One can resolve this paradox by including in the consideration the diagrams with the $\Delta$ field in the intermediate state. Cancellation of the “wrong” part of $T_{\pi NN}$, proportional to $N_c^1$, occurs if in the large $N_c$ limit the $\Delta$-N mass splitting goes to 0, and if the ratio $g_{\pi N \Delta}/g_{\pi NN}$ has a certain value [7,8]. Repeating the argument for scattering off the $\Delta$, and yet higher spin and isospin states, one gets a set of ratios of axial vector coupling constants between the whole tower of the nucleon, $\Delta$, $I = J = 5/2$ states, etc.

One should remark here that the set (I) of consistency conditions has been known for a very long time in context of strong-coupling theories [22] — it just follows from unitarity. Thus, it holds for strong-coupling models as different as the Chew-Low theory [31], and the Skyrme model [23].

The second set (denoted by (II)) of LNCC may be derived by examining the inelastic process $\pi N \rightarrow \pi \pi N$. By reconciling the $N_c$ counting at the hadronic level with Witten’s rules [2], one obtains the result that the corrections to ratios of axial vector coupling constants from the values given by (I) vanish at the level $1/N_c$, and start at the level $1/N_c^2$. To our knowledge, this result has not been known before Ref. [7]. It adds more significance to conditions (I), since it means that expected corrections to the large $N_c$ values enter at the
level of, say, 10%.

The third set of consistency conditions, (III), concerns matrix element of other operators, e.g. isovector magnetic moments [3,6], and is obtained in Ref. [8] by considering the pion loop dressing of appropriate vertices.

III. ADLER-WEISBERGER SUM RULE AND $N_c$-COUNTING

In this section we turn to the main goal of this paper, which is an alternative derivation of LNCC (I)–(III) starting from low-energy sum rules. Consider the Adler-Weisberger sum rule [14,15] for the pion-nucleon scattering:

$$1 - g_A^2 = \frac{2 F_\pi^2}{\pi} \int_{m_\pi}^\infty \frac{d\omega}{\omega^2} \sqrt{\omega^2 - m_\pi^2} \sigma^{(-)}(\omega),$$

(1)

where $\omega$ is the pion LAB energy, $F_\pi = 93 MeV$ is the pion decay constant, $m_\pi$ is the pion mass, and $\sigma^{(-)}(\omega) = \sigma_{\pi^-p} - \sigma_{\pi^+p}$ is the total charge-exchange cross section for the $\pi$-$N$ scattering. It was noticed several years ago that if Witten’s counting rules are used naively in Eq. (1), then inconsistency follows [17–21]. Indeed, the left-hand-side of Eq. (1) scales as $N_c^2$, and, naively, the right-hand-side scales at most as $N_c$, since the cross section scales at most as $N_c^0$. The resolution of this paradox, quite reminiscent of the paradox which led to relations (I) in the derivation of Ref. [7,8], is hidden in the role of the $\Delta$ resonance.

Let us first glance at Witten’s derivation of the $N_c$ rules for scattering amplitudes. The idea is essentially contained in Fig. 1. From the optical theorem, the cross section is related to the imaginary part of the forward scattering amplitude. In the first case (Fig. 1(a)) the external pion lines are attached to the same quark line, and the resulting scaling of the cross section is $N_c \times (1/\sqrt{N_c})^2 \sim N_c^0$, where the first factor comes from combinatorics (number of lines), and the second factor comes from the pion-quark coupling constant squared. In the second case (Fig. 1(b)) the external pion lines are attached to different quark lines, and at least one gluon exchange between these lines is necessary. The resulting scaling of the cross section is $N_c(N_c+1)/2 \times (1/\sqrt{N_c})^2/N_c \sim N_c^0$, where the first factor comes from combinatorics.
(number of pairs), the second factor comes from the pion-quark coupling constant squared, and the last factor comes from the gluon exchange. Thus, the amplitude, or the cross section, scales as $N_c^0$. There is one instance, however, where the above “finger counting” fails. This is the case when the energy denominator of the intermediate state vanishes, as happens for the case of $\Delta$ in the large-$N_c$ limit. Additional powers of $N_c$ may arise in this resonant case. We will see later in a greater detail why for the processes involving the $\Delta$ the naive counting does not work, and what should be done in this case.

Returning to the AW sum rule, we note that it was proposed in Refs. [17–21,32,33] how to resolve the paradox of the $N_c$ counting of Eq. (1). The contribution of the $\Delta$ to the sum rule has to be extracted out of the dispersive integral. Note, that in the large-$N_c$ limit, the $\Delta$ is a very narrow resonance. There are two cases we can consider: 1) finite pion mass, and 2) the pion mass is set to zero (cases 1) and 2) correspond to two orderings of the chiral and large-$N_c$ limits). In the first case the $\Delta$ becomes absolutely stable in the large $N_c$-limit (it is infinitely narrow), since the pion-nucleon decay channel is energetically forbidden ($M_\Delta - M_N \sim 1/N_c$, $m_\pi \sim N_c^0$). In the second case, the $\Delta$ decay width is

$$\Gamma_\Delta \sim \frac{g_A^2}{F_\pi^2} \left( \frac{M_\Delta^2 - M_N^2}{M_\Delta} \right)^3 \sim \frac{1}{N_c^2},$$

hence the resonance is narrow compared to $N_c$ scaling of all other relevant scales (e.g. the resonance position). Thus, in both cases 1) and 2) it is legitimate to pull out the contribution of the $\Delta$ resonance from the integral in Eq. (1). Using the narrow resonance approximation we arrive at the following form of the AW sum rule: [17–21]

$$1 - g_A^2 = -(g_A^*)^2 + \frac{2F_\pi^2}{\omega^2} \int_{m_\pi}^{\infty} d\omega \sqrt{\omega^2 - m_\pi^2} \sigma^{(-)}_{\text{background}}(\omega),$$

where

$$g_A^* = \frac{2F_\pi g_\pi N}{3M_\Delta} \frac{M_N + M_\Delta}{2M_\Delta} = \frac{2F_\pi g_\pi N}{3M_N} + O(1/N_c) = \frac{2g_\pi N}{3g_\pi NN} g_A + O(1/N_c).$$

We have used the Goldberger-Treiman relation in the last equality. We call the remaining cross section in the integrand of Eq. (3) the background cross section; it includes all processes apart from the $\Delta$ resonance contribution.
The proper $N_c$-counting of the background contribution to Eq. (3) is essential for our results. According to Witten’s counting, the cross section may scale at most as $N_c^0$. In the next section we will show that, in fact, it is suppressed by one additional power of $N_c$, and

$$\sigma_{\text{background}}^{(-)} \sim 1/N_c.$$  

(5)

Before showing Eq. (5), however, let us assume for a moment that it holds and list the immediate consequences of this scaling. Using Eq. (5) in Eq. (3), and the fact that $F^2_\pi \sim N_c$, we obtain

$$g_A^2 = (g_A^*)^2 + \mathcal{O}(N_c^0).$$  

(6)

Next, we expand $g_A$ and $g_A^*$ in decreasing powers of $N_c$, and denote the coefficients in this expansion by superscripts corresponding to the power of $N_c$, i.e.

$$g_A = g_A^{(1)} N_c + g_A^{(0)} + g_A^{(-1)} N_c^{-1} + ..., \quad (7)$$

and similarly for $g_A^*$. Then, Eq. (8) leads to the following relations:

$$\frac{g_A^{*(1)}}{g_A^{(1)}} = 1, \quad \frac{g_A^{*(0)}}{g_A^{(0)}} = 1.$$  

(8)

It also immediately gives the condition (here we use the Goldberger-Treiman relation)

$$\frac{g_{\pi N\Delta}^{(0)}}{g_{\pi N\Delta}^{(1)}} = \frac{g_{\pi NN}^{(0)}}{g_{\pi NN}^{(1)}}.$$  

(9)

which means that the ratio of the first $N_c$-subleading term to the leading term is the same for $g_{\pi NN}$ and $g_{\pi N\Delta}$. One also obtains

$$\frac{g_A^*}{g_A} = 1 + \mathcal{O}(1/N_c^2).$$  

(10)

The first equation in (8) is nothing else but the consistency condition (I) of Ref. [7,8]. The second equation in (8), and Eqs. (9-10) are consistency conditions (II).

Considering Adler-Weisberger sum rules for scattering of pions on $\Delta$’s, and yet higher $I = J$ resonances, we obtain the consistency conditions (I) and (II) of Ref. [7,8] for all
member of the $I = J$ tower. Thus, conditions (I) and (II) can alternatively be derived in a rather straightforward manner starting from the Adler-Weisberger sum rule. Note, that if instead of the scaling of Eq. (6) we had hastily used the weaker statement that $\sigma_{\text{background}}^{(-)}$ scales at most as $N_c^0$, then conditions (II) would not have followed.

**IV. $N_c$ COUNTING OF $\sigma^{(-)}$**

We now return to the question of the scaling of the background cross section, Eq. (6). Since this is the crucial point of this paper, we will discuss it twice, using 1) the quark point of view and 2) the hadronic point of view. In the first case we will just use the Witten counting rules for the charge exchange cross section. In the second case we will present the arguments at the level of hadronic physics.

**A. Quark level**

Using isospin invariance we can write $\sigma^{(-)} = \sigma_{\pi^-p} - \sigma_{\pi^-n}$. Consider the situation presented in Fig. 2. In the large-$N_c$ world, the proton is represented by a state with $(N_c + 1)/2$ up quarks and $(N_c - 1)/2$ down quarks (Fig. 2(a)), and the neutron has one more down and one less up quark (Fig. 2(b)). In addition to the flavor composition, the states in Fig. 4 have some spin configuration. Examples of various possible scattering processes are depicted in (Figs. 2(a–f)).

First let us consider the difference of processes from Figs. 2(a) and (b). The $\pi^-$-quark interaction is proportional to $\sigma \cdot q \overrightarrow{\tau}$, where $q$ is the momentum transferred to the quark. From this form it is clear that the forward amplitude of the elementary pion-quark scattering does not depend on the spin of the quark. Thus, we get the following $N_c$ behavior for the difference of processes of Fig. 2(a) and (b):

$$
\sigma_{\text{background}}^{(-)} \sim \frac{N_c + 1}{2} \left( \frac{1}{\sqrt{N_c}} \right)^2 - \frac{N_c - 1}{2} \left( \frac{1}{\sqrt{N_c}} \right)^2 \sim 1/N_c,
$$

(11)
where the first factor in each component comes from combinatorics (number of \(u_p\)-quark lines), and the second factor comes from the square of the quark-pion coupling constant. In simple words, the cancellation of the leading \(N_c\) power is due to the fact that the proton has one more \(u_p\) quark than the neutron.

A similar inspection may be done for other processes in Fig. 2. Again, for the forward amplitude there is no dependence on the spins of the quarks. For the case of Figs. 2(c) and (d) we find that

\[
\sigma^{(-)}_{\text{background}} \sim \frac{(N_c + 1)(N_c - 1)}{8} \left( \frac{1}{\sqrt{N_c}} \right)^2 \frac{1}{N_c} - \frac{(N_c - 1)(N_c - 3)}{8} \left( \frac{1}{\sqrt{N_c}} \right)^2 \frac{1}{N_c} \sim \frac{1}{N_c},
\]

(12)

where the first factor in each component comes from combinatorics (number of \(u_p\)-quark pairs), the second factor comes from the square of the quark-pion coupling constant, and the last factor comes from the gluon exchange. Repeating this exercise for all other possible processes we always get the cancellation of the leading part, and thus obtain Eq. (11) for the case of counting at the quark level.

**B. Hadronic level**

In this subsection we discuss Eq. (11) from the point of view of hadronic physics. In the sum rule, we need the background cross section at all energies. The energy may be divided into three regions: low energies, close to the pion threshold, the *resonance* region \((\sim 0.5 - 4 \text{ GeV})\), and the high-energy *Regge* tail. Let us begin from the high-energy end. It is well established that at high energies the *Regge exchange phenomenology* works very well for charge-exchange processes. In our case, the relevant Reggeon is the \(\rho\) Reggeon. The exchange is shown in Fig. 3(a). The QCD structure of the Reggeon is illustrated in Fig. 3(b). It is clear from that figure that the \(N_c\) scaling of this \(t\)-channel \(\rho\)-Reggeon exchange is the same as the scaling of the usual \(\rho\)-meson exchange. We have \(g_{\rho\pi\pi} \sim g_{\rho NN} \sim N_c^{-1/2}\), and similarly \(g_{R\rho\pi\pi} \sim g_{R\rho NN} \sim N_c^{-1/2}\). Note that this is consistent with the universality hypothesis, which also works remarkably well for the Regge phenomenology [34]. At this
point one may also resort to explicit expressions from the Regge phenomenology. As shown in Ref. [34], the expression for the cross section described by the process of Fig. 3 has, for large energies \( \omega \), the form

\[
\sigma_{\text{Regge}}^{(-)} = \pi g_{\rho \pi \pi}(0) g_{\rho NN}(0) \left( \frac{M_N \omega}{s^2 N_\pi N_N} \right)^{\alpha_0 - 1},
\]  

(13)

where \( s \) and \( \Lambda \) are some \( N_c \) independent scales, \( N_\pi \sim N_c^0 \) is the average number of constituents in the pion, \( N_N \sim N_c \) is the average number of constituents in the nucleon, and \( \alpha_0 \) is the \( \rho \)-Reggeon intercept, which scales as \( N_c^0 \) \( [35] \). Hence, the process of Fig. 3 scales as \( 1/N_c \), as anticipated, and the high-energy tail contribution obeys Eq. (6).

The intermediate energy range is the realm of various hadronic resonances. To the extent they are narrow, they yield the following contribution to the AW sum rule \([7,36]\):

\[
1 - g_A^2 = -(g_A^*)^2 + F_\pi^2 \sum_{N^*} \left[ \frac{1}{3} \right] g_{\pi N N^*}^2 (M_{N^*}^2 - M_N^2)^{2j - 3} \left( M_{N^*} \pm M_N \right)^2 [j + 1/2]!^2 / \left( M_N M_{N^*} \right)^{2j - 1} / (2j + 1)!^2,
\]  

(14)

where \( g_A^* \) is the \( \Delta \) resonance contribution \([4]\), \( N^* \) denotes any of the resonances with spin \( j \), and isospin 1/2 or 3/2. The upper and lower levels in the bracket correspond to isospin 1/2 and 3/2 contributions, respectively, and the sign factor corresponds to parity \((-1)^{j \pm 1/2}\).

Let us first concentrate on the contribution of the Roper resonance to the right-hand-side of Eq. (14), following Refs. \([24,37]\). In the large-\( N_c \) limit the mean-field description of baryons is valid, and the Roper may be viewed as a single particle excitation of the nucleon: one quark is promoted to an excited state. For simplification of the argument, let us use hedgehog baryons. The ground state of a hedgehog baryon consists of \( N_c \) quarks which occupy the same space-spin-isospin state \( q \), and in the first-quantized notation we may write \( |H\rangle = |qqq...\rangle \).

The hedgehog Roper baryon is obtained by promoting one quark to a radially excited state \( q^* \). Since any of the quarks can be excited, we have the following space-spin-isospin symmetrized wave function: \( |H^*\rangle = 1/\sqrt{N_c}(|q^*qq...\rangle + |qq^*q...\rangle + |qqq^*...\rangle + ...\). Consider a one-body current \( J \), \( e.g. \) the pion source current. We find \( \langle H | J | H \rangle = N_c \langle q | j | q \rangle \), and \( \langle H^* | J | H \rangle = \sqrt{N_c} \langle q^* | j | q \rangle \). For the case of the pion source current we have
\langle q \mid j \mid q \rangle \sim \langle q^* \mid j \mid q \rangle \sim 1/\sqrt{N_c}, and we see that the \(H^*-\)pion coupling constant is suppressed by the factor \(\sqrt{N_c}\) compared to the \(H\)-pion coupling constant.

The same result holds for the case of states of good spin and isospin. One has \(g_{\pi NN}/(2M) \sim \sqrt{N_c}\) and \(g_{\pi N_{Roper}}/(2M) \sim N_c^0\). Similarly, for other spin and parity resonances (apart from the \(\Delta(1232)\)) the result is

\[
\frac{g_{\pi NN^*}}{M_N^{j-1/2}} \sim N_c^0 \quad \text{for parity } P = (-1)^{j+1/2} \\
\frac{g_{\pi NN^*}}{M_N^{j+1/2}} \sim N_c^0 \quad \text{for parity } P = (-1)^{j-1/2}
\]

(15)

The presence of powers of \(M_N\) comes from the standard definition of the couplings \[36,38\]. In the \(P = (-1)^{j-1/2}\) case an extra power of \(M_N\) comes from the nonrelativistic reduction of the pseudoscalar matrix element.

Using Eq. (15) in Eq. (14) we find that the contribution of any spin and parity resonance to the right-hand-side of the AW sum rule scales as \(N_c\), one power too much compared to what was found in Sec. [\text{IV A}] and what is needed for LNCC (II). Therefore, as suggested in Ref. \[24\], a cancellation must occur between the resonance contributions to the AW sum rule (see Fig. 4). This is an interesting observation. In fact, reversing the logic of this paper and assuming that LNCC (II) hold, as derived in Refs. \[7,8\], we see that this has to happen. Thus, LNCC extend also to excited baryon couplings. It is outside the scope of this paper to attempt to demonstrate LNCC for excited states in specific large-\(N_c\) models.

A final remark we would like to make to end the discussion of resonances concerns their widths. We noted before that \(\Gamma_\Delta \sim N_c^{-2}\) (or is strictly 0 is the \(N_c\)-limit is taken before the chiral limit). For the other resonances Eq. (15) yields \(\Gamma^* \sim N_c^0\). This is not a “narrow” width, since it is of the same order as the mass splittings between the resonances. Therefore our use of Eq. (14) is somewhat questionable. However, this is an expected behavior. The resonances are wide enough in order to fill continuously the cross section, which at large energies may be described by the Regge phenomenology. Therefore one could say that duality requires the resonance width to scale at least as \(N_c^0\), and this is precisely what
happens.

There are other possible hadronic processes which may occur in pion-nucleon scattering at low and intermediate energies. Examples are depicted in Fig. 5. The diagram in Fig. 5(a) superficially scales as $N_c^0$. This is because the magnetic $\rho$-nucleon coupling scales as $\sqrt{N_c}$, and $g_{\rho\pi}\pi \sim 1/\sqrt{N_c}$. Due to the cancellation of diagrams with $N$ and $\Delta$ states the contribution to the charge-exchange cross section scales as $1/N_c$. Similarly, the diagram in Fig. 5(b) scales as $1/N_c$ due to cancellation from the contribution with the $\Delta$ resonance. For these cancellations to occur one has to use for the coupling the large-$N_c$ result of Eq. (9). Since this is what we want to prove, in this place we are just checking the consistency of the scheme. Once LNCC hold, processes such as in Fig. 5(a) preserve them.

An analogous consistency check can be done for various low-energy processes which may be important near the pionic threshold. For instance, we can construct diagrams with pion loops. By considering examples, we can convince ourselves that if all possible $I = J$ states are included in intermediate states, and if the coupling constants satisfy the consistency conditions, then these conditions are preserved. A simple example is the contribution to $Im(T_{\pi NN})$ with one pion loop. The corresponding diagram superficially scales as $N_c^2$. Assuming the large $N_c$ consistency conditions, and including the diagrams with the $\Delta$ and $I = J = 5/2$ resonances reduces the scaling by two powers of $N_c$. An additional power of $N_c$ is canceled by taking the difference in the charge-exchange cross section, such that finally the contribution to the AW sum rule from this process is of the order $N_c^0$.

To briefly summarize, we have demonstrated that also at the level of hadronic diagrams the crucial result of Eq. (9) follows. This, however, requires additional cancellations (or LNCC) between the $\pi NN^*$ coupling constants.
V. CABIBBO-RADICATI SUM RULE AND $N_c$-COUNTING

Now, we will turn our attention to the CR sum rule [16], and show how the large-$N_c$ consistency conditions for the ratios of isovector magnetic moments can be obtained. According to the large-$N_c$ rules, $\mu_{I=1} \sim N_c$ [39,40]. The CR sum rule has the form

$$\left(\mu_{I=1}^2\right)^2 \frac{2}{\pi} \int \frac{d\omega}{\omega} \left(\sigma_{3/2}^{l=1}(\omega) - 2\sigma_{1/2}^{l=1}(\omega)\right) + \frac{e^2}{3} \langle r \rangle^2, \quad (16)$$

where $\sigma_{3/2}^{l=1}$ and $\sigma_{1/2}^{l=1}$ are the total photoproduction cross sections for scattering of isovector photons off protons in total isospin $3/2$ and $1/2$ states, respectively. The quantity $\mu_N$ is the nuclear magneton, $e$ is the unit of the electric charge, and $\langle r \rangle^2$ is related to the isovector electric charge of the nucleon: $\frac{1}{6} \langle r \rangle^2 = -\frac{dG_{I=1}(q^2)}{dq^2}|_{q^2=0} + \frac{1}{8M^2}$ [16]. The left-hand-side of the sum rule scales as $N_c^2$, while the right-hand-side scales, according to the naive Witten’s prescription [2], scales at most as $N_c$ (of course, $\mu_N$, as a unit, scales as $N_c^0$, and $\langle r \rangle \sim N_c^0$).

Again, as in the case of the AW sum rule, the resolution of this paradox is hidden in the role of the $\Delta$. Before repeating the analysis of the previous sections for the present case, let us first rewrite the integrand of Eq. (16) in a more convenient fashion. Let us think of the isovector photon as of the neutral component of a massless “$\rho$-meson” [16]. Then, using the isospin symmetry, we have the following identity:

$$\sigma_{3/2}^{l=1} - 2\sigma_{1/2}^{l=1} = e^2 / g_\rho^2 (\sigma_{\rho^+p} - \sigma_{\rho^-p}). \quad (17)$$

Hence, the integrand in the sum-rule may be viewed as the charge-exchange cross section for the “massless $\rho$”-nucleon scattering. Note Eq. (17) is exact — it is merely a change of notation. Plugging Eq. (17) into Eq. (16) we obtain the form very reminiscent of the AW sum rule. The former case involved the pion-nucleon charge exchange scattering, the present case has the pion replaced by the “$\rho$”.

Now, we may pull the $\Delta$ contribution outside of the integral. Introducing notation analogous to Eq. (3) we may write

$$\left(\mu_{I=1}^2 - \mu^{*2}\right)^2 \frac{2e^2}{\pi \mu_N^2 g_\rho^2} \int \frac{d\omega}{\omega} \left(\sigma_{\rho^+p} - \sigma_{\rho^-p}\right)_{\text{background}} + O(N_c^0), \quad (18)$$
where $\mu^*$ is proportional to $\mu_{N\Delta}$. Using the large-$N_c$ result $\mu_{N\Delta} = \frac{1}{\sqrt{2}} \mu^{I=1}$, we obtain the desired cancellation of the leading-$N_c$ powers in Eq. (18). This is equivalent to the consistency condition (III) of Refs. [7,8], where it was found that in the large-$N_c$ limit matrix elements of the isovector magnetic moment are proportional to matrix elements of the axial vector current. 

However, one can go one step farther, as in the case of the AW sum rule. Repeating the arguments of Sec. [IV] we arrive at the result $(\sigma_{\rho^+ p} - \sigma_{\rho^- p})_{\text{background}} \sim N_c^{-1}$, and since $g_\rho \sim 1/\sqrt{N_c}$, the right-hand-side of Eq. (18) scales as $N_c^0$. This, in direct analogy to the discussion in Sec. [II], leads to the consistency condition

$$\frac{\mu^*)}{\mu^{I=1}} = 1 + O(1/N_c^2), \tag{19}$$

which means that the ratios of isovector anomalous magnetic couplings acquire corrections to the large-$N_c$ values at the level $O(1/N_c^2)$. We have therefore a completely parallel behavior of axial vector and isovector magnetic couplings in the large-$N_c$ limit. To our knowledge, Eq. (19) has not been realized in earlier works.

Note that Eq. (19), as well as Eq. (10) are realized in the quark model [5]. Also, these relations are obeyed by the recently calculated rotational $1/N_c$ corrections to solitons in the Nambu–Jona-Lasinio model [14,13].

VI. OTHER RESULTS

There is yet another implication of our approach — large-$N_c$ consistency rules for scattering lengths. In Ref. [28], the Goldberger-Miyazawa-Oehme sum rule [24] was considered:

$$\frac{2}{3} \left( \frac{1}{m_\pi} + \frac{1}{M_N} \right) (a_{1/2} - a_{3/2}) = A_N - A_\Delta + \frac{1}{2\pi^2} \int_{m_\pi}^\infty \frac{d\omega}{\sqrt{\omega^2 - m_\pi^2}} \sigma^{(-)}, \tag{20}$$

where $a_{1/2}$ and $a_{3/2}$ denote the pion-nucleon scattering lengths in isospin 1/2 and 3/2 channels, respectively, and the nucleon and $\Delta$ pole contributions are given by

$A_N = \frac{1}{4\pi} \left( \frac{g_A}{F_\pi} \right)^2 + O(1/N_c), \quad A_\Delta = -\frac{1}{4\pi} \left( \frac{g_A}{F_\pi} \right)^2 + O(1/N_c). \tag{21}$
Using Eq. (6) in Eq. (21) we find, reverting the worries of reference [26], that

\[ a_{1/2} - a_{3/2} \sim 1/N_c. \]  

(22)

The above relation has been found to hold in the Skyrme model [27–29].

Finally, we would like to comment on the large-$N_c$ scaling of the $\rho$-nucleon coupling. If universality is to be satisfied, then $\rho$ couples to isospin, and its (electric) coupling to the nucleon scales in the same manner as the coupling to the quark, $g_{\rho NN} \sim N_c^{-1/2}$. Consider, however, the hadronic diagram of Fig. 6. Without the inclusion of $\Delta$, this diagram scales as $N_c^{1/2}$. The contribution of the $\Delta$ with couplings given by the consistency conditions (3) produce the required cancellation, and we recover the desired scaling.

\section*{VII. CONCLUSION}

The purpose of this paper was to discuss the large-$N_c$ consistency conditions using a different method than in the original derivation of Refs. [7,8]. In conclusion, we list our points:

- Using the Adler-Weisberger sum rule we have alternatively derived the large-$N_c$ consistency conditions for the axial vector couplings of pions to baryons. The procedure involves a separation of the $\Delta$-resonance contribution, and a careful $N_c$-counting of the remaining cross section.

- Examination of contributions of resonances other than $\Delta(1232)$ indicates the need of additional consistency relations for the couplings of excited states. This important point requires a further investigation in the framework of available large-$N_c$ models.

- Using the Cabibbo-Radicati sum rule we have found that the ratios of isovector magnetic couplings acquire corrections to the large-$N_c$ values of the order $1/N_c^2$. This is in complete analogy to the behavior of the axial vector couplings.
• We have pointed out implications of the large-$N_c$ consistency conditions for the difference of isospin $1/2$ and $3/2$ pion-nucleon scattering length. We have also shown how the $N_c$ counting of the dressed $\rho$-nucleon vertex is preserved.

• The final point concerns the order of the chiral limit and the large-$N_c$ limit. Although these limits do not commute [44], it has no effect on the large-$N_c$ consistency conditions. In our derivation, or in the original derivation of Refs. [7,8] the limits may be taken in any order, and the results for LNCC do not change.

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### TABLE I. $N_c$-counting rules used in this paper

| quantity                              | symbol           | $N_c$-counting |
|---------------------------------------|------------------|----------------|
| QCD coupling constant                 | $g$              | $N_c^{-1/2}$   |
| pion decay constant                   | $F_\pi$          | $N_c^{1/2}$    |
| quark-meson coupling constant         | $g_{\text{meson}-\text{quark}}$ | $N_c^{-1/2}$   |
| meson mass                            | $M_{\text{meson}}$ | $N_c^0$        |
| $n$-meson vertex                      | $g_{\text{meson}}^{(n)}$ | $N_c^{(2-n)/2}$ |
| baryon mass                           | $M_{\text{baryon}}$ | $N_c$          |
| axial vector coupling constant        | $g_A$            | $N_c$          |
| pion-nucleon coupling constant        | $g_{\pi NN}$     | $N_c^{3/2}$    |
| $\Delta$-nucleon mass splitting      | $M_\Delta - M_N$ | $N_c^{-1}$     |
| Roper-nucleon mass splitting          | $M_{N^*} - M_N$  | $N_c^0$        |
| pion-nucleon-Roper coupling constant  | $g_{\pi NN^*}$   | $N_c$          |
| $\rho$-nucleon coupling constant     | $g_{\rho NN}$    | $N_c^{-1/2}$   |
| $\rho$-Reggeon–nucleon coupling constant | $g_{R_\rho NN}$ | $N_c^{-1/2}$   |
| $\rho$-Reggeon intercept             | $\alpha_0$       | $N_c^0$        |
| isovector magnetic moment             | $\mu^{I=1}$      | $N_c$          |
FIGURES

FIG. 1. A visual representation of Witten’s $N_c$ counting for the imaginary part of the forward pion-nucleon amplitude. (a) External pion lines are attached to the same quark line. (b) External pion lines are attached to different quark lines.

FIG. 2. $N_c$-counting for the imaginary part of the charge-exchange pion-nucleon scattering amplitude. The proton in diagrams (a), (c) and (e) consists of $(N_c + 1)/2$ up quarks and the neutron in diagrams (b), (d) and (f) consists of $(N_c - 1)/2$ down quarks. (a-b) Example of a diagram where the external pions are attached to the same quark line. (c-d) Example of the diagram where the external pions are attached to two different quark lines of the same flavor. Addition pion is being exchanged. (e-f) Example of the diagram where the external pions are attached to two different quark lines of a different flavor. Addition gluon is being exchanged.

FIG. 3. Exchange of the $\rho$-Reggeon (a) and its typical QCD anatomy (b). The $\rho$-Reggeon coupling constants scale with $N_c$ in the same way as the $\rho$-meson coupling constants.

FIG. 4. Excited baryon contributions to the pion-proton scattering cross section. The leading-$N_c$ pieces in the difference of diagrams (a) and (b) have to cancel.

FIG. 5. Examples of possible processes in the pion-nucleon scattering at intermediate energies. In charge-exchange cross section cancellation occurs between diagrams with $N$ and $\Delta$ in the intermediate states, and the cross section scales as $1/N_c$.

FIG. 6. One-pion loop contribution to the isovector charge form factor of the nucleon. Cancellation between the $N$ and $\Delta$ contributions occurs, and the electric $\rho$-nucleon coupling scales as $N_c^{-1/2}$. 
Figure 1

Figure 2 (a-b)
Figure 2 (c-f)
Figure 3

Figure 4
Figure 5

Figure 6