Integrable Dispersive Chains and Their Multi-Phase Solutions.

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September 19, 2018

Abstract

In this paper we construct multi-phase solutions for integrable dispersive chains associated with the three-dimensional linearly-degenerate Mikhailëv system of first order. These solutions are parameterized by infinitely many arbitrary parameters. As byproduct we describe multi-phase solutions for finite component dispersive reductions of these integrable dispersive chains.

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1 Introduction

Construction of one-phase solutions for dispersive multi-dimensional nonlinear systems based on the ansatz “travelling wave reduction”, i.e. the dependent functions \(u_k(x, t, y, z, \ldots)\) should be replaced by \(u_k(\theta)\), where the phase \(\theta = k_1 x + k_2 t + k_3 y + \ldots\) and \(k_m\) are constants. This ansatz leads to systems of ordinary differential equations. Usually this ansatz allows to find global solutions, i.e. periodic solutions, which are very important in different applications in physics and mathematics. For instance, the two-dimensional Korteweg–de Vries equation or the three-dimensional Kadomtsev–Petviashvili equation have such solutions. Moreover, these two equations are integrable by the Inverse Scattering Transform, i.e. they have infinitely many multi-phase solutions.

Recently the authors of the paper [12] showed that also dispersionless three-dimensional systems can possess infinitely many global solutions. For instance, the authors considered the Mikhalëv system (see [18])

\[
a_{1,t} = a_{2,x}, \quad a_{1} a_{2,x} + a_{1,y} = a_{2} a_{1,x} + a_{2,t},
\]

and constructed several explicit global solutions for this system. This system is integrable by the method of hydrodynamic reductions (see [20]). Its Lax pair is (see again [20])

\[
p_t = [(\lambda + a_1)p]_x, \quad p_y = [(\lambda^2 + a_1 \lambda + a_2)p]_x,
\]

where \(\lambda\) is an arbitrary parameter. The method of hydrodynamic reductions means (see [11]) that two families of commuting hydrodynamic type systems \((N\) is an arbitrary natural number)

\[
r_i^t = (r_i + a_1)r_i^x, \quad r_i^y = [(r_i^y)^2 + a_1 r_i^y + a_2]r_i^x, \quad i = 1, 2, \ldots, N
\]

have infinitely many conservation laws (2), where now (!) \(a_1(r), a_2(r)\) and \(p(\lambda, r)\) such that (see [20])

\[
a_1 = \sum_{m=1}^{N} f'_m(r^m), \quad a_2 = \sum_{m=1}^{N} (r^m f'_m(r^m) - f_m(r^m)) + \frac{1}{2} a_1^2, \quad p = \exp \sum_{m=1}^{N} \int \frac{f''_m(r^m)dr^m}{r^m - \lambda},
\]

where \(f_k(r^k)\) are arbitrary functions. In the particular case \((\alpha_k\) and \(\gamma_m\) are arbitrary constants)

\[
f_m(r^m) = -\frac{1}{2}(r^m)^2 + \alpha_m r^m + \gamma_m,
\]

hydrodynamic type systems are called linearly-degenerate (i.e. their characteristic velocities do not depend on corresponding Riemann invariants; see, for instance [21]). Then (for simplicity here we denote \(\eta_1 = \Sigma \alpha_m\) and \(\eta_2 = \eta_1^2/2 - \Sigma \gamma_m\))

\[
a_1 = -\sum_{m=1}^{N} r^m + \eta_1, \quad a_2 = \frac{1}{2} \left( \sum_{m=1}^{N} r^m \right)^2 - \frac{1}{2} \sum_{m=1}^{N} (r^m)^2 - \eta_1 \sum_{m=1}^{N} r^m + \eta_2,
\]
and
\[ p = \prod_{m=1}^{N} (r^m - \lambda)^{-1}. \]

Just in this particular case, hydrodynamic type systems (3) are compatible with (see [10] and [21])
\[ r^k_x = \frac{g_k(r^k)}{\prod_{m \neq k} (r^k - r^m)}, \quad k = 1, 2, \ldots, N, \]
where \( g_k(r^k) \) are arbitrary functions. Then the extended system (see (3))
\[ r^k_x = \frac{g_k(r^k)}{ \prod_{m \neq k} (r^k - r^m) }, \quad r^k_t = \frac{(r^k + a_1)g_k(r^k)}{ \prod_{m \neq k} (r^k - r^m) }, \quad r^k_y = \frac{[(r^k)^2 + a_1r^k + a_2]g_k(r^k)}{ \prod_{m \neq k} (r^k - r^m) } \]
is integrable in quadratures (see [10]). Then integration of the above equations leads to the implicit solution
\[ x + \eta_1 t + \eta_2 y = \sum_{n=1}^{N} \int r^n \frac{\lambda^{N-1} d\lambda}{g_n(\lambda)}, \quad t + \eta_1 y = \sum_{n=1}^{N} \int r^n \frac{\lambda^{N-2} d\lambda}{g_n(\lambda)}, \quad (5) \]
\[ y = \sum_{n=1}^{N} \int r^n \frac{\lambda^{N-3} d\lambda}{g_n(\lambda)}, \quad \pi_k = \sum_{n=1}^{N} \int r^n \frac{\lambda^{N-k-1} d\lambda}{g_n(\lambda)}, \quad k = 3, \ldots, N-1, \]
where \( \pi_k \) are arbitrary constants.

Our target in this paper is to select such functions \( g_k(\lambda) \), that these solutions simultaneously are multi-phase solutions of the so called \( M \)th dispersive integrable chains (see [22]) and their infinitely many dispersive integrable reductions.

So, in this paper we consider integrable dispersive chains (see details in [22])
\[ u_{k,t} = u_{k+1,x} - \frac{1}{2} u_1 u_{k,x} - u_k u_{1,x} + \frac{1}{4} \delta^M_k u_{1,xxx}, \quad k = 1, 2, \ldots, (6) \]
for all natural numbers \( M \). Here \( u_k \) are unknown functions (or field variables) and \( \delta^k_m \) is the Kronecker delta. The simplest example \( (M = 1) \) is the Korteweg–de Vries chain
\[ u_{1,t} = u_{2,x} - \frac{3}{2} u_1 u_{1,x} + \frac{1}{4} u_{1,xxx}, \quad u_{k,t} = u_{k+1,x} - \frac{1}{2} u_1 u_{k,x} - u_k u_{1,x}, \quad k = 2, 3, \ldots, (7) \]
The simplest constraint \( u_2 = 0 \) yields the remarkable Korteweg–de Vries equation
\[ u_{1,t} = -\frac{3}{2} u_1 u_{1,x} + \frac{1}{4} u_{1,xxx}. \]
The next constraint \( u_3 = 0 \) implies another well-known integrable system: the Ito system
\[ u_{1,t} = u_{2,x} - \frac{3}{2} u_1 u_{1,x} + \frac{1}{4} u_{1,xxx}, \quad u_{2,t} = -\frac{1}{2} u_1 u_{2,x} - u_2 u_{1,x}. \]
If $M = 2$, the constraint $u_3 = 0$ leads to the Kaup–Boussinesq system

\[ u_{1,t} = u_{2,x} - \frac{3}{2} u_1 u_{1,x}, \quad u_{2,t} = -\frac{1}{2} u_1 u_{2,x} - u_2 u_{1,x} + \frac{1}{4} u_{1,xxx} \]

connected with the nonlinear Schrödinger equation by an appropriate transformation (see, for instance, [2]).

Multi-phase solutions for these integrable systems are associated with hyperelliptic Riemann surfaces. $N$ phase solutions of the Korteweg–de Vries equation are parameterised by $2N + 1$ arbitrary constants\(^1\), $N$ phase solutions of the Ito system and of the nonlinear Schrödinger equation are parameterised by $2N + 2$ arbitrary constants. In comparison with finite-component reductions, $N$ phase solutions of the dispersive integrable chains depend on infinitely many arbitrary parameters. In this paper we utilise a simple construction of multi-phase solutions based on several first original papers where the finite-gap integration was established, i.e. [19], [9], [14]. For our convenience we describe these multi-phase solutions simultaneously for first $N - 1$ commuting flows belonging to corresponding integrable dispersive hierarchies.

In 1919 French mathematician Jules Drach (see [8]) formulated the following problem\(^2\): let us consider the linear equation

\[ \psi_{xx} = (\lambda + u) \psi, \]

where the function $\psi$ depends on an independent variable $x$ and on an arbitrary parameter $\lambda$, while the function $u$ depends just on an independent variable $x$. Under the substitution $\psi = \exp(\int qdx)$, one can obtain well-known Riccati equation

\[ q_x + q^2 = \lambda + u. \]

The Goal: to select all functions $u(x)$ such that this Riccati equation (with an arbitrary parameter $\lambda$!) is integrable in quadratures.

Jules Drach derived the system of ordinary differential equations (cf. (4))

\[ r_k^x = 2 \frac{\sqrt{S(r^k)}}{\prod_{m\neq k} (r^k - r^m)}, \quad k = 1, 2, \ldots, N, \]

(9)

where $u = 2\Sigma r^m - \Sigma \beta_m$ and ($\beta_k$ are arbitrary parameters)

\[ S(\lambda) = \prod_{m=1}^{2N+1} (\lambda - \beta_m). \]

Then he presented a general solution of this system (9) in quadratures\(^3\). These results of Jules Drach were forgotten for many years. The Korteweg–de Vries equation is determined by the Lax pair

\[ \psi_{xx} = (\lambda + u) \psi, \quad \psi_t = \left(\lambda - \frac{1}{2} u\right) \psi_x + \frac{1}{4} u_x \psi. \]

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\(^1\)Here we ignore phase shifts.

\(^2\)The translation of these both papers to English language can be found in [7] from page 445, to Russian language can be found in [6] from page 20.

\(^3\)The solution found by Jules Drach is exactly coincides with (5), if to fix $t$ and $y$ to constants.
Its solution in a similar form was found independently by B.A. Dubrovin in 1975, but also evolution with respect to the time variable $t$ was presented in [9].

In this paper we select all functions $u_k(x)$ such that the Riccati equation ($M = 0, 1, 2, ...$)

$$q_x + q^2 = \lambda^M \left(1 + \frac{u_1}{\lambda} + \frac{u_2}{\lambda^2} + \frac{u_3}{\lambda^3} + ...\right)$$

is integrable in quadratures. We find dependencies $u_k(r)$, where the functions $r^k(x)$ satisfy system (9) again as in the case of the Korteweg–de Vries equation. However, in this case the function $S(\lambda)$ depends on infinitely many arbitrary parameters$^4$.

Three-dimensional integrable linearly-degenerate systems possess simultaneously hydrodynamic and dispersive reductions (see [15], [16], [17], [20], [22]), while three-dimensional integrable quasilinear systems possess just infinitely many hydrodynamic reductions, see details in [13], [11].

For $M$th dispersive integrable chains associated with the Mikhalëv system we construct infinitely many multi-phase solutions, which depend on arbitrary number of arbitrary parameters, but these $N$ phase solutions also depend on two non-negative integers $M$ and $K$. This is a biggest difference with two-dimensional integrable dispersive systems: while they have just one $N$ phase solution for every natural number $N$, $M$th dispersive integrable chains associated with the Mikhalëv system possess $N$ phase solutions for every pair of non-negative integers $M$ and $K$. This situation is similar to the famous Kadomtsev–Petviashvili equation, which also has infinitely many multi-phase solutions (i.e. infinitely many one-phase solutions, infinitely many two-phase solutions, etc.). The crucial difference between the Mikhalëv system and the Kadomtsev–Petviashvili equation is that the first of them is dispersionless, while the second one is dispersive$^5$. Nevertheless, both of them possess infinitely many dispersive reductions, like the Korteweg–de Vries equation and the Kaup–Boussinesq system. Since such two-dimensional integrable dispersive systems have multi-phase solutions, we can compute multi-phase solutions for every dispersive reduction of $M$th dispersive integrable chains associated with the Mikhalëv system. Thus, they are simultaneously are multi-phase solutions for the Mikhalëv system (5).

The paper is organised as follows: we discuss integrable dispersive chains (6), and their common properties like conservation laws and commuting flows in Section 2. Then in Section 3 we construct multi-phase solutions for these integrable dispersive chains (6). In Section 4 we extract infinitely many integrable dispersive reductions and discuss their multi-phase solutions. In Section 5 we consider the exceptional case $M = 0$. Finally in Conclusion 6 we discuss ..........

$^4$Integrability in quadratures of this Riccati equation (except the case $M = 0$) was considered in [1], [23]. However, the authors presented their construction without integration constants, which play a significant role in description of multi-phase solutions.

$^5$This means that one can widely use the method of hydrodynamic reductions [13], [11] and the generalised hodograph method [24] for the Mikhalëv system.
2 Integrable Dispersive Chains

If the function $\psi$ satisfies the pair of linear equations in partial derivatives

$$\psi_{xx} = u\psi, \quad \psi_t = a\psi_x - \frac{1}{2}a_x\psi,$$  \hspace{1cm} (10)

the compatibility condition $(\psi_{xx})_t = (\psi_t)_{xx}$ implies the relationship

$$u_t = \left( -\frac{1}{2}\partial^3_x + 2u\partial_x + u_x \right) a.$$ \hspace{1cm} (11)

The choice $(M = 1, 2, ..., \text{see details in [22])}

$$u = \lambda^M \left( 1 + \frac{u_1}{\lambda} + \frac{u_2}{\lambda^2} + \frac{u_3}{\lambda^3} + ... \right)$$ \hspace{1cm} (12)

leads to infinitely many commuting integrable dispersive chains. Here $\lambda$ is the so called spectral parameter.

Indeed, one can introduce infinitely many linear equations

$$\psi_{t_k} = a^{(k)}\psi_x - \frac{1}{2}a^{(k)}_x\psi,$$ \hspace{1cm} (13)

where $(k = 1, 2, ...)$

$$a^{(k)} = \lambda^k + \lambda^{k-1}a_1 + \ldots + \lambda a_{k-1} + a_k.$$ \hspace{1cm} (14)

For instance, the compatibility condition $(\psi_{xx})_{t_1} = (\psi_{t_1})_{xx}$ yields integrable dispersive chains (6)

$$u_{k,t_1} = u_{k+1,x} - \frac{1}{2}u_1 u_{k,x} - u_k u_{1,x} + \frac{1}{4}\delta^M u_{1,xxx}, \quad k = 1, 2, ...,$$

where

$$a_1 = -\frac{1}{2}u_1;$$

the compatibility condition $(\psi_{xx})_{t_2} = (\psi_{t_2})_{xx}$ implies their first higher commuting flows

$$u_{k,t_2} = u_{k+2,x} - \frac{1}{2}u_1 u_{k+1,x} - u_{k+1} u_{1,x} + \frac{1}{4}\delta^M u_{1,xxx}$$ \hspace{1cm} (15)

$$+a_2 u_{k,x} + 2u_k a_{2,x} - \frac{1}{2}\delta^M a_{2,xxx},$$

where

$$a_2 = \frac{3}{8}u_1^2 - \frac{1}{2}u_2 - \frac{1}{8}\delta^M u_{1,xx}.$$  

Next expression (see [22]) is more complicated:

$$a_3 = -\frac{1}{2}u^3 + \frac{3}{4}u_1^2u^2 - \frac{5}{16}(u^1)^3 + \frac{1}{32}\delta^M (10u_1^1 u_{1,x} + 5(u_1^1)^2 - u_{1,xxx} - 4u_{2,xx}) - \frac{1}{8}\delta^2 M u_{1,xx}.$$ 

Nevertheless, every $M$th dispersive chain can be written via these expressions $a_k$ in a very compact form (see [22] again)

$$a_{k,t_1} = a_{k+1,x} + a_1 a_{k,x} - a_k a_{1,x}, \quad k = 1, 2, ...$$ \hspace{1cm} (16)
Here we emphasise that a corresponding natural number $M$ and higher order derivatives are hidden inside of differential polynomials $a_k(u, u_x, u_{xx}, \ldots)$. Higher commuting flows also take hydrodynamic form in these coordinates $a_m$. For instance, (15) becomes

$$a_{k,t_2} = a_{k+2,x} + a_1 a_{k+1,x} + a_2 a_{k,x} - a_k a_{2,x} - a_{k+1} a_{1,x}. \quad (17)$$

**Remark:** Eliminating the derivative $a_{3,x}$, Mikhailëv system (1) can be obtained from first two equations of (16) and the first equation of (17).

Integrable dispersive chains (6) as well as their higher commuting flows have generating equations of conservation laws (see (14))

$$p_{tk} = (a^{(k)} p)_x, \quad k = 1, 2, \ldots \quad (18)$$

Here the generating function of conservation law densities $p = 1/\varphi$, where the function $\varphi = \psi \psi^{+}$, and $\psi, \psi^{+}$ are two linearly independent solutions of the first equation in (10) such that $\psi \psi^{+} - \psi^{+} \psi = 1$. Indeed, an equivalent form of Lax pair (10) is (we remind the function $u$ is determined by (12))

$$\varphi_{xxx} = 4u \varphi_{x} + 2\varphi u_{x}, \quad (19)$$

$$\varphi_{tk} = a^{(k)} \varphi_{x} - a^{(k)} \varphi. \quad (20)$$

Finally introducing another function $p = 1/\varphi$, (20) takes the conservative form (18), while the generating function $p(\lambda)$ of conservation law densities satisfies the nonlinear equation

$$(p^{-1})_{xxx} = 4u(p^{-1})_{x} + 2p^{-1} u_{x}. \quad (21)$$

If $\lambda \to \infty$, the asymptotic behaviour of both solutions of the linear equation (see the first equation in (10) and (12))

$$\psi_{xx} = \lambda^M \left( 1 + \frac{u_1}{\lambda} + \frac{u_2}{\lambda^2} + \frac{u_3}{\lambda^3} + \ldots \right) \psi$$

is: $\psi \to \exp(\lambda^{M/2} x)$, $\psi^{+} \to \exp(-\lambda^{M/2} x)$. Thus, the asymptotic behaviour of $\varphi$ starts from the unity,$^6$ i.e.

$$\varphi = 1 + \frac{a_1}{\lambda} + \frac{a_2}{\lambda^2} + \frac{a_3}{\lambda^3} + \frac{a_4}{\lambda^4} + \ldots \quad (22)$$

Corresponding asymptotic behaviour of generating function of conservation law densities also starts from the unity, i.e.

$$p = 1 + \frac{\sigma_1}{\lambda} + \frac{\sigma_2}{\lambda^2} + \frac{\sigma_3}{\lambda^3} + \frac{\sigma_4}{\lambda^4} + \ldots, \quad (23)$$

where $p\varphi = 1$, i.e.

$$a_1 + \sigma_1 = 0, \quad a_2 + a_1\sigma_1 + \sigma_2 = 0, \quad a_3 + a_2\sigma_1 + a_1\sigma_2 + \sigma_3 = 0, \ldots \quad (24)$$

$^6$We remind that the function $\varphi$ satisfies to linear equations (19) and (20). Thus, the function $\varphi$ is determined up to an arbitrary constant factor. This means that we fix this factor to unity for our further convenience here.
Substituting the expansion (23) into generating equations of conservation laws (18), one can express infinitely many local conservation laws ($a_0 = 1$)

$$(\sigma_k)_{tm} = (\sum_{n=0}^{m} a_n \sigma_{k+m-n})_x, \quad k, m = 1, 2, ...$$

Taking into account relationships (24), fluxes of these conservation laws can be expressed via $\sigma_m$ only. For instance

$$(\sigma_k)_{t_1} = (\sigma_{k+1} - \sigma_1 \sigma_k)_x, \quad (\sigma_k)_{t_2} = (\sigma_{k+2} - \sigma_1 \sigma_{k+1} + (\sigma_1^2 - \sigma_2) \sigma_k)_x,$$

$$(\sigma_k)_{t_3} = (\sigma_{k+3} - \sigma_1 \sigma_{k+2} + (\sigma_1^2 - \sigma_2) \sigma_{k+1} - (\sigma_1^3 - 2\sigma_1 \sigma_2 + \sigma_3) \sigma_k)_x.$$

Substituting expansion (23) into (21) allows to find all conservation law densities $\sigma_k$ as differential polynomials with respect to the field variables $u_k$. For instance,

$$\sigma_1 = \frac{1}{2} u_1, \quad \sigma_2 = \frac{1}{2} u_2 - \frac{1}{8} u_1^2 + \frac{1}{8} \delta_1^M u_{1,xx},$$

$$\sigma_3 = \frac{1}{2} u_3 - \frac{1}{4} u_1 u_2 + \frac{1}{16} u_1^3 + \frac{1}{32} \delta_1^M (u_{1,x})^2 + \frac{1}{8} \left[ \delta_2^M u_1 + \delta_1^M \left( u_2 - \frac{3}{4} u_1 - \frac{1}{4} u_{1,xx} \right) \right]_{xx}.$$  

### 3 Multi Phase Solutions

In this section we consider multi-phase solutions for integrable dispersive chains (6) and their first $N - 2$ commuting flows (see (13) and (14)).

As usual, construction of $N$-phase solutions for integrable systems associated with linear system (10) (see also (12), (14)) is based on the crucial observation: the $\psi$ function (see (10)) as well as the field variables $u_k$ no longer depend on the time variable $t_N$ of $N$th commuting flow (13). This means that we are looking for an ansatz for the $\psi$ function in the form\(^7\)

$$\psi(t_0, t_1, ..., t_{N-1}, t_N, t_{N+1}, ...) = e^{\mu N} \tilde{\psi}(t_0, t_1, ..., t_{N-1}),$$

where $\mu$ is an arbitrary constant. In this case $N$th equation from (13) becomes

$$\mu \tilde{\psi} = \tilde{a}^{(N)} \tilde{\psi}_x - \frac{1}{2} \tilde{a}^{(N)} \tilde{\psi},$$

where (see (14))

$$\tilde{a}^{(N)} = \lambda^N + \lambda^{N-1} \tilde{a}_1 + ... + \lambda \tilde{a}_{N-1} + \tilde{a}_N = a^{(N)} + \kappa_1 a^{(N-1)} + ... + \kappa_{N-1} a^{(1)} + \kappa_N.$$  

\(^7\)Indeed, one can look for the ansatz $\psi(t_0, t_1, ..., t_{N-1}, t_N, t_{N+1}, ...) = \chi(t_N) \tilde{\psi}(t_0, t_1, ..., t_{N-1})$, where $\chi(t_N)$ is not yet known function. Then $N$th equation from (13) yields $\chi^{(N)} \tilde{\psi} = a^{(N)} \tilde{\psi}_x - \frac{1}{2} a^{(N)} \tilde{\psi}$. Since the functions $\tilde{\psi}$ and $a^{(N)}$ do not depend on $t_N$, the ratio $\chi^{(N)} \tilde{\psi} = \chi(t_N) \tilde{\psi}_x - \frac{1}{2} a^{(N)} \tilde{\psi}$. Since the functions $\tilde{\psi}$ and $a^{(N)}$ do not depend on $t_N$, the ratio $\chi^{(N)} \tilde{\psi}$ also does not depend on $t_N$. This means, that $\chi(t_N) = \exp(\mu t_N)$, where $\mu$ is an arbitrary constant. Below we show that $\tilde{\psi}$ also does not depend on higher time variables $t_{N+1}, t_{N+2}, ...$
and \( \kappa_m \) are arbitrary constants. Indeed, corresponding relationship (11)

\[
u_{t_N} = \left( -\frac{1}{2} \frac{\partial^2}{\partial x^2} + 2u \frac{\partial}{\partial x} + u \right) \tilde{a}^{(N)}
\]

reduces to the “stationary” form (with respect to the time variable \( t_N \), because also all field variables \( u_k \) do not depend on this time variable \( t_N \))

\[
- \frac{1}{2} \tilde{a}_{xxx}^{(N)} + 2u \tilde{a}_x^{(N)} + \tilde{a}^{(N)} u_x = 0.
\]  (27)

This linear equation on function \( \tilde{a}^{(N)} \) is determined up to any linear combination of lower expressions \( \tilde{a}^{(k)} \). Thus, the parameters \( \kappa_m \) play an important role in construction of corresponding \( N \)-phase solutions.

So, equation (27) exactly coincides with (19), where \( \varphi \to \Phi = \tilde{a}^{(N)} \). Thus, the “stationary” reduction \( \psi_{t_N} = 0 \) leads to the polynomial ansatz (see (26))

\[
\Phi(\lambda, r) = \prod_{m=1}^{N} \left( \lambda - r^m(x, t) \right)
\]  (28)

for linear system (19), (20). Taking into account (25) the first equation in (10) implies the first integral of (27):

\[
\mu^2 = u(\tilde{a}^{(N)})^2 + \frac{1}{4}(\tilde{a}_x^{(N)})^2 - \frac{1}{2} \tilde{a}^{(N)} \tilde{a}_{xx}^{(N)}.
\]  (29)

Indeed, substituting (see (25))

\[
\tilde{\psi}_x = \left( \frac{\mu}{\tilde{a}^{(N)}} + \frac{1}{2} \tilde{a}_x^{(N)} \right) \tilde{\psi}
\]

into the first equation in (10), we obtain

\[
\left( \frac{\mu}{\tilde{a}^{(N)}} + \frac{1}{2} \tilde{a}_x^{(N)} \right) \tilde{\psi}_x + \left[ \frac{1}{2} \left( \frac{\tilde{a}_x^{(N)}}{\tilde{a}^{(N)}} \right) x - \mu \frac{\tilde{a}_x^{(N)}}{(\tilde{a}^{(N)})^2} \right] \tilde{\psi} = u \tilde{\psi}.
\]

Finally, eliminating \( \tilde{\psi}_x \) from both above equations, one can obtain first integral (29).

So, we are looking for polynomial solutions (of a degree \( N \) with respect to the spectral parameter \( \lambda \)) of the first integral (19)

\[
2\Phi \Phi_{xx} - \Phi_x^2 = 4\lambda^M \left( 1 + \frac{u_1}{\lambda} + \frac{u_2}{\lambda^2} + \frac{u_3}{\lambda^3} + \ldots \right) \Phi^2 - 4S(\lambda),
\]  (30)

where \( \mu^2 = S(\lambda) \) is an “integration constant”, see (29).

Substitution (28) into (30) leads to (see (12))

\[
u(\lambda, r, r_x, r_{xx}) \equiv \lambda^M \left( 1 + \frac{u_1}{\lambda} + \frac{u_2}{\lambda^2} + \frac{u_3}{\lambda^3} + \ldots \right) = \frac{\Phi_{xx}}{2\Phi} - \frac{\Phi_x^2}{4\Phi^2} + S(\lambda) \frac{1}{\Phi^2}
\]
\[
= \frac{1}{2} \left( \ln \prod_{n=1}^{N} (\lambda - r^n) \right)_{xx} + \frac{1}{4} \left[ \left( \ln \prod_{n=1}^{N} (\lambda - r^n) \right)_{xx} \right]^2 + \frac{S(\lambda)}{\prod_{n=1}^{N} (\lambda - r^n)^2}, \tag{31}
\]
where \((s_k\) are integration constants)
\[
S(\lambda) = \lambda^{2N+M} \left( 1 + \frac{s_1}{\lambda} + \frac{s_2}{\lambda^2} + \frac{s_3}{\lambda^3} + ... \right).	ag{32}
\]
So, we have (30) with polynomial ansatz (28) and similar equation (equivalent to (19))
\[
2\varphi_{xx} - \varphi_x^2 = 4\lambda^M \left( 1 + \frac{u_1}{\lambda} + \frac{u_2}{\lambda^2} + \frac{u_3}{\lambda^3} + ... \right) \varphi^2 - 4\lambda^M, \tag{33}
\]
where coefficients \(a_k\) are determined by (22). Eliminating \(u\) from (30) and (33), one can obtain
\[
2 \frac{\Phi_{xx}}{\Phi} - \frac{\Phi_x^2}{\Phi^2} + 4 \frac{S(\lambda)}{\Phi} = 2 \frac{\varphi_{xx}}{\varphi} - \frac{\varphi_x^2}{\varphi^2} + 4 \frac{\lambda^M}{\varphi^2}.
\]
This means
\[
\varphi = \lambda^{-N} \left( 1 + \frac{s_1}{\lambda} + \frac{s_2}{\lambda^2} + \frac{s_3}{\lambda^3} + ... \right)^{-1/2} \Phi. \tag{34}
\]
Thus, all dependencies \(a_k(r)\) can be found by substitution (22) and (28) into the above equation, i.e. \((a_0 = 1, \eta_0 = 1)\)
\[
a_n = \sum_{m=0}^{n} \eta_{n-m} \tilde{a}_m, \quad n = 1, 2, ..., N; \tag{35}
\]
\[
a_n = \sum_{m=0}^{N} \eta_{n-m} \tilde{a}_m, \quad n = N + 1, N + 2, ..., \tag{36}
\]
where parameters \(\eta_m\) can be found from the expansion
\[
1 + \frac{\eta_1}{\lambda} + \frac{\eta_2}{\lambda^2} + \frac{\eta_3}{\lambda^3} + ... = \left( 1 + \frac{s_1}{\lambda} + \frac{s_2}{\lambda^2} + \frac{s_3}{\lambda^3} + ... \right)^{-1/2}.
\]
Hence,
\[
\eta_p = \sum_{k_1, \ldots, k_m} (-1)^{k_1+\ldots+k_m} \frac{(2k_1 + \ldots + 2k_m - 1)!}{2^{2k_1+\ldots+2k_m-1} (k_1 + \ldots + k_m - 1)! k_1! \ldots k_m!} s_{i_1}^{k_1} \ldots s_{i_m}^{k_m},
\]
where it is assumed that \(i_1, k_1, \ldots, i_m, k_m\) run over all \(2m\)-tuples such that \(i_1k_1 + \ldots + i_mk_m = p\) and \(i_1, \ldots, i_m\) are pairwise different. For instance,
\[
\begin{align*}
\eta_1 &= -\frac{1}{2} s_1, \\
\eta_2 &= -\frac{1}{2} s_2 + \frac{3}{4} s_1^2, \\
\eta_3 &= -\frac{1}{2} s_3 + \frac{3}{4} s_1 s_2 - \frac{5}{16} s_1^3, \\
\eta_4 &= -\frac{1}{2} s_4 + \frac{3}{4} s_1 s_3 + \frac{3}{8} s_2^2 - \frac{15}{16} s_1^2 s_2 + \frac{35}{128} s_1^4.
\end{align*}
\]
Also (see (14) and (28)) all parameters $\kappa_m$ in (26) can be found from the inverse formula
\[
1 + \frac{\kappa_1}{\lambda} + \frac{\kappa_2}{\lambda^2} + \frac{\kappa_3}{\lambda^3} + \ldots = \left(1 + \frac{s_1}{\lambda} + \frac{s_2}{\lambda^2} + \frac{s_3}{\lambda^3} + \ldots\right)^{1/2}.
\] (37)
Hence,
\[
\kappa_p = \sum_{k_1, \ldots, k_m} \frac{(-1)^{k_1+\ldots+k_m+1}}{2^{2k_1+\ldots+2k_m-1}} \frac{(2k_1 + \ldots + 2k_m - 2)!}{(k_1 + \ldots + k_m - 1)! k_1! \ldots k_m!} \sum_{i=1}^{k_1} \ldots \sum_{i_m=1}^{k_m} s_i^{k_1} \ldots s_i^{k_m}.
\]
Here it is assumed that $i_1, k_1, \ldots, i_m, k_m$ run over all 2m-tuples such that $i_1 k_1 + \ldots + i_m k_m = p$ and $i_1, \ldots, i_m$ are pairwise different. For instance,
\[
\begin{align*}
\kappa_1 &= \frac{1}{2} s_1, \\
\kappa_2 &= \frac{1}{2} s_2 - \frac{1}{8} s_1^2, \\
\kappa_3 &= \frac{1}{2} s_3 - \frac{1}{4} s_1 s_2 + \frac{1}{16} s_1^3, \\
\kappa_4 &= \frac{1}{2} s_4 - \frac{1}{4} s_1 s_3 - \frac{1}{8} s_1^2 s_2 + \frac{3}{16} s_1^2 s_2 - \frac{5}{128} s_1^4.
\end{align*}
\]
Following J. Drach [8] and B.A. Dubrovin [9], we consider the limit $\lambda \to r^i(x,t)$ of (30). This straightforward computation yields the $N$ component system (cf. (4) and (9))
\[
r^k_x = 2 \prod_{m \neq k} (r^k - r^m), \quad k = 1, 2, \ldots, N.
\] (38)
Substituting these expressions back into (31), one can obtain\(^8\)
\[
u(\lambda, r) = \frac{S(\lambda)}{\prod_{n=1}^{N}(\lambda - r^n)^2} + \sum_{n=1}^{N} \frac{1}{\lambda - r^n} \left(2 \sum_{m \neq n} \frac{1}{r^n - r^m} - \frac{1}{\lambda - r^n} - \frac{S'(r^n)}{S(r^n)} \right) \frac{S(r^n)}{\prod_{s \neq n}(r^n - r^s)^2}.
\]
Taking into account (32) this equality becomes
\[
u(\lambda, r) = \frac{S(\lambda)}{\prod_{n=1}^{N}(\lambda - r^n)^2} + \sum_{p=0}^{\infty} s_p Q_{2N+M-p},
\] (39)
where ($k = 0, \pm 1, \pm 2, \ldots$)
\[
Q_k(\lambda, r) = \sum_{n=1}^{N} \frac{1}{\lambda - r^n} \left(2 \sum_{m \neq n} \frac{r^n}{r^n - r^m} - \frac{r^n}{\lambda - r^n} - k\right) \frac{(r^n)^{k-1}}{\prod_{q \neq n}(r^n - r^q)^2}.
\] (40)
Now we introduce the power sum symmetric polynomials
\[
c_k(r) = \frac{1}{k} \sum_{m=1}^{N} (r^m)^k, \quad c_k(R) = \frac{1}{k} \sum_{m=1}^{N} (R^m)^k, \quad k = 1, 2, \ldots,
\]
\(^8\)First order derivatives of expressions in parentheses lead to all possible combinations $r^i_x, r^k_x$. However, taking into account also second order derivatives, these products will be cancelled simultaneously.
where }R^n = 1/r^n.

Lemma: The coefficients }u_m(r)} of the expansion (31)

\[ u(\lambda, r) = \lambda^M \left( 1 + \frac{u_1(r)}{\lambda} + \frac{u_2(r)}{\lambda^2} + \frac{u_3(r)}{\lambda^3} + \ldots \right), \quad \lambda \to \infty \]

are determined by

\[ u_m(r) = \sum_{k=0}^{m} s_{m-k} \sum_{n=0}^{k} B_n B_{n-k}, \quad 1 \leq m \leq M; \quad (41) \]

\[ u_{m+M}(r) = \sum_{k=1}^{\infty} s_{2N+M+k+m-1} \sum_{n=1}^{k} B_{-n} B_{n-1-k}, \quad m \geq 1, \quad (42) \]

where }B_k(r)} are Bell polynomials, i.e.

\[ B(\lambda, r) = 1 + \frac{B_1(r)}{\lambda} + \frac{B_2(r)}{\lambda^2} + \frac{B_3(r)}{\lambda^3} + \ldots = \exp \left( \frac{c_1(r)}{\lambda} + \frac{c_2(r)}{\lambda^2} + \frac{c_3(r)}{\lambda^3} + \ldots \right); \quad (43) \]

\[ B_{-1} = \prod_{m=1}^{N} R_m^{m} \equiv \prod_{m=1}^{N} (r^m)^{-1}. \quad (44) \]

and all other negative }B_k(r)} are proportional to Bell polynomials up to the common factor }B_{-1} \quad \text{(expressible at } \lambda \to 0) \]

\[ B(\lambda, r) = B_{-1} + \lambda B_{-2} + \lambda^2 B_{-3} + ... = B_{-1} \exp(\lambda c_1(R) + \lambda^2 c_2(R) + \lambda^3 c_3(R) + ...). \]

Examples:

\[ B_1 = c_1(r), \quad B_2 = c_2(r) + \frac{1}{2} c_1^2(r), \quad B_3 = c_3(r) + c_1(r) c_2(r) + \frac{1}{6} c_1^3(r), \ldots; \quad (45) \]

\[ B_{-2} = B_{-1} c_1(R), \quad B_{-3} = B_{-1} \left( c_2(R) + \frac{1}{2} c_1^2(R) \right), \quad (46) \]

\[ B_{-4} = B_{-1} \left( c_3(R) + c_1(R) c_2(R) + \frac{1}{6} c_1^3(R) \right), \ldots \]

Proof: At first we introduce the generating function

\[ Q(\lambda, \zeta, r) = \sum_{n=1}^{N} \frac{1}{\lambda - r^n} \frac{1}{\zeta - r^n} \prod_{q \neq n} \frac{1}{(r^n - r^q)^2} \left( 2 \sum_{m \neq n} \frac{1}{r^m - r^n} - \frac{1}{\lambda - r^n} - \frac{1}{\zeta - r^n} \right), \]

such that (see (40))

\[ Q(\lambda, \zeta, r) = \sum_{k=0}^{\infty} Q_k(\lambda, r) \zeta^{-k-1}, \quad \zeta \to \infty; \]

\[ Q(\lambda, \zeta, r) = -\sum_{k=1}^{\infty} Q_{-k}(\lambda, r) \zeta^{-k-1}, \quad \zeta \to 0. \]
This generating function can be presented in the form
\[ Q(\lambda, \zeta, r) = \frac{G(\lambda, r) - G(\zeta, r)}{\lambda - \zeta}, \]
where
\[ G(\lambda, r) = \sum_{n=1}^{N} \frac{1}{\Pi_{q \neq n} (r^n - r^q)^2 (\lambda - r^n)^2} - 2 \sum_{n=1}^{N} \frac{1}{\Pi_{q \neq n} (r^n - r^q)^2} \frac{1}{\lambda - r^n} \sum_{m \neq n} \frac{1}{r^n - r^m}. \]

However the above expression is nothing but a partial fraction decomposition of the product
\[ G(\lambda, r) = \prod_{n=1}^{N} (\lambda - r^n)^{-2}. \]

Thus, all expressions \( Q_k(\lambda, r) \) can be found by expansion with respect to the parameter \( \zeta \) from the generating function
\[ Q(\lambda, \zeta, r) = 1 + \frac{1}{\lambda - \zeta} \left( \frac{1}{\Pi_{n=1}^{N} (\lambda - r^n)^2} - \frac{1}{\Pi_{n=1}^{N} (\zeta - r^n)^2} \right). \]

Now we introduce the generating function (see (28))
\[ B(\lambda, r) = \frac{1}{\Phi(\lambda, r)} = \prod_{m=1}^{N} (\lambda - r^m)^{-1}. \]

So, if \( \lambda \to \infty \), then \( (B_0 \equiv 1) \)
\[ B(\lambda, r) = \lambda^{-N} \left( 1 + \frac{B_1}{\lambda} + \frac{B_2}{\lambda^2} + \frac{B_3}{\lambda^3} + \ldots \right), \quad (47) \]

where
\[ B_m = \sum_{n=1}^{N} \frac{(r^n)^{N+m-1}}{\Pi_{s \neq n} (r^m - r^s)}. \]

These functions \( B_k(r) \) are Bell polynomials (45).

Then \( (\zeta \to \infty) \)
\[ Q(\lambda, \zeta, r) = \frac{B^2(\zeta, r) - B^2(\lambda, r)}{\zeta - \lambda} \]
\[ = \sum_{n=0}^{\infty} \frac{1}{\zeta^{n+2N+1}} \sum_{m=0}^{n} \left( \sum_{k=0}^{m} B_k B_{m-k} \right) \lambda^{n-m} - B^2(\lambda, r) \sum_{n=0}^{\infty} \frac{\lambda^n}{\zeta^{n+1}}. \]

If \( \lambda \to 0 \), then
\[ B(\lambda, r) = B_{-1} + \lambda B_{-2} + \lambda^2 B_{-3} + \ldots, \]
where
\[ B_{-m} = -\sum_{n=1}^{N} \frac{(r^n)^{-m}}{\Pi_{s \neq n} (r^n - r^s)}. \]
The function $B_{-1}$ is determined by (44), and the other functions $B_{-k}(r)$ are determined by (46).

Then $(\zeta \to 0)$

$$Q(\lambda, \zeta, r) = \frac{B^2(\lambda, r) - B^2(\zeta, r)}{\lambda - \zeta}$$

$$= \sum_{m=0}^{\infty} \zeta^m \left[ B^2(\lambda, r) \lambda^{-m} - \sum_{k=0}^{m} \left( \sum_{n=0}^{k} B_{-n-1}B_{n-k} \right) \lambda^{k-m} \right].$$

The equality (39) we rewrite in the form

$$u(\lambda, r) = \frac{S(\lambda)}{\prod_{n=1}^{N}(\lambda - r^n)^2} + \sum_{k=0}^{2N+M} s_{2N+M-k}Q_k + \sum_{k=1}^{\infty} s_{2N+M+k}Q_{-k}, \quad (48)$$

where

$$Q_k(\lambda, r) = -B^2(\lambda, r)\lambda^k, \quad k = 0, 1, ..., 2N - 1,$$

$$Q_k(\lambda, r) = \sum_{m=0}^{k-2N} \left( \sum_{n=0}^{m} B_nB_{m-n} \right) \lambda^{k-2N-m} - B^2(\lambda, r)\lambda^k, \quad k = 2N, 2N + 1, ...,$$

$$Q_{-k}(\lambda, r) = \sum_{m=0}^{k-1} \left( \sum_{n=0}^{m} B_{-n-1}B_{n-1-m} \right) \lambda^{m-k} - B^2(\lambda, r)\lambda^{-k}, \quad k = 1, 2, ...$$

Thus (48) becomes

$$u(\lambda, r) = \sum_{k=0}^{M} s_{M-k} \sum_{m=0}^{k} \left( \sum_{n=0}^{m} B_nB_{m-n} \right) \lambda^{k-m}$$

$$+ \sum_{k=1}^{\infty} s_{2N+M+k} \sum_{m=1}^{k} \left( \sum_{n=1}^{m} B_{-n}B_{n-1} \right) \lambda^{m-k-1}.$$
invariants \( (\partial_i a^{(k)}_i(r) = 0, \text{no summation here!}) \). Taking into account (38), we obtain the system

\[
 r^i_x = \frac{2\sqrt{S(r^i)}}{\prod_{m \neq i}(r^i - r^m)}, \quad r^i_t = 2a^{(k)}_i(r)\frac{\sqrt{S(r^i)}}{\prod_{m \neq i}(r^i - r^m)}, \quad k = 1, 2, ..., N - 1. \tag{49}
\]

Taking into account (14) and (35), we obtain

\[
 a^{(k)}_i(r) = \bar{a}^{(k)}_i(r) + \eta_1 \bar{a}^{(k-1)}_i(r) + \ldots + \eta_{k-1} \bar{a}^{(1)}_i(r) + \eta_k.
\]

Introducing (instead of independent variables \( t_k \)) \( N \) phases \( \theta_k \) such that (here \( \eta_0 = 1 \) and \( t_0 \equiv x \))

\[
 \theta_k = \sum_{m=k}^{N-1} \eta_{m-k} t_m, \quad k = 0, 1, ..., N - 1,
\tag{50}
\]

system (49) becomes

\[
 r^i_\theta = 2 \bar{a}^{(k)}_i(r)\frac{\sqrt{S(r^i)}}{\prod_{m \neq i}(r^i - r^m)}, \quad k = 0, 1, 2, ..., N - 1.
\]

This system also can be written in the Hamiltonian form\(^\text{9}\)

\[
 r^i_\theta = \frac{\partial H_k}{\partial \mu_i}, \quad (\mu_i)\theta_k = -\frac{\partial H_k}{\partial r^i}, \quad i = 1, 2, ..., N, \quad k = 0, 1, ..., N - 1,
\]

where Hamilton functions are (here \( \bar{a}^{(0)}_i(r) = 1 \))

\[
 H_k = \sum_{i=1}^{N} \bar{a}^{(k)}_i(r) \mu_i^2 + V_k(r) = \sum_{i=1}^{N} \frac{\bar{a}^{(k)}_i(r)}{\prod_{m \neq i}(r^i - r^m)}(\mu_i^2 - S(r^i)).
\]

By virtue of the identities

\[
 \sum_{m=1}^{N} \frac{\partial r^i}{\partial \theta_m} \frac{\partial \theta_m}{\partial r^k} = \delta^i_k, \quad \sum_{m=1}^{N} \frac{\partial r^m}{\partial \theta_i} \frac{\partial \theta_k}{\partial r^m} = \delta^i_k,
\]

one can find all particular derivatives \( \partial \theta_i / \partial r^k \). Their straightforward integration implies multi-phase solutions written in the implicit form\(^\text{10}\)

\[
 \theta_k = \frac{1}{2} \sum_{n=1}^{N} \int_{r^n}^{r^k} \frac{\lambda^{N-k-1}d\lambda}{\sqrt{S(\lambda)}}, \quad k = 0, 1, ..., N - 1.
\tag{51}
\]

for integrable dispersive chains (6) and their first \( N - 2 \) commuting flows altogether, where (see (50))

\[
 t_k = \sum_{m=k}^{N-1} \kappa_{m-k} \theta_m.
\]

\(^9\)This Hamiltonian part is based on ideas of S. Alber, see, for instance, [2]. Later the same approach was applied in [5] for multi-component dispersive reductions also considered in our paper.

\(^{10}\)This is nothing but the well-known Hamilton–Jacobi approach.
Remark: Multi-phase solution (51) coincides with multi-phase solution (5) for Mikhalëv system (1) in the case (see (32))

$$g_k(\lambda) = 2\sqrt{S(\lambda)}, \quad k = 0, 1, ..., N - 1,$$

(52)

if we keep first three independent variables \(x, t, y\) only (i.e. we fix higher time variables to constants). The minor difference between both constructions (the finite-gap integration and the method of hydrodynamic reductions) appears in the one-phase and the two-phase solutions only. Two-phase solution (51) for \(M\)th disperive chain (6) is (see (50))

$$\theta_0 = x + \eta_1 t = \frac{1}{2} \sum_{n=1}^{2} \int_{r^n} \frac{\lambda d\lambda}{\sqrt{S(\lambda)}}, \quad \theta_1 = t = \frac{1}{2} \sum_{n=1}^{2} \int_{r^n} \frac{d\lambda}{\sqrt{S(\lambda)}},$$

(53)

while the two-phase solution (see (5)) for Mikhalëv system (1) is (see (52))

$$\theta_0 = x + \eta_1 t + \eta_2 y = \frac{1}{2} \sum_{n=1}^{2} \int_{r^n} \frac{\lambda d\lambda}{\sqrt{S(\lambda)}}, \quad \theta_1 = t + \eta_1 y = \frac{1}{2} \sum_{n=1}^{2} \int_{r^n} \frac{d\lambda}{\sqrt{S(\lambda)}},$$

(54)

This means that two-phase solution (53) of \(M\)th disperive chain (6) can be obtained from two-phase solution (54) of Mikhalëv system (1) by the stationary reduction \(y \to \text{const}\). One-phase solution of \(M\)th disperive chain (6) can be obtained directly from (5)

$$\theta_0 = x + \eta_1 t + \eta_2 y = \frac{1}{2} \int_{r^1} \frac{d\lambda}{g(\lambda)}$$

by reduction (52) and by the stationary reduction \(y \to \text{const}\). In this particular case, plenty important formulas are simplified. For instance, (41), (42) reduce to the form

$$u_m = \sum_{k=0}^{m} (k+1)s_{m-k}(r^1)^k, \quad 1 \leq m \leq M; \quad u_m = \sum_{k=2}^{\infty} (k-1)s_{m+k}(r^1)^{-k}, \quad m > M.$$

Also \(a_k = \eta_k - \eta_{k-1}r^1\) and 

$$\sigma_s = \sum_{m=0}^{s} \kappa_m (r^1)^{s-m}.$$

Remark: Conservation law densities \(\sigma_k(r)\) are Bell polynomials. Indeed, (34) can be rewritten in the form \((\varphi \to 1/p, \text{see (22) and (23); } \Phi(\lambda, r) \to 1/B(\lambda, r), \text{see (47)})

$$p(\lambda, r) = \lambda^N \left(1 + \frac{s_1}{\lambda} + \frac{s_2}{\lambda^2} + \frac{s_3}{\lambda^3} + ...\right)^{1/2}B(\lambda, r).$$

Thus (we remind that \(\kappa_0 = 1\) and \(B_0 = 1\),

$$\sigma_k = \sum_{m=0}^{k} \kappa_m B_{k-m}.$$
Taking into account relationships (24), all functions $a_k(r)$ can be also expressed via Bell polynomials, for instance,

\[
\begin{align*}
a_1 &= -B_1 - \frac{1}{2}s_1, \\
a_2 &= -B_2 + B_1^2 + \frac{1}{2}s_1 B_1 - \frac{1}{2}s_2 + \frac{3}{8}s_1^2, \\
a_3 &= -B_3 + 2B_1B_2 - B_1^3 + (\frac{1}{2}s_2 - \frac{3}{8}s_1^2)B_1 + \frac{1}{2}s_1(B_2 - B_1^2) \\
&- \frac{1}{2}s_3 + \frac{3}{8}s_1s_2 - \frac{5}{16}s_1^3, \\
\end{align*}
\]

\[
\begin{align*}
a_4 &= -B_4 + 2B_1B_3 + B_2^2 - 3B_1^2B_2 + B_1^3 + (\frac{3}{8}s_1^2 - \frac{1}{2}s_2)(B_2 - B_1^2) \\
&+ \frac{1}{2}s_1(B_3 - 2B_1B_2 + B_1^3) + (\frac{1}{2}s_3 - \frac{3}{4}s_1s_2 + \frac{5}{16}s_1^3)B_1 \\
&- \frac{1}{2}s_4 + \frac{3}{4}s_1s_3 + \frac{3}{8}s_1^2 - \frac{15}{16}s_1^2s_2 + \frac{35}{128}s_1^4. \\
\end{align*}
\]

4 Multi-Component Dispersive Reductions

In this Section we consider several integrable dispersive systems\textsuperscript{11} which can be extracted as dispersive reductions of dispersive chains (6).

A wide class of integrable dispersive systems (see [22])

\[
u_{m,t} = u_{m+1,x} - u_m u_{1,x} - \frac{1}{2} u_1 u_{m,x}, \quad m = 1, \ldots, M - 1, \tag{55}
\]

\[
u_M,t = -u_M u_{1,x} - \frac{1}{2} u_1 u_{M,x} + \sum_{k=1}^{K} w_{k,x} + \frac{1}{4} u_{1,xxx},
\]

\[
w_{k,t} = \left( \epsilon_k - \frac{1}{2} u_1 \right) w_{k,x} - w_k u_{1,x},
\]

embedded into dispersive chains (6) is selected by the rational ansatz (see (12))

\[
u = \lambda^M + \sum_{m=1}^{M} \lambda^{M-m} u_m + \sum_{k=1}^{K} \frac{w_k}{\lambda - \epsilon_k}, \tag{56}
\]

where $\epsilon_k$ are pairwise distinct arbitrary constants and $w_k$ are new field variables, such that (see (12))

\[
u_{M+m} = \sum_{k=1}^{K} (\epsilon_k)^{m-1} w_k, \quad m = 1, 2, \ldots
\]

**Remark:** This rational ansatz (56) also can be written in the factorised form

\[
u = \frac{\prod_{m=1}^{M+K} (\lambda - q^m)}{\prod_{k=1}^{K} (\lambda - \epsilon_k)},
\]

\textsuperscript{11}Most of these integrable dispersive systems associated with the Energy Dependent Schrödinger operator were introduced in [4], where their multi-Hamiltonian structures were constructed. See also [15]. Multi-phase solutions (51), (57) and their connection with the Jacobi theta-function was investigated in [3].
where \( q^m \) are pairwise distinct field variables. Then integrable dispersive systems (55) take the form (see [22])

\[
q'_t = (q^i + a_1)q_x + \frac{1}{2} \prod_{m \neq i}^{K} \frac{K(q^i - \epsilon)}{2(q^i - q^m)} a_{1,xxx},
\]

where

\[
a_1 = \frac{1}{2} \left( \sum_{m=1}^{M+K} q^m - \sum_{k=1}^{K} \epsilon_k \right).
\]

In this generic case\(^{12}\) Laurent series (32) reduces to the rational form

\[
S(\lambda) = \prod_{m=1}^{2N+M+K} (\lambda - \beta_m) \prod_{k=1}^{K} (\lambda - \epsilon_k).
\]

(57)

This means that corresponding multi-phase solutions of dispersive system (55) are parameterised by a finite number of arbitrary constants \( \beta_k \) only. The first \( N \) functions \( u_m(r) \) are determined by (41), while the functions \( w_k(r) \) take the form

\[
w_k = \prod_{m=1}^{2N+M+K} (\epsilon_k - \beta_m) \prod_{s \neq k}^{N} (r^n - \epsilon_k)^{-2}
\]

\[
= \prod_{m=1}^{2N+M+K} (\epsilon_k - \beta_m) \prod_{s \neq k}^{\infty} (\epsilon_k)^n \sum_{p=0}^{n} B_{-1-p} B_{-n-1+p}.
\]

Plenty potentially interesting sub-cases can be investigated separately, for instance, the such a case \( \epsilon_2 = \epsilon_1 \). One can easily make an appropriate computation. In this paper we omit detailed investigation of infinitely many such particular cases. Here we just mention, that if all parameters \( \epsilon_k \) vanish, a simplest set of dispersive reductions is selected by the constraint \( u_{M+K+1} = 0 \), where \( K = 0, 1, 2, \ldots \) In this case (see (12))

\[
u = \lambda^M \left( 1 + \frac{u_1}{\lambda} + \frac{u_2}{\lambda^2} + \cdots + \frac{u_{M+K}}{\lambda^{M+K}} \right),
\]

and Laurent series (32) will be truncated and reduces to the rational form (see (57))

\[
S(\lambda) = \lambda^{-K} \prod_{m=1}^{2N+M+K} (\lambda - \beta_m).
\]

**Examples:**

\(^{12}\)The generic case means that the parameters \( \epsilon_k \) are pairwise distinct and the field variables \( q^k \) are pairwise distinct.
1. If $K = 0$, the equation of the Riemann surface is pure polynomial

$$\mu^2 = S(\lambda) = \prod_{m=1}^{2N+M} (\lambda - \beta_m).$$

Thus, if $M = 1$, the Korteweg–de Vries equation (7) has $N$-phase solutions (51), where (see (41))

$$u_1(r) = 2 \sum_{n=1}^{N} r^n - \sum_{m=1}^{2N+1} \beta_m;$$

if $M = 2$, the Kaup–Boussinesq system (7) has $N$-phase solutions (51), where (see (41))

$$u_1(r) = 2 \sum_{n=1}^{N} r^n - \sum_{m=1}^{2N+2} \beta_m,$$

$$u_2(r) = \sum_{n=1}^{N} (r^n)^2 + 2 \left( \sum_{n=1}^{N} r^n \right)^2 - 2 \left( \sum_{m=1}^{2N+1} \beta_m \right) \sum_{n=1}^{N} r^n + \frac{1}{2} \left( \sum_{m=1}^{2N+2} \beta_m \right)^2 - \frac{1}{2} \sum_{m=1}^{2N+2} \beta_m^2.$$

2. If $K \geq 1$, the equation of the Riemann surface is rational

$$\mu^2 = S(\lambda) = \lambda^{-K} \prod_{m=1}^{2N+M+K} (\lambda - \beta_m).$$

(58)

Thus $N$-phase solutions (51) can be written in the form

$$\theta_k = \frac{1}{2} \sum_{n=1}^{N} \int r^n \frac{\lambda^{N+K/2-k-1} d\lambda}{\sqrt{\tilde{S}(\lambda)}}, \quad k = 0, 1, ..., N-1,$$

where

$$\tilde{S}(\lambda) = \prod_{m=1}^{2N+M+K} (\lambda - \beta_m).$$

For instance, if $M = 1$ and $K = 1$, the Ito system (8) has $N$-phase solutions (51)

$$\theta_k = \frac{1}{2} \sum_{n=1}^{N} \int r^n \frac{\lambda^{N-k-1/2} d\lambda}{\sqrt{\tilde{S}(\lambda)}}, \quad k = 0, 1, ..., N-1,$$

where

$$u_1(r) = 2 \sum_{n=1}^{N} r^n - \sum_{m=1}^{2N+2} \beta_m, \quad u_2(r) = \left( \prod_{m=1}^{2N+2} \beta_m \right) \prod_{n=1}^{N} (r^n)^{-2},$$

(59)
and
\[ \tilde{S}(\lambda) = \prod_{m=1}^{2N+2} (\lambda - \beta_m). \]

**Remark:** In the generic case (57) constants \( \kappa_m \) and \( \eta_m \) can be expressed via \( \beta_k \) as Bell polynomials, i.e. \( \eta_k = C_k(\beta, \epsilon) \), where
\[
C_k(\beta, \epsilon) = \frac{1}{2k} \left( \frac{2N+M+K}{\beta_m^k} - \frac{K}{\epsilon_m^k} \right). \tag{60}
\]

Indeed, taking into account (36), one can obtain (we remind that \( \eta_0 = 1 \) and \( s_0 = 1 \))
\[
\sum_{m=0}^{\infty} \eta_m \lambda^{-m} = \left( \sum_{m=0}^{\infty} s_m \lambda^{-m} \right)^{-1/2} = \prod_{k=1}^{K} (1 - \epsilon_k / \lambda)^{1/2} \prod_{m=1}^{2N+M+K} (1 - \beta_m / \lambda)^{-1/2}
\]
\[= \exp \left[ \frac{1}{2} \sum_{k=1}^{K} \ln(1 - \epsilon_k / \lambda) - \frac{1}{2} \sum_{m=1}^{2N+M+K} \ln(1 - \beta_m / \lambda) \right]
\]
\[= \exp \left[ \sum_{m=1}^{\infty} \frac{1}{2m} \left( \sum_{k=1}^{\infty} \left( \beta_k \right)^m - \sum_{k=1}^{K} \left( \epsilon_k \right)^m \right) \lambda^{-m} \right].
\]

This means (see (60) and cf. (43))
\[
\sum_{m=0}^{\infty} \eta_m \lambda^{-m} = \exp \left( \sum_{m=1}^{\infty} C_m \lambda^{-m} \right).
\]

Correspondingly (see (37)),
\[
\sum_{m=0}^{\infty} \kappa_m \lambda^{-m} = \exp \left( - \sum_{m=1}^{\infty} C_m \lambda^{-m} \right).
\]

Thus, for all natural numbers \( n \), we have
\[
\eta_n = \sum_{k_1! \cdot \ldots \cdot k_m!} \frac{1}{k_1! \cdot \ldots \cdot k_m!} C_{i_1}^{k_1} C_{i_2}^{k_2} \ldots C_{i_m}^{k_m},
\]
\[
\kappa_n = \sum_{k_1! \cdot \ldots \cdot k_m!} (-1)^{k_1+k_2+\ldots+k_m} C_{i_1}^{k_1} C_{i_2}^{k_2} \ldots C_{i_m}^{k_m},
\]
where the summation runs over all partitions
\[ n = \underbrace{i_1 + \ldots + i_1}_{k_1} + \underbrace{i_2 + \ldots + i_2}_{k_2} + \ldots + \underbrace{i_m + \ldots + i_m}_{k_m} \]
of the number \( n \) as a sum of \( m \) distinct positive natural numbers \( i_1, i_2, \ldots, i_m \), where \( m \) is arbitrary.

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5 The Exceptional Case $M = 0$

In the particular case $M = 0$ the compatibility conditions $(\psi_{i_1})_{xx} = (\psi_{x})_{t_1}, (\psi_{i_2})_{xx} = (\psi_{xx})_{t_2}$ (see (10), (12), (14)) lead to Camassa–Holm dispersive commuting chains

$$u_{k,t_1} = u_{k+1,x} + a_1 u_{k,x} + 2 u_k a_{1,x},$$

$$u_{k,t_2} = u_{k+2,x} + a_1 u_{k+1,x} + 2 u_{k+1} a_{1,x} + a_2 u_{k,x} + 2 u_k a_{2,x},$$

where the functions $a_1, a_2$ are connected with $u_1, u_2$ via $(\xi_1, \xi_2$ are arbitrary constants)

$$u_1 = \frac{1}{2} a_{1,xx} - 2 a_1 + \xi_1, \quad u_2 = \frac{1}{2} a_{2,xx} - 2 a_2 - \frac{1}{2} a_1 a_{1,xx} - \frac{1}{4} (a_{1,x})^2 + 3 a_1^2 - 2 \xi_1 a_1 + \xi_2.$$

The simplest reduction $u_2 = 0$ yields the Camassa–Holm equation\(^{13}\)

$$u_{1,t_1} = a_1 u_{1,x} + 2 u_1 a_{1,x}, \quad u_1 = \frac{1}{2} a_{1,xx} - 2 a_1 + \xi_1.$$

In comparison with the general case $M > 0$, in this exceptional case $M = 0$, the conservation law densities $\sigma_k$ no longer can be found from (19) as differential polynomials with respect to the field variables $u_m$. Here we obtain $u_m$ as differential polynomials with respect to the conservation law densities $\sigma_k$:

$$u_1 = -\frac{1}{2} \sigma_{1,xx} + 2 \sigma_1 + \xi_1, \quad u_2 = \frac{1}{2} (\sigma_1^2 - \sigma_2)_{xx} - \frac{1}{2} \sigma_1 \sigma_{1,xx} - \frac{1}{4} (\sigma_{1,x})^2 + \sigma_1^2 + 2 \sigma_2 + 2 \xi_1 \sigma_1 + \xi_2,$n

$$u_3 = -\frac{1}{2} (\sigma_3 - 2 \sigma_1 \sigma_2 + \sigma_1^3)_{xx} + 2 \sigma_3 + 2 \sigma_1 \sigma_2 + \frac{3}{2} \sigma_1 (\sigma_{1,x})^2 - \frac{1}{2} \sigma_{1,xx} \sigma_{2,x}$$

$$+ \left( \sigma_1^2 - \frac{1}{2} \sigma_2 \right) \sigma_{1,xx} - \frac{1}{2} \sigma_1 \sigma_{2,xx} + \xi_1 (\sigma_{1,x}^2 + 2 \sigma_2) - 2 \xi_2 \sigma_1 + \xi_3, ...$$

Moreover, in this exceptional case $M = 0$, formula (41) is no longer applicable. So, multi-phase solutions for the Camassa–Holm chains are determined by (42) and (51), i.e.

$$u_m(r) = \sum_{k=1}^{\infty} s_{2N+k+m-1} \sum_{n=1}^{k} B_n B_{n-1-k}, \quad \theta_k = \frac{1}{2} \sum_{n=1}^{N} \lambda^{N-k-1} d\lambda, \quad k = 0, 1, ..., N - 1,$$

where (see (32))

$$S(\lambda) = \lambda^{2N} \left( 1 + \frac{s_1}{\lambda} + \frac{s_2}{\lambda^2} + \frac{s_3}{\lambda^3} + ... \right).$$

Thus, multi-phase solutions for Camassa–Holm equation (62) take the form (here $K = 1$, cf. (59))

$$u_1(r) = \left( \prod_{m=1}^{2N+1} \beta_m \right) \prod_{n=1}^{N} (r^n)^{-2}, \quad \theta_k = \frac{1}{2} \sum_{n=1}^{N} \int r^n \frac{\lambda^{N-k-1/2} d\lambda}{\sqrt{S(\lambda)}}, \quad k = 0, 1, ..., N - 1,$$

\(^{13}\)Constants $\xi_m$ are not important. They can be removed by appropriate Galilean transformation. For instance, in the case of the Camassa–Holm equation, the constant $\xi_1$ can be removed by the transformation $z = x + \xi_1 t/2, y = t$ and $a_1(x, t) = w(y, z) + \xi_1/2$. 

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where (see (58))
\[
\tilde{S}(\lambda) = \prod_{m=1}^{2N+1} (\lambda - \beta_m).
\]

Camassa–Holm chain (61) also (as well as the general case \( M > 0 \)) possesses two-dimensional reductions (cf. (55))
\[
w_{k,t} = (\epsilon_k + a_1)w_{k,x} + 2w_k a_{1,x},
\]
selected by rational ansatz (see (56))
\[
u = 1 + \sum_{k=1}^{K} \frac{w_k}{\lambda - \epsilon_k},
\]
where the function \( a_1 \) is determined by the constraint
\[
\sum_{k=1}^{K} w_{k,x} - \frac{1}{2} a_{1,xxx} + 2a_{1,x} = 0.
\]

This dispersive system (63) possesses multi-phase solutions presented in implicit form (cf. previous Section)
\[
w_k = \frac{\prod_{m=1}^{2N+K} (\epsilon_k - \beta_m)}{\prod_{s\neq k} (\epsilon_k - \epsilon_s)} \prod_{n=1}^{N} (r^n - \epsilon_k)^{-2} = \frac{\prod_{m=1}^{2N+K} (\epsilon_k - \beta_m)}{\prod_{s\neq k} (\epsilon_k - \epsilon_s)} \sum_{n=0}^{\infty} (\epsilon_k)^n \sum_{p=0}^{n} B_{-1-p} B_{-n-1+p},
\]
\[
\theta_k = \frac{1}{2} \sum_{n=1}^{N} \int r^n \frac{\lambda^{N-k-1} d\lambda}{\sqrt{S(\lambda)}}, \quad k = 0, 1, ..., N - 1,
\]
where (cf. (57))
\[
S(\lambda) = \frac{\prod_{m=1}^{2N+K} (\lambda - \beta_m)}{\prod_{k=1}^{K} (\lambda - \epsilon_k)}.
\]

6 Conclusion

Three-dimensional linearly-degenerate Mikhalëv system (1) possesses \( N \) component two-dimensional hydrodynamic reductions parameterised by \( N \) arbitrary functions of a single variable. Each of them possesses a general solution parameterised by \( N \) arbitrary functions of a single variable. However, just \( N \) component two-dimensional linearly degenerate hydrodynamic reductions possess global solutions parameterised by an arbitrary number of constants. In this paper we describe connection of global solutions of Mikhalëv system (1) and multi-phase solutions for integrable two-dimensional dispersive systems associated with the energy dependent Schrödinger operator.

Moreover, the finite-gap (or algebro-geometric) method of integration was developed and applied for finite component systems only. In this paper we successfully extended this approach to infinite component systems. For \( M \)th dispersive integrable chains associated
with the Mikhalëv system we constructed infinitely many multi-phase solutions, which depend on infinite number of arbitrary parameters. Also we derived the so called “trace formulas” for coefficients of the energy dependent Schrödinger operator \( u_k(r) \) as well as for conservation law densities \( \sigma_k(r) \) and for coefficients \( a_k(r) \) of evolution with respect to higher time variables. This approach allows to reconsider multi-phase solutions of multi-component dispersive reductions from unified point of view.

**Acknowledgements**

MM gratefully acknowledges the support from GAČR under project P201/12/G028. MVP’s work was partially supported by the grant of Presidium of RAS “Fundamental Problems of Nonlinear Dynamics” and by the RFBR grant 17-01-00366. MVP also thanks V.E. Adler, A.V. Aksenov, L.V. Bogdanov, Yu.V. Brezhnev, E.V. Ferapontov, G.A. El, A.Ya. Maltsev, V.G. Marikhin, V.B. Matveev, A.E. Mironov, A.O. Smirnov, A.I. Zenchuk for important discussions.

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