On Wilson loops and $Q\bar{Q}$-potentials from the AdS/CFT relation at $T \geq 0$

H. Dorn, H.-J. Otto

Abstract

We give a short introduction to and a partial review of the work on the calculation of Wilson loops and $Q\bar{Q}$-potentials via the conjectured AdS/CFT duality. Included is a discussion of the relative weight of the stringy correction to the target space background versus the correction by the quantum fluctuations of the string world sheet.

---

1 Based on talks at the conferences “32nd International Symposium Ahrenshoop on the Theory of Elementary Particles” Buckow, September 1-5, 1998 and ”Quantum Aspects of Gauge Theories, Supersymmetry and Unification”, Corfu, 20-26 September 1998

2 e-mail: dorn@physik.hu-berlin.de otto@physik.hu-berlin.de
On Wilson loops and $Q\bar{Q}$-potentials from the
AdS/CFT relation at $T \geq 0$

Harald Dorn and Hans-Jörg Otto

Institut f. Physik, Humboldt-Universität Berlin, Invalidenstr. 110, D-10115 Berlin,
Germany

Abstract. We give a short introduction to and a partial review of the work on the
calculation of Wilson loops and $Q\bar{Q}$-potentials via the conjectured AdS/CFT duality.
Included is a discussion of the relative weight of the stringy correction to the target
space background versus the correction by the quantum fluctuations of the string world
sheet.

1 Wilson loops in gauge theory

Wilson loops $W[C] = \text{tr } P \exp(i \int_C A_\mu dx^\mu)$ play an extremely crucial role in var-
ious aspects of gauge field dynamics. In the context of this talk we have in mind
the attempts to encode gauge field dynamics in equations for functional deriva-
tives of $W$ and in particular the evaluation of quark-antiquark potentials from
the Wilson loop for rectangular closed contours $C$.

The static potential $V(L)$ between external colour sources (heavy quarks)
separated by a distance $L$ is related to the Wilson loop for a rectangular $L \times t$
contour in Euclidean pure gauge theory by [2]

$$V(L) = -\lim_{t \to \infty} \frac{1}{t} \log \langle W[C] \rangle.$$  \hspace{1cm} (1)

A linear confining potential corresponds to the famous area law.

In $T > 0$ equilibrium thermodynamics, described by $D = 3 + 1$ dimensional
Euclidean QFT with a periodic dimension of period $\frac{1}{T}$, the role of the Wil-
son loops is twofold. At first the expectation value $\langle W[C] \rangle$ for a closed contour
wrapping the compactified direction (Wilson line, Polyakov loop) is an order
parameter for confinement/deconfinement. Secondly, one can read off the $Q\bar{Q}$-
potential from the correlation function of two Wilson lines related to contours
$C_1$ and $C_2$ separated by the distance $L$

$$V(L, T) = -\log \langle W[C_1]W[C_2] \rangle.$$  \hspace{1cm} (2)

2 AdS/CFT representation of Wilson loops at $T = 0$

The AdS/CFT duality conjecture [3] in its most familiar case states the equiva-
ience of type IIB string theory on $AdS_5 \times S^5$ in the presence of $N$ units of flux
of the RR 5-form to $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills field theory in $M_4$. The couplings of both sides are related by

$$\frac{1}{2\pi}(\frac{1}{g} + i\frac{\chi}{2\pi}) = \frac{1}{g_{YM}} + i\frac{\Theta}{8\pi}.$$ 

The main motivation for the conjecture comes from the equivalence of the isometry group of $AdS_5 \times S^5$ and the $\mathcal{N} = 4$ superconformal group in $M_4$ and the dynamics of open strings in the presence of $N$ copies of D3-branes. To get some intuition for the calculation of Wilson loops via this duality we now shortly comment on this string dynamics. The metric associated to $N$ such copies at coinciding position is given by

$$ds^2 = f^{-\frac{1}{2}} dx_{||}^2 + f^{\frac{1}{2}}(dr^2 + r^2 d\Omega_5^2),$$

$$f = 1 + \frac{4\pi gN\alpha'^2}{r^4}, \quad r = |x_\perp|.$$  

(3)

Enumerating the D3-branes by a Chan-Paton index the dynamics of open strings with Chan-Paton charges at their ends yields an $U(N)$ gauge theory on the branes which decouples from the bulk of 10-dimensional target space in the limit $\alpha' \to 0$. Separating a subset of branes realizes the stringy version of the Higgs effect. E.g. $U(N+1) \to U(N) \times U(1)$ is manufactured by separating one brane from the remaining $N$ branes. A string stretching between the separated brane and the other ones mimics a quark in the unbroken $U(N)$. The distance in $u = \frac{r}{\alpha'}$ plays the role of a mass scale.

The metric (3) interpolates between flat 10-dim Minkowski space at $r \to \infty$ and $AdS_5 \times S^5$ in the near brane region $r \to 0$. Performing the decoupling limit $\alpha' \to 0$ at fixed $u$ one tests the near brane region. The metric expressed in $x_{||}, u$ and the $S^5$-variables becomes with $R = (4\pi gN)^{\frac{1}{4}} = (2g_{YM}^2N)^{\frac{1}{4}}$

$$ds^2 = \alpha' \left(\frac{u^2}{R^2} dx_{||}^2 + \frac{R^2}{u^2} du^2 + R^2 d\Omega_5^2\right) = \alpha' G_{MN} dx^M dx^N.$$  

(4)

The precise mapping for correlation functions of local operators has been given in ref.[4] and those for the nonlocal Wilson loops in ref.[5]. The analogous recipe for Wilson loops is

$$\langle W[C]\rangle = Z[C],$$ 

(5)

where $C$ is the contour for the Wilson loop in $M_4 \times S^5$, realized at the boundary of $AdS_5$ at $u \to \infty$ and $Z[C]$ is the string partition function for world sheets approaching the given $C$ at $u \to \infty$.

The factor $\alpha'^{-1}$ in the Nambu-Goto action is cancelled by the factor $\alpha'$ in eq.(4), i.e.

$$S_{NG} = \frac{1}{2\pi} \int d^2z \sqrt{\det(G_{MN}\partial_{\mu}x^M \partial_{\nu}x^N)}.$$  

(6)

The string partition function can be approximated by the AdS-area of the stationary surface of lowest order topology for small string coupling $g$ and small curvature of the target space (i.e. large $R$). We call this the classical approximation. On the YM side of the duality this limit corresponds to the 't Hooft limit $N \to \infty$, $g_{YM} \to 0$ at large 't Hooft coupling $g_{YM}^2N$. 

The factor $\frac{1}{2\pi} \int d^2z \sqrt{\det(G_{MN}\partial_{\mu}x^M \partial_{\nu}x^N)}$ contributes to the volume of the target space.
Even in this classical approximation the calculation of the Wilson loop for generic contours is a highly nontrivial task. Up to now results are available only for circular loops [6, 7] and in the limiting case for rectangles needed to evaluate $Q\bar{Q}$ potentials via (1). In the last case one needs the leading $t$-behaviour of a $L \times t$ rectangle. This becomes translation invariant in the $t$ coordinate and instead solving a partial differential equation one can restrict to an ordinary one. The potential between static $Q$ and $\bar{Q}$ separated by a distance $L$ and for $Q$ and $\bar{Q}$ both having constant orientation in internal $S^5$, with relative angle $\Delta \Theta$, has been calculated in ref. [5]

$$V(L, \Delta \Theta) = -\frac{2}{\pi} \left( \frac{2g_\text{YM}^2 N}{L} \right)^{\frac{1}{2}} \left[ 2\int_1^\infty \frac{dy}{y^2 \sqrt{(y^2 - 1)(y^2 + 1 - l^2)}} \right]^2.$$ (7)

The quantity $l$ is a monotonic function of $\Delta \Theta$ with $l(0) = 0$ and $l(\pi) = 1$.

The potential is Coulombic. Hence the AdS/CFT conjecture passed a further consistency check, since the $\frac{1}{L}$ behaviour is dictated by the conformal invariance of $N = 4$ SYM. The calculation involved a subtraction of the energy stored in two strings stretching from $u = 0$ to $u = \infty$. The regularised version of (7) obtained by positioning $C$ at $u = \Lambda$ has Coulomb behaviour for $L \cdot \Lambda \gg 1$. This is another manifestation of the IR/UV relation within the AdS/CFT correspondence [8].

Formula (7), valid for large 't Hooft coupling, is nonperturbative from the SYM point of view. The SYM perturbative potential is Coulombic as well, but has a factor $g_\text{YM}^2 N$ instead of $\left( \frac{g_\text{YM}^2 N}{L} \right)^{\frac{1}{2}}$. Higher order corrections with respect to the AdS curvature should interpolate between both regimes. It has been argued in ref. [6] that the same situation concerning the coupling constant dependence appears for the factors in front of singularities due to cusps of the Wilson loop contour $C$.

A last comment concerns the dependence on $\Delta \Theta$. For opposite orientation of $Q$ and $\bar{Q}$ in $S^5 (\Delta \Theta = \pi)$ the static force vanishes. This is in agreement with a corresponding BPS argument [5].

3 $T > 0$ and attempts to make contact with QCD

To describe the situation with non-zero temperature the metric (4) has to be replaced by the corresponding near brane limit of the metric of a set of $N$ coinciding non-extremal D3-branes leading to [9, 10, 11]

$$ds^2 = \alpha' \left( \frac{u^2}{R^2} [h(u) dx_0^2 + dx_i^2] + \frac{R^2}{u^2 h(u)} du^2 + R^2 d\Omega_5^2 \right),$$ (8)

with $h(u) = 1 - \frac{u_T^4}{u^4}$, $u_T = \pi R^2 T$, $x_0$ periodic with period $\frac{1}{T}$, $i = 1, 2, 3$ and $R$ as before. In these papers the stationary string world sheet is constructed for two qualitatively different situations.

In the so-called time-like case the boundary at $u = \infty$ is given by two lines at constant $x_i$, separated by a distance $L$ and wrapping the compact dimension
On Wilson loops and $Q\bar{Q}$-potentials from the AdS/CFT relation at $T \geq 0$. Under the AdS/CFT duality it is mapped to the correlation function of two Wilson lines and via (2) to the static $Q\bar{Q}$-potential in 3-dim space in a heat bath at temperature $T$.

In the second case, called space-like, the boundary at $u = \infty$ is a rectangle extending in the space dimensions $x_i$ only. It becomes of particular interest in connection with the proposal \cite{9} to use the breaking of SUSY by the periodic/antiperiodic boundary conditions to reach nonsupersymmetric gauge theory for $T \rightarrow \infty$, i.e. in the limit of decoupling compactified dimension. The resulting “QCD” lives in $(2 + 1)$ space-time dimensions at zero temperature.

For the time-like case one gets a $Q\bar{Q}$-potential \cite{10, 11} which behaves like $L$ at small $L$ and can be estimated numerically for generic $L$. There appears a critical distance $L_{\text{crit}}$ beyond which the $Q\bar{Q}$ force vanishes identically. It has been argued that this total screening should be absent if all quantum corrections are taken into account \cite{12}.

In the space-like case the $Q\bar{Q}$-potential turns out to be linear in $L$ for large $L$. The factor $\sigma$ in front of $L$, tentatively called QCD string tension, is equal to $\frac{\pi R^2 T^2}{2}$ \cite{10, 11}. Expressing $R$ in terms of $g_{\text{YM}}$ and $N$ and taking into account the relation to the Yang-Mills coupling in the dimensionally reduced theory ($g_{\text{YM}, 3}^2 = T g_{\text{YM}}^2$) one gets

$$
\sigma = \frac{\pi R^2 T^2}{2} = \frac{\pi}{\sqrt{2}} g_{\text{YM}, 3}^2 N^\frac{1}{2} T^\frac{3}{2}.
$$

(9)

The whole setup has been generalised to D-branes of arbitrary dimension. In particular, to end up with “QCD"$_4$ one has to start with a set of D4-branes. The tension then turns out to be $8 \frac{\pi R^2 T^2}{\pi g_{\text{YM}, 4}^2 N T^2}$ \cite{10, 12}.

Only for $T \rightarrow \infty$, where the compactified dimension decouples, there is a chance to make contact with renormalised QCD. $T \rightarrow \infty$ and finite tension $\sigma$ requires $R \rightarrow 0$. In this limit one obviously runs outside the range of applicability of the classical approximation. Reaching QCD requires substantial progress in calculating the higher order $\frac{1}{R}$-corrections. Another problem for reaching realistic QCD is connected with the possibility of a phase transition in varying $N$ \cite{9, 12, 13}.

The background (8) has been used for a calculation of glueball masses \cite{14, 15}, too. There it turns out that the mass of the KK excitations due to the compactification are of the order of the glueball masses. This obstacle might be overcome by using modified backgrounds containing a second scale \cite{17} as it is done in ref.\cite{15}. The Wilson loop calculation in such multi-scale backgrounds has been included in ref.\cite{16}. They also yield linear confining potentials.

A last point, which has been raised \cite{18} in connection with a comparison to QCD, is the absence of any universal Lüscher-type $\frac{1}{R}$ term \cite{19} in the “QCD" potentials derived so far. As argued in \cite{20, 21} such a term should be connected with the quantum fluctuations of the string world sheet, similar to the original situation in \cite{19}.
4 Corrections to the classical approximation

Leaving aside the contributions from higher genus string world surfaces, there are for fixed (lowest order) genus two sources of corrections to the classical approximation.

At first the target space background which was constructed out of a D3-brane solution of type IIB supergravity should be replaced by the similar construction based on D3-brane solutions of the stringy effective action to all orders in $\alpha'$. For $T = 0$ there are no such corrections [23]. For $T > 0$ they have been studied in ref. [22]. Their nextleading term has a factor $(\alpha')^3 L_{\text{throat}}^{-6}$ relative to the leading term. Expressing $L_{\text{throat}}$ by our parameter $R$, $\alpha'$ cancels and the relative weight is $R^{-6}$, up to a numerical factor.

The second source of corrections are quantum fluctuations of the string world sheet. Their dependence on $R$ is most easily estimated by changing variables in (8) by $u = \frac{R^2}{v}$, resulting in

$$ds^2 = \alpha' R^2 \left( \frac{1 - \pi^4 T^4 v^4}{v^2} dx_0^2 + \frac{1}{v^2} dx_i^2 + \frac{1}{v^2 (1 - \pi^4 T^4 v^4)} dv^2 + d\Omega_5^2 \right).$$

The boundary of the string world sheet now is at $v = 0$, the other coordinates describing the contour $\mathcal{C}$ are not changed. For $T = 0$ this metric is a conformal flat version of $AdS_5 \times S^5$. Now $R^2$ appears as an overall factor in front of the Nambu-Goto action $S_{\text{NG}}$ for the metric given by (10) without the factor $\alpha' R^2$. For $R \to \infty$ the expansion of the string partition function $Z[\mathcal{C}]$ has the form ($R^{-\infty}$ put into the normalization)

$$Z[\mathcal{C}] = e^{-R^2 S_{\text{NG}}[x_{\text{class}}]} \left( \det \frac{\delta^2 S_{\text{NG}}}{\delta x_{\text{class}} \delta x_{\text{class}}} \right)^{-\frac{1}{2}} (1 + O(R^{-\frac{5}{2}})).$$

The relative strength of the nextleading (determinant) contribution in comparison to the leading classical contribution is up to numerical factors $R^{-2}$. Therefore, assuming the absence of numerical peculiarities, the contribution from quantum fluctuations should dominate the corrections due to stringy corrections of the classical target space background.

Although more important, up to now the quantum fluctuations have not been calculated due to technical difficulties. On the other side the extension of the classical calculations of refs. [5, 10, 11] is straightforward and has been done in our paper [24]. We used the metric $(h(u) = 1 - \frac{u_T^4}{u})$

$$ds^2 = \alpha' \left( \frac{u^2 h(u)}{R^2} e^{\gamma A} dx_0^2 + \frac{u^2}{R^2} e^{\gamma C} dx_i^2 + \frac{R^2}{u^2 h(u)} e^{\gamma B} du^2 + R^2 e^{\gamma D} d\Omega_5^2 \right),$$

with $u_T = \pi R^2 T (1 + 15\gamma)^{-1}$ and $A, B, C, D$ polynomials in $\frac{u_T}{u}$ with known coefficients. We took $\gamma = 0$ to describe the uncorrected classical approximation or $\gamma = \frac{1}{6} \zeta(3) R^{-6}$ to include the first nontrivial stringy correction [22].
Our main focus was both the inclusion of the $\Delta \Theta$ internal space dependence into the $T > 0$ calculation and studying the effect of switching on the nextleading background correction.

In the timelike case we found a critical line in the $(L, \Delta \Theta)$-plane beyond which the $Q\bar{Q}$-force is screened. The absence of a force for all distances at antipodal internal orientation $\Delta \Theta = \pi$, known from $T = 0$, holds to continue also for $T > 0$. This is a nontrivial fact, since due to broken SUSY the absence of a force is no longer guaranteed by a BPS argument. The critical line is driven to larger values of $L$ if the correction is switched on $(\gamma > 0)$, see fig. 1. This is a movement in the right direction in the sense of ref. [12].

In the spacelike case we found for $L \to \infty$

$$V(\Delta \Theta, L) = \frac{\pi R^2 T^2}{2} (1 - \frac{265}{8} \gamma) \cdot L + \frac{1}{4\pi} R^2 (\Delta \Theta)^2 (1 + \frac{15}{8} \gamma) \cdot \frac{1}{L} + O(1).$$

The tension of the "QCD"-string is independent of $\Delta \Theta$. The nextleading term in the $L$-asymptotics is $\propto \frac{1}{L}$. Due to its dependence on the YM-coupling via $R$ and its sign it is no Lüscher-type [19] term. Furthermore, there is in QCD no place for the $\Delta \Theta$ dependence, which is a remnant of $N = 4$ SUSY. Hence it is gratifying that the term under discussion drops out in the limit $T \to \infty$, $R \to 0$ which has been identified before as the limit necessary to make contact with renormalised QCD.

**Fig. 1** $\frac{\pi}{2} T \cdot L_{\text{crit}}$ as a function of $\frac{1}{\pi} \cdot \Delta \Theta$. The lowest line is for $\gamma = 0$, the others for $\gamma = 0.01, 0.02, 0.05$ respectively.

References

1. Yu.M. Makeenko, A.A. Migdal, *Phys. Lett.* **B88** (1979) 135  
A.M. Polyakov, *Nucl. Phys.* **B164** (1980) 171  
2. K.G. Wilson, *Phys. Rev.* **D10** (1974) 2445  
3. J. Maldacena, *Adv.Theor.Math.Phys.* **2** (1998) 231, hep-th/9711200
4. S.S. Gubser, I.R. Klebanov, A.M. Polyakov, *Phys.Lett.* B428 (1998) 105, hep-th/9802109  
   E. Witten, *Adv.Theor.Math.Phys.* 2 (1998) 253, hep-th/9802150  
5. J. Maldacena, *Phys.Rev.Lett.* 80 (1998) 4859, hep-th/9803002  
   S.-J. Rey, J.-T. Yee, hep-th/9803001  
6. D.J. Gross, Talk at Strings 98, http://www.itp.ucsb.edu/online/strings98/gross/  
7. D. Berenstein, R. Corrado, W. Fischler, J. Maldacena, hep-th/9809188  
8. L. Susskind, E. Witten, hep-th/9805114  
9. E. Witten, *Adv.Theor.Math.Phys.* 2 (1998) 505, hep-th/9803131  
10. A. Brandhuber, N. Itzhaki, J. Sonnenschein, S. Yankielowicz, *Phys.Lett.* B434 (1998) 36, hep-th/9803137 and *J.High Energy Phys.* 06(1998)001, hep-th/9803263  
11. S.-J. Rey, S. Theisen, J.-T. Yee, *Nucl.Phys.* B527 (1998) 171, hep-th/9803135  
12. D.J. Gross, H. Ooguri, *Phys.Rev.* D58 (1998) 106002, hep-th/9805129  
13. M. Li, hep-th/9807196  
14. C. Csaki, H. Ooguri, Y. Oz, J. Terning, hep-th/9806021  
   R.de Mello Koch, A. Jevicki, M. Mihailescu, J. P. Nunes, *Phys.Rev.* D58 (1998) 105009, hep-th/9806125  
15. C. Csaki, Y. Oz, J. Russo, J. Terning, hep-th/9810186, J. Minahan, hep-th/9811156  
16. Y. Kinar, E. Schreiber, J. Sonnenschein, hep-th/9811192  
17. J. Russo, hep-th/9808117, K. Sfetsos, hep-th/9811167  
18. J. Greensite, P. Olesen, *J.High Energy Phys.* 08(1998)009, hep-th/9806235  
19. M. Lüscher, K. Symanzik, P. Weisz, *Nucl.Phys.* B173 (1980) 365  
   M. Lüscher, *Nucl.Phys.* B180 (1981) 317  
20. R. Kallosh, A.A. Tseytlin, *J.High Energy Phys.* 10(1998)016, hep-th/9808088  
21. Y. Kinar, E. Schreiber, J. Sonnenschein, hep-th/9809133  
22. S.S. Gubser, I.R. Klebanov, A.A. Tseytlin, *Nucl.Phys.* B534 (1998) 202, hep-th/9805156  
   J. Pawelczyk, S. Theisen, *J.High Energy Phys.* 09(1998)010, hep-th/9808126  
23. T. Banks, M.B. Green, *J.High Energy Phys.* 05(1998)002, hep-th/9804170  
24. H. Dorn, H.-J. Otto, *J.High Energy Phys.* 09(1998)021, hep-th/9807093