Dynamics of position disordered Ising spins with a soft-core potential

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We theoretically study magnetization relaxation of Ising spins distributed randomly in a d-dimension homogeneous and Gaussian profile under a soft-core two-body interaction potential \( \propto 1/[1 + (r/R_c)\alpha] \) (\( \alpha \geq d \)), where \( r \) is the inter-spin distance and \( R_c \) is the soft-core radius. The dynamics starts with all spins polarized in the transverse direction. In the homogeneous case, an analytic expression is derived at the thermodynamic limit, which starts as \( \propto \exp(-kt^2) \) with a constant \( k \) and follows a stretched-exponential law at long time with an exponent \( \beta = d/\alpha \). In between an oscillating behaviour is observed with a damping amplitude. For Gaussian samples, the degree of disorder in the system can be controlled by the ratio \( l_p/R_c \) with \( l_p \) the mean inter-spin distance and the magnetization dynamics is investigated numerically. In the limit of \( l_p/R_c \ll 1 \), a coherent many-body dynamics is recovered for the total magnetization despite of the position disorder of spins. In the opposite limit of \( l_p/R_c \gg 1 \), a similar dynamics as that in the homogeneous case emerges at later time after an initial fast decay of the magnetization. We obtain a stretched exponent of \( \beta \approx 0.18 \) for the asymptotic evolution with \( d = 3, \alpha = 6 \), which is different from that in the homogeneous case (\( \beta = 0.5 \)).

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I. INTRODUCTION

Disorder plays an essential role in determining both equilibrium and non-equilibrium properties of a many-body system, e.g. glassy phase and dynamics in magnets [1], localization phenomenon of transports [2], and novel materials by disorder engineering [3–5]. While knowing very details of a disordered system is difficult and not necessary, understanding its universal behaviour starting from a microscopic Hamiltonian is important to pin down the underlying physics. For example, many relaxations in glass materials (normal or spin-type) follow a simple stretched-exponential law \( \propto \exp[-(\gamma t)^\beta] \), \( \beta < 1 \) [1, 6]. Klafter and Shlesinger found that a scale-invariant distribution of relaxation times was the common underlying structure for three different physical models showing a stretched-exponential decay [7], which was generalized to closed quantum systems by Schultzen and coworkers recently [8].

For a disordered spin-1/2 system, recent studies confirmed a stretched-exponential decay of magnetization in both Ising [8] and Heisenberg [9, 10] models, where the pairwise spin interaction exhibits a pow-law dependence on the inter-spin distance \( r \), \( J(r) \propto 1/r^\alpha \) with \( \alpha \geq d \) in the \( d \)-dimension. The scale invariance is guaranteed since pairwise contribution to the relaxation dynamics is determined by \( J(r)t \), which is invariant under the following rescaling of space and time: \( r \to \lambda r \) and \( t \to \lambda^\alpha t \).

How would the stretched-exponential law change if the scale invariance is broken? Here we consider a specific type of pairwise interactions in an Ising Hamiltonian, namely a soft-core potential \( J(r) \propto 1/[1 + (r/R_c)\alpha] \) with \( \alpha \geq d \) and \( R_c \) the soft-core radius, reducing to the power-law behaviour at large \( r \). We have studied two different situations: (i) For homogeneously distributed spins, an analytical formula is derived for the magnetization relaxation at the thermodynamic limit, which features three different regions in the time axis: The dynamics starts as \( \propto \exp(-kt^2) \), followed by an oscillating decay, and eventually obeys an stretched-exponential law. (ii) For a spatially inhomogeneous sample, e.g. Gaussian distributed, a coherent many-body dynamics is observed in small-spatial-size system while disorder-induced relaxation is recovered for large spatial sizes. Our investigation concerning the soft-core potential is inspired by Rydberg dressing in cold-atom experiments (for recent reviews see [11, 12]) and both studied situations can be readily tested there; A uniform gas can be prepared via box potentials [13, 14] and a Gaussian distribution of atoms is obtained with a harmonic trap [15].

The article is organized as follows: We derive and discuss the analytical result for homogeneous samples in Sec. II B. The inhomogeneous situation is numerically investigated in Sec. II C and Sec. III concludes the paper.

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FIG. 1: Soft-core interaction potential between two spin-up particles. The soft-core potential in Eq. (2) is plotted as a function of inter-spin distance $r$ with $\alpha = 3$ and $6$ (solid curves). As a comparison, the power-law interactions $J_0/(r/Rc)^{\alpha}$ are also shown.

II. DISORDERED ISING MODEL WITH A SOFT-CORE POTENTIAL

A. The Ising Hamiltonian and its dynamics

A general Ising Hamiltonian for $N$ spin-1/2 particles reads

$$\hat{H}_{\text{Ising}} = \frac{1}{2} \sum_{i,j}^N J_{ij} \hat{\sigma}_{i}^{z} \hat{\sigma}_{j}^{z},$$

where $\hat{\sigma}_{i}^{z}$ is the Pauli $z$ operator and $J_{ij}$ is the coupling strength between spins $i$ and $j$. $J_{ij}$ takes a form of the soft-core potential

$$J_{ij} \equiv J(r_{ij}) = \frac{J_0}{1 + (r_{ij}/R_c)^{\alpha}},$$

where the long-range part ($r_{ij} \gg R_c$) has a power-law form ($\propto 1/r_{ij}^{\alpha}$) and the short-range ($r_{ij} \ll R_c$) interaction is almost a constant $J_0$, as seen in Fig. 1. Such a potential is not invariant under the spatial scaling $r \rightarrow \lambda r$ in general, while it is approximately invariant at large $r \gg R_c$. We will see later that this leads to a stretched-exponential relaxation for long-time dynamics both in the analytic solution of a homogeneous sample and the numerical results of a Gaussian one.

We focus on dynamics of the mean magnetization $\langle \hat{S}_{z}(t) \rangle = \langle \sum_{i=0}^{N} \hat{\sigma}_{i}^{z}(t) \rangle / N$ with an initial state that all spins are polarized in the $+x$ direction $|\phi_{0}\rangle = |+\rangle^{\otimes N}$ with $\hat{\sigma}_{x}|+\rangle = +1|+\rangle$, i.e. $\langle \hat{S}_{z}(0) \rangle = 1$. Emch [16] and Radin [17] have obtained an analytical expression for $\langle \hat{S}_{z}(t) \rangle$ with the initial state $|\phi_{0}\rangle$, which reads as

$$\langle \hat{S}_{z}(t) \rangle = \sum_{i=1}^{N} 1/N \prod_{j \neq i} \cos(J_{ij}t).$$

All following analytical and numerical results are based on the above equation.

B. Homogeneous samples: the thermodynamic limit

We consider a system of $N$ spins uniformly distributed in a spherical volume $V$ in the $d$ dimension. Following the same derivation procedure in Ref. [8], by replacing the ensemble average with an average over all possible configurations of placing $N - 1$ spins around a reference one at $r_1 = 0$, Eq. (3) can be transformed to

$$\langle \hat{S}_{z}(t) \rangle = \int_{V} dr_{2} \cdots dr_{N} P(r_{2}, \ldots, r_{N}) \prod_{j=2}^{N} \cos(J_{ij}t)$$

$$= \frac{1}{V} \int_{V} dr \cos[J(r)t]^{N-1}$$

$$= \frac{d}{r_0^d} \int_{0}^{r_0} r^{d-1} dr \cos[J_0t/1+(r/R_c)^{\alpha}]^{N-1}$$

Here $P(r_{2}, \ldots, r_{N}) = 1/V^{N-1}$ is the probability of placing the $N - 1$ spins at positions $r_{2}, \ldots, r_{N}$, respectively, and $J(r)$ takes the form in Eq. (2).
For the power-law interaction \((\propto 1/r^\alpha)\) a short-distance cutoff has to be introduced to avoid the divergence of interaction strength for further simplifying Eq. (4) \([8]\), which is not necessary for the soft-core potential considered here. By introducing a new variable \(y = J_0 t/[1 + (r/R_c)^\alpha]\) and integrating by parts, Eq. (4) can be written as
\[
\langle \hat{S}_x(t) \rangle = \left[ 1 - \frac{\pi^{d/2} \rho R_c^d}{\Gamma(d/2 + 1)} N \int_{y_0}^{J_0 t} (J_0 t/y - 1)^{\beta_0} \sin y dy \right]^{N-1}
\]
where \(N = \rho \nu = \rho \pi^{d/2} r_0^d/\Gamma(d/2 + 1)\), \(\beta_0 = d/\alpha\), and \(y_0 = J_0 t/[1 + (r_0/R_c)^\alpha]\). Here \(\rho\) is the particle density and \(\Gamma(x)\) is the Gamma function. In the thermodynamic limit \((N, r_0 \to \infty\) and \(\rho\) is a constant\), the integral \(I(J_0 t; \beta_0) = \int_{y_0}^{J_0 t} (J_0 t/y - 1)^{\beta_0} \sin y dy\) is finite only if \(\beta_0 \leq 1\) and the above equation gives
\[
\langle \hat{S}_x(t) \rangle = \exp[-FI(J_0 t; \beta_0)] \quad ,
\]
where \(F = \pi^{d/2} \rho R_c^d/\Gamma(d/2 + 1)\).

Let us first consider \(\beta_0 = 1\),
\[
I(J_0 t; 1) = J_0 t \mathrm{Si}(J_0 t) + \cos(J_0 t) - 1 ,
\]
where \(\mathrm{Si}(x) = \int_0^x \sin(t)/tdt\) is the sine integral function. At short times \((J_0 t \ll 1)\),
\[
I(J_0 t; 1) \sim 1/2(J_0 t)^2 
\]
for long times \((J_0 t \gg 1)\).

The asymptotic form of Eq. (8b) for \(\beta_0 \to 1\) actually coincides with Eq. (7b). Thus, for \(\beta_0 \leq 1\) the initial dynamics of \(\langle \hat{S}_x(t) \rangle\) follows \(\exp[-kt^2]\) with \(k = J_0^2 F(1-\beta_0)B(1-\beta_0, 1+\beta_0)/2\). In the long-time limit, the second term inside the square bracket in Eq. (8c) is a stretched exponential \(\exp[-(\gamma t)^\beta]\) with \(\beta = \beta_0\) and \(\gamma = J_0 F \cos(\beta_0 \pi/2) \Gamma(1-\beta_0)^{1/\beta_0}\). As a specific example, we show plots of Eq. (6) with \(I(J_0 t; \beta_0)\) from Eqs. (8a), (8b), and (8c) in Fig. 2(b) for \(\beta_0 = 0.5\). Other than the two limits discussed before, a damped oscillating decay is observed in between, which to a large extent can be captured by the neglected second term inside the square bracket in Eq. (8c). This oscillating decay signatures the breakdown of scale invariance with the soft-core potential.

while
\[
I(J_0 t; 1) \sim \pi/2 J_0 t - 1
\]
for long times \((J_0 t \gg 1)\). In Fig. 2(a), all three formulae are plotted as a function of evolution time and Eqs. (7b) and (7c) describe excellently the asymptotic dynamics at short and long times, respectively. Specifically, a stretched-exponential decay, \(\exp[-(\gamma t)^\beta]\) with \(\beta = \beta_0 = 1, \gamma = J_0 F \pi/2\), is seen for the long-time dynamics.

Next we look at \(\beta_0 < 1\), the integral \(I(J_0 t; \beta_0)\) is
\[
I(J_0 t; \beta_0) = J_0 t B(1-\beta_0, 1+\beta_0) \mathbb{3}[1 F_1(1-\beta_0, 2; iJ_0 t)] ,
\]
where \(B(x, y)\) is the Euler beta function, \(1 F_1(a, b, z)\) is the Kummer confluent hypergeometric function, and \(\mathbb{3}[z]\) gives the imaginary part of \(z\). More details can be found in Appendix III. The asymptotic behaviors of Eq. (8a) are
\[
I(J_0 t; \beta_0) \sim 1/2(J_0 t)^2(1-\beta_0)B(1-\beta_0, 1+\beta_0) 
\]
for short times \((J_0 t \ll 1)\), and

\[
I(J_0 t; \beta_0) \sim (J_0 t)^{\beta_0} \cos(\beta_0 \pi/2) \Gamma(1-\beta_0) - (J_0 t)^{-2\beta_0} \cos(\beta_0 \pi/2) \Gamma(1+\beta_0) 
\]
\(\text{C. Gaussian samples: a numerical study}\)

To extend the above analytic result for the homogeneous case, we numerically investigate the magnetization relaxation for an inhomogeneously distributed spin sample (Gaussian distributed) in this section, where the degree of disorder can be tuned. We focus on dynamics of the magnetization \(\langle \hat{S}_x(t) \rangle\) under the setting specified in Sec. II A, however, with spin positions \(r = (x, y, z)\) randomly distributed in a three-dimension Gaussian distribution \((d = 3)\)
\[
G(r) = \frac{1}{(2\pi)^{3/2} w_x w_y w_z} \exp(-x^2/2w_x^2 - y^2/2w_y^2 - z^2/2w_z^2) ,
\]
where \(w_\eta\) is the Gaussian waist in \(\eta\) direction \((\eta \in \{x, y, z\})\). This distribution of spins could be realized with ultracold atoms trapped in harmonic traps \([15]\). The mean particle density is \(\rho = N/(8\pi^{3/2} w_x w_y w_z)\) with the total particle number \(N\) and for simplicity we assume \(w_x = w_y = w_z \equiv w\), giving rise to a mean inter-spin dis-
tance of \( l_p \equiv \rho^{-1/3} = 2\sqrt{\pi}w/N^{1/3} \) and its corresponding interaction strength \( J_p = J(l_p) = J_0/[1 + (\pi d/2/F)(d/2 + 1)^{1/\beta_0}] \) in Eq. (2) with \( F, \beta_0 \) defined in Sec. II.B. For following numeric calculation, we fix the total spin number \( N = 100 \) and \( \alpha = 6 \) (\( \beta_0 = 0.5 \)).

For the soft-core potential in Eq. (2), \( R_c \) separates the interaction-strength randomness into two different regimes according to the ratio \( l_p/R_c \) for the above Gaussian sample. We show in Fig. 3 the distribution of pair interaction strengths with 100 spins randomly distributed according to Eq. (9) for three different values of \( l_p/R_c: 0.1, 1, \) and 5. When \( l_p \) is much smaller than \( R_c \) \( [l_p/R_c = 0.1 \) in Fig. 3(a)], \( J(r) \) is almost the constant \( J_0 \) for all pairs, hence randomness is minimized. Otherwise when \( l_p/R_c \geq 1 \), the distribution of \( J(r) \) spans over several orders of magnitude, as seen in Figs. 3(b, c). Thus effects arising from disorder are expected to be important in this regime.

In Fig. 4 we present the numerical results from Eq. (3) for \( l_p/R_c \in (0.1, 0.2) \) \( (\rho \sim 10^{14} - 10^{15} \text{ cm}^{-3} \) for \( R_c = 1 \mu \text{m} \)), coined high-density regime. In this regime, the system behaves like a all-to-all interacting one with a single interaction strength \( J_0 [18] \), recovering a coherent many-body dynamics. In this regime, the system behaves like a all-to-all interacting one with a single interaction strength \( J_0 [18] \), recovering a coherent many-body dynamics. In this regime, the system behaves like a all-to-all interacting one with a single interaction strength \( J_0 [18] \), recovering a coherent many-body dynamics. In this regime, the system behaves like a all-to-all interacting one with a single interaction strength \( J_0 [18] \), recovering a coherent many-body dynamics. In this regime, the system behaves like a all-to-all interacting one with a single interaction strength \( J_0 [18] \), recovering a coherent many-body dynamics. In this regime, the system behaves like a all-to-all interacting one with a single interaction strength \( J_0 [18] \), recovering a coherent many-body dynamics. In this regime, the system behaves like a all-to-all interacting one with a single interaction strength \( J_0 [18] \), recovering a coherent many-body dynamics. In this regime, the system behaves like a all-to-all interacting one with a single interaction strength \( J_0 [18] \), recovering a coherent many-body dynamics. In this regime, the system behaves like a all-to-all interacting one with a single interaction strength \( J_0 [18] \), recovering a coherent many-body dynamics. In this regime, the system behaves like a all-to-all interacting one with a single interaction strength \( J_0 [18] \), recovering a coherent many-body dynamics. In this regime, the system behaves like a all-to-all interacting one with a single interaction strength \( J_0 [18] \), recovering a coherent many-body dynamics. In this regime, the system behaves like a all-to-all interacting one with a single interaction strength \( J_0 [18] \), recovering a coherent many-body dynamics. In this regime, the system behaves like a all-to-all interacting one with a single interaction strength \( J_0 [18] \), recovering a coherent many-body dynamics.
are different from the values obtained analytically for a homogeneous sample with $\beta_0 = 0.5$ in Sec. II B, where $\beta = \beta_0 = 0.5, \gamma \approx 1.57J_0F^{1/\beta_0}$.

### III. CONCLUSION

In conclusion, we have considered magnetization relaxation of homogeneous and inhomogeneous samples of Ising spins with a soft-core pairwise potential. We have derived an analytic formula describing the whole dynamics in the homogeneous case, with three distinct relaxation regions in the time axis. The short-time dynamics follows $\exp(-kt^2)$ and stretched-exponential laws are found at long-time dynamics. As conjectured by Klafter and Shlesinger, this law arises from a scale-invariant distribution of relaxation times, which is only approximately fulfilled in the long-time limit since the soft-core potential in general is not scale-invariant. The breakdown of scale invariance is indicated by an oscillating feature in the relaxation between the short- and long-time limit.

Similar behaviours emerge for large Gaussian samples compared to the soft-core radius, where strong disorder presents in the system. However, for small Gaussian samples a coherent many-body dynamics is found since all spins interact with each other with an almost constant interaction strength. A smooth change from the coherent regime to the strongly disordered regime can be realized via tuning the Gaussian size of the sample. Our results in both homogeneous and inhomogeneous situations may stimulate experimental interests in the cold-atom community and may also be generalized to other types of interaction potentials.

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Appendix

Analytic derivation in a homogeneous sample

To derive Eq. (8a) from $I(J_0t; \beta_0) = \int_{J_0}^{J_0t} (J_0t/y - 1)^{\beta_0} \sin ydy$ for $\beta_0 < 1$, we first introduce a new variable $x = 1/y$ in the later integral, resulting in

$$I(J_0t; \beta_0) = \int_{1/(J_0t)}^{1/y_0} (J_0tx - 1)^{\beta_0} x^{-2} \sin(x^{-1})dx$$

$$\xrightarrow{y_0 \to 0} \frac{(J_0t)^{\beta_0}}{2i} \int_{1/(J_0t)}^{\infty} (x - \frac{1}{J_0t})^{\beta_0} x^{-2} (e^{ix^{-1}} - e^{-ix^{-1}})dx$$

$$= \frac{(J_0t)^{\beta_0}}{2i} B(1 - \beta_0, 1 + \beta_0) (J_0t)^{1-\beta_0} \{F_1(1 - \beta_0, 2, iJ_0t) - F_1(1 - \beta_0, 2, -iJ_0t)\}$$

$$= J_0t B(1 - \beta_0, 1 + \beta_0) \Im[F_1(1 - \beta_0, 2, J_0t)]$$

Here we have used a integral formula listed in Ref. [21], which reads

$$\int_{m}^{\infty} x^{v-1}(x - m)^{\mu-1}e^{b/x}dx = B(1 - \mu - v, \mu)m_{1}^{\mu+v-1}F_1(1 - \mu - v, 1 - v, b/m),$$

and is valid for $m > 0, 0 < \Re(\mu) < \Re(1-v)$. $\Re(z)$ represents the real part of $z$. The asymptotic behavior of the Kummer confluent hypergeometric function $\text{F}_1(a, b, z)$ at large $|z|$ is $\text{F}_1(a, b, z) \sim \Gamma(b)[e^{z(a-b)/\Gamma(a)} + (-z)^{-a}/\Gamma(b-a)]$, which gives rise to Eq. (8c).

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