On the second cohomology of semidirect products

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Abstract

Let $G$ be a group which is the semidirect product of a normal subgroup $N$ and a subgroup $T$, and let $M$ be a $G$-module with not necessarily trivial $G$-action. Then we embed the simultaneous restriction map

$$
\text{res} = (\text{res}_N^G, \text{res}_T^G)^t : H^2(G,M) \to H^2(N,M)^T \times H^2(T,M)
$$

into a natural five term exact sequence consisting of one and two-dimensional cohomology groups of the factors $N$ and $T$. The elements of $H^2(G,M)$ are represented in terms of group extensions of $G$ by $M$ constructed from extensions of $N$ and $T$.

Introduction. The low dimensional cohomology groups $H^n(G,M)$, $n \leq 2$, of a group $G$ with coefficients in a $G$-module $M$ crucially occur in many fields, in algebra as well as in geometry. In fact, they reflect the structure of $G$ (and of $M$ if the $G$-action on it is non trivial) in a subtle way which is far from being understood in general. If $G$ admits a proper normal subgroup $N$ it can be viewed as an extension

$$
1 \to N \to G \to Q \to 1,
$$

(1)

and one wishes to express the cohomology of $G$ in terms of the cohomology of the simpler “pieces” $N$ and $Q$. Formally, the Lyndon-Hochschild-Serre spectral sequence (referred to as LHSSS in the sequel) $H^n(Q,H^{n-p}(N,M)) \Rightarrow H^n(G,M)$ solves this problem, computing certain filtration quotients of $H^n(G,M)$ provided one can manage to compute the corresponding differentials; those concerning
$H^2(G, M)$ were determined by Huebschmann [4], in terms of automorphism groups of group extensions and of 2-fold crossed extensions, data which, however, are not easy to control in general. Also, knowing the filtration quotients of $H^n(G, M)$ does not amount to knowing its group structure completely unless $M$ is a vector space, and one often needs to represent the elements of the abstract group $H^n(G, M)$ by either explicit cocycles or group extensions (for $n = 2$). Another approach to the study of $H^2(G, M)$ consists in embedding it into exact sequences involving the cohomology groups of $N$ and $Q$; the so-called “fundamental exact sequence” derived from the LHSSS being

\[ 0 \to H^1(Q, M^N) \xrightarrow{inf} H^1(G, M) \xrightarrow{res} H^1(N, M)^Q \xrightarrow{d_2} H^2(Q, M^N) \xrightarrow{inf} H^2(G, M)_1 \xrightarrow{tr} H^1(Q, H^1(N, M)) \xrightarrow{d_2} H^3(Q, M^N) \]  \tag{2} \]

where $H^2(G, M)_1 = \text{Ker}(res^G_N : H^2(G, M) \to H^2(N, M))$. (We remark that in [3] we offer an elementary conceptual construction and proof of this exact sequence which, unlike the one in [5] concerning the first five terms, does not invoke automorphism groups). A different extension of the first five terms of (2), embedding the full group $H^2(G, M)$ instead of only $H^2(G, M)_1$, is given by Huebschmann in [5], as follows:

\[ H^2(Q, M^N) \xrightarrow{inf} H^2(G, M) \to \text{Xpext}(G, N; M) \xrightarrow{\Delta} H^3(Q, M^N) \xrightarrow{inf} H^3(G, M) \]  \tag{3} \]

where the group $\text{Xpext}(G, N; M)$ consists of equivalent classes of crossed pairs introduced in that paper. We point out that according to both sequences (2) and (3), the study of $H^2(G, M)$ involves a three-dimensional cohomology group of $Q$.

If the extension (1) splits, i.e. if $G$ is the semidirect product of $N$ and some subgroup $T$ isomorphic with $Q$, the situation is slightly better, at least in special cases. For example, if $T$ is free, the LHSSS amounts to a short exact sequence

\[ 0 \to H^1(T, H^1(N, M)) \to H^2(G, M) \xrightarrow{res} H^2(N, M)^T \to H^3(G, M)_1 \to 0 \]  \tag{4} \]

If $T$ is arbitrary, but $G$ acts trivially on $M$, $H^2(G, M)$ contains $H^2(T, M)$ as a canonical direct factor, and the complementary piece $H^2(G, M)_2 = \text{Ker}(res^G_T : H^2(G, M) \to H^2(T, M))$ fits into an exact sequence

\[ 0 \to H^1(T, \text{Hom}(N, M)) \to H^2(G, M)_2 \xrightarrow{res} H^2(N, M)^T \to H^2(T, \text{Hom}(N, M)) \to H^3(G, M)_2 \]  \tag{5} \]

due to Tahara [7] who also provides a construction of the elements of $H^2(G, M)_2$ in terms of suitable cocycles. Moreover, these results determine $H^2(G, M)$ out of only 1- and 2-dimensional cohomology groups of $N$ and $T$, in contrast with the sequences (2) and (3). When $M$ is a non trivial $G$-module, however, $H^2(T, M)$ is no longer a direct factor of $H^2(G, M)$ if $M \neq M^N$, and no general description
of the latter group in terms of first and second cohomology groups of $N$ and $T$ seems to be known. This is now provided in the present paper, by embedding the “simultaneous restriction map”

$$\text{res} = (\text{res}_N^G, \text{res}_T^G)^t : H^2(G, M) \rightarrow H^2(N, M)^T \times H^2(T, M)$$

into an exact sequence which generalizes both sequences (4) and (5), as follows.

$$H^1(T, M) \xrightarrow{\partial N} H^1(T, \text{Der}(N, M)) \xrightarrow{\tau} H^2(G, M) \xrightarrow{\text{res}} H^2(N, M)^T \times H^2(T, M) \xrightarrow{\phi} H^2(T, \text{Der}(N, M))$$

(6)

Here $\text{Der}(N, M)$ denotes the group of derivations from $N$ to $M$, which can be easily determined using Fox differential calculus, by means of the Jacobian associated to a presentation of $N$, see [1]. Thus the two terms left of $H^2(G, M)$ are easily accessible to computation. The maps in sequence (6) are described in theorem 1.1 below. Note that our sequence, unlike the preceding ones, invokes the group $\text{Der}(N, M)$ instead of its quotient $H^1(N, M)$; this may be considered as the price to pay for avoiding the appearance of a cohomology group of dimension three.

As did Tahara in his work, we also construct the elements of $H^2(G, M)$ out of those of the other groups, but not in form of cocycles but of group extensions of $G$ by $M$, the basic idea being to somehow lift the semidirect product decomposition of $G$ to any group $E$ fitting into an extension $0 \rightarrow M \rightarrow E \xrightarrow{\pi} G \rightarrow 1$. In fact, $E$ turns out to be an “amalgamated semidirect product” $E_N \rtimes_M E_T$ where $E_N = \pi^{-1}N$ and $E_T = \pi^{-1}T$; so sequence (6) arises from studying the appropriate actions of $E_T$ on $E_N$, by using automorphisms of group extensions as Huebschmann did in the cited papers, but in a different way.

We also point out that a description of $H^2(G, M)$ for $G = N \rtimes T$ is given in [2], in terms of generators and relations computed from compatible presentations of $G$, $N$ and $T$.

1 The main result

Throughout this paper, $G$ denotes a group and $M$ a $G$-module, i.e. an abelian group endowed with a not necessarily trivial action $\psi : G \rightarrow \text{Aut}(M)$. As usual, $M^G$ denotes the subgroup of elements of $M$ which are invariant under the action of $G$. Moreover, $(C^*(G, M), \partial_G^*)$ denotes the standard complex of normalized cochains on $G$ with values in $M$, i.e. $C^n(G, M)$ is the abelian group of all functions $\beta : G^{\times n} \rightarrow M$ annihilating any tuple $(g_1, \ldots, g_n)$ where $g_i = 1$ for some $i$, and the differential $\partial_G^n : C^n(G, M) \rightarrow C^{n+1}(G, M)$ is given by the formula

$$\partial_G^n(\beta) (g_1, \ldots, g_{n+1}) = g_1 \beta(g_2, \ldots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i \beta(g_1, \ldots, g_i g_{i+1}, \ldots, g_n)$$
By definition, \( H^n(G, M) = H^n(C^*(G, M), \partial_T^n) \). Denote the group of \( n \)-cocycles of \((C^*(G, M), \partial_T^n)\) by \( Z^n(G, M) = \ker(\partial_T^n) \); in particular, \( \text{Der}(G, M) = \ker(\partial_T^1) \) is the set of derivations from \( G \) to \( M \), i.e. the set of all functions \( d : G \rightarrow M \) such that \( d(gg') = gd(g') + d(g) \) for \( g, g' \in G \). If \( N \) and \( T \) are subgroups of \( G \) such that \( N \) is normal then the action of \( T \) on \( N \) by conjugation, \( t \mapsto tnt^{-1} \), induces an action of \( T \) on the complex \((C^*(N, M), \partial_N^*)\), given by \((t \beta)(n_1, \ldots, n_k) = t\beta(t^{-1}n_1, \ldots, t^{-1}n_k)\) for \( \beta \in C^k(N, M) \). Thus \( T \) acts on \( H^*(N, M) \). The following elementary construction turns out to be crucial in the sequel. If \( \Gamma \) is a group then a homomorphism of \( \Gamma \)-modules \( f : N \rightarrow N' \) gives rise to the composite homomorphism

\[
\omega^n(f) : H^n(\Gamma, \text{coker}(f)) \xrightarrow{\omega_n} H^{n+1}(\Gamma, \text{Im}(f)) \xrightarrow{\omega_{n+1}} H^{n+1}(\Gamma, \text{Ker}(f))
\]

where \( \omega_n \) and \( \omega_{n+1} \) are the connecting homomorphisms associated with the obvious short exact sequences of \( \Gamma \)-modules. In particular, for \( \Gamma = T \) and \( f = \partial_N^1 : C^1(N, M) \rightarrow Z^2(N, M) \) given by \( \partial_N^1 \) we get the map

\[
\omega_T^0(\partial_N^1) : H^2(N, M)^T \rightarrow H^2(T, \text{Der}(N, M)).
\]

The following conceptual construction of this map will be provided in the proof of Proposition 2.2. Let \( z \in H^2(N, M)^T \) be represented by a group extension \( e : M \twoheadrightarrow E \xrightarrow{i} N \) of \( N \) by \( M \), see section 2. Then \( \omega_T^0(\partial_N^1)(z) \) is represented by the restriction to \( T \) of the class of the extension

\[
0 \rightarrow \text{Der}(N, M) \rightarrow \text{Aut}_G(e) \rightarrow G \rightarrow 1
\]

constructed by Huebschmann in [5]. More explicitly, relation (14) below provides the following description of this class in terms of cocycles: Let \( z \) be represented by a 2-cocycle \( \beta \in C^2(N, M) \). Then for \( t \in T \) there exists \( \gamma(t) \in C^1(N, M) \) such that

\[
t\beta - \beta = \partial_N^1(\gamma(t)). \tag{7}
\]

We thus get a map \( \gamma \in C^1(T, C^1(N, M)) \). Its image \( \partial_T^1(\gamma) \in Z^2(T, C^1(N, M)) \) actually takes values in \( \text{Der}(N, M) \xrightarrow{i_1} C^1(N, M) \), and we have

\[
\omega_T^0(\partial_N^1)(z) = [i_1^{-1}\partial_T^1(\gamma)]. \tag{8}
\]

We are now ready to state our main result.

**Theorem 1.1** Let \( G \) be the semidirect product of a normal subgroup \( N \) and a subgroup \( T \), and let \( M \) be a \( G \)-module. Then sequence (6) in the introduction is exact, where the maps are defined as follows. For \( d \in \text{Der}(T, \text{Der}(N, M)) \) the class \( \tau[d] \) is represented by the 2-cocycle \( \beta_d : G \times G \rightarrow M \), \( \beta_d(n, nt') = nd(t)(n't) \) for \( n, n' \in N \), \( t, t' \in T \), and the map \( \phi \) is given by \( \phi = (\omega_T^0(\partial_N^1), \partial_T^n) \).

The proof will occupy the rest of the paper.
2 Automorphisms of group extensions

We first recall some basic facts about group extensions and homomorphisms between them.

An extension of $G$ by $M$ is a short exact sequence of groups

$$
0 \rightarrow M \xrightarrow{i} E \xrightarrow{\pi} G \rightarrow 1
$$

(which we will mostly write in the shorter form $M \xrightarrow{i} E \xrightarrow{\pi} G$) such that the given action of $G$ on $M$ coincides with the one induced by conjugation in $E$, i.e., $^e(i(m)) = i(\pi(e)m)$ for $e \in E$, $m \in M$. Two group extensions $\mathcal{E}, \mathcal{E}'$ are said to be congruent if there is a map (and hence an isomorphism) from $E$ to $E'$ commuting with the injections of $M$ and the projections to $G$. Congruence classes of extensions of $G$ by $M$ form an abelian group which is canonically isomorphic with $H^2(G, M)$, see [H IV.3]. Finally, if $f : \Gamma \rightarrow G$ is a homomorphism we denote by $f^*M$ the $\Gamma$-module which is $M$ as an abelian group endowed with the action of $\Gamma$ given by pulling back the action of $G$ via $f$.

**Proposition 2.1** Let $\mathcal{E}_k : M \xrightarrow{i_k} E_k \xrightarrow{\pi_k} G$, $k = 1, 2$, be two group extensions of $G$ by $M$, and let $f \in \text{Hom}(G_1, G_2)$ and $\alpha \in \text{Hom}_{G_1}(M_1, f^*M_2)$. Then the diagram of unbroken arrows

$$
\begin{array}{ccc}
\mathcal{E}_2 : & M_2 & \xrightarrow{i_2} E_2 \xrightarrow{\pi_2} G_2 \\
& \uparrow \alpha & \uparrow \epsilon \\
\mathcal{E}_1 : & M_1 & \xrightarrow{i_1} E_1 \xrightarrow{\pi_1} G_1 \\
\end{array}
$$

admits a filler $\epsilon$ (i.e. a group homomorphism from $E_1$ to $E_2$ rendering the diagram commutative) if and only if $\alpha, \epsilon_1 \in \text{Hom}(E_1, M_2)$ and $\alpha^*\epsilon_1 = f^*(\epsilon_2)$ in $H^2(G_1, f^*M_2)$. Moreover, the group $\text{Der}(G_1, f^*M_2)$ acts simply and transitively on the set $X_{(f, \alpha)}$ of all such fillers, by $(d + \epsilon)(e_1) = i_2(\pi_1(e_1))\epsilon(e_1)$ for $d \in \text{Der}(G_1, f^*M_2)$, $\epsilon \in X_{(f, \alpha)}$ and $e_1 \in E_1$. \qed

Now let $\mathcal{E} : M \xrightarrow{i} E \xrightarrow{\pi} G$ be an extension of $G$ by $M$. Consider the following subgroups of $\text{Aut}(E)$ and of $\text{Aut}(G) \times \text{Aut}(M)$, resp.

$$
\text{Aut}^M(E) = \{ \epsilon \in \text{Aut}(E) \mid \epsilon(iM) = iM \}
$$

$\text{Aut}(G, M) = \{(f, \alpha) \in \text{Aut}(G) \times \text{Aut}(M) \mid \forall (g, m) \in G \times M, \alpha(gm) = f(g)\alpha(m) \}$

A homomorphism $\rho : \text{Aut}^M(E) \rightarrow \text{Aut}(G, M)$ is defined by $\rho(\epsilon) = (\epsilon_G, \epsilon_M)$ where $\epsilon_G$ and $\epsilon_M$ are induced by $\epsilon$. Moreover, the group $\text{Aut}(G, M)$ acts on $(C^*(G, M), \partial_G)$ by automorphisms of complexes where

$$
(f, \alpha)\beta = \alpha_*((f^{-1})^*)^*\beta
$$

(10)
for $\beta \in C^n(G, M)$. We write $\alpha_* f^* [\beta]$ for the induced action on $H^n(G, M)$.

**Corollary 2.2** There is an exact sequence of groups

$$0 \rightarrow \text{Der}(G, M) \xrightarrow{(-) + \text{id}} \text{Aut}^M(E) \xrightarrow{\rho} \text{Aut}(G, M) \xrightarrow{\mathcal{O}} H^2(G, M) \quad (11)$$

where $(-) + \text{id}$ is a homomorphism and $\mathcal{O} = \partial^0_{\text{Aut}(G, M)} [\mathcal{E}]$ is the inner derivation associated with the element $[\mathcal{E}]$ of the $\text{Aut}(G, M)$-module $H^2(G, M)$. More explicitly, $\mathcal{O}$ is given by $\mathcal{O}(f, \alpha) = \alpha_* f^* [\mathcal{E}] - [\mathcal{E}]$. □

We also need to determine the cohomology class of the group extension

$$\text{Aut}(E) : 0 \rightarrow \text{Der}(G, M) \xrightarrow{(-) + \text{id}} \text{Aut}^M(E) \xrightarrow{\rho} \text{Ker}(\mathcal{O}) \rightarrow 1 \quad (12)$$

obtained from sequence (11). It is easy to check that the action of $\text{Ker}(\mathcal{O})$ on $\text{Der}(G, M)$ induced by conjugation in $\text{Aut}^M(E)$ coincides with the natural action given by (10), i.e., $(f, \alpha)d = \alpha df$.

**Proposition 2.3** The class of the extension (12) in $H^2(\text{Ker}(\mathcal{O}), \text{Der}(G, M))$ is given by the element $\omega^0_{\text{Ker}(\mathcal{O})}(\partial^1_N)[\mathcal{E}]$.

More explicitly, let $\beta : G \times G \rightarrow M$ be a 2-cocycle representing the extension $\mathcal{E}$. Then for $(f, \alpha) \in \text{Ker}(\mathcal{O})$ there exists $\gamma(f, \alpha) \in C^1(G, M)$ such that

$$(f, \alpha)\beta - \beta = \partial^1_G(\gamma(f, \alpha)). \quad (13)$$

We thus get a map $\gamma \in C^1(\text{Ker}(\mathcal{O}), C^1(G, M))$. Its image $\partial^1_{\text{Ker}(\mathcal{O})}(\gamma) \in Z^2(\text{Ker}(\mathcal{O}), C^1(G, M))$ actually takes values in $\text{Der}(G, M) \subset C^1(G, M)$, and we have

$$[\text{Aut}(\mathcal{E})] = [i_1^{-1} \partial^1_{\text{Ker}(\mathcal{O})}(\gamma)]. \quad (14)$$

**Proof:** Evaluating the maps in equation (13) on the couple $(f(g), f(g'))$ for $(g, g') \in G^2$ we get the following relation.

$$\alpha \beta(g, g') - \beta(f(g), f(g')) = f(g)\gamma(f, \alpha)(f(g')) - \gamma(f, \alpha)(f(gg')) + \gamma(f, \alpha)(f(g)) \quad (15)$$

Next we use $\beta$ to replace $\mathcal{E}$ by the congruent extension

$$\mathcal{E}' : 0 \rightarrow M \xrightarrow{i'} E' \xrightarrow{\pi'} G \rightarrow 1$$

where $E' = M \times G$ endowed with the group law $(m, g)(m', g') = (m + gm', \beta(g, g')$, $gg')$, $i'(m) = (m, 1)$ and $\pi'(m, g) = g$. It is clear that extension $\text{Aut}(\mathcal{E})$ is congruent with $\text{Aut}(\mathcal{E}')$, so we may replace it by the latter. We construct a normalized set-theoretic section $s : \text{Ker}(\mathcal{O}) \rightarrow \text{Aut}^M(E')$ of $\rho$, as follows. For...
\((f, \alpha) \in \text{Ker}(\mathcal{O})\) define a map \(s(f, \alpha) : E' \to E', \ s(f, \alpha)(m, g) = (\alpha(m) + \gamma(f, \alpha)f(g), f(g))\). We must check that \(s(f, \alpha)\) is a homomorphism; the third (and crucial) equality in the following calculation follows from \([15]\).

\[
s(f, \alpha)(m, g')(m', g') = s(f, \alpha)(m + gm' + \beta(g, g'), gg') = (\alpha(m) + \alpha(gm') + \alpha\beta(g, g') + \gamma(f, \alpha)f(gg'), f(gg'))
\]

\[
= (\alpha(m) + f(g)\alpha(m') + \gamma(f, \alpha)f(g) + f(g)\gamma(f, \alpha)f(g') + \beta(f(g), f(g')), f(g'))
\]

\[
= (\alpha(m) + \gamma(f, \alpha)f(g) + f(g))(\alpha(m') + \gamma(f, \alpha)f(g'), f(g'))
\]

\[
= (s(f, \alpha)(m, g))(s(f, \alpha)(m', g'))
\]

Moreover, the diagram

\[
\begin{array}{ccc}
M & \xrightarrow{i} & E' \\ \downarrow\alpha & & \downarrow\pi' \\ M & \xrightarrow{i} & E' \\
\end{array}
\]

\[
\begin{array}{ccc}
& & \xrightarrow{f} \\
\downarrow s(f, \alpha) & & \\
& & G
\end{array}
\]

obviously commutes, whence \(s(f, \alpha) \in \text{Aut}^M(E')\) and \(\rho(s(f, \alpha)) = (f, \alpha)\). Thus the extension \(\text{Aut}(E')\) is represented by the 2-cocycle \(\beta' \in Z^2(\text{Ker}(\mathcal{O}), \text{Der}(G, M))\) defined by

\[
\beta'((f, \alpha), (f', \alpha')) = ((-1) + \text{id})^{-1}(s(f, \alpha) \circ s(f', \alpha') \circ s(f f', \alpha \alpha')^{-1})
\]

But

\[
s(f, \alpha) \circ s(f', \alpha')(m, g) = s(f, \alpha)(\alpha'(m) + \gamma(f', \alpha')f'(g), f'(g))
\]

\[
= (\alpha\alpha'(m) + \alpha\gamma(f', \alpha')f'(g) + \gamma(f, \alpha)ff'(g), ff'(g))
\]

while

\[
(\beta'((f, \alpha), (f', \alpha')) + \text{id}) \circ s(ff', \alpha\alpha')(m, g)
\]

\[
= (\beta'((f, \alpha), (f', \alpha')) + \text{id})(\alpha\alpha'(m) + \gamma(ff', \alpha\alpha')ff'(g), ff'(g))
\]

\[
= (\beta'((f, \alpha), (f', \alpha'))ff'(g), 1)(\alpha\alpha'(m) + \gamma(ff', \alpha\alpha')ff'(g), ff'(g))
\]

\[
= (\beta'((f, \alpha), (f', \alpha'))ff'(g) + \alpha\alpha'(m) + \gamma(ff', \alpha\alpha')ff'(g), ff'(g))
\]

Thus

\[
(\beta'((f, \alpha), (f', \alpha'))ff'(g) = \alpha\gamma(f', \alpha')f^{-1}ff'(g) - \gamma(ff', \alpha\alpha')ff'(g) + \gamma(f, \alpha)ff'(g)
\]
whence
\[
\beta'(f, \alpha), (f', \alpha') = (f, \alpha)\gamma(f', \alpha') - \gamma(ff', \alpha\alpha') + \gamma(f, \alpha)
\]
\[
= \partial^1_{\text{Ker}(\mathcal{O})}(\gamma)((f, \alpha), (f', \alpha'))
\]
This shows that the map \(\partial^1_{\text{Ker}(\mathcal{O})}(\gamma) \in Z^2(\text{Ker}(\mathcal{O}), C^1(G, M))\) actually takes values in \(\text{Der}(G, M)\) \(\overset{\partial_1}{\rightarrow} C^1(G, M)\), so \(\beta' = (\iota_{1*})^{-1}\partial^1_{\text{Ker}(\mathcal{O})}(\gamma)\). Now recall that \(\omega^0_{\text{Ker}(\mathcal{O})}(\tilde{\partial}^1_N) = \omega_0\omega_1\) where
\[
H^0(\text{Ker}(\mathcal{O}), H^2(G, M)) \overset{\omega_0}{\rightarrow} H^1(\text{Ker}(\mathcal{O}), \text{Im}(\partial^1_G)) \overset{\omega_1}{\rightarrow} H^2(\text{Ker}(\mathcal{O}), \text{Der}(G, M))
\]
are the connecting homomorphisms induced by the short exact sequences of \(\text{Ker}(\mathcal{O})\)-modules
\[
0 \rightarrow \text{Im}(\partial^1_G) \overset{\iota_0}{\rightarrow} Z^2(G, M) \overset{q_2}{\rightarrow} H^2(G, M) \rightarrow 0
\]
\[
0 \rightarrow \text{Der}(G, M) \overset{\iota_1}{\rightarrow} C^1(G, M) \overset{\tilde{\partial}^1_N}{\rightarrow} \text{Im}(\partial^1_G) \rightarrow 0
\]
where \(q_2\) is the canonical projection and \(\tilde{\partial}^1_G\) is given by \(\tilde{\partial}^1_G\). So \([\text{Aut}(\mathcal{E}')] = [\beta'] = [(\iota_{1*})^{-1}\partial^1_{\text{Ker}(\mathcal{O})}(\gamma)] = [(\iota_{1*})^{-1}\partial^1_{\text{Ker}(\mathcal{O})}(\tilde{\partial}^1_N)^{-1}\partial^1_G(\gamma)] = \omega_1[\tilde{\partial}^1_G(\gamma)].\) But
\[
[\tilde{\partial}^1_G(\gamma)] = [(\iota_{0*})^{-1}\partial^1_G(\gamma)] \quad \text{since} \quad \partial^1_G = \iota_0\tilde{\partial}^1_N
\]
\[
= [(\iota_{0*})^{-1}\partial^0_{\text{Ker}(\mathcal{O})}(\beta)] \quad \text{by} \quad (13)
\]
\[
= [(\iota_{0*})^{-1}\partial^0_{\text{Ker}(\mathcal{O})}q_2^{-1}[\mathcal{E}]]
\]
\[
= \omega_0[\mathcal{E}]
\]
So \([\text{Aut}(\mathcal{E})] = [\text{Aut}(\mathcal{E}')] = \omega_1\omega_0[\mathcal{E}],\) as asserted. \(\square\)

3 Extensions of semidirect products

From now on we suppose that \(G = N \rtimes T,\) writing \(N \overset{\iota_N}{\rightarrow} G,\) \(T \overset{\iota_T}{\rightarrow} G,\) and \(\varphi : T \rightarrow \text{Aut}(N)\) for the action given by conjugation in \(G.\)

**Definition 3.1** Let \(E_N : M \overset{\pi_N}{\rightarrow} N \overset{\iota_N}{\rightarrow} E_N\) and \(E_T : M \overset{\pi_T}{\rightarrow} T \overset{\iota_T}{\rightarrow} E_T\) be group extensions and \(\tilde{\varphi} : E_T \rightarrow \text{Aut}^M(E_N)\) be a homomorphism. We say that the triple \((E_N, E_T, \tilde{\varphi})\) is realizable if there exists an extension \(\mathcal{E} : M \overset{\iota}{\rightarrow} E \overset{\pi}{\rightarrow} G\)
and a commutative diagram

\[
\begin{array}{cccccc}
E_N: & M & \xrightarrow{i_N} & E_N & \xrightarrow{\pi_N} & N \\
\parallel & \downarrow{i_1} & \parallel & \downarrow{i_N} \\
E: & M & \xrightarrow{i} & E & \xrightarrow{\pi} & G \\
\parallel & \downarrow{i_2} & \parallel & \downarrow{i_T} \\
E_T: & M & \xrightarrow{i_T} & E_T & \xrightarrow{\pi_T} & T \\
\end{array}
\]

(16)

such that for \(e_T \in E_T, e_N \in E_N\)

\[
\tilde{\varphi}(e_T)(e_N) = i_1^{-1}\left( i_2(e_T)i_1(e_N) \right).
\]  

(17)

**Proposition 3.2** A triple \((E_N, E_T, \tilde{\varphi})\) is realizable if and only if the following diagram commutes.

\[
\begin{array}{ccc}
M & \xrightarrow{i_T} & E_T & \xrightarrow{\pi_T} & T \\
\downarrow{-\theta_N} & \downarrow{\tilde{\varphi}} & \downarrow{(\varphi, \psi_T)} \\
\text{Der}(N, M) & \xrightarrow{(-)+id} & \text{Aut}^M(E_N) & \xrightarrow{\rho} & \text{Aut}(N, M) \\
\end{array}
\]

(18)

The proof requires the following, certainly well-known construction.

Let \(\Gamma\) be a group. Recall that a \(\Gamma\)-group is a group \(\Lambda\) endowed with a homomorphism \(\alpha: \Gamma \rightarrow \text{Aut}(\Lambda)\); we write \(\gamma \cdot \lambda = \alpha(\gamma)(\lambda)\). A homomorphism of \(\Gamma\)-groups is a homomorphism between \(\Gamma\)-groups which is \(\Gamma\)-equivariant.

**Proposition 3.3** Let \(K \xleftarrow{f} \Lambda \xrightarrow{g} \Gamma\) where \(f\) is a homomorphism of \(\Gamma\)-groups and where \(g\) is a precrossed module, i.e. a homomorphism of \(\Gamma\)-groups where \(\Gamma\) acts on itself by conjugation. Furthermore, suppose that \(g(\lambda) \cdot \kappa = f(\lambda)\kappa\) for all \((\lambda, \kappa) \in \Lambda \times K\). Then the amalgamated semi-direct product group \(K \rtimes_\Lambda \Gamma = K \rtimes \Gamma / \{(f(\lambda), g(\lambda)^{-1}) | \lambda \in \Lambda\}\) is defined and has the following universal property: for any commutative diagram of homomorphisms of \(\Gamma\)-groups

\[
\begin{array}{ccc}
\Lambda & \xrightarrow{g} & \Gamma \\
\downarrow{f} & & \downarrow{h_\Gamma} \\
K & \xrightarrow{h_K} & \Omega \\
\end{array}
\]
Proof of Proposition 3.2: Suppose that \( f \) and \( g \) are injective, then so is \( qj_2 \) resp. \( qj_1 \).

Let \( \Gamma \) act on \( \Omega \) via \( \gamma \cdot \omega = h_{\Gamma}(\gamma)\omega \), there is a unique homomorphism \( (h_K, h_{\Gamma}) : K \rtimes A \Gamma \to \Omega \) such that \( (h_K, h_{\Gamma})qj_1 = h_K \) and \( (h_K, h_{\Gamma})qj_2 = h_{\Gamma} \) where \( j_1(\kappa) = (\kappa, 1) \) and \( j_2(\gamma) = (1, \gamma) \), and \( q : K \times \Gamma \to K \rtimes A \Gamma \) is the canonical projection. Moreover, if \( f \) resp. \( g \) is injective, then so is \( qj_2 \) resp. \( qj_1 \).

Proof of Proposition 3.2: Suppose that \( (\mathcal{E}_N, \mathcal{E}_T, \tilde{\varphi}) \) is realizable. Then the right hand square of diagram (18) commutes since the vertical maps both are induced by conjugation in \( E \). The left hand square also commutes since

\[
(-\partial_N^B(m) + \text{id})(e_N) = i_N(-\partial_N^B(m)(\pi_N e_N))e_N = i_N(m - \pi_N(e_N)m)e_N = i_N(m) (e Ni_N(m)^{-1})e_N = i_N(m) e_N_i_N(m)^{-1} = i^{-1}_N (i_N \tilde{\varphi} e_N) \tilde{\varphi} i_N (e_N) = \tilde{\varphi} (i_N e_N).
\]

Conversely, let \( (\mathcal{E}_N, \mathcal{E}_T, \tilde{\varphi}) \) such that diagram (18) commutes. Then the maps \( \mathcal{E}_N \xrightarrow{i_N} M \xrightarrow{i_T} \mathcal{E}_T \) satisfy the hypothesis of Proposition 3.3 where \( \mathcal{E}_T \) acts on \( \mathcal{E}_N \) via \( \tilde{\varphi} \). Indeed, \( i_N \) is \( \mathcal{E}_T \)-equivariant since by commutativity of the right hand square of diagram (18),

\[
i_N(\pi_T(e_T)m) = i_N \psi_T(\pi_T e_T(m) = i_N \tilde{\varphi}(e_T) m(m) = i_N \left(i^{-1}_N \tilde{\varphi}(e_T) i_N m \right) = \tilde{\varphi}(e_T)(i_N m),
\]

while commutativity of left hand square of (18) implies

\[
\tilde{\varphi}(i_N m)(e_N) = i_N \left((-\partial_N^B m)(\pi_N e_N) \right) e_N = i_N (m - \pi_N (e_N)m)e_N = i_N (m) (e N (m)^{-1})e_N = i_N (m)e_N .
\]

Thus the amalgamated semidirect product \( E = E_N \rtimes \tilde{\varphi} E_T / \{ (i_N(m), i_T(m)^{-1}) \mid m \in M \} \) is defined, as well as the homomorphism \( \pi = (i_N \pi_N, i_T \tilde{\pi}_T) : E \to G \); in fact, \( i_N \pi_N \) is \( \mathcal{E}_T \)-equivariant as

\[
\pi_N \tilde{\varphi}(e_T)(e_N) = \tilde{\varphi}(e_T) \pi_N (\pi_N e_N) = \varphi(\pi_T e_T) (\pi_N e_N) = i_T \pi_T (e_T) i_N \pi_N (e_N).
\]
Putting \( i_k = qj_k, \ k = 1, 2, \) and \( i = i_1i_N = i_2i_T : M \to E \) we obtain a commutative diagram (16). The sequence

\[
\mathcal{E}(E_N, E_T, \tilde{\varphi}) : M \xrightarrow{i} E \xrightarrow{\pi} G
\]

is an extension: \( i \) is injective as \( i_1 \) (see Proposition 3.3) and \( i_N \) are; \( \pi \) is surjective as \( N \) and \( T \) are in its image; and if \((e_N, e_T) \in \text{Ker}(\pi), \ \pi_N e_N = \pi_T e_T = 1\) as \( N \cap T = \{1\} \), whence \((e_N, e_T) = (i_N m_1, i_T m_2)\) for some \( m_1, m_2 \in M \). But then

\[
q(e_N, e_T) = q(i_N m_1 + \tilde{\varphi}(i_T m_2)(i_N m_2), 1) = q(i_N m_1 + i_N m_2, 1) \quad \text{by (19)}
\]

It remains to check that \( q^{e_N, e_T}i(m) = i(\pi q(e_N, e_T)m) \). Indeed, in \( E_N \rtimes_{\tilde{\varphi}} E_T \),

\[
(e_N, e_T)j_1i_N(m)(e_N, e_T)^{-1} = (e_N, e_T)(i_N m, 1)(e_N, e_T)^{-1}
\]

\[
= (e_N \tilde{\varphi}(e_T)(i_N m) \tilde{\varphi}(e_T)^{-1}(e_N^{-1}), 1)
\]

\[
= (e_N \pi_N e_N^{-1}(e_T(m)) e_N^{-1}, 1) \quad \text{(right hand square of (15))}
\]

\[
= (e_N \pi_N e_N^{-1}(e_T(m)), 1)
\]

\[
= (i_N \pi_N e_N^{-1}(e_T(m)), 1)
\]

\[
= j_1i_N(\pi q(e_N, e_T)m).
\]

Finally, condition (17) is satisfied by definition of the semidirect product, whence \((E_N, E_T, \tilde{\varphi})\) is realizable.

\[\square\]

**Proposition 3.4** (a) If \((E_N, E_T, \tilde{\varphi})\) is realizable then the restricted extensions \( \mathcal{E}(E_N, E_T, \tilde{\varphi})_N \) and \( \mathcal{E}(E_N, E_T, \tilde{\varphi})_T \) are congruent with \( E_N \) and \( E_T \), resp.

(b) Any extension \( \mathcal{E} \) of \( G \) by \( M \) is congruent with \( \mathcal{E}(E_N, E_T, \tilde{\varphi}) \) where \( \tilde{\varphi} \) is given by (17).

**Proof:** Assertion (a) is immediate from diagram (16). Now if \( \mathcal{E} : M \xrightarrow{i} E \xrightarrow{\pi} G \) is any extension then the inclusions of \( E_N \) and \( E_T \) into \( E \) induce a surjective homomorphism \( \xi : E_N \rtimes_{\tilde{\varphi}} E_T \to E \) whose kernel is \( \{(i_N(m), i_T(m)^{-1}) \mid m \in M\} \), so \( \xi \) induces the desired congruence from \( \mathcal{E}(E_N, E_T, \tilde{\varphi}) \) to \( \mathcal{E} \).

\[\square\]

Now let \( E_N : M \xrightarrow{i_N} M \rtimes N \xrightarrow{\pi_N} N \) and \( E_T : M \xrightarrow{i_T} M \rtimes T \xrightarrow{\pi_T} T \) be the canonical split extensions. Then the bottom sequence in (18) (which is exact
by Proposition 2.2 is split by means of the canonical section \( s : \text{Aut}(N, M) \to \text{Aut}^M(M \times N), \ s(f, \alpha) = \alpha \times f. \) Hence we have a commutative diagram of homomorphisms with short exact rows

\[
\begin{array}{cccccc}
M & \xrightarrow{i_T} & M \rtimes T & \xrightarrow{\pi_T} & T \\
\downarrow{-\partial^0_N} & & \downarrow{-\partial^0_N \times (\varphi,\psi)^t} & & \downarrow{(\varphi,\psi)^t} \\
\text{Der}(N, M) & \xrightarrow{(-) + \text{id}} & \text{Der}(N, M) \times \text{Aut}(N, M) & \xrightarrow{\rho} & \text{Aut}(N, M) \\
\end{array}
\]  

(21)

where \( \zeta(d, (f, \alpha)) = (d + \text{id}) \circ (\alpha \times f). \) Put \( \tilde{\varphi}_0 = \zeta(\partial^0_N \times (\varphi, \psi)^t) : M \rtimes T \to \text{Aut}^M(M \rtimes N). \) Now let \( d \in \text{Der}(T, \text{Der}(N, M)) \) and \( \tilde{\varphi}_d = d + \tilde{\varphi}_0, \) see Proposition 2.1. Then

\[
\tilde{\varphi}_d(0, t)(m, n) = (d(t) + \text{id}) \circ \tilde{\varphi}_0(0, t)(m, n) \\
= (d(t) + \text{id}) \circ (\psi(t) \times \varphi(t))(m, n) \\
= (d(t) + \text{id})(tm^t, n) \\
= i_N d(t) \pi_N(tm^t, n)(tm^t, n) \\
= (d(t)(n), 1)(tm^t, n) \\
= (d(t)(n) + tm^t, n) \\
\]

(22)

**Proposition 3.5** Let \( d \in \text{Der}(T, \text{Der}(N, M)). \) Then the 2-cochain \( \beta_d : G \times G \to M, \beta_d(nt, n't') = nd(t)(n') \) is a 2-cocycle representing the extension \( \mathcal{E}(E_N, E_T, \tilde{\varphi}_d). \) Moreover, the following properties are equivalent.

1. \( [\beta_d] = 0 \) in \( H^2(G, M); \)
2. there exist derivations \( D_N : N \to M \) and \( D_T : T \to M \) such that \( \beta_d \) is the coboundary of the function \( D : G \to M \) defined by \( D(nt) = nD_T(t) + D_N(n); \)
3. there exist derivations \( D_N : N \to M \) and \( D_T : T \to M \) such that \( d = \partial^0_T(D_N) - \partial^0_N(D_T). \)

**Proof:** Abbreviate \( E_d = (M \rtimes N) \rtimes \tilde{\varphi}_d(M \rtimes T)/\{(i_N(m), i_T(m)^{-1})(m) \mid m \in M\}. \) Then a normalized set-theoretic section \( \sigma \) of \( \pi : E \to G \) is given by \( \sigma(nt) = \)
$$q((0, n), (0, t)).$$ Then

$$\sigma(nt)\sigma(n't') = q\left(\left((0, n), (0, t)\right)\left((0, n'), (0, t')\right)\right)$$

$$= q\left((0, n)\tilde{\varphi}_d(0, t)(0, n'), (0, t)(0, t')\right)$$

$$= q\left((0, n)(d(t)'n'), (0, tt')\right) \quad \text{by (22)}$$

$$= q\left((nd(t)'n'), (0, tt')\right)$$

$$= q\left((nd(t)'n'), 1\right)q\left((0, n')\right)$$

$$= i(nd(t)'n')\sigma(ntn't')$$

Hence the 2-cocycle representing $\mathcal{E}(E_N, E_T, \tilde{\varphi}_d)$ associated to $\sigma$ is $\beta_d$. So it remains to prove the asserted equivalences. First note that the implication (2) \(\Rightarrow\) (1) is plain, and that $\beta_d$ is the coboundary of a function $D : G \to M$ iff

$$\forall(n, t), (n', t') \in N \times T \text{ one has }$$

$$D(ntn't') = ntD(nt') + D(nt) - nd(t)'n'.$$  \hspace{1cm} (23)

Noting that for $t = 1$ or $n' = 1$ one has $d(t)'n' = 0$ we may take $t = t' = 1$ or $n = n' = 1$ in (23) to see that the restriction of $D$ to $N$ and to $T$, denoted by $D_N$ and $D_T$, resp., are both derivations. Moreover, taking $t = n' = 1$ in (23) we get $D(nt') = nD(t') + D(n) = ndT(t') + D_N(n)$, whence (1) implies (2). Now let $D_N \in \text{Der}(N, M)$ and $D_T \in \text{Der}(T, M)$ and define $D : G \to M$ by $D(nt) = nD_T(t) + D_N(n)$. Then we have the following equivalences:

$$D \text{ satisfies } (23)$$

$$\iff \begin{cases} n(n')D_T(tt') + D_N(n'(n')) \\
= nt(n'D_T(t') + D_N(n')) + nD_T(t) + D_N(n) - nd(t)'n') \\
end{cases}$$

$$\iff \begin{cases} ntn'D_T(t') + n(n')D_T(t) + nD_N(n)D_T(t) + D_N(n) \\
= ntn'D_T(t') + ntD_N(n') + nD_T(t) + D_N(n) - nd(t)'n') \\
end{cases}$$

$$\iff nd(t)'n' = ntD_N(n') - nD_N(n') + n(1 - t'n')D_T(t)$$

$$\iff d(t)'n' = tD_N(n') - D_N(n') - (t'n' - 1)D_T(t).$$

13
Putting \( n = \ell n \) we see that \( D \) satisfies (23) iff \( \forall (n, t) \in N \times T \),

\[
d(t)(n) = tD_N(t^{-1}n) - D_N(n) - (n - 1)D_T(t)
\]

\[
= ((t - 1)D_N(n) - \partial^0_N(D_T(t))(n)
\]

\[
= \left( \partial^0_T(D_N)(t) - \partial^0_{N*}(D_T)(t) \right)(n)
\]

\[
= \left( \partial^0_T(D_N) - \partial^0_{N*}(D_T) \right)(t)(n).
\]

Hence \( (2) \Leftrightarrow (3) \). \( \square \)

To prove our main result it now suffices to assemble all the above propositions, as follows.

**Proof of theorem 1.1:** Let \( d \in \Der(T, \Der(N,M)) \). If \( d \) is inner, i.e. if \( d = \partial^0_T(D_N) \) for some \( D_N \in \Der(N,M) \), we can take \( D_T = 0 \) in Proposition 3.3 to see that \( \beta_d = 0 \), so \( \tau \) is welldefined. Moreover, \( \tau[d] = 0 \) iff \( [d] \in \Im(\partial^0_{N*}) \), again by Proposition 3.3. Therefore sequence (6) is exact in \( H^2(G,M) \) first note that \( \text{res} \circ \tau[d] = (\text{res}^G_N[\mathcal{E}(E_N,E_T,\tilde{\varphi}_d)],\text{res}^G_N[\mathcal{E}(E_N,E_T,\tilde{\varphi}_d)]) = ([E_N],[E_T]) = (0,0) \) by Propositions 3.5, 3.4(a) and by construction of \( \mathcal{E}(E_N,E_T,\tilde{\varphi}_d) \). Now let \( E \) be some extension of \( G \) by \( M \) such that \( \text{res}[\mathcal{E}] = 0 \). By Proposition 3.4(b) \( \mathcal{E} \) is congruent with \( \mathcal{E}(E_N,E_T,\tilde{\varphi}); \) as \( E_N \) and \( E_T \) are split we may replace them by the canonical split extensions. The triple \( \mathcal{E}(E_N,E_T,\tilde{\varphi}) \) being realizable \( \tilde{\varphi} \) fits into the commutative diagram (18) by Proposition 3.2 so by Proposition 2.1 \( \tilde{\varphi} = \tilde{\varphi}_d \) for some \( d \in \Der(T, \Der(N,M)) \). Thus \( [\mathcal{E}] = [\mathcal{E}(E_N,E_T,\tilde{\varphi}_d)] = [\beta_d] = \tau[d] \) by Proposition 3.3. Thus sequence (6) is exact in \( H^2(G,M) \). Finally, let \( x = ([E_N],[E_T]) \in H^2(N,M)^T \times H^2(T,M) \). By Proposition 3.2 \( x \in \Im(\text{res}) \) iff there exists a homomorphism \( \tilde{\varphi} : E_T \to \text{Aut}^M(E_N) \) fitting into the commutative diagram (18). Now \( \Im((\varphi,\psi_T)) \subset \text{Ker}(\mathcal{O}) \) since \( [E_N] \) is \( T \)-invariant, so by Proposition 2.1 a filler \( \tilde{\varphi} \) of (18) exists iff \( (\varphi,\psi_T)^*[\text{Aut}(E_N)] = - (\partial^0_T)[E_T] = 0 \) in \( H^2(T,(\varphi,\psi_T)^*\Der(N,M)) = H^2(T,\Der(N,M)) \) by definition of the \( T \)-action on \( C^*(N,M) \). But

\[
(\varphi,\psi_T)^*[\text{Aut}(E_N)] = (\varphi,\psi_T)^*\omega_1\omega_0[E_N] \quad \text{by Proposition 2.3}
\]

\[
= \omega_1\omega_0(\varphi,\psi_T)^*[E_N] \quad \text{by naturality of connecting maps}
\]

\[
= \omega_1\omega_0[E_N]
\]

So \( x \in \text{Ker}(\text{res}) \) iff \( \phi(x) = 0 \), which concludes the proof. \( \square \)

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