Fourier series with the continuous primitive integral

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Alexiewicz norm

\[ f : [a, b] \rightarrow \mathbb{R} \]

**Lebesgue integral**

\[ f \in L^1 \iff \text{there is } F \in AC \text{ such that } F'(x) = f(x) \text{ a.e.} \]

**Henstock–Kurzweil integral**

\[ f \in HK \iff \text{there is } F \in ACG^* \text{ such that } F'(x) = f(x) \text{ a.e.} \]

\[ C^1 \subsetneq AC \subsetneq ACG^* \subsetneq C \]

**Alexiewicz norm**

\[ \| f \| = \sup_{\alpha < \beta} \left| \int_\alpha^\beta f \right| = \sup_{\alpha < \beta} |F(\alpha) - F(\beta)| \]

- \( L^1, HK \) are incomplete normed linear spaces with the Alexiewicz norm
- for completion use \( C([a, b]) \) as primitives
Test functions and distributions

Test functions
\( \mathcal{D}(\mathbb{R}) = C_c^\infty(\mathbb{R}) \) = smooth functions with compact support

Convergence \( \phi_n \to 0 \)
There is compact \( K \) such that \( \text{supp}(\phi_n) \subset K \) for all \( n \geq 1 \).
For each \( m \geq 0 \), \( \phi_n^{(m)} \to 0 \) uniformly on \( K \) as \( n \to \infty \).

Distributions
\( \mathcal{D}'(\mathbb{R}) = \) dual space of \( \mathcal{D}(\mathbb{R}) \)
If \( T \in \mathcal{D}'(\mathbb{R}) \) then \( T : \mathcal{D}(\mathbb{R}) \to \mathbb{R} \). \( \langle T, \phi \rangle \in \mathbb{R} \) for \( \phi \in \mathcal{D}(\mathbb{R}) \)

Continuous linear functionals on \( \mathcal{D}(\mathbb{R}) \)
\( (\forall \phi, \psi \in \mathcal{D}(\mathbb{R})) (\forall a, b \in \mathbb{R}) \langle T, a\phi + b\psi \rangle = a\langle T, \phi \rangle + b\langle T, \psi \rangle \)
\( \phi_n \to 0 \) in \( \mathcal{D}(\mathbb{R}) \) \( \Rightarrow \langle T, \phi_n \rangle \to 0 \) in \( \mathbb{R} \)
Distributions

\[ f \in L^p_{loc}(1 \leq p \leq \infty) \Rightarrow \langle T_f, \phi \rangle = \int_{-\infty}^{\infty} f \phi \text{ is a distribution} \]

**Dirac distribution** \[ \langle \delta, \phi \rangle = \phi(0) \]

**Derivative** \[ \langle T', \phi \rangle = -\langle T, \phi' \rangle; \quad T \in D'(\mathbb{R}), \ \phi \in D(\mathbb{R}) \]

**Composition with smooth bijection** \[ \psi : \mathbb{R} \to \mathbb{R} \]
\[ \langle T \circ \psi, \phi \rangle = \left\langle T, \frac{\phi \circ \psi^{-1}}{\psi' \circ \psi^{-1}} \right\rangle \]

**Translations** \[ \langle \tau_x T, \phi \rangle = \langle T, \tau_{-x} \phi \rangle \text{ for } \phi \in D(\mathbb{R}) \text{ where } \tau_x \phi(y) = \phi(y - x). \]

**Periodic distributions** \[ \langle \tau_p T, \phi \rangle = \langle T, \phi \rangle \text{ for some } p > 0 \text{ and all } \phi \in D(\mathbb{R}). \]
Continuous primitive integral

Primitives

\[ \mathcal{B}_c(\mathbb{T}) = \{ F \in C(\mathbb{R}) \mid F(-\pi) = 0, \text{ if } y \in [-\pi, \pi) \text{ then } F(x) = F(y) + nF(\pi) \text{ when } x = y + 2n\pi \text{ for } n \in \mathbb{Z} \} \]

If \( x \in \mathbb{R} \) and \( n \in \mathbb{Z} \) then \( F(x + 2n\pi) = F(x) + nF(\pi) \) and \( F(x) = (x - x \mod 2\pi)F(\pi)/(2\pi) + F(x \mod 2\pi) \).

\( \mathcal{B}_c(\mathbb{T}) \) is a Banach space under the uniform norm

\[ \|F\|_{\mathbb{T},\infty} = \sup_{|\alpha - \beta| \leq 2\pi} |F(\alpha) - F(\beta)| \]
Continuous primitive integral

\[ \mathcal{A}_c(\mathbb{T}) = \{ f \in \mathcal{D}'(\mathbb{R}) \mid f = F' \text{ for some } F \in \mathcal{B}_c(\mathbb{T}) \} \]

\[ \int_a^b f = F(b) - F(a); \quad F \in \mathcal{B}_c(\mathbb{T}), \; F' = f \in \mathcal{A}_c(\mathbb{T}) \]

Action of \( f \in \mathcal{A}_c(\mathbb{T}) \) on \( \phi \in \mathcal{D}(\mathbb{R}) \) is given by
\[ \langle f, \phi \rangle = \langle F', \phi \rangle = -\langle F, \phi' \rangle = -\int_{-\infty}^{\infty} F(x)\phi'(x) \, dx. \]

**Alexiewicz norm** of \( f \in \mathcal{A}_c(\mathbb{T}) \)
\[ \| f \|_T = \sup_{|I| \leq 2\pi} \left| \int_I f \right| = \max_{|\beta - \alpha| \leq 2\pi} |F(\beta) - F(\alpha)| = \| F \|_{T,\infty} \]

**linear isometry**
\[ \mathcal{A}_c(\mathbb{T}) \leftrightarrow \mathcal{B}_c(\mathbb{T}) \quad f \leftrightarrow F \]
Integration by parts

Bounded variation
If \( g : \mathbb{R} \rightarrow \mathbb{R} \) is periodic then its variation over \( \mathbb{T} \) is
\[
Vg = \sup \sum |g(s_i) - g(t_i)| \quad \text{where the supremum is taken over all disjoint intervals } \{(s_i, t_i)\} \subset (-\pi, \pi).
\]

Write \( \mathcal{BV}(\mathbb{T}) \) for the periodic functions with finite variation. This is a Banach space under the norm \( \|g\|_{\mathcal{BV}} = \|g\|_{\infty} + Vg \).

Let \( f \in \mathcal{A}_c(\mathbb{T}) \) and \( g \in \mathcal{BV}(\mathbb{T}) \) then
\[
\int_{-\pi}^{\pi} fg = F(\pi)g(\pi) - \int_{-\pi}^{\pi} F(t) \, dg(t)
\]

Hölder inequality
\[
\left| \int_{-\pi}^{\pi} fg \right| \leq \|f\|_{\mathbb{T}} \|g\|_{\mathcal{BV}}
\]
Fourier coefficients

\[ \hat{f}(n) = \int_{-\pi}^{\pi} f(t)e^{-int} dt = (-1)^n F(\pi) + in \int_{-\pi}^{\pi} F(t)e^{-int} dt \]

\[ |\hat{f}(n)| \leq |F(\pi)| + |n| \int_{-\pi}^{\pi} |F| \leq 4\sqrt{2} |n| \|f\|_T \]

**Riemann–Lebesgue lemma**

\( \hat{f}(n) = o(n) \) as |n| \( \to \infty \)

**Theorem**

For \( j \in \mathbb{N} \), let \( f, f_j \in A_c(\mathbb{T}) \) such that \( \|f_j - f\|_T \to 0 \) as \( j \to \infty \). Then for each \( n \in \mathbb{Z} \) we have \( \hat{f}_j(n) \to \hat{f}(n) \) as \( j \to \infty \). The convergence need not be uniform in \( n \in \mathbb{Z} \).
**Convolution**

\[ f \ast g(x) = \int_{-\pi}^{\pi} f(x - t)g(t) \, dt; \quad f \in \mathcal{A}_c(\mathbb{T}) \text{ and } g \in \mathcal{BV}(\mathbb{T}) \]

**Theorem**

Let \( f \in \mathcal{A}_c(\mathbb{T}) \) and let \( g \in \mathcal{BV}(\mathbb{T}) \).

(a) \( f \ast g \in C(\mathbb{T}) \)
(b) \( f \ast g = g \ast f \)
(c) \( \|f \ast g\|_\infty \leq \|f\|_T \|g\|_{\mathcal{BV}} \)
(d) For \( y \in \mathbb{R} \) we have \( \tau_y(f \ast g) = (\tau_y f) \ast g = f \ast (\tau_y g) \).
(e) If \( h \in L^1(\mathbb{T}) \) then \( f \ast (g \ast h) = (f \ast g) \ast h \in C(\mathbb{T}) \).
(f) \( \hat{f} \ast \hat{g}(n) = \hat{f}(n)\hat{g}(n) \) for all \( n \in \mathbb{Z} \).
Summability kernels

**Theorem**

Let $f \in A_c(\mathbb{T})$. For each $n \in \mathbb{N}$, let $k_n \in BV(\mathbb{T})$ such that $\int_{-\pi}^{\pi} k_n = 1$ and $\lim_{n \to \infty} \int_{|s|>\delta} |k_n(s)| \, ds = 0$ for each $0 < \delta \leq \pi$. Suppose there is $M \in \mathbb{R}$ so that $\|k_n\|_1 \leq M$ for all $n \in \mathbb{N}$. Then $\|f \ast k_n - f\|_{\mathbb{T}} \to 0$ as $n \to \infty$.

**Fejér kernel**

$$k_n(t) = \frac{1}{2\pi} \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n + 1}\right) e^{ikt} = \frac{1}{2\pi(n+1)} \left\{ \frac{\sin[(n+1)t/2]}{\sin(t/2)} \right\}^2$$
**Convergence**

**Lemma**

Let $f \in \mathcal{A}_c(\mathbb{T})$. Let $e_n(x) = e^{-inx}$. Then $f \ast e_n(x) = \hat{f}(n)e^{inx}$.

Let $g(t) = \sum_{-n}^{n} a_k e_k(t)$ for a sequence $\{a_k\} \subset \mathbb{R}$.

Then $f \ast g(x) = \sum_{-n}^{n} a_k \hat{f}(k)e^{ikx}$.

**Theorem**

The trigonometric polynomials are dense in $\mathcal{A}_c(\mathbb{T})$.

For $f \in \mathcal{A}_c(\mathbb{T})$

$$f \ast k_n(t) = \frac{1}{2\pi} \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) \hat{f}(k)e^{ikt}.$$

And $\lim_{n \to \infty} \|f \ast k_n - f\|_\mathbb{T} = 0$.

If $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$ then $f = 0$. 

Dirichlet kernel

\[ D_n(t) = \sum_{k=-n}^{n} e^{ikt} = \frac{\sin[(n + 1/2)t]}{\sin(t/2)} \]

This is not a summability kernel
\[ \|D_n\|_1 \sim \left(\frac{4}{\pi^2}\right) \log(n) \quad \text{as } n \to \infty. \]

**Theorem**

Let \( f \in L^1(\mathbb{T}) \). Then \( \|f \ast D_n - f\|_\mathbb{T} \to 0 \) as \( n \to \infty \).

It is known that \( \|f \ast D_n - f\|_1 \) need not converge to 0.
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