Recursion operator in a noncommutative Minkowski phase space

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Abstract

A recursion operator for a geodesic flow, in a noncommutative (NC) phase space endowed with a Minkowski metric, is constructed and discussed in this work. A NC Hamiltonian function $H_{nc}$ describing the dynamics of a free particle system in such a phase space, equipped with a noncommutative symplectic form $\omega_{nc}$ is defined. A related NC Poisson bracket is obtained. This permits to construct the NC Hamiltonian vector field, also called NC geodesic flow. Further, using a canonical transformation induced by a generating function from the Hamilton-Jacobi equation, we obtain a relationship between old and new coordinates, and their conjugate momenta. These new coordinates are used to re-write the NC recursion operator in a simpler form, and to deduce the corresponding constants of motion. Finally, all obtained physical quantities are re-expressed and analyzed in the initial NC canonical coordinates.

Keywords: Noncommutative Minkowski phase space, recursion operator, geodesic flow, Nijenhuis torsion, constants of motion.

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1 Introduction

In the last few decades there was a renewed interest in completely integrable Hamiltonian systems, the concept of which goes back to Liouville in 1897 [14] and Poincaré in 1899 [18]. They are dynamical systems admitting a Hamiltonian description and possessing sufficiently many constants of motion, so that they can be integrated by quadratures. Some qualitative features of these systems remain true in some special classes of infinite-dimensional Hamiltonian systems expressed by nonlinear evolution equations as, for instance, Korteweg-de Vries and Sine-Gordon [25].

A relevant progress in the study of these systems with an infinite-dimensional phase manifold $M$ was the introduction of the Lax Representation [13]. It played an important role in formulating the Inverse Scattering Method [1], one of the most remarkable result of theoretical physics in last decades. This method allows the integration of nonlinear dynamics, both with finitely or infinitely many degrees of freedom, for which a Lax representation can be given [8], this being both of physical and mathematical relevance [5].

Another progress, in the analysis of the integrability, was the important remark that many of these systems are Hamiltonian dynamics with respect to two compatible symplectic structures [15], [3], [25], this leading to a geometrical interpretation of the so-called recursion operator [15]. For more details, see [20] and references therein. A description of integrability working both for systems with finitely many degrees of freedom and for field theory can be given in terms of invariant, diagonalizable mixed $(1,1)$-tensor field, having bidimensional eigenspaces and vanishing Nijenhuis torsion. A natural approach to integrability is to try to find sufficient conditions for the eigenvalues of the recursion operator to be in involution. Thereby, a new characterization of integrable Hamiltonian systems is given by De Filippo et al through the following Theorem [6]:

**Theorem 1.** Let $X$ be a dynamical vector field on a $2n$-dimensional manifold $M$. If the vector field $X$ admits a diagonalizable mixed $(1,1)$-tensor field $T$ which is invariant under $X$, has a vanishing Nijenhuis torsion and has doubly degenerate eigenvalues with nowhere vanishing differentials, then there exist a symplectic structure and a Hamiltonian function $H$ such that the vector field $X$ is separable, Hamiltonian vector field of $H$, and $H$ is completely integrable with respect to the symplectic structure.


Such a $(1,1)$-tensor field $T$ is called a recursion operator of $X$.

In a particular case of $\mathbb{R}^{2n}$, a recursion operator can be constructed as follows \cite{22}:

**Lemma 1.** Let us consider vector fields

$$X_l = -\frac{\partial}{\partial x_{n+l}}, \quad l = 1, \ldots, n$$

on $\mathbb{R}^{2n}$ and let $T$ be a $(1,1)$-tensor field on $\mathbb{R}^{2n}$ given by

$$T = \sum_{i=1}^{n} x_i \left( \frac{\partial}{\partial x_i} \otimes dx_i + \frac{\partial}{\partial x_{n+i}} \otimes dx_{n+i} \right).$$

Then, we have that the Nijenhuis torsion $N_T$ and the Lie derivative $\mathcal{L}_{X_i}$ of $T$ are vanishing, i.e.

$$\mathcal{L}_{X_i} T = 0.$$

That is the $(1,1)$-tensor field $T$ is a recursion operator of $X_i, (l = 1, \ldots, n)$.

On the other hand, this $(1,1)$-tensor field $T$ is used as an operator which generate enough constants of motion \cite{12}. Based on Theorem 1, a series of investigations was done (see e.g. \cite{6}, \cite{7}, \cite{12}, \cite{17}, \cite{20}, \cite{26}, \cite{10}, \cite{21}, \cite{22}, \cite{23} and references therein). One of powerful methods of describing completely integrable Hamiltonian systems with involutive Hamiltonian functions or constants of motion uses the recursion operator admitting a vanishing Nijenhuis torsion.

Recently, in 2015, Takeuchi constructed recursion operators of Hamiltonian vector fields of geodesic flows for some Riemannian and Minkowski metrics \cite{21}, and obtained related constants of motion. Further, he used five particular solutions of the Einstein equation in the Schwarzschild, Reissner-Nordström, Kerr, Kerr-Newman, and FLRW metrics, and showed that the Hamiltonian functions of the associated corresponding geodesic flows form a system of variables separation equations. Then, he constructed recursion operators inducing the complete integrability of the Hamiltonian functions. In the present work, we investigate the same problem in a deformed Minkowski phase space.

This paper is organized as follows. In Section 2, we consider a noncommutative Minkowski phase space, and define the NC Hamiltonian function and symplectic form, as well as the corresponding NC Poisson bracket. In Section 3, we construct the associated NC recursion operator for the NC Hamiltonian vector field of the geodesic flow, and obtain related constants of motion. In Section 4, we end with some concluding remarks.

## 2 Noncommutative Minkowski phase space

The noncommutativity between space-time coordinates was first introduced by Snyder \cite{19}. Later, Alain Connes developed the noncommutative geometry \cite{3} and applied it to various physical situations \cite{4}. Since then, the noncommutative geometry remained a very active research subject in several domains of theoretical physics and mathematics.

Noncommutativity between phase space variables is here understood by replacing the usual product with the $\beta$--star product, also known as the Moyal product law between two arbitrary functions of position and momentum, as follows \cite{24}, \cite{16}, \cite{11}:

$$\langle f *_{\beta} g \rangle(q,p) = f(q_i, p_i) \exp \left( \frac{\beta}{2} \gamma^{ab}_{cd} \frac{\partial}{\partial q^c} \frac{\partial}{\partial p^d} \right) \bigg|_{(q_i, p_i) = (q_j, p_j)} g(q_j, p_j),$$

(2)
where
\[ \beta_{ab} = \begin{pmatrix} \alpha_{ij} & i & i \\ -\delta_{ij} - \gamma_{ij} & \lambda_{ij} \\ \end{pmatrix}, \] (3)

\( \alpha \) and \( \lambda \) are antisymmetric \( n \times n \) matrices which represent the noncommutativity in coordinates and momenta, respectively; \( \gamma \) is some combination of \( \alpha \) and \( \lambda \). The \( *_{\beta} \) deformed Poisson bracket can be written as
\[ \{ f, g \}_{\beta} = f *_{\beta} g - g *_{\beta} f. \] (4)

So, we can show that :
\[ \{ q_i, q_j \}_{\beta} = \alpha_{ij}, \{ q_i, p_j \}_{\beta} = \delta_{ij} + \gamma_{ij}, \{ p_i, q_j \}_{\beta} = -\delta_{ij} - \gamma_{ij}, \{ p_i, p_j \}_{\beta} = \lambda_{ij}. \] (5)

Now, consider the following transformations :
\[ q'_i = q_i - \frac{1}{2} \sum_{j=1}^{n} \alpha_{ij} p_j, \quad p'_i = p_i + \frac{1}{2} \sum_{j=1}^{n} \lambda_{ij} q_j, \] (6)

where \( q'_i \) and \( p'_i \) obey the same commutation relations as in (5), but with respect to the usual Poisson bracket :
\[ \{ q'_i, q'_j \} = \alpha_{ij}, \{ q'_i, p'_j \} = \delta_{ij} + \gamma_{ij}, \{ p'_i, q'_j \} = -\delta_{ij} - \gamma_{ij}, \{ p'_i, p'_j \} = \lambda_{ij}, \] (7)

with \( q_i \) and \( p_j \) satisfying the following commutation relations :
\[ \{ q_i, q_j \} = 0, \quad \{ q_i, p_j \} = \delta_{ij}, \quad \{ p_i, p_j \} = 0. \] (8)

In our framework, we consider the noncommutative Minkowski phase space with the metric defined by
\[ ds^2 = -dq_1^2 + dq_2^2 + dq_3^2 + dq_4^2, \] (9)

where \( q'_1 \) is time coordinate; \( q'_2, q'_3, q'_4 \) are space coordinates. The tensor metric is given by
\[ g^i_j = g^{ij} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \] (10)

and the equation of geodesic is
\[ \frac{d^2 q^\mu}{dt^2} + \Gamma^\mu_{\nu \lambda} \frac{dq^\nu}{dt} \frac{dq^\lambda}{dt} = \frac{d^2 q^\mu}{dt^2} = 0, \quad (\mu = 1, 2, 3, 4), \] (11)

with the Christoffel symbols \( \Gamma^\mu_{\nu \lambda} = 0 \).

Set:
\[ q'_i = q_i - \frac{1}{2} \sum_{j=1}^{4} \alpha_{ij} p_j, \quad p'_i = p_i + \frac{1}{2} \sum_{j=1}^{4} \lambda_{ij} q_j, \quad \lambda_{ij} = \alpha_{1j} = 0, \quad p_1 > 0. \] (12)

Then, the commutation relations (7) become :
\[ \{ q'_i, q'_j \} = \alpha_{ij}, \{ q'_i, p'_j \} = \delta_{ij} + \gamma_{ij}, \{ p'_i, q'_j \} = -\delta_{ij} - \gamma_{ij}, \{ p'_i, p'_j \} = \lambda_{ij}. \] (13)

2.1 NC Hamilton function and NC symplectic form

The NC Hamiltonian function \( \mathcal{H}_{nc} \) describing the dynamics of a free particle system, in the considered NC Minkowski phase space is defined as follows :
\[ \mathcal{H}_{nc} := \frac{1}{2} \left( -p_1^2 + \sum_{k=2}^{4} p_k^2 \right). \] (11)

Using equations (12), we get
\[ \mathcal{H}_{nc} = \frac{1}{2} \left[ -p_1^2 + \sum_{k=2}^{4} \left( p_k + \frac{1}{2} \sum_{j=2}^{4} \lambda_{kj} q_j \right) \right]^2. \] (14)

3
Proposition 1. The exterior derivative of the Hamiltonian function $H_{nc}$ is given by

$$dH_{nc} = -p_1 dq_1 + \sum_{k=2}^{4} \omega_k dq_k + \frac{1}{2} \sum_{k,i=2}^{4} \lambda_{ik} \Omega_i dq_k,$$

where

$$\omega_k = \left( p_k + \frac{1}{2} \sum_{j=2}^{4} \lambda_{kj} q_j \right)$$

and

$$\Omega_i = \left( p_i + \frac{1}{2} \sum_{j=2}^{4} \lambda_{ij} q_j \right).$$

The NC symplectic form is now defined by

$$\omega_{nc} := \sum_{i=1}^{4} dp'_i \wedge dq'_i = dp'_1 \wedge dq'_1 + \sum_{k=2}^{4} dp'_k \wedge dq'_k.$$

Proposition 2. Considering the NC Minkowski phase space, the symplectic form associated with the Hamiltonian function $H_{nc}$ is given by

$$\omega_{nc} = \sum_{\nu=1}^{4} \theta_{\nu} dp_{\nu} \wedge dq_{\nu},$$

where

$$\theta_{\nu} = \sum_{\mu=1}^{4} \left( \delta_{\mu\nu} + \frac{1}{4} \lambda_{\mu\nu} \alpha_{\mu\nu} \right) \neq 0, \quad \delta_{\mu\nu} = \begin{cases} 0, & \text{if } \mu \neq \nu \\ 1, & \text{if } \mu = \nu. \end{cases}$$

2.2 NC Poisson bracket and NC Hamiltonian vector field

Proposition 3. The bracket given by

$$\{ f, g \}_{nc} = \sum_{\nu=1}^{4} \theta_{\nu}^{-1} \left( \frac{\partial f}{\partial p_{\nu}} \frac{\partial g}{\partial q_{\nu}} - \frac{\partial f}{\partial q_{\nu}} \frac{\partial g}{\partial p_{\nu}} \right)$$

is a Poisson bracket which respects the symplectic form $\omega_{nc}$, where $f$ and $g$ are arbitrary differentiable coordinate functions on the NC Minkowski phase space.

Proposition 4. In the NC Minkowski phase space, the Hamiltonian vector field is given by

$$X_{H_{nc}} = -p_1 \frac{\partial}{\partial q_1} + \sum_{k=2}^{4} \theta_k^{-1} \left( \omega_k \frac{\partial}{\partial q_k} - \frac{1}{2} \sum_{i=2}^{4} \lambda_{ik} \Omega_i \frac{\partial}{\partial p_k} \right),$$

where

$$\omega_k = \left( p_k + \frac{1}{2} \sum_{j=2}^{4} \lambda_{kj} q_j \right)$$

and

$$\Omega_i = \left( p_i + \frac{1}{2} \sum_{j=2}^{4} \lambda_{ij} q_j \right).$$
3 NC Recursion operator

In this section, we construct the recursion operator for the geodesic flow in the NC Minkowski phase space, and derive the constants of motion. We consider the Hamilton-Jacobi equation \([2]\) for the Hamiltonian function \((14)\), and introduce a generating function \(W_{nc}\) satisfying the following relations:

\[
p = \frac{\partial W_{nc}}{\partial q} \quad \text{and} \quad P = -\frac{\partial W_{nc}}{\partial Q}.
\]

This allows us to obtain a relationship between the \((p, q)\) and \((P, Q)\) coordinates. Then, using Lemma 1, we build a recursion operator for the NC Hamiltonian vector field \(X_{H_{nc}}\).

3.1 NC Hamilton-Jacobi equation and generating function

The NC Hamilton-Jacobi equation is a nonlinear equation given by

\[
E_{nc} = H_{nc}(q, \frac{\partial W_{nc}}{\partial q}).
\]

Thus,

\[
E_{nc} = \frac{1}{2} \left\{ -\left( \frac{\partial W_{nc}}{\partial q_1} \right)^2 + \sum_{k=2}^{4} \left( \frac{\partial W_{nc}}{\partial q_k} + \frac{1}{2} \sum_{j=2}^{4} \lambda_{kj} q_j \right)^2 \right\},
\]

where \(E_{nc}\) is a constant. Setting \(W_{nc} = \sum_{i=1}^{4} W_{nc}^i(q_i)\), where \(W_{nc}^i(q_i) = a_i q_i\) and \(a_i (i = 1, 2, 3, 4)\) are constants, not depending on \(q_i\), leads to \(a_i = \frac{\partial W_{nc}^i}{\partial q_i}\), and \((22)\) becomes

\[
2E_{nc} = -a_1^2 + \sum_{k=2}^{4} \left( a_k + \frac{1}{2} \sum_{j=2}^{4} \lambda_{kj} q_j \right)^2.
\]

Assume now \(\sum_{k=2}^{4} \left( a_k + \frac{1}{2} \sum_{j=2}^{4} \lambda_{kj} q_j \right)^2 - 2E_{nc} > 0\). Then,

\[
a_1 = \pm \sqrt{\sum_{k=2}^{4} \left( a_k + \frac{1}{2} \sum_{j=2}^{4} \lambda_{kj} q_j \right)^2 - 2E_{nc}}.
\]

Considering the future domain yields

\[
a_1 = \sqrt{\sum_{k=2}^{4} \left( a_k + \frac{1}{2} \sum_{j=2}^{4} \lambda_{kj} q_j \right)^2 - 2E_{nc}},
\]

and

\[
W_{nc} = \left( \sum_{k=2}^{4} \left( a_k + \frac{1}{2} \sum_{j=2}^{4} \lambda_{kj} q_j \right)^2 - 2E_{nc} \right) q_1 + \sum_{k=2}^{4} a_k q_k.
\]

Now, we introduce a generating function by using the above solution such that \(W_{nc} = W_{nc}(q_1, q_2, q_3, q_4, Q_1, Q_2, Q_3, Q_4)\) becomes

\[
W_{nc} = \left( \sum_{k=2}^{4} Q_k^2 - 2Q_1 \right) q_1 + \sum_{k=2}^{4} \left( Q_k - \frac{1}{2} \sum_{j=2}^{4} \lambda_{kj} q_j \right) q_k.
\]
where

\[
\left( \sum_{k=2}^{4} Q_k^2 - 2Q_1 \right) > 0, \quad Q_1 = E_{nc},
\]  

and

\[
Q_k = a_k + \frac{1}{2} \sum_{j=2}^{4} \lambda_{kj} q_j, \quad (k = 2, 3, 4).
\]

Thanks to the canonical transformations (21), we obtain the following relationship between the canonical coordinate system \((P, Q)\) and the original coordinate system \((p, q)\) :

\[
\begin{align*}
\begin{cases}
p_1 = \sum_{k=2}^{4} Q_k^2 - 2Q_1 \\
p_k = Q_k \\
q_1 = P_1 \\
q_k = -P_k - Q_k P_1
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
P_1 = \frac{q_1}{p_1} \\
P_k = -\frac{p_k q_1}{p_1} - q_k \\
Q_1 = H_{nc} \\
Q_k = p_k,
\end{cases}
\end{align*}
\]

where \(k = 2, 3, 4\).

### 3.2 (1,1)-tensor field \(T\) as recursion operator

In the \((P, Q)\) coordinate systems, the Hamiltonian vector field is defined by

\[
X_{H_{nc}} := \{ H_{nc}, \cdot \}_{nc} = \sum_{\nu=1}^{4} \theta_{\nu}^{-1} \left( \frac{\partial H_{nc}}{\partial P_{\nu}} \frac{\partial}{\partial Q_{\nu}} - \frac{\partial H_{nc}}{\partial Q_{\nu}} \frac{\partial}{\partial P_{\nu}} \right).
\]

Setting \(H_{nc} = Q_1\) and \(\theta_1 = 1\) transforms the NC Hamiltonian vector field \(X_{H_{nc}}\) and symplectic form \(\omega_{nc}\) into the forms

\[
X_{H_{nc}} = -\frac{\partial H_{nc}}{\partial Q_1} \frac{\partial}{\partial P_1} = -\frac{\partial}{\partial P_1},
\]

and

\[
\omega_{nc} = \sum_{\nu=1}^{4} \theta_{\nu} dP_{\nu} \wedge dQ_{\nu},
\]

respectively. A tensor field \(T\) of \((1,1)\)-type can then be expressed as :

\[
T = \sum_{\nu=1}^{4} Q_\nu \left( \frac{\partial}{\partial P_{\nu}} \otimes dP_{\nu} + \frac{\partial}{\partial Q_{\nu}} \otimes dQ_{\nu} \right).
\]

Letting \(x_\nu = Q_\nu\) and \(x_{\nu+4} = P_\nu\), where \(\nu = 1, 2, 3, 4\), affords the tensor field

\[
T = \sum_{i,j=1}^{8} T_j^i \frac{\partial}{\partial x^i} \otimes dx^j,
\]

with \(x \equiv (Q_1, ..., Q_4, P_1, ..., P_4)\). The matrix \((T_j^i)\) is given by

\[
(T_j^i) = \begin{pmatrix}
A & O \\
O & A
\end{pmatrix}, \quad (A_j^i) = \begin{pmatrix}
Q_1 & 0 & 0 & 0 \\
0 & Q_2 & 0 & 0 \\
0 & 0 & Q_3 & 0 \\
0 & 0 & 0 & Q_4
\end{pmatrix}.
\]

Then, by Lemma \[1\] \(T\) satisfies \(\mathcal{L}_{X_{H_{nc}}} T = 0\), the Nijenhuis torsion \(\mathcal{N}_T\) of \(T\) is vanishing, i.e. \(\mathcal{N}_T = 0\), and \(\text{deg}Q_1 = 2\). Hence, \(T\) is a recursion operator of the Hamiltonian vector field \(X_{H_{nc}}\). The constants of motion \(T r(T^l), \; (l = 1, 2, 3, 4)\), of the geodesic flow are :

\[
T r(T^l) = 2(Q_1^l + Q_2^l + Q_3^l + Q_4^l), \quad l = 1, 2, 3, 4.
\]
Proposition 5. Assume:

(1) \( \lambda_{1\mu} = \alpha_{1\mu} = 0, \quad \mu = 1, 2, 3, 4; \)

(2) \( \lambda_{\nu\mu} = \lambda_{\mu\nu} = 0, \quad \) for every \( \nu, \mu = 2, 3, 4. \)

Then, the geodesic flow has a recursion operator \( T \) given by

\[
T = \sum_{\mu, \nu = 1}^{4} \left( M_{\nu}^\mu \frac{\partial}{\partial q_{\nu}} \otimes dq_{\mu} + N_{\nu}^\mu \frac{\partial}{\partial p_{\nu}} \otimes dp_{\mu} + L_{\nu}^\mu \frac{\partial}{\partial q_{\nu}} \otimes dp_{\mu} + R_{\nu}^\mu \frac{\partial}{\partial p_{\nu}} \otimes dq_{\mu} \right),
\]  

(32)

where

\[
M = \begin{pmatrix}
\mathcal{H}_{nc} & \frac{p_2}{p_1} (p_2 - \mathcal{H}_{nc}) & \frac{p_3}{p_1} (p_3 - \mathcal{H}_{nc}) & \frac{p_4}{p_1} (p_4 - \mathcal{H}_{nc}) \\
q_1 \mathcal{H}_{nc} S_2 & p_2 & 0 & 0 \\
q_1 \mathcal{H}_{nc} S_3 & 0 & p_3 & 0 \\
q_1 \mathcal{H}_{nc} S_4 & 0 & 0 & p_4
\end{pmatrix},
\]

\[
N = \begin{pmatrix}
\mathcal{H}_{nc} & 0 & 0 & 0 \\
\frac{p_2}{p_1} V_2 & p_2 & 0 & 0 \\
\frac{p_3}{p_1} V_3 & 0 & p_3 & 0 \\
\frac{p_4}{p_1} V_4 & 0 & 0 & p_4
\end{pmatrix},
\]

\[
L = \begin{pmatrix}
0 & q_1 \frac{p_2}{p_1} (p_2 - \mathcal{H}_{nc}) & q_1 \frac{p_3}{p_1} (p_3 - \mathcal{H}_{nc}) & q_1 \frac{p_4}{p_1} (p_4 - \mathcal{H}_{nc}) \\
q_1 \frac{p_2}{p_1} V_2 & 0 & 0 & 0 \\
q_1 \frac{p_2}{p_1} V_3 & 0 & 0 & 0 \\
q_1 \frac{p_2}{p_1} V_4 & 0 & 0 & 0
\end{pmatrix},
\]

\[
R = \begin{pmatrix}
0 & 0 & 0 & 0 \\
\mathcal{H}_{nc} S_2 & 0 & 0 & 0 \\
\mathcal{H}_{nc} S_3 & 0 & 0 & 0 \\
\mathcal{H}_{nc} S_4 & 0 & 0 & 0
\end{pmatrix},
\]

with \( V_k = p_k - \mathcal{H}_{nc} - \frac{\mathcal{H}_{nc}}{2p_k} \sum_{j=2}^{4} \lambda_{kj} q_j \) and \( S_k = -\frac{1}{2} \sum_{i=2}^{4} \lambda_{ik} \Omega_{vi} \) \( (k = 2, 3, 4). \)

The constants of motion in the original coordinate system \( (p, q) \) are \( Tr(T^l) \), \( (l = 1, 2, 3, 4) : \)

\[
Tr(T^l) = 2\mathcal{H}_{nc}^l + 2(p_2^l + p_3^l + p_4^l) = \frac{1}{2} (-p_1^2 + \sum_{k=2}^{4} \omega_k^2)^l + 2(p_2^l + p_3^l + p_4^l).
\]  

(33)
4 Concluding remarks

In this paper, we have constructed a recursion operator of a Hamiltonian vector field for a geodesic flow in a noncommutative Minkowski phase space, and computed the associated constants of motion. For the vanishing deformation parameter $\beta$, the NC Minkowski phase space turns to be the usual one, and all the results displayed in this work are reduced to the particular cases examined in [21] and [22].

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