COCENTER OF $p$-ADIC GROUPS, II: INDUCTION MAP

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Abstract. In this paper, we study some relation between the cocenter $\bar{H}(G)$ of the Hecke algebra $\mathcal{H}(G)$ of a connected reductive group $G$ over a nonarchimedean local field and the cocenter $\bar{H}(M)$ of its Levi subgroups $M$.

Given any Newton component of $\bar{H}(G)$, we construct the induction map $\bar{i}$ from the corresponding Newton component of $\bar{H}(M)$ to it. We show that this map is surjective. This leads to the Bernstein-Lusztig type presentation of the cocenter $\bar{H}(G)$, which generalizes the work [13] on the affine Hecke algebras. We also show that the map $\bar{i}$ we constructed is adjoint to the Jacquet functor and in characteristic 0, the map $\bar{i}$ is an isomorphism.

Introduction

0.1. Let $G$ be a connected reductive group over a nonarchimedean local field $F$ of arbitrary characteristic and $G = \mathbb{G}(F)$. Let $R$ be an algebraically closed field of characteristic not equal to $p$, where $p$ is the characteristic of residue field of $F$. Let $\mathcal{H}_R$ be the Hecke algebra of $G$ over $R$ and $\bar{\mathcal{H}}_R = \mathcal{H}_R/[\mathcal{H}_R, \mathcal{H}_R]$ be its cocenter. Let $\mathfrak{R}(G)_R$ be the $R$-vector space with basis the isomorphism classes of irreducible smooth admissible representations of $G$ over $R$. Then we have the trace map

$$\text{Tr}_R : \bar{\mathcal{H}}_R \to \mathfrak{R}(G)_R^*.$$ 

On the representation side, we have the induction functor and the Jacquet functor

$$i_{M,R} : \mathfrak{R}(M)_R \to \mathfrak{R}(G)_R, \quad r_{M,R} : \mathfrak{R}(G)_R \to \mathfrak{R}(M)_R,$$

where $M$ is a Levi subgroup of $G$.

What happens on the cocenter side?

The functor adjoint to the induction functor $i_M$ is the restriction map $\bar{r}_{M,R} : \bar{\mathcal{H}}(G)_R \to \bar{\mathcal{H}}(M)_R$. It can be expressed explicitly via the Van Dijk’s formula. In this paper, we investigate the functor $\bar{i}_{M,R} : \bar{\mathcal{H}}_R(M) \to \bar{\mathcal{H}}_R(G)$, which is adjoint to the Jacquet functor $r_{M,R} : \mathfrak{R}(G)_R \to \mathfrak{R}(M)_R$.

0.2. We first describe the properties we expect for the map $\bar{i}_{M,R}$ and then discuss the approach toward it.

First, instead of working over various algebraically closed fields $R$, it is desirable to have the map $\bar{i}_M$ defined on the integral form $\bar{\mathcal{H}}$ (the cocenter of the Hecke algebra of $\mathbb{Z}[\frac{1}{p}]$-valued functions). Such map, if exists, provides not only a uniform approach to the map $\bar{i}_{M,R}$ for all $R$, but also some useful information on the mod-$l$
representations (see Theorem [4] in the introduction and a future work [6] for some results in this direction).

Second, in [11], we introduced the Newton decomposition. Roughly speaking,
\[ G = \sqcup G(v) \text{ and } H = \bigoplus H(v), \]
where \( v \) runs over the set of dominant rational coweights of \( G \). Such description is expected to play an important role in the representation theory of \( p \)-adic groups. In order to relate the Newton decomposition with the representations, we would like to know that the Newton decomposition is compatible with the map \( \tilde{i}_M \).

0.3. Now we discuss several approaches in the literature towards the understanding of the map \( \tilde{i}_M \).

Over \( \mathbb{C} \), the spectral density Theorem of Kazhdan [14] asserts that the trace map \( \text{Tr}_C: \tilde{H}_C \rightarrow \mathfrak{R}(G)^*_C \) is injective. Hence the map \( \tilde{i}_{M,C} \) is uniquely determined by the adjunction formula
\[ \text{Tr}_C^M(f, r_{M,C}(\pi)) = \text{Tr}_C^G(\tilde{i}_{M,C}(f), \pi). \]
However, if \( R \) is of positive characteristic, the trace map \( \text{Tr}_R \) may not be injective and thus the map \( \tilde{i}_{M,R} \) is not uniquely determined by the adjunction formula.

In those cases, one may use the categorical description of the cocenter to give a definition of \( \tilde{i}_{M,R} \). Bernstein’s second adjointness theorem implies that the map \( \tilde{i}_{M,R} \) defined in this way is adjoint to the Jacquet functor (see [7, (1.8)]). However, it is not clear that this map preserves the integral structure (see some discussion in [7, §4.27]). Also it is not clear if this description is compatible with the Newton decomposition.

0.4. A different, but more explicit approach is given by Bushnell in [2].

Note that the induction functor \( i_{M,R} \) on the representations of \( M \) depends not only on the Levi subgroup \( M \), but also on the parabolic subgroup \( P \) with Levi factor \( M \). However, when passing to the Grothendieck group of the representations, the dependence of \( P \) disappears. On the other hand, the Jacquet functor \( r_{M,R} \), even if one passes to the Grothendieck groups of the representations, still depend on the choice of parabolic subgroup.

Let \( v \) be a rational coweight. Then \( v \) determines a Levi subgroup \( M = M_v \) and the parabolic subgroup \( P_v = MN_v \). Let \( \mathcal{K} \) be a “nice” open compact subgroup of \( G \) (e.g. the \( n \)-th congruent subgroup \( \mathcal{Z}_n \) of an Iwahori subgroup) and \( \mathcal{K}_M = \mathcal{K} \cap M \). Bushnell introduced the \( P_v \)-positive elements of \( M \) and the subalgebra \( H^v(M, \mathcal{K}_M) \) of \( H(M, \mathcal{K}_M) \), consisting of compactly supported \( \mathcal{K}_M \)-biinvariant functions supported in the \( P_v \)-positive elements. Then he proves that

(a) The algebra \( H(M, \mathcal{K}_M) \) is isomorphic to the localization of \( H^v(M, \mathcal{K}_M) \) at a strongly positive element \( f_z \).

(b) The map
\[ j_{v,\mathcal{K}}: H^v(M, \mathcal{K}_M) \longrightarrow H(G, \mathcal{K}), \delta_{\mathcal{K}_M m \mathcal{K}_M} \longrightarrow \delta_{P_v}(m)^{-\frac{1}{2}} \mu_G(\mathcal{K}) \mu_M(\mathcal{K}_M) \delta_{\mathcal{K}_M m \mathcal{K}} \]
is an injective algebra homomorphism.

(c) The map \( j_{v,\mathcal{K}} \) is adjoint to the Jacquet functor \( r_{M,\mathcal{K},R}: \mathfrak{R}_{\mathcal{K}}(G)_R \rightarrow \mathfrak{R}_{\mathcal{K} \cap M}(M)_R \) relative to \( P_v \). Here \( \mathfrak{R}_{\mathcal{K}}(G)_R \subset \mathfrak{R}(G)_R \) consists of representations generated by their \( \mathcal{K} \)-fixed vectors.
Moreover, Bushnell’s map \( j_{v,K} \) also preserves the integral structure of the Hecke algebra.

0.5. It is tempting to apply Bushnell’s result to the cocenter of Hecke algebras. However, there are several obstacles.

If \( K \) is the Iwahori or pro-\( p \) Iwahori subgroup, then the map \( j_{v,K} \) extends to an algebra homomorphism \( H(M, K \cap M) \to H(G, K) \). In this case, the localization of Hecke algebra \( H^v(M, K \cap M) \) is consistent with the Bernstein-Lusztig presentation ([10] and [18]). However, as pointed out in [2], these are essentially the only cases of this kind. Thus one may only use \( j_{v,K} \) to deduce the induction map from part of the cocenter of \( H(M) \) to the cocenter of \( H(G) \).

The Newton strata of \( M \) with integral dominant Newton points are positive, but the strata with rational (but not integral) Newton point may not be positive for any parabolic \( P \). Those strata are not in the domain of the maps \( j_{v,K} \).

Also if one fixes \( M \) and \( P \), the maps \( j_{v,K} \) are not compatible with the change of open compact subgroups \( K \), even at the cocenter level (see §2.5). Thus the maps \( j_{v,K} \) does not induce a well-defined map \( \tilde{i}_v : \tilde{H}(M; v) \to \tilde{H} \).

0.6. The idea behind Bushnell’s map \( j_{v,K} \) is to enlarge the open compact subset \( K_M m \mathcal{K}_M \) of \( M \) to the open compact subset \( K m \mathcal{K} \) of \( G \) by multiplying the open compact subgroup \( K \).

Let \( v \) be a rational coweight and \( P = MN_v \) be the associated parabolic subgroup. The elements in the Newton stratum \( M(v) \) may not be \( P_v \)-positive, but a sufficiently large power of it is \( P_v \)-positive. One may enlarge an open compact subset inside \( M(v) \) by multiplying a suitable open compact subgroup of \( G \) to obtain an open compact subset of \( G \). Unlike the situation in [2], the lack of \( P_v \)-positivity condition prevents us to give an explicit open compact subgroup of \( G \) that works in our situation. We have to use sufficiently small open compact subgroup of \( G \). Since \( v \) is strictly positive with respect to \( N_v \), we finally show that our construction is independent of the choice of such open compact subgroups. We have

**Theorem A.** Let \( v \) be a rational coweight and \( M = M_v \). Let \( \bar{v} \) be the \( G \)-dominant coweight associated to \( v \). Then

1. **[Theorem 3.1]** The map
   \[
   \delta_{mK_M} \mapsto \delta_{P_v(m)}^{-\frac{1}{2}} \mu_M(K_M) \delta_{mK_M K} + [H, H]
   \]
   for sufficiently small open compact subgroup \( K \) of \( G \) gives a well-defined map
   \[\tilde{i}_v : \tilde{H}(M; v) \to \tilde{H} \].

2. **[Theorem 4.1]** The image of \( \tilde{i}_v \) equals \( \tilde{H}(G; \bar{v}) \).

3. **[Theorem 6.5]** If moreover, \( \text{char}(F) = 0 \), then the map \( \tilde{i}_v \) gives a bijection between \( \tilde{H}(M; v) \) and \( \tilde{H}(G; \bar{v}) \).

**Theorem B** *(Theorem 5.2).* Let \( v \) be a rational coweight and \( M = M_v \). Then for any \( f \in \bar{H}_R(M; v) \) and \( \pi \in \mathcal{R}(G)_R \), we have the following adjunction formula

\[
\text{Tr}^M_R(f, r_{v,R}(\pi)) = \text{Tr}^G_R(\tilde{i}_v(f), \pi).
\]

Here \( r_{v,R} : \mathcal{R}(G)_R \to \mathcal{R}(M)_R \) is the Jacquet functor relative to \( P_v \).
0.7. Now we discuss some applications. In [11], we introduced the rigid cocenter $H^{\text{rig}} = \bigoplus H(v)$, where $v$ runs over rational central coweights.

Now for any standard Levi subgroup $M$, we introduce the $+\text{-rigid}$ part $\check{H}(M)^{+\text{,rig}} = \bigoplus \check{H}(M; v)$, where $v$ runs over rational dominant coweights with $M = M_v$. We then have the well-defined map

$$\tilde{i}_M^+ = \bigoplus_v \tilde{i}_v^+ : \check{H}(M)^{+\text{,rig}} \rightarrow \check{H}.$$ 

As an application of Theorem A and the Newton decomposition of $\check{H}$ (see [11, Theorem 3.1]), we have

**Theorem C.** We have the decomposition of the cocenter $\check{H}$ into $+\text{-rigid}$ parts:

$$\check{H} = \bigoplus M \text{ is a standard Levi subgroup } \tilde{i}_M^+ (\check{H}(M)^{+\text{,rig}}).$$

For affine Hecke algebras, such decomposition is first obtained in [13] via an elaborate analysis on the minimal length elements in the affine Weyl groups of $G$ and its Levi subgroups $M$. In loc.cit., such decomposition is called the Bernstein-Lusztig presentation of the cocenter of affine Hecke algebras, since the explicit expression of $\tilde{i}_M^+$ there is given in terms of the Bernstein-Lusztig presentation. Although there is no Bernstein-Lusztig type presentation for $H$, we follow [13] and still call the decomposition in Theorem C the Bernstein-Lusztig presentation of the cocenter $\check{H}$. It is also worth mentioning that the proof in this paper does not involve the elaborate analysis on the minimal length elements as in [13], but based on the compatibility between the change of different open compact subgroups $K$ of $G$.

Theorem C asserts that the rigid cocenters of Levi subgroups form the “building blocks” of the whole cocenter $\check{H}$. We also show that that they are compatible with the trace map in the following way.

**Theorem D (Theorem 6.1).** Let $R$ be an algebraically closed field of characteristic not equal to $p$. Then we have

$$\ker \text{Tr}_R = \bigoplus M \text{ is a standard Levi subgroup } \tilde{i}_M^+ (\ker \text{Tr}_R^M \cap \check{H}_R(M)^{+\text{,rig}}).$$

If $R = \mathbb{C}$, we have the spectral density theorem and the kernel of the trace map is zero. Theorem D is trivial in this case. However, if $R$ is of positive characteristic, especially when the spectral density theorem fails, then Theorem D would provide useful information toward the understanding of those representations.

0.8. The outline of the proof is as follows. In [12] we introduce the notion of quasi-positive elements and we use some remarkable properties on the minimal length elements established in [12] to show that any element in the Newton stratum $M(v)$ is quasi-positive. Then in [8], we use the quasi-positivity to show that the map in Theorem A(1) is well-defined and factors through $\check{H}(M; v)$. This proves part (1) of Theorem A.

As to part (2) of Theorem A, we first prove in Proposition 4.2 that $M(v) \subset G(\bar{v})$. Then by the admissibility of Newton strata ([11, Theorem 3.2]), any open compact subset $X$ of $M(v)$ enlarged by a sufficiently small open compact subgroup is still contained in $G(\bar{v})$. This shows that the image of $\tilde{i}_v$ is contained in $\check{H}(G; \bar{v})$. The key ingredients in the proof of surjectivity are

- The notation of $P$-alcove elements introduced in [8].
• The Iwahori-Matsumoto presentation of $\tilde{H}(G; \tilde{v})$ ([11, Theorem 4.1]).

By the quasi-positivity, for any $f \in H(M; v)$, $f^l \in H^v(M)$ for sufficiently large $l$. Theorem [A] follows from the adjunction formula proved in [2], the comparison between $i_v(f)^l$ with $j_v, \ast (f^l)$ and a trick of Casselman [4].

Finally, the injectivity in part (3) of Theorem A follows from the adjunction formula (Theorem B), the spectral density theorem and the freeness of the cocenter $\bar{H}$ (which is only known in the case of $\text{char}(F) = 0$).

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1. **Preliminary**

1.1. Let $G$ be a connected reductive group over a nonarchimedean local field $F$ of arbitrary characteristic. Let $G = G(F)$. We fix a maximal $F$-split torus $A$ and an alcove $a_C$ in the corresponding apartment, and denote by $T$ the associated Iwahori subgroup.

Let $Z = Z_G(A)$. We denote by $W_0 = N_G(A)/Z(F)$ the relative Weyl group and $\tilde{W} = N_G(A)/Z_0$ the Iwahori-Weyl group, where $Z_0$ is the unique parahoric subgroup of $Z(F)$.

We fix a special vertex of $a_C$ and identify $\tilde{W}$ as

$$\tilde{W} \cong X_s(Z)_{\text{Gal}(\bar{F}/F)} \rtimes W_0 = \{ t^\lambda w; \lambda \in X_s(Z)_{\text{Gal}(\bar{F}/F)}, w \in W_0 \}.$$  

We have a semidirect product

$$\tilde{W} = W_a \rtimes \Omega,$$

where $W_a$ is the affine Weyl group associated to $\tilde{W}$ and $\Omega$ is the stabilizer of the alcove $a_C$ in $\tilde{W}$. Let $\tilde{S}$ be the set of affine simple reflections of $W_a$ determined by the fundamental alcove $a_C$. The groups $W_a$ and $\tilde{W}$ are equipped with a Bruhat order $\preceq$ and a length function $\ell$.

The subgroup $\Omega$ of $\tilde{W}$ is the subgroup consisting of length-zero elements.

1.2. For any $K \subset \tilde{S}$, let $W_K$ be the subgroup of $\tilde{W}$ generated by $s \in K$. Let $K^{\tilde{W}}$ be the set of elements $w \in \tilde{W}$ of minimal length in the cosets $W_Kw$.

Let $\Phi = \Phi(G, A)$ be the set of roots of $G$ relative to $A$ and $\Phi^+$ be the set of positive roots so that $a_C$ is contained in the antidominant chamber of $V$ determined by $\Phi^+$. Let $R = \{ \alpha \}$ be the set of affine roots on $\mathcal{A}$. We choose a normalization of the valuation on $F$ so that if $\alpha \in R$, then so is $\alpha \pm 1$ (see [11, §5.2.23]). For any $n \in \mathbb{N}$, let $\mathcal{I}_n$ be the $n$-th Moy-Prasad subgroup associated to the barycenter of $a_C$ [15]. This is the subgroup of $G$ generated by the $n$-th congruence subgroup of $Z(F)$ and the affine root subgroup $X_{a+n}$ for $\alpha \in R_+$. For any $n \in \mathbb{N}$ and a subgroup $G'$ of $G$, we set $G'_n = G' \cap \mathcal{I}_n$. We write $\mathcal{I}_{G'}$ for $G' \cap \mathcal{I}$. 
1.3. Let $\mu_G$ be the Haar measure on $G$ such that the pro-$p$ Iwahori subgroup $\mathcal{I}'$ has volume 1. As in \cite[Section 1]{[11]}, we denote by $H = H(G)$ the Hecke algebra of locally constant, compactly supported, $K \times K$-invariant $\mathbb{Z}^{[\frac{1}{p}]}$-valued functions on $G$. We have

$$H = \lim_{\longrightarrow} H(G, K),$$

where $K$ runs over open compact subgroups of $G$ and $H(G, K)$ is the space of compactly supported, $K \times K$-invariant $\mathbb{Z}^{[\frac{1}{p}]}$-valued functions on $G$, i.e., $H(G, K) = \bigoplus_{g \in K \backslash G / K} \delta_{gK} \mathbb{Z}^{[\frac{1}{p}]}/KgK$, where $\delta_{gK}$ is the characteristic function on $KgK$.

We define the action of $G$ on $H$ by $xf(g) = f(x^{-1}gx)$ for $f \in H$, $x, g \in G$. By \cite[Proposition 1.1]{[11]}, the commutator $[H, H]$ of $H$ is the $\mathbb{Z}^{[\frac{1}{p}]}$-submodule of $H$ spanned by $f - xf$ for $f \in H$ and $x \in G$. Let $\tilde{H} = H/[H, H]$ be the cocenter of $H$.

1.4. Now we recall the Newton decomposition introduced in \cite{[11]}.

Set $V = X_*(\mathbb{Z}_{\text{Gal}(\bar{F}/F)} \otimes \mathbb{R})$ and $V_+$ be the set of dominant elements in $V$. For any $w \in \tilde{W}$, there exists a positive integer $l$ such that $w^l = l^\lambda$ for some $\lambda \in X_*(\mathbb{Z}_{\text{Gal}(\bar{F}/F)})$. We set $\nu_w = \lambda/l \in V$ and $\tilde{\nu}_w$ to be the unique dominant in the $W_0$-orbit of $\nu_w$. The element $\nu_w$ and $\tilde{\nu}_w$ are independent of the choice of $l$.

Let $\mathfrak{N} = \Omega \times V_+$. We have a map (see \cite[§2.1]{[11]})

$$\pi = (\kappa, \tilde{\nu}) : \tilde{W} \rightarrow \mathfrak{N}, \quad w \mapsto (wW_a, \tilde{\nu}_w).$$

We denote by $\tilde{W}_{\min}$ be the subset of $\tilde{W}$ consisting of elements of minimal length in their conjugacy classes. For any $\nu \in \mathfrak{N}$, we set

$$X_{\nu} = \bigcup_{w \in \tilde{W}_{\min} : \pi(w) = \nu} \tilde{W} : \mathcal{I} \quad \text{and} \quad G(\nu) = G \cdot_{\tilde{\nu}} X_{\nu}.$$ 

Here $\cdot_{\tilde{\nu}}$ means the conjugation action of $G$. Let $H(\nu)$ be the submodule of $H$ consisting of functions supported in $G(\nu)$ and let $\tilde{H}(\nu)$ be the image of $H(\nu)$ in the cocenter $\tilde{H}$. The Newton decomposition of $\tilde{H}$ is established in \cite[Theorem 3.1 (2)]{[11]}.

**Theorem 1.1.** We have that

$$\tilde{H} = \bigoplus_{\nu \in \mathfrak{N}} \tilde{H}(\nu).$$

In this paper, we are mainly interested in the $V$-factor of $\mathfrak{N}$. For any $\nu \in V_+$, we also set $G(\nu) = \bigcup_{\tau \in \Omega} G(\nu)$, $H(\nu) = \bigoplus_{\tau \in \Omega} H(\nu)$ and $\tilde{H}(\nu) = \bigoplus_{\tau \in \Omega} \tilde{H}(\nu)$.

1.5. Let $M$ be a semistandard Levi subgroup of $G$, i.e., a Levi subgroup of some parabolic subgroup of $G$ that contains $Z$. Let $\mathcal{I}_M = \mathcal{I} \cap M$ be the Iwahori subgroup of $M$ and $\tilde{W}(M)$ be the Iwahori-Weyl group of $M$. We denote by $S(M)$ the set of affine simple reflections of $\tilde{W}(M)$ determined by the Iwahori subgroup $\mathcal{I}_M$.

We may regard $\tilde{W}(M)$ as a subgroup of $\tilde{W}$ in a natural way. However, the length function $\ell_M$ on $\tilde{W}(M)$ does not equal to the restriction of $\tilde{W}$ of the length function $\ell$ on $\tilde{W}$.

Let $\Omega_M$ be the subgroup of $\tilde{W}(M)$ consisting of length-zero elements with respect to the length function $\ell_M$. We have $\Omega_M \cong \tilde{W}(M)/W_a(M)$, where $W_a(M)$ is the affine Weyl group of the subgroup of $\tilde{W}(M)$. We have $W_a(M) \subset W_a$ and
thus a natural map $\Omega_M \cong \hat{W}(M)/W_a(M) \to \hat{W}/W_a \cong \Omega$. Let $V_M^M$ be the set of $M$-dominant elements in $V$. We set $\mathcal{N}_M = \Omega_M \times V_M^M$ and we have a map $\pi_M = (\kappa_M, \bar{v}_M) : \hat{W}(M) \to \mathcal{N}_M$.

We also have a natural map $\mathcal{N}_M \to \mathfrak{g}$ sending $(\tau, v)$ to $(\tau', \bar{v})$, where $\tau'$ is the image of $\tau$ in $\Omega$ and $\bar{v}$ is the unique $(G)$-dominant element in the $W_v$-orbit of $v$.

Let $\mu_M$ be the Haar measure on $M$ such that the pro-$p$ Iwahori subgroup of $M$ has volume 1. Let $H(M)$ be the Hecke algebra of $M$ and $\hat{H}(M)$ be its cocenter. For any $\nu_M \in \mathcal{N}_M$, we denote by $\hat{H}(M; \nu_M)$ the corresponding Newton component of $H(M)$. By Theorem [11], we have

$$\hat{H}(M) = \oplus_{\nu_M \in \mathcal{N}_M} \hat{H}(M; \nu_M).$$

2. Quasi-positive elements

2.1. The semistandard Levi may be described as the centralizer of elements in $V$. For any $v \in \mathfrak{g}$, we set $\Phi_v = \{a \in \Phi; \langle a, v \rangle = 0\}$ and $\Phi_v^+ = \{a \in \Phi; \langle a, v \rangle > 0\}$.

Let $M_v \subset G$ be the Levi subgroup generated by $Z$ and $U_a(F)$ for $a \in \Phi_v^0$ and $N_v \subset G$ be the unipotent subgroup generated by $U_a(F)$ for $a \in \Phi_v$. Set $P_v = M_v N_v$.

Then $P_v$ is a semistandard parabolic subgroup and $M_v$ is a Levi subgroup of $P_v$. We denote by $P_v^- = M_v N_v^-$ the opposite parabolic. Let $\mu_{N_v}, \mu_{N_v^-}$ be the Haar measures on $N_v$ and $N_v^-$ respectively such that $\mu_G(nmn^-) = \mu_{N_v}(n)\mu_{M_v}(m)\mu_{N_v^-}(n^-)$ for $n \in N_v, m \in M_v, n^- \in N_v^-$. For $m \in M_v$, set $\delta_v(m) = \frac{\mu_{N_v}(mN_v^am^{-1})}{\mu_{N_v}(N_v^a)}$. For $\nu = (\tau, v) \in \mathfrak{g}$, we may also write $M_v$ for $M_v, N(\nu)$ for $N_v$ and $N^-(\nu)$ for $N_v^-$.

If $v$ is dominant, then $P_v$ is a standard parabolic subgroup of $G$ and $M_v$ is a standard Levi subgroup of $G$.

2.2. Let $v \in \mathfrak{g}$. Following [3, Definition 6.5 & Definition 6.14], we call an element $m \in M_v$ a $(P_v, I_n)$-positive element if

$$mN_v m^{-1} \subset N_v, \text{ and } m^{-1}N_v^- m \subset N_v^-.$$

We call an element $z$ in the center of $M_v$ a strongly $P_v$-positive element if the sequences $z^n N_v^0 z^{-n}, z^{-n} N_v^- z^n$ both tend monotonically to 1 as $n \to \infty$.

Following [2, §3.1], let $H^v(M_v, M_v, n)$ be the subalgebra of $H(M_v, M_v, n)$ of functions with support consisting of $(P_v, I_n)$-positive elements. The following result is proved in [2, Proposition 5].

Proposition 2.1. The map $\delta_{M_v, n} : H^v(M_v, M_v, n) \to \hat{H}(G, I_n)$ defines an injective algebra homomorphism

$$j_{v, n} : H^v(M_v, M_v, n) \hookrightarrow H(G, I_n).$$

The formula we have here differs from [2] by the factor $\delta_v(m)^{-\frac{1}{2}}$, since in [2] the map is adjoint to the (unnormalized) Jacquet functor while we consider the (normalized) Jacquet functor.

By [2, §3.2], $H(M_v, M_v, n) = S^{-1} H^v(M_v, M_v, n)$ is the localization of $H^v(M_v, M_v, n)$, where $S = (\delta_{M_v, n} M_v, n)$ is the the multiplicative closed set of the function $\delta_{M_v, n} M_v, n$ with a strongly $P_v$-positive element $z$. It is pointed out in [2, Remark 5] that the map $j_{v, n}$ does not extend to an algebra homomorphism $H(M_v, M_v, n) \to H(G, I_n)$ for $n > 0$. 


2.3. Let $v \in V$ be a rational coweight and $M = M_v$. For any $l \in \mathbb{N}$ with $lv \in X_* (Z)$, the element $lv$ is strongly $P_v$-positive. However, in general, the element in $M(v)$ may not be $(P_v, \ast)$-positive. Therefore, one cannot deduce a map from $H(M; v)$ to $H$ via the map $j_{v,n}$.

Example 2.2. Let $G$ be split $GL_5$ and $M = GL_3 \times GL_2$. Let $v = (\frac{2}{3}; \frac{2}{3}; \frac{2}{3}; \frac{1}{2}; \frac{1}{2})$. Then $M = M_v$. The element $w = f^{(1,1,0,1,0)}(132)(45)$ of $W$ has Newton point $v$. However, $w(e_1 - e_3) = e_5 - e_2 - 1$ is a negative affine root. Therefore the element $\tilde{w}$ is not $(P_v, \ast)$-positive.

2.4. To overcome the difficulty, we introduce the quasi-positive elements.

An element $m \in M_v$ is called $P_v$ quasi-positive if there exists $l \in \mathbb{N}$ such that

(a) $m^l N_{v,n} m^{-l} \subset N_{v,n+1}$, and $m^{-1} N_{v,n} m^l \subset N_{v,n+1}$ for any $n \in \mathbb{N}$.

For any $n \in \mathbb{N}$, $w \in \tilde{W}$ and $g \in I \tilde{w} I$, we have $g I_{n+\ell(w)} g^{-1} \subset I_n$. So

(b) Let $w \in \tilde{W}(M)$ and $m \in I_M \tilde{w} I_M$. If $m$ satisfies (a), then we have

$m^n N_{v,n'} + (l-1)(w)m^{-n} \subset N_{v,n'}$, and $m^{-n} N_{v,n'+(l-1)(w)} m^n \subset N_{v,n'}$ for any $n, n' \in \mathbb{N}$.

We first discuss some properties on the quasi-positive elements.

Proposition 2.3. Let $v \in V$ and $M = M_v$. Let $w \in \tilde{W}(M)$ and $m \in I_M \tilde{w} I_M$.

Suppose that $m$ satisfies the inclusion relation (2.2) (a).

(1) For any $n \in \mathbb{N}$, any element in $m I_{n+\ell(l-w)} I_n$ is conjugate by an element in $I_{n+\ell(l-w)} I_n$ to an element in $m I_{n+\ell(l-w)} I_n$.

(2) For any $n, n' \in \mathbb{N}$ and $g \in I_{n+\ell(l-w)} I_n$, we have

$$\delta_m g I_{n+\ell(l-w)} I_n + g' \equiv \delta_m I_{n+\ell(l-w)} I_n + g' \mod [H, H].$$

Proof. (1) We first show that

(a) For any $i \in \mathbb{N}$, any element in $m M_{n+\ell(l-w)} I_n + (l-w) I_{n+i}$ is conjugate by $I_{n+i}$ to an element in $m M_{n+\ell(l-w)} I_n + (l-w) I_{n+i}$.

Note that any element in $m M_{n+\ell(l-w)} I_{n+\ell(l-w)} I_{n+i}$ is conjugate by $I_{n+\ell(l-w)} I_{n+i}$ to an element of the form $u'g'$ with $u' \in N_{v,n+\ell(l-w)} I_{n+i}$, $g' \in m M_{n+\ell(l-w)} I_{n+i}$ and $u' N_{v,n+\ell(l-w)} I_{n+i}$. By (2.3) (b), $g u g^{-1} \in N_{v,n+i}$. We have $(u', g u g^{-1}) \in (I_{n+\ell(l-w)} I_{n+i}, I_{n+i}) \subset I_{n+\ell(l-w)} I_{n+i+1}$. Now we have

$$u' g u g^{-1} g \in (g u g^{-1}) u' I_{n+\ell(l-w)} I_{n+i+1}.$$ So $u'g$ is conjugate by $I_{n+i}$ to an element in

$$u' I_{n+\ell(l-w)} I_{n+i+1} g (g u g^{-1}) = u' I_{n+\ell(l-w)} I_{n+i+1} (g^2 u (g^2)^{-1}) g = u' (g^2 u (g^2)^{-1}) I_{n+\ell(l-w)} I_{n+i+1} g = (g^2 u (g^2)^{-1}) u' I_{n+\ell(l-w)} I_{n+i+1} g.$$ By the same procedure, for any $l \in \mathbb{N}$, $u'g$ is conjugate by $I_{n+i}$ to an element in

$$(g^l u (g^l)^{-1}) u' I_{n+\ell(l-w)} I_{n+i+1} g.$$ By (2.3) (a), $g^l u (g^l)^{-1} \in I_{n+\ell(l-w)} I_{n+i+1}$. Hence $u'g$ is conjugate by $I_{n+i}$ to an element in $u' I_{n+\ell(l-w)} I_{n+i+1} g = u' g I_{n+\ell(l-w)} I_{n+i+1} g$. By the same argument, any element in $u' g I_{n+\ell(l-w)} I_{n+i+1} g$ is conjugate by $I_{n+i}$ to an element in $g I_{n+\ell(l-w)} I_{n+i+1}$.

(a) is proved.

Let $g_0 \in m M_{n+\ell(l-w)} I_n$. By (a), we may construct inductively an element $z_i \in I_{n+i}$ for $i \in \mathbb{N}$ such that $g_{i+1} := z_i^{-1} g_i z_i$ is contained in $m M_{n+\ell(l-w)} I_{n+\ell(l-w) + i}$.
The convergent product \( z := z_1 z_2 \cdots \) is a well-defined element in \( \mathcal{I}_n \) and \( z^{-1} g z \in mM_{n+1}(l-1)(l(w)) \).

(2) By part (1), there exists \( h \in \mathcal{I}_{n+n'} \) such that \( h m g h^{-1} \in mM_{n+1}(l-1)(l(w)) \). We have \( (\mathcal{I}_{n+n'}, M_{n+1}(l-1)(l(w))) \subset (\mathcal{I}_{n+n'}(l-1)(l(w))) \). Therefore \( M_{n+1}(l-1)(l(w)) \mathcal{I}_{n+n'+1}(l-1)(l(w)+n') \) is a subgroup of \( \mathcal{I} \) and is stable under the conjugation action of \( \mathcal{I}_{n+n'} \). Thus \( h m g M_{n+1}(l-1)(l(w)) \mathcal{I}_{n+n'+1}(l-1)(l(w)+n')h^{-1} = mM_{n+1}(l-1)(l(w)) \mathcal{I}_{n+n'+1}(l-1)(l(w)+n') \). The statement is proved.

2.5. We say that \( m \in M \) is \( P_v \) strictly positive if for any \( n \in \mathbb{N} \), we have

\[
mN_{v,n} m^{-1} \subset N_{v,n+1}, \text{ and } m^{-1} N_{v,n} m \subset N_{v,n+1}.
\]

We denote by \( H^v(M) \) the subalgebra of \( H(M) \) consisting of functions with support consisting of \( P_v \) strictly positive elements. Note that the limit of the support of \( j_{v,n}(\delta Z_0) \) for \( v \) dominant regular, as \( n \) goes to infinite, is just \( Z_0 \) itself, but the support of \( j_{v,n}(\delta Z_0) \) for each \( n \) contains of non-split regular semisimple elements. Thus the maps \( \{j_{v,n}\} \) are not compatible with the natural maps \( H^v(M, M_n) \to H^v(M, M_{n+1}) \).

However, we have the following compatibility result for \( P_v \) strictly positive part.

**Corollary 2.4.** Let \( n \in \mathbb{N} \). Then the following diagram commutes

\[
\begin{array}{ccc}
\tilde{H}^v(M, M_n) & \xrightarrow{j_{v,n}} & \tilde{H}(G, \mathcal{I}_n) \\
\downarrow & & \downarrow \\
\tilde{H}^v(M, M_{n+1}) & \xrightarrow{j_{v,n+1}} & \tilde{H}(G, \mathcal{I}_{n+1}).
\end{array}
\]

**Proof.** Let \( m \in M \) be \( P_v \) strictly positive. Then \( \delta_{M_n M_n} \in H^v(M, M_n) \subset H^v(M, M_{n+1}). \) By definition,

\[
j_{v,n+1}(\delta_{M_n M_n}) = \delta_v(m) - \frac{1}{2} \frac{\mu_M(M_{n+1})}{\mu_G(I_n)} \delta_{I_{n+1}M_n M_n I_{n+1}}.
\]

Note that \( I_{n+1}M_n = M_n I_{n+1} \) is a subgroup of \( \mathcal{I} \). We have

\[
I_n m I_n = \cup \{(i_1, i_2, i'_1, i'_2) : i_1 i'_1 I_{n+1} M_n m M_n I_{n+1} i'_2 i_2 \},
\]

where \( \{(i_1, i_2, i'_1, i'_2) \} \subset N_n \times N_n \times N_n^* \times N_n^* \) is a finite subset. By Proposition 2.3 (2), for \( i_1, i_2 \in N_n \) and \( i'_1, i'_2 \in N_n^* \), we have

\[
\delta_{i_1 i'_1 I_{n+1} M_n m M_n I_{n+1} i'_2 i_2} \equiv \delta_{I_{n+1} M_n M_n I_{n+1}} \mod [H, H].
\]

Thus

\[
j_{v,n}(\delta_{M_n M_n}) \equiv \delta_v(m) - \frac{1}{2} \frac{\mu_M(M_n)}{\mu_G(I_n)} \mu_G(I_{n+1}) \mu_G(I_{n+1} m M_n I_{n+1}) \delta_{I_{n+1} M_n M_n I_{n+1}} \mod [H, H].
\]

It remains to show that

\[
\frac{\mu_M(M_{n+1})}{\mu_G(I_{n+1})} = \frac{\mu_M(M_n)}{\mu_G(I_n)} \frac{\mu_G(I_{n+1} m M_n I_{n+1})}{\mu_G(I_{n+1} m M_n I_{n+1})}.
\]

Suppose that \( m \in I_M w I_M \) for some \( w \in W(M) \). By Lemma 4.6,

\[
\frac{\mu_G(I_n m I_n)}{\mu_G(I_n)} = \frac{\mu_G(I_{n+1} m I_{n+1})}{\mu_G(I_{n+1})} = q^v(w), \quad \frac{\mu_M(M_n m M_n)}{\mu_M(M_n)} = \frac{\mu_M(M_{n+1} m M_{n+1})}{\mu_M(M_{n+1})} = q^v(M).\]
Now we have
\[
\begin{align*}
\frac{\mu_M(M_n)}{\mu_G(I_n)} & = \frac{\mu_M(M_n)}{\mu_G(I_{n+1})} \quad \frac{\mu_G(I_n m I_n)}{\mu_G(I_{n+1} m I_{n+1})} \\
& = \frac{\mu_M(M_n)}{\mu_G(I_{n+1})} \quad \frac{\mu_G(I_n m I_n)}{\mu_G(I_{n+1} m I_{n+1})} \\
& = \frac{\mu_M(M_n)}{\mu_G(I_{n+1})} \quad \frac{\mu_M(M_{n+1} m I_{n+1})}{\mu_G(I_{n+1} m I_{n+1})} \\
& = \frac{\mu_M(M_n)}{\mu_G(I_{n+1})} \quad \frac{\mu_M(M_{n+1} m I_{n+1})}{\mu_G(I_{n+1} m I_{n+1})} \\
& = \frac{\mu_M(M_n)}{\mu_G(I_{n+1})} \quad \frac{\mu_M(M_{n+1})}{\mu_G(I_{n+1})}.
\end{align*}
\]

The statement is proved. □

**Proposition 2.5.** Let \( v \in V \) be a rational coweight and \( M = M_v \). Let \( w \in \hat{W}(M) \). Then there exists a positive integer \( i_{v,w} \) such that for any \( m \in I_M \hat{w} I_M \cap M(v) \) and \( n \geq i_{v,w} \), we have
\[
m_{v,w} v_n(m_{v,w}^{-1}) \subset N_{v,n+1}, \quad (m_{v,w})^{-1} N_{v,n} m_{v,w} \subset N_{v,n+1}.
\]

2.6. The proof relies on some remarkable properties of the Iwahori-Weyl group, which we recall here.

For \( w, w' \in \hat{W} \) and \( s \in \hat{S} \), we write \( w \overset{s}{\rightarrow} w' \) if \( w' = sws \) and \( \ell(w') \leq \ell(w) \). We write \( w \rightarrow w' \) if there is a sequence \( w = w_0, w_1, \ldots, w_n = w' \) of elements in \( \hat{W} \) such that for any \( 1 \leq k \leq n \), \( w_{k-1} \overset{s_k}{\rightarrow} w_k \) for some \( s_k \in \hat{S} \). We write \( w \approx w' \) if \( w \rightarrow w' \) and \( w' \rightarrow w \). It is easy to see that if \( w \rightarrow w' \) and \( \ell(w) = \ell(w') \), then \( w \approx w' \).

We have that
(a) If \( w \overset{s}{\rightarrow} w' \) and \( \ell(w) = \ell(w') \), then for any \( g \in I\hat{w}I \), there exists \( g' \in I\hat{s}I \) such that \( g'g(g')^{-1} \in I\hat{w}'I \).

(b) If \( w \overset{s}{\rightarrow} w' \) and \( \ell(w') < \ell(w) \), then for any \( g \in I\hat{w}I \), there exists \( g' \in I\hat{s}I \) such that \( g'g(g')^{-1} \in I\hat{w}'I \sqcup I\hat{s}\hat{w}I \).

An element \( w \in \hat{W} \) is called straight if \( \ell(w^n) = n\ell(w) \) for any \( n \in \mathbb{N} \). A triple \((x, K, u)\) is called a standard triple if \( x \in \hat{W} \) is straight, \( K \subset \hat{S} \) with \( W_K \) finite, \( x \in \hat{K}W \) and \( \text{Ad}(x)(K) = K \), and \( u \in \hat{W}_K \). By definition,
(c) For any \( n \in \mathbb{N} \) and \( g_1, \ldots, g_n \in I\hat{w}I \), we have \( g_1g_2\cdots g_n \in (I\hat{w}I)(I\hat{w}^nI) \).

It is proved in [12, Theorem A & Proposition 2.7] that

**Theorem 2.6.** For any \( w \in \hat{W} \), there exists a standard triple \((x, K, u)\) such that \( ux \in \hat{W}_{\min} \) and \( w \rightarrow ux \). In this case, \( \pi(w) = \pi(x) \).

Following [12, §4.3], we write \( w \overset{s}{\rightarrow} w' \) if either \( w \overset{s}{\rightarrow} w' \) or \( w' = sw \) and \( \ell(w') > \ell(sws) \), and we write \( w \rightarrow w' \) if there exists a sequence \( w = w_0, w_1, \ldots, w_n = w' \) of elements in \( \hat{W} \) such that for any \( 1 \leq k \leq n \), \( w_{k-1} \overset{s_k}{\rightarrow} w_k \) for some \( s_k \in \hat{S} \). It is easy to see that if \( w \in \hat{W}_{\min} \) and \( w \rightarrow w' \), then \( w \approx w' \).

We show that

**Lemma 2.7.** Let \( w \in \hat{W} \) and \( g \in I\hat{w}I \). Then there exists a standard triple \((x, K, u)\), a sequence \( w = w_0, w_1, \ldots, w_n = ux \) of distinct elements in \( \hat{W} \) and a sequence \( g = g_0, g_1, \ldots, g_n \) of elements in \( G \) such that
\[
\begin{enumerate}
\item \( ux \in \hat{W}_{\min} \);
\item \( \text{for any } 0 \leq k \leq n, \ g_k \in I\hat{w}_kI \);
\end{enumerate}
\]
Remark 2.8. By definition, if $w \rightarrow w'$, then $w' \in wW_a$ and $\ell(w') \leq \ell(w)$. In particular, the length of the sequence is at most $\sharp\{x \in W_a; \ell(x) \leq \ell(w)\}$.

Proof. We argue by induction on $\ell(w)$.

If $w \in \hat{W}_{\text{min}}$, by Theorem 2.6 there exists a standard triple $(x, K, u)$ with $ux \in \hat{W}_{\text{min}}$ and a sequence $w = w_0, w_1, \ldots, w_n = ux$ of distinct elements in $\hat{W}$ such that for any $1 \leq k \leq n$, $w_{k-1} \overset{k}{\rightarrow} w_k$ for some $s_k \in \hat{S}$. Since $w \in \hat{W}_{\text{min}}$, we have $\ell(w_k) = \ell(w)$ for all $k$. Now the statement follows from (2.6) (a).

If $w \notin \hat{W}_{\text{min}}$, then by Theorem 2.6 there exists a sequence $w = w_0, w_1, \ldots, w_n$ of distinct elements in $\hat{W}$ such that $\ell(w) = \ell(w_n)$, for any $1 \leq k \leq n$, $w_{k-1} \overset{k}{\rightarrow} w_k$ for some $s_k \in \hat{S}$ and there exists $s \in \hat{S}$ with $sw_{n}s < w_n$. Then we have $\ell(w_k) = \ell(w)$ for all $k$. By (2.6) (a), for any $1 \leq k \leq n$, there exists $h_k \in \mathcal{I}_k\mathcal{I}$ such that $g_k = h_kg_{k-1}h_k^{-1}$. By (2.6) (b), there exists $h_{n+1} \in \mathcal{I}\mathcal{I}$ such that $h_{n+1}g_nh_{n+1}^{-1} \in \mathcal{I}\hat{w}_{n+1}\mathcal{I}$ with $w_{n+1} \in \{sw, sws\}$. Now the statement follows from inductive hypothesis on $w_{n+1}$. \hfill $\square$

2.7. Proof of Proposition 2.5. Let $N_0 = \sharp\{w' \in W_a(M); \ell(M(w')) \leq \ell(M(w))\}$. By Lemma 2.7 and remark 2.8 there exists a standard triple $(x, K, u)$ of $W(M)$ and an element $h \in \cup_{z \in W_a(M); \ell(z) \leq N_0}\mathcal{I}_M\mathcal{I}_M$ such that $ux \in \hat{W}(M)_{\text{min}}$, $w \rightarrow ux$ and $h \hat{m}h^{-1} \in \mathcal{I}_M\hat{u}x\mathcal{I}_M$.

Let $i$ be a positive integer with $iv \in X_\ast(Z)$. Then $x^i = t^iv \in \hat{W}$ represents a central element in $M$. By (2.6) (c), for any $l \in \mathbb{N}$,

$$ (h \hat{m}h^{-1})^l \in (\mathcal{I}_M W_K \mathcal{I}_M)(\mathcal{I}_M t^iv \mathcal{I}_M). $$

Let $N_1 = \max_{K \subset \hat{S}(M); W_K}$ finite $\sharp W_K$. Let $i_{v, w} = (2N_0 + N_1 + 1)i$. Then for any $\alpha \in \Phi_{v, +}$, $(i_{v, w} + \alpha) \geq 2N_0 + N_1 + 1$. Note that $m_{v, w} = h^{-1}(g_1g_2)h$ with $h \in \cup_{w' \in W(M); \ell(w') \leq N_0}\mathcal{I}_M\hat{u}w'\mathcal{I}_M$, $g_1 \in \cup_{w' \in W(M); \ell(w') \leq N_1}\mathcal{I}_M\hat{u}w'\mathcal{I}_M$ and $g_2 \in \mathcal{I}_M t^iv, w' \mathcal{I}_M$. So

$$ m_{v, w} N_{v, n}(m_{v, w})^{-1} = h^{-1}g_1g_2hN_{v, n}h^{-1}g_1^{-1}h $$
$$ \quad \subset h^{-1}g_1g_2N_{v, n}N_{v, n}g_2^{-1}g_1^{-1}h $$
$$ \quad \subset h^{-1}g_1N_{v, n}N_{v, n}(2N_0 + N_1 + 1)g_1^{-1}h $$
$$ \subset h^{-1}N_{v, n}N_{v, n}(2N_0 + N_1 + 1)N_{v, n} $$
$$ \subset N_{v, n}N_{v, n}(2N_0 + N_1 + 1)N_{v, n} = N_{v, n+1}. $$

Similarly, $m^{-i_{v, w}} N_{v, n} m^{i_{v, w}} \subset N_{v, n+1}^-.$

3. THE MAP $\tilde{i}_\nu$

We define the induction map $\tilde{i}_\nu$, which is the main object in this paper.

Theorem 3.1. Let $M$ be a semistandard Levi subgroup of $G$ and $\nu \in \mathcal{R}_M$ with $M = M_\nu$. Then

1. For $m \in M$ and an open compact subgroup $K_M$ of $\mathcal{I}_M$ with $mK_M \subset \mathcal{M}(\nu)$, the map

$$ \delta_{mK_M} \mapsto \delta_\nu(m)^{-\frac{1}{2}} \frac{\mu_M(K_M)}{\mu_G(K_M)} \delta_{mK_M} + [H, H] $$
from $H(M; \nu)$ to $\bar{H}(\bar{\nu})$ is independent the choice of sufficiently small open compact subgroup $K$ of $G$

(2) The map $i_\nu : H(M; \nu) \to \bar{H}$ defined above induces a map

$$i_\nu : H(M; \nu) \to \bar{H}.$$ 

Remark 3.2. Unlike the map $j_{\nu,n}$, the map $\tilde{i}_\nu$ does not send $\bar{H}(M, M_n; \nu)$ to $H(G, \mathcal{I}_n, \bar{\nu})$. One needs to replace $\mathcal{I}_n$ by a smaller open compact subgroup of $G$. However, by the Iwahori-Matsumoto presentation of $\bar{H}(M, M_n; \nu)$ ([11] Theorem 4.1) and Proposition [2.5] there exists a positive integer $n'$ (depending on $\nu$) such that $\tilde{i}_\nu : H(M, M_n; \nu) \to H(G, \mathcal{I}_{n+n'}; \bar{\nu})$ for any $n \in \mathbb{N}$.

Proof. (1) Let $\nu$ be the $V$-factor of $\nu$. Let $w \in \hat{W}(M)$ with $m \in \mathcal{I}_M w \mathcal{I}_M$. Let $i_{\nu,w}$ be an positive integer in Proposition 2.5. Let $l$ be a multiple of $i_{\nu,w}(w)$ with $M_l \subset K_M$. By Proposition 2.3 (2), for any $n \in \mathbb{N}$ and $g \in \mathcal{I}_l$, we have

$$\delta_{m'gM_l \mathcal{I}_{l+n}} \equiv \delta_{m'M_l \mathcal{I}_{l+n}} \mod [H, H].$$

Let $K, K' \subset$ be open compact subgroups of $G$ with $K, K' \subset \mathcal{I}_l$. Let $n \in \mathbb{N}$ with $\mathcal{I}_{l+n} \subset K, K'$. Now we have

$$\delta_{mK_M K} = \sum_{m' \in mK_M K} \delta_{m'M_l K} \equiv \sum_{m' \in mK_M K} \frac{\mu_G(M_K)}{\mu_G(M_l \mathcal{I}_{l+n})} \delta_{m'M_l \mathcal{I}_{l+n}}$$

$$\equiv \frac{\mu_G(M_K)}{\mu_G(M_l \mathcal{I}_{l+n})} \delta_{mK_M K} \mod [H, H].$$

As $K_M$ is stable under the right multiplication of $M_l$, we have $\mu_G(K_M \mathcal{I}_{l+n}) = \overline{\delta}(K_M/M_l) \mu_G(M_l \mathcal{I}_{l+n})$ and $\mu_G(M_K \mathcal{I}_{l+n}) = \mu_G(K_M \mathcal{I}_{l+n})$. Thus for any $n \in \mathbb{N}$, we have

$$\frac{\mu_G(K_M)}{\mu_G(K_M \mathcal{I}_{l+n})} \delta_{mK_M K} \equiv \frac{\mu_G(K_M)}{\mu_G(K_M \mathcal{I}_{l+n})} \delta_{mK_M K} \mod [H, H].$$

Similarly, $\frac{\mu_G(K_M)\delta_{mK_M K'}}{\mu_G(K_M \mathcal{I}_{l+n})} \delta_{mK_M K} \equiv \mu_G(K_M)\delta_{mK_M \mathcal{I}_{l+n}} \mod [H, H]$. Part (1) is proved.

(2) By [11] §3.3 (2), $[H(M), H(M)] = \bigoplus_{\nu \in \mathcal{R}_M} ([H(M), H(M)] \cap H(M))$, the kernel of the map $H(M) \nu \to H(M)\bar{\nu}$ is spanned by $\delta_{mK_M} - h \delta_{mK_M}$ for $h, m \in M$ and open compact subgroup $K_M$ of $\mathcal{I}_M$ such that $mK_M \subset M_\nu$. It remains to prove that $i_\nu(\delta_{mK_M}) = i_\nu(h \delta_{mK_M})$.

Set $m' = hm^{-1}$ and $K'_M = hK_M h^{-1}$. By part (1), there exists a sufficiently small open compact subgroup $\mathcal{K}$ of $G$ such that

$$i_\nu(\delta_{mK_M}) \equiv \delta_\nu(m) \frac{1}{\mu_G(K_M)\delta_{mK_M K}} \mod [H, H],$$

$$i_\nu(\delta_{m'K'_M}) \equiv \delta_\nu(m') \frac{1}{\mu_G(K'_M)\delta_{m'K'_M K'}} \mod [H, H].$$

Here $K'_M = hK_M h^{-1}$.

We have $\delta_{m'K'_M K} = \delta_{h(mK_M K)h^{-1}} \equiv \delta_{mK_M K} \mod [H, H]$. Part (2) is proved. □

3.1. In the rest of this section, we show that the maps $\tilde{i}_\nu$ are compatible with conjugating the Levi subgroups.

For any semistandard Levi subgroup $M$, we have a natural projection

$$X_+(Z)_{\text{Gal}(F/F)}/Z\Phi_M^\nu \cong \Omega_M$$
Proposition 3.3. Let \( \nu \in \mathcal{N}_M \). The natural action of \( W_0 \) on \( X_*(Z)_{\text{Gal}(F/F)} \times V \) induces the following commutative diagram for any \( w \in W_0 \)

\[
\xymatrix{
X_*(Z)_{\text{Gal}(F/F)} \times V \ar[r]^-w \ar[d] & X_*(Z)_{\text{Gal}(F/F)} \times V \\
\mathcal{N}_M \ar[r] & \mathcal{N}_{wM\hat{w}^{-1}}.
}
\]

We denote the induced map \( \mathcal{N}_M \to \mathcal{N}_{wM\hat{w}^{-1}} \) still by \( w \). If moreover, \( w \in W_M \), i.e. \( w \) sends the positive roots of \( M \) to the positive roots of \( \hat{w}M\hat{w}^{-1} \), then we have \( \hat{w}I_M\hat{w}^{-1} = I_{wM\hat{w}^{-1}} \). By definition, the \( M \)-fundamental alcove is the unique \( M \)-alcove that contains the \( G \)-fundamental alcove. Since the conjugation by \( \hat{w} \) sends the Iwahori-subgroup of \( M \) to the Iwahori-subgroup of \( \hat{w}M\hat{w}^{-1} \), it also sends the \( M \)-fundamental alcove to the \( \hat{w}M\hat{w}^{-1} \)-fundamental alcove, and thus induces a length-preserving map from \( \hat{W}(M) \) to \( \hat{W}(\hat{w}M\hat{w}^{-1}) \). In particular, the conjugation by \( w \) sends the minimal length elements of \( \hat{W}(M) \) (with respect to \( \ell_M \)) to the minimal length elements of \( \hat{W}(\hat{w}M\hat{w}^{-1}) \) (with respect to \( \ell_{wM\hat{w}^{-1}} \)). Therefore, by the definition of Newton strata, we have that

(a) Let \( M \) be a semistandard Levi subgroup \( M \) and \( \nu \in \mathcal{N}_M \). Let \( w \in W_0 \) and \( M' = \hat{w}M\hat{w}^{-1} \), then

\[
\hat{w}M(\nu)\hat{w}^{-1} = M'(w(\nu)).
\]

**Proposition 3.3.** Let \( M \) be a semistandard Levi subgroup and \( \nu \in \mathcal{N}_M \) and \( w \in W_0 \). Then for any \( m \in M' \) and an open compact subgroup \( K_M \) of \( I_M \) with \( mK_M \subseteq M' \) and \( \hat{w}K_M\hat{w}^{-1} \subseteq I_{wM\hat{w}^{-1}} \), we have

\[
i_{\nu}(\delta_{mK_M}) = i_{w(\nu)}(\delta_{w^{m}K_M\hat{w}^{-1}}) \in \hat{H}.
\]

**Proof.** The proof is similar to the proof of Theorem 3.1 (2).

Set \( M' = \hat{w}M\hat{w}^{-1} \), \( m' = \hat{w}m\hat{w}^{-1} \) and \( K_{M'} = \hat{w}K_M\hat{w}^{-1} \). By Theorem 3.1 (1), there exists a sufficiently small open compact subgroup \( K \) of \( G \) such that

\[
i_{\nu}(\delta_{mK_M}) \equiv \delta_{\nu}(m)^{-\frac{1}{2}} \frac{\mu_M(K_M)}{\mu_G(K_M)} \delta_{mK_MK} \mod [H, H],
\]

\[
i_{w(\nu)}(\delta_{mK_{M'}}) \equiv \delta_{w(\nu)}(m')^{-\frac{1}{2}} \frac{\mu_M(K_{M'})}{\mu_G(K_{M'}K')} \delta_{m'K_{M'}K'} \mod [H, H].
\]

Here \( K' = \hat{w}K^{-1} \).

We have \( \delta_{m'K_{M'}K'} = \delta_{w^{mK_MK}\hat{w}^{-1}} \equiv \delta_{mK_MK} \mod [H, H] \). The statement is proved.

**Corollary 3.4.** Let \( M \) be a semistandard Levi subgroup of \( G \) and \( \nu \in \mathcal{N}_M \) with \( M = M_\nu \). Then for any \( w \in W_0 \),

\[
\text{Im}(i_{\nu} : \bar{H}(M; \nu) \to \bar{H}) = \text{Im}(i_{w(\nu)} : \bar{H}(\hat{w}M\hat{w}^{-1}; w(\nu)) \to \bar{H}).
\]

4. THE IMAGE OF THE MAP \( \bar{t}_\nu \)

The main result of this section is

**Theorem 4.1.** Let \( M \) be a semistandard Levi subgroup and \( \nu \in \mathcal{N}_M \) with \( M = M_\nu \). Then the image of the map \( \bar{t}_\nu : \bar{H}(M; \nu) \to \bar{H} \) equals \( \bar{H}(\nu) \).

We first compare the Newton strata of \( G \) and its Levi subgroups.
Proposition 4.2. Let $M$ be a semistandard Levi subgroup and $\nu \in \mathcal{X}$. Then we have $M(\nu) \subset G(\bar{v})$.

Proof. The idea is similar to the proof of [11, Theorem 2.1].

By [11, Remark 2.6], after conjugating by a suitable element in $W_0$, we may assume that $M$ is a standard Levi subgroup. Since $M = M_\nu$, the $V$-factor of $\nu$ is $G$-dominant. By the Newton decomposition of $G$ ([11, Theorem 2.1]), it suffices to prove that $M(\nu) \cap G(\nu') = \emptyset$ for any $\nu' \in \mathcal{X}$ with $\nu' \neq \bar{v}$.

Let $\nu = (\tau, v)$ and $\nu' = (\tau', v')$. If the image of $\tau$ in $\Omega$ does not equal to $\tau'$, then $M(\nu) \cap G(\nu') = \emptyset$. Now we assume that the $\Omega$-factor matches. Since $\nu' \neq \bar{v}$, we have $\nu' \neq v$.

By [11, Remark 2.6],

$$M(\nu) = \cup_{(x, K, u) \in \mathcal{X} \cap W_0} p_{x, K, u}(\mathcal{U}_M) \cdot \bar{x},$$

where $(x, K, u)$ runs over standard triples of $\mathcal{U}$ such that $ux \in \mathcal{U}(M)_{\min}$ and $\pi(x) = \nu$. $(x', K', u')$ runs over standard triples of $\mathcal{U}$ such that $u'x' \in \mathcal{U}(M)_{\min}$ and $\pi(x') = \nu'$.

If $M(\nu) \cap G(\nu') \neq \emptyset$, then there exists standard triples $(x, K, u)$ and $(x', K', u')$ as above and $h \in \mathcal{U}_M \cdot \bar{x}$, $h' \in \mathcal{U}_M \cdot \bar{x}'$, $g \in G$ such that $ghg^{-1} = h'$. For any $n \in \mathbb{N}$, we have $g h^n g^{-1} = (h')^n$. By [11, Remark 2.6], we have $h^n \in (\mathcal{U}_M \bar{u}) \mathcal{U}_M$, $(h')^n \in (\mathcal{U}_M \bar{u}) \mathcal{U}_M$.

Let $l > 0$ with $l v \cdot l v' \in \mathcal{X}$. Suppose that $g \in \mathcal{U} \bar{x}$ for some $z \in \mathcal{U}$. Then for any $n \in \mathbb{N}$, we have

$$\mathcal{U} \bar{x} \mathcal{U} l v \mathcal{U} \bar{x} \mathcal{U} l v' \mathcal{U} \cap \mathcal{U} \mathcal{U} l v \mathcal{U} \bar{x} \mathcal{U} l v' \mathcal{U} \neq \emptyset.$$

Similar to the argument in [11, §2.6], this is impossible for $n \gg 0$. The statement is proved.

Corollary 4.3. The image of the map $\tilde{i}_\nu$ is contained in $\tilde{H}(\bar{v})$.

Proof. Let $m \in M$ and $K_M$ be an open compact subgroup of $\mathcal{I}_M$ with $m K_M \subset M(\nu)$. By Proposition 4.2, $m K_M \subset G(\bar{v})$. Let $\mathcal{X}$ be an open compact subset of $G$ with $m K_M \subset X$. By [11, Theorem 3.2], there exists $n \in \mathbb{N}$ such that $\mathcal{X} \cap G(\bar{v})$ is stable under the right multiplication by $\mathcal{I}$. In particular, $m K_M \mathcal{I}_n \subset G(\bar{v})$. Thus $\tilde{i}_\nu(\delta m K_M) \subset \tilde{H}(\bar{v})$.

4.1. In order to prove the other direction, we use the notion of alcove elements in [8] and [9].

Let $w \in \mathcal{U}$. We may regard $w$ as an affine transformation. Let $p : \text{Aff}(V) = V \ltimes GL(V) \to GL(V)$ be the natural projection map. Let $v \in V$. We say that $w$ is a $v$-alcove element if

- $p(w)(v) = v$;
- $N_v \cap w \mathcal{I} \mathcal{I}_w^{-1} \subset N_v \cap \mathcal{I}$.

Note that the first condition implies that $w M_v w^{-1} = M_v$. We have the following result.

Theorem 4.4. Let $w \in \mathcal{U}$. If $w$ is a $v_w$-alcove element, then any element in $\mathcal{U} w \mathcal{I}$ is conjugate by $\mathcal{I}$ to an element in $w \mathcal{I} M_v w$.
Proposition 2.3 (1), \( H \sim \) Here any by an element in \( a \) exists a root \( \nu \) in \( a \). By the definition of \( N \) in \( a \), \( \delta \) can be proved in the same way.

Now we start with an element in \( \dot{\mu} \). By Corollary 4.3, the image of \( \bar{\nu} \) and conjugates the given element to an element in \( \dot{\mu} M \) (and that the conjugator can be taken to be small when \( r \) is large).

If \( I[r] \subset I_M I[r + 1] \), then we may absorb the \( I_M \) part into \( i_M \). Otherwise, there exists a root \( a \) outside \( M \) such that \( I[r] = X_{\alpha + \epsilon} I[r + 1] \) and \( X_{\alpha + \epsilon} \subset I[r + 1] \) for any \( \epsilon > 0 \).

We show that each element \( \dot{w} i_M [r] \) with \( i_M \in I_M \) and \( i[r] \in I[r] \) is conjugate by an element in \( I \) to an element in \( \dot{\mu} I_M I[r + 1] \) (and that the conjugator can be taken to be small when \( r \) is large).

Let \( \nu \) be a positive integer in Proposition 2.5. By definition, \( [\nu : w] \in M \) and \( \nu' = \pi_M [w] \in M' \).

Let \( i_{\nu', w} \) be a positive integer in Proposition 2.5. By definition, \( H_w \) is spanned by \( \delta_{g, n} \) for \( g \in \dot{\mu} I \) and \( n > i(\nu', w) \ell(w) \). By the proof of Theorem 3.1 (1), for any \( n > i(\nu', w) \ell(w) \) and \( g \in \dot{\mu} I_M \), \( \delta_{g, n} + [H, H] \) is contained in the image of \( i_{\nu', \nu} \).

Let \( g \in \dot{\mu} I \). By Theorem 4.3 there exists \( i \in I \) and \( g' \in \dot{\mu} I_M \) such that \( g = ig' i^{-1} \). Then

\[ \delta_{g, n} = \delta_{g' I, n} i^{-1} \equiv \delta_{g, n} \mod [H, H]. \]

Therefore \( H_w \) is contained in the image of \( i_{\nu', \nu} \). By Proposition 3.3 \( H_w \) is also contained in the image of \( i_{\nu'} \).
5. Adjunction with the Jacquet Functor

5.1. Let \( R \) be an algebraically closed field of characteristic \( \neq p \). Set \( H_R = H \otimes_{\mathbb{Z}[\frac{1}{p}]} R, \) \( H_R = H \otimes_{\mathbb{Z}[\frac{1}{p}]} R \) and \( H_R(v) = H(v) \otimes_{\mathbb{Z}[\frac{1}{p}]} R \). Recall that \( \mathcal{R}(G)_R \) is the \( R \)-vector space with basis the isomorphism classes of irreducible smooth admissible representations of \( G \) over \( R \). We consider the trace map

\[
\text{Tr}_R^G : \overline{H}_R \to \mathcal{R}(G)_R^*.
\]

Similarly, for any semistandard Levi subgroup \( M \), we have

\[
\text{Tr}_R^M : \overline{H}_R(M) \to \mathcal{R}(M)_R^*.
\]

Let \( v \in V \) and \( M = M_v \). Let \( r_{v,R} : \mathcal{R}(G)_R \to \mathcal{R}(M)_R \) be the (normalized) Jacquet functor. Note that the Jacquet functor does not only depend on the Levi \( M \), but also depends on the direction \( v \) (or equivalently, the parabolic subgroup \( P_v \) with Levi factor \( M \)). The following result is proved by Bushnell in [2, Corollary 1].

Proposition 5.1. Let \( n \in \mathbb{N} \). Let \( v \in V \) and \( M = M_v \). Then for any \( f \in H_R^0(M,M_n) \), and \( \pi \in \mathcal{R}(G)_R \), we have

\[
\text{Tr}_R^M(f, r_{v,R}(\pi)) = \text{Tr}_R^G(j_{v,n}(f), \pi).
\]

The main result of this section is the following adjunction formula.

Theorem 5.2. Let \( M \) be a semistandard Levi subgroup and \( v \in \mathcal{R}_M \). Suppose that \( M = M_v \). Then for any \( f \in H_R(M; v) \) and \( \pi \in \mathcal{R}(G)_R \), we have

\[
\text{Tr}_R^M(f, r_{v,R}(\pi)) = \text{Tr}_R^G(\tilde{\iota}_v(f), \pi).
\]

5.2. Let \((x, K, u)\) be a standard triple of \( \hat{W}(M) \) such that the Newton point of \( x \) is \( v \). Let \( i \) be the smallest positive integer with \( tv \in X_v(Z) \). Let \( i \in \mathbb{N} \) such that for any \( \alpha \in \Phi_{v,+}, (iv, \alpha) \geq \sharp W_K + (i-1)\ell(x) + 1 \). Let \( l \geq i \). Then \( l = i', j \) for some \( i' \geq i \) and \( 0 \leq j < i \). Then for any \( m_1, \ldots, m_l \in I_M \hat{u}\hat{x}I_M \), by [26] (c), we have

\[
m_1m_2 \ldots m_l \in (I_M W_K I_M)(I_M \hat{u}\hat{x}I_M)(I_M t^{i'u}I_M).
\]

Note that for \( g \in I t^{i'u}I, gN_n g^{-1} \subset N_{n+\sharp W_K+(i-1)\ell(x)+1} \). Also \( (I W_K I)(I \hat{x}I) \subset \bigcup_{u \in \hat{W}, \ell(u) \leq \sharp W_K+(i-1)\ell(x)} I \hat{u}I \). Thus \((m_1 \ldots m_l)N_n(m_1 \ldots m_l)^{-1} \subset N_{n+1}. Similarly \((m_1 \ldots m_l)^{-1} N_n^{-}(m_1 \ldots m_l) \subset N_{n+1} \)

(a) Let \( l \geq ii \) and \( m_1, \ldots, m_l \in I_M \hat{u}\hat{x}I_M \), then \( m_1m_2 \ldots m_l \) is \( P_v \) strictly positive element.

Moreover, for any \( n, l' \in \mathbb{N} \) and \( m_1, \ldots, m_l' \in I_M \hat{u}\hat{x}I_M \), we have

\[
(m_1 \ldots m_l)N_{n+\sharp W_K+(i-1)\ell(x)}(m_1 \ldots m_l)^{-1} \subset N_n,
\]

\[
(m_1 \ldots m_l)^{-1} N_{n+\sharp W_K+(i-1)\ell(x)}(m_1 \ldots m_l) \subset N_n^{-}.
\]

One deduces that

(b) Let \( n, l' \in \mathbb{N} \), and \( g_1, \ldots, g_v \in N_{n+\sharp W_K+(i-1)\ell(x)}I_M \hat{u}\hat{x}I_M N_{n+\sharp W_K+(i-1)\ell(x)}^{-} \). Then \( g_1 \ldots g_v \in N_nM N_n^{-} \).
represent the element $l$

We have the following commutative diagram

$suffices to prove that for $M$

Proof of Theorem 5.2. By [11, Theorem 4.1 & §4.6], it suffices to prove it for locally constant functions on $M$, supported in $M \hat{u} \hat{v} M$, where $(x, K, u)$ is a standard triple of $\hat{W}(M)$ and the Newton point of $x$ is $v$.

Let $n > 2W_K + (i - 1)\ell(x)$ such that $\pi \in R_{\mathcal{I}_n}(G)$. It is enough to consider the function $f = \delta_{M_m M_n}$, where $m \in M \hat{u} \hat{v} M$.

Let $n' \gg n$ and $\hat{f} = \frac{\delta_v(m)^{-\frac{1}{2}}}{\mu_N(N_v)^{-\frac{1}{2}} \delta_{n', M_m M_n N_{n'}}}$. By Theorem 3.1 (1), $\hat{f}$ represents the element $\tilde{i}_v(f) \in \hat{H}$. By Casselman’s trick [4, Corollary 4.2], it suffices to prove that for $l \gg 0$, $\text{Tr}_R(f^l, r_v(\pi)) = \text{Tr}_R(\hat{f}^l, \pi)$.

Let $p_M : (M_n m M_n)^l \rightarrow M$ and $p_G : (N_{n'} M_n M_n N_{n'})^l \rightarrow G$ be the multiplication map. Since $l \gg 0$, by [5,2](a) and (b), any element in $\text{Im}(p_M)$ is $p_v$ strictly positive and

$$\text{Im}(p_G) \subset N_n \text{Im}(p_M) N_n^\perp \cong N \times \text{Im}(p_M) \times N^{-}.$$  

We have the following commutative diagram

$$
\begin{array}{ccc}
(N_{n'}, M_n m M_n N_{n'})^l & \xrightarrow{p_G} & \text{Im}(p_G) \\
pr \downarrow & & \downarrow pr_1 \\
(M_n m M_n)^l & \xrightarrow{p_M} & \text{Im}(p_M),
\end{array}
$$

where $pr : N \times M \times N^{-} \rightarrow M$ is the projection map and $pr_1$ is the restriction of $pr$ to $\text{Im}(p_G)$.

Let $m' \in \text{Im}(p_M)$. Then

$$\mu_{G'}(p_G^{-1} pr_1^{-1}(M_n m' M_n)) = \mu_{G'}((pr)^{-1} p_M^{-1}(M_n m' M_n)) = \mu_N(N_{n'})^{-1} \mu_{N^{-} (n')} \mu_{M'}(p_M^{-1}(M_n m' M_n)).$$

By Proposition 2.3 (2), $\delta_{\mathcal{I}_{n'} M_n m' M_n M_{n'}} \equiv \delta_{\mathcal{I}_{n'} M_n m' M_n M_{n'}} \mod [H, H]$ for any $i \in N_n$ and $i' \in N_{n'}$. Thus

$$\hat{f}^l = \frac{\delta_v(m)^{-\frac{1}{2}}}{\mu_N(N_v)^{-\frac{1}{2}} \delta_{n', M_n M_{n'}}} \sum_{m' \in M_n \backslash M / M_n} \mu_{G'}(p_G^{-1} pr_1^{-1}(M_n m' M_n)) \delta_{pr_1^{-1}(M_n m M_n)}$$

$$\equiv \sum_{m' \in M_n \backslash M / M_n} \delta_v(m)^{-\frac{1}{2}} \mu_M(p_M^{-1}(M_n m' M_n)) \delta_{pr_1^{-1}(M_n m M_n)}$$

$$\equiv \sum_{m' \in M_n \backslash M / M_n} \delta_v(m)^{-\frac{1}{2}} \mu_M(p_M^{-1}(M_n m' M_n)) \delta_{\mathcal{I}_{n'} M_n m' M_n M_{n'}} \mod [H, H].$$

On the other hand,

$$f^l = \sum_{m' \in M_n \backslash M / M_n} \frac{\mu_M(p_M^{-1}(M_n m' M_n))}{\mu_M(M_n m' M_n)} \delta_{M_n m' M_n}.$$  

By Corollary 2.3, we have

$$j_{v, n}(f^l) \equiv j_{v, n'}(f^l)$$

$$= \sum_{m' \in M_n \backslash M / M_n} \delta_v(m)^{-\frac{1}{2}} \mu_M(p_M^{-1}(M_n m' M_n)) \mu_M(M_n) \mu_G(\mathcal{I}_{n'}) \delta_{\mathcal{I}_{n'} M_n m' M_n M_{n'}} \mod [H, H].$$
Since the elements in $M_\mu m'M_\nu$ are $P$, strictly positive, we have $\mathcal{I}_n M_\mu m'M_\nu = N_\mu (M_\mu m'M_\nu) N_\nu$ and
\[
\mu_G(\mathcal{I}_n M_\mu m'M_\nu) = \mu_N(N_\mu)\mu_{N-,N_\mu} M_\mu m'M_\nu = \frac{\mu_G(\mathcal{I}_n)}{\mu_M(M_\mu m')} M_\mu m'M_\nu.
\]
So $\tilde{f} \equiv j_{\nu,n}(f^I) \mod [H, H]$ and $\text{Tr}_{\tilde{R}}^M(f^I, r_\nu(\pi)) = \text{Tr}_{\tilde{R}}^G(\tilde{f}^I, \pi).

6. The kernel of the trace map

6.1. Let $M$ be a semistandard Levi subgroup of $G$. Let $M^0$ be the subgroup of $G$ generated by the parahoric subgroups of $M$. Then we have $M/M^0 \cong \Omega_M$. Let $\Psi(M)_R = \text{Hom}_\mathbb{Z}(M/M^0, R^\times)$ be the torus of unramified characters of $M$.

Let $i_{M,R}: \mathcal{R}(M)_R \to \mathcal{R}(G)_R$ be the induction functor. Then for any $\sigma \in \mathcal{R}(M)_R$ and $f \in \tilde{H}_R$, the map
\[
\Psi(M)_R \to R, \quad \chi \mapsto \text{Tr}_{\tilde{R}}(f, i_{M,R}(\sigma \circ \chi))
\]
is an algebraic function over $\Psi(M)_R$.

6.2. Let $v \in V$ and $M = M_v$. Recall that
\[(a) \quad \hat{H}(M; v) = \bigoplus_{\nu_M \in \mathcal{R}_M; \nu=(\tau_M, v)} \text{ for some } \tau_M \in \Omega_M \hat{H}(M; v),
(b) \quad \hat{H}(\bar{v}) = \bigoplus_{\nu \in \mathcal{R}; \nu=(\tau, \bar{v})} \text{ for some } \tau \in \Omega \hat{H}(\nu).
\]

Note that if $\tau_M, \tau'_M \in \Omega_M$ are mapped under $\kappa$ to the same element in $\Omega$, then they differ by a central cocharacter of $M$. By the definition of the map $\pi = (\kappa, \bar{v})$, if both $(\tau_M, v)$ and $(\tau'_M, v)$ are in the image of $\pi_M$ and that $\kappa(\tau_M) = \kappa(\tau'_M)$, then $\tau_M = \tau'_M$. In other words, there is a natural bijection between the components appear on the right hand sides of (a) and (b). We define
\[
\tilde{i}_v = \bigoplus_{\nu_M \in \mathcal{R}_M; \nu=(\tau_M, v)} \text{ for some } \tau \in \Omega_M \tilde{i}_v : \hat{H}(M; v) \to \hat{H}(\bar{v}).
\]

**Theorem 6.1.** Let $v \in V$ and $M = M_v$. Let $f \in \tilde{H}(\bar{v})$. If $\text{Tr}_{\tilde{R}}^G(f, i_{M,R}(\sigma)) = 0$ for all $\sigma \in \mathcal{R}(M)_R$, then $f \in \tilde{i}_v(\ker \text{Tr}_{\tilde{R}}^M)$.

**Proof.** For $\sigma \in \mathcal{R}(M)_R$ and $\chi \in \Psi(M)_R$, the map
\[
\chi \mapsto \text{Tr}_{\tilde{R}}^G(\tilde{i}_v(f), i_{M,R}(\sigma \circ \chi))
\]
is an algebraic function on $\chi$. We consider its “positive part”, i.e. the linear combination of the terms $\langle \chi, \lambda \rangle$ for dominant coweight $\lambda$. It is obvious that if an algebraic function is zero, then its “positive part” is also zero.

By the Mackey formula [10 §5.5], we have
\[
\text{Tr}_{\tilde{R}}^G(\tilde{i}_v(f), i_{M,R}(\sigma \circ \chi)) = \text{Tr}_R^M(f, r_{M,R} \circ i_{M,R}(\sigma \circ \chi))
= \sum_{w \in M W M} \text{Tr}_R^M(f, i_{M,M \setminus w M,R} \circ \bar{w} \circ r_{M \setminus m w^{-1} M,R}^M(\sigma \circ \bar{w})).
\]

As $w \in M W M$ and $M = M_v$, $w(v)$ is dominant if and only if $w = 1$. Therefore the “positive part” of $\text{Tr}_{\tilde{R}}^G(\tilde{i}_v(f), i_{M,R}(\sigma \circ \chi))$ is $\text{Tr}_R^M(f, \sigma \circ \chi)$.

Therefore if $\text{Tr}_{\tilde{R}}^G(f, i_{M,R}(\sigma)) = 0$ for any $\sigma \in \mathcal{R}(M)_R$ and $\chi \in \Psi(M)_R$, then $\text{Tr}_R^M(f, \sigma \circ \chi) = 0$ for any $\sigma \in \mathcal{R}(M)_R$ and $\chi \in \Psi(M)_R$. Hence $f \in \ker \text{Tr}_R^M$. \(\square\)
Corollary 6.2. Let \( v \in V \) and \( M = M_v \). Then
\[
\tilde{\iota}_v^{-1}(\ker \Tr^G_R |_{H_R(\tilde{\nu})}) = \ker \Tr^M_R |_{H_R(M;v)}.
\]

Proof. If \( f \in \ker \Tr^M_R \), then \( \Tr^M_R(f, r_{M,R}(\pi)) = 0 \) for any \( \pi \in \mathfrak{H}(G)_R \). By Theorem 5.2 \( \Tr^G_R(\tilde{\iota}_v(f), \pi) = 0 \). Thus \( \tilde{\iota}_v(f) \in \ker \Tr^G_R \). The other direction follows from Theorem 6.1. \( \square \)

Theorem 6.3. We have \( \ker \Tr^G_R = \bigoplus_{v \in V_+} (\ker \Tr^G_R \cap \tilde{H}_R(v)) \).

Remark 6.4. In general, \( \bigoplus_{v \in \mathfrak{H}} \ker \Tr^G_R \cap \tilde{H}_R(v) \langle \ker \Tr^G_R \rangle \). However, the equality may not hold. For example, if \( \Omega = \{1, \tau\} \) is finite of order 2 and characteristic of \( R \) is also 2, then for any \( \lambda \in X_+(Z)_+ \) and \( f \in H(\lambda) \), we have \( f + \tau f \in \ker \Tr^G_R \).

Proof. The idea is similar to the proof of [23 Theorem 7.1].

Let \( f = \sum_{v \in V_+} a_v f_v \in \ker \Tr^G_R \), where \( f_v \in H_v \) and \( a_v \in R \). Let \( M \) be a minimal standard Levi subgroup such that \( a_v \neq 0 \) for some \( v \in V_+ \) with \( M = M_v \). Then for \( \sigma \in R(M) \) and \( \chi \in \Psi(M)_R \), we have
\[
\text{tr}_{M,R}(f, i_{M,R}(\sigma \circ \chi)) = \sum_{v \in V_+, M = M_v} a_v \text{tr}_{R}(f_v, i_{M,R}(\sigma \circ \chi)) + \sum_{v \in V_+, M \neq M_v} a_v \text{tr}_{R}(f_v, i_{M,R}(\sigma \circ \chi)).
\]

This is an algebraic function on \( \Psi(M)_R \). Note that in (a), the first part is more regular in \( \Psi(M)_R \) than the second part. Therefore we have
\[
\sum_{v \in V_+, M = M_v} a_v \text{tr}_{R}(f_v, i_{M,R}(\sigma \circ \chi)) = 0
\]
for all \( \sigma \in R(M) \) and \( \chi \in \Psi(M)_R \). As an algebraic function on \( \Psi(M)_R \), the “leading term” of \( \text{tr}_{R}(f_v, i_{M,R}(\sigma \circ \chi)) \) is a multiple of \( \langle v, \chi \rangle \). Hence \( a_v \text{tr}_{R}(f_v, i_{M,R}(\sigma \circ \chi)) = 0 \) for every \( v \in V_+ \) with \( M = M_v \). By Theorem 6.1
\[
a_v f_v \in \tilde{\iota}_v(\ker \Tr^M_R |_{R(M;v)}). \quad \square
\]

Finally, we have

Theorem 6.5. Assume that \( \text{char}(F) = 0 \). Let \( M \) be a semistandard Levi subgroup and \( \nu \in \mathfrak{H}_M \) with \( M = M_\nu \). Then the map
\[
\tilde{\iota}_\nu : \tilde{H}(M;\nu) \simarrow \tilde{H}(\tilde{\nu})
\]
is an isomorphism.

Proof. Let \( f \in \ker \tilde{\iota}_\nu \). Set \( \tilde{f} = f \otimes 1 \in \tilde{H}_C(M;\tilde{\nu}) \). By Theorem 6.1 (2), we have \( \tilde{f} \in \ker \Tr^M_C \). By the spectral density theorem [14 Theorem 0], \( \tilde{f} = 0 \in \tilde{H}(M)_C \). By [17], \( \tilde{H}(M) \) is free. Hence \( f = 0 \in H(M) \). \( \square \)

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