Use of even Grassmann variables to construct effective actions for mesons

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In the framework of an approach to bosonization based on the use of fermionic composites as fundamental variables, a quadratic action in even Grassmann variables with the quantum numbers of the pions has been constructed. It includes the $\sigma$-field in order to be invariant under $[SU(2)]_L \otimes [SU(2)]_R$ transformations over the quarks. This action exhibits the Goldstone phenomenon reducing its symmetry to the O(3) isospin invariance. The model has been investigated in the Stratonovitch–Hubbard representation, in which form it is reminiscent of the Gell-Mann–Lévy model. By the saddle point method a renormalizable expansion in inverse powers of the index of nilpotency of the mesonic fields (which is 24), is generated. The way it might be used in a new perturbative approach to QCD is outlined.

1. Introduction

Effective Grassmann actions for even composites are in general polynomials of degree equal to the index of nilpotency $\Omega$ of the composites [1]. QCD in strong coupling [2–4], the model defined in Ref. [5] and the pairing model [6] provide already examples where the action shows one and the same peculiar feature, namely the wave operator becomes static in the formal continuum limit, but, nonetheless it gives rise to a non trivial propagator. We have considered a lattice action of this type for the pion, invariant under $[SU(2)]_L \otimes [SU(2)]_R$ transformations over the quarks. This model exhibits the Goldstone phenomenon by reducing its symmetry to the O(3) isospin invariance. In the presence of an explicit breaking a squared mass for the pion is generated proportional to the breaking parameter.

2. A quadratic action for the pions

The composites with the quantum numbers of the pions are

$$\vec{\pi} = i a^2 \vec{\lambda} \gamma_5 \tau \lambda.$$(1)

In the above definition the $\tau_k$'s are the Pauli matrices and the sum over the colour, isospin and spin indices $a$, $f$ and $\beta$ of the quark field $\lambda^a_{f,\beta}$ is understood. The power of the lattice spacing $a$ has been introduced in order to give the fields the canonical dimension of a boson. To formulate a chiral invariant model we need also the field

$$\sigma = a^2 \vec{\lambda} \lambda.$$ (2)

All these composites have [1] index of nilpotency $\Omega = 24$.

The chiral transformations over the quarks

$$\delta \lambda = \frac{i}{2} \gamma_5 \vec{\sigma} \cdot \vec{\alpha} \lambda, \quad \delta \vec{\lambda} = \frac{i}{2} \vec{\gamma} \gamma_5 \vec{\pi} \cdot \vec{\alpha} \lambda$$ (3)

induce $O(4)$-transformations over the mesons

$$\delta \sigma = \vec{\alpha} \cdot \vec{\pi}, \quad \delta \vec{\pi} = -\vec{\alpha} \sigma.$$ (4)

A quadratic action with the above symmetry must therefore be of the form

$$S_C = -a^{-2} \left[ \frac{1}{2} (\vec{\sigma}, A \vec{\pi}) + \frac{1}{2} (\sigma, A \sigma) + (m, \sigma) \right],$$ (5)

where a breaking term has been included. The scalar product is defined as

$$(f, g) = a^4 \sum_x f(x) g(x).$$ (6)

The powers of the lattice spacing $a$ have been introduced assuming the wave-operator $A$ to be dimensionless.
The definition of $A$ that we will use is
\[ A = \frac{\rho^4}{-a^2 \Box + \rho^2}. \] (7)
which becomes, for small lattice spacing,
\[ A \sim \rho^2 (\rho^2 + a^2 \Box). \] (8)

This, as previously announced, is static. The first two terms in the action (8) have dimension 6 and we are therefore free to add them to the QCD action. The symmetry breaking term has instead dimension 4 (indeed it is of the same form of the quark mass term already present in the QCD action). We assume
\[ m = \sqrt{\Omega} \rho m_{\pi}^2 \] (9)
so that it formally vanishes in the continuum limit and it can be added to the QCD action.

Let us make clear that the partition function is
\[ Z_C = \int [d\bar{\chi} d\chi] \exp(-S_C). \] (10)

3. Breaking of the chiral invariance

In order to investigate our model we introduce the auxiliary fields $\bar{\chi}$ and $\chi_0$ by the Stratonovich–Hubbard transform. The partition function can then be written
\[ Z_C = \frac{1}{(\det A)^2} \int \left[ \frac{d\bar{\chi}}{\sqrt{2\pi}} \right] \left[ \frac{d\chi_0}{\sqrt{2\pi}} \right] \exp(-S_X) \] (11)
where
\[ S_X = \frac{1}{2a^2} \left[ (\bar{\chi}, A^{-1} \chi) + (\chi_0, A^{-1} \chi_0) \right] - \frac{\Omega}{2a^4} (1, \ln D(\bar{\chi}, \chi_0)) \] (12)
and
\[ D(\bar{\chi}, \chi_0) = a^2 \left[ (m + \chi_0)^2 + \chi^2 \right]. \] (13)

This action is $O(4)$ invariant for vanishing quark mass and reminds us of the Gell-Mann–Lévy model [1].

Since $\Omega$ is a rather large number we can apply the saddle-point method and evaluate the partition function as a series in inverse powers of this parameter. The minimum of $S_X$ is achieved for $\bar{\chi} = 0$ and
\[ \bar{\chi}_0 = -ma \pm \sqrt{m^2 a^2 + 4\Omega \rho^2}, \] (14)

where the + (respectively −) sign has to be chosen when $m > 0$ (respectively $m < 0$), which we shall assume to be the case.

Now we assume
\[ (am)^2 < 4\Omega \rho^2 \] (15)
so that
\[ \bar{\chi}_0 \approx a^{-1}[\sqrt{\Omega \rho} - \frac{1}{2}am]. \] (16)

The second derivatives of $S_X$ at the minimum are
\[ \begin{align*}
\frac{\partial^2 S_X}{\partial \chi_0(x) \partial \chi_0(y)} & \bigg|_{x = \bar{x}} = a^2 \left[ A^{-1}_{xy} + \delta_{xy} \frac{\Omega}{D} \right] \\
& = a^4 \left( -\Box + \frac{2\rho^2}{a^2} - m \frac{\rho}{a\sqrt{\Omega}} \right) \\
\frac{\partial^2 S_X}{\partial \chi_k(x) \partial \chi_k(y)} & \bigg|_{x = \bar{x}} = \delta_{hk} a^2 \left[ A^{-1}_{xy} - \delta_{xy} \frac{\Omega}{D} \right] \\
& = \delta_{hk} \frac{a^4}{\rho^4} \left( -\Box + m \frac{\rho}{a\sqrt{\Omega}} \right).
\end{align*} \] (17)

The propagator of the pion field to leading order
\[ \langle \pi_k(x) \pi_k(y) \rangle = \frac{1}{a^2} \left( \frac{1}{-\Box + m_{\pi}^2} \right)_{x,y} \] (18)
turns out to have the canonical form.

The $\sigma$-field acquires the nonvanishing expectation value
\[ \langle \sigma \rangle = \frac{1}{V} \frac{\partial}{\partial (a^2 m)} \log Z_C \approx a^{-1} \rho^{-1} \sqrt{\Omega}, \] (19)
and its mass is divergent according to
\[ m_{\sigma}^2 = 2a^{-2} \rho^2 - m_{\pi}^2 = 2M_1 - m_{\pi}^2. \] (20)

It is a consequence of the inequality (15) that $m_{\sigma} >> m_\pi$.

4. The $\Omega$-expansion

In order to formulate the $\Omega$-expansion it is convenient to introduce the fields which correspond to rescaled fluctuations
\[ \theta_0 = \rho^{-2}(\chi_0 - \bar{\chi}_0), \quad \theta_k = \rho^{-2}\chi_k. \] (21)
In terms of these fields the function $D$ takes the form

$$D = C^2 F$$

where

$$C = \sqrt{\Omega \rho^2 + \frac{1}{4} a^2 m^2 + \frac{1}{2} am} \approx \sqrt{\Omega \rho + \frac{1}{2} am}$$

$$F = 1 + \frac{\rho^2}{C} 2a \theta_0 + \frac{\rho^4}{C^2} a^2 (\theta_0^2 + \bar{\theta}^2).$$

By expanding the $\ln D$ we rewrite the action $S_\chi$ as a series

$$S_\chi = \sum_{n=2}^{\infty} S^{(n)}.$$  

The term $S^{(n)}$, for $n > 2$, is a homogeneous polynomial of degree $n$ in the $\theta$-fields proportional to $\rho^{2n} a^n / C^n$. The first three terms are

$$S^{(2)} = a^4 \sum_x \frac{1}{2} \bar{\theta} (-\Box + m_\pi^2) \bar{\theta}$$

$$+ \frac{1}{2} \theta_0 (-\Box + m_\pi^2) \theta_0$$

$$S^{(3)} = \frac{\rho^6}{C^3} a^4 \sum_x \frac{1}{a} \left( \frac{2}{3} \theta_0^3 - 2 \theta_0 \bar{\theta}^2 \right)$$

$$S^{(4)} = \frac{\rho^8}{C^4} a^4 \sum_x -\frac{1}{2} \theta_0^4 - \frac{1}{2} (\bar{\theta}^2)^2 + 3 \theta_0^2 \bar{\theta}^2.$$  

$S^{(n)}$ turns out to be of the order $\Omega^{-\frac{n}{2}}$, for $n > 2$. We actually have then an expansion in inverse powers of $\sqrt{\Omega}$, but the first correction is of order $\Omega^{-\frac{3}{2}}$.

We can now investigate the renormalizability of this expansion. This requires $\rho^2 / C$ not to diverge in the continuum limit, which is, indeed, the case.

We should mention that by a different dependence of $\rho$ and $m$ on the lattice spacing we can get a *truly free action* for the pions, all the interaction terms giving vanishing contribution to the $n$-point functions. The resulting action, however, cannot be added to the QCD action as an irrelevant operator, because it is not accompanied by the necessary powers of the lattice spacing. Therefore it cannot be used to set up a new perturbative approach to QCD, which is our main motivation. In such a case one can regard such a result as the construction of a simple model, whose main ingredient is the compositness of the mesonic fields.

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