Channeling of Positrons through Periodically Bent Crystals: on Feasibility of Crystalline Undulator and Gamma-Laser

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Abstract. The electromagnetic radiation generated by ultra-relativistic positrons channelling in a crystalline undulator is discussed. The crystalline undulator is a crystal whose planes are bent periodically with the amplitude much larger than the interplanar spacing. Various conditions and criteria to be fulfilled for the crystalline undulator operation are established. Different methods of the crystal bending are described. We present the results of numeric calculations of spectral distributions of the spontaneous radiation emitted in the crystalline undulator and discuss the possibility to create the stimulated emission in such a system in analogy with the free electron laser. A careful literature survey covering the formulation of all essential ideas in this field is given. Our investigation shows that the proposed mechanism provides an efficient source for high energy photons, which is worth to study experimentally.

1. Introduction

We discuss a new mechanism of generation of high energy photons by means of a planar channeling of ultra-relativistic positrons through a periodically bent crystal. The feasibility of this scheme was explicitly demonstrated in [1, 2]. In these papers as well as in our subsequent publications [3]-[13] the idea of this new type of radiation, all essential conditions and limitations which must be fulfilled to make possible the observation of the effect and a crystalline undulator operation were formulated in a complete and adequate form for the first time. A number of corresponding novel numerical results

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were presented to illustrate the developed theory, including, in particular, the calculation of the spectral and angular characteristics of the new type of radiation.

The aim of this paper is to review the results obtained so far in this newly arisen field as well as to carry out a historical survey of the development of all principal ideas and related phenomena. The necessity of this is motivated by the fact that the importance of the ideas suggested and discussed in [1]-[13] has been also realized by other authors resulting in a significant increase of the number of publications in the field within the last 3-5 years [14]-[25] but, unfortunately, often without proper citation [18]-[25]. We review all publications known to us which are relevant to the subject of our research.

The main phenomenon to be discussed in this review is the radiation formed in a crystalline undulator. The term 'crystalline undulator' (introduced but not at all elaborated clearly in [26]) stands for a system which consists of two essential parts: (a) a crystal whose crystallographic planes are bent periodically, and (b) a bunch of ultra-relativistic positively charged particles undergoing planar channeling in the crystal. In such a system there appears, in addition to a well-known channeling radiation, the radiation of an undulator type which is due to the periodic motion of channeling particles which follow the bending of the crystallographic planes. The intensity and the characteristic frequencies of this undulator radiation can be easily varied by changing the energy of beam particles and the parameters of crystal bending.

![Schematic representation of spontaneous and stimulated radiation in a periodically bent crystal.](image)

**Figure 1.** Schematic representation of spontaneous and stimulated radiation in a periodically bent crystal. The y- and z-scales are incompatible!

The mechanism of the photon emission by means of a crystalline undulator is illustrated by figure 1. Short comments presented below aim to focus on the principal
features of the scheme relevant to the subject of the review. At the moment we do not elaborate all the important details but do this in section 3.

The \((yz)\)-plane in the figure is a cross section of an initially linear crystal, and the \(z\)-axis represents the cross section of a midplane of two neighbouring non-deformed crystallographic planes (not drawn in the figure) spaced by the interplanar distance \(d\). Two sets of black circles denote the nuclei which belong to the periodically bent neighbouring planes which form a periodically bent channel. The amplitude of the bending, \(a\), is defined as a maximum displacement of the deformed midplane (thick solid line) from the \(z\)-axis. The quantity \(\lambda\) stands for a spatial period of the bending. In principle, it is possible to consider various shapes, \(y(z)\), of the periodically bent midplane. In this review we will mainly discuss the harmonic form of this function, \(y(z) = a \sin(2\pi z/\lambda)\). For further referencing let us stress here that we will mainly consider the case when the quantities \(d\), \(a\) and \(\lambda\) satisfy strong double inequality: \(d \ll a \ll \lambda\). Typically \(d \sim 10^{-8}\) cm, \(a \sim 10 \ldots 10^2 d\), and \(a \sim 10^{-5} \ldots 10^{-4}\) \(\lambda\).

Open circles in figure 1 denote the channeled ultra-relativistic particles. Provided certain conditions are met, the particles, injected into the crystal, will undergo channeling in the periodically bent channel. Thus, the trajectory of a particle (represented schematically by the dashed line) contains two elements. Firstly, there are oscillations inside the channel due to the action of the interplanar potential, - the channeling oscillations. This mode is characterized by a frequency \(\Omega_{\text{ch}}\) dependent on the projectile type, energy, and the parameters of the interplanar potential. Secondly, there are oscillations because of the periodicity of the distorted midplane, - the undulator oscillations, whose frequency is \(\omega_0 \approx 2\pi c/\lambda\) (\(c\) is the velocity of light which approximately is the velocity of an ultra-relativistic particle).

Spontaneous emission of photons which appears in this system is associated with both of these oscillations. Typical frequency of the emission due to the channeling oscillations is \(\omega_{\text{ch}} \approx 2\gamma^2 \Omega_{\text{ch}}\) where \(\gamma\) is the relativistic Lorenz factor \(\gamma = \varepsilon/mc^2\). The undulator oscillations give rise to the photons with frequency \(\omega \approx 4\gamma^2 \omega_0/(2 + p^2)\) where the quantity \(p\), a so-called undulator parameter, is related to the amplitude and the period of bending, \(p = 2\pi \gamma (a/\lambda)\).

If strong inequality \(\omega_0 \ll \Omega_{\text{ch}}\) is met than the frequencies of the channeling radiation and the undulator radiation are also well separated, \(\omega \ll \omega_{\text{ch}}\). In this case the characteristics of the undulator radiation (the intensity and spectral-angular distribution) are practically independent on the channeling oscillations but depend on the shape of the periodically bent midplane.

For \(\omega_0 \ll \Omega_{\text{ch}}\) the scheme presented in figure 1 leads to the possibility of generating a stimulated undulator emission. This is due to the fact, that photons emitted at the points of the maximum curvature of the midplane travel almost parallel to the beam and thus, stimulate the photon generation in the vicinity of all successive maxima and
minima. In [2] we demonstrated that it is feasible to consider emission stimulation within the range of photon energies $\hbar \omega = 10 \ldots 10^4$ keV (a Gamma-laser). These energies correspond to the range $10^{-8} \ldots 10^{-4}$ µm of the emission wavelength, which is far below than the operating wavelengths in conventional free-electron lasers (both existing and proposed) based on the action of a magnetic field on a projectile [27, 28, 29].

However, there are essential features which distinguish a seemingly simple scheme presented in figure 1 from a conventional undulator. In the latter the beam of particles and the photon flux move in vacuum whereas in the proposed scheme they propagate through a crystalline media. The interaction of both beams with the crystal constituents makes the problem much more complicated from theoretical, experimental and technical viewpoints. Taking into consideration a number of side effects which accompany the beams dynamics, it is not at all evident a priori that the effect will not be smeared out. Therefore, to prove that the crystalline undulator as well as the radiation formed in it both are feasible it is necessary to analyze the influence, in most cases destructive, of various related phenomena. Only on the basis of such an analysis one can formulate the conditions which must be met and define the ranges of parameters (which include the bunch energy, the types of projectiles, the amplitude and the period of bendings, the crystal length, the photon energy) within which all the criteria are fulfilled. In full this accurate analysis was carried out very recently and the feasibility of the crystalline undulator and the Gamma-laser based on it were demonstrated in an adequate form for the first time in [1, 2] and in [3]-[13].

From the viewpoint of this compulsory programme which had to be done in order to draw a conclusion that the scheme in figure 1 can be transformed from the stage of a purely academic idea up to an observable effect and an operating device we review critically some of the recent publications as well as the much earlier ones [26, 30, 31, 32, 33, 34, 35, 36]. This is done in section 3.

Prior to start with the discussion of various aspects of the electromagnetic radiation from a beam of charged particles channeling in a periodically bent crystal let us briefly mention the phenomena closely related to our main problem. These are: channeling in straight and in bent crystals, channeling radiation and undulator radiation.

2. Channeling, channeling radiation, undulator radiation

We do not pretend to cover the whole range of problems concerning the channeling effect, the channeling radiation and the radiation formed in the undulators. This section is devoted solely for a brief description of the effects closely related to the main subject of this review.
2.1. Channeling

The basic effect of the channeling process in a straight crystal is that a charged particle can penetrate at an anomalously large distance when traveling nearly parallel to a crystallographic plane (the planar channeling) or an axis (the axial channeling) and experiencing the collective action of the electrostatic field of the lattice ions. The latter is repulsive at small distances for positively charged particles and, therefore, they are steered into the interatomic region, while negatively charged projectiles move in the close vicinity of ion strings or planes.

Channeling was discovered in the early 1960s by computer simulations of ion motion in crystals of various structure [37]. Large penetration lengths were obtained for low-energy (up to 10 keV) ions incident along crystallographic directions of low indexes. These calculations allowed to explain the results of earlier experiments. Later, a comprehensive theoretical study [38] introduced the important model of continuum potential for the interaction energetic charged projectiles and lattice atoms arranged in strings and planes. Using this approach the criterion for a stable channeling was formulated. According to it the beam of particles becomes trapped by the interplanar or axial potential if the incident angle of the beam with respect to the crystal plane or the axis is smaller than the so-called critical Lindhard angle $\theta_L$. These concepts were subsequently widely used to interpret channeling experiments for low energy projectiles ($\varepsilon < 1$ GeV), see, e.g., [39], and for high energies of the particles ($\varepsilon = 1 \ldots 10^2$ GeV) [40, 41, 42, 43, 44]. In recent years channeling of charged particles in straight nano-tubes, usually carbon nano-tubes, has been intensively investigated [45]-[52]. We mention these papers for the sake of completeness but do not pretend to review them, since this field is not related directly to the main subject of our research.

Lindhard [38] was also the first who theoretically described the dechanneling process, - the phenomenon of a gradual increase of the transverse energy of a channeled particle due to inelastic collisions with nuclei and electrons of the crystal. As a consequence, initially channeled particle during its passage through the crystal gains the transverse energy higher than the continuum potential barrier. At this point the particle leaves the channel and is, basically, lost for the channeling process. The scale which defines the (average) interval for a particle to penetrate into a crystal until its dechannels is called a dechanneling length, $L_d$. This quantity depends on the parameters of a crystal (these include the charge of nuclei, the interatomic spacing, the mean atomic radius, the amplitude of thermal vibrations) and the parameters of a channeled particle, - the energy and the charge. It is important to note that for negative and for positive projectiles the dechanneling occurs in different regimes. As mentioned above, the interplanar (or axial) potential is repulsive at small distances for positively charged particles and is attractive for negatively ones. Therefore, negatively charged particles tend to channel in the
regions around the nuclei whereas positive particles are pushed away. Consequently, the number of collisions with the crystal constituents is much larger for negatively charge particles and they dechannel faster. Typically, the dechanneling lengths of positive charges exceed those for negative ones (of the same energy and charge modulus) by the order of magnitude or more [16, 53]. This statement is valid, both for axial and planar channeling, and for various pairs of positive/negative particles of the same charge modulus: for $e^+/e^−$ [54], $\pi^+/\pi^−$ [55], and $p/\bar{p}$ [56].

2.2. Channeling radiation

Channeling of charged particles in crystals is accompanied by the channeling radiation [57, 58]. This specific type of electromagnetic radiation arises due to the transverse motion of the particle inside the channel under the action of the interplanar field (the channeling oscillations, see figure 1). The phenomena of channeling radiation of a charged projectile in a linear crystal, see e.g. [41, 42, 53, 59, 60, 61, 62], as well as in a ‘simple’ (i.e. non-periodic, one-arc) bent channel [63, 64, 65], are known, although in the latter case the theoretical and experimental data are scarce, at least up to now.

For the purpose of this review it is important to mention several well-established features of the channeling radiation. Firstly, as well as in any other radiative process of a charge moving in an external potential (e.g. bremsstrahlung [66]), the intensity of the channeling radiation is inversely proportional to the squared mass of the projectile. Consequently, a channeled electron/positron emits $(m_p/m_e)^2$ times more intensively than a proton with the same value of relativistic factor $\gamma$. Secondly, for both electrons and positrons the intensity of radiation in the channeling mode greatly exceeds (by more than an order of magnitude) that by the same projectile in an amorphous medium [54, 67]. Finally, the radiative energy loss is of a channeled electron is noticeably higher than that of a positron of the same energy. This is valid for all energy ranges (from several MeV up to hundreds of GeV) of the projectile [16, 41, 67, 68, 69, 70]. Physical reason for this is that electrons channel in vicinity of the crystal nuclei, therefore they are accelerated stronger and radiated more energetic photons than positrons which channel in the region of a weaker field.

The study of the channeling radiation initially proposed for particles moving in crystals was later extended to the case of nanotubes [45, 46, 47, 48].

2.3. Channeling in bent crystals

The channeling process in a bent crystal takes place [71] if the maximal centrifugal force, acting on the projectile because of the channel bending, is less than the force due to the interplanar field [30, 65, 71, 72]:

$$m\gamma v^2/R < q U_{\text{max}}'. \quad (1)$$
Here $m$ and $v$ are the mass and velocity of the projectile, $\gamma$ is its relativistic factor and $R$ is a curvature radius of the bent channel, $q$ is the charge of the projectile and the quantity $U'_{\text{max}}$ stands for the maximum gradient of the interplanar field.

Provided this condition is fulfilled, the beam of positively charged channeling particles at each instant moves inside the channel, especially parallel to the bent crystal midplane as it does in the case of the channeling in a straight crystal. Since it first theoretical prediction [71] and experimental support [73] the idea to deflect high-energy beams of charged particles by means of a planar channeling in bent crystals had attracted a lot of attention worldwide and still is of a great interest. Indeed, the beams (in particular, those of ultra-relativistic protons and heavy ions) can be steered by crystals much more efficiently than by means of external macroscopic electric or magnetic fields. In recent experiments with 450 GeV protons [74, 75] the efficiency of the particle beam deflection was reported on the level of 60% [75] and of 85% for the 70-GeV protons beam [76]. The progress in deflecting of heavy-ions beams was reported recently as well [77]. We refer to the reviews papers which cover state-of-the-art in this field, and describe in detail all the stages of the development of the idea as well as a number of phenomena which are of current interest [40, 42, 43, 44, 78, 79, 80].

Deflection of the beams of negatively charged particles by planar channeling is strongly suppressed because of the increased role of the dechanneling [43, 82]. The deflection of beams of particles, both positive and negative, during axial channeling was simulated numerically [81]. The experimental study of this effect has demonstrated small efficiency of extracting the beam particles [80, 82].

Another possible application of bent crystal concerns its possible use to focus beams of ultra-relativistic heavy-ions [83] or protons [84]. To this end a crystal is needed in which the crystal axes are no longer parallel, but are slanted more and more the farther away they are from the axis of the beam. Then the bending angle of the particles far away from the beam axis would be largest and a general focusing effect will result. Such a crystal can in principle be produced by varying the nickel to copper (or Sb to Bi) ratio in a mixed crystal [83]. Similar idea was suggested also for producing periodically bent crystals, see section 3 for more detail.

Recently, it was suggested to use bent nanotubes to steer beams of charged particle [50, 52].

2.4. Undulator radiation

The theory and also various practical implementations of the undulator radiation, i.e. the radiation emitted by a charge moving in spatially periodic static magnetic fields (a magnetic undulator), or in a static macroscopic electric field (electrostatic undulator), or in a laser field (a laser-based undulator), etc. have long history [85, 86] and are
well elaborated [28, 60, 61, 87, 88, 89]. The most important feature of the undulator radiation, which clearly distinguishes it from other types of electromagnetic radiation formed by a charge moving in external fields, is in a peculiar form of the spectral-angular distribution. Namely, for each value of the emission angle $\theta$ (measured with respect to the undulator axis) the spectral distribution consists of a set of narrow, powerful and equally spaced peaks (harmonics). The peak intensity is proportional to the square of total number of the undulator periods, $N^2$. This factor reflects the constructive interference of radiation emitted from each of the undulator periods and is typical for any system which contains $N$ coherent emitters (e.g. [90]). In an ideal undulator, i.e. the one in which a particle follows either a sinusoidal periodic trajectory (a planar undulator) or a helical one (a helical undulator) under the action of a non-dissipative external force, and for a fixed $\theta$ the widths of all peaks are the same and proportional to $1/N$. The coherence is lost if one integrates the spectral-angular distribution over the emission angle. Indeed, although the spectral distribution as a function of photon frequency $\omega$ contains maxima (which are noticeably widened as compared to those in spectral-angular distribution) the intensity in which is proportional to $N$.

These two features of the undulator radiation, the high intensity in the peaks of spectral-angular distribution and the narrow widths, are important for a successful operation of free-electron lasers (FEL), the devices which transform the spontaneous undulator radiation into the stimulated one. Since the first discovery of the FEL operational principle [91] both the theory and practical implementation of FELs have been well elaborated (see, e.g. [28, 29, 61, 88, 89, 92]).

In ideal conditions a bunch of particle undergoing channeling in a linear crystal can be considered as an undulator (a natural undulator). Indeed, due to the periodicity of the trajectory the characteristics of the channeling radiation are close to those of the undulator radiation [53]. However, the characteristics of the radiation emitted in natural undulator are masked in experiment (e.g. [41]) because of the distribution of the beam particles in the transverse energy, and in the incident angle. Additionally, in the natural undulator the emission peaks become noticeably wider due to a deviation of an interplanar potential from a harmonic one. This leads to the dependence of the undulator period on the amplitude of the channeling oscillations which, in turn, defines the frequency of the emitted radiation.

3. **Channeling in a periodically bent crystal: feasibility of a crystalline undulator**

In this section we describe, more thoroughly, the crystalline undulator, formulate the conditions, which must be fulfilled for its operation, and present the detailed literature survey covering the development of all essential aspects of this important idea. Also we
review theoretical methods to be used for an adequate description of this phenomenon. The idea of a gamma-laser based on the crystalline undulator is discussed in section 5.

3.1. The conditions to be met in a crystalline undulator

As it was noted in section 1 in a crystalline undulator the beam of channeled particles and the emitted photons propagate in a media. Therefore, prior to drawing a conclusion that the scheme illustrated by figure 1 is not of academic interest but can be made realistic and to represent a new type of an undulator, one has to understand to what extent general characteristics of the undulator radiation (high intensity, high degree of monochromaticity of the spectral-angular distribution) are influenced by the presence of a crystalline media.

To fulfill this programme and to establish the ranges of various parameters within which the operation of the crystalline undulator is feasible we had analyzed the following basic problems:

- Planar or axial channeling? Positive or negative particles? [1, 2]
- Condition for the stable channeling in a periodically bent crystal [1, 2].
- Crystalline undulator preparation: static and dynamic bending [1, 2, 10, 11].
- Large and small amplitude regimes [2, 3, 4, 5, 8].
- Dechanneling and photon attenuation and the length of a crystalline undulator [2, 5, 7, 8].
- Energy losses and the shape of a crystalline undulator [3, 10, 11].

3.1.1. Planar channeling of positively charged particles. Keeping in mind that negatively charged particles are steered along crystallographic axes and planes much less efficiently than positively charged ones and that the effect of axial channeling in bent crystals has not been observed so far, we focus our discussion on the planar channeling of positively charged particles and, in particular, on the channeling of light projectiles, positrons, in periodically bent crystals. This system is the most appropriate for the creation of the crystalline undulator operating in the high energy photon regime. Note, that from theoretical viewpoint the generalization of the treatment of the undulator radiation in crystalline undulators to the case of axial channeling is straightforward. However, small values of the dechanneling lengths of projectiles in this case make such a discussion purely academic and bring it far beyond any realistic experimental opportunities.

The principal difference in the behaviour of positively and negatively charged particles in a crystalline undulator was realized for the first time in [1, 2]. This fact has determined, to a great extent, the main focuses of our subsequent publications [3]-[13]. In some of the earlier publications on the subject [26, 31, 32, 36], the authors
did not distinguish clearly the cases of electron and positron channeling in a crystalline undulator and often discussed channeling of electrons rather than positrons. They did not analyze the strong condition, limiting the operation of a crystalline undulator, which originates from the dechanneling of particles during their passage through periodically bent crystals. Below we consider this important condition in more detail when discussing the dechanneling process. Following our conclusions made in [1, 2, 3, 4, 5], the authors of [14] in their later publications [18, 20] also began to focus on the channeling of positively charged particles in crystalline undulators. This remark concerns not only this particular issue, but rather most of the physics of the processes taking place in crystalline undulators, which was analyzed and clarified in [1, 2] and our subsequent publications. For example, all physical conditions for the operation of a crystalline undulator outlined on page 113 in [18] are taken from our earlier work [2] but without any reference.

3.1.2. Stable channeling in a periodically bent crystal. The condition for channeling in a periodically bent crystal is subject to the general criterion (1) for the channeling process in a bent crystal [71] (see also [30, 32, 44, 65, 93]), and can be fulfilled by a proper choice of the projectile energy and the maximal curvature of the channel. For the first time the limiting role of the channeling criterion on the parameters of the crystalline undulator was elucidated in [1, 2, 4]. In the earlier publications on the subject this analysis was not carried out.

Similar to the case of an one-arc bent crystal, a stable channeling of an ultra-relativistic positively charged particle in a periodically bent crystal occurs if the maximum centrifugal force, \( F_{cf} \approx m\gamma c^2/R_{\min} \), is less than the maximal force due to the interplanar field, \( F_{int} = qU'_{\max} \) [1, 2, 30, 32]. More specifically [2, 3, 4], the ratio \( C = F_{cf}/F_{int} \) is better to keep smaller than 0.1. If otherwise the phase volume of the trajectories, corresponding to the channeling mode, becomes significantly reduced. The inequality \( C \ll 1 \) relates the energy of an ultra-relativistic particle, \( \varepsilon = m\gamma c^2 \), the parameters of the bending (these define the quantity \( R_{\min} \)), and the characteristics of the planar potential.

Choosing a harmonic shape \( y(z) = a\sin\left(\frac{2\pi z}{\lambda}\right) \) for the periodically bent channel (see figure 1), one derives \( R_{\min} = \lambda^2/4\pi^2 a \). Thus, the decrease in \( R_{\min} \) and, consequently, the increase in the maximum centrifugal acceleration of the particle in the channel is achieved by decreasing \( \lambda \) and increasing \( a \). The criterion for a stable channeling implies the following relationship between the parameters \( a, \lambda, \varepsilon \) and \( U'_{\max} \):

\[
C \equiv (2\pi)^2 \frac{\varepsilon}{qU'_{\max}} \frac{a}{\lambda^2} \ll 1. \tag{2}
\]

For the first time, the limiting role of the channeling criterion (2) on the parameters of the crystalline undulator was elucidated in [1, 2, 4].
Provided (2) is fulfilled, the projectile, incident at the angle smaller than the Lindhard angle $\theta_L$ with respect to the mid-plane, is trapped in the channel, see figure 1. The passage of an ultra-relativistic particle gives rise to the emission of photons due to the curvature of the trajectory, which becomes periodic reflecting the periodicity of the channel midplane. This radiation is enhanced because of the coherent emission from similar parts of the trajectory, and, as a result, it may dominate over the channeling radiation [1, 2, 3, 4].

3.1.3. Large and small amplitude regimes. There are two essentially different regimes of the radiation formation in a periodically bent crystals. These regimes are defined by the magnitude of the ratio $a/d$.

It was demonstrated in [1, 2, 4, 30] that the undulator radiation and the channeling radiation are well separated provided the condition $a \gg d$ is fulfilled, see figure 1. Typically, spacing between the planes, which are characterized by the low values of the Miller indices such as the (100), (110) and (111) planes, lies within the range $0.6 \ldots 2.5$ Å (see, e.g. [39, 44, 61]). Therefore, the condition is fulfilled for the bending amplitudes $a \geq 10$ Å. A similar separation of the two radiative mechanisms takes place in non-periodically bent crystals, where the curvature of the channel leads to an additional synchrotron-type radiation by a channeling particle [64, 65]. This component of the emission transforms into the undulator radiation in the periodically bent channel.

In the limit $a \gg d$, which we call a large-amplitude regime, the separation of the two radiative mechanisms occurs because the frequency of the channeling oscillations, $\Omega_{ch} \sim c(qU'_{\text{max}}/d \varepsilon)^{1/2}$, is much higher than the frequency, $\omega_0$, of the transverse oscillations caused by the periodicity of the channel $\omega_0 = 2\pi c/\lambda$. Consequently, the characteristic frequencies of the channeling and the undulator radiation can be estimated as $\omega_{ch} \approx \gamma^2 \Omega_{ch} \sim \gamma^2 c(qU'_{\text{max}}/d \varepsilon)^{1/2}$ and $\omega_u \sim \gamma^2 \omega_0 = 2\pi c\gamma^2/\lambda$. Then, the ratio $\omega_u^2/\omega_{ch}^2$ can be written in the following form

$$
\frac{\omega_u^2}{\omega_{ch}^2} \sim \frac{(2\pi)^2 \frac{d \varepsilon}{\lambda^2} qU'_{\text{max}}}{C d} \ll 1.
$$

This relation shows that if both conditions, $C \ll 1$ and $a \gg d$, are fulfilled, then the characteristic frequencies are well separated. Moreover, if one is only interested in the spectral distribution of the undulator radiation, one may disregard the channeling oscillations and assume that the projectile moves along the centerline of the bent channel [1, 2]. Additionally, as it will be demonstrated below in the paper, in the high-amplitude regime the intensity of the undulator radiation is higher than that of the channeling radiation [2, 3, 4]. Therefore, we may state that the limit $a \gg d$ is essential to call the undulator-type radiation due to the periodic structure of the crystal bending as a new phenomenon and to consider it as a new source of the emission within the $X$- and $\gamma$-range.
For $a \sim 10d$ and for $C \leq 0.1$ equation (3) yields $\omega / \omega_{\text{ch}} \sim 0.1$. Therefore, the two spectra are well separated and the crystalline undulator radiation, both spontaneous and stimulated, can be treated independently from the ordinary channeling radiation.

The importance of the large-amplitude regime was noted in [30] and independently in [1, 2]. In the papers [1, 2, 4, 5] the detailed qualitative and quantitative analysis of this condition was carried out, and the calculation of realistic spectral and angular distribution of the undulator radiation were performed for the first time. The first calculation of the total radiative spectra (i.e. the spectra in which both ordinary channeling radiation and the crystalline undulator radiation are taken into account) for the $\varepsilon = 500$ MeV positrons channeling in the silicon along the (110) crystallographic planes for different $a/d$ ratios has been carried out in [4]. These calculations have supported the qualitative arguments, formulated above, on the possibility to separate radiations emitted via two different mechanisms. Calculations, performed in [1, 2, 4, 5], have also demonstrated that the intensity of the crystalline undulator radiation can be made much larger than that of the channeling radiation. We consider this example in more detail in section 4.

In the large-amplitude regime, in addition to the condition (2), it is very important to consider other conditions under which the periodically bent crystal may serve as a crystalline undulator. These other conditions appear when one takes into account the destructive role of the dechanneling effect and the photon attenuation. The restrictive influence of these phenomena on the parameters of the crystalline undulator is discussed in detail in section 3.1.5. Here we want to note that a comprehensive quantitative analysis of all the conditions, which must be met, was performed in [1, 2, 3, 4, 5]. In contrast such a full and important discussion was omitted in the papers [26, 30, 31, 32, 36, 59].

Our analysis, carried out in [2, 4, 5], has shown, in particular, that the optimal range of the amplitude values, where all the necessary conditions can be fulfilled, is $a \sim (10 \ldots 100) d \approx (10^{-7} \ldots 10^{-6})$ cm. We want to point out that exactly this range of $a$ was discussed in later publications [18, 19, 20, 21, 22, 23, 24, 25] without proper citation of our results. The values of $a$ mentioned above are much lower than the interval $a \sim (10^{-5} \ldots 10^{-4})$ cm considered in [30]. In section 3.3 we demonstrate that the parameters, used in [30] to characterize the crystal bending, do not lead to the emission of undulator-type radiation.

In the papers [2, 4, 5] we demonstrated that in the low-amplitude regime, when $a < d$, the intensity of the undulator radiation is smaller than that of the channeling radiation. Moreover, in the limit $a \ll d$ the undulator radiation becomes less intensive than the background bremsstrahlung radiation. Hence, it is highly questionable whether the crystalline undulator radiation can be considered as a new phenomenon in the limit of low $a$. However, the low-amplitude regime allows to consider a a resonant coupling
of two mechanisms of the photon emission, the channeling radiation and the undulator radiation. Indeed, for \( a \ll d \) and by a proper choice of the parameter \( C \) (see (2)) it is possible to make the frequency of the undulator radiation \( \omega_u \) comparable or equal to that of the channeling radiation \( \omega_{ch} \). If \( \omega_u \sim \omega_{ch} \), then the intensity of the channeling radiation can be resonantly enhanced even in the case when the undulator radiation is much less intensive. This very interesting phenomenon was considered in a series of papers [31, 32, 33, 34, 35, 36, 94, 95]. The first two from this list discussed the modification of the channeling radiation spectrum in the case of an electron and/or a positron channeling in a superlattice. In the papers [33, 34, 35, 36, 94, 95] the parametric resonant enhancement of the channeling radiation emitted by positrons in the presence of either transverse or longitudinal supersonic field was considered. The authors of the cited papers investigated neither the crystalline undulator nor the undulator radiation formed in it. Therefore, we are not going to discuss further the results obtained in these papers, because the subject of their research is absolutely different from the topic of this review.

3.1.4. Crystalline undulator preparation: static and dynamic bending The term 'undulator' implies that the number of periodic elements (i.e. the number of undulator periods, \( N \)) is large. Only in this limit the radiation formed during the passage of a bunch of relativistic particles through a periodic system bears the features of an undulator radiation (narrow, well-separated peaks in spectral-angular distribution) rather than those of a synchrotron radiation. Hence, the following strong inequality, which entangles the period \( \lambda \) and the length of a crystal \( L \) must be met in the crystalline undulator [1, 2]:

\[
N = \frac{L}{\lambda} \gg 1.
\] (4)

The parameter \( \lambda \), together with the amplitude \( a \) and the energy of the particle, define other quantities, which are called the undulator frequency \( \omega_0 \) and parameter \( p \), and which are commonly used when characterizing an undulator:

\[
\omega_0 = 2\pi \frac{c}{\lambda}, \quad p = 2\pi \gamma \frac{a}{\lambda}.
\] (5)

Let us note the proportionality of the parameter \( p \) to the relativistic factor \( \gamma \). This dependence is typical for a crystalline undulator, but is absent for the undulators (both helical and planar) based on the action of magnetic field. In the latter case the undulator parameter is equal to \( p_B = qB\lambda_B/2\pi mc^2 \) (see e.g. [89, 96]), where \( B \) is the amplitude value of the magnetic induction and \( \lambda_B \) is the period of the magnetic field.

In turn, the quantities \( \omega_0 \), \( p \) and \( N \) define the parameters of the spontaneous undulator radiation, which are the characteristic frequencies \( \omega_k = k\omega_1 \), \( k = 1, 2, \ldots \) (harmonics) and the natural width, \( \Delta \omega \), of the peaks. In a perfect planar undulator,
where the trajectory of the particle has a harmonic form \( y(z) = a \sin\left(\frac{2\pi z}{\lambda}\right) \), the frequency \( \omega_1 \) of the fundamental harmonic and the width are given by (see, e.g., [89]):

\[
\begin{align*}
\omega_1 &= \frac{4\gamma^2 \omega_0}{2 + 2\gamma^2 \vartheta^2 + p^2}, \\
\frac{\Delta \omega}{\omega_1} &= \frac{2}{N}.
\end{align*}
\]  

(6)

Here \( \vartheta \) is the emission angle measured with respect to the \( z \)-direction (see figure 1). Note that the relative width, \( \Delta \omega/\omega_1 \), is inversely proportional to \( N \).

From the general theory of a planar undulator (see, e.g., [61, 89]) it is known, that the magnitude of the undulator parameter, \( p \), defines the number of the harmonics in which the radiation is effectively emitted. In the case \( p < 1 \) the radiation is mainly emitted into the first harmonic. In the limit \( p \gg 1 \) the number of harmonics increases proportionally to \( p^3 \), and, the spectrum of emission acquires the form of a synchrotron radiation rather than an undulator radiation. In [1, 2] it was demonstrated that in a crystalline undulator both possibilities can be realized. However, to stay away from the synchrotron limit, \( p \to \infty \), it is desirable to consider moderate values of the undulator parameters \( p \sim 1 \). This condition, accompanied with the inequality (4) ensures that the spectrum of radiation formed in the undulator will be presented by several powerful, narrow and well-separated peaks. Taking into account that the undulator parameter is proportional to the relativistic factor, which in ultra-relativistic case is much larger than one (typically, \( \gamma = 10^2 \ldots 10^4 \) in crystalline undulators), the condition \( p \sim 1 \) results in a strong inequality \( \lambda \gg a \) [1, 2]. Combined with the large-amplitude regime, which is explained in section 3.1.3, we found that three quantities, \( d, a \) and \( \lambda \), which characterize the crystalline undulator must satisfy the following strong double inequality:

\[ d \ll a \ll \lambda. \]  

(7)

This condition, clearly stated in [1, 2] and used by us in further publications, together with the conditions (2) and (4), are essential for the effective operation of the crystalline undulator. Let us note, that the last inequality in (7) ensures, also, the deformation of a crystal is the elastic one and does not destroy the crystalline structure.

The periodic bending of the crystal can be achieved either dynamically or statically. In [1]-[9] the main focus of our studies was made on the dynamic bending by means of a high-amplitude \( (a \gg d) \) transverse acoustic wave (AW), although the possibility of the static bending was pointed out as well. The general formalism, developed in these papers for the description of the crystalline undulator and the radiation formed in it, does not depend on the method of the crystal bending. The analysis, which was performed in the cited papers, allowed us to establish the ranges of the AW amplitude and frequency within which the conditions (2), (4) and (7) as well as other very important conditions (which are described below in this section) are fulfilled. This was done for different
types of projectiles, wide ranges of their energies and for a variety of crystals. Thus, we
proved theoretically the feasibility of the construction of a micro-undulator by means of a
periodically bent crystal and the possibility of generation of spontaneous and stimulated
photon emission in such a system.

The monochromatic transverse AW (either standing or running) of the large
amplitude, transmitted along a crystallographic plane, allows to achieve a harmonic
shape $y(z) = a \sin\left(\frac{2\pi z}{\lambda}\right)$ for the midplane. One of the possibilities which can be
used to do this is to place a piezo sample atop the crystal and to generate the radio
frequency AW to excite the oscillations. The important feature of the dynamic scheme
of the crystal bending is that the time period of the AW must exceed greatly the time
of flight $\tau = L/c$ of a bunch of particles through the crystal. Then, on the time scale
of $\tau$, the shape of the crystal bending doesn’t change, so that all particles of the bunch
channel inside the same undulator. Thus, for the AW frequencies $\nu \leq 100$ MHz, one
gets $L \ll 70$ cm, which is more than well-fulfilled for any realistic $L$-value (for more
details see [2]).

The idea to create an undulator by transmitting a high-amplitude transverse AW
through a crystalline structure was mentioned for the first time in [26, 30] (see, also,
[59]). However, we state that in these papers neither the feasibility of the crystalline
undulator nor its theory were developed in a complete and adequate form, because not
all essential regimes, conditions and limitations were understood and elucidated. As a
result, few statements and estimates, made in [26, 30], turn out to be not quite correct,
as it is shown in sections 3.1.5 and 3.3. This is the reason why the idea of the crystalline
undulator based on the action of AW has not been attracting attention during the period
from 1980 till late 90’s, when our first publications [1, 2] appeared.

There were other papers [31, 32, 33, 34, 35, 36], published prior to our first works
[1], devoted to the problem of radiation by ultra-relativistic particles (electrons and
positrons) channeled in the crystal in the presence of ultrasonic wave. However, in
all these publications the low-amplitude regime $a < d$ (and, in some cases, the limit
$a \ll d$) was discussed. Therefore, the main focus of these studies was made on various
aspects of the influence of the additional undulator-type oscillations of the particles on
the spectrum of the channeling radiation rather than on the properties of the crystalline
undulator.

The important feature of a dynamically bent crystal by means of an AW is that
it allows to consider an undulator with the parameters $N$ and $p$ varying over a wide
range, which is determined not only by the projectile’s energy but also by the AW
frequency and amplitude. The latter two quantities can easily be tuned resulting in the
possibility of varying significantly the intensity and shape of the angular distribution of
the radiation (for a detailed discussion and an number of concrete examples see [1, 2]).
As demonstrated in [1, 2], the parameters of the crystalline undulator based on the
AW are inaccessible in conventional undulators, where the periodicity of the motion of charged particles is achieved by applying periodic magnetic fields or the laser field [29, 60, 87].

The advantage of the static channel is that its parameters are fixed and thus, the projectile moves along the fixed trajectory as well. To calculate the characteristics of the emitted radiation one needs to know only the number of the periods and the local curvature radius. The disadvantage is that when fixing the number of periods the parameters of the system can be varied only by changing the energy of the particle. This makes the photon generation less tuneable.

For the first time the idea of a static crystalline undulator was proposed in [31, 32] and implied the use of a superlattice made of two constituents, which have different, but close, lattice spacings. However, these papers were devoted only to the study of the low-amplitude regime $a < d$. As mentioned above, in this limit the intensive undulator radiation does not appear.

It is feasible, by means of modern technology (like molecular beam epitaxy or chemical vapor deposition, see the references in [97, 98]), to grow the crystal with its channels been statically bent according to a particular pattern. The usage of static methods to produce periodically bent crystals with $a \gg d$ was initially suggested in [1, 2] and later was discussed in [16], where the idea to construct a crystalline undulator based on graded composition strained layers was proposed. Earlier, the same idea was exploited in [83], where the possibility to create a crystalline lens for the focusing of a beam of charged particles through a bent crystal was discussed. Experimentally, the possibility of a 3 MeV proton beam bending by the Si$_{1-x}$Ge$_x$ graded composition strained layers was demonstrated in [97].

In our papers [10, 11] we developed further the ideas of [16, 83, 97] and demonstrated, for the first time, that it is possible to obtain periodically bent channels with arbitrary shapes $y(z)$.

In particular, it was described in detail, that to obtain a pure sine form of the channel profile, one starts with a pure silicon substrate and adds Si$_{1-x}$Ge$_x$ layers with continuously increasing Ge content. This results in bending of the (110) channels in the direction of the (100) channels. The periodicity of the shape requires the change of the direction of the bending toward the (010) channels. This, in turn, can be achieved by reducing $x$ until it reaches 0. The last (within the first period) crystal layer consists of pure silicon, so that the second period can be built up on top of the first in the same manner. To be captured by the bent channel, the positron beam should be directed towards the (110) channel of the substrate. The crystal strain is strongest after half a period, when the germanium content reaches its maximum. The thickness of the layers corresponding to half a period must be smaller than the critical thickness $h_c$ [97].

In [10, 11] we developed the formalism and carried out the corresponding calculations
which demonstrated, how one can built up a crystal, the channels of which are bent periodically with arbitrary shapes \( y(z) \).

In the papers by Avakian et al. [19, 20], published within the same time interval as our papers [10, 11], somewhat different approach, based on the use of graded composition strained superlattices \( \text{Si}_{1-x}\text{Ge}_x \), for constructing the sine profile of crystal channels was described. In the more recent publication [21] of this group the authors proceeded further with the study of the possibility to construct crystalline undulators by means of gradient crystals. We want to point out that in this paper, which appeared after [10, 11], the authors completely omitted the citation of our papers. Indeed, the publication [11] was ignored at all, whereas the citation of [10] (labeled as Ref. [7] in [21]) was done in an extremely negligent and misleading way which excludes any possibility for a reader to find our paper. Moreover, in the introductory part of their paper (p. 496) the authors of [21], when reviewing ‘...a few proposals [2-5] for constructing micro-crystalline undulators...’, found it possible to include [16] and their own publications [18, 19, 20] in the citation list but to ignore our papers [1]-[11]. This very selective and misleading style of citation one finds in all publications of this group, see Refs. [18, 19, 20, 21]. We find this style as totally unacceptable, especially in the field which has become studied intensively after our first papers [1, 2]. Unfortunately, this is not the only example of a selective citation. It refers also to the papers by Bellucci et al. [22, 23, 24] and to a very recent paper [25], where the authors claimed that ‘the theory of radiation in micro-undulators is developed’ (see the abstract).

The periodic bending of crystallographic planes can also be achieved by making regular defects either in the crystal volume or on its surface [99]. Then, the crystalline planes in the vicinity of the defects become periodically bent. The practical realization of this idea was achieved in [22] by giving periodic micro-scratches to one face of a silicon crystal by means of a diamond blade. Paying tribute to the fact that this was the first attempt of the actual construction of the micro-undulator, we, nevertheless, want to point out that the introductory part of the paper [22] significantly misinterprets the history of the crystalline undulator idea. Moreover, at the stage of preparation of the subsequent paper [24] one of the authors of [22] was aware of our work in this field, but found it possible to himself not to include our papers in the citation list.

3.1.5. Restrictions due to the dechanneling effect and the photon attenuation. As was pointed out in [1, 2, 5, 7, 8], two physical phenomena, the dechanneling effect and the photon attenuation, lead to severe limitation of the allowed values of the crystalline undulator.

If the dechanneling effect is neglected, one may unrestrictedly increase the intensity of the undulator radiation by considering larger \( N \)-values. In reality, random scattering of the channeling particle by the electrons and nuclei of the crystal leads to a gradual
increase of the particle energy associated with the transverse oscillations in the channel. As a result, the transverse energy at some distance from the entrance point exceeds the depth of the interplanar potential well, and the particle leaves the channel. Consequently, the volume density \( n(z) \) of the channeled particles decreases with the penetration distance, \( z \), and, roughly, satisfies the exponential decay law [44]

\[
n(z) = n(0) \exp(-z/L_d),
\]

where \( n(0) \) is the volume density at the entrance. The dechanneling length \( L_d(C) \), which is dependent on the parameter \( C \) (see (2)), depends also on the particle energy, mass and charge, on the parameters of the channel (its width and the distribution of electron charge in the channel), and on the charge of the crystal nuclei.

The dechanneling phenomenon introduces a natural upper limit on the length of a crystalline undulator: \( L \leq L_d(C) \). Indeed, one can consider the limit \( L \gg L_d(C) \). However, as it was demonstrated in [5, 7], the intensity of the undulator radiation in this case is not defined by the expected number of the undulator periods \( L/\lambda \) but rather is formed in the undulator of the effective length \( L_d(C) \). Therefore, it is important to carry out realistic estimates of the quantity \( L_d(C) \) in order to understand what are the limitations on the parameters of the undulator imposed by dechanneling.

In a model approach, presented in [2], we used the following expression for the dechanneling length in a periodically bent crystal:

\[
L_d(C) = (1-C)^2 L_d(0) \tag{9}
\]

where \( L_d(0) \) is the dechanneling length in a straight crystal. For a positively charged projectile a good estimate for \( L_d(0) \) is [5, 44]:

\[
L_d(0) = \gamma \frac{256}{9\pi^2} \frac{M}{Z} \frac{a_{TF}}{r_0} \frac{d}{\Lambda} \tag{10}
\]

where \( r_0 = 2.8 \times 10^{-13} \text{ cm} \) is the electron classical radius, \( Z, M \) are the charge and the mass of a projectile measured in units of elementary charge and electron mass, \( a_{TF} \) is the Thomas-Fermi atomic radius. The quantity \( \Lambda \) stands for a 'Coulomb logarithm' which characterizes the ionization losses of an ultra-relativistic particle in amorphous media (see e.g. [66, 100]):

\[
\Lambda = \begin{cases} 
\ln \frac{2\varepsilon}{I} - 1 & \text{for a heavy projectile} \\
\ln \frac{\sqrt{2\varepsilon}}{\gamma^{1/2}I} - 23/24 & \text{for } e^+ 
\end{cases} \tag{11}
\]

with \( I \) being the mean ionization potential of an atom in the crystal.

In channeling theory it is more common [43, 44] to relate \( L_d(0) \) to the parameter \( pv \) (\( p \) is the momentum of a projectile, and in the ultra-relativistic case \( pv \approx \varepsilon \)) rather than to express it via the relativistic factor \( \gamma \) as in (10). However, explicit dependence
of the dechanneling length on $\gamma$ is more convenient when discussing the parameters of the crystalline undulator and its radiation [2, 5].

The dependences $L_d(0)$ on $\gamma$ for a positron and a proton are illustrated by figure 2. It is seen that in the case of a positron (the solid curves) the quantity $L_d(0)$ varies within $5 \times 10^{-4} \ldots 0.3$ cm for $\gamma$ within $10 \ldots 10^4$. The dechanneling length of a positron with energy within the GeV range ($\gamma \sim 10^4$) does not exceed several millimeters. For a proton (the dashed curves) of the same $\gamma$ the magnitude of $L_d(0)$ is enhanced by the factor $\approx 10^3$. This is largely due to the factor $M \approx 2000$ (see (10)). Some additional correction originates from the difference of the Coulomb logarithms, equation (11). To obtain the values of $L_d(0)$ for a heavy ion one can multiply the dashed curves by the factor $\approx A/Z \approx 2.5$.

![Figure 2](image-url)

**Figure 2.** Dependence of $L_d(0)$ on relativistic factor $\gamma$ calculated from (10) -(11) for a positron (solid lines) and a proton (dashed lines) and for (110) channels in various crystals: the black lines stand for $C$, the red lines - $Si$, the green lines - $Ge$, the blue lines - $W$.

Using the results presented in figure 2 one can write the following estimate of the
dechanneling length for various projectiles [13]:

\[
L_d(C) \sim (1 - C)^2 \gamma \times \begin{cases} 
(2.5 \ldots 5) \times 10^{-5} \text{ cm} & \text{for } e^+ \\
(0.05 \ldots 0.1) \text{ cm} & \text{for } p \\
(0.1 \ldots 0.25) \text{ cm} & \text{for a heavy ion}
\end{cases}
\] (12)

For heavy projectiles formulae (10)-(11) are in good agreement with measured values of \( L_d(0) \) in a wide range of \( \gamma \) [43, 44]. In the case of a positron channeling in [5] we tested equations (9) and (10) against more rigorous calculation of the dechanneling lengths based on the simulation procedure of the positron channeling in straight and periodically bent crystals. The approach developed and described in detail in [5] is based on the simulation of the trajectories and the dechanneling process of an ultra-relativistic positron. This was done by solving the three-dimensional equations of motion which account for: (i) the interplanar potential; (ii) the centrifugal potential due to the crystal bending; (iii) the radiative damping force; (iv) the stochastic force due to the random scattering of projectile by lattice electrons and nuclei. Note that the radiation damping force becomes very significant at sufficiently large energies of positrons (see section 3.2 for more details). Simultaneously with simulating the trajectories of the channeled particles, we calculated the total spectrum of the radiation, including its undulator and channeling parts. This was done for all trajectories, including those which corresponded to the over-barrier particles. Such a study was carried out for the first time in [4, 5]. We point out that the referencing to these papers of ours was not made in the later publications by Bellucci et al [22, 23, 24], where the development of a similar approach was mentioned.

We analyzed the dependence of \( L_d \) on the energy of the projectile, the type of crystal and crystallographic plane, the parameter \( C \) and the ratio \( a/d \) (see (2) and (3)). Some results of these calculations, corresponding to 5 GeV positrons channeling along the (110) plane in a Si crystal, are presented in figures 3 and 4.

In table 1 the results of our calculation [5] of the dechanneling lengths are compared with those estimated from (9). It can be concluded that the approximate formula (9) adequately reproduces the values of \( L_d(C) \) for a positron in the range \( C = 0 \ldots 0.2 \). As noted in [5], some discrepancy between the calculated, \( L_d^c \), and the estimated \( L_d \), dechanneling lengths can be attributed to the fact that the quantity \( L_d(0) \), defined in (10), was obtained by using the Lindhard planar potential (see, e.g., [39]) and, additionally, several simplifying assumptions were made concerning the electron charge distribution in the channel (see [44]). On the other hand, in [5] the numerical procedure was based on the Molière approximation for both the potential and the charge distribution. More discussion on the comparison of the model (9) and the data obtained numerically one finds in [5]. In the cited paper similar calculations were performed also for Ge and W crystals.
Figure 3. The calculated dependences $n(z)/n(0)$ versus penetration distance $z$ for 5 GeV positrons channeling along the (110) in Si crystal for various values of the parameter $C$ [5]. The $a/d$ ratio equals 10. The interplanar distance is $d = 1.92$ Å.

Combining conditions (2), (4), (7) and $L \leq L_d(C)$ one can find, fixing a crystal and energy $\varepsilon$, the allowed ranges of the parameters $a$ and $\lambda$ [2, 4, 5]. Figure 5 illustrates this for the case of $\varepsilon = 0.5$ GeV positrons planar channeling in Si along the (110) crystallographic planes. The diagonal straight lines correspond to various values of the parameter $C$. The curved lines correspond to various values of the number of undulator periods $N$ related to the dechanneling length $L_d$ through $N = L_d/\lambda$. The horizontal lines mark the values of the amplitude equal to $d$ and to $10d$, where $d = 1.92$ Å is the interplanar spacing for the (110) planes in Si. The vertical line marks the value $\lambda = 2.335 \times 10^{-3}$ cm, for which the spectra (see section 4) were calculated.

Figure 5 illustrates that the restrictions, imposed by the dechanneling effect on the length of a crystalline undulator, are very severe, especially in the case of a projectile positron. Therefore, the influence of the dechanneling must be studied very accurately in
order to produce realistic predictions on the parameters of the crystalline undulator and the undulator emission. For the first time this comprehensive analysis was performed in [1, 2, 4, 5]. In the earlier papers the dechanneling effect was either completely ignored (see [30, 31, 32, 36]) or its role was estimated erroneously, as in [26]. In particular, in the latter paper, where the authors did not explicitly distinguish the electron and positron channeling, the following comment was made: ‘We recall that the channeling length in centimeters is approximately $L_0 = \varepsilon \, (\text{GeV})\ldots$’ (see page 650 in [26]). This is absolutely incorrect. Indeed, as it is seen from figures 2-5 and table 1, the dechanneling lengths for a positron are, at least, an order of magnitude less, not mentioning the electron case, when the values of $L_d$ are even more lower. As a result, the estimated values of $\lambda$, $a$ and $\omega$, which were suggested in [26, 30], for a crystalline undulator based on a $\varepsilon = 1 \, \text{GeV}$ positron channeling in Si, lay far away from the regions allowed for feasible crystalline
Table 1. Dechanneling lengths for 5 GeV positron channeling along the (110) planes for various crystals and for various values of the parameter \( C \). The \( a/d \) ratio equals 10 except for the case \( C = 0 \) (the straight channel). The quantity \( L_d^c \) presents the results of our calculations published in [5]. \( N_d^c = L_d^c / \lambda \) is the corresponding number of the undulator periods, \( L_d^e \) is the dechanneling length estimated according to (9)-(11), \( N_d^e = L_d^e / \lambda \). For each \( C \) the value of \( \lambda \) was derived from (2). Other parameters are: \( \hbar \omega_1 \) is the energy of the first harmonic of the undulator radiation for the forward emission, \( p \) is the undulator parameter.

| \( C \) | \( \lambda \) | \( L_d^e \) | \( L_d^c \) | \( N_d^e \) | \( N_d^c \) | \( \omega_1 \) | \( p \) |
|-------|--------|--------|--------|--------|--------|--------|--------|
| 0.00  | -      | 0.312  | 0.463  | -      | -      | -      | -      |
| 0.05  | 100.9  | 0.281  | 0.430  | 25     | 39     | 1.38   | 1.08   |
| 0.10  | 77.1   | 0.253  | 0.393  | 32     | 51     | 1.42   | 1.53   |
| 0.15  | 63.0   | 0.225  | 0.321  | 35     | 51     | 1.37   | 1.87   |
| 0.20  | 54.5   | 0.200  | 0.223  | 36     | 41     | 1.31   | 2.16   |
| 0.25  | 48.8   | 0.175  | 0.170  | 35     | 35     | 1.24   | 2.42   |
| 0.30  | 44.5   | 0.153  | 0.102  | 34     | 23     | 1.18   | 2.65   |

Figure 5. The range of parameters \( a \) and \( \lambda \) for a bent Si(110) crystal at \( \varepsilon = 500 \) MeV.
undulators. In section 3.3 below we discuss in more detail the parameters suggested in [26, 30].

The propagation of photons emitted in a crystalline undulator is strongly influenced by a variety of processes occurring in a crystal. These are the atomic and the nuclear photoeffects, the coherent and incoherent scattering on electrons and nuclei, the electron-positron pair production (in the case of high energy photons). All these processes lead to the decrease in the intensity of the photon flux as it propagates through the crystal:

$$I(z) = I(0) \exp(-z/L_a(\omega))$$  \hspace{1cm} (13)

where $I(0)$ is the initial intensity, $I(z)$ is that which remains after traversal of the distance $z$. A quantitative parameter, which we introduced in (13) to account for all these effects, can be called the attenuation length, $L_a(\omega)$. It is related to the mass attenuation coefficient $\mu(\omega)$ as $L_a(\omega) = 1/\mu(\omega)$ [101, 102, 103]. Thus, $L_a(\omega)$ defines the scale within which the intensity of a photon flux decreases by a factor of $e$.

The mass attenuation coefficients are tabulated for all elements and for a wide range of photon frequencies. Figure 6 represents the dependences $L_a(\omega)$ for a variety of crystals over the broad range of photon energies. For photon energies $30 \text{ eV} < \hbar \omega < 1 \text{ keV}$ the mass attenuation coefficients (which are mainly due to the atomic photoeffect) can be
found in [101]. The corresponding values of \(L_a(\omega)\) are lower than the minimum values in figure 6. However, for \(\hbar \omega \ll I_0\) (where \(I_0\) is the ionization potential of the crystal atom, and \(I_0 \leq 10\) eV for most of crystals) there is no photoabsorption and, therefore, the attenuation is defined solely by elastic photon scattering, i.e. is comparatively weak. Using these arguments and the data presented in figure 6 can be summarized in the form convenient for a quick estimation of \(L_a(\omega)\):

\[
L_a(\omega) \approx \begin{cases} 
\infty & \text{for } \hbar \omega \ll I_0 \leq 10\text{ eV} \\
\ll 10^{-2}\text{ cm} & \text{for } \hbar \omega = 10^{-2}\ldots10\text{ keV} \\
= 0.01\ldots10\text{ cm} & \text{for } \hbar \omega > 10\text{ keV}
\end{cases}
\tag{14}
\]

The quantities \(L_d(C)\) and \(L_a(\omega)\) introduced above in this section define the effective upper limit of the crystal length \(L\) which can be used to calculate the number of undulator periods \(N\) [2]:

\[
L < \min \{L_d(C), L_a(\omega)\} \tag{15}
\]

3.2. Energy losses and shape of crystalline undulator.

The coherence of the radiation, emitted from similar parts of the trajectory of a particle in the crystalline undulator, takes place if the energy of the channeling particle does not change noticeably during with the penetration distance, at least, on the scale of the dechanneling length. For ultra-relativistic projectiles the main source of energy losses are the radiative losses [61, 66]. Therefore, it is important to establish the range of energies of channeling particles for which the parameters of undulator radiation formed in a perfect periodic crystalline structure are stable. For the first time, the importance of the restrictive role of the radiative losses was realized in [2]. Later, in [3], we carried out a comprehensive theoretical and numerical analysis of the radiative loss of energy, \(\Delta \epsilon\), of ultra-relativistic positrons channeling in crystalline undulators. General formalism described in [3] is applicable for the calculation of the total losses, which account for the contributions of both the undulator and the channeling radiation. We analyzed the relative importance of the two mechanisms for various values of \(a, \lambda,\) and \(\varepsilon\). We established the ranges of energies for positrons, in which relative radiative losses, \(\Delta \epsilon/\varepsilon\), are small (lower than 1 per cent) for a variety of crystals and crystallographic planes. The results of these calculations are illustrated in figure 7, where the dependences \(\Delta \epsilon/\varepsilon\) versus relativistic factor \(\gamma\) are presented for a positron channeling in (110) channel of Si and for several values of the parameter \(C\). The analysis, performed in [3] for the crystals LiH, C, Si, Ge, Fe and W, demonstrated that for a perfectly shaped crystalline undulator (i.e., the one in which the midplane is modulated as \(y(z) = a \sin(2\pi z/\lambda)\), see figure 7) the radiative energy losses become large if the initial energy of the positron bunch is \(\varepsilon > 10\) Gev. For lower energies of positrons, when the relativistic factor satisfies
the inequality
\[ \gamma \leq 10^4, \tag{16} \]
the radiative losses are small, \( \Delta \varepsilon < 0.01 \varepsilon \).

The condition (16) establishes the upper limit of positron energies which is meaningful to use to generate the stable undulator radiation in the ideal crystalline undulator. In the high-energy regime, when \( \varepsilon > 10 \) Gev, the gradual decrease of the positron energy strongly influences the stability of the parameters of the undulator radiation. However, in [10, 11] we demonstrated, for the first time, that the coherence and the monochromaticity of the undulator radiation in the high-energy regime can be maintained if the amplitude and the period of the bent channel are made dependent on the penetration distance \( z \), i.e. \( a = a(z) \), and \( \lambda = \lambda(z) \). We derived the equations for these dependences and found the corresponding solutions. The method of preparation of the crystals the midplanes of which are shaped as \( y(z) = a(z) \sin(2\pi z/\lambda(z)) \) was described in detail. This method is based on the crystal growing by means of molecular

\[ \]
beam epitaxy or chemical vapor deposition of a crystal with graded strained layers \[97\]. As an example, we considered a pure silicon substrate on which a Si_{1-x}Ge_{x} layers are added. Here x = x(z) is the germanium content in the layer, and it is varied during the growing process in order to achieve the desired shape of the channels.

3.3. **Feasibility of a crystalline undulator**

Let us summarize all the conditions which must be fulfilled in order to treat a crystalline undulator as a feasible scheme for devising on its basis new sources of electromagnetic radiation. These conditions are:

\[
\begin{align*}
C &= \left(\frac{2\pi}{qU_{\max}'}\right)^2 \frac{a}{\lambda^2} \ll 1 \quad \text{stable channeling} \\
\frac{d}{a} &\ll 1 \ll \lambda \quad \text{large-amplitude regime} \\
N &\approx \frac{L}{\lambda} \gg 1 \quad \text{large number of undulator periods} \\
L &\leq \min\left[\frac{C}{\lambda^3}, \frac{1}{\omega}\right] \quad \text{account for the dechanneling and photon attenuation} \\
\frac{\Delta \varepsilon}{\varepsilon} &\ll 1 \quad \text{low radiative losses}
\end{align*}
\]

As a supplement to this system one must account for the formulae (5) and (6), which define the parameters of the undulator and the frequencies of the undulator radiation. Provided all conditions (17) are met for a positively charged particle channeling through a periodically bent crystal then

- within the length L the particle experiences stable planar channeling between two adjacent crystallographic planes,
- the characteristic frequencies of the undulator radiation and the ordinary channeling radiation are well separated,
- the intensity of the undulator radiation is essentially higher than that of the ordinary channeling radiation,
- the emission spectrum is stable towards the radiative losses of the particle.

For each type of the projectile and its energy, for a given crystal and crystallographic plane the analysis of the system (17) is to be carried out in order to establish the ranges of a, \lambda and \omega within which the operation of the crystalline undulator is possible.

Most of these important conditions were realized and carefully investigated for the first time in \[1\]-[11], where the realistic numerical calculations of the characteristics of the radiation formed in crystalline undulator were performed as well. We consider the set of analytical and numerical results obtained by us in the cited papers as a proof of the statement that the scheme illustrated in figure 1 can be transformed from the stage of a purely academic idea up to an observable effect and an operating device.

For a positron channeling, in particular, we found the the optimal regime in which
the spontaneous undulator radiation is most stable and intensive, and demonstrated that this regime is realistic. This regime is characterized by the following ranges of the parameters: \( \gamma = (1 \ldots 10) \times 10^3, a/d = 10 \ldots 50, C = 0.01 \ldots 0.2 \), which are common for all the crystals which we have investigated. These ranges ensure that the energy of the first harmonic \( \omega_1 \) (see (6)) lies within the interval 50 \ldots 150 keV and the length of the undulator can be taken equal to the dechanneling length because of the inequality \( L_d(C) < L_a(\omega) \).

The importance of exactly this regime of operation of the crystalline undulator was later realized by other authors. In particular, in recent publications by Bellucci et al [22, 23, 24], where the first practical realization of the crystalline undulator was reported, the parameters chosen for a Si crystal were as follows: \( \varepsilon = 0.5 \ldots 0.8 \) GeV for a positron (i.e. \( \gamma = (1 \ldots 1.6) \times 10^3 \)), \( a = 20 \ldots 150 \) Å (i.e. \( a/d = 10 \ldots 80 \)), \( L = L_d \). These are exactly the values for which we predicted the strong undulator effect. However, in these papers, where the authors mention all the conditions (17) and stress their importance, there is no proper reference to our works. Instead, our paper [2], labeled as Ref. [10] in [22], was cited as follows: ‘With a strong world-wide attention to novel sources of radiation, there has been broad theoretical interest [4-12] in compact crystalline undulators...’ (page 034801-1 in the cited paper). This was the only referencing to the paper [2], in which we clearly formulated, for the first time, the conditions (17) and carried out a detailed analysis aimed to prove why this regime is most realistic. None of it was done in the papers [26, 30, 31, 32, 36] (labeled in [22] as Refs. [4],[6],[7],[8] and [9], correspondingly). Moreover, we state that one will fail to construct a crystalline undulator basing on the estimates presented in [26, 30, 31, 32, 36].

In what follows we carry out critical analysis of the statements and the estimates made in the cited papers.

Historically, the paper by Kaplin et al [26] was the first one, where the idea of a crystalline undulator based on the action of the transverse acoustic wave was presented. However, a number of ambiguous or erroneous statements makes it impossible to accept the thesis that the concept of a crystalline undulator was correctly described in this two-page paper. To be precise in our critics, below we use the exact citations taken from the English edition of [26]. In the citations the italicizing is made by us.

Our first remark concerns the type of a projectile which the authors propose to use in the undulator. The first paragraph of the paper contains:

‘Radiation by relativistic electrons and positrons, which occurs during channeling in single crystals, has been observed experimentally and is being extensively studied at the present time1-4.’

This is the only place in the text where the term ‘positron’ is used. In the rest of the paper the projectile is called either ‘a particle’, or a ‘relativistic electron’ as in one before the last paragraph of the paper (page 651). Thus, it is absolutely unclear to the reader,
which particle is to be used. For a positron it is possible to construct an undulator, however if an electron is considered, then the rest of the paper does not make any sense.

The concept of a periodically bent crystal and its parameters is formulated as follows (page 650, right column):

'Stil higher intensity can be achieved by using instead of a uniformly curved crystal one deformed in such a way that the radiation from different portions of the particle trajectory adds coherently. This can be accomplished by giving a crystalline plate a wavelike shape in such a way that the sagitta $A$ satisfies the relation $4A\gamma/\lambda_0 < 1$ in relation to the quarter period $\lambda_0$ of the bending. For large values of the dechanneling depth $L_0$ this will provide a high radiated power from the crystalline undulator (wiggler). For rather thin crystalline plates with a simple bend one can produce $\lambda_0 \sim 4 \, \text{mm}$ ...

We recall that the channeling depth in centimeters is approximately $L_0 = E \,(\text{GeV})$, as follows from experiments.'

Note, that no citation is made when referring to the experiments which result in $L_0 \,(\text{cm}) = E \,(\text{GeV})$. For a positron (see section 3.1.5) this relation overestimates the dechanneling length by more than an order of magnitude, for an electron it is even farther from the reality. Therefore, the idea to construct an undulator for a positron with the period $\lambda = 4\lambda_0 = 1.6 \, \text{cm}$ is absolutely unrealistic.

The parameters of the undulator based on the action of the acoustic wave are presented in the left column on page 651:

'To obtain radiation in the optical region in a transparent crystal or to generate very hard $\gamma$ rays, it has been proposed to use ultrasonic vibrations to deform the crystal lattice... For example, one can obtain $\gamma$ rays with the energy up to $\omega = 0.14 - 14 \, \text{MeV}$ for $\varepsilon = 1 \, \text{GeV}$ and $\lambda_0 = 10 - 0.1 \, \text{µm}$. '

Note, that none of the following characteristics, - the type of the projectile, the crystal, the acoustic wave amplitude (in our notations 'sagitta $A$' is called 'amplitude $a$'), are specified. Assuming that the positron channeling is implied, let us analyze the above mentioned values from the viewpoint of the condition for a stable channeling, equation (2) (see also (17)). The parameter $C$ can be written in the form: $C \approx 40/\lambda^2 (\varepsilon d/qU'_{\text{max}}) (a/d)$ with $\lambda$ in $\mu$m, $\varepsilon$ in GeV, $d$ in $\text{Å}$, and $qU'_{\text{max}}$ in GeV/cm. Let us estimate the ratio $a/d$ for the range $\lambda = 4\lambda_0 = 0.4 - 40 \, \mu$m and for (110) planes in Si and W, for which $d_{\text{Si}} = 1.92 \, \text{Å}$, $d_{\text{W}} = 2.24 \, \text{Å}$, $(qU'_{\text{max}})_{\text{Si}} = 6.9 \, \text{Gev/cm}$, $(qU'_{\text{max}})_{\text{W}} = 57 \, \text{Gev/cm}$ [61]. For $\varepsilon = 1 \, \text{GeV}$ and the lowest $\lambda$-value one gets $C \approx 250(\varepsilon d/qU'_{\text{max}}) (a/d)$, which means that, for both crystals, to satisfy the condition $C \ll 1$ it is necessary to consider $a \ll d$. Thus, this is a low-amplitude regime, for which the intensity of the undulator radiation is negligibly small. The upper limit of $\lambda$ is more realistic to ensure the condition $C \ll 1$ for the amplitudes $a \gg d$. However, this analysis is not performed by the authors.

Our final remark concerns the statement (the last paragraph in the left column on
A lattice can be deformed elastically up to $A = 1000$ Å... This is true, but when referring to the crystalline undulator with the amplitude $a = 10^{-5}$ cm one has to supply the reader (and a potential experimentalist) with the estimates of the corresponding values of $\lambda$ and $N$. Let us carry out these estimates (note, this was not done in the paper). The channeling condition (2) can be written as follows:

$$\lambda = \frac{\lambda_{\min}}{\sqrt{C}} > \lambda_{\min}, \quad (18)$$

where $\lambda_{\min}$ is the absolute minimum of $\lambda$ (for given $a$, $\varepsilon$ and a crystal) which corresponds to $C = 1$ (i.e. to the case when the dechanneling length $L_d(C)$ effectively equals to zero, see (9)). It is equal to

$$\lambda_{\min} = 2\pi \sqrt{a} \left( \frac{\varepsilon}{q U''_{\max}} \right)^{1/2}. \quad (19)$$

For a 1 GeV positron channeling in Si and W crystals along the (110) plane, which is plane bent periodically with $a = 10^{-5}$ cm, the values of $\lambda_{\min}$ are: $7.6 \times 10^{-3}$ cm for Si and $7.5 \times 10^{-3}$ cm for W. These values already exceed the upper limit of 40 µm mentioned by Kaplin et al. Choosing the length of the crystal to be equal to the dechanneling length and using equation (9) to estimate $L_d(C)$ one estimates the number of undulator periods $N = L_d(C)/\lambda = C^{1/2}(1 - C)^2 L_d(0)/\lambda_{\min}$. The largest value of $N$ is achieved when $C = 0.2$, giving $C^{1/2}(1 - C)^2 \approx 0.29$. Hence, $N \leq N_{\max} = 0.29 L_d(0)/\lambda_{\min}$. Using formulae (10) and (11) one calculates the dechanneling lengths in straight crystals: $L_d(0) = 6.8 \times 10^{-2}$ cm $L_d(0) = 3.9 \times 10^{-3}$ cm for W. Finally, one derives that the ‘undulator’ suggested in the cited paper contains $N \leq 2.6$ periods in the case of Si, and $N \leq 1.5$ for a tungsten crystal.

Thus, because of the inconsistent and ambiguous character of the paper [26] we cannot agree with the statement, the feasibility of a crystalline undulator was demonstrated in this paper in a manner, sufficient to stimulate the experimental study of the phenomenon.

None of the essential conditions, summarized in (17), were analyzed in [26]. For the first time such an analysis was carried out in [1, 2] and developed further in our subsequent publications. In this connection we express disagreement with utterly negligent and unbalanced style of citation adopted by Avakian et al in [18] and other publications [19, 20, 21] by this group, and by Bellucci et al [22, 23, 24].

Much of our critics expressed above in connection with [26] refers also to the paper by Baryshevsky et al [30]. The main point of ours is: the concept of the crystalline undulator based on the action of an acoustic wave was not convincingly presented. From the text of the paper it is not at all clear what channeling regime, axial or planar, should be used. The only reference to the regime is made in last part of the paper, on
page 63, which is devoted to the quantum description of the spectral distribution of the undulator radiation. This part starts with the sentence: 'Let us consider, for example, planar channeling'. The question on whether the axial channeling is also suitable for a crystalline undulator is left unanswered by the authors. Neither is it clearly stated what type of a projectile is considered. Indeed, in all parts of the paper, where the formalism is presented, the projectile is called as a 'particle'. The reference to a positron is made in the introductory paragraph, where the effect of channeling radiation is mentioned, and on page 62, where the numerical estimates of the intensity of the undulator radiation are presented. The limitations due to the dechanneling effect are not discussed. As a consequence, the regime, for which the estimates are made, hardly can be called the undulator one. Indeed, on page 62 the ratio of the undulator to the channeling radiation intensities is estimated for a 1 GeV positron channeled in Si (*presumably*, the planar channeling is implied). The amplitude of the acoustic wave (labeled as \( r_{\perp}^* \)) is chosen to be equal to \( 10^{-5} \) cm. The period \( \lambda \) is not explicitly written by the authors. However, they indicate the frequency of the acoustic wave, \( f = 10^7 \) s\(^{-1} \). Hence, the reader can deduce that \( \lambda = v/f = 4.65 \times 10^{-2} \) cm, if taking the value \( v = 4.65 \times 10^5 \) cm/s for the sound velocity in Si [104]. The values of \( \varepsilon, a \) and \( \lambda \), together with the maximal gradient of the interplanar field \( (qU'_{\text{max}})_{\text{Si}} = 6.9 \) GeV/cm [61], allows one to calculate \( C = 2.65 \times 10^{-2} \) (see (17)), and, consequently, to estimate the dechanneling length \( L_d(C) = 6.47 \times 10^{-2} \) cm. As a result, we find that the number of the undulator periods in the suggested system is \( N = 1.4 \), which is not at all \( N \gg 1 \) as it is implied by the authors (this is explicitly accent by them in the remark in the line just below their equation (2) on page 62). Another point of critics is that the classical formalism, used to derive the equation (2), is applicable only for the dipole case, i.e. when the undulator parameter is small, \( p^2 \ll 1 \). However, the estimates which are made refer to a strongly non-dipolar regime: \( p^2 = (2\pi\gamma a/\lambda)^2 = 7.3 \). As a consequence, the estimate of the energy of the largest emitted harmonic, carried out by the authors on page 62, is totally wrong. Exactly in their regime the harmonics with low number will never emerge from the crystal due to the photon attenuation.

Papers [31, 32, 36] considered only the case of small amplitudes, \( a \ll d \), when discussing the channeling phenomenon in periodically bent crystalline structures. As a result, in [31, 36] the attention was paid not to the undulator radiation (the intensity of which is negligibly small in the low-amplitude regime, see section 4), but to the influence of the periodicity of the channel bending on the spectrum of the channeling radiation. Similar studies were carried out in [33, 34, 35, 94, 95]. These effects are irrelevant from the viewpoint of the crystalline undulator problem discussed here. Another issue, which we want to point out, is that the authors of [31, 36] did not distinguish between the cases of an electron and a positron channeling. The limitations due to the dechanneling effect were not discussed. In [32] the idea of using a superlattice (or a crystal bent by
means of a low-amplitude acoustic wave) as an undulator for a free electron laser was explored. The main focus was made on the regime when the undulator radiation is strongly coupled with the ordinary channeling radiation. This regime is different from the subject of the present discussion. The essential role of the large-amplitude regime of the crystalline undulator was not demonstrated in these papers.

3.4. Quasi-classical description of the crystalline undulator problem

The important issue of the study of the radiation formed in a crystalline undulator concerns the choice of the formalism used to describe the phenomenon. This point could have been regarded as merely a technical one but it is not so. Contrary to the case of conventional undulators, based on the action of magnetic fields, the physics of crystalline undulators is, basically, a newly arisen field of research. Therefore, any theoretical study of the effect, which pretends to go a bit farther than purely academic considerations, must contain a great part of numerical analysis and numerical data on the basis of which real experimental investigations can be envisaged. In turn, to obtain the reliable data it is necessary to choose a theoretical tool which allows, on the one hand, to treat adequately all principal physical phenomena involved into the problem, and, on the other hand, to carry out numerical analysis of the obtained analytical expressions. In the crystalline undulator problem there are three basic phenomena which must be accurately described. These are: (i) the motion of an ultra-relativistic particle in an external (strong) field, (ii) the process of photon emission by the particle, (iii) the problem of the radiative recoil, which results in the radiative losses of the projectile.

The most rigorous approach to tackle (theoretically) these problems is the one based on quantum electrodynamics (see, e.g. [66]), where the amplitude of the process is described in terms of a single free-free matrix element of the photon emission taken between the initial and final states of an ultra-relativistic particle in the interplanar field. The main (technical) limitation of this approach appear due to the fact that in the ultra-relativistic limit, when \( \gamma \gg 1 \) the number of the energy levels related to the transverse motion in the effective potential increases significantly. Consequently, an accurate description, i.e. numerical calculations, of the particle dynamics becomes a formidable task. It is exactly this sort of difficulties which resulted in the absence of any numerical analysis and data for the emission spectra in the papers [30, 31, 36], where the radiation formed in the crystalline undulator was treated in terms of quantum electrodynamics.

Another option is to study the problem within the framework of classical electrodynamics (see, e.g. [90]). This method was used in the early works [26, 32], and was later applied in [14, 18, 25]. In connection to the crystalline undulator problem the purely classical description is valid if (i) the characteristic energy of the projectile
in an external field, $\hbar \tilde{\omega}$, is much less than its total energy, $\varepsilon = m\gamma c^2$, and, (ii) the radiative recoil, i.e. the change of the projectile energy due to the photon emission, is neglected. The first condition is well fulfilled in the case of the ultra-relativistic particle channeling in a crystal. Indeed, typical values of $\hbar \tilde{\omega}$ are equal, in the order of magnitude, to the depth of the interplanar potential well. The latter varies from several eV, for the crystals of made of light elements (e.q. LiH crystal, see [105, 106]), up to $10^2$ eV for heavy crystals like W (see, e.g., [44]). Therefore, $\hbar \tilde{\omega}/\varepsilon \ll \gamma^{-1} \ll 1$. The role of the radiative recoil is described by the ratio $\hbar \omega/\varepsilon$. Purely classical description implies that $\hbar \omega \rightarrow 0$. Although practical implementation of the classical treatment is comparatively simple, in application to the crystalline undulator it is fully approved in the so-called dipole-limit (see e.g. [60, 61]), when the undulator parameter is small, $p < 1$, and all the undulator emission occurs in the fundamental harmonic. Such an assumption leads to a considerable narrowing of the parameters of the crystalline radiation and, also, disregards the possibility to generate the emission in higher harmonics. Another disadvantage of the approach based on classical mechanics and electrodynamics is that it fails when the energy of the projectile becomes sufficiently large. For a positron this means $\varepsilon \leq 10$ GeV. For this energies the probability of the emission of the gamma-quanta of energy $\hbar \omega \leq \varepsilon$ via the mechanism of the channeling radiation cannot be neglected.

The third approach, which can be used in studying of the radiative processes occurring in external fields in ultra-relativistic domain, was developed by Baier and Katkov in the late 1960s’ [107], and was called by the authors ‘the operator quasi-classical method’. The details of this formalism can be found also in [61, 66].

From the practical viewpoint, the advantage of the quasi-classical method is that it justifies the classical description of the motion of an ultra-relativistic particle in an external field (i.e., the use of the trajectories rather than the wavefunctions), and, simultaneously, takes into account the effect of the radiative recoil. Thus, the quasi-classical approach neglects the the terms $\hbar \tilde{\omega}/\varepsilon$, but it explicitly takes into account the quantum corrections due to the radiative recoil in the whole range of the emitted photon energies, except for the extreme high energy tail of the spectrum. Using this method the spectra of photons and electron-positron pairs in linear crystals were successfully described [61]. It was also applied to the problem of a synchrotron-type radiation emitted by an ultra-relativistic projectile channeling in a non-periodically bent crystal [64, 65].

In [1]-[11] this general formalism of Baier and Katkov was used for theoretical and numerical description of the spectral and angular distribution of the crystalline undulator radiation, total photon emission spectra and the radiative energy losses of positrons channeling through the periodically bent crystal.
4. **Crystalline undulator radiation**

To illustrate the crystalline undulator radiation phenomenon, let us consider the spectra of spontaneous radiation emitted during the passage of positrons through periodically bent crystals. The results presented below clearly demonstrate the validity of the statements made in [1, 2, 3, 4, 5] and summarized in section 3.3 above, that the properties of the undulator radiation can be investigated separately from the ordinary channeling radiation.

![Graph showing spectral distribution of the total radiation emitted in the forward direction](image)

**Figure 8.** Spectral distribution of the total radiation emitted in the forward direction ($\vartheta = 0^\circ$) for $\varepsilon = 0.5$ GeV ($\gamma \approx 10^3$) positron channeling in Si along the (110) crystallographic planes calculated at different $a/d$ ratios. Other parameters are given in the text. The crystal length is $L = 3.5 \times 10^{-2}$ cm.

The calculated spectra of the radiation emitted in the forward direction (with respect to the $z$-axis, see figure 1) in the case of $\varepsilon = 0.5$ GeV planar channeling in Si along
(110) crystallographic planes and for the photon energies from 45 keV to 1.5 MeV are presented in figures 8 [4]. The ratio $a/d$ was varied within the interval $a/d = 0 \ldots 10$ (the interplanar spacing is 1.92 Å). The case $a/d = 0$ corresponds to the straight channel. The period $\lambda$ used for these calculations equals to $2.33 \times 10^{-3}$ cm. The number of undulator periods and crystal length were fixed at $N = 15$ and $L = N \lambda = 3.5 \times 10^{-2}$ cm. These data are in accordance with the values allowed by (17) (see also figure 5).

The spectra correspond to the total radiation, which accounts for the two mechanisms, the undulator and the channeling. They were calculated using the quasiclassical method [61, 107]. Briefly, to evaluate the spectral distribution the following procedure was adopted (for more details see [4, 8, 11]). First, for each $a/d$ value the spectrum was calculated for individual trajectories of the particles. These were obtained by solving the relativistic equations of motion with both the interplanar and the centrifugal potentials taken into account. We considered two frequently used [39] analytic forms for the continuum interplanar potential, the harmonic and the Molière potentials calculated at the temperature $T = 150$ K to account for the thermal vibrations of the lattice atoms. The resulting radiation spectra were obtained by averaging over all trajectories. Figures 8 correspond to the spectra obtained by using the Molière approximation for interplanar potential.

The first graph in figure 8 corresponds to the case of zero amplitude of the bending (the ratio $a/d = 0$) and, hence, presents the spectral dependence of the ordinary channeling radiation only. The asymmetric shape of the calculated channeling radiation peak, which is due to the strong anharmonic character of the Molière potential, bears close resemblance with the experimentally measured spectra [108]. The spectrum starts at $\hbar \omega \approx 960$ keV, reaches its maximum value at 1190 keV, and steeply cuts off at 1200 keV. This peak corresponds to the radiation into the first harmonic of the ordinary channeling radiation (see e.g. [53]), and there is almost no radiation into higher harmonics.

Increasing the $a/d$ ratio leads to the modifications in the radiation spectrum. The changes which occur are: (i) the lowering of the channeling radiation peak, (ii) the gradual increase of the intensity of undulator radiation due to the crystal bending.

The decrease in the intensity of the channeling radiation is related to the fact that the increase of the amplitude $a$ of the bending leads to lowering of the allowed maximum value of the channeling oscillations amplitude $a_c$ (this is measured with respect to the centerline of the bent channel) [3, 44]. Hence, the more the channel is bent, the lower the allowed values of $a_c$ are, and, consequently, the less intensive is the channeling radiation, which is proportional to $a_c^2$ [61].

The undulator radiation related to the motion of the particle along the centerline of the periodically bent channel is absent in the case of the straight channel (the graph $a/d = 0$), and is almost invisible for comparatively small amplitudes (see the graph
for $a/d = 1$). Its intensity, which is proportional to $(a/d)^2$, gradually increases with the amplitude $a$. For large $a$ values ($a/d \sim 10$) the intensity of the first harmonic of the undulator radiation becomes larger than that of the channeling radiation. The undulator peak is located at much lower energies, $\hbar \omega^{(1)} \approx 90$ keV, and has the width $\hbar \Delta \omega \approx 6$ keV which is almost 40 times less than the width of the peak of the channeling radiation.

![Figure 9](image)

**Figure 9.** Comparison of different approximations for the interplanar potentials used to calculate the total radiative spectrum in vicinity of the first harmonic of the undulator radiation. The ratio $a/d = 10$, other parameters as in figure 8.

It is important to note that the position of sharp undulator radiation peaks, their narrow widths, and the radiated intensity are, practically, insensitive to the choice of the approximation used to describe the interplanar potential. In addition, provided the first two conditions from (17) are fulfilled, these peaks are well separated (in the photon energy scale) from the peaks of the channeling radiation. Therefore, if one is only interested in the spectral distribution of the undulator radiation, one may disregard the channeling oscillations and to assume that the projectile moves along the centerline of the bent channel [1, 2]. This statement is illustrated by 9 [4] where we compare the results of different calculations of the radiative spectrum in vicinity of the first harmonic of the undulator radiation in the case $a/d = 10$. All parameters are the same
as in figure 8. The filled and open circles represent the results of evaluation of the total spectrum of radiation accompanied by numerical solution of the equations of motion for the projectile within the Molière (filled circles) and the harmonic (open circles) approximations for the interplanar potential. The solid line corresponds to the undulator radiation only. For the calculation of the latter it was assumed that the trajectory of a positron, \( y(z) = a \sin(2\pi \lambda / z) \), coincides with the centerline of the bent channel (see figure 1). It is clearly seen that the more sophisticated treatment has almost no effect on the profile of the peak obtained by means of simple formulae describing purely undulator radiation [1, 2]. Moreover, the minor changes in the position and the height of the peak can be easily accounted for by introducing the effective undulator parameter and (in the case of the harmonic approximation) the effective undulator amplitude [3].

![Graph](image)

**Figure 10.** Comparison of the experimentally measured spectrum [67, 108] and the results of the calculation [8, 11] for 6.7 GeV positrons in Si(110).

To check the numerical method, which was developed in [4] for the calculation of the total emission spectrum of ultra-relativistic positrons in a crystalline undulator, we calculated the spectrum of the channeling radiation for 6.7 GeV positrons in Si(110) integrated over the emission angles. Figure 10 shows the experimental data [67, 108] and the results of our calculations [8, 11] normalized to the experimental data at the right wing of the spectrum. The height of the first harmonic is overestimated in our
calculations. The calculations performed in [67] gave a similar result. This disagreement arises likely due to the neglect of multiple collisions which were accounted for neither in [8, 11] nor in [67]. However, the shape and the location of the first harmonic of the channeling radiation are described quite well.

![Graphical representation](image)

**Figure 11.** Spectral distribution (in $10^9$ sr$^{-1}$) of the undulator radiation at $\theta = 0$ for 5 GeV positron channeling along periodically bent (110) planes in Si (figures (a) and (b)) and W (figures (c) and (d)) crystals. The $a/d$ ratio is equal to 10. Other parameters used are presented in table 2. The upper figures (a) and (c) reproduce $\langle dE_N/\hbar \omega d\omega d\Omega_n \rangle$ in the wide ranges of $\omega$ and correspond to $N = 4N_d$. The numbers enumerate the harmonics (in the case of the forward emission the radiation occurs only in odd harmonics). The profiles of the first harmonic peak (figures (b) and (d)) are plotted for $N = 4N_d$ (solid lines), $N = 2N_d$ (dotted lines), $N = N_d$ (dashed lines), $N = N_d/2$ (long-dashed lines).

The intensity and the profile of the peaks of the undulator radiation are defined, to a great extent, by the magnitude of the dechanneling length. In [5] a more sophisticated, than in [1, 2], theoretical and numerical analysis of this influence was presented. In particular, we solved the following problems: (a) simple analytic expression was
evaluated for spectral-angular distribution of the undulator radiation which contains, as a parameter, the dechanneling length $L_d$, (b) the simulation procedure of the dechanneling process of a positron in periodically bent crystals was presented, (c) the dechanneling lengths were calculated for 5 GeV positrons channeling in Si, Ge and W crystals along the periodically bent crystallographic planes, (d) the spectral-angular and spectral distributions of the undulator radiation formed in crystalline undulator were calculated in a broad range of the photon energies and for various $a$, $\lambda$ and $C$.

Table 2. The values of the parameter $C$ (see (2)), undulator period $\lambda$, the dechanneling length $L_d(C)$, the number of undulator periods $N_d = L_d(C)/\lambda$ within $L_d(C)$, the undulator parameter $p$ (see (5)), and the fundamental harmonic energy (see (6 with $\vartheta = 0^\circ$)) used for the calculation [5] of the spectra presented in figures 11(a)-(d).

| Crystal | $C$ (µm) | $\lambda$ (cm) | $L_d(C)$ (µm) | $N_d$ | $p$ | $\hbar\omega_1$ (MeV) |
|---------|---------|-------------|-------------|-------|----|----------------|
| Si      | 0.15    | 63.0        | 0.321       | 51    | 1.87 | 1.37          |
| W       | 0.05    | 42.2        | 0.637       | 151   | 3.26 | 0.89          |

To illustrate the results obtained in [5], in figures 11(a)-(d) we present the spectral distribution of the undulator radiation emitted along the undulator axis, $\hbar^{-1}\langle dE_N/d\omega \, d\Omega_n \rangle_{\vartheta=0^\circ}$, for 5 GeV positron channeling along (110) planes in Si and W crystals. The spectra correspond to the ratio $a/d$, where $d = 1.92$ Å for Si and $d = 2.45$ Å for W. The values of other parameters, used in the calculations, are given in table 2. The values of the dechanneling lengths, $L_d(C)$, were obtained in [5] by means of the simulation procedure of the dechanneling process of a positron in periodically bent crystals.

The upper figures, 11(a) and (c), illustrate the spectral distributions in Si and W over a wide range of emitted photon energy, and corresponds to the crystal length, $L$, exceeding the dechanneling length by a factor of 4: $L = 4L_d(C)$. Each peak corresponds to emission into the odd harmonics, the energies of which follow from the relation $\omega_k = k\omega_1$, $k = 1, 3, \ldots$. The difference in the magnitudes of the undulator parameters for Si and W (see table 2) explains number of the harmonics visible in the spectra. It is seen that all harmonics are well separated: the distance $2\hbar\omega_1$ between two neighbouring peaks is 2.74 MeV for Si and 1.78 MeV in the case of W, whilst the width of each peak $\hbar\Delta\omega$ is $\approx 8.7$ keV for Si and $\approx 2.5$ keV for W.

Figures 11(b) and (d) exhibit, in more detail, the structure of the first-harmonic peaks. For the sake of comparison we plotted the curves corresponding to different values of the undulator periods. It is seen that for $N > N_d$ the intensity of the peaks is no longer proportional to $N^2$, as it is in the case of the ideal undulator without the
dechanneling of the particles [60]. For both Si and W crystals, the intensities of the radiation calculated at \( N \to \infty \) exceed those at \( N = 4N_d \) (the thick full curves in the figures) only by several per cent. Thus, the full curves correspond to almost saturated intensities which are the maximal ones for the crystals used, projectile energies and the parameters of the crystalline undulator. For a more detailed discussion see paper [5].

5. Stimulated emission from a crystalline undulator

As demonstrated in [1, 2], the scheme illustrated by figure 1 allows to consider a possibility to generate stimulated emission of high energy photons by means of a bunch of ultra-relativistic positrons moving in a periodically bent channel. The photons, emitted in the forward direction (\( \vartheta = 0 \)) at the points of the maximum curvature of the bent channel, travel parallel to the beam and, thus, stimulate the photon generation in the vicinity of all successive maxima and minima. This mechanism of the radiation stimulation is similar to that known for a free-electron laser (see, e.g. [92]), in which the periodicity of a trajectory of an ultra-relativistic projectile is achieved by applying a spatially periodic magnetic field. Also from the theory of FEL it is known [91], that the stimulation occurs at the frequencies of the harmonics of the spontaneous emission, \( \omega_k = k\omega_1, \ k = 1,2,\ldots \). The frequency of fundamental harmonic, \( \omega_1 \), is defined in (6). In [1, 2] and, also, in a more recent paper [13] it was shown, that it is possible to separate the stimulated photon emission in the crystalline undulator from the ordinary channeling radiation in the regime of large bending amplitudes \( a \gg d \). This scheme of the stimulated photon emission allows to generate high energy photons up to MeV region and, thus, we call it as a Gamma-laser. As a further step in developing the ideas proposed in these papers, the study, carried out in [18], was devoted to the investigation of the influence of the beam energy spread on the characteristics of the stimulated emission in crystalline undulators.

In the regime of low amplitudes, \( a < d \), the idea of using a periodically bent crystal as an undulator for a free electron laser was explored in [32]. In this regime the intensity of the undulator radiation is relatively small compared with the channeling radiation. However, it is possible to match the undulator frequency to that of the channeling motion. This results in a resonant coupling of the emissions via the two mechanisms, which leads to the enhancement of the gain factor.

Let us review the results obtained in [1, 2, 13]. To do this we first outline the derivation of the general expression for the gain factor in an undulator, and, after accounting for the conditions (17), estimate gain for the crystalline undulator. For the sake of simplicity we consider the stimulated emission for the fundamental harmonic only, and, also, consider the emission in the forward direction. In the formulae below, we use the notation \( \omega \) instead of \( \omega_1 \) for the fundamental harmonic frequency.
The gain factor, \( g(\omega) \), defines the increase in the total number, \( N \), of the emitted photons at a frequency \( \omega \) due to stimulated emission by the particles of the beam:

\[
dN = g(\omega)N \, dz.
\]

The general expression for the quantity \( g(\omega) \) is

\[
g(\omega) = n \left[ \sigma_e(\varepsilon, \varepsilon - \hbar \omega) - \sigma_a(\varepsilon, \varepsilon + \hbar \omega) \right],
\]

where \( \sigma_e(\varepsilon, \varepsilon - \hbar \omega) \) and \( \sigma_a(\varepsilon, \varepsilon + \hbar \omega) \) are the cross sections of, correspondingly, the spontaneous emission and absorption of the photon by a particle of the beam, \( n \) stands for the volume density (measured in cm\(^{-3}\)) of the beam particles. By using the known relations between the cross sections \( \sigma_e,a \) and the spectral-angular intensity of the emitted radiation \([66]\), one derives the following expression for the gain:

\[
g = -\left(2\pi\right)^3 \frac{c^2}{\hbar^2} n \frac{d}{d\varepsilon} \int_{0}^{\Delta \omega} \frac{dE}{d\omega \, d\Omega} \, \Delta \omega \, \Delta \Omega.
\]

Here \( dE/d\omega \, d\Omega \) is the spectral-angular intensity of the radiation, \( \Delta \omega \) is the width of the first harmonic peak, and \( \Delta \Omega \) is the effective cone (with respect to the undulator axis) into which the emission of the \( \omega \)-photon occurs. Note that expression (21) is derived under the assumption that the photon energy is small compared to the energy of the particle, \( \hbar \omega \ll \varepsilon \).

For an undulator of the length \( L \) the total increase in the number of photons is

\[
N = N_0 e^{G(\omega)L},
\]

where \( G(\omega) = g(\omega)L \) is the total gain on the scale \( L \). The expression for \( G(\omega) \) follows from (21) (the details of derivation one finds in [2, 61]):

\[
G(\omega) = n \left(2\pi\right)^3 r_0 \frac{Z^2}{M} \frac{L^3}{\gamma^3 \lambda} \cdot \begin{cases} 1 & \text{if } p^2 > 1 \\ p^2 & \text{if } p^2 < 1 \end{cases}
\]

where \( r_0 = 2.8 \cdot 10^{-13} \) cm is the electron classical radius, \( Z \) and \( M \) are the charge and the mass of a projectile in the units of elementary charge and electron mass. Note the strong inverse dependence on \( \gamma \) and \( M \) which is due to the radiative recoil, and the proportionality of the gain to \( L^3 \) and to the squared charge of the projectile \( Z^2 \).

The main difference, of a principal character, between a conventional FEL and a FEL-type device based on a crystalline undulator is that in the former the bunch of particles and the photon flux both travel in vacuum whereas in the latter they propagate in a crystalline medium. Consequently, in a conventional FEL one can, in principle, increase infinitely the length of the undulator \( L \). This will result in the increase of the total gain and the number of undulator periods \( N \), (4). The limitations on the magnitude of \( L \) in this case are mainly of a technological nature.

The situation is different for a crystalline undulator, where the dechanneling effect and the photon attenuation lead to the decrease of \( n \) and of the photon flux density with the penetration length and, therefore, result in the limitation of the allowed \( L \)-values.
The reasonable estimate of \( L \) is given by the condition (15). In turn, this condition, together with the estimate (14), defines the ranges of photon energies for which the operation of a crystalline undulator is realistic. These ranges are:

- **High-energy photons**: \( \hbar \omega > 10 \text{ keV} \) when \( L_a > 0.01 \text{ cm} \);
- **Low-energy photons**: \( \hbar \omega < I_0 \leq 10 \text{ eV} \).

In the regime of high-energy photons (the gamma-laser regime) the stimulation of the emission must occur during a single pass of the bunch of the particles through the crystal. Indeed, for such photon energies there are no mirrors, and, therefore, the photon flux must develop simultaneously with the bunch propagation. In the theory of FEL this principle is called ‘Self-Amplified Spontaneous Emission’ (SASE) [92, 109] and is usually referred as the FEL operation in the high gain regime, which implies that \( G(\omega) > 1 \) to ensure that the exponential factor in (22) is large. In this case the quantity \( N_0 \) denotes the number of photons which appear due to the spontaneous emission at the entrance part of the undulator.

For \( \hbar \omega < I_0 \leq 10 \text{ eV} \) there is no principal necessity to go beyond 1 for the magnitude of \( G(\omega) \) during a single pass. Indeed, for such photons there is a possibility to use mirrors to reflect the photons. Therefore, the emitted photons, after leaving the undulator can be returned back to the entrance point to be used for further stimulation of the emission by the incoming projectiles.

Below we present the results of numerical calculations of the parameters of the undulator (the first harmonic energy and the number of periods) and of the volume density \( n \) of the bunch particles needed to achieve \( G(\omega) = 1 \). The calculations were performed for relativistic positrons, muons, protons and heavy ions and took into account all the conditions summarized in (17). The results presented correspond to the cases of the lowest values of \( n \) needed to ensure \( G(\omega) = 1 \) and which, simultaneously, produce the largest available values of \( N \).

### 5.1. High-energy photons: the gamma-laser regime

Detailed analysis of the conditions (17) and demonstrates, that to optimize the parameters of the stimulated emission in the photon energy range \( \hbar \omega > 10 \text{ keV} \) in the case of a positron channeling in a periodically bent crystal one should consider the following ranges of parameters: \( \gamma = (1...5) \times 10^3 \), \( a/d = 10...20 \), \( C = 0.1...0.3 \), which are common for all the crystals which we have investigated. For these ranges the energy of the first harmonic (see (6)) lies within the interval 50...150 keV, and the length of the undulator can be taken equal to the dechanneling length because of the inequality \( L_d(C) < L_a(\omega) \), valid for such \( \omega \).

Results of calculations are presented in figure 12, where the dependences of the first harmonic energy, \( \hbar \omega \), the number of undulator periods, \( N \), and the ratio \( G(\omega)/n \) versus
the parameter $C$ (see (2)) are presented for various crystals. The data correspond to the ratio $a/d = 20$ except for the case of Si for which $a/d = 10$.

![Graph](image)

**Figure 12.** First harmonic energy, $\omega$, number of undulator periods, $N$, and the ratio $G(\omega)/n$ in $\text{cm}^3$ versus $C$ for 0.5 GeV positron channeling in various channels as indicated.

For each crystal the curves $\hbar \omega$ and $G(\omega)/n$ were truncated at those $C$ values for which the number of undulator periods becomes less than 10 (see the graph in the middle). It is seen from the bottom graph, that $G(\omega)/n$ is a rapidly varying function of $C$ (note the log scale of the vertical axis). For all the channels this function attains its maximum value $\approx 10^{-21} \text{ cm}^3$ at $C \approx 0.1$. The maximum value of $G(\omega)/n$ defines the magnitude of the volume density of a positron bunch needed to achieve total gain $G(\omega) = 1$. Then it follows from the graph that to achieve the emission stimulation within the range $\hbar \omega = 50 \ldots 150$ keV on the basis of the SASE mechanism it is necessary to reach the value $n = 10^{21} \text{ cm}^{-3}$ for a positron bunch of the energy of several GeV.
At first glance the idea of using a crystalline undulator based on the channeling of heavy positively charged particles looks very attractive. Indeed, as it is seen from (9), (10) and (12), the dechanneling length for a heavy particle is $M/Z \gg 1$ times larger than that for a positron with the same value of $\gamma$. This factor, being cubed in (23), could lead to a noticeable increase of the total gain (over-forcing, in the cases of $\mu^+$ and $p$, the small multiplier $Z^2/M$).

However, as it is seen from (2) the allowed undulator period $\lambda$ increases with a projectile mass: $\lambda > \lambda_{\text{min}} \propto \sqrt{M}$. In turn, this results in a decrease in the first harmonic energy $\omega \approx 4\pi c \gamma^2/\lambda \propto 1/\sqrt{M}$ (in the case of a heavy projectile the undulator parameter is small, and the term $p^2$ can be disregarded when calculating $\omega$, see (6)). For realistic values of relativistic factor, $\gamma \leq 10^3$, this results in the following restriction on the photon energy:

$$\hbar \omega \leq \hbar \omega_{\text{max}} \approx \begin{cases} 50 \text{ keV} & \text{for } \mu^+ \\ 10 \text{ keV} & \text{for } p \\ < 1 \text{ keV} & \text{for a heavy ion} \end{cases}$$

For a proton and an ion the range of photon energies is exactly the one where the attenuation is very strong. Therefore, the crystal length is defined by a small value of $L_a(\omega)$, see figure 6. In the case of $\mu^+$ the upper limit of $\hbar \omega$ is higher but, nevertheless, it leads to a condition $L_a(\omega) \ll L_d(C)$, so that the crystal length also must be chosen as $L = L_a(\omega)$. Although in such conditions it is possible to construct an undulator with sufficiently large number of periods, the total gain factor becomes very small:

$$G(\omega) \sim n \cdot \begin{cases} 10^{-22} & \text{for } \mu^+ \\ 10^{-26} & \text{for } p \end{cases}$$

Therefore, it is not realistic to consider the stimulated emission from a heavy projectile in the high-energy photon range.

5.2. Low-energy photons

For $\hbar \omega \leq I_0$ the photon attenuation becomes small and the length of the undulator is defined by the dechanneling length of a particle.

To illustrate the regime of low-energy stimulated emission during the positron channeling, in figure 13 we present the dependences of $N$ and $G(\omega)/n$ on the relativistic factor. The data correspond to a fixed ratio $a/d = 5$ and to a fixed energy of the first harmonic, $\hbar \omega = 5$ eV, which is lower than the atomic ionization potentials for all crystals indicated in the figures. The undulator length was chosen to be equal to the dechanneling length, which is the minimum from $L_d(C)$ and $L_a(\omega)$. Two graphs in figure 13 demonstrate, that although the number of the undulator periods is, approximately, independent on the type of the crystal, the magnitude of the total gain is quite sensitive
Figure 13. Values of $N$ and $G(\omega)/n$ versus $\gamma$ for a positron-based crystalline undulators in a low-$\omega$ region calculated for various channels as indicated.

to the choice of the channel. The highest values of $G(\omega)/n$ (and, correspondingly, the lowest densities $n$ needed to achieve $G(\omega) = 1$) can be achieved for heavy crystals.

Analysis of (23) together with (17) shows, that, in the case of a heavy projectile, to obtain the largest possible values of the total gain $G(\omega)$ during a single pass through a crystal the following regime can be considered: (a) moderate values of the relativistic factor, $\gamma \sim 10 \ldots 100$; (b) $C = 0.25$ which turns out to be the optimal value of $C$; (c) $Z \gg 1$, i.e. the best choice is to use a bunch of heavy ions.

In figure 14 we present the dependences of $N$ and $G(\omega)/n$ on $\gamma$, lying within the range specified above. All curves, which were obtained for different $a/d$ ratios, refer to the case of $U^{+92}$ channeling in $W$ along the (110) crystallographic planes. The energy of the emitted photon is fixed at 5 eV. It is seen from the graphs that for all the crystals the most optimal range of relativistic factor is $\gamma \sim 10 \ldots 30$ where both the number of the undulator periods and the magnitude of $G(\omega)/n$ noticeably exceed the corresponding
values in the case of a positron channeling, see figure 13.

Similar analysis, carried out for the case of a proton channeling, demonstrates that for the same value of $\gamma$ the magnitudes of $G(\omega)/n$ are several times higher than those for a heavy ion.

Results presented in this section show, that the stimulated emission in the low-$\omega$ range ($h\omega < I_0 \sim 10$ eV) can be discussed for all types of positively-charged projectiles. In this case to achieve the value $G(\omega) = 1$ within the SASE mode it is necessary to consider the densities from $n \approx 5 \times 10^{18}$ cm$^{-3}$ for heavy ion beams up to $n \approx 5 \times 10^{21}$ cm$^{-3}$ for a positron beam. However, this large numbers can be reduced by orders of magnitudes if one considers the multi-pass mode of the FEL. Indeed, there exist the mirrors which allow to reflect photons of the energy $h\omega < I_0 \sim 10$ eV. Thus, the emitted
photons can be returned back to the entrance point and used further to stimulate the emission generated by the particles of the long bunch. The number of the passes, equal approximately to $L_{bunch}/L$ (here $L_{bunch}$ is the bunch length), can be very large (up to $10^4$). Therefore, volume density can be reduced by the factor $L/L_{bunch} \ll 1$.

6. Conclusions

In this paper we have discussed the feasibility of the crystalline undulator and Gamma-laser based on it. We have presented the detailed review covering the development of all essential aspects of these important ideas.

Firstly, we note that it is absolutely realistic to use a crystalline undulator for generating spontaneous radiation in a wide range of photon energies. The parameters of such an undulator, being subject to the restrictions mentioned in section 3.1, can be easily tuned by varying the shape function, the energy and the type of a projectile and by choosing different channels. The large range of energies available in modern colliders for various charged particles, both light and heavy, together with the wide range of frequencies and bending amplitudes in crystals allow to generate the crystalline undulator radiation with the energies from eV up to the MeV region.

Secondly, it is feasible to obtain stimulated emission by means of a crystalline undulator. For a single-pass laser (SASE mode) high volume densities are needed: the stimulated emission in the high-$\omega$ range ($\bar{\hbar}\omega > 10$ keV) demands high volume densities of positrons, $n \geq 10^{20}$ cm$^{-3}$. For this values of $n$ a FEL device based on a crystalline undulator can be operated in a single-pass (SASE) mode.

Stimulated emission in the low-$\omega$ range ($\bar{\hbar}\omega < I_0 \sim 10$ eV) can be discussed for all types of positively-charged projectiles. In this case the large values of the beam densities required for the lasing effect can be reduced by orders of magnitudes if one considers the multi-pass mode of the FEL. Indeed, there exist the mirrors which allow to reflect photons of the energy $\bar{\hbar}\omega < I_0 \sim 10$ eV. Thus, the emitted photons can be returned back to the entrance point and used further to stimulate the emission generated by the particles of the long bunch. The number of the passes, equal approximately to $L_{bunch}/L$ (here $L_{bunch}$ is the bunch length), can be very large (up to $10^4$). Therefore, beam density at which the lasing effect arises can be reduced by the factor $L/L_{bunch} \ll 1$.

The crystalline undulators discussed in this paper can serve as a new efficient source for the coherent high energy photon emission. As we have pointed out, the present technology is nearly sufficient to achieve the necessary conditions to construct not only crystalline undulator, but also the stimulated photon emission source. The parameters of the crystalline undulator and the Gamma-laser based on it differ substantially from what is possible to achieve with the undulators constructed on magnetic fields.

This review clearly demonstrates that experimental efforts are needed for the
verification of numerous theory predications outlined in this review. Such efforts will certainly make this field of endeavour even more fascinating than as it is now and possibly will lead to the practical development of a new type of tunable and monochromatic radiation sources.

Finally, we mention that not all theoretical issues for the described system have been solved so far. Thus, the analyses of dynamics of a high-density positon beam channeling through a periodically bent crystal in presence of an induced high intensity photon flux is to be performed in a greater detail. This and many more other interesting theoretical problems are still open for future investigation.

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