A SIMPLE MONTE CARLO METHOD FOR
SOLVING OF NAVIER-STOKES EQUATIONS.

E. Ostrovsky\textsuperscript{a}, L. Sirota\textsuperscript{b}

\textsuperscript{a} Corresponding Author. Department of Mathematics and computer science, Bar-Ilan University, 84105, Ramat Gan, Israel.
E-mail: eugostrovsky@list.ru
eugeneiostrovsky@gmail.com

\textsuperscript{b} Department of Mathematics and computer science. Bar-Ilan University, 84105, Ramat Gan, Israel.
E-mail: sirota3@bezeqint.net

Abstract.

We offer a simple Monte-Carlo method for solving of the multidimensional initial value and non-homogeneous problem for the Navier-Stokes Equations in whole space when the initial function and right hand side belong to the correspondent Sobolev-Lebesgue-Riesz space.

Keywords and phrases: Multivariate Navier-Stokes (NS) equations, Riesz integral transform, method Monte-Carlo, Gaussian and uniform distribution, polygonal beta distribution, depending trial method, random vectors generation, Sobolev-Lebesgue-Riesz spaces, initial value problem, Helmholtz-Weyl and Riesz projection, divergence, Laplace operator, heat equation and kernel, random processes and fields (r.p.; r.f.), Central Limit Theorem (CLT) in Banach spaces, non-asymptotical estimations for r.f., Young inequality, lifespan of solution.

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1 Introduction. Notations. Statement of problem.

The mild solution $u = u(x, t)$ of a Navier-Stokes equation in the whole space $x \in \mathbb{R}^d$ during its lifetime $t \in [0, T]$, $0 < T = \text{const} \leq \infty$ may be represented as a limit as $n \to \infty, n = 0, 1, 2, \ldots$ in appropriate space-time norm of the following recursion:

$$ u_{n+1}(x, t) = u_0(x, t) + G[u_n, u_n](x, t), n = 1, 2, \ldots $$

(1.0) where $u_0(x, t)$ is the well-known solution of linear heat equation with correspondent initial value and right-hand side and $G[u, v]$ is bilinear unbounded pseudo-differential operator, including space-time convolution, Riesz’s transform, see [5], [6], [10], [12], [13], [17], [18], [40] etc.
We offer a simple depending trial Monte-Carlo method (stochastic modelling), without method based on solving of non-linear system of algebraic equations offered in [39], for multiple parametric integrals computation emerging in (1.0) such that by optimizing of the proportion between the amount of random variables with different degrees of integrals the speed of convergence of a random approximation $u_{n,N}$ to $u_n$ as $N \to \infty$ is optimal: for many important space-times norms $|| \cdot ||$, for example mixed Bochner’s type anisotropic Lebesgue-Riesz $L_{r,p}(R^d, [0,T])$ norms or Lebesgue-continuous norm

$$
\forall n = 1, 2, \ldots \Rightarrow E||u_{n,N} - u_n|| \leq K(n) N^{-1/2}, \quad (1.0a)
$$

analogously to the work [55] devoted to the linear integral equations. Here $N$ denotes the general amount of elapsed random variables. The proof of this proposal used CLT and LIL in the considered spaces, see [56], [54], [52], [66], [69], [70], [71], [74] etc.

**Detail description of problem.**

We consider in this article the initial value problem for the multivariate Navier-Stokes (NS) equations

$$
\partial u_t - 0.5 \Delta u + (u \cdot \nabla) u = \nabla P, \; x \in R^d, \; d \geq 3, \; t > 0; \quad (1.1)
$$

$$
\text{Div}(u) = 0, \; x \in R^d, \; t > 0; \quad (1.2)
$$

$$
u(x, 0) = a(x), \; x \in R^d. \quad (1.3)
$$

Here as ordinary

$$
x = (x_1, x_2, \ldots, x_k, \ldots, x_d) \in R^d; \; ||x|| := \sqrt{\sum_{j=1}^{d} x_j^2},
$$

$$
\partial g(x) = \text{grad} g(x) = \{\partial g/\partial x_i\},
$$

and

$$u = u(t) = u(t, \cdot) = u(x, t) = \{u_1(x, t), u_2(x, t), \ldots, u_d(x, t)\}
$$

denotes the (vector) velocity of fluid in the point $x$ at the time $t$, $P$ is represents the pressure.

Equally:

$$
\partial u_i/\partial t = 0.5 \sum_{j=1}^{d} \partial_{x_j}^2 u_i - \sum_{j=1}^{d} u_j \partial_{x_j} u_i + \partial_{x_i} P,
$$

$$
\sum_{j=1}^{d} \partial_{x_j} u_j = 0, \; u(x, 0) = a(x),
$$

$$\text{Div} u = \text{Div} \bar{u} = \text{Div}\{u_1, u_2, \ldots, u_d\} = \sum_{k=1}^{d} \frac{\partial u_k}{\partial x_k} = 0$$
in the sense of distributional derivatives.

As long as

\[ P = \sum_{j,k=1}^{d} R_j R_k (u_j \cdot u_k), \]

where \( R_k = R_k^{(d)} \) is the \( k^{th} d \) dimensional Riesz transform:

\[ R_k^{(d)} [f](x) = c(d) \lim_{\epsilon \rightarrow 0+} \int_{||y|| > \epsilon} ||y||^{-d} \Omega_k(y) f(x - y) \, dy, \]

\[ c(d) = -\frac{\pi^{(d+1)/2}}{\Gamma \left( \frac{d+1}{2} \right)} \Omega_k(x) = x_k/||x||, \]

the system (1.1) - (1.3) may be rewritten as follows:

\[ \partial u_t = 0.5 \Delta u - (u \cdot \nabla) u + Q \cdot \nabla \cdot (u \otimes u), \quad x \in R^d, \quad t > 0; \quad (1.4) \]

\[ \text{Div}(u) = 0, \quad x \in R^d, \quad t > 0; \quad (1.5) \]

\[ u(x,0) = a(x), \quad x \in R^d, \quad (1.6) \]

where \( Q \) is multidimensional Helmholtz-Weyl projection operator, i.e., the \( d \times d \) matrix pseudo-differential operator in \( R^d \) with the matrix symbol

\[ a_{i,j}(\xi) = \delta_{i,j} - \xi_i \xi_j/||\xi||^2, \quad \delta_{i,j} = 1, i = j; \delta_{i,j} = 0, i \neq j. \]

The consistent regularization of the Riesz transform looks as follows:

\[ R_{k,\epsilon}^{(d)} [f](x) = c(d) \int_{R^d} [\epsilon + ||y||^{-d}] \Omega_k(y) f(x - y) \, dy, \]

herewith

\[ ||R_{k,\epsilon}^{(d)} [f] - R_k^{(d)} [f]||_p \leq C(p) \epsilon ||f||_p, \]

see [35], chapters 4,5.

Note that the last representation of the Riesz’s potential may be used by its Monte Carlo computation, if we will use the polar coordinates and the density of applied random variables to be proportional to the kernel \( \epsilon + ||y||^{-d} \).

At the same considerations may be provided for the NS equations with external density of force \( f = f(x,t) \):

\[ \partial u_t = 0.5 \Delta u - (u \cdot \nabla) u + Q \cdot \nabla \cdot (u \otimes u) + f(x,t), \quad x \in R^d, \quad t > 0; \]

\[ u(x,0) = a(x), \quad x \in R^d. \]

see [12] - [15], [20], [26], [29], [39].

More detail, the considered here problem may be rewritten as follows:
\[ u(x, t) = e^{0.5t\Delta}a(x) + G[u, u](t) \overset{def}{=} u_0(x, t) + G[u, u](t) + v[f](x, t), \]

where

\[ v[f](x, t) = v(x, t) = v = \int_0^t ds \int_{\mathbb{R}^d} w_{t-s}(x - y) f(y, s) dy = \int_0^t w_{t-s}(\cdot) * f(\cdot, s) ds. \]

We will denote further for simplicity

\[ \int f(y) dy = \int f = \int_{\mathbb{R}^d} f(y) dy, \]

\[ w \odot u(x, t) \overset{def}{=} \int_0^t \int_{\mathbb{R}^d} w_{t-s}(x - y) u(y, s) ds dy = \int_0^t \int w_{t-s}(x - y) u(y, s) ds dy. \]

We will understand henceforth as a capacity of the solution (1.4)-(1.6) the vector-function

\[ u = \vec{u} = \{ u_1(x, t), u_2(x, t), \ldots, u_d(x, t) \} \]

the so-called mild solution, see [28].

Namely, the vector-function \( u = u(\cdot, t) = u(x, t) \) satisfies almost everywhere in the time \( t \) the following non-linear integral-differential equation:

\[
\begin{align*}
    u(t) &= e^{0.5t\Delta}a + \int_0^t e^{0.5(t-s)\Delta} \left( (u \cdot \nabla) u(s) + Q \cdot \nabla \cdot (u \otimes u)(s) \right) ds \\
    &= e^{0.5t\Delta}a + G[u, u](t) \overset{def}{=} u_0(x, t) + G[u, u](t),
\end{align*}
\]

the operator \( \exp(0.5t\Delta) \) is the classical convolution integral operator with heat kernel:

\[ u_0(x, t) := e^{0.5t\Delta}[u](x, t) = w_t(x) * a(x), \]

where \( G(u, u) \overset{def}{=} F(u, u) = F(u), \)

\[
F(u, v) = \int_0^t e^{0.5(t-s)\Delta} B[u, v](x, s) ds = \int_0^t \int_{\mathbb{R}^d} w_{t-s}(x - y) B[u, v](y, s) ds dy, \]

\[ B(u, v) := (u \cdot \nabla) v(s) + Q \cdot \nabla \cdot (u \otimes v)(s), \]

\[ w_t(x) \overset{def}{=} (2\pi t)^{-d/2} \exp \left( -\frac{|x|^2}{2t} \right), \]

The convolution between two functions \( r = f(t), g(t) \) defined on the set \( R_+ = (0, \infty) \) is defined as usually

\[ f \odot g(t) := \int_0^t f(t - s) g(s) ds = g \odot f(t) \]
("time-wise" convolution) and between two, of course, measurable vector-functions \( u(x), v(x) \) defined on the whole space \( x \in \mathbb{R}^d \)

\[
u * v(x) = \int_{\mathbb{R}^d} u(x - y) \, v(y) \, dy,
\]

("space-wise" or "coordinate-wise" convolution).

The authors hope that this notations does not follow the confusion.

More results about the existence, uniqueness, numerical methods, and a priory estimates in the different Banach function spaces: Lebesgue-Riesz \( L_p \), Morrey, Besov for this solutions see, e.g. in [1]- [41]. The first and besides famous result belong to J.Leray [24]; it is established there in particular the **global in time** solvability and uniqueness of NS system in the space \( L_2(\mathbb{R}^d) \) and was obtained a very interest a 

The quantitative estimations for solution and lifespan of NS equations in some rearrangement invariant spaces see, e.g. in [60], [61].

T.Kato in [17] proved in particular that if the initial function \( a = a(x) \) belongs to the space \( L_d(\mathbb{R}^d) \) (in our notations), then there exists a positive time value \( T > 0 \) (lifespan of solution) such that the solution of NS system \( u = u(x, t) \) there exists for \( t \in (0, T) \), is smooth and satisfy some a priori integral estimates.

Furthermore, if the norm \( ||a||_{L_d(\mathbb{R}^d)} \) is sufficiently small, then \( T = \infty \), i.e. the solution \( u = u(x, t) \) is global.

The **upper** estimate for the value \( T \), conditions for finite-time blow-up and asymptotical behavior of solution as \( t \to T - 0 \) see in the articles [43], [44], [2], [1], [45], [27], [31], [32] etc.

With regards the numerical methods for solving of NS equations, we note only the classical monograph [39] and articles [46], [40], [33].

**Our purpose in this short report is to offer some modification of the optimal Monte - Carlo method for solving of NS equations during the lifespan of solution** \( T \).

The essence of the proposed method can be explained very simply: we will write the approximation \( u_n(x, t) \) in the **explicit view** through multiple sums of multiple parametric integrals of increasing dimension, to calculate which may be used the so-called depending trial Monte Carlo method.

With regard to the Monte-Carlo method for the **function computation** (in the other terms, depending trials method) see [53], [55], [63], [66].

Note that Monte-Carlo method can not give a very high precision, but it is very simple. For instance, it does not use the solving of system of (non-linear!) algebraic equations and following is stable (robust).
2 Some Notations, with Clarification. The essence of the method.

As ordinary, for the measurable function \( x \rightarrow u(x), \ x \in \mathbb{R}^d \)

\[
||u||_p = \left[ \int_{\mathbb{R}^d} |u(x)|^p \, dx \right]^{1/p}.
\] (2.1)

**Multidimensional case.**

Let \( u = \vec{u} = \{u_1(x), u_2(x), \ldots, u_d(x)\} \) be measurable vector-function: \( u_k : \mathbb{R}^d \rightarrow \mathbb{R} \). We can define as ordinary the \( L_p \), \( p \geq 1 \) norm of the function \( u \) by the following way:

\[
||u||_p := \max_{k=1,2,\ldots,d} ||u_k||_p, \ p \geq 1.
\] (2.2)

We present now some important results belonging to T.Kato [17]; see also the article of H.Fujita and T.Kato [10]. Let us consider the following recursion:

\[
u_{n+1}(x,t) = u_0(x,t) + G[u_n, u_n](t) = u_0(x,t) + G[u_n, u_n](t)
\] (2.3)

with initial condition for iterations

\[
u_0(x,t) = (2\pi t)^{-d/2} \int_{\mathbb{R}^d} \exp \left( -\frac{||x - y||^2}{2t} \right) a(y) \, dy + \\
\int_0^t ds \int_{\mathbb{R}^d} (2\pi t)^{-d/2} \exp \left( -\frac{||x - y||^2}{2(t-s)} \right) f(y,s) \, dy.
\]

So, the recursion (2.3) may be rewritten as follows:

\[
u_{n+1}(x,t) = u_0(x,t) + (2\pi t)^{-d/2} \int_0^t ds \int_{\mathbb{R}^d} \exp \left( -\frac{||x - y||^2}{2(t-s)} \right) F[u_n](y,s) \, dy.
\] (2.4)

H.Fujita and T.Kato [10], [17] proved that if \( a(\cdot) \in L_d(\mathbb{R}^d) \), \( \text{Div} a = 0 \), then there exists a positive value \( T = T(||a||_d) \) (lifespan of solution) such that the iteration sequence \( u_n(\cdot, \cdot) \) converges in the senses (1.9) to the uniquely defined solution of NS equations.

Furthermore, if the norm \( ||a||_d \) is sufficiently small, then \( T = \infty \) (global solution).

The quantitative lower bound for lifespan \( T \) and some quantitative a priori estimation in Grand Lebesgue Spaces (GLS) for solution \( u = u(x,t) \) see, e.g. in [60], [61].

Moreover, for all the values \( \delta \in (0, 1) \) there exist a constants \( q_1 = q_1(\delta), q_2 \in (0, 1) \) such that

\[
||u_n - u||_{B((0,T), L_{d/\delta})} \leq C_1(a; d, \delta) q_1^n
\]

and
\[ ||\nabla u_n - \nabla u||_{B((0, T), L_d)} \leq C_2(a; d) q_2^2. \]

**A.** Note that if \( f(x, t) = 0 \) then
\[
u_0(x, t) = u_0(x, t) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \exp \left( -\frac{||z||^2}{2} \right) a(x + z\sqrt{t}) \, dz = E[a(x + \xi \cdot \sqrt{t})], \tag{2.5}\]
where the random vector \( \xi \) has a Gaussian \( d \) - dimensional standard distribution.

This imply that the random vector \( \xi \) has a Gaussian \( d \) - dimensional distribution with parameters
\[ E\xi = 0, \quad \text{Var} \xi = I_d \]
be an unit matrix of a size \( d \times d \).

Let \( \{\xi_i\}, \, i = 1, 2, \ldots, N \) be a sequence of independent Gaussian \( d \) - dimensional standard distributed random vectors. The Monte-Carlo approximation \( u_{0,N} = u_{0,N}(x, t) \) in that its modification which is called ”depending trial method” [53] for \( u_0(x, t) \) has a view
\[
u_{0,N}(x, t) = u_{0,N}[a](x, t) := N^{-1} \sum_{i=1}^{N} a(x + \xi_i \cdot \sqrt{t}). \tag{2.6}\]

**B.** Analogously, let us consider the \( d \) - dimensional heat equation with zero initial values but with external force:
\[
\partial_t u^{(0)}(x, t) = 0.5 \Delta u^{(0)}(x, t) + f(x, t), \quad u^{(0)}(x, 0) = 0. \tag{2.7}\]

Then \( u^{(0)}(x, t) = u^{(0)}[f](x, t) = \]
\[
\int_0^t w_{t-s} * f](x, s) \, ds = \int_0^t ds \int_{\mathbb{R}^d} (2\pi(t-s))^{-d/2} \exp \left( -\frac{||x-y||^2}{2(t-s)} \right) f(y, s) \, dy =
\]
\[
t \int_0^1 dv \int_{\mathbb{R}^d} (2\pi)^{-d/2} e^{-||z||^2/2} f(x + z\sqrt{t}, t \cdot v) \, dz =
\]
\[ E \left[ t f(x + \eta\sqrt{t} \sqrt{(1 - \tau), t \cdot \tau}) \right], \tag{2.8}\]
where the random \( \text{vector} \eta \) has a Gaussian \( d \) - dimensional standard distribution, the random \( \text{variable} \tau \) is uniformly distributed on the unit segment \([0, 1]\).

Let \( \{\eta_i\}, \, i = 1, 2, \ldots, N \) be a sequence of independent Gaussian \( d \) - dimensional standard distributed random vectors and \( \{\tau_i\}, \, i = 1, 2, \ldots, N \) be a sequence of independent and independent on the \( \{\eta_i\} \) uniform distributed on the segment \([0, 1]\) random variables.

The Monte-Carlo approximation \( u_{N}^{(0)} = u_{N}^{(0)}(x, t) = u_{N}^{(0)}[f](x, t) \) in that its modification which is called ”depending trial method” for \( u^{(0)}(x, t) \) has a view
\[ u_N^{(0)}(x, t) = u_N^{(0)}[f](x, t) := \frac{t}{N} \left[ \sum_{i=1}^{N} f(x + \eta_i \sqrt{t} \sqrt{(1 - \tau_i)}, t \cdot \tau_i) \right]. \quad (2.9) \]

Of course, the "initial" function \( u_0(x, t) \) may be computed by means of deterministic methods: finite differences, finite elements method etc., as well as the Riesz's transform computation.

C. The expression for the member \( G[u] \) or for the \( G[u_n] \) is alike to the one in the formula (2.8) with replacing \( f := F = F[u](x, t) \).

Let us define the following iteration sequence:

\[ u_{n+1, N(n+1)}(x, t) = u_{0, N(0)}[a](x, t) + \]

\[ \frac{t}{N(n+1)} \left[ \sum_{i=1}^{N(n+1)} F[u_{n, N(n)}] \left( x + \eta_i^{(n+1)} \sqrt{t} \sqrt{(1 - \tau_i^{(n+1)}), t \cdot \tau_i^{(n+1)}} \right) \right]. \quad (2.10) \]

Here \( n = 0, 1, 2, \ldots, L; \) the number \( L \) is the total number of iterations,

\[ (N(0), N(1), N(2), \ldots, N(L)) = (N^{(L)}(0), N^{(L)}(1), N^{(L)}(2), \ldots, N^{(L)}(L)) \]

is the sequence of integer numerical vectors ever-increasing dimension \( L + 1, \ L = 1, 2, \ldots; \ L \to \infty \) such that

\[ \forall n = 0, 1, 2, \ldots L \Rightarrow \lim_{L \to \infty} N^{(L)}(n) = \infty. \]

The random vectors \( \eta_i^{(n+1)} \) have \( d- \) dimensional standard Gaussian distribution, the random variables \( \tau_i^{(n+1)} \) are uniformly distributed in the unit segment \([0, 1]\) and all the introduced random vectors and variables are independent.

The total number of spent random variables, i.e. including the vector coordinates \( M = M(L) \) may be calculated as follows:

\[ M = M(L) = 2N^{(L)}(0) + \sum_{d=1}^{L} (d + 1)N^{(L)}(d). \quad (2.11) \]

But it is very hard to error estimate of this procedure, especially in important Banach functional norms for the computed function.

A. Let us consider now the alternative approach.

Namely, we denote

\[ J_m(h) = \partial w \otimes \partial w \otimes \ldots \otimes \partial w \otimes h = \]

\[ \int_0^t ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_{m-1}} ds_m \int_{R^m} (2\pi)^{-dm/2}(t-s_1)^{-d/2}(s_1-s_2)^{-d/2} \ldots (s_{m-1}-s_m)^{-d/2} \times \]
\[
\exp \left[ -\frac{||x - y_1||^2}{2(t-s_1)} - \frac{||y_1 - y_2||^2}{2(s_1 - s_2)} - \ldots - \frac{||y_{m-1} - y_m||^2}{2(s_{m-1} - s_m)} \right] \times \\
\begin{pmatrix}
x - y_1 \\ t - s_1
\end{pmatrix}, \begin{pmatrix}
y_1 - y_2 \\ s_1 - s_2
\end{pmatrix}, \ldots, \begin{pmatrix}
y_{m-1} - y_m \\ s_{m-1} - s_m
\end{pmatrix} \cdot h(y, s) \, dy.
\]

We make the change of variables in the interior integral as follows:

\[
y_1 = x + z_1\sqrt{t - s_1}, \ y_2 = y_1 + z_2\sqrt{s_1 - s_2}, \ldots, \ y_m = y_{m-1} + z_m\sqrt{s_{m-1} - s_m},
\]

with Jacobian

\[
[(t-s_1)(s_1-s_2)\ldots(s_{m-1}-s_m)]^{d/2},
\]

and after this - the substitution \( s_j = t \cdot \tau_j \) with Jacobian \( t^{m/2} \). The resulting transform may be written briefly as follows

\[
(y, s) = Y(z, \tau) = Y_{x,t}(z, \tau), \ y, z \in (R^d)^m = R^{dm}, \ s, \tau \in (R_+)^m,
\]

where the values \( x \in R^d, \ t \in R_+^1 \) be a fix (temporarily).

We obtain: \( J_m(h) \cdot t^{-m/2}(2\pi)^{dm/2} = \)

\[
\int_{S(m)} \int_{R^{dm}} \frac{1}{(1 - \tau_1)^{1/2}} \frac{1}{(\tau_1 - \tau_2)^{1/2}} \ldots \frac{1}{(\tau_{m-1} - \tau_m)^{1/2}} \times \\
\exp \left[ \frac{||z_1||^2}{2} - \frac{||z_2||^2}{2} - \ldots - \frac{||z_m||^2}{2} \right] \cdot \tilde{h}(z, \tau),
\]

where \( S(m) \) is an \( m \) - dimensional unit polygon (simplex) of a form \( S(m) = \)

\[
\{ \tau = \tilde{\tau} = (\tau_1, \tau_2, \ldots, \tau_m) : 0 < \tau_1 < 1, 0 < \tau_2 < \tau_1, 0 < \tau_3 < \tau_2, \ldots, 0 < \tau_m < \tau_{m-1} \},
\]

and

\[
\tilde{h}(z, \tau) = \hat{h}_{x,t}(z, \tau) = \prod_{j=1}^{m} z_j \cdot h(Y(z, \tau)) = \prod_{j=1}^{m} z_j \cdot h(Y_{x,t}(z, \tau)).
\]

It is easy to calculate

\[
\int \ldots \int_{S(m)} \frac{ds_1 ds_2 \ldots ds_m}{s_1^{\alpha_1} (s_2 - s_1)^{\alpha_2} (s_3 - s_2)^{\alpha_3} \ldots (s_m - s_{m-1})^{\alpha_m}} = \\
\frac{\prod_{k=1}^{m} \Gamma(1 - \alpha_k)}{\Gamma(1 + \sum_{k=1}^{m} (1 - \alpha_k))}, \ 0 \leq \alpha_k < 1.
\]

Denote \( W_m = W(m) = \pi^{m/2}/\Gamma(1 + m/2) \) - the volume of an unit ball of the Euclidean space \( R^m \),

\[
H_m(s) = [(1 - s_1)(s_1 - s_2)(s_2 - s_3)\ldots(s_{n-1} - s_n)]^{-1/2} / W_m \overset{def}{=} 
\]
\[ R_m(s)/W_m, \ s \in S(m). \quad (2.14) \]

Evidently, \( \lim_{m \to \infty} W_m = 0 \).

Recall in addition to the article [67] that the generating function for the sequence

\[ W_m(\beta) = \frac{1}{\Gamma(1 + n\beta)}, \]

i.e. the function

\[ ML_\beta(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1 + n\beta)}, \ z \in \mathbb{C} \]

is named as Mittag - Leffler function.

Therefore, the function \( s \to H_m(s), \ s \in S(m) \) could be chosen as a density of distribution of a random variable (vector), say, \( \kappa = \kappa_m \) with support on the simplex \( S(m) \):

\[ P(\kappa_m \in G) = \int_G H_m(s) \ ds \overset{\text{def}}{=} \mu_m(G). \]

This random vector \( \kappa = \kappa_m = \bar{\kappa} = \bar{\kappa}_m \) with values in the polygon \( S(m) \) is a particular case of the so - called polygonal Beta distribution, written: \( \text{Law}(\kappa) = PB(1/2, m) \) if it has a density \( H_m(s), \ s \in S(m) \), see e.g. [67].

Note that \( \text{dim} \kappa_m = m \). The most economical way of generation of this distribution, such that for each value \( \kappa_m \) is elapsed exact \( m \) random variables uniform distributed on the interval \((0, 1)\), is described in the aforementioned article [67].

We can offer the following probabilistic representations for the integral \( J_m(h) \):

\[ J_m(h) \cdot t^{-m/2}/W_m = \mathbf{E} \bar{h}(\zeta, \kappa_m), \]

where the distribution of the r.v. \( \kappa_m \) was described before, the random vector \( \zeta = \{\zeta_1, \zeta_2, \ldots, \zeta_m\} \) consists on the \( m \) independent centered Gaussian standard distributed matrices of the size \( d \times d \):

\[ f_\zeta(z_1, z_2, \ldots, z_m) = (2\pi)^{-dm/2} \exp \left[ -\frac{||z_1||^2}{2} - \frac{||z_2||^2}{2} - \ldots - \frac{||z_m||^2}{2} \right] \quad (2.15) \]

and the random vectors \((\zeta, \kappa_m)\) are independent.

In the sequel the notation \( ||A||^2 \) for the \( d \times d \) matrix \( A = \{a_{i,j}\} \) with real entries denotes

\[ ||A||^2 \overset{\text{def}}{=} \sum_{i=1}^{d} \sum_{j=1}^{d} a_{i,j}^2. \]

We denote also for the positive semi - definite matrix \( A = \{a_{i,j}\} \)

\[ [A] := \max_i a_{i,i}. \]
Note that for the r.v. \((\zeta, \kappa_m)\) generation need \(m + dm = m(d + 1)\) uniform distributed on the set \([0, 1]\) r.v.

Let \(N(m) = N(m, n)\) be arbitrary positive integer numbers. The Monte Carlo approximation \(J_{m,N(m)}(h)\) for the integral \(J_m(h)\), the so-called ”depending trial method”, see [53], [55] has the form

\[
J_{m,N(m)}(h) := t^{m/2} \cdot W_m \cdot \frac{1}{N(m)} \sum_{i=1}^{N(m)} \tilde{h}(\zeta_i, \kappa_{m,i}),
\]

(2.16)

where the r.v. \((\zeta_i, \kappa_{m,i})\) are independent copies of \((\zeta, \kappa_m)\).

Of course, this estimate is unbiased: \(\mathbf{E}J_{m,N(m)} = J_m(h)\). Let us estimate the variation of \(J_{m,N(m)}(h)\). Evidently,

\[
[\text{Var}] J_{m,N(m)}(h) \leq t^m \cdot W_m \cdot \frac{1}{N(m)} \cdot J_m(\tilde{h}^2),
\]

(2.17)

and following

\[
[\text{Var}] J_{m,N(m)}(h) \leq t^m \cdot W_m^2 \cdot \frac{1}{N(m)} \cdot \sup_{y,s} h^2(y, s).
\]

(2.17a)

Note that the integral \(J_m(\tilde{h}^2)\) may be estimated as well as the source integral \(J_m(h)\).

B. We denote and consider now the following integral

\[
I_m(h) = w \odot w \odot \ldots \odot w \odot h =
\]

\[
\int_0^t ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_{m-1}} ds_m \int_{R^{dm}} (2\pi)^{-dm/2} (t-s_1)^{-d/2} (s_1-s_2)^{-d/2} \ldots (s_{m-1}-s_m)^{-d/2} \times
\]

\[
\exp \left[ -\frac{||x - y_1||^2}{2(t-s_1)} - \frac{||y_1 - y_2||^2}{2(s_1-s_2)} - \ldots - \frac{||y_{m-1} - y_m||^2}{2(s_{m-1}-s_m)} \right] \times h(y, s) dy.
\]

(2.18)

We have \(I_m(h) \cdot t^{-m}(2\pi)^{dm/2} =
\]

\[
\int_{S(m)} d\tau \int_{R^{dm}} d\tau \times \exp \left[ -\frac{||z_1||^2}{2} - \frac{||z_2||^2}{2} - \ldots - \frac{||z_m||^2}{2} \right] \cdot \tilde{h}(z, \tau),
\]

(2.19)

where as before \(S(m)\) is an \(m\) – dimensional unit polygon (simplex), \((y, s) = Y(z, \tau)\) is at the same substitution and \((z, \tau) = Y(y, s)\), \(\tilde{h}(z, \tau) = h(Y(y, s))\).

Since the volume of the simplex \(S(m)\) is equal to \(1/m!\), the expression for the integral \(I_m(h)\) obeys a following probabilistic representation:

\[
t^{-m} \cdot m! \cdot I_m(h) = \mathbf{E}\tilde{h}(\zeta, \nu_m),
\]

(2.20)
where the r.v. \( \zeta \) has as before multidimensional centered standard Gaussian (Normal) distribution in the space \( R^d_m \), the random vector \( \nu_m \) has an uniform distribution in the polygon \( S(m) \) and the r.v. \((\zeta, \nu_m)\) are independent.

Recall that this distribution is a particular case of general Polygonal Beta distribution, see [67].

Naturally, the Monte Carlo approximation for \( I_m(h) \) has a form

\[
I_{m,N(m)}(h) := t^m \cdot \frac{1}{m!} \cdot \frac{1}{N(m)} \sum_{i=1}^{N(m)} \tilde{h}(\zeta_i, \nu_{m,i}),
\]

(2.21)

where \( N(m) \) is non-random positive integer number, \((\zeta_i, \nu_{m,i})\) are independent copies of the r.v. \((\zeta, \nu_m)\).

This estimate is unbiased: \( \mathbb{E} I_{m,N(m)} = I_m(h) \). Let us estimate the variation of the random value \( I_{m,N(m)} \).

\[
[\text{Var}] I_{m,N(m)}(h) \leq t^{2m} \cdot \frac{1}{m!} \cdot \frac{1}{N(m)} \cdot I_m(\tilde{h}^2) \quad (2.22)
\]

and following

\[
[\text{Var}] I_{m,N(m)}(h) \leq t^{2m} \cdot \frac{1}{m!} \cdot \frac{1}{N(m)} \cdot \sup_{y,s} h^2(y, s). \quad (2.22a)
\]

Note that the integral \( I_m(\tilde{h}^2) \) may be estimated as before as well as the source integral \( I_m(h) \).

C. Mixed case.

We introduce for simplicity two operators:

\[
T[w](x,t) := [w \circ g](x,t), \quad T[\partial w](x,t) := [\partial w \circ g](x,t),
\]

(2.23)

and consider the following multiple convolution:

\[
K[h] = K[h](y, s) = K_{m_1,m_2}[h](y, s) := T_{m_1}^{m_1} T_{\partial m_2}^{m_2}[h](y, s).
\]

(2.24)

Here \( m_1, m_2 \) be non - negative integer numbers and we denote (temporarily) \( m = m_1 + m_2 \), and suppose \( m \geq 1 \); the case \( m = 0 \) is trivial.

We offer and investigate the Monte - Carlo approximation for \( K[h] \) computation alike in the pilcrow A and in the pilcrow B.

Figuratively speaking, the pilcrow C is synthesis of the subsections A and B.

We obtain after at the same linear changing of variables the following expression for the function \( K(\cdot) \)

\[
K_{m_1,m_2}[h] = (2\pi)^{dm/2} \cdot t^{m_1/2 + m_2} \cdot \frac{1}{S(m_1)} \cdot \frac{1}{S(m_2)} \cdot \int_{R^d m} dz R_m(\tau) \times \exp \left( -0.5 \sum_{i=1}^{m} ||z_i||^2 \right) \cdot \tilde{h}(z, \tau, \theta),
\]

\[ \]
which admit the next probabilistic representation

\[ K_{m_1,m_2}[h] = t^{m_1/2+m_2} \cdot W(m_1) \cdot (1/m_2!) \cdot \mathbb{E} \tilde{h}(\zeta, \kappa_{m_1}, \nu_{m_2}), \quad (2.25) \]

where as before the random vectors \((\zeta, \kappa_{m_1}, \nu_{m_2})\) are (common) independent, the random vector \(\zeta\) is normal centered standard blocky distributed in the space \(\mathbb{R}^m\), the r.v. \(\kappa_{m_1}\) has the Polygonal Beta distribution with index \((0.5)\) in the simplex \(S(m_1)\) and the r.v. \(\nu_{m_2}\) is uniformly distributed inside the other simplex \(S(m_2)\).

The Monte Carlo approximation \(K_{m_1,m_2,N(m)}[h]\) for the function \(K_{m_1,m_2}[h]\) is clear:

\[ K_{m_1,m_2,N(m)}[h] := t^{m_1/2+m_2} \cdot W(m_1) \cdot (1/m_2!) \cdot \frac{1}{N(m)} \cdot \sum_{i=1}^{N(m)} \tilde{h}(\zeta_i, \kappa_{m_1,i}, \nu_{m_2,i}), \quad (2.26) \]

where \((\zeta_i, \kappa_{m_1,i}, \nu_{m_2,i})\) are independent copies of \((\zeta, \kappa_{m_1}, \nu_{m_2})\).

The approximation \(K_{m_1,m_2,N(m)}[h]\) has the following variation estimate

\[ \text{Var}(K_{m_1,m_2,N(m)}[h]) \leq t^{m_1+2m_2} \cdot W^2(m_1) \cdot (1/m_2!) \cdot \sup_{y,s} h^2(y,s)/N(m), \quad (2.27) \]

which allows in turn a very simple but rough estimate

\[ \text{Var}(K_{m_1,m_2,N(m)}[h]) \leq \max(t^m, t^{2m}) \cdot W^2(m) \cdot \sup_{y,s} h^2(y,s)/N(m). \quad (2.28) \]

Note that the number \(N(m)\) dependent only on the number \(m\), but not on the individual numbers \(m_1, m_2\). Therefore, the general amount of standard uniformly distributed on the interval \((0, 1)\) random numbers for \(K_{m_1,m_2,N(m)}[h]\) computations in (2.26) is equal to \(N(m) \cdot (m + dm) = N(m) \cdot m \cdot (d + 1)\).

### 3 Auxiliary facts. Non-linear recursions.

#### A. Polynomial recursion.

Let us introduce the following sequence of polynomials \(\{P_n(z)\}\), \(n = 0, 1, 2, \ldots\): \(P_0(z) = 0, P_1(z) = z\) and by recursion over \(n\):

\[ P_{n+1}(z) = z + P_n^2(z), \quad n = 1, 2, \ldots \]

For instance,

\[ P_2(z) = z + z^2 + 2z^3 + z^4; \quad P_3(z) = z + z^2 + 2z^3 + 5z^4 + 6z^5 + 6z^6 + 4z^7 + z^8. \]

Evidently, \(P_n(0) = 0\) and \(\text{deg}(P_n) = 2^n\). Further, we conclude by means of induction
$$P_n(z) = \sum_{m=1}^{2^n} A(m, n) z^n,$$

where the coefficients \(\{A(m, n)\}\) are positive and integer.

We derive substituting the value \(z = 1\) and denoting

$$P_n(1) = \sum_{m=1}^{2^n} A(m, n) =: \tilde{P}(n)$$

the following recursion:

$$\tilde{P}(0) = 0, \quad \tilde{P}(1) = 1, \quad \tilde{P}(n + 1) = 1 + (\tilde{P}(n))^2.$$ 

Suppose \(0 < z < 1/4\); then the sequence \(\{P_n(z)\}, n = 1, 2, \ldots\) monotonically increases and converges uniformly in the ball

\(\{z : |z| \leq 1/4 - \epsilon_0\}, \quad \epsilon_0 = \text{const} \in (0, 1/4)\)

to the analytic function \(P(z)\) which satisfies the equation

$$P(z) = 1 + z P^2(z), \quad \Leftrightarrow \quad P(z) = \frac{1 - \sqrt{1 - 4z}}{2z}; \quad P(0) = 1.$$ 

This function has a Taylor’s expression

$$P(z) = 1 + \sum_{M=1}^{\infty} \frac{(2M)!}{M! (M+1)!} z^M.$$ 

As long as

$$\frac{(2M)!}{M! (M+1)!} \leq \pi^{-1/2} M^{-1/2} (M + 1)^{-1} 2^{2M} e^{5/(24M)}, \quad M \geq 1,$$

the series for the function \(P(z)\) converge inside the closed ball \(|z| \leq 1/4\).

We conclude also

$$\sup_n A(M, n) = \lim_{n \to \infty} A(M, n) = \frac{(2M)!}{M! (M+1)!}.$$ 

Moreover, let us introduce the following relation of equivalence \(E(n), n = 2, 3, \ldots,\) more exactly, the sequence of relations, between the polynomials \(P = P(z)\) and \(Q = Q(z)\):

$$P \overset{E(n)}{\sim} Q \Leftrightarrow \forall k \leq n - 1 \Rightarrow P = Q \mod z^k.$$ 

We deduce then by induction:

$$P_{n+1} \overset{E(n-1)}{\sim} P_n,$$

and as a consequence

$$\forall k \leq n - 1 \Rightarrow A(k, n - 1) = A(k, n).$$
B. Non-linear numerical recursion.

The following numerical recurrence relation (dynamical system) with quadratic non-linearity

$$D(n + 1) = 1 + d^2 \cdot D^2(n)$$

with initial condition $D(0) = 1$ is investigated with some numerical examples in the important case $d = 3$ in particular in the article [68].

For instance, it was obtained there the speed of increase of solution $D(n)$ as $n \to \infty$ and bilateral exact bounds.

For example:

$D(0) = 1$, $D(1) = 10$, $D(2) = 901$, $D(3) = 811,802$, $D(4) = 659,022,487,205$, $D(5) = 434,310,638,641,864,388,712,026$;

$$\forall k, l = 1, 2, \ldots \Rightarrow 1 \leq \frac{9D(k + l)}{[9D(l)]^{2k}} \leq \left[1 + \frac{1}{9D^2(l)}\right]^{2k-1},$$

$$\forall k \geq 1 \Rightarrow \lim_{l \to \infty} \frac{9D(k + l)}{[9D(l)]^{2k}} = 1.$$  

It is proved also in [68] that $D(n)$ is number of independent summands for the $n^{th}$ iteration $u_n$, $\partial u_n$.

4 Iterations.

Notations. $v = v_0 = u_0 \cdot \partial u_0$; $|||u(\cdot, \cdot)|||$ :=

$$\max_{i=1,2,\ldots,d} \max_{j=1,2,\ldots,d} \sup_{x,t} \max\{|\max(|x|, 1) u_i(x, t) |, |\max(|x|, 1)|\partial u_i/\partial x_j|\}. \quad (4.0)$$

The mild solution $u = u(x, t)$ of a Navier-Stokes equation in the whole space $x \in \mathbb{R}^d$ throughout its lifetime $t \in [0, T], T = \text{const} \leq \infty$ may be represented as a limit as $n \to \infty$, $n = 0, 1, 2, \ldots$ the following recursion:

$$u_{n+1}(x, t) = u_0(x, t) + G[u_n, u_n](x, t), n = 0, 1, 2, \ldots,$$

where $u_0(x, t)$ is the solution of heat equation with correspondent initial $a(x)$ value and right-hand side $f(x, t)$:

$$\partial u_0/\partial t = 0.5 \Delta u_0 + f(x, t), \quad u_0(x, 0+) = a(x)$$

and $G[u, v]$ is bilinear unbounded pseudo-differential operator, [17].
The iterative recursion may be written as follows:

\[ u_{n+1} = u_0 + w \odot v_n, \quad \partial u_{n+1} = \partial u_0 + \partial w \odot v_n, \]

where

\[ v_n := u_n \cdot \partial u_n, \quad n = 0, 1, 2, \ldots \]

For instance,

\[ u_1 = u_0 + w \odot v, \quad \partial u_1 = \partial u_0 + \partial w \odot v, \quad (4.1) \]

\[ u_2 = u_0 + w \odot v_1, \quad \partial u_2 = \partial u_0 + \partial w \odot v_1, \]

and we obtain after substitution

\[ u_2 = u_0 + w \odot v + w \odot [u_0 \cdot (\partial w \odot v)] + \]

\[ w \odot \{ [w \odot v] \cdot \partial u_0 \} + w \odot [(w \odot v) \cdot (\partial w \odot v)], \quad (4.2a) \]

\[ \partial u_2 = \partial u_0 + \partial w \odot v + \partial w \odot [u_0 \cdot (\partial w \odot v)] + \]

\[ \partial w \odot \{ [w \odot v] \cdot \partial u_0 \} + \partial w \odot [(w \odot v) \cdot (\partial w \odot v)]. \quad (4.2b) \]

It follows from the inductions method that

\[ u_n = u_0 + \sum_{r=1}^{D(n)} Q_r(u_0), \quad (4.3a) \]

\[ \partial u_n = \partial u_0 + \sum_{r=1}^{D(n)} \partial Q_r(u_0), \quad (4.3b) \]

where

\[ Q_r(u_0) = Q_r(u_0, \partial u_0, v, \ldots) \]

is homogeneous relative the source function \( u_0(\cdot, \cdot) \) non-linear operator acting on the continuous differentiable functions defined on the semi-space \( \mathbb{R}^d \times \mathbb{R}_+ \) into itself.

Of course, we offer to compute each integral in (4.3a) and in (4.3b) by means of the Monte Carlo method. We discuss many technical details.

Note first of all that every member \( Q_r(\cdot) \) in (4.3a) (and analogously in (4.3b) ) has a form

\[ Q_r(\cdot) = Q_{r, l_1, l_2}(\cdot) = Q^{(n)}_{r, l_1, l_2}(\cdot) = T^{l_1}_{w} T^{l_2}_{\partial w}[h], \]

with appropriate function \( h = h[u_0] = h_{l_1, l_2}[u_0](y, s) \), where

\[ l := l_1 + l_2 = \deg_{u_0}(h) - 1. \quad (4.4) \]
The last notion \( \text{deg}_{u_0}[h] \overset{\text{def}}{=} k \) implies by definition that

\[ h[\lambda u_0] = \lambda^k h[u_0], \quad \lambda = \text{const} \in R. \]

The expression (4.3a) and equally (4.3b) may be rewritten as follows.

\[ u_n = u_0 + \sum_{k=1}^{2^n} L_k[u_0], \]

where

\[ L_k[u_0] = L_k^{(n)}[u_0] = \sum \sum_{l_1,l_2:t_1+l_2=k} Q^{(n)}_{r_1,l_2}[h]. \quad (4.5) \]

Note that the amount of summands in the right-hand side of the expression (4.5) is equal to \( A(k,n) \).

Each member in (4.5) has the degree \( k - 1 \) relative the function \( u_0(\cdot,\cdot) \) and may be computed by means of the Monte Carlo method in accordance to the second section.

We offer to give for all the members into (4.5) computation at the same amount \( N(k) = N(k,n) \) random (quasi-random) independent vector variables, so that the \textit{general} amount the standard (uniformly distributed) r.v. (spending) for approximation \( u_n \approx u_{n,N} \) Monte-Carlo computation is equal to

\[ N = \sum_{k=1}^{2^n} A(k,n) \cdot N(k,n) \cdot d(k+1). \quad (4.6) \]

Notice that at the same random variables may be used also for the \( \partial u_n \) Monte Carlo computation, for the sake of saving.

We give now the rough variation estimate for \( u_{n,N} \) approximation based on the formulae 2.28. Namely,

\[ [\text{Var}](u_{n,N}) \leq \sum_{k=1}^{2^n} \max(t^k, t^{2k}) \cdot W^2(k) \cdot A(k,n) \cdot |||u_0|||^2 / N(k,n). \quad (4.7) \]

and at the same estimate is true for \( \partial u_n \approx \partial u_{n,N} \) Monte Carlo computation.

\textbf{Remark 4.1.} Recall that the functions \( u, \partial u \) and following \( u_n, \partial u_n \) are vector and moreover matrix functions. For instance, \( \partial u^{(j)}_i = \partial u_i / \partial x_j \).

But for the \( \partial u_i / \partial x_j \) by means of offered here method can be used, for the sake of saving, at the same random variables as by computation \( u_n = u_{n,i} \).

\section{Subject of optimization}

It seems quite reasonable the following statement of constrained optimization problem. Let the general amount of spending standard distributed r.v. \( N \) be a given “great” number, for example, \( N = 10^6 - 10^8 \).

Subject of minimization:
\[ Z := Z(N(1), N(2), \ldots, N(2^n)) \overset{def}{=} \sum_{k=1}^{2^n} W^2(k) \cdot A(k, n) \cdot \|u_0\|^{2k}/N(k) \quad (5.1) \]

This function is weakly proportional to the upper estimation for the variance \( \text{Var}(u_{n,N}) \) in (4.7), moreover:

\[ \text{Var}(u_{n,N}) \sim Z(N(1), N(2), \ldots, N(2^n)), \quad t \in (0, T), \quad T = \text{const} \in (0, \infty). \]

Restriction:

\[ Y := Y(N(1), N(2), \ldots, N(2^n)) \overset{def}{=} \sum_{k=1}^{2^n} A(k, n) \cdot N(k) \cdot d(k + 1) = N. \quad (5.2) \]

So, we get to the following problem of constrained optimization:

\[ Z(N(1), N(2), \ldots, N(2^n)) \to \min \quad Y(N(1), N(2), \ldots, N(2^n)) = N, \quad N(k) \geq 1. \quad (5.3) \]

We find by means of Lagrange factor method neglecting to start an integer variables:

\[ N_0(k) = \frac{N \cdot d^{-3/2} \cdot W(k) \cdot \|u_0\|^{k}}{\sum_{r=1}^{2^n} W(r) \cdot \|u_0\|^{r}}, \]

up to around to nearest integer number. Herewith

\[ \min_{\{N(k)\}} \text{Var}(u_{n,N}) \asymp \frac{d^{3/2}}{N} \cdot \sum_{m=1}^{2^n} W(m) \cdot A(m, n) \cdot \|u_0\|^m \times \]

\[ \sum_{r=1}^{2^n} (r + 1)^{1/2} \cdot W(r) \cdot A(r, n) \cdot \|u_0\|^r. \quad (5.4) \]

Some slight simplification: as \( N \to \infty \)

\[ \min_{\{N(k)\}} \text{Var}(u_{n,N}) \asymp \frac{d^{3/2}C(n)}{N}, \quad (5.5a) \]

and analogously

\[ \min_{\{N(k)\}} \text{Var}(\partial u_{n,N}) \asymp \frac{d^{3/2}C(n)}{N}. \quad (5.5b) \]

The last estimates imply that the speed of convergence \( u_{n,N} \) to \( u_n \) as \( N \to \infty \) is equal \( N^{-1/2} \), as in the classical Monte Carlo method.

For the linear integral equations this effect was detected in [53], [55].
6 Concluding remarks.

A. Functional approach.

In order to estimate the random error, i.e. the deviation $u_{n,N} - u_n$ in some space - time norm $|| \cdot || = || \cdot ||_{X,T}$, more exactly, to estimate the value

$$Q_{n,N}(v) \overset{def}{=} P(\sqrt{N}||u_{n,N} - u_n||_{X,T} > v), \ n = 2, 3, \ldots, \ v = \text{const} > 0, \quad (6.0)$$

we to use the Central Limit Theorem (CLT) in the correspondent Banach space $(K, || \cdot ||_{X,T})$, in accordance with which there exists a limit

$$\lim_{N \to \infty} Q_{n,N}(v) \overset{def}{=} Q_{n,\infty}(v), \quad (6.1)$$

where

$$Q_{n,\infty}(v) = P(||\xi(\cdot, \cdot)|| > v), \quad (6.2)$$

$\xi(x,t)$ is centered Gaussian distributed random field with values in the space $K$, (mod $P$).

For the space of continuous functions it is proved, e.g. in [53], [56], [54], [55], [57], [48], [52], [74]; in the classical Lebesgue - Riesz spaces $L_p$ - in [58]; in the mixed Lebesgue-Riesz spaces-in [69], [70]; in the mixed hybrid Lebesgue-continuous spaces- in [71] etc.

The behavior as $v \to \infty$ of the probability $Q_{n,\infty}(v)$, asymptotical as well as non - asymptotical is obtained in many works, see, e.g. [72]:

$$Q_{n,\infty}(v) \sim C(X,T) \ v^{\kappa-1} \ \exp(-v^2/(2\sigma^2)), \ C(X,T), \kappa, \sigma^2 = \text{const} > 0. \quad (6.3)$$

Equating the approximation of a value $Q_{n,\infty}(v)$ with $v \geq 3\sigma$, in the right - hand of (6.3) to the value $\delta$ :

$$C(X,T) \ v(\delta)^{\kappa-1} \ \exp(-v(\delta)^2/(2\sigma^2)) = \delta, \ v(\delta) \geq 3\sigma, \quad (6.4)$$

where $1 - \delta$ is reliability of the confidence interval, for instance, 0.95 or 0.99, we obtain the asymptotical confidence region for the function $u_n(x,t)$ in the norm $|| \cdot ||$ of the form

$$||u_n - u_{n,N}|| \leq \frac{v(\delta)}{\sqrt{N}}, \quad (6.5)$$

B. General optimization.

The inequality (1.0a) follows from (6.2) - (6.5). Moreover, it may be proved under simple condition that there exists finite function $K = K_p(n), \ p > 1$ for which

$$\forall n = 1, 2, \ldots \Rightarrow [E||u_{n,N} - u_n||^p]^{1/p} \leq K_p(n) \ N^{-1/2},$$
and analogous conclusion may be obtained for the $G(\psi)$ norm for the norm difference $||u_{n,N} - u_n||$.

The accuracy calculation show us that the constant $K(n)$ in (1.0a) is proportional to the value $D(n)$, where $D(n)$ is introduced and investigated in the third section, and obviously the relation $\sqrt{N} >> D(n)$ should be performed.

The common error $||u - u_{n,N}||$, including the deterministic part $\leq q^n$ and probabilistic part $\leq D(n)/\sqrt{N}$ does not exceed the value

$$\Delta := C \left( q^n + \frac{D(n)}{\sqrt{N}} \right).$$

(6.6)

It appears naturally the following qualitative optimization problem by fixed great value $N$:

$$q^n + \frac{D(n)}{\sqrt{N}} \rightarrow \min_n : D(n) << \sqrt{N}.$$  

(6.7)

The practical computation taking into account the rate of increasing of the sequence $\{D(n)\}$ show us that the optimal value $n$ is 4 - 5.

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