Schrödinger cats and quantum complementarity

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Complementarity tells us we cannot know precisely the values of all the properties of a quantum object at the same time: the precise determination of one property implies that the value of some other (complementary) property is undefined. E.g., the precise knowledge of the position of a particle implies that its momentum is undefined. Here we show that a Schrödinger cat has a well defined value of a property that is complementary to its “being dead or alive” property. Then, thanks to complementarity, it has an undefined value of the property “being dead or alive”.

In other words, the cat paradox is explained through quantum complementarity: of its many complementary properties, any quantum system, such as a cat, can have a well defined value only of one at a time. Schrödinger’s cat has a definite value of a property which is complementary to “being dead or alive”, so it is neither dead nor alive. Figuratively one can say it is both dead and alive. While this interpretation only uses textbook concepts (the Copenhagen interpretation), apparently it has never explicitly appeared in the literature. We detail how to build an Arduino based simulation of Schrödinger’s experiment based on these concepts for science outreach events.

The Schrödinger’s cat argument was published by Schrödinger’s in [1] and devised during a discussion with Einstein. It was a provocation to show that quantum effects cannot be naively hidden in the microscopic realm, but through linearity (and entanglement) they can affect also macroscopic systems such as cats, giving rise, in Schrödinger’s words, to “ridiculous cases”. While quantum mechanics is by now well established, the debate on its interpretation is still very much open: a huge number of competing interpretations vie for an explanation of the quantum formulas. The Schrödinger cat is a perfect testbed for such explanations [2].

In this paper, we give a very simple interpretation to the cat, the “complementarity interpretation”, based on textbook concepts, namely on the Copenhagen interpretation. We then show how to use it to create a simulated experiment which is effective to communicate to the general public concepts such as the superposition principle and quantum complementarity. Even though our explanation could have been proposed by Bohr himself (who famously championed quantum complementarity), Bohr never explicitly addressed the cat [3]. In his discussions with Schrödinger [4], he did address classicality, but only limiting himself to the measurement apparatuses, which is not what Schrödinger’s argument embodies. Surprisingly, to our knowledge the proposal presented in this paper (Schrödinger’s cat through complementarity) has not appeared in the English literature previously [5].

The outline of the paper follows. We start by explaining quantum complementarity and how it is embodied in quantum mechanics. We then recall Schrödinger’s argument and show how quantum complementarity can be used to make sense of the superposition. We conclude detailing how one can construct a simulation of Schrödinger’s experiment in a cardboard box controlled through an Arduino microcontroller with a simulated cat.

Quantum complementarity:— All systems are described by a collection of their properties. For example, the state of a featureless particle is described by two properties, its position and its momentum. A spin 1/2 is described by the components $J_x$, $J_y$, $J_z$ of its angular momentum. A cat is described by a myriad of complicated properties (color, weight, furiness, position, etc.). One of these is its “being dead or alive”: it can have two distinct values ‘dead’ and ‘alive’ (we will neglect intermediate ‘moribund’ values).

Properties in quantum mechanics are described by observables, such as the position observable $X$. The states of quantum systems are described by vectors $|\psi\rangle$, and the state of systems which have a definite value of a property are eigenstates $|x\rangle$ of the observable. The corresponding eigenvalue $x$ is the value of the property in such state. States are vectors because of the superposition principle: any state is a linear combination of eigenvectors of some observable. Also any eigenstate of an observable is a linear combination of eigenstates of other observables. The Born rule tells us that, if we prepare a system in a (nontrivial) superposition of eigenstates of the observable we are measuring, we will get a probabilistic answer: that observable property is not well defined in that system state. (The probabilities are given by the square moduli of the linear combination coefficients. We discuss below whether such probabilities are due to ignorance of something, or to the fact that this something is undefined.)

Complementarity, hence, follows by joining superposition with the Born rule. Consider two properties: observable $\hat{H}$ and observable $\hat{S}$. Suppose that the $\hat{H}$ eigenstates are all nontrivial superpositions of $\hat{S}$ eigenstates and that the state of the system is in an eigenvector of the $\hat{S}$ observable. Then property $\hat{S}$ will have a definite value, but $\hat{H}$ will not. Complementarity [6] [7] can be loosely stated as: “if the value of one property is determined exactly, the values of some other properties will be undefined”. E.g. if the position of the particle is exactly
known, its momentum is completely undefined: position eigenvectors are equal superpositions of all momentum eigenvectors\(^1\). Complementarity is usually vaguely defined in textbooks (if at all), but it is just an aspect of the superposition principle (which is similarly vaguely defined [8]).

The cat argument:— Schrödinger suggests closing a cat in a perfectly isolated box, which contains an “internal device” which opens a poison vial if an atom decays. He then suggest to use an atom that has a half-life of one hour and to wait for one hour. What will happen? Of course, the atom has probability one half of decaying, so the cat has probability one half of dying. At first sight, this is not paradoxical at all.

The paradox emerges if one analyzes more carefully the predictions of quantum mechanics: since the atom is a quantum system, it evolves through the Schrödinger equation which describes a deterministic evolution: after one hour the atom has an equal probability amplitude of being decayed or non decayed. Namely, the state of the atom is an equally weighted superposition: \( \frac{1}{\sqrt{2}} (|\text{decayed} \rangle + e^{i \varphi} |\text{non decayed} \rangle) \) where \( \varphi \) is a phase that depends on the details of the evolution (we will choose \( \varphi = 0 \) for simplicity). Since the box is perfectly isolated, the evolution of the whole box can be described through the Schrödinger equation\(^2\), and the cat (through the above device) inherits the properties of the atom. Namely, the box prepares the cat in the state
\[
\frac{1}{\sqrt{2}} (|\text{dead} \rangle + |\text{alive} \rangle) ,
\]
where for simplicity of notation, the two kets \(|\text{dead} \rangle\) and \(|\text{alive} \rangle\) represent the (entangled) state of all the degrees of freedom in the box: all the atoms and photons that compose the cat, its fleas, the poison vial, the radioactive atom, the molecules of air in the cat’s lungs and in the rest of the box and so on\(^3\). This state is a superposed state (“smear psi-function”, in the words of Schrödinger), which is a situation physically distinguishable from the situation in which the box contains a cat which is dead or alive with probability one half. This last situation is described by a mixed state of the form:
\[
\frac{1}{2} (|\text{dead} \rangle \langle \text{dead}| + |\text{alive} \rangle \langle \text{alive}|) .
\]
The two states \([1]\) and \([2]\) are distinguishable only if one looks at a property that is complementary to “being dead or alive”, as discussed below. If, instead we only consider the property of “being dead or alive”: we cannot distinguish them, they both describe a box containing a cat that is dead or alive with probability one half.

Consider two complementary observables: the first observable “being dead or alive” \( \hat{H} \) has the two orthogonal states \(|\text{dead} \rangle\) and \(|\text{alive} \rangle\) as eigenstates, corresponding to the two possible values of the cat’s health; the second observable \( \hat{S} \) is defined as the one with eigenstates \(|+\rangle = (|\text{dead} \rangle + |\text{alive} \rangle)/\sqrt{2}\) and \(|-\rangle = (|\text{dead} \rangle - |\text{alive} \rangle)/\sqrt{2}\) and eigenvalues +1 and −1 respectively. For lack of better words, we will call this property “plus or minus”\(^6\). The observable \( \hat{S} \) can be measured by using some quantum interferometric experiment \([12]\), where the \(|+\rangle\) and \(|-\rangle\) states refer to constructive and destructive interference respectively. Of course, the apparatus which measures such an observable will be insanely impractical (understatement) \([12]\), but there is no in-principle reason why it cannot be built. The \( \hat{S} \) property and the \( \hat{H} \) property are complementary, since the eigenvalues of one are superpositions of the eigenvalues of the other, and viceversa.

Since the cat is in the state \([1]\), the measurement of the “plus or minus” \( \hat{S} \) observable will have outcome +1 with certainty. Thus, the cat possesses a definite value (plus) of the \( \hat{S} \) “plus or minus” property, which is complementary to \( \hat{H} \) “being dead or alive”. Thus it is neither dead nor alive. As a figure of speech, it is customary to say that it is “dead AND alive at the same time” (since it’s neither).

This is the paradoxical situation: we have experience of dead cats or of alive cats. We can even easily think of cats in a box of which we do not know if they are dead or alive, a situation described by \([2]\). However, we cannot even imagine a quantum-superposed (“smear”) cat that is neither dead nor alive, described by the state \([1]\). It would not even remotely look like a cat: it would look like some interference pattern in an enormously complicated interferometer (even though it is a cat: the interferometer is showing a strange property of the cat that is complementary to its usual properties). It is so inconceivable to think of such a “cat” that we do not have appropriate words to describe the situation (except in

\(^1\) Similarly, if the \( z \) component \( J_z \) of the angular momentum of a spin \( \frac{1}{2} \) particle is exactly known, the \( x \) and \( y \) components are undefined: the \( J_z = \pm \frac{1}{2} \) eigenstate \(|J_z = \pm \frac{1}{2} \rangle\) is an equally weighted superposition \(|J_z = \pm \frac{1}{2} \rangle\) of the eigenstates relative to both possible values \( \pm \frac{1}{2} \) of the \( z \) component \( J_z \) (and similarly for \( J_y \)).

\(^2\) We emphasize that, from the point of view of anyone outside the box, the evolution of the cat is not a measurement. Indeed, a perfectly isolated evolution cannot be seen as a measurement since the bare minimum that any quantum measurement must satisfy is that it must provide an outcome. An isolated evolution cannot provide any outcome by definition. Moreover, the quantum evolution postulate asserts that isolated systems evolve according to the Schrödinger equation. In this paper we will not be concerned of the point of view of the cat [9,10].

\(^3\) While the nomenclature “quantum entanglement” was coined by Schrödinger in the cat paper [11] (and was discovered by Einstein, Podolsky, Rosen [11]), it is not mentioned in the exposition of the cat and appears much later in the paper. Entanglement is not really necessary to understand the cat argument, unless one wants to discuss separately the single degrees of freedom of the contents of the box.
the language of mathematics) and we resort to somewhat inappropriate “dead AND alive” statements. The reason we do not have experience of such cats is due to the incredible complexity of the experiment necessary to measure the $\hat{S}$ property in a cat. This, in turn, is due to decoherence: the almost unavoidable interaction with the environment of macroscopic systems such as a cat means that some of the cat’s properties (such as $\hat{H}$) are evident, whereas others (such as $\hat{S}$) are not. This selection of evident properties (technically, einselection [13]) is due to the fact that local interactions with superpositions of states localized in macroscopically different positions (such as a superposition of a dead and alive cat) will decohere rapidly: the environment becomes immediately entangled with the system. Instead, localized states such as a dead or an alive cat will not decohere at all. This is the modern view of the emergence of classicality from a quantum world. In other words, the reason why we never see superposed cats is, according to quantum mechanics, only due to the practical difficulty in isolating a cat from its environment and in carrying out the appropriate interferometric experiment, and not because of any fundamental reason: sophisticated enough experiments can achieve it in practice [14, 15], although it would be vastly impractical on the scale of a cat [12].

In summary, using quantum complementarity, the cat paradox is readily explained: There exist a property $\hat{S}$ “plus or minus” that is complementary to $\hat{H}$ “dead or alive”. Since the cat has a definite value of the “plus or minus” property (“plus”, in this case), then it cannot have a definite value of the complementary property “dead or alive”.

Complementarity and the Bell theorem:— One may be tempted to dismiss quantum complementarity by stating that a cat in the state $(|\text{dead}\rangle + |\text{alive}\rangle)/\sqrt{2}$ has also a definite value of the “dead or alive” property but, somehow, we are ignorant of it. Namely, we can argue that complementarity is not a limitation on the values of the properties that an object can possess, but rather on the fact that a quantum measurement can extract only one of them at a time (and, possibly, perturbs the values of the complementary ones).

In order to maintain such position, however, one needs to give up Einstein locality (technically the “no signaling condition”). Indeed, Bell’s theorem [16] tells us that quantum mechanics is incompatible with local hidden (i.e. unknown) values. A simple exposition of Bell’s theorem and its implications is in [17]. This means that, if one wants to retain Einstein locality, values of complementary properties are not even defined in a quantum system. One cannot say that they are defined, but unknown. Of course, one could retain unknown values at the cost of locality, as is done in Bohmian mechanics, where the hidden variables are notoriously nonlocal (signaling) [4]. In this paper we take the point of view of Copenhagen quantum mechanics.

In fact, by measuring complementary observables on entangled states, one can observe [18–20] correlations among the measurement outcomes which are incompatible (Bell’s theorem) with any possible prescription that assigns pre-determined values to these properties, unless one postulates that the measurement of one property somehow is able to nonlocally change the value of a property of the correlated distant system. Most physicists (not all!) are unwilling to abandon Einstein locality, because of the causality problems that would emerge if one could access these hypothetical “instantaneously” [5] propagating hidden variables and because one would have to introduce a preferred reference frame which would violate the postulate of relativity that all inertial frames are equivalent. Then Bell’s theorem forces us to admit that if one observable is well defined, then the values of complementary ones are not even defined. I emphasize that “undefined” is very different from “unknown”. While an undefined quantity is clearly also unknown, the opposite is not true. For Copenhagen, the state [1] represents a state where the “being dead or alive” property is undefined, whereas the state [2] represents a state where it is unknown [6].

Then the Schrödinger cat, thanks to complementarity, has an undefined value of whether it is alive or dead. A quite weird situation for the poor cat!

Public outreach:— In this section we detail how one can create a simple Arduino-based simulation of the cat experiment using a cardboard box.

On the box front (see Fig. 1) there are a series of input switches. On the left a press-button switch simulates

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4 At the cost of stating the obvious, quantum mechanics is nonlocal in the quantum sense (i.e. it contains nonlocal correlations), but it is an interpretation-dependent statement whether it is nonlocal in the Einstein sense [17]. Bohmian hidden variables are nonlocal in the Einstein sense (their knowledge would allow signaling), but they are absent in the Copenhagen interpretation which can then be considered local in the Einstein sense [21–23], thanks to complementarity.

5 This word is meaningless unless one specifies which reference frame it refers to.

6 Historically, at the time the cat paper was published, Einstein [11] and Schrödinger [1] were convinced that quantum mechanics was incomplete. Einstein’s stance referred to the fact that the quantum state should only refer to information about an ensemble of systems, and not about a single system [26–27]. He apparently never proposed that deterministic hidden variables should be used to describe single systems, and criticized Bohm for doing so [28]. Schrödinger’s stance (11, Sec.13) seems to veer in the direction that hidden variables (not present in Copenhagen’s interpretation) might keep track of the values of complementary observables. Bohr, of course, dissented [7]. Bell’s theorem (formulated after Einstein’s and Schrödinger’s deaths) showed that any hidden variables that lead to the statistical predictions of quantum mechanics would violate Einstein’s locality, arguably a vindication of Bohr’s point of view.
the preparation of the cat state, namely the activation
of Schrödinger’s infernal machine. After pressing it, the
display announces that the cat is in the “plus” state, the
one described by the state \( |1 \rangle \) relative to the eigenvalue
+1 of the \( \hat{S} \) operator.

Then, using a selector switch to the right of the press-
button, the “experimenter” can choose which of the
two complementary properties to measure: either the
“dead/alive” or the “plus/minus” property. The first
refers to the measurement of the cat in the \( |\text{dead} \rangle \) and
\( |\text{alive} \rangle \) basis, the second is a measurement of \( \hat{S} \), namely
the \( (|\text{dead}\rangle+|\text{alive}\rangle)/\sqrt{2} \) and \( (|\text{dead}\rangle-|\text{alive}\rangle)/\sqrt{2} \) basis.
A led lights up to confirm the choice of measurement: a
green led for the \( \hat{S} \) measurement and a white led for the
dead/alive measurement.

Finally, a last press-button to the right simulates the
activation of the measurement of the previously chosen
property. The display returns the measurement outcome.
If the cat state is in one of the eigenstates of the chosen
measurement, then that is the outcome. E.g. if the cat
is in an \( |\text{alive} \rangle \) state and the “dead/alive” measurement
is selected, the outcome will be “alive”. Otherwise, the
outcome is chosen at random with uniform distribution
and is the cat’s state is updated to the eigenstate relative
to the obtained outcome (collapse of the state).

After the outcome is presented, the display shows the
current state of the cat, and one can perform a new mea-
surement or one can reprepare the cat in the state \( |1 \rangle \) by
pressing the first button.

A switch-sensor determines whether the box is opened.
In this case, the measured property (indicated by the led)
is automatically switched to the “dead/alive” property,
and the cat is measured in the \( |\text{dead} \rangle \) or \( |\text{alive} \rangle \) basis.

Clearly, this is not a historically accurate reproduc-
tion of Schrödinger’s proposal, since he never advocated
for complementary properties. Nonetheless, the goal is
science outreach: explaining in a clear fashion quantum
complementarity and superposition, rather than histori-
cal accuracy.

The box is powered by an Arduino nano microcon-
troller (but any Arduino variant will work). The instruc-
tions to build it, the software and an illustrative movie
of its operation can be found here [28]. The cost is about
25-30 dollars (or euros), excluding the power source. It
can be powered through any powerbank or usb charger.
An approximate bill of materials of the main components:
Arduino nano: 4$; LCD display ST7920: 8$; cardboard
box: 5$; stuffed cat: 8$. Since the box is intended for an
Italian public, all labels on the box and messages on the
display are in Italian, but they can be readily translated
to any local language.

Conclusions:— In conclusion, we have presented a
simple interpretation of Schrödinger’s cat, based on
quantum complementarity. The superposed cat has a
definite value of a property \( \hat{S} \) with eigenstates \( (|\text{dead} \rangle \pm
|\text{alive} \rangle)/\sqrt{2} \) which is, hence, complementary to the prop-
erty “being dead or alive”. Such interpretation uses only
standard textbook concepts. A simulated Schrödinger
cat experiment is presented, together with the indica-
tions of how to cheaply replicate it. It has been tested in
several science outreach occasions.

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