Quantum Coulomb gases

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1 Introduction

Ordinary matter is composed of electrons (negatively charged) and nuclei (positively charged) interacting via electromagnetic forces. The electric potential between two such particles of charges $Q$ and $Q'$ located at $r$ and $r'$ in $\mathbb{R}^3$ is $\frac{QQ'}{|r - r'|}$. The Coulomb potential poses two difficulties: (1) The local singularity and (2) its long range. One has to understand why the local singularity does not cause instabilities and why the long range does not have strong macroscopic effects. One of the first
major triumphs of the theory of quantum mechanics is the explanation it gives of
the stability of the hydrogen atom (and the complete description of its spectrum)
and of other microscopic quantum Coulomb systems. It, surprisingly, took nearly
forty years before the question of stability of everyday macroscopic objects was
even raised. The rigorous answer to the question came shortly thereafter in what
came to be known as the *Theorem on Stability of Matter* proved first by Dyson
and Lenard and later, in a more simple and transparent way by Lieb and Thirring.
Since these seminal works, the proof of stability of matter has been extended to
several different settings, including relativistic systems and systems in the presence
of dynamic electromagnetic fields.

We will, in particular, discuss the importance of particle statistics, i.e., whether
the particles are bosons or fermions. Using Bogolubov’s theory for Bose gases
Dyson concluded that charged bosonic particles would be macroscopically unstable.
That Bogolubov’s theory gives the correct description of charged Bose systems
was proved in a series of papers by the author in collaboration with E.H. Lieb.

Having proved stability of matter the next question is whether one can establish
the thermodynamics of charged systems, i.e., the existence of the thermodynamic
limit. This was originally achieved by Lieb and Lebowitz. We will describe a new
approach to the existence of the thermodynamic limit, which applies to many dif-
ferent quantum Coulomb systems, possibly in the presence of an underlying lattice
structure. This generalizes earlier work of Fefferman.

An important theme in these notes is the use of functional inequalities, among
which a prominent role is played by the Lieb-Thirring inequality.

Another important theme will be how to control the screening of the Coulomb
potential.

The notes are organized as follows. In Section 2 we derive the classical Hamil-
tonian for charged particles interacting with electromagnetic fields. In Section 3
we discuss different ways of quantizing the system. We may quantize the particles
and leave the fields unquantized or quantize both the particles and the fields. Moreover,
we may consider the particles relativistically or classically. This will give a variety
of different models. In Section 4 we discuss stability both in the sense of stability
of individual atoms and in the sense of stability of macroscopic matter. Finally, in
Section 5 we discuss situations where stability fails, in particular the case of bosonic
matter.

### 2 Classical “point” charges

We consider $N$ classical particles with charges $Q_1, \ldots, Q_N \in \mathbb{R}$ situated at points
$r_1, \ldots, r_N \in \mathbb{R}^3$. Dimension 3 is of course the physical space dimension, but a nat-
ural questions would be whether the discussion of charged system could be general-
ized to other dimensions. It is however not entirely clear what the correct physical
questions would be and we will there restrict the discussion to dimension 3.
We would ultimately like to consider the particles as point charges. Unfortunately this will however lead to divergencies. In order to avoid this we will initially assume that each particle is given by a charge distribution $Q_i \chi_R(r - r_i)$ where $\chi_R(r) = R^{-3} \chi(r/R)$ and $\chi \in C_0(\mathbb{R}^3)$ (a continuous compactly supported function) with $\int_{\mathbb{R}^3} \chi = 1$.

We will start the discussion of this system of charged particles from the Lagrangian of the particles and the electromagnetic field. It is

$$L_R(r_j, A, V) = \sum_{j=1}^{N} \left( T^*_j(\dot{r}_j) - Q_j \dot{r}_j \cdot A * \chi_R(r_j) - Q_j V * \chi_R(r_j) \right)$$

$$+ \frac{1}{8\pi} \int (|\partial_t A + \nabla V|^2 - |\nabla \times A|^2).$$

where

- $\dot{r}_j$ denotes the velocity of particle $j$.
- $T^*_j(v)$ is the Legendre transform of the kinetic energy $T_j(p)$ as a function of momentum $p$. We assume that the kinetic energy functions $T_j$ are convex functions on $\mathbb{R}^3$. For a non-relativistic particle of mass $m_j$ we have $T_j(p) = \frac{1}{2} m_j p^2$ and thus $T^*_j(v) = \frac{1}{2} m_j v^2$ and for a relativistic particle it is $T_j(p) = \sqrt{p^2 + m_j^2} - m_j$ and hence $T^*_j(v) = m_j - m_j \sqrt{1 - v^2}$. We are using units in which the speed of light is 1 (a convention we will use throughout these notes).
- $A$ is the vector potential and the magnetic field is $B = \nabla \times A$
- $V$ is the electric potential. The electric field is

$$E = -\partial_t A - \nabla V.$$

In order to go to a Hamiltonian description we will choose Coulomb gauge or more precisely require that

$$\nabla \cdot \partial_t A = 0,$$

i.e., we assume that $\nabla \cdot A$ is independent of time. We will also for simplicity assume that $A$ decays sufficiently fast that we are allowed to ignore boundary terms when integrating by parts.

With the Coulomb gauge condition we see that

$$E_\perp = -\partial_t A$$

is the divergence free part of the electric field. The total electric field is $E = E_\perp - \nabla V$ and we have

$$\int |E|^2 = \int |E_\perp|^2 + \int |\nabla V|^2.$$

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1 Strictly speaking, even if we use a relativistic kinetic energy, this Lagrangian is not relativistically invariant. The reason is that we consider the particles as rigid bodies, which do not Lorenz contract as they move. We will here ignore this additional complication. The Lagrangian in the form given here is that of the Abraham model of charged particles [32].
The electric potential $V$ is not a dynamic variable in the sense that that $\partial_t V = \dot{V}$ does not occur in $L_R$. The equation for $V$ is

$$0 = \frac{\delta L_R}{\delta V} = -\frac{1}{4\pi}\Delta V - \sum_{j=1}^N Q_j \chi_R(r - r_j),$$

where we have used the Coulomb gauge condition. We recognize the equation for $V$ as Gauss' law

$$4\pi \sum_{j=1}^N Q_j \chi_R(r - r_j) = -\Delta V = -\nabla \cdot (\partial_t A + \nabla V) = \nabla \cdot E.$$

In the Hamiltonian formalism this is a constraint equation. The solution is

$$V(r) = \sum_{j=1}^N Q_j \int \chi_R(r) * |r - r_j|^{-1} dr.$$

The canonical variable dual to $r_j$ is

$$p_j = \nabla_j L_R = \nabla T_j^*(\dot{r}_j) - Q_j A * \chi_R(r_j).$$

The canonical variable dual to $A$ is

$$\frac{\delta L_R}{\delta \partial_A} = -\frac{1}{4\pi} \partial_t A = -(4\pi)^{-1} E_\perp,$$

due to the Coulomb gauge condition. We then find the Hamiltonian function (where we assume that $T_j$ is convex such that the double Legendre transform of $T_j^{**} = T_j$)

$$H_R(r_j, p_j, A, E_\perp) = \sum_{j=1}^N p_j \dot{r}_j - \frac{1}{4\pi} \int \partial_t A \cdot E_\perp - L_R(r_j, A, V)$$

$$= \sum_{j=1}^N T_j \left( p_j + Q_j A * \chi_R(r_j) \right) + \sum_{j=1}^N Q_j V * \chi_R(r_j)$$

$$+ \frac{1}{8\pi} \int (|E_\perp|^2 + |\nabla \times A|^2) - \frac{1}{8\pi} \int |\nabla V|^2$$

$$= \sum_{j=1}^N T_j \left( p_j + Q_j A * \chi_R(r_j) \right) + \frac{1}{2} \sum_{j=1}^N Q_j V * \chi_R(r_j)$$

$$+ \frac{1}{8\pi} \int (|E_\perp|^2 + |\nabla \times A|^2)$$

$$= \sum_{j=1}^N T_j \left( p_j + Q_j A * \chi_R(r_j) \right)$$
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\[ + \frac{1}{2} \sum_{j=1}^{N} \sum_{i=1}^{N} Q_j Q_i \int \frac{\chi_k(r-r_i)\chi_k(r'-r_j)}{|r-r'|} dr dr' \]
\[ + \frac{1}{8\pi} \int (|E_\perp|^2 + |B|^2). \]

If we subtract the (divergent) self-energy

\[ \sum_{j=1}^{N} \frac{Q_j^2}{2} \int \int \frac{\chi_k(r)\chi_k(r')}{|r-r'|} dr dr' = R^{-1} \sum_{j=1}^{N} \frac{Q_j^2}{2} \int \int \frac{\chi(r)\chi(r')}{|r-r'|} dr dr' \]

and assume that \( A \) is continuous at \( r_j \), we see that as \( R \) tends to zero \( H_R \) converges pointwise to the Hamiltonian

\[ H(r_j, p_j, A, E_\perp) = \sum_{j=1}^{N} T_j (p_j + Q_j A(r_j)) + \sum_{1 \leq i < j \leq N} \frac{Q_i Q_j}{|r_i - r_j|} \]
\[ + \frac{1}{8\pi} \int (|E_\perp|^2 + |B|^2). \] (1)

Here \( r_j \) and \( p_j \) are dual canonical variables as are the field variables \( A \) and \( -\frac{1}{4\pi} E_\perp \). Unfortunately, this Hamiltonian is only formal and suffers from singularities (the field \( A \) that solves Hamilton’s equations will be singular at \( r_j \)) leading to severe difficulties in describing the motion of classical point charges.

If external fields \( V_{ex} \) and \( A_{ex} \) are present, the energy is

\[ \sum_{j=1}^{N} \left( T_j (p_j + Q_j (A + A_{ex})(r_j)) + Q_j V_{ex}(r_j) \right) + \sum_{1 \leq i < j \leq N} \frac{Q_i Q_j}{|r_i - r_j|} \]
\[ + \frac{1}{8\pi} \int (|E_\perp|^2 + |B|^2). \]

3 Charged quantum gases

We now discuss the quantization of the Hamiltonian of charged point particles. We emphasize that it has not been possible to define a fully relativistically invariant and causal theory of quantum electrodynamics (QED) and all the models we describe here are at best approximations to such a theory (if it exists). All models discussed here are mathematically well defined (except when otherwise stated explicitly).

There are several levels of quantization that may be considered.

- We can leave the fields \( A \) and \( E_\perp \) classical and quantize the particles, i.e., describe them by a square integrable wave function \( \psi(r_1, \ldots, r_N) \).
- We may quantize the particles and the fields, i.e., describe the particles in terms of a wave function \( \psi \) and turn the fields into operator valued functions \( A \) and \( E_\perp \). This would require introducing some cut-off regularization in the fields.
We may second quantize the particles, i.e., let also $\psi$ be an operator valued function. This procedure is necessary if we consider relativistic particles described by the Dirac operator.

### 3.1 Quantized particles and classical fields

The variables of the system are the 3-dimensional vector fields $A, E_\perp$ (assumed to satisfy appropriate regularity and decay properties, at least implying that $E_\perp$ and $B = \nabla \times A$ are square integrable) and the wave function $\psi$, which is a normalized function in $\bigotimes_{j=1}^N [L^2(\Omega)]^{\nu_j}$, where $\Omega \subseteq \mathbb{R}^3$ (say an open set) and $\nu_j$ is a positive integer counting the number of internal degrees of freedom of particle $j$, (e.g. a particle of spin $s$ would correspond to $\nu_j = 2s + 1$). We shall write

$$\psi = \psi(r_1, s_1, \ldots, r_N, s_N), \quad r_j \in \Omega, \quad s_j = 1, \ldots, \nu_j.$$  

The energy of the system is given by

$$E_N(\psi, A, E) = \langle \psi, H_N(A, E_\perp)\psi \rangle,$$

where $\langle \psi, \phi \rangle$ refers to the inner product of $\psi, \phi \in \bigotimes_{j=1}^N [L^2(\Omega)]^{\nu_j}$ and $H_N$ is the (unbounded) operator (depending on $A$ and $E_\perp$)

$$H_N(A, E_\perp) = \sum_{j=1}^N \left( T_j(-i\nabla_j + Q_j(A + A_{\text{ex}})(r_j)) + Q_j V_{\text{ex}}(r_j) \right) + \sum_{1 \leq i < j \leq N} \frac{Q_i Q_j}{|r_i - r_j|} + \frac{1}{8\pi} \int (|E_\perp|^2 + |B|^2).

We will throughout be using units in which the reduced Planck constant $\hbar = 1$. The last integral above acts as a ($A$ and $E_\perp$ dependent) scalar in the Hilbert space. The Hamiltonian $H(A, E_\perp)$ depends also on the exterior fields $V_{\text{ex}}$ and $A_{\text{ex}}$, but we suppress this in the notation as these fields usually remain fixed. In fact, we will mostly, and unless otherwise explicitly stated, assume that the exterior fields vanish, i.e., $V_{\text{ex}} = 0$ and $A_{\text{ex}} = 0$.

The expectation value $E_N(\psi, A, E_\perp)$ is not defined for all $\psi$ in the Hilbert space

$$\bigotimes_{j=1}^N [L^2(\Omega)]^{\nu_j}.$$

We will here avoid the discussion of domains of self-adjointness of the operator $H(A, E_\perp)$. We will instead restrict attention to $\psi$ in the the subspace of smooth functions with compact support, i.e., $[C^\infty_0(\Omega^N)]^{\nu_1 \cdots \nu_N} \subseteq \bigotimes_{j=1}^N [L^2(\Omega)]^{\nu_j}$.

One of the main issues we will discuss in these notes is the question of stability, i.e., whether $E_N(\psi, A, E_\perp)$ is bounded from below independently of $\psi$ (normalized),
If such a lower bound holds the operator $H(A,E^\perp)$ has a self-adjoint Friedrichs extension and we are actually making claims about this extension. From our point of view the only complication due to considering the restriction to $C^\infty_0$ is that a possible ground state (a state achieving the lowest possible energy) is most likely not represented by an element in $C^\infty_0$, but only by an element in the Friedrichs extended domain. We shall, however, not be concerned with the actual ground states, but only the energy, so we ignore this issue.

Since the three coordinates of $-i\nabla_j + Q_jA(r_j)$ correspond to in general non-commuting operators, we must discuss the meaning of $T_j(-i\nabla_j + Q_jA(r_j))$. We will, in fact, only consider examples where the functions $T_j(p)$ can be written in terms of (possibly matrix-valued) polynomial expressions of $p$ in such a way that the meaning of $T_j$ (at least on a suitable domain) will be clear. The examples we will consider are

- Non-relativistic kinetic energy operators, where $T_j(p) = (2m_j)^{-1}p^2$, i.e., the operator is
  \[ T_j(-i\nabla_j + Q_jA(r_j)) = (2m_j)^{-1}(-i\nabla_j + Q_jA(r_j))^2. \]  
  We will refer to particles with this kinetic energy as non-relativistic particles. This is the kinetic energy used when treating non-relativistic atoms and molecules or ordinary matter.

- Relativistic kinetic energy operators, where $T_j(p) = (p^2 + m_j^2)^{1/2} - m_j$, i.e., the operator is
  \[ T_j(-i\nabla_j + Q_jA(r_j)) = ((-i\nabla_j + Q_jA(r_j))^2 + m_j^2)^{1/2} - m_j. \]  
  (5)
  The square root of an operator is here defined in the spectral theoretic sense\(^2\). We will refer to particles with this kinetic energy as relativistic (or sometimes pseudo-relativistic) particles. Both relativistic and non-relativistic particles may have internal degrees of freedom corresponding to $\nu_j$, being greater than one.

- The non-relativistic and relativistic Pauli-operators. These are operators acting on two-component vector valued functions given by inserting the operator $\sigma_j \cdot (-i\nabla_j + Q_jA(r_j))$ into the kinetic energy functions above. Here $\sigma = (\sigma^1, \sigma^2, \sigma^3)$ is the vector of $2 \times 2$ Pauli matrices
  \[ \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]
  (The subscript $j$ on $\sigma$ above indicates that it acts on the internal degrees of freedom of particle $j$.) Thus in this case $\nu_j = 2$. The resulting kinetic energy operators are
  \[ T_j(-i\nabla_j + Q_jA(r_j)) = (2m_j)^{-1}(\sigma_j \cdot (-i\nabla_j + Q_jA(r_j))^2 \]  
  \[ (6) \]  

\(^2\) The operator inside the square root is defined as a self-adjoint operator by Friedrichs extending it from the domain of smooth functions with compact support.
for non-relativistic Pauli particles and
\[
T_j \left( -i \nabla_j + Q_j \mathbf{A}(r_j) \right) = \left( \mathbf{\sigma} \cdot \left( -i \nabla_j + Q_j \mathbf{A}(r_j) \right) \right)^2 + m_j^2 \right)^{1/2} - m_j \quad (7)
\]
for relativistic Pauli particles. For the Pauli operator we have the Lichnerowicz’ formula
\[
\left( \mathbf{\sigma} \cdot \left( -i \nabla_j + Q_j \mathbf{A}(r_j) \right) \right)^2 = \left( -i \nabla_j + Q_j \mathbf{A}(r_j) \right)^2 + Q_j \mathbf{\sigma} \cdot \mathbf{B}(r_j) \quad (8)
\]
and we see that the Pauli operator includes the coupling of the particle spin to the magnetic field.

• We could also consider the $4 \times 4$ Dirac operator
\[
T_j(p) = \mathbf{\alpha} \cdot p + m_j \mathbf{\beta},
\]
i.e.,
\[
T_j (-i \nabla_j + Q_j \mathbf{A}(r_j)) = \mathbf{\alpha} \cdot (-i \nabla_j + Q_j \mathbf{A}(r_j)) + m_j \mathbf{\beta}_j
\]
where $\mathbf{\alpha}$ and $\mathbf{\beta}$ are standard $4 \times 4$ Dirac matrices, e.g.,
\[
\mathbf{\alpha} = \begin{pmatrix} 0 & \mathbf{\sigma} \\ \mathbf{\sigma} & 0 \end{pmatrix}, \quad \mathbf{\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]
using a $2 \times 2$-block notation. Thus in this case $v_j = 4$. In contrast to the other types of operators the Dirac operator, however, is not positive, in fact, not bounded below and we will therefore not be able to treat it unless we second quantize the particle fields. We will discuss this briefly below. A different approach to deal with the unboundedness from below of the Dirac operator is to restrict to the subspace of $L^2(\mathbb{R}^3)^4$ which corresponds to the positive spectral subspace of the Dirac operator. This approach is called the no-pair theory, but we will not discuss it further here.

### 3.2 Statistics of identical particles

Until now all particles have been considered as distinguishable, but if we have identical particles the issue of particle statistics plays an important role.

The $N$-particle space for $N$-identical particles moving in $\Omega \subset \mathbb{R}^3$ and with $v$ internal degrees of freedom is $\mathcal{H}_N = \bigotimes^N [L^2(\Omega)]^v$. On $\mathcal{H}_N$ we define the orthogonal projections $P_N^\pm$
\[
(P_N^\pm \psi)(r_1, s_1, \ldots, r_N, s_N) = \frac{1}{N!} \sum_{\sigma \in S_N} (\pm 1)^{\sigma} \psi(r_{\sigma^{-1}(1)}, s_{\sigma^{-1}(1)}, \ldots, r_{\sigma^{-1}(N)}, s_{\sigma^{-1}(N)}).
\]
They project onto the symmetric (+) or antisymmetric (−) subspaces. We denote these subspaces $\mathcal{H}_N^\pm = P_N^\pm \mathcal{H}_N$. We will also use the notation $H_N^\pm = \Lambda_N^\pm L^2(\Omega)^\vee$.

In case of $N$ identical particles, i.e., if the operators $T_j$ and the charges $Q_j$ are the same for all $j = 1, \ldots, N$, the Hamiltonian $H_N$ in (3) maps the subspaces $\mathcal{H}_N^\pm$ to themselves and it makes sense to restrict to these subspaces. We will write

$$E_N^\pm(\psi, A, E_\perp)$$

(9)

to emphasize that we restrict to $\psi \in \mathcal{H}_N^\pm$. In the symmetric case (+) we say that we have a system of $N$ bosons in the antisymmetric case (−) we say that we have a system of $N$ fermions.

It is of course also possible to have mixtures of several species of identical particles being fermions or bosons or even several species of identical particles together with a number of distinguishable particles. We leave it to the reader to work out the structure of the underlying Hilbert space and the Hamiltonian in the general case. We will look at specific examples later.

### 3.3 Grand canonical picture

It is often useful to consider a situation where the particle number is not specified at the outset, but where we would instead ask what the optimal particle number is in a given situation, e.g., what number of particles minimizes the energy. This picture is referred to as the grand canonical picture. The optimal particle number may if necessary be adjusted by adding a term $\mu$ times the particle number to the Hamiltonian. Such a parameter $\mu$ is called a chemical potential. We note that this is not the same as adding a constant to the exterior electric potential $V_{\text{ext}}$ as such a constant will multiply the total charge of the system.

In order to treat variable particle number we define the bosonic or fermionic Fock spaces

$$\mathcal{F}^\pm = \mathcal{F}^\pm((L^2(\Omega))^\vee) = \bigoplus_{N=0}^\infty \mathcal{H}_N^\pm,$$

with the convention that $\mathcal{H}_0^\pm = \mathbb{C}$. The element $1 \in \mathbb{C} = \mathcal{H}_0^\pm$ is referred to as the vacuum vector and will be denoted by $|0\rangle$. For a normalized vector $\psi \in \mathcal{F}^\pm$ we may write $\psi = \bigoplus_{N=0}^\infty \psi_N$, where $\psi_N \in \mathcal{H}_N^\pm$ with $\sum_{N=0}^\infty \|\psi_N\|^2 = 1$. We say that such a vector represents a grand canonical state and we define the grand canonical energy (with chemical potential $\mu$ included)

$$\mathcal{E}^\pm(\mu, \psi, A, E_\perp) = \sum_{N=0}^\infty \mathcal{E}_N^\pm(\psi_N, A, E_\perp) + \mu N\|\psi_N\|^2.$$  

(10)

As before this energy is not defined for all $\psi$, but we restrict to $\psi$ corresponding to finitely many particles, i.e., $\psi = \bigoplus_{N=0}^M \psi_N$ for some finite integer $M$ and where each $\psi_N$ is in $C_0^\infty$. 
Again it is possible to consider several species of identical particles in the grand canonical picture. We leave it to the reader to write down the Hilbert space and the energy (see also below).

### 3.4 Second quantization and quantization of fields

We shall here give a brief introduction to second quantization and discuss how to quantize particle fields and the electromagnetic fields.

For \( f \in L^2(\Omega)^y \) we define the annihilation operator \( \tilde{a}(f) : \mathcal{H}_N \to \mathcal{H}_{N-1} \) for \( N = 1, \ldots \) by

\[
(\tilde{a}(f)\psi)(r_1,s_1,\ldots,r_{N-1},s_{N-1}) = \sqrt{N} \sum_{j=1}^y \int \overline{f(r_j,s_j)} \psi(r_1,s_1,\ldots,r_{N-1},s_{N-1}) dr_N.
\]

The adjoint of this operator is \( \tilde{a}^+(f) : \mathcal{H}_{N-1} \to \mathcal{H}_N \) given by

\[
(\tilde{a}^+(f)\psi)(r_1,s_1,\ldots,r_{N-1},s_{N-1}) = \sqrt{N} f(r_{N-1},s_{N-1}) \psi(r_1,s_1,\ldots,r_{N-1},s_{N-1}).
\]

We define the bosonic and fermionic annihilation operators \( a_\pm(f) : \mathcal{H}_N^\pm \to \mathcal{H}_{N-1}^\pm \) as the restriction of \( \tilde{a}(f) \) to the respective subspaces, i.e., \( a_\pm(f) = \tilde{a}(f)|_{\mathcal{H}_N^\pm} \). The adjoints are \( a_\mp(f) : \mathcal{H}_{N-1}^\pm \to \mathcal{H}_N^\pm \) given by \( a_\pm(f) = p_N^\pm \tilde{a}^+(f)|_{\mathcal{H}_{N-1}^\pm} \).

We may extend \( a_\pm(f) \) and \( a_\mp(f) \) to operators on the subspace of the Fock spaces \( \mathcal{F}_\pm \) corresponding to finite particle numbers, i.e., span \( \bigoplus_{M=0}^N \bigoplus_{N=0}^M \mathcal{H}_N^\pm \). They cannot be extended as bounded operators on the full Fock spaces.

The extended operators satisfy the famous commutation (\( + \)) and anti-commutation (\( - \)) relations

\[
[a_\pm(f), a_\pm^+(g)] = (f,g)_{L^2(\Omega)^y} I
\]

where \([A,B]_\pm = AB \mp BA \) and \( I \) is the identity of \( L^2(\Omega)^y \) (or rather its restriction to the subspace corresponding to finite particle numbers).

If \( \{f_j\} \) is an orthonormal basis in \( L^2(\Omega)^y \) we define the operator valued distributions

\[
\phi_\pm(r,s) = \sum_{j=1}^\infty f_j(r,s) a_\pm(f_j), \quad \phi_\pm^+(r,s) = \sum_{j=1}^\infty \overline{f_j(r,s)} a_\pm^+(f_j),
\]

allowing us to write

\[
a_\pm(f) = \sum_{s=1}^y \int \overline{f(r,s)} \phi_\pm(r,s) dr, \quad a_\pm^+(f) = \sum_{s=1}^y \int f(r,s) \phi_\pm^+(r,s) dr.
\]

for \( f \in L^2(\Omega)^y \). Formally we have

\[
[\phi_\pm(r,s), \phi_\pm^+(r',s')] = \delta_{ss'} \delta(r-r').
\]
We refer to \( \phi \) and \( \phi^* \) as field operators.

Using these field operators we may write the grand canonical Hamiltonian
\( \bigoplus_{N=0}^{\infty}(H_N + \mu N) \), corresponding to identical fermions or bosons, formally as
\[
\bigoplus_{N=0}^{\infty}(H_N(A, E_{\perp}) + \mu N) = \sum_{s,s'} \int \phi^*_s (r,s) \left[ T^s_{s'}(-i\nabla_r + QA(r)) + \mu \delta_{ss'} \right] \phi_{s'}(r,s') dr \\
+ \frac{1}{2} \sum_{s,s'} \int \phi^*_s(r,s) \phi^*_s(r',s') \frac{Q^2}{|r-r'|} \phi_{s}(r',s') \phi_{s}(r,s) dr dr' \\
+ \frac{1}{8\pi} \int (|E_{\perp}|^2 + |B|^2),
\]
where \( T^{s's'}_{s} = 1, \ldots, \nu \) refer to the matrix components of the kinetic energy operator. It is left as an exercise to the reader to check that the ordering of \( \phi^* \) and \( \phi \) exactly gives the correct Hamiltonian with no-self interactions.

In this formalism it is easy to write down the grand canonical operator corresponding to \( K \) different species of either fermions or bosons. For \( j = 1, \ldots, K \) let \( T_j \), \( Q_j \), and \( v_j \) represent the kinetic energy function, the charge, and the internal degrees of freedom of species \( j \) which is either a fermion or a boson. The relevant Hilbert space is \( \mathcal{H} = \bigotimes_{j=1}^{K} \mathcal{F}_j (L^2(\Omega))^{v_j} \) where \( \mathcal{F}_j \) is the Fock space for species \( j \). Denoting the field operators for species \( j \) by \( \phi_j \) and \( \phi_j^* \) the corresponding grand canonical Hamiltonian (with chemical potential \( \mu \) included) is
\[
H(\mu, A, E_{\perp}) = \sum_{j=1, s,s' = 1}^{K} \int \phi^*_j (r,s) \left[ T^s_{s'}(-i\nabla_r + QA(r)) + \mu \delta_{ss'} \right] \phi_{j}(r,s') dr \\
+ \frac{1}{2} \sum_{j,s,s' = 1}^{K} \sum_{i=1}^{v_j} \int \phi^*_j(r,s) \phi^*_j(r',s') \frac{Q_j Q_{i}}{|r-r'|} \phi_{j}(r',s') \phi_{j}(r,s) dr dr' \\
+ \frac{1}{8\pi} \int (|E_{\perp}|^2 + |B|^2).
\]
The subspace on which \( H(\mu, A, E_{\perp}) \) is defined is the space corresponding to finitely many particles and where the restriction to each particle sector is a smooth function with compact support. Although it is hopefully clear what this means it is rather complicated to write it down explicitly. For the convenience of the reader we will nevertheless do this now. The subspace of \( \mathcal{H} \) corresponding to \( N_j \) particles of species \( j \), \( j = 1, \ldots, K \) is \( \bigotimes_{j=1}^{K} P_j (\bigotimes_{i=1}^{N_j} L^2(\Omega))^{v_i} \), where \( P_j \) refers to the relevant projection corresponding to the statistics of species \( j \). We may consider this space a subspace of \( [L^2(\Omega^{N_1+\ldots+N_K})]^{v_1 \ldots v_k} \). The subspace of smooth functions with compact support is \( [C_0^\infty(\Omega^{N_1+\ldots+N_K})]^{v_1 \ldots v_k} \). Thus the subspace on which we define \( H(\mu, A, E_{\perp}) \) is
\[ \mathcal{D} = \text{span} \bigcup_{M_1, \ldots, M_j=0}^\infty M_1 \bigotimes \cdots \bigotimes M_j = 0 \bigotimes \left[ C_0^\infty(\Omega_{N_1+N_k}) \bigotimes \nu_{N_1} \cdots \nu_{N_k} \right] \cap L^2(\mathbb{R}^3)^{N_j} \]

The energy of the system with particles being in a state represented by \( \psi \in \mathcal{D} \) is denoted
\[
E(\mu, \psi, A, E_\perp) = \langle \psi, H(\mu, A, E_\perp) \psi \rangle.
\]

It is easy to check that this agrees with the definition \((10)\) in the case of only one species.

### 3.5 Quantization of the electromagnetic field

We will briefly discuss how to quantize the electromagnetic field. We will remain in Coulomb gauge and quantize such that \( \nabla \cdot A = 0 \). This is most conveniently done in momentum space.

For \( k \in \mathbb{R}^3 \) choose \( e_1(k), e_2(k) \in \mathbb{R}^3 \) such that \( e_1(k), e_2(k), k \) form an orthonormal basis. \( e_1, e_2 \) cannot be chosen continuously, but this will not cause problems for what we want to say.

Let \( \phi(r, \lambda), \lambda = 1, 2 \) be a bosonic field operator with two internal degrees of freedom. They are field operators for the light quanta, i.e., photons. Define the Fourier transformed operators (Of course they are also simply bosonic field operators)
\[
\hat{\phi}(k, \lambda) = (2\pi)^{-3/2} \int e^{-ikr} \phi(r, \lambda) dr,
\]
\[
\hat{\phi}^*(k, \lambda) = (2\pi)^{-3/2} \int e^{ikr} \phi^*(r, \lambda) dr.
\]

We define the quantized magnetic vector potential as the operator valued distribution
\[
A(r) = (2\pi)^{-3/2} \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \sqrt{\frac{2\pi}{|k|}} e_\lambda(k) (e^{ikr} \hat{\phi}(k, \lambda) + e^{-ikr} \hat{\phi}(k, \lambda)) dk
\]
and the transversal electric field as
\[
E_\perp(r) = i(2\pi)^{-1/2} \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \sqrt{\frac{|k|}{2\pi}} e_\lambda(k) (e^{ikr} \hat{\phi}(k, \lambda) - e^{-ikr} \hat{\phi}(k, \lambda)) dk.
\]

We then find the commutator between the conjugate variables
\[
\left[ A_i(r), -\frac{1}{4\pi} E_{\perp,j}(r', r) \right] = P_{ij}(r, r'),
\]
where \( P(r, r') \) is the \( 3 \times 3 \)-matrix valued integral kernel of the projection in \( L^2(\mathbb{R}^3)^3 \) projecting onto divergence free vector fields.

A straightforward (formal) calculation gives for the field energy
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\[
\frac{1}{8\pi} \int_{\mathbb{R}^3} |E_{\perp}(r)|^2 + |\nabla \times A(r)|^2 \, dr = \frac{1}{2} \sum_{\lambda} \int_{\mathbb{R}^3} |k| (\hat{\phi}^* (k, \lambda) \hat{\phi} (k, \lambda) \\
+ \hat{\phi} (k, \lambda) \hat{\phi}^* (k, \lambda)) \, dk.
\]

This expression however is infinite and we must normal order it to get a well-defined operator:

\[
\sum_{\lambda} \int_{\mathbb{R}^3} |k| \hat{\phi}^* (k, \lambda) \hat{\phi} (k, \lambda) \, dk.
\]

This is the field energy operator of the electromagnetic field on the Fock space \( \mathcal{F}^+ (L^2(\mathbb{R}^3))^2) \).

### 3.6 Non-relativistic QED

We may now write down the Hamiltonian of non-relativistic QED, i.e., of the quantized electromagnetic field coupled to quantized non-relativistic particles. The particles will be described by the non-relativistic kinetic energies (4) or (6), but since A is now an operator valued distribution, these operators will not make sense unless we again introduce the extended charge distribution of the particles. The grand canonical non-relativistic QED Hamiltonian for \( K \) species of identical particles is then (ignoring for simplicity the chemical potential)

\[
H = \sum_{j=1}^K \nu_j \sum_{s,s'=-1}^1 \int \phi_j^* (r,s) T^{s's}_j (-i \nabla_r + Q_j A * \chi_{\Omega}(r)) \phi_j (r,s') \, dr \\
+ \sum_{j=1}^K \frac{Q_j Q_j}{2} \sum_{s=1}^{\nu_j} \sum_{s'=1}^{\nu_j} \int \phi_j^* (r,s) \phi_j^* (r',s') \frac{1}{|r-r'|} \phi_j (r',s') \phi_j (r,s) \, dr \, dr' \\
+ \sum_{\lambda} \int_{\mathbb{R}^3} |k| \hat{\phi}^* (k, \lambda) \hat{\phi} (k, \lambda) \, dk.
\]

This operator is defined on the Hilbert space

\[
\left( \bigotimes_{j=1}^K \mathcal{F}_j (L^2(\Omega))^\nu_j \right) \bigotimes \mathcal{F}^+ ((L^2(\mathbb{R}^3))^2).
\]

The operators \( \phi_j \) are field operators for the particles and \( \phi \) is the field operator for the photons. The energy may be calculated in a state represented by a \( \Psi \) in the subspace of the Hilbert space consisting of \( C_0^\infty \) functions of finitely many particles and photons (we will not write this explicitly this time). The energy is denoted

\[
E_{\text{NRQED}} (\Psi) = \langle \Psi, H \Psi \rangle.
\]
As written now the model depends on the regularization parameter \( R \). The limit as \( R \) tends to 0 is not well understood and will require at least to renormalize the bare mass and charges of the particles.

### 3.7 Relativistic QED Hamiltonian

As already emphasized a Hamiltonian (or for that matter any non-perturbative) formulation of QED is non-existent. Here we simply write down the formal expression for the Hamiltonian for the electron-positron field (with charge \( e \)) interacting with the electromagnetic field:

\[
H_{\text{QED}} = \sum_{a,b=1}^{4} \int \phi_e^\dagger(r,a) \left( -i \nabla + e A(r) \right) \phi_e(r,b) dr \\
+ \frac{e^2}{8} \sum_{a,b=1}^{4} \int \left[ \phi_e(a,r), \phi_e^\dagger(a,r) \right]_+ \left[ \phi_e(b,r'), \phi_e^\dagger(b,r') \right]_+ \frac{dr dr'}{|r-r'|} \\
+ \sum_{\lambda} \int_{\mathbb{R}^3} |k|^2 \hat{\phi}^\dagger(k,\lambda) \hat{\phi}(k,\lambda) dk.
\]

Here \( \phi_e \) refers to the fermionic field operator for the electron-positron field and \( \phi \) is the bosonic field operator for the photon field. The operator \( A \) is given by (13). Note that we have not distinguished between electrons and positrons, but that the operator is written in a charge conjugation invariant way as the density is written as the commutator \( \frac{1}{2} \sum_{a=1}^{4} \left[ \phi_e(a,r), \phi_e^\dagger(a,r) \right]_+ \).

The operator \( H_{\text{QED}} \) is ill-defined unless regularizations are introduced and even in this case it is very difficult to analyze. The no-photon situation was studied in the mean-field approximation in [15].

### 4 Stability

In the previous section we discussed how to define the energy of states of charged quantum gases in different models.

We have introduced the fixed particle number (or canonical) energy \( \mathcal{E}_N(\psi,A,E_\perp) \) in (2) (or the bosonic or fermionic analogs in (7)) or the grand canonical energy \( \mathcal{E}(\mu,\psi,A,E_\perp) \) in (12). We also defined the non-relativistic QED energy \( \mathcal{E}_\text{NRQED}(\Psi) \) in (15).

We will say that a system is **stable of the first kind** or **canonically stable** if the energy \( \mathcal{E}_N(\psi,A,E_\perp) \) is bounded below independently of \( A, E_\perp \), and normalized \( \psi \). In this case we will call the infimum of \( \mathcal{E}_N(\psi,A,E_\perp) \) the **ground state energy** regardless of whether an actual minimizer (a ground state) exists or not. Thus the canonical ground state energy of the system is
\[ E_N(\Omega) = \inf \left\{ E_N(\psi, A, E) \mid \psi \in \left( \bigotimes_{j=1}^N L^2(\Omega)^{y_j} \right) \cap C_0^\infty(\Omega^N)^{y_1 \ldots y_N}, \| \psi \| = 1, \right. \]
\[ \left. A, E_\perp \in C_0^\infty(\mathbb{R}^3; \mathbb{R}^3) \right\}. \]

Note that we are restricting the particles to be in the set \( \Omega \) whereas \( A \) and \( E \) are unrestricted vector fields in \( \mathbb{R}^3 \). It is immediate to see that we might take \( E_\perp = 0 \) in the infimum, this will however not be the case for quantized fields below.

The ground state energy of course depends on the types of particles in the system. We are suppressing this dependence in order not to overburden the notation.

The ground state is the state of the system at absolute zero temperature. It is of course also of interest to study quantum gases at positive temperature corresponding to minimizing the free energy we shall however not do this here.

We could also have chosen to consider the purely static Coulomb potential and set \( A = 0 \), but as we shall see the inclusion of \( A \) does not really change the treatment in the non-relativistic (and non-Pauli) case from the points of view discussed here.

We say that a system satisfies stability of the second kind or stability of matter if \( N^{-1}E_N(\Omega) \) is bounded below independently of \( N \) for all (open or in some cases sufficiently regular) \( \Omega \subset \mathbb{R}^3 \). This is the version of stability mainly studied in [21].

We will here use a slightly stronger notion which we refer to as grand canonical stability. We define the grand canonical ground state energy as

\[ E(\mu, \Omega) = \inf \left\{ E(\mu, \psi, A, E_\perp) \mid \psi \in \mathcal{D}, \| \psi \| = 1, A, E_\perp \in C_0^\infty(\mathbb{R}^3; \mathbb{R}^3) \right\}, \]

where \( E(\mu, \psi, A, E_\perp) \) was defined in (12). It of course depends on the species of particles.

We say that a system is grand canonically stable (with chemical potential \( \mu \)) if

\[ \inf_{\Omega \subset \mathbb{R}^3} |\Omega|^{-1}E(\mu, \Omega) > -\infty. \]

The infimum here is over all open sets \( \Omega \) with bounded volume \( |\Omega| \) (or possibly sufficiently regular sets if necessary, but we will not consider such cases here).

The original proof of stability of matter is due to Dyson and Lenard [7, 8] and later by a simpler method by Lieb and Thirring [28]. We will present a proof of grand canonical stability in a simple case relying on a combination of the two approaches.

For grand canonically stable systems it is of interest to consider whether the thermodynamic limit

\[ \lim_{\Omega \to \mathbb{R}^3} |\Omega|^{-1}E(\mu, \Omega) \] (16)

exists. The limit \( \Omega \to \mathbb{R}^3 \) can be given a precise meaning in different ways. Here we shall simply take the simple situation of the family of scaled copies \( L\Omega \) of a fixed set \( \Omega \) and let the real parameter \( L \) tend to infinity.
4.1 Stability of the first kind for non-relativistic particles

We shall here prove the stability of the first kind for non-relativistic particles, i.e., particles with the kinetic energy (4).

Theorem 1 (Non-relativistic stability of the first kind). For all $\psi \in \mathcal{C}_0^\infty(\Omega^N_{\nu_1,\ldots,\nu_N})$ and all vector fields $A, E \in \mathcal{C}_0^\infty(\mathbb{R}^3; \mathbb{R}^3)$ we have

$$\langle \psi, \left( \sum_{j=1}^N \frac{1}{2m_j} (-i\nabla + Q_j A(r_j))^2 + \sum_{1 \leq i < j \leq N} \frac{Q_i Q_j}{|r_i - r_j|} + \frac{1}{8\pi} \int (|E_\perp|^2 + |B|^2) \right) \psi \rangle \geq -C \|\psi\|^2,$$

where the constant $C > 0$ depends only on the number of particles $N$ and their properties, i.e., on $\nu_1, \ldots, \nu_N \in \mathbb{N}, m_1, \ldots, m_N > 0$ and $Q_1, \ldots, Q_N \in \mathbb{R}$.

This theorem follows easily from the diamagnetic inequality and the Sobolev inequality (see [20]).

Theorem 2 (Diamagnetic Sobolev inequality). For all $f \in \mathcal{C}_0^\infty(\mathbb{R}^3)$ and all $A \in \mathcal{C}_0^\infty(\mathbb{R}^3; \mathbb{R}^3)$ there is a constant $C > 0$ such that

$$\int_{\mathbb{R}^3} |(-i\nabla + A) f|^2 \geq \int_{\mathbb{R}^3} |\nabla f|^2 \geq C \left( \int_{\mathbb{R}^3} |f|^6 \right)^{1/3}.$$

An immediate corollary of this result (using simply Hölder’s inequality) is the following bound on one-body Schrödinger operators.

Corollary 1 (Lower bound on Schrödinger operator). For all $f \in \mathcal{C}_0^\infty(\mathbb{R}^3), A \in \mathcal{C}_0^\infty(\mathbb{R}^3; \mathbb{R}^3), 0 \leq V_1 \in L^{5/2}(\mathbb{R}^3), \text{ and } 0 \leq V_2 \in L^\infty(\mathbb{R}^3)$ we have

$$\langle f, ((-i\nabla + A)^2 - V_1 - V_2) f \rangle_{L^2} \geq -C \left( \int V_1^{5/2} + \|V_2\|_\infty \right) \|f\|_{L^2}^2.$$

We leave it to the reader to prove Theorem 1 from this corollary and the observation that $|r|^{-1} \in L^{5/2}(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$.

The stability of the first kind holds even if the field energy

$$\frac{1}{8\pi} \int |E_\perp|^2 + |B|^2$$

is ignored. Moreover, it also holds if $A$ is quantized, i.e., if we replace $A(r)$ by the operator (13). This last statement follows since $A(r)$ is a commuting family (indexed by $r$) and thus may be considered as a classical field.
4.2 Grand canonical stability

We turn to the question of grand canonical stability. We will study this in the simple special case of two species of identical fermions with opposite charges. For grand canonical stability it is not necessary that all particles are fermions. It is, in fact, enough that all particles with one sign of the charge, i.e., say, all negatively charged particles form a collection of finitely many species of fermions. Stability of matter in this more general setting was proved in [7, 8, 28] (see also [18]) and the case of grand canonical stability was treated in [17].

One of the main ingredients in the proof of grand canonical stability is the use of the celebrated Lieb-Thirring inequality [28] (see also [21]) which replaces the Sobolev inequality which we used in the proof of stability of the first kind.

Theorem 3 (Lieb-Thirring inequality).
Assume $0 \leq V \in L^{5/2}(\mathbb{R}^3)$ and $A \in C_0(\mathbb{R}^3; \mathbb{R}^3)$ then for all $N$ we have on the anti-symmetric subspace $\mathcal{H}_N = \Lambda^N[L^2(\Omega)]^\nu$ the operator inequality

$$
\sum_{j=1}^N \left( \frac{1}{2m} (-i \nabla_j + A(r_j))^2 - V(r_j) \right) \geq -Cm^{3/2} \nu \int V^{5/2},
$$

for a universal constant $C > 0$. In particular, it is independent of the number $N$ of particles.

Note the apparent similarity between the Lieb-Thirring inequality and the Corollary 1 to the Sobolev inequality. The important difference is that Corollary 1 would only imply that

$$
\sum_{j=1}^N \left( \frac{1}{2m} (-i \nabla_j + A(r_j))^2 - V(r_j) \right) \geq -Cm^{3/2} N \int V^{5/2},
$$

which, in fact, holds on all of $\mathcal{H}_N$ (left as an exercise for the reader). The lower bound with a constant independent of $N$ holds only on the fermionic subspace.

The Lieb-Thirring inequality relates the energy of a gas of independent particles to the corresponding classical energy. The classical energy (ignoring internal degrees of freedom) would indeed be

$$
\int \int \frac{1}{2m} (p^2 + A(r))^2 - V(r) dr dp = -\frac{8\pi}{15} m^{3/2} \int V^{5/2}.
$$

As a consequence of the Lieb-Thirring inequality we have the following lower bound on the kinetic energy of $N$ fermions confined to move in a bounded volume.

Corollary 2. If the open set $\Omega$ has finite volume $|\Omega|$ then in $\Lambda^N[L^2(\Omega)]^\nu$ we have a universal constant $C > 0$ such that
\begin{equation}
\sum_{j=1}^{N} \frac{1}{2m} (-i\nabla_j + A(r_j))^2 \geq C m^{-1} v^{-2/3} N^{8/3} |\Omega|^{-2/3} .
\end{equation}

**Proof.** If we use the Lieb-Thirring inequality with a constant potential \( V \) we obtain
\[
\sum_{j=1}^{N} \frac{1}{2m} (-i\nabla_j + A(r_j))^2 \geq NV - C m^{3/2} v V^{5/2} |\Omega|
\]
which gives the estimate above after optimization in \( V \).

We now consider the situation with two species of identical non-relativistic fermions with masses \( m_{\pm} > 0 \) and charges \( \pm Q_{\pm} \) where \( Q_{\pm} > 0 \). For simplicity we assume that there are no internal degrees of freedom, i.e., \( \nu_{\pm} = 1 \). In this case the Hamiltonian with particle numbers \( N_{\pm} \) for the two species is
\[
H_{N_{+}, N_{-}} = \sum_{j=1}^{N_{+}} \frac{1}{2m_{+}} (-i\nabla_j + Q_{+} A(r_j))^2 + \sum_{j=N_{+}+1}^{N_{+}+N_{-}} \frac{1}{2m_{-}} (-i\nabla_j - Q_{-} A(r_j))^2 + V_C
\]
\[
+ \frac{1}{8\pi} \int \left( |E_{\perp}|^2 + |B|^2 \right)
\]
where the Coulomb energy is
\[
V_C = -\sum_{j=1}^{N_{+}} \sum_{N_{+}+1}^{N_{+}+N_{-}} \frac{Q_{+} Q_{-}}{|r_i - r_j|} + \sum_{1 \leq i < j \leq N_{+}} \frac{Q_{+}^2}{|r_i - r_j|} + \sum_{N_{+} < i < j \leq N_{+}+N_{-}} \frac{Q_{-}^2}{|r_i - r_j|}.
\]

Note that we have numbered the positively charged particles \( 1, \ldots, N_{+} \) and the negatively charged particles \( N_{+} + 1, \ldots, N_{+} + N_{-} \). The Hamiltonian acts on the subspace
\[
D = \left( \bigwedge^{N_{+}} L^2(\Omega) \otimes \bigwedge^{N_{-}} L^2(\Omega) \right) \cap C^{\infty}_0(\Omega^{N_{+} + N_{-}}).
\]

**Theorem 4 (Simple case of grand canonical stability).**

The grand canonical energy in the finite volume set \( \Omega \subseteq \mathbb{R}^3 \)
\[
E(\mu, \Omega) = \inf \left\{ \langle \psi, H_{N_{+}, N_{-}} \psi \rangle + \mu (N_{+}, N_{-}) \mid \psi \in D, \|\psi\| = 1, E_{\perp}, A \in C^{\infty}_0(\mathbb{R}^3; \mathbb{R}^3) \right\}
\]
satisfies the stability bound
\[
E(\mu, \Omega) \geq -C(\mu, m_{\pm}, Q_{\pm}) |\Omega|,
\]
for a constant \( C(\mu, m_{\pm}, Q_{\pm}) > 0 \) depending only on \( \mu, m_{\pm}, Q_{\pm} \).

**Proof.** We define the distance from particle \( j \) to the nearest particle of the opposite charge, i.e.,
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\[ \delta_j = \delta_j(r_1, \ldots, r_{N_+ + N_-}) \]
\[ = \begin{cases} 
\min_{i=N_+ + 1, \ldots, N_+} |r_i - r_j|, & \text{if } j = 1, \ldots, N_+ \\
\min_{i=1, \ldots, N_+} |r_i - r_j|, & \text{if } j = N_+ + 1, \ldots, N_+ + N_- \end{cases} \]

Let \( \chi_j = \frac{6}{\pi \delta^2} \mathbb{1}_{B(r_j, \delta_j/2)} \), where \( B(r_j, \delta_j/2) \) denotes the ball centered at \( r_j \) with radius \( \delta_j/2 \) and \( \mathbb{1}_{B(r_j, \delta_j/2)} \) is its characteristic function. Note that \( \int \chi_j = 1 \).

We will use the following two observations:

**Observation 1:**
\[ \sum_{j=1}^{N_+} \sum_{i=N_+ + 1}^{N_+ + N_-} Q_+ \frac{Q_0}{|r_i - r_j|} = \sum_{j=1}^{N_+} \sum_{i=N_+ + 1}^{N_+ + N_-} Q_+ \int \frac{\chi_j(r) \chi_j(r')}{|r - r'|} dr dr' \]

**Observation 2:**
\[ \sum_{1 \leq i < j \leq N_+} \frac{Q_+^2}{|r_i - r_j|} \geq \sum_{1 \leq i < j \leq N_+} Q_+^2 \int \frac{\chi_j(r) \chi_j(r')}{|r - r'|} dr dr' \]
and likewise for the \( Q_-^2 \)-terms.

The observations follow from Newton’s Theorem:
\[ \frac{6}{\pi \delta^2} \int_{|r'| < \delta/2} |r - r'|^{-1} d r' = \begin{cases} |r|^{-1}, & \text{if } |r| > \delta/2 \\
\delta^{-1} (3 - 4|r|^{-2}), & \text{if } |r| < \delta/2 \leq |r|^{-1}. \end{cases} \]

From the two observations above we arrive at the following lower bound on the Coulomb energy
\[ V_C \geq \frac{1}{2} \int \frac{\rho(r) \rho(r')}{|r - r'|} dr dr' - \frac{12}{5} \sum_{j=1}^{N_+} Q_+^2 \delta_j^{-1} - \frac{12}{5} \sum_{j=N_+ + 1}^{N_+ + N_-} Q_-^2 \delta_j^{-1}, \]
where we introduced the smeared charge density
\[ \rho(r) = \sum_{j=1}^{N_+} Q_+ \chi_j(r) - \sum_{j=N_+ + 1}^{N_+ + N_-} Q_- \chi_j(r) \]
and used that
\[ \int \frac{\chi_j(r) \chi_j(r')}{|r - r'|} dr dr' = \frac{12}{5} \delta_j^{-1}. \]

Using now the positive type (i.e., positivity of the Fourier transform) of the Coulomb kernel we find
\[ V_C \geq - \frac{12}{5} \sum_{j=1}^{N_+} Q_+^2 \delta_j^{-1} - \frac{12}{5} \sum_{j=N_+ + 1}^{N_+ + N_-} Q_-^2 \delta_j^{-1}. \]

A similar application of the positive type of the Coulomb kernel goes back to an early paper of Onsager [31], who might have been the first to address the issue.
of grand canonical stability. Better lower bounds on the Coulomb energy can be derived by more sophisticated use of the same ideas (see e.g. [11, 29, 21]).

We are led to the following lower bound on the Hamiltonian

\[ H_{N_+, N_-} \geq H_{N_+} + H_{N_-} + \mu (N_+ + N_-) \]

where

\[ H_{N_+} = \sum_{j=1}^{N_+} \frac{1}{2m_+} (-i\nabla_j + Q_j A(r_j))^2 - \frac{12}{5} \sum_{j=1}^{N_+} Q_j^2 \delta_j^{-1} \]

and likewise for \( H_{N_-} \). Observe now that for \( j = 1, \ldots, N_+ \) the length \( \delta_j \) depends on the position \( r_j \) and the positions \( r_{N_++1}, \ldots, r_{N_++N_-} R \) of the negatively charged particles but not on the positions of the other positively charged particles. In other words we may write

\[-\frac{12}{5} \sum_{j=1}^{N_+} Q_j^2 \delta_j^{-1} = -\frac{12}{5} \sum_{j=1}^{N_+} Q_j^2 \delta(r_j)^{-1},\]

where \( \delta(r) = \min_{i=N_++1, \ldots, N_++N_-} |r_i - r| \). We thus have a potential parameterized by the positions of the negatively charged particles. This observation allows us to use the Lieb-Thirring inequality Theorem 3. If we choose a parameter \( R \) (to be optimized over) and divide the space into the region where \( \delta(r) < R \) (a union of \( N_- \) possibly intersecting balls of radius \( R \)) and \( \delta(r) > R \) we obtain from the Lieb-Thirring inequality

\[ H_{N_+} \geq \frac{1}{2} \sum_{j=1}^{N_+} \frac{1}{2m_+} (-i\nabla_j + Q_j A(r_j))^2 \]

\[-\frac{1}{2} \sum_{j=1}^{N_+} \left( -i\nabla_j + Q_j A(r_j) \right)^2 \]

\[ \geq C Q_j^5 m_+^{3/2} \left( N_- \int_{|r|<R} |r|^{-5/2} dr + R^{-5/2} |\Omega| \right) \]

\[ \geq C m_+^{-1} N_+^{5/3} |\Omega|^{-2/3} - C Q_j^5 m_+^{3/2} (N_- R^{1/2} + |\Omega| R^{-5/2}) \]

\[ = C m_+^{-1} N_+^{5/3} |\Omega|^{-2/3} - C Q_j^5 m_+^{3/2} N_-^{5/6} |\Omega|^{1/6}, \]

where we saved half of the kinetic energy in the first inequality and estimated it by Corollary 2 in the second inequality. Finally, we optimized over the parameter \( R > 0 \).

Since the corresponding estimate holds for \( H_{N_-} \) we finally get the lower bound

\[ H_{N_+, N_-} \geq C m_+^{-1} N_+^{5/3} |\Omega|^{-2/3} + C m_+^{-1} N_-^{5/3} |\Omega|^{-2/3} \]

\[ -C Q_j^5 m_+^{3/2} N_-^{5/6} |\Omega|^{1/6} - C Q_j^5 m_+^{3/2} N_+^{5/6} |\Omega|^{1/6} + \mu (N_+ + N_-) \]

\[ \geq -C(\mu, m_\pm, Q_\pm) |\Omega|, \]

where we have minimized in \( N_\pm \). We leave it to the reader to determine the exact form of the constant \( C(\mu, m_\pm, Q_\pm) \).
The same proof would work also if periodic external electric and magnetic fields were present, e.g., a situation describing a crystal structure.

As should also be clear from the proof the field energy

\[
\frac{1}{8\pi} \int |E_\perp|^2 + |B|^2
\]

plays no role for stability in the present case. Moreover, as in the case discussed for stability of the first kind we could also have considered \( A \) quantized.

### 4.3 Existence of the thermodynamic limit

We will briefly discuss existence of the thermodynamic limit \([16]\). This was first proved by Lieb and Lebowitz \([22]\) for the case of several species of particles where all the species of, say, negatively charged particles are fermions. The method does not allow for an exterior periodic potential or magnetic field. In particular, the method does work in the case where the nuclei are confined to a periodic crystal arrangement. This case was later treated by Fefferman in \([9]\). In \([16, 17]\) an abstract method was developed to conclude existence of thermodynamic limits for Coulomb systems in great generality including periodic background potentials.

Indeed, the method relies on establishing general abstract properties of the energy function that implies existence of the thermodynamic limit.

We will just give a brief overview of the method. For the details and more precise definitions and assumptions we refer to \([16, 17]\).

Let \( \mathcal{M} = \{ \Omega \subset \mathbb{R}^3 \text{ open and bounded} \} \) and consider a map \( E : \mathcal{M} \to \mathbb{R} \) with the following properties. Given a function \( \alpha : [0, \infty) \) with \( \lim_{\ell \to \infty} \alpha(\ell) = 0 \), a subset \( \mathcal{R} \subseteq \mathcal{M} \) of sufficiently regular sets, constants \( \kappa, \delta > 0 \), and a reference set \( \triangle \in \mathcal{R} \), such that

\[
(A1) \quad (\text{Normalization}). \quad E(\emptyset) = 0.
\]

\[
(A2) \quad (\text{Stability}). \quad \forall \Omega \in \mathcal{M}, \quad E(\Omega) \geq -\kappa |\Omega|.
\]

\[
(A3) \quad (\text{Translation Invariance}). \quad \forall \Omega \in \mathcal{R}, \forall z \in \mathbb{Z}^3, \quad E(\Omega + z) = E(\Omega).
\]

\[
(A4) \quad (\text{Continuity}). \quad \forall \Omega, \Omega' \in \mathcal{R}, \text{ with } \Omega' \subseteq \Omega \text{ and } d(\partial \Omega, \partial \Omega') > \delta, \quad E(\Omega) \leq E(\Omega') + \kappa |\Omega \setminus \Omega' + |\Omega| \alpha(\ell)|\Omega|).
\]

\[
(A5) \quad (\text{Subaverage Property}). \quad \forall \Omega \in \mathcal{M}, \text{ we have}
\]

\[
E(\Omega) \geq \frac{1}{|\ell\triangle|} \int_{\mathbb{R}^3 \times SO(3)} E(\Omega \cap g \cdot (\ell\triangle)) \, d\lambda(g) - |\Omega| \alpha(\ell)
\]

where \( d\lambda \) is the Haar-measure of \( \mathbb{R}^3 \times SO(3) \), \( |\Omega| \) is the volume of \( \Omega \) and \( |\Omega| := \inf\{ |\tilde{\Omega}|, \quad \Omega \subseteq \tilde{\Omega}, \quad \tilde{\Omega} \in \mathcal{R} \} \) is a regularized volume.
\[
\lim_{\ell \to \infty} |\ell \Delta|^{-1} E(g\ell \Delta)
\]
exists for all \(g \in \mathbb{R}^3 \times SO(3)\), i.e., it exists for all rotations or translations of the reference set \(\Delta\). Under slightly more restrictive assumptions which we will not repeat here the limit holds for a very large class of regular sets.

We see that (A2) is grand canonical stability. The difficult property to establish for Coulomb systems is (A5). For \(\Delta\) being a simplex it is a consequence of the following result of Graf and Schenker [14] generalizing a somewhat similar estimate by Conlon, Lieb and Yau [5]:

**Theorem 5 (Graf-Schenker inequality).**

Let \(\Delta\) be a simplex in \(\mathbb{R}^3\). There exists a constant \(C\) such that for any \(N \in \mathbb{N}\), \(Q_1, \ldots, Q_N \in \mathbb{R}\), \(r_1, \ldots, r_N \in \mathbb{R}^3\) and any \(\ell > 0\),

\[
\sum_{1 \leq i < j \leq N} \frac{Q_i Q_j}{|r_i - r_j|} \geq \frac{1}{|\ell \Delta|} \int_{\mathbb{R}^3 \times SO(3)} \sum_{1 \leq i < j \leq N} \frac{Q_i Q_j 1_g\ell \Delta(r_i) 1_g\ell \Delta(r_j)}{|r_i - r_j|} d\lambda(g) - \frac{C}{\ell} \sum_{j=1}^N Q_j^2.
\]

This inequality follows by proving that the function

\[
F(r, r') = \int_{\mathbb{R}^3 \times SO(3)} 1_g\ell \Delta(r_i) 1_g\ell \Delta(r_j) d\lambda(g),
\]

is of the form \(F(r, r') = g(|r - r'|)\) where \(g\) is such that \(|r|^{-1}(1 - g(|r|))\) has positive Fourier transform. Recall that for a function \(f\) of positive type

\[
\sum_{1 \leq i < j \leq N} Q_i Q_j f(r_i - r_j) \geq - \sum_{j=1}^N Q_j^2 f(0).
\]

**5 Instability**

**5.1 Examples of instability of the first kind**

As an example of a system that can show instability of the first kind we consider two relativistic particles with masses \(m_1 = m_2 = 1\) and charges \(Q_1 = -1\) and \(Q_2 = Q > 0\). The kinetic energy is given by (5) and we simply set \(A = 0\). Thus the Hamiltonian is

\[
H = \sqrt{-\Delta_1 + 1} - 1 + \sqrt{-\Delta_2 + 1} - 1 - \frac{Q}{|r_1 - r_2|}
\]

acting on the smooth compactly supported functions in \(L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3)\). Let \(\psi \in C_0^\infty(\mathbb{R}^6)\) be normalized, i.e., its square integral is one. and define \(\psi_\ell(r_1, r_2) = \)
\[ \ell^{-3} \psi(r_1/\ell, r_2/\ell) \] for \( \ell > 0 \). Note that \( \psi_\ell \) is still normalized for all \( \ell \). Then

\[ \langle \psi_\ell, H \psi_\ell \rangle = \ell^{-1} \left\langle \psi, \left( \sqrt{-\Delta_1 + \ell^2} - \ell + \sqrt{-\Delta_2 + \ell^2} - \ell - \frac{Q}{|r_1 - r_2|} \right) \psi \right\rangle . \]

Thus if we let \( \ell \) tend to zero

\[ \lim_{\ell \to 0} \ell \langle \psi_\ell, H \psi_\ell \rangle = \left\langle \psi, \left( \sqrt{-\Delta_1 + \sqrt{-\Delta_2} - \frac{Q}{|r_1 - r_2|} \right) \psi \right\rangle . \]

If \( Q \) is large enough we find that the right side is negative and hence for such a \( Q \)

\[ \lim_{\ell \to 0} \langle \psi_\ell, H \psi_\ell \rangle = -\infty \]

and the system is not stable of the first kind.

On the other hand, if the negatively charged particles belong to a finite number of fermionic species and if the number of fermionic species, the maximal negative charge and the maximal positive charge satisfy appropriate bounds, then stability of matter holds [4, 12, 21, 24, 29].

A similar situation happens for non-relativistic particles interacting with magnetic fields according to the Pauli operator (6). Consider as an example a Hamiltonian for two particles of mass \( m = 1 \) and charges \( Q_1 = Q > 0 \) and \( Q_2 = -Q < 0 \):

\[
H(\mathbf{A}) = \frac{1}{2}(\mathbf{\sigma} \cdot (-i\nabla_1 - QA(r_1)))^2 + \frac{1}{2}(\mathbf{\sigma} \cdot (-i\nabla_2 + QA(r_2)))^2 - \frac{Q^2}{|r_1 - r_2|} + \frac{1}{8\pi} \int |\nabla \otimes \mathbf{A}|^2 ,
\]

where we have chosen \( E = 0 \) (which is the energetically best choice). The instability in this case relies on the existence (see [13, 30]) of a non-zero \( \tilde{\psi} \in L^2(\mathbb{R}^3) \) and a magnetic field \( \mathbf{A} \) with \( \int |\nabla \otimes \mathbf{A}|^2 < \infty \) such that

\[ \frac{1}{2}(\mathbf{\sigma} \cdot (-i\nabla_1 - \tilde{\mathbf{A}}(r_1)))^2 \tilde{\psi} = 0. \]

We may assume that \( \tilde{\psi} \) is normalized. If for \( \ell > 0 \) we set

\[ \psi_\ell(r_1, r_2) = \ell^{-3} \tilde{\psi}(r_1/\ell) \tilde{\psi}(r_2/\ell) \]

(which is also normalized) and \( \mathbf{A}_\ell(r) = (Q\ell)^{-1}\tilde{\mathbf{A}}(r/\ell) \) we obtain for the energy expectation

\[ \ell \langle \psi_\ell, H(\mathbf{A}_\ell) \psi_\ell \rangle = -\left\langle \psi_{\ell-1}, \frac{Q^2}{|r_1 - r_2|} \psi_{\ell-1} \right\rangle + \frac{1}{8\pi Q^2} \int |\nabla \otimes \tilde{\mathbf{A}}|^2 . \]

Again we see that if \( Q \) is large enough the right side is negative and hence for such a \( Q \) we have as before \( \lim_{\ell \to \infty} \langle \psi_\ell, H(\mathbf{A}_\ell) \psi_\ell \rangle = -\infty \). As for the relativistic case
stability of matter also holds in this case under appropriate conditions \[10, \ 25\]. This problem with a quantized field has been treated in \[3, \ 11\], the relativistic case with classical fields is considered in \[26\], and the relativistic case with quantized field in \[23\].

5.2 Fermionic instability of the second kind

As the final topic of these notes we will discuss instability of the second kind. We will first make a very simple general remark about instability of many-body systems with attractive interactions which has nothing to do with charged systems and holds even for fermions.

**Theorem 6 (Fermionic instability for attractive 2-body potentials).** Assume that the potential \( W : \mathbb{R}^n \to \mathbb{R} \) satisfies \( W(r) \leq -c < 0 \) for all \( r \) in a ball around the origin. Consider the \( N \)-body operator

\[
H_N = \sum_{j=1}^{N} -\frac{1}{2} \Delta_j + \sum_{1 \leq i < j \leq N} W(r_i - r_j)
\]

acting in the fermionic Hilbert space \( \bigwedge^N L^2(\mathbb{R}^n) \). If \( n \geq 3 \) then \( H_N \) cannot be stable of the second kind, i.e., we can find a a sequence of normalized vectors \( \psi_N \in \bigwedge^N L^2(\mathbb{R}^n) \) such that

\[
\lim_{N \to \infty} N^{-1} \langle \psi_N, H_N \psi_N \rangle = -\infty.
\]

**Proof.** Assume that \( W(r) \leq -c < 0 \) on the ball of radius \( R \) centered at the origin. Define \( \psi_N \) as the (normalized) Slater determinant

\[
\psi_N(r_1, \ldots, r_N) = (N!)^{-1/2} \det(u_j(r_i))_{i,j=1}^{N}
\]

where \( u_j, j = 1, \ldots, N \) are orthonormalized eigenfunctions corresponding to the \( N \) lowest eigenvalues of the negative Laplacian with Dirichlet boundary conditions for the largest cube centered at the origin and contained in the ball of radius \( R \). We extend the functions to be 0 outside the cube. The functions \( u_j \) are explicit and can be written in terms of sines and cosines. It is a simple straightforward calculation to show that in all dimensions \( n \) there is a constant \( C_n \) such that

\[
\langle \psi_N, \sum_{j=1}^{N} (-\frac{1}{2} \Delta_j) \psi_N \rangle \leq C_n N^{(n+2)/n} R^{-2}.
\]

Comparing with Corollary \[2\] (written for the case \( n = 3 \)) we see that there is always a similar lower bound.

Thus
\begin{equation*}
N^{-1} \langle \psi_N, H_N \psi_N \rangle \leq C_n N^{2/n} R^{-2} - \frac{1}{2} (N - 1)c.
\end{equation*}

We see that instability occurs when \(n > 2\).

### 5.3 Instability of bosonic matter

For matter consisting of charged particles we have discussed that the fermionic property ensures grand canonical stability. In this final section we will show that the fermionic property is indeed a necessity as stability fails for bosons.

We consider two species of bosons with masses \(m_\pm = 1\), \(Q_+ = -Q_- = 1\), \(A = E_\perp = 0\). We describe them by the standard Schrödinger kinetic energy (4). If we have \(N_+\) positively charged particles and \(N_-\) negatively charged particles we may write the Hamiltonian as

\begin{equation*}
H_{N_+, N_-} = \sum_{j=1}^{N_+ + N_-} \frac{1}{2} \Delta_j + \sum_{1 \leq i < j \leq N_+ + N_-} \frac{e_i e_j}{|r_i - r_j|},
\end{equation*}

where \(e_j = 1\) if \(j = 1, \ldots, N_+\) and \(e_j = -1\) if \(j = N_+ + 1, \ldots, N_+ + N_-\). The Hilbert space is

\[
\mathcal{H}_{N_+, N_-} = P_{N_+} \bigotimes L^2(\mathbb{R}^3) \otimes P_{N_-} \bigotimes L^2(\mathbb{R}^3).
\]

This system is not stable of the second kind, in fact, the energy behaves like the number of particles to the \(7/5\)-th power. The following precise asymptotics was conjectured by Dyson in [6].

**Theorem 7 (Dyson’s formula).**

Let

\[
E(N) = \inf_{N_+ + N_- = N} \inf \{ \langle \psi, H_{N_+, N_-} \psi \rangle \mid \psi \in \mathcal{H}_{N_+, N_-} \cap C^\infty_0(\mathbb{R}^{3(N_+ + N_-)}), \|\psi\| = 1 \}
\]

then as \(N \to \infty\)

\[
\lim_{N \to \infty} \frac{E(N)}{N^{7/5}} = \inf \left\{ \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \Phi|^2 - I_0 \int_{\mathbb{R}^3} \Phi^{5/2} \mid 0 \leq \Phi, \int_{\mathbb{R}^3} \Phi^2 = 1 \right\},
\]

with \(I_0\) given by

\[
I_0 = \left( \frac{2}{\pi} \right)^{3/4} \int_0^\infty \left( 1 + x^4 - x^2 (x^4 + 2)^{1/2} \right) dx = \frac{4^{5/4} \Gamma(3/4)}{5 \pi^{1/4} \Gamma(5/4)}.
\]

From the Sobolev inequality (Theorem2, it follows that the inf on the right of (19) is finite. In [6] Dyson proved an upper bound on \(E(N)\) of the form \(-cN^{7/5}\) and thus indeed proved the instability of the second kind. In [5] a lower bound of the form \(-CN^{7/5}\) was established thus concluding that \(7/5\) is the correct power. The theorem
was finally proved in \cite{27,33}. In \cite{19} Lieb proved that if the positively charged particles have infinite mass then the energy is much smaller, indeed, bounded above by $-CN^{5/3}$ a corresponding lower bound had already been proved in \cite{7,8}.

The proof of Theorem 7 relies on an application of Bogolubov’s theory of superfluidity \cite{2}. The charged system, in fact, forms a superfluid state.

Dyson’s formula \cite{19} is proved by establishing the corresponding two inequalities. Establishing the lower bound is technically very involved and is beyond the scope of these notes. It is the content of the paper \cite{27}. We will here give a brief sketch of the proof of the upper bound from \cite{33}. The upper bound is proved by finding an appropriate trial state. Here we are guided by Bogolubov’s theory.

It turns out that it is significantly easier to write down a grand canonical trial state than a canonical state. We are, however, interested in a canonical state. This will not be a serious problem as we will eventually be able to show that the state we construct is sharply peaked around the average particle number. We will ignore this point here and simply work with the grand canonical state. We refer the reader to \cite{33} for details.

Another simplification is to consider the two species of bosons as one species with two internal degrees of freedom corresponding to the two signs of the charge. Constructing a trial state in this space will correspond to averaging over states with different numbers of positively and negatively charged particles.

We are thus considering the Fock space $\mathcal{F}^+ = \mathcal{F}^+ (L^2(\mathbb{R}^3)^2)$. We write a function $f \in L^2(\mathbb{R}^3)^2$, as $f = f(r,e)$, where $e = \pm 1$ is the sign of the charge. Let $|0\rangle$ be the vacuum vector in $\mathcal{F}^+$.

In constructing a bosonic trial state the first guess is to put all particles in the same one-particle state, i.e., to have a condensate. Let this state be represented by the (normalized) vector $\xi \in L^2(\mathbb{R}^3)^2$. Introduce first the normalized grand canonical vector $|\Xi\rangle = \exp\left(-\frac{N}{2} + \sqrt{N}a_\xi^\dagger \langle \xi \rangle \right)|0\rangle = \sum_{n=0}^{\infty} e^{-N/2} \frac{N^{n/2}}{n!} a_\xi^\dagger (\xi)^n |0\rangle$.

The corresponding state is an average over states with varying occupation in the condensate $\xi$. The average particle number in $\xi$ is $\langle \Xi | a_\xi^\dagger (\xi) a_\xi (\xi) |\Xi\rangle = N$ and the variance is also $\langle \Xi | (a_\xi^\dagger (\xi) a_\xi (\xi))^2 |\Xi\rangle - \langle \Xi | a_\xi^\dagger (\xi) a_\xi (\xi) |\Xi\rangle^2 = N$.

Thus this state is peaked around particle number $N$ with a standard deviation $\sqrt{N}$.

There is a unitary operator $U$ on $\mathcal{F}^+$ such that $U^* a_\xi^\dagger(f) U = a_\chi^\dagger(f) + \sqrt{N} \langle \xi, f\alpha \rangle$.

Using this unitary we may also write $|\Xi\rangle = U|0\rangle$.

A pure condensate like this will however not give the correct state. It is important to build pair excitations too. This is achieved as follows. Let $\{f_\alpha\}_{\alpha=0}^{\infty}$ be an orthonormal family in $L^2(\mathbb{R}^3)^2$ (they will represent the pair states). The normalized
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vector \( \Psi \in \mathcal{F}^+ \) representing our trial state may be abstractly written

\[
\Psi = \prod_{\alpha=0}^{\infty} (1 - \lambda_\alpha^2)^{1/4} \exp \left( \sum_{\alpha=0}^{\infty} -\frac{\lambda_\alpha}{2} (a_\alpha^* (f_\alpha) - \sqrt{N} (\xi, f_\alpha))^2 \right) |\Xi\rangle \tag{21}
\]

\[
= U \prod_{\alpha=0}^{\infty} (1 - \lambda_\alpha^2)^{1/4} \exp \left( \sum_{\alpha=0}^{\infty} -\frac{\lambda_\alpha}{2} a_\alpha^* (f_\alpha)^2 \right) |0\rangle.
\]

We have introduced parameters \( 0 < \lambda_\alpha < 1 \) with \( \sum_{\alpha=0}^{\infty} \lambda_\alpha^2 < \infty \) to control the occupations in the pair states. For simplicity we will assume that \( \xi \) and \( \{ f_\alpha \}_{\alpha=0}^{\infty} \) are real functions.

We encode the information about the pair states in the positive semi-definite trace class operator on \( L^2(\mathbb{R}^3)^2 \)

\[
\gamma = \sum_{\alpha=0}^{\infty} \frac{\lambda_\alpha^2}{1 - \lambda_\alpha} |f_\alpha\rangle \langle f_\alpha|.
\]

In terms of this operator a lengthy but straightforward calculation shows that

\[
\langle \Psi, (a_\alpha^* (f) - \sqrt{N} (\xi, f)) (a_\alpha (g) - \sqrt{N} (g, \xi)) | \Psi \rangle = \langle g, \gamma f \rangle,
\]

(the inner product on the left is in \( \mathcal{F}^+ \) and the one on the right is in \( L^2(\mathbb{R}^3)^2 \)) and \( \langle \Psi, (a_\alpha^* (f) - \sqrt{N} (\xi, f)) | \Psi \rangle = 0 \). In particular,

\[
\langle \Psi, a_\alpha^* (f) a_\alpha (g) | \Psi \rangle = \langle g, (N|\xi\rangle |\xi\rangle + \gamma f \rangle.
\]

Or equivalently using the field operators from Section 3.4

\[
\langle \Psi, \Phi (r, e)^* \Phi (r', e') | \Psi \rangle = N\xi \langle r, e \rangle \xi (r', e') + \gamma (r, e; r', e') \tag{23}
\]

where \( \gamma (r, e; r', e') \) is the integral kernel of \( \gamma \). Likewise,

\[
\langle \Psi, a_\alpha^* (f) a_\alpha^* (g) | \Psi \rangle = \langle g, \left( N|\xi\rangle \langle \xi| - \sqrt{\gamma (\gamma + 1)} \right) f \rangle,
\]

or

\[
\langle \Psi, \Phi (r, e)^* \Phi (r', e') \rangle = N\xi \langle r, e \rangle \xi (r', e') - \sqrt{\gamma (\gamma + 1)} (r, e; r', e'). \tag{24}
\]

Moreover, the state represented by \( \Psi \) satisfies Wick’s formula, which for the 4-point function reads

\[
\langle \Psi, \prod_{j=1}^{4} (a_\alpha^* (g_j) - \sqrt{N} (g_j, \xi)^\#) \rangle = \langle \Psi, \prod_{j=1,2} (a_\alpha^* (g_j) - \sqrt{N} (g_j, \xi)^\#) \langle \Psi, \prod_{j=3,4} (a_\alpha^* (g_j) - \sqrt{N} (g_j, \xi)^\#) \rangle.
\]
\[ + \left\langle \Psi, \prod_{j=1,3} (a^\dagger_j(g_j) - \sqrt{N}(g_j, \xi_j)^\dagger) \Psi \right\rangle \left\langle \Psi, \prod_{j=2,4} (a^\dagger_j(g_j) - \sqrt{N}(g_j, \xi_j)^\dagger) \Psi \right\rangle 
\]

\[ + \left\langle \Psi, \prod_{j=1,4} (a^\dagger_j(g_j) - \sqrt{N}(g_j, \xi_j)^\dagger) \Psi \right\rangle \left\langle \Psi, \prod_{j=2,3} (a^\dagger_j(g_j) - \sqrt{N}(g_j, \xi_j)^\dagger) \Psi \right\rangle. \]

Here # refers to either a * (interpreted as complex conjugation on scalars) or no *. In particular, since \( \xi \) is real this gives

\[
\left\langle \Psi, (\phi(r,e)^* - \sqrt{N}(\xi^*)(r,e)) (\phi'(r',e')^* - \sqrt{N}(\overline{\xi})(r',e')) \middle| r_i - r_j \right\rangle (r_i, e_i) (r_j, e_j) = \left( \gamma(r_i, e_i; r_j, e_j) \right) \left( \phi(r_i, e_i) \phi'(r_j, e_j) \right) \]

 Armed with these identities we can calculate the expectation of the energy in the state represented by \( \Psi \).

First we will explain, for the special case of the charged Bose system, how to choose the condensate function \( \xi \) and the trace class operator \( \gamma \). More precisely, we will specify their charge dependence. We set

\[
\xi(r,e) = \sqrt{\frac{1}{2}} \xi_0(r),
\]

where \( \xi_0 \) is a real normalized function in \( L^2(\mathbb{R}^3) \). Thus the condensate function does not depend on the charge. The operator \( \gamma \) on \( L^2(\mathbb{R}^3)^2 = L^2(\mathbb{R}^3 \otimes \mathbb{C}^2) \) will be chosen to have the form

\[
\gamma = \gamma_0 \otimes \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},
\]

where \( \gamma_0 \) is a positive trace-class operator on \( L^2(\mathbb{R}^3) \). Put differently, the integral kernel of \( \gamma \) is chosen to be

\[
\gamma(r, e; r', e') = \frac{1}{2} e e' \gamma_0(r; r').
\]

The charge part of this operator is a rank one operator and thus we also have

\[
\sqrt{\gamma(r+1)} = \sqrt{\gamma_0(\gamma_0+1) \otimes \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}.
\]

It is now straightforward to calculate the expectation of the Coulomb potential in the state represented by \( \Psi \). From \([23, 27]\) we obtain

\[
\left\langle \Psi, \bigoplus_{M=0}^\infty \sum_{1 \leq i < j \leq M} \frac{e_i e_j}{|r_i - r_j|} \Psi \right\rangle =
\]

\[
\left\langle \Psi, \prod_{j=1,3} (a^\dagger_j(g_j) - \sqrt{N}(g_j, \xi_j)^\dagger) \Psi \right\rangle \left\langle \Psi, \prod_{j=2,4} (a^\dagger_j(g_j) - \sqrt{N}(g_j, \xi_j)^\dagger) \Psi \right\rangle 
\]

\[ + \left\langle \Psi, \prod_{j=1,4} (a^\dagger_j(g_j) - \sqrt{N}(g_j, \xi_j)^\dagger) \Psi \right\rangle \left\langle \Psi, \prod_{j=2,3} (a^\dagger_j(g_j) - \sqrt{N}(g_j, \xi_j)^\dagger) \Psi \right\rangle. \]
\[ \left\langle \Psi, \sum_{ee' = \pm 1} \int ee' \phi(r,e)\phi(r',e')^* |r-r'|^{-1} \phi(r,e)\phi(r',e') \Psi \right\rangle = N \text{Tr}_{L^2(\mathbb{R}^3)}(\mathcal{K}(\gamma_0 - \sqrt{\gamma_0(\gamma_0+1)})) \].

Here, \( \mathcal{K} \) is the operator with integral kernel

\[ \mathcal{K}(r,r') = \xi_0(r)|r-r'|^{-1} \xi_0(r') \].

The total energy expectation is

\[ \left\langle \Psi, \bigoplus_{N_i, N_\downarrow = 0} H_{N_i, N_\downarrow} \Psi \right\rangle = \frac{N}{2} \int |\nabla \xi_0|^2 + \frac{1}{2} \text{Tr}(-\Delta \gamma_0) + N \text{Tr}(\mathcal{K}(\gamma_0 - \sqrt{\gamma_0(\gamma_0+1)})) \].

The final step in the argument is to minimize the above expression over \( \gamma_0 \). More precisely, this is done in a semiclassical approximation. We will only sketch this argument. The rigorous argument can again be found in [33]. We assume that \( \gamma_0 \) is the quantization of a classical symbol \( f(r,p) \geq 0 \). The semiclassical approximation to the energy is then

\[ \frac{N}{2} \int |\nabla \xi_0|^2 + (2\pi)^{-3} \int \frac{p^2}{2} f(r,p) + 4\pi N|p|^2 \xi_0(r)^2 \left(f(r,p) - \sqrt{f(r,p)(f(r,p)+1)}\right) dr dp. \]

Minimizing this expression over \( f(r,p) \) and performing the \( p \) integration gives

\[ \frac{N}{2} \int |\nabla \xi_0|^2 - I_0 N^{5/4} \int \xi_0(r)^{5/2} dr, \]

where \( I_0 \) is given in (20). If we introduce the rescaling \( \xi_0(r) = N^{3/10} \Phi(N^{1/5} r) \), where \( \Phi \) is also normalized then the energy expression above becomes

\[ N^{7/5} \left( \int |\nabla \Phi|^2 - I_0 \int \Phi^{5/2} \right), \]

which is exactly the expression conjectured by Dyson for the energy.

Note that the instability is also reflected in the shrinking of the linear dimension of the state with increasing \( N \). According to the scaling of \( \xi_0 \) above, the linear dimension of the state behaves like \( \sim N^{-1/5} \).
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