Chiral Hall effect in strained Weyl semimetals

Shiva Heidari and Reza Asgari

School of Physics, Institute for Research in Fundamental Sciences, IPM, Tehran, 19395-5531, Iran
ARC Centre of Excellence in Future Low-Energy Electronics Technologies, UNSW Node, Sydney 2052, Australia
School of Nano Science, Institute for Research in Fundamental Sciences, IPM, Tehran, 19395-5531, Iran

In this paper, the chiral Hall effect of strained Weyl semimetals without any external magnetic field is proposed. The electron-phonon coupling emerges in the low-energy fermionic particle through a pseudogauge potential. We show, by using chiral kinetic theory, that the chiral Hall effect emerges as a response of a real time-varying electric field in the presence of the structural distortion and it leads to a spatial chirality and charges separation in the system. We also show that the coupling of the electrons to acoustic phonons as a gapless excitation leads to emerging an optical absorption peak at \( \omega = \omega_{el} \) where \( \omega_{el} \) is defined as a characteristic frequency associated to the pseudomagnetic field.

I. INTRODUCTION

Weyl semimetals as a new type of symmetry breaking topological semimetals have been attracted attention recently [1–7]. Its non-degenerate band topology is described by closing the energy gap at certain points in the Brillouin zone, which are not at symmetry points, when either time-reversal or spatial inversion symmetry has been broken. The Weyl equation, originating from the Dirac equation in the zero mass limits in the odd spatial dimension, describes Weyl fermions as the low-energy bulk quasiparticles. The energy eigenstates of spin one-half Weyl fermions are eigenstates of the helicity operator \( k \cdot \sigma/2 \) and their corresponding eigenvalues with the opposite sign so-called chirality. The chirality associated with each node in Weyl semimetals (WSMs) can be considered as a charge of each monopole in \( k \)-space originates from the topological nature of WSMs. The topological notion of WSMs can be understood as the (local) topological protection of such monopole charges, that can be only added or removed in pair [8]. According to the Noether’s theorem, for every symmetry that exists in a physical system, there is a corresponding conservation law. The quantities that are conserved in classical mechanics but are not in a quantum theory called quantum anomalies. Chiral anomaly in WSMs can be obtained from an axionic term in the electromagnetic response of WSMs where the chiral current is no longer conserved \( \partial_\mu J^\mu_b = \frac{e^2}{2\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \) [9–12]. The RHS of this equation is proportional to \( E \cdot B \) implying parallel electric and magnetic fields leads to the intervalley charge pumping caused by the chemical potential imbalance at each node with different chirality [8].

Strain, on the other hand, couples to the low-energy electronic sector through the elastic gauge fields [13–15]. The general form of the axial gauge potential induced by lattice distortion can be described by the displacement vector \( u \):

\[
A^i_b = u_{ij} b^j
\]

where \( u_{ij} = \frac{1}{2} (\partial_i u_j + \partial_j u_i) \) is the strain tensor [16] and \( b^i \) is the nodal separation vector in momentum space.

It turns out that the electron-phonon coupling arising from experiencing space-time variation by Weyl fermions emerges in the system when electromagnetic fields are applied [17, 18]. If the axial four-vector \( b_\mu \), in other words, varies smoothly in space and time, the axial- or pseudo-gauge fields \( A^i_b \) emerges in the low-energy Hamiltonian couples with electrons. Since \( b_\mu \) varies over the whole sample the average of its corresponding pseudomagnetic filed \( (B^i) \) vanishes [14, 15, 19]. Recently it has been predicted that electron-phonon coupling in systems with time-reversal symmetry breaking leads to the nonzero Hall viscosity [13, 20, 21].

The corresponding pseudomagnetic and pseudoelectric fields are defined as \( B_5 = \nabla \times b \) and \( E_5 = (\partial_b - v_F \nabla) b_0 \). The space dependence of \( b \) can be achieved by applying strain [13, 18], propagating the chiral sound wave [15, 22], inhomogeneous magnetization [17, 23] or at the interface between two WSMs with different vector \( b \) [19, 24, 25]. The semiclassical description of the equation of motion for the electrons and their corresponding wave-packet in each node is powerful enough to describe anomalies, in the presence of \( E_5 \) and \( B_5 \). The covariant anomaly dictates the non-conservation of chiral charge in the presence of both real and pseudo fields for massless fermions; \( \partial_\mu J^\mu = \frac{e^2}{2\pi^2} (E \cdot B_5 + E_5 \cdot B) \) [17]. It is surprising because in a real physical system charge must be conserved. This non-conservation problem can be released by adding the so-called Bardeen polynomial to \( J^\mu \) and \( J^\mu_5 \) leading to the consistent version of the covariant anomaly equation, \( \partial_\mu J^\mu = 0 \) [26–28]. This additional term is the Hall current proportional to Weyl node separation [29] is complemented by the Chern-Simons current and is not captured by the semiclassical formalism. Accordingly, in the presence of both chiral gauge fields \( (B_5) \) and ordinary electric field \( E \) the chiral anomaly appears in the bulk of WSMs with the same sign of the anomaly at each node (two right-moving modes in bulk). In contrast to the chiral anomaly induced by real \( E \cdot B \), the bulk non-conservation of charge coming from the mixing term of the pseudo- and real fields can pump charge between the
bulk and the edge of the system where $B_5$ changes its sign [15, 26]. Similar studies have been performed on propagating acoustic waves (induce $E_5$) along the direction of the external magnetic field causes charge redistribution between bulk and the boundary [15, 22]. The boundaries always hosted a finite axial magnetic field $B_5$ where $b$ jumps from zero to a finite value or vice versa [14, 19].

The anomalous transport induced by axial fields or axial chemical potential has been studied to gain much attention [30] more recently. In addition to the chiral magnetic effect (CME) induced by chirality imbalance between two nodes [11, 31], chiral pseudo magnetic effect [14], chiral torsional effect [15], and visco-Hall current [21] can be induced in strained Weyl semimetals owing to the presence of the pseudomagnetic and pseudoelectric fields. In this paper, we discuss a generation of a transverse chiral Hall current (CHE) [32, 33] induced by the pseudomagnetic field.

We are also interested to investigate the impact of a pseudomagnetic filed on the optical conductivity of a doped WSMs in the limit of $\hbar \omega, k_B T \ll \mu$ where the intraband transitions are dominated. The intraband component of the optical conductivity usually gives rise to the Drude contribution which is proportional to the Dirac $\delta$-function in a clean limit and has a Lorentzian shape in the presence of disorders. For a system in which the interband transition is vital, the quantum effects are important, the optical absorption has an extra contribution and we assume that in our system all interband transitions are negligible. In two-band systems with dispersion $\epsilon(k) \propto k^2$ Kubo formula for interband transitions (ac mode) provide a general expression for the real part of the optical conductivity $\sigma_{\text{inter}}^{\text{ac}}(\omega) \propto \omega^{d-2}$ [34], where $d$ is the spatial dimension of the system, e.g. graphene has a constant interband conductivity [35] and 3D Dirac [36–38] and Weyl materials [34, 36, 39] exhibit a linear dependence of photon frequency ($\omega$) which is an important signature of these materials.

In this paper, we explore the CHE in the chiral system in the presence of the pseudomagnetic and the external electric field and in particular, concentrate on intraband transitions as a consequence of coupling light to the free carriers. In the case of our study, the emergent of the acoustic chiral phonons as a gapless low-energy excitation can have an impact on intraband dispersion. Low-energy acoustic phonons can not affect on interband transitions because of ($\hbar \omega_{\text{ph}} < 2|\mu|$ where $\mu$ is the chemical potential), and only optical phonons with ($\omega_{\text{ph}}^{\text{opt}} > 2|\mu|$) can lead to extra interband transitions. Making use of the chiral kinetic theory represents that the coupling of the electrons to acoustic phonons leads to emerging an optical absorption peak at the cyclotron frequency of the massless fermions in the pseudomagnetic field. We show that the chiral Hall current emerges as a response of a real time-varying electric field in the presence of $B_5$ induced by strain. Since this current is perpendicular to both fields $E$ and $B_5$ it is known as the CHE leads to spatial chirality separation in the case of zero axial chemical potential, and causes both spatial chirality and charge separation in the case of nonzero axial chemical potential. The emerging of transverse chiral Hall current as an optical response in the presence of the pseudomagnetic field, which to our best knowledge has not been discussed in the context of WSMs, is our main findings in this paper.

The rest of this paper is organized as follows. We begin with a description of our theoretical formalism in Sec. II, followed by the details of the chiral kinetic theory, Boltzmann transport theory in WSMs and optical current in terms of the external direction. Analytical expressions of the optical conductivity tensor are obtained for strained WSMs in the presence of an external electric field. In Sec. III we summarize our main findings.

II. MODEL AND THEORETICAL FORMALISM

We consider a doped WSM system exerted by a mechanical force. The low-energy effective Hamiltonian in the continuum limit in the vicinity of the nodal points is given by

$$\hat{H} = \hbar \nu_F (q_x + \chi A_3^{\text{el}}) \sigma_x - \hbar \nu_F (q_y + \chi A_3^{\text{el}}) \sigma_y + \hbar \nu_F (q_x + \chi A_3^{\text{el}}) \sigma_z$$

where the parameter $\chi$ denotes the chirality and strain couples with the low-energy electron excitations and for the sake of simplicity, we assume that the cartesian components of the Fermi velocity are the same. The pseudomagnetic field is obtained by $B^{\text{el}} = \nabla \times A^{\text{el}}$ where the axial gauge potential, $A_3^{\text{el}}$, can be obtained by the Weyl lattice displacement vector. For our purposes, we assume a kind of the lattice distortion which leads to a nonzero pseudomagnetic field. We would like to investigate the response of the system to a real time-varying electric field, therefore, we will discuss in the following subsections the chiral kinetic theory and semiclassical Boltzmann theory in WSMs.

A. Chiral kinetic theory

Chiral kinetic theory is the standard semiclassical Boltzmann theory to describe the behavior of Weyl fermions near at a finite chemical potential. We are interested in a doped WSMs, where the Fermi level $\mu$ crosses the conduction band. In this regime, according to the Pauli blocking, the range of frequency lower than $2\mu$ is blocked and all the dynamics coming from the processes in the vicinity of the Fermi surface and therefore only intraband transitions have dominated the process. The chiral kinetic theory works at a finite chemical potential where interband transitions are negligible. The semiclassical Boltzmann formalism obeys the semiclassical description where the wave packet of the Bloch functions in contrast to the plane wave has a Gaussian shape that spreads in the real and momentum spaces. Most importantly, WSMs show non-trivial and anomalous transport properties through semiclassical Boltzmann formalism owing
We note that the group velocity is independent of the chirality where
\[ v = \nabla_k \epsilon_k = v \hat{k}[1 - 2B^{cl} \cdot \Omega] + vB^{cl}(\hat{k} \cdot \Omega) \] (5)
We note that the group velocity is independent of the chirality, i.e. both modes are right-moving in a bulk in contrast to applying a real magnetic field where each mode disperses oppositely [14, 15].

The nonequilibrium distribution function \( f_X(t, p, x) \) for a given chirality \( \chi \) is obtained as the solution of the semiclassical Boltzmann equation
\[
\frac{\partial f_X}{\partial t} + \hat{x} \cdot \frac{\partial f_X}{\partial x} + \hat{k} \cdot \frac{\partial f_X}{\partial k} = I_{coll}(f_X)
\] (6)
where \( I_{coll}(f_X) \) is a collision integral. We focus on a very low temperature \( k_B T \ll \mu \) and finite frequencies where the collisionless limit \( \omega \tau \gg 1 \) (\( \tau \) is the shortest relaxation time) valid in this regime [40] and the thermally excited carriers can be ignored. Therefore the RHS of Eq. 6 vanishes. For the sake of simplicity, we consider the linear response regime where corrections to the equilibrium distribution function \( f^{eq} \) are linear in \( E \). We therefore choose \( f_X = f^{eq} + \frac{\partial f^{eq}}{\partial x} \delta f_X e^{-i\omega + iq \cdot r} \), and one obtains
\[
(1 - \frac{\hat{k} \cdot B^{cl}}{2|k|^2}) \partial_t \delta f_X + [v - \chi (v \cdot \Omega)] B^{cl} \cdot \partial_r \delta f_X + [eE - \chi v \times B^{cl} - \chi e(E \cdot B^{cl}) \Omega] \cdot \partial_k f^{eq} - \chi v \times B^{cl} \cdot \partial_k \delta f_X = 0
\] (7)
Substituting \( v \) from Eq. 5 into Eq. 7 and keeping the linear order of the electric field terms, the above expression becomes
\[
i[(1 + \kappa)\omega - v(q \cdot \hat{k})] \delta f_X + \chi v(\hat{k} \times B^{cl}) \cdot \partial_k \delta f_X = ev[\hat{k} \cdot E - (E \cdot B^{cl})(\Omega \cdot \hat{k})](1 + 2\kappa)
\] (8)
where we define \( \kappa = \frac{\hat{k} \cdot B^{cl}}{2|k|^2} \).

Making use of the spherical coordinate and considering the pseudomagnetic field along the \( z \) axis and \( \varphi \) being the azimuthal angle of the momentum \( k \), the above equation can be written as
\[
\chi \omega^{cl} \partial_\varphi \delta f_X + i[(1 + \kappa)\omega - v(q \cdot \hat{k})] \delta f_X = ev[\hat{k} \cdot E - (E \cdot B^{cl})(\Omega \cdot \hat{k})]
\] (9)
with \( \omega^{cl} = \frac{vB^{cl}}{1} \) is a cyclotron frequency of the massless fermions in the pseudomagnetic field. In the long-wavelength limit \( q \ll 1 \), the analysis is significantly simplified because one can neglect the dependence of \( \delta f_X \) on the wavevector \( q \). We make use of the standard parametrization of vectors in the spherical coordinates and then decompose \( \delta f_X \) into harmonics as \( \delta f_X = \delta f_0 + \delta f_\varphi e^{i\varphi} + \delta f_\chi e^{-iq \cdot r} \), and obtain the following solutions which are linear in \( E \)
\[
\delta f^{0}_X = -iev \left( \frac{\cos \theta - \alpha}{\omega(1 + \alpha \cos \theta)} \right) E \cdot \hat{\varphi}
\]
\[
\delta f^+_{\chi} = -iev \left( \frac{\sin \theta}{1 + \alpha \cos \theta} \right) \omega + \chi \omega^{cl} E_-
\]
\[
\delta f^-_{\chi} = -iev \left( \frac{\sin \theta}{1 + \alpha \cos \theta} \right) \omega - \chi \omega^{cl} E_+
\] (10)
where \( E_+ = \frac{E_x + iE_y}{2}, E_- = \frac{E_x - iE_y}{2}, \kappa = \alpha \cos \theta \) with \( \alpha = \frac{B^{cl}}{2|k|^2} \) and accordingly \( \omega^{cl} = 2|\alpha| |\mu| \) are defined.

C. Optical current density

1. The electric field is perpendicular to the pseudomagnetic field: \( E \perp B^{cl} \)

In this stage, once the nonequilibrium distribution function is obtained, we can calculate the optical quantities by knowing the direction of the external electric field. The total and chiral current can be obtained as
where
\[ \mathcal{D}\hat{x}_\chi = v\dot{k}_\chi(1-2\alpha \cos \theta) + va^2 \cos \theta \hat{n} + eE \times \Omega \chi \] (12)
in which \( \hat{n} \) is the unit vector pointing along the pseudomagnetic field \( B^\parallel \).

The current \( J^\chi \) can be further simplified using Eq. 12. Keeping terms up to the linear order of \( E \) and considering \( E \perp B^\parallel \), the current of each node reads as
\[ J^\chi_{i,\perp} = -ev \int \frac{d^3k}{(2\pi)^3} \hat{k}_\chi(1-2\kappa)\delta f^\chi \frac{\partial f^\chi}{\partial \epsilon_k} \] (13)
with the change of variable \( u = \cos \theta \). We are also interested to consider the pseudomagnetic field as a small perturbation, therefore we can evaluate the integral in the regime of small \( \alpha \) where pseudo-Landau level quantization is unimportant.

In generic, a Weyl semimetal consists of an even number of the Weyl points [8]. We study the case of two Weyl points so our results can be easily generalized to other Weyl systems with more than two nodes.

Once the total current \( J_i = \sum_\chi J^\chi_i \) is obtained, the optical conductivity tensor which is defined by
\[ J_i = \sigma_{ij}E_j \] (15)
can be calculated. The longitudinal optical conductivity is therefore given by
\[ \hat{\sigma}_{xx} = \hat{\sigma}_{yy} = -\frac{ie\mu^2}{3v^2}(\omega^2 - \omega_{el}^2) \left[ \frac{1}{\omega^2 - \omega_{el}^2} + \frac{3}{5} \alpha^2 \omega^2 \right] \] (16)
and the transverse optical response per each node \( \chi \) is
\[ \hat{\sigma}_{xy} = -\hat{\sigma}_{yx} = \frac{\chi e\mu^2}{6\pi v^2} \omega_{el} \left[ \frac{1}{\omega^2 - \omega_{el}^2} + \frac{7}{5} \alpha^2 \omega^2 \right] \] (17)
where \( \mu_\chi \) is the chemical potential at node \( \chi \). The hat-label here denotes that these optical conductivity expressions are complex-valued numbers. In order to specify the real and imaginary part of \( \hat{\sigma} \) we add the small quantity \( i\gamma \) to the frequency \((\omega \rightarrow \omega - i\gamma)\). The quantity of \( \gamma \) prevents vanishing of the denominator of the conductivity. The off-diagonal elements of the optical conductivity tensor cancel identically each other (for a balanced charge if \( \mu_+ = \mu_- \)) after summing over the chirality and hence they have no contribution to the total charge current. In the following, we will demonstrate that a spectacular feature occurs for \( \mu_+ \neq \mu_- \). It is worth mentioning that our results are valid in ac mode, i.e. the frequency interval \((0 < \omega < 2\mu)\) which induce intraband transitions apart from dc Drude response at zero frequency.

The general form of the optical response, using Eq. 16, to the time-varying electric field would be
\[ \text{Re}[\sigma^\chi_{xx}(\omega)] = \sigma_{dc} \delta(\omega) + \text{Re}[\hat{\sigma}_{xx}] \] (18)
where \( \sigma_{dc} = e\mu^2/3\pi^2 v_\gamma \), and the longitudinal conduc-
itivity diverges in a pristine system where \( \gamma \to 0 \). We consider a small broadening \( \gamma \) for \( \delta(\omega) \) and the second part is obviously valid for the range of \( 0 < \omega < 2 \mu \) originates from the pseudomagnetic field. Fig. 1(a) illustrates the low-frequency optical response, in units of \( \omega_{el} \), in the presence of strain. The longitudinal optical conductivity shows an absorption peak centered at \( \omega = \omega_{el} \) with accumulating of the chiral electrons and furthermore the conventional Drude peak at \( \omega = 0 \), with a small broadening \( \gamma \). When the strength of \( \gamma \) is increased, the height of the main peak in the optical conductivity increases. The real part of the optical conductivity shows the excitations in the system and it displays considerable structure at \( \omega_{el} \). This leads us to imagine that around \( \omega_{el} \), \( \sigma_{\perp}^{L} \) is due to the topmost accumulated electron levels. It should be worth mentioning that this chirality accumulation can be detected by the imbalanced absorbance of the circularly polarized light [41].

Dielectric function is an important physical quantity to explore. In general, the dielectric function is a complex valued-function, and the imaginary part of the dielectric function determines the amount of absorption inside the medium. Most importantly, including quantum interlayer contributions leads to increasing the imaginary part of the dielectric function, \( \epsilon_{1} \), and it turns out that the effects of quantum mechanics are very vital in systems in which interlayer transitions play an important role. There is a well-known connection between the real part of the dielectric function, \( \epsilon_{1} \), and the imaginary part of the optical conductivity.

\[
\epsilon_{1}(\omega) = \epsilon_{b} - \frac{4\pi Im \sigma_{\perp}^{L}(\omega)}{\omega} \tag{19}
\]

where \( \epsilon_{b} \) is the high-frequency dielectric constant of the WSM. It is worth mentioning that the real part of the dielectric function or the imaginary part of the conductivity describes the reflection of light (an elastic process).

Fig. 1(b) represents the real part of the dielectric function as a function of frequency. We ought to note that at the onset of transparency at the plasmon frequency we have \( \epsilon_{1}(\omega)=0 \). Since the function is always negative, which leads to the metal exhibits reflectivity, there is no plasmon mode of the system in the range of considered parameters.

In contrast to the total charge current \( J \) the axial (chiral) current \( J^{\parallel} \) will not vanish even if \( \mu_{+} = \mu_{-} \)

\[
J_{ij}^{\parallel} = \sum_{x} \chi_{ij}^{x} E_{x} = \\
= \frac{e^{2} \mu_{x}^{2} \omega_{el}(\frac{1}{\omega^{2} - \omega_{el}^{2}} + \frac{7}{5} \alpha^{2} \omega_{el}^{2} \omega^{2} + \frac{4}{9} \omega_{el}^{2}})}{6\pi^{2} v \omega^{2} - \omega_{el}^{2}}, \tag{20}
\]

This nonzero axial current induced by structure deformation through \( B^{el} \) and coupled to the time-varying electric field interpreted as the Chiral Hall Effect [32] and it vanishes in the absence of the pseudomagnetic field in
the chiral fermion system. In the case of \( \mu_+ = \mu_- \) the current associated with right-handed fermions (positive projection of spin on momentum: \( \chi = + \)) is the same in value but in the opposite direction of the left-handed ones (\( \chi = - \)) [Fig. 2(a)]. This extrinsic chiral Hall current is a current of the chirality without flowing a net charge. In this case carriers with opposite chirality diffuse in the opposite direction perpendicular to both electric and pseudomagnetic fields lead to the chirality separation in space [Fig. 2(c)]. This phenomenon is analogous to the spin Hall effect where the spin-orbit interaction causes the spin separation.

In order to have a nonzero Hall conductivity in the system, another prerequisite condition might be fulfilled for which we get \( \mu_+ \neq \mu_- \). A moderate inversion symmetry breaking mechanism can provide the condition, for instance a small electrostatic potential difference between the top and bottom surface of a WSM unit cell can basically generate a chemical potential difference between nodes [42]. Note that the impact of the potential is only to reduce the distance between the Weyl nodes in momentum space and therefore the low-energy model Hamiltonian in the system is unchanged. Generically, in the case of \( \mu_+ \neq \mu_- \) in WSMs, the chemical potential at the plus node (\( \chi = + \)) is pushed up by an amount \( Q_0 \) while \( \mu \) at the minus node is pushed down by the same amount. In this situation, the current associated with the right-handed and left-handed fermions are in the opposite direction but with different values [Fig. 2(b) and Fig. 2(d)]. Accordingly, the currents associated with each node do not compensate each other and the system develops a net charge. As a result, the CHE leads to both charge and chirality separation at the same time. Charge and chirality separation along and perpendicular \( E \) field has been discussed in the literature based on the kinetic theory quadratic in both electric and magnetic fields [43, 44], however we are proposing the CHE of strained WSMs without any external magnetic field.

In addition, a physical quantity which related to the Nernst parameter is the bulk Hall angle. Having calculated the conductivity tensor, the bulk Hall angle can also be obtained as

\[
\Theta_H = \tan^{-1}\left(Re\left[\frac{\sigma_{xy}(\omega)}{\sigma_{xx}(\omega)}\right]\right)
\]  

(21)

In the dc case, the \( \tan(\Theta_H) \) is \(-2eQ_0\omega_0/\gamma\mu_0\) showing that the Hall angle diverges at small chemical potential. When the character of chiral particle changes from \( \mu < 0 \) to \( \mu > 0 \), the Hall angle also changes from positive to negative. Fig. 3 shows the Hall angle as a function of frequency. In the case that \( \mu_+ = \mu_- \) or \( Q_0 = 0 \) the only nonzero optical response is along the electric field (longitudinal \( \sigma_{xx} \)) and then the Hall angle vanishes throughout all the frequency range. If \( Q_0 \neq 0 \) the transverse optical response (\( \sigma_{xy} \)) becomes nonzero leads to a finite Hall angle. The Hall angle decreases monotonically by frequency increment meaning that the longitudinal response dominates rather than the transverse one.

2. The electric field is parallel to the pseudomagnetic field: \( E \parallel B^{el} \)

In a similar manner of the ordinary chiral magnetic effect (CME) in the presence of a real \( B \) field, the pseudomagnetic field also generates a chiral pseudomagnetic effect. In contrast to the ordinary CME, which might vanish in equilibrium (\( \mu_5 = 0 \)), the chiral pseudomagnetic effect is allowed in the equilibrium [14, 15, 19], but the left-moving modes near the boundary and the right-moving modes in the bulk compensate each other after summing over the entire region. Therefore it leads to an electrical charge moving between regions of the opposite \( B^{el} \) via the anomalous Hall effect (AHE). In particular, we suppose the sign of \( B^{el} \) changes in the region where \( b_u \) vanishes rapidly (two nodes annihilate each other) over a short distance which implies the surface of WSMs. Therefore, we consider \( B^{el} > 0 \) in the bulk and \( B^{el} < 0 \) in the surface.

When an electric field \( E \) is applied parallel to the pseudomagnetic field, according to the covariant anomaly equation, a charge density imbalance between bulk and surface is induced to the system [15]

\[
\delta \rho = \frac{e^2\tau}{2\pi^2} E \cdot B^{el} \tag{22}
\]

Here, \( \tau \) is the electron relaxation rate. This charge density imbalance can relax through the processes which the right-moving modes in the bulk scatter back to the left-moving modes near the boundary. The induced bulk electron density gets disturbed among all the empty bulk states above the Fermi energy and leads to a small shift in the chemical potential \( \mu \rightarrow \mu + \delta \mu \). In the limit of \( k_BT \ll \delta \mu \ll \mu \) the shift in the chemical potential \( \delta \mu \) of the bulk states is given by

\[
\delta \mu \approx \frac{2\pi^2 v_f^3}{\mu^2} \delta \rho \tag{23}
\]

which is valid in the semiclassical limit where pseudo-Landau level quantization is unimportant. With this changing of the chemical potential, the population of the right-moving modes in the bulk and the left-moving modes in the surface rises with the same amount of \( \mu \pm = \mu_0 \pm + \delta \mu \).

In the collisionless limit, we suppose the particle path is straight not affected by a circular motion owing to the pseudomagnetic field and therefore the relaxation time \( \tau \) is field independent. The corresponding conductivity associated to the charge pumping from the boundary to the bulk, the chiral torsional effect (CTE), is

\[
\sigma_{CTE} = \frac{e^4 v_f^3 \tau}{8\pi^2 \mu^2} (B^{el})^2 \tag{24}
\]

that gives the electrical current \( J_{CTE} = \sigma_{CTE} E_z \) which cannot be captured by the CKT. The main consequence of the above equation is the enhancement of the bulk
transport current coming directly from the response of the system to the electric field parallel to the pseudomagnetic field \((E \parallel B_{cl})\). The parallel current \(J_{||}^X\) in the frame of the CKT is given by

\[
J_{||}^X = e v \int \frac{d^3k}{(2\pi)^3} (-\frac{\partial f^{(eq)}}{\partial \epsilon_k}) (\cos \theta (1 - 2\alpha \cos \theta)) + \alpha^2 \cos^2 \theta) \delta f_0 - e \int \frac{d^3k}{(2\pi)^3} f^{(eq)} v \epsilon_k (1 - 2\alpha \cos \theta) \tag{25}
\]

The first integral corresponds to the response of the system to the time-varying electric field and the second integral leads to the chiral pseudomagnetic effect causes spatial charge separation along the \(B_{cl}\) field. From the previous expression, we can read off the parallel component of the conductivity tensor

\[
\sigma_{zz} = \frac{ie^2 \mu^2}{4\pi^2 \omega^2} \sum_x \int_{-1}^{1} du (u(1 - 2\alpha u) + \alpha^2 u^2) (u - \alpha) \frac{1 + \alpha u}{1 + \alpha u} \tag{26}
\]

As before, we expand the integrand up to the second-order in \(\alpha\). After integrating upon \(u\) and collecting the terms, the longitudinal optical conductivity along the pseudomagnetic field coming from CKT is given by

\[
\sigma_{zz} = \frac{\sigma_{dc}}{1 + i \omega \tau} (1 + \frac{24}{5} \alpha^2) \tag{27}
\]

The correction term is proportional to \(\alpha^2 \propto (B_{cl})^2\) having a positive contribution to the longitudinal conductivity in the strained WSM system. Taking into account the contribution of the chiral torsion effect Eq. 24, we finally have

\[
\tilde{\sigma}_{zz} = \sigma_{zz} + \sigma_{CTE}
\]

\[
\tilde{\sigma}_{zz} = \frac{\sigma_{dc}}{1 + i \omega \tau} (1 + \frac{24}{5} \alpha^2) + \frac{e^4 v^3}{8\pi^3} \frac{\tau_r}{\mu^2} (B_{cl})^2 \tag{28}
\]

The first term is the correction to the Drude conductivity. This originates from the CKT by utilizing the semiclassical Boltzmann formalism and the second term stems from the chiral pseudomagnetic effect which cannot be captured by semiclassical formalism. We also find that the chiral pseudomagnetic effect \((E \parallel B_{cl})\) leads to an enhancement of the longitudinal conductivity \(\sigma \propto B_{cl}^2\) in Dirac and Weyl semimetals. In analogous to the chiral magnetic effect, this has a positive contribution to a magnetococonductivity [19].

III. CONCLUSION

In summary, we have considered strained and doped Weyl semimetals in the presence of an external electric field. We have explored the CHE in the chiral system focusing on intraband transitions as a consequence of coupling light to the free carriers. The chiral kinetic theory is utilized to calculate the optical conductivity tensor. When a moderate inversion symmetry is broken in Weyl semimetals the two Weyl nodes with opposite chirality would split in energy and therefore the chemical potential in each node could be different. Accordingly, the structural deformation together with the time-varying external electric filed leads to emerging the charring Hall effect in doped WSMs.

We have also discussed the emerging of transverse chiral Hall current as an optical response in the presence of the pseudomagnetic field, which to our best knowledge, has not been discussed in the context of WSMs. We have shown that the CHE current emerges as a response of a real time-varying electric field in the presence of strain and a spatial chirality and charges separation occur in the system. The longitudinal conductivity peak occurs at the cyclotron frequency of the massless fermions in the pseudomagnetic field.

Finally, we would like to note that the imbalanced absorbance of the left- and right-handed circularly polarized light is a technique to determine the surface chirality charge of a pristine system. Our findings here can be explored by making use of this experiment technique.

IV. ACKNOWLEDGEMENT

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