Techniques of evaluation of QCD low-energy physical quantities with running coupling with infrared fixed point

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(Dated: May 11, 2014)

PACS numbers: 11.10.Hi, 12.38.Cy, 12.38.Aw

I. INTRODUCTION

One of the main problems in QCD is to understand the theory at low (hadronic) scales $|q| \lesssim 1$ GeV. The usual perturbative QCD (pQCD) running coupling $a(Q^2) = \alpha_s(Q^2)/\pi$ is expected to get modified at low spacelike momenta $0 < Q^2 \lesssim 1$ GeV$^2$ so that, instead of having unphysical (Landau) singularities it remains smooth and finite there, due to infrared (IR) fixed point. This behavior is suggested by: Gribov-Zwanziger approach, Dyson-Schwinger equations (DSE) and other functional methods, lattice calculations, light-front holographic mapping AdS/CFT modified by a dilaton background, and by most of the analytic (holomorphic) QCD models. All such couplings, $\mathcal{A}(Q^2)$, differ from the pQCD couplings $a(Q^2)$ at $|Q| \gtrsim 1$ GeV by nonperturbative (NP) terms, typically by some power-suppressed terms $\sim 1/Q^{2N}$.

Evaluations of low-energy physical QCD quantities in terms of such $\mathcal{A}(Q^2)$ couplings (with IR fixed point) at a level beyond one-loop are usually performed with (truncated) power series in $\mathcal{A}(Q^2)$. We argue that such an evaluation is not correct, because the NP terms in general get out of control as the number of terms in the power series increases. The series consequently become increasingly unstable under the variation of the renormalization scale, and have a fast asymptotic divergent behavior compounded by the renormalon problem. We argue that an alternative series in terms of logarithmic derivatives of $\mathcal{A}(Q^2)$ should be used. Further, a Padé-related resummation based on this series gives results which are renormalization scale independent and show very good convergence. Timelike low-energy observables can be evaluated analogously, using the integral transformation which relates the timelike observable with the corresponding spacelike observable.

One of the main problems in QCD is to understand the theory at low (hadronic) scales $|q| \lesssim 1$ GeV. The usual perturbative QCD (pQCD) running coupling $a(Q^2) = \alpha_s(Q^2)/\pi$, where $q^2 = −Q^2$ is the squared momentum transfer, suffers from Landau singularities at low scales: $|Q^2| \lesssim 1$ GeV$^2$ and $Q^2 \not< 0$. These singularities can be called unphysical for the following reason: the spacelike observables $d(Q^2)$, which are expected to be evaluated as a function of $a(\kappa Q^2)$ (with $\kappa \sim 1$), do not have such singularities. In fact, $d(Q^2)$ are analytic functions of $Q^2$ in the entire complex plane with the exception of the negative axis $Q^2 < 0$ (where $M_{thr}^2 \sim 10^{-1}$ GeV$^2$ is a squared threshold scale), see Fig. 1, this property following from the basic principles of quantum field theories such as locality, unitarity and microcausality.

The mentioned analyticity properties of a realistic QCD coupling $\mathcal{A}(Q^2)$ are supported by Gribov-Zwanziger approach [1], calculations involving Dyson-Schwinger equations (DSE) for gluon and ghost propagators and vertices [2, 3], stochastic quantization [4], functional renormalization group equations [5], and by lattice calculations [6, 7]. Most of these calculations suggest that $\mathcal{A}(Q^2)$ is analytic functions of $Q^2$ in the entire complex plane (with the exception of the negative axis $Q^2 < 0$, where $M_{thr}^2 \sim 10^{-1}$ GeV$^2$ is a squared threshold scale), see Fig. 1. The spacelike observables $d(Q^2)$ are considered to be functions of $\mathcal{A}(\kappa Q^2)$ (with $\kappa \sim 1$), then $\mathcal{A}(Q^2)$ $[\mathcal{A}(\kappa Q^2)]$ should reflect the aforementioned analyticity properties of $d(Q^2)$. It is interesting that imposition of such analyticity properties on $\mathcal{A}(Q^2)$ almost always leads to an IR fixed point for $\mathcal{A}(Q^2)$ as well [9–18]. This is also true for the perturbation theory in the confining QCD background in the large-$N_c$ limit [19].

One may also ask whether there exists a renormalization scheme in which a purely perturbative coupling $a(Q^2)$ is an analytic function of $Q^2$ in the mentioned sense. In Refs. [20] it was shown that it is difficult to construct such models. Namely, in order to have analyticity of $a(Q^2)$ and simultaneously the reproduction of the measured value of the effective charge for the semihadronic $\tau$ decay ratio $r_\tau \approx 0.20$ ($V + A$ channel), the required schemes are such that they result in power series for observables such that, after a few finite terms, the further terms appear to be uncontrollably large.

On the other hand, there exist acceptable analytic models of $\mathcal{A}(Q^2)$ which practically merge with the perturbative coupling (in the same scheme) at high $|Q^2| > \Lambda^2$, i.e., $\mathcal{A}(Q^2) - a(Q^2) \sim (\Lambda^2/Q^2)^N$ (where $\Lambda^2 \sim 1$ GeV$^2$, with $N$ large, e.g., $N = 4, 5$), cf. Refs. [16–18, 21]. For example, the model [18] has $N = 5$ and it reproduces the correct value of $r_\tau$. In such models, due to large $N$ the Operator Product Expansion (OPE) approach can be used and interpreted in the same way as in pQCD (pQCD+OPE), cf. Ref. [22]. Nonetheless, at low energies $|Q^2| < 1$ GeV$^2$ the nonperturbative contributions become appreciable, the theory differs there appreciably from pQCD.
The coupling $\mathcal{A}(Q^2)$ in some of the models with IR fixed point may have Landau singularities within the complex $Q^2$-plane outside the negative $Q^2$-semiaxis, such as, e.g., the model of Ref. [29] (cf. the comments on that coupling in Ref. [23]). However, in general, it is reasonable to assume that in most of the models with IR fixed point the analyticity requirement is fulfilled as well, or can be made fulfilled.

We will present several frameworks with IR fixed point. In such frameworks it is usually assumed in the literature that the series in powers of $a(Q^2)$ for physical quantities can be used unchanged, with the replacements $a(Q^2)^n \rightarrow \mathcal{A}(Q^2)^n$, where $a(Q^2)$ and $\mathcal{A}(Q^2)$ are in the same renormalization scheme in the perturbative sense. We will argue that this assumption is not correct in IR fixed point scenarios, as it leads in general to increasingly stronger renormalization scale dependence of the result when the number of terms in the truncated series increases. Further, such a series shows tendency to strong divergence, especially for low-momenta physical quantities, partly as a consequence of the fast growth of the coefficients of the series due to renormalons (cf. Ref. [24] and references therein).

In Sec. II we present several frameworks with IR fixed point. In Sec. III we present a construction method for a nonpower series, in terms of the logarithmic derivatives $\tilde{A}_n(Q^2) \propto d^n/d(\ln Q^2)^{n-1}$, and argue that the correct approach for the evaluation of spacelike QCD physical quantities in the frameworks with IR fixed point is in terms of $\tilde{A}_n(Q^2)$ and not $\mathcal{A}(Q^2)^n$. In Sec. IV we present numerical evidence for this, using as a test case a specific spacelike physical quantity (massless Adler function) in the leading-$\beta_0$ (LB) approximation to very high orders, and in the full case (“LB+beyondLB”) to the available orders. We apply evaluations in the usual pQCD (where the running coupling has unphysical/Landau singularities) and in three chosen scenarios with IR fixed point. Specifically, in Sec. IV A we present the renormalization scale dependence of various evaluations in the various scenarios, and in Sec. IV B the convergence/divergence properties of such evaluations when the truncation order $N$ increases. In Sec. V B we apply, in addition, a resummed version of the logarithmic derivatives approach, namely a resummation based on a generalization of the diagonal Padé resummation. We show that the latter method is superior to all others in the IR fixed point scenarios. In Sec. VI we then argue that for the timelike physical quantities $\Gamma(s)$ ($s = -Q^2 > 0$) the evaluation should proceed via the integral transformations which relate them with the corresponding spacelike quantities, where the latter are evaluated with the mentioned approaches. In Sec. VI we summarize the presented results.

II. IR FIXED POINT SCENARIOS

The simplest case of freezing comes from the use of the the one-loop perturbative coupling with the replacement $Q^2 \rightarrow Q^2 + m^2$ where $m$ is a constant mass (of the order of meson masses)

$$\mathcal{A}^{(m)}(Q^2) = \frac{1}{\beta_0 \ln \left(\frac{Q^2 + m^2}{\Lambda^2}\right)},$$

where $\beta_0 = (1/4)(11 - 2N_f/3)$. It was obtained in Ref. [25] as a consequence of the use of nonperturbative QCD background, and is $m \sim 1$ GeV. It was also used in Refs. [26, 27] for an analysis of structure functions (with $m \approx 0.8$ GeV). Similar construction was made in Ref. [28]. This coupling is analytic, in the sense that it has singularities in the complex $Q^2$-plane on the negative semiaxis only: a pole at $Q^2 = \Lambda^2 - m^2$ ($< 0$), and a cut at $Q^2 < -m^2$. At $Q^2 \rightarrow 0$ the coupling freezes at the positive value $[\beta_0 \ln(m^2/\Lambda^2)]^{-1}$. At large $|Q^2| > \Lambda^2$ it tends to
one-loop pQCD coupling and differs from it by
\[ A^{(m)}(Q^2) - a^{(1-\epsilon)}(Q^2) \sim \frac{m^2}{Q^2 \ln^2(Q^2/\Lambda^2)}. \] (2)

In Figs. 2(a), (b) we present the beta function \( \beta(A^{(m)}(Q^2)) = Q^2 dA^{(m)}(Q^2)/dQ^2 \), and the running coupling \( A^{(m)}(Q^2) \)

at positive \( Q^2 \), where we chose \( N_f = 3 \), \( m = 0.8 \) GeV and \( \Lambda \) such that \( A^{(m)}(0) = 0.3/\pi \) (\( \Lambda \approx 0.078 \) GeV). These curves are qualitatively representative of any IR fixed point scenario: the freezing of the running coupling \( A^{(m)}(Q^2) \) at low \( Q^2 \) (where \( Q^2 \) is on logarithmic scale), and the beta function achieves zero at \( A^{(m)} = A^{(m)}(0) | = 0.3/\pi \approx 0.0955 \) in this specific case. We note that the beta function in this case is

\[ \beta(A^{(m)}) = -\beta_0(A^{(m)})^2 \left[ 1 - \frac{m^2}{2\Lambda^2} \exp \left( -\frac{1}{\beta_0 A^{(m)}} \right) \right], \] (3)

which is a function of \( A^{(m)} \) nonanalytic at \( A^{(m)} = 0 \), implying that this scenario is not of the pQCD-type since the beta function has a nonperturbative contribution \( \sim \exp(-1/\beta_0 A^{(m)}) \).

A range of models with similar running of the coupling is suggested by extensive analyses of the Dyson-Schwinger equations for the gluon and ghost propagators and vertices \[23, 24\] and by other functional methods \[14, 15\].

At higher \( |Q^2| (> \Lambda^2) \), when going beyond the one-loop level, the multiplicative renormalizability suggests that the replacement \( Q^2 \rightarrow (Q^2 + \rho m(Q^2)^2) \) should be made in the perturbative coupling \[29\] (cf. Refs. \[28, 30\] when \( m \) is constant)

\[ A^{(DS,n-\ell)}(Q^2) \approx a^{(n-\ell)}(Q^2 + \rho m(Q^2)^2). \] (4)

The dynamical mass \( m(Q^2) \) of the DSE-approaches introduces nonperturbative effects which are felt at \( |Q^2| > \Lambda^2 \) as

\[ A^{(DS)}(Q^2) - a(Q^2) \sim \frac{m(Q^2)^2}{Q^2 \ln^2(Q^2/\Lambda^2)}. \] (5)

Coupling with IR fixed point is suggested also by AdS/CFT correspondence modified by a (positive-sign) dilaton background \[8\]

\[ A^{(AdSmod)}(Q^2) = A^{(AdS)}(Q^2) g_+(Q^2) + a^{(fit)}(Q^2) g_-(Q^2), \] (6)

where at low \( Q < 0.8 \) GeV predominates the AdS-part

\[ A^{(AdS)}(Q^2) = A^{(AdS)}(0)e^{-Q^2/(4\kappa^2)}, \] (7)

with \( \kappa = 0.54 \) GeV; and \( A^{(AdS)}(0) = 1 \) is the IR fixed point in \( g_1 \) (Bjorken sum rule) effective charge scheme.\(^1\) On the other hand, \( a^{(fit)}(Q^2) \) is obtained by fit to the data for \( Q > 0.8 \) GeV. \( g_\pm(Q^2) \) are smeared step functions, e.g.,

\(^1\) It turns out that the same coupling can be obtained also in the negative-sign dilaton scenario; the five-dimensional coupling is defined in both cases as \( g_c^{-2}(z) = e^\phi(z) g_c^{-2} \) where \( \phi(z) = \kappa^2 z^2 \); the sign of the dilaton affects neither the running coupling nor the mass spectrum, but becomes important for the calculation of the bulk-to-boundary propagator in the AdS space, Ref. \[31\].
\( g_{\pm}(Q^2) = 1/(1 + e^{\pm (Q^2 - Q_0^2)/\tau^2}) \) with \( Q_0 = 0.8 \text{ GeV} \) and \( \tau = \kappa \). At large \( |Q^2| > \kappa^2 \) the difference between this coupling and the perturbative coupling is very small

\[
A^{(\text{AdSmod.})}(Q^2) - a(Q^2) \sim \frac{e^{-Q^2/\kappa^2}}{\ln(Q^2/\Lambda^2)} \quad (|Q^2| \gg \kappa^2).
\]

Another case is the Analytic Perturbation Theory (APT) coupling [9][11], which is obtained by “minimally” analytizing the perturbative \( n \)-loop coupling \( a(Q^2) \). The construction of the APT coupling is the following. The pQCD coupling has singularities on the semiaxis \( Q^2 < \Lambda^2 \), where the (Landau) cut \( 0 < Q^2 < \Lambda^2 \) starts at the branching point \( \Lambda^2 \), and is unphysical in the aforementioned sense. Application of the Cauchy theorem to the function \( a(Q^2)/(Q^2 - Q^2) \) to an appropriate closed contour (avoiding the cuts) in the complex \( Q^2 \)-plane, leads to the following dispersion relation for \( a(Q^2) \)

\[
a(Q^2) = \frac{1}{\pi} \int_{\sigma = -\Lambda^2 - i\varepsilon}^{\infty} \frac{d\sigma \rho^{(\text{pt})}(\sigma)}{(\sigma + Q^2)}, \quad (\eta \to +0),
\]

where \( \rho^{(\text{pt})}(\sigma) \) is the pQCD discontinuity function of \( a \) along the cut axis: \( \rho^{(\text{pt})}(\sigma) = \text{Im}(-\sigma - i\varepsilon) \). The APT procedure consists in the elimination, in the above integral, of the contributions of the Landau cut \( 0 < (-\sigma) \leq \Lambda^2 \), leading to the APT analytic analog of \( a \) (see Fig. 3)

\[
A^{(\text{APT})}(Q^2) = \frac{1}{\pi} \int_{\sigma = 0}^{\infty} \frac{d\sigma \rho^{(\text{pt})}(\sigma)}{(\sigma + Q^2)}.
\]

The APT analogs of powers \( a^n (\nu \text{ a real exponent}) \) is obtained in the same way

\[
A^{(\text{APT})}(Q^2) = \frac{1}{\pi} \int_{\sigma = 0}^{\infty} \frac{d\sigma \rho^{(\text{pt})}(\sigma)}{(\sigma + Q^2)}.
\]

where \( \rho^{(\text{pt})}(\sigma) = \text{Im}a^n(-\sigma - i\varepsilon) \). The underlying pQCD coupling \( a(Q^2) \) can run at any \( n \)-loop level and can be in any chosen renormalization scheme; the corresponding renormalization group equation (RGE) is

\[
\frac{\partial a(\ln Q^2; \beta_2, \ldots)}{\partial \ln Q^2} = - \sum_{j=0}^{n-1} \beta_j a^{j+2}(\ln Q^2; \beta_2, \ldots),
\]

where the first two beta coefficients are universal \( [\beta_0 = (1/4)(11 - 2N_f/3), \beta_1 = (1/16)(102 - 38N_f/3)] \), and the other coefficients \( \beta_k (k \geq 2) \) characterize the perturbative renormalization scheme. It turns out that the APT coupling has IR fixed point: \( A(0) = 1/\beta_0 (\approx 0.44 \text{ if } N_f = 3) \). At one-loop level, it is particularly simple:

\[
A^{(\text{APT},1-\ell)}(Q^2) = \frac{1}{\beta_0} \left[ \frac{1}{\ln z} - \frac{1}{(z - 1)} \right], \quad (z \equiv Q^2/\Lambda^2).
\]

FIG. 3: Left-hand Figure: the integration path for the integrand \( a_{4\nu}(Q^2)/(Q^2 - Q^2) \) leading to the dispersion relation [9] for \( a_{4\nu}(Q^2) \). Right-hand Figure: the integration path for the same integrand, leading to the dispersion relation [11] for the APT coupling \( A^{(\text{APT})}(Q^2) \).
Explicit expressions for $A_{\nu}^{\text{(APT)}}$ at one-loop level also exist and were constructed and used in Ref. [13]

\[ A_{\nu}(Q^2)^{\text{(APT,1-\ell)}} = \frac{1}{\beta_0} \left( \frac{1}{\ln^2(z)} - \frac{L_{-\nu+1}(1/z)}{\Gamma(\nu)} \right), \]

where $z \equiv Q^2/\Lambda^2$ and $L_{-\nu+1}(z)$ is the polylogarithm function of order $-\nu + 1$. Extensions to higher loops were performed via expansions of the one-loop result [14] [Fractional APT (FAPT)]. For a review of FAPT, see Refs. [22], and mathematical packages for numerical calculation are given in Refs. [33].

Another analytic model, based on the minimal analyticity of the function $d\ln a(Q^2)/d\ln Q^2$, Refs. [34], leads to a coupling with no freezing in the IR.

It turns out that the APT coupling differs from the pQCD coupling by terms $\sim (\Lambda^2/Q^2)^4$ at large $|Q^2| > \Lambda^2$

\[ A^{\text{(APT)}}(Q^2) - a(Q^2) \sim \left( \frac{\Lambda^2}{Q^2} \right)^4, \]

which may be appreciable even at high energies. An extension of the APT coupling at one-loop, such that the difference between it and the pQCD coupling is $\sim (\Lambda^2/Q^2)^p$, was proposed by Webber [16]

\[ A^{\text{(W,1-\ell)}}(Q^2) = \frac{1}{\beta_0} \left[ \frac{1}{\ln z} + \frac{1}{1-z} \frac{z+b}{1+b} \left( \frac{1+c}{z+c} \right)^p \right], \]

where $z \equiv Q^2/\Lambda^2$ and specific values of parameters were chosen such that the model gives good agreement with a range of data on power corrections: $b = 1/4$, $c = 4$, and $p = 4$. The coupling has IR fixed point, $A^{\text{(W,1-\ell)}}(0) = 1/(2\beta_0) \approx 0.22$. In this model, the difference from the pQCD coupling is

\[ A^{\text{(W,1-\ell)}}(Q^2) - a^{(1-\ell)}(Q^2) \sim \left( \frac{\Lambda^2}{Q^2} \right)^4. \]

A somewhat related is the model of Alekseev [21] for the coupling $A(Q^2)$ (called synthetic coupling), which is a modification of the APT model $A$ (at any loop-level)

\[ A^{\text{(AL)}}(Q^2) = A^{\text{(APT)}}(Q^2) + \frac{1}{\beta_0} \left[ \frac{c\Lambda^2}{Q^2} - \frac{d \Lambda^2}{Q^2 + m_g^2} \right], \]

where the three parameters $c$, $d$ and the effective gluon mass $m_g$ were determined by the requirement

\[ A^{\text{(AL)}}(Q^2) - a(Q^2) \sim \left( \frac{\Lambda^2}{Q^2} \right)^3 \]

and by the string tension parameter in the IR. However, in this case, there is no IR fixed point, due to the term $\sim 1/Q^2$ in the constructed coupling.

Yet another approach is based on the general dispersive relation for analytic couplings,

\[ A(Q^2) = \frac{1}{\pi} \int_{\sigma=0}^{\infty} d\sigma \rho(\sigma), \]

where $\rho$ is the discontinuity function of $A$: $\rho(\sigma) = \text{Im}A(-\sigma - i\epsilon)$. In Refs. [17] [18] this discontinuity function was approximated at high momenta $\sigma \geq M_0^2$ ($\gtrsim 1 \text{ GeV}^2$) by its pQCD analog $\rho^{(pt)}(\sigma) = \text{Im}a(-\sigma - i\epsilon)$: in the unknown low-energy regime, $0 < \sigma < M_0^2$ it was approximated by either one (Ref. [17]) or two delta functions (Ref. [18])

\[ \rho^{(16)}(\sigma) = \frac{\pi F_1^2}{Q^2 + M_1^2} + \Theta(\sigma - M_0^2) \rho^{(pt)}(\sigma), \]

\[ \rho^{(25)}(\sigma) = \frac{\pi F_1^2}{Q^2 + M_1^2} + \pi F_2^2 \delta(\sigma - M_2^2) + \Theta(\sigma - M_0^2) \rho^{(pt)}(\sigma). \]

The parameters of the delta functions and the pQCD-onset scale $M_0$ were adjusted so that the correct value of the semihadronic tau decay ratio $r_\tau \approx 0.20 (V + A \text{ channel})$ was reproduced and that the difference from the pQCD coupling at high $|Q^2| > \Lambda^2$ is as strongly suppressed as possible

\[ A^{(16)}(Q^2) = \frac{F_1^2}{Q^2 + M_1^2} + \frac{1}{\pi} \int_{M_0^2}^{\infty} d\sigma \frac{\rho^{(pt)}(\sigma)}{(Q^2 + \sigma)}, \]

\[ A^{(25)}(Q^2) = \frac{F_1^2}{Q^2 + M_1^2} + \frac{F_2^2}{Q^2 + M_2^2} + \frac{1}{\pi} \int_{M_0^2}^{\infty} d\sigma \frac{\rho^{(pt)}(\sigma)}{(Q^2 + \sigma)}. \]
The resulting deviations from pQCD at high $|Q^2| > \Lambda^2$ are

\[ A^{(1\delta)}(Q^2) - a(Q^2) \sim \left( \frac{\Lambda^2}{Q^2} \right)^3, \]  
\[ A^{(2\delta)}(Q^2) - a(Q^2) \sim \left( \frac{\Lambda^2}{Q^2} \right)^5. \]  

(23a)  

(23b)

The suppression of deviation from pQCD coupling, at high $|Q^2|$, may be regarded as preferred because then OPE can be used and interpreted in such models in the same way as OPE in pQCD – that the higher dimensional nonperturbative terms $\sim 1/(Q^2)^N$ ($N \leq N_{cr}$, with $N_{cr} = 4$ in $2\delta$ model) have purely IR origin. In APT, in view of the significant difference [15], such interpretation is not possible, and part of the nonperturbative terms $\sim 1/(Q^2)^N$ ($N = 1, 2, \ldots$) comes from the leading-twist contribution and has UV origin. Both models ($1\delta$, $2\delta$) have IR fixed point, with $A(0) \leq 1$.

### III. PROBLEMS WITH POWER SERIES FOR SPACELIKE PHYSICAL QUANTITIES IN IR FIXED POINT SCENARIOS, AND SOLUTION

If we regard a spacelike physical quantity $d(Q^2)$, such as current correlators (Adler function, etc.) or structure function sum rules, the usual evaluation in pQCD is via the power series

\[ d(Q^2)_{pt} = a(\kappa Q^2) + \sum_{n=1}^{\infty} d_n(\kappa) a(\kappa Q^2)^{n+1}, \]  

(24)

where $\mu^2 = \kappa Q^2$ is a renormalization scale ($\kappa \sim 1$). The dependence on other renormalization scheme parameters ($\beta_2, \beta_3, \ldots$) has been suppressed in the notation. Unless this series is the leading-$\beta_0$ resummation or some other partial resummation, the series is known only up to certain order $\sim a^N$ (usually $N = 3$ or 4)

\[ d(Q^2; \kappa)_{pt}^{[N]} = a(\kappa Q^2) + \sum_{j=1}^{N-1} d_j(\kappa) a(\kappa Q^2)^{j+1}. \]  

(25)

As a consequence of truncation, the truncated series has unphysical dependence on the renormalization scale (RS) parameter $\kappa$. However, the more terms are included, the weaker is the RS dependence (at high $|Q^2|$) generally

\[ \frac{\partial d_{pt}^{[N]}}{\partial \ln \kappa} = K_N a(\kappa Q^2)^{N+1} + K_{N+1} a(\kappa Q^2)^{N+2} + \cdots \sim a^{N+1}, \]  

(26)

where $K_N, K_{N+1}, \ldots$ are specific coefficients determined by the original truncated series coefficients $d_n(\kappa)$ ($n = 1, \ldots, N - 1$).\(^2\) If we now have a model where the coupling $A(Q^2)$ has nonperturbative contributions (such as any of the aforementioned IR fixed point scenarios), we have

\[ A(Q^2) - a(Q^2) = T_{NP}(Q^2), \]  

(27)

where the term $T_{NP}(Q^2)$ is nonperturbative, i.e., at $|Q^2| > \Lambda^2$ it is a function of $a(Q^2)$, $T_{NP}(Q^2) = F(a(Q^2))$, which is nonanalytic at $a = 0$. For example,

\[ T_{NP}(Q^2) \sim \left( \frac{\Lambda^2}{Q^2} \right)^n \approx \exp \left[ - \frac{n}{\beta_0 a(Q^2)} \right], \]  

(28a)

\[ T_{NP}(Q^2) \sim \exp \left( - \frac{Q^2}{K^2} \right) \sim \exp \left[ - \left( \frac{\Lambda^2}{K^2} \right) e^{1/\beta_0 a(Q^2)} \right]. \]  

(28b)

If applying now the power series [24] in the IR fixed point scenarios

\[ d(Q^2; \kappa)_{pt, A}^{[N]} = A(\kappa Q^2) + \sum_{j=1}^{N-1} d_j(\kappa) A(\kappa Q^2)^{j+1}, \]  

(29)

\(^2\) The RS dependence at low $\kappa Q^2$ is, however, strengthened by the large value of $a(\kappa Q^2)$, as seen from Eq. (26).
In virtually all IR fixed point (analytic) models we have:

\[ |A| \]

Therefore, in general

\[ |A| \]

the inclusion of more terms in this power series tends to make the result increasingly more RS-dependent\(^3\) or the RS dependence becomes more erratic, due to the inclusion of the effects of the NP terms \( \sim T_{NP}(\kappa Q^2)^m \). This can be numerically verified. These aspects are also reflected in the fact that the beta function in all the aforementioned IR fixed point scenarios, \( \beta(A(Q^2)) \equiv \partial A(Q^2)/\partial \ln Q^2 \), cannot be presented fully with a power expansion in \( A \), i.e., \( \beta(A) \) contains also terms which are nonanalytic in \( A \).

All this suggests that the analog of the power \( a^n \) is not \( A^n \), but rather a nonpower expression \( A_n \). Within the context of APT\(^9\), this has been noted by the authors of APT, and the construction Eq. (11) really gives \( A_n^{(APT)} \neq (A_1^{(APT)})^n \). However, in general models with finite \( A(0) \), the APT-type of construction cannot be made since it uses only the pQCD couplings (and their discontinuities).

It turns out that the construction of \( A_n \), the analog of \( a^n \), can be made in such general frameworks with IR fixed point, via a detour by construction of logarithmic derivatives, \([20, 35, 36]\). In pQCD these are

\[
\bar{a}_{n+1}(Q^2) = (-1)^n \frac{\partial^n a(Q^2)}{\beta_0^n n!} \frac{\partial (\ln Q^2)^n}{\partial (\ln Q^2)^n}, \quad (n = 1, 2, \ldots). \tag{30}
\]

We note that \( \bar{a}_{n+1}(Q^2) = a(Q^2)^{n+1} + O(a^{n+2}) \) by RGE \( \partial a(Q^2)/\partial \ln Q^2 = \beta(a(Q^2)) \), where beta function \( \beta(a) \) has the pQCD expansion as given in Eq. (12). The analytization is a linear operation. Therefore

\[
(a(Q^2))_n = \mathcal{A}(Q^2) = \left( \frac{\partial a(Q^2)}{\partial \ln Q^2} \right) _n = \frac{\partial A(Q^2)}{\partial \ln Q^2}. \tag{31}
\]

Therefore, in general

\[
\bar{a}_{n+1}(Q^2) = \bar{\mathcal{A}}_{n+1}(Q^2), \tag{32}
\]

where

\[
\bar{\mathcal{A}}_{n+1}(Q^2) = (-1)^n \frac{\partial^n A(Q^2)}{\beta_0^n n!} \frac{\partial (\ln Q^2)^n}{\partial (\ln Q^2)^n}. \quad (n = 1, 2, \ldots). \tag{33}
\]

In virtually all IR fixed point (analytic) models we have: \( |\mathcal{A}(Q^2)| > |\bar{\mathcal{A}}_2(Q^2)| > |\bar{\mathcal{A}}_3(Q^2)| > \cdots \) for any \( Q^2 \) (not just when \( |Q^2| \) is large). This is an empirical observation which could possibly be proven under specific conditions.

The basic relation (32) then requires reexpression of the power series (24) as a series in logarithmic derivatives \( \bar{a}_{n+1}(Q^2) \) (“modified” perturbation series, mpt)

\[
d(Q^2)_{\text{mpt}} = a(\kappa Q^2) + \sum_{n=1}^{\infty} \bar{a}_{n}(\kappa) \bar{a}_{n+1}(\kappa Q^2). \tag{34}
\]

This leads, after the analytization term-by-term, to the “modified” analytic (man) series

\[
d(Q^2)_{\text{man}} = A(\kappa Q^2) + \sum_{n=1}^{\infty} \bar{a}_{n}(\kappa) \bar{\mathcal{A}}_{n+1}(\kappa Q^2). \tag{35}
\]

This is the basic expression for evaluation of \( d(Q^2) \) in IR fixed point scenarios. Incidentally, also the mpt truncated series

\[
d(Q^2, \kappa)_{\text{mpt}}^{[N]} = a(\kappa Q^2) + \sum_{j=1}^{N-1} \bar{a}_{j}(\kappa) \bar{a}_{j+1}(\kappa Q^2), \tag{36}
\]

has RS dependence due to truncation, similar to the dependence of the truncated pt series, but even simpler

\[
\frac{\partial d_{\text{mpt}}^{[N]}}{\partial \ln \kappa} = -\beta_0 N \bar{a}_{N-1}(\kappa) \bar{a}_{N+1}(\kappa Q^2). \tag{37}
\]

\(^3\) On the other hand, the couplings at low momenta are in general smaller than in the underlying pQCD, and this effects tends to make the RS dependence of the truncated power series smaller than in pQCD.
The truncated modified analytic series is
\[
d(Q^2;\kappa)|_{\text{man}}^{[N]} = A(\kappa Q^2) + \sum_{j=1}^{N-1} \tilde{a}_j(\kappa) \tilde{A}_{j+1}(\kappa Q^2) .
\] (38)

The mpt series \([24]\) is just a reorganization of the original perturbation (pt) series \([24]\), so it is also RS-independent. In conjunction with the recurrence relation \(\partial a_n(\kappa Q^2)/\partial \ln \kappa = -\beta_0 n a_{n+1}(\kappa Q^2)\) which follows from the definition \([30]\), we obtain simple differential relations between \(d_n(\kappa)\):
\[
\frac{d}{d \ln \kappa} \tilde{a}_n(\kappa) = n \beta_0 \tilde{a}_{n-1}(\kappa) \quad (n = 1, 2, \ldots) .
\] (39)

\((d_0(\kappa) = \tilde{d}_0(\kappa) = 1\) by definition). Integrating them, the renormalization scale dependence of the coefficients \(\tilde{a}_n\) is particularly simple
\[
\tilde{a}_n(\kappa) = \tilde{a}_n(1) + \sum_{k=1}^{n} \binom{n}{k} \beta_0^k \ln^k(\kappa) \tilde{a}_{n-k}(1) .
\] (40)

\((\kappa \equiv \mu^2/Q^2; \ d_0 = \tilde{d}_0 = 1\) ). The coefficients \(\tilde{a}_n(\kappa)\) are obtained from \(d_k(\kappa)'s\) \((k \leq n)\) in the following way. First we express the logarithmic derivatives \(\tilde{a}_{n+1}\) in terms of the powers \(a_{k+1}\), at a given scale \(Q^2\) or \(\mu^2 = \kappa Q^2\), using the RGE relations in pQCD for these powers [RGE \([12]\) and its derivatives]
\[
\tilde{a}_2 = a^2 + c_1 a^3 + c_2 a^4 + \cdots ,
\]
\[
\tilde{a}_3 = a^3 + \frac{5}{2} c_1 a^4 + \cdots , \quad a_4 = a^4 + \cdots , \quad \text{etc.} ,
\]
where we use the notation \(c_j \equiv \beta_j/\beta_0\). We now invert them
\[
a^2 = \tilde{a}_2 \equiv a^2 - c_1 a^3 + \left(\frac{5}{2} c_1^2 - c_2\right) a_4 + \cdots ,
\]
\[
a^3 = \tilde{a}_3 \equiv a_3 - \frac{5}{2} c_1 a_4 + \cdots , \quad a^4 = a_4 + \cdots , \quad \text{etc.}
\]
Replacing these relations into the original perturbation expansion \([24]\) for \(d(Q^2)\), the coefficients \(\tilde{a}_n(\kappa)\) of the reorganized ("modified") expansions \([34], [35]\) can be read off
\[
\tilde{d}_1(\kappa) = d_1(\kappa) , \quad \tilde{d}_2(\kappa) = d_2(\kappa) - c_1 d_1(\kappa) ,
\]
\[
\tilde{d}_3(\kappa) = d_3(\kappa) - \frac{5}{2} c_1 d_2(\kappa) + \left(\frac{5}{2} c_1^2 - c_2\right) d_1(\kappa) , \quad \text{etc.}
\]
Now we perform analytization, Eqs. \([32]-[33]\), in relations \([42a]-[42b]\) term-by-term. In this way we obtain the (IR fixed point) analogs of integer powers \(a^n\), \(\tilde{A}_n = (a^n)_{\text{an}}\)
\[
\tilde{A}_2 \equiv (a^2)_{\text{an}} = \tilde{A}_2 - c_1 \tilde{A}_3 + \left(\frac{5}{2} c_1^2 - c_2\right) \tilde{A}_4 + \cdots ,
\]
\[
\tilde{A}_3 \equiv (a^3)_{\text{an}} = \tilde{A}_3 - \frac{5}{2} c_1 \tilde{A}_4 + \cdots , \quad \tilde{A}_4 \equiv (a^4)_{\text{an}} = \tilde{A}_4 + \cdots , \quad \text{etc.}
\]
This allows us to reexpress the "modified" analytic series \([35]\) in a form which is in close analogy with the original perturbation series \([24]\)
\[
d(Q^2)_{\text{an}} = A(\kappa Q^2) + \sum_{n=1}^{\infty} d_n(\kappa) A_{n+1}(\kappa Q^2) .
\] (45)

This series is \(\kappa\)-independent since it coincides with the series \(d(Q^2)_{\text{an}}\) of Eq. \([35]\). The truncated series is
\[
d(Q^2;\kappa)|_{\text{an}}^{[N]} = A(\kappa Q^2) + \sum_{n=1}^{N-1} d_n(\kappa) A_{n+1}(\kappa Q^2) .
\] (46)
When we truncate the relations \[44\] at \(\tilde{A}_N\), it is straightforward to check that the truncated series \[38\] coincides with the truncated series \[40\]

\[
d(Q^2;\kappa)^{[N]}_{an} = d(Q^2;\kappa)^{[N]}_{man}.
\]

The quantities \(\tilde{A}_n\) and \(\tilde{\alpha}_n\) have the same RS dependence relations (just interchanging \(\tilde{A}_n \leftrightarrow \tilde{\alpha}_n\)); and the quantities \(A_n\) and \(\alpha^n\) have the same RS-dependence relations (just interchanging \(\tilde{A}_n \leftrightarrow \alpha^n\)). This implies that the structure of the RS-dependence of the truncated pt and mpt series in pQCD, Eqs. \[26\] and \[37\], survives in its analytic form for the truncated analytic \[46\] and modified analytic series \[38\]

\[
\frac{\partial d^{[N]}_{an}}{\partial \ln \kappa} = K_N A_{N+1}(\kappa Q^2) + K_{N+1} A_{N+2}(\kappa Q^2) + \cdots. \tag{48a}
\]

\[
\frac{\partial d^{[N]}_{man}}{\partial \ln \kappa} = -\beta_0 N \tilde{d}_{N-1}(\kappa) \tilde{A}_{N+1}(\kappa Q^2), \tag{48b}
\]

When the truncations in the construction of \(A_n\)’s, Eqs. \[44\], are made at \(\tilde{A}_N\), we have the coincidence of the two truncated series, Eq. \[47\], and then the right-hand side of Eq. \[48a\] can be written in the simpler form of the right-hand side of Eq. \[48b\].

These relations, in conjunction with the aforementioned hierarchy \(|A(Q^2)| > |\tilde{A}_2(Q^2)| > \cdots\) and hierarchy \(|A(Q^2)| > |A_0(Q^2)| > |A_1(Q^2)| > \cdots\), at all \(Q^2\) (not just high \(Q^2\)), suggest that the truncated analytic series \(d^{[N]}_{man}(Q^2;\kappa)\), Eq. \[38\], and \(d^{[N]}_{an}(Q^2;\kappa)\), Eq. \[46\], have in general weaker RS dependence when the number of terms increases,\(^4\) or that the RS dependence is more under control (less erratic) than in the case of truncated series in powers of \(A\). This is true even for low-energy quantities (i.e., when \(|Q^2|\) is low), in contrast to the case of perturbative truncated series \(d(Q^2;\kappa)^{[N]}_{pt}\) and \(d(Q^2;\kappa)^{[N]}_{mpt}\).

Further, the described construction is applicable even in the scenarios without IR fixed point, as long as the analyticity of \(A(Q^2)\) is valid in the complex \(Q^2\)-plane outside the semiaxis \(|Q^2| \leq 0\), e.g, the model of Refs. \[34\].

All the above considerations can be extended in the same spirit to the case of the subleading renormalization scheme dependence, i.e., dependence on the scheme parameters \(c_j = \beta_j / \beta_0\) (\(j = 2, 3, \ldots\)). We refer to Appendix A for a few details about this aspect.

The construction of \(\tilde{A}_n\) and \(A_n\) was demonstrated here for integer \(n\). However, for noninteger \(n = \nu\) these quantities can also be obtained \[37\], via an analytic continuation of the general formulas in \(n \rightarrow \nu\). We refer to Appendix B for some of the details of the construction of \(\tilde{A}_\nu\) and \(A_\nu\).

IV. NUMERICAL EVIDENCE: THE CASE OF ADLER FUNCTION

A. Renormalization scale dependence of truncated series

Here we will illustrate the arguments of the previous Section numerically, in the case of a specific massless spacelike observable, for the truncated power series and the truncated series in logarithmic derivatives, within various IR fixed point frameworks. We will consider the massless Adler function. The effective charge of the (massless) Adler function is defined as

\[
d_{\text{Adl}}(Q^2) = -(2\pi^2)\frac{d\Pi(Q^2)}{d\ln Q^2} - 1, \tag{49}
\]

whose pQCD power expansion (pt) is

\[
d_{\text{Adl}}(Q^2)_{\text{pt}} = a(Q^2) + d_1 a(Q^2)^2 + \cdots, \tag{50}
\]

and where \(\Pi(Q^2) = \Pi_V(Q^2) + \Pi_A(Q^2) = 2\Pi_V(Q^2)\), in the massless case, is the correlator of the nonstrange charged hadronic currents

\[
\Pi_{\mu\nu}(q) = i \int d^4x \ exp(iq \cdot x)(TV_{\mu}(x)V_{\nu}(0)^\dagger) = (q_\mu q_\nu - g_{\mu\nu} q^2)\Pi_V(Q^2), \tag{51}
\]

\(^4\) The renormalon growth of the coefficients \(\tilde{d}_N\) with increasing \(N\) eventually increases the RS dependence.
where: \( V_\mu = \pi \gamma_\mu d \). The leading-\( \beta_0 \) (LB) part of this spacelike quantity is known to all orders

\[
d_{\text{Adl}}^{(\text{LB})} (Q^2)_{\text{mpt}} = \int_0^\infty \frac{dt}{t} F_d(t) a(tQ^2e^C) = a(Q^2) + \tilde{d}_n^{(\text{LB})} a_{2}(Q^2) + \cdots + \tilde{d}_n^{(\text{LB})} a_{n+1}(Q^2) + \cdots
\] (52a)

\[
= a(Q^2) + \tilde{d}_n^{(\text{LB})} a(Q^2)^2 + \cdots + \tilde{d}_n^{(\text{LB})} a(Q^2)^{n+1} + \cdots
\] (52b)

\[
= a(Q^2) + \tilde{d}_n^{(\text{LB})} a(Q^2)^2 + \cdots + \tilde{d}_n^{(\text{LB})} a(Q^2)^{n+1} + \cdots
\] (52c)

where \( F_d(t) \) is the distribution function of the Adler function. It was obtained in Ref. [38] on the basis of the LB expansion coefficients \( \tilde{d}_n^{(\text{LB})} \equiv \tilde{d}_{n,n}^0 \) obtained from the LB Borel transform of Refs. [39, 40] (cf. also [41]). The value of \( a(tQ^2e^C) \) is independent of the scaling convention (\( \Lambda \) definition). Here we take the \( \overline{\text{MS}} \) scaling convention: \( \mathcal{C} = -5/3 \). We refer to Appendix C for details on the formulas (52). The couplings \( a \) and \( \tilde{a}_{n+1} \) in Eqs. (52) are considered here to be general (\( N \)-loop) couplings, and in the IR fixed point frameworks they are replaced by \( A \) and \( \tilde{A}_{n+1} \), respectively.

We stress that what was obtained in Refs. [39, 40] are the coefficients \( \tilde{d}_n^{(\text{LB})} \) of the rearranged perturbation expansion (52b), i.e., the complete LB part of the expansion in logarithmic derivatives (31). These coefficients in general differ from the coefficients \( d_n^{(\text{LB})} \) of the perturbation expansion (52a) in powers of \( a(Q^2) \) which in general contain also some contributions beyond large-\( \beta_0 \) (only at one-loop level \( d_{n,0}^{(\text{LB})} = \tilde{d}_{n}^{(\text{LB})} \)). The result (52a) is exactly renormalization scale independent; the scheme dependence (i.e., dependence on the scheme coefficients \( c_j = \beta_j/\beta_0, j = 2, \ldots, N - 1 \)) appears, though, if the coupling \( a(tQ^2e^C) \) there runs according to the \( N \)-loop RGE (\( N \geq 3 \)). Nonetheless, we will consider the quantity (52a) as a useful (quasi)observable and will use it to test various evaluations of this quantity. These evaluations will be based on the (artificially assumed) knowledge of only a finite number of terms in the expansion (52b), i.e., resummations of truncated series where the couplings \( a (A) \) and \( \tilde{a}_{n+1} (\tilde{A}_{n+1}) \) in these series are taken to be general (\( N \)-loop) couplings. The fact that all the terms of that series are known allows us to evaluate the “exact” value of this quasiobservable and compare it with the results of resummations of the truncated series. This will give us indications of the quality of various resummation methods, especially in frameworks with IR fixed point where we replace \( a \to A \) and \( \tilde{a}_{n+1} \to \tilde{A}_{n+1} \) in the above expressions Eqs. (52). Since the resummation methods based on given truncated series, they in general do not reproduce the correct large-\( n \) behavior as dictated by the renormalon structure. We are interested in the numerical efficiency of such methods, i.e., the renormalization-scale (in)dependence and the convergence behavior of the resummed results.

We mention here that there exist various other models for the Adler function coefficients, such as the one used in Ref. [19] which captures main features of the renormalon growth and reproduces the full first three coefficients \( (d_1, d_2, d_3) \), and renormalon models of Refs. [42]. Furthermore, an approach which allows generalization of the expression (52) beyond the large-\( \beta_0 \) approximation can be found in Ref. [43]. In this work, we chose the large-\( \beta_0 \) expression (52a) as the test case because of the practical simplicity of the evaluation of the “exact” values, i.e., of the integral (52a).

The coefficients \( \tilde{d}_n^{(\text{LB})} \) can be represented as logarithmic moments of the distribution function \( F_d(t) \) of the Adler function

\[
\tilde{d}_n^{(\text{LB})} = (-\beta_0)^n \int_0^\infty d(ln t) \ln^n (te^C) F_d(t) .
\] (53)

For simplicity, we perform the evaluations in the \( c_2 = c_3 = \ldots = 0 \) renormalization scheme, where the pQCD running coupling \( a(\kappa Q^2) \) has formally the two-loop form and is expressed with the Lambert function \( W(z) \), cf. Refs. [44, 45]. Only in the analytic QCD model with two deltas (2\( \delta \)anQCD), Ref. [18], we will use for the renormalization scheme the preferred central Lambert scheme of the model (with \( N_f = 3 \)): \( c_2 = -4.76, c_j = c_{2j-1}/c_{2j-2}^2 \) \((j = 3, 4, \ldots, \)) , where the exact solution of the underlying pQCD coupling is also known in terms of the Lambert function (Ref. [44], cf. also Ref. [46]). We refer for some more details on this to Appendix D.

We vary the renormalization scheme \( \mu^2 = \kappa Q^2 \), and perform evaluations in various IR fixed point frameworks, with \( N_f = 3 \):

1. A representative case of freezing – the case with constant effective gluon mass \( m \), Refs. [25, 27], Eq. (4), applied to the coupling (D1) in the spirit of Ref. [29]

\[
\mathcal{A}^{(m)} (\mu^2) = a(\mu^2 + m^2) ,
\] (54)

where we take \( m = 0.8 \text{ GeV} \) and \( \Lambda_L = 0.487 \text{ GeV} \), giving at \( \mu^2 = m_c^2 \) the value \( \mathcal{A}(m_c^2) = 0.293/\pi \).
FIG. 4: The effective charge of the massless Adler function \(d_{\text{Adl}}(Q^2)\), at leading-\(\beta_0\) (LB), for \(Q^2 = 1 \text{ GeV}^2\), as a function of the (squared) spacelike renormalization scale \(\mu^2\): (a) in pQCD [Eq. (\ref{pQCD})] (the upper left-hand Figure); and in IR fixed point frameworks: (b) with the constant effective gluon mass \(m = 0.8 \text{ GeV}\) (the upper right-hand Figure); (c) the (fractional) analytic perturbation theory (F)APT (the lower left-hand Figure); and (d) the analytic model 2\(\delta\)anQCD which has, in the discontinuity function of \(A(Q^2)\), two delta functions in the low-\(\sigma\) regime (the lower right-hand Figure). The truncations are made at \(\sim A^4(\tilde A_4)\) and \(\sim A^6(\tilde A_6)\).

2. The (fractional) analytic perturbation theory (F)APT case, Refs. [9][11][13][15], Eq. (\ref{APT}). The APT scale is fixed at \(\Lambda_L(\text{APT}) = 0.572 \text{ GeV}\), and \(N_f = 3\), giving the value \(A_1^{(\text{APT})}(m_2^2) = 0.295/\pi\).

3. The analytic QCD case with two deltas (2\(\delta\)anQCD) in the low-\(\sigma\) region for the analyticity function, Ref. [18], Eqs. (22b) and (23b). This model is numerically very close to pQCD coupling [13], with the exception of the regime \(|\mu^2| < 1 \text{ GeV}^2\). The input values of the model are the central ones used in Ref. [18] (among them: \(c_2 = -4.76\), \(\Lambda_L = 0.260 \text{ GeV}\)) and give the value \(A_1^{(2\delta)}(m_2^2) = 0.291/\pi\).

Furthermore, the first three full (i.e., LB+beyondLB) coefficients \(d_1, d_2\), and \(d_3\) (\(\Rightarrow \tilde d_1, \tilde d_2, \tilde d_3\)) of the Adler function are now exactly known [11][53]

\[
\begin{align*}
    d_{\text{Adl}}(Q^2)^{[4]}_{\text{pt}} &= a(Q^2) + d_1 a(Q^2)^2 + d_2 a(Q^2)^3 + d_3 a(Q^2)^4, \\
    d_{\text{Adl}}(Q^2)^{[4]}_{\text{impt}} &= a(Q^2) + \tilde d_1 \tilde a_2(Q^2) + \tilde d_2 \tilde a_3(Q^2) + \tilde d_3 \tilde a_4(Q^2).
\end{align*}
\]

That is why we can also evaluate the full Adler function, but truncated at order 4 (TS[4]) or lower, in any scheme and at any scale \(\mu^2 = \kappa Q^2\), for example in pQCD and in the aforementioned four IR fixed point frameworks.

The results of the LB Adler function, truncated at order 4 and 6, as power series and as series in logarithmic derivatives, for \(Q^2 = 1 \text{ GeV}^2\), are presented as functions of the squared (spacelike) renormalizations scale \(\mu^2 = \kappa Q^2\) in Figs. 4 for the pQCD case and for the three considered aforementioned IR fixed point frameworks. Truncations are made at \(\sim A^4\) and \(\sim A^6\) for power series, and at \(\tilde A_4\) and \(\tilde A_6\) for the series in logarithmic derivatives.

Furthermore, the analogous results based on the truncated series (55) with full (LB+bLB) coefficients, are given in Figs. 5 for pQCD and for the three considered IR fixed point cases. Truncations are made at \(\sim A^3(\tilde A_3)\) and \(\sim A^4(\tilde A_4)\).
These figures show how the arguments of the previous Section manifest themselves in practice. In the IR fixed point frameworks, the truncated power expansions have increasingly strong renormalization scale dependence, due to the wrong incorporation of the nonperturbative contributions at higher orders there. This effect is stronger when \( Q^2 \) values are lower. On the other hand, the truncated series in logarithmic derivatives, in the IR fixed point frameworks, have weaker scale dependence, and this dependence in general does not get stronger when the number of terms in the truncated series increases. Furthermore, these figures indicate that the power series has divergent behavior already at relatively low orders, in contrast to the series in logarithmic derivatives.

On the other hand, in pure pQCD scenario, the two types of truncated series give comparable results and it is not clear which one is better, as demonstrated also in Ref. \[54\].

### B. Convergence properties

In this Subsection we present, for the leading-\( \beta_0 \) (LB) Adler function as a test case, the convergence properties of truncated series in powers, in logarithmic derivatives, and of a resummed version of the latter series based on a generalized diagonal Padé (dPA) method. The latter method was introduced in Ref. \[48\] in the context of pQCD, and was motivated by the dPA resummation approach and its renormalization scale independence in the one-loop approximation \[49\]. It was later applied to analytic QCD frameworks in Refs. \[47\] and \[22\]. It consists in the following expression:

\[
\hat{G}_{d}^{[M/M]}(Q^2) = \sum_{j=1}^{M} \tilde{\alpha}_j \, A_j(Q^2),
\]

\[ (56) \]
where the scale parameters \( \kappa_j \) and the coefficients \( \tilde{\alpha}_j = (\tilde{\alpha}_1 + \ldots + \tilde{\alpha}_M = 1) \) are determined uniquely from the known truncated series of the observable \( d(Q^2) \) up to \( a_{2M} \) \( (\sim a^{2M}) \)

\[
d(Q^2; \mu^2)_{\text{part}}^{[2M]} = a(\mu^2) + \sum_{j=1}^{2M-1} \tilde{d}_j (\mu^2/Q^2) \tilde{\alpha}_{j+1}(\mu^2). \tag{57}
\]

The mentioned parameters \( \kappa_j \) and \( \tilde{\alpha}_j \) are obtained by regarding the series \( (57) \) in logarithmic derivatives as formally a (truncated) series in powers of one-loop coupling \( [\tilde{\alpha}_{j+1}(\mu^2) \mapsto a_{1\ell}(\mu^2)^{j+1}] \)

\[
\tilde{d}(Q^2; \mu^2)_{\text{pt}}^{[2M]} = a_{1\ell}(\mu^2) + \sum_{j=1}^{2M-1} \tilde{d}_j (\mu^2/Q^2) a_{1\ell}(\mu^2)^{j+1}. \tag{58}
\]

and constructing for it the diagonal Padé (dPA) \( [M/M] \) which\(^5\) is then decomposed in a linear combination of simple fractions (in Mathematica software \[50\]), the command “Apart” achieves this decomposition

\[
[M/M]_d(a_{1\ell}(\mu^2)) = \sum_{j=1}^{M} \tilde{\alpha}_j \frac{x}{1 + \tilde{u}_j x} \bigg|_{x=a_{1\ell}(\mu^2)}. \tag{59}
\]

\(^5\) \( [M/M]_d \) is by definition a ratio of two polynomials in \( a_{1\ell}(\mu^2) \) of order \( M \) each, and whose coefficients are determined by the condition:

\[
[M/M]_d - \tilde{d}(Q^2; \mu^2)_{\text{pt}}^{[2M]} \sim a_{1\ell}^{2M+1}.
\]
Each simple fraction $x/(1 + \tilde{u}_j x)$ [with: $x = a_{1\ell}(\mu^2)$] can be written as $a_{1\ell}(\kappa_j Q^2)$, i.e.,

$$[M/M]\_\varepsilon(a_{1\ell}(\mu^2)) = \sum_{j=1}^{M} \tilde{a}_j \ a_{1\ell}(\kappa_j Q^2), \quad \text{where } \kappa_j Q^2 = \mu^2 \exp(\tilde{u}_j/\beta_0). \quad (60)$$

This procedure gives us the mentioned parameters $\tilde{a}_j$ and $\kappa_j$; it turns out that they are exactly-independent of the chosen renormalization scale $\mu^2$, Refs. [22, 48], and that the resummed conformal approximant $G_d^{[M/M]}(Q^2)$, Eq. (56), fulfills the basic requirement of the approximant of order $N = 2M$, Ref. [47]

$$d(Q^2) - G_d^{[M/M]}(Q^2) = O(\bar{A}_{2M+1}) = O(A_{2M+1}). \quad (61)$$

We stress that the approximant (56) is applicable in the general case of $N$-loop RGE-running of $\mathcal{A}(\mu^2)$, and the relation (61) is valid in such general case.\(^6\) This is what makes this approximant so attractive theoretically, as pointed out in Refs. [22, 47, 48]. It is thus not the direct dPA method of Ref. [49], but a nontrivial generalization thereof, which takes into account the general $N$-loop RGE-running of the couplings and gives exactly renormalization scale independent results. The crucial part in the construction of this method is the formal replacement $\tilde{a}_{j+1} \rightarrow a_{j+1} + 1$ [Eqs. (57)–(58)] as an intermediate step, and the use of dPA on the formal power series to obtain the scale and weight parameters $\kappa_j$ and $\tilde{a}_j$.

As shown in Refs. [22, 47], these approximants work very well in practice in the analytic QCD frameworks.

Using the LB Adler function, Eqs. (52) and (53), as a test case,\(^7\) at $Q^2 = 1 \text{ GeV}^2$, we present in Figs. 6 the results of the evaluation of this “quasi-observable” as a function of the truncation order $N$ in the case of pQCD coupling $a$.

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\(^6\) It is valid for any RGE running with a given beta function, and this function can contain terms nonanalytic in $\mathcal{A}$, as it happens in the models with IR fixed point.

\(^7\) The authors of Refs. [59] evaluated the Adler function and related quantities in pQCD framework using nonpower expansions based on conformal transformations and renormalon structure of the Borel transform, and using as test quantities renormalon models for Adler function of Refs. [22].
Eq. (D1), as: truncated power series, truncated series in logarithmic derivatives, and the generalized dPA Eq. (56) (in that case: \( N = 2M = 2, 4, \ldots \)). We can see that the power series and the series in logarithmic derivatives increase with increasing \( N \) above the exact value\(^8\), while the generalized dPA oscillates uncontrollably around it.

In Figs. 7 we present the corresponding results for the gluon effective mass case: \( m = 0.8 \text{ GeV} \) case of Eq. (54); in Figs. 8 the results for the APT model of Eq. (11) and of the 2\( \delta \)anQCD model of Eqs. (22b) and (23b).

We can see that, in contrast to pQCD, any framework with IR fixed point gives for the series in logarithmic derivatives a clearly better convergence properties than for the power series. The power series, although having usually \( A(Q^2) < 1 \), is badly divergent, in part due to a renormalon growth of the coefficients \( \tilde{d}_n \). However, we note that the series in logarithmic derivatives also has a (one-loop) renormalon growth of the coefficients \( \tilde{d}_n \). And both the power terms and the logarithmic derivatives, at any \( Q^2 \), have the hierarchy: \( A(Q^2) > A(Q^2)^2 > A(Q^2)^3 > \ldots \) and \( A(Q^2) > |\tilde{A}_2(Q^2)| > |\tilde{A}_3(Q^2)| > \ldots \). Nonetheless, the logarithmic derivatives \( \tilde{A}_n(Q^2) \) have alternating signs at large \( n \), which indicates why the series in logarithmic derivatives has a better convergence (less severe divergence) behavior than the power series.

The results of the Figures further indicate that the generalized dPA method works very well in all the frameworks with IR fixed point, it gives a clearly convergent behavior when the truncation order \( N = 2M \) increases.

![Figure 9](image)

**FIG. 9**: Analogous results as those of Figs. 6, 8 but for the truncated series based on the full (LB+beyondLB) coefficients, cf. Eqs. (55): for pQCD and for the three considered IR fixed point frameworks.

Finally, in Fig. 9 we present analogous results as in Figs. 6, 8 for pQCD and the three considered IR fixed point scenarios, but this time with the known complete (LB+beyondLB) coefficients \( \tilde{d}_n \) \((\bar{d}_n)\), cf. Eq. (55). Since only up to \( d_4 \) (\( \bar{d}_4 \)) coefficients are known exactly, the results are shown only up to the order \( N = 4 \). Also in this (LB+beyondLB) case, we can see that in the IR fixed point frameworks the series in logarithmic derivatives behave significantly better than the corresponding power series; and that the generalized dPA method is often even better. These Figures include also the result of the LB resummation \( \text{i.e.}, \) the integral (52a)\(^9\) with the three known beyond-LB terms added (here

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\(^8\) The “exact” value is here taken as the Principal Value of the integral (52a) which has ambiguity due to Landau singularities of pQCD coupling. No such ambiguity problems appear in the other considered cases, because they have IR fixed point.

\(^9\) In the case of pQCD, the LB-integral has ambiguity due to the Landau singularities, and we took the Principal Value in this case.
added in the form of logarithmic derivatives). The latter method is also considered as probably competitive with the 
generalized dPA method, at least at the considered order ($N = 4$).

On the other hand, in pQCD it appears to be impossible to identify a method that is better than the other methods.

V. EVALUATION OF TIMELIKE PHYSICAL QUANTITIES IN IR FIXED POINT SCENARIOS

The extension of the described formalism to the evaluation of timelike physical quantities $\mathcal{T}(s) \ (s = -Q^2 > 0)$ is 
based on the assumption of existence of an integral transformation which relates such timelike quantities with the 
(corresponding) spacelike quantities $\mathcal{F}(Q^2)$. The latter are evaluated in the aforedescribed way, for any complex $Q^2$, 
and the integral trasformation is applied on them.

Often the integral transformation is the same as when $\mathcal{T}(s)$ is the ($e^+e^- \to$ hadrons) ratio $R(s)$ and $\mathcal{F}(Q^2)$ is the 
Adler function (log-derivative of the quark-current correlator)

$$\mathcal{F}(Q^2) = Q^2 \int_{0}^{\infty} \frac{d\sigma}{(\sigma + Q^2)^2} \ . \tag{62}$$

The inverse transformation is

$$\mathcal{T}(\sigma) = \frac{1}{2\pi i} \int_{\gamma_{\sigma-i\varepsilon}} \frac{dQ^2}{Q^2} \mathcal{F}(Q^2) \ , \tag{63}$$

where the integration contour is in the complex $Q^2$-plane encircling the singularities of the integrand, e.g., path $C_1$ or 
$C_2$ of Fig. 10. Let us consider the case when the truncated perturbation expansion of the (massless) spacelike quantity

\[ \mathcal{F}(Q^2) \] 
in pQCD is of the form

$$\mathcal{F}(Q^2; \kappa)^{[N]} = a(\kappa Q^2) + \sum_{n=1}^{N-1} \mathcal{F}_n(\kappa) a(\kappa Q^2)^{n+1} \ . \tag{64}$$

In IR fixed point scenarios this implies the following nonpower expansion, as explained in the previous Section:

$$\mathcal{F}(Q^2; \kappa)^{[N]}_{\text{lnm}} = \mathcal{A}(\kappa Q^2) + \sum_{n=1}^{N-1} \bar{\mathcal{F}}_n(\kappa) \bar{\mathcal{A}}_{n+1}(\kappa Q^2) = \mathcal{A}(\kappa Q^2) + \sum_{n=1}^{N-1} \mathcal{F}_n(\kappa) \mathcal{A}_{n+1}(\kappa Q^2) \ . \tag{65}$$

The application of the integral transformation to this expression then gives the desired result

$$\mathcal{T}(\sigma; \kappa)^{[N]}_{\text{lnm}} = \frac{1}{2\pi i} \int_{\gamma_{\sigma-i\varepsilon}} \frac{dQ^2}{Q^2} \mathcal{F}(Q^2; \kappa)^{[N]}_{\text{lnm}} \ , \tag{66}$$

This can be performed term-by-term, leading to

$$\mathcal{T}(\sigma; \kappa)^{[N]}_{\text{lnm}} = \mathcal{A}(\kappa \sigma) + \sum_{n=1}^{N-1} \mathcal{F}_n(\kappa) \mathcal{A}_{n+1}(\kappa \sigma) \ , \tag{67}$$
where the timelike (Minkowskian) couplings \( \mathfrak{A}_n(\sigma) \) (with: \( \mathfrak{A} \equiv \mathfrak{A}_1 \) and \( \mathcal{A} \equiv \mathcal{A}_1 \)) are defined as

\[
\mathfrak{A}_n(\sigma) \equiv \frac{1}{2\pi i} \int_{-\sigma - i\varepsilon}^{-\sigma + i\varepsilon} \frac{dQ^2}{Q^2} \mathcal{A}_n(Q^2),
\]

(68)

and the inverse transformation is

\[
\mathcal{A}_n(\kappa Q^2) = \kappa Q^2 \int_0^\infty \frac{d\sigma \mathfrak{A}_n(\sigma)}{(\sigma + \kappa Q^2)^2} \quad (n = 1, 2, \ldots).
\]

(69)

We can also apply the generalized dPA method mentioned in Sec. IV B to evaluate \( \mathcal{F}(Q^2; \kappa)^{[2M]} \)

\[
\mathcal{F}(Q^2; \kappa)^{[2M]} \rightarrow \mathcal{G}_F(Q^2) = \sum_{j=1}^M \bar{\alpha}_j \mathcal{A}(\kappa_j Q^2),
\]

(70a)

\[
\Rightarrow \mathcal{T}(\sigma; \kappa)^{[2M]} \rightarrow \mathcal{G}_T(\sigma) = \sum_{j=1}^M \bar{\alpha}_j \mathfrak{A}(\kappa_j \sigma).
\]

(70b)

For example, to calculate the effective charge \( \mathcal{T}(s) = r_{e^+e^-}(s) \) of the \( e^+e^- \rightarrow \text{hadrons} \) ratio \( R(s) \), we apply the mentioned evaluation to the effective charge \( \mathcal{F}(Q^2) = d(Q^2) \) \((= \mathcal{A}(Q^2) + \mathcal{O}(\mathcal{A}_2))\) of the Adler function \( d(Q^2) \), for complex \( Q^2 = s \exp(i\phi) \), and integrate this expression in the contour integral \([63]\).

Another example is the effective charge \( r_\tau \) of the strangeless \( V + A \) semihadronic \( \tau \) decay ratio \( R_\tau \). After removing the effects of nonzero quark masses, this quantity can be expressed in terms of the effective charge of the Adler function \( d_{Adl}(Q^2) \), defined in Eqs. \([49]-[51]\), as the following contour integral: \([56],[57]\).

\[
r_\tau = \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\phi \ (1 + e^{i\phi})^3 (1 - e^{i\phi}) \ d_{Adl}(Q^2) = m_\tau^2 e^{i\phi}.
\]

(71)

The mass-dependent timelike observables can be evaluated analogously as the mentioned mass-independent ones. Furthermore, we can encounter cases of such observables which have pQCD expansion in noninteger powers [cf. Eqs. \([B2],[B3]\)], such as is the partial decay width of the Higgs to \( bb \); for an application in this case, cf. Ref. \([37]\) in a general IR fixed-point frameworks, and Ref. \([14]\) in the fractional APT (FAPT).

**VI. SUMMARY**

We considered specific aspects of the problem of evaluation of QCD effects at hadronic (low) momenta \( \lesssim 1 \text{ GeV} \). Most of the lattice calculations and calculations using Dyson-Schwinger equations and/or other functional methods indicate that the QCD running coupling freezes to a finite value at low momenta, i.e., that it has an IR fixed point. Various models with IR fixed point were considered here. We argued, on theoretical grounds, that the perturbation expansions of low-momentum spacelike physical quantities can be used in such IR fixed point frameworks, provided that the naive power series is abandoned and replaced by the corresponding nonpower series in logarithmic derivatives. We then numerically compared both types of expansion in the case of the massless Adler function, for three different frameworks of IR fixed point scenario. The numerical results consistently indicate that the mentioned nonpower expansion should be used, in order to have nonperturbative effects at higher orders incorporated correctly. The correct incorporation of these terms results in a weaker renormalization scale dependence and in significantly better convergence properties of the series.

Moreover, a resummation method based on the (truncated) series in logarithmic derivatives, namely a generalization of the diagonal Padé resummation method proposed in Ref. \([13]\) and later extended and applied in IR fixed point scenarios \([22],[47]\), results in very improved convergence properties of the evaluated series in all considered IR fixed point scenarios.

If, however, we work in the usual pQCD framework, with the running coupling which suffers from the unphysical (Landau) singularities, the mentioned methods of evaluations for low-momentum spacelike physical quantities show no improvement with respect to the usual truncated power series.

In addition, we argued how the described formalisms can be extended to evaluations of low-energy timelike physical quantities.

On the other hand, it is unrealistic to assume that the IR fixed point coupling is universal and, at the same time, it gives us all the nonperturbative effects via the mentioned evaluations. It appears to be more realistic to assume that
such a coupling, while being universal, incorporates in the mentioned series only part of the nonperturbative effects, and that other nonperturbative effects can be added, for example via higher-twist terms of OPE.\footnote{On the other hand, approaches exist which eliminate the unphysical singularities, and include nonperturbative effects, directly in the specific (spacelike) observables, cf. Refs. \[58–62\] (see also \[63, 64\]).} If we want to apply OPE in such IR fixed point scenarios, and if at the same time we want to maintain the ITEP School interpretation of the OPE (that the higher-twist terms are exclusively of IR origin, Refs. \[65, 66\]), it is preferable that the running coupling differs very little from the pQCD coupling at high squared momenta $Q^2$, by $\sim (\Lambda^2/Q^2)^N$ where $N$ is large. Such models do exist, e.g., Refs. \[17, 18\] for $N = 3, 5$, respectively; OPE in $\tau$ decay physics has been applied with the model of Ref. \[18\] in Ref. \[22\].

Acknowledgments

This work was supported in part by FONDECYT (Chile) Grant No. 1130599.

Appendix A: Subleading scheme dependence in scenarios with the coupling with IR freezing

When we consider the subleading renormalization scheme dependence, the basic pQCD relations are those of the $c_j$-dependence of the pQCD coupling $a$, i.e., “scheme RGEs” \[67\]

\[
\frac{\partial a}{\partial c_2} = a^3 + \mathcal{O}(a^5) \Rightarrow \frac{\partial a^2}{\partial c_2} = 2a^4 + \cdots ,
\]

\[\text{(A1a)}\]

\[
\frac{\partial a}{\partial c_3} = \frac{1}{2}a^4 + \cdots .
\]

\[\text{(A1b)}\]

For the IR fixed point scenarios (or any analytic model of $A$), we can define the same scheme dependence, under the correspondence $a^n \leftrightarrow A_n$

\[
\frac{\partial A}{\partial c_2} = A_3 + \mathcal{O}(A_5) \Rightarrow \frac{\partial A_2}{\partial c_2} = 2A_4 + \cdots ,
\]

\[\text{(A2a)}\]

\[
\frac{\partial A}{\partial c_3} = \frac{1}{2}A_4 + \cdots .
\]

\[\text{(A2b)}\]

These differential equations can be rewritten in terms of $\tilde{A}_n$’s using the relations \[44\]. All the scheme dependence relations in pQCD now carry over to IR fixed point scenarios, under the correspondence $a^n \leftrightarrow A_n$ (or equivalently $\tilde{a}_n \leftrightarrow A_n$)

\[
\frac{\partial d_{\text{pt}}^{[N]}}{\partial c_j} = K_N ^{(j)} a^{N+1}(\kappa Q^2) + K_{N+1} ^{(j)} a^{N+2}(\kappa Q^2) + \cdots , \Rightarrow \quad (A3a)
\]

\[
\frac{\partial d_{\text{man}}^{[N]}}{\partial c_j} = K_N ^{(j)} A_{N+1}(\kappa Q^2) + K_{N+1} ^{(j)} A_{N+2}(\kappa Q^2) + \cdots , \quad (A3b)
\]

and analogously for $d_{\text{man}}^{[N]}$.

Appendix B: Analogs of $a^n$ and $\tilde{a}_n$ for noninteger $n = \nu$ in IR fixed point scenarios

In the scenarios with IR freezing of the coupling $A$, for noninteger $n = \nu$ the quantities $\tilde{A}_n$ and $A_{\nu}$ were obtained in Ref. \[37\]. The idea was to perform the analytic continuation of the most general form of the formulas for $\tilde{A}_n$ and in $A_{\nu}$ in $n$ ($\mapsto \nu$).

For this, a dispersion relation for the logarithmic derivatives $\tilde{A}_{n+1}(Q^2)$ of Eq. \[33\] was obtained, by applying the logarithmic derivatives on the dispersion relation \[20\] for $A(Q^2)$. The obtained dispersion relation was then written
in a form involving the polylogarithm function of order \(-n\)
\[
\tilde{A}_{n+1}(Q^2) = \frac{1}{\pi^2} \left[ (-1)^n \int_0^\infty \frac{d\sigma}{\sigma} \rho(\sigma) \text{Li}_n(-\sigma/Q^2) \right],
\]
where we recall that \(\rho(\sigma) = \text{Im}\mathcal{A}(-\sigma - i\epsilon)\). Analytic continuation in \(n \rightarrow \nu\) then gives simply
\[
\tilde{A}_{\nu+1}(Q^2) = \frac{1}{\pi^2} \left[ (-1)^\nu \int_0^\infty \frac{d\sigma}{\sigma} \rho(\sigma) \text{Li}_{\nu}(\text{Im} - \frac{\sigma}{Q^2}) \right] \quad (-1 < \nu),
\]
where \(\nu\) can now be noninteger. The couplings \(A_\nu\), which are (in IR fixed point scenario) analogs of the noninteger powers \(\alpha^\nu\), can then be obtained as a linear combination of the quantities \(\tilde{A}_{\nu+m}\) \((m = 0, 1, 2, \ldots)\), via a generalization of the relations (44) to any integer \(n\) and then replacing \(n \rightarrow \nu\)
\[
A_\nu \equiv \tilde{A}_\nu + \sum_{m \geq 1} \tilde{k}_m(\nu)\tilde{A}_{\nu+m} \quad (\nu > 0).
\]

The coefficients \(\tilde{k}_m(\nu)\) involve Gamma functions \(\Gamma(x)\) and their derivatives (up to \(m\) derivatives) at the values \(x = 1, \nu + 1, \nu + 2, \ldots, \nu + m\), cf. App. A of Ref. [37].

It turns out that in the (fractional) APT model of Refs. [9, 10, 13–15], the (fractional) power analogs \(A_{\nu}^{\text{APT}}\), Eq. (11), constructed entirely from the discontinuities of the pQCD coupling \(a^\nu\), coincide with the result of the approach described here, for the corresponding special (APT) case: \(\rho(\sigma) = \rho^{(\text{pt})}(\sigma)\), i.e., when \(\text{Im}\mathcal{A}(-\sigma - i\epsilon) = \text{Im}a(-\sigma - i\epsilon)\).

Appendix C: Distribution function of the leading-\(\beta_0\) part of Adler function

Here we review the formalism of the distribution function for the leading-\(\beta_0\) part of dimensionless renormalization scheme invariant QCD quantities \(d(Q^2)\), as developed in Ref. [38], but using here the notations of the present paper. Further, an emphasis will be made here which is slightly different from that of Ref. [38]: namely, the distribution function \(F_d(t)\), which generates the leading-\(\beta_0\) parts of the perturbation coefficients of the quantity \(d(Q^2)\), will appear in the integral over momenta Eq. (52a) whose integrand includes the general \((N\text{-loop})\) coupling \(a(tQ^2e^C)\), and not necessarily one-loop coupling \(a_{1\ell}(Q^2)\). The formalism can be applied to spacelike and timelike quantities. Throughout this Appendix, we can replace the (usual) perturbative quantities \(a\) and \(\tilde{a}_{n+1}\) by the (holomorphic) quantities of the IR fixed point scenarios \(A\) and \(\tilde{A}_{n+1}\), cf. Eqs. (30)-(33).

In general, the coefficients \(\tilde{d}_n(k)\) of the reorganized perturbation series of \(d(Q^2)\), Eqs. (34)-(35), can be written as a sum of powers of the number of active flavors \(N_f\), the latter indicating the number of loops of (massless) quark flavors
\[
\tilde{d}_n(k) = C_{n,n}(k)N_f^n + C_{n,n-1}(k)N_f^{n-1} + \ldots + C_{n,0},
\]
where the leading-\(N_f\) and the leading-\(\beta_0\) (LB) coefficients are simply related [68]
\[
c_{n,n} = (-6)^n C_{n,n},
\]
due to the relation \(N_f = -6\beta_0 + 33/2\). It is interesting that the dependence on the renormalization scale \(\mu (\kappa \equiv \mu^2/Q^2)\) of the LB coefficient \(\tilde{d}_n^{\text{LB}}(k) = c_{n,n}(\kappa)\beta_0^n (c_{0,0} = 1)\) is the same as for the entire coefficients \(\tilde{d}_n(k)\) [cf. Eqs. (39)-(40)]
\[
\frac{dc_{n,n}(\kappa)}{d\ln \kappa} = nc_{n-1,n-1}(\kappa) \Rightarrow
\]
\[
c_{n,n}(\kappa_2) = \sum_{p=0}^{n} \left( \begin{array}{c} n \\ p \end{array} \right) \ln^{n-p} \left( \frac{\kappa_2}{\kappa_1} \right) c_{p,p}(\kappa_1),
\]
(for the above equality, see, for example, Ref. [35]). The LB part of the reorganized perturbation series [cf. Eqs. (34)-(35)] is
\[
d^{\text{LB}}(Q^2)_{\text{mpt}} = a(\kappa Q^2) + \sum_{n=1}^{\infty} c_{n,n}(\kappa) \beta_0^n \tilde{a}_{n+1}(\kappa Q^2).
\]
This expression can be written as an integral over the squared momenta $-k^2 = tQ^2$ of the form (52a)

$$d^{(\text{LB})}(Q^2)_{\text{mpt}} = \int_0^\infty \frac{dt}{t} F_d(t) \, a(tQ^2 e^C) ,$$  \hspace{1cm} (C5)

where $C = -5/3$ in $\overline{\text{MS}}$ scheme (scaling convention). We now review how the distribution function $F_d(t)$ which appears in this integral is obtained from the knowledge of the LB coefficients $c_{n,n}$, following Ref. [38] (with adapted notations).

If in the LB-series the logarithmic derivatives $\bar{a}_{n+1}(\kappa Q^2)$ are formally replaced by powers $a(\kappa Q^2)^{n+1}$ (note: $\bar{a}_{n+1} \neq a^{n+1}$ if general $N$-loop running with $N \geq 2$), we name the new formal LB quantity as $\bar{d}^{(\text{LB})}$

$$\bar{d}^{(\text{LB})}(Q^2; \kappa^2)^{(\text{LB})} = a(\kappa Q^2) + \sum_{n=1}^{\infty} c_{n,n}(\kappa) \beta_0^n a^{n+1}(\kappa Q^2) .$$  \hspace{1cm} (C6)

It is in general renormalization scale dependent, unless $a(\mu^2)$ runs according to one-loop. The Borel transform of this quantity is

$$B_d(b; \kappa) = 1 + \frac{\bar{d}_1^{(\text{LB})}(\kappa)}{1!} b + \frac{\bar{d}_2^{(\text{LB})}(\kappa)}{2!} b^2 + \ldots$$ \hspace{1cm} (C7a)

$$= 1 + \frac{c_{1,1}(\kappa)}{1!} b + \frac{c_{2,2}(\kappa)}{2!} b^2 + \ldots .$$ \hspace{1cm} (C7b)

The idea is to relate this quantity with the distribution function $F_d(t)$ via an integral relation, and to invert that relation in order to obtain $\hat{F}_d$. The scale dependence of the quantity (C7) is obtained immediately from the relations (C3a)

$$B_d(b; \kappa) = B_d(b; 1) \, \kappa^b .$$ \hspace{1cm} (C8)

However, the renormalization scale $\mu^2$ (eq. $\kappa Q^2$) has scaling convention dependence ($\Lambda$ definition), also called $\Lambda$ scheme or $C$-dependence. This is reflected by the fact that the RGE running of the coupling $a(\mu^2)$ is scaling convention dependent $[67]$, while the quantity $a(\mu^2 e^C)$ has no such dependence, where $C = -5/3, -5/3 + \gamma_E - \ln(4\pi)$, and 0, in the scaling convention frameworks $\overline{\text{MS}}, \text{MS}$, and $V$, respectively. Specifically, we have $a(\mu^2) = f(\mu^2/\Lambda^2)$ where $\Lambda$ is different in different scaling conventions, and we have the following relations:

$$\mu^2 = \mu_{(0)}^2 e^C , \quad \Lambda^2 = \Lambda_{(0)}^2 e^C \Rightarrow$$ \hspace{1cm} (C9a)

$$a(\kappa Q^2) = a(\kappa e^{-C} Q^2; C = 0) , \quad \bar{a}_{n+1}(\kappa Q^2) = \bar{a}_{n+1}(\kappa e^{-C} Q^2; C = 0) ,$$ \hspace{1cm} (C9b)

$$\bar{d}_n(\kappa) = \bar{d}_n(\kappa e^{-C}; C = 0) , \quad c_{n,n}(\kappa) = c_{n,n}(\kappa e^{-C}; C = 0) ,$$ \hspace{1cm} (C9c)

where, as usual, $\kappa \equiv \mu^2/Q^2$, and the subscript “(0)” denotes the $V$ scaling convention ($C = 0$). The last identity implies that the Borel transform (C7b) has, in addition to the $\kappa$-dependence (C5), also $C$-dependence

$$B_d(b; \kappa) = B_d(b; \kappa e^{-C}; C = 0) = B_d(b; 1; C = 0)(\kappa e^{-C})^b = B_d(b; e^C)(\kappa e^{-C})^b .$$ \hspace{1cm} (C10)

This leads to the following definition of the renormalization scale independent and $C$-independent Borel transform $\hat{B}_d(b)$:

$$\tilde{B}_d(b) \equiv B_d(b; \kappa e^{-C})^{-b}$$ \hspace{1cm} (C11a)

$$= B_d(b; \kappa = e^C) = 1 + \frac{c_{1,1}(e^C)}{1!} b + \frac{c_{2,2}(e^C)}{2!} b^2 + \ldots .$$ \hspace{1cm} (C11b)

In the integral (C5) the ($C$-independent) quantity $a(tQ^2 e^C)$ can be expanded around the point $\ln \mu^2$, using the definitions (30)

$$a(tQ^2 e^C) = a(\mu^2) + (-\beta_0) \ln \left( \frac{tQ^2}{\mu^2} e^C \right) \bar{a}_2(\mu^2) + (-\beta_0)^2 \ln^2 \left( \frac{tQ^2}{\mu^2} e^C \right) \bar{a}_3(\mu^2) + \ldots .$$ \hspace{1cm} (C12)

\footnote{If $a(tQ^2 e^C)$ RGE-evolves at a $N$-loop level with $N > 2$, then $a(tQ^2 e^C)$ and thus the quantity $d^{(\text{LB})}(Q^2)$ acquire the renormalization scheme dependence, i.e., dependence on the beta-parameters $c_j \equiv \beta_j/\beta_0$ ($j = 2, \ldots, N - 1$).}
Using this in Eq. (C5) and exchanging there the order of summation and integration, leads to the series Eq. (C4), with the following relation between the LB-coefficients $c_{n,n}$ and the searched for distribution function $F_d$ ($c_{n,n}$ are assumed known)

$$c_{n,n}(\kappa) = (-1)^n \int_0^\infty d(t) \ln^n (t\kappa^{-1}e^c) F_d(t) ,$$

(C13)

where $n = 0,1,2,\ldots$ and $c_{0,0} = 1$ [when $\kappa = 1$ this gives Eq. (53)]. Inserting these expressions into the expansion of the Borel transform $B_d(b;\kappa)$, and exchanging the order of integration and summation, leads to

$$B_d(b;\kappa) = \int_0^\infty d(t) F_d(t) \left[ 1 + \sum_{n=1}^\infty \frac{1}{n!} (-1)^n \ln^n (t\kappa^{-1}e^c)b^n \right] = (\kappa e^{-c})b \int_0^\infty dt F_d(t) t^{-b-1} ,$$

(C14)

which can be rewritten in terms of the invariant Borel transform (C11) as

$$\hat{B}_d(t) = \int_0^\infty dt F_d(t) t^{-b-1} .$$

(C15)

This is the Mellin transform of $F_d$, and the inverse transformation then gives the distribution function $F_d$

$$F_d(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} db \hat{B}_d(t)b^b ,$$

(C16)

where the real value $c$ is any value between the first UV and the first IR renormalon pole of $d(Q^2)$ in $b$ plane: $-k < c < K$ (for the Adler function: $-k = -1$, $K = 2$). Therefore, if the LB coefficients $c_{n,n}$ and thus $\hat{B}_d(t)$ are known, Eq. (C16) gives the distribution function $F_d$ of the quantity $d^{\text{LB}}(Q^2)$ [Eq. (C4)] which appears in the momentum-integral (C5) and is the complete LB part of the full quantity $d(Q^2)$.

For the QCD Adler function, the large-$N_f$ coefficients $c_{n,n}$, Eq. (C14), with $\kappa = e^c$, can be deduced from Ref. [39] where their generating function was obtained in the context of large-$N$ expansion in QED. The LB coefficients $c_{n,n}$ are then obtained by Eq. (C2), giving

$$c_{n,n}(e^c) = \frac{3}{4} C_F \left( \frac{d}{db} \right)^n P(1 - b)|_{b=0} ,$$

(C17)

where $C_F = (N_c^2 - 1)/(2N_c) = 4/3$, and $P(x)$ is the trigamma function obtained in Ref. [39] by extension of the analysis of Ref. [70]

$$P(x) = \frac{32}{3(1 + x)} \sum_{k=2}^\infty \frac{(-1)^k}{(k^2 - x^2)} .$$

(C18)

Eq. (C17) gives, via Eq. (C11b), the invariant Borel transform for the LB Adler function

$$\hat{B}_d(b) = \frac{3}{4} C_F P(1 - b) .$$

(C19)

The function $P(x)$ is thus the generator of the Adler function LB coefficients $c_{n,n}(\kappa)$ with $\kappa (\equiv \mu^2/Q^2) = e^c$

$$P(1 - b) = 1 + \left( \frac{23}{6} - 4\xi_3 \right) b + \frac{1}{2!} 6 (3 - 2\xi_3) b^2 + \frac{1}{3!} \left( \frac{201}{2} - 42\xi_3 - 60\xi_5 \right) b^3 + \left( \frac{1305}{2} - 180\xi_3 - 360\xi_5 \right) b^4 + \ldots .$$

(C20)

---

12 Cf. also Ref. [39] where $c_{n,n}$ were obtained for the QCD Adler function explicitly in $\overline{\text{MS}}$ scheme and at $\kappa = 1$. The LB coefficients $c_{n,n}(\kappa = 1)$ are then obtained by Eq. (C2), and then the coefficients $c_{n,n}(\kappa = e^c)$ ($C = -5/3$) by (C3).

13 It can be expressed as a combination of the Hurwitz zeta functions $\xi(s,y)$ with $s = 2$:

$$P(x) = \frac{2}{3} \frac{1}{x(1 + x)} \left[ \xi \left( 2, \frac{1}{2} (2 - x) \right) - \xi \left( 2, \frac{1}{2} (3 - x) \right) - \xi \left( 2, \frac{1}{2} (2 + x) \right) + \xi \left( 2, \frac{1}{2} (3 + x) \right) \right] .$$
Using the results (C18)-(C19) in Eq. (C16) with $c = 1$, and $z = -i(1 - b)$, and exchanging the order of integration and summation, leads to

$$F_d(t) = \frac{4C_F}{\pi} t \sum_{k=2}^\infty (-1)^k \int_{-\infty}^{\infty} dz \exp(-iz\ln t) \frac{1}{1 + iz(k^2 + z^2)^2} \frac{k}{1 + iz(k^2 + z^2)^2}$$

$$= \frac{2C_F}{\pi} t \sum_{k=2}^\infty (-1)^k \left( \frac{d}{dk} \int_{-\infty}^{\infty} dz \exp(-iz\ln t) \right) \frac{1}{(z - i)(z - i)(z + ik)(z + ik)} .$$

(C21)

In the integration over $z$ the Cauchy formula can be used: when $0 < t < 1$, the integration path is closed in the upper half plane; when $t > 0$, it is closed in the lower half plane. This gives the following result for the (LB) Adler distribution function:

$$F_d(t)_{(t<1)} = 2C_F t \sum_{k=2}^\infty \left[ t(-1)^k \left( \frac{1}{(k-1)^2} - \frac{1}{(k+1)^2} \right) + \ln(t)(-1)^kt^k \left( \frac{1}{(k-1)^2} - \frac{1}{k^2} \right) - (-1)^kt^k \left( \frac{1}{(k-1)^2} - \frac{1}{k^2} \right) \right]$$

$$= 2C_F t \left[ -t \ln(t) + (1 + t) \ln(1 + t) \ln(t) + \frac{7}{4}t + (1 + t)\text{Li}_2(-t) \right] ,$$

(C22a)

$$F_d(t)_{(t>1)} = 2C_F t \sum_{k=2}^\infty \left[ \ln(t)(-1)^kt^k \left( \frac{1}{k} - \frac{1}{(k+1)^2} \right) + (-1)^kt^k \left( \frac{1}{k^2} - \frac{1}{(k+1)^2} \right) \right]$$

$$= 2C_F \left[ \frac{1}{2} - t \right] \ln(t) - t(1 + t) \ln(t) \left( 1 + \frac{1}{t} \right) + \left( \frac{3}{4} + t \right) + t(1 + t)\text{Li}_2\left( -\frac{1}{t} \right) \right] .$$

(C22b)

Use has been made of the expansion of the polylogarithm function

$$\text{Li}_2(-t) = \sum_{k=1}^\infty (-1)^k \frac{t^k}{k^2} \quad (|t| < 1) .$$

(C23)

As mentioned, the result (C22) was first obtained in Ref. [44], the function $\hat{\alpha}(\tau)$ there is $4F_d(\tau)/\tau$ here. The integration [45] is applied here, however, for a general $N$-loop coupling $a(tQ^2e^C)$; in the case of IR fixed point scenarios, $a \to A$ and the integral is now convergent, apart from being renormalization scale and $C$-independent. If the usual MS-like coupling $a_{pt}(tQ^2e^C)$ is applied, the integral becomes ambiguous due to the Landau (cut and pole) singularities of $a_{pt}(tQ^2e^C)$ at low $t$, and an integration prescription is necessary; in such cases, we use the principal value prescription

$$d^{(LB)}(Q^2){(\text{min)}}^{pt} = \text{Re} \int_{i\epsilon}^{i\epsilon+\infty} \frac{dt}{t} F_d(t) a_{pt}(tQ^2e^C) , \quad (\epsilon \to +0) .$$

(C24)

Appendix D: Exact solutions of RGE in terms of Lambert function

In the scheme where $c_2 = c_3 = \ldots = 0$, the pQCD running coupling $a(nQ^2)$ has formally the two-loop form and there is an exact solution of the RGE in this case, cf. Refs. [44], [45] 14

$$a(nQ^2) = -\frac{1}{c_1} \frac{1}{[1 + W_{x1}(z)]} .$$

(D1)

Here, $Q^2 = |Q^2| \exp(i\phi)$; $W_{-1}$ and $W_{+1}$ are the branches of the Lambert function for $0 \leq \phi < +\pi$ and $-\pi < \phi < 0$, respectively, and $z$ is defined as

$$z = -\frac{1}{c_1 e} \left( \frac{\kappa|Q^2|}{\Lambda^2_{L}} \right)^{-\beta_0/c_1} \exp(-i\beta_0\phi/c_1) ,$$

(D2)

where $\Lambda_L$ is the Lambert QCD scale. 15

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14 The authors of Ref. [69] demonstrated that in this scheme the beta function does not factor out in the generalized Crewther relation.

15 $\Lambda_L \sim \Lambda$, where $\Lambda$ is the usual $\overline{\text{MS}}$ scale appearing in the expansion of pQCD coupling $a(Q^2)$ in inverse powers of logarithms $L \equiv \ln(Q^2/\Lambda^2)$: $a(Q^2) = 1/(\beta_0 L) - (c_1/\beta_0^2) (\ln L/\mathcal{L}^2) + O(\ln^2 L/\mathcal{L}^2)$. It can be checked that in the $c_2 = c_3 = \ldots = 0$ renormalization scheme with $N_f = 3$ we have: $\Lambda_L = \Lambda/0.72882$. We use $\Lambda_L = 0.487$ GeV, thus $\Lambda = 0.355$ GeV.
In the analytic QCD model with two deltas (2δanQCD), Ref. [18], however, we use for the renormalization scheme the preferred central Lambert scheme of the model (with \( N_f = 3 \)): \( c_2 = -4.76, \ c_3 = c_3^{-1}/c_1^{-2} \ (j = 3, 4, \ldots) \). The exact solution of the underlying pQCD coupling is also known in this case, again in terms of the Lambert function (Ref. [44], cf. also Ref. [46]).

\[
a(sQ^2) = -\frac{1}{c_1} \left[ 1 - \frac{1}{c_2} \frac{c_1^4}{c_4} + W_1(z) \right], \quad \left( A_L = 0.260 \text{ GeV} ; \ \rho'(\sigma) = \text{Im} \ a(-\sigma - ie) \right), \quad (D3)
\]

and \( z \) is again defined by Eq. (D2).

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