On the convex hull of symmetric stable processes

JÜRGEN KAMPF∗, GÜNTER LAST†

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Abstract

Let $\alpha \in (1,2]$ and $X$ be an $\mathbb{R}^d$-valued $\alpha$-stable process with independent and symmetric components starting in 0. We consider the closure $S_t$ of the path described by $X$ on the interval $[0,t]$ and its convex hull $Z_t$. The first result of this paper provides a formula for certain mean mixed volumes of $Z_t$ and in particular for the expected first intrinsic volume of $Z_t$. The second result deals with the asymptotics of the expected volume of the stable sausage $Z_t + B$ (where $B$ is an arbitrary convex body with interior points) as $t \to 0$.

Keywords: stable process, convex hull, mixed volume, intrinsic volume, stable sausage, Wiener sausage, mean body

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1 Introduction and main results

For fixed $\alpha \in (1,2]$ and fixed integer $d \geq 1$ we consider an $\mathbb{R}^d$-valued stochastic process $X \equiv (X(t))_{t \geq 0} = (X_1(t), \ldots, X_d(t))_{t \geq 0}$, defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$, such that the components $X_j := (X_j(t))_{t \geq 0}$, $j \in \{1, \ldots, d\}$, are independent $\alpha$-stable symmetric Lévy processes with scale parameter 1 starting in 0. The characteristic function of $X_j(t)$ is given by

$$\mathbb{E} \exp[isX_j(t)] = \exp[-|s|^\alpha], \quad s \in \mathbb{R}, t \geq 0,$$

(1.1)

cf. [8] Section 1.3. This implies that $X$ is self-similar in the sense that $(X(st))_{s \geq 0} \overset{d}{=} t^{1/\alpha}X$ for any $t > 0$, see [8] Example 7.1.3 and [4] Chapter 15. We assume that $X$ is right-continuous with left-hand limits (rcll). For $t \geq 0$, let $S_t$ be the closure of the path $S_t^0 := \{X(s) : 0 \leq s \leq t\}$ and let $Z_t$ denote the convex hull of $S_t$. These are random closed sets. We abbreviate $Z := Z_1$. By self-similarity

$$Z_t \overset{d}{=} t^{1/\alpha}Z, \quad t > 0.$$
If $\alpha = 2$ then $X$ is a standard Brownian motion. A classical result of \cite{12} for planar Brownian motion says that

$$E V_1(Z) = \sqrt{2\pi}, \quad (1.3)$$

where $V_1(K)$ denotes half the circumference of a convex set $K \subset \mathbb{R}^2$. Our first aim in this paper is to formulate and to prove such a result for arbitrary $\alpha \in (1, 2]$ and arbitrary dimension $d$. In fact we also consider more general geometric functionals. A convex body (in $\mathbb{R}^d$) is a non-empty compact and convex subset of $\mathbb{R}^d$. We let $V(K_1, \ldots, K_d)$ denote the mixed volumes of convex bodies $K_1, \ldots, K_d \subset \mathbb{R}^d$ \cite[Section 5.1]{9}. These functionals are symmetric in $K_1, \ldots, K_d$ and we have for any convex bodies $K, B \subset \mathbb{R}^d$

$$V_d(K + tB) = \sum_{j=0}^{d} \binom{d}{j} t^{d-j} V(K[j], B[d-j]), \quad t \geq 0, \quad (1.4)$$

where $V_d$ is Lebesgue measure, $tB := \{tx : x \in B\}$, $B + C := \{x + y : x \in B, y \in C\}$ is the Minkowski sum of two sets $B, C \subset \mathbb{R}^d$, and $V(K[j], B[d-j])$ is the mixed volume of $K_1, \ldots, K_d$ in case $K_1 = \ldots = K_j = K$ and $K_{j+1} = \ldots = K_d = B$. The $j$th intrinsic volume $V_j(K)$ of a convex body $K$ is given by

$$V_j(K) = \frac{\binom{d}{j}}{\kappa_{d-j}} V(K[j], B^d[d-j]), \quad j = 0, \ldots, d, \quad (1.5)$$

where $B^d$ is the Euclidean unit ball in $\mathbb{R}^d$, and $\kappa_j$ is the $j$-dimensional volume of $B^j$. In particular, $V_d(K)$ is the volume of $K$, $V_{d-1}(K)$ is half the surface area, $V_{d-2}(K)$ is proportional to the integral mean curvature, $V_1(K)$ is proportional to the mean width of $K$, and $V_0(K) = 1$. (If $d = 2$ then $V_1(K)$ has been discussed at (1.3).) A geometric interpretation of $V(K_1, \ldots, K_d)$ in the case $K_1 = \ldots = K_{d-1} = B$ is provided by (1.4):

$$V(B, \ldots, B, K) = \lim_{r \to 0} r^{-1} (V_d(B + rK) - V_d(B)). \quad (1.6)$$

For any $p \geq 1$ define $B_p := \{u \in \mathbb{R}^d : \|u\|_p \leq 1\}$ as the unit ball with respect to the $L_p$-norm $\|(u_1, \ldots, u_d)\|_p := (|u_1|^p + \ldots + |u_d|^p)^{1/p}$. Finally we introduce the constant

$$c_\alpha := \frac{\alpha}{2} E|X_1(1)|, \quad (1.7)$$

Since $\alpha > 1$, this constant is finite, see \cite[Property 1.2.16]{8}. A direct calculation shows that

$$c_2 = \sqrt{\frac{2}{\pi}}. \quad (1.8)$$

In the case of $\alpha < 2$ we are not aware of an explicit expression for $c_\alpha$.

**Theorem 1.1.** Let $K_1, \ldots, K_{d-1} \subset \mathbb{R}^d$ be convex bodies. Then

$$E V(K_1, \ldots, K_{d-1}, Z) = c_\alpha V(K_1, \ldots, K_{d-1}, B_{\alpha'}), \quad (1.9)$$

where $1/\alpha + 1/\alpha' = 1$. 

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Remark 1.2. By the scaling relation (1.2) and the homogeneity property of mixed volumes [9, (5.1.24)] the identity (1.9) can be generalized to

\[
\mathbb{E}V(K_1, \ldots, K_{d-1}, Z_t) = c_\alpha t^{1/\alpha}V(K_1, \ldots, K_{d-1}, B_{\alpha'}). 
\] (1.10)

A similar remark applies to all results of this paper.

The proof of Theorem 1.1 relies on the fact that

\[
\mathbb{E}h(Z, u) = h(B_{\alpha'}, u), \quad u \in S^{d-1},
\] (1.11)

where \(S^{d-1}\) denotes the unit sphere,

\[h(K, u) := \max\{\langle x, u \rangle : x \in K\}, \quad u \in S^{d-1},\]

is the support function of a convex body \(K\), and \(\langle \cdot, \cdot \rangle\) denotes the Euclidean scalar product on \(\mathbb{R}^d\). This means that \(B_{\alpha'}\) is the mean body of \(Z\) \cite[p. 146]{10}, or the selection expectation of \(Z\) \cite[Theorem 2.1.22]{7}.

The next corollary provides a direct generalization of (1.3).

**Corollary 1.3.** Assume that \(X\) is a standard-Brownian motion in \(\mathbb{R}^d\). Then

\[
\mathbb{E}V_1(Z) = \frac{d\sqrt{2}\Gamma\left(\frac{d-1}{2} + 1\right)}{\Gamma\left(\frac{d}{2} + 1\right)}
\] (1.12)

In the case of Brownian motion it is possible to calculate the expectation of the second intrinsic volume \(V_2(Z)\) of \(Z\).

**Proposition 1.4.** Assume that \(X\) is a standard-Brownian motion in \(\mathbb{R}^d\). Then

\[
\mathbb{E}V_2(Z) = (d - 1)\frac{\pi}{2}.
\]

Our second theorem deals with the asymptotic behaviour of the expected volume of the stable sausage \(S_t + B\) as \(t \to 0\), where \(B\) is a convex body. Our result complements classical results on the asymptotic behaviour of \(\mathbb{E}V_d(S_t + B)\) as \(t \to \infty\), cf. \cite{11} for the case of Brownian motion and \cite{3} for the case of more general stable processes.

**Theorem 1.5.** Let \(B\) be a convex body with non-empty interior and let \(\alpha'\) be as in Theorem 1.1. Then

\[
\lim_{t \to 0} t^{-1/\alpha}(\mathbb{E}V_d(S_t + B) - V_d(B)) = dc_\alpha V(B, \ldots, B, B_{\alpha'}).
\]

In the case \(\alpha = 2\) the random set \(S_t + B\) is known as Wiener sausage. Even then Theorem 1.5 seems to be new:

**Corollary 1.6.** Assume that \(X\) is a Brownian motion and let \(B\) be a convex body with non-empty interior. Then

\[
\lim_{t \to 0} t^{-1/2}(\mathbb{E}V_d(S_t + B) - V_d(B)) = \frac{d\sqrt{2}}{\sqrt{\pi}}V(B, \ldots, B, B^d).
\]

In the case \(B = B^d\) the limit equals \(2\sqrt{2}\pi^{(d-1)/2}/\Gamma(d/2)\).
Remark 1.7. In the special case $d = 3$ and $\alpha = 2$ we have (see [11])

$$
\mathbb{E} V_3(S_t + rB^3) = \frac{4}{3} \pi r^3 + 4\sqrt{2\pi r^2}\sqrt{t} + 2\pi rt
$$

(1.13)

for any $r, t \geq 0$. The term constant in $t$ clearly allows a geometric interpretation as $V_3(rB^3)$. Now we are able to give a geometric interpretation of the coefficient of $\sqrt{t}$ as well.

From (1.13) we get

$$
\lim_{t \to 0} t^{-1/2}(\mathbb{E} V_3(S_t + rB^3) - V_3(rB^3)) = \lim_{t \to 0} t^{-1/2}(4\sqrt{2\pi r^2}\sqrt{t} + 2\pi rt) = 4\sqrt{2\pi r^2}.
$$

On the other hand from the proof of Theorem 1.5 one can see

$$
\lim_{t \to 0} t^{-1/2}(\mathbb{E} V_3(S_t + rB^3) - V_3(rB^3)) = 3\mathbb{E} V(rB^3, rB^3, Z).
$$

By (1.5) and the homogenity property of mixed volumes (see e.g. [9, (5.1.24)]) we have

$$
3V(rB^3, rB^3, Z) = r^2\kappa_2 V_1(Z).
$$

Altogether this is

$$
4\sqrt{2\pi r^2} = r^2\kappa_2 \mathbb{E} V_1(Z).
$$

2 Proofs

We need the following measurability property of the closure $S_t$ of $\{X(s) : 0 \leq s \leq t\}$ and its convex hull $Z_t$, refering to [7, 10] for the notion of a random closed set.

Lemma 2.1. For any $t \geq 0$, $S_t$ and $Z_t$ are random closed sets.

Proof: To prove the first assertion it is enough to show that $\{S_t \cap G = \emptyset\}$ is measurable for any open $G \subset \mathbb{R}^d$, see [10, Lemma 2.1.1]. But since $X$ is rccl it is clear that $S_t \cap G = \emptyset$ if $X(u) \notin G$ for all rational numbers $u \leq t$. The second assertion is implied by [10, Theorems 12.3.5,12.3.2].

The previous lemma implies, for instance, that $V(K_1, \ldots, K_{d-1}, Z_t)$ and $V_d(S_t + B)$ are random variables, see e.g. [9, p. 275] and [10, Theorem 12.3.5 and Theorem 12.3.6].

Proof of Theorem 1.3: By [9, (5.1.18)] we have that

$$
V(K_1, \ldots, K_{d-1}, K) = \frac{1}{d} \int_{S_{d-1}} h(K, u) S(K_1, \ldots, K_{d-1}, du) \quad (2.1)
$$

holds for every convex body $K \subset \mathbb{R}^d$, where $S(K_1, \ldots, K_{d-1}, \cdot)$ is the mixed area measure of $K_1, \ldots, K_{d-1}$, see [9, Section 4.2]. From (2.1) and Fubini’s theorem we obtain that

$$
\mathbb{E} V(K_1, \ldots, K_{d-1}, Z) = \frac{1}{d} \int_{S_{d-1}} \mathbb{E} h(Z, u) S(K_1, \ldots, K_{d-1}, du). \quad (2.2)
$$
For any \( u \in S^{d-1} \) we have
\[
\mathbb{E} h(Z, u) = \mathbb{E} \max \{ \langle x, u \rangle : x \in Z \} \\
= \mathbb{E} \sup \{ \langle x, u \rangle : x \in S^d_1 \} \\
= \mathbb{E} \sup \{ \langle X(s), u \rangle : s \in [0, 1] \}.
\]
It follows directly from (1.1) that the process \( \langle X, u \rangle \) has the same distribution as \( \|u\|_\alpha X_1 \).
By [1, Theorem 4a], \( \sup \{ X_1(s) : s \in [0, 1] \} \) has a finite expectation. Differentiating equation (7b) in [1] (Spitzer’s identity in continuous time), one can easily show that
\[
\mathbb{E} \sup \{ X_1(s) : s \in [0, 1] \} = \alpha \mathbb{E} X_1(1)^+,
\]
where \( a^+ := \max \{0, a\} \) denotes the positive part of a real number \( a \). Since \( X_1(1) \) has a symmetric distribution and \( \mathbb{P}(X_1(1) = 0) = 0 \) (stable distributions have a density) we have \( \mathbb{E}|X_1(1)| = 2\mathbb{E}X_1(1)^+ \) and it develops that \( \mathbb{E} h(Z, u) = c_\alpha \|u\|_\alpha \), with \( c_\alpha \) given by (1.7). Inserting this result into (2.2) gives
\[
\mathbb{E} \nu V(K_1, \ldots, K_{d-1}, Z) = \frac{c_\alpha}{d} \int_{S^{d-1}} \|u\|_\alpha S(K_1, \ldots, K_{d-1}, du).
\]
By [9, Remark 1.7.8], \( \|u\|_\alpha \) is the support function of the polar body
\[
B^*_\alpha := \{ x \in \mathbb{R}^d : \langle x, u \rangle \leq 1 \text{ for all } u \in B_\alpha \}
\]
of \( B_\alpha \). Using the Hölder inequality, it is straightforward to check that \( B^*_\alpha = B_{\alpha'} \), where \( 1/\alpha + 1/\alpha' = 1 \). Using this fact as well as (2.1) in (2.3), we obtain the assertion (1.9). \( \square \)

**Proof of Corollary 1.3:** Since \( \alpha = 2 \) we have \( \alpha' = 2 \) and \( B_{\alpha'} = B^d \). By Theorem 1.1 and (1.5),
\[
\mathbb{E} V_1(Z) = \mathbb{E} \frac{d}{\kappa_{d-1}} V(B^d, \ldots, B^d, Z) = \frac{c_2 d}{\kappa_{d-1}} V(B^d, \ldots, B^d, B^d) = \frac{c_2 d \kappa_d}{\kappa_{d-1}},
\]
where we have used that \( V(B^d, \ldots, B^d) = V_d(B^d) \). Using (1.8) and the well-known formula \( \kappa_d = \pi^{d/2}/\Gamma(d/2 + 1) \) in (2.4), we obtain the result. \( \square \)

**Proof of Proposition 1.4:** By Kubota’s formula (see e.g. [9, (5.3.27)]) we have
\[
V_2(Z) = \frac{d(d-1)\kappa_d}{2\kappa_2 \kappa_{d-2}} \int_{G_2} V_2(Z|L) \nu_2(dL),
\]
where \( G_2 \) denotes the set of all 2-dimensional linear subspaces of \( \mathbb{R}^d \), \( \nu_2 \) is the Haar measure on \( G_2 \) with \( \nu_2(G_2) = 1 \) and \( Z|L \) denotes the image of \( Z \) under the orthogonal projection onto the linear subspace \( L \). By Fubini’s theorem,
\[
\mathbb{E} V_2(Z) = \frac{d(d-1)\kappa_d}{2\kappa_2 \kappa_{d-2}} \int_{G_2} \mathbb{E} V_2(Z|L) \nu_2(dL).
\]
The spherical symmetry of Brownian motion implies that \( \mathbb{E} V_2(Z|L) \) does not depend on \( L \). Assume that \( L = \{(x_1, x_2, 0, \ldots, 0) : x_1, x_2 \in \mathbb{R}\} \). Now it is clear from the definition of...
the $d$-dimensional Brownian motion that the random closed set $Z|L$ is the convex hull of a Brownian path in $L$. By Remark (a) in [2, p. 149] (see also [6]) we have $\mathbb{E}V_2(Z|L) = \pi/2$. Therefore,

$$\mathbb{E}V_2(Z) = \frac{d(d - 1)\kappa_d \pi}{2\kappa_2\kappa_{d-2}}$$

and the result follows by a straightforward calculation.

\[ \square \]

**Proof of Theorem 1.5.** By self-similarity and the dominated convergence theorem, on whose conditions we will comment below, we have

$$\lim_{t \to 0} t^{-1/\alpha}(\mathbb{E}V_d(S_t + B) - V_d(B)) = \lim_{t \to 0} t^{-1/\alpha}(\mathbb{E}V_d(t^{1/\alpha}S_1 + B) - V_d(B)) = \lim_{t \to 0} t^{-1}(\mathbb{E}V_d(tS_1 + B) - V_d(B)) = \mathbb{E}\lim_{t \to 0} t^{-1}(V_d(tS_1 + B) - V_d(B)). \quad (2.5)$$

In order to justify the application of the dominated convergence theorem, put

$$Y_j = \sup\{X_j(s) : s \in [0, 1]\}, \quad \tilde{Y}_j = \inf\{X_j(s) : s \in [0, 1]\}, \quad j = 1, \ldots, d.$$ 

As noted in the proof of Theorem 1.1, $Y_j$ has a finite expectation. Since $-\tilde{Y}_j$ has the same distribution as $Y_j$, $\tilde{Y}_j$ has a finite expectation as well. From (1.4) we obtain for all $t \in (0, 1]$ that

$$t^{-1}V_d(tS + B) - V_d(B) \leq t^{-1}(V_d(tZ + B) - V_d(B))$$

$$= \sum_{j=0}^{d-1} \binom{d}{j} t^{d-j-1} V(B[j], Z[d - j])$$

$$\leq \sum_{j=0}^{d} \binom{d}{j} V(B[j], Z[d - j])$$

$$= V_d(Z + B).$$

Furthermore,

$$Z + B \subset \times_{j=1}^{d} [\tilde{Y}_j - h_B(-e_j), Y_j + h_B(e_j)],$$

where $e_j$ denotes the $j$th unit vector. It follows that

$$t^{-1}(V_d(tS_1 + B) - V_d(B)) \leq \prod_{j=1}^{d} (Y_j + h_B(e_j) - \tilde{Y}_j + h_B(-e_j)), \quad t \in (0, 1].$$

This is a product of independent random variables with finite expected values and hence has finite expected value.

By [5, Corollary 3.2 (2)] we have

$$\lim_{t \to 0} t^{-1}(V_d(tS_1 + B) - V_d(B)) = \int_{S^{d-1}} h(Z, u) S(B, \ldots, B, du),$$

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and using Theorem 1.1 we conclude from (2.5)

\[ \lim_{t \to 0} t^{-1/\alpha} \mathbb{E}(V_d(S_t + B) - V_d(B)) = \mathbb{E} \int_{S^{d-1}} h(Z, u) S(B, \ldots, B, du) \]
\[ = d\mathbb{E}V(B, \ldots, B, Z) \]
\[ = dc_{\alpha} V(B, \ldots, B, B_{\alpha}). \]

\[ \square \]

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