Numerical bifurcation analysis of two coupled FitzHugh-Nagumo oscillators

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Abstract The behavior of neurons can be modeled by the FitzHugh-Nagumo oscillator model, consisting of two nonlinear differential equations, which simulates the behavior of nerve impulse conduction through the neuronal membrane. In this work, we numerically study the dynamical behavior of two coupled FitzHugh-Nagumo oscillators. We consider unidirectional and bidirectional couplings, for which Lyapunov and isoperiodic diagrams were constructed calculating the Lyapunov exponents and the number of the local maxima of a variable in one period interval of the time-series, respectively. By numerical continuation method the bifurcation curves are also obtained for both couplings. The dynamics of the networks here investigated are presented in terms of the variation between the coupling strength of the oscillators and other parameters of the system. For the network of two oscillators unidirectionally coupled, the results show the existence of Arnold tongues, self-organized sequentially in a branch of a Stern-Brocot tree and by the bifurcation curves it became evident the connection between these Arnold tongues with other periodic structures in Lyapunov diagrams. That system also present multistability shown in the planes of the basin of attractions.

Key words. FitzHugh-Nagumo oscillator – Periodic structures – Lyapunov diagrams – Isoperiodic diagrams – Neuronal systems – Bifurcation curves.

1 Introduction

The FitzHugh-Nagumo (FHN) oscillator model \cite{1,2} is a mathematical description of the qualitative features of the nervous impulse conduction in the neural cell, namely the membrane potential dynamics, the resting bias of the excitable membrane and the existence of an action potential threshold. The two-dimensional FHN oscillator is described by the variables \( u \) and \( w \), that represent the potential of the cell membrane performing the excitability of the system, and the recovery variable after an action potential, respectively. Following Campbell and Waite \cite{3}, such model is represented by the following differential equations

\begin{align}
\frac{du}{dt} &= c(w + u - \frac{1}{3}a^3 + J(t)), \\
\frac{dw}{dt} &= -\frac{1}{c}(u - a + bw),
\end{align}

where \( J(t) \) is a time-dependent function that represents the external stimulus, and \( a, b, \) and \( c \) are parameters \cite{1,2}. Such system can produce chaotic motion only if an external stimulus \( J(t) \) is applied \cite{4}. Oscillatory chaotic behaviors can also be induced considering the coupling of FHN oscillators in a network \cite{5,6,7}.

Here, we consider two coupled FHN oscillators with two types of coupling, namely unidirectional and bidirectional. Our main interest is to obtain the bifurcation curves and consequently in describing the dynamics of the variables regarding the coupling strength and other system parameters in the parameter-planes. The parameter-planes are diagrams when two parameters of the system are varied and the other are kept constant. The colors codify some quantity that can be computed on the model. Usually, this quantity is the Lyapunov exponent \cite{8,9,10,11}, periods \cite{9,12,13}, or other invariant quantities \cite{9,14}. That procedure allows us to identify regions of periodic and chaotic behavior, and recently it is applied in several models \cite{9,15,16,17}. A general feature is observed in these parameter-planes, the existence of shrimp-shaped periodic structures embedded in chaotic domains \cite{16,17}.

In this paper we investigate the dynamics of two coupled FHN oscillators as a function of the type of coupling, without external stimulus. The dynamics is investigated in the parameter-planes, namely via Lyapunov and isoperiodic diagrams \cite{9,12,13,16,19}, and we compare their structural changes as the type of coupling. It is shown the limitation of the Lyapunov diagrams to distinguish the quasi-periodic from periodic oscillations. For two FHN oscillators unidirectionally coupled we report periodicity domains similar to the Arnold tongues. The role of the coupling in FHN-networks was studied in recent papers \cite{20,21,22}, but with different approaches of those
studied here. In those papers, the interesting is in the syn-chronization phenomena, and in our work we carry out a systematic dynamical study of the two FHN oscillators with respect to the coupling type.

This paper is organized as follows. In Section 2 the network models are introduced, and in the Section 3 we present the numerical results concerning with each coupling type. In Section 4 we present the conclusions of our work.

2 FitzHugh-Nagumo network models

In this section we present the FHN-network models for two types of coupling between two oscillators, namely unidirectional and bidirectional coupling.

Following Campbell and Waite [3], for the unidirectional coupling case between two oscillators, the FHN-network model is given by

\[
\begin{align*}
\frac{dx_1}{dt} &= c(y_1 + x_1 - \frac{1}{3}x_1^3) + \gamma(x_1 - x_2), \\
\frac{dy_1}{dt} &= -\frac{1}{c}(x_1 - a + by_1), \\
\frac{dx_2}{dt} &= c(y_2 + x_2 - \frac{1}{3}x_2^3), \\
\frac{dy_2}{dt} &= -\frac{1}{c}(x_2 - a + by_2),
\end{align*}
\] (2)

where \(x_i, y_i, i = 1, 2\), represent the voltage across the cell membrane, and the recovery state of the resting membrane of a neuron, respectively. On the other hand \(a, b, \) and \(c\) are parameters, and \(\gamma\) is the coupling strength between the network elements.

For bidirectional coupling case between two oscillators, the FHN-network model can be written as

\[
\begin{align*}
\frac{dx_1}{dt} &= c(y_1 + x_1 - \frac{1}{3}x_1^3) + \gamma(x_1 - x_2), \\
\frac{dy_1}{dt} &= -\frac{1}{c}(x_1 - a + by_1), \\
\frac{dx_2}{dt} &= c(y_2 + x_2 - \frac{1}{3}x_2^3) + \gamma(x_2 - x_1), \\
\frac{dy_2}{dt} &= -\frac{1}{c}(x_2 - a + by_2),
\end{align*}
\] (3)

Variables and parameters have the same meaning as in system (2).

In the next section we present and discuss the results obtained from the numerical solutions of those equations whose behaviors depend on the four above-mentioned parameters. The results are essentially presented in two-dimensional diagrams, using the Lyapunov exponents and the periods as the invariant quantities, here namely Lyapunov and isoperiodic diagrams, respectively, as reported previously [12][13]. Bifurcation curves obtained by numerical continuation method [23] are also presented for both network models.

The Lyapunov diagram plots are obtained for the following parameter combinations: \(\gamma \times a\) with \(b\) and \(c\) fixed, \(\gamma \times b\) with \(a\) and \(c\) fixed, and \(\gamma \times c\) with \(a\) and \(b\) fixed, for the systems (2) and (3). They were constructed using the Lyapunov exponents numerically calculated for these systems. To evaluate the Lyapunov spectra, the Eqs. (2) and (3) are numerically solved by the Runge-Kutta method with fixed time step in 1.0 \times 10^{-1} and 5.0 \times 10^{-5} integration steps, via the algorithm proposed in Ref. [24] for each parameter pair discretized in a grid of 500 \times 500 values. We performed other tests using smaller time steps and the results are practically unchanged. Therefore, we obtain 2.5 \times 10^5 Lyapunov spectra for each two-dimensional diagrams, where each spectrum has the number of values of Lyapunov exponents equal to the number of dimensions of the network.

The isoperiodic diagram plots are also computed for the two coupled FHN oscillator models. To evaluate the period for each time-series with a fixed set of parameters, we use the Runge-Kutta method again, but now with variable time step and removed a transient time equal to 5 \times 10^5 and 1 \times 10^6 integration steps to find the period obtained by the maxima of the time-series with a precision of 1 \times 10^{-3}. The parameter pairs were discretized in a grid of 10^3 \times 10^3 values.

To obtain the bifurcation curves in the parameter-planes, we used the numerical continuation method. For this purpose, the MATCONT package [23] was used, and Hopf, saddle-node, Neimark-Sacker, and period-doubling curves were obtained unveiling the bifurcation structures of the systems (2) and (3). The bifurcation curves are used extensively to unveil the bifurcation structures of dynamical systems [20][21][22][23][24][25]. In one of the first papers on this matter [27], Barrio et al. performed a detailed study of bifurcations in the three-parametric three-dimensional Rössler model. Among other results, the authors overlapped bifurcation curves, obtained by numerical continuation method [23], on two-dimensional parameter-spaces of the model, and some periodic structures might be described by those curves, as also be done recently in Ref. [30], for a four-dimensional Chua model.

3 Numerical results

We begin our numerical investigation analyzing the parameter-planes of systems (2) and (3) in a large range of parameter values. The main goal here is to know how is the dynamical behavior of the systems regarding the bifurcation curves and Lyapunov exponents in those parameter values. Based in the results presented in Ref. [3], we show in Fig. 1, the parameter-planes of the systems (2) ((a)-(c)) and (3) ((d)-(f)) for the largest Lyapunov exponent with bifurcation curves overlapped. The set of initial conditions used is \((x_1, y_1, x_2, y_2) = (-0.1, 0.1, 0.5, -0.3)\), and the fixed parameters used are \((a, b, c) = (0.7, 0.4, 2.0)\), as the case. The cyan and magenta curves correspond to Hopf bifurcations and the green curve corresponds to limit-point bifurcation. The white, black and yellow regions correspond to equilibrium points, periodic and chaotic motions, respectively, and codify the sign of
The largest Lyapunov exponent of the spectrum. For example, white corresponds to negative, black to null and yellow to positive largest exponent. In Fig. 1 the Hopf bifurcations occur when an equilibrium point (white regions) lose its stability and a limit cycle (black regions) born. This behavior is corroborated by the largest Lyapunov exponents, white and black regions in the diagrams of Fig. 1. As can also be observed, the chaotic behaviors (yellow regions) are small regions in the expanded diagrams and the arrows indicate where those regions are localized.

The equations used to describe the dynamical behavior of the FHN-network with two unidirectional coupled oscillators are given by the system (2). For the same set of initial conditions used in Fig. 1, we show in Figs. 2(a)-(c) the Lyapunov diagrams for the following planes: (a) $\gamma \times a$ with $b$ and $c$ fixed, (b) $\gamma \times b$ with $a$ and $c$ fixed, and (c) $\gamma \times c$ with $a$ and $b$ fixed. Such diagrams are amplifications of the small regions indicated by the arrows in Figs. 1(a)-(c), and are constructed with the largest Lyapunov exponent of the spectrum codified in colors. For example, in Fig. 2 yellow-red-blue colors identify chaotic behavior, since for points taken from these regions the largest exponent of the spectrum is positive. In Figs. 2(d)-(f) we present the Lyapunov diagrams for the largest exponents of the spectrum for two bidirectionally coupled models, described by system (3). Such diagrams are amplifications of the small regions indicated by the arrows in Figs. 1(d)-(f). As the unidirectional case, the Lyapunov spectrum has four exponents, and the diagrams shown in Figs. 2(d)-(f) were constructed with the largest one. The construction and interpretation of colors follow the same of Figs. 2(a)-(c).

All diagrams in Fig. 2(a)-(f) show large periodic regions (regions in black) and chaotic windows (yellow-red-blue regions) with periodic structures (in black) embedded on them. As far as our knowledge, this feature is commonly found in almost every nonlinear dissipative dynamical systems [8,9,10,11,12,13,14,15,16,17,18,19,31]. Fig. 2 show the existence of paths crossing the chaotic regions in which the variables $x_i$ and $y_i$ have periodic behaviors. Moreover, there exist regions in which these variables have chaotic oscillations, depending of the parameter combinations. It is important to observe that for the single FHN oscillator without external forcing, namely a single neuron without external stimulus, system (1), the variables can not present chaotic motion.

The limitations of the Lyapunov diagram shown in Fig. 2 are related to the distinction between periodic (in black) and chaotic (yellow-red-blue) regions, and to observe the presence of periodic structures, as cuspidal, non-cuspidal and shrimp-like structures [9,19]. In such diagrams is not possible to infer the periods of the structures and their organization rules, as period-adding cascades [12,13,16], and Stern-Brocot trees [12,13,16]. Following recent works about isoperiodic diagrams [12,13,16], that are parameter-planes using the numbers of spikes of the time series (periods), we also constructed these diagrams for regions inside the boxes of Figs. 2(a) and (c).

Figure 3 shows the Lyapunov and the isoperiodic diagrams for the boxes A and B in Fig. 2. In these diagrams it is clear the limitations of the Lyapunov diagram in both cases. The isoperiodic diagram in Fig. 3(c) unveils hidden regions and pieces of periodic structures that are not perceived in Fig. 3(a). For example, on the left top in

**Figure 1.** Bifurcation curves in the parameter-planes for two FHN oscillators with unidirectional coupling, system (2) [top line (a)-(c), and for two FHN oscillators with bidirectional coupling, system (3) [bottom line (d)-(f)]. (a) $\gamma \times a$ plane, with $b = 0.4$ and $c = 2.0$. (b) $\gamma \times b$ plane with $a = 0.7$ and $c = 2.0$. (c) $\gamma \times c$ plane with $a = 0.7$ and $b = 0.4$. (d) $\gamma \times a$ plane, with $b = 0.4$ and $c = 2.0$. (e) $\gamma \times b$ plane with $a = 0.7$ and $b = 2.0$. (f) $\gamma \times c$ plane with $a = 0.7$ and $b = 0.4$. The colors white, black, and yellow represent the largest Lyapunov exponent and correspond to equilibrium points, periodic and chaotic motion, respectively. The arrows indicate small regions that present chaotic motion and are enlarged in Fig. 2.
organizing in a circular layers pattern, in which each layer
odic structures can be visualized. Here, the structures self-
B in Fig. 2(a), and another pattern formation of peri-
pectrum (see the insets in Fig. 4).

III). In each plot we also show the respective Lyapunov
in Fig. 4 three attractors in three different points of the
tinct behaviors presented in Figs. 3(a) and (c), we show
example: $7 + 13 \rightarrow 20 + 6 \rightarrow 20 + 13 \rightarrow 19 + 6 + 11 \rightarrow 17 + 5 + 11 \rightarrow$
branches of periodic structures embedded on it. Moreover, the organization pattern of these periodic structures delimits the quasi-periodic region, upper black region in Fig. 3(c), with the chaotic region, lower black region in Fig. 3(c). This organization pattern is similar to the Arnold’s tongues [32, 33], and its periodicity is similar to branches of Stern-Brocot tree observed in a variety of continuous-time systems [12, 13, 16, 34]. Basically, in Fig. 3(c) the organization rule of the periods can be summarized as follows.

The primary structures follow a period-adding cascade: $7 \rightarrow 6 \rightarrow 5 \rightarrow 4 \rightarrow 3 \rightarrow 2$. The sum of the periods of two consecutive primary structures is equal of the period of secondary structures that are between these two primary structures. For example: $7 + 6 \rightarrow 13 + 6 + 5 \rightarrow 11 + 5 + 4 \rightarrow 9 + 4 + 3 \rightarrow 7, 3 + 2 \rightarrow 5$ (see Fig. 3(c)). The sum of the period of a primary structure with the consecutive secondary structure is equal of the period of tertiary structure between these primary and secondary structures. For example: $7 + 13 \rightarrow 20 + 6 + 13 \rightarrow 19 + 6 + 11 \rightarrow 17 + 5 + 11 \rightarrow 16 + 5 + 9 \rightarrow 14, 4 + 9 \rightarrow 13, 4 + 7 \rightarrow 11, 3 + 7 \rightarrow 10, 3 + 5 \rightarrow 8, 2 + 5 \rightarrow 7$.

To corroborate the differences between the three distinct behaviors presented in Figs. 3(a) and (c), we show in Fig. 4 three attractors in three different points of the Lyapunov diagram of Fig. 3(a) (see the points I, II, and III). In each plot we also show the respective Lyapunov spectrum (see the insets in Fig. 4).

Figures 2(b) and (d) show the amplification of the box B in Fig. 2(a), and another pattern formation of periodic structures can be visualized. Here, the structures self-organizing in a circular layers pattern, in which each layer is composed by two connected structure with same period. A period-adding cascade occurs in direction of the inner layers.

Figures 2(b) and (c) show the two other planes, (b) $\gamma \times h$, and (c) $\gamma \times c$ for the largest Lyapunov exponents of the system 2. In Fig. 2(b) the similarities with the $\gamma \times a$ plane (see Fig. 2(a)), are evident with the two types of organization patterns of structures shown in Fig. 3. In $\gamma \times c$ plane (see Fig. 2(c)), we observe the same periodic structures similar to the Arnold’s tongues observed previously in the $\gamma \times a$ plane (see Fig. 3). For example, Figs. 5(a) and (b) are amplifications of the Lyapunov and isoperiodic diagrams, respectively, for the box in Fig. 2(c). Bifurcation curves are also overlapped in the Lyapunov diagram of Fig. 5(a), where the cyan and blue curves are saddle-node bifurcations, and the green curves are Neimark-Sacker bifurcation curves. It is clear to observe the same organization pattern presented in Fig. 5(a) and (c), quasi-periodic regions (lower left and right portions of the diagram) with branches of periodic structures embedded on it, and other branches of these structures embedded on chaotic regions (higher portion of the diagram). The same organization rule of periods of these structures was also observed. To corroborate this statement, in Fig. 6, we show amplifications of Fig. 5 and we observe similarities with Figs. 3(a) and (c). In Figs. 6(b) and (d) the same organization rule of the Arnold tongues in Figs. 6(a) and (c), or in Figs. 3(a) and (c), is observed for the periodic structures of Figs. 6(b) and (d).

Another remarkable result is the agreement between the bifurcation curves in Fig. 3(a) and the periodic structures. In addition, there exist a connection between the lower left Arnold tongues and the central periodic structures. Initially, in Fig. 3(a) it is clear to see two saddle-node bifurcation curves (cyan curves) bordering two central periodic structures (the central yellow and red peri-

Figure 2. Lyapunov diagrams for the largest exponent of the spectra codified by colors, as the color bar at the right side, for two FHN oscillators with unidirectional coupling, system 2 [top line (a)-(c)], and for two FHN oscillators with bidirectional coupling, system 3 [bottom line (d)-(f)]. The diagrams are amplifications of small regions located by the arrows in Fig. 1. Periodic/quasi-periodic domains are represented by black colors, and chaotic ones are yellow-red-blue colors. The boxes in panels (a) and (c) delimit the regions magnified in Figs. 3 and 5, respectively.
Figure 3. Unidirectional coupling case: (a) and (b) Lyapunov diagrams for the largest exponents, and (c) and (d) isoperiodic diagrams, both for the boxes A and B, respectively, in Fig. 2(a). In (a) and (b) the black-yellow-red colors codify the largest Lyapunov exponent, and in (c) and (d) the top color bars codify the periods. Black corresponds to quasi-periodic/chaotic behaviors or periods greater than 20.

Figure 4. Attractors of three distinct points in Fig. 3(a), showing (a) Torus-2 (point I), (b) periodic (point II), and (c) chaotic (point III) behaviors. The Lyapunov spectrum for each attractor is also shown.

odic structures in Fig. 5(b)) at their lower portions. Those saddle-node bifurcations occur when, as the control parameters are varied, the chaotic attractors become unstable and born stable periodic attractors. Those features are corroborated by the isoperiodic diagram in Fig. 5(b). Following the second cyan curve (the curve that border the red structure in Fig. 5(b)) from right to left, we see that after to border the lower portion of the periodic structure this curve reach the respective left Arnold tongue of same period (the red Arnold tongue in Fig. 5(b)). Due to multistability the yellow Arnold tongue was hidden by the lower blue periodic region (see Fig. 5(b)). Moreover, due to numerical precision to obtain those bifurcation curves, how smaller is the periodic structure, more difficult is to obtain the curves. Therefore, for period-4 structure and on, we have not accuracy to numerically obtain their saddle-node bifurcation curves. However, our results show that the formation of the central set of periodic structures is connected with the formation of the left set of Arnold tongues.

The blue saddle-node bifurcation curve in Fig. 5(a) delimits the blue periodic region in the bottom of the
isoperiodic diagram of Fig. 5(b). The horizontal green Neimark-Sacker curve in Fig. 5(a) (in the bottom) shows the line where born the Arnold tongues, corroborated by the isoperiodic diagram in Fig. 5(b), where in the bottom of the diagram we see the branches of the Arnold tongues in the quasiperiodic domains (lower black regions in Fig. 5(b)). The diagonal green Neimark-Sacker curve in Fig. 5(a) has a strict connection with the multistability presented by the system (2) and with a family of Torus that emerges very close to the curve. To show this feature, in Figs. 5(a) and (b) we present in more details a zoom of Fig. 5(b), with the green Neimark-Sacker and cyan saddle-node bifurcation curves for two different initial conditions: (a) \((x_1, y_1, x_2, y_2) = (-0.1, 0.1, 0.5, -0.3)\), and (b) \((x_1, y_1, x_2, y_2) = (-0.1, 0.5, 0.1, -0.3)\). By contrasting the diagrams in (a) and (b) we see the effect of the initial condition and the role of the green Neimark-Sacker curve, for different initial conditions the black periodic border increases until near the limit of the green curve destroying chaotic regions and periodic structures, and very close to the green curve a Tori domain emerge. The Tori domain is observed in Fig. 5(c) by plotting the Lyapunov diagram with the second largest exponent. In this case, the periodic domains are in white, Tori (quasiperiodicity) and chaotic domains through those bifurcation curves.

Regarding the two coupled FHN oscillators with bidirectional coupling, Figs. 2(d)-(f), the dynamics does not display the richness of details shown in Figs. 2(a)-(c), for the two coupled FHN oscillators with unidirectional coupling. Comparing the dynamics, presented in the Lyapunov diagrams of Fig. 2, for both coupling, it is possible to conclude that the unidirectional coupling presents a more subtlety in the dynamics concerning the presence of self-organized periodic structures embedded in quasi-periodic and chaotic regions in the three planes of parameters, namely \(\gamma \times a\), \(\gamma \times b\), and \(\gamma \times c\).

4 Conclusions

In this paper we have investigated the dynamics of FHN oscillators composed by two coupled models, using unidirectional and bidirectional coupling. Each oscillator is modeled by a set of two autonomous nonlinear first-order ordinary differential equations that describes the dynamics of a nerve impulse through the neuronal membrane, namely FitzHugh-Nagumo model. Our main goal is the investigation of the influence of the coupling strength between the oscillators with the type of coupling. For this purpose, we constructed the Lyapunov and isoperiodic diagrams, which are parameter-planes with the Lyapunov exponents and periods codified by colors, respectively, for the two cases of the FHN-networks. By contrasting those diagrams in each case, we observe the changes in the dynamical behaviors of these models. We showed that \(\gamma\) also has an important role in the dynamics of the FHN-networks. In addition, we have obtained the bifurcation curves by numerical continuation method, and overlapped those curves in the Lyapunov diagrams, unveiling the existence of a connection between different types of periodic structures through those bifurcation curves.
Regarding the two coupled FHN oscillators for the unidirectional coupling, *i.e.*, the coupling only occurs in one of the four variables \((x_i, y_i; i = 1, 2)\), namely \(x_1\) in system (2). The dynamics for this case in the Lyapunov and isoperiodic diagrams presents the coexistence of periodic, quasi-periodic, and chaotic oscillations, and the presence of stability domains (periodic structures) embedded in quasi-periodic and chaotic domains. The dynamical behavior presented in this system resembles with the dynamics of systems that show the presence of Arnold tongues. The organization rule of periods, shown in Figs. 3(b) and 5(c), seems to be a branch of the Stern-Brocot tree previously reported in a three-dimensional version of the FHN model [12]. A circular organization of periodic structures in layers was also observed in a portion of the parameter-plane (Figs. 3(b) and 5(c)).

For the bidirectional case, the coupling occurs in two of the four variables \((x_i, y_i; i = 1, 2)\), namely \(x_1\) and \(x_2\) in system (3). In this case, the dynamics change drastically with the disappearance of the quasi-periodic domain and of the Arnold tongue-like structures, surviving the chaotic domains and some periodic structures (see Fig. 2). Among some examples of discrete- and continuous-time systems that present Arnold tongues in the parameter-planes [32,33,34], forced oscillators show a large spectrum of applications and Arnold tongues are abundantly observed in those systems. In our work we characterize the Arnold tongues in the parameter-planes of the unidirectional coupling case that, indeed, can be interpreted as a forced oscillator, being \((x_2, y_2)\) the master and \((x_1, y_1)\) the slave in system (2). The primary Arnold tongues and periodic structures shown in the isoperiodic diagrams of Figs. 3(c), 5(b) and (d) are organized by period-adding cascades with period-1 being the factor of the adding. An interesting verification is that the master oscillates in cycle-1 driving the slave with a period-1 oscillations, which is the factor of the period-adding cascades of the slave.

For networks of two models, and for both coupling cases, the Lyapunov diagrams are slightly different over wide regions (see Fig. 2) but a general feature was observed in some regions of these diagrams, namely the existence of periodic structures embedded in chaotic regions. These sets of periodic structures are presented in a wide range of nonlinear systems [9,15,16,17]. An exception is in high-dimensional systems with more than three-dimensions, where hyperchaotic behaviors can occur. In these systems, on the hyperchaotic regions there are no periodic structures or the structures are malformed or shapeless [18].

Relating the study presented here with the synchronization regimes in the two coupled FHN systems, it was also observed that, for both systems (unidirectional and bidirectional coupling) initializing in the same initial co-
Figure 7. Unidirectional coupling case: (a) and (b) Lyapunov diagrams for the $\gamma \times c$ plane for the largest exponents for two distinct initial conditions, and in (c) Lyapunov diagram for the second largest exponent for the same initial condition in (b). Cyan and green curves are saddle-node and Neimark-Sacker bifurcations, respectively.

Figure 8. (a) and (b) planes of the four-dimensional basin of attractions for the point P in Figs. 7(a) or (b). Black color is the basin of periodic attractors, and red one for chaotic attractors.

ditions, lead to synchronized states. On the other hand, initializing both systems (unidirectional and bidirectional coupling) in different initial conditions, lead to non-synchronized states in the periodic domains of the parameter-planes of Fig. 1. Moreover, chaotic synchronization was not observed in the chaotic domains covered in our study.

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