FAST AND ACCURATE COMPUTATION OF TIME-DOMAIN ACOUSTIC SCATTERING PROBLEMS WITH EXACT NONREFLECTING BOUNDARY CONDITIONS

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Abstract. This paper concerns with fast and accurate computation of exterior wave equations truncated via exact circular or spherical nonreflecting boundary conditions (NRBCs, which are known to be nonlocal in time and space). We first derive analytic expressions for the underlying convolution kernels involving inverse Laplace transform of the logarithmic derivative of a modified Bessel function, which allow us to rapidly and accurately evaluate the temporal convolution with \(O(N_t)\) operations over \(N_t\) successive time steps. To handle the nonlocality in space, we introduce the notion of boundary perturbation, which enables us to handle general bounded scatters by solving a sequence of wave equations in an annulus or a spherical shell. We present and analyze an efficient spectral-Galerkin and Newmark’s time integration solver for the truncated wave equation in the regular domain. Moreover, we provide ample numerical results to show the high accuracy of the NRBCs and efficiency of the proposed scheme.

1. Introduction

Wave propagation and scattering problems in unbounded media arise from diverse application areas such as acoustics, aerodynamics, electromagnetics, antenna design, oceanography and among others (see, e.g., \([17, 38, 10]\)). A surge of research has been devoted to their numerical studies that include the boundary element methods (cf. \([7]\)), infinite element methods (cf. \([16]\)), perfectly matched layers (PML) (cf. \([5]\)), nonreflecting boundary condition methods (cf. \([26, 21]\)), and among others (cf. \([29, 42]\)). An essential ingredient for the latter approach is to truncate an unbounded domain to a bounded domain by imposing an exact or approximate nonreflecting (absorbing or transparent) boundary condition at the outer artificial boundary, where the NRBC is devised to prevent spurious wave reflection from the artificial boundary (cf. the review papers \([17, 18]\) and the references therein). The frequency-domain approaches for e.g., the time-harmonic Helmholtz problems have been intensively investigated, while the time-domain simulations, which are capable of capturing wide-band signals and modeling more general material inhomogeneities and nonlinearities (cf. \([3, 9]\)), have been relatively less studied.

Although several types of NRBCs based on different principles have been proposed (see, e.g., \([14, 4, 44, 39, 40, 19, 20, 21, 18]\)), a longstanding issue on time-domain computation is...
the efficient treatment for the NRBCs that can scale and integrate well with the solver for the underlying truncated problem (cf. [43, 23]). In practice, if an accurate NRBC is imposed, the artificial boundary could be placed as close as possible to the scatter that can significantly reduce the computational cost. Here, we restrict our attention to the exact NRBC on the circular or spherical artificial boundary. One major difficulty lies in that such a NRBC is global in space and time in nature, since it involves the Fourier/spherical harmonic transform in space, and history dependence in time induced by a convolution. The convolution kernel, termed as nonreflecting boundary kernel (NRBK) in [2], is the inverse Laplace transform of an expression that includes the logarithmic derivative of a modified Bessel function of integer order in 2-D and of fractional order in 3-D. The rapid computation of the NRBK and fast convolution in time are of (independent) relevance and interest. Alpert, Greengard and Hagstrom [2] proposed a rational approximation of the logarithmic derivative with a least square implementation, which allows for a reduction of the summation of the poles from \( O(\nu) \) to \( O(\log \nu \log \frac{1}{\varepsilon}) \) (where \( \nu \gg 1 \) is the order of the modified Bessel function and \( \varepsilon \) is a given tolerance), and a rapid evaluation of the NRBK recursively in time via the summation of exponentials. Jiang and Greengard [24, 25] further discussed the interesting applications to Schrödinger equations in one and two dimensions. Li [27] introduced a more accurate low order approximation of the three-dimensional NRBK at a slightly expensive cost, where the observation that the Laplace transform of the three-dimensional NRBK is exactly a rational function lies at the heart of this algorithm. However, in many cases, the expressions are not rational. For instance, the two-dimensional NRBK also integrates the contributions from the brach-cut along the negative real axis (see Theorem 2.1 below). Lubich and Schädle [28] developed some fast algorithm for the temporal convolution with \( O(N_t \log N_t) \) operations (over \( N_t \) successive time steps) arising from NRBCs with non-rational expressions for other equations (e.g., Schrödinger equations and damped wave equations).

In this paper, we derive an analytic expression for the NRBK based on a direct inversion of the Laplace transform by the residue theorem (see Theorem 2.1 below). In fact, Sofronov [40] derived similar formulas by working on much more complicated expressions of the kernel in terms of Tricomi’s confluent hypergeometric functions. The NRBK is expressed in terms of the zeros of the modified Bessel function with an additional improper integral in the two-dimensional case. The presence of the temporal variable in the exponentials allows to evaluate the temporal convolution recursively and rapidly with \( O(N_t) \) operations almost without extra memory for the history. Moreover, the analytic expression provides a useful apparatus for the stability and convergence analysis. It is worthwhile to remark that Chen [6] reformulated the two-dimensional wave problem into a first-order system in time and showed the well-posedness of the truncated problem with an alternative formulation of the NRBC. In this paper, we provide a direct proof for both two and three dimensions.

It is known that the non-locality of NRBC in space can be efficiently handled by Fourier/spherical harmonic expansions when the scatter is a disk or a ball. Recently, a systematic approach, based on the boundary perturbation technique (also called the transformed field expansion (TFE) method (cf. [32])), has been developed in [33, 15, 34] for time-harmonic Helmholtz equations in exterior domains with general bounded obstacles, under which the whole algorithm boils down to solving a sequence of Helmholtz equations in a 2-D annulus or a 3-D spherical shell. In
this paper, we highlight that this notion can be extended to time-domain computation, though it has not been investigated before as far as we know. Consequently, it is only necessary to develop fast solver for the wave equations with NRBC in regular separable domains. In this paper, we propose an efficient spectral-Galerkin method with Newmark’s time integration for the truncated wave equations in an annulus or a spherical shell, and provide ample numerical results to show the efficiency of the solver and high accuracy of NRBC from several angles.

The rest of the paper is organized as follows. In Section 2, we sketch the derivation of NRBC, and collect some properties of zeros of the modified Bessel function. Then, we derive the analytic formula for NRBC via the residue theorem, and provide a fast convolution of NRBC. In Section 3, we present some properties of NRBC and analyze well-posedness of the truncated wave equation. In Section 4, we outline the notion of the TFE method and propose an efficient spectral-Galerkin and Newmark’s time integration scheme for the truncated wave problem in regular separated domains. We provide ample numerical results in Section 5 to show the high accuracy of NRBC and efficiency of the proposed scheme.

2. Evaluation of nonreflecting boundary kernels

In this paper, we consider the time-domain acoustic scattering problem with sound-soft boundary conditions on the bounded obstacle:

\[
\partial_t^2 U = c^2 \Delta U + F, \quad \text{in } \Omega_\infty := \mathbb{R}^d \setminus \bar{D}, \quad t > 0, \quad d = 2, 3; \tag{2.1}
\]

\[
U = U_0, \quad \partial_t U = U_1, \quad \text{in } \Omega_\infty, \quad t = 0; \tag{2.2}
\]

\[
U = G, \quad \text{on } \Gamma_D, \quad t > 0; \quad \partial_t U + c \partial_n U = o(|x|^{(1-d)/2}), \quad |x| \to \infty, \quad t > 0. \tag{2.3}
\]

Here, \(D\) is a bounded obstacle (scatter) with Lipschitz boundary \(\Gamma_D\), \(c > 0\) is a given constant, and the radiation condition (2.3), where \(n = x/|x|\), corresponds to the well-known Sommerfeld radiation condition in the frequency domain. Assume that the data \(F, U_0\) and \(U_1\) are compactly supported in a 2-D disk or a 3-D ball \(B\) of radius \(b\).

A common way is to reduce this exterior problem to the problem in a bounded domain by imposing an exact or approximate NRBC at the artificial boundary \(\Gamma_b := \partial B\). In what follows, we shall focus on the wave equation truncated by the exact circular or spherical NRBC:

\[
\partial_t^2 U = c^2 \Delta U + F, \quad \text{in } \Omega := B \setminus \bar{D}, \quad t > 0, \quad d = 2, 3; \tag{2.4}
\]

\[
U = U_0, \quad \partial_t U = U_1, \quad \text{in } \Omega, \quad t = 0; \tag{2.5}
\]

\[
U = G, \quad \text{on } \Gamma_D, \quad t > 0; \tag{2.6}
\]

\[
\partial_r U = T_d(U), \quad \text{at } r = b, \quad t > 0, \tag{2.7}
\]

where \(T_d(U)\) is the so-called time-domain DtN map.

2.1. Formulation of the time-domain DtN map. We briefly review the derivation and formulation of the NRBC (2.7) (see, e.g., [21, 3]). Consider the “auxiliary” exterior problem:

\[
\partial_t^2 U^e = c^2 \Delta U^e, \quad \text{in } \Omega_{\text{ext}} := \mathbb{R}^d \setminus \bar{B}, \quad t > 0, \quad d = 2, 3; \tag{2.8}
\]

\[
U^e|_{r=b} = U|_{r=b}, \quad t > 0; \quad U^e|_{t=0} = \partial_t U^e|_{t=0} = 0, \quad \text{in } \Omega_{\text{ext}};
\]
together with the radiation condition (2.3) at infinity, where the Dirichlet data \( U_{r=b} \) is taken from the interior problem (2.4)-(2.7). The problem (2.8) can be solved analytically by using Laplace transform in time and separation of variables in space.

Let \( u = \mathcal{L}[U^*] \) be the Laplace transform of \( U^* \), defined by
\[
 u(x, s) = \mathcal{L}[U^*](x, s) = \int_0^\infty e^{-st}U^*(x, t) \, dt, \quad s \in \mathbb{C}, \quad \text{Re}(s) > 0, \quad x \in \Omega_{\text{ext}},
\]
and likewise for \( \psi = \mathcal{L}[U|_{r=b}] \). Applying Laplace transform to (2.8) leads to
\[
 -c^2 \Delta u + su = 0, \quad \text{in} \ \Omega_{\text{ext}}, \quad s \in \mathbb{C}, \quad \text{Re}(s) > 0;
\]
\[
 u|_{r=b} = \psi; \quad c\partial_r u + su = o(r^{1-\delta/2}), \quad d = 2, 3,
\]
which can be solved by separation of variables.

In the 3-D case, the equation (2.10) admits the series solution:
\[
 u(r, \theta, \phi, s) = \sum_{n=0}^\infty \frac{k_n(sr/c)}{k_n(sb/c)} \sum_{|m|=0}^n \hat{\psi}_{nm}(s) Y_n^m(\theta, \phi),
\]
where \((r, \theta, \phi) \in [b, \infty) \times [0, \pi] \times [0, 2\pi)\) is the spherical coordinates, and \(\{Y_n^m\}\) are the spherical harmonics, which are orthonormal as defined in the book [30]. In (2.11), \(k_n\) is the modified spherical Bessel function of the second kind, defined by
\[
k_n(z) = \sqrt{\frac{2}{\pi z}} K_{n+1/2}(z), \quad n \geq 0,
\]
where \(K_{n+1/2}\) is the modified Bessel function of the second kind (see, e.g., [1, 45]), and \(\hat{\psi}_{nm}\) are the spherical harmonic expansion coefficients of the Laplace transformed Dirichlet data in (2.10):
\[
 \mathcal{L}[U|_{r=b}] = \psi(\theta, \phi, s) = \sum_{n=0}^\infty \sum_{|m|=0}^n \hat{\psi}_{nm}(s) Y_n^m(\theta, \phi).
\]
Therefore, the Neumann data at \(\Gamma_b\) (in the Laplace transformed domain) can be obtained by
\[
 \frac{\partial u}{\partial r} \bigg|_{r=b} = \sum_{n=0}^\infty \frac{s}{c} \frac{k_n(sb/c)}{k_n(sr/c)} \sum_{|m|=0}^n \hat{\psi}_{nm}(s) Y_n^m(\theta, \phi).
\]
Applying the inverse Laplace transform on both sides of (2.14) leads to the time-domain DtN map:
\[
 \frac{\partial U^*}{\partial r} \bigg|_{r=b} = T_d(U) := -\frac{1}{c} \frac{\partial U}{\partial t} \bigg|_{r=b} + \frac{1}{c} \sum_{n=0}^\infty \sum_{|m|=0}^n \omega_n(t) * \partial_t \hat{U}_{nm}(b, t) Y_n^m(\theta, \phi),
\]
where \(\{\hat{U}_{nm}\}\) are the spherical harmonic expansion coefficients of \(U|_{r=b}\), “*” is the usual convolution operation, and
\[
 \omega_n(t) := \mathcal{L}^{-1} \left[ 1 + \frac{k_n(sb/c)}{k_n(sr/c)} \right] (t), \quad n \geq 0.
\]
By requiring \(\partial_r U = \partial_r U^*\) at \(r = b\), we derive the NRBC with \(d = 3\) in (2.7).
Remark 2.1. To derive (2.15) from (2.14), we used the property $\mathcal{L}^{-1}[\hat{s}\psi_{nm}] = \partial_t \hat{U}_{nm}(b, t)$, and added 1 to the ratio $k_n^2/b$ in $\omega_n(t)$ for the purpose of removing its singular part. Indeed, recall the asymptotic formula for fixed $\nu \geq 0$ and large $|z|$ (see Formula 9.7.2 of [1]):

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \left\{ 1 + \frac{4\nu^2 - 1}{8z} + O(z^{-2}) \right\},$$  \hspace{1cm} (2.17)

for $|\arg z| < 3\pi/2$, and the recurrence relation:

$$zK'_\nu(z) = \nu K_\nu(z) - zK_{\nu+1}(z).$$  \hspace{1cm} (2.18)

One verifies that

$$\frac{K'_\nu(z)}{K_\nu(z)} \sim -1 - \frac{1}{2z} + O(z^{-2}),$$  \hspace{1cm} (2.19)

and

$$\frac{k_n'(z)}{k_n(z)} = -\frac{1}{2z} + \frac{K'_{n+1/2}(z)}{K_{n+1/2}(z)} \sim -1 - \frac{1}{z} + O(z^{-2}).$$  \hspace{1cm} (2.20)

Therefore, $1 + k_n^2/b$ tends to zero as $|z| \to \infty$. □

Similarly, in the 2-D case, applying the method of separation of variables to solve (2.10) leads to the series solution outside $B$:

$$u(r, \phi, s) = \sum_{|n|=0}^\infty \frac{K_n(sb/c)}{K_n(sb/c)} \hat{\psi}_n(s)e^{in\phi},$$  \hspace{1cm} (2.21)

where $(r, \phi) \in [b, \infty) \times [0, 2\pi)$ is the polar coordinate, $K_n$ is the modified Bessel function of the second kind of order $n$ (cf. [1]), and $\{\hat{\psi}_n\}$ are the Fourier expansion coefficients of $\psi$ in (2.10). Thus, we can obtain the DtN map for $d = 2$:

$$T_d(U) = -\frac{1}{c} \left. \frac{\partial U}{\partial t} \right|_{r=b} + \frac{1}{c} \sum_{|n|=0}^\infty \omega_n(t) + \partial_t \hat{U}_n(b, t)e^{in\phi},$$  \hspace{1cm} (2.22)

where

$$\omega_n(t) := \mathcal{L}^{-1} \left[ 1 + \frac{K'_n(sb/c)}{K_n(sb/c)} \right](t).$$  \hspace{1cm} (2.23)

Remark 2.2. Since $K_{-n}(z) = K_n(z)$ (see Formula 9.6.6 in [1]), we have $\omega_{-n}(t) = \omega_n(t)$. Thus, for $d = 2$, it suffices to consider $\omega_n(t)$ with $n \geq 0$. □

It is seen that the time-domain DtN maps (2.15) and (2.22) involve the convolution of the temporal derivative of the Dirichlet data, and accordingly, the NRBC (2.7) corresponds to the DtN boundary condition of the Helmholtz equation in the frequency domain (see, e.g., [11, 30]). Alternatively, we can reformulate (2.15) and (2.22) as the convolution involving only the Dirichlet data on $\Gamma_b$ as in [21, 3]:

$$T_d(U) = \begin{cases} \left( -\frac{1}{c} \left. \frac{\partial U}{\partial t} \right|_{r=0} - \frac{U}{2r} \right)_{r=b} + \sum_{|n|=0}^{\infty} \sigma_n(t) * \hat{U}_n(b, t)e^{in\phi}, & d = 2, \\ \left( -\frac{1}{c} \left. \frac{\partial U}{\partial t} \right|_{r=0} - \frac{U}{r} \right)_{r=b} + \sum_{n=0}^{\infty} \sum_{|n|=0}^{n} \sigma_{n+1/2}(t) * \hat{U}_{nm}(b, t)Y_n^m(\theta, \phi), & d = 3, \end{cases}$$  \hspace{1cm} (2.24)

where for $d = 2$,

$$\sigma_n(t) := \mathcal{L}^{-1} \left[ \frac{s}{c} + \frac{1}{2b} + \frac{s K'_n(sb/c)}{c K_n(sb/c)} \right],$$  \hspace{1cm} (2.25)
and for \( d = 3 \),
\[
\sigma_{n+1/2}(t) := \mathcal{L}^{-1}\left\{ \frac{s}{c} + \frac{1}{b} + \frac{s K_n'(sb/c)}{c K_n(sb/c)} \right\} = \mathcal{L}^{-1}\left\{ \frac{s}{c} + \frac{1}{2b} + \frac{s K_n'(sb/c)}{c K_{n+1/2}(sb/c)} \right\}. \tag{2.26}
\]

Once again, in view of (2.19) and (2.20), we removed the singular part by subtracting the first two terms of the ratio \( sK'_n/(cK_n) \).

In fact, these two nonreflecting boundary kernels have the following relation.

**Lemma 2.1.** We have
\[
\omega_n(t) = \begin{cases} 
-\frac{c}{2b} + c \int_0^t \sigma_n(\tau)d\tau, & \text{for } d = 2, \\
-\frac{c}{b} + c \int_0^t \sigma_{n+1/2}(\tau)d\tau, & \text{for } d = 3.
\end{cases} \tag{2.27}
\]

**Proof.** We first consider the kernels for \( d = 2 \). It follows from (2.23), (2.25) and the properties of Laplace transform that
\[
\omega_n(t) = \mathcal{L}^{-1}\left\{ \frac{s}{c} + \frac{1}{2b} + \frac{s K_n'(sb/c)}{c K_n(sb/c)} \frac{c - \frac{c}{2bs}}{s} \right\} = \sigma_n(t) - \mathcal{L}^{-1}\left\{ \frac{c}{s} - \frac{c}{2b} \mathcal{L}^{-1}\left\{ \frac{1}{s} \right\} \right\}
\]

Similarly, we can derive (2.27) for \( d = 3 \). \( \square \)

**Remark 2.3.** We find that the use of the NRBC involving \( \omega_n \) is more convenient, if one reformulates (2.4) into a first-order (with respect to the time variable) system (cf. [6]), and it is also more suitable for analysis, while the latter involving \( \sigma_n \) is more appropriate for computation.

In the above, we derived exact boundary conditions for circular and spherical boundaries. They are in nature global in time and space, and involve the inverse Laplace transform of an expression including the logarithmic derivative of the modified Bessel function. To solve the truncated problem (2.4)-(2.7) efficiently, several tasks remain:

(i) We need to invert the Laplace transform to compute the NRBKs \( \sigma_n \) and \( \omega_n \).
(ii) We need to deal with the history dependence of the temporal convolution efficiently.
(iii) We need to handle the nonlocality of the NRBC in space effectively.

The rest of the paper will be centered around these issues.

### 2.2. Evaluation of the NRBKs

Our starting point is to invert the Laplace transform via evaluating the Bromwich’s contour integral:
\[
\sigma_\nu(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \left( \frac{s}{c} + \frac{1}{b} + \frac{s K_\nu(sb/c)}{c K_\nu(sb/c)} \right) e^{st} ds = \frac{c}{2b^2 \pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} F_\nu(z) e^{zt/b} dz, \tag{2.28}
\]

for \( \nu = n, n+1/2, n \geq 0 \), where
\[
F_\nu(z) = z + \frac{1}{2} + z \frac{K'_\nu(z)}{K_\nu(z)}, \tag{2.29}
\]

and \( \gamma \) is the Laplace convergence abscissa, which is a generic constant greater than the real part of any singularity of \( F_\nu(z) \).

In order to use the residue theorem to evaluate (2.28), we need to understand the behavior of the poles of \( F_\nu(z) \), i.e., the zeros of \( K_\nu(z) \).
Lemma 2.2. Let \( \nu \geq 0 \) be a real number.

(i) All zeros of \( K_\nu(z) \) lie in the second and third quadrants with \( \text{Re}(z) < 0 \), and appear in complex conjugate pairs.

(ii) The total number of zeros of \( K_\nu(z) \) is the even integer nearest to \( \nu - 1/2 \), if \( \nu - 1/2 \) is not an integer, or exactly \( \nu - 1/2 \), if \( \nu - 1/2 \) is an integer.

(iii) All zeros of \( K_n(z) \) and \( K_{n+1/2}(z) \) are simple, and lie approximately along the left half of the boundary of an eye-shaped domain around \( z = 0 \) (see Figure 2.1).

(iv) \( K_n(z) \) has no zero on the negative real axis, and \( K_{n+1/2}(z) \) has one zero on the negative real axis, only when \( n \) is odd.

Proof. The properties (i) and (ii) can be found from Page 511 of [45]. In particular, the property that the zeros appear in conjugate pairs follows from the identity \( K_\nu(z) = \bar{K}_\nu(z) \) for real \( \nu \) (see Formula 9.6.32 of [1]), where \( \bar{z} \) stands for the complex conjugate of \( z \).

We now turn to the property (iii). As a consequence of (i), it suffices to consider the zeros of \( K_\nu(z) \) in the third quadrant and on the negative real axis (i.e., with \( -\pi \leq \arg z < -\pi/2 \)) of the complex plane. According to Formula 9.6.4 of [1], we have the following relation between \( K_\nu(z) \) and the Hankel function of the first kind:

\[
K_\nu(z) = \frac{\pi i}{2} e^{\pi i \nu} H^{(1)}_\nu(iz), \quad -\pi < \arg z \leq \frac{\pi}{2}, \tag{2.30}
\]

which implies that all zeros of \( K_\nu(z) \) in the third quadrant (i.e., with \( -\pi < \arg z < -\pi/2 \)) are obtained by rotating all zeros of \( H^{(1)}_\nu(z) \) in the fourth quadrant (i.e., with \( -\pi/2 < \arg z < 0 \)) by an angle \(-\pi/2\). Recall that the zeros of \( H^{(1)}_\nu(z) \) in the forth quadrant lie approximately along the (left half) boundary of an eye-shaped domain around \( z = 0 \) (see Figure 9.6 and Page 441 of [1]), whose boundary curve intersects the real axis at \( z = n \) and the imaginary axis at the points \( z = -ina \), where \( a = \sqrt{t_0^2 - 1} \approx 0.66274 \) and \( t_0 \approx 1.19968 \) is the positive root of \( \coth t = t \).

Finally, we find from Page 511 of [45] that \( K_n(z) \) has no zero on the negative real axis, and from Page 441 of [1] that \( K_{n+1/2}(z) \) has one zero on the negative real axis, only when \( n \) is odd. \( \square \)

For convenience of presentation, let \( M_\nu \) be the total number of zeros of \( K_\nu(z) \) with \( \nu = n, n + 1/2 \), that is,

\[
M_\nu = \begin{cases} 
\text{the largest even integer nearest to } n - 1/2, & \text{for } K_n(z), \\
n, & \text{for } K_{n+1/2}(z). 
\end{cases} \tag{2.31}
\]

We arrange the zeros, denoted by \( \{z_j^{\nu}\}_{j=1}^{M_\nu} \), counterclockwise along the boundary curve of the eye-shaped domain from the second quadrant to the third quadrant. Thus, we have

\[
z_j^{\nu} = \bar{z}_{M_\nu - j + 1}^{\nu}, \quad 1 \leq j \leq M_\nu; \quad \text{Re}(z_{j+1}^{\nu}) < \text{Re}(z_j^{\nu}) < 0, \quad 1 \leq j \leq \lfloor M_\nu/2 \rfloor - 1. \tag{2.32}
\]

We plot in Figure 2.1 some samples of zeros of \( K_n(z) \) (left) and \( K_{n+1/2}(z) \) for various \( n \), and visualize that the zeros are located symmetrically in the second and third quadrants, and sitting on the left half boundary of an eye-shaped domain as predicted in Lemma 2.2 (iii).

To have more insights on the distribution of the zeros, we quote the following asymptotic estimates, which can be obtained from the relation (2.30) and the results on the zeros of the Hankel function in [12].
Distributions of zeros \( \{ z_j^\nu \}_{j=1}^{M_\nu} \) of \( K_\alpha(z) \) (left) and \( K_{\nu+1/2}(z) \) (right) for various \( n \). For a given \( n \), the zeros sit on the left half boundary of an eye-shaped domain that intersects the imaginary axis approximately at \( \pm ni \), and the negative real axis at \(-na\) with \( a \approx 0.66274 \) (see the dashed coordinate grids).

**Lemma 2.3.** For large real \( \nu \), the zeros \( \{ z_j^\nu \} \) of \( K_\nu(z) \) in the second quadrant have the asymptotic estimate:

\[
z_j^\nu = \tilde{z}_j^\nu + O(\nu^{-1}),
\]

where

\[
z_j^\nu = \left\{ \frac{\sqrt{3}}{2} \left( \frac{\nu}{\pi} \right)^{1/3} a_j + \frac{3\sqrt{3}}{20} \left( \frac{1}{4\nu} \right)^{1/3} a_j^2 \right\} + i \left\{ \left[ \nu \right]^{1/3} a_j - \frac{1}{2} \left( \frac{1}{4\nu} \right)^{1/3} 3a_j^2 \right\},
\]

for fixed \( j \geq 1 \), where \( a_j \) is the \( j \)th negative zero of the Airy function with the bounds (cf. [22]):

\[
-\left\{ \frac{3\pi}{8} (4j - 1) + \frac{3}{2} \arctan \frac{5}{18\pi(4j - 1)} \right\}^{2/3} \leq a_j \leq -\left\{ \frac{3\pi}{8} (4j - 1) \right\}^{2/3}.
\]

**Remark 2.4.** As a consequence of (2.33)-(2.34), we find \( a_1 \approx -2.329 \) and

\[
z_1^\nu \approx -1.601\nu^{1/3} + 0.888\nu^{-1/3} + i \left\{ \nu - 0.924\nu^{1/3} - 0.513\nu^{-1/3} \right\}.
\]

In Figure 2.2, we plot the errors \( E_1^\nu = |z_1^\nu - \tilde{z}_1^\nu| \) and \( E_{J_0}^\nu = |z_{J_0}^\nu - \tilde{z}_{J_0}^\nu| \) (\( \nu = n \) on the left and \( \nu = n + 1/2 \) on the right), where \( z_{J_0}^\nu \) is the zero nearest to \(-na/2\) in the second quadrant. It is seen that the asymptotic estimate (2.33) provides a fairly good approximation for \( j \) much less than \( n \).

With the above understanding of the poles of the integrand \( F_\nu(z) \) of the Brownwich’s contour integral (2.29), we now present the exact formula for the NRBKs \( \sigma_\nu(t) \) with \( \nu = n, n + 1/2 \).

**Theorem 2.1.** Let \( \nu = n, n + 1/2 \) with \( n \geq 0 \), and let \( \{ z_j^\nu \}_{j=1}^{M_\nu} \) be the zeros of \( K_\nu(z) \). Then

- for \( d = 2 \),

\[
\sigma_n(t) = \frac{c}{b^2} \left\{ \sum_{j=1}^{M_\nu} z_j^\nu e^{ctz_j^\nu/b} + (-1)^n \int_{0}^{+\infty} \frac{e^{-ctr/b}}{K_\nu^2(r) + \pi^2 F_\nu^2(r)} dr \right\},
\]

where

\[
F_\nu(z) = \frac{d_\nu}{2} \left\{ \frac{\nu}{\pi} \right\}^{1/2} \left( \frac{\nu}{\pi} \right)^{1/3} a_j + \frac{3\sqrt{3}}{20} \left( \frac{1}{4\nu} \right)^{1/3} a_j^2 \right\}.
\]
Figure 2.2. Zeros against their asymptotic estimates: $E_1^n, E^n_0$ (left) and $E_1^{n+1/2}, E_1^{n+1/2}$ (right) for various $n \in \{6, 110\}$.

bullet for $d = 3$,

$$
\sigma_\nu(t) = \frac{c}{b^2} \sum_{j=1}^{M_\nu} \Re(z_\nu^j) e^{c t z_\nu^j / b}, \quad \nu = n + \frac{1}{2},
$$

where $I_n(z)$ is the modified Bessel function of the first kind (cf. [1]).

We sketch the proof of this theorem in Appendix A by applying the residue theorem to the Bromwich’s contour integral (2.28). We remark that Sofronov [40] derived similar formulas by using the residue theorem on a much more complicated expression of the integrand in terms of Tricomi’s confluent hypergeometric functions. However, the formulas in the above theorem are more compact and informative.

Observe that the three-dimensional NRBK $\sigma_{n+1/2}$ is purely a summation of exponentials in $t$, as the integrand $F_{n+1/2}$ in (2.29) turns out to be a rational function of $s$. However, in the two-dimensional case, $F_n$ is not a rational function, so the exact expression (2.35) includes an additional improper integral, which counts the contribution from the branch-cut along the negative real axis (see the proof in Appendix A). As commented in [28], the compression of $F_\nu$ via rational approximation (i.e., with a much lower order than $\nu$, see [2, 27]) is workable for the 3-D case, but this may not be the best choice for the 2-D case.

Remark 2.5. Based on a delicate study of the logarithmic derivative of the Hankel function $H^{(1)}_\nu(z)$, Alpert et al. [2] (see Theorem 4.1 and Lemma 4.2 in [2]) obtained the following formula:

$$
\frac{\partial}{\partial z} \frac{H^{(1)}_\nu(z)}{H^{(1)}_\nu(z)} = iz - \frac{1}{2} + \sum_{j=1}^{N_\nu} \frac{h_{\nu,j}}{z - h_{\nu,j}} - \frac{1}{\pi} \int_0^\infty \frac{\pi \cos(\nu \pi)}{\cos^2(\nu \pi) K^{2}_{\nu}(r) + (\pi I_{\nu}(r) + \sin(\nu \pi) K_{\nu}(r))^2} \frac{1}{ir + z} dr,
$$

for any $\nu \neq n + 1/2$, where $h_{\nu,1}, h_{\nu,2}, \ldots, h_{\nu,N_\nu}$ are the zeros of $H^{(1)}_\nu(z)$, which number $N_\nu$. Interestingly, the formula (2.35) can be derived from (2.37), which is justified in Appendix B. □
Next, we present some mathematical consequences of Theorem 2.1. Since the zeros \( \{z_j^\nu\}_{j=1}^{M_\nu} \) appear in complex conjugate pairs (cf. Lemma 2.2), we can express \( \sigma_\nu(t) \) in terms of the zeros in the second quadrant.

**Corollary 2.1.** Let \( \{z_j^\nu = x_j^\nu + iy_j^\nu\}_{j=1}^{[M_\nu/2]} \) be the zeros of \( K_\nu(z) \) in the second quadrant (i.e., \( x_j^\nu < 0 \) and \( y_j^\nu > 0 \)). Then we have

- for \( d = 2 \),
  \[
  \sigma_\nu(t) = \frac{c}{b^2} \left\{ 2 \sum_{j=1}^{[M_\nu/2]} e^{jtx_j^\nu} \left\{ x_j^n \cos(\beta ty_j^n) - y_j^n \sin(\beta ty_j^n) \right\} + \int_0^\infty \frac{(-1)^n e^{-\nu r} dr}{K_n^2(r) + \pi^2 I_n^2(r)} \right\},
  \]
- for \( d = 3 \),
  \[
  \sigma_\nu(t) = \alpha(t) + \frac{2c}{b^3} \sum_{j=1}^{[M_\nu/2]} e^{jtx_j^\nu} \left\{ x_j^n \cos(\beta ty_j^n) - y_j^n \sin(\beta ty_j^n) \right\}, \quad \nu = n + \frac{1}{2},
  \]

where \( \beta = c/b \), and \( \alpha(t) = 0 \), if \( n = 2k \), and \( \alpha(t) = \frac{c x}{b} e^{cx^t/b} \), if \( n = 2k + 1 \) (where \( x \) is the zero of \( K_{n+1/2}(z) \) located at the negative real axis).

We see that \( \sigma_\nu \) is real-valued. Thanks to Lemma 2.1, we obtain from Theorem 2.1 the following exact expressions of the NRNKs \( \omega_n(t) \) in (2.16) and (2.23).

**Corollary 2.2.** We have

- for \( d = 2 \),
  \[
  \omega_n(t) = -\frac{c}{2b} + \frac{c}{b} \left\{ \sum_{j=1}^{M_\nu} (e^{jtx_j^\nu/b} - 1) + (-1)^n \int_0^\infty \frac{1 - e^{-\nu r/b} dr}{r\{K_n^2(r) + \pi^2 I_n^2(r)\}} \right\}, \quad \nu = n + \frac{1}{2},
  \]
- for \( d = 3 \),
  \[
  \omega_n(t) = \frac{c}{b} \sum_{j=1}^{M_\nu} (e^{jtx_j^\nu/b} - 1), \quad \nu = n + \frac{1}{2}.
  \]

Moreover, like Corollary 2.1, we can express \( \omega_n(t) \) in terms of zeros in the second quadrant.

### 2.3. Computation of the improper integral in (2.35)

The computation of the two-dimensional NRBK requires to evaluate the improper integral involving the kernel function:

\[
W_n(r) := \frac{1}{K_n^2(r) + \pi^2 I_n^2(r)} := \frac{1}{G_n(r)}, \quad n \geq 0, \quad r > 0,
\]

whose important properties are characterized below.

**Lemma 2.4.** For any \( n \geq 0 \) and any real \( r > 0 \), we have

(i) \( G_n(r) \) is a convex function of \( r \), and \( W_n(r) \) attains its maximum value at a unique point.

(ii) For large \( n \), we have the uniform asymptotic estimate:

\[
W_n(n\kappa) \sim \frac{n\sqrt{1 + \kappa^2}}{\pi} \text{sech}(2n\Theta) := \widetilde{W}_n(n\kappa), \quad \kappa > 0,
\]

where

\[
\Theta = \Theta(\kappa) := \sqrt{1 + \kappa^2} + \ln \frac{\kappa}{1 + \sqrt{1 + \kappa^2}}.
\]
Approximately, the maximum value of $W_n(r)$ attains at $r = na$ with $a \approx 0.66274$ being the root of $\Theta$, and the maximum value is approximately $n\sqrt{1+a^2}/\pi \approx 0.38187n$.

**Proof.** (i). We find from Page 374 of [1] that for a given $n$, $K_n(r), I_n(r) > 0$, and $K_n(r)$ (resp. $I_n(r)$) is monotonically descending (resp. ascending) with respect to $r$. From the series representation (see Formula 9.6.10 of [1]):

$$I_n(r) = \frac{r^n}{2n} \sum_{k=0}^{\infty} \frac{r^{2k}}{2^{2k} k!(n+k)!}, \quad n \geq 0,$$

we conclude that $I''_n(r) > 0$. Moreover, since $K_n(r)$ satisfies (see Formula 9.6.1 of [1])

$$r^2 K''_n(r) + r K'_n(r) - (r^2 + n^2) K_n(r) = 0,$$

we have $K''_n(r) > 0$. Therefore, a direct calculation shows that $G''_n(r) > 0$, so $G_n(r)$ is a convex function of $r$.

One verifies that $G_n(0+) = G_n(+\infty) = +\infty$ for all $n$, which follows from (2.17) and the following asymptotic properties (see [1] again):

$$K_{\nu}(r) \sim \begin{cases} -\ln r, & \text{if } \nu = 0, \\ \Gamma(\nu) (\frac{r}{2})^{-\nu}, & \text{if } \nu > 0, \end{cases} \quad I_{\nu}(r) \sim \frac{1}{\Gamma(\nu+1)} (\frac{r}{2})^{\nu}, \quad \nu \geq 0,$$

for $0 < r \ll 1$, and

$$I_{\nu}(r) \sim \sqrt{\frac{1}{2\pi r}} e^r, \quad r \gg 1, \quad \nu \geq 0.$$  \tag{2.44}

Since $G_n(r)$ is convex, $G_n(r)$ attains its minimum value at a unique point $r_0$. Thanks to the fact $G''_n(r) = -W'_n(r)/W^2_n(r)$, we conclude that $W_n(r)$ has a unique maximum value at the same point $r_0$.

(ii). Recall the uniform asymptotic formulas for large orders (see Formulas (9.7.7)-(9.7.11) of [1]):

$$K_n(n\kappa) \sim \sqrt{\frac{\pi}{2n}} e^{-n\Theta}, \quad I_n(n\kappa) \sim \frac{1}{\sqrt{2\pi n}} (1 + \kappa^2)^{1/4} e^{n\Theta},$$

$$K'_n(n\kappa) \sim -\sqrt{\frac{\pi}{2n}} \frac{1}{\kappa} e^{-n\Theta}, \quad I'_n(n\kappa) \sim \frac{1}{\sqrt{2\pi n}} (1 + \kappa^2)^{1/4} \frac{e^{n\Theta}}{\kappa},$$

which, together with (2.40), leads to the asymptotic estimate (2.41). Thus, the maximum value of $W_n(r)$ approximately attains at the unique root of $\Theta(\kappa)$, which turns out to be $a \approx 0.66274$ as in Figure 2.1 (see the caption), and the maximum value is about $n\sqrt{1+a^2}/\pi \approx 0.38187n$. \hfill \Box

**Remark 2.6.** Notice that $a$, the unique root of $\Theta$, coincides with the constant in the proof of Lemma 2.2, where $a = \sqrt{t_0^2 - 1}$ and $t_0$ is the positive root of $\coth t = t$. Indeed, since

$$\coth \sqrt{1+a^2} = \frac{e^{2\sqrt{1+a^2}} + 1}{e^{2\sqrt{1+a^2}} - 1} = \sqrt{1+a^2},$$

we have

$$\sqrt{1+a^2} = \frac{1}{2} \ln \frac{\sqrt{1+a^2} + 1}{\sqrt{1+a^2} - 1} = -\ln \frac{a}{1 + \sqrt{1+a^2}},$$

which means $a$ is a root of $\Theta$. \hfill \Box
We depict in Figure 2.3 (left) the graph of $\Theta(\kappa)$, and highlight the zero point $(a, 0)$. Observe that $\Theta(\kappa)$ grows like $\kappa$, when $\kappa > a$. We also plot in Figure 2.3 (right) several sample graphs of $W_n$ (solid lines) and the asymptotic estimate $\tilde W_n$ (‘•’) for $n = 5, 15, 30, 45$, and particularly mark the asymptotic maximum point and the maximum value $(na, n\sqrt{1 + a^2}/\pi)$ of $W_n(r)$ obtained in Lemma 2.4 (ii). Observe that even for small $n$, the asymptotic estimate provides a very accurate approximation of $W_n$.

Those properties presented in Lemma 2.4 greatly facilitate the computation of the improper integral in (2.35). We truncate $(0, \infty)$ to a narrow interval $[L_1, L_2] := [(a - \delta_1)n, (a + \delta_2)n]$ around the maximum point $r = na$:

$$
\int_0^\infty e^{-\beta r} W_n(r) \, dr \approx \int_{L_1}^{L_2} e^{-\beta r} W_n(r) \, dr,
$$

(2.46)

where $\delta_1$ and $\delta_2$ are chosen such that for a preassigned tolerance $\varepsilon > 0$,

$$
\delta_1 = \sup \tilde W_n((a - \delta)n) < \varepsilon, \quad \delta_2 = \inf \tilde W_n((a + \delta)n) < \varepsilon.
$$

In Table 2.1, we list some values of $\delta_1$ and $\delta_2$ with $\varepsilon = 10^{-13}$ for various $n$, and see that the length of the interval of interest is about 18.

**Table 2.1.** Truncation of the improper integral.

| $n$ | $\delta_1$ | $\delta_2$ | $L_2 - L_1$ | $W_n(L_1)$ | $W_n(L_1)$ | $W_n(L_2)$ | $W_n(L_2)$ |
|-----|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 10  | 0.51        | 1.22        | 17.4        | 9.9885e-14  | 9.9998e-14  | 9.9261e-14  | 9.9998e-14  |
| 20  | 0.34        | 0.55        | 17.9        | 9.9897e-14  | 9.9996e-14  | 9.9709e-14  | 9.9996e-14  |
| 30  | 0.25        | 0.36        | 18.2        | 9.9847e-14  | 9.9988e-14  | 9.9947e-14  | 9.9995e-14  |
| 40  | 0.20        | 0.25        | 18.4        | 9.9894e-14  | 9.9968e-14  | 9.9955e-14  | 9.9993e-14  |
| 50  | 0.17        | 0.21        | 18.5        | 9.9998e-14  | 9.9850e-14  | 9.9953e-14  | 9.9997e-14  |
2.4. Rapid evaluation of the temporal convolution. Remarkably, the presence of the time variable \( t \) in the exponentials in (2.35) and (2.36) allows us to eliminate the burden of history dependence of the global temporal convolution easily. More precisely, given a function \( g(t) \), we define

\[
 f(t; r) := e^{-ctr/b} * g(t) = \int_0^t e^{-c(t-\tau)r/b}g(\tau)d\tau.
\]

One verifies readily that

\[
 f(t + \Delta t; r) = e^{-c\Delta tr/b}f(t; r) + \int_t^{t+\Delta t} e^{-c(t+\Delta t-\tau)r/b}g(\tau)d\tau,
\]

so \( f(t; r) \) can march in \( t \) with step size \( \Delta t \) recursively. This enables us to compute the time convolution rapidly. For example, in the 2-D case,

\[
 [\sigma_n * g](t) = \int_0^t \sigma_n(t-\tau)g(\tau)d\tau = \frac{c}{b^2} \sum_{j=1}^{M_n} z_j^n \int_0^t e^{c(t-\tau)z_j^n/b}g(\tau)d\tau
 + \frac{(-1)^n c}{b^2} \int_0^\infty \frac{1}{K_0^2(r)} \int_0^t e^{-c(t-\tau)r/b} g(\tau)d\tau dr
 = \frac{c}{b^2} \sum_{j=1}^{M_n} z_j^n f(t; -z_j^n) + \frac{(-1)^n c}{b^2} \int_0^+ \int_0^+ f(t; r) W_n(r) dr.
\]

Thanks to (2.47), \([\sigma_n * g](t + \Delta t)\) can be computed recursively from the previous step and the history dependence is then narrowed down to \([t, t + \Delta t]\). We also refer to Subsection 4.4 for more detailed discussions.

3. A priori estimates

In this section, we analyze the well-posedness of the truncated problem (2.4)-(2.7), and provide a priori estimates for its solution. These results will be useful for the convergence analysis in the forthcoming section.

We first recall the Plancherel or Parseval results for the Laplace transform.

**Lemma 3.1.** Let \( s = s_1 + is_2 \) with \( s_1, s_2 \in \mathbb{R} \). If \( f, g \) are Laplace transformable, then

\[
 \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathcal{L}[f](s) \mathcal{L}[g^\star](s) ds_2 = \int_{0}^{+\infty} e^{-2s_1 t} f(t) \bar{g}(t) dt, \quad \forall s_1 > \gamma,
\]

where \( \gamma \) is the abscissa of convergence for both \( f \) and \( g \), and \( \bar{g} \) is the complex conjugate of \( g \).

**Proof.** This identity can be proved by following the same lines as for Formula (2.46) in [8] (also see [13]). \( \square \)

The following properties are also indispensable for the analysis.

**Lemma 3.2.** Let \( s = s_1 + is_2 \) with \( s_1, s_2 \in \mathbb{R} \). Then we have

\[
 \text{Re} \left( \frac{Z_n'(sb/c)}{Z_n(sb/c)} \right) \leq 0, \quad \text{Re} \left( \frac{Z_n''(sb/c)}{Z_n(sb/c)} \right) \leq 0, \quad \forall s_1 > 0,
\]

and

\[
 \text{Im} \left( \frac{Z_n'(sb/c)}{Z_n(sb/c)} \right) \leq 0, \quad \forall s_1 > 0, \ s_2 \geq 0,
\]

where \( Z_n(z) = K_n(z) \) or \( k_n(z) \).
Proof. The results with $Z_n = K_n$ were proved in Chen [6]. We next prove (3.2) and (3.3) with $Z_n = k_n$ by using a similar argument. Multiplying the first equation of (2.10) by $\bar{u}$ and integrating over $\Omega_{b,\rho} := B_\rho \setminus B_\rho$, where $B_\rho$ is a ball of radius $\rho > b$, the imaginary part of the resulting equation reads

$$2s_1s_2 \int_{\Omega_{b,\rho}} |u|^2 d{x^2} - e^2 \text{Im} \int_{\partial \Omega_{b,\rho}} \frac{\partial u}{\partial n} \bar{u} d\gamma = 0,$$

where $n$ is the unit normal of $\partial \Omega_{b,\rho}$. Since $s_1 > 0$, multiplying (3.4) by $s_2$ yields

$$s_2 \text{Im} \int_{(r=b)} \frac{\partial u}{\partial r} \bar{u} d\gamma \leq s_2 \text{Im} \int_{(r=\rho)} \frac{\partial u}{\partial r} \bar{u} d\gamma. \quad (3.5)$$

It is clear that $u = k_n(sr/c)/k_n(sb/c)\tilde{\psi}_{nm}(s)Y_n^m(\theta, \phi)$ (where $\tilde{\psi}_{nm}$ is the same as in (2.11)) is a solution of (2.10), so using the orthogonality of $\{Y_n^m\}$, we obtain from (3.5) that

$$-\frac{b}{c} \text{Im}\left(s_2s_n\frac{k_n'(sb/c)}{k_n(sb/c)}\right)|\tilde{\psi}_{nm}|^2 \leq \frac{\rho}{c} \text{Im}\left(s_2s_n\frac{k_n'(sp/c)}{k_n(sb/c)}\right)|\tilde{\psi}_{nm}|^2. \quad (3.6)$$

We find from (2.12) and (2.17) that $k_n'(sp/c)$ decays exponentially if $s_1 > 0$, so the right hand side of (3.5) tends to zero as $\rho \to +\infty$. Thus, letting $\rho \to +\infty$ in (3.6) leads to

$$\text{Im}\left(s_2s_n\frac{k_n'(sb/c)}{k_n(sb/c)}\right) \leq 0. \quad (3.7)$$

If $s_2 \geq 0$, then we obtain (3.3). Next, we prove the first inequality of (3.2). Recall the formula (see Lemma 2.3 of [6]):

$$|K_{n+1/2}(sr)|^2 = \frac{1}{2} \int_0^{+\infty} e^{-\frac{|s|^2 r^2}{2} - \frac{r^2}{2} \frac{\gamma_1^2 + i \gamma_2^2}{|s|^2}} K_{n+1/2}(\gamma) d\gamma, \quad s_1 > 0,$$

which implies that $|K_{n+1/2}(sr)|^2$ is monotonically descending with respect to $r$. The property $\frac{d}{dr}|K_{n+1/2}(sr)|^2 \leq 0$, together with (2.20), implies $\text{Re}\left(s_2s_n\frac{k_n'(sb/c)}{k_n(sb/c)}\right) \leq 0$. Denoting $s_n\frac{k_n'(sb/c)}{k_n(sb/c)} = \gamma_1 + i \gamma_2$ with $\gamma_1, \gamma_2 \in \mathbb{R}$, we know from (3.7) that $\gamma_1 \leq 0$ and $s_2 \gamma_2 \leq 0$. Therefore,

$$\text{Re}\left(s_n\frac{k_n'(sb/c)}{k_n(sb/c)}\right) = \text{Re}\left(\frac{\gamma_1 + i \gamma_2}{s}\right) = \frac{1}{|s|^2} (s_1 \gamma_1 + s_2 \gamma_2) \leq 0.$$

This ends the proof. \qed

With the above preparations, we can derive the following important property.

Theorem 3.1. For any $v \in L^2(0, T)$, we have

$$\int_0^T |\omega_n * v|(t)\tilde{v}(t)dt \leq \int_0^T |v(t)|^2 dt, \quad \forall T > 0, \quad n \geq 0, \quad (3.8)$$

where $\omega_n$ is the NRBK given by (2.16) for $d = 3$ and (2.23) for $d = 2$, respectively.

Proof. Let $\tilde{v} = v \mathbb{1}_{[0, T]}$, where $\mathbb{1}_{[0, T]}$ is the characteristic function of $[0, T]$. Then we obtain from (3.1) that for $d = 3$,

$$\int_0^T e^{-2s_1 t} |\omega_n * v|(t)\tilde{v}(t)dt = \int_0^{+\infty} e^{-2s_1 t} |\omega_n * \tilde{v}|(t)\tilde{v}(t)dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[\frac{k_n'(sb/c)}{k_n(sb/c)} + 1\right] |\mathcal{L}[\tilde{v}](s)|^2 ds_2$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} k_n'(sb/c) |\mathcal{L}[\tilde{v}](s)|^2 ds_2 + \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\mathcal{L}[\tilde{v}](s)|^2 ds_2.$$
It is clear that by (3.1) with \( f = g = \tilde{v} \),
\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} |L[\tilde{v}](s)|^2 ds = \int_0^T e^{-2s_1^* t} |\tilde{v}(t)|^2 dt = \int_0^T e^{-2s_1^* t} |v(t)|^2 dt.
\]
Using (2.18) and the properties \( K_\nu(z) = K_\nu(\bar{z}) \) and \( L[\tilde{v}](\bar{s}) = L[\tilde{v}](s) \), we find
\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} k_n'(s)(sb/c) |L[\tilde{v}](s)|^2 ds = \frac{1}{\pi} \int_0^{\infty} \Re \left( \frac{k_n'(sb/c)}{k_n(sb/c)} \right) |L[\tilde{v}](s)|^2 ds
\]
\[
= \frac{1}{\pi} \int_0^{\infty} \Re \left( \frac{k_n'(sb/c)}{k_n(sb/c)} \right) s^2 |L[\tilde{v}](s)|^2 ds.
\]
Thus, we conclude from Lemma 3.2 and the above identities that for \( s_1 > 0 \),
\[
\int_0^T e^{-2s_1^* t} |\omega_n \ast v(t)|^2 dt \leq \int_0^T e^{-2s_2^* t} |v(t)|^2 dt.
\]
Notice that the asymptotic formulas in (2.43) are also valid for complex \( r \) (see Formula 9.6.9 of [1]), which, together with Lemma 2.2, implies that \( k_n(sb/c) \) is analytic for all \( \Re(s) \geq 0 \) but \( |s| \neq 0 \), and \( \lim_{s \to 0+} |s|^\nu k_n(sb/c) \) exists for all \( s_2 \geq 0 \). Hence, the integral in (3.9) is finite. Finally, letting \( s_1 \to 0^+ \) in (3.10) leads to the desired result for \( d = 3 \).

The result (3.8) with \( d = 2 \) can be proved in a similar fashion.

A consequence of Theorem 3.1 and Lemma 2.1 is as follows.

**Corollary 3.1.** Suppose that \( v' \in L^2(0,T) \) with \( v(0) = 0 \). Then we have
\[
\int_0^T [\sigma_\nu \ast v(t)]v'(t)dt \leq \frac{1}{c} \int_0^T |v'(t)|^2 dt + \frac{d - 1}{4b} |v(T)|^2,
\]
for \( \nu = n, n + 1/2 \), where \( \sigma_\nu(t) \) is the NRBK defined in (2.25) and (2.26) for \( d = 2, 3 \), respectively.

**Proof.** By Lemma 2.1, we have
\[
\omega_n'(t) = c\sigma_\nu(t), \quad \omega_n(0) = -\frac{(d-1)c}{2b}.
\]
Thus, we obtain from integration by parts and the fact \( v(0) = 0 \) that
\[
[\omega_n \ast v'](t) = -\frac{(d-1)c}{2b} v(t) + c[\sigma_\nu \ast v](t).
\]
By Theorem 3.1 with \( v' \) in place of \( v \),
\[
\int_0^T [\omega_n \ast v'](t)v'(t)dt = -\frac{(d-1)c}{2b} \int_0^T v(t)v'(t)dt + c \int_0^T [\sigma_\nu \ast v](t)v'(t)dt
\]
\[
= -\frac{(d-1)c}{4b} |v(T)|^2 + c \int_0^T [\sigma_\nu \ast v](t)v'(t)dt \leq \int_0^T |v'(t)|^2 dt.
\]
This gives (3.11). \( \square \)

Now, we are ready to analyze the stability of the solution of the truncated problem (2.4)-(2.7). To this end, we assume that the scatter \( D \) is a simply connected domain with Lipschitz boundary \( \Gamma_D \), and the Dirichlet data \( G = 0 \) on \( \Gamma_D \). Denote \( X := \{ U \in H^1(\Omega) : U|_{\Gamma_D} = 0 \} \), and let \( (\cdot, \cdot)_{L^2(\Omega)} \) and \( || \cdot ||_{L^2(\Omega)} \) be the inner product and norm of \( L^2(\Omega) \), respectively.
Let $U$ be the solution of (2.4)-(2.7) with $G = 0$. If $U_0 \in H^1(\Omega)$, $U_1 \in L^2(\Omega)$ and $F \in L^1(0,T;L^2(\Omega))$ for any $T > 0$, then we have $\nabla U \in L^\infty(0,T;H^1(\Omega))$, $\partial_t U \in L^\infty(0,T;L^2(\Omega))$, and there holds
\[
\|\partial_t U\|_{L^\infty(0,T;L^2(\Omega))} + c\|\nabla U\|_{L^\infty(0,T;L^2(\Omega))} \leq C(\|U_1\|_{L^2(\Omega)} + c\|\nabla U_0\|_{L^2(\Omega)} + \|F\|_{L^1(0,T;L^2(\Omega))}),
\] (3.12)
where $C$ is a positive constant independent of any functions and $c,b$.

**Proof.** Multiplying (2.4) by $2\partial_t U$ and integrating over $\Omega$, we derive from the Green’s formula that for any $t > 0$,
\[
\frac{d}{dt}\left(\|\partial_t U\|_{L^2(\Omega)}^2 + c^2\|\nabla U\|_{L^2(\Omega)}^2\right) - 2c^2 \int_{\Gamma_n} T_d(U)\partial_t U\,d\gamma = 2(F,\partial_t U)_{L^2(\Omega)}. \tag{3.13}
\]
Integrating the above equation over $(0,t)$, we find that for any $t > 0$,
\[
\begin{align*}
\|\partial_t U\|_{L^2(\Omega)}^2 + c^2\|\nabla U\|_{L^2(\Omega)}^2 - 2c^2 & \int_0^t \int_{\Gamma_n} T_d(U)\partial_t U\,d\gamma\,d\tau \\
& \leq 2\int_0^t (F,\partial_t U)_{L^2(\Omega)}\,d\tau + \|U_1\|_{L^2(\Omega)}^2 + c^2\|\nabla U_0\|_{L^2(\Omega)}^2 \\
& \leq 2\|\partial_t U\|_{L^\infty(0,T;L^2(\Omega))}\|F\|_{L^1(0,T;L^2(\Omega))} + \|U_1\|_{L^2(\Omega)}^2 + c^2\|\nabla U_0\|_{L^2(\Omega)}^2.
\end{align*}
\] (3.14)
We next show that for any $t > 0$,
\[
\int_0^t \int_{\Gamma_n} T_d(U)\partial_t U\,d\gamma\,d\tau \leq 0. \tag{3.15}
\]
For $d = 3$, it follows from (2.15), Theorem 3.1 and the orthogonality of $\{Y_n^m\}$ that
\[
\begin{align*}
\int_0^t \int_{\Gamma_n} T_d(U)\partial_t U\,d\gamma\,d\tau & = -\frac{1}{c} \int_0^t \|\partial_t U\|_{L^2(\Gamma_n)}^2\,d\tau \\
& \quad + \frac{1}{c} \sum_{n=0}^\infty \sum_{|m|=0}^n \int_0^t \left[\omega_n + \partial_t \hat{U}_{nm}(b,\tau)\right] \partial_t \bar{U}_{nm}(b,\tau)\,d\tau \\
& \quad + \sum_{n=0}^\infty \sum_{|m|=0}^n \int_0^t \left|\partial_t \hat{U}_{nm}(b,\tau)\right|^2\,d\tau.
\end{align*}
\] (3.8)
This verifies (3.15) with $d = 3$. Similarly, one can justify (3.15) with $d = 2$.

Consequently, the estimate (3.12) follows from (3.14), (3.15) and the Cauchy-Schwarz inequality. \hfill \square

**Remark 3.1.** Several remarks are in order. (i) If the DtN map $T_d(U)$ is given by (2.24), we can use Corollary 3.1 to verify (3.15). (ii) A similar estimate for $d = 2$ was derived by [6] with a slightly different argument. (iii) The estimate (3.12) indicates that the solution of the truncated problem does not grow in time. Moreover, if the source term $F \equiv 0$ in (2.1) and (2.4), we find from (3.13)-(3.15) the conservation of the energy:
\[
\frac{d}{dt} E(t) = 0, \quad t > 0; \quad E(0) = E_0 \quad \Rightarrow \quad E(t) = E_0, \quad \forall t \geq 0. \tag{3.16}
\]
where \( E_0 = \int_\Omega \left( |U_1|^2 + c^2 |\nabla U_0|^2 \right) dx \), and
\[
E(t) = \int_\Omega \left( |\partial_t U|^2 + c^2 |\nabla U|^2 \right) dx - 2c^2 \int_0^t \int_{\Gamma_b} T_d(U) \partial_t U d\gamma d\tau. \]

4. Spectral-Galerkin approximation and Newmark’s time integration

This section is devoted to numerical approximation of the truncated problem (2.4)-(2.7) with a focus on the treatment for the exact NRBC (2.7). Note that if the scatter is a disk/ball (i.e., \( \Omega \) in (2.4)-(2.7) is an annulus/spherical shell), the nonlocality of the NRBC in space becomes local in the space of Fourier/spherical harmonic coefficients in polar/spherical coordinates. Correspondingly, the truncated problem can be reduced to a sequence of one-dimensional problems with mixed boundary conditions at the outer artificial boundary. In order to deal with the general scatter, we resort to the transformed field expansion method (TFE) (cf. [32]), which has been successfully applied to time-harmonic Helmholtz equations (cf. [33, 15, 34]). The use of this method allows us to solve a sequence of truncated problems (2.4)-(2.7) in regular domains. Accordingly, it is essential to construct fast and accuracy solvers for (2.4)-(2.7) in an annulus or a spherical shell, which is the theme of this section. We shall report the combination with the TFE method in a future work as the implementation is much involved.

4.1. Notion of TFE. We outline the TFE method for (2.4)-(2.7) with \( d = 2 \) and
\[
\Gamma_D = \{ r = b_0 + \eta(\phi) : 0 \leq \phi < 2\pi \}, \quad b > b_0 + \max_{\phi \in [0,2\pi]} |\eta(\phi)|,
\]
where \( r(\phi) = b_0 + \eta(\phi) \) is the parametric equation of the boundary of the scatter \( D \).

- Make a change of variables
\[
r' = \frac{(b - b_0)r - b\eta(\phi)}{(b - b_0) - \eta(\phi)}, \quad \phi' = \phi,
\]
which maps \( \Omega \) to the annulus \( \Omega_0 = \{ (r', \phi') : b_0 < r' < b, \ 0 \leq \phi' < 2\pi \} \). To simplify the notation, we still use \( U, F, r, \phi \) etc. to denote the transformed functions or variables.
Then the problem (2.4)-(2.7) becomes
\[
\partial_t^2 U = c^2 \Delta U + F + J(\eta, U), \quad \text{in} \ \Omega_0, \ t > 0;
U = U_0, \quad \partial_t U = U_1, \quad \text{in} \ \Omega_0, \ t = 0;
U|_{r=b_0} = G, \quad t > 0; \quad \left( \partial_t U - T_d(U) \right)|_{r=b} = L(\eta, U)|_{r=b}, \quad t > 0,
\]
where \( J(\eta, U) \) and \( L(\eta, U) \) contain differential operators with nonconstant coefficients.

- To solve the transformed problem efficiently, we adopt the boundary perturbation technique by viewing the obstacle as a perturbation of the disk. More precisely, we write \( \eta = \varepsilon \zeta \) and expand the solution \( U \) as
\[
U(r, \phi, t; \varepsilon) = \sum_{l=0}^{\infty} U_l(r, \phi, t)\varepsilon^l,
\]
Solve the above equation for \( i \) with Galerkin method to approximate (that where
\[ T(4.2). \]

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form the interval (\( \sigma \)) and (\( \nu \)) denote the Fourier/spherical harmonic expansion coefficients of (in 3-D) transform, reduces to a sequence of one-dimensional problems (for brevity, we use \( u \) to denote the expansion coefficients of \( U \), and likewise, we use \( u_0, u_1 \) and \( f \) to denote the expansion coefficients of \( U_0, U_1 \) and \( F \), respectively):

\[
\frac{\partial^2 u}{\partial t^2} - \frac{c^2}{r^{d-1}} \frac{\partial}{\partial r} \left( r^{d-1} \frac{\partial u}{\partial r} \right) + c^2 \beta_n \frac{u}{r^2} = f, \quad b_0 < r < b, \ t > 0; \\
u|_{t=0} = u_0, \quad \frac{\partial u}{\partial t} \bigg|_{t=0} = u_1, \quad b_0 < r < b; \quad u|_{r=b_0} = 0, \ t > 0; \\
\left( \frac{1}{c} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial r} + \frac{d-1}{2r} u \right) \bigg|_{r=b} = \int_0^t \sigma_\nu(t-\tau) u(b, \tau) d\tau, \ t > 0,
\]

where \( \beta_n = n^2, n(n+1) \) and \( \nu = n, n + 1/2 \) for \( d = 2, 3 \), respectively. Notice that in view of (215) and (222), the boundary condition at \( r = b \) can be replaced by

\[
\left( \frac{1}{c} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial r} \right) \bigg|_{r=b} = \frac{1}{c} \omega_n(t) \ast \delta_i u(b, t), \ t > 0.
\]

Since \( \sigma_\nu \) and \( \omega_n \) are real, the real and imaginary parts of \( u \) can be decoupled, and we assume that \( u \) is real.

4.2. Prototype equation and dimension reduction. Under this notion, the whole algorithm boils down to solving a sequence of prototype equations. More precisely, we consider

\[
\frac{\partial^2 U}{\partial t^2} = c^2 \Delta U + F, \quad \text{in} \ \Omega_0, \ t > 0; \\
U = U_0, \quad \partial_t U = U_1, \quad \text{in} \ \Omega_0, \ t = 0; \\
U|_{r=b_0} = 0, \ t > 0; \quad (\partial_t U - T_d(U))|_{r=b} = \tilde{L}(\eta, U_{l-1}), \ t > 0.
\]

- Solve the above equation for \( l = 0, 1, \cdots \), and sum up the series by using a Padé approximation.

4.3. Spectral-Galerkin approximation in space. We now apply the Legendre-spectral-Galerkin method to approximate (4.18) in space. For convenience of implementation, we transform the interval \((b_0, b)\) to the reference interval \( I = (-1, 1) \) by \( r = \frac{b-b_0}{2} x + \frac{b+b_0}{2} \) with \( x \in I \), and denote the transformed functions by \( v(x, t) = u(r, t), h(x, t) = f(r, t) \) and \( v_i(x) = u_i(r) \) with \( i = 0, 1 \), respectively. Then (4.18) can be reformulated as

\[
\frac{\partial^2 v}{\partial t^2} - \frac{c^2}{(x + c_0)^{d-1}} \frac{\partial}{\partial x} \left( (x + c_0)^{d-1} \frac{\partial v}{\partial x} \right) + c^2 \beta_n \frac{v}{(x + c_0)^2} = h, \quad x \in I, \ t > 0; \\
v(x, 0) = v_0(x), \quad \frac{\partial v}{\partial t}(x, 0) = v_1(x), \quad x \in I; \quad v(-1, t) = 0, \ t > 0; \\
\left( \frac{1}{c} \frac{\partial v}{\partial t} + \frac{2}{b-b_0} \frac{\partial v}{\partial x} + \frac{d-1}{2b} v \right)(1, t) = \int_0^t \sigma_\nu(t-\tau) v(1, \tau) d\tau, \ t > 0,
\]

and likewise for the data. Formally, we obtain the sequence of equations after collecting the terms of \( \varepsilon^l \) (see [33]):

\[
\partial_t^2 U_l = c^2 \Delta U_l + F_l + \tilde{J}(\eta, U_{l-4}, \cdots, U_{l-1}), \quad \text{in} \ \Omega_0, \ t > 0; \\
U_l = U_{0,t}, \quad \partial_t U_l = U_{1,t}, \quad \text{in} \ \Omega_0, \ t = 0; \\
U_l|_{r=b_0} = G_l, \quad t > 0; \quad (\partial_t U_l - T_d(U_l))|_{r=b} = \tilde{L}(\eta, U_{l-1}), \ t > 0.
\]
where the constants \( \tilde{c} = \frac{2\pi}{b-b_0} \) and \( c_0 = \frac{b+b_0}{b-b_0} \).

Let \( V_N := \{ \psi \in P_N : \psi(-1) = 0 \} \), where \( P_N \) is the set of all algebraic polynomials of degree at most \( N \). The semi-discretization Legendre spectral-Galerkin approximation of (4.20) is to find \( v_N(x,t) \in V_N \) for all \( t > 0 \) such that for all \( w \in V_N \),

\[
(\tilde{v}_N, w)_{\infty} + \tilde{c}(1 + c_0)^{d-1} \tilde{v}_N(1,t)w(1) + \tilde{c}^2(\partial_x v_N, \partial_x w)_{\infty} + \tilde{c}^2 \beta_n (v_N(x + c_0)^{-2}, w)_{\infty}
+ \frac{2c^2}{b-b_0}(1 + c_0)^{d-1}\left( \frac{d-1}{2b}v_N(1,t) - \sigma_v(t) * v_N(1,t) \right)w(1) = (I_N h, w)_{\infty},
\]

(4.21)

\( v_N(x,0) = v_{0,N}(x), \quad \tilde{v}_N(x,0) = v_{1,N}(x), \quad x \in I, \)

where \((\cdot, \cdot)_{\infty}\) is the (weighted) inner product of \( L^2_{\infty}(I) \) with the weight function \( \varpi = (x+c_0)^{-1} \), \( \tilde{v} \) denotes \( \partial_x^2 v \) or \( \frac{d^2 v}{dx^2} \) as usual, \( I_N \) is the interpolation operator on \( (N+1) \) Legendre-Gauss-Lobatto points, and \( v_{0,N}, v_{1,N} \in P_N \) are the approximations of the initial values.

Like Theorem 3.2, we have the following \textit{a priori} estimates for the solutions of (4.20) and (4.21).

**Theorem 4.1.** Let \( v_N \) be the solution of (4.21). Then for all \( t > 0 \),

\[
\|\partial_t v_N\|_{L^2_{\infty}(I)}^2 + \tilde{c}^2(\|\partial_x v_N\|_{L^2_{\infty}(I)}^2 + \beta_n \|v_N/(x+c_0)\|_{L^2_{\infty}(I)}^2)
\leq C\left( \|v_{1,N}\|_{L^2_{\infty}(I)}^2 + \tilde{c}^2(\|\partial_x v_{0,N}\|_{L^2_{\infty}(I)}^2 + \beta_n \|v_{0,N}/(x+c_0)\|_{L^2_{\infty}(I)}^2) + \|I_N h\|_{L^2_{\infty}(I)}^2 \right),
\]

(4.22)

where \( C \) is a positive constant independent of \( N, \tilde{c} \) and \( b \).

This estimate holds for (4.20) with \( v, v_N, v_1, h \) in place of \( v_N, v_{0,N}, v_{1,N}, I_N h \), respectively.

**Proof.** Taking \( w = \partial_t v_N \) in (4.21), and integrating the resulted identity with respect to \( t \), we use Corollary 3.1 and the argument similar to that for Theorem 3.2 to derive the estimates. \( \square \)

**Remark 4.1.** Equipped with Theorem 4.1, we can analyze the convergence of the semi-discretized scheme (4.21) as on Page 341 of [37]. \( \square \)

We next examine the linear system of (4.21). As shown in [35, 37], it is advantageous to construct basis functions satisfying the underlying homogeneous Dirichlet boundary conditions by using compact combinations of orthogonal polynomials. Let \( L_l(x) \) be the Legendre polynomial of degree \( l \) (see, e.g., [41]), and define \( \phi_k(x) = L_k(x) + L_{k+1}(x) \). Then \( \phi_k(-1) = 0 \) and \( V_N = \text{span}\{ \phi_k : 0 \leq k \leq N - 1 \} \). Note that \( \phi_k(1) = 2 \).

Setting

\[
v_N(x,t) = \sum_{j=0}^{N-1} \hat{v}_j(t) \phi_j(x), \quad v(t) = (\hat{v}_0, \hat{v}_1, \cdots, \hat{v}_{N-1})^t,
\]

\[
m_{ij} = (\phi_j, \phi_i)_{\infty}, \quad s_{ij} = (\phi_j', \phi_i)_{\infty}, \quad \tilde{m}_{ij} = (\phi_j(x + c_0)^{-2}, \phi_i)_{\infty},
\]

\[
h_i = (I_N h, \phi_i)_{\infty}, \quad \tilde{h}(t) = (h_0, h_1, \cdots, h_{N-1})^t, \quad 1 = (1, 1, \cdots, 1)^t,
\]

we obtain the system:

\[
M \tilde{v} + \alpha E \tilde{v} + \tilde{c}^2 (S + \beta_n \tilde{M}) v + \mu E v - c_0 \left( \sum_{j=0}^{N-1} \hat{v}_j \right) 1 = h,
\]

(4.24)

\[
v(0) = v_0, \quad \tilde{v}(0) = v_1,
\]

where \( M = (m_{ij}), S = (s_{ij}), \tilde{M} = (\tilde{m}_{ij}) \) and \( E = 11^t \) is an \( N \times N \) matrix of all ones. In (4.24), \( v_0 \) and \( v_1 \) are column-\( N \) vectors of the expansion coefficients of \( v_{0,N} \) and \( v_{1,N} \) in terms of \{\phi_k\}.
and the constants
\[ \alpha = \frac{8c}{b-b_0}(1+c_0)^{d-1}, \quad \mu = \frac{4c^2(d-1)}{b(b-b_0)}(1+c_0)^{d-1}. \]

4.4. **Newmark’s time integration.** To discretize the second-order ordinary differential system (4.24) with history dependence due to the temporal convolution, we resort to the implicit second-order Newmark’s scheme, which has wide applications in the field of structural mechanics (see [31, 46]). For clarity of presentation, we first consider the general system:

\[ A\ddot{u} + B\dot{u} + Cu = f, \quad (4.25) \]

where \(A, B, C\) are the mass, damping and mass matrices, respectively, \(f\) is the applied load vector, and \(u\) is usually the unknown displacement vector. Let \(\Delta t\) be the time-stepping size, and let \(\{u^m, \dot{u}^m, \ddot{u}^m\}\) be the approximation of \(\{u, \dot{u}, \ddot{u}\}\) at \(t = t_m = m\Delta t\), and \(f^m = f(t_m)\). Given \(u^0\) and \(\dot{u}^0\), we obtain the numerical solution \(u^{m+1}\) by the Newmark’s scheme through

\[ u^{m+1} = u^m + \Delta t\dot{u}^m + \frac{1}{2}\Delta t^2\ddot{u}^m, \]
\[ \dot{u}^{m+1} = \dot{u}^m + (1-\vartheta)\Delta t\dot{u}^m + \vartheta\Delta t\ddot{u}^m, \]

(4.26) (4.27)

together with the system (4.25) collocated at \(t_{m+1}\):

\[ A\ddot{u}^{m+1} + B\dot{u}^{m+1} + Cu^{m+1} = f^{m+1}. \quad (4.28) \]

Eliminating \(u^{m+1}\) and \(\dot{u}^{m+1}\) from (4.28) by using (4.26) and (4.27), we obtain \(u^{m+1}\) from (4.28) by solving

\[ (A + \vartheta\Delta tB + \theta\Delta t^2C)\ddot{u}^{m+1} = f^{m+1} - B(u^m + (1-\vartheta)\Delta t\dot{u}^m) - C(u^m + \Delta t\dot{u}^m + \frac{1}{2}\Delta t^2\ddot{u}^m), \]

(4.29)

and update \(u^{m+1}\) and \(\dot{u}^{m+1}\) by (4.26) and (4.27), respectively. If the parameters satisfy

\[ \vartheta \geq \frac{1}{2} \quad \text{and} \quad \theta \geq \frac{1}{4}\left(\frac{1}{2} + \vartheta\right)^2, \quad (4.30) \]

the above Newmark’s scheme is of second-order and unconditionally stable (see [46]).

We now apply the scheme to the system (4.24). Compared with (4.25), the entries of the damping matrix \(B = \alpha E\) (induced by the NRBC) are all \(\alpha\), which make the related matrix-vector multiplication in (4.29) fairly simple and efficient. However, some care has to be taken to deal with the convolution term in (4.24):

\[ \left[\sigma_{\nu} \ast \sum_{j=0}^{N-1} \hat{v}_j(t_{m+1})\right] = \sum_{j=0}^{N-1} \int_{t_m}^{t_{m+1}} \sigma_{\nu}(t_{m+1} - \tau)\hat{v}_j(\tau)d\tau + \sum_{j=0}^{N-1} \int_{0}^{t_m} \sigma_{\nu}(t_{m+1} - \tau)\hat{v}_j(\tau)d\tau \]
\[ \approx \frac{\Delta t}{2}\left(\sigma_{\nu}(0) \sum_{j=0}^{N-1} \hat{v}_j^{n+1} + \sigma_{\nu}(\Delta t) \sum_{j=0}^{N-1} \hat{v}_j^n\right) + \frac{\Delta t}{2} \sum_{j=0}^{N-1} \int_{0}^{t_m} \sigma_{\nu}(t_{m+1} - \tau)\hat{v}_j(\tau)d\tau \]
\[ \approx \frac{\Delta t}{2}\sigma_{\nu}(0)E\hat{v}^{m+1} + \hat{g}^m, \]

where we used the Trapezoidal rule (of order \(O(\Delta t^2)\)) to approximate the integral over \((t_m, t_{m+1})\), and denoted by \(\hat{g}^m\) the approximation of the remaining terms. Note that \(\hat{g}^m\) depends on the history \(\hat{v}^l(0 \leq l \leq m)\), but fortunately, it can be evaluated recursively and rapidly as described in Subsection 2.4. Moreover, only the history data \(\sum_{j=0}^{N-1} \hat{v}_j^l (0 \leq l \leq m)\) need to be stored, and no
any matrix-vector multiplication is involved. As a result, the burden of the history dependence can be eliminated.

We summarize the full scheme for (4.20) as follows.

Algorithm for (4.21):
(i) Set $v^0 = v_0$ and $\dot{v}^0 = v_1$, and compute $\ddot{v}^0$ from the system (4.24) by
\[
\ddot{v}^0 = M^{-1}\{\varrho^0 - \alpha E\dot{v}^0 - \tilde{c}^2(S + \beta_n \tilde{M})v^0 - \mu Ev^0\},
\]
where the convolution term vanishes at $t = 0$.
(ii) For $m \geq 0$, we compute $v^{m+1}$ by the Newmark’s scheme with
\[
A = M, \quad B = \alpha E, \quad C = \tilde{c}^2(S + \beta_n \tilde{M}) + \left(\mu - c_0 \frac{\Delta t}{2} \sigma_\nu(0)\right)E,
\]
and $f^{m+1} = h^{m+1} + c_0 g^m$.

Remark 4.2. The computational cost of the full algorithm lies in solving (4.29) with (4.32). Some remarks are in order.

- If $a = 0$ (i.e., $\Omega$ in (4.17) is a disk or a ball), the matrices $M, \tilde{M}$ and $S$ are sparse with small bandwidth under the basis $\{\varphi_k\}$, and their entries can be evaluated exactly by using the properties of Legendre polynomials [36]. However, if $b_0 \neq 0, d = 2$ and $n \neq 0$, the matrix $\tilde{M}$ becomes full, due to the rational coefficient $1/(x + c_0)$ with $c_0 = \frac{b + b_0}{b - b_0} > 1$. In this case, we can reformulate (4.21) with the weight function $\varpi = (x + c_0)^2$ (rather than $\varpi = x + c_0$), so that the matrix $\tilde{M}$ is sparse with finite bandwidth.
- The matrices $B$ and $C$ in (4.32) appear to be full, but since all entries of $E$ are one, the coefficient matrix of (4.29) can be reduced to a band matrix as $M, \tilde{M}$ and $S$ with $O(N)$ operations. Hence, the direct inversion of (4.32) is applicable, if $N$ is moderate large. Moreover, notice that $Ev = (\sum \tilde{v}_j) 1$, so the matrix-vector multiplication in (4.29) can be carried out in $O(N)$ operations. Indeed, the presence of the global DtN boundary condition does not increase the computational cost. □

5. Numerical results

In this section, we present various numerical results to show the accuracy of the NRBC and convergence of the proposed algorithm.

5.1. Testing problem and setup. We examine the NRBC and the scheme via the following problem with an exact solution.

**Proposition 5.1.** The exterior problem (2.1)-(2.3) with $d = 2, D = \{x \in \mathbb{R}^2 : |x| < b_0\}, F = U_0 = U_1 \equiv 0$, and $U|_{r=b_0} = G$, admits the exact solution:
\[
U(r, \phi, t) = \sum_{|n| = 0}^{\infty} \hat{U}_n(r, t)e^{in\phi}, \quad r > b_0, \quad \phi \in [0, 2\pi), \quad t > 0,
\]
(5.1)
where
\[
\tilde{U}_n(r, t) = \begin{cases} 
0, & t < \beta_0, \\
H_n(r, t) \ast \hat{G}_n(t - \beta_0) + \sqrt{b_0/r}\hat{G}_n(t - \beta_0), & t \geq \beta_0,
\end{cases}
\] (5.2)
with \(\beta_0 = (r - b_0)/c\), \(\hat{G}_n(t)\) being the Fourier expansion coefficient of \(G(\phi, t)\), and
\[
H_n(r, t) = \frac{c}{b_0} \sum_{j=1}^{M_n} \frac{K_{n+1}(rz_j^0/b_0)}{K_n(z_j^0)} e^{rz_j^0/b_0} \\
+ (-1)^n \frac{c}{b_0} \int_0^\infty I_n(rp/b_0)K_n(\rho) - K_n(rp/b_0)I_n(\rho)\rho/K_n^2(\rho) + \frac{1}{2}I_n^2(\rho) e^{-\epsilon p/\sigma_0 d\rho}.
\] (5.3)

Proof. We sketch its derivation in Appendix C.

Remark 5.1. Notice that \(\tilde{U}_n\) satisfies the one-dimension problem:
\[
\frac{\partial^2 \tilde{U}_n}{\partial t^2} - \frac{c^2}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \tilde{U}_n}{\partial r} \right) + c^2 n^2 \frac{\tilde{U}_n}{r^2} = 0, \quad b_0 < r < b, \quad t > 0;
\] (5.4)
\[
\tilde{U}_n|_{t=0} = \frac{\partial \tilde{U}_n}{\partial t} \bigg|_{t=0} = 0, \quad b_0 < r < b; \quad \tilde{U}_n|_{r=b} = \hat{G}_n, \quad t > 0;
\] (5.5)
\[
\left( \frac{1}{c} \frac{\partial \tilde{U}_n}{\partial t} + \frac{\partial \tilde{U}_n}{\partial r} + \frac{1}{2} \hat{G}_n \right) \bigg|_{r=b} = \int_0^t \sigma_n(t-\tau)\hat{U}_n(b, \tau) d\tau, \quad t > 0,
\] (5.6)
for any \(b > b_0\).

We take the Dirichlet data:
\[
G(\phi, t) = A_1 e^{-i((b_0 \cos x_s + y_s)^2 + (b_0 \sin \phi - y_s)^2)} \sin^p(\omega t),
\] (5.7)
where \(A_1 = 10\), \(t = 0.1\), \(x_s = y_s = 2.1\), \(b_0 = 2\), and \(\omega, p\) are to be specified later. We compute the Fourier coefficient
\[
\hat{G}_n(t) = \frac{A_1}{2\pi} \sin^p(\omega t) \int_0^{2\pi} e^{-i((b_0 \cos x_s + y_s)^2 + (b_0 \sin \phi - y_s)^2)} e^{-i n \phi} d\phi,
\]
via the fast Fourier transform with sufficient Fourier points so that the error of numerical integration can be neglected.

The first important issue is to compute the exact solution and \(\sigma_n\) very accurately. We point out that the convolution in (5.2) can be computed exactly, and the improper integral in (5.3) can be treated by a similar manner as the improper integral of \(\sigma_n(t)\) in Subsection 2.3. We adopt a Newton's iteration method to compute the zeros of \(K_n(z)\) accurately for moderate large \(n\), and use a (composite) Legendre-Gauss quadrature to evaluate the integral with high precision.

For the readers' reference, we tabulate in Table 5.1 some samples of \(\sigma_n(t)\) at \(t = 0.1, 2\) with \(b = 3, c = 5\).

| \(n\) | \(\sigma_n(0.1)\) | \(\sigma_n(2)\) | \(n\) | \(\sigma_n(0.1)\) | \(\sigma_n(2)\) |
|-----|----------------|----------------|-----|----------------|----------------|
| 0   | 5.922020676764810e-2 | 1.127355855271518e-2 | 5   | -5.39429665508512e-2 | 2.65211425230534e-3 |
| 1   | -1.770775250905292e-1 | -1.840758830245609e-2 | 6   | -7.50579837812085e-2 | -1.62840928792232e-3 |
| 2   | -8.76675435408012e-1 | -3.515167498632986e-3 | 7   | -9.78660490985682e-3 | 1.032611050422504e-3 |
| 3   | -2.102104067208546e-1 | 6.198828109004315e-3 | 8   | -12.1421492386835e-2 | -6.79371907642434e-4 |
| 4   | -3.358153424430011e-1 | -4.319820513209325e-3 | 9   | -14.07330651171318e-2 | 4.62860007197164e-4 |
We plot in Figure 5.1 the exact solution with $p = 2, b_0 = 2, b = 4$ and two different $\omega$ at various $t$. We see that the Dirichlet data $G(\phi, t)$ acts as a dynamical wave-maker.

**Figure 5.1.** Time evolution of exact solution for $\omega = 10\pi$ and $\omega = 20\pi$.

5.2. **Accuracy of the NRBC.** Now, we examine the accuracy of the NRBC by computing the error:

$$\max_{|\phi| \leq 2\pi} \left| \partial_t U_M - T_d(U_M) \right| \leq \sum_{|n| = 0}^{M} e_n(t),$$

with

$$e_n(t) := \left| \left( \frac{1}{c} \partial_t \hat{U}_n + \partial_r \hat{U}_n + \frac{\hat{U}_n}{2r} \right) \right|_{r = b} - \sigma_n(t) \ast \hat{U}_n(b, t).$$

In (5.8), $T_d(U_M)$ is defined in (2.24) with $d = 2$, and $U_M(b, \phi, t) = \sum_{|n| = 0}^{M} \hat{U}_n(b, t)e^{in\phi}$ is the truncation of the exact solution in Proposition 5.1. We choose the same parameters as those for the exact solution in Figure 5.1, and take $M = 32$ so that $|\hat{U}_n(b, t)|$ are sufficiently small for all modes $|n| \leq M$ and all $t$ of interest. Note that the differentiations in (5.9) are performed exactly on the exact solution.

In Table 5.2, we tabulate the errors:

$$E^{(1)}_M(t) = \max_{|n| \leq M} e_n(t), \quad E^{(2)}_M(t) = \sum_{|n| = 0}^{M} e_n(t),$$

at some samples of $t$, compute the errors at the outer artificial boundary $r = b$, where the exact solution has the magnitude as large as possible.

We see from Table 5.2 that in all tests, the errors are extremely small. This validates the formula for $\sigma_n$ in Theorem 2.1 and high accuracy of the numerical treatment.
Table 5.2. The errors $E^{(1)}_M(t)$ and $E^{(2)}_M(t)$.

| $t$ | $\omega = 10\pi, b = 2.2$ | $\omega = 10\pi, b = 2.75$ | $\omega = 20\pi, b = 2.38$ | $\omega = 20\pi, b = 2.87$ |
|-----|---------------------|---------------------|---------------------|---------------------|
| 0.5 | $2.336e-16$ | $7.026e-16$ | $1.148e-16$ | $6.215e-17$ |
| 1   | $4.684e-16$ | $8.406e-16$ | $2.099e-16$ | $9.021e-16$ |
| 5   | $2.567e-16$ | $4.386e-16$ | $1.131e-15$ | $7.251e-16$ |
| 10  | $7.772e-16$ | $1.020e-15$ | $2.207e-15$ | $1.457e-15$ |
| 15  | $1.390e-15$ | $1.731e-15$ | $3.236e-15$ | $2.331e-15$ |
| 20  | $1.512e-15$ | $1.890e-15$ | $4.273e-15$ | $3.02e-08$ |

5.3. Numerical tests for the spectral-Galerkin-Newmark’s scheme. Hereafter, we provide some numerical results to illustrate the convergence of the numerical scheme described in Section 4. In the following computations, we take the Dirichlet data given by (5.7) with $A_1 = 10, t = 0.1, x_s = y_s = 2.1$ and $b_0 = 2$ as before.

Given a cut-off number $M > 0$, we compute the numerical solutions $\{\hat{U}^N_n\}$ of (5.4)-(5.6) for the modes $|n| \leq M$ by using the spectral-Galerkin and Newmark’s time integration scheme. The full numerical solution of the problem in Proposition 5.1 is then defined as $U^N_M(r, \phi, t) = \sum_{|n| = 0}^M \hat{U}^N_n(r, t)e^{in\phi}$, and the exact solution $U_M(r, \phi, t) = \sum_{|n| = 0}^M \hat{U}_n(r, t)e^{in\phi}$ is evaluated as before. Once again, we choose $M$ such that the Fourier coefficient of the exact solution $|\hat{U}_n| \leq 10^{-16}$ for $|n| > M$. The numerical errors are measured by

$$\hat{E}^N_M(t) = \max_{|n| \leq M} \|\hat{U}^N_n(\cdot, t) - \hat{U}_n(\cdot, t)\|_{L^2,N}, \quad E^N_M(t) = \max_{|n| \leq M} \|\hat{U}^N_n(\cdot, t) - \hat{U}_n(\cdot, t)\|_{L^\infty,N},$$

where $\| \cdot \|_{L^2,N}$ is the discrete $L^2$-norm associated with the Legendre-Gauss-Lobatto interpolation, and $\| \cdot \|_{L^\infty,N}$ is the corresponding discrete maximum norm.

To test the second-order convergence of the Newmark’s scheme, we choose $N = 50$ so that the error of the spatial discretization is negligible. We provide in Table 5.3 the numerical errors and the order of convergence for various $t$ with $M = 15, b = 5, \omega = \pi$, and $p = 6$. As expected, we observe a second-order convergence of the time integration.

| $\Delta t$ | $E^N_M(t)$ | order | $E^N_M(t)$ | order | $\Delta t$ | $E^N_M(t)$ | order | $E^N_M(t)$ | order |
|------------|-------------|-------|-------------|-------|------------|-------------|-------|-------------|-------|
| 1e-05      | 1.21e-05    | 1.93e-05 | 1.99994     | 3     | 1e-03      | 1.23e-05    | 2.01e-05 | 1.99997     |       |
| 5e-04      | 3.02e-06    | 4.83e-06 | 1.99994     | 3     | 5e-04      | 3.07e-06    | 5.02e-06 | 1.99997     |       |
| 1e-04      | 1.21e-07    | 1.99998 | 1.99998     | 1e-04 | 1.23e-07   | 2.01e-07    | 1.99998 | 2.01e-07    | 1.99998 |
| 5e-05      | 3.02e-08    | 4.83e-08 | 1.99998     | 5e-05 | 3.07e-08   | 5.02e-08    | 2.00004 | 2.00004     |       |
| 1e-03      | 1.23e-05    | 2.01e-05 | 1.99998     | 4     | 1e-03      | 1.23e-05    | 2.01e-05 | 1.99998     |       |
| 5e-04      | 3.07e-06    | 5.02e-06 | 1.99993     | 5e-04 | 3.07e-06   | 5.02e-06    | 2.00004 | 2.00004     |       |
| 1e-04      | 1.23e-07    | 2.01e-07 | 1.99993     | 5e-04 | 1.23e-07   | 2.01e-07    | 1.99998 | 2.01e-07    | 1.99998 |
| 5e-05      | 3.07e-08    | 5.02e-08 | 1.99993     | 5e-04 | 3.07e-08   | 5.02e-08    | 2.00004 | 2.00004     |       |

Next, we fix the time step size $\Delta t = 10^{-5}$ and choose different $N$ to check the accuracy in spatial discretization. The convergence behavior is illustrated in Table 5.4. With a small number of modes in space, we observe a fast decay of the errors, which is typical for the spectral approximation.
Table 5.4. The errors $\hat{E}_N^{N}(t)$ and $\tilde{E}_N^{N}(t)$.

| $t$  | $N = 8$     | $N = 10$     | $N = 16$     | $N = 32$     |
|------|-------------|-------------|-------------|-------------|
|      | $\hat{E}_N^{N}(t)$ | $\hat{E}_N^{N}(t)$ | $\hat{E}_N^{N}(t)$ | $\hat{E}_N^{N}(t)$ |
| 0.5  | 2.349e-04   | 2.687e-04   | 2.253e-05   | 8.224e-07   |
| 1.0  | 3.877e-04   | 4.179e-04   | 1.708e-05   | 1.833e-07   |
| 1.5  | 3.276e-04   | 3.596e-04   | 5.965e-06   | 3.072e-08   |
| 2.0  | 4.105e-04   | 4.368e-04   | 1.092e-05   | 7.353e-09   |
| 2.5  | 3.239e-04   | 3.607e-04   | 5.724e-06   | 2.381e-09   |
| 3.0  | 4.113e-04   | 4.380e-04   | 1.095e-05   | 1.404e-09   |
| 3.5  | 3.238e-04   | 3.608e-04   | 5.747e-06   | 1.554e-09   |
| 4.0  | 4.113e-04   | 4.380e-04   | 1.095e-05   | 1.269e-09   |

Finally, we plot in Figure 5.2 the numerical solution (red nets) against the exact solution (blue smooth surface), where the exact solution in (a) and (b) is in the annulus: $2 \leq r \leq 10$, while the numerical solution within $2 \leq r \leq 5$. Figure 5.2(c) shows the wave propagation through the artificial boundary at $b = 5$, $0 \leq \phi \leq 2\pi$ for $0 \leq t \leq 4$ (numerical solution) against $0 \leq t \leq 4.5$ (exact solution). We see that the red nets and blue surfaces agree well in these plots. Moreover, we observe the waves pass the boundary transparently. This shows that the proposed scheme is very stable and has high-order of accuracy.

Concluding remarks

We proposed in this paper analytic and accurate numerical means for the time-domain wave propagation with exact and global nonreflecting boundary conditions. We derived the analytic expressions of the involved convolution kernels and presented highly accurate methods for their evaluations. We analyzed the stability of the solution to the truncated problem and provided efficient numerical schemes. Ample numerical results were given to demonstrate the features of the method. We shall report the combination of the methods with the boundary perturbation technique in a future work and the methods and ideas will be useful to study the Maxwell equations and elastic wave propagations.
We first consider $d = 2$. Observe from (2.29) that $F_n(z)$ contains the logarithmic derivative of $K_n(z)$ (a multi-valued function, see Formula 9.6.11 of [1]), so it is a multi-valued function with branch points at $z = 0$ and infinity. The contour $L$ for the Bromwich's contour integral (2.28) is depicted in Figure A.1 (left) with the branch-cut along the negative real axis.

![Figure A.1](image-url)

**Figure A.1.** The contour $L$ used for the inverse Laplace transform. Left: $d = 2$. Right: $d = 3$.

We know from Lemma 2.2 that $F_n(z)$ has a finite number of simple poles in the second and third quadrants, but not on the negative real axis. Therefore, for any $n \geq 0$, it follows from the residue theorem that

\[
\lim_{R \to +\infty} \lim_{r \to 0^+} \int_L F_n(z)e^{czt/b}dz = 2\pi i \sum_{j=1}^{M_n} \text{Res} \left[ F_n(z)e^{czt/b}, z_j^n \right] = 2\pi i \sum_{j=1}^{M_n} \lim_{z \to z_j^n} \left( (z - z_j^n)e^{czt/b} \left[ z + \frac{1}{2} + z \frac{K_n'(z)}{K_n(z)} \right] \right) \]

(A.1)

Thus, by (2.28) and (A.1),

\[
\frac{2b^2\pi i}{c} \sigma_n(t) = 2\pi i \sum_{j=1}^{M_n} e^{ctz_j^n/b}z_j^n - \lim_{R \to +\infty} \lim_{r \to 0^+} \left[ \int_{BC} F_n(z)e^{czt/b}dz + \int_{CD} F_n(z)e^{czt/b}dz + \int_{EF} F_n(z)e^{czt/b}dz + \int_{GA} F_n(z)e^{czt/b}dz \right] .
\]

(A.2)

In view of (2.19), we find from the Jordan’s lemma (cf. [13, 8]) and a direct calculation that

\[
\lim_{R \to +\infty} \lim_{r \to 0^+} \left[ \int_{BC} F_n(z)e^{czt/b}dz + \int_{DEF} F_n(z)e^{czt/b}dz + \int_{GA} F_n(z)e^{czt/b}dz \right] = 0, \quad (A.3)
\]
since each contour integral tends to zero. Thus, we only need to evaluate the integrals along the line segments $\overline{CD}$ and $\overline{FG}$. We have

$$
\lim_{R \to +\infty \atop r \to 0^+} \left[ \int_{\overline{CD}} F_n(z)e^{czt/b}dz + \int_{\overline{FG}} F_n(z)e^{czt/b}dz \right] = \int_0^\infty \left[ F_n(re^{i\pi}) - F_n(re^{-i\pi}) \right] e^{-ctr/b}dr \quad (A.4)
$$

By Formula 9.6.31 of [1],

$$
K_n(re^{\pi i}) = e^{-n\pi i}K_n(r) - \pi i I_n(r), \quad K_n(re^{-\pi i}) = e^{n\pi i}K_n(r) - \pi i I_n(r), \quad (A.5)
$$

which, together with (2.18), implies

$$
\frac{K_n'(re^{\pi i}) - K_n'(re^{-\pi i})}{K_n(re^{\pi i}) - K_n(re^{-\pi i})} = \frac{K_{n+1}(re^{\pi i}) - K_{n+1}(re^{-\pi i})}{K_n(re^{\pi i}) - K_n(re^{-\pi i})}
$$

$$
= \frac{e^{(n+1)\pi i}I_{n+1}(r)}{e^{n\pi i}I_n(r)} = \frac{e^{-n\pi i}I_n(r)}{K_n(re^{\pi i}) - K_n(re^{-\pi i})} \quad (A.6)
$$

where we used the Wronskian identity (see Formula 9.6.15 of [1]):

$$
I_n(z)K_{n+1}(z) + I_{n+1}(z)K_n(z) = z^{-1}. \quad (A.7)
$$

A combination of (A.2)-(A.4) and (A.6) leads to

$$
\sigma_n(t) = \frac{c}{b^2} \left\{ \sum_{j=1}^{M_n} \left( z_j^n e^{ctz_j^n/b} + \int_0^\infty \frac{(-1)^ne^{-ctr/b}}{K_n(r) + \pi^2 I_n^2(r)}dr \right) \right\}, \quad (A.8)
$$

which is the expression (2.35).

Now, we turn to the three dimensional case. It is important to notice that the kernel function $F_{n+1/2}(z)$ is not a multi-valued complex function, as opposed to the two dimensional case. Indeed, although $K_{n+1/2}(z)$ is multi-valued, in view of the formula (see Page 80 of [45]):

$$
K_{n+1/2}(z) = \sqrt{\frac{\pi}{2z}} \sum_{k=0}^{n} \frac{(n+k)!e^{-z}}{k!(n-k)!(2z)^k}, \quad \forall n \geq 0, \quad (A.9)
$$

the fact $1/\sqrt{z}$ can be eliminated from the ratio $K_{n+1/2}'/K_{n+1/2}$. Thanks to this property, we use the contour in Figure A.1 (right), and the derivation becomes much simpler. By the residue theorem and Jordan’s lemma, we have

$$
\lim_{R \to +\infty} \oint_{\gamma} F_{n+1/2}(z)e^{czt/b}dz = \int_{\gamma - \infty i}^{\gamma + \infty i} F_{n+1/2}(z)e^{czt/b}dz
$$

$$
= \frac{2\pi i}{\pi} \sum_{j=1}^{M_n} \text{Res} \left[ F_{n+1/2}(z)e^{czt/b}, z_j^\nu \right] = \frac{2\pi i}{\pi} \sum_{j=1}^{M_n} e^{cz_j^\nu/b}z_j^\nu, \quad (A.10)
$$

for $\nu = n + 1/2$. This leads to the formula (2.36).
APPENDIX B. JUSTIFICATION FOR REMARK 2.5

Thanks to (2.30), we have
\[ \frac{z H_v^{(1)'}(z)}{H_v^{(1)}(z)} = -iz \frac{K_v(-iz)}{K_v(-iz)}. \]
Thus, by (2.37) with \( \nu = n \) and \( -iz = bs/c \), we find
\[ \frac{s}{c} + \frac{1}{2b} + \frac{s K_v'(bs/c)}{c K_v(bs/c)} = \frac{-ic}{b^2} \sum_{n=1}^{N_\nu} \frac{h_{n,j}}{s + \imath \omega_{n,j}} + \frac{c}{b^2} \int_0^\infty \frac{(-1)^n}{K_n^2(r) + \pi^2 T_n^2(r)} \frac{1}{\frac{sr}{c} + s} \, dr. \]
Applying the inverse Laplace transform to both sides of the above identity leads to
\[ \mathcal{L}^{-1} \left[ \frac{s}{c} + \frac{1}{2b} + \frac{s K_v'(bs/c)}{c K_v(bs/c)} \right] = \frac{c}{b^2} \sum_{n=1}^{N_\nu} (-i\omega_{n,j}) e^{-\imath \omega_{n,j} t} + \frac{c}{b^2} \int_0^\infty \frac{(-1)^n}{K_n^2(r) + \pi^2 T_n^2(r)} \frac{1}{r} e^{-rt/b} \, dr, \]
which turns out to be (A.8), since \( -i\omega_{n,j} \) are zeros of \( K_n(z) \) and \( N_\nu = M_n \). □

APPENDIX C. PROOF OF PROPOSITION 5.1

As shown in Subsection 2.1, this problem can be solved by Laplace transform and separation of variables. Indeed, like (2.21), the Fourier coefficients \( \{ \hat{U}_n \} \) in (5.1) can be expressed as
\[ \hat{U}_n(r, t) = \mathcal{L}^{-1} \left( \frac{K_n(sr/c)}{K_n(sb_0/c)} \right)(t) \ast \hat{G}_n(t), \quad r > b_0. \tag{C.1} \]
We next sketch the evaluation of the inverse Laplace transform by using the residue theorem as in Appendix A for \( \sigma_n(t) \). Using (2.17), we find
\[ \frac{K_n(sr/c)}{K_n(sb_0/c)} \sim \sqrt{\frac{b_0}{r}} e^{-s\beta_0}, \]
where \( \beta_0 = (r - b_0)/c \). In order to use the Jordan’s lemma (cf. [13, 8]), we write
\[ \mathcal{L}^{-1} \left( \frac{K_n(sr/c)}{K_n(sb_0/c)} \right)(t) = \mathcal{L}^{-1} \left( e^{-\beta_0 s} \left\{ e^{\beta_0 s} \frac{K_n(sr/c)}{K_n(sb_0/c)} - \sqrt{\frac{b_0}{r}} \right\} \right) + \mathcal{L}^{-1} \left( e^{-\beta_0 s} \sqrt{\frac{b_0}{r}} \right)(t) \tag{C.2} \]
\[ = \bar{H}_n(r, t - \beta_0) U_{\beta_0}(t) + \sqrt{\frac{b_0}{r}} \delta(t - \beta_0), \]
where \( \bar{H}_n(r, t) \) is \( \mathcal{L}^{-1} \left( e^{\beta_0 s} \frac{K_n(sr/c)}{K_n(sb_0/c)} - \sqrt{\frac{b_0}{r}} \right)(t) \), \( U_{\beta_0}(t) \) is the unit step function (which takes 1 for \( t \geq \beta_0 \), and 0 for \( t < \beta_0 \)), and \( \delta(t) \) is the Dirac delta function. By applying the residue theorem and Jordan’s lemma to the Brownwich’s contour integral:
\[ \bar{H}_n(r, t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\beta_0 s} \frac{K_n(sr/c)}{K_n(sb_0/c)} - \sqrt{\frac{b_0}{r}} \, e^{ts} \, ds, \tag{C.3} \]
we obtain (5.3) by using the contour in Figure A.1 (left) and the same argument as in Appendix A. Denoting \( \tilde{H}_n(r, t) := \bar{H}_n(r, t - \beta_0) \), the expression (5.2) follows from (C.2). We leave the details to the interested readers. □

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