ON THE RAMIFIED CLASS FIELD THEORY OF RELATIVE CURVES

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Abstract. We generalize Deligne’s approach to tame geometric class field theory to the case of a relative curve, with arbitrary ramification.

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1. Introduction

Let $X \to S$ be a relative curve, i.e. a smooth morphism of schemes of relative dimension 1, with connected geometric fibers, which is Zariski-locally projective over $S$. Let $Y \hookrightarrow X$ be a relative effective Cartier divisor over $S$ (cf. 4.10), and let $U$ be the complement of $Y$ in $X$.

The pairs $(L, \alpha)$, where $L$ is an invertible $O_X$-module and $\alpha$ is a rigidification of $L$ along $Y$, are parametrized by an $S$-group scheme $Pic_S(X,Y)$, the relative rigidified Picard scheme (cf. 4.8). The Abel-Jacobi morphism

$$\Phi : U \to Pic_S(X,Y)$$

is the morphism which sends a section $x$ of $U$ to the pair $(O(x), 1)$, cf. 4.14. We prove the following relative version of the main theorem of geometric global class field theory:

**Theorem 1.1.** (Th. 5.3) Let $\Lambda$ be a finite ring of cardinality invertible on $S$, and let $F$ be an étale sheaf of $\Lambda$-modules, locally free of rank 1 on $U$, with ramification bounded by $Y$ (cf. 5.2). Then, there exists a unique (up to isomorphism) multiplicative étale sheaf of $\Lambda$-modules $G$ on $Pic_S(X,Y)$, locally free of rank 1, such that the pullback of $G$ by $\Phi$ is isomorphic to $F$.

The notion of multiplicative locally free $\Lambda$-module of rank 1 is defined in 2.5, and it corresponds to isogenies $G \to Pic_S(X,Y)$ with constant kernel $\Lambda^\times$.

The case where $S$ is the spectrum of a perfect field is originally due Serre and Lang, cf. ([La56], 6) and [Se59]. Their proof relies on the Albanese property of Rosenlicht’s generalized Jacobians [Ro54]. Another proof was given by Deligne (unpublished) in the tamely ramified case. We generalize Deligne’s approach to allow arbitrary ramification and an arbitrary base $S$. This generalization is inspired by notes by Alain Genestier (unpublished) on arithmetic global class field theory.

Deligne’s approach has the advantage over Serre and Lang’s to yield an explicit geometric construction of the isogeny over $Pic_S(X,Y)$ corresponding to a local system of rank 1 over $U$. This
feature of Deligne’s approach carries over to ours, and is in fact crucial in order to handle the case of an arbitrary base \( S \).

The author was informed during the preparation of this manuscript that Daichi Takeuchi has independently obtained a different proof of 1.1 in the case where \( S \) is the spectrum of a perfect field, also by generalizing Deligne’s approach to handle arbitrary ramification.

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Notation and conventions. We fix a universe \( \mathcal{U} \) ([SGA4], I.0). Throughout this paper, all sets are assumed to belong to \( \mathcal{U} \) and we will use the term “topos” as a shorthand for “\( \mathcal{U} \)-topos” ([SGA4], IV.1.1). The category of sets belonging to \( \mathcal{U} \) is simply denoted by Sets.

For any integers \( e, d \) we denote by \([e, d]\) the set of integers \( n \) such that \( e \leq n \leq d \) and by \( \Sigma_d \) the group of bijections of \([1, d]\) onto itself.

In this paper, all rings are unital and commutative. For any ring \( A \), we denote by \( \text{Alg}_A \) the category of \( A \)-algebras. For any scheme \( S \), we denote by \( \text{Sch}_{/S} \) the category of \( S \)-schemes. We denote by \( S_{\text{ét}} \) (resp. \( S_{\text{ét}}^\text{big} \)) the small étale topos (resp. big étale topos) of a scheme \( S \), i.e. the topos of sheaves of sets for the étale topology ([SGA4], VII.1.2) on the category of étale \( S \)-schemes (resp. on \( \text{Sch}_{/S} \)), and by \( S_{\text{fppf}} \) the big fppf topos of \( S \), i.e. the topos of sheaves of sets for the fppf topology on \( \text{Sch}_{/S} \) ([SGA4], VII.4.2). If \( f : X \to S \) is a morphism of schemes, then we denote by \((f^{-1}, f_*)\) the induced morphism of topos from \( X_{\text{Ét}} \) to \( S_{\text{Ét}} \). The symbol \( f^* \) will exclusively denote the pullback functor from \( \mathcal{O}_S \)-modules to \( \mathcal{O}_X \)-modules.

2. Preliminaries

2.1. Let \( E \) be a topos and let \( G \) be an abelian group in \( E \). We denote by \( GE \) the category of objects of \( E \) endowed with a left action of \( G \). If \( X \) is an object of \( E \), we denote by \( E_{/X} \) the topos of \( X \)-objects in \( E \). If \( X \) is considered as an object of \( GE \) by endowing it with the trivial left \( G \)-action, then we have \( (GE)_{/X} = G(E_{/X}) \) and this category will be simply denoted by \( GE_{/X} \).

Definition 2.2. A \( G \)-torsor over an object \( X \) of \( E \) is an object \( P \) of \( GE_{/X} \) such that \( P \to X \) is an epimorphism and the morphism
\[
G \times_X P \to P \times_X P
\]
\[
(g, p) \mapsto (g \cdot p, p)
\]
is an isomorphism. We denote by \( \text{Tors}(X, G) \) the full subcategory of \( GE_{/X} \) whose objects are the \( G \)-torsors over \( X \). If \( f : Y \to X \) is a morphism in \( E \), we denote by \( f^{-1} : \text{Tors}(X, G) \to \text{Tors}(Y, G) \) the functor which associates \( f^{-1} P = P \times_X f^* Y \) to a \( G \)-torsor \( P \) over \( X \).

The category \( \text{Tors}(X, G) \) is monoidal, with product
\[
P_1 \otimes P_2 = G_2 \setminus P_1 \times_X P_2,
\]
where \( G_2 \) is the kernel of the multiplication morphism \( G \times G \to G \), and where \( G_2 \to G \times G \) acts diagonally on \( P_1 \times_X P_2 \). The neutral element for this product is the trivial \( G \)-torsor over \( X \), namely \( G \times X \), and each \( G \)-torsor \( P \) over \( X \) is invertible with respect to \( \otimes \), with inverse given by
\[
P^{-1} = \text{Hom}_{GE_{/X}}(P, G \times X),
\]
where \( \text{Hom}_{G/E/X} \) denotes the internal Hom functor in \( G/E/X \).

**Example 2.3.** If \( G = \Lambda^\times \) for some ring \( \Lambda \) in \( E \), then the monoidal category \( \text{Tors}(X,G) \) is equivalent to the groupoid of locally free \( \Lambda \)-modules of rank 1 in \( E/X \). The equivalence is given by the functor which sends an object \( P \) of \( \text{Tors}(X,G) \) to the \( \Lambda \)-module \( G \backslash (\Lambda \times P) \), where the action of \( G = \Lambda^\times \) on \( \Lambda \times P \) is given by the formula \( g \cdot (\lambda,p) = (g\lambda,g \cdot p) \). The functor which sends a locally free \( \Lambda \)-module \( M \) of rank 1 of \( E/X \) to the \( G \)-torsor of isomorphisms of \( \Lambda \)-modules from \( M \) to \( \Lambda \) defines a quasi-inverse to the latter functor.

2.4. Let \( E \) be a topos, and let us denote by 1 its terminal object. Let us consider an exact sequence

\[
1 \to G \xrightarrow{i} P \xrightarrow{\tau} Q \to 1
\]

of abelian groups in \( E \). The morphism

\[
G \times_Q P \to P \times_Q P
\]

\[
(g,p) \mapsto (i(g) + p, p)
\]

is an isomorphism, so that \( P \) is a \( G \)-torsor over \( Q \). Moreover, the multiplication morphism

\[
P \times P \to P
\]

factors through \( G_2 \backslash P \times P \), where \( G_2 \to G \times G \) is the kernel of the multiplication morphism of \( G \), acting diagonally on \( P \times P \). We thus obtain a morphism

\[
p_1^{-1} P \otimes p_2^{-1} P \to m^{-1} P
\]

of \( G \)-torsors over \( Q \times Q \), where \( p_1 \) and \( p_2 \) are the canonical projections and \( m \) is the multiplication morphism of \( Q \).

The following definition is inspired by ([MB85], I.2.3):

**Definition 2.5.** Let \( G \) be an abelian group of \( E \) and let \( Q \) be a commutative semigroup of \( E \) (with or without identity). Let \( m : Q \times Q \to Q \) be the multiplication morphism of \( Q \). A **multiplicative \( G \)-torsor** over \( Q \) is a \( G \)-torsor \( P \to Q \), together with an isomorphism \( \theta : p_1^{-1} P \otimes p_2^{-1} P \to m^{-1} P \) of \( G \)-torsors over \( Q \times Q \) where \( p_1 \) and \( p_2 \) are the canonical projections, which satisfy the following two properties.

- **Symmetry:** if \( \sigma \) is the involution of \( Q \times Q \) which switches the two factors, then the isomorphism

\[
p_2^{-1} P \otimes p_1^{-1} P \to \sigma^{-1}(p_1^{-1} P \otimes p_2^{-1} P) \xrightarrow{\theta^{-1}} \sigma^{-1} m^{-1} P \to m^{-1} P
\]

is the composition of \( \theta \) with the canonical isomorphism \( p_2^{-1} P \otimes p_1^{-1} P \to p_1^{-1} P \otimes p_2^{-1} P \).

- **Associativity:** if \( q_i : Q \times Q \to Q \) (resp. \( q_{ij} : Q \times Q \to Q \times Q \) for \( i \in \{1,3\} \) (resp. on the \( i \)-th and \( j \)-th factors for \( (i,j) \in \{1,3\}^2 \) such that \( i < j \)) and if \( m_3 : Q \times Q \times Q \to Q \) is the multiplication morphism, then the diagram of \( G \)-torsors over \( Q \times Q \times Q \)
are morphisms of descent for the fibered category of multiplicative commutative semigroups of $Q$. If

\[
\text{Remark 2.6. If } G = \Lambda^\times \text{ for some ring } \Lambda \text{ in } E, \text{ we use the term "multiplicative locally free } \Lambda\text{-module of rank 1" as a synonym for "multiplicative } G\text{-torsor", when we want to emphasize the locally free } \Lambda\text{-module of rank 1 corresponding to a given } G\text{-torsor, rather than the } G\text{-torsor itself (cf. 2.3).}
\]

**Proposition 2.7.** Let $G$ be an abelian group in $E$, let $Q$ be a commutative semigroup in $E$ and let $I$ be an ideal of $Q$. If the projection morphisms $Q \times I \to Q$ and $I \times I \to I$ onto the first factors are morphisms of descent for the fibered category of multiplicative $G$-torsors (cf. 2.5), then the restriction functor

\[
\text{Tors}^\otimes(Q, G) \to \text{Tors}^\otimes(I, G)
\]

is fully faithful.

Let $i : I \to Q$ be the canonical injection morphism. Let $p_1$ and $p_2$ be the projection morphisms of $Q \times I$ onto its first and second factors respectively, and let $m : Q \times I \to I$ be the multiplication morphism. Let $(P, \theta)$ and $(P', \theta')$ be multiplicative $G$-torsors over $Q$. We have an isomorphism

\[
\beta_P : p_1^{-1}P \xrightarrow{(\text{id} \times m)^{-1}\theta} m^{-1}P \otimes p_2^{-1}P^{-1},
\]

and similarly for $P'$. If $\alpha : i^{-1}P \to i^{-1}P'$ is a morphism of multiplicative $G$-torsors over $I$, then $\beta_P^{-1}(m^{-1}\alpha \otimes p_2^{-1}\alpha)\beta_P$ is an isomorphism from $p_1^{-1}P$ to $p_1^{-1}P'$, which is compatible with the canonical descent datum for $p_1$ associated to $p_1^{-1}P$ and $p_1^{-1}P'$. Since $p_1$ is a morphism of descent for the fibered category of multiplicative $G$-torsors, there is a unique morphism $\gamma : P \to P'$ of multiplicative $G$-torsors over $Q$ such that $p_1^{-1}\gamma = \beta_P^{-1}(m^{-1}\alpha \otimes p_2^{-1}\alpha)\beta_P$. The restriction of $p_1^{-1}\gamma$ to $I \times I$ is the pullback of $\alpha$ by the first projection, which is a morphism of descent for the fibered category of multiplicative $G$-torsors, so that the restriction of $\gamma$ to $I$ is $\alpha$.

**Proposition 2.8.** Let $G$ be an abelian group in $E$, and let $\rho : M \to Q$ be a morphism of commutative semigroups in $E$. If $\rho$ (resp. $\rho \times \rho$ and $\rho \times \rho \times \rho$) is a morphism of effective descent (resp. of descent) for the fibered category of $G$-torsors, then $\rho$ is a morphism of effective descent for the fibered category of multiplicative $G$-torsors.

A descent datum of multiplicative $G$-torsors for $\rho$ yields a descent datum of $G$-torsors for $\rho$, hence a $G$-torsor over $Q$ by hypothesis. Since $\rho \times \rho$ and $\rho \times \rho \times \rho$ are morphisms of descent for the fibered category of $G$-torsors, the structure of multiplicative $G$-torso descends as well. Details are omitted.
Proposition 2.9. Let \( G \) and \( Q \) be abelian groups in \( E \). The groupoid \( \text{Tors}^\circ(Q,G) \) of multiplicative \( G \)-torsors over \( Q \) is equivalent as a monoidal category to the groupoid of extensions of \( Q \) by \( G \) in \( E \), with the Baer sum as a monoidal structure.

We have already seen how to associate a multiplicative \( G \)-torsor to an extension of \( Q \) by \( G \). This construction is functorial, and the corresponding functor is an equivalence by ([MB85], I.2.3.10).

Corollary 2.10. Let \( G \) and \( Q \) be abelian groups in \( E \). The group of isomorphism classes of multiplicative \( G \)-torsors over \( Q \) is isomorphic to the group \( \text{Ext}^1(Q,G) \) of isomorphism classes of extensions of \( Q \) by \( G \) in \( E \).

2.11. Let \( S \) be a scheme, let \( X \) be an \( S \)-scheme, and let \( G \) be a finite abelian group. Let \( P \) be a \( G \)-torsor over \( X \) in \( S_{\text{\acute{e}t}} \). Since \( P \to X \) is an epimorphism in \( S_{\text{\acute{e}t}} \), there is an \( \text{\acute{e}tale} \) cover \((X_i \to X)_{i \in I}\) such that for each \( i \in I \), the morphism \( X_i \to X \) factors through \( P \to X \). In particular, for each \( i \in I \) the \( G \)-torsor \( P \times_X X_i \to X_i \) is isomorphic to the trivial \( G \)-torsor \( G \times X_i \to X_i \), so that \( P \times_X X_i \) is representable by a finite \( \text{\acute{e}tale} \) \( X_i \)-scheme. By \( \text{\acute{e}tale} \) descent of affine morphisms, we obtain:

Proposition 2.12. Let \( G \) be a finite abelian group, let \( S \) be a scheme, and let \( P \) be a \( G \)-torsor over an \( S \)-scheme \( X \) in \( S_{\text{\acute{e}t}} \). Then the \( \text{\acute{e}tale} \) sheaf \( P : \text{Sch}_S \to \text{Sets} \) is representable by a finite \( \text{\acute{e}tale} \) \( X \)-scheme.

The topos \((S_{\text{\acute{e}t}})_S \times X \) coincides with \( X_{\text{\acute{e}t}} \). The category of \( G \)-torsors over \( X \) in \( S_{\text{\acute{e}t}} \) is therefore equivalent to the category of \( G \)-torsors over the terminal object in \( X_{\text{\acute{e}t}} \), and Proposition 2.12 yields:

Corollary 2.13. Let \( G \) be a finite abelian group, let \( S \) be a scheme, and let \( X \) be an \( S \)-scheme. Then the category of \( G \)-torsors over \( X \) in \( S_{\text{\acute{e}t}} \) is equivalent to the category of \( G \)-torsors over the terminal object in \( X_{\text{\acute{e}t}} \).

2.14. Let \( S \) be a scheme, and let \( G \) be a finite abelian group. Let \( Q \) be a commutative \( S \)-group scheme, and let \( M \) be a sub-\( S \)-semigroup scheme of \( Q \).

Proposition 2.15. Assume that the morphism

\[
\rho : M \times_S M \to Q \\
(x,y) \mapsto xy^{-1}
\]

is faithfully flat and quasi-compact, and that \( M \) is flat over \( S \). Then the restriction functor

\[
\text{Tors}^\circ(Q,G) \to \text{Tors}^\circ(M,G),
\]

is an equivalence of categories.

Let \((P,\theta)\) be a multiplicative \( G \)-torsor over \( M \). For \( i \in [1,4] \), let \( r_i \) be the projection of \( R = (M \times_S M) \times_{\rho,Q,\rho} (M \times_S M) \) onto its \( i \)-th factor. Similarly, for \( i,j \in [1,4] \) such that \( i < j \), we denote by \( r_{ij} : R \to M \times M \) the projection on the \( i \)-th and \( j \)-th factors. We then have a sequence of isomorphisms

\[
(r_1^{-1}P \otimes r_2^{-1}P^{-1}) \otimes (r_3^{-1}P \otimes r_4^{-1}P^{-1})^{-1} \to r_{14}^{-1}(p_1^{-1}P \otimes p_2^{-1}P) \otimes r_{23}^{-1}(p_1^{-1}P \otimes p_2^{-1}P)^{-1} \\
\Rightarrow r_{14}^{-\theta \otimes (r_{23}^{-\theta})^{-1}} \to (mr_{14})^{-1}P \otimes ((mr_{23})^{-1}P)^{-1},
\]

of \( G \)-torsors over \( R \), where \( m : M \times_S M \to M \) is the multiplication of \( M \). Since \( mr_{14} = mr_{23} \), the latter \( G \)-torsor is canonically trivial. We thus obtain an isomorphism

\[
\psi : r_1^{-1}P \otimes r_2^{-1}P^{-1} \to r_3^{-1}P \otimes r_4^{-1}P^{-1},
\]
of $G$-torsors over $R$. The associativity of $\theta$ (cf. 2.5) implies that $\psi$ is a cocycle, i.e. $(p_1^{-1}P \otimes p_2^{-1}P^{-1}, \psi)$ is a descent datum for $\rho$. By Proposition 2.12 and since faithfully flat and quasicompact morphisms of schemes are of effective descent for the fibered category of affine morphisms, the conditions of Proposition 2.8 are satisfied, and thus there exists a multiplicative $G$-torsor $P'$ over $Q$ and an isomorphism $\alpha: \rho^{-1}P' \rightarrow p_1^{-1}P \otimes p_2^{-1}P^{-1}$ such that $\psi$ is given by the composition

$$r_1^{-1}P \otimes r_2^{-1}P^{-1} \rightarrow_{r_1^{-1}\alpha^{-1}} (pr_{12})^{-1}P' = (pr_{34})^{-1}P' \rightarrow_{r_3^{-1}\alpha} r_3^{-1}P \otimes r_4^{-1}P^{-1}.$$  

The association $P \mapsto P'$ then defines a functor from Tors$^\otimes(M, G)$ to Tors$^\otimes(Q, G)$. For any multiplicative $G$-torsor $U$ over $Q$, we have an isomorphism $U \rightarrow (U \times Q M)$ by functoriality, which is functorial in $U$.

We now construct, for any multiplicative $G$-torsor $(P, \theta)$ over $M$, an isomorphism $P \rightarrow P' \times Q M$ of multiplicative $G$-torsors which is functorial in $P$. Let $\nu: M \times S M \rightarrow M \times S M$ be the morphism which sends a section $(x, y)$ to $(xy, y)$. We have an isomorphism

$$(\rho \nu)^{-1}P' \overset{\nu^{-1}}{\longrightarrow} \nu^{-1}(p_1^{-1}P \otimes p_2^{-1}P^{-1}) \rightarrow m^{-1}P \otimes p_2^{-1}P^{-1} \overset{\theta^{-1}}{\rightarrow} p_1^{-1}P.$$  

The diagram

$$\begin{array}{ccc}
M \times S M & \xrightarrow{\nu} & M \times S M \\
\downarrow{\rho} & & \downarrow{\rho} \\
M & \xrightarrow{p_1} & Q
\end{array}$$

is commutative, hence $(\rho \nu)^{-1}P'$ is isomorphic to $p_1^{-1}(P' \times Q M)$. We thus obtain an isomorphism

$$\beta: p_1^{-1}P \rightarrow p_1^{-1}(P' \times Q M),$$

of multiplicative $G$-torsors. The morphism $\beta$ is compatible with the canonical descent data for $p_1$ associated to $p_1^{-1}P$ and $p_1^{-1}(P' \times Q M)$. Since $p_1$ is a covering for the fpqc topology, Proposition 2.8 applies, hence there is a unique isomorphism $\gamma: P \rightarrow P' \times Q M$ of multiplicative $G$-torsors such that $\beta = p_1^{-1}\gamma$. The construction of this isomorphism of multiplicative $G$-torsors is functorial in $P$, hence the result.

2.16. Let $A$ be a ring. If $M$ is an $A$-module, we denote by $\underline{M}$ the functor $B \mapsto M \otimes_A B$ from Alg$_A$ to Sets.

**Definition 2.17.** ([SGA4], XVII 5.5.2.2) Let $M$ and $N$ be $A$-modules. A **polynomial map** from $M$ to $N$ is a morphism of functors $\underline{M} \rightarrow \underline{N}$. A polynomial map $f: \underline{M} \rightarrow \underline{N}$ is **homogeneous of degree** $d$ if for any $A$-algebra $B$, any element $\lambda$ of $B$ and any element $m$ of $M(B)$, we have $f(\lambda m) = \lambda^d f(m)$.

For each integer $d$ and any $A$-module $M$, let $\text{TS}^d_A(M) = (M \otimes_A d)^{\otimes d}$ be the $A$-module of symmetric tensors of degree $d$ with coefficients in $M$. If $M$ is a free $A$-module with basis $(e_i)_{i \in I}$, then we have a decomposition

$$(2.17.1) \quad \text{TS}^d_A(M) = \bigoplus_{\beta: [1,d] \rightarrow I} A e_{\beta(1)} \otimes \cdots \otimes e_{\beta(d)} = \bigoplus_{\alpha: I \rightarrow \mathbb{N}} A e_{\alpha},$$

where $\sum_{i \in I} \alpha(i) = d$.
where we have set
\[ e_\alpha = \sum_{\beta: [1,d] \rightarrow I \forall i, \beta^{-1}(\{i\}) = \alpha(i)} e_{\beta(1)} \otimes \cdots \otimes e_{\beta(d)}. \]

In particular \( TS^d_A(M) \) is a free \( A \)-module, and its formation commutes with base change by any ring morphism \( A \rightarrow B. \)

**Proposition 2.18.** Let \( M \) be a flat \( A \)-module and let \( d \geq 0 \) be an integer. Then \( TS^d_A(M) \) is a flat module, and for any \( A \)-algebra \( B \) the canonical homomorphism
\[ TS^d_A(M) \otimes_A B \rightarrow TS^d_B(M \otimes_A B) \]
is bijective.

Any flat \( A \)-module is a filtered colimit of finite free modules. We have already seen that the conclusion of Proposition 2.18 holds whenever \( M \) is free, hence the conclusion in general since the functor \( TS^d_A \) commutes with filtered colimits.

**Proposition 2.19.** Let \( M \) be a flat \( A \)-module and let \( d \geq 0 \) be an integer. Let \( \gamma_d: M \rightarrow TS^d_A(M) \) be the functor which sends, for any \( A \)-algebra \( B \), an element \( m \) of \( M(B) \) to the element \( m \otimes \gamma_d \) of \( TS^d_B(M \otimes_A B) = TS^d_A(M \otimes_A B) \) (cf. 2.18). Then, for any homogeneous polynomial map \( f: M \rightarrow N \) of degree \( d \), there is a unique \( A \)-linear homomorphism \( \tilde{f}: TS^d_A(M) \rightarrow N \) such that \( f = \gamma_d \).
for any $A$-algebra $B$ and any element $m = \sum_{i \in I} b_i e_i$ of $M(B)$. Using the decomposition (2.17.1), we also have
\[
\gamma_d(m) = \sum_{\beta \vdash [1,d] \rightarrow I} \otimes_{j=1}^{d} b_j e_{\beta(j)} = \sum_{\alpha : I \rightarrow N_{|\alpha|=d}} b_\alpha e_\alpha.
\]

The conclusion of Proposition 2.19 is achieved by taking $\tilde{f}$ to be the unique morphism of $A$-modules from $TS^d_A(M)$ to $N$ which sends $e_\alpha$ to $f_\alpha$.

2.20. Let $A \rightarrow C$ be a ring morphism such that $C$ is a finitely generated projective $A$-module of rank $d$. For any $A$-algebra $B$ and any element $m$ of $C(B)$, we set
\[
N_{C/A}(c) = \det_A(m_c),
\]
where $m_c$ is the $A(B)$-linear endomorphism of $C(B)$ induced by the multiplication by $c$. This defines a homogeneous polynomial map $N_{C/A} : C \rightarrow A$ of degree $d$ (cf. 2.17). By 2.19, there is a unique morphism of $A$-modules $\varphi : TS^d_A(C) \rightarrow A$ such that $N_{C/A} = \varphi \gamma_d$.

**Proposition 2.21** ([SGA4], XVII 6.3.1.6). The morphism of $A$-modules $\varphi : TS^d_A(C) \rightarrow A$ is a morphism of $A$-algebras.

Let $x$ be an element of $C$, and let us consider the morphism of $A$-modules $f : y \rightarrow \varphi(\gamma_d(x)y)$ from $TS^d_A(C)$ to $A$. For any $A$-algebra $B$ and any element $c$ of $C(B)$, we have
\[
f(\gamma_d(c)) = \varphi(\gamma_d(x)\varphi(c)) = \varphi(\gamma(x)c) = N_{C/A}(xc) = N_{C/A}(x)N_{C/A}(c).
\]

by the multiplicativity of determinants, so that $f(\gamma_d(c)) = N_{C/A}(x)\varphi(\gamma_d(c))$. By the uniqueness statement in 2.19, we obtain $f = N_{C/A}(x)\varphi$, i.e. for all $y$ in $TS^d_A(C)$ we have
\[
(2.21.1) \quad \varphi(\gamma(x)y) = N_{C/A}(x)\varphi(y).
\]

For any $A$-algebra $B$, one can apply this argument to the morphism $B \rightarrow C(B)$ instead of $A \rightarrow C$. Thus (2.21.1) also holds for any element $x$ of $C(B)$ and any element $y$ of $TS^d_A(C)(B) = TS^d_A(C)(B)$ (cf. 2.18). Now, let $y$ be an element of $TS^d_A(C)$ and let us consider the morphism of $A$-modules $g : z \rightarrow \varphi(zy)$ from $TS^d_A(C)$ to $A$. We have proved that $g\gamma_d = \varphi(y)N_{C/A}$, hence $g = \varphi(y)\varphi$ by 2.19. Thus $\varphi$ is a morphism of rings. Since $\varphi$ is also $A$-linear, it is a morphism of $A$-algebras.

2.22. Let $S$ be a scheme.

**Definition 2.23** ([SGA1], V.1.7).

- Let $T$ be an object of a category $C$ endowed with a right action of a group $\Gamma$. We say that the quotient $T/\Gamma$ exists in $C$ if the covariant functor
\[
C \rightarrow \text{Sets}
\]
\[
U \mapsto \text{Hom}_C(T, U)^\Gamma
\]
is representable by an object of $C$.

- Let $T$ be an $S$-scheme. An action of a finite group $\Gamma$ on $T$ is admissible if there exists an affine $\Gamma$-invariant morphism $f : T \rightarrow T'$ such that the canonical morphism $\mathcal{O}_{T'} \rightarrow f_*\mathcal{O}_T$ induces an isomorphism from $\mathcal{O}_{T'}$ to $(f_*\mathcal{O}_T)^\Gamma$. 
Proposition 2.24 ([SGA1], V.1.3). Let $T$ be an $S$-scheme endowed with an admissible right action of a finite group $\Gamma$. If $f:T \rightarrow T'$ is an affine $\Gamma$-invariant morphism such that the canonical morphism $\mathcal{O}_{T'} \rightarrow f_\ast \mathcal{O}_T$ induces an isomorphism from $\mathcal{O}_{T'}$ to $(f_\ast \mathcal{O}_T)^\Gamma$, then the quotient $T/\Gamma$ exists and is isomorphic to $T'$.

Proposition 2.25 ([SGA1], V.1.8). Let $T$ be an $S$-scheme endowed with a right action of a finite group $\Gamma$. Then, the action of $\Gamma$ on $T$ is admissible if and only if $T$ is covered by $\Gamma$-invariant affine open subsets.

Proposition 2.26 ([SGA1], V.1.9). Let $T$ be an $S$-scheme endowed with an admissible right action of a finite group $\Gamma$, and let $S'$ be a flat $S$-scheme. Then, the action of $\Gamma$ on the $S'$-scheme $T \times_S S'$ is admissible, and the canonical morphism

$$(T \times S')/\Gamma \rightarrow (T/\Gamma) \times_S S'$$

is an isomorphism.

Let $X$ be an $S$-scheme and let $d \geq 0$ be an integer. The group $\mathfrak{S}_d$ of permutations of $[1,d]$ acts on the right on the $S$-scheme $X^{\times s d} = X \times_S \cdots \times_S X$ by the formula

$$(x_i)_{i \in [1,d]} \cdot \sigma = (x_{\sigma(i)})_{i \in [1,d]}.$$

Proposition 2.27. If $X$ is Zariski-locally quasi-projective over $S$, then the right action of $\mathfrak{S}_d$ on $X^{\times s d}$ is admissible. In particular, the quotient $\text{Sym}_S^d(X) = X^{\times s d}/\mathfrak{S}_d$ exists in the category of $S$-schemes.

Since $X$ is Zariski-locally quasi-projective over $S$, any finite set of points in $X$ with the same image in $S$ is contained in an affine open subset of $X$. Thus $X^{\times s d}$ is covered by open subsets of the form $U^{\times s d}$ where $U$ is an affine open subset of $X$ whose image in $S$ is contained in an affine open subset of $S$. These particular open subsets are affine and $\mathfrak{S}_d$-invariant, so that the action of $\mathfrak{S}_d$ on $X^{\times s d}$ is admissible by Proposition 2.25.

Remark 2.28. If $X = \text{Spec}(B)$ and $S = \text{Spec}(A)$ are affine, then for any $S$-scheme $T$ we have

$$\text{Hom}_{\text{Sch}/S}(X^{\times s d}, T)^{\mathfrak{S}_d} = \text{Hom}_{\text{Alg}_A}(\Gamma(T, \mathcal{O}_T), B^{\mathfrak{S}_d})^{\mathfrak{S}_d}$$

$$= \text{Hom}_{\text{Alg}_A}(\Gamma(T, \mathcal{O}_T), T^{\mathfrak{S}_d}_A(B)),$$

cf. 2.16. Thus $\text{Sym}_S^d(X)$ is representable by the $S$-scheme $\text{Spec}(T^{\mathfrak{S}_d}_A(B))$.

Proposition 2.29. If $X$ is flat and Zariski-locally quasi-projective over $S$, then $\text{Sym}_S^d(X)$ is flat over $S$. Moreover, for any $S$-scheme $S'$, the canonical morphism

$$\text{Sym}_S^d(X \times_S S') \rightarrow \text{Sym}_S^d(X) \times_S S'$$

is an isomorphism.

This follows from Remark 2.28 and from Proposition 2.18.

Proposition 2.30. Let $G$ be a finite abelian group, let $P$ be a $G$-torsor over an $S$-scheme $X$ in $S_{\text{et}}$. Assume that $P$ and $X$ are endowed with right actions from a finite group $\Gamma$ such that the morphism $P \rightarrow X$ is $\Gamma$-equivariant, and that the following properties hold:

(a) The right $\Gamma$-action on $P$ commutes with the left $G$-action.

(b) The right $\Gamma$-action on $X$ is admissible (cf. 2.23), and the quotient morphism $X \rightarrow X/\Gamma$ is finite.

(c) For any geometric point $\bar{x}$ of $X$, the action of the stabilizer $\Gamma_{\bar{x}}$ of $\bar{x}$ in $\Gamma$ on the fiber $P_{\bar{x}}$ of $P$ at $\bar{x}$ is trivial.
Then the action of $\Gamma$ on $P$ is admissible, and $P/\Gamma$ is a $G$-torsor over $X/\Gamma$ in $\mathcal{S}_{\text{et}}$.

If $U$ is a $\Gamma$-invariant affine open subset of $X$, then its inverse image in $P$ is a $\Gamma$-invariant affine open subset of $P$ since the morphism $P \to X$ is affine by Proposition 2.12. Thus the right action of $\Gamma$ on $P$ is admissible by Proposition 2.25. It is sufficient to show that for any flat morphism $Y \to X/\Gamma$ from the spectrum of a strictly henselian local ring to $X/\Gamma$, the fiber product $P/\Gamma \times_{X/\Gamma} Y$ is a (trivial) $G$-torsor over $Y$. Let $Z$ be the fiber product $X \times_{X/\Gamma} Y$. By Proposition 2.26, the actions of $\Gamma$ on $Z$ and $P \times_X Z$ are admissible, so that the quotients $Z/\Gamma$ and $(P \times_X Z)/\Gamma$ exist, and the canonical morphisms

$$Z/\Gamma \to Y,$$

$$(P \times_X Z)/\Gamma \to P/\Gamma \times_{X/\Gamma} Y,$$

are isomorphisms. By replacing $P$ and $X$ by $P \times_X Z$ and $Z$, we can therefore assume that $X/\Gamma$ is the spectrum of a strictly henselian local ring. In this case, since the morphism $X \to X/\Gamma$ is finite by (b), the scheme $X$ is a finite disjoint union of spectra of strictly henselian local rings. In particular, the set $\pi_0(X)$ of connected components of $X$ is finite, and we have

$$X = \coprod_{C \in \pi_0(X)} C.$$

The group $\Gamma$ acts on the finite set $\pi_0(X)$ on the right. Let $C_1, \ldots, C_r$ be a set of representatives for the orbits of $\Gamma$ on $\pi_0(X)$, and let $\bar{c}_1, \ldots, \bar{c}_r$ be their respective closed points. For each $i \in [1, r]$, an element of $\Gamma$ fixes $\bar{c}_i$ if and only if it stabilizes $C_i$, so that the stabilizer of $C_i$ in $\pi_0(X)$ coincides with the stabilizer $\Gamma_{\bar{c}_i}$ of the geometric point $\bar{c}_i$ of $X$. We therefore have a decomposition

$$X = \prod_{i=1}^r \prod_{\gamma \in \Gamma_{\bar{c}_i}} \gamma(C_i).$$

Correspondingly, the quotient $X/\Gamma$ admits the following decomposition:

$$X/\Gamma = \prod_{i=1}^r C_i/\Gamma_{\bar{c}_i}.$$

Since $X/\Gamma$ is connected, we must have $r = 1$, i.e. $X/\Gamma = C_1/\Gamma_{\bar{c}_1}$. In a similar fashion, we have

$$P/\Gamma = \left( \prod_{\gamma \in \Gamma_{\bar{c}_1}} P \times_X \gamma(C_1) \right)/\Gamma = (P \times_X C_1)/\Gamma_{\bar{c}_1}.$$

Since $C_1$ is the spectrum of a strictly henselian local ring, the $G$-torsor $P \times_X C_1$ is trivial. Let $e : C_1 \to P \times_X C_1$ be a section of this $G$-torsor. For each element $\gamma$ of $\Gamma_{\bar{c}_1}$, the section $(e \circ \gamma^{-1}) \cdot \gamma$ takes the form $g_\gamma \cdot e$, where $g_\gamma$ is a section of the trivial $G$-torsor over $C_1$. The corresponding map $g_\gamma : C_1 \to G$ is locally constant, hence constant since $C_1$ is connected. But $g_\gamma$ takes the value 1 at the closed point of $C_1$ by (c), so that $g_\gamma$ is the unit section. We conclude that $(e \circ \gamma^{-1}) \cdot \gamma = e$, and thus that the isomorphism

$$G \times C_1 \to P \times_X C_1$$

$$(g, c) \mapsto (ge(c), c),$$

is $\Gamma_{\bar{c}_1}$-equivariant, where $\Gamma_{\bar{c}_1}$ acts on $G \times C_1$ by the formula $(g, c) \cdot \gamma = (g, c\gamma)$. We therefore obtain an isomorphism

$$G \times C_1/\Gamma_{\bar{c}_1} \to (P \times_X C_1)/\Gamma_{\bar{c}_1}.$$
so that \((P \times X C_1)/\Gamma_{\xi_1} = P/\Gamma\) is a (trivial) \(G\)-torsor over \(C_1/\Gamma_{\xi_1} = X/\Gamma\).

2.31. Let \(S\) be a scheme, let \(X\) be an \(S\)-scheme and let \(d \geq 1\) be an integer. Let \(G\) be a finite abelian group, and let \(P \to X\) be a \(G\)-torsor over \(X\) in \(S_{\text{\acute{e}t}}\). By 2.12, the sheaf \(P\) is representable by a finite \(\text{\acute{e}tale}\) \(X\)-scheme.

For each \(i \in [1, d]\) let \(p_i : X^{\times s d} \to X\) be the projection on \(i\)-th factor, and let us consider the \(G\)-torsor

\[
p_i^{-1} P \otimes \cdots \otimes p_d^{-1} P = G_d \setminus P^{\times s d}
\]

over \(X^{\times s d}\), where \(G_d \subseteq G^d\) is the kernel of the multiplication morphism \(G^d \to G\). By 2.12, the object \(G_d \setminus P^{\times s d}\) of \(S_{\text{\acute{e}t}}\) is representable by an \(S\)-scheme which is finite \(\text{\acute{e}tale}\) over \(X^{\times s d}\). The group \(S_d\) acts on the right on \(G_d \setminus P^{\times s d}\) by the formula

\[
(p_i)_{i \in [1, d]} \cdot \sigma = (p_{\sigma(i)})_{i \in [1, d]}.
\]

This action of \(S_d\) commutes with the left action of \(G\) on \(G_d \setminus P^{\times s d}\).

**Proposition 2.32.** If \(X\) is Zariski-locally quasi-projective on \(S\), then the right action of \(S_d\) on \(G_d \setminus P^{\times s d}\) is admissible (cf. 2.23), so that the quotient \(P^{[d]}\) of \(G_d \setminus P^{\times s d}\) by \(S_d\) exists in \(\text{Sch}/S\). Moreover, the canonical morphism \(P^{[d]} \to \text{Sym}^d_S(X)\) is a \(G\)-torsor, and the morphism

\[
p_i^{-1} P \otimes \cdots \otimes p_d^{-1} P \to r^{-1} P^{[d]}
\]

where \(r : X^{\times s d} \to \text{Sym}^d_S(X)\) is the canonical projection, is an isomorphism of \(G\)-torsors over \(X^{\times s d}\).

By 2.27 and 2.30, it is sufficient to show that if \(\bar{x} = (\bar{x}_i)_{i=1}^d\) is a geometric point of \(X^{\times s d}\), then the stabilizer of \(\bar{x}\) in \(S_d\) acts trivially on \((G_d \setminus P^{\times s d})_{\bar{x}}\). Assume that the finite set \(\{\bar{x}_i \mid i \in [1, d]\}\) has exactly \(r\) distinct elements \(\bar{y}_1, \ldots, \bar{y}_r\), where \(\bar{y}_j\) appears with multiplicity \(d_j\). Then the stabilizer of \(\bar{x}\) in \(S_d\) is isomorphic to the subgroup \(\prod_{j=1}^r S_{d_j}\) of \(S_d\). For each \(j \in [1, r]\), the \(G\)-torsor \(P_{\bar{y}_j}\) is trivial, and if \(e\) is a section of this torsor then \((e)^{d_j}_{i=1}\) is a section of \(G_{d_j} \setminus P_{\bar{y}_j}^{d_j}\) which is \(S_{d_j}\)-invariant.

The action of \(S_{d_j}\) on \(G_{d_j} \setminus P_{\bar{y}_j}^{d_j}\) is therefore trivial, so that the action of \(\prod_{j=1}^r S_{d_j}\) on the \(G\)-torsor

\[
(G_d \setminus P^{\times s d})_{\bar{x}} = G_r \setminus \left(\prod_{j=1}^r G_{d_j} \setminus P_{\bar{y}_j}^{d_j}\right)
\]

is trivial as well.

**Proposition 2.33.** If \(X\) is flat and Zariski-locally quasi-projective on \(S\), then for any \(S\)-scheme \(S'\), the canonical morphism

\[
(P \times_S S')^{[d]} \to P^{[d]} \times_S S'
\]

is an isomorphism.

By Proposition 2.29, the canonical morphism

\[
\text{Sym}^d_S(X \times_S S') \to \text{Sym}^d_S(X) \times_S S'
\]

is an isomorphism. Thus the second morphism in the composition

\[
(P \times_S S')^{[d]} \to (P^{[d]} \times_S S') \times_{\text{Sym}^d_S(X) \times_S S'} \text{Sym}^d_S(X \times_S S') \to P^{[d]} \times_S S'
\]

is an isomorphism, while the first morphism is a morphism of \(G\)-torsors, hence an isomorphism.
3. Geometric Local Class Field Theory

Let $k$ be a perfect field, and let $L$ be a completely discretely valued extension of $k$ with residue field $k$. We denote by $\mathcal{O}_L$ its ring of integers, and by $\mathfrak{m}_L$ the maximal ideal of $\mathcal{O}_L$.

3.1. Let us consider the functor

$$\mathcal{O}_L : \text{Alg}_k \to \text{Alg}_{\mathcal{O}_L},$$

$$A \mapsto \lim_n A \otimes_k \mathcal{O}_L / \mathfrak{m}_L^n,$$

with values in the category of $\mathcal{O}_L$-algebras.

**Proposition 3.2.** The functor $\mathcal{O}_L$ is representable by a $k$-scheme.

Indeed, if $\pi$ is a uniformizer of $L$, then we have an isomorphism $k((t)) \to L$ which sends $t$ to $\pi$, so that the functor $\mathcal{O}_L$ is isomorphic to the functor $A \mapsto A[[t]]$, which is representable by an affine space over $k$ of countable dimension.

**Corollary 3.3.** The functor $L = \mathcal{O}_L \otimes \mathcal{O}_L$ is representable by an ind-$k$-scheme.

We can assume that $L$ is the field of Laurent series $k((t))$. In this case, we have

$$\mathcal{L}(A) = A((t)) = \colim_n t^{-n}A[[t]]$$

for any $k$-algebra $A$, and for each integer $n$ the functor $A \mapsto t^{-n}A[[t]]$ is representable by a $k$-scheme, cf. 3.2.

**Proposition 3.4.** Let $G$ (resp. $H$) be the functor from $\text{Alg}_k$ to the category of groups which associates to a $k$-algebra $A$ the subgroup $G(A)$ of $A((t))^\times$ consisting of Laurent series of the form $1 + \sum_{r>0} a_r t^{-r}$ where $a_r$ is a nilpotent element of $A$ for each $r > 0$ and vanishes for $r$ large enough (resp. of Laurent series of the form $1 + \sum_{r>0} a_r t^{-r}$ where $a_r$ belongs to $A$ for each $r > 0$). Let $\mathbb{Z}$ be the functor which sends a $k$-algebra $A$ to the group of locally constant functions $\text{Spec}(A) \to \mathbb{Z}$. Then for any uniformizer $\pi$ of $L$, the morphism

$$G_{m,k} \times \mathbb{Z} \times G \times H \to \mathbb{L}^\times,$$

$$(a, n, g, h) \mapsto a\pi^n g(\pi)h(\pi),$$

is an isomorphism of group-valued functors.

Let $A$ be a $k$-algebra. By ([ICC13], 0.8), every invertible element $u$ of $A((t))$ uniquely factors as $u = t^n f(t) h(t)$ where $f(t)$ and $h(t)$ are elements of $A[[t]]^\times$ and $G(A)$ respectively, and $n : \text{Spec}(A) \to \mathbb{Z}$ is a locally constant function. Moreover, there is a unique factorisation $f(t) = ag(t)$ where $a$ and $g(t)$ belong to $A^\times$ and $H(A)$ respectively, hence the result.

**Corollary 3.5.** The functor $\mathbb{L}^\times$ is representable by an ind-$k$-scheme. Moreover, its restriction to the category of reduced $k$-algebras is representable by a reduced $k$-scheme.

The groups $\mathbb{Z}$ and $H$ from Proposition 3.4 are representable by reduced $k$-schemes, and so is $G_{m,k}$. Moreover, the group $G$ from 3.4 is the filtered colimit of the functor $n \mapsto G_n$, where $G_n$ is the functor which associates to a $k$-algebra $A$ the subset $G_n(A)$ of $A((t))^\times$ consisting of Laurent series of the form $1 + \sum_{r=1}^{n} a_r t^{-r}$ where $a_r^n = 0$ for each $r \in [1, n]$. For each $n$, the functor $G_n$ is representable by an affine $k$-scheme. Thus $G$ is representable by an ind-$k$-scheme, and so is $\mathbb{L}^\times$ by 3.4. The last assertion of Corollary 3.5 follows from the fact that $G(A)$ is the trivial group for any reduced $k$-algebra $A$. 



Corollary 3.6. Let \( d \geq 0 \) be an integer, and let \( U^{(d)}_L \) be the subfunctor \( 1 + m^d_L O_L \) (resp. \( O_L^\times \)) of \( L^\times \) for \( d \geq 1 \) (resp. for \( d = 0 \)). Then the functor
\[
L^\times / U^{(d)}_L : \text{Alg}_k \to \text{Alg}_{O_L}
\]
\[
A \mapsto L^\times (A)/U^{(d)}_L (A),
\]
is representable by an ind-\( k \)-scheme. Moreover, its restriction to the category of reduced \( k \)-algebras is representable by a reduced \( k \)-scheme.

According to Proposition 3.4, it is sufficient to show that \((G_{m,k} \times H)/U^{(d)}_{k(i\ell)}\) is representable by a reduced \( k \)-scheme. The case \( d = 0 \) is clear, while for \( d \geq 1 \), we have for any \( k \)-algebra \( A \) a bijection
\[
A^\times \times A^{[1,d-1]} \to (G_{m,k} \times H)(A)/U^{(d)}_{k(i\ell)}(A)
\]
\[
(a_i)_{0 \leq i \leq d-1} \mapsto \sum_{i=0}^{d-1} a_it^i,
\]
hence the result.

3.7. From now on, we consider \( \text{Spec}(L) \), \( L^\times \) and \( L^\times / U^{(d)}_L \) for each integer \( d \geq 0 \) as objects of the topos \( \text{Spec}(k)_{\text{ét}} \). Let \( \pi \) be a uniformizer of \( L \). We denote by \( \Pi \) the element of \( L(k) \) corresponding to \( \pi \) via the canonical identification \( L \simeq L(k) \). Thus the functor \( L^\times \) is given by
\[
L^\times : A \in \text{Alg}_k \mapsto A(\Pi)^\times.
\]
In the particular, the Laurent series \( (\Pi - \pi)^{-1}\Pi = -\sum_{n \geq 1} \pi^{-n}\Pi^n \) defines an \( L \)-point of \( L^\times \). We denote by \( \varphi : \text{Spec}(L) \to L^\times \) the corresponding morphism.

Theorem 3.8 ([Su13], Th. A (1)). Let \( G \) be a finite abelian group. The functor
\[
\text{Tors}^\otimes(L^\times, G) \to \text{Tors}(\text{Spec}(L), G)
\]
\[
P \mapsto \varphi^{-1}P
\]
is an equivalence of categories (cf. 2.2, 2.5).

In the case where \( k \) is algebraically closed, Serre constructed in [Se61] an equivalence
\[
\text{Tors}(\text{Spec}(L), G) \to \text{Tors}^\otimes(L^\times, G).
\]
In [Su13], Suzuki shows that the functor from Theorem 3.8 is a quasi-inverse to Serre’s functor when \( k \) is algebraically closed, and extends the result to arbitrary perfect residue fields. His proof of Theorem 3.8 relies on the Albanese property of the morphism \( \varphi \), previously established by Contou-Carrère.

Let \( L^{\text{sep}} \) be a separable closure of \( L \), and let \( G_L \) be the Galois group of \( L^{\text{sep}} \) over \( L \), so that the small étale topos of \( \text{Spec}(L) \) is isomorphic to the topos of sets with continuous left \( G_L \)-action. By 2.13, the category of \( G \)-torsors over \( \text{Spec}(L) \) in \( \text{Spec}(k)_{\text{ét}} \) is isomorphic to the category of \( G \)-torsors in the small étale topos \( \text{Spec}(L)_{\text{ét}} \). Correspondingly, for each finite abelian group \( G \), the group of isomorphism classes of the category \( \text{Tors}(\text{Spec}(L), G) \) is isomorphic to the group of continuous homomorphisms from \( G_L \) to \( G \).

We denote by \( (G_L^j)_{j \geq 1} \) the ramification filtration of \( G_L \) ([Se68], IV.3), so that \( G_L^{-1} = G_L \) and \( G_L^0 \) is the inertia subgroup of \( G_L \), while \( G_L^+ = \cup_{j > 0} G_L^j \) is the wild inertia subgroup of \( G_L \).
Definition 3.9. Let $G$ be a finite abelian group and let $d \geq 0$ be a rational number. A $G$-torsor over $\text{Spec}(L)$ (in $\text{Spec}(k)_\text{ét}$), corresponding to a continuous homomorphism $\rho : G_L \to G$, is said to have \textbf{ramification bounded by $d$} if $\rho(G^d_L) = \{1\}$. A $G$-torsor over $\text{Spec}(L)$ with ramification bounded by 0 (resp. 1) is said to be unramified (resp. tamely ramified).

Remark 3.10. If $P \to \text{Spec}(L)$ is a $G$-torsor in $\text{Spec}(k)_\text{ét}$, then we have a finite decomposition

$$P = \prod_{i \in I} \text{Spec}(L_i),$$

where each $L_i$ is a finite separable extension of $L$. The $G$-torsor $P$ has ramification bounded by $d$ if and only if for each $i$ the extension $L_i/L$ has ramification bounded by $d$, in the sense $G^d_L$ acts trivially on the finite set $\text{Hom}_{L_i}(\text{Spec}(L_i), L^\text{sep})$.

Proposition 3.11. Let $G$ be a finite abelian group, let $d \geq 0$ be an integer, and let $P$ be a multiplicative $G$-torsor $P$ over $L^\times$ (cf. 2.5). Assume that $k$ is algebraically closed. Then $\varphi^{-1}P$ has ramification bounded by $d$ (cf. 3.9) if and only if $P$ is the pullback of a multiplicative $G$-torsor over $L^\times/U^d_L$ (cf. 3.6).

This follows from ([Se61], 3.2 Th. 1) and from the compatibility of $\varphi^{-1}$ with Serre’s construction ([Su13], Th. A (2)).

3.12. Let $\pi$ and $\varphi$ be as in 3.7. Let $K$ be a closed sub-extension of $k$ in $L$, such that $K \to L$ is a finite extension of degree $d$. Since $L$ is a finite free $K$-algebra of rank $d$, we have a canonical morphism of $K$-schemes

$$\psi : \text{Spec}(K) \to \text{Sym}^d_K(\text{Spec}(L))$$

by 2.21.

Proposition 3.13. The composition

$$\text{Spec}(K) \xrightarrow{\psi} \text{Sym}^d_K(\text{Spec}(L)) \to \text{Sym}^d_K(\text{Spec}(L)) \xrightarrow{\text{Sym}^d_K(\varphi)} \text{Sym}^d_K(L^\times) \to L^\times,$$

where the last morphism is given by the multiplication, corresponds to the $K$-point $P_{\pi}(\Pi)^{-1} \Pi^d$ of $L^\times$, where the polynomial $P_{\pi}$ is the characteristic polynomial of the $K$-linear endomorphism $x \mapsto \pi x$ of $L$.

We first describe the morphism $\psi$. The scheme $\text{Sym}^d_K(\text{Spec}(L))$ is the spectrum of the $k$-algebra $\text{TS}^d_K(L)$ of symmetric tensors of degree $d$ in $L$, cf. 2.27. The elements $e_i = \pi^{i-1}$ for $i = 1, \ldots, d$ form a $K$-basis of $L$, so that we have a decomposition

$$\text{TS}^d_K(L) = \bigoplus_{\alpha : [1,d] \to \mathbb{N}, \sum_i \alpha(i) = d} K e_\alpha,$$

where we have set (cf. 2.16)

$$e_\alpha = \sum_{\beta : [1,d] \to [1,d], \forall i, \beta^{-1}(i)) = \alpha(i)} e_{\beta(1)} \otimes \cdots \otimes e_{\beta(d)}.$$

Let us write the norm polynomial as

$$N_{L/K} \left( \sum_{i=1}^{d} x_i e_i \right) = \sum_{\alpha : [1,d] \to \mathbb{N}, \sum_i \alpha(i) = d} f_\alpha x^\alpha,$$
where $x^\alpha = x_1^{\alpha(1)} \ldots x_d^{\alpha(d)}$, and the $f_\alpha$’s are uniquely determined elements of $K$. The morphism $\operatorname{TS}_K^d(L) \to K$ corresponding to $\psi$ is the unique $K$-linear homomorphism which sends $e_\alpha$ to $f_\alpha$ (cf. 2.19 and its proof).

Next we describe the composition

$$\operatorname{Sym}_K^d(\operatorname{Spec}(L)) \to \operatorname{Sym}_K^d(\operatorname{Spec}(L)) \xrightarrow{\operatorname{Sym}_K^d(\varphi)} \operatorname{Sym}_K^d(\mathbb{L}^\times) \to \mathbb{L}^\times.$$ 

Its precomposition with the projection $\operatorname{Spec}(L)^\times \times \kappa^d \to \operatorname{Sym}_K^d(\operatorname{Spec}(L))$ corresponds to the element of $L^\otimes \kappa^d(\mathbb{P})^\times$ given by the formula

$$\prod_{i=1}^d \left( (\mathbb{P} - 1 \otimes (i-1) \otimes \pi \otimes 1^{d-i})^{-1} \mathbb{P} \right) = P(\mathbb{P})^{-1}P^d,$$

where the polynomial $P(\mathbb{P})$ can be computed as follows:

$$P(\mathbb{P}) = \frac{d}{d-1} \left( \prod_{i=1}^d \left( \mathbb{P} - 1 \otimes (i-1) \otimes \pi \otimes 1^{d-i} \right) \right)
= \sum_{r=0}^{d-1} (-1)^r \mathbb{P}^{d-r} \sum_{|\{i_1, \ldots, i_d\}|=r} \pi^{i_1} \otimes \cdots \otimes \pi^{i_d}
= \sum_{r=0}^{d} (-1)^r e_\alpha \mathbb{P}^{d-r},$$

where $e_r : [1, d] \to \mathbb{N}$ is the map which sends 1 and 2 to $d - r$ and $r$ respectively, and any $i > 2$ to 0. The image of $P(\mathbb{P})$ by $\psi$ in $K[\mathbb{P}]$ is the polynomial

$$\sum_{r=0}^{d} (-1)^r f_\alpha \mathbb{P}^{d-r} = N_{L[\mathbb{P}]/K[\mathbb{P}]} (\mathbb{P}e_1 - e_2).$$

Since $e_1 = 1$ and $e_2 = \pi$, we obtain 3.13.

**Proposition 3.14.** Let $G$ be a finite abelian group, and let $Q$ be a $G$-torsor over $\operatorname{Spec}(L)$ (in $\operatorname{Spec}(k)_{\Et}$) of ramification bounded by $d$ (cf. 3.9). Then $\psi^{-1}Q^d$ (cf. 2.32) is tamely ramified on $\operatorname{Spec}(K)$.

Let $K'$ be the maximal unramified extension of $K$ in a separable closure of $K$. The formation of $\operatorname{Sym}_K^d(\operatorname{Spec}(L))$ is compatible with any base change by Proposition 2.26 or by Proposition 2.29, and so is the formation of $\varphi$. Moreover, a $G$-torsor over $\operatorname{Spec}(K)$ is tamely ramified if and only if its restriction to $\operatorname{Spec}(K')$ is tamely ramified. By replacing $K$ and $L$ by $K'$ and $K' \otimes_K L$ respectively, we can assume that the residue field $k$ is algebraically closed.

Let $P$ be the multiplicative $G$-torsor on $\mathbb{L}^\times$ (cf. 2.5) associated to $Q$ (cf. 3.8), so that $Q$ is isomorphic to $\varphi^{-1}P$. Then $\psi^{-1}Q^d$ is isomorphic to the pullback of $P$ along the composition

$$\operatorname{Spec}(K) \xrightarrow{\psi} \operatorname{Sym}_K^d(\operatorname{Spec}(L)) \to \operatorname{Sym}_K^d(\operatorname{Spec}(L)) \xrightarrow{\operatorname{Sym}_K^d(\varphi)} \operatorname{Sym}_K^d(\mathbb{L}^\times) \to \mathbb{L}^\times$$

considered in 3.13. By 3.13, this composition corresponds to the $K$-point of $\mathbb{L}^\times$ given by $P_\pi(\mathbb{P})^{-1}P^d$, where $P_\pi$ is the characteristic polynomial of $\pi$ acting $K$-linearly by multiplication on $L$. Let us
consider the morphism of pointed sets
\[ \rho : \mathbb{L}^\times(K) \to H^1(Spec(K)_{\text{Et}}, G) \]
\[ \nu \to \nu^{-1}P \]
where an element \( \nu \) of \( \mathbb{L}^\times(K) \) is identified to a morphism \( \text{Spec}(K) \to \mathbb{L}^\times \). If \( \nu_1 \) and \( \nu_2 \) are elements of \( \mathbb{L}^\times(K) \), then using the isomorphism \( \theta : p_1^{-1}P \otimes p_2^{-1}P \to m^{-1}P \) from 2.5, we obtain isomorphisms
\[ (\nu_1 \nu_2)^{-1}P \leftrightarrow (\nu_1 \times \nu_2)^{-1}m^{-1}P \leftrightarrow (\nu_1 \times \nu_2)^{-1}(p_1^{-1}P \otimes p_2^{-1}P) \leftrightarrow \nu_1^{-1}P \otimes \nu_2^{-1}P. \]
Thus \( \rho \) is an homomorphism of abelian groups.

We have to prove that \( \rho(\nu) \) is the isomorphism class of a tamely ramified \( G \)-torsor over \( \text{Spec}(K) \), where \( \nu = \nu G \). Since \( \rho \) is an Eisenstein polynomial, it can be written as \( \nu = p \nu G \). Since \( \rho \) is an homomorphism of abelian groups, and an exact sequence
\[ \text{Ext}(\nu G, \pi^0cR) = (1 + c^{-1}dR \nu G)^{-1}, \]
so that \( \rho(\nu) = \rho(c)^{-1} \rho(\nu_1)^{-1} \rho(\nu_2)^{-1} \).

Since \( R \) has ramification bounded by \( d \) (cf. 3.9), the restriction of \( \rho \) to \( \mathbb{U}_L(d)(K) \) is trivial (cf. 3.11). In particular, \( \rho(\nu_2) \) is trivial since \( \nu_2 \) belongs to \( \mathbb{U}_L(d)(K) \).

The element \( \nu_2 \) belongs to \( \mathbb{L}^\times(\mathcal{O}_K) \), so that the morphism \( \eta_1 : \text{Spec}(K) \to \mathbb{L}^\times \) factors through \( \text{Spec}(\mathcal{O}_K) \). This implies that \( \rho(\nu_1) \) is the isomorphism class of an unramified \( G \)-torsor over \( \text{Spec}(K) \).

We have to prove that \( \rho(c) \) is the isomorphism class of a tamely ramified \( G \)-torsor over \( \text{Spec}(K) \).

Since \( c \) belongs to \( K^\times = \mathcal{G}_{m,k}(K) \subseteq \mathbb{L}^\times(K) \), this is a consequence of the following lemma:

**Lemma 3.15.** Let \( T \) be a multiplicative \( G \)-torsor over the \( k \)-group scheme \( \mathcal{G}_{m,k} \) (cf. 2.5). Then \( T \) is tamely ramified at 0 and \( \infty \).

Let \( G_k \) be the constant \( k \)-group scheme associated to \( k \). By 2.9, there is a structure of \( k \)-group scheme on \( T \) and an exact sequence
\[ (3.15.1) \quad 1 \to G_k \to T \to \mathcal{G}_{m,k} \to 1 \]
in \( \text{Spec}(k)_{\text{Et}} \), such that the structure of \( G \)-torsor on \( T \) is given by the action of its subgroup \( G \) by translations. Since the fppf topology is finer than the étale topology on \( \text{Sch}_{k} \), the sequence \( 3.15.1 \) remains exact in the topos \( \text{Spec}(k)_{\text{fppf}} \). In particular, we obtain a class in the group \( \text{Ext}^1_{\text{fppf}}(\mathcal{G}_{m,k}, G_k) \) of extensions of \( \mathcal{G}_{m,k} \) by \( G_k \) in \( \text{Spec}(k)_{\text{fppf}} \).

Let \( n = |G| \). In the topos \( \text{Spec}(k)_{\text{fppf}} \) we have an exact sequence
\[ (3.15.2) \quad 1 \to \mu_{n,k} \to \mathcal{G}_{m,k} \to G_k \to 1, \]
where \( \mu_{n,k} \) is the \( k \)-group scheme of \( n \)-th roots of unity. By applying the functor \( \text{Hom}(\cdot, G_k) \), we obtain an exact sequence
\[ \text{Hom}(\mu_{n,k}, G_k) \to \text{Ext}^1_{\text{fppf}}(\mathcal{G}_{m,k}, G_k) \to \text{Ext}^1_{\text{fppf}}(\mathcal{G}_{m,k}, G_k). \]
Since \( n = |G| \), the group \( \text{Ext}^1_{\text{fppf}}(\mathcal{G}_{m,k}, G_k) \) is annihilated by \( n \), so that the homomorphism \( \delta \) above is surjective. Thus the exact sequence (3.15.1) in \( \text{Spec}(k)_{\text{fppf}} \) is the pushout of (3.15.2) along an homomorphism \( \mu_{n,k} \to G_k \). Let \( n' \) be the largest divisor of \( n \) which is invertible in \( k \). Then the largest étale quotient of \( \mu_{n,k} \) is the epimorphism \( \mu_{n,k} \to \mu_{n',k} \) given by \( x \mapsto x^{-n'} \). In particular, the homomorphism \( \mu_{n,k} \to G_k \) factors through \( \mu_{n',k} \), so that (3.15.1) is the pushout of the extension
\[ 1 \to \mu_{n',k} \to \mathcal{G}_{m,k} \to G_k \to 1. \]
along an homomorphism $\mu_{n',k} \to G_k$. Since the morphism $\mathbb{G}_{m,k} \xrightarrow{n'} \mathbb{G}_{m,k}$ is tamely ramified above $0$ and $\infty$, so is the morphism $T \to \mathbb{G}_{m,k}$.

4. Rigidified Picard schemes of relative curves

4.1. Let $f : X \to S$ be a smooth morphism of schemes of relative dimension $1$, with connected geometric fibers of genus $g$, which is Zariski-locally projective over $S$.

**Proposition 4.2.** The canonical homomorphism $O_S \to f_*O_X$ is an isomorphism.

If $S$ is locally noetherian, then $O_X$ is cohomologically flat over $S$ in dimension $0$ by ([EGA3], 7.8.6). This means that for any quasi-coherent $O_S$-module $M$, the canonical homomorphism $f_*f^*O_X \otimes_{O_S} M \to f_*f^*M$ is an isomorphism. This implies that the formation of $f_*O_X$ commutes with arbitrary base change: if $f' : X \times_S S' \to S'$ is the base change of $f$ by a morphism of schemes $S' \to S$, then the canonical morphism $f_*O_X \otimes_{O_S} O_{S'} \to f'_*O_X \otimes_{O_{S'}} O_{S'}$ is an isomorphism, cf. ([EGA3], 7.7.5.3). By applying this result to the inclusion $\text{Spec}(\kappa(s)) \to S$ of a point $s$ of $S$, we obtain that $f_*(O_X)_s \otimes_{O_{S,s}} \kappa(s)$ is isomorphic to $H^0(X_s, O_{X_s}) = \kappa(s)$. Since $f_*(O_X)$ is a coherent $O_S$-module, Nakayama’s lemma yields that the canonical morphism $O_S \to f_*(O_X)$ is an epimorphism. It is also injective since $f$ is faithfully flat, hence the result.

In general one can assume that $S$ is affine and that $X$ is projective over $S$, in which case there is a noetherian scheme $S_0$, a morphism $S \to S_0$ and a smooth projective $S_0$-scheme $X_0$ with geometrically connected fibers such that $X$ is isomorphic to the $S$-scheme $X_0 \times_{S_0} S$, cf. ([EGA4], 8.9.1, 8.10.5(xiii), 17.7.9). We have already seen that in this case the canonical homomorphism $O_{S_0} \to f_*O_{X_0}$ is an isomorphism, and that the formation of $f_*O_{X_0}$ commutes with arbitrary base change. In particular, both morphisms in the sequence

$$O_S \to f_*O_{X_0} \otimes_{O_{S_0}} O_S \to f_*O_X$$

are isomorphisms.

**Proposition 4.3.** Let $d \geq 2g - 1$ be an integer, and let $L$ be an invertible $O_X$-module with degree $d$ on each fiber of $f$. Then, the $O_S$-module $f_*L$ is locally free of rank $d - g + 1$, the higher direct images $R^j(f_*L)$ vanish for $j > 0$, and the formation of $f_*L$ commutes with arbitrary base change: if $f' : X' \to S'$ is the base change of $f$ by a morphism $S' \to S$, then the canonical homomorphism $f_*L \otimes_{O_S} O_{S'} \to f'_*(L \otimes_{O_X} O_{X'})$ is an isomorphism.

We first assume that $S$ is locally noetherian. For each point of $s$ of $S$ and for each integer $i$, the Riemann-Roch theorem for smooth projective curves implies that the $k(s)$-vector space $H^i(X_s, L_s)$ is of dimension $d - g + 1$ for $i = 0$, and vanishes otherwise. This implies that $R^j f_*(L \otimes_{O_X} f^*N)$ vanishes for any integer $j > 0$ and any $O_S$-module $N$ by the proof of ([EGA3], 7.9.8). Let

$$0 \to N \to M \to P \to 0$$

be an exact sequence of $O_S$-modules. Since $f$ is flat and since $L$ is a flat $O_X$-module, the sequence

$$0 \to L \otimes_{O_X} f^*N \to L \otimes_{O_X} f^*M \to L \otimes_{O_X} f^*P \to 0$$

is exact as well. Since $R^1 f_*(L \otimes_{O_X} f^*N)$ vanishes, the sequence

$$0 \to f_*(L \otimes_{O_X} f^*N) \to f_*(L \otimes_{O_X} f^*M) \to f_*(L \otimes_{O_X} f^*P) \to 0$$

is exact. The $O_X$-module $L$ is therefore cohomologically flat over $S$ in dimension $0$, cf. ([EGA3], 7.8.1). By ([EGA3], 7.8.4(d)) the $O_S$-module $f_*L$ is locally free, and the formation of $f_*L$ commutes with arbitrary base change. By applying the latter result to the inclusion $\text{Spec}(\kappa(s)) \to S$ of a
point $s$ of $S$ and by using that $H^0(X_\alpha, \mathcal{L}_s)$ is of dimension $d - g + 1$ over $\kappa(s)$, we obtain that the locally free $\mathcal{O}_X$-module $f_*\mathcal{L}$ is of constant rank $d - g + 1$.

In general one can assume that $S$ is affine and that $X$ is projective over $S$, in which case there is a noetherian scheme $S_0$, a morphism $S \to S_0$, a smooth projective $S_0$-scheme $X_0$, and an invertible $\mathcal{O}_{X_0}$-module $\mathcal{L}_0$ such that $X$ is isomorphic to the $S$-scheme $X_0 \times_{S_0} S$ and $\mathcal{L}$ is isomorphic to the pullback of $\mathcal{L}_0$ by the canonical projection $X_0 \times_{S_0} S \to X_0$, cf. ([EGAIV], 8.9.1, 8.10.5(xiii), 17.7.9). We have seen that the $\mathcal{O}_{S_0}$-module $f_0_*\mathcal{L}$ is locally free of rank $d - g + 1$, and that its formation commutes with arbitrary base change. By performing the base change by the morphism $S \to S_0$, we obtain that $f_*\mathcal{L}$ is a locally free $\mathcal{O}_S$-module of rank $d - g + 1$ and that the formation of $f_*\mathcal{L}$ commutes with arbitrary base change.

4.4. Let $f : X \to S$ be as in 4.1. The relative Picard functor of $f$ is the sheaf of abelian groups $\text{Pic}_S(X) = R^1f_{Fppf}^*\mathbb{G}_m$ in $S_{Fppf}$. Alternatively, $\text{Pic}_S(X)$ is the sheaf of abelian groups on $S$ associated to the presheaf which sends an $S$-scheme $T$ to $\text{Pic}(X \times_S T)$, the abelian group of isomorphism classes of invertible $\mathcal{O}_{X \times_S T}$-modules. For any $S$-scheme $S'$, we have $(S_{Fppf})_{/S'} = S'_{Fppf}$, and we thus have:

**Proposition 4.5.** For any $S$-scheme $S'$, the canonical morphism

$$\text{Pic}_S(X \times_S S') \to \text{Pic}_S(X) \times_S S'$$

is an isomorphism in $S'_{Fppf}$.

The elements of $\text{Pic}(X \times_S T)$ which are pulled back from an element of $\text{Pic}(T)$ yield trivial classes in $\text{Pic}_S(X)(T)$, since invertible $\mathcal{O}_T$-modules are locally trivial on $T$ (for the Zariski topology, and thus for the fppf-topology). This yields a sequence

$$(4.5.1) \quad 0 \to \text{Pic}(T) \to \text{Pic}(X \times_S T) \to \text{Pic}_S(X)(T) \to 0,$$

which is however not necessarily exact. The following is Proposition 4 from ([BLR90], 8.1), whose assumptions are satisfied by 4.2:

**Proposition 4.6.** If $f$ has a section, then the sequence $\text{(4.5.1)}$ is exact for any $S$-scheme $T$.

By a theorem of Grothendieck ([BLR90], 8.2.1) the sheaf $\text{Pic}_S(X)$ is representable by a separated $S$-scheme. By ([BLR90], 9.3.1) the $S$-scheme $\text{Pic}_S(X)$ is smooth of relative dimension $g$, and there is a decomposition

$$\text{Pic}_S(X) = \coprod_{d \in \mathbb{Z}} \text{Pic}_S^d(X),$$

into open and closed subschemes, where $\text{Pic}_S^d(X)$ is the fppf-sheaf associated to the presheaf

$$\text{Sch}_{/S}^{fpp} \to \text{Sets}$$

$$T \mapsto \{ \mathcal{L} \in \text{Pic}(X \times_T T) | \forall \bar{t} \to T, \deg_{X_{\bar{t}}}(\mathcal{L}_{\bar{t}}) = d \}.$$

Here the condition $\deg_{X_{\bar{t}}}(\mathcal{L}_{\bar{t}}) = d$ runs over all geometric points $\bar{t} \to T$ of $T$.

4.7. Let $f : X \to S$ be as in 4.1, and let $i : Y \hookrightarrow X$ be a closed subscheme of $X$, which is finite locally free over $S$ of degree $N \geq 1$. A $Y$-rigidified line bundle on $X$ is a pair $(\mathcal{L}, \alpha)$ where $\mathcal{L}$ is a locally free $\mathcal{O}_X$-module of rank $1$ and $\alpha : \mathcal{O}_X \to i^*\mathcal{L}$ is an isomorphism of $\mathcal{O}_Y$-modules. Two $Y$-rigidified line bundles $(\mathcal{L}, \alpha)$ and $(\mathcal{L}', \alpha')$ are equivalent if there is an isomorphism $\beta : \mathcal{L} \to \mathcal{L}'$ of $\mathcal{O}_X$-modules such that $(i^*\beta)\alpha = \alpha'$. If such an isomorphism $\beta$ exists, then it is unique. Indeed, any other such isomorphism would take the form $\gamma \beta$ for some global section $\gamma$ of $\mathcal{O}_X^\times$ such that $i^*\gamma = 1$. Since $f_*\mathcal{O}_X = \mathcal{O}_S$ (cf. 4.2), we have $\gamma = f^*\delta$ for some global section $\delta$ of $\mathcal{O}_S^\times$. Since the
restriction of $\delta$ along the finite flat surjective morphism $Y \to S$ is trivial, one must have $\delta = 1$ as well, hence $G = 1$.

**Proposition 4.8.** Let $\text{Pic}_S(X, Y)$ be the presheaf of abelian groups on $\text{Sch}^{fp}_S$ which maps a finitely presented $S$-scheme $T$ to the set of isomorphism classes of $Y_T$-rigidified line bundles on $X_T$. Then, the presheaf $\text{Pic}_S(X, Y)$ is representable by a smooth separated $S$-scheme of relative dimension $N + g - 1$.

We first consider the case where $N = 1$:

**Lemma 4.9.** The conclusion of Proposition 4.8 holds if $N = 1$.

Indeed, if $N = 1$ then $Y$ is the image of a section $x : S \to X$ of $f$. For any finitely presented $S$-scheme $T$, we have a morphism

$$\text{Pic}(X \times_S T) \to \text{Pic}_S(X, x)(T)$$

$$\mathcal{L} \to (\mathcal{L} \otimes (f^*x^*\mathcal{L})^{-1}, \text{id}).$$

The kernel of this homomorphism consists of all invertible $\mathcal{O}_{X \times_S T}$-modules which are given by the pullback of an invertible $\mathcal{O}_T$-module. Moreover, any isomorphism class $(\mathcal{L}, \alpha)$ in $\text{Pic}_S(X, x)(T)$ is the image of $\mathcal{L}$ by this morphism, hence its surjectivity. We conclude by 4.6 that the canonical projection morphism

$$\text{Pic}_S(X, x) \to \text{Pic}_S(X)$$

$$(\mathcal{L}, \alpha) \to \mathcal{L},$$

is an isomorphism of presheaves of abelian groups on $\text{Sch}^{fp}_S$. This yields Lemma 4.9 since $\text{Pic}_S(X)$ is a smooth separated $S$-scheme of relative dimension $g$ (cf. 4.4).

We now prove Proposition 4.8. Since $X \times_S Y \to Y$ has a section $x = (i \times \text{id}_Y) \circ \Delta_Y$ where $\Delta_Y : Y \to Y \times_S Y$ is the diagonal morphism of $Y$, we deduce from Lemma 4.9 and its proof that the canonical projection morphism

$$\text{Pic}_Y(X \times_S Y, x) \to \text{Pic}_Y(X \times_S Y) = \text{Pic}_S(X) \times_S Y$$

sending a pair $(\mathcal{L}, \alpha)$ to the class of $\mathcal{L}$ is an isomorphism. Let $Z$ be the $Y$-scheme $\text{Pic}_Y(X \times_S Y, x)$, and let $(\mathcal{L}_u, \alpha_u)$ be the universal $x$-rigidified line bundle on $X \times_S Z$. The morphism $Y \times_S Z \to Z$ is finite locally free of rank $N$, so that the pushforward $\mathcal{A}$ (resp. $\mathcal{M}$) of $\mathcal{O}_{Y \times_S T}$ (resp. $i_Z^*\mathcal{L}_u$) is a locally free $\mathcal{O}_Z$-algebra of rank $N$ (resp. a locally free $\mathcal{O}_Z$-module of rank $N$). Let $\lambda : \mathcal{M} \to \mathcal{O}_Z$ be the surjective $\mathcal{O}_Z$-linear homomorphism corresponding to $\alpha_u^{-1} : x^*_Z\mathcal{L}_u \to \mathcal{O}_Z$.

Let $T$ be a $Y$-scheme, and let $(\mathcal{L}, \beta)$ be a $Y_T$-rigidified line bundle on $X_T$. The section $x_T : T \to X_T$ uniquely factors through $Y_T$ and we still denote by $x_T$ the corresponding section of $Y_T$. The pair $(\mathcal{L}, x_T^*\beta)$ is then an $x_T$-rigidified line bundle on $X_T$, so that there is a unique morphism $z : T \to Z$ such that $(\mathcal{L}, x_T^*\beta)$ is equivalent to the pullback by $z$ of $(\mathcal{L}_u, \alpha_u)$. Let us assume that $(\mathcal{L}, x_T^*\beta)$ is equal to this pullback. Then $\beta$ is a section of $i_Z^*\mathcal{L} = z^*\mathcal{M}$ over $T$ such that $(z^*\lambda)(\beta) = 1$ and $z^*\mathcal{M} = (z^*\mathcal{A})\beta$. Conversely, any such section produces a $Y_T$-rigidification of $\mathcal{L}$ on $X_T$. The functor $\text{Pic}_S(X, Y) \times_S Y = \text{Pic}_Y(X \times_S Y, Y \times_S Y)$ is therefore isomorphic to the functor

$$\text{Sch}^{fp}_{/S} \to \text{Sets}$$

$$T \mapsto \{(z, \beta) \mid z \in Z(T), \beta \in \Gamma(T, z^*\mathcal{M}), \lambda(m) = 1 \text{ and } \mathcal{M} = A_T m\}.$$ 

This implies that $\text{Pic}_S(X, Y) \times_S Y$ is representable by a relatively affine $Z$-scheme, smooth of relative dimension $N - 1$ over $Z$. By fpfp-descent of affine morphisms of schemes along the fpfp-cover $\text{Pic}_S(X) \times_S Y \to \text{Pic}_S(X)$, this implies the representability of $\text{Pic}_S(X, Y)$ by an $S$-scheme,
which is relatively affine and smooth of relative dimension $N - 1$ over $\text{Pic}_S(X)$. Since $\text{Pic}_S(X)$ is separated and smooth of relative dimension $g$ over $S$ (cf. 4.1), the $S$-scheme $\text{Pic}_S(X, Y)$ is separated and smooth of relative dimension $g + N - 1$.

4.10. Let $f : X \to S$ be as in 4.1, and let $i : Y \hookrightarrow X$ be a closed subscheme of $X$, which is finite locally free over $S$ of degree $N \geq 1$. A $Y$-trivial effective Cartier divisor of degree $d$ on $X$ is a pair $(\mathcal{L}, \sigma)$ such that $\mathcal{L}$ is a locally free $\mathcal{O}_X$-module of rank 1 and $\sigma : \mathcal{O}_X \to \mathcal{L}$ is an injective homomorphism such that $i^*\sigma$ is an isomorphism and such that the closed subscheme $\mathcal{V}(\sigma)$ of $X$ defined by the vanishing of the ideal $\sigma\mathcal{L}^{-1}$ of $\mathcal{O}_X$ is finite locally free of rank $d$ over $S$. Two $Y$-trivial effective divisors $(\mathcal{L}, \sigma)$ and $(\mathcal{L}', \sigma')$ are equivalent if there is an isomorphism $\beta : \mathcal{L} \to \mathcal{L}'$ of $\mathcal{O}_X$-modules such that $\beta \sigma = \sigma'$. As in 4.7, if such an isomorphism exists then it is unique.

**Proposition 4.11.** The map $(\mathcal{L}, \sigma) \mapsto (V(\sigma) \hookrightarrow X)$ is a bijection from the set of equivalence classes of $Y$-trivial effective Cartier divisors of degree $d$ on $X$ onto the set of closed subschemes of $U$ which are finite locally free of degree $d$ over $S$.

Let $(\mathcal{L}, \sigma)$ be a $Y$-trivial effective divisor of degree $d$ on $X$. The ideal $\mathcal{I} = \sigma\mathcal{L}^{-1}$ is an invertible ideal of $\mathcal{O}_X$ such that the vanishing locus $V(\mathcal{I})$ is finite locally free of rank $d$ over $S$ and is contained in $U$. The pair $(\mathcal{L}, \sigma)$ is equivalent to $(\mathcal{I}^{-1}, 1)$, and $\mathcal{I}$ is uniquely determined by $V(\mathcal{I})$. Conversely, for any closed subscheme $Z$ of $U$ which is finite locally free of rank $d$ over $S$, the scheme $Z$ is proper over $S$ hence closed in $X$ as well, and its defining ideal $\mathcal{I}$ in $\mathcal{O}_{X_T}$ is invertible by ([BLR90], 8.2.6(ii)). The pair $(\mathcal{I}^{-1}, 1)$ is then a $Y$-trivial effective Cartier divisor of degree $d$ on $X$.

**Proposition 4.12.** Let $d$ be an integer and let $\text{Div}^{d,+}_S(X, Y)$ be the functor which to an $S$-scheme $T$ associates the set of equivalence classes of $Y_T$-trivial effective Cartier divisors of degree $d$ on $X_T$. Then $\text{Div}^{d,+}_S(X, Y)$ is representable by the $S$-scheme $\text{Sym}_S^d(U)$, the $d$-th symmetric power of $U = X \setminus Y$ over $S$ (cf. 2.22). In particular $\text{Div}^{d,+}_S(X, Y)$ is smooth of relative dimension $d$ over $S$.

By Proposition 4.11, the functor $\text{Div}^{d,+}_S(X, Y)$ is isomorphic to the functor which sends an $S$-scheme $T$ to the set of closed subschemes of $U_T$ which are finite locally free of rank $d$ over $T$. In other words, $\text{Div}^{d,+}_S(X, Y)$ is isomorphic to the Hilbert functor of $d$-points in the $S$-scheme $U$.

If $x$ is a $T$-point of $U$, we denote $\mathcal{O}(-x)$ the kernel of the homomorphism $\mathcal{O}_{X \times U_T} \to x_*\mathcal{O}_T$, which is an invertible ideal sheaf, and by $\mathcal{O}(x)$ its dual, which is endowed with a section $1_x : \mathcal{O}_T \hookrightarrow \mathcal{O}(x)$. The morphism

$$\text{Sym}_S^d(U) \to \text{Div}^{d,+}_S(X, Y)$$

$$(x_1, \ldots, x_d) \mapsto \left(\bigotimes_{i=1}^d \mathcal{O}(x_i), \prod_{i=1}^d 1_{x_i}\right)$$

is then an isomorphism of fppf-sheaves by ([SGA4], XVII.6.3.9), hence Proposition 4.12.

**Remark 4.13.** Let $T$ be an $S$-scheme. Let $Z$ be a closed subscheme of $U_T$ which is finite locally free of rank $d$ over $T$, therefore defining a $T$-point of $\text{Div}^{d,+}_S(X, Y) = \text{Sym}_T^d(U)$ by Proposition 4.11. By ([SGA4], XVII.6.3.9), this $T$-point is given by the composition

$$T \to \text{Sym}_T^d(U_T) \to \text{Sym}_T^d(U),$$

where the first morphism is the canonical morphism from Proposition 2.21.
Proposition 4.14. Let $d \geq N + 2g - 1$ be an integer, and let $\text{Pic}^d_S(X, Y)$ be the inverse image of $\text{Pic}^d_S(X)$ by the natural morphism $\text{Pic}_S(X, Y) \to \text{Pic}_S(X)$. Then the Abel-Jacobi morphism

$$\Phi_d : \text{Div}^{d,+}_S(X, Y) \to \text{Pic}^d_S(X, Y)$$

$$(\mathcal{L}, \sigma) \mapsto (\mathcal{L}, i^*\sigma)$$

is surjective smooth of relative dimension $d - N - g + 1$ and it has geometrically connected fibers.

Let $Z$ be the scheme $\text{Pic}^d_S(X, Y)$, and let $(\mathcal{L}_u, \alpha_u)$ be the universal $Y$-rigidified line bundle of degree $d$ on $X_Z$. By ([BLR90], 8.2.6(ii)), the closed subscheme $Y_Z$ of $X_Z$ is defined by an invertible ideal sheaf $\mathcal{I}$.

Let $\mathcal{E}$ be the pushforward of $\mathcal{M} = \mathcal{L}_u \otimes_{\mathcal{O}_{X_Z}} \mathcal{I}$ by the morphism $f_Z : X_Z \to Z$. By 4.3, the $\mathcal{O}_Z$-module $\mathcal{E}$ is locally free of rank $d - N - g + 1$, and for any morphism $T \to Z$ the canonical homomorphism

$$\mathcal{E} \otimes_{\mathcal{O}_Z} \mathcal{O}_T \to f_{T,*}(\mathcal{M} \otimes_{\mathcal{O}_{X_Z}} \mathcal{O}_X)$$

is an isomorphism, where $f_T : X_T \to T$ is the base change of $f$ by the morphism $T \to S$. We thus obtain an isomorphism

$$E \cong E', \tag{4.14.1}$$

of functors on the category of $Z$-schemes, where $E$ is the functor $T \mapsto \Gamma(T, \mathcal{E} \otimes_{\mathcal{O}_Z} \mathcal{O}_T)$ and $E'$ is the functor $T \mapsto \Gamma(X_T, \mathcal{M} \otimes_{\mathcal{O}_{X_Z}} \mathcal{O}_X)$. Let $\mathcal{F}$ be the pushfoward of $\mathcal{L}_u$ by the morphism $f_Z$. By the same argument, we obtain that the $\mathcal{O}_Z$-module $\mathcal{F}$ is locally free of rank $d - g + 1$, and that we have an isomorphism

$$F \cong F', \tag{4.14.2}$$

of functors on the category of $Z$-schemes, where $F$ is the functor $T \mapsto \Gamma(T, \mathcal{F} \otimes_{\mathcal{O}_Z} \mathcal{O}_T)$ and $F'$ is the functor $T \mapsto \Gamma(X_T, \mathcal{L}_u \otimes_{\mathcal{O}_{X_Z}} \mathcal{O}_X)$. Let us consider the exact sequence

$$0 \to \mathcal{M} \to \mathcal{L}_u \to \mathcal{L}_u \otimes_{\mathcal{O}_{X_Z}} \mathcal{O}_Y \to 0.$$ 

Since $R^1f_{Z,*}\mathcal{M} = 0$ by 4.3, we obtain an exact sequence

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0,$$

where $\mathcal{G}$ is a locally free $\mathcal{O}_Z$-module of rank $N$. Together with (4.14.1) and (4.14.2), this yields an exact sequence

$$0 \to E' \to F' \xrightarrow{b} G \to 0,$$

of $Z$-group schemes in $\text{Z}_{\text{fpf}}$, where $G$ is the functor $T \mapsto \Gamma(T, \mathcal{G}_T \otimes_{\mathcal{O}_Z} \mathcal{O}_T)$. The section $\alpha_u$ of $\mathcal{G}$ over $Z$ corresponds to a morphism $\alpha_u : Z \to G$, and we have an isomorphism

$$\text{Div}^{d,+}_S(X, Y) \to F' \times_{b, G, \alpha_u} Z$$

$$(\mathcal{L}, \sigma) \mapsto (\sigma, (\mathcal{L}, i^*\sigma)).$$

Since $b$ is an $E'$-torsor over $G$ in $\text{Z}_{\text{fpf}}$, we obtain that $\text{Div}^{d,+}_S(X, Y)$ is an $E'$-torsor in $\text{Z}_{\text{fpf}}$. Since $E'$ is isomorphic to $E$ by (4.14.1), it is smooth of relative dimension $d - N - g + 1$ over $Z$ with geometrically connected fibers, hence the conclusion of Proposition 4.14.
5. Geometric Global Class Field Theory

5.1. Let $f : X \to S$ be a smooth morphism of schemes of relative dimension 1, with connected geometric fibers of genus $g$, which is Zariski-locally projective over $S$, and let $i : Y \hookrightarrow X$ be a closed subscheme of $X$ which is finite locally free over $S$ of degree $N \geq 1$. Let $j : U \to X$ be the open complement of $Y$. Let $\Lambda$ be a finite ring whose cardinality is invertible on $S$.

**Definition 5.2.** A locally free $\Lambda$-module $F$ of rank 1 in $U_{\acute{e}t}$ has **ramification bounded by $Y$ over $S$** if for any geometric point $\bar{x}$ of $Y$ with image $\bar{s}$ in $S$, the restriction of $F$ to $\text{Spec}(\mathcal{O}_{X,\bar{x}}) \times_{X_s} U_s$ has ramification bounded by the multiplicity of $Y_{\bar{x}}$ at $\bar{s}$ (cf. 3.9).

**Theorem 5.3.** Let $F$ be a locally free $\Lambda$-module of rank 1 in $U_{\acute{e}t}$ with ramification bounded by $Y$ over $S$ (cf. 5.2). Then, there is a unique (up to isomorphism) multiplicative locally free $\Lambda$-module $G$ of rank 1 on the $\Lambda$-group scheme $\text{Pic}_S(X,Y)$ (cf. 2.6) such that the pullback of $G$ by the Abel-Jacobi morphism

$$U \to \text{Pic}_S(X,Y),$$

which sends $x$ to $(\mathcal{O}(x), 1)$, is isomorphic to $F$.

In Sections 5.4 and 5.10, we study the restriction of the locally free $\Lambda$-module $F^{[d]}$ of rank 1 on $\text{Div}_{\Lambda}^{d,+}(X,Y)$ (cf. 2.32 and 4.12) to a geometric fiber of the Abel-Jacobi morphism (cf. 4.14)

$$\Phi_d : \text{Div}_{\Lambda}^{d,+}(X,Y) \to \text{Pic}^d(X,Y)$$

$$(\mathcal{L}, \sigma) \mapsto (\mathcal{L}, i^*\sigma).$$

This study will enable us to prove Theorem 5.3 in Section 5.17.

5.4. Let $k$ be an algebraically closed field, let $X$ be a smooth connected projective curve of genus $g$ over $k$ and let $i : Y \to X$ be an effective Cartier divisor of degree $N$. Let $\mathcal{L}$ be a line bundle of degree $d \geq N+2g$ on $X$, and let $V$ be the $(d-N-g+1)$-dimensional affine space over $k$ associated to the $k$-vector space $V = H^0(X, \mathcal{L}(-Y))$, i.e. $V$ is the spectrum of the symmetric algebra of the $k$-module $\text{Hom}_k(V,k)$. We denote by $\sigma$ the section of $p_1^*\mathcal{L}(Y)$ on $X \times_k V$, where $p_1 : X \times_k V \to X$ is the first projection, given by the composition of the universal section $\mathcal{O}_{X \times_k V} \to V \otimes_k \mathcal{O}_{X \times_k V}$ with the adjunction morphism $V \otimes_k \mathcal{O}_{X \times_k V} \to p_1^*\mathcal{L}(Y)$.

Let $\tau$ be a global section of $\mathcal{L}$ on $X$ such that $i^*\tau : \mathcal{O}_Y \to i^*\mathcal{L}$ is an isomorphism.

**Lemma 5.5.** There exists an element $s$ of $V$ such that $s$ and $\tau$ have disjoint divisors.

Let $x$ be a point of $X$ such that $\tau(x) = 0$. Then $x$ does not belong to $Y$ since $i^*\tau$ is nowhere vanishing, so that the subspace of $V$ given by

$$H^0(X, \mathcal{L}(-Y-x)) = \{s \in V \mid s(x) = 0\}$$

has codimension 1 in $V$ (by the requirement $d \geq N+2g$). Since $k$ is infinite, the union of finitely many codimension 1 subspaces of $V$ cannot be equal to $V$, hence Lemma 5.5.

**Lemma 5.6.** The ideal $\mathcal{I} = (\sigma, p_1^*\tau)p_1^*\mathcal{L}^{-1}$ of $\mathcal{O}_{X \times_k V}$ is Zariski-locally generated by a regular sequence of length 2.

Let $U = \text{Spec}(A)$ be an affine open subset of $X$ together with a trivialization $\mathcal{L}|_U = \mathcal{O}_{U,e}$ of $\mathcal{L}$ on $U$. Let $s_1, \ldots, s_m$ be a $k$-basis of $V$ such that $s_1$ and $\tau$ have disjoint divisors (cf. 5.5), and let us write $U \times_k V = \text{Spec}(A[[T_1, \ldots, T_m]])$ where $\sigma = \sum_{i=1}^m T_i s_i$. On the affine scheme $U$, we can write $\tau = \tilde{\tau} e$ and $s_i = \tilde{s}_i e$, for some elements $\tilde{\tau}, \tilde{s}_1, \ldots, \tilde{s}_m$ of $A$. On $U \times_k V$, the ideal $(\sigma, p_1^*\tau)p_1^*\mathcal{L}^{-1}$
is generated by \( \tilde{\tau} \) and \( \sum_{i=1}^{m} T_i \tilde{s}_i \). The element \( \tilde{\tau} \) is a not a zero divisor since it is non zero and \( A[\{T_i\}_{1 \leq i \leq m}] \) is an integral domain. Since \( \tilde{s}_1 \) is not a zero divisor in \( A/(\tilde{\tau}) \), the element

\[
\sum_{i=1}^{m} T_i \tilde{s}_i = \tilde{s}_1 T_1 + \left( \sum_{i=2}^{m} T_i \tilde{s}_i \right)
\]

of \( A/(\tilde{\tau})[\{T_i\}_{1 \leq i \leq m}] = (A/(\tilde{\tau})[\{T_i\}_{2 \leq i \leq m}])[T_1] \) is not a zero divisor. Since it is not a unit either, we obtain that \( (\tilde{\tau}, \sum_{i=1}^{m} T_i \tilde{s}_i) \) is a regular sequence in \( A[\{T_i\}_{1 \leq i \leq m}] \), hence Lemma 5.6.

**Definition 5.7.** We denote by \( b : W \to X \times_k V \) the blow-up of the ideal \( \mathcal{I} \) of \( \mathcal{O}_{X \times_k V} \) (cf. 5.6), and by \( a : W \to \mathbb{P}^1_V \) the morphism induced by the pair of sections \( (b^* \sigma : b^* p_i^* \tau) \) which generate the invertible \( \mathcal{O}_W \)-module \( \mathcal{I}\mathcal{O}_W \otimes_{\mathcal{O}_W} b^* p_i^* \mathcal{L} \). We also denote by \( a^o \) and \( b^o \) the restrictions of \( a \) and \( b \) to \( W^o = a^{-1}(\mathbb{P}^1_V \setminus \{0 : 1\}_V) \).

**Lemma 5.8.** The morphism \( a^o : W^o \to \mathbb{P}^1_V \setminus \{0 : 1\}_V \) is finite locally free of rank \( d \).

Let us write \( \mathbb{P}^1_V \setminus \{0 : 1\}_V = \mathbb{A}^1_V = \text{Spec} \mathcal{O}_V[t] \). By Lemma 5.6 and ([Stacks], 0BIQ), the \( X \times_k V \)-scheme \( W^o \) is isomorphic to the closed subscheme of \( X \times_k V \times_k \mathbb{A}^1_k \) defined by the vanishing of \( p^*_i \tau - t \sigma \). For each \( k \)-point \((v, z) \) of \( V \times_k \mathbb{A}^1_k \), the section \( \tau - z \sigma \), of \( \mathcal{L} \) over \( X \) induces an injective \( \mathcal{O}_X \)-linear homomorphism \( \mathcal{O}_X \to \mathcal{L} \), which is an isomorphism on \( Y \). By ([Stacks], 046Z), this implies that \( a^o \) is flat. Since \( a^o \) is proper and quasi-finite, it is finite by ([Stacks], 02OG). Thus \( a^o \) is finite locally free. Moreover, its fibers have rank \( d \), hence Lemma 5.8.

**Lemma 5.9.** Let \( \varphi : \mathbb{A}^1_V \to \text{Sym}_V^d(U^o) \) be the morphism obtained from the finite locally free morphism \( a^o \) by Proposition 2.21. For each element \( t \) of \( k \), the morphism from \( V \) to \( \text{Div}_V^d(X, Y) = \text{Sym}_V^d(U) \) which sends \( s \) to \( (\mathcal{L}, \tau - ts) \) is the composition of the morphism

\[
\mathbb{A}^1_V \xrightarrow{\varphi} \text{Sym}_V^d(U^o) \to \text{Sym}_V^d(U^o) \xrightarrow{\text{Sym}_V^d(b^o)} \text{Sym}_V^d(U \times_k V) \to \text{Sym}_V^d(U).
\]

with the section \( t : V \to \mathbb{A}^1_V \).

We have a commutative diagram

\[
\begin{array}{ccc}
\mathbb{A}^1_V & \xrightarrow{\varphi} & \text{Sym}_V^d(U^o) \\
\uparrow & & \uparrow \\
V & \xrightarrow{t} & \text{Sym}_V^d(U^o \times_k U V).
\end{array}
\]

By Lemma 5.6 and ([Stacks], 0BIQ), or by the proof of Lemma 5.8, the scheme \( W^o \times_k \mathbb{A}^1_k \) is isomorphic to the closed subscheme of \( U \times_k V \) defined by the vanishing of \( p^*_i \tau - t \sigma \). This closed subscheme of \( U \times_k V \) is finite locally free of rank \( d \) by Lemma 5.8, and thus corresponds to a \( V \)-point of \( \text{Sym}_V^d(U) \) by 4.11 and 4.12. By Remark 4.13, this \( V \)-point is given by the composition

\[
V \to \text{Sym}_V^d(W^o \times_k \mathbb{A}^1_k, V) \to \text{Sym}_V^d(U \times_k V) \to \text{Sym}_V^d(U).
\]

If \( \mathcal{J} \) is the defining ideal of \( W^o \times_k \mathbb{A}^1_k \) in \( U \times_k V \), then this \( V \)-point of \( \text{Sym}_V^d(U) = \text{Div}_V^d(X, Y) \) is given by \( (\mathcal{J}^{-1}, 1) \), which is equivalent to \( (p^*_i \mathcal{L}, p^*_i \tau - t \sigma) \), hence the conclusion.
5.10. Let $k, X, i, \mathcal{L}, N, V$ be as in 5.4. Let $\Lambda$ be a finite ring of cardinality invertible in $k$, and let $j : U \to X$ be the open complement of $Y$. Let $\mathcal{F}$ be a locally free $\Lambda$-module of rank 1 in $U_{\text{Et}}$. Let

$$\varphi : \mathbb{A}^1_V \to \text{Sym}^d_{\mathbb{A}^1_V}(W^\circ),$$

be the morphism considered in Lemma 5.9, and let $\mathcal{G}$ be the pullback by $\varphi$ of $(b^{(d-1)p^{-1}}f)[d]$ (cf. 2.32).

**Proposition 5.11.** Let $\mathcal{F}, \mathcal{G}$ be as above. If $\mathcal{F}$ has ramification bounded by $Y$ (cf. 5.2), then the locally free $\Lambda$-module $\mathcal{G}$ is constant on all geometric fibers of the morphism $\mathbb{A}^1_V \to V$.

Let $\bar{v} \to V$ be a geometric point of $V$, and let us denote by $\sigma_{\bar{v}}$ the corresponding section of $\mathcal{L}(-Y)$ on $X_{\bar{v}}$. By Propositions 2.29 and 2.33, the formation of $\varphi$ and $\mathcal{G}$ is compatible with the base change by $\bar{v} \to V$. By 5.6 and ([Stacks], 0BIQ), the fiber $W_{\bar{v}}$ is the closed subscheme of $X \times_k \mathbb{P}^1_{\bar{v}}$ defined by the vanishing of $p_1^*\tau - t\sigma_{\bar{v}}$. If $\sigma_{\bar{v}} = 0$, then we have

$$W_{\bar{v}} = \text{div}(\tau) \times_k \mathbb{P}^1_{\bar{v}} \cup X_{\bar{v}} \times_{\bar{v}} [0 : 1] \hookrightarrow X_{\bar{v}} \times_{\bar{v}} \mathbb{P}^1_{\bar{v}},$$

so that the pullback of $\mathcal{F}$ to $W^\circ_{\bar{v}} = \text{div}(\tau) \times_k \mathbb{A}^1_{\bar{v}}$ is constant, which implies that $\mathcal{G} = \varphi^{-1}(b^{(d-1)p^{-1}}f)[d]$ is constant on $\mathbb{A}^1_{\bar{v}}$.

We now assume that $\sigma_{\bar{v}} \neq 0$. Since $\sigma_{\bar{v}}$ vanishes on the empty divisor $Y_{\bar{v}}$ and $\tau$ does not, the sections $\sigma_{\bar{v}}$ and $\tau$ are $k(\bar{v})$-linearly independent in $H^0(X_{\bar{v}}, \mathcal{L})$. Let $D$ be the greatest divisor on $X_{\bar{v}}$ such that $D \leq \text{div}(\sigma_{\bar{v}})$ and $D \leq \text{div}(\tau)$. We can then write $\sigma_{\bar{v}} = \tilde{\sigma}1_D$ and $\tau = \tilde{\tau}1_D$, where $1_D$ is the canonical section of $\mathcal{O}(D)$ and $\tilde{\sigma}, \tilde{\tau}$ are global sections of $\mathcal{L}(-D)$ on $X_{\bar{v}}$ without common zeroes. Thus $f = [\tilde{\tau} : \tilde{\sigma}]$ is a well defined non constant morphism from $X_{\bar{v}}$ to $\mathbb{P}^1_{\bar{v}}$, and we have

$$W_{\bar{v}} = D \times_{\bar{v}} \mathbb{P}^1_{\bar{v}} \cup \text{Graph}(f) \hookrightarrow X_{\bar{v}} \times_{\bar{v}} \mathbb{P}^1_{\bar{v}}.$$

Let $K = k(\bar{v})((t^{-1}))$ and let $\eta = \text{Spec}(K) \to \mathbb{A}^1_{\bar{v}}$ be a punctured formal neighbourhood of $\infty$. Let us form the following commutative diagram:

$$\begin{array}{ccc}
\mathbb{A}^1_{\bar{v}} & \xrightarrow{\varphi_{\bar{v}}} & \text{Sym}^d_{\mathbb{A}^1_{\bar{v}}}(W^\circ_{\bar{v}}) \\
\eta & \xrightarrow{\eta} & \text{Sym}^d_{\eta}(W^\circ_{\bar{v}} \times_{\mathbb{A}^1_{\bar{v}}} \eta). \\
\end{array}$$

Since the divisor of $\tau$ is contained in $U_{\bar{v}}$, so is $D$. In particular, we have

$$W^\circ_{\bar{v}} \times_{\mathbb{A}^1_{\bar{v}}} \eta = W_{\bar{v}} \times_{\mathbb{P}^1_{\bar{v}}} \eta = D \times_{\bar{v}} \eta \cup \text{Graph}(f) \times_{\mathbb{P}^1_{\bar{v}}} \eta = D \times_{\bar{v}} \eta \cup X \times_{f, \mathbb{P}^1_{\bar{v}}} \eta.$$

The divisors $D \times_{\bar{v}} \eta$ and $X_{\bar{v}} \times_{f, \mathbb{P}^1_{\bar{v}}} \eta$ of $W_{\bar{v}} \times_{\mathbb{A}^1_{\bar{v}}} \eta$ are disjoint, since the former lies over closed points of $X$, while the latter lies over the generic point of $X$. We thus have a decomposition

$$W^\circ_{\bar{v}} \times_{\mathbb{A}^1_{\bar{v}}} \eta = D \times_{\bar{v}} \eta \amalg X \times_{f, \mathbb{P}^1_{\bar{v}}} \eta = \coprod_i \text{Spec}(L_i)$$

where $L_i$ is either of the form $K[T]/(T^{d_i})$ if $\text{Spec}(L_i)$ is a connected component of $D \times_{\bar{v}} \eta$, or a field extension of degree $d_i$ of $K$ if $\text{Spec}(L_i)$ is a connected component of $X \times_{f, \mathbb{P}^1_{\bar{v}}} \eta$. In the former case, the restriction of $\mathcal{F}$ to $\text{Spec}(L_i)$ is constant, while in the latter case, we have the further information that the restriction of $\mathcal{F}$ to $\text{Spec}(L_i)$ has ramification bounded by $d_i$ (cf. 3.9), since the ramification indices of $f$ above $\infty$ are lower bounded by the multiplicities of $Y_{\bar{v}}$ and $\mathcal{F}$ has
ramification bounded by these multiplicities by assumption. Moreover, we have $\sum_i d_i = d$, and the morphism $\eta \mapsto \text{Sym}_{\eta}^{d}(W^{\circ}_{\bar{v}} \times_{A^1_{\eta}} \eta)$ factors through the canonical morphism
\[
\prod_{i} \text{Sym}_{\eta}^{d_i}(\text{Spec}(L_i)) \to \text{Sym}_{\eta}^{d}(W^{\circ}_{\bar{v}} \times_{A^1_{\eta}} \eta).
\]

By 3.14, we obtain that the restriction of $G_{\bar{v}}$ to $\eta$ is tamely ramified. Since the tame fundamental group of $A^1_{\eta}$ is trivial, we conclude that $G_{\bar{v}}$ is a constant étale $\Lambda$-module on $A^1_{\eta}$.

**Remark 5.12.** While the proof of Proposition 3.14, which constitutes the core of the proof of Proposition 5.11 above, uses geometric local class field theory, it should be noticed that its statement does not refer to it. This explains why no form of local-global compatibility is required in the proof of Proposition 5.11.

**Corollary 5.13.** Let $F, G$ be as above (cf. (5.10)). Assume that $F$ has ramification bounded by $Y$. Then $G$ is the pullback by the projection $A^1_{V} \to V$ of a locally free $\Lambda$-module of rank 1 on $V_{\text{ét}}$.

This follows from Proposition 5.11, from Corollary 2.13 and from the following lemma:

**Lemma 5.14.** Let $g : T' \to T$ be a quasicompact smooth compactifiable morphism of schemes of relative dimension $\delta$ with geometrically connected fibers, and let $G$ be an étale sheaf of $\Lambda$-modules on $T'_{\text{ét}}$ which is constant on each geometric fiber of $g$. Then $G$ is isomorphic to the pullback by $g$ of an étale sheaf of $\Lambda$-modules on $T_{\text{ét}}$.

By ([SGA4], XVIII 3.2.5) the functor $Rg_{!*}$ on the derived category of $\Lambda$-modules on $T$ admits the functor $g' : K \mapsto g^*K(\delta)[2\delta]$ as a right adjoint. Let us apply the functor $\mathcal{H}^0$ to the adjunction morphism $G \to g^!Rg_{!*}G$. The morphism
\[
G \to \mathcal{H}^0(g^!Rg_{!*}G) = g^*R^{2\delta}g_{!*}G(\delta)
\]
is an isomorphism, as can be seen by checking the stalks at geometric points with the proper base change theorem.

**Proposition 5.15.** Let $F$ be as above (cf. (5.10)). Assume that $F$ has ramification bounded by $Y$. Then the pullback of $F^{[d]}$ (cf. 2.32) by the morphism
\[
V \to \text{Div}_{k}^{d,+}(X,Y),
\]
which sends a section $s$ of $V$ to $(\mathcal{L},\tau - s)$, is a constant étale sheaf.

For each element $t$ of $k$, let $\alpha_t$ be the morphism from $V$ to $\text{Div}_{k}^{d,+}(X,Y) = \text{Sym}_{k}^{d}(U)$ which sends a section $s$ of $V$ to $(\mathcal{L},\tau - ts)$. Then $\alpha_t$ is the composition of the morphism
\[
A^1_{V} \xrightarrow{s_t} \text{Sym}_{A^1_{V}}^{d}(W^{\circ}) \to \text{Sym}_{V}^{d}(W^{\circ}) \xrightarrow{\text{Sym}_{V}^{d}(b^s)} \text{Sym}_{V}^{d}(U \times_k V) \to \text{Sym}_{k}^{d}(U),
\]
with the section $t : V \to A^1_{V}$ by Lemma 5.9, so that $\alpha_t^{-1}F^{[d]}$ coincides with $t^{-1}G$. By Corollary 5.13, the sheaf $\alpha_1^{-1}F^{[d]} = 1^{-1}G$ is isomorphic to $0^{-1}G = \alpha_0^{-1}F^{[d]}$. However, the morphism $\alpha_0$ is constant, so that $\alpha_0^{-1}F^{[d]}$ is a constant étale sheaf. Thus the sheaf $\alpha_t^{-1}F^{[d]}$ is constant as well.

**Corollary 5.16.** Let $F$ be as above. Assume that $F$ has ramification bounded by $Y$. Then the locally free $\Lambda$-module $F^{[d]}$ on $\text{Div}_{k}^{d,+}(X,Y)_{\text{ét}}$ is constant on the fiber at $(\mathcal{L},i^*\tau)$ of the morphism
\[
\Phi_{d} : \text{Div}_{k}^{d,+}(X,Y) \to \text{Pic}_{k}(X,Y)
\]
from 4.14.
Indeed, the morphism
\[ V \to \text{Div}^{d,+}_k(X, Y), \]
which sends \( \sigma \) to \((\mathcal{L}, \tau - \sigma)\), is an isomorphism from \( V \) to the fiber of \( \Phi_d \) over the \( k \)-point \((\mathcal{L}, i^* \tau)\), cf. 4.14. The conclusion then follows from Proposition 5.15.

5.17. We now prove Theorem 5.3. Let \( \mathcal{F} \) be a locally free \( \Lambda \)-module of rank 1 over \( U_{\text{ét}} \). The family \((\mathcal{F}[d])_{d \geq 0}\) of locally free \( \Lambda \)-modules of rank 1 yields a multiplicative étale \( \Lambda \)-module of rank 1 on the \( S \)-semigroup scheme
\[ \text{Div}^+_S(X, Y) = \coprod_{d \geq 0} \text{Div}^{d,+}_S(X, Y). \]

For each integer \( d \geq N + 2g \), Corollary 5.16 implies that the locally free \( \Lambda \)-module \( \mathcal{F}[d] \) of rank 1 on \( \text{Div}^{d,+}_S(X, Y) \) (cf. 2.32 and 4.12) is constant on the geometric fibers of the smooth surjective morphism (cf. 4.14)
\[ \Phi_d : \text{Div}^{d,+}_S(X, Y) \to \text{Pic}^d_S(X, Y) \]
\[ (\mathcal{L}, \sigma) \mapsto (\mathcal{L}, i^* \sigma). \]

This morphism satisfies the conditions of Lemma 5.14 by Proposition 4.14. We can therefore apply Lemma 5.14, and we obtain a locally free \( \Lambda \)-module \( \mathcal{G}_d \) of rank 1 over \( \text{Pic}^d_S(X, Y) \) such that \( \Phi_d^{-1} \mathcal{G}_d \) is isomorphic to \( \mathcal{F}[d] \). By Proposition 2.8, the family \((\mathcal{G}_d)_{d \geq N + 2g}\) yields a multiplicative locally free \( \Lambda \)-module of rank 1 on the \( S \)-semigroup scheme
\[ M = \coprod_{d \geq N + 2g} \text{Pic}^d_S(X, Y). \]

Since the morphism
\[ \rho : M \times_S M \to \text{Pic}_S(X, Y) \]
\[ (x, y) \mapsto xy^{-1} \]
is faithfully flat and quasicom pact, we can apply Proposition 2.15, which yields a multiplicative locally free \( \Lambda \)-module \( \mathcal{G} \) of rank 1 over \( \text{Pic}^d_S(X, Y) \) whose restriction to \( \text{Pic}^d_S(X, Y) \) coincides with \( \mathcal{G}_d \) for \( d \geq N + 2g \). The families \((\mathcal{F}[d])_{d \geq 0}\) and \((\Phi_d^{-1} \mathcal{G}_d)_{d \geq 0}\) yield multiplicative locally free \( \Lambda \)-modules of rank 1 on the \( S \)-semigroup scheme \( \text{Div}^+_S(X, Y) = \coprod_{d \geq 0} \text{Div}^{d,+}_S(X, Y) \), whose restrictions to the ideal
\[ I = \coprod_{d \geq N + 2g} \text{Div}^{d,+}_S(X, Y) \]
of \( \text{Div}^+_S(X, Y) \) are isomorphic. We obtain by Proposition 2.7 an isomorphism from \( \mathcal{F}[d] \) to \( \Phi_d^{-1} \mathcal{G}_d \) for each \( d \geq 0 \). In particular, the locally free \( \Lambda \)-module \( \Phi_1^{-1} \mathcal{G}_1 \) of rank 1 is isomorphic to \( \mathcal{F} \).

References

[BLR90] S. Bosch, W. Lutkebohmert and M. Raynaud, Nérond Models, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag.

[CC13] C. Contou-Carrère, Jacobienne locale d’une courbe formelle relative, Rendiconti del Seminario Matematico della Università di Padova 130 (2013), pp.1-106.

[EGA3] A. Grothendieck, J. Dieudonné. Éléments de géométrie algébrique: III. Étude cohomologique des faisceaux coherents. Publications Mathématiques de l’IHÉS 11-17, 1961-1963.

[EGA4] A. Grothendieck, J. Dieudonné. Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas. Publications Mathématiques de l’IHÉS 20-24-28-32, 1964-1967.

[La56] S. Lang, Sur les séries L d’une variété algébrique, Bulletin de la S. M. F. 84 (1956), p. 385-407.

[MB85] L. Moret-Bailly, Pinceaux de variétés abéliennes, Astérisque 129 (1985).
[Ro54] M. Rosenlicht, Generalized Jacobian Varieties, Annals of Mathematics 59 (1954), p. 505-530.
[Se59] J.-P. Serre, Groupes algébriques et corps de classes, Hermann (Paris), 1959.
[Se61] J.-P. Serre, Sur les corps locaux à corps résiduel algébriquement clos, Bulletin de la S. M. F. 89 (1961), p. 105-154.
[Se68] J.-P. Serre, Corps locaux, Hermann (Paris), 1968.
[SGA1] A. Grothendieck, Revêtements étals et groupe fondamental (SGA 1), Séminaire de Géométrie Algébrique du Bois Marie 1960/61, Springer-Verlag, Lecture Notes in Mathematics 224, 1971.
[SGA4] M. Artin, A. Grothendieck, J. L. Verdier, Théorie des topos et cohomologie étale des schémas (SGA 4), Séminaire de Géométrie Algébrique du Bois Marie 1963/64, Springer-Verlag, Lecture Notes in Mathematics 305, 1972.
[Stacks] The Stacks Project Authors, Stacks Project, http://stacks.math.columbia.edu, 2017.
[Su13] T. Suzuki, Some remarks on the local class field theory of Serre and Hazewinkel, Bulletin de la S.M.F. 141 (2013), p. 1-24.

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