Does Considering Quantum Correlations Resolve the Information Paradox?

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Abstract

The absence of consideration of subtle correlations in radiation process is the reason why Hawking’s semiclassical analysis is often criticized. There has been speculations that accounting for such quantum correlations would eventually invalidate Hawking’s result that black hole evolution is non-unitary. However, it has been recently showed that considering small deviations from Hawking’s analysis does not help significantly to bypass the information paradox—the irreversible loss of information in the evolution of a black hole. This paper generalizes the above result by parameterizing the amount of deviation from Hawking’s analysis that is required to resolve the paradox. With a more rigorous and non-trivial bound than that appeared in literature before, it is confirmed that information retrieval indeed requires ‘not-so-small’ deviation from the Hawking state. In connection to this result, a previously proposed toy model of black hole evaporation is generalized in this paper, and it is showed that allowing quantum correlation in the evaporation process fails to ensure information leakage from the black hole. Finally, using the newly developed generalized bound and its parameters, it is showed that, the recent claim that the assumptions of black hole complementarity are inconsistent, is true even with a ‘relaxed’ definition of ‘innocuous’ horizon.

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I. INTRODUCTION

After Bekenstein had argued in favor of associating entropy with black holes [1], a series of results [2], [3], [4] converged to a complete formulation of black hole thermodynamics as presented in [5]. Following Hawking’s results [4], Bekenstein proposed a linear relationship between the black hole entropy and the surface area of black hole event horizon [6] and formulated the generalized second law of thermodynamics. Thermodynamic features of a black hole suggested that black holes should radiate like any black body, though the idea is in clear contradiction with the classical description of a black hole. In solving the conundrum, Hawking introduced a semiclassical approach introducing quantum mechanical effects on the matter field while still ignoring the quantum gravitational effects. His analysis in [7] yielded a bizarre result that a black hole radiates as a black body of temperature $\frac{\hbar c}{2\pi k}$. This phenomenon, called ‘Hawking radiation’, is attributed to the vacuum fluctuations near the event horizon causing a flux of thermal radiation to be observed by an asymptotic observer. Energy of the radiation must be attributed to the mass of the black hole to ensure conservation of energy. However, such a process violates the underlying principle of quantum evolutions- unitarity. Reasonably, it can be assumed that a black hole is formed from some matter in pure state. On the other hand, radiation is thermal, characterized by the black hole temperature and described by a mixed state [8], [9]. Two black holes with identical temperature would radiate identical radiation irrespective of the matter that forms it, causing irretrievable loss of information. Violation of unitarity by Hawking’s semiclassical results is known as the black hole information paradox.

Out of many advocated approaches that might resolve the paradox, one pragmatic view is that small correction to the leading order semiclassical result accumulates over time so that by the end of its lifetime, black hole has radiated away most of the information [10]. However, this speculation has been invalidated in [11] by clearly showing that the paradox cannot actually be bypassed using ‘small’ departure from the semiclassical analysis. This result has been exemplified by some toy models in [12] and [13] considering ‘small’ correction to the Hawking state in the form of weak correlation between quanta produced in successive steps of black hole radiation. The present authors generalize the toy model in [13] allowing arbitrary, small corrections and reassure the same result.
Then the authors quantify the notion of ‘correction’ to the semiclassical analysis that is required to avoid the paradox and ensure that black hole evolution is unitary. A rigorous and nontrivial bound on the ‘correction’ factor is established for possible recovery of information from the black hole.

A recent paper [14] has raised the debate that the assumptions of black hole complementarity [15] are not consistent. Specifically, the notion that horizon is ‘innocuous’ to infalling observer, and the requirement that information leaks out of the black hole, are at odds under the assumption that LQFT is the accurate low curvature physics. In this paper, the present authors introduce a ‘relaxed’ notion of ‘innocuous’ horizon using the framework of generalized ‘correction’ and show that even relaxing the conditions does not resolve the above stated inconsistency.

The organization of the paper is as follows. Section II introduces the leading order formulation of the black hole information paradox. In section III, the authors present the motivation to look for resolving the paradox using ‘small’ corrections. Section IV discusses the arguments why the problem cannot be avoided incorporating ‘small’ corrections. In section V a simple toy model supporting the results of the previous section is briefly analyzed. Section VI produces a rigorous and nontrivial generalization of the bound on correction factors required to ensure information retrieval from an evaporating black hole. In section VII, the authors show that the inconsistency of the assumptions of black hole complementarity cannot be avoided using an even more relaxed definition for ‘innocuous’ horizon. Finally, section VIII summarizes the conclusion of the paper.

II. LEADING ORDER FORMULATION OF THE BLACK HOLE INFORMATION PARADOX

In this section we present the leading order formulation as presented in [11] and subsequently followed up in [12], [16], [17], [18] to investigate different information theoretic aspects of the information paradox. It can be showed [8, 9] that the joint system of the particle pair near the horizon can be given as-

\[ |\Psi_{\text{pair}}\rangle = C e^{\beta c^\dagger b^\dagger} |0\rangle_c |0\rangle_b \]

where \( \beta \) is a number of order unity, \( c^\dagger \) and \( b^\dagger \) are creation operators, \( |0\rangle \) represents the vacuum state and \( c, b \) represent the ingoing and outgoing quanta respectively. This state
is generally an entangled one, indicated by a trivially non-vanishing von Neumann entropy of any of the subsystems. Leading order analysis makes the following assumptions—

(a) The black hole geometry can be foliated by a set of spacelike slices \[19\], \[11\]. Evolution of states over these slices can be explained by local quantum field theory as long as black hole dimensions are much larger than Planckian dimensions.

(b) The state of the collapsing shell is represented by the state vector \(|\Psi\rangle_M\). During the evolution, state of this matter usually remains unchanged, i.e. \(|\Psi\rangle_M \rightarrow |\Psi\rangle_M\). However, allowing some local operator act on this subspace alone while leaving other subspaces unchanged will also produce the essentially same result.

(c) State of the newly created pair can be approximated as

\[
|\Psi\rangle_{\text{pair}} = \frac{1}{\sqrt{2}}(|0\rangle_c|0\rangle_b + |1\rangle_C|1\rangle_b)
\]

This state is a simplified version of \[11\] considering the newly evolved pair to be maximally entangled. It should be noted that instead of using an infinite dimensional Hilbert space as in \[11\], this simplified state reflects essentially the same physics on a much simpler qubit space. The entropy of entanglement associated to any of the subspaces is given by

\[
S_{\text{ent}} = \log 2
\]

(d) At each successive step of pair creation the initial matter and the earlier created quanta move along the spacelike slice by distances of the order of \(R\), the radius of the black hole event horizon. Stretched spacelike slice causes a new pair to evolve according to \[2\] while the earlier qubits and the matter state moves farther along the spacelike slice. We ignore any influence of these quanta over the newly created pair and describe the entire system as a tensor product state-

\[
|\Psi\rangle = |\Psi\rangle_M \otimes |\Psi\rangle_1 \otimes |\Psi\rangle_2 \otimes \ldots \otimes |\Psi\rangle_N
\]

where each of the \(|\Psi\rangle_i\) state is given by \[2\]. After emission of \(N\) pairs, the entanglement entropy between the ingoing and outgoing quanta is given by

\[
S_{\text{ent}} = N \log 2
\]

It is this monotonically increasing entangle entropy that lies at the heart of the paradox. If the remnant scenario is discarded as advocated in \[10\], \[20\] and \[21\], the black
hole completely vanishes and only the outgoing quanta remain as a thermal radiation with nonzero entropy despite starting with a pure state with zero entanglement entropy. One intriguing feature of this analysis is the increasing dimension of the Hilbert space. The black hole starts with a large but finite entropy and hence should be constrained to a quantum mechanical description within a Hilbert space of finite dimension \[22\]. Unitary evolution preserves entropy which is in general a function of the dimension of the Hilbert space, though dimensions of individual subspaces can decrease as advocated in \[22\]. This might intuitively imply an inevitable departure from unitarity. However, it has been showed in \[16\] that dimension can be effectively preserved in this framework as well. One should consider the additional \(2k\) degrees of freedom after \(k\) steps of pair production as auxiliary and not physical ones- a mathematical tool introduced only to conveniently represent the phenomenon. Since we can always imagine an ancillary to purify a mixed state by enlarging the Hilbert space, we might expect a restoration of unitarity over an enlarged Hilbert space provided that the radiation quanta at the end of the evaporation are in a pure state. Similar models involving nonlocal modes of information transfer have been presented in \[22\], \[23\].

### III. MOTIVATION OF SMALL CORRECTIONS TO LEADING ORDER

The leading order analysis captures the coarse grained physical reality- the salient features of the contradiction between black hole radiation and unitarity. However, it ignores any possible correlation between the outgoing radiation and the earlier ones or the black hole forming matter. Associating back reaction, quantum gravity effects or small perturbations to the Schwarzschild geometry might modify the state \(2\) of the particle pair. However, incorporating such details should bring about only some small correction. This is because the horizon is still a low curvature region and LQFT should provide a reasonably accurate picture of quantum processes near the horizon. Despite the predicted smallness of the correction, it has been widely speculated that these corrections might build up in course of time to eventually restore unitarity \[24\]. Such speculations rely on the large number of radiated quanta over the lifetime of a black hole. One simple example might illustrate the effect of accumulating corrections in departure from predicted result. Let us assume that a particular process generates photons in a definite polarization state,
say the horizontally polarized state-

\( \hat{\rho} = |\rightarrow\rangle\langle \rightarrow | \)  

(6)

This is a pure state and hence possesses zero entanglement entropy. However, some error in the process might cause this state to be slightly modified as a mixed one.

\( \hat{\rho}' = (1 - \epsilon) |\rightarrow\rangle\langle \rightarrow | + \epsilon |\uparrow\rangle\langle \uparrow | \)  

(7)

Here \( \epsilon \) is a small positive number, much less than unity. Entanglement entropy associated with this state is given by \( S(\hat{\rho}') = -((1 - \epsilon) \log (1 - \epsilon) + \epsilon \log \epsilon) \) which is close to zero. However, after \( N \) such photons are emitted, entanglement entropy of the joint system could be considerably large where the entanglement entropy of the joint system of \( N \) unmodified photons will be still zero. This departure can also be quantified in measures of closeness e.g. fidelity. \( \hat{\rho}' \) closely resembles \( \hat{\rho} \) as \( F(\hat{\rho}, \hat{\rho}') = 1 - \epsilon \). However, we find the system of \( N \) photons exhibit only low fidelity with the joint state of \( N \) unmodified photons as \( N \) becomes large.

\( F(\hat{\rho}^{\otimes N}, \hat{\rho}'^{\otimes N}) = (1 - \epsilon)^N \to 0 \)  

(8)

As accumulated small corrections over a large number of states can yield significant departure from a desired state, it is speculative to consider such processes in case of black hole radiation. In literature, alternative formulations of the radiation spectrum of black holes consider such corrections. For example, particle production has been associated to a tunneling picture in [25], [26], [27] and it has been advocated that non thermal corrections might lead to information leakage at least at late times. Such excursions lead to the curious insight- if small corrections can truly contribute to information transfer. In [11], an abstract mathematical formalism has been established to investigate the implications of small corrections to the leading order analysis which has been followed up in [16] and [17] to generalize the result. The following sections review the mathematical results from [11] and elaborate the results from [17] to provide stronger support to the fact that small corrections actually cannot restore unitarity in the context of black hole radiation.
IV. SMALL CORRECTION TO HAWKING STATE - MATHUR’S BOUND

Let us assume that the created pair at each time-step of evolution is not invariably in the Hawking state, rather it can be in any state of the space spanned by the basis states

\[ S^{(1)} = \frac{1}{\sqrt{2}} |0\rangle_{c_{n+1}} |0\rangle_{b_{n+1}} + \frac{1}{\sqrt{2}} |1\rangle_{c_{n+1}} |1\rangle_{b_{n+1}} \]  

and

\[ S^{(2)} = \frac{1}{\sqrt{2}} |0\rangle_{c_{n+1}} |0\rangle_{b_{n+1}} - \frac{1}{\sqrt{2}} |1\rangle_{c_{n+1}} |1\rangle_{b_{n+1}}. \]

Here we deliberately choose to avoid the subspace spanned by the states \(|0\rangle_{c_{n+1}} |0\rangle_{b_{n+1}}\) and \(|1\rangle_{c_{n+1}} |0\rangle_{b_{n+1}}\), because there is not much physical explanation for pair creation in such states. Moreover, a four dimensional space considering all these four states as basis states has been considered in [12], and it shows no result essentially different from that obtained from a two dimensional analysis.

The complete system consists of the matter \(M\), inside quanta \(c_i\) and outside quanta \(b_i\). Let us choose a basis \(|\psi_i\rangle\) for the subsystem comprising matter \(M\) and inside quanta \(c_i\) and another basis \(|\chi_i\rangle\) for the radiation subsystem comprising of the \(b_i\) quanta. Then the state of the complete system can be expressed as

\[ |\Psi_{M,c,b}(t_n)\rangle = \sum_{m,n} C_{m,n} \psi_m \chi_n. \]  

We can always perform Schmidt decomposition to express this state as

\[ |\Psi_{M,c,b}(t_n)\rangle = \sum_i C_i \psi_i \chi_i. \]

At next time-step of evolution, the \(b_i\) quanta move farther apart from the vicinity of the hole. Since the hole can no longer influence their evolution, we consider that no further evolution takes place for the outgoing quanta. The created pair can be in a superposition of the states \(S^{(1)}\) and \(S^{(2)}\). Hence the state \(\psi_i\) can evolve into

\[ \psi_i \rightarrow \psi_i^{(1)} S^{(1)} + \psi_i^{(2)} S^{(2)} \]

where the state \(\psi_i\) has been expressed as the tensor product of the state \(\psi_i^{(i)}\) representing \(\{M, c_i\}\) subsystem and \(S^{(i)}\) representing the newly created pair. Since \(S^{(1)}\) and \(S^{(2)}\) are orthonormal states, unitarity requires that

\[ \|\psi_i^{(1)}\|^2 + \|\psi_i^{(2)}\|^2 = 1. \]
In leading order case, newly created pair is invariably in the state $S^{(1)}$; hence $\psi_i^{(1)} = \psi_i$ and $\psi_i^{(2)} = 0$.

Now,

$$|\Psi_{M,c}^{(1)}\rangle = \sum_i C_i [\psi_i^{(1)} S^{(1)} + \psi_i^{(2)} S^{(2)}] \chi_i$$

$$= \left[ \sum_i C_i \psi_i^{(1)} \chi_i \right] S^{(1)} + \left[ \sum_i C_i \psi_i^{(2)} \chi_i \right] S^{(2)}$$

$$= \Lambda^{(1)} S^{(1)} + \Lambda^{(2)} S^{(2)}$$

where $\Lambda^{(1)} = \sum_i C_i \psi_i^{(1)} \chi_i$, $\Lambda^{(2)} = \sum_i C_i \psi_i^{(2)} \chi_i$.

Entanglement entropy of the $\{b\}$ quanta

$$S_{bn} = - \text{tr} \hat{\rho}_{bn} \log \hat{\rho}_{bn}$$

$$= \sum_i |C_i|^2 \log |C_i|^2 = S_0.$$  

Since earlier emitted outside quanta can no longer be influenced, we have the same entanglement entropy of $\{b\}$ quanta at time-step $t_{n+1}$.

Now, entanglement entropy of the pair $(b_{n+1}, c_{n+1})$ with the rest of the system is given by

$$S(b_{n+1}, c_{n+1}) = - \text{tr} \hat{\rho}_{b_{n+1}, c_{n+1}} \log \hat{\rho}_{b_{n+1}, c_{n+1}}.$$  

(18)

Density matrix for the system $(b_{n+1}, c_{n+1})$ is

$$\hat{\rho}_{b_{n+1}, c_{n+1}} = \begin{pmatrix} \langle \Lambda^{(1)} | \Lambda^{(1)} \rangle & \langle \Lambda^{(1)} | \Lambda^{(2)} \rangle \\ \langle \Lambda^{(2)} | \Lambda^{(1)} \rangle & \langle \Lambda^{(2)} | \Lambda^{(2)} \rangle \end{pmatrix}.$$  

(19)

Again, normalization of $|\Psi_{M,c}^{(1)}\rangle$ requires

$$\| \Lambda^{(1)} \|^2 + \| \Lambda^{(2)} \|^2 = 1.$$  

Mathur defined the correction to leading order Hawking state to be ‘small’ in the sense that $\| \Lambda^{(2)} \|^2 < \epsilon$, where $\epsilon \ll 1$ [11]. This definition implies that there is very small admixture of the $S^{(2)}$ state with the $S^{(1)}$ state when new particle pairs are generated. Under such ‘small’ departure from leading order semi-classical Hawking analysis, Mathur showed that entanglement entropy at each time-step increases by at least $\log 2 - 2 \epsilon$ [11]. Since $\epsilon$ is a very small number, by definition, there is still order unity increase in entanglement entropy at each time-step, when ‘small’ corrections are allowed. This
result has been exemplified by a simple model incorporating small correlations between quanta created in consecutive steps \[12\]. Some other toy models of evaporation have been studied in \[13\], which also conform to this result. We shall discuss briefly about these models in the next section. We shall also illustrate Mathur’s bound in a toy model that incorporates a more general correlation compared to the model presented in \[12\].

V. A SIMPLE TOY MODEL INCORPORATING SMALL CORRECTION

In this section we shall discuss a simple toy model of black hole evaporation that incorporate small correction to leading order Hawking analysis. This is a simple generalization of the model presented in \[12\] that abandons the ‘symmetry’ of impacts of early radiated quanta on the new pair and allows arbitrary but small corrections to the new state. In \[12\] it has been assumed that the probability amplitude of particle creation at any step in the same state as the previous step is increased by a small factor and in the different state than the previous step is decreased by a small factor. In initial step there is no pair.

At step 1, pair is created in ‘Hawking state’.

\[
|\psi\rangle_{n=1} = \frac{1}{\sqrt{2}}|0\rangle_{c_1}|0\rangle_{b_1} + \frac{1}{\sqrt{2}}|1\rangle_{c_1}|1\rangle_{b_1} \tag{20}
\]

There is equal probability of no creation (\(|0\rangle_{c_1}|0\rangle_{b_1}\)) or pair creation (\(|1\rangle_{c_1}|1\rangle_{b_1}\)) at this step. If no pair is created at this step, the probability amplitude of no pair creation at the next step is \(\frac{1}{\sqrt{2}}e^a\) and the probability amplitude of pair creation at next step is \(\frac{1}{\sqrt{2}}e^b\); here \(a\) is a very small positive quantity and \(b\) is a very small negative quantity.

\[
|0\rangle_{c_1}|0\rangle_{b_1} \rightarrow \frac{1}{\sqrt{2}}e^a|0\rangle_{c_2}|0\rangle_{b_2} + \frac{1}{\sqrt{2}}e^b|1\rangle_{c_2}|1\rangle_{b_2} \tag{21}
\]

Similarly, if we have pair creation step \(n = 1\), then we have a symmetric correction to the new pair, i.e. the same factor that modified the state \(|1\rangle_{c_1}|1\rangle_{b_1}\) will appear with the state \(|0\rangle_{c_1}|0\rangle_{b_1}\) and vice versa.

\[
|1\rangle_{c_1}|1\rangle_{b_1} \rightarrow \frac{1}{\sqrt{2}}e^b|0\rangle_{c_2}|0\rangle_{b_2} + \frac{1}{\sqrt{2}}e^a|1\rangle_{c_2}|1\rangle_{b_2} \tag{22}
\]

By unitarity, we have

\[
e^{2a} + e^{2b} = 2 \tag{23}
\]

This model illustrates the failure of small corrections to restore unitarity. We now show that this result holds even if corrections are arbitrary and small but not 'symmetric'.
Though the generalization is obvious, the results are trickier to obtain. Suppose at each successive stage, the probability amplitude for pair production is slight, but arbitrarily, modified by the history of earlier stage. Instead of (21) and (22), we let the general evolution to take the form

\[
|00\rangle_1 \rightarrow |00\rangle_1 \otimes \left( \frac{e^{a_0}}{\sqrt{2}} |00\rangle_2 + \frac{e^{b_0}}{\sqrt{2}} |11\rangle_2 \right) \tag{24}
\]

\[
|11\rangle_1 \rightarrow |11\rangle_1 \otimes \left( \frac{e^{a_1}}{\sqrt{2}} |00\rangle_2 + \frac{e^{b_1}}{\sqrt{2}} |11\rangle_2 \right) \tag{25}
\]

By unitarity, we have

\[
e^{2a_0} + e^{2b_0} = e^{2a_1} + e^{2b_1} = 2 \quad \tag{26}
\]

The general state of the first two emitted quanta is given as

\[
|\psi\rangle_2 = \frac{1}{\sqrt{2}} |00\rangle_1 \otimes \left( \frac{e^{a_0}}{\sqrt{2}} |00\rangle_2 + \frac{e^{b_0}}{\sqrt{2}} |11\rangle_2 \right) + \frac{1}{\sqrt{2}} |11\rangle_1 \otimes \left( \frac{e^{a_1}}{\sqrt{2}} |00\rangle_2 + \frac{e^{b_1}}{\sqrt{2}} |11\rangle_2 \right) \tag{27}
\]

Let, \( S^{(n)} \) denote the entanglement entropy of the outgoing quanta after \( n \) pairs have been emitted. It is straightforward to calculate \( S^{(2)} \).

\[
S^{(2)} = 2 \log 2 - \frac{1}{2^2} \left( 2a_0 e^{2a_0} + 2b_0 e^{2b_0} + 2a_1 e^{2a_1} + 2b_1 e^{2b_1} \right) \tag{28}
\]

Let us assume,

\[
S_{2,0} = \frac{1}{2^2} \left( 2a_0 e^{2a_0} + 2b_0 e^{2b_0} \right) \tag{29}
\]

\[
S_{2,1} = \frac{1}{2^2} \left( 2a_1 e^{2a_1} + 2b_1 e^{2b_1} \right) \tag{30}
\]

Significance of the terms \( S_{2,0} \) and \( S_{2,1} \) will be explored shortly. In general, we can write the joint state of \( n \) pairs as

\[
|\psi\rangle_n = |\psi\rangle_{n-1}^{(1)} \otimes |00\rangle_n + |\psi\rangle_{n-1}^{(2)} \otimes |11\rangle_n \tag{31}
\]

Hence, by (24-25), the joint state of all the produced quanta after the emission of \( n + 1 \) pairs becomes

\[
|\psi\rangle_{n+1} = |\psi\rangle_{n-1}^{(1)} \otimes |00\rangle_n \otimes \left( \frac{e^{a_0}}{\sqrt{2}} |00\rangle_{n+1} + \frac{e^{b_0}}{\sqrt{2}} |11\rangle_{n+1} \right) \\
+ |\psi\rangle_{n-1}^{(2)} \otimes |11\rangle_n \otimes \left( \frac{e^{a_1}}{\sqrt{2}} |00\rangle_{n+1} + \frac{e^{b_1}}{\sqrt{2}} |11\rangle_{n+1} \right) \tag{32}
\]
Analysis of this model becomes complex because of the intricate form of $|\psi\rangle_{n+1}$. After manually calculating entropy relations for some early values of $n$, a recursive relation for $S^{(n)}$ can be coined with the form

$$S^{(n+1)} = S^{(n)} + \log 2 - \text{correction terms} \quad (33)$$

Our objective is to find the form of the correction terms, to be denoted by $C(n + 1)$ from now on. Fortunately, this can be done from intuitive understanding. It can be easily identified that $|\psi\rangle_{n+1}$ is actually a superposition of states corresponding to the $n + 1$ bit binary strings in the chosen basis. Now, we could divide these strings in four disjoint classes.

- $0\rightarrow0\ldots$ (followed by all possible $n - 1$ bit strings)
- $0\rightarrow1\ldots$ (followed by all possible $n - 1$ bit strings)
- $1\rightarrow0\ldots$ (followed by all possible $n - 1$ bit strings)
- $1\rightarrow1\ldots$ (followed by all possible $n - 1$ bit strings)

From (24-25), it can be identified that the first transition takes place with a probability $\frac{e^{2a_0}}{2}$. Similarly, the rest of the transitions are identified with probabilities $\frac{e^{2a_1}}{2}, \frac{e^{2b_0}}{2}, \frac{e^{2b_1}}{2}$ respectively. Now we shall define the following quantities,

- $S_{n,0} = \text{Correction due to } n \text{ qubit states that start with a 0}$
- $S_{n,1} = \text{Correction due to } n \text{ qubit states that start with a 1}$

Hence,

$$C(n + 1) = S_{n,0} \frac{e^{2a_0}}{2} + S_{n,1} \frac{e^{2b_0}}{2} + S_{n,0} \frac{e^{2a_1}}{2} + S_{n,1} \frac{e^{2b_1}}{2}$$

$$= \frac{1}{2} \left\{ S_{n,0} \left( e^{2a_0} + e^{2a_1} \right) + S_{n,1} \left( e^{2b_0} + e^{2b_1} \right) \right\} \quad (34)$$

It is possible to define the terms $S_{n,0}$ and $S_{n,1}$ recursively as well. Each $n$ bit binary string will be followed by a $n - 1$ bit string starting with either 0 or 1. As stated earlier, for each case we have a probability of transition of the form $\frac{e^{2a_i}}{2}$ or $\frac{e^{2b_i}}{2}$ depending on the combination of the first and the second bit. Therefore,

$$S_{n,0} = S_{n-1,0} \frac{e^{2a_0}}{2} + S_{n-1,1} \frac{e^{2b_0}}{2} \quad (35)$$

$$S_{n,1} = S_{n-1,0} \frac{e^{2a_1}}{2} + S_{n-1,1} \frac{e^{2b_1}}{2} \quad (36)$$
Assuming \( a_i \) and \( b_i \) to be contributing only small corrections to the leading order state, it can be deduced that \( |a_i| \ll 1, |b_i| \ll 1 \). Expanding the exponential terms up to the first order in the normalization condition implies

\[
e^{2a_i} + e^{2b_i} = 2
\]

\[
(1 + 2a_i) + (1 + 2b_i) = 2
\]

\[
\Rightarrow a_i \approx -b_i
\]

(37)

From (29),

\[
S_{2,0} = \frac{1}{2^2} (2a_0e^{2a_0} - 2a_0e^{-2a_0})
\]

\[
= \frac{1}{2^2} (2a_0(1 + 2a_0) - 2a_0(1 - 2a_0))
\]

\[
= 2a_0^2
\]

\[
\Rightarrow S_{2,0} = 2a_0^2 \ll \log 2
\]

(38)

And similarly it can be shown from (30) that

\[
S_{2,1} \ll \log 2
\]

(39)

Now that we have proved that both \( S_{2,0} \) and \( S_{2,1} \) are much smaller than \( \log 2 \), we shall apply mathematical induction to prove that this holds for the correction terms \( S_{n,0} \) and \( S_{n,1} \) for every value of \( n \). Let this be true for some \( n = m \), i.e.

\[
\frac{S_{m,0}}{\log 2} = \epsilon_{m,0} \ll 1
\]

(40)

\[
\frac{S_{m,1}}{\log 2} = \epsilon_{m,1} \ll 1
\]

(41)

We shall now prove that this is true for \( n = m + 1 \) as well.

\[
\frac{S_{m+1,0}}{\log 2} = \epsilon_{m,0} \frac{e^{2a_0}}{2} + \epsilon_{m,1} \frac{e^{2b_0}}{2} < \max(\epsilon_{m,0}, \epsilon_{m,1}) \ll 1
\]

(42)

\[
\frac{S_{m+1,1}}{\log 2} = \epsilon_{m,0} \frac{e^{2a_1}}{2} + \epsilon_{m,1} \frac{e^{2b_1}}{2} < \max(\epsilon_{m,0}, \epsilon_{m,1}) \ll 1
\]

(43)

This proves that \( S_{n,0} \) and \( S_{n,1} \) are much smaller than \( \log 2 \) for every value of \( n \). Now we use these expressions from (42, 43) in (34) to obtain the expression for the correction term
\[ C(n + 1), \]
\[ C(n + 1) = \frac{1}{2} \left\{ S_{n,0} \left( e^{2a_0} + e^{2a_1} \right) + S_{n,1} \left( e^{2b_0} + e^{2b_1} \right) \right\} \]
\[ < \max(\epsilon_{n-1,0}, \epsilon_{n-1,1}) \frac{e^{2a_0} + e^{2a_1} + e^{2b_0} + e^{2b_1}}{2} \log 2 \]
\[ = 2 \max(\epsilon_{n-1,0}, \epsilon_{n-1,1}) \log 2 \ll \log 2 \]  

(44)

This implies that the change in entanglement entropy is of the same order as \( \log 2 \) despite allowing small correlations to build up with each pair of emitted particles. This result establishes the invalidity of the ‘class’ of models that advocate small modifications in the probability amplitude of pair production in successive states. In essence, this result nullifies any significant impact of small back reaction effects or non-thermal factors incorporated in the evolution picture in terms of modified probability amplitude and hence, pronounces a much significant result than the model in \([12]\).

VI. GENERALIZATION OF MATHUR’S BOUND

In this section we review the generalization of Mathur’s bound as presented in \([17]\). This generalization is motivated with a view to exploring the effects of introducing corrections of arbitrary magnitude to the leading order analysis and explore, if any, the possibilities of restoration of unitarity. Similar generalization has been demonstrated in \([16]\) but \([11]\) and \([16]\) only establishes trivial lower bounds to the change of entanglement entropy. The results in \([17]\) establish nontrivially stronger upper and lower bounds to \( \Delta S \). To facilitate the derivation, we first derive two lemmas leading to the derivation of the final result as a theorem.

Let us introduce the correction parameters and corresponding quantities first:

\[ \langle \Lambda^{(2)}|\Lambda^{(2)} \rangle = \epsilon^2, \]  

(45)

\[ \langle \Lambda^{(1)}|\Lambda^{(1)} \rangle = 1 - \epsilon^2, \]  

(46)

\[ \langle \Lambda^{(1)}|\Lambda^{(2)} \rangle = \langle \Lambda^{(2)}|\Lambda^{(1)} \rangle = \epsilon, \]  

(47)

\[ \gamma^2 = 1 - 4 \left[ \epsilon^2(1 - \epsilon^2) - \epsilon^2 \right]. \]  

(48)

In general \( \epsilon \) should be a complex quantity and \( \langle \Lambda^{(1)}|\Lambda^{(2)} \rangle = \langle \Lambda^{(2)}|\Lambda^{(1)} \rangle^* \). However, in \([17]\) we assume a real \( \epsilon \) for the sake of simplicity, though letting it have complex values will give the same result.
Lemma 1. Entanglement entropy of the newly created pair is given by

\[ S(p) \leq \sqrt{1 - \gamma^2} \log 2. \]

**Proof.** Reduced density matrix for the pair

\[ \hat{\rho}_p = \begin{bmatrix} 1 - \epsilon^2 & \epsilon^2 \\ \epsilon^2 & \epsilon^2 \end{bmatrix}. \] (49)

Eigenvalues of this matrix are: \( \lambda_1 = \frac{1 + \gamma^2}{2} \) and \( \lambda_2 = \frac{1 - \gamma^2}{2} \). Hence entanglement entropy of the pair is

\[ S(p) = -\text{tr} \hat{\rho}_p \log \hat{\rho}_p = -\sum_{i=1}^{2} \lambda_i \log \lambda_i \]

\[ = \log 2 - \frac{1}{2} \left[ (1 + \gamma) \log(1 + \gamma) + (1 - \gamma) \log(1 - \gamma) \right]. \] (50)

It can be shown easily for \( 0 \leq x \leq 1 \) that

\[ (1 - x^2) \log 2 \leq \log 2 - \frac{1}{2} \left[ (1 + x) \log(1 + x) \right. \]

\[ \left. + (1 - x) \log(1 - x) \right] \leq \sqrt{1 - x^2} \log 2. \] (51)

Now, the result follows from (51). \( \square \)

Lemma 2.

\[ (1 - 4\epsilon_2^2) \log 2 \leq S(b_{n+1}) = S(c_{n+1}) \leq \sqrt{1 - 4\epsilon_2^2} \log 2 \]

**Proof.** The complete state of the system after creation of the first pair is

\[ |\Psi_{M,c, \psi_b(t_{n+1})}\rangle = \left[ |0\rangle_{c_{n+1}} |0\rangle_{b_{n+1}} \frac{1}{\sqrt{2}} \left( \Lambda^{(1)} + \Lambda^{(2)} \right) \right] \]

\[ + \left[ |1\rangle_{c_{n+1}} |1\rangle_{b_{n+1}} \frac{1}{\sqrt{2}} \left( \Lambda^{(1)} - \Lambda^{(2)} \right) \right]. \] (52)

Now, the reduced density matrix describing \( c_{n+1} \) or \( b_{n+1} \) quanta is

\[ \hat{\rho}_{b_{n+1}} = \hat{\rho}_{c_{n+1}} \]

\[ = \begin{bmatrix} \frac{1}{2} \langle (\Lambda^{(1)} + \Lambda^{(2)}) |(\Lambda^{(1)} + \Lambda^{(2)}) \rangle & 0 \\ 0 & \frac{1}{2} \langle (\Lambda^{(1)} - \Lambda^{(2)}) |(\Lambda^{(1)} - \Lambda^{(2)}) \rangle \end{bmatrix} \]

\[ = \begin{bmatrix} \frac{1 + 2\epsilon_2}{2} & 0 \\ 0 & \frac{1 - 2\epsilon_2}{2} \end{bmatrix}. \] (54)
Then, entanglement entropy of the $c_{n+1}$ or $b_{n+1}$ quanta is

$$S(b_{n+1}) = S(c_{n+1}) = \log 2 - \frac{1 + 2\epsilon_2}{2}\log(1 + 2\epsilon_2) - \frac{1 - 2\epsilon_2}{2}\log(1 - 2\epsilon_2). \quad (55)$$

Now the result follows directly from (51).

With the use of these lemmas, we now prove our desired result as a theorem.

**Theorem 1.** Change of entanglement entropy from time-step $t_n$ to $t_{n+1}$ is restricted by the following bound:

$$1 - 4\epsilon_2 - \sqrt{1 - \gamma^2} \leq \frac{\Delta S}{\log 2} \leq \sqrt{1 - 4\epsilon_2^2} \quad (56)$$

where $\Delta S = S(b_{n+1}, \{b\}) - S(\{b\})$.

**Proof.** Let us assume $A = \{b\}, B = b_{n+1}, C = c_{n+1}$.

Using **strong subadditivity inequality** we have,

$$S(A) + S(C) \leq S(A, B) + S(B, C)$$

$$\Rightarrow S(\{b\}) + S(c_{n+1}) \leq S(\{b\}, b_{n+1}) + S(b_{n+1}, c_{n+1})$$

$$\Rightarrow \Delta S \geq (1 - 4\epsilon_2^2)\log 2 - \sqrt{1 - \gamma^2}\log 2. \quad (57)$$

Inequality (57) follows from using Lemma 1 and Lemma 2.

Now using **subadditivity inequality** we have,

$$S(A) + S(B) \geq S(A, B)$$

$$\Rightarrow S(\{b\}) + S(b_{n+1}) \geq S(\{b\}, b_{n+1})$$

$$\Rightarrow \Delta S \leq \sqrt{1 - 4\epsilon_2^2}\log 2 \quad (58)$$

Inequality (58) follows from Lemma 2.

The result follows from combining (57) and (58).
In establishing the upper and lower bounds of change in entanglement entropy, we have dropped the concept of ‘smallness’ of the parameters opening the window for a wide range of speculated effects that may cause departure from the leading order analysis of black hole evaporation. Implication of the result \((56)\) is twofold– (a) in putting an upper bound on the change in entanglement entropy and (b) in providing a measure of ‘correction’ that needs to be incorporated in any hope to restore unitarity in the process of Hawking radiation.

**A. Nontriviality and Large Corrections**

The result obtained in \([11]\) and its subsequent generalization in \([16]\) do not engage in any argument that address large corrections. As a result, both results perform poorly asymptotically. We can establish the nontriviality of \((56)\) by looking at its limiting characteristics, at \(\epsilon \to 0\) and \(\epsilon \to 1\). In both cases, it is straightforward to calculate that both limits approach unity representing maximal increase of entropy, as expected from the model considered. Nontriviality may further be inferred from the maximum difference of the lower and upper bound in \((56)\), as we expect this difference to never exceed \(2 \log 2\).

Denoting this quantity as \(D_{\Delta S}\),

\[
D_{\Delta S} = \log 2 \left( \sqrt{1 - 4\epsilon_2^2} + \sqrt{4(\epsilon^2(1 - \epsilon^2) - \epsilon_2^2) - (1 - 4\epsilon_2^2)} \right) \quad (59)
\]

It is straightforward to note that for any value of \(\epsilon_2\), \(D_{\Delta S}\) is maximized when \(\epsilon^2(1 - \epsilon^2)\) is maximum i.e. \(\frac{1}{4}\). Then \((56)\) reduces to

\[
\frac{D_{\Delta S}}{\log 2} = 2 \sqrt{1 - 4\epsilon_2^2} - (1 - 4\epsilon_2^2) \quad (60)
\]

It is straightforward to show that maximization of \(D_{\Delta S}\) requires \(\frac{dD_{\Delta S}}{de_2} = 0 \Rightarrow \epsilon_2 = 0\) implying

\[
\frac{D_{\Delta S}}{\log 2} \leq 1 \quad (61)
\]

and hence establishing nontriviality of the bound as expected.

**B. Necessary Condition for Information Retrieval**

The result in \((56)\) facilitates us to derive the necessary condition for retrieving information from Hawking radiation. Retrieval of information requires monotonic decrease of
entanglement entropy after half of the black hole lifetime. This implies a necessarily negative lower bound in (56) implying

$$4\epsilon_2^2(1 - 4\epsilon_2^2) > 1 - 4\epsilon^2(1 - \epsilon^2)$$

(62)

after replacing $\gamma^2 = 1 - 4[\epsilon^2(1 - \epsilon^2) - \epsilon_2^2]$ and some manipulation.

RHS of (62) must be less than the maximum value of the function in LHS, i.e.

$$1 - 4\epsilon^2(1 - \epsilon^2) < \frac{1}{4}$$

$$\Rightarrow \frac{1}{2} < \epsilon < \frac{\sqrt{3}}{2}$$

(63)

This is the cherished necessary condition for retrieval of information from black hole radiation. $\epsilon$ is the parameter that introduces correction to the leading order state in terms of admixture of the state $S^{(2)}$ with the leading order state $S^{(1)}$. This result not only reinforces the arguments in [11] and [16]—small corrections fail to restore unitarity, but also provides a speculative bound for reducing entropy by allowing ‘somewhat large’ correction. However, this is only a necessary condition and any parametric bound that actually causes entanglement to decrease should be checked for sufficiency as well.

We can obtain a tighter bound on the other parameter $\epsilon_2$. From (62)

$$16\epsilon_2^4 - 4\epsilon_2^2 + 1 - 4\epsilon^2(1 - \epsilon^2) < 0.$$  

(64)

Solving (64) we get the desired bound for $\epsilon_2$ in terms of $\epsilon$:

$$1 - \sqrt{16\epsilon^2(1 - \epsilon^2) - 3} < 8\epsilon_2^2 < 1 + \sqrt{16\epsilon^2(1 - \epsilon^2) - 3}.$$  

(65)

Compared to the trivial bound on $\epsilon_2$ in terms of the Cauchy-Schwarz inequality i.e. $\epsilon_2 \leq \epsilon\sqrt{1 - \epsilon^2}$, (65) restricts the value of $\epsilon_2$ with a lower bound. Recall that $\epsilon_2$ represents the overlap between $\Lambda^{(1)}$ and $\Lambda^{(2)}$. The value of $\epsilon_2$ depends on the value of $\epsilon$. It is possible for $\epsilon_2$ to become zero for certain value of $\epsilon$, but for other allowed values of $\epsilon$ it must be a positive number. This implies that $\Lambda^{(1)}$ and $\Lambda^{(2)}$ are not necessarily orthogonal in all cases and the overlap between them cannot be arbitrarily small.
VII. MATHUR’S ARGUMENT AND THE FIREWALL

The firewall argument presented in [14] caused a major paradigm shift in the analysis of black hole information paradox. It has been argued that the well sought postulates of black hole complementarity as introduced in [15] contradict and the most conservative solution to this conundrum lies in abandoning the *innocuous* horizon for an infalling observer. The argument in [14] is actually an extended interpretation of the results in [11]. For the sake of completeness, we first review the firewall argument in the mathematical framework we have presented so far.

Innocuous horizon for an infalling observer requires that $S(b_{n+1} \cup c_{n+1}) = 0$ and $b_{n+1}$ is maximally entangled with $c_{n+1}$. On the other hand, the unitary evolution from the perspective of an asymptotic observer implies that late radiation should be entangled with early radiation and hence after halfway radiation $S_{b_{n+1}} < S_n \Rightarrow \Delta S < 0$. It has been proved in [11] and re-established in [14] by invoking the strong subadditivity relation that these requirements are at odds and innocuous horizon necessarily implies that within small corrections

$$\Delta S \approx \log 2$$  \hspace{1cm} (66)

We incorporate the results obtained so far to imply a stronger restatement of the argument. The arguments in [14] ignore corrections to the leading order analysis. Nearly maximal entanglement between the ingoing and outgoing quanta is a result of leading order analysis carried out by Hawking. We have incorporated anticipated corrections to the leading order state in terms of the correction parameters $\epsilon$ and $\epsilon^2$. Smallness of such corrections have been proven inadequate to overcome the inconsistencies. However, our work keeps open the window of arbitrary corrections. Still, vacuums are coordinate dependent artifacts in quantum field theory and a the number operator in a freely falling observer’s frame does not commute with that of a distant observer’s. So, even in a complete theory of quantum gravity these quanta are likely to be somewhat entangled. So instead of assuming maximally entangled pairs at the innocuous horizon, we assume somewhat entangled pairs to be generated.

Purity of the newly evolved pair is also a leading order phenomenon and in essence represents the same physical reality as does the so called *no hair theorem*. Inclusion of effects like back reaction, small quantum gravity effects, perturbations to the symmetric
geometry or modest nonlocal effects are likely to cause departure from absolute purity of the newly generated pair near the horizon. However, for large black holes the horizon is essentially a low curvature region and physics there should replicate the results predicted by LQFT with considerable accuracy. Hence, though we allow arbitrary corrections in terms of an admixture of two mutually orthogonal candidates for the new pair, \( S^{(1)} \) and \( S^{(2)} \), we want it to be in a ‘nearly pure’ state.

So, we characterize a broader class of innocuous horizons by the following assumptions—

A. Newly evolved pairs are generated in nearly pure states

B. The ingoing and outgoing quanta are somewhat entangled

Now we show that even under these relaxed assumptions, entanglement entropy fails to decrease. Let us note that the outgoing Hawking mode is invariably entangled with its ingoing partner irrespective of the magnitude of the correction parameter \( \epsilon_2 \). We can readily see it by noting the trace of \( \hat{\rho}^2_{n+1} \)

\[
\text{tr} \left( \hat{\rho}^2_{n+1} \right) = \frac{1}{2} \left( 1 + 4\epsilon_2^2 \right)
\]

which is less than unity unless \( \epsilon_2 \neq \frac{1}{2} \). Now, purity of the newly evolved pair is marked by the equality in Cauchy-Schwartz inequality. We assume that pairs are generated in nearly pure states, i.e., \( \epsilon^2 (1 - \epsilon^2) - \epsilon_2^2 = \delta \sim 0 \). Putting this result in (56) gives the following bound

\[
1 - 4\epsilon_2^2 - 2\sqrt{\delta} \leq \frac{\Delta S}{\log 2} \leq \sqrt{1 - 4\epsilon_2^2}
\]

This reveals that \( \Delta S \) is necessarily positive for typical values of \( \epsilon_2 \). Hence, we reach the same conclusion as in [11] and [14] but with a more relaxed set of assumptions. This result also follows from the bound in (65), which allows a more restrictive set of values for \( \epsilon_2 \) than allowed by the Cauchy-Schwartz inequality.

**VIII. CONCLUSION**

The failure of ‘small’ corrections to resolve the black hole information paradox is reinforced in this paper with the generalization of Mathur’s bound and this result is also demonstrated in the case of a generalized toy model of black hole evaporation. Information cannot leak out of a black hole unless we allow ‘not-so-small’ to somewhat ‘large’ correction to the Hawking state. One of the major outcomes of this paper is to precisely
quantify the amount of correction that would allow information retrieval from a black hole. The parameters used in this analysis rigorously bound the change in entanglement entropy that would result if such correction to leading order Hawking analysis is adopted in any evaporation model. It remains open to debate what implication a ‘not-so-small’ correction would have on the other assumptions of the theory, but this generalized bound would help to quantify any change in entanglement entropy that evaporation models deviating from leading order Hawking analysis would bring. Besides, this generalization also reinforces the finding that information leakage from a black hole with ‘innocuous’ horizon is not possible. With an even ‘relaxed’ definition of such horizon adopted by the present authors, it is showed that entanglement entropy fails to decrease by any significant amount.

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