Enumeration for spanning trees and forests of join graphs based on the combinatorial decomposition

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Abstract
This paper discusses the enumeration for rooted spanning trees and forests of the labelled join graphs $K_m + H_n$ and $K_m + K_{n,p}$, where $H_n$ is a graph with $n$ isolated vertices.

Keywords: spanning tree, spanning forest, join graph, enumeration
Mathematics Subject Classification: 05C05 05C30
DOI:10.5614/ejgta.2016.4.2.5

1. Introduction

In this paper we consider the enumeration problem of rooted spanning trees and forests of two labelled join graphs. In [2], the number of spanning forests of the labelled complete bipartite graph $K_{m,n}$ on $m$ and $n$ vertices has been enumerated by combinatorial method. In [1] and [3], it has been given the enumeration of spanning trees of the complete tripartite graph $K_{m,n,p}$ on $m$, $n$ and $p$ vertices and the complete multipartite graph, respectively. In [4], by using the multivariate Lagrange inverse, the number of spanning forests of the labelled complete multipartite graph was derived. And, in [5], it has been found the asymptotic number of labeled spanning forests of the complete bipartite graph $K_{m,n}$ as $m \to \infty$ when $m \leq n$ and $n = o(m^{6/5})$.

Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with disjoint vertex sets, we let $G_1 + G_2$ denote the join of $G_1$ and $G_2$, that is, the graph $G_1 + G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup E(V_1, V_2))$ where $E(V_1, V_2) = \{(i, j) | i \in V_1, j \in V_2\}$, $(i, j)$ denotes an edge between two vertices $i \in V_1, j \in V_2$. 

Received: 21 April 2014, Revised: 11 July 2015, Accepted: 2 August 2016.
Clearly, by the definition of a join graph, the complete bipartite graph $K_{m,n}$ is a join graph $H_{m} + H_{n}$ and the complete tripartite graph $K_{m,n,p}$ is a join graph $H_{m} + H_{n} + H_{p}$, where $H_{m}$, $H_{n}$ and $H_{p}$ are graphs with $m$ isolated vertices, $n$ isolated vertices and $p$ isolated vertices, respectively.

The goal of this paper first is to give a combinatorial proof of the enumeration for the spanning trees and forests of a labelled join graph $K_{m} + H_{n}$, where $K_{m}$ is the complete graph on $m$ vertices and $H_{n}$ is the graph with $n$ isolated vertices. Second, this paper also gives a combinatorial proof of the enumeration for the spanning trees and all forests of another labelled join graph $K_{m} + K_{n,p}$, where $K_{n,p}$ is the complete bipartite graph on $n$ vertices and $p$ vertices.

2. Enumeration for spanning trees and forests of a join graph $K_{m} + H_{n}$

Let $V(G)$ denote the vertex set of graph $G$. Throughout this paper, we will consider only the labelled graphs. In this section, we consider a join graph $K_{m} + H_{n}$ where $K_{m}$ is the complete graph on the vertex set $\{x_{1}, x_{2}, \cdots, x_{m}\}$.

**Lemma 2.1.** The number $f(m, l)$ of the labelled spanning forests of $K_{m}$ with $l$ roots is

$$f(m, l) = \left(\begin{array}{c} m \\ l \end{array}\right)lm^{m-l-1}. \tag{1}$$

**Proof** Let $X = V(K_{m}) = \{x_{1}, x_{2}, \cdots, x_{m}\}$ be the vertex set of $K_{m}$ and $\{x_{i_1}, x_{i_2}, \cdots, x_{i_l}\}$ be the given root set of $K_{m}$. There are $\left(\begin{array}{c} m \\ l \end{array}\right)$ ways to choose the $l$ roots in $V(K_{m})$. Also, let $X' = X \setminus \{x_{i_1}, x_{i_2}, \cdots, x_{i_l}\}$ be a subset of $X$, and $X''$ be another copy of $X'$ and let $x'' \in X''$ denote copy of $x' \in X'$. Take the complete bipartite graph $K_{m,m-l}$ with the partition $(X, X'')$ of its vertex set. Consider the subgraph $G$ of $K_{m,m-l}$ that contains only the directed edges of the form $(x', x'')$, $x' \in X'$, $x'' \in X''$. The number of the components of $G$ is equal to $m-l$ and $G$ is a forest of $K_{m-l,m-l} = (X', X'')$. Let $D(m, |\{x_{i_1}, x_{i_2}, \cdots, x_{i_l}\}|; m-l, 0)$ be the set of the labelled spanning forests of $K_{m,m-l} = (X, X'')$ with $l$ roots $x_{i_1}, x_{i_2}, \cdots, x_{i_l} \in X$ and $D^{*}(K_{m}; x_{i_1}, x_{i_2}, \cdots, x_{i_l})$ be the set of the labelled spanning forests of $K_{m}$ with $l$ roots $x_{i_1}, x_{i_2}, \cdots, x_{i_l} \in X$. Now any spanning forest in $D(m, |\{x_{i_1}, x_{i_2}, \cdots, x_{i_l}\}|; m-l, 0)$ containing $G$ gives rise to a spanning forest in $D^{*}(K_{m}; x_{i_1}, x_{i_2}, \cdots, x_{i_l})$ by contracting the edges $(x', x'')$, $x' \in X'$, $x'' \in X''$.

Conversely, any forest in $D^{*}(K_{m}; x_{i_1}, x_{i_2}, \cdots, x_{i_l})$ can be extended to a forest in $D(m, |\{x_{i_1}, x_{i_2}, \cdots, x_{i_l}\}|; m-l, 0)$ containing $G$ by inserting vertex $x'' \in X''$ after $x' \in X'$. Therefore, from $G$, we will construct the rooted spanning forests of $K_{m,m-l}$ with $l$ roots in $X$ as follows.

For any fixed integer $t \in [0, m-l-1]$, add $t$ edges consecutively to $G$ as follows. At each step we add an edge of the form $(v, x')$ between $x' \in X'$ and a (unique)vertex $v \in X''$ of out-degree zero in any component not containing $x'$ in the graph already constructed. The number of components decreases by one each time such an edge is added.

Since $|X'| = m-l$ and the number of components not containing $x'$ in the graph $G$ is $m-l-1$, there are $(m-l)(m-l-1)$ choices for the first such edge. Similarly, there are $(m-l)(m-l-2)$ choices for the second edge, etc., and $(m-l)(m-l-t)$ choices for the $t$th edge.

The order in which the $t$ edges are added to $G$ is immaterial, so it follows that there are

$$\frac{(m-l)(m-l-1)![(m-l)(m-l-2)]!\cdots[(m-l)(m-l-t)]!}{t!} = \binom{m-l-1}{t}(m-l)^t$$

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Every graph we obtained will have \( m - l - t \) (weakly) connected components each of which has a unique vertex in \( X'' \) of out-degree zero. Link edges from \( m - l - t \) vertices of out-degree zero in these components to \( l \) given roots \( x_{i_1}, x_{i_2}, \ldots, x_{i_l} \), there are \( l^{m-l-t} \) ways. Hence,

\[
f(m, l) = \binom{m}{l} \sum_{t=0}^{m-l-1} \binom{m-l-1}{t} l^{m-l-t}(m-l)^t = \binom{m}{l} l m^{m-l-1}. \]

Let \( D(m, l) \) be the set of the labelled spanning forests of \( K_m \) with \( l \) roots, i.e.,

\[
f(m, l) = |D(m, l)|. \tag{2}\]

**Theorem 2.1.** The number \( g(m, n) \) of the labelled spanning trees of \( K_m + H_n \) is

\[
g(m, n) = m^{n-1}(m+n)^{m-1}. \tag{3}\]

**Proof.** Let \( V(K_m) = \{x_1, x_2, \ldots, x_m\} \), \( V(H_n) = \{y_1, y_2, \ldots, y_n\} \) be the vertex sets of \( K_m, H_n \), respectively, and \( y_1 \in V(H_n) \) be the given root of \( K_m + H_n \). Let \( D(m, 0; n, \{y_1\}) \) be the set of the labelled spanning trees of \( K_m + H_n \) with root \( y_1 \) and \( T(m, n) \) be the set of the labelled spanning trees of \( K_m + H_n \). Clearly, \( |T(m, n)| = |D(m, 0; n, \{y_1\})| \).

From every graph \( F \in D(m, l) \), we will construct the rooted spanning trees of \( K_m + H_n \) as follows. Link an edge \((y, x)\) between every \( y \in V(H_n) \setminus \{y_1\} \) and some \( x \in V(F) \). There are \( m^{n-1} \) ways. Notice that the obtained graph \( G \) has \( l \) (weakly) connected components each of which has a unique vertex in \( V(K_m) \) of out-degree zero.

Now, for any fixed integer \( t \), let \( G' \) denote a graph obtained by adding \( t \) edges consecutively to \( G \) as follows. At each step we add an edge of the form \((x, y)\) where \( y \) is any vertex of \( y \in V(H_n) \setminus \{y_1\} \) and \( x \in V(K_m) \) is a vertex of out-degree zero in any component not containing \( y \) in the graph already constructed. The number of components decreases by one each time such an edge is added.

Since \( |V(H_n) \setminus \{y_1\}| = n - 1 \) and the number of components not containing \( y \) in the graph \( G \) already constructed is \( l - 1 \), there are \((n - 1)(l - 1)\) choices for the first such edge. Similarly, there are \((n - 1)(l - 2)\) choices for the second edge, \( \ldots \), and \((n - 1)(l - t)\) choices for the \( t \)th edge, where, \( 0 \leq t \leq l - 1 \), because the number of components in the graph \( G \) is \( l \). The graph \( G' \) thus constructed has \( l - t \) components each of which has a unique vertex in \( V(K_m) \) of out-degree zero and the remaining vertices all have out-degree one; if we add edges from these vertices of out-degree zero to \( y_1 \), we obtain a tree \( T' \in D(m, 0; n, \{y_1\}) \) that contains \( G \) and in which the in-degree of \( y_1 \) equals to \( l - t \). The order in which the \( t \) edges are added to \( G \) to form \( G' \) is immaterial, so it follows that there are

\[
\frac{[(n - 1)(l - 1)][(n - 1)(l - 2)] \cdots [(n - 1)(l - t)]}{t!} = \binom{l - 1}{t} (n - 1)^t
\]

rooted spanning trees \( T' \) for fixed integer \( t \). This implies that there are

\[
\sum_{t=0}^{l-1} \binom{l - 1}{t} (n - 1)^t = n^{l-1}
\]
spanning trees $T$ in $D(m, 0; n, \{|y_i\}|)$ that contain $G$. Hence, by (2) and Lemma 2.1, we have

$$g(m, n) = |D(m, 0; n, \{|y_i\}|)| = \sum_{l=1}^{m} |D(m, l)| n^{l-1}m^{n-1}$$

$$= \sum_{l=1}^{m} \binom{m}{l} ln^{m-l-1}l^{n-1}m^{n-1} = m^{n-1}(m+n)^{n-1}$$
as desired. $\square$

**Theorem 2.2.** The number $g(m, l; n, k)$ of the labelled spanning forests of $K_m + H_n$ with $l$ roots in $K_m$ and $k$ roots in $H_n$ is

$$g(m, l; n, k) = \binom{m}{l} \binom{n}{k} m^{n-k-1}(m+n)^{m-l-1}(ln+mk+ln-kl).$$

**Proof** Let $V(H_n) = \{y_1, y_2, \cdots, y_n\}$ be the vertex set of $H_n$ and $\{y_1, y_2, \cdots, y_k\}$ be the given root set of $H_n$. There are $\binom{n}{k}$ ways to choose the $k$ roots in $V(H_n)$. Let $V(K_m) = \{x_1, x_2, \cdots, x_m\}$ be the vertex set of $K_m$ and $V' = V(H_n) \setminus \{y_1, y_2, \cdots, y_k\}$ be a subset of $V(H_n)$.

From every graph $F \in D(m, s)(s \geq l)$, we will construct the rooted spanning forests of $K_m + H_n$ with $l$ roots in $K_m$ and $k$ roots in $H_n$ as follows. Link an edge $(y, v)$ between every $y \in V'$ and some $v \in V(F)$. There are $m^{n-k}$ ways. Notice that the obtained graph $G$ has $s$ (weakly) connected components each of which has a unique vertex in $V(K_m)$ of out-degree zero and the remaining vertices all have out-degree one.

As in the proof of former theorem, link an edge $(v, y)$ between $y \in V'$ and a vertex $v \in V(K_m)$ of out-degree zero in any component not containing $y$ in the graph already constructed, we repeat this procedure $i$ times, where, $0 \leq i \leq s-l$, because the required forests have $l$ roots in $V(K_m)$.

There are

$$\frac{(n-k)(s-1)[(n-k)(s-2)]\cdots[(n-k)(s-1)]}{i!} = \binom{s-1}{i}(n-k)^i$$

ways.

Every graph $G'$ we obtained will have $s-i$ components each of which has a unique vertex in $V(K_m)$ of out-degree zero. Now, choose the $s-i-l$ vertices of out-degree zero in these $s-i$ components and link edges from these $s-i-l$ vertices to $k$ roots $y_{i1}, y_{i2}, \cdots, y_{ik}$. There are

$$\binom{s-i}{s-i-l} = \binom{s-i}{l}$$

ways.

Therefore, by (5) and (6), the number of the rooted spanning forests of $K_m + H_n$ which are obtained from $F$ is equal to

$$\sum_{i=0}^{s-l} \binom{s-1}{i} \binom{s-i}{l} (n-k)^i k^{s-i-l} = \binom{s}{l} n^{s-l} - \binom{s}{l} \frac{s-l}{s} n^{s-l-1}(n-k).$$
Hence, by (2), (7) and Lemma 2.1, the number \( g(m, l; n, k) \) of the labelled spanning forests of \( K_m + H_n \) with \( l \) roots in \( K_m \) and \( k \) roots in \( H_n \) is as follows.

\[
g(m, l; n, k) = \binom{n}{k} \sum_{s=0}^{m} |D(m, s)| m^{n-k} \sum_{i=0}^{s-k} \binom{s-i}{l} (n-k)^i k^{s-i-l}
\]

\[
= \binom{n}{k} \sum_{s=0}^{m} \binom{m}{s} s m^{m-s-1} m^{n-k} \left[ \binom{s}{l} n^{s-l} - \frac{s}{s} \right] n^{s-l-1} (n-k)
\]

\[
= \binom{m}{l} \binom{n}{k} m^{n-k-1} (m+n)^{m-1} (lm+mk+ln-kl).
\]

We get the required result. \( \Box \)

**Corollary 2.1.** The number \( S(m, n) \) of all spanning forests of the join graph \( K_m + H_n \) is equal to

\[
S(m, n) = (m+n+1)^m (m+1)^{n-1}.
\]  

**Proof** By Theorem 2.2,

\[
S(m, n) = \sum_{l=0}^{m} \sum_{k=0}^{n} g(m, l; n, k)
\]

\[
= \sum_{l=0}^{m} \sum_{k=0}^{n} \binom{m}{l} \binom{n}{k} m^{n-k-1} (m+n)^{m-1} (lm+mk+ln-kl)
\]

\[
= (m+n+1)^m (m+1)^{n-1}.
\]

Thus, this corollary is true. \( \Box \)

**3. Enumeration for spanning trees and forests of a join graph** \( K_m + K_{n, p} \)

In this section, we consider another join graph \( K_m + K_{n, p} \), where \( K_m \) is the complete graph and \( K_{n, p} \) is the complete bipartite graph. We will show how to count the number of the spanning trees of a join graph \( K_m + K_{n, p} \). Clearly, \( K_m + K_{n, p} = (K_m + H_n) + H_p \). Let \( D(m, l; n, k) \) be the set of the labelled spanning forests of \( K_m + H_n \) with \( l \) roots in \( K_m \) and \( k \) roots in \( H_n \), i.e.,

\[
g(m, l; n, k) = |D(m, l; n, k)|.
\]  

**Theorem 3.1.** The number \( g(m, n, p) \) of the spanning trees of \( K_m + K_{n, p} \) is equal to

\[
g(m, n, p) = (m+n)^{p-1} (m+p)^{n-1} (m+n+p)^m.
\]  

**Proof** Let \( V(K_m + H_n) = \{x_1, x_2, \cdots, x_m; y_1, y_2, \cdots, y_n\} \) be the vertex set of \( K_m + H_n \) and \( V(H_p) = \{z_1, z_2, \cdots, z_p\} \) be the vertex set of \( H_p \). Let \( z_1 \in V(H_p) \) be the given roots of \( K_m + K_{n, p} \) and \( Z' = V(H_p) \setminus \{z_1\} \), \( D(m, 0; n, 0; p, |\{z_1\}|) \) be the set of the labelled spanning trees of \( K_m + K_{n, p} \) with root \( z_1 \). Clearly,

\[
g(m, n, p) = |D(m, 0; n, 0; p, |\{z_1\}|)|.
\]
We shall obtain the spanning trees in \( D(m, 0; n, 0; p, |\{z_1\}|) \) from every graph \( F \in D(m, l; n, k) \). As in the proof of former theorem, link an edge \((z, v)\) between every \( z \in Z'\) and some \( v \in V(F) \). There are \((m + n)^{p-1}\) ways. Notice that the obtained graph \( G \) has \( l + k \) (weakly) connected components each of which has a unique vertex in \( V(K_m) \cup V(H_n)\) of out-degree zero and the remaining vertices all have out-degree one.

For any fixed integer \( t \) such that \( 0 \leq t \leq l + k - 1 \), link an edge \((v, z)\) between \( z \in Z'\) and a vertex \( v \in V(K_m) \cup V(H_n)\) of out-degree zero in any component not containing \( z \) in the graph already constructed, we repeat this procedure \( t \) times.

There are
\[
\frac{(p - 1)(l + k - 1)!}{t!} \cdots \frac{(p - 1)(l + k - t)!}{t!} = \binom{l + k - 1}{t} (p - 1)^t
\]
ways. Therefore, the number of the spanning trees which are obtained from \( F \) is equal to
\[
\sum_{t=0}^{l+k-1} \binom{l + k - 1}{t} (p - 1)^t = p^{l+k-1}.
\]

Hence, by (9) and Theorem 2.2,
\[
g(m, n, p) = \left| D(m, 0; n, 0; p, |\{z_1\}|) \right| \\
= \sum_{l=0}^{m} \sum_{k=0}^{n} |D(m, l; n, k)| p^{l+k-1} (m + n)^{p-1} \\
= \sum_{l=0}^{m} \sum_{k=0}^{n} \binom{m}{l} \binom{n}{k} m^{n-k-1} (m + n)^{m-l-1} (l m + k m + l n - l k) p^{k+t-1} (m + n)^{p-1} \\
= (m + n)^{p-1} (m + p)^{n-1} (m + n + p)^{m}.
\]

Therefore, we get the required result. \( \square \)

**Theorem 3.2.** The number \( S(m, n, p) \) of all spanning forests of the join graph \( K_m + K_{n,p} \) is equal to
\[
S(m, n, p) = (m + n + p + 1)^{m+1} (m + n + 1)^{p-1} (m + p + 1)^{n-1}.
\]

**Proof** Let \( B(p, r) \) denote the set of spanning forests of the join graph \( K_m + K_{n,p} = (K_m + H_n) + H_p \) which \( r \) roots are in \( V(H_p) \) and remaining roots are in \( V(K_m) \) or \( V(H_n) \).

From every graph \( F \in D(m, l; n, k) \), we will construct the rooted spanning forests of \((K_m + H_n) + H_p\) with \( r \) roots in \( V(H_p) \) as follows. Let \( z_{i_1}, z_{i_2}, \ldots, z_{i_r} \in V(H_p) \) be root vertices. The number of ways to select \( r \) roots in \( V(H_p) \) is equal to \( \binom{p}{r} \). Let \( Z' = V(H_p) \setminus \{z_{i_1}, z_{i_2}, \ldots, z_{i_r}\} \). Link an edge \((z, v)\) between every \( v \in Z'\) and some \( v \in V(F) \). There are \((m + n)^{p-r}\) ways. Notice that the obtained graph \( G \) has \( l + k \) (weakly) connected components each of which has a unique vertex in \( V(K_m) \cup V(H_n)\) of out-degree zero and the remaining vertices all have out-degree one.

As in the proof of former theorem, for any fixed integer \( t \) such that \( 0 \leq t \leq l + k - 1 \), link an edge \((v, z)\) between \( z \in Z'\) and a vertex \( v \in V(K_m) \cup V(H_n)\) of out-degree zero in any component
not containing $z$ in the graph already constructed, we repeat this procedure $t$ times. There are
\[
\frac{[(p - r)(l + k - 1)][(p - r)(l + k - 2)] \cdots [(p - r)(l + k - t)]}{t!} = \binom{l + k - 1}{t} (p - r)^t
\]
ways.

The graph $G'$ thus constructed has $l + k - t$ components each of which has a unique vertex in $V(K_m) \cup V(H_n)$ of out-degree zero and the remaining vertices all have out-degree one; if we add edges from some vertices of these vertices of out-degree zero to $z_{i_1}, z_{i_2}, \ldots, z_{i_r} \in Z$, we obtain a forest in $B(p, r)$ that contains $G$. There are $(r + 1)^{l+k-t}$ ways. Therefore, this implies that there are
\[
\sum_{t=0}^{l+k-1} \binom{l + k - 1}{t} (p - r)^t (r + 1)^{l+k-t} = (r + 1)(p + 1)^{l+k-1}
\]
forests in $B(p, r)$ that contain $G$. Hence, by (9) and Theorem 2.2,
\[
S(m, n, p) = \sum_{l=0}^{m} \sum_{k=0}^{n} |D(m, l; n, k)| \sum_{r=0}^{p} \binom{p}{r} (m + n)^{p-r}(r + 1)(p + 1)^{l+k-1}
\]
\[
= \sum_{l=0}^{m} \sum_{k=0}^{n} \binom{m}{l} \binom{n}{k} m^{n-k-1}(m + n)^{m-l-1}(lm + mk + ln - lk)
\]
\[
\sum_{r=0}^{p} \binom{p}{r} (m + n)^{p-r}(r + 1)(p + 1)^{l+k-1}
\]
\[
= (m + n + p + 1)^{m+1}(m + n + 1)^{p-1}(m + p + 1)^{n-1}.
\]
Thus, this theorem is true. □

Acknowledgement

The author thanks the referees for very careful readings and many constructive comments that greatly improve the presentation of this paper.

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