ON M-SECTORIAL EXTENSIONS OF SECTORIAL OPERATORS

YU. M. ARLINSKIĬ AND A. B. POPOV

Abstract. In our article [15] description in terms of abstract boundary conditions of all \( m \)-accretive extensions and their resolvents of a closed densely defined sectorial operator \( S \) have been obtained. In particular, if \( \{ \mathcal{H}, \Gamma \} \) is a boundary pair of \( S \), then there is a bijective correspondence between all \( m \)-accretive extensions \( \tilde{S} \) of \( S \) and all pairs \( (Z, X) \), where \( Z \) is a \( m \)-accretive linear relation in \( \mathcal{H} \) and \( X : \operatorname{dom}(Z) \rightarrow \operatorname{ran}(S_F) \) is a linear operator such that:

\[
\|Xe\|^2 \leq \Re (Z(e), e)_{\mathcal{H}} \quad \forall e \in \operatorname{dom}(Z).
\]

As is well known the operator \( S \) admits at least one \( m \)-sectorial extension, the Friedrichs extension. In this paper, assuming that \( S \) has non-unique \( m \)-sectorial extension, we established additional conditions on a pair \( (Z, X) \) guaranteeing that corresponding \( \tilde{S} \) is \( m \)-sectorial extension of \( S \). As an application, all \( m \)-sectorial extensions of a nonnegative symmetric operator in a planar model of two point interactions are described.


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INTRODUCTION

Let $\mathcal{H}$ be a complex Hilbert space with the inner product $(\cdot, \cdot)$. We use the symbols $\text{dom}(T)$, $\text{ran}(T)$, $\ker(T)$ for the domain, the range, and the null-subspace of a linear operator $T$. The resolvent set of a linear operator $T$ is denoted by $\rho(T)$. The linear space of bounded operators acting between Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ is denoted by $L(H_1, H_2)$ and the Banach algebra $\mathcal{L}(\mathcal{H})$ by $\mathcal{L}(\mathcal{H})$. A linear operator $T$ in a complex Hilbert space $\mathcal{H}$ is called accretive if its numerical range $W(T) \overset{\text{def}}{=} \{(Tu, u) : u \in \text{dom}(T), \|u\| = 1\}$ is contained in the closed right half-plane, i.e.,

$$\text{Re} (Tu, u) \geq 0$$

for all $u \in \text{dom}(T)$.

An accretive operator $T$ is called maximal accretive or $m$-accretive, if one of the following equivalent conditions holds [28, 37, 38]:

1) $T$ is closed and has no accretive extensions in $\mathcal{H}$;
2) resolvent set $\rho(T)$ contains a point from an open left half-plane;
3) $T$ is a closed densely defined operator and its adjoint $T^*$ is an accretive operator;
4) the operator $-T$ generates one-parameter contractive semigroup $U(t) = \exp(-tT)$, $t \geq 0$.

One can prove the following equality for an arbitrary $m$-accretive operator $T$:

$$\ker(T) = \ker(T^*). \quad (0.1)$$

Let $\alpha \in [0, \pi/2)$. Denote by the $\Theta(\alpha)$ the sector in the complex plane

$$\Theta(\alpha) \overset{\text{def}}{=} \{ z \in \mathbb{C} : |\arg z| \leq \alpha \} .$$

A linear operator $S$ is called sectorial with the vertex at the origin and the semi-angle $\alpha$ [28] if $W(S) \subseteq \Theta(\alpha)$. Clearly, $S$ is sectorial if and only if:

$$|\text{Im} (Sx, x)| \leq \tan \alpha \text{Re} (Sx, x),$$

for all $x \in \text{dom}(S)$. In particular, if $\alpha = 0$, then $(Sx, x) \geq 0$ for all $x \in \text{dom}(S)$, i.e., $S$ is symmetric and nonnegative operator. In the sequel we will use the word “sectorial” only for sectorial operators and sectorial sesquilinear forms with vertex at the origin. In addition, if semi-angle of sectorial operator $S$ is $\alpha$ we will call $S$ $\alpha$-sectorial operator. A linear operator $S$ is called $m$-sectorial, if it is sectorial and $m$-accretive. If $T$ is $m$-sectorial operator and if $\gamma \in (\alpha, \pi/2)$ then

$$\lambda \in \mathbb{C} \setminus \Theta(\gamma) \Rightarrow ||(T - \lambda I)^{-1}|| \leq \frac{1}{|\lambda| \sin (\gamma - \alpha)}, \quad (0.2)$$

and the one-parameter semigroup $U(t) = \exp(-tT)$, $t \geq 0$, admits a holomorphic contractive continuation into the interior of the sector $\Theta(\pi/2 - \alpha)$ [28].

It is well-known that there is a one-to-one correspondence between closed densely defined sectorial forms and $m$-sectorial operators. This correspondence is given by the First and the Second Representations Theorems [28]. We will denote by $T[u, v]$ the closed form associated with $m$-sectorial extension $T$ and by $\mathcal{D}(T)$ its domain.
In the present paper we continue to study $m$-accretive extensions of a densely defined closed sectorial operator $S$. It is well known [28], that $S$ admits at least one $m$-sectorial extension $S_F$, the Friedrichs extension, which is associated with the closure of sesquilinear form $(Sf,g)$, $f,g \in \text{dom}(S)$. In [7, 8, 9, 10, 11, 13], the boundary triplets methods have been applied for a description of all $m$-accretive, $m$-sectorial extensions, and their resolvents for sectorial operators $S$ satisfying condition

$$\text{dom}(S^*) \subseteq D[S_N],$$

where $S_N$ is “extremal” $m$-sectorial extension of $S$, called the Kreĭn-von Neumann extension [7, 8]. Such extension is an analog of the “soft” (“the Kreĭn”, “the Kreĭn-von Neumann”) nonnegative selfadjoint extension of a nonnegative symmetric operator, discovered by M.G. Kreĭn in [30, 31]. Recall that $S$ is called nonnegative if $(Sf,f) \geq 0$ for all $f \in \text{dom}(S)$. Observe that condition (0.3) holds true if for sectorial $S$ the equality $\text{dom}(S^*_F) + \text{dom}(S^*_N) = \text{dom}(S^*)$ is satisfied. The latter occurs, for instance, if $S$ is coercive, i.e., $\text{Re} (Sf,f) \geq m||f||^2$ for all $f \in \text{dom}(S)$, where $m > 0$.

In our recent paper [15] in the general case of an arbitrary closed densely defined sectorial operator $S$ we propose a new approach for the problem of parametrization of all $m$-accretive extensions. Our methods is applicable, in particular, for sectorial operator $S$, having a unique $m$-sectorial extension ($S_F = S_N$). In this paper, assuming $S_F \neq S_N$, we apply our method for a description of all $m$-sectorial extensions.

Let $A$ be a densely defined closed symmetric operator in $\mathcal{H}$. Extensions $\tilde{A}$ of $A$ possessing property

$$A \subset \tilde{A} \subset A^*$$

are called quasi-selfadjoint (proper, intermediate) extensions of $S$. The problem of existence and description of all quasi-selfadjoint $m$-accretive extensions of a nonnegative symmetric operator via linear-fractional transformation has been solved in [16] and via abstract boundary conditions in [35, 29, 5, 23, 22]. We refer on this matter to the survey [18] where one can find information about various approaches to the extension problem of nonnegative symmetric operators. Notice that in [14], developing the method proposed in [17], an intrinsic parametrization of domains of all $m$-accretive and $m$-sectorial quasi-selfadjoint extensions of nonnegative $A$ have been obtained.

In the present paper we use the approach of [15] for such extensions. Applications to nonnegative symmetric operator in a planar model of two-centers point interactions are given. In one-center point interaction planar model the corresponding nonnegative symmetric operator admits a unique nonnegative selfadjoint extension [11], [24], hence, the Friedrichs extension is unique among all quasi-selfadjoint $m$-accretive extensions [11] and all $m$-sectorial extensions [7]. In our paper [15] we described all $m$-accretive extensions for this case. In the case of two and more centers, the Friedrichs extension is a non-unique element of the set of all nonnegative selfadjoint extensions, therefore, there are non-selfadjoint $m$-accretive quasi-selfadjoint extensions and $m$-sectorial extensions.
1. Preliminaries

1.1. Sectorial forms and operators. The basic definitions and results concerning sesquilinear forms can be found in [28]. If \( \tau \) is a closed densely defined sectorial form in the Hilbert space \( \mathcal{H} \), then by the First Representation Theorem [30] [28], there exists a unique \( m \)-sectorial operator \( T \) in \( \mathcal{H} \) associated with \( \tau \) in the following sense: \( (Tu,v) = \tau[u,v] \), for all \( u \in \text{dom}(T) \) and for all \( v \in \text{dom}(\tau) \). The adjoint operator \( T^* \) is associated with the adjoint form \( \tau^*[u,v] := \overline{\tau[v,u]} \). The nonnegative selfadjoint operator \( T_R \) associated with the real part \( \tau_R[u,v] := (\tau[u,v] + \tau^*[u,v]) / 2 \) of the form \( \tau \) and is called the real part of \( T \). According to the Second Representation Theorem [28] the equality \( \text{dom}(\tau) = \text{dom}(T^{1/2}) \) holds. Moreover,

\[
\tau[u,v] = ((I+iG)T^2_R u, T^2_R v), \quad u, v \in \text{dom}(\tau),
\]

where \( G \) is a bounded selfadjoint operator in the subspace \( \text{ran}(T_R) \) and \( ||G|| \leq \tan \alpha \) iff \( \tau \) is \( \alpha \)-sectorial. It follows that

\[
\text{dom}(T) = \{ u \in \text{dom}(\tau) : (I+iG)T^{1/2}_R u \in \text{dom}(\tau) \},
\]

\[
Tu = T^{1/2}_R (I+iG)T^{1/2}_R u, \quad u \in \text{dom}(T).
\]

In the sequel we will use the following notations for a \( m \)-sectorial operator \( T \):

\[
\text{D}[T] \overset{\text{def}}{=} \text{dom}(T^{1/2}), \quad \text{R}[T] \overset{\text{def}}{=} \text{ran}(T^{1/2}).
\]

Also, for a \( m \)-sectorial operator \( T \) we denote by

\[
\hat{T} = T|\text{ran}(T), \quad \hat{T}_R = T_R|\text{ran}(T).
\]

Equality (0.1) yields that \( \ker(\hat{T}) = \ker(\hat{T}_R) = \{0\} \). From (1.1) it follows for \( \lambda = -a + ib, a, b \in \mathbb{R}, a > 0 \) (see [27] [8])

\[
(T - \lambda I)^{-1} = (T_R + aI)^{-1/2} (I + iG(\lambda))^{-1} (T + aI)^{-1/2};
\]

\[
G(\lambda) = T^{1/2}_R (T_R + aI)^{-1/2} GT^{1/2}_R (T + aI)^{-1/2} - b(T_R + aI)^{-1}.
\]

The latter equalities imply the following statement.

**Proposition 1.1** ([8]). If \( T = T^{1/2}_R (I+iG)T^{1/2}_R \) is an \( m \)-\( \alpha \)-sectorial operator in the Hilbert space \( \mathcal{H} \), and \( \gamma \in (\alpha, \pi/2) \), then

\[
\text{R}[T] = \left\{ f \in \mathcal{H} : \sup_{x \in \text{dom}(T)} \frac{|(f,x)|^2}{\text{Re}(Tx,x)} < \infty \right\} =
\]

\[
= \left\{ f \in \mathcal{H} : \lim_{\lambda \to \alpha} \left| \left( (T - \lambda I)^{-1} f, f \right) \right| < \infty \right\} ; \quad (1.2)
\]

\[
\lim_{\lambda \to \alpha} \left( (T - \lambda I)^{-1} f, g \right) = \hat{T}^{-1}[f,g] = ((I+iG)^{-1}\hat{T}_R^{-1/2} f, \hat{T}_R^{-1/2} g), \quad f, g \in \text{R}[T]; \quad (1.3)
\]
\[ \lim_{\lambda \to 0} T^{1/2}_R (T - \lambda I)^{-1} T^{1/2}_R g = (I + iG)^{-1} g; \quad g \in D[T] \ominus \ker(T). \] (1.4)

1.2. The Friedrichs and Kreĭn-von Neumann m-sectorial extensions. Let \( S \) be an \( \alpha \)-sectorial operator. It is well known [28], that the form \( (Su, v) \), \( u, v \in \operatorname{dom}(S) \) is closable. We will denote by \( S[u, v] \) its closure. The domain of the form \( S[u, v] \) is denoted by \( D[S] \). With the closed form \( S[u, v] \) is associated the maximal \( \alpha \)-sectorial operator \( S_F \), which is called the Friedrichs extension of \( S \) [28]. So \( D[S] = D[S_F] \) and \( S_F[u, v] = S[u, v] \) for all \( u, v \in D[S] \). Let \( S_{FR} \) be the real part of \( S_F \). Clearly, \( D[S] = \operatorname{dom}(S^{1/2}_{FR}) \). We will use the representations

\[
S[u, v] = S_F[u, v] = ((I + iG_F)S^{1/2}_{FR}u, S^{1/2}_{FR}v), \quad u, v \in D[S] = \operatorname{dom}(S^{1/2}_{FR}),
\]

\[
S_F f = S^{1/2}_{FR}(I + iG_F)S^{1/2}_{FR} f, \quad f \in \operatorname{dom}(S_F),
\]

\[
S_{FR} g = S^{1/2}_{FR}(I - iG_F)S^{1/2}_{FR} g, \quad g \in \operatorname{dom}(S^*_{F}).
\]

It follows from the definition of the closure of the form \( (Su, v) \), that

\[
\operatorname{R}[S_F] = \operatorname{ran}(S^{1/2}_{FR}) = \left\{ f \in \mathcal{H} : \sup_{\varphi \in \operatorname{dom}(S)} \frac{|(f, \varphi)|^2}{\operatorname{Re}(S^* \varphi, \varphi)} < \infty \right\}. \tag{1.5}
\]

In the case of a nonnegative symmetric operator \( S (\alpha = 0) \) it was discovered by M.G. Kreĭn [30] that the set of all its nonnegative selfadjoint extensions has a minimal element (in the sense of associated closed quadratic forms). This minimal element \( S_N \) is defined in [30] by means of linear–fractional transformation. Another (equivalent) definitions of \( S_N \) are given in [3] and in [20]. If \( \alpha \neq 0 \), then the corresponding m-sectorial analog of such extremal extension also exists [7, 8] and can be defined similarly, see [7, 8, 13]. We preserve the same notation \( S_N \) and the name Kreĭn-von Neumann extension in the general case of non necessarily symmetric sectorial operator \( S \). We notice that interesting applications of Kreĭn-von Neumann extension of nonnegative symmetric operator can be found in [19].

The domain of closed sesquilinear form associated with Kreĭn-von Neumann extension of \( \alpha \)-sectorial operator \( S \) \( S \) is given by (see [8])

\[
D[S_N] = \left\{ u \in \mathcal{H} : \sup_{\varphi \in \operatorname{dom}(S)} \frac{|(u, S \varphi)|^2}{\operatorname{Re}(S^* \varphi, \varphi)} < \infty \right\}. \tag{1.6}
\]

This is an analog of the formula established by T. Ando and K. Nishio in [3] for the case of nonnegative symmetric operator \( S (\alpha = 0) \).

Let

\[
\mathcal{N}_\lambda \stackrel{\text{def}}{=} \mathcal{H} \ominus \ker(S - \bar{\lambda} I)
\]

be the defect subspace of a linear operator \( S \). If \( S \) closed and densely defined, then

\[
\mathcal{N}_\lambda = \ker(S^* - \lambda I).
\]

It is easy to see, that if \( \hat{S} \) is an extension of \( S \) with nonempty resolvent set, then for all \( \lambda, z \in \rho(\hat{S}^*) \)

\[
(\hat{S}^* - \lambda I)(\hat{S}^* - zI)^{-1}\mathcal{N}_\lambda = (I + (z - \lambda)(\hat{S}^* - zI)^{-1})\mathcal{N}_\lambda = \mathcal{N}_z. \tag{1.7}
\]
Note, that from (1.5) and (1.6) it follows that
\[ D[S_N] \cap \mathfrak{N}_\lambda = R[S_F] \cap \mathfrak{N}_\lambda. \]  
(1.8)

For the operators \( S_F, S_N \), and for an arbitrary \( m \)-sectorial extension \( \tilde{S} \) of \( S \) the following relations are valid [7, 8]:
\[ D[S] \cap \mathfrak{N}_\lambda = \{ 0 \}, \]
(1.9)
\[ D[S] = D[\tilde{S}] = D[\tilde{S}] \subseteq D[S_N], \quad R[S_N] \subseteq R[\tilde{S}] \subseteq R[S_F], \]  
(1.10)
\[ \tilde{S}[f, v] = S_N[f, v] \quad \forall f \in D[S], v \in D[\tilde{S}], \]
(1.11)
\[ S_N[f, v] = (f, S^*v) \quad \forall f \in D[S], v \in \text{dom}(S^*) \cap D[S_N], \]
(1.12)
\[ \text{dom}(S^*_F) = D[S] \cap \text{dom}(S^*). \]
(1.13)

If \( S \) is coercive, then
\[ \text{dom}(S_N) = \text{dom}(S) + \ker(S^*), \quad S_N | \ker(S^*) = 0, \]
\[ D[S_N] = D[S] + \ker(S^*). \]
(1.14)

The operator \( S \) has a unique \( m \)-sectorial extension if and only if, for some \( \lambda \in \rho(S^*_F) \) (then for all \( \lambda \in \rho(S^*_F) \)):
\[ \sup_{x \in \text{dom}(S)} \frac{|\langle f, x \rangle^2}{\Re \langle Sx, x \rangle} = \infty \quad \forall f \in \mathfrak{N}_\lambda \setminus \{ 0 \}. \]
(1.15)

From (1.5), (1.6), and (1.14) it follows that
\[ S_N \neq S_F \iff D[S_N] \cap \mathfrak{N}_\lambda \neq \{ 0 \} \iff R[S_F] \cap \mathfrak{N}_\lambda \neq \{ 0 \}, \lambda \in \rho(S^*_F). \]
(1.16)

Taking into account (1.15), (1.2), (1.3) we get for \( \mu \in \mathbb{C} \setminus \Theta(\alpha) \)
\[ \varphi_\mu \in \mathfrak{N}_\mu \cap D[S_N] \iff \lim_{\lambda \to \alpha} \left| \left( (S_F^* - \lambda I)^{-1} \varphi_\mu, \psi_\mu \right) \right| < \infty, \]
(1.17)

and for \( \gamma \in (\alpha, \pi/2) \)
\[ \lim_{\lambda \to 0} \left( (S^*_F - \lambda I)^{-1} \varphi_\mu, \psi_\mu \right) = \left( (I - iG_F)^{-1} \tilde{S}_{FR}^{-1/2} \varphi_\mu, \tilde{S}_{FR}^{-1/2} \psi_\mu \right), \]
\[ \varphi_\mu, \psi_\mu \in \mathfrak{N}_\mu \cap D[S_N]. \]

Fix \( z \in \rho(S^*_F) \) and define a linear manifold \( \mathfrak{L} \):
\[ \mathfrak{L} \overset{\text{def}}{=} D[S] + \mathfrak{N}_z, \quad z \in \rho(S^*_F). \]
(1.18)

Then \( \mathfrak{L} \) does not depend on the choice of \( z \in \rho(S^*_F) \) and, clearly, \( \text{dom}(S^*) \subseteq \mathfrak{L} \). We will denote by \( P_{z,F} \) and \( P_{\gamma} \) the skew projectors in \( \mathfrak{L} \) onto \( D[S] \) and \( \mathfrak{N}_z \), corresponding to the decomposition (1.17). If \( z = i \), these projectors we will denote by \( P_F \) and \( P_i \), respectively.
Let us consider the form $\hat{S}_z[u, v]$ defined on the linear manifold $\mathcal{L}$

$$\hat{S}_z[u, v] = S[\mathcal{P}_z F u, \mathcal{P}_z F v] - z(\mathcal{P}_z F u, \mathcal{P}_z F v), \quad \forall z \in \mathbb{C} \setminus \Theta(\alpha).$$

The following relations have been established in [3]:

$$D[S_N] = \left\{ u \in \mathcal{L} : \lim_{z \to \mathbb{C} \setminus \Theta(\alpha)} \left| \frac{\hat{S}_z[u]}{z^\gamma} \right| < \infty \right\},$$

$$S_N[u, v] = \lim_{z \to \mathbb{C} \setminus \Theta(\alpha)} \hat{S}_z[u, v], \quad u, v \in D[S_N], \quad \gamma \in (\alpha; \pi/2),$$

$$S_N[u, v] = \left( I + iG_F \right) \left( S_{FR}^{1/2} \mathcal{P}_z F u + z(I - iG_F)^{-1} \hat{S}_{FR}^{-1/2} \mathcal{P}_z u \right),$$

$$\left( S_{FR}^{1/2} \mathcal{P}_z F v + z(I - iG_F)^{-1} \hat{S}_{FR}^{-1/2} \mathcal{P}_z v \right), \quad u, v \in D[S_N]. \quad (1.18)$$

### 1.3. Boundary triplets and abstract boundary conditions for quasi-selfadjoint extensions of nonnegative symmetric operator

Let $A$ be a closed densely defined symmetric operator in $\mathcal{H}$. Recall the definition of a boundary triplet (boundary value space) [25] for $A^*$.

**Definition 1.2.** A triplet $\{\mathcal{H}, \Gamma_1, \Gamma_0\}$ is called boundary triplet of $A^*$ if $\mathcal{H}$ is a Hilbert space and $\Gamma_0, \Gamma_1$ are bounded linear operators from the Hilbert space $H_+ = \text{dom}(S^*)$ with the graph norm into $\mathcal{H}$ such that the map $\Gamma = \{\Gamma_0, \Gamma_1\}$ is a surjection from $H_+$ onto $H^2 = \mathcal{H} \oplus \mathcal{H}$ and the Green identity holds:

$$(A^* f, g) - (f, A^* g) = (\Gamma_1 f, \Gamma_0 g)_\mathcal{H} - (\Gamma_0 f, \Gamma_1 g)_\mathcal{H} \quad \forall f, g \in H_+. \quad (1.19)$$

In the sequel for descriptions of extensions in terms of the abstract boundary conditions the linear relations will be used. One can find basic notions, and properties related to these objects in, for instance, [4, 39, 25, 22, 13]. The formulas

$$\text{dom}(\tilde{A}) = \left\{ u \in \text{dom}(A^*): \Gamma u \in \mathcal{T} \right\}, \quad \tilde{A} = A^* \upharpoonright \text{dom}(\tilde{A}) \quad (1.20)$$

give a one-to-one correspondence between all quasi-selfadjoint extensions $\tilde{A}$ of $A$ ($A \subset \tilde{A} \subset A^*$) and all linear relations $\Gamma$ in $\mathcal{H}$. Moreover $\tilde{A}^* \leftrightarrow \Gamma^*$. Therefore, an extension $\tilde{A}$ is a selfadjoint one if and only if the relation $\Gamma$ is a selfadjoint in $\mathcal{H}$.

As it was shown in [21, 22] the operators $A_0, A_1$ defined as follows

$$A_k = A^* \upharpoonright \text{Ker} \Gamma_k, \quad k = 0, 1$$

are mutually transversal selfadjoint extensions of $A$, i.e.,

$$\text{dom}(A^*) = \text{dom}(A_0) + \text{dom}(A_1).$$

The function $\Gamma_0(\lambda) := (\Gamma_0 \upharpoonright \mathcal{N}_\lambda)^{-1}$ [21] is the $\gamma$-field corresponding to $A_0$ [32, 33], i.e.,

$$\text{ran}(\Gamma_0(\lambda)) = \mathcal{N}_\lambda,$$
\[ \Gamma_0(\lambda) = \Gamma(z) + (\lambda - z)(A_0 - zI)^{-1}\Gamma_0(z). \]

Note that as a consequence of (1.19) one can obtain the equality
\[ \Gamma_0(\lambda) = (\Gamma_1(A_0 - \lambda I)^{-1})^* . \]

V. Derkach and M. Malamud \[21, 22\] define the Weyl function \( M_0(\lambda) \) by the equality
\[ M_0(\lambda) = \Gamma_1\Gamma_0(\lambda). \]

The Nevanlinna class operator valued function \( M_0 \) is Krein-Langer \( Q \)-function \[32, 33\], and the following identity
\[ M_0(\lambda) - M_0(z) = (\lambda - z)\Gamma_0^*(\overline{\lambda})\Gamma_0(\lambda) \]
holds. In terms of boundary triplet the connection between a quasi-selfadjoint extension \( \tilde{A}_T \) defined by relations (1.20) and its resolvent is given by the Krein resolvent formula
\[ (\tilde{A}_T - \lambda I)^{-1} = (A_0 - \lambda I)^{-1} + \Gamma_0(\lambda)\left( \tilde{T} - M_0(\lambda) \right)^{-1} \Gamma_0^*(\overline{\lambda}), \]
\[ \lambda \in \rho(A_0) \cap \rho(\tilde{A}_T). \]

The following theorem has been established by V. Derkach and M. Malamud (see \[21, 22, 23, 34\]).

**Theorem 1.3.** Let \( A \) be a closed nonnegative symmetric operator and let \( \{ \mathcal{H}, \Gamma_1, \Gamma_0 \} \) be a boundary triplet of \( A^* \) such that \( A_0 = A_F(= A^*| \text{Ker} \Gamma_0) \). Then \( A \) has a non-unique nonnegative selfadjoint extension if and only if
\[ \mathcal{D}_0 = \left\{ h \in \mathcal{H} : \lim_{x \uparrow 0} (M_0(x)h, h)_{\mathcal{H}} < \infty \right\} \neq \{ 0 \}, \]
and the quadratic form
\[ \tau[h] = \lim_{x \uparrow 0} (M_0(x)h, h)_{\mathcal{H}}, \mathcal{D}[\tau] = \mathcal{D}_0 \]
is bounded from below. Define by \( M_0(0) \) the selfadjoint linear relation in \( \mathcal{H} \) associated with \( \tau \). Then the Krein-von Neumann extension \( A_N \) can be defined by the boundary condition
\[ \text{dom}(A_N) = \left\{ u \in \text{dom}(A^*) : \langle \Gamma_0^*u, \Gamma_1^*u \rangle \in M_0(0) \right\}. \]
The relation \( M_0(0) \) is also the strong resolvent limit of \( M_0(x) \) when \( x \to -0 \). Moreover, \( A_0 \) and \( A_N \) are disjoint iff \( \mathcal{D}_0 = \mathcal{H} \) and transversal iff \( \mathcal{D}_0 = \mathcal{H} \). In there is a one-to-one correspondence given by (1.20) between \( m \)-accretive extensions \( \tilde{A}_T \) and \( m \)-accretive linear relations \( \tilde{T} \) satisfying the condition
\[ \text{dom}(\tilde{T}) \subseteq \mathcal{D}_0, \text{ Re}(\tilde{T} x, x) \geq \tau[x], x \in \text{dom}(\tilde{T}). \]
The extension \( \tilde{A}_T \) is \( m \)-\( \alpha \)-sectorial iff the form
\[ (\tilde{T} x, y) - \tau[x, y] \]
is \( \alpha \)-sectorial.
2. Abstract boundary conditions for \emph{m}-accretive extensions of sectorial operators

Next, we recall some definitions and results established in [15]. A sesquilinear form

\[ \tau[u, v] \stackrel{\text{def}}{=} S_F R[\mathcal{P}_{-1,F} u, \mathcal{P}_{-1,F} v] + (\mathcal{P}_{-1} u, \mathcal{P}_{-1} v), \ u, v \in \mathfrak{L} \]

is a nonnegative and closed [15] in the Hilbert space \( \mathfrak{H} \). So, we can consider the linear manifold \( \mathfrak{L} \) as a Hilbert space with the inner product

\[ (u, v)_\tau = \tau(u, v) + (u, v)_{\mathfrak{H}}. \]

\textbf{Definition 2.1 (15).} A pair \( \{\mathcal{H}, \Gamma\} \) is called boundary pair of \( S \), if \( \mathcal{H} \) is a Hilbert space and \( \Gamma \in \mathfrak{L}(\mathfrak{L}, \mathcal{H}) \) is such that \( \ker(\Gamma) = D[S] \), \( \text{ran}(\Gamma) = \mathcal{H} \).

Let

\[ \gamma(\lambda) = (\Gamma | \mathfrak{R}_\lambda)^{-1}, \ \lambda \in \rho(S_F^*). \]

Then \( \gamma(\lambda) \in \mathfrak{L}(\mathcal{H}, \mathfrak{H}) \) for all \( \lambda \in \rho(S_F^*) \). The operator-function \( \gamma(\lambda) \) is called \( \gamma \)-field of the operator \( S \) associated with the boundary pair \( \{\mathcal{H}, \Gamma\} \). Clearly, \( \gamma(\lambda) \) maps \( \mathcal{H} \) onto \( \mathfrak{R}_\lambda \). Hence \( S^* \gamma(\lambda) = \lambda \gamma(\lambda) \) and

\[ \ker(\gamma^*(\lambda)) = \text{ran}(S - \lambda I) \]

The following relations are valid:

\[ \gamma(\lambda) = \gamma(z) + (\lambda - z)(S_F^* - \lambda I)^{-1}\gamma(z), \quad (2.1) \]

\[ \mathcal{P}_\lambda F u = u - \gamma(\lambda)\Gamma u, \ u \in \mathfrak{L}, \]

\[ \mathcal{P}_F \gamma(\lambda)e = (\lambda - i)(S_F^* - \lambda I)^{-1}\gamma(i)e, \quad \mathcal{P}_i \gamma(\lambda)e = \gamma(i)e, \ e \in \mathcal{H}. \]

Define on \( \mathfrak{L} \) one more sesquilinear form \( l[u, v] \):

\[ l[u, v] = S_F [\mathcal{P}_F u, \mathcal{P}_F v] - i(\mathcal{P}_i u, \mathcal{P}_F v) - i(\mathcal{P}_F u, \mathcal{P}_i v) - i(\mathcal{P}_i u, \mathcal{P}_i v). \quad (2.2) \]

Due to the equality

\[ \text{Re} \ l[u] = \text{Re} \ S[\mathcal{P}_F u] = \left\| S_{FR}^{1/2} \mathcal{P}_F u \right\|^2, \ u \in \mathfrak{L}, \]

the form \( l[u, v] \) is accretive. Moreover,

\[ \inf_{\varphi \in D[S]} \{\text{Re} \ l[u - \varphi]\} = 0, \quad \forall u \in \mathfrak{L}, \]

and \( l[\varphi, v] = (\varphi, S^* v) \) for all \( \varphi \in D[S], v \in \text{dom}(S^*) \).

Relations (1.15) and (2.2) imply the following representation of the form \( S_N[u, v] \):

\[ S_N[u, v] = l[u, v] \]

\[ + \left[ i \left( \gamma(i) \Gamma u, \gamma(i) \Gamma v \right) + \left( (I - iG_F)^{-1} \hat{S}_{FR}^{-1/2} \gamma(i) \Gamma u, \hat{S}_{FR}^{-1/2} \gamma(i) \Gamma v \right) \right] \]

\[ + 2l \left( (I - iG_F)^{-1} \hat{S}_{FR}^{-1/2} \gamma(i) \Gamma u, S_{FR}^{1/2} \mathcal{P}_F v \right), \ u, v \in D[S_N]. \quad (2.3) \]
Definition 2.2 ([15]). The triplet \( \{ \mathcal{H}, G, \Gamma \} \) is called a boundary triplet for the operator \( S^* \) if \( \{ H, \Gamma \} \) is a boundary pair for \( S \) and \( G \): \( \text{dom}(S^*) \to \mathcal{H} \) is a linear operator such that the relation
\[
(l^*[u,v] = (S^*u,v) - (Gu,\Gamma v)_H, \quad \forall u \in \text{dom}(S^*), \forall v \in \mathcal{L}) \tag{2.4}
\]
is valid.

It is shown in [15] that there exists a unique operator \( G : \text{dom}(S^*) \to \mathcal{H} \) such that, \( \{2.4\} \) holds and, moreover,
\[
Gu = \gamma(i)(S^* - iI)u.
\]

Next, we define operator-functions \( Q(\lambda) \in \mathcal{L}(\mathcal{H}), G(\lambda) \in \mathcal{L}(\mathcal{H}, \mathcal{H}), \Phi(\lambda) \in \mathcal{L}(\mathcal{H}, \mathcal{H}), q(\lambda) \in \mathcal{L}(\mathcal{H}, \mathcal{H}), \lambda \in \rho(S_F) \) associated with the boundary triplet for the operator \( S^* \), see [15]:
\[
Q(\lambda) \overset{\text{def}}{=} G(\lambda), \quad q(\lambda) \overset{\text{def}}{=} (G(S_F^* - \bar{\lambda}I)^{-1})^*, \quad G(\lambda) \overset{\text{def}}{=} \left(S_{FR}^{1/2}P_F\gamma(\lambda)\right)^*, \quad \Phi(\lambda) \overset{\text{def}}{=} \left(S_{FR}^{1/2}(S_F^* - \bar{\lambda}I)^{-1}\right)^*.
\]

The following identities are valid [15]:
\[
Q(\lambda) = \gamma'(i)(S_F^* - iI)\gamma(\lambda) = (\lambda - i)\gamma'(i)\gamma(\lambda), \tag{2.5}
\]
\[
\Phi(\lambda) - \Phi(z) = (\lambda - z)(S_F - \lambda I)^{-1}\Phi(z) = (\lambda - z)(S_F - zI)^{-1}\Phi(\lambda),
\]
\[
G(\lambda) - G(z) = (\lambda - z)\gamma'(\bar{z})\Phi(\lambda),
\]
\[
q(\lambda) - q(z) = (\lambda - z)(S_F - \lambda I)^{-1}q(z),
\]
\[
Q(\lambda) - Q(z) = (\lambda - z)q'(\bar{\lambda})\gamma(z).
\]

Observe that the function \( Q(\lambda) \) is an analog of the Weyl function [1.22] corresponding to a boundary triplet of the adjoint to a symmetric operator, while \( q(\lambda) \) is an analog of the function in [1.21].

Let \( L \) be a linear operator in \( \mathcal{L} \) defined as follows:
\[
\text{dom}(L) = \text{dom}(S_F) + \mathfrak{N}, \quad L(u_F + u_i) = S_Fu_F - iu_i, \quad u_F \in \text{dom}(S_F), u_i \in \mathfrak{N}. \tag{2.6}
\]

Then \( L \) is closed, and
\[
(Lu, \varphi) = l[u, \varphi], \quad \forall u \in \text{dom}(L), \varphi \in D[S],
\]
\[
\text{ker}(L - \lambda I) = \text{ran}(q(\lambda)), \quad \forall \lambda \in \rho(S_F),
\]
\[
\text{dom}(L) = \text{dom}(S_F) + \text{ran}(q(\lambda)), \quad \forall \lambda \in \rho(S_F).
\]

Definition 2.3 ([15]). Let \( S \) be a densely defined sectorial operator and let \( \{ \mathcal{H}, \Gamma \} \) be a boundary pair for \( S \). A triplet \( \{ \mathcal{H}, G_*, \Gamma \} \) is called a boundary triplet for \( L \) if \( G_* : \text{dom}(L) \to \mathcal{H} \) is a linear operator such that
\[
(l[u,v] = (Lu,v) - (G_*u,\Gamma v)_H, \forall u \in \text{dom}(L), \forall v \in \mathcal{L}).
\]
The operator $G_\ast$ is uniquely defined \cite{15} and, moreover, for each $\lambda \in \rho(S_F)$
\[ G_\ast f = \gamma^*(\lambda)(S_F - \lambda I)f, \quad f \in \text{dom}(S_F), \]
\[ G_\ast q(\lambda)e = Q^*(\lambda)e, \quad e \in \mathcal{H}. \] (2.7)

Thus, given a boundary pair $\{\mathcal{H}, \Gamma\}$ for an operator $S$, the boundary triplets corresponding to it are $\{\mathcal{H}, G, \Gamma\}$ for $S^\ast$ and $\{\mathcal{H}, G_\ast, \Gamma\}$ for $L$, and we have the abstract Green formula
\[ (Lu, v) - (u, S^\ast v) = (G_\ast u, \Gamma v)_\mathcal{H} - (\Gamma u, Gv)_\mathcal{H}, \quad \forall u \in \text{dom}(L), \forall v \in \text{dom}(S^\ast). \]

Let $\tilde{S}$ be an $m$-accretive extension of $S$. The following inclusions are established in \cite{15}:
\[ \text{dom}(\tilde{S}) \subseteq \mathcal{L}, \]
\[ \tilde{S}u + \lambda \mathcal{P}_\lambda u \in \text{ran}(S_F^{1/2}) = \text{R}[S_F], \quad \lambda \in \rho(S_F^*). \] (2.8)

The next two theorems follow from (2.8).

**Theorem 2.4** (\cite{15}). Let $S$ be a densely defined closed sectorial operator. Let $\{\mathcal{H}, \Gamma\}$ be a boundary pair for $S$ and $\{\mathcal{H}, G, \Gamma\}$ be a corresponding boundary triplet for $S^\ast$. If $\tilde{S}$ is an $m$-accretive extension of $S$, then there exist linear operators $Z : \text{dom}(\tilde{S}) \to \mathcal{H}$ and $X : \text{dom}(X) = \Gamma \text{dom}(\tilde{S}) \to \text{ran}(S_F)$, such that:

1) $\text{dom}(S) \subseteq \ker(Z)$;
2) $(\tilde{S}u, v) = l[u, v] + (Zu, \Gamma v)_\mathcal{H} + 2(\lambda \mathcal{P}_\lambda u, S_F^{1/2} \mathcal{P}_F v), \quad \forall u \in \text{dom}(\tilde{S}), \quad v \in \mathcal{L}$
3) $Z = \{(\Gamma u, Zu), u \in \text{dom}(\tilde{S})\}$ — is an $m$-accretive linear relation in $\mathcal{H}$;
4) $\|X e\|^2 \leq \text{Re} \langle Z(e), e \rangle_\mathcal{H}$ for all $e \in \text{dom}(Z) = \Gamma \text{dom}(\tilde{S})$

**Theorem 2.5** (\cite{15}). There is a bijective correspondence between all $m$-accretive extensions $\tilde{S}$ of $S$ and all pairs $(Z, X)$, where $Z$ is an $m$-accretive linear relation in $\mathcal{H}$ and $X : \text{dom}(Z) \to \text{ran}(S_F)$ is a linear operator such that:
\[ \|X e\|^2 \leq \text{Re} \langle Z(e), e \rangle_\mathcal{H} \quad \forall e \in \text{dom}(Z). \] (2.9)

This correspondence is given by the boundary conditions for the domain and the action of $\tilde{S}$ as follows: for all $\text{Re} \lambda < 0$

\[ \text{dom}(\tilde{S}) = \left\{ u \in \mathcal{L} : \begin{array}{c} 1) u - (q(\lambda) - 2\Phi(\lambda)X)\Gamma u \in \text{dom}(S_F); \\
2) G_\ast(u + 2\Phi(\lambda)X)\Gamma u \in (Z + 2\mathcal{G}(\lambda)X)\Gamma u \end{array} \right\}. \] (2.10)

Set
\[ \mathcal{W}(\lambda) := Z - \mathcal{Q}^*(\lambda) + 2\mathcal{G}(\lambda), \quad \lambda \in \rho(S_F). \] (2.11)

Then

1) a number $\lambda \in \rho(S_F)$ is a regular point of $\tilde{S}$ if and only if $\mathcal{W}^{-1}(\lambda) \in \text{L}(\mathcal{H})$, 

and,
\[ (\hat{S} - \lambda I)^{-1} = (S_F - \lambda I)^{-1} + (q(\lambda) - 2\Phi(\lambda)X)W^{-1}(\lambda)\gamma^*(\lambda), \quad (2.12) \]
\[
\text{dom}(\hat{S}) = \left( I + (q(\lambda) - 2\Phi(\lambda)X)W^{-1}(\lambda)\gamma^*(\lambda)(S_F - \lambda I) \right) \text{dom}(S_F), \quad (2.13) \]
\[
\hat{Su} = (S_F - \lambda I)f + \lambda u \quad (2.14) \]
for
\[ u = \left( I + (q(\lambda) - 2\Phi(\lambda)X)W^{-1}(\lambda)\gamma^*(\lambda)(S_F - \lambda I) \right)f, \quad f \in \text{dom}(S_F), \quad (2.15) \]

2) a number \( \lambda \in \rho(S_F) \) is an eigenvalue of \( \hat{S} \) if and only if
\[ \ker(W(\lambda)) \neq \{0\}, \]

and,
\[ \ker(\hat{S} - \lambda I) = (q(\lambda) - 2\Phi(\lambda)X) \ker(W(\lambda)). \]

**Remark 2.6.** Relations \((2.10)\) remain valid for all \( \lambda \in \rho(\hat{S}) \cap \rho(S_F) \). The resolvent formula \((2.12)\) is an analog of the resolvent formula \((1.24)\).

Let \( S \) be a densely defined closed sectorial operator. Define for all \( z \in \mathbb{C}, \text{Re} \ z \leq 0 \) a linear operator \( S_z \) as follows [9 10]:
\[
\text{dom}(S_z) = \text{dom}(S) + \mathfrak{N}_z, \quad S_zh = S\varphi - z\varphi_z, \quad h = \varphi + \varphi_z \in \text{dom}(S_z). \quad (2.16) \]

**Proposition 2.7** [9 10]. The operator \( S_z \) is \( m \)-accretive extension of \( S \).

**Proof.** Proposition has been proved in [9 10] for \( \text{Re} \ z < 0 \). Let us prove the statement for \( z = ix, \ x \in \mathbb{R} \). Let \( g = \varphi + \varphi_{ix}, \ \varphi \in \text{dom}(S), \ \varphi_{ix} \in \mathfrak{N}_{ix} \). Then
\[ (S_{ix}g, g) = (S\varphi - ix\varphi_{ix}, \varphi + \varphi_{ix}) \]
\[ = (S\varphi, \varphi) - ix\|\varphi_{ix}\|^2 - 2i\text{Im}(ix(\varphi_{ix}, \varphi)). \]
Hence \( \text{Re}(Sg, g) = \text{Re}(S\varphi, \varphi) \geq 0 \) for all \( g \in \text{dom}(S_{ix}) \). Furthermore, one can verify that
\[
\begin{cases}
\text{dom}(S_{ix}^*) = (S_F^* - ixI)^{-1}(S + ixI) \text{dom}(S) + \mathfrak{N}_{ix}, \\
S_{ix}^* ((S_F^* - ixI)^{-1}(S + ixI)f + \varphi_{ix}) = S_{ix}^*(S_F^* - ixI)^{-1}(S + ixI)f + ix\varphi_{ix}, \\
f \in \text{dom}(S), \ \varphi_{ix} \in \mathfrak{N}_{ix}
\end{cases}
\]
and
\[ \text{Re}(S_{ix}^*h, h) = \text{Re}(S_F^*(S_F^* - ixI)^{-1}(S + ixI)f, (S_F^* - ixI)^{-1}(S + ixI)f) \geq 0. \]
for
\[ h = (S_F^* - ixI)^{-1}(S + ixI)f + \varphi_{ix}, \ f \in \text{dom}(S), \ \varphi_{ix} \in \mathfrak{N}_{ix}. \]
This means that \( S_{ix}^* \) is accretive. Thus, \( S_{ix} \) and \( S_{ix}^* \) are accretive. It follows that \( S_{ix} \)

is \( m \)-accretive. \( \square \)
Note that in general from (2.16) it follows for Re $z \leq 0$ that
\[
\text{dom}(S_z^*) = \{ g \in \text{dom}(S^*): (S^* + zI)g \in \text{ran}(S - zI) \},
\]
\[
S_z^* = S^*|_{\text{dom}(S_z^*)}.
\]
In addition, for the boundary operators in the boundary triplets in Definitions 2.2 and 2.3 the equalities are valid
\[
\ker(G) = \text{dom}(S_z^*), \quad \ker(G_*) = \text{dom}(S_l).
\]

**Remark 2.8.** It is proved in [10, 12] that

1) for each $\gamma \in [0, \pi/2)$ the equalities are valid:
\[
s-R\text{-lim}_{z \to \gamma} S_z = S_N, \quad s-R\text{-lim}_{z \to \gamma} S_z = S_F,
\]
where $s-R\text{-lim}$ is the strong resolvent limit [28];

2) the following conditions are equivalent:
   (a) $S_z$ is $m$-sectorial operator for one (then for all) $z$, Re $z < 0$;
   (b) $\text{dom}(S^*) \subset \text{dom}(S_N)$, where $S_N$ is the Krein–von Neumann extension of $S$.

Next we give expressions for pairs $\langle Z_z, X_z \rangle$ corresponding to $S_z$, Re $z \leq 0$ in accordance with Theorem 2.4.

**Proposition 2.9.** $Z_z$ is the graph of the operator $Z_z = -Q(z)$, $\text{dom}(Z_z) = \mathcal{H}$ and $X_z = -G^*(\bar{z})$. In addition, for $u \in \text{dom}(S_z)$, $v \in \mathcal{L}$
\[
(S_zu, v) = l[u, v] - (Q(z)\Gamma u, \Gamma v)_{\mathcal{H}} - 2(G^*(\bar{z})\Gamma u, S_{FR}^{1/2}P_Fv). \tag{2.17}
\]

**Proof.** Define for $u \in \text{dom}(S_z)$
\[
Z_zu := \gamma^*(i)(S_z + iI)u, \\
M_zu := \frac{1}{2} \left( S_{FR}^{1/2}(S_zu + iP_1u) - (I + iG)S_{FR}^{1/2}P_Fu \right). \tag{2.18}
\]
Observe that from (2.18) one obtains the inclusions $\text{dom}(S) \subseteq \ker(Z)$ and $\text{dom}(S) \subseteq \ker(M_z)$. In addition, due to definition of $\mathcal{L}$ (1.17), Definition 2.1 of a boundary pair, and (2.16), one obtains the equality
\[
\Gamma \text{dom}(S_z) = \mathcal{H}.
\]
According to the proof of Theorem 2.4 (see [15]), the relations
\[
Z_z = \{ (\Gamma u, Z_zu), \ X_z\Gamma u = M_zu, \ u \in \text{dom}(S_z) \}
\]
hold. Then, taking into account that $u = \gamma(z)\Gamma u$ and relations (2.4), (2.5), (2.16), we have
\[
Z_zu = \gamma^*(i)(S_z + iI)\gamma(z)\Gamma u = \gamma^*(i)(-z\gamma(z)\Gamma u + i\gamma(z)\Gamma u) = \\
= -(z - i)\gamma^*(i)\gamma(z)\Gamma u = -Q(z)\Gamma u.
\]
Let $\Gamma u = e$, then $u = \varphi + \gamma(z)e$, $\varphi \in \text{dom}(S)$, and
\[
X_z\Gamma u = M_zu = M_z\gamma(z)e =
\]
Proof. 1) Clearly

\[ \frac{1}{2} \left( S_{FR}^{-1/2} (S_{z}\gamma(z)e + iP_{\gamma}(z)e) - (I + iG_{F})S_{FR}^{1/2}P_{F}\gamma(z)e \right) = \]

\[ = \frac{1}{2} \left( S_{FR}^{-1/2} (-\gamma(z)e + i\gamma(i)e) - (I + iG_{F})S_{FR}^{1/2}P_{F}\gamma(z)e \right) = \]

\[ = \frac{1}{2} \left( S_{FR}^{-1/2} (-S^{*}\gamma(z)e + S^{*}\gamma(i)e) - (I + iG_{F})S_{FR}^{1/2}P_{F}\gamma(z)e \right) = \]

\[ = \frac{1}{2} \left( -S_{FR}^{-1/2} S_{FR}^{*}P_{F}\gamma(z)e - (I + iG_{F})S_{FR}^{1/2}P_{F}\gamma(z)e \right) = \]

\[ = \frac{1}{2} \left( -(I - iG_{F})S_{FR}^{1/2}P_{F}\gamma(z)e - (I + iG_{F})S_{FR}^{1/2}P_{F}\gamma(z)e \right) = \]

\[ = -S_{FR}^{1/2}P_{F}\gamma(z)e = -G^{*}(\bar{z})\Gamma u. \]

Equality (2.17) follows from Theorem 2.4 \qed

3. m-SECTORIAL EXTENSIONS

By Theorem 2.5 there is a bijective correspondence between all m-accretive extensions \( S' \) of \( S \) and all pairs \( (Z, X) \) satisfying condition (2.9). Our main goal is to establish additional conditions which guarantee that corresponding m-accretive extension \( S' \) is sectorial.

Next, we will need the following auxiliary result:

**Lemma 3.1.** 1) If \( T \) is a m-accretive operator and \( \beta \in (0, \pi/2) \), then:

\[ \lim_{z \to 0, \pi/2+\beta \leq |\arg z| \leq \pi} z(T - zI)^{-1}h = \begin{cases} -h, & h \in \ker(T) \\ 0, & h \in \text{ran}(T) \end{cases} \quad (3.1) \]

2) If \( T \) is m-\( \alpha \)-sectorial and \( \beta \in (\alpha, \pi/2) \), then

\[ \lim_{z \to 0, z \in \mathbb{C} \setminus \Theta(\beta)} z(T - zI)^{-1}h = \begin{cases} -h, & h \in \ker(T) \\ 0, & h \in \text{ran}(T) \end{cases} \quad (3.2) \]

**Proof.** 1) Clearly

\[ h \in \ker(T) \Rightarrow (T - zI)^{-1}h = -\frac{h}{z} \quad \text{for all} \ z \in \rho(T) \setminus \{0\}. \]

Therefore

\[ \lim_{z \to 0, \pi/2+\beta \leq |\arg z| \leq \pi} z(T - zI)^{-1}h = -h. \]

Now let, \( h \in \text{ran}(T) \). Then \( h = T\varphi, \varphi \in \text{dom}(T) \) and

\[ z(T - zI)^{-1}h = z(T - zI)^{-1}T\varphi = \]

\[ = z(T - zI)^{-1}(T - zI + zI)\varphi = z\varphi - z^2(T - zI)^{-1}\varphi. \]

Taking into account that

\[ \|(T - zI)^{-1}\| \leq \frac{1}{|\text{Re } z|}, \quad \text{Re } z < 0, \]
and $|\text{Re} \, z| \geq |z| \sin \beta$ for $\pi/2 + \beta \leq |\arg z| \leq \pi$, we get for all $\varphi \in \text{dom}(T)$ that
$$
\lim_{z \to 0, \pi/2 + \beta \leq |\arg z| \leq \pi} z(T - zI)^{-1}T\varphi = 0.
$$
Further, since $\text{ran}(T)$ is dense in $\text{ran}(T)$ and
$$
\|z(T - zI)^{-1}\| \leq \frac{1}{\sin \beta}, \pi/2 + \beta \leq |\arg z| \leq \pi,
$$
then
$$
\lim_{z \to 0, \pi/2 + \beta \leq |\arg z| \leq \pi} z(T - zI)^{-1}h
$$
for all $h \in \text{ran}(T)$. Thus (3.1) is valid.

2) Relation (3.2) follows from (0.2). $\square$

**Proposition 3.2.** Let $S$ be a densely defined closed $\alpha$-sectorial operator, $\gamma(z)$ its $\gamma$-field, corresponding to the boundary pair $\{H, \Gamma\}$ of $S$. Suppose $S_F \neq S_N$. Then for all $e \in H$ such that, $\gamma(\lambda)e \in D[S_N]$:
$$
\lim_{z \to 0, z \in \mathbb{C} \setminus \Theta(\beta)} z\gamma(z)e = 0,
$$
where $\beta \in (0, \pi/2)$.

**Proof.** Let $\gamma(\lambda)e \in D[S_N]$. Since $D[S_N] \cap \mathcal{M}_\alpha = R[S_F] \cap \mathcal{M}_\alpha$, then $\gamma(\lambda)e \in R[S_F]$. Since $R[S_F] = \text{ran}(S_F) = \text{ran}(S_F^*)$, from Lemma 3.1 and (2.1) we have:
$$
\lim_{z \to 0, z \in \mathbb{C} \setminus \Theta(\beta)} z\gamma(z)e = \lim_{z \to 0, z \in \mathbb{C} \setminus \Theta(\beta)} \left( z\gamma(\lambda)e + (z - \lambda)z(S_F^* - zI)^{-1}\gamma(\lambda)e \right) = 0. \quad \square
$$

**Theorem 3.3.** Let $S$ be a densely defined closed sectorial operator, $\gamma(z)$ its $\gamma$-field, corresponding to the boundary pair $\{H, \Gamma\}$ of $S$. Define a set in $H$:
$$
D_0 := \left\{ e \in H : \lim_{z \to 0, z \in \mathbb{C} \setminus \Theta(\alpha)} |(Q(z)e, e)_H| < \infty \right\}. \quad (3.3)
$$
Then
$$
\gamma(\mu)D_0 = \mathcal{M}_\mu \cap D[S_N].
$$
for all $\mu \in \mathbb{C} \setminus \Theta(\alpha)$ and
$$
D_0 = \Gamma D[S_N].
$$
Moreover, the following limits exist
$$
\Omega_0[e, g] := -\lim_{z \to 0, z \in \mathbb{C} \setminus \Theta(\beta)} (Q(z)e, g), e, g \in D_0,
$$
$$
X_0e := -\lim_{z \to 0, z \in \mathbb{C} \setminus \Theta(\beta)} G^*(\bar{z})e, e \in D_0, \beta \in (\alpha, \pi/2),
$$
and
\[ \Omega_0[e, g] = i (\gamma(i)e, \gamma(i)g) + \left( (I - iG_F)^{-1} \hat{S}_{FR}^{-1/2} \gamma(i)e, \hat{S}_{FR}^{-1/2} \gamma(i)g \right) = i (\gamma(i)e, \gamma(i)g) + S_{FR}^{-1} [\gamma(i)e, \gamma(i)g], \ e, g \in D_0, \]
\[ X_0e = i (I - iG_F)^{-1} \hat{S}_{FR}^{-1/2} \gamma(i)e, \ e \in D_0. \]

**Proof.** Let \( e \in \mathcal{H} \). Then using (2.4) and (2.5) we have for \( z \in \mathbb{C} \setminus \Theta(\alpha) \)
\[ (Q(z)e, e)_{\mathcal{H}} = (z - i)(\gamma(z)e, \gamma(i)e) \]
\[ = (z - i)(\gamma(i)e + (z - i)((S_F^* - zI)^{-1} \gamma(i)e, \gamma(i)e) \]
Hence
\[ ((S_F^* - zI)^{-1} \gamma(i)e, \gamma(i)e) = -\frac{1}{z - i} \gamma(i)e, \gamma(i)e) + \frac{1}{(z - i)^2} (Q(z)e, e)_{\mathcal{H}}. \]

The latter equality and (1.2) yields
\[ \lim_{z \to 0, \ z \in \mathbb{C} \setminus \Theta(\alpha)} |(Q(z)e, e)_{\mathcal{H}}| < \infty \]
\[ \iff \lim_{z \to 0, \ z \in \mathbb{C} \setminus \Theta(\alpha)} \left| (S_F^* - zI)^{-1} \gamma(i)e, \gamma(i)e \right| < \infty \]
\[ \iff \gamma(i)e \in R[S_F] \cap \mathcal{M}_k. \]

Let \( D_0 \) be defined by (3.3). Then, using (1.3), (1.10), and Corollary 3.2 one obtains
\[ e \in D_0 \iff \gamma(i)e \in \mathcal{M}_k \cap D[S_N]. \]
Hence \( \gamma(\mu)D_0 = \mathcal{M}_k \cap D[S_N] \) for all \( \mu \in \mathbb{C} \setminus \Theta(\alpha) \). Observe that \( D_0 \) is a linear manifold. Equality (1.9) yields that \( \Gamma D[S_N] = D_0 \).

Notice that the equality
\[ \gamma(z) = \gamma(i) + (z - i)(S_F^* - zI)^{-1} \gamma(i), \]
the inclusion \( \gamma(i)D_0 \subseteq \text{ran}(S_F^*), \) and applying Proposition 3.2 leads to
\[ \lim_{z \to 0, \ z \in \mathbb{C} \setminus \Theta(\beta)} z \gamma(z)e = 0, \ e \in D_0 \]
for \( \beta \in (\alpha, \pi/2) \). Applying equality (1.3), we get the rest equalities in Theorem. \( \square \)

Clearly the form \( \Omega_0[e, g] \) can also be rewritten as follows:
\[ \Omega_0[e, g] = i (\gamma(i)e, \gamma(i)g) - i \left( X_0e, \hat{S}_{FR}^{-1/2} \gamma(i)g \right), \ e, g \in D_0. \]

Using expressions for \( \Omega_0 \) and \( X_0 \), by straightforward calculations one can deduce that
\[ \text{Re} \Omega_0[e] = \|(I + iG_F)^{-1} \hat{S}_{FR}^{-1/2} \gamma(i)e\|^2 = \|X_0e\|^2, \ e \in D_0. \]

(3.4)

It follows that the sesquilinear form \( \Omega_0[e, g] \) is accretive, and, moreover, the form
\[ \text{Re} \Omega_0 \] is closed in the Hilbert space \( \mathcal{H} \). Observe that the form
\[ t_0[e, g] := \Omega_0[e, g] - i(\gamma(i)e, \gamma(i)g) = \left( (I - iG_F)^{-1} \hat{S}_{FR}^{-1/2} \gamma(i)e, \hat{S}_{FR}^{-1/2} \gamma(i)g \right) \]
is closed and sectorial in $H$. Let the linear relation $\mathcal{Z}_0$ be associated with $t_0$ by the First Representation Theorem (see [39] for nondensely defined closed sectorial forms). Then define

$$Z_0 = \mathcal{Z}_0 + iP_{\mathcal{Z}_0}\gamma^*(i)\gamma(i),$$

where $P_{\mathcal{Z}_0}$ is the orthogonal projection in $H$ onto the subspace $\overline{\mathcal{D}}_0$. The linear relation $Z_0$ is $m$-accretive and associated with the form $\Omega_0$ in the sense

$$(Z_0e, g)_H = \Omega_0[e, g] \text{ for all } e \in \text{dom}(Z_0) \text{ and all } g \in \mathcal{D}_0.$$

**Theorem 3.4.** Let $\{\mathcal{H}, \Gamma\}$ be a boundary pair of $S$. Then the pair $\langle Z_0, X_0 \rangle$ corresponds to the Krein-von Neumann extension $S_N$ of the operator $S$ in accordance with Theorem [2,4].

**Proof.** It follows from (2.3) and from Theorem 3.3 that

$$S_N[u, v] = l[u, v] + \Omega_0[\Gamma u, \Gamma v] + 2(X_0\Gamma u, S_{FR}^{1/2}P_{FR}v), \ u, v \in D[S_N]. \tag{3.5}$$

Let the pair $(Z_N, X_N)$ corresponds to $S_N$ in accordance with Theorem [2,4] $\text{dom}(Z_N) = \text{dom}(S_N)$, $\text{dom}(X_N) = \Gamma \text{dom}(S_N)$. Then

$$(S_Nu, v) = l[u, v] + (Z_Nu, \Gamma v)_{\mathcal{H}} + 2(X_N\Gamma u, S_{FR}^{1/2}P_{FR}v), \ u \in \text{dom}(S_N), \ v \in \mathcal{L}. \tag{3.6}$$

Then (3.5) and (3.6) imply for $v \in D[S]$ that

$$(X_0\Gamma u, S_{FR}^{1/2}_Rv) = (X_N\Gamma u, S_{FR}^{1/2}_Rv).$$

Hence $X_N = X_0| \Gamma \text{dom}(S_N)$. Further

$$\Omega_0[\Gamma u, \Gamma v] = (Z_Nu, \Gamma v)_{\mathcal{H}}, \ u \in \text{dom}(S_N), \ v \in D[S_N].$$

Therefore, $m$-accretive linear relation

$$Z_N = \{\{\Gamma u, Z_Nu\}, \ u \in \text{dom}(S_N)\}$$

is associated with the form $\Omega_0$. It follows the equality

$$Z_N = Z_0.$$

**Remark 3.5.** If the set $\mathcal{D}_0$ in Theorem 3.3 is trivial, then the operator $S$ admits a unique $m$-sectorial extension, namely the Friedrichs extension $S_F$.

Let

$$S_N[u, v] = \left((I + iG_N)S_{NR}^{1/2}u, S_{NR}^{1/2}v\right), \ u, v \in D[S_N].$$

Since $S_N[u, v] = S_F[u, v]$, for all $u, v \in D[S]$, there exists an isometry $U_F$ mapping $\text{ran}(S_F)$ onto $\text{ran}(S_N)$ such that (see [7,8])

$$S_{NR}^{1/2}u = U_FS_{FR}^{1/2}u, \ u \in D[S],$$

$$G_NU_F = U_FG_F,$$

$$S_{NR}^{1/2}\varphi_\mu = \mu U_(I - iG_F)^{-1} \tilde{S}_{FR}^{-1/2} \varphi_\mu, \ \varphi_\mu \in \mathcal{M}_\mu \cap D[S_N].$$

It follows that

$$S_{NR}^{1/2}u = U_FS_{FR}^{1/2}P_{FR}u + U_FX_0\Gamma u, \tag{3.7}$$
Description of all closed sesquilinear forms associated with \( m \)-sectorial extensions of operator \( S \) in the terms of boundary pair has been obtained in [7].

**Definition 3.6 ([7]).** A pair \( \{ \mathcal{H}', \Gamma' \} \) is called boundary pair of the operator \( S \), if \( \mathcal{H}' \) is a Hilbert space, and \( \Gamma' : D[S_N] \to \mathcal{H}' \) is a linear operator such that \( \ker(\Gamma') = D[S] \), \( \text{ran}(\Gamma') = \mathcal{H}' \).

Since \( D[S] \) is a subspace in \( D[S_N] \), the boundary pairs \( \{ \mathcal{H}', \Gamma' \} \) for operator \( S \) exist.

**Theorem 3.7 ([7, 11]).** Let \( \{ \mathcal{H}', \Gamma' \} \) be a boundary pair of the operator \( S \) in the sense of Definition 3.6. Then the formula

\[
\tilde{S}[u,v] = S_N[u,v] + \omega'[\Gamma'u, \Gamma'v] + 2(X\Gamma'u, S_{N_H}v),
\]

\( u, v \in D[\tilde{S}] = \Gamma'^{-1}D[\omega'] \) (3.8)

establish a bijective correspondence between all closed forms associated with \( m \)-sectorial extensions \( \tilde{S} \) of \( S \) and all pairs \( \langle \omega', X' \rangle \), where

1) \( \omega' \) is a closed and sectorial sesquilinear in the Hilbert space \( \mathcal{H}' \);
2) \( X' : \text{dom}(\omega') \to \text{ran}(S) \) is a linear operator, such that for some \( \delta \in [0, 1) \):

\[
\|X'e\|^2 \leq \delta^2 \text{Re} \omega'[e],
\]

for all \( e \in \text{dom}(\omega') \).

Let \( \{ \mathcal{H}, \Gamma \} \) be a boundary pair of the operator \( S \) in the sense of Definition 2.1. Set

\[
\mathcal{H}' = D_0 (= \text{dom}(\Omega_0)),
\]

\[
(e, g)_{\mathcal{H}'} = (e, g)_{\mathcal{H}} + \text{Re} \Omega_0[e, g] = (e, g)_{\mathcal{H}} + (X_0e, X_0g),
\]

\[
\Gamma' = \Gamma \upharpoonright D[S_N] = \Gamma \upharpoonright (D[S] + \gamma(i)D_0).
\]

Then \( \mathcal{H}' \) is a Hilbert space w.r.t. the inner product \( (\cdot, \cdot)_{\mathcal{H}'} \) and \( \{ \mathcal{H}', \Gamma' \} \) is boundary pair of the operator \( S \) in the sense of Definition 3.6. Note that

1) the operators \( X_0 \) and \( \gamma(\lambda) \) are continuous from \( \mathcal{H}' \) into \( \mathcal{H} \),
2) the sesquilinear form \( \Omega_0 \) is continuous in \( \mathcal{H}' \).

Further, using Theorem 2.4 and representation (3.5) for the form \( S_N[u, v] \), we are going to established additional conditions on the pairs \( \langle \mathcal{Z}, X \rangle \) that determine \( m \)-sectorial extensions of the operator \( S \) in accordance with Theorem 2.5.

**Theorem 3.8.** Let \( \{ \mathcal{H}, \Gamma \} \) be a boundary pair of \( S \). Then the pair \( \langle \mathcal{Z}, X \rangle \) determines an \( m \)-sectorial extension \( \tilde{S} \) of \( S \), see Theorem 2.7 and Remark 2.6, if and only if the following conditions are fulfilled:

1) \( \text{dom}(\mathcal{Z}) \subseteq D_0; \)
2) the sesquilinear form

\[
\omega[e, g] = (Z\mathcal{Z}, g)_{\mathcal{H}} - \Omega_0[e, g] - 2((X - X_0)e, X_0g)
\]

\[
= (Z\mathcal{Z}, g)_{\mathcal{H}} + \Omega_0'[e, g] - 2(Xe, X_0g),
\]

\( e, g \in \text{dom}(\mathcal{Z}) = \Gamma \text{dom}(\tilde{S}) \) (3.10)

is sectorial and admits a closure in the Hilbert space \( \mathcal{H}' \);
3) \[ \|(X - X_0)e\|^2 \leq \delta^2 \text{Re} \omega[e], \ e \in \text{dom}(\mathcal{Z}) \] for some \( \delta \in [0, 1) \).

Moreover, the closed sesquilinear form associated with \( S \) is given by

\[ \tilde{S}[u, v] = l[u, v] + Z[\Gamma u, \Gamma v] + 2(\overline{X} \Gamma u, S_{FR}^{1/2} P_F v), \]

\[ u, v \in D[\tilde{S}] = \Gamma^{-1} \text{dom}(\overline{\omega}), \]

where \( \overline{X} \) is continuous extension of \( X \) on the domain \( \text{dom}(\overline{\omega}) \) of the closure \( \overline{\omega} \) of \( \omega \)

and

\[ Z[e, g] := \overline{\omega}[e, g] - \overline{\Omega}_0[e, g] + 2(\overline{X} e, X_0 g), \ e, g, \in \text{dom}(\overline{\omega}). \] (3.12)

**Proof.** Let \( \tilde{S} \) be an \( m \)-sectorial extension of \( S \) determined by the pair \( \langle Z, X \rangle \) in accordance with Theorem 2.4. Note, that since \( \tilde{S} \) is \( m \)-sectorial extension of \( S \), we have (see (1.10)) \( \text{dom}(\tilde{S}) \subset D[\tilde{S}] \subset D[S_N] \), and \( \Gamma \text{ dom}(\tilde{S}) \) is a core of the linear manifold \( \Gamma D[\tilde{S}] \). Then

\[ (\tilde{S} u, v) = l[u, v] + (Z \Gamma u, \Gamma v)_H + 2(X \Gamma u, S_{FR}^{1/2} P_F v), \ u, v \in \text{dom}(\tilde{S}). \]

Using (3.5), one obtains:

\[ (\tilde{S} u, v) = S_N[u, v] + (Z \Gamma u, \Gamma v)_H - \overline{\Omega}_0[\Gamma u, \Gamma v] + 2((X - X_0) \Gamma u, S_{FR}^{1/2} P_F v), \ u, v \in \text{dom}(\tilde{S}). \]

From (3.7) \( S_{FR}^{1/2} P_F v = U_F S_{NR}^{1/2} - X_0 \Gamma v \). Hence,

\[ (\tilde{S} u, v) = S_N[u, v] + (Z \Gamma u, \Gamma v)_H - \overline{\Omega}_0[\Gamma u, \Gamma v] - 2((X - X_0) \Gamma u, X_0 \Gamma v) + 2(U_F (X - X_0) \Gamma u, S_{NR}^{1/2} v) \]

\[ = S_N[u, v] + \omega[\Gamma u, \Gamma v] + 2(\overline{X} \Gamma u, S_{NR}^{1/2} v), \ u, v \in \text{dom}(\tilde{S}), \]

where \( \omega \) is given by (3.10) and \( \overline{X} = U_F (X - X_0) \). From Theorem 3.7 it follows that \( \omega \) is sectorial form, \( \text{dom}(\omega) = \text{dom}(Z) \subset D_0 = \mathcal{H}' \) and

\[ ||\overline{X} e||^2 = ||(X - X_0) e||^2 \leq \delta^2 \text{Re} \omega[e] \]

for all \( e \in \text{dom}(\mathcal{Z}) \), where \( \delta \in [0, 1) \). Moreover, the form \( \omega \) admits closure \( \overline{\omega} \) in the Hilbert space \( \mathcal{H}' \), and \( \overline{X} \) can be extended on \( \text{dom}(\overline{\omega}) \) by continuity as a linear operator from \( \text{dom}(\overline{\omega}) \) with the inner product

\[ (e, g)_{\overline{\omega}} = (e, g)_{H'} + \text{Re} \overline{\omega}[e, g]. \]

Since \( X_0 \) is continuous from \( \mathcal{H}' \) into \( \mathcal{Y} \), the operator \( X \) admits a continuation \( \overline{X} \) on \( \text{dom}(\overline{\omega}) \). It follows that the form \( Z \) given by (3.12) is well defined and the closed form \( \tilde{S}[u, v] \) associated with \( \tilde{S} \) is of the form (3.11).

Conversely, let conditions (1)–(3) of the theorem be fulfilled. Denote by \( \overline{\omega} \) the closure in the Hilbert space \( \mathcal{H}' \) of the sesquilinear form \( \omega \) given by (3.10), and by \( \overline{X} \) the continuation of the operator \( \overline{X} = U_F (X - X_0) \) on \( \text{dom}(\overline{\omega}) \), which exists due condition (2). Then, by Theorem 3.7 the pair \( (\overline{\omega}, \overline{X}) \) determines by (3.5) a closed
sectorial form $\tilde{S}[u,v]$ associated with some $m$-sectorial extension $\tilde{S}$ of the operator $S$. □

**Remark 3.9.** We can rewrite condition (3) of Theorem 3.8 in slightly different form. Let us find the real part of the form $\omega[e,e]$. We have:

$$\omega[e,e] = (Ze,e)_H - \Omega_0[e,e] - 2((X - X_0)e, X_0e).$$

Using (3.4), we obtain:

$$\text{Re}\,\omega[e,e] = (Ze,e)_H - \|X_0e\|^2 + 2\|X_0e\|^2 - 2\text{Re}(Xe, X_0e) = \text{Re}(Ze,e)_H + \|X_0e\|^2 - 2\text{Re}(Xe, X_0e) = \text{Re}(Ze,e)_H + \|(X - X_0)e\|^2 - \|Xe\|^2.$$ 

Then the inequalities

$$\|(X - X_0)e\|^2 \leq \delta^2\text{Re}\,\omega[e,e] = \delta^2 \left(\text{Re}(Ze,e)_H + \|(X - X_0)e\|^2 - \|Xe\|^2\right)$$

and $0 \leq \delta < 1$ imply

$$M\|(X - X_0)e\|^2 \leq \text{Re}(Ze,e)_H - \|Xe\|^2,$$

where $M = \frac{1 - \delta^2}{\delta^2} > 0$.

Thus, condition 3 can be rewritten as

$$\text{Re}(Ze,e)_H - \|Xe\|^2 \geq M\|(X - X_0)e\|^2, \quad M > 0.$$

### 4. Nonnegative Symmetric Operator and Its Quasi-Selfadjoint $m$-Accretive Extensions

In this section we will consider a densely defined closed nonnegative symmetric operator $A$ and parameterize all its quasi-selfadjoint $m$-accretive extensions in terms of abstract boundary conditions. We will use a boundary pair and boundary triplets defined in Definitions 2.1, 2.2, and 2.3. In this case if $\{H, \Gamma\}$ is the boundary pair for $A$ in the sense of Definition 2.1 then the sesquilinear form $\Omega_0$ and the linear operator $X_0$ defined on the linear manifold $D_0 = \Gamma D[A_N]$ (see Theorem 3.3) are of the form

$$\Omega_0[e,g] = i (\gamma(i)e, \gamma(i)g) + \left(\hat{A}_F^{-1/2} \gamma(i)e, \hat{A}_F^{-1/2} \gamma(i)g\right)$$

$$X_0e = i\hat{A}_F^{-1/2} \gamma(i)e, \quad e, g \in D_0.$$

In addition, from (1.18) it follows that

$$A_N[u,v] = \left(\hat{A}_F^{1/2} \mathcal{P}_{z,F}u + z \hat{A}_F^{-1/2} \mathcal{P}_{z}u, \hat{A}_F^{1/2} \mathcal{P}_{z,F}v + z \hat{A}_F^{-1/2} \mathcal{P}_{z}v\right)$$

$$= \left(\hat{A}_F^{1/2} (u - \gamma(z)\Gamma u) + z \hat{A}_F^{-1/2} \gamma(z)\Gamma u), \hat{A}_F^{1/2} (v - \gamma(z)\Gamma v) + z \hat{A}_F^{-1/2} \gamma(z)\Gamma v)\right),$$

$$u, v \in D[A_N] = D[A_F] + (\mathcal{N}_z \cap \text{ran}(A_F^{1/2})) = D[A_F] + \gamma(z)D_0. \quad (4.1)$$
It is established in [6] (see also [14]) that the following assertions are equivalent for \(m\)-accretive extension \(\tilde{A}\) of \(A\):

(i) \(A\) is quasi-selfadjoint extension;

(ii) \(\text{dom}(\tilde{A}) \subseteq \text{D}[A_N]\) and \(\text{Re}\ (\tilde{A}f, f) \geq A_N[f]\) for all \(f \in \text{dom}(\tilde{A})\).

Observe that the operator \(L\) defined in (2.6) is of the form

\[\text{dom}(L) = \text{dom}(A^*), \quad Lu = A^*u - 2iu_i,\]

where \(u = u_F + u_i, \ u_F \in \text{dom}(A_F), \ u_i \in \mathfrak{N}_i\). If \(\{\mathcal{H}, \Gamma\}\) is a boundary pair for \(A\) (see Definition 2.1), then

\[Lu = A^*u - 2i\gamma(i)\Gamma u, \ u \in \text{dom}(A^*).\]

**Proposition 4.1.** Let \(A\) be a closed densely defined nonnegative symmetric operator in \(\mathfrak{S}\) and let \(\{\mathcal{H}, \Gamma\}\) be its boundary pair (in the sense of Definition 2.1). Assume \(\mathcal{D}_0 \neq \{0\}\). Then a pair \(\langle Z, X \rangle\) determines a quasi-selfadjoint \(m\)-accretive extension \(\tilde{A}\) of \(A\) in accordance with Theorem 2.5 if and only if the following conditions hold true

1) \(\text{dom}(Z) \subseteq \mathcal{D}_0,\)

2) \(X = X_0|_{\text{dom}(Z)} = i\tilde{A}_F^{-1/2}\gamma(i)|_{\text{dom}(Z)}\).

**Proof.** Let \(\tilde{A}\) be a quasi-selfadjoint \(m\)-accretive extension of the operator \(A\). Then \(\text{dom}(\tilde{A}) \subseteq \text{D}[A_N]\). By Theorem 2.4 this implies the inclusion \(\text{dom}(Z) \subseteq \Gamma \text{D}[A_N] = \mathcal{D}_0\). Taking into account the decomposition \(\text{dom}(A^*) = \text{dom}(A_F) + \mathfrak{N}_i\), from (2.18) for \(\text{dom}(A) \ni u = u_F + u_i, \ u_F \in \text{dom}(A_F), \ u_i \in \mathfrak{N}_i\) we have

\[X\Gamma u = M u = \frac{1}{2} \left( \tilde{A}_F^{-1/2}(\tilde{A}u + iP_iu) - (I + iG_F)A_F^{1/2}P_Fu \right) = \frac{1}{2} \left( \tilde{A}_F^{-1/2}(A^*u + iu_i) - A_F^{1/2}u_F \right) = \frac{1}{2} \left( \tilde{A}_F^{-1/2}(A_Fu_F + 2iu_i) - A_F^{1/2}u_F \right) = i\tilde{A}_F^{-1/2}\gamma(i)\Gamma u = X_0\Gamma u.\]

Now consider a pair \(\langle Z, X \rangle\), where \(Z\) is \(m\)-accretive linear relation in \(\mathcal{H}\) such that

(a) \(\text{dom}(Z) \subseteq \mathcal{D}_0\) and (b) \(\text{Re}\ (Ze, e)_{\mathcal{H}} \geq ||X_0e||^2\) for all \(e \in \text{dom}(Z)\). This pair determines an \(m\)-accretive extension \(\tilde{A}\). Let us prove that \(\tilde{A} \subseteq A^*\). Note that for all \(u \in \mathfrak{L}, v \in \mathfrak{S}\)

\[(\Phi(\lambda)X_0\Gamma u, v) = i \left( \tilde{A}_F^{1/2}\gamma(i)\Gamma u, A_F^{1/2}(A_F - \lambda I)^{-1}v \right) = i \left( (A_F - \lambda I)^{-1}\gamma(i)\Gamma u, v \right).\]

So,

\[\Phi(\lambda)X_0\Gamma u = i(A_F - \lambda I)^{-1}\gamma(i)\Gamma u \subset \text{dom}(A_F).\]

Using (4.2) one gets

\[q(\lambda) - 2\Phi(\lambda)X_0 = \gamma(i) + (\lambda + i)(A_F - \lambda I)^{-1}\gamma(i) - 2i(A_F - \lambda I)^{-1}\gamma(i) = \gamma(i) + (\lambda - i)(A_F - \lambda I)^{-1}\gamma(i) = \gamma(\lambda).\]
From boundary conditions (2.10) for \( u \in \mathfrak{L} \) we have:

\[
u \in \text{dom}(\tilde{A}) \Rightarrow u - (q(\lambda) - 2\Phi(\lambda)X_0)\Gamma u \in \text{dom}(A_F) \Rightarrow u - \gamma(\lambda)\Gamma u \in \text{dom}(A_F),\]

and, therefore, \( u \in \text{dom}(A_F) + \mathfrak{N} = \text{dom}(A^*) \). Further, for \( u = \mathcal{P}_{A,F}u + \mathcal{P}_\lambda u \)

\[
\tilde{A}u = A_F(u - (q(\lambda) - 2\Phi(\lambda)X_0)\Gamma u) + \lambda(q(\lambda) - 2\Phi(\lambda)X_0)\Gamma u = \\
= A_F(u - \gamma(\lambda)\Gamma u) + \lambda \gamma(\lambda)\Gamma u = \\
= A_F\mathcal{P}_{A,F}u + \lambda \mathcal{P}_\lambda u = A^*(\mathcal{P}_{A,F}u + \mathcal{P}_\lambda u).
\]

So, \( \tilde{A} \subseteq A^* \).

\[\square\]

**Theorem 4.2.** Let \( \{\mathcal{H}, \Gamma\} \) and \( \{\mathcal{H}, G_\gamma, \Gamma\} \) be a boundary pair for \( A \) and the corresponding boundary triplet for \( L \), see Definition [2.3]. Assume \( \mathcal{D}_0 \neq \{0\} \). Then there is a bijective correspondence between all \( m \)-accretive quasi-selfadjoint extensions \( \tilde{A} \) of \( A \) and all \( m \)-accretive linear relations \( Z \) in \( \mathcal{H} \) such that \( \text{dom}(Z) \subseteq \mathcal{D}_0 \) and:

\[
\text{Re}(Ze, e) \geq \|\tilde{A}^{1/2} \gamma(i)e\|^2, \quad \forall e \in \text{dom}(Z).
\]

This correspondence is given by

\[
\text{dom}(\tilde{A}) = \{u \in \text{dom}(A^*): G_\gamma u \in (Z - 2i \gamma^*(i)\gamma(i))\Gamma u\},
\]

(4.4)

Moreover,

1) a number \( \lambda \in \rho(A_F) \) is a regular point of \( \tilde{A} \) if and only if

\[
\left( Z - \frac{\lambda + i}{\lambda - i} Q(\lambda) \right)^{-1} \in \mathcal{L}(\mathcal{H}),
\]

and,

\[
(\tilde{A} - \lambda I)^{-1} = (A_F - \lambda I)^{-1} + \gamma(\lambda) \left( Z - \frac{\lambda + i}{\lambda - i} Q(\lambda) \right)^{-1} \gamma^*(\lambda);
\]

(4.5)

2) a number \( \lambda \in \rho(A_F) \) is an eigenvalue of \( \tilde{A} \) if and only if

\[
\ker \left( Z - \frac{\lambda + i}{\lambda - i} Q(\lambda) \right) \neq \{0\},
\]

and,

\[
\ker(\tilde{A} - \lambda I) = \gamma(\lambda) \ker \left( Z - \frac{\lambda + i}{\lambda - i} Q(\lambda) \right).
\]

**Proof.** We will use (2.10). Due to (2.3), the boundary condition 1) in (2.10) is fulfilled. Let us transform boundary condition 2). Due to (2.7) we have for \( \lambda \in \rho(A_F) \)

\[
G_\gamma(f + q(\lambda)e) = \gamma^*(\lambda)(A_F - \lambda I)f + \mathcal{Q}^*(\lambda)e, \quad f \in \text{dom}(A_F), \quad e \in \text{dom}(A_F).
\]

So, we have

\[
G_\gamma(u + 2\Phi(\lambda)X\Gamma u) = G_\gamma(u + (q(\lambda) - \gamma(\lambda))\Gamma u) =
\]

...
Further, using that

\[ = G_s(u + 2i(A_F - \lambda I)^{-1}\gamma(i)\Gamma u) = \]

\[ = G_s u + 2i\gamma^*(\tilde{\lambda})\gamma(i)\Gamma u = \]

\[ = \gamma^*(\tilde{\lambda})(A_F - \lambda I)\mathcal{P}_{A_F}u + Q^*(\tilde{\lambda})\Gamma u. \]

On the other hand,

\[ \textbf{W}(\lambda) = \textbf{Z} - Q^*(\tilde{\lambda}) + 2\mathcal{G}(\lambda)X_0 \]

\[ = \textbf{Z} - (\lambda + i)\gamma^*(\tilde{\lambda})\gamma(i) + 2(\lambda + i)\gamma^*(\tilde{\lambda})\Phi(\lambda)X_0 \]

\[ = \textbf{Z} - (\lambda + i)(\gamma^*(i) + (\lambda + i)\gamma^*(i)(A_F - \lambda I)^{-1})\gamma(i) - 2(\lambda + i)i\gamma^*(i)(A_F - \lambda I)^{-1}\gamma(i) \]

\[ = \textbf{Z} - (\lambda + i)\gamma^*(i)(I + (\lambda + i)(A_F - \lambda I)^{-1} - 2i(A_F - \lambda I)^{-1})\gamma(i) \]

\[ = \textbf{Z} - (\lambda + i)\gamma^*(i)\gamma(\lambda) = \textbf{Z} - \frac{\lambda + i}{\lambda - i}Q(\lambda). \]

Then

\[ \textbf{Z} + 2\mathcal{G}(\lambda)X_0 = \textbf{Z} + Q^*(\tilde{\lambda}) - \frac{\lambda + i}{\lambda - i}Q(\lambda). \]

So, for the boundary condition 2) from from (2.10) one has

\[ G_s u + 2i\gamma^*(\tilde{\lambda})\gamma(i)\Gamma u \in \left( \textbf{Z} + Q^*(\tilde{\lambda}) - \frac{\lambda + i}{\lambda - i}Q(\lambda) \right)\Gamma u \]

\[ \iff G_s u \in \left( \textbf{Z} + Q^*(\tilde{\lambda}) - \frac{\lambda + i}{\lambda - i}Q(\lambda) - 2i\gamma^*(\tilde{\lambda})\gamma(i) \right)\Gamma u \]

\[ \iff G_s u \in \left( \textbf{Z} + (\lambda + i)\gamma^*(\tilde{\lambda})\gamma(i) - \frac{\lambda + i}{\lambda - i}Q(\lambda) - 2i\gamma^*(\tilde{\lambda})\gamma(i) \right)\Gamma u \]

\[ \iff G_s u \in \left( \textbf{Z} + \frac{\lambda - i}{\lambda + i}Q^*(\tilde{\lambda}) - \frac{\lambda + i}{\lambda - i}Q(\lambda) \right)\Gamma u. \]

Further, using that \( Q(\lambda) = (\lambda - i)\gamma^*(i)\gamma(i) \), we get

\[ (\textbf{Z} + (\lambda - i)\gamma^*(\tilde{\lambda})\gamma(i) - (\lambda + i)\gamma^*(i)\gamma(\lambda))\Gamma u \]

\[ = \left( \textbf{Z} + (\lambda - i)(\gamma^*(i) + (\lambda + i)\gamma^*(i)(A_F - \lambda I)^{-1})\gamma(i) \right. \]

\[ - (\lambda + i)\gamma^*(i)(\gamma(i) + (\lambda - i)(A_F - \lambda I)^{-1}\gamma(i)))\Gamma u \]

\[ = \left( \textbf{Z} + \gamma^*(i)(\lambda - i)I + (\lambda^2 + 1)(A_F - \lambda I)^{-1} \right. \]

\[ - ((\lambda + i)I + (\lambda^2 + 1)(A_F - \lambda I)^{-1})\gamma(i)\Gamma u \]

\[ = \left( \textbf{Z} - 2i\gamma^*(i)\gamma(i) \right)\Gamma u. \]

\[ \square \]

Remark 4.3. The boundary condition (4.4) also can be written for any \( \lambda \in \rho(\tilde{A}) \cap \rho(A_F) \) as

\[ \text{dom}(\tilde{A}) = \left\{ u \in \text{dom}(A^*) : \gamma^*(\tilde{\lambda})(A_F - \lambda I)(u - \gamma(\lambda)\Gamma u) \in \left( \textbf{Z} - \frac{\lambda + i}{\lambda - i}Q(\lambda) \right)\Gamma u \right\}, \]

and

\[ \tilde{A}u = A^*u = A_F(u - \gamma(\lambda)\Gamma u) + \lambda\gamma(\lambda)\Gamma u. \]
From Theorems 3.8, 4.2 we obtain

**Corollary 4.4.** Let \( Z \) be \( m \)-accretive linear relation, corresponding to a quasi-selfadjoint \( m \)-accretive extension \( \tilde{A} \) of \( A \) by the Theorem 4.2. Then extension \( \tilde{A} \) is a sectorial (nonnegative) if and only if

1) \( \text{dom}(Z) \subseteq \mathcal{H}'(= \text{dom}(\Omega_0) = \mathcal{D}_0) \);

2) the form \( \tilde{\omega}[e, g] = (Ze, g)_{\mathcal{H}} - \Omega_0[e, g] \) is sectorial (nonnegative).

**Remark 4.5.** The form \( \tilde{\omega} \) admits a closure in the Hilbert space \( \mathcal{H}' \) defined by (3.9).

Actually, since \( \tilde{\omega}[e, g] = (Ze, g)_{\mathcal{H}} - \Omega_0[e, g] \) is sectorial, the form \( \eta[e, f] = (Ze, g)_{\mathcal{H}} - i(\gamma(i)e, \gamma(i)f) \), \( e, f \in \mathcal{H}'(= \mathcal{D}_0) \), is sectorial as well. If

\[
\lim_{n \to \infty} e_n = 0 \quad \text{in} \quad \mathcal{H},
\]

\[
\lim_{m, n \to \infty} \tilde{\omega}[e_n - e_m] = 0,
\]

then

\[
\lim_{n \to \infty} e_n = 0 \quad \text{in} \quad \mathcal{H}, \quad \lim_{n \to \infty} \text{Re} \Omega[e_n] = \lim_{n \to \infty} ||X_0e_n||^2 = 0,
\]

\[
\lim_{n \to \infty} \gamma(i)e_n = 0 \quad \text{in} \quad \mathcal{F}.
\]

Since linear relation \( Z \) is \( m \)-accretive and \( Z - i\gamma^*(i)\gamma(i) \) is sectorial, we get \( \lim_{n \to \infty} (Ze_n, e_n)_{\mathcal{H}} = 0 \) (see [28]).

Next we will find relationships between

- a boundary triplet \( \{\mathcal{H}, \Gamma_1, \Gamma_0\} \) for \( A^* \) given by Definition 1.2 and boundary triplets \( \{\mathcal{H}, G, \Gamma\} \), \( \{\mathcal{H}, G^*, \Gamma\} \) of Definitions 2.2 and 2.3;

- parameterizations of quasi-selfadjoint \( m \)-accretive extensions given by Theorem 1.3 and Theorem 4.2.

Let \( \{\mathcal{H}, \Gamma_1, \Gamma_0\} \) be a boundary triplet of \( A^* \) (see Definition 1.2) such that \( \text{ker}(\Gamma_0) = \text{dom}(A_F) \). Then

1) since \( \text{dom}(A_F) \) is a core of \( D[A] \) and \( \text{ker}(\Gamma_0) = \text{dom}(A_F) \), we can define a boundary pair \( \{\mathcal{H}, \overline{\Gamma}_0\} \) where \( \overline{\Gamma}_0 \) is a continuation of \( \Gamma_0 \) onto \( \mathcal{L} = D[A] \oplus \mathfrak{N}_i \) from \( \text{dom}(A^*) = \text{dom}(A_F) \oplus \mathfrak{N}_i \);

2) it follows that

\[
\gamma(\lambda) = (\overline{\Gamma}_0 \mathfrak{N}_\lambda)^{-1} = \Gamma_0(\lambda);
\]

3) because relation (1.23) can be rewritten as

\[
M_0(\lambda) - M_0(z) = (\lambda - z)\gamma^*(\bar{z})\gamma(\lambda),
\]

using (2.25), one gets

\[
Q(\lambda) = (\lambda - i)\gamma^*(i)\gamma(\lambda) = \frac{\lambda - i}{\lambda + i} (M_0(\lambda) - M_0(-i));
\]

so,

\[
M_0(\lambda) - M_0(-i) = \frac{\lambda + i}{\lambda - i} Q(\lambda);
\]

(4.6)
4) equation (4.6) yields that the linear manifolds $D_0$ in Theorems 1.3 and Theorem 3.3 coincide and
\[
\tau[h, g] = (M_0(-i) h, g)_\mathcal{H} + \Omega_0[h, g], \quad h, g \in D_0;
\]
5) comparing resolvent formulas (1.24) and (4.5) we get that the linear relation
$Z$ from Theorem 1.2 and the linear relation $\tilde{T}$ from Theorem 1.3 (see (1.20), (1.25)) are connected by the equality
\[
Z = \tilde{T} - M(-i). \tag{4.7}
\]

Proposition 4.6. Let $\{\mathcal{H}, \Gamma\}$ be a boundary pair for nonnegative symmetric operator $A$. Let $\hat{A}$ be a quasi-selfadjoint $m$-accretive extension of $A$ and let $Z$ be the corresponding linear relation in $\mathcal{H}$ (see Theorem 1.2). Then
\[
Z^* + 2i\gamma^*(i)\gamma(i)
\]
corresponds to the adjoint extension $\hat{A}^*$.

Proof. The proof it easy, if we recall that to the adjoint extension $\hat{A}^*$ corresponds the adjoint linear relation $\tilde{T}^*$. Since
\[
\tilde{T} = Z + M(-i).
\]
Then
\[
\tilde{T}^* = Z^* + M^*(-i).
\]
Again, it follows from (4.7), equality $M^*(z) = M(\bar{z})$, and (1.23) that the adjoint extension $\hat{A}^*$ corresponds to
\[
\tilde{T}^* - M(-i) = Z^* + M^*(-i) - M(-i) = Z^* + 2i\gamma^*(i)\gamma(i). \quad \square
\]

5. $m$-SECTORIAL EXTENSIONS OF A SYMMETRIC OPERATOR IN THE MODEL OF TWO POINT INTERACTIONS ON A PLANE

Let $y_1, y_2 \in \mathbb{R}^2$. Consider in the Hilbert space $L_2(\mathbb{R}^2)$ the operator $A$ given by:
\[
\text{dom}(A) = \left\{ f(x) \in W_2^2(\mathbb{R}^2) : f(y_k) = f(y_2) = 0, \quad k = 1, 2 \right\}, \quad Af = -\Delta f, \tag{5.1}
\]
where $x = (x_1, x_2) \in \mathbb{R}^2$, $W_2^2(\mathbb{R}^2)$ is a Sobolev space, and
\[
\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}
\]
is Laplacian.

The operator $A$ is a densely defined closed nonnegative symmetric with defect indices $(2, 2)$ [2]. Such operators are basic in the models of point interactions [2]. In the case of one point the corresponding operator
\[
\text{dom}(A_y) = \left\{ f(x) \in W_2^2(\mathbb{R}^2) : f(y) = 0 \right\}, \quad A_y f = -\Delta f
\]
adopts a unique nonnegative selfadjoint extension $[1] [24]$, the free Hamiltonian:
\[
\text{dom}(A_F) = W_2^2(\mathbb{R}^2), \quad A_F f = -\Delta f,
\]
Therefore, $A_y$ has no $m$-sectorial and quasi-selfadjoint $m$-accretive extensions. All $m$-accretive extensions of $A_y$ have been described in [15]. For two and more point interactions the relation $A_F \neq A_N$ holds [1]. In this section we apply Theorems 2.5 and [3,8] for a parametrization of all $m$-sectorial extensions of the operator $A$. It is convenient to use the Fourier transform and the momentum representation of $A$:

$$
\hat{A}\hat{f}(p) = |p|^2\hat{f}(p),
$$

$$
dom(\hat{A}) = \left\{ \hat{f}(p) \in L_2(\mathbb{R}^2, dp) : \begin{align*}
1) |p|^2\hat{f}(p) &\in L_2(\mathbb{R}^2, dp), \\
2) \int_{\mathbb{R}^2} \hat{f}(p)e^{ipy} dp &= \int_{\mathbb{R}^2} \hat{f}(p)e^{ipy} dp = 0.
\end{align*} \right\}
$$

For a one-center point interaction this method has been used in [15]. In this paper we omit details in the momentum representation and present final results in the coordinate representation.

The Friedrichs extension of the operator $A$ is the free Hamiltonian $A_F$ and $A_F^{1/2} = (-\Delta)^{1/2}$ is a pseudodifferential operator of the form:

$$
dom(A_F^{1/2}) = D[A_F] = W_2^1(\mathbb{R}^2),
$$

$$
A_F^{1/2} f(x) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} |p| \exp(i(x - y)p) f(y) dy dp,
$$

where $W_k^2(\mathbb{R}^2), k = 1, 2$, are the Sobolev spaces. Note that, see [2], the resolvent is of the form

$$
(A_F - \lambda I)^{-1}f(x) = \frac{i}{4} \int_{\mathbb{R}^2} H^{(1)}_0(\sqrt{\lambda}|x - y|) f(y) dy, \quad f \in L_2(\mathbb{R}^2),
$$

$$
\lambda \in \mathbb{C} \setminus [0, +\infty), \quad \text{Im} \sqrt{\lambda} > 0,
$$

where $H^{(1)}_0(\cdot)$ denotes the Hankel function of first kind and order zero [36]. It is well known [2] that

$$
\mathcal{M}_\lambda = \left\{ \frac{\pi i}{2} \sum_{k=1}^2 H^{(1)}_0(\sqrt{\lambda}|x - y_k|) c_k, \ c_1, c_2 \in \mathbb{C} \right\},
$$

$$
\lambda \in \mathbb{C} \setminus [0, +\infty), \quad \text{Im} \sqrt{\lambda} > 0
$$

is the defect subspace of $A$, corresponding to $\lambda$. Therefore, for the linear manifold $\mathcal{L}$ defined by (1.17) we have

$$
\mathcal{L} = W_2^1(\mathbb{R}^2) + \mathcal{M}_\lambda
$$

$$
= \left\{ f(x) + \frac{\pi i}{2} \sum_{k=1}^2 H^{(1)}_0(\sqrt{\lambda}|x - y_k|) c_k, \ f \in W_2^1(\mathbb{R}^2), \ c_1, c_2 \in \mathbb{C} \right\},
$$

where $\lambda$ is a number from $\mathbb{C} \setminus [0, +\infty)$. Now, let $\mathcal{H} = \mathbb{C}^2$ and set

$$
\Gamma \left( f(x) + \frac{\pi i}{2} \sum_{k=1}^2 H^{(1)}_0(\sqrt{\lambda}|x - y_k|) c_k \right) = \tilde{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \in \mathbb{C}^2, \ f(x) \in W_2^1(\mathbb{R}^2).
Then from the equality \( H_0^{(1)}(\sqrt{\lambda}|x|) = H_0^{(2)}(\sqrt{\lambda}|x|) \) [36] it follows that
\[
\gamma(\lambda)c = \frac{\pi i}{2} \sum_{k=1}^{2} H_0^{(1)}(\sqrt{\lambda}|x-y_k|)c_k, \quad \vec{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \in \mathbb{C}^2,
\]
\[
\gamma^*(\bar{\lambda})h(x) = -\frac{\pi i}{2} \int_{\mathbb{R}^2} h(x)H_0^{(2)}(\sqrt{\lambda}|x-y_1|)dx.
\]

Set \( r = |y_1 - y_2|, \)
\[
H(\lambda, r) = H_0^{(1)}(\sqrt{\lambda}r) - H_0^{(1)}(e^{3\pi i/4}r).
\]

From (2.5), using unitarity of the Fourier transform, one can derive that the matrix \( Q(\lambda) \) in the standard basis is of the form:
\[
Q(\lambda) = \frac{\lambda - i}{\lambda + i} \begin{bmatrix} -\ln(\lambda i) & \pi iH(\lambda, r) \\ \pi iH(\lambda, r) & -\ln(\lambda i) \end{bmatrix}.
\]

Hence,
\[
Q^*(\bar{\lambda}) = \frac{\bar{\lambda} + i}{\bar{\lambda} - i} \begin{bmatrix} -\ln\left(\frac{\lambda}{i}\right) & -\pi iH(\lambda, r) \\ -\pi iH(\lambda, r) & -\ln\left(\frac{\lambda}{i}\right) \end{bmatrix}.
\]

Now we will find the subspace \( D_0 \) and the sesquilinear form \( \Omega_0[\cdot, \cdot] \) (see Theorem 3.3).

\[
(Q(\lambda)\vec{c}, \vec{d}) = \frac{\lambda - i}{\lambda + i} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}^* \begin{bmatrix} -\ln(\lambda i) & \pi iH(\lambda, r) \\ \pi iH(\lambda, r) & -\ln(\lambda i) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{\lambda - i}{\lambda + i} 
\]
\[
\left( (c_1 \bar{d}_1 + c_2 \bar{d}_2) \ln(\lambda i) + (c_2 \bar{d}_1 + c_1 \bar{d}_2)\pi i \left( H_0^{(1)}(\sqrt{\lambda}r) - H_0^{(1)}(e^{3\pi i/4}r) \right) \right).
\]

Taking into account the asymptotic behavior [36]
\[
H_0^{(1)}(\lambda) = 1 + \frac{2i}{\pi} \left( \ln\left(\frac{\lambda}{2}\right) + \gamma \right) + o(\lambda), \quad \lambda \to 0,
\]
where \( \gamma \) is Euler’s constant, we see that
\[
D_0 := \left\{ e \in H : \lim_{z \to 0, z \in \mathbb{C}\setminus(0, +\infty)} |(Q(z)e, e)_H| < \infty \right\} = \left\{ \begin{bmatrix} \zeta \\ -\bar{\zeta} \end{bmatrix} \in \mathbb{C}^2 : \zeta \in \mathbb{C} \right\}.
\]

Let
\[
\vec{c}_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.
\]

Then
\[
\Omega_0[\zeta \bar{c}_0, \eta \bar{c}_0] = -\zeta \eta \lim_{\lambda \to 0} (Q(\lambda) \bar{c}_0, \bar{c}_0)
\]
\[
= \pi \zeta \eta \lim_{\lambda \to 0} \left(-2 \ln(\lambda i) - 2\pi i (H_0^{(1)}(\sqrt{\lambda}r) - H_0^{(1)}(e^{3\pi i/4}r)) \right)
\]
\[
= 2\pi \zeta \eta \lim_{\lambda \to 0} \left(-\ln(\lambda i) - \pi i \left(1 + \frac{2i}{\pi} \left(\ln \left(\frac{\sqrt{\lambda}r}{2}\right) + \gamma\right) - H_0^{(1)}(e^{3\pi i/4}r)\right) \right)
\]
\[
= 2\pi \zeta \eta \lim_{\lambda \to 0} \left(-\ln(\lambda i) - \pi i + 2 \ln \left(\frac{\sqrt{\lambda}r}{2}\right) + 2\gamma + \pi i H_0^{(1)}(e^{3\pi i/4}r) \right)
\]
\[
= 4\pi \zeta \eta \left(\ln \frac{r}{2} - \frac{3\pi i}{4} + \gamma + \frac{\pi i}{2} H_0^{(1)}(e^{3\pi i/4}r) \right) = \omega_0 \cdot \zeta \eta,
\]
where
\[
\omega_0 = 4\pi \left(\ln \frac{r}{2} - \frac{3\pi i}{4} + \gamma + \frac{\pi i}{2} H_0^{(1)}(e^{3\pi i/4}r) \right).
\]

From the latter equality one can obtain that
\[
\text{Re} \Omega_0[\zeta \bar{c}_0, \eta \bar{c}_0] = \text{Re} \omega_0 \cdot \zeta \eta = 4\pi \left(\ln \frac{r}{2} + \gamma + \text{ker}(r)\right) \zeta \eta,
\]
where the functions \(\text{ker}(\cdot)\) and \(\text{kei}(\cdot)\) are Kelvin functions [36, p.268], i.e., the real and imaginary parts of the function \(\frac{\pi i}{2} H_0^{(1)}(e^{3\pi i/4} \cdot)\), respectively:
\[
\text{ker}(r) + i \text{kei}(r) = \frac{\pi i}{2} H_0^{(1)}(e^{3\pi i/4}r).
\]

For the operator-functions \(\Phi(\lambda), G(\lambda), Q^*(\bar{\lambda})\), and \(q(\lambda)\) on \(\mathcal{D}_0 = \text{dom}(\Omega_0)\) we have:
\[
\Phi(\lambda)X \begin{bmatrix} \zeta \\ -\zeta \end{bmatrix} = \frac{\zeta}{4\pi^2} \int \int \frac{|p|}{|p|^2 - \lambda} \exp (i(x - y)p)g(y)dydp,
\]
\[
G(\lambda)X \begin{bmatrix} \zeta \\ -\zeta \end{bmatrix} = -\frac{\pi i(\lambda + i)\zeta}{2} \left[ \int \int \Phi(\lambda)(f(x))H_0^{(2)}(e^{3\pi i/4}|x - y_1|) dx \right] + i \frac{\lambda + i}{\lambda - i} \pi \left[ -\ln \left(\frac{\lambda}{i}\right) + \pi i H(\lambda, |y_1 - y_2|) \right] \begin{bmatrix} \zeta \\ -\zeta \end{bmatrix},
\]
\[
Q^*(\bar{\lambda}) \begin{bmatrix} \zeta \\ -\zeta \end{bmatrix} = \frac{\lambda + i}{\lambda - i} \pi \left[ -\ln \left(\frac{\lambda}{i}\right) + \pi i H(\lambda, |y_1 - y_2|) \right] \begin{bmatrix} \zeta \\ -\zeta \end{bmatrix},
\]
\[
q(\lambda) \begin{bmatrix} \zeta \\ -\zeta \end{bmatrix} = \frac{\pi i}{2} \int \frac{1}{\lambda - \zeta} \left( (i + \lambda)(H_0^{(1)}(\sqrt{\lambda}|x - y_2|) - H_0^{(1)}(\sqrt{\lambda}|x - y_1|)) + 2i \left( H_0^{(1)}(e^{\pi i/4}|x - y_1|) - H_0^{(1)}(e^{\pi i/4}|x - y_2|) \right) \right).
\]

Now we find the operator \(X_0e = i\hat{A}_F^{-1/2} \gamma \gamma_0, e \in \mathcal{D}_0\) from Theorem 3.3. As was mentioned above it is convenient to use the momentum representation. Let \(\hat{\gamma}(\lambda) = \)
\( \mathcal{F}\gamma(\lambda) \), where

\[
\hat{f}(p) = (\mathcal{F}f)(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} f(x)e^{-ix\cdot p}dx, \quad p = (p_1, p_2).
\]
is the Fourier transform of \( f(x) \in L_2(\mathbb{R}^2, dx) \). Then

\[
\hat{\gamma}(\lambda)\hat{c} = \sum_{k=1}^{2} c_k \frac{e^{-ipy_k}}{|p|^2 - \lambda}, \quad \forall \hat{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \in \mathbb{C}^2.
\]

Hence,

\[
\hat{X}_0 \begin{bmatrix} \zeta \\ -\zeta \end{bmatrix} = \mathcal{F}X_0 \begin{bmatrix} \zeta \\ -\zeta \end{bmatrix} = i\hat{A}_F^{-1/2}\hat{\gamma}(i) \begin{bmatrix} \zeta \\ -\zeta \end{bmatrix} = \frac{i(e^{-ipy_1} - e^{-ipy_2})}{|p|(|p|^2 - i)} \zeta,
\]

So, \( \hat{X}_0 \begin{bmatrix} \zeta \\ -\zeta \end{bmatrix} = \hat{g}_0(p)\zeta \), where

\[
\hat{g}_0(p) = \frac{i(e^{-ipy_1} - e^{-ipy_2})}{|p|(|p|^2 - i)}.
\]

(5.2)

Getting back to the coordinate representation, we obtain, using [40], [26, p.671], that

\[
g_0(x) = \mathcal{F}^{-1}\hat{g}_0(p) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{i(e^{ip(x-y_1)} - e^{ip(x-y_2)})}{|p|(|p|^2 - i)} dp =
\]

\[
i \int_{0}^{+\infty} \frac{J_0(\rho|x-y_1|) - J_0(\rho|x-y_2|)}{\rho^2 - i} d\rho =
\]

\[
= \frac{\pi i}{2\sqrt{-i}} \left( I_0(\sqrt{-i}|x-y_1|) - L_0(\sqrt{-i}|x-y_1|) \right) - \frac{\pi i}{2\sqrt{-i}} \left( I_0(\sqrt{-i}|x-y_2|) - L_0(\sqrt{-i}|x-y_2|) \right) =
\]

\[
= -\frac{\pi}{2} e^{3\pi i/4} \left( M_0(e^{-\pi i/4}|x-y_1|) - M_0(e^{-\pi i/4}|x-y_2|) \right),
\]

where \( I_0(\cdot) \) is the Bessel function and \( L_0(\cdot), M_0(\cdot) \) are modified Struve functions [36, p.288]. So,

\[
X_0 \begin{bmatrix} \zeta \\ -\zeta \end{bmatrix} = g_0(x)\zeta,
\]

where

\[
g_0(x) = -\frac{\pi}{2} e^{3\pi i/4} \left( M_0(e^{-\pi i/4}|x-y_1|) - M_0(e^{-\pi i/4}|x-y_2|) \right).
\]

According to (3.4) we have

\[
\|g_0(x)\|_{L_2(\mathbb{R}^2)}^2 = \text{Re} \omega_0 = 4\pi \left( \ln \frac{r}{2} + \gamma + \text{ker}(r) \right).
\]

**Remark 5.1.** Since \( \|g_0(x)\|_{L_2(\mathbb{R}^2)}^2 = \|\hat{g}_0(p)\|_{L_2(\mathbb{R}^2)}^2 \) (the unitarity of the Fourier transform), expression (5.2) for \( \hat{g}_0(p) \) gives

\[
\|\hat{g}_0(p)\|_{L_2(\mathbb{R}^2)}^2 = 4\pi \int_{0}^{\infty} \frac{1 - J_0(\rho \rho)}{\rho(\rho^2 + 1)} d\rho.
\]
Thus, the form \( \omega \), since

\[
\langle \zeta, \eta, z \rangle = \left( \int \frac{1 - J_0(r \rho)}{\rho(\rho^2 + 1)} d\rho \right) = \frac{1}{4\pi} \Re \omega_0 = \left( \ln \frac{r}{2} + \gamma + \ker(r) \right).
\]

In order to describe all \( m \)-sectorial extensions of \( A \) we need to define pairs \( \langle Z, X \rangle \) satisfying conditions 3), 4) from Theorem 2.4 and conditions 1)–3) of Theorem 3.8.

Since \( Z \) is \( m \)-accretive linear relation in \( C^2 \) and \( \text{dom}(Z) \subseteq D_0 \), there are only two possible cases:

1) \( Z = \left\{ \zeta \right\}, \text{z} \cdot \zeta, z \in C, \Re z \geq 0; \)

2) \( Z = \{0, C^2\} \). As has been mentioned in [15] this linear relation corresponds to the Friedrichs extension \( A_F \) of \( A \).

In the first case the operator \( X \), acting from \( \text{dom}(Z) \) into \( L_2(\mathbb{R}^2) \), takes the form

\[
X \left[ \begin{array}{c} \zeta \\ \zeta \end{array} \right] = \zeta g(x), \text{where a function } g(x) \in L_2(\mathbb{R}^2) \text{ satisfies the condition}
\]

\[
\|g(x)\|_{L_2(\mathbb{R}^2)}^2 = \int_{\mathbb{R}^2} |g(x)|^2 dx \leq 2\Re z. (5.3)
\]

For the form \( \omega[\cdot, \cdot] \) defined by (3.10) we have

\[
\omega[\zeta \bar{c}_0, \eta \bar{c}_0] = (Z \zeta \bar{c}_0, \eta \bar{c}_0) - \Omega_0[\zeta \bar{c}_0, \eta \bar{c}_0] = 2((X - X_0)\zeta \bar{c}_0, X_0 \eta \bar{c}_0) - 2((X - X_0)\zeta \bar{c}_0, X_0 \eta \bar{c}_0)
\]

\[
= \left( 2z - \omega_0 - 2 \int_{\mathbb{R}^2} (g(x) - g_0(x)) \bar{g}_0(x) dx \right) \zeta \bar{\eta}. (5.4)
\]

\[
\Re \omega[\zeta \bar{c}_0] = \left( 2\Re z + \int_{\mathbb{R}^2} |g(x) - g_0(x)|^2 dx - \int_{\mathbb{R}^2} |g(x)|^2 dx \right) |\zeta|^2.
\]

Thus, the form \( \omega[\cdot, \cdot] \) is determined by the number

\[
w_{\{z, g(x)\}} = 2z - \omega_0 - 2 \int_{\mathbb{R}^2} (g(x) - g_0(x)) \bar{g}_0(x) dx. (5.5)
\]

Clearly, the form \( \omega[\cdot, \cdot] \) is sectorial iff

\[
\Re w_{\{z, g(x)\}} = 2\Re z + \int_{\mathbb{R}^2} |g(x) - g_0(x)|^2 dx - \int_{\mathbb{R}^2} |g(x)|^2 dx > 0 \quad \text{or } w_{\{z, g(x)\}} = 0. (5.6)
\]

Remark 5.2. Due to \( 2\Re z - \int_{\mathbb{R}^2} |g(x)|^2 dx \geq 0 \) the equality \( w_{\{z, g(x)\}} = 0 \) implies that \( g(x) = g_0(x) \) almost everywhere and \( z = \omega_0/2 \).

Further, condition 3) from Theorem 5 takes the form

\[
M \int_{\mathbb{R}^2} |g(x) - g_0(x)|^2 dx \leq 2\Re z - \int_{\mathbb{R}^2} |g(x)|^2 dx,
\]
where $M > 0$. The latter inequality can be simplified as follows

$$2\text{Re } z - \int_{\mathbb{R}^2} |g(x)|^2 dx > 0. \tag{5.7}$$

So, conditions (5.3), (5.6) are satisfied. Note, that in this case linear relation $W(\lambda)$, see (2.11), is of the form

$$W(\lambda) = \left\langle \begin{bmatrix} \zeta \\ -\zeta \end{bmatrix}, \left( z - \frac{\lambda + i}{\lambda - i} \pi \left( -\ln \left( \frac{\lambda}{i} \right) + \pi i H(\lambda, |y_1 - y_2|) \right) \right) \right\rangle \cdot \left\langle \begin{bmatrix} \zeta \\ -\zeta \end{bmatrix} \right\rangle$$

Clearly, $\ker(W(\lambda))$ for all $\lambda \in \rho(A_F) = \mathbb{C} \setminus [0, +\infty)$. Then

$$W^{-1}(\lambda) = \left\langle \begin{bmatrix} \zeta \\ \eta \end{bmatrix}, \frac{1}{w_{(z,g(x))}(\lambda)} \begin{bmatrix} \zeta - \eta \\ -\zeta + \eta \end{bmatrix} \right\rangle,$$

where

$$w_{(z,g(x))}(\lambda) = 2 \left( z - \frac{\lambda + i}{\lambda - i} \pi \left( -\ln \left( \frac{\lambda}{i} \right) + \pi i H(\lambda, |y_1 - y_2|) \right) \right)$$

$$- \pi i(\lambda + i) \int_{\mathbb{R}^2} \Phi(\lambda)(g(x)) \left( H_0^2(e^{3\pi i/4}|x - y_1|) - H_0^2(e^{3\pi i/4}|x - y_2|) \right) dx.$$

Clearly, $\ker(W(\lambda)) \neq \{0\}$ iff $w_{(z,g(x))}(\lambda) = 0$ and

$$\ker(W(\lambda)) = \text{dom}(W(\lambda)) = \left[ \begin{bmatrix} \eta \\ -\eta \end{bmatrix} \right], \ \eta \in \mathbb{C}.$$

Let an $m$-sectorial extension $\tilde{A}$ of $A$ be defined by a pair $\langle z, g(x) \rangle$, satisfying (5.7), see Theorem (3.3). Since $\tilde{A}$ is $m$-sectorial extension and

$$G(-i) = 0, \ \mathcal{Q}^*(-i) = 0, \ q(-i) = \gamma(i),$$

it is suitable to take $\lambda = -i$ and apply Theorem 2.5, Remark 2.6 and equalities (2.13), (2.14), (2.15). Then,

$$W = W(-i) = \mathbb{Z} = \left\langle \begin{bmatrix} \zeta \\ \zeta \end{bmatrix}, z \cdot \begin{bmatrix} \zeta \\ -\zeta \end{bmatrix} \right\rangle \oplus \left\langle \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \eta \\ \eta \end{bmatrix} \right\rangle,$$

$$W^{-1} = \left\langle \begin{bmatrix} \zeta \\ \eta \end{bmatrix}, \frac{1}{2z} \begin{bmatrix} \zeta - \eta \\ -\zeta + \eta \end{bmatrix} \right\rangle.$$

By (2.13)

$$\text{dom}(\tilde{A}) = \left( I + (q(-i) - 2\Phi(-i)X)W^{-1}(-i)\gamma^*(-i)A_F + iI \right) \text{dom}(A_F).$$

Further, let $\delta(x), \ x = (x_1, x_2)$ be the Dirac delta. Then $\delta(x) \in W_2^{-2}(\mathbb{R}^2)$ \cite{2}. Since $\mathcal{F}(\delta(x)) = 1/2\pi$, then $\mathcal{F}^{-1}(1) = 2\pi \delta(x)$. So, if $\mathcal{F}(h(x)) = \hat{h}(p)$ and $h(x) \in \text{dom}(A_F) = W_2^2(\mathbb{R}^2)$, then

$$\int_{\mathbb{R}^2} (e^{ipy_1} - e^{ipy_2})\hat{h}(p)dp = 2\pi(h(y_1) - h(y_2)).$$
Using the latter equality and the Fourier transform we obtain that
\[
W^{-1}(-i)\gamma^*(-i)(A_F + iI)h(x) = \frac{\pi(h(y_1) - h(y_2))}{z}.
\]
If \(h(x) \in \text{dom}(A_F)\), then
\[
\text{dom}(\tilde{A}) = \left\{ h(x) + \frac{\pi(h(y_1) - h(y_2))}{z} \times \left( \frac{\pi i}{2} \left( H_0^{(1)}(e^{3\pi i/4}|x - y_1|) - H_0^{(1)}(e^{3\pi i/4}|x - y_2|) \right) - 2\Phi(-i)(g(x)) \right) \right\}.
\]

Then applying Theorems 2.3, 3.8 we arrive at the following statement.

**Theorem 5.3.** There is a bijective correspondence between all \(m\)-sectorial extensions \(\tilde{A}\) (except Friedrichs and Kreĭn-von Neumann extensions) of \(A\) given by (5.1) and all pairs \((z, g(x))\), where \(z \in \mathbb{C}\) and a function \(g(x) \in L_2(\mathbb{R}^2)\) are such that:
\[
||g(x)||_{L_2(\mathbb{R}^2)}^2 < 2\text{Re} z.
\]

This correspondence is given by the relations:
\[
\text{dom}(\tilde{A}) = \left\{ u(x) = h(x) + \frac{\pi(h(y_1) - h(y_2))}{z} \times \left( \frac{\pi i}{2} \left( H_0^{(1)}(e^{3\pi i/4}|x - y_1|) - H_0^{(1)}(e^{3\pi i/4}|x - y_2|) \right) - 2\Phi(-i)(g(x)) \right), \right. \\
\left. h(x) \in W_2^2(\mathbb{R}^2) \right\}
\]

\[
\tilde{A}u(x) = -\Delta h(x) - \frac{\pi(h(y_1) - h(y_2))}{z} \times \left( \frac{\pi i}{2} \left( H_0^{(1)}(e^{3\pi i/4}|x - y_1|) - H_0^{(1)}(e^{3\pi i/4}|x - y_2|) \right) - 2\Phi(-i)(g(x)) \right).
\]

Moreover,

1) a number \(\lambda \in \mathbb{C} \setminus [0, +\infty)\) is a regular point of \(\tilde{A}\) if and only if \(w(z, g(x))(\lambda) \neq 0\) and,
\[
(\tilde{A} - \lambda I)^{-1}h(x) = \frac{i}{4} \int_{\mathbb{R}^2} H_0^{(1)}(\sqrt{|x - y|})f(y)dy + \frac{1}{w(z, g(x))} \times \left( \frac{\pi i}{2} \frac{1}{i - \lambda} \left( (i + \lambda)(H_0^{(1)}(\sqrt{|x - y_2|}) - H_0^{(1)}(\sqrt{|x - y_1|})) + 2i (H_0^{(1)}(e^{\pi i/4}|x - y_1|) - H_0^{(1)}(e^{\pi i/4}|x - y_2|)) \right) - 2\Phi(\lambda)(g(x)) \right)
\]
Krein-von Neumann extensions) and all complex numbers between all

Let Corollary 5.4. Moreover, an extension \( \tilde{A} \) and is nonnegative selfadjoint if and only if

\[
\ker(\tilde{A} - \lambda I) = \left( \frac{\pi i}{2} \frac{1}{i - \lambda} \left( (i + \lambda)(H_0^{(1)}(\sqrt{\lambda}|x - y_2|) - H_0^{(1)}(\sqrt{\lambda}|x - y_1|) \right) + 2i \left( H_0^{(1)}(e^{\pi i/4}|x - y_1|) - H_0^{(1)}(e^{\pi i/4}|x - y_2|) \right) \right) \eta, \quad \eta \in \mathbb{C}.
\]

**Corollary 5.4.** Let \( A \) be given by (5.1). Then there is a bijective correspondence between all \( m \)-accretive quasi-selfadjoint extensions \( \tilde{A} \) of \( A \) (except Friedrichs and Krein-von Neumann extensions) and all complex numbers \( z \in \mathbb{C} \) such that:

\[
\text{Re } z \geq 2\pi \left( \ln \frac{|y_1 - y_2|}{2} + \gamma + \ker(|y_1 - y_2|) \right).
\]

Moreover, an extension \( \tilde{A} \) is \( m \)-sectorial if and only if

\[
\text{Re } z > 2\pi \left( \ln \frac{|y_1 - y_2|}{2} + \gamma + \ker(|y_1 - y_2|) \right),
\]

and is nonnegative selfadjoint if and only if

\[
\text{Im } z = \pi (-3\pi + 4 \text{kei}(|y_1 - y_2|))
\]

The correspondence is given by relations

\[
\text{dom}(\tilde{A}) = \left\{ \begin{array}{l}
\{ u(x) = h(x) + \frac{\pi(h(y_1) - h(y_2))}{z} \\
\quad \times \left( \frac{\pi i}{2} \left( H_0^{(1)}(e^{\pi i/4}|x - y_1|) - H_0^{(1)}(e^{\pi i/4}|x - y_2|) \right) \right), \\
\quad h(x) \in W_2^3(\mathbb{R}^2) \}
\end{array} \right.,
\]

\[
\tilde{A}u(x) = -\Delta h(x) + \frac{\pi^2(h(y_1) - h(y_2))}{2z} \times \\
\quad \times \left( H_0^{(1)}(e^{\pi i/4}|x - y_1|) - H_0^{(1)}(e^{\pi i/4}|x - y_2|) \right). \]

Moreover,

1) a number \( \lambda \in \mathbb{C} \setminus [0, +\infty) \) is a regular point of \( \tilde{A} \) if and only if

\[
w(z, \lambda) = z - \pi \ln(\lambda i) - \pi^2i(H_0^{(1)}(\sqrt{\lambda}|y_1 - y_2|) - H_0^{(1)}(e^{\pi i/4}|y_1 - y_2|)) \neq 0
\]

and,

\[
(\tilde{A} - \lambda I)^{-1}h(x) = \frac{i}{4} \int_{\mathbb{R}^2} H_0^{(1)}(\sqrt{\lambda}|x - y|) f(y) dy \\
+ \frac{\pi^2}{8w(z, \lambda)} \times \left( H_0^{(1)}(\sqrt{\lambda}|x - y_1|) - H_0^{(1)}(\sqrt{\lambda}|x - y_2|) \right) \\
\times \int_{\mathbb{R}^2} \left( H_0^{(2)}(\sqrt{\lambda}|x - y_1|) - H_0^{(2)}(\sqrt{\lambda}|x - y_2|) \right) h(x) dx.
\]
2) a number \( \lambda \in \mathbb{C} \setminus [0, +\infty) \) is an eigenvalue of \( \tilde{A} \) if and only if \( w(z, \lambda) = 0 \) and,

\[
\ker(\tilde{A} - \lambda I) = (H_0^{(1)}(\sqrt{\lambda}|x - y_1|) - H_0^{(1)}(\sqrt{\lambda}|x - y_2|)) \eta, \quad \eta \in \mathbb{C}.
\]

**Remark 5.5.** One can obtain a description of the Kreĭn-von Neumann extension \( A_N \) of \( A \) from relations (5.8), (5.9) by substituting

\[
2z = \omega_0 = 4\pi \left( \ln \frac{|y_1 - y_2|}{2} - \frac{3\pi i}{4} + \gamma + \frac{\pi i}{2} H_0^{(1)}(e^{3\pi i/4}|y_1 - y_2|) \right).
\]

It follows from (5.8) that form \( A_N[u, v] \) associated with the Kreĭn-von Neumann extension \( A_N \) takes the form

\[
D[A_N] = \begin{cases} 
  u(x) = h(x) + \frac{\pi i}{2} \left( H_0^{(1)}(e^{\pi i/4}|x - y_1|) - H_0^{(1)}(e^{\pi i/4}|x - y_2|) \right) \omega_1, \\
  h(x) \in W^1_2(\mathbb{R}^2), \quad \omega \in \mathbb{C}
\end{cases}
\]

and if

\[
u(x) = h_2(x) + \frac{\pi i}{2} \left( H_0^{(1)}(e^{\pi i/4}|x - y_1|) - H_0^{(1)}(e^{\pi i/4}|x - y_2|) \right) \omega_2,
\]

where \( h_1(x), h_2(x) \in W^1_2(\mathbb{R}^2), \quad \omega_1, \omega_2 \in \mathbb{C} \), then

\[
A_N[u, v] = \int_{\mathbb{R}^2} \nabla h_1(x) \nabla h_2(x) \, dx \\
- \frac{\pi \omega_1 \omega_2}{2} \int_{\mathbb{R}^2} h_1(x) \left( H_0^{(1)}(e^{\pi i/4}|x - y_1|) - H_0^{(1)}(e^{\pi i/4}|x - y_2|) \right) \, dx \\
- \frac{\pi \omega_1 \omega_2}{2} \int_{\mathbb{R}^2} \left( H_0^{(1)}(e^{\pi i/4}|x - y_1|) - H_0^{(1)}(e^{\pi i/4}|x - y_2|) \right) \nabla h_2(x) \, dx \\
+ 4\pi \left( \ln \frac{|y_1 - y_2|}{2} + \gamma + \ker \left| y_1 - y_2 \right| \right) \cdot \omega_1 \omega_2.
\]

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