Toward a unified description of hadro- and photoproduction: $S$–wave $\pi$– and $\eta$–photoproduction amplitudes

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The Chew-Mandelstam parameterization, which has been used extensively in the two-body hadronic sector, is generalized in this exploratory study to the electromagnetic sector by simultaneous fits to the π – and η photoproduction $S$–wave multipole amplitudes for center-of-mass energies from the pion threshold through 1.61 GeV. We review the Chew-Mandelstam parameterization in detail to clarify the theoretical content of the SAID hadronic amplitude analysis and to place the proposed, generalized SAID electromagnetic amplitudes in the context of earlier employed parameterized forms. The parameterization is unitary at the two-body level, employing four hadronic channels and the γN electromagnetic channel. We compare the resulting fit to the MAID parameterization and find qualitatively agreement though, numerically, the solution is somewhat different. Applications of the extended parameterization to global fits of the photoproduction data and to global fits of the combined hadronic and photoproduction data are discussed.

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I. INTRODUCTION

Most of our knowledge of the excited baryons has come from fits to hadronic scattering data, in particular pion-nucleon scattering, for which there exists an accurate and nearly complete database extending through and above the resonance region. Sufficient polarization observables exist to constitute complete measurements over a significant kinematic interval. The range of hadroproduction data, including $\pi N \rightarrow \pi N$, $\pi N \rightarrow \eta N$, $\pi N \rightarrow \omega N$, and other inelastic processes including, for example, strangeness production, have also been used to constrain theoretical models and phenomenological parameterizations of the scattering and reaction amplitudes.

Currently, however, a renaissance is underway in meson production and resonance physics with reaction data issuing from a number of precision electromagnetic facilities. Collaborative theoretical and phenomenological efforts have started to analyze these data in ways consistent with some subset of constraints imposed by quantum field theory upon the reaction amplitudes. The quality and quantity of data in electromagnetic induced reactions is becoming sufficient to rival and possibly surpass the hadroproduction data. Since the electromagnetic reactions proceed mainly through the hadronic channels, the new data offers the possibility of “back-constraining” the hadronic amplitudes, conventionally determined only in fits to the hadroproduction data.

It is in this context that we have completed an exploratory study of the $S$–wave $\pi$– and $\eta$–photoproduction multipoles in the “Chew-Mandelstam” approach, related to the $N/D$ representation, to the electromagnetic reaction amplitude. The novel concept, which provokes and permits this exploratory study, is the generalization of the Chew-Mandelstam approach to the electromagnetic sector. We have developed a new form for the amplitude that incorporates multichannel hadronic rescattering effects in a complete manner consistent with unitarity. The near-term objective is to develop a framework in which to analyze the hadro- and electroproduction reactions simultaneously in a global framework.

Recent experimental observations of the photoproduction of the η meson from the proton have yielded measurements of the unpolarized differential cross section [1–3] and photon beam asymmetry [4–5] of high precision. Forthcoming measurements from the CLAS Collaboration at Jefferson Lab [6] and Mainz [7] will rival, if not surpass, the precision of the existing measurements.

Several interesting features of η meson physics motivate these measurements and their theoretical interpretation in various fields of nuclear physics, astrophysics, and particle physics. The possibility that the η-nucleon interaction may be attractive [8–9] suggests the existence of bound states of the η meson with nuclei. Certain resonances, the $S_{11}(1535)$ $N^*$ resonance in particular, are significantly coupled to the ηN channel, and the photoproduction of this final state provides an independent method to probe the isospin $T = \frac{1}{2}$ resonance spectrum and its couplings [10].

The strong interactions of the π and η mesons require multichannel descriptions which respect unitarity in the relevant channel space in order to obtain a realistic description of the data. The Chew-Mandelstam $K$-matrix approach [11–13] is an effective parameterization of the observed reaction data since the elements of the Chew-Mandelstam $K$ matrix may be assumed to be real if the couplings to neglected, open channels are small.

Several relatively recent $K$ matrix analyses of the coupled $\pi N$, $\eta N$, and $\gamma N$ channels [14–17] have been successful in obtaining reasonable parameterizations of the
two-body partial wave amplitudes\textsuperscript{[13].} The purpose of the present work is to investigate the extent to which a description of the $\pi$ photoproduction $E_{0+}^{1/2} (S_{11})$ amplitude and the modulus of the $\eta$ photoproduction amplitude yields an $\eta$ photoproduction multipole with a resonant phase. Various calculations\textsuperscript{[16] [19] [21]}, indicate that the modulus of the $\eta$ photoproduction amplitude near threshold is fairly model-independent, being reproduced in a range of calculational models or schemes. In the present work, we take as input hadronic $T$ matrix elements, determined in realistic ($\chi$-squared per datum $\sim$ 1) fits to data\textsuperscript{[22]}, as discussed in the following sections.

In Sec. II, we review in some detail the Chew-Mandelstam form\textsuperscript{[13]} of the parameterization. The purpose of this review is to establish the theoretical considerations that motivate the amplitude parameterizations used in the SAID program, to place these amplitudes in the context of other hadronic amplitude parameterization schemes and to lay the groundwork for future improvements. Section II\textsuperscript{[11]} gives the results for the fits to the isospin $T = \frac{1}{2}$ $\pi$-photoproduction amplitude, $E_{0+}$, and the modulus of the $\eta$ photoproduction amplitude, $E_{0+}$. The conclusions are given in Sec. IV. We find, in this exploratory study, an $\eta$ photoproduction multipole having a resonant shape, qualitatively similar to a Breit-Wigner form, and similar to other calculations\textsuperscript{[10, 21, 22]}. There is, however, significant deviation from the simple Breit-Wigner form.

II. CHEW-MANDELSTAM PARAMETERIZATION

Previous work in the determination of the $\eta$ photoproduction amplitudes\textsuperscript{[14] [16]} has shown that an approach which includes the coupling of the electromagnetic channel to the $\pi N$ and $\eta N$ channels in the region of energies near the center-of-mass energy, $W = 1535$ MeV gives a reasonably good description of the data and a plausible form for the amplitudes. However, as our ultimate objective is the simultaneous parameterization of hadro- and photoproduction scattering and reaction observables, we will go beyond the two-channel treatment for this study of the $E_{0+}^{1/2}$ multipole amplitude.

A. Unitarity constraint

The form of the Chew-Mandelstam parameterization, which we employ in this study follows as a consequence of the analytic structure imposed by the unitarity\textsuperscript{[21, 22]} of the $S$ matrix in the physical region, $W > m_t + m_i$, where $W$ is the center-of-mass energy and $m_t$ and $m_i$ are the masses of the incident and target particles. Confining our attention to two-particle initial and final states, the $S$ matrix is defined as

\[ S_{\alpha\beta}(E) = \langle k_{\alpha}\sigma | S | k_{\beta}\sigma \rangle \]

\[ = \delta^{(3)}(k_{\alpha} - k_{\beta})\delta_{\alpha\beta} + 2i\pi\delta(E_{\alpha} - E_{\beta}) \langle k_{\alpha}\sigma | T | k_{\beta}\sigma \rangle \]  

(2)

where $k_{\alpha, \beta}$ are the final and initial relative momenta, respectively, $E_{\alpha} = E_{\beta} = W$ is the center-of-mass energy, and the labels $\alpha$ and $\beta$ denote the particle species, spins, and internal quantum numbers, such as isospin. The initial and final energies, $E_{\beta}$ and $E_{\alpha}$, respectively are related to the on-shell relative momenta for channel $\alpha$, $\vec{k}_\alpha$ as

\[ W = E_{\alpha,1} + E_{\alpha,2} \]  

(3)

\[ E_{\alpha,i} = \sqrt{k_{\alpha,i}^2 + m_{\alpha,i}^2}. \]  

(4)

The on-shell relative momentum may be expressed in terms of the center-of-mass energy, $W$, as

\[ \vec{k}_\alpha = \frac{1}{2W} \sqrt{W - m_{\alpha+}} \sqrt{W - m_{\alpha-}} \times \sqrt{W + m_{\alpha+}} \sqrt{W + m_{\alpha-}}, \]  

(5)

with $m_{\alpha\pm} = m_{\alpha,1} \pm m_{\alpha,2}$.

The scattering operator, $S$, is unitary

\[ S^\dagger S = SS^\dagger = 1 \]  

(6)

and if we restrict our analysis to energies where just two-particle channels contribute, we obtain

\[ \sum \int d^3k_\sigma \langle k_{\sigma}\alpha | S^\dagger | k_{\sigma}\sigma \rangle \langle k_{\sigma}\sigma | S | k_{\beta}\beta \rangle \]

\[ = \delta^{(3)}(k_{\alpha} - k_{\beta})\delta_{\alpha\beta}. \]  

(7)

Substitution of Eq. (2) into the relation above yields the unitarity constraint on $T$

\[ T_{\alpha\beta} - T_{\alpha\beta}^\dagger = 2\pi i \sum \int d^3k_\sigma T_{\alpha\sigma}^\dagger \delta(E_{\alpha} - E_{\sigma})T_{\sigma\beta}. \]  

(8)

Effecting the integration on $k_\sigma \equiv |k_\sigma|$ gives

\[ T_{\alpha\beta} - T_{\alpha\beta}^\dagger = 2\pi \sum \int d\Omega_\sigma T_{\alpha\sigma}^\dagger \theta(W - m_{\sigma+})\rho_\sigma T_{\sigma\beta} \]

(9)

where

\[ \rho_\sigma(\vec{k}_\sigma) = \frac{\pi \vec{k}_\sigma E_{\sigma 1} E_{\sigma 2}}{\sqrt{W}}. \]  

(10)

The presence of the Heaviside step function, $\theta(W - m_{\sigma+})$ is a consequence of the fact that, over the range of integration $k_\sigma > 0$, the argument of the $\delta$ function has a solution, $E_{\alpha} - E_{\sigma} = 0$ only when $W > m_{\sigma+}$. Equation
where we have assumed that, due to the time-reversal invariance of the strong interaction, $T_{\alpha\beta} = T_{\beta\alpha}$. The violation of the Cauchy-Riemann equations at threshold indicates the presence of a branch point. We distinguish between the dynamical singularities at each threshold opening $m_{\sigma+}$ and kinematical singularities, due to the presence of kinematical factors such as $\bar{k}_\sigma$. The kinematical singularities are removed from the unitarity constraint by considering $T'_{\alpha\beta} = \sqrt{\rho}_\alpha T_{\alpha\beta} \sqrt{\rho}_\beta$.

We may transform to the partial wave representation and write

$$T'_{\alpha\beta} - T^*_{\alpha\beta} = 2i \sum_\sigma T^*_{\alpha\sigma} \theta(W - m_{\sigma+}) T'_{\sigma\beta} \quad (13)$$

where the $T'_{\alpha\beta}$ now represent the partial wave amplitudes. Casting this relation as a matrix equation

$$\frac{1}{2i} [T' - T^*] = T^* \theta(W - M_+) T', \quad (14)$$

where $M_{+,\alpha} = m_{\sigma+} \delta_{\sigma\alpha}$, and multiplying from the left by $[T^*]^{-1}$ and from the right by $T'^{-1}$ gives

$$\text{Im} \ T'^{-1} = -\theta(W - M_+), \quad (15)$$

a diagonal matrix. Since this equation isolates the imaginary part of the inverse-$T$ matrix, we write

$$T'^{-1} = \text{Re} \ T'^{-1} + i \text{Im} \ T'^{-1}, \quad (16) = K'^{-1} - i \theta(W - M_+), \quad (17)$$

where we’ve defined $\text{Re} \ T'^{-1} = K'^{-1}$ and $K'_{\alpha\beta} = \sqrt{\rho}_\alpha K_{\alpha\beta} \sqrt{\rho}_\beta$. Multiplying from one side by $T'$ and the other by $K'$ gives the Heitler integral equation

$$T' = K' + K' i \theta(W - M_+) T'. \quad (18)$$

This is the starting point for the Chew-Mandelstam parameterization of the reaction amplitude.

We emphasize that, in the physical region, the unitarity relation is satisfied by the imaginary part of $T'^{-1}$. Therefore the Heitler $K$ matrix is analytic, except for possible isolated poles throughout the physical region. This is apparent if we consider a dynamical equation of, for example, the Lippmann-Schwinger form:

$$T = V + VG_0 T, \quad (19)$$

$$G_0 = \mathcal{P} \frac{1}{E - H_0} - i \pi \delta(E - H_0). \quad (20)$$

Here $V$ is the interaction part of the full Hamiltonian, $E = W$, $H_0$ is the free-particle Hamiltonian, and $\mathcal{P}$ denotes the Cauchy principal value prescription. Substitution of Eq. (20) into Eq. (19) gives $T = K + iK\delta(E - H_0)T$ where

$$K = V + VP \frac{1}{E - H_0} K. \quad (21)$$

The Cauchy principal value prescription in this equation yields a kernel which is completely continuous in the physical region. The spectrum of the kernel therefore possesses no eigenvalues in the continuum and $K$ is analytic (other than possible poles) there. The $K$ matrix may possess singularities in other regions of the complex energy plane. In fact, the interaction $V$ possesses singularities in regions outside the physical region. In particular, there is a branch point at some value $W < 0$. We intend to neglect singularities in the region $W < 0$ for the purposes of the present study and avoid a detailed discussion of them here. A description is available in the literature. Inclusion of singularities in the region $W < 0$ will be explored in subsequent investigations.

The partial wave amplitude is therefore known to have the following singularities. There are branch points in the physical region at the channel-opening thresholds as in Eq. (9), branch points in the region $W < 0$, and possible poles consistent with causality. An efficient parameterization following Ref. (27), which encodes these singularities, involves the factorization of the partial wave amplitude. This is referred to as the “$N/D$” approach. We will use the $N/D$ language to clarify the nature of the singularities of the $T$ matrix which are included and those neglected in our Chew-Mandelstam approach.

**B. Relation to $N/D$ approach**

The $N/D$ approach has been used to analyze a variety of reactions. As our long term objective is the generalization of the existing method used to parameterize the hadronic and electromagnetic amplitudes, we collect here some of the relevant equations of the $N/D$ approach. The $T$ matrix is written in the factorized form

$$T(W) = D^{-1}(W) N(W) \quad (22)$$

where $N$ and $D$ are $N_{ch} \times N_{ch}$ arrays, where $N_{ch}$ is the number of included two-body channels. This relation has been shown to be consistent with the requirement of time-reversal invariance in Ref. (31). The relations

$$\text{Im} D(W) = N(W) \text{Im} T^{-1}(W) \quad W > m_i + m_t \quad (23) \\text{Im} N(W) = 0 \quad W > m_i + m_t \quad (24) \\text{Im} N(W) = D(W) \text{Im} T(W) \quad W < 0 \quad (25) \\text{Im} D(W) = 0 \quad W < 0 \quad (26)$$

give the essential content of the $N/D$ approach. They state that the function $D$ has branch points only in the
physical, $W > m_i + m_t$ region and that $N$ has only unphysical, $W < 0$ branch points. These relations determine the following dispersion relation (or Hilbert transform) representation for $D$

$$D(W) = \sum_{i=1}^{n_p} D(W; W_i) - \frac{1}{\pi} \prod_{i=1}^{n_p} (W - W_i)$$

$$\times \int_{W_i}^{\infty} dW' \frac{N(W')\rho(W')}{(W' - W) \prod_{j=1}^{n'} (W' - W_j)},$$

with $n_p$ subtractions. Here, $D(W; W_i)$ is a polynomial of order $n_p$, $W \in \mathbb{C}$, and $W_i$ is the lowest production threshold. Here, we show the polynomial ambiguity of the Hilbert transform explicitly to allow for the possibility that the parameterization includes several subtraction points.

Using the relation $T = ND^{-1}$ in the physical region, the numerator factor, $N$ can be shown to satisfy the integral equation

$$N(W) = K \left\{ \sum_i D(W; W_i) - \frac{1}{\pi} \prod_{i=1}^{n_p} (W - W_i) \right\}$$

$$\times \int_{W_i}^{\infty} dW' \frac{N(W')\rho(W')}{(W' - W) \prod_{j=1}^{n'} (W' - W_j)}$$

(28)

where $f$ denotes the Cauchy principal value integral, and the Heitler $K$ matrix, $K$ is defined by Eq.(18), with $K = \rho^{-\frac{1}{2}}K'\rho^{-\frac{1}{2}}$.

### C. Chew-Mandelstam parameterization

The preceding discussion of unitarity and the $N/D$ approach provides the context for our present parameterization. The Chew-Mandelstam parameterization developed here is similar to those of Refs. 11-12 and 13. We consider Eq.(17) and rewrite it, confining our attention to the $S$ wave multipole as

$$T^{-1} = K^{-1} - i\tilde{\rho}$$

$$= (K^{-1} + \text{Re} C) - (\text{Re} C + i\tilde{\rho})$$

$$= K^{-1} - C,$$

(30)

where $\tilde{\rho} = \rho\theta(W - M_\pi)$ and $\text{Im} C = \tilde{\rho} = \theta(W - M_\pi)\rho$. The transition matrix is given in terms of the “Chew-Mandelstam” (CM) $K$ matrix, $\bar{K}$ by

$$T = \bar{K} + \bar{K}C\bar{T}.$$  

(31)

Equation (31) fixes our Chew-Mandelstam parameterization. In the language of the preceding section, we have neglected the $W < 0$ branch points of $N$ and made the approximation $N(W) = \bar{K}(W)$, an entire function. The “Chew-Mandelstam” function, $C_\alpha$, is determined solely by the unitarity constraint, Eq.(15) since Eq.(30) is equivalent to taking $D = 1 - \bar{K}C$. Then the Chew-Mandelstam function is given by a Cauchy integral over the discontinuity of $C_\alpha$ in the physical region

$$C_\alpha(W) = \int_{W_i}^{\infty} dW' \frac{\rho_\alpha(W')}{W' - W}$$

$$- \int_{W_i}^{\infty} dW' \frac{\rho_\alpha(W')}{W' - W_s},$$

(32)

where we have made one subtraction, $0 \leq W_s < W_i$. Defining $\bar{\rho}_\alpha = \frac{W - W_s}{W - W_s}$, we can rewrite Eq.(32) as

$$C_\alpha(W) = \int_{0}^{1} \frac{dx}{\pi} \frac{\rho(x)}{x - \bar{\rho}_\alpha(W)}. $$

(33)

The relationship between the Heitler $K$ matrix and the CM $K$ matrix, $\bar{K}$ is given by

$$K = \bar{K} + \bar{K}[\text{Re} C]K.$$  

(34)

This demonstrates a possible advantage of using the CM $K$ matrix. If we consider a polynomial parameterization of a given CM $K$ matrix element

$$\bar{K}_{\alpha\beta} = \sum_{n=0}^{n_{\alpha\beta}} c_{\alpha\beta,n} \bar{\rho}_\alpha^m$$

(35)

where $n_{\alpha\beta}$ are channel dependent integers controlling the order of the polynomial (polynomials typically less than fifth order are used) and $\bar{\rho}_\alpha$ is a possibly channel dependent linear function of the center-of-mass energy, $W$ then we see, by solving Eq.(34) for $K$

$$K = \frac{1}{1 - \bar{K}[\text{Re} C]}\bar{K} $$

(36)

that poles may appear in the $K$ matrix. Attempts to relate the $K$ matrix poles to resonances have been made. Here, we simply point out that, though $K$ matrix poles are not simply related to $T$ matrix poles, Eq.(34) shows that one need not explicitly include pole terms in $\bar{K}$ in order to have poles in $K$. Parameterizing $\bar{K}(W) = N(W)$ as a polynomial, as noted, neglects singularities in the unphysical region, $W < 0$. The branch points there and discontinuities across their associated branch cuts are determined by the production mechanisms relevant for the reaction considered.

There are at least two reasons why polynomials may provide a reasonable starting point for a realistic parameterization of multichannel scattering and reaction amplitudes. The unitarity branch points, given their physical nature, largely determine the gross structure of the amplitudes in the physical region. This leads in an obvious way to the supposition that more distant singularities in the complex $W$ plane associated, in particular, with the branch points in the unphysical region may be less important. Experience has also confirmed this to be true. The existing said parameterizations of $\pi N$ elastic scattering, the $\pi N \to \eta N$ reaction, $\pi^-$ photoproduction and electroproduction, and other reactions all reveal that a realistic description with $\chi^2$ per datum in the range of $1 - 3$ is possible with the polynomial approximation for $\bar{K}$. 


TABLE I: Normalized (left of each column pair) and un-normalized (right of each column pair) $\chi^2$-per-datum for the SP06 [22] and FA02 [50] solutions of said, KA84 [51], EBAC [52], and Giessen [53]. The energy ranges of the four groups are from threshold to 2.5, 2.9, 1.91, and 2.0 GeV, respectively. [54]

| $\pi^+ p \rightarrow \pi^+ p$ | SP06 | FA02 | KA84 |
|-----------------------------|------|------|------|
| $\pi^- p \rightarrow \pi^- p$ | 2.0  | 6.1  | 2.1  | 8.8  | 5.0  | 24.9 |
| $\pi^- p \rightarrow \pi^0 n$ | 1.9  | 6.2  | 2.0  | 6.6  | 9.1  | 51.9 |
| $\pi^- p \rightarrow p n$    | 2.0  | 4.0  | 1.9  | 5.9  | 4.4  | 8.8  |
| $\pi^- p \rightarrow \eta n$  | 2.5  | 9.6  | 2.5  | 10.5 | –    | –    |

III. RESULTS

The Chew-Mandelstam parameterization for the $T$ matrix, described in the preceding section, has been applied recently [22] to a coupled-channel fit for the $\pi N$ elastic scattering and $\pi N \rightarrow \eta N$ reaction. It gives a realistic description of the data with $\chi^2$ per datum better than any other parameterization or model, to our knowledge. The $\chi^2$ per datum is shown in Table I against other parameterizations and model calculations for which we possess sufficient amplitude information to perform such an analysis [54]. The current SAIID parameterization used in this fit is given as

$$T_{\alpha\beta} = \sum_{\sigma} [1 - K C]^{-1}_{\alpha\sigma} K_{\sigma\beta}$$

(37)

where $\alpha$, $\beta$, and $\sigma$ are channel indices for the considered channels, $\pi N$, $\pi \Delta$, $\rho N$, and $\eta N$. This parameterization has been discussed in Refs. [13, 22, 50]. Given the success of this approach in the hadronic two-body sector, the application to the study of meson photoproduction is warranted.

The central result of the current exploratory study is to show that this form can be extended to include the electromagnetic channel,

$$T_{\alpha\gamma} = \sum_{\sigma} [1 - K C]^{-1}_{\alpha\sigma} K_{\sigma\gamma}$$

(38)

where $\gamma$ denotes the electromagnetic channel, $\gamma N$. Note that Eqs. (37) and (38) share the common factor, $[1 - K C]^{-1}_{\alpha\sigma}$ which encodes, at least qualitatively speaking, the hadronic channel coupling (or rescattering) effects.

The form Eq. (38) for photoproduction should be contrasted with that currently employed in the $\pi$-
photoproduction studies of Refs. [15, 17, 18].

\[ T_{\gamma\pi} = A(W)(1 + iT_{\pi\pi}(W)) + iB(W)T_{\pi\pi}(W) \]  

(39)

where the "structure functions" \( A(W) \) and \( B(W) \) are parameterized as polynomials in the energy, \( W \), \( T_{\gamma\pi} = T_{\pi N,\gamma N} \) and \( T_{\pi\pi} = T_{\pi N,\pi N} \), and the factor \( A(W) \) contains a contribution from tree-level Born diagrams. This satisfies Watson’s theorem [57] (as does Eq. (38)), and is derived via the considerations discussed in Ref. [58]. While resulting in a realistic description of the data and being comparable, at least qualitatively, with other parameterizations such as MAID [23] for \( \pi \)-photoproduction, it does not satisfy the full multichannel unitarity constraint imposed by Eq. (9). This deficiency led us to consider the form in Eq. (38), which manifestly satisfies the multichannel unitarity constraint, Eq. (9).

The need to include the multichannel unitarity effects of Eq. (38) have also become apparent in difficulties faced in attempts to parameterize the \( \eta \)-photoproduction reaction using forms [59] similar to Eq. (39). Forms of this type, used in fits to the \( \eta \)-photoproduction data alone, yielded an \( S \)-wave multipole without a clearly resonant shape, even while yielding fits to the observed data with realistic \( \chi^2 \) per datum on the order of 2 to 4. An example of such a fit employing Eq. (39) is shown in Fig. (4). Near values of the center-of-mass energy \( W \simeq 1535 \) the amplitude in Fig. (4) is decidedly not resonant. This is also clear in the Argand plot of Fig. (5). Here we have shown the comparison of the fit forms used in Ref. [59] (with energies marked by triangles) This difficulty was an early motivation for the present study. Expectation of resonant behavior for \( \eta \)-photoproduction, \( \gamma N \rightarrow \eta N \) in the \( S \)-wave can be argued straightforwardly. For example, since the electromagnetic coupling to the \( \pi N \) channel is large, the \( \gamma N \rightarrow \eta N \) reaction may proceed via the \( \pi N \rightarrow \eta N \) amplitude of Fig. (1), or through direct resonance production. We therefore anticipate the hadronic subprocess will drive a significant resonant effect in the (isoscalar) electromagnetic transition.

Several other studies have determined that a resonant structure near \( W \sim 1535 \) MeV is consistent with the reaction data. These works include those in Refs. [10, 19, 21, 23]. We should note that all of these works have assumed the \( S_{11} \) wave to be resonant, usually by including a Breit-Wigner or similar term explicitly into their formalism. We do not make this assumption in using Eq. (38).

In light of the study of Ref. [59] and the necessity of including the full multichannel unitarity for the purposes of obtaining a global description of the hadro- and photoproduction data, we have carried out an exploratory study to determine the efficacy of doing such a fit within the Chew-Mandelstam parameterization, Eq. (38). In the present study we perform a coupled-channel fit of the modulus \( |E_{0+}^\eta(W)| \) and (the real and imaginary parts of) the existing SAID and MAID \( \pi \)-photoproduction amplitudes, \( E_{0+}^\eta \) in the \( S_{11}(1535) \) resonance region, compared in Fig. (2). The fit was carried out by taking the factor \( 1 - \mathcal{K}C_{-1} \) in Eq. (38), as determined in the hadronic study of Ref. [22] and adjusting the parameters of \( \mathcal{K}_{\gamma\pi} \) (discussed in detail in the subsections below). The phase of the \( E_{0+} \) multipole in this study gives a resonant wave and encourages us to continue with this approach, as dis-
FIG. 6: The predicted values for the real (solid curve) and imaginary (dashed curve) for $E_{\eta}^{0+}$ versus the energy, $W$. The modulus $|E_{\eta}^{0+}|$ (dotted curve), the real (dot-dashed curve), and the imaginary (double dot-dashed curve) parts of the $\pi$–photoproduction, $E_{\pi}^{0+}$, were fit to pseudodata generated from the MAID solution[60] with the parameterized form Eq.(38) using 7 parameters (see text).

The decision to fit the modulus $|E_{\eta}^{0+}|$ is based on empirical considerations. The MAID[23] parameterization by the Dubna-Mainz-Taipei Collaboration (DMT) and the model calculations in Refs.[16, 19, 21, 61] agree at the few-percent level on the modulus of the low-energy $\eta$–photoproduction amplitude, $|E_{\eta}^{0+}(W)|$. This is anticipated on the grounds that, in the $S_{11}(1535)$ resonance region, the differential cross section is largely angle dependent and therefore dominated by the $S$–wave production. It also indicates that the production is largely resonant, but we do not make this common assumption.

While the modulus $|E_{\eta}^{0+}|$ appears to be known at the level of a few percent, the $\pi$–photoproduction $S_{11}$ amplitude is, surprisingly, not very well determined through different parameterizations. Figure (2) shows the SAID[22] and MAID[23] results for $E_{\pi}^{0+}$. Given this discrepancy, we have also carried out the fit described above with the modulus $|E_{\eta}^{0+}|$ and the MAID parameterization.

FIG. 7: The predicted values for the real (solid curve) and imaginary (dashed curve) for $E_{\eta}^{0+}$ versus the energy, $W$. The modulus $|E_{\eta}^{0+}|$ (dotted curve), the real (dot-dashed curve), and the imaginary (double dot-dashed curve) parts of the $\pi$–photoproduction, $E_{\pi}^{0+}$, were fit to pseudodata generated from the MAID solution[60] with the parameterized form Eq.(38) using 14 parameters (see text).

FIG. 8: The real (solid curves) and imaginary (dashed curve) for $E_{\eta}^{0+}$ from the 7-parameter fit in Fig.(6) compared with the $\eta$–MAID solution[23], marked by squares.

FIG. 9: The real (solid curves) and imaginary (dashed curve) for $E_{\eta}^{0+}$ from the 14-parameter fit in Fig.(7) compared with the $\eta$–MAID solution[23], marked by squares.
Figure 3 shows the result of fitting the modulus $|E_{0+}^\pi|$ and the real and imaginary parts of thesaid $E_{0+}^\pi$ multipole using an eight-parameter fit. The Chew-Mandelstam $K$ matrix was assumed to have the form

$$K_{\sigma\gamma}(W) = c_{\sigma\gamma,0} + c_{\sigma\gamma,1}z_{\sigma\gamma}$$  \hspace{1cm} (40)$$

taking $n_{\alpha\beta} = 1$, in Eq. (35), for $\alpha$ and $\beta$ taking values in the set of four channels, $\pi N, \pi \Delta, \rho N$, and $\eta N$. The energy variable $z_{\alpha\beta}$ is

$$z_{\alpha\beta} = W - W_{t,\alpha},$$  \hspace{1cm} (41)$$

where the threshold masses, $W_{t,\alpha}$, are $m_\pi + m_N$, $2m_\pi + m_N$, $2m_\pi + 2m_N$, and $m_\eta + m_N$ for $\alpha = \pi N, \pi \Delta, \rho N$, and $\eta N$, respectively, and $W_{t,\alpha}$ is taken to be the lower of the thresholds for channels $\alpha$ and $\beta$. The eight parameters were varied in the fit to a total of 113 pseudodata points including the modulus $|E_{0+}^\eta|$ over the energy range 1490 MeV $\leq W \leq 1610$ MeV and the amplitudes $E_{0+}^\pi$ over the energy range 1120 MeV $\leq W \leq 1610$. The $\chi^2$ per datum over for the fits to the pseudodata, generated with the SAID interactive code facility [62], were less than one in all of the fits made in this work including those in the region 1120 MeV $\leq W \leq 1490$ MeV which are not displayed in order to keep the figures manageable and focus attention on the $S_{11}(1535)$ resonance region. The pseudodata were assigned 5% errors in the fit.

**B. Fit with MAID $E_{0+}^\pi$**

The graphs in Figs. 6 and 7 used seven and fourteen parameters, respectively, to fit the $|E_{0+}^\eta|$ and $E_{0+}^\pi$ amplitudes from MAID [22]. The seven-parameter fit in Fig. 6 is the minimal set of parameters needed to obtain a $\chi^2$ per datum $\lesssim 1$. The parameters used in Eq. (35) for this fit were $c_{\gamma n_1}$, $n_1 = 0,1,2$, $c_{\gamma n_2}$, where $n_2 = 0,1$, and $c_{\gamma n_3}$ and $c_{\gamma n_4}$. The quality is degraded at the higher energy end of the fit region for the imaginary part of $E_{0+}^\pi$. Nearly perfect agreement is obtained if we use a fourteen parameter form for $K_{\sigma\gamma}$. The parameters used in Eq. (35) for this fit were $c_{\pi\gamma,n_1}$, $c_{\Delta\gamma,n_2}$, $c_{\rho\gamma,n_3}$, where $n_1 = 0,1,2,3$ and $c_{\gamma n_4}$ and $c_{\gamma n_5}$. Note, from Figs. 6 and 7, that the fit giving the better representation of the MAID $E_{0+}^\pi$ amplitude is similarly closer to the MAID $E_{0+}^\eta$-result. This is somewhat surprising perhaps, since although the MAID pion- and eta-photproduction use the same pole positions in both amplitudes, these parameterizations are not constrained by unitarity.

**IV. CONCLUSION AND ONGOING WORK**

We reviewed the implication of unitarity on the analytic structure of the single meson production scatterings and reaction amplitudes. The non-analyticities in the regions $W > 0$ and $W < 0$, the right- and left-hand cuts, respectively were demonstrated to be properly accounted for by the $N/D$ approach. We related the Chew-Mandelstam $K$-matrix parameterization to the $N/D$ approach, showing that the parameterization of the $\overline{K}$ matrix neglects the effects of the distant left-hand cut. The purpose of this review is to place the long-used SAID amplitudes in the context of other hadronic amplitude parameterization schemes and to lay the groundwork for future improvements to the existing parameterization forms.

Using the Chew-Mandelstam $K$ matrix $\overline{K}$, we performed a simultaneous coupled-channel fit of the $\eta-$ photoproduction $S_{11}$ multipole modulus, $|E_{0+}^\pi|$ and the $\pi-$photoproduction amplitude, $E_{0+}^\pi$. The parameterization was restricted only to the CM $K$ matrix elements $\overline{K}_{\sigma\gamma}$ in Eq. (38), while the $[1 - \overline{K}C]^{-1}$ factors were taken from the existing SAID fits to the hadronic data. The anticipated resonant structure for the phase of the $E_{0+}^\pi$ multipole was demonstrated in fits to both SAID and MAID amplitudes.

The results of the exploratory study indicate that this is a reasonable approach toward the objective of determining a complete set of scattering and reaction amplitudes for $\pi N \rightarrow \pi N$, $\pi N \rightarrow \eta N$, $\gamma N \rightarrow \pi N$, and $\gamma N \rightarrow \eta N$ processes in a multichannel unitary formalism. The first stage in this procedure, demonstrating that coupled-channel simultaneous fits of the $\pi-$ and $\eta-$photoproduction reactions for a single partial wave ($S_{11}$) is possible, has been completed. The next phase, consists of a fit to the $\pi-$photoproduction reaction observables. Following this, a simultaneous fit to the reaction observables for the $\pi-$ and $\eta-$photoproduction reactions will be performed. As a practical matter, these two phases will be completed using the $[1 - \overline{K}C]^{-1}$ “rcattering” factors determined in separate fits to the hadronic scattering and reaction data. The final phase of the study will be a simultaneous fit to both the hadronic and electromagnetic scattering and reaction observables and will constitute, at least for two-body unitarity, a global description of the hadro- and photoproduction amplitudes.

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The said interactive suite of codes is available by secure shell interface via:

```
ssh -C -X 
said@said.phys.gwu.edu.
```

[62] S. L. Adler, Phys. Rev. 137, B1022 (1965).