Fractional conservation laws in optimal control theory

Abstract  Using the recent formulation of Noether’s theorem for the problems of the calculus of variations with fractional derivatives, the Lagrange multiplier technique, and the fractional Euler-Lagrange equations, we prove a Noether-like theorem to the more general context of the fractional optimal control. As a corollary, it follows that in the fractional case the autonomous Hamiltonian does not define anymore a conservation law. Instead, it is proved that the fractional conservation law adds to the Hamiltonian a new term which depends on the fractional-order of differentiation, the generalized momentum, and the fractional derivative of the state variable.

Keywords  Fractional derivatives · Optimal control · Noether’s theorem · Conservation laws · Symmetry

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1 Introduction

The concept of symmetry plays an important role both in Physics and Mathematics. Symmetries are described by transformations of the system, which result in the same object after the transformation is carried out. They are described mathematically by parameter groups of transformations. Their importance ranges from fundamental and theoretical aspects to concrete applications, having profound implications in the dynamical behavior of the systems, and in their basic qualitative properties.

Another fundamental notion in Physics and Mathematics is the one of conservation law. Typical application of conservation laws in the calculus of variations and optimal control is to reduce the number of degrees of freedom, and thus reducing the problems to a lower dimension, facilitating the integration of the differential equations given by the necessary optimality conditions.

Emmy Noether was the first who proved, in 1918, that the notions of symmetry and conservation law are connected: when a system exhibits a symmetry, then a conservation law can be obtained. One of the most important and well known illustration of this deep and rich relation, is given by the conservation of energy in Mechanics: the autonomous Lagrangian $L(q, \dot{q})$, correspondent to a mechanical system of conservative points, is invariant under time-translations (time-homogeneity symmetry), and

$$\frac{d}{dt}[L(q, \dot{q}) - \partial_2 L(q, \dot{q}) \cdot \dot{q}] = 0 \quad (1)$$

follows from Noether’s theorem, i.e., the total energy of a conservative closed system always remain constant in time, “it cannot be created or destroyed, but only transferred from one form into another”. Expression (1) is valid along all the Euler-Lagrange extremals $q(\cdot)$ of an autonomous problem of the calculus of variations. The conservation law (1) is known in the calculus of variations as the 2nd Erdmann necessary condition; in concrete applications, it gains different interpretations: conservation of energy in Mechanics; income-wealth law in Economics; first law of Thermodynamics; etc. The literature on Noether’s theorem is vast, and many extensions of the classical results of Emmy Noether are now available for the more general setting of optimal control (see [7,29,30] and references therein). Here we remark that in all those results conservation laws always refer to problems with integer derivatives.

Nowadays fractional differentiation plays an important role in various fields: physics (classic and quantum mechanics, thermodynamics, etc), chemistry, biology, economics, engineering, signal and image processing, and control theory [2,13,16]. Its origin goes back three centuries, when in 1695 L’Hopital and Leibniz exchanged some letters about
the mathematical meaning of \(\frac{d^n}{dt^n}\) for \(n = \frac{1}{2}\). After that, many famous mathematicians, like J. Fourier, N. H. Abel, J. Liouville, B. Riemann, among others, contributed to the development of the Fractional Calculus \([13,20,27]\).

The study of fractional problems of the Calculus of Variations and respective Euler-Lagrange type equations is a subject of current strong research. F. Riewe \([25,26]\) obtained a version of the Euler-Lagrange equations for problems of the Calculus of Variations with fractional derivatives, that combines the conservative and non-conservative cases. In 2002 O. Agrawal proved a formulation for variational problems with right and left fractional derivatives in the Riemann-Liouville sense \([1]\). Then, these Euler-Lagrange equations were used by D. Baleanu and T. Avkar to investigate problems with Lagrangians which are linear on the velocities \([5]\). In \([14,15]\) fractional problems of the calculus of variations with symmetric fractional derivatives are considered and correspondent Euler-Lagrange equations obtained, using both Lagrangian and Hamiltonian formalisms. In all the above mentioned studies, Euler-Lagrange equations depend on left and right fractional derivatives, even when the problem depend only on one type of them. In \([17]\) problems depending on symmetric derivatives are considered for which Euler-Lagrange equations include only the derivatives that appear in the formulation of the problem. In \([18,19]\) Riemann-Liouville fractional integral functionals, depending on a parameter \(\alpha\) but not on fractional-order derivatives of order \(\alpha\), are introduced and respective fractional Euler-Lagrange type equations obtained. More recently, the authors have used the results of \([1]\) to generalize the classical Noether’s theorem for the context of the Fractional Calculus of Variations \([11]\). Differently from \([11]\), where the Lagrangian point of view is considered, here we adopt an Hamiltonian point of view. Fractional Hamiltonian dynamics is a very recent subject but the list of publications has become already a long one due to many applications in mechanics and physics. \([4,6,10,22,23,24,28]\). We extend the previous optimal control Noether results of \([29,30]\) to the wider context of fractional optimal control \([24,28]\). This is accomplished by means (i) of the fractional version of Noether’s theorem \([11]\). (ii) and the Lagrange multiplier rule \([8]\). As a consequence of our main result, it follows that the “total energy” (the autonomous Hamiltonian) of a fractional system is not conserved: a new expression appears (cf. Corollary \([4,1]\) which also depends on the fractional-order of differentiation, the adjoint variable, and the fractional derivative of the state trajectory.

2 Fractional Derivatives

We briefly recall the definitions of right and left Riemann-Liouville fractional derivatives, as well as their main properties \([1,20,27]\).

**Definition 2.1** Let \(f\) be a continuous and integrable function in the interval \([a, b]\). For all \(t \in [a, b]\), the left Riemann-Liouville fractional derivative \(D_t^{\alpha} f(t)\), and the right Riemann-Liouville fractional derivative \(D_t^{\alpha} f(t)\), of order \(\alpha\), are defined in the following way:

\[
D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \int_a^t (t-\theta)^{n-\alpha-1} f(\theta) d\theta, \quad (2)
\]

where \(n \in \mathbb{N}, n-1 \leq \alpha < n\), and \(\Gamma\) is the Euler gamma function.

**Remark 2.1** If \(\alpha\) is an integer, then from \((2)\) and \((3)\) one obtains the standard derivatives, that is,

\[
D_t^{\alpha} f(t) = \left( \frac{d}{dt} \right)^{\alpha} f(t), \quad (4)
\]

\[
D_t^{\alpha} f(t) = \left( -\frac{d}{dt} \right)^{\alpha} f(t). \quad (5)
\]

**Theorem 2.1** Let \(f\) and \(g\) be two continuous functions on \([a, b]\). Then, for all \(t \in [a, b]\), the following properties hold:

1. for \(p > 0\),

\[
aD_t^{p} (f(t) + g(t)) = aD_t^{p} f(t) + aD_t^{p} g(t); \quad (8)
\]

2. for \(p \geq q \geq 0\),

\[
aD_t^{p} (aD_t^{-q} f(t)) = aD_t^{p-q} f(t); \quad (9)
\]

3. for \(p > 0\),

\[
aD_t^{p} (aD_t^{-p} f(t)) = f(t)
\]

(fundamental property of the Riemann-Liouville fractional derivatives).

**Remark 2.2** In general, the fractional derivative of a constant is not equal to zero.

**Remark 2.3** The fractional derivative of order \(p \geq 0\) of function \((t-a)^v\), \(v > -1\), is given by

\[
aD_t^{p} (t-a)^v = \frac{\Gamma(v+1)}{\Gamma(-p+v+1)}(t-a)^{v-p}. \quad (10)
\]

**Remark 2.4** When one reads “Riemann-Liouville fractional derivative” in the literature, it is usually meant (implicitly) the left Riemann-Liouville fractional derivative. In Physics, \(t\) often denotes the time-variable, and the right Riemann-Liouville fractional derivative of \(f(t)\) is interpreted as a future state of the process \(f(t)\). For this reason, right derivatives are usually neglected in applications: the present state of a process does not depend on the results of the future development. Following \([3\], and differently from \([11]\), in this work we focus on problems with left Riemann-Liouville fractional derivatives only. This has the advantage of simplifying greatly the theory developed in \([11]\), making possible the generalization of the results to the fractional optimal control setting.

We refer the interested reader in additional background on fractional theory, to the comprehensive book \([27]\).
3 Preliminaries

In [11] a formulation of the Euler-Lagrange equations is given for problems of the calculus of variations with fractional derivatives.

Let us consider the following fractional problem of the calculus of variations: to find function \( q(t) \) that minimizes the integral functional

\[
I[q(t)] = \int_a^b L(t, q(t), \alpha q'(t)) \cdot dt,
\]

where the Lagrangian \( L : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) is a \( C^2 \) function with respect to all its arguments, and \( 0 < \alpha \leq 1 \).

Remark 3.1 In the case \( \alpha = 1 \), problem (4) is reduced to the classical problem

\[
I[q(t)] = \int_a^b L(t, q(t), q'(t)) dt \to \min.
\]

Theorem 3.1 (cf. [11]) If \( q \) is a minimizer of problem (4), then it satisfies the fractional Euler-Lagrange equations:

\[
\partial_aL(t, q, \alphaq') + \partial_bL(t, q, \alpha q')q = 0.
\]

The following definition is useful in order to introduce an appropriate concept of fractional conservation law. We recall that the classical Noether’s conservation laws are always a sum of products (as assumed in (6)) and that the fractional rule for differentiation of a product, in the sense of Riemann-Liouville, is enough complex (see e.g. [27]). With respect to this, our operator \( \partial_t^{\alpha} (f, g) \) is useful. This operator was introduced in [11] and we refer the reader to this reference for several illustrative examples and remarks. Here we just mention that the operator \( \partial_t^{\alpha} (f, g) \) has resemblances with the classical Poisson bracket (cf. Remark 3.4).

Definition 3.1 (cf. [11]) Given two functions \( f, g \) of class \( C^1 \) in the interval \([a, b]\), we define the following operator:

\[
\partial_t^{\alpha} (f, g) = f_aD^\alpha_t \cdot \partial_t g - g_aD^\alpha_t \cdot \partial_t f, \quad t \in [a, b].
\]

Remark 3.2 For \( \alpha = 1 \), operator \( \partial_t^{1} (f, g) \) is reduced to

\[
\partial_t^{1} (f, g) = f_aD^1_t \cdot \partial_t g - g_aD^1_t \cdot \partial_t f = fg + fg = \frac{d}{dt}(fg).
\]

Remark 3.3 The linearity of the operators \( \partial_t^{\alpha} \cdot \partial_t \) and \( \partial_t^{\alpha} \cdot \partial_b \) imply the linearity of the operator \( \partial_t^{\alpha} \).

Definition 3.2 (cf. [11]) We say that \( C_f(t, q, \alpha q') \) is a fractional conservation law if and only if it is possible to write \( C_f \) in the form of a sum of products,

\[
C_f(t, q, d) = \sum_{i=1}^r C^1_f(t, q, d) \cdot C^2_f(t, q, d)
\]

for some \( r \in \mathbb{N} \), and for each \( i = 1, \ldots, r \) the pair \( C^1_i \) and \( C^2_i \) satisfy one of the following relations:

\[
\partial_t^{\alpha} (C^1_i(t, q, \alpha q'), C^2_i(t, q, \alpha q')) = 0
\]

or

\[
\partial_t^{\alpha} (C^1_i(t, q, \alpha q'), C^2_i(t, q, \alpha q')) = 0
\]

along all the fractional Euler-Lagrange extremals (i.e. along all the solutions of the fractional Euler-Lagrange equations (5)). We then write \( \mathcal{D} \{ C_f(t, q, \alpha q') \} = 0 \).

Remark 3.4 For \( \alpha = 1 \) (7) and (8) coincide, and

\[
\mathcal{D} \{ C(t, q, \alpha q') \} = 0
\]

is reduced to

\[
\frac{d}{dt} \{ C(t, q(t), \dot{q}(t)) \} = 0 \Leftrightarrow C(t, q(t), \dot{q}(t)) \equiv \text{constant},
\]

which is the standard meaning of conservation law, i.e. a function \( C(t, q, \dot{q}) \) preserved along all the Euler-Lagrange extremals \( q(t), t \in [a, b] \), of the problem. This implies that if \( (p(t), q(t)) \) is a solution to the classical Hamilton-Jacobi equations of motion, then \( C \) defines a conservation law of the Hamiltonian equations with Hamiltonian \( H \) if \( \{ H, C \} = 0 \) or \( \{ C, H \} = 0 \), where \( \{ \cdot, \cdot \} \) denotes the canonical Poisson bracket operator. In the more general fractional context, the Hamilton-Jacobi equations were recently derived in [18,19].

Definition 3.3 (cf. [11]) Functional (4) is said to be invariant under the one-parameter group of infinitesimal transformations

\[
\begin{align*}
\bar{\alpha} &= \alpha + \varepsilon \tau(t, q) + o(\varepsilon), \\
\bar{q} &= q(t) + \varepsilon \xi(t, q) + o(\varepsilon),
\end{align*}
\]

if, and only if,

\[
\int_a^b L(t, q(t), \alpha q'(t)) dt = \int_{t(a)}^{t(b)} \bar{L}(\bar{t}, \bar{q}(\bar{t}), \alpha \bar{q}'(\bar{t})) d\bar{t}
\]

for any subinterval \( [t_a, t_b] \subseteq [a, b] \).

Remark 3.5 Having in mind that condition (10) is to be satisfied for any subinterval \( [t_a, t_b] \subseteq [a, b] \), we can rid off the integral signs in (10). This is done in the new Definition 3.3.

The next theorem provides an extension of the classical Noether’s theorem to Fractional Problems of the Calculus of Variations.

Theorem 3.2 (cf. [11]) If functional (4) is invariant under (9), then

\[
\left[ L(t, q, \alpha q') - \alpha \partial_3L(t, q, \alpha q') \cdot \alpha \partial_3^1 q \right] \tau(t, q) + \partial_3L(t, q, \alpha q') \cdot \xi(t, q)
\]

is a fractional conservation law (cf. Definition 3.2).
4 Main Results

Using Theorem 4.2, we obtain here a Noether’s Theorem for the fractional optimal control problems introduced in [3]:

\[ I[q(\cdot), u(\cdot)] = \int_a^b L(t, q(t), u(t)) dt \rightarrow \min, \quad \text{(11)} \]

\[ aD_\alpha^a q(t) = \phi (t, q(t), u(t)), \]

together with the initial condition \( q(a) = q_a \). In problem (11), the Lagrangian \( L : [a, b] \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R} \) and the velocity vector \( \phi : [a, b] \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) are assumed to be \( C^1 \) functions with respect to all the arguments. In agreement with the calculus of variations, we also assume that the admissible control functions take values on an open set of \( \mathbb{R}^m \).

**Definition 4.1** A pair \((q(\cdot), u(\cdot))\) satisfying the fractional control system \( aD_\alpha^a q(t) = \phi (t, q(t), u(t)) \) of problem (11), \( t \in [a, b] \), is called a process.

**Theorem 4.1** (cf. (13)-(15) of [3]) If \((q(\cdot), u(\cdot))\) is an optimal process for problem (11), then there exists a co-vector function \( p(\cdot) \) such that the following conditions hold:

- the Hamiltonian system

\[ \begin{bmatrix} aD_\alpha^a q(t) = \partial_3 \mathcal{H}(t, q(t), u(t), p(t)) \\
\partial_2 \mathcal{H}(t, q(t), u(t), p(t)) \end{bmatrix} = \begin{bmatrix} \partial_3 \mathcal{H}(t, q(t), u(t), p(t)) \\
\partial_2 \mathcal{H}(t, q(t), u(t), p(t)) \end{bmatrix} = 0; \]

with the Hamiltonian \( \mathcal{H} \) defined by

\[ \mathcal{H}(t, q, u, p) = L(t, q, u) + p \cdot \phi (t, q, u). \quad \text{(12)} \]

**Remark 4.1** In classical mechanics, the Lagrange multiplier \( p \) is called the generalized momentum. In the language of optimal control, \( p \) is known as the adjoint variable.

**Definition 4.2** Any triplet \((q(\cdot), u(\cdot), p(\cdot))\) satisfying the conditions of Theorem 4.1 will be called a fractional Pontryagin extremal.

For the fractional problem of the calculus of variations (4) one has \( \phi (t, q, u) = u \Rightarrow \mathcal{H} = L + p \cdot u \), and we obtain from Theorem 4.1 that

\[ aD_\alpha^a q = u, \]

\[ aD_\alpha^a p = \partial_3 L, \]

\[ \partial_3 \mathcal{H} = 0 \Rightarrow p = -\partial_3 L \Rightarrow \partial_3 aD_\alpha^a p = -\partial_3 aD_\alpha^a L. \]

Comparing the two expressions for \( aD_\alpha^a p \), one arrives to the Euler-Lagrange differential equations [3]: \( \partial_3 L = -\partial_\alpha^a \partial_3 L. \)

We define the notion of invariance for problem (11) in terms of the Hamiltonian, by introducing the augmented functional as in [3]:

\[ I[q(\cdot), u(\cdot), p(\cdot)] = \int_a^b \left[ \mathcal{H}(t, q(t), u(t), p(t)) - p(t) \cdot aD_\alpha^a q(t) \right] dt, \quad \text{(13)} \]

where \( \mathcal{H} \) is given by (12).

**Remark 4.2** Theorem 4.1 is easily applied obtaining the necessary optimality condition (5) to problem (11).

**Definition 4.3** A fractional optimal control problem (11) is said to be invariant under the \( \epsilon \)-parameter local group of transformations

\[ \begin{bmatrix} \tilde{t} = t + \epsilon \tau (t, q(t), u(t), p(t)) \circ \epsilon \cdot \xi, \\
\tilde{q}(t) = q(t) + \epsilon \xi (t, q(t), u(t), p(t)) + o(\epsilon), \\
\tilde{p}(t) = p(t) + \epsilon \xi (t, q(t), u(t), p(t)) + o(\epsilon) \end{bmatrix}, \quad \text{if, and only if,} \]

\[ \mathcal{H}(\tilde{t}, \tilde{q}(\tilde{t}), \tilde{p}(\tilde{t})) = \mathcal{H}(t, q(t), p(t)) \quad \text{is a fractional conservation law, that is,} \]

\[ \mathcal{D} \left\{ [H - (1 - \alpha) p(t) \cdot aD_\alpha^a q(t)] \tau - p(t) \cdot \xi \right\} = 0 \quad \text{along all the fractional Pontryagin extremals.} \]

**Remark 4.3** For \( \alpha = 1 \), the fractional optimal control problem (11) is reduced to the classical optimal control problem

\[ I[q(\cdot), u(\cdot)] = \int_a^b L(t, q(t), u(t)) dt \rightarrow \min, \]

\[ q(t) = \phi (t, q(t), u(t)) \]

and we obtain from Theorem 4.2 the optimal control version of Noether’s theorem [29]: invariance under a one-parameter group of transformations (14) imply that

\[ C(t, q, u, p) = \mathcal{H}(t, q, u, p) \tau - p \cdot \xi \quad \text{(17)} \]

is constant along any Pontryagin extremal (one obtains (17) from (16) setting \( \alpha = 1 \)).

**Proof** The fractional conservation law (16) is obtained applying Theorem 3.2 to the augmented functional (13). \( \square \)

Theorem 4.2 provides a new interesting insight for the fractional autonomous variational problems. Let us consider the autonomous fractional optimal control problem, i.e. the situation when the Lagrangian \( L \) and the fractional velocity vector \( \phi \) do not depend explicitly on time \( t \):

\[ I[q(\cdot), u(\cdot)] = \int_a^b L(q(t), u(t)) dt \rightarrow \min, \quad aD_\alpha^a q(t) = \phi (q(t), u(t)). \quad \text{(18)} \]

**Corollary 4.1** For the autonomous problem (18) the following fractional conservation law holds:

\[ \mathcal{D} \left\{ \mathcal{H} - (1 - \alpha) p(t) \cdot aD_\alpha^a q(t) \right\} = 0. \quad \text{(19)} \]
Remark 4.4 In the classical framework of optimal control theory one has $\alpha = 1$ and our operator $\mathcal{D}$ coincides with $\frac{d}{dt}$. We then get from [19] the classical result: the Hamiltonian $\mathcal{H}$ is a preserved quantity along any Pontryagin extremal of the problem.

Proof The Hamiltonian $\mathcal{H}$ does not depend explicitly on time, and it is easy to check that [18] is invariant under time-translations: invariance condition (15) is satisfied with $\bar{t} = t + \epsilon$, $\bar{q}(t) = q(t)$, $\bar{u}(t) = u(t)$ and $\bar{p}(\bar{t}) = p(t)$. In fact, given that $d\bar{t} = dt$, (15) holds trivially proving that $a\mathcal{D}\alpha q(\bar{t}) = a\mathcal{D}\alpha q(t)$:

$$
\begin{align*}
\alpha D^\alpha q(\bar{t}) &= \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \int_0^\bar{t} (\bar{t} - \theta)^{n-\alpha-1} \bar{q}(\theta) d\theta \\
&= \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \int_0^{\bar{t}+\epsilon} (\bar{t} + \epsilon - \theta)^{n-\alpha-1} \bar{q}(\theta) d\theta \\
&= \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \int_0^{\bar{t}+\epsilon} (\bar{t} + \epsilon - \theta)^{n-\alpha-1} \bar{q}(\theta) d\theta \\
&= a\mathcal{D}\alpha q(t + \epsilon) = a\mathcal{D}\alpha q(\bar{t}) = a\mathcal{D}\alpha q(t) .
\end{align*}
$$

Using the notation in (14), one has $\tau = 1$ and $\xi = \sigma = \zeta = 0$. Conclusion (19) follows from Theorem 4.2.

5 Illustrative Examples

We begin by illustrating our results with two Lagrangians that do not depend explicitly on the time variable $t$. These two examples are borrowed from [13] §4.1 and [24] §3.1, where the authors write down the respective fractional Euler-Lagrange equations. Here, we use our Corollary 4.1 to obtain new fractional conservation laws.

Example 5.1 We begin by considering a simple fractional problem of the calculus of variations (see [11] Example 1) and [24] §3.1):

$$
I[q(\cdot)] = \frac{1}{2} \int_0^1 \left( a\mathcal{D}\alpha q(t) \right)^2 dt \longrightarrow \min , \quad \alpha > \frac{1}{2} .
$$

Equation (12) takes the form

$$
\mathcal{H} = -\frac{1}{2} p^2 .
$$

We conclude from Corollary 4.1 that

$$
P^2 \left( 1 - 2\alpha \right)
$$

is a fractional conservation law.

Example 5.2 Let us now consider the following fractional optimal control problem [3] §4.1:

$$
I[q(\cdot)] = \frac{1}{2} \int_0^1 \left[ q^2(t) + u^2(t) \right] dt \longrightarrow \min , \quad a\mathcal{D}\alpha q(t) = -q(t) + u(t) ,
$$

under the initial condition $q(0) = 1$. The Hamiltonian (12) has the form

$$
\mathcal{H} = \frac{1}{2} (q^2 + u^2) + p(-q + u).
$$

From Corollary 4.1 it follows that

$$
\frac{1}{2} (q^2 + u^2) + \alpha p(-q + u)
$$

is a fractional conservation law.

For $\alpha = 1$, the fractional conservation laws (22) and (24) give conservation of energy.

Finally, we give an example of an optimal control problem with three state variables and two controls ($n = 3, m = 2$). The problem is inspired in [12] Example 2].

Example 5.3 We consider the following fractional optimal control problem:

$$
\int_a^b \left( u_1(t)^2 + u_2(t)^2 \right) dt \longrightarrow \min ,
$$

\begin{equation}
\left\{ \begin{array}{ll}
\alpha \mathcal{D}\alpha q_1(t) = u_1(t) \cos(q_3(t)) , \\
\alpha \mathcal{D}\alpha q_2(t) = u_1(t) \sin(q_3(t)) , \\
\alpha \mathcal{D}\alpha q_3(t) = u_2(t) .
\end{array} \right.
\end{equation}

For $\alpha = 1$ the control system (26) serves as model for the kinematics of a car and (25)–(26) reduces to Example 2 of [12]. From Corollary 4.1 one gets that

$$
u_1^2 + u_2^2 + p_1 \left( u_1 \cos(q_3) - (1 - \alpha) u_2 q_1 \right)
$$

$$
+ p_2 \left( u_1 \sin(q_3) - (1 - \alpha) u_2 q_1 \right) + p_3 \left( u_2 - (1 - \alpha) u_2 q_3 \right)
$$

is a fractional conservation law.

Main difficulty of our approach is related with the computation of the invariance transformations. To illustrate this issue, let us consider problem (11) with

$$
L(t, q, u) = L(t, u) , \quad \varphi(t, q, u) = \varphi(t, u) .
$$

In the classical case, since $q$ does not appear both in $L$ and $\varphi$, such a problem is trivially invariant under translations on the variable $q$, i.e. condition (15) is verified for $\alpha = 1$ with $\bar{t} = t$, $\bar{q}(t) = q(t) + \epsilon$, $\bar{u}(t) = u(t)$ and $\bar{p}(\bar{t}) = p(t)$. In the fractional case this is not in general true: we have $d\bar{t} = dt$, but condition (15) is not satisfied since $a\mathcal{D}\alpha q(\bar{t}) = a\mathcal{D}\alpha q(t) + a\mathcal{D}\alpha \epsilon$ and the second term on the right-hand side is in general not equal to zero (Remark 2.2).
6 Conclusions

The fractional Euler-Lagrange equations are a subject of strong current study [15,18,29,14,17,25,26] because of its numerous applications. In [11] a fractional Noether’s theorem is proved.

The fractional Hamiltonian perspective is quite recent subject, being investigated in a serious of publications [4,10,21,22,23,24,28]. One can say, however, that the fractional variational theory is still in its childhood. Much remains to be done. This is particularly true in the area of fractional optimal control where results are a rarity. The main study of fractional optimal control problems seems to be [3], where the Euler-Lagrange equations for fractional optimal control problems (Theorem 4.1) are obtained, using the traditional approach of the Lagrange multiplier rule. Here we use the Lagrange multiplier technique to derive, from the results in [11], a new Noether-type theorem for fractional optimal control systems. Main result generalizes the results of [29]. As an application, we have considered the fractional autonomous problem, proving that the Hamiltonian defines a conservation law only in the integer case $\alpha = 1$.

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