Anomaly-free $W$-gravity Theories

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ABSTRACT

We give a review of some recent developments in the quantisation of $W$-gravity theories. In particular, we discuss the construction of anomaly-free $W_\infty$ and $W_3$ gravities.

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1. Introduction

Two-dimensional gravity has provided a rich and fascinating field of study in the last few years. One can view two-dimensional gravity as being the gauge theory of the Virasoro algebra in two dimensions. Since higher-spin extensions of the Virasoro algebra exist, it is natural to investigate the corresponding two-dimensional gauge theories. Many of these higher-spin algebras, known as $W$ algebras, have been discovered. The first, called $W_3$, contains a current of spin 3 in addition to the usual spin-2 current, the holomorphic energy-momentum tensor, of the Virasoro algebra [1]. Subsequently, generalisations to $W_N$ algebras were obtained, which contain currents of each spin $s$ in the interval $2 \leq s \leq N$ [2]. A general feature of these extended algebras is that they are non-linear, in the sense that the operator product of two currents may produce terms that must be viewed as composite operators, built from products of the fundamental currents in the algebra. The reason for this is that to leading order, the OPE of currents with spins $s$ and $s'$ gives a quantity of spin $s'' = s + s' - 2$, which may exceed $N$ if $N > 2$. This argument for the occurrence of non-linearities breaks down if $N = \infty$, and indeed a linear algebra of this kind, known as $W_\infty$, exists [3]. In most other respects, $W_\infty$ is qualitatively similar to the finite-$N$ algebras. For example, they all have non-trivial central terms in the OPE of any pair of equal-spin currents. There is another related algebra, known as $W_{1+\infty}$, which has currents of each spin $1 \leq s \leq \infty$ [3,4]. All of the $W_N$ algebras, with $N$ finite or infinite, admit contractions to algebras that may, in a sense to be clarified below, be thought of as classical limits. In the case of $W_\infty$, the corresponding contracted algebra is the $w_\infty$ algebra found in [5]. For $W_{1+\infty}$, its contraction, $w_{1+\infty}$, is isomorphic to the algebra of area-preserving diffeomorphisms on a cylinder [3,5].

In this article, we shall review the construction and quantisation of two $W$-gravity theories. The first of these is based upon the classical two-dimensional gauge theory of the $w_\infty$ algebra [6]. We shall see that in order to quantise the theory, it is necessary to renormalise the classical currents that generate the classical $w_\infty$ symmetry. After the dust has settled, one finds that the renormalised currents generate the full $W_\infty$ algebra [7]. Thus the classical $w_\infty$ gravity that one starts with ends up as quantum $W_\infty$ gravity. The second example that we shall consider here is that of $W_3$ gravity. One might think that the non-linearities of the $W_3$ algebra would make the analysis of this case much more difficult. Up to a point, this is true. However, the essential qualitative features are in fact remarkably similar to those of the $W_\infty$ case. The starting point in this second example is the two-dimensional gauge theory of a contraction of the $W_3$ algebra [8,9]. It would therefore be more appropriate to call this classical $w_3$ gravity, rather than $W_3$ gravity. Again, we shall see that the quantisation process requires that the classical $w_3$ currents must be renormalised. In the end, the renormalised currents generate the full $W_3$ algebra [10,11].
2. Classical and Quantum $W_\infty$ Gravity

The classical theory of $w_\infty$ gravity was constructed in [6]. In its simplest form, one can consider a chiral gauging of $w_\infty$; i.e. one gauges just one copy of the algebra, in, say, the left-moving sector of the two-dimensional theory. We shall discuss non-chiral gaugings in more detail later. For now we just remark that, thanks to an ingenious trick introduced in [9], involving the use of auxiliary fields, the treatment of the non-chiral case can be essentially reduced to two independent copies of the chiral case.

As our starting point, let us consider the free action $S = 1/\pi \int d^2 z L$ for a single scalar field in two dimensions, where $L$ is given by

$$L = \frac{1}{2} \bar{\partial} \varphi \partial \varphi. \tag{2.1}$$

Here, we use coordinates $z = x^-$ and $\bar{z} = x^+$ on the (Euclidean-signature) worldsheet. This action is invariant under the semi-rigid spin-$s$ transformations

$$\delta \varphi = \sum_{s \geq 2} k_s (\partial \varphi)^{s-1}, \tag{2.2}$$

where the parameters $k_s$ are functions of $z$, but not $\bar{z}$. These transformations are generated by the currents

$$v_s(z) = \frac{1}{s} (\partial \varphi)^s. \tag{2.3}$$

At the classical level, these currents generate the $w_\infty$ algebra. In operator-product language, this means that they close on the $w_\infty$ algebra at the level of single contractions. To keep track of the orders it is useful to introduce Planck’s constant, so that the OPE of the field $\varphi$ is

$$\partial \varphi(z) \partial \varphi(w) \sim \frac{\hbar}{(z-w)^2}. \tag{2.4}$$

The OPEs of the currents are then given by

$$\hbar^{-1} v_s(z) v_{s'}(w) \sim (s + s' - 2) \frac{v_{s+s'-2}(w)}{(z-w)^2} + (s-1) \frac{\partial v_{s+s'-2}(w)}{z-w} + O(\hbar). \tag{2.5}$$

The $\hbar$-independent terms on the right-hand side are precisely those for the $w_\infty$ algebra.

The classical semi-rigid $w_\infty$ symmetry (2.2) of (2.1) can be gauged by introducing a gauge field $A_s$ for each current $v_s$. Thus we find that the Lagrangian

$$L = \frac{1}{2} \bar{\partial} \varphi \partial \varphi - \sum_{s \geq 2} A_s v_s \tag{2.6}$$
is invariant under local $w_\infty$ transformations [6], where the gauge fields are assigned the transformation rules:

$$\delta A_s = \bar{\partial} k_s - \sum_{s'=2}^s \left( (s'-1) A_{s'} \partial k_{s-s'+2} - (s-s'+1) k_{s-s'+2} \partial A_{s'} \right). \quad (2.7)$$

When one is presented with a classical theory it is natural, when considering its quantisation, to begin by contemplating what might go wrong. In the case of a gauge theory, with classical local symmetries, the obvious danger is that these might become anomalous at the quantum level. Indeed, in the case of two-dimensional gravity, the gauge theory of the Virasoro algebra, we know that anomaly freedom requires that the matter fields should generate the Virasoro algebra with central charge $c = 26$, in order to cancel the central-charge contribution of $-26$ from the ghosts for the gauge fixing of the spin-2 gauge field (the metric). We can certainly expect to meet analogous anomalies in the higher-spin generalisations that we are considering here. In fact, potentially-worse things could also happen:

Commonly, as for example in the case of the Virasoro algebra and two-dimensional gravity, one has matter currents that are quadratic in matter fields. These, by construction, generate the gauge algebra at the classical (single-contraction) level. Upon quantisation, higher numbers of contractions must also be taken into account, corresponding to Feynman diagrams with closed loops. If the currents are at most quadratic in matter fields, then the “worst case” is to have two contractions between a pair of currents. This corresponds therefore to a pure $c$-number term in the OPE of currents; in fact, the central term in the Virasoro algebra. It is these terms that in fact save the critical two-dimensional gravity theory from anomalies, by cancelling against anomalies from the ghost sector.

Things are potentially worse in the case of $w_\infty$ gravity because now the currents (2.3) involve arbitrarily-high powers of the matter field $\varphi$. Thus at the quantum level, one might get matter-dependent anomalies, associated with diagrams corresponding to multiple contractions of the currents (2.3) that still have some matter fields left uncontracted. Of course the question of whether a theory is actually anomalous is really a cohomological one, in the sense that the crucial question is whether or not it is possible to introduce finite local counterterms, and $h$-dependent renormalisations of the classical transformation rules in such a way as to remove the apparently-anomalous contributions of the kind we have been considering. Only if such an attempt fails can the theory be said to be anomalous.

In general, the process of quantising a theory, and introducing counterterms and renormalisations of the transformation rules order by order in $h$ to remove potential anomalies, can be a complicated and tedious one. Fortunately, in our two-dimensional example the process of removing potential matter-dependent anomalies can be accomplished in one fell swoop. All that we have to do is to find quantum renormalisations of the classical currents (2.3) such that at the full quantum level (arbitrary numbers of contractions in the OPEs) they close on an algebra. This algebra, whatever it turns out to be, will be the quantum
renormalisation of the original classical $w_\infty$ algebra. In fact, as we shall see, it is precisely $W_\infty [7]$.

The problem, then, boils down to the fact that the currents (2.3) do not close on any algebra at the quantum level. We must therefore seek $\hbar$-dependent modifications of them, with a view to achieving quantum closure. The most general plausible modifications would consist of higher-order terms added to (2.3) in which the same number of derivatives (to give the same spin $s$) are distributed over smaller numbers of $\varphi$ fields. From (2.4), we see that $\varphi$ has the dimensions of $\sqrt{\hbar}$, and so the modifications will be of the form of power series in $\sqrt{\hbar}$. Thus we may try an ansatz of the form

$$V_s = \frac{1}{s} (\partial\varphi)^s + \alpha_s \sqrt{\hbar} (\partial\varphi)^{s-2} \partial^2 \varphi + \beta_s \hbar (\partial\varphi)^{s-3} \partial^3 \varphi + \gamma_s \hbar (\partial\varphi)^{s-4} (\partial^2 \varphi)^2 + O(\hbar^{3/2}), \quad (2.8)$$

for constant coefficients $\alpha_s, \beta_s, \gamma_s, \cdots$ to be determined. Requiring quantum closure of the OPE algebra for these currents will then give conditions on this infinite number of coefficients. They will not be determinable uniquely, since one is always free to make redefinitions of currents of the form $V_s \rightarrow V_s + \partial V_{s-1} + \cdots$. However, if, for convenience and without loss of generality, we demand that the currents should all be quasiprimary with respect to the energy-momentum tensor $V_2$, then the result is unique. The expressions for the first few renormalised currents (spins 2, 3 and 4) are [7]:

$$V_2 = \frac{1}{2} (\partial\varphi)^2 + \frac{1}{2} \sqrt{\hbar} \partial^2 \varphi,$$

$$V_3 = \frac{1}{3} (\partial\varphi)^3 + \frac{1}{2} \sqrt{\hbar} \partial\varphi \partial^2 \varphi + \frac{1}{12} \hbar \partial^3 \varphi,$$

$$V_4 = \frac{1}{4} (\partial\varphi)^4 + \frac{1}{2} \sqrt{\hbar} (\partial\varphi)^2 \partial^2 \varphi - \frac{1}{20} \hbar (\partial^2 \varphi)^2 + \frac{1}{5} \hbar \partial\varphi \partial^3 \varphi + \frac{1}{60} \hbar^{3/2} \partial^4 \varphi. \quad (2.9)$$

These renormalised currents can be recognised as the currents of the $W_\infty$ algebra in the following way: We know that there is a realisation of $W_{1+\infty}$ in terms of bilinear currents built from a free complex fermion [12]. We also know that one can make a one-parameter family of rotations of the basis for $W_{1+\infty}$, retaining the quasiprimary nature of all the currents, such that for one member of the family, the spin-1 current can be truncated out, leaving currents of spins $\geq 2$ that generate $W_\infty [13, 14]$. The first few such currents are:

$$V_2 = \partial \bar{\psi} \psi,$$

$$V_3 = \frac{1}{2} \partial^2 \bar{\psi} \psi - \frac{1}{2} \partial \bar{\psi} \partial \psi,$$

$$V_4 = \frac{1}{5} \partial^3 \bar{\psi} \psi - \frac{3}{8} \partial^2 \bar{\psi} \partial \psi + \frac{1}{5} \partial \bar{\psi} \partial^2 \psi. \quad (2.10)$$

It is then straightforward to see that the currents (2.9) are nothing but the bosonisation of the fermionic currents (2.10) [15, 7], where we write

$$\psi = : e^{\varphi/\sqrt{\hbar}} :,$$

$$\bar{\psi} = : e^{-\varphi/\sqrt{\hbar}} :. \quad (2.11)$$
Having obtained renormalised currents that close on an algebra (the \( W_\infty \) algebra) at the full quantum level, we are now guaranteed to have a quantum theory with no matter-dependent anomalies. The prescription for writing down the counterterms and renormalisations of the transformation rules (2.2) and (2.7) necessary to make this anomaly freedom manifest is very straightforward. For the counterterms, we simply replace the classical currents \( v_s \) in the Lagrangian (2.6) by the renormalised currents \( V_s \), of which the first few are given by (2.9). The \( \hbar \)-dependent terms are the necessary counterterms. For the transformation rules, we use the ones that are generated by the renormalised currents. For \( \varphi \), this means we have

\[
\delta \varphi = \hbar^{-1} \sum_{s \geq 2} \oint \frac{dz}{2\pi i} k_s(z)V_s(z)\varphi(w).
\]  
(2.12)

For \( A_s \), we will have

\[
\delta A_s = \bar{\partial} k_s + \delta A_s,
\]  
(2.13)

where \( \delta A_s \) is such that

\[
\sum_{s \geq 2} \int \left( \delta A_s V_s + A_s \delta V_s \right),
\]  
(2.14)

with \( \delta V_s \) given by

\[
\delta V_s = \hbar^{-1} \sum_{s' \geq 2} \oint \frac{dz}{2\pi i} k_{s'}(z)V_{s'}(z)V_s(w).
\]  
(2.15)

The \( \hbar \)-independent terms in these transformation rules are precisely the original classical ones (2.2) and (2.7). The \( \hbar \)-dependent terms are the renormalisations necessary, together with the counterterms, to make the absence of matter-dependent anomalies manifest. The fact that the potential matter-dependent anomalies are actually removable by this means, i.e. that they are cohomologically trivial, is a consequence the fact that it is possible to renormalise the classical currents (2.3) to give currents that do close at the quantum level.

So far, we have been concerned here only with the question of matter-dependent anomalies. This is an issue that does not even arise for usual formulations of two-dimensional gravity, since the Virasoro symmetry is usually realised linearly. We must still face the analogue of the anomaly that one \( \textit{does} \) meet in two-dimensional gravity, namely the universal anomaly that is cancelled by choosing a \( c = 26 \) matter realisation in order to cancel against the \(-26\) contribution to the total central charge coming from the gravity ghosts. For \( W_\infty \) gravity, we face a more serious-looking problem, since now there will be ghosts associated with the fixing of the gauge symmetry for each of the gauge-fields \( A_s \). The ghosts for a spin-\( s \) gauge field contribute

\[
c_{gh}(s) = -12s^2 + 12s - 2
\]  
(2.16)

to the ghostly central charge. Summing over all \( s \geq 2 \) would seem to imply that the total ghostly central charge is \( c_{gh}(\text{tot}) = -\infty \). At best, an infinity of matter fields seem to be needed, and even then, the process of cancelling the universal anomaly could be a delicate
one. There is, however, a different approach that one could take. The sum of $c_{gh}(s)$ over all $s \geq 2$ can be regularised using zeta-function techniques, to give

$$c_{gh}(\text{tot}) = -\sum_{n \geq 0} \left( 6(n + \frac{3}{2})^2 - \frac{1}{2} \right)$$

$$= -6\zeta(-2, \frac{3}{2}) + \frac{1}{2}\zeta(0, \frac{3}{2})$$

$$= 2,$$

(2.17)

where $\zeta(s, a)$ is the generalised Riemann zeta function, defined by analytic continuation from $\zeta(s, a) = \sum_{n \geq 0} (n + a)^{-s}$ in the region $\mathcal{R}(s) > 1$. Of course this regularisation procedure looks somewhat arbitrary, but actually there is very strong evidence to suggest that there is an underlying justification for it. The ghost anomalies for $W_\infty$ have been analysed in detail in a BRST approach in [16,17]. Not only does one have an anomaly in the spin-2 sector, with central charge given by (2.17), but also there are related anomalies in all the higher-spin sectors, corresponding to the central terms in the OPEs of each higher-spin current with itself. The relative values of all these central terms are related in the $W_\infty$ algebra, with just one overall scale parameter free (the central charge $c$). Thus for the ghost currents to provide a realisation of the algebra, it is necessary that all the higher-spin anomalies must regularise in a self-consistent way. This was examined up to the spin-18 level in [17], and it was found that the only plausible regularisation scheme was the one that gives $c_{gh}(\text{tot}) = 2$. One may hope that there is some (higher-dimensional?) explanation for this that will eventually emerge.

Assuming for now that the $c_{gh}(\text{tot}) = 2$ result is to be taken seriously for $W_\infty$, it follows that the cancellation of the universal anomalies will occur provided that the matter realisation of $W_\infty$ has central charge $c_{\text{mat}} = -2$. Remarkably, this is precisely what we have for our single-scalar realisation! One can easily check from (2.9) that the background-charge term is precisely such as to give $c = -2$. Thus, in a regularised sense at least, the $W_\infty$ gravity theory that we have constructed is free of all anomalies [7]. This includes not only the spin-2 anomaly, of the form $(c_{gh}(\text{tot}) + c_{\text{mat}}) \int k_2 \partial^3 A_2$, but also the higher-spin anomalies, of the form $C_s \int k_s \partial^{s+1} A_s$. For the same reasons as discussed above, all of the coefficients $C_s$ will vanish simultaneously, provided that $c_{\text{mat}}$ is equal to $-2$.

For now, the possible cancellation of the regularised universal anomalies should perhaps be viewed as an amusing observation that may ultimately turn out to have some deeper underlying explanation. Perhaps the more important lesson to be derived from looking at the quantisation of classical $w_\infty$ gravity is that the key requirements are that one should be able to renormalise the classical currents so that they close on an algebra at the full quantum level. This ensures the absence of matter-dependent anomalies. Furthermore, if the central charge for the matter currents is chosen to cancel that from the ghosts for the gauge fields, then the universal anomalies will cancel also. These desiderata can be summarised succinctly in one equation: we require that the BRST operator $Q$ should be nilpotent.
3. Classical and Quantum $W_3$ gravity

The philosophy for quantising classical $w_3$ gravity is essentially the same as that of the previous section. The starting point is the classical matter Lagrangian \[ L = \frac{1}{2} \partial \varphi_i \partial \varphi_i - hT - BW, \] (3.1)

where $\varphi_i$ denotes a set of $n$ real matter fields; $h$ and $B$ are spin-2 and spin-3 gauge fields; and the spin-2 and spin-3 matter currents $T$ and $W$ are given by

\[ T = \frac{1}{2} \partial \varphi_i \partial \varphi_i, \]
\[ W = \frac{1}{3} d_{ijk} \partial \varphi_i \partial \varphi_j \partial \varphi_k. \] (3.2)

The quantity $d_{ijk}$ is a totally-symmetric constant tensor that satisfies

\[ d_{(ij} \delta_{lm)} = \mu \delta_{(ij} \delta_{lm)}. \] (3.3)

At the classical level (single contractions), the currents generate what we may call the $w_3$ algebra,

\[ h^{-1} T(z) T(w) \sim \frac{\partial T}{z - w} + \frac{2T}{(z-w)^2} + O(h), \]
\[ h^{-1} T(z) W(w) \sim \frac{\partial W}{z - w} + \frac{3W}{(z-w)^2} + O(h), \]
\[ h^{-1} W(z) W(w) \sim \frac{\partial \Lambda}{z - w} + \frac{2\Lambda}{(z-w)^2}, \] (3.4)

where $\Lambda$ is the composite current \[ \Lambda = 2\mu(TT). \] (3.5)

This $w_3$ algebra is a classical limit of the full $W_3$ algebra, which is given below.

Various possible solutions for the tensor $d_{ijk}$, satisfying (3.3), have been found [18]. They fall into two categories. The first consists of solutions for an arbitrary number of scalars $n$, with the components of the (totally symmetric) tensor $d_{ijk}$ given by

\[ d_{111} = n, \quad d_{1ab} = -n \delta_{ab}; \quad (a = 2, \ldots, n). \] (3.6)

This satisfies (3.3) with $\mu = n^2$. The second category of solution relies upon the abnormalities and perversities of the Jordan algebras. There are four such solutions, with $n = 5, 8, 14$ and 26 scalars, corresponding to invariant tensors of Jordan algebras over the reals, complex numbers, quaternions and octonions respectively [18]. For the complex case, with $n = 8$, the $d_{ijk}$ tensor coincides with the symmetric $d_{ijk}$ tensor of $SU(3)$.

As in the case of the currents (2.3) that generate the $w_\infty$ contraction of $W_\infty$ classically, so also here the currents (3.2) generate the $w_3$ contraction of $W_3$ classically. At the full
quantum level of multiple contractions in the operator-product expansion, one finds that the classical $w_3$ currents (3.2) fail to close on any algebra. This is the signal for potential trouble with matter-dependent anomalies upon quantisation of the theory. The remedy is again to look for quantum renormalisations of the currents (3.2) to give currents that do generate an algebra at the quantum level. Modulo the freedom to make field redefinitions, the answer, as in the $w_\infty$ case, is unique. In this case, it turns out that the resulting algebra on which the renormalised currents close is $W_3$. This algebra takes the form [1]

\begin{align}
\hbar^{-1} T(z)T(w) &\sim \frac{\partial T(w)}{z-w} + \frac{2T(w)}{(z-w)^2} + \frac{c/2}{(z-w)^4} \\
\hbar^{-1} T(z)W(w) &\sim \frac{\partial W}{z-w} + \frac{3W(w)}{(z-w)^2} \\
\hbar^{-1} W(z)W(w) &\sim \frac{1}{z-w} \left( \frac{1}{15} \hbar \partial^3 T(w) + \frac{16}{22 + 5c} \partial \Lambda(w) \right) \\
&+ \frac{1}{(z-w)^2} \left( \frac{3}{10} \hbar \partial^2 T(w) + \frac{32}{22 + 5c} \Lambda(w) \right) \\
&+ \hbar^2 \frac{\partial^2 T(w)}{(z-w)^3} + \hbar \frac{2T(w)}{(z-w)^2} + \hbar^2 \frac{c/3}{(z-w)^4} + \hbar^2 \frac{c/3}{(z-w)^6} \tag{3.7c}
\end{align}

In (3.7c), $\Lambda$ is a composite current:

$$\Lambda = (TT) - \frac{3}{10} \hbar \partial^2 T,$$  

(3.8)

where the normal ordering is taken with respect to the modes of the currents $T$, according to the prescription [19]

$$(AB)(w) \equiv \oint \frac{dz}{z-w} A(z)B(w).$$  

(3.9)

The possible renormalisations of the currents (3.2) can be parametrised by

\begin{align}
T &= \frac{1}{2} \partial \varphi^i \partial \varphi^i + \sqrt{\hbar} \alpha_i \partial^2 \varphi^i, \\
W &= \frac{1}{4} d_{ijk} \partial \varphi^i \partial \varphi^j \partial \varphi^k + \sqrt{\hbar} e_{ij} \partial \varphi^i \partial^2 \varphi^j + \hbar f_i \partial^3 \varphi^i, \tag{3.10}
\end{align}

The requirement that the currents generate the $W_3$ algebra gives a set of conditions on the coefficients $d_{ijk}$, $\alpha_i$, $e_{ij}$ and $f_i$ that may be found in [18]. The upshot is that the general family of classical currents, with $d_{ijk}$ given by (3.6) for the case of $n$ scalars, can be successfully renormalised to give currents that close at the quantum level, on the $W_3$ algebra [18]. We shall give the form of the renormalised currents below. For the four exceptional cases based on the Jordan algebras, however, it appears that it is not possible to renormalise the currents with $\hbar$-dependent corrections so as to achieve closure [18,20]. This includes the special 8-scalar realisation based on the totally-symmetric $d_{ijk}$ tensor of $SU(3)$. There is thus no sense in which the currents in these exceptional cases could be said to be $W_3$. 

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currents. Consequently the quantum theory of \( w_3 \) gravity for any of the four exceptional cases will definitely suffer from anomalies. On the other hand, for the general family of \( n \)-scalar realisations with \( d_{ijk} \) given by (3.6), as we shall see below, all anomalies can be cancelled.

Having established that the matter currents for the \( n \)-scalar realisations (3.2), (3.6) can be renormalised to give currents that close, on the \( W_3 \) algebra, we are now able to proceed to the next stage in the quantisation procedure, by looking for a nilpotent BRST charge. Fortunately the construction has already been carried out for \( W_3 \) [21]. Despite the non-linearities of the algebra, it turns out that the structure of the BRST charge is quite similar to that for a linear algebra. The main difference is that now the ghost currents involve matter currents. Thus we still have [21]

\[
Q = \oint dz \left( c(T + \frac{1}{2} T_{gh}) + \gamma(W + \frac{1}{2} W_{gh}) \right),
\]

(3.11)

where \( T \) and \( W \) are the matter currents that generate the \( W_3 \) algebra; \( T_{gh} \) and \( W_{gh} \) are the ghost currents; and the ghost-antighost pairs \((c,b)\) and \((\gamma,\beta)\) correspond respectively to the \( T \) and \( W \) generators. For nilpotency, we can see from (2.16) that the central charge for the matter currents must be

\[
c = 26 + 74 = 100.
\]

(3.12)

From now on, we shall assume therefore that the matter currents satisfy the \( W_3 \) algebra (3.7a-c) with \( c = 100 \). The ghost currents \( T_{gh} \) and \( W_{gh} \) are then given by [21]

\[
\begin{align*}
T_{gh} &= -2b \partial c - \partial b c - 3\beta \partial \gamma - 2\partial \beta \gamma \\
W_{gh} &= -\partial \beta c - 3\beta \partial c - \frac{8}{261} [\partial(b \gamma T) + b \partial \gamma T] \\
&\quad + \frac{25}{6 \cdot 261} h \left( 2\gamma \partial^3 b + 9\partial \gamma \partial^2 b + 15\partial^2 \gamma \partial b + 10\partial^3 \gamma b \right).
\end{align*}
\]

(3.13a)

(3.13b)

Note that the spin-3 ghost current \( W_{gh} \) involves the spin-2 matter current \( T \). This looks intuitively reasonable; one can view the non-linear terms on the right-hand side of (3.7c) as being like a linear algebra but with \( T \)-dependent structure “constants.” These structure constants then appear in the construction of the ghost currents. Note also that the ghost currents need not, and indeed do not, satisfy the \( W_3 \) algebra [21,22]. It is shown in [21,22] that, provided the matter central charge is given by (3.12), the BRST operator (3.11) is indeed nilpotent.

The remaining ingredient needed for constructing anomaly-free \( W_3 \) gravity is a matter realisation of the \( W_3 \) algebra with central charge \( c = 100 \). A 2-scalar realisation, with background charge that can be tuned to give, in particular, \( c = 100 \), was obtained in [2] by making use of the quantum Miura transformation. The most general known realisations in terms of scalar fields are the \( n \)-scalar realisations found in [18], which correspond to the
renormalisations (3.10) of the classical currents (3.2) with \(d_{ijk}\) given by (3.6). At \(c = 100\), these take the form

\[
T = T_X + \frac{1}{2}(\partial \varphi_1)^2 + \frac{1}{2}(\partial \varphi_2)^2 + \sqrt{h}(\alpha_1 \partial^2 \varphi_1 + \alpha_2 \partial^2 \varphi_2) \tag{3.14a}
\]

\[
W = \frac{2}{\sqrt{261}} \left\{ \frac{1}{3}(\partial \varphi_1)^3 - \partial \varphi_1 (\partial \varphi_2)^2 + \sqrt{h}(\alpha_1 \partial \varphi_1 \partial^2 \varphi_1 - 2\alpha_2 \partial \varphi_1 \partial^2 \varphi_2 - \alpha_1 \partial \varphi_2 \partial^2 \varphi_2) \right. \\
+ h(\frac{1}{3} \alpha_1^2 \partial^3 \varphi_1 - \alpha_1 \alpha_2 \partial^3 \varphi_2) - 2\partial \varphi_1 T_X - \alpha_1 \sqrt{h} \partial T_X \bigg\}, \tag{3.14b}
\]

where \(T_X\) is a stress tensor for \(D = n - 2\) scalar fields without background charges,

\[
T = \frac{1}{2} \sum_{\mu=1}^{D} \partial X^\mu \partial X^\mu, \tag{3.15}
\]

and the background charges \(\alpha_1\) and \(\alpha_2\) for \(\varphi_1\) and \(\varphi_2\) are given by

\[
\begin{align*}
\alpha_1^2 &= -\frac{49}{8} \\
\alpha_2^2 &= \frac{1}{12}(D - \frac{49}{2}).
\end{align*} \tag{3.16}
\]

These conditions on the background charges ensure that the matter central charge satisfies

\[
c = D + (1 - 12\alpha_1^2) + (1 - 12\alpha_2^2) = 100. \tag{3.17}
\]

Note that no matter how many scalar fields one chooses, including \(n = 100\), it is necessary to have background charges in order to achieve \(c = 100\).

Now that we have obtained a nilpotent BRST operator, and appropriate matter realisations of the \(W_3\) algebra, it is completely straightforward to write down a Lagrangian for anomaly-free \(W_3\) gravity. We shall not give the detailed result here; it may be found in [10]. Here, we just remark that it is obtained from the general BRST prescription:

\[
\mathcal{L} = \frac{1}{2} \bar{\partial} \varphi^i \partial \varphi^i - hT - BW + \delta \left( b(h - h_{\text{back}}) + \beta(B - B_{\text{back}}) \right), \tag{3.18}
\]

where \(\delta\) denotes the BRST variation, which can be deduced from the BRST operator (3.11), and \(h_{\text{back}}\) and \(B_{\text{back}}\) denote background gauge-fixed values for the spin-2 and spin-3 gauge fields \(h\) and \(B\). Thus one has [10]

\[
\mathcal{L} = \frac{1}{2} \bar{\partial} \varphi^i \partial \varphi^i - b\bar{\partial}c - \beta\bar{\partial}g \\
+ \pi_b(h - h_{\text{back}}) + \pi_\beta(B - B_{\text{back}}) - h(T + T_{\text{gh}}) - B(W + W_{\text{gh}}), \tag{3.19}
\]

where \(\delta b = \pi_b\) and \(\delta \beta = \pi_\beta\). As in section 2, the \(h\)-independent terms in (3.19) (with matter currents \(T\) and \(W\) given by (3.14a,b), and ghost currents \(T_{\text{gh}}\) and \(W_{\text{gh}}\) given by (3.13a,b))
represent the classical Lagrangian (plus ghost Lagrangian), and the $\hbar$-dependent terms correspond to counterterms necessary for the explicit removal of the potential anomalies.

4. Discussion

In this paper, we have reviewed some of the aspects of the quantisation of $W_\infty$ and $W_3$ gravities. Our discussion has been concerned entirely with chiral $W$ gravity, in the sense that we have considered gaugings only of a single (left-moving) copy of the algebra. As remarked at the beginning of section 2, it is completely straightforward to extend all of the discussions in this paper to the non-chiral case by exploiting the ingenious trick, introduced in [9], of using additional, auxiliary, fields. Thus, for example, for $W_3$ gravity we introduce auxiliary fields $J^i$ and $\tilde{J}^i$, and write the classical Lagrangian as

$$L = -\frac{1}{2} \partial \phi^i \partial \phi^i + J^i \tilde{J}^i + J^i \tilde{J}^i + J^i \tilde{J}^i,$$

where the tilded variables refer to a second (right-moving) copy of the gauge algebra. The equations of motion for the auxiliary fields are

$$J^i = \partial \phi^i - \hbar \tilde{J}^i - B_{ijk} J^j J^k - \frac{1}{2} \hbar \tilde{J}^i \tilde{J}^i - \frac{1}{3} \tilde{B}_{ijk} \tilde{J}^i \tilde{J}^j \tilde{J}^k,$$

$$\tilde{J}^i = \partial \phi^i - \hbar J^i - B_{ijk} J^j J^k,$$

which can be recursively solved to give $J^i$ and $\tilde{J}^i$ as non-polynomial expressions in $\phi^i$ and the gauge fields. Upon quantisation, one finds that the auxiliary fields have the propagators

$$J^i(z)J^j(w) \sim \frac{\hbar \delta^{ij}}{(z-w)^2},$$

$$\tilde{J}^i(z)\tilde{J}^j(\bar{w}) \sim \frac{\hbar \delta^{ij}}{(z-\bar{w})^2},$$

$$J^i(z)\tilde{J}^j(\bar{w}) \sim 0.$$

Thus the whole problem has been cloven into separate left-moving and right-moving sectors. The left-moving matter and ghost currents are now constructed, at the full quantum level, by replacing $\partial \phi^i$ in (3.10), etc, by $J^i$. Similarly, one uses $\tilde{J}^i$ in the construction of analogous right-moving currents. Full details may be found in [11].

An obvious application for anomaly-free $W_3$ gravity is in the construction of the $W_3$ extension of string theory, i.e. $W_3$ strings. The idea is that the equations of motion for the spin-2 and spin-3 gauge fields impose the vanishing of the spin-2 and spin-3 currents. At the quantum level, these conditions can be interpreted, as in ordinary string theory, as operator constraints on physical states. By interpreting the scalar fields in the matter realisations (3.14a,b) as spacetime coordinates, one arrives at a first-quantised description of $W_3$-string
excitations in an \( n \)-dimensional spacetime. Because of the necessity for background charges one does not have the full \( SO(1,n-1) \) Lorentz group acting, but only \( SO(1,n-3) \). The issues arising in the analysis of the spectrum of \( W_3 \) strings are quite involved. Preliminary discussions were given in \cite{11}, and a more extensive analysis is contained in \cite{23}. Here, we just summarise a couple of the main results.

One can see from the form of the realisations (3.14a,b) that the scalar \( \varphi_1 \) is on a very special footing. In fact all the remaining scalars (\( \varphi_2 \) and \( X^\mu \)) appear only via their stress tensor. (In the case of \( \varphi_2 \), it has a background-charge contribution in the stress tensor.) Thus, in some sense the scalar \( \varphi_1 \) is the only one which is intrinsically “non-stringy” in nature. The spin-3 current \( W \) can be written as a sum of terms that involve only \( \varphi_1 \), plus terms involving the total stress tensor \( T \). It is then rather easy to see at the classical level that having imposed the \( T = 0 \) constraint, the \( W \) constraint reduces to the statement that \( \varphi_1 = \text{constant} \) \cite{11}. At the first-quantised level, this becomes the statement that physical states cannot involve any \( \alpha_{-n} \) creation operators in the \( \varphi_1 \) direction \cite{23}. The conclusion from this is that the sole effect of the \( W \) constraint is to “freeze out” the \( \varphi_1 \) coordinate: One is left with a theory that looks remarkably like ordinary string theory. The generalisation to \( W_3 \) strings introduces the new feature of the non-stringy coordinate \( \varphi_1 \), but this is then removed again by the new \( W \) constraint, so it seems that the Lord giveth, and the Lord taketh away. Unfortunately, it seems that one is left with a string-like theory that lacks certain of the attractive features of normal string theory; in particular, it seems not to describe any massless states \cite{23}. Of course one may hope that this aspect is not generic to all higher-spin extensions. In particular, it would be interesting to see what happens for a supersymmetric extension of the \( W_3 \) algebra. This is currently under investigation \cite{24}.

There are other aspects of the quantisation of \( W \)-gravity theories that we have not touched on here. In particular, there is the very interesting problem of constructing the \( W_3 \) analogue of the Polyakov induced action of two-dimensional gravity \cite{25}. To do this, one wants to choose a matter realisation with non-critical value for the central charge, so that the universal anomaly “brings to life” the Liouville field (and its higher-spin analogues). Considerable progress in this area has been made \cite{26}. There seems to be a certain sense in which there is really no such thing as a “non-critical” theory, since the Liouville fields will always come to the rescue and make up the deficit in the central charge. It would be interesting to see whether there is ultimately a convergence of the critical and non-critical approaches. Similar issues have also been considered for \( W_\infty \) and \( W_{1+\infty} \) gravity. In particular, it has been shown that the hidden \( SL(2,R) \) Kac-Moody symmetry of light-cone two-dimensional gravity \cite{27} generalises to \( SL(\infty,R) \) for the \( W_\infty \) case, and \( GL(\infty,R) \) for the \( W_{1+\infty} \) case \cite{28}.

Two-dimensional gravity has proved to be an exceptionally interesting field of study. \( W \) gravity promises to be an even richer one.
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