ON THE REGULARITY OF GEODESIC RAYS ASSOCIATED TO TEST CONFIGURATIONS

D.H. Phong* and Jacob Sturm†

Abstract

Geodesic rays of class $C^{1,1}$ are constructed for any test configuration of a positive line bundle $L \to X$, using resolution of singularities. The construction reduces to finding a subsolution of the corresponding Monge-Ampère equation. Geometrically, this is accomplished by the use a positive line bundle on the resolution which is trivial outside of the exceptional divisor.

1 Introduction

Let $X$ be a compact complex manifold, $L \to X$ an ample line bundle, and $\mathcal{H}$ the space of Kähler metrics $\omega$ on $X$ with $\omega \in c_1(L)$. The work of Mabuchi [12], Semmes [15] and Donaldson [8] shows that $\mathcal{H}$ is a infinite-dimensional non-positively curved symmetric space with respect to a natural Riemannian structure. Moreover, according to the program proposed by Donaldson [8], the existence of metrics of constant scalar curvature in $\mathcal{H}$ is linked to the existence of geodesic rays in $\mathcal{H}$, and the uniqueness of such metrics is linked to the existence of geodesic segments in $\mathcal{H}$ (which we shall refer to as Donaldson rays and Donaldson segments). On the other hand, the conjecture of Yau-Tian-Donaldson [22, 19, 9] gives a necessary and sufficient criterion for the existence of constant scalar curvature metrics which can be formulated in terms of Bergman geodesics. (Recall that the space $\mathcal{H}_k$ of Bergman metrics is just the space of pullbacks of the Fubini-Study metric by the Kodaira imbeddings defined by bases of $H^0(X, L^k)$.) Thus it is natural to look for a direct link between Bergman segments/rays and Donaldson segments/rays.

This program was carried out for geodesics segments in [13]: If $h_0, h_1 \in \mathcal{H}$, then we showed that there is a sequence of Bergman geodesics $h_t(k)$, with $0 \leq t \leq 1$, whose limit is a $C^{1,1}$ Donaldson geodesic $h_t$ joining $h_0$ to $h_1$. The proof of this theorem consists of three main steps:

1) The limit of the $h_t(k)$ is a weak geodesic $h_t$ joining $h_0$ to $h_1$.
2) There exists some $C^{1,1}$ geodesic $\tilde{h}_t$ joining $h_0$ to $h_1$.
3) The two geodesics are equal: $h_t = \tilde{h}_t$.

---

*Research supported in part by National Science Foundation grants DMS-02-45371 and DMS-05-14003
The first step was carried out in [13]. The main ingredients are the Tian-Yau-Zelditch [23] expansion and the monotonicity theorem of Bedford-Taylor [3].

The second step is due to Chen [5]. The main ingredients are the interior $C^2$ estimates of Yau [21], the general theory of Caffarelli, Kohn, Nirenberg, and Spruck [7], and the boundary $C^2$ estimate of Guan [10] for the Monge-Ampère equation. The last part makes use of a blow-up analysis of the solution.

The third step is also in [13]. The main ingredient is the pluri-potential capacity theory of Bedford-Taylor [4].

Now we turn to the program for geodesic rays. We would like to implement the same three steps which were used to solve the geodesic segment problem. To describe the results, we first let $\mathcal{H}$ be the space of positively curved metrics on $L$, so that $\mathcal{H}/\mathbb{R}$ can be identified with $\mathcal{H}$ (it is somewhat easier to formulate the results for $\mathcal{H}$). Step 1) for geodesics rays was carried out in [14]. We showed that if $h_0 \in \mathcal{H}$, and if $\rho : \mathcal{C}^k \to \text{Aut}(\mathcal{L} \to \mathcal{X} \to \mathbb{C})$ is a test configuration $T$, then there exists a weak geodesic ray emanating from $h_0$ and “pointing in the direction” of $T$. This ray is defined as a limit of a certain natural sequence of Bergman rays $h_t(k)$ with $t \in [0, \infty)$. The proof uses again the Tian-Yau-Zelditch expansion and the Bedford-Taylor monotonicity theorem. The main new ingredients are Donaldson’s imbedding theorem for test configurations (which is spelled out in detail in [14]) and a careful analysis of the asymptotic expansion of the Bergman rays as $t \to \infty$.

The purpose of this note is to carry out step 2) for geodesic rays. We shall show that if $h_0 \in \mathcal{H}$ and if $T$ is a test configuration, then there is a $C^{1,1}$ geodesic ray $\tilde{h}_t$ emanating from $h_0$ with the following property: $\tilde{h}_t$ extends to a solution of the Dirichlet problem for the Monge-Ampère equation on a resolution of singularities $\tilde{\mathcal{X}} \to \mathcal{X}$. To do this, we first construct a subsolution of the associated Monge-Ampère equation. The existence of a subsolution is not automatic, since the underlying manifolds are not strictly pseudo-convex. After the subsolution is constructed, the arguments of [7, 10, 5] are used to construct the geodesic $\tilde{h}_t$.

In order to complete the program for geodesics rays, we need to implement step 3), which is the identification of $h_t$ with $\tilde{h}_t$. This will be considered in a forthcoming paper.

Recently, there has been considerable activity in the study of Donaldson geodesics. In [1], Arezzo-Tian have constructed families of analytic geodesic rays near infinity for test configurations with smooth central fiber. In [6], Chen has constructed geodesic rays parallel to a given one, under various assumptions. In [16] Song and Zelditch have provided a sharp analysis of the approximation in [13] of Donaldson geodesics by Bergman geodesics, for toric varieties. In particular, prior to our present work, they [17] have shown that the optimal regularity for geodesic rays constructed in this manner for toric varieties in general is $C^{1,1}$. 


2 Statement of theorem

Let \( X \) be a compact complex manifold and \( L \rightarrow X \) an ample line bundle. Recall that a test configuration \( T \) for \((X, L)\) is a homomorphism
\[
\rho : C^\times \rightarrow \text{Aut}(L \rightarrow X \rightarrow C) \tag{2.1}
\]
where \( \pi : X \rightarrow C \) is a proper flat morphism, \( L \rightarrow X \) is a line bundle which is ample on the fibers and \( \rho \) is an action of \( C^\times \) on \( L \rightarrow X \rightarrow C \) which covers the standard action of \( C^\times \) on \( C \), and which satisfies the following additional property: \( L_1 = L \) and \( X_1 = X \) (where, for \( w \in C \), we let \( X_w = \pi^{-1}(w) \) and \( L_w = L|_{X_w} \)). We shall say \( T \) is trivial if \( X = X \times C \) and \( \rho \) is the trivial homomorphism.

Fix a resolution of singularities \( p : \tilde{X} \rightarrow X \rightarrow C \). \tag{2.2}

According to [2, 20] (see also [11]), the resolution can be chosen to be equivariant, i.e., the homomorphism \( (2.1) \) lifts into a homomorphism \( \tilde{\rho} : C^\times \rightarrow \text{Aut}(p^*L \rightarrow \tilde{X} \rightarrow C) \), and all the diagrams commute.

Let \( D = \{ w \in C : |w| \leq 1 \}, D^\times = \{ w \in D : w \neq 0 \} \),
\[
M = X \times D^\times, \tag{2.3}
\]
\( \pi_{D^\times} : M \rightarrow D^\times \) the projection onto the second component, \( \pi_X : M \rightarrow X \) the projection onto the second component, and \( T \) a test configuration as above. Let \( X_D = \pi^{-1}(D) \) and \( X_D^\times = \pi^{-1}(D^\times) \). We observe there is a biholomorphic map \( \kappa : M \rightarrow X_D^\times \) given by \( (z, w) \rightarrow \rho(w)z \). The map \( (l, w) \mapsto \rho(w)l \) defines an isomorphism \( \pi_X^*L = \kappa^*(L|_{X^\times}) \).

Thus we may view \( M \) as an open subset of \( X_D^\times \). Moreover, if \( p : \tilde{X}_D \rightarrow X_D \rightarrow D \) is a resolution of singularities, then \( \tilde{X}_D^\times = p^{-1}X_D^\times \rightarrow X_D^\times \) is biholomorphic, so we may view \( M \) as an open subset of \( \tilde{X}_D \) as well. In order to avoid cumbersome notation, we shall write \( X, \tilde{X} \) for \( X_D \) and \( \tilde{X}_D \) when there is no fear of confusion.

Now fix \( h_0 \in \tilde{H} \) and let \( \omega_0 \in H \) be its curvature. Let \( M = X \times D^\times, \pi_M : M \rightarrow D^\times \) the projection onto the second component and
\[
\Omega_0 = \pi_M^*\omega_0. \tag{2.4}
\]

**Theorem 1** There exists a unique \( C^{1,1} \) function \( \Psi : M \rightarrow R \) satisfying:

(a) The current \( \Omega_0 + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \Psi \) is non-negative on \( M \).

(b) It solves the following Dirichlet problem for the completely degenerate Monge-Ampère equation on \( M \)
\[
(\Omega_0 + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \Psi)^{n+1} = 0 ; \quad \Psi|_{X \times \partial D} = 0 \tag{2.5}
\]
(c) The current $\Omega_0 + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \Psi$ is the restriction to $\tilde{X}^\times$ of a non-negative current $\Omega_{\tilde{X}}$, which is a solution of the equation

$$\Omega_{\tilde{X}}^{n+1} = 0$$

(2.6)

on the smooth manifold $\tilde{X}$. Moreover, $\Omega_{\tilde{X}} = \Omega' + \partial \bar{\partial} \Psi'$ where $\Omega'$ is a smooth Kähler metric on $\tilde{X}$ and $\Psi' : \tilde{X} \to \mathbb{R}$ is a $C^{1,1}$ function.

(d) The function $\Psi$ is invariant under the rotations on $D$. In particular, let $\phi_t = \Psi(z, e^{-t})$, $t \in [0, \infty)$. Then $\tilde{h}_t = h_0 e^{-\phi_t}$ is a $C^{1,1}$ geodesic ray in $\tilde{H}$ which emanates from $h_0$, i.e., $\phi_t$ satisfies the geodesic equation

$$\dddot{\phi}_t - g^{jk}_{\phi_t} \dot{\phi}_j \dot{\phi}_k = 0 \quad \text{on } X \times [0, \infty).$$

(2.7)

where $g_{\phi_t}$ is the Kähler metric $-\partial \bar{\partial} \log \tilde{h}_t$. If $T$ is a non-trivial test configuration, then the geodesic ray is infinite.

### 3 Proof of theorem

We divide the proof of the theorem into several steps. The first step is the following lemma, which is an extension of the results of Chen [5] to the case of general complex manifolds with boundary:

**Lemma 1** Let $\tilde{X}$ be a compact complex manifold with smooth boundary $\partial \tilde{X}$. If $\tilde{X}$ admits a smooth Kähler metric $\Omega$, then the Dirichlet problem

$$(\Omega + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \Phi)^{n+1} = 0 \quad \text{on } \tilde{X}, \quad \Phi|_{\partial \tilde{X}} = 0$$

(3.1)

admits a unique $C^{1,1}$ solution $\Phi$ which is $\Omega$-plurisubharmonic, i.e., $\Omega + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \Phi \geq 0$.

**Proof of Lemma 1:** Following Chen [5], we let $t \in [0, 1]$ and consider the equation

$$(\Omega + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \Phi_t)^{n+1} = t \cdot \Omega^n \quad \text{on } \partial \tilde{X}$$

(3.2)

Then (3.2) has a solution when $t=1$, and, since $\Omega$ is a positive definite $(1,1)$-form, the equation is elliptic and the set of $t$ for which a solution exists is open. Thus, to prove the theorem, it suffices to show that that $||\Phi_t||_{C^2}$ is uniformly bounded in $t$. This follows in turn from the estimates of [7], [10], [21] and [5]. The uniqueness statement is a general property of the Dirichlet problem for complex Monge-Ampère equations (see e.g. [13], Theorem 6). Q.E.D.
We note that the condition that $\Omega$ is a Kähler metric implies the existence of a subsolution $\Phi$ of the Dirichlet problem,

$$\left(\Omega + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \Phi\right)^{n+1} > 0 \text{ on } M; \quad \Phi|_{\partial M} = 0$$

(3.3)

namely $\Phi = 0$. From this point of view, the formulation of Lemma 1 is modeled on the earlier result of Guan [10], reducing the solution of the complex Monge-Ampère equation to finding a subsolution.

We shall apply Lemma 1 to the case when $\tilde{X}$ is the equivariant resolution of the test configuration $T$ introduced in the previous section, with boundary $\partial \tilde{X} = (p\pi)^{-1}(\partial D)$. Thus we need to construct a Kähler metric $\Omega$ on $\tilde{X}$, in fact a metric with some precise boundary conditions. To do this, we prove next the following lemma:

**Lemma 2** There exists a line bundle $M \to \tilde{X}$ and an integer $m > 0$ with the following properties:

1. $p^*L^m \otimes M \to \tilde{X}$ is ample, that is, it has a metric of positive curvature.
2. $M|_{\tilde{X}^\times}$ is trivial, that is, there is a meromorphic section $\mu : \tilde{X} \to M$ whose restriction to $\tilde{X}^\times$ is holomorphic and nowhere vanishing.

Before establishing the lemma, we recall the definition of a blow-up. If $W$ is a scheme, and if $Z \subseteq W$ is a closed subscheme with ideal sheaf $I_Z \subseteq O_W$, then $B(W; Z)$, the blow-up of $W$ along $Z$, is the scheme Proj $(\oplus_{d=0}^\infty I_Z^d)$. If $W$ and $Z$ are both smooth varieties, then $B(W; Z)$ is also a smooth variety whose underlying set is the disjoint union $B(W; Z) = (W \setminus Z) \cup P(E)$. Here $E \to Z$ is the vector bundle $E = TW|_Z/TZ$. If we let $p : B(W; Z) \to W$ be the projection map, then $p$ is biholomorphic over the set $W \setminus Z$.

In order to describe the complex structure in a neighborhood of points in $p^{-1}(Z)$, let $a \in Z$ and let $U \subseteq W$ be a small open set containing $a$. Choose local coordinates $(z_1, \ldots, z_n)$ for $W$ centered at $a$ with the property $Z \cap U = \{z_1 = \cdots = z_{n-d} = 0\}$. Let $B_U = \{(z, y) \in U \times \mathbb{P}^{n-d-1} : y_iz_j - y_jz_i = 0 \text{ for all } 1 \leq i, j \leq n-d\}$. Then $B_U$ is a submanifold of $U \times \mathbb{P}^{n-d-1}$. Define $\phi_U : B_U \to B(W; Z)$ as follows: $\phi_U(z, y) = z$ if $z \notin Z$ and $\phi_U(z, y) = \sum_{j=1}^{n-d} y_j \frac{\partial}{\partial y_j}$ if $z \in Z$. Then $\phi_U$ maps $B_U$ biholomorphically onto the open set $p^{-1}(U) \subseteq B(W; Z)$. This defines the complex structure on $B(W; Z)$.

**Proof of Lemma 2.** The map $p : \tilde{X} \to X$ is a composition of blow-ups with smooth centers. Thus, by induction, we may assume $p$ is a single blow-up: Let $W \subseteq \mathbb{P}^N$ be a smooth variety and $Z \subseteq W$ a smooth subvariety. Let $L = O(1)|_W$ and $p : W' = Bl(W; Z) \to W$ the blow-up of $W$ with center $Z$. We wish to show that there exists a line bundle $M \to W'$ such that $p^*L^m \otimes M \to W'$ is ample and $M_{W' \setminus E}$ is trivial, where $E \subseteq W'$ is the exceptional divisor.
Let \( p_1 : Bl(P^N, Z) \to P^N \) be the blow up of \( P^N \) along \( Z \). Since \( Bl(W, Z) \subseteq Bl(P^N, Z) \) is the closure of \( p_1^{-1}(W \setminus Z) \subseteq Bl(P^N, Z) \), we may assume \( W = P^N \).

Thus we let \( Z \subseteq P^N \) be a smooth subvariety of dimension \( d \). Then \( Z \) is a local complete intersection. This means that there exists an integer \( k > 0 \), polynomials \( F_1, \ldots, F_r \) of degree \( k \), and polynomials \( f_1, \ldots, f_m \) of degree \( k \), with the following properties: The \( F_s \) have no common zeros and for every \( s \) with \( 1 \leq s \leq r \), there exists \( 1 \leq i_1 \leq \cdots \leq i_{N-d} \) such that \( f_{i_1}, \ldots, f_{i_{N-d}} \) generate the ideal sheaf of \( Z \cap U_s \) where \( U_s \subseteq P^N \) is the affine open set defined by \( F_s \neq 0 \). In other words, if \( G \) is a homogeneous polynomial whose degree is a multiple of \( k \), and if \( G \) vanishes on \( Z \), then there exists and integer \( a \geq 1 \) and homogeneous polynomials \( A_1, \ldots, A_{N-d} \) such that

\[
F_s^a G = A_1 f_{i_1} + \cdots + A_{N-d} f_{i_{N-d}}
\]

Note that \( m \geq N - d \). Let

\[
\mathcal{B} = \{ (x, y) \in P^N \times P^{m-1} : Y_i f_j(x) - Y_j f_i(x) = 0, 1 \leq i, j \leq m \} \tag{3.4}
\]

If \( m = N - d \) (i.e., \( Z \) is a complete intersection) then, by the definition of blow-up, \( \mathcal{B} = Bl(P^N; Z) \). In general, let \( \mathcal{B}_0 \subseteq P^N \times P^{m-1} \) be the set \( \mathcal{B}_0 = \{ (x, y) \in \mathcal{B} : x \notin Z \} \). Thus \( \mathcal{B}_0 \to P^N \setminus Z \) is biholomorphic. Let \( \mathcal{B}_0 \subseteq P^N \times P^{m-1} \) be the closure of \( \mathcal{B}_0 \) and \( P : \mathcal{B}_0 \to P^N \) projection onto the first component. We claim that \( \mathcal{B}_0 = Bl(P^N; Z) \). To see this, define \( \Theta : \mathcal{B}(P^N; Z) \setminus p^{-1}(Z) \to \mathcal{B}_0 \setminus P^{-1}(Z) \) as follows: \( \Theta = P^{-1} \circ p \). We must show that \( \Theta \) uniquely extends to a biregular map \( \Theta : \mathcal{B}(P^N; Z) \to \mathcal{B}_0 \).

To prove that \( \Theta \) has a unique extension, we may work locally: Fix \( z \in Z \). Choose \( F_s \), such that \( F_s(z) \neq 0 \), and choose \( i_1 < \cdots < i_{N-d} \) as above. Without loss of generality, we may assume \( i_j = j \). We shall write \( F = F_s \) and \( U = U_s \). Then

\[
p^{-1}(U \setminus Z) = \{ (x, y) \in (U \setminus Z) \times P^{N-d-1} : y_i f_j(x) - y_j f_i(x) = 0, 1 \leq i, j \leq N - d \} \tag{3.5}
\]

\[
P^{-1}(U \setminus Z) = \{ (x, y) \in (U \setminus Z) \times P^{m-1} : Y_i f_j(x) - Y_j f_i(x) = 0, 1 \leq i, j \leq m \} \tag{3.6}
\]

We must examine the map \( \Theta : p^{-1}(U \setminus Z) \to P^{-1}(U \setminus Z) \). If \( \Theta(x, y) = (x', Y) \) then \( x' = x \) and \( Y_i = y_i \) if \( 1 \leq i \leq N - d \). Moreover, on the open set \( y_l \neq 0 (1 \leq l \leq N - d) \), we have

\[
Y_i = \frac{y_i f_i(x)}{f_l(x)} = \frac{y_i \sum_{l=1}^{N-d} A_i f_l(x)}{F^*(x) f_l(x)} = \frac{\sum_{l=1}^{N-d} A_i(x) y_l(x)}{F^*(x)} \tag{3.7}
\]

Note that \( f_l(x) \neq 0 \) in (3.7) since \( y_l f_l = y_l f_i \) so if \( f_l(x) = 0 \) then \( f_i(x) = 0 \) for all \( 1 \leq i \leq N - d \) which implies \( x \in Z \), and contradiction.

Thus (3.7) gives the formula for \( \Theta(x, y) \) when \( x \notin Z \). On the other hand, the denominator \( F^*(x) \) does not vanish if \( x \in Z \) and thus (3.7) defines a unique extension of \( \Theta \) from
$p^{-1}(U \setminus Z)$ to $p^{-1}(U)$. The map $\Theta$ is clearly biregular. In fact, its inverse is given explicitly by $(x', Y) \rightarrow (x, y)$ where $x = x'$ and $y_i = Y_i$ for $1 \leq i \leq N - d$.

Now we can finish the proof of Lemma 2: We must show that there exists a line bundle $\mathcal{M} \rightarrow B(\mathbb{P}^N; Z)$ such that $p^*O_{\mathbb{P}^N}(m) \otimes \mathcal{M} \rightarrow B(\mathbb{P}^N; Z)$ is ample, for some $m > 0$, and such that $\mathcal{M}|_{B(\mathbb{P}^N; Z)\setminus p^{-1}(Z)}$ is trivial. Since $B = B(\mathbb{P}^N; Z) = \mathcal{B}_0 \subseteq \mathbb{P}^N \times \mathbb{P}^{m-1}$, we see that $p^*O_{\mathbb{P}^N}(1) \otimes q^*O_{\mathbb{P}^{m-1}}(1) \rightarrow \mathcal{B}$ is ample, where $q : B \rightarrow \mathbb{P}^{m-1}$ the the projection onto the second factor. Let $\mathcal{M}^{-1} = p^*O_{\mathbb{P}^N}(k) \otimes q^*O_{\mathbb{P}^{m-1}}(-1)$ and let $s$ be the section of $\mathcal{M}^{-1}$ defined by $s = \frac{f(x)}{y_j}$ on the open set $y_j \neq 0$, then $s$ is a global section which vanishes precisely on the exceptional divisor $D = p^{-1}(Z) \subseteq \mathcal{B}$. Thus $\mathcal{M}$ is trivial on the complement of the exceptional divisor. Moreover

$$p^*O_{\mathbb{P}^N}(k + 1) \otimes M = p^*O_{\mathbb{P}^N}(1) \otimes q^*O_{\mathbb{P}^{m-1}}(1)$$

This proves Lemma 2.

Now let $T$ and $M$ be as in Lemma 2, and let $\omega_0 \in c_1(L)$ be a fixed Kähler metric and let $h_0 \in \mathcal{H}$ be a hermitian metric on $L$ such that $\omega_0$ is the curvature of $h_0$.

**Lemma 3** There exists a Kähler metric $\Omega$ on $\tilde{X}$ with the following properties:

(i) $m\Omega$ is the curvature of a hermitian metric $H$ on $p^*L^m \otimes \mathcal{M}$.

(ii) $\rho(w)^*(\Omega|_{X_w}) = \omega_0$ for all $w \in \partial D$.

(iii) $\Omega$ is invariant under the group action.

**Proof.** According to Lemma 2, there is a metric $H_1$ on $p^*L^m \otimes \mathcal{M}$ with positive curvature $\Omega_1$. We shall modify $H_1$ in a neighborhood of $\partial \tilde{X}$ in order to produce a new positively curved metric whose boundary value is $\omega_0$: Let $s$ be a meromorphic section of $\mathcal{M}$ which is holomorphic and nowhere vanishing on $\tilde{X}^\times$. We define a metric $H_2$ on $\pi_X^*L^m = L^m \times D^\times$ as follows:

$$H_2(\ell, w) = H_1(p^*(\rho(w)\ell) \otimes s))$$

Since $H_2$ and $h_0^m$ are two metrics on the same line bundle $L^m \times D^\times$, there exists a smooth function $\Psi : M = X \times D^\times \rightarrow \mathbb{R}$ such that $H_2 = h_0^m e^{-\Psi(x,w)}$.

Let $\eta : D^\times \rightarrow [0, 1]$ be a smooth function such that $\eta(w) = 1$ if $|w| \leq \frac{1}{3}$ and $\eta(w) = 0$ if $|w| \geq \frac{2}{3}$. Let

$$H_3 = h_0^m e^{-\eta(w)\Psi(x,w)}$$

Then $H_3 = h_0$ if $|w| \geq \frac{2}{3}$ and $H_3 = H_2$ if $|w| \leq \frac{1}{3}$. Moreover, the curvature of $H_3$ is positive on the fibers $X \times \{w\}$ for all $w \in D^\times$.

Let $\alpha > 0$ be a large positive number, and define

$$H_4 = H_3 e^{-\alpha(|w|^2 - 1)}$$

7
Then $H_4(\ell, w) = H_5(p^*(\rho(w)\ell) \otimes s)$ for some unique smooth metric $H_5$ on $p^*\mathcal{L}^n \otimes \mathcal{M}$, and $H$ satisfies properties (i) and (ii) by construction. Finally, we average $H_5$ over the $S^1$ action to obtain the desired metric $H$. Q.E.D.

**Proof of theorem.** We now turn to the proof of Theorem 1: Fix $\Omega$ as in Lemma 3. By Lemma 1, there is a unique $\Omega$-plurisubharmonic, $C^{1,1}$ invariant solution $\Phi$ on $\tilde{\mathcal{X}}$ of the equation (3.1). Now, restricted to $\tilde{\mathcal{X}} \sim X \times D$, the metric $\Omega$ can be expressed as

$$\Omega = \Omega_0 + \frac{\sqrt{-1}}{2} \partial\bar{\partial}\Phi_0$$

(3.11)

for some smooth function $\Phi_0 \in C^\infty(X \times \bar{D})$, with $\Phi_0 = 0$ on $X \times \partial D$. The equation (3.1) on $\tilde{\mathcal{X}}$ can be written as

$$(\Omega_0 + \frac{\sqrt{-1}}{2} \partial\bar{\partial}(\Phi_0 + \Phi))^{n+1} = 0,$$

(3.12)

Set $\Psi = \Phi_0 + \Phi$. Then $\Psi$ satisfies conditions (a) and (b) of the theorem. The uniqueness stated in (c) is a consequence of the uniqueness of solutions of the Monge-Ampère equation on $\tilde{\mathcal{X}}$ stated in Lemma 1. Clearly, the invariance of $\Omega$ and of $\Phi$ results in the invariance of $\Psi$. It is now a well-known fact that, for invariant functions $\Psi$ on $X \times D$, the Monge-Ampère equation in (2.5) reduces precisely to the geodesic equation (2.7). Q.E.D.
References

[1] Arezzo, C., and Tian, G., “Infinite geodesic rays in the space of Kähler potentials”, Ann. Sci. Norm. Sup. Pisa Sci. (5) 2 (2003) 617-630, arXiv: math.DG/0210389.

[2] Bierstone, E. and P. Milman, “Canonical desingularization in characteristic zero by blowing up the maximal stata of a local invariant. Inv. Math. 128 (1997), 207-302

[3] Bedford, E. and B.A. Taylor, “The Dirichlet problem for a complex Monge-Ampère equation”, Invent. Math. 37 (1976), 1-44.

[4] Bedford, E. and B.A. Taylor, “A new capacity for plurisubharmonic functions”, Acta Math. 149 (1982), 1-40.

[5] Chen, X.X., “The space of Kähler metrics”, J. Differential Geom. 56 (2000), 189-234.

[6] Chen, X.X., “Space of Kähler metrics III: On the lower bound of the Calabi energy and geodesic distance”, arXiv: math.DG/0606228

[7] Caffarelli, L., J.J. Kohn, L. Nirenberg, and J. Spruck, “The Dirichlet problem for non-linear second-order elliptic equations II. Complex Monge-Ampère and uniformly elliptic equations”, Comm. Pure Appl. Math. 38 (1985) 209-252.

[8] Donaldson, S.K., “Symmetric spaces, Kähler geometry, and Hamiltonian dynamics”, Amer. Math. Soc. Transl. 196 (1999) 13-33.

[9] Donaldson, S.K., “Scalar curvature and stability of toric varieties”, J. Differential Geom. 62 (2002) 289-349

[10] Guan, B., “The Dirichlet problem for the complex Monge-Ampère operator”, Comm. Anal. Geom. 6 (1998) 687-703

[11] Hauser, H., “The Hironaka theorem on resolution of singularities”, Bull. Am. Math. Soc. 40 (2003) 323-403

[12] Mabuchi, T., “Some symplectic geometry on compact Kähler manifolds I”, Osaka J. Math. 24 (1987) 227-252.

[13] Phong, D.H. and J. Sturm, “The Monge-Ampère operator and geodesics in the space of Kähler potentials”, Inventiones Math. 166 (2006) 125-149, arXiv: math.DG/0504157.

[14] Phong, D.H. and J. Sturm, “Test Configurations for K-Stability and Geodesic Rays” preprint, math.DG/0606423

[15] Semmes, S., “Complex Monge-Ampère equations and symplectic manifolds”, Amer. J. Math. 114 (1992) 495-550.

[16] Song, J. and S. Zelditch, “Bergman metrics and geodesics in the space of Kähler metrics on toric varieties”, arXiv:0707.3082 (math.CV)
[17] Song, J. and S. Zelditch, “Test configurations and geodesic rays on toric varieties”, in preparation.

[18] Tian, G., “On a set of polarized Kähler metrics on algebraic manifolds”, J. Differential Geom. 32 (1990) 99-130

[19] Tian, G., “Kähler-Einstein metrics with positive scalar curvature”, Inventiones Math. 130 (1997) 1-39

[20] Villamayor, O., “Constructiveness of Hironaka’s resolution”, Ann. Scient. Ec. Norm. Sup. Paris 22 (1989) 1-32

[21] Yau, S.T., “On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation I”, Comm. Pure Appl. Math. 31 (1978) 339-411.

[22] Yau, S.T., “Open problems in geometry”, Proc. Symp. Pure Math. 54, Amer. Math. Soc. Providence, RI (1993) 1-28

[23] Zelditch, S., “The Szegö kernel and a theorem of Tian”, Int. Math. Res. Notices 6 (1998) 317-331.

* Department of Mathematics
Columbia University, New York, NY 10027

† Department of Mathematics
Rutgers University, Newark, NJ 07102