A Note on State Decomposition Independent Local Invariants

Ting-Gui Zhang\textsuperscript{1}, Naihuan Jing\textsuperscript{2,3}, Xianqing Li-Jost\textsuperscript{1}, Ming-Jing Zhao\textsuperscript{1}, Shao-Ming Fei\textsuperscript{1,4}

\textsuperscript{1} Max-Planck-Institute for Mathematics in the Sciences, 04103 Leipzig, Germany
\textsuperscript{2} Department of Mathematics, North Carolina State University, Raleigh, NC 27695, USA
\textsuperscript{3} School of Sciences, South China University of Technology, Guangzhou 510640, China
\textsuperscript{4} School of Mathematical Sciences, Capital Normal University, Beijing 100048, China

\textsuperscript{†}E-mail: tinggui333@163.com

Abstract. We derive a set of invariants under local unitary transformations for arbitrary dimensional quantum systems. These invariants are given by hyperdeterminants and independent from the detailed pure state decompositions of a given quantum state. They also give rise to necessary conditions for the equivalence of quantum states under local unitary transformations.

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I. INTRODUCTION

Invariants under local unitary transformations are tightly related to the discussions on nonlocality - a fundamental phenomenon in quantum mechanics, to the quantum entanglement and classification of quantum states under local transformations. In recent years many approaches have been presented to construct invariants of local unitary transformations. One method is developed in terms of polynomial invariants Refs. \cite{9,20}, which allows in principle to compute all the invariants of local unitary transformations, though it is not easy to perform operationally. In Ref. \cite{16}, a complete set of 18 polynomial invariants is presented for the locally unitary equivalence of two qubit-mixed states. Partial results have been obtained for three qubits states \cite{13}, tripartite pure and mixed states \cite{1}, and some generic mixed states \cite{3,4,22}. Recently the local unitary equivalence problem for multiqubit \cite{12} and general multipartite \cite{14} pure states has been solved.

However, generally one still has no operational ways to judge the equivalence of two arbitrary dimensional bipartite or multipartite mixed states under local unitary transformations. An effective way to deal with the local equivalence of quantum states is to find the
complete set of invariants under local unitary transformations. Nevertheless usually these invariants depend on the detailed expressions of pure state decompositions of a state. For a given state, such pure state decompositions are infinitely many. Particularly when the density matrices are degenerate, the problem becomes more complicated. Since in this case even the eigenvector decompositions of a given state are not unique.

In this note, we give a way of constructing invariants under local unitary transformations such that the invariants obtained in this way are independent from the detailed pure state decompositions of a given state. They give rise to operational necessary conditions for the equivalence of quantum states under local unitary transformations. We show that the hyperdeterminants, the generalized determinant for higher dimensional matrices [8], can be used to construct such invariants. The hyperdeterminant is in fact closely related to the entanglement measure like concurrence [2, 10, 21, 23, 25] and 3-tangle [7]. It has also been used in classification of multipartite pure state [15, 17, 24]. By employing hyperdeterminants, we construct some trace invariants that are independent of the detailed pure state decompositions of a given state. These trace invariants are a priori invariant under local unitary transformations.

II. STATE DECOMPOSITION INDEPENDENT LOCAL INVARIANT

Let $H_1$ and $H_2$ be $n$ and $m$-dimensional complex Hilbert spaces, with $\{ |i\rangle \}_{i=1}^n$ and $\{ |j\rangle \}_{j=1}^m$ the orthonormal basis of spaces $H_1$ and $H_2$ respectively. Let $\rho$ be an arbitrary mixed state defined on $H_1 \otimes H_2$,

$$\rho = \sum_{i=1}^I p_i |v_i\rangle \langle v_i|,$$

where $|v_i\rangle$ is a normalized bipartite pure state of the form:

$$|v_i\rangle = \sum_{k,l=1}^{n,m} a_{kl}^{(i)} |kl\rangle, \quad \sum_{k,l=1}^{n,m} a_{kl}^{(i)} a_{kl}^{(i)*} = 1, \quad a_{kl}^{(i)} \in \mathbb{C},$$

where $*$ denotes complex conjugation. Denote $A_i$ the matrix with entries given by the coefficients of the vector $\sqrt{p_i} |v_i\rangle$, i.e. $(A_i)_{kl} = (\sqrt{p_i} a_{kl}^{(i)})$ for all $i = 1, \cdots, I$. Define $I \times I$ matrix $\Omega$ such that $(\Omega)_{ij} = tr(A_i A_j^\dagger)$, $i, j = 1, \cdots, I$, where $\dagger$ stands for transpose and complex conjugation.
The pure state decomposition (1) of a given mixed state $\rho$ is not unique. For another decomposition:

$$\rho = \sum_{i=1}^{I} q_i |\psi_i\rangle\langle\psi_i|,$$

with

$$|\psi_i\rangle = \sum_{k,l=1}^{n,m} b_{kl}^{(i)} |kl\rangle, \quad \sum_{k,l=1}^{n,m} b_{kl}^{(i)} b_{kl}^{(i)*} = 1, \quad b_{kl}^{(i)} \in \mathbb{C},$$

equation (2)

one similarly has matrices $B_i$ with entries $(B_i)_{kl} = (\sqrt{q_i} b_{kl}^{(i)})$, $i = 1, \cdots, I$, and $I \times I$ matrix $\Omega'$ with entries

$$(\Omega')_{ij} = \text{tr}(B_i B_j^\dagger), \quad i, j = 1, \cdots, I.$$ 

A quantity $F(\rho)$ is said to be invariant under local unitary transformations if $F(\rho) = F((u_1 \otimes u_2)\rho (u_1 \otimes u_2)^\dagger)$ for any unitary operators $u_1 \in SU(n)$ and $u_2 \in SU(m)$. Generally $F(\rho)$ may depend on the detailed pure state decomposition. We investigate invariants $F(\rho)$ that are independent on the detailed decompositions of $\rho$. That is, expression in Eq. (1) and expression in Eq. (2) give the same value of $F(\rho)$ for a given state $\rho$. These kind of invariants are of special significance in determining the equivalence of two density matrices under local unitary transformations.

Two density matrices $\rho$ and $\tilde{\rho}$ are said to be equivalent under local unitary transformations if there exist unitary operators $u_1$ (resp. $u_2$) on the first (resp. second) space of $H_1 \otimes H_2$ such that

$$\tilde{\rho} = (u_1 \otimes u_2)\rho (u_1 \otimes u_2)^\dagger.$$ 

(3)

A necessary condition that (3) holds is that the local invariants have the same values $F(\rho) = F(\tilde{\rho})$. Therefore if the expression of the invariants $F(\rho)$ do not depend on the detailed pure state decomposition, one can easily compare the values of the invariants between $F(\rho)$ and $F(\tilde{\rho})$. Otherwise one has to verify $F(\rho) = F(\tilde{\rho})$ by surveying all the possible pure state decompositions of $\rho$ and $\tilde{\rho}$. In particular, when $\rho$ is degenerate, even the eigenvector decomposition is not unique, which usually gives rise to the main problem in finding an operational criterion for local equivalence of quantum states. In fact, we have presented a complete set of invariants in [26]. However, these invariants depend on the eigenvectors of a state $\rho$. When the state is degenerate, this set of invariants is no longer efficient as criterion of local equivalence.
We set out to discuss how to find parametrization independent local unitary invariants. First of all we give an elementary result that the determinant can be used to give invariants that are independent from the choice of the pure state decomposition.

**Theorem 1:** The coefficients $F_i(\Omega)$, $i = 1, 2, \ldots, I$, of the characteristic polynomials of the matrix $\Omega$,

$$
\det(\Omega - \lambda E) = \lambda^I + \lambda^{I-1}F_1(\Omega) + \cdots + \lambda F_{I-1}(\Omega) + F_I(\Omega) = \sum_{i=0}^{I} \lambda^{I-i}F_i(\Omega),
$$

where $E$ is the $I \times I$ unit matrix, $F_0(\Omega) = 1$, $\det$ denotes the determinant, have the following properties:

(i) $F_i(\Omega)$ are independent of the pure state decompositions of $\rho$;

(ii) $F_i(\Omega)$ are invariant under local unitary transformations, $i = 1, \ldots, I$.

**Proof:** (i) If Eq. (1) and Eq. (2) are two different representations of a given mixed state $\rho$, we have $B_i = \sum_j U_{ij} A_j$ for some unitary operator $U$. Consequently, $\Omega'_{ij} = tr(B_iB_j^\dagger) = tr\left[ \sum_{k,l} U_{ik} A_k U_{jl}^* A_l^\dagger \right] = \sum_{k,l} U_{ik} U_{jl}^* tr(A_k A_l^\dagger) = \sum_{k,l} U_{ik} U_{jl}^* \Omega_{kl} = (U\Omega U^\dagger)_{ij}$, i.e. $\Omega' = U\Omega U^\dagger$. Therefore $\det(\Omega' - \lambda E) = \det(U\Omega U^\dagger - \lambda E) = \det(\Omega - \lambda E)$. Thus the matrices $\Omega$ and $\Omega'$ have the same characteristic polynomials. Namely $F_i(\Omega) = F_i(\Omega')$. Therefore $F_i(\Omega)$ are invariants under the pure state decomposition transformations.

(ii) Let $P \otimes Q \in SU(n) \otimes SU(m)$. Under the local unitary transformations one has

$$
\tilde{\rho} = (P \otimes Q) \rho (P \otimes Q)^\dagger = \sum_{i=1}^{I} p_i (P \otimes Q) |v_i\rangle \langle v_i|(P \otimes Q)^\dagger = \sum_{i=1}^{I} p_i |w_i\rangle \langle w_i|,
$$

with

$$
|w_i\rangle = P \otimes Q |v_i\rangle = \sum_{k,l}^{n,m} a_{kl}^{(i)\dagger} |kl\rangle, \quad \sum_{k,l}^{n,m} a_{kl}^{(i)\dagger} a_{kl}^{(i)\dagger*} = 1, \quad a_{kl}^{(i)\dagger} \in \mathbb{C}.
$$

Denote $(A'_i)_{kl} = \sqrt{p_i} a_{kl}^{(i)\dagger}$. We have

$$
A'_i = PA_i Q^T.
$$

Therefore $tr(A_i A_j^\dagger) = tr(A'_i A'_j^\dagger)$ and $\Omega(\rho) = \Omega(\tilde{\rho})$. Hence $F_i(\Omega(\rho)) = F_i(\Omega(\tilde{\rho}))$, and $F_i(\Omega)$, $i = 1, \ldots, I$, are invariant under local unitary transformations. \qed
In particular, the invariants \( F_1 = \sum \text{tr}(\sum_i A_i A_i^\dagger) \) and \( F_I = \det(\Omega) \). For the case \( I = 2 \), one has
\[
\Omega = \begin{pmatrix}
\text{tr}(A_1 A_1^\dagger) & \text{tr}(A_1 A_2^\dagger) \\
\text{tr}(A_2 A_1^\dagger) & \text{tr}(A_2 A_2^\dagger)
\end{pmatrix}
\]
and \( F_1 = \text{tr}(A_1 A_1^\dagger) + \text{tr}(A_2 A_2^\dagger), \ F_2 = \text{tr}(A_1 A_1^\dagger)\text{tr}(A_2 A_2^\dagger) - \text{tr}(A_1 A_2^\dagger)\text{tr}(A_2 A_1^\dagger) \).

**Remark:** The number of local invariants \( F_i \) is uniquely determined by the rank \( r \) of the mixed state \( \rho \), i.e. \( I = r \). Therefore we only need to calculate the invariants corresponding to the eigenvector decomposition. Because for arbitrary pure state decomposition \( \rho = \sum_j q_j |\psi_j\rangle \langle \psi_j| \) with \( J > r \), the above determinant is the same as that of the eigenvector decomposition of \( \rho = \sum_i p_i |\phi_i\rangle \langle \phi_i| \) after adding \( J - r \) zero vectors. The determinant \( \det(\Omega' - \lambda E) \) of the eigenvector decomposition of \( \rho \) after adding \( J - r \) zero vectors and \( \det(\Omega - \lambda E) \) of \( \rho = \sum_j q_j |\psi_j\rangle \langle \psi_j| \) without \( J - r \) zero vectors have the relation: \( \det(\Omega' - \lambda E) = \lambda^{J-r} \det(\Omega - \lambda E) \). This means that the number of independent local invariants given by \( \Omega' \) does not depend on the number of pure states in the ensemble of a given \( \rho \). Therefore if two mixed states \( \rho \) and \( \tilde{\rho} \) have different ranks, they are not local unitary equivalent. If their ranks are the same, one only needs to calculate the corresponding invariants with respect to the same numbers \( I \) of pure states in the pure state decompositions.

In fact for a quantum state \( \rho \) in eigenvector decomposition \( \rho = \sum \lambda_i |\psi_i\rangle \langle \psi_i| \), the corresponding matrix \( \Omega \) is a diagonal one with \( \rho \)'s eigenvalues \( \lambda_i \) as the diagonal entries. In this case the local invariants from Theorem 1 are just the coefficients of the characteristic polynomial of the quantum state \( \rho \). Theorem 1 shows that these coefficients are local invariants and independent from the detailed pure state decompositions. But the easy approach employed in Theorem 1 can be generalized to construct more local invariants that are independent of the detailed pure state decompositions by using hyperdeterminant \( \mathbb{H} \).

In order to derive more parametrization independent quantities we consider the multilinear form \( f_A : V \otimes \cdots \otimes V \rightarrow \mathbb{C} \) given by
\[
f_A(e_{i_1}, \cdots, e_{i_s}, e_{j_1}, \cdots, e_{j_s}) = \text{tr}(A_{i_1} A_{j_1}^\dagger \cdots A_{i_s} A_{j_s}^\dagger), \quad (6)
\]
where \( e_i \ (1 \leq i \leq I) \) are standard basis elements in \( V = \mathbb{C}^I \). The multilinear form \( f \) can also be written as a tensor in \( V^* \otimes \cdots \otimes V^* \):
\[
f_A = \sum_{i\leq j} \text{tr}(A_{i_1} A_{j_1}^\dagger \cdots A_{i_s} A_{j_s}^\dagger) e_{i_1}^* \otimes \cdots \otimes e_{i_s}^* \otimes e_{j_1}^* \otimes \cdots \otimes e_{j_s}^*, \quad (7)
\]
where \( e^*_i \) are standard 1-forms on \( \mathbb{C}^I \) such that \( e^*_i(e_j) = \delta_{ij} \), and \( \underline{i} = (i_1, \ldots, i_s), \underline{j} = (j_1, \ldots, j_r), 1 \leq i_p, j_p \leq I \). In general we call the 2s-dimensional matrix or hypermatrix \( A = (A_{\underline{i}\underline{j}}) = (tr(A_{i_1}A_{j_1}^\dagger \cdots A_{i_s}A_{j_s}^\dagger)) \) formed by the coefficients of \( \underline{7} \) the hypermatrix of the multilinear form \( f_A \) relative to the standard basis.

The Cayley hyperdeterminant \( \text{Det}(A) \) \( \underline{8} \) is defined to be the resultant of the multilinear form \( f_A \), that is, \( \text{Det}(A) \) is the normalized integral equation of the hyperplane given by the multilinear form \( f_A \). It is known that \( \underline{8} \) the hyperdeterminant exists for a given format and is unique up to a scalar factor, if and only if the largest number in the format is less than or equal to the sum of the other numbers in the format. Hyperdeterminants enjoy many of the properties of determinants. One of the most familiar properties of determinants, the multiplication rule \( \text{det}(AB) = \text{det}(A)\text{det}(B) \), can be generalized to the situation of hyperdeterminants as follows. Given a multilinear form \( f(x^{(1)}, \ldots, x^{(r)}) \) and suppose that a linear transformation acting on one of its components using an \( n \times n \) matrix \( B, y_r = Bx_r \). Then

\[
\text{Det}(f, B) = \text{Det}(f)\text{det}(B)^{N/n},
\]

where \( N \) is the degree of the hyperdeterminant. Therefore we have the following result.

**Lemma 1** The hyperdeterminant of format \( (k_1, \ldots, k_r) \) is an invariant under the action of the group \( SL(k_1) \otimes \cdots \otimes SL(k_r) \), and subsequently also invariant under \( SU(k_1) \otimes \cdots \otimes SU(k_r) \).

**Proof:** For \( (A, B, \ldots, C) \in SL(k_1) \otimes \cdots \otimes SL(k_r) \), it follows from Eq. \( \underline{8} \) that

\[
\text{Det}((A^{(1)} \cdot B^{(2)} \cdots C^{(r)})f) = \text{Det}(f)\text{det}(A)^{N/k_1}\text{det}(B)^{N/k_2} \cdots \text{det}(C)^{N/k_r} = \text{Det}(f). \quad (9)
\]

The three-dimensional hyperdeterminant of the format \( 2 \times 2 \times 2 \) is known as the Cayley’s hyperdeterminant \( \underline{5} \). In this case the hyperdeterminant of a hypermatrix \( A \) with components \( a_{ijk}, i, j, k \in \{0, 1\} \), is given by

\[
\text{Det}(A) = a_{000}^2a_{111}^2 + a_{001}^2a_{110}^2 + a_{010}^2a_{101}^2 + a_{100}^2a_{011}^2 - 2a_{000}a_{001}a_{110}a_{111} - 2a_{000}a_{010}a_{101}a_{111} - 2a_{001}a_{011}a_{100}a_{111} - 2a_{010}a_{011}a_{101}a_{100} + 4a_{000}a_{011}a_{101}a_{110} + 4a_{001}a_{010}a_{100}a_{111}. \quad (10)
\]
This hyperdeterminant can be written in a more compact form by using the Einstein convention and the Levi-Civita symbol $\varepsilon^{ij}$, with $\varepsilon^{00} = \varepsilon^{11} = 0, \varepsilon^{01} = -\varepsilon^{10} = 1$; and $b_{kn} = (1/2)\varepsilon^{il} \varepsilon^{jm} a_{ij} a_{km}$. $Det(A) = (1/2)\varepsilon^{il} \varepsilon^{jm} b_{ij} b_{lm}$. The four-dimensional hyperdeterminant of the format $2 \times 2 \times 2 \times 2$ has been given in Ref. [15].

For the general mixed state $\rho$ in Eq. (1), we can define a hypermatrix $\Omega_s$ with entries

$$
(\Omega_s)_{i_1 i_2 \ldots i_s j_1 j_2 \ldots j_s} = tr(A_{i_1} A_{j_1}^\dagger A_{i_2} A_{j_2}^\dagger \cdots A_{i_s} A_{j_s}^\dagger),
$$

for $i_k, j_k = 1, \ldots, I$, $s \geq 1$, $0 \leq i_j \leq k_j$. The format of $\Omega_s$ is $I \times \cdots \times I$.

**Theorem 2:** $Det(\Omega_s - \lambda E)$, with $E = (E_{i_1 j_1}, \ldots, E_{i_s j_s}) = (\delta_{i_1 j_1}, \delta_{i_2 j_2}, \ldots, \delta_{i_s j_s})$, is independent of the pure state decompositions of $\rho$. It is also invariant under local unitary transformations of $\rho$. In particular, all coefficients of polynomial $Det(\Omega_s - \lambda E)$ are local invariants independent from the pure state decompositions and are invariance under local unitary transformations.

**Proof:** We first show that it is independent from the pure state decomposition of $\rho$. Let Eq. (1) and Eq. (2) be two different representations of a given mixed state $\rho$. We have

$$
(\Omega'_s)_{i_1 i_2 \ldots i_s j_1 j_2 \ldots j_s} = tr(B_{i_1} B_{j_1}^\dagger B_{i_2} B_{j_2}^\dagger \cdots B_{i_s} B_{j_s}^\dagger) = tr \left[ \sum_{i_1' j_1', \ldots, i_s' j_s'} U_{i_1} U_{i_1'}^* A_{i_2} U_{j_2}^* A_{j_2}^\dagger \cdots U_{i_s} U_{i_s'}^* A_{j_s} U_{j_s}^* A_{j_s}^\dagger \right] = ((U \otimes U \otimes \cdots \otimes U)(\Omega_s)(U^\dagger \otimes U^\dagger \otimes \cdots \otimes U^\dagger))_{i_1 i_2 \ldots i_s j_1 j_2 \ldots j_s}.
$$

Therefore $\Omega'_s = (U \otimes U \otimes \cdots \otimes U)\Omega_s(U^\dagger \otimes U^\dagger \otimes \cdots \otimes U^\dagger)$. Using the action, the associated multilinear form $f_\omega$ is acted upon by $U \otimes U \otimes \cdots \otimes U$ and $U^\dagger \otimes U^\dagger \otimes \cdots \otimes U^\dagger$ as follows:

$$(U_{(1)} \cdots U_{(s)} \cdot U_{(1)}^* \cdots U_{(s)}^*) f_\omega$$

Using the formula under the action (9) we get $Det(\Omega'_s - \lambda E) = Det(\Omega_s - \lambda E)$, and thus $Det(\Omega_s - \lambda E)$ does not depend on the detailed pure state decompositions of a given $\rho$. Note that in general we don’t know the exact formula for the hyperdeterminant, but we can still derive its invariance abstractly.

On the other hand, under local unitary transformations $\tilde{\rho} = (P \otimes Q) \rho (P \otimes Q)^\dagger$ for some local unitary operators $P \otimes Q \in SU(n) \otimes SU(m)$, similar to the proof of the second part of the Theorem 1 and using Lemma 8 in [8], it is easy to get $\Omega_s = \Omega'_s$. Therefore $Det(\Omega_s - \lambda E)$ is invariant under local unitary transformations. Moreover, following [15], the invariant polynomials are invariance under local unitary transformations. \[\square\]
As the application of our theorems we now give two interesting examples.

Example 1: Consider two mixed states $\rho_1 = \text{diag}\{1/2, 1/2, 0, 0\}$ and $\rho_2 = \text{diag}\{1/2, 0, 1/2, 0\}$. $\rho_1$ has a pure state decomposition with

$$A_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix}.$$ 

While $\rho_2$ has a pure state decomposition with

$$B_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}.$$ 

We have the corresponding matrices $(\Omega(\rho_1))_{i,j} = \text{tr}(A_iA_j^\dagger)$ and $(\Omega(\rho_2))_{i,j} = \text{tr}(B_iB_j^\dagger)$, $i, j = 0, 1$. From Theorem 1 one can find that these two states have the same values of the invariants in Eq. (4), $F_i(\Omega(\rho_1)) = F_i(\Omega(\rho_2))$.

We now consider further the four-dimensional hyperdeterminant of the format $2 \times 2 \times 2 \times 2$ [15]. Let $(\Omega(\rho_1))_{ijkl} = \text{tr}(A_iA_j^\dagger A_kA_l^\dagger) \equiv a_r$, $r = 0, \ldots, 15$, where $r = 8i + 4j + 2k + l$. From Ref. [15], one invariant with degree 4 is given by

$$N(\rho_1) = \text{det} \begin{pmatrix} a_0 & a_1 & a_8 & a_9 \\ a_2 & a_3 & a_{10} & a_{11} \\ a_4 & a_5 & a_{12} & a_{13} \\ a_6 & a_7 & a_{14} & a_{15} \end{pmatrix} = \frac{1}{256}.$$ 

However for $\rho_2$ we have $N(\rho_2) = 0$. Therefore $\rho_1$ and $\rho_2$ are not equivalent under local unitary transformations.

In Ref. [6], the Ky Fan norm of the realignment matrix of the quantum states $\mathcal{N}(\rho)$ is proved to be invariant under local unitary operations. By calculation we find $\mathcal{N}(\rho_1) = \mathcal{N}(\rho_2) = \frac{1}{\sqrt{2}}$. This means the Ky Fan norm of the realignment matrix can not recognize that $\rho_1$ and $\rho_2$ are not equivalent under local unitary transformations. Therefore Theorem 2 has its superiority over it with respect to these two states.

Example 2: Let two mixed states $\sigma_1 = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$ and $\sigma_2 = \text{diag}\{2/3, 0, 0, 1/3\}$. Then
\(\sigma_1\) has a pure state decomposition with
\[
C_0 = \begin{pmatrix} 1/\sqrt{3} & 0 \\ 0 & 0 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0 & 1/\sqrt{3} \\ 1/\sqrt{3} & 0 \end{pmatrix}.
\]

While \(\sigma_2\) has a pure state decomposition with
\[
D_0 = \begin{pmatrix} \sqrt{2}/\sqrt{3} & 0 \\ 0 & 0 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1/\sqrt{3} \end{pmatrix}.
\]

We have the corresponding matrices \((\Omega(\sigma_1))_{i,j} = tr(C_iC_j^\dagger)\) and \((\Omega(\sigma_2))_{i,j} = tr(D_iD_j^\dagger)\), \(i, j = 0, 1\). From Theorem 1 one can find that these two states have the same values of the invariants in Eq. (4), \(F_i(\Omega(\sigma_1)) = F_i(\Omega(\sigma_2))\).

We also consider further the four-dimensional hyperdeterminant of the format \(2 \times 2 \times 2 \times 2\). Also let \((\Omega(\sigma_1))_{ijkl} = tr(C_iC_j^\daggerC_kC_l^\dagger) \equiv a_{sr}, r = 0, \cdots, 15,\) where \(s = 8i + 4j + 2k + l\). From Ref. [13], the another invariant with degree 4 is given by
\[
M(\sigma_1) = det \begin{pmatrix} a_0 & a_8 & a_2 & a_{10} \\ a_1 & a_9 & a_3 & a_{11} \\ a_4 & a_{12} & a_6 & a_{14} \\ a_5 & a_{13} & a_7 & a_{15} \end{pmatrix} = \frac{1}{6561}.
\]

However for \(\sigma_2\) we have \(M(\sigma_2) = 0\). Therefore \(\sigma_1\) and \(\sigma_2\) are not equivalent under local unitary transformations. From example 2, one can see that the spectra of their reduced one-qubit density matrices have the same value. Therefore, only by the spectra of their reduced one-qubit density matrices can not judge the equivalence of given states.

Our results can be generalized to multipartite case. Let \(H_1, H_2, \cdots, H_m\) be \(n_1, n_2, \cdots, n_m\)-dimensional complex Hilbert spaces with \(\{|k_1\rangle\}_{k_1=1}^{n_1}, \{|k_2\rangle\}_{k_2=1}^{n_2}, \cdots, \{|k_m\rangle\}_{k_m=1}^{n_m}\) the orthonormal basis of \(H_1, H_2, \cdots, H_m\) respectively. Let \(\rho\) be an arbitrary mixed state defined on \(H_1 \otimes H_2 \otimes \cdots \otimes H_m\), \(\rho = \sum_{i=1}^{l} p_i |v_i\rangle \langle v_i|\), where \(|v_i\rangle\) is a multipartite pure state of the form: \(|v_i\rangle = \sum_{k_1,k_2,\cdots,k_m = 1}^{n_1,n_2,\cdots,n_m} a_{k_1k_2\cdots k_m}^{(i)} |k_1k_2\cdots k_m\rangle\), \(a_{k_1k_2\cdots k_m}^{(i)} \in \mathbb{C}\). Now we view \(|v_i\rangle\) as bipartite pure state under the partition between the first \(l\) subsystems and the rest, \(1 \leq l < n\). Then \(A_i = (\sqrt{p_i} a_{k_1k_2\cdots k_m}^{(i)})\) can be regarded as the \(N_1 \times N_2\) matrix with \(N_1 = n_1 \times n_2 \times \cdots \times n_l\) and \(N_2 = n_{l+1} \times n_{l+2} \times \cdots \times n_m\) for all \(i = 1, \cdots, I\). We define the matrix \(\Omega_s\) with entries \((\Omega_s)_{i_1i_2\cdots i_s,j_1j_2\cdots j_s} = tr(A_{i_1}A_{j_1}^\dagger \cdots A_{i_s}A_{j_s}^\dagger)\), for \(i_k,j_k = 1, \cdots, I, s \geq 1\). Then we have that \(Det(\Omega_s - \lambda E)\) does not depend on the pure states decompositions and is invariant under local unitary transformations.
III. CONCLUSION

We have investigated the invariants under local unitary transformations for arbitrary dimensional quantum systems. These invariants are independent of the detailed pure state decompositions of a given state. They give rise to the necessary conditions for the equivalence of quantum states under local unitary transformations. These invariants may be also used in characterizing quantum correlations such as quantum entanglement \cite{11} and quantum discord \cite{19}, since all these quantities are at least the invariants under local unitary transformations and are independent of the pure state decompositions.

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