Distribution of spin correlation strengths in multipartite systems

Bing Yu1 · Naihuan Jing2 · Xianqing Li-Jost3

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Abstract
For a two-qubit state, the isotropic strength measures the degree of isotropic spin correlation. The concept of isotropic strength is generalized to multipartite qudit systems, and the strength distributions for tripartite and quadripartite qudit systems are thoroughly investigated. We show that the sum of relative isotropic strengths of any three-qudit state over \( d \)-dimensional Hilbert space cannot exceed \( d - 1 \), which generalizes the case \( d = 2 \). The trade-off relations and monogamy-like relations of the sum of spin correlation strengths for pure three- and four-partite systems are derived. Moreover, the bounds of spin correlation strengths among different subsystems of a quadripartite state are used to analyze quantum entanglement.

1 Introduction
Let \( \rho \) be a two-qubit state on \( \mathcal{H}_2^A \otimes \mathcal{H}_2^B \) in the Bloch form [1]

\[
\rho_{AB} = \frac{1}{4} \left( I \otimes I + \sum_{i=1}^{3} a_i \sigma_i \otimes I + \sum_{j=1}^{3} b_j I \otimes \sigma_j + \sum_{i,j=1}^{3} R_{ij} \sigma_i \otimes \sigma_j \right),
\]

where \( \sigma_1, \sigma_2, \sigma_3 \) are the Pauli spin matrices, \( a_i = \text{tr}(\rho_{AB} \sigma_i \otimes I) \), \( b_j = \text{tr}(\rho_{AB} I \otimes \sigma_j) \), and \( R_{ij} = \text{tr}(\rho_{AB} \sigma_i \otimes \sigma_j) \). The spin correlation matrix \( R = (R_{ij}) \) and the...
vectors $a = (a_1, a_2, a_3)^t$ and $b = (b_1, b_2, b_3)^t$ characterize the two-qubit state in an essential way. The three quantities are closely related to the intensity of quantum correlations [2–4], and they are also utilized in several fundamental concepts such as quantum entanglement [5–15], quantum discord [16–20], EPR steering [21–26] and Bell nonlocality [27–30].

For multipartite quantum state $\rho$, the spin correlation matrices among any two-partite substates also reveal intrinsic properties of the quantum phenomena. For three-qubit pure state $\rho_{ABC} \in \mathcal{H}_A^2 \otimes \mathcal{H}_B^2 \otimes \mathcal{H}_C^2$, its three reduced two-qubit states $\rho_{AB}$, $\rho_{BC}$ and $\rho_{AC}$, and the associated spin correlation matrices $R^{AB}$, $R^{BC}$ and $R^{AC}$ have figured prominently in the recent interesting work of Cheng and Hall [31]. Therein they introduced the isotropic strength $s_{iso}^{AB}$ as the average of the three eigenvalues of the matrix $R^{AB}R^{ABt}$ and showed that the sum of the three isotropic strengths $s_{iso}^{AB}$, $s_{iso}^{BC}$ and $s_{iso}^{AC}$ satisfies the amazing identity $s_{iso}^{AB} + s_{iso}^{BC} + s_{iso}^{AC} = 1$, which can be used to deduce the volume monogamy relation of quantum steering ellipsoids [25] and strong monogamy relations for Bell nonlocality [31].

In this work, we generalize the spin correlation matrix of the two-qubit systems to the general qudit system in order to reveal fundamental properties of the quantum system. Moreover, we would like to understand how variance of the Hilbert space affects the situation, in the hope to learn the fundamental local unitary invariance. We generalize the isotropic strength from the three-qubit system to the tripartite and quadripartite qudit systems and investigate the distributions of spin correlation strengths. For pure tripartite qudit systems $\mathcal{H}_A^d \otimes \mathcal{H}_B^d \otimes \mathcal{H}_C^d$, based on the purity of the reduced state of the pure three-qudit state, we obtain that the sum of the isotropic strengths for arbitrary three-qudit state over $d$-dimensional Hilbert space cannot exceed $d - 1$, which is a generalization of one main identity in [31]. For pure quadripartite qudit systems, we give the trade-off relations, monogamy relations of the spin correlation strengths similarly in the tripartite case.

For a multipartite system, our bounds of the spin correlation strengths among different subsystems can be utilized to analyze the intrinsic quantum entanglement. We first give necessary conditions of a pure four-qudit state being biseparable by using (2.7) and Corollary 2.2 [cf. (2.10)]. After that, we generalize Vicente–Huber’s method [11] of detecting genuine multipartite entanglement (GME) to all quantum four-qudit states.

This paper is organized as follows. In Sect. 2, we generalize the isotropic strength of the pure three-qubit systems to the tripartite qudit systems and show the sum of isotropic strengths can not exceed $d - 1$. We obtain the trade-off relation and other interesting properties of isotropic strengths. In Sect. 3, we extend the results to pure quadripartite qudit systems and present the trade-off relation for the spin correlation strengths. In Sect. 4, we show how the distribution of spin correlation strengths is used to detect quantum entanglement. Conclusion and summary are given in Sect. 5.

2 Distribution of spin correlation strengths for tripartite state

Let $\rho_{AB}$ be the density matrix of a bipartite state on the tensor product $\mathcal{H}_d^A \otimes \mathcal{H}_d^B$, where $\mathcal{H}_d$ is a $d$-dimensional Hilbert space. Let $\lambda_i$ be the Gell-Mann basis elements (self-
dual) on \( \mathcal{H}_d \) normalized as \( \text{tr}(\lambda_i \lambda_j) = d \delta_{ij} \) and \( \lambda_0 = I \). Denote \( \lambda = (\lambda_1, \ldots, \lambda_{d^2-1}) \). Then \( \rho_{AB} \) can be written in the Bloch form

\[
\rho_{AB} = \frac{1}{d^2} \left( I \otimes I + \sum_{i=1}^{d^2-1} a_i \lambda_i \otimes I + \sum_{j=1}^{d^2-1} b_j I \otimes \lambda_j + \sum_{i,j=1}^{d^2-1} R_{ij} \lambda_i \otimes \lambda_j \right),
\]

(2.1)

where \( a_i = \text{tr}(\rho_{AB} \lambda_i \otimes I) \), \( b_j = \text{tr}(\rho_{AB} I \otimes \lambda_j) \), and \( R_{ij} = \text{tr}(\rho_{AB} \lambda_i \otimes \lambda_j) \). The vectors

\[
a = (a_1, a_2, \ldots, a_{d^2-1})^t, \quad b = (b_1, b_2, \ldots, b_{d^2-1})^t
\]

(2.2)

are the Bloch vectors of the reduced states \( \rho_A \) and \( \rho_B \), respectively:

\[
\rho_A = \frac{1}{d} (I + a \cdot \lambda) = \frac{1}{d} \left( I + \sum_{i=1}^{d^2-1} a_i \lambda_i \right).
\]

(2.3)

The matrix \( R = R^{AB} = (R_{ij})_{(d^2-1) \times (d^2-1)} \) is called the spin correlation matrix of \( \rho_{AB} \).

Clearly, \( a \), \( b \) and \( R^{AB} \) are local unitary invariants of the state \( \rho^{AB} \). In particular, if \( \rho^{AB} \) is pure, then \( \text{tr}\rho_A^2 = \text{tr}\rho_B^2 \) implies that \( a^2 = b^2 \). Also \( a^2 \leq d - 1 \) due to the fact that \( \text{tr}\rho^2_A \leq 1 \).

The spin correlation matrix \( R^{AB} \) reveals some of the characteristic properties of \( \rho^{AB} \). Consider the eigenvalues of \( R^{AB} R^{AB \dagger} \) arranged in descending order: \( s_1 \geq s_2 \geq \ldots \geq s_{d^2-1} \). Generalizing the qubit case [31], we define the isotropic strength of the density matrix \( \rho_{AB} \) as the average of the eigenvalues:

\[
s_{iso}^{AB} = \frac{1}{d^2 - 1} \sum_{k=1}^{d^2-1} s_k = \frac{||R^{AB}||^2}{d^2 - 1},
\]

(2.4)

where \( ||R|| = \sqrt{\text{tr}(RR\dagger)} \) is the Frobenius norm.

Now we consider a general pure tripartite state \( \rho_{ABC} = |\psi\rangle \langle \psi| \) on \( \mathcal{H}_d^A \otimes \mathcal{H}_d^B \otimes \mathcal{H}_d^C \), where \( \langle \psi | \psi \rangle = 1 \). Its Bloch form relative to tensor products of the Gell-Mann basis \( \lambda_i \) is

\[
\rho_{ABC} = \frac{1}{d^3} \left( I \otimes I \otimes I + \sum_{i=1}^{d-1} a_i \lambda_i \otimes I \otimes I + \sum_{j=1}^{d-1} b_j I \otimes \lambda_j \otimes I + \sum_{l=1}^{d-1} c_l I \otimes I \otimes \lambda_l \\
+ \sum_{i,j=1}^{d^2-1} R_{ij}^{AB} \lambda_i \otimes \lambda_j \otimes I + \sum_{j,l=1}^{d-1} R_{jl}^{BC} I \otimes \lambda_j \otimes \lambda_l + \sum_{i,l=1}^{d^2-1} R_{il}^{AC} \lambda_i \otimes I \otimes \lambda_l \\
+ \sum_{i,j,l=1}^{d^2-1} R_{ijl}^{ABC} \lambda_i \otimes \lambda_j \otimes \lambda_l \right),
\]

(2.5)
where the vectors $a$, $R^{AB}$, $R^{ABC}$, etc., are taken as column vectors with the indices arranged in the lexicographical order. Each entry of the component vector is given by trace function, for example, $R^{ABC}_{ijl} = \text{tr}(\rho_{ABC}\lambda_i \otimes \lambda_j \otimes \lambda_l)$. For uniformity the vectors $a, b, c$ are also denoted by $R^A, R^B, R^C$, respectively.

The two-partite reduced states are $\rho_{AB} = \text{tr}_C \rho_{ABC}$, etc. See (2.1). So $R_{AB}$, $R_{BC}$ and $R_{AC}$ still denote the spin correlation matrices, respectively. It follows from the Schmidt decomposition that the purity of tripartite state implies that any bipartition of the pure state $\rho_{ABC}$ satisfies $\text{tr}(\rho_{AB}^2) = \text{tr}(\rho_{C}^2)$, $\text{tr}(\rho_{BC}^2) = \text{tr}(\rho_{A}^2)$ and $\text{tr}(\rho_{AC}^2) = \text{tr}(\rho_{B}^2)$.

Invoking the purity of any bipartition of the pure three-qudit state $\rho_{ABC}$, we calculate the isotropic strengths as follows.

\[
\begin{align*}
\alpha_{iso}^{AB} &= \frac{d(1 + c^2) - 1 - a^2 - b^2}{d^2 - 1}, \\
\alpha_{iso}^{BC} &= \frac{d(1 + a^2) - 1 - b^2 - c^2}{d^2 - 1}, \\
\alpha_{iso}^{AC} &= \frac{d(1 + b^2) - 1 - a^2 - c^2}{d^2 - 1}. \\
\end{align*}
\]

(2.6)

Note that

\[
\|R_{ABC}\|^2 = \alpha_{iso}^{AB}(d^2 - 1) \leq d - 1 + d c^2 \leq d^2 - 1, \tag{2.7}
\]

where the second inequality comes from $c^2 \leq d - 1$, similarly the inequality holds for $\|R_{AC}\|^2$ and $\|R_{BC}\|^2$ as well.

Subsequently for a pure tripartite state

\[
\alpha_{iso}^{AB} + \alpha_{iso}^{BC} + \alpha_{iso}^{AC} = \frac{3d - 3 + (d - 2)(a^2 + b^2 + c^2)}{d^2 - 1}.
\]

(2.8)

**Lemma 2.1** For a pure tripartite qudit state $\rho_{ABC}$, the invariants satisfy the following relation:

\[
a^2 + b^2 + c^2 + \frac{\|R_{ABC}\|^2}{d - 1} = (d + 2)(d - 1) \tag{2.9}
\]

where $a$, $b$ and $c$ are the Bloch vectors of the reduced states $\rho_A$, $\rho_B$ and $\rho_C$, respectively, and $\|R_{ABC}\|$ is the Euclidean norm.

**Proof** For a pure tripartite state $\rho_{ABC}$ given as in (2.5), $\text{tr}\rho_{ABC}^2 = 1$. Then $1 + a^2 + b^2 + c^2 + (d^2 - 1)(\alpha_{iso}^{AB} + \alpha_{iso}^{BC} + \alpha_{iso}^{AC}) + \|R_{ABC}\|^2 = d^3$. Using (2.8), we see that the invariants satisfy the relation (2.9). \qed

**Corollary 2.2** For any tripartite state $\rho$, one has the following bound:

\[
\|R_{ABC}\|^2 \leq (d - 1)^2(d + 2). \tag{2.10}
\]
In fact, Lemma 2.1 says that \( \| R_{ABC} \| \leq (d - 1)\sqrt{d + 2} \) for a pure state \( \rho \). Then for a general state \( \rho = \sum \alpha \rho_{\alpha} \) with \( \sum \alpha \rho_{\alpha} = 1 \), the convex property of the Euclidean norm implies that

\[
\| R_{ABC} \| = \| \sum \alpha \rho_{\alpha} R_{ABC}(\rho_{\alpha}) \| \leq \sum \alpha \rho_{\alpha} \| R_{ABC}(\rho_{\alpha}) \| \leq \sum \alpha \rho_{\alpha} (d - 1)\sqrt{d + 2} = (d - 1)\sqrt{d + 2}.
\] (2.11)

Using Lemma 2.1, we can derive the upper bound of the sum of the isotropic strengths of the pure three-qudit state \( \rho_{ABC} \) and the trade-off relation about isotropic strengths immediately.

**Theorem 2.3** For a pure tripartite qudit state \( \rho_{ABC} \), the sum of isotropic strengths has the following trade-off relation:

\[
s_{iso}^{AB} + s_{iso}^{AC} + s_{iso}^{BC} = d - 1 - \frac{d - 2}{(d + 1)(d - 1)^2} \| R_{ABC} \|^2.
\] (2.12)

Thus \( \| R_{ABC} \|^2 \) can be viewed as a measure of the tripartite spin correlation strength.

Note that when \( d = 2 \), (2.12) reduces to \( s_{iso}^{AB} + s_{iso}^{AC} + s_{iso}^{BC} = 1 \), which is one of the key relations discovered in [31] for pure three-qubit state.

From (2.6), it follows that

\[
\frac{1}{d - 1} a^2 - s_{iso}^{BC} = \frac{1}{d - 1} b^2 - s_{iso}^{AC} = \frac{1}{d - 1} c^2 - s_{iso}^{AB}.
\] (2.13)

We remark that a pure tripartite state has the following bounds: \( \frac{3}{d + 1} \leq s_{iso}^{AB} + s_{iso}^{AC} + s_{iso}^{BC} \leq d - 1 \).

**Theorem 2.4** Let \( \rho_{ABC} \) be a general tripartite quantum state and \( s_{iso}^{AB}, s_{iso}^{AC}, s_{iso}^{BC} \) the relative isotropic strengths of the reduced states. One has that \( s_{iso}^{AB} + s_{iso}^{AC} + s_{iso}^{BC} \leq d - 1 \).

**Proof** It suffices to consider the pure state. The upper bound is clear from Theorem 2.3.

Theorems 2.3–2.4 generalize the trade-off relations from three qubits to general quantum tripartite systems.

### 3 Distribution of spin correlation strengths for quadripartite state

In this section, we generalize the trade-off relations of spin correlation strengths to any quadripartite quantum state on \( \mathcal{H}_d^A \otimes \mathcal{H}_d^B \otimes \mathcal{H}_d^C \otimes \mathcal{H}_d^D \).

Let \( \rho_{ABCD} \) be a pure four-qudit state on \( \mathcal{H}_d^{\otimes 4} \) in the Bloch form similarly to (2.5), then its reduced states \( \rho_{ABC} \) and \( \rho_D \) can be written, respectively, as in (2.5) and (2.3).
It follows from the purities of the bipartition reduced states that \( \text{tr}(\rho_{AB}^2) = \text{tr}(\rho_{CD}^2) \), \( \text{tr}(\rho_{AC}^2) = \text{tr}(\rho_{BD}^2) \), and \( \text{tr}(\rho_{AD}^2) = \text{tr}(\rho_{BC}^2) \). The following relations are then easily seen for the isotropic strengths of bipartitions:

\[
\begin{align*}
    s_{iso}^{AB} - s_{iso}^{CD} &= \frac{-(a^2 + b^2) + c^2 + d^2}{d^2 - 1}, \\
    s_{iso}^{AC} - s_{iso}^{BD} &= \frac{-(a^2 + c^2) + b^2 + d^2}{d^2 - 1}, \\
    s_{iso}^{AD} - s_{iso}^{BC} &= \frac{-(a^2 + d^2) + b^2 + c^2}{d^2 - 1}.
\end{align*}
\]

Similarly for the bipartitions \((ABC, D), (ACD, B), (BCD, A)\) and \((ABD, C)\), one also has identical purities for any pair of the reduced states \(\{\rho_{ABC}, \rho_D\}, \{\rho_{ACD}, \rho_B\}, \{\rho_{BCD}, \rho_A\}\) and \(\{\rho_{ABD}, \rho_C\}\). Subsequently, we have that

\[
\begin{align*}
    s_{iso}^{BC} + s_{iso}^{AC} + s_{iso}^{AB} &= \frac{d^2(1 + d^2) - (1 + a^2 + b^2 + c^2) - \|R^{ABC}\|^2}{d^2 - 1}, \\
    s_{iso}^{CD} + s_{iso}^{AC} + s_{iso}^{AD} &= \frac{d^2(1 + b^2) - (1 + a^2 + c^2 + d^2) - \|R^{ACD}\|^2}{d^2 - 1}, \\
    s_{iso}^{BC} + s_{iso}^{CD} + s_{iso}^{BD} &= \frac{d^2(1 + a^2) - (1 + b^2 + c^2 + d^2) - \|R^{BCD}\|^2}{d^2 - 1}, \\
    s_{iso}^{AB} + s_{iso}^{AD} + s_{iso}^{BD} &= \frac{d^2(1 + c^2) - (1 + a^2 + b^2 + d^2) - \|R^{ABD}\|^2}{d^2 - 1}.
\end{align*}
\]

where \(\|R^{ABC}\|^2\) is the tripartite spin correlation strength of the reduced state \(\rho_{ABC} = \text{tr}_D(\rho_{ABC})\), and the other spin correlation strengths \(\|R^{ACD}\|^2\), \(\|R^{BCD}\|^2\) or \(\|R^{ABD}\|^2\) are defined similarly.

Simple calculation leads to the relation between the four tripartite spin correlation strengths, for example,

\[
(\phi^2 - 1)\phi^2 = (\phi^2 - 1)b^2 - \|R^{ACD}\|^2 = (\phi^2 - 1)c^2 - \|R^{ABD}\|^2 = (\phi^2 - 1)d^2 - \|R^{ABC}\|^2.
\]

Therefore, we have that

\[
\begin{align*}
    \|R^{ACD}\|^2 - \|R^{BCD}\|^2 &= (d^2 - 1)(b^2 - a^2), \\
    \|R^{ABC}\|^2 - \|R^{ABD}\|^2 &= (d^2 - 1)(d^2 - c^2), \\
    \|R^{ABD}\|^2 - \|R^{ACD}\|^2 &= (d^2 - 1)(c^2 - b^2).
\end{align*}
\]

**Theorem 3.1** For a pure quadripartite state \(\rho_{ABCD}\) over \(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \mathcal{H}_D\), the isotropic strengths of the state satisfy the following trade-off relation,
\[ s_{iso}^{AB} + s_{iso}^{AC} + s_{iso}^{AD} + s_{iso}^{BC} + s_{iso}^{CD} + s_{iso}^{BD} \]
\[ = \frac{(-d^2 + 3)(d^2 - 1) + (d^2 - 2)(a^2 + b^2 + c^2 + d^2) + \|R^{ABCD}\|^2}{d^2 - 1}, \]  
\[ (3.7) \]

where \( \|R^{ABCD}\| \) is the Euclidean norm of the vector \( R^{ABCD} \).

As \( \|R^{ABCD}\|^2 \) is a measure of the quadripartite spin correlation strength, the trade-off relation implies that the quadripartite spin correlations are tied up with relative isotropic strengths of the four-partite qudit state \( \rho_{ABCD} \).

**Corollary 3.2** For a pure quadripartite state \( \rho_{ABCD} \), the sum of isotropic strengths also satisfies the trade-off relation,

\[ s_{iso}^{AB} + s_{iso}^{AC} + s_{iso}^{AD} + s_{iso}^{BC} + s_{iso}^{CD} + s_{iso}^{BD} \]
\[ = 2(d^2 - 1) - (a^2 + b^2 + c^2 + d^2) - 2((d^2 - 1)a^2 - \|R^{BCD}\|^2) \]
\[ = \frac{d^2 - 1 + (d^2 - 3)a^2 - \|R^{BCD}\|^2}{d^2 - 1}. \]  
\[ (3.8) \]

**Proof** The above trade-off relation can be obtained easily by combining (3.2) with (3.3).

**Corollary 3.3** For any pure four-qudit state \( \rho_{ABCD} \), the sum of isotropic strengths also satisfies the monogamy relation:

\[ s_{iso}^{AB} + s_{iso}^{AC} + s_{iso}^{AD} = \frac{d^2 - 1 + (d^2 - 3)a^2 - \|R^{BCD}\|^2}{d^2 - 1}. \]  
\[ (3.9) \]

**Remark** Similar monogamy relations are easily obtained when taking one of other particles \( B, C \) and \( D \) as a central one. For example, when particle \( B \) is treated as central, the following monogamy relation follows immediately,

\[ s_{iso}^{BA} + s_{iso}^{BC} + s_{iso}^{BD} = \frac{d^2 - 1 + (d^2 - 3)b^2 - \|R^{ACD}\|^2}{d^2 - 1}. \]  
\[ (3.10) \]

Note that for \( d = 2 \) it follows from Theorem 2.4 that each equality in (3.2) cannot exceed 1 and sum of isotropic strengths for the four-qubit pure state satisfies the trade-off relation:

\[ s_{iso}^{AB} + s_{iso}^{AC} + s_{iso}^{AD} + s_{iso}^{BC} + s_{iso}^{CD} + s_{iso}^{BD} \]
\[ = \frac{12 + a^2 - \|R^{BCD}\|^2 + b^2 - \|R^{ACD}\|^2 + c^2 - \|R^{ABD}\|^2 + d^2 - \|R^{ABC}\|^2}{6} \]
\[ \leq 2. \]  
\[ (3.11) \]

In this case, one also has that

\[ a^2 + b^2 + c^2 + d^2 \leq \|R^{BCD}\|^2 + \|R^{ACD}\|^2 + \|R^{ABD}\|^2 + \|R^{ABC}\|^2. \]  
\[ (3.12) \]
4 Quadripartite entanglement detection

In this section, we discuss how our bounds of spin correlation strengths among different subsystems are applied in detecting entanglement for quadrupartite quantum states. We consider the quadrupartite space $\otimes^4_{k=1} \mathcal{H}_k$, where $\mathcal{H}_k \simeq \mathbb{C}^d$ is the space for the $k$th particle.

A density matrix $\rho$ on $\otimes^4_{k=1} \mathcal{H}_k$ can be expressed in the Bloch form similar to the tripartite case (2.5). To streamline the notation, all vectors in the Bloch form of $\rho^{ABCD}$ will be denoted as $R^A$, $R^{AB}$, $R^{ABC}$, $R^{ABCD}$, etc. For example, the previous vectors $a$, $b$, $c$ etc will be denoted as $R^A$, $R^B$, $R^C$, etc. To express our results, we now rearrange the Bloch vectors $R^A$, $R^{AB}$, $R^{ABC}$ in a matrix form. As tensor functions, $\rho \rightarrow R^{i_1 \cdots i_4}(\rho)$ is convex linear, i.e., $R^{i_1 \cdots i_4}((c\rho_1 + (1-c)\rho_2) = cR^{i_1 \cdots i_4}(\rho_1) + (1-c)R^{i_1 \cdots i_4}(\rho_2)$.

The realignments are in one-to-one correspondence to bipartitions of the index set $\{1234\}$ or $\{A, B, C, D\}$. If the $k$th particle is grouped with $l$th particle, we use underlined indices to indicate such a realignment. For instance, when the 1st and 3rd particles are grouped together, the column vector $R^{ABC}$ is converted to a square matrix via

$$R^{ABC}_{i l} = \sum_{i,j,l,m=1}^{d^2-1} R^{ABCD}_{i j l m} |i l \rangle \langle j m| ,$$

(4.1)

where $|i l\rangle$ (resp. $|j m\rangle$) represents column indices (resp. row indices) in lexicographical order. We will take the freedom to use the same notation for the matrix $R^{ABCD}_{i j l m}$ as well. Recall that the Ky Fan $k$-norm $\|S\|_k$ of an $m \times n$ matrix $S$ is defined as the sum of the $k$th partial sum of the singular values, i.e., $\|S\|_k = \sum_{i=1}^k \alpha_i$, where $\alpha_i (1 \leq \ldots \leq \min(m,n))$ are the singular values of $S$ in decreasing order.

If a pure state $|\Psi\rangle \in \otimes^4_{k=1} \mathcal{H}_k$ can be decomposed as $|\Psi\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle \otimes |\Psi_3\rangle \otimes |\Psi_4\rangle$, where $|\Psi_k\rangle$ is a pure state in $k$th subsystem, then $|\Psi\rangle$ is called fully separable. A pure state $|\Psi\rangle$ is biseparable provided that it can be written as $|\Psi\rangle = |\Psi\rangle_T \otimes |\Psi\rangle_{\tilde{T}}$, where $T$ denotes some subset of subsystems and its complement is $\tilde{T}$. We now derive some useful bounds for the tensor $R^{ABCD}$, which will be used in detecting multipartite entanglement.

Note that there are seven matrix forms (or realignments) of the tensor $R^{ABCD}$, i.e., seven partitions into two subsets of two particles or partitions into one vs. three particles, namely the $2 \times 2$-matrices $\{R^{ABCD}_{i j l m}, R^{ABCD}_{i j l m}, R^{ABCD}_{i j l m}\}$, and the $1 \times 3$ or $3 \times 1$-rectangular matrices $\{R^{ABCD}_{i j l m}, R^{ABCD}_{i j l m}, R^{ABCD}_{i j l m}\}$.

We first present upper bounds for the matrix form $R^{ABCD}_{i j l m} = \sum_{i,j,l,m} R^{ABCD}_{i j l m} |i j \rangle \langle l m|$ of the tensor $R^{ABCD}$ in the pure biseparable four-qudit state. The bounds for the other two particles vs two particles matrix forms follow easily by using similar discussion. Moreover, if $\rho^{ABCD} = \rho^{AB} \otimes \rho^{CD}$, then $\text{Tr}(\rho^{ABCD} \lambda_i \otimes \lambda_j \otimes \lambda_l \otimes \lambda_m) = \text{Tr}(\rho^{AB} \lambda_i \otimes \lambda_j) \text{Tr}(\rho^{CD} \lambda_l \otimes \lambda_m)$, i.e., $R^{ABCD}_{i j l m} = R^{AB}_{i j} R^{CD}_{l m}$.
Lemma 4.1 Assume that a pure four-qudit state is biseparable. Then one has that

1) If the state is fully separable, i.e., for partition \( i \mid j \mid l \mid m \)

\[
\| R^{ABCD}_{ijlm} \|_k \leq (d - 1)^2; \tag{4.2}
\]

2) If the state is biseparable as one vs. three particles,
   (i) for the bipartite partition \( i \mid j \mid l \mid m \) (\( j \mid i \mid l \mid m \), similarly)

\[
\| R^{ABCD}_{ijlm} \|_k \leq \sqrt{k}(d - 1)\sqrt{(d - 1)(d + 2)}; \tag{4.3}
\]
   (ii) for the bipartite partition \( i \mid j \mid l \mid m \) (\( i \mid j \mid m \), similarly)

\[
\| R^{ABCD}_{ijlm} \|_k \leq \sqrt{k}(d - 1)\sqrt{(d - 1)(d + 2)}; \tag{4.4}
\]

3) If the state is separable into two subsystems of two particles,
   (i) for the bipartite partition \( i \mid j \mid l \mid m \)

\[
\| R^{ABCD}_{ijlm} \|_k \leq d^2 - 1; \tag{4.5}
\]
   (ii) for the bipartite partition \( i \mid j \mid l \mid m \) (\( i \mid j \mid m \), similarly)

\[
\| R^{ABCD}_{ijlm} \|_k \leq \sqrt{k}(d^2 - 1); \tag{4.6}
\]

Proof We have already shown that \( \| R^A \| \leq \sqrt{d - 1}, \| R^{AB} \| \leq \sqrt{d^2 - 1} \) (cf. (2.7)) and \( \| R^{ABC} \| \leq (d - 1)\sqrt{d + 2} \) [cf. (2.10)]. Also for any matrix \( S \), one has that \( \| S \|_k \leq \sqrt{k} \| S \| \).

(1) Therefore, for partition \( i \mid j \mid l \mid m \)

\[
\| R^{ABCD}_{ijlm} \|_k = \| R^A_i \| \| R^B_j \| \cdot (R^C_l \otimes R^D_m) \|_k
= \| R^A_i \| \| R^B_j \| \| R^C_l \| \| R^D_m \| \leq (d - 1)^2. \tag{4.7}
\]

(2) (i) As for bipartite partition \( i \mid j \mid l \mid m \)

\[
\| R^{ABCD}_{ijlm} \|_k = \| R^A_i \| \| R^{BCD}_{jlm} \|_k \leq \sqrt{k} \| R^A_i \| \| R^{BCD}_{jlm} \|_k
\leq \sqrt{k}(d - 1)\sqrt{(d - 1)(d + 2)}, \tag{4.8}
\]

and similarly one can see it for \( j \mid i \mid l \).
(ii) For bipartite partition $i|j|m$, we have that
\[
\|R_{ijlm}^{ABCD}\|_k = \|R_{ij}^{AB} \otimes R_{lm}^D\|_k \\
\leq \sqrt{k} (d-1) \sqrt{(d-1)(d+2)},
\]  
(4.9)

and the same holds for $i|m|j$.

(3) (i) Now for bipartite partition $i|j|m$:
\[
\|R_{ijlm}^{ABCD}\|_k = \|R_{ij}^{AB} \cdot R_{lm}^{CD}\|_k \\
= \|R_{ij}^{AB}\| \|R_{lm}^{CD}\| \\
\leq d^2 - 1.
\]  
(4.10)

(ii) for bipartite partition $i|m|j$:
\[
\|R_{ijlm}^{ABCD}\|_k = \|R_{ij}^{AC} \otimes R_{lm}^{BD}\|_k \\
\leq \sqrt{k} (d-1) \sqrt{d^2 - 1}.
\]  
(4.11)

For $d = 2$, one has that
\[
\|R_{ijlm}^{ABCD}\|_k \leq (d-1) \sqrt{d^2 - 1}
\] due to the fact that the singular values of $A \otimes B$ are products of those of $A$ and $B$. The same inequality holds for $i|m|j|l$ similarly.

One can see that $\|R_{ijlm}^{ABCD}\|_k \leq (d-1) \sqrt{d^2 - 1}$ for tripartite partition $i|j|m$ and $\|R_{ijlm}^{ABCD}\|_k \leq \sqrt{k}(d-1) \sqrt{d^2 - 1}$ for partitions $i|l|jm$ and $i|m|jl$. It is clear that these two bounds are strictly weaker than the upper bounds in (4.3–4.6), which means that if a pure state is tripartite separable, it must be biseparable.

Note that if one considers the bipartition of one particle vs three particles, the matrix satisfies that $\|R_{ijlm}^{ABCD}\|_k \leq \sqrt{k}(d^2 - 1)$, which is weaker than (4.5). Thus we do not take these matrix forms into account.

To detect genuine multipartite entanglement, we define the average matrix $k$-norm of all two vs two partitions $\|M_{22}(R^{ABCD})\|_k = \left( \|R_{ijlm}^{ABCD}\|_k + \|R_{ijlm}^{ABCD}\|_k + \|R_{ijlm}^{ABCD}\|_k \right) / 3$. The following theorem gives a lower bound for this average norm.
Theorem 4.2 Let $\rho$ be a four-qudit quantum state. If the average $k$-norm satisfies the inequality

$$\| M_{22}(R^{ABCD}) \|_k > \max \left\{ \frac{d^2 - 1 + 2\sqrt{k}(d^2)}{3}, \sqrt{k}(d - 1)^{\sqrt{(d - 1)(d + 2)}} \right\}$$

(4.13)

for some integer $k \in [1, \ldots, (d^2 - 1)^2]$, then $\rho$ is genuinely multipartite entangled.

Proof Assume that $\rho$ is bipartite separable along the bipartite partition $i \mid jlm$, then for each $k$

$$\| M_{22}(R^{ABCD}) \|_k = \frac{1}{3} \left( \| R^{ABCD}_{ijlm} \|_k + \| R^{ABCD}_{ijlm} \|_k + \| R^{ABCD}_{ijlm} \|_k \right)$$

$$= \frac{1}{3} \left( \sum_{\alpha} p_{\alpha} R^{ABCD}_{ijlm}(\rho_{\alpha}) \|_k + \sum_{\alpha} p_{\alpha} R^{ABCD}_{ijlm}(\rho_{\alpha}) \|_k \right)$$

$$\leq \frac{1}{3} \sum_{\alpha} p_{\alpha} \left( \| R^{ABCD}_{ijlm}(\rho_{\alpha}) \|_k + \| R^{ABCD}_{ijlm}(\rho_{\alpha}) \|_k + \| R^{ABCD}_{ijlm}(\rho_{\alpha}) \|_k \right)$$

$$\leq \sum_{\alpha} p_{\alpha} \left( \sqrt{k}(d - 1)^{\sqrt{(d - 1)(d + 2)}} \right)$$

$$= \sqrt{k}(d - 1)^{\sqrt{(d - 1)(d + 2)}}.$$  (4.14)

where the second inequality uses (4.3) and (4.4).

If the biseparable is along the bipartite partition $ij \mid l m$, we can use the bounds (4.5) and (4.6) to derive that $\| M_{22}(R^{ABCD}) \|_k \leq \frac{d^2 - 1 + 2\sqrt{k}(d^2)}{3}$. Thus, if $\| M_{22}(R^{ABCD}) \|_k > \max \left\{ \frac{d^2 - 1 + 2\sqrt{k}(d^2)}{3}, \sqrt{k}(d - 1)^{\sqrt{(d - 1)(d + 2)}} \right\}$ for some $k$, the quantum state $\rho$ is genuinely multipartite entangled. \qed

Example Let $\rho = (1 - p) |\Psi\rangle \langle \Psi| + \frac{p}{16} I \in \mathcal{H}_2^{\otimes 4}$, where $|\Psi\rangle = \frac{1}{2} (|0101\rangle + |1010\rangle + |1001\rangle + |0101\rangle)$. It follows from Theorem 4.2 that the quantum state $\rho$ is genuinely entangled for the white noise tolerance of $p < \frac{2}{5}$.

Remark The distribution of spin correlation strengths among different subsystems can lead to many interesting applications. For example, Wang et al. [30] gave an upper bound for the sum of tripartite spin correlation strength and obtained a trade-off relation of the Svetlichny inequality for any multipartite qubits systems by using the upper bound. From Theorem 2.4, we have $s_{iso}^{AB} + s_{iso}^{AC} + s_{iso}^{BC} \leq 1$ for three-qubit state $\rho_{ABC}$. Cheng et al. have derived the steering ellipsoid volumes monogamy relation $v_{B|A}^{2/3} + v_{C|A}^{2/3} \leq 1$ [25]. From Corollary 3.3, we have $s_{iso}^{AB} + s_{iso}^{AC} + s_{iso}^{AD} = \frac{3}{3} - \frac{3}{3} \frac{R^{ABCD}}{2}$, while it is known that the monogamy relation $v_{B|A}^{2/3} + v_{C|A}^{2/3} + v_{D|A}^{2/3} \leq 1$ holds for pure four-qubit state $\rho_{ABCD}$ by using the conjecture $s_{iso}^{AB} + s_{iso}^{AC} + s_{iso}^{AD} \leq 1$ (cf
[25]), and the validity of this volume monogamy relation holds for all four-qubit states by numerical simulation.

5 Summary and discussion

Spin correlation strengths reveal intrinsic property of bipartite qubits. In this paper, we have generalized the isotropic strength from two-qubit states to three- and four-qudit systems and show that they are also useful concepts for multipartite states and can help analyze quantum correlations. In particular, for the tripartite and quadripartite qudit systems, we have obtained the trade-off and various internal bounding relations of the spin correlations strengths among different subsystems of a multipartite state. We have employed distributions of spin correlations strengths to investigate quantum entanglement for four-partite qudit systems. We also obtained a criterion to detect genuine multipartite entanglement for any four-qudit state, which generalizes Vicente–Huber’s result for the four-qubit state.

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