QUIVERS WITH RELATIONS FOR SYMMETRIZABLE CARTAN MATRICES IV: CRYSTAL GRAPHS AND SEMICANONICAL FUNCTIONS

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Abstract. We generalize Lusztig’s nilpotent varieties, and Kashiwara and Saito’s geometric construction of crystal graphs from the symmetric to the symmetrizable case. We also construct semicanonical functions in the convolution algebras of generalized preprojective algebras. Conjecturally these functions yield semicanonical bases of the enveloping algebras of the positive part of symmetrizable Kac-Moody algebras.

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1. Introduction and main results

1.1. Introduction. There is a remarkable geometric universe relating the representation theory of quivers and preprojective algebras with the representation theory of symmetric Kac-Moody algebras. This includes the realization of the enveloping algebra $U(n)$ of the positive part $n$ of a symmetric Kac-Moody algebra $\mathfrak{g}$ as an algebra of constructible functions on varieties of modules over path algebras $S$ and over preprojective algebras $L_1, L_2$. The latter leads to the construction of a semicanonical basis $S$ of $U(n)$ due to Lusztig $L_2$. The elements of $S$ are parametrized by the irreducible components of varieties of modules over preprojective algebras. Furthermore, closely linked with varieties of modules over preprojective algebras, there is a geometric realization of the crystal graph $B(-\infty)$ of the quantized enveloping algebra $U_q(n)$ due to Kashiwara and Saito $[KS]$. This crystal graph controls the decompositions of tensor products of irreducible integrable highest weight $\mathfrak{g}$-modules, and it encodes all crystals graphs and characters of these modules.

Many geometric constructions for symmetric Kac-Moody algebras, especially the construction of Lusztig’s semicanonical basis, do not exist for non-symmetric Kac-Moody algebras. Nandakumar and Tingley $[NT]$ recently realized $B(-\infty)$ in the symmetrizable
case via varieties of modules over preprojective algebras associated with species. In the non-symmetric cases, their construction cannot be carried out over algebraically closed fields, especially not over $\mathbb{C}$. There exists also a folding technique, which sometimes allows to transfer results from the symmetric cases to the non-symmetric ones.

In our setting, symmetric and symmetrizable cases are dealt with uniformly. We generalize Lusztig’s nilpotent varieties, and Kashiwara and Saito’s geometric construction of the crystal graph $B(-\infty)$ from the symmetric to the symmetrizable case. We also construct semicanonical functions in the convolution algebras of generalized preprojective algebras. Conjecturally these functions yield semicanonical bases of the enveloping algebras $U(\mathfrak{n})$.

In the symmetric cases with minimal symmetrizer, we recover as a special case Lusztig’s semicanonical basis, and Kashiwara and Saito’s construction of $B(-\infty)$.

1.2. Main results. We now describe our results in more detail. Let $C \in M_n(\mathbb{Z})$ be a symmetrizable generalized Cartan matrix, and let $D$ be a symmetrizer of $C$. Let $\Pi = \Pi(C, D)$ be the associated preprojective algebra as defined in [GLS1]. We assume throughout that our ground field $K$ is algebraically closed. For $d \in \mathbb{N}^n$, let $\text{nil}_E(\Pi, d)$ be the variety of $E$-filtered $\Pi$-modules with dimension vector $d$.

Let $G(d)$ be the product of linear groups, which acts on $\text{nil}_E(\Pi, d)$ by conjugation. For $d = (d_1, \ldots, d_n)$ and $D = \text{diag}(c_1, \ldots, c_n)$ define $d/D := (d_1/c_1, \ldots, d_n/c_n)$. Let $q_{DC}$ be the quadratic form associated with $1/2DC$.

**Theorem 1.1.** For each irreducible component $Z$ of $\text{nil}_E(\Pi, d)$ we have $$\dim(Z) \leq \dim G(d) - q_{DC}(d/D).$$

Let $\text{Irr}(\text{nil}_E(\Pi, d))^{\text{max}}$ be the set of irreducible components of $\text{nil}_E(\Pi, d)$ of maximal dimension $\dim G(d) - q_{DC}(d/D)$.

Assume that $C$ is symmetric and $D$ is the identity matrix. Then $\Pi$ is a classical preprojective algebra associated with a quiver $Q$, the $\text{nil}_E(\Pi, d)$ are Lusztig’s nilpotent varieties, $\dim G(d) - q_{DC}(d/D)$ is the dimension of the affine space of representations of the quiver $Q$ with dimension vector $d$, and all irreducible components of $\text{nil}_E(\Pi, d)$ have the same dimension $\dim G(d) - q_{DC}(d/D)$.

Let $\mathfrak{n}(C)$ be the positive part of the symmetrizable Kac-Moody algebra $\mathfrak{g}(C)$ associated with $C$. Let $B(-\infty)$ be the crystal graph of the quantized enveloping algebra $U_q(\mathfrak{n}(C))$.

The following theorem is our first main result.

**Theorem 1.2.** Let $\Pi = \Pi(C, D)$, and set $$\mathcal{B} := \bigsqcup_{d \in \mathbb{N}^n} \text{Irr}(\text{nil}_E(\Pi, d))^{\text{max}}.$$ Then there are isomorphisms of crystals $$(\mathcal{B}, \text{wt}, \bar{e}_i, \bar{f}_i, \varphi_i, \varepsilon_i) \cong (\mathcal{B}, \text{wt}, \bar{e}_i^*, \bar{f}_i^*, \varphi_i^*, \varepsilon_i^*) \cong B(-\infty).$$

The operators and maps $\text{wt}, \bar{e}_i, \bar{f}_i, \varphi_i, \varepsilon_i$ (and their $\ast$-versions) appearing in Theorem 1.2 are defined in a module theoretic way in the fashion of Kashiwara and Saito’s geometric realization of $B(-\infty)$, see also Nandakumar and Tingley [NT]. Kashiwara and Saito only work with symmetric Kac-Moody algebras ($C$ symmetric and $D$ the identity matrix), and Nandakumar and Tingley need to work over fields which are not algebraically closed in case $C$ is non-symmetric. For $C$ symmetric and $D$ the identity matrix, Theorem 1.2 coincides with Kashiwara and Saito’s result.
For $K = \mathbb{C}$ the field of complex numbers, let $\widetilde{F}(\Pi)$ be the convolution algebra of constructible functions on the representation varieties $\text{rep}(\Pi, d)$, and let

$$\widetilde{M}(\Pi) = \bigoplus_{d \in \mathbb{N}_0} \widetilde{M}(\Pi)_d$$

be the subalgebra generated by the characteristic functions $\{\tilde{\theta}_i := 1_{E_i} | 1 \leq i \leq n\}$. (Here $E_i$ is a free $K[X]/(X^{c_i})$-module of rank 1, which can be seen as a $\Pi$-module in a natural way.) We assume that all constructible functions are constant on $G(d)$-orbits. The elements in $\widetilde{M}(\Pi)_d$ are constructible functions $\text{nil}_{E}(\Pi, d) \to \mathbb{C}$. In general, the functions $\tilde{\theta}_i$ do not satisfy the Serre relations. For a constructible function $f: \text{nil}_{E}(\Pi, d) \to \mathbb{C}$ and an irreducible component $Z$ of $\text{nil}_{E}(\Pi, d)$ let $\rho_Z(f)$ be the generic value of $f$ on $Z$.

**Theorem 1.3.** For $K = \mathbb{C}$ and $\Pi = \Pi(C, D)$, the convolution algebra $\widetilde{M}(\Pi)$ contains a set

$$\tilde{S} := \{\tilde{f}_Z | Z \in \mathcal{B}\}$$

of constructible functions such that for each $Z' \in \mathcal{B}$ we have

$$\rho_{Z'}(\tilde{f}_Z) = \begin{cases} 1 & \text{if } Z = Z', \\ 0 & \text{otherwise.} \end{cases}$$

Define

$$\mathcal{M}(\Pi) := \widetilde{M}(\Pi)/\mathcal{I}$$

where $\mathcal{I}$ is the ideal generated by the Serre relations $\{\tilde{\theta}_{ij} | 1 \leq i, j \leq n \text{ with } c_{ij} \leq 0\}$ where

$$\tilde{\theta}_{ij} := \text{ad}(\tilde{\theta}_i)^{1-c_{ij}}(\tilde{\theta}_j).$$

Let

$$\theta_i := \tilde{\theta}_i + \mathcal{I} \quad \text{and} \quad f_Z := \tilde{f}_Z + \mathcal{I}$$

be the residue classes of $\tilde{\theta}_i$ and $\tilde{f}_Z$ in $\mathcal{M}(\Pi)$. For a constructible function $f: \text{nil}_{E}(\Pi, d) \to \mathbb{C}$ let

$$\text{supp}(f) := \{M \in \text{nil}_{E}(\Pi, d) | f(M) \neq 0\}$$

be the support of $f$. By Theorem 1.1 we have $\dim \text{supp}(f) \leq \dim G(d) - q_{DC}(d/D)$.

**Conjecture 1.4.** Let $K = \mathbb{C}$ and $\Pi = \Pi(C, D)$. For $0 \neq f \in \widetilde{M}(\Pi)_d \cap \mathcal{I}$ we have

$$\dim \text{supp}(f) < \dim G(d) - q_{DC}(d/D).$$

The conjecture above is supported by Corollary 6.5. The examples discussed in Section 8.2.4 illustrate certain subtleties.

The next theorem is our second main result.

**Theorem 1.5.** Let $K = \mathbb{C}$, $\Pi = \Pi(C, D)$ and $n = n(C)$. Assume that Conjecture 1.4 is true. Then the following hold:

(i) There is a Hopf algebra isomorphism

$$\eta_{\Pi}: U(n) \to \mathcal{M}(\Pi)$$

defined by $e_i \mapsto \theta_i$.

(ii) Via the isomorphism $\eta_{\Pi}$, the set

$$\mathcal{S} := \{f_Z | Z \in \mathcal{B}\}$$

is a $\mathbb{C}$-basis of $U(n)$. 
(iii) For $0 ≠ f ∈ \tilde{M}(Π)_d$ the following are equivalent:
(a) $f ∈ \mathcal{I}$;
(b) $\dim \text{supp}(f) < \dim G(d) - q DC(d/D)$.

We have $\mathcal{I} ≠ 0$ if and only if $c_{ij} < 0$ and $c_i ≥ 2$ for some $1 ≤ i, j ≤ n$.

One should expect that $S$ (seen as a subset of $U(n)$ via $η_Π$) does not depend on the symmetrizer $D$.

Suppose that $C$ is symmetric and that $D$ is the identity matrix. Then $\tilde{M}(Π) = M(Π)$ and the Hopf algebra isomorphism $U(n) → M(Π)$ can be obtained by combining [L1] Lemma 12.11 with either [KS] or [S], see [L2]. Furthermore, $\tilde{S} = S$ is exactly Lusztig’s [L2] semicanonical basis.

1.3. The paper is organized as follows. In Section 2 we recall definitions and results on preprojective algebras and their representation varieties. In Section 3 we generalize Lusztig’s construction of certain fibre bundles from the classical nilpotent varieties to our more general setup. The proof of Theorem 1.1 is contained in Section 4. We also show that generically the modules in maximal irreducible components are crystal modules. (These modules are defined in Section 1.2.) Section 5 contains the proof of Theorem 1.2. The convolution algebra $M(Π)$ is defined in Section 6. Section 7 contains the proof of Theorems 1.3 and 1.5. Assuming that Conjecture 1.4 is true, we also show that the semicanonical bases of the enveloping algebras $U(n)$ induce semicanonical bases of all irreducible integrable highest weight modules. Section 8 contains the classification of maximal irreducible components for the Dynkin cases, and also examples of Dynkin type $A_2$, $B_2$ and $G_2$.

1.4. Notation. By a module we mean a finite-dimensional left module, unless mentioned otherwise. For maps $f : X → Y$ and $g : Y → Z$ the composition is denoted by $gf : X → Z$. A module $M$ over an algebra $A$ is rigid if $\text{Ext}^1_A(M, M) = 0$. For a module $M$, let $M^m$ be the direct sum of $m$ copies of $M$.

For a constructible subset $X$ of a quasi-projective variety, let $\text{Irr}(X)$ be the set of irreducible components of $X$.

Let $\mathbb{N}$ be the natural numbers, including 0.

2. Quivers with relations associated with symmetrizable Cartan matrices

In this section, we recall some definitions and results from [GLS1].

2.1. The preprojective algebras $Π(C, D)$. A matrix $C = (c_{ij}) ∈ M_n(\mathbb{Z})$ is a symmetrizable generalized Cartan matrix provided the following hold:

(C1) $c_{ii} = 2$ for all $i$;
(C2) $c_{ij} ≤ 0$ for all $i ≠ j$;
(C3) $c_{ij} ≠ 0$ if and only if $c_{ji} ≠ 0$;
(C4) There is a diagonal integer matrix $D = \text{diag}(c_1, \ldots, c_n)$ with $c_i ≥ 1$ for all $i$ such that $DC$ is symmetric.

The matrix $D$ appearing in (C4) is called a symmetrizer of $C$. The symmetrizer $D$ is minimal if $c_1 + \cdots + c_n$ is minimal.
From now on, let $C = (c_{ij}) \in M_n(\mathbb{Z})$ be a symmetrizable generalized Cartan matrix. Throughout, let

$$I := \{1, \ldots, n\}.$$  

An orientation of $C$ is a subset $\Omega \subset I \times I$ such that for all $(i, j) \in I \times I$ the following are equivalent:

(i) $\{(i, j), (j, i)\} \cap \Omega \neq \emptyset$;
(ii) $|\{(i, j), (j, i)\} \cap \Omega| = 1$;
(iii) $c_{ij} < 0$.

The opposite orientation of an orientation $\Omega$ is defined as $\Omega^* := \{(j, i) \mid (i, j) \in \Omega\}$. Let $\Omega := \Omega \cup \Omega^*$. Define $\Omega(i) := \{j \in I \mid (i, j) \in \Omega\} = \{j \in I \mid (j, i) \in \Omega\} = \{j \in I \mid c_{ij} < 0\}$.

For $(i, j) \in \Omega$ set

$$\text{sgn}(i, j) := \begin{cases} 1 & \text{if } (i, j) \in \Omega, \\ -1 & \text{if } (i, j) \in \Omega^*. \end{cases}$$

For all $c_{ij} < 0$ define

$$g_{ij} := \gcd(c_{ij}, c_{ji}), \quad f_{ij} := |c_{ij}|/g_{ij}.$$  

Let $Q := Q(C) := (I, Q_1, s, t)$ be the quiver with the set of vertices $I = \{1, \ldots, n\}$ and with the set of arrows

$$Q_1 := \{\alpha_{ij}^{(g)} : j \rightarrow i \mid (i, j) \in \Omega, 1 \leq g \leq g_{ij}\} \cup \{\varepsilon_i : i \rightarrow i \mid i \in I\}.$$  

(Thus we have $s(\alpha_{ij}^{(g)}) = j$ and $t(\alpha_{ij}^{(g)}) = i$ and $s(\varepsilon_i) = t(\varepsilon_i) = i$, where $s(a)$ and $t(a)$ denote the starting and terminal vertex of an arrow $a$, respectively.) If $g_{ij} = 1$, we also write $\alpha_{ij}$ instead of $\alpha_{ij}^{(1)}$.

For $Q = Q(C)$ and a symmetrizer $D = \text{diag}(c_1, \ldots, c_n)$ of $C$, we define an algebra

$$\Pi := \Pi(C, D, \Omega) := K\overline{Q}/\mathcal{T}$$

where $K\overline{Q}$ is the path algebra of $\overline{Q}$ and $\mathcal{T}$ is the ideal defined by the following relations:

(P1) For each $i$ we have

$$\varepsilon_i^{g_{ij}} = 0.$$  

(P2) For each $(i, j) \in \Omega$ and each $1 \leq g \leq g_{ij}$ we have

$$\varepsilon_i^{f_{ij}^{(g)}} \alpha_{ij}^{(g)} = \alpha_{ij}^{(g)} \varepsilon_j^{f_{ij}^{(g)}}.$$  

(P3) For each $i$ we have

$$\sum_{j \in \Omega(i)} \sum_{g=1}^{g_{ij}} \sum_{f=0}^{f_{ij}-1} \text{sgn}(i, j) \varepsilon_i^{f_{ij}^{(g)}} \alpha_{ij}^{(g)} \alpha_{ji}^{(g)} \varepsilon_j^{f_{ij}^{(g)}} \varepsilon_j^{f_{ij}^{(g)}-1} = 0.$$  

We call $\Pi$ a preprojective algebra of type $C$. These algebras generalize the classical preprojective algebras associated with quivers, see [GLS1] for details. Up to isomorphism, the algebra $\Pi := \Pi(C, D) := \Pi(C, D, \Omega)$ does not depend on the orientation $\Omega$ of $C$. Let $\text{rep}(\Pi)$ be the category of finite-dimensional $\Pi$-modules.

Define bilinear forms

$$\langle -, - \rangle : \mathbb{Z}^n \times \mathbb{Z}^n \rightarrow \mathbb{Z}$$
by \((\alpha_i, \alpha_j) := c_{ij}\), and
\[\langle-x, -x\rangle: \mathbb{Z}^n \times \mathbb{Z}^n \rightarrow \mathbb{Z}\]
by \((\alpha_i, \alpha_j) := c_i c_{ij}\). (Here \(\alpha_1, \cdots, \alpha_n\) denotes the standard basis of \(\mathbb{Z}^n\).) Let
\[q_{DC}: \mathbb{Z}^n \rightarrow \mathbb{Z}\]
be the associated quadratic form defined by \(q_{DC}(x) := \langle x, x \rangle / 2\).

For \(i \in I\) let \(S_i\) be the 1-dimensional simple \(\Pi\)-module associated with the vertex \(i\), and let \(E_i\) be the \(c_i\)-dimensional uniserial \(\Pi\)-module associated with \(i\). Let
\[H_i := K[\varepsilon_i]/(\varepsilon_i^{c_i}),\]
and let \(e_i \in \Pi\) be the idempotent associated with \(i\). For each \(\Pi\)-module \(M\) the space \(e_i M\) is naturally a free \(H_i\)-module. We have \(E_i = e_i E_i\), and \(e_i E_i\) is free of rank 1 as an \(H_i\)-module. A \(\Pi\)-module \(M\) is locally free if \(e_i M\) is a free \(H_i\)-module for all \(i\). The rank of a free \(H_i\)-module \(M_i\) is denoted by \(\text{rank}(M_i)\). For a locally free \(\Pi\)-module \(M\) let \(\text{rank}(M) := (\text{rank}(e_1 M), \ldots, \text{rank}(e_n M))\) be the rank vector of \(M\). A \(\Pi\)-module \(M\) is \(E\)-filtered (resp. \(S\)-filtered) if there exists a chain
\[0 = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_t = M\]
of submodules \(U_i\) of \(M\) such that for each \(1 \leq k \leq t\) we have \(U_k / U_{k-1} \cong E_{i_k}\) (resp. \(U_k / U_{k-1} \cong S_{i_k}\)) for some \(i_k \in I\). Let \(\text{nil}_E(\Pi) \subseteq \text{rep}(\Pi)\) be the subcategory of \(E\)-filtered \(\Pi\)-modules. Note that each \(E\)-filtered \(\Pi\)-module is locally free. The converse of this statement is in general wrong. We refer to [GLS1] for further details.

2.2. Representation varieties (quiver version). Let \(\Pi = \Pi(C, D)\). For a dimension vector \(d = (d_1, \ldots, d_n)\) let
\[\overline{\mathcal{H}}(d) := \prod_{a \in \overline{Q}_1} \text{Hom}_K(K^{d_{i(a)}}, K^{d_{i(a)}}),\]
and let \(\text{rep}(\Pi, d)\) be the varieties of \(\Pi\)-modules with dimension vector \(d\). By definition, the points in \(\text{rep}(\Pi, d)\) are tuples
\[(M(a))_a \in \overline{\mathcal{H}}(d)\]
satisfying the equations
\[M(\varepsilon_i)^{c_i} = 0, \quad M(\varepsilon_i)^{f_{ij}}M(\alpha_{ij}^{(g)}) = M(\alpha_{ij}^{(g)})M(\varepsilon_j)^{f_{ij}},\]
\[\sum_{j \in \overline{I}(i)} \sum_{g = 1}^{g_{ij}} \sum_{f = 0}^{f_{ij} - 1} \text{sgn}(i, j)M(\varepsilon_i)^{f}M(\alpha_{ij}^{(g)})M(\alpha_{ji}^{(g)})M(\varepsilon_j)^{f_{ij} - 1 - f} = 0\]
for all \(i \in I\), \((i, j) \in \overline{\Omega}\) and \(1 \leq g \leq g_{ij}\). The group
\[G(d) := \prod_{i \in I} \text{GL}_K(d_i)\]
acts on \(\text{rep}(\Pi, d)\) by conjugation. For a module \(M \in \text{rep}(\Pi, d)\) let \(\mathcal{O}(M) := G(d)M\) be its \(G(d)\)-orbit. The \(G(d)\)-orbits are in bijection with the isomorphism classes of modules in \(\text{rep}(\Pi, d)\). For \(M \in \text{rep}(\Pi, d)\) we have
\[\dim \mathcal{O}(M) = \dim G(d) - \dim \text{End}_\Pi(M)\]
Let \(\text{rep}_{1,f}(\Pi, d) \subseteq \text{rep}(\Pi, d)\) be the subvarieties of locally free modules, and let \(\text{nil}_E(\Pi, d) \subseteq \text{rep}_{1,f}(\Pi, d)\) be the subset of \(E\)-filtered \(\Pi\)-modules. Using the same technique as in the
proof of [CBS] Theorem 1.3(i), one shows that nil$_E(\Pi, d)$ is a constructible subset of rep$_{1f}(\Pi, d)$.

2.3. Representation varieties (species version). Let $\Pi = \Pi(C, D) = \Pi(C, D, \Omega)$. For a tuple $M = (M_1, \ldots, M_n)$ with $M_i \in \operatorname{rep}(H_i)$ let

$$H(M) := \prod_{(i,j) \in \Omega} \operatorname{Hom}_{H_i}(iH_j \otimes_j M_j, M_i)$$

and

$$\overline{H}(M) := \prod_{(i,j) \in \Omega} \operatorname{Hom}_{H_i}(iH_j \otimes_j M_j, M_i).$$

Here $iH_j$ are the $H_i$-$H_j$-bimodules defined in [GLS1]. Using the results in [GLS1] we see that $(\dim \overline{H}(M))/2 = \dim H(M)$.

For $M \in \overline{H}(M)$ let $M_{ij} : iH_j \otimes_j M_j \to M_i$ be the corresponding homomorphisms in $\operatorname{Hom}_{H_i}(iH_j \otimes_j M_j, M_i)$.

Let

$$G(M) := \prod_{i \in I} \operatorname{GL}_{H_i}(M_i)$$

where $\operatorname{GL}_{H_i}(M_i)$ is the group of $H_i$-linear automorphisms of $M_i$. The group $G(M)$ acts by conjugation on $\overline{H}(M)$. We call $M$ locally free if each $M_i$ is a free $H_i$-module. In this case, let

$$\operatorname{rank}(M) := (\operatorname{rank}(M_1), \ldots, \operatorname{rank}(M_n))$$

be the rank vector of $M$. The total rank of $M$ is defined as $\operatorname{rank}(M_1) + \cdots + \operatorname{rank}(M_n)$.

For $M \in \overline{H}(M)$ let

$$M_{i,\text{in}} := (\operatorname{sgn}(i,j)M_{ij})_j : \bigoplus_{j \in \Omega(i)} iH_j \otimes_j M_j \to M_i$$

and

$$M_{i,\text{out}} := (M^{j')}_{ij} : M_i \to \bigoplus_{j \in \Omega(i)} iH_j \otimes_j M_j$$

be defined as in [GLS1, Section 5]. Let

$$\operatorname{rep}(\Pi, M) := \{ M \in \overline{H}(M) \mid M_{i,\text{in}} \circ M_{i,\text{out}} = 0 \}.$$ 

We can see $\operatorname{rep}(\Pi, M)$ as the affine variety of $\Pi$-modules $M$ with $e_iM = M_i$ for $i \in I$. The $G(M)$-action on $\overline{H}(M)$ restricts to $\operatorname{rep}(\Pi, M)$. The isomorphism classes of $\Pi$-modules $M$ with $e_iM = M_i$ for all $i$ are in bijection with the $G(M)$-orbits in $\operatorname{rep}(\Pi, M)$. For a module $M \in \operatorname{rep}(\Pi, M)$ let $\mathcal{O}(M) := G(M)M$ be its $G(M)$-orbit. For $M \in \operatorname{rep}(\Pi, M)$ we have

$$\dim \mathcal{O}(M) = \dim G(M) - \dim \operatorname{End}_\Pi(M).$$

For $M$ locally free, let

$$\operatorname{nil}_E(\Pi, M) := \Pi(M)$$

be the subset of $E$-filtered modules in $\operatorname{rep}(\Pi, M)$. This is a constructible subset of $\operatorname{rep}(\Pi, M)$. 
For a rank vector \( r = (r_1, \ldots, r_n) \) define \( M(r) := (H_{r_1}^{r_1}, \ldots, H_{r_n}^{r_n}) \) and set
\[
\begin{align*}
H(r) &:= H(M(r)), \\
\overline{H}(r) &:= \overline{H}(M(r)).
\end{align*}
\]
then
\[
\begin{align*}
\text{rep}(\Pi, r) &:= \text{rep}(\Pi, M(r)), \\
\text{nil}_E(\Pi, r) &:= \Pi(r) := \Pi(M(r)).
\end{align*}
\]
Set \( G(r) := G(M(r)) \). (We always denote rank vectors in bold letters, like \( r \), and dimension vectors in ordinary letters, like \( d \).)

Obviously, each variety \( \Pi(M) \) is isomorphic to \( \Pi(r) \) where \( r = \text{rank}(M) \). We sometimes just identify \( \Pi(M) \) and \( \Pi(r) \).

**2.4. Relating the quiver version and the species version.** We have the obvious projection
\[
\begin{align*}
\text{rep}(\Pi, d) &\xrightarrow{\varepsilon_{\Pi}} \prod_{i \in I} \text{rep}(H_i, d_i).
\end{align*}
\]
For \( M = (M_1, \ldots, M_n) \in \prod_{i \in I} \text{rep}(H_i, d_i) \) we have
\[
\varepsilon^{-1}_{\Pi}(M) \cong \text{rep}(\Pi, M).
\]
This follows from the considerations in [GLSII Section 5]. We see that \( \varepsilon_{\Pi} \) is a fibre bundle. We identify the fibre \( \varepsilon^{-1}_{\Pi}(M) \) with \( \text{rep}(\Pi, M) \). For \( M \in \text{rep}(\Pi, M) \) we have
\[
G(M)M = G(d)M \cap \text{rep}(\Pi, M).
\]
Assume now that \( M \) is locally free. Recall that \( d/D = (d_1/c_1, \ldots, d_n/c_n) \), and note that \( \text{rank}(M) = d/D \). An easy calculation shows that
\[
\dim H(M) = \dim G(M) - q_{DC}(d/D).
\]
For a closed \( G(M) \)-stable subset \( Z \) of \( \text{rep}(\Pi, M) \) of dimension \( \dim G(M) + m \) for some \( m \in Z \), the correspond subset \( G(d)Z \) of \( \text{rep}(\Pi, d) \) has dimension \( \dim G(d) + m \).

**2.5. Convolution algebras.** In this section, assume that \( K = \mathbb{C} \). Let
\[
\tilde{\mathcal{F}}(\Pi) := \bigoplus_{d \in \mathbb{N}^n} \tilde{\mathcal{F}}(\Pi)_d
\]
be the convolution algebra associated with \( \Pi \), where \( \tilde{\mathcal{F}}(\Pi)_d \) is the \( \mathbb{C} \)-vector space of constructible functions \( \text{rep}(\Pi, d) \to \mathbb{C} \). Recall that a map
\[
f: \text{rep}(\Pi, d) \to \mathbb{C}
\]
is a **constructible function** if the following hold:
\[
\begin{align*}
\text{(i)} \quad &\text{Im}(f) \text{ is finite;} \\
\text{(ii)} \quad &\text{For each } m \in \mathbb{C}, \text{ the preimage } f^{-1}(m) \text{ is a constructible subset of } \text{rep}(\Pi, d); \\
\text{(iii)} \quad &f \text{ is constant on } G(d)\text{-orbits.}
\end{align*}
\]
For \( M \in \text{rep}(\Pi) \) define \( 1_M \in \tilde{\mathcal{F}}(\Pi) \) by
\[
1_M(N) := \begin{cases} 
1 & \text{if } M \cong N, \\
0 & \text{otherwise.}
\end{cases}
\]
For \( f, g \in \tilde{F}(\Pi) \) the product \( f * g \) is defined by
\[
(f \ast g)(M) := \sum_{m \in \mathbb{C}} m \chi(\{ U \subseteq M \mid f(U)g(M/U) = m \})
\]
where \( M \in \text{rep}(\Pi) \) and \( \chi \) denotes the topological Euler characteristic.

For \( i \in I \) let
\[
\tilde{\theta}_i := 1_{E_i}.
\]
Let
\[
\tilde{\mathcal{M}}(\Pi) = \bigoplus_{d \in \mathbb{N}^n} \tilde{\mathcal{M}}(\Pi)_d
\]
be the subalgebra of \( \tilde{F}(\Pi) \) generated by \{ \tilde{\theta}_i \mid i \in I \}, where
\[
\tilde{\mathcal{M}}(\Pi)_d := \tilde{F}(\Pi)_d \cap \tilde{\mathcal{M}}(\Pi).
\]

For \( f \in \tilde{\mathcal{M}}(\Pi)_d \) let
\[
\text{supp}(f) := \{ M \in \text{rep}(\Pi, d) \mid f(M) \neq 0 \}
\]
be the support of \( f \). We have \( \text{supp}(f) \subseteq \text{rep}_{\mathbb{L}}(\Pi, d) \). Using the same arguments as in the proof of [GLS2, Proposition 4.7], we get that \( \tilde{\mathcal{M}}(\Pi) \) is a Hopf algebra, which is isomorphic to the enveloping algebra \( U(\mathcal{P}(\tilde{\mathcal{M}}(\Pi))) \) of the Lie algebra \( \mathcal{P}(\tilde{\mathcal{M}}(\Pi)) \) of primitive elements in \( \tilde{\mathcal{M}}(\Pi) \). A constructible function \( f \in \tilde{\mathcal{M}}(\Pi)_d \) is in \( \mathcal{P}(\tilde{\mathcal{M}}(\Pi)) \) if and only if \( \text{supp}(f) \) consists just of indecomposable modules. The comultiplication in \( \tilde{\mathcal{M}}(\Pi) \) is given by
\[
\tilde{\theta}_i \mapsto \tilde{\theta}_i \otimes 1 + 1 \otimes \tilde{\theta}_i.
\]

For a dimension vector \( d \) with \( \text{rep}_{\mathbb{L}}(\Pi, d) \neq \emptyset \) let \( r := d/D \) be the associated rank vector. Alternatively, we can define \( \tilde{\mathcal{M}}(\Pi) \) using constructible functions \( \text{rep}(\Pi, n) \rightarrow \mathbb{C} \). (Condition (iii) in the definition of a constructible function is replaced by demanding that \( f \) is constant on \( G(r) \)-orbits.) It is straightforward to check that the two definitions yield canonically isomorphic algebras. In this article, we mainly work with the varieties \( \text{rep}(\Pi, r) \) (the species version) instead of the varieties \( \text{rep}(\Pi, d) \) (the quiver version). Our main results in the introduction and also the examples collection in Section 8 are formulated using the quiver version, whereas the rest of the article (especially the proofs) are based on the more convenient species version.

2.6. Hom-Ext formulas. The following result is proved in [GLS1, Theorem 12.6]. It generalizes [CB2, Lemma 1].

**Lemma 2.1.** For \( M, N \in \text{rep}_{\mathbb{L}}(\Pi) \) the following hold:

(i) \( \dim \text{Ext}^1_{\Pi}(M, N) = \dim \text{Ext}^1_{\Pi}(N, M) \);

(ii) \( \dim \text{Ext}^1_{\Pi}(M, N) = \dim \text{Hom}_{\Pi}(M, N) + \dim \text{Hom}_{\Pi}(N, M) - \langle \text{rank}(M), \text{rank}(N) \rangle \).

**Corollary 2.2.** For \( M \in \text{rep}_{\mathbb{L}}(\Pi) \) and \( i \in I \) we have
\[
\dim \text{Ext}^1_{\Pi}(E_i, M) = \dim \text{Hom}_{\Pi}(E_i, M) + \dim \text{Hom}_{\Pi}(M, E_i) - c_i \langle \text{rank}(M), \alpha_i \rangle.
\]

**Proof.** Let \( \text{rank}(M) = (m_1, \ldots, m_n) \). We have
\[
\langle \text{rank}(M), \alpha_i \rangle = 2m_i + \sum_{j \in \Pi(i)} c_{ij} m_j
\]
and
\[
\langle \text{rank}(M), \alpha_i \rangle = 2c_i m_i + \sum_{j \in \Pi(i)} c_i c_{ij} m_j.
\]
Now the result follows from Lemma 2.1(ii). □

Let \( M \in \text{rep}(\Pi) \). For each \( i \in I \) let
\[
\widetilde{M}_i := \bigoplus_{j \in \Pi(i)} \iota H_j \otimes_j M_j.
\]
As before, let
\[
M_{i,\text{in}} := (\text{sgn}(i, j)M_{ij})_j : \bigoplus_{j \in \Pi(i)} \iota H_j \otimes_j M_j \to M_i
\]
and
\[
M_{i,\text{out}} := (M_{ji}')_j : M_i \to \bigoplus_{j \in \Pi(i)} \iota H_j \otimes_j M_j.
\]

For \( M \in \text{rep}(\Pi) \) and \( i \in I \) let \( \text{sub}_i(M) \) (resp. \( \text{fac}_i(M) \)) be the largest submodule (resp. factor module) of \( M \) such that each composition factor of \( \text{sub}_i(M) \) (resp. \( \text{fac}_i(M) \)) is isomorphic to \( S_i \).

Let \( \text{top}(M) \) be the largest semisimple factor module of \( M \), and let \( \text{top}_i(M) \) be the largest semisimple factor module of \( M \) such that each composition factor of \( \text{top}_i(M) \) is isomorphic to \( S_i \).

**Lemma 2.3.** For \( M \in \text{rep}(\Pi) \) the following hold:

(i) \( \dim \text{Hom}_\Pi(E_i, M) = \dim \text{Ker}(M_{i,\text{out}}) = \dim \text{sub}_i(M) \);
(ii) \( \dim \text{Hom}_\Pi(M, E_i) = \dim (\text{Cok}(M_{i,\text{in}})) = \dim \text{fac}_i(M) \);
(iii) If \( M \) is locally free, then
\[
\dim \text{Ext}^1_\Pi(M, E_i) = \dim(\widetilde{M}_i) - \dim \text{Im}(M_{i,\text{in}}) - \dim \text{Im}(M_{i,\text{out}})
= \dim(\text{Ker}(M_{i,\text{in}})/\text{Im}(M_{i,\text{out}})).
\]

**Proof.** We have \( \text{Ker}(M_{i,\text{out}}) = \text{sub}_i(M) \) and \( \text{Cok}(M_{i,\text{out}}) = \text{fac}_i(M) \). The \( H_i \)-module \( H_i = E_i \) is indecomposable projective-injective in \( \text{rep}(H_i) \). Thus \( \dim \text{Hom}_\Pi(E_i, M) \) and \( \dim \text{Hom}_\Pi(M, E_i) \) are the dimensions of \( \text{sub}_i(M) \) and \( \text{fac}_i(M) \), respectively. This proves (i) and (ii).

For \( M \in \text{rep}(\Pi) \) we have a sequence
\[
0 \to \text{sub}_i(M) \xrightarrow{\iota} M_i \xrightarrow{M_{i,\text{out}}} \widetilde{M}_i \xrightarrow{M_{i,\text{in}}} M_i \xrightarrow{\pi} \text{fac}_i(M) \to 0
\]
of \( H_i \)-linear maps, where \( \iota \) is a monomorphism, \( \pi \) is an epimorphism, \( \text{Im}(\iota) = \text{Ker}(M_{i,\text{out}}) \), \( \text{Im}(M_{i,\text{in}}) = \text{Ker}(\pi) \) and \( \text{Im}(M_{i,\text{out}}) \subseteq \text{Ker}(M_{i,\text{in}}) \). Observe that
\[
\dim(\widetilde{M}_i) - \dim \text{Im}(M_{i,\text{in}}) - \dim \text{Im}(M_{i,\text{out}}) = \dim(\text{Ker}(M_{i,\text{in}})/\text{Im}(M_{i,\text{out}})).
\]
We have
\[
\dim \text{Im}(M_{i,\text{out}}) := \dim(M_i) - \dim(\text{sub}_i(M)),
\]
\[
\dim \text{Im}(M_{i,\text{in}}) := \dim(M_i) - \dim(\text{fac}_i(M)).
\]
It follows that
\[
\dim(\text{Ker}(M_{i,\text{in}})/\text{Im}(M_{i,\text{out}})) = \dim(\widetilde{M}_i) - 2 \dim(M_i) + \dim(\text{sub}_i(M) + \dim \text{fac}_i(M)).
\]
We know that
\[
\dim \widetilde{M}_i = \sum_{j \in \Pi(i)} c_j |cji|m_j
\]
where \((m_1, \ldots, m_n) = \text{rank}(M)\). Here we used that

\[ iH_j \otimes_j M_j \cong H_j^{[c_j|m_j]} \]

We have

\[(\text{rank}(M), \text{rank}(E_i)) = (\text{rank}(M), \alpha_i) = 2 \dim(M_i) - \dim(\tilde{M}_i)\]

Combining the above equalities with (i) and (ii) and with Lemma 2.4 ii) we get the formula (iii).

\[
3. \text{ Lusztig's bundle construction}
\]

3.1. Partitions and \(H_k\)-modules. For \(m \geq 0\) let \(P_m\) be the set of partitions with entries bounded by \(m\). (These are tuples \(p = (p_1, \ldots, p_t)\) of integers with \(m \geq p_1 \geq \cdots \geq p_t \geq 0\). We identify \((p_1, \ldots, p_t, 0, \ldots, 0)\) with \((p_1, \ldots, p_t)\).) For a partition \(p = (p_1, \ldots, p_t)\) and \(k \geq 0\) let \(p(k) := \{|1 \leq i \leq t \mid p_i = k\}\) be the number of entries equal to \(k\), and let \(\text{length}(p) := \{|1 \leq i \leq t \mid p_i \neq 0\}|\). For \(p, m \geq 0\) we also write \((p^m) = (p, \ldots, p)\) for the partition with entries equal to \(p\). Similarly, for a partition \((p_1, \ldots, p_t)\) and \(m_1, \ldots, m_t \geq 0\) we define \((p_1^{m_1}, \ldots, p_t^{m_t})\) in the obvious way.

For \(k \in I\) the isomorphism classes of finite-dimensional \(H_k\)-modules can be parametrized by \(P_{ck}\) in the obvious way. For \(p \in P_{ck}\) let \(H_k^p\) be an \(H_k\)-module corresponding to \(p\). Vice versa, for \(M \in \text{rep}(H_k)\) let \(p(M) \in P_{ck}\) be the partition associated with \(M\).

3.2. Stratifications of \(\Pi(M)\). Let \(\Pi = \Pi(C, D)\) and let \(M = (M_1, \ldots, M_n)\) be locally free.

Recall that for \(M \in \text{rep}(\Pi)\), \(\text{fac}_k(M)\) is the largest factor module \(M/U\) such that each composition factor of \(M/U\) is isomorphic to \(S_k\). Similarly, \(\text{sub}_k(M)\) is the largest submodule \(U \subseteq M\) such that each composition factor of \(U\) is isomorphic to \(S_k\).

Recall that

\[ \Pi(M) = \text{nil}_E(\Pi, M). \]

For \(p \in P_{ck}\) let

\[ \Pi(M)^{k,p} := \{M \in \Pi(M) \mid \text{fac}_k(M) \cong H_k^p\} \]

and

\[ \Pi(M)_{k,p} := \{M \in \Pi(M) \mid \text{sub}_k(M) \cong H_k^p\}. \]

For the special case \(p = (c_k^p)\) we define

\[ \Pi(M)^{k,p} := \Pi(M)^{k,p} \quad \text{and} \quad \Pi(M)_{k,p} := \Pi(M)_{k,p}. \]

In the following, we prove some results involving the varieties \(\Pi(M)^{k,p}\). We leave it as an easy exercise to formulate and prove the corresponding dual results for \(\Pi(M)_{k,p}\).

**Lemma 3.1.** The following hold:

(a) \(\Pi(M)^{k,p}\) is a locally closed \(G(M)\)-stable subvariety of \(\Pi(M)\).
(b) \(\Pi(M)^{k,0}\) is open in \(\Pi(M)\).
(c) For \(M \neq 0\) we have

\[ \Pi(M) = \bigcup_{p \in P_{ck}} \Pi(M)^{k,p}. \]
such that \( \pi \) is a fibre bundle. The irreducible components of \( F \) are open and dense in \( V \) and \( B \) is a base of \( F \). We recall some classical concepts particular, we use freely elementary concepts from algebraic geometry like dimension, irreducible components and morphisms between varieties. We recall some classical concepts.

(a): For \( 1 \leq i \leq c_k \), recall that \( H_k^{(i)} \) denotes the uniserial \( H_k \)-module of length \( i \). For \( M \in \Pi(M) \) the numbers
\[
\dim \Hom_{\Pi}(M, H_k^{(i)})
\]
with \( 1 \leq i \leq c_k \) determine \( \text{fac}_k(M) \). It follows that \( \Pi(M)^{k,p} \) is a finite intersection of locally closed sets. This yields the result.

(b): This follows directly from the upper semicontinuity of the map \( \dim \Hom_H(M, -) \).

(c): By definition each non-zero \( M \in \Pi(M) \) has a chain
\[
0 = U_0 \subset U_1 \subset \cdots \subset U_t = M
\]
such that for \( 1 \leq k \leq t \) we have \( U_j/U_{j-1} \cong E_{ij} \) for some \( i_j \in I \). Wit \( k := i_t \) we get
\[
M \in \Pi(M)^{k,p}
\]
where \( p \) is a partition of the form \( p = (c_k, \ldots) \). This proves (c).

By upper semicontinuity, for each \( Z \in \text{Irr}(\Pi(M)) \) there exists a dense open subset \( U_Z \subseteq Z \) such that for all \( k \in I \) and all \( M, N \in U \) we have \( \text{sub}_k(M) \cong \text{sub}_k(N) \) and \( \text{fac}_k(M) \cong \text{fac}_k(N) \). Let \( \text{sub}_k(Z) := \text{sub}_k(M) \) and \( \text{fac}_k(Z) := \text{fac}_k(M) \) for some \( M \in U \).

Again, by upper semicontinuity it follows that for each \( Z \in \text{Irr}(\Pi(M)) \) and \( k \in I \) there exists a unique \( p, q \in P_{c_k} \) such that
\[
Z^{k,p} := Z \cap \Pi(M)^{k,p} \quad \text{and} \quad Z_{k,q} := Z \cap \Pi(M)_{k,q}
\]
are open and dense in \( Z \).

We say that a \( \Pi \)-module \( M \) is generic in \( Z \), if \( M \) is contained in a sufficiently small dense open subset of \( Z \) defined by a finite set of suitable open conditions. The context will always imply which conditions are meant. For example, we often demand that \( M \in Z \) with \( \text{sub}_k(M) \cong \text{sub}_k(Z) \) and \( \text{fac}_k(M) \cong \text{fac}_k(Z) \) for all \( k \).

3.3. **Fibre bundles and principal \( G \)-bundles.** All varieties considered are algebraic varieties over the algebraically closed field \( K \), and our topology is the Zarisky topology. In particular, we use freely elementary concepts from algebraic geometry like dimension, irreducible components and morphisms between varieties. We recall some classical concepts from topology in our setting.

A morphism between varieties
\[
\pi: B \to V
\]
is a fibre bundle with fibre \( F \), if \( V \) has an open covering \( (V_i)_{i \in I} \) together with isomorphisms
\[
\tau_i: V_i \times F \to \pi^{-1}(V_i)
\]
such that \( \pi \tau_i(v, f) = v \) for all \( (v, f) \in V_i \times F \). In particular, we have \( \pi^{-1}(v) \cong F \) for all \( v \in V \), and our fibre bundles are always locally trivial in the Zarisky topology. Thus, if \( F \) is irreducible, there is a natural bijection between the irreducible components of \( V \) and the irreducible components of \( B \), and we have
\[
\dim(B) = \dim(V) + \dim(F).
\]

Let \( \phi: U \to V \) be another morphism of varieties, then the pullback
\[
\phi^*(B) := \{(u, b) \in U \times B \mid \phi(u) = \pi(b)\},
\]
together with the projection

$$\phi^*(\pi) : \phi^*(B) \to U$$

defined by \((u, b) \mapsto u\) is again a fibre bundle. In particular, it is easy to see how to trivialize \((\phi^*(B), \phi^*(\pi))\) over the open subsets \(\phi^{-1}(V_i) \subseteq U\) with fibre \(F\).

Let now \(G\) be an algebraic group which acts (algebraically) on \(B\) from the right, such that \(\pi(b \cdot g) = \pi(b)\) for all \(b \in B\) and \(g \in G\). We say that the fibre bundle \(\pi : B \to V\) is a principal \(G\)-bundle if \(G\) acts freely and transitively on the fibres of \(\pi\). In this case, all fibres \(\pi^{-1}(v)\) are isomorphic to \(G\) as a variety. Again, it is easy to see that the pullback of a principal \(G\)-bundle is again a principal \(G\)-bundle.

3.4. Grassmannians of submodules of fixed type. In this section, we fix some \(k \in I\), and set \(c := c_k\). Let

\[ A := H_k = K[\varepsilon_k]/(\varepsilon_k^c). \]

For a partition \(p \in P_c\) with we define the \(A\)-module

\[ A^p := H^p_k. \]

For \(A\)-modules \(M\) and \(U\) let

\[ \text{Gr}_U(M) := \{V \subseteq M \mid V \cong U\} \]

be the quasi-projective variety of \(A\)-submodules \(V\) of \(M\) which are isomorphic to \(U\). Similarly, let

\[ \text{Gr}^U(M) := \{V \subseteq M \mid M/V \cong U\} \]

be the quasi-projective variety of \(A\)-factor modules \(M/V\) of \(M\) which are isomorphic to \(U\). (Factor modules \(M/V\) are defined via submodules \(V\), so we can think of \(\text{Gr}^U(M)\) as a variety of factor modules.)

Consider the open subset

\[ \text{Inj}_A(U, M) := \{f \in \text{Hom}_A(U, M) \mid f \text{ is injective}\} \subset \text{Hom}_A(U, M). \]

Following Haupt [H, Section 3.1], we consider

\[ \text{Inj}_A(U, M) \to \text{Gr}_U(M), \quad f \mapsto \text{Im}(f). \]

It is easy to see that this is a principal \(\text{Aut}_A(U)\)-bundle with \(\text{Aut}_A(U)\) acting on \(\text{Inj}_A(U, M)\) by precomposition. Now, \(\text{Aut}_A(U)\) and \(\text{Inj}_A(U, M)\) are, as open subsets in a vector space, smooth and irreducible. If \(\text{Gr}_U(M)\) is non-empty, then \(\text{Gr}_U(M)\) is smooth and irreducible, and we have

\[ \dim \text{Gr}_U(M) = \dim \text{Hom}_A(U, M) - \dim \text{End}_A(U), \]

see [H Theorem 3.1.1]. Similarly, if \(\text{Gr}^U(M)\) is non-empty, then \(\text{Gr}^U(M)\) is smooth and irreducible with

\[ \dim \text{Gr}^U(M) = \dim \text{Hom}_A(M, U) - \dim \text{End}_A(U). \]

For the special case \(U = A^p\) and \(M = A^b\) we have \(\text{Gr}_U(M) \neq \emptyset\) if and only if \(b \geq \text{length}(p)\). In this case, we get

\[ d(p, b) := \dim \text{Gr}_U(M) = \sum_{i=1}^{t} p_i(b + 1 - 2i) \]

where \(p = (p_1, \ldots, p_t)\).
3.5. Two-step flags of submodules as fibre bundles. For $A$-modules $U_1, U_2, M$ let
\[ \text{Gr}^{U_2}_1(M) := \{(V_1, V_2) \in \text{Gr}_{U_1}(M) \times \text{Gr}_{U_2}(M) \mid V_1 \subseteq V_2 \} \]
be the variety of 2-step chains $(0 \subseteq V_1 \subseteq V_2 \subseteq M)$ of submodules of $M$ with $V_1 \cong U_1$ and $M/V_2 \cong U_2$. This is a closed subset of $\text{Gr}_{U_1}(M) \times \text{Gr}_{U_2}(M)$.

Let $p = (p_1, \ldots, p_t)$ with $p_t \geq 1$, and let $b \geq t$. Let $U \in \text{Gr}_A(A^b)$. We obviously get $A^b/U \cong A^q$ for
\[ q \deq (c^{b-t}, c-p_t, c-p_{t-1}, \ldots, c-p_d) \quad \text{with} \quad d \deq \min\{1 \leq i \leq t \mid p_i < c\}. \tag{3.2} \]

If $p = (c^t)$, we have just $q = (c^{b-t})$. We have an obvious isomorphism
\[ \text{Gr}^{A^t}(A^q) \cong \{ V \subseteq A^b \mid U \subseteq V \text{ and } A^b/V \cong A^r \}. \]
Clearly, $\text{Gr}^{A^t}(A^q)$ is non-empty if and only if $r \leq b-t$. In this case, it is smooth and irreducible of dimension
\[ \dim \text{Gr}^{A^t}(A^q) = \dim \text{Hom}_A(A^q, A^r) - \dim \text{End}_A(A^r) = \dim \text{Hom}_A(V, A^r) \]
for any $V \in \text{Gr}^{A^t}(A^q)$. In view of \[H \] Theorem 3.1.1] we only need to show the last equality. For each $V \in \text{Gr}^{A^t}(A^q)$ there is a short exact sequence
\[ 0 \to V \to A^q \to A^r \to 0 \]
of $A$-modules. Since $A^r$ is a projective $A$-module, this sequence splits. Applying the functor $\text{Hom}_A(-, A^r)$ to this sequence yields the result.

Lemma 3.2. The restriction of the projection
\[ \text{Gr}_A(A^b) \times \text{Gr}^{A^t}(A^b) \to \text{Gr}_A(A^b) \]
defined by $(U, V) \mapsto U$ to $\text{Gr}_A^{A^t}(A^b)$ yields a fibre bundle
\[ \pi : \text{Gr}_A^{A^t}(A^b) \to \text{Gr}_A(A^b) \]
with fibre $\text{Gr}^{A^t}(A^q)$ with $q$ as in (3.2). In particular, the fibre is smooth and irreducible.

Note, that our claim about the type of the fibre is clear, however the local triviality seems not to be so obvious. We will see this in the next section.

3.6. Proof of Lemma 3.2

3.6.1. Notation. Let us write the partition $p$ as
\[ p = (c^{p_0}, (c-1)^{p_1}, \ldots, 1^{p_c-1}) \quad \text{and set} \quad p'_c \deq b-l. \]
With this notation we can rewrite (3.1) as
\[ d(p, b) = \sum_{1 \leq i < j \leq b} (p_i - p_j) = \sum_{0 \leq i < j \leq c} (j-i)p_i'p_j', \]
Recall, that this is the dimension of $\text{Gr}_A(A^b)$.

Next, we define
\[ p_j'' \deq \sum_{i=0}^{j-1} p_i' \quad \text{for} \quad 0 \leq j \leq c. \]
Thus in particular $p_0'' = 0$ and $p_c'' = t$, and we have
\[ p''_{c-p_j} < j \leq p''_{c-p_j+1} \quad \text{for} \quad 1 \leq j \leq t. \]
Finally we set $j_+ := p''_{c-p_j+1} + 1$ for all $1 \leq j \leq t$. 
3.6.2. **Affine charts for \( \text{Gr}_A^p(A^b) \).** Let \( U \in \text{Gr}_A^p(A^b) \). For an appropriate \( A \)-basis \( v := (v_1, \ldots, v_b) \) of \( A^b \) we have

\[
U = \bigoplus_{j=1}^t A e^{c-p_j} v_j.
\]

We may set

\[
V_i := \bigoplus_{j=p_i' + 1}^b Av_j \quad \text{for} \quad 1 \leq i \leq c,
\]

and consider the open subset

\[
\mathcal{O}_U^V := \{ U' \in \text{Gr}_A^p(A^b) \mid e^{-i} U' \cap V_i = 0 \text{ for } 1 \leq i \leq c \}
\]

of \( \text{Gr}_A^p(A^b) \), which clearly contains \( U \).

Imitating the description of the open Schubert cells in ordinary Grassmannians we see that each element \( U' \in \mathcal{O}_U^V \) has a unique set of generators in normal form with respect to the chosen basis:

\[
g_i' = e^{c-p_i} v_i(U') \quad \text{for} \quad 1 \leq i \leq t, \quad \text{where}
\]

\[
v_i(U') = v_i + \sum_{j=i+}^b \left( \sum_{k=0}^{p_i - p_i - 1} a_j^{(k)}(U') e^k \right) v_j
\]

where

\[
a_{ji}(U') := a_{ji}(U') \in A
\]

Altogether we showed:

**Lemma 3.3.** With

\[
I(p, b) := \{(i, j, k) \in \mathbb{Z}^3 \mid 1 \leq i \leq l, \, i_+ \leq j \leq b, \, 0 \leq k \leq p_i - p_j - 1 \}
\]

we have an isomorphism of varieties

\[
\mathcal{O}_U^V \to K^I(p, b)
\]

defined by

\[
U' \mapsto (a_{ji}^{(k)}(U'))_{(i, j, k) \in I(p, b)}.
\]

We leave it as an exercise to verify directly that \( I(p, b) \) has exactly \( d(p, b) \) elements.

3.6.3. **Local trivialization.** For \( U' \in \mathcal{O}_U^V \) we define \( g_{U'} \in \text{Aut}_A(A^b) \) by

\[
g_{U'}(v_i) := \begin{cases} v_i(U') & \text{if } 1 \leq i \leq l, \\ v_i & \text{if } l < i \leq b. \end{cases}
\]

Note that \( \text{Aut}_A(A^b) \) acts naturally on \( \text{Gr}_A^p(A^b) \) and on \( \text{Gr}_A^p(A^b) \) as an algebraic group, and we trivially have \( g_{U'}(U) = U' \) for all \( U' \in \mathcal{O}_U^V \). Thus, we obtain the required local trivialization of

\[
\pi: \text{Gr}_A^p(A^b) \to \text{Gr}_A^p(A^b)
\]

on the open neighbourhood \( \mathcal{O}_U^V \) by

\[
\mathcal{O}_U^V \times \pi^{-1}(U) \to \pi^{-1}(\mathcal{O}_U^V), \quad (U', V) \mapsto g_{U'}(V).
\]

Here it is clear, that the map \( \mathcal{O}_U^V \to \text{Aut}_A(A^b) \) defined by \( U' \mapsto g_{U'} \) is a morphism of varieties.
3.7. Bundle construction. Let $\Pi = \Pi(C, D)$, and let $M = (M_1, \ldots, M_n)$ with $M_i$ a free $H_i$-module for all $i$.

We fix now some $k \in I$ and $U = (U_1, \ldots, U_n)$ with $U_i \subseteq M_i$ a free $H_i$-submodule of $M_i$ with $U_i = M_i$ for all $i \neq k$. Let

$$J := \prod_{i \in I} \text{Hom}_{H_i}(U_i, M_i)$$
and

$$J_0 := \{(f_i) \in J \mid f_i \text{ is injective for all } i \in I\}.$$

With $M$ and $U$ defined as above, let $p$ and $q$ be partitions in $\mathcal{P}_{c_k}$. We assume that $p = (c_r^k, q_1, \ldots, q_t)$ and $q = (q_1, \ldots, q_t)$ with $r \geq 1$. Assume that $M_k/U_k \cong E_r^k$.

We fix a direct sum decomposition

$$M_k = U_k \oplus T_k$$
of $H_k$-modules. Such a decomposition exists, since $U_k$ is by assumption free. Note that $T_k$ is also a free $H_k$-module.

Let

$$Y := Y^{k,q,p}$$
be the variety of all triples

$$(U, M, f) \in \Pi(U)^{k,q} \times \overline{H}(M) \times J_0$$
such that for all $(i, j) \in \overline{\Omega}$ the diagram

$$\begin{array}{ccc}
iH_j \otimes_j U_j & \xrightarrow{U_{ij}} & U_i \\
1 \otimes f_j & & f_i \\
iH_j \otimes_j M_j & \xrightarrow{M_{ij}} & M_i
\end{array}$$
commutes and such that for all $i \in I$ we have $M_{i,\text{in}} \circ M_{i,\text{out}} = 0$. Note that for $(U, M, f) \in Y$ we have $M \in \Pi(M)$.

On $Y$ we have a free $G(U)$-action defined by

$$g \cdot (U, M, f) := ((g_i U_{ij} (1 \otimes g_j^{-1}))_{(i, j) \in \overline{\Omega}}, M, (f_i \circ g_i^{-1})_i).$$

We define a diagram

$$\begin{array}{ccc}
\Pi(U)^{k,q} \times J_0 & \xrightarrow{p'} & \Pi(M)^{k,p} \\
\downarrow & & \downarrow \\
Y & \xrightarrow{p''} & \Pi(U)^{k,q} \times J_0
\end{array}$$
by $p'(U, M, f) := (U, f)$ and $p''(U, M, f) := M$. The maps $p'$ and $p''$ are of central importance. We apply now the findings of the previous sections to describe them in more detail.

Lemma 3.4. With the notation above, $p'$ is a vector bundle with fibres isomorphic to $K^m$ with

$$m = \sum_{(j, k) \in \overline{\Omega}} \dim \text{Hom}_{H_k}(T_{k, k} \otimes_j M_j) - \dim \text{Hom}_{H_k}(T_{k, k} U_k').$$

Proof. The canonical projection

$$\Pi(U)^{k,q} \times \overline{H}(M) \times J_0 \to \Pi(U)^{k,q} \times J_0$$
is obviously a vector bundle. One also checks easily that \( Y \) is a closed subset of \( \Pi(U)^{k,q} \times \overline{P}(M) \times J_0 \). We fix \((U, f) \in \Pi(U)^{k,q} \times J_0\). Let
\[
\mathcal{F} := \{ M \in \overline{P}(M) \mid (U, M, f) \in Y \} = p''((p')^{-1}((U, f))).
\]
We have to show that \( \mathcal{F} \cong K^m \) for some \( m \) which is independent of \((U, f)\). Set \( U'_k := \text{Im}(U_{k,\text{in}}) \). Note that \( p(U_k/U'_k) = q \) and \( p(M_k/U'_k) = p \). In other words, we have \( M_k/U'_k \cong H_k^P \).

Define
\[
\eta: \bigoplus_{j \in \Pi(k)} \text{Hom}_{H_k}(T_k, k H_j \otimes_j M_j) \to \text{Hom}_{H_k}(T_k, f_k(U'_k))
\]
by
\[
(T^\lor_{jk})_j \mapsto \sum_{(k,j) \in \Pi} \text{sgn}(k,j)U_{kj}T^\lor_{jk}
\]
and let
\[
\mathcal{F}' := \ker(\eta).
\]
Recall that \( U_j = M_j \) for all \( j \neq k \). Since \( f_k \) is a monomorphism, we have
\[
\dim \text{Hom}_{H_k}(T_k, f_k(U'_k)) = \dim \text{Hom}_{H_k}(T_k, U'_k).
\]
Clearly, \( \eta \) is \( K \)-linear. Since \( T_k \) is a free \( H_k \)-module (and therefore projective as an \( H_k \)-module) we get that \( \eta \) is surjective. Thus we get \( \mathcal{F}' \cong K^m \) with
\[
m := \sum_{(j,k) \in \Pi} \dim \text{Hom}_{H_k}(T_k, k H_j \otimes_j M_j) - \dim \text{Hom}_{H_k}(T_k, U'_k).
\]
Let
\[
\mu: \mathcal{F} \to \mathcal{F}'
\]
be defined by
\[
M \mapsto (M'_k|_{T_k})_{(j,k) \in \Pi}.
\]
This is obviously an isomorphism of \( K \)-vectorspaces. \(\square\)

We need one more auxiliary variety
\[
Y'' := \{ (M'_k, M) \in \text{Gr}^{T_k}(M_k) \times \Pi(M)^{k,p} \mid M'_k \supseteq \text{Im}(M_{k,\text{in}}) \}
\]
where we recall that \( \text{Gr}^{T_k}(M_k) \) is the Grassmannian of \( H_k \)-submodules \( U \) of \( M_k \) such that \( M_k/U \cong T_k \cong H_k^P \). We have two natural morphisms
\[
p'_1: Y \to Y'', \quad (U, M, f) \mapsto (\text{Im}(f_k), M),
p'_2: Y'' \to \Pi(M)^{k,p}, \quad (M'_k, M) \mapsto M.
\]
Obviously, we have \( p'' = p'_2 \circ p'_1 \).

**Lemma 3.5.** With the above notation we have:

(a) \( p'_1: Y \to Y'' \) is a \( G(U) \)-principal bundle.
(b) \( p'_2: Y'' \to \Pi(M)^{k,p} \) is a fibre bundle with fibre \( \text{Gr}^{T_k}(H_k^P) \). In particular, this fibre is smooth, irreducible and of dimension
\[
\dim \text{Hom}_{H_k}(H_k^P, T_k) = \dim_{H_k} \text{End}_{H_k}(T_k).
\]
(c) \( p'' \) is a fibre bundle with fibres isomorphic to
\[
G(U) \times \text{Gr}^{T_k}(H_k^P).
\]
Proof. (a): It is easy to see that
\[ \text{Im}_k: J_0 \to \text{Gr}^k(M_k), \quad (f_i)_{i \in I} \mapsto \text{Im}(f_k) \]
is a principal \(G(U)\)-bundle since \(\text{Gr}^k(M_k) \cong \text{Gr}_{U_k}(M_k)\), see also Section 3.4. Now, consider the morphism
\[ \phi_1: Y'' \to \text{Gr}^k(M_k), \quad (M'_k, M) \mapsto M'_k. \]
We observe that \(Y \cong \phi_1^*(J_0)\), the pullback of a \(G(U)\)-principal bundle, see Section 3.3. In fact, it follows directly from the definitions that \(\phi_1^*(J_0)\) can be identified with
\[ Y'' := \{ (M, f) \in \Pi(M)^k \times J_0 | \text{Im}(f_k) \supset \text{Im}(M_{k, \text{in}}) \}. \]
Clearly, for each \((M, f) \in Y''\) there exists a unique \(U \in \Pi(U)^k\) with \(f \in \text{Hom}_{\Pi}(U, M)\).

(b): There exists a unique partition \(p^*\) such that \(\text{Im}(M_{k, \text{in}}) \cong H^k_{p^*}\) for all \(M \in \Pi(M)^k\). With this notation we consider the fibre bundle
\[ \pi: \text{Gr}_{H^k_{p^*}}(M_k) \to \text{Gr}_{H^k_{p^*}}(M_k) \]
with fibre \(\text{Gr}^k(H^k_{p^*})\), see Lemma 3.2. By construction, we have the natural morphism
\[ \phi_2: \Pi(M) \to \text{Gr}_{H^k_{p^*}}(M_k), \quad M \mapsto \text{Im}(M_{k, \text{in}}). \]
It follows directly from the definitions that
\[ \phi_2^*(\text{Gr}^k(M_k)) = Y''. \]
Thus
\[ p_2'': Y'' \to \Pi(M)^k \]
is a fibre bundle with the requested type of fibre. This proves (b).

Part (c) is a direct consequence of (a) and (b) and the fact that \(p'' = p_2'' \circ p_1''\). \(\square\)

Lemma 3.6. The following hold:

(a) If \(Z'\) is an irreducible component of \(\Pi(U)^k\), then
\[ Z := p''((p')^{-1}(Z' \times J_0)) \]
is an irreducible component of \(\Pi(M)^k\).

(b) The map \(Z' \mapsto Z\) defines a bijection
\[ \text{Irr}(\Pi(U)^k) \to \text{Irr}(\Pi(M)^k). \]

(c) We have
\[ \dim(Z) - \dim(Z') = \dim(H(M)) - \dim(H(U)). \]

Proof. Recall that we have two maps
\[ \begin{array}{ccc}
Y & \xrightarrow{p''} & \Pi(M)^k \\
\Pi(U)^k \times J_0 & \xrightarrow{p'} & \Pi(U)^k, \\
\end{array} \]
defined by \(p'(U, M, f) := (U, f)\) and \(p''(U, M, f) := M\). The statements (a) and (b) follow immediately from combining Lemma 3.3 and Lemma 3.5(c). Also from these lemmas we get that
\[ \dim(Z) + \dim(G(U) + \dim \text{Hom}_{H_k}(T_k, U_k/U_k') = \dim(Z') + \dim(J_0) + m. \]
with
\[ m = \sum_{(j,k) \in \Omega} \dim \text{Hom}_{H_k}(T_k, kH_j \otimes_j M_j) - \dim \text{Hom}_{H_k}(T_k, U_k'). \]

One easily checks that
\[ \dim(J_0) - \dim G(U) = \dim \text{Hom}_{H_k}(U_k, T_k) = \dim \text{Hom}_{H_k}(T_k, U_k) = \dim \text{Hom}_{H_k}(T_k, U_k') + \dim \text{Hom}_{H_k}(T_k, U_k/U_k'). \]

Furthermore, we have
\[ \dim H(M) = \sum_{(j,i) \in \Omega} \dim \text{Hom}_{H_j}(jH_i \otimes_j M_i, M_j) = \sum_{(j,i) \in \Omega} \dim \text{Hom}_{H_i}(M_i, iH_j \otimes_j M_j), \]
\[ \dim H(U) = \sum_{(j,i) \in \Omega} \dim \text{Hom}_{H_j}(jH_i \otimes_j U_i, U_j) = \sum_{(j,i) \in \Omega} \dim \text{Hom}_{H_i}(U_i, iH_j \otimes_j U_j). \]

Thus we have and
\[ \dim H(M) - \dim H(U) = \sum_{(j,k) \in \Omega} \dim \text{Hom}_{H_k}(T_k, kH_j \otimes_j M_j). \]

Combining the above equalities we obtain Thus we get
\[ \dim(Z) - \dim(Z') = \dim(J_0) - \dim G(U) - \dim \text{Hom}_{H_k}(T_k, U_k/U_k') + m = \sum_{j \in \Omega(k)} \dim \text{Hom}_{H_k}(T_k, kH_j \otimes_j M_j) = \dim H(M) - \dim H(U). \]

This proves (c). \(\square\)

3.8. **Comparision to Lusztig’s bundle construction.** In the classical case \((C\) symmetric and \(D\) the identity matrix), Lusztig constucted bundles

\[
\begin{array}{cc}
Y & \\
p' & p'' & \\
\Pi(U)_{k,0} \times J_0 & \Pi(M)_{k,p} & \\
\end{array}
\]

with \(p \geq 1\) and \(\text{rank}(M/U) = p\alpha_k\). (Here \(p\) stands for the partition \((c_k^p)\).) Lusztig does not consider the situation

\[
\begin{array}{cc}
Y & \\
p' & p'' & \\
\Pi(U)_{k,q} \times J_0 & \Pi(M)_{k,p} & \\
\end{array}
\]

with \(p > q \geq 1\) and \(\text{rank}(M/U) = (p - q)\alpha_k\). Thus for the classical case, one can see our construction as a refinement of Lusztig’s bundle construction. Another important difference is that in our setup
from Section 3.7 the closures (in $\Pi(M)$) of the irreducible components of $\Pi(M)^{k,p}$ are in general not irreducible components of $\Pi(M)$. In general, this will only be the case for maximal components of $\Pi(M)^{k,p}$. Some examples of this kind can be found in Section 8.2.5.

3.9. The maps $e_{k,r}, f_{k,r}, e^*_k, f^*_k$. Let $p, q \in P_{ck}$ be partitions of the form

$$p = (e^r_k, q_1, \ldots, q_t) \quad \text{and} \quad q = (q_1, \ldots, q_t),$$

with $r \geq 1$. Then Lemma 3.6(b) and its dual yield bijections

$$\bigsqcup_{r \in \mathbb{N}_n} \text{Irr}(\Pi(r)^{k,p}) \begin{array}{c} \xrightarrow{f^*_k} \\ \xleftarrow{e^*_k} \end{array} \bigsqcup_{r \in \mathbb{N}_n} \text{Irr}(\Pi(r - r\alpha_k)^{k,q})$$

and

$$\bigsqcup_{r \in \mathbb{N}_n} \text{Irr}(\Pi(r)_{k,p}) \begin{array}{c} \xrightarrow{f_{k,r}} \\ \xleftarrow{e_{k,r}} \end{array} \bigsqcup_{r \in \mathbb{N}_n} \text{Irr}(\Pi(r - r\alpha_k)_{k,q})$$

with $f^*_{k,r} = (e^*_{k,r})^{-1}$ and $f_{k,r} = (e_{k,r})^{-1}$.

The following lemma is a straightforward consequence of Lemma 3.6

**Lemma 3.7.** We have

$$(f^*_{k,1})^r = f^*_{k,r}, \quad (e^*_{k,1})^r = e^*_{k,r}, \quad (f_{k,1})^r = f_{k,r}, \quad (e_{k,1})^r = e_{k,r}.$$ 

3.10. The functions $\varphi_i$ and $\varphi^*_i$. For $M \in \text{rep}(H_i)$ let $[M : E_i]$ be the maximal number $p \geq 0$ such that there exists a direct sum decomposition $M = U \oplus V$ with $U \cong E_i^p$. Define functions

$$\varphi_i, \varphi^*_i : \Pi(r) \to \mathbb{Z}$$

by

$$\varphi_i(M) := [\text{sub}_i(M) : E_i] \quad \text{and} \quad \varphi^*_i(M) := [\text{fac}_i(M) : E_i].$$

We obviously get $\varphi_i(M) = p$ for all $M \in \Pi(r)_{i,p}$, and $\varphi^*_i(M) = p$ for all $M \in \Pi(r)^{i,p}$.

4. Maximal irreducible components and crystal modules

4.1. Maximal irreducible components.

**Theorem 4.1.** For each $Z \in \text{Irr}(\Pi(M))$ we have

$$\dim(Z) \leq \dim H(M).$$

**Proof.** Let $Z \in \text{Irr}(\Pi(M))$. There exists some $k \in I$ with $\varphi^*_k(Z) > 0$. Thus there is a partition $p = (e^r_k, q_1, \ldots, q_t)$ with $r \geq 1$ such that $Z^{k,p} = Z \cap \Pi(M)^{k,p}$ is dense in $Z$. Furthermore, we have $Z^{k,p} \in \text{Irr}(\Pi(M)^{k,p})$. Let $Z'$ be the corresponding component of $\Pi(\mathcal{U})^{k,q}$, where $q := (q_1, \ldots, q_t)$ and $\mathcal{U}$ is defined as in Section 3.7. By induction we know that $\dim(Z') \leq \dim H(\mathcal{U})$. By Lemma 3.6(c) we know that

$$\dim(Z) - \dim(Z') = \dim H(M) - \dim H(\mathcal{U}).$$

This implies

$$\dim(Z) - \dim H(M) = \dim(Z') - \dim H(\mathcal{U}) \leq 0.$$

This finishes the proof. \qed
An irreducible component \( Z \) of \( \Pi(M) \) is **maximal** if \( \dim(Z) = \dim H(M) \). We denote the set of maximal irreducible components of \( \Pi(M) \) by \( \text{Irr}(\Pi(M))^{\text{max}} \). Similarly, let \( \text{Irr}(\Pi(M)^k,p)^{\text{max}} \) and \( \text{Irr}(\Pi(M)_{k,p})^{\text{max}} \) be the sets of irreducible components of \( \Pi(M)^k,p \) and \( \Pi(M)_{k,p} \) of dimension \( \dim H(M) \), respectively.

We can embed \( H(M) \) into \( \Pi(M) \) in the obvious way. By Theorem 4.1, \( H(M) \) is then a maximal irreducible component of \( \Pi(M) \). Thus \( \text{Irr}(\Pi(M))^{\text{max}} \) is non-empty. However, the sets \( \text{Irr}(\Pi(M)^k,p)^{\text{max}} \) and \( \text{Irr}(\Pi(M)_{k,p})^{\text{max}} \) can be empty, depending on the partition \( p \).

### 4.2. Crystal modules.

For \( M \in \text{rep}(\Pi) \) and \( i \in I \) there are canonical short exact sequences

\[
0 \to K_i(M) \to M \to \text{fac}_i(M) \to 0
\]

and

\[
0 \to \text{sub}_i(M) \to M \to C_i(M) \to 0.
\]

Here \( K_i(M) \) is the unique submodule of \( M \) with \( M/K_i(M) \cong \text{fac}_i(M) \), and \( C_i(M) = M/\text{sub}_i(M) \) is the unique factor module of \( M \).

We say that \( M \in \text{nil}_E(\Pi) \) is a **crystal module** if \( \text{fac}_i(M) \) and \( \text{sub}_i(M) \) are locally free for all \( i \), and if \( K_i(M) \) and \( C_i(M) \) are crystal modules for all \( i \in I \). By definition the trivial module \( 0 \) is a crystal module.

Clearly, if \( M \in \text{nil}_E(\Pi) \) is a crystal module, then we have \( \dim \text{sub}_i(M) = c_i \varphi_i(M) \) and \( \dim \text{fac}_i(M) = c_i \varphi_i^*(M) \).

For \( i, j \in I \) there is a canonical homomorphism \( f: \text{sub}_j(M) \to \text{fac}_i(M) \) defined by \( u \mapsto u + K_i(M) \).

**Lemma 4.2.** Let \( M \in \text{rep}(\Pi) \). For \( i \in I \) and each submodule \( U \subseteq M \) we have

\[
\text{fac}_i(M/U) \cong \text{fac}_i(M)/(U + K_i(M)).
\]

**Proof.** We have canonical short exact sequences

\[
0 \to K_i(M) \to M \to \text{fac}_i(M) \to 0
\]

and

\[
0 \to K_i(M/U) \to M/U \to \text{fac}_i(M/U) \to 0.
\]

We use that submodules of a factor module \( M/V \) are in bijection with submodules \( W \) of \( M \) with \( V \subseteq W \subseteq M \). In this way, we can interpret \( U + K_i(M) \) as a submodule of \( \text{fac}_i(M) \), and \( K_i(M/U) \) as a submodule of \( M \).

We get the obvious inclusions displayed in the following diagram:

\[
\begin{array}{ccc}
M & \xrightarrow{K_i(M/U)} & U + K_i(M) \\
\downarrow & & \downarrow & & \downarrow \\
U & \xrightarrow{K_i(M)} & K_i(M)
\end{array}
\]

There is an epimorphism

\[
\pi: M/U \to \text{fac}_i(M)/(U + K_i(M))
\]
defined by \( m + U \mapsto m + (U + K_i(M)) \). Since all composition factors of the image of \( \pi \) are isomorphic to \( S_i \), the epimorphism \( \pi \) factors through \( \text{fac}_i(M/U) \). This yields an epimorphism

\[
\pi': \text{fac}_i(M/U) \to \text{fac}_i(M)/(U + K_i(M)).
\]

We obviously have \( U \leq K_i(M/U) \). We also have \( K_i(M) \leq K_i(M/U) \), since all composition factors of \( M/K_i(M/U) \) are isomorphic to \( S_i \), and \( M/K_i(M) \) is the unique maximal factor module with this property. It follows that \( U + K_i(M) \leq K_i(M/U) \). Thus for dimension reasons, \( \pi' \) has to be an isomorphism.

**Lemma 4.3.** For \( i, j \in I \) and \( M \in \text{nil}_E(\Pi) \) a crystal module the following hold:

(i) If the canonical homomorphism

\[
f: \text{sub}_j(M) \to \text{fac}_i(M)
\]

is non-zero, then \( i = j \) and \( M \) has a direct summand isomorphic to \( E_i \).

(ii) If the canonical homomorphism

\[
f: \text{sub}_j(M) \to \text{fac}_i(M)
\]

is zero, then

\[
\text{fac}_i(M) \cong \text{fac}_i(C_j(M)) \quad \text{and} \quad \text{sub}_j(M) = \text{sub}_j(K_i(M)).
\]

**Proof.** We first prove (i). If \( i \neq j \), then the canonical homomorphism \( f: \text{sub}_j(M) \to \text{fac}_i(M) \) is obviously zero. Thus assume that \( i = j \) and that \( f: \text{sub}_j(M) \to \text{fac}_i(M) \) is non-zero. We know that \( \text{sub}_i(M) \) and \( \text{fac}_i(M) \) are free \( H_i \)-modules. Let \( p: \text{fac}_i(M) \to \text{top}(\text{fac}_i(M)) \) be the canonical projection of \( \text{fac}_i(M) \) onto its top. Note that \( \text{top}(\text{fac}_i(M)) = \text{top}_i(M) \). By Lemma 4.2 we have

\[
\text{fac}_i(M/\text{sub}_i(M)) \cong \text{fac}_i(M)/f(\text{sub}_i(M)).
\]

Now suppose that \( pf = 0 \). Then we get

\[
\text{top}_i(\text{fac}_i(M)) \cong \text{top}_i(\text{fac}_i(M)/f(\text{sub}_i(M))).
\]

Together with our assumption that \( f(\text{sub}_i(M)) \neq 0 \), this implies that \( \text{fac}_i(M/\text{sub}_i(M)) \) is not free, a contradiction to our assumption that \( M \) is a crystal module. Thus we proved that \( pf \neq 0 \). This implies that there is a submodule \( U \) of \( \text{sub}_i(M) \) with \( U \cong E_i \) and \( f(U) \cong E_i \). This yields a homomorphism \( g: \text{fac}_i(M) \to U \) with \( g f U = 1_U \), where \( \iota_U : U \to \text{sub}_i(M) \) denotes the inclusion. We have \( f = f_2 f_1 \) with the obvious homomorphisms \( f_1 : \text{sub}_i(M) \to M \) and \( M \to \text{fac}_i(M) \). We get \( (gf_2)(f_1 \iota_U) = 1_U \). This shows that \( f_1 \iota_U : U \to M \) is a split monomorphism. It follows that \( M \) has a direct summand isomorphic to \( E_i \). This finishes the proof of (i). Part (ii) is straightforward. \( \Box \)

Let

\[
\text{nil}_E^\text{cr}(\Pi, M) = \Pi(M)^\text{cr}
\]

be the subset of crystal modules in \( \text{nil}_E(\Pi, M) = \Pi(M) \).

An irreducible component \( Z \in \text{Irr}(\Pi(M)) \) is a **crystal component** if it contains a dense open subset of crystal modules.

**Proposition 4.4.** For \( Z \in \text{Irr}(\Pi(M)) \) the following are equivalent:

(i) \( Z \) is maximal.

(ii) \( Z \) is a crystal component.
Proof. (ii) $\implies$ (i): Let $\mathbf{M} = (M_1, \ldots, M_n)$ be locally free. Suppose $Z \in \text{Irr}(\Pi(\mathbf{M}))$ is a crystal component. By Lemma 3.1(c) there exists some $k \in I$ and some $p > 0$ such that $\varphi_k^*(Z) = p$. Now choose $\mathbf{U} = (U_1, \ldots, U_n)$ such that $U_i = M_i$ for all $i \neq k$, and $U_k$ is a free $H_k$-submodule of $M_k$ such that $M_k/U_k \cong E^p_k$. Let $(Z^{k,p})' \in \text{Irr}(\Pi(U)^{k,0})$ be the irreducible component corresponding to $Z^{k,p} := Z \cap \Pi(M)^{k,p}$ under the bijection $\text{Irr}(\Pi(M)^{k,p}) \to \text{Irr}(\Pi(U)^{k,0})$ from Lemma 3.6(b). Finally, let $Z'$ be the closure of $(Z^{k,p})'$ in $\Pi(U)$. It follows that $Z' \in \text{Irr}(\Pi(U))$, since $\Pi(U)^{k,0}$ is non-empty and open in $\Pi(U)$. It is straightforward that the component $Z'$ is again a crystal component. By induction, $Z'$ is maximal, i.e. $\dim(Z') = \dim H(U)$. Now Lemma 3.6 implies that $\dim(Z) = \dim H(M)$. In other words, $Z$ is maximal.

(i) $\implies$ (ii): Let $\mathbf{M} = (M_1, \ldots, M_n)$ be locally free. Assume that $Z \in \text{Irr}(\Pi(\mathbf{M}))$ is maximal, and that $Z$ is not a crystal component. Let $r := \text{rank}(\mathbf{M})$ be minimal such that such a $Z$ exists.

By minimality, it follows that $\text{fac}_k(Z)$ or $\text{sub}_k(Z)$ is not free for some $k$. Without loss of generality we assume that $\text{fac}_1(Z)$ is not free. Again by minimality, we know that $\varphi_1^*(Z) = 0$, i.e. $\text{fac}_1(Z)$ does not have a direct summand isomorphic to $E_1$.

There exists some $s \in I$ such that $\varphi_s(Z) > 0$, i.e. $\text{sub}_s(Z)$ contains a direct summand isomorphic to $E_s$. Now choose $\mathbf{U} = (U_1, \ldots, U_n)$ such that $U_i = M_i$ for all $i \neq s$, and $U_s = M_s/U$ is a free $H_s$-submodule of $M_s$ with $U \cong E_s$.

There is a partition $\mathbf{p} = (s_1, q_1, \ldots, q_t)$ such that $Z_{s,\mathbf{p}} := Z \cap \Pi(M)_{s,\mathbf{p}}$ is open and dense in $Z$. We have $Z_{s,\mathbf{p}} \in \text{Irr}(\Pi(M)_{s,\mathbf{p}})$. Set $\mathbf{q} := (q_1, \ldots, q_t)$.

Under the bijection $\text{Irr}(\Pi(M)_{s,\mathbf{p}}) \to \text{Irr}(\Pi(U)_{s,\mathbf{q}})$ from the dual of Lemma 3.6(b), let $Z'_{s,\mathbf{p}} \in \text{Irr}(\Pi(U)_{s,\mathbf{q}})$ be the irreducible component corresponding to $Z_{s,\mathbf{p}}$. Let $Z'$ be the closure of $Z'_{s,\mathbf{p}}$ in $\Pi(U)$.

The dual of Lemma 3.6 yields $\mathbf{U}$ and an irreducible component $Z'_{s,\mathbf{q}}$ of $\Pi(U)_{s,\mathbf{q}}$ corresponding to $Z$. Let $Z'$ be the closure of $Z_{s,\mathbf{q}}$ in $\Pi(U)$. By the dual of Lemma 3.6(c) we know that $Z'$ is a maximal irreducible component of $\Pi(U)$. Furthermore, by induction $Z'$ is a crystal component. In particular, this implies that $\text{fac}_1(Z')$ is free.

Let $M$ be generic in $Z$. There is a short exact sequence

$$0 \to E_s \xrightarrow{f} M \to M' \to 0$$

with $M'$ generic in $Z'$. This implies that $s = 1$. (Otherwise $\text{fac}_1(M) \cong \text{fac}_1(M')$ and therefore $\varphi_1(Z) = \text{fac}_1(Z')$, a contradiction.) The short exact sequence above is non-split. (Otherwise $\text{fac}_1(M) = \text{fac}_1(Z)$ would contain a direct summand isomorphic to $E_1$, a contradiction.) In other words, we have $\text{Ext}^1_{\Pi}(M', E_1) \neq 0$.

Without loss of generality we assume that $f: E_1 \to M$ is just an inclusion map and that

$$M_1 = U_1 \oplus E_1.$$ 

By Lemma 4.2 we have

$$\text{fac}_1(M') \cong \text{fac}_1(M/E_1) \cong \text{fac}_1(M)/(E_1 + K_1(M)).$$

We have $E_1 + K_1(M) = p(E_1)$, where

$$p: M \to \text{fac}_1(M)$$

is the obvious canonical epimorphism. Since $\text{fac}_1(M')$ is free, and $\text{fac}_1(M)$ is not, this implies $p(E_1) \neq 0$. Since $\text{fac}_1(M)$ does not contain a free direct summand, and $\text{fac}_1(M')$
is free, we even get \( p(E_1) = \text{fac}_1(M) \) and therefore \( \text{fac}_1(M') = 0 \). In particular, \( \text{fac}_1(M) \) is isomorphic to a proper factor module of \( E_1 \).

We have \( M = (M_{ij}) \in \Pi(M) \) and \( M' = (M'_{ij}) \in \Pi(U) \) with \( M'_{ij} = M_{ij} \) for all \( (i, j) \) with \( i \neq 1 \) and \( j \neq 1 \). Furthermore, we have \( M_{1,\text{out}} \mid U_1 = M'_{1,\text{out}} \) and \( M_{1,\text{out}} \mid E_1 = 0 \). (For the last equality we used that \( E_1 \) is a submodule of \( M \).) In particular, we have \( \text{Im}(M_{1,\text{out}}) = \text{Im}(M'_{1,\text{out}}) \).

By induction we know that \( M' \) is a crystal module. This implies that \( \text{Im}(M'_{1,\text{out}}), \text{Ker}(M'_{1,\text{in}}) \) and therefore also \( \text{Ker}(M'_{1,\text{out}})/\text{Im}(M'_{1,\text{out}}) \) are free \( H_1 \)-modules.

We now describe the \( H_1 \)-linear maps

\[
M_{1,\text{in}}: \tilde{M}_1 \to M_1 \quad \text{and} \quad M'_{1,\text{in}}: \tilde{M}_1 \to U_1
\]

where

\[
\tilde{M}_1 = \bigoplus_{j \in \mathbb{N}(1)} 1 H_j \otimes_j M_j.
\]

We have a decomposition

\[
\tilde{M}_1 = \text{Im}(M'_{1,\text{out}}) \oplus V \oplus W
\]

into a direct sum of \( H_1 \)-modules, where \( \text{Im}(M'_{1,\text{out}}) \oplus V = \text{Ker}(M'_{1,\text{in}}) \). (Here we used that \( \text{Im}(M'_{1,\text{out}}), \text{Ker}(M'_{1,\text{in}}) \) and \( \text{Im}(M'_{1,\text{in}}) \cong W \) are free \( H_1 \)-modules. It follows also that \( V \) is free.) We have

\[
V \cong \text{Ker}(M'_{1,\text{in}})/\text{Im}(M'_{1,\text{out}}) \cong \text{Ext}^1_{\Pi}(M', E_1) \neq 0.
\]

(For the last isomorphism we used Lemma 2.3(iii).) Using both decompositions \( \tilde{M}_1 = \text{Im}(M'_{1,\text{out}}) \oplus V \oplus W \) and \( M_1 = U_1 \oplus E_1 \) we can write \( M_{1,\text{in}}: \tilde{M}_1 \to M_1 \) as a matrix

\[
M_{1,\text{in}} = \begin{pmatrix} 0 & 0 & f_{13} \\ 0 & f_{22} & f_{23} \end{pmatrix}: \text{Im}(M'_{1,\text{out}}) \oplus V \oplus W \to U_1 \oplus E_1
\]

where the \( f_{ij} \) are \( H_1 \)-module homomorphisms, and \( M'_{1,\text{in}}: \tilde{M}_1 \to U_1 \) is given by the matrix

\[
M'_{1,\text{in}} = \begin{pmatrix} 0 & 0 & f_{23} \end{pmatrix}: \text{Im}(M'_{1,\text{out}}) \oplus V \oplus W \to U_1.
\]

Since \( \text{fac}_1(M') = 0 \), we get that \( f_{13}: W \to U_1 \) is an isomorphism.

We now define a new \( \Pi \)-module \( \overline{M} \) by replacing \( f_{22}: V \to E_1 \) by an \( H_1 \)-linear map \( \overline{f}_{22}: V \to E_1 \) of maximal rank. Thus \( \overline{f}_{22} \) is an epimorphism, since \( V \) is non-zero and free. It is clear that \( \overline{M} \) is indeed a \( \Pi \)-module. (Using that \( M_{1,\text{out}} \mid E_1 = 0 \) and that \( M_{i,\text{in}} \circ M_{i,\text{out}} = 0 \) for all \( i \), we get that \( \overline{M}_{i,\text{in}} \circ \overline{M}_{i,\text{out}} = 0 \) for all \( i \).) Since \( f_{13} \) and \( \overline{f}_{22} \) are both epimorphisms, we get that \( \overline{M}_{1,\text{in}} \) is an epimorphism. This means that \( \text{fac}_1(\overline{M}) = 0 \).

We have \( \overline{M}/E_1 = M' \). Since \( M' \) is generic in \( Z' \), we get that \( \overline{M} \) is also contained in \( Z \). (Here we used again Lemma 3.6.) This is a contradiction to \( M \) being generic in \( Z \), since \( \text{fac}_1(M) \neq 0 \) and \( \text{fac}_1(\overline{M}) = 0 \). Thus we got a contradiction to our assumption that \( \text{fac}_1(M) \) is not free.

So we proved that \( \text{fac}_i(M) \) is free for all \( i \). Dually one shows that \( \text{sub}_i(M) \) is free for all \( i \). Thus by induction, \( Z \) is a crystal component. \( \square \)

**Corollary 4.5.** \( \Pi(M)^{\text{cr}} \) is equidimensional of dimension \( \dim H(M) \).

**Corollary 4.6.** For a partition \( p \in \mathcal{P}_c \) which is not of the form \( p = (c^p_k) \) for some \( p \), we have \( \dim \Pi(M)^{k,p} < \dim H(M) \) and \( \dim \Pi(M)^{k,p} < \dim H(M) \).

Examples of non-maximal irreducible components can be found in Section 8.
5. Geometric construction of crystal graphs

This section follows very closely \[NT\], which on the other hand is based on \[KS\].

5.1. Kac-Moody algebras. Let \( C = (c_{ij}) \in M_n(\mathbb{Z}) \) be a symmetrizable generalized Cartan matrix. Recall that \( I = \{1, \ldots, n\} \).

Let \( \mathfrak{h} \) be a \( \mathbb{C} \)-vector space of dimension \( 2n - \text{rank}(C) \), and let \( \{\alpha_1, \ldots, \alpha_n\} \subset \mathfrak{h}^* \) and \( \{\alpha_1^\vee, \ldots, \alpha_n^\vee\} \subset \mathfrak{h} \) be linearly independent subsets of the vector spaces \( \mathfrak{h}^* \) and \( \mathfrak{h} \), respectively, such that

\[
\alpha_i(\alpha_j^\vee) = c_{ij}
\]

for all \( i, j \). (Here \( \mathfrak{h}^* = \text{Hom}_\mathbb{C}(\mathfrak{h}, \mathbb{C}) \) is the dual space of \( \mathfrak{h} \).)

Now \( \mathfrak{g} = (\mathfrak{g}, [\cdot, \cdot]) \) is the Lie algebra over \( \mathbb{C} \) generated by \( \mathfrak{h} \) and the symbols \( e_i \) and \( f_i \) \( (i \in I) \) satisfying the following defining relations:

\[
\begin{align*}
(\text{i}) & \ [h, h'] = 0 \text{ for all } h, h' \in \mathfrak{h}; \\
(\text{ii}) & \ [h, e_i] = \alpha_i(h)e_i \text{ and } [h, f_i] = -\alpha_i(h)f_i \text{ for all } i \text{ and all } h \in \mathfrak{h}; \\
(\text{iii}) & \ [e_i, f_j] = \delta_{ij}\alpha_i^\vee \text{ for all } i, j; \\
(\text{iv}) & \ (\text{ad}(e_i))^{1-c_{ij}}(e_j) = 0 \text{ for all } i \neq j; \\
(\text{v}) & \ (\text{ad}(f_i))^{1-c_{ij}}(f_j) = 0 \text{ for all } i \neq j.
\end{align*}
\]

(For \( x, y \in \mathfrak{g} \) and \( m \geq 1 \) we set \( \text{ad}(x)(y) := \text{ad}(x)^1(y) := [x, y] \) and \( \text{ad}(x)^{m+1}(y) := \text{ad}(x)^m([x, y]). \) The Lie algebra \( \mathfrak{g} \) is the Kac-Moody algebra associated with \( C \). As a general reference on Kac-Moody algebras, we refer to Kac’s book \[Ka\].

Let \( \mathfrak{n} = \mathfrak{n}(C) \) be the Lie subalgebra of \( \mathfrak{g} \) generated by \( e_i \) \( (i \in I) \). Then \( U(\mathfrak{n}) \) is the associative \( \mathbb{C} \)-algebra with generators \( e_i \) \( (i \in I) \) subject to the relations

\[
(\text{ad} e_i)^{-c_{ij}}(e_j) = 0
\]

for all \( i \neq j \). (Here we interpret \( [x, y] \) as a commutator \( xy - yx. \))

Let \( \mathfrak{h}^* = \mathfrak{h}^*_1 \oplus \mathfrak{h}^*_2 \) be a vector space decomposition, where \( \mathfrak{h}^*_1 \) is just the subspace with basis \( \{\alpha_1, \ldots, \alpha_n\} \), and \( \mathfrak{h}^*_2 \) is any direct complement of \( \mathfrak{h}^*_1 \) in \( \mathfrak{h}^* \). Let

\[
\langle -,- \rangle : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}
\]

be the standard bilinear form, defined by \( \langle \alpha_i, \alpha_j \rangle := \alpha_i(\alpha_j^\vee) = c_{ji}, \langle \alpha_i, x \rangle := x(\alpha_i^\vee) \), \( \langle x, \alpha_i \rangle := x(\alpha_i^\vee) \), and \( \langle x, y \rangle := 0 \) for all \( x, y \in \mathfrak{h}^*_2 \) and \( i, j \in I \). (Identifying the \( \alpha_i \) with the standard basis of \( \mathbb{Z}^n \), this definition of \( \langle -,- \rangle \) is compatible with the bilinear form defined in Section 2.1)

Finally, let us fix a basis \( \{\varpi_j \mid 1 \leq j \leq 2n - \text{rank}(C)\} \) of \( \mathfrak{h}^* \) such that

\[
\varpi_j(\alpha_i^\vee) = \delta_{ij}, \quad (i \in I, \ 1 \leq j \leq 2n - \text{rank}(C)).
\]

The \( \varpi_j \) are the fundamental weights. Note that for \( i \in I \) we have

\[
\alpha_i = \sum_{j \in I} c_{ji} \varpi_j.
\]

We denote by

\[
P := \{\nu \in \mathfrak{h}^* \mid \langle \nu, \alpha_i \rangle \in \mathbb{Z} \text{ for all } i \in I\}
\]

the integral weight lattice, and we set

\[
P^+ := \{\nu \in P \mid \langle \nu, \alpha_i \rangle \geq 0 \text{ for all } i \in I\}.
\]
The elements in $P^+$ are called **dominant integral weights**. We have

$$P = \bigoplus_{j \in I} \mathbb{Z} \omega_j \oplus \bigoplus_{j=1}^{2n-\text{rank}(C)} \mathbb{C} \omega_j$$

and

$$P^+ = \bigoplus_{j \in I} \mathbb{N} \omega_j \oplus \bigoplus_{j=1}^{2n-\text{rank}(C)} \mathbb{C} \omega_j.$$

For

$$\lambda = \sum_{j \in I} a_j \omega_j + \sum_{j=1}^{2n-\text{rank}(C)} a_j \omega_j$$

in $P$, we have

$$a_j = \langle \lambda, \alpha_j \rangle$$

for $1 \leq j \leq n$. Let

$$R := \bigoplus_{i \in I} \mathbb{Z} \alpha_i$$

be the **root lattice**, and set

$$R^+ := \bigoplus_{i \in I} \mathbb{N} \alpha_i.$$

### 5.2. Crystals

As before, let $C$ be a symmetrizable generalized Cartan matrix with symmetrizer $D$, and let $P$ be the associated integral weight lattice.

Following [K1, Section 7.2], a **crystal** is a tuple $(B, \text{wt}, \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i)$ where $B$ is a set and

$$\text{wt}: B \rightarrow P, \quad \tilde{e}_i, \tilde{f}_i: B \rightarrow B \cup \{\emptyset\}, \quad \varepsilon_i, \varphi_i: B \rightarrow \mathbb{Z}$$

with $i \in I$ are maps such that for all $i \in I$ and all $b \in B$ the following hold:

1. **(cr1)** $\varphi_i(b) = \varepsilon_i(b) + \langle \text{wt}(b), \alpha_i \rangle$;
2. **(cr2)** $\varphi_i(\tilde{e}_i(b)) = \varphi_i(b) + 1, \quad \varepsilon_i(\tilde{e}_i(b)) = \varepsilon_i(b) - 1, \quad \text{wt}(\tilde{e}_i(b)) = \text{wt}(b) + \alpha_i$;
3. **(cr3)** For all $b, b' \in B$ the following are equivalent:
   - (a) $\tilde{f}_i(b) = b'$;
   - (b) $\tilde{e}_i(b') = b$.

Kashiwara [K1] also allows the values of $\varepsilon_i$ and $\varphi_i$ to be $-\infty$. This assumption is not needed here.

A **lowest weight crystal** is a crystal with a distinguished element $b_- \in B$ (the **lowest weight element**) such that the following hold:

1. **(cr4)** For each $b \in B$ there exists a sequence $(i_1, \ldots, i_t)$ with $i_k \in I$ for all $1 \leq k \leq t$ such that
   $$b_- = \tilde{f}_{i_1} \cdots \tilde{f}_{i_t}(b).$$
2. **(cr5)** For each $b \in B$ and $i \in I$ we have
   $$\varphi_i(b) = \max\{m \mid \tilde{f}_i^m(b) \neq \emptyset\}.$$
Proposition 5.1. Fix a set $B$ with operators 

$$
\tilde{e}_i, \tilde{f}_i, \tilde{e}_i^*, \tilde{f}_i^*: B \to B \cup \{\emptyset\}.
$$

Assume $(B, \tilde{e}_i, \tilde{f}_i)$ and $(B, \tilde{e}_i^*, \tilde{f}_i^*)$ are both lowest weight crystals with the same lowest weight element $b_-$, where the other data is determined by setting $\text{wt}(b_-) = 0$. Assume further that for all $i, j \in I$ and all $b \in B$ the following hold:

(i) $\tilde{e}_i(b), \tilde{e}_i^*(b) \neq \emptyset$.

(ii) If $i \neq j$, then $\tilde{e}_j \tilde{e}_i(b) = \tilde{e}_j \tilde{e}_i^*(b)$.

(iii) For all $b \in B$ we have $\varphi_i(b) + \varphi_i^*(b) - \langle \text{wt}(b), \alpha_i \rangle \geq 0$.

(iv) If $\varphi_i(b) + \varphi_i^*(b) - \langle \text{wt}(b), \alpha_i \rangle = 0$, then $\tilde{e}_i(b) = \tilde{e}_i^*(b)$.

(v) If $\varphi_i(b) + \varphi_i^*(b) - \langle \text{wt}(b), \alpha_i \rangle \geq 1$, then $\varphi_i(\tilde{e}_i(b)) = \varphi_i(b)$ and $\varphi_i^*(\tilde{e}_i(b)) = \varphi_i^*(b)$.

(vi) If $\varphi_i(b) + \varphi_i^*(b) - \langle \text{wt}(b), \alpha_i \rangle \geq 2$, then $\tilde{e}_i \tilde{e}_i^*(b) = \tilde{e}_i^* \tilde{e}_i(b)$.

Then $(B, \tilde{e}_i, \tilde{f}_i) \cong (B, \tilde{e}_i^*, \tilde{f}_i^*) \cong B(-\infty)$.

5.3. Geometric crystal operators. As before, let 

$$
B = \bigsqcup_{r \in \mathbb{N}^n} \text{Irr}(\Pi(r))^\text{max}.
$$

We set 

$$
B_r := \text{Irr}(\Pi(r))^\text{max}.
$$

We know that $Z \cap \Pi(r)^r$ is dense in $Z$ for each $Z \in B_r$. The operators $e_{i,r}, f_{i,r}, e_{i,r}^*, f_{i,r}^*$ defined in Section 3.9 yield bijections  

$$
\tilde{f}_{i,r}: \bigsqcup_{r \in \mathbb{N}^n} \text{Irr}(\Pi(r)^i,p)^\text{max} \to \bigsqcup_{r \in \mathbb{N}^n} \text{Irr}(\Pi(r)^i,q)^\text{max}
$$

and 

$$
\tilde{f}_{i,r}^*: \bigsqcup_{r \in \mathbb{N}^n} \text{Irr}(\Pi(r)^i,p)^\text{max} \to \bigsqcup_{r \in \mathbb{N}^n} \text{Irr}(\Pi(r)^i,q)^\text{max}.
$$

where $r := p - q \geq 1$. For $Z \in B$ we set 

$$
\tilde{f}_i(Z) := \begin{cases} 
\tilde{f}_{i,1}(Z) & \text{if } \varphi_i(Z) \geq 1, \\
\emptyset & \text{otherwise},
\end{cases}
$$

and 

$$
\tilde{f}_i^*(Z) := \begin{cases} 
\tilde{f}_{i,1}^*(Z) & \text{if } \varphi_i^*(Z) \geq 1, \\
\emptyset & \text{otherwise}.
\end{cases}
$$

Similarly, we have bijections 

$$
e_{i,r}: \bigsqcup_{r \in \mathbb{N}^n} \text{Irr}(\Pi(r)^i,q)^\text{max} \to \bigsqcup_{r \in \mathbb{N}^n} \text{Irr}(\Pi(r)^i,p)^\text{max}
$$

and 

$$
e_{i,r}^*: \bigsqcup_{r \in \mathbb{N}^n} \text{Irr}(\Pi(r)^i,q)^\text{max} \to \bigsqcup_{r \in \mathbb{N}^n} \text{Irr}(\Pi(r)^i,p)^\text{max}
$$

where $r := p - q \geq 1$. For $Z \in B$ we set 

$$
\tilde{e}_i(Z) := e_{i,1}(Z)
$$

and 

$$
\tilde{e}_i^*(Z) := e_{i,1}^*(Z).
$$

Thus, we defined maps 

$$
\tilde{f}_i, \tilde{f}_i^*: B \to B \cup \{\emptyset\} \text{ and } \tilde{e}_i, \tilde{e}_i^*: B \to B.
$$

Note that our definition of the crystal operators is slightly different from the one used in [KS], see also [NT]. The reason is that we are working with a refined version of Lusztig’s bundle construction, see our discussion in Section 5.8.
For $Z \in \text{Irr}(\Pi(r))^{\text{max}}$ define
\[
\begin{align*}
\varphi_i(Z) &:= \min\{\varphi_i(M) \mid M \in Z\}, \\
\varepsilon_i(Z) &:= \varphi_i(Z) - \langle \text{wt}(Z), \alpha_i \rangle, \\
\varphi_i^*(Z) &:= \min\{\varphi_i^*(M) \mid M \in Z\}, \\
\varepsilon_i^*(Z) &:= \varphi_i^*(Z) - \langle \text{wt}(Z), \alpha_i \rangle.
\end{align*}
\]
(In the definition of $\text{wt}(Z)$, we identify the rank vector $r = (r_1, \ldots, r_n)$ with $r_1 \alpha_1 + \cdots + r_n \alpha_n \in \mathbb{R}^+ \subset P$.)

5.4. **The $*$-operator.** For a matrix $A$ let $^tA$ denote its transpose. Let $\Pi$ and $\mathcal{B}$ be defined as before. For a representation $M = (M(\varepsilon_i), M(\alpha_{ij}^g)) \in \text{nil}_E(\Pi, d)$ let
\[
S(M) := (S(M(\varepsilon_i)), S(M(\alpha_{ij}^g))) \in \text{nil}_E(\Pi, d)
\]
where
\[
S(M(\varepsilon_i)) := ^tM(\varepsilon_i) \quad \text{and} \quad S(M(\alpha_{ij}^g)) := ^tM(\alpha_{ji}^g).
\]
For each dimension vector $d$, we get an automorphism $S_d$ of the variety $\text{nil}_E(\Pi, d)$ defined by $S_d(M) := S(M)$. This construction yields an automorphism $S_r$ of $\Pi(r)$ for each rank vector $r$. The automorphism $S_r$ induces a permutation
\[
*_r : \mathcal{B}_r \rightarrow \mathcal{B}_r.
\]
This yields a permutation
\[
*: \mathcal{B} \rightarrow \mathcal{B}.
\]
For all $i \in I$ we get
\[
*_e_i = e_i^*, \quad _*e_i^* = e_i, \quad _*f_i = f_i^*, \quad _*f_i^* = f_i.
\]

5.5. **Examples.** Let $\Pi = \Pi(C, D)$ with
\[
C = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.
\]
Thus $C$ is of Dynkin type $B_2$ and $D$ is minimal. Let $Z \in \text{Irr}(\Pi((2, 1))^{\text{max}}$ be the maximal irreducible component with generic $\Pi$-module
\[
M = \frac{1}{2} \oplus \frac{1}{2}.
\]
(Each number stands for a basis vector of $M$, with $i$ belonging to $e_i M$. At the same time, $i$ represents a composition factor isomorphic to $S_i$. The module $M$ is a direct sum of two serial modules, whose composition series look as indicated.) The following picture illustrates how the various operators $\tilde{e}_k, \tilde{f}_k, \tilde{e}_k^*, \tilde{f}_k^*$ act on $Z$. 

![Diagram showing various operators acting on a module with basis vectors labeled $\frac{1}{2}, \frac{1}{1}, \frac{1}{3}$, etc.]
We also have $\hat{f}_2^*(Z) = \emptyset$. We have
\[
\varphi_1^*(Z) = 2, \quad \varphi_2^*(Z) = 0, \quad \varphi_1(Z) = 1, \quad \varphi_2(Z) = 1,
\]
\[
\dim \operatorname{Ext}_1^i(M, E_1) = c_1(\varphi_1(Z) + \varphi_1^*(Z) - \langle \omega(Z), \alpha_1 \rangle) = 2(1 + 2 - 3) = 0,
\]
\[
\dim \operatorname{Ext}_1^i(M, E_2) = c_2(\varphi_2(Z) + \varphi_2^*(Z) - \langle \omega(Z), \alpha_2 \rangle) = 1(1 + 0 - (-2)) = 3.
\]

Let $Z' \in \operatorname{Irr}(B(2, 2))$ be the maximal irreducible component with generic $B$-module
\[
M' = \frac{1}{2} \oplus \frac{2}{2}.
\]

5.6. **Realization of $B(-\infty)$.** The formula in the following lemma is an analogue of the formula in [NT] Lemma 3.16.

**Lemma 5.2.** Let $Z \in B$, and let $M$ be generic in $Z$. Then we have
\[
\dim \operatorname{Ext}_1^i(M, E_i) = c_i(\varphi_i(Z) + \varphi_i^*(Z) - \langle \omega(Z), \alpha_i \rangle).
\]

**Proof.** This follows from Corollary [2.2] Lemma 2.3 and the definitions of $\varphi_i(Z)$ and $\varphi_i^*(Z)$. $\square$

The next lemma is an analogue of [NT] Proposition 3.17.

**Lemma 5.3.** For $Z \in B$ and $i, j \in I$ the following hold:

(i) $\tilde{e}_i(Z), \tilde{e}_j^*(Z) \neq \emptyset$.
(ii) If $i \neq j$, then $\tilde{e}^*_i \tilde{e}_j(Z) = \tilde{e}^*_j \tilde{e}_i(Z)$.
(iii) For all $Z \in B$ we have $\varphi_i(Z) + \varphi_i^*(Z) - \langle \omega(Z), \alpha_i \rangle \geq 0$.
(iv) If $\varphi_i(Z) + \varphi_i^*(Z) - \langle \omega(Z), \alpha_i \rangle = 0$, then $\tilde{e}_i(Z) = \tilde{e}_i^*(Z)$.
(v) If $\varphi_i(Z) + \varphi_i^*(Z) - \langle \omega(Z), \alpha_i \rangle \geq 1$, then $\varphi_i(\tilde{e}_i^*(Z)) = \varphi_i(Z)$ and $\varphi_i^*(\tilde{e}_i(Z)) = \varphi_i^*(Z)$.
(vi) If $\varphi_i(Z) + \varphi_i^*(Z) - \langle \omega(Z), \alpha_i \rangle \geq 2$, then $\tilde{e}_i \tilde{e}_i^*(Z) = \tilde{e}_i^* \tilde{e}_i(Z)$.

**Proof.** Throughout, let $Z \in B$, and let $M \in Z$ be generic. In particular, we assume that the maps $\varphi_i$ and $\varphi_i^*$ take minimal values on $M$.

(i): This follows from the definition of $\tilde{e}_i$ and $\tilde{e}_i^*$ combined with Lemma 3.6

(ii): Let $Z_1 := \tilde{e}_i^* \tilde{e}_j(Z)$ and $Z_2 := \tilde{e}_j \tilde{e}_i^*(Z)$. Since $i \neq j$, the canonical homomorphisms
\[
\text{sub}_j(Z_k) \to \text{fac}_i(Z_k)
\]
with $k = 1, 2$ are both zero. This implies
\[
\tilde{f}_i^* \tilde{f}_j(Z_k) = \tilde{f}_j \tilde{f}_i^*(Z_k) = Z
\]
for $k = 1, 2$. Here we used Lemma 4.3(ii). Since $\tilde{f}_p \tilde{e}_p = 1_B$ and $\tilde{f}_p^* \tilde{e}_p = 1_B$ for all $p \in I$ we get that $Z_1 = Z_2$.

(iii): This follows directly from Lemma 5.2.

(iv): Assume that $\varphi_i(Z) + \varphi_i^*(Z) - \langle \omega(Z), \alpha_i \rangle = 0$. Then Lemma 5.2 yields that
\[
\operatorname{Ext}_1^i(M, E_i) = \operatorname{Ext}_1^i(E_i, M) = 0.
\]
This implies that
\[ \hat{c}_i(Z) = \tilde{c}_i^*(Z) = Z \oplus \mathcal{O}(E_i). \]
(Here we used the notion of direct sums of irreducible components from \text{[CBS].})

(v): Assume that \( \varphi_i(Z) + \varphi_i^*(Z) - (\langle w_t(Z), \alpha_i \rangle \geq 1. \) Then Lemma \ref{lem:5.2} implies that \( \dim \text{Ext}_1^\Pi(M, E_i) > 0. \) Let \( Z' := \hat{c}_i(Z). \) There is a short exact sequence
\[ 0 \to E_i \to M' \to M \to 0 \]
with \( M' \) generic in \( Z'. \) This sequence is non-split, since \( \text{Ext}_1^\Pi(M, E_i) \neq 0. \) Applying \( \text{Hom}_\Pi(-, E_i) \) we get
\[ \dim \text{fac}_i(M') - \dim \text{fac}_i(M) < \dim(E_i). \]
Since both \( \text{fac}_i(M') \) and \( \text{fac}_i(M) \) are free (using that \( M \) and \( M' \) are crystal modules), this inequality implies that \( \text{fac}_i(M') \cong \text{fac}_i(M) \) and therefore \( \text{fac}_i(Z') \cong \text{fac}_i(Z). \) This implies
\[ \varphi_i^*(\hat{c}_i(Z)) = \varphi_i^*(Z). \]
The other equality in (ii) is proved dually, working with \( Z' = \hat{c}_i(Z) \) instead of \( Z' = \hat{c}_i(Z). \)

(vi): Assume that \( \varphi_i(Z) + \varphi_i^*(Z) - \langle w_t(Z), \alpha_i \rangle \geq 2. \) Consider a generic \( M' \) in \( \hat{c}_i(Z) \) and a generic \( M'' \) in \( \tilde{c}_i^*(Z). \) We claim that the canonical homomorphism from \( f'' : \text{sub}_b(M'') \to \text{fac}_i(M'') \) is trivial. By Lemma \ref{lem:4.3}(i) it is enough to show that \( E_i \) is not a direct summand of \( M''. \) First, note that \( E_i \) cannot be a summand of \( M. \) Namely, if \( M = E_i \oplus N, \) then, since \( M \) is generic, this would imply \( \text{Ext}_1^\Pi(M, E_i) = 0, \) which is false by Lemma \ref{lem:5.2}. Consequently, since \( \text{Ext}_1^\Pi(E_i, M) > 0, \) a generic \( M' \in \hat{c}_i(Z) \) also doesn’t contain \( E_i \) as a direct summand. Thus we get a non-split short exact sequence
\[ 0 \to E_i \to M' \to M \to 0. \]
Applying \( \text{Hom}_\Pi(-, E_i) \) and keeping in mind that \( \text{Ext}_1^\Pi(E_i, E_i) = 0 \) we get
\[ \dim \text{Ext}_1^\Pi(M', E_i) \geq \dim \text{Ext}_1^\Pi(E_i, M) - c_i > 0. \]
For the second inequality we used that \( \varphi_i(Z) + \varphi_i^*(Z) - \langle w_t(Z), \alpha_i \rangle \geq 2. \) Now the same argument as before shows that \( M'' \) does not contain \( E_i \) as a direct summand. Thus we proved that \( f'' = 0. \) Now we can proceed as in the proof of part (ii). This finishes the proof.

Finally, the following theorem is an analogue of [NT, Theorem 3.18].

**Theorem 5.4.** We have
\[ (\mathcal{B}, \text{wt}, \hat{c}_i, \tilde{f}_i, \varepsilon_i, \varphi_i) \cong (\mathcal{B}, \text{wt}, \tilde{c}_i^*, \tilde{f}_i^*, \varepsilon_i^*, \varphi_i^*) \cong B(-\infty). \]

**Proof.** The set \( \mathcal{B} \) of maximal irreducible components together with either set of operators \( (\text{wt}, \hat{c}_i, \tilde{f}_i, \varepsilon_i, \varphi_i) \) or \( (\text{wt}, \tilde{c}_i^*, \tilde{f}_i^*, \varepsilon_i^*, \varphi_i^*) \) defined in Section 5.3 is a crystal. (In (cr1) we just define \( \varepsilon_i(Z) := \varphi_i(Z) - \langle w_t(Z), \alpha_i \rangle. \) The first and third equalities in (cr2) are clearly satisfied for \( \mathcal{B}. \) These together with (cr1) imply the second equality of (cr2). To check (cr3) is straightforward with the help of Lemma \ref{lem:5.6}.

For any \( 0 \neq Z \in \mathcal{B}, \) there exist \( i \) and \( j \) such that \( \tilde{f}_i(Z) \neq 0 \) and \( \tilde{f}_j^*(Z) \neq 0. \) We also know that in these cases we have \( \text{wt}(\tilde{f}_i(Z)) = w_t(Z) - \alpha_i \) and \( \text{wt}(\tilde{f}_j^*(Z)) = w_t(Z) - \alpha_j. \) For \( b_- \) we take the (unique) irreducible component \( Z_- \) of \( \Pi(0). \) (The variety \( \Pi(0) \) is just a point.) Together with the definitions of \( \varphi_i \) and \( \varphi_i^*, \) this implies that the crystals \( (\mathcal{B}, \text{wt}, \hat{c}_i, \tilde{f}_i, \varepsilon_i, \varphi_i) \) and \( (\mathcal{B}, \text{wt}, \tilde{c}_i^*, \tilde{f}_i^*, \varepsilon_i^*, \varphi_i^*) \) are both lowest weight crystals.

The conditions of Proposition \ref{prop:5.1} are all satisfied by Lemma \ref{lem:5.3} This yields isomorphisms of crystals \( B(-\infty) \cong (\mathcal{B}, \text{wt}, \tilde{c}_i^*, \tilde{f}_i^*, \varepsilon_i^*, \varphi_i^*). \) \( \square \)
5.7. Littlewood-Richardson coefficients. Let $\Pi = \Pi(C, D)$, $g = g(C)$ and $B$ be defined as before.

For $\lambda \in P^+$ a dominant integral weight, let $V(\lambda)$ be the associated irreducible integrable highest weight $g$-module with highest weight $\lambda$.

One of the main applications of crystal graphs is the calculation of tensor product multiplicities. More precisely, it is well known that the tensor product multiplicities

$$c_{\lambda, \mu}^{\nu} := \dim \left[ V(\lambda) \otimes V(\mu) : V(\nu) \right]$$

can be expressed in terms of crystal graphs. The numbers $c_{\lambda, \mu}^{\nu}$ are called Littlewood-Richardson coefficients.

For $\lambda \in P^+$ define

$$B_{\lambda} := \{ Z \in B \mid \varphi_i(Z) \leq a_i \text{ for every } i \in I \},$$

$$B_{\lambda}^* := \{ Z \in B \mid \varphi_i^*(Z) \leq a_i \text{ for every } i \in I \},$$

where $a_i := \langle \lambda, \alpha_i \rangle$.

The permutation $*: B \to B$ yields equalities

$$*(B_{\lambda}) = B_{\lambda}^* \quad \text{and} \quad *(B_{\lambda}^*) = B_{\lambda}.$$

Define

$$B_{\lambda, \mu}^{\nu} := \{ Z \in B_{\lambda}^* \cap B_{\mu} \mid \wt(Z) = \lambda + \mu - \nu \}.$$  

An example can be found in Section 8.2.6.

Using our description of $B(-\infty)$, this gives the following result.

**Proposition 5.5.**

$$c_{\lambda, \mu}^{\nu} = |B_{\lambda, \mu}^{\nu}|.$$

**Proof.** Let $B(\lambda)$ denote the crystal graph of $V(\lambda)$ with highest weight vertex $b_{\lambda}$ of weight $\lambda$. It is known [K1] Proposition 4.2 that

$$c_{\lambda, \mu}^{\nu} = |\{ b \in B(\lambda) \mid \wt(b) = \nu - \mu \text{ and } \varepsilon_i(b) \leq \langle \mu, \alpha_i \rangle \text{ for every } i \in I \}|.$$

It is also known that $B(\lambda)$ can be realized as a subgraph of $B \equiv B(-\infty)$. More precisely, it follows from [K1] Proposition 8.2 that there is a unique injective map

$$\iota_{\lambda}: B(\lambda) \to B(-\infty)$$

sending $b_\lambda$ to the lowest weight element of $B(-\infty)$ and satisfying

$$\iota_{\lambda} \varepsilon_i = \tilde{f}_i \iota_{\lambda}, \quad \varepsilon_i(b) = \varphi_i(\iota_{\lambda}(b)), \quad \wt(\iota_{\lambda}(b)) = \lambda - \wt(b), \quad (b \in B(\lambda)).$$

Moreover, we have

$$\iota(B(\lambda)) = B_{\lambda}^*.$$  

This shows that the sets

$$\iota_{\lambda}(\{ b \in B(\lambda) \mid \wt(b) = \nu - \mu \text{ and } \varepsilon_i(b) \leq \langle \mu, \alpha_i \rangle \text{ for all } i \})$$

and

$$B_{\lambda, \mu}^{\nu} = \{ Z \in B_{\lambda}^* \cap B_{\mu} \mid \wt(Z) = \lambda + \mu - \nu \}$$

are equal. □
6. Convolution algebras

In this section, assume that $K = \mathbb{C}$.

6.1. The convolution algebra $\mathcal{M}(\Pi)$. Let $\Pi = \Pi(C, D)$ and define the convolution algebra $\tilde{\mathcal{F}}(\Pi)$ as in Section 2.5. For $c_{ij} \leq 0$ we define

$$\tilde{\theta}_{ij} := \text{ad}(\tilde{\theta}_i)^{1-c_{ij}}(\tilde{\theta}_j) \in \tilde{\mathcal{M}}(\Pi).$$

Let $\mathcal{I}$ be the ideal in $\tilde{\mathcal{M}}(\Pi)$ generated by the functions $\tilde{\theta}_{ij}$ with $c_{ij} \leq 0$. Define

$$\mathcal{M}(\Pi) := \tilde{\mathcal{M}}(\Pi)/\mathcal{I}.$$

For $r \in \mathbb{N}^n$ set

$$\mathcal{I}_r := \mathcal{I} \cap \tilde{\mathcal{M}}(\Pi)_r \quad \text{and} \quad \mathcal{M}(\Pi)_r := \mathcal{M}(\Pi) \cap \tilde{\mathcal{M}}(\Pi)_r.$$

We get

$$\mathcal{I} = \bigoplus_{r \in \mathbb{N}^n} \mathcal{I}_r \quad \text{and} \quad \mathcal{M}(\Pi) = \bigoplus_{r \in \mathbb{N}^n} \mathcal{M}(\Pi)_r.$$

Let $\theta_i := \tilde{\theta}_i + \mathcal{I}$ be the residue class of $\tilde{\theta}_i$ in $\mathcal{M}(\Pi)$. It follows immediately, that we have a surjective algebra homomorphism

$$U(n) \to \mathcal{M}(\Pi)$$

defined by $e_i \mapsto \theta_i$.

6.2. Serre relations. In contrast to [GLS3, Proposition 3.10], the functions $\tilde{\theta}_i$ do not in general satisfy the Serre relations.

**Lemma 6.1.** For $\Pi = \Pi(C, D)$ assume that $c_{ij} < 0$ and $c_i \geq 2$ for some $i, j \in I$. Then there exists an indecomposable locally free $\Pi$-module $X(i, j)$ with rank vector $(1-c_{ij})\alpha_i + \alpha_j$.

**Proof.** Recall that $g_{ij} = |\gcd(c_{ij}, c_{ji})|$, $f_{ij} = |c_{ij}|/g_{ij}$ and $c_ic_{ij} = c_jc_{ji}$. It follows that $f_{ij} \leq c_j$. Without loss of generality assume $c_{12} < 0$ and $c_1 \geq 2$. For each $1 \leq f \leq f_{12}$ and $1 \leq g \leq g_{12}$ let $E_{1f}^{(g)}$ be a copy of $E_1$ with basis $\{b_1^{(g)}(i), \ldots, b_c^{(g)}(i)\}$ such that

$$\varepsilon_1 b_{1f}^{(g)} = \begin{cases} b_{i-1f}^{(g)} & \text{if } i \geq 2, \\ 0 & \text{otherwise}. \end{cases}$$

Furthermore, let $\{b_1, \ldots, b_c\}$ be a basis of another copy of $E_1$ such that

$$\varepsilon_1 b_i = \begin{cases} b_{i-1} & \text{if } i \geq 2, \\ 0 & \text{otherwise}. \end{cases}$$

Let $a_1, \ldots, a_c$ be a basis of $E_2$ such that

$$\varepsilon_2 a_i = \begin{cases} a_{i-1} & \text{if } i \geq 2, \\ 0 & \text{otherwise}. \end{cases}$$

For $1 \leq f \leq f_{12}$ and $1 \leq g \leq g_{12}$ define

$$a_{12}^{(g)} a_{c2-f+1} := b_{1f}^{(g)}$$

and

$$a_{21}^{(g)} b_1 := a_1.$$
It is easy to check that thus defines a locally free Π-module $X(1, 2)$ with $\text{rank}(X(1, 2)) = (1 - c_{12})\alpha_1 + \alpha_2$. Note that $X(1, 2)$ is a tree module in the sense of Crawley-Boevey [CB1]. In particular, $X(1, 2)$ is indecomposable. This finishes the proof. □

**Proposition 6.2.** For $\Pi = \Pi(C, D)$ the following are equivalent:

(i) The functions $\widetilde{\theta}_1, \ldots, \widetilde{\theta}_n$ satisfy the Serre relations.

(ii) $I = 0$.

(iii) If $c_{ij} < 0$ for some $i, j \in I$, then $c_i = 1$.

**Proof.** It is obvious that (i) and (ii) are equivalent.

(i) $\implies$ (iii): Assume $c_{ij} < 0$ and $c_i \geq 2$ for some $i, j \in I$. For $X(i, j)$ as defined in the proof of Lemma 6.1 it is straightforward to check that $\widetilde{\theta}_{ij}(X(i, j)) \neq 0$.

Thus $\widetilde{\theta}_1, \ldots, \widetilde{\theta}_n$ do not satisfy the Serre relations.

(iii) $\implies$ (i),(ii): Suppose (iii) holds. We can assume that $Q(C)$ is connected. If $n \geq 2$, then $C$ is symmetric, and $D$ is the identity matrix. Thus $\Pi(C, D)$ is a classical preprojective algebra associated with a quiver. If $n = 1$, then $\Pi(C, D) = K[\varepsilon_1]/(\epsilon_1^2)$. In the first case, Lusztig [L1] proved that $\widetilde{\theta}_1, \ldots, \widetilde{\theta}_n$ satisfy the Serre relations. In the second case, $I = 0$, since there are no Serre relations. □

6.3. **Example.** Let $\Pi = \Pi(C, D)$ where

$$C = \begin{pmatrix} 2 & -6 \\ -2 & 2 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}. $$

We have $c_1 = 2$, $c_2 = 6$, $f_{12} = 3$ and $g_{12} = 2$. The $\Pi$-module $X(1, 2)$ constructed in the proof of Lemma 6.1 looks as follows:

(The numbers in the picture correspond to basis vectors of $X(1, 2)$ with $i$ being in $e_i X(1, 2)$. The arrows indicate how the arrows of the quiver of $\Pi$ act on the basis vectors.) We get

$$\widetilde{\theta}_{12}(X(1, 2)) = \text{ad}(\tilde{\theta}_1)^7(\tilde{\theta}_2)^7(X(1, 2)) = -7\tilde{\theta}_1^6\tilde{\theta}_2\tilde{\theta}_1(X(1, 2)) = -7 \cdot (6!) = -(7!).$$

Thus we see that $I \neq 0$.

As a smaller example, one could also take the preprojective algebra $\Pi$ of type $B_2$ with minimal symmetrizer together with the module $X$ displayed in Section 8.2.4.

6.4. **The support of the Serre relations.**

**Lemma 6.3.** Suppose $c_{ij} \leq 0$. Then there is no indecomposable crystal module $M \in \text{nil}_E(\Pi)$ with $\text{rank}(M) = (1 - c_{ij}, 1)$. 
Proof. Without loss of generality assume $c_{i2} \leq 0$. Let $r = (1 - c_{i2}, 1)$, and let $M \in \text{nil}_E(\Pi)$ be a crystal module with $\text{rank}(M) = r$.

We consider the maps

$$M_1 \xrightarrow{M_{1,\text{out}}} \widetilde{M}_1 \xrightarrow{M_{1,\text{in}}} M_1$$

as defined in Section 2.3. The maps $M_{1,\text{out}}$ and $M_{1,\text{in}}$ are $H_1$-module homomorphisms with $\text{Im}(M_{1,\text{out}}) \subseteq \text{Ker}(M_{1,\text{in}})$. Since $M$ is a crystal module, we know that $M_{1,\text{out}}$ and $M_{1,\text{in}}$ are split, i.e. their images, kernels and cokernels are free $H_1$-modules and therefore direct summands. As $H_1$-modules, we have $M_1 \cong H_1^{1-c_{i2}}$ and $\widetilde{M}_1 = 1 H_2 \otimes 2 H_2 \cong H_1^{-c_{i2}}$.

Let $r_1 := \text{rank}(\text{Im}(M_{1,\text{out}}))$ and $r_2 := \text{rank}(\text{Im}(M_{1,\text{in}}))$. Since $\text{Im}(M_{1,\text{out}}) \subseteq \text{Ker}(M_{1,\text{in}})$, we get $r_1 + r_2 \leq -c_{i2}$. Let $C$ be a submodule of $M_1$ such that $\text{Im}(M_{1,\text{in}}) \oplus C = M_1$. Thus $C \cong \text{Cok}(M_{1,\text{in}})$. We have $\text{rank}(\text{Ker}(M_{1,\text{out}})) = (1 - c_{i2}) - r_1$ and $\text{rank}(C) = (1 - c_{i2}) - r_2$. Thus $\text{Ker}(M_{1,\text{out}}) \cap C$ contains a free submodule $U$ isomorphic to $H_1$. (Here we use the following fact: If $V_1$ and $V_2$ are free submodules of $H_1^m$ with $\text{rank}(V_1) + \text{rank}(V_2) \geq m + 1$, then $V_1 \cap V_2$ contains a free submodule $V$ with $\text{rank}(V) = 1$. Namely, there is a short exact sequence

$$0 \to V_1 \cap V_2 \to V_1 \oplus V_2 \xrightarrow{f} V_1 + V_2 \to 0$$

with $f(v_1, v_2) := v_1 - v_2$. We have $\text{dim top}(V_1 + V_2) \leq m$, since $V_1 + V_2 \subseteq H_1^m$. The module $V_1 \oplus V_2$ is a projective $H_1$-module with $\text{dim top}(V_1 \oplus V_2) \geq m + 1$. Thus $V_1 \cap V_2$ contains a direct summand isomorphic to $H_1$. It follows that $U$ is a direct summand of $M$. Thus the $\Pi$-module $M$ is decomposable. This finishes the proof. \qed

Corollary 6.4. Suppose $c_{i2} \leq 0$. For each crystal module $M \in \Pi(r)$ we have

$$\widetilde{\theta}_{ij}(M) = 0.$$

Proof. Since $\widetilde{\theta}_{ij}$ is defined as an iterated Lie bracket of the generators $\widetilde{\theta}_i$ and $\widetilde{\theta}_j$, it is a primitive element in the Hopf algebra $\tilde{\mathcal{M}}(\Pi)$. Thus the support of $\widetilde{\theta}_{ij}$ consists of indecomposable $\Pi$-modules. Let $r = (1 - c_{i2}, 1)$, and let $M \in \Pi(r)^{\text{cr}}$. By Lemma 6.3 we know that $M$ is decomposable. Thus we get $\theta_{ij}(M) = 0$. \qed

Corollary 6.5. For $c_{ij} \leq 0$ and $r = (1 - c_{ij}\alpha_i + \alpha_j)$, we have

$$\text{dim sup}(\widetilde{\theta}_{ij}) < \text{dim} H(r).$$

One can see Corollary 6.5 as a first step towards a proof of Conjecture 1.4.

7. Semicanonical bases

In this section, assume that $K = \mathbb{C}$.

7.1. Semicanonical functions. This section follows very closely Lusztig [L2]. Most of Lusztig's proofs translate almost literally to our more general setup.

Let $Z \in \text{Irr}(\Pi(r))$. Then for each $f \in \mathcal{M}(\Pi)$ there exists a unique $c \in \mathbb{Z}$ such that $f^{-1}(c) \cap Z$ contains a dense open subset of $Z$. The map $f \mapsto c$ yields a linear map

$$\rho_Z : \mathcal{M}(\Pi) \to \mathbb{Z}.$$
Lemma 7.1. Let \( Z \in \mathrm{Irr}(\Pi(\mathbf{r}))^{\text{max}} \). There exists some \( \tilde{f}_Z \in \tilde{\mathcal{M}}(\Pi)_{\mathbf{r}} \) such that for each \( Z' \in \mathrm{Irr}(\Pi(\mathbf{r}))^{\text{max}} \) we have
\[
\rho_{Z'}(\tilde{f}_Z) = \begin{cases} 
1 & \text{if } Z = Z', \\
0 & \text{otherwise.}
\end{cases}
\]

Proof. We argue by induction on \( \mathbf{r} = (r_1, \ldots, r_n) \). When \( \mathbf{r} = 0 \), the result is trivial. Hence we may assume that \( \mathbf{r} \neq 0 \) and that the result is known for all smaller rank vectors. (This is the first induction hypothesis.) For our \( \mathbf{r} \) we fix \( i \in I \) and we shall prove the following:

(a) The lemma holds for any \( Z \in \mathrm{Irr}(\Pi(\mathbf{r}))^{\text{max}} \) such that \( \varphi_i^*(Z) > 0 \).

We argue by descending induction on \( \varphi_i^*(Z) \). Since \( \varphi_i^*(Z) \leq r_i \), we may assume that \( \varphi_i^*(Z) = p > 0 \) and that (a) holds for any \( \tilde{Z} \in \mathrm{Irr}(\Pi(\mathbf{r}))^{\text{max}} \) such that \( \varphi_i^*(\tilde{Z}) > p \). (Thus is the second induction hypothesis.)

Note that
\[
Z^{i,p} := Z \cap \Pi(\mathbf{r})^{i,p}
\]
is open and dense in \( Z \). Using the results in Section 3, we get that \( Z^{i,p} \in \mathrm{Irr}(\Pi(\mathbf{r})^{i,p})^{\text{max}} \).

By Lemma 3.6, \( Z^{i,p} \) corresponds to some \( Z_1 \in \mathrm{Irr}(\Pi(\mathbf{s})^{i,0}) \) with \( \mathbf{s} := \mathbf{r} - p\mathbf{a}_i \).

\[
\begin{array}{ccc}
Y & \xrightarrow{p''} & \Pi(\mathbf{r})^{i,p} \\
\Pi(\mathbf{s})^{i,0} \times J_0 & \xrightarrow{p'} & \end{array}
\]

Let \( \overline{Z}_1 \) be the closure of \( Z_1 \) in \( \Pi(\mathbf{s}) \). Theorem 4.1 implies that \( \overline{Z}_1 \in \mathrm{Irr}(\Pi(\mathbf{s}))^{\text{max}} \).

By the first induction hypothesis, there exists \( \tilde{g} \in \tilde{\mathcal{M}}(\Pi)_{\mathbf{s}} \) such that \( \rho_{\overline{Z}_1}(\tilde{g}) = 1 \) and \( \rho_{Z_2}(\tilde{g}) = 0 \) for any \( Z_2 \in \mathrm{Irr}(\Pi(\mathbf{s}))^{\text{max}} \setminus \{\overline{Z}_1\} \). In other words, we have
\[
\tilde{g} = \tilde{f}_Z^{i,0}.
\]

For each \( M \in \Pi(\mathbf{r})^{i,p} \) there is a uniquely determined submodule \( U \) of \( M \) such that \( M/U \cong E_{i,p}^f \). We obviously have \( U \in \Pi(U)^{i,0} \) for some locally free \( U = (U_1, \ldots, U_n) \) with \( \mathrm{rank}(U) = \mathbf{s} \). We identify \( \Pi(U) \) and \( \Pi(\mathbf{s}) \) and consider \( U \) as an element in \( \Pi(\mathbf{s}) \).

Let
\[
\tilde{g}^{i,p} : \Pi(\mathbf{r})^{i,p} \to Z
\]
be defined by \( \tilde{g}^{i,p}(M) := \tilde{g}(U) \).

Let
\[
\tilde{f} := \tilde{g} \ast 1_{E_{i,p}^f} \in \tilde{\mathcal{M}}(\Pi)_{\mathbf{r}}.
\]

From the definitions we see that

(b) \( \tilde{f}_{|\Pi(\mathbf{r})^{i,p}} = \tilde{g}^{i,p} \);

(c) If \( \tilde{f}(M) \neq 0 \) for some \( M \in \Pi(\mathbf{r}) \), then \( M \in \Pi(\mathbf{r})^{i,p'} \) for some partition \( \mathbf{p}' \) with \( \mathbf{p}'(c_i) \geq p \).

Using (b) and the definitions we see that \( \rho_{Z}(\tilde{f}) = 1 \) and \( \rho_{Z'}(\tilde{f}) = 0 \) for all \( Z' \in \mathrm{Irr}(\Pi(\mathbf{r}))^{\text{max}} \setminus \{Z\} \) such that \( \varphi_i^*(Z') = p \).

Using (c), we see that \( \rho_{Z'}(\tilde{f}) = 0 \) for all \( Z' \in \mathrm{Irr}(\Pi(\mathbf{r}))^{\text{max}} \) such that \( \varphi_i^*(Z') < p \). By the second induction hypothesis, for all \( Z' \in \mathrm{Irr}(\Pi(\mathbf{r}))^{\text{max}} \) such that \( \varphi_i^*(Z') > p \)
we can find a function \( \tilde{f}_{Z'} \in \tilde{\mathcal{M}}(\Pi) \) such that \( \rho_{Z'}(\tilde{f}_{Z'}) = 1 \) and \( \rho_Z(\tilde{f}_{Z'}) = 0 \) for any \( \tilde{Z} \in \text{Irr}(\Pi(\mathbf{r}))^{\text{max}} \setminus \{Z'\} \).

Let \( \tilde{f}_Z := \tilde{f} - \sum_{Z'} \rho_{Z'}(\tilde{f}) \tilde{f}_{Z'} \)

where \( Z' \) runs over all irreducible components in \( \text{Irr}(\Pi(\mathbf{r}))^{\text{max}} \) with \( \varphi^*_i(Z) > p \).

We have \( \tilde{f}_Z \in \tilde{\mathcal{M}}(\Pi) \). It is clear that \( \tilde{f}_Z \) satisfies the requirements of the lemma. Thus (a) is proved (assuming the first induction hypothesis). Now, by Lemma 3.1(c) we know that any \( Z \in \text{Irr}(\Pi(\mathbf{r}))^{\text{max}} \) satisfies \( \varphi^*_i(Z) > 0 \) for some \( i \). Hence the lemma holds for \( Z \) (assuming the first induction hypothesis). This provides the induction step. The lemma is proved. \( \Box \)

Let us stress that the inductive construction of the maps \( \tilde{f}_Z \) in the proof of Lemma 7.1 involves the choice of some \( i \) with \( \varphi^*_i(Z) > 0 \).

**Theorem 7.2.** For each \( \mathbf{r} \in \mathbb{N}^n \) we have

\[
\dim(U(\mathbf{n})_\mathbf{r}) = |\text{Irr}(\Pi(\mathbf{r}))^{\text{max}}|.
\]

**Proof.** This follows from our geometric realization of the crystal graph \( B(-\infty) \) (see Theorem 5.4) combined with the groundbreaking results in [K2]. \( \Box \)

Recall that

\[
\mathcal{B} = \bigcup_{\mathbf{r} \in \mathbb{N}^n} \text{Irr}(\Pi(\mathbf{r}))^{\text{max}}.
\]

Slightly rephrasing Lemma 7.1, we proved the following theorem.

**Theorem 7.3.** The convolution algebra \( \tilde{\mathcal{M}}(\Pi) \) contains a set

\[
\tilde{\mathcal{S}} := \{\tilde{f}_Z \mid Z \in \mathcal{B}\}
\]

of constructible functions such that for each \( Z' \in \mathcal{B} \) we have

\[
\rho_{Z'}(\tilde{f}_Z) = \begin{cases} 
1 & \text{if } Z = Z', \\
0 & \text{otherwise}.
\end{cases}
\]

Recall that \( \mathcal{I} \) is the ideal in \( \tilde{\mathcal{M}}(\Pi) \) generated by the elements \( \tilde{\theta}_{ij} \) with \( c_{ij} \leq 0 \), and that

\[
\mathcal{M}(\Pi) := \tilde{\mathcal{M}}(\Pi)/\mathcal{I}.
\]

As mentioned in Section 2.5, the convolution algebra \( \tilde{\mathcal{M}}(\Pi) \) is a Hopf algebra with comultiplication \( \tilde{\mathcal{M}}(\Pi) \to \tilde{\mathcal{M}}(\Pi) \otimes \tilde{\mathcal{M}}(\Pi) \) defined by \( \tilde{\theta}_i \to \tilde{\theta}_i \otimes 1 + 1 \otimes \tilde{\theta}_i \). Furthermore, \( \tilde{\mathcal{M}}(\Pi) \) is isomorphic to the universal enveloping algebra \( U(\mathcal{P}(\mathcal{M}(\Pi))) \) of the Lie algebra \( \mathcal{P}(\mathcal{M}(\Pi)) \) of primitive elements in \( \tilde{\mathcal{M}}(\Pi) \).

The surjective algebra homomorphism \( \tilde{\mathcal{M}}(\Pi) \to \mathcal{M}(\Pi) \) defined by \( \tilde{\theta}_i \to \theta_i \) yields a Hopf algebra structure on \( \mathcal{M}(\Pi) \) with comultiplication defined by \( \theta_i \to \theta_i \otimes 1 + 1 \otimes \theta_i \).
7.2. **Proof of Theorem 1.5.** Let

\[ B_r := \text{Irr}(\Pi(r))^{\text{max}}, \]
\[ \tilde{S}_r := \tilde{\mathcal{S}}(C, D)_r := \{ f_Z | Z \in B_r \} \subset \tilde{\mathcal{M}}(\Pi)_r, \]
\[ S_r := S(C, D)_r := \{ f_Z | Z \in B_r \} \subset M(\Pi)_r \text{ where } f_Z := \tilde{f}_Z + I. \]

We have disjoint unions

\[ B = \bigcup_{r \in \mathbb{N}^n} B_r, \quad \tilde{S} := \tilde{\mathcal{S}}(C, D) = \bigcup_{r \in \mathbb{N}^n} \tilde{S}_r, \quad S := S(C, D) = \bigcup_{r \in \mathbb{N}^n} S_r. \]

**Theorem 7.4.** Assume that Conjecture 1.4 is true. For \( \Pi = \Pi(C, D), \ n = n(C) \) and \( S = S(C, D) \) the following hold:

(i) There is a Hopf algebra isomorphism

\[ \eta_\Pi : U(n) \to M(\Pi) \]

defined by \( e_i \mapsto \theta_i \).

(ii) Via the isomorphism \( \eta_\Pi \), \( S_r \) is a \( \mathbb{C} \)-basis of \( U(n)_r \), and \( S \) is a \( \mathbb{C} \)-basis of \( U(n) \).

(iii) For \( 0 \neq f \in \tilde{\mathcal{M}}(\Pi) \) the following are equivalent:

(a) \( f \in I \).

(b) \( f \) has non-maximal support.

**Proof.** There is a surjective algebra homomorphism

\[ \eta_\Pi : U(n) \to M(\Pi) \]

defined by \( e_i \mapsto \theta_i \). (Dividing \( \tilde{\mathcal{M}}(\Pi) \) by the ideal \( I \) forces the algebra generators \( \theta_i \) of \( M(\Pi) \) to satisfy the Serre relations.) It is also clear that \( \eta_\Pi \) induces a surjective \( K \)-linear map

\[ \eta_{\Pi, r} : U(n)_r \to M(\Pi)_r. \]

As an immediate consequence of Theorem 7.3 the set \( \tilde{S}_r \) is linearly independent in \( \tilde{\mathcal{M}}(\Pi)_r \). Theorem 5.4 implies that

\[ |\tilde{S}_r| = \dim U(n)_r. \]

Assume that

\[ f := \sum_{Z \in B_r} \lambda_Z f_Z = 0 \]

for some \( \lambda_Z \in K \). It follows that

\[ \sum_{Z \in B_r} \lambda_Z \tilde{f}_Z \in I. \]

By our assumption that Conjecture 1.4 holds, it follows that \( \lambda_Z = 0 \) for all \( Z \).

It follows that the set \( S_r \) is linearly independent in \( M(\Pi)_r \). So for dimension reasons,

\[ \eta_{\Pi, r} : U(n)_r \to M(\Pi)_r \]

is an isomorphism of \( \mathbb{C} \)-vector spaces, and therefore \( \eta_\Pi \) is an algebra isomorphism.

It also follows that \( S_r \) is a \( \mathbb{C} \)-basis of \( U(n)_r \), and \( S \) is a \( \mathbb{C} \)-basis of \( U(n) \). Thus we proved (ii).

As a \( K \)-vector space we get a direct sum decomposition

\[ \tilde{\mathcal{M}}(\Pi)_r = U_r \oplus I_r \]
where \( \mathcal{U}_r \) is the subspace generated by \( \widetilde{S}_r \). Each function in \( \mathcal{U}_r \) has maximal support, and by our assumption that Conjecture 1.4 holds, each function in \( \mathcal{I}_r \) has non-maximal support.

Clearly, for each sum \( h := f + g \) with \( f \in \mathcal{U}_r \) and \( g \in \mathcal{I}_r \), we have that \( h \) has non-maximal support if and only if \( f = 0 \). This finishes the proof of (iii).

The enveloping algebra \( U(n) \) is a Hopf algebra with comultiplication \( U(n) \rightarrow U(n) \otimes U(n) \) defined by \( e_i \mapsto e_i \otimes 1 + 1 \otimes e_i \). The algebra isomorphism \( \eta_{\Pi}: U(n) \rightarrow M(\Pi) \) is obviously a Hopf algebra isomorphism. This finishes the proof of (i). \( \square \)

For \( \Pi = \Pi(C, D) \) and \( n = n(C) \) we call \( S = S(C, D) \) the semicanonical basis of \( U(n) \).

**Proposition 7.5.** Assume that Conjecture 1.4 is true. Let \( \widetilde{S} = \{ \widetilde{f}_Z \mid Z \in \mathcal{B} \} \) and \( \widetilde{G} = \{ \widetilde{g}_Z \mid Z \in \mathcal{B} \} \) be subsets of \( \widetilde{M}(\Pi) \) satisfying

\[
\rho_{Z'}(\widetilde{f}_Z - \widetilde{g}_Z) = \begin{cases} 1 & \text{if } Z = Z', \\ 0 & \text{otherwise} \end{cases}
\]

for all \( Z, Z' \in \mathcal{B} \). Then \( \widetilde{f}_Z - \widetilde{g}_Z \in \mathcal{I} \).

**Proof.** By definition we have

\[ \rho_{Z'}(\widetilde{f}_Z - \widetilde{g}_Z) = 0 \]

for all \( Z' \in \mathcal{B} \). This implies \( \dim \text{supp}(\widetilde{f}_Z - \widetilde{g}_Z) < \dim H(r) \) for all \( Z \in \mathcal{B}_r \). By Theorem 7.4(iii) we get \( \widetilde{f}_Z - \widetilde{g}_Z \in \mathcal{I} \). \( \square \)

### 7.3. Semicanonical bases for irreducible integrable highest weight modules

Let \( \Pi = \Pi(C, D) \), \( g = g(C) \), \( n = n(C) \) and \( \mathcal{B} \) be defined as before. Assume that Conjecture 1.4 is true.

Recall that for \( \lambda \in P^+ \) a dominant integral weight, \( V(\lambda) \) denotes the irreducible integrable highest weight \( g \)-module with highest weight \( \lambda \).

In view of Theorem 7.4, we can then identify \( \mathcal{M}(\Pi) \) with \( U(n) \), and we consider the semicanonical basis \( S = S(C, D) \) of \( \mathcal{M}(\Pi) \) as a basis of \( U(n) \).

Let \( \lambda \in P^+ \) be a dominant integral weight. Fix a highest weight vector \( v_\lambda \in V(\lambda) \). Furthermore, let \( x \mapsto x^{-} \) denote the algebra automorphism of \( U(g) \) defined by

\[ e_i^- := f_i, \quad f_i^- := e_i, \quad h^- := -h, \quad (i \in I, \ h \in \mathfrak{h}). \]

We then have a surjective homomorphism of \( U(n) \)-modules

\[ \pi_\lambda: U(n) \rightarrow V(\lambda) \]

defined by \( x \mapsto x^{-}v_\lambda \).

**Proposition 7.6.** Assume that Conjecture 1.4 is true. For each \( \lambda \in P^+ \) the following hold:

(i) \( \pi_\lambda(f_Z) = 0 \) if and only if \( Z \notin \mathcal{B}_\lambda^* \).

(ii) \( \mathcal{S}_\lambda := \{ \pi_\lambda(f_Z) \mid Z \in \mathcal{B}_\lambda^* \} \) is a basis of \( V(\lambda) \).
Proof. This is similar to [L2, Section 3], so we will only sketch the argument. It follows from the proof of Lemma 7.1 that for every \( Z \in B \),
\[
f_Z \epsilon_i^p/p! = f_{Z'} + \sum_{Z''} \mu_{Z''} f_{Z''},
\]
where \( Z' = (\tilde{e}_i^*)^p(Z) \), and the sum is over \( Z'' \) with \( \varphi_i^*(Z'') > \varphi_i^*(Z) \). (The function \( 1_{E_i^p} \) in \( G_{II} \) corresponds to \( e_i^p/p! \) in \( U(n) \).) This implies that the left ideal \( U(n)e_i^p \) is contained in the subspace spanned by \( \{ f_Z \mid \varphi_i^*(Z) \geq d \} \). More generally, if \( d = (d_i) \in \mathbb{N}^I \), we have
\[
\sum_{i \in I} U(n)e_i^{d_i} \subseteq W_d := \text{Span}\{ f_Z \mid \varphi_i^*(Z) \geq d_i \text{ for some } i \in I \}.
\]
Conversely, consider \( f_Z \in S \) such that \( \varphi_i^*(Z) = p \). Using again the proof of Lemma 7.1, we get that
\[
f_Z = f_{Z'} \epsilon_i^p/p! + \sum_{Z''} \nu_{Z''} f_{Z''},
\]
where \( Z' = (\tilde{f}_i^*)^p(Z) \), and the sum is over \( Z'' \) with \( \varphi_i^*(Z'') > \varphi_i^*(Z) \). Using descending induction on \( p \), it follows that
\[
\sum_{i \in I} U(n)e_i^{d_i} \supseteq W_d.
\]
Hence the left ideal \( \sum_{i \in I} U(n)e_i^{d_i} \) coincides with the subspace \( W_d \) spanned by a subset of \( S \).

Now it is known that
\[
\text{Ker}(\pi_\lambda) = \sum_{i \in I} U(n)e_i^{a_i+1}.
\]
Therefore \( \text{Ker}(\pi_\lambda) = W_d \) with \( d = (a_i + 1) \), that is,
\[
\text{Ker}(\pi_\lambda) = \text{Span}\{ f_Z \mid Z \notin B_\lambda^* \}
\]
and the proposition follows. \( \square \)

8. Examples

8.1. Maximal irreducible components for the Dynkin cases. Let \( \Pi = \Pi(C, D) = \Pi(C, D, \Omega) \). Let \( Q = Q(C, \Omega) = (I, Q_1, s, t) \) be the full subquiver of \( \overline{Q}(C) = (I, \overline{Q}_1, s, t) \) with arrow set
\[
Q_1 = \{ \alpha_{ij}^{(g)} \in \overline{Q}_1 \mid (i, j) \in \Omega, 1 \leq g \leq g_{ij} \} \cup \{ \varepsilon_i \mid i \in I \}.
\]
Let \( H = H(C, D, \Omega) \) be the subalgebra of \( \Pi \) given by \( Q(C, \Omega) \). Thus we have
\[
H = KQ/J
\]
where \( KQ \) is the path algebra of \( Q \) and \( J \) is the ideal defined by the following relations:

(H1) For each \( i \) we have the nilpotency relation
\[
\varepsilon_i^{c_i} = 0.
\]
(H2) For each \( (i, j) \in \Omega \) and each \( 1 \leq g \leq g_{ij} \) we have the commutativity relation
\[
\varepsilon_i^{f_{ij}} \alpha_{ij}^{(g)} = \alpha_{ij}^{(g)} \varepsilon_j^{f_{ij}}.
\]
There is an obvious embedding \( \text{rep}(H) \to \text{rep}(\Pi) \). Thus each \( H \)-module can be seen as a \( \Pi \)-module. Let \( TC^+ : \text{rep}(H) \to \text{rep}(H) \) denote the twisted Coxeter functor defined in [GLS1].

As before, for a dimension vector \( d \) let \( \text{rep}(H, d) \) and \( \text{rep}(\Pi, d) \) be the varieties of representation of \( H \)-modules \( \Pi \)-modules with dimension vector \( d \), respectively.

Let \( \text{rep}_l.f.(H, d) \subseteq \text{rep}(H, d) \) and \( \text{rep}_l.f.(\Pi, d) \subseteq \text{rep}(\Pi, d) \) denote the subvarieties of locally free modules.

Let

\[ \pi_H : \text{rep}(\Pi, d) \to \text{rep}(H, d) \]

be the obvious restriction map.

**Proposition 8.1.** For each \( M \in \text{rep}_l.f.(H, d) \) we have

\[ \pi_H^{-1}(M) \cong \text{Hom}_H(M, TC^+(M)). \]

**Proof.** Using [GLS1] one can adapt the construction in [R] to obtain the result. \( \square \)

Recall from [GLS1] that for all \( M \in \text{rep}_l.f.(H) \) we have a functorial isomorphism

\[ TC^+(M) \cong \text{D Ext}^1_H(M, H) \cong \tau_H(M). \]

**Proposition 8.2.** For \( M \in \text{rep}_l.f.(H, d) \) we have

\[ \dim \pi_H^{-1}(\mathcal{O}(M)) = \dim \text{rep}_l.f.(H, d). \]

**Proof.** The proof is based in Proposition 8.1. For \( M \in \text{rep}_l.f.(H, d) \) we have

\[ \dim \mathcal{O}(M) + \dim \text{Hom}_H(M, TC^+(M)) = \dim \mathcal{O}(M) + \dim \text{Ext}^1_H(M, M) \]
\[ = \sum_{i \in I} d_i^2 - \dim \text{End}_H(M) + \dim \text{Ext}^1_H(M, M) \]
\[ = \sum_{i \in I} d_i^2 - \sum_{i \in I} c_i a_i^2 + \sum_{(j, i) \in \Omega} c_i |c_{ij}| a_i a_j \]
\[ = \dim \text{rep}_l.f.(H, d). \]

Here \( (a_1, \ldots, a_n) \) is the rank vector of \( M \) The first equality follows since \( TC^+(M) \cong \tau_H(M) \) for \( M \in \text{rep}_l.f.(H) \) and by the Auslander-Reiten formulas. The second equality is just the general formula for orbit dimensions in representation varieties of algebras, the third equality holds by [GLS1], Proposition 4.1 and the last equality follows from [GLS2], Proposition 3.1. The result follows. \( \square \)

Assume now that \( C \) is of Dynkin type. We assume also that the orientation \( \Omega \) is acyclic, i.e. that for each sequence \( (i_1, i_2, i_3, \ldots, i_t, i_{t+1}) \) with \( t \geq 1 \) and \( (i_s, i_{s+1}) \in \Omega \) for all \( 1 \leq s \leq t \) we have \( i_1 \neq i_{t+1} \).

For each positive root \( \alpha \in \Delta^+(C) \) there is a (unique) indecomposable preprojective \( H \)-module \( M_\alpha \) with \( \text{rank}(M_\alpha) = \alpha \), see [GLS1]. For a **Kostant partition** \( \nu = (n_\alpha) \in \mathbb{N}^{\Delta^+(C)} \) let

\[ M_\nu := \bigoplus_{\alpha \in \Delta^+(C)} M_\alpha^{n_\alpha} \]

be the preprojective \( H \)-module associated with \( \nu \), and let

\[ d(\nu) := \sum_{\alpha \in \Delta^+(C)} n_\alpha \text{dim}(M_\alpha). \]
Furthermore, set
\[ Z_\nu := \pi_H^{-1}(O(M_\nu)) \subseteq \text{rep}(\Pi, d(\nu)). \]

**Lemma 8.3.** Let \( \Pi = \Pi(C, D) \) and \( H = H(C, D, \Omega) \). For each Kostant partition \( \nu \in \Delta^+(C) \) we have
\[ Z_\nu \in \text{Irr}(\text{nil}_E(\Pi, d(\nu)))^{\max}. \]

**Proof.** By definition we have \( Z_\nu \subseteq \text{rep}(\Pi, d(\nu)) \). We know that \( \dim(Z_\nu) = \dim \text{rep}(H, d(\nu)) \).

It remains to show that each \( X \in Z_\nu \) is \( E \)-filtered.

For brevity let \( F := TC^+ \), where \( T \) is the twist functor and \( C^+ \) is the Coxeter functor, see [GLS1]. We know that the category \( \text{rep}(\Pi) \) can be identified with the category of \( H \)-module homomorphisms \( f: M \to F(M) \). For \( M \in \text{rep}(H) \) we have \( M \cong (0: M \to F(M)) \).

Given such an \( f \) let \( (M, f) \) be the corresponding \( \Pi \)-module.

Now assume that \( M = M_\nu \) is a preprojective \( H \)-module. Thus we have
\[ M \cong \bigoplus_{\alpha \in \Delta^+(C)} M_{\alpha}^{n_{\alpha}} \]
for some \( n_{\alpha} \geq 0 \). There exists some \( \beta \) with \( n_\beta \neq 0 \) such that \( \text{Hom}_H(M_\beta, \tau_H(M)) = 0 \).

It follows that \( 0: M_{\beta}^{n_\beta} \to F(M_{\beta}^{n_\beta}) \) is a submodule of \( (0: M \to F(M)) \) with factor module of the form \( f: M/M_{\beta}^{n_\beta} \to F(M/M_{\beta}^{n_\beta}) \). The \( \Pi \)-module \( 0: M_{\beta}^{n_\beta} \to F(M_{\beta}^{n_\beta}) \) is \( E \)-filtered, since \( M_\beta \) is \( E \)-filtered. Now the result follows by induction. \( \square \)

**Theorem 8.4.** Let \( \Pi = \Pi(C, D) \) with \( C \) of Dynkin type. For \( Z \in \text{Irr}(\text{nil}_E(\Pi, d)) \) the following are equivalent:

(i) \( Z \) is maximal.
(ii) \( Z = Z_\nu \) for some Kostant partition \( \nu = (n_\alpha) \in \mathbb{N}^{\Delta^+(C)} \) with \( d(\nu) = d \).

**Proof.** Let \( M_\nu \) be a preprojective \( H \)-module in the sense of [GLS1], and let \( r = \text{rank}(M_\nu) \).

By Lemma 8.3 we have \( Z_\nu \in \text{Irr}(\text{nil}_E(\Pi, d(\nu)))^{\max} \).

For preprojective \( H \)-modules \( M_\nu \) and \( N_\mu \) we clearly have \( Z_\nu = Z_\mu \) if and only if \( M_\nu \cong M_\mu \).

By our geometric realization of \( B(-\infty) \) we know that
\[ \dim \text{U}(n)_r = |\text{Irr}(\text{nil}_E(\Pi, d))^{\max}|. \]

Furthermore, the number of isomorphism classes of preprojective \( H \)-modules \( M \) with \( \text{rank}(M) = r \) is exactly \( \dim \text{U}(n)_r \). This follows from [GLS1 Section 11.2]. This finishes the proof. \( \square \)

8.2. **Type \( B_2 \).**

8.2.1. **The preprojective algebra of type \( B_2 \).** For the whole Section 8.2 let \( \Pi = \Pi(C, D) = \Pi(C, D, \Omega) \) with
\[ C = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \]
and \( \Omega = \{(1, 2)\} \). Set \( H = H(C, D, \Omega) \) and \( H^* = H(C, D, \Omega^*) \). Thus \( C \) is a Cartan matrix of Dynkin type \( B_2 \), and the symmetrizer \( D \) is minimal. We have \( \Pi = K \overline{Q}/T \) where \( \overline{Q} = \overline{Q}(C) \) is the quiver

\[
\varepsilon_1 \quad \begin{array}{c} \alpha_{21} \\ \alpha_{12} \end{array} \quad 1 \quad \begin{array}{c} \alpha_{21} \\ \alpha_{12} \end{array} \quad 2
\]

and \( T \) is generated by the set

\[
\{ \varepsilon_1^2, \alpha_{12}\alpha_{21}\varepsilon_1 + \varepsilon_1\alpha_{12}\alpha_{21}, -\alpha_{21}\alpha_{12} \}.
\]

Thus \( \Pi \) is a finite-dimensional special biserial algebra. The modules and the AR-quiver of a special biserial algebra can be determined combinatorially, see for example [BR]. The indecomposable \( \Pi \)-modules are either projective-injective, or string modules, or band modules. The band modules are locally free, but they are not \( E \)-filtered.

The indecomposable projective \( \Pi \)-modules are shown in Figure 1. (The arrows indicate when an arrow of the algebra \( \Pi \) acts with a non-zero scalar on a basis vector.)

![Figure 1. The indecomposable projective \( \Pi(C, D) \)-modules for type \( B_2 \).](image)

All results in Section 8.2 can be proved by using the classification of finite-dimensional indecomposable \( \Pi \)-modules.

8.2.2. Irreducible components. Up to isomorphism there are 8 indecomposable rigid \( \Pi \)-modules, namely

\[
\begin{align*}
P_1 &= \frac{1}{2} \frac{1}{1} \frac{1}{2}, & P_2 &= \frac{1}{2} \frac{1}{1} \frac{1}{2}, & E_1 &= \frac{1}{1} \frac{1}{1}, & E_2 &= 2, \\
T_1 &= \frac{1}{2} \frac{1}{1} \frac{1}{2}, & T_2 &= \frac{1}{2} \frac{1}{1} \frac{1}{2}, & T_3 &= \frac{1}{2} \frac{1}{1} \frac{1}{2}, & T_4 &= \frac{1}{2} \frac{1}{1} \frac{1}{2}.
\end{align*}
\]

The following is a complete list of basic maximal rigid \( \Pi \)-modules:

\[
\begin{align*}
P_1 \oplus P_2 \oplus E_1 \oplus T_1, & \quad P_1 \oplus P_2 \oplus E_1 \oplus T_2, & \quad P_1 \oplus P_2 \oplus E_2 \oplus T_3, \\
P_1 \oplus P_2 \oplus E_2 \oplus T_4, & \quad P_1 \oplus P_2 \oplus T_1 \oplus T_3, & \quad P_1 \oplus P_2 \oplus T_2 \oplus T_4.
\end{align*}
\]

Let \( R_1 \oplus R_2 \oplus R_3 \oplus R_4 \) be one of these modules. Then \( R_1^{a_1} \oplus R_2^{a_2} \oplus R_3^{a_3} \oplus R_4^{a_4} \) is a rigid \( \Pi \)-module for all \( a_1, a_2, a_3, a_4 \geq 0 \), and we obtain all rigid \( \Pi \)-modules in this way.

For \( Z \in \text{Irr} (\text{nil}_E(\Pi, d)) \) the following are equivalent:

(i) \( Z \) is maximal.

(ii) \( Z = \mathcal{O}(R) \) with \( R \in \text{rep}_{l,f}(\Pi, d) \) rigid.

Recall that the dimension of an orbit \( \mathcal{O}(M) \) for \( M \in \text{rep}_{l,f}(\Pi, d) \) can be computed by the formula

\[
\dim \mathcal{O}(M) = \dim G(d) - \dim \text{End}_\Pi(M).
\]
For modules of small dimension, it is an easy exercise to compute \( \dim \text{End}_\Pi(M) \).

We have \( H = KQ/I \) where \( Q = Q(C, \Omega) \) is the quiver
\[
\begin{array}{c}
\varepsilon_1 \\
\alpha_1 \\
1 \\
\end{array} \rightarrow \begin{array}{c}
\varepsilon_2 \rightarrow 2
\end{array}
\]
and \( I \) is generated by \( \{ \varepsilon_1^2 \} \). The indecomposable locally free \( H \)-modules are
\[
E_1 = \frac{1}{1}, \quad E_2 = 2, \quad T_2 = \frac{2}{1}, \quad T_4 = \frac{1}{1}, \quad X_1 = \frac{1}{1}, \quad 2,
\]
and the indecomposable locally free \( H^* \)-modules (apart from \( E_1 \) and \( E_2 \)) are
\[
T_1 = \frac{1}{2}, \quad T_3 = 2 \frac{1}{2}, \quad X_2 = 2 \frac{1}{1}.
\]
Furthermore, we define certain indecomposable locally free \( \Pi \)-modules:
\[
\begin{array}{ccc}
X: & 1 & \rightarrow & 1 \\
& & 2 & \rightarrow & 1 \\
1 & \rightarrow & 1
\end{array} \quad \quad \quad \begin{array}{ccc}
M(\lambda): & 1 & \rightarrow & 2 \\
& \downarrow & \lambda & \downarrow 2 \\
1 & \rightarrow & 1
\end{array}
\]
where \( \lambda \in K^* \). The module \( X \) is obviously \( E \)-filtered. The modules \( M(\lambda) \) are band modules sitting at the bottom of a \( K^* \)-family of 1-tubes in the Auslander-Reiten quiver of \( \Pi \). Note that none of the modules \( X_1, X_2, X, M(\lambda) \) is a crystal module.

Using CBS it is possible to determine all irreducible components of \( \text{nil}_E(\Pi, d) \) for all \( d \). Here we just discuss one example. Let \( d = (4, 1) \). We have \( \dim G(d) = 17 \) and \( \dim G(d) - q_{DC}(d/D) = 12 \). There are three locally free \( H \)-modules (up to isomorphism) with dimension vector \( d \):
\[
M_1 = \frac{1}{1} \oplus \frac{1}{1} \oplus 2, \quad M_2 = \frac{1}{1} \oplus \frac{2}{1}, \quad M_3 = \frac{1}{1} \oplus \frac{1}{1} \oplus 2.
\]
Denote by \( Z_{M_1}, Z_{M_2}, Z_{M_3} \), respectively, the closures of the preimages of their orbits under \( \pi_H : \text{rep}_{1, f}(\Pi, d) \rightarrow \text{rep}_{1, f}(H, d) \).

We have
\[
\text{rep}_{1, f}(\Pi, d) = Z_{M_1} \cup Z_{M_2} \cup Z_{M_3},
\]
where
\[
\begin{aligned}
Z_{M_1} &= \mathcal{O}(T_1 \oplus E_1) \cup \mathcal{O}(X_2 \oplus E_1) \cup \mathcal{O}(E_1^2 \oplus E_2) = \text{rep}(H^*, d), \\
Z_{M_2} &= \mathcal{O}(T_2 \oplus E_1) \cup \mathcal{O}(X_1 \oplus E_1) \cup \mathcal{O}(E_1^2 \oplus E_2) = \text{rep}(H, d), \\
Z_{M_3} &= \bigsqcup_{\lambda \in K^*} \mathcal{O}(M(\lambda) \oplus E_1) \cup \mathcal{O}(X) \cup \mathcal{O}(X_1 \oplus E_1) \cup \mathcal{O}(X_2 \oplus E_1) \cup \mathcal{O}(E_1^2 \oplus E_2).
\end{aligned}
\]
The orbits \( \mathcal{O}(T_1 \oplus E_1) \) and \( \mathcal{O}(T_2 \oplus E_1) \) have dimension 12, and we have
\[
Z_{M_1} = \overline{\mathcal{O}(T_1 \oplus E_1)} \quad \text{and} \quad Z_{M_2} = \overline{\mathcal{O}(T_2 \oplus E_1)}.
\]
Each orbit \( \mathcal{O}(M(\lambda) \oplus E_1) \) has dimension 11, so their union also has dimension 12, and we have
\[
Z_{M_3} = \bigsqcup_{\lambda \in K^*} \overline{\mathcal{O}(M(\lambda) \oplus E_1)}.
\]
Hence
\[
\dim Z_{M_1} = \dim Z_{M_2} = \dim Z_{M_3} = \dim(\text{rep}_{1, f}(H, d)) = 12.
\]
The orbit \( \mathcal{O}(X) \) as dimension 11. We have
\[
\text{nil}_E(\Pi, d) = Z_{M_1} \cup Z_{M_2} \cup Z_{M_3}',
\]
where
\[ Z'_{M_3} = \mathcal{O}(X) \sqcup \mathcal{O}(X_1 \oplus E_1) \sqcup \mathcal{O}(X_2 \oplus E_1) \sqcup \mathcal{O}(E_1^2 \oplus E_2) \]
is an irreducible component of \( \text{nil}_E(\Pi, d) \) of non-maximal dimension 11, and we have
\[ Z'_{M_3} = \mathcal{O}(X). \]
(The fact that \( X \) cannot be contained in any of the components \( Z_{M_1} \) or \( Z_{M_2} \) can be shown by a simple semicontinuity argument.)

We consider now the enveloping algebra \( U(\mathfrak{n}) \) for type \( B_2 \). Then the dimension of \( U(\mathfrak{n})_{(2,1)} \) is two, which is perfectly in line with \( \text{nil}_E(\Pi, d) \) having exactly two maximal components. (The rank vector \((2,1)\) corresponds to the dimension vector \( d = (4,1) \).

8.2.3. Semicanonical basis. Now assume that \( K = \mathbb{C} \). We get
\[ \frac{1}{2} \tilde{\theta}_2 \ast \tilde{\theta}_1 \ast \tilde{\theta}_1 = 1_{Z_{M_1}}, \quad \frac{1}{2} \tilde{\theta}_1 \ast \tilde{\theta}_1 \ast \tilde{\theta}_2 = 1_{Z_{M_2}}, \quad \tilde{\theta}_1 \ast \tilde{\theta}_1 \ast \tilde{\theta}_1 = 1_{Z_{M_1}} + 1_{Z_{M_2}} + 1_X, \]
so the Serre relation is verified up to the function \( 1_X \in \mathcal{I} \).

The images of \( \frac{1}{2} \tilde{\theta}_2 \ast \tilde{\theta}_1 \ast \tilde{\theta}_1 \) and \( \frac{1}{2} \tilde{\theta}_1 \ast \tilde{\theta}_1 \ast \tilde{\theta}_2 \) in \( \mathcal{M}(\Pi) \) form the semicanonical basis \( \mathcal{S}_r \) of \( \mathcal{M}(\Pi)_r \), where \( r = (2,1) = d/D \). They evaluate to 1 at the generic point of one of the two maximal irreducible components of \( \text{nil}_E(\Pi, d) \), and to 0 at the generic point of the other.

8.2.4. Examples of constructible functions with non-maximal support. Again we assume that \( K = \mathbb{C} \). We define indecomposable \( \Pi \)-modules \( X \), \( Y_1 \) and \( Y_2 \) as follows:

\[
\begin{align*}
X : & \quad 1 \longrightarrow 1 \\
& \quad 2 \longrightarrow 1 \\
& \quad 1 \longrightarrow 1
\end{align*}
\]
\[
\begin{align*}
Y_1 : & \quad 1 \longrightarrow 1 \\
& \quad 2 \longrightarrow 2 \\
& \quad 1 \longrightarrow 1 \\
& \quad 1 \longrightarrow 2
\end{align*}
\]
\[
\begin{align*}
Y_2 : & \quad 2 \longrightarrow 1 \\
& \quad 1 \longrightarrow 1
\end{align*}
\]

An easy calculation shows that
\[ \tilde{\theta}_{12} = -2 \cdot 1_X. \]

We have \( \text{supp}(\tilde{\theta}_{12}) = \mathcal{O}(X) \). Furthermore, one can check that
\[ \tilde{\theta}_{12} \ast \tilde{\theta}_2 = -2(1_{Y_1} + 1_{Y_2} + 1_X \oplus E_2) \]
This implies
\[ \text{supp}(\tilde{\theta}_{12} \ast \tilde{\theta}_2) = \mathcal{O}(Y_1) \sqcup \mathcal{O}(Y_2) \sqcup \mathcal{O}(X \oplus E_2). \]

Let \( M = P_1 \oplus E_1 \). We have
\[ ((\tilde{\theta}_{12} \ast \tilde{\theta}_2) \ast \tilde{\theta}_1)(M) = \sum_{m \in \mathbb{C}} m \chi(\{ U \subset M \mid M/U \cong E_1, (\tilde{\theta}_{12} \ast \tilde{\theta}_2)(U) = m \}). \]
One easily sees that \( M \) does not have any submodules isomorphic to \( Y_2 \) or \( X \oplus E_2 \). Furthermore, one can check that we have isomorphisms of varieties
\[ \{ U \subset M \mid M/U \cong E_1, (\tilde{\theta}_{12} \ast \tilde{\theta}_2)(U) = -2 \} \cong \{ U \subset M \mid M/U \cong E_1, U \cong Y_1 \} \cong \mathbb{C}^*. \]
Since \( \chi(\mathbb{C}^*) = 0 \), we get
\[ ((\tilde{\theta}_{12} \ast \tilde{\theta}_2) \ast \tilde{\theta}_1)(M) = 0. \]
Note that the closure of \( \mathcal{O}(M) \) is a maximal irreducible component.
All three functions $\tilde{\theta}_{12}$, $\tilde{\theta}_{12} \ast \tilde{\theta}_2$ and $\tilde{\theta}_{12} \ast \tilde{\theta}_2 \ast \tilde{\theta}_1$ have non-maximal support. However, our calculation above in a small case like $B_2$ shows that this is a non-trivial fact which depends on the vanishing of some Euler characteristic.

As before, we define

$$X_1: \begin{array}{ccc}
2 & \downarrow & 2 \\
1 & \downarrow & 1
\end{array} \quad \text{and} \quad T_4: \begin{array}{ccc}
2 & \downarrow & 2 \\
1 & \downarrow & 1
\end{array}$$

In $\overline{\mathcal{F}}(\Pi)$ we get

$$1_{X_1} \ast 1_{E_2} = 1_{T_4} + 2 \cdot 1_{X_1 \oplus E_2}.$$ 

The function $1_{X_1}$ has non-maximal support, and $1_{E_2}$ and $1_{T_4} + 2 \cdot 1_{X_1 \oplus E_2}$ have maximal support. (But note that $1_{X_1}$ does not belong to $\mathcal{M}(\Pi)$.) In particular, in $\overline{\mathcal{F}}(\Pi)$ the functions with non-maximal support do not form an ideal.

8.2.5. Bundle construction. We keep the notation introduced in Sections 8.2.2 and 8.2.4. We study the bundles

$$\begin{array}{ccc}
p' \downarrow & & p'' \downarrow \\
\Pi((2,1))^{2,(0)} \times J_0 & & \Pi((2,2))^{2,(1)}
\end{array}$$

We have

$$\text{Irr}(\Pi((2,1))^{2,(0)}) = Z_1 \cup Z_2$$

where

$$Z_1 := \overline{\mathcal{O}(T_1 \oplus E_1)} \cap \Pi((2,1))^{2,(0)} \quad \text{and} \quad Z_2 := \overline{\mathcal{O}(X)} \cap \Pi((2,1))^{2,(0)}.$$ 

The component $Z_1$ is maximal, and $Z_2$ is non-maximal. We have

$$p''(p')^{-1}(Z_1 \times J_0) = \overline{\mathcal{O}(P_2 \oplus E_1)} \cap \Pi((2,2))^{2,(1)} \in \text{Irr}(\Pi((2,2))^{2,(1)}),$$

$$p''(p')^{-1}(Z_2 \times J_0) = \overline{\mathcal{O}(Y_1)} \cap \Pi((2,2))^{2,(1)} \in \text{Irr}(\Pi((2,2))^{2,(1)}).$$

We have $\mathcal{O}(Y_1) \subset \overline{\mathcal{O}(P_1)}$, thus $\overline{\mathcal{O}(Y_1)}$ cannot be in $\text{Irr}(\Pi((2,2)))$. Furthermore, we get

$$\overline{\mathcal{O}(P_1)} \in \text{Irr}(\Pi((2,2)))^{\text{max}},$$

$$\overline{\mathcal{O}(P_1)} \cap \Pi((2,2))^{1,(2)} \in \text{Irr}(\Pi((2,2)))^{1,(2)}^{\text{max}},$$

$$\overline{\mathcal{O}(P_1)} \cap \Pi((2,2))^{2,(1)} = \overline{\mathcal{O}(Y_1)} \cap \Pi((2,2))^{2,(1)} \in \text{Irr}(\Pi((2,2)))^{2,(1)}.$$ 

Next, we study the bundles

$$\begin{array}{ccc}
p' \downarrow & & p'' \downarrow \\
\Pi((1,1))^{1,(1)} \times J_0 & & \Pi((2,1))^{1,(2,1)}
\end{array}$$

Then $\overline{\mathcal{O}(X_1)} \in \text{Irr}(\Pi((1,1)))^{1,(1)}$ and

$$p''(p')^{-1}(\overline{\mathcal{O}(X_1)} \times J_0) = \overline{\mathcal{O}(X)} \cap \text{Irr}(\Pi((2,1)))^{1,(2,1)}.$$ 

We have $\overline{\mathcal{O}(X_1)} \notin \text{Irr}(\Pi((1,1)))$ and $\overline{\mathcal{O}(X)} \in \text{Irr}(\Pi(2,1)).$
8.2.6. Crystal graphs and Littlewood-Richardson coefficients. In Figure 2 we display part of the geometric crystal graph \((B, \tilde{e}_i) \equiv (B(-\infty), \tilde{e}_i)\) of type \(B_2\). (Each box in the figure contains a crystal module over \(\Pi\). The orbit closure of this \(\Pi\)-module is a maximal irreducible component.)

We have

\[
\begin{align*}
\alpha_1 &= 2\omega_1 - 2\omega_2, & \omega_1 &= \alpha_1 + \alpha_2, \\
\alpha_2 &= -\omega_1 + 2\omega_2, & \omega_2 &= 1/2\alpha_1 + \alpha_2.
\end{align*}
\]

In Figure 2 we display the geometric crystal graph \((B^*_{\omega_1+\omega_2}, \tilde{e}_i) \equiv (B(\omega_1 + \omega_2), \tilde{e}_i)\) of the simple representation \(V(\omega_1 + \omega_2)\) over the simple complex Lie algebra \(g\) of type \(B_2\), and we display the geometric crystal graph \((B_{2\omega_2}, \tilde{e}_i^*)\).

Set \(\lambda = \omega_1 + \omega_2\) and \(\mu = 2\omega_2\). The possible \(\nu \in P^+\) with \(\lambda + \mu - \nu \in R^+\) are

\[
\{\omega_1 + 3\omega_2, 2\omega_1 + \omega_2, 3\omega_2, \omega_1 + \omega_2, \omega_2\}.
\]

For \(\lambda + \mu - \nu\) we get the elements

\[
\{0, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, 2\alpha_1 + 3\alpha_2\}.
\]

The components in \(B^*_\lambda \cap B_\mu\) have a double frame. We get the tensor product decomposition

\[
V(\omega_1 + \omega_2) \otimes V(2\omega_2) \cong V(\omega_1 + 3\omega_2) \oplus V(2\omega_1 + \omega_2) \oplus V(3\omega_2) \oplus V(\omega_1 + \omega_2)^2 \oplus V(\omega_2).
\]

(The two copies of \(V(\omega_1 + \omega_2)\) in this decomposition come from the fact we have two irreducible components with rank vector \(\alpha_1 + 2\alpha_2\) in \(B^*_\lambda \cap B_\mu\).)

![Figure 2](image)

**Figure 2.** The first four layers of the geometric crystal graph \((B, \tilde{e}_i) \equiv (B(-\infty), \tilde{e}_i)\) of type \(B_2\).

8.3. Type \(G_2\). Let \(\Pi = \Pi(C, D)\) with

\[
C = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Thus \(C\) is a Cartan matrix of Dynkin type \(G_2\), and \(D\) is minimal. We have

\[
\Pi = K\overline{Q}/T
\]

where \(\overline{Q} = \overline{Q}(C)\) is the quiver

\[
\begin{tikzpicture}
\node (1) at (0,0) {1};
\node (2) at (1,0) {2};
\node (3) at (0.5,1) {1};
\node (4) at (1.5,1) {2};
\node (5) at (0.75,2) {1+2};
\node (6) at (1.25,2) {2+2};
\node (7) at (0,2) {1+1};
\node (8) at (1,2) {2+1};
\draw (1) to (3);
\draw (2) to (4);
\draw (3) to (5);
\draw (4) to (6);
\draw (5) to (7);
\draw (6) to (8);
\end{tikzpicture}
\]

\[\varepsilon_1 \quad 1 \quad \frac{\alpha_{21}}{\alpha_{12}} \quad 2\]
and $T$ is generated by the set
\[ \{ \varepsilon_3^3, \alpha_1 \alpha_2 \varepsilon_1^2 + \varepsilon_1 \alpha_1 \alpha_2 \varepsilon_1 + \varepsilon_1^2 \alpha_1 \alpha_2, -\alpha_2 \varepsilon_1 \alpha_2 \}. \]

In Figure 4, we display part of the geometric crystal graph $(\mathcal{B}, \tilde{e}_i) \equiv (B(-\infty), \tilde{e}_i)$ of type $G_2$. One of the components has a double frame. This component does not have a dense orbit, but it contains a dense $K^*$-family of orbits of $\Pi$-modules $Q(\lambda)$ with $\lambda \in K^*$, which we define as follows:

\[ Q(\lambda) = \begin{array}{c}
\lambda \\
\downarrow \\
1 \\
\downarrow \\
2 \\
\end{array} \quad \cong \quad \begin{array}{c}
\lambda \\
\downarrow \\
1 \\
\downarrow \\
1 \\
\end{array} \quad \begin{array}{c}
\lambda \\
\downarrow \\
2 \\
\downarrow \\
2 \\
\end{array} \]

Note that $Q(\lambda)$ is $E$-filtered.
8.4. **Type** $A_2$. Let $\Pi = \Pi(C, D)$ with

$$C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$  

Thus $C$ is a Cartan matrix of Dynkin type $A_2$. For the minimal symmetrizer

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

each irreducible component of $\text{nil}_E(\Pi, d)$ is maximal. This is no longer true if $D$ is non-minimal. From now on assume that

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$  

Thus we have $\Pi = \Pi(C, D) = K\overline{Q}/\overline{I}$ where $\overline{Q} = \overline{Q}(C)$ is the quiver

$$\varepsilon_1 \bigcup \begin{array}{c} 1 \\ \alpha_{21} \\ \alpha_{12} \end{array} \bigcup \begin{array}{c} 2 \\ \varepsilon_2 \end{array}$$

and $\overline{I}$ is generated by the set

$$\{\varepsilon_1^2, \varepsilon_2^2, \varepsilon_1\alpha_{12} - \alpha_{12}\varepsilon_2, \varepsilon_2\alpha_{21} - \alpha_{21}\varepsilon_1, \alpha_{12}\alpha_{21}, -\alpha_{21}\alpha_{12}\}.$$  

The preprojective algebra $\Pi$ is a finite-dimensional special biserial algebra.

Up to isomorphism there are 4 indecomposable rigid $\Pi$-modules, namely

$$P_1 = \begin{array}{c} 1 \\ \frac{1}{2} \\ 2 \end{array}, \quad P_2 = \begin{array}{c} 2 \\ \frac{1}{2} \\ 1 \end{array}, \quad E_1 = \begin{array}{c} \frac{1}{2} \\ 1 \end{array}, \quad E_2 = \begin{array}{c} 2 \\ \frac{1}{2} \end{array}.$$  

Let $d = (4, 2)$. We have $\dim G(d) = 20$ and $\dim G(d) - q_{DC}(d/D) = 14$. We define an indecomposable locally free $\Pi$-module $X$ as follows:

$$X = \begin{array}{c} 1 \rightarrow 1 \\ 2 \rightarrow 2 \\ 1 \rightarrow 1 \end{array}$$

The module $X$ is obviously $E$-filtered. The variety $\text{nil}_E(\Pi, d)$ has 3 irreducible components, namely

$$Z_1 := \mathcal{O}(P_1 \oplus E_1), \quad Z_2 := \mathcal{O}(P_2 \oplus E_1), \quad Z_3 := \mathcal{O}(X).$$
We have \( \dim(Z_1) = \dim(Z_2) = 14 \) and \( \dim(Z_3) = 13 \).

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