The Free Energy Formula

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Abstract. The Onsager formula for the free energy of the two dimensional Ising model with periodic boundary conditions is proved.

1 Introduction

Onsager and Kaufman [1], [2], [3] proposed the formula for the partition function of the two dimensional Ising model. Another heuristic method for the calculation of the partition function of the two dimensional Ising model is proposed by Kac and Ward [4]. Their idea consists of the construction of the special matrix $A$ whose determinant is connected with the partition function of the two dimensional Ising model. They considered simultaneously two formulae: the determinant of the matrix $A$ is proportional to the partition function of the Ising model and it is proportional to the square of the partition function. For the proof of the first formula they used a topological statement. Sherman [5] constructed a counter-example for this statement. Hurst and Green [6] proposed to use for the calculation of the Ising model partition function not a determinant but a Pfaffian of some special matrix. This method was improved in the papers [7], [8]. The proof proposed by McCoy and Wu [9] is considered as the most mathematically correct.

We consider a rectangular lattice on the plane formed by the points with integral Cartesian coordinates $x = j$, $y = k$, $0 \leq j \leq N$, $0 \leq k \leq M$, and the corresponding horizontal and vertical edges in which the opposite sides of the entire rectangular are identified. In other words, we consider a rectangular lattice on a torus. McCoy and Wu [9] proved the following formula for the partition function of the Ising model

$$Z = (2 \cosh \beta E_1 \cosh \beta E_2)^{MN} 1/2 (-\text{Pf} \, \tilde{A}_1 + \text{Pf} \, \tilde{A}_2 + \text{Pf} \, \tilde{A}_3 + \text{Pf} \, \tilde{A}_4),$$

(1.1)

where $\beta$ is the inverse temperature and $E_1(E_2)$ is horizontal (vertical) interaction energy. McCoy and Wu could not calculate the Pfaffians $\text{Pf} \, \tilde{A}_i$. They calculated the determinants $\det \tilde{A}_i$ only. Since

$$\text{Pf} \, \tilde{A}_i = \pm (\det \tilde{A}_i)^{1/2},$$

(1.2)

*This work is supported in part by the Russian Foundation for Basic Research (Grant No. 96 - 01 - 00167)
it is necessary to calculate the sign in the formula (1.2). These signs are calculated in [9] by using the heuristic method.

In the paper [10] a general formula of type (1.1) is obtained for the lattices placed on an arbitrary orientable surface. In this paper we prove that for the torus case the general formula [10] is the following formula

\[ Z = (-2 \cosh \beta E_1 \cosh \beta E_2)^{MN} \frac{1}{2} (-\text{Pf} \bar{A}_1 + \text{Pf} \bar{A}_2 + \text{Pf} \bar{A}_3 + \text{Pf} \bar{A}_4), \]  

(1.3)

We will calculate the Pfaffians Pf \( \bar{A}_i \) and we will prove the Onsager formula for the free energy of the two dimensional Ising model. It is easy to guess our formulae for the Pfaffians by using the formulae for det \( \bar{A}_i \) given in the book [9] and the equality

\[ (1 + z_1^2)(1 + z_2^2) - 2z_1(1 - z_2^2) - 2z_2(1 - z_1^2) = (z_1z_2 + z_1 + z_2 - 1)^2. \]  

(1.4)

2 Homological Formula

We denote by \( G(N, M) \) the graph described in the previous section. Let the function \( \sigma \) on the vertices of the graph \( G(N, M) \) takes the values in the multiplicative group \( \mathbb{Z}_2 = \{1, -1\} \). The energy \( H(\sigma) \) for the Ising model with zero magnetic field can be expressed in the form

\[ H(\sigma) = -\sum_{j=1}^N \sum_{k=1}^M (E_1 \sigma(j, k) \sigma(j + 1, k) + E_2 \sigma(j, k) \sigma(j, k + 1)), \]  

(2.1)

where the numbers 1 and \( N + 1 \) are identified and the numbers 1 and \( M + 1 \) are identified also. The partition function of the Ising model with zero magnetic field is defined as follows

\[ Z(N, M) = \sum_\sigma \exp\{-\beta H(\sigma)\}, \]  

(2.2)

where the summing runs over the group of the functions \( \sigma \) on the vertices of the graph \( G(N, M) \) taking values in the group \( \mathbb{Z}_2 \). By performing the Fourier transformation [11], ([9], Chapter 5) we have

\[ Z(N, M) = (2 \cosh \beta E_1 \cosh \beta E_2)^{MN} \sum_{\xi} \prod_{i=1}^2 \left( \tanh \beta E_i \right)^{k_i(\xi)}, \]  

(2.3)

The function \( \xi \) on the edges of the graph \( G(N, M) \) is called the cycle with the coefficients in the group \( \mathbb{Z}_2 \) if the product of all values of the function \( \xi \) on the edges connecting an arbitrary vertex of \( G(N, M) \) to other vertices is equal to 1. The number \( k_1(\xi)(k_2(\xi)) \) is the total number of all horizontal (vertical) edges on which the cycle \( \xi \) takes the value \(-1\).

By a dimer configuration on the graph \( G \) we mean a system of edges \( U = \{(p_k, q_k)\} \) satisfying the following conditions: the edges of \( U \) are pairwise disjoint; all vertices of the graph \( G \) are covered by the edges of the dimer configuration \( U \).

Due to [10] we will establish a one - to - one correspondence between the elements of the group of cycles with coefficients in \( \mathbb{Z}_2 \) and the dimer configurations on another, expanded graph. Every vertex of the initial graph \( G(N, M) \) is connected with four other vertices. In the new graph we replace every vertex with two new vertices and add to the old four edges
one new edge connecting two new vertices. We assume that new vertices are the end points of two old edges and one new edge. Thus every vertex is connected with exactly three other vertices. By applying this construction for every vertex of the graph $G(N, M)$ we obtain a new graph $tr(G(N, M))$. Let a cycle $\xi$ be given on edges of the graph $G(N, M)$. Then it is given on the old edges of the graph $tr(G(N, M))$. By using the condition $\partial \xi = -1$ we continue this function $\xi$ as $\mathbb{Z}_2$ valued function $\hat{\xi}$ on the edges of the graph $tr(G(N, M))$. Namely for every vertex of the graph $tr(G(N, M))$ the product of the function $\hat{\xi}$ values on the edges connecting this vertex with three other vertices of the graph $tr(G(N, M))$ is equal to $-1$. The function $\xi$ is a cycle and every vertex of the graph $G(N, M)$ corresponds with two vertices of the graph $tr(G(N, M))$. Therefore the extension $\hat{\xi}$ on the edges of the graph $tr(G(N, M))$ is defined uniquely. Conversely the restriction of every function $\hat{\xi}$ on the edges of the graph $tr(G(N, M))$, satisfying the condition $\partial \hat{\xi} = -1$, to the edges of the graph $G(N, M)$ defines a cycle on the edges of the graph $G(N, M)$ with coefficients in the group $\mathbb{Z}_2$.

We change every vertex of the graph $tr(G(N, M))$ by a triangle each vertex of which is an end point of one edge of the graph $tr(G(N, M))$ and an end point of the two edges of triangle. We obtain a new graph $cl(G(N, M))$. The set of the functions $\hat{\xi}$ on the edges of the graph $tr(G(N, M))$, satisfying condition $\partial \hat{\xi} = -1$, is in one - to - one correspondence with the set of all dimer configurations on the graph $cl(G(N, M))$. Indeed, let us include into the dimer configuration $U(\xi)$ all the edges on which the function $\hat{\xi}$ takes the value $-1$. Then three or one such edges come to every triangle. In the first case all vertices of the triangle are covered by the edges which do not intersect each other and some edges of the triangle does not belong to the dimer configuration $U(\xi)$. In the second case one edge of the dimer configuration $U(\xi)$ comes to one vertex of the triangle, and we include into the dimer configuration the opposite side of the triangle. Now again all vertices of the triangle are covered by the edges from the dimer configuration $U(\xi)$ which do not intersect each other.

Let the dimer configuration $U$ on the graph $cl(G(N, M))$ be given. Let us define the function $\hat{\xi}(U)$ by taking it is equal to 1 on the edges of the graph $tr(G(N, M))$ which do not belong to the dimer configuration $U$ and by taking it equal to $-1$ on the edges of the graph $tr(G(N, M))$ which belong to the dimer configuration $U$. If one side of a triangle belongs to the dimer configuration $U$, then one edge from the dimer configuration $U$ comes to a triangle. If the sides of a triangle do not belong to the dimer configuration $U$, then three edges from the dimer configuration $U$ come to a triangle. In both cases the function $\hat{\xi}(U)$ satisfies the condition $\partial \hat{\xi} = -1$.

One - to - one correspondence allows us to rewrite the partition function (2.3) in another form. Every dimer configuration $U$ on the graph $cl(G(N, M))$ corresponds with a cycle $\xi$ on the edges of the graph $G(N, M)$ with coefficients in the group $\mathbb{Z}_2$: $(p, q) \in U \cap G(N, M)$ if and only if $\xi((p, q)) = -1$. Therefore the relation (2.3) can be rewritten in the following form

$$Z(N, M) = (2 \cosh \beta E_1 \cosh \beta E_2)^{MN} \sum_{U} \prod_{i=1}^{2} \left( \tanh \beta E_i \right)^{#(U \cap G(N, M))_i},$$

(2.4)

where summation runs over all dimer configurations $U$ on the graph $cl(G(N, M))$. $#(U \cap G(N, M))_1$, $(#(U \cap G(N, M))_2)$ denotes the total number of horizontal (vertical) common edges in the dimer configuration $U$ and in the graph $G(N, M)$.

With each vertex of the graph $G(N, M)$ there corresponds a cluster of the graph $cl(G(N, M))$: two triangles connected by one edge. We place one triangle under another such that their bases are horizontal. Let vertex opposite to the base of the upper triangle be set
below the base and the vertex opposite to the base of the lower triangle be set above the base. The vertical edge connects these two vertices. The right vertex of the upper triangle is denoted by 1. The left vertex of the lower triangle is denoted by 2. The left vertex of the upper triangle is denoted by 3. The right vertex of the lower triangle is denoted by 4. The upper vertex of the lower triangle is denoted by 5. The lower vertex of the upper triangle is denoted by 6. With all six vertices there correspond two numbers \((j, k)\), \(1 \leq j \leq N, 1 \leq k \leq M\) corresponding to the old vertex of the graph \(G(N, M)\). The vertices \((1, j, k)\) and \((6, j, k)\); \((1, j, k)\) and \((3, j, k)\); \((3, j, k)\) and \((6, j, k)\); \((5, j, k)\) and \((6, j, k)\); \((2, j, k)\) and \((4, j, k)\); \((2, j, k)\) and \((5, j, k)\); \((4, j, k)\) and \((5, j, k)\) are connected by the edges of one cluster. The vertices \((1, j, k)\) and \((2, j + 1, k)\) are connected by the horizontal edges of the graph \(G(N, M)\). We consider the numbers 1 and \(N + 1\) identical. The vertices \((3, j, k)\) and \((4, j, k + 1)\) are connected by the vertical edges of the graph \(G(N, M)\). The numbers 1 and \(M + 1\) are considered identical.

Let an orientation of every edge of the graph \(\text{cl}(G(N, M))\) be given. If an edge is defined by its end points \((p, q)\), then an orientation is given by a function \(\phi(p, q)\) taking only two values 0, 1 and satisfying the following condition

\[
\phi(p, q) + \phi(q, p) = 1 \mod 2. \tag{2.5}
\]

Let us define the function \(z(p, q) = z(q, p)\) on the oriented edges \((p, q)\): \(z(p, q) = 1\) if the vertices \(p, q\) belong to the same cluster and are connected by an edge; \(z(p, q) = \tanh \beta E_1\) \((z(p, q) = \tanh \beta E_2)\) if the vertices \(p, q\) belong to the neighboring clusters and are connected by a horizontal (vertical) edge; \(z(p, q) = 0\) if the vertices \(p, q\) are not connected by an edge.

Let the linear ordering of vertices of the graph \(\text{cl}(G(N, M))\) be given

\[
n(i, j, k) = i + 6(k - 1) + 6M(j - 1) \tag{2.6}
\]

where \(1 \leq i \leq 6, 1 \leq j \leq N, 1 \leq k \leq M\). The function \((2.6)\) establishes the correspondence between the vertices of the graph \(\text{cl}(G(N, M))\) and the numbers \(1, \ldots, 6MN\).

Let us define \(6MN \times 6MN\) - matrix

\[
A(\phi)_{n(p)n(q)} = \exp\{i\pi \phi(p, q)\} z(p, q) \tag{2.7}
\]

where the numbers \(p = (i, j, k)\) are indices for the vertices of the graph \(\text{cl}(G(N, M))\). Due to the relation \((2.5)\) the matrix \((2.7)\) is antisymmetric. Hence we can define its Pfaffian

\[
Pf A(\phi) = ((3MN)!)^{-1} 2^{-3MN} \sum_{\pi \in S_{6MN}} (-1)^{\sigma(\pi)} A(\phi)^{\pi(1)\pi(2)} \cdots A(\phi)^{\pi(6MN - 1)\pi(6MN)} \tag{2.8}
\]

where \(\pi\) is an arbitrary permutation of the numbers \(1, \ldots, 6MN\) and \(\sigma(\pi)\) is its parity.

The definition \((2.8)\) can be expressed as

\[
Pf A(\phi) = \sum_U (-1)^{\sigma(U)} \prod_{(i, j) \in U} A(\phi)_{i, j} \tag{2.9}
\]

where \(U\) is an ordered subdivision of the numbers \(1, \ldots, 6MN\) into pairs \(\{(i_1, j_2), \ldots, (i_{3MN}, j_{3MN})\}\), \(1 = i_1 < \cdots < i_{3MN} \leq 6MN, i_p < j_p, p = 1, \ldots, 3MN\). The number \(\sigma(U) = 0, 1\) respectively with the parity of the permutation mapping the numbers \((i_1, j_1, \ldots, i_{3MN}, j_{3MN})\) into the numbers \((1, \ldots, 6MN)\). Due to the relation \((2.7)\)
and the definition of the function $z(p, q)$ the summation in (2.9) runs only such subdivisions $U$ which correspond with the dimer configurations on the graph $cl(G(N, M))$. Let $U = \{(p_1, q_1), ..., (p_{3MN}, q_{3MN})\}$ be a dimer configuration on the graph $cl(G(N, M))$. By using an orientation function $\phi$ satisfying the condition (2.7) we define a function on the dimer configurations on the graph $cl(G(N, M))$

$$\tau[\phi](U) = \sigma(n(p_1), n(q_1), ..., n(p_{3MN}), n(q_{3MN})) + \sum_{j=1}^{3MN} \phi(p_j, q_j) \mod 2 \quad (2.10)$$

where $\sigma(n(p_1), n(q_1), ..., n(p_{3MN}), n(q_{3MN})) = 0, 1$ respectively with the parity of the permutation mapping the numbers $(n(p_1), n(q_1), ..., n(p_{3MN}), n(q_{3MN}))$ into the numbers $(1, ..., 6MN)$. The function $\phi(p, q)$ satisfies the condition (2.8). Hence the the right hand side of the equality (2.10) is independent of an ordering of the vertices in the edges $(p_j, q_j)$ although every summand in (2.10) depends on this ordering. The right hand side of the equality (2.10) is also independent of an ordering of the edges $(p_j, q_j)$.

The substitution of the definitions (2.7) and (2.10) into the right hand side of the equality (2.9) yields

$$\text{Pf } A(\phi) = \sum_U (-1)^{\tau[\phi](U)} z(U) \quad (2.11)$$

where

$$z(U) = \prod_{(p, q) \in U} z(p, q) \quad (2.12)$$

and the summation in (2.11) runs over all dimer configurations on the graph $cl(G(N, M))$.

The expression (2.11) differs from the sum (2.4) by the multiplier $(-1)^{\tau[\phi](U)}$ only. For any dimer configuration $U_0$ on the graph $cl(G(N, M))$ we have

$$\text{Pf } A(\phi) = (-1)^{\tau[\phi](U_0)} \sum_U (-1)^{\tau[\phi](U_0)+\tau[\phi](U)} z(U). \quad (2.13)$$

Let us study the function $\tau[\phi](U_1) + \tau[\phi](U_2)$ of a pair of the dimer configurations $U_1, U_2$ on the graph $cl(G(N, M))$. Let us consider the symmetric difference

$$U_1 \triangle U_2 = (U_1 \cup U_2) \setminus (U_1 \cap U_2). \quad (2.14)$$

The dimer configuration definition implies that this set consists of the finite number of closed broken lines which do not intersect each other, that is, we can write

$$U_1 \triangle U_2 = \cup_{i=1}^{s} \gamma_i. \quad (2.15)$$

Every broken line $\gamma_i$ does not intersect itself. The edges from $U_1$ and $U_2$ are included in each broken line alternatively. Due to Theorem 2 from [10] for any dimer configurations $U_1, U_2$ on the graph $cl(G(N, M))$

$$\tau[\phi](U_1) + \tau[\phi](U_2) = s + \sum_{i=1}^{s} \phi(\gamma_i) \mod 2 \quad (2.16)$$

where the broken lines $\gamma_i$ are given by the relation (2.15) and

$$\phi(\gamma_i) = \sum_{(p, q) \in \gamma_i} \phi(p, q) \mod 2. \quad (2.17)$$
This sum does not depend on the orientation chosen on the broken line \( \gamma_i \) since the total number of edges in the broken line \( \gamma_i \) is even.

Let us define on the graph \( cl(G(N, M)) \) special orientation function

\[
\phi((1,j,k),(6,j,k)) = \phi((6,j,k),(3,j,k)) = 0 \mod 2 \\
\phi((3,j,k),(1,j,k)) = \phi((5,j,k),(4,j,k)) = 0 \mod 2 \\
\phi((4,j,k),(2,j,k)) = 0 \mod 2 \\
\phi((5,j,k),(6,j,k)) = 0 \mod 2 \\
\phi((1,j,k),(2,j + 1,k)) = 0 \mod 2 \\
\phi((3,j,k),(4,j,k + 1)) = 0 \mod 2. 
\]

(2.18)

In the equalities (2.18), the numbers 1 and \( N + 1 \), 1 and \( M + 1 \) are identified. The graph \( cl(G(N, M)) \) consists of the triangles and 12 - angles. The relations (2.18) have the property: if a broken line \( \gamma \) is a boundary of a triangle or 12 - angle and the counter- clockwise orientation is chosen, then

\[
\phi(\gamma) = 1 \mod 2. 
\]

(2.19)

**Theorem 2.1** Let on the graph \( cl(G(N, M)) \) the orientation function (2.18) be given. Let domain on the torus be bounded by the broken lines \( \gamma_1, \ldots, \gamma_s \) and these broken lines coincide with the symmetric difference of two dimer configurations on the graph \( cl(G(N, M)) \). Then

\[
s + \sum_{i=1}^{s} \phi(\gamma_i) = 0 \mod 2. 
\]

(2.20)

**Proof.** In view of the condition of the theorem the domain on the torus is bounded by broken lines \( \gamma_1, \ldots, \gamma_s \) which do not intersect each other. We stretch the auxiliary disk \( d_i \) on each broken line \( \gamma_i \). In general, we obtain the new closed orientable surface \( P \). The domain consists of the faces \( \sigma \) (triangles and 12 - angles). We choose the orientation which induces the counter- clockwise orientation on the broken lines \( \partial \sigma \). Let us choose an orientation of each auxiliary disk \( d_i \) coherently with the orientation of the boundary of the domain.

Since the set of closed broken lines \( \gamma_1, \ldots, \gamma_s \) coincide with the symmetric difference of two dimer configurations, the domain contains an even number of vertices, the number of edges and the number of vertices for each broken line \( \gamma_i \) are also even. Therefore the number of vertices \( v \) of the constructed cell complex for an orientable closed surface \( P \) is even. Let \( f \) and \( e \) denote the total numbers of faces and edges of the orientable surface \( P \). By using the skew symmetry condition (2.5) and by summing over all faces of the surface \( P \) including the auxiliary disks \( d_i \) we have

\[
\sum_{\sigma \in P} (1 + \phi(\partial \sigma)) = f + e = v + \chi = 0 \mod 2, 
\]

(2.21)

since Euler characteristic \( \chi = 2(1 - g) \) of the orientable surface \( P \) is even.

If the orientation function \( \phi \) on the graph \( cl(G(N, M)) \) is given by the relations (2.18), then for any face of the surface \( P \) excepting the auxiliary disks the relation (2.19) holds. Hence the left hand sides of the equalities (2.20) and (2.21) coincide. The theorem is proved.

**Theorem 2.2** Let the dimer configuration \( U_0 \) on the graph \( cl(G(N, M)) \) correspond to the cycle \( \xi_0 \) with the coefficients in the multiplicative group \( \mathbb{Z}_2 \) which takes the value 1 on all
edges of the graph \( G(N, M) \). Let the function \( \tau[\phi](U) \) be given by the relation (2.10) for the orientation function \( \phi \) defined by the equalities (2.18). Then

\[
\tau[\phi](U_0) = MN \mod 2. \tag{2.22}
\]

Proof. The dimer configuration \( U_0 \) does not contain the edges connecting different clusters. In every cluster \((j, k)\) the dimer configuration \( U_0 \) contains three edges: \(((1, j, k), (3, j, k))\), \(((2, j, k), (4, j, k))\) and \(((5, j, k), (6, j, k))\). In according to (2.4)

\[
\sigma(n(1, 1, 1), n(3, 1, 1), n(2, 1, 1), n(4, 1, 1), n(5, 1, 1), n(6, 1, 1), \ldots, n(1, N, M), n(3, N, M), n(2, N, M), n(4, N, M), n(5, N, M), n(6, N, M)) =
\]

\[MN \mod 2 \tag{2.23}
\]

since there is one disorder: \( n(3, j, k), n(2, j, k) \) in any of \( MN \) clusters.

The definitions (2.18) and the relation (2.3) imply

\[
\sum_{j=1}^{N} \sum_{k=1}^{M} \left[ \phi((1, j, k), (3, j, k)) + \phi((2, j, k), (4, j, k)) + \phi((5, j, k), (6, j, k)) \right] = 0 \mod 2. \tag{2.24}
\]

The substitution of the equalities (2.23) and (2.24) into the equality (2.10) for the dimer \( U_0 \) yields the equality (2.22). The theorem is proved.

The number (2.22) is the total number of the vertices of the graph \( G(N, M) \) or it is the total number of the clusters of the graph \( tr(G(N, M)) \). The formulae (1.1) and (1.3) differ each other by the multiplier \((-1)^{\tau[\phi](U_0)}\).

We denote by \( \mathbb{Z}_2^{add} \) the group of modulo 2 residuals. The modulo 2 residuals are multiplied each other and the group \( \mathbb{Z}_2^{add} \) is a field. The graph \( cl(G(N, M)) \) divides the torus into the faces: the triangles and 12 - angles. The cell complex \( P(cl(G(N, M))) \equiv P(G) \) is called the set consisting of the cells (vertices, edges, faces). To every cell \( s_i^p \) there corresponds the natural number \( p \) (dimension). For the vertices \( p = 0 \), for the edges \( p = 1 \) and for the faces \( p = 2 \). To every pair of the cells \( s_i^p, s_j^{p-1} \) there corresponds the number \((s_i^p : s_j^{p-1}) \in \mathbb{Z}_2^{add}\) (incidence number). If the cell \( s_j^{p-1} \) is included into the boundary of the cell \( s_i^p \), then the incidence number \((s_i^p : s_j^{p-1}) = 1\). Otherwise the incidence number \((s_i^p : s_j^{p-1}) = 0\). For any pair of the cells \( s_i^2, s_j^0 \) the incidence numbers satisfy the condition

\[
\sum_m (s_i^2 : s_m^1)(s_m^1 : s_j^0) = 0 \mod 2. \tag{2.25}
\]

Indeed, if the vertex \( s_j^0 \) is not contained in the boundary of the triangle or 12 - angle \( s_i^2 \), then the condition (2.25) is fulfilled. If the vertex \( s_j^0 \) is included into the boundary of the face \( s_i^2 \), then it is included into the boundaries of three edges \( s_m^1 \) two of which are included into the boundary of the face \( s_i^2 \). The condition (2.25) is fulfilled again.

A cochain \( c^p \) of the complex \( P(G) \) with the coefficients in the group \( \mathbb{Z}_2^{add} \) is a function on the \( p \) dimensional cells taking values into the group \( \mathbb{Z}_2^{add} \). Usually the cell orientation is considered and the cochains are the antisymmetric functions: \( c^p(-s^p) = -c^p(s^p) \). However, \(-1 = 1 \mod 2\) and we can neglect the cell orientation for the coefficients in the group \( \mathbb{Z}_2^{add} \). The cochains form an Abelian group

\[
(c^p + c^p)(s_i^p) = c^p(s_i^p) + c^p(s_i^p) \mod 2. \tag{2.26}
\]
It is denoted by \( C^p(P(G), \mathbb{Z}_2^{add}) \). The mapping
\[
\partial c^p(s_{i}^{p-1}) = \sum_j (s_{j}^{p} : s_{i}^{p-1}) c(s_{j}^{p}) \mod 2
\]  
(2.27)
defines the homomorphism of the group \( C^p(P(G), \mathbb{Z}_2^{add}) \) into the group \( C^{p-1}(P(G), \mathbb{Z}_2^{add}) \). It is called the boundary operator. The mapping
\[
\partial^* c^p(s_{i}^{p+1}) = \sum_j (s_{j}^{p+1} : s_{i}^{p}) c(s_{j}^{p}) \mod 2
\]  
(2.28)
defines the homomorphism of the group \( C^p(P(G), \mathbb{Z}_2^{add}) \) into the group \( C^{p+1}(P(G), \mathbb{Z}_2^{add}) \). It is called the coboundary operator. The condition (2.25) implies \( \partial \partial = 0 \), \( \partial^* \partial^* = 0 \). A kernel \( \mathbb{Z}_p(P(G), \mathbb{Z}_2^{add}) \) of the homomorphism \( \partial : C^p(P(G), \mathbb{Z}_2^{add}) \to C^{p-1}(P(G), \mathbb{Z}_2^{add}) \) is called a group of cocycles \( U \) with coefficients in the group \( \mathbb{Z}_2 \). It is denoted by \( \mathbb{Z}_2 \). It is possible to introduce the bilinear form on \( C^p(P(G), \mathbb{Z}_2^{add}) \)
\[
\langle f^p, g^p \rangle = \sum_i f^p(s_{i}^{p}) g^p(s_{i}^{p}) \mod 2.
\]  
(2.29)
The definitions (2.27) and (2.28) imply
\[
\langle f^p, \partial^* g^{p-1} \rangle = \langle \partial f^p, g^{p-1} \rangle \mod 2
\]  
(2.30)
\[
\langle f^p, \partial g^{p+1} \rangle = \langle \partial^* f^p, g^{p+1} \rangle \mod 2.
\]  
(2.31)

We identify every set of cells \( \{ s_{i}^{p} \} \) with a cochain taking value 1 on these cells and which takes value 0 on other cells. For example, the symmetric difference (2.15) of two dimer configurations \( U_1, U_2 \) on the graph \( d(G(N,M)) \) is identified with the cochain which takes the value 1 on the edges belonging to the broken lines \( \gamma_i \) and which takes the value 0 on the other edges of the graph \( d(G(N,M)) \).

In according to the relation (2.13) it is necessary to study the function
\[
\psi(U \triangle U_0) = \tau[\phi](U) + \tau[\phi](U_0) \mod 2
\]  
(2.32)
where the function \( \tau[\phi](U) \) is defined by the relation (2.10) by using the orientation function (2.18). For the dimer configuration \( U_0 \) we take one corresponding to the cycle \( \xi_0 \) with the coefficients in the group \( \mathbb{Z}_2 \) which takes value 1 on all edges of the graph \( G(N,M) \). (We consider on the graph \( G(N,M) \) the cycles with the coefficients in the multiplicative group \( \mathbb{Z}_2 \). We hope that making use of the cycles with the coefficients in the different groups \( \mathbb{Z}_2 \) and \( \mathbb{Z}_2^{add} \) on the different graphs does not put us to confusion.) Due to relations (2.13), (2.10)
\[
U \triangle U_0 = \sum_{i=1}^{s} \gamma_i
\]  
(2.33)
\[ \psi(U \triangle U_0) = s + \sum_{i=1}^{s} \gamma_i \mod 2. \]  

(2.34)

**Theorem 2.3** The function \( \psi(U \triangle U_0) \), given by the relations (2.32) - (2.34), depends on the equivalence class of the cycle \( U \triangle U_0 \) in the homology group \( H_1(P(G), \mathbb{Z}_2^{add}) \) only.

**Proof.** Due to relation (2.33) the symmetric difference \( U \triangle U_0 \) is the set of the broken closed lines. Hence it is identified with the cycle from the group \( Z_1(P(G), \mathbb{Z}_2^{add}) \). Thus the symmetric difference \( U \triangle U_0 \) generates some equivalence class in the homology group \( H_1(P(G), \mathbb{Z}_2^{add}) \). Let for the dimer configurations \( U_1 \) and \( U_2 \) on the graph \( cl(G(N, M)) \) the symmetric differences \( U_1 \triangle U_0 \) and \( U_2 \triangle U_0 \) generate the same equivalence class in the homology group. Let us prove \( \psi(U_1 \triangle U_0) = \psi(U_2 \triangle U_0) \). Indeed in this case the symmetric difference \((U_1 \triangle U_0) \triangle (U_2 \triangle U_0)\) is a boundary of some domain on the torus. Due to Lemma 1 from [10]

\[ (U_1 \triangle U_0) \triangle (U_2 \triangle U_0) = U_1 \triangle U_2. \]

(2.35) Therefore the symmetric difference \( U_1 \triangle U_2 \) is a boundary of some domain on the torus. Now the definition (2.32) implies

\[ \psi(U_1 \triangle U_0) + \psi(U_2 \triangle U_0) = \tau[\phi](U_1) + \tau[\phi](U_2) \mod 2. \]

(2.36) It follows from the equalities (2.13), (2.16) and equality (2.20) from Theorem 2.1 that the right hand side of the equality (2.36) equals 0. Thus the values \( \psi(U_1 \triangle U_0) \) and \( \psi(U_2 \triangle U_0) \) coincide. The theorem is proved.

**Theorem 2.4** The symmetric difference \( U \triangle U_0 \) of the dimer configurations \( U, U_0 \) on the graph \( cl(G(N, M)) \) generates all elements of the homology group \( H_1(P(G)), \mathbb{Z}_2^{add} \).

**Proof.** Let the dimer configuration \( U_0 \) consist of the edges \(((3, j, 1), (6, j, 1)), ((4, j, 1), (5, j, 1)), j = 1, ..., N; ((1, j, 1), (2, j + 1, 1)), j = 1, ..., N - 1; ((1, N, 1), (2, 1, 1)); ((1, j, k), (3, j, k)), ((2, j, k), (4, j, k)), ((5, j, k), (6, j, k)), 1 \leq j \leq N, 2 \leq k \leq M \). Let us remind that the dimer configuration \( U_0 \) corresponding to the cycle \( \zeta_0 \) with the coefficients in the group \( \mathbb{Z}_2 \) which takes value 1 on all edges of the graph \( G(N, M) \) consists of the following edges of the graph \( cl(G(N, M)) \): \(((1, j, k), (3, j, k)), ((2, j, k), (4, j, k)), ((5, j, k), (6, j, k)), 1 \leq j \leq N, 1 \leq k \leq M \). Thus the symmetric difference \( U_0 \triangle U_0 \) is the closed non-intersecting itself non-homological to zero broken line which goes around the torus horizontally.

Let the dimer configuration \( U_0 \) consist of the edges: \(((1, k, 1), (6, k, 1)), ((2, k, 1), (5, k, 1)), k = 1, ..., M; ((3, k, 1), (4, 1, k + 1)), k = 1, ..., M - 1; ((3, 1, M), (4, 1, 1)); ((1, j, k), (3, j, k)), ((2, j, k), (4, j, k)), ((5, j, k), (6, j, k)), 2 \leq j \leq N, 1 \leq k \leq M \). Thus the symmetric difference \( U_0 \triangle U_0 \) is the closed non-intersecting itself non-homological to zero broken line which goes around the torus vertically.

Let the dimer configuration \( U_0 \) consist of the edges: \(((5, 1, 1), (6, 1, 1)), ((3, 1, 1), (6, 1, 1)), ((4, j, 1), (5, j, 1)), 2 \leq j \leq N; ((1, 1, k), (6, 1, k)), ((2, 1, k), (5, 1, k)), 2 \leq k \leq M; ((1, j, 1), (2, j + 1, 1)), j = 1, ..., N - 1; ((1, N, 1), (2, 1, 1)); ((3, 1, M), (4, 1, 1)); ((1, j, k), (3, j, k)), ((2, j, k), (4, j, k)), ((5, j, k), (6, j, k)), 2 \leq j \leq N, 2 \leq k \leq M \). Thus the symmetric difference \( U_0 \triangle U_0 \) is the closed non-intersecting itself non-homological to zero broken line which goes around the torus horizontally and vertically.

At last \( U_0 \triangle U_0 = \emptyset \). The equivalence classes of the symmetric differences \( U_0 \triangle U_0, U_0 \triangle U_0, U_0 \triangle U_0 \) as the closed curves on the torus represent all elements of the homology group \( H_1(P(G), \mathbb{Z}_2^{add}) \). The theorem is proved.

Let us define the cochain \( a^* \in C^1(P(G), \mathbb{Z}_2^{add}) \) in the following way:
The cochain $a^*$ equals 0 on all other edges of the graph $cl(G(N,M))$. By making use of the definition (2.28) it is easy to verify $\partial^*a^* = 0$ i.e. $a^* \in Z^1(P(G),Z_2^{add})$. Due to definition (2.29) the linear function $\langle a^*, f \rangle$ is given on the group $Z_1(P(G),Z_2^{add})$. The equality (2.31) and the equality $\partial^*a^* = 0$ imply that this function is equal to zero on the group $B_1(P(G),Z_2^{add})$. Hence the linear function $\langle a^*, f \rangle$ defines the function on the homology group $H_1(P(G),Z_2^{add})$. We denote by $U_a \triangle U_0$ the cochain which takes the value 1 on all edges of the broken line $U_a \triangle U_0$. It is equal to zero on all other edges of the graph $cl(G(N,M))$. The cochains $U_b \triangle U_0$, $U_{a+b} \triangle U_0$ and $U_0 \triangle U_0$ are defined analogously. The definition (2.29) implies

\[
\langle a^*, U_a \triangle U_0 \rangle = 1 \mod 2, \quad \langle a^*, U_b \triangle U_0 \rangle = 0 \mod 2, \\
\langle a^*, U_{a+b} \triangle U_0 \rangle = 1 \mod 2, \quad \langle a^*, U_0 \triangle U_0 \rangle = 0 \mod 2.
\] (2.37)

We define the cochain $b^* \in C^1(P(G),Z_2^{add})$ in the following way: $b^*)((3,j, M), (4,j, 1))) = 1, 1 \leq j \leq N$. The cochain $b^*$ equals 0 on all other edges of the graph $cl(G(N,M))$. By making use of the definition (2.28) it is easy to verify $\partial^*b^* = 0$ i.e. $b^* \in Z^1(P(G),Z_2^{add})$. Due to definition (2.29) the linear function $\langle b^*, f \rangle$ is given on the group $Z_1(P(G),Z_2^{add})$. The equality (2.31) and the equality $\partial^*b^* = 0$ imply that this function is equal to zero on the group $B_1(P(G),Z_2^{add})$. Hence the linear function $\langle b^*, f \rangle$ defines the function on the homology group $H_1(P(G),Z_2^{add})$. The definition (2.29) implies

\[
\langle b^*, U_a \triangle U_0 \rangle = 0 \mod 2, \quad \langle b^*, U_b \triangle U_0 \rangle = 1 \mod 2, \\
\langle b^*, U_{a+b} \triangle U_0 \rangle = 1 \mod 2, \quad \langle b^*, U_0 \triangle U_0 \rangle = 0 \mod 2.
\] (2.38)

Let the orientation function $\phi$ satisfy the skew symmetry condition (2.3). Let us define three new orientation functions on the directed edges

\[
(\phi + a^*)((p,q)) = \phi(p,q) + a^*(p,q) \mod 2,
\] (2.39)

\[
(\phi + b^*)((p,q)) = \phi(p,q) + b^*(p,q) \mod 2,
\] (2.40)

\[
(\phi + a^* + b^*)((p,q)) = \phi(p,q) + a^*(p,q) + b^*(p,q) \mod 2.
\] (2.41)

Since $a^*(q,p) = a^*(p,q)$, $b^*(q,p) = b^*(p,q)$, the orientation functions $\phi + a^*$, $\phi + b^*$, $\phi + a^* + b^*$ satisfy the skew symmetry condition (2.3).

**Theorem 2.5** Let the orientation function (2.13) be given on the graph $cl(G(N,M))$. The relation (2.7) defines four antisymmetric $6MN \times 6MN$ - matrices: $A(\phi)$, $A(\phi + a^*)$, $A(\phi + b^*)$, $A(\phi + a^* + b^*)$. Then the following relation for the partition function (2.2) holds

\[
Z(N,M) = (-2 \cosh 2E_1 \cosh 2E_2)^{MN/2} \left[ -\Pf A(\phi) + \Pf A(\phi + a^*) + \Pf A(\phi + b^*) + \Pf A(\phi + a^* + b^*) \right].
\] (2.42)

**Proof.** The relation (2.4) may be rewritten as

\[
Z(N,M) = (2 \cosh 2E_1 \cosh 2E_2)^{MN} \sum_U z(U)
\] (2.43)

where the monomial $z(U)$ is given by the equality (2.12) and the summation in (2.43) runs over all dimer configurations on the graph $cl(G(N,M))$. 

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The equalities (2.10), (2.18) and (2.32) define the function \( \psi(U \triangle U_0) \) of the symmetric difference \( U \triangle U_0 \) of the dimer configurations \( U, U_0 \) on the graph \( cl(G(N, M)) \). The dimer configuration \( U_0 \) corresponds with the cycle \( \xi_0 \) with the coefficients in the group \( \mathbb{Z}_2 \) which takes the value 1 on all edges of the graph \( G(N, M) \). Due to Theorem 2.3 the function \( \psi(U \triangle U_0) \) depends on the equivalence class of the cycle \( U \triangle U_0 \) in the homology group \( H_1(P(G), \mathbb{Z}_2^{add}) \) only. Hence it is sufficient to calculate the function \( \psi(U \triangle U_0) \) on the following symmetric differences: \( U_a \triangle U_0, U_b \triangle U_0, U_{a+b} \triangle U_0 \) and \( U_0 \triangle U_0 \). It is easy to verify

\[
\psi(U_a \triangle U_0) = \psi(U_b \triangle U_0) = \psi(U_{a+b} \triangle U_0) = 1 \text{ mod } 2, \psi(U_0 \triangle U_0) = 0 \text{ mod } 2. \tag{2.44}
\]

By making use of the equalities (2.37), (2.38) and (2.44) it is possible to verify the following relation

\[
(-1)\psi(U \triangle U_0) = 1/2(-1 + (-1)^{a^*, U \triangle U_0} + (-1)^{b^*, U \triangle U_0} + (-1)^{(a^*, U \triangle U_0) + (b^*, U \triangle U_0)}) \tag{2.45}
\]

for four symmetric differences: \( U_a \triangle U_0, U_b \triangle U_0, U_{a+b} \triangle U_0, U_0 \triangle U_0 \). All terms of the equality (2.45) depends on the equivalence class of the cycle \( U \triangle U_0 \) in the homology group \( H_1(P(G), \mathbb{Z}_2^{add}) \) only. Hence if the equality (2.45) holds for the symmetric differences: \( U_a \triangle U_0, U_b \triangle U_0, U_{a+b} \triangle U_0, U_0 \triangle U_0 \), it holds for the symmetric difference \( U \triangle U_0 \) where \( U \) is an arbitrary dimer configuration on the graph \( cl(G(N, M)) \).

For any dimer configuration \( U \) on the graph \( cl(G(N, M)) \) we introduce the cochain \( U \in C^1(P(G), \mathbb{Z}_2^{add}) \) which equals 1 on all edges from the dimer configuration \( U \) and which equals 0 on all other edges of the graph \( cl(G(N, M)) \). Due to definition the cochains \( a^* \) and \( b^* \) are equal to zero on all edges from the dimer configuration \( U_0 \). Hence

\[
\langle a^*, U \triangle U_0 \rangle = \langle a^*, U \rangle \text{ mod } 2, \langle b^*, U \triangle U_0 \rangle = \langle b^*, U \rangle \text{ mod } 2. \tag{2.46}
\]

Let us multiply by \((-1)^{\psi(U \triangle U_0)}\) the equality (2.45). Then it follows from the equalities (2.32), (2.46) and from Theorem 2.2 that

\[
1 = (-1)^{MN}1/2[-(-1)^{\tau[\phi(U)]} + (-1)^{\tau[\phi(U)] + (a^*, U)} + (-1)^{\tau[\phi(U)] + (b^*, U)} + (-1)^{\tau[\phi(U)] + (a^*, U) + (b^*, U)}]. \tag{2.47}
\]

Now substituting the relation (2.47) into the equality (2.43) and making use of the equalities (2.7), (2.10), (2.11) we obtain the equality (2.42). The theorem is proved.

For the calculation of the Pfaffians it is necessary to define the matrices \( A(\phi), A(\phi + a^*), A(\phi + b^*), A(\phi + a^* + b^*) \) explicitly. The relations (2.5) - (2.7) and (2.18) imply

\[
A(\phi)_{n(1,j_1,k_1)n(2,j_2,k_2)} = -A(\phi)_{n(2,j_2,k_2)n(1,j_1,k_1)} = \tanh \beta E_1(\delta_{j_1+1,j_2} + \delta_{j_1,k_2})\delta_{k_1,k_2} \tag{2.48}
\]

\[
A(\phi)_{n(1,j_1,k_1)n(3,j_2,k_2)} = -A(\phi)_{n(3,j_2,k_2)n(1,j_1,k_1)} = -\delta_{j_1,j_2}\delta_{k_1,k_2} \tag{2.49}
\]

\[
A(\phi)_{n(1,j_1,k_1)n(6,j_2,k_2)} = -A(\phi)_{n(6,j_2,k_2)n(1,j_1,k_1)} = \delta_{j_1,j_2}\delta_{k_1,k_2} \tag{2.50}
\]

\[
A(\phi)_{n(2,j_1,k_1)n(4,j_2,k_2)} = -A(\phi)_{n(4,j_2,k_2)n(2,j_1,k_1)} = -\delta_{j_1,j_2}\delta_{k_1,k_2} \tag{2.51}
\]

\[
A(\phi)_{n(2,j_1,k_1)n(5,j_2,k_2)} = -A(\phi)_{n(5,j_2,k_2)n(2,j_1,k_1)} = \delta_{j_1,j_2}\delta_{k_1,k_2} \tag{2.52}
\]

\[
A(\phi)_{n(3,j_1,k_1)n(4,j_2,k_2)} = -A(\phi)_{n(4,j_2,k_2)n(3,j_1,k_1)} = \tanh \beta E_2\delta_{j_1,j_2}(\delta_{k_1+1,k_2} + \delta_{k_1M}\delta_{k_2}) \tag{2.53}
\]
All other elements of the matrix $A(\phi)$ are equal to zero.

In view of the definition the cochain $a^*$ equals 1 on the edges $((1, N, k), (2, 1, k))$, $1 \leq k \leq M$ and it is equal to zero on all other edges. The equalities (2.4) - (2.6) and (3.1) imply

$$A(\phi + a^*)_{n(1, j_1, k_1)n(2, j_2, k_2)} = -A(\phi + a^*)_{n(2, j_2, k_2)n(1, j_1, k_1)} = \tanh \beta E_1(\delta_{j_1+1, j_2} - \delta_{j_1 N} \delta_{j_2 k_1} \delta_{k_1 k_2})$$  \quad (2.57)

All the other elements of the matrices $A(\phi)$ and $A(\phi + a^*)$ coincide. These non-zero elements are given by the relations (2.4) - (2.6).

Due to definition the cochain $b^*$ equals 1 on the edges $((3, j, M), (4, j, 1))$, $1 \leq j \leq N$, and it is equal to zero on all other edges. The equalities (2.7) - (2.18) imply

$$A(\phi + b^*)_{n(3, j_1, k_1)n(4, j_2, k_2)} = -A(\phi + b^*)_{n(4, j_2, k_2)n(3, j_1, k_1)} = \tanh \beta E_2(\delta_{j_1, j_2} - \delta_{k_1, k_2} \delta_{k_1 k_2})$$  \quad (2.58)

All the other elements of the matrices $A(\phi)$ and $A(\phi + b^*)$ coincide. These non-zero elements are given by the relations (2.48) - (2.52), (2.54) - (2.56).

In view of the relations (2.4) - (2.7) and (2.18) and the definitions of the cochains $a^*$, $b^*$ we have

$$A(\phi + a^* + b^*)_{n(1, j_1, k_1)n(2, j_2, k_2)} = -A(\phi + a^* + b^*)_{n(2, j_2, k_2)n(1, j_1, k_1)} = \tanh \beta E_1(\delta_{j_1+1, j_2} - \delta_{j_1 N} \delta_{j_2 k_1} \delta_{k_1 k_2})$$  \quad (2.59)

$$A(\phi + a^* + b^*)_{n(3, j_1, k_1)n(4, j_2, k_2)} = -A(\phi + a^* + b^*)_{n(4, j_2, k_2)n(3, j_1, k_1)} = \tanh \beta E_2(\delta_{j_1, j_2} - \delta_{k_1, k_2} \delta_{k_1 k_2})$$  \quad (2.60)

All the other elements of the matrices $A(\phi)$ and $A(\phi + a^* + b^*)$ coincide. These non-zero elements are given by the relations (2.48) - (2.52), (2.54) - (2.56).

In the next section we calculate Pfaffians of the matrices (2.48) - (2.60).

### 3 Pfaffians

For the natural numbers $-N < j \leq N$ the following relation is valid

$$N^{-1} \sum_{j' = 1}^N \exp\{i2\pi N^{-1} jj'\} = \delta_{j0} + \delta_{j N}.$$  \quad (3.1)

The relations (2.48) - (2.52) and (3.1) imply

$$A(\phi)_{n(i_1, j_1, k_1)n(i_2, j_2, k_2)} = \sum_{i_1', i_2' = 1}^6 \sum_{j_1' = 1}^N \sum_{k_1' = 1}^M \sum_{j_2' = 1}^N \sum_{k_2' = 1}^M$$

$$B(\phi)_{n(i_1', j_1', k_1')n(i_2', j_2', k_2')} A(\phi)_{n(i_1', j_1', k_1')n(i_2', j_2', k_2')} B(\phi)_{n(i_2', j_2', k_2')n(i_2, j_2, k_2)}$$  \quad (3.2)
where
\[ B(\phi)_{n(\nu', j', k') n(\nu, j, k)} = \delta_{pp'} \exp\{i2\pi(N^{-1} j' + M^{-1} k'k)\} \] (3.3)

\[ \tilde{A}(\phi)_{n(1, j_1, k_1) n(2, j_2, k_2)} = -\tilde{A}(\phi)_{n(2, j_2, k_2) n(1, j_1, k_1)} = \tanh \beta E_1 \exp\{i2\pi N^{-1} j_1\} \delta(j_1, j_2, k_1, k_2) \] (3.4)

\[ \tilde{A}(\phi)_{n(3, j_1, k_1) n(4, j_2, k_2)} = -\tilde{A}(\phi)_{n(4, j_2, k_2) n(3, j_1, k_1)} = \tanh \beta E_2 \exp\{i2\pi M^{-1} k_1\} \delta(j_1, j_2, k_1, k_2) \] (3.5)

\[ \tilde{A}(\phi)_{n(1, j_1, k_1) n(3, j_2, k_2)} = -\tilde{A}(\phi)_{n(3, j_2, k_2) n(1, j_1, k_1)} = \tilde{A}(\phi)_{n(1, j_1, k_1) n(6, j_2, k_2)} = -\tilde{A}(\phi)_{n(6, j_2, k_2) n(1, j_1, k_1)} = \tilde{A}(\phi)_{n(2, j_1, k_1) n(4, j_2, k_2)} = -\tilde{A}(\phi)_{n(4, j_2, k_2) n(2, j_1, k_1)} = \tilde{A}(\phi)_{n(3, j_1, k_1) n(5, j_2, k_2)} = -\tilde{A}(\phi)_{n(5, j_2, k_2) n(3, j_1, k_1)} = \tilde{A}(\phi)_{n(4, j_1, k_1) n(5, j_2, k_2)} = -\tilde{A}(\phi)_{n(5, j_2, k_2) n(4, j_1, k_1)} = \tilde{A}(\phi)_{n(5, j_1, k_1) n(6, j_2, k_2)} = -\tilde{A}(\phi)_{n(6, j_2, k_2) n(5, j_1, k_1)} = \delta(j_1, j_2, k_1, k_2) \] (3.6)

\[ \delta(j_1, j_2, k_1, k_2) = (MN)^{-1}(\delta_{N-j_1, j_2} + \delta_{j_1, N}\delta_{j_2, N})(\delta_{M-k_1, k_2} + \delta_{k_1, M}\delta_{k_2, M}) \] (3.7)

All the other elements of the matrix \( \tilde{A}(\phi) \) are equal to zero.

Due to Proposition 1 from ([12], chap. IX, sect. 5)

\[ \text{Pf}(B^T \tilde{A}(\phi)B) = \det B \text{Pf} \tilde{A}(\phi). \] (3.8)

In view of the definition (3.3) the matrix \( B = I_6 \otimes C_N \otimes C_M \), where \( I_6 \) is the identity 6 \times 6 \ - matrix and \( N \times N \ - matrix (C_N)_{kj} = \exp\{i2\pi N^{-1} k j\} \). Therefore

\[ \det B = (\det C_N)^{6M}(\det C_M)^{6N}. \] (3.9)

For the determinant of the matrix \( (\tilde{C}_N)_{jk} = \exp\{i2\pi N^{-1}(N-j)k\} \) we have

\[ \det \tilde{C}_N = (-1)^{N-1} \det C_N \] (3.10)

where \([r]\) denotes the integral part of the real number \( r \). The relation (3.1) implies

\[ \tilde{C}_N C_N = NI_N \] (3.11)

where \( I_N \) is the identity \( N \times N \ - matrix \). It follows from the relations (3.10), (3.11)

\[ (\det C_N)^2 = (-1)^{N-1}N^N. \] (3.12)

The substitution of the equality (3.12) into the equality (3.3) yeilds

\[ \det B = (-1)^{M[N-1]+[N-1]}(MN)^3N. \] (3.13)
Instead of the definition (2.8) we make use of the definition from ([12], chap. IX, sect. 5). Let $e_k, k = 1, \ldots, 6MN$, be the basis of the space where the matrix $A(\phi)$ acts. Then

$$\wedge^{3MN} \left( \sum_{j,k=1}^{6MN} \tilde{A}(\phi)_{jk} e_j \wedge e_k \right) = 2^{3MN} (3MN)! \text{Pf} \tilde{A}(\phi) e_1 \wedge \cdots \wedge e_{6MN}. \quad (3.14)$$

We introduce $6 \times 6$ - matrix

$$(D(j, k))_{i_1i_2} = MN \tilde{A}(\phi)_{n(i_1,j,k),n(i_2,N-j,M-k)}$$

$$(D(N, k))_{i_1i_2} = MN \tilde{A}(\phi)_{n(i_1,N,k),n(i_2,N,M-k)}$$

$$(D(j, M))_{i_1i_2} = MN \tilde{A}(\phi)_{n(i_1,j,M),n(i_2,N-j,M)}$$

$$(D(N, M))_{i_1i_2} = MN \tilde{A}(\phi)_{n(i_1,N,M),n(i_2,N,M)} \quad (3.15)$$

where $1 \leq i_1 \leq 6, 1 \leq i_2 \leq 6, 1 \leq j < N, 1 \leq k < M$. In view of the definitions (3.4) - (3.7) the matrix (3.15) is anti - Hermitian: $(D(j, k))_{i_1i_2} = -(D(j, k))_{i_2i_1}$. For the natural numbers $j = \frac{N}{2}, N, k = \frac{M}{2}, M$ the matrix $D(j, k)$ is real and therefore antisymmetric. Hence its Pfaffian is defined for these natural numbers $j = \frac{N}{2}, N, k = \frac{M}{2}, M$

$$\text{Pf} D(j, k) = z_1 z_2 \exp\{i 2\pi (N^{-1} j + M^{-1} k)\} + z_1 \exp\{i 2\pi N^{-1} j\} + z_2 \exp\{i 2\pi M^{-1} k\} - 1 \quad (3.16)$$

where $z_i = \tanh \beta E_i, i = 1, 2$. The determinant of the matrix $D(j, k)$ is defined for any numbers $1 \leq j \leq N, 1 \leq k \leq M$

$$\det D(j, k) = (1 + z_1^2)(1 + z_2^2) - 2z_1(1 - z_2^2) \cos(2\pi N^{-1} j) - 2z_2(1 - z_1^2) \cos(2\pi M^{-1} k). \quad (3.17)$$

The equality (1.4) coincides with the relation

$$\det D(N, M) = (\text{Pf} D(N, M))^2. \quad (3.18)$$

By making use of the definitions (3.4) - (3.7) and the Pfaffian definition (3.14) it is possible to show that the Pfaffian of the $6MN \times 6MN$ - matrix $\tilde{A}(\phi)$ is proportional to the product of all Pfaffians of $6 \times 6$ - matrices $D(j, k)$ for the natural numbers $j = \frac{N}{2}, N, k = \frac{M}{2}, M$ and the product of all determinants of the matrices $D(j, k), 1 \leq j \leq N, 1 \leq k \leq M$, except $j = \frac{N}{2}, N, k = \frac{M}{2}, M$ simultaneously.

Let us introduce the products

$$P_{11}(l_1, l_2; m_1, m_2) = \prod_{j=l_1}^{l_2} \prod_{k=m_1}^{m_2} [(1 + z_1^2)(1 + z_2^2) -$$

$$-2z_1(1 - z_2^2) \cos(2\pi N^{-1} j) - 2z_2(1 - z_1^2) \cos(2\pi M^{-1} k)] \quad (3.19)$$

$$P_{21}(l_1, l_2; m_1, m_2) = \prod_{j=l_1}^{l_2} \prod_{k=m_1}^{m_2} [(1 + z_1^2)(1 + z_2^2) -$$

$$-2z_1(1 - z_2^2) \cos(\pi N^{-1}(2j - 1)) - 2z_2(1 - z_1^2) \cos(2\pi M^{-1} k)] \quad (3.20)$$
\[ P_{12}(l_1, l_2; m_1, m_2) = \prod_{j=l_1}^{l_2} \prod_{k=m_1}^{m_2} [(1 + z_1^2)(1 + z_2^2) - \]
\[ -2z_1(1 - z_2^2)\cos(2\pi N^{-1}j) - 2z_2(1 - z_1^2)\cos(\pi M^{-1}(2k - 1))] \]  
(3.21)

\[ P_{22}(l_1, l_2; m_1, m_2) = \prod_{j=l_1}^{l_2} \prod_{k=m_1}^{m_2} [(1 + z_1^2)(1 + z_2^2) - \]
\[ -2z_1(1 - z_2^2)\cos(\pi N^{-1}(2j - 1)) - 2z_2(1 - z_1^2)\cos(\pi M^{-1}(2k - 1))] \]  
(3.22)

Calculating the Pfaffian Pf \( \tilde{A}(\phi) \) and making use of the relations (3.2), (3.8) and (3.13) we have:

- if the numbers \( N, M \) are odd, then
\[ \text{Pf} A(\phi) = F_{11}(z_1, z_2)(z_1z_2 + z_1 + z_2 - 1), \]  
(3.23)

where \( z_i = \tanh \beta E_i, i = 1, 2 \) and
\[ F_{11}(z_1, z_2) = P_{11}(0, \frac{N - 1}{2}; 1, \frac{M - 1}{2})P_{11}(1, \frac{N - 1}{2}; 0, \frac{M - 1}{2}); \]  
(3.24)

- if the number \( N \) is odd and the number \( M \) is even, then
\[ \text{Pf} A(\phi) = F_{12}(z_1, z_2)(-z_1z_2 + z_1 - z_2 - 1)(z_1z_2 + z_1 + z_2 - 1), \]  
(3.25)

\[ F_{12}(z_1, z_2) = P_{11}(0, \frac{N - 1}{2}; 1, \frac{M}{2} - 1)P_{11}(1, \frac{N - 1}{2}; 0, \frac{M}{2}); \]  
(3.26)

- if the number \( N \) is even and the number \( M \) is odd, then
\[ \text{Pf} A(\phi) = F_{13}(z_1, z_2)(-z_1z_2 - z_1 + z_2 - 1)(z_1z_2 + z_1 + z_2 - 1), \]  
(3.27)

\[ F_{13}(z_1, z_2) = P_{11}(0, \frac{N}{2}; 1, \frac{M - 1}{2})P_{11}(1, \frac{N}{2} - 1; 0, \frac{M - 1}{2}); \]  
(3.28)

- if the numbers \( N, M \) are even, then
\[ \text{Pf} A(\phi) = F_{14}(z_1, z_2)(z_1z_2 - z_1 - z_2 - 1)(-z_1z_2 - z_1 + z_2 - 1) \times \]
\[ (-z_1z_2 + z_1 - z_2 - 1)(z_1z_2 + z_1 + z_2 - 1), \]  
(3.29)

\[ F_{14}(z_1, z_2) = P_{11}(0, \frac{N}{2}; 1, \frac{M}{2} - 1)P_{11}(1, \frac{N}{2} - 1; 0, \frac{M}{2}). \]  
(3.30)

Analogously the definitions (2.49) - (2.57) imply:

- if the numbers \( N, M \) are odd, then
\[ \text{Pf} A(\phi + a^*) = F_{21}(z_1, z_2)(-z_1z_2 - z_1 + z_2 - 1), \]  
(3.31)

\[ F_{21}(z_1, z_2) = P_{21}(1, \frac{N + 1}{2}; 1, \frac{M - 1}{2})P_{21}(1, \frac{N - 1}{2}; 0, \frac{M - 1}{2}); \]  
(3.32)

- if the number \( N \) is odd and the number \( M \) is even, then
\[ \text{Pf} A(\phi + a^*) = F_{22}(z_1, z_2)(z_1z_2 - z_1 - z_2 - 1)(-z_1z_2 - z_1 + z_2 - 1), \]  
(3.33)
\[ F_{22}(z_1, z_2) = P_{21} \left( 1, \frac{N + 1}{2}; 1, \frac{M}{2} - 1 \right) P_{21} \left( 1, \frac{N - 1}{2}; 0, \frac{M}{2} \right); \quad (3.34) \]

if the number \( N \) is even and the number \( M \) is odd, then

\[ PfA(\phi + a^* ) = F_{23}(z_1, z_2) = P_{21} \left( 1, \frac{N}{2}; \frac{M - 1}{2} \right) P_{21} \left( 1, \frac{N}{2}; 0, \frac{M - 1}{2} \right); \quad (3.35) \]

if the numbers \( N, M \) are even, then

\[ PfA(\phi + a^* ) = F_{24}(z_1, z_2) = P_{21} \left( 1, \frac{N}{2}; 1, \frac{M}{2} - 1 \right). \quad (3.36) \]

Similarly the definitions (2.48) - (2.52), (2.54) - (2.56) and (2.58) imply:

if the numbers \( N, M \) are odd, then

\[ PfA(\phi + a^* ) = F_{31}(z_1, z_2) = F_{31}(z_1, z_2) = P_{12} \left( 0, \frac{N - 1}{2}; 1, \frac{M - 1}{2} \right) P_{12} \left( 1, \frac{N - 1}{2}; 1, \frac{M}{2} \right); \quad (3.37) \]

if the number \( N \) is odd and the number \( M \) is even, then

\[ PfA(\phi + b^* ) = F_{32}(z_1, z_2) = P_{12} \left( 0, \frac{N - 1}{2}; 1, \frac{M}{2} - 1 \right) P_{12} \left( 1, \frac{N - 1}{2}; 1, \frac{M}{2} \right); \quad (3.38) \]

if the number \( N \) is even and the number \( M \) is odd, then

\[ PfA(\phi + b^* ) = F_{33}(z_1, z_2) = F_{33}(z_1, z_2) = P_{12} \left( 0, \frac{N}{2}; 1, \frac{M - 1}{2} \right) P_{12} \left( 1, \frac{N}{2}; 1, \frac{M}{2} \right); \quad (3.39) \]

if the numbers \( N, M \) are even, then

\[ PfA(\phi + b^* ) = F_{34}(z_1, z_2) = P_{12} \left( 0, \frac{N}{2} - 1; 1, \frac{M}{2} \right) P_{12} \left( 1, \frac{N}{2}; 1, \frac{M}{2} \right). \quad (3.40) \]

Similarly the definitions (2.49) - (2.52), (2.54) - (2.58), (2.59, (2.60) imply:

if the numbers \( N, M \) are odd, then

\[ PfA(\phi + a^* + b^* ) = F_{41}(z_1, z_2) = F_{41}(z_1, z_2) = P_{22} \left( 1, \frac{N + 1}{2}; 1, \frac{M - 1}{2} \right) P_{22} \left( 1, \frac{N - 1}{2}; 1, \frac{M + 1}{2} \right); \quad (3.41) \]

if the number \( N \) is odd and the number \( M \) is even, then

\[ PfA(\phi + a^* + b^* ) = F_{42}(z_1, z_2) = P_{22} \left( 1, \frac{N + 1}{2}; 1, \frac{M}{2} \right) P_{22} \left( 1, \frac{N}{2}; 1, \frac{M}{2} \right). \quad (3.42) \]
if the number $N$ is even and the number $M$ is odd, then

$$\text{Pf} A(\phi + a^* + b^*) = F_{43}(z_1, z_2) = P_{22}(1, \frac{N}{2}, 1, \frac{M - 1}{2}) P_{22}(1, \frac{N}{2}, 1, \frac{M + 1}{2});$$

(3.46)

if the numbers $N, M$ are even, then

$$\text{Pf} A(\phi + a^* + b^*) = F_{44}(z_1, z_2) = \left( P_{22}(1, \frac{N}{2}, 1, \frac{M}{2}) \right)^2.$$

(3.47)

The formulae (3.23) - (3.47) are in accordance with the formulae [9] for $\det \bar{A}_{i}$. The formulae (1.4), (3.23) - (3.47) imply

$$\det A(\phi) = P_{11}(1, N; 1, M)$$

$$\det A(\phi + a^*) = P_{21}(1, N; 1, M)$$

$$\det A(\phi + b^*) = P_{12}(1, N; 1, M)$$

$$\det A(\phi + a^* + b^*) = P_{22}(1, N; 1, M).$$

(3.48)

The expressions (3.48) coincide with the expressions for $\det \bar{A}_4, i = 1, \ldots, 4$, given in the book [3].

### 4 Free Energy

The following lemma is the straightforward consequence of the relation (1.4).

**Lemma 4.1** For the variables $z_i = \tanh \beta E_i$, $\beta > 0$, $E_i \neq 0$, $i = 1, 2$ the polynomial

$$(1 + z_1^2)(1 + z_2^2) - 2z_1(1 - z_2^2) \cos \phi_1 - 2z_2(1 - z_1^2) \cos \phi_2 > 0$$

(4.1)

if at least one of the following conditions

$$\cos \phi_i = \text{sgn} z_i, i = 1, 2,$$

(4.2)

is not valid.

**Proof.** Since $0 < |z_i| < 1$, $i = 1, 2$, the following inequality holds

$$(1 + z_1^2)(1 + z_2^2) - 2z_1(1 - z_2^2) \cos \phi_1 - 2z_2(1 - z_1^2) \cos \phi_2 \geq (1 + z_1^2)(1 + z_2^2) - 2|z_1|(1 - z_2^2) - 2|z_2|(1 - z_1^2)$$

(4.3)

Due to the equality (1.4) the right hand side of the inequality (1.3) is equal to

$$\left( |z_1 z_2| + |z_1| + |z_2| - 1 \right)^2.$$

(4.4)

The inequality (1.3) may be an equality if and only if both conditions (1.2) are valid. The lemma is proved.

**Lemma 4.2** The range of values for the variables $z_i = \tanh \beta E_i$, $\beta > 0$, $E_i \neq 0$, $i = 1, 2$ is divided into eight two dimensional domains:

$$(1 + z_1)(1 + z_2) > 2, (1 - z_1)(1 + z_2) < 2,$$

$$(1 + z_1)(1 - z_2) < 2, (1 - z_1)(1 - z_2) < 2,$$

$$z_1 > 0, z_2 > 0;$$

(4.5)
\[(1 + z_1)(1 + z_2) < 2, (1 - z_1)(1 + z_2) < 2, \]
\[(1 + z_1)(1 - z_2) < 2, (1 - z_1)(1 - z_2) < 2, \]
\[z_1 > 0, z_2 > 0; \quad (4.6)\]

\[(1 + z_1)(1 + z_2) < 2, (1 - z_1)(1 + z_2) > 2, \]
\[(1 + z_1)(1 - z_2) < 2, (1 - z_1)(1 - z_2) < 2, \]
\[z_1 < 0, z_2 > 0; \quad (4.7)\]

\[(1 + z_1)(1 + z_2) < 2, (1 - z_1)(1 + z_2) < 2, \]
\[(1 + z_1)(1 - z_2) < 2, (1 - z_1)(1 - z_2) < 2, \]
\[z_1 < 0, z_2 > 0; \quad (4.8)\]

\[(1 + z_1)(1 + z_2) < 2, (1 - z_1)(1 + z_2) < 2, \]
\[(1 + z_1)(1 - z_2) > 2, (1 - z_1)(1 - z_2) < 2, \]
\[z_1 > 0, z_2 < 0; \quad (4.9)\]

\[(1 + z_1)(1 + z_2) < 2, (1 - z_1)(1 + z_2) < 2, \]
\[(1 + z_1)(1 - z_2) < 2, (1 - z_1)(1 - z_2) < 2, \]
\[z_1 > 0, z_2 < 0; \quad (4.10)\]

\[(1 + z_1)(1 + z_2) < 2, (1 - z_1)(1 + z_2) < 2, \]
\[(1 + z_1)(1 - z_2) < 2, (1 - z_1)(1 - z_2) > 2, \]
\[z_1 < 0, z_2 < 0; \quad (4.11)\]

\[(1 + z_1)(1 + z_2) < 2, (1 - z_1)(1 + z_2) < 2, \]
\[(1 + z_1)(1 - z_2) < 2, (1 - z_1)(1 - z_2) < 2, \]
\[z_1 < 0, z_2 < 0; \quad (4.12)\]

and into four one dimensional domains:

\[(1 + z_1)(1 + z_2) = 2, (1 - z_1)(1 + z_2) < 2, \]
\[(1 + z_1)(1 - z_2) < 2, (1 - z_1)(1 - z_2) < 2, \]
\[z_1 > 0, z_2 > 0; \quad (4.13)\]

\[(1 + z_1)(1 + z_2) < 2, (1 - z_1)(1 + z_2) = 2, \]
\[(1 + z_1)(1 - z_2) < 2, (1 - z_1)(1 - z_2) < 2, \]
\[z_1 < 0, z_2 > 0; \quad (4.14)\]

\[(1 + z_1)(1 + z_2) < 2, (1 - z_1)(1 + z_2) < 2, \]
\[(1 + z_1)(1 - z_2) = 2, (1 - z_1)(1 - z_2) < 2, \]
\[z_1 > 0, z_2 < 0; \quad (4.15)\]
\[(1 + z_1)(1 + z_2) < 2, (1 - z_1)(1 + z_2) < 2, (1 + z_1)(1 - z_2) < 2, (1 - z_1)(1 - z_2) = 2, \]
\[z_1 < 0, z_2 < 0. \quad (4.16)\]

**Proof.** Let two sets of numbers \(\epsilon_i = \pm 1, \sigma_i = \pm 1, i = 1, 2\) be given. Let us prove that if these two sets do not coincide, then two inequalities

\[(1 + \epsilon_1 z_1)(1 + \epsilon_2 z_2) \geq 2 \]
\[(1 + \sigma_1 z_1)(1 + \sigma_2 z_2) \geq 2 \quad (4.17)\]

are not compatible for the variables \(z_i = \tanh \beta E_i, i = 1, 2\). Indeed, let \(\epsilon_1 \neq \sigma_1\), i.e. \(\epsilon_1 \sigma_1 = -1\). Multiplying the inequalities (4.17) we obtain the inequality
\[(1 - z_1^2)(1 + \epsilon_2 z_2)(1 + \sigma_2 z_2) \geq 4 \quad (4.18)\]

Since \(z_i = \tanh \beta E_i, \beta > 0, E_i \neq 0, i = 1, 2\), then \(0 < |z_i| < 1\) and
\[0 < 1 - z_1^2 < 1, 0 < 1 + \epsilon_2 z_2 < 2, 0 < 1 + \sigma_2 z_2 < 2. \quad (4.19)\]

The inequalities (4.18) and (4.19) are not compatible. The case \(\epsilon_2 \neq \sigma_2\) is considered similarly.

Let us prove that for the variables \(z_i = \tanh \beta E_i, \beta > 0, E_i \neq 0, i = 1, 2\), the first inequality (4.17) implies \(\epsilon_i z_i > 0, i = 1, 2\). Indeed, for the variables \(z_i\), the inequality \(0 < 1 + \epsilon_i z_i < 2\) is valid. Hence the first inequality (4.17) implies \(1 + \epsilon_i z_i > 1\) or \(\epsilon_i z_i > 0\). The lemma is proved.

**Theorem 4.3** For the partition function (2.2) the following relation is valid
\[\beta^{-1} \lim_{M,N \to \infty} (MN)^{-1} \ln Z(N, M) = \]
\[-\beta^{-1} \ln(2 \cosh \beta E_1 \cosh \beta E_2) - 1/2 \beta^{-1}(2\pi)^{-2} \int_{0}^{2\pi} d\theta_1 \int_{0}^{2\pi} d\theta_2 \ln[(1 + z_1^2)(1 + z_2^2) - 2z_1(1 - z_2^2) \cos \theta_1 - 2z_2(1 - z_1^2) \cos \theta_2] \quad (4.20)\]

where the variables \(z_i = \tanh \beta E_i, \beta > 0, E_i \neq 0, i = 1, 2\).

**Proof.** Let the numbers \(N, M\) be odd. Then the homological formula (2.42) and the equalities (3.23), (3.31), (3.37), (3.43) imply
\[Z(N, M) = (2 \cosh \beta E_1 \cosh \beta E_2)^{MN} \times \]
\[1/2[F_{11}(z_1, z_2)((1 + z_1)(1 + z_2) - 2) - F_{21}(z_1, z_2)((1 - z_1)(1 + z_2) - 2) - \]
\[-F_{31}(z_1, z_2)((1 + z_1)(1 - z_2) - 2) - F_{41}(z_1, z_2)((1 - z_1)(1 - z_2) - 2)]. \quad (4.21)\]

Since the function \(\cos x\) is monotone decreasing on the interval \([0, \pi]\), we have
\[\cos(2\pi N^{-1}j) < \cos(\pi N^{-1}(2j - 1)), 1 \leq j \leq \frac{N - 1}{2}, \]
\[\cos(\pi N^{-1}(2(j + 1) - 1)) < \cos(2\pi N^{-1}j), 0 \leq j \leq \frac{N - 3}{2}. \quad (4.22)\]

It follows from the relations (3.24), (3.32), (3.38), (3.44) and the inequalities (4.22) that for \(z_1 > 0, z_2 > 0\)
\[F_{21}(z_1, z_2) < \]
\[P_{21}(\frac{N + 1}{2}, \frac{N + 1}{2}; 1, \frac{M - 1}{2})(P_{11}(0, 0; 1, \frac{M - 1}{2}))^{-1}F_{11}(z_1, z_2) \quad (4.23)\]
\[
F_{21}(z_1, z_2) > P_{21}(1, 1; 0, \frac{M - 1}{2})P_{21}(\frac{N + 1}{2}, \frac{N + 1}{2}; 1, \frac{M - 1}{2}) \times \\
(P_{11}(\frac{N - 1}{2}, \frac{N - 1}{2}; 0, \frac{M - 1}{2})P_{11}(\frac{N - 1}{2}, \frac{N - 1}{2}; 1, \frac{M - 1}{2}))^{-1}F_{11}(z_1, z_2)
\] (4.24)

\[
F_{31}(z_1, z_2) < P_{12}(\frac{N - 1}{2}; \frac{M + 1}{2}, \frac{M + 1}{2})(P_{11}(1, \frac{N - 1}{2}; 0, 0))^{-1}F_{11}(z_1, z_2)
\] (4.25)

\[
F_{31}(z_1, z_2) > P_{12}(0, \frac{N - 1}{2}; 1, 1)P_{12}(1, \frac{N - 1}{2}; \frac{M + 1}{2}, \frac{M + 1}{2}) \times \\
(P_{11}(0, \frac{N - 1}{2}; \frac{M - 1}{2}, \frac{M - 1}{2})P_{11}(1, \frac{N - 1}{2}; \frac{M - 1}{2}, \frac{M - 1}{2}))^{-1}F_{11}(z_1, z_2)
\] (4.26)

\[
F_{41}(z_1, z_2) < P_{22}(1, \frac{N - 1}{2}; \frac{M + 1}{2}, \frac{M + 1}{2})P_{22}(\frac{N + 1}{2}, \frac{N + 1}{2}; 1, \frac{M - 1}{2}) \times \\
(P_{11}(1, \frac{N - 1}{2}; 0, 0)P_{11}(0, 0; 1, \frac{M - 1}{2}))^{-1}F_{11}(z_1, z_2)
\] (4.27)

\[
F_{41}(z_1, z_2) > P_{22}(1, \frac{N - 1}{2}; 1, 1)P_{22}(1, 1; \frac{M + 1}{2}, \frac{M + 1}{2}) \times \\
P_{22}(1, \frac{N - 1}{2}; \frac{M + 1}{2}, \frac{M + 1}{2})P_{22}(\frac{N + 1}{2}, \frac{N + 1}{2}; 1, \frac{M - 1}{2}) \times \\
(P_{11}(0, \frac{N - 3}{2}; \frac{M - 1}{2}, \frac{M - 1}{2})P_{11}(\frac{N - 1}{2}, \frac{N - 1}{2}; 0, \frac{M - 3}{2}))(\frac{N - 1}{2}; \frac{N - 1}{2}; \frac{M - 1}{2}))^{-1} \times \\
(P_{11}(1, \frac{N - 1}{2}; \frac{M - 1}{2}, \frac{M - 1}{2})P_{11}(\frac{N - 1}{2}, \frac{N - 1}{2}; 1, \frac{M - 1}{2}))^{-1}F_{11}(z_1, z_2).
\] (4.28)

The relation (4.21) and the inequalities (4.23), (4.23), (4.27) imply for the domains (4.5), (4.6) and (4.13)

\[
2Z(N, M)(2 \cosh \beta E_1 \cosh \beta E_2)^{-MN}F_{11}(z_1, z_2))^{-1} < \\
(1 + z_1)(1 + z_2) - 2)
P_{21}(\frac{N + 1}{2}, \frac{N + 1}{2}; \frac{1}{2}, \frac{M - 1}{2})P_{11}(0, 0; 1, \frac{M - 1}{2}))^{-1} - \\
-((1 - z_1)(1 + z_2) - 2)P_{21}(\frac{N + 1}{2}, \frac{N + 1}{2}; \frac{1}{2}, \frac{M - 1}{2})P_{11}(0, 0; 1, \frac{M - 1}{2}))^{-1} - \\
-((1 + z_1)(1 - z_2) - 2)P_{21}(\frac{N + 1}{2}, \frac{N + 1}{2}; \frac{1}{2}, \frac{M - 1}{2})P_{11}(0, 0; 1, \frac{M - 1}{2}))^{-1} - \\
-((1 - z_1)(1 - z_2) - 2)P_{21}(\frac{N - 1}{2}, \frac{M + 1}{2}; \frac{1}{2}, \frac{M - 1}{2})P_{22}(\frac{N - 1}{2}, \frac{M + 1}{2}; \frac{1}{2}, \frac{M - 1}{2}) \times \\
P_{11}(1, \frac{N - 1}{2}; 0, 0)P_{11}(0, 0; 1, \frac{M - 1}{2}))^{-1}(4.29)
\]

By making use of the inequalities (4.22) we have

\[
P_{11}(1, \frac{N - 1}{2}; 0, 0)(P_{22}(1, \frac{N - 1}{2}; \frac{M + 1}{2}, \frac{M + 1}{2}))^{-1} < \\
((1 + z_1^2)(1 + z_2^2) + 2z_1(1 - z_2^2) \cos(\pi N^{-1}) - 2z_2(1 - z_1^2)) \times \\
((1 + z_1^2)(1 + z_2^2) - 2z_1(1 - z_2^2) \cos(\pi N^{-1}) + 2z_2(1 - z_1^2))^{-1}(4.30)
\]

\(20\)
In view of Lemma 4.1 the right hand sides of the equalities (4.30) - (4.33) are bounded. By dividing the inequality (4.29) by

\[ P_{22}(N, N - 1, M, M + 1) < (1 + z_1^2)(1 + z_2^2) - 2z_1(1 - z_2^2) - 2z_2(1 - z_1^2) \cos(\pi N^{-1}) \times \]

\[ (1 + z_1^2)(1 + z_2^2) + 2z_1(1 - z_2^2) - 2z_2(1 - z_1^2) \cos(\pi M^{-1}))^{-1} \]

(4.31)

\[ P_{21}(N, N - 1, M, M + 1) < (1 + z_1^2)(1 + z_2^2) - 2z_1(1 - z_2^2) - 2z_2(1 - z_1^2) \cos(\pi N^{-1}) + 2z_2(1 - z_1^2) \times \]

\[ (1 + z_1^2)(1 + z_2^2) - 2z_1(1 - z_2^2) \cos(\pi N^{-1}) + 2z_2(1 - z_1^2))^{-1} \]

(4.32)

\[ P_{12}(N, N - 1, M, M + 1) < (1 + z_1^2)(1 + z_2^2) - 2z_1(1 - z_2^2) \cos(\pi N^{-1}) + 2z_2(1 - z_1^2) \times \]

\[ (1 + z_1^2)(1 + z_2^2) - 2z_1(1 - z_2^2) \cos(\pi N^{-1}) + 2z_2(1 - z_1^2))^{-1} \]

(4.33)

and by using the inequalities (4.30) - (4.33) we have

\[-\beta^{-1} \lim_{M,N \to \infty} (MN)^{-1} \ln Z(N, M) \geq -\beta^{-1} \ln(2 \cosh \beta E_1 \cosh \beta E_2) - 1/2 \beta^{-1}(2\pi)^{-2} \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \]

\[ \ln[(1 + z_1^2)(1 + z_2^2) - 2z_1(1 - z_2^2) \cos \theta_1 - 2z_2(1 - z_1^2) \cos \theta_2] \]

(4.34)

for the domains (4.5), (4.6) and (4.13).

The relation (4.21) and the inequalities (4.24), (4.26), (4.28) imply for the domains (4.5), (4.6) and (4.13)

\[ 2Z(N, M)(2 \cosh \beta E_1 \cosh \beta E_2)^{-MN}(F_{11}(z_1, z_2))^{-1} > \]

\[ ((1 + z_1)(1 + z_2) - 2) \times \]

\[ -((1 - z_1)(1 + z_2) - 2)P_{21}(1, 1; 0, M - 1) \times \]

\[ (P_{11}(N - 1, N - 1; 0, M - 1))^{-1} - \]

\[ -((1 + z_1)(1 - z_2) - 2)P_{12}(0, N - 1; 1, 1) \times \]

\[ (P_{11}(N - 1, M - 1; 1, M - 1))^{-1} - \]

\[ -((1 - z_1)(1 - z_2) - 2)P_{22}(1, N - 1; 1, 1) \times \]

\[ (P_{11}(N - 1, M - 1; 0, M - 1))^{-1} \]

(4.35)
By using the inequalities (4.22) it is easy to obtain for the domains (4.5), (4.6), (4.13) for large numbers $N,M$ the inequalities similar to the inequalities (4.30) - (4.33) for the products

\begin{align*}
P_{21}(1,1;0, N-1) & \times P_{21}(0, N-1; 0, M-1) \\
(P_{11}(N-1, N-1; 0, M-1))^{-1} & \\
(P_{11}(N-1, M-1; 0, M-1) & \times (P_{11}(N-1, M-1; 0, M-1))^{-1} \\
\times (P_{11}(N-1, N-1; 0, M-1))^{-1} \times (P_{11}(N-1, M-1; 0, M-1))^{-1}.
\end{align*}

Therefore the inequality (4.35) implies the following inequality

\begin{align*}
-\beta^{-1} \lim_{M,N \to \infty} (MN)^{-1} \ln Z(N,M) & \leq \\
-\beta^{-1} \ln(2 \cosh \beta E_1 \cosh \beta E_2) - 1/2 \beta^{-1}(2\pi)^{-2} \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \\
\ln[(1 + z_1^2)(1 + z_2^2) - 2z_1(1 - z_2^2) \cos \theta_1 - 2z_2(1 - z_1^2) \cos \theta_2]
\end{align*}

(4.36)

for the domains (1.35), (1.36), (1.38). The equality (4.20) for the domains (1.35), (1.40), (1.43) follows from the inequalities (4.34), (4.36).

Let the numbers $N,M$ be odd and $z_1 < 0$, $z_2 > 0$. The relations (3.24), (3.32), (3.38), (3.44) and inequalities (4.22) imply

\begin{align*}
F_{11}(z_1, z_2) & < \\
P_{11}(0, 0; 1, N-1) \times P_{21}(1, N-1; 0, M-1) \times
\end{align*}

\begin{align*}
(P_{11}(N-1, N-1; 0, M-1))^{-1} \times (P_{11}(N-1, M-1; 0, M-1))^{-1}.
\end{align*}

(4.37)
\[ F_{11}(z_1, z_2) > P_{11}(1, \frac{N - 1, N - 1}{2}; 1, \frac{M - 1}{2})P_{11}(\frac{N - 1}{2}, \frac{N - 1}{2}; 0, \frac{M - 1}{2}) \times \\
( P_{21}(\frac{N + 1}{2}, \frac{N + 1}{2}; 1, \frac{M - 1}{2})P_{21}(1, 1; 0, \frac{M - 1}{2}))^{-1} F_{21}(z_1, z_2) \] (4.38)

\[ F_{31}(z_1, z_2) < P_{12}(0, 0; 1, \frac{M - 1}{2})P_{12}(1, \frac{N - 1}{2}; \frac{M + 1}{2}, \frac{M + 1}{2}) \times \\
( P_{21}(1, \frac{N - 1}{2}; 0, 0)P_{21}(\frac{N + 1}{2}, \frac{N + 1}{2}; 1, \frac{M - 1}{2}))^{-1} F_{21}(z_1, z_2) \] (4.39)

\[ F_{31}(z_1, z_2) > P_{21}(1, \frac{N - 1}{2}; 0, 0)P_{12}(1, \frac{N - 1}{2}; \frac{M + 1}{2}, \frac{M + 1}{2}) \times \\
( P_{12}(\frac{N - 1}{2}, \frac{N - 1}{2}; 1, \frac{M - 1}{2}))^2(P_{21}(1, \frac{N - 1}{2}; \frac{M - 1}{2}, \frac{M - 1}{2}))^{-2} \times \\
( P_{21}(1, 1; 0, \frac{M - 3}{2})P_{21}(\frac{N + 1}{2}, \frac{N + 1}{2}; 1, \frac{M - 1}{2}))^{-1} F_{21}(z_1, z_2) \] (4.40)

\[ F_{41}(z_1, z_2) < P_{22}(1, \frac{N - 1}{2}; \frac{M + 1}{2}, \frac{M + 1}{2}) \times \\
( P_{21}(1, \frac{N - 1}{2}; 0, 0))^{-1} F_{21}(z_1, z_2) \] (4.41)

\[ F_{41}(z_1, z_2) > P_{21}(1, \frac{N + 1}{2}; 0, 0)P_{22}(1, \frac{N - 1}{2}; \frac{M + 1}{2}, \frac{M + 1}{2}) \times \\
( P_{21}(1, \frac{N + 1}{2}; \frac{M - 1}{2}, \frac{M - 1}{2})P_{21}(1, \frac{N - 1}{2}; \frac{M - 1}{2}, \frac{M - 1}{2}))^{-1} F_{21}(z_1, z_2). \] (4.42)

By making use of the relation (4.21) and the inequalities (4.38), (4.39), (4.41) it is possible to obtain the inequality (4.34) for the domains (4.7), (4.8), (4.14). Similarly the relation (4.21) and the inequalities (4.37), (4.40), (4.42) imply the inequality (4.36) for the domains (4.7), (4.8), (4.14). The inequalities (4.34), (4.36) for the domains (4.7), (4.8), (4.14) imply the equality (4.20) for these domains.

The proof of the equality (4.20) for odd numbers \( N, M \) and for the domains (4.9), (4.10), (4.15) is quite similar to the proof of the equality for odd numbers \( N, M \) and for the domains (4.7), (4.8), (4.14).

Let the numbers \( N, M \) be odd and \( z_1 < 0, z_2 < 0 \). The relations (3.24), (3.32), (3.38), (3.44) and the inequalities (4.22) imply

\[ F_{11}(z_1, z_2) < P_{11}(1, \frac{N - 1}{2}; 0, 0)P_{11}(0, 0; 1, \frac{M - 1}{2}) \times \\
( P_{22}(1, \frac{N - 1}{2}; \frac{M + 1}{2}, \frac{M + 1}{2})P_{22}(\frac{N + 1}{2}, \frac{N + 1}{2}; 1, \frac{M - 1}{2}))^{-1} F_{21}(z_1, z_2) \] (4.43)

\[ F_{11}(z_1, z_2) > P_{11}(1, \frac{N - 1}{2}; \frac{M - 1}{2}, \frac{M - 1}{2})P_{11}(0, \frac{N - 3}{2}; \frac{M - 1}{2}, \frac{M - 1}{2}) \times \\
P_{11}(\frac{N - 1}{2}, \frac{N - 1}{2}; 1, \frac{M - 1}{2})P_{11}(\frac{N - 1}{2}, \frac{N - 1}{2}; 0, \frac{M - 3}{2}) \times \\
( P_{22}(1, \frac{N - 1}{2}; 1, 1)P_{22}(1, \frac{N - 1}{2}; \frac{M + 1}{2}, \frac{M + 1}{2}))^{-1} \times \\
( P_{22}(\frac{N + 1}{2}, \frac{N + 1}{2}; 1, \frac{M - 1}{2})P_{22}(1, 1; 1, \frac{M - 1}{2}))^{-1} F_{21}(z_1, z_2) \] (4.44)
\[
F_{21}(z_1, z_2) < P_{21}(1, \frac{N-1}{2}, 0, 0; 0, 0) P_{22}(1, \frac{N-1}{2}, \frac{M+1}{2}, \frac{M+1}{2}))^{-1} F_{41}(z_1, z_2) \]  
(4.45)

\[
F_{21}(z_1, z_2) > P_{21}(1, \frac{N-1}{2}, \frac{M-1}{2}, \frac{M-1}{2}) P_{21}(1, \frac{N+1}{2}, \frac{M-1}{2}, \frac{M-1}{2}) \times \n(P_{22}(1, \frac{N+1}{2}, 1, 1) P_{22}(1, \frac{N-1}{2}, \frac{M+1}{2}, \frac{M+1}{2})^{-1} F_{41}(z_1, z_2) \]  
(4.46)

\[
F_{31}(z_1, z_2) < P_{12}(0, 0; 1, \frac{M-1}{2}) P_{22}(\frac{N+1}{2}, \frac{N+1}{2}; 1, \frac{M-1}{2}))^{-1} F_{41}(z_1, z_2) \]  
(4.47)

\[
F_{31}(z_1, z_2) > P_{12}(\frac{N-1}{2}, \frac{N-1}{2}; 1, \frac{M-1}{2}) P_{12}(\frac{N-1}{2}, \frac{N-1}{2}; 1, \frac{M-1}{2}) \times \n(P_{22}(1, 1; 1, M+1) P_{22}(\frac{N+1}{2}, \frac{N+1}{2}; 1, \frac{M-1}{2}))^{-1} F_{41}(z_1, z_2). \]  
(4.48)

The inequalities (4.43) - (4.48) imply the inequalities (4.34), (4.36) and therefore the equality (4.20) for the domains (4.11), (4.12), (4.16). The equality (4.20) is proved for the odd numbers \(N, M\).

The cases when \(N\) is odd, \(M\) is even; when \(N\) is even, \(M\) is odd and when \(N, M\) are even are analogous to the case when the numbers \(N, M\) are odd. The theorem is proved.

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