Yet another hysteresis model

Sergey Langvagen
Chernogolovka, Moscow region, Russia
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INTRODUCTION

The problem of hysteresis modeling is a subject of persistent interest. One of the most popular, Preisach model, was proposed more than 65 years ago [1]. It was initially based on some hypothesis concerning physical mechanisms of magnetization, but now it is considered mainly as a mathematical tool for describing various hysteresis phenomena. Preisach model has many attractive features such as simplicity, flexibility and ability to represent most important properties of real hysteresis systems [2, 3, 4]. There are known numerous applications of Preisach model in different areas of physics. The postulates of Preisach model lead to the return-point memory [5, 6].

In this article we introduce a simple hysteresis model, which we call state vector model. It uses very different postulates but is similar to Preisach model in that it is compliant with the return-point memory and can be easily analysed.

DESCRIPTION OF THE MODEL

Let us assume that states of a system can be represented by the “state vector” \( x \) with components \( x_\alpha \) equal to +1 or −1, where \( \alpha = 1 \ldots N \), and \( N \) is some large integer.

Suppose that the magnetization \( M \) (and other physical values related to the system) depend on \( x \):

\[
M = M(x_1, \ldots, x_N). \tag{1}
\]

The behavior of the state vector \( x \) is postulated as follows.

1. When the magnetic field \( H \) increases, negative components \( x_\alpha \) change the sign from −1 to +1 one by one in order of increasing \( \alpha \).

2. When the magnetic field \( H \) decreases, positive components \( x_\alpha \) change the sign from +1 to −1 one by one in order of increasing \( \alpha \).

3. Each state \( x = (x_1, x_2, \ldots, x_N) \) of the system has a value of the external magnetic field

\[
H = H(x_1, \ldots, x_N) \tag{2}
\]

near which the state is stable.

Note that Eq. (3) must not contradict the rules 1, 2. While \( x \) changes according to the rules 1 and 2, the point defined by Eq. (1), (3) form a trace on the \( HM \)-plane. Due to the above postulates the system in our model exactly obeys the return-point memory: this will be discussed in the next section.

The model admits following interpretation in terms of the “ensemble of domains”. We may consider components \( x_\alpha \) as indicating the magnetic state of the “domains”, and assume that each of them reverses its sign in one Barkhausen jump. In this case for the specimen of unit volume instead of Eq. (3) holds

\[
M = \sum_{\alpha=1}^{N} m_\alpha x_\alpha,
\]

where \( m_\alpha > 0 \) and \( x_\alpha \) denote the absolute value of the magnetic moment and its direction respectively for each domain.

Due to the hysteresis curve symmetry, it is natural to assume that \( M(-x) = -M(x) \) and \( H(-x) = -H(x) \), which means that if the state \( x \) is stable in the external field \( H \), the state \( -x \) is stable in the field \( -H \).

All components \( x_\alpha \) become equal to +1 in a large positive field, and equal to −1 in a large negative field. Demagnetized state can be obtained as the result of applying to the system alternating magnetic field of slowly decreasing magnitude, which gives the state vector with “disordered” components. We may take as “pure” demagnetized state the pair of states \((+1, -1, +1, \ldots, \pm 1)\) and \((-1, +1, -1, \ldots, \pm 1)\). The behavior of the state vector is illustrated in Fig. 1.

In sequel the following “continuous” formulation of the model will be convenient. Let us arrange all \( x_\alpha \) on the interval \([0, 1]\) in the order of increasing \( \alpha \) (Fig. 2). Function \( x(\xi) : [0, 1] \to \{-1, 1\} \) determines the state of the system as follows: if \( x(\xi) \) equals to +1 (−1) on some interval, then all \( x_\alpha \) are equal to +1 (−1) on this interval. Instead of the rules 1 and 2, which determine the behavior of components \( x_1, \ldots, x_N \), we have the following rules for \( x(\xi) \).
The process starts from the demagnetized state \((O)\). On the descending curve \(OA\) the magnetic field decreases, and \(x_α\) become equal to \(-1\) one by one from left to right. After the point \(A\) the magnetic field increases; starting from this point the components \(x_α\) become equal to \(+1\) one by one, also from left to right. After the point \(B\) the magnetic field decreases again and \(x_α\) become equal to \(-1\) in the same succession, and so on.

1*. When the magnetic field \(H\) increases, the leftmost point where \(x(ξ)\) changes the sign from \(+1\) to \(-1\), moves from left to right.

2*. When the magnetic field \(H\) decreases, the leftmost point where \(x(ξ)\) changes the sign from \(-1\) to \(+1\), moves from left to right.

To express the rules in a simple form, we have assumed that \(x(ξ)\) always changes the sign at \(ξ = 0\). With \(x(ξ)\) the right sides of Eq. (3) can be rewritten in the form of functionals

\[
M = M[x(ξ)], \quad H = H[x(ξ)].
\]

Here \(H[x(ξ)]\) must not contradict 1*, 2*.

RETURN-POINT MEMORY

Let us show that the model can be obtained as a consequence of the return-point memory [3].

Return-point memory. — Suppose the system is evolved under field \(H(t)\), where \(0 < t < T\) and \(H(0) \leq H(t) \leq H(T)\) or \(H(0) \geq H(t) \geq H(T)\). Then for a given initial state of the system the final state depends only on \(H(T)\), and is independent of the time \(T\) or the history \(H(t)\).

Besides of the return-point memory the existence of states with some special properties is required. We suppose that a system has two states \(A\) and \(B\), with corresponding fields \(H_A\) and \(H_B\), such that: i) the system evolves from the state \(A\) to the state \(B\), if the magnetic field monotonically increases from \(H_A\) to \(H_B\); ii) the system evolves from the state \(B\) to the state \(A\), if the magnetic field monotonically decreases from \(H_B\) to \(H_A\).

In the remaining part of this section we will usually assume that the initial state of the system is \(A\), all inputs \(H(t)\) are continuous and piecewise monotonic and for each of them holds \(H(0) = H_A\), \(H_A \leq H(t) \leq H_B\). We shall restrict our consideration to the set of states \(\Sigma_{AB}\) reachable from \(A\) by applying \(H(t)\) that satisfy the above conditions.

Note that if the initial state is \(A\), \(H(0) = H_A\) and \(H_A \leq H(t) \leq H_B\), then \(H(T) = H_A\) implies that the final state of the system is \(A\), and \(H(T) = H_B\) implies that the final state of the system is \(B\).

The last statement concerning the final state \(B\) follows directly from the return-point memory. To ensure that it is true for the final state \(A\) consider the input \(H(t)\) that is applied to the system in the state \(B\) and is composed of \(H(t)\) before which the magnetic field decreases monotonically from \(H_B\) to \(H_A\). It is clear that \(H(t)\) and \(\tilde{H}(t)\) have the same final states; from the return-point memory follows that the final state of \(\tilde{H}(t)\) is \(A\).

Consider some input \(H(t)\), \(H_A \leq H(t) \leq H_B\), \(H(0) = H_A\), that is applied to the system in the state \(A\). Let \(H(t)\) increase from \(H_0 = H_A\) to \(H_1\), then decrease from \(H_1\) to \(H_2\), increase from \(H_2\) to \(H_3\) and so on, where \(H_1, \ldots, H_{n-1}\) are successive maxima and minima of \(H(t)\), and \(H_n = H(T)\).

Prime inputs. — Let us call a piecewise monotonic input \(H(t)\) prime, if \(H(0) = H_A\), \(H_A \leq H(t) \leq H_B\), and one of the following is true: (i) the input is invariable \(H(t) \equiv H_A\); (ii) the input \(H(t)\) is monotonically increasing; (iii) for all successive maxima and minima holds \(|H_k - H_{k+1}| < |H_k - H_{k-1}|\) where \(k = 1, \ldots, n-1\) and \(n \geq 1\).
It is reasonable do not distinguish prime inputs with identical values \(H_0, \ldots, H_n\), because according to the return-point memory such inputs give the same final states. With this agreement any prime input is determined by the values \(H_0, \ldots, H_n\) where \(H_0 = H_A\); the case \(n = 0\) corresponds to the invariable input.

Lemma. — Each state in the set \(\Sigma_{AB}\) can be obtained from the state \(A\) by applying a prime input.

Proof. — For any arbitrary state \(s\) in \(\Sigma_{AB}\) exists piecewise monotonic input \(H(t)\) with the final state \(s\), such that \(H(0) = H_A\), \(H_A \leq H(t) \leq H_B\), \(0 \leq t \leq T\). If \(H(t) = H_A\) at some point \(t'\), then \(H(t)\) may be replaced on the interval \([0, t']\) with invariant function \(H(t) = H_A\). In the case \(t' = T\) we have the invariable input that is prime and has the the final state \(A\). Otherwise, we may assume that \(H(t)\) does not remain constant on any subinterval, because, as follows from the return-point memory, on each interval of monotony \(H(t)\) can be replaced with monotonically increasing or decreasing function without affecting the final state. This means that an arbitrary state \(s \neq A\) can be obtained by applying \(H(t)\) that increases from \(H_0 = H_A\) to \(H_1\), then decreases from \(H_1\) to \(H_2\), increases from \(H_2\) to \(H_3\) and so on, where \(H_A < H_k \leq H_B\) for all \(k = 1, \ldots, n\), and \(n\) is the number of monotony intervals. If \(H(t)\) is non-prime, then there are four successive extrema \(H_{k-1}, H_k, H_{k+1}, H_{k+2}\) such that \(|H_{k-1} - H_k| > |H_k - H_{k+1}|, |H_k - H_{k+1}| \leq |H_{k+1}, H_{k+2}|\). According to the return-point memory, \(H(t)\) on the interval where \(H\) changes from \(H_{k-1}\) to \(H_{k+2}\) can be replaced with monotonic function without affecting the final state. Such replacement can be repeated not more than finite number of times, because the number of monotony intervals is finite, and finally we get a prime input with the final state \(s\). The lemma is proved.

After applying a prime input to the system in the state \(A\) some definite state in \(\Sigma_{AB}\) is obtained, which can be assigned to the prime input. According to the lemma, such correspondence between prime inputs and states defines the mapping of the set of prime inputs on the set \(\Sigma_{AB}\).

Let us consider now the set \(X\) of functions \(x(\xi) : [0, 1] \to \{-1, 1\}\) which change the sign in a finite number of points. We can establish one-to-one correspondence between prime inputs and functions \(x(\xi)\) in the following way. Let us assign to the invariable input \(H(t) = H_A\) the function \(x(\xi) = -1\). For prime inputs with \(n \geq 1\) let us define the points

\[
\xi_1 = |H_1 - H_0| / |H_B - H_A|,
\xi_2 = |H_2 - H_1| / |H_B - H_A|,
\ldots
\xi_n = |H_n - H_{n-1}| / |H_B - H_A|.
\]

According to the definition of prime inputs we have \(0 < \xi_n < \ldots < \xi_2 < \xi_1 \leq 1\).

Let us assign to a given prime input the function \(x(\xi)\) that changes its sign from +1 to −1 at the point \(\xi_1\), from −1 to +1 at the point \(\xi_2\) and so on. The above determines \(x(\xi)\) for any given prime input and vice versa. It is illustrated in Fig. 3 from which it is clear that \(x(\xi)\) changes according to the rules \(1^*\) and \(2^*\) of “continuous” state vector model, when the magnetic field decreases or increases after the final point of a prime input. For the final value of the magnetic field holds

\[
H = \frac{H_A + H_B}{2} + \frac{H_B - H_A}{2} \int_0^1 x(\xi) d\xi. \quad (4)
\]

The right side of this formula is a functional that we denote as \(H[x(\xi)]\). As the result we have the following proposition.

Proposition. — The set \(X\) of functions \(x(\xi)\) can be mapped on the set of states \(\Sigma_{AB}\) and \(H[x(\xi)]\) exists, such that if \(x(\xi)\) changes according to the rules \(1^*\) and \(2^*\), the state that corresponds to \(x(\xi)\) changes in the same way as the state of the system under varying external field \(H = H[x(\xi)]\).

Once \(x(\xi)\) determines the state of the system, its physical properties can be represented as the functionals with argument \(x(\xi)\): in particular, \(M = M[x(\xi)]\).

As follows from Proposition, the model is applicable to any system that is compliant to the return-point memory and has the states \(A\) and \(B\) with above mentioned properties. The last condition is normally satisfied: we can take as \(A\) and \(B\) the states at the vertices of the limiting or any minor hysteresis loop. Note that \(x(\xi) = -1\) corresponds to the state \(A\) and \(x(\xi) = +1\) corresponds to the state \(B\). For the symmetric hysteresis loop \(H_A = -H_B\) and, according to Eq. (4), \(H[x(\xi)]\) is antisymmetric; from the symmetry of hysteresis curves with respect to the origin of the \(HM\)-plane follows antisymmetry of \(M[x(\xi)]\).
We have proved the sufficiency of the return-point memory for simulation by the model. The necessity also can be shown. Namely, it can be proved that any \( H[x(\xi)] \) which continuously and monotonically increases and decreases when \( x(\xi) \) changes according to the rules 1* and 2* correspondingly, is compliant with the return-point memory.

**FUNCTIONALS \( H[x(\xi)], M[x(\xi)] \)**

Different forms of functionals \( H[x(\xi)], M[x(\xi)] \) are suitable for hysteresis simulation. Let us consider some of the simplest.

**Linear functionals**

Let us try as \( H[x(\xi)], M[x(\xi)] \) linear functionals:

\[
M = \int_0^1 m(\xi)x(\xi) \, d\xi, \\
H = \int_0^1 h(\xi)x(\xi) \, d\xi,
\]

where \( m(\xi), h(\xi) \) are some continuous positive functions.

Consider \( x(\xi) \) such that \( x(\xi) = -1 \) for all \( \xi \) on the interval \( 0 < \xi \leq \xi_1 \) for some given \( \xi_1 \). According to the model postulates it means that the magnetic field was decreasing before. Let the magnetic field begin to increase, and denote \( \xi' \) the point where \( x(\xi) \) changes the sign from +1 to -1, assuming \( \xi' \leq \xi_1 \). In this case

\[
2 \int_0^{\xi'} d\xi = |\Delta M|, \\
2 \int_0^{\xi'} h(\xi) \, d\xi = |\Delta H|,
\]

where \( \Delta M \) and \( \Delta H \) are changes of \( M \) and \( H \) starting from the origin of new ascending hysteresis branch; the same equations are true in the case of any descending branch.

We can see that from Eq. (6), (7) follow that all the ascending hysteresis branches and all the descending hysteresis branches are congruent, and can be described with the same function \( \varphi \), which is determined by Eq. (6):

\[
\Delta H = \pm \varphi(\pm \Delta M).
\]

Here “+” corresponds to ascending and “−” to descending branch.

Let us divide the interval \([0, 1]\) into \( 2^n \) equal subintervals, and define the state \( x^{(0)} \) so that \( x^{(0)}(\xi) = -1 \) on the odd subintervals and \( x^{(0)}(\xi) = +1 \) on the even ones. In accordance with the model postulates we may consider \( x^{(0)} \) (or \( -x^{(0)} \)) as the demagnetized state, which is obtained via a demagnetization process performed as the consequence of \( n \) demagnetization cycles. From Eq. (6), (7) and the continuity of \( m(\xi) \) and \( h(\xi) \) follow that \( H(x^{(0)}) \rightarrow 0, M(x^{(0)}) \rightarrow 0 \) when \( n \rightarrow \infty \). On the initial magnetization curve instead of Eq. (6), (7) holds

\[
\int_0^{\xi'} m(\xi) \, d\xi = |M|, \\
\int_0^{\xi'} h(\xi) \, d\xi = |H|,
\]

and

\[
H = \pm \frac{1}{2} \varphi(\pm 2M).
\]

Two first terms of Taylor series for \( \varphi^{-1} \) in Eq. (8), (9) correspond to Rayleigh relations \[3\]; this lead us to a conclusion that the linear approximation of the functionals is suitable in the case of small fields.

**One nonlinear expression for \( H[x(\xi)] \)**

Let us consider as an example following expression

\[
H = a(M) \int_0^1 h(\xi)x(\xi) \, d\xi + b(M),
\]

assuming that \( M[x(\xi)] \) is represented by the right side of Eq. (8). Here \( a(M) \) and \( b(M) \) are some functions; due to the hysteresis loop symmetry \( a(M) \) must be even and \( b(M) \) must be odd.

From Eq. (8), (9) we can found explicit equations for branches of the hysteresis curve and the initial magnetization curve

\[
H = \pm a(M)\varphi(\pm (M - M_0)) + b(M) + a(M)\frac{H_0 - b(M_0)}{a(M_0)},
\]

\[
H = \frac{1}{2}a(M)\varphi(2M) + b(M) \quad (M > 0).
\]

Here \( \varphi \) is determined by \( m(\xi), h(\xi) \) in the same way as it was considered previously, and \( M_0, H_0 \) denote coordinates of the beginning of a hysteresis branch.

These formulae were applied for hysteresis simulation of low-alloyed electrical steel. Experimental and simulated curves are shown in Fig. 4.

**CONNECTION TO PREISACH MODEL**

The Preisach distribution \( p(h_u, h_c) \) describes density of the “domains” on the Preisach plane. Each domain has a shifted square hysteresis loop; \( h_c \) is the coercive force of the domain and \( h_u \) denotes the shift. Function \( p(h_u, h_c) \) must be positive, characterized by the symmetry \( p(h_u, h_c) = -p(-h_u, h_c) \), and normalized to unity.

Let the magnetic field \( H \) always remains in the interval \([-H_m, H_m]\), where \( H_m \) is as large as desired but fixed; we assume also that \( p(h_u, h_c) \) is a finite function with support in the triangle with vertices \((0, H_m), (0, -H_m), (H_m, 0)\).

Magnetization \( M \) is expressed as the integral over all domains

\[
M = M_s \int_{D_c} dh_u dh_c p(h_u, h_c) - M_s \int_{D_-} dh_u dh_c p(h_u, h_c),
\]
FIG. 4: Experimental hysteresis curves of low-alloyed electrical steel (solid lines) and simulated curves (dashed lines). The simulation is equally accurate in the intermediate and in the small fields.

where $D_+$ and $D_-$ are regions with positive and negative domain orientation respectively. The boundary $h_u = b(h_c)$ between these regions is a broken line, made of segments with positive and negative slope $db(h_c)/dh_c = \pm 1$.

The magnetization and the magnetic field are expressed via $b(h_c)$:

$$H = b(0), \quad M = 2M_s \int_0^{h_m} db(h_c) \int_0^{b(h_c)} p(h_u, h_c) dh_u. \quad (11)$$

Let us define

$$x(\xi) = -\frac{1}{H_m} \frac{d}{d\xi} b(\xi H_m), \quad 0 \leq \xi \leq 1. \quad (12)$$

Function $x(\xi)$ equals to $\pm 1$ and can determine the state of the Preisach ensemble:

$$b(h_c) = H_m \int_{h_c/H_m}^{1} x(\xi)d\xi. \quad (13)$$

FIG. 5: A geometric illustration of the connection between Preisach model and the state vector model.

Now Eq. (11) can be rewritten in the form of functionals Eq. (14):

$$H = \tilde{H}[x(\xi)], \quad M = \tilde{M}[x(\xi)], \quad (14)$$

where

$$\tilde{H}[x(\xi)] = H_m \int_0^1 x(\xi) d\xi, \quad (15)$$

and

$$\tilde{M}[x(\xi)] = 2M_s \int_0^{h_m} dh_c \int_0^{b(h_c)} p(h_u, h_c) dh_u. \quad (16)$$

The boundary $b(h_c)$ changes in a well-known manner, what can be expressed in terms of $x(\xi)$. This gives exactly the rules 1*, 2*, and in conjunction with Eq. (14) we get the state vector hysteresis model with the functionals of special form defined by Eq. (15), (16). Connections between two hysteresis models is illustrated in Fig. 5.

The traditional Preisach model has some disadvantages such as zero value of the turning point susceptibility and the congruency property [7]. Limitations of the original Preisach model take much weaker form in its modifications. It is possible to overcome partially the congruency property [8] by introducing internal mean field $H_{MF}(M)$, which depends on the magnetization. This extension of Preisach model corresponds to the functionals

$$H = \hat{H}[x(\xi)] - H_{MF}(\tilde{M}[x(\xi)]), \quad M = \hat{M}[x(\xi)]. \quad (17)$$

Another variant, the product Preisach model [8], also can be represented in a form of the state vector hysteresis model. In this case we have

$$H = \tilde{H}[x(\xi)], \quad M = G(\beta \hat{H}[x(\xi)] + \hat{M}[x(\xi)]),$$

where $G$ is “transformation function”, and the constant $\beta$ provides non-zero value of the turning point susceptibility.
DISCUSSION AND CONCLUSIONS

The return-point memory is the only essential condition on a system, that is necessary for simulation by the model. This is an obvious advantage comparative to Preisach model, which requires an extra condition, congruency, not exhibited by majority of systems [7].

Preisach model can be represented as a special case of the model, with a particular form of the functionals $H[x(\xi)], M[x(\xi)]$. Explicit expressions for $H[x(\xi)], M[x(\xi)]$ are found in the cases of traditional Preisach model and some of its extensions. From our consideration follows that a rather complex functional Eq. (16) that corresponds to Preisach model is not necessary for hysteresis simulation; it can be replaced with much simpler functionals that may not require two-dimensional integration (see as an example Eq. (5), (10) and Fig. 4).

It is worth noting that a general approach for representing states of a system was established as a consequence of the return-point memory. This grants the right to use the model for any physical value $y$ that can be expressed as a function of state; corresponding functional $y[x(\xi)]$ determines its behavior under varying input. Note that the model does not comprises any other properties except the return-point memory, and compliance with thermodynamics must be considered as a restriction on the functionals.

Different forms of the functionals could be proposed for different kinds of materials. Actually a class of models is defined, depending on a particular form of the functionals. Due to extra flexibility of the model simpler calculations and more precise simulation can be expected, which may assist in systematizing experimental data.

* Electronic address: lang@ezan.ac.ru

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