Noncompact KK theory of gravity: stochastic treatment for a nonperturbative inflaton field in a de Sitter expansion

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Abstract

We study a stochastic formalism for a nonperturbative treatment of the inflaton field in the framework of a noncompact Kaluza-Klein (KK) theory during an inflationary (de Sitter) expansion, without the slow-roll approximation.

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I. INTRODUCTION

Stochastic inflation model is one of the very few that solves almost all of the well-known cosmological problems. Since the differential microwave radiometer (DMR) mounted on the Cosmic Background Explorer satellite (COBE) first detected temperature anisotropies in the cosmic background radiation (CBR), we have the possibility to directly probe the initial density perturbation. The fact that the resulting energy density fluctuations ($\delta \rho / \rho \approx 10^{-5}$) fit the scaling spectrum predicted by the inflation model, suggests that they had indeed their origin in the quantum fluctuations of the “inflaton” scalar field during the inflationary era. Although in principle this problem is of a quantal nature, the fact that under certain conditions—which are made precise in [1–3]—the inflaton field can be considered as classical largely simplifies the approach, by allowing a Langevin-like stochastic treatment. The most widely accepted approach assume that the inflationary phase is driving by a quantum scalar field $\phi$ with a potential $V(\phi)$. Within this perspective, the stochastic inflation proposes to describe the dynamics of this quantum field on the basis of a splitting of $\phi$ in a homogeneous and an inhomogeneous components. Usually the homogeneous one $\phi_c(t)$ is interpreted as a classical field that arises from a coarsed-grained average over a volume larger than the observable universe, and plays the role of a global order parameter [4]. All information

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on scales smaller than this volume, such as the density fluctuations, is contained in the inhomogeneous component. Although this theory is widely used and accepted as general one needs to make the approximation \( \langle \rho \rangle \approx \dot{\phi}^2 + V(\phi_c) \) to can make some calculations in a linear expansion for the scalar potential \( V(\varphi) \) around its classical background field \( \phi_c(t) \) [3]. It was Starobinsky the first one to derive a Fokker-Planck equation for the transition probabilities \( P(\phi_L, t|\phi'_L, t') \) in comoving coordinates [5] from the stochastic equation for the dynamics of the inflaton field. \( P(\phi_L, t|\phi'_L, t') \) provides us with statistical information about the relative number of “domains” (metastable vacuum configurations) that having a typical value \( \phi'_L \) of the coarse-grained inflaton field, evolve in a time interval \( (t - t') \) towards a new configuration with a typical value \( \phi_L \).

The main aim of this work consists to make a consistent coarse-graining treatment for the scalar field dynamics on cosmological scales during the inflationary epoch. For simplicity, as an example, we shall study a de Sitter expansion for the universe, but the formalism here developed can be used to study other more realistic inflationary models. The strategy consists to starts from a 5D globally flat metric and an action for a purely kinetic quantum scalar field minimally coupled to gravity, which define a 5D vacuum state. The metric we consider can be mapped to a 5D generalized Friedman-Roberton-Walker (FRW) on which we make a foliation by considering the fifth (spatial-like) coordinate as a constant. As a result of this foliation we obtain an effective 4D FRW metric and an effective 4D density Lagrangian in which appears an effective term which depends on the fifth coordinate and is not kinetic in 4D. This term is identified as a 4D scalar potential or source, and has an origin purely geometric. The idea that matter in four dimensions (4D) can be explained from a 5D Ricci-flat (\( R_{AB} = 0 \)) Riemannian manifold is a consequence of the Campbell’s theorem. It says that any analytic \( N \)-dimensional Riemannian manifold can be locally embedded in a \( (N + 1) \)-dimensional Ricci-flat manifold. This is of great importance for establishing the generality of the proposal that 4D field equations with sources can be locally embedded in 5D field equations without sources [6,7]. The advantage of this propose is that provides an exact (nonperturbative) treatment for the 4D dynamics of the inflaton field (the scalar field) with back-reaction effects included [8]. Furthermore, it is possible to make a consistent treatment for the effective 4D dynamics of the universe in other models governed by a single scalar field. In this paper we consider a general version of the Kaluza - Klein theory in 5D, where the extra dimension is not assumed to be compactified. In other words, this means that the cylinder condition in the fifth coordinate of the original Kaluza - Klein theory is relaxed. From the mathematical viewpoint, this means that the 5D metric tensor is allowed to depend explicitly on the fifth coordinate. Without cylindricity, there is no reason to compactify the fifth dimension, so this approach is properly called noncompactified.

II. REVIEW OF THE FORMALISM

We consider an action

\[
I = -\int d^4x d\psi \sqrt{\frac{g}{g_0}} \left[ \frac{(5)R}{16\pi G} + (5)\mathcal{L}(\varphi, \varphi, A) \right],
\]

(1)
for a scalar field \( \varphi \), which is minimally coupled to gravity. Here, \( |^{(5)} g | = \psi^8 e^{6N} \) is the absolute value of the determinant for the 5D metric tensor with components \( g_{AB} \) (\( A, B \) take the values 0, 1, 2, 3, 4) and \( |^{(5)} g_0 | = \psi_0^8 e^{6N_0} \) is a constant of dimensionalization determined by \( |^{(5)} g | \) evaluated at \( \psi = \psi_0 \) and \( N = N_0 \). Furthermore, \( ^{(5)} R \) is the 5D Ricci scalar and \( G \) is the gravitational constant. In this work we shall consider \( N_0 = 0 \), so that \( ^{(5)} g_0 = \psi_0^8 \). Here, the index “0” denotes the values at the end of inflation (i.e., when \( \ddot{b} = 0 \)). With the aim to describe a manifold in apparent vacuum the Lagrangian density \( \mathcal{L} \) in (1) must be only kinetic in origin

\[
^{(5)} \mathcal{L}(\varphi, \varphi, A) = \frac{1}{2} g^{AB} \varphi_A \varphi_B, \tag{2}
\]

and the 5D metric should be globally flat [9–11]

\[
dS^2 = \psi^2 dN^2 - \psi^2 e^{2N} dr^2 - d\psi^2, \tag{3}
\]

where \( dr^2 = dx^2 + dy^2 + dz^2 \). The coordinates \((N, \vec{r})\) are dimensionless and the fifth coordinate \( \psi \) has spatial units. The metric (3) describes a flat 5D manifold in apparent vacuum \((G_{AB} = 0)\), so that the Ricci scalar must be zero: \(^{(5)} R = 0\). We consider a diagonal metric because we are dealing only with gravitational effects, which are the important ones for the global evolution for the universe [12].

The equation of motion for the scalar quantum field \( \varphi \) is

\[
\left( 2\psi \frac{\partial \varphi}{\partial N} + 3\psi^2 \right) \frac{\partial \varphi}{\partial N} + \psi^2 \frac{\partial^2 \varphi}{\partial N^2} - \psi^2 e^{-2N} \nabla^2_r \varphi - 4\psi^3 \frac{\partial \varphi}{\partial \psi} - 3\psi^4 \frac{\partial N}{\partial \psi} \frac{\partial \varphi}{\partial \psi} - \psi^4 \frac{\partial^2 \varphi}{\partial \psi^2} = 0, \tag{4}
\]

where \( \frac{\partial N}{\partial \psi} \) and \( \frac{\partial \psi}{\partial N} \) are zero because the coordinates \((N, \vec{r}, \psi)\) are independent.

The eq. (4) can be written as

\[
\dddot{\varphi} + 3 \dot{\varphi} - e^{-2N} \nabla^2_r \varphi - \left[ 4\psi \frac{\partial \varphi}{\partial \psi} + \psi^2 \frac{\partial^2 \varphi}{\partial \psi^2} \right] = 0, \tag{5}
\]

where the overstar denotes the derivative with respect to \( N \) and \( \varphi \equiv \varphi(N, \vec{r}, \psi) \). The commutator between \( \varphi \) and \( \Pi^N = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_N} = g^{NN} \varphi_N \) is given by

\[
[\varphi(N, \vec{r}, \psi), \Pi^N(N, \vec{r}, \psi')] = i g^{NN} \left| \frac{(5)}{(5)} g_0 \right| \delta^{(3)}(\vec{r} - \vec{r}') \delta(\psi - \psi'), \tag{6}
\]

where \( \left| \frac{(5)}{(5)} g_0 \right| \) is the inverse of the renormalized volume of the manifold (3) and \( g^{NN} = \psi^{-2} \). Hence, the commutator between \( \varphi \) and \( \dot{\varphi} \) will be

\[
[\varphi(N, \vec{r}, \psi), \dot{\varphi}(N, \vec{r}, \psi')] = i \left| \frac{(5)}{(5)} g_0 \right| \delta^{(3)}(\vec{r} - \vec{r}') \delta(\psi - \psi'). \tag{7}
\]

By means of the transformation \( \varphi = \chi e^{-3N/2} \left( \frac{\psi}{\psi} \right)^2 \) we obtain the 5D generalized Klein-Gordon like equation for \( \chi(N, \vec{r}, \psi) \) and the commutator between \( \chi \) and \( \dot{\chi} \):
\[ \chi - \left[ e^{-2N} \nabla_r^2 + \left( \psi^2 \frac{\partial^2}{\partial \psi^2} + \frac{1}{4} \right) \right] \chi = 0, \]  
\[ \chi(N, r, \psi), \chi(N, r, \psi') = i \delta^{(3)}(r - r') \delta(\psi - \psi'). \]  

The redefined field \( \chi \) can be written in terms of a Fourier expansion
\[ \chi(N, r, \psi) = \frac{1}{(2\pi)^{3/2}} \int d^3 k_r \int dk_\psi \left[ a_{k_r k_\psi} e^{i(k_\psi \cdot \vec{r} + k_\psi \cdot \rho)} \xi_{k_r k_\psi} (N, \psi) + a_{k_r k_\psi}^\dagger e^{-i(k_\psi \cdot \vec{r} + k_\psi \cdot \rho')} \xi_{k_r k_\psi}^* (N, \psi) \right], \]
where the asterisk denotes the complex conjugate and \( (a_{k_r k_\psi}, a_{k_r k_\psi}^\dagger) \) are respectively the annihilation and creation operators which satisfy the following commutation expressions
\[ \left[ a_{k_r k_\psi}, a_{k_r' k_\psi'}^\dagger \right] = \delta^{(3)} \left( \vec{k}_r - \vec{k}_r' \right) \delta \left( \vec{k}_\psi - \vec{k}_\psi' \right), \]
\[ \left[ a_{k_r k_\psi}, a_{k_r' k_\psi'} \right] = \left[ a_{k_r' k_\psi}, a_{k_r k_\psi'} \right] = 0. \]

The expression (9) complies if the modes are renormalized by the following condition:
\[ \xi_{k_r k_\psi} \left( \xi_{k_r k_\psi}^\dagger \right)^* - \left( \xi_{k_r k_\psi} \right)^* \xi_{k_r k_\psi}^\dagger = i. \]

This equation provides the renormalization for the complete set of solutions on all the spectrum \( (k_r, k_\psi) \). On the other hand, the dynamics for the modes \( \xi_{k_r k_\psi} (N, \psi) \) is well described by the equation
\[ \ddot{\xi}_{k_r k_\psi} + k^2_r e^{-2N} \xi_{k_r k_\psi} + \psi^2 \left( k^2_\psi - 2i k_\psi \frac{\partial}{\partial \psi} - \frac{\partial^2}{\partial \psi^2} - \frac{1}{4 \psi^2} \right) \xi_{k_r k_\psi} = 0. \]

To solve this equation we can make
\[ \xi_{k_r k_\psi} (N, \psi) = \xi_{k_r}^{(1)} (N) \xi_{k_\psi}^{(2)} (\psi), \]
where the dynamics for \( \xi_{k_r}^{(1)} (N) \) and \( \xi_{k_\psi}^{(2)} (\psi) \) are given by the following differential equations
\[ \ddot{\xi}_{k_r}^{(1)} + \left[ k^2_r e^{-2N} - \alpha \right] \xi_{k_r}^{(1)} = 0, \]
\[ \frac{d^2 \xi_{k_\psi}^{(2)}}{d\psi^2} + 2i k_\psi \frac{d\xi_{k_\psi}^{(2)}}{d\psi} - \left( k^2_\psi - \frac{1/4 - \alpha}{\psi^2} \right) \xi_{k_\psi}^{(2)} = 0. \]

being \( \alpha \) a dimensionless constant.

The solution of (14) can be written as \[13]\]
\[ \xi_{k_r k_\psi} (N, \psi) = \frac{i \sqrt{\pi}}{2} e^{-i k_\psi \cdot \vec{r}} H^{(2)}_{\nu} [k_r e^{-N}] = e^{-i k_\psi \cdot \vec{r}} \tilde{\xi}_{k_r} (N), \]
where \( H^{(1,2)} [x(N)] = J_{\nu} [x(N)] \pm i Y_{\nu} [x(N)] \) are the Hankel functions, \( J_{\nu} [x(N)] \) and \( Y_{\nu} [x(N)] \) are the first and second kind Bessel functions with \( \nu = \sqrt{\alpha} = 1/2 \) and \( x(N) = k_r e^{-N} \).

Furthermore the function \( \tilde{\xi}_{k_r} (N) \) is given by
\[ \bar{\xi}_{k_r}(N) = \frac{i\sqrt{\pi}}{2} \mathcal{H}^{(2)}_{1/2} [k_r e^{-N}] . \]  

(19)

In other words, \( \xi_{k_r k_\psi}(N, \psi) = e^{-i \vec{k}_\psi \cdot \vec{\psi}} \bar{\xi}_{k_r}(N) \), where \( \bar{\xi}_{k_r}(N) \) is a solution of

\[ \mathbf{\ddot{\xi}}_{k_r} + \left( k_r^2 e^{-2N} - \frac{1}{4} \right) \xi_{k_r} = 0, \]

(20)

such that the renormalization condition for \( \bar{\xi}_{k_r}(N) \) becomes

\[ \bar{\xi}_{k_r} \left( \bar{\xi}_{k_r}^* \right)^* - \left( \bar{\xi}_{k_r} \right)^* \bar{\xi}_{k_r}^* = i. \]

(21)

Note the discrepancy of the result (18) with whole obtained in [8], which relies in the particular choice of the vacuum. The vacuum here used is general and solves the problem of the earlier work in the sense that now there is not dependence on \( \psi \) in \( \chi \).

Hence, the field \( \chi \) in eq. (10) can be rewritten as

\[ \chi(N, \vec{r}, \psi) = \chi(N, \vec{r}) = \frac{1}{(2\pi)^{3/2}} \int d^3 k_r \int d k \psi \left[ a_{k_r k_\psi} e^{i \vec{k}_r \cdot \vec{r}} \bar{\xi}_{k_r}(N) + a_{k_r k_\psi}^* e^{-i \vec{k}_r \cdot \vec{r}} \xi_{k_r}^*(N) \right] . \]

(22)

Finally, the field \( \varphi \) is given by

\[ \varphi(N, \vec{r}, \psi) = e^{-\frac{3N}{2}} \left( \frac{\psi_0}{\psi} \right)^2 \chi(N, \vec{r}) , \]  

(23)

with \( \chi(N, \vec{r}) \) given by eq. (22). Note that exponentials \( e^{\pm i \vec{k}_\psi \cdot \vec{\psi}} \) disappear in \( \chi(N, \vec{r}) \) and there is not dependence on the fifth coordinate \( \psi \) in this field. This is a very important fact that say us that the field \( \varphi(N, \vec{r}, \psi) \) propagates only on the 3D spatially isotropic space \( r(x, y, z) \), but not on the additional space-like coordinate \( \psi \).

### III. COARSE-GRANNING IN 5D

To study the large scale evolution of the field \( \varphi \) on large 3D spatial scales, we can introduce the field \( \chi_L \)

\[ \chi_L(N, \vec{r}, \psi) = \frac{1}{(2\pi)^{3/2}} \int d^3 k_r \int d k \psi \theta(\epsilon k_0(N) - k_r) \left[ a_{k_r k_\psi} e^{i \vec{k}_r \cdot \vec{r}} \bar{\xi}_{k_r}(N) + c.c. \right] , \]

(24)

where c.c. denotes the complex conjugate of the first term inside the brackets and \( k_0 = \sqrt{\alpha e^N} \) is the \( N \)-dependent wavenumber (related to the 3D spatially isotropic, homogeneous and flat space \( r^2 = x^2 + y^2 + z^2 \)), which separates the long \(( k_r^2 \ll k_0^2 \) and short \(( k_r^2 \gg k_0^2 \) sectors. Modes with \( k_r/k_0 < \epsilon \) are referred to as outside the horizon.

If the short wavelength modes are described with the field \( \chi_S \)

\[ \chi_S(N, \vec{r}, \psi) = \frac{1}{(2\pi)^{3/2}} \int d^3 k_r \int d k \psi \theta(k_r - \epsilon k_0(N)) \left[ a_{k_r k_\psi} e^{i \vec{k}_r \cdot \vec{r}} \bar{\xi}_{k_r}(N) + c.c. \right] , \]

(25)
such that \( \chi = \chi_L + \chi_S \), hence the equation of motion for \( \chi_L \) will be approximately

\[
\dddot{\chi}_L - \left( \frac{k_0(N)}{a} \right)^2 \chi_L = \epsilon \left[ \dddot{\chi}_0 \eta(N, \vec{r}, \psi) + \dot{k}_0 \kappa(N, \vec{r}, \psi) + 2 \dot{k}_0 \gamma(N, \vec{r}, \psi) \right],
\]  

(26)

where the stochastic operators \( \eta, \kappa \) and \( \gamma \) are given respectively by

\[
\eta(N, \vec{r}, \psi) = \frac{1}{(2\pi)^{3/2}} \int d^3k_r \int dk \psi \delta(\epsilon k_0(N) - k_r) \left[ a_{k_r \psi} e^{ik_r \cdot \vec{r}} \xi_{k_r}(N) + c.c. \right],
\]  

(27)

\[
\kappa(N, \vec{r}, \psi) = \frac{1}{(2\pi)^{3/2}} \int d^3k_r \int dk \psi \delta(\epsilon k_0(N) - k_r) \left[ a_{k_r \psi} e^{ik_r \cdot \vec{r}} \xi_{k_r}(N) + c.c. \right],
\]  

(28)

\[
\gamma(N, \vec{r}, \psi) = \frac{1}{(2\pi)^{3/2}} \int d^3k_r \int dk \psi \delta(\epsilon k_0(N) - k_r) \left[ a_{k_r \psi} e^{ik_r \cdot \vec{r}} \xi_{k_r}^*(N) + c.c. \right].
\]  

(29)

The equation (26) can be rewritten as

\[
\dddot{\chi}_L - \alpha \chi_L = \epsilon \left[ \frac{d}{dN} \left( k_0 \eta(N, \vec{r}, \psi) \right) + k_0 \gamma(N, \vec{r}, \psi) \right].
\]  

(30)

This is a second order stochastic equation that can be written as two first order stochastic ones by introducing the auxiliary field \( u = \dot{x}_L - \epsilon k_0 \eta \)

\[
\dot{u} = \alpha \chi_L + \epsilon k_0 \gamma,
\]  

(31)

\[
\dot{x}_L = u + \epsilon k_0 \eta.
\]  

(32)

In the system (31), (32) the role of the noise \( \gamma \) can be minimized if \( \left( k_0 \right)^2 \langle \gamma^2 \rangle \ll \left( k_0 \right)^2 \langle \eta^2 \rangle \), which holds if the following condition holds

\[
\frac{\dot{\xi}_{k_r} \xi_{k_r}^*}{\xi_{k_r} \dot{\xi}_{k_r}^*} \ll 1.
\]  

(33)

In such case the system (31), (32) can be approximated to

\[
\dot{u} = \alpha \chi_L,
\]  

(34)

\[
\dot{x}_L = u + \epsilon k_0 \eta.
\]  

(35)

This system represents two Langevin equations with a noise \( \eta \) which is gaussian and white in nature

\[
\langle \eta \rangle = 0,
\]  

\[
\langle \eta^2 \rangle = \frac{\epsilon (k_0)^2}{2\pi^2 k_0} \int dk \psi \xi_{k_0} \xi_{k_0}^* \delta(N - N').
\]  

(36)

The equation that describes the dynamics of the transition probability \( P \left[ \chi_L^{(0)}, u^{(0)} | \chi_L, u \right] \) from a configuration \( (\chi_L^{(0)}, u^{(0)}) \) to \( (\chi_L, u) \) is a Fokker-Planck one.
\[
\frac{\partial P}{\partial N} = -u \frac{\partial P}{\partial \chi_L} - \alpha \chi_L \frac{\partial P}{\partial u} + \frac{1}{2} D_{11} \frac{\partial^2 P}{\partial \chi^2_L}, \tag{38}
\]

where \( D_{11} = \frac{1}{2} \left( \epsilon k_0^* \right)^2 [f dN \langle \eta^2 \rangle] \) is the diffusion coefficient related to the variable \( \chi_L \) due to the stochastic action of the noise \( \eta \). Explicitly

\[
D_{11} = \frac{\epsilon^3 (k_0)^2}{4 \pi^2} k_0^* \int dk \psi \xi \bar{\xi} \epsilon k_0 \bar{\epsilon} k_0, \tag{39}
\]

which is divergent.

**IV. Ponce de Leon Metric and 4D de Sitter Expansion**

In order to describe the metric (3) in physical coordinates we can make the following transformations:

\[
t = \psi_0 N, \quad R = \psi_0 r, \quad \psi = \psi, \tag{40}
\]

such that we obtain the 5D metric

\[
dS^2 = \left( \frac{\psi}{\psi_0} \right)^2 \left[ dt^2 - e^{2t/\psi_0} dR^2 \right] - d\psi^2, \tag{41}
\]

where \( t \) is the cosmic time and \( R^2 = X^2 + Y^2 + Z^2 \). This metric is the Ponce de Leon one [14], and describes a 3D spatially flat, isotropic and homogeneous extended (to 5D) FRW metric in a de Sitter expansion [15].

To study the de Sitter evolution of the universe on the 4D spacetime we can take a foliation \( \psi = \psi_0 \) in the metric (41), such that the effective 4D metric results

\[
dS^2 \rightarrow ds^2 = dt^2 - e^{2t/\psi_0} dR^2, \tag{42}
\]

which describes 4D globally isotropic and homogeneous expansion of a 3D spatially flat, isotropic and homogeneous universe that expands with a Hubble parameter \( H = 1/\psi_0 \) (in our case a constant) and has a 4D scalar curvature \((4)^R = 6(\dot{H} + 2H^2)\). Note that in this particular case the Hubble parameter is constant so that \( \dot{H} = 0 \).

The 4D energy density \( \rho \) and the pressure \( p \) are

\[
8\pi G \langle \rho \rangle = \frac{3}{\psi_0^2}, \tag{43}
\]

\[
8\pi G \langle p \rangle = -\frac{3}{\psi_0^2}, \tag{44}
\]

where \( G = M_p^{-2} \) is the gravitational constant and \( M_p = 1.2 \times 10^{19} \text{ GeV} \) is the Planckian mass. Furthermore, the universe describes a vacuum equation of state: \( p = -\rho \), such that

\[
\langle \rho \rangle = \left\langle \frac{\dot{\varphi}^2}{2} + \frac{a_0^2}{2a^2} \left( \nabla \varphi \right)^2 + V(\varphi) \right\rangle, \tag{45}
\]
where the brackets denote the 4D expectation vacuum and the cosmological constant $\Lambda$ gives the vacuum energy density $\langle \rho \rangle = \frac{\Lambda}{8\pi G}$. Thus, $\Lambda$ is related with the fifth coordinate by means of $\Lambda = \frac{3}{\psi_0^2}$ [14]. Furthermore, the 4D Lagrangian is given by

$$
(4) \mathcal{L}(\varphi, \varphi, \mu) = -\sqrt{\frac{1}{2} g^{\mu\nu} \dot{\varphi}_\mu \dot{\varphi}_\nu + V(\varphi)},
$$

where the effective potential for the 4D FRW metric [16], is

$$
V(\varphi) = -\frac{1}{2} g^{\psi\psi} \dot{\varphi}_\psi \dot{\varphi}_\psi = \frac{1}{2} \left( \frac{\partial \varphi}{\partial \psi} \right)^2 |_{\psi=\psi_0}.
$$

In our case this potential takes the form

$$
V(\varphi) = \frac{2}{\psi_0^2} \varphi^2(t, \vec{R}, \psi_0),
$$

where $k_{\psi_0}$ is the wavenumber for $\psi = \psi_0$. Notice this potential has a geometrical origin and assume different representations in different frames. In our case the observer is in a frame $U^\psi = 0$, because we are taking a foliation $\psi = \psi_0$ on the 5D metric (41). Furthermore, the effective 4D motion equation for $\varphi$ is

$$
\ddot{\varphi} + \frac{3}{\psi_0} \dot{\varphi} - e^{-2t/\psi_0} \nabla^2_R \varphi - \left[ \frac{4 \psi}{\psi_0^2} \frac{\partial \varphi}{\partial \psi} + \frac{\psi^2}{\psi_0^2} \frac{\partial^2 \varphi}{\partial \psi^2} \right] |_{\psi=\psi_0} = 0,
$$

which means that the effective derivative (with respect to $\varphi$) for the potential, is

$$
V'(\varphi)|_{\psi=\psi_0} = \frac{2}{\psi_0^2} \varphi(\vec{R}, t, \psi_0).
$$

Now we can make the following transformation:

$$
\varphi(\vec{R}, t) = e^{-\frac{3t}{2\psi_0}} \chi(\vec{R}, t).
$$

Note that now $\varphi \equiv \varphi(\vec{R} = \psi_0 r, t = \psi_0 N, \psi = \psi_0) = e^{-3t/(2\psi_0)} \chi(\vec{R}, t)$, where [see eq. (22)]

$$
\chi(t, \vec{R}) = \chi(t = \psi_0 N, \vec{R} = \psi_0 r, \psi = \psi_0):
$$

$$
\chi(\vec{R}, t) = \frac{1}{(2\pi)^{3/2}} \int d^3k_R \int dk_\psi \left[ a_{k_R k_\psi} e^{ik_R \vec{R}} \bar{\xi}_{k_R}(t) + c.c. \right] \delta(k_\psi - k_{\psi_0}).
$$

Hence, we obtain the following 4D Klein-Gordon equation for $\chi$

$$
\ddot{\chi} - \left[ e^{-\frac{2t}{\psi_0}} \nabla^2_R + \frac{1}{4\psi_0^2} \right] \chi = 0.
$$

The equation of motion for the time dependent modes $\bar{\xi}_{k_R}(t)$ is

$$
\ddot{\bar{\xi}}_{k_R} + \left[ k_R^2 e^{-\frac{2t}{\psi_0}} - \frac{1}{4\psi_0^2} \right] \bar{\xi}_{k_R} = 0.
$$

It is important to notice that eq. (54) is exactly the equation (20) with the variables transformation (40), on the hypersurface $\psi = \psi_0$. 

8
A. 4D stochastic dynamics for \( \chi_L \) in a de Sitter expansion

As was made in 5D, we can define the fields \( \chi_L(t, \vec{R}) \) and \( \chi_S(t, \vec{R}) \), which describe respectively the long and short wavelength sectors of the field \( \chi \).

\[
\chi_L(t, \vec{R}) = \frac{1}{(2\pi)^{3/2}} \int d^3k_R \int \theta(\epsilon k_0(t) - k_R) \left[ a_{k_R \kappa_0} e^{ik_R \cdot \vec{R}} \xi_{k_R}(t) + c.c. \right] \delta(k_R - \kappa_0),
\]

\[
\chi_S(t, \vec{R}) = \frac{1}{(2\pi)^{3/2}} \int d^3k_R \int \theta(k_R - \epsilon k_0(t)) \left[ a_{k_R \kappa_0} e^{ik_R \cdot \vec{R}} \xi_{k_R}(t) + c.c. \right] \delta(k_R - \kappa_0),
\]

where \( k_0(t) = \frac{1}{2\psi_0} e^{t/\psi_0} \). The field that describes the dynamics of \( \chi \) on the infrared sector \( (k_R^2 \ll k_0^2) \) is \( \chi_L \). Its dynamics obeys the Kramers-like stochastic equation

\[
\ddot{\chi}_L - \frac{\epsilon}{4\psi_0^2} \dot{\chi}_L = \epsilon \left[ \frac{d}{dt} \left( \dot{k}_0 \eta(t, \vec{R}) \right) + \dot{k}_0 \gamma(t, \vec{R}) \right],
\]

where the stochastic operators \( \eta, \kappa \) and \( \gamma \) are

\[
\eta = \frac{1}{(2\pi)^{3/2}} \int k^3 k_R \delta(\epsilon k_0 - k_R) \left[ a_{k_R \kappa_0} e^{ik_R \cdot \vec{R}} \xi_{k_R}(t) + c.c. \right],
\]

\[
\gamma = \frac{1}{(2\pi)^{3/2}} \int k^3 k_R \delta(\epsilon k_0 - k_R) \left[ a_{k_R \kappa_0} e^{ik_R \cdot \vec{R}} \xi_{k_R}(t) + c.c. \right].
\]

This second order stochastic equation can be rewritten as two Langevin stochastic equations

\[
\dot{u} = \frac{\epsilon}{4\psi_0^2} \chi_L + \dot{k}_0 \gamma,
\]

\[
\dot{\chi}_L = u + \epsilon \dot{k}_0 \eta,
\]

where \( u = \dot{\chi}_L - \dot{k}_0 \gamma \). The condition to can neglect the noise \( \gamma \) with respect to \( \eta \), now holds

\[
\frac{\dot{\xi}_{k_R} \dot{\xi}^*_{k_R}}{\xi_{k_R} \xi^*_{k_R}} \ll \left( \frac{\dot{k}_0}{k_0} \right)^2,
\]

on super Hubble scales. Notice this result is exactly the same in eq. (33), with the transformation (40). For a de Sitter expansion eq. (62) becomes \( k_R/k_0 < \epsilon \ll 1 \). It means that the noise \( \gamma \) can be neglected on scales \( k_R \ll \frac{\epsilon^{t/\psi_0}}{\psi_0} \) (i.e., on super Hubble scales). The Fokker-Planck equation for the transition probability \( P(\chi_L(0), u(0)|\chi_L, u) \) is

\[
\frac{\partial P}{\partial t} = -u \frac{\partial P}{\partial \chi_L} - \frac{\epsilon^{2\psi_0}}{4\psi_0^2} \chi_L \frac{\partial P}{\partial u} + D_{11}(t) \frac{\partial^2 P}{\partial \chi_L^2},
\]

where \( D_{11}(t) = \frac{\epsilon^{3} k_0 k^2}{4\pi^2} \left| \dot{\xi}_{k_0} \right|^2 \). Hence, the equation of motion for \( \langle \chi_L^2 \rangle = \int d\chi_L du \chi_L^2 P(\chi_L, u) \) is

\[
\frac{d}{dt} \langle \chi_L^2 \rangle = D_{11}(t) \simeq \frac{\epsilon^{2} \psi_0}{32\pi^{2}\psi_0^3}.
\]
In order to return to the original field $\varphi_L = e^{-\frac{2t}{\psi_0}} \chi_L$ the equation (64) can be rewritten as

$$\frac{d}{dt} \langle \varphi_L^2 \rangle = -\frac{3}{\psi_0} \langle \varphi_L^2 \rangle + \frac{e^2}{32\pi^2 \psi_0^2},$$

which has the following solution

$$\langle \varphi_L^2 \rangle = \frac{e^2}{96\pi^2 \psi_0^2} \left[ 1 + C e^{-\frac{3t}{\psi_0}} \right],$$

where $C$ is a constant of integration. When $\frac{2t}{\psi_0} \ll 1$, one obtains [for $e^2(1 + C) = 24$]

$$\frac{\langle \varphi_L^2 \rangle}{\psi_0} \ll 1 \approx \frac{H^2}{4\pi^2} \left[ 1 - 3Ht \right].$$

However, after the end of inflation, when $\frac{2t}{\psi_0} \gg 1$, it becomes

$$\frac{\langle \varphi_L^2 \rangle}{\psi_0} \gg 1 \approx \frac{e^2 H^2}{96\pi^2},$$

which is valid only for $e^2 \ll 1$.

In order to understand better this result in the context of the inflaton field fluctuations $\varphi(\vec{R}, t)$, we can make the following semiclassical approach:

$$\varphi(\vec{R}, t) = \langle \varphi(\vec{R}, t) \rangle + \phi(\vec{R}, t),$$

where $\langle \varphi \rangle = \phi_c(t)$ and $\langle \phi \rangle = 0$. With this representation one obtains

$$\langle \varphi^2 \rangle = \phi_c^2 + \langle \phi^2 \rangle,$$

where $\phi_c(t)$ is the solution of the equation

$$\ddot{\phi}_c + \frac{3}{\psi_0} \dot{\phi}_c + \frac{2}{\psi_0^2} \phi_c = 0.$$ 

The general solution of the differential equation (71) is

$$\phi_c(t) = \phi_c^{(0)} e^{-\frac{3t}{\psi_0}} \left( 1 + C_1 e^{-\frac{3t}{\psi_0}} \right),$$

where $C_1$ is a constant of integration and $\phi_c^{(0)} = \phi_c(t = t_i)$, being $t_i$ the time when inflation starts. Note that after inflation ends $\phi_c(t \to \infty) \to 0$. Hence, after inflation one obtains the following result:

$$\frac{\langle \varphi^2 \rangle}{\psi_0} \gg 1 \approx \frac{\langle \phi^2 \rangle}{\psi_0} \gg 1,$$

which means that for $\frac{2t}{\psi_0} \gg 1$ the following approximation is fulfilled:
\[
\left\langle \phi_L^2 \right\rangle \gtrsim \left\langle \varphi_L^2 \right\rangle \gtrsim \frac{\epsilon^2 H^2}{96\pi^2}.
\]

This is an important result which says us that the expectation value for the second momenta of the field \(\varphi_L\) at the end of inflation is approximately given by the expectation value for the inflaton field fluctuations on cosmological scales (for \(\epsilon^2 \ll 1\)).

We can estimate the amplitude of density energy fluctuations on cosmological scales

\[
\left| \frac{\delta \rho}{\rho} \right|_{\text{end}} \sim \frac{\left\langle V'(\varphi) \right\rangle}{\left\langle V(\varphi) \right\rangle} \left\langle \phi_L^2 \right\rangle^{1/2} \sim \frac{\phi_c}{\left\langle \phi_L^2 \right\rangle} \left\langle \phi_L^2 \right\rangle^{1/2}.
\]

In order to obtain \(\left| \frac{\delta \rho}{\rho} \right|_{\text{end}} \approx 10^{-5}\), the value of \(\phi_c\) at this moment should be (taking \(\epsilon = 10^{-3}\))

\[
\phi_{c,\text{end}} \approx \frac{0.66 \times 10^{-10}}{\psi_0} = 0.66 \times 10^{-10} H.
\]

Finally, we can estimate the initial value for \(\phi_c\): \(\phi_c^{(0)}\). If we consider \(t_{\text{end}} \approx 10^{10} G^{1/2}\), we obtain

\[
\phi_c^{(0)} \approx M_P,
\]

for \(H \lesssim 10^{-10} M_P\) (or \(\psi_0 \gtrsim 10^{10} G^{1/2}\)). Hence, the value of \(\phi_c\) when inflation starts assumes sub Planckian values.

V. CONCLUSIONS

In this work we have developed a stochastic treatment for the effective 4D inflaton field from a KK theory of gravity without the hypothesis of a slow-roll regime. In this framework the long-wavelength modes of the inflaton field reduces to a quantum system subject to a quantum noise originated by the short-wavelength sector. In this approach, the effective 4D potential is quadratic in \(\varphi\) and has a geometrical origin. Hence, as in STM theory of gravity 4D source terms are induced from a 5D vacuum and the fifth dimension (here a space-like dimension) is noncompact. In our theory the 5D vacuum is represented by a 5D globally flat metric (related with a \(R_{AB} = 0\) manifold) and a purely kinetic density Langrangian for a quantum scalar field minimally coupled to gravity. Since the treatment for the scalar field is nonperturbative a very important feature of this formalism is that back reaction effects are included in the calculations in a consistent manner. An important result here obtained is that the expectation value for the second momenta for the field \(\varphi_L\) at the end of inflation is approximately given by the expectation value for the inflaton field fluctuations on cosmological scales, being both determinated by the value of the fifth coordinate on which we take the foliation: \(\psi = \psi_0\). Furthermore, the initial value for the background (and spatially homogeneous) inflaton field take sub Planckian values. This fact is very important because solves the problem of initial conditions in other treatments of chaotic inflation, in which \(\phi_c^{(0)}\) assumes trans Planckian values.

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