Lines pinning lines

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Abstract

A line $\ell$ is a transversal to a family $F$ of convex polytopes in $\mathbb{R}^3$ if it intersects every member of $F$. If, in addition, $\ell$ is an isolated point of the space of line transversals to $F$, we say that $F$ is a pinning of $\ell$. We show that any minimal pinning of a line by polytopes in $\mathbb{R}^3$ such that no face of a polytope is coplanar with the line has size at most eight. If in addition the polytopes are pairwise disjoint, then it has size at most six.

1 Introduction

A line transversal to a family $F$ of disjoint compact convex objects is a line that meets every object of the family. Starting with the classic work of Grünbaum, Hadwiger, and Danzer in the 1950’s, geometric transversal theory studies properties of line transversals and conditions for their existence. There is a sizable body of literature on line transversals in two dimensions. Of particular interest are so-called “Helly-type” theorems, such as the following theorem already conjectured by Grünbaum in 1958, but proven in this form by Tverberg only in 1989: If $F$ is a family of at least five disjoint translates of a compact convex figure in the plane such that every subfamily of size five has a line transversal, then $F$ has a line transversal. More background on geometric transversal theory can be found in the classic survey of Danzer et al. \cite{4}, or in the more recent ones by Goodman et al. \cite{6}, Eckhoff \cite{5}, Wenger \cite{15}, or Holmsen \cite{8}.

Much less is known about line transversals in three dimensions. Cheong et al. \cite{3} showed a Helly-type theorem for pairwise disjoint congruent Euclidean balls, Borcea et al. \cite{2} generalized this to families of arbitrary disjoint balls (with an additional ordering condition). On the other hand, Holmsen and Matoušek \cite{9} proved that no such theorem can exist for convex polytopes: For every $n > 2$ they construct a convex polytope $K$ and a family $F$ of $n$ disjoint translates of $K$ such that $F$ has no line transversal, but $F \setminus \{F\}$ has a transversal for every $F \in F$.

If a line transversal $\ell$ to a family $F$ cannot move without missing some $F \in F$, then we call the line $\ell$ pinned by $F$. Here we consider small continuous movements of $\ell$ in the vicinity of its current position. (We obviously exclude a translation of $\ell$ parallel to itself, which moves the points on the line but does not change the line as a whole.) In other words, $\ell$ is pinned if $\ell$ is an isolated point in the space of line transversals to $F$. We will define an appropriate space $\mathcal{L}$ of lines in Section 2.

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The line shown on the left is pinned by the three convex figures. Whenever a family \( \mathcal{F} \) of convex objects in the plane pins a line \( \ell \), there are three objects in \( \mathcal{F} \) that already pin \( \ell \); this fact was already used by Hadwiger [7]. Or, put differently, any minimal pinning of a line by convex figures in the plane is of size three. Here, a minimal pinning is a family \( \mathcal{F} \) pinning a line \( \ell \) such that no proper subset of \( \mathcal{F} \) pins \( \ell \). The Helly-type theorems for balls in three dimensions are based on a similar result: any minimal pinning of a line by pairwise disjoint balls in \( \mathbb{R}^3 \) has size at most five [2].

In this paper we prove the analogous result for convex polytopes, with a restriction:

**Theorem 1.** Any minimal pinning of a line by possibly intersecting convex polytopes in \( \mathbb{R}^3 \), no facet of which is coplanar with the line, has size at most eight. The number reduces to six if the polytopes are pairwise disjoint.

In the light of the construction by Holmsen and Matoušek, which shows that there can be no Helly-type theorem for line transversals to convex polytopes, Theorem 1 is perhaps surprising. Any Helly-type theorem can be considered as a guarantee for the existence of a small certificate: Tverberg’s theorem, for instance, says that whenever a family of translates in the plane does not admit a line transversal, we can certify this fact by exhibiting only five objects from the family that already do not admit a line transversal. (This interpretation of Helly-type theorems is the basis for their relation to LP-type problems [1].) Holmsen and Matoušek’s construction, on the other hand, shows that there are families \( \mathcal{F} \) of translates of a convex polytope that do not admit a line transversal, but such that it is impossible to certify this fact by a small subfamily of \( \mathcal{F} \). Now, Theorem 1 can be interpreted as follows: Whenever there is a family \( \mathcal{F} \) of convex polytopes and a line \( \ell \) (with the non-coplanarity restriction) such that there is a neighborhood of \( \ell \) that contains no other line transversal to \( \mathcal{F} \), then there is a subfamily \( \mathcal{G} \subset \mathcal{F} \) of size at most eight that admits no other line transversal in a neighborhood of \( \ell \). So, while the absence of a line transversal globally cannot be witnessed by a small certificate, locally this is the case.

Consider a family \( \mathcal{F} \) of convex polytopes pinning a line \( \ell_0 \). A polytope whose interior intersects \( \ell_0 \) cannot contribute to a minimal pinning, so we can assume that each element of \( \mathcal{F} \) is tangent to \( \ell_0 \). The simplest case is when \( \ell_0 \) intersects a polytope \( F \) in a single point interior to an edge \( e \) of \( F \) (see the figure on the left). In that case, the condition that a line \( \ell \) near \( \ell_0 \) intersects \( F \) is characterized by the “side” (a notion we formalize in Section 2) on which \( \ell \) passes the oriented line \( e \) supporting the edge \( e \). Since pinning is a local property, it follows that we can ignore the polytopes that constitute the family \( \mathcal{F} \), and speak only about the lines supporting their relevant edges. We say that a family \( \mathcal{F} \) of oriented lines in \( \mathbb{R}^3 \) pins a line \( \ell_0 \) if there is a neighborhood of \( \ell_0 \) such that for any oriented line \( \ell \neq \ell_0 \) in this neighborhood, \( \ell \) passes on the left side of some line in \( \mathcal{F} \). We will prove the following theorem:

**Theorem 2.** Any minimal pinning of a line \( \ell_0 \) by lines in \( \mathbb{R}^3 \) has size at most eight. If no two of the lines are simultaneously coplanar and concurrent with \( \ell_0 \), then it has size at most six.

We will actually give a full characterization of families of lines that minimally pin a line \( \ell_0 \) when either all lines are orthogonal to \( \ell_0 \), or when they do not contain pairs of lines that are at the same time concurrent and coplanar with \( \ell_0 \).

When \( \ell_0 \) lies in the plane of a facet of a polytope, the condition that a line close to \( \ell_0 \) intersects the polytope becomes a disjunction of two sidedness constraints with respect to lines. Thus it cannot be handled by the methods of Theorem 2. In fact, the non-coplanarity condition
is necessary: We describe a construction that shows that no pinning theorem holds for collections of polytopes when they are allowed both to intersect each other and to touch the line in more than just a single point:

**Theorem 3.** There exist arbitrarily large minimal pinnings of a line by convex polytopes in $\mathbb{R}^3$.

We do not know if the condition that the convex polytopes be pairwise disjoint is sufficient to guarantee a bounded minimal pinning number. In fact, we are not aware of any construction of a minimal pinning by pairwise disjoint convex objects in $\mathbb{R}^3$ of size larger than six.

Pinning is related to the concept of **grasping** used in robotics: an object is immobilized, or grasped, by a collection of contacts if it cannot move without intersecting (the interior of) one of the contacts. The study of objects (“fingers”) that immobilize a given object has received considerable attention in the robotics community [10] [11], and some Helly-type theorems are known for grasping. For instance, Mishra et al.’s work [12] implies Helly-type theorems for objects in two and three dimensions that are grasped by point fingers. We give two examples of how our pinning theorems naturally translate into Helly-type theorems for grasping a line. First, we can interpret a family $F$ of lines pinning a line $\ell_0$ by considering all lines as solid cylinders of zero radius. Each constraint “cylinder” touches the “cylinder” $\ell_0$ on the left. As a result, it is impossible to move $\ell_0$ in any way (except to rotate it around or translate it along its own axis), because it would then intersect one of the constraints, so $\ell_0$ is grasped by $F$. Second, consider a family $F = \{P_1, \ldots, P_n\}$ of polytopes such that $P_i$ is tangent to $\ell_0$ in a single point interior to an edge $e_i$, and let $\tilde{P}_i$ denote the mirror image of $P_i$ with respect to the plane spanned by $\ell_0$ and $e_i$. Since $F$ pins $\ell_0$ if and only if $\tilde{F}$ grasps it, our pinning theorems directly translate into Helly-type theorems for grasping a line by polytopes.

**Outline of the paper.** Our main technical contributions are our pinning theorem for lines (Theorem 2), a construction of arbitrarily large minimal pinnings of a line by overlapping polytopes with facets coplanar with the line (Theorem 3), and a classification of minimal pinnings of a line by lines (Theorems 6 and 7).

Sections 2 and 3 are devoted to the proof of the general case of Theorem 2. The idea of the proof is the following. We consider a parameterization of the space of lines by a quadric $\mathcal{M} \subset \mathbb{R}^5$, where a fixed line $\ell_0$ is represented by the origin $0 \in \mathbb{R}^5$, and the set of lines passing on one side of a given line $g$ is recast as the intersection of $\mathcal{M}$ with a halfspace $\overline{U}_g$. This associates with a family $F$ of lines a polyhedral cone $C$; $C$ represents all lines not passing on the left side of some element of $F$. Then $F$ pins $\ell_0$ if and only if the origin $0$ is an isolated point of $C \cap \mathcal{M}$. We give a characterization of the cones $C$ such that the origin is isolated in $C \cap \mathcal{M}$ in terms of the trace of $C$ on the hyperplane tangent to $\mathcal{M}$ in the origin; this is our **Isolation Lemma** (Lemma 3). The Isolation Lemma allows us to analyze the geometry of the intersection $C \cap \mathcal{M}$ when $F$ is a pinning of $\ell_0$, and to find a subfamily of size at most eight that suffices to pin.

The subsequent two sections discuss the various configurations of lines that form minimal pinnings, first for constraints that are perpendicular to the pinned line (Section 4) and then for the general case (Section 5). Theorems 1, 2, and 3 are then proven in Section 6.

**2 Lines and constraint sets**

**Sides of lines.** Throughout this paper, all lines are oriented unless specified otherwise. Given two non-parallel lines $\ell_1$ and $\ell_2$ with direction vectors $d_1$ and $d_2$, we say that $\ell_2$ **passes to the right of** $\ell_1$ if $\ell_2$ can be translated by a positive multiple of $d_1 \times d_2$ to meet $\ell_1$, or, equivalently, if

$$\det \begin{pmatrix} p_1 & p'_1 & p_2 & p'_2 \\ 1 & 1 & 1 & 1 \end{pmatrix} < 0,$$  

(1)
The common transversals of the three remaining lines where $p_i$ and $p_i'$ are points on $\ell_i$ such that $p_ip_i'$ is a positive multiple of $d_i$.

**Constraints.** Throughout the paper, $\ell_0$ will denote the line to be pinned. A line meeting $\ell_0$ in a single point represents a constraint on $\ell_0$. A line $\ell$ satisfies a constraint $g$ if and only if $\ell$ meets $g$ or passes to the right of $g$. Consequently, the line $\ell_0$ is pinned by a family $F$ of constraints if there is a neighborhood of $\ell_0$ such that $\ell_0$ is the only line in this neighborhood satisfying all constraints in $F$.

We call a constraint *orthogonal* if it is orthogonal to $\ell_0$. We will give an example of eight orthogonal constraints that form a minimal pinning configuration, showing that the constant eight in Theorem 2 is best possible. Four generically chosen non-oriented lines $g_1, \ldots, g_4$ meeting $\ell_0$ and perpendicular to it will have at most two common transversals: if no two among $g_1, g_2, g_3$ are coplanar or concurrent, their transversals define a hyperbolic paraboloid and it suffices to choose $g_4$ not lying on this surface. Now, let $g_i^+$ and $g_i^-$ denote the two oriented lines supported by $g_i$ (see Figure 1). Since a line satisfies $g_i^+$ and $g_i^-$ if and only if it meets $g_i$, the eight constraints $g_1^+, g_1^-, \ldots, g_4^+, g_4^-$ pin $\ell_0$. Suppose we remove one of the eight constraints, say $g_1^+$. The common transversals of the three remaining lines $g_2, g_3, g_4$ form a hyperbolic paraboloid. By our construction, $g_1$ intersects this quadric surface transversely (since all four lines are orthogonal to $\ell_0$, $g_1$ cannot be tangent to the quadric, and since the four lines have at most two transversals, $g_1$ cannot lie in the quadric). Thus, the quadric of transversals of $g_2, g_3, g_4$ contains lines on both sides of $g_1$, and $\ell_0$ is no longer pinned. Therefore the eight oriented lines form a minimal pinning, as claimed.

**A space of lines.** We choose a coordinate system where $\ell_0$ is the positive $z$-axis, and denote by $\mathcal{L}$ the family of lines whose direction vector makes a positive dot-product with $(0,0,1)$. Since pinning is a local property, we can decide whether $\ell_0$ is pinned by considering only lines in $\mathcal{L}$ as alternate positions for $\ell$. (The constraint lines are of course not restricted to $\mathcal{L}$.) We identify $\mathcal{L}$ with $\mathbb{R}^4$ using the intersections of a line with the planes $z = 0$ and $z = 1$: the point $u = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4$ represents the line $\ell(u)$ passing through the points $(u_1, u_2, 0)$ and $(u_3, u_4, 1)$. The line $\ell_0$ is represented by the origin in $\mathbb{R}^4$.

We describe a constraint using the following three parameters: $g(\lambda, \alpha, \delta)$ denotes the constraint that meets $\ell_0$ in the point $(0,0,\lambda)$, makes slope $\delta$ with the plane $z = \lambda$, and projects into the $xy$-plane in a line making an angle $\alpha$ with the positive $x$-axis. On the line $g(\lambda, \alpha, \delta)$ we can choose two points $(0,0,\lambda)$ and $(\cos \alpha, \sin \alpha, \lambda + \delta)$. By Figure 1, a line $\ell(u) \in \mathcal{L}$ satisfies $g(\lambda, \alpha, \delta)$
within the space of lines \( \mathcal{L} \) if and only if
\[
\begin{vmatrix}
0 & \cos \alpha & u_1 & u_3 \\
0 & \sin \alpha & u_2 & u_4 \\
\lambda & \lambda + \delta & 0 & 1 \\
1 & 1 & 1 & 1
\end{vmatrix} \leq 0.
\]
The set \( \mathcal{U}_g \subset \mathbb{R}^4 \) of lines in \( \mathcal{L} \) satisfying the constraint \( g = g(\lambda, \alpha, \delta) \) is thus
\[
\mathcal{U}_g = \{ u \in \mathbb{R}^4 \mid \zeta_g(u) \leq 0 \},
\]
where \( \zeta_g : \mathbb{R}^4 \to \mathbb{R} \) is defined as \( \zeta_g(u) = \delta(u_2u_3 - u_1u_4) + \eta_g \cdot u \) and
\[
\eta_g = \eta(\lambda, \alpha) = \begin{pmatrix}
(1 - \lambda) \sin \alpha \\
-(1 - \lambda) \cos \alpha \\
\lambda \sin \alpha \\
-\lambda \cos \alpha
\end{pmatrix}.
\]

**Pinning by orthogonal constraints.** As a warm-up, we consider pinning by *orthogonal* constraints. If \( g \) is an orthogonal constraint, then its parameter \( \delta = 0 \), so the set \( \mathcal{U}_g \) is the halfspace \( \eta_g \cdot u \leq 0 \) in \( \mathbb{R}^4 \). This reduces the problem of pinning by orthogonal constraints to the following question: when is the intersection of halfspaces in \( \mathbb{R}^4 \) reduced to a single point? It is well-known that this is equivalent to the fact that the normal vectors surround the origin as an interior point of their convex hull:

**Proposition 1.** A finite collection of halfspaces whose boundaries contain the origin intersects in a single point if and only if their outer normals contain the origin in the interior of their convex hull.

Here, the interior of the convex hull refers to the interior in the ambient space, not the relative interior (in the affine hull).

**Proof.** If the intersection of the halfspaces contains a point \( a \neq 0 \), then all normal vectors \( n \) must have \( n \cdot a \leq 0 \). This implies that all normal vectors \( n \), and hence their convex hull, lie in the halfspace \( \{ x \mid x \cdot a \leq 0 \} \). Therefore, the convex hull contains no neighborhood of the origin 0.

Conversely, if the origin is on the boundary or outside the convex hull, there must be a (weakly) separating hyperplane \( \{ a \cdot x = b \} \) such that \( a \cdot n \leq b \) for all normal vectors \( n \), and \( a \cdot x \geq b \) for \( x = 0 \). This implies \( a \cdot n \leq 0 \) for all normal vectors \( n \), and thus the point \( a \neq 0 \) lies in all halfspaces.

To analyze *minimal* pinnings, we make use of the following classic theorem of Steinitz:

**Theorem 4** (Steinitz). If a point \( y \) is interior to the convex hull of a set \( X \subset \mathbb{R}^d \), then it is interior to the convex hull of some subset \( Y \) of at most \( 2d \) points of \( X \). The size of \( Y \) is at most \( 2d - 1 \) unless the convex hull of \( X \) has \( 2d \) vertices that form \( d \) pairs \((x, x')\) with \( y \) lying on the segment \( xx' \).

Equivalently, if the intersection of a family \( H \) of halfspaces in \( \mathbb{R}^d \) is a single point, then there is a subfamily of \( H \) of size at most \( 2d \) whose intersection is already a single point.

For a proof of Steinitz’s Theorem that includes the second statement, we refer to Robinson [14 Lemma 2a]. Theorem 4 immediately implies the following lemma.

**Lemma 1.** Any minimal pinning of \( \ell_0 \) by orthogonal constraints has size at most eight.
The example of Figure 1 shows that the bound is tight. By inspection of the formula for \( \eta_g \) we observe that two normals \( \eta_{g_1} \) and \( \eta_{g_2} \) are linearly dependent if and only if the orthogonal constraints \( g_1 \) and \( g_2 \) are the same line up to orientation. The second statement in Theorem 4 implies therefore that the set of the four pairs of parallel constraints in Figure 1 is in fact the only example of a minimal pinning by eight orthogonal constraints.

Linearizing the constraint sets. If the constraint \( g \) is not orthogonal, then the boundary \( \zeta_g(u) = 0 \) of the set \( U_g \) is a quadric through the origin. Since \( \text{grad} \zeta_g(0) = \eta_g \), the vector \( \eta_g \) is the outward normal of \( U_g \) at the origin, and we will call it the normal of \( g \). Rather than analyzing the geometry of the intersection of the volumes \( U_g \) bounded by quadrics, we linearize these sets by embedding \( \mathcal{L} \) into five-dimensional space. This is based on the observation that the functions \( \zeta_g(u) \) have, up to multiplication by a scalar, the same nonlinear term \( u_2u_3 - u_1u_4 \). The map \( \psi: \mathbb{R}^4 \rightarrow \mathbb{R}^5 \) defined as

\[
\psi: (u_1, u_2, u_3, u_4) \mapsto (u_1, u_2, u_3, u_4, u_2u_3 - u_1u_4)
\]

identifies \( \mathcal{L} \) with the quadratic surface

\[
\mathcal{M} = \{ (u_1, u_2, u_3, u_4, u_5) \mid u_5 = u_2u_3 - u_1u_4 \} \subset \mathbb{R}^5.
\]

The image of \( U_g \) under this transformation is

\[
\{ (u_1, \ldots, u_5) \in \mathcal{M} \mid \delta u_5 + \eta_g \cdot (u_1, \ldots, u_4) \leq 0 \},
\]

This leads us to define the five-dimensional halfspace

\[
\tilde{U}_g = U_g(\lambda, \alpha, \delta) = \{ (u_1, \ldots, u_5) \in \mathbb{R}^5 \mid \delta u_5 + \eta_g \cdot (u_1, \ldots, u_4) \leq 0 \}.
\]

We have \( \psi(U_g) = \tilde{U}_g \cap \mathcal{M} \). If \( \mathcal{F} \) is a family of constraints, then the set of lines in \( \mathcal{L} \) satisfying all constraints in \( \mathcal{F} \) is identified with

\[
\bigcap_{g \in \mathcal{F}} (\tilde{U}_g \cap \mathcal{M}) = \left( \bigcap_{g \in \mathcal{F}} \tilde{U}_g \right) \cap \mathcal{M}. \tag{3}
\]

Cones and the Isolation Lemma. We restrict our attention to convex polyhedral cones with apex at the origin, that is, intersections of finitely many closed halfspaces whose bounding hyperplanes go through the origin. We will simply refer to them as cones. By (3), the set of lines satisfying a family \( \mathcal{F} \) of constraints is represented as the intersection of a cone \( C \) and \( \mathcal{M} \). The family \( \mathcal{F} \) pins \( \ell_0 \) if and only if the origin is isolated in \( C \cap \mathcal{M} \). Our next step is to characterize cones \( C \) such that the origin is isolated in \( C \cap \mathcal{M} \).

Let \( T = \{ u_5 = 0 \} \) denote the hyperplane tangent to \( \mathcal{M} \) at the origin. Let \( T^> = \{ u_5 > 0 \} \) and \( T^< = \{ u_5 < 0 \} \) be the two open halfspaces bounded by \( T \). Similarly, let \( \mathcal{M}^> = \{ u_5 > u_2u_3 - u_1u_4 \} \) and \( \mathcal{M}^< = \{ u_5 < u_2u_3 - u_1u_4 \} \) denote the open regions above and below \( \mathcal{M} \). We define \( T^>, T^<, \mathcal{M}^> \), and \( \mathcal{M}^< \) analogously; refer to Figure 2. The following lemma classifies how a ray \( \{ tu \mid t > 0 \} \) starting in the origin may be positioned with respect to \( \mathcal{M} \).

Lemma 2. Let \( u \in \mathbb{R}^5 \) be a non-zero vector.

(i) If \( u \in T \cap \mathcal{M} \), then the line \( \{ tu \mid t \in \mathbb{R} \} \) lies in \( \mathcal{M} \);
(ii) If \( u \in T \cap \mathcal{M}^> \), then the line \( \{ tu \mid t \in \mathbb{R} \} \) is contained in \( \mathcal{M}^> \cup \{0\} \);
(iii) If \( u \in T \cap \mathcal{M}^< \), then the line \( \{ tu \mid t \in \mathbb{R} \} \) is contained in \( \mathcal{M}^< \cup \{0\} \);
(iv) If \( u \in T^> \), there is some \( \varepsilon > 0 \) such that for all \( t \in (0, \varepsilon) \), \( tu \in \mathcal{M}^> \);
(v) If \( u \in T^< \), there is some \( \varepsilon > 0 \) such that for all \( t \in (0, \varepsilon) \), \( tu \in \mathcal{M}^< \).
The upper bound follows from \( C \) we have \( T = \{ u_1 u_4 = u_2 u_3 \} \) is represented as two intersecting lines.

**Proof.** Since \( \mathcal{M} \) is a quadric and \( T \) is its tangent hyperplane in the origin, any line in \( T \) through the origin either lies in \( \mathcal{M} \) or meets it only in the origin and otherwise stays completely on one side of it. This can be seen directly from the formula for \( \mathcal{M} \cap T \), and it implies statements (i)–(iii). Define now the function \( f : x \mapsto x_5 - (x_2 x_3 - x_1 x_4) \), whose zero set is \( \mathcal{M} \); statements (iv)–(v) follow from the observation that \( f(t u) = t u_5 - t^2 (u_2 u_3 - u_1 u_4) \), and so the sign of \( f(t u) \) is determined, for small \( t \), by the sign of \( u_5 \).

We can now characterize the cones whose intersection with \( \mathcal{M} \) contains the origin as an isolated point.

**Lemma 3** (Isolation Lemma). Let \( C \) be a cone in \( \mathbb{R}^5 \). The origin is an isolated point of \( C \cap \mathcal{M} \) if and only if either

(i) \( C \) is a line intersecting \( T \) transversely,

(ii) \( C \) is contained in \( T^+ \cup (T \cap \mathcal{M}^+) \cup \{0\} \), or

(iii) \( C \) is contained in \( T^- \cup (T \cap \mathcal{M}^-) \cup \{0\} \).

**Proof.** Assume that the origin is isolated in \( C \cap \mathcal{M} \). First, by Lemma 2 (i), the intersection of \( C \) and \( T \cap \mathcal{M} \) is exactly the origin. Thus, \( C \subseteq T^+ \cup (T \cap \mathcal{M}^+) \cup T^- \cup (T \cap \mathcal{M}^-) \cup \{0\} \). Assume, for a contradiction, that \( C \) contains both a point \( u \in T^+ \cup (T \cap \mathcal{M}^+) \) and a point \( v \in T^- \cup (T \cap \mathcal{M}^-) \), and \( C \) is not a line. Since \( C \) is not a line, we can ensure that the segment \( w = u \) does not contain the origin, by perturbing \( u \) if necessary. By Lemma 2(ii)–(v), there is an \( \varepsilon > 0 \) such that for \( t \in (0, \varepsilon) \), \( u + t w \in \mathcal{M}^+ \) and \( v + t w \in \mathcal{M}^- \). Let \( w_0 \neq 0 \) be the point of \( \mathcal{M} \) on the segment joining \( tu \) and \( tv \); observe that \( w_0 \in C \), by convexity, and \( w_0 \) tends to 0 as \( t \) goes to 0, which contradicts the assumption that the origin is isolated in \( C \cap \mathcal{M} \). The condition is therefore necessary.

A line intersecting a quadric transversely meets it in at most two points, so condition (i) is sufficient. Assume condition (ii) holds. If \( C = \{0\} \) we are done. Otherwise, the set \( A = \{ u \in \mathcal{M}^\perp | \|u\| = 1 \} \) is compact and nonempty. Let \( B = \{ u \in \mathcal{M}^\perp | \|u\| \leq 1 \} \). Since \( \mathcal{M}^\perp \) and \( T \) are closed, \( B \) is compact, and since \( 0 \in B \), it is nonempty. Since \( C \subseteq T^\perp \cup (T \cap \mathcal{M}^\perp) \cup \{0\} \), we have \( C \cap \mathcal{M}^\perp \cap T = \{0\} \), and thus \( A \) and \( B \) are disjoint. Let \( \tau > 0 \) be the distance between \( A \) and \( B \). For any \( u = (u_1, \ldots, u_5) \in C \cap \mathcal{M} \setminus \{0\} \), we claim that

\[
\tau \|u\| \leq u_5 \leq \frac{1}{2} \|u\|^2.
\]  

(4)

The upper bound follows from \( u_5 = u_2 u_3 - u_1 u_4 \) and the inequality \( xy \leq \frac{x^2 + y^2}{2} \) for \( x, y \in \mathbb{R} \). For the lower bound, let \( v = u/\|u\| \). Since \( C \) is a cone, \( v \in C \) and so \( v \in A \). By assumption, \( u \in C \cap \mathcal{M} \setminus \{0\} \subseteq T^\perp \), and thus \( u_5 > 0 \). This implies that the projection \( u' = (u_1, \ldots, u_4, 0)\)
of \( u \) on \( T \), is in \( \mathcal{M} \). Therefore, by Lemma \( \text{(iii)} \), \( v' = u'/\|u'\| \in \mathcal{M} \). Since \( \|v'\| \leq \|v\| = 1 \), we have \( v' \in B \). Thus \( \|v - v'\| \geq \tau \), and so \( u_5 = \|u - u'\| = \|v - v'\| \cdot \|u\| \geq \tau \|u\| \), completing the proof of \( \text{(4)} \). Now, \( \text{(4)} \) implies that any point \( u \in C \cap \mathcal{M} \) other than the origin satisfies \( \|u\| \geq 2\tau \). This proves that the origin is isolated in \( C \cap \mathcal{M} \). Condition (ii) is thus sufficient, and the same holds for condition (iii) by symmetry.

The tangent hyperplane. The hyperplane \( T \) tangent to \( \mathcal{M} \) in the origin plays a special role in the Isolation Lemma. There is a geometric reason for this: \( \mathcal{M} \cap T \) represents exactly those lines of \( L \) that meet \( \ell_0 \). Indeed, for \( \ell(u) \) to meet \( \ell_0 \), the two-dimensional points \((u_1, u_2), (0, 0)\), and \((u_3, u_4)\) have to lie on a line, which is the case if and only if \( u_2u_3 - u_1u_4 = 0 \). But that is equivalent to \( \psi(u) \in T \).

**Lemma 4.** If \( g \) is a constraint, then \( T \cap \mathcal{M} \cap \partial\mathcal{U}_g \) is the union of two two-dimensional linear subspaces.

**Proof.** Since \( \mathcal{M} \cap T \) are the lines in \( L \) that meet \( \ell_0 \) and \( \mathcal{M} \cap \partial\mathcal{U}_g \) are the lines in \( L \) that meet \( g \), \( T \cap \mathcal{M} \cap \partial\mathcal{U}_g \) are exactly those lines of \( L \) that meet both \( \ell_0 \) and \( g \). There are two families of such lines, namely the lines through the point \( g \cap \ell_0 \), and the lines lying in the plane spanned by \( \ell_0 \) and \( g \). Each family is easily seen to be represented by a two-dimensional linear subspace in \( \mathbb{R}^5 \).

Relation to Plücker coordinates. A classic parameterization of lines in space is by means of Plücker coordinates \([13]\). These coordinates map lines in \( \mathbb{P}^4(\mathbb{R}) \) to the Plücker quadric \( \{x_1x_4 + x_2x_5 + x_3x_6 = 0\} \subset \mathbb{P}^5(\mathbb{R}) \). In particular, the oriented line through \((u_1, u_2, 0)\) and \((u_3, u_4, 1)\) has Plücker coordinates

\[
(u_3 - u_1, u_4 - u_2, 1, u_2, -u_1, u_1u_4 - u_2u_3),
\]

and so the transformation \( \mathbb{R}^5 \to \mathbb{P}^5(\mathbb{R}) \) defined as

\[
\psi(u) = (u_3 - u_1, u_4 - u_2, 1, u_2, -u_1, -u_5)
\]

clearly maps \( \psi(u) \) to the Plücker coordinates of the line \( \ell(u) \). It follows that our manifold \( \mathcal{M} \) is the image of the Plücker quadric under a mapping that sends \( \ell_0 \) to the origin, the hyperplane tangent to the Plücker quadric at \( \ell_0 \) to \( \{u_5 = 0\} \), and the lines orthogonal to \( \ell_0 \) to infinity. Our 5-dimensional affine representation has the advantage that \( \mathcal{M} \) admits a parameterization of the form \( u_5 = f(u_1, \ldots, u_4) \) where \( f \) is a homogeneous polynomial of degree two. This is instrumental in our proof of the Isolation Lemma.

We showed that the lines meeting a constraint \( g \) are represented by (the intersection of \( \mathcal{M} \) and) a hyperplane in \( \mathbb{R}^5 \). This is no coincidence: the Plücker correspondence implies that this is true for the set of lines meeting any fixed line \( g \). Similarly, given any point \( u \in \mathcal{M} \), the hyperplane tangent to \( \mathcal{M} \) at \( u \) intersects \( \mathcal{M} \) in exactly those lines that meet the line represented by \( u \)—we saw this only for the special hyperplane \( T \). (Of course, both properties can be easily verified in our setting without resorting to Plücker coordinates.)

3 Minimal pinnings by lines have size at most eight

Let \( \langle X \rangle \) denote the linear hull of a set \( X \subset \mathbb{R}^d \), that is, the smallest linear subspace of \( \mathbb{R}^d \) containing \( X \). A \( j \)-space is a \( j \)-dimensional linear subspace. We start with four lemmas on cones. (Recall that we defined a cone as the intersection of halfspaces whose bounding hyperplanes go through the origin.)
Lemma 5. Let $C = \bigcap_{h \in H} h$ be a cone defined by a family $H$ of halfspaces in $\mathbb{R}^d$. Then

$$\langle C \rangle = \bigcap_{h \in H} h = \bigcap_{h \in H} \partial h.$$ 

Proof. Clearly $\langle C \rangle \subseteq \bigcap_{h \in H} (\langle C \rangle \subseteq h)$. To show the reverse inclusion, we first pick an arbitrary point $x$ in the relative interior of $C$. Note that for any halfspace $h \in H$, $x \in \partial h$ implies $\langle C \rangle \subseteq h$.

Let $y \in \bigcap_{h \in H} (\langle C \rangle \subseteq h)$, and consider the segment $xy$. We show that a neighborhood of $x$ on this segment lies in $C$, implying $y \in \langle C \rangle$. Indeed, consider $h \in H$. If $h$ contains $\langle C \rangle$ then segment $xy$ is entirely contained in $h$. If $h$ does not contain $\langle C \rangle$ then $x$ lies in the interior of $h$, and a neighborhood of $x$ on the segment $xy$ is in $h$.

The second equality follows from the fact that any linear subspace (in particular, $\langle C \rangle$) that is contained in $h$ must also be contained in $\partial h$. \hfill \Box

We also use the following extension of Steinitz’s Theorem:

Lemma 6. If the cone defined by a family $H$ of halfspaces in $\mathbb{R}^d$ is a $j$-space, then $2d - 2j$ of these halfspaces already define $E$.

Proof. Every halfspace $h \in H$ contains $E$, so its bounding hyperplane contains $E$. Every $h \in H$ can be decomposed as the Cartesian product of $E$ and $h$’s orthogonal projection on the $(d-j)$-space $F$ orthogonal to $E$. A subfamily of $H$ intersects in exactly $E$ if and only if their orthogonal projections intersect in exactly the origin. Since the projection of $H$ on $F$ is a collection of halfspaces that intersect in a single point, by Steinitz’s Theorem some $2(d-j)$ of these sets must already intersect in a single point, and the statement follows. \hfill \Box

In the next two lemmas we consider cones that lie entirely in $T^\geq$, or even in $T^\geq \cup \{0\}$. Since we need these lemmas for arbitrary dimension, we define $T_d = \{ x \in \mathbb{R}^d \mid x_d = 0 \}$, $T_d^\geq = \{ x \in \mathbb{R}^d \mid x_d > 0 \}$, and $T_d^\leq = T_d^\geq \cup T_d = \{ x \in \mathbb{R}^d \mid x_d \geq 0 \}$.

Lemma 7. Let $C = \bigcap_{h \in H} h \subseteq T_d^\geq$ be a cone defined by a family $H$ of halfspaces in $\mathbb{R}^d$. Then

$$\bigcap_{h \in H} h \subseteq T_d^\geq.$$ 

Proof. Assume there is a point $y \in T_d^\geq$ in $\bigcap_{h \in H} (\langle C \cap T_d \rangle \subseteq h)$. Pick a point $x$ in the relative interior of $C \cap T_d$. We consider the segment $xy$, and show that a neighborhood of $x$ on this segment lies in $C$, a contradiction to $C \subset T_d^\geq$. Indeed, if $(C \cap T_d) \not\subseteq h$ we have $y \in h$ and so $xy$ lies entirely in $h$. On the other hand, if $(C \cap T_d) \not\subseteq h$, then $x$ lies in the interior of $h$, and a neighborhood of $x$ lies in $h$. \hfill \Box

Lemma 8. Let $C = \bigcap_{h \in H} h$ be a cone defined by a family $H$ of halfspaces in $\mathbb{R}^d$, with $d \geq 2$, such that no $h \in H$ is bounded by the hyperplane $T_d$. If $C$ is contained in $T_d^\geq \cup \{0\}$ then there is a subfamily $H' \subset H$ of size at most $2d - 2$ such that the cone $C' = \bigcap_{h \in H'} h$ defined by $H'$ is contained in $T_d^\geq \cup \{0\}$.

Proof. The cone $C$ is nonempty, but it does not intersect the hyperplane $F = \{ x_d = -1 \}$. Helly’s theorem thus implies that there is a subset $H_d \subset H$ of size at most $d$ such that $C_d \cap F = \emptyset$, where $C_d = \bigcap_{h \in H_d} h$. Since $C_d$ is a cone, this implies $C_d \subset T_d^\geq$. If $C_d \subset T^\geq \cup \{0\}$, we are done.

Otherwise, let $E = \langle C \cap T_d \rangle$. Since $C_d \subset T_d^\geq$, $C \cap T_d$ is a face of $C_d$. Since no halfspace in $H_d$ is bounded by $T_d$, this face cannot be $(d-1)$-dimensional, and so the dimension $k$ of $E$ satisfies $1 \leq k \leq d - 2$. For $d = 2$, we already obtain a contradiction, establishing the induction basis for
an inductive proof. So let $d > 2$ and assume that the statement holds for dimensions $2 \leq j < d$. Let $K = \{ h \in H_d \mid E \subset h \}$. By Lemma 4 applied to the cone $C_d \cap T_d$, we have $E = \bigcap_{h \in K} h \cap T_d$, and by Lemma 7 we have $\bigcap_{h \in K} h \subset T_d^\circ$. For $h \in K$, we define $h$ as the projection of $h$ on the $(d - k)$-space orthogonal to $E$. Let $\hat{T}^\circ_d$ be the projection of $T^\circ_d$. We have $\bigcap_{h \in K} \hat{h} \subset \hat{T}^\circ_d \cup \{0\}$, and since $2 \leq d - k < d$, the induction hypothesis implies that there is a subset $K' \subset K$ of size at most $2(d - k) - 2$ such that $\bigcap_{h \in K'} \hat{h} \subset \hat{T}^\circ_d \cup \{0\}$. This implies $\bigcap_{h \in K} h \subset T^\circ_d \cup E$. By the original assumption, we have $\bigcap_{h \in E} h \cap E = \{0\}$, and Steinitz’s theorem (Theorem 4) inside the subspace $E$ implies that there is a subset $K'' \subset H$ of size at most $2k$ such that $\bigcap_{h \in K''} h \cap E = \{0\}$. Setting $H' = K' \cup K''$ we have $\bigcap_{h \in H'} h \subset T^\circ_d \cup \{0\}$ with $|H'| \leq 2(d - k) - 2 + 2k = 2d - 2$, completing the inductive step. 

We are now ready to prove the first half of Theorem 2.

**Lemma 9.** Any minimal pinning $F$ of a line by constraints has size at most eight.

**Proof.** Let $F$ be a minimal pinning of the line $\ell_0$ by constraints, let $H$ be the family of halfspaces $\{ U_g \mid g \in F \}$, and let $C$ denote the cone $\bigcap_{h \in H} h$. Since $F$ pins $\ell_0$, the origin is an isolated point of $C \cap \mathcal{M}$ and we are in one of the cases (i)–(iii) of the Isolation Lemma. In case (i), $C$ is a line. By Lemma 6, $C$ is then equal to the intersection of at most eight halfspaces from $H$, implying that $F$ has cardinality at most eight. Without loss of generality, we can now assume that we are in case (ii) of the Isolation Lemma, that is $C \subseteq T^\circ \cup (T \cap \mathcal{M}^\circ) \cup \{0\}$. (Case (iii) is symmetric.)

Let $E = \langle \mathcal{C} \cap T \rangle$ and denote the dimension of $E$ by $k$. As in the proof of Lemma 8 we observe that since $T$ cannot be the boundary of a halfspace in $H$, we have $E \neq T$ and so $0 \leq k \leq 3$. Let $H' \subset H$ be the set of $h \in H$ with $E \subset h$. For $h \in H'$, we define $h$ as the projection of $h$ on the $(5 - k)$-space orthogonal to $E$. Let $\hat{T}$, $\hat{T}^\circ$, and $\hat{\mathcal{M}}$ be the projections of $T$, $T^\circ$, and $T \cap \mathcal{M}$, respectively, inside $E$.

By Lemma 7 we have $\bigcap_{h \in H'} h \subset \hat{T}^\circ$. Applying Lemma 5 to the cone $C \cap T$ inside the 4-space $T$, we have $\bigcap_{h \in H'} h \cap T = E$. Together this implies that $\bigcap_{h \in H'} \hat{h} \subset \hat{T}^\circ \cup \{0\}$. Applying Lemma 8 in the $(5 - k)$-dimensional subspace orthogonal to $E$, we have a subfamily $\mathcal{K} \subset H'$ of size at most $2(5 - k) - 2 = 8 - 2k$ such that $\bigcap_{h \in \mathcal{K}} h \subset T^\circ \cup \{0\}$, implying that $\bigcap_{h \in \mathcal{K}} h \subset T^\circ$ and $\bigcap_{h \in \mathcal{K}} h \cap T = E$.

We have thus found a small subset $\mathcal{K}$ of constraints that prevent $C$ from entering $\hat{T}^\circ \cap \hat{\mathcal{M}}$. By case (ii) of the Isolation Lemma, we still have to ensure that the part of the cone that lies within $T$ does not enter $\mathcal{M}^\circ$, by an appropriate set of additional constraints. We give a direct geometric argument in the $k$-dimensional subspace $E$, arguing separately for each possible value of $k$:

- If $k = 0$, then $E = \{0\}$ and the at most $8 - 0 = 8$ constraints in $K$ already pin $\ell_0$.
- If $k = 1$, then $E$ is a line contained in $T$. By Lemma 2 (ii), that line intersects $\mathcal{M}$ in a single point, which is the origin. Thus the at most $8 - 2 = 6$ constraints in $K$ already pin $\ell_0$.
- If $k = 2$ then $C \cap T$ is a plane, a halfplane, or a convex wedge lying in the 2-space $E$. We can pick at most two constraints $h_1, h_2$ in $H$ such that $(\bigcap_{h \in \mathcal{K}} h) \cap h_1 \cap h_2 \cap T = C \cap T$. Lemma 3 now implies that the at most $8 - 4 + 2 = 6$ constraints $\mathcal{K} \cup \{h_1, h_2\}$ pin $\ell_0$, since their intersection is contained in $T^\circ \cup (T \cap C) \subseteq T^\circ \cup (T \cap \mathcal{M}^\circ) \cup \{0\}$.
- If $k = 3$, applying Lemma 5 to the cone $C \cap T$, we find that $E = \bigcap_{h \in H'} (h \cap T)$. By Lemma 6 the 3-space $E$ is the intersection of two and halfspaces in $H$. This implies that $E$ is of the form $E = T \cap \partial U_{g_0}$ with $g_0 \in F$. Then, by Lemma 4 $E \cap \mathcal{M}$ is the union of two 2-spaces $E_1$ and $E_2$ that intersect along a line $f$. These two 2-spaces partition the 3-space $E$. 

[10]
into four quadrants; since \(C \cap T\) intersects \(M\) only in the origin, it must be contained in one of these quadrants. We project \(C \cap T\) along \(f\) and obtain a two-dimensional wedge. The boundaries of this wedge are projections of edges of the three-dimensional cone \(C \cap T\). Each edge is defined by at most two constraints of the three-dimensional cone \(C \cap T\), and thus we can find at most four constraints \(K'\) of \(C \cap T\) that define the same projected wedge, and thus ensure that \(\bigcap_{h \in K \cup K'} h \cap T \subseteq (T \cap \mathcal{M}) \cup \{0\}\). Adding these edges to \(K\), we obtain a family of at most \(8 - 6 + 4 = 6\) constraints that pin \(\ell_0\).

All cases in the proof can actually occur—we will see examples in Section 5 when we understand the geometry of pinning configurations better. The example of Figure 1 shows that the constant eight is indeed best possible. However, a look at the proof of Lemma 9 shows that in the case were \(C \cap T \neq \{0\}\) (that is, when \(k = 1, 2, 3\), a minimal pinning has size at most six. We will make use of this fact later, in the proof of Lemma 19.

The first statement of Theorem 1 (that is, the general case) follows from Lemma 9. The reader interested only in obtaining a finite bound for minimal pinnings by polytopes can skip Sections 3 and 4.

4 Minimal pinning configurations by orthogonal constraints

In this section, we geometrically characterize minimal pinning configurations for a line by orthogonal constraints. We consider our representation of linespace by \(\mathbb{R}^4\), where the volume of lines satisfying an orthogonal constraint is a halfspace having the origin on its boundary. Recall that a family of halfspaces through the origin intersects in a single point if and only if the convex hull of their normal vectors contains the origin in its interior. We say that a set of points in \(\mathbb{R}^d\) surrounds the origin if the origin is in the interior of their convex hull. A set \(N\) minimally surrounds the origin if \(N\) surrounds the origin, but no proper subset of \(N\) does.

We first give a description of minimal families of points surrounding the origin in \(\mathbb{R}^4\) as the union of simplices surrounding the origin in a linear subspace (Theorem 5). We then characterize such simplices that can be realized by normals of orthogonal constraints (Lemma 16), and our classification follows (Theorem 6).

4.1 Points minimally surrounding the origin in \(\mathbb{R}^4\)

A point set \(S\) surrounds the origin in a linear subspace \(E\) if \(S\) spans \(E\) and the origin lies in the relative interior of \(\text{conv}(S)\). We note that this is true if and only if every point \(y \in E\) can be written as \(y = \sum_{x \in S} \lambda_x x\) with all \(\lambda_x \geq 0\).

A simplex of dimension \(k\), or \(k\)-simplex, is a set of \(k + 1\) affinely independent points in \(\mathbb{R}^d\); we also say segment for \(k = 1\), triangle for \(k = 2\), and tetrahedron for \(k = 3\). We say that a simplex \(N\) is critical if it surrounds the origin in its linear hull \(\langle N \rangle\). Observe that if \(S\) and \(T\) are two critical simplices then \(S \not\subset T\).

**Lemma 10.** Let \(N\) be a point set with \(0 \in \text{conv}(N)\). Then \(N\) contains a critical simplex.

**Proof.** We simply take a simplex \(S\) in \(N\) of smallest dimension such that \(0 \in \text{conv}(S)\). Such a simplex must exist by Carathéodory’s Theorem. \(\square\)

**Lemma 11.** Let \(A\) be a set that minimally surrounds the origin.

(i) For any critical simplex \(X \subset A\), no point of \(A \setminus X\) lies in the linear hull of \(X\).

(ii) If a critical simplex \(X \subset A\) contains \(k\) points of a critical \(k\)-simplex \(Y \subset A\) then \(X = Y\).
Thus, if \( X \) with 
and

(i) Let \( \text{conv} \beta z \) must have 
and so \( q = 0 \) can be written as \( q = \sum_{a \in A} \mu_a a + \mu_a, \)

and so \( q \) can be expressed as a non-negative sum of the elements of \( A' \). But then \( A' \) surrounds the origin, and \( A \) is not minimal, a contradiction that proves statement (i).

(ii) If \( Y \subset A \) is a critical simplex then any point \( p \in Y \) lies in the linear hull of \( Y \setminus \{p\} \). Thus, if \( X \subset A \) is another critical simplex that contains \( Y \setminus \{p\} \), then \( p \in \langle X \rangle \), and (i) implies \( p \in X \). Thus \( Y \subset X \), and hence \( Y = X \).

Our goal is to describe sets minimally surrounding the origin as unions of (not necessarily disjoint) critical simplices. Our first step is the following “decomposition” lemma.

**Lemma 12.** Let \( A \) be a critical simplex of dimension at most \( d - 1 \), let \( E = \langle A \rangle \) be its linear hull, and let \( B \) be a set of points in \( \mathbb{R}^d \).

(i) \( A \cup B \) surrounds the origin in \( \mathbb{R}^d \) if and only if the orthogonal projection of \( B \) on the orthogonal complement \( E^\perp \) of \( E \) surrounds the origin in \( E^\perp \).

(ii) If \( A \cup B \) minimally surrounds the origin in \( \mathbb{R}^d \) then the orthogonal projection of \( B \) on the orthogonal complement \( E^\perp \) of \( E \) minimally surrounds the origin in \( E^\perp \).

(iii) If \( \text{conv}(B) \cap E \neq \emptyset \), and \( \text{conv}(X) \cap E = \emptyset \) for every \( X \subset B \), then \( B \) is contained in a critical simplex of \( A \cup B \).

**Proof.** (i) Let \( \pi \) denote the orthogonal projection on \( E \). If \( A \cup B \) surrounds the origin, then every point \( y \in E^\perp \subset \mathbb{R}^d \) can be written as \( y = \sum_{x \in A \cup B} \lambda_x x \) with \( \lambda_x \geq 0 \). Since \( \pi(x) = 0 \) for \( x \in A \), we have

\[
y = \pi(y) = \sum_{x \in A \cup B} \lambda_x \pi(x) = \sum_{x \in B} \lambda_x \pi(x),
\]

so \( \pi(B) \) surrounds the origin in \( E^\perp \).

Assume now that \( \pi(B) \) surrounds the origin in \( E^\perp \), and consider a point \( y \in \mathbb{R}^d \). Since \( \pi(y) \in E^\perp \), we can write \( \pi(y) = \sum_{x \in B} \lambda_x \pi(x) \) with \( \lambda_x \geq 0 \). Let \( w = y - \pi(y) + \sum_{x \in B} \lambda_x (\pi(x) - x) \).

Since \( w \in E \), we can express \( w \) as \( w = \sum_{x \in A} \mu_x x \) with \( \mu_x \geq 0 \). Thus

\[
y = [y - \pi(y)] + \pi(y) = w - \sum_{x \in B} \lambda_x (\pi(x) - x) + \sum_{x \in B} \lambda_x \pi(x) = \sum_{x \in A} \mu_x x + \sum_{x \in B} \lambda_x x,
\]

and \( A \cup B \) surrounds the origin.

(ii) If \( B \) has a proper subset \( C \) such that \( \pi(C) \) already surrounds the origin in \( E^\perp \) then \( A \cup C \) surrounds the origin in \( \mathbb{R}^d \) by claim (i). Thus, if \( A \cup B \) minimally surrounds the origin in \( \mathbb{R}^d \) then \( \pi(B) \) minimally surrounds the origin in \( E^\perp \).

(iii) Let \( A \) be the set of facets of \( A \), and let \( x \) be an arbitrary point in \( \text{conv}(B) \cap E \). Then \( \bigcup_{F \in A} \text{conv}(F \cup \{x\}) \) covers \( \text{conv}(A) \), and so \( 0 \in \text{conv}(F \cup \{x\}) \) for some facet \( F \) of \( A \). This implies \( 0 \in \text{conv}(F \cup B) \), so by Lemma 10 some simplex \( T \subset B \cup F \) is critical. We will conclude the proof by showing that \( B \subset T \). Let \( X = B \cap T \). Since \( 0 \in \text{conv}(T) \subset \text{conv}(F \cup X) \), we can write \( 0 = \alpha y + \beta z \), where \( \alpha, \beta \geq 0 \), \( y \in \text{conv}(F) \) and \( z \in \text{conv}(X) \). Since \( 0 \notin \text{conv}(F) \) we must have \( \beta z \neq 0 \). From \( 0 \in E \) and \( \alpha y \in E \), we have \( \beta z \in E \), which implies \( z \in E \), and so \( \text{conv}(X) \cap E \neq \emptyset \). By assumption, we now have \( X = B \) and therefore \( B \subset T \).
We will now describe any minimal set surrounding the origin in up to four dimensions as the union of critical simplices. As a warm-up exercise, we state without proof the following lemma about minimal sets surrounding the origin in one or two dimensions, since this will be used in the proof for four dimensions. (The lemma can easily be proven directly, or along the lines of the proof of Theorem 5 below.)

**Lemma 13.** A point set that minimally surrounds the origin in \( \mathbb{R}^1 \) consists of a positive and a negative point. A point set that minimally surrounds the origin in \( \mathbb{R}^2 \) is either a triangle with the origin as an interior point, or a convex quadrilateral with the origin as the intersection point of the diagonals.

\[
\begin{array}{cccc}
\text{(2)} & \text{(3)} & \text{(4)} & \text{(5)} \\
\text{(6)} & \text{(7)} & \text{(*)} & \text{(8)}
\end{array}
\]

Figure 3: Combinatorial description of nongeneric minimal sets of points surrounding the origin in \( \mathbb{R}^4 \) according to Theorem 5.

We now turn to point sets minimally surrounding the origin in \( \mathbb{R}^4 \). The generic case (1) is a 4-simplex surrounding the origin; the remaining cases (2)–(8) are depicted in diagram form in Figure 3. The case (*) has a special label as we will see in Theorem 6 that it cannot be realized by normals of orthogonal constraints. Note that we are not claiming that the critical simplices shown are all critical simplices of the point set (although we are not aware of a situation that has additional critical simplices).

**Theorem 5.** A set \( S \) minimally surrounds the origin in \( \mathbb{R}^4 \) if and only if the linear hull of \( S \) is \( \mathbb{R}^4 \) and one of the following holds: (i) \( |S| = 5 \) and \( S \) is a critical 4-simplex, or (ii) \( |S| = 6 \) and \( S \) is the union of two critical simplices, each of dimension at most three, or (iii) \( |S| = 7 \) and \( S \) is the union of three critical simplices: \( k \geq 1 \) critical triangles having a single point in common and \( 3 - k \) disjoint critical segments, or (iv) \( |S| = 8 \) and \( S \) is the disjoint union of four critical segments.

**Proof.** We first prove that these cases exhaust all possibilities. Let \( S \) be a minimal set of points that surrounds 0. By Lemma 10 some simplex of \( S \) is critical. Let \( A \) be such a simplex with maximal cardinality, and denote by \( B = S \setminus A \) the remaining points of \( S \). Let \( E = \langle A \rangle \) denote the linear hull of \( A \) and \( E^\perp \) the orthogonal complement of \( E \). Let \( G \) be the affine hull of \( B \). The linear hull of \( E \cup G \) is \( \mathbb{R}^4 \), as otherwise \( S \) cannot surround the origin. We consider various cases depending on the cardinality of \( A \).

If \( |A| = 5 \), then \( A \) surrounds the origin and by minimality of \( S \) we are in case (i).
If \(|A| = 4\), then \(E\) is a 3-space and \(E^\perp\) is a line. By Lemmas 12 (ii) and 13, \(B\) consists of exactly two points, one on each side of \(E\). Since \(B \cap E = \emptyset\) and \(\text{conv}(B)\) meets \(E\), Lemma 12 (iii) implies that \(B\) is contained in some critical simplex \(T \subset S\). Since \(A\) was chosen with maximal cardinality, we have \(|T| \leq 4\), and we are in case (ii) (cases (3)--(5) of Figure 3).

If \(|A| = 2\), any critical simplex of \(S\) has size exactly two. It is easy to see that there must be exactly four critical segments, and we are in case (iv) (case (8) of Figure 3).

It remains to deal with the case where \(|A| = 3\), so that \(E\) is a 2-space. Lemmas 12 (ii) and 13 imply that \(B\) consists of three or four points.

If \(|B| = 3\), then \(B\) is a triangle and \(G\) is a 2-flat that intersects \(E\) in a single point \(x\) interior to \(\text{conv}(B)\). Since no edge of the triangle \(B\) meets \(E\), by Lemma 12 (iii) \(B\) is contained in a critical simplex \(T\) of \(S\). Since \(|T| \leq 3\) we have \(T = B\) and \(S\) is the disjoint union of two critical triangles. We are thus in case (ii) (case (2) of Figure 3).

If \(|B| = 4\), then by Lemma 13 the orthogonal projection of \(B\) on \(E^\perp\) consists of two critical segments. Thus, \(B\) consists of two disjoint pairs, say \(B_1\) and \(B_2\), whose convex hulls intersect \(E\). By Lemma 12 (iii), each \(B_i\) is contained in a critical simplex of \(S\). Since \(A\) is of maximal cardinality, we have that \(B_i\) either is a critical segment or is contained in a critical triangle \(T_i = B_i \cup \{a_i\}\). If at least one \(B_i\) is a critical segment, then we are in case (iii) (case (6) or (7) of Figure 3).

Assume finally that both \(B_i\) are contained in a critical triangle. If \(a_1 \neq a_2\), then \(\{a_1, a_2\} = E\) and \((T_1 \cup T_2) = \mathbb{R}^3\). Since \((T_1)\) and \((T_2)\) are two-dimensional, \((T_1) \cap (T_2) = \{0\}\). It follows that the orthogonal projection of \(T_2\) on \((T_1)^\perp\) is two-dimensional and therefore surrounds the origin in that subspace. Lemma 12 (i) then implies that \(T_1 \cup T_2 \subseteq S\) surrounds the origin, contradicting the minimality of \(S\). Hence, \(a_1 = a_2\) and \(S\) is the union of three critical triangles with exactly one common point, and we are in case (iii) (case (*) of Figure 3).

We now turn to the converse. We first argue that every point set \(S\) according to one of the types (i)--(iv) must surround the origin because \((S) = \mathbb{R}^4\) and \(S\) is the union of critical simplices: If \(S\) does not surround the origin, then there is a closed halfspace \(h\) through the origin such that \(S \subset h\). Since \((S) = \mathbb{R}^4\), there must be a point \(p \in S\) in the interior of \(h\). But then the critical simplex \(T \subset S\) containing \(p\) cannot surround the origin in \((T)\), a contradiction.

We proceed to argue the minimality of \(S\): no proper subset of \(S\) surrounds the origin. We distinguish the four cases of the theorem.

In case (i), \(|S| = 5\) and \(S\) is a critical 4-simplex. By definition it surrounds the origin and none of its faces does.

In case (iv), \(|S| = 8\) and \(S\) is the disjoint union of four critical segments. Since \((S) = \mathbb{R}^4\), the directions of these segments are linearly independent, and \(S\) minimally surrounds the origin.

In case (ii), \(|S| = 6\). Assume that there is a subset \(R \subseteq S\) that surrounds the origin. Since \(|R| \leq 5\), our necessary condition implies that \(R\) is a 4-simplex. Let \(p\) be the point in \(S\) not in \(R\). Since \(S\) is of type (ii), it can be written as \(S = A \cup B\) where \(A\) and \(B\) are critical simplices of dimension at most 3. Since neither \(A\) nor \(B\) can be contained in \(R\), they both contain \(p\) and are not disjoint. Without loss of generality, we can assume that \(A\) is a tetrahedron and \(B\) is a triangle or a tetrahedron. Let \(A' = A \setminus \{p\}\) and \(B' = B \setminus \{p\}\) and consider the ray \(r\) originating in 0 with direction \(\vec{p0}\). Since \(A = A' \cup \{p\}\) is a critical simplex, \(r\) intersects the relative interior of \(\text{conv}(A')\) in a point \(x_A\); similarly, \(r\) intersects the relative interior of \(\text{conv}(B')\) in a point \(x_B\).

Put \(E = (A)\) and consider the situation in the 3-space \(E\) (see Figure 4). There exists a 2-plane \(\Pi\) that strictly separates, in \(E\), \(A' \cup \{x_B\}\) from the origin. The affine hull of \(B'\) is either a line intersecting \(E\) in \(x_B\) (if \(B\) is a triangle) or a 2-plane intersecting \(E\) in the line \(qx_B\) (if \(B\) is a tetrahedron), where \(q = A' \cap B'\). (Indeed, if the intersection of the affine hull of \(B'\) with \(E\) has higher dimension, then \(B' \subset E\) and \(E = (R)\), contradicting our assumption that \(R\) surrounds
the origin.) In either case, there exists a 3-space in $\mathbb{R}^4$ that strictly separates the origin from $A' \cup B' = R$, so $R$ cannot surround the origin, completing the argument in case (ii).

Finally, in case (iii) $|S| = 7$. Then $S$ can be written as $S = A \cup B \cup C$, where $\{A, B, C\}$ consists of $k \geq 1$ critical triangles sharing exactly one vertex $p$ and $3-k$ disjoint critical segments. Assume that there exists a proper subset $R$ of $S$ that minimally surrounds the origin. First consider the case where $p \in R$, and note that this implies that any point in $S \setminus R$ belongs to a unique simplex from $\{A, B, C\}$. If $R$ has size 5 it contains one critical simplex from $\{A, B, C\}$, and that contradicts our necessary condition that $R$ be a critical 4-simplex. Thus $R$ must have size 6 and contains two critical simplices from $\{A, B, C\}$, say $A$ and $B$. Our necessary condition implies that $R$ can be written $R = X \cup Y$ where $X$ and $Y$ are critical simplices of dimension at most 3. Since $|A| \leq 3$, $X$ or $Y$ must contain $|A| - 1$ points from $A$. Without loss of generality let this be $X$. Lemma 11 (ii) implies that $X = A$. Then $Y$ contains all points from $B$ except possibly $p$, and so $Y = B$ by Lemma 11 (ii). Since $A \cup B$ has size at most 5, whereas $X \cup Y = R$ has size 6, we get a contradiction. Now consider the case where $p \notin R$. Let $r$ be the ray originating in 0 with direction vector $\vec{p}$ of $0$. Let $Z \in \{A, B, C\}$ and observe that $Z' = Z \cap R$ is either a single point, a critical segment, or a non-critical segment. Moreover, if $Z'$ is a non-critical segment then $Z = Z' \cup \{p\}$ is a triangle and since $Z$ is critical, $r$ intersects the relative interior of $\text{conv}(Z')$. Let $G$ be a 3-space passing through 0, parallel to the segments in $\{A', B', C'\}$ and passing through the points in $\{A', B', C'\}$ (if any). The halfspace in $\mathbb{R}^4$ bounded by $G$ and containing $r$ has 0 on its boundary and contains $R$. This implies that $R$ cannot surround 0, a contradiction.

Finally, the minimally surrounding sets in $\mathbb{R}^3$ can easily be obtained from Theorem 5.

**Lemma 14.** If a set $S \subset \mathbb{R}^3$ minimally surrounds the origin in $\mathbb{R}^3$ then either (i) $S$ is a critical tetrahedron, or (ii) $S$ is the union of two critical triangles sharing one point, or (iii) $S$ is the disjoint union of a critical triangle and a critical segment, or (iv) $S$ consists of three disjoint critical segments.

**Proof.** $S$ minimally surrounds the origin in $\mathbb{R}^3$ if and only if $S$ can be extended to a point set minimally surrounding the origin in $\mathbb{R}^4$ by adding a single critical segment in a transverse direction. We thus obtain all possible configurations from Figure 3 by taking all configurations containing a critical segment, and deleting it. Cases (5), (6), (7), and (8) of Figure 3 are possible, and lead to the four cases of the lemma.

**4.2 Critical simplices formed by constraints**

We now discuss how critical simplices can be formed by the normal vectors of a family of constraints. A necessary condition for a family of normals to form a critical simplex is that they be linearly dependent, but that any proper subset be linearly independent.
We call a family of constraints dependent if their normal vectors are linearly dependent. Our first step in rewriting Theorem 5 in terms of constraints is to give a geometric interpretation of dependent families of constraints.

We say that four or more lines $\ell_1, \ldots, \ell_k$ are in hyperboloidal position if they belong to the same family of rulings of a nondegenerate quadric surface. In the case of orthogonal constraints, which are parallel to a common plane, that quadric surface is always a hyperbolic paraboloid.

**Lemma 15.** Two orthogonal constraints are dependent if and only if they are identical except possibly for their orientation.

Three orthogonal constraints are dependent if and only if (i) two of them are, or if (ii) the constraints are coplanar with $\ell_0$, or if (iii) the constraints are concurrent with $\ell_0$.

Four orthogonal constraints are dependent if and only if (i) three of them are, or if (ii) the constraints are in hyperboloidal position, or if (iii) two of the lines are concurrent with $\ell_0$ and the other two are coplanar with $\ell_0$.

Proof. Let $g_1, \ldots, g_k$ be orthogonal constraints, for $k \in \{2, 3, 4\}$. The normals $\eta_{g_1}, \ldots, \eta_{g_k}$ are linearly dependent if and only if they span a linear subspace $E$ of dimension less than $k$. This is equivalent to saying that the orthogonal complement $F$ of $E$ has dimension larger than $4 - k$. Since $F = \{ u \in \mathbb{R}^4 \mid \zeta_0(u) = 0 \text{ for } 1 \leq i \leq k \}$, $F$ is the space of lines in $\mathcal{L}$ that meet all $k$ constraints $g_1, \ldots, g_k$. If the normals are linearly independent, then $F$ has dimension $4 - k$. If any $k - 1$ normals are linearly independent, but all $k$ are linearly dependent, then any $k - 1$ normals already span the subspace $E$, and its complement is $F$. This is equivalent to saying that every constraint must meet all the lines meeting the remaining $k - 1$ constraints.

Consider first two orthogonal constraints $g_1$ and $g_2$. They are dependent if and only if every line meeting $g_1$ also meets $g_2$. This happens if and only if $g_1$ and $g_2$ are the same line except possibly for their orientation.

Consider now three orthogonal constraints $g_1, g_2, g_3$, and assume that no two are dependent. Then all three are dependent if and only if every line meeting two constraints also meets the third. This is impossible if the constraints are pairwise skew. If two are coplanar with $\ell_0$, say $g_1$ and $g_2$, then their common transversals are exactly the lines in this plane. For $g_3$ to meet all of them, $g_3$ has to lie in the same plane as well. Finally, if two constraints, say $g_1$ and $g_2$, meet in a point $p \in \ell_0$ then the common transversals are exactly the lines through $p$. For $g_3$ to meet all of them, $g_3$ has to contain $p$ as well, and the three lines are concurrent with $\ell_0$.

Finally, consider four orthogonal constraints $g_1, \ldots, g_4$, and assume that no three are dependent. Then every three constraints have a one-dimensional family of common transversals. The four constraints are dependent if and only if every constraint meets every transversal to the other three. If the lines are pairwise skew, then $g_4$ must lie in the hyperbolic paraboloid formed by the transversals to $g_1, g_2, g_3$, and the four lines are in hyperboloidal position. Otherwise, two constraints must be coplanar with $\ell_0$ or must meet on $\ell_0$. If two constraints, say $g_1$ and $g_2$, meet in $p \in \ell_0$ then the remaining constraints cannot contain $p$. The set of transversals to $g_1, g_2, g_3$ is the set of lines through $p$ and $g_3$. For $g_4$ to meet all these lines, $g_4$ has to lie in the plane spanned by $p$ and $g_3$, and so $g_3, g_4, \ell_0$ are coplanar. If no two constraints meet, then two must be coplanar, say $g_1$ and $g_2$, and the constraints $g_3$ and $g_4$ intersect the plane spanned by $g_1, g_2, \ell_0$ in two distinct points $p_1$ and $p_2$. But there is only one line through $p_1$ and $p_2$, contradicting the assumption that $g_1, \ldots, g_4$ are dependent, and so this case cannot occur. \[\square\]

We call a family of constraints a block if their normals form a critical simplex. (We will use blocks to build pinning configurations.) The following lemma characterizes blocks geometrically and introduces names for the various types: A $k$-block consists of $k$ constraints. $3^\parallel$-blocks and $4^\parallel$-blocks contain lines that intersect, while the lines in $3^\perp$-blocks and $4^\perp$-blocks do not intersect. Figure 5 shows all blocks of size at most four.
Lemma 16. A family of orthogonal constraints forms a block if and only if it contains no proper dependent subfamily and the constraints form one of the following configurations:

- **2-block** the two orientations of the same line;
- **3∥-block** three coplanar constraints with alternating orientations (so the lines are $g(\lambda_1, \alpha, 0)$, $g(\lambda_2, \alpha + \pi, 0)$ and $g(\lambda_3, \alpha, 0)$ with $\lambda_1 < \lambda_2 < \lambda_3$);
- **3×-block** three constraints concurrent with $\ell_0$ whose direction vectors positively span $\ell_0^\perp$;
- **4∥-block** four constraints in hyperboloidal position oriented such that only lines lying in the quadric satisfy all four constraints;
- **4×-block** four constraints $\{g_1, \ldots, g_4\}$, where $g_1$ and $g_2$ lie in a common plane $\Pi$ containing $\ell_0$, $g_3$ and $g_4$ meet in a point $p \in \ell_0$, and $g_1, \ldots, g_4$ are oriented such that the lines satisfying them are exactly the lines in $\Pi$ passing through $p$;
- **5-block** five constraints oriented such that only $\ell_0$ satisfies them all.

Proof. Let $F$ be a set of $k + 1$ constraints, let $N = \{ \eta_g \mid g \in F \}$ be the set of their normals, and let $E \subset \mathbb{R}^4$ be the set of lines satisfying the constraints in $F$. Then $N$ forms a critical $k$-simplex if and only if $E$ is a linear subspace of dimension $4 - k$.

By construction, this is true for the configurations of constraints described in the lemma. It follows that these configurations are indeed blocks.

It remains to show that if $F$ is a block, then it falls into one of the six cases. Since every subset of $N$ must be linearly independent, we must have $k \leq 4$. Since $N$ must be linearly dependent, we obtain a first necessary condition from Lemma 15.

If $k = 1$, then we are in the first case of Lemma 15. $F$ consists of two constraints $\{g_1, g_2\}$ that are either equal or the two orientations of the same line $g$. Two equal points cannot form a critical segment, so $F$ is a 2-block.
If \( k = 2 \), we are in the second case of Lemma 15: \( \mathcal{F} \) consists of three constraints \( \{g_1, g_2, g_3\} \) that are either coplanar or concurrent with \( \ell_0 \). If the \( g_i \) are coplanar, then \( E \) contains the lines in the plane \( \Pi \) spanned by the \( g_i \). This set is already a two-dimensional linear subspace, and so \( E \) must be equal to this set. If two constraints met consecutively by \( \ell_0 \) have the same orientation then one of them is redundant, and \( E \) contains other lines; and so the orientations must alternate and \( \mathcal{F} \) is a 3\( \times \)-block.

If the \( g_i \) are concurrent in some point \( p \) on \( \ell_0 \), then \( E \) contains all the lines through \( p \). Again, this set is a two-dimensional linear subspace, and therefore equal to \( E \). Let \( \Pi \) denote the plane containing the three constraints. A line \( \ell \) satisfies a constraint \( g_i \) if and only if \( \ell \cap \Pi \) lies in the closed halfplane of \( \Pi \) to the right of \( g_i \). The set of points of \( \Pi \) to the right of all three \( g_i \) is reduced to \( \{p\} \) if and only if the direction vectors of the \( g_i \) positively span \( \ell_0^{\perp} \). This is a 3\( \times \)-block.

If \( k = 3 \), we are in the third case of Lemma 15: \( \mathcal{F} \) consists either of four constraints in hyperboloidal position, or two lines concurrent with \( \ell_0 \) and two other lines coplanar with \( \ell_0 \).

If the \( g_i \) are in hyperboloidal position, then any line in the other family of rulings of the quadric containing the \( g_i \) satisfies all constraints. This set is a line in \( \mathbb{R}^4 \), and so must be identical to \( E \). It follows that \( \mathcal{F} \) is a 4\( \parallel \)-block. Otherwise, \( \mathcal{F} \) consists of two lines \( g_1 \) and \( g_2 \) coplanar with \( \ell_0 \), and two lines \( g_3 \) and \( g_4 \) meeting in a point \( p \in \ell_0 \). The set \( E \) already contains all lines lying in the plane spanned by \( g_1 \cup \{p\} \) and passing through \( p \). This set is a line in \( \mathbb{R}^4 \), and therefore identical to \( E \). And so \( \mathcal{F} \) is a 4\( \times \)-block.

Finally, if \( k = 4 \), then \( N \) is a 4-simplex surrounding the origin. The five constraints \( \mathcal{F} \) are satisfied only by \( \ell_0 \), and so \( \mathcal{F} \) is a 5-block.

**Remark.** Given five orthogonal constraints such that no four are dependent, we can always orient them (that is, reverse some of them) so that they pin \( \ell_0 \), obtaining a 5-block.

### 4.3 Characterization of minimal pinning configurations

Combining the descriptions of Theorem 5 with the characterization of Lemma 16, we obtain the following characterization of minimal pinnings of a line by orthogonal constraints. The numbering of cases corresponds to the cases in Figure 3.

**Theorem 6.** A family of orthogonal constraints minimally pins \( \ell_0 \) if and only if it forms one of the following configurations:

1. A single 5-block;
2a. Two disjoint 3\( \parallel \)-blocks defining distinct planes;
2b. Two disjoint 3\( \times \)-blocks meeting \( \ell_0 \) in distinct points;
3a. Two 4\( \parallel \)-blocks sharing two constraints and defining distinct quadrics;
3b. Two 4\( \times \)-blocks sharing two constraints, such that their coplanar pairs define distinct planes or their concurrent pairs define distinct points;
3c. A 4\( \parallel \)-block and a 4\( \times \)-block sharing two constraints;
4a. A 4\( \parallel \)-block and a 3\( \parallel \)-block sharing one constraint;
4b. A 4\( \parallel \)-block and a 3\( \times \)-block sharing one constraint;
4c. A 4\( \times \)-block and a 3\( \parallel \)-block sharing one constraint such that they define distinct planes;
4d. A 4\( \times \)-block and a 3\( \times \)-block sharing one constraint such that their concurrent pairs meet \( \ell_0 \) in distinct points;
5a. A 4\( \parallel \)-block and a disjoint 2-block, where the 2-block constraints are not contained in the quadric defined by the 4\( \parallel \)-block;
5b. A 4\( \times \)-block and a disjoint 2-block, where the 2-block constraints are neither coplanar with the coplanar pair nor concurrent with the concurrent pair of the 4\( \times \)-block;
(6a) A 3*-block and two 2-blocks, where the four 2-block constraints do not all meet, and where no 2-block constraint is contained in the plane defined by the 3*-block;
(6b) A 3*-block and two 2-blocks, where the four 2-block constraints do not all meet, and where no 2-block constraint goes through the common point of the 3*-block;
(7) A 3*-block and a 3*-block sharing one constraint, and a disjoint 2-block that does not lie in the plane of the 3*-block and does not go through the common point of the 3*-point;
(8) Four disjoint 2-blocks whose supporting lines are not in hyperboloidal position (that is, all orientations of four lines with finitely many common transversals).

Proof. Let $F$ be a family of orthogonal constraints and let $N = \{ \eta_g \mid g \in F \}$ be the corresponding family of normals. Recall that $F$ minimally pins $\ell_0$ if and only if $N$ minimally surrounds the origin in $\mathbb{R}^4$.

We assume first that $F$ is of one of the 16 types described in the theorem. For each of the types, we can argue that the line $\ell_0$ is pinned, since the sets of lines satisfying each of the blocks have only $\ell_0$ as a common element. Since $\ell_0$ is pinned, $N$ spans $\mathbb{R}^4$. For all the 16 types, the critical simplices of $N$ corresponding to the blocks of $F$ are of the form described in Theorem 5 so $N$ minimally surrounds the origin, and $F$ minimally pins $\ell_0$.

It remains to argue the reverse. We assume that $F$ minimally pins $\ell_0$, that is $N$ minimally surrounds the origin. Then $N$ is of one of the types described in Theorem 5 and shown in Figure 2.

If $|N| = 5$, then $N$ is a critical 4-simplex, and so $F$ is a 5-block—this is case (1).
If $|N| = 8$, then $N$ is the disjoint union of four critical segments, and so $F$ is the disjoint union of four 2-blocks. Their supporting lines cannot be in hyperboloidal position (as then $F$ would not be pinning at all), and so we have case (8).

We now consider $|N| = 6$. If $N$ is the disjoint union of two critical triangles, then we must be in case (2a) or (2b), as the union of a 3*-block and a 3*-block cannot pin. If $N$ consists of a two critical tetrahedra sharing two constraints, we are in cases (3a)–(3c). If $N$ consists of a critical tetrahedron and a critical triangle sharing one constraint, we are in cases (4a)–(4d). If $N$ consists of critical tetrahedron and a disjoint critical segment, we have cases (5a)–(5b). The additional conditions in cases (2a)–(3b) and (4c)–(4d) hold as otherwise $F$ would not be pinning at all.

We finally turn to the case $|N| = 7$. Here we need an additional observation: If two 3*-blocks share a constraint, then they both define the same plane. That implies that the critical triangles formed by the normals span the same 2-space, and so these triangles cannot appear together in a point set minimally surrounding the origin by Lemma 11. It follows that $F$ cannot contain two 3*-blocks sharing a constraint, and the same reasoning applies to two 3*-blocks. Therefore $F$ cannot contain three 3-blocks sharing a common constraint, and if it contains two 3-blocks sharing a constraint, then these must be a 3*-block and a 3*-block. This implies that we must be in cases (6a), (6b) or (7). The additional conditions again hold as otherwise $F$ would not be pinning.

5 Minimal pinning configurations by general constraints

We now return to constraints that are not necessarily orthogonal. As we saw before, the boundary $\zeta_g(u) = 0$ of the set $U_g$ is a quadric through the origin, with outward normal $\eta_g$ in the origin. Interestingly, this normal $\eta_g$ only depends on $\lambda$ and $\alpha$, but not on the slope $\delta$. If we consider a constraint $g(\lambda, \alpha, \delta)$ for varying $\delta$, the shape of $\partial U_g$ changes, but its outward normal in the origin remains the same. In particular, when $\delta$ reaches zero, the quadratic term in $\zeta_g$ vanishes, and we have $\zeta_g(u) = \eta_g \cdot u$. In other words, linearizing the volume $U_g$ corresponds to replacing $g$ by its projection $g^\perp$ on the plane perpendicular to $\ell_0$ in $\ell_0 \cap g$; $U_g^\perp$ is the halfspace with outer normal
\( \eta_g \) through the origin. We call \( g^\perp \) the orthogonalized constraint of \( g \), and denote by \( F^\perp \) the family of orthogonalized constraints of \( F \). (Note that \( F^\perp \) can have smaller cardinality than \( F \).)

**First order pinning.** As in the previous section, we call a family of constraints \( F \) dependent if their normals are linearly dependent. Since the observation above shows that the normal to a constraint \( g \) is identical to the normal of \( g^\perp \), \( F \) is dependent if and only if \( F^\perp \) is dependent. For a characterization of dependent families, we can simply refer to Lemma 15, but that leaves a question: What does it mean geometrically for four constraints if their orthogonalized constraints are in hyperboloidal position? The following lemma clarifies this. Given three pairwise skew lines \( g_1, g_2, g_3 \), we let \( B(g_1, g_2, g_3) \) be the quadric formed by the transversals to the three lines.

**Lemma 17.** Two constraints are dependent if and only if they are at the same time coplanar and concurrent with \( \ell_0 \).

Three constraints are dependent if and only if (i) two of them are, or if (ii) the constraints are coplanar with \( \ell_0 \), or if (iii) the constraints are concurrent with \( \ell_0 \).

Four constraints \( g_1, g_2, g_3, g_4 \) are linearly dependent if and only if (i) three of them are, or if (ii) \( g_4 \) is tangent to the quadric \( B(g_1, g_2, g_3) \) at \( \ell_0 \cap g_4 \), or if (iii) two of the lines are concurrent with \( \ell_0 \) and the other two are coplanar with \( \ell_0 \).

**Proof.** The normals of constraints are linearly dependent if and only if the normals of their orthogonalized constraints are. This immediately implies the statements for two and three constraints, and nearly implies the statement for four constraints. It remains to show that \( g_1^\perp, \ldots, g_4^\perp \) are in hyperboloidal position if and only if \( g_4 \) is tangent to the quadric \( B(g_1, g_2, g_3) \) at \( \ell_0 \cap g_4 \).

We can assume that the lines \( g_1, g_2, g_3 \) are pairwise skew, so the quadric \( B = B(g_1, g_2, g_3) \) is defined. Since \( \ell_0 \) is contained in \( B \), for any point \( p \in \ell_0 \) there is a unique constraint contained in \( B \) and passing through \( p \). In particular, let \( g^* = g(\lambda_1, \alpha^*, \delta^*) \) denote the constraint contained in \( B \) through \( p = \ell_0 \cap g_4 = (0, 0, \lambda_4) \). Since the curve \( \bigcap_{i=1}^3 \partial U_{g_i} \) is contained in \( \partial U_{g^*} \), its tangent in the origin, which is orthogonal to the normals \( \eta_{g_1}, \ldots, \eta_{g_3} \), is contained in the tangent plane to \( U_{g^*} \) at the origin. This implies that \( \eta_{g^*} \) lies in the linear subspace \( E \) spanned by \( \eta_{g_1}, \ldots, \eta_{g_3} \). Let \( w \in \mathbb{R}^4 \) be such that \( E = \{ w \cdot u = 0 \mid u \in \mathbb{R}^4 \} \). Consider another constraint \( g = g(\lambda_4, \alpha, \delta) \) through \( p \). The normal \( \eta_g \) lies in \( E \) if and only if

\[
0 = w \cdot \eta_g = w \cdot \eta(\lambda_4, \alpha) = \left( \begin{array}{cc} w_1(1 - \lambda_4) + w_3\lambda_4 & w_2(1 - \lambda_4) - w_4\lambda_4 \end{array} \right), \left( \begin{array}{c} \sin \alpha \\ \cos \alpha \end{array} \right),
\]

that is when (\( \sin \alpha, \cos \alpha \)) is orthogonal to a fixed vector. Modulo \( \pi \), that is, up to reversal of the line, there is only one solution for \( \alpha \), namely when \( \alpha = \alpha^* \).

It follows that the normals of \( g_1, g_2, g_3 \) and \( g \) are linearly dependent if and only if \( g \) is coplanar and concurrent with \( g^* \) and \( \ell_0 \). This is true if and only if \( g \) is tangent to \( B \) in \( p \).

By approximating the volumes \( U_g \) to first order, we get:

**Lemma 18.** Let \( F \) be a family of constraints. If \( F^\perp \) pins \( \ell_0 \) then \( F \) pins \( \ell_0 \). If \( F \) pins \( \ell_0 \) and no four constraints in \( F \) are dependent, then \( F^\perp \) pins \( \ell_0 \).

**Proof.** Since the sets \( U_g \) are bounded by algebraic surfaces of constant degree, the origin 0 is isolated in the intersection of such volumes if and only if there exists no smooth path moving away from 0 inside that intersection. If the tangent vector at 0 to a smooth path \( \gamma \) makes a positive dot product with \( \eta_g \), then \( \gamma \) locally exits \( U_g \). If \( F^\perp \) pins \( \ell_0 \), then the normals to \( F^\perp \) surround the origin and any vector must make a positive dot product with the normal to at least one of the constraints in \( F^\perp \). Since these are also the normals of the constraints in \( F \), no
Lemma 19. If \( B \) is a quadric, that is contained in a tangent counted twice), and thus lies in \( g \). In concreteness, we choose a smooth path can move away from 0 inside \( \cap_{g \in F} U_g \), and \( F \) also pins \( \ell_0 \). The same argument shows that if \( F \) pins \( \ell_0 \), then \( \cap_{g \in F'} U_{g'} \) must have empty interior. In that case, if it is not a single point then four of the normals to the constraints in \( F' \) are linearly dependent, and the statement follows.

Lemma 18 implies that the minimal pinning examples we gave in the previous section are surprisingly robust: We can start with a family of orthogonal constraints pinning \( \ell_0 \), rotate each constraint by changing its \( \delta \)-parameter arbitrarily, and the resulting family will still pin \( \ell_0 \). However, the lemma does not exclude the possibility that there are pinnings that are not robust in this sense, and indeed this is the case. We saw in the previous section that minimal pinnings by orthogonal constraints consist of between five and eight lines. Surprisingly, it is possible to pin using only four non-orthogonal constraints, as we will see below.

Our Isolation Lemma analyzes the intersection of the cone \( C = \cap_{g \in F} U_g \) with \( M \) near 0 in terms of the trace of \( C \) on the hyperplane \( T \). Perhaps surprisingly, this trace has a simple geometric interpretation. Consider the map \( \phi : \mathbb{R}^4 \to \mathbb{R}^3 \) with \( \phi(u_1, u_2, u_3, u_4) = (u_1, u_2, u_3, u_4, 0) \). Clearly, \( \phi \) defines a bijection between \( S \) and the hyperplane \( T = \{ u_5 = 0 \} \). Moreover, given a constraint \( g \), a line \( \ell(u) \) satisfies \( g^\perp \) if and only if \( \phi(u) \in U_g \). Thus, \( \phi^{-1}(C \cap T) \) represents, in our \( \mathbb{R}^4 \) parameterization of \( S \), those lines that satisfy \( F^\perp \). In particular, \( F^\perp \) pins \( \ell_0 \) if and only if \( C \cap T = \{ 0 \} \). In the proof of Lemma 9 we observed that if \( F \) minimally pins \( \ell_0 \) and \( C \cap T \neq \{ 0 \} \) then \( F \) has size at most six. This implies:

**Lemma 19.** If \( F \) is a minimal set of constraints pinning \( \ell_0 \) such that \( F^\perp \) does not pin \( \ell_0 \) then \( F \) has size at most six.

**Examples of higher-order pinnings.** In the proof of Lemma 9 we had considered two cases, namely where \( C \cap T = \{ 0 \} \) (this includes the case where \( C \) is a line intersecting \( M \) transversely), and where \( E = \langle C \cap T \rangle \) is a \( k \)-space with \( 1 \leq k \leq 3 \). In the former case only \( \ell_0 \) satisfies \( F^\perp \), so \( F^\perp \) pins \( \ell_0 \). In the latter case, however, all lines in \( \phi^{-1}(C \cap T) \) satisfy \( F^\perp \), and so \( F^\perp \) does not pin. We give a few such examples, also showing that dimensions 1, 2, and 3 are all possible for \( E = \langle C \cap T \rangle \).

First, consider the four constraints

\[
\begin{align*}
g_0 &= \{ (-t, 0, 0) \mid t \in \mathbb{R} \}, \\
g_1 &= \{ (t, t, 1) \mid t \in \mathbb{R} \}, \\
g_2 &= \{ (-t, -2t, 2) \mid t \in \mathbb{R} \}, \quad \text{and} \\
g_3 &= \{ (t, 3t, 3) \mid t \in \mathbb{R} \},
\end{align*}
\]

oriented in the direction of increasing \( t \). The four constraints lie in the quadric \( B \) defined by the equation \( y = xz \) and have alternate orientations, so they form a 4\(^1\)-block. A line \( \ell \) satisfies the four constraints if and only if it lies in the other family of rulings of \( B \). These lines are of the form \( \{ (a, at, t) \mid t \in \mathbb{R} \} \), for \( a \in \mathbb{R} \). We now replace \( g_0 \) by \( g_0' \), by rotating the constraint slightly around the origin inside the plane \( y = 0 \) (the tangent plane to \( B \) in the origin). For concreteness, we choose \( g_0' \) to be the line \( \{ (-t, 0, -t/100) \} \). This means that for points on \( g_0' \), we have \( y - xz = 0 - t^2/100 < 0 \) for \( t \neq 0 \). This implies that \( g_0' \) touches \( B \) in the origin, and otherwise lies entirely in the volume \( y < xz \) bounded by \( B \). In order to satisfy the four constraints \( g_0', g_1, g_2, g_3 \), a line near \( \ell_0 \) would have to intersect \( B \) at least three times (points of tangency counted twice), and thus lies in \( B \) as \( B \) is a quadric. A line close to \( \ell_0 \), but distinct from it, that is contained in \( B \) must violate \( g_0' \). It follows that the family \( F = \{ g_0', g_1, g_2, g_3 \} \) pins \( \ell_0 \). Since \( F^\perp = \{ g_0, g_1, g_2, g_3 \} \) does not pin \( \ell_0 \), this example already shows that the condition of
independence in the second statement of Lemma 18 is necessary. Note that in this example the space $E = (C \cap T)$ is one-dimensional (it is the set of transversals to $F^\perp$).

Second, consider the family $F$ consisting of the six lines $g_1 = g(0, -\pi/2, -1)$, $g_2 = g(0, \pi/2, -1)$, $g_3 = g(1, 0, 1)$, $g_4 = g(1, \pi, 1)$, $g_5 = g(2, 3\pi/4, 0)$, and $g_6 = g(2, -\pi/4, 0)$. The family $F^\perp$ consists of three 2-blocks, that is all orientations of three lines. The set of lines satisfying $F^\perp$ is the set of transversals to these three lines, again a one-dimensional family. We can verify that $F$ pins $\ell_0$ using the Isolation Lemma. The halfspaces corresponding to the lines are:

\begin{align*}
\tilde{U}_{g_1} : & -u_1 - u_5 \leq 0, \\
\tilde{U}_{g_2} : & u_1 - u_5 \leq 0, \\
\tilde{U}_{g_3}: & -u_4 + u_5 \leq 0, \\
\tilde{U}_{g_4}: & u_4 + u_5 \leq 0, \\
\tilde{U}_{g_5}: & -u_1 - u_2 + 2u_3 + 2u_4 \leq 0, \text{ and} \\
\tilde{U}_{g_6}: & u_1 + u_2 - 2u_3 - 2u_4 \leq 0.
\end{align*}

The constraints $g_1$ and $g_2$ imply $u_5 \geq 0$, and constraints $g_3$ and $g_4$ imply $u_5 \leq 0$. Together they enforce $u_5 = 0$ and then $u_1 = u_4 = 0$. Plugging this into the last two constraints we obtain $u_2 = 2u_3$. Since the 1-space $\{(0, 2t, t, 0, 0) \mid t \in \mathbb{R}\}$ intersects $\mathcal{M}$ in the origin only, $F$ pins $\ell_0$. None of the constraints is redundant, as can be checked by showing that for each $g_i$ there is a point in $T \cap \mathcal{M}$ satisfying all but this constraint.

Third, replace the lines $g_5$ and $g_6$ in the previous family by $g'_5 = g(2, -3\pi/4, 0)$ and $g'_6 = g(3, \pi/4, 0)$. This produces two different halfspaces

\begin{align*}
\tilde{U}_{g'_5}: & u_1 - u_2 - 2u_3 + 2u_4 \leq 0, \text{ and} \\
\tilde{U}_{g'_6}: & -2u_1 + 2u_2 + 3u_3 - 3u_4 \leq 0.
\end{align*}

Plugging $u_1 = u_2 = u_5 = 0$ into these constraints we obtain $-u_2/2 \leq u_3 \leq -u_2/3$. This is a two-dimensional wedge in the $(u_2, u_3)$-plane. It follows that $E = (C \cap T)$ is the 2-space $u_1 = u_4 = u_5 = 0$. The intersection $E \cap \mathcal{M}$ consists of the two 1-spaces $u_2 = 0$ and $u_3 = 0$, which intersect the wedge $-u_2/2 \leq u_3 \leq -u_2/3$ in the origin only. It follows that $F = \{g_1, g_2, g_3, g_4, g'_5, g'_6\}$ pins $\ell_0$. Again, we can check minimality by verifying that no constraint is redundant.

Finally, we consider the family $F = \{g_1, g_2, g_3', g_4', g_5', g_6\}$, where $g'_3 = g(-1, \pi - \tau_3, 0)$, $g'_4 = g(-1/2, -\tau_2, 0)$, $g'_5 = g(1/4, \tau_2, 0)$, and $g'_6 = g(1/3, \pi + \tau_3, 0)$, with $\tau_2 = \arctan 2 \approx 63.4^\circ$ and $\tau_3 = \arctan 3 \approx 71.6^\circ$. The corresponding halfspaces are

\begin{align*}
\tilde{U}_{g_1}: & -u_1 - u_5 \leq 0, \\
\tilde{U}_{g_2}: & u_1 - u_5 \leq 0, \\
\tilde{U}_{g_3'}: & 6u_1 + 2u_2 - 3u_3 - u_4 \leq 0, \\
\tilde{U}_{g_4'}: & -6u_1 - 3u_2 + 2u_3 + u_4 \leq 0, \\
\tilde{U}_{g_5'}: & 6u_1 - 3u_2 + 2u_3 - u_4 \leq 0, \text{ and} \\
\tilde{U}_{g_6'}: & -6u_1 + 2u_2 - 3u_3 + u_4 \leq 0.
\end{align*}

The first two constraints again ensure $u_5 \geq 0$. To construct $C \cap T$, we note that $u_5 = 0$ implies $u_1 = 0$. Plugging $u_1 = u_5 = 0$ into the remaining four constraints, we obtain a three-dimensional cone with apex at the origin that lies in the octant $u_2, u_3, u_4 > 0$. It follows that $E = (C \cap T)$ is the 3-space $u_4 = u_5 = 0$. The intersection $E \cap \mathcal{M}$ consists of the two 2-spaces $u_2 = 0$ and $u_3 = 0$, both of which intersect $C$ only in the origin. It follows that $F$ pins $\ell_0$, and minimality is verified by checking that no constraint is redundant.
The unique nondegenerate minimal pinning of higher order. Call a pair of constraints \{g_1, g_2\} degenerate if they are at the same time coplanar and concurrent with \ell_0. This is equivalent to \(g_1^\perp = g_2^\perp\) or \(g_1^\perp, g_2^\perp\) forming a 2-block. We do not attempt to give a full characterization of all possible pinning configurations \(\mathcal{F}\) where \(\mathcal{F}^\perp\) does not pin. However, we observe that all but one of the examples given above contained degenerate pairs of constraints. We can show that the example we gave is in fact unique in this respect.

**Theorem 7.** Let \(\mathcal{F}\) be a minimal set of constraints pinning \(\ell_0\) not containing degenerate pairs and such that \(\mathcal{F}^\perp\) does not pin \(\ell_0\). Then \(\mathcal{F}\) consists of exactly four constraints and \(\mathcal{F}^\perp\) is a \(4\)-block.

**Proof.** Let \(C = \bigcap_{g \in \mathcal{F}} \bar{U}_g\). Since \(\mathcal{F}^\perp\) does not pin, we have \(C \cap T \neq \{0\}\). Let \(E = \langle C \cap T\rangle\), and define \(\mathcal{G} = \{g \in \mathcal{F} \mid E \subset U_g\}\). By Lemma 5 applied to the cone \(C \cap T\), we have \(\bigcap_{g \in \mathcal{G}} U_g \cap T = E\).

Since \(\phi^{-1}(U_g \cap T) = U_{g^\perp}\), the set \(D = \phi^{-1}(E)\) represents, in our parameterization of \(\mathcal{L}\) by \(\mathbb{R}^4\), the set of lines satisfying \(G^\perp\). In other words, \(G^\perp\) is a family of orthogonal constraints such that the set \(D\) of lines satisfying \(G^\perp\) is a \(k\)-dimensional subspace, where \(k \in \{1, 2, 3\}\).

We now argue that \(k = 1\) and that \(D\) is the set of lines satisfying a \(4\)-block. Let \(N\) denote the set of normals to the constraints in \(G^\perp\), and observe that \(N\) surrounds the origin in the \((4 - k)\)-dimensional subspace orthogonal to \(D\). Since \(\mathcal{F}\) contains no degenerate pair, \(G^\perp\) contains no 2-block, and so \(N\) contains no critical segment. This immediately implies \(k \neq 3\). If \(k = 2\), then \(N\) surrounds the origin in a 2-space. Since \(N\) contains no critical segment, it must contain a critical triangle, and by Lemma 16 \(D\) is the set of lines satisfying a 3-block. \(D\) is thus either the set of lines lying in fixed plane \(\Pi \supset \ell_0\), or the set of lines through a fixed point \(p \in \ell_0\). In both cases, all such lines meet \(\ell_0\). However, if \(\ell(u)\) meets \(\ell_0\) then \(\psi(u) \in T\) and \(\psi(u) = \phi(u)\). It follows that \(C \cap T \subset E \subset \mathcal{M}\), and \(\mathcal{F}\) cannot pin \(\ell_0\), a contradiction. It follows that \(k = 1\), so \(D\) is one-dimensional and \(N\) surrounds the origin in the 3-space orthogonal to \(D\). Since \(N\) contains no critical segment, there are two cases by Lemma 14. First, \(D\) could be the set of lines satisfying two 3-blocks sharing one constraint, which is the set of lines lying in a fixed plane and going through a fixed point. All such lines meet \(\ell_0\), a contradiction. Second, \(D\) could be the set of lines satisfying a 4-block. If this is a \(4^\times\)-block, then again all lines in \(D\) meet \(\ell_0\), a contradiction. We have thus established that \(D\) is the set of lines satisfying a \(4\)-block, that is one family of rulings of a quadric \(B\), while \(G^\perp\) is a subset of the other family of rulings of \(B\).

To conclude, we will now show that \(\mathcal{F}\) has size four. Since \(\mathcal{F}\) pins \(\ell_0\) and \(C \cap T \neq \{0\}\), we are in case (ii) or (iii) of the Isolation Lemma. Without loss of generality, we assume case (ii), and have \(C \subset T^\perp \cup \{\mathcal{M}^\perp \cap T\} \cup \{0\}\). This implies \(C \cap T \subset \mathcal{M}^\perp \cup \{0\}\), and by Lemma 2 (ii) we have \(E \subset \mathcal{M}^\perp \cup \{0\}\). By Lemma 7 we have \(\bigcap_{g \in \mathcal{G}} U_g \subset T^\perp\), and since \(\bigcap_{g \in \mathcal{G}} U_g \cap T = E \subset \mathcal{M}^\perp \cup \{0\}\), this implies \(\bigcap_{g \in \mathcal{G}} U_g \subset T^\perp \cup \{\mathcal{M}^\perp \cap T\} \cup \{0\}\). By Lemma 3 the family \(G\) pins \(\ell_0\), and by minimality of \(\mathcal{F}\) we have \(\mathcal{F} = G\), and so the constraints in \(\mathcal{F}^\perp\) are in hyperboloidal position.

Let \(Y = \{u_5 \leq -1\}\). Since \(\bigcap_{g \in \mathcal{F}} U_g \subset T^\perp\), we have \(\bigcap_{g \in \mathcal{F}} U_g \cap Y = \emptyset\). Since \(E \subset \partial U_g\) for \(g \in \mathcal{F}\) and \(E\) is parallel to \(\partial Y\), we can project all halfspaces on the 4-space orthogonal to \(E\) and apply Helly’s theorem there to obtain a five-element subset of \(\{U_g \mid g \in \mathcal{F}\} \cup \{Y\}\) with empty intersection. Since \(Y\) must be one of the five elements, there is a four-element subset \(\mathcal{F}_1 \subset \mathcal{F}\) with \(\bigcap_{g \in \mathcal{F}_1} U_g \subset T^\perp\). Consider the cone \(C_1 = \bigcap_{g \in \mathcal{F}_1} \bar{U}_g\) and let \(E_1 = \langle C_1 \cap T\rangle\). If \(E_1\) is one-dimensional, then \(E = E_1\), and \(\mathcal{F}_1\) pins \(\ell_0\), implying that \(\mathcal{F} = \mathcal{F}_1\). Since \(\mathcal{F}^\perp\) is in hyperboloidal position, it is a \(4\)-block. It remains to consider the case that \(E_1\) is at least two-dimensional. Let \(\mathcal{F}_2 = \{g \in \mathcal{F}_1 \mid E_1 \subset U_g\}\). By Lemma 3 we have \(\bigcap_{g \in \mathcal{F}_2} U_g \cap T = E_1\). The normals of the constraints in \(\mathcal{F}^\perp_2\) would have to surround the origin in a 1-space or 2-space, but this is impossible for constraints in hyperboloidal position without a 2-block. □
6 Proofs of Theorems 1, 2, and 3

We can now finish the proof of Theorem 2, which states that any minimal pinning of a line by constraints has size at most eight. The bound reduces to six if no two constraints are simultaneously concurrent and coplanar with \( \ell_0 \), that is form a degenerate pair.

**Proof of Theorem 2.** The first statement was proven in Lemma 9. For the second statement, consider be a minimal pinning \( F \) of \( \ell_0 \) by constraints, no two forming a degenerate pair. If \( F^\perp \) does not pin \( \ell_0 \) then by Lemma 9 we have that \( F \) has size at most six. If \( F^\perp \) pins \( \ell_0 \) but is not minimal, then some subfamily \( G \subseteq F \) is such that \( G^\perp \) pins \( \ell_0 \); Lemma 18 yields that \( G \) pins \( \ell_0 \), a contradiction. If \( F^\perp \) is a minimal pinning of \( \ell_0 \), since it cannot contain a 2-block (otherwise \( F \) would contain a degenerate pair) it must have size at most six by Theorem 6. □

We now return to families of convex polytopes that pin a line, and prove Theorem 1, which asserts that any minimal pinning of a line by polytopes in \( \mathbb{R}^3 \) has size at most eight if no facet of a polytope is coplanar with the line. The number reduces to six if, in addition, the polytopes are pairwise disjoint.

**Proof of Theorem 1.** Consider a family \( F \) of convex polytopes pinning the line \( \ell_0 \), such that no facet of a polytope is coplanar with \( \ell_0 \). Then, for each polytope \( F \in F \), \( \ell_0 \) intersects \( F \) either in the interior of an edge \( e^F \) or in a vertex \( v \). In the former case, a line \( \ell \) in a neighborhood of \( \ell_0 \) intersects \( F \) if and only if it satisfies a constraint supporting \( e^F \). In the latter case, there are exactly two silhouette edges \( e_1^F \) and \( e_2^F \) incident to \( v \) in the direction of \( \ell_0 \), and a line \( \ell \) in a neighborhood of \( \ell_0 \) intersects \( F \) if and only if it satisfies the two constraints supporting \( e_1^F \) and \( e_2^F \). It follows that the family of these constraints pins \( \ell_0 \), and so Theorem 2 implies that \( \ell_0 \) is already pinned by eight of the constraints. The corresponding at most eight polytopes pin \( \ell_0 \). If we now make the additional assumption that the polytopes are pairwise disjoint, then no two constraints can be coplanar and concurrent with \( \ell_0 \). It then follows from the second statement in Theorem 2 that six constraints suffice to pin the line, and the statement for pairwise disjoint polytopes follows. □

Last, we give a construction of arbitrarily large minimal pinnings of a line by polytopes in \( \mathbb{R}^3 \), proving Theorem 3.

**Proof of Theorem 3.** We again identify \( \ell_0 \) with the \( z \)-axis. We first pick two polytopes \( D_1 \) and \( D_2 \) such that their common transversals in the vicinity of \( \ell_0 \) are precisely the lines intersecting the \( y \)-axis. Similarly, we pick two polytopes \( D_3 \) and \( D_4 \) that restrict the transversals to pass through the line \( \{ (t, 0, 1) \mid t \in \mathbb{R} \} \), as in Figure 6. A line \( \ell(u) \) meets all four polytopes if and

![Figure 6: Four polytopes that restrict transversals to a two-dimensional set.](image-url)
As a result, the family \( F \) witnessed by the path \( j \) we can clearly make 
\[ \ell(u) = (u_2, u_3) \]
the wedge corresponding to \( \xi_{v,w} \) vector points \( P_0 \) shows that 
\[ P_0 \leq \in L \]
one of the \( F \) pins \( \ell(u) \) such lines that intersect all \( W \xi \) the wedges cover 
the \( 2u_2 \) and \( 2u_3 \leq 2u_2 \).

Consider two angles, \( \beta \) and \( \theta \), with \( 0 < \beta < \theta < \pi/2 \). Let \( v = (v_x, v_y) = (\cos \beta, \sin \beta) \), 
w = (w_x, w_y) = (cos \theta, \sin \theta) \) be two unit vectors, and define the unbounded polyhedral wedge 
\[ F(v, w) = \{(x, y, z) \mid v_x x + v_y y \leq 0 \text{ and } w_x x + w_y y \leq 0 \} \]. The left-hand side of Figure 7 shows 
a projection along \( \ell_0 \). A line \( \ell(u) \) with \( u_2, u_3 > 0 \) and \( u_4 = 0 \) misses \( F(v, w) \) if and only if 
the vector \( (u_2, u_3) \in \mathbb{R}^2 \) falls in the (closed, counterclockwise) acute angular interval 
\( \xi_{v,w} = [\beta, \theta] \) between \( v \) and \( w \). In other words, the set of lines intersecting \( F(v, w) \) looks like the gray shape 
in the \( u_2u_3 \)-plane depicted on the right hand side of Figure 7, namely the plane with the closed 
 wedge corresponding to \( \xi_{v,w} \) removed.

We pick \( n \) pairs of vectors \((v^1, w^1), (v^2, w^2), \ldots, (v^n, w^n)\), with the property that together 
the wedges cover \( W \), and such that the middle vector \( v^i + w^i \) lies in \( W \) but does not lie in 
\( \xi_{v^j,w^j} \) for \( j \neq i \). The family \( \{F_1, F_2, \ldots, F_n\} \) of shapes \( F_i = F(v^i, w^i) \) has the property that \( \ell_0 \) is 
the only line in \( W \) intersecting the entire family, but for any \( 1 \leq i \leq n \) there is an entire sector of 
such lines that intersect all \( F_i \) with \( j \neq i \). It follows that the family \( \mathcal{F} = \{D_1, \ldots, D_6, F_1, \ldots, F_n\} \) 
pins \( \ell_0 \), but has no pinning subfamily of size smaller than \( n \) (some \( D_i \) could be redundant, but 
none of the \( F_i \) is).

In the final step, we crop the polyhedral wedges to create a family of bounded polytopes 
with the same property. Since pinning is determined by lines in a neighborhood of \( \ell_0 \) only, 
we can clearly make \( D_1, \ldots, D_6 \) bounded. For each \( F_i \), select the following “linear” path \( \gamma_i(t) \) 
in \( \mathcal{L} \) starting at the origin: 
\[ \gamma_i(t) = (u_1(t), u_2(t), u_3(t), u_4(t)) = (0, v_x^i + w_y^i, v_y^i + w_x^i, 0) \cdot t, \]
for \( 0 \leq t \leq 1 \). In the projection on the \( xy \)-plane, the line \( \gamma_i(t) \) is, for any \( t > 0 \), perpendicular to 
the vector \( v^i + w^i \). Therefore \( \gamma_i(t) \) does not intersect \( F_i \), but by construction it intersects each \( F_j \) 
with \( j \neq i \). Let \( P_{ij}(t) \) be the point where \( \gamma_i(t) \) enters or exits \( F_j \). A straightforward calculation 
shows that \( P_{ij}(t) \) moves linearly away from \( \ell_0 \) along a line perpendicular to \( \ell_0 \).

We can therefore crop each \( F_j \) to a bounded polytope, ensuring that it still contains all 
points \( P_{ij}(t) \) for \( i \neq j \) and \( 0 < t \leq 1 \), and hence intersects all lines on the paths \( \gamma_i(t) \) for \( i \neq j \). 
As a result, the family \( \mathcal{F} \) still pins \( \ell_0 \), but for any \( F_i \) the family \( \mathcal{F} \setminus \{F_i\} \) is not a pinning as 
witnessed by the path \( \gamma_i \).

Figure 7: The polyhedron \( F(v, w) \) and the set of lines intersecting it.
7 Concluding remarks

We have shown that minimal pinnings by convex polytopes have bounded size if the line \( \ell_0 \) is not coplanar with a facet of a polytope. It seems that this condition can be slightly relaxed: if \( \ell_0 \) intersects a polytope in a vertex \( v \) and lies in the plane of a facet incident to \( v \), then one can argue separately in \( T^\geq \) and \( T^\leq \). However, we do not know if the result generalizes any further.

If \( \ell_0 \) meets the relative interiors of two different edges of a polytope facet (and is thus coplanar with this facet), then a line \( \ell \) near \( \ell_0 \) intersects the polytope if and only if it satisfies one of the two line constraints that support the front edge and the back edge of the facet. It follows that the set of lines intersecting the polytope is the union of two „halfspaces.” Since this is not a convex shape, our techniques do not seem to apply.

We have shown that intersecting convex polytopes can have minimal pinnings of arbitrary size. We conjecture that pairwise disjoint convex polytopes have bounded pinning number. In fact, as mentioned in the introduction, we are not aware of any construction of a minimal pinning by arbitrary pairwise disjoint compact convex objects of size larger than six.

Lemma 9 is a key lemma. The most difficult case is when \( E \) is three-dimensional, and in fact the proof only works because \( E \cap 2\mathbb{R} \) must consist of two intersecting 2-spaces. The proof would not go through if \( E \cap \mathbb{R} \) were allowed to be, for example, a circular cone. Perhaps this is an indication that there might be no corresponding result in higher dimensions.

We saw in the introduction that a family of lines pinning a line can be considered as a grasp of that line. In grasping, one often considers form closure, which means that the object is immobilized even with respect to infinitesimally small movements. For instance, an equilateral triangle with a point finger at the midpoint of every edge is immobilized, as it cannot be moved in any way, but it is not in form closure because an infinitesimal rotation around its center is possible. It is easy to see that all grasps listed in Theorem 6 are form closure grasps in this sense. The grasp caused by a 4-pinning, however, is not a form-closure grasp, as \( \ell_0 \) can be moved infinitesimally in the quadric defined by three of the lines.

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