A DISCRETE ITÔ CALCULUS APPROACH TO HE’S FRAMEWORK FOR MULTI-FACTOR DISCRETE MARKETS

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Abstract. In the present paper, a discrete version of Itô’s formula for a class of multi-dimensional random walk is introduced and applied to the study of a discrete-time complete market model which we call He’s framework. The formula unifies continuous-time and discrete-time settings and by regarding the latter as the finite difference scheme of the former, the order of convergence is obtained. The result shows that He’s framework cannot be of order 1 scheme except for the one dimensional case.

1. Introduction

In He (1990), the binomial tree approach by Cox, Ross and Rubinstein (1979) is generalized to a multi-nomial one and limit theorems concerning pricing kernels and hedging strategies are established. The main feature of He’s multi-nomial tree framework is that each approximating market itself is arbitrage-free and complete.

In the present paper, a new insight to He’s framework, which leads to further applications, will be introduced. The insight comes from a discrete version of Itô’s formula. As is the case with continuous-time models, our discrete Itô formula relates the value process of a contingent claim to a difference equation. This means that the formula enables a discrete version of so-called partial differential equation (PDE) approach to the pricing-hedging problems in the literature of mathematical finance; we do not use the usual martingale argument.

Further, if a continuous-time limit exists, then the discrete equations obtained via our Itô formula can be seen as explicit finite difference approximations of the limit PDE, and we can obtain the order of convergence by using the standard argument of the finite difference scheme.

Key words and phrases. discrete Itô formula, finite difference scheme, discrete-time multi-asset market.

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The contributions of the present paper are:

- a multi-dimensional version of discrete Itô formula [Theorem 3.1] which enables the discrete PDE approach.
- the order of convergence of the value functions of European options within He’s framework [Theorem 4.2], which is proved to be $O(N^{-1/2})$ in general and $O(N^{-1})$ in single risky asset cases. Here $N$ is the number of time-discretization steps.

The point is that completeness makes it slow; as He’s framework is based on completeness of market. The first observation from discrete-Itô formula shows that approximations by the discrete market model of He’s framework are always a kind of finite-difference approximation of a PDE, while the second observation says that the convergence is much slower when $n \geq 2$ than an approximation by an Euler-Maruyama scheme of first order.

This paper is organized as follows. In section 2 a quick review of He’s framework will be presented. In section 3 the Szabados-Fujita formula and the discrete PDE framework will be introduced. In section 4 a limit theorem will be established. In section 5 the relations with the group theory will be explained. Finally in section 6 proofs of the theorems in the present paper will be undertaken.

**Remark 1.1.** This paper is motivated by the textbook Fujita (2002), where he gives a very nice description of from CRR to Black-Scholes argument by using his discrete Itô formula. His (and our) approach would be very instructive for those who are not familiar with higher mathematics.

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## 2. He’s Framework: An Overview

In essence, He (1990) approximated $n$-dimensional Brownian motions by a system of mutually orthogonal martingales of finite states—$(n+1)$ states at each step.

Let us briefly review He’s framework. Let $(e_{i,j})_{0 \leq i,j \leq n}$ be an $(n+1) \times (n+1)$-orthogonal matrix such that $e_{0,j} > 0$ for $j = 0, 1, ..., n$, and define

$$E := \left\{ e_j = \frac{1}{e_{0,j}} (e_{1,j}, ..., e_{n,j}) \in \mathbb{R}^n : j = 0, ..., n \right\}.$$  

Let $\tau \equiv (\tau^1, ..., \tau^n)$ be a random variable taking values in $E$ with

$$P(\tau = e_j) = e_{0,j}^2, \quad j = 0, 1, ..., n.$$
Then, \(^1\)

\[
E[\tau^i] = 0, \quad i = 1, ..., n, \quad \text{and} \quad \text{Cov}(\tau^i, \tau^j) = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j). \end{cases}
\]

Let \(\tau_1, ..., \tau_... \) be independent copies of \(\tau\). Define a sequence of \(\mathbb{R}^n\) valued stochastic processes \(\{X^N_t\}\) by

\[
X^N_t = X_0 + N^{-1/2} \sum_{u=1}^{[Nt]} \tau_u
\]

for a given initial point \(X_0 \in \mathbb{R}^n\). By (2.2), components of \(X^N_t - X^N_0\) are mutually orthogonal martingales, and therefore, the martingale central limit theorem (see Ethier and Kurtz (1986) for example) ensures that the law of \(X^N\) converges weakly to the \(n\)-dimensional Wiener measure as \(N \to \infty\).

Fix \(N \in \mathbb{N}\). For \(T > 0\), we denote \(T_N = [TN]/N\). For a subinterval \(I\) of \([0, \infty)\), we denote \(I_N = I \cap \{k/N : k = 0, 1, 2, \ldots\}\).

In our market there are \((n + 1)\)-securities whose prices are given by \(S^j_N(t, X^N_t)\) for \(j = 0, 1, ..., n\), where \(h^j\)'s are real functions defined on \([0, T]_N \times \mathbb{R}^n\) such that the following \((n + 1) \times (n + 1)\)-matrix

\[
H^N(t, x) := \begin{pmatrix}
h^{0,0}(t, x + N^{-1/2}e_0) & \cdots & h^{0,N}(t, x + N^{-1/2}e_0) \\
h^{1,0}(t, x + N^{-1/2}e_1) & \cdots & h^{1,N}(t, x + N^{-1/2}e_1) \\
\vdots & \ddots & \vdots \\
h^{n,0}(t, x + N^{-1/2}e_n) & \cdots & h^{n,N}(t, x + N^{-1/2}e_n)
\end{pmatrix}
\]

is invertible for arbitrary \((t, x) \in [0, T]_N \times \mathbb{R}^n\).

Suppose that at time \(t = k/N\) we have \(\theta^j_t \equiv \theta^j_t(\tau_1, ..., \tau_k)\) amount of \(j\)-th security for each \(j \in \{0, 1, ..., n\}\). The cost of the portfolio at time \(t\) is

\[
v^N(t, \tau_1, ..., \tau_k) := \sum_j h^{j, N}(t, X^N_t) \theta^j_t,
\]

and at time \(t + N^{-1}\) the value of the portfolio becomes

\[
v^N(t + N^{-1}, \tau_1, ..., \tau_k, \tau_{k+1}) := \sum_j h^{j, N}(t, X^N_{t+N^{-1}}) \theta^j_t,
\]
or equivalently

\[
\begin{pmatrix}
v^N(t + N^{-1}, \tau_1, ..., \tau_k, N^{-1/2}e_0) \\
v^N(t + N^{-1}, \tau_1, ..., \tau_k, N^{-1/2}e_1) \\
\vdots \\
v^N(t + N^{-1}, \tau_1, ..., \tau_k, N^{-1/2}e_n)
\end{pmatrix} = H^N(t + N^{-1}, x)
\begin{pmatrix}
\theta^0_t \\
\theta^1_t \\
\vdots \\
\theta^N_t
\end{pmatrix}.
\]

\(^1\)Note that the converse is true; any random variable \(\tau\) satisfying (2.2) is, if it is defined on a finite set, constructed in the above way from such an orthogonal matrix. He (1990) treated only the uniform cases of \(e_{0,0} = \cdots = e_{0,n} = 1/\sqrt{n + 1}\).
If the portfolio is self-financed, then \( c^N(t, \cdot) = v^N(t, \cdot) \). Since we have assumed that \( H^N \) is invertible, we have by combining (2.3) and (2.4),

(2.5)
\[
v^N(t, x) = (h^{0,N}(t, x), \ldots, h^{n,N}(t, x)) H^N(t + N^{-1}, x)^{-1} \begin{pmatrix}
v^N(t + N^{-1}, x, N^{-1/2}e_0) \\
v^N(t + N^{-1}, x, N^{-1/2}e_1) \\
\vdots \\
v^N(t + N^{-1}, x, N^{-1/2}e_n)
\end{pmatrix};
\]

\( t \in [0, T_N)_N, \ x \in E^{[N]} \).

If the terminal value (to be hedged) \( \Phi : E^N \to \mathbb{R} \) is dependent only on \( X^N_T \), then \( v^N(T - N^{-1}, \cdot) \) depends only on \( X^N_{t-N^{-1}} \), etc, etc, and finally we have the following recursive equation, which has a unique solution:

(2.6)
\[
v^N(T_N, x) = \Phi^N(x); \ x \in \mathbb{R}^n,
\]
\[
v^N(t, x) = (h^{0,N}(t, x), \ldots, h^{n,N}(t, x)) H^N(t + N^{-1}, x)^{-1} \begin{pmatrix}
v^N(t + N^{-1}, x + N^{-1/2}e_0) \\
v^N(t + N^{-1}, x + N^{-1/2}e_1) \\
\vdots \\
v^N(t + N^{-1}, x + N^{-1/2}e_n)
\end{pmatrix};
\]

\( t \in [0, T_N)_N, \ x \in \mathbb{R}^n \).

Here all we can say is that \( v(t, X^N_t) \) is the replication cost (at time \( t \)) of an European option whose pay-off is described by \( \Phi(X^N_T) \), where \( \Phi^N : \mathbb{R}^{n+1} \to \mathbb{R} \).

As is well known, absence of arbitrage opportunities is equivalent to the positivity of the state price (see e.g. Duffie (1996)). In other words, denoting \( h^N(t, x) = (h^{0,N}(t, x), \ldots, h^{n,N}(t, x)) \),

(2.7)
\[h^N(t, x) H^N(t + N^{-1}, x)^{-1} \text{ is strictly positive.}\]
Under the hypothesis of (2.7), the unique solution \( v^N(t, x) \) is the unique fair price at time \( t \in [0, T) \) and the state \( x \in \mathbb{R}^n \) with \( S^N_t = h^N(t, x) \) of the European option whose pay-off is \( \Phi^N(X^N_T) \). Note that the invertibility of \( H \) is equivalent to completeness of the market.

The above derivation of (2.6) is also valid for any Markov process\(^2\) \( Z^N \) replacing \( X^N \). In fact He (1990) modeled the price vector

\( Z^N_{t+N^{-1}} - Z^N_t = \sum_{j=0}^{n} F_j(Z^N_t) r^N_{t+N^{-1}} \),

---

\( ^2 \)In general it is represented by some \( F_j, j = 0, 1, \ldots, n \) as
\( S_t = (S^1_t, ..., S^n_t) \) directly (meaning \( h^j \)'s are identity maps) by an Euler-Maruyama approximation of a stochastic differential equation. Here we have changed the setting as above. The differences is that we have preserved the structure of so-called recombining tree: if we consider \( S_{k/N} \) as a function of \( \tau_1, ..., \tau_k \), we have

\[
S_{k/N}(e_{i_1}, ..., e_{i_k}) = S_{k/N}(e_{i_{\sigma(1)}}, ..., e_{i_{\sigma(k)}})
\]

for arbitrary permutation \( \sigma \in \mathfrak{S}_k \).

The reasons for this modification are: (i) Euler-Maruyama approximations by finite-points random variables using Monte-Carlo are not practical, (ii) nor is solving an equation like (2.6) without recombining structure of (2.8).

In fact, it relaxes quite a lot computational complexity, by which we mean how many times we need to solve the one-step linear equation (2.6) to obtain the value for \( v^N(t, x) \). In other words, it is the number of possible states

\[
\mathcal{X}(t, x, Z) := \{ y \in \mathbb{R}^n : \mathbb{P}(Z^N_t = x, X^N_T = y) > 0 \}
\]

In general we have \( \sharp \mathcal{X}(T - kN^{-1}, x, Z) = (n + 1)^k \). Even if \( Z \) is an Euler-Maruyama approximation of a solution to SDE, almost always this is the case. However, the symmetry (2.8), which comes from that of \( X_t \), reduces it dramatically. More precisely, we have the following.

**Proposition 2.1.**

\[
\sharp \mathcal{X}(T - kN^{-1}, x, X^N) = \frac{(k + n)!}{k!n!}.
\]

**Proof.** Since \( \{e_1, ..., e_n\} \) spans \( n \)-dimensional subspace in \( \mathbb{R}^{n+1} \), they have no linear dependence other than \( e_1 + \cdots + e_n = 0 \). Therefore, the number is equal to that of solutions to

\[
x_1 + x_2 + \cdots + x_{n+1} = k, \quad x_j \in \mathbb{Z}_+, j = 1, ..., n + 1,
\]

which is exactly \( (n + k)!/k!n! \). \( \square \)

**Remark 2.2.** Denoting by \( A(t, x) \) the sum of all the components of \( h^N(t, x)H^N(t + N^{-1}, x)^{-1} \), the value process of money market account is given by

\[
\prod_{k}^{[Nt]} 1/A(k/N, X^N_{(k-1)/N}).
\]

In particular, **positive interest rate** is equivalent to \( A(t, x) < 1 \) for arbitrary \((t, x)\). with a convention of \( \tau^0 \equiv 1 \). This is because \( 1, \tau^1, ..., \tau^n \) forms an orthonormal basis of the space of random variables on generated by \( \tau \). In particular, a discrete approximation of an SDE by a Markov chain always has an Euler-Maruyama representation.
3. A Discrete Itô formula and discrete PDE

Let us introduce a discrete version of Itô’s formula for the process $X^N \equiv (X_{N,1}, \ldots, X_{N,n})$.

**Theorem 3.1.** (i) For a function $f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$, we have

\[
    f(t, X^N_t) - f(0, X_0) = \sum_{u=1}^{[Nt]} \left( \sum_{k=1}^n \partial^N_k f(u/N, X^N_{(u-1)/N}) (X^N_{u/N} - X^N_{(u-1)/N}) 
    + \left( \frac{1}{2} \Delta^N + \partial^N_t \right) f(u/N, X^N_{(u-1)/N}) / N \right),
\]

where

\[
    \partial^N_k f(\cdot, x) = \sqrt{N} \sum_{j=0}^n f(\cdot, x + N^{-1/2} e_j) e_{0,j} e_{k,j}
\]

\[
    \Delta^N f(\cdot, x) = 2N \sum_{j=0}^n \{ f(\cdot, x + N^{-1/2} e_j) - f(\cdot, x) \} e_{0,j}^2,
\]

\[
    \partial^N_t f(t, \cdot) = N(f(t, \cdot) - f(t - N^{-1}, \cdot)).
\]

(ii) If $f$ is in $C^{1,2}$ in a neighborhood of $(t, x)$, then letting $N \to \infty$, we have

\[
    \partial^N_j f(t, x) \to \frac{\partial}{\partial x_j} f(t, x), \quad \Delta^N f(t, x) \to \Delta f(t, x), \quad \partial^N_t f(t, x) \to \frac{\partial}{\partial t} f(t, x).
\]

Here $\Delta$ is the Laplacian in $\mathbb{R}^n$. (iii) Further, for fixed $t \in [0, T]$, if $f(t, \cdot)$ is in $C^3$ in an open set $U \subset \mathbb{R}^n$, then for every compact subset $K \subset U$, there exists a positive constant $C_K$ depending only on $f(t, \cdot)$ such that

\[
    \max_j \left| \partial^N_j f(t, x) - \frac{\partial}{\partial x_j} f(t, x) \right| + \left| \Delta^N f(t, x) - \Delta f(t, x) \right| \leq C_K N^{-1/2}
\]

for all $x$ in $K$. (iv) The order of convergence cannot be improved for general $f \in C^{1,4}$ when $n \geq 2$. (v) For the case of $n = 1$, it can be improved to be $N^{-1}$, provided that $f \in C^{1,4}$.

A proof of Theorem 3.1 will be given in section 6.1

**Remark 3.2.** This version of Itô’s formula is different from those for jump semimartingales which, for example, is appearing in Protter (2004), different in that ours gives the Doob decomposition of $f(t, X_t)$. This version of discrete Itô’s formula was introduced by Fujita (2003) for the case of $n = 1$. Kudzhma (1982) and Szabados (1990) also studied discrete Itô formulas as discrete-analogues of the standard one, which point of view is what we share in this paper. It is true that it should
be called Kudzhma-Szabados-Fujita formula, but here the term discrete Itô formula is preferred since the true name is too long and confusing.

We claim that the recursive equation (2.6) defines a discrete PDE with respect to these differentials of (3.2). Define

$$\Sigma^N(t, x) := \begin{pmatrix}
    h_0^N(t-N^{-1}, x) & \cdots & h_n^N(t-N^{-1}, x) \\
    \partial_1^N h_0^N(t, x) & \cdots & \partial_1^N h_n^N(t, x) \\
    \vdots & \ddots & \vdots \\
    \partial_n^N h_0^N(t, x) & \cdots & \partial_n^N h_n^N(t, x)
\end{pmatrix}.$$ 

Theorem 3.3. Let us assume that the market is arbitrage-free and complete. Namely, the existence of $(H^N)^{-1}$ and (2.7) are assumed. Then, $\Sigma^N$ is always invertible and $v^N$ satisfies the following discrete PDE.

$$v^N(T_N, x) = \Phi^N(x); \quad x \in \mathbb{R}^n,$$

$$\partial_t^N v^N + \frac{1}{2} \Delta^N v^N - \langle b^N, \nabla^N v^N \rangle - c^N(1^N v^N) = 0;$$

$$t \in (0, T_N), \quad x \in \mathbb{R}^n.$$ 

Here $1^N v^N(t, x) = v^N(t-N^{-1}, x)$ and $(c^N, b^N) = (\partial_t^N h^N + \frac{1}{2} \Delta^N h^N)[\Sigma^N]^{-1}.$

A proof of Theorem 3.3 will be given in section 6.2.

The equation (3.5) can be obtained directly by using the discrete Itô’s formula (3.1) if we a priori assume that $\Sigma^N$ is invertible. Let us write $dY_t := Y_t - Y_{t-N^{-1}}$ for a process $Y$, $dt := 1/N$, $\nabla^N = (\partial_1^N, ..., \partial_n^N)$, $V_t := v^N(t, X_t^N)$, and so on. If we have $dV_t = \sum_{j=1}^{n+1} \theta^j_t dh^j_t$ and $V_t = \sum_{j=1}^{n+1} \theta^j_{t+1} h^j_t$, then $\theta$ is the hedging strategy and the problem is settled. This can be done quite easily in a parallel way with the continuous-time cases. In fact, we have

$$dV = \nabla v^N \cdot dX^N + \left( \partial_t^N v^N + \frac{1}{2} \Delta^N v^N \right) dt = \sum_{j=1}^{n+1} \theta^j_t dh^j_t,$$

$$dh^j_t = \nabla h^j_t \cdot dX^N + \left( \partial_t^N h^j_t + \frac{1}{2} \Delta^N h^j_t \right) dt,$$

and $v^N = \sum_{j=1}^{n+1} \theta^j_{t+1} h^j_t$, hence $\theta = (\theta^1, ..., \theta^{n+1}) = (\Sigma^N)^{-1}(v^N(t-N^{-1}), \nabla v^N)$.

Note that the above argument can be applied to the case of $N = \infty$, where $X^\infty$ is the standard Brownian motion, $dt$ is the standard one, and so on. The corresponding standard PDE shares the algebraic structure with the discrete ones. Since the second assertion (ii) of the above theorem 3.1 can be seen as consistency of the difference operators of (3.2) in the context of finite difference method (see e.g. Richtmyer and Morton (1994)), $v^N$ converges to a solution $v^\infty$ to the
PDE at least when \( v^\infty \) is regular enough. We will make a detailed study about this topic in the next section.

4. Limit Theorem

The solution \( v^N(t, \cdot) : \mathbb{R}^n \to \mathbb{R} \) is solved inductively for each \( t \in [0, T_N] \), and for each \( x \in \mathbb{R}^n \), the function \( v^N(\cdot, x) \) on \([0, T]\) can be extended to a piecewise-constant function on \([0, T]\). We choose such an extension on \([0, T] \times \mathbb{R}^n\) and denote it by the same symbol.

Here we assume the followings to establish our limit theorem.

Assumption 4.1. (i) The market is arbitrage-free and complete; i.e. we assume (2.7) and invertibility of \( H^N \). (ii) There exist bounded measurable functions \( b : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n \) and \( \Phi : \mathbb{R}^n \to \mathbb{R} \) and a continuous function \( c : [0, T] \times \mathbb{R}^n \to \mathbb{R} \) such that

\[
\sup_N \sup_{t \in [0, T], x \in \mathbb{R}^n} N^{1/2} \left( |b - b^N| + |c - c^N| + |\Phi - \Phi^N| \right) < \infty.
\]

and (iii) they are regular enough to allow the following partial differential equation to have a bounded solution in \( C^{1,3} \) whose first order derivatives are also bounded.

\[
\frac{\partial v}{\partial t} + \frac{1}{2} \Delta v - \langle b(t, x), \nabla v \rangle - c(t, x)v = 0, \quad t \in [0, T], \quad v(T, x) = \Phi(x),
\]

where \( \Delta \) is the Laplacian of \( \mathbb{R}^n \) and \( \nabla \) is the gradient operator in \( \mathbb{R}^n \).

(iv) We also assume that the interest rate is positive (See Remark 2.2).

As we pointed out in Remark 2.2, the assumption (iv) is equivalent to \( A < 1 \), and hence, as we shall show in the proof of Theorem 3.3, is equivalent to the positivity of the first component of

\[
(\partial_N + \frac{1}{2} \Delta_N)[\Sigma N]^{-1}.
\]

This in turn implies that \( c \) is positive.

Under the Assumption 4.1, we have the following

Theorem 4.2. The solutions \( v^N \) to (3.5) converges uniformly on compact intervals of \([0, T] \times \mathbb{R}^n\) to the solution \( v \in C^{1,3} \) to (4.2) in an order of \( N^{-1/2} \). For the one dimensional case, the order can be improved to \( N^{-1} \) provided that \( v \) is in \( C^{1,4} \) and the order in (4.1) is replaced with \( N^{-1} \).

A proof of Theorem 4.2 will be given in section 6.3.

Remark 4.3. Our scope covers as a special case the Black-Scholes economy by setting \( h^j,N(t, x) \equiv S_0^j e^{(\sigma_j x - \mu_j) t} \) for \( j = 1, \ldots, n \) and \( h^{n+1,N}(t, x) \equiv e^{rt} \), where \( S_0, r \in \mathbb{R}_+, \mu_j \in \mathbb{R} \) and \( \Sigma = [\sigma_1, \ldots, \sigma_n] \) is a \( n \times n \) positive definite matrix.

Remark 4.4. Since we are working on a reference measure which is not necessarily a risk neutral measure nor so-called physical measure, \( W \) can be a diffusion process other than Brownian motions under those
measures. Roughly speaking, $W$ can be a solution (in the weak sense!) to a stochastic differential equation whose diffusion coefficients are constant functions. In one dimensional cases, by scaling we can work on any diffusion whose diffusion coefficient is monotone and smooth.

5. **Supplementary Remark: relations to Group Representation**

We remark here that a specification of $\mathcal{E}$ can be done with the help of group representation theory.

Let us recall the basics of group representation theory (see e.g. Serre (1978)). Let $G$ be a compact abelian group, and let $\hat{G}$ be its dual group. The members of $\hat{G}$ are often called **characters**, which forms an orthonormal basis of $L^2(G; \mathbb{C})$; the space of square integrable functions on $G$ with respect to its Haar measure. Since $L^2(G; \mathbb{C})$ is a complex vector space, we need to modify it to get an orthonormal basis over real field. One candidate is obtained by the transform $\varphi: \mathbb{C} \rightarrow \mathbb{R}$ defined by $\varphi(x+iy) = x+y$. It is easy to check that $\{\varphi(\chi) : \chi \in \hat{G}\}$ is an orthogonal basis of $L^2(G; \mathbb{R})$. The group $\hat{G}$ always contains a unit, which corresponds to 1. Thanks to Peter-Weyl Theorem, the above argument is extended to non-abelian groups.

In particular, a choice of group $G$ with $|G| = n + 1$ gives us an $\mathcal{E}$. The simplest choice may be the cyclic group $C_{n+1}$. In this case, $\tau = (\tau^1, \ldots, \tau^n)$ is obtained by taking $\tau^k = \varphi(\eta^k)$ where $\eta$ is a uniformly distributed random variable taking values in $(n+1)$-th units of root. The fundamental theorem of finitely generated abelian groups says that the characters are always taking values in a set of the units of root. Therefore, the **scenarios** are generated by a random walk on a ring of integers of an algebraic number field. The easiest case ($n = 2$) is studied in Akahori (2003).

6. **Proofs**

6.1. **A Proof of Theorem 3.1** Let $L(\tau)$ be a linear space of $\tau$-measurable real valued random variables. Since $\tau$ takes only $(n+1)$-distinct values, the dimension of $L(\tau)$ is $(n+1)$. On the other hand, as a matter of course, the coordinate maps $\tau^1, \ldots, \tau^n$ are members of $L(\tau)$. The moment condition (2.2) says that $\{\tau^1, \ldots, \tau^n\}$ and constant function 1 are mutually orthogonal with respect to the inner product $\langle x, y \rangle = \mathbb{E}[xy]$. Hence $\{1, \tau^1, \ldots, \tau^n\}$ is an orthonormal basis of $L(\tau)$. 

Orthogonal expansion of \( f(t, x + N^{-1/2} \tau) \) with respect to the basis \( \{1, \tau^1, \ldots, \tau^n\} \) are as follows:

\[
f(t, x + N^{-1/2} \tau) = \sum_{k=1}^{n} \mathbb{E}[f(t, x + N^{-1/2} \tau) \tau^k] + \mathbb{E}[f(t, x + N^{-1/2} \tau)]
\]

\[
= \sum_{k=1}^{n} \left( \sum_{j=0}^{n} P(\tau = e_j)f(t, x + N^{-1/2}e_j) \frac{\delta_{k,j}}{e_{0,j}} \right) \tau^k
\]

\[
+ \sum_{j=0}^{n} P(\tau = e_j)f(t, x + N^{-1/2}e_j)
\]

\[
= \sum_{k=1}^{n} \left( \sum_{j=0}^{n} f(t, x + N^{-1/2}e_j) e_{0,j} e_{k,j} \right) \tau^k
\]

\[
+ \sum_{j=0}^{n} e_{0,j}^2 f(t, x + N^{-1/2}e_j)
\]

\[
= \sum_{k=1}^{n} \frac{\partial_k^N f(t, x)}{\sqrt{N}} \tau^k + \frac{1}{2N} \Delta^N f(t, x) + f(t, x).
\]

Substituting \( X_u^N \) for \( x \) and \( X_{u+N-1}^{N,k} - X_u^{N,k} \) for \( \tau^k/\sqrt{N} \), we have

\[
f(u + N^{-1}, X_u^{N}) - f(u, X_u^N)
\]

\[
= \sum_{k=1}^{n} \partial_k^N f(u + N^{-1}, X_u^{N}) (X_{u+N-1}^{N,k} - X_{u}^{N,k})
\]

\[
+ \frac{1}{2} \Delta^N f(u + N^{-1}, X_u^N)/N.
\]

By summing up (6.1) for \( u = 0, 1/N, 2/N, \ldots, ([Nt] - 1)/N \), we obtain (3.1).

Let us consider next the following formal Taylor expansion of \( f(t, x + N^{-1/2} \tau) \) with respect to \( N^{-1/2} \tau \):

\[
f(t, x + N^{-1/2} \tau)
\]

\[
= f(t, x) + \frac{1}{\sqrt{N}} \langle \nabla f, \tau \rangle + \frac{1}{\sqrt{N}} \langle \nabla f \otimes \nabla f, \tau \otimes \tau \rangle
\]

\[
+ \cdots + \frac{1}{N^{m/2}} \langle (\nabla f)^{\otimes m \text{ times}}, (\tau \otimes \tau)^{\otimes m \text{ times}} \rangle + \cdots
\]

Recalling (or observing the proof given above) that

\[
\partial_k^N f(t, x) = \sqrt{N} \mathbb{E}[f(t, x + N^{-1/2} \tau) \tau^k],
\]

and

\[
\Delta^N f(t, x) = 2N \mathbb{E}[f(t, x + N^{-1/2} \tau) - f(t, x)],
\]
we have the following formal expansions:
\[
\partial_k^N f(t, x) = \sqrt{N} \mathbb{E}[f(t, x)\tau^k] + \mathbb{E}[\langle \nabla f, \tau \rangle \tau^k] \\
+ \frac{1}{2\sqrt{N}} \mathbb{E}[\langle \nabla f \otimes \nabla f, \tau \otimes \tau \rangle \tau^k] \\
+ \frac{1}{2N} \mathbb{E}[\langle \nabla f \otimes \nabla f \otimes \nabla f, \tau \otimes \tau \otimes \tau \rangle \tau^k] + \cdots + \cdots
\]
and
\[
\frac{1}{2} \Delta^N f(t, x) = +\sqrt{N} \mathbb{E}[\langle \nabla f, \tau \rangle] + \frac{1}{2} \mathbb{E}[\langle \nabla f \otimes \nabla f, \tau \otimes \tau \rangle] \\
+ \frac{1}{6\sqrt{N}} \mathbb{E}[\langle \nabla f \otimes \nabla f \otimes \nabla f, \tau \otimes \tau \otimes \tau \rangle] \\
+ \frac{1}{24N} \mathbb{E}[\langle (\nabla f)^{\otimes 4}, \tau^{\otimes 4} \rangle] + \cdots + \cdots .
\]

Now the assertions (ii) and (iii) are verified since for \( f(t, \cdot) \in C^k \) the expansion up to \( k \)-th term is valid.

The assertion (v) is verified by looking at the case \( P(\tau = \pm 1) = 1/2 \) where \( \mathbb{E}[\tau^3] = 0 \). The assertions (iv) is a consequence of the following lemma.

\[\square\]

**Lemma 6.1.** Suppose that \( \tau : \Omega \to \mathbb{R}^n \) satisfies (2.2) and \( \#\Omega = n+1 \). Then if \( n \geq 2 \), there exists \( (i, j, k) \in \{1, \ldots, n\}^3 \) such that \( \mathbb{E}[\tau^i \tau^j \tau^k] \neq 0 \).

**Proof.** Denote \( D = (e_{i,j})_{0 \leq i, j \leq n} \) and let
\[
D_k = \text{diag}[e_{k,0}/e_{0,0}, e_{k,1}/e_{0,1}, \ldots, e_{k,n}/e_{0,n}], \quad k = 1, \ldots, n.
\]
Then one will find that the \((i, j)\)-th component of \( D^* D_k D \) is given by
\[
d_{i,j} = \sum_{l=0}^{n} \frac{e_{i,l} e_{k,l} e_{j,l}}{e_{0,l}} = \mathbb{E}[\tau^i \tau^j \tau^k],
\]
where conventionally \( \tau^0 \equiv 1 \) and \( d_{i,j} \) for \( 0 \leq i, j \leq n \) are numbered as follows:
\[
D^* D_k D = \begin{pmatrix}
\frac{d_{0,0}}{e_{0,0}} & \frac{d_{0,1}}{e_{0,1}} & \cdots & \frac{d_{0,n}}{e_{0,n}} \\
\frac{d_{1,0}}{e_{0,0}} & \frac{d_{1,1}}{e_{0,1}} & \cdots & \frac{d_{1,n}}{e_{0,n}} \\
\vdots & \ddots & \vdots & \vdots \\
\frac{d_{n,0}}{e_{0,0}} & \frac{d_{n,1}}{e_{0,1}} & \cdots & \frac{d_{n,n}}{e_{0,n}}
\end{pmatrix}.
\]

If we assume \( \mathbb{E}[\tau^i \tau^j \tau^k] = 0 \) for all \( (i, j, k) \in \{1, \ldots, n\}^3 \), then for arbitrary fixed \( k \) we have \( d_{i,j} = 0 \) for every \( (i, j) \in \{1, \ldots, n\}^2 \). Since \( d_{0,j} = \mathbb{E}[\tau^k \tau^j] = \delta_{k,j} \), we notice that \( \text{rank} D^* D_k D = \text{rank} D_k = 2 \). This implies, since \( D_k \) is a diagonal matrix, \( e_{k,j} = 0 \) except for exactly two \( j \)’s, for which we write \( k_+ \) and \( k_- \).

We may assume without loss of generality \( e_{k,k_-} < 0 < e_{k,k_+} \) since \( \mathbb{E}[\tau^k] = e_{k,k_-} e_{0,k_-} + e_{k,k_+} e_{0,k_+} \) must be zero. This in turn implies \( \{k_-, k_+\}, k = 1, \ldots, n \) must be disjoint to fulfill \( \mathbb{E}[\tau^k \tau^{k'}] = 0 \) for \( k \neq k' \).

Hence finally we notice that \( 2n \leq n + 1 \). This implies \( n = 1 \). \[\square\]
6.2. Proof of Theorem 3.3. We will write
\[ \tilde{f}(x) = (f(x + N^{-1/2}e_0), \ldots, f(x + N^{-1/2}e_n)) \]
for \( f : \mathbb{R}^n \to \mathbb{R} \). Note that \( \tilde{f} \) is a map to \( \mathbb{R}^{n+1} \).

As in the above proof we denote \( \mathcal{D} = (e_{i,j})_{0 \leq i,j \leq n} \). Then we have
\[
\mathcal{D} \tilde{f}(x) = \left( f(x) + (2N)^{-1} \nabla f(x), N^{-1/2} \partial_1 f(x), \ldots, N^{-1/2} \partial_n f(x) \right).
\]

Since \( \mathcal{D} \mathcal{H}^N = \mathcal{D} \mathcal{h}^N = \Sigma^N + (a, 0, \ldots, 0)^* \) for some \( a = a(t, x) \), we have
\[
(6.2) \quad \Sigma^N [\mathcal{D} \mathcal{H}^N(t, x)]^{-1} = \left( \pi_1^N(t, x) \ldots \pi_{n+1}^N(t, x) \right)_{0}^{N/2}I_{n},
\]
where \( 0 = (0, \ldots, 0)^* \in \mathbb{R}^n \), \( I_n \) is the unit \( n \times n \) matrix, and
\[
(6.3) \quad \pi_1^N = (\pi_1^N, \ldots, \pi_{n+1}^N) = (1^N h^N)[\mathcal{D} \mathcal{H}^N]^{-1} = (1^N h^N)[H^N]^{-1} D^{-1}.
\]

Since \( D^{-1} = D^* \), we have \( \pi_1^N = 1^N A \), the sum of the components of \( (1^N h^N)[H^N]^{-1} \), which is strictly positive by the assumption (i), and hence \( \Sigma^N \) is invertible.

Using (6.2), we have
\[
(6.4) \quad \Sigma^N [\mathcal{D} \mathcal{H}^N(t, x)]^{-1} [\pi_1^N \mathcal{D} \tilde{v}^N, \partial_1^N, \ldots, \partial_n^N]^* = (1^N, \nabla^N)^* \nu^N.
\]

Here we use the equality \( (1^N h^N)[H^N]^{-1} \tilde{v}^N = 1^N \nu^N \) which comes from (2.6). Hence we have
\[
(6.5) \quad \mathcal{D} \tilde{v}^N = \mathcal{D} \mathcal{H}^N [\Sigma^N]^{-1} (1^N, \nabla^N)^* \nu^N.
\]

In particular, we have the following relation from the first component of the above (6.5):
\[
(6.6) \quad \nu^N + \frac{1}{2N} \Delta^N \nu^N = \left( h^N + \frac{1}{2N} \Delta^N h^N \right) [\Sigma^N]^{-1} (1^N, \nabla^N)^* \nu^N.
\]

Since
\[
\nu^N = 1^N \nu^N + N^{-1} \partial_1^N \nu^N, \quad h^N = 1^N h^N + N^{-1} \partial_1^N h^N
\]
and
\[
(1^N h^N)[\Sigma^N]^{-1} = (1, 0, \ldots, 0),
\]
we have
\[
(6.9) \quad \partial_1^N \nu^N + \frac{1}{2} \Delta^N \nu^N = \left( \partial_1^N h^N + \frac{1}{2} \Delta^N h^N \right) [\Sigma^N]^{-1} (1^N, \nabla^N)^* \nu^N.
\]

This is exactly (3.5). \( \square \)
6.3. **Proof of Theorem 4.2** The following proof is a routine-work in the context of finite difference method.

First we will show that our scheme is stable. Let \( u^N \) be the unique solution of the following difference equation.

\[
    u^N(T_N, x) = \Psi^N(x); \quad x \in \mathbb{R}^n,
\]

(6.10) \[ \partial_t^N u^N + \frac{1}{2} \Delta^N u^N - \langle b^N, \nabla^N u^N \rangle - c^N(1^N u^N) = g^N; \]

\[ t \in (0, T_N), \quad x \in \mathbb{R}^n. \]

where \( g^N \) and \( \Psi^N \) are given functions on \( \mathbb{R}_+ \times \mathbb{R}^n \) and \( \mathbb{R}^n \) respectively. We claim that

(6.11) \[ \sup_{x \in \mathbb{R}^n} |u^N(t, x)| \leq \sup_{(s, y) \in [t, T] \times \mathbb{R}^n} \left\{ (T - t)|g^N(s, y)| + |\Psi^N(s, y)| \right\} \]

for every \( t \in [0, T]_N \). This inequality shows the stability of our scheme. To prove (6.11), we first remark that the equation in (6.10) can be rewritten as

\[
    1^N u^N = 1^N g^N / N + (1^N h^N)[H^N]^{-1} \tilde{u}^N,
\]

which comes from Theorem 3.3. By the positivity assumption on \( h^N(t, x)[H^N(t + N^{-1}, x)]^{-1} \), we see that \( \frac{1}{N} h^N(t, x)[H^N(t + N^{-1}, x)]^{-1} \) defines a transition probability of a time-inhomogeneous Markov chain \((Y^N_t, \mathbf{P}^x_t)_{t \in [0, T], x \in \mathbb{R}^n}: \mathbf{P}^x_t(Y^N_t = x) = 1 \)

\[ = \text{the } j\text{-th component of} \]

\[
    A(t, x)^{-1} h^N(t, x)[H^N(t + N^{-1}, x)]^{-1}.
\]

Denoting the expectation with respect to \( \mathbf{P}^x_t \) by \( \mathbf{E}^x_t \), we have

(6.13) \[ u^N(t, x) = \frac{1}{N} g^N(t, x) + \mathbf{E}^x_t \left[ A(t, Y^N_t) u^N(t + N^{-1}, Y^N_{t+N^{-1}}) \right]. \]

By iterating (6.13) and by the Markov property, we have

\[
    u^N(t, x) = \frac{1}{N} \sum_{s \in [t, T]_N} \mathbf{E}^x_t \left[ g^N(s, Y^N_s) \prod_{u \in [t, s]_N} A(u, Y^N_u) \right]
\]

(6.14) \[ + \mathbf{E}^x_t \left[ \Psi^N(Y^N_{T_N}) \prod_{u \in [t, T_N]} A(u, Y^N_u) \right]. \]

(This is a discrete version of Feynman-Kac formula.) By the assumption of \( 0 < A < 1 \), we obtain (6.11).

Next, we will show that

(6.15) \[ \sup_{N} \sup_{(t, x) \in [0, T]_N \times \mathbb{R}^n} \left| N^{1/2} g^N(t, x) \right| < \infty \]

13
where
\begin{equation}
(6.16) \quad g^N := \partial'_t \nu + \frac{1}{2} \Delta \nu - \langle b^N, \nabla \nu \rangle - c^N (1^N \nu)
\end{equation}
for the solution \( \nu \) to \((4.2)\). Since
\begin{equation}
(6.17) \quad g^N = g^\nu - \frac{\partial \nu}{\partial t} - \frac{1}{2} \Delta \nu + \langle b, \nabla \nu \rangle + c \nu,
\end{equation}
we have
\begin{equation}
(6.18) \quad |g^N| \leq \left| \frac{\partial \nu}{\partial t} - \partial'_t \nu \right| + \frac{1}{2} |\Delta \nu - \Delta^N \nu|
+ |b^N| |\nabla \nu - \nabla^N \nu| + |\nabla \nu| |b - b^N|
+ |c^N| |\nu - 1^N \nu| + |\nu| |c - c^N|.
\end{equation}
By the Assumption \textbf{4.1} and the consistency \textbf{(3.4)}, we obtain \textbf{(6.15)}.

Finally, by combining \textbf{(6.11)} and \textbf{(6.15)}, and by the uniform continuity of \( \nu \), we have the desired result since \( \nu - \nu^N \) is the solution to \textbf{(6.10)} with \( \Psi^N(x) = \Phi(x) - \Phi^N(x) \) and \( g^N \) given by \textbf{(6.10)}. \qed

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