Lie algebra automorphisms as Lie-point symmetries and the solution space for Bianchi type I, II, IV, V vacuum geometries

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Abstract
Lie-group symmetry analysis for systems of coupled, nonlinear ordinary differential equations is performed in order to obtain the entire solution space to Einstein’s field equations for vacuum Bianchi spacetime geometries. The symmetries used are the automorphisms of the Lie algebra of the corresponding three-dimensional isometry group acting on the hyper-surfaces of simultaneity for each Bianchi type, as well as the scaling and the time reparametrization symmetry. A detailed application of the method is presented for Bianchi type IV. The result is the acquisition of the general solution of type IV in terms of sixth Painlevé transcendent $P_{VI}$, along with the known pp-wave solution. For Bianchi types I, II, V the known entire solution space is attained and very briefly listed, along with two new type V solutions of Euclidean and neutral signature and a type I pp-wave metric.

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1. Introduction

The exploitation of the group of automorphisms in order to obtain a unified treatment of spatially homogeneous Bianchi spacetime geometries has a rather long history, dating back to the early 1960s [1]. In 1979, Harvey [2] found the automorphisms of all three-dimensional Lie algebras, while the corresponding results for the four-dimensional Lie algebras have been presented in [3]. In Jantzen’s tangent space approach the automorphism matrices are considered as the means for achieving a convenient parametrization of a full scale-factor matrix in terms of a desired, diagonal matrix [4–6]. Siklos used these time-dependent automorphisms as a tool for the proper choice of variables aiming at a simplification of the ensuing equations [7], while Samuel and Ashtekar were the first to look upon automorphisms from a space viewpoint [8]. The notion of time-dependent automorphism inducing diffeomorphisms, i.e.
coordinate transformations mixing space and time in the new spatial coordinates and inducing automorphic motions on the scale-factor matrix, the lapse, and the shift vector have been developed in [9]. The use of these covariances enables one to set the shift vector to zero without destroying manifest spatial homogeneity. At this stage one can use the rigid automorphisms, i.e. the remaining gauge symmetry, as Lie-point symmetries of Einstein’s field equations in order to reduce the order of these equations and ultimately completely integrate them [10].

The methodology applied in this work is the Lie-group symmetry analysis of differential equations. For a detailed account of this approach see, e.g., [11–13]. Just to put it simply, a symmetry of a system of differential equations is a transformation mapping of any solution of the system to another solution. In other words, the symmetry group of the system is the group-transforming solutions of the system to other solutions. In fact, it is the largest local group of transformations acting on the system variables. Such groups are Lie groups depending on continuous parameters and consisting of either point transformations, acting on the systems space of independent and dependent variables or, more generally, contact transformations that act also on all first derivatives of the dependent variables. The theory can be also generalized to the so-called higher order (Lie–Backlund) symmetries, by including in the transformation of the independent variables derivatives of the dependent. In this work, we use all these three kinds. In general, the aforementioned transformations are nonlinear and the main benefit of the Lie-group symmetry analysis comes from the replacement of the nonlinear symmetry conditions of the system investigated by ‘linear’ conditions associated with the infinitesimal generators of the symmetries. To this purpose, it is necessary first to determine a class of general admissible variable transformations and then to search for special members of this class under which the system of differential equations remains invariant. It is rather evident that the degree of generality of the admissible transformations is proportional to the number of the existing symmetries. One should further stress that one of the main ingredients of the method lies in the notion of ‘prolongation’ of a group action on the space of derivatives of the system-dependent variables up to any finite order. One is thus able to deal with differential equations of any order. In closing this short introduction, we must point out that the knowledge of a symmetry group of a higher order differential equation has much the same consequences as the knowledge of a symmetry group of the corresponding system of first-order differential equations. In fact, we will make use of the following two important theorems (for a proof, see [12]).

**Theorem 1.1.** Let

\[ \frac{dy^\nu}{dx} = F^\nu(x, y), \quad \nu = 1, \ldots, q \]  

(1.1)

be a first-order system of q differential equations and suppose that G is a one-parameter symmetry group of the system. Then, there exists a change of variables \((t, u) = \psi(x, y)\) under which the system can be written as

\[ \frac{du^\nu}{dt} = H^\nu(t, u^1, \ldots, u^{q-1}), \quad \nu = 1, \ldots, q \]  

(1.2)

so that the original system is reduced to a new system of \(q - 1\) differential equations for \(u^1, u^2, \ldots, u^{q-1}\) together with the integral

\[ u^q(t) = \int H^q(t, u^1(t), \ldots, u^{q-1}(t)) \, dt + c. \]  

(1.3)

**Theorem 1.2.** If the system of q first-order differential equations considered in theorem 1.1 admits an r-parameter solvable symmetry group, then the solution of the system can be found by quadrature from the solution of a reduced system of \(q - r\) first-order differential equations.
If the original system is invariant under a q-parameter solvable symmetry group, then its general solution can be found by quadrature alone.

The paper is organized as follows: in section 2, we give our method. In section 3, the detailed application of the method to Bianchi type IV is presented. In section 4, the results of the method’s application to Bianchi types I, II and V are briefly listed along with three new solutions. Finally, some discussion and concluding remarks are given in section 5.

2. The method

It is known that the line element for spatially homogeneous spacetime geometries with a simply transitive action of the corresponding isometry group [14, 15] assumes the form
\[ ds^2 = (N^\alpha N_\alpha - N^2) dt^2 + 2N_\alpha \sigma^\mu \sigma^\nu dt dx^\mu dx^\nu \]  
(2.1)
with the base invariant 1-forms \( \sigma^\alpha \) defined from
\[ d\sigma^\alpha = C^\alpha_\beta \sigma^\beta \wedge \sigma^\gamma \Leftrightarrow \alpha_{ij} - \alpha_{ji} = 2C^\alpha_\beta \sigma^i \sigma^j. \]  
(2.2)
(Small Latin letters denote world space indices while small Greek letters count the different basis 1-forms; both types of indices range over the values of 1, 2, 3).

At this stage, we deem it pertinent to give a further explanation of the use of the term Bianchi types: we mean it to indicate each one of the nine different three-dimensional Lie groups (as in the original Bianchi’s work). Therefore, the (simply transitive) action of the group on a 3d hyper-surface can not a priori determine the nature of the hyper-surface in which it acts. This is determined by the solution of the Einstein field equations (EFEs).

The use of (2.1) in the EFEs results in (see, e.g., [16, 9])
\[ E_\alpha^\beta = K_\alpha^\beta - K^2 - R = 0 \]  
(2.3)
\[ E^a = K^a_\alpha C^\alpha_\mu - K^\mu_\alpha C^\alpha_\mu = 0 \]  
(2.4)
\[ E_{a\beta} = \tilde{K}_{a\beta} + N(2K^\mu_\beta K_\alpha^\mu - K_{a\beta}) + 2N^\mu (K_\alpha^\mu C^\nu_\beta + K^\nu_\beta C^\nu_\alpha) - NR_{a\beta} = 0 \]  
(2.5)
where
\[ K_{a\beta} = -\frac{1}{2N} \left( \gamma_{a\beta} + 2\gamma_{a\beta} C^\nu_\alpha N^\nu + 2\gamma_{a\beta} C^\nu_\mu N^\nu \right) \]  
(2.6)
is the extrinsic curvature of the three-dimensional hyper-surface and
\[ R_{a\beta} = C_{\alpha\mu} C_{\mu \nu} \gamma_{a\alpha} \gamma_{b\beta} \gamma_{c\nu} \gamma_{d\mu} + 2C_{\alpha\beta} C_{\mu \rho} \gamma_{a\alpha} \gamma_{b\beta} \gamma_{c\rho} \gamma_{d\mu} + 2C_{\alpha\beta} C_{\mu \rho} \gamma_{a\alpha} \gamma_{b\beta} \gamma_{c\rho} \gamma_{d\mu} + 2C_{\alpha\beta} C_{\mu \rho} \gamma_{a\alpha} \gamma_{b\beta} \gamma_{c\rho} \gamma_{d\mu} + 2C_{\alpha\beta} C_{\mu \rho} \gamma_{a\alpha} \gamma_{b\beta} \gamma_{c\rho} \gamma_{d\mu} \]  
(2.7)
is its Ricci tensor.

In [9], particular spacetime-coordinate transformations have been found, which reveal as symmetries of (2.3)–(2.5) the following induced transformations of the dependent variables \( N, N_\alpha, \gamma_{a\beta} \):
\[ \tilde{N} = N, \quad \tilde{N}_\alpha = \Lambda^\alpha_\rho (N_\rho + \gamma_{\rho\alpha} P^\alpha), \quad \tilde{\gamma}_{\mu\nu} = \Lambda^\alpha_\mu \Lambda^\beta_\nu \gamma_{a\alpha} \]  
(2.8)
where the matrix \( \Lambda \) and the triplet \( P^\alpha \) must satisfy
\[ \Lambda^\alpha_\rho C^\rho_\beta = C^\alpha_\mu \Lambda^\mu_\beta \Lambda^\nu_\gamma \]  
(2.9a)
\[ 2P^\mu C^\mu_\alpha \Lambda^\alpha_\beta = \Lambda^\nu_\beta. \]  
(2.9b)
These transformations were first presented in [4], see also the discussion on p 3586 of [9] and \( \Lambda, P^\alpha \) describe the action of the Automorphism group on the various components of the line element.

For all Bianchi types, this system of equations admits solutions that contain three arbitrary functions of time plus several constants depending on the automorphism group of each type. The three functions of time are distributed among \( \Lambda \) and \( P^\alpha \) (which also contains derivatives of these functions). So one can use this freedom either to simplify the form of the scale-factor matrix or to set the shift vector \( N^\alpha \) to zero. The second action can always be taken, since, for every Bianchi type, all three functions appear in \( P^\alpha \).

In this work, we adopt the latter point of view. Having used the freedom stemming from the three arbitrary functions in order to set the shift vector to zero, there is still a remaining ‘gauge’ freedom consisting of a constant \( \Lambda_\alpha^\beta \) (automorphism group matrices of the Lie group defined by the structure constants \( C^\gamma_{\alpha\beta} \). Indeed, the system (2.9) accepts the solution \( \Lambda_\alpha^\beta = \text{const.}, P^\alpha = 0 \). The latter are also known in the literature as rigid symmetries [17].

The quadratic constraint (2.3) can be used in order to define the lapse function \( N^2 \), since it is algebraically contained in it. In the type I case, where the quadratic constraint does not determine the lapse, the well-known Taub time gauge \( N^2 \) equals the determinant of \( \gamma_{\alpha\beta} \) can be used instead. One can furthermore see, using the definition of \( K_{\alpha\beta} \), that the dynamical equations (2.5) involve only \( N^2 \) and not \( N \). Thus \( N^2 \) is of a nominal value (not a positive-definite function), a fact that will enable the method to produce all hyper-surface homogeneous spacetimes, even the ones with Euclidean or neutral signature.

The generators of the corresponding motions \( \vec{\gamma}_{\mu\nu} = \Lambda^{\alpha}_{\mu} \Lambda^{\beta}_{\nu} \gamma_{\alpha\beta} \) induced in the space of the dependent variables spanned by \( \gamma_{\alpha\beta} \) (the lapse is given in terms of \( \gamma_{\alpha\beta}, \dot{\gamma}_{\alpha\beta} \) by algebraically solving the quadratic constraint equation) are

\[
X_I = \lambda_I^\rho \gamma_{\rho\beta} \frac{\partial}{\partial \gamma_{\alpha\beta}},
\]

with \( \lambda_I^\rho \) satisfying

\[
\lambda_I^\rho C^\rho_{\beta\gamma} = \lambda_I^\rho C^\rho_{\alpha\beta} + \chi_I^\alpha C^{\rho\alpha}. \tag{2.11}
\]

These generators define a Lie algebra and each one of them induces, through its integral curves, a transformation on the configuration space spanned by the \( \gamma_{\alpha\beta} \) [18]. If a generator is brought to its normal form (i.e. \( \frac{\partial}{\partial z_i} \)), then the EFEs, written in terms of the new dependent variables, will not explicitly involve \( z_i \). They thus become a first-order system in the function \( \dot{z}_i \) [11]. If the aforesaid Lie algebra is Abelian, then all generators can be brought to their normal form simultaneously. If the Lie algebra is non-Abelian, then we can diagonalize in one step those generators corresponding to any eventual Abelian subgroup. The rest of the generators (not brought in their normal form) continue to define a symmetry of the reduced system of the field equations if the Lie algebra of the \( X_I,S \) is solvable [12]. One can thus repeat the previous step by choosing one of these remaining generators and bring it to its normal form. This choice will of course depend upon the simplifications brought to the system at the previous level. Finally, if the Lie algebra does not contain any Abelian subgroup, one can always choose one of the generators, bring it to its normal form, reduce the system of equations and search for its symmetries (if there are any). Lastly, two further symmetries of (2.3)–(2.5) are also present and can be used in conjunction with the constant automorphisms. The time reparametrization \( t \rightarrow t + \alpha \), owing to the non-appearance of time in these equations (the system being autonomous), and the scaling by a constant \( \gamma_{\alpha\beta} \rightarrow \lambda \gamma_{\alpha\beta} \) (homothety) as can be straightforwardly verified. Hence, in every Bianchi type there are, added to the \( X_I(t) \)
generators, also the following generators:

\[ Y_1 = \frac{\partial}{\partial t} \]

\[ Y_2 = \gamma_{11} \frac{\partial}{\partial \gamma_{11}} + \gamma_{12} \frac{\partial}{\partial \gamma_{12}} + \gamma_{13} \frac{\partial}{\partial \gamma_{13}} + \gamma_{22} \frac{\partial}{\partial \gamma_{22}} + \gamma_{23} \frac{\partial}{\partial \gamma_{23}} + \gamma_{33} \frac{\partial}{\partial \gamma_{33}}, \]

These generators commute among themselves, as well as with the \( X_I \)\(_{\alpha} \)s:

\[ [X_I, Y_\alpha] = 0 \quad \{ I = 1, 2, 3, 4 \mid \alpha = 1, 2 \}. \]  

### 3. Bianchi type IV

For this type, the structure constants are

\[ C_{11} = -C_{13} = C_{23} = -C_{32} = \frac{1}{2} \]

\[ C_{13} = -C_{32} = \frac{1}{2} \]

\[ C_{\alpha \beta} = 0 \quad \text{for all other values of } \alpha \beta \gamma. \]

Using these values in the defining relation (2.2) of the 1-forms \( \sigma_\alpha \) we obtain

\[ \sigma_\alpha = \begin{pmatrix} 0 & e^{-x} & xe^{-x} \\ 0 & 0 & -e^{-x} \\ 1 & 0 & 0 \end{pmatrix}. \]

The corresponding vector fields \( \xi_\alpha \) (satisfying \([\xi_\alpha, \xi_\beta] = C_{\alpha \beta \gamma} \xi_\gamma\)) with respect to which the Lie derivative of the above 1-forms is zero are

\[ \xi_1 = -\partial_x, \quad \xi_2 = \partial_z, \quad \xi_3 = \partial_x + (y-z)\partial_y + z\partial_z. \]

The time-depended AIDs are described by

\[ \Lambda_{\alpha \beta} = \begin{pmatrix} P(t) & P(t) \ln(cP(t)) & x(t) \\ 0 & P(t) & y(t) \\ 0 & 0 & 1 \end{pmatrix} \]

and

\[ P^\alpha = \begin{pmatrix} x(t) & \left( \ln \frac{x(t)}{P(t)} \right)' - y'(t) & y(t) \left( \ln \frac{y(t)}{P(t)} \right)' - \left( \ln P(t) \right)' \end{pmatrix} \]

where \( P(t), x(t) \) and \( y(t) \) are arbitrary functions of time. As we have already remarked the three arbitrary functions appear in \( P^\alpha \) and thus can be used to set the shift vector to zero.

The remaining symmetry of the EFEs is, consequently, described by the constant matrix:

\[ M = \begin{pmatrix} e^t & s_2 & s_3 \\ 0 & e^{s_2} & s_4 \\ 0 & 0 & 1 \end{pmatrix} \]

where the parametrization has been chosen so that the matrix becomes identity for the zero value of all parameters.
Thus, the induced transformation on the scale-factor matrix is $\tilde{y}_{\alpha\beta} = M^\mu_\alpha M^\nu_\beta y_{\mu\nu}$, which explicitly reads

$$
\begin{align*}
\tilde{y}_{11} &= e^{2s_1} y_{11} \\
\tilde{y}_{12} &= e^{s_1} s_2 y_{12} + e^{2s_1} y_{12} \\
\tilde{y}_{13} &= e^{s_1} (s_3 y_{11} + s_4 y_{12} + y_{13}) \\
\tilde{y}_{22} &= e^{s_1} y_{22} + 2 e^{s_1} s_2 y_{12} + s_3^2 y_{11} \\
\tilde{y}_{23} &= e^{s_1} (s_3 y_{12} + s_4 y_{22} + y_{23}) + s_2 (s_3 y_{11} + y_{13} + s_4 y_{12}) \\
\tilde{y}_{33} &= s_3^2 y_{11} + s_3 (y_{12} + y_{13}) + s_4^2 y_{22} + 2 s_4 (s_3 y_{12} + y_{23}) + y_{33}.
\end{align*}
$$

(3.7)

The previous equations define a group of transformations $\mathcal{G}$ of dimension $r = \dim(\text{Aut}(\text{IV})) = 4$. The four generators of the group are

$$
\begin{align*}
X_1 &= 2 \gamma_{11} \frac{\partial}{\partial \gamma_{11}} + 2 \gamma_{12} \frac{\partial}{\partial \gamma_{12}} + \gamma_{13} \frac{\partial}{\partial \gamma_{13}} + 2 \gamma_{22} \frac{\partial}{\partial \gamma_{22}} + \gamma_{23} \frac{\partial}{\partial \gamma_{23}} \\
X_2 &= \gamma_{12} \frac{\partial}{\partial \gamma_{12}} + 2 \gamma_{13} \frac{\partial}{\partial \gamma_{13}} + \gamma_{23} \frac{\partial}{\partial \gamma_{23}} \\
X_3 &= \gamma_{13} \frac{\partial}{\partial \gamma_{13}} + \gamma_{12} \frac{\partial}{\partial \gamma_{12}} + 2 \gamma_{33} \frac{\partial}{\partial \gamma_{33}} \\
X_4 &= \gamma_{23} \frac{\partial}{\partial \gamma_{23}} + \gamma_{12} \frac{\partial}{\partial \gamma_{12}} + 2 \gamma_{33} \frac{\partial}{\partial \gamma_{33}}.
\end{align*}
$$

(3.8)

The algebra $\mathfrak{g}$, that corresponds to the group $\mathcal{G}$, has the following table of non-zero commutators:

$$
\begin{align*}
[X_1, X_2] &= 0, & [X_1, X_3] &= X_3, & [X_1, X_4] &= X_4, \\
[X_2, X_3] &= 0, & [X_2, X_4] &= X_3, & [X_3, X_4] &= 0.
\end{align*}
$$

(3.12)

As it is evident from the above commutators (3.12) the group is non-Abelian, so we cannot diagonalize at the same time all the generators. However, if we calculate the derived algebra of $\mathcal{G}$, we have

$$
\mathfrak{g}' = \{X_1, X_2 \} : X_4, X_8 \in \mathfrak{g'} \Rightarrow \mathfrak{g'} = \{X_1, X_4\}
$$

(3.13)

and furthermore, its second-derived algebra reads

$$
\mathfrak{g}'' = \{X_1, X_2 \} : X_8 \in \mathfrak{g}'' \Rightarrow \mathfrak{g}'' = \{0\}.
$$

(3.14)

Thus, the group $\mathcal{G}$ is solvable since the $\mathfrak{g}''$ is zero. As it is evident $X_3, X_4, X_2$ generate an Abelian subgroup, and we can, therefore, bring them to their normal form simultaneously. The appropriate transformation of the dependent variables is

$$
\begin{align*}
\gamma_{11} &= e^{u_1} \\
\gamma_{12} &= e^{u_1} u_2 \\
\gamma_{13} &= e^{u_1} \left( -e^{u_1} u_2^3 + u_3 + u_2 u_4 \right) \\
\gamma_{22} &= e^{u_1 + u_4} \\
\gamma_{23} &= e^{u_1} (u_2 (u_3 - 1) + e^{u_1} u_5) \\
\gamma_{33} &= e^{u_1} (e^{-2u_1} + u_4^2 - e^{-u_1} u_2^2 (2u_3 - 1) + 2 u_2 u_5 (u_3 - 1) + e^{u_1} u_5^2).
\end{align*}
$$

(3.15)
In these coordinates, the generators \( Y_2, X_A \) assume the form
\[
Y_2 = \frac{\partial}{\partial u_1}, \quad X_4 = \frac{\partial}{\partial u_5}, \quad X_3 = \frac{\partial}{\partial u_3},
\]
\[
X_2 = \frac{\partial}{\partial u_2} + (e^{-u_1}u_2 - u_3) \frac{\partial}{\partial u_3} + 2e^{-u_1}u_2 \frac{\partial}{\partial u_4} + e^{-2u_1}(e^{u_1} - 2u_2^2) \frac{\partial}{\partial u_5},
\]
\[
X_1 = 2 \frac{\partial}{\partial u_1} - u_3 \frac{\partial}{\partial u_3} + (e^{-u_1}u_2 - u_3) \frac{\partial}{\partial u_5} + \frac{\partial}{\partial u_6}.
\]  
(3.16)

Evidently, a first look at (3.15) gives the feeling that it would be hopeless even to write down the Einstein equation. However, the simple form of the first three of the generators (3.16) ensures us that these equations will be of first order in the functions \( \dot{u}_1, \dot{u}_3 \) and \( u_5 \).

### 3.1. Description of the solution space

Before we begin solving the Einstein equations, a few comments for the possible values of the functions \( u_i, i = 1, \ldots, 6 \) will prove very useful.

The determinant of \( \gamma_{\alpha\beta} \) is
\[
\det[\gamma_{\alpha\beta}] = e^{3u_1 - 2u_2}(e^{u_1} - u_2^2),
\]
so we must have \( e^{u_1} > u_2^2 \).

The two linear constraint equations, written in the new variables (3.15), give
\[
E_1 = 0 \Rightarrow e^{u_1} \dot{u}_3 + u_2^2 \dot{u}_4 + u_2(e^{u_1} \dot{u}_5 - \dot{u}_2) = 0
\]
\( \text{(18)} \)

\[
E_2 = 0 \Rightarrow (3e^{u_1} + u_2)\dot{u}_2 - e^{u_1}(1 + 3u_2)\dot{u}_3 - (u_2 + 3e^{u_1})(u_2\dot{u}_4 + e^{u_1}\dot{u}_5) = 0.
\]
\( \text{(19)} \)

Solving this system for the functions \( \dot{u}_3, \dot{u}_5 \) we have
\[
\dot{u}_3 = 0, \quad \dot{u}_5 = e^{-u_1}(\dot{u}_2 - u_2\dot{u}_4)
\]
\( \text{(20)} \)

yielding to
\[
u_3 = k_3, \quad u_5 = k_5 + e^{-u_1}u_2.
\]
\( \text{(21)} \)

Now, these values of \( u_3, u_5 \) make \( \gamma_{13}, \gamma_{23} \) functionally dependent upon \( \gamma_{11}, \gamma_{12}, \gamma_{22} \) (see (3.15)). It is thus possible to set these two components to zero by means of an appropriate constant automorphism.

*Without loss of generality, we can start our investigation of the solution space for type IV vacuum Bianchi cosmology from a block-diagonal form of the scale-factor matrix (and, of course, zero shift)*

\[
\gamma_{\alpha\beta} = \begin{pmatrix}
\gamma_{11} & 0 & 0 \\
0 & \gamma_{22} & 0 \\
0 & 0 & \gamma_{33}
\end{pmatrix}
\]
\( \text{(22)} \)

Note that this conclusion could have not been reached off mass-shell, due to the fact that the time-dependent automorphism (3.4) does not contain the necessary two arbitrary functions of time of the (13) and (23) components (besides the fact that all the freedom in arbitrary functions of time has been used to set the shift to zero). As we have earlier remarked, since the algebra (3.12) is solvable, the remaining (reduced) generators \( X_1, X_2 \) (corresponding to diagonal constant automorphisms) as well as \( Y_2 \) continue to define a Lie-point symmetry of the reduced EFEs and can thus be used for further integration of this system of equations.
The remaining (reduced) automorphism generators are

\[ X_1 = \gamma_{11} \frac{\partial}{\partial \gamma_{11}} + \gamma_{12} \frac{\partial}{\partial \gamma_{12}} + \gamma_{22} \frac{\partial}{\partial \gamma_{22}} \]

\[ X_2 = \gamma_{11} \frac{\partial}{\partial \gamma_{12}} + 2\gamma_{12} \frac{\partial}{\partial \gamma_{22}} \]  
(3.23)

The appropriate change of dependent variables which brings these generators—along with \( Y_2 \)—into normal form, is described by the following scale-factor matrix:

\[ Y_{\alpha\beta} = \begin{pmatrix} e^{u_1 + 2u_6} & e^{u_1 + 2u_6}u_2 & 0 \\ e^{u_1 + 2u_6}u_2 & e^{u_1 + 2u_6}(u_2^2 + u_4) & 0 \\ 0 & 0 & e^{u_1} \end{pmatrix} . \]  
(3.24)

The generators are now reduced to

\[ Y_2 = \frac{\partial}{\partial u_1}, \quad X_2 = \frac{\partial}{\partial u_2}, \quad X_1 = \frac{\partial}{\partial u_6} \]  
(3.25)

indicating that the system will be of first order in the derivatives of these variables. The remaining variable \( u_4 \) will enter (along with \( \dot{u}_4, \ddot{u}_4 \) explicitly in the system and is therefore advisable (if not mandatory) to be used as the time parameter, i.e. to effect the change of time coordinate

\[ t \rightarrow u_4(t) = s, \; u_1(t) \rightarrow u_1(t(s)), \; u_2(t) \rightarrow u_2(t(s)), \; u_6(t) \rightarrow u_6(t(s)) . \]  
(3.26)

This choice of time will of course be valid only if \( u_4 \) is not a constant. We are thus led to consider two cases according to the constancy or non-constancy of this variable.

3.1.1. Case I: \( u_4(t) = k_4 \). In these variables the first two linear constraint equations are identically satisfied, and the determinant of \( Y_{\alpha\beta} \) is

\[ \text{det}[Y_{\alpha\beta}] = e^{3u_1 + 4u_6}k_4 , \]

so we must have \( k_4 > 0 \). The third linear constraint reads

\[ E_3 = 0 \Rightarrow \dot{u}_2 + 4k_4\dot{u}_6 = 0 \Rightarrow u_2 = k_2 - 4k_4u_6 . \]  
(3.27)

Substituting this value into the quadratic constraint equation \( E_0 \) we obtain for the lapse function

\[ N^2 = \frac{k_4}{12k_4 + 1} e^{u_1}(3u_1^2 + 8u_1\dot{u}_6 - 4(4k_4 - 1)\ddot{u}_6^2) . \]  
(3.28)

If we substitute this value of the lapse function into the equations of motion (2.5), we are left with the unknown functions \( u_1, u_6 \). The strategy we follow is to solve one of (2.5) for a second derivative of some function, say \( \ddot{u}_6 \) and replace the result into the rest of the equations. In order to do that, we must ensure that the coefficient of \( \ddot{u}_6 \) does not equal to zero. Looking at \( E_{11} = 0 \), the coefficient of \( \ddot{u}_6 \) is proportional to

\[ \ddot{u}_1 (\dddot{u}_1 + 2(4k_4 + 1)\dddot{u}_6) \Rightarrow \{ u_1 = k_1 \text{ or } u_1 = k_1 - 2u_6 - 8k_4u_6 \} . \]

Thus, we are forced to examine the above equalities before we solve \( E_{11} = 0 \), for \( \ddot{u}_6 \).

The first possibility \( u_1 = k_1 \) yields an inconsistency, so we are left with the second, i.e.

\[ u_1 = k_1 - 2u_6 - 8k_4u_6 . \]  
(3.29)

The above choice satisfies all the spatial equations \( E_{\alpha\beta} = 0 \) and gives the lapse function

\[ N^2 = 16e^{k_1-2(4k_4+1)u_6}k_4^2\ddot{u}_6^2 . \]  
(3.30)
Redefining the constants \( k_1 = \ln \kappa^2 \), \( k_4 = \frac{-\mu}{(\mu - 1)} \), choosing a time parametrization \( u_6 = (\mu - 1) \ln(t) \), and using the automorphism matrix (3.6) with entries \( s_1 = 0 \), \( s_2 = -k_2 \), \( s_3 = 0 \), \( s_4 = 0 \) we arrive at the line element
\[
\mathrm{d}s^2 = -\mu^2 (\mathrm{d}t)^2 + \mu^2 (\sigma^1)^2 + 2t^2 \mu \ln t \sigma^1 \sigma^2 + \mu^2 \ln^2 t - \frac{1}{4(\mu - 1)} (\sigma^3)^2 + t^2 (\sigma^3)^2
\]  
(3.31)

with \( t > 0 \), \( 0 < \mu < 1 \). In the above line element, we have dropped the constant \( \kappa \) since this line element admits a homothetic vector field
\[
H = \partial_x + (\mu z - y(\mu - 1)) \partial_y - z(\mu - 1) \partial_z.
\]  
(3.32)

Line element (3.31) was first derived by Harvey and Tseytlin [19] and admits besides the three Killing fields (3.3) three more, namely
\[
\xi_4 = e^{-x/\mu} \frac{\partial}{\partial t} + \frac{e^{-x/\mu}}{t} \partial_x 
\]  
(3.33)
\[
\xi_5 = ye^{-x/\mu} \frac{\partial}{\partial t} + \frac{ye^{-x/\mu}}{t} \partial_y + f_1 \partial_y + f_2 \partial_z 
\]  
(3.34)
\[
\xi_6 = ze^{-x/\mu} \frac{\partial}{\partial t} + \frac{ze^{-x/\mu}}{t} \partial_x + f_2 \partial_y + f_3 \partial_z 
\]  
(3.35)

with
\[
f_1 = \frac{\mu e^{(2\mu - 1)x/\mu} t^{-2\mu + 1}}{(-2\mu + 1)^3}
\]
\[
\quad \times (4\mu^2 (2\mu - 1)^2 (\mu - 1) \ln^2 t - 8\mu (\mu - 1)(2\mu - 1)(-\mu + (2\mu - 1)x) \ln t - \mu + 4(\mu - 1)(\mu + (1 - 2\mu)x)^2)
\]
\[
f_2 = \frac{4\mu (\mu - 1)e^{(2\mu - 1)x/\mu} t^{-2\mu + 1}}{(-2\mu + 1)^2}
\]
\[
\quad \times (\mu (2\mu - 1) \ln t + \mu + (1 - 2\mu)x)
\]
\[
f_3 = \frac{4\mu (\mu - 1)e^{(2\mu - 1)x/\mu} t^{-2\mu + 1}}{-2\mu + 1}
\]  
(3.36)

There is thus a \( G_6 \) symmetry group acting (of course, multiply transitively) on each \( V_3 \) of this metric. However, it is interesting to note that we have not imposed the extra symmetry from the beginning, but rather it emerged as a result of the investigation process.

Having ensured that the coefficient of \( \dot{u}_6 \) at \( E_{11} = 0 \) is not zero, we can solve this equation for \( \dot{u}_6 \) and insert the result into the rest of the spatial equations. But doing that we end up with a zero lapse, indicating that the only solution for this case is described by the line element (3.31).

### 3.1.2. Case II: \( u_4(t) = t \)

In this case, the determinant of the scale-factor matrix equals to
\[
\det [\gamma_{\alpha \beta}] = e^{3u_3 + 4\alpha u_4 t},
\]
so we must demand that \( t > 0 \) in order for \( \gamma_{\alpha \beta} \) to be positive defined.

The first two linear constraints are identically zero while the third one \( E_3 = 0 \) can be used to define the function \( u_2 \):
\[
E_3 = 0 \Rightarrow \dot{u}_2 + 4t \dot{u}_6 + 1 = 0 \Rightarrow u_2 = k_2 - t - 4 \int t \dot{u}_6 \, \mathrm{d}t,
\]  
(3.37)
and the quadratic constraint $E_{ii} = 0$ defines the lapse function $N^2$:

$$N^2 = \frac{e^{\tau}}{12t + 1}(3tu_t^2 + 8tu_tu_\phi - 4t(4t - 1)u_\phi^2 - 2(4t - 1)u_\phi + 2u_t - 1).$$  \hspace{1cm} (3.38)

Substituting the above values of the lapse $N^2$ and the function $u_t$ in equation $E_{33} = 0$ we find the coefficient of $u_\phi$ is proportional to

$$\dot{u}_1(4t\dot{u}_1 - (4t - 1)(4tu_\phi + 1)),\]

a quantity that can be safely regarded different from zero, since its nihilism leads either to zero lapse or to inconsistency when combined with the rest of the dynamical equations. Thus, we can solve $E_{33} = 0$ for $u_\phi$ and substitute it to $E_{11} = 0$. In order to solve this equation for $\ddot{u}_1$, we must be assured that its coefficient does not equal to zero. Setting this coefficient equal to zero we arrive to the following equation:

$$\dot{u}_1 = 1 + (4t - 1)\dot{u}_6,$$

which is unacceptable because it leads to inconsistency. After solving equation $E_{11} = 0$ for $\ddot{u}_1$, we finally arrive to the following polynomial system of first order in $\dot{u}_1$, $\dot{u}_6$:

$$\dot{u}_1 = \langle \dot{u}_1 | B_1 | \dot{u}_6 \rangle, \quad \dot{u}_6 = \langle \dot{u}_1 | B_2 | \dot{u}_6 \rangle,$$

where we have used the notation $\langle \dot{u}_1 | = (1 \dot{u}_1 \dot{u}_2 \dot{u}_3)$ and $| \dot{u}_6 \rangle = \langle \dot{u}_6 |$ with the $4 \times 4$ matrices $B_1$, $B_2$ given by

$$B_1 = \frac{1}{12t + 1} \begin{pmatrix}
-4t - 1 & -32t^2 + 2t \\
24t^2 + 1 & 16t^2 + 4t \\
12t^2 - t & -16t \\
-6t & 0
\end{pmatrix},$$

$$B_2 = \frac{1}{2t(12t + 1)} \begin{pmatrix}
12t + 3 & 144t^2 + 8t - 6 & 16t(36t^2 + 7t - 1) & 16t^2(48t^2 + 8t - 1) \\
-4 & -24t & -32^2 & 0 \\
-6t & -12t^2 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.$$  

Due to the form of $B_1$, $B_2$ (their components are rational functions of the time $t$), system (3.39) can be partially integrated with the help of the following Lie–Bäcklund transformation:

$$\dot{u}_1 = \frac{(48t^2 + 16t + 1)\dot{r}(t) - 2(12t - 1)\tan r(t)}{4\sqrt{7}(12t + 1)}$$ \hspace{1cm} (3.40a)

$$\dot{u}_6 = -\frac{\sqrt{7}(12t + 1)\dot{r}(t) + 6\sqrt{7}\tan r(t) + 24t + 2}{8t(12t + 1)}$$ \hspace{1cm} (3.40b)

yielding the single second-order ODE for the function $r(t)$:

$$\ddot{r} = \left(\frac{\tan r + \sqrt{7}}{2}\right)^2 - 2\left(\frac{6t + 1}{\tan r + \sqrt{7}}\right)\ddot{r} + \frac{36t^2 \tan^2 r + 36\sqrt{7} \tan r - 12t - 1}{\sqrt{7}(12t + 1)^2}.$$ \hspace{1cm} (3.41)

This equation contains all the information concerning the unknown part of the solution space of the type IV vacuum cosmology. Unfortunately, it does not possess any Lie-point symmetries that can be used to reduce its order and ultimately solve it. However, its form can be substantially simplified through the use of new dependent and independent variable $(\rho, u(\rho))$ according to $r(t) = \arcsin \frac{u(\rho)}{\sqrt{\rho^2 - 1}}$, $t = \frac{1}{\sqrt{\rho^2 - 1}}$, thereby obtaining the equation

$$\ddot{\rho} = \frac{1 - u^2}{\sqrt{6(\rho - 1)}(\rho^2 - \rho^2 - 1)} \Rightarrow \ddot{\rho} = \frac{(1 - u^2)^2}{6(\rho - 1)(\rho^2 - \rho^2 - 1)}.$$ \hspace{1cm} (3.42)
This equation is a special case of the general equation

$$
\ddot{u}^2 = \frac{(1 - u^2)^2}{(\kappa + \lambda \rho)(\rho^2 - u^2 - 1)}
$$

(3.43)

with the values $\kappa = -6$, $\lambda = 6$. The general solution of (3.43) was first given in [20] and can be obtained as follows. We first apply the contact transformation:

$$
u(\rho) = -\frac{8}{\lambda}y(\xi) + \frac{4(2\xi - 1)}{\lambda}y'(\xi) \quad \rho = -\frac{\kappa}{\lambda} + \frac{4}{\lambda}y'(\xi)
$$

(3.44)

$$
\ddot{\nu}(\rho) = 2\xi - 1 \quad \ddot{\nu}(\rho) = \frac{\lambda}{2y'(\xi)}
$$

which reduces it to

$$
\xi^2 (\xi - 1)^2 y'' = -4y' (\xi y' - y)^2 + 4y'^2 (\xi y' - y) - \frac{\kappa}{2} y'^2 + \frac{\kappa^2 - \lambda^2 - 16}{16} y'.
$$

(3.45)

This equation is a special form of the equation SD-Ia, appearing in [21], where a classification of all second-order second-degree ordinary differential equations was performed. The general solution of (3.45) is obtained with the help of the sixth Painlevé transcendent $w := P_{VI}(\alpha, \beta, \gamma, \delta)$ and reads

$$
y = \frac{\xi^2 (\xi - 1)^2}{4w(w - 1)(w - \xi)} \left( w' - \frac{w(w - 1)}{\xi(\xi - 1)} \right)^2 + \frac{1}{8} (1 + \sqrt{2\alpha})^2 (1 - 2w) - \frac{\beta}{4} \left( 1 - \frac{2\xi}{w} \right) + \frac{\gamma}{4} \left( 1 - \frac{2w - 2(\xi - 1)}{w - 1} \right) + \frac{\lambda}{8} \left( 1 - \frac{2w - 2(\xi - 1)}{w - 1} \right),
$$

(3.46)

where the sixth Painlevé transcendent $w := P_{VI}(\alpha, \beta, \gamma, \delta)$ is defined by the ODE:

$$
w'' = \frac{1}{2} \left( \frac{1}{w - 1} + \frac{1}{w} + \frac{1}{w - \xi} \right) w'^2 - \left( \frac{1}{\xi - 1} + \frac{1}{\xi} + \frac{1}{w - \xi} \right) w' + \frac{w(w - 1)(w - \xi)}{\xi^2 (\xi - 1)^2} \left( \alpha + \beta \frac{\xi}{w^2} + \gamma \frac{(\xi - 1)}{(w - 1)^2} + \delta \frac{\xi}{(w - \xi)^2} \right).
$$

(3.47)

The values of the parameters $(\alpha, \beta, \gamma, \delta)$ of the Painlevé transcendent can be obtained from the solution of the following system:

$$
\alpha - \beta + \gamma - \delta \pm \sqrt{2\alpha} + 1 = -\frac{\kappa}{2}
$$

(3.48a)

$$
(\beta + \gamma) (\alpha + \delta \pm \sqrt{2\alpha}) = 0
$$

(3.48b)

$$
(\gamma - \beta) (\alpha - \delta \pm \sqrt{2\alpha} + 1) + \frac{1}{4} (\alpha - \beta - \gamma + \delta \pm \sqrt{2\alpha})^2 = \frac{\kappa^2 - \lambda^2 - 16}{16}
$$

(3.48c)

$$
\frac{1}{4} (\gamma - \beta) (\alpha + \delta \pm \sqrt{2\alpha} + 1) + \frac{1}{4} (\beta + \gamma)^2 (\alpha - \delta \pm \sqrt{2\alpha} + 1) = 0.
$$

(3.48d)

Inserting in (3.48) the values of $\kappa = -6$, $\lambda = 6$ for type IV, we have 24 solutions (counting multiplicities) of this system. In order for the parameters $(\alpha, \beta, \gamma, \delta)$ to be real numbers we end up only with three possibilities

$$
(\alpha, \beta, \gamma, \delta) = \left( \frac{1}{2}, -\frac{3}{2}, \frac{3}{2}, \frac{1}{2} \right)
$$

(3.49a)

$$
(\alpha, \beta, \gamma, \delta) = (2 + \sqrt{3}, 0, 0, -1)
$$

(3.49b)

$$
(\alpha, \beta, \gamma, \delta) = (2 - \sqrt{3}, 0, 0, -1).
$$

(3.49c)
Gathering all the pieces the final form of the general line element describing Bianchi type IV vacuum cosmology is
\[
\begin{aligned}
\text{d}s^2 &= \kappa^2 \left( -\frac{e^{u_1(\xi)}}{4\xi(\xi - 1)} (\text{d}\xi)^2 + \sqrt{\xi} (\xi - 1) y'(\xi) (\sigma^1)^2 + 2 \sqrt{\xi} (\xi - 1) y'(\xi) u_2(\xi) \sigma^1 \sigma^2 \\
&+ \frac{1}{4} \sqrt{\xi} (\xi - 1) y'(\xi) (4u_2^2(\xi) y'(\xi) + 1)(\sigma^2)^2 + e^{u_1(\xi)}(\sigma^3)^2 \right)
\end{aligned}
\] (3.50)

where
\[
\begin{aligned}
u_1'(\xi) &= \frac{1 - 2\xi - 2y(\xi)}{2\xi(\xi - 1)}, \quad \nu_2'(\xi) = \frac{y(\xi)}{2\xi y'(\xi)(\xi - 1)}
\end{aligned}
\] (3.51)

and \(y(\xi)\) is given by (3.46).

The above line element does not admit a homothetic field, thus the constant \(\kappa\) is essential and together with the two constants of integration inherent in \(P_V\) indicates the generality of the solution see e.g. [14].

4. Solution space for types I, II, V

In this section, we summarize the results of applying the method in Bianchi types I, II, V. These types can be considered as ‘easier’ than the rest, a characterization which can be supported both by the time of the discovery of their solution [25, 29, 30] and the fact that their generalizations with a cosmological constant is also known. Despite all this, the reproduction of all the above metrics by a single method is, in our opinion, worth reporting even in a ‘telegraphic’ sort of presentation. Moreover, along with the known solutions, three new metrics are given; one of Euclidean, one of neutral and one of Lorentzian signature are obtained. The detailed calculations can be found in [24].

4.1. Bianchi type I

In this model, the structure constants, the basis 1-forms and the Killing fields are
\[
\begin{aligned}
C_{\beta\gamma}^\alpha &= 0 \quad \text{for every value of } \alpha, \beta, \gamma \\
\sigma^1 &= \text{d}x, \quad \sigma^2 = \text{d}y, \quad \sigma^3 = \text{d}z
\end{aligned}
\] (4.1a)
\[
\xi_1 = \partial_1, \quad \xi_2 = \partial_2, \quad \xi_3 = \partial_3
\] (4.1c)

The ensuing metrics are as follows.

- Kasner metrics.
\[
\text{d}s^2 = -e^{(1+\alpha+\beta)\tau} \text{d}t^2 + e^\tau \text{d}x^2 + e^\beta \text{d}y^2 + e^\alpha \text{d}z^2
\] (4.2)

where the constants \((\alpha, \beta)\) satisfy \(\alpha + \beta + \alpha \beta = 0\). The above metric was first given, although in a different form, in [25] and admits a homothety
\[
H = 2(\alpha + 1)\partial_1 + \alpha^2 x\partial_x + y\partial_y + (\alpha + 1)^2 z\partial_z.
\] (4.3)

The metric is particularly interesting for the values \((\alpha, \beta) = (1, -1/2)\) or \((\alpha, \beta) = (-1/2, 1)\): in addition to the three Killing fields (4.1c), there is a fourth of the form
\[
\xi_4 = y\partial_x - x\partial_y.
\] (4.4)

The pair of values \((\alpha, \beta) = (0, 0)\) leads the metric (4.2) to the standard Minkowski form.
In this model, the structure constants, the basis 1-forms and the Killing fields are

- Harrison metrics

\[ d\Omega^2 = e^{(2\lambda + \beta^\prime - 1)\tau} d\tau^2 + e^{\tau/\beta} dx^2 - e^{\lambda \tau} \sin \tau dy^2 + 2 e^{\lambda \tau} \cos \tau dy dz + e^{\lambda \tau} \sin \tau dz^2 \]  

(4.5)

which possesses a homothety produced by the field

\[ H = -4\lambda \partial_t - 4\lambda^2 x \partial_x + (y(-\lambda^2 + 1) + 2\lambda z) \partial_y - (z(\lambda^2 - 1) + 2\lambda y) \partial_z. \]  

(4.6)

This metric was first given, although produced in a different way, in [26]. Also in this case there are special values of the constant \( \lambda \) for which we have a fourth Killing field:

\[ \lambda = \frac{\sqrt{3}}{2} \Rightarrow \xi_4 = 6 \partial_x + 2\sqrt{3}x \partial_x - (\sqrt{3}y + 3z) \partial_y + (3y - \sqrt{3}z) \partial_z, \]

\[ \lambda = -\frac{\sqrt{3}}{2} \Rightarrow \xi_4 = 6 \partial_x - 2\sqrt{3}x \partial_x + (\sqrt{3}y - 3z) \partial_y + (3y + \sqrt{3}z) \partial_z, \]

while there is no homothety. Finally, it is worth noting that in metric (4.5) the hyper-surface \( t = \text{const.} \) is space-like.

Another obtained member of the Harrison families is

\[ d\Omega^2 = e^{(2\lambda + 3)\tau} d\tau^2 + e^\tau dx^2 + \tau dx^2 e^{\lambda \tau} dy^2 + 2 e^{\lambda \tau} dy dz \]  

(4.7)

This metric admits the homothetic field

\[ H = 2\partial_t + 2x \lambda \partial_x + (\lambda + 1) y \partial_y + (z(\lambda + 1) - y) \partial_z. \]  

(4.8)

In this case too, the hyper-surface \( t = \text{const.} \) is space-like. Furthermore, for the value \( \lambda = 0 \) the metric (4.7) describes a pp-wave, since the Killing field \( u = \xi_3 = \partial_z \) has zero covariant derivative and zero measure:

\[ \lambda = 0 \Rightarrow u^\alpha u_\alpha = 0 \land u^\alpha_i = 0 \]  

(4.9)

- New metric. The metric given below is, to the best of our knowledge, new:

\[ d\Omega^2 = dt^2 + 2 t^2 dx^2 + dy^2 - 4 \tau dx dy + 4 dx dz. \]  

(4.10)

This metric admits the homothetic field \( H = t \partial_t + y \partial_y + 2z \partial_z \), describes a pp-wave since for the Killing field \( u = \xi_3 = \partial_z \), we have \( u^\alpha u_\alpha = 0 \land u^\alpha_i = 0 \) and the hyper-surface \( t = \text{const.} \) is space-like. It can also be proven that this metric is different from the pp-wave ensuing by setting \( \lambda = 0 \) in (4.7): indeed the tensor

\[ \Pi_{\alpha \beta \gamma \delta} = R^{\lambda}_{\alpha \beta} R_{\gamma \delta \lambda \ell} \]

vanishes identically for metric (4.10) but not for metric (4.7) with \( \lambda = 0 \).

4.2. Bianchi type II

In this model, the structure constants, the basis 1-forms and the Killing fields are

- \( C_{23}^1 = -C_{32}^1 = \frac{1}{2} \)

\[ C_{\beta \gamma}^\alpha = 0 \quad \text{for all the other values of } \alpha, \beta, \gamma \]  

(4.12a)

\[ \sigma^1 = \partial_x + dy, \quad \sigma^2 = \partial_z, \quad \sigma^3 = dx \]  

(4.12b)

\[ \xi_1 = \partial_x, \quad \xi_2 = -x \partial_y + \partial_z, \quad \xi_3 = \partial_y \]  

(4.12c)
In this model, the structure constants, the basis 1-forms and the Killing fields are

4.3. Bianchi type V

In this model, the structure constants, the basis 1-forms and the Killing fields are

\[ C_{13} = -C_{13} = C_{23} = -C_{32} = \frac{1}{7} \]
\[ C_{\beta\gamma} = 0 \quad \text{for all the other values of } \alpha, \beta, \gamma \] (4.16a)

\[ \sigma^1 = e^{-\tau} dz, \quad \sigma^2 = e^{-\tau} dy, \quad \sigma^3 = dx \] (4.16b)

\[ \xi_1 = \partial_t, \quad \xi_2 = \partial_x, \quad \xi_3 = \partial_y + y \partial_z + z \partial_z. \] (4.16c)

**Joseph metrics.**

\[ ds^2 = \kappa^2 \left( \frac{1}{2} \cosh^2 \tau \, d\tau^2 + e^{-\sqrt{3} \tau} \cosh (\sigma^1)^2 + e^{\sqrt{3} \tau} \cosh (\sigma^3)^2 \right). \] (4.17)

This metric was first derived in [30] and does not admit a homothety, hence the constant \( \kappa \) is essential.

**Flat space parametrization.**

\[ ds^2 = -dt^2 + \tau^2 (\sigma^1)^2 + \tau^2 (\sigma^2)^2 + \tau^2 (\sigma^3)^2. \] (4.18)

This metric even though describes a flat spacetime is ‘new’ in the sense that it is a parametrization of itself, as a type V cosmological line element. In [23], at p 194 the Milne form of flat spacetime is considered only as a type VII\(_{h}\) parametrization.

**New neutral signature metric.**

\[ ds^2 = -\frac{1}{4t} \, dt^2 + t \, dz^2 - e^{-2z} (t - 1) \, dy^2 + 2e^{-2z} \, dy \, dz + e^{-2z} (t + 1) \, dz^2. \] (4.19)

This metric has neutral signature \((- - + +)\), admits a homothetic field \( H = 2t \, \partial_t + z \, \partial_y + y \, \partial_z.\)
• **New Euclidean signature metric.**

\[
ds^2 = \lambda^2 \left( \frac{1}{4} \text{sech}^2 \xi \ d\xi^2 + e^{-\sqrt{3} \xi} \ \text{sech} \ (\sigma^1) \ d\sigma^1 + e^{\sqrt{3} \xi} \ \text{sech} \ (\sigma^2) \ d\sigma^2 + \text{sech} \ (\sigma^3) \ d\sigma^3 \right). \tag{4.20}
\]

This metric does not admit a homothetic field, thus the constant \( \lambda \) is essential.

5. **Discussion**

This work completes the first phase of the programme initiated in [10], utilizing the automorphisms of the various Bianchi types as Lie-point symmetries of the corresponding Einstein’s field equations with the aim of uncovering their solution space. The power of the method lies in the fact that it constitutes a semi-algorithm which, if successfully applied, results in the acquisition of the entire space of solutions. This successful implementation had, so far, been carried out in the case of Bianchi type III ([20]) and VII\(_h\) ([22]), while this paper covers types I, II, IV and V. In all cases considered, where the general solution is expected to have three essential constants (III, IV and VII\(_h\)), it is given in terms of the sixth Painlevé transcendent \( \text{P}_{VI} \), along the way with all the known particular solutions. It is noteworthy that these known metrics are rediscovered in a systematic way and without any extra assumption, in contrast to how they were originally obtained. The case of types I, II and V is characteristic. The general solutions, not considered as such at the time of their first derivation and containing one or two essential constants, were produced with the aid of various simplifying ansatzen in a time scale of half a century; [25, 26, 29, 30]. Here, they are comprehensively re-acquired along with the solutions not attributed to anyone else which, to the best of our knowledge, are new: the Lorentzian type I pp-wave metric (4.10), the type V metrics (4.18), (4.20) (Euclidean signature) and (4.19) (neutral signature). Metric (4.18) even though describes a flat spacetime is ‘new’ in the sense that it is a parametrization of itself, as a type V cosmological line element. In [23], at p 194 the Milne form of flat spacetime is considered only as a type VII\(_h\) parametrization. The production of metrics with Euclidean signature may, at first sight, strike as odd; since our staring point is a line element of Lorentzian signature. However, it is made possible by allowing the lapse to be determined through the quadratic constrained equation instead of prescribing it by an \textit{ab initio} choice of time gauge.

In the remaining Bianchi types VIII, IX and the exceptional VI\(_h\) the number of existing automorphisms is not sufficient to allow our method to reduce the problem to a single second-order ODE in one unknown function, but rather to a third-order one. We strongly suspect it to be an equivalent form of the Chazy-type equations. However, the task of proving it involves the search for an appropriate Lie–Bäcklund transformation which is highly non-trivial and non-algorithmic. We plan to return if and when there is something concrete to be reported.

Some directions for future work include the application of the method in the presence of matter sources and/or in higher dimensions.

References

[1] Heckmann O and Schücking E 1962 Gravitation, An Introduction to Current Research ed L Witten (New York: Wiley)
[2] Harvey A 1979 J. Math. Phys. 20 251
[3] Christodoulakis T, Papadopoulos G O and Dimakis A 2003 J. Phys. A: Math. Gen. 36 427
[4] Jantzen R T 1979 Commun. Math. Phys. 64 211
[5] Jantzen R T 1982 J. Math. Phys. 23 1137
[6] Uggla C, Jantzen R T and Rosquist 1995 Phys. Rev. D 51 5522
[7] Siklos S T C 1980 Phys. Lett. A 76 19
[8] Samuel J and Ashtekar A 1991 Class. Quantum Grav. 8 2191
[9] Christodoulakis T, Kofinas G, Korfiatis E, Papadopoulos G O and Paschos A 2001 J. Math. Phys. 42 3580
[10] Christodoulakis T and Terzis P A 2006 J. Math. Phys. 47 102502
[11] Stephani H 1989 Differential Equations: Their Solution Using Symmetries ed M A H MacCallum (Cambridge: Cambridge University Press)
[12] Olver P J 2000 Applications of Lie Groups to Differential Equations 2nd edn (Berlin: Springer)
[13] Bluman G W and Anco S C 2002 Symmetry and Integration Methods for Differential Equations (Berlin: Springer)
[14] Stephani H, Kramer D, MacCallum M, Hoenselaers C and Herlt E 2003 Exact Solutions of Einstein’s Field Equations 2nd edn (Cambridge: Cambridge University Press)
[15] Ellis G F R and MacCallum M A H 1969 Commun. Math. Phys. 12 108
[16] Ryan M P Jr and Shepley L C 1975 Homogeneous Relativistic Cosmologies (Princeton: Princeton University Press)
[17] Coussaert O and Hennaux M 1993 Class. Quantum Grav. 10 1607
[18] Christodoulakis T, Korfiatis E and Papadopoulos G O 2002 Commun. Math. Phys. 226 377
[19] Harvey A and Tsoubelis D 1977 Phys. Rev. D 15 2734–7
[20] Christodoulakis T and Terzis P A 2007 Class. Quantum Grav. 24 875
[21] Cosgrove C M and Scoufis G 1993 Stud. Appl. Math. 88 25–87
[22] Terzis P A and Christodoulakis T 2009 Gen. Rel. Grav. 41 469–95
[23] Wainwright J and Ellis G F R (ed) 1989 Dynamical Systems in Cosmology (Cambridge: Cambridge University Press)
[24] Terzis P A and Christodoulakis T 2010 arXiv:1007.1561v1 [gr-qc]
[25] Kasner E 1921 Am. J. Math. 43 217
[26] Harrison B K 1959 Phys. Rev. 116 1285
[27] Lorenz-Petzold D 1983 Acta Phys. Pol. B 14 791
[28] Valent G and Yahia H B 2007 Class. Quantum Grav. 24 255
[29] Taub A H 1951 Ann. Math. 53 472
[30] Joseph V 1966 Proc. Camb. Phil. Soc. 62 87