ON THE RIEMANN ZETA-FUNCTION
AND THE DIVISOR PROBLEM II

ALEKSANDAR IVIĆ
Central European Journal of Mathematics 3(2) (2005), 203-214

Abstract. Let $\Delta(x)$ denote the error term in the Dirichlet divisor problem, and $E(T)$ the error term in the asymptotic formula for the mean square of $|\zeta(\frac{1}{2} + it)|$. If $E^*(t) = E(t) - 2\pi \Delta^*(t/2\pi)$ with $\Delta^*(x) = -\Delta(x) + 2\Delta(2x) - \frac{1}{2}\Delta(4x)$, then we obtain

$$\int_0^T |E^*(t)|^5 \, dt \ll \varepsilon T^{2+\varepsilon}$$

and

$$\int_0^T |E^*(t)|^\frac{544}{75} \, dt \ll \varepsilon T^{\frac{601}{225}+\varepsilon}.$$ 

It is also shown how bounds for moments of $|E^*(t)|$ lead to bounds for moments of $|\zeta(\frac{1}{2} + it)|$.

1. Introduction and statement of results

This work is the continuation of [8], where several aspects of the connection between the divisor problem and $\zeta(s)$, the zeta-function of Riemann, were investigated. As usual, let

$$\Delta(x) = \sum_{n \leq x} d(n) - x (\log x + 2\gamma - 1)$$

(1.1)

denote the error term in the Dirichlet divisor problem, and

$$E(T) = \int_0^T |\zeta(\frac{1}{2} + it)|^2 \, dt - T \left( \log \left( \frac{T}{2\pi} \right) + 2\gamma - 1 \right)$$

(1.2)
where $d(n)$ is the number of divisors of $n$, $\gamma = -\Gamma'(1) = 0.577215\ldots$ is Euler’s constant. Instead of $\Delta(x)$ we work with the modified function $\Delta^*(x)$ (see M. Jutila [10]), where

$$\Delta^*(x) := -\Delta(x) + 2\Delta(2x) - \frac{1}{2}\Delta(4x).$$

(1.3)

M. Jutila (op. cit.) investigated both the local and global behaviour of the difference

$$E^*(t) := E(t) - 2\pi \Delta^*(\frac{t}{2\pi}),$$

and in particular he proved that

$$\int_0^T (E^*(t))^2 \, dt \ll T^{4/3} \log^3 T.$$  

(1.4)

In [8] this bound was complemented with the new bound

$$\int_0^T (E^*(t))^4 \, dt \ll \varepsilon T^{16/9+\varepsilon};$$  

(1.5)

neither (1.4) or (1.5) seem to imply each other. Here and later $\varepsilon$ denotes positive constants which are arbitrarily small, but are not necessarily the same ones at each occurrence. Our first aim is to obtain another bound for moments of $|E^*(t)|$. This is given by

**THEOREM 1.** We have

$$\int_0^T |E^*(t)|^5 \, dt \ll \varepsilon T^{2+\varepsilon}.$$  

(1.6)

From (1.4), (1.6) and Hölder’s inequality for integrals, it follows that

$$\int_0^T |E^*(t)|^4 \, dt = \int_0^T |E^*(t)|^{2/3} |E^*(t)|^{10/3} \, dt \leq \left( \int_0^T |E^*(t)|^2 \, dt \right)^{1/3} \left( \int_0^T |E^*(t)|^5 \, dt \right)^{2/3} \ll \varepsilon T^{16/9+\varepsilon},$$

which implies (1.5). This means that (1.6) and (1.4) together are stronger than (1.5). Another result of a more general nature (for the definition and properties of exponent pairs see e.g., [3] or [6, Chapter 2]) is contained in
THEOREM 2. Let \((\kappa, \lambda)\) be an exponent pair such that \(2\lambda \leq 1 + \kappa\), and

\[
V \geq T^{\frac{1 + \lambda - 2\kappa}{3(2-\kappa)} + \varepsilon}.
\]  

(1.7)

Let \(t_r \in [T, 2T] \ (r = 1, \ldots, R)\) be points such that \(|t_r - t_s| \geq V \ (r \neq s)\) and \(|E^*(t_r)| \geq V \ (r = 1, \ldots, R)\). Then

\[
R \ll \varepsilon T^{1+\varepsilon} V^{-3} + T^{\frac{1+4\kappa+\lambda}{3\kappa}} \varepsilon V^{-\frac{3\kappa+2}{\kappa}}.
\]  

(1.8)

From Theorem 2 we can obtain specific bounds for moments of \(|E^*(t)|\), provided we choose the exponent pair \((\kappa, \lambda)\) appropriately. The optimal choice of the exponent pair is hard to determine, since several conditions have to hold (see e.g., (5.5)). However, by trying some of the standard exponent pairs one can obtain a bound which is not far from the optimal bound that the method allows. For instance, with the exponent pair \((\kappa, \lambda) = (75/197, 104/197)\) (this exponent pair arises, in the terminology of exponent pairs, as \((75/197, 104/197) = BA^3 BA^3 B(0,1)\)) we can obtain

THEOREM 3. We have

\[
\int_0^T |E^*(t)|^{544} \, dt \ll \varepsilon T^{601} + \varepsilon.
\]  

(1.9)

One of the main reasons for investigating power moments of \(|E^*(t)|\) is the possibility to use them to derive results on power moments of \(|\zeta(\frac{1}{2} + it)|\), which is one of the main themes in the theory of \(\zeta(s)\). A result in this direction is given by

THEOREM 4. Let \(k \geq 1\) be a fixed real, and let \(c(k)\) be such a constant for which

\[
\int_0^T |E^*(t)|^k \, dt \ll \varepsilon T^{c(k) + \varepsilon}.
\]  

(1.10)

Then we have

\[
\int_0^T |\zeta(\frac{1}{2} + it)|^{2k+2} \, dt \ll \varepsilon T^{c(k)+\varepsilon}.
\]  

(1.11)

The constant \(c(k)\) must satisfy

\[
c(k) \geq 1.
\]  

(1.12)
This is obvious if \( k \) is an integer, as it follows from [6, Theorem 9.6]. If \( k \) is not an integer, then this result yields \( (p = \frac{2k+2}{2k+2} > 1) \)

\[
T \ll \int_0^T |\zeta(\frac{1}{2} + it)|^{2k+2} \, dt \leq \left( \int_0^T |\zeta(\frac{1}{2} + it)|^{2k+2} \, dt \right)^{1/p} T^{1-1/p}
\]

by Hölder’s inequality for integrals. After simplification (1.12) easily follows again.

**Corollary 1.** We have

\[
\int_0^T |\zeta(\frac{1}{2} + it)|^{12} \, dt \ll \varepsilon T^{2+\varepsilon}.
\]

(1.13)

This follows from Theorem 1 and Theorem 4 (with \( k = 5 \)), and is the well-known result of D.R. Heath-Brown [2], who had \( \log^{17} T \) in place of \( T^\varepsilon \) on the right-hand side of (1.13).

**Corollary 2.** We have

\[
\int_0^T |\zeta(\frac{1}{2} + it)|^{1238} \frac{1}{25} \, dt \ll \varepsilon T^{\frac{691}{1225} + \varepsilon}.
\]

(1.14)

This follows from Theorem 3 and Theorem 4 (with \( k = \frac{544}{75} \)). The bound (1.14) does not follow from (1.13) (and the strongest pointwise estimate for \( |\zeta(\frac{1}{2} + it)| \)), but on the other hand (1.13) does not follow from (1.14). In principle, (1.14) could be used for deriving zero-density bounds for \( \zeta(s) \) (see e.g., [6, Chapter 10]), but very likely its use would lead to very small improvements (if any) of the existing bounds.

**Acknowledgement.** I wish to thank Prof. Matti Jutila for valuable remarks.

2. **The necessary lemmas**

In this section we shall state the lemmas which are necessary for the proof of Theorem 1.

**Lemma 1** (O. Robert–P. Sargos [11]). Let \( k \geq 2 \) be a fixed integer and \( \delta > 0 \) be given. Then the number of integers \( n_1, n_2, n_3, n_4 \) such that \( N < n_1, n_2, n_3, n_4 \leq 2N \) and

\[
|n_1^{1/k} + n_2^{1/k} - n_3^{1/k} - n_4^{1/k}| < \delta N^{1/k}
\]

is, for any given \( \varepsilon > 0 \),

\[
\ll \varepsilon N^\varepsilon(N^4 \delta + N^2).
\]

(2.1)
This Lemma was crucial in obtaining the asymptotic formulas for the third and fourth moment of $\Delta(x)$ in [9].

**LEMMA 2.** Let $T^\varepsilon \ll G \ll T/\log T$. Then we have

$$E^*(T) \leq \frac{2}{\sqrt{\pi G}} \int_0^\infty E^*(T + u) e^{-u^2/G^2} \, du + O(\varepsilon GT^\varepsilon), \quad (2.2)$$

and

$$E^*(T) \geq \frac{2}{\sqrt{\pi G}} \int_0^\infty E^*(T - u) e^{-u^2/G^2} \, du + O(\varepsilon GT^\varepsilon). \quad (2.3)$$

Lemma 2 follows on combining Lemma 2.2 and Lemma 2.3 of [8].

The next lemma is F.V. Atkinson’s classical explicit formula for $E(T)$ (see [1], [6] or [7]).

**LEMMA 3.** Let $0 < A < A'$ be any two fixed constants such that $AT < N < A'T$, and let $N' = N'(T) = T/(2\pi) + N/2 - (N^2/4 + NT/(2\pi))^{1/2}$. Then

$$E(T) = \Sigma_1(T) + \Sigma_2(T) + O(\log^2 T), \quad (2.4)$$

where

$$\Sigma_1(T) = 2^{1/2}(T/(2\pi))^{1/4} \sum_{n \leq N} (-1)^n d(n)n^{-3/4} e(T, n) \cos(f(T, n)), \quad (2.5)$$

$$\Sigma_2(T) = -2 \sum_{n \leq N'} d(n)n^{-1/2}(\log(T/(2\pi n))^{-1} \cos(T \log(T/(2\pi n)) - T + \pi/4), \quad (2.6)$$

with

$$f(T, n) = 2T \arcsinh(\sqrt{\pi n/(2T)}) + \sqrt{2\pi nT + \pi^2 n^2 - \pi/4}$$

$$= -\frac{1}{4} \pi + 2\sqrt{2\pi nT} + \frac{1}{6} \sqrt{2\pi^3 n^{3/2} T^{-1}/2} + a_5 n^{-5/2} T^{-3/2} + a_7 n^{-7/2} T^{-5/2} + \ldots, \quad (2.7)$$

$$e(T, n) = (1 + \pi n/(2T))^{-1/4} \left\{ (2T/\pi n)^{1/2} \arcsinh(\sqrt{\pi n/(2T)}) \right\}^{-1} \quad (2.8)$$

$$= 1 + O(n/T) \quad (1 \leq n < T),$$

and $\arcsinh x = \log(x + \sqrt{1 + x^2})$.
LEMMA 4 (M. Jutila [10]). If $A \in \mathbb{R}$ is a constant, then we have

$$
\cos \left( \sqrt{8\pi nT} + \frac{1}{6}\sqrt{2\pi^4 n^3 T^{-1/2}} + A \right) = \int_{-\infty}^{\infty} \alpha(u) \cos(\sqrt{8\pi n} \sqrt{T} + u) + A \, du,
$$

where $\alpha(u) \ll T^{1/6}$ for $u \neq 0$,

$$
\alpha(u) \ll T^{1/6} \exp(-bT^{1/4}|u|^{3/2})
$$

for $u < 0$, and

$$
\alpha(u) = T^{1/8}u^{-1/4} \left( d \exp(ibT^{1/4}u^{3/2}) + \bar{d} \exp(-ibT^{1/4}u^{3/2}) \right) + O(T^{-1/8}u^{-7/4})
$$

for $u \geq T^{-1/6}$ and some constants $b > 0$ and $d$.

3. The proof of Theorem 1

The proof is on the lines of [8]. We seek an upper bound for $R$, the number of points $\{t_r\} \in [T, 2T]$ such that $|E^*(t_r)| \geq V \geq T^\varepsilon$ and $|t_r - t_s| \geq V$ for $r \neq s$. We consider separately the points where $E^*(t_r)$ is positive or negative. Suppose the first case holds (the other one is treated analogously). Then from Lemma 2 we have

$$
V \leq E^*(t_r) \leq \frac{2}{\sqrt{\pi G}} \int_0^\infty E^*(t_r + u) e^{-u^2/G^2} \, du + O(GT^\varepsilon).
$$

The integral on the right-hand side is simplified by Atkinson’s formula (Lemma 3) and the truncated formula for $\Delta^*(x)$ (see [8, eq. (6)]), as in [8]. We take $G = cV T^{-\varepsilon}$ (with sufficiently small $c > 0$) to make the $O$-term in (3.1) $\leq \frac{1}{2}V$, raise everything to the fourth power and sum over $r$. By Hölder’s inequality we obtain

$$
RV^4 \ll V^{-1}T^\varepsilon \max_{|u| \leq G \log T} \int_{T/2}^{2T} \varphi(t) \left( \Sigma_4^4(X, N; u) + \Sigma_5^4(X, N; u) + \Sigma_6^4(X; u) \right) dt,
$$

with the notation introduced in (2.7), (2.8) and [8]:

$$
\Sigma_4(X, N; u) := t^{1/4} \sum_{X < n \leq N} (-1)^n d(n) n^{-3/4}e(t + u, n) \cos(f(t + u, n)),
$$

$$
\Sigma_5(X, N; u) := t^{1/4} \sum_{X < n \leq N} (-1)^n d(n) n^{-3/4} \cos(\sqrt{8\pi n} (t + u) - \pi/4),
$$

$X < N$.
\[
\Sigma_6(X; u) := \sum_{n \leq X} t^{-1/4}(-1)^n d(n)n^{3/4} \cos(\sqrt{8\pi n}(t + u) - \pi/4).
\] (3.4)

Here we have \(X = T^{1/3 - \varepsilon}, N = TG^{-2} \log T\), and \(\varphi(t)\) is a smooth, nonnegative function supported in \([T/2, 5T/2]\), such that \(\varphi(t) = 1\) when \(T \leq t \leq 2T\). The basic idea is that the contributions of \(\Sigma_6(X; u)\) and \(\Sigma_5(X, N; u)\) will be approximately equal at \(X\), and the same will be true of \(\Sigma_4(X, N; u)\) as well. In the latter case, as was discussed in detail in [8], one has to use Lemma 4 to deal with the complications arising from the presence of \(\cos(f(t + u, n))\) in (3.3). The difference from [8] is that the choice \(G = cV T^{-\varepsilon}\) leads directly to (3.2), which is in a certain sense optimal, while in [8] the choice was \(N = T^{5/9}\). Proceeding now as in [8] (here Lemma 1 with \(k = 2\) was crucial) we obtain

\[
RV^4 \ll_{\varepsilon} V^{-1}T^\varepsilon(T^{3/2}N^{1/2} + T^2X^{-1} + T^{-1/2}X^{13/2} + X^5)
\]

\[
\ll_{\varepsilon} V^{-1}T^\varepsilon(T^2V^{-1} + T^{5/3})
\]

\[
\ll_{\varepsilon} T^{2+\varepsilon}V^{-2},
\] (3.5)

since \(V < T^{1/3}\) in view of the best known estimates for \(\Delta(x)\) and \(E(t)\). Namely with suitable \(C > 0\) one has (see M.N. Huxley [3], [4])

\[
\Delta(x) \ll x^{131/416} \log^C x, \quad 131/416 = 0.3149038 \ldots,
\]

\[
E(T) \ll T^{72/227} \log^C T, \quad 72/227 = 0.3171806 \ldots.
\] (3.6)

Therefore (3.5) yields the large values estimate

\[
R \ll_{\varepsilon} T^{2+\varepsilon}V^{-6},
\]

and Theorem 1 easily follows, as in [5] or [6, Chapter 13] for moments of \(\Delta(x)\).

4. The proof of Theorem 2

We start again from (3.1), choosing \(G = cVT^{-\varepsilon} (\ll \frac{1}{2}V), T = t_r\), so that we have

\[
E^* (t_r) \geq V, \quad E^* (t_r) \ll G^{-1} \int_0^\infty E^* (t_r + u)e^{-(u/G)^2} \, du
\] (4.1)

in case \(E^* (t_r) > 0\), and the case of negative values is analogous. We relabel the points for which (4.1) holds in the sense that it will hold for \(r = 1, \ldots, R\). The proof is similar to the proof of (13.52) of Theorem 13.8 of [6]. To remove the function \(d(n)\) from the sums in (3.3)–(3.4) we use the inequality (see the Appendix of [6])

\[
\sum_{r \leq R} \|(\xi, \phi_r)\|^2 \leq \|\xi\|^2 \max_{r \leq R} \sum_{s \leq R} |(\phi_r, \phi_s)|,
\] (4.2)
where for two complex vector sequences \( a = \{a_n\}_{n=1}^{\infty}, \ b = \{b_n\}_{n=1}^{\infty} \) the inner product is defined as

\[
(a, b) = \sum_{n=1}^{\infty} a_n \bar{b}_n.
\]

We shall also use (3.3)–(3.4) with \( N = TG^{-2} \log T \). We shall consider separately the points where \( |\sum| \gg V \) when \( \sum \) equals \( \Sigma_4(X, N; u), \Sigma_5(X, N; u) \) or \( \Sigma_6(X; u) \) (\(|u| \leq G \log T\)), as the case may be. Taking the maximum over \(|u| \leq G \log T\) over the whole sum, we may relabel the points such that they are called again \( t = t_r, r \leq R \).

Moreover, let \( R_0 \) denote the number of such \( t_r \)'s (in each case) lying in an interval of length \( T_0 \), where \( T_0 \) is a function of \( V \) and \( T \) that will be determined later. Thus \( V \leq T_0 \) has to hold and

\[
R \ll R_0(1 + T/T_0), \tag{4.3}
\]

As in the proof of Theorem 2, the choice of \( X \) will be

\[
X = T^{1/3-\varepsilon},
\]

when the largest term in \( \sum_6 \) is approximately equal to the smallest term in \( \sum_4 \) and \( \sum_5 \). This choice exploits the specific structure of the function \( E^*(t) \), and leads to a better bound than was possible for large values of \( \Delta(x) \) in Chapter 13 of [6]. Namely in the latter case the maximum occurred at \( n = TG^{-2} \log T \), but in our case \( X = T^{1/3-\varepsilon} < TG^{-2} \log T \), since \( V < T^{1/3-\varepsilon} \) must hold in view of (3.6). For example, from (3.4) and (4.2) (in case \( |\sum_6| \gg V \) holds) we obtain

\[
R_0V^2 \ll \frac{\log^2 T}{\sqrt{T}} \max_{|u| \leq G \log T, M \leq X/2} \left\{ \sum_{r \leq R_0} \sum_{M < n \leq 2M} (-1)^n d(n)n^{3/4}e^{i\sqrt{8\pi n(t_r+u)}} \right\}^2
\]

\[
\ll \frac{\log^2 T}{\sqrt{T}} \max_{|u| \leq G \log T, M \leq X/2, r \leq R_0} M^{5/2} \log^4 M \left( M + \sum_{s \leq R_0, s \neq r} \sum_{M < n \leq 2M} e^{i\sqrt{8\pi n(t_r+u)-\sqrt{8\pi n(t_s+u)}}} \right), \tag{4.4}
\]

which corresponds to (13.60) of [6]. If we set

\[
f(x) = \sqrt{8\pi x}(\sqrt{t_r+u} - \sqrt{t_s+u}),
\]

then we can use the first derivative test (Lemma 2.1 of [6]) to deduce that the
contribution of \( x = n \) (in the last sum in (4.4)) for which \(|f'(x)| < 1/2\) is

\[
\ll \sum_{s \leq R_0, s \neq r} \frac{\sqrt{M}}{|t_r - t |} \ll \sqrt{M} T \sum_{s \leq R_0, s \neq r} \frac{1}{|t_r - t_s|} \ll \sqrt{MTV^{-1} \log T},
\]

since \(|t_r - t_s| \geq V\) if \( r \neq s \). The contribution of \(|f'(x)| \geq 1/2\) is estimated by the theory of exponent pairs. The portion of the last sum in (4.4) is, in this case,

\[
\ll R_0 \left( |t_r - t_s|(MT)^{-1/2} \right)^\kappa M^\lambda \ll R_0 T_0^\kappa M^{\lambda - \kappa/2} T^{-\kappa/2},
\]

(4.6) since \(|t_r - t_s| \leq T_0\). Therefore from (4.4)–(4.6) it follows that

\[
R_0 V^2 \ll T^{-1/2} X^{5/2} \log^6 T + X^3 V^{-1} \log^7 T + R_0 T_0^\kappa X^{5/2 + \lambda - \frac{1}{2}} T^{-\frac{1}{2}} - \frac{1}{2} \log^6 T
\]

\[
\ll T^{2/3} \log^6 T + TV^{-1} \log^7 T + R_0 T_0^\kappa T^{1+\lambda-2\kappa} \log^6 T.
\]

(4.7)

The contribution of large values of \(|\sum_4|\) and \(|\sum_5|\) is estimated analogously. We proceed, similarly as in (4.7), to obtain in these cases

\[
R_0 V^2 \ll T^{1/2} \log^2 T \max_{|u| \leq G \log T, X \leq M \leq T^{1+\varepsilon} V^{-2}} \times
\]

\[
\times \sum_{r \leq R_0} \left| \sum_{M < n \leq 2M} (-1)^n d(n) n^{-3/4} e^{i \sqrt{8 \pi n (t_r + u)}} \right|^2
\]

\[
\ll T^{1/2} \log^2 T \max_{|u| \leq G \log T, r \leq R_0, X < M \leq T^{1+\varepsilon} V^{-2}} M^{-1/2} \log^4 M \left( M + \sum_{s \leq R_0, s \neq r} \sum_{M < n \leq 2M} e^{i \sqrt{8 \pi n (\sqrt{t_r + u} - \sqrt{t_s + u})}} \right)
\]

\[
\ll \varepsilon T^{1+\varepsilon} V^{-1} + R_0 T_0^\kappa T^{1/2 - \kappa/2} \log^5 T \max_{X < M \leq T^{1+\varepsilon} V^{-2}} M^{\lambda - \kappa/2 - 1/2}.
\]

(4.8)

The hypothesis in the formulation of the theorem was that

\[
2 \lambda \leq \kappa + 1,
\]

(4.9) hence by combining (4.7) and (4.8) it follows that

\[
R_0 V^2 \ll \varepsilon T^{1+\varepsilon} V^{-1} + R_0 T_0^\kappa T^{1+\lambda-2\kappa} + \varepsilon,
\]

(4.10)
since \( T^{2/3} \leq TV^{-1} \) because \( V \leq T^{1/3} \) has to hold. If we choose
\[
T_0 = V^2 T^{2\kappa-1-\lambda-\frac{2\varepsilon}{\kappa}}
\] (4.11)
then (4.10) reduces to \( R_0 V^2 \ll \varepsilon T^{1+\varepsilon} V^{-1} \), and the condition \( T_0 \geq V \) becomes
\[
V \geq T^{\frac{1+\lambda-2\kappa}{3\kappa}+\varepsilon},
\] (4.12)
which is (1.7). Therefore (4.10) gives
\[
R \ll R_0 (1 + T/T_0) \ll \varepsilon T^{1+\varepsilon} V^{-3} + T^{\frac{1+4\kappa+\lambda}{3\kappa}+\varepsilon} V^{-\frac{3\kappa+2}{\kappa}},
\]
thereby completing the proof of Theorem 2.

5. The proof of Theorem 3
With the choice \((\kappa, \lambda) = (75/197, 104/197)\) it is seen that (1.7) and (1.8) of Theorem 2 reduce to
\[
R \ll T^\varepsilon (TV^{-3} + T^{\frac{601}{957} V^{-\frac{619}{75}}}) \quad (V \geq T^{\frac{151}{957}+\varepsilon}, \frac{151}{957} = 0.157784\ldots).
\] (5.1)
Let
\[
J_V(T) = \left\{ t \in [T, 2T] : V \leq |E^*(t)| < 2V \right\},
\]
and write
\[
\int_T^{2T} |E^*(t)|^{\frac{619}{75}} \, dt \ll \varepsilon \log T \max_{V \geq T^\varepsilon} \int_{J_V(T)} |E^*(t)|^{\frac{619}{75}} \, dt + T^{1+\varepsilon}.
\] (5.2)
For \( V \leq T^{\frac{151}{957}+\varepsilon} \) we have, on using (1.6) of Theorem 1,
\[
\int_{J_V(T)} |E^*(t)|^{\frac{544}{75}} \, dt = \int_{J_V(T)} |E^*(t)|^6 |E^*(t)|^{\frac{160}{75}} \, dt
\ll \varepsilon T^{2+\frac{160}{75}+\frac{151}{957}+\varepsilon} \leq T^{\frac{601}{957}}.
\] (5.3)
Suppose now that \( V \geq T^{\frac{151}{957}+\varepsilon} \), and divide \([T, 2T]\) into subintervals of length \( V \) (the last of these subintervals may be shorter). Let \( |E^*(\tau_j)| \) be the supremum of \( |E^*(t)| \) in the \( j \)th of these subintervals, and let further \( t_1, \ldots, t_{R_V} \) denote the \( \tau_j \)'s with even or odd indices such that the intersection of the \( j \)th subinterval and \( J_V(T) \) is non-empty. Then \(|t_r - t_s| \geq V\) for \( r \neq s\), and (5.1) gives
\[
R_V \ll \varepsilon T^\varepsilon (TV^{-3} + T^{\frac{601}{957} V^{-\frac{619}{75}}}) \ll \varepsilon T^{\frac{601}{957}+\varepsilon} V^{-\frac{619}{75}}
\] (5.4)
On the Riemann zeta-function and the divisor problem II

for

\[ V \leq T^{\frac{188}{591}}, \quad 188/591 = 0.3181049 \ldots. \]  

(5.5)

But in view of (3.6) it is seen that (5.5) is always satisfied (the choice of our exponent pair was made to ensure that this is indeed the case), and we obtain from (5.4)

\[ \int_{J_V(T)} |E^*(t)|^{\frac{188}{591}} \, dt \ll R_V V^{1+\frac{188}{591}} \ll T^{\frac{601}{225}+\varepsilon} V^{-\frac{619}{75} V^{\frac{1}{3}}} = T^{\frac{601}{225}+\varepsilon}. \]  

(5.6)

Theorem 3 follows then from (5.2), (5.3) and (5.6), on replacing \( T \) by \( T^{2-j} \) and summing over \( j = 1, 2, \ldots \).

6. The proof of Theorem 4

To prove Theorem 4 it is enough to prove that

\[ R \ll T^{c(k)+\varepsilon} V^{-2k-2}, \]  

(6.1)

where \( R \) is the number of points \( t_r \in [T, 2T] \) (\( r = 1, \ldots, R \)), such that \( |\zeta(\frac{1}{2} + it_r)| \geq V \) with \( |t_r - t_s| \geq 1 \) for \( r \neq s \) and \( V \geq T^{\varepsilon} \). We denote actually by \( R \) the number of points with even and odd indices, so that the intervals \( [t_r - \frac{1}{3}, t_r + \frac{1}{3}] \) are disjoint. Then we have, using Theorem 1.2 of [7] with \( k = 2, \delta = \frac{1}{3}, \)

\[ RV^2 \leq \sum_{r=1}^{R} |\zeta(\frac{1}{2} + it_r)|^2 \ll \log T \sum_{r=1}^{R} \int_{t_r - \frac{1}{3}}^{t_r + \frac{1}{3}} |\zeta(\frac{1}{2} + it)|^2 \, dt \ll \log T \sum_{j=1}^{J} \int_{\tau_j-G}^{\tau_j+G} |\zeta(\frac{1}{2} + it)|^2 \, dt, \]  

(6.2)

where \( \tau_j \in [T-G, T+G] \) (\( j = 1, \ldots, J \)) is a system of points such that \( |\tau_j - \tau_\ell| \geq 2G \) for \( j \neq \ell \) and \( T^{\varepsilon} \leq G = G(T) \ll T \). By the definition of \( E^*(t) \) we have

\[ \int_{\tau_j-G}^{\tau_j+G} |\zeta(\frac{1}{2} + it)|^2 \, dt = E(\tau_j + G) - E(\tau_j - G) + O(G \log T) \]

\[ = E^*(\tau_j + G) - E^*(\tau_j - G) + 2\pi \Delta^* \left( \frac{\tau_j + G}{2\pi} \right) - 2\pi \Delta^* \left( \frac{\tau_j - G}{2\pi} \right) + O(G \log T) \]

\[ = E^*(\tau_j + G) - E^*(\tau_j - G) + O(\varepsilon(GT^{\varepsilon})). \]

Here we used the fact that

\[ \Delta^*(x) - \Delta^*(y) \ll \varepsilon x^\varepsilon (x - y + 1) \quad (1 \ll y \leq x), \]
which follows from (1.1), (1.3) and \(d(n) \ll \varepsilon n^{\varepsilon}\). This arithmetic property of \(d(n)\) is essential, since it makes it possible to connect the large values of \(|\zeta(\frac{1}{2} + it)|\) to sums of values of \(E^*(t)\), and hence to exploit the special structure of the function \(E^*(t)\). If we worked only with \(E(t)\), we would obtain Theorem 4, where (1.10) has \(E^*(t)\) replaced by \(E(t)\). However, the existing estimates for the moments of \(|E(t)|\) (see [5] and Chapter 13 of [6]) are not as strong as the moments of \(|E^*(t)|\) (cf. (1.6) and (1.9)).

Returning to the proof, note that (6.2) yields

\[
RV^2 \ll_{\varepsilon} \log T \left\{ \sum_{j=1}^{J} (E^*(\tau_j + G) - E^*(\tau_j - G)) \right\} + RGT^\varepsilon,
\]

giving

\[
RV^2 \ll_{\varepsilon} \log T \left\{ \sum_{j=1}^{J} (E^*(\tau_j + G) - E^*(\tau_j - G)) \right\}
\]

(6.3)

with

\[
G = cV^2 T^{-\varepsilon}
\]

(6.4)

and sufficiently small \(c > 0\). If we use Lemma 2 we may replace \(\sum_j E^*(\tau_j + G)\) by its majorant

\[
\frac{2}{\sqrt{\pi G}} \int_0^\infty \sum_{j=1}^{J} E^*(\tau_j + G + u) e^{-u^2/G^2} du + RGT^\varepsilon,
\]

and similarly for the sum with \(E^*(\tau_j - G)\). By Hölder’s inequality we have (since \(J \leq R\))

\[
\int_0^\infty \sum_{j=1}^{J} E^*(\tau_j + G + u) e^{-u^2/G^2} du
\]

\[
\ll \int_0^\infty e^{-u^2/G^2} \left( \sum_{j=1}^{J} |E^*(\tau_j + G + u)|^k \right)^{\frac{1}{k}} du
\]

\[
\ll \left( \int_0^\infty e^{-u^2/G^2} \sum_{j=1}^{J} |E^*(\tau_j + G + u)|^k du \right)^{\frac{1}{k}} G^{1 - \frac{1}{k}}
\]

\[
\ll (GR)^{1 - \frac{1}{k}} \left( \int_{T/2}^{5T/2} |E^*(t)|^k dt \right)^{\frac{1}{k}} \log T,
\]

(6.5)
by breaking the system of points $\tau_j$ into $\ll \log T$ subsystems with $|\tau_j - \tau_\ell| \geq G \log T$
for $\ell \neq j$. From (1.10) and (6.3)–(6.5) it follows that

$$RV^2 \ll_\varepsilon T^\varepsilon (RV^2)^{1-k} \cdot T^{\varepsilon(k)} + \varepsilon V^{-2},$$

which on simplifying yields

$$R \ll_\varepsilon T^{\varepsilon(k)} + \varepsilon V^{-2k-2},$$

(6.6)

and (6.6) implies easily (1.11) of Theorem 4. By the same method one also obtains

$$\gamma \leq c(k)/(k + 1) \text{ for every } k \geq 1,$$

if

$$\gamma := \inf \{g \geq 0 : E^*(t) \ll T^g\},$$

but better bounds for $\gamma$ can be derived from short interval results on $E^*(t)$, provided
they can be obtained. The existing results make it hard to even conjecture what
should be the true value of $\gamma$. 
References

[1] F.V. Atkinson, The mean value of the Riemann zeta-function, Acta Math. 81(1949), 353-376.
[2] D.R. Heath-Brown, The twelfth power moment of the Riemann zeta-function, Quart. J. Math. (Oxford) 29(1978), 443-462.
[3] M.N. Huxley, Area, Lattice Points and Exponential Sums, Oxford Science Publications, Clarendon Press, Oxford, 1996.
[4] M.N. Huxley, Exponential sums and lattice points III, Proc. London Math. Soc. 387(2003), 591-609.
[5] A. Ivić, Large values of the error term in the divisor problem, Invent. Math. 71(1983), 513-520.
[6] A. Ivić, The Riemann zeta-function, John Wiley & Sons, New York, 1985 (2nd ed. Dover, Mineola, New York, 2003).
[7] A. Ivić, The mean values of the Riemann zeta-function, LNs 82, Tata Inst. of Fundamental Research, Bombay (distr. by Springer Verlag, Berlin etc.), 1991.
[8] A. Ivić, On the Riemann zeta-function and the divisor problem, Central European J. Math. (2)(4) (2004), 1-15.
[9] A. Ivić and P. Sargos, On the higher moments of the error term in the divisor problem, to appear.
[10] M. Jutila, Riemann’s zeta-function and the divisor problem, Arkiv Mat. 21(1983), 75-96 and II, ibid. 31(1993), 61-70.
[11] O. Robert and P. Sargos, Three-dimensional exponential sums with monomials, J. reine angew. Math. (in print).