Dvoretzky’s theorem and the complexity of entanglement detection

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Eprint arxiv:1510.00578, 17p.
We present a lower bound for complexity of entanglement detection which (ultimately) relies on the 1961 Dvoretzky’s theorem, a fundamental result from Asymptotic Geometric Analysis asserting that high-dimensional convex sets typically look round when we observe only their section with a randomly chosen subspaces of smaller dimension.
We present a lower bound for complexity of entanglement detection which (ultimately) relies on the 1961 Dvoretzky’s theorem, a fundamental result from Asymptotic Geometric Analysis asserting that high-dimensional convex sets typically look round when we observe only their section with a randomly chosen subspaces of smaller dimension.

More on the interface between Asymptotic Geometric Analysis and Quantum Information Theory can be found in the forthcoming book

G. Aubrun and S. Szarek, *Alice and Bob meet Banach*

of which a preliminary version is available via Aubrun’s web page.
Outline

- Notation
- Background: Gurvits, Horodecki, Størmer, Skowronek
- The main result
- Strategy behind the proof:
  - Tangible version of Dvoretzky’s theorem (Milman 1971)
  - Face/vertex counting (Figiel–Lindenstrauss–Milman 1977)
  - Bounds on vertical and facial complexity of sets of quantum states
- Sketch of the proof
- Conclusions
Before we proceed... an announcement

Fall 2017 (Sep. 4 - Dec. 15): Trimester on
ANALYSIS IN QUANTUM INFORMATION THEORY
at the Institut Henri Poincaré in Paris
Pre-school, September 4-8, Cargèse, Corsica
http://www.ihp.fr/en/activities/trimester-thematic/calendar
Organizers: G. Aubrun, B. Collins, I. Nechita, S. Szarek
Don’t be shy and let one of us know if you are interested!
Entanglement

A quantum state on a finite-dimensional complex Hilbert space $\mathcal{H}$ is a positive operator of trace 1. Denote by $D = D(\mathcal{H})$ the set of states on $\mathcal{H}$. Note that

$$D(\mathcal{H}) = \text{conv}\{|\psi\rangle\langle\psi| : \psi \in \mathcal{H}, |\psi| = 1\}.$$
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When $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ is a bipartite Hilbert space, the set of separable states is the subset $\text{Sep}(\mathcal{H}) \subset D(\mathcal{H})$ defined as

$$\text{Sep} = \text{Sep}(\mathcal{H}) := \text{conv}\{ |\psi_1 \otimes \psi_2\rangle\langle\psi_1 \otimes \psi_2| : \psi_i \in \mathcal{H}_i, |\psi_i| = 1 \}.$$ 

Elements of $D \setminus \text{Sep}$ are called entangled states.
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The dichotomy between entanglement vs. separability is fundamental in quantum theory. Entanglement is indispensable for most protocols of quantum information theory/cryptography/computing/teleportation...
Certifying/witnessing entanglement

Since entanglement is defined as non-membership in a (closed) convex set, it follows from the Hahn–Banach separation theorem that for every entangled state $\rho$, there is a linear form $f$ such that $f \leq a$ on $\text{Sep}$ and $f(\rho) > a$. Such $f$ certifies, or witnesses, the entanglement of $\rho$. 

Equivalently, $\text{Sep}$ is the intersection of half-spaces, which leads to a natural scheme for approximating $\text{Sep}$ by polytopes. A naive – but meaningful – way of measuring complexity of entanglement detection would be determining how well $\text{Sep}$ can be approximated by a polytope with $\leq N$ faces. More generally, it is known (Gurvits 2003) that deciding whether a state is entangled or separable is, in general, NP-hard. Later refinements have been due to Ioannou (2007), Gharibian (2010) and others; in particular some upper bounds were supplied by Brandão et al (2011).
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The Horodecki criterion

A more structured scheme of witnessing entanglement is given by the following ($M_d$ stands for the space of $n \times n$ complex matrices).

Theorem (The Horodecki criterion 1996)

A state $\rho \in \text{Sep}(\mathbb{C}^d \otimes \mathbb{C}^d)$ is entangled if and only if there is a positive map $\Phi : M_d \rightarrow M_d$ such that the operator $(\text{Id} \otimes \Phi)\rho$ is not positive semi-definite (one says that $\Phi$ witnesses the entanglement of $\rho$).
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A map $M_d \to M_d$ is completely positive (CP) if it is a positive linear combination of maps of the form $\Phi_A : X \mapsto AXA^\dagger$. CP maps cannot be witnesses since $\text{Id} \otimes \Phi_A = \Phi_{I \otimes A}$ is also positive.
Størmer’s theorem

In dimension 2 the cone of positive maps has a rather simple structure.

Theorem (Størmer 1963)

Any positive map $\Phi : \mathbb{M}_2 \rightarrow \mathbb{M}_2$ can be written as $\Phi = \Phi_1 + \Phi_2 \circ T$, where $\Phi_1, \Phi_2$ are CP maps and $T$ is the transposition on $\mathbb{M}_2$.

It follows that a state $\rho$ on $\mathbb{C}^2 \otimes \mathbb{C}^2$ is separable if and only if $(\text{Id} \otimes T)(\rho)$ is positive. The transposition is a universal witness.
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**Theorem (Skowronek, unpublished)**

Let \( d \geq 3 \). Let \( \mathcal{F} \) be a family of positive maps \( M_d \to M_d \) such that any entangled state \( \rho \) on \( \mathbb{C}^d \otimes \mathbb{C}^d \) is witnessed by an element of \( \mathcal{F} \). Then \( \mathcal{F} \) is infinite.

In fact, any closed universal family of witnesses must be uncountable.
Our main result

Denote by \( \bullet \) homotheties with respect to \( \rho_* := \frac{I_H}{\dim \mathcal{H}} \): \[
\bullet\langle t \rho := t\rho + (1 - t)\rho_*.\]

Say that a state \( \rho \) is **robustly entangled** if \( \frac{1}{2} \bullet \rho \) is entangled. Robustly entangled states remain entangled in the presence of randomizing noise.
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Suppose that \( \Phi_1, \ldots, \Phi_N \) are positive maps on \( M_d \) such that, for any robustly entangled state \( \rho \) on \( \mathbb{C}^d \otimes \mathbb{C}^d \), there is an index \( i \) such that \( (\text{Id} \otimes \Phi_i)(\rho) \) is not positive semi-definite. Then \( N \geq \exp(cd^3/\log d) \) for some universal constant \( c > 0 \).
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This shows that the set of separable states is complex, and not because of some fine features of its boundary. Results about NP-hardness of entanglement detection have usually focused on boundary effects.
Let $K \subset \mathbb{R}^n$ be a convex compact set with 0 in the interior. Define the vertical and facial dimensions of $K$ as

$$\dim_V(K) := \log \inf \{ \#\text{vertices}(P) : K \subset P \subset 4K \}$$

$$\dim_F(K) := \log \inf \{ \#\text{facets}(P) : K \subset P \subset 4K \}$$

where the infima run over all polytopes $P \subset \mathbb{R}^n$. The affine invariants $\dim_F$ and $\dim_V$ are measures of complexity.
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The \textit{affine invariants} \( \dim_F \) and \( \dim_V \) are measures of complexity. These are \textit{dual} concepts since \( \dim_F(K) = \dim_V(K^\circ) \), where \( K^\circ := \{ x \in \mathbb{R}^n : \langle x, y \rangle \leq 1, \forall y \in K \} \) is the \textit{polar} of \( K \).
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One has $\dim_F(K) = O(n)$ and $\dim_V(K) = O(n)$ if (say) the origin is the center of mass of $K$.

If $E \subset \mathbb{R}^n$ is a linear subspace, then $\dim_F(K \cap E) \leq \dim_F(K)$. 
Vertical and facial dimensions of convex bodies

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We have $\dim_F(B_2^n) = \dim_V(B_2^n) = \Theta(n)$ (where $B_2^n$ is the Euclidean ball).
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Another parameter is the asphericity of $K$ defined as

$$a(K) := \inf \{ R/r : rB_2^n \subset K \subset RB_2^n \}.$$
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For any convex body $K \subset \mathbb{R}^n$ containing the origin in the interior we have

$$\dim_F(K) \cdot \dim_V(K) \cdot a(K)^2 = \Omega(n^2).$$
The Figiel–Lindenstrauss–Milman bound

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This result is a consequence of the tangible version of Dvoretzky’s theorem due to Milman, which gives a sharp formula for the dimension of almost Euclidean sections of convex bodies.

If $K \subset \mathbb{R}^n$ is a convex body such that $rB_2^n \subset K$ and $M = M(K)$ denotes the average of the "norm" $\| \cdot \|_K$ over the sphere, then $K$ has lots of almost Euclidean sections of dimension $k = \Omega(nr^2M^2)$. 
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Thus the facial dimension of $K$ exceeds $ck$. Applying the same argument to $K^\circ$ and using the inequality $M(K)M(K^\circ) \geq 1$ yields the FLM bound.
The FLM bound – examples

We illustrate the FLM bound on some examples where it is sharp up to polylog factors

\[ \dim_V(K) \cdot \dim_F(K) \geq c \left( \frac{n}{a(K)} \right)^2 \]

| $K$       | dimension | $a(K)$ | $\dim_V(K)$ | $\dim_F(K)$ |
|-----------|-----------|--------|-------------|-------------|
| $B_2^n$   | $n$       | 1      | $\Theta(n)$ | $\Theta(n)$ |
| $[-1,1]^n$| $n$       | $\sqrt{n}$ | $\Theta(n)$ | $\Theta(\log n)$ |
| $\Delta_n$| $n$       | $n$    | $\Theta(\log n)$ | $\Theta(\log n)$ |

Recall that $B_2^n$ is the $n$-dimensional Euclidean ball, while $\Delta_n$ is the $n$-dimensional simplex.
Quantum-related examples

And here are some more examples related to entanglement detection.

| $K$       | dimension | $a(K)$ | $\text{dim}_V(K)$ | $\text{dim}_F(K)$ |
|-----------|-----------|--------|-------------------|-------------------|
| $\text{D}(\mathbb{C}^m)$ | $m^2 - 1$ | $m - 1$ |                    |                   |
| $\text{Sep}(\mathbb{C}^d \otimes \mathbb{C}^d)$ | $d^4 - 1$ | $d^2 - 1$ |                    |                   |

The value of $a(\text{D})$ is elementary to compute; the value of $a(\text{Sep})$ is due to Gurvits–Barnum (2002).
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- There are $\varepsilon$-nets in the sphere of $\mathbb{C}^d$ with $(2/\varepsilon)^{2d} = e^{\Theta(d \log(2/\varepsilon))}$ elements.
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- There are $\varepsilon$-nets in the sphere of $\mathbb{C}^d$ with $(2/\varepsilon)^{2d} = e^{\Theta(d \log(2/\varepsilon))}$ elements.
- For a well chosen (e.g., random) $\frac{1}{10}$-net $\mathcal{N}$ in the unit sphere of $\mathbb{C}^m$ and $P = \text{conv}\{\ketbra{\psi}{\psi} : \psi \in \mathcal{N}\}$, we have $\frac{1}{4} \bullet D \subset P \subset D$. 
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- Similarly, for any $\frac{1}{10d}$-net $\mathcal{N}'$ in the unit sphere of $\mathbb{C}^d$ and $P' = \text{conv}\{|\psi \otimes \varphi\rangle\langle\psi \otimes \varphi| : \psi, \varphi \in \mathcal{N}'\}$, we have $\frac{1}{4} \cdot \text{Sep} \subset P' \subset \text{Sep}$. 
We now complete the table.

| $K$               | dimension       | $a(K)$ | $\dim V(K)$ | $\dim F(K)$ |
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The set $D$ is self-dual (or, more precisely, $D^\circ = (−m) \bullet D$), so its facial dimension equals its vertical dimension.
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| $\text{Sep}(\mathbb{C}^d \otimes \mathbb{C}^d)$ | $d^4 - 1$ | $d^2 - 1$ | $\Theta(d \log d)$ | $\Omega(d^3 / \log d)$ |

The set $D$ is self-dual (or, more precisely, $D^\circ = (-m) \cdot D$), so its facial dimension equals its vertical dimension.

The lower bound on the facial dimension of $\text{Sep}$ follows from the Figiel–Lindenstrauss–Milman inequality

$$\dim_V(\text{Sep}) \cdot \dim_F(\text{Sep}) \geq c \left( \frac{d^4 - 1}{d^2 - 1} \right)^2 > cd^4$$
Quantum-related examples, II

We now complete the table.

| $K$              | dimension | $a(K)$ | $\dim_V(K)$ | $\dim_F(K)$ |
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However, it is conceivable that we actually have $\dim_F(\text{Sep}) = \Theta(d^4)$. 
Sketch of the proof of the theorem

Let $\Phi_1, \ldots, \Phi_N$ be $N$ positive maps on $\mathbb{M}_d$ with the property that for every robustly entangled state $\rho$, there exists an index $i$ such that $(\Phi_i \otimes \text{Id})(\rho)$ is not positive. This hypothesis is equivalent to the following inclusion

$$\bigcap_{i=1}^N \{ \rho \in D(\mathbb{C}^d \otimes \mathbb{C}^d) : (\text{Id} \otimes \Phi_i)(\rho) \text{ is PSD} \} \subset 2 \cdot \text{Sep}.$$
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Next, for simplicity, let us assume first that each $\Phi_i$ is trace-preserving, i.e., $\Phi_i(\rho_*) = \rho_*$. Consider the convex body

$$K = D \cap \bigcap_{i=1}^{N} (\text{Id} \otimes \Phi_i)^{-1}(D)$$

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which satisfies $\text{Sep} \subset K \subset 2 \cdot \text{Sep}$. Note the trace-preserving condition assures that for all $i$’s we are $\bullet$-dilating with respect to the same point.
Sketch of the proof of the theorem, II

Since the facial dimension of $D(\mathbb{C}^d \otimes \mathbb{C}^d)$ is of order $d^2$, there exists a polytope $P$ with at most $\exp(Cd^2)$ facets such that $\frac{1}{2} \bullet D \subset P \subset D$. Then the polytope

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satisfies $\frac{1}{2} \bullet \text{Sep} \subset \frac{1}{2} \bullet K \subset Q \subset K \subset 2 \bullet \text{Sep}$. Since

$$\#\text{facets}(P_1 \cap P_2) \leq \#\text{facets}(P_1) + \#\text{facets}(P_2),$$

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The general situation (without the trace-preserving restriction) is handled similarly starting with the assumption that $(1 - \frac{1}{2d}) \cdot D \subset P \subset D$. 
Conclusion

We illustrated the complexity of robust entanglement by showing that super-exponentially many positive maps are needed to detect it – at least if used non-adaptively/without reflection.
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The proof is via a facet-counting argument (even if the set of separable states is not a polytope itself) and ultimately relies on the bound due to Figiel–Lindenstrauss–Milman which asserts that – between (i) the number of vertices, (ii) the number of facets, and (iii) asphericity – complexity must lie somewhere.

Can this approach be used to handle other problems in complexity theory?

Some other directions in which this work can be continued are:

• Upper bounds; in particular, what is the order of $d^{F(Sep)}$?

• Less/more robust entanglement, i.e., replacing $\frac{1}{2}$ with $\varepsilon \in (0, 1)$?

• What if we use witnesses $\Phi : M_d \to M_m$, where $m = poly(d)$?

• The multipartite or "unbalanced" ($H = C^d \otimes C^m$) setting
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