Research Article

Analysis of One-Dimensional Consolidation for Unsaturated Soils under Piecewise Cyclic Loading

Yuanchun Huang,1 Sidong Shen,1 Lei Wang,1 Tianyi Li,1 and Xianlei Fu2

1School of Urban Railway Transportation, Shanghai University of Engineering Science, Shanghai 201620, China
2Jiangsu Key Laboratory of Urban Underground Engineering & Environmental Safety Institute of Geotechnical Engineering, Southeast University, Nanjing 210096, China

Correspondence should be addressed to Yuanchun Huang; yuanchun_huang@163.com

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1. Introduction

Traditional researches upon the soil foundations are mainly based on saturated soil mechanics, and comprehensive conclusions have been made to guide practical engineering [1]. However, the soils are always unsaturated in natural stratum [2], in which the air and water phases are widely included. Different from that, in saturated soils, the consolidation problem of soils under the unsaturated condition is much more complex to solve, because of the coexistence of air and water phases [3]. Thus, the saturated soil mechanics cannot be applied to study the consolidation of unsaturated ground, or unaccepted errors will occur. Moreover, cyclic loading is often endured by some foundations of buildings or structures (i.e., highways, storage tanks or storage yards, etc.). For example, triangular cyclic loading can represent repetitive process of loading and unloading. In fact, relative research has been conducted in the field of saturated soils with piecewise cyclic loading. For instance, by taking piecewise cyclic loading into consideration, Liang et al. [4] studied the 1D consolidation of saturated foundation under the semipermeable boundary conditions. However, there is not a systematic research framework upon the consolidation of unsaturated soils under piecewise cyclic loading. Therefore, it is of great significance to explore the effects of piecewise cyclic loading on the 1D consolidation of unsaturated soil, since unsaturated soil is widely distributed all over the world.

So far, the consolidation theory of unsaturated soils has been studied by peer scholars. By considering the existence of occluded bubbles, Biot [5] first raised a general consolidation theory of soils under the partially saturated condition. However, there is not a systematic research framework upon the consolidation of unsaturated soils under piecewise cyclic loading. Therefore, it is of great significance to explore the effects of piecewise cyclic loading on the 1D consolidation of unsaturated soil, since unsaturated soil is widely distributed all over the world.
dimensional conditions. Lorent and Khalili [9] applied the mixture theory to three-phase media and put forward a determined three-dimensional model of the constitutive relation relationship of unsaturated soils. Specifically, on basis of the theory of two stress variables of unsaturated soils proposed by Fredlund and Morgenstern [10], Fredlund [10] and Fredlund and Hasan [11] proposed a systematic consolidation theory of unsaturated soils. Wang et al. [12] developed permeameter function for measurement of coefficient of permeability of unsaturated sandy specimens. Zhang et al. [13] simulated the moisture migration of unsaturated clay embankments in southern China considering stress state by a numerical method.

Current solutions for analyzing the consolidation of unsaturated foundations are mostly based on the research framework by Fredlund and his coworkers. The authors of [14–17] and Li et al. [18] studied the 1D consolidation behaviors of unsaturated stratum by analytical and semi-analytical solutions considering various boundary conditions and external loadings. Zhou and Zhao [19] studied 1D consolidation problem of unsaturated foundation under different initial conditions and time-varying loading by a simplified analytical method. Ho et al. [20] obtained analytical solutions for 1D consolidation of unsaturated soils under step loading by adopting the eigenfunction expansion method. The authors of [21–23] investigated a series of semianalytical solutions for 1D consolidation of soils with semipermeable drainage boundary and exponentially time-growing drainage boundary. Zhao et al. [24] investigated the 1D consolidation characteristics of unsaturated foundation considering the semipermeable boundary condition with explicit solutions. By taking the lateral and vertical drainage, Huang et al. [25] presented the general analytical solutions of tow-dimensional consolidation of unsaturated soils. Li et al. [26] and Huang et al. [3, 25] analyzed the axisymmetric consolidation of unsaturated foundation with an analytical method. In summary, a lot of studies on the consolidation of unsaturated soil have been conducted with the continuous loading (i.e., instantaneous, exponential, ramp, and sinusoidal loading). However, the more practical piecewise cyclic loadings are rarely considered in the investigation of the consolidation behavior in unsaturated soils.

This paper aims to derive semianalytical solutions to analyze the consolidation of unsaturated foundation under piecewise cyclic loading (namely, triangular and trapezoidal cyclic loadings). In the main derivative process, the Laplace transform method [27] is applied to transfer the loading functions into continuous expressions in the Laplace domain. The verification work is carried out by comparing the results of this paper and those in the existing literature. Finally, the 1D consolidation characteristics of unsaturated soils under piecewise cyclic loading are investigated against ratio $k_a/k_w$, depth $z$ and loading parameters.

2. Mathematical Modeling

According to the dissipation of air and water in unsaturated soils, the calculation modeling of the 1D consolidation is shown in Figure 1. In this model, the soil layer is considered as an infinitely horizontal layer with thickness $h$ under the external loading $q(t)$ varying with time. $k_a$ and $k_w$ are the permeability coefficients of air and water phases in the soil layer, respectively. $z$ is the vertical coordinate.

2.1. Basic Assumptions. The following basic assumptions are adopted in this paper [20, 25, 27]:

(1) The flow of air and water phases are independent and continuous
(2) Solid particles and water phase are incompressible
(3) The dissipation of excess pore pressures and the deformation of soil occur only in the vertical direction
(4) The permeability coefficients kept constant in the consolidation process

2.2. Governing Equations. According to the widely accepted consolidation theory of unsaturated soils [11], governing equations are listed as follows:

$$\frac{\partial u_a}{\partial t} = -C_a \frac{\partial u_a}{\partial t} - C_a \frac{\partial u_a}{\partial z} + C_a \frac{\partial q}{\partial t},$$

$$\frac{\partial u_w}{\partial t} = -C_w \frac{\partial u_w}{\partial t} - C_w \frac{\partial u_w}{\partial z} + C_w \frac{\partial q}{\partial t},$$

where $t$ is time and the intermediate coefficients in equations (1) and (2) are defined as

$$C_a = \frac{m_{1k}^a - m_{1w}^a}{m_{2w}^a} - \left( \frac{u_{atm} n_0 (1 - S_{r0})}{(T_{atm})^2} \right),$$

$$C_w = \frac{k_w}{\gamma_w m_2},$$

$$C_a = \frac{k_a}{\gamma_a m_2},$$

$$C_a = \frac{k_a}{\gamma_a m_2},$$

where $C_a$ and $C_w$ are interactive constants of air and water phases, respectively; $C_a^a$ and $C_a^w$ are coefficients of volume and vertical stress variations of air phase, respectively; $C_w^a$ and $C_w^w$ are coefficients of volume and vertical stress variations of water phase, respectively; $q$ is the total vertical stress; $u_a$ and $u_w$ denote the excess pore air and water.
2.2. Initial Conditions

\[ u_a(z, 0) = u_{a0}, \]
\[ u_w(z, 0) = u_{w0}, \]

where \( u_{a0} \) and \( u_{w0} \) are initial excess pore air and water pressures.

2.2.2. Boundary Conditions. The top and bottom surfaces are assumed to be completely permeable and impermeable to air and water phases, respectively, and details are listed as follows [14, 28]:

Top boundary:
\[ u_a(0, t) = 0, \]
\[ u_w(0, t) = 0. \]

Bottom boundary:
\[ \frac{\partial u_a(h, t)}{\partial z} = 0, \]
\[ \frac{\partial u_w(h, t)}{\partial z} = 0. \]

2.3. Applied Loading. The schematic diagram of triangular cyclic loading is expressed in equations (9a) and (9b) and illustrated in Figure 2, respectively:

Triangular cyclic loading:
\[ q(t) = \begin{cases} q_0 + at, & 0 < t < T, \\ q_{\text{max}} - a(t - T), & T < t < 2T, \\ q_{\text{max}} - b(t - 3T), & 3T < t < 4T, \\ q(t + 4nT) = q(t), & \end{cases} \]

where \( n = 1, 2, \ldots \), \( q_0 \) and \( q_{\text{max}} \) are the initial and the maximum loading, \( a \) is the loading rate of triangular cyclic loading, and \( 2T \) represents the time of one cyclic change of the loading.

Trapezoidal cyclic loading:

The schematic diagram of trapezoidal cyclic loading is shown in Figure 3, and the expression of trapezoidal cyclic loading is as follows:

\[ q(t) = \begin{cases} q_0 + bt, & 0 < t < T, \\ q_{\text{max}}, & T < t < 3T, \\ q_{\text{max}} - b(t - 3T), & 3T < t < 4T, \\ q(t + 4nT) = q(t), & \end{cases} \]

where \( b \) denotes the loading rate of trapezoidal cyclic loading.

3. Derivation of Semianalytical Solutions

Performing the Laplace transform on equations (1) and (2) gives

\[ \tilde{u}_a(z, s) = \frac{C_w}{sC_w} \frac{\partial^2 \tilde{u}_a(z, s)}{\partial z^2} - \frac{1}{C_w} \tilde{u}_a(z, s) \]
\[ + \frac{u_{a0} + C_wu_{a0}}{sC_w} + \frac{C_w}{sC_w} Q(s), \]

where \( Q(s) \) is the result of Laplace transform of \( \partial q / \partial t \) upon time \( t \).

Substituting equation (11) into equation (2) gives

\[ d_1 \frac{\partial^4 \tilde{u}_a(z, s)}{\partial z^4} + d_2 \frac{\partial^2 \tilde{u}_w(z, s)}{\partial z^2} + d_3 \tilde{u}_w(z, s) - d_4 = 0, \]
where

\[ a_1 = \frac{C_s^2C_w^2}{sC_w}, \]
\[ a_2 = C_s^2 + \frac{C_s^2}{C_w}, \]
\[ a_3 = s(1 - C_w), \]
\[ a_4 = \left( \frac{1 - C_w}{C_w} \right) \frac{C_w}{C_w} \frac{C_w}{C_w} + \left( C_w - \frac{C_w}{C_w} \right) Q(s). \]

According to the general solution of the fourth-order differential equation (12), we have

\[ \bar{u}_w(z, s) = C_1 e^{\xi z} + C_2 e^{-\xi z} + D_1 e^{\eta z} + D_2 e^{-\eta z} - \frac{a_4}{a_4} \]

where

\[ \xi = \sqrt{(-a_2 - \sqrt{a_2^2 - 4a_1a_3})/(2a_1)}, \quad \eta = \sqrt{(-a_2 + \sqrt{a_2^2 - 4a_1a_3})/(2a_1)}, \quad \text{and} \quad C_1, C_2, D_1, \text{and} \ D_2 \]

are arbitrary functions of \( s \), to be determined by the boundary conditions.

Conducting the first-order derivative of equation (14) on \( z \), we have

\[ \frac{\partial \bar{u}_w(z, s)}{\partial z} = C_1 \xi e^{\xi z} - C_2 \xi e^{-\xi z} + D_1 \eta e^{\eta z} - D_2 \eta e^{-\eta z}. \] (15)

By taking the second-order derivative upon \( z \), equation (14) can be rearranged as

\[ \frac{\partial^2 \bar{u}_w(z, s)}{\partial z^2} = C_1 \xi^2 e^{\xi z} + C_2 \xi^2 e^{-\xi z} + D_1 \eta^2 e^{\eta z} + D_2 \eta^2 e^{-\eta z}. \] (16)

Combining equations (11), (15), and (16) gives

\[ \bar{u}_w(z, s) = C_1 a_5 e^{\xi z} + C_2 a_5 e^{-\xi z} + D_1 a_6 e^{\eta z} + D_2 a_6 e^{-\eta z} + a_7. \] (17)

where

\[ a_5 = \frac{C_s^2 \xi^2 - 1}{C_w}, \]
\[ a_6 = \frac{C_s^2 \eta^2 - 1}{C_w}, \]
\[ a_7 = \frac{a_4^0}{s} + \frac{a_4}{a_4} + \frac{a_4^0}{a_4} \frac{C_w^2}{sC_w} + \frac{C_w^2}{sC_w}. \]

Performing the first derivative of equation (17) of \( z \) gives

\[ \frac{\partial \bar{u}_w(z, s)}{\partial z} = C_1 a_5 \xi e^{\xi z} - C_2 a_5 \xi e^{-\xi z} + D_1 a_6 \eta e^{\eta z} - D_2 a_6 \eta e^{-\eta z}. \] (19)

Substitute equations (14), (15), (17), and (19) into equations (5)–(8) and solve a set of equations of \( C_1, C_2, D_1, \text{and} \ D_2 \). Then, substituting the expressions of \( C_1, C_2, D_1, \text{and} \ D_2 \) into equations (14) and (17) has

\[
\bar{u}_w(z, s) = \frac{(-a_4 a_{5} - a_{3} a_{2}) a_{3} \sinh [\xi (h - z)]}{a_{3} (a_{5} - a_{5}) (1 + e^{2h}) \cosh (\xi h)} + \frac{(-a_4 a_{5} + a_{3} a_{2}) a_{3} \sinh [\eta (h - z)]}{a_{3} (a_{5} - a_{5}) (1 + e^{2h}) \cosh (\eta h)} + a_{7},
\]

(20)

\[
\bar{u}_w(z, s) = \frac{(-a_4 a_{5} - a_{3} a_{2}) a_{3} \sinh [\xi (h - z)]}{a_{3} (a_{5} - a_{5}) (1 + e^{2h}) \cosh (\xi h)} + \frac{(-a_4 a_{5} + a_{3} a_{2}) a_{3} \sinh [\eta (h - z)]}{a_{3} (a_{5} - a_{5}) (1 + e^{2h}) \cosh (\eta h)} - \frac{a_4}{a_4}.
\]

(21)

\[
Q(s) = \frac{2q_{\text{max}}}{sT \left( \cot (sT/2) + \coth (3sT/2) \right)} - q_0.
\]

(23)

After substituting equations (22) and (23) into intermediate variables \( a_4 \) and \( a_7 \), the semianalytical solutions for excess pore pressures subjected to different piecewise cyclic loading in Laplace domain can be obtained.

Based on the constitutive modeling of unsaturated soil, we have...
\[
\frac{\partial e_r}{\partial t} = m_{1k} \frac{\partial (\sigma - u_a)}{\partial t} + m_2 \frac{\partial (u_a - u_w)}{\partial t}, \tag{24}
\]

where \( m_{1k}^r = m_{1k}^a + m_{1k}^w, m_2^r = m_2^a + m_2^w \).

Combining equation (24) into the equation of settlement for unsaturated soils produces

\[
\bar{w}(s) = (m_2^r - m_{1k}^r) \int_0^h \bar{u}_a dz - m_2^r \int_0^h \bar{u}_w dz + \frac{h}{s} \left[ (m_{1k}^r - m_2^r)u_a^0 + m_2^r u_w^0 + m_{1k}^r Q(s) \right]. \tag{25}
\]
Equations (20), (21), and (25) are the final semianalytical solutions for excess pore pressures and settlement of 1D consolidation of unsaturated foundation in the Laplace domain.

4. Verification and Example

In this part, the obtained solutions are firstly proofed reliable, and a case study is presented to evaluate the consolidation characteristics of the 1D unsaturated foundation under various cyclic loadings. It is assumed that the unsaturated stratum is infinite along the horizontal direction, and the necessary values of parameters are adopted from those in Qin et al. [16] and Wang et al. [21]: \( h = 8 \) m, \( n = 50\% \), \( S_{o0} = 80\% \), \( m'_k = -2.5 \times 10^{-4} \) kPa\(^{-1}\), \( m'_w = 0.4m'_k \), \( m''_k = 0.2m'_k \), \( m''_w = 4m'_k \), \( k_w = 10^{-10} \) m/s, \( \gamma_w = 9.807 \) kN/m\(^3\), \( g = 9.807 \) m/s\(^2\), and \( u_{atm} = 101.3 \) kPa.

4.1. Verification. The validation work is conducted with an instantaneous loading \((q = 100 \) kPa\) acting uniformly on the foundation surface, producing the initial excess pore pressures \( u'_0 = 20 \) kPa and \( u''_0 = 40 \) kPa [11]. Afterwards, the current solutions (CS) in this paper are compared with the existing solutions (ES) in literature [16].
The comparison results of the proposed solutions and those in Qin et al. [16] are shown in Figure 4. It can be observed that the obtained results in this paper agree well with those in the existing literature; thus, it can be concluded that the current solutions are reliable.

4.2. Example and Analysis. An example is applied to investigate the consolidation patterns of the unsaturated soils against the ratios of $k_a/k_w$, the depth $z$ and the loading parameters under different piecewise cyclic loadings. In the discussion section, the ratio $k_a/k_w$ keeps constant (i.e., $k_a/k_w = 1$), and the effects of different $k_a/k_w$ and loading parameters on consolidation are investigated at $z = 6$ m. The initial loading is $q_0 = 100$ kPa, and the time for a cyclic change is $T = 40$ days, when the influences of different depths and ratio $k_a/k_w$ are taken into consideration.

Figures 5 and 6 show the dissipation of excess pore-air pressure under piecewise cyclic loading (i.e., triangular and trapezoidal cyclic loadings) with different loading parameters, and the figures are partially enlarged to demonstrate the details of the curves. It can be analyzed that the dissipation curves do not significantly change with loading parameters, and excess pore-air pressure dissipates almost along the same path in the latter stage. Excess pore-air pressure is positively correlated with cyclic loading, and the pressure reaches the maximum when the loading increases to the peak value. Moreover, the smaller the loading parameter is, the earlier the curves oscillate. The shapes and periods of the dissipation curves are consistent with those of the piecewise cyclic loading.

Figures 7 and 8 show the dissipation of excess pore-water pressures under piecewise cyclic loading against different loading parameters. Similarly, the shapes and periods of the dissipation curves are consistent with those of the piecewise cyclic loading. Excess pore-water pressure is positively correlated with cyclic loading. From Figures 7 and 8, the dissipation curve shows nonlinear increase with the linear increase of loading. The smaller the loading period is, the earlier the cyclical change occurs, the faster the excess pore-water pressure increases, and the larger the values of the maximum and the minimum are. As the water in the unsaturated soils drains, the maximum values of the excess pore-water pressure start to decrease after $10^7$ s.

Figures 9 and 10 show the settlement curves under piecewise cyclic loading against different loading parameters. Overall, the curves of settlement show obvious periodicity under the cyclic loading. The curves of settlement subjected to triangular and trapezoidal cyclic loading show nonlinear increase with the linear increase of loading in the early stage. As the settlement gradually increases under
external loading, the maximum value of the settlement also increases, while settlement tends to be stable in the latter stage. The smaller the loading period is, the earlier the settlement stabilizes, and the faster the curve changes. Amplitudes of the settlement curves under various loading parameters are almost the same.

Variations of excess pore-air pressure under piecewise cyclic loading by adopting different ratios $k_d/k_w$ are demonstrated in Figures 11 and 12. The figures are partially enlarged to show the curves over a period of $10^7$ to $10^8$ s. In general, the ratio $k_d/k_w$ has great influence on the dissipation of excess pore-air pressure, and larger air permeability (or $k_d/k_w$) means faster dissipation rate of the excess pore-air pressure and earlier occurrence of dissipation.

Dissipation curves of excess pore-water pressures under different piecewise cyclic loading against different values of $k_d/k_w$ are shown in Figures 13 and 14. It can be observed that $k_d/k_w$ has little impact on the variation of excess pore-water pressure, because $k_w$ remains constant. In terms of the dissipation curves of excess pore-water pressure subjected to triangular and trapezoidal cyclic loading, the dissipation curve shows nonlinear increase at the beginning as the loading increases. The dissipation curves also show the same periodicity with the cyclic loading, and excess pore-water pressure positively correlates with cyclic loading. In addition, smaller $k_d/k_w$ leads to larger maximum and minimum values of dissipation curve.

The curves of settlement changing with time under piecewise cyclic loading with different $k_d/k_w$ are shown in
Figures 15 and 16. It is obvious that a nonlinear increase occurs with the linear increase of loading in the early stage, while the effect of $k_a/k_w$ on the settlement curves is less significant. Larger $k_a/k_w$ gives rise to larger maximum value of settlement. All the curves of settlement under cyclic loading show periodic changes matching the cyclic loading.

Figures 17 and 18 illustrate the variation of excess pore-air pressure under different depths $z$. Overall, depth $z$ has great influence on the excess pore-air pressure, and the smaller the $z$, the earlier and the faster the air pressure to dissipate. Moreover, the excess pore-air pressure has obvious differentiation in the early stage. Different $z$ has little influence on dissipation, and the excess pore-air pressure dissipates along almost the same path in the latter period.

Figures 19 and 20 demonstrate the dissipation of excess pore-water pressure under different piecewise cyclic loading. The periodic variation of excess pore-water pressure is consistent with those of piecewise cyclic loading. And smaller depth $z$ leads to smaller maximum and minimum values. In terms of the dissipation curves of excess pore-water pressures subjected to triangular and trapezoidal cyclic loading, the dissipation curves show nonlinear increase with the linear increase of cyclic loading at early stage. In latter stages, there is a clear downward trend in the dissipation curve of excess pore-water pressure with smaller depth $z$.

5. Conclusion

In this paper, semianalytical solutions concerning the one-dimensional consolidation of unsaturated foundation in Laplace domain are obtained under piecewise cyclic loading, and verification work is then carried out to prove the correctness of semianalytical solutions. Examples are presented to analyze the consolidation properties of 1D consolidation characteristics for unsaturated soils under triangular and trapezoidal piecewise cyclic loadings. Main conclusions of this paper are as follows:

1. **Cyclic loading types**
   - The excess pore pressures and settlement of 1D unsaturated foundation change periodically under the piecewise cyclic loading, and the shape and period of dissipation curve are matched with those of the piecewise cyclic loading.

2. **Loading parameters**
   - Different loading parameters represent different periods of piecewise cyclic loading, which will affect the time for excess pore pressure to reach maximum value. Moreover, various loading parameters have little influence on excess pore-air pressure. Therefore, it is necessary to make proper construction plan to avoid accidents (i.e., sudden settlement) in real engineering.

3. **Permeability coefficients**
   - The ratio $k_a/k_w$ has great influence on the dissipation of excess pore-air pressure, but it has little influence on the dissipation of excess pore-air pressure.

**Data Availability**

All data, models, or code generated or used during the study are available from the corresponding author (Yuanchun Huang, e-mail: yuanchun_huang@163.com) upon request.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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