Description of Nonlinear Phenomena in the Atmospheric Dynamics through Linear Wave type Equations

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Abstract

The paper tackles with a procedure which allow to extend some linear, wave type models currently used in describing phenomena appearing in atmosphere to the study of nonlinear models. More concretely, we present a practical way to generate the largest class of (1 + 1)-dimensional second order partial differential equations (pdes) of a given form which could be reduced to an imposed ordinary wave type equation. This class generalize the ordinary differential equation describing the equatorial trapped waves generated in a continuously stratified ocean and will be obtained following the Lie symmetry and similarity reduction procedures. Moreover, some concrete nonlinear second order differential equations will be proposed as possible candidates for replacing more complicated, nonintegrable systems, as the Rossby type equation.

Keywords: Nonlinear dynamical systems, Lie symmetries, Similarity reduction procedure, Rossby type symmetries.

1 Introduction

A rich variety of complex phenomena occurring in many physical fields, including the atmospheric dynamics, are described by linear differential equations which allow a simple handling of the constraints which appear in the system’s evolution. Although, the linearized models often do not adequately describe the dynamics of the processes as a whole and it is very simple to shift the system to a region in which the linear behavior is no longer valid. This is why, in order to capture the real behavior, to accurately estimate and control the complex systems in all their regimes, nonlinear models must be defined. In this case, the linear differential equations could appear as approximations to the nonlinear systems, valid under restricted conditions.

The price to be paid when nonlinearity is taken into consideration appears in the investigation of the exact solutions of the attached equations. There are not standard methods of solving nonlinear differential equations, they are usually depend on the form of the equations and on their particular symmetries. Many interesting nonlinear models have been proposed over the last years and a lot of methods have been developed in order to find solutions of equations describing these nonlinear phenomena. Some of the most important methods are the inverse scattering method, the Darboux and Bäcklund transformations, the Hirota bilinear method, the Lie symmetry analysis, etc. By applying these methods, many types of specific solutions have been obtained. For example, solitary waves or solitons, which have no analogue for linear partial differential equations, are very important for the nonlinear dynamical systems.

In this paper we shall concentrate our attention to the Lie group method. It is well-known that this method is a powerful and direct approach to construct many types of exact solutions of nonlinear differential equations, such as soliton solutions, power series solutions, fundamental solutions, and so on. The existence of the operators associated with the Lie group of infinitesimal transformations allows the reduction of equations to simpler ones. The similarity reduction method for example is an important way of transforming a (1 + 1)–dimensional pde into an ordinary differential one. We shall concretely consider the inverse symmetry problem and
we shall generate the largest class of second order \((1 + 1)\)-dimensional pdes which generalize the ordinary, wave type, differential equation describing the equatorial trapped waves generated in a continuously stratified ocean. Practically, four types of waves appears in this case and have to be found among the solutions of the equation: Kelvin waves, Rossby waves, inertia-gravity waves and mixed Rossby-gravity waves. In the Boussinesq approximation and on an equatorial \(\beta\)-plane, the equation which describe the \(m\)-th oscillation mode of the wave’s vertical velocity \(\phi_m(z)\) on the direction \(z\) has the form \([23]\):

\[
\frac{\partial^2 \phi_m(z)}{\partial z^2} + \frac{N^2_m(z)}{C^2_m} \phi_m(z) = 0 \tag{1}
\]

One consider for the velocity the boundary conditions:

\[
\phi_m(z = -H) = 0 \quad \text{(ocean floor)}
\]
\[
\phi_m(z = 0) = 0 \quad \text{(ocean surface)}
\]

In the equation \([1]\), \(C_m\) is a constant and \(N_m(z)\) represents the "buoyancy" frequency. Measurements made during El Niño events \([23]\) show that the buoyancy frequency \(N_m(z)\) has strong variations with the water depth close to the surface and practically vanishes for higher depths. We notice that in the first case (at the surface, \(z \in [0, 300] \text{ m}\) one can approximate \(N_m(z)\) with an averaged value around \(\overline{N}(z) = 2 \cdot 10^{-4} \text{ m} \cdot \text{s}^{-1}\)). So, the equation \([1]\) can be linearized in one of the following forms:

\[
\ldots \phi(z) = 0; z \geq 300 \tag{2}
\]
\[
\ldots \phi(z) + k^2 \phi(z) = 0; z \in [0, 300]; k \equiv \frac{N}{C} = \text{const.} \tag{3}
\]

In this paper, we shall consider the two previous wave type equations and we shall see how they can be extended towards \((1 + 1)\)-second order differential equations with the same group of symmetries as the initial ordinary wave type equations have.

The outline of this paper is as follows: after this introductory notes, in Section 2, we shall obtain the general determining system for a chosen class of \((1+1)\)-dimensional models. The system will be generated by using the Lie symmetry approach and by asking for an imposed form of the similarity reduction equation. More exactly, we shall generate a class of \((1 + 1)\)-second order differential equations which by similarity reduction come to the wave forms \([2] and \([3]\). The general results of the second section will be particularized in Section 3, when concrete examples of two dimensional equations with similar solutions as the ordinary wave equations \([2] and \([3]\) will be generated. Moreover, we shall compute the form of the second order partial differential equation which admit an imposed form of symmetry, specific for the two dimensional Rossby type equation. So we shall be able to replace the study of this last strongly nonintegrable equation with a simpler class of equations observing similar symmetries. Some concluding remarks will end the paper.

2 Determining equations for the Lie symmetry group

Let us consider the class of general dynamical systems described in a \((1 + 1)\)-dimensional space \((x, t)\) by a second order partial differential equation of the form:

\[
u_t = A(x, t)u_{xx} + B(x, t)u_x + C(x, t)u \Leftrightarrow \Omega(x, t, u, u_x, u_t, u_{2x}) = 0 \tag{4}\]

Our aim is to select from the general dynamical systems described by \([1]\) the class of differential equations which admit a similarity reduction to wave type equations of the form \([2] and \([3]\). As feed-back, the solution of the wave equations will be used in order to obtain a solution for \([1]\). The procedure that will be followed firstly implies to obtain the system of determining equations for
the Lie symmetry group of (1). Then, an additional system of partial differential equations will be
generated by imposing that (1) possess a reduced similarity equation of the wave type. Finally, this
last system and the Lie determining equations will be solved and the coefficient functions \( A(x, t), B(x, t), C(x, t) \) will be obtained.

In this section we shall apply the Lie symmetry approach for the equation (1). Let us consider
a one-parameter Lie group of infinitesimal transformations:

\[
\bar{x} = x + \varepsilon \xi(t, x, u), \quad \bar{t} = t + \varepsilon \varphi(t, x, u), \quad \bar{u} = u + \varepsilon \eta(t, x, u) \tag{5}
\]

with a small parameter \( \varepsilon \ll 1 \). The Lie symmetry operator associated with the above group of
transformations can be written as:

\[
U(x, t, u) = \varphi(x, t, u) \frac{\partial}{\partial t} + \xi(x, t, u) \frac{\partial}{\partial x} + \eta(x, t, u) \frac{\partial}{\partial u} \tag{6}
\]

The second order equation \( \Omega(x, t, u, u_x, u_t, u_{2x}) = 0 \) of the form (1) is invariant under the action of
the operator (6) if and only if the following condition [11] is verified:

\[
U^{(2)}(\Omega) \big|_{\Omega=0} = 0 \tag{7}
\]

where \( U^{(2)} \) is the second extension of the generator (6). A concrete computation shows that the
coefficient functions from (1) and (6), \( A(x, t), B(x, t), C(x, t), \varphi(x, t, u), \xi(x, t, u), \eta(x, t, u) \), must satisfy the equation:

\[
(\varphi A_t + \xi A_x)u_{2x} + (\varphi B_t + \xi B_x)u_x + \varphi C_t u + \xi C_x u + C \eta + B \eta^x - \eta^t + A \eta^{2x} = 0 \tag{8}
\]

The coefficient functions \( \eta^x, \eta^t, \eta^{2x} \) appear in the process of extension of \( U \) towards \( U^{(2)} \) and their
general expressions are given in [11]. Using these expressions in (8) and asking for the vanishing of
the coefficients of each monomial in the derivatives of \( u(t,x) \), we obtain the following differential system:

\[
\begin{align*}
\varphi_x &= 0; \quad \varphi_u = 0; \quad \xi_u = 0; \quad \eta_{2u} = 0; \quad \varphi A_t + \xi A_x + A \varphi_t - 2A \xi_x = 0; \\
- \varphi B_t - \xi B_x + B \xi_x - \xi_t - B \varphi_t - 2A \eta_{xu} + A \xi_{2x} &= 0 \tag{9} \\
- \varphi C_t u - \xi C_x u - C \eta - B \eta_x + \eta_t + C \eta_{u} u - \varphi_t C u - A \eta_{2x} &= 0
\end{align*}
\]

The first four equations of the system (9) lead, for coefficient functions \( \varphi(x, t, u), \xi(x, t, u), \eta(x, t, u) \),
to the following reduced dependences:

\[
\varphi = \varphi(t), \quad \xi = \xi(x,t), \quad \eta = M(x,t)u \tag{10}
\]

Consequently, the remaining equations of (9) become:

\[
\begin{align*}
\varphi A_t + \xi A_x + A \varphi_t - 2A \xi_x &= 0; \\
- \varphi B_t - \xi B_x + B \xi_x - \xi_t - B \varphi_t - 2A M_x + A \xi_{2x} &= 0 \tag{11} \\
- \varphi C_t - \xi C_x - B M_x + M_t - \varphi_t C - A M_{2x} &= 0
\end{align*}
\]

with 6 unknown functions: \( A(x, t), B(x, t), C(x, t) \) provided by the evolutionary equation (1) and \( \varphi(t), \xi(x, t), \eta(x, t, u) \) introduced by the symmetry group of transformations (5) and described by
the relations (10).
3 Similarity reduction procedure

Let us consider now the similarity reduction procedure. In this section, some particular choices for the system (11) will be considered. We shall find equations describing concrete dynamical systems which admit reduction through the similarity procedure to ordinary wave type equations of the form (2) and (3). For the moment, we restrict the forms (10) of the infinitesimals $\varphi(t), \xi(x,t), \eta(x,t,u)$ to the following separable expressions:

$$
\varphi = \varphi(t), \quad \xi = \xi(x,t) = \xi_1(x)\xi_2(t), \quad \eta = M(x,t)u = M_1(x)M_2(t)u
$$

(12)

The Lie operator (6) becomes:

$$
U(x,t,u) = \varphi(t)\frac{\partial}{\partial t} + \xi_1(x)\xi_2(t)\frac{\partial}{\partial x} + M_1(x)M_2(t)u\frac{\partial}{\partial u}
$$

(13)

The general expressions of the invariants could be found if we should consider the characteristic equations associated with the new generator (13). These equations are:

$$
dt = \frac{dx}{\xi_1(x)\xi_2(t)} = \frac{du}{M_1(x)M_2(t)u}
$$

(14)

By integrating the previous equations, two invariants are obtained with the following expressions:

$$
I_1 = \exp\left(\int \frac{1}{\xi_1(x)}dx - \int \frac{\xi_2(t)}{\varphi(t)}dt\right), \quad I_2 = u\exp\left(-\frac{M_2(t)}{\xi_2(t)}\int \frac{M_1(x)}{\xi_1(x)}dx\right)
$$

(15)

In the similarity reduction procedure two similarity variables have to be considered:

$$
I_1 = z, \quad I_2 = \phi(z)
$$

(16)

The invariants (16) allow us, by an appropriate change of coordinates $\{u, x, t\} \to \{z, \phi(z)\}$, to reduce the initial (1+1) dimensional equation (11) to an ordinary differential equation of the form:

$$
\Omega'[z, \phi(z), \phi'(z), ...] = 0
$$

(17)

3.1 Homogeneous wave type equation

Our aim is now to select from the general dynamical systems described by (11) the class of differential equations for which the equation (17) can be reduced at a wave type equation of the form (2):

$$
\frac{d^2\phi(z)}{dz^2} = 0 \Leftrightarrow \phi(z) = az + b
$$

(18)

where $a$ and $b$ are arbitrary constants.

The previous solution, written in terms of the initial variable $(x,t)$, leads to the following form of the solution $u(x,t)$ of (11):

$$
u(x,t) = a\exp\left(\int \frac{1}{\xi_1(x)}dx - \int \frac{\xi_2(t)}{\varphi(t)}dt\right) + b\exp\left(-\frac{M_2(t)}{\xi_2(t)}\int \frac{M_1(x)}{\xi_1(x)}dx\right)
$$

(19)

For convenience reasons, we shall impose the following relations to be valid:

$$
\frac{\xi_2(t)}{\varphi(t)} = q, \quad \frac{M_2(t)}{\xi_2(t)} = v, \quad \int \frac{1}{\xi_1(x)}dx \equiv P(x), \quad \int \frac{M_1(x)}{\xi_1(x)}dx \equiv R(x)
$$

(20)

with $q, v$ arbitrary constants.
In terms of notations (20), the infinitesimals (12) and the solution (19) become:

\[ \varphi = \varphi(t), \xi = \frac{\varphi(t)}{P(x)}, \eta = q\frac{\varphi(t)}{P(x)}u \] (21)

\[ u(x, t) = [a \exp (P(x) - qt) + b] \exp (vR(x)) \] (22)

The solution (22) must verify the equation (4) which describes the analyzed model. This condition generates a differential system of the form:

\[
\begin{align*}
0 &= q + 2vA(x, t)\ddot{R}(x)\dot{P}(x) + v^2A(x, t)[\dot{R}(x)]^2 + A(x, t)\ddot{P}(x) + A(x, t)[\dot{P}(x)]^2 + \\
&+ vA(x, t)\dot{R}(x) + B(x, t)\dot{P}(x) + vB(x, t)\dot{R}(x) + C(x, t) \\
0 &= v^2A(x, t)[\dot{R}(x)]^2 + vA(x, t)\dot{R}(x) + vB(x, t)\dot{R}(x) + C(x, t) \tag{23}
\end{align*}
\]

For an unitary analysis, it is necessary to describe the differential system (11), obtained in the previous subsection, in terms of the functions \( P(x) \) and \( R(x) \) introduced by (20). Using the expressions (21) we obtain the following differential system:

\[ \varphi A_x P_x^2 + q\varphi A_x P_x + \varphi_t AP_x^2 + 2q\varphi AP_x = 0 \]
\[ \varphi B_x P_x^2 + q\varphi B_x P_x + \varphi_t BP_x^2 + q\varphi_t P_x^2 + \varphi_t^2 P_x^2 + \\
+ 2vq\varphi AR_x P_x^3 - 2vq\varphi AR_x P_x^2 P_x^2 + q\varphi AP_x^2 - 2q\varphi AP_x P_x^2 = 0 \] (24)
\[ \varphi C_x P_x^4 + q\varphi C_x P_x^3 + q\varphi B_x R_x P_x^3 - q\varphi B_x R_x P_x^2 P_x^2 + \varphi_t CP_x^4 + q\varphi CR_x P_x^4 - \\
+ vq\varphi AR_x P_x^2 P_x^2 - 2vq\varphi AR_x P_x P_x^2 P_x^2 + +2vq\varphi AR_x P_x P_x^2 - q\varphi_t R_x P_x^3 = 0 \]

**Conclusion:** Our problem is to find the class of \((1 + 1)\) evolutionary equations of type (4) which could be reduced by the similarity approach to an ordinary wave type equation. Solving this problem is equivalent with searching the solutions of the system described by equations (23) and (24).

**Remark 1:** The system (23-24) can be solved following two paths: (i) by choosing a concrete dynamical system, that is to say concrete expressions for the functions \( A(x, t) \), \( B(x, t) \), \( C(x, t) \) and trying to find out if this equation admits or not solution of the type (22). Now the unknown functions of the system are \( \varphi(t) \), \( P(x) \), \( R(x) \) defined by (20); (ii) by considering \( A(x, t) \), \( B(x, t) \), \( C(x, t) \) as unknown functions and by choosing \( \varphi(t) \), \( P(x) \), \( R(x) \). This is the way we shall follow in the next section.

**Remark 2:** In the case (ii) the general solutions obtained by computational way can be expressed as:

\[
\begin{align*}
A(x, t) &= F \left( \frac{qt - P(x)}{q} \right) \exp [G(x)] \\
B(x, t) &= \left[ -2F \left( \frac{qt - P(x)}{q} \right) \left( \frac{1}{2} [\dot{P}(x)]^2 + v\dot{R}(x)\dot{P}(x) + \frac{1}{2} \ddot{P}(x) \right) \exp [-G(x)] - q \right] \frac{\dot{P}(x)}{P(x)} \\
C(x, t) &= v \left[ F \left( \frac{qt - P(x)}{q} \right) \left[ \ddot{R}(x)\dot{P}(x) + \dot{P}(x)(-\ddot{R}(x) + \dot{R}(x)(v\dot{R}(x) + \dot{P}(x))) \right] \exp [-G(x)] + q\ddot{R}(x) \right] \frac{\dot{P}(x)}{P(x)} \tag{25}
\end{align*}
\]

where

\[ G(x) = -\int x 2(D(2)(P)(a)\varphi \left( \frac{P(a) + qt - P(a)}{q} \right) q + [D(P)(a)]^2 D(\varphi) \left( \frac{P(a) + qt - P(a)}{q} \right) \frac{\dot{P}(a)}{P(a)} \right) \frac{\dot{P}(a) + qt - P(a)}{q} q \] (26)

These solutions are valid for arbitrary constants \( q, v \) and for an arbitrary function \( F \left( \frac{qt - P(x)}{q} \right) \).
3.2 Harmonic oscillators

Let us consider now that the similarity reduction equation is an ordinary oscillator type equation of the form \(3\):

\[
\frac{d^2\phi(z)}{dz^2} + k^2\phi(z) = 0
\]

It has the solution:

\[
\phi(z) = a\sin(kz) + b\cos(kz)
\]

Here \(k, a\) and \(b\) are arbitrary constants.

To write down the previous solution in terms of the initial variable \(u(x, t)\) means that the solution of \(4\) should have the form:

\[
\begin{align*}
  u(x, t) &= a\exp\left(\frac{M_2(t)}{\xi_2(t)}\int \frac{M_1(x)}{\xi_1(x)} dx\right)\sin\left(\sqrt{k}\int \frac{1}{\xi_1(x)} dx - \int \frac{\xi_2(t)}{\varphi(t)} dt\right) \\
  &\quad + b\exp\left(\frac{M_2(t)}{\xi_2(t)}\int \frac{M_1(x)}{\xi_1(x)} dx\right)\cos\left(\sqrt{k}\int \frac{1}{\xi_1(x)} dx - \int \frac{\xi_2(t)}{\varphi(t)} dt\right)
\end{align*}
\]

For convenience reasons, we shall impose again the following relations to be valid:

\[
\frac{\xi_2(t)}{\varphi(t)} = q, \quad \frac{M_2(t)}{\xi_2(t)} = v, \quad \int \frac{1}{\xi_1(x)} dx = P(x), \quad \int \frac{M_1(x)}{\xi_1(x)} dx = R(x)
\]

with \(q, v\) arbitrary constants.

In terms of notations \(30\), the infinitesimals \(12\) and the solution \(29\) become:

\[
\begin{align*}
  \varphi &= \varphi(t), \quad \xi = q\frac{\varphi(t)}{P(x)}, \quad \eta = qv\frac{\varphi(t)\dot{R}(x)}{P(x)}u
  \\
  u(x, t) &= [a\sin(k\exp(P(x) - qt)) + b\cos(k\exp(P(x) - qt))]\exp(vR(x))
\end{align*}
\]

The solution \(32\) must verify the equation \(4\) which describes the analyzed model. This condition generates the vanishing of the coefficient function \(A(x, t)\) and two other differential equations of the form:

\[
\begin{align*}
  A(x, t) &= 0 \\
  q + B(x, t)\dot{P}(x) &= 0 \\
  vB(x, t)\dot{R}(x) + C(x, t) &= 0
\end{align*}
\]

For an unitary analysis, it is again necessary to describe the general differential system \(11\) obtained in the previous section, in terms of the functions \(P(x)\) and \(R(x)\) introduced by \(30\). Taking into account the equations \(31\), we obtain the following differential system:

\[
\begin{align*}
  0 &= \varphi\dot{B}x^4 + q\varphi Bx^3 + q\varphi B\dot{P}_x P_x^2 + q\varphi\dot{P}_x P_x^3 + \varphi\dot{B}P_x^4 \\
  0 &= \varphi\dot{C}x^4 + q\varphi Cx^3 + q\varphi B\dot{R}_x P_x^3 - q\varphi\dot{R}_x P_x^2 + \varphi\dot{C}P_x^4 - q\varphi\dot{R}_x P_x^3
\end{align*}
\]

The system \(33\)-\(34\) can be solved following two paths: (i) by choosing a concrete dynamical system, that is to say concrete expressions for the functions \(B(x, t), C(x, t)\) and trying to find out
if this equation admits or not solution of the type \(32\). Now the unknown functions of the system are \(\varphi(t), P(x), R(x)\) defined by \(30\); (ii) by considering \(B(x,t), C(x,t)\) as unknown functions and by choosing \(\varphi(t), P(x), R(x)\).

This second case is the way we are interested in to follow and, in this case, the general solutions obtained by computational way can be expressed as:

\[
B(x,t) = \frac{-q}{P(x)}, \quad C(x,t) = \frac{v q \dot{R}(x)}{P(x)}
\]

or in terms of the coefficient functions \(\varphi(t), \xi(x,t), M(x,t)\) which appear in the general Lie symmetry operator \((??)\), in the equivalent forms:

\[
B(x,t) = \frac{-\xi(x,t)}{\varphi(t)}, \quad C(x,t) = \frac{M(x,t)}{\varphi(t)}
\]

### 3.3 Rossby type symmetries

The equation for coupled gravity, inertial and Rossby waves in a rotating, stratified atmosphere using the \(\beta\)-plane approximation (which simplifies the spherical geometry whilst retaining the essential dynamics) and the Boussinesq approximation which filters out higher frequency acoustic waves can be written in \((2 + 1)-\)dimensions in the form \((24)\):

\[
\frac{\partial}{\partial t} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u_y = -\beta \frac{\partial u_y}{\partial x}
\]

As we mentioned, this equation describes the coupling between the inertial, the gravity and the Rossby waves, but also the shallow water in an ocean of depth \(H\). It was proven \((25)\) that reducing the model to \((1 + 1)-\)dimensions, \((x, t)\), it admits some very simple Lie symmetries of the form:

\[
\varphi(t) = ct + c_1, \quad \xi(x,t) = cx + f(t), \quad \eta(u) = -3cu
\]

with \(f(t)\) arbitrary function and \(c_0, c\) arbitrary constants. Despite this fact, it is still difficult to find explicit solutions for the equation \((36)\). This is why, we shall consider another approach: we shall impose the Lie symmetries \((37)\) to our general equation \((1)\) and we shall try to find the class of equations which observe them. This means that we have in fact to impose the Rossby symmetries \((37)\) to the system \((11)\). It will take the form:

\[
(c t + c_1) A_t + (c x + c_2) A_x + 3 c A = 0
\]

\[
-(c t + c_1) B_t - (c x + c_2) A_x - 2 c B = 0
\]

\[
-(c t + c_1) C_t - (c x + c_2) C_x - c C = 0
\]

with the unknown functions \(A(x,t), B(x,t), C(x,t)\).

This system admits the solutions:

\[
A(x,t) = \frac{F(x(\varphi(t)) - c_2 t)}{(ct + c_1)^3} = \frac{F(x(t)) - c_2 t}{[\varphi(t)]^3}
\]

\[
B(x,t) = \frac{G(x(\varphi(t)) - c_2 t)}{(ct + c_1)^2} = \frac{G(x(t)) - c_2 t}{[\varphi(t)]^2}
\]

\[
C(x,t) = \frac{H(x(\varphi(t)) - c_2 t)}{ct + c_1} = \frac{H(x(t)) - c_2 t}{\varphi(t)}
\]

with \(F, G, H\) arbitrary functions of their arguments. These expressions give us equations of the form \((1)\) which are equivalent from the point of view of their symmetries with the Rossby equation.
4 Conclusions

The problem of finding exact solutions for nonlinear differential equations plays an important role in the study of nonlinear dynamics. There are many ways of tackling with it. One of them is based on the Lie symmetry method. This method supposes to find the symmetries of the system and, on this basis, to try to determine the general or some particular solutions of the equations. There is a direct approach in which the symmetries of a given equation are obtained, but also an inverse problem has been formulated [10]. A step forward for this latter approach is represented by the use of similarity reduction, a procedure which allows the reduction of the number of degrees of freedom and, by that, simplifies the problem of solving the equation. This paper used this approach and determined a class of $(1+1)$ dimensional second order differential equations which can be reduced to ordinary wave-type equations with simple solutions. Using the Lie symmetry and the similarity reduction procedures, some particular cases of the equation (4) arise as good candidates of equations which could be used as generalization of the linear wave type equations describing complex atmospheric phenomena. Moreover, following our method, we were able to write down the solutions of these equations, solutions which otherwise could be derived by computational methods, but in a very complicated form. Another interesting results of our paper consisted in the fact that a complicated, nonintegrable equation, the Rossby equation, could be replaced by another, simpler equation, which have similar symmetries. The paper is important both by these results, but also as a methodological approach in finding exact solutions through similarity reduction procedure. We have shown how, starting from a particular form of solution for the reduced equation we could recover the solution of a most complicated problem, defined in a space with more than one dimensions. We tackled out a particular case, looking only for linear solutions of the reduced equation and considering that the coefficient functions appearing in the symmetry operators are separable. The problem can be extended for other cases, too.

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