MULTIPLICATIVE NAMBU STRUCTURES ON LIE GROUPOIDS

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Abstract. We study some properties of coisotropic submanifolds of a manifold with respect to a given multivector field. Using this notion, we generalize the results of Weinstein [17] from Poisson bivector field to Nambu-Poisson tensor or more generally to any multivector field. We also introduce the notion of Nambu-Lie groupoid generalizing the concepts of both Poisson-Lie groupoid and Nambu-Lie group. We show that the infinitesimal version of Nambu-Lie groupoid is the notion of weak Lie-Filippov bialgebroid as introduced in [1]. Next we introduce coisotropic subgroupoids of a Nambu-Lie groupoid and these subgroupoids corresponds to, so called coisotropic subalgebroids of the corresponding weak Lie-Filippov bialgebroid.

1. Introduction

In [17], Weinstein introduced the notion of coisotropic submanifold of a Poisson manifold generalizing the notion of Lagrangian submanifold of symplectic manifold. A submanifold $C$ of a Poisson manifold $(P, \pi)$ is called coisotropic, if $\pi^\#(TC)^0 \subset TC$, or, equivalently, $\pi(\alpha, \beta) = 0$, for all $\alpha, \beta \in (TC)^0$, where $(TC)^0$ is the conormal bundle of $C$. Moreover, Weinstein proved the following results.

1. A map $\phi : P_1 \to P_2$ between Poisson manifolds is a Poisson map if and only if its graph is a coisotropic submanifold of $P_1 \times P_2^-$, where $P_2^-$ stands for the manifold $P_2$ with opposite Poisson structure.

2. If $\phi : P \to Q$ is a surjective submersion from a Poisson manifold $P$ to some manifold $Q$, then $Q$ has a Poisson structure for which $\phi$ is a Poisson map if and only if

$$\{(x, y)|\phi(x) = \phi(y)\} \subset P \times P$$

is a coisotropic submanifold of $P \times P^-$. To define the coisotropic submanifold of a Poisson manifold, one does not require the Poisson tensor to be closed, that is, $[\pi, \pi] = 0$, where $[\ ,\ ]$ denotes the Schouten bracket on multivector fields. Therefore, the notion of coisotropic submanifolds make sense for any bivector field, or more generally, for any multivector field. Explicitly, if $M$ is a smooth manifold and $\Pi \in \mathcal{X}^n(M) = \Gamma \wedge^n TM$ be an $n$-vector field on $M$, then a submanifold $C \hookrightarrow M$ is called coisotropic with respect to $\Pi$ if

$$\Pi^\sharp(\bigwedge^{n-1}(TC)^0) \subset TC \iff \Pi(\alpha_1, \ldots, \alpha_n) = 0, \text{ for all } \alpha_1, \ldots, \alpha_n \in (TC)^0,$$

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where $\Pi^2 : \wedge^{n-1} T^* M \to TM$ is the bundle map induced by $\Pi$.

Nambu-Poisson manifolds are generalization of Poisson manifolds. Recall that a Nambu-Poisson manifold of order $n$ is a manifold $M$ equipped with an $n$-vector field $\Pi$ such that the induced bracket on functions satisfies the Fundamental identity (Definition 2.1). The $n$-vector field $\Pi$ of a Nambu-Poisson manifold is referred to as the associated Nambu tensor.

Coisotropic submanifolds of a Nambu-Poisson manifold $M$ are those submanifolds which are coisotropic with respect to the Nambu tensor $\Pi$.

In the present paper, we study some basic properties of coisotropic submanifolds of a manifold with respect to a given multivector field and generalize the results of Weinstein to the case of multivector field. More precisely, we prove the following results (Propositions 3.5 and 3.10).

1. Let $(M, \Pi_M)$ and $(N, \Pi_N)$ be two manifolds with $n$-vector fields and $\phi : M \to N$ be a smooth map. Then $\phi_* \Pi_M = \Pi_N$ if and only if its graph $\text{Gr}(\phi) := \{(m, \phi(m)) | m \in M\}$ is a coisotropic submanifold of $M \times N$ with respect to $\Pi_M \oplus (-1)^{n-1} \Pi_N$.

2. Let $(M, \Pi_M)$ be a manifold with an $n$-vector field and $\phi : M \to N$ be a surjective submersion. Then $N$ has an (unique) $n$-vector field $\Pi_N$ such that $\phi_* \Pi_M = \Pi_N$ if and only if $R(\phi) := \{(x, y) \in M \times M | \phi(x) = \phi(y)\}$ is a coisotropic submanifold of $M \times M$ with respect to $\Pi_M \oplus (-1)^{n-1} \Pi_M$.

Poisson Lie group is a Lie group equipped with a Poisson structure such that the group multiplication map is a Poisson map. Equivalently, a Lie group equipped with a Poisson structure is a Poisson Lie group if the Poisson bivector field is multiplicative [8]. These definitions have no natural extension when one wants to define Poisson groupoid. Nevertheless, the notion of coisotropic submanifolds of Poisson manifolds was used by Weinstein [17] to introduce the notion of Poisson groupoid. Recall that a Poisson groupoid is a Lie groupoid $G \rightrightarrows M$ with a Poisson structure on $G$ such that the graph of the groupoid (partial) multiplication map is a coisotropic submanifold of $G \times G \times G$.

In [18], P. Xu gave an equivalent formulation of Poisson groupoid which generalizes the multiplicativity condition for Poisson Lie group. More generally, in [13], the authors introduced the notion of multiplicative multivector fields on a Lie groupoid. Given a Lie groupoid $G \rightrightarrows M$, an $n$-vector field $\Pi \in \mathfrak{X}^n(G)$ is called multiplicative, if the graph of the groupoid multiplication is a coisotropic submanifold of $G \times G \times G$ with respect to $\Pi \oplus \Pi \oplus (-1)^{n-1} \Pi$. In this terminology, a Poisson groupoid is a Lie groupoid equipped with a multiplicative Poisson tensor.

In the present paper, we extend this approach to the case of Lie groupoid with a Nambu structure. We introduce the notion of a Nambu-Lie groupoid as a Lie groupoid with a Nambu structure $\Pi$ such that the Nambu tensor $\Pi$ is multiplicative (Definition 4.4). When $G$ is a Lie group, this definition coincides with the definition of Nambu-Lie group given by Vaisman [16]. Using results proved in [13] for multiplicative multivector fields on Lie
groupoid, we deduce the following facts which are parallel to the case of Poisson groupoid. Suppose \((G \rightrightarrows M, \Pi)\) is a Nambu-Lie groupoid, then

1. \(M \hookrightarrow G\) is a coisotropic submanifold of \(G\);
2. the groupoid inversion map \(\iota : G \to G\) is an anti Nambu-Poisson map;
3. there is a unique Nambu-Poisson structure \(\Pi_M\) on \(M\) for which the source map is a Nambu-Poisson map (Proposition 4.6).

It is well known that for a Nambu-Poisson manifold \(M\) of order \(n\), the space of 1-forms admits a skew-symmetric \(n\)-bracket which satisfies the Fundamental identity modulo some restriction ([1, 4, 16]). Moreover, the bracket on forms and the de-Rham differential of the manifold satisfy a compatibility condition similar to that of a Lie bialgebroid. This motivates the authors [1] to introduce a notion of weak Lie-Filippov bialgebroid of order \(n\). If \(M\) is a Nambu-Poisson manifold of order \(n\), then \((TM, T^*M)\) provides such an example. Roughly speaking, a weak Lie-Filippov bialgebroid of order \(n\) \((n > 2)\) over \(M\) is a Lie algebroid \(A \to M\) together with a skew-symmetric \(n\)-ary bracket on the space of sections of the dual bundle \(A^* \to M\) and a bundle map \(\rho : \bigwedge^{n-1} A^* \to TM\) satisfying some conditions (cf. Definition 5.1). Moreover it is proved in [1] that, if \((A, A^*)\) is a weak Lie-Filippov bialgebroid of order \(n\) over \(M\), then there is an induced Nambu-Poisson structure of order \(n\) on the base manifold \(M\).

In the present paper, we prove that weak Lie-Filippov bialgebroids are infinitesimal form of Nambu-Lie groupoids. Explicitly, if \(C\) is a coisotropic submanifold of a Nambu-Poisson manifold \((M, \Pi)\), then we show that the \(n\)-ary bracket on the space of 1-forms on \(M\) restricts to the sections of the conormal bundle \((TC)^0 \to C\) and the induced bundle map \(\Pi^\sharp : \bigwedge^{n-1} T^*M \to TM\) maps \(\bigwedge^{n-1} (TC)^0\) to \(TC\) (Proposition 5.3). Therefore, if \(G \rightrightarrows M\) is a Nambu-Lie groupoid of order \(n\) whose Lie algebroid is \(AG \to M\), then as \(M\) is a coisotropic submanifold of \(G\), the space of sections of the dual bundle \(A^*G \cong (TM)^0 \to M\) is equipped with a skew-symmetric \(n\)-ary bracket. Moreover, there is a bundle map \(\bigwedge^{n-1} A^*G \to TM\) so that the pair \((AG, A^*G)\), with the above data, satisfies the conditions of a weak Lie-Filippov bialgebroid. Thus we prove the following (cf. Theorem 5.7).

1.1. Theorem. Let \((G \rightrightarrows M, \Pi)\) be a Nambu-Lie groupoid of order \(n\) with Lie algebroid \(AG \to M\). Then \((AG, A^*G)\) forms a weak Lie-Filippov bialgebroid of order \(n\) over \(M\).

Finally, we compare the Nambu-Poisson structures on the base manifold \(M\) induced from the Nambu-Lie groupoid \(G \rightrightarrows M\) and the weak Lie-Filippov bialgebroid \((AG, A^*G)\) (cf. Proposition 5.9).

Next, we introduce the notion of a coisotropic subgroupoid \(H \rightrightarrows N\) of a Nambu-Lie groupoid \((G \rightrightarrows M, \Pi)\). To study the infinitesimal form of a coisotropic subgroupoid, we introduce the notion of coisotropic subalgebroid of a weak Lie-Filippov bialgebroid. Then we show that the Lie algebroid of a coisotropic subgroupoid \(H \rightrightarrows N\) is a coisotropic subalgebroid of the corresponding weak Lie-Filippov bialgebroid \((AG, A^*G)\) (cf. Proposition 6.6).

Organization. The paper is organized as follows. In section 2, we recall basic definitions and conventions. In section 3, we study some properties of coisotropic submanifolds of a
manifold with respect to a given multivector field and in section 4 we introduce Nambu-Lie groupoids and study some of its basic properties. In section 5, we show that the infinitesimal object corresponding to Nambu-Lie groupoid is weak Lie-Filippov bialgebroid and in section 6 we introduce coisotropic subgroupoids of a Nambu-Lie groupoid and study their infinitesimal.

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## 2. Preliminaries

In this section, we recall some basic preliminaries from \([2,9,18]\) and fix the notations that will be used throughout the paper.

Nambu-Poisson manifolds are \(n\)-ary generalizations of Poisson manifolds introduced by Takhtajan \([14]\).

### 2.1. Definition.

Let \(M\) be a smooth manifold. A **Nambu-Poisson structure** of order \(n\) on \(M\) is a skew-symmetric \(n\)-multilinear bracket

\[
\{ \ldots, \} : C^\infty(M) \times [n] \times C^\infty(M) \to C^\infty(M)
\]

satisfying the following conditions:

1. **Leibniz rule:**
   \[
   \{f_1, \ldots, f_{n-1}, fg\} = f_1\{f_1, \ldots, f_{n-1}, g\} + \{f_1, \ldots, f_{n-1}, f\}g;
   \]

2. **Fundamental identity:**
   \[
   \{f_1, \ldots, f_{n-1}, \{g_1, \ldots, g_n\}\} = \sum_{k=1}^n\{g_1, \ldots, g_{k-1}, f_1, \ldots, f_{n-1}, g_k\}, \ldots, g_n\};
   \]

for all \(f, g, f_1, g_1, \ldots, f_{n-1}, g_1, \ldots, g_n \in C^\infty(M)\). A manifold together with a Nambu-Poisson structure of order \(n\) is called a **Nambu-Poisson manifold of order \(n\)**. Thus the space of smooth functions with this bracket forms a Nambu-Poisson algebra. A Nambu-Poisson manifold of order 2 is nothing but a Poisson manifold \([15]\). Since the bracket above is skew-symmetric and satisfies Leibniz rule, there exists an \(n\)-vector field \(\Pi \in \mathcal{X}^n(M)\), such that

\[
\{f_1, \ldots, f_n\} = \Pi(df_1, \ldots, df_n),
\]

for all \(f_1, \ldots, f_n \in C^\infty(M)\). Given any \((n-1)\) functions \(f_1, \ldots, f_{n-1} \in C^\infty(M)\), the **Hamiltonian vector field** associated to these functions is denoted by \(X_{f_1 \ldots f_{n-1}}\) and is defined by

\[
X_{f_1 \ldots f_{n-1}}(g) = \{f_1, \ldots, f_{n-1}, g\}.
\]

Note that the Fundamental identity, in terms of Hamiltonian vector fields, is equivalent to the condition

\[
[X_{f_1 \ldots f_{n-1}}, X_{g_1 \ldots g_{n-1}}] = \sum_{k=1}^{n-1}X_{g_1 \ldots [f_1, \ldots, f_{n-1}, g_k], \ldots, g_n}.
\]

for all \(f_1, \ldots, f_{n-1}, g_1, \ldots, g_{n-1} \in C^\infty(M)\). The Fundamental identity can also be rephrased as

\[
\mathcal{L}_{X_{f_1 \ldots f_{n-1}}} \Pi = 0,
\]
for all \( f_1, \ldots, f_{n-1} \in C^\infty(M) \), which shows that every Hamiltonian vector field preserves the Nambu tensor. A Nambu-Poisson manifold is often denoted by \((M, \{ , , \})\) or simply by \((M, \Pi)\).

2.2. Example. (1) Let \( M \) be an orientable manifold of dimension \( n \) and \( \nu \) be a volume form on \( M \). Define an \( n \)-bracket \( \{ , , \} \) on \( C^\infty(M) \) by the following identity

\[
df_1 \wedge \cdots \wedge df_n = \{ f_1, \ldots, f_n \} \nu.
\]

Then \( \{ , , \} \) defines a Nambu-Poisson structure of order \( n \) on \( M \). Let \( \Pi_\nu \in \Gamma \wedge^n TM \) denotes the associated Nambu-Poisson tensor. If \( \Pi \in \Gamma \wedge^n TM \) is any Nambu-Poisson structure of order \( n \) such that \( \Pi \neq 0 \) at every point, then there exists a volume form \( \nu' \) on \( M \) such that \( \Pi = \Pi_{\nu'} \). If \( M = \mathbb{R}^n \) and \( \nu = dx_1 \wedge \cdots \wedge dx_n \) is the standard volume form, then one recovers the Nambu structure on \( \mathbb{R}^n \) originally discussed by Y. Nambu [12].

(2) Let \( \Pi \) be any \( n \)-vector field on an oriented manifold \( M \) of dimension \( n \). Then \( \Pi \) defines a Nambu structure of order \( n \) on \( M \) (see [5]).

(3) Let \( M \) be a manifold of dimension \( m \) and \( X_1, \ldots, X_n \) be linearly independent vector fields such that \([X_i, X_j] = 0\) for all \( i, j = 1, \ldots, n \). Then the \( n \)-vector field \( \Pi = X_1 \wedge \cdots \wedge X_n \) defines a Nambu structure of order \( n \).

(4) Let \((M, \{ , , \})\) be a Nambu-Poisson manifold of order \( n \). Suppose \( k \leq n - 2 \) and \( F_1, \ldots, F_k \in C^\infty(M) \) be any fixed functions on \( M \). Define a \((n-k)\)-bracket \( \{ , , \}' \) on \( C^\infty(M) \) by

\[
\{ f_1, \ldots, f_{n-k} \}' = \{ F_1, \ldots, F_k, f_1, \ldots, f_{n-k} \}
\]

for \( f_1, \ldots, f_{n-k} \in C^\infty(M) \). Then \( \{ , , \}' \) defines a Nambu-structure of order \((n-k)\) on \( M \). This Nambu structure is called the subordinate Nambu structure of \((M, \{ , , \})\) with subordinate function \( F_1, \ldots, F_k \).

(5) If \( \Pi_i \) is a Nambu structure of order \( n_i \) on a manifold \( M_i \) \( (i = 1, 2) \), then \( \Pi = \Pi_1 \wedge \Pi_2 \) is a Nambu structure of order \( n_1 + n_2 \) on \( M_1 \times M_2 \) [2].

More examples of Nambu structures can be found in [6,16].

Let \((M, \Pi)\) be a Nambu-Poisson manifold of order \( n \). For each \( m \in M \), let \( D_mM \subset T_mM \) be the subspace of the tangent space at \( m \) generated by all Hamiltonian vector fields at \( m \). Since the Lie bracket of two Hamiltonians is again a Hamiltonian, therefore \( D \) defines a \((\text{singular}) \) integrable distribution whose leaves are either \( n \)-dimensional submanifolds endowed with a volume form or just singletons [2].

2.3. Definition. Let \((M, \Pi_M)\) and \((N, \Pi_N)\) be two manifolds with \( n \)-vector fields. A smooth map \( \phi : M \to N \) is called \((\Pi_M, \Pi_N)\)-map if the induced brackets on functions satisfies:

\[
\{ \phi^* f_1, \ldots, \phi^* f_n \}_M = \phi^* \{ f_1, \ldots, f_n \}_N
\]
for all $f_1, \ldots, f_n \in C^\infty(N)$, or equivalently, $\phi_*\Pi_M = \Pi_N$. The map $\phi$ is called an anti $(\Pi_M, \Pi_N)$-map if

$$\{\phi^* f_1, \ldots, \phi^* f_n\}_M = (-1)^{n-1} \phi^* \{f_1, \ldots, f_n\}_N$$

for all $f_1, \ldots, f_n \in C^\infty(N)$. A $(\Pi_M, \Pi_N)$-map $\phi : (M, \Pi_M) \to (N, \Pi_N)$ between Nambu-Poisson manifolds of the same order $n$ is called a Nambu-Poisson map or a $N$-$P$-map.

2.4. Remark. The condition for a $(\Pi_M, \Pi_N)$-map can also be expressed in terms of the induced bundle maps as

$$\Pi^1_{N,\phi(m)} = T_m\phi \circ \Pi^1_{M,m} \circ T^*_m\phi$$

for each $m \in M$, where $\Pi^1_M : \bigwedge^{n-1} T^*M \to TM$ is the induced bundle map and is given by

$$\langle \beta, \Pi^1_M (\alpha_1 \wedge \cdots \wedge \alpha_{n-1}) \rangle = \Pi_M (\alpha_1, \ldots, \alpha_{n-1}, \beta)$$

for all $\alpha_1, \ldots, \alpha_{n-1}, \beta \in T^*_x M$, $x \in M$.

2.5. Definition. A Lie groupoid over a smooth manifold $M$ is a smooth manifold $G$ together with the following structure maps:

1. two surjective submersions $\alpha, \beta : G \to M$, called the source map and the target map respectively;
2. a smooth partial multiplication map

$$G_{(2)} = \{(g, h) \in G \times G | \beta(g) = \alpha(h)\} \to G, \ (g, h) \mapsto gh;$$

3. a smooth unit map $\epsilon : M \to G, \ x \mapsto \epsilon_x$;
4. and a smooth inverse map $i : G \to G, \ g \mapsto g^{-1}$ with $\alpha(g^{-1}) = \beta(g)$ and $\beta(g^{-1}) = \alpha(g)$

such that, the following conditions are satisfied

(i) $\alpha(gh) = \alpha(g)$ and $\beta(gh) = \beta(h);
(ii) (gh)k = g(hk), \text{ whenever the multiplications make sense};
(iii) $\alpha(\epsilon_x) = \beta(\epsilon_x) = x, \forall x \in M;
(iv) \epsilon_{\alpha(g)}g = g \text{ and } g\epsilon_{\beta(g)} = g, \forall g \in G;
(v) gg^{-1} = \epsilon_{\alpha(g)} \text{ and } g^{-1}g = \epsilon_{\beta(g)}, \forall g \in G.$

A Lie groupoid $G$ over $M$ is denoted by $G \rightrightarrows M$ when all the structure maps are understood.

2.6. Remark. Note that the smooth structure on $G_{(2)}$ comes from the fact that

$$G_{(2)} = (\beta \times \alpha)^{-1}(\Delta_M),$$

where $\beta \times \alpha : G \times G \to M \times M, \ (g, h) \mapsto (\beta(g), \alpha(h))$ and $\Delta_M = \{(m, m) | m \in M\} \subset M \times M$ is the diagonal submanifold of $M \times M$. Then these conditions imply that the inverse map $i : G \to G, \ g \mapsto g^{-1}$ is also smooth [9]. Moreover, $\alpha$-fibers and $\beta$-fibers are submanifolds of $G$ as both $\alpha$ and $\beta$ are surjective submersions.
2.7. Definition. Given a Lie groupoid $G \rightrightarrows M$, define an equivalence relation $\sim'$ on $M$ by the following: two points $x, y \in M$ are said to be equivalent, written as $x \sim y$, if there exists an element $g \in G$ such that $\alpha(g) = x$, $\beta(g) = y$. The quotient $M/\sim$ is called the orbit set of $G$.

2.8. Definition. Given two Lie groupoids $G_1 \rightrightarrows M_1$ and $G_2 \rightrightarrows M_2$, a morphism between Lie groupoids is a pair $(F, f)$ of smooth maps $F : G_1 \to G_2$ and $f : M_1 \to M_2$ which commute with all the structure maps of $G_1$ and $G_2$. In other words,

$$\alpha_2 \circ F = f \circ \alpha_1, \quad \beta_2 \circ F = f \circ \beta_1,$$

and $F(g_1 h_1) = F(g_1) F(h_1)$

for all $(g_1, h_1) \in (G_1)_{(2)}$.

2.9. Definition. Let $G \rightrightarrows M$ be a Lie groupoid. A Lie subgroupoid of it is a Lie groupoid $H \rightrightarrows N$ together with injective immersions $i : H \to G$ and $i_0 : N \to M$ such that $(i, i_0)$ is a Lie groupoid morphism.

2.10. Definition. Let $G \rightrightarrows M$ be a Lie groupoid. A submanifold $\mathcal{K}$ of $G$ is called a bisection of the Lie groupoid, if $\alpha|_\mathcal{K} : \mathcal{K} \to M$ and $\beta|_\mathcal{K} : \mathcal{K} \to M$ are diffeomorphisms.

The existence of local bisections through any point $g \in G$ is always guaranteed. The space of bisections $\mathcal{B}(G)$ form an infinite dimensional (Fréchet) Lie group under the multiplication of subsets induced from the partial multiplication of $G$. Note that the left (right) multiplication is defined only on $\alpha$-fibers ($\beta$-fibers), therefore, we cannot define a diffeomorphism of $G$ using left (right) multiplication by an element, like a Lie group. However we can do so by using bisection instead of an element. Given a bisection $\mathcal{K} \in \mathcal{B}(G)$, let $l_\mathcal{K}$ and $r_\mathcal{K}$ be the diffeomorphisms on $G$ defined by

$$l_\mathcal{K}(h) = gh, \text{ where } g \in \mathcal{K} \text{ is the unique element such that } \beta(g) = \alpha(h)$$

and

$$r_\mathcal{K}(h) = hg', \text{ where } g' \in \mathcal{K} \text{ is the unique element such that } \alpha(g') = \beta(h).$$

2.11. Remark. Suppose $\mathcal{K}$ is any (local) bisection of $G$ through $g \in G$. Then the restriction of the map $l_\mathcal{K}$ to $\alpha^{-1}(\beta(g))$ is the left translation $l_g$ by $g$:

$$l_g : \alpha^{-1}(\beta(g)) \to \alpha^{-1}(\alpha(g)), \; h \mapsto gh.$$

Then we have the following result [18].

2.12. Proposition. Let $G \rightrightarrows M$ be a Lie groupoid and $P$ be an $n$-vector field on $G$. Suppose for any $g \in G$ with $\beta(g) = u$, $P$ satisfies $P(g) = (l_g)_* P(e_u)$, where $G \in \mathcal{B}(G)$ is any arbitrary bisection through the point $g$. Then $P$ is left invariant.

2.13. Definition. A Lie algebroid $(A, [\; , \; ], a)$ over a smooth manifold $M$ is a smooth vector bundle $A$ over $M$ together with a Lie algebra structure $[\; , \; ]$ on the space $\Gamma A$ of the smooth sections of $A$ and a bundle map $a : A \to TM$, called the anchor, such that

1. the induced map $a : \Gamma A \to \mathcal{X}^1(M)$ is a Lie algebra homomorphism, where $\mathcal{X}^1(M)$ is the usual Lie algebra of vector fields on $M$.
(2) For any $X, Y \in \Gamma A$ and $f \in C^\infty(M)$, we have
$$[X, fY] = f[X, Y] + (a(X)f)Y.$$  

We may denote a Lie algebroid simply by $A$, when all the structures are understood. Any Lie algebra is a Lie algebroid over a point with zero anchor. The tangent bundle of any smooth manifold is a Lie algebroid with usual Lie bracket of vector fields and identity as anchor.

**Lie algebroid of a Lie groupoid.** Given a Lie groupoid $G \rightrightarrows M$, its Lie algebroid consists of the vector bundle $AG \to M$ whose fiber at $x \in M$ coincides with the tangent space at the unit element $\epsilon_x$ of the $\alpha$-fiber at $x$. Then the space of sections of $AG$ can be identified with the left invariant vector fields $X_{\text{inv}}(G) = \{ X \in \Gamma(T^\alpha G) = \Gamma(\ker(d\alpha))| X_{gh} = (l_g)_*X_h, \forall (g, h) \in G(2) \}$ on $G$. Since the space of left invariant vector fields on $G$ is closed under the Lie bracket, therefore it defines a Lie bracket on $\Gamma AG$. The anchor $a$ of $AG$ is defined to be the differential of the target map $\beta$ restricted to $AG$.

Let $AG$ be the Lie algebroid of the Lie groupoid $G \rightrightarrows M$. Given any $X \in \Gamma AG$, let $\vec{X}$ be the corresponding left invariant vector field on $G$. Then there exists an $\epsilon > 0$ and a 1-parameter family of transformations $\phi_t (|t| < \epsilon)$, generated by $\vec{X}$ ([9]). Suppose each $\phi_t$ is defined on all of $M$, where $M$ is identified with a closed embedded submanifold of $G$ via the unit map. We denote the image of $M$ via $\phi_t$ by $\exp tX$. Then $\exp tX$ is a bisection of the groupoid (for all $|t| < \epsilon$) and satisfies 1-parameter group like conditions, namely
$$\exp(t + s)X = \exp tX \cdot \exp sX, \quad \text{whenever} \quad |t|, |s|, |t + s| < \epsilon,$$
where on the right hand side, we used the multiplication of bisections.

3. **Coisotropic submanifolds**

Let $M$ be a manifold and $\Pi \in \mathcal{A}^n(M)$ be a $n$-vector field on $M$. Let
$$\Pi^2 : \bigwedge^{n-1} T^*M \to TM$$
be the induced bundle map given by
$$(\beta, \Pi^2(\alpha_1 \wedge \cdots \wedge \alpha_{n-1})) = \Pi(\alpha_1, \ldots, \alpha_{n-1}, \beta)$$
for all $\alpha_1, \ldots, \alpha_{n-1}, \beta \in T^*_xM$, $x \in M$.

We recall the following definition from [13].

3.1. **Definition.** A submanifold $C \hookrightarrow M$ is said to be **coisotropic** with respect to $\Pi$, if
$$\Pi^2(\bigwedge^{n-1}(TC)_x^0) \subset TC$$
where
$$(TC)_x^0 = \{ \alpha \in T^*_xM | \alpha(v) = 0, \forall v \in T_xC \}, \quad x \in C,$$
or equivalently,
\[ \Pi_x(\alpha_1, \ldots, \alpha_n) = 0, \forall \alpha_i \in (TC)^0_x, \ x \in C. \]

We have the following easy observation for coisotropic submanifolds of a Nambu-Poisson manifold.

3.2. **Proposition.** Let \((M, \Pi)\) be a Nambu-Poisson manifold of order \(n\) and \(C\) be a closed embedded submanifold of \(M\). Let \(\mathcal{I}(C) = \{ f \in C^\infty(M) | f|_C \equiv 0 \}\) denote the vanishing ideal of \(C\). Then the followings are equivalent:

1. \(C\) is a coisotropic submanifold;
2. \(\mathcal{I}(C)\) is a Nambu-Poisson subalgebra;
3. for every \(f_1, \ldots, f_{n-1} \in \mathcal{I}(C)\), the Hamiltonian vector field \(X_{f_1 \ldots f_{n-1}}\) is tangent to \(C\).

**Proof.**

(1) \(\Rightarrow\) (2) Let \(f_1, \ldots, f_n \in \mathcal{I}(C)\). Then for any \(x \in C\), \(d_x f_i \in (TC)^0_x\), for all \(i = 1, \ldots, n\). Now since \(C\) is a coisotropic submanifold, we have
\[ \{ f_1, \ldots, f_n \}(x) = \Pi_x(d_x f_1, \ldots, d_x f_n) = 0, \ \forall x \in C. \]
Hence \(\{ f_1, \ldots, f_n \} \in \mathcal{I}(C)\). Therefore \(\mathcal{I}(C)\) is a Nambu-Poisson subalgebra.

(2) \(\Rightarrow\) (3) Let \(f_1, \ldots, f_{n-1} \in \mathcal{I}(C)\) and \(x \in C\). Let \(\alpha \in (TC)^0_x\). Then there exists a function \(g\) vanishing on \(C\) such that \(d_x g = \alpha\). Since \(\mathcal{I}(C)\) is a Nambu-Poisson subalgebra, we have
\[ \{ f_1, \ldots, f_{n-1}, g \}(x) = 0. \]
Thus,
\[ X_{f_1 \ldots f_{n-1}}|_x(\alpha) = X_{f_1 \ldots f_{n-1}}|_x(d_x g) = \{ f_1, \ldots, f_{n-1}, g \}(x) = 0 \]
and consequently, \(X_{f_1 \ldots f_{n-1}}\) is tangent to \(C\).

(3) \(\Rightarrow\) (1) Let \(x \in C\) and \(\alpha_1, \ldots, \alpha_n \in (TC)^0_x\). Then there exist functions \(f_1, \ldots, f_n \in \mathcal{I}(C)\) such that \(d_x f_i = \alpha_i, \ \forall i = 1, \ldots, n\). Therefore,
\[ \Pi_x(\alpha_1, \ldots, \alpha_n) = \Pi_x(d_x f_1, \ldots, d_x f_n) = X_{f_1 \ldots f_{n-1}}|_x(d_x f_n) = 0. \]
Hence \(C\) is a coisotropic submanifold of \(M\). \(\square\)

3.3. **Proposition.** Let \((M, \Pi_M)\) and \((N, \Pi_N)\) be two Nambu-Poisson manifolds of same order \(n\) and \(C \hookrightarrow N\) be a coisotropic submanifold of \(N\) with respect to \(\Pi_N\). If \(\phi : M \rightarrow N\) is a Nambu-Poisson map transverse to \(C\), then \(\phi^{-1}(C)\) is a coisotropic submanifold of \(M\) with respect to \(\Pi_M\). (the result holds true for manifolds with \(n\)-vector fields such that \(\phi_* \Pi_M = \Pi_N\)).

**Proof.** Since \(\phi\) is transverse to \(C\), therefore \(\phi^{-1}(C)\) is a submanifold of \(M\). Moreover
\[ T(\phi^{-1}(C)) = (T\phi)^{-1}TC. \]
Therefore $T(\phi^{-1}(C))^0 = (T\phi)^*(TC)^0$. Observe that
\[
T\phi(\Pi_M^2(\bigwedge^{n-1} T(\phi^{-1}(C))^0)) = T\phi(\Pi_M^2((T\phi)^* \bigwedge^{n-1} (TC)^0))
= \Pi_N^2(\bigwedge^{n-1} (TC)^0) \subseteq TC.
\]
Thus,
\[
\Pi_M^2(\bigwedge^{n-1} T(\phi^{-1}(C))^0) \subseteq (T\phi)^{-1}TC = T(\phi^{-1}(C))
\]
and hence $\phi^{-1}(C)$ is coisotropic with respect to $\Pi_M$. \hfill \Box

3.4. **Proposition.** Let $\phi : (M, \Pi_M) \to (N, \Pi_N)$ be a Nambu Poisson map between two Nambu-Poisson manifolds $(M, \Pi_M)$ and $(N, \Pi_N)$ and $C \hookrightarrow M$ be a coisotropic submanifold of $M$. Assume that $\phi(C)$ is a submanifold of $N$. Then $\phi(C)$ is a coisotropic submanifold of $N$ (the result holds true for manifolds with $n$-vector fields such that $\phi_*\Pi_M = \Pi_N$).

**Proof.** We have $T(\phi(C)) \supseteq T(\phi(TC))$ and $(T\phi)^*(T(\phi(C)))^0 \subseteq (TC)^0$. Therefore,
\[
\Pi_N^2(\bigwedge^{n-1} T(\phi(C))^0) = T\phi(\Pi_M^2((T\phi)^* \bigwedge^{n-1} T(\phi(C))^0)) \quad \text{(since $\phi$ is a N-P map)}
\[
\subseteq T\phi(\Pi_M^2(\bigwedge^{n-1} (TC)^0))
\subseteq T\phi(TC) \quad \text{(since $C \hookrightarrow M$ is coisotropic)}
\subseteq T(\phi(C))
\]
which shows that $\phi(C)$ is a coisotropic submanifold of $N$. \hfill \Box

Using the terminology of coisotropic submanifold with respect to any multivector field allows us to extend the results of Weinstein [17] from Poisson bivector field to Nambu-Poisson tensor or more generally to any multivector field.

3.5. **Proposition.** Let $(M, \Pi_M)$ and $(N, \Pi_N)$ be two manifolds with $n$-vector fields and $\phi : M \to N$ be a smooth map. Then $\phi$ is a $(\Pi_M, \Pi_N)$-map, that is $\phi_*\Pi_M = \Pi_N$ if and only if its graph
\[
\text{Gr}(\phi) = \{(m, \phi(m)) \mid m \in M\}
\]
is a coisotropic submanifold of $M \times N$ with respect to $\Pi_M \oplus (-1)^{n-1}\Pi_N$.

**Proof.** Let $C = \text{Gr}(\phi) \subset M \times N$. Then $C$ is a closed embedded submanifold of $M \times N$. Note that, a tangent vector to the graph consist of a pair $(v_m, (T\phi)(v_m))$, where $m \in M$, $v_m \in T_m M$. Therefore, $(TC)^0$ consists of a pair of covectors $(-(T\phi)^* \psi, \psi)$, where $\psi \in T^*_{\phi(m)} N$. Therefore, $\text{Gr}(\phi)$ is a coisotropic submanifold of $M \times N$ with respect to $\Pi_M \oplus (-1)^{n-1}\Pi_N$ if and only if $(\Pi_M^2 \times (-1)^{n-1}\Pi_N^2)$ maps $(-(T\phi)^* \psi_1, \psi_1) \land \cdots \land (-(T\phi)^* \psi_{n-1}, \psi_{n-1})$ into $TC$, for all $\psi_1, \ldots, \psi_{n-1} \in T^*_{\phi(m)} N$ and $m \in M$. In other words,
\[
(T\phi) \left( \Pi_M^2(-(T\phi)^* \psi_1, \ldots, -(T\phi)^* \psi_{n-1}) \right) = (-1)^{n-1}\Pi_N^2(\psi_1, \ldots, \psi_{n-1})
\]
that is,
\[(T\phi)(\Pi_M((T\phi)^*\psi_1, \ldots, (T\phi)^*\psi_{n-1})) = \Pi_N(\psi_1, \ldots, \psi_{n-1}).\]
This is equivalent to the condition that $\phi$ is a $(\Pi_M, \Pi_N)$-map.

3.6. **Definition.** Let $(M, \Pi_M)$ be a Nambu-Poisson manifold of order $n$ and $\phi : M \to N$ be a smooth surjective map. If there exist a Nambu-Poisson structure $\Pi_N$ (of order $n$) on $N$ which makes $\phi$ into a Nambu-Poisson map, then $\Pi_N$ is called the Nambu-Poisson structure *coinduced* by the mapping $\phi$.

The following is a characterization of coinduced Nambu-Poisson structure.

3.7. **Proposition.** Let $(M, \Pi_M)$ be a Nambu-Poisson manifold of order $n$ and $\phi : M \to N$ be a smooth surjective map from $M$ to some manifold $N$. Then $N$ has a Nambu-Poisson structure coinduced by $\phi$ if and only if for all $f_1, \ldots, f_n \in C^\infty(N)$, the function \(\{\phi^*f_1, \ldots, \phi^*f_n\}_M\) is constant along the fibers of $\phi$.

**Proof.** Let $f_1, \ldots, f_n \in C^\infty(N)$. If the function \(\{\phi^*f_1, \ldots, \phi^*f_n\}_M\) is constant along the $\phi$-fibers, then there exists a function on $N$, which we denote by \(\{f_1, \ldots, f_n\}_N\) such that \(\{\phi^*f_1, \ldots, \phi^*f_n\}_M = \phi^*\{f_1, \ldots, f_n\}_N\). Clearly this bracket defines a coinduced Nambu-Poisson structure on $N$.

Conversely, suppose that there is a Nambu-Poisson bracket \(\{\ldots, \ldots\}_N\) on $N$ coinduced by $\phi$. Then for any $y \in N$,
\[
\{\phi^*f_1, \ldots, \phi^*f_n\}_M(\phi^{-1}\{y\}) = (\phi^*\{f_1, \ldots, f_n\}_N)(\phi^{-1}y) = \{f_1, \ldots, f_n\}_N(y)
\]
proving \(\{\phi^*f_1, \ldots, \phi^*f_n\}_M\) is constant along the $\phi$-fibers. □

3.8. **Remark.** Let $(M, \Pi_M)$ be a manifold with an $n$-vector field and $\phi : M \to N$ be a smooth map. Then there exists an $n$-vector field $\Pi_N$ on $N$ such that $\phi$ is a $(\Pi_M, \Pi_N)$-map if and only if for all $f_1, \ldots, f_n \in C^\infty(N)$, the function \(\{\phi^*f_1, \ldots, \phi^*f_n\}_M\) is constant along the fibers of $\phi$.

3.9. **Proposition.** Let $(M, \Pi_M)$ be a Nambu-Poisson manifold and $\phi : M \to N$ be a surjective submersion with connected fibers. Let $\ker \phi_*(m)$ is spanned by local Hamiltonian vector fields (that is, $\ker \phi_*(m) \subset D_m M$, for all $m \in M$). Then $N$ has a Nambu-Poisson structure coinduced by $\phi$.

**Proof.** Since $\phi$ is a submersion, the fibers of $\phi$ are submanifolds of $M$. Then for $y \in N$, $\phi^{-1}\{y\} = C$ is a submanifold of $M$. Let $g_1, \ldots, g_{n-1}$ be locally defined functions on $M$ such that $X_{g_1 \ldots g_{n-1}} \in \ker \phi_*$. Let $f_1, \ldots, f_n \in C^\infty(N)$. To prove that \(\{\phi^*f_1, \ldots, \phi^*f_n\}\) is constant on the fibers, it is enough to prove that
\[
X_{g_1 \ldots g_{n-1}}\{\phi^*f_1, \ldots, \phi^*f_n\} = 0.
\]
Note that
\[ X_{g_1 \cdots g_{n-1}}(\phi^* f_1, \ldots, \phi^* f_n) = \sum_{k=1}^{n} (\phi^* f_k) X_{g_1 \cdots g_{n-1}}(\phi^* f_k) + (\phi^* f_n) \]
and the functions \( \phi^* f_i \) are constant along the fibers. Hence by the Proposition 3.7, there exists a coinduced Nambu-Poisson structure on \( N \).

\[ \square \]

3.10. **Proposition.** Let \((M, \Pi_M)\) be a manifold with an \( n \)-vector field and \( \phi : M \to N \) be a surjective submersion. Then \( N \) has an (unique) \( n \)-vector field \( \Pi_N \) such that \( \phi \) is a \((\Pi_M, \Pi_N)\)-map if and only if \( R(\phi) = \{(x, y) \in M \times M | \phi(x) = \phi(y)\} \) is a coisotropic submanifold of \( M \times M \) with respect to \( \Pi_M \oplus (-1)^{n-1} \Pi_M \).

**Proof.** Note that \( R(\phi) = (\phi \times \phi)^{-1}(\Delta_N) \), where \( \Delta_N \) is the diagonal of \( N \times N \). Since \( \phi \) is surjective submersion \( R(\phi) \) is a submanifold of \( M \times M \). Moreover, for \((x, y) \in R(\phi)\)
\[ T_{(x,y)}(R(\phi)) = \{(X, Y) \in T_x M \times T_y M | (T\phi)_x(X) = (T\phi)_y(Y)\}. \]
Therefore, \( T(R(\phi))^0 \) consists of covectors \(-(T\phi)^*_x \psi, (T\phi)^*_y \psi)\), where \( \psi \in T^*_x N \).

Thus, \( R(\phi) \) be a coisotropic submanifold of \( M \times M \) with respect to \( \Pi_M \oplus (-1)^{n-1} \Pi_M \) if and only if for all \( \psi_1, \ldots, \psi_{n-1} \in T^*_x N \) and \( (x, y) \in R(\phi) \), \( \Pi^*_M \oplus (-1)^{n-1} \Pi^*_M \) maps
\[ -(T\phi)^*_x \psi_1, (T\phi)^*_y \psi_1) \wedge \cdots \wedge -(T\phi)^*_x \psi_{n-1}, (T\phi)^*_y \psi_{n-1} \]
into \( T(R(\phi)) \). That is
\[ (T\phi)_x \Pi^*_M ((T\phi)^*_x \psi_1, \ldots, (T\phi)^*_x \psi_{n-1}) = (1)^{n-1}(T\phi)_y \Pi^*_M ((T\phi)^*_y \psi_1, \ldots, (T\phi)^*_y \psi_{n-1}), \]
or equivalently,
\[ (T\phi)_x \Pi^*_M ((T\phi)^*_x \psi_1, \ldots, (T\phi)^*_x \psi_{n-1}) = (T\phi)_y \Pi^*_M ((T\phi)^*_y \psi_1, \ldots, (T\phi)^*_y \psi_{n-1}) \]
holds. Let \( f_1, \ldots, f_n \in C^\infty(N) \) and \( x \in M \). Then
\[ \{\phi^* f_1, \ldots, \phi^* f_n\}_M(x) = \Pi^*_M (d_x(\phi^* f_1) \wedge \cdots \wedge d_x(\phi^* f_{n-1}), d_x(\phi^* f_n)) \]
\[ = \Pi^*_M ((T\phi)^*_x \psi_1 \wedge \cdots \wedge (T\phi)^*_x \psi_{n-1}), (T\phi)^*_y \psi_n) \]
\[ = \langle (T\phi)_x \Pi^*_M ((T\phi)^*_x \psi_1 \wedge \cdots \wedge (T\phi)^*_x \psi_{n-1}), \psi_n \rangle \]
where \( \psi_i = d_{\phi(x)} f_i = d_{\phi(y)} f_i \in T^*_N, \) for all \( 1 \leq i \leq n \). It follows from the Equation (1) that the function \( \{\phi^* f_1, \ldots, \phi^* f_n\}_M \) is constant along the \( \phi \)-fibers if and only if \( R(\phi) \) is a coisotropic submanifold of \( M \times M \) with respect to \( \Pi^*_M \oplus (-1)^{n-1} \Pi^*_M \). Hence the result follows by the Remark 3.8. The uniqueness follows from the surjectivity of \( \phi \).

\[ \square \]

4. **Nambu-Lie groupoids**

In this section, we recall the definition of multiplicative multivector fields on Lie groupoid ([13]) and define Nambu-Lie groupoid (of order \( n \)) as a Lie groupoid with a multiplicative \( n \)-vector field which is also a Nambu-Poisson tensor.
4.1. **Definition.** Let $G \Rightarrow M$ be a Lie groupoid and $\Pi \in \mathcal{X}^n(G)$ be an $n$-vector field on $G$. Then $\Pi$ is called *multiplicative* if the graph of the groupoid multiplication

$$\{(g, h, gh) \in G \times G \times G \mid \beta(g) = \alpha(h)\}$$

is a coisotropic submanifold of $G \times G \times G$ with respect to $\Pi \oplus \Pi \oplus (-1)^{n-1} \Pi$.

Then we have the following characterization of multiplicative multivector fields [13]:

4.2. **Theorem.** Let $G \Rightarrow M$ be a Lie groupoid and $\Pi \in \mathcal{X}^n(G)$ be an $n$-vector field on $G$. Then $\Pi$ is multiplicative if and only if the following conditions are satisfied.

1. $\Pi$ is an affine tensor. In other words
   $$\Pi(gh) = (r_h)_* \Pi(g) + (l_g)_* \Pi(h) - (r_h)_* (l_g)_* \Pi(u)$$
   where $u = \beta(g) = \alpha(h)$ and $\mathcal{G}, \mathcal{H}$ are (local) bisections through the points $g, h$ respectively.

2. $M$ is a coisotropic submanifold of $G$ with respect to $\Pi$.

3. For all $g \in G$, $\alpha_\ast \Pi(g)$ and $\beta_\ast \Pi(g)$ depend only on the base points $\alpha(g)$ and $\beta(g)$ respectively.

4. For all $f, f' \in C^\infty(M)$, the $(n - 2)$-vector field $\iota_{d(\alpha_\ast f) \wedge d(\beta_\ast f')} \Pi$ is zero. In other words,
   $$\{\cdots, \alpha_\ast f, \beta_\ast f'\} = 0.$$

5. For all $f_1, \ldots, f_k \in C^\infty(M)$, $\iota_{d(\beta_\ast f_1) \wedge \cdots \wedge d(\beta_\ast f_k)} \Pi$ is a left invariant $(n - k)$-vector field on $G$, $1 \leq k < n$.

4.3. **Remark.** Suppose $G$ be a Lie group considered as a Lie groupoid over a point. Then the conditions (3) - (5) of the Theorem 4.2 are satisfied automatically. The condition (2) implies that $\Pi(e) = 0$ (where $e$ is the identity element of the group), which together with condition (1) implies that $\Pi$ satisfies the usual multiplicativity condition

$$\Pi(gh) = (r_h)_* \Pi(g) + (l_g)_* \Pi(h).$$

4.4. **Definition.** A *Nambu-Lie groupoid of order $n$* is a Lie groupoid $G \Rightarrow M$ with a multiplicative Nambu tensor $\Pi \in \mathcal{X}^n(G)$ of order $n$.

A Nambu Lie groupoid (of order $n$) will be denoted by $(G \Rightarrow M, \Pi)$.

4.5. **Example.** (1) Poisson groupoids [17] are examples of Nambu-Lie groupoids with $n = 2$.

(2) Any Lie groupoid with zero Nambu structure is a Nambu-Lie groupoid.

(3) Let $(G, \Pi)$ be a Nambu-Lie group (of order $n$) [16]. Thus $G$ is a Lie group equipped with a Nambu structure $\Pi$ of order $n$ on $G$ such that

$$\Pi(gh) = (r_h)_* \Pi(g) + (l_g)_* \Pi(h)$$

for all $g, h \in G$. Note that the right hand side of the above equality is equal to $m_*(\Pi(g), \Pi(h))$, where $m_* : \wedge^n T_{(g,h)}(G \times G) \to \wedge^n T_{gh}G$ is the map induced by the
multiplication map \( m : G \times G \to G \). Therefore,
\[
\Pi(gh) = m_*(\Pi(g), \Pi(h)).
\]
Thus, the group multiplication map \( m : G \times G \to G \) is a \((\Pi \oplus \Pi, \Pi)\)-map. Therefore, by the Proposition 3.5, the graph of the group multiplication map is a coisotropic submanifold of \( G \times G \times G \) with respect to \( \Pi \oplus \Pi \oplus (-1)^{n-1} \Pi \). Hence \((G, \Pi)\) is a Nambu-Lie groupoid over a point. Conversely, if \((G, \Pi)\) is a Nambu-Lie groupoid over a point, then the group multiplication map \( m : G \times G \to G \) is a \((\Pi \oplus \Pi, \Pi)\)-map. Hence \((G, \Pi)\) is a Nambu-Lie group in the sense of [16]. One can also see the equivalence between Nambu-Lie groupoid over a point and Nambu-Lie group by using Remark 4.3.

For a Poisson groupoid the following facts are well known [17].

- The groupoid inversion map is a anti-Poisson map.
- The Poisson structure on the total space induces a Poisson structure on the base such that the source map is a Poisson map and the target map is a anti-Poisson map.

In the next proposition we generalize the above facts to the Nambu-Poisson setting.

4.6. **Proposition.** Let \((G \rightrightarrows M, \Pi)\) be a Nambu-Lie groupoid. Then

1. The inverse map \( i : G \to G, \ g \mapsto g^{-1} \) is an anti-Nambu Poisson map.
2. There is a unique Nambu-Poisson structure on \( M \) which we denote by \( \Pi_M \) for which \( \alpha \) is a Nambu-Poisson map and \( \beta \) is an anti-Nambu-Poisson map.

**Proof.** (1) It is proved in [13] that given a Lie groupoid \( G \rightrightarrows M \) with multiplicative \( n \)-vector field \( \Pi \in X^n(G) \), the groupoid inversion map \( i : G \to G \) satisfies
\[
i_*\Pi = (-1)^{n-1}\Pi.
\]
Hence the result follows as \( \Pi \) is a Nambu tensor.

(2) Let \( f_1, \ldots, f_n \in C^\infty(M) \) be any functions on \( M \). Then for any \( g \in G \), we have
\[
\{\alpha^*f_1, \ldots, \alpha^*f_n\}(g) = \Pi(g)(d_g(\alpha^*f_1), \ldots, d_g(\alpha^*f_n)) = \Pi(g)(\alpha^*(d_{\alpha(g)}f_1), \ldots, \alpha^*(d_{\alpha(g)}f_n)) = \alpha_*\Pi(g)(d_{\alpha(g)}f_1, \ldots, d_{\alpha(g)}f_n).
\]
Since \( \alpha_*\Pi(g) \) depends only on the value of \( \alpha(g) \), it follows that the function \( \{\alpha^*f_1, \ldots, \alpha^*f_n\} \) is constant on the \( \alpha \)-fibers. Therefore, by the Proposition 3.7, there exists a Nambu-Poisson structure \( \Pi_M \) with the induced bracket denoted by \( \{ \ldots, \} \) on \( M \) for which \( \alpha \) is a Nambu-Poisson map. Since \( \beta = \alpha \circ i \) and \( i \) is anti Nambu-Poisson, therefore \( \beta \) is an anti Nambu-Poisson map.

4.7. **Remark.** Consider the map \((\alpha, \beta) : G \to M \times M \). Since we have \( \alpha_*\Pi = \Pi_M \) and \( \beta_*\Pi = (-1)^{n-1}\Pi_M \), using property \((4)\) of the Theorem 4.2 we obtain
\[
(\alpha, \beta)_*\Pi = \alpha_*\Pi \oplus \beta_*\Pi = \Pi_M \oplus (-1)^{n-1}\Pi_M.
\]
4.8. **Proposition.** Let \((G \rightrightarrows M, \Pi)\) be a Nambu-Lie groupoid. If the orbit space \(M/\sim\) is a smooth manifold, then \(M/\sim\) carries a Nambu-Poisson structure such that the projection \(q : M \to M/\sim\) is a Nambu-Poisson map.

**Proof.** Let \(\Pi_M\) be the induced Nambu structure on the base \(M\). For the projection map \(q : M \to M/\sim\), we have

\[
\mathcal{R}(q) = \{(x,y) \in M \times M | q(x) = q(y)\} = \{(\alpha(g), \beta(g)) | g \in G\} = (\alpha, \beta)(G).
\]

Consider \(G\) as a coisotropic submanifold of \(G\) with respect to \(\Pi\) and also consider the map \((\alpha, \beta) : G \to M \times M\). By the above Remark we have \((\alpha, \beta)_\ast \Pi = \Pi_M \oplus (-1)^{n-1} \Pi_M\). Therefore, by the Proposition 3.4, \(\mathcal{R}(q) = (\alpha, \beta)(G)\) is a coisotropic submanifold of \(M \times M\) with respect to \(\Pi_M \oplus (-1)^{n-1} \Pi_M\). Hence the result follows from the Proposition 3.10. \(\Box\)

5. **Infinitesimal form of Nambu-Lie groupoid**

The aim of this section, is to study the infinitesimal form of a Nambu-Lie groupoid. We show that if \((G \rightrightarrows M, \Pi)\) is a Nambu-Lie groupoid of order \(n\) with Lie algebroid \(AG \to M\), then \((AG, A^\ast G)\) forms a weak Lie-Filippov bialgebroid of order \(n\) introduced in [1]. Before proceeding further, let us briefly recall from [1] the notion of a weak Lie-Filippov bialgebroid.

Lie bialgebroids are generalization of both Poisson manifolds and Lie bialgebras. Recall that a Lie bialgebroid, introduced by Mackenzie and Xu [10] is also the infinitesimal form of a Poisson groupoid. It is defined as a pair \((A, A^\ast)\) of Lie algebroids in duality, where the Lie bracket of \(A\) satisfies the following compatibility condition expressed in terms of the differential \(d^\ast\) on \(\Gamma(\bigwedge \cdot A)\)

\[
d^\ast[X,Y] = [d^\ast X, Y] + [X, d^\ast Y],
\]

for all \(X, Y \in \Gamma A\).

We note that if \(M\) is a Poisson manifold then the Lie algebroid structures on \(TM\) and \(T^\ast M\) form a Lie bialgebroid. On the other hand, it is well known [7,10] that if \((A, A^\ast)\) is a Lie bialgebroid over a smooth manifold \(M\) then there is a canonical Poisson structure on the base manifold \(M\).

Thus it is natural to ask the following question which was posed in [1]:

*Does there exist some notion of bialgebroid associated to a Nambu-Poisson manifold of order \(n > 2\)*?

To answer this question, the authors [1] introduced the notion of weak Lie-Filippov bialgebroid.

It is well known [4,16] that for a Nambu-Poisson manifold \(M\) of order \(n \geq 2\), the space \(\Omega^1(M)\) of 1-forms admits an \(n\)-ary bracket, called *Nambu-form bracket*, such that the bracket satisfies almost all the properties of an \(n\)-Lie algebra (also known as Filippov algebra of order \(n\)) bracket [3] except that the Fundamental identity is satisfied only in a restricted sense as described below.
Let \((M, \{\ldots,\})\) be a Nambu-Poisson manifold of order \(n\) with associated Nambu-Poisson tensor \(\Pi\). Then one can define the Nambu form-bracket on the space of 1-forms

\[ [\ldots] : \Omega^1(M) \times \cdots \times \Omega^1(M) \to \Omega^1(M) \]

by the following

\[
[a_1, \ldots, a_n] = \sum_{k=1}^{n} (-1)^{n-k} \mathcal{L}_{\Pi(a_1 \wedge \cdots \wedge \hat{a}_k \wedge \cdots \wedge a_n)} a_k - (n-1)d(\Pi(a_1, \ldots, a_n))
\]

for \(a_i \in \Omega^1(M), i = 1, \ldots, n\). Here \(\hat{a}_k\) in a monomial \(a_1 \wedge \cdots \wedge \hat{a}_k \wedge \cdots \wedge a_n\) means that the symbol \(a_k\) is missing in the monomial. The above bracket satisfies the following properties ([16]).

1. The bracket is skew-symmetric.
2. \([df_1, \ldots, df_n] = d\{f_1, \ldots, f_n\}\).
3. \([a_1, \ldots, a_{n-1}, f a_n] = f[a_1, \ldots, a_{n-1}, a_n] + \Pi^2(a_1 \wedge \cdots \wedge a_{n-1})(f)a_n\).
4. The bracket satisfies the Fundamental identity

\[
[a_1, \ldots, a_{n-1}, [\beta_1, \ldots, \beta_n]] = \sum_{k=1}^{n} [\beta_1, \ldots, \beta_{k-1}, [a_1, \ldots, a_{n-1}, \beta_k], \ldots, \beta_n]
\]

whenever the 1-forms \(a_i \in \Omega^1(M)\) are closed, \(1 \leq i \leq n-1\) and for any \(\beta_j\).

5. \([\Pi^1(a_1 \wedge \cdots \wedge a_{n-1}), \Pi^2(\beta_1 \wedge \cdots \wedge \beta_{n-1})] = \sum_{k=1}^{n-1} \Pi^2(\beta_1 \wedge \cdots \wedge [a_1, \ldots, a_{n-1}, \beta_k] \wedge \cdots \wedge \beta_{n-1})\)

for closed 1-forms \(a_i \in \Omega^1(M)\) and for any 1-forms \(\beta_j\).

The Nambu-form bracket on \(\Omega^1(M)\), together with the usual Lie algebroid structure on \(TM\) yields an example of a notion called a weak Lie-Filippov algebroid pair of order \(n\), \(n > 2\), on a smooth vector bundle (cf. Definition 5.5, [1]).

In order to classify such structures, the authors formulate a notion of Nambu-Gerstenhaber algebra of order \(n\). It turns out, weak-Lie-Filippov algebroid pair structures of order \(n\), \(n > 2\), on a smooth vector bundle \(A\) over \(M\), are in bijective correspondence with Nambu-Gerstenhaber brackets of order \(n\) on the graded commutative, associative algebra \(\Gamma \Lambda^* A^*\) of multisections of \(A^*\), where \(A^*\) is the dual bundle (cf. Definition 5.7, Theorem 5.8, [1]).

Moreover, for a Nambu-Poisson manifold \(M\) of order \(n > 2\), the Nambu-Gerstenhaber bracket on \(\Omega^1(M)\), extending the Nambu-form bracket on \(\Omega^1(M)\) satisfies certain suitable compatibility condition similar to the compatibility condition of a Lie bialgebroid. This motivates the authors to introduce the notion of a weak Lie-Filippov bialgebroid structure of order \(n\) on a smooth vector bundle.

5.1. Definition. A weak Lie-Filippov bialgebroid of order \(n > 2\) over a smooth manifold \(M\) consists of a pair \((A, A^*)\), where \(A\) is a smooth vector bundle over \(M\) with dual bundle \(A^*\) satisfying the following properties:
(1) $A$ is a Lie algebroid with $d_A$ being the differential of the Lie algebroid cohomology of $A$ with trivial representation;

(2) the space of smooth sections $\Gamma A^*$ admits a skew-symmetric $n$-ary bracket

\[ [\ldots, ] : \Gamma A^* \times \cdots \times \Gamma A^* \longrightarrow \Gamma A^* \]

satisfying

\[ [\alpha_1, \ldots, \alpha_{n-1}, [\beta_1, \ldots, \beta_n]] = \sum_{k=1}^{n} [\beta_1, \ldots, \beta_{k-1}, [\alpha_1, \ldots, \alpha_{n-1}, \beta_k], \ldots, \beta_n] \]

for all $d_A$-closed sections $\alpha_i \in \Gamma A^*$, $1 \leq i \leq n-1$ and for any sections $\beta_j \in \Gamma A^*$, $1 \leq j \leq n$;

(3) there exists a vector bundle map $\rho: \bigwedge^{n-1} A^* \longrightarrow TM$, called the anchor of the pair $(A, A^*)$, such that the identity

\[ [\rho(\alpha_1 \wedge \cdots \wedge \alpha_{n-1}), \rho(\beta_1 \wedge \cdots \wedge \beta_{n-1})] = \sum_{k=1}^{n-1} \rho(\beta_1 \wedge \cdots \wedge [\alpha_1, \ldots, \alpha_{n-1}, \beta_k] \wedge \cdots \wedge \beta_{n-1}) \]

holds for all $d_A$-closed sections $\alpha_i \in \Gamma A^*$, $1 \leq i \leq n-1$ and for any sections $\beta_j \in \Gamma A^*$, $1 \leq j \leq n-1$;

(4) for all sections $\alpha_i \in \Gamma A^*$, $1 \leq i \leq n$ and any $f \in C^\infty(M)$,

\[ [\alpha_1, \ldots, \alpha_{n-1}, f\alpha_n] = f[\alpha_1, \ldots, \alpha_{n-1}, \alpha_n] + \rho(\alpha_1 \wedge \cdots \wedge \alpha_{n-1})(f)\alpha_n \]

holds;

(5) the following compatibility condition holds:

\[ d_A[\alpha_1, \ldots, \alpha_n] = \sum_{k=1}^{n} [\alpha_1, \ldots, d_A\alpha_k, \ldots, \alpha_n], \]

for any $\alpha_i \in \Gamma A^*$, $1 \leq i \leq n$, where the bracket $[\ldots, ]$ on the right hand side is the graded extension of the bracket on $\Gamma A^*$.

A weak Lie-Filippov bialgebroid (of order $n$) over $M$ is denoted by $(A, A^*)$ when all the structures are understood. A Lie bialgebroid is a Lie-Filippov bialgebroid of order 2 such that the conditions (2) and (3) of the above definition has no restriction on $\alpha$.

In [1], the authors have shown that for a Nambu-Poisson manifold $M$ of order $n > 2$, the pair $(TM, T^*M)$ is a weak Lie-Filippov bialgebroid of order $n$ (cf. Corollary 6.3, [1]). It is also proved that if $(G, \Pi)$ is a Nambu-Lie group [16] of order $n$ with its Lie algebra $\mathfrak{g}$, then $(\mathfrak{g}, \mathfrak{g}^*)$ forms a (weak) Lie-Filippov bialgebroid of order $n$ over a Point.

It is known that the base of a Lie bialgebroid carries a natural Poisson structure. In [1] it has been extended to the Nambu-Poisson set up.

5.2. Proposition. ([1]) Let $(A, A^*)$ be a weak Lie-Filippov bialgebroid (of order $n$) over $M$. Then the bracket

\[ \{f_1, \ldots, f_n\}_{(A, A^*)} := \rho(d_Af_1 \wedge \cdots \wedge d_Af_{n-1})f_n \]

defines a Nambu-Poisson structure of order $n$ on $M$. 


It is known that, given a coisotropic submanifold $C$ of a Poisson manifold $M$, the conormal bundle $(TC)^0 \to C$ is a Lie subalgebroid of the cotangent Lie algebroid $T^*M$ \cite{17}. If $M$ is a Nambu-Poisson manifold of order $n$ ($n \geq 3$), the cotangent bundle $T^*M$ is not a Filippov algebroid. However we have the following useful result.

5.3. Proposition. Let $C$ be a closed embedded coisotropic submanifold of a Nambu-Poisson manifold $(M, \Pi)$ of order $n$. Then

1. the bundle map $\Pi^2 : \wedge^{n-1} T^*M \to TM$ maps $\wedge^{n-1}(TC)^0$ to $TC$;
2. the Nambu-form bracket on the space of 1-forms $\Omega^1(M)$ can be restricted to the sections of the conormal bundle $(TC)^0 \to C$.

Proof. The assertion (1) follows from the definition of coisotropic submanifold. To prove (2), let $\alpha_1, \ldots, \alpha_n \in \Gamma(TC)^0$. We extend them to 1-forms on $M$, which we denote by the same notation. Let $X \in \mathcal{X}(M)$ be such that $X\big|_C$ is tangent to $C$. From the definition of Nambu-form bracket on 1-forms, we have

$$\langle [\alpha_1, \ldots, \alpha_n], X \rangle = \sum_{k=1}^{n} (-1)^{n-k} \langle \mathcal{L}_{\Pi^2(\alpha_1 \wedge \cdots \wedge \alpha_k \wedge \cdots \wedge \alpha_n)} \alpha_k, X \rangle - (n - 1) \langle d(\Pi(\alpha_1, \ldots, \alpha_n)), X \rangle.$$

Observe that

$$\langle \mathcal{L}_{\Pi^2(\alpha_1 \wedge \cdots \wedge \alpha_k \wedge \cdots \wedge \alpha_n)} \alpha_k, X \rangle = \langle \mathcal{L}_{\Pi^2(\alpha_1 \wedge \cdots \wedge \alpha_k \wedge \cdots \wedge \alpha_n)} \alpha_k, X \rangle - \langle \alpha_k, [\Pi^2(\alpha_1 \wedge \cdots \wedge \alpha_k \cdots \wedge \alpha_n), X] \rangle.$$

This is zero on $C$, because,

- $\langle \alpha_k, X \rangle$ is zero on $C$;
- $\Pi^2(\alpha_1 \wedge \cdots \wedge \alpha_k \wedge \cdots \wedge \alpha_n)$ and $X$ are both tangent to $C$ and hence their Lie bracket is also tangent to $C$. Thus its pairing with $\alpha_k$ vanish on $C$.

Note that $\Pi^2(\alpha_1 \wedge \cdots \wedge \alpha_{n-1})|_C$ is tangent to $C$ and $\alpha_n|_C \in (TC)^0$. As a consequence, the function

$$\Pi(\alpha_1, \ldots, \alpha_n) = \langle \alpha_n, \Pi^2(\alpha_1 \wedge \cdots \wedge \alpha_{n-1}) \rangle$$

is zero on $C$. Therefore, the differential $d(\Pi(\alpha_1, \ldots, \alpha_n))$ restricted to $C$ is in $(TC)^0$, which in turn implies that the second term of the right hand side also vanish on $C$. Hence

$$[\alpha_1, \ldots, \alpha_n]|_C \in (TC)^0.$$

One can check that the restriction to $C$ does not depend on the chosen extension. Hence it defines a bracket on the sections of the conormal bundle $(TC)^0 \to C$.

5.4. Remark. (1) Let $m_0 \in M$ such that $\Pi(m_0) = 0$. Then \{m_0\} is a coisotropic submanifold of $M$. In this case, the conormal structure becomes $T^*_{m_0} M$, which is a Filippov algebra.

(2) The Nambu structure of a Nambu-Lie group $G$ vanishes at the identity element and therefore the dual $g^*$ of the Lie algebra $g$ of $G$ has a Filippov algebra structure \cite{16}.
5.5. Remark. Let \((G \rightrightarrows M, \Pi)\) be a Nambu-Lie groupoid of order \(n\) with Lie algebroid \(AG \to M\). By the Proposition 5.3, we see that the space of sections of the conormal bundle \(A^*G = (TM)^0 \to M\) admits a skew-symmetric \(n\)-bracket \([\ldots]\) and there exists a bundle map

\[
\rho := \Pi^2 \mid_{\Lambda^{n-1}(TM)^0} : \bigwedge^{n-1} A^*G = \bigwedge^{n-1} (TM)^0 \to TM,
\]

as \(M\) is a coisotropic submanifold of \(G\).

Let \((G \rightrightarrows M, \Pi)\) be a Nambu-Lie groupoid of order \(n\) with Lie algebroid \(AG \to M\). Let \(f \in C^\infty(M)\). Then by part (5) of the Theorem 4.2, \(\iota_{\delta^2 G} \Pi\) is a left invariant \((n-1)\)-vector field on \(G\). Therefore, there exists an \((n-1)\)-multisection \(\delta^0_\Pi(f) \in \Gamma \wedge^{n-1} AG\) of the Lie algebroid \(AG\) such that

\[
\iota_{\delta^2 G} \Pi = \delta^0_\Pi(f).
\]

Then we have the following result.

5.6. Proposition. Let \((G \rightrightarrows M, \Pi)\) be a Nambu-Lie groupoid of order \(n\) and \(AG \to M\) be its Lie algebroid. Then for any \(X \in \Gamma AG\),

\[
\mathcal{L}_X \Pi := [\tilde{X}, \Pi]
\]

is a left invariant \(n\) vector field on \(G\), where \(\tilde{X}\) is the left invariant vector field on \(G\) corresponding to \(X\). Moreover \(\mathcal{L}_X \Pi\) corresponds to the \(n\)-multisection \(-\delta^1_\Pi(X) \in \Gamma \wedge^n AG\), that is,

\[
\mathcal{L}_X \Pi = -\delta^1_\Pi(X)
\]

where \(\delta^1_\Pi(X) \in \Gamma \wedge^n AG\) is given by

\[
\delta^1_\Pi(X)(\alpha_1, \ldots, \alpha_n) = \sum_{k=1}^n (-1)^{n-k} \Pi^2(\alpha_1 \wedge \ldots \wedge \hat{\alpha}_k \wedge \ldots \wedge \alpha_n)(X(\alpha_k)) - X([\alpha_1, \ldots, \alpha_n])
\]

for \(\alpha_1, \ldots, \alpha_n \in \Gamma A^*G = \Gamma(TM)^0\).

Proof. Let \(X_t = \exp t X\) be the one-parameter family of bisections generated by \(X \in \Gamma AG\). Let \(g \in G\) with \(\beta(g) = u\). Let \(u_t = (\exp t X)(u)\) be the integral curve of \(\tilde{X}\) starting from \(u\). If \(G\) is any (local) bisection through \(g\), then from the multiplicativity condition of \(\Pi\) (cf. Theorem 4.2), we have

\[
\Pi(g u_t) = (r_{X_t^*} \Pi)(g) + (l_g)_* \Pi(u_t) - (r_{X_t} \Pi)(l_g)_* \Pi(u).
\]

Therefore,

\[
(r_{X_t^*} \Pi)(g u_t) - \Pi(g) = (r_{X_t} \Pi)(l_g)_* \Pi(u_t) - (l_g)_* \Pi(u).
\]

Taking derivative at \(t = 0\), one obtains

\[
(\mathcal{L}_X^\Pi)(g) = (l_g)_* ((\mathcal{L}_X^\Pi)(u)).
\]

Therefore, \(\mathcal{L}_X^\Pi\) is left invariant by the Proposition 2.12 and hence it corresponds to some \(n\)-multisection of \(AG\). To show that \(\mathcal{L}_X^\Pi\) corresponds to \(-\delta^1_\Pi(X) \in \Gamma \wedge^n AG\), we have
to check that $\mathcal{L}\hat{\Pi}(X)$ coincide on the unit space $M$ (both being left invariant). Since both of them are tangent to $\alpha$-fibers, it is enough to show that they coincide on the conormal bundle $(TM)^0$. Let $\alpha_1, \ldots, \alpha_n$ be any sections of $(TM)^0$ and $\hat{\alpha}_1, \ldots, \hat{\alpha}_n$ be their respective extensions to one forms on $G$. Observe that

$$\left(\mathcal{L}_{\hat{\Pi}}(X)\right)_{|M}(\alpha_1, \ldots, \alpha_n)$$

$$= \left\langle \hat{\Pi}, \left[d(\hat{\Pi}(\hat{\alpha}_1, \ldots, \hat{\alpha}_n))\right] \right\rangle - \sum_{k=1}^{n} \hat{\Pi}(\hat{\alpha}_1, \ldots, \hat{\alpha}_k, \mathcal{L}_{\hat{\Pi}}(\alpha_1, \ldots, \alpha_n)) \right\rangle_{|M}$$

$$= \left\langle \hat{\Pi}, \hat{\Pi}(\alpha_1, \ldots, \alpha_n) \right\rangle - \sum_{k=1}^{n} (-1)^{n-k} \left\langle \hat{\Pi}^2(\hat{\alpha}_1 \wedge \cdots \wedge \hat{\alpha}_k \wedge \cdots \wedge \hat{\alpha}_n), \mathcal{L}_{\hat{\Pi}}(\alpha_1, \ldots, \alpha_n) \right\rangle_{|M}$$

(from the Equation (2))

$$= \left\langle X, [\alpha_1, \ldots, \alpha_n] \right\rangle - \sum_{k=1}^{n} (-1)^{n-k} \left\langle \hat{\Pi}^2(\alpha_1 \wedge \cdots \wedge \alpha_k \wedge \cdots \wedge \alpha_n), \mathcal{L}_{\hat{\Pi}}(\alpha_1, \ldots, \alpha_n) \right\rangle_{|M}$$

(using Cartan formula)

$$= - \delta^1_{\hat{\Pi}}(X)(\alpha_1, \ldots, \alpha_n).$$

To make our notation simple, let us denote $\delta^0_{\hat{\Pi}}, \delta^1_{\hat{\Pi}}$ by the same symbol $\delta_{\hat{\Pi}}$. We extend $\delta_{\hat{\Pi}}$ to the graded algebra $\Gamma^\Lambda\wedge A$ of multisections of $AG$ by the following rule

$$\delta_{\hat{\Pi}}(P \wedge Q) = \delta_{\hat{\Pi}}(P) \wedge Q + (-1)^{|P|(n-1)}P \wedge \delta_{\hat{\Pi}}(Q)$$

for $P \in \Gamma^\Lambda[|P|] A, Q \in \Gamma^\Lambda[|Q|] A$. Then the operator

$$\delta_{\hat{\Pi}} : \Gamma^\Lambda \bigwedge^k AG \to \Gamma^\Lambda \bigwedge^{k+n-1} AG$$

satisfies

$$\delta_{\hat{\Pi}}([P, Q]) = [\delta_{\hat{\Pi}}(P), Q] + (-1)^{(|P|-1)(n-1)}[P, \delta_{\hat{\Pi}}(Q)].$$

Note that the operator $\delta_{\hat{\Pi}}$ need not satisfy condition $\delta_{\hat{\Pi}} \circ \delta_{\hat{\Pi}} = 0$.

We known that, Lie bialgebroids are infinitesimal form of Poisson groupoids. More precisely, given a Poisson groupoid $G \rightrightarrows M$ with Lie algebroid $AG$, it is known that its dual bundle $A^*G$ also carries a Lie algebroid structure and $(AG, A^*G)$ forms a Lie bialgebroid. In the next theorem we show that weak Lie-Filippov bialgebroids are infinitesimal form of Nambu-Lie groupoids.

5.7. Theorem. Let $(G \rightrightarrows M, \Pi)$ be a Nambu-Lie groupoid of order $n$ with Lie algebroid $AG \to M$. Then $(AG, A^*G)$ forms a weak Lie-Filippov bialgebroid of order $n$ over $M$. 
Proof. From the Remark 5.5, we have the space of sections of the bundle $A^*G = (TM)^0 \to M$ admits a skew-symmetric $n$-bracket $[,]\ldots[,]$ and there exists a bundle map

$$\rho : \bigwedge^{n-1} A^*G \to TM.$$

Let $\alpha_1, \ldots, \alpha_{n-1} \in \Gamma(TM)^0 = \Gamma(A^*G)$ with $d_A \alpha_i = 0$, for all $i = 1, \ldots, n-1$. Let $\bar{\alpha}_1, \ldots, \bar{\alpha}_{n-1}$ be, respectively, their extensions to left invariant 1-forms on $G$ such that $d\bar{\alpha}_i = 0$, for all $i = 1, \ldots, n-1$. Then the conditions (2) and (3) of the Definition 5.1 of a weak Lie-Filippov algebroid pair follows from the weak Lie-Filippov bialgebroid structure $(TG, T^*G)$ (Note that $G$ is a Nambu-Poisson manifold).

Let $f \in C^\infty(M)$. Then observe that

$$[\bar{\alpha}_1, \ldots, \bar{\alpha}_{n-1}, (\beta^* f)\bar{\alpha}_n] = (\beta^* f)[\bar{\alpha}_1, \ldots, \bar{\alpha}_{n-1}, \bar{\alpha}_n] + \Pi^1(\bar{\alpha}_1 \wedge \cdots \wedge \bar{\alpha}_{n-1})(\beta^* f)\bar{\alpha}_n.$$

Since $(\beta^* f)\bar{\alpha}_n = \tilde{f}\alpha_n$, by assertion (2) of the Proposition 5.3, we get

$$[\alpha_1, \ldots, \alpha_{n-1}, f\alpha_n] = f[\alpha_1, \ldots, \alpha_n] + \rho(\alpha_1 \wedge \cdots \wedge \alpha_{n-1})(f)\alpha_n$$

proving condition (4) of the Definition 5.1. Moreover the compatibility condition of the weak Lie-Filippov bialgebroid (condition (5) of the Definition 5.1) follows from the observation that for any $\alpha \in \Gamma(A^*G) = \Gamma(TM)^0$ and any left invariant extension $\tilde{\alpha} \in \Omega^1(G)$, we have

$$d_A \alpha = (d\tilde{\alpha})|_M.$$

Thus, $(AG, A^*G)$ is a weak Lie-Filippov bialgebroid of order $n$. \hfill \Box

5.8. Remark. If $(G, \Pi)$ is a Nambu-Lie group with Lie algebra $\mathfrak{g}$, the dual vector space $\mathfrak{g}^*$ carries a Filippov algebra structure [16]. Moreover the pair $(\mathfrak{g}, \mathfrak{g}^*)$ forms a (weak) Lie-Filippov bialgebra ([1,16]). The Lie-Filippov bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ is the infinitesimal form of the Nambu-Lie group $(G, \Pi)$. A Lie-Filippov bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ can also be seen a Lie algebra $\mathfrak{g}$ together with a Filippov algebra structure on the dual vector space $\mathfrak{g}^*$ such that the map $\delta : \mathfrak{g} \to \bigwedge^n \mathfrak{g}$ dual to the Filippov bracket on $\mathfrak{g}^*$, defines a 1-cocycle of $\mathfrak{g}$ with respect to the adjoint representation on $\bigwedge^n \mathfrak{g}$.

We have seen that given a Nambu-Lie groupoid of order $n$, there is an induced Nambu-Poisson structure on the base manifold (cf. Proposition 4.6). On the other hand, given a weak Lie-Filippov bialgebroid, there is an induced Nambu-Poisson structure on the base (cf. Theorem 5.2). The next proposition compares these Nambu-Poisson structures on the base induced from the Nambu Lie groupoid and its infinitesimal.

5.9. Proposition. Let $(G \Rightarrow M, \Pi)$ be a Nambu-Lie groupoid (of order $n$) with associated weak Lie-Filippov bialgebroid $(AG, A^*G)$. Then the induced Nambu structures on $M$ coming from the Nambu-Lie groupoid and the weak Lie-Filippov bialgebroid are related by

$$\{\ldots,\} = (-1)^{n-1}\{(AG, A^*G)\}.$$

Proof. For any functions \(f_1, \ldots, f_n \in C^\infty(M)\), we have
\[
\{(f_1, \ldots, f_n)\}_{(AG, A^*G)} = \Pi|_M (d_A f_1 \wedge \cdots \wedge d_A f_{n-1}) f_n \\
= \Pi(d(\beta^* f_1) \wedge \cdots \wedge d(\beta^* f_{n-1}))|_M f_n \\
= \Pi(\beta^* f_1, \ldots, \beta^* f_{n-1}, \beta^* f_n)|_M \\
= (-1)^{n-1} (\beta^* \{f_1, \ldots, f_n\})|_M \\
= (-1)^{n-1} \{f_1, \ldots, f_n\}_M.
\]

\[\square\]

5.10. Remark. It is known that under some connectedness and simply connectedness assumption, any Lie bialgebra integrates to a Poisson-Lie group \([8]\), and any Lie bialgebroid integrates to a Poisson groupoid \([11]\). These results does not hold in the context of Nambu structures of order \(\geq 3\). Let \(G\) be a connected and simply-connected Lie group with Lie algebra \(\mathfrak{g}\). Given a Lie-Filippov bialgebra structure \((\mathfrak{g}, \mathfrak{g}^*)\) on \(\mathfrak{g}\), the 1-cocycle \(\delta: \mathfrak{g} \to \wedge^n \mathfrak{g}\) dual to the Filippov algebra bracket on \(\mathfrak{g}^*\) integrates a multiplicative \(n\)-vector field \(\Pi\) on the Lie group. However this \(n\)-vector field (for \(n \geq 3\)) need not be a Nambu tensor \([16]\), that is, need not be locally decomposable. Thus (weak) Lie-Filippov bialgebra does not integrate to a Nambu-Lie group in general.

6. Coisotropic subgroupoids of a Nambu-Lie groupoid

In this final section, we introduce the notion of coisotropic subgroupoid of a Nambu-Lie groupoid and study the infinitesimal object corresponding to it.

6.1. Definition. Let \((G \Rightarrow M, \Pi)\) be a Nambu-Lie groupoid of order \(n\). Then a subgroupoid \(H \Rightarrow N\) is called a coisotropic subgroupoid if \(H\) is a coisotropic submanifold of \(G\) with respect to \(\Pi\).

6.2. Example. (1) For \(n = 2\), that is, when \(G \Rightarrow M\) is a Poisson groupoid, this notion is same as the coisotropic subgroupoid of a Poisson groupoid introduced in \([18]\).

(2) Let \((G, \Pi)\) be a Nambu-Lie group. Then a subgroup of \(G\) is called coisotropic if it is also a coisotropic submanifold of \(G\). Any coisotropic subgroup of \(G\) is a coisotropic subgroupoid over a point.

(3) Let \((G \Rightarrow M, \Pi)\) be a Nambu-Lie groupoid. Then by the Proposition 4.6, there exist an induced Nambu-structure on \(M\) for which the source map \(\alpha\) is a Nambu-Poisson map. Let \(N \Rightarrow M\) be a coisotropic submanifold of \(M\) with respect to this induced Nambu structure. Consider the restriction \(G|_N := \alpha^{-1}(N) \cap \beta^{-1}(N)\), then \(G|_N \Rightarrow N\) is a coisotropic subgroupoid.

(4) Let \(G \Rightarrow M\) be a Nambu-Lie groupoid. If the set of all elements of \(G\) which has same source and target, is a submanifold of \(G\), then it is a coisotropic subgroupoid.

Note that, the infinitesimal object corresponding to a Nambu-Lie groupoid \((G \Rightarrow M, \Pi)\) is the weak Lie-Filippov bialgebroid \((AG, A^*G)\). Therefore it is natural to ask how the Lie algebroid of a coisotropic subgroupoid \(H \Rightarrow N\) is related to the weak Lie-Filippov
bialgebroid \((AG, A^*G)\). To answer this question, we introduce a notion of coisotropic subbialgebroid of a weak Lie-Filippov bialgebroid and show that infinitesimal forms of coisotropic subgroupoids of a Nambu-Lie groupoid \((G \rightrightarrows M, \Pi)\) appear as coisotropic subalgebroids of the corresponding weak Lie-Filippov bialgebroid \((AG, A^*G)\).

6.3. Definition. Let \((A, A^*)\) be a weak Lie-Filippov bialgebroid of order \(n\) over \(M\). Then a Lie subalgebroid \(B \to N\) of \(A \to M\) is called a coisotropic subalgebroid if the anchor \(\rho : \bigwedge^{n-1} A^* \to TM\) and the \(n\)-bracket \([\ldots]\) on \(\Gamma A^*\) satisfy the following properties.

1. The anchor \(\rho\) maps \(\bigwedge^{n-1} B^0 \to TN\).
2. If \(\alpha_1, \ldots, \alpha_n \in \Gamma A^*\) with \(\alpha_i|_N \in B^0\) for all \(i\), then \([\alpha_1, \ldots, \alpha_n]|_N \in B^0\).
3. If \(\alpha_1, \ldots, \alpha_n \in \Gamma A^*\) with \(\alpha_i|_N \in B^0\) for all \(i\) and \(\alpha_n|_N = 0\), then \([\alpha_1, \ldots, \alpha_n]|_N = 0\).

where \(B_x^0 = \{\gamma \in A_x^* | \gamma(v) = 0, \forall v \in B_x\}\), is the annihilator of \(B_x\), \(x \in N\).

6.4. Example. Let \(M\) be a Nambu-Poisson manifold, then \((TM, T^*M)\) is a weak Lie-Filippov bialgebroid over \(M\). Let \(N \rightrightarrows M\) be a coisotropic submanifold. Then from the Proposition 5.3, it follows that the tangent bundle \(TN \to N\) is a coisotropic subalgebroid.

It is known that (Proposition 5.2, see also [1]), the base of a weak Lie-Filippov bialgebroid carries a Nambu structure. The next Proposition shows that the base of a coisotropic subalgebroid is a coisotropic submanifold with respect to this induced Nambu structure.

6.5. Proposition. Let \((A, A^*)\) be a weak Lie-Filippov bialgebroid over \(M\) and \(B \to N\) be a coisotropic subalgebroid. Then \(N\) is a coisotropic submanifold of \(M\).

Proof. Let \(a : A \to TM\) denote the anchor of the Lie algebroid \(A\) and \(\rho : \bigwedge^{n-1} A^* \to TM\) be the anchor of pair \((A, A^*)\). We first show that, \(a^*(TN)^0 \subseteq B^0\). This is true because,

\[\langle a^*\xi_x, v \rangle = \langle \xi_x, a(v) \rangle = 0\] for \(\xi_x \in (TN)_x^0\) and \(v \in B_x\).

Let \(\Pi^{\sharp}_{(A, A^*)}\) be the induced Nambu structure on \(M\) coming from the weak Lie-Filippov bialgebroid \((A, A^*)\). Then the induced map \(\Pi^{\sharp}_{(A, A^*)} : \bigwedge^{n-1} A^*M \to TM\) is given by

\[\Pi^{\sharp}_{(A, A^*)} = \rho \circ \bigwedge^{n-1} a^*\].

Therefore, for any \(\xi_1, \ldots, \xi_{n-1} \in (TN)^0\), we have

\[\Pi^{\sharp}_{(A, A^*)}(\xi_1, \ldots, \xi_{n-1}) = \rho(a^*\xi_1, \ldots, a^*\xi_{n-1}) \in TN\]
as \(a^*\xi_i \in B^0\) and \(B\) is a coisotropic subalgebroid. Therefore \(N\) is a coisotropic submanifold of \(M\).

The next proposition shows that the infinitesimal object corresponding to coisotropic subgroupoids are coisotropic subalgebroids.

6.6. Proposition. Let \((G \rightrightarrows M, \Pi)\) be a Nambu-Lie groupoid with weak Lie-Filippov bialgebroid \((AG, A^*G)\). Let \(H \rightrightarrows N\) be a coisotropic subgroupoid of \(G \rightrightarrows M\) with Lie algebroid \(AH \to N\). Then \(AH \to N\) is a coisotropic subalgebroid.

Proof. Since \(H \rightrightarrows N\) is a Lie subgroupoid of \(G \rightrightarrows M\), therefore \(AH \to N\) is a Lie subalgebroid of \(AG \to M\). We claim that the anchor \(\rho = \Pi^{\sharp}_{(A, A^*)}|_{\bigwedge^{n-1} (TM)^0} = \Pi^{\sharp}_{(A, A^*)}|_{\bigwedge^{n-1} (AG)^0}\) of the weak Lie-Filippov bialgebroid \((AG, A^*G)\) maps \(\bigwedge^{n-1} (AH)^0\) to \(TN\). First observe that, for any
$x \in N, (AH)_x^0 = (TM)_x^0 \cap (TH)_x^0$ and $T_xN = T_xM \cap T_xH$. Therefore, $\rho$ maps $\wedge^{n-1}(AH)^0$ to

$$\Pi^2(\wedge^{n-1}(TM)^0) \cap \Pi^2(\wedge^{n-1}(TH)^0) \subseteq TM \cap TH \cong TN,$$

here we have used the fact that $M$ and $H$ are both coisotropic submanifolds of $G$.

Let $\alpha_1, \ldots, \alpha_n \in \Gamma^*G = \Gamma(TM)^0$ such that $\alpha_i|_N \in (AH)^0$, for all $i = 1, \ldots, n$. Let $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n$ be one forms on $G$ extending $\alpha_1, \ldots, \alpha_n$ and are conormal to $H$. Then by the Proposition 5.3, the 1-form $[\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n]$ is conormal to both $M$ and $H$, as $M$ and $H$ are both coisotropic submanifolds of $G$. Therefore,

$$[\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n]_N \in (TM)^0 \cap (TH)^0 \cong (AH)^0.$$

Verification of the last condition of the Definition 6.3 is similar. Hence $AH \to N$ is a coisotropic subalgebroid of $(AG, A^*G)$. □

### 6.7. Corollary

Let $(G \rightrightarrows M, \Pi)$ be a Nambu-Lie groupoid and $H \rightrightarrows N$ be a coisotropic subgroupoid. Then $N$ is a coisotropic submanifold of $M$.

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