Intersection Homology of Linkage Spaces

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Given $\ell = (\ell_1, \ldots, \ell_n) \in \mathbb{R}^n$ with $\ell_i > 0$, we want to look at closed linkages in some euclidean space $\mathbb{R}^d$ up to rotation

$$
\mathcal{M}_d(\ell) = \left\{ (z_1, \ldots, z_n) \in (S^{d-1})^n \left| \sum_{i=1}^n \ell_i z_i = 0 \right. \right\} \bigg/ \text{SO}(d)
$$

and study how the topology depends on the length vector $\ell$. 
If $J \subset \{1, \ldots, n\}$ define
\[
H_J = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{j \in J} x_j = \sum_{i \not\in J} x_i\right\}
\]

Then
\[
\mathbb{R}^n - \bigcup H_J
\]
consists of finitely many components, called *chambers*. If $\ell$ sits in one of these chambers, it is called *generic*. This is in fact equivalent to $\mathcal{M}_1(\ell) = \emptyset$. If $\ell$ and $\ell'$ are in the same chamber, then $\mathcal{M}_d(\ell) \cong \mathcal{M}_d(\ell')$.

Also, if we define for $\sigma \in \Sigma_n$ the length vector $\ell^\sigma$ by
\[
\ell^\sigma = (\ell_{\sigma 1}, \ldots, \ell_{\sigma n})
\]

it is clear that $\mathcal{M}_d(\ell^\sigma) \cong \mathcal{M}_d(\ell)$.

One may therefore ask whether these are the only conditions leading to the same spaces.
Theorem (Farber, Hausmann, S)

Let $\ell, \ell'$ be generic, such that $\mathcal{M}_d(\ell) \simeq \mathcal{M}_d(\ell')$ for $d = 2$ or $3$.

1. If $d = 2$, then $\ell$ and $\ell'$ are in the same chamber up to permutation.

2. If $d = 3$, and $n \geq 5$, then $\ell$ and $\ell'$ are in the same chamber up to permutation.

The proof requires a good understanding of the cohomology ring. In particular, knowing the homology is not enough.

For $d \geq 4$, work of Schoenberg ('69) shows that $\mathcal{M}_d(\ell^n)$ is homeomorphic to a sphere for $n = d + 1$ and to a disc for $n \leq d$.

Definition

Call $J \subset \{1, \ldots, n\}$

- $\ell$-short, if $\sum_{j \in J} \ell_j < \sum_{i \notin J} \ell_i$
- $\ell$-long, if $\sum_{j \in J} \ell_j > \sum_{i \notin J} \ell_i$
For generic $\ell$, knowing the short subsets determines the chamber that $\ell$ is in.

**Definition**

Let $\ell \in \mathbb{R}^n$ be a length vector and $d \geq 2$. Then $\ell$ is called $d$-regular, if

$$\bigcap_{J \in \mathcal{L}^d(\ell)} J \neq \emptyset$$

where $\mathcal{L}^d(\ell)$ are the subsets $J \subset \{1, \ldots, n\}$ with $d - 1$ elements that are $\ell$-long. If $\mathcal{L}^d(\ell) = \emptyset$, we let the intersection above be $\{1, \ldots, n\}$.

**Theorem**

*Let $d \geq 4$ be even, $\ell, \ell' \in \mathbb{R}^n$ be generic, $d$-regular length vectors. If $\mathcal{M}_d(\ell)$ and $\mathcal{M}_d(\ell')$ are homeomorphic, then $\ell$ and $\ell'$ are in the same chamber up to a permutation.*
The proof uses Intersection Homology rather than Cohomology.

The spaces $\mathcal{M}_d(\ell)$ are pseudomanifolds with stratification coming from

$$\emptyset \subset \mathcal{N}_2(\ell) \subset \mathcal{N}_3(\ell) \subset \cdots \subset \mathcal{N}_{d-2}(\ell) \subset \mathcal{M}_d(\ell).$$

Here $\mathcal{N}_i(\ell)$ is the image of the 2:1 map $\mathcal{M}_i(\ell) \to \mathcal{M}_d(\ell)$ induced by inclusion. Furthermore, the dimension of $\mathcal{M}_d(\ell)$ is

$$d^n_d = (n-3)(d-1) - \frac{(d-2)(d-3)}{2},$$

If $\ell = (\ell_1, \ldots, \ell_n)$ is an ordered length vector, and $J \subset \{1, \ldots, n-1\}$, let $\ell_J$ be the length vector which fixes the links of $J \cup \{n\}$ into the last link. Hence

$$\mathcal{M}_d(\ell_J) \subset \mathcal{M}_d(\ell).$$
If we choose the perversity

\[ p^k = (2k, 3k, \ldots, (d - 2)k), \]

we get a well defined element

\[ [\mathcal{M}_d(\ell_J)] \in I^{p^{|J|}} H_{d - |J|}^{n - |J|}(\mathcal{M}_d(\ell)) \]

Also, intersection homology has an intersection pairing, which leads to a product

\[ \cap : I^{p^k} H_{d - k}^{n - k}(\mathcal{M}_d(\ell)) \times I^{p^l} H_{d - l}^{n - l}(\mathcal{M}_d(\ell)) \to I^{p^{k+l}} H_{d - (k+l)}^{n - (k+l)}(\mathcal{M}_d(\ell)) \]

We can then define a ring

\[ IH^*(\mathcal{M}_d(\ell)) \]

generated by the elements of \( I^{p^1} H_{d}^{n - 1}(\mathcal{M}_d(\ell)) \) which is a homeomorphism invariant of the pseudomanifold.
For $d$ even it can be shown that the ring $IH^*(\mathcal{M}_d(\ell); \mathbb{Q})$ is generated by the elements

$$[\mathcal{M}_d(\ell\{1\})], \ldots, [\mathcal{M}_d(\ell\{n-1\})]$$

and therefore turns $IH^*(\mathcal{M}_d(\ell); \mathbb{Q})$ into a quotient of an exterior algebra.

The products satisfy

$$[\mathcal{M}_d(\ell_J)] \cdot [\mathcal{M}_d(\ell_K)] = [\mathcal{M}_d(\ell_{J \cup K})] \text{ for } J \cap K = \emptyset.$$

and we can show that all $[\mathcal{M}_d(\ell_J)]$ are linearly independent, as long as $J \cup \{n\}$ is short.

So $IH^*(\mathcal{M}_d(\ell); \mathbb{Q})$ is a so-called exterior face ring, and the Theorem can be derived in the same way as the cases $d = 2$ and $d = 3$. 
Why do we need $d$-regularity? If $J \cup \{n\}$ is short, but has too many elements, then $M_d(\ell_J)$ is topologically a disc, and is not detected by homology. If $\ell$ is $d$-regular, then $M_d(\ell_J)$ will either be a pseudomanifold or empty, but never a disc.

If $d$ is odd, we do not get a quotient of an exterior algebra. Furthermore, there is an extra generator in $IH^*(M_d(\ell))$.

It may be possible that $IH^*(M_d(\ell))$ only depends on the parity of $d$, in which case one can use the knowledge of the case $d = 3$ to extend the theorem to all odd $d \geq 5$, but so far this is only a conjecture.