On an interpolative Schrödinger equation and an alternative classical limit

K. R. W. Jones

Physics Department,
University of Queensland,
St Lucia 4072, Brisbane, Australia.
(Revised version 13/10/92)

Abstract

We introduce a simple deformed quantization prescription that interpolates the classical and quantum sectors of Weinberg’s nonlinear quantum theory. The result is a novel classical limit where $\hbar$ is kept fixed while a dimensionless mesoscopic parameter, $\lambda \in [0, 1]$, goes to zero. Unlike the standard classical limit, which holds good up to a certain timescale, ours is a precise limit incorporating true dynamical chaos, no dispersion, an absence of macroscopic superpositions and a complete recovery of the symplectic geometry of classical phase space. We develop the formalism, and discover that energy levels suffer a generic perturbation. Exactly, they become $E(\lambda^2\hbar)$, where $\lambda = 1$ gives the standard prediction. Exact interpolative eigenstates can be similarly constructed. Unlike the linear case, these need no longer be orthogonal. A formal solution for the interpolative dynamics is given, and we exhibit the free particle as one exactly soluble case. Dispersion is reduced, to vanish at $\lambda = 0$. We conclude by discussing some possible empirical signatures, and explore the obstructions to a satisfactory physical interpretation.

PACS numbers: 03.65.Bz, 02.30.+g, 03.20.+i, 0.3.65.Db

* Author’s note: Archival version of an old preprint restored from obsolete electronic media. This was first submitted to Phys. Rev. D back in 1992 but rejected as being of “insufficient interest”. It describes a generalized dynamical system which contains an exact embedding of the classical Hamiltonian point mechanics alongside the entire non-relativistic quantum theory. The two are joined by a one-parameter family of deformed dynamics in a dimensionless parameter with $\hbar$ kept constant throughout. Originally, the work was presented as a toy model called $\lambda$–dynamics for the purpose of illustrating the absurdity of the Copenhagen Interpretation conception of a “classical domain” sitting alongside a “quantum domain”. The relevance of the work today is primarily mathematical. This preprint will be superceded by a more contemporary study of this system in relation to the Renormalization Group and the connection between classical and quantum dynamics. This work posted under Creative Commons 3.0 - Attribution License.
I. INTRODUCTION

It is generally thought that classical dynamics is a limiting case of quantum dynamics. Certainly, the subclass of coherent states admit a rigorous reduction of quantum dynamics to classical dynamics as $\hbar \to 0$[1]. However, as many authors have noted[2], this result does not hold for all quantum states. To see this most clearly imagine that we are in the deep semiclassical regime. I may choose two coherent states centered about different points in phase space. These follow the classical trajectories over some finite time interval with an error, and dispersion, that can be made as small as one pleases. Any linear combination of these is also a solution of the Schrödinger evolution, but it need not follow any classical trajectory[3].

Ordinarily we solve this problem by prohibiting the appearance of such states at the classical level[4]. This can be partially justified using measurement as a means to remove coherences[5]. For practical purposes the dilemma is of no consequence[6]. We are not forbidden to use the old theory, when appropriate[7]. The problem is thus one of consistency (for example, Ford et al.[8] argue that quantum suppression of dynamical chaos[9] spells trouble for the correspondence principle). If quantum theory is universal then why does it not give us a clean, simple, and chaotic reduction[10]?

In this paper we outline such a reduction. To do this we must pay a heavy price and forsake the assumption of universality. Keeping that which is good, we require a generalized theory which contains both classical and quantum dynamics. The only candidate we know of is Weinberg’s nonlinear quantum theory[11, 12]. Elsewhere we used this to recast exact Hamiltonian classical mechanics[13]. Here we develop a way to pass smoothly between both regimes. Our motivation is curiosity; to find a nice way to do this, irrespective of what it might mean. However, where possible we have attempted to interpret the formalism as physical theory. This is fraught with interpretative difficulty, but some generic empirical signatures can be extracted.

To set the scene, quantum theory is superbly successful. In looking around to find trouble’s mark, we can think of no place but the classical regime (gravitation, the most classical theory, remains the hardest uncraked nut). It is at the interface between the microworld and the macroworld that aesthetic dissaffection arises, for it is here that quantum stochasticity and measurement prove necessary. Most “resolutions”, “new interpretations”, whatever...,
depart little from the orthodox theory. Here our philosophy is to first enlarge quantum
dynamics and then seek a natural way to blend the classical and quantum components
together. The interpolative dynamics is then put forward as a candidate to describe a
regime that borders the cut we customarily make in everyday calculations. We make a
guess at some kind of general theoretical structure in which to think around the questions.
Without evidence that quantum theory fails we can do no more. Why do it then? Because
when no alternative exists we are unlikely to find any failure.

II. CLASSICAL MECHANICS IN WEINBERG’S THEORY

Unlike regular classical mechanics, the carbon copy within Weinberg’s theory employs
wavefunctions, $\psi$ and the commutation relation $[\hat{q}, \hat{p}] = i\hbar$. To form it we take any classical
function, say $H(q, p)$, and turn it into a Weinberg observable\[14\] via the ansatz

$$h_0(\psi, \psi^*) \equiv \langle \psi | H(\langle \hat{q} \rangle, \langle \hat{p} \rangle) | \psi \rangle,$$

(1)

where $\langle \hat{q} \rangle \equiv \langle \psi | \hat{q} | \psi \rangle / n$, $\langle \hat{p} \rangle \equiv \langle \psi | \hat{p} | \psi \rangle / n$, with $n = \langle \psi | \psi \rangle$. Commutators are then replaced
by the Weinberg bracket,

$$[g, h]_W \equiv g * h - h * g,$$

(2)

where $g * h = \delta \psi g \delta \psi^* h$ and $\delta \psi$, and $\delta \psi^*$ are shorthand for functional derivatives\[15\]. Canonical
commutators then translate to: $[\langle \hat{q} \rangle, \langle \hat{q} \rangle]_W = 0$, $[\langle \hat{p} \rangle, \langle \hat{p} \rangle]_W = 0$, and $[\langle \hat{q} \rangle, \langle \hat{p} \rangle]_W = i\hbar / n$. The
equation of motion now reads,

$$i\hbar \frac{dg}{dt} = [g, h]_W.$$

(3)

Taking the special functionals (1) one shows\[13\] that $[g, h]_W = i\hbar n \{G, H\}_PB$. Then, since
the dynamics is norm preserving, we have $dn / dt = 0$ and (3) reduces to

$$\frac{dG}{dt} = \{G, H\}_PB \equiv \partial_{\langle \hat{q} \rangle} G \partial_{\langle \hat{p} \rangle} H - \partial_{\langle \hat{p} \rangle} G \partial_{\langle \hat{q} \rangle} H.$$

(4)

Hitherto, noncommutativity was thought to embody the essential difference between the
classical and quantum theories. Now we see things differently, (3) reduces to (4) for any
value of $\hbar$.

What, then, is the fundamental difference? To see this, we simply compare the classical
functional ansatz (1) to the Weinberg analogue of canonical quantization,

$$h_1(\psi, \psi^*) \equiv \langle \psi | \hat{H}(\hat{q}, \hat{p}) | \psi \rangle.$$

(5)
Now equation (3) reduces to the familiar result

\[ i\hbar \frac{d}{dt} \langle \psi | \hat{G} | \psi \rangle = \langle \psi | [\hat{G}, \hat{H}] | \psi \rangle. \]

(6)

Clearly, Weinberg’s theory is general enough to embrace both standard quantum theory and a novel wave version of Hamiltonian classical mechanics.

Our point of departure for an alternative classical limit is the recovery of a familiar result. Comparing (1) and (5), we write

\[ h_1 = h_0 \left( 1 + \frac{h_1 - h_0}{h_0} \right). \]

(7)

At any \( \hbar \) the classical approximation is good for those \( \psi \) such that \( (h_1 - h_0)/h_0 \ll 1 \). Two features deserve explicit note: the smaller is \( \hbar \) the better is the approximation for a given \( \psi \); and, for all non-zero \( \hbar \), there exist states such that the criterion fails.

III. AN INTERPOLATIVE DOMAIN?

Some functionals \( h(\psi, \psi^*) \) are classical, of form (1), others are quantal, of form (5), while most are neither. Since both sectors are disjoint for all \( \hbar \) we seek an interpolation which joins them. In physical terms, we imagine that the correspondence principle is to be taken literally. Thus we speculate that, some objects, composed of many quantum particles, act as a collective mesoparticle[16], with a center of mass dynamics that is neither strictly quantum nor strictly classical, but some curious blend of both. For simplicity, we assume that a one–particle equation can do this many–particle job.

IV. DEFORMED QUANTIZATION

A. The mathematical notion

To formulate this concept we generalize the central idea of canonical quantization and postulate a map which sends any classical phase space function \( H(q, p) \) into a one–parameter family of interpolative Weinberg observables \( h_\lambda(\psi, \psi^*) \). Symbolically, we write

\[ Q_\psi^\lambda \vdash H(q, p) \xmapsto{\lambda} h_\lambda(\psi, \psi^*), \]

(8)
and call $Q^\lambda_\psi$ a *deformed quantization*. Imposing (1) and (5) as known boundary conditions, we interpret $\lambda \in [0, 1]$ as a dimensionless index of mesoscopic effects.

Since $\lambda$ is to govern emergence of classical behaviour we expect it to depend upon some function of particle size, mass, number, or mixture thereof. There is no way to guess this. Some authors suggest that gravity could have something to do with it[17]. Here we pick $\lambda(m) \equiv 1/(1+(m/m_P)^\alpha)$, for some $\alpha > 0$, where $m_P = 2.177 \times 10^{-5} \text{g}$ is the Planck mass to illustrate how the proposal might work[18]. However, we emphasize that $\lambda$ is an adjustable parameter which cannot be fixed within this framework.

**B. The specific proposal**

With only the boundary conditions known we cannot fix (8) uniquely. However, since the ansatz (1) contains only expectations, and (5) only operators, it is suggestive to deform the particle coordinates via the simple convex combination[19]:

\[
\hat{q}_\lambda \equiv \lambda \hat{q} + (1-\lambda)\langle \hat{q} \rangle, \\
\hat{p}_\lambda \equiv \lambda \hat{p} + (1-\lambda)\langle \hat{p} \rangle.
\]

(9)

(10)

This prescription is unique among linear combinations once we impose the physical constraints: $q_\lambda \equiv \langle \hat{q}_\lambda \rangle = \langle \hat{q} \rangle$, and $p_\lambda \equiv \langle \hat{p}_\lambda \rangle = \langle \hat{p} \rangle$. These enforce invariance of both the center of mass coordinates, and the canonical Weinberg bracket relations under deformation.

Having chosen the deformed operators we now select the obvious generalization of canonical quantization:

\[
Q^\lambda_\psi \vdash H(q, p) \mapsto h_\lambda(\psi, \psi^*) = \langle \psi | \hat{H}^\lambda | \psi \rangle,
\]

(11)

where $\hat{H}^\lambda \equiv \hat{H}(\hat{q}_\lambda, \hat{p}_\lambda)$, and, for definiteness, we assume that $\hat{q}_\lambda$ and $\hat{p}_\lambda$ are Weyl–ordered[20]. As we now show, (11) gives an interpolative dynamical system with some interesting properties. For inessential simplicity we treat only systems with one classical degree of freedom. The generalization is straightforward.
V. THE REDUCED WEINBERG BRACKET

A. A general reduction lemma

Of fundamental importance is the effect of the ansatz (11) upon the bracket (2). We begin with a computation for the more general class of functionals

\[ h(\psi, \psi^*) \equiv \langle \psi | \hat{H}'(\hat{q}, \langle \hat{q} \rangle; \hat{p}, \langle \hat{p} \rangle) | \psi \rangle, \]  

(12)

with \( H'(q_1, q_2; p_1, p_2) \) an auxiliary c-number function. Applying the chain rule first, the functional derivative of this expands to

\[ \delta \psi h = \langle \psi | \hat{H}' + \langle \psi | \partial_{\langle \hat{q} \rangle} \hat{H}' | \psi \rangle \delta \psi \langle \hat{q} \rangle + \langle \psi | \partial_{\langle \hat{p} \rangle} \hat{H}' | \psi \rangle \delta \psi \langle \hat{p} \rangle. \]  

(13)

Evaluating \( \delta \psi \langle \hat{q} \rangle \) and \( \delta \psi \langle \hat{p} \rangle \) gives the bra–like pair:

\[ \delta \psi \langle \hat{q} \rangle = \langle \psi | (\hat{q} - \langle \hat{q} \rangle)/n, \]  

(14)

\[ \delta \psi \langle \hat{p} \rangle = \langle \psi | (\hat{p} - \langle \hat{p} \rangle)/n. \]  

(15)

Taking hermitian adjoints of (13), (14) and (15) gives the ket–like quantities \( \delta_{\psi^*} h, \delta_{\psi^*} \langle \hat{q} \rangle \), and \( \delta_{\psi^*} \langle \hat{p} \rangle \). Using these rules it becomes a simple matter to expand

\[ [g, h]_W = \delta_{\psi^*} g \delta_{\psi^*} h - \delta_{\psi^*} g \delta_{\psi^*} h. \]

In reducing the expansion it is helpful to identify like terms and to make frequent use of (14), (15) and their adjoints. Of special utility is a family of results like

\[ \delta_{\psi^*} \langle \hat{q} \rangle \hat{H}' | \psi \rangle - \langle \psi | \hat{H}' \delta_{\psi^*} \langle \hat{q} \rangle = i\hbar \langle \psi | \partial_{\langle \hat{p} \rangle} \hat{H}' | \psi \rangle /n, \]

where \( \partial_{\hat{q}} \equiv [\bullet, \hat{p}] / i\hbar \), and \( \partial_{\hat{p}} \equiv [\hat{q}, \bullet] / i\hbar \). Then, after some cancellation using canonical bracket relations, and some rearrangement, we find that

\[ [g, h]_W = \langle \psi | [\hat{G}', \hat{H}'] | \psi \rangle \]

\[ + i\hbar \left\{ \langle \psi | \partial_{\langle \hat{q} \rangle} \hat{G}' | \psi \rangle \langle \psi | \partial_{\langle \hat{p} \rangle} \hat{H}' | \psi \rangle - \langle \psi | \partial_{\langle \hat{p} \rangle} \hat{G}' | \psi \rangle \langle \psi | \partial_{\langle \hat{q} \rangle} \hat{H}' | \psi \rangle \right\} /n \]

\[ + i\hbar \left\{ \langle \psi | \partial_{\hat{q}} \hat{G}' | \psi \rangle \langle \psi | \partial_{\hat{p}} \hat{H}' | \psi \rangle - \langle \psi | \partial_{\hat{p}} \hat{G}' | \psi \rangle \langle \psi | \partial_{\hat{q}} \hat{H}' | \psi \rangle \right\} /n \]

\[ + i\hbar \left\{ \langle \psi | \partial_{\langle \hat{q} \rangle} \hat{G}' | \psi \rangle \langle \psi | \partial_{\langle \hat{p} \rangle} \hat{H}' | \psi \rangle - \langle \psi | \partial_{\langle \hat{p} \rangle} \hat{G}' | \psi \rangle \langle \psi | \partial_{\langle \hat{q} \rangle} \hat{H}' | \psi \rangle \right\} /n. \]  

(16)

This expression is rather more general than is required, but displays the essential origin of our next result.
B. Reduction for interpolative observables

To treat the interpolative case (11) we choose

\[ H' = H(\lambda q_1 + (1 - \lambda)q_2, \lambda p_1 + (1 - \lambda)p_2). \]

Then, \( \partial_q H' = \lambda \hat{H}_q^\lambda \), \( \partial_{\langle \hat{q} \rangle} H' = (1 - \lambda) \hat{H}_q^\lambda \), \( \partial_p H' = \lambda \hat{H}_p^\lambda \), and \( \partial_{\langle \hat{p} \rangle} H' = (1 - \lambda) \hat{H}_p^\lambda \), where \( \hat{H}_q^\lambda \) and \( \hat{H}_p^\lambda \) denote the quantized classical partials of \( H(q,p) \). Thus (16) becomes

\[
[g_\lambda, h_\lambda]_W = \langle \psi | [\hat{G}^\lambda, \hat{H}^\lambda] | \psi \rangle + i\hbar (1 - \lambda^2) \left\{ \langle \psi | \hat{G}^\lambda_q | \psi \rangle \langle \psi | \hat{H}_p^\lambda | \psi \rangle - \langle \psi | \hat{G}^\lambda_p | \psi \rangle \langle \psi | \hat{H}_q^\lambda | \psi \rangle \right\}/n. \tag{17}
\]

Thus the ansatz (11) collects the three residual terms of (16) into a “mean-field” Poisson bracket[21]. The scale factor \( (1 - \lambda^2) \) now controls the mixture of quantum and classical effects[22].

VI. AN INTERPOLATIVE SCHRÖDINGER EQUATION

A. The equation of motion for expectation values

Substituting (17) into (3), and using the property that \( dn/dt = 0 \), now gives

\[
\frac{d\langle G^\lambda \rangle}{dt} \equiv \langle [\hat{G}^\lambda, \hat{H}^\lambda] \rangle/i\hbar + (1 - \lambda^2) \left\{ \langle \hat{G}^\lambda_q \rangle \langle \hat{H}_p^\lambda \rangle - \langle \hat{G}^\lambda_p \rangle \langle \hat{H}_q^\lambda \rangle \right\}, \tag{18}
\]

where \( \langle \bullet \rangle \equiv \langle \psi | \bullet | \psi \rangle/n \). This provides an interpolative analogue of the standard Schrödinger picture equation of motion for expectation values.

Of course, at \( \lambda = 0 \) all deformed operators commute and the first term vanishes. We are thus left with the second term alone and (4) drops out directly. The other limit \( \lambda = 1 \) kills the second term, operators revert to their standard canonical quantizations and (6) results. So the commutator term is certainly “quantum” and the bracket term is certainly “classical”.
B. An interpolative Ehrenfest theorem

Applying (18) to the coordinate operators now gives an interpolative Ehrenfest–type theorem[23]:

\[
\frac{d\langle \hat{q}_\lambda \rangle}{dt} = +\langle \hat{H}_p^\lambda \rangle \tag{19}
\]

\[
\frac{d\langle \hat{p}_\lambda \rangle}{dt} = -\langle \hat{H}_q^\lambda \rangle. \tag{20}
\]

Using this we obtain valuable insight about how wave propagation is affected by \( \lambda \). For instance, choosing \( H(q,p) = p^2/2m + V(q) \), we find that the only change appears in the force term. After some rearrangement, this reads

\[
\frac{d\langle \hat{p}_\lambda \rangle}{dt} = -\langle V_q((\hat{q}) + \lambda[\hat{q} - \langle \hat{q} \rangle])\rangle
\]

\[= -\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \langle (\hat{q} - \langle \hat{q} \rangle)^k \rangle \partial_q^{k+1} V(\langle \hat{q} \rangle). \tag{21}\]

Looking at this we see that \( \lambda \) controls the range at which the wavefunction \( \psi \) probes the potential \( V(q) \). At the classical extreme, wavepackets feel only the classical force at their centre, whereas, in the quantum extreme, this is averaged over space[16].

C. The interpolative wave equation

Consider now Weinberg’s generalized Schrödinger equation[24]

\[
i\hbar \frac{d\psi}{dt} = \delta_{\psi^*} h.\]

Although nonlinear, standard Hilbert space methods are easily adapted by using the definition (11), along with the hermitian adjoints of (13), (14) and (15), to introduce an effective Hamiltonian operator,

\[
\hat{H}^\lambda_{\text{eff}}(\psi, \psi^*) \equiv \hat{H}^\lambda +
\]

\[
(1 - \lambda) \left\{ \langle \hat{H}_q^\lambda(\hat{q} - \langle \hat{q} \rangle) + \langle \hat{H}_p^\lambda(\hat{p} - \langle \hat{p} \rangle) \right\}. \tag{22}\]

such that \( \delta_{\psi^*} h = \hat{H}^\lambda_{\text{eff}}(\psi, \psi^*)|\psi\rangle \). This operator defines the interpolative Schrödinger equation[25],

\[
i\hbar \frac{d|\psi\rangle}{dt} = \hat{H}^\lambda_{\text{eff}}(\psi, \psi^*)|\psi\rangle. \tag{23}\]
One verifies easily that \( \lambda = 1 \) returns the ordinary linear Schrödinger equation. However, for \( \lambda \neq 1 \) the operator is generally \( \psi \)-dependent.

This property is responsible for the failure of many standard results, such as the superposition principle, preservation of the \textit{global} inner product between distant states, and the orthogonality of eigenvectors for self-adjoint operators. Proofs of these assume that \( \hat{H} \) is the \textit{same} for any quantum state.

Choosing \( H(q, p) = \frac{p^2}{2m} + V(q) \), we set \( \hat{q} = q \) and \( \hat{p} = -i\hbar \partial_q \). Then defining,

\[
Q(t) = n^{-1} \int_{-\infty}^{\infty} \psi(q, t)^* q \psi(q, t) \, dq
\]

\[
P(t) = n^{-1} \int_{-\infty}^{\infty} \psi(q, t)^* \{-i\hbar \partial_q \psi(q, t)\} \, dq,
\]

\[
F(t) = n^{-1} \int_{-\infty}^{\infty} \psi(q, t)^* V_q (\lambda q + (1 - \lambda)Q(t)) \psi(q, t) \, dq,
\]
equations (22) and (23) yield the explicit nonlinear integrodifferential wave equation,

\[
i\hbar \frac{\partial \psi(q, t)}{\partial t} = \frac{1}{2m} \left\{-\lambda^2 \hbar^2 \partial_q^2 - 2i\hbar(1 - \lambda^2)P(t)\partial_q - (1 - \lambda^2)P^2(t)\right\} \psi(q, t)
\]

\[
+ \{V(\lambda q + (1 - \lambda)Q(t)) + (1 - \lambda)F(t)(q - Q(t))\} \psi(q, t).
\]

The nonlinearity of (23) lies in those terms carrying state dependent parameters (24), (25) and (26). Given the complexity of this form, abstract operator techniques are preferable. Calculations with the explicit equation are hideous.

Of particular interest is the case \( \lambda = 0 \). From (22) we compute the effective classical Hamiltonian

\[
\hat{H}_{\text{eff}}^0(\psi, \psi^*) \equiv H(\langle \hat{q} \rangle, \langle \hat{p} \rangle) +
H_q(\langle \hat{q} \rangle, \langle \hat{p} \rangle)(\hat{q} - \langle \hat{q} \rangle) + H_p(\langle \hat{q} \rangle, \langle \hat{p} \rangle)(\hat{p} - \langle \hat{p} \rangle).
\]

Combining (23) and (27) now gives us a classical Schrödinger equation.

\[ \text{VII. A CLASSICAL SCHröDINGER EQUATION} \]

From (4), we know that all solutions \( \psi(t) \) must have expectations, \( Q(t) \equiv \langle \hat{q} \rangle \) and \( P(t) \equiv \langle \hat{p} \rangle \), that precisely follow the classical trajectories of any chosen \( H(q, p) \), for all time, and for all values of \( \hbar \). We now construct the explicit solution.
A. An heuristic overview

For short time intervals, $\Delta t$, we can assume that (27) is constant. In the simplest approximation, we let $\psi_{t_0}$ be the initial wavefunction, and construct the infinitesimal unitary propagator

$$\hat{U}_{\Delta t} \approx \exp \left\{ -\frac{i\Delta t}{\hbar} \hat{H}_{\text{eff}}(\psi_{t_0}, \psi_{t_0}^*) \right\}.$$ \hspace{1cm} (28)

Then, since (27) is linear in $\hat{q}$ and $\hat{p}$, it follows that (28) is a member of the Heisenberg–Weyl group[26]. Operators of this type assume the general form,

$$\hat{U}[Q,P; S] \equiv \exp \left\{ \frac{i}{\hbar} \left[S \hat{1} + P\hat{q} - Q\hat{p}\right]\right\},$$ \hspace{1cm} (29)

and obey the operator relations:

$$\hat{U}^\dagger[Q,P; S]\hat{q}\hat{U}[Q,P; S] = \hat{q} + Q\hat{1},$$ \hspace{1cm} (30)

$$\hat{U}^\dagger[Q,P; S]\hat{p}\hat{U}[Q,P; S] = \hat{p} + P\hat{1}.$$ \hspace{1cm} (31)

Comparing (28) and (29), and rewriting the definition (27) in the form,

$$\hat{H}_{\text{eff}}^0 = -\{QH_q + PH_p - H\} \hat{1} + H_q\hat{q} + H_p\hat{p},$$ \hspace{1cm} (32)

now gives the approximate result

$$|\psi_{t_0+\Delta t}\rangle \approx \hat{U}_{\Delta t}|\psi_{t_0}\rangle$$

$$= \hat{U}[+H_q\Delta t, -H_q\Delta t; \Delta S]|\psi_{t_0}\rangle,$$

with $\Delta S = \{Q(t_0)H_q + P(t_0)H_p - H\}\Delta t$. Invoking (30) and (31), it follows that:

$$Q(t_0 + \Delta t) \approx Q(t_0) + H_p(Q(t_0), P(t_0))\Delta t,$$

$$P(t_0 + \Delta t) \approx P(t_0) - H_q(Q(t_0), P(t_0))\Delta t.$$

These considerations show how the effective Hamiltonian (27) propagates any wave $\psi$ along classical trajectories, as expected from equation (4).

B. The exact treatment

Suppose we construct the operator $U[Q(t), P(t); S(t)]$ using parameters $Q(t)$ and $P(t)$ that are obtained from solving Hamilton’s equations for the initial conditions, $Q(t_0)$, and
\( P(t_0) \). Specifically, we demand that,

\[
\begin{align*}
\dot{P}(t) &= -\partial_q H(Q,P), \\
\dot{Q}(t) &= +\partial_p H(Q,P),
\end{align*}
\]

(33)  
(34)  
for all \( t \geq t_0 \). Then, choosing \( \psi_0 \) to be an arbitrary state with both coordinate expectation values equal to zero, we construct the trial solution

\[
|\psi_t\rangle = U[Q(t), P(t); S(t)]|\psi_0\rangle, \quad t \geq t_0.
\]

Equation (4) is now trivially satisfied. To verify (23) note that \( \hat{U}[t] \) determines,

\[
\hat{\mathcal{H}}(t) \equiv i\hbar \left\{ \frac{d}{dt} \hat{U}[t] \right\} \hat{U}^\dagger[t],
\]

(35)  
Then, using the Weyl multiplication rule[26],

\[
\hat{U}[Q_2, P_2; S_2] \hat{U}^\dagger[Q_1, P_1; S_1] = e^{i/2\hbar(P_1Q_2 - Q_1P_2)} \times \\
\hat{U}[Q_2 - Q_1, P_2 - P_1; S_2 - S_1],
\]

(36)  
and (35), we compute:

\[
\hat{H}_\text{eff} = i\hbar \lim_{\delta t \to 0} \frac{\hat{U}[Q(t + \delta t), P(t + \delta t); S(t + \delta t)] \hat{U}^\dagger[Q(t), P(t); S(t)] - \hat{1}}{\delta t}
\]

\[
= i\hbar \lim_{\delta t \to 0} \frac{e^{i\delta t/2\hbar(P\dot{Q} - Q\dot{P})} \hat{U}^\dagger[Q\delta t, P\delta t; S\delta t] - \hat{1}}{\delta t}
\]

\[
= - \{(P\dot{Q} - Q\dot{P})/2 + \dot{S}\} - \dot{P}\hat{q} + \dot{Q}\hat{p}.
\]

Comparing this to (32), we first pick out (33) and (34) as necessary conditions. Then, looking at the constant term, we solve for \( \dot{S} \) to obtain \( \dot{S} = 1/2(P\dot{Q} - Q\dot{P}) - H \). Integrating \( \dot{S} \) now gives the exact classical propagator,

\[
\hat{U}[t] = \exp \left\{ \frac{i}{\hbar} \left[ \phi(t)\hat{1} + P(t)\hat{q} - Q(t)\hat{p} \right] \right\},
\]

(37)  
where \( Q(t) \) and \( P(t) \) obey (33), and the phase factor \( \phi(t) \) reads

\[
\phi(t) = \int_{t_0}^t \left( \frac{P\dot{Q} - Q\dot{P}}{2} \right) - H(Q,P) \, d\tau.
\]

(38)  
Unlike ordinary classical mechanics, our wave version has an extra degree of freedom; a phase factor. As one might have expected[27], this phase records the classical action. However, unlike linear theory, the phase–to–action correspondence is now exact.
C. Phase anholonomy effects

Interestingly, (38) contains a simple Berry phase\(^{28}\). To isolate this we employ the Aharanov–Anandan\(^{29}\) formula, \(\dot{\gamma}(t) = i \langle \tilde{\psi}| \{d/dt|\tilde{\psi} \rangle \}, \) where \(\tilde{\psi}\) is a ray–space trajectory. If \(|\tilde{\psi}(0)\rangle\) is any state with vanishing coordinate expectations, then a ray path can be parametrized as \(|\tilde{\psi}(t)\rangle = \hat{U}[Q(t), P(t); 0]|\tilde{\psi}(0)\rangle\), to give,

\[\dot{\gamma}(t) = i \langle \tilde{\psi}(0)| \hat{U}^\dagger[t]\left\{ \frac{d}{dt}\hat{U}[t] \right\} |\tilde{\psi}(0)\rangle\]

\[= \langle \tilde{\psi}(0)| (P\dot{Q} - Q\dot{P})/2 - \dot{P}\tilde{q} + \dot{Q}\tilde{p}|\tilde{\psi}(0)\rangle /\hbar\]

\[= (P\dot{Q} - Q\dot{P})/2\hbar.\]

On a closed loop \(\Gamma\), we find \(\int_0^T P\dot{Q} dt = +f_\Gamma P dQ\), and \(\int_0^T Q\dot{P} dt = -f_\Gamma P dQ\), where \(T\) is the circuit time and signs are fixed by the sense of traversal. Thus,

\[\gamma(\Gamma) = + \frac{1}{\hbar} \int_\Gamma P dQ. \tag{39}\]

This explicit relationship suggests that geometric phases upon closed loops might well be interpreted as the natural action variables of quantum mechanics.

D. Explicit wavefunction solutions

Returning to (37), we now seek explicit wavefunction solutions. Passing to the Schrödinger representation, \(\hat{q} \mapsto q\), and \(\hat{p} \mapsto -i\hbar\partial_q\), we note the standard result\(^{26}\),

\[U[Q, P; 0]\psi(q) = e^{-iPQ/2\hbar} e^{iPq/\hbar}\psi(q - Q). \tag{40}\]

Then, given any state \(\psi_0(q)\) with both expectation values equal to zero, equation (37) yields

\[\psi(q, t) = e^{i\phi(t)/\hbar} e^{-iP(t)Q(t)/2\hbar} e^{iP(t)q/\hbar}\psi_0(q - Q(t)).\]

Looking at this we see directly that all waves propagate without dispersion. The arbitrary wave envelope \(\psi_0(q)\) preserves its shape while being moved around in Hilbert space via its expectation value parameters \(Q(t)\) and \(P(t)\). Therefore, no interference or tunnelling is possible in this limit. A wave–packet must reflect or pass a barrier with certainty, just as a point particle does in ordinary classical mechanics. Suppose we fire a packet at a double slit. Then it must go through either one or the other slit, or it must strike the
slit screen and return. Hence it is possible to view interference and diffraction phenomena as products of linear dynamics. Pick the right kind of nonlinear propagation, and they evaporate altogether[30].

E. The recovery of classical phase space

Since the wave aspects are frozen out, we can now build a faithful analogue of classical phase space. To define this, we introduce the coordinate map,

\[ \Pi : \mathcal{H} \mapsto \mathbb{R}^2 \text{ where } \Pi[\psi] = (\langle \hat{q} \rangle, \langle \hat{p} \rangle). \]

The appropriate mathematical object involves a partition of Hilbert space into disjoint sets of wavefunctions which share identical coordinate expectations. These sets are defined as the \( \Pi \)-induced equivalence classes,

\[ \tilde{\psi}(Q, P) = \{ \psi \in \mathcal{H} \mid \Pi[\psi] = (Q, P) \in \mathbb{R}^2 \}. \]

One can now treat the labels \((Q, P)\) as points, just like in ordinary classical phase space. Each emblazons a bag of \( \Pi \)-equivalent wavefunctions. We think of the classical limit as a dynamical regime where \( \psi \) does not matter, only its parameters \((Q, P)\). The original classical Hamiltonian \( H(q, p) \) now determines, via the ansatz (1), and equations, (23), and (27), a symplectomorphism of this phase space[31, 32].

VIII. THE INTERPOLATIVE PROPAGATOR

A. The Liouville equation

Introducing a Liouville operator \( \mathcal{L}_h \equiv [\bullet, \hbar]_W \), such that \( \mathcal{L}_h \circ g \equiv [g, \hbar]_W \) with iterated “powers”: \( \mathcal{L}_h^{k+1} \circ g = [\mathcal{L}_h^k \circ g, \hbar]_W \), we can obtain a formal solution to (3) via exponentiation of the “tangent vector” identity \( \frac{d}{dt} \equiv \mathcal{L}_h/\hbar i \). Thus,

\[ g_t = \exp \{ -i(t - t_0)\mathcal{L}_h/\hbar \} \circ g_{t_0}, \quad (41) \]

where \( L_{\Delta t} \equiv e^{-i(t-t_0)\mathcal{L}_h/\hbar} \) is the Liouville propagator. Now, \( \mathcal{L}_h \circ (f + g) = \mathcal{L}_h \circ f + \mathcal{L}_h \circ g \), so \( L_{\Delta t} \) is a linear operator on the vector space of Weinberg observables. However, because \( \mathcal{L}_h \) depends, via \( \hbar \), upon \( \psi \), the object \( L_{\Delta t} \) is usually a nonlinear operator when acting on
wavefunctions. Therefore, one must be exceedingly careful to distinguish the trivial pseudo–
superposition

\[(f + g)_t(\psi, \psi^*) = f_t(\psi, \psi^*) + g_t(\psi, \psi^*),\]

which is always valid, from the special *trajectorial* superposition property

\[(\psi + \phi)(t) = \psi(t) + \phi(t).\]

This is valid when \(h(\psi, \psi^*)\) is a linear functional in both slots[33], but fails in general (one
sees this easily from (23), if \(\hat{H}\) depends upon \(\psi\) then we cannot add operators for different
states).

**B. The classical propagator**

Using the identity \([g_0, h_0]_W = i\hbar n\{G, H\}_PB\), valid for functionals of type (1), and the fact
that \(dn/dt = 0\), we recover the well–known classical result:

\[G_t = G_{t_0} + \{G_{t_0}, H_{t_0}\}_PB(t - t_0) + \frac{1}{2!}\{\{G_{t_0}, H_{t_0}\}_PB, H_{t_0}\}_PB(t - t_0)^2 + \ldots.\]

Similarly, one can use (41) to expand a formal solution for the classical Schrödinger equation.
Here there is no need given the exact solution (37).

**C. The quantum propagator**

For quantum functionals, as defined by (4), we invoke the identity \([g_1, h_1]_W = \langle \psi|[[\hat{G}, \hat{H}]]|\psi\rangle\), and (41) becomes:

\[\langle \hat{G}\rangle_t = \langle \hat{G}\rangle_{t_0} + \langle [\hat{G}, \hat{H}]\rangle_{t_0}(t - t_0)/i\hbar + \frac{1}{2!}\langle [[[\hat{G}, \hat{H}]], \hat{H}]\rangle_{t_0}(t - t_0)^2/(i\hbar)^2 + \ldots.\]

Similarly, using \([\psi, h_1]_W = \hat{H}|\psi\rangle\), one gets \(|\psi_t\rangle = e^{-i(t-t_0)\hat{H}/\hbar}|\psi_{t_0}\rangle\). In this special case the
propagator does not depend upon \(|\psi_{t_0}\rangle\).

This property encodes the superposition principle. All complexity lies in the propagator,
which happens to be independent of the initial condition for linear theory. More generally
this is not the case. Treating function–valued curves \(\psi(t)\) as “trajectories”, the overlap:

\[D(\psi, \psi') = 1 - |\langle \psi | \psi' \rangle|^2 \text{ where } D \in [0, 1],\]  

(42)
need not be constant in time. Divergence, and the possibility of strong divergence (i.e. “ex-
ponential”, in some sense), is thus permitted in the nonlinear sector. To formalize this notion
one can look to extend the KS–entropy, or the classical Lyapunov exponent to Weinberg’s
theory[35] via use of the metric (42) (see [36], for its properties).

D. Dynamical chaos in the interpolative regime?

In the interpolative case, an explicit computation of the iterated bracket (17) is pro-
hibitive. Nevertheless, the existence of a formal solution permits direct study of the formal
computability properties of both the classical and quantal dynamics. Ford et al.’s algorithm-
ic information theory approach[8] to the study of “quantum chaos” might extend in this
direction.

On the numerical front, one needs to ascertain when, and how, exactly, quantum sup-
pression of chaos is switched off. Certainly, it must happen at some \( \lambda \in [0, 1] \). Since (17)
has a Poisson bracket contribution for every \( \lambda \neq 1 \), this is the candidate chaos factory[34].

IX. INTERPOLATIVE EIGENSTATES

A. The fundamental variational principle

Weinberg has generalized the eigenstates of linear quantum theory as stationary points
of the normalized observables via the simple variational principle[37],

\[
\delta \left( \frac{h(\psi, \psi^*)}{n(\psi, \psi^*)} \right) = 0, \tag{43}
\]

which is equivalent to[11]:

\[
\delta_{\psi^*} \left( \frac{h}{n} \right) = \frac{1}{n} \delta_{\psi^*} h - \frac{h}{n^2} \delta_{\psi^*} n = 0, \tag{44}
\]

\[
\delta_\psi \left( \frac{h}{n} \right) = \frac{1}{n} \delta_\psi h - \frac{h}{n^2} \delta_\psi n = 0. \tag{45}
\]

In the linear case this reduces to the familiar result \( \hat{H}\left|\psi\right\rangle = E\left|\psi\right\rangle \). Given the form of (22),
we expect a similar result for the special interpolative observables (11).
B. Some preliminary observations

Suppose, first of all, that \( \psi \) is a stationary point of the Weinberg observable \( h(\psi, \psi^*) \). Then, if \( a(\psi, \psi^*) \) is any other Weinberg observable, we can use the definitions (2), (44) and (45) to compute

\[
[a, h]_W = \delta_\psi a \delta_\psi^* h - \delta_\psi h \delta_\psi^* a
\]

\[
= \frac{h}{n} (\delta_\psi a \delta_\psi^* n - \delta_\psi n \delta_\psi^* a)
\]

\[
= \frac{h}{n} [a, n]_W = 0. \tag{46}
\]

This property generalizes the obvious fact that an eigenstate \( \psi \) of the linear operator \( \hat{H} \), must return \( \langle \psi | [\hat{A}, \hat{H}] | \psi \rangle = 0 \), for all \( \hat{A} \).

As an immediate consequence of (46) we deduce, via the expressions (19) and (20), that

\[
\langle \hat{H}^{\lambda}_q \rangle = 0, \quad \text{and} \quad \langle \hat{H}^{\lambda}_p \rangle = 0, \tag{47}
\]

of necessity.

C. The interpolative eigenvalue equation

To construct the stationarity conditions for (22), we substitute \( \delta_\psi^* h = \hat{H}^{\lambda}_{\text{eff}} | \psi \rangle \) into (44), identify \( \delta_\psi^* n = | \psi \rangle \), and obtain the eigenvalue equation

\[
\hat{H}^{\lambda}_{\text{eff}} | \psi \rangle = \langle \hat{H}^{\lambda}_{\text{eff}} \rangle | \psi \rangle. \tag{48}
\]

Combining (47) with (22) we see that

\[
(1 - \lambda) \left\{ \langle H^{\lambda}_q \rangle (\hat{q} - \langle \hat{q} \rangle) + \langle H^{\lambda}_p \rangle (\hat{p} - \langle \hat{p} \rangle) \right\} \equiv 0, \tag{49}
\]

which reduces (48) to

\[
\hat{H}^{\lambda} | \psi \rangle = E^{\lambda} | \psi \rangle, \tag{50}
\]

with the deformed eigenvalue,

\[
E^{\lambda} \equiv \langle \hat{H}^{\lambda}_{\text{eff}} \rangle = \langle \hat{H}^\lambda \rangle.
\]

So (48) implies (50). Passing in the other direction, we assume that \( \lambda \neq 0 \), and notice that:

\[
\lambda \langle \hat{H}^{\lambda}_q \rangle = \langle [\hat{H}^\lambda, \hat{p}] \rangle / i\hbar, \quad \text{and} \quad \lambda \langle \hat{H}^{\lambda}_p \rangle = \langle [\hat{q}, \hat{H}^\lambda] \rangle / i\hbar,
\]

16
whence (50) implies (47), (49), and thus (48).

To treat the exceptional point $\lambda = 0$, we invoke (47) alone, and deduce that the classical stationary states of the deformed dynamical system comprise all $\psi$ such that $\langle \hat{q} \rangle$ and $\langle \hat{p} \rangle$ lie at a fixed point of the classical Hamiltonian flow (as one might have guessed). Clearly, such states have infinite degeneracy, with a deformed eigenvalue that is precisely the classical energy at the fixed point.

D. The general solution via linear quantum theory

Equation (50) is simpler than (48), but there remains a bothersome difficulty in that

$$\hat{H}^\lambda = \hat{H}(\lambda \hat{q} + (1 - \lambda)\langle \hat{q} \rangle, \lambda \hat{p} + (1 - \lambda)\langle \hat{p} \rangle).$$

(51)

Although the expectation values are stationary, we have to solve (50) self-consistently.

To fix this trouble, we bootstrap from solutions of the simpler, linear, eigenvalue problem,

$$\hat{H}(\lambda \hat{q}, \lambda \hat{p}) |\psi\rangle = E^\lambda |\psi\rangle.$$  

(52)

Defining the new operators: $\hat{q}' \equiv \lambda \hat{q}$, and $\hat{p}' \equiv \lambda \hat{p}$, we observe that $[\hat{q}', \hat{p}'] = i\hbar'$ with $\hbar' = \lambda^2 \hbar$. Equation (52) is, therefore, just the standard eigenvalue problem with a rescaled value of $\hbar$.

Given a parametric family of $\hbar$–dependent eigenstates $\psi(q; \hbar)$, eigenvalues $E(\hbar)$, and eigenstate expectations, $Q(\hbar)$, and $P(\hbar)$, for the ordinary Schrödinger problem, we identify:

$$\hbar' \mapsto \hbar' = \lambda^2 \hbar$$

$$\hat{q}' \mapsto q' = \lambda q$$

$$\hat{p}' \mapsto -i\hbar' \partial_q = -i(\lambda^2 \hbar) \partial_{(\lambda q)} = \lambda(-i\hbar \partial_q).$$

Thus the solution to (52) is obtained by applying the rescalings $q \mapsto \lambda q$ and $\hbar \mapsto \lambda^2 \hbar$ to the known solutions for the $\lambda = 1$ problem. Imposing the constraint, $\int_{-\infty}^{\infty} \psi(\lambda q)\psi^*(\lambda q) \, dq = 1$, now fixes the renormalized quantities:

$$\langle q|\psi_\lambda \rangle = \lambda^{1/2} \psi(\lambda q; \lambda^2 \hbar),$$

(53)

$$E^\lambda = E(\lambda^2 \hbar),$$

(54)

$$Q^\lambda = \langle \psi_\lambda|\hat{q}|\psi_\lambda \rangle = Q(\lambda^2 \hbar)/\lambda,$$

(55)

$$P^\lambda = \langle \psi_\lambda|\hat{p}|\psi_\lambda \rangle = P(\lambda^2 \hbar)/\lambda.$$  

(56)

17
Using these expressions we can construct a solution to the general problem (50).

First we form, after (29), and using (55) and (56), the Weyl operator,
\[
\hat{V} \equiv \hat{U}[(1 - \lambda)Q^\lambda, (1 - \lambda)P^\lambda].
\] (57)

Applying this to both sides of (52) gives,
\[
\hat{V}^\dagger \hat{H}(\lambda \hat{q}, \lambda \hat{p}) \hat{V}^\dagger |\psi_\lambda\rangle = E^\lambda \hat{V}^\dagger |\psi_\lambda\rangle.
\]

Thus we can identify,
\[
|\psi'_\lambda\rangle = \hat{V}^\dagger |\psi_\lambda\rangle
\] (58)
as an eigenstate of the new operator, \(\hat{V}^\dagger \hat{H}(\lambda \hat{q}, \lambda \hat{p}) \hat{V}\) with the eigenvalue \(E^\lambda\) unchanged.

Using (30), (31) and (58) we compute:
\[
\langle \psi'_\lambda | \hat{q} | \psi'_\lambda \rangle / n = \langle \psi_\lambda | \hat{q} - (1 - \lambda)Q^\lambda | \psi_\lambda \rangle / n = \lambda Q^\lambda,
\] (59)
\[
\langle \psi'_\lambda | \hat{p} | \psi'_\lambda \rangle / n = \langle \psi_\lambda | \hat{p} - (1 - \lambda)P^\lambda | \psi_\lambda \rangle / n = \lambda P^\lambda.
\] (60)

Similarly,
\[
\hat{V}^\dagger \hat{H}(\lambda \hat{q}, \lambda \hat{p}) \hat{V} = \hat{H}(\lambda [\hat{q} + (1 - \lambda)Q^\lambda], \lambda [\hat{p} + (1 - \lambda)P^\lambda]).
\]

Combining these relations, and comparing to (51), we verify that solves (50) self–consistently.

To pass in the other direction, we start with a solution to (50), pick \(\hat{V}\) as the inverse of (57), with \(Q^\lambda\) and \(P^\lambda\) determined from (59) and (60), and obtain, via (58), a solution of (52).

Making use of (53), (55), (56) and the disentanglement relation (40),
\[
\psi_\lambda(q) = \lambda^{1/2} e^{-i(1 - \lambda)P^\lambda q^2/(\beta \hbar)} \times \psi(\lambda q + (1 - \lambda)Q^\lambda \hbar, \lambda^2 \hbar),
\] (61)
where all indicated functions are obtained as solutions to the standard Schrödinger problem\( (\lambda = 1)\).

Recall the harmonic oscillator wavefunctions[38],
\[
\psi_n(q, \hbar) = (2^n n!)^{-1/2} (\beta / \pi)^{1/4} e^{-\beta q^2/2} H_n(q^{1/2}),
\] (62)
where \(H_n(z) = (-1)^n e^{z^2} (d^n / dz^n) e^{-z^2}\), with \(\beta(\hbar) = m \omega / \hbar\). Since the position and momentum expectations of these vanish, it is easy to verify that (62) are invariant under the transformation (61).
Although (61) looks singular at \( \lambda = 0 \), this need not always be the case. As a matter of curiosity, we wonder which class of Hamiltonians have eigenstates that are fixed points of this abstract mapping.

**E. Degeneracies and the failure of orthogonality**

Some minor trouble arises if (52) is degenerate. Then (57) must be applied, in turn, to each member of the invariant subspace associated with \( E^\lambda \), so as to generate a corresponding interpolative eigensubspace. Thus one can think of the solutions to (50) as being constructed by applying the nonlinear mapping (57) to the entire Hilbert space. Evidently, the usual linear eigenvector orthogonality relations are preserved, if, and only if, all eigenvectors of (52) happen to share identical coordinate expectations. Although the form of \( E^\lambda \) suggests, on first sight, that we are merely taking \( \hbar \to 0 \) via a circuitous route, the failure of orthogonality shows that the two approaches are, in fact, fundamentally different. One distinguishes this limit from the standard classical limit via the modification to eigenfunctions (examine (61)).

Another clear distinguishing feature is that we cannot superpose the nonlinear eigen-solutions

\[
|\psi'_\lambda(t)\rangle = e^{-i(t-t_0)E^\lambda/h}|\psi'_\lambda(t_0)\rangle,
\]

to get a solution of (23).

**F. A connection between quantum eigenstates and fixed points of the classical Hamiltonian flow?**

Given that \( \lambda = 0 \) eigenstates lie at fixed points of the classical Hamiltonian flow, we conjecture that:

\[
\lim_{\lambda \to 0} E(\lambda^2 \hbar) = E^0_{f.p.},
\]

(63)

\[
\lim_{\lambda \to 0} Q(\lambda^2 \hbar) = Q^0_{f.p.},
\]

(64)

\[
\lim_{\lambda \to 0} P(\lambda^2 \hbar) = P^0_{f.p.}.
\]

(65)

Two problems confound a proof. Firstly, continuity of the defining variational problem, (43), is essential, but the infinite degeneracy of solutions at \( \lambda = 0 \) contradicts this. Secondly, at
this same point the *auxilliary* problem, (52), is obviously singular. So the known $\lambda = 0$ behaviour need not always connect with the above limits.

For example, parity arguments applied to the quartic double well potential, $V(q) = (q^2 - 1)(q^2 + 1)$, show that (64) fails. Eigenstates have vanishing expectation so the two stable fixed points are missed out. Either conditions of broken symmetry must obtain, or the correct statement is more subtle.

For exact single fixed point problems, the limit (63) is easily verified[39]. The harmonic oscillator obeys it,

$$E^\lambda = \lambda^2 \hbar \omega (n + 1/2) \to E^0 = 0,$$

as does the hydrogen atom,

$$E_n^\lambda = -\frac{Z^2 e^4 m_e}{2n^2 \lambda^2 \hbar^2} \to E^0 = -\infty,$$

(if we treat the origin as a fixed point). A soluble example with two fixed points is Calogero’s problem[40],

$$\left\{ -\alpha \frac{\partial^2}{\partial q^2} + \beta q^2 + \gamma q^{-2} \right\} \psi(q) = E\psi(q),$$

with the eigenfunctions[41],

$$\psi(q) = (\kappa q)^{a+1/2} e^{-\kappa^2 q^2/2} L_n^a(\kappa^2 q^2), \quad n = 0, 1, 2, \ldots$$

where $\kappa = (\beta/\alpha)^{1/4}$, $a = 1/2(1 + 4\gamma/\alpha)^{1/2}$, and $4\gamma/\alpha > -1$. The classical fixed points lie at $q = \pm (\gamma/\beta)^{1/4}$, with energy $2(\gamma/\beta)^{1/2}$. Taking Calogero’s eigenvalue formula

$$E_n = (\alpha \beta)^{1/2}(2 + 2a + 4n),$$

we let $\alpha \to 0$ and verify (63).

Thus the energy result seems quite general. Indeed one can take the EBK semiclassical quantization rule[42], $\oint \Gamma p dq = 2\pi \hbar (n + \alpha/4)$ and deduce that, as $\hbar \to 0$, the symplectic area enclosed by the classical periodic orbits $\Gamma_n(h)$ must vanish. Now we assume that a continuously parametrized family of periodic orbits with this property must converge upon some classical fixed point. Then EBK connects a quantized energy level with the action parameter labelling the “disappearing torus”. It appears that integrable Hamiltonians must respect (63).
X. UNCERTAINTY PRODUCTS AND DISPERSION

A. Generalized dispersion

To develop a generalized uncertainty relation we recall the usual definition, \( \Delta_a^2 \equiv \langle \psi | (\hat{A} - \langle \hat{A} \rangle)^2 | \psi \rangle \), where \( \hat{A} \) is a linear operator. Then for \( a \equiv \langle \psi | \hat{A} | \psi \rangle \), we observe that

\[
\Delta_a^2 = a \star a - a^2 / n,
\]

(66)

where \( a \star a \equiv \delta_\psi a \delta_\psi \). If we assume that \( a \) commutes with all its \( \star \)-product powers, then, Weinberg argues[43], the usual probability interpretation is retained. Thus \( a \star a \) is the average of the square, \( a^2 \) the average squared, and (66) is a generalized dispersion observable.

B. A generalized uncertainty principle?

Given a second observable \( b \), whose \( \star \)-powers again commute, we treat \( \delta_\psi a \) and \( \delta_\psi b \) as kets, their adjoints as bras, and set

\[
|\alpha\rangle = \delta_\psi a - a/n\delta_\psi n, \quad \text{and} \quad |\beta\rangle = \delta_\psi b - b/n\delta_\psi n.
\]

Substituting these into the Schwartz inequality[54], \( \langle \alpha | \alpha \rangle \langle \beta | \beta \rangle \geq |\langle \alpha | \beta \rangle|^2 \), we collect \( \star \)-products to obtain the inequality

\[
(a \star a - a^2 / n)(b \star b - b^2 / n) \geq |(a \star b - ab/n)|^2.
\]

(67)

Working on the right hand side, we have

\[
a \star b - ab/n = \frac{1}{2} [a, b]_W + \frac{1}{2} [a, b]_W^+ - ab/n,
\]

with \( [a, b]_W^+ \equiv a \star b + b \star a \). Taking the square norm, we observe that \( 1/2[a, b]_W \) is pure imaginary, while \( 1/2[a, b]_W^+ - ab/n \), is pure real. Given that the real term vanishes on the minimum uncertainty states, (67) permits the simpler, weakened, form

\[
\Delta_a^2 \Delta_b^2 \geq \frac{1}{4} |[a, b]_W|^2.
\]

(68)

Although this inequality bears a striking resemblance to the standard Heisenberg–Robertson relation[45], it is only properly motivated if \( a \) and \( b \) are observables whose \( \star \)-powers commute.
Caution is advisable since the right and left members of (67) need not be invariant under general nonlinear canonical transformations.

Although (67) has the formal properties of dispersion, its physical interpretation is unclear. If dispersion depends upon the coordinate system, we can make little of it, except perhaps to distinguish the value zero as being special.

C. A simple example: coordinate functionals

For a simple example, we take the deformed coordinate functionals $q_\lambda$ and $p_\lambda$. Since these commute with their $\star$-powers, we have

$$\Delta^2_{q_\lambda} \Delta^2_{p_\lambda} \geq \frac{1}{4} |[q_\lambda, p_\lambda]|^2 = \frac{\hbar^2}{4}.$$  

Thus deformation preserves the generalized uncertainty principle (68), and coordinate dispersions are seen to obey the usual interpretative rules.

D. Wider validity?: classical observables

Interestingly, the general stationarity conditions (44) and (45) imply, via (66), that dispersion must vanish for generalized stationary states. This is the most cogent physical reason for believing that (67) may be of general significance.

For example, using (13) we compute,

$$\Delta^2_{h_0} = (\partial_q H)^2 \Delta^2_q + 2(\partial_q H)(\partial_p H)\Delta^2_{qp} + (\partial_p H)^2 \Delta^2_p,$$

where,

$$\Delta^2_{qp} \equiv \frac{1}{2}\langle \psi | (\hat{p} - \langle \hat{p} \rangle)(\hat{q} - \langle \hat{q} \rangle) + (\hat{q} - \langle \hat{q} \rangle)(\hat{p} - \langle \hat{p} \rangle)|\psi \rangle.$$  

Thus classical dispersion is just a “quantized” version of gaussian quadrature error analysis. Dispersion vanishes at classical fixed points, as does the right hand member of (68) for quantities in involution (i.e. with zero Poisson bracket).

More generally the interpolative dispersion does not seem to have any ready interpretation. We therefore doubt that the concept is useful, except as a means to study the spreading of quantum states under evolution.
E. Interpolative dynamics of dispersion

Since the generalized dispersions formed via rule (66) are again homogeneous of degree one, we can use the evolution equation (3). For instance, from (9), (10), and the formula (17) we compute \([\Delta^2_{q\lambda\hbar}, h\lambda]\) and \([\Delta^2_{p\lambda\hbar}, h\lambda]\), to obtain:

\[
\frac{d\Delta^2_{q\lambda\hbar}}{dt} = +\lambda n \left\{ \langle [\hat{q}, \hat{H}_{\lambda}]^+ \rangle - 2\langle \hat{q} \rangle \langle \hat{H}_{\lambda} \rangle \right\},
\]

\[
(69)
\]

\[
\frac{d\Delta^2_{p\lambda\hbar}}{dt} = -\lambda n \left\{ \langle [\hat{p}, \hat{H}_{\lambda}]^+ \rangle - 2\langle \hat{p} \rangle \langle \hat{H}_{\lambda} \rangle \right\}.
\]

\[
(70)
\]

No matter what the chosen state \(\psi\), or Hamiltonian \(H\), dispersion is smoothly switched off as \(\lambda \rightarrow 0\).

XI. THE INTERPOLATIVE FREE PARTICLE

To illustrate the preceding formal material we consider the interpolative free particle Hamiltonian:

\[
\hat{H}_{\lambda}^{\text{eff}} = \frac{\hat{p}^2}{2m} + \frac{\langle \hat{p} \rangle}{m} (\hat{p} - \langle \hat{p} \rangle).
\]

\[
(71)
\]

Using either the propagator formula (41), or the fact that the momentum \(P_0 = \langle \hat{p} \rangle\) is a constant of the motion (via equations (19) and (20)), we see that the free particle propagator is just

\[
\hat{U}_{\Delta t} = \exp \left\{ -\frac{i\Delta t}{\hbar} (ap^2 + bp + c\hat{1}) \right\},
\]

\[
(72)
\]

where, from (71), the constants \(a, b\) and \(c\) read:

\[
a = \frac{\lambda^2}{2m}, \quad b = \frac{(1 - \lambda^2)P_0}{m}, \quad \text{and} \quad c = \frac{(\lambda^2 - 1)P_0^2}{m}.\]

\[
(73)
\]

The problem is now easily solved using the deformed free particle Green’s function,

\[
K_{\lambda}(q', q; \Delta t) \equiv \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} e^{-i\Delta t(ap^2 + bp + c\hat{1})/\hbar} e^{i(q' - q)p/\hbar} dp,
\]

\[
(74)
\]

such that,

\[
\psi(q', t_0 + \Delta t) = \int_{-\infty}^{\infty} K_{\lambda}(q', q; \Delta t)\psi(q, t_0) dq.
\]

\[
(75)
\]

Evaluating (76) we get,

\[
K_{\lambda}(q', q; \Delta t) = (\pi/\gamma)^{-1/2} e^{-ix}\gamma^{[q-(q'-\delta)]^2},
\]

\[
(76)
\]

23
where,
\[ \gamma = \frac{1}{4a\hbar \Delta t}, \quad \delta = b\Delta t, \quad \text{and} \quad \kappa = c\Delta t/\hbar. \quad (77) \]

Choosing an initial gaussian at the origin,
\[ \psi(q, t_0) = \left(\frac{\pi}{2\alpha}\right)^{-1/4} e^{-\alpha q^2 + i\beta q}, \quad (78) \]

with appropriate width and momentum parameters,
\[ \alpha = \frac{1}{4\sigma_q^2}, \quad \text{and} \quad \beta = \frac{P_0}{\hbar}, \quad (79) \]

we substitute (76) and (78) into (75), and compute the evolved gaussian state,
\[ \psi(q, t_0 + \Delta t) = \left(\frac{\pi}{2\alpha}\right)^{-1/4} e^{-\alpha q^2 + i\beta q} \exp \left\{ -\frac{\gamma^2[q - (\delta + \beta/2\gamma)]^2}{(\alpha - i\gamma)} \right\}, \quad (80) \]

where primes are now dropped. Next we form,
\[ |\psi(q, t_0 + \Delta t)|^2 = \left(\frac{\pi(\alpha^2 + \gamma^2)}{2\alpha \gamma}\right)^{-1/2} \exp \left\{ -\frac{2\alpha \gamma^2[q - (\delta + \beta/2\gamma)]^2}{(\alpha^2 + \gamma^2)} \right\}, \quad (81) \]

and use (73), (77) and (79), to pick out the evolved packet centre and dispersion formulæ:
\[ q_0(t_0 + \Delta t) = \frac{P_0\Delta t}{m}, \quad (82) \]
\[ \sigma_q^2(t_0 + \Delta t) = \sigma_q^2(t_0) \left\{ 1 + \frac{\lambda^4 \hbar^2 (\Delta t)^2}{4m^2 \sigma_q^4(t_0)} \right\}. \quad (83) \]

We check that interpolative particles propagate at the desired classical velocity $P_0/m$. Moreover, as with the energies $E^\lambda$, the formula (83) is identical to the standard linear one, except that $\hbar$ is replaced by $\lambda^2 \hbar$. Compare the $\lambda = 0$ behaviour with standard quantum theory.

For any mass $m$, there exists some time interval $\Delta t_c$, such that a particle will eventually disperse so as to fill the entire known universe. Ordinarily, we dispense with this difficulty by stating that the interval is far too long to matter, and that particles are, in any case, localized by measurements long before the situation gets out of hand. In contrast, the limit (83) offers greater descriptive (not prescriptive) power in that we can hang the value $\lambda = 0$ upon this circumstance.

**XII. PROSPECTS FOR EMPIRICAL TEST**

**A. Where does linearity apply, for sure?**

There have been numerous stringent tests of quantum linearity performed upon microscopic systems. Each of these has yielded a null result[46]. Bollinger et al[46], have bounded
the Weinberg nonlinearity in Beryllium nuclei spin–precession experiments at less than 4 parts in $10^{-27}$. Other indirect tests, such as the atomic version of Young’s double slit experiment[47], and inversion tunnelling in small molecules like Ammonia provide strong evidence against nonlinearity in atomic scale systems.

**B. How might nonlinearity emerge?**

The quantum dynamics of isolated systems observed in today’s laboratory must therefore be linear to a very high degree of precision. If nonlinearity lies somewhere, then it seems that one must look for its effects in a *new* place. Either that, or one argues that this *exact* version of Hamiltonian classical dynamics, formulated as a *wave* theory for *any* value of $\hbar$, is just a bizarre mathematical accident, put there expressly to tease us.

A clear question emerges. Is quantum theory *always* linear with an *approximate* classical limit; or is there a more general nonlinear theory which is linear for small systems and progressively nonlinear until we recover an *exact* classical limit?

Two distinct physical interpretations appear possible. Either the $\psi$–dependent operators express a statistical result that should then be traced to environment–induced fluctuations (decoherence[5]); or, since (3) is deterministic, the nonlinearity might reflect a purely causal coupling to the environment (a back–reaction or self–energy effect). In either case, it seems *plausible* that nonlinearity should become larger the less isolated, and more entangled, a quantum system becomes.

**C. In search of a mesoscopic “elementary particle”**

Most elementary particles have internal structure. However, if empirical energy scale is decoupled from the internal degrees of freedom, then we can exploit a structureless one–particle approximation.

In particle physics one reveals internal structure by building a higher energy accelerator. To test any one–particle wave equation one needs an inverted version of this program. The goal is to screen the known internal degrees of freedom and get the detector energies *low* enough (or sideband them on a more accessible frequency).

To make a mesoscopic “elementary particle” we could take a spherical macromolecule, or
perhaps a microsphere[48]. Then we charge it, or magnetize it, and find an ingenious way to measure this and weigh it[49]. Then we give the particle a moment of some kind, put it in a well and couple it to coherent radiation in an accessible range (probably microwaves). Then it is feasible, in principle, to resolve the quantized energy levels. Nobody does this now because it seems impossible to get the thermal background cool enough, or the characteristic frequencies high enough, to be able to resolve the levels of a particle in, say, the microgram range. The lighter our particle the easier the experiment, but the further we are likely to be from the classical regime.

D. A possible empirical signature

Suppose we can do this at some mass (or size) scale. Given the standard prediction for energy levels $E(\hbar)$, one needs to use the spectroscopic data, along with the known particle mass etc., to measure Planck’s constant (assuming the radiation law $\Delta E = h\nu$). If this were to exhibit a monotonic decrease as one passes to more classical systems, then has evidence for a perturbative energy level shift, like the $E(\lambda^2\hbar)$ effect. Because the classical and quantal Weinberg energy functionals differ, one might expect something similar for any interpolative scheme. Our investigation is thus helpful, if only to show that any observed discrepancy of this kind deserves careful attention.

XIII. THEORETICAL DIFFICULTIES

Given that experimental tests of the validity of exact linear quantum theory in the classical domain are so very difficult; we now highlight some of the severe problems the nonlinear theory generates. It may be that strong exclusions can be found via this route.

A. The free nature of $\lambda$

This is the most obvious problem. Without positive empirical evidence one cannot fix $\lambda$. The only thing we learn is what kind of effects one might need to look for. There does not seem to be any way around this problem. Remember also that (11) is just a postulate. Canonical quantization is not the only route to generalization.
B. Lack of manifest algebraic closure

From (16), we see that the interpolative observables (11), do not manifestly comprise a subalgebra, except at $\lambda = 0, 1$. This ugly mathematical feature strongly suggests that the interpolation is unphysical. A subalgebra may show up using coordinate free methods (i.e. write (11) in terms of $\star$–products and ordinary products). However, this fact, and the general complexity of the interpolative domain, leads us to conclude that (11) has no fundamental physical content, other than as a guide to formulating empirical questions.

At a deeper level we obtain a sieve: “What existence and uniqueness constraints apply to a one–parameter family of Weinberg subalgebras which joins the classical and quantum regimes?”.

C. Problems with measurement: a provisional probabilistic interpretation

Weinberg has emphasized[50] that generalization of the probability interpretation to nonlinear observables is defeated by non–associativity of the functional $\star$–product. Nor can we use the Hilbert space inner product, since this is not a canonical invariant in the nonlinear sector of the theory.

How else might we get a probability interpretation? Since problems arise due to nonlinearity, the natural place to look for the “right” idea is in this sector. It is much easier to specialize a working result; than to generalize from a special one.

Classical statistical physics employs densities $\rho(q, p)$ on phase space. Liouville’s theorem preserves normalization and Hamilton’s equations determine evolution of the ensemble. One can then discuss classical measurement as a stochastic diffusive process superimposed upon the dynamics, and justify statistical mechanics via the ergodic hypothesis[51].

Classical expectations are phase space averages

$$\bar{f} = \int \rho(q, p) f(q, p) \, dpdq,$$

where $\rho(q, p)$ is stationary.

Since Weinberg’s theory specializes the Hamiltonian formalism (homogeneity is a constraint upon the hamiltonian) we can try and carry this over directly. The key is to find an invariant measure upon quantum states.
Canonical invariance of the symplectic form $dp \wedge dq$ (and thus its exterior powers), implies Liouville’s theorem\[52\]. In Weinberg’s theory we identify the corresponding canonically invariant symplectic form $\sum_{k=1}^{D} d\psi_j^* \wedge d\psi_j$. Taking exterior powers of this we get Liouville’s theorem, and an induced invariant measure on the projective Hilbert space of normalized states.

Exploiting canonical invariance of the norm $n$, we focus on functionals that are homogeneous of degree $p$, and define the measure\[53\]:

$$\int F(\psi, \psi^*) d\tilde{\Omega} \equiv \frac{\Gamma(D)}{\Gamma(D+p)} \int F(\psi, \psi^*) e^{-n} \prod_{j=1}^{D} \pi^{-1} d\text{Re}[\psi_j] d\text{Im}[\psi_j], \quad (85)$$

where $d\tilde{\Omega}$ emphasizes the analogy with solid angle. To get a good $D \to \infty$ limit, we set $p = 0$ on the right hand side (divide $F$ by $n^p$ when taking the average).

The formula (85) is immediately recognized as the standard functional measure of path integrals\[54\], or the theory of gaussian random fields\[55\].

Now let $\rho(\psi, \psi^*)$ be any positive Weinberg observable satisfying,

$$\int \rho(\psi, \psi^*) d\tilde{\Omega} = 1. \quad (86)$$

The uniform density becomes $n(\psi, \psi^*)$, the norm functional. A nontrivial example is,

$$\rho_N(\psi, \psi^*) = n^{1-N} \frac{\Gamma(D+N)}{\Gamma(N)\Gamma(D)} |\langle \psi|\phi \rangle|^2N, \quad (86)$$

where the factor $n^{1-N}$ makes this homogeneous of degree one, so that (3) applies. On averaging we set this to $n^{-N}$. As $N \to \infty$, (86) peaks strongly about $\phi$. This density plays the role of a delta function on states.

To see this, we use the formula\[56\],

$$\int |\langle \phi|\psi \rangle|^2 f(|\langle \omega|\psi \rangle|^2) d\tilde{\Omega} \equiv \frac{1}{D-1} (1 - |\langle \phi|\omega \rangle|^2) \int f(|\langle \psi|\omega \rangle|^2) d\tilde{\Omega} \equiv \frac{1}{D-1} \int |\langle \psi|\omega \rangle|^2 f(|\langle \psi|\omega \rangle|^2) d\tilde{\Omega}. \quad (87)$$

Defining generalized quantum averages in the classical fashion,

$$\bar{a} = \int \rho(\psi, \psi^*) a(\psi, \psi^*) d\tilde{\Omega}, \quad (88)$$

we choose the bilinear functional,

$$a_1(\psi, \psi^*) = |\langle \psi|\hat{A}|\psi \rangle| = \sum_{j=1}^{D} |\langle \psi|\omega_j \rangle|^2, \quad (89)$$
where $A_j$ are the eigenvalues of $\hat{A}$, and $|\omega_j\rangle$ its eigenvectors. Substituting (89) and (86) into (88), we use (87) and the definition (85) to verify that,

$$\int \rho_N^\phi(\psi, \psi^*) a_1(\psi, \psi^*) d\hat{\Omega}_\psi = \sum_{j=1}^D \left\{ \frac{D}{(D-1)(D+N)} + \frac{N}{D+N} \left( 1 - \frac{D}{N(D-1)} \right) A_j |\langle \phi | \omega_j \rangle|^2 \right\}. \quad (91)$$

Keeping $D$ fixed, and taking $N \to \infty$, we recover the desired result

$$\bar{a}_1 = \int \rho_\infty^\phi(\psi, \psi^*) a_1(\psi, \psi^*) d\hat{\Omega}_\psi = \langle \phi | \hat{A} | \phi \rangle. \quad (92)$$

Thus quantal expectation values can be reinterpreted as phase space averages with respect to the delta distribution $\rho_\infty^\phi$.

If we let $D \to \infty$, in heuristic fashion, and choose classical Weinberg functionals, then (85) induces the standard Liouville measure over the phase space of coordinate expectations, and we recover the classical result (84).

More generally, one consider $\rho(\psi, \psi^*)$ as defining the density matrix,

$$\hat{\rho} = \int \rho(\psi, \psi^*) |\psi\rangle \langle \psi| d\hat{\Omega}_\psi. \quad (93)$$

Linearity of the trace operation and positivity of the probability density ensures that, $\hat{\rho} > 0$ and $\text{Tr}[\hat{\rho}] = 1$. This connection is many–to–one, so that $\rho(\psi, \psi^*)$ is a “hidden”, or “indeterminable”, representation of $\hat{\rho}$. Nevertheless,

$$\bar{a}_1 = \int \rho(\psi, \psi^*) a_1(\psi, \psi^*) d\hat{\Omega}_\psi = \text{Tr}[\hat{\rho} \hat{A}].$$

Pure states become delta function ensembles, whereas smeared densities generate the mixed states.

Rephrasing quantum averages in this fashion, we acquire a common statistical language for both classical and quantum physics. Next we need to incorporate a generalized theory of measurement. Thusfar we can only do this by appeal to the known result. Nevertheless, our hope is that a suitably generalized perspective might reveal (94) as a special case, whose inner product nature is accidental to the linear sector, but somehow necessary.

For a quantum system in state $\phi$ subjected to a complete measurement with the operator $\hat{A}$, we have the jump process $\phi \mapsto \omega_j$ occurring with conditional probability,

$$p(\omega_j | \phi) = |\langle \phi | \omega_j \rangle|^2. \quad (94)$$
Post measurement, we have a probability density peaked as delta function spikes on each of the eigenvectors, with weight given by rule (94):

\[ \rho(\psi, \psi^*) = \sum_{k=1}^{D} |\langle \phi | \omega_j \rangle|^2 \rho_{\omega_j}^2(\psi, \psi^*). \]  

(95)

Substituting this into the rule (88), we verify that \( \bar{a} = \langle \phi | \hat{A} | \phi \rangle \). This is the same as the result (92), but the underlying distribution over states is different.

How can we understand this? Although (86), with \( N \rightarrow \infty \), and (95) generate exactly the same statistics, their dynamical properties under (3) are different. If \( \hat{A} \) were the Hamiltonian then (95) is a stationary probability density. In contrast, the density (86) must be time-dependent, unless \( \phi \) happens to be an eigenstate of \( \hat{A} \).

Thus the stationary states of the hamiltonian flow appear rather special to the quantum case, they are associated with stationary probability densities which are sums of delta functions upon these. This suggests that we should incorporate (94) as a dynamical result[60], albeit via stochastic dynamics[57].

Master equations, either classical, or quantal, are the canonical examples of this paradigm[58]. They encapsulate stochastic evolution of individual ensemble members via a deterministic equation of Fokker–Planck type. Adopting this formal route, we postulate the generalized nonlinear quantum master equation[59]:

\[ \frac{d\rho}{dt} = \frac{1}{i\hbar} [\rho, h]_W + \frac{\Gamma}{(i\hbar)^2} [\rho, a]_W, a_W, \]  

(96)

where \( h \) is the “free evolution” and \( a \) is the “measurement functional”, while \( \Gamma \) is a phenomenological parameter (zero if measurement is switched off).

Given (96) we must solve for the stationary probability density \( \rho_\infty \) (defined as the limit \( t \rightarrow \infty \)) generated from a chosen initial condition \( \rho_0 \). Assuming existence of \( \rho_\infty \), the averaging rule (88) provides the statistical prediction.

From (96) the stationarity condition reads \( \dot{\rho} = 0 \). To recast this we introduce Liouville operators: \( \mathcal{L}_h \equiv [\bullet, h]_W \), and \( \mathcal{L}_a \equiv [\bullet, a]_W \), to get:

\[ (\mathcal{L}_h - \mathcal{L}_a \circ \mathcal{L}_a) \circ \rho = 0. \]  

(97)

This specifies the kernel of a linear operator,

\[ \mathcal{L}_M \equiv \mathcal{L}_h - \mathcal{L}_a \circ \mathcal{L}_a, \]  

(98)
on the space of Weinberg functionals (the operator acts in the adjoint representation of this Lie algebra). Thus we identify $L_M$ as the formal “measurement operator” which is to describe an $a$–measurement performed upon a system undergoing $\hbar$–evolution.

Conveniently, linearity of (98) implies a spectral theory. Intuitively, we expect the spectrum of (98) to determine the decay rate of $\rho_0$ to $\rho_\infty$, and also the class of initial conditions $\rho_0$, upon which a given measurement will be good (in the sense that we get to a stationary density, or arbitrarily close to it, in a finite interaction time). If $\rho^k_\infty$ is a finite set $k \in [1, M]$ of stationary densities, then so too is the linear combination,

$$\rho_\infty = \sum_{k=1}^{M} w_k \rho^k_\infty,$$

provided only that the weights $w_k$ sum to unity. Thus a measurement theory somewhat analogous to that of linear quantum theory exists, even when orthogonality is relaxed (recovery of this, on the space of density functionals, would require (98) to be self–adjoint).

The generalization is certainly suggestive. Significantly, the equation (96) is not a hamiltonian flow, but it has the desired physical property of being expressed purely via canonically invariant Weinberg brackets.

Unsolved problems aside, a consistent, and inclusive, statistical interpretation of nonlinear quantum theory is conceivable via appeal to stochastic dynamics.

D. Thermodynamic constraints

Although $\hbar$ is fixed, harmonic oscillator energy levels have the reduced spacing $\lambda^2 \hbar \omega$ and the $n$th stationary solution now reads:

$$|n_t\rangle = e^{-i\lambda^2 \omega(n+1/2)(t-t_0)}|n_{t_0}\rangle.$$  \hspace{1cm} (100)

However, from (19) and (20) one verifies that a gaussian wave packet oscillates at the classical frequency $\omega$, for all $\lambda$. Thus we encounter the bizarre circumstance that a transition between two stationary states suggests the photon frequency $\nu = \lambda^2 \omega$, while the classical radiation frequency remains $\omega$.

We could try and fix this by letting $\nu = \omega$ so that the photon energies become $\lambda^2 \hbar \nu$. However, that leads to the horrible consequence that photons must either, be confined to a single $\lambda$–sector, or, change their frequency at each interaction with matter. Worse still, the
deformed Planck black body factor (excluding degeneracy),

\[
e^{-\lambda^2 \hbar \nu / kT} \left( 1 - e^{-\lambda^2 \hbar \nu / kT} \right),
\]

(101)
detonates at \( \lambda = 0 \). Presto, an ultraviolet catastrophe! The only way to salvage this disaster is to postulate that photons are always \( \lambda = 1 \) particles. To defend that, superposition of light waves is regularly observed at the classical level, whereas that of matter waves is not.

Thus a deformed harmonic oscillator must have two characteristic frequencies. This curious property offends cherished physical intuition. However, as shown in great depth by Weinberg[61], such behaviour is common. Moreover, because the state preparations differ, it would be impossible to observe both frequencies in a single experiment. Evidently, there is no ambiguity or contradiction, a situation not unlike wave–particle duality. How one could ever detect this is a problem, unless perhaps thermodynamics can do it for us via some modification of specific heats. Certainly, the deformed black body rule would predict this; but given the historical importance of that problem it is hard to believe that there is any discrepancy lurking in the data. Currently it is assumed that material and radiative oscillators must be quantized in the same way. Certainly radiation must be consistently quantized, because it mediates interaction between material particles. That leaves us in some doubt as to whether material oscillators must obey the same rule of consistency. Clarification of this issue is probably the most powerful constraint upon any modified quantum theory. Nobody would reject thermodynamics.

XIV. CONCLUSION

In summary, we have embedded both Hamiltonian classical mechanics and linear quantum theory as two disjoint dynamical sectors of Weinberg’s generalized nonlinear theory. To explore the idea of a mesoscopic regime we then studied one technique for interpolation. Although not fully constrained, our method is simple, general, and has some desirable physical features. The result is an alternative classical limit whereby quantal evolution is smoothly transformed into classical evolution as we vary a single dimensionless control parameter \( \lambda \). Significantly, this works for any value of \( \hbar \).

At the level of mathematical physics, we have a new tool for comparing classical and quantal dynamics. This can be put to immediate use in studies of “quantum chaos”. The
ability to turn dynamical chaos on and off via \( \lambda \), whatever the magnitude of \( \hbar \), provides a new probe of the origin of dynamical chaos suppression, and the potential for exposing some interesting phenomena in the transition regime. We will return to study this later, with a parting comment that the interpretative problems have no bearing upon this pursuit.

Concerning the working hypothesis that nonlinearity emerges at the classical level, we stress that the evident success of linear quantum theory for microscopic systems is not in dispute. Rather we imagine that a complex of atomic systems, a whole molecule, a block of solid, glass of beer, cat, flea on cat, or ribbon of its DNA, has gotten complicated enough that the dynamics for the \( \psi \) of its centre of mass is described by a nonlinear theory.

This attempt at a physical interpretation is imprecisely formulated. The mathematics is unwieldy, and devoid of predictive power. Given its complexity, we do not believe that the interpolative technique has any fundamental physical content. Nevertheless, the one-particle assumption at least enables us to compute deformed energies \( E(\lambda^2\hbar) \), and show that the free particle has uniformly suppressed dispersion as \( \lambda \to 0 \). Thus we settle upon the view that the proper role of interpolative dynamical studies is to guide tests of the universality of linear quantum theory.

Fundamental questions of this nature demand careful scrutiny. Indeed, the idea of emergent nonlinearity, bizarre as it may be, is consistent with both the observed linearity of isolated atomic scale systems and the fact that classical mechanics describes the familiar world of our senses so well. In the current climate one is led to reject a complete recovery of classical theory, because it implies that there is a nonlinear regime, and so linear quantum theory could not be considered universal. We suggest that if our prejudices demand that we invent reasons to ignore simple mathematical facts, then physics is in very serious trouble.

XV. ACKNOWLEDGMENTS

Portions of this work were carried out at the University of Melbourne, Australia; University of Houston, Texas; University of Texas at Austin; Institute of Advanced Study Princeton; and my current address. I am grateful to: B.H.J. McKellar, A.G. Klein, S. Adler, S. Weinberg, S.C. Moss, and M. Eisner for their hospitality, and for useful discussions. Conversation or correspondence with: A.J. Davies, S. Dyrting, O. Bonfim, N.E. Frankel, Z. Ficek, G.J. Milburn, V. Kowalenko, H. Wiseman, R. Volkas and I.C. Percival sharpened the
ideas, and provided encouragement. Support from the University of Melbourne, A.G. Klein, B.H.J. McKellar and N.E. Frankel, via a Visiting Research Fellowship, and a Special Studies Travel Grant are further acknowledged, along with an A.R.C. postdoctoral fellowship.

[1] G.A. Hagedorn, Commun. Math. Phys. 71, 77 (1980).
[2] For example, see: L.G. Yaffe, Rev. Mod. Phys. 54, 407 (1982).
[3] If $\psi_1(t)$ and $\psi_2(t)$ are solutions then so too is $\psi(t) = \alpha \psi_1(t) + \beta \psi_2(t)$, even if these states are non–orthogonal. The normalisation is time independent; absorb it into $\alpha$ and $\beta$. Then, using results from Ref. [1], one can choose states such that $\langle \hat{A} \rangle_{\psi_1}(t)$ and $\langle \hat{A} \rangle_{\psi_2}(t)$ follow the classical trajectories. But now

$$\langle \hat{A} \rangle_{\psi}(t) = |\alpha|^2 \langle \hat{A} \rangle_{\psi_1}(t) + |\beta|^2 \langle \hat{A} \rangle_{\psi_2}(t) + 2 \text{Re}[\alpha^* \beta \langle \psi_1(t) | \hat{A} | \psi_2(t) \rangle],$$

which need not follow any classical trajectory.

[4] The paradox of Schrödinger’s cat is thus avoided by the practical assertion that non–local, alive and dead or correlated singlet felines are unpreparable.

[5] W.H. Zurek, Phys. Rev. D24, 1516 (1981); W.H. Zurek, Phys. Rev. D26, 1862 (1982); E. Joos and H.D. Zeh, Z. Phys. B–Cond. Matt. 59, 223 (1985); M. Gell–Mann and J.B. Hartle, in Complexity, Entropy and the Physics of Information edited by W.H. Zurek (Addison–Wesley, Redwood CA, 1991); R. Omnés, Rev. Mod. Phys. 64, 339 (1992).

[6] Philosophical work is resurgent: B. D’Espagnat, Reality and the Physicist (Cambridge, London, 1989); H. Krips, The Metaphysics od Quantum Theory (Clarendon Press, Oxford, 1987); M. Redhead, Incompleteness Nonlocality and Realism (Clarendon Press, Oxford, 1989); J.M. Jauch, Are Quanta Real? A Galilean Dialogue (Indiana Press, Bloomington, 1989).

[7] Bohr insisted that classical theory is required to describe the final stage of observation [N. Bohr, in Quantum Theory and Measurement, edited by J.A. Wheeler and W.H. Zurek (Princeton, New Jersey 1983); N. Bohr, Atomic theory and the Description of Nature (Cambridge, London, 1934). Also, L. Landau and E. Lifschitz, Quantum Mechanics (Pergamon Press, London, 1958)]. Axiomatic theory leads to an infinite regress of uncommitted alternatives (or the explosive universal parallelism of Everett [H. Everett, in The Many Worlds Interpretation of Quantum Mechanics edited by B. De Witt and N. Graham (Princeton, New Jersey, 1973)].

34
Heisenberg’s cut must be executed to crystalize a definite observed phenomenon. There is no dispute about probabilities, only about their origin (the problem of hidden variables [D. Bohm, Phys. Rev. 85, 166, 180 (1952); J.S. Bell, Speakable and Unspeakable in Quantum Mechanics (Cambridge, London 1987)]. One can also formulate dynamical models for the stochastic transition using external noise sources (somewhat like the assumption of molecular chaos in statistical mechanics). See: D. Bohm and J. Bub, Rev. Mod. Phys., 38, 453 (1966); G.C. Ghirardi, A. Rimini and T. Weber, Phys. Rev. D34, 470 (1986); G.C. Ghirardi, P. Pearle and A. Rimini, Phys. Rev. A42, 78 (1990); P. Pearle, Phys. Rev. D13, 857 (1976); P. Pearle, J. Stat. Phys. 41, 719 (1985); N. Gisin, Helv. Phys. Acta 54, 457 (1981); N. Gisin, Phys. Rev. Lett. 52, 1657 (1984); L. Diosi, J. Phys. A. 21, 2885 (1988); C.M. Caves and G.J. Milburn, Phys. Rev. D36, 5543 (1987); G.J. Milburn, Phys. Rev. A44, 5401 (1991); N. Gisin and I.C. Percival, Phys. Lett. A167, 315 (1992). On the hidden variables front it is known from EPR–experiments [A. Aspect, P. Grangier and G. Roger, Phys. Rev. Lett. 49, 91 (1982); and A. Aspect, J. Dalibard and G. Roger, Phys. Rev. Lett. 49, 1804 (1982)] that any successfull hidden variable theory would have to be of non–local character. Most working physicists find this idea repugnant.

[8] J. Ford, G. Mantica and G.H. Ristow, Physica D50 493 (1991); and J. Ford and M. Ilg, Phys. Rev. A45, 6165 (1992). Their basic idea is that quantum evolution is not “complex” enough to replicate classical dynamical chaos (in an algorithmic sense).

[9] Quantum chaos is generally suppressed. See: B. Eckhardt, Phys. Rep. 163, 205 (1988); and references therein. M.C. Gutzwiller, Chaos in Classical and Quantum Mechanics (Springer–Verlag, New York 1990); and F. Haake, Quantum Signatures of Chaos (Springer–Verlag, Berlin, 1991).

[10] Berry has described how standard quantum theory does not permit such a reduction because of nonanalyticity in \( \hbar \) at the origin (i.e. wavefunctions etc, generally have an essential singularity at \( \hbar = 0 \)) [M.V. Berry, in Les Houches school on Chaos and Quantum Physics session 52 (North Holland, Amsterdam 1991)]. Here we achieve the reduction by regaining exact classical theory at all non–zero \( \hbar \) using a nonlinear quantum theory. A different method is to reconstruct Hamilton–Jacobi theory using a modified Schrödinger equation. See: R. Schiller, Phys. Rev. 125, 1100 (1962); R. Schiller, Phys. Rev. 125, 1109 (1962); R. Schiller, Phys. Rev. 125, 1116 (1962); N. Rosen, Am. J. Phys. 32, 597 (1964); N. Rosen, Am. J. Phys. 33, 146 (1965).
This does not fit readily with an identifiable generalized theory, and is thus limited in scope. Moreover, single particle description is impossible in this framework, since each wavefunction encodes a whole family of trajectories via Hamilton’s principal function.

[11] S. Weinberg, Ann. Phys. (N.Y.) 194, 336 (1989).
[12] S. Weinberg, Phys. Rev. Lett. 62, 485 (1989).
[13] K.R.W. Jones, Phys. Rev. D45, R2590 (1992).

[14] This class consists of all real–valued functionals of \( \psi = (\psi_1, \ldots, \psi_d) \) such that \( h(\lambda \psi, \psi^*) = \lambda h(\psi, \psi^*) = h(\psi, \lambda \psi^*) \), for all complex \( \lambda \), or, equivalently (Ref.[11]),

\[
\frac{\partial h}{\partial \psi_k} \psi_k = h = \frac{\partial h}{\partial \psi_k^*} \psi_k^*,
\]

with summation over \( k \) implicit. Although the norm \( n = \psi_k^* \psi_k \) is invariant, there is no invariant meaning for the global inner product. To motivate Hilbert space methods we observe that homogeneity implies,

\[
h = \psi_k^* \frac{\partial^2 h}{\partial \psi_k^* \partial \psi_l} \psi_l,
\]

whatever the chosen coordinate system. Thus, at each \( \psi \), \( h \) fixes an Hermitian form, a local inner product, a local orthonormal basis and, consequently, a tangent Hilbert space \( \mathcal{H} \), its dual, and a space of linear operators \( \mathcal{L}(\mathcal{H}) \) acting on these. The local inner product is not invariant under general canonical transformations. It seems that demanding such invariance characterizes the usual linear theory [see the analysis by R. Cirelli, A. Mania and L. Pizzocchero, Int. J. Mod. Phys. A6, 2133 (1991)]. This has important interpretational consequences (we can’t use the projection postulate). However, all of our computations can still be carried out in Hilbert space in a representation independent fashion (this is like fixing a system of Euclidean coordinates in classical mechanics for the purpose of displaying the motion). (N.B. locality means the mathematical kind in the space of all \( \psi \); physically these objects are non–local.)

[15] All computations follow from the bilinear result \( \delta_{\psi} \langle \psi | \hat{A} | \psi \rangle = \langle \psi | \hat{A} \hat{A}^\dagger | \psi \rangle \), where \( \hat{A} \) is any linear operator. This device, from Ref.[13], permits direct comparison with standard theory. As per Ref.[14] inner products apply between quantities defined at the same \( \psi \).

[16] This hypothesis was posed in the context of a nonlinear wave theory by I. Bialynicki-Birula and J. Mycielski, Ann. Phys. N.Y. 100, (1976). Null results for their log–nonlinear wave equation include: C.G. Shull, D.K. Attwood, J. Arthur and M.A. Horne, Phys. Rev. Lett. 44, 765 (1980); and R. Gähler, A.G. Klein, and A. Zeilinger, Phys. Rev. 23, 1611 (1981). Because
our wave equation recovers exact classical theory, which is known to be empirically accurate in a certain domain, the theory posed here is much harder to exclude outright.

[17] R. Penrose, in Quantum Concepts in Space and Time, edited by C.J. Isham and R. Penrose (Oxford, Oxford, 1986); R. Penrose, The Emperor’s New Mind, (Oxford, Oxford, 1989) pp367–370; N. Rosen, Found. Phys. 16, 687 (1986); and A. Peres, Nucl. Phys. 48, 622 (1963).

[18] For instance, if we take a “typical” $A \approx 50$ atom, we get $m/m_P = 0(10^{-17})$. For $\alpha = 1$ we find the spectral perturbation is sub–Lamb shift ($\lambda$ is almost unity so the order of $\hbar$ does not matter). For $\alpha = 2$, it is $O(10^{-34})$, (cf. Bollinger et al. Ref.[46]). This mischief continues ad infinitum. Pick any $f$ such that $f(0) = 1$ and $f(\infty) = 0$ with $\lambda(m) = f(m/m_P)$. The free nature of $\lambda$ is an very serious defect. However, the Copenhagen interpretation shares a similar inability to pin down Heisenberg’s cut. Bell called this situation the shifty split [J.S. Bell, Phys. World, 3, 33 (1990)].

[19] Expectations generate “particle”–like evolution and operators “wave”–like evolution. Varying the mix effectively controls wave–particle duality. The uncertainty principle stands, so precise measurability is not implied. More generally we can take an arbitrary Lie algebra of operators $\hat{A}_k$, and replace all commutators $[\hat{A}_j, \hat{A}_k] = iC^l_{jk}\hat{A}_l$ by their Weinberg bracket equivalents. The $C^l_{jk}$ are preserved by $\lambda$–deformation. Taking $\lambda \to 0$ we get a “classical limit” for any quantum system, even those with no classical analogue.

[20] Deformed Weyl–ordered quantization is defined via the obvious generalization,

$$Q_\psi^\lambda \circ H(q, p) \equiv (2\pi\hbar)^{-2} \int_{\mathbb{R}^4} e^{i[\sigma(\hat{p}_\lambda - p) + \tau(\hat{q}_\lambda - q)]/\hbar} H(q, p) d\sigma d\tau dq dp,$$

of the standard Weyl operator fourier transform. One checks easily that: $\partial_q Q_\psi^\lambda = \lambda Q_\psi^\lambda \partial_q$ and $\partial_q Q_\psi^\lambda = \lambda Q_\psi^\lambda \partial_q$. For the standard theory ($\lambda = 1$) see: H. Weyl, Z. Physik. 46, 1 (1927); H. Weyl, The Theory of Groups and Quantum Mechanics (Dover, New York, 1950) pp272–280; N.H. McCoy, Proc. Nat. Acad. Sci. 18, 674 (1932); K.E. Cahill and R.J. Glauber, Phys. Rev. 177, 1857, 1882 (1969); G.S. Agarwal and E. Wolf, Phys. Rev. D2, 2161, 2187, 2206 (1970); F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, Ann. Phys. (N.Y.) 111, 61, 111 (1978); and M. Hillery, R.F. O’Connell, M.O. Scully and E.P. Wigner, Phys. Rep. 106, 121 (1984).

[21] Connections between nonlinearity, mean–field theory, and/or dynamical chaos have been ex-
amined in many places. For example, Primas [H. Primas, in *Sixty–Two Years of Uncertainty* edited by A.I. Miller (Plenum, New York, 1990)], discusses this in connection with early work by Onsager [L. Onsager, J. Amer. Chem. Soc. **58**, 1486 (1936)]. More recently, see P. Boná, Comenius University Report, Faculty of Mathematics and Physics Report No. Ph10-91, 1991 (unpublished), and references therein. In connection with chaos, see D. David, D.D. Holm, and M.V. Tratnik, Phys. Lett. **138A**, 29 (1989); W.M. Zhang, D.H. Feng, J.M. Yuan and S.H. Wang, Phys. Rev. **A40**, 438 (1989); and W.M. Zhang, D.H. Feng and J.M. Yuan, Phys. Rev. **A42**, 7125 (1990). The factorization algorithm common to much of this work, and given detailed study in Ref. [2] often generates dynamical chaos in what began as a non–chaotic quantum model. This is because the $O(1/N)$ error control of large $N$ limits is rapidly overcome in any chaotic regime of the classical system. This can have important, sometimes dire, consequences for studies that seek to match theory to experiment via this approximation.

[22] To place both terms on equal footing in $\lambda$ and $\hbar$ one adapts Moyal’s calculus [J.E. Moyal, Proc. Camb. Phil. Soc. **45**, 99 (1949)] to prove that,

$$\langle [\hat{G}^\lambda, \hat{H}^\lambda] \rangle / i\hbar = \lambda^2 \langle \{G, H\}^\lambda_M \rangle,$$

where $\{G, H\}^\lambda_M$ denotes the deformed Weyl–quantization of the Moyal bracket,

$$\{\bullet, \bullet\}_M = \frac{2}{\hbar} \sin \left( \frac{\hbar}{2} \left[ \frac{\partial}{\partial Q} \frac{\partial}{\partial P} - \frac{\partial}{\partial P} \frac{\partial}{\partial Q} \right] \right),$$

of the classical functions $G$ and $H$. See also, T. F. Jordan and E.C.G Sudarshan, Rev. Mod. Phys. **33**, 515 (1961); and G.A. Baker Jr., Phys. Rev. **109**, 2198 (1958).

[23] Compare, Messiah Ref.[1]

[24] Ref.[11] equation 2.12, we use $\hbar \neq 1$

[25] A Schrödinger equation of this type, with a $\psi$–dependent Hermitian operator, appears in T. Kibble, *Commun. Math. Phys.* **64**, 73 (1978). We stumbled across it in: K.R.W. Jones, University of Melbourne Report No. UM-P-91/47 (unpublished) 1991.

[26] W.M. Zhang, D.H. Feng and R. Gilmore, Rev. Mod. Phys. **62** (1990), 867; A. Perelomov, *Generalized Coherent States and Their Applications* (Springer, Berlin, 1986). J.R. Klauder and B.S. Skagerstam, *Coherent States: Applications in Physics and Mathematical Physics* (World Scientific, Singapore, 1985).
[27] P.A.M. Dirac, Phys. Zeit. der Sowjet. 3 64 (1933); R.P. Feynmann, Rev. Mod. Phys. 20 267 (1948). The reprints appear in: Selected Papers on Quantum Electrodynamics edited by J. Schwinger (Dover, New York, 1958).

[28] M.V. Berry, Proc. Roy. Soc. Lond. A392, 45 (1984); M.V. Berry, in Geometric Phases in Physics edited by A. Shapere and F. Wilczek (World Scientific, Singapore, 1989).

[29] Y. Aharonov and J. Anandan, Phys. Rev. Lett. 58, 1593 (1987); see also, J. Anandan and L. Stodolsky, Phys. Rev. D35, 2597 (1987).

[30] This is why we adopt the interpretation noted at Ref.[19]. The \(Q(t)\) and \(P(t)\) are not precisely measurable, but they can “guide” \(\psi\) along a classical path.

[31] V.I. Arnol’d, Mathematical Methods of Classical Mechanics 2nd edn. (Springer–Verlag, Berlin, 1989) chap 8.

[32] Elsewhere, we have derived the classical Schrödinger equation [K.R.W. Jones, University of Melbourne Report No. UM-P-91/45, 1991 (unpublished)]. In 1927 Weyl proved there is but one projective representation of the Abelian group of translations on the plane [H. Weyl, The Theory of Groups and Quantum Mechanics (Dover, New York, 1950) pp272–280]. Exploiting this fact we can rewrite Hamilton’s equations in the operator form,

\[
i\hbar \frac{d}{dt} \hat{U}[Q, P] = \hat{H}(Q, P) \hat{U}[Q, P],
\]

where \(\hat{U}[Q, P]\) is a member of the Heisenberg–Weyl group and the Hamiltonian reads,

\[
\hat{H}(Q, P) = H + H_Q(\hat{q} - Q) + H_P(\hat{p} - P),
\]

with \(H(Q, P)\) the classical Hamiltonian. The solution is the operator–valued trajectory,

\[
\hat{U}[Q, P] = e^{\frac{i}{\hbar} \int L dt} \hat{U}[Q, P],
\]

where, \(\int L dt = \int (P \dot{Q} - Q \dot{P})/2 - H(Q, P) \, dt\) and \(Q(t)\), and \(P(t)\) solve Hamilton’s equations. This is verified by differentiation. Then we invoke the Stone–von Neumann theorem [M.H. Stone, Proc. Nat. Acad. Sci. 16, 172 (1932); J. von Neumann, Math. Ann. 104 570 (1931)], and note that any Hilbert space which carries an irrep. of the Heisenberg–Weyl group is unitarily equivalent to the standard Schrödinger representation. We then place a ket on the right to get the wave evolution. Thus the projective revision of classical theory automatically gives us: some constant \(\hbar\), wavefunctions, canonical commutation relations, and the classical Schrödinger equation. P. Boná (private communication, see Ref.[21]) informs me that he has obtained a similar result.
It is a folk prejudice of the quantum chaos community that the linearity of quantum theory has nothing to do with chaos suppression because the classical Liouville equation has this trivial linearity property. Wider study of nonlinear quantum theory, via numerical simulations, should help decide the matter.

Given that strobe maps are so useful as test examples, it might be interesting to study one-parameter families of nonlinear Floquet maps defined by,

\[ |\psi_{n+1}\rangle = U^\mu_n (|\psi_n\rangle) \equiv e^{-i\frac{\Delta t}{\hbar} H_{\text{eff}}^{1-\mu}(\psi_n,\psi_n^*)}|\psi_n\rangle, \]

where \( t = n\Delta t, n = 0,1,2\ldots \) and \( \mu = 1 - \lambda \) is the nonlinearity parameter [compare: M.J. Feigenbaum, in *Universality in Chaos* edited by P. Cvitanović (Adam Hilger, Bristol, 1986)]. Of interest is the fact that quantum systems with suppressed chaos must be perturbed via \( \mu \) towards their nonintegrable classical counterparts.

L.E. Reichl, *The transition to chaos in conservative classical systems: quantum manifestations* (Springer–Verlag, New York, 1992).

J.P. Provost and G. Vallee, Commun. Math. Phys. **76**, 289 (1980).

Ref. 11 §3

P.M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw–Hill, New York, 1953).

We can find no single–fixed point exactly soluble problem which does not, but these are terribly un–representative examples.

F. Calogero, J. Math. Phys. **10**, 2197 (1969); and F. Calogero, J. Math. Phys. **12**, 419 (1971).

M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions* (National Bureau of Standards, Washington, 1964) eq. 22.6.18 p781.

See Eckhardt Ref.[9] pp.224-235; V.P. Maslov and M.V. Fedoriuk, *Semi–Classical approximations in quantum mechanics* (Reidel, Holland, 1981); J.B. Keller, Ann. Phys. (N.Y.) **4**, 180 (1958); and references therein.

Ref. 11 §5

L.I. Schiff, *Quantum Mechanics* 3rd edn. (McGraw-Hill, New York, 1968).

W. Heisenberg, *The Physical Principles of the Quantum Theory* (University of Chicago, Chicago, 1930); and H.P. Robertson, *Phys. Rev.* **34**, (1929).

Numerous high precision tests of linearity have been performed using the Weinberg theory (see Ref.[12] for the proposal). Early examples include: J.J. Bollinger, D.J. Heinzen, W.M.
Itano, S.L. Gilbert and D.J. Wineland, Phys. Rev. Lett. 63, 1031 (1989); T.E. Chupp and R.J. Hoare, Phys. Rev. Lett. 64, 2261 (1990); R.L. Walsworth, I.F. Silvera, E.M. Mattison and R.F.C. Vessot, Phys. Rev. Lett. 64, 2599 (1990).

[47] O. Carnal and J. Mlynek, Phys. Rev. Lett. 66, 2689 (1991); D.W. Keith, C.R. Ekstrom, Q.A. Turchette, and D.E. Pritchard, Phys. Rev. Lett. 66, 2693 (1991).

[48] Microspherules have been developed for use in guided drug delivery, P. Guiot and P. Couvreur, *Polymeric Nanoparticles and Microspheres* (CRC Press, Florida, 1986). They can be manufactured down to 1–100µm. At a notional specific gravity of unity (they are prepared in suspension), this corresponds to $m \approx 10^{-9}–10^{-15}$ kg. For a natural frequency of $10^9$ Hz (microwaves), we need a “spring constant” $k = \omega^2 m \approx 10^3–10^6$ Nm$^{-1}$. Over one particle radius (a simple measure of stress) this is $10^{-3}–10^2$ N. This may be feasible at the lower end. Single–atom trapping technology might scale for this purpose [see: H. Dehmelt, Rev. Mod. Phys. 62, 525 (1990); W. Paul, Rev. Mod. Phys. 62, 531 (1990); and N.F. Ramsey, Rev. Mod. Phys. 62, 541 (1990)].

[49] For true elementary particles the requisite parameters can be measured in different experiments, for a composite particle it becomes very much harder.

[50] Ref. 11 §5

[51] L.D. Landau and E.M. Lifschitz, *Statistical Physics* (Pergamon, Oxford, 1989). One must keep the ideas put forward here distinct from quantum statistical mechanics. Measurement probabilities differ from thermodynamic ones.

[52] See Ref. [31] pp206–207.

[53] K.R.W Jones, Ann. Phys. (N.Y.) 207, 140 (1991).

[54] L. Schulman, *Techniques and Applications of Path Integration* (Wiley, New York, 1981).

[55] S.P. Gudder, *Stochastic Methods in Quantum Mechanics* (North–Holland, Amsterdam, 1979).

[56] K.R.W. Jones, J. Phys. A 24, 121 (1991); K.R.W. Jones, J. Phys. A. 24, 1237 (1991).

[57] This idea is the basis of much current work on measurement modelling. The approach is particularly useful in quantum optics. See, for example: Gisin and Percival Ref. [7]; and H.M. Wiseman and G.J. Milburn, University of Queensland, Preprint (1992). An early example, is Bohm and Bub, Ref. [7]. They sought to interpret their model as a non–local hidden variables theory.

[58] H. Risken, *The Fokker–Planck Equation* (Springer–Verlag, Berlin, 1989).
S. Dyrting (private communication) 1992, told me of this possibility. The idea of elevating such formal double–commutator type equations to fundamental status is briefly explored in G.J. Milburn, Phys. Rev. A44, 5401 (1991); and references therein. Milburn shows how such behaviour can be made to emerge from a “shortest tick of the universal clock” postulate.

This may involve non–local hidden variables traced to the unknown wavefunction of the environment. J. Polchinski, Phys. Rev. Lett. 66, 397 (1991) has pointed out some subtle difficulties of nonlinear theories in relation to EPR–type experiments. However, if the quantum statistics are correctly recovered, then singlet states cannot be used for superluminal communication. See the discussion in R.J. Glauber, Ann. N.Y. Acad. Sci. 480, 336 (1986).

Many characteristic effects of this kind are developed in Ref. 11 §6. Although our interpolation does not lie in the class considered by Weinberg, the stringent exclusions made there appear to extend to this work also.