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DARMON’S POINTS AND QUATERNIONIC SHIMURA VARIETIES

JÉRÔME GÄRTNER

Abstract. In this paper, we generalize a conjecture due to Darmon and Logan (see [DL03] and [Dar04], chapter 8) in an adelic setting. We study the relation between our construction and Kudla’s works on cycles on orthogonal Shimura varieties. This relation allows us to conjecture a Gross-Kohnen-Zagier theorem for Darmon’s points.

1. Introduction

The theory of complex multiplication gives a collection of Heegner points on elliptic curves over \( \mathbb{Q} \), which are defined over class fields of imaginary quadratic fields. These points allowed to prove Birch and Swinnerton-Dyer’s conjecture over \( \mathbb{Q} \) for analytic rank 1 curves, thanks to the work of Gross-Zagier and Kolyvagin.

Let us briefly recall the construction of Heegner points. If \( E \) is an elliptic curve over \( \mathbb{Q} \) then we know that \( E \) is modular. Let \( N \) be the conductor of \( E \). There exists a modular form \( f \in S_2(N) \) such that \( L(E, s) = L(f, s) \). Denote by \( \Phi_N : \Gamma_0(N) \backslash \mathcal{H} \rightarrow E(\mathbb{C}) \) the modular uniformization which is obtained by taking the composition of the map \( z_0 \in \mathcal{H} \mapsto e^{2\pi i f(z)/z} \) (here \( c \) denotes the Manin constant) with the Weierstrass uniformization. Let \( z_0 \in \mathcal{H} \cap K \), where \( K/\mathbb{Q} \) is an imaginary quadratic field. A Heegner point is given essentially by \( 2 \) modulo periods of \( f \). It is the Abel-Jacobi image of \( z_0 \) in \( \mathbb{C}/\mathcal{A}_E \cong E(\mathbb{C}) \). The theory of complex multiplication shows that these points are defined over class fields of \( K \).

In [Dar04], Darmon gives a conjectural construction of Stark-Heegner points, which is a generalization of classical Heegner points. These points should help us to understand, on one hand the Birch and Swinnerton-Dyer conjecture, on the other hand Hilbert’s twelfth problem.

In more concrete terms, assume that \( F \) is a totally real number field of narrow class number 1. Let \( \tau_j \) be its archimedean places, and \( K/F \) some quadratic “ATR” extension (i.e. \( K \) has exactly one complex place). Darmon defines a collection of points on elliptic curves \( E/F \) which are expected to be defined over class fields of \( K \). In this case, the (conjectural, but partially proved by Skinner - Wiles) modularity of \( E \) gives the existence of a Hilbert modular form \( f \) on \( \mathcal{H} \) whose periods appear as a tensor product of periods of \( E_{\tau_j} = E \otimes_{F, \tau_j} \mathbb{C} \). The construction explained in [DL03] can be seen as an exotic Abel-Jacobi map.

In this paper, we generalize Darmon’s construction by removing the hypothesis “ATR” on \( K \) (but we assume that \( K \) is not CM) and the technical hypothesis that \( F \) has narrow class number 1. We replace the Hilbert modular variety used in the “ATR” case by a general quaternionic Shimura variety and define a suitable Abel-Jacobi map. We are able to specify the invariants of the quaternion algebra using local epsilon factors and to give a conjectural Gross-Zagier formula for these points. We conclude the paper by establishing a relation to Kudla’s study of cycles on orthogonal Shimura varieties, in order to give a Gross-Kohnen-Zagier type conjecture.

Let us summarize the main construction of this paper. Let \( F \) be a totally real field of degree \( d \) and let \( \tau_1, \ldots, \tau_d \) be its archimedean places. Fix \( r \in \{2, \ldots, d\} \), and a quadratic extension \( K/F \) such that the set of archimedean places of \( F \) that split completely in \( K \) is \( \{\tau_2, \ldots, \tau_r\} \). Let \( B/F \) be a quaternion algebra which splits at \( \tau_1, \ldots, \tau_r \) and ramifies at \( \tau_{r+1}, \ldots, \tau_d \). Let \( G = \text{Res}_{F/Q} B^r \). We will denote by \( \text{Sh}_H(G, X) \) the quaternionic Shimura variety of level \( H \) (a compact open subgroup of \( G(\mathbb{A}_f) \)) whose complex points are given by

\[
\text{Sh}_H(G, X)(\mathbb{C}) = \text{G}(\mathbb{Q})/(\mathbb{C} \otimes \mathbb{R})^r \times G(\mathbb{A}_f)/H.
\]

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Fix an embedding \( q : K \hookrightarrow B \). There is an action of \( (K \otimes \mathbb{R})_+^\times / (F \otimes \mathbb{R})^\times \) on \( (C \otimes \mathbb{R})^r \).

By considering a suitable orbit of this action, we obtain a real cycle \( T_b \) of dimension \( r - 1 \) on \( \text{Sh}_H(G, X)(\mathbb{C}) \). Using the theorem of Matsushima and Shimura, we deduce that there exists an \( r \)-cycle \( \Delta_b \) on \( \text{Sh}_H(G, X)(\mathbb{C}) \) such that \( \partial \Delta_b \) is an integral multiple of \( \mathcal{J} \).

Let \( E/F \) be an elliptic curve, assumed modular, i.e., there exists a Hilbert modular eigenform \( \tilde{\varphi} \) satisfying \( L(E, s) = L(\tilde{\varphi}, s) \). We will assume that this form corresponds to an automorphic form \( \varphi \) on \( B \) by the Jacquet-Langlands correspondence. There exists a holomorphic differential form \( \omega_\varphi \) of degree \( r \) on \( \text{Sh}_H(G, X)(\mathbb{C}) \) naturally attached to \( \varphi \). In general, the set of periods of \( \omega_\varphi \) is a dense subset of \( C \). Fix some character \( \beta \) of the set of connected components of \( (K \otimes \mathbb{R})_+^\times / (F \otimes \mathbb{R})^\times \).

Following Darmon we define a modified differential form \( \omega^\beta_\varphi \) whose periods are, assuming Yoshida’s period conjecture, a lattice, homothetic to some sublattice of the Neron lattice of \( E/F \).

The image of (a suitable multiple of) the complex number \( \int_{\Delta_b} \omega^\beta_\varphi \) in \( C/\Lambda_E \) is independent of the choice of \( \Delta_b \). Hence it defines by Weierstrass uniformization a point \( P^b_\beta \) in \( E(\mathbb{C}) \). We conjecture

**Conjecture (5.1.1).** \( P^b_\beta = \Phi \left( \int_{\Delta_b} \omega^\beta_\varphi \right) \in E(\mathbb{C}) \) lies in \( E(K^{ab}) \) and

\[
\forall a \in A^\times_K \quad \text{rec}_K(a) P^b_\beta = \beta(a_\infty) P^b_{\hat{\chi}(a)b}.
\]

Let us assume this conjecture is true and denote by \( K_b^+ \) the field of definition of \( P^b_\beta \). Let \( \pi = \pi(\varphi) \) be the automorphic representation generated by \( \varphi \); fix a character \( \chi : \text{Gal}(K_b^+ / K) \to \mathbb{C}^\times \). Denote by \( \varepsilon(\pi \times \chi, \frac{1}{2}) \) the sign in the functional equation of the Rankin-Selberg \( L \)-function \( L(\pi \times \chi, s) \) and by \( \eta_K : F_b^e / F^e \text{N}_K/F(K_b^+) \to \{\pm 1\} \) the quadratic character of \( K_b^+ / F \). The following proposition proves that \( U \) is uniquely determined by \( K_b^+ \) and the isogeny class of \( E/F \).

**Proposition (5.3.1).** Let \( b \in \hat{B}^\times \) and assume conjecture 5.1.1. If

\[
\eta(P^b_\beta) = \sum_{\sigma \in \text{Gal}(K_b^+ / K)} \chi(\sigma) \otimes P^b_\beta \in E(K_b^+) \otimes \mathbb{Z}[\chi]
\]

is not torsion, then:

\[
\forall v \mid \infty \quad \eta_{K,v}(-1) \varepsilon(\pi_v \times \chi_v, \frac{1}{2}) = \text{inv}_v(B_v) \quad \text{and} \quad \varepsilon(\pi \times \chi, \frac{1}{2}) = -1.
\]

The last part of this paper is focused on a conjecture in the spirit of the Gross-Kohnen-Zagier theorem. Assume that \( E(F) \) has rank 1. Denote by \( P_1 \) some generator modulo torsion. For each totally positive \( t \in OF \) such that \( (t) \) is square free and prime to \( d_K/F \), denote by \( K[t] \) the quadratic extension \( K[t] = F(\sqrt{-D_t}) \), where \( D_t \in F \) satisfies \( \tau_0(D_t) > 0 \) if and only if \( j \in \{1, r + 1, \ldots, d\} \).

Let \( P_{t,1} \) be Darmon’s point obtained for \( K[t] \) and \( b = 1 \), and set

\[
P_t = \text{Tr}_{K[t] / F} P_{t,1}.
\]

The point \( P_t \) is in \( E(F) \) and there exists some integer \( |P_t| \in \mathbb{Z} \) such that \( P_t = [P_t]P_b \). In the spirit of conjecture 5.3 of [DT08] we conjecture that:

**Conjecture (6.3.5).** There exists a Hilbert modular form \( g \) of level \( 3/2 \) such that the \( [P_t]s \) are proportional to some Fourier coefficients of \( g \).

In our attempt to adapt Yuan, Zhang and Zhang’s proof in the CM case [YZZ09] to prove this conjecture, we obtained a relation between Darmon’s points and Kudla’s program, see Proposition 5.5.3.2.

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## 2. Quaternionic Shimura varieties

In this section we recall some properties of Shimura varieties associated to quaternion algebras. The standard references are Reimann’s book [Rei97] and [Mil05]. The content of this section is more or less the transcription to Shimura varieties of what is done for curves in [CV07] and [Nek07].
Let $F$ be a totally real field of degree $d = [F : \mathbb{Q}]$ and $\tau_1, \ldots, \tau_d$ its archimedean places. Denote by $\overline{\mathbb{Q}} \subset \mathbb{C}$ the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ so $\tau_j : F \hookrightarrow \overline{\mathbb{Q}}$. Fix $r \in \{2, \ldots, d\}$ and a finite set $S_B$ of non-archimedean primes satisfying

$$|S_B| \equiv d - r \mod 2.$$ 

Let $B$ be the unique quaternion algebra over $F$ ramified at the set

$$\text{Ram}(B) = \{\tau_{r+1}, \ldots, \tau_d\} \cup S_B.$$ 

For each $j \in \{1, \ldots, d\}$ we put $B_{\tau_j} = B \otimes_{F, \tau_j} \mathbb{R}$. It is not necessary but more convenient to fix for each $j \in \{\tau_1, \ldots, \tau_r\}$ an $\mathbb{R}$-algebra isomorphism

$$B_{\tau_j} \xrightarrow{\sim} M_2(\mathbb{R}).$$ 

The constructions given in this paper are independent on the choice of these isomorphisms, as in the author’s PhD thesis [Gär11].

Let $G$ be the algebraic group over $\mathbb{Q}$ satisfying $G(\mathbb{A}) = (B \otimes_{\mathbb{Q}} \mathbb{A})^\times$ for every commutative $\mathbb{Q}$-algebra $\mathbb{A}$. We will denote by $\text{nr} : G(\mathbb{A}) \longrightarrow (F \otimes_{\mathbb{Q}} \mathbb{A})^\times$ the reduced norm and by $Z$ the center of $G$. For $j \in \{1, \ldots, d\}$ let $G_j$ be the algebraic group over $\mathbb{R}$ given by $G_j = G \otimes_{F, \tau_j} \mathbb{R}$; thus $G_{\mathbb{R}}$ decomposes as $G_1 \times \cdots \times G_d$. For any abelian group $A$, denote by $\widehat{A}$ the group $A \otimes \mathbb{Z}$.

Let $X$ be the $G(\mathbb{R})$-conjugacy class of the morphism $h : S = \text{Res}_{\mathbb{C}/\mathbb{R}}(G_{m, \mathbb{C}}) \longrightarrow G(\mathbb{R}) = G_1(\mathbb{R}) \times \cdots \times G_d(\mathbb{R})$ defined by

$$x + iy \mapsto \begin{pmatrix} x & y \\ -y & x \end{pmatrix}, \ldots, \begin{pmatrix} x & y \\ -y & x \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ \vdots & \vdots \end{pmatrix}_{r \text{ times}}, \begin{pmatrix} 1 & 1 \\ \vdots & \vdots \end{pmatrix}_{d - r \text{ times}}.$$

The set $X$ has a natural complex structure [Mil90] and the following map is an holomorphic isomorphism between $X$ and $(\mathbb{C} \setminus \mathbb{R})^r$:

$$ghg^{-1} \mapsto g \cdot (i, \ldots, i) = \begin{pmatrix} a_1i + b_1 \\ c_1i + d_1 \end{pmatrix}, \ldots, \begin{pmatrix} a_r + b_r \\ c_r + d_r \end{pmatrix},$$

where $g = (g_1, \ldots, g_d) \in G(\mathbb{R})$ and for $j \in \{1, \ldots, r\}$ $g_j$ is identified with $\begin{pmatrix} a_j \\ c_j \\ b_j \\ d_j \end{pmatrix}$.

Quaternionic Shimura varieties. Let $H$ be an open-compact subgroup of $\hat{B}^\times$. The quaternionic Shimura varieties considered in this paper are algebraic varieties $\text{Sh}_H(G, X)$ whose complex points are given by

$$\text{Sh}_H(G, X)(\mathbb{C}) = B^\times / (X \times \hat{B}^\times / H),$$

where the left-action of $B^\times$ and the right-action of $H$ are given by

$$\forall k \in B^\times \forall h \in H \forall (x, b) \in X \times \hat{B}^\times \quad k \cdot (x, b) \cdot h = (kx, kbh).$$

Such Shimura varieties are defined over some number field called the reflex field. In our case this number field is

$$F' = \mathbb{Q} \left( \sum_{j=1}^r \tau_j(\alpha), \ \alpha \in F \right) \subset \overline{\mathbb{Q}} \subset \mathbb{C}.$$ 

We will denote by $[x, b]_H$ the element of $\text{Sh}_H(G, X)(\mathbb{C})$ represented by $(x, b)$ and by $[x, b]_{H_{\overline{F}^\times}}$ the corresponding element of the modified variety $\text{Sh}_H(G/Z, X)(\mathbb{C}) = B^\times / (X \times \hat{B}^\times / HZ)$.

Remark 2.1.1. The complex Shimura varieties are compact whenever $B \neq M_2(F)$. The Hilbert modular varieties used by Darmon in [Dar04] chapter 7 and 8 are obtained when $B = M_2(F)$ and $r = d$.

The Shimura varieties form a projective system $\{\text{Sh}_H(G, X)\}_H$ indexed by open compact subgroups in $\hat{B}^\times$. The transition maps $\text{pr} : \text{Sh}_H(G, X) \rightarrow \text{Sh}_{H'}(G, X)$ are defined on complex points by

$$[x, b]_H \rightarrow [x, b]_{H'}.$$
There is an action of $\tilde{B}^\times$ on the projective system $\{\text{Sh}_H(G, X)\}_H$. The right multiplication by $g \in B^\times$ induces an isomorphism $[g] : \text{Sh}_H(G, X) \xrightarrow{\sim} \text{Sh}_H(G, X)_{g^{-1}Hg}$, defined on complex points by

$$[g][x, b]_H = [x, bg]_{g^{-1}Hg}.$$ 

Complex conjugation. Fix $j \in \{1, \ldots, r\}$. Let $h_j : S \to G_j, \mathbb{R}$ be the morphism obtained by composing $h$ with the $j$-th projection $G_j \to G_j, \mathbb{R}$ and $X_j$ the $G_j(\mathbb{R})$-conjugacy class of $h_j$. For $x_j = g_j h_j g_j^{-1} \in X_j$, the set $\text{Im}(g_j h_j g_j^{-1})$ is a maximal anisotropic $\mathbb{R}$-torus in $G_j(\mathbb{R})$. The map $\ell_j : x_j \mapsto \text{Im}(x_j)$ satisfies $|\ell_j^{-1}(\ell_j(x_j))| = 2$, thus there exists a unique antiholomorphic and $G_j, \mathbb{R}$-equivariant involution $t_j : X_j \to X_j$ such that

$$\forall x_j \in X_j \quad \ell_j^{-1}(\ell_j(x_j)) = \{x_j, t_j(x_j)\}.$$ 

More precisely, under the identification $X_j \xrightarrow{\sim} \mathbb{C} \times \mathbb{R}$, the map $\ell_j$ satisfies $\ell_j(x + iy) = \left\{ \begin{array}{ll} x & y \\ -y & x \end{array} \right\}$ and $\ell_j^{-1}(\ell_j(x + iy)) = \{x + iy, x - iy\}$. Note that the map $t_j$ can be extended to complex points of the Shimura varieties by $t_j([x, b]_H) = [x_j, t_j(x)]_H$; $t_j$ acts trivially on $X_k$ for $k \neq j$.

Differential forms. In this section we recall some facts concerning differential forms on Shimura varieties. We will denote by $\Omega_H = \Omega^r_{H, F}$ the sheaf of differentials of degree $r$ on $\text{Sh}_H(G, X)$ and by $\Omega^r_H$ the sheaf of holomorphic $r$-differentials on $\text{Sh}_H(G, X)(\mathbb{C})$, provided that $\text{Sh}_H(G, X)$ is smooth. Recall that the GAGA principle gives us the following isomorphism between global sections

$$\Gamma(\text{Sh}_H(G, X), \Omega^r_H) \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\sim} \Gamma(\text{Sh}_H(G, X)(\mathbb{C}), \Omega^r_H).$$

Notice that in general, $\text{Sh}_H(G, X)$ is not smooth. In this last case we will fix some integer $n \geq 3$ such that for each $p$ in $\text{Ram}(B)$ we have $p \nmid n\mathbb{O}_F$ and for each $v \mid n\mathbb{O}_F$ isomorphisms $\iota_v : B_v \xrightarrow{\sim} M_2(F_v)$.

The group

$$H' = \{ (h_v) \in H, \text{ s.t. } \forall v \mid n\mathbb{O}_F h_v \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ mod } n\mathbb{O}_F \}$$

is of finite index in $H$ and $\text{Sh}_{H'}(G, X)$ is smooth. The map $\text{Sh}_{H'}(G, X) \to \text{Sh}_H(G, X)$ is a finite covering. We define $\Omega_H = \frac{1}{H' - H} \sum_{\sigma \in H'/H'} \sigma \Omega_{H'} = (\Omega_{H'})^H$. By abuse of language, we shall call an element of $\Gamma(\Omega_H) = \Gamma(\text{Sh}_H(G, X), \Omega_H) = \left( \bigoplus_{\sigma \in H'/H'} \sigma \Gamma(\text{Sh}_H(G, X), \Omega_{H'}) \right)$ a global $r$-form on $\text{Sh}_H(G, X)$. Remark that the space of global holomorphic $r$-forms $\text{lim} \Gamma(\Omega^r_H)$ is equipped with a canonical action of $\tilde{B}^\times$ given by pull-backs $[g]^\times$.

Let $\varepsilon \in \{\pm 1\}^r$ and denote by $\Gamma((\Omega^r_H)^\varepsilon)$ the space of $r$-forms on $\text{Sh}_H(G, X)(\mathbb{C})$ which are holomorphic (resp. anti-holomorphic) in $z_j$ if $\varepsilon_j = +1$ (resp. if $\varepsilon_j = -1$). The maps $t_j$ pulled-back on $\Gamma((\Omega^r_H)^\varepsilon)$ satisfy

$$t_j^* : \Gamma((\Omega^r_H)^\varepsilon) \to \Gamma((\Omega^r_H)^{\varepsilon})$$

where $\varepsilon'_k = \varepsilon_k$ for $k \neq j$ and $\varepsilon'_j = -\varepsilon_j$.

When $\sigma \in \prod_{j=2}^{r+1} \{\pm 1\}$ we will define $e_j \in \{0, 1\}$ by $e_j = (-1)^{\varepsilon_j}$ and $t_\sigma^* \in \prod_{j=2}^{r+1} (t_j^*)^{e_j}$. Let

$$\beta : \prod_{j=2}^{r+1} \{\pm 1\} \to \{\pm 1\} \quad \text{be a character and } \omega \in \Gamma((\Omega^r_H)^\varepsilon).$$

We shall denote by $\omega^\beta$ the element

$$\omega^\beta = \sum_{\sigma \in (\pm 1)^{r+1}} -\beta(\sigma) t_\sigma^*(\omega) \in \bigoplus_{\sigma \in H'/H'} \Gamma((\Omega^r_H)^\varepsilon).$$

Automorphic forms. Let $S^H_{2, \ldots, 2, 0, \ldots, 0}(B^\times_{\Lambda})$ of functions $\varphi : B^\times_{\Lambda} \cong G(\mathbb{R}) \times \tilde{B}^\times \to \mathbb{C}$ satisfying the following properties:

1. $\forall g \in B^\times \forall b \in B^\times_{\Lambda} \varphi(bg) = \varphi(b)$,
2. $\forall \mathbf{g} \in (\mathbb{R}^+)^{r+1} \times G_{r+1}(\mathbb{R}) \times \cdots \times G_{2}(\mathbb{R}) \subset G(\mathbb{R}) \forall b \in B^\times_{\Lambda} \varphi(bg) = \varphi(b)$,
3. $\forall \mathbf{h} \in H \forall b \in B^\times_{\Lambda} \varphi(bh) = \varphi(b)$,
4. $\forall \mathbf{g} \in B^\times_{\Lambda}(\theta_1, \ldots, \theta_r) \in \mathbf{R}^r \varphi\left( g \left( \begin{array}{ccc} \cos \theta_1 & -\sin \theta_1 \\ \sin \theta_1 & \cos \theta_1 \end{array} \right), \ldots, \left( \begin{array}{ccc} \cos \theta_r & -\sin \theta_r \\ \sin \theta_r & \cos \theta_r \end{array} \right) \right) = e^{-2\theta_1} \cdots e^{-2\theta_r} \varphi(g)$,
(5) For all \( g \in B^x_A \), the map
\[
(x_1 + iy_1, \ldots, x_r + iy_r) \mapsto \frac{1}{y_1 \cdots y_r} \varphi \left( g \left\{ \begin{array}{c}
y_1 \\
0
\end{array} \right., \ldots, \begin{array}{c}
x_1 \\
0
\end{array} \right) \right), 1, \ldots, 1
\]
is holomorphic on \( \mathcal{H}^r \) where \( \mathcal{H} \) denotes the Poincaré upper-half plane.

Remark that we do not need any assumption to obtain cuspidal forms as \( \hat{B} \) will be assumed to differ from \( M_2(F) \).

There is an action of \( \hat{B}^x \) on \( S_2 = \bigcup_{\mathcal{H}} S^H_2 \) defined by
\[
\forall g \in \hat{B}^x, \ \forall \varphi \in S_2, \ \forall x \in B^x_A \ \ g \cdot \varphi(x) = \varphi(xg);
\]
thus \( S^H_2 \) is the space of \( H \)-invariant functions in \( S_2 \).

By modifying the properties 4 and 5 above we obtain the following new definition:

**Definition 2.1.2.** Let \( \varepsilon : \{\tau_1, \ldots, \tau_r\} \to \{\pm 1\} \) and \( \varepsilon_i = \varepsilon(\tau_i) \). The space \( (S^H_2)^x \) is the space of maps \( \varphi : B^x_A \simeq G(\mathbb{R}) \times \hat{B}^x \to \mathbb{C} \) satisfying 1-3 above and

4'. for all \( g \in B^x_A \) and \((\theta_1 \ldots \theta_r) \in \mathbb{R}^r\)
\[
\varphi \left( g \left\{ \begin{array}{c}
\cos \theta_1 \\
\sin \theta_1
\end{array} \right., \ldots, \begin{array}{c}
\cos \theta_r \\
\sin \theta_r
\end{array} \right) \right), 1, \ldots, 1
\]

5'. for all \( g \in B^x_A \) the map
\[
(x_1 + iy_1, \ldots, x_r + iy_r) \mapsto \frac{1}{y_1 \cdots y_r} \varphi \left( g \left\{ \begin{array}{c}
y_1 \\
0
\end{array} \right., \ldots, \begin{array}{c}
x_1 \\
0
\end{array} \right) \right), 1, \ldots, 1
\]
is holomorphic (resp. anti-holomorphic) in \( z_j = x_j + iy_j \in \mathcal{H} \) if \( \varepsilon_j = 1 \) (resp. \( \varepsilon_j = -1 \)).

We will denote by \( S^x_2 \) (resp. \( (S^x_2)^F \)) the space of elements in \( S_2 \) (resp. \( S^H_2 \)) which are \( \hat{F}^x \)-invariant.

We are now able to affirm the existence of relations between \( r \)-forms on \( \text{Sh}_H(G, X)(\mathbb{C}) \) and automorphic forms:

**Proposition 2.1.3.** There exist bijections compatible with the \( \hat{B}^x \)-action between the following spaces:

\[
\frac{\Gamma(\Omega^m_+) \Gamma((\Omega^m_+)^\varepsilon)}{\Gamma(\Omega^m_+)^{\varepsilon_1}} \text{ and } S^H_2 \quad \frac{\Gamma(\text{Sh}_H(G/Z, X)(\mathbb{C}),(\Omega^m_+)^\varepsilon)}{\Gamma(\text{Sh}_H(G/Z, X)(\mathbb{C}),(\Omega^m_+)^{\varepsilon_1})} \text{ and } (S^x_2)^H \cdot \hat{F}^x
\]

This statement is completely analogous to section 3.6 of [CV07], see [Gär11], Propositions 1.2.2.4
and 1.2.2.5 for more details.

Matsushima-Shimura theorem. The decomposition of the cohomology of quaternionic Shimura varieties given by Matsushima-Shimura theorem will be useful in the following sections. Let us recall this result when \( B \neq M_2(F) \) [MS63] and [Pre90]. Denote by \( h^+_F \) the narrow class number of \( F \).

**Theorem 2.1.4.** Let \( m \in \{0, \ldots, 2r\} \). We have the following decomposition:

\[
H^m(\text{Sh}_H(G, X)(\mathbb{C}), \mathbb{C}) \simeq \begin{cases}
\left( \text{Vect} \left( \bigwedge_{1 \in \varepsilon \subset \{1, \ldots, r\}} \frac{dz_1 \wedge \ldots \wedge dz_r}{y_1} \right) \right)^s & \text{if } m \neq r \\
\left( \text{Vect} \left( \bigwedge_{1 \in \varepsilon \subset \{1, \ldots, r\}} \frac{dz_1 \wedge \ldots \wedge dz_r}{y_1} \right) \right)^s \bigoplus \bigoplus_{\varepsilon \in \{\pm 1\}^r} (S^x_2)^H & \text{if } m = r
\end{cases}
\]

and

\[
H^m(\text{Sh}_H(G/Z, X)(\mathbb{C}), \mathbb{C}) \simeq \begin{cases}
\left( \text{Vect} \left( \bigwedge_{1 \in \varepsilon \subset \{1, \ldots, r\}} \frac{dz_1 \wedge \ldots \wedge dz_r}{y_1} \right) \right)^s' & \text{if } m \neq r \\
\left( \text{Vect} \left( \bigwedge_{1 \in \varepsilon \subset \{1, \ldots, r\}} \frac{dz_1 \wedge \ldots \wedge dz_r}{y_1} \right) \right)^s' \bigoplus \bigoplus_{\varepsilon \in \{\pm 1\}^r} (S^x_2)^{H \cdot \hat{F}^x} & \text{if } m = r,
\end{cases}
\]

where \( s \) (resp. \( s' \)) is the number of connected components of \( \text{Sh}_H(G, X)(\mathbb{C}) \) (resp. of \( \text{Sh}_H(G/Z, X)(\mathbb{C}) \)).
3. Periods

3.1. Yoshida’s conjecture. Let \( E/F \) be an elliptic curve, assumed modular in the sense that there exists a cuspidal, parallel weight two Hilbert modular form \( \tilde{\varphi} \in S_2(\text{GL}_2(F_\Lambda)) \) satisfying \( L(E, s) = L(\tilde{\varphi}, s) \). We shall assume that the automorphic representation generated by \( \tilde{\varphi} \) is obtained by the Jacquet-Langlands correspondence from \( \varphi \in S_{\text{cusp}}^2(\text{GL}_2(F_\Lambda)) \).

Denote by \( \pi = \pi_\infty \otimes \pi_f \) the automorphic representation of \( B_\Lambda^\vee/F_\Lambda^\vee \) generated by \( \varphi \). We shall assume until section 3.3, only for simplicity, that \( \dim \pi_f^J = 1 \).

Let \( M = h^1(E) \) be the motive over \( F \) with coefficients in \( \mathbb{Q} \) associated to \( E \). Yoshida [Yos94] conjectures the existence of a rank 2 motive \( M' \) over the reflex field \( F' \), with coefficients in \( \mathbb{Q} \), satisfying \( M' = \bigotimes_{\{\tau_1, \ldots, \tau_r\}} \text{Res}_{F/F'} M \). This motivic conjecture is the following:

**Conjecture 3.1.1** (Yoshida, [Yos94]). The motive \( M' \) over \( F' \) is isomorphic to the motive associated to the part \( H^*(\text{Sh}_{\text{rig}}(G, X))(E) \) of the cohomology for which Hecke eigenvalues are the same as \( E \).

While looking at the \( \ell \)-adic realization, this conjecture is in fact the Langlands cohomological conjecture. This case is known, up to semi-simplification, thanks to Brylinski and Labesse in the case \( B = M_2(F) \) [BL84], Langlands in the case \( B \neq M_2(F) \) for primes of good reduction, [Lan79] and Reimann (- Zink) [Rei97, RZ91] for a more general cases.

Recall the following decompositions given by Yoshida in [Yos94] section 5.1, when we focus on \( \tau' : F' \hookrightarrow \mathbb{C} \) induced by \( \tilde{\tau} : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C} \).

**Betti cohomology.** There exists an isomorphism of \( \mathbb{Q} \)-vector spaces

\[
\mathcal{J} : M_{B, \tau_j} \simeq \bigotimes_{j=1}^{r} M_{B, \tau_j}
\]

de Rham cohomology. The map

\[
\mathcal{J} : M_{dR}^j \rightarrow \left( \bigotimes_{j=1}^{r} (M_{dR} \otimes_{F, \tau_j} \overline{\mathbb{Q}}) \right)^{\text{Gal}(\overline{\mathbb{Q}}/F')}
\]

is an isomorphism of \( F' \)-vector-spaces. The right hand side is a tensor product of \( \overline{\mathbb{Q}} \)-vector spaces and the action of \( \sigma \in \text{Gal}(\overline{\mathbb{Q}}/F') \) is given by \( \bigotimes_{\{\tau_1, \ldots, \tau_r\}} (x_\tau \otimes_{F, \tau_s} a_s) \mapsto \bigotimes_{\{\tau_1, \ldots, \tau_r\}} (x_\tau \otimes_{F, \tau_s} \sigma(a_s)) \).

**Comparison isomorphisms.** Let \( I = \bigotimes_{j=1}^{r} I_{\tau_j} \), where

\[
I_{\tau_j} : M_{B, \tau_j} \otimes_{\mathbb{Q}} \mathbb{C} \rightarrow M_{dR} \otimes_{F, \tau_j} \mathbb{C}
\]

are isomorphisms of \( \mathbb{C} \)-vector spaces, and \( I' \) be the following isomorphism over \( \mathbb{C} \):

\[
I' : M_B^j \otimes_{\mathbb{Q}} \mathbb{C} \rightarrow M_{dR}^j \otimes_{F'} \mathbb{C}.
\]

The maps \( I \circ (\mathcal{J} \otimes_{\mathbb{Q}} \text{id}_\mathbb{C}) \) and \( (\mathcal{J} \otimes_{F'} \text{id}_\mathbb{C}) \circ I' \) satisfy:

\[
(*) \quad I \circ (\mathcal{J} \otimes_{\mathbb{Q}} \text{id}_\mathbb{C}) = (\mathcal{J} \otimes_{F'} \text{id}_\mathbb{C}) \circ I' : M_B^j \otimes_{\mathbb{Q}} \mathbb{C} \rightarrow \bigotimes_{k=1}^{r} (M_{dR} \otimes_{F, \tau_k} \mathbb{C}).
\]

Yoshida’s period conjecture consists of the isomorphisms \( \mathcal{J} \), \( \mathcal{J} \), \( I \) and \( I' \) satisfying (*) . It is the Hodge-de Rham realization of the motivic conjecture above.

**Complex conjugation.** Let \( c_{\tau_j} \) be the complex conjugation on \( M_{B, \tau_j} \). We will need the following hypothesis, which allows us to compare \( c_{\tau_j} \) with \( t_{\tau_j}^* \) on \( M_{dR}^j \otimes_{F'} \mathbb{C} \).

**Hypothesis 3.1.2.** The action of \( t_{\tau_j}^* \) on \( M_{dR}^j \otimes_{F'} \mathbb{C} \) corresponds via the isomorphism

\[
(\mathcal{J} \otimes_{\mathbb{Q}} \text{id}_\mathbb{C}) \circ (I')^{-1} : M_{dR}^j \otimes_{F'} \mathbb{C} \rightarrow M_B^j \otimes_{\mathbb{Q}} \mathbb{C} \rightarrow \bigotimes_{k=1}^{r} (M_{B, \tau_k} \otimes_{\mathbb{Q}} \mathbb{C})
\]

to the action of \( c_{\tau_j} \) on \( M_{B, \tau_j} \).
3.2. Lattices and periods. Fix some $\omega_\varphi \neq 0$ in $F^r M_{dR}'$. By definition of $M'$, there exists a finite set of places $S$ of $F$ such that for $v \notin S$, $T_v \omega_\varphi = a_v(E) \omega_\varphi$.

Let $\Omega_{E/F}$ be the sheaf of differentials on $E/F$. Fix $\eta \neq 0 \in H^0(E, \Omega_{E/F}) = F^1 M_{dR}$. For $j \in \{1, \ldots, n\}$, let

$$\eta_j = \eta \otimes_{F, \tau_j} 1 \in H^0 \left( E \otimes_{F, \tau_j} \widehat{\mathbb{Q}}, \Omega_{(E \otimes_{F, \tau_j} \widehat{\mathbb{Q}})} \right) = (F^1 M_{dR}) \otimes_{F, \tau_j} \widehat{\mathbb{Q}}.$$ 

Then

$$\bigotimes_{j=1}^r \eta_j \in \left( \bigotimes_{j=1}^r (F^1 M_{dR}) \otimes_{F, \tau_j} \widehat{\mathbb{Q}} \right) \text{Gal}(\widehat{\mathbb{Q}}/F') = \mathcal{J} (F^r M_{dR}')$$

and there exists $\alpha \in F^r \times$ such that

$$\mathcal{J} (\alpha \omega_\varphi) = \eta_1 \otimes \cdots \otimes \eta_r.$$

Let $j \in \{1, \ldots, r\}$ and $E_j = E \otimes_{F, \tau_j} \mathbb{C}$. We shall denote by $H_1(E_j, \mathbb{Z})^\pm$ the eigenspaces of the complex conjugation action on $H_1(E_j, \mathbb{Z})$. Then

$$\left\{ \int_{\Gamma} \eta_j, \ \eta \in H_1(E_j, \mathbb{Z})^\pm \right\} = Z \Omega_j^\pm,$$

where $\Omega_j^+ \in \mathbb{R} \setminus \{0\}$ and $\Omega_j^- \in i \mathbb{R} \setminus \{0\}$ are determined up to a sign. We fix the signs by imposing, e.g., $\text{Re} \left( \Omega_j^+ \right) > 0$ and $\text{Im} \left( \Omega_j^- \right) > 0$.

Fix a character $\beta : \{1\} \times \prod_{j=2}^r \{\pm 1\} \to \{\pm 1\}$, and write $\beta = \prod_{j=2}^r \beta_j$. We set

$$\omega_\varphi^\beta = \left( \sum_{\sigma \in \{1\} \times \prod_{j=2}^r \{\pm 1\}} \beta(\sigma) \eta_\sigma^* \right) \omega_\varphi = \prod_{j=2}^r (1 + \beta_j(-1)t_j^*) \omega_\varphi$$

and

$$\Omega^\beta = \prod_{j=2}^r \Omega_j^{\beta_j(-1)}.$$

The following identities

$$\left( \bigotimes_{j=1}^r M_{B, \tau_j} \right) \otimes_{\mathbb{Q}} \mathbb{C} = \bigotimes_{j=1}^r \text{Hom}_{\mathbb{Z}}(H_1(E_j, \mathbb{Z}), \mathbb{C}) = \text{Hom}_{\mathbb{Z}} \left( \bigotimes_{j=1}^r H_1(E_j, \mathbb{Z}), \mathbb{C} \right)$$

and Yoshida’s conjecture show that the image of $\alpha \omega_\varphi^\beta$ under the map

$$(\mathcal{J} \otimes_{\mathbb{Q}} \text{id}_{\mathbb{C}}) \circ I^{-1} = I^{-1} \circ (\mathcal{J} \otimes_{F'} \text{id}_{\mathbb{C}}) : M_{dR}' \otimes_{F'} \mathbb{C} \to \left( \bigotimes_{j=1}^r M_{B, \tau_j} \right) \otimes_{\mathbb{Q}} \mathbb{C}$$

is identified with the linear form

$$(1) \left\{ \bigotimes_{j=1}^r H_1(E_j, \mathbb{Z}), \mathbb{C} \right\} \to \bigotimes_{\Gamma_1 \otimes \cdots \otimes \Gamma_r} \eta_j$$

Hypothesis 3.1.2 allows us to be more explicit. Let $\Gamma_1 \otimes \cdots \otimes \Gamma_r \in \bigotimes_{j=1}^r H_1(E_j, \mathbb{Z})$, then

$$\left( \bigotimes_{\Gamma_1 \otimes \cdots \otimes \Gamma_r} \prod_{j=1}^r (1 + \beta_j(-1)t_j^*) \eta_j \right) = \left( \int_{\Gamma_1} \eta_1 \right) \prod_{j=2}^r \int_{\Gamma_j} (1 + \beta_j(-1)t_j^*) \eta_j$$

and

$$= \left( \int_{\Gamma_1} \eta_1 \right) \prod_{j=2}^r \int_{\Gamma_j + \beta_j (-1)c_j \Gamma_j} \eta_j.$$
are commensurable. Thus there exists $\xi \in \mathbb{Z} \setminus \{0\}$ such that

$$\xi \text{Im} (H_r(\text{Sh}_H(G/Z, X)(C), \mathbb{Z}) \rightarrow (M'_B)^*) \subset \mathcal{G}^r \left( \bigotimes_{j=1}^r z H_1(E_j, \mathbb{Z}) \right).$$

This proves the following proposition:

**Proposition 3.2.1.** Under the hypothesis made in this section (E is modular, the multiplicity one in Yoshida’s motivic conjecture and 3.1.2), there exist $\alpha \in F^\times$ and $\xi \in \mathbb{Z} \setminus \{0\}$ such that

$$\forall \gamma \in H_r(\text{Sh}_H(G, X)(C), \mathbb{Z}), \forall \beta : \prod_{j=2}^r \{\pm 1\} \rightarrow \{\pm 1\}, \quad \xi \int_{\gamma} \alpha \omega^\beta \in \Lambda_1 \Omega^\beta.$$

3.3. **General case.** When $m_H(\pi) = \dim \pi^H_f(\phi) > 1$ Yoshida’s conjecture is the following

**Conjecture 3.3.1.** The motive $H'(\text{Sh}_H(G, X))^{(E)}$ is isomorphic to

$$\bigotimes_{\{\tau_1, \ldots, \tau_r\}} \text{Res}_{F/F'} M^{(m_H(\pi))}.$$

In general the motive $H'(\text{Sh}_H(G, X))^{(E)}$ has rank $\neq 2^r$. We shall provide Betti and de Rham realizations of a submotive $M' \subset H'(\text{Sh}_H(G, X))^{(E)}$ of rank $2^r$ and an isomorphism $M' \sim \bigotimes_{\{\tau_1, \ldots, \tau_r\}} \text{Res}_{F/F'} M.$

We need $0 \neq \omega_\phi \in F' H_{\text{dR}}^r(\text{Sh}_H(G/Z, X)/F')^{(E)}$ satisfying de Rham cohomology. The $F'$-vector space

$$M'^{\text{dR}} := \left( \bigoplus_{\sigma \in \{\pm 1\}^r} C t_\phi^*(\omega_\phi \otimes 1) \right) \cap H^r_{\text{dR}}(\text{Sh}_H(G/Z, X)/F')^{(E)}$$

has dimension $2^r$.

Thus

$$F' M'^{\text{dR}} := M'^{\text{dR}} \cap F' H_{\text{dR}}^r(\text{Sh}_H(G/Z, X)/F')^{(E)} = F' \omega_\phi.$$ 

Betti cohomology. Let

$$I' : H^r_B(\text{Sh}_H(G/Z, X)(C), \mathbb{Q})^{(E)} \otimes \mathbb{Q} C \sim H^r_{\text{dR}}(\text{Sh}_H(G/Z, X)/F')^{(E)} \otimes F' C.$$

The $\mathbb{Q}$-vector space

$$M'_B := I'^{-1}(M'^{\text{dR}} \otimes F' C) \cap H^r_B(\text{Sh}_H(G/Z, X)(C), \mathbb{Q})^{(E)}$$

has dimension $2^r$.

**Definition 3.3.2.** An element $\omega_\phi \in F' H_{\text{dR}}^r(\text{Sh}_H(G/Z, X)/F')^{(E)}$ is said rational if it satisfies the equations above.

Comparison isomorphisms. There exist isomorphisms

$$\mathcal{F} : M'_B \sim \bigotimes_{\tau_j} M_{B, \tau_j},$$

$$\mathcal{F} : M'^{\text{dR}} \sim \left( \bigotimes_{j=1}^r (M_{\text{dR}} \otimes_{F, \tau_j} \overline{Q})^{\text{Gal}(\overline{Q}/F')} \right),$$

and

$$I_{\tau_j} : M_{B, \tau_j} \otimes \mathbb{Q} C \sim M_{\text{dR}} \otimes_{F, \tau_j} C.$$

Set $I = \bigotimes_{j=1}^r I_{\tau_j}$. We have

$$(*) \quad I \circ (\mathcal{F} \otimes \mathbb{id}_C) = (\mathcal{F} \otimes F' \mathbb{id}_C) \circ I' : M'_B \otimes \mathbb{Q} C \sim \bigotimes_{j=1}^r (M_{\text{dR}} \otimes_{F, \tau_j} C).$$

As in Proposition 3.2.1 we have
Proposition 3.3.3. Let $\omega_\varphi \in F^rH^r_{\text{dR}}(\text{Sh}_H(G/Z,X)/F^r)^{(E)}$ be rational. If $E$ is modular and if Yoshida’s conjecture is true, then there exist $\alpha \in F^r\times$ and $\xi \in \mathbb{Z} \smallsetminus \{0\}$ such that

$$\forall \gamma \in H_\gamma(\text{Sh}_H(G,X)(\mathbb{C}),\mathbb{Z}), \forall \beta : \prod_{j=2}^r \{\pm 1\} \rightarrow \{\pm 1\}, \quad \xi \int_\gamma \alpha \omega_\varphi^* \in \Lambda_1 \Omega^\beta.$$ 

Example. Let $H_1, H_2 \subset \hat{B}^\times$ be compact open subgroups such that there exists $g \in \hat{B}^\times$ satisfying $g^{-1}H_1g \subset H_2$. Let $\omega_\varphi \in F^rH^r_{\text{dR}}(\text{Sh}_H(G/Z,X)/F^r)^{(E)}$ be rational. Let us explain a way to obtain $\omega_\varphi \in F^rH^r_{\text{dR}}(\text{Sh}_H(G/Z,X)/F^r)^{(E)}$ rational.

Let

$$\text{pr} : \text{Sh}_{g^{-1}H_1g}(G/Z,X) \rightarrow \text{Sh}_{H_2}(G/Z,X)$$

be the map given by

$$[x,b]_{g^{-1}H_1g} \mapsto [x,b]_{H_2}$$

and

$$[g] : \text{Sh}_{H_1}(G/Z,X) \rightarrow \text{Sh}_{g^{-1}H_1g}(G/Z,X)$$

by

$$[x,b]_{H_1} \mapsto [x,bg]_{g^{-1}H_1g}.$$ 

Let $\text{pr}_g : \text{Sh}_{H_1}(G/Z,X) \rightarrow \text{Sh}_{H_2}(G/Z,X)$ be the composition of $\text{pr}$ with $[g]$. Choose $\theta_g \in \mathbb{Q}$. Set

$$\omega_\varphi := \sum_{g \in \hat{B}^\times} \theta_g \text{pr}_g^*(\omega_\varphi),$$

s.t. $g^{-1}H_1g \subset H_2$

$$(M'_1)_{\text{dR}} = \left( \sum g \theta_g \text{pr}_g^* \right)(M'_2)_{\text{dR}}$$

and

$$(M'_1)_B = \left( \sum g \theta_g \text{pr}_g^* \right)(M'_2)_B.$$ 

Proposition 3.3.4. If $\omega_\varphi \neq 0$, then the map

$$\sum_{g \in \hat{B}^\times} \theta_g \text{pr}_g^*$$

is injective on $\bigoplus_{\sigma \in \{\pm 1\}^r} \text{Ct}_\sigma^*(\omega_\varphi \otimes 1)$

and $\omega_\varphi \in F^rH^r_{\text{dR}}(\text{Sh}_H(G/Z,X)/F^r)^{(E)}$ is rational.

Proof. Assume that $\omega = \sum_{\sigma \in \{\pm 1\}^r} \lambda_\sigma t_{\sigma}^* \omega_\varphi \in \bigoplus_{\sigma \in \{\pm 1\}^r} \text{Ct}_\sigma^*(\omega_\varphi \otimes 1)$ (where $\lambda_\sigma \in \mathbb{C}$) is such that $\sum g \theta_g \text{pr}_g^*(\omega) = 0$. We have the following equalities:

$$\sum_g \theta_g \text{pr}_g^* \omega = \sum_g \theta_g \text{pr}_g^* \sum_{\sigma} \lambda_\sigma t_{\sigma}^* \omega_\varphi = \sum_{\sigma} \lambda_\sigma t_{\sigma}^* \sum_g \theta_g \text{pr}_g^* \omega_\varphi = \sum_{\sigma} \lambda_\sigma t_{\sigma}^* \omega_\varphi,$$

Thus

$$\sum_{\sigma} \lambda_\sigma t_{\sigma}^* \omega_{\varphi_1} = 0 \in \bigoplus_{\sigma \in \{\pm 1\}^r} \text{Ct}_{\sigma}^* \omega_{\varphi_1},$$

and

$$\forall \sigma \in \{\pm 1\}^r \quad \lambda_\sigma t_{\sigma}^* \omega_{\varphi_1} = 0.$$ 

Hence $\forall \sigma \in \{\pm 1\}^r \lambda_\sigma \neq 0$. The map $\sum_{g \in \hat{B}^\times} \theta_g \text{pr}_g^*$ commutes with $T_v$, $v \notin S$ and is an isomorphism $\bigoplus \text{Ct}_\sigma^*\omega_\varphi \rightarrow \bigoplus \text{Ct}_\sigma^*\omega_\varphi$. Hence $\omega_\varphi \in \left( \bigoplus_{\sigma \in \{\pm 1\}^r} \text{Ct}_\sigma^*(\omega_{\varphi_1} \otimes 1) \right) \cap F^rH^r_{\text{dR}}(\text{Sh}_H(G/Z,X)/F^r)^{(E)}$ is rational.
4. Toric orbits

Let $K/F$ be a quadratic extension satisfying the following properties:

1. The places $\tau_2, \ldots, \tau_r$ of $F$ are split in $K$.
2. The places $\tau_1, \tau_{r+1}, \ldots, \tau_d$ are ramified in $K$.

Thanks to the Skolem-Noether theorem, there exists an $F$-embedding $q : K \hookrightarrow B$, unique up to conjugacy. We will denote by $q_j$ (resp. $\tilde{q}_j q_A$) the induced embedding $K \hookrightarrow B_{\tau_j}$ (resp. $\tilde{K} \hookrightarrow \tilde{B}$, $K_A \hookrightarrow B_A$). For each place $v$ of $F$, set $K_v = K \otimes_F F_v$.

4.1. Cycles on $X$. Let $T = \text{Res}_{K/Q}(G_m)/\text{Res}_{F/Q}(G_m)$. Thanks to Hilbert’s Theorem 90 we have

$$T(A) = (K \otimes Q)A/(F \otimes Q)A$$

for every $Q$-algebra $A$.

Fix an embedding $q : T \hookrightarrow G/Z(G)$. The group $T(R)$ is identified with $\prod_{j=1}^d K^\times_{\tau_j}/F_{\tau_j}^\times$ which allows us to define $q_j : K^\times_{\tau_j}/F_{\tau_j}^\times \hookrightarrow G_{\tau_j}$.

Let $\pi_0(T(R))$ be the set of connected components of $T(R)$ and denote by $T(R)^o$ the component of the identity. Fix a multi-orientation on $T(R)^o = \prod_{j=1}^d (K^\times_{\tau_j}/F_{\tau_j}^\times)^o$ (i.e. an orientation of each factor $(K^\times_{\tau_j}/F_{\tau_j}^\times)^o$) and remark that

$$\pi_0(T(R)) = T(R)/T(R)^o \simeq \prod_{j=2}^r \{\pm 1\}.$$  

We will focus on the orbits in $X$ under the action of $q(T(R)^o)$ by conjugation.

**Proposition 4.1.1.** Let $\mathcal{O}$ be an orbit of $q(T(R)^o)$ in $X$. Then $\mathcal{O}$ decomposes into a product of orbits in $X_j$ under $q_j(T(R)^o)$ and is multi-oriented.

**Proof.** The first part of this assertion follows from the natural decomposition $X = X_1 \times \ldots X_r$. The orbit $\mathcal{O}$ decomposes into orbits under $q_j((K^\times_{\tau_j}/F_{\tau_j}^\times)^o)$. For $j = 1$, $q_j((K^\times_{\tau_1}/F_{\tau_1}^\times)^o) \simeq S^1$ or a point and the orientation does not change. For $j \in \{2, \ldots, r\}$, $q_j((K^\times_{\tau_j}/F_{\tau_j}^\times)^o) \simeq R^+_1$. The action of $R^+_j$ on itself by multiplication does not change the orientation. Hence the multi-orientation induced on $\mathcal{O}$ by $T(R)^o$ is well-defined.

\[\square\]

In the following sections we shall fix some $q(T(R)^o)$-orbit $\mathcal{O}$ whose projection on $X_1$ is a point.

**Proposition 4.1.2.** $\mathcal{O}$ is a connected multi-oriented submanifold of real dimension $r-1$.

**Proof.** Recall that $\mathcal{O}$ is decomposed as $\mathcal{O} = \{z_1\} \times \mathcal{O}_2 \times \cdots \times \mathcal{O}_r$. Fix $x \in X$ such that $\mathcal{O} = q(T(R)^o) \cdot x$. Then for $j \in \{2, \ldots, r\}$ we have $\mathcal{O}_j = q_j((K^\times_{\tau_j}/F_{\tau_j}^\times)^o) \cdot \text{pr}_j(x)$. The group $q_j((K^\times_{\tau_j}/F_{\tau_j}^\times)^o)$ is naturally identified with $R^+_1$ and $\mathcal{O}_j$ is a connected oriented manifold of real dimension one.

\[\square\]

As a corollary, we have the following decomposition:

$$\mathcal{O} = \{z_1\} \times \gamma_2 \times \cdots \times \gamma_r,$$

when $z_1$ is one of the two fixed points in the action of $q_1(T(R)^o)$ on $X_1$ and $\gamma_j$ is an oriented connected submanifold of real dimension one in $X_j$.

When we use the identification of $X$ with $(\mathbb{C} \setminus \mathbb{R})^r$, the action of $T(R)$ on $X$ by conjugation is an action of $\text{PGL}_2(R)$ on $(\mathbb{C} \setminus \mathbb{R})^r$ by homography. Let $z \in K \setminus F$. For $j \in \{2, \ldots, r\}$ the matrix $q_j(z)$ is hyperbolic with exactly two fixed points in $\mathbb{P}^1(R)$, $z_j$ and $z'_j$. The manifold $\gamma_j$ is then a circle arc in the Poincaré upper half-plane joining $z_j$ to $z'_j$ (or a line if $z'_j = \infty$). Figure 1 gives some examples of what could the $\gamma_j$s be in the case of circle arcs.
4.2. Tori on $\text{Sh}_H(G/Z,X)(C)$. Let $b \in \hat{B}^\times$. We will denote by $\mathcal{R}_b^o$ the following subset of $\text{Sh}_H(G/Z,X)(C)$

$$
\mathcal{R}_b^o = \left\{ [x,b]_{H\hat{F}_x} \in \mathcal{F}^o \right\}.
$$

**Proposition 4.2.1.** $\mathcal{R}_b^o$ is an oriented torus of real dimension $r-1$.

**Proof.** Let $x, x' \in \mathcal{F}^o$ and $b \in \hat{B}^\times$. We know that

$$
[x,b]_{H\hat{F}_x} = [x',b]_{H\hat{F}_x} \iff \exists k \in B^\times \text{ and } h \in H\hat{F}_x \quad (kx', kbh) = (x, b)
$$

Since the projection of $\mathcal{F}^o$ on $X_1$ is a point, we have $k \in B \cap q_1(K_{r_1}) = q_1(K)$ and $k \in q(K^\times) \cap bH\hat{F}_x b^{-1}$.

Thus the stabilizer $\mathcal{W}$ of $\mathcal{R}_b^o$ under the action of $q(K^\times)$ is

$$
\mathcal{W} = q(K^\times) \cap (bH\hat{F}_x b^{-1})
$$

which is commensurable with $O_{K,+}^\times/O_F^\times$. This quotient has rank $r-1$ over $Z$ as a consequence of Dirichlet’s units theorem:

$$
O_{K,+}^\times/O_F^\times \simeq \text{torsion} \times Z^{r-1},
$$

and the torsion is finite. The action of $T(R)$ on $\mathcal{F}^o$ is given by $\prod_{j=2}^r (K_{r_j}/F_{r_j}^\times)^o$ and there is an isomorphism

$$
\prod_{j=2}^r (K_{r_j}/F_{r_j}^\times)^o \xrightarrow{\sim} R^{r-1}.
$$

The image $\tilde{\mathcal{O}}$ of $O_{K,+}^\times/O_F^\times$ in $R^{r-1}$ is isomorphic to $Z^s$ with $s \leq r-1$. Denote by $\tilde{O}^\times_K$ the image of $O^\times_K$ in $(K \otimes R)^{\times, N_{K/Q}=1}$. As

$$
\prod_{j \notin \{2, \ldots, r\}} K_{r_j}/F_{r_j}^\times
$$

are compact, $R^{r-1}/\tilde{O}$ is compact. Thus, the image of $O_{K,+}^\times/O_F^\times$ in $R^{r-1}$ is a lattice.

The set $\mathcal{F}_b^o$ is a principal homogeneous space under

$$
q(K^\times)/\mathcal{W} \simeq (R/Z)^{r-1}.
$$

It is a real torus in $\text{Sh}_H(G/Z,X)(C)$ of dimension $r-1$, which is oriented by the fixed multi-orientation on $\mathcal{F}^o$.

□

For each $u \in \pi_0(T(R))$ and $b \in \hat{B}^\times$ let

$$
\mathcal{R}_b^u = \left\{ [q(u) \cdot x, b]_{H\hat{F}_x} \in \mathcal{F}^o \right\}.
$$

It is a real oriented torus of dimension $r-1$.  

---

**Figure 1.** Case of circle arcs.
Proposition 4.2.2. The set

$$\{ \tilde{T}_b^u \mid b \in \tilde{B}^\times, \ u \in \pi_0(T(R)) \}$$

does not depend on the choice of \( q : K \hookrightarrow B \).

Proof. Let \( \tilde{q} : K \hookrightarrow B \) be another embedding. Thanks to the Skolem-Noether theorem there exists \( \alpha \in B^\times \) such that

$$\forall k \in K \quad \tilde{q}(k) = \alpha q(k)\alpha^{-1}.$$ 

Let \( x_0 \in X \), and assume that \( \tilde{T}^\circ = q(T(R)^0) \cdot x_0 \). We have \( \tilde{T}^\circ := \tilde{q}(T(R)^0) \cdot \alpha(x_0) = \alpha \cdot T^\circ \) and for each \( u \in \pi_0(T(R)) \)

$$\alpha \cdot q(u) \cdot \tilde{T}^\circ = \tilde{q}(uT(R)^0) \cdot \alpha \cdot x_0.$$ 

Let \( b \in \tilde{B}^\times \). As \( \alpha \in B^\times \) we have

$$\tilde{T}_b^u := \left[ \tilde{q}(u) \tilde{T}^\circ, b \right]_H \tilde{F}_x = \left[ \alpha \cdot q(u) \cdot \tilde{T}^\circ, b \right]_H \tilde{F}_x = \left[ q(u) \cdot \tilde{T}^\circ, \alpha^{-1} \cdot b \right]_H \tilde{F}_x = \tilde{T}^u_{\alpha^{-1}b}.$$ 

The map \( b \mapsto \alpha^{-1}b \) is a bijection. Thus

$$\{ \tilde{T}_b^u, \ b \in \tilde{B}^\times, \ u \in \pi_0(T(R)) \} = \{ \tilde{T}_b^u, \ b \in \tilde{B}^\times, \ u \in \pi_0(T(R)) \}.$$ 

Action of \( \text{Gal}(K^{ab}/K) \). Let us denote by \( K^{ab} \) the maximal abelian extension of \( K \) and by \( \text{rec}_{K} : K_{\text{ab}}^X / K^X \rightarrow \text{Gal}(K^{ab}/K) \) the reciprocity map normalized by letting uniformizers correspond to geometric Frobenius elements.

The group \( K_{\text{ab}}^X \) acts on \( \{ \tilde{T}_b^u \mid b \in \tilde{B}^\times, \ u \in \pi_0(T(R)) \} \) by

$$\forall a = (a_{\infty}, a_{f}) \in K^X_{\text{ab}} = K^X_{\text{ab}} \times \tilde{K}^\times \forall b \in \tilde{B}^\times \quad a \cdot \tilde{T}_b^u = \tilde{T}^{(a_{\infty})u}_{\hat{a}_{\infty}b}.$$ 

The action of \( k \in K^X \) is trivial; as \( q(k) \in B^\times \), the definition of \( \text{Sh}_H(G/Z, X)(C) \) gives:

$$k \cdot \tilde{T}_b^u = [q(k)q(u)\tilde{T}^\circ, \tilde{q}(k)b]_{H\tilde{F}_x} = [q(u)\tilde{T}^\circ, b]_{H\tilde{F}_x} = \tilde{T}_b^u.$$ 

The action of \( F_{\hat{A}}^X \) is trivial. For \( a = (a_{\infty}, a_{f}) \in F_{\hat{A}}^X \), and \( b \in \tilde{B}^\times \), \( \tilde{q}(a_f)b = b\tilde{q}(a_f) \) and

$$q(a_{\infty})q(u)\tilde{T}^\circ = q(u)\tilde{T}^\circ$$

hence

$$a \cdot \tilde{T}_b^u = [q(a_{\infty})q(u)\tilde{T}^\circ, \tilde{q}(a_f)b]_{H\tilde{F}_x} = [q(u)\tilde{T}^\circ, b]_{H\tilde{F}_x} = \tilde{T}_b^u.$$ 

4.3. Special cycles on \( \text{Sh}_H(G/Z, X)(C) \). In this section we construct some \( r \)-chain on \( \text{Sh}_H(G/Z, X)(C) \).

Proposition 4.3.1. The homology class \( [\tilde{T}^\circ_b] \in H_{r-1}(\text{Sh}_H(G/Z, X)(C), \mathbb{Z}) \) of \( \tilde{T}^\circ_b \) is torsion.

Proof. Let us denote by \( \text{pr} \) the map

$$pr : X \times \{ b \} \longrightarrow \text{Sh}_H(G/Z, X)(C).$$ 

\( \tilde{T}^\circ_b \) is in the image of \( \text{pr} \) and

$$\text{pr}^{-1}(\tilde{T}^\circ_b) = \{(z_1) \times \gamma_2 \times \cdots \times \gamma_r \} \times \{ b \}.$$ 

Let \( \omega \in H^{r-1}(\text{Sh}_H(G/Z, X)(C), \mathbb{C}) \). Thanks to the Matsushima-Shimura theorem, \( \omega = \omega_{\text{univ}} + \omega_{\text{cusp}} \). As \( r - 1 \neq r \) we know that \( \omega = \omega_{\text{univ}} \).

- If \( r - 1 \) is odd, then \( H^{r-1}(\text{Sh}_H(G/Z, X)(C), \mathbb{C}) = \{ 0 \} \).
- If \( r - 1 = 2s \) is even, \( \omega \) is the pull-back of \( \bigwedge_{j=2}^{r} \omega^{(j)} \), where

$$\omega^{(j)} = 1 \quad \text{or} \quad \frac{dx_j \wedge dy_j}{y_j^2}.$$ 

With the notations of the proof of Proposition 4.2.1, \( \tilde{T}^\circ_b \) is a principal homogeneous space under \( \mathbb{H} \). Fix a fundamental domain \( \mathbb{W} \) of \( \mathbb{H} \) in \( \gamma_2 \times \cdots \times \gamma_r \). The incompatibility of degrees gives

$$\int_{\tilde{T}^\circ_b} \omega = \int_{\mathbb{W}} \omega^{(2)} \wedge \cdots \wedge \omega^{(r)} = 0,$$
\[ \forall \omega \in H^{r-1}(\text{Sh}_H(G/Z, X)(\mathbb{C}), \mathbb{C}) \quad \int_{\mathcal{F}_b^\circ} \omega = 0. \]

This proves that \([\mathcal{F}_b^\circ] = 0 \in H_r(\text{Sh}_H(G/Z, X)(\mathbb{C}), \mathbb{C})\) and \([\mathcal{F}_b^\circ] \in H_r(\text{Sh}_H(G/Z, X)(\mathbb{C}), \mathbb{Z})\) is torsion. \(\square\)

**Definition 4.3.2.** Let \(n \in \mathbb{Z}_{>0}\) be the exponent of \(H_{r-1}(\text{Sh}_H(G/Z, X)(\mathbb{C}), \mathbb{Z})_{\text{tors}}\). Then
\[ n[\mathcal{F}_b^\circ] = \partial \Delta_b^\circ \]
for some piece-wise differentiable \(r\)-chain \(\Delta_b^\circ\).

Proposition 3.2.1 proves that the value of
\[ \left( \frac{1}{\Omega^\beta \xi^\alpha} \int_{\Delta_b^\circ} \omega^\beta \right) \in \mathbb{C} \]
modulo \(\Lambda_1\) does not depend on the particular choice of \(\Delta_b^\circ\). If \(T(\mathbb{R})^\circ\) is fixed, then we have the following proposition.

**Proposition 4.3.3.** Let \(\mathcal{F}^\circ\) and \(\mathcal{F}'^\circ\) be two special cycles such that \(pr_1(\mathcal{F}^\circ) = pr_1(\mathcal{F}'^\circ) = \{z_1\}\). Assume that \(pr_j(\mathcal{F}^\circ)\) and \(pr_j(\mathcal{F}'^\circ)\) lie in the same connected component of \(X_j\) for each \(j \in \{2, \ldots, r\}\). Let \(n\) be the exponent of \(H_{r-1}(\text{Sh}_H(G/Z, X)(\mathbb{C}), \mathbb{Z})_{\text{tors}}\) and let \(\Delta_b^\circ\) and \(\Delta_b'^\circ\) satisfy
\[ n[\mathcal{F}_b^\circ] = \partial \Delta_b^\circ \quad \text{and} \quad n[\mathcal{F}_b'^\circ] = \partial \Delta_b'^\circ. \]

Then we have
\[ \int_{\Delta_b^\circ} \omega^\beta = \int_{\Delta_b'^\circ} \omega^\beta \pmod{\xi^{-1} \alpha^{-1} \Omega^\beta \Lambda_1}. \]

**Proof.** Our hypothesis allows us to decompose \(\Delta_b^\circ - \Delta_b'^\circ\)
\[ \Delta_b^\circ - \Delta_b'^\circ = pr(\{z_1\} \times \mathcal{C}) + \mathcal{D}, \]
where \(\mathcal{D}\) is a cycle with \(\partial \mathcal{D} = 0\) and \(pr\) is the map
\[ pr : \left\{ \begin{array}{ccc} X & \longrightarrow & \text{Sh}_H(G/Z, X)(\mathbb{C}) \\ x & \longmapsto & [x, b]_{H\hat{F}^*} \end{array} \right. \]

Let us show that \(\int_{\Delta_b^\circ - \Delta_b'^\circ} \omega^\beta \in \xi^{-1} \alpha^{-1} \Omega^\beta \Lambda_1\). We have
\[ \omega^\beta = \sum_\varepsilon \omega_\varepsilon \in \bigoplus_{\varepsilon : \{\tau_1, \ldots, \tau_r\} \rightarrow \{\pm 1\}^r} \Gamma(\text{Sh}_H(G/Z, X)(\mathbb{C}), (\Omega^\tau_{hF})^\varepsilon).\]
Each \( \omega_{z} \in \Gamma(\text{Sh}_{H}(G/Z, X)(C), (\Omega^{\text{an}}_{H})^{c}) \) satisfies
\[
pr^{*}(\omega_{z}) = dz_{1} \wedge \omega'_{z}
\]
We have
\[
\int_{pr\{z_{1}\} \times C} \omega_{z} = \int_{\{z_{1}\} \times C} dz_{1} \wedge \omega'_{z} = 0,
\]
thus
\[
\int_{\{z_{1}\} \times C} \omega'_{z} = 0.
\]
Thanks to Proposition 3.2.1 we have
\[
\int_{D} \omega_{\beta} \phi \in \xi^{-1} \alpha^{-1} \Omega^{\beta} \Lambda^{1}
\]
and the result follows.

\[\square\]

Corollary 4.3.4. The value modulo \( \Lambda_{1} \) of
\[
\left( \frac{1}{1^{p}}\xi \int_{\Delta^{u}_{\phi}} \omega^{\beta}_{\phi} \right) \in \mathbf{C}
\]
depends neither on the choice of \( \mathcal{F}^{\circ} \) whose projection on \( X_{1} \) is \( \{z_{1}\} \) nor on \( \Delta^{u}_{\phi} \) satisfying \( n[\mathcal{F}^{\circ}] = \partial \Delta^{u}_{\phi} \).

Definition 4.3.5. We set
\[
J_{\beta}^{b} = \frac{1}{1^{p}}\xi \int_{\Delta^{u}_{\phi}} \omega^{\beta}_{\phi} \text{ (mod } \Lambda_{1}) \in \mathbf{C}/\Lambda_{1},
\]
the image of \( \mathcal{F}^{\circ} \) by an exotic Abel-Jacobi map.

Properties of \( J_{\beta}^{b} \). For each \( u \in \pi_{0}(T(R)) \) let \( \Delta^{u}_{\phi} \) be some piece-wise differentiable chain satisfying
\[
n\left[ q(u) \cdot \mathcal{F}^{\circ}, b \right]_{H} \mathcal{F}^{\circ} = \partial \Delta^{u}_{\phi}.
\]

Proposition 4.3.6. We have
\[
J_{\beta}^{b} = \frac{1}{1^{p}}\xi \alpha \sum_{u \in \pi_{0}(T(R))} \beta(u) \int_{\Delta^{u}_{\phi}} \omega_{\phi} \text{ (mod } \Lambda_{1}).
\]

Proof. Let us identify \( \pi_{0}(T(R)) \) with \( \prod_{j=2}^{r} \{\pm 1\} \) and assume that the image of \( T(R)^{\circ} \) is \( (1, \ldots, 1) \).

\[
\omega_{\phi}^{\beta} = \sum_{u \in \pi_{0}(T(R))} \beta(u) t^{*}_{u}(\omega_{\phi}).
\]
The chains \( t_{u} \Delta^{u}_{\phi} \) and \( \Delta^{u}_{\phi} \) are in the same connected component. Thus using 4.3.3, we have
\[
\int_{t_{u} \Delta^{u}_{\phi}} \omega_{\phi} = \int_{\Delta^{u}_{\phi}} \omega_{\phi}
\]
and the result follows.

Recall that \( z_{1} \in X_{1} \) is fixed by \( q(K_{\alpha}) \).

Proposition 4.3.7. Let \( \mathcal{F}^{\circ} \) and \( \mathcal{F}^{\circ} \) be two \( q(T(R))^{\circ} \)-orbits such that \( pr_{1}(\mathcal{F}^{\circ}) = pr_{1}(\mathcal{F}^{\circ}) = \{z_{1}\} \). There exists a unique \( u \in \pi_{0}(T(R)) \) such that, for all \( j \in \{2, \ldots, r\} \),
\[
pr_{j}(\mathcal{F}^{\circ}) \text{ and } pr_{j}(q(u) \cdot \mathcal{F}^{\circ})
\]
are in the same connected component of \( X_{j} \).

If \( J_{\beta}^{b} \in \mathbf{C}/\Lambda_{1} \) denotes the value obtained from \( \mathcal{F}^{\circ} \), we have
\[
J_{\beta}^{b} = \beta(u) J_{\beta}^{b}.
\]
Proof. Let \( x, x' \in X \) be such that \( \mathcal{T}_x = q(T(R)^\circ) \cdot x \) (resp. \( \mathcal{T}_x = q(T(R)^\circ) \cdot x' \)). There exists \( u \in \pi_0(T(R)) \) such that for all \( j \in \{1, \ldots, r\} \), \( \text{pr}_j(q(u) \cdot x) \) and \( \text{pr}_j(x') \) are in the same connected component of \( X_j \). As \( \mathcal{T}_x = q(u) \cdot \mathcal{T}_x \), the chain \( \Delta^u_0 \) whose boundary up to torsion is \( [\mathcal{T}_x, b]_{H^\mathcal{F}_x} \), equals \( \Delta^u_0 \). Thus

\[
\sum_{u' \in \pi_0(T(R))} \beta(u') \int_{\Delta^u_0} \omega_{x'} = \sum_{u' \in \pi_0(T(R))} \beta(u') \int_{\Delta^u_0} \omega_{x'} = \beta(u) \sum_{u'' \in \pi_0(T(R))} \beta(u'') \int_{\Delta^u_0} \omega_{x'}.
\]

\(\square\)

Let \( q, q' : K \hookrightarrow B \) be two embeddings and \( x \in X \), \( \mathcal{T}_x = q(T(R)^\circ) \cdot x \) (resp. \( \mathcal{T}_x = q'(T(R)^\circ) \cdot x' \)). There exists \( a \in B^\times \) such that

\[
q' = a q a^{-1}
\]

thanks to the Skolem-Noether theorem. For each \( j \in \{1, \ldots, r\} \), \( \text{pr}_j(\mathcal{T}_x) \) and \( \text{pr}_j(\mathcal{T}_x) \) are in the same connected component of \( X_j \) if and only if \( \tau_j(\text{nr}(a)) > 0 \).

Using 4.3.7 we obtain

**Proposition 4.3.8.** If

\[
\alpha = (\text{sgn} \circ \tau_j(\text{nr}(a)))_{j \in \{1, \ldots, r\}} \in \{\pm 1\}^{r-1},
\]

then

\[
J_b^\beta = \beta(\alpha) J_b^\beta.
\]

Let \( N_{B^\times}(K^{\times}) \) be the normalizer of \( K^{\times} \) in \( B^\times \). Let \( a \in N_{B^\times}(K^{\times}) \setminus K^{\times} \). After multiplying \( a \) by an element in \( K^{\times} \) we may assume

\[
\forall j \in \{2, \ldots, r\} \quad \tau_j(\text{nr}(a)) > 0.
\]

We have

\[
\text{pr}_1(q(a) \cdot \mathcal{T}_\circ) = t_1(z_1)
\]

and

\[
\forall j \in \{2, \ldots, r\} \quad \text{pr}_j(q(a) \cdot \mathcal{T}_\circ) = \text{pr}_j(\mathcal{T}_\circ)
\]

but the orientations of \( \text{pr}_j(q(a) \cdot \mathcal{T}_\circ) \) and \( \text{pr}_j(\mathcal{T}_\circ) \) are not the same.

Thus

\[
[t_1, \mathcal{T}_\circ, b]_{H^\mathcal{F}_x} = [q(a) \mathcal{T}_\circ, b]_{H^\mathcal{F}_x} = [\mathcal{T}_\circ, q(a)^{-1} b]_{H^\mathcal{F}_x},
\]

but the orientations differ by \((-1)^{r-1}\). Hence

**Proposition 4.3.9.** The tori \( \mathcal{T}_\circ \) and \( t_1 \mathcal{T}_\circ \) \( q(a)^{-1} b \) are the same up to orientation.

5. **Generalized Darmon’s points**

5.1. **The main conjecture.** Let \( \Phi_1 : \mathbf{C}/\Lambda_1 \rightarrow E_1(\mathbf{C}) \) be the Weierstrass uniformization; i.e. the inverse of \( \Phi_1 \) is the Abel-Jacobi map for the differential \( \eta_1 \). For each \( a_\infty \in K^{\times}_{\infty} \), fix some \( r \)-chain \( q(a_\infty) : \Delta^\beta_b \) satisfying \( [q(a_\infty) \cdot \Delta^\beta_b] = q(a_\infty) \cdot \Delta^\beta_b \) and denote by \( \beta(a_\infty) \) the following sign

\[
\beta(a_\infty) = \prod_{j=2}^r \beta \left( \text{sgn} \left( \prod_{w \mid \tau_j} a_\infty, w \right) \right).
\]

**Conjecture 5.1.1.** The point

\[
P_b^\beta = \Phi_1 \left( \frac{1}{2^2} \xi_\alpha \int_{\Delta^\beta_b} \omega_{x'} \right) = \Phi_1(J_b^\beta) \in E_1(\mathbf{C})
\]

lies in \( E(K^{ab}) \) and

\[
\forall a = (a_\infty, a_f) \in K^{\times}_{\mathbf{A}} \quad \text{rec}_{K}(a) P_b^\beta = \Phi_1 \left( \frac{\xi_\alpha}{\Omega \mathcal{T}_x} \int_{q(a_\infty) \cdot \Delta^\beta_{q(a_f)^{-1} b}} \omega_{x'} \right) = \beta(a_\infty) P_b^\beta q(a_f)^{-1}.
\]
Remark 5.1.2. The choice of \( z_1 \in \mathfrak{X}_1^{R}(K_{\tau_1}) \) fixes a morphism \( h_1 : S \to G_{1,R} \), hence a morphism \( C^\times = S(R) \to G_{1,R}(R) = B_{\tau_1}^\times = (B \otimes_{F,\tau_1} R)^\times \) satisfying \( h_1(C^\times) = q_1(K_{\tau_1}) \). This fixes an embedding \( \tau_{1,K} : K \to C \) such that the following diagram

\[
\begin{array}{ccc}
C^\times & \xrightarrow{h_1} & (B \otimes_{F,\tau_1} R)^\times \\
\tau_{1,K} & \downarrow & \downarrow q_1 \\
(K \otimes_{F,\tau_1} R)^\times & & \end{array}
\]

commutes. We may fix \( \tilde{\tau}_1 : K^{ab} \to C \) above \( \tau_{1,K} \), such that

\[
\begin{array}{ccc}
F & \xrightarrow{\tau_1} & R \\
\downarrow \tau_{1,K} & & \downarrow \tilde{\tau}_1 \\
K & & K^{ab}
\end{array}
\]

commutes. Moreover the isomorphism

\[
\begin{cases}
\text{Gal}(K^{ab}/K) \\ \sigma \mapsto \tilde{\tau}_1 \circ \sigma \circ \tilde{\tau}_1^{-1}
\end{cases}
\]

\[
\text{Gal}(K^{ab}/K) \xrightarrow{\sim} \text{Gal}(\tilde{\tau}_1(K^{ab})/\tau_{1,K}(K))
\]

does not depend on the choice of \( \tilde{\tau}_1 \). If \( \tilde{\tau}'_1 \) is another embedding above \( \tau_{1,K} \), then \( \tilde{\tau}'_1 = \tilde{\tau}_1 \circ \sigma' \) with \( \sigma' \in \text{Gal}(K^{ab}/K) \) and

\[
\forall \sigma \in \text{Gal}(K^{ab}/K) \quad \tilde{\tau}'_1 \circ \sigma \circ \tilde{\tau}'_1^{-1} = \tilde{\tau}_1 \circ \sigma' \circ \sigma^{-1} \circ \tilde{\tau}_1^{-1} = \tilde{\tau}_1 \circ \sigma \circ \tilde{\tau}_1^{-1}
\]

because \( \text{Gal}(K^{ab}/K) \) is commutative. Hence the Galois action of 5.1.1 does not depend on the particular choice of \( \tilde{\tau}_1 \).

Remark 5.1.3. Using conjecture 5.1.1, we obtain

\[
\forall a_\infty \in K^\times_1 \quad \text{rec}_K(a_\infty)P_\beta = \beta(a_\infty)P_\beta^\infty.
\]

\[
\forall a \in F^\times_1 \quad \text{rec}_K(a)P_\beta = P_\beta.
\]

5.2. Field of definition. Let \( B^\times_+ = \{ b \in B^\times \mid \forall j \in \{ 2, \ldots, r \}, \tau_j(nr(b)) > 0 \} \). It is diagonally embedded in \( (B \otimes R)^\times \). Set

\[
K_\beta^+ = (K^{ab})^{\text{rec}_K(q^{-1}_A(bHF^x b^{-1}B^+_\tau))} \quad \text{and} \quad K_\beta := (K^{ab})^{\text{rec}_K(q_A^{-1}(bHF^x b^{-1}B^\times))} \subseteq K_\beta^+.
\]

Note that \( K_\beta \) and \( K_\beta^+ \) depend on the choice of \( q : K \to B \).

Proposition 5.2.1. The point \( P_\beta^\infty \) is defined over \( K_\beta^+ : P_\beta^\infty \in E(K_\beta^+) \).

Proof. Let \( a = (1, bhf^{-1}b)(a_\infty, 1_f) \in q^{-1}_A(bHF^x b^{-1}B^+_\tau) \) with \( f \in F^\times \) and \( h \in H \). We have

\[
\text{rec}(a)P_\beta = \text{rec}(q^{-1}_A((1, bhf^{-1}b))P_\beta^\infty = P_\beta^\infty = P_\beta = P_\beta^\infty
\]

\[
\square
\]

Remark that \( \text{rec}_K \) induces a surjection

\[
\mathcal{R} : \pi_0(T(R)) \to \frac{(K \otimes Q R)^\times}{(F \otimes Q R)^\times(K \otimes Q R)^\times_+} \simeq \prod_{j=2}^r \{ \pm 1 \} \to \text{Gal}(K_\beta^+/K_\beta).
\]

Thus, we have

Proposition 5.2.2. The points \( P_\beta^\infty \) lie in \( K_\beta^+ = (K_\beta^+)_{\mathcal{R}K(\beta)} \).

Remark 5.2.3. As \( \text{Ker} \beta \) has index 2 in \( \prod_{j=2}^r \{ \pm 1 \} \), the field \( K_\beta^\infty \) has degree 1 or 2 over \( K_\beta \).
Assume that the conductor $N$ of $E$ decomposes as $N = N_+N_-$ with $N = p_1 \ldots p_r$, $p_i$ distinct prime ideals of $O_F$ and $t \equiv d - r \equiv 2 \pmod 2$. If $\text{Ram}(B) = \{\tau_1, \ldots, \tau_d\} \cup \{p_1, \ldots, p_r\}$ and $H = (R \otimes \widehat{Z})^\times$ where $R \subset B$ is an Eichler order of level $N_+$, then $K_b$ is a ring class field of conductor $f_b$ and $K_b^+$ a ring class field of conductor $f_{b\infty}$, where $f_{\infty} = \prod_{j=1}^r \tau_j$.

5.3. Local invariants of $B$. Let $\pi$ be the irreducible automorphic representation of $B^\times_A$ generated by $\varphi$ and

$$\eta_K = \eta_{K/F} : F^\times_K/F^\times K(F^\times_K) \to \{\pm 1\}$$

the quadratic character of $K/F$. For each place $v$ of $F$ let $\text{inv}_v(B_v) = \{\pm 1\}$ be the invariant of $B$:

$$\text{inv}_v(B_v) = 1 \text{ if and only if } B_v \simeq M_2(F_v).$$

Fix $b \in \mathcal{B}^\times$ and a character

$$\chi : \text{Gal}(K_b^+ / K) \to \mathbb{C}^\times,$$

which will be identified with

$$K_b^\times \text{rec} \xrightarrow{\sim} \text{Gal}(K^{ab} / K) \to \text{Gal}(K_b^+ / K) \xrightarrow{\chi} \mathbb{C}^\times.$$

Let $L(\pi \times \chi, s)$ be the Rankin-Selberg $L$-function, see [Jac72] page 132 and [JL70] section 12. This function admits, since $\pi$ has trivial central character, a holomorphic extension to $\mathbb{C}$ satisfying

$$L(\pi \times \chi, s) = \varepsilon(\pi, \chi, s)L(\pi \times \chi, 1 - s).$$

In this section, we prove the following

**Proposition 5.3.1.** Let $b \in \mathcal{B}^\times$ and assume conjecture 5.1.1. If

$$\varepsilon(\pi_b^\beta) = \sum_{\chi \in \text{Gal}(K_b^+ / K)} \chi(\sigma) \otimes P_b^\beta \in E(K_b^+) \otimes \mathbb{Z}[\chi]$$

is not torsion, then $\beta = \chi_{\infty}$,

$$\forall v \neq \tau_1 \quad \eta_{K,v}(-1) \varepsilon(\pi_v \times \chi_v, \frac{1}{2}) = \text{inv}_v(B_v) \quad \text{and} \quad \varepsilon(\pi \times \chi, \frac{1}{2}) = -1.$$

We shall use the following theorem ([Tun83] and [Sai03]).

**Theorem 5.3.2.** The equality $\eta_{K,v}(-1)\varepsilon(\pi_v \times \chi_v, \frac{1}{2}) = \text{inv}_v(B_v)$ holds if and only if there exists a non-zero invariant linear form

$$\ell_v : \pi_v \times \chi_v \to \mathbb{C}$$

unique up to a scalar satisfying

$$\forall a \in K_b^\times \forall u \in \pi_v \quad \ell_v(q_v(a)u) = \chi_v(a)^{-1}\ell_v(u)$$

i.e. $\ell_v$ is $q(K_b^\times)$-invariant.

**Proof.** (of Proposition 5.3.1) We follow the proof of [AN10], Proposition 2.6.2.

Let $S'$ be a finite set of finite places of $F$ containing the places where $B$, $\pi$ or $K_b^+ / F$ ramify, and such that the map $r = (r_v : K_b^\times \to \text{Gal}(K_b^+ / K))_{v \in S'}$ obtained by composition

$$r : \prod_{v \in S'} K_v^\times \to K_A^\text{rec} \xrightarrow{\chi} \text{Gal}(K^{ab} / K) \to \text{Gal}(K_b^+ / K)$$

is surjective.

For each $v \in S'$ let

$$j_v : \left\{ \begin{array}{ll} K_v & \leftarrow B_v \ni b_v \\ k & \mapsto b_v^{-1}q_v(k)b_v \end{array} \right.$$  

and

$$j = (j_v)_{v \in S'} : \prod_{v \in S'} K_v \leftarrow \prod_{v \in S'} B_v.$$  

As $S'$ does not contain any archimedean place of $F$,

$$\forall a \in \prod_{v \in S'} K_v^\times \quad [\mathcal{S}, \tilde{q}(a)b]_{HF^\times} = [\mathcal{S}, bj(a)]_{HF^\times}$$  

and

$$\forall a \in \prod_{v \in S'} K_v^\times \forall b \in \mathcal{B}^\times \quad \text{rec}_K(a)P_b^\beta = P_{\tilde{q}(a)b}^\beta = P_{bj(a)}^\beta.$$
Let \((K_\nu)^\nu \subset K_\nu^\nu\) be the inverse image of \((K_\nu^\nu/\mathcal{O}_K)^{\text{Gal}(K/F)} \subset K_\nu^\nu/\mathcal{O}_K^\nu\).

We have

\[
K_\nu^\nu/\mathcal{O}_K^\nu F_\nu^\nu \xrightarrow{\sim} \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } v \text{ is inert in } K/F \\ \mathbb{Z} & \text{if } v \text{ splits in } K/F, \end{cases}
\]

the quotient \((K_\nu^\nu)^\nu/F_\nu^\nu\) is compact and

\[
D_v := K_v^\nu/(K_\nu^\nu)^\nu \xrightarrow{\sim} \begin{cases} \mathbb{Z} & \text{if } v \text{ splits in } K/F \\ 0 & \text{otherwise,} \end{cases}
\]

\[
(K_\nu^\nu)^\nu/\mathcal{O}_K^\nu F_\nu^\nu \xrightarrow{\sim} \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } v \text{ ramifies in } K/F \\ 0 & \text{otherwise.} \end{cases}
\]

For each \(v \in S', C_v = \mathcal{O}_K^\nu \cap \text{Ker}(r_v)\) is an open subgroup of \(\mathcal{O}_K^\nu\) and \(V_\nu = (K_\nu^\nu)^\nu/\mathcal{O}_K^\nu C_v\) is finite.

Let \(V_\nu\) be the following subset of \(K_\nu^\nu/\mathcal{O}_K^\nu C_v:\)

\[\bullet \text{ if } v \text{ does not split in } K/F, \quad V_\nu = K_\nu^\nu/\mathcal{O}_K^\nu C_v \quad \text{and } V_v := V_\nu.\]

\[\bullet \text{ If } v \text{ splits in } K/F, \quad \text{we fix some section of } K_\nu^\nu \twoheadrightarrow K_v^\nu/(K_\nu^\nu)^\nu \twoheadrightarrow \mathbb{Z}. \text{ Hence } K_v^\nu = \{K_\nu^\nu\}_v \times D_v \text{ and there exists } n_v \geq 1 \text{ such that } \text{Ker}(r_v|_{D_v}) = n_v D_v.\]

Fix a set of representatives \(D'_v \subset D_v\) of \(D_v/n_v D_v\) and set \(V'_v = V_\nu D'_v \subset K_v^\nu/\mathcal{O}_K^\nu C_v\).

Let \(V = \prod_{v \in S'} V_v \subset \prod_{v \in S'} K_v^\nu/\mathcal{O}_K^\nu C_v\), which is stable under multiplication by the abelian group \(V^\oplus = \prod_{v \in S'} V_v^\oplus\) and such that \(V \twoheadrightarrow \prod_{v \in S'} K_v^\nu/\mathcal{O}_K^\nu C_v \twoheadrightarrow \text{Gal}(K^+/K)\) is surjective with fibers of cardinality \(\frac{|V|}{|\text{Gal}(K^+/K)|}\).

We have

\[
\frac{|V|}{|\text{Gal}(K^+/K)|} \int_{P^\nu_b} \chi = \sum_{\sigma \in \text{Gal}(K^+/K)} \chi(\sigma) \cdot P^\nu_b = \sum_{a \in V} \chi(a) \cdot P^\nu_b(a).
\]

Fix some open-compact subgroup \(H_1 \subset \bigcap_{a \in V} j(a)Hj(a)^{-1}\). Using the maps

\[\text{Sh}_{H_1}(G/Z, X) \xrightarrow{\langle j(a) \rangle} \text{Sh}_{j(a)^{-1}H_1j(a)}(G/Z, X) \xrightarrow{\text{pr}} \text{Sh}_{H}(G/Z, X),\]

we have

\[
\sum_{a \in V} \chi(a) \int_{\Delta_b(a)} \omega^\beta = \sum_{a \in V} \chi(a) \int_{\Delta_b} \langle j(a) \rangle^* \omega^\beta = \int_{\Delta_b} \sum_{a \in V} \chi(a) \langle j(a) \rangle^* \omega^\beta = \int_{\Delta_b} \omega^\beta,
\]

where

\[
\omega^\beta := \sum_{a \in V} \chi(a) \langle j(a) \rangle^* \omega^\beta_a.
\]

Whenever \(\frac{|V|}{|\text{Gal}(K^+/K)|} \int_{P^\nu_b} \chi = \sum_{a \in V} \chi(a) \otimes P^\nu_b(a) \in \mathbb{Z}[\chi] \otimes \mathbb{Z} E(K_b^+) \subset \mathbb{Z}[\chi] \otimes \mathbb{Z} C/\Lambda_1\) is not torsion, there exists \(\sigma : \mathbb{Z}[\chi] \twoheadrightarrow C\) such that

\[
\xi \int_{\Delta_b} \sum_{a \in V} \sigma \chi(a) \langle j(a) \rangle^* \omega^\beta_a \notin \mathbb{Q}[\chi] \cdot \Lambda_1,
\]

where \(\sigma \chi = \sigma \circ \chi\). The vector

\[
\sigma \omega_1 = \sum_{a \in V} \sigma \chi(a) \langle j(a) \rangle^* \omega^\beta_a \in \pi^{H_1} \cap \Gamma(\text{Sh}_{H_1}(G/Z, X), \Omega_{H_1}),
\]

is non-zero and invariant under \(j(\prod_{v \in S'} (K_\nu^\nu)^\nu)\). Moreover,

\[
\forall a \in \prod_{v \in S'} (K_\nu^\nu)^\nu \quad j(a) \omega_1 = \sigma^{-1}(a) \omega_1.
\]
Let

$$
\sigma_{\ell_{S'}}: \bigotimes_{v \in S'} \sigma_{\pi_v} = \bigotimes_{v \in S'} \pi_v \longrightarrow \mathbb{C}(\sigma_{\chi^{-1}})
$$

be the $j(\prod_{v \in S'} (K_v)^{s_v})$-invariant projection on $\mathbb{C}\omega_1$.

Assume that $v \in S'$ does not split in $K$. In this case $(K_v)^{s_v} = K_v$ and $\sigma_{\ell_{S'}}$ induces a $q_v(K_v)$-invariant linear form $\sigma_{\ell_v}: \pi_v \to \mathbb{C}(\sigma_{\chi_v^{-1}})$. We have $q_v(\omega_1,v) \neq 0$, where

$$
\omega_1,v = \sum_{a_v \in V_v} \sigma_{\chi_v} \circ r_v(a_v)\left[:f_v(a_v)\right]^{*} \omega_v.
$$

As $\varepsilon_v(\pi_v \times \sigma_{\chi_v}, \frac{1}{2})$ is independent of $\sigma: \mathbb{Z}[\chi] \rightarrow \mathbb{C}$, Theorem 5.3.2 shows that

$$
\eta_{K,v}(-1)\varepsilon_v(\pi_v \times \chi_v, \frac{1}{2}) = \text{inv}_v(\mathcal{B}_v).
$$

When $v \in S'$ splits in $K$ or $v \notin S' \cup S_{\infty}$, the equality

$$
\eta_{K,v}(-1)\varepsilon_v(\pi_v \times \chi_v, \frac{1}{2}) = 1 = \text{inv}_v(\mathcal{B}_v)
$$

follows from calculations which may be found for example in [Nek06] Proposition 12.6.2.4.

Global sign. If $v = \tau_j$ is an archimedean place, then $\varepsilon(\pi_v \times \chi_v, \frac{1}{2}) = 1$. Moreover $\eta_{K,v}(-1) = 1$ if and only if $j \in \{2, \ldots, r\}$ and $\text{inv}_v(\mathcal{B}_v) = 1$ if and only if $j \in \{1, \ldots, r\}$. Thus

$$
\eta_{K,v}(-1)\text{inv}_v(\mathcal{B}_v) = \begin{cases}
-1 \times 1 & \text{if } j = 1 \\
1 \times 1 & \text{if } j \in \{2, \ldots, r\} \\
-1 \times -1 & \text{if } j > 1.
\end{cases}
$$

and

$$
\forall j \in \{1, \ldots, d\} \quad \varepsilon_v(\pi_v \times \chi_v, \frac{1}{2}) = \eta_{K,v}(-1)\text{inv}_v(\mathcal{B}_v) \times \begin{cases}
-1 & \text{if } j = 1 \\
1 & \text{if } j > 1.
\end{cases}
$$

Hence

$$
\varepsilon(\pi \times \chi, \frac{1}{2}) = -\prod_v \eta_{K,v}(-1)\text{inv}_v(\mathcal{B}_v) = -1.
$$

5.4. Global invariant linear form and a conjectural Gross-Zagier formula. For any open subgroup $H' \subset H$, $b \in \hat{B}^\times$ and $u \in \pi_0(T(R))$ fix $\Delta_{H',b}^u \in C^\tau(\text{Sh}_H(G/Z,X)(\mathbb{C}), \mathcal{Q})$ such that

$$
\partial\Delta_{H',b}^u = [\mathcal{F}_{H',b}^u],
$$

where $\mathcal{F}_{H',b} = \{[g(u)x, b]_{H',\hat{F}^\times}, \ x \in \mathcal{F}^\circ\}$.

Recall that

$$
\forall u' \in \pi_0(T(R)) \quad t_{u'}\Delta_{H',b}^u = \Delta_{H',b}^{u'u'}.
$$

Let $\pi_\infty$ be the archimedean part of $\pi$. Fix $\varphi_\infty \in \pi_\infty$ a lowest weight vector of weight $(2, \ldots, 2, 0, \ldots, 0)$ of $\pi_\infty$ and $\omega_\varphi$ such that $\omega_\varphi = \varphi_\infty \otimes \varphi_f \in \pi_\infty \otimes \pi_f \subset S_2(B_\lambda^\times)$.

Let us denote by $\mathcal{Q}\pi_f$ the sub $\mathcal{Q}[\hat{B}^\times]$-module of $\pi_f$ generated by $\varphi_f$.

Proposition 5.4.1. The space $\mathcal{Q}\pi_f$ is a $\mathbb{Q}$-vector space and $\mathcal{Q}\pi_f \otimes \mathcal{Q} \longrightarrow \pi_f$ is surjective.

Proof. The space $\text{Im}(\mathcal{Q}\pi_f \otimes \mathcal{Q} \longrightarrow \pi_f)$ is a zero subvector space of $\pi_f$ invariant under $B_\lambda^\times$. As $\pi_f$ is irreducible, we have $\text{Im}(\mathcal{Q}\pi_f \otimes \mathcal{Q} \longrightarrow \pi_f) = \pi_f$ and $\mathcal{Q}\pi_f \otimes \mathcal{Q} \longrightarrow \pi_f$ is surjective.

Fix $\eta \neq 0 \in H^0(E, \Omega_{E/F})$. There exists $\alpha \in F^\times$ such that

$$
\mathfrak{f}(\alpha\omega_\varphi) = \eta.
$$

Fix a continuous character of finite order $\chi: K_\lambda^\times/K_\lambda F_\lambda^\times \longrightarrow \mathbb{Z}[\chi]^\times$. Let $H' \subset H$ be any open compact subgroup of $\hat{B}^\times$ satisfying $\chi(q_\lambda^{-1}(H' F_\lambda^\times)) = 1$. Assume that there exists $b_0 \in \hat{B}^\times$ such that $b_0^{-1}H'b_0 \subset H$. Let $\text{pr}_{b_0}$ be the map $\text{Sh}_H(G/Z,X) \longrightarrow \text{Sh}_H(G/Z,X)$ defined on complex points by

$$
[x, b]_{H',\hat{F}^\times} \mapsto [x, b b_0]_{H',\hat{F}^\times}.
$$
Proposition 5.4.2. If $b_0^{-1}H'b_0 \subset H$ for some $b_0 \in \hat{B}^\times$, then
\[
\forall Z' \subset C^r(Sh_{H'}(G/Z,X)(C), Z) \quad \int_{Z'} \text{pr}_{b_0}^*(\omega_{\varphi}^{\lambda_1}) \in Q\alpha^{-1}\Omega^\times \Lambda_1.
\]

Proof. Let $Z = \text{pr}_{b_0}(Z') \subset C^r(Sh_H(G/Z,X)(C), Z)$. We have
\[
\int_{Z'} \text{pr}_{b_0}^*\omega_{\varphi}^{\lambda_1} = \text{deg}(\text{pr}_{b_0} : Z' \rightarrow Z) \int_{Z} \omega_{\varphi}^{\lambda_1}.
\]
Thanks to Proposition 3.3.3, we have $\int_{Z} \omega_{\varphi}^{\lambda_1} \in Q\alpha^{-1}\Omega^\times \Lambda_1$ hence $\int_{Z'} \text{pr}_{b_0}^*\omega_{\varphi}^{\lambda_1} \in Q\alpha^{-1}\Omega^\times \Lambda_1$. \qed

Denote by $\text{pr} : Sh_{H'_1}(G/Z, X) \rightarrow Sh_H(G/Z, X)$ the natural projection, and by $(K \otimes R)^{\times}_+$ the set of elements of $(K \otimes R)^{\times}$ whose norm to $F$ is positive at each place of $F$. We have $\pi_0(T(R)) = (F \otimes R)^{\times}_+(K \otimes R)^{\times}_+$. The following formula
\[
\ell_\chi(\omega') = \frac{1}{[H : H']} \delta_{H', H} \sum_{a \in \mathfrak{A}^\times/(\mathfrak{A}^H \cap K \otimes R)^{\times}_+} \chi(a) \otimes \int_{\Delta_{H, q_{\varphi}(\chi)}} \omega' \quad (\mod Q(\chi) \otimes Q\alpha^{-1}\Omega^\times \Lambda_1),
\]
where $\delta_{H', H} = \Delta^H_{H', H}(\varphi(a), a_f) = [\mathfrak{F}_{H, q_{\varphi}(\chi)}]$, is independent of the specific choice of $\Delta_{H', q_{\varphi}(\chi)}$; we can assume that $\omega' = \text{pr}_{b_0}^*(\omega_{\varphi})$ for some $b_0 \in \hat{B}^\times$; decompose each $a \in K^\times_{\chi}/q_{\chi}(H') \mathfrak{A}^H/(K \otimes R)^{\times}_+$ as $a = (a_f, 1_{\infty})(1_f, a_{\infty})$. Remark that
\[
K^\times_{\chi}/q_{\chi}(H') \mathfrak{A}^H/(K \otimes R)^{\times}_+ = \hat{R}^\times /q_{\chi}^{-1}(H' \mathfrak{A}^H) \times (K \otimes R)^{\times}_+/(K \otimes R)^{\times}_+,
\]
hence $a_f \in \hat{R}^\times /q_{\chi}^{-1}(H' \mathfrak{A}^H)$ and $a_{\infty} \in (K \otimes R)^{\times}_+/(K \otimes R)^{\times}_+$. Thanks to Proposition 5.4.2, the following formula
\[
\sum_{a_{\infty} \in K_{\chi}^\times} \chi(\alpha(\infty)) \int_{\Delta_{H, q_{\varphi}(\chi)}} \omega' = \sum_{a_{\infty} \in K_{\chi}^\times} \chi(\alpha(\infty)) \int_{\Delta_{H, q_{\varphi}(\chi)}} \omega_{\varphi}^{\lambda_1} \cdot \text{pr}_{b_0}^*(\omega_{\varphi})
\]
does not depend on the specific choice of $\Delta_{H, q_{\varphi}(\chi)}$.

Thus, the expression of $\ell_\chi(\omega')$ above defines a linear form
\[
\ell_\chi : S_{\mathfrak{S}_2}^{H'} \cap Q|\hat{B}^\times|_{\omega_{\varphi}} \rightarrow Q(\chi) \otimes Q(\mathcal{C}/Q\alpha^{-1}\Omega^\times \Lambda_1).
\]

To simplify the notations, let
\[
\delta_{H', H} = \deg(\mathcal{F}_{H, b} \xrightarrow{\text{pr}} \mathcal{F}_{H, b}) \quad \text{and} \quad W_{H'} = K^\times_{\chi}/q_{\chi}^{-1}(H' \mathfrak{A}^H)/(K \otimes R)^{\times}_+.
\]

Thus
\[
\ell_\chi(\omega') = \frac{1}{[H : H']} \delta_{H', H} \sum_{a \in W_{H'}} \chi(a) \otimes \int_{\Delta_{H, q_{\varphi}(\chi)}} \omega'.
\]

Proposition 5.4.3. (1) Let $\mathfrak{S}_2 \subset H' \subset H$ be open compact subgroups such that $\chi(q_{\chi}^{-1}(H' \mathfrak{A}^H)) = 1$ and $\text{pr}_{b_0}^*$ the map $\text{pr}_{b_0}^* : S_{\mathfrak{S}_2}^{H'}(B_{\mathfrak{A}}^H) \rightarrow S_{\mathfrak{S}_2}^{H'}(B_{\mathfrak{A}}^H)$.

If $\omega' \in S_{\mathfrak{S}_2}^{H'}(B_{\mathfrak{A}}^H) \cap Q|\hat{B}^\times|_{\omega_{\varphi}}$, then $\ell_\chi(\omega') = \ell_\chi(\text{pr}_{b_0}^*(\omega'))$ and $\ell_\chi$ defines a linear form on $Q|\hat{B}^\times|_{\omega_{\varphi}}$.

(2) We have
\[
\forall a \in \hat{R}^\times \forall \omega \in Q|\hat{B}^\times|_{\omega_{\varphi}} \quad \ell_\chi([q_{\varphi}(a)]\omega) = \chi_f(a)^{-1} \ell_\chi(\omega).
\]

(3) If $\chi$ factors through $\text{Gal}((K^\times_{\chi}/K)$ and if $P_{b_0}^\beta = \text{Pr}_1 \left(\int_{\Delta_{H, b}} \omega_{\varphi}^{\beta}\right) \otimes 1 \in C/Q\Lambda_1$, then
\[
\ell_\chi(P_{b_0}^{\infty}) = \sum_{\text{Gal}((K^\times_{\chi}/K)} \chi(\sigma) \otimes \sigma(P_{b_0}^{\infty}) \in Q(\chi) \otimes Q E(K^\times_{\chi}) \subset Q(\chi) \otimes Q (C/Q\Lambda_1).
equals $\Phi_1(\ell_\chi([b]^*\omega_\varphi))$, up to a non-zero rational factor.

Proof. Proof of 1. Let $a \in \hat{K}^\times$. We have $\text{pr}(\Delta_{H''} \hat{\varphi}(a)) = \Delta_{H''} \hat{\varphi}(a)$ and

$$
\int_{\Delta_{H''} b} \text{pr}^{*}\omega' = \text{deg}(\mathcal{F}_{H''} b \rightarrow \mathcal{F}_{H''} b) \int_{\Delta_{H''} b} \omega' = \delta_{H'' H} \int_{\Delta_{H''} b} \omega'.
$$

As $\chi(q^{-1}_A(H' F_\mathcal{A}^\times)) = 1$, we have (thanks to Proposition 5.4.2)

$$
\ell_\chi(\text{pr}^{*}\omega') = \frac{1}{[H : H'']} \sum_{a \in W_{H''}} \chi(a) \otimes \int_{\Delta_{H''} \hat{\varphi}(a)} \text{pr}^{*}\omega' \pmod{\mathbf{Q}(\chi) \otimes \mathbf{Q} \alpha^{-1}\Omega^{\infty} A_1}
$$

$$
= \frac{\delta_{H'' H}}{[H : H'']} \sum_{a \in W_{H''}} \chi(a) \otimes \int_{\Delta_{H''} \hat{\varphi}(a)} \omega' \pmod{\mathbf{Q}(\chi) \otimes \mathbf{Q} \alpha^{-1}\Omega^{\infty} A_1}
$$

$$
= \frac{[H' : H'']}{[H : H'']} \frac{\delta_{H'' H}}{[H : H'']} \sum_{a \in W_{H''}} [H' : H''] \chi(a) \otimes \int_{\Delta_{H''} \hat{\varphi}(a)} \omega' \pmod{\mathbf{Q}(\chi) \otimes \mathbf{Q} \alpha^{-1}\Omega^{\infty} A_1}
$$

$$
= \ell_\chi(\omega').
$$

Proof of 2. Assume $H''$ is sufficiently small such that $[\hat{\varphi}(a)]^{*}\text{pr}^{*}\omega \in S_2^{H''}$. We have

$$
\ell_\chi([\hat{\varphi}(a)]^{*}\omega) = \ell_\chi([\hat{\varphi}(a)]^{*}\text{pr}^{*}\omega)
$$

$$
= \frac{1}{[H : H'']} \sum_{a' \in W_{H''}} \chi(a') \otimes \int_{\Delta_{H''} \hat{\varphi}(a')} [\hat{\varphi}(a)]^{*}\text{pr}^{*}\omega \pmod{\mathbf{Q}(\chi) \otimes \mathbf{Q} \alpha^{-1}\Omega^{\infty} A_1}
$$

$$
= \frac{1}{[H : H'']} \sum_{a' \in W_{H''}} \chi(a') \otimes \int_{\Delta_{H''} \hat{\varphi}(a')} \text{pr}^{*}\omega \pmod{\mathbf{Q}(\chi) \otimes \mathbf{Q} \alpha^{-1}\Omega^{\infty} A_1}
$$

$$
= \frac{1}{[H : H'']} \sum_{a'' \in W_{H''}} \chi(a'' a^{-1}) \otimes \int_{\Delta_{H''} \hat{\varphi}(a'')} \text{pr}^{*}\omega \pmod{\mathbf{Q}(\chi) \otimes \mathbf{Q} \alpha^{-1}\Omega^{\infty} A_1}
$$

$$
= \chi_f(a)^{-1} \frac{1}{[H : H'']} \sum_{a'' \in W_{H''}} \chi(a'') \otimes \int_{\Delta_{H''} \hat{\varphi}(a'')} \text{pr}^{*}\omega \pmod{\mathbf{Q}(\chi) \otimes \mathbf{Q} \alpha^{-1}\Omega^{\infty} A_1}
$$

$$
= \chi_f(a)^{-1} \ell_\chi(\text{pr}^{*}\omega)
$$

$$
= \chi_f(a)^{-1} \ell_\chi(\omega).
$$

Proof of 3. As $\omega_\varphi \in S_2(B_\mathcal{A}^\times) = \bigcup_H S_2^H(B_\mathcal{A}^\times)$, there exists $H'$ sufficiently small such that

$$
\omega_\varphi \in S_2^{H'} 	ext{ and } [b]^*\omega_\varphi \in S_2^{H'}.
$$

Let $m = [K_\mathcal{A}^\times / q^{-1}_A(H' F_\mathcal{A}^\times)(K \otimes \mathbb{R})^\times : \text{Gal}(K_\mathcal{A}^\times / K)]$ and $\nu = \frac{1}{[H : H'] \deg(\mathcal{F}_{H'} \rightarrow \mathcal{F}_H)}$. We have:
\[ \ell(\chi \circ [b]^* \omega_\varphi) = \nu \sum_{a \in \mathbb{A}_\mathbb{K}^+ \setminus \mathbb{A}_\mathbb{K}^+ (\mathcal{O}_\mathbb{K} \otimes \mathbb{R})^+} \chi_f(a_f) \chi_\infty(a_\infty) \otimes \int_{\Delta^{\infty}_{(a_\infty) \setminus q_{(a_f)}}} [b]^* \omega_\varphi \pmod{Q(\chi) \otimes \mathbb{Q} \alpha^{-1} \Omega^{\infty} \Lambda_1} \]

\[ = \nu \sum_{a_f} \chi_f(a_f) \otimes \sum_{a_\infty} \chi_\infty(a_\infty) \text{rec}_K(a_f) \cdot \int_{\Delta^{\prime}_{H_v \otimes b}} t_{\text{rec}_K(a_\infty)} \omega_\varphi \pmod{Q(\chi) \otimes \mathbb{Q} \alpha^{-1} \Omega^{\infty} \Lambda_1} \]

\[ = \nu m \sum_{\sigma \in \text{Gal}(K_v^+/K)} \chi(\sigma) \otimes \int_{\Delta^{\prime}_{H_v \otimes b}} \chi_\infty(a_\infty) t_{\text{rec}_K(a_\infty)} \omega_\varphi \pmod{Q(\chi) \otimes \mathbb{Q} \alpha^{-1} \Omega^{\infty} \Lambda_1} \]

hence

\[ \varepsilon(P_b^{\infty}) = \Phi_1(\ell([b]^* \omega_\varphi)). \]

Let us consider the Néron-Tate height \( h_{\text{NT}} : E(K^{ab}) \times E(K^{ab}) \to \mathbb{R} \) extended to an hermitian form

\[ h_{\text{NT}} : E(K^{ab}) \otimes \mathbb{C} \times E(K^{ab}) \otimes \mathbb{C} \to \mathbb{C}. \]

Recall the condition

\[ \forall v \neq \tau_1 \quad \varepsilon(\pi_v \times \chi, v, 1/2) \eta_{K,v}(-1) = \text{inv}_v(B) \]

from Proposition 5.3.2: if 2 fails, then \( P_{b^{\infty}} \in E(K^{ab}) \) is torsion.

In general, there should be some \( k(b, \omega_\varphi) \in \mathbb{C} \) such that

\[ \forall \sigma : Q(\chi) \to \mathbb{C} \quad h_{\text{NT}}(\varepsilon(\sigma(P_b^{\infty}))) = k(b, \omega_\varphi)L'(\pi \times \sigma^{\chi}, 1/2), \]

as in Gross-Zagier, Zhang and Yuan-Zhang-Zhang [GZ86, Zha01, YZZ09].

This formula explains the following conjecture:

**Conjecture 5.4.4.** Let \( K_\chi = (K^{ab})^{\text{Ker}(\chi)} \) be the extension of \( K \) trivializing \( \chi \). If

\[ \forall v \neq \tau_1 \quad \varepsilon(\pi_v \times \chi, v, 1/2) \eta_{K,v}(-1) = \text{inv}_v(B), \]

then there exists \( b \in \hat{B}^\infty \) such that \( k(b, \omega_\varphi) \neq 0 \) and we have the following equivalences:

\[ \ell(\chi) \neq 0 \iff \exists b \in \hat{B}^\infty \text{ such that } K_\chi \subset K_b^+ \text{ and } \sigma^{\chi}(P_b^{\infty}) \in \mathbb{Z} \otimes \text{E}(K_b^+) \text{ is not torsion} \]

\[ \iff \exists \sigma : Q(\chi) \to \mathbb{C} \quad L'(\pi \times \sigma^{\chi}, 1/2) \neq 0 \]

\[ \iff \forall \sigma : Q(\chi) \to \mathbb{C} \quad L'(\pi \times \sigma^{\chi}, 1/2) \neq 0. \]

6. A RELATION TO KUDLA’S PROGRAM

The theorem of Gross-Kohnen-Zagier asserts that the positions of the traces to \( \mathbb{Q} \) of classical Heegner points are given by the Fourier coefficients of some Jacobi form. The geometric proof of Zagier explained for example in [Zag85] has been recently generalized by Yuan, Zhang and Zhang in [YZZ09] using a result of Kudla-Millson [KM90]. In this section we establish a relation between Darmon’s construction and Kudla’s program. This is a first step in an attempt to apply the arguments of Zagier [Zag85] and Yuan-Zhang and Zhang’s [YZZ09] to Darmon’s points.
6.1. Some computations. Let us fix a modular elliptic curve $E/F$ of conductor $N = N_+ N_-$. Assume $\text{Ram}(B) = \{\tau_{r+1}, \ldots, \tau_d\} \cup \{v \mid N_-\}$ and that the quadratic extension $K/F$ satisfies the following hypothesis

\[ \forall v \mid N_+ \text{ splits in } K \quad \forall v \mid N_- \text{ is inert in } K. \]

In particular, the relative discriminant $d_{K/F}$ is prime to $N$. Let $R$ be an Eicher order of $B$ of level $N_+$. Identify $K$ with its image in $B$ by $q$ and assume $K \cap R = \mathcal{O}_K$, $H = \hat{R}^\times$ (which implies that $\dim \pi_H = 1$).

Recall that $h_z$ defines an embedding $\tau_{1/K} : K \hookrightarrow \mathbb{C}$ and denote by $c$ the non-trivial element of $\text{Gal}(K/F)$. Assume that Conjecture 5.1.1 is true for $\beta = 1$ and let $P = \text{Tr}_{K_1^+}/K P_1 \in E(K)$.

**Proposition 6.1.1.** If $\varepsilon$ is the global sign of $E/F$, i.e. $\Lambda(E/F, s) = \varepsilon \Lambda(E/F, 2 - s)$, where $\Lambda$ is the completed $L$-function of $E/F$, then $c(P) = -\varepsilon P$.

**Proof.** Assume that $K = F(i)$ and $B = K(j)$, with $i^2 = a \in F^\times$, $j^2 = b \in F^\times$ and $ij = -ji$. Recall that

\[ \mathcal{I}_1 = [z_1] \times z_2 \times \cdots \times z_r \]

with $\mathcal{I}^\circ = \{z_1\} \times z_2 \times \cdots \times z_r$. Thus

\[ c(\mathcal{I}) = \left[\left\{z_1\right\} \times z_2 \times \cdots \times z_r, 1\right]_{HF^\times} = (-1)^{r-1}[z_1(\mathcal{I}), 1]_{HF^\times} \]

and

\[ c(\mathcal{I}) = (-1)^{r-1}[\mathcal{I}, j]_{HF^\times} \]

since $j \in B$. This shows that $c(P) = (1)^{r-1}P_j$. We will write $P_j$ using only $P_i$. We will make the following abuse of language. For each place $v$ of $F$, $j_v$ shall denote the element $(1, \ldots, 1, j_v, 1, \ldots) \in B^\times$ and we will use the following lemma

**Lemma 6.1.2.** Let $b \in \hat{B}^\times$ and $v$ a place of $F$. When $v \mid N_+$, set $k_v \in K_v^\times$ corresponding to

\[ \begin{bmatrix} 1 & 0 \\ \varpi_v & 1 \end{bmatrix} \]

where $\varpi_v$ is an uniformizer of $K_v$. If $b_v = 1$, then

\[ P_{b_j} = \begin{cases} -\varepsilon_v P_b & \text{if } v \mid N_- \\ \varepsilon_v \text{rec}_K(k_v^{-1}) P_b & \text{if } v \mid N_+ \\ P_b & \text{if } v \nmid N \end{cases} \]

**Proof.** (of the lemma)

For each $v$ inert in $K/F$ we have

\[ \text{inv}_v(B) = 1 \iff B_v \simeq M_2(F_v) \iff b \in \text{Ker}(\mathcal{O}_{B_v}/(K_v^\times)) \iff 2 \mid \text{ord}_v(b) \]

As $j = -j$, we have $\text{nr}(j) = -j^2 = -b$ and

\[ \text{inv}_v(B) = 1 \iff 2 \mid \text{ord}_v(\text{nr}(j_v)). \]

If $v \mid N_-$ then $H_v = \mathcal{O}_{B_v}$, where $\mathcal{O}_{B_v}$ is the unique maximal order in $B_v$ hence $H_v \leq B_v$ and $B_v/H_v^\times \simeq \mathbb{Z}$ by choosing some uniformizer. As $H_v$ is normal in $B_v$, the map

\[ [j_v] : \text{Sh}_H(G/Z, X)(C) \rightarrow \text{Sh}_{j_v^{-1}}(G/Z, X)(C) \]

is well-defined on $\text{Sh}_H(G/Z, X)(C)$. Thus $[\mathcal{I}, b_j]_{HF^\times} = [j_v][\mathcal{I}, b]_{HF^\times}$ and

\[ \int_{\Delta^\times_{b_j}} \omega_{\varphi} = \int_{\Delta^\times_{b}} [j_v]^{*} \omega_{\varphi} = \int_{\Delta^\times_{b}} \pi_v(j_v) \omega_{\varphi}. \]

Decompose $\pi = \pi(\varphi) = \mathcal{O}_v^{\times} \varpi_v$. We have

\[ \pi_v : B_v^\times \overset{\text{nr}}{\rightarrow} F_v^\times \overset{\text{ord}_v}{\rightarrow} \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \sim \{\pm 1\}. \]

Let us denote by $\alpha$ the following unramified character

\[ \alpha : F_v^\times \overset{\text{ord}_v}{\rightarrow} \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \sim \{\pm 1\}. \]
satisfying \( \pi_v = \alpha \circ \text{nr} \).

As \( v \mid N_\pm \), \( E \) has multiplicative reduction in \( v \). The character \( \alpha \) is trivial if and only if \( E \) has split multiplicative reduction in \( v \), i.e. \( \varepsilon_v = -1 \).

Hence
\[
[j_v]^* \omega_\varphi = \alpha(\text{nr}(j_v))\omega_\varphi = \left\{ \begin{array}{ll} \omega_\varphi & \text{if } \alpha = 1 \\ (-1)^\text{ord}_v(\text{nr}(j_v))\omega_\varphi & \text{otherwise.} \end{array} \right.
\]

As \( v \mid N_-, \) \( v \in \text{Ram}(B) \) is inert in \( K/F \) and \( \text{inv}_v(B) = -1 \), thus \( 2 \not| \text{ord}_v(\text{nr}(j_v)) \). Hence
\[
[j_v]^* \omega_\varphi = \alpha(\text{nr}(j_v))\omega_\varphi = \left\{ \begin{array}{ll} \omega_\varphi & \text{if } \alpha = 1 \\ -\omega_\varphi & \text{otherwise} \end{array} \right.
\]

and \( P_{b_{j_v}} = -\varepsilon_vP_b \).

If \( v \mid N_+ \), then we fix some uniformizer \( \varpi_v \) of \( F_v \) and an isomorphism \( B_v \cong M_2(F_v) \) which identifies \( K_v \) with the set of diagonal matrices and \( R_v \) with \( \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(O_{F,v}) \mid \varpi_v^{\text{ord}_v(N_+)} \mid c \right\} \).

As \( \text{inv}_v(B_v) = 1 \), \( j_v \) is a local norm. There exists \( k_v \in K_v \) such that \( j_v = N_{K_v/K}(k_v) \). We may assume that \( j_v^2 = 1 \). Moreover \( j_v \) is in the normalizer of \( K_v^\times \) in \( B_v^\times \) we thus identify \( j_v \) to \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).

Set
\[
W_v = \begin{pmatrix} 1 & 0 \\ \varpi_v^{\text{ord}_v(N_+)} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \varpi_v^{\text{ord}_v(N_+)} & 0 \end{pmatrix} = k_vj_v.
\]

This matrix is in the normalizer of \( R_v \) in \( B_v \). As \( W_v \) normalize \( H_v \),
\[
[\mathcal{F}_v, b_{j_v}]_{H^\times} = [\mathcal{F}_v, b_{k_v}^{-1}W_v]_{H^\times} = [W_v][\mathcal{F}_v, b_{k_v}^{-1}]_{H^\times}.
\]

Decompose \( \omega_\varphi = \otimes_{v|N_+} \omega_v \otimes \omega' \), where \( \omega_v \) satisfies \( (W_v)^*\omega_v = \varepsilon_v\omega_v \); then
\[
\int_{\Delta_v^*} \omega_\varphi = \varepsilon_v \int_{\Delta_v} \omega_\varphi.
\]

As \( b_v = 1 \),
\[
P_{b_{j_v}} = \varepsilon_v\text{rec}_K(k_v^{-1})P_b.
\]

If \( v \mid N \), then by a similar calculation we obtain
\[
P_{b_{j_v}} = \text{rec}_K(k_v^{-1})P_b.
\]

End of the proof of Proposition 6.1.1. Lemma 6.1.2 implies that
\[
c(P_1) = (-1)^{r-1} \prod_{v|N_-} (\varepsilon_v) \prod_{v|N_+} \varepsilon_v\text{rec}_K(k_v^{-1})P_1
\]
and
\[
\forall a \in K^\times \ A c \text{rec}_K(a)P_1 = (-1)^{r-1} \prod_{v|N_-} (\varepsilon_v) \prod_{v|N_+} \varepsilon_v\text{rec}_K(k_v^{-1})\text{rec}_K(a)P_1.
\]

As \( P \in E(K) \), we know that \( \text{rec}_K(k^{-1})P = P \). Thus
\[
c(P) = (-1)^{r-1} \prod_{v|N_-} (\varepsilon_v) \prod_{v|N_+} \varepsilon_v P = (-1)^{r-1} (-1)^{[v|N_-]} \prod_{v|\infty} \varepsilon_v P.
\]

We have to show that \(-1)^{r-1} \prod_{v|N_-} (\varepsilon_v) \prod_{v|N_+} \varepsilon_v = -\varepsilon. \) For each \( v \mid \infty \) we have \( \varepsilon_v = -1 \).

Since \( \prod_{v|\infty} = (-1)^d \), the sign in equation (3) is
\[
(-1)^d \prod_{v \mid \infty} \varepsilon_v (-1)^{r-1} (-1)^{[v|N_-]} = (-1)^d (\varepsilon_v)^{-1} (-1)^{[v|N_-]}.
\]

Recall that \( \{ v \mid N_- \} = \text{Ram}(B) \cap S_f \). As \( |\text{Ram}(B)| \) is even, we have
\[
(-1)^{[v|N_-]} = (-1)^{|\text{Ram}(B) \cap S_w|} = (-1)^{d-r}.
\]

Hence
\[
c(P) = (-1)^d \varepsilon (-1)^{r-1} (-1)^{[v|N_-]} P = -\varepsilon P.
\]
Remark 6.1.3. The above computations are a particular case of a result of Prasad, [Pra96] Theorem 4, which asserts that if Hom$_{K^\times}(\pi_v, 1) \neq \{0\}$, then the non trivial element in $N_{B^\times} \backslash K^\times$ acts on Hom$_{K^\times}(\pi_v, 1)$ by multiplication by $\inv_v(B)\varepsilon_v = \inv_v(B)\varepsilon(\pi_v, \frac{1}{2}) \in \{\pm\}$. 

6.2. Orthogonal Shimura manifolds. Until the end of this paper we shall assume $h_F^+ = 1$.

Let us recall some definitions used by Kudla [Kud97] in the particular case $r = 1$. Let $n \in \mathbb{Z}_{\geq 0}$ and let $(V, Q)$ be a quadratic space over $F$ of dimension $n + 2$. We assume that the signature of $V \otimes_{F, \tau} R$ is

$$(n, 2) \times (n + 1, 1)^{r-1} \times (n + 2, 0)^{d-r}.$$ 

Denote by $D$ the symmetric space of $G = \text{Res}_{F/Q}\text{GSpin}(V)$. $D$ is the product of the oriented symmetric spaces of $V_j = V \otimes_{\tau_j, F} R$. Thus $D = D_1 \times \ldots D_d$, where $D_j$ is the set of oriented positive subspaces in $V_j$ of maximal dimension. For each $x \in V$ let $x_j$ be the image of $x$ in $V_j$. Assume that $Q(x)$ is totally positive. Set $V_x = x^+ = G_x = \text{Res}_{F/Q}\text{GSpin}(V_x)$ and for each $j \in \{1, \ldots, d\}$

$$D_{x_j} = \{z \in D_j \mid z \perp x_j\}.$$ 

We shall focus on the following real cycle on the Shimura manifold $G(Q) \backslash D \times G(\hat{Q})/H$. 

Definition 6.2.1. Let $H$ be an open compact subgroup in $G(\hat{Q})$ and $g \in G(\hat{Q})$. The cycle $Z(x; g; H)$ is defined to be the image of the map

$$Z(x; g; H) : \begin{cases} G_x(Q) \backslash D_x \times G_x(\hat{Q})/H^g_x \to G(Q) \backslash D \times G(\hat{Q})/H \\ G_x(Q)(y, u)H^g_x \to G(Q)(y, ug)H\hat{F}^\times, \end{cases}$$

where $H^g_x$ denotes $G_x(\hat{Q}) \cap gHg^{-1}$.

Example (including Proposition 6.2.2) : Fix $D_0 \in F$ satisfying

$$\begin{cases} \tau_j(D_0) > 0 & \text{if } j \in \{1, r + 1, \ldots, d\} \\ \tau_j(D_0) < 0 & \text{if } j \in \{2, \ldots, r\} \end{cases}$$

Set

$$(V, Q) = (B^{Tr=0}, D_0 \cdot nr).$$

$(V \otimes_{F, \tau_j} R, \tau_j \circ D_0 \cdot nr)$ has signature

$$\begin{cases} (1, 2) & \text{if } j = 1 \\ (2, 1) & \text{if } j \in \{2, \ldots, r\} \\ (3, 0) & \text{if } j \in \{r + 1, \ldots, d\}. \end{cases}$$

Let $G = \text{Res}_{F/Q}\text{GSpin}(V)$. The action of $B^\times$ on $V$ by conjugation induces an isomorphism

$$B^\times \xrightarrow{\sim} \text{GSpin}(V), \quad b \mapsto (v \mapsto bvb^{-1}),$$

thus $G \simeq \text{Res}_{F/Q}(B^\times)$. 

Let $x \in V$ such that $Q(x) \gg 0$, and denote by $x_1$ its image in $V \otimes_{F, \tau_j} R$. Denote by $K$ the quadratic extension $F + Fx$ and $T = \text{Res}_{K/Q}(G_m)/\text{Res}_{F/Q}(G_m)$ as above. Let $q$ be the inclusion $\text{K} \hookrightarrow V \twoheadrightarrow B$.

Proposition 6.2.2. The set

$$D_x = D_{x_1} \times \cdots \times D_{x_r}$$

is a $q(T(R))$-orbit in $D$ whose projection on $D_1$ is a point.

Proof. As $x \in V$, $\text{Tr}(x) = 0$ and $x^2 = -nr(x) = -\frac{Q(x)}{D_0} \in F^\times$. Let $j \in \{1, \ldots, r\}$. We have $\tau_j(Q(x)) > 0$ hence $\tau_j(x^2)\tau_j(D_0) < 0$. Thus $\tau_1$ ramifies in $K$ and $\tau_2, \ldots, \tau_r$ are split. Moreover $q_1(K^\times)$ fixes $x_1$ by definition of $K$. 

$\square$
Let us focus on the general case when $V$ has dimension $n$. Fix $t \in F$ satisfying $\forall j \in \{1, \ldots, r\} \sigma_j(t) > 0$. $G(\hat{Q})$ acts on $\Omega_t = \{x \in V(F) \mid Q(x) = t\}$ by conjugation.

Let $\varphi$ be a Schwartz function on $V(\hat{F})$. Assume $\Omega_t \neq \emptyset$ and fix $x \in \Omega_t$. Denote by $Z(y, \varphi; H)$ the following sum

$$Z(t, \varphi; H) = \sum_{g \in G_\chi(\hat{Q}) \backslash G(\hat{Q}) / H} \varphi(g^{-1} \cdot x)Z(x, g; H).$$

Proposition 4.3.1 showed that for $n = 1$ $[Z(x, g; H)] = 0 \in H_{r-1}(\text{Sh}_H(G/Z, X)(\mathbb{C}), \mathbb{C})$. A natural invariant to consider is the refined class

$$\{Z(t, \varphi; H)\} = \omega \mapsto J^t_\omega \in \text{Im}(H_\omega(\text{Sh}_H(G/Z, X)(\mathbb{C}), \mathbb{Z}) \to \text{Harm}(\text{Sh}_H(G/Z, X)(\mathbb{C}), \mathbb{C}))).$$

where $\text{Harm}(\text{Sh}_H(G/Z, X)(\mathbb{C}))$ is the set of harmonic differential forms on $\text{Sh}_H(G/Z, X)(\mathbb{C})$.

In order to adapt the work of Yuan, Zhang and Zhang, we need the following conjecture

**Conjecture 6.2.3.** In the situation of the above example $(V, Q) = (B^{Tr=0}, D_0 \cdot \text{nr})$, the sum

$$\sum_{t \in \text{O}_F, t \geq 0} \{Z(t, \varphi; H)\} q^t$$

is a Hilbert modular form of weight $3/2$.

In [YZZ09], the authors work by induction. To apply their method we would need to prove that the refined classes $\{Z(t, \varphi; H)\}$ are compatible with the tower of varieties attached to quadratic spaces $V_2 \hookrightarrow V$ of signature $(n, 2) \times (n + 1, 1)^{r-1} \times (n + 2, 0)^{d-r}$ (in which case a generalization of [KM90] should imply that $\sum_{t \in \text{O}_F} \{Z(t, \varphi; H)\} q^t$ is a Hilbert modular form of weight $\frac{3}{2} + 1$ with coefficients in $H^{r+1}(\text{Sh}_H(G/Z, X)(\mathbb{C}), \mathbb{C}))$.

### 6.3. A Gross-Kohnen-Zagier-type conjecture.

The Bruhat-Tits tree. In this section we recall some basic facts about the Bruhat-Tits tree (see [CL] and [Vig80]).

Let $v$ be a finite place of $F$. The vertices of the Bruhat-Tits tree of $\text{PGL}_2(F_v)$ are the maximal orders of $\text{M}_2(F_v)$. Such maximal orders are endomorphism rings of lattices in $F_v^2$ ([Vig80], lemme 2.1). There is an oriented edge between two vertices $\text{O}_1$ and $\text{O}_2$ if and only if there exist $L_1, L_2$ lattices in $F_v^2$ such that $\text{O}_1 = \text{End}(L_1), L_2 \subset L_1$ and $L_1 / L_2 \cong \text{O}_{F_v} / \mathbb{E}_v \text{O}_{F_v}$. The intersection of the source and the target of paths of length $n$ correspond to level $v^n$ Eichler orders.

Fix some quadratic extension $K/F$. This data allow us to organize the Bruhat-Tits tree. Let $\Psi : K_v \hookrightarrow \text{M}_2(F_v)$ be a $F_v$-embedding of $K_v$. Let $\text{M}_0(N)$ be the set of matrices in $\text{M}_2(F_v)$ which are upper triangular modulo $N$. If

$$\Psi(\text{O}_{K_v}) = \Psi(K_v) \cap \text{M}_0(N),$$

we say that $\Psi$ has level $N$. We can organize the vertices of the tree in "levels", by privileging a direction. Each level corresponds to a level of embedding relatively to $\text{O}_{K_v}$ i.e. to orders which are in the same orbit under $K_v^n$. The maximal orders in $\text{PGL}_2(F_v)$ which are maximally embedded are on the bottom of the tree.

Figures 2, 3 and 4 illustrate the dependence on the ramification type of $v$ in $K$.

Darmon’s points, Kudla’s program and a Gross-Kohnen-Zagier-type theorem. Recall that $H = (R \otimes \hat{Z})^\times$, where $R$ is an Eichler order of $B$ of level $N_+$ and that $K = F + Fx$ satisfies the following Heegner hypothesis.

**Hypothesis 6.3.1.** Each prime $p | N_+$ splits in $K$ and each prime $p | N_-$ is inert in $K$.

The group $G_t$ is isomorphic to $K^\times$ and $Z(x, 1; H)$ is the image of $K^\times \backslash \text{D}_x \times \hat{K}^\times / H$ in $\text{Sh}_H(G, X)(\mathbb{C})$. Note that

$$Z(x, 1; H) = \mathcal{J}_1^t + t_1(\mathcal{J}^t_1),$$

where $\mathcal{J}_1^t = [\cup_{u \in \text{O}_0(\mathbb{F}(R))} q(u) \cdot \mathcal{J}_0^t, 1]_{H^{\hat{F}_x}}$.

Let $\varphi = 1_{R^{Tr=0}}$ and $\psi$. We are able to prove an analogue of Proposition A.1.1 of [Kud04] when $N = 1, B = \text{M}_2(F), R = \text{M}_2(\text{O}_F), t = Q(x) = D_{\text{nr}}(x) \in F$ and $K = F + Fx$ is such that $K \cap R = \text{O}_K$ and $\text{O}_K = \text{O}_F + Fx$. Set $c_1(\mathcal{J}_1^t) = \{[t_1(x), b]_{H^{\hat{F}_x}}, b \in \hat{B}^\times\}$. 
Proposition 6.3.2. If $N = 1$, $r = d$, $B = M_2(F)$, $H = \hat{R}^\times$ with $R = M_2(O_F)$ and if $O_K = O_F + O_{F,x}$, then $Z(t, \varphi; H)$ is equal to

$$Z(x, 1; H) = \mathcal{R}_1^1 + c_1(D_1^1) = \mathcal{R}_1^1 - \varepsilon D_1^1.$$ 

Remark 6.3.3. Under the strong hypotheses above, $\varepsilon = (-1)^d$ and the cycle obtained is zero when $d$ is even.

Proof. By definition

$$Z(t, \varphi; H) = \sum_{g \in \hat{K}^\times \backslash \hat{B}^\times / \hat{R}^\times} 1_{\hat{R}^{\text{Tr}=0}}(g^{-1} \cdot x) Z(x, g; H).$$

We have to determine $g \in \hat{K}^\times \backslash \hat{B}^\times / \hat{R}^\times$ satisfying $g^{-1}xg \in \hat{R}^{\text{Tr}=0}$, i.e. $x \in g\hat{R}^{\text{Tr}=0}g^{-1}$. As $F^\times \subset K^\times$,

$$\hat{K}^\times \backslash \hat{B}^\times / \hat{R}^\times = \prod_v \hat{K}_v^\times \backslash \hat{B}_v^\times / \hat{R}_v^\times = \prod_v \hat{K}_v^\times \backslash \hat{B}_v^\times / F_v^\times \hat{R}_v^\times.$$
This allows us to work locally with $K_v^\times \backslash B_v^\times /F_v^\times R_v^\times$, which is identified to the $K_v^\times$-orbits of maximal orders of $\text{PGL}_2(F_v)$. This gives the following condition, $x_v \in g_v R_v g_v^{-1}$.

First let us consider those $g_v \in B_v^\times \backslash R_v^\times F_v^\times$ satisfying $x_v \in g_v R_v g_v^{-1}$. The ring $g_v R_v g_v^{-1}$ is a maximal order containing $x_v$. Using the fact that $O_K = O_F + O_F x$, we have

$$x_v \in g_v R_v g_v^{-1} \iff g_v R_v g_v^{-1} \cap K_v = O_{K_v}.$$ 

Hence the maximal order $g_v R_v g_v^{-1}$ is maximally embedded in $K_v$. It is identified to a vertex at the lowest level of the Bruhat-Tits tree. As each vertex at the same level is in the same $K_v^\times$-orbit, we have

$$\forall v \quad g_v = 1 \in K_v^\times \backslash B_v^\times /F_v^\times R_v^\times.$$ 

Thus $Z(t, \varphi; H) = Z(x, 1; H)$ and as $D_{x_1}$ is a set of two points, $Z(x, 1; H)$ is identified with $\mathcal{F}_1 + c_1(\mathcal{F}_1) = \mathcal{F}_1 - \varepsilon \mathcal{F}_1$, thanks to Proposition 6.1.1.

We now consider the case when $N = N_+ N_- \neq 1$ is prime to $d_{K/F}$. The following proposition is true even if $B \neq M_2(F)$ but we still assume that $R$ is an Eichler order of level $N_+$ and $O_K = O_F + O_F x$.

**Proposition 6.3.4.** Let $N$ be the conductor of $E$. If $N$ is prime to $d_{K/F}$, then

$$Z(t, \varphi; H) = \prod_{v \mid N} (1 + \text{inv}_v(B) \varepsilon_v) Z(x, 1; H).$$ 

**Proof.** The proof is analogous to the proof of Proposition 6.3.2. Let us first compute the number of terms in $Z(t, \varphi; H)$. We need to determine for each $v$ the number of $K_v^\times$-orbits of oriented paths of length $\text{ord}_v(N_+)$ in the Bruhat-Tits tree; this is equal to the number of $g_v$ such that $x_v \in g_v R_v g_v^{-1}$.

- If $v \nmid N$ then the same argument as in Proposition 6.3.2 shows that there is only one orbit.
- If $v \mid N_-$, $B_v$ is ramified and $v$ is inert in $K$. Hence $K_v^\times \backslash B_v^\times /F_v^\times R_v^\times = \{1, \pi_v\}$ where $\pi_v \in B_v^\times$ is an element whose reduced norm has order 1 at $v$; $\pi_v$ corresponds to the Atkin-Lehner involution.
- If $v \mid N_+$, $v$ splits in $K$. Denote by $v^\delta$ the level of the order $R_v$. Each Eichler order of level $v^\delta$ is the intersection of the origin and the target of an oriented path of length $\delta$. By hypothesis those orders are maximally embedded in $K_v$ and the path corresponding to $g_v R_v g_v^{-1}$ is contained in the lowest level of the tree. As $K_v^\times$ acts by translations on this level, there are exactly two $K_v^\times$-orbits corresponding to $g_v$ depending on the orientation. We have $g_v^+ = g_v$ and $g_v^-$ which are exchanged by the Atkin-Lehner involution corresponding to $$(0 \quad \pi_v)

\begin{pmatrix}
1 & 0
\end{pmatrix}.$$ 

Let $n$ be the number of prime ideals in the decomposition of $N$. The sum $Z(t, \varphi; H)$ has $2^n$ factors. Let $W$ be the sets of these factors. By definition $Z(x, g; H) = [g] Z(x, 1; H)$. Using Proposition 6.1.1 we obtain

$$Z(t, \varphi; H) = \sum_{g \in W} [g] Z(x, 1; H) = \prod_{v \mid N} (1 + \text{inv}_v(B) \varepsilon_v) Z(x, 1; H).$$

\[\square\]

Let us conclude this paper by another conjecture. Assume that $E(F)$ has rank 1. Denote by $P_0$ some generator of $E(F)$ modulo torsion. For each $t \in O_F$ totally positive such that $(t)$ is square free and prime to $d_{K/F}$, denote by $K[t]$ the quadratic extension

$$K[t] = F(\sqrt{-D_0 t}),$$

which satisfies the hypothesis used to build Darmon’s points. Let $P_{t, 1}$ be Darmon’s point obtained for $K[t]$ and $b = 1$, and set

$$P_t = \text{Tr}_{K[t]/F} P_{t, 1}.$$ 

The point $P_t$ lies in $E(F)$ and there exists an integer $[P_t] \in \mathbb{Z}$ such that $P_t = [P_t] P_0$ modulo torsion.

Proposition 6.3.4 together with Conjecture 6.2.3 suggest the following (as in Conjecture 5.3 of [DT08]).
Conjecture 6.3.5. There exists some Hilbert modular form $g$ of level $3/2$ such that the $[P_1]$s are proportional to some Fourier coefficients of $g$.

Remark 6.3.6. Using the analogy with the Gross-Kohnen-Zagier theorem, the integers $[P_1]$ should be (proportional to) square roots of $L(E_{-D_0},1)$, where $E_{-D_0}$ is the twist of $E$ by $-D_0$.

Let us end this paper with two open questions.

Question 6.3.7. Does Bruinier’s generalization of Borcherds products [Bru] give anything interesting in this situation?

It is natural to expect that results of Cornut and Vatsal [CV07, CV05] hold also for Darmon’s points.

Question 6.3.8. Would it be possible to deduce such a result from suitable equidistribution properties for the real tori $\mathcal{T}_0^n$?
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