SPECIAL VALUES OF $L$-FUNCTIONS OF ONE-MOTIVES 
OVER FUNCTION FIELDS

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Abstract. The purpose of this paper is to give a formula for the leading 
coefficient at $s = 1$ of the $L$-function of one-motives over function fields 
in terms of Weil-étale cohomology, generalizing the Weil-étale version 
of the Birch and Swinnerton-Dyer conjecture in the authors’ previous 
work. As a consequence we express the Tamagawa number of a torus 
troduced by Ono-Oesterlé in terms of Weil-étale cohomology, and re-
prove their Tamagawa number formula.

1. Introduction

Let $S$ be a proper smooth geometrically connected curve over a finite 
field $\mathbb{F}_q$ with function field $K$, and let $M = [X \to G]$ be a 1-motive over $K$, 
that is, a lattice $X$ and a semi-abelian variety $G$ over $K$ placed in degrees 
$-1$ and $0$, respectively. Consider the Hasse-Weil $L$-function $L(M, s)$ of the $l$-adic representation $V_l(M)(-1)$ over $K$

$$L(M, s) = \prod_v \det(1 - \varphi_v N(v)^{-s} | V_l(M)(-1)^{I_v})^{-1},$$

where $l$ is a prime different from the characteristic $p$ of $K$, $v$ runs through 
the places of $S$, $I_v$ is the inertia group at $v$, $\varphi_v$ is the geometric Frobenius at $v$, and $N(v)$ the order of the residue field $k(v)$ at $v$. Denote the Néron model and the connected Néron model of $G$ over $S$ by $G$ and $G^0$, respectively, 
and let $\text{Lie } G^0$ be the Lie algebra of $G^0$ (a locally free sheaf). Let $\mathcal{X} = j_*X$, 
where $j: \text{Spec } K \hookrightarrow S$ is the inclusion. Define $\mathcal{X}^\Delta$ by the fiber product

$$
\begin{array}{ccc}
\mathcal{X}^\Delta & \longrightarrow & G^0 \\
\downarrow & & \downarrow \\
\mathcal{X} & \longrightarrow & G,
\end{array}
$$
and let $\mathcal{M}^\Delta$ be the complex of étale sheaves $[\mathcal{X}^\Delta \to \mathcal{G}^0]$ on $S$. If we set

$$r_M := - \sum_i (-1)^i \cdot \dim H_i^W(S, \mathcal{M}^\Delta) \otimes \mathbb{Q},$$

then

$$r_M = \text{rk } A(K) - \text{rk } X(K) - \text{rk } Y(K) + \sum_v \text{rk}(\mathcal{X}/\mathcal{X}^\Delta)(k(v)),$$

where $A$ is the abelian variety quotient of $G$ and $Y$ is the character module of the torus part of $G$. Note that $r_M$ can be negative. We give the following formula in the spirit of Lichtenbaum [Lic09].

**Theorem 1.1.** Assume that the Tate-Shafarevich group $\Sha(A)$ of $A$ is finite. Then the groups $H^*_W(S, \mathcal{M}^\Delta)$ are finitely generated, $L(M, s)$ has a zero of order $r_M$ at $s = 1$, and

$$\lim_{s \to 1} \frac{L(M, s)}{(s - 1)^{r_M}} = (-1)^{\text{rk } X(K)} \cdot \chi_W(S, \mathcal{M}^\Delta)^{-1} \cdot q^{\chi(S, \text{Lie } G^0)} \cdot (\log q)^{r_M}.$$

Here $\chi_W(S, \mathcal{M}^\Delta)$ is the Euler characteristic of the complex $H^*_W(S, \mathcal{M}^\Delta)$ with differential the cup product with a generator $e \in H^1_W(S, \mathbb{Z}) \cong \mathbb{Z}$.

This includes the formula for abelian varieties in [GS20] and implies a formula for tori as a special case. Note that the left-hand side depends on the map $X \to G$ just as the right-hand side does; in fact, we have

$$L(M, s) = L(\mathcal{X}^\Delta, s - 1) \cdot L(T, s) \cdot L(A, s),$$

where $L(\mathcal{X}^\Delta, s)$ is the $L$-function of $\mathcal{X}^\Delta$ defined below. In particular, the theorem is more subtle than just combining formulas for abelian varieties, tori, and lattices, and we need a result for $\mathbb{Z}$-constructible sheaves $\mathcal{Z}$ on $S$. Let

$$L(Z, s) = \prod_v \det(1 - \phi_v N(v)^{-s} | Z_{\phi} \otimes \mathbb{Q}_l)^{-1},$$

be the $L$-function of the $l$-adic sheaf $Z \otimes \mathbb{Q}_l$ on $S$, where $Z_{\phi}$ is the stalk of $Z$ at a geometric point lying over $v$. The Weil-étale cohomology groups $H^*_W(S, \mathcal{Z})$ are finitely generated, and we define $r_Z$ as in (1.1) and $\chi_W(S, \mathcal{Z})$ as above.

**Theorem 1.2.** The function $L(Z, s)$ has a pole of order $r_Z$ at $s = 0$, and

$$\lim_{s \to 0} L(Z, s) \cdot s^{r_Z} = (-1)^{\text{rk } Z(K)} \cdot \chi_W(S, \mathcal{Z}) \cdot (\log q)^{-r_Z}.$$

Similar formulas in the number field case were given by Tran [Tra15, Tra16]. The proof uses Artin’s induction theorem to reduce to the case of $\mathcal{Z} = \mathcal{Z}$. To prove Theorem 1.1 for $M = T$ a torus, we apply Theorem 1.2 to $j_* Y$, where $Y = \text{Hom}(T, G_m)$ is the character module of $T$, and use duality for Weil-étale cohomology of [Gei12] as well as the functional equation

$$L(T, 1 - s) = q^{-\chi(S, \text{Lie } (T^0))} \cdot L(j_* Y, s).$$
The proof of Theorem 1.1 is completed by combining the cases of constructible sheaves, tori, and abelian varieties.

As a by-product, we are able to express the Ono-Oesterlé Tamagawa number \( \tau(T) \) of a torus in terms of global invariants:

\[
\tau(T) = \frac{\# \text{Cl}(T^0)_{\text{tor}} \cdot q^{\chi(S, \text{Lie } T^0)}}{\# T^0(S) \cdot \rho(T) \cdot (\log q)^{k Y(S)} \cdot \text{Disc}(h_T)}
\]

and reprove the Tamagawa number formula of Ono [Ono63] and Oesterlé [Oes84]

\[
\tau(T) = \# H^1(K, Y) \cdot \# \Pi(T).
\]

Here \( \rho(T) \) is the value in Theorem 1.1 for \( M = T \),

\[
\text{Cl}(T^0) = \frac{\text{T} \cdot \text{A}_K}{\text{T}(K) + T^0(\mathcal{O}_{A_K})} \approx \bigoplus_v \pi_0(T_v)(k(v)) \div T(K),
\]

and \( h_T \) is the pairing

\[
h_T : \text{Cl}(T^0) \times Y(K) \to \text{Cl}(G_m) = \text{Pic}(S) \to \mathbb{Z}.
\]

The object \( M^\Delta \) is functorial in \( M \). It is closely related to, but different from, the Néron model \( M \) of \( M \) in the sense of [Suz19], whose cohomology groups \( H^*_W(S, M) \) are not finitely generated in general. We are planing to discuss the duality of \( M^\Delta \) and its Weil-étale cohomology, as well as their relations to the functional equation for \( L(M, s) \), in a forthcoming paper.

It would be desirable to unify Theorems 1.1 and 1.2 in terms of “constructible 1-motives” and their \( L \)-functions. As a first step, Pepin Lehalleur [PL19] defined constructible 1-motives with \( \mathbb{Q} \)-coefficients, but one would need to define a refinement with \( \mathbb{Z} \)-coefficients in order to formulate a special value formula, and this is especially difficult for the \( p \)-integral structure.

After the first version of this paper was uploaded to the arXiv, A. Morin [Mor22] gave a number field version of Theorem 1.2, improving on Tran’s work.

**Notation.** Throughout the paper \( k = \mathbb{F}_q \) is a finite field of characteristic \( p \) and \( S \) a proper, smooth, and geometrically connected curve over \( k \) of genus \( g \) with function field \( K \). For a place \( v \) of \( K \) (or a closed point of \( S \)), we denote the completed, henselian, and strict henselian local ring of \( S \) at \( v \) by \( \mathcal{O}_v, \mathcal{O}^h_v \), and \( \mathcal{O}^{sh}_v \), and their fraction field by \( K_v, K^h_v \), and \( K_v^{sh} \), respectively. Denote the residue field of \( \mathcal{O}_v \) by \( k(v) \), the degree of \( v \) by \( \deg(v) = [k(v) : k] \), and \( N(v) = \#k(v) = q^{\deg(v)} \). The adele ring of \( K \) is denoted by \( A_K \) and its subring of integral adeles by \( \mathcal{O}_{A_K} \).

For an abelian group \( G \), denote its torsion part by \( G_{\text{tor}} \) and its torsion-free quotient by \( G/\text{tor} \). If we have a pairing \( \varphi : G \times H \to \mathbb{Z} \) between finitely generated abelian groups, then the discriminant of \( \varphi/\text{tor} : G/\text{tor} \times H/\text{tor} \to \mathbb{Z} \) is denoted by \( \text{Disc}(\varphi) \). A lattice over a field is a finitely generated free
abelian group equipped with a continuous action of the absolute Galois group of the field (which necessarily factors through a finite group).

The Néron models we consider are Néron lift (locally finite type) models in the terminology of [BLR90, Chapter 10]. The connected Néron model means the part of the Néron model with connected fibers, usually denoted by $G^0$ if $G$ denotes the Néron model. For a group scheme $G$ locally of finite type over a field, we denote the étale group scheme of connected components of $G$ by $\pi_0(G)$ ([DG70, Chapter II, §5, No. 1, Proposition 1]).

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2. Weil-étale cohomology of tori and lattices

We recall the Weil-étale cohomology groups [Lic05], [Gei04] of varieties over a finite field. See also [GS20, Section 5] for another survey.

Let $\mathbb{F} = \mathbb{F}_q$ be an algebraic closure of $k = \mathbb{F}_q$. Denote the $q$-th power (arithmetic) Frobenius map by $\phi \in \text{Gal}(\overline{k}/k)$. Denote the Weil group of $k$ by $W = \langle \phi \rangle \subset \text{Gal}(\overline{k}/k)$ and the category of $W$-modules by $\text{Mod}_W$. Let $X$ be a proper smooth variety over $k$ and $\overline{X}$ be the base change of $X$ to $\overline{k}$. Denote the category of sheaves of abelian groups on the small étale site $X_{\text{et}}$ by $\text{Ab}(X_{\text{et}})$ and its bounded derived category by $D^b(X_{\text{et}})$. For a sheaf $F \in \text{Ab}(X_{\text{et}})$, we denote its inverse image to $X_{\text{et}}$ by $F_{\overline{X}}$. Hence we have a left exact functor

$$\text{Ab}(X_{\text{et}}) \to \text{Mod}_W, \quad F \mapsto \Gamma(\overline{X}, F)$$

and its right derived functor

$$D^b(X_{\text{et}}) \to D^b(\text{Mod}_W), \quad F \mapsto R\Gamma(\overline{X}, F).$$

The group $\Gamma(\overline{X}, F)$ has a natural action of $\phi$. Composing it with the group cohomology functor $R\Gamma(W, \cdot)$, we obtain a triangulated functor

$$R\Gamma_W(X, \cdot) : D^b(X_{\text{et}}) \to D^b(\text{Ab}), \quad F \mapsto R\Gamma(W, R\Gamma(\overline{X}, F)).$$

and denote the $n$-th cohomology of $R\Gamma_W(X, \cdot)$ by $H^n_W(X, \cdot)$.

Let $e \in H^1_W(k, \mathbb{Z}) \cong \text{Hom}(W, \mathbb{Z})$ be the homomorphism sending $\phi$ to $1$. For any $F \in D^b(X_{\text{et}})$, the cup product with $e$ defines a homomorphism $e : H^n_W(X, F) \to H^{n+1}_W(X, F)$. Since $e \cup e = 0$, we obtain a complex $(H^n_W(X, F), e)$. It is exact after tensoring with $\mathbb{Q}$ by [Gei04 Corollary 5.2], hence the cohomology groups of this complex are torsion. If the groups $H^n_W(X, F)$ are finitely generated for all $n$ and zero for almost all $n$, then the Euler characteristic of the complex $(H^n_W(X, F), e)$ is thus well-defined.

We denote this Euler characteristic by $\chi_W(X, F)$:

$$\chi_W(X, F) = \chi(H^n_W(X, F), e) = \prod_n (\# H^n_W(X, F), e)^{(-1)^n}.$$
Weil-étale cohomology does not depend on the choice of $k$:

**Proposition 2.1.** Let $k'/k$ be a finite field contained in the field of constants of $X$, and let $W$ and $W'$ be the Weil groups of $k$ and $k'$, respectively. Then the natural morphism

$$R\Gamma_W(X, \cdot) \to R\Gamma_{W'}(X, \cdot)$$

of functors $D^b(\text{Sch}_k) \to D^b(\text{Ab})$ is an isomorphism.

*Proof.* Fix an algebraic closure $\overline{k'} = \overline{k}$ of $k'$ (or $k$). Then $R\Gamma(X \times_k \overline{k'}, \cdot)$ is canonically isomorphic to $R\Gamma(X \times_k \overline{k}, \cdot) \otimes_{\mathbb{Z}[W']} \mathbb{Z}[W]$ as functors $D^b(\text{Sch}_k) \to D^b(\text{Mod}_W)$. This implies the result. \hfill \Box

Note however that the cup product with a generator of $H^1_W(k, \mathbb{Z})$ and a generator of $H^1_{W'}(k', \mathbb{Z})$ on these isomorphic functors differ by a factor of $[k' : k]$.

From now on we assume that the base is a smooth, proper, and geometrically connected curve $S$ over $k$. Let $K$ be the function field of $S$, $T/K$ be a torus and $Y = \text{Hom}_K(T, \mathbb{G}_m)$ its character lattice. Let $\mathcal{T}$ and $\mathcal{T}^0$ be the Néron and connected Néron models over $S$, respectively. Let $\mathcal{Y} = j_s Y$ and $\mathcal{Y} = \tau_{\leq 1} R j_* Y$, where $j : \text{Spec } K \hookrightarrow S$ is the inclusion and $\tau_{\leq 1}$ is the truncation functor in degrees $\leq 1$ (in the cohomological grading).

By [Suz19, Definition 4.8], the natural pairing $T \times Y \to \mathbb{G}_m$ over $K$ canonically extends to a morphism

(2.1) $\mathcal{T}^0 \otimes^L \mathcal{Y} \to \mathbb{G}_m$

in $D^b(\text{Sch}_k)$. Denote the sheaf-Hom functor for $S_{\text{et}}$ by $\mathcal{H}om_{S_{\text{et}}}$.

**Proposition 2.2.** The induced morphism

$$\mathcal{T}^0 \to R \mathcal{H}om_{S_{\text{et}}} (\mathcal{Y}, \mathbb{G}_m)$$

in $D^b(\text{Sch}_k)$ is an isomorphism.

*Proof.* We can check this at stalks. The morphism pulled back to $K_{et}$ is nothing but the duality between $T$ and $Y$. Hence it is enough to show that for any place $v \in S$, the induced morphism

$$\mathcal{T}^0(\mathcal{O}_{v,et}^{sh}) \to R \mathcal{H}om_{\mathcal{O}_{v,et}^{sh}} (\mathcal{Y}, \mathbb{G}_m)$$

in $D^b(\text{Ab})$ is an isomorphism. Denote by $j : \text{Spec } K_v^{sh} \hookrightarrow \text{Spec } \mathcal{O}_{v,et}^{sh}$ and $i : \text{Spec } k(v) \hookrightarrow \text{Spec } \mathcal{O}_{v,et}^{sh}$ the inclusions. Set $\mathcal{Y}^0 = j_Y Y$ and $\mathcal{Y} = \tau_{\leq 1} R j_* Y$. By [Suz19, Proposition 4.14], we have a canonical morphism of distinguished triangles

$$
\begin{array}{ccc}
\mathcal{T}^0(\mathcal{O}_{v,et}^{sh}) & \longrightarrow & T(K_v^{sh}) \\
\downarrow & & \downarrow \\
R \mathcal{H}om_{\mathcal{O}_{v,et}^{sh}} (\mathcal{Y}, \mathbb{G}_m) & \longrightarrow & R \mathcal{H}om_{\mathcal{O}_{v,et}^{sh}} (\mathcal{Y}^0, \mathbb{G}_m) \\
& & \longrightarrow \\
& & R \mathcal{H}om_{\mathcal{O}_{v,et}^{sh}} (i_* \mathcal{Y}, \mathbb{G}_m)[1].
\end{array}
$$
(There is actually a shifted term $T^0(\mathcal{O}^s_v)[1]$ next to $\pi_0(\mathcal{T}_v)(k(v))$, a similar term for the lower row and another commutative square next to the right square.) By the adjunction and the duality between $T$ and $Y$, the middle map can be identified with the isomorphism

$$T(K^s_v) \cong R\Gamma(K^s_v, T) \cong R\text{Hom}_{K^s_v, \text{et}}(Y, G_m)$$

since $H^n(K^s_v, T) = 0$ for $n \geq 1$ by [Ser79, Chapter X, Section 7, “Application”]. For the right vertical morphism, we use the exact sequence

$$0 \rightarrow G_m \rightarrow G_m \rightarrow i_*Z \rightarrow 0$$

in $\text{Ab}(\mathcal{O}^s_v)$, where $G_m$ is the Néron model of $G_m$. Since $H^n(K^s_v, G_m) = 0$ for $n \geq 1$, we have $Rj_*G_m \cong G_m$, hence

$$R\text{Hom}_{\mathcal{O}^s_v, \text{et}}(i_*\bar{\mathcal{Y}}_v, G_m) \cong R\text{Hom}_{K^s_v, \text{et}}(j^*i_*\bar{\mathcal{Y}}_v, G_m) = 0.$$ 

Therefore

$$R\text{Hom}_{\mathcal{O}^s_v, \text{et}}(i_*\bar{\mathcal{Y}}_v, G_m)[1] \cong R\text{Hom}_{\mathcal{O}^s_v, \text{et}}(i_*\bar{\mathcal{Y}}_v, i_*Z) \cong R\text{Hom}_{\text{Ab}}(\bar{\mathcal{Y}}_v, Z) = R\text{Hom}_{\text{Ab}}(\tau_{\leq 1}R\Gamma(K^s_v, Y), Z).$$

Therefore the right vertical morphism in the above diagram is

$$\pi_0(\mathcal{T}_v)(k(v)) \rightarrow R\text{Hom}_{\text{Ab}}(\tau_{\leq 1}R\Gamma(K^s_v, Y), Z).$$

This is an isomorphism by [Suz19, Theorem B (5)]. Therefore the left vertical morphism in the above diagram is also an isomorphism. □

**Theorem 2.3.** The groups $H^n_W(S, T^0)$ as well as the group $H^n_W(S, \check{Y})$ are finitely generated for all $n$, and zero for $n \not= 0, 1, 2, 3$. Moreover, the pairing

$$R\Gamma_W(S, \mathcal{T}^0) \otimes \mathbb{L} R\Gamma_W(S, \check{Y}) \rightarrow R\Gamma_W(S, G_m) \rightarrow \mathbb{Z}[-2]$$

induced by (2.1) is perfect.

**Proof.** This follows from [Gei12, Proposition 2.6, Theorem 4.2 and Corollary 4.3] because the cohomology groups and the pairing agrees with the one in [Gei12] by Proposition 2.2. □

**Corollary 2.4.** We have perfect pairings

$$H^1_W(S, \mathcal{T}^0)_{\text{tor}} \times H^2(W, \check{Y})_{\text{tor}} \rightarrow \mathbb{Z}$$

of finitely generated free abelian groups as well as perfect pairings

$$H^0_W(U, \mathcal{T}^0)_{\text{tor}} \times H^3_W(U, \check{Y})_{\text{tor}} \rightarrow \mathbb{Q}/\mathbb{Z}$$

of finite abelian groups.

**Lemma 2.5.** We have

$$\dim H^0(S, \check{Y}) \otimes \mathbb{Q} = \text{rk} Y(K),$$

and all other étale cohomology groups of $\check{Y}$ are torsion.
Proof. The first statement follows from \( H^0(S, \tilde{Y}) = H^0(K, Y) \), and second from the long exact sequence
\[
\ldots \to H^i(S, \tilde{Y}) \to H^i(K, Y) \to H^i(S, \tau_{\geq 1} R^j_* Y) \to \ldots
\]
because the sheaves \( R^n j_* Y \) as well as Galois cohomology are torsion for \( n > 0 \). \( \square \)

Proposition 2.6.
\[
\text{rk } H^i_W(S, \tilde{Y}) = \begin{cases} 
\text{rk } Y(K) & i = 0, 1 \\
0 & \text{otherwise}
\end{cases}
\]
\[
\text{rk } H^i_W(S, T^0) = \begin{cases} 
\text{rk } Y(K) & i = 1, 2 \\
0 & \text{otherwise}
\end{cases}
\]

Proof. We know the groups in question are finitely generated. The statement for \( H^i_W(S, \tilde{Y}) \) follows from the Lemma and the isomorphism between \( H^i_W(S, \tilde{Y}) \otimes \mathbb{Q} \) and \( (H^i(S, \tilde{Y}) \oplus H^{i-1}(S, \tilde{Y})) \otimes \mathbb{Q} \) ([Gei04, Corollary 5.2]). The statement for \( H^i_W(S, T^0) \) follows by duality. \( \square \)

We summarize the above finiteness results (where \( \text{fg} \) stands for finitely generated):

| \( i \) | 0 | 1 | 2 | 3 |
|----|----|----|----|----|
| \( H^i_W(S, \tilde{Y}) \) | \( \text{fg free} \) | \( \text{fg finite} \) | \( \text{finite} \) | \( \text{finite} \) |
| \( H^i_W(S, T^0) \) | \( \text{finite} \) | \( \text{fg} \) | \( \text{fg} \) | \( 0 \) |

Here the freeness of \( Y(K) \) reflects to the vanishing of (the torsion of) \( H^3_W(S, T^0) \).

Corollary 2.7. The secondary Euler characteristic of the torus and the lattice are given by
\[
\begin{align*}
\rho_T &:= - \sum_i (-1)^i \cdot \text{rk } H^i_W(S, T^0) = - \text{rk } Y(K) \\
\rho_Y &:= - \sum_i (-1)^i \cdot \text{rk } H^i_W(S, \tilde{Y}) = \text{rk } Y(K)
\end{align*}
\]

Proposition 2.8.
\[
\chi_W(S, T^0) = \chi_W(S, \tilde{Y})^{-1}.
\]

Proof. The cup product with the element \( e \in H^1_W(k, \mathbb{Z}) \) induces a natural transformation \( e: R\Gamma_W(S, \cdot) \to R\Gamma_W(S, \cdot)[1] \). The associativity of cup product shows that the diagram
\[
\begin{array}{ccc}
R\Gamma_W(S, T^0) \otimes L R\Gamma_W(S, \tilde{Y}) & \xrightarrow{id \otimes e} & R\Gamma_W(S, T^0) \otimes L R\Gamma_W(S, \tilde{Y})[1] \\
\downarrow_{e \otimes \text{id}} & & \downarrow_{\cup}
\end{array}
\]
\[
\begin{array}{ccc}
R\Gamma_W(S, T^0)[1] \otimes L R\Gamma_W(S, \tilde{Y}) & \xrightarrow{\cup} & R\Gamma_W(S, T^0 \otimes L \tilde{Y})[1]
\end{array}
\]
is commutative. Hence the diagram

\[ R\Gamma_W(S, \mathcal{T}^0) \xrightarrow{\sim} R\text{Hom}(R\Gamma_W(S, \mathcal{Y})[2], \mathbb{Z}) \]

\[ \downarrow e \quad \quad \quad \quad \downarrow e \]

\[ R\Gamma_W(S, \mathcal{T}^0)[1] \xrightarrow{\sim} R\text{Hom}(R\Gamma_W(S, \mathcal{Y})[1], \mathbb{Z}) \]

is commutative, from which the result follows. \qed

3. \text{L}-VALUES FOR $\mathbb{Z}$-CONSTRUCTIBLE SHEAVES

Let $\mathcal{Z}$ be a $\mathbb{Z}$-constructible étale sheaf on $S$ \cite[Chapter II, Section 0]{Mil06}. The groups $H^i_W(S, \mathcal{Z})$ are finitely generated by \cite[Proposition 2.6]{Gei12}. In this section, we prove a formula for the leading coefficient of the $\text{L}$-function of $\mathcal{Z}$ at $s = 0$ in terms of $\chi_W(S, \mathcal{Z})$. Similar formulas in the number field case were given by Tran \cite{Tra15}, \cite{Tra16}.

We define $L(\mathcal{Z}, s)$ by the Euler product

\[ \prod_v \det(1 - \varphi_v N(v)^{-s} \mid \mathcal{Z}_v \otimes \mathbb{Q})^{-1}, \]

where $v$ runs through the places of $S$, $\mathcal{Z}_v$ is the stalk of $\mathcal{Z}$ at a geometric point lying over $v$ and $\varphi_v$ is the geometric Frobenius at $v$. This function agrees with the $\text{L}$-function of the $l$-adic sheaf $\mathcal{Z} \otimes \mathbb{Q}_l$ on $S$ \cite[Definition 5.3.7]{Kah18}, where $l \neq p$ is any prime. In particular, it is equal to the value at $t = q^{-s}$ of the rational function

\[ \prod_{i=0}^{2} \det(1 - \varphi t \mid H^i(S, \mathcal{Z} \otimes \mathbb{Q}_l))^{-1}^{(-1)^{i+1}}, \]

where $\varphi$ is the geometric Frobenius of $k$ and the cohomology is taken as the continuous cohomology \cite[Corollary 5.3.11]{Kah18}. Let $j: \text{Spec} K \hookrightarrow S$ be the inclusion, $Y$ the torsion-free quotient of the generic fiber $j^* \mathcal{Z}$ of $\mathcal{Z}$ and $Y'$ the dual lattice of $Y$. We set

\[ L(Y, s) := L(j_* Y, s). \]

Then $L(Y, s)$ agrees with the Artin $\text{L}$-function

\[ \prod_v \det(1 - \phi_v N(v)^{-s} \mid Y'(K_v^{sh}) \otimes \mathbb{Q})^{-1}, \]

of $Y'$, where $\phi_v$ is the arithmetic Frobenius at $v$. The function $L(Y, s)$ is also the Hasse-Weil $\text{L}$-function

\[ \prod_v \det(1 - \varphi_v N(v)^{-s} \mid (Y(K_v^{\text{sep}}) \otimes \mathbb{Q}_l)_{I_v})^{-1}, \]

of the $l$-adic representation $Y(K_v^{\text{sep}}) \otimes \mathbb{Q}_l$ over $K$, where $I_v$ is the inertial group at $v$. The functions $L(\mathcal{Z}, s)$ and $L(Y, s)$ are equal up to finitely many
Euler factors since the kernel and the cokernel of the natural morphism $Z \to j_*Y$ are concentrated at finitely many closed points. Let

$$r_Z = - \sum i \cdot (-1)^i \cdot \text{rk} H^i_W(S, Z).$$

**Theorem 3.1.** We have

$$\lim_{s \to 0} L(Z, s) \cdot s^{r_Z} = \pm \chi_W(S, Z) \cdot (\log q)^{-r_Z}. \tag{3.1}$$

We will see in Proposition 4.3 that the sign $\pm$ is $(-1)^{\text{rk} Z(K)}$.

**Proof.** We will proceed in several steps.

**Step 1:** If $Z = Z$, then $L(Z, s)$ is the zeta function of $S$, and (3.1) reduces to [Gei04, Theorem 9.1, Proposition 9.2].

**Step 2:** If $Z$ is constructible, then both sides of (3.1) are 1.

This is clear for the left-hand side. For the right-hand side, the constructibility of $Z$ implies that the groups $H^i(S, Z)$ are finite by [Mil80, Chapter VI, Theorem 2.1], hence $\#H^i(S, Z)^G = \#H^i(S, Z)_G$, which implies that $\chi_W(S, Z) = 1$ in view of the short exact sequences

$$0 \to H^{i-1}(\bar{S}, Z)_G \to H^i_W(S, Z) \to H^i(\bar{S}, Z)^G \to 0.$$ 

**Step 3:** Equation (3.1) holds if $Z$ is supported on closed points of $S$.

We may assume that $Z$ is supported at a single place $v$. Write $Z = i_{v*}Z_v$, where $i_v: v \hookrightarrow S$ is the inclusion. The number $r := r_Z$ is the rank of $Z_v(k(v))$. The left-hand side is

$$\lim_{s \to 0} \det(1 - N(v)^{-s})^{-r} \cdot (\deg(v) \cdot \log q)^{-r} = \chi_W(k(v), Z_v) \cdot (\deg(v) \cdot \log q)^{-r},$$

where $W_v$ is the Weil group of $k(v)$. Let $e_v \in H^1_W(k(v), Z)$ be the generator corresponding to the arithmetic Frobenius of $k(v)$. Since $H^1_W(k, Z) \to H^1_W(k, Z)$ is multiplication by $\deg(v)$, we obtain

$$\chi_W(S, Z) = \chi_W(k(v), Z_v) = \chi_W(k(v), Z_v) \cdot \deg(v)^{-r}.$$ 

Hence both sides of (3.1) are $\chi_W(S, Z) \cdot (\log q)^{-r}$.

**Step 4:** Let $K'/K$ be a finite separable extension. Denote the normalization of $S$ in $K'$ by $S'$. Assume that $Z$ is the pushforward of a $\mathbb{Z}$-constructible sheaf $Z'$ on $S'$. Then (3.1) for $Z'$ over $S'$ implies (3.1) for $Z$ over $S$.

Denote the constant field of $K'$ by $k'$, its order by $q'$, and the morphism $S' \to S$ by $\pi$. Since $\pi$ is finite, we have $R^n\pi_* = 0$ for $n \geq 1$. Hence by [Mil80, Chapter VI, Lemma 13.8 (c)], we have

$$\det(1 - \varphi t | H^i(S, Z \otimes \mathbb{Q}_l)) = \det(1 - \varphi t | H^i(S', Z' \otimes \mathbb{Q}_l)).$$
for any \( i \), where \( S' = S' \times_k k \) as before. Taking the alternating product over \( i \), evaluating it at \( t = q^{-s} \) and noting that the \( L \)-function is independent of the choice of a constant field, we have \( L(Z, s) = L(Z', s) \).

Also \( H^1_{W'}(S, Z) \cong H^1_{W'}(S', Z') \) by Proposition [2.1] where \( W' \) is the Weil group of \( k' \). In particular, \( r_Z = r_{Z'} \). Let \( e' \in H^1_{W'}(k', \mathbb{Z}) \) be the generator corresponding to the \( q \)-th power arithmetic Frobenius. Then \( e = [k' : k]e' \) via the homomorphism \( H^1_{W'}(k, \mathbb{Z}) \to H^1_{W'}(k', \mathbb{Z}) \). This implies

\[
\chi_W(S, Z) = \chi_{W'}(S', Z') \cdot [k' : k]^{-r_{Z'}}.
\]

As \( q' = q^{[k' : k]} \), we get the result.

**Step 5:** Let \( n \geq 1 \) be an integer. If (3.1) holds for \( Z^n \), then it holds for \( Z \).

This follows by taking the \( n \)-th roots of the real numbers on both sides of (3.1).

**Step 6:** We now finish the proof. We first observe that both of the sides of (3.1) are multiplicative in \( Z \) with respect to short exact sequences. In particular, we may assume that \( Z \) is torsion-free by Step 2. Denote the generic fiber of \( Z \) by \( Z_K \). By the Artin induction theorem [Swa60] Corollary 4.4, Proposition 4.1], there exist an integer \( n \geq 1 \), finite separable extensions \( K'_1, \ldots, K'_m, K'_1', \ldots, K'_n' \), a finite Galois module \( N_K \) over \( K \) and an exact sequence

\[
0 \to \bigoplus_i \pi_{K'_i/K_s} \mathbb{Z} \to \bigoplus_j \pi_{K'_j/K_s} \mathbb{Z} \to N_K \to 0
\]

of Galois modules over \( K \), where \( \pi_{K'_i/K} \) is the morphism \( \text{Spec } K'_i \to \text{Spec } K \) and \( \pi_{K'_j/K} \) are similarly defined. By spreading out, this sequence can be obtained as the generic fiber of an exact sequence

\[
0 \to \bigoplus_i \pi_{U'_i/U_s, \ast} \mathbb{Z} \to \bigoplus_j \pi_{U'_j/U_s, \ast} \mathbb{Z} \to N_U \to 0
\]

of étale sheaves over some dense open subscheme \( U \subseteq S \), where \( Z_U \) is the restriction of \( Z \) to \( U \); \( U'_i \) and \( U''_j \) denote the normalization of \( U \) in \( K'_i \) and \( K''_j \), respectively; \( \pi_{U'_i/U} \) and \( \pi_{U''_j/U} \) are the morphism \( U'_i \to U \) and \( U''_j \to U \), respectively; and \( N_U \) is a finite étale group scheme over \( U \). Denote the inclusion map \( U \hookrightarrow S \) by \( \iota \). By the exact sequence

\[
0 \to \iota_* Z_U \to Z \to \bigoplus_{v \not\in U} i_{v, \ast} Z_v \to 0
\]

and Step 3, (3.1) for \( \iota_* Z_U \) and for \( Z \) are equivalent. By the exact sequence (3.2), Steps 2 and 5 and the exactness of \( \iota_* \), it is enough to show the proposition for \( \iota_* \pi_{U'_i/U_s, \ast} \mathbb{Z} \) and \( \iota_* \pi_{U''_j/U_s, \ast} \mathbb{Z} \). If \( \iota'_i : U'_i \hookrightarrow S'_i \) is the inclusion into the smooth compactification, then \( \iota'_i(\pi_{U'_i/U_s, \ast} \mathbb{Z}) \cong \pi_{S'_i/S_s, \ast}(\iota'_{i, \ast} \mathbb{Z}) \). Finally, (3.1) for \( \pi_{S'_i/S_s, \ast}(\iota'_{i, \ast} \mathbb{Z}) \) follows from Steps 1, 2, and 4. \( \square \)
4. Functional equations and $L$-values for tori

We will determine the sign and express the exponential term appearing in the functional equation relating the $L$-functions of $T$ and of $Y$ in terms of $\chi(S, \text{Lie } T^0)$. This will allow us to give a formula for the leading coefficient of the $L$-function of $T$ at $s = 1$ in terms of $\chi_w(S, T^0)$.

Let $r = \text{rk } Y(K)$, $d = \dim T$, $Y'$ be the dual lattice of $Y$, and define

$$L(T, s) := L(Y', s).$$

More explicitly, by the discussion in Section 3, we know that $L(T, s)$ is the Artin $L$-function of $Y$ and also the Hasse-Weil $L$-function of the $l$-adic representation $Y' \otimes_{\mathbb{Z}} \mathbb{Q}_l \cong V_l(T)(-1)$ over $K$ (where $l \neq p$ is any prime),

$$L(T, s) = \prod_v \det(1 - \varphi_v | N(v)^{-s} | V_l(T)(-1)_{I_v})^{-1},$$

where $\varphi_v$ is the geometric Frobenius at $v$ and $I_v$ is the inertia group at $v$.

Let $f(Y)$ be the Artin conductor of the Galois representation $Y \otimes_{\mathbb{Z}} \mathbb{Q}_l$ over $K$ as defined in [Ser79, Chapter VI, Section 3]. It is an effective divisor on $S$. Denote its multiplicity at a place $v$ by $f(Y|_{D_v})$ (where $D_v$ is the decomposition group at $v$), so that

$$f(Y) = \sum_v f(Y|_{D_v}) \cdot v.$$  

The degree $f(Y)$ of $f(Y)$ is given by

$$f(Y) = \sum_v \deg(v) f(Y|_{D_v}).$$

The functional equation for Artin $L$-functions [Kah18 Theorem 4.4.8] in this case says that

$$L(T, 1 - s) = \pm q^{((2g - 2)d + f(Y))(s - 1/2)} \cdot L(Y, s).$$

We have

$$L(Y, s) = \prod_{i=0}^{2} \det(1 - \varphi q^{-s} | H^1(S, Y \otimes \mathbb{Q}_l))(-1)^{i+1}$$

by [Kah18 Corollary 5.3.11], where $\varphi$ is the geometric Frobenius of $k$. Each term $\det(1 - \varphi t | H^1(S, Y \otimes \mathbb{Q}_l))$ is a polynomial with integer coefficients in $t$ with constant term 1 whose reciprocal roots are Weil $q$-numbers of weight $i$ [Kah18 Theorem 5.5.9].

Proposition 4.1. We have

$$\det(1 - \varphi t | H^1(S, Y \otimes \mathbb{Q}_l)) = \det(1 - \varphi t | H^1(C, \mathbb{Q}_l))$$

for some abelian variety $C$ over $\mathbb{F}_q$. 

Proof. Let $K'$ be a finite Galois extension of $K$ with Galois group $G$ that trivializes $Y$. Denote the normalization of $S$ in $K'$ by $S'$. Denote the constant field of $K'$ by $k'$. Let $U \subseteq S$ be a dense open subscheme over which $Y$ is a lattice. Let $U' \subseteq S'$ be the inverse image of $U$ in $S'$. Denote the inclusion map $\text{Spec} K' \hookrightarrow S'$ by $j'$. Set $Y = j^*_s(Y \times_K K')$, $\overline{S}' = S' \times_k \overline{k}$ and $\overline{U}' = U' \times_k \overline{k}$. The long exact sequence for cohomology with compact support for $\overline{U} \hookrightarrow \overline{S}'$ yields an exact sequence

$$\bigoplus_{v \in S' \setminus U} Y(K_v^{sh} \otimes_K K') \otimes \mathbb{Q}_l \to H^1_c(\overline{U}', Y' \otimes \mathbb{Q}_l) \to H^1(\overline{S}', Y' \otimes \mathbb{Q}_l) \to 0.$$ 

This sequence remains exact after taking $G$-invariants since $G$ is finite and the groups are $\mathbb{Q}_l$-vector spaces. Comparing the resulting exact sequence with the similar exact sequence for $U \hookrightarrow S$, we know that

$$H^1(\overline{S}, Y \otimes \mathbb{Q}_l) \cong H^1(\overline{S}', Y' \otimes \mathbb{Q}_l)^G.$$ 

The Jacobian variety $J_{S'/k}$ of $S'/k$ is isomorphic to the Weil restriction of the Jacobian $J_{S'/k'}$ from $k'$ to $k$. In particular, it is an abelian variety over $k$ with a natural action of $G$ by group scheme morphisms over $k$. Consider the abelian variety $J_{S'/k} \otimes_{\mathbb{Z}} Y(K') \cong J_{S'/k}^d$ over $k$. The tensor product of the $G$-actions on $J_{S'/k}$ and on $Y(K')$ defines a $G$-action on $J_{S'/k} \otimes_{\mathbb{Z}} Y(K')$ by group scheme morphisms over $k$. Let $C$ be the maximal reduced and connected subgroup scheme of the $G$-invariant part of $J_{S'/k} \otimes_{\mathbb{Z}} Y(K')$. It is an abelian variety over $k$. Thus to prove the proposition it suffices to observe that the $l$-adic Tate module of $C$ is isomorphic to $H^1(\overline{S'}, Y' \otimes \mathbb{Q}_l)^G(1)$ as a Gal$(\overline{k}/k)$-module and

$$H^1(\overline{S}, Y \otimes \mathbb{Q}_l) \cong H^1(\overline{S'}, Y' \otimes \mathbb{Q}_l)^G \cong H^1(\overline{C}, \mathbb{Q}_l).$$

$\square$

Proposition 4.2. The sign of the functional equation (4.1) is positive.

Proof. Recall again that $L(T, s) = L(Y', s)$ and $r = \text{rk} Y(K) = \text{rk} Y(K')$. The function $L(Y, s)$ is real-valued for real $s$, positive for large real $s$ (by the Euler product) and has a pole of order $r$ at $s = 0$ and $s = 1$ by Theorem 3.1 and (4.1). The only other possible zero or pole are at $s = 1/2$. Hence it is enough to show that $L(Y, s)$ has a zero of even order at $s = 1/2$. This order is equal to the order of zero of the function $\text{det}(1 - \varphi q^{-s} | H^1(\overline{S}, Y \otimes \mathbb{Q}_l))$ at $s = 1/2$. But this function is a polynomial with $\mathbb{Z}$-coefficients in $q^{-s}$ of even degree by Proposition 4.1.

$\square$

Proposition 4.3. The sign in the formula (3.1) is $(-1)^{\text{rk} Z(K)}$.

Proof. Denote the generic fiber of $Z$ by $Z_K$. The two functions $L(Z, s)$ and $L(Z_K, s) (= L(j_*Z_K, s))$ differ only by finitely many Euler factors of weight zero (namely polynomials in $q^{-s}$ with roots of unity roots). Hence they have the same zeros and poles for positive $s$. By the proof of Proposition 4.2 we know that the function $L(Z_K, s)$ has a pole of order $\text{rk} Z(K)$ at $s = 1$, a
zero of even order at \( s = 1/2 \), and does not have a zero or pole for other positive values of \( s \). This implies the result.

**Proposition 4.4.**

\[
\frac{f(Y)}{2} = -\deg(\text{Lie} T^0).
\]

**Proof.** Recall that \( f(Y) = \sum_{v \in S} \deg(v) f(Y|_{D_v}) \). Let \( K' \) be a finite Galois extension of \( K \) that trivializes \( Y \). Let \( k' \) be the field of constants of \( K' \). Let \( S' \) be the normalization of \( S \) in \( K' \), and \( T'^0 \) the connected Néron model over \( S' \) of \( T \times_K K' \). For each \( v \in S \), fix a place \( v' \) of \( S' \) above \( v \). By [CY01, Theorem (12.1)], we have

\[
\frac{f(Y|_{D_v})}{2} = \frac{1}{e_{v'/v}} \text{length}_{\mathcal{O}_{v'}} \frac{\text{Lie}(T^0) \otimes_{\mathcal{O}_{S'}} \mathcal{O}_{v'}}{\text{Lie}(T^0) \otimes_{\mathcal{O}_S} \mathcal{O}_{v'}},
\]

where \( e_{v'/v} \) is the ramification index of \( \mathcal{O}_{v'}/\mathcal{O}_v \) and \( \text{length}_{\mathcal{O}_{v'}} \) denotes the length of \( \mathcal{O}_{v'} \)-modules. The length of the cokernel of a full rank embedding of finite free modules is invariant under taking the top exterior power and inverts when taking duals. Hence the right-hand side is equal to

\[
\frac{1}{e_{v'/v}} \text{length}_{\mathcal{O}_{v'}} \frac{\text{det}(\text{Lie}(T^0))^* \otimes_{\mathcal{O}_S} \mathcal{O}_{v'}}{\text{det}(\text{Lie}(T'^0))^* \otimes_{\mathcal{O}_{S'}} \mathcal{O}_{v'}}.
\]

where \( \text{det} \) denotes the top exterior power and \( * \) denotes the dual line bundle.

Let \( \omega \in \text{det}(\text{Lie}(T))^*(K) \) be a non-zero invariant top degree differential form on \( T/K \). Then \( f(Y)/2 \) can be written as the number

\[
\sum_{v \in S} \frac{\deg(v)}{e_{v'/v}} \text{length}_{\mathcal{O}_{v'}} \frac{\text{det}(\text{Lie}(T^0))^* \otimes_{\mathcal{O}_S} \mathcal{O}_{v'}}{\omega \mathcal{O}_{v'}},
\]

(4.2)

minus the number

\[
\sum_{v \in S} \frac{\deg(v)}{e_{v'/v}} \text{length}_{\mathcal{O}_{v'}} \frac{\text{det}(\text{Lie}(T'^0))^* \otimes_{\mathcal{O}_{S'}} \mathcal{O}_{v'}}{\omega \mathcal{O}_{v'}},
\]

(4.3)

where the length of \( L/M \) for two finite free \( \mathcal{O}_{v'} \)-modules \( L \subset M \) means the negative of the length of \( M/L \). The number (4.2) is equal to

\[
\sum_{v \in S} \deg(v) \text{length}_{\mathcal{O}_v} \frac{\text{det}(\text{Lie}(T^0))^* \otimes_{\mathcal{O}_S} \mathcal{O}_v}{\omega \mathcal{O}_v} = -\deg(\text{det}(\text{Lie}(T^0))).
\]

Similarly the number (4.3) is equal to

\[
\frac{[k' : k]}{[K' : K]} \sum_{v' \in S'} \deg(v') \text{length}_{\mathcal{O}_{v'}} \frac{\text{det}(\text{Lie}(T'^0))^* \otimes_{\mathcal{O}_{S'}} \mathcal{O}_{v'}}{\omega \mathcal{O}_{v'}}
\]

\[
= -\frac{[k' : k]}{[K' : K]} \cdot \deg(\text{det}(\text{Lie}(T'^0))),
\]

where the degree is relative to the field of constants \( k' \) of \( K' \). The group \( T'^0 \) is a finite product of copies of \( G_m \) over \( S' \). Hence the degree of its Lie algebra is zero. This proves the proposition. \( \square \)
Proposition 4.5.
\[ L(T, 1 - s) = q^{-\chi(S, \text{Lie}(T^0))(2s-1)} \cdot L(Y, s). \]

Proof. This follows from (4.1), Propositions 4.2 and 4.4, and the Riemann-Roch formula
\[ \chi(S, \text{Lie}(T^0)) = (1 - g)d + \deg(\text{Lie}(T^0)). \]

\[ \square \]

Recall from Corollary 2.7 that \( r_T = -r = -\text{rk}_Y(K). \)

Theorem 4.6.
\[ \lim_{s \to 1} \frac{L(T, s)}{(s - 1)^r_T} = \chi_W(S, T^0)^{-1} \cdot q^{\chi(S, \text{Lie} T^0)} \cdot (\log q)^{r_T}. \]

Proof. We have
\[ \chi_W(S, T^0)^{-1} = \chi_W(S, \tilde{Y}) = \chi_W(S, Y) \]
by Proposition 2.8 and the constructibility of \( R^1 j_* Y. \) Hence the result follows from Theorem 3.1, Propositions 4.3 and 4.5.

\[ \square \]

5. Calculations of the Weil-étale Euler characteristic

The goal of this section is to express the Weil-étale Euler characteristic \( \chi_W(S, T^0) \) in terms of classical invariants. Define
\[ \Phi_T(k) := \bigoplus_v \pi_0(T_v)(k(v)) \]
and let
\[ T(\mathbb{A}_k) = \lim_{U \subseteq S} \prod_{v \in U} T^0(\mathcal{O}_v) \times \prod_{v \notin U} T(K_v) \]
be the restricted direct product of \( T(K_v) \) with respect to the connected components \( T^0(\mathcal{O}_v) \). For \( i_v: v \hookrightarrow \mathcal{O}_v \) the inclusion, we have an exact sequence
\[ 0 \to T^0 \to T \to i_{v,*} \pi_0(T_v) \to 0 \]
of fpf sheaves over \( \mathcal{O}_v^h \). Taking sections over \( \mathcal{O}_v \), we obtain an exact sequence
\[ 0 \to T^0(\mathcal{O}_v) \to T(K_v) \to \pi_0(T_v)(k(v)) \to 0. \]

because \( T(\mathcal{O}_v) = T(K_v) \) and \( H^n(\mathcal{O}_v, T^0) \cong H^n(k(v), T_v^0) = 0 \) for \( n \geq 1 \) by [Mil80, Chapter III, Remark 3.11 (b)] and Lang’s theorem. We also have an exact sequence
\[ 0 \to T^0(\mathcal{O}_v^h) \to T(K_v^h) \to \pi_0(T_v^h)(k(v)) \to 0 \]

since \( T(\mathcal{O}_v^h) = T(K_v^h) \) and \( H^n(\mathcal{O}_v^h, T^0) \cong H^n(k(v), T_v^0) = 0 \) similarly. Taking the restricted direct product of (5.2), we obtain an exact sequence
\[ 0 \to T^0(\mathcal{O}_K) \to T(\mathbb{A}_K) \to \Phi_T(k) \to 0. \]
Define $\text{Cl}(\mathcal{T}^0)$ to be the quotient
\[ \frac{T(\mathcal{A}_K)}{T(K) + \mathcal{T}^0(\mathcal{O}_{h_K})} \approx \Phi_T(k), \]
and let
\[ \text{III}(T) = \text{Ker} \left( H^1(K, T) \to \prod H^1(K_v, T) \right) \]
be the Tate-Shafarevich group of $T$. We note that the Tate-Shafarevich group does not change if we use $H^1(K^h_v, T)$ instead of $H^1(K_v, T)$ since $H^1(K^h_v, T) \cong H^1(K_v, T)$.

**Proposition 5.1.** There exist a canonical exact sequence and a canonical isomorphism
\[ 0 \to \text{Cl}(\mathcal{T}^0) \to H^1_W(S, \mathcal{T}^0) \to \text{III}(T) \to 0, \]
\[ H^1_W(S, \tilde{Y}) \cong H^1(K, Y). \]

**Proof.** The exact sequence
\[ 0 \to H^1(S, \mathcal{T}^0) \to H^1_W(S, \mathcal{T}^0) \to \mathcal{T}^0(S) \otimes \mathbb{Q} \]
and the finiteness of $\mathcal{T}^0(S)$ (Proposition 2.6) shows $H^1(S, \mathcal{T}^0) \cong H^1_W(S, \mathcal{T}^0)$.

The localization sequence gives an exact sequence
\[ T(K) \to \bigoplus_v H^1_v(\mathcal{O}^h_v, \mathcal{T}^0) \to H^1(S, \mathcal{T}^0) \to H^1(K, T) \to \bigoplus_v H^2_v(\mathcal{O}^h_v, \mathcal{T}^0). \]

Now consider the analogous sequence for $\text{Spec} \mathcal{O}^h_v$. The vanishing of $H^n(\mathcal{O}^h_v, \mathcal{T}^0)$ for $n \geq 1$ gives an isomorphism $H^1(K_v^h, T) \cong H^2_v(\mathcal{O}^h_v, \mathcal{T}^0)$ as well as a short exact sequence
\[ 0 \to \mathcal{T}^0(\mathcal{O}^h_v) \to T(K_v^h) \to H^1_v(\mathcal{O}^h_v, \mathcal{T}^0) \to 0, \]
which implies $H^1_v(\mathcal{O}^h_v, \mathcal{T}^0) \cong \pi_0(T_v(k(v)))$ by comparing to (5.3).

For the isomorphism, we use that $H^1(S, \tilde{Y}) \cong H^1(K, Y)$ as above. Hence it is enough to show that the natural map $H^1(S, \tilde{Y}) \to H^1(K, Y)$ is an isomorphism, which follows from $H^n_v(\mathcal{O}^h_v, \tilde{Y}) = 0$ for $n \leq 2$. For this, it is enough to show that $H^n_v(\mathcal{O}^{sh}_v, \tilde{Y}) = 0$ for $n \leq 2$ which follows because
\[ H^n_v(\mathcal{O}^{sh}_v, \tilde{Y}) \cong \begin{cases} H^n(K^{sh}_v, Y) & n = 0, 1; \\ 0 & n \neq 0, 1. \end{cases} \]
by definition of $\tilde{Y}$. \hfill \Box

Denote the first map in the exact sequence in Proposition 5.1 by $\text{cl}_T$:
\[ \text{cl}_T: \text{Cl}(\mathcal{T}^0) \hookrightarrow H^1_W(S, \mathcal{T}^0). \]

As seen in the proof of Proposition 5.1, $\text{cl}_T$ is the composite of the natural maps
\[ \frac{T(\mathcal{A}_K)}{T(K) + \mathcal{T}^0(\mathcal{O}_{h_K})} \approx \bigoplus_v H^1_v(\mathcal{O}^h_v, \mathcal{T}^0) \to H^1(S, \mathcal{T}^0) \cong H^1_W(S, \mathcal{T}^0). \]
We denote the map induced on the torsion-free quotients by $\text{cl}_T/\text{tor}$:

$$\text{cl}_T/\text{tor} : \text{Cl}(T^0)/\text{tor} \to H^1_W(S, T^0)/\text{tor}.$$ 

By the functoriality of class groups $\text{Cl}$ and connected Néron models, we have a natural map

$$(5.4) \quad Y(K) = \text{Hom}_K(T, G_m) \cong \text{Hom}_S(T^0, G_m) \to \text{Hom}(\text{Cl}(T^0), \text{Cl}(G_m)).$$

Hence we have a canonical pairing

$$(5.5) \quad h_T : Cl(T^0) \times Y(K) \to \text{Cl}(G_m) = \text{Pic}(S)^{\text{deg}} \mathbb{Z}.$$ 

Recall the isomorphism $Y(K) \cong H^0_W(S, \hat{Y})$ and the morphism $e : H^0_W(S, \hat{Y}) \to H^1_W(S, \hat{Y})$. We denote the composite $Y(K) \to H^1_W(S, \hat{Y})$ by $e$ by abuse of notation.

**Proposition 5.2.** The composite

$$\text{Cl}(T^0) \times Y(K) \xrightarrow{\text{cl}_T \times e} H^1_W(S, T^0) \times H^1_W(S, \hat{Y}) \to \mathbb{Z},$$

where the last (perfect) pairing is the pairing of Corollary 2.4, agrees with $h_T$.

**Proof.** The pairing $T^0 \otimes^L \hat{Y} \to G_m$ over $S$ in (2.4) induces a commutative diagram

$$
\begin{array}{ccc}
T(K) \times Y(K) & \longrightarrow & K^x \\
\downarrow & & \downarrow \\
\bigoplus_v H^1_v(O_v^h, T^0) \times Y(K) & \longrightarrow & \bigoplus_v H^1_v(O_v^h, G_m),
\end{array}
$$

where the vertical maps are coboundary maps of localization sequences. On the cokernels of the vertical maps, this diagram induces a pairing

$$\text{Cl}(T^0) \times Y(K) \to \text{Cl}(G_m).$$

This agrees with the map (5.4). Hence the naturality of the cup product shows that the diagram

$$
\begin{array}{ccc}
\text{Cl}(T^0) \times Y(K) & \longrightarrow & \text{Cl}(G_m) = \text{Pic}(S) \\
\downarrow_{\text{cl}_T \times \text{id}} & & \downarrow \\
H^1_W(S, T^0) \times Y(K) & \longrightarrow & H^1_W(S, G_m) = \text{Pic}(S)
\end{array}
$$

is commutative, where the upper horizontal pairing is $h_T$. Also consider the commutative diagram

$$
\begin{array}{ccc}
H^1_W(S, T^0) \times Y(K) & \longrightarrow & H^1_W(S, G_m) = \text{Pic}(S) \\
\downarrow_{\text{id} \times e} & & \downarrow e \\
H^1_W(S, T^0) \times H^1_W(S, \hat{Y}) & \longrightarrow & H^2_W(S, G_m) = \mathbb{Z}
\end{array}
$$
where the commutativity of the right square follows from the geometric connectivity of $S$ over $k$. Combining these two diagrams, we get the result.

With these preparations we can determine the Euler characteristic of the torus.

**Proposition 5.3.**

$$
\chi(H^*_W(S, T^0)/tor, e)^{-1} = \frac{\# \text{Coker}(cl_T/tor)}{\text{Disc}(h_T)}.
$$

**Proof.** The only non-trivial map for this Euler characteristic is

$$
e: H^1_W(S, T^0)/tor \to H^2_W(S, T^0)/tor
$$

by Proposition 5.6. Its linear dual is

$$e: Y(K) \to H^1_W(S, \tilde{Y})/tor$$

by Corollary 2.4 and the proof of Proposition 2.8. Hence Proposition 5.2 gives the result.

**Proposition 5.4.** Denote the alternating product of the orders of $H^*_W(S, T^0)_{tor}$ by $\chi(H^*_W(S, T^0)_{tor})$. Then

$$
\chi(H^*_W(S, T^0)_{tor})^{-1} = \frac{\# \text{Cl}(T^0)_{tor} \cdot \# \text{III}(T)}{\# T^0(S) \cdot \# H^1(K, Y) \cdot \# \text{Coker}(cl_T/tor)}.
$$

**Proof.** We have an exact sequence of finite groups

$$0 \to \text{Cl}(T^0)_{tor} \xrightarrow{cl_T} H^1_W(S, T^0)_{tor} \to \text{III}(T) \to \text{Coker}(cl_T/tor) \to 0$$

by Proposition 5.1. Also

$$\# H^2_W(S, T^0)_{tor} = \# H^1_W(S, \tilde{Y})_{tor} = \# H^1(K, Y)$$

by Corollary 2.4 and Proposition 5.1. Therefore

$$\chi(H^*_W(S, T^0)_{tor})^{-1} = \frac{\# H^1_W(S, T^0)_{tor}}{\# H^0_W(S, T^0)_{tor} \cdot \# H^2_W(S, T^0)_{tor}} = \frac{\# \text{Cl}(T^0)_{tor} \cdot \# \text{III}(T)}{\# T^0(S) \cdot \# H^1(K, Y) \cdot \# \text{Coker}(cl_T/tor)}.$$

**Proposition 5.5.**

$$
\chi_W(S, T^0)^{-1} = \frac{\# \text{Cl}(T^0)_{tor} \cdot \# \text{III}(T)}{\# T^0(S) \cdot \# H^1(K, Y) \cdot \text{Disc}(h_T)}.
$$

**Proof.** The number $\chi_W(S, T^0)$ is the product of $\chi(H^*_W(S, T^0)/tor, e)$ and $\chi(H^*_W(S, T^0)_{tor})$ by the proof of [Gei04, Theorem 9.1]. Therefore Propositions 5.3 and 5.4 give the result.
6. A Weil-étale Tamagawa number formula for tori

In this section, we express the Tamagawa number of $T$, defined by Ono in [Ono61] and redefined (in the function field case) by Oesterlé [Oes84], in terms of arithmetic-geometric invariants defined without using Haar measures. We use this to reprove Ono-Oesterlé’s Tamagawa number formula [Ono63], [Oes84].

We begin by recalling the Tamagawa number from [Ono61, Sections 3.1–3.5], [Oes84, Chapter I]. Set

$$d = \dim(T), \quad r = \text{rk}(Y(K)), \quad g = \text{genus } S.$$ 

The function $L(T, s)$ has a pole of order $r$ at $s = 1$ by Theorem 4.6. We define

$$\rho(T) = \lim_{s \to 1} L(T, s)(s - 1)^r.$$ 

We denote the sum of the valuations $\mathfrak{A}_K \twoheadrightarrow \bigoplus_v \mathbb{Z} \twoheadrightarrow \mathbb{Z}$ by deg. A character $\chi \in Y(K) = \text{Hom}_K(T, G_m)$ induces a homomorphism $\bar{\chi} : T(\mathbb{A}_K) \to \mathfrak{A}_K$.

In other words, $T(\mathbb{A}_K)^1$ is the inverse image of the left kernel of the pairing $h_T$ in (5.5) under the quotient map $T(\mathbb{A}_K) \to \text{Cl}(T^0)$. This group contains $T(K)$ and $T^0(\mathcal{O}_{\mathbb{A}_K})$. Recall that the quotient $\text{Cl}(T^0)$ of $T(\mathbb{A}_K)$ by $T(K) + T^0(\mathcal{O}_{\mathbb{A}_K})$ is finitely generated by Propositions 5.1 and 2.3.

**Proposition 6.1.** In the exact sequence

$$0 \to \frac{T(\mathbb{A}_K)^1}{T(K) + T^0(\mathcal{O}_{\mathbb{A}_K})} \to \text{Cl}(T^0) \to \frac{T(\mathbb{A}_K)}{T(\mathbb{A}_K)^1} \to 0,$$

the first term is the torsion part of $\text{Cl}(T^0)$. In particular, $T(\mathbb{A}_K)^1$ is the inverse image of $\text{Cl}(T^0)_{\text{tor}}$ under the surjection $T(\mathbb{A}_K) \twoheadrightarrow \text{Cl}(T^0)$.

**Proof.** The quotient $T(\mathbb{A}_K)^1/T(K)$ is compact by [Ono61] Theorem 3.1.1]. Hence the first term is finite. The third term is torsion-free by definition. □

Let $\omega$ be a non-zero invariant differential form on $T/K$ of maximal degree. It induces a canonical Haar measure on $\text{Lie}(T)(K_v)$ and $T(K_v)$ for each $v$, which we denote by $\mu_v$. Set

$$P_v(T, t) = \det(1 - \varphi_v t | V_v(T)(-1)^{l_v}).$$

We define the Tamagawa measure $\mu_{T(\mathbb{A}_K)}$ on $T(\mathbb{A}_K)$ by

$$\mu_{T(\mathbb{A}_K)} = \frac{1}{\rho(T) \cdot q^{(g-1)d}} \prod_v P_v(T, N(v)^{-1})^{-1} \mu_v.$$ 

The infinite product evaluated on open compact subgroups absolutely converges [Ono61, Section 3.3] and hence defines a Haar measure on $T(\mathbb{A}_K)$.
which does not depend on the choice of $\omega$ by the product formula. The composite map
\[
T(\mathbb{A}_K) \to \text{Cl}(T^0) \xrightarrow{h_T} \text{Hom}(Y(K), \mathbb{Z}) \xrightarrow{n \to q^n} \text{Hom}(Y(K), \mathbb{R}_>^\times)
\]
is denoted by $\vartheta$ in [Oes84, Chapter I, Section 5.5]. Recall from [Ono61, Section 3.5] and [Oes84, Chapter I, Definition 5.12] that the Tamagawa number $\tau(T)$ is defined as
\[
\tau(T) = \frac{\mu_{T(\mathbb{A}_K)}(T(\mathbb{A}_K)(K)^1/T(K))}{(\log q)^r \cdot \text{Disc}(h_T)}
\]
The correction factor $\text{Disc}(h_T)$ was introduced by Oesterlé [Oes84, Chapter I, Definition 5.9 (b)] and did not appear in [Ono61]. It can be non-trivial [Oes84, Chapter I, Remark 5.7].

**Proposition 6.2.**
\[
\tau(T) = \frac{\# \text{Cl}(T^0)_{\text{tor}} \cdot q^{\chi(S,\text{Lie}^0)}}{\# T^0(S) \cdot \rho(T) \cdot (\log q)^r \cdot \text{Disc}(h_T)}.
\]

**Proof.** By Proposition 6.1, we have an exact sequence
\[
0 \to T^0(O_{\mathbb{A}_K}) \xrightarrow{} T(\mathbb{A}_K) \xrightarrow{} T^0(K) \to \text{Cl}(T^0)_{\text{tor}} \to 0.
\]
Hence $\tau(T) \cdot (\log q)^r \cdot \text{Disc}(h_T) = \mu_{T(\mathbb{A}_K)}(T(\mathbb{A}_K)(K)^1/T(K))$ can be written as
\[
\frac{\# \text{Cl}(T^0)_{\text{tor}}}{\# T^0(S) \cdot \rho(T)} \cdot \mu_{T(\mathbb{A}_K)}(T^0(O_{\mathbb{A}_K}))
\]
\[
= \frac{\# \text{Cl}(T^0)_{\text{tor}}}{\# T^0(S) \cdot \rho(T) \cdot q^{(g-1)d} \cdot \prod_v P_v(T, N(v)^{-1})^{-1} \mu_v(T^0(O_v))}
\]
To calculate the factors in the product term, we have
\[
P_v(T, N(v)^{-1}) = \frac{\# T^0(k(v))}{N(v)^d}
\]
(see the proof of [GS20, Proposition 4.1] for example) and
\[
\mu_v(T^0(O_v)) = \# T^0(k(v)) \cdot \mu_v(T^0(p_v))
\]
\[
= \# T^0(k(v)) \cdot \mu_v(\text{Lie}(T^0)(p_v))
\]
\[
= \frac{\# T^0(k(v))}{N(v)^d} \cdot \mu_v(\text{Lie}(T^0)(O_v))
\]
where $T^0(p_v)$ denotes the kernel of the reduction map $T^0(O_v) \to T^0(k(v))$ and $\text{Lie}(T^0)(p_v)$ similarly. Hence
\[
P_v(T, N(v)^{-1})^{-1} \mu_v(T^0(O_v)) = \mu_v(\text{Lie}(T^0)(O_v))
\]
\[
= N(v)^{-v(\omega)}
\]
where \( v(\omega) \) is the order of zero at \( v \) of \( \omega \) as a rational section of \( \det(\text{Lie}(\mathcal{T}^0))^* \). Therefore

\[
\prod_v P_v(T, N(v)^{-1})^{-1} \mu_v(\mathcal{T}_v(0)) = \prod_v q^{-\deg(v) \cdot v(\omega)} = q^{\deg(\text{Lie}(\mathcal{T}^0))} = q^{\chi(S, \text{Lie} \mathcal{T}^0) - (1-g)d},
\]

where the last equality is the Riemann-Roch theorem. \( \Box \)

**Proposition 6.3.**

\[
\tau(T) = \frac{\#H^1(K, Y)}{\#\Pi(T)}.
\]

**Proof.** This follows from Theorem 4.6, Propositions 5.5 and 6.2. \( \Box \)

This reproves Ono-Oesterlé’s Tamagawa number formula [Ono63, Section 5, Main theorem], [Oes84, Chapter IV, Corollary 3.3].

### 7. 1-Motives

Let \( M \) be a 1-motive over \( K \). The goal of this section is to combine the results of this paper for tori and lattices with the results of for abelian varieties of [GS20] to obtain a formula for the \( L \)-function of \( M \) at \( s = 1 \).

More precisely, we define a model \( M^\Delta \) over \( S \) such that, assuming the finiteness of the Tate-Shafarevich group of the abelian variety component of \( M \), the groups \( H^*_W(S, M^\Delta) \) are finitely generated, and the leading coefficient of the \( L \)-function of \( M \) at \( s = 1 \) can be expressed in terms of \( H^*_W(S, M^\Delta) \).

Let \( M = [X \to G] \) be a 1-motive over \( K \), where \( X \) and \( G \) are a lattice and a semi-abelian variety over \( K \) placed in degree \(-1\) and \( 0 \), respectively. Let \( T \) be the torus part of \( G \) and \( A \) the abelian variety quotient of \( G \). Let \( Y \) be the character lattice of \( T \). Denote \( \mathcal{X} = j_* X \), where \( j : \text{Spec} \, K \to S \). Denote the Néron and connected Néron model of \( G \) over \( S \) by \( \mathcal{G} \) and \( G^0 \), respectively. Define \( \mathcal{X}^\Delta \) by the fiber product

\[
\begin{array}{ccc}
\mathcal{X}^\Delta & \longrightarrow & G^0 \\
\downarrow & & \downarrow \\
\mathcal{X} & \longrightarrow & \mathcal{G}.
\end{array}
\]

Note that despite the notation, the scheme \( \mathcal{X}^\Delta \) depends not only on \( \mathcal{X} \) but the whole data \( M = [X \to G] \). The fiber \( \mathcal{X}_v^\Delta \) of \( \mathcal{X}^\Delta \) at a place \( v \) is the kernel of the morphism \( \mathcal{X}_v \to \pi_0(\mathcal{G}_v) \) over \( k(v) \). Therefore the group of geometric points of \( \mathcal{X}_v^\Delta \) is the kernel of the map \( X(K_v^{sh}) \to \pi_0(\mathcal{G}_v)(k(v)) \).

**Proposition 7.1.** The scheme \( \mathcal{X}^\Delta \) is a \( \mathbb{Z} \)-constructible étale subsheaf of \( \mathcal{X} \) such that the quotient \( \mathcal{X}/\mathcal{X}^\Delta \) is supported on finitely many closed points of \( S \).
Proof. It is enough to show that the morphism $\mathcal{X}_v \to \pi_0(\mathcal{G}_v)$ over $k(v)$ is zero for almost all places $v$. We have an exact sequence

$$0 \to \mathcal{G}^0 \to \mathcal{G} \to \bigoplus_v i_{v,*} \pi_0(\mathcal{G}_v) \to 0$$

on $S_{\text{ct}}$. We want to show that the induced morphism $\mathcal{X} \to \bigoplus_v i_{v,*} \pi_0(\mathcal{G}_v)$ factors through a finite partial direct sum. Let $U \subseteq S$ be a dense open subscheme such that $\mathcal{X}$ is a lattice over $U$. Let $U'/U$ be a finite étale Galois covering with Galois group $G$ trivializing $\mathcal{X}$. Then the morphism $\mathcal{X} \times_S U \to \bigoplus_{v \notin U} i_{v,*} \pi_0(\mathcal{G}_v)$ over $U$ corresponds to a $G$-module homomorphism $\mathcal{X}(U') \to \bigoplus_{v \notin U} \pi_0(\mathcal{G}_v)(U' \times_U k(v))$. Since $\mathcal{X}(U')$ is finitely generated, this homomorphism indeed factors through a finite partial direct sum. \hfill \Box

Example 7.2.

1. Take $X = \mathbb{Z}$ and $G = \mathbb{G}_m$, so that the morphism $X \to G$ corresponds to a non-zero rational function $f \in K^X$. Then for any place $v$, the map $\mathcal{X}(K^X_v) \to \pi_0(\mathcal{G}_v)(k(v))$ is the map $\mathbb{Z} \to \mathbb{Z}$ given by multiplication by the valuation $v(f)$ of $f$. Therefore $\mathcal{X}_v^\Delta \neq \mathcal{X}_v = \mathbb{Z}$ if and only if $f$ has a zero or pole at $v$, in which case $\mathcal{X}_v^\Delta = 0$. Therefore

$$\mathcal{X}/\mathcal{X}^\Delta = \bigoplus_{v \in \Sigma} i_{v,*} \mathbb{Z},$$

where $\Sigma$ is the set of places where $f$ has a zero or pole.

2. Take $X = \mathbb{Z}$, and assume $G = A$. Then the morphism $X \to G$ corresponds to a rational point $a \in A(K)$. For any place $v$, the map $\mathcal{X}(K^X_v) \to \pi_0(\mathcal{G}_v)(k(v))$ corresponds to the image $a_v \in \pi_0(\mathcal{A}_v)(k(v))$ of $a$. Note that $\pi_0(\mathcal{A}_v)(k(v))$ is a finite group. Therefore $\mathcal{X}_v^\Delta \subseteq \mathcal{X}_v$ is given by the finite index subgroup $n_v \mathbb{Z} \subset \mathbb{Z}$, where $n_v$ is the order of $a_v$. We have $n_v = 1$ for almost all $v$ since $A$ has good reduction almost everywhere and so $\pi_0(\mathcal{A}_v) = 0$ for almost all $v$. Thus

$$\mathcal{X}/\mathcal{X}^\Delta = \bigoplus_v i_{v,*} (\mathbb{Z}/n_v \mathbb{Z})$$

is constructible in this case.

Define $\mathcal{M}^\Delta$ to be the complex

$$\mathcal{M}^\Delta := [\mathcal{X}^\Delta \to \mathcal{G}^0].$$

The $l$-adic representation $V_l(M)$ over $K$ associated with $M$ fits in the exact sequence

$$0 \to V_l(G) \to V_l(M) \to X \otimes \mathbb{Q}_l \to 0.$$  

Define $L(M, s)$ to be the Hasse-Weil $L$-function of $V_l(M)(-1)$.

Example 7.3. The sheaf $\mathcal{X}/\mathcal{X}^\Delta$ non-trivially contributes to $L(M, s)$ and $\chi_W(S, \mathcal{M}^\Delta)$ in general. To see this, first we have

$$L(X, s) = L(\mathcal{X}^\Delta, s) \cdot L(\mathcal{X}/\mathcal{X}^\Delta, s),$$

Define $M^\Delta$ to be the complex

$$M^\Delta := [\mathcal{X}^\Delta \to \mathcal{G}].$$

The $l$-adic representation $V_l(M)$ over $K$ associated with $M$ fits in the exact sequence

$$0 \to V_l(G) \to V_l(M) \to X \otimes \mathbb{Q}_l \to 0.$$  

Define $L(M, s)$ to be the Hasse-Weil $L$-function of $V_l(M)(-1)$.
\[
\chi_W(S, \mathcal{X}) = \chi_W(S, \mathcal{X}^\Delta) \cdot \chi_W(S, \mathcal{X}/\mathcal{X}^\Delta).
\]

In Example 7.2 (1), we have
\[
L(\mathcal{X}/\mathcal{X}^\Delta, s) = \prod_{v \in \Sigma} (1 - N(v)^{-s})^{-1}.
\]

This has a pole of order \(\#\Sigma\) at \(s = 0\). We have
\[
\lim_{s \to 0} L(\mathcal{X}/\mathcal{X}^\Delta, s) \cdot s^{\#\Sigma} = \prod_{v \in \Sigma} \left(\log N(v)\right)^{-1} = \prod_{v \in \Sigma} \left(\deg(v)\right)^{-1} \cdot (\log q)^{-\#\Sigma}.
\]

Also
\[
\chi_W(S, \mathcal{X}/\mathcal{X}^\Delta) = \prod_{v \in \Sigma} (\deg(v))^{-1}.
\]

This is consistent with Theorem 3.1.

**Proposition 7.4.** The complex of sheaves on \(S_{\text{et}}\)
\[
(7.1) \quad 0 \to T^0 \to G^0 \to A^0 \to 0
\]
is exact at \(T^0\) and \(A^0\), and its cohomology at \(G^0\) is an étale skyscraper sheaf with finite stalks.

Note that the exactness of the sequence is not considered in the category of group schemes over \(S\). For example, the morphism \(T^0 \to G^0\) may not be a closed immersion as noted in [Cha00, Remark 4.8 (b)].

**Proof.** For any \(v \in S\), the stalk of \(R^1 j_* T\) at \(\overline{v} = \text{Spec} k(v)\) is \(H^1(K_v^{sh}, T)\), which is zero by [Ser79, Chapter X, Section 7, “Application”]. Hence we have an exact sequence \(0 \to T \to G \to A \to 0\) in \(\text{Ab}(S_{\text{et}})\). In particular, the morphism \(T^0 \to G^0\) is injective in \(\text{Ab}(S_{\text{et}})\). For almost all \(v\), the sequence (7.1) pulled back to \(O_v\) is exact. Let \(C_v\) and \(D_v\) the cohomology of the complex
\[
0 \to T^0(O_v^{sh}) \to G^0(O_v^{sh}) \to A^0(O_v^{sh}) \to 0
\]
in the middle and on the right, respectively. It suffices to show that \(C_v\) is finite and \(D_v\) is zero. It follows from the diagram
\[
\begin{array}{ccccccc}
0 & & 0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & T^0(O_v^{sh}) & \longrightarrow & G^0(O_v^{sh}) & \longrightarrow & A^0(O_v^{sh}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & T(O_v^{sh}) & \longrightarrow & G(O_v^{sh}) & \longrightarrow & A(O_v^{sh}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \pi_0(T_v)(k(v)) & \longrightarrow & \pi_0(G_v)(k(v)) & \longrightarrow & \pi_0(A_v)(k(v)) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0, & & 0,
\end{array}
\]
with exact columns and exact middle row that

\[ C_v \cong \ker \left( \pi_0(T_v)(k(v)) \to \pi_0(G_v)(k(v)) \right) \]

\[ D_v \cong H \left( \pi_0(T_v)(k(v)) \to \pi_0(G_v)(k(v)) \to \pi_0(A_v)(k(v)) \right) \]

The groups \( \pi_0(T_v)(k(v)) \) and \( \pi_0(G_v)(k(v)) \) are finitely generated by [HNT11, Proposition 3.5] for example. Therefore \( C_v \) and \( D_v \) are finitely generated, and do not change if \( O_v^{sh} \) is replaced by the maximal unramified extension \( O_v^{ur} \) of \( O_v \) or its completion \( \hat{O}_v^{ur} \). The group \( G^0(O_v^{ur}) \) is the union of profinite subgroups \( G^0(O_L) \), where \( L \) runs through finite subextensions of \( K_v^{ur}/K_v \).

Similar statements hold for \( T^0(O_v^{ur}) \) and \( A^0(O_v^{ur}) \). Hence \( C_v \) and \( D_v \) are unions of profinite subgroups and thus they are finite.

It remains to prove that \( D_v = 0 \). Let \( n = \# \text{Aut}(D_v) \). Then \( \varphi^n_v \) acts trivially on \( D_v = A^0(\hat{O}_v^{ur})/G^0(\hat{O}_v^{ur}) \), so that \( \varphi^n_v - 1 \) is the zero map on \( D_v \). But \( \varphi^n_v - 1 \) is surjective on \( A^0(\hat{O}_v^{ur}) \) by Lang’s theorem. □

Recall that \( Y \) denotes the character lattice of \( T \).

**Corollary 7.5.**

1. The groups \( H^*_W(S, M^\Delta) \) have finite ranks (namely, they become finite-dimensional after \( \otimes \mathbb{Q} \)) and we have

\[
r_M := - \sum_i (-1)^i \cdot i \cdot \text{rk} H^i_W(S, M^\Delta)
= r_T + r_A - r_{X^\Delta}
= \text{rk} A(K) - \text{rk} X(K) - \text{rk} Y(K)
\]

+ \( \sum_v \text{rk}(\mathcal{X}/\mathcal{X}^\Delta)(k(v)) \).

2. Assuming the finiteness of the Tate-Shafarevich group of the abelian variety component of \( M \), the groups \( H^*_W(S, M^\Delta) \) are finitely generated abelian groups.

Note that \( r_T = - \text{rk} Y(K) \) by Corollary [2.7] and \( r_A = \text{rk} A(K) \) by [GS20].

**Proof.** By Proposition 7.4, we have a long exact sequence

\[
\cdots \to H^i_W(S, T^0) \to H^i_W(S, G^0) \to H^i_W(S, A^0) \to \cdots
\]

up to finite abelian groups. If III(\( A \)) is finite, then the groups \( H^*_W(S, A^0) \) are finitely generated by [GS20] Theorem 1.1]. As \( M^\Delta = [\mathcal{X}^\Delta \to G^0] \), the result follows.

**Proposition 7.6.** We have

\[
\chi(S, \text{Lie} G^0) = \chi(S, \text{Lie} T^0) + \chi(S, \text{Lie} A^0).
\]
Proof. By the Riemann-Roch formula, it is enough to show that
\[ \deg \operatorname{Lie} G^0 = \deg \operatorname{Lie} T^0 + \deg \operatorname{Lie} A^0. \]
For this, it is enough to show that
\[ \det \operatorname{Lie} G^0 \cong \det \operatorname{Lie} T^0 \otimes \mathcal{O}_S \det \operatorname{Lie} A^0 \]
as line subbundles of the rank one \( K \)-vector space
\[ \det \operatorname{Lie} G \cong \det \operatorname{Lie} T \otimes_K \det \operatorname{Lie} A. \]
But this is [Cha00, Theorem 4.1, Question 8.1]. \( \square \)

**Proposition 7.7.** We have
\[ L(M, s) = L(X^\Delta, s - 1) \cdot L(T, s) \cdot L(A, s). \]

**Proof.** For each place \( v \), we have an exact sequence
\[ 0 \to V_i(G)^I_v \to V_i(M)^I_v \to (X \otimes \mathbb{Q}_l)^I_v \to H^1(K_v^{ur}, V_i(G)). \]
The last term contains the \( l \)-adic completion of \( G(K_v^{ur}) \) tensored with \( \mathbb{Q}_l \). We have a commutative diagram
\[ \begin{array}{ccc}
X(K_v^{ur}) & \longrightarrow & G(K_v^{ur}) \\
\downarrow & & \downarrow \\
(X \otimes \mathbb{Q}_l)^I_v & \longrightarrow & H^1(K_v^{ur}, V_i(G)).
\end{array} \]
Hence the lower horizontal morphism factors through \( \pi_0(G_v)(\overline{k(v)}) \otimes \mathbb{Q}_l \) and is equal to the map \( X(K_v^{ur}) \to \pi_0(G_v)(\overline{k(v)}) \) tensored with \( \mathbb{Q}_l \). The kernel of \( X(K_v^{ur}) \to \pi_0(G_v)(\overline{k(v)}) \) is \( X^\Delta(\overline{k(v)}) \) by definition. Therefore we have an exact sequence
\[ 0 \to V_i(G)^I_v \to V_i(M)^I_v \to X^\Delta(\overline{k(v)}) \otimes \mathbb{Q}_l \to 0. \]
This implies
\[ L(M, s) = L(G, s) \cdot L(X^\Delta, s - 1). \]
Proposition 7.3 implies that the sequence
\[ 0 \to V_i(T)^I_v \to V_i(G)^I_v \to V_i(A)^I_v \to 0 \]
is exact since \( V_i(G)^I_v \cong V_i(G^0(\mathcal{O}_s^{sh})) \) and so on. Hence \( L(G, s) = L(T, s) \cdot L(A, s). \) \( \square \)

**Theorem 7.8.** Assume that \( \text{III}(A) \) is finite. Then the groups \( H^*_W(S, \mathcal{M}^\Delta) \) are finitely generated and
\[ \lim_{s \to 1} \frac{L(M, s)}{(s - 1)^r_M} = (-1)^{\chi_Y(K)} \cdot \chi_W(S, M^\Delta)^{-1} \cdot q^{\chi_{S, \operatorname{Lie} G^0}} \cdot (\log q)^r_M. \]
Proof. The finiteness of $\text{III}(A)$ implies the finite generation of $H^*_W(S,A^0)$ by [GS20, Theorem 1.1]. We have

$$\chi_W(S,M^\Delta) = \chi_W(S,X^\Delta)^{-1} \cdot \chi_W(S,G^0) = \chi_W(S,X^\Delta)^{-1} \cdot \chi_W(S,T^0) \cdot \chi_W(S,A^0)$$

by Proposition 7.4. With Propositions 7.5, 7.6 and 7.7, the theorem reduces to Theorem 4.6 for $T$, Theorem 3.1 for $X^\Delta$ and the Weil-étale BSD formula [GS20, Theorem 1.1]

$$\lim_{s \to 1} \frac{L(A,s)}{(s-1)^{\text{rk} A(K)}} = \chi_W(S,A^0)^{-1} \cdot q^{\chi(S,\text{Lie} A^0)} \cdot (\log q)^{\text{rk} A(K)}$$

for $A$. □

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