Convolutional codes from unit schemes

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Abstract

Algebraic methods for the construction, design and analysis of series of convolutional codes are developed. The methods use row or block structures of matrices. New codes and infinite series of codes are presented. Codes with specific properties are designed and analysed and examples of various types are given. Free distances and lower bounds on free distances are proved algebraically and these can be at or near the maximum free distances attainable for their types. Methods for designing series of LDPC (low density parity check) convolutional codes are derived. Design methods for self-dual and dual-containing convolutional codes are included.

1 Introduction

Convolutional codes are error-correcting codes which are used extensively in many applications including digital video, radio, mobile communication, and satellite/space communications.

There is a huge and expanding literature on convolutional codes; basic constructions and properties may be found in [1], [12], [13], [14] and many others. The paper [14] is an often quoted source of information on convolutional codes wherein is mentioned the lack of algebraic methods for their construction and that many of the existing ones have been found by search methods, which limits their size and availability. Algebraic implementations, considerations and properties of convolutional codes are our main concerns here. For a nice detailed analysis of the different definitions of convolutional codes found in the literature see [13]. We are interested in algebraic formulations.

A convolutional code may equivalently and conveniently be described as follows; details may be found in [4] and [18]. A convolutional code \( C \) over a field \( F \) with parameters \((n, r, \delta; \mu)\) is a submodule of \( F[z]^n \) generated by a reduced basic matrix \( G[z] = (g_{ij}) \in F[z]^{r\times n} \) where \( n \) is the length, \( \delta = \sum_{i=1}^{k} \delta_i \) is the degree where \( \delta_i = \max_{1 \leq j \leq r} \deg g_{ij} \), and \( \mu = \max_{1 \leq i \leq n} \delta_i \) is the memory. The parameters \((n, r, \delta; \mu, d_f)\) will also be used for such codes when \( d_f \) denotes the free distance and \((n, r, \delta; \mu, \geq d_f)\) will be used to denote a code with free distance \( \geq d_f \).

A convolutional code may equivalently and conveniently be described as follows; details may be found in [4] and [18]. A convolutional code \( C \) of length \( n \) and dimension \( k \) is a direct summand of \( F[z]^n \) of rank \( k \). Here \( F[z] \) is the polynomial ring over \( F \) and \( F[z]^n = \{(v_1, v_2, \ldots, v_n) : v_i \in F[z]\} \). Suppose \( V \) is a submodule of \( F[z]^n \) and that \( \{v_1, \ldots, v_r\} \subset F[z]^n \) forms a generating set for \( V \). Then \( V = \text{im} \ M = \{uM : u \in F[z]^r\} \) where \( G[z] = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_r \end{pmatrix} \in F[z]^{r\times n} \). This \( G[z] \) is called a generating matrix of \( V \). A generating matrix \( G[z] \in F[z]_{r\times n} \) having rank \( r \) is called a generator or encoder matrix of \( C \). A matrix \( H \in F[z]_{n \times (n-k)} \) satisfying \( C = \ker H = \{v \in F[z]^n : vH = 0\} \) is said to be a control matrix or check matrix of the code \( C \).

The maximum distance attainable by an \((n, r)\) linear code is \((n - r + 1)\) and this is known as the Singleton bound of the linear code, [14]. A linear code \((n, r, n - r + 1)\) attaining the bound is called an mds (maximum distance separable) code. By [17] the maximum free distance attainable by an \((n, r; \delta; \mu)\) code is \((n - r)(\delta/k + 1) + \delta + 1\). (The Singleton bound corresponds to the case \( \delta = 0 \), that is, to a system with zero memory; a linear code.) The bound, \((n - r)(\lceil \delta/k \rceil + 1) + \delta + 1\), on convolutional codes will be referred to as the generalised Singleton bound, and is denoted by GSB.

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For a given generator matrix \( G[z] \in F[z]_{r \times n} \), a codeword is of the form \( u(z)G[z] \) with \( u(z) \in F[z]^r \). Now \( u(z) \) will be referred to as the \textit{inputted word} to the codeword. Then \( u(z) = \sum \alpha_i z^i \) for vectors \( \alpha_i \in F^r \); the \( \alpha_i \) are the \textit{components} of \( u(z) \). The \textit{support} of \( u(z) \) is the number of non-zero (vector) components of \( u(z) \).

In [17] a non-constructive proof is given that for any prime \( p \) and positive integers \( k < n, \delta \) there exists a rate \( k/n \) convolutional code of degree \( \delta \) over some suitably big field whose free distance is equal to the GSB. In [28] explicit examples over suitably big fields are given.

Attributes of the methods here include:

- The constructions are relatively easy to describe and implement.
- The free distances and lower bounds on the free distances may often be proven algebraically.
- Codes are constructed in which the distance of the codeword increases with the support of an inputted word.
- Codes may be explicitly designed over fields of prime order in which cases modular arithmetic may be used.
- A number of different codes with the same parameters can be derived from a single unit scheme type.
- The methods allow the design of series of ‘good’ convolutional codes.
- For given \( n \) and prime \( q \) there exists a unit scheme over a finite field of characteristic \( q \) from which ‘good’ convolutional codes of length \( p \geq n \) may be designed.
- Convolutional codes may be decoded by known techniques, such as Viterbi decoding or sequential decoding, [11] Chapter 11, and algebraic decoding algorithms are known to exist, see [5], [19], and [24] for some of those introduced here.
- Self-dual and dual-containing convolutional codes may be designed and analysed by choosing the unit systems to have particular properties. Dual-containing codes are important for the construction of quantum codes, [2].
- Series of LDPC (low density parity check) Convolutional Codes may be designed and analysed by the methods. These may be designed so that the check/control matrices have no short cycles.

### 1.1 Organisation

Section 1.2 gives further background material. Section 2 describes the general methods and proves results on these. Section 3 discusses the Chebotarëv property for matrices, showing how using such matrices leads to good free distances for constructed codes. Section 3.3 gives fundamental examples which have intrinsic interest and can also serve as prototypes for longer length schemes; this section could now be consulted in order to see the scope of the constructions. Section 3.4 provides details on the designs given in Section 3.3.

Section 4 generalises the row structure design method to a block design structure method. Section 4.2 describes the design methods for LDPC convolutional codes. Section 4.3 deals with the design and resulting properties of certain self-dual and dual-containing codes.

Some of the block cases overlap some of those in [16].

So as to reduce the length of the paper some proofs are omitted, particularly some of the algebraic proofs of the distances attained.

### 1.2 Background

Background on algebra may be found in [15]. Recall the following from [20]: further details may be found in expanded book form within [21]. Let \( R_{n \times n} \) denote the ring of \( n \times n \) matrices with entries from \( R \), a ring usually a field but not restricted to such. Suppose \( UV = 1 \) in \( R_{n \times n} \). Taking any \( r \)
rows of $U$ as a generator matrix defines an $(n,r)$ code and the check matrix is obtained by deleting the corresponding columns of $V$. Let such a code be denoted by $C_r$. When $R$ is a field and the matrix $V$ has the property that the determinant of any square submatrix of $V$ is non-zero then any such code $C_r$ is an mds $(n,r,n-r+1)$ linear code, see [3] for details.

Convolutional codes have been constructed in [10] using $UV = 1$ in which $\{U,V\}$ are polynomial or group rings and further details on these may be obtained in book format at [22].

1.2.1 mds linear codes

One of the methods in [3] for the construction of series of mds linear codes essentially choose rows of certain Fourier matrices where the Fourier matrices, over a finite or an infinite field in which it exists, have the property that the determinant of any square submatrix is non-zero. Finite fields over which Fourier matrices exist with this property are determined in [3].

Theorem 1.1 Suppose $UV = 1$ in $F_{n \times n}$ where $F$ is a field and that the determinant of any square submatrix of $V$ is non-zero. Then a code $C_r$ formed using any $r$ rows of $U$ has distance $(n-r+1)$; it is thus an $(n,r,n-r+1)$ mds code.

Proof: The proof follows for example from Theorem 3.2.2/Corollary 3.2.3 of [1] as the check matrix is obtained using certain columns of $V$, see [20], and any square submatrix of $V$ has non-zero determinant. □

Choosing any $r$ rows of $U$ allows the construction of many different $(n,r)$ codes from an $(n \times n)$ unit system $UV = I$ and when $V$ satisfies the conditions as in Theorem 1.1 each code is mds. In Section 2 and in Section 3 convolutional codes are constructed from a system $UV = 1$. When $V$ has this subdeterminant property the convolutional codes constructed will have good free distances.

2 Convolutional code design from unit schemes

Suppose that $UV = 1$ in $F_{n \times n}$. Denote the rows of $U$ by $\{e_0,e_1,\ldots,e_{n-1}\}$ and the columns of $V$ by $\{f_0,f_1,\ldots,f_{n-1}\}$. Then $e_if_j = \delta_{ij}$, the Kronecker delta. Choose any $r$ rows of $U$ to form an $r \times n$ matrix $E$ which then has rank $r$. Any order on the rows may be chosen with which to construct the $E$ and there are $\left(\begin{array}{c}n \\ r \end{array}\right)$ ways of choosing $r$ rows. Suppose $\{E_0,E_1,\ldots,E_s\}$ are $s$ such $r \times n$ matrices. Define

$$G[z] = E_0 + E_1z + \ldots + E_sz^s. \quad (1)$$

Consider $G[z]$ to be a generating matrix for a convolutional code. Not every choice for the $E_i$ will be suitable or useful. It is desirable to choose the $E_i$ so that $G[z]$ is noncatastrophic. For definition of noncatastrophic see [1] or [14] or others. Here it is sufficient to note, see [13], that $G[z]$ is noncatastrophic if it has a polynomial right inverse, that is if there exists a polynomial $n \times r$ matrix $H[z]$ of rank $r$ such that $G[z]H[z] = I_r$, the identity $r \times r$ matrix.

For a given $n \times n$ system $UV = 1$ and a given $r < n$ the method allows the construction of codes of rate $r/n$.

Say $e_j \in E_i$ if $E_i$ contains the row $e_j$ and $e_j \not\subset E_i$ to mean that $E_i$ does not contain the row $e_j$. Define $e_i^T = f_i$ and $E_j^T$ to be the transpose of $E_j$ with $e_i^T$ replaced by $f_i$. For example if $E = \begin{pmatrix} e_0 \\ e_1 \\ e_3 \end{pmatrix}$ then $E^T = (f_0,f_1,f_3) = (e_0^*,e_1^*,e_3^*)$.

When $E_0$ is formed from $e_i$ none of which are contained in any of the other $E_i,i \geq 1$ then the resulting matrix will be noncatastrophic; this is done by showing it has a one-sided polynomial inverse.

Proposition 2.1 Suppose $G[z] = E_0 + E_1z + \ldots + E_sz^s$ and that $e_j \in E_0$ implies $e_j \not\subset E_i$ for $i \neq 0$. Then $G[z]$ is a noncatastrophic generator matrix for a convolutional $(n,r)$ code.

Proof: By [14] if there exists a polynomial matrix $H[z]$ such that $G[z]H[z] = I_r$, the identity $r \times r$ matrix, then $G[z]$ is noncatastrophic and it must then have rank $r$. Now $e_if_j = \delta_{ij}$ and thus $E_iE_0^* = \delta_{0i}I_r$ as no row of $E_i$ is a row of $E_0$ for $i \neq 0$. 1 then follows that $G[z]E_0^* = I_r$. □
We can do better:

**Theorem 2.1** Suppose there exists a row $e$ such that $e \in E_0$ but $e \not\in E_i$ for any $i \geq 1$. Then $G[z]$ is a noncatastrophic generator matrix for a convolutional code.

**Proof:** We shall assume that $E_0$ has its first row entry which is not in any other $E_i$, $i \geq 1$. The proof for any other cases is similar. Thus $E_0 = \begin{pmatrix} e \\ e_{i_1} \\ \vdots \\ e_{i_t} \end{pmatrix}$. Here $t + 1 = r$ where each $E_i$ has size $r \times n$. We show there exists a $H[z]$ such that $G[z]H[z] = I_r \times r$.

Consider $G[z](f, f_1, \ldots, f_w)$ and then construct $H[z]$ as follows. Let $H_0[z] = (f, f_1, \ldots, f_w)$. Suppose $H_{k-1}[z]$ has been constructed for $1 \leq k \leq t$. Let $e_{i_k}$ occur in $E_{m_1}, E_{m_2}, \ldots, E_{m_w}$ at rows $k_1, k_2, \ldots, k_w$ respectively. For each $1 \leq i \leq w$ let $t_i[z] = \alpha_i z^{m_i}$ where $\alpha_i$ is the $n \times r$ vector with columns consisting of $0$'s except for the $k_i$ column which is $f$. Then let $T_k[z] = \sum_{i=1}^{w} t_i[z]$ and $H_k[z] = H_{k-1}[z] - T_k[z]$. Define $H'[z] = H_1[z]$. Then $G[z]H'[z] = I_r \times r$. Thus $G[z]$ is a noncatastrophic generator matrix for a convolutional code.

(In short order the proof is as follows. Consider $G[z](f, f_1, \ldots, f_w)$ and where $f_{i_k}$ appears in a $E_j$ for $j \geq 1$ use the $f$ corresponding to the unique $e \in E_0$ in a position corresponding to the position of the $f_{i_k}$ in $E_j$ together with zero matrices to give a correct size matrix which will offset the multiplication of $E_j$ with $f_{i,k}$.)

A control matrix for such a noncatastrophic generator matrix may explicitly be derived.

Here is an illustrative example. Suppose $U, V \in F_{7 \times 7}$ satisfies $UV = I_7$ where $U = \begin{pmatrix} e_0 \\ e_1 \\ \vdots \\ e_6 \end{pmatrix}$ and $V = (f_0, f_1, \ldots, f_5) = (e_0, \ldots, e_6)$ are the row and column structures of $U, V$ respectively. Consider $G[z] = \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{pmatrix} = \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 + e_5 z^2 \end{pmatrix}$. Then $G[z]$ is a noncatastrophic generator matrix of rank 2 for a convolutional code by Proposition 2.1. Set $H[z] = (f_2, f_3, f_4, f_5, f_6) - (f_0, f_1, 0, 0, 0)z - (0, 0, f_0, f_1, 0)z^2$ where 0 denotes the zero $7 \times 1$ vector. Then $H[z]$ is a $7 \times 5$ matrix with $G[z]H[z] = 0_{2 \times 5}$. A left inverse for $H[z]$ is easily constructed and so in particular it has rank 5.

There are other cases besides those in Theorem 2.1 for which $G[z]$ is noncatastrophic.

Permuting the rows in (1) gives generator matrices for different convolutional codes. Some of these are equivalent but not many; unless the same permutation is made on each of the $E_i$ non-equivalent codes are obtained.

### 2.0.2 A general design method

Suppose it is desired for given $n, r, \delta, \mu$ with $\delta = r\mu, r < n$ to design a $(n, r, \delta; \mu)$ code. Find a suitable $n \times n$ system $U, V$ with $UV = 1$. Choose $E_0, E_1, \ldots, E_{\mu}$ where each $E_i$ consists of $r$ rows of $U$. To ensure that the resultant generating matrix is noncatastrophic it is enough by Theorem 2.1 to reserve one of the rows to be in $E_0$ and not in any other $E_i$. These $E_i$ can be different choices of $r$ rows of $U$ if $\mu > \binom{n-1}{r}$. If $\mu > \binom{n-1}{r}$ then choose the $E_i$ for $1 \leq i \leq \binom{n-1}{r}$ to be different choices then permutations of the rows of an $E_i$ ($i \geq 1$) already chosen $m$ are used and if (in the unlikely case that these are not sufficient $E_i$) repeats of the choices of the $E_i$ constructed may then be used. Eventually $E_0, E_1, \ldots, E_{\mu}$ will be selected. Form $G[z] = E_0 + E_1 z + \ldots + E_{\mu} z^\mu$ and $G[z]$ is a noncatastrophic generator matrix for a convolutional $(n, r, \delta; \mu)$ code where $\delta = r\mu$.

An algebraic proof of the bounds on the free distances may sometimes be given. Proofs for the free distances depend on Theorem 5.2 and in particular on its Corollary 5.3.

Matrices, necessarily invertible, with additional properties such as being unitary or orthogonal may also be chosen to design suitable convolutional codes with specific properties (such as for example being LDPC); see Section 4 below for developments.
Most of the convolutional codes in use and available have been constructed on a case by case basis and/or by computer; computer techniques are now often beyond the range of computers.

See also Section 3.2 for specific design methods using matrices satisfying the Chebotarëv property. These have good distance properties.

So find your favourite matrix, Fourier or otherwise, satisfying the Chebotarëv property or not or with specific suitable properties and begin designing (good) convolutional codes by the methods.

## 3 Chebotarëv Property

Consider cases where $UV = 1$ and $V$ has the Chebotarëv property. This property is defined in section 3.1 below. Certain Fourier matrices have this property. Then the matrix formed from any $r$ rows of $U$ is a generator matrix for an mds $(n, r, n-r+1)$ (linear) code. [8].

Using $U, V$ where $V$ has the Chebotarëv property with which to construct convolutional codes by the method of Section 2 will give good free distances. The free distances and/or lower bounds on the free distances may often be proven algebraically and the codes are relatively easy to implement and simulate.

### 3.1 Fourier matrices

Suppose a primitive $n^{th}$ root of unity, $\alpha$, exists in a field $K$. The Fourier $n \times n$ matrix $F_n$ over $K$ is

$$F_n = \begin{pmatrix}
1 & 1 & 1 & \ldots & 1 \\
1 & \alpha & \alpha^2 & \ldots & \alpha^{(n-1)} \\
1 & \alpha^2 & \alpha^4 & \ldots & \alpha^{2(n-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \alpha^{n-1} & \alpha^{2(n-1)} & \ldots & \alpha^{(n-1)(n-1)}
\end{pmatrix}.$$  

Defining

$$F_n^* = \begin{pmatrix}
1 & 1 & 1 & \ldots & 1 \\
1 & \alpha^{-1} & \alpha^{-2} & \ldots & \alpha^{-(n-1)} \\
1 & \alpha^{-2} & \alpha^{-4} & \ldots & \alpha^{-(2(n-1))} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \alpha^{-(n-1)} & \alpha^{-(2(n-1))} & \ldots & \alpha^{-(n-1)(n-1)}
\end{pmatrix}.$$  

then $F_nF_n^* = nI$ which gives a formula for the inverse of $F_n$. As $n$ must then divide $(|K| - 1)$, the inverse of $\alpha$ exists in $K$.

It is clear that $\alpha^{-1}$ is a primitive $n^{th}$ root of 1 and the matrix $F_n^*$ is also a Fourier matrix over $K$. Now $F_n$ may be normalised by dividing by $\sqrt{n}$, when it exists, to give a unitary matrix but this is not necessary for our purposes.

Chebotarëv’s Theorem is proved for Fourier matrices over the complex numbers. It has been proved and reproved a number of times, see for example [29], [25], [30] or [3]. It has been termed an uncertainty principle in [30].

**Theorem 3.1 (Chebotarëv)**

Suppose that $\omega \in \mathbb{C}$ is a primitive $p^{th}$ root of unity where $p$ is a prime. Let $V$ be the Fourier matrix with $(i,j)$-entry equal to $\omega^{ij}$, for $0 \leq i, j \leq p - 1$. Then all square submatrices of $V$ have nonzero determinant.

Say a matrix $M$ over the ring $K$ has the Chebotarëv property if the determinant of any square submatrix is non-zero. It is clear that if $M$ has the Chebotarëv property then so does $\alpha M$ for any non-zero scalar $\alpha$. In [8] it is shown, using Chebotarëv’s theorem, that the Fourier matrix $F_n$ over a field $K$ of characteristic 0 has the Chebotarëv property for a prime $n$.

Using results of [6], Fourier matrices with the Chebotarëv property over fields of characteristic $\neq 0$ are developed in [8]. These are used in [8] to construct series of mds (maximal distance separable) linear codes over these fields.

The following is a corollary of Theorem 1.1.
Theorem 3.2 Suppose $U$ is an $n \times n$ matrix with rows $\{e_0, e_1, \ldots, e_{n-1}\}$ with $UV = I$ and that $V$ satisfies the Chebotarëv property. Then the code generated by any $r$ of the rows of $U$ is an mds $(n, r, n-r+1)$ linear code.

Corollary 3.1 A non-zero linear combination of $r$ rows of $U$ has distance (= number of non-zero entries) greater than or equal to $n-r+1$.

In another paper [24] efficient decoding techniques for many of the convolutional codes derived here, and for the mds codes of [8], are developed. The decoding algebraic methods are based on techniques of error location as for example in [26].

Any collection of $r$ rows of $U$ may be used to generate an mds $(n, r, n-r+1)$ code when $UV = 1$ and $V$ satisfies the Chebotarëv property. Hence there are $\binom{n}{r}$ mds $(n, r, n-r+1)$ different codes derived from the single unit.

3.2 Code design using matrices with Chebotarëv property

In [8] Fourier matrices over finite fields which satisfy the Chebotarëv property are constructed.

Let $p, q$ be unequal primes where the order of $q \mod p$ is $\phi(p) = p - 1$, and (hence as shown in [8]) $x^{p-1} + x^{p-2} + \ldots + 1$ is irreducible over $GF(q)$. Then the Fourier matrix $F_p$ over $GF(q^{\phi(p)})$ exists and satisfies the Chebotarëv property. There are clearly many such examples and particular ones are given in the various sections of [8].

A Germain prime $p$ is a prime such that $2p + 1$ is also a prime. In such a case the prime $2p + 1$ is called the (corresponding) safe prime. Let $p$ be a Germain prime so that $2p + 1 = q$ is also a prime. It is shown in [8] that the Fourier matrix $F_p$ exists in $GF(q) = Z_q$ and satisfies the Chebotarëv property. Algebra in matrices over $Z_q$ is modular arithmetic.

Suppose now for given $n$ and given prime $q$ it is desired to construct good convolutional codes of length $\geq n$ over a field of characteristic $q$. Let $p$ be the first prime $\geq n$ for which for which the order of $p \mod q$ is $q - 1$. Then construct the Fourier matrix $F_p$ over $GF(q^{\phi(p)})$. Now $F_p$ satisfies the Chebotarëv property and may be used to construct convolutional codes of length $p$ and of varying rates.

For example suppose it is required to construct convolutional codes of length $\geq 16$ over a field of characteristic 2. Now the order of 2 $\mod 17$ is 8 and not 16 but the order of 2 $\mod 19$ is 18 = 19 − 1. Hence construct $F_{19}$ over $GF(2^{18})$ and then construct convolutional codes of length 19 of varying rates over $GF(2^{18})$.

It is unknown if there exist infinitely many Germain primes but it is conjectured that this is the case. ‘Big’ Germain primes are known. Suppose it is desired to design convolutional codes of length $\geq n$ over a field of prime order. Let $p$ be the smallest Germain prime $\geq n$ if such exists. Then construct the Fourier matrix $F_p$ over $Z_{2p+1}$ and from this design convolutional codes of varying rates.

3.3 Series of examples

The following are examples of the types of series of convolutional codes that can be constructed using Fourier matrices satisfying the Chebotarëv property:

1. Length 3: Series of $(3, 1, 2; 2, 9)$ codes attaining the GSB. Series of $(3, 2, 2; 1, 4)$ codes where inputted words of support $\geq 2$ have distance $\geq 5$; the GSB here is 5. The fields here can be $GF(2^2), GF(2^4)$ or $GF(7) = Z_7$. see [8].

2. Length 5:

(a) Series of $(5, 2, 2; 1, 8)$ codes. The GSB for such codes is 10. For an inputted word of support $\geq 2$ the distance of the codeword is $\geq 10$; indeed for a word of support $t$ the code word then has distance $\geq 10 + 2(t - 1)$.

(b) Series of $(5, 2, 4; 2, \geq 12)$ codes. The GSB for such codes is 14.

(c) Series of $(5, 3, 3; 1, \geq 6)$ codes. The GSB for such codes is 8; a codeword for an inputted word of support $\geq 2$ has distance
3.3.1 Basic cases

An initial case is to take $s = n - 1$ in equation (1) and use all the rows of a Fourier matrix satisfying Chebotarëv property when this is available. Thus define $G[z] = e_0 + e_1z + \ldots + e_{n-1}z^{n-1}$ where $F_n$ has rows $e_i$ and satisfies the property. This is a $(n, 1, n - 1; n - 1)$ convolutional code with degree and memory $(n - 1)$. The rate is not very good particularly for large $n$ but indeed the maximum free distance is attained. The maximal free distance attainable by such a code is by \cite{17} $(n - r)(\lceil \delta/r \rceil + 1) + \delta + 1 = (n - 1)(n - 1 + 1) + n - 1 + 1 = n^2$.

**Theorem 3.3** The free distance of the code generated by $G[z]$ above is $n^2$.

The proof is omitted. Note that each $e_i$ generates a $(n, 1, n)$ code and that any non-zero combination of $r$ of the $e_i$ has distance $\geq (n - r + 1)$. The general proof then depends on Corollary 3.1 and is done by showing directly that $f[z]G[z]$ has distance $\geq n^2$ for a polynomial $f[z]$.

By Proposition \cite{2.1} or Theorem 2.1 $G[z]$ is noncatastrophic. It is fairly easy to show this directly and we give an independent proof and produce the control/check matrix.

(d) Series of $(5, 1, 4; 4, 25)$ codes which attain the GSB. The fields here can be $GF(11) = Z_{11}$ or $GF(7^4)$, see \cite{3}.

3. Length 7:

(a) Series of $(7, 3, 3; 1, \geq 10)$ codes. The GSB for such codes is 12. An inputted word of support $t$ has corresponding codeword of distance $\geq 10 + 2(t - 1)$.

(b) Series of $(7, 2, 4; 2, 18)$ codes. The GSB for such codes is 20.

(c) Series of $(7, 2, 6; 3, 24)$ codes. The GSB for such codes is 27.

(d) Series of $(7, 1, 6; 6, 49)$ codes. The GBS is attained but the rate is not good.

The fields here can be $GF(3^6)$ or $GF(5^6)$, see \cite{3}.

4. Length 11:

(a) Series of $(11, 5, 5; 1, \geq 14)$ codes; the GSB for such codes is 18. Inputted words of support $t$ have distance $\geq 14 + 2(t - 1)$.

(b) Series of $(11, 3, 6; 2, \geq 29)$ codes. Series of $(11, 4, 4; 1, 16)$ codes; the GSB here is 19.

(c) Series of $(11, 4, 8; 2, 24)$ codes; the GSB for such codes is 30.

(d) Series of $(11, 5, 10; 2, \geq 20)$ codes; the GSB for such is 29.

5. General $n$ with memory 1: Suppose the Fourier matrix $F_n$ exists over $K$ and satisfies the Chebotarëv property; see \cite{3} for examples of such $n$ and such fields.

Let $k = \lfloor n/2 \rfloor$. Then series of $(n, k; k; 1, \geq n + 3)$ codes are constructed. An inputted word of support $t$ has codeword of distance $n + 3 + 2(t - 1)$. The GSB for such $(n, k; k; 1)$ codes is $3\lceil n/2 \rceil$. The rate is approximately a half.

6. Large examples in modular arithmetic: Use for example Germain primes; these are primes $p$ such that $2p + 1$ is also a prime. Fourier $p \times p$ matrices over $GF(2p + 1)$ satisfying the Chebotarëv property are constructed in \cite{8}. Let $K = GF(227) = Z_{227}$ and $n = 113$. Here 113 is a Germain prime and $2 \times 113 + 1 = 227$ is the corresponding safe prime. Then explicit series of $(113, 56, 56; 1, \geq 116)$ codes exist over $Z_{227}$ and others may be designed.

The GSB used in the above is calculated using $(n - r)(\lceil \delta/r \rceil + \delta + 1)$.

These are some examples of the possible constructions. More details on these are given in Section 3.3.
Lemma 3.1. Let $G[z] = e_0 + e_1 z + \ldots + e_{n-1} z^{n-1}$ as above.

(i) Define $H(z) = e_0$. Then $G[z] H(z) = 1$.

(ii) Define $K[z] = (e_0, e_1^*, \ldots, e_{n-1}^*) - (e_0^*, 0 \ldots, 0) - (0, e_1^*, \ldots, e_{n-1}^*) z - \ldots - (0, 0, \ldots, e_{n-1}^*) z^{n-1}$ where

$\mathbb{G}$ is the $n \times 1$ zero matrix. Then $G[z] K[z] = 0_{1 \times (n-1)}$ and $K[z]$ has rank $n - 1$.

Proof: These follow by direct multiplication on noting $e_i e_j^* = \delta_{ij}$ (Kronecker delta).

Corollary 3.2. $G[z]$ is noncatastrophic.

By permuting the $e_i$ we may obtain $n!$ such different codes.

3.4 Cases: Convolutional Codes from Fourier with Chebotarëv property

3.4.1 Length 3

Suppose the Fourier $3 \times 3$ matrix $F_3$ exists over $K$ and satisfies the Chebotarëv property. As pointed out in [10] cases of such $K$ are $GF(2^2)$, $GF(5^2)$, $GF(7) = \mathbb{Z}_7$. Denote the rows of $F_3$ by $\{e_0, e_1, e_2\}$.

Define $G[z] = e_0 + e_1 z + e_2 z^2$. Then $G[z]$ is a noncatastrophic generator matrix for a $(3,1,3;3,9)$ convolutional code which is MDS.

Consider $GF(2^2)$. Here note that $x^4 + x^3 + x^2 + x + 1$ is irreducible over $\mathbb{Z}_2$ and if $\alpha$ is a primitive element then choose $\omega = \alpha^3$.

In $GF(11)$ it is seen that the order of 2 is 10 with $2^{10} = 1$ and let $\omega = 2^2 = 4$ to get a primitive $5^{th}$ root of unity.

In all cases $F_3 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega & \omega^3 \\ 1 & \omega^3 & \omega^4 & \omega \end{pmatrix}$ and its inverse is such that the determinant of every submatrix non-zero.

Explicitly when for example $K = GF(11)$ then $F_3 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 4 & 4^2 & 4^3 \\ 1 & 4^2 & 4 & 4^3 \\ 1 & 4^3 & 4^2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 4 & 5 & 9 \\ 1 & 5 & 3 & 4 \\ 1 & 9 & 4 & 3 \end{pmatrix}$

is a Fourier matrix over $GF(11)$ which has the Chebotarëv property. The entries are elements of $\mathbb{Z}_{11}$. Let the rows of $F_3$ be denoted by $\{e_0, e_1, e_2, e_3, e_4\}$.

By Theorem 5.3, the code generated by $G[z] = e_0 + e_1 z + e_2 z^2 + e_3 z^3 + e_4 z^4$ is a $(5,1,4;4,25)$ code attaining the GSB and is noncatastrophic.

Consider $E_0 = \begin{pmatrix} e_0 \\ e_1 \end{pmatrix}$, $E_1 = \begin{pmatrix} e_2 \\ e_3 \end{pmatrix}$ and let $G[z] = E_0 + E_1 z$. This is a $(5,2)$ with $\delta = 2, \mu = 1$ which can have at best free distance $(n - r)(\delta/r + 1) + \delta + 1 = 3(2) + 2 + 1 = 9$ by [77].

Proposition 3.1. (i) The matrix $G[z]$ is noncatastrophic.

(ii) The code generated by $G[z]$ has $d_{free} = 8$.

(iii) For any inputted word $f(z)$ of support $\geq 2$ the codeword $f(z) G[z]$ has distance $\geq 10$. 

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Proof: (i) \((E_0 + E_1z)(e_0', e_1') = I_2\)

(ii) For \(1 \times 2\) vectors \(\alpha, f(z) = (\alpha_0 + \alpha_1 z + \ldots + \alpha_r z^r)(E_0 + E_1 z) = \alpha_0 E_0 + \ldots + \alpha_r E_1 z^{r+1}\). Now \(E_0, E_1\) both generate \((5, 2, 4)\) codes and thus free distance of \(f(z)\) is \(\geq 4 + 4 = 8\). It is easy to show that the actual distance 8 is attained by considering \(\alpha_0 G[z]\).

(iii) The proof of this is similar and the general proof is omitted. It is noted that a linear code with generator matrix consisting of rows \(\{e_0, e_1, e_2, e_3\}\) is a \((5, 4, 2)\) code and thus a non-zero linear combination of these has distance \(\geq 2\). A zero combination of these puts restrictions on the vectors which increases the distances at other components.

\[\] 

3.4.3 Length 7

Consider a Fourier \(7 \times 7\) matrix \(F_7\) over a field \(K\) which satisfies the Chebotarëv property. Examples of such fields are given in \([8]\). Denote the rows of \(F_7\) by \(\{e_0, e_1, \ldots, e_6\}\).

Setting \(G[z] = e_0 + e_1 z + e_2 z^2 + \ldots + e_6 z^6\) gives a \((7, 1, 6; 6, 49)\) by Theorem 3.3. Setting \(E_i = \begin{pmatrix} e_i \\ e_{i+1} \end{pmatrix}\) for \(i = 0, 1, 2\) and then letting \(G[z] = E_0 + E_1 z + E_2 z^2\) gives a \((7, 2, 4; 2, 18)\) code. The maximum free distance of such a \((7, 2, 4; 2)\) code is \((5)(4/2 + 1) + 4 + 1 = 20\). By permuting the \(e_i\) we may obtain \(7\) such codes. How many of these are equivalent?

Suppose we wish to obtain a memory 3 code from this structure. Let \(E_0 = \begin{pmatrix} e_0 \\ e_1 \\ e_2 \end{pmatrix}\), \(E_1 = \begin{pmatrix} e_3 \\ e_4 \\ e_5 \end{pmatrix}\), \(E_2 = \begin{pmatrix} e_6 \\ e_7 \\ e_8 \end{pmatrix}\). Note here that \(E_2, E_3\) have \(e_5\) as common. Set \(G[z] = E_0 + E_1 z + E_2 z^2 + E_3 z^3\). Then \(G[z]\) is noncatastrophic follows from Theorem 2.1 but as easily seen \(G[z] E_1^\ast = I_{2 \times 2}\). It may be shown that this is a \((7, 2, 6; 3, 24)\) code. The maximum free distance of such a \((7, 2, 6; 3)\) code is by \([17]\) \((5)(6/2 + 1) + 6 + 1 = 27\).

Setting \(E_0 = \begin{pmatrix} e_0 \\ e_1 \\ e_2 \end{pmatrix}\), \(E_1 = \begin{pmatrix} e_3 \\ e_4 \\ e_5 \end{pmatrix}\) and \(G[z] = E_0 + E_1 z\) gives a noncatastrophic matrix generator of a \((7, 3, 3; 1, 10)\) code. The maximum free distance of such a code is \(4(3/3 + 1) + 3 + 1 = 12\). Also note that \(f(z) G[z]\) has distance \(\geq 12\) for any \(f(z)\) of support \(\geq 2\). Permuting the \(e_i\) gives \(7\) such codes.

Setting \(E_0 = \begin{pmatrix} e_0 \\ e_1 \\ e_2 \end{pmatrix}\), \(E_1 = \begin{pmatrix} e_3 \\ e_4 \\ e_5 \end{pmatrix}\), \(E_2 = \begin{pmatrix} e_6 \\ e_7 \\ e_8 \end{pmatrix}\) and then \(G[z] = E_0 + E_1 z + E_2 z^2\), gives a \((7, 3, 6; 2)\) code. \(G[z]\) is noncatastrophic from Theorem 2.1. The free distance is 15. The maximum free distance of such a code is \((4)(6/3 + 1) + 6 + 1 = 17\).

3.4.4 \(GF(23) = \mathbb{Z}_{23}\)

The next case taken is that of a Fourier matrix over a finite field in which the entries are elements of \(\mathbb{Z}_{23}\).

Now 11 is a Germain prime with safe prime \(23 = 11 \times 2 + 1\) and so by \([8]\) the Fourier matrix \(F_{11}\) over \(\mathbb{Z}_{23}\) exists and satisfies Chebotarëv property.

Denote the rows of the Fourier matrix by \(\{e_0, e_1, \ldots, e_{10}\}\).

Then

1. \(G[z] = e_0 + e_1 z + \ldots + e_{10} z^{10}\) is a noncatastrophic matrix for a \((11, 1)\) code which has the mds free distance \(11^2 = 121\). There are \(11!\) similar such codes obtained by permuting the order of \(0, 1, \ldots, 10\).

2. Let \(E_0 = \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix}\), \(E_1 = \begin{pmatrix} e_5 \\ e_6 \\ e_7 \\ e_8 \\ e_9 \end{pmatrix}\).
Proof: distance A

Let \( \{e_0, e_1, \ldots, e_{10} \} \) be a convolutional code of free distance 14. The maximum distance for such a \((11,5,5)\) convolutional code is \((n-r)(\delta/r+1)+\delta+1 = (6/2) + 5 + 1 = 18\). An inputted word of memory \( \geq 1 \) has distance at least 16. An inputted word of memory \( \geq 2 \) has distance at least 18. Now 11! such codes of this type.

3. Define \( E_0 = \begin{pmatrix} e_0 \\ e_1 \\ e_2 \end{pmatrix}, E_1 = \begin{pmatrix} e_3 \\ e_4 \\ e_5 \end{pmatrix}, E_2 = \begin{pmatrix} e_6 \\ e_7 \\ e_8 \end{pmatrix} \). Define \( G[z] = E_0 + E_1 z + E_2 z^2 \). Then \( G[z] \) is a noncatastrophic generator matrix for a \((11,3,6)\) convolutional code. Its free distance is 27. The maximum free distance for such an \((11,3,6)\) code is \( (n-r)(\delta/r+1)+\delta+1 = 6(3/3+1)+6+1 = 31 \). Now 11! such codes may be constructed by permuting the order of 0, 1, 2, \ldots, 10.

4. Define \( E_i = \begin{pmatrix} e_i \\ e_{i+1} \end{pmatrix} \) for \( i = 0, 1, 2, 3, 4 \). Define \( G[z] = E_0 + E_1 z + E_2 z^2 + E_3 z^3 + E_4 z^4 \). Then \( G[z] \) is a noncatastrophic generator matrix for a \((11,2,8)\) code and it has free distance 50. The maximum distance for such a \((11,2,8)\) code is \( (9)(8/2) + 8 + 1 = 54 \). If the inputted word has memory \( \geq 1 \) then the distance is 52.

In \( GF(23) \) a primitive element is 5 and so \( 5^2 = 2 \) is an element of order 11 from which the Fourier matrix \( F_{11} \) over \( GF(23) \) can be constructed. This gives

\[
F_{11} = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & 2 & 2^2 & \ldots & 2^{10} \\ 1 & 2^2 & 2^4 & \ldots & 2^{20} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 1 & 2^{10} & 2^{20} & \ldots & 2^{100} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & 2 & 4 & \ldots & 12 \\ 1 & 4 & 14 & \ldots & 6 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 1 & 12 & 6 & \ldots & 2 \end{pmatrix}
\]

Consider \( E_0 = \begin{pmatrix} e_0 \\ e_1 \end{pmatrix}, E_1 = \begin{pmatrix} e_2 \\ e_3 \end{pmatrix} \) and let \( G[z] = E_0 + E_1 z \). This is a \((5,2)\) with \( \delta = 2, \mu = 1 \) which can have at best free distance \( (n-k)(\delta/k+1) + \delta + 1 = 3(2) + 2 + 1 = 9 \).

**Proposition 3.2**

(i) The code \( G[z] \) is noncatastrophic.

(ii) The code \( G[z] \) has \( d_{free} = 8 \).

(iii) For any inputted word \( f(z) \) of support \( \geq 2 \) the corresponding codeword has distance \( \geq 9 \).

Let \( G[z] = E_0 + E_1 z + E_2 z^2 \). The free distance of the code generated by \( G[z] \) is 27. Note that a non-zero linear combination of any three of \( \{e_0, \ldots, e_{10}\} \) has distance at least 9, a non-zero linear combination of any 6 of these has distance at least 6 and a non-zero linear combination of 9 of these has distance at least 3 by Corollary 3.11.

3.4.5 A large example:

Consider \( GF(227) \). By [23] the Fourier matrix \( F_{113} \) exists over \( GF(227) \) and satisfies the Chebotarëv property since 113 is a Germain prime with matching safe prime 227. This for example enables the construction of \( \binom{113}{56} \) (different) mds \((113,56,58)\) codes over \( \mathbb{F}_{227} \). The number \( \binom{113}{57} \) is of order \( 10^{12} \) (order \( 2^{100} \)).

**Lemma 3.2** Let \( A, B \) be mds linear \((n,r,n-r+1)\) codes. Then \( G[z] = A + B z \) is generator matrix for a convolutional code of free distance \( d_{free} \geq (n-r+1) \).

**Proof:** Consider \( H[z] = a_0 + a_1 z + \ldots + a_r z^r \) for \( 1 \times r \) vectors \( a_i \) and assume \( a_0 \neq 0, a_t \neq 0 \). Then \( H[z]G[z] = a_0 A + (R[z]) + a_1 B z^{t+1} \) where \( R[z] \) is a polynomial of degree at least 1 and at most \( t \). Now distance \( A \geq (n-r+1) \), distance \( B \geq (n-r+1) \). Thus \( d_{free}, H[z]G[z] \geq (n-r+1)+(n-r+1) = 2(n-r+1) \). □

Thus memory 1 codes rate (nearly) 1/2 with good free distances may be formed from the rows \( \{e_0, e_1, \ldots, e_{n-1}\} \) of matrix whose inverse satisfies the Chebotarëv condition as for example certain Fourier matrices as follows: Set \( r = \lfloor \frac{n}{2} \rfloor \). Let \( E_0 \) be \( r \times n \) formed by having as rows \( r \) of \( e_0, e_1, \ldots e_{n-1} \).
and let $E_1$ be formed by taking $r$ other different of $e_0, e_1, \ldots, e_{n-1}$ as rows. Form $G[z] = E_0 + E_1 z$. By (2.1) $G[z]$ is the generator matrix of a noncatastrophic convolutional code of type $(n, r)$. Its rank is $r$ and its degree $\delta$ and memory $\mu$ are $r$. The mds of such a code is $(n - r)(\delta/r + 1) + \delta + 1 = (n - r)2 + r + 1 = 2n - 2r + r + 1 = 2n - r + 1 = 2(n - r + 1) + r - 1$. The code is less than its possible mds by $r - 1$.

It is clear that many cases and examples may be determined from the general constructions.

4 Block constructions

The methods using rows of matrices in order to construct convolutional codes are now generalised to using blocks of (invertible) matrices. This leads in particular to the algebraic construction of LDPC convolutional codes and self-dual and dual-containing convolutional codes.

(The block constructions may be considered as cases of row constructions where rows can be put together to form blocks of the same size.)

Some of the cases here overlap some of those in [10].

We begin with an illustrative example. Let $U, V$ be $2n \times 2n$ matrices with $UV = 1$ and $U, V$ have block representations as follows: $U = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}$, $V = \begin{pmatrix} C & D \\ I & 0 \end{pmatrix}$ where $A, B$ are block $n \times 2n$ matrices and $C, D$ are block $2n \times n$ matrices. Then $UV = 1$ implies $\begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix} \begin{pmatrix} C & D \\ I & 0 \end{pmatrix} = \begin{pmatrix} AC & AD \\ BC & BD \end{pmatrix} = I_{2n \times 2n}$ and hence $AC = I_{n \times n}, AD = 0_{n \times n}, BC = I_{n \times n}, BD = 0_{n \times 2n}$.

Define $G[z] = A + Bz$; this is a $n \times 2n$ matrix. Then

$$G[z]C = I_{n \times n}G(z)(D - Cz) = 0_{n \times n}$$

Hence $G[z]$ is of rank $n$ with left inverse $C$ and so is a noncatastrophic generator matrix for a $(2n, n; n, 1)$ convolutional code $C$. Also the check/control matrix is $(D - Cz)$ which is also of rank $n$ as may be shown by producing a right inverse for it. Now if $V$ is a low density matrix we have constructed a low density convolutional code; see section 4.2 below for relevant definitions of low density. Let $d(W)$ denote the distance of a linear code generated by $W$.

**Lemma 4.1** $d_{free}C \geq d(A) + d(B)$.

4.1 General block method

The general block method is as follows. It is similar in principle to the row method of Section 2, but has certain constructions in mind.

Suppose that $UV = 1$ in $F_{sn \times sn}$ and that $U, V$ have block structures $U = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_s \end{pmatrix}$ and $V = \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_s \end{pmatrix}$ where the $A_i$ are $n \times sn$ matrices and the $B_j$ are $sn \times n$ matrices. Then $A_i B_j = \delta_{ij}I_n$.

Choose any $r$ of these blocks of $U$ to form an $rn \times sn$ matrix $E$ which then has rank $rn$; any order on the blocks may be chosen with which to construct the $E$. There are $s^r$ ways of choosing $r$ blocks. Suppose $\{E_0, E_1, \ldots, E_s\}$ are $s$ such $rn \times sn$ matrices. Define

$$G[z] = E_0 + E_1 z + \ldots + E_s z^s.$$ 

(2)

Consider $G[z]$ to be the generating matrix for a convolutional code.

Say that $A \in E$ if $A$ occurs as a block in forming $E$ and $A \notin E$ if $A$ does not occur as a block in $E$.

The following may be proved in a similar manner to Proposition 2.1 in Section 2.

**Proposition 4.1** Suppose $G[z] = E_0 + E_1 z + \ldots + E_s z^s$ and that $A_j \in E_0$ implies $A_j \notin E_i$ for $i \neq 0$. Then $G[z]$ is a noncatastrophic generator matrix for a convolutional $(n, r)$ code.

The following may be proved in a similar manner to Theorem 2.1 in Section 2.
Theorem 4.1 Suppose there exists a block $A$ such that $A \in E_0$ but $A \not\in E_i$ for any $i \geq 1$. Then $G[x]$ is a noncatastrophic generator matrix for a convolutional code.

There are other cases where the $G[z]$ is also noncatastrophic.

We illustrate the method for a unit with 3 blocks. Suppose $UV = 1$ are $3n \times 3n$ matrices with

$U = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$, $V = (D, E, F)$ where $A, B, C$ are $n \times 3n$ matrices and $D, E, F$ are $3n \times n$ matrices. This then gives:

$AD = I_{n \times n}, AE = 0_{n \times n}, AF = 0_{n \times n}$
$BD = 0_{n \times n}, BE = I_{n \times n}, BF = 0_{n \times n}$
$CD = 0_{n \times n}, CE = 0_{n \times n}, CF = I_{n \times n}$

Define $G[z] = A + Bz + Cz^2$. Then $G[z]D = I_{n \times n}, G[z](E - Dz) = 0$. Thus $G[z]$ is a noncatastrophic generator matrix a $(3n, n, 2n; 2)$ convolutional code $D$ with check matrix $(E - Dz)$. Let $d(Y, X)$ denotes the distance of the linear code with generator matrix $\begin{pmatrix} Y \\ X \end{pmatrix}$.

Proposition 4.2 $d_{free}D \geq \min\{d(A) + d(A, B) + d(B, C) + d(C), d(A) + d(B) + d(C)\}$.

We can generate $(3n, 2n)$ codes from the unit system as follows. Define $G[z] = \begin{pmatrix} A \\ B \end{pmatrix} + \begin{pmatrix} B \\ C \end{pmatrix} z$.

Proposition 4.3 (i) $G[z]((D, E) - (0_{3n \times 0}, Dz)) = I_{2n}$.

(ii) $G[z](F - Ez + Dz^2) = 0_{2n \times n}$.

Corollary 4.1 If $G[z]$ is a noncatastrophic generator matrix for a $(3n, 2n, 2n; 1)$ convolutional code $K$ with check matrix $H[z] = F - Ez + Dz^2$.

See below for details on LDPC codes but it’s worth mentioning the following at this stage.

Corollary 4.2 $K$ is an LDPC convolutional code when $V$ is low density. If further $V$ has no short cycles then neither does the $K$.

Proposition 4.4 $d_{free}K \geq d(A, B) + d(B, C)$.

Use $G[z] = \begin{pmatrix} A \\ B \\ C \end{pmatrix} z + \begin{pmatrix} C \\ B \end{pmatrix} z^2$ to construct a $(3n, 2n, 4n; 2)$ convolutional code.

4.2 LDPC convolutional codes

Suppose now $UV = 1$ where $V$ is of low density. Say $V$ has low density if and only if $V$ has a small number of elements, compared to its size, in each row and column. Then convolutional codes constructed by the row method of Section 4 or the block method of Section 4 must necessarily be LDPC (low density parity check) convolutional codes. By ensuring that $V$ has no short cycles, which can be done by methods of $[23]$, the LDPC convolutional codes constructed will have no short cycles in their check matrices. It is known that LDPC codes with no short cycles in their check matrices perform well.

The paper $[23]$ gives methods for constructing classes of matrices $U, V$ of arbitrary size over various fields with $UV = 1$ where $V$ is of low density. Such matrices may for example be obtained, $[23]$, from group ring elements $u, v$ with $uv = 1$ in which the support of $v$ as a group ring element is small; corresponding matrices may be obtained by mapping the group ring into a ring of matrices as per $[9]$. Matrices of arbitrary size and over many fields including $GF(2) = \mathbb{Z}_2$ satisfying the conditions may be obtained in this manner. It may also be ensured in the construction that $V$ has no short cycles thus ensuring the codes obtained do not have short cycles in their check matrices.
Since low density implies the length must comparatively be long, to actually write out examples explicitly is more difficult in a research paper but as shown in [23] many such constructions may be formulated.

The following is an example taken from [23] and was used to construct industry standards LDPC linear codes. Here we show how to construct LDPC convolutional codes from the structure.

Consider $\mathbb{Z}_2(C_{204} \times C_4)$ where $C_{204}$ is generated by $g$ and $C_4$ is generated by $h$. Set $v = g^{204-75} + h(g^{204-13} + g^{204-111} + g^{204-168}) + h^2(g^{204-29} + g^{204-34} + g^{204-170}) + h^3(g^{204-27} + g^{204-180})$.

The support of $v$ is 9 and is a low density group ring element. As shown in [23] $v$ has no short cycles. Its inverse $u$ may be easily found but has large support and so is not written out. The matrices corresponding to $u, v$ are denoted $U, V$ (see [23]), which in this case are circulant-by-circulant, and have the forms

$$U = \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{pmatrix}, \quad V = \begin{pmatrix} B_0, B_1, B_2, B_3 \end{pmatrix}$$

where $A_i$ are $204 \times 816$ matrices and $B_j$ are (low density) $816 \times 204$ matrices with $A_iB_j = \delta_{ij}I_{204 \times 204}$. These blocks may now be used to construct LDPC convolutional of various types.

For example $G[z] = A_0 + A_1z + A_2z^2 + A_3z^3$ is a noncatastrophic generator matrix for an LDPC convolutional $(816, 204, 612; 3)$ code with check low density matrix, with no short cycles, $H[z] = (B_1, B_2, B_3) + (B_0, 0, 0)z + (0, B_0, 0)z^2 + (0, 0, B_0)z^3$ where 0 is the zero $816 \times 204$ matrix.

Also $G[z] = (A_0 + A_1 + A_2 + A_3)z$ gives an LDPC $(816, 612, 612; 1)$ code with check low density matrix $H[z] = (B_1 + B_2z + B_3z^2 + B_0z^3).

Similarly $(816, 408, 404; 2)$ LDPC convolutional codes may be produced. Also any permutation of $\{0, 1, 2, 3\}$ may be used in $G[z]H[z]$ and rows of $U$ and corresponding columns of $V$ may also be permuted for further constructions. Higher memory may also be produced but make sure the conditions of Theorem 4.2 are satisfied in the selection.

The group ring constructions of [23] allow the construction of many series of these LDPC convolutional codes.

### 4.3 Self-dual

A convolutional code with generator matrix $G[z]$ which necessarily has size $n \times 2n$ is said to a self-dual code if $G[z]G[z^{-1}]^\top = 0_{n \times n}$.

Consider as before $UV = 1$ with $UV = \begin{pmatrix} A \\ B \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} AC \\ BC \end{pmatrix} = I_{2n \times 2n}$. Then $G[z] = A + Bz$ has check matrix $D - Cz$ as $G[z](D - Cz) = 0_{n \times n}$. We require $(D - Cz)$ to be some $z$ or $z^{-1}$ multiple of $G[z]^\top$. Suppose now $U$ is an orthogonal matrix and then get $C = AT, D = BT$. Now $D - Cz = B^T - A^Tz$ and then $(D - Cz)z^{-1} = -A^T + Bz^{-1}$. In characteristic 2, $-A = A$ and in this case $G[z^{-1}]^\top$ is a check matrix (of rank $n$) for the code and hence the code is self-dual.

**Lemma 4.2** Suppose $U = \begin{pmatrix} A \\ B \end{pmatrix}$ is an orthogonal matrix in characteristic 2. Then the convolutional code with generator matrix $G[z] = A + Bz$ is a self-dual code.

Methods are developed in [11] in which to construct orthogonal matrices of arbitrary long sizes over various fields including fields of characteristic 2. The methods involves constructing such elements in a group ring and then finding the corresponding matrices as per the embedding of the group ring into a ring of matrices.

Here is an example. Consider $\mathbb{Z}_2C_4$ where $C_4$ is the cyclic group generated by $a$. Let $u = a + a^2 + a^3$. Then $u^2 = 1$, and $u^3 = u$. Thus taking the corresponding matrix, $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, 

$$U = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}.$$
we have $UU^T = 1 = I_{4 \times 4}$.

Then $G[z] = A + Bz$ is the generator matrix of a self-dual convolutional $(4, 2, 2; 1)$ code. It is easy to check that the free distance of the code is 4; both $A, B$ are generator matrices for $(4, 2, 2)$ linear codes.

The next example follows methods developed in [11]. Consider $Z_2D_8$ where $D_8 = \langle a, b | a^4 = 1 = b^2, ba = a^{-1}b \rangle$ is the dihedral group of order 8. Let $u = 1 + b + ba$. Then $u^2 = 1, u^T = u$. The matrix of $u$ is

$$
U = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}
$$

and $UU^T = 1$.

Then $G[z] = A + Bz$ is the generator matrix for a self-dual convolutional $(8, 4, 4; 1)$. Its free distance is 6; each of $A, B$ generates a $(8, 4, 3)$ linear code.

Further examples may be generated similar to methods in [11] by finding group ring elements with $u^2 = 1, u^T = u$ and then going over to the matrix representation as per [9]. Using for example $Z_2D_{16}$ will lead to a $(16, 8, 8; 1, 10)$ self-dual convolutional code.

It is known that self-dual and dual-containing codes leads to the construction of quantum codes, see [2].

### 4.4 Dual-containing systems

**Dual-containing convolutional codes** may be constructed as follows.

Let $U$ be an orthogonal $4n \times 4n$ matrix $UU^* = 1$ with $U = \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix}$ and $U^* = (A^*, B^*, C^*, D^*)$ for blocks $A, B, C, D$ of size $n \times 4n$. Define

$$
G[z] = \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} z.
$$

Then $G[z]$ is a noncatastrophic generator matrix for a convolutional $(4n, 3n, 3n; 1)$ code.

Now $G[z] \{ D^* - C^* z + B^* z^2 - A^* z^3 \} = 0_{3n \times n}$ as may easily be verified. Let $H[z^{-1}] = A^* - B^* z^{-1} + C^* z^{-2} - D^* z^{-3}$. Then $H[z^{-1}] z^3 = D^* - C^* z + B^* z^2 - A^* z^3$. Thus $H[z^{-1}]$ is a check matrix for the code also. To show that code with generator $G[z]$ is dual containing it is necessary to show that the code generated by $H[z]$ is contained in the the code $C$ generated by $G[z]$.

Suppose the characteristic of the field is 2. Then $H[z] = \{(I_n, [A, B], 0, 0, I_n) z^2 \} G[z]$ and hence code generated by $H[z] \subseteq C$. Thus $C$ is dual-containing.

Other dual-containing codes may be obtained by permuting $\{A, B, C, D\}$ in $G[z]$.

Here is a specific example. Consider $u = a + a^4 + a^7$ in $Z_2C_8$ where $C_8$ is generated by $a$. Then $u^4 = 1, u^2 \neq 1, u^T = u$. Thus $u^2 = a^2 + a^6$ is orthogonal matrix and is symmetric. The matrix of $u^2$ has the form $U = \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix}$ and $U$ is orthogonal and symmetric. Then $G[z] = \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} z$ determines a dual-containing $(8, 6, 6; 1)$ convolutional code.

$$
A = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}
$$
\[
B = \begin{pmatrix}
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0
\end{pmatrix}, \\
C = \begin{pmatrix}
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1
\end{pmatrix}, \\
D = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1
\end{pmatrix}.
\]

The free distance is 4.

We may also construct self-dual codes from this set-up. Again \(U\) is an orthogonal \(4n \times 4n\) matrix \(UU^* = 1\) with \(U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\) and \(U^* = (A^*, B^*, C^*, D^*)\) for blocks \(A, B, C, D\) of size \(n \times 4n\). Define

\[
G[z] = \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} + \begin{pmatrix} C \\ D \end{pmatrix} z.
\]

Then \(G[z]\{\{C^*, D^*\} - (A^*, B^*)z\} = 0_{2n \times 2n}\) and \(G[z]\) is a generator matrix for convolutional code which is self-dual in characteristic 2. By permuting the \(\{A, B, C, D\}\) other different self-dual convolutional codes may be obtained. Note that \(\{A, B, C, D\}\) must be different as \(U\) is invertible and this this gives 4! self-dual convolutional codes. The distances of the codes depend on the distances of of the linear codes generated by \(\{A, B, C, D\}\).

Here are examples of dual-containing code in characteristic 3. Suppose \(U\) with rows \(e_0, e_1, e_2\) is an orthogonal matrix. Then \(UV = I\) where \(V\) has columns \(e_0^*, e_1^*, e_2^*\). Consider \(G[z] = \begin{pmatrix} e_0 \\ e_1 \\ e_2 \end{pmatrix} + \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} z\). Then \(G[z]\{e_0^2 - e_1^2 z + e_0^3 z^2\} = 0_{2 \times 1}\) and thus \(H[z] = e_0 - e_1 z + e_2 z^2\) satisfies \(G[z]H[z]^\ast = 0_{2 \times 1}\). Now \((1, 0) + (0, 1)zG[z] = e_0 + e_1 z + e_1 z + e_2 z^2 = e_0 - e_1 z + e_2 z^2\) in characteristic 3. Thus the code generated by \(H[z]\) is contained in the code generated by \(G[z]\). Thus the code generated by \(G[z]\) is dual-containing.

Larger rate dual-containing codes may also be constructed. Here we indicate how this can be constructed for rate \(7/8\). The process may be continued. Let \(U\) be an orthogonal \(8n \times 8n\) matrix \(UU^* = 1\) with \(U = \begin{pmatrix} A_0 & A_1 & \ldots & A_7 \\ \vdots & \vdots & \ddots & \vdots \\ A_7 & \ldots & A_1 & A_0 \end{pmatrix}\) and \(U^* = (A_0^*, A_1^*, \ldots, A_7^*)\) for blocks \(A_i\) of size \(n \times 8n\). Define

\[
G[z] = \begin{pmatrix} A_0 \\ A_1 \\ \vdots \\ A_6 \\ A_7 \end{pmatrix} + \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_6 \\ A_7 \end{pmatrix} z \quad (3)
\]

Then \(G[z]\{A_0^2 - A_1^2 z + \ldots + A_7^2 z\} = 0_{7n \times n}\). It may then be shown that \(G[z]\) is a generator matrix for a convolutional \((8n, 7n, 7n; 1)\) code which is dual-containing when the characteristic is 2.

Specifically For example consider \(FC_{16}\), with \(F\) of characteristic 2 and \(C_{16}\) generated by \(a\). Let \(u = a + a^2 + a^8 + a^9 + a^{10}\). Then \(u^2 = 1, u^7 = u\). The matrix \(U\) corresponding to \(u\) as per the isomorphism in [9] is circulant and has the form \(\begin{pmatrix} A_0 \\ A_1 \\ \vdots \\ A_7 \end{pmatrix}\) for \(2 \times 16\) matrices \(A_i\). The resulting \(G[z]\) in equation [3] is a dual-containing convolutional \((16, 14, 14; 1, 4)\) code; the rate is \(7/8\). By taking pairs of the \(A_i\) together and forming for example \(G[z] = \begin{pmatrix} A_0 \\ A_1 \\ \vdots \\ A_7 \\ A_8 \\ A_9 \\ \vdots \\ A_{15} \end{pmatrix} + \begin{pmatrix} A_2 \\ A_3 \\ \vdots \\ A_6 \\ A_7 \\ A_8 \\ \vdots \\ A_{15} \end{pmatrix} z\) give \((16, 12, 12; 1, 8)\) dual-containing convolutional codes.
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