Adapting the Euler–Lagrange equation to study one-dimensional motions under the action of a constant force

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Abstract
The Euler–Lagrange equations (ELE) are very important in the theoretical description of several physical systems. In this work we have used a simplified form of ELE to study one-dimensional motions under the action of a constant force. From the use of the definition of partial derivative, we have proposed two operators, here called mean delta operators, which may be used to solve the ELE in a simplest way. We have applied this simplification to solve three simple mechanical problems in which the particle is under the action of the gravitational field: a free falling body, the Atwood’s machine and the inclined plane. The proposed simplification can be used to introduce the lagrangian formalism in teaching classical mechanics in introductory physics courses.

1. Introduction
In introductory physics courses, the student is presented, in general, to the newtonian formalism to describe the dynamics of a rigid body. In this formalism, the concept of force is used and the three Newton’s laws depict all relevant features to understand the motion of the body. This fact requires students to think in force before to progress to concepts such as energy, momentum and least action principle. The understanding of the newtonian formalism requires the student to deal with the vectorial character of force, which must bring difficulties in solving problems where vector decomposition is demanded. In this context, the concept of force has received several criticisms. For example, Wilczec [1] has argued that the force, given by the Newton’s second law, has no independent meaning. Furthermore, Jammer [2] has suggested that the concept of force is in the end its life cycle.

The main alternative to describe the motion of a rigid body without recourse to the concept of force is the lagrangian formulation to classical mechanics, which describes the dynamical evolution of a mechanical system from the concept of least action principle. The motion equations obtained from the least action principle agree with the Newton’s second law [3] (without use the concept of force). It is worth noting that the lagrangian formalism is not important only in classical mechanics, but also in several areas of physics such as quantum field theory [4] and...
condensed matter field theory [5]. In addition, the lagrangian formulation for field theory is behind
the Noether’s theorem, which connects symmetries and conservation laws [4]. In this context,
the sooner the students are presented to this formalism, the sooner they will be able to study
and discuss several relevant and current topics in physics.

Despite its importance, no attention is
devoted to the lagrangian formalism for classical
mechanics in the beginning of physics courses
at undergraduate level. In fact, it is not a simple
task to introduce this theme without speaking on
the concepts of partial derivative and least action
principle, which requires a mathematical knowl-
dge that an ordinary student in introductory
courses did not reach yet. However, Curtis has
called attention to the importance of a qualitative
discussion of modern physics themes, including
the least action principle, in introductory courses
of physics [6]. Indeed, recent works have been
devoted to provide students with information on
relevant and relatively advanced topics in Physics
[6]. For example, Organtini [7] and Cid [8] discuss
the possibility to introduce the Higgs mechanism
to undergraduate students. The study of surface
plasmon resonance, which is often confusing for
undergraduate students, was proposed from the
link between classical concepts of resonance and
undergraduate students, was proposed from the
link between classical concepts of resonance and
the solution of problems [9]. In addition, Bezerra
et al has proposed that the introduction of the
concept of magnetic dipoles can be given from a
Taylor expansion of the Biot–Savart law to obtain,
explicitly, the dominant contribution of the magn-
etic field at distant points, identifying the magnetic
dipole moment of the distribution [10].

In this context, aiming to obtain a simple
description of a mechanical system, we propose a
simplification of the ELE to analyze the dynamics
of particles under the action of a constant force.
This mathematical formulation can be adopted to
present, in introductory physics courses, the impor-
tance of the Lagrangian formalism to the descrip-
tion of the mechanics of a rigid body. Despite the
criticisms on the ‘overmathematicalization’ in phys-
ics education and the proposition that math must be
the last thing to be taught, Taber argues that the
mathematical concepts are indispensable in physics
courses [11]. Thus, we believe that this work must
bring new contributions and ways to introduce mod-
ern topics of physics in introductory courses.

This work is divided as follows: in section 2
we perform a brief review about the Lagrange
formalism for classical mechanics; section 3
presents the simplification of the ELE and three
simple examples are developed in section 4. Finally, section 5 brings the conclusions and
prospects.

2. The Euler–Lagrange equations

In the lagrangian formalism, the mechanical sys-
tem is described by \( N \) generalized coordinates
and \( N \) generalized velocities. The system evolves
from a configuration in the time \( t_1 \) to another
in the time \( t_2 \). The ask to be answered is: How
does the system evolve from the configuration
1 for the configuration 2? In the newtonian for-
alism, this evolution is given by the Newton’s
second law. In the lagrangian one, the Hamilton’s
principle (also called least action principle)
describes the dynamical evolution of the system.
The least action principle states that the evolution
of the system from configuration 1 to configura-
tion 2 is such that the action is a minimum.

The concept of action is associated with the
lagrangian \( L \) of the system, which is a function
of the generalized coordinates \( (q) \), generalized
velocities \( (\dot{q}) \) and time \( (t) \), that is, \( L \equiv L(q, \dot{q}, t) \).
The action is defined as

\[
S = \int_{t_1}^{t_2} L(q, \dot{q}, t) \, dt. \tag{1}
\]

From the Hamilton’s principle, we have that
\( \delta S = 0 \), and then

\[
\delta S = \int_{t_1}^{t_2} \sum_{i=1}^{N} \left( \frac{\partial L}{\partial q_i} \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i(t) \, dt = 0. \tag{2}
\]

Once the \( \delta q_i \) are arbitrary functions of \( t \), if there
is no links between the \( q_i \)’s, they will be indepen-
dent and

\[
\frac{\partial L}{\partial q_i} - \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \dot{q}_i} \right) = 0, \quad \tag{3}
\]

where we have omitted the \( i \) sub index. Above
equation is the ELE, which gives us the time evo-
lution of the dynamical properties of the system.
This equation is as important to the lagrangian
formalism as the Newton’s second law is to the
newtonian one. In fact, if the lagrangian is defined
as \( L = T - V \), where \( T \) is the kinetic energy and

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V is the potential energy, the obtained ELE is equivalent to the Newton’s second law. For proofing it in the case of conservative forces, we will consider a particle in a region where there is an interaction potential

\[ L = \frac{1}{2} m v^2 - V(r). \]  

(4)

By assuming that the generalized coordinates are the cartesian coordinates, we have that

\[ \frac{\partial L}{\partial \dot{x}_i} = \frac{\partial V}{\partial x_i}, \]

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} = m \ddot{x}_i. \]  

(5)

Thus, the ELE is evaluated as

\[ m \ddot{x}_i = -\frac{\partial V}{\partial x_i}, \]  

(6)

which, as expected, is the Newton’s second law for conservative forces. Now, we are in conditions to simplify the ELE for one-dimensional motions under the action of a constant force and study it for some particular problems.

3. Simplification of the ELE

From equation (3), one can note that the resolution of the ELE equation for a particular mechanical system is obtained from solving two partial derivatives, which is not a simple task for most students. In this context, a simplification of equation (3) can be useful to introduce the lagrangian formalism in introductory courses. Aiming to simplify the ELE for one-dimensional problems, without lost of generality, we will adopt the x-axis as the direction of the motion and will start our analyses from the concept of partial derivative.

Given an arbitrary function \( f(x, y) \), the partial derivatives of \( f \) in relation to \( x \) and \( y \) at the point \((x_0, y_0)\) are defined, respectively, by

\[ \frac{\partial f}{\partial x}(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \]  

(7)

and

\[ \frac{\partial f}{\partial y}(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}. \]  

(8)

In this way, we have that

\[ \frac{\partial L}{\partial v} = \lim_{v \to v_0} \frac{L(x_0, v) - L(x_0, v_0)}{v - v_0} \]  

(9)

and

\[ \frac{\partial L}{\partial x} = \lim_{x \to x_0} \frac{L(x, v_0) - L(x_0, v_0)}{x - x_0}, \]  

(10)

which are the partial derivative of \( L \) in relation to \( v \) and \( x \), respectively.

Now, assuming that a particle is moving under the action of a constant force, we will define two operators, \( \Delta_x \) and \( \Delta_v \), here called mean delta operators, which act at the lagrangian in the below form

- the \( \Delta_x \) operator acts in the lagrangian terms that depend only on the \( x \) coordinates. That is
  \[ \Delta_x L = L(x) - L(x_0); \]  

- the \( \Delta_v \) operator, which acts in the lagrangian terms that depends only on \( v \). That is
  \[ \Delta_v L = L(v) - L(v_0). \]  

(11)

(12)

where \( L(x) \) and \( L(v) \) are the terms in the lagrangian depending only on \( x \) and \( v \), respectively. Therefore

\[ \frac{\Delta_x L}{\Delta v} = \frac{L(v) - L(v_0)}{v - v_0} \]  

(13)

and

\[ \frac{\Delta_v L}{\Delta x} = \frac{L(x) - L(x_0)}{x - x_0} \]  

(14)

As well as the mean velocity of a particle is defined as \( v_m = \Delta x/\Delta t \), we will call the equations (13) and (14) as the mean lagrangian in relation to \( v \) and \( x \), respectively. With that definitions, we will rewrite the equation (3) as

\[ \frac{\Delta}{\Delta t} \left( \frac{\Delta_x L}{\Delta v} \right) - \frac{\Delta_v L}{\Delta x} = 0, \]  

(15)

where \( \Delta x = x - x_0, \Delta v = v - v_0 \) and \( \Delta t = t - t_0 \). From equation (15) we can conclude that the temporal variation of the mean lagrangian with relation to \( v \) is equal to the variation of the mean lagrangian in relation to \( x \). Here, the equation (15) is the simplification of the ELE describing the motion of bodies under the action of a constant force.
After the presentation of the simplification of the ELE equation, we can study three simple examples for proving the validity of this approximation in which we consider particles in the presence of a gravitational field: free fall body, the Atwood’s machine and the inclined plan.

4. Three simple examples

4.1. Free fall body

A free fall body consists in a particle having mass $m$ and subject only to a gravitational field $g$. In this case, the lagrangian is given by

$$L = \frac{1}{2}mv^2 - mgy,$$

where $y$ is the height in relation to the ground. In this case, the lagrangian components depending on the position and on the velocity are given by $L(v) = \frac{1}{2}mv^2$ and $L(y) = -mgy$. Obviously, $L(v_0) = \frac{1}{2}mv_0^2$ and $L(y_0) = -mgy_0$, where $y_0$ and $v_0$ are respectively the initial height and initial velocity of the particle. Thus

$$\Delta L = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = \frac{1}{2}m(v + v_0)$$

and

$$\Delta y = -mgy + mgy_0 = -mg,$$

where we have used the factorization property $(v^2 - v_0^2) = (v + v_0)(v - v_0)$.

Aiming to continue our analysis, we will define $\Delta y/v = \Delta t$, as the average of velocities of two arbitrary points along the trajectory. For uniformly varied rectilinear motions the average of velocities is equal to the mean velocity of the particle. In order to prove it, we will start from the definition of mean velocity, given by $\bar{v}_m = \Delta x/\Delta t$. In the case of a particle subject to a constant force, the motion is uniformly varied and the Torricelli’s equation leads to

$$\Delta x = \frac{(v + v_0)(v - v_0)}{2a} = \frac{(v + v_0)\Delta t}{2} \Rightarrow \bar{v}_m = \frac{v + v_0}{2} = \bar{v}_u.$$ (19)

And then

$$\frac{\Delta}{\Delta t} \left( \frac{\Delta L}{\Delta v} \right) = m\frac{\Delta v_u}{\Delta t}.$$ (20)

It is worth noting that in the above equation we are not considering the variation of the mean velocity of a particle, but the variation of the average of velocities between two arbitrary points along the particle trajectory, which changes if we consider two different excerpts during the body’s motion. For example, when we compare the average of velocities during the first and second halves of the motion of a free fall body, we note that they have different values. In fact, if some body is dropped from a height $h$, in a place where the gravitational field is $g$, when it reaches the height $h/2$, its speed is $v_1 = \sqrt{2gh}$. The initial velocity of this body is $v_0 = 0$ and then, in the first half of the fall, averaged velocity is

$$\bar{v}_m = \frac{v_1 + v_0}{2} = \frac{\sqrt{2gh}}{2}.$$ (21)

However, when it reaches the ground, the velocity of the body is $v_2 = \sqrt{2gh}$, in such way that during the second half of the trajectory, the averaged velocity is

$$\bar{v}_m = \frac{v_2 + v_1}{2} = \frac{\sqrt{3gh}}{2}.$$ (22)

Therefore, there is a variation of the averaged velocities of the particle along the trajectory. Finally, from the definition of mean acceleration, we have that $a = \Delta v_u/\Delta t$. In this way, from equation (15), we have $a = -g$.

Since the gravitational field is constant next to the Earth surface, the particle realizes an uniformly accelerated rectilinear motion, with velocity and position given by

$$v = v_0 - gt, y = y_0 + v_0t - \frac{1}{2}gt^2,$$

and that the motion equations obtained from the Newton’s laws, where the force is given by $f = -mg\hat{y}$, where $\hat{y}$ is the unitary vector pointing along the gravitational field direction.

4.2. The Atwood’s machine

The Atwood’s machine consists in a simple mechanical system that can be used to introduce problems involving ropes and chains. Despite it is an ancient problem, it will be considered here by pedagogical reasons due to its easy physical interpretation and usefulness in studying variable mass systems [12].
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The Atwood’s machine consists in two blocks of masses $m_1$ and $m_2$, which are connected by a massless rope with length $\ell$ passing over a frictionless pulley of negligible mass and radius $R$. By taking $m_1 > m_2$ and $\ell = \pi R + x_1 + x_2$ (see figure 1), we obtain

$$L(v_1) = \frac{1}{2} m_1 v_1^2$$ and $$L(x_1) = -m_1 g x$$ (24)

and

$$L(v_2) = \frac{1}{2} m_2 v_2^2$$ and $$L(x_2) = -m_2 g (\ell - x),$$ (25)

where we have used the fact that when the block 1 downs a distance $x$, the block 2 rises $\ell - x$. Once the blocks are connected by a rope with constant length $\ell$, their velocities $v_1$ and $v_2$ are equal during all the motion, that is, $v_1 = v_2 \equiv v$. Thus, the total lagrangian of the system is

$$L = \frac{1}{2} (m_1 + m_2) v^2 - m_1 g x - m_2 g (\ell - x).$$ (26)

At the time $t_0$, we have:

$$L_0 = \frac{1}{2} (m_1 + m_2) v_0^2 - m_1 g x_0 - m_2 g (\ell - x_0).$$ (27)

The acceleration of the system can be obtained from equation (15). In this case, we obtain

$$\frac{\Delta v}{\Delta t} = \frac{\frac{1}{2} (m_1 + m_2) v^2 - \frac{1}{2} (m_1 + m_2) v_0^2}{v - v_0} = \frac{1}{2} (m_1 + m_2) \left( \frac{v + v_0}{v - v_0} \right)$$

(28)

and

$$\frac{\Delta L}{\Delta x} = \frac{-(m_1 g x - m_2 g (\ell - x) + (m_1 g x_0 + m_2 g (\ell - x_0))}{x - x_0} = g (m_2 - m_1).$$ (29)

By substituting equations (28) and (29) in equation (15), we obtain

$$\left(m_1 + m_2\right) \frac{\Delta v}{\Delta t} - g (m_1 - m_2) = 0 \Rightarrow a = \left( \frac{m_2 - m_1}{m_2 + m_1} \right) g.$$ (30)

Then, as expected, we have found the acceleration predicted by the Newton’s law [12]. Once the acceleration is a constant, the motion equations are given by equation (23) with the replacement of $g$ by $g (m_2 - m_1)/(m_2 + m_1)$.

4.3. The Inclined Plan

The last example to be discussed in this paper is the inclined plan, which consists in a block with mass $m$ on a plan inclined by an angle $\theta$, in the presence of a gravitational field $g$ (see figure 2). Supposing that the block is left from the rest, we have that
and in this case, we have that
\[
\Delta v_L = \frac{1}{2} m v^2 = \Delta v_u
\]
(32)
and,
\[
\Delta x_L = \frac{-mgx \sin \theta}{x} = -mg \sin \theta.
\]
(33)
Finally, from equation (15), the acceleration is evaluated as
\[
a = -g \sin \theta,
\]
(34)
which is the acceleration predicted by the Newton’s law, as it should be. Once the angle is not variable, the acceleration of the block is constant. It is important to note that we have obtained this result without the use of the concept of forces and without the need of vectors decomposition.

5. Conclusions and prospects
In this work, we have proposed that the lagrangian formalism can be presented in introductory physics courses. In this context, from introducing two operators, here called mean delta operators, we have simplified the ELE in such way that it may be solved by students in introductory physics courses even when they do not know mathematical techniques to solve partial derivatives. We have used the described simplification to solve three examples: the free fall body, the Atwood’s machine and the inclined plan. In the three cases, the obtained motion equations agree with that predicted by the Newton’s second law, as it should be.

The proposed simplification has the limitation to be applied only to one-dimensional problems in which the particle is under the action of a constant force. However, this work opens new possibilities in the discussions about the removal of the force concept from mechanical courses, once this one is in the end of its life cycle [2]. Among the perspectives of this work, we can cite the possibility to extend this formulation for more complex problems, such as two-dimensional motions and fluidodynamics [13].

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