Dynamic Critical Behavior of Percolation Observables in the 2d Ising Model

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Abstract. We present preliminary results of our numerical study of the critical dynamics of percolation observables for the two-dimensional Ising model. We consider the (Monte-Carlo) short-time evolution of the system obtained with a local heat-bath method and with the global Swendsen-Wang algorithm. In both cases, we find qualitatively different dynamic behaviors for the magnetization and $\Omega$, the order parameter of the percolation transition. This may have implications for the recent attempts to describe the dynamics of the QCD phase transition using cluster observables.

INTRODUCTION

The study of the dynamic critical behavior of simple statistical models might be of relevance for understanding non-equilibrium effects in hot QCD, such as the effects due to heating and cooling of matter produced in heavy-ion collisions. The possible connection [1] between the deconfinement transition in QCD and the percolation phenomenon [2] has received renewed attention recently [3] and the dynamics of cluster observables has been investigated using hysteresis methods [4]. As a preparation for the study of the continuous-spin $O(4)$ model, whose magnetic transition is expected to be in the same universality class of the chiral phase transition in two-flavor QCD, we consider here the short-time dynamics of the 2d Ising model and focus on the dynamic critical behavior of percolation observables.

For many physical systems, a suitable definition of cluster provides a mapping of the physical phase transition into the geometric problem of percolation, allowing a better understanding of how the transition is induced in the system. For Ising and $O(N)$ spin models this mapping is well understood [3, 6], whereas in QCD it may be harder to define, even in the pure-gauge case [3]. Given a definition for a cluster on the lattice, the order parameter in percolation theory is the stress of the percolating cluster $\Omega$, defined by

$$\Omega \equiv \begin{cases} 0, & T > T_c \\ \Delta/V, & T \leq T_c \end{cases}$$

(1)

where $\Delta$ is the volume of the percolating cluster$^1$ and $T_c$ is the critical temperature.

$^1$ Note that a percolating cluster is a set of spins connected from the first to the last row of the lattice.
SHORT-TIME (MONTE CARLO) DYNAMICS

Using renormalization-group theory, it can be shown [7] that the early time evolution of an order parameter (e.g. the magnetization $M$) already displays universal critical behavior, given by

$$M(t, \epsilon, m_0, L) = L^{-\beta/\nu} \mathcal{M}(tL^{-z}, \epsilon L^{1/\nu}, m_0L^{x_0}),$$

(2)

where $\epsilon \equiv (T - T_c)/T_c$ and $\mathcal{M}$ is a universal function. We thus expect for $T = T_c$ a power-law behavior at early times $M(t)_{\epsilon \to 0} \sim m_0^\theta$, with $\theta = (x_0 - \beta/\nu)/z$. In principle, we would assume that the percolation order parameter $\Omega$ should have a similar behavior. The time evolution for the heat-bath and cluster algorithms is described below.

The heat-bath dynamics [8] consists in choosing the two possible directions of each Ising spin according to the exact conditional probability given by its nearest neighbors. Each spin $S_i$ is chosen UP with probability $p_i$, or DOWN with probability $1 - p_i$, where

$$p_i = \frac{1}{1 + \exp\left(-2\beta \sum_{j} S_j\right)}$$

(3)

and the sum is over nearest neighbors of $S_i$. After a certain number of iterations the spin configuration obeys the Boltzmann distribution. In the heat-bath method, since the updates are local, this transient time becomes considerably large at criticality.

The Swendsen-Wang (cluster) algorithm is obtained from the Ising-model Hamiltonian by writing the partition function as

$$\mathcal{Z} = \sum_{\{S\}} \sum_{\{n\}} \left\{ \prod_{\langle i,j \rangle} p_{ij} \delta_{S_i S_j} \right\} \left\{ \prod_{\langle i,j \rangle} (1 - p_{ij}) \right\},$$

(4)

where $p_{ij} = 1 - \exp(-2J\beta)$ is the probability of having a link between two nearest-neighbor sites of equal spin value. This link is represented by $n_{ij}$ and determines the clusters that will be associated with percolation at the critical temperature [9]. The dynamics then consists of global moves in which the spins of a cluster are flipped together, with probability $1/2$.

NUMERICAL RESULTS

In order to study the short-time dynamics we simulate at $T = T_c$ and force the system to have an initial magnetization $m_0$. We let the system evolve in time and look for power-law behavior of the order parameters $M$ and $\Omega$ as a function of the (Monte Carlo) time. Each temporal sequence is generated from a different random seed, i.e. each sequence has a different initial spin configuration. The time history is then obtained from an average over all the generated sequences. We have performed Monte Carlo simulations with 50000 seeds and 5000 sweeps for three initial magnetizations ($m_0 = 0.02, 0.03, 0.04$), for six lattice volumes, for both heat-bath and Swendsen-Wang algorithms.
We obtain that a power-law fit works very well for $M$, yielding the literature value \[10\] for the exponent $\theta$. However, as can be seen in Fig. 1, the percolation order parameter $\Omega$ does not show a power-law behavior, being consistent with an exponential behavior of the type $\exp(-\tau/t)$, where $\tau$ is essentially the relaxation time to equilibrium. This seems to hold also for the case of the Swendsen-Wang dynamics. We thus see that although the equilibrium behaviors of $M$ and $\Omega$ are equivalent, the two order parameters show qualitatively different dynamic critical behavior.

**ACKNOWLEDGMENTS**

This work was partially supported by CNPq, CAPES and FAPESP (Brazilian financial agencies).

**REFERENCES**

1. Baym, G., *Physica*, A96, 131 (1979).
2. Stauffer, D., and Aharony, A., *Introduction to Percolation Theory*, Taylor & Francis, 1992.
3. Satz, H., *Comp. Phys. Comm.*, 147, 46 (2002).
4. Berg, B.A., Heller, U.M., Meyer-Ortmanns, H., and Velytsky, A., *Phys. Rev.*, D69, 034501 (2004).
5. Wanzeller, W.G., Cucchieri, A., Krein, G., and Mendes, T., *Braz. J. Phys.*, 34, 247 (2004).
6. Blanchard, P., et al., *J. Phys.*, A33, 8603 (2000).
7. Janssen, H.K., Schaub, B., and Schmittmann, B., *Z. Phys.*, B73, 539 (1989).
8. Landau, D. P., and Binder, K., *Monte Carlo Simulations in Statistical Physics*, Cambridge, 2000.
9. Sokal, A., *Monte Carlo Methods in Statistical Mechanics: Foundations and New algorithms*, 1996, URL [http://citeseer.nj.nec.com/sokal96monte.html](http://citeseer.nj.nec.com/sokal96monte.html)
10. Zang, J.-B., et al., *Phys. Lett.*, A262, 226 (1999).