On solvability of $p$-harmonic type equations in grand Sobolev spaces

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Abstract. In this paper with the help of variational method existence and uniqueness of solution of $p$-harmonic type equations in grand Sobolev spaces is studied.

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1. Introduction and preliminary notes

It is well known that the existence and uniqueness of Dirichlet problem for $p$-harmonic equations

$$\text{div} \left( |\nabla u|^{p-2} \nabla u \right) = \text{div} f,$$

$$u|_{\partial G} = 0$$

in Sobolev and grand Sobolev spaces were studied, e.g., in [1, 2] see also [4–7, 10–13]. Namely, in these papers the different problems for $p$-harmonic equations were considered. Similar and various problems of partial differential equations in grand Sobolev, Besov and Morrey type spaces were studied in [8, 9, 14–16, 18–23] and others. Most of these papers were used the variational methods. Evidently, in the above-mentioned papers only $p$-harmonic equations (1) was considered.

In this paper we consider Dirichlet problem for $p$-harmonic type equation has a form

$$\text{div} \left( |\nabla u|^{p-q} \nabla u \right) = \text{div} f,$$

$$u|_{\partial G} = \varphi|_{\partial G},$$

where $1 < p < \infty$; $2 \leq q < \infty$; $\varphi \in W^{1}_{p}(G)$, $f \in L_{(p-\varepsilon)'}(G)$, $(p-\varepsilon)' = \frac{p-\varepsilon}{p-\varepsilon-1}$ and $G$ in $\mathbb{R}^n$ is a bounded domain.

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Definition 1. ([6, 17, 23]) Denote by $W^{1}_{p}(G)$ the grand Sobolev space of locally summable functions $u$ on $G$ having the weak partial derivatives $D^{1}_{x_{i}}u$ $(i = 1, 2, \ldots, n)$ with the finite norm
\[
\|u\|_{W^{1}_{p}(G)} = \|u\|_{L^{p}(G)} + \|\nabla u\|_{L^{p}(G)},
\]
where
\[
\|u\|_{L^{p}(G)} = \sup_{0 < \varepsilon < p-1} \left( \frac{\varepsilon}{|G|} \int_{G} |u(x)|^{p-\varepsilon} dx \right)^{1/(p-\varepsilon)}
\]
and $|G|$ is the Lebesgue measure of $G$.

We note that the correct choice of space for problem (3)-(4) is the grand Lebesgue space (or grand Sobolev space).

In this paper using the variational method an existence and uniqueness of solution to Dirichlet problem for $p-$ harmonic type equations (3)- (4) in grand Sobolev spaces is studied.

A weak solution for the problem (3)-(4) on $G$ is a function $u(x) \in W^{1}_{p}(G)$, if $u - \varphi \in W_{p}^{1}(G)$ such that
\[
\sum_{i=1}^{n} \int_{G} |\nabla u|^{p-q} u_{x_{i}} \vartheta_{x_{i}} dx = \sum_{i=1}^{n} \int_{G} f \vartheta_{x_{i}} dx, \tag{5}
\]
for every $\vartheta \in W_{p}^{1}(G)$.

2. Main results

In this section we prove the existence and uniqueness of weak solution (5) for the problem (3)-(4).

Theorem 1. Let $G \subset \mathbb{R}^{n}$ is bounded domain, $1 < p < \infty$; $2 \leq q < \infty$; $g, h \in W^{1}_{p-(q-2)}(G)$, $\varphi \in W^{1}_{p}(G)$ and $f \in L^{1}_{(p-\varepsilon)}$. Then the Dirichlet problem for pharmonic type equation (3) has a unique weak solutions in $W^{1}_{p}(G)$.

Proof. Since functions $g$ and $h \in W^{1}_{p-(q-2)}(G)$, then we consider the bilinear functional as the form
\[
F(g, h) = \sum_{i=1}^{n} \int_{G} |\nabla g|^{p-q} g_{x_{i}} h_{x_{i}} dx - \sum_{i=1}^{n} \int_{G} f h_{x_{i}} dx =
\]
\[
= I(g, h) - \sum_{i=1}^{n} \int_{G} f h_{x_{i}} dx = I(g, h) - (f, h), \tag{6}
\]
since \( f \in L_{(p-\varepsilon)'}(G) \), \( (p-\varepsilon)' = \frac{p-\varepsilon}{p-\varepsilon-1} \). Consequently, we have

\[
|I(g,g)| = |I(g)| = \left| \sum_{i=1}^{n} \int_{G} |\nabla g|^{p-q} g_{x_i} g_{x_i} \, dx \right| \leq \sum_{i=1}^{n} \int_{G} |\nabla g|^{p-q} |g_{x_i}| |g_{x_i}| \, dx = \sum_{i=1}^{n} \int_{G} |\nabla g|^{p-q}|g_{x_i}|^2 \, dx = \int_{G} |\nabla g|^{p-(q-2)} \, dx < \infty,
\]

\[
|I(g)| \leq \|\nabla g\|_{L_{p-(q-2)}(G)}^{p-(q-2)}.
\]
Consequently, for every \( q - 2 < \varepsilon < p - 1 \) function \( g \in W^{1}_{p}(G) \) and

\[
\|g\|_{W^{1}_{p}(G)} \leq C_{1} \|g\|_{W^{1}_{p-(q-2)}(G)},
\]

and, note that

\[
\|\nabla g\|_{L_{p}(G)} \leq C_{2} |I(g)|, \tag{7}
\]

where \( C_{1} \) and \( C_{2} \) are constants independents on function \( g \).

The variational problem is stated as follows. Find a function \( g \in W^{1}_{p}(G) \) such that which gives the minimum value to the integral \( F(g) \) and is unique. The Euler-Lagrange equation for the variational problem (6) under consideration is the equation (3). With the help of the inequality (7), we have

\[
|F(g,g)| = |F(g)| = \left| I(g) - \sum_{i=1}^{n} \int_{G} f g_{x_i} \, dx \right| \geq |I(g)| - \sum_{i=1}^{n} \int_{G} f g_{x_i} \, dx \geq |I(g)| - \sum_{i=1}^{n} \int_{G} f g_{x_i} \, dx \geq |I(g)| - \sum_{i=1}^{n} \int_{G} f g_{x_i} \, dx \geq |I(g)| - \sum_{i=1}^{n} \int_{G} |f| |g_{x_i}| \, dx \geq C_{3} \|g\|_{W^{1}_{p}(G)}^{p-\varepsilon} - \|g\|_{L_{p}(G)}^{p-\varepsilon} - \|f\|_{L_{(p-\varepsilon)'}(G)} \|\nabla g\|_{L_{p}(G)} \geq C_{4} \|g\|_{W^{1}_{p}(G)} = M_{0},
\]

\( C_{3} \) and \( C_{4} \) are constants independent on the function \( g(x) \).

This means that \( F(g) \) is lower bounded on \( W^{1}_{p}(G) \) show that there exists \( g_{0} \in W^{1}_{p}(G) \) such that \( F(g_{0}) = \min_{g \in W^{1}_{p}(G)} F(g) \). Fix some sequence \( \{g_{m}\} \in W^{1}_{p}(G) \) \((m = 1, 2, \ldots)\) such that \( \lim_{m \to \infty} F(g_{m}) = r_{0} \). Let \( \sigma > 0 \) choose \( m_{\sigma} \) so for \( m \geq m_{\sigma} \) and \( s = 1, 2, \ldots \) it holds \( F(g_{m+s}) < r_{0} + \sigma \). Then noting that \( \frac{1}{2} (g_{m+s} + g_{m}) \in W^{1}_{p}(G) \) we have
Since minimum to the functional and hence it follows that 

\[ r \]

So 
\[ W \]

on the segment \( 1 \theta \) solution of the problem (3)-(4). By 

\[ g \]

a function in the spaces 
\[ \| g \|_{W_p^1(G)} \leq 2 \left( \frac{1}{p} \right) \frac{1}{p-1}. \] This means that the sequence \( \{ g_m \} \) is fundamental in the spaces \( W_p^1(G) \), consequently in view of completeness the spaces \( W_p^1(G) \) there exist a function \( g_0 \in W_p^1(G) \) such that \( \lim_{m \to \infty} \| g_m - g_0 \|_{W_p^1(G)} = 0 \). By theorem on trace in 
\[ W_p^1(G), (3), p.143 \], we get 
\[ W_p^1(G) \to W_{p-\varepsilon}^1(G) \to L_{t-\varepsilon}(G_k), \quad G_k = G \cap \mathbb{R}^k, \ p < t \leq \infty, \ 1 \leq k \leq n. \]

So 
\[ \| F(g_m) - F(g_0) \| \leq C \| g_m - g_0 \|_{W_p^1(G)} \]

and hence it follows that 
\[ r_0 = \lim_{m \to \infty} F(g_m) = F(g_0). \] Show that the function delivering minimum to the functional \( F(g) \) is unique and satisfies equation (3) in the space \( W_p^1(G) \). Since \( g \in W_p^1(G) \) and \( F(g_0) = r_0 \), we have 
\[ 0 \leq I \left( \frac{g - g_0}{2} \right) = \frac{1}{2} F(g) + \frac{1}{2} F(g_0) - F \left( \frac{g + g_0}{2} \right) \leq \frac{r_0}{2} + \frac{r_0}{2} - r_0 = 0, \]
\[ I(g - g_0) = 0. \]

By \( \| g_m - g_0 \|_{W_p^1(G)} \to 0, \ m \to \infty, \) it follows that the function \( g \) coincides with \( g_0 \) as an element of the space \( W_p^1(G) \). Again from the theorem on trace in space \( W_p^1(G) \), we have 
\[ \| (g_m - g_0) \|_{\partial G} \leq C \| g_m - g_0 \|_{W_p^1(G)} \to 0, \ m \to \infty. \]

Since 
\[ \| g_m \|_{\partial G} - \varphi \|_{\partial G} \to 0, \ m \to \infty, \]
therefore 
\[ \| g_0 \|_{\partial G} - \varphi \|_{\partial G} \to 0 \ m \to \infty. \]

Taking into account the condition \( \frac{d}{d \mu} (F(g_0 + \mu \omega))_{\mu=0} = 0 \), show that the function \( g_0 \in W_p^1(G) \), minimizing the integral \( F(g) \) satisfies the following equation 
\[ I(g_0, \omega) - (f, \omega) = 0. \] (8)

Now prove that the function \( g_0 \in W_p^1(G) \) minimizing the integral \( F(g) \) is the weak solution of the problem (3)-(4). By \( \theta(t) \) we denote some monotonically decreasing function on the segment \( 1 t \leq 1 \) and having the following properties 
\[ \theta \left( \frac{1}{2} + 0 \right) = 1, \ \theta \left( 1 - 0 \right) = -1, \ \theta^{(s)} \left( \frac{1}{2} + 0 \right) = (1 - 0) = 0, \ s = 1, 2, \ldots . \]
The function\
\[ \gamma(t) = \begin{cases} \theta'(t), & \frac{1}{2} \leq t \leq 1, \\ 0, & -\infty < t < \frac{1}{2}, \quad 1 < t < \infty \end{cases} \]
is infinitely differentiable and finite on the real line. Note that the function \( \gamma \) satisfy condition\
\[ \gamma^{(s)} \left( \frac{1}{2} + 0 \right) = \gamma(1 - 0), \quad (s = 1, 2, \ldots). \]

Let \( \delta > 0 \) and let \( G_\delta = \{ y : \rho(y, R^n \setminus G) > \delta \} \) be arbitrary point of the domain \( G \), and \( r = \rho(x, x_0) \). There \( \rho(x, x_0) \) is the Euclidean distance between \( x \) and \( x_0 \), where \( x \in G \) and \( x_0 \) be a fixed point in \( G \). Following Sobolev [24], we introduce the function\
\[ \omega(x) = \gamma \left( \frac{r}{l_1} \right) - \gamma \left( \frac{r}{l_2} \right), \]
for \( 0 < l_1 < l_2 < \delta \). It is obvious that \( \omega(x) \) is a infinitely differentiable finite function with a support lying on a annular domain \( \frac{l_1}{2} < r < l_2 \). Therefore \( \omega \in C_0^\infty(G) \) and \( D^s \omega|_{\partial G} = 0 \) for all \( s = 1, 2, \ldots \). Then from (8) by definition of the weak derivative it follows that\
\[ \int_G K \left( \frac{r}{l_1} \right) g(x) \, dx = \int_G K \left( \frac{r}{l_2} \right) g(x) \, dx, \]
(9)
where\
\[ K \left( \frac{r}{l_i} \right) = \text{div} \left( \left| \nabla \gamma \left( \frac{r}{l_i} \right) \right|^{p-q} \nabla \gamma \left( \frac{r}{l_i} \right) \right) - \text{div} f, \quad i = 1, 2. \]

Note that the function \( K \left( \frac{r}{l_i} \right) \) having all properties of kernel. Namely, the following properties hold:
1) \( K \) is infinitely differentiable function with support in the ball \( r \leq l_i \);
2) The function \( K \) and all its derivatives on sphere \( R = h \) are zero;
3)\
\[ \frac{1}{\tau_n l_i^n} \int_G K \left( \frac{r}{l_i} \right) \, dx = 1, \]
where\
\[ \tau_n = \frac{2 \pi^{\frac{n}{2}}}{\Gamma \left( \frac{n}{2} \right)} \int_0^1 \xi^{n-1} K(\xi) \, d\xi. \]

Then for the function \( g_0(x) \) we can constructed Sobolev’s averaging \( g_{0,i}(x) \), \( i = 1, 2 \) on the ball \( l_i \) \( (i = 1, 2) \) with centered at the point \( x \) as\
\[ g_{0,i}(x) = \frac{1}{\tau_n l_i^n} \int_{R^n} K \left( \frac{|z-x|}{l_i} \right) g_0(z) \, dz, \quad i = 1, 2. \]

The we can rewrite equality (9) in the form \( g_{0,i}(x) = g_{0,l_2}(x) \). Consequently, for \( l < \delta \)\
\[ g_{0,l}(x) = g_0(x). \]
Since the average functions $g_{0,i}(x)$, $i = 1, 2$ are continuous and has continuous derivatives for any order, then $g_0(x)$ also is a kernel. Integrating by parts in the equality $I(g_0, \omega) - (f, \omega) = 0$, whence is the limit case

$$\sum_{i=1}^{n} \int_{G} \omega(x) \frac{\partial}{\partial x_i} \left( |\nabla g_0|^{p-q} \frac{\partial}{\partial x_i} g_0(x) \right) \, dx = \sum_{i=1}^{n} \int_{G} \omega(x) \frac{\partial}{\partial x_i} f(x) \, dx .$$

Hence by the arbitrariness of the functions $\omega(x)$ it follows that

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( |\nabla g_0|^{p-q} \frac{\partial}{\partial x_i} g_0(x) \right) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(x)$$

i.e

$$\text{div} \left( |\nabla g_0|^{p-q} \nabla g_0 \right) = \text{div} f .$$

Thus, solution of the variational problem (5) from the class $W^1_p(G)$ is also solution of Dirichlet problem (3)-(4) and this solution is unique.

3. Conclusion

In conclusion, we note that for a $p$-harmonic type equation in the grand Sobolev space, a result is obtained on the existence and uniqueness of a weak solution.

References

[1] G Afrouzi and A Hadjian. Non Trivial Solutions for $p$- Harmonic type Equations via a local minimum theorem for functionals. Taiwanese Journal of Math., 19(6):1731–1742, 2015.

[2] G Aronsson and P Lingvist. On $p$- Harmonic Functions in the Plane and their Stream Functions. Jour. of Diff. Equations, 74:157–188, 1988.

[3] O V Besov, V P Ilyin, and S M Nikolskii. Integral Representations Functions and Embedding Theorems. M. Nauka, Moscow, 1996.

[4] G Boccardo. Non linear elliptic and parabolic equations involving measure data. Jour. Func. Anal., 87:149–169, 1989.

[5] Y Deng and H Pi. Multiple solutions for $p$- harmonic type equations. Nonlinear Anal., 71:4952–4959, 2009.

[6] A Fiorenza, M R Formica, and A Gogatishvili. On grand and small Lebesgue and Sobolev spaces and some applications to PDEs. Diff. Equat. and Applic., 10(1):21–46, 2018.
[7] A Fiorenza and C Sbordone. Existence and uniqueness results for solutions of nonlinear equations with right hand side in $L^1$. *Stud. Math.*, 127(3):4959–4969, 1998.

[8] S Gala, Q Liu, and M A Ragusa. A new regularity criterion for the nematic liquid crystal flows. *Applicable Analysis*, 91(9):1741–1747, 2012.

[9] S Gala and M A Ragusa. Logarithmically improved regularity criterion for the Boussinesq equations in Besov spaces with negative indices. *Applicable Analysis*, 95(6):1271–1279, 2016.

[10] L Greco, T Iwaniec, and C Sbordone. Inverting the $p$-harmonic operator. *Manuscripta Math.*, 92(2):249–258, 1997.

[11] L Greco and A Verde. A reguliraty property of $p$-harmonic functions. *Annal. Academ. Scien. Fennicae Math.*, 25:317–323, 2000.

[12] H Luio and M Parviainen. Gradient walks and $p$-harmonic functions. *Proc. Amer. of the Math. Soc.*, 145:4313–4324, 2017.

[13] J Manfredi. $P$-harmonic functions in the plane. *Proc. of the Amer. Math. Soc.*, 103(2):473–479, 1988.

[14] A M Najafov. Problem on smoothness of solution of one class of hypoelliptic equations. *Proc. A. Razm. Math. Inst.*, 140:131–139, 2006.

[15] A M Najafov. The Differential Properties of Functions from Sobolev-Morrey type Spaces of Fractional Order. *Jour. Math. Res.*, 7(3):1–10, 2015.

[16] A M Najafov and A T Orujova. On the Solution of a Class of Partial Differential Equations. *Electron. Jour. Qual. Theory Diff. Equ.*, 2017(44):1–9, 2017.

[17] A M Najafov and N R Rustamova. Some Differential Properties of Anisotropic Grand Sobolev-Morrey spaces. *Trans. A. Razm. Math. Inst.*, 172:82–89, 2018.

[18] A M Najafov, N R Rustamova, and S T Alekberli. On Solvability of a Quasi-Elliptic Partial Differential Equations. *Jour. of Ellip. and Parab. Equ.*, 2019(5):175187, 2019.

[19] N S Papageorgiou and A Scapellato. Nonlinear Robin problems with general potential and crossing reaction. *Rend. Lincei-Mat. Appl.*, 30:1–29, 2019.

[20] N S Papageorgiou and A Scapellato. Concave-Convex Problems for the Robin p-Laplacian Plus an Indefinite Potential. *Mathematics*, 8(3,421):1–27, 2020.

[21] S Polidoro and M A Ragusa. Harnack Inequality for Hypoelliptic Ultraparabolic Equations with a Singular Lower Order Term. *Revista Matematica Iberoamericana*, 24(3):1011–1046, 2008.

[22] M A Ragusa and A Tachikawa. Regularity for Minimizers for Functionals of Double Phase with Variable Exponents. *Advances in Nonlinear Analysis*, 9(1):710–728, 2020.
[23] C Sbordone. Grand Sobolev Spaces and their Applications to Variational Problems. 
   *Le Matematiche (Catania)*, 51(2):335–347, 1996.

[24] S L Sobolev. *Some Applications of Functional Analysis in Mathematical Physics.* 
   Novosibirsk, Russian, 1950.