Root systems and Weyl groupoids for Nichols algebras

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Abstract
Motivated by the work of Kac and Lusztig, we define a root system for a large class of semisimple Yetter–Drinfeld modules over an arbitrary Hopf algebra which admits the symmetry of the Weyl groupoid introduced by Andruskiewitsch and the authors. The obtained combinatorial structure fits perfectly into an existing framework of generalized root systems associated to a family of Cartan matrices and provides novel insight into Nichols algebras. We demonstrate the power of our construction with new results on Nichols algebras over finite non-abelian simple groups and symmetric groups.

Introduction
In [16], Kac defines the Lie algebra $\mathfrak{g}(A)$ associated to a symmetrizable Cartan matrix $A = (a_{ij})_{1 \leq i,j \leq \theta}$ by generators and relations, where the relations are not given explicitly, but they are determined by dividing out an ideal with a certain universal property. Similarly, Lusztig [19] defines the braided algebra $f$, that is, the plus part of the quantum deformation of the universal enveloping algebra of $\mathfrak{g}(A)$, by dividing out from the free algebra an ideal defined by a universal property (the radical of a bilinear form). One can show after a considerable amount of work that the relations are the Serre relations.

Let $k$ be a field. Using the language of braided vector spaces and braided categories, Lusztig’s definition can be formulated as follows. Let $V$ be a vector space over $k$ with basis \{x_1, \ldots, x_\theta\} and define a braiding $c : V \otimes V \to V \otimes V$, $c(x_i \otimes x_j) = q^{d_i a_{ij}} x_j \otimes x_i$ for all $i, j$, where $q$ is the deformation parameter and $(d_i a_{ij})$ is the symmetrized Cartan matrix. The braiding $c$ has a categorical explanation. Let $G$ be the free abelian group with basis $K_1, \ldots, K_\theta$ and let $G \mathcal{YD}$ be the category of Yetter–Drinfeld modules over $G$, that is, of $G$-graded vector spaces that are $G$-modules, where each $G$-homogeneous part is stable under the action of $G$. Then $V \in G \mathcal{YD}$, where each $x_i$ has degree $K_i$ and where the action is given by $K_i \cdot x_j = q^{d_i a_{ij}} x_j$ for all $i, j$, and $V$ is a braided vector space as an object of the braided category $G \mathcal{YD}$. Then

$$f = T(V)/I_V,$$

where the free algebra $T(V)$ is a braided Hopf algebra such that the elements of $V$ are primitive, and $I_V$ is the largest coideal spanned by elements of $\mathbb{N}$-degree at least 2. In later terminology, $B(V) = T(V)/I_V$ is called the Nichols algebra of $V$. Note that $V = k x_1 \oplus \cdots \oplus k x_\theta$ is a direct sum of one-dimensional Yetter–Drinfeld modules $k x_i$, and the Nichols algebra of the irreducible pieces $k x_i$ is easy to compute as a (truncated) commutative polynomial ring.

Yetter–Drinfeld modules can be defined over any Hopf algebra $H$ over $k$ with bijective antipode instead of the group algebra of $G$. For details we refer to Section 1. It is a fundamental problem in Hopf algebra theory to understand $B(V)$ for arbitrary objects $V$ in the category.

Received 4 May 2009; revised 7 November 2009; published online 19 February 2010.

2000 Mathematics Subject Classification 17B37, 16W30, 20F55.

The work of I.H. was supported by DFG within a Heisenberg fellowship at the University of Munich.
\( H \mathcal{YD} \) of Yetter–Drinfeld modules over \( H \) (see the pioneering work [23] of Rosso and the survey article [5] of Andruskiewitsch and the second author). Let \( \theta \geq 1 \). In this paper we study the Nichols algebra
\[
\mathcal{B}(M_1 \oplus \ldots \oplus M_\theta), \quad M_1, \ldots, M_\theta \in \overset{H}{\mathcal{YD}}
\]
of the direct sum of finitely many finite-dimensional irreducible Yetter–Drinfeld modules \( M_1, \ldots, M_\theta \).

Let \( \mathcal{F}_\theta \) denote the set of \( \theta \)-tuples of finite-dimensional irreducible objects in \( \overset{H}{\mathcal{YD}} \) and let \( \mathcal{X}_\theta \) denote the set of \( \theta \)-tuples of isomorphism classes of finite-dimensional irreducible objects in \( \overset{H}{\mathcal{YD}} \). For each \( M = (M_1, \ldots, M_\theta) \in \mathcal{F}_\theta \) let \( [M] = ([M_1], \ldots, [M_\theta]) \in \mathcal{X}_\theta \) denote the corresponding \( \theta \)-tuple of isomorphism classes. Let \( \{\alpha_1, \ldots, \alpha_\theta\} \) denote the standard basis of \( \mathbb{Z}^\theta \).

Let \( M = (M_1, \ldots, M_\theta) \in \mathcal{F}_\theta \) and \( \mathcal{B}(M) = \mathcal{B}(M_1 \oplus \ldots \oplus M_\theta) \). We say that the Nichols algebra \( \mathcal{B}(M) \) is decomposable (see Definition 6.1) if there exist a totally ordered index set \( (L, \leq) \) and a family \( (W_i)_{i \in L} \) of finite-dimensional irreducible \( \mathbb{N}_0^\theta \)-graded objects in \( \overset{H}{\mathcal{YD}} \) such that
\[
\mathcal{B}(M_1 \oplus \ldots \oplus M_\theta) \simeq \bigotimes_{i \in L} \mathcal{B}(W_i)
\]
as \( \mathbb{N}_0^\theta \)-graded objects in \( \overset{H}{\mathcal{YD}} \), where \( \deg M_i = \alpha_i \) for \( 1 \leq i \leq \theta \).

In equation (0.2) the isomorphism classes of the Yetter–Drinfeld modules \( W_i \) and their degrees in \( \mathbb{Z}^\theta \) are uniquely determined by Lemma 4.7, and we define the positive roots \( \Delta^{[M]}_+ \) and the roots \( \Delta^{[M]} \) of \( [M] \) by
\[
\Delta^{[M]} = \{\deg(W_i) \mid i \in L\}, \\
\Delta^{[M]}_+ = \Delta^{[M]} \cup -\Delta^{[M]}_+.
\]
The tensor product decomposition in equation (0.2) should be viewed as a weak form of a PBW-basis of \( \mathcal{B}(M_1 \oplus \ldots \oplus M_\theta) \), where the \( W_i \) are the generators of a PBW-basis.

Our first main result is the following existence theorem.

**Theorem I** (see Theorem 4.5(1)). Assume that all finite tensor powers of the direct sum \( M_1 \oplus \ldots \oplus M_\theta \) are semisimple objects in \( \overset{H}{\mathcal{YD}} \). Then \( \mathcal{B}(M) \) is decomposable.

Note that the assumption of Theorem I is satisfied if \( H \) is a semisimple and cosemisimple Hopf algebra, for example, the group algebra of a finite group over a field of characteristic zero, or if all the \( M_i \) are one-dimensional. The main ingredients of the proof are a generalization by Graña and the first author [10] of Kharchenko’s result on PBW-bases of braided Hopf algebras of diagonal type [17], and Theorem 3.5, which says that certain braided Hopf algebras are isomorphic as a Yetter–Drinfeld module to a Nichols algebra.

Our goal now is to actually construct \( L \) and the \( W_i \) in case \( \mathcal{B}(M) \) is decomposable and \( M \) satisfies an appropriate local finiteness condition. To this end we introduce reflections of \( M \). Their definition and the proof of their basic properties are the main result of the paper [2] of Andruskiewitsch and the authors. Let \( N = (N_1, \ldots, N_\theta) \in \mathcal{F}_\theta \), and \( 1 \leq i \leq \theta \). We say that \( N \) is \( i \)-finite (see Definition 6.4) if for all \( j \in \{1, \ldots, \theta\} \) with \( j \neq i \), we have
\[
-a_{ij}^N := \sup\{m \in \mathbb{N}_0 \mid (\text{ad}_i N_j)^m(N_j) \neq 0\} < \infty.
\]
Let \( a_{ii}^N = 2 \). By Proposition 6.5, the Yetter–Drinfeld module \( (\text{ad}_i N_j)^m(N_j) \) can be computed directly as a certain subobject of \( N_i \otimes_m N_j \). If \( N \) is \( i \)-finite, then define \( s_i^N \in \text{Aut}(\mathbb{Z}^\theta) \) by
\[
 s_i^N(\alpha_j) = \alpha_j - a_{ij}^N \alpha_i \quad \text{for all } 1 \leq j \leq \theta.
\]
Following [2], we define the $i$th reflection $R_i : \mathcal{F}_\theta \to \mathcal{F}_\theta$ by $R_i(N) = N$ if $N$ is not $i$-finite, and by $R_i(N) = (N_1', \ldots, N_\theta')$ if $N$ is $i$-finite and
\[
N_j' = \begin{cases} (\text{ad}_e(N_i)^{-a_{ij}}(N_j)) & \text{if } j \neq i, \\ N_i^* & \text{if } j = i. \end{cases}
\]
Note that $[R_i(N)] = [R_i(P)]$ in $\mathcal{X}_\theta$ for all $N, P \in \mathcal{F}_\theta$ with $[N] = [P]$. Thus we may define $r_i : \mathcal{X}_\theta \to \mathcal{X}_\theta$ by
\[
r_i([N]) = [R_i(N)].
\]
Let
\[
\mathcal{F}_\theta(M) = \{ R_{i_1} \cdots R_{i_n}(M) \in \mathcal{F}_\theta \mid n \in \mathbb{N}_0, 1 \leq i_1, \ldots, i_n \leq \theta \},
\]
\[
\mathcal{X}_\theta(M) = \{ r_{i_1} \cdots r_{i_n}([M]) \in \mathcal{X}_\theta \mid n \in \mathbb{N}_0, 1 \leq i_1, \ldots, i_n \leq \theta \}.
\]

Our second main achievement is to associate to $M$ a Cartan scheme $\mathcal{C}$ and a generalized root system $\mathcal{R}$ of type $\mathcal{C}$ in the sense of Yamane and the first author [9, 15]. This is done similarly to the definition of the root system of a Kac–Moody Lie algebra $\mathfrak{g}(A)$ in [16]. Instead of one Cartan matrix $A$ we have to deal with a family of Cartan matrices and instead of the Weyl group with the Weyl groupoid of $\mathcal{R}$. In contrast to Lusztig’s approach, the Cartan matrices are not given a priori. In Section 5 we recall the definition of generalized root systems. To define our root system we need the following finiteness condition on the braided adjoint action.

We say that $M$ admits all reflections (see Definition 6.9), if $N$ is $i$-finite for all $N \in \mathcal{F}_\theta(M)$ and all $1 \leq i \leq \theta$. If $M$ admits all reflections, we let $A^N = (a_{ij}^N)_{1 \leq i, j \leq \theta}$ and $A^[N] = A^N$ for all $N \in \mathcal{F}_\theta(M)$. Then we also write $s_i^N = s_i^N$ for all $i = \{i, \ldots, \theta\}$ and $N \in \mathcal{F}_\theta(M)$.

**Theorem II** (see Theorems 6.10 and 6.11). Assume that $M$ admits all reflections and that $\mathcal{B}(M)$ is decomposable. Then $\mathcal{B}(N)$ is decomposable for all $N \in \mathcal{F}_\theta(M)$, and the following hold.

(i) the tuple $\mathcal{C}(M) = (\{1, \ldots, \theta\}, \mathcal{X}_\theta(M), (r_{i_1}|_{\mathcal{X}_\theta(M)})_{1 \leq i \leq \theta}, (A^X)_{X \in \mathcal{X}_\theta(M)})$ is a Cartan scheme;

(ii) $\mathcal{R}(M) = (\mathcal{C}(M), (\Delta^X)_X \in \mathcal{X}_\theta(M))$ is a root system of type $\mathcal{C}(M)$.

In particular, the matrices $A^N$, with $N \in \mathcal{F}_\theta(M)$, are generalized Cartan matrices, and the Weyl groupoid $\mathcal{W}(M)$ of $\mathcal{R}(M)$ is defined in a natural way generalizing the Weyl group of a Cartan matrix. For all $N \in \mathcal{F}_\theta(M)$ and $1 \leq i \leq \theta$, the reflections $s_i^N \in \text{Aut}(2^\theta)$ defined by equation (0.3) map $\Delta^{[N]}$ onto $\Delta^{[R_i(N)]}$. The Weyl groupoid $\mathcal{W}(M)$ is an important combinatorial invariant. In principle, $\mathcal{W}(M)$ can be computed. We note that the Yetter–Drinfeld module $V$ generating Lusztig’s algebra $\mathfrak{f}$ (see equation (0.1)) admits all reflections.

For any algebra $B$ let $\text{GK dim } B$ denote its Gelfand–Kirillov dimension; see [18] or [21, § 8]. Generalizing the work of Rosso [23, Lemma 20] we prove the following.

**Theorem III** (see Corollary 6.17). Assume that $\mathcal{B}(M)$ is decomposable and that $\text{GK dim } \mathcal{B}(M_1 \oplus \ldots \oplus M_\theta) < \infty$. Then $M$ admits all reflections, and for all $N = (N_1, \ldots, N_\theta) \in \mathcal{F}_\theta(M)$, we have
\[
\text{GK dim } \mathcal{B}(M_1 \oplus \ldots \oplus M_\theta) = \text{GK dim } \mathcal{B}(N_1 \oplus \ldots \oplus N_\theta).
\]

The effect of the root system on the Nichols algebra is tremendous. Its dimension and (multivariate) Hilbert series are controlled by the set of roots, the Nichols algebras of $M_1, \ldots, M_\theta$, and their images under reflections. The latter are much easier to calculate than
the Nichols algebra itself. As for Kac–Moody Lie algebras, the situation is best if all roots are real. This is the case when the Weyl groupoid is finite, in particular when the Nichols algebra is finite-dimensional. In Theorem IV(1), we show how to calculate the real. This is the case when the Weyl groupoid is finite, in particular when the Nichols algebra is finite-dimensional. In particular, Hopf algebras generated by skew-primitive and group-like elements (as the quantum groups $U_q(g(A))$) are pointed. From our theory, we expect a deep impact on the further analysis of pointed Hopf algebras and finite-dimensional Hopf algebras in particular.

**THEOREM IV** (see Theorem 7.2). Assume that $\mathcal{B}(M)$ is decomposable, $M$ admits all reflections, and the Weyl groupoid $\mathcal{W}(M)$ is finite. Then the following hold.

1. Let
   \[
   L = \{ \lambda \in \mathbb{N}_0^\theta \mid \exists n \geq 0, 1 \leq i, i_1, \ldots, i_n \leq \theta : \lambda = \sum_i R_{i_1}(M) R_{i_2}(M) \ldots R_{n-1}(M) R_n(M) (\alpha_i) \}\.
   \]
   For any $\lambda \in L$ choose $1 \leq i, i_1, \ldots, i_n \leq \theta$, with
   \[
   \lambda = \sum_i R_{i_1}(M) R_{i_2}(M) \ldots R_{n-1}(M) R_n(M) (\alpha_i),
   \]
   and define $W_\lambda = N_i$ with $\deg W_\lambda = \lambda$, where
   \[
   R_{i_n} \ldots R_{i_2} R_1(M) = (N_1, \ldots, N_\theta).
   \]
   Then $L = \Delta_+^{[M]}$ is finite, and
   \[
   \mathcal{B}(M_1 \oplus \ldots \oplus M_\theta) \simeq \bigotimes_{\lambda \in L} \mathcal{B}(W_\lambda)
   \]
   as $\mathbb{N}_0^\theta$-graded objects in $\mathcal{YD}^H$, where $\deg M_i = \alpha_i$ for $1 \leq i \leq \theta$.

2. For all $1 \leq i, j \leq \theta$, with $i \neq j$, and $m \geq 1$ the Yetter–Drinfeld module $(\text{ad}_c M_i)^m(M_j)$ is irreducible or 0.

By different methods, we show in [14] that in Theorem IV one can omit the assumption that $\mathcal{B}(M)$ is decomposable. In particular, if $k$ has finite characteristic, then Theorem IV holds without any assumption on semisimplicity of tensor powers of $M_1 \oplus \ldots \oplus M_\theta$. Note also that $M$ admits all reflections whenever $\mathcal{B}(M)$ is finite-dimensional.

Nichols algebras appear naturally inside of the associated graded Hopf algebra of pointed Hopf algebras [3, 5]. Recall that a Hopf algebra is pointed if its simple comodules are one-dimensional. In particular, Hopf algebras generated by skew-primitive and group-like elements (as the quantum groups $U_q(g(A))$) are pointed. From our theory, we expect a deep impact on the further analysis of pointed Hopf algebras and finite-dimensional Hopf algebras in particular.

Our root system generalizes the construction of the first author [11] in the case of diagonal braidings, where $H$ is the group algebra of an abelian group over a field of characteristic zero. In this setting, finite-dimensional Nichols algebras have been classified in [13]. These results allowed one to complete the classification of a large class of finite-dimensional pointed Hopf algebras with abelian group of group-like elements [6].

The classification of finite-dimensional pointed Hopf algebras with non-abelian group of group-like elements is not known, but see [1, 7] and the references therein. In the few examples, the group seems to be close to being abelian. This observation is supported by our results in Section 8, where the field is assumed to be algebraically closed of characteristic zero. In particular, in Corollaries 8.3 and 8.4 we show that the Nichols algebra of a non-simple Yetter–Drinfeld module over any non-abelian simple group or over the symmetric group $S_n$, with $n \geq 3$, is infinite-dimensional. For arbitrary finite groups $G$ we prove in Theorem 8.6 the following necessary condition for finiteness of the dimension of the Nichols algebra: If $i \neq j$ and if $s \in G$
and $t \in G$ are the degrees of non-zero homogeneous elements in $M_i$ and $M_j$, respectively, then

$$(st)^2 = (ts)^2 \quad \text{in } G.$$ 

We believe that these rather immediate consequences of the existence of the root system form just the tip of the iceberg.

1. **Preliminaries**

Let $H$ be a Hopf algebra over $k$ with bijective antipode $S$. Recall that a (left) Yetter–Drinfeld module over $H$ (see [5; 22, §10.6]), is an $H$-module $V$ equipped with a left comodule structure $\delta : V \to H \otimes V$, with $v \mapsto v_{(-1)} \otimes v_{(0)}$, such that $\delta(h \cdot v) = h_{(1)}v_{(-1)}S(h_{(3)}) \otimes h_{(2)} \cdot v_{(0)}$ for all $h \in H$ and $v \in V$.

Let $\mathcal{H}_H^Y$ and $\mathcal{H}_H^Y^{fd}$ denote the category of Yetter–Drinfeld modules and finite-dimensional Yetter–Drinfeld modules over $H$, respectively. The categories $\mathcal{H}_H^Y$ and $\mathcal{H}_H^Y^{fd}$ are braided with the braiding

$$c : X \otimes Y \longrightarrow Y \otimes X, \quad c(x \otimes y) = x_{(-1)} \cdot y \otimes x_{(0)}$$

for all $X, Y \in \mathcal{H}_H^Y$, $x \in X$, and $y \in Y$. Braided bialgebras and Hopf algebras in this paper are always bialgebras and Hopf algebras in the braided category $\mathcal{H}_H^Y$.

If $B \in \mathcal{H}_H^Y$ is a braided Hopf algebra, $x, y \in B$, and $x$ is a primitive element, then let $(\text{ad}_{x,y})(y) = xy - (x_{(-1)} \cdot y)x_{(0)}$.

Recall that a braided bialgebra is connected, if its coradical is $k1$. Let $\mathcal{H}_H^\circ$ denote the category of connected $\mathbb{N}_0$-graded braided Hopf algebras in $\mathcal{H}_H^Y$, which are generated as an algebra by elements of degree 1. The morphisms of $\mathcal{H}_H^\circ$ should be the maps of $\mathbb{N}_0$-graded braided Hopf algebras. For any $R \in \mathcal{H}_H^\circ$ and $n \in \mathbb{N}_0$ we write $R(n)$ for the homogeneous component of $R$ of degree $n$. Let $P(R)$ be the set of primitive elements of $R \in \mathcal{H}_H^\circ$. Let $\mathcal{H}_H^{fr}$ be the full subcategory of $\mathcal{H}_H^\circ$ consisting of those braided Hopf algebras $R$, for which $\dim R(1) < \infty$.

**Definition 1.1.** Let $R \in \mathcal{H}_H^\circ$. The maximal coideal of $R$ contained in $(R^+)^2 = \bigoplus_{n \geq 2} R(n)$ is denoted by $\mathcal{J}_R$. Let $\pi_R : R \to R/\mathcal{J}_R$ denote the canonical projection.

If $P(R) = R(1)$, then $R$ is called the Nichols algebra of $R(1)$ (see [5]). Then we write $R = \mathcal{B}(V)$, where $V = R(1) \in \mathcal{H}_H^Y$.

Let $R \in \mathcal{H}_H^\circ$. Then $\mathcal{J}_R$ is an $\mathbb{N}_0$-graded Hopf ideal of $R$, and

$$R/\mathcal{J}_R \simeq \mathcal{B}(R(1)).$$

(1.1)

In order to avoid possible confusion caused by several brackets, we write $\mathcal{B}^k(R(1))$, where $k \in \mathbb{N}_0$, for the $\mathbb{N}_0$-homogeneous component of $\mathcal{B}(R(1))$ of degree $k$.

Let $V \in \mathcal{H}_H^Y$. The tensor algebra $T(V)$ is a braided Hopf algebra in $\mathcal{H}_H^Y$ such that $V \subset P(T(V))$. Another description of the ideal $\mathcal{J}_T(V)$ is as the sum of the kernels of the quantum symmetrizer $S_n \in \text{End}(V^\otimes n)$, with $n \geq 2$, introduced by Woronowicz [25; see 20, 10.4.13; 24]. Explicitly, $S_n$ can be defined with the help of the quantum shuffle maps $S_{k,1} \in \text{End}(V^\otimes k+1)$, $k \geq 1$, as follows. We write $c_{i,i+1}$, if we apply the braiding to the $i$th and $(i+1)$th components of a tensor product of Yetter–Drinfeld modules. Then

$$S_{n-1,1} = \sum_{k=1}^n c_{n-1,n}c_{n-2,n-1} \cdots c_{k,k+1}.$$  

(1.2)

$$S_n = (S_{1,1} \otimes \text{id})(S_{2,1} \otimes \text{id}) \cdots (S_{n-2,1} \otimes \text{id})S_{n-1,1}.$$  

(1.3)
For any $\mathbb{Z}^n$-graded vector space $X$, where $n \in \mathbb{N}$, let $X_\gamma$ denote the homogeneous component of degree $\gamma \in \mathbb{Z}^n$ of $X$.

2. Approximations of Nichols algebras

In this section we define and study, for each $i \in \mathbb{N} \cup \{\infty\}$, a functor $F_i : \mathcal{H} \to \mathcal{H}$ and a natural transformation $\lambda_i$ from $\text{id}$ to $F_i$. The functor $F_i$ converts primitive elements of degree at most $i$ to elements of degree 1, and $\lambda_i$ maps primitive elements of degree 2, 3, $\ldots$, $i$ to zero.

**Definition 2.1.** Let $R = \bigoplus_{n=0}^\infty R(n) \in \mathcal{H}$ and $i \in \mathbb{N} \cup \{\infty\}$. Let

$$P_i(R) = \bigoplus_{n=1}^i R(n) \cap P(R),$$

(2.1)

$$P_{i-1}(R) = 0, \quad R_0 = k, \quad R_k = (k \oplus P_i(R))^k \subset R \quad \text{for all } k \in \mathbb{N},$$

(2.2)

$$P_i(R) = \bigoplus_{k=0}^\infty R_k / R_{k-1}.$$  

(2.3)

Let $\lambda_i : R \to F_i(R)$ be the linear map defined by

$$\lambda_i : R(k) \ni x \mapsto x + R_{k-1} \in R_k/R_{k-1} \quad \text{for all } k \in \mathbb{N}_0.$$  

(2.4)

For later use we define

$$P'_i(R) = \bigoplus_{n=2}^i R(n) \cap P(R).$$

(2.5)

**Remark 2.2.** Let $R = \bigoplus_{n=0}^\infty R(n) \in \mathcal{H}$ and $i \in \mathbb{N} \cup \{\infty\}$. Then $R_k/R_{k-1} \in \mathcal{H}^{\mathcal{YD}}$ for all $k \in \mathbb{N}_0$, and $F_i(R) \in \mathcal{H}^{\mathcal{YD}}$ with algebra and coalgebra structure induced by those of $R$, and $\mathbb{N}_0$-grading given by the decomposition in equation (2.3). If $i \in \mathbb{N}$ and $R \in \mathcal{H}^{\mathcal{YD}}$, then $\dim R(k) < \infty$ for all $k \in \mathbb{N}$, since $R(k) = R(1)^k$. Therefore in this case $P_i(R)$ is finite-dimensional, and hence $F_i(R) \in \mathcal{H}^{\mathcal{YD}}$.

Let $R, S \in \mathcal{H}$ and let $f : R \to S$ be a morphism. Then $f$ induces a morphism $F_i(f) : F_i(R) \to F_i(S)$ in $\mathcal{H}^{\mathcal{YD}}$. Thus $F_i$ is a covariant functor.

The map $\lambda_i : R \to F_i(R)$ is well-defined, since $R(k) = R(1)^k$ and $R(1) \subset P_i(R)$, and it is a morphism in $\mathcal{H}^{\mathcal{YD}}$. Moreover, ker $\lambda_i$ is the ideal of $R$ generated by $P'_i(R)$, and Im $\lambda_i$ is the subalgebra of $F_i(R)$ generated by $R(1) \subset F_i(R)(1)$.

**Proposition 2.3.** Let $R \in \mathcal{H}$ and $i \in \mathbb{N} \cup \{\infty\}$. Let $k(R(1))$ be the subalgebra of $F_i(R)$ generated by $R(1) \subset F_i(R)(1)$ and let $P'_i(R)$ be the ideal of $F_i(R)$ generated by $P'_i(R) \subset F_i(R)(1)$. Then $k(R(1)) \in \mathcal{H}$, $(P'_i(R))$ is an $\mathbb{N}_0$-graded Hopf ideal of $F_i(R)$ in $\mathcal{H}^{\mathcal{YD}}$, and $F_i(R) = k(R(1)) \oplus (P'_i(R))$.

**Proof.** Clearly, $k \oplus R(1)$ is a subcoalgebra and Yetter–Drinfeld submodule of $F_i(R)$, and hence $k \oplus R(1) \in \mathcal{H}^{\mathcal{YD}}$. Further, $(P'_i(R)) \in \mathcal{H}^{\mathcal{YD}}$ since $P'_i(R) \in \mathcal{H}^{\mathcal{YD}}$, and $(P'_i(R))$ is an $\mathbb{N}_0$-graded coideal of $F_i(R)$ since $P'_i(R)$ is an $\mathbb{N}_0$-graded coideal of $F_i(R)$. The algebra $F_i(R)$ is generated by $R(1) \oplus P'_i(R)$, and therefore $k(R(1)) + (P'_i(R)) = F_i(R)$. It remains to show that the sum is direct. Since $k(R(1))$ and $(P'_i(R))$ are $\mathbb{N}_0$-graded, it suffices to consider $\mathbb{N}_0$-homogeneous components. Let $k \geq 0$ and $x \in k(R(1)) \cap (P'_i(R)) \cap F_i(R)(k)$. Since $x \in k(R(1))$, there exists a representant $x \in R(k)$ of $x \in R_k/R_{k-1}$. Further, $x \in (P'_i(R))$, and hence the set
Remark 2.4. Let \( R \in \mathcal{H}_H \) and let \( R(1) = V \oplus W \) be a decomposition in \( \mathcal{H}_H \mathcal{YD} \). Let \( k(V) \) be the subalgebra of \( R \) generated by \( V \) and let \( (W) \) be the ideal of \( R \) generated by \( W \). Then in general, the sum \( R = k(V) + (W) \) is not direct.

For example, let \( H = k(\mathbb{Z}/2\mathbb{Z})^2 \) and assume that \( g, h \in (\mathbb{Z}/2\mathbb{Z})^2 \) generate \( (\mathbb{Z}/2\mathbb{Z})^2 \) as a group. Let \( V, W \in \mathcal{H}_H \mathcal{YD} \) with \( V = kx \) and \( W = ky \), and assume that the coaction and the action of \( H \) satisfy

\[
\delta(x) = g \otimes x, \quad \delta(y) = h \otimes y,
\]

\[
g \cdot x = -x, \quad g \cdot y = y, \quad h \cdot x = x, \quad h \cdot y = -y.
\]

Then \( \mathcal{J} = (xy - yx, x^2 - y^2) \) is an \( \mathbb{N}_0 \)-graded Hopf ideal in \( T(V \oplus W) \), where \( V \oplus W \subset P(T(V \oplus W)) \). Since \( x^2 \in k(V) \cap (W) \subset T(V \oplus W)/\mathcal{J} \), the sum \( k(V) + (W) \) in \( T(V \oplus W)/\mathcal{J} \) is not direct.

Lemma 2.5. Let \( R \in \mathcal{H}_H, k \in \mathbb{N}_0 \), and \( i \in \mathbb{N} \) such that \( i > k \). If \( R(m) \cap P(R) = 0 \) for all \( m \) with \( 2 \leq m \leq k \), then \( \lambda_i(R)(m) \cap P(\lambda_i(R)) = 0 \) for all \( m \) with \( 2 \leq m \leq k + 1 \).

Proof. The condition \( R(m) \cap P(R) = 0 \) for all \( m \) with \( 2 \leq m \leq k \) is equivalent to \( P_i(R) = \bigoplus_{n=k+1} R(n) \cap P(R) \). Since \( \lambda_i(R) \simeq R/\ker \lambda_i \), the claim follows from the end of Remark 2.2.

Proposition 2.6. Let \( m \in \mathbb{N} \) and \( i_2, i_3, \ldots, i_m \in \mathbb{N} \) such that \( i_n \geq n \) for all \( n \leq m \). Let \( R \in \mathcal{H}_H \) and

\[
\lambda = \lambda_{i_m} \ldots \lambda_{i_3} \lambda_{i_2} : R \to F_{i_m} \ldots F_{i_3} F_{i_2}(R).
\]

Then \( \ker \lambda \cap R(k) = \mathcal{J}_R \cap R(k) \) for all \( k \) with \( 0 \leq k \leq m \).

Proof. By Lemma 2.5 and induction on \( m \) one obtains that \( \lambda(R(k)) \) has, if \( 2 \leq k \leq m \), no primitive elements besides 0. Since \( R \) is generated by \( R(1) \) and \( \lambda(R) \simeq R/\ker \lambda \), we obtain that \( \ker \lambda \cap R(k) = \mathcal{J}_R \cap R(k) \) for \( 0 \leq k \leq m \). Now if \( i \leq m \), then \( P_i(F_{i_m} \ldots F_{i_3} F_{i_2}(R)) \cap \lambda(R) = 0 \), and hence \( \lambda_i : \lambda(R) \to \lambda_i \lambda(R) \) is an isomorphism. Otherwise \( i > m \), and the claim follows from the equalities

\[
\ker \lambda \cap R(k) = \ker \lambda_i \lambda \cap R(k) = \mathcal{J}_R \cap R(k), \quad 0 \leq k \leq m,
\]

shown in the first part of the proof.

3. Braided Hopf algebras and Yetter–Drinfeld modules

In Theorem 3.5 we give a criterion for a braided Hopf algebra to be isomorphic to a Nichols algebra in \( \mathcal{H}_H \mathcal{YD} \). First, we show that \( F_i(R) \simeq R \) in \( \mathcal{H}_H \mathcal{YD} \) for all \( R \in \mathcal{H}_H \) which are semisimple objects in \( \mathcal{H}_H \mathcal{YD} \).
Lemma 3.1. Let $R$ be a coalgebra in $H_H YD$ and let $R = \bigcup_{n=0}^{\infty} R_n$ be a coalgebra filtration of $R$ in $H_H YD$. Suppose that $R_n$ is a semisimple object in $H_H YD$ for all $n \in \mathbb{N}_0$. Let $\gr R = \bigoplus_{n=0}^{\infty} R(n)$, where $R_{n-1} = 0$ and $R(n) = R_n/R_{n-1}$ for all $n \in \mathbb{N}_0$. Then the coalgebra $\gr R$ is a semisimple object in $H_H YD$, and $\gr R \simeq R$ as objects in $H_H YD$.

Proof. For each $n \in \mathbb{N}_0$ the sequence
\[
0 \longrightarrow R_{n-1} \longrightarrow R_n \longrightarrow R(n) \longrightarrow 0
\]
in $H_H YD$ is exact. Thus $R_n \simeq R(n) \oplus R_{n-1}$ in $H_H YD$ and $R(n)$ is semisimple for all $n \in \mathbb{N}_0$, since $R_n$ is semisimple for all $n \in \mathbb{N}_0$. By induction one obtains for all $n \in \mathbb{N}_0$ that $R_n \simeq \bigoplus_{i=0}^{n} R(i)$ in $H_H YD$. For all $n \in \mathbb{N}_0$, the isomorphism $R_n \simeq \bigoplus_{i=0}^{n} R(i)$ can be extended to an isomorphism $R_{n+1} \simeq \bigoplus_{i=0}^{n+1} R(i)$, since $R_{n+1} \simeq R_n \bigoplus R(n+1)$. Thus the lemma follows from
\[
R = \bigcup_{n \in \mathbb{N}} R_n \simeq \bigcup_{n \in \mathbb{N}} \bigoplus_{i=0}^{n} R(i) \simeq \bigoplus_{i=0}^{\infty} R(i) = \gr R.
\]

Lemma 3.2. Let $R$ be an algebra in $H_H YD$ that is generated by a subspace $U \subset H_H YD$. Let $R_{-1} = 0$, $R_0 = \mathbb{k}$, $R_n = (U + \mathbb{k})^n \subset R$ for all $n \in \mathbb{N}$, and $\gr R = \bigoplus_{n=0}^{\infty} R_n/R_{n-1}$. If $R_n$ is semisimple in $H_H YD$ for all $n \in \mathbb{N}_0$, then $\gr R \simeq R$ as objects in $H_H YD$.

Proof. Since $\mathbb{k} \otimes W \simeq W$ for all $W \in H_H YD$, it follows that $R_{n-1}$ is a subobject of $R_n$ in $H_H YD$ for all $n \in \mathbb{N}_0$. Thus the arguments in the proof of Lemma 3.1 yield the claim.

Corollary 3.3. Let $R \in H_H H$ and $i \in \mathbb{N}$. If $R(n)$ is semisimple in $H_H YD$ for all $n \in \mathbb{N}$, then $F_i(R)(n)$ is semisimple in $H_H YD$ for all $n \in \mathbb{N}$, and $F_i(R) \simeq R$ as objects in $H_H YD$.

Proof. This is clear by the definition of $F_i$ and by Lemma 3.2.

Recall that, for any embedding $\iota_U^V : U \subset V$ in $H_H YD$, there exists a canonical embedding $B(\iota_U^V) : B(U) \rightarrow B(V)$ of braided Hopf algebras in $H_H YD$ such that $B(\iota_U^V)|_U = \iota_U^V$.

Lemma 3.4. Let $R \in H_H H$ and $i \in \mathbb{N}$ such that all finite tensor powers of $R(1)$ are semisimple in $H_H YD$. Let $\iota_R : B(R(1)) \rightarrow R$ be an $\mathbb{N}_0$-graded splitting of the exact sequence
\[
0 \longrightarrow \mathcal{F}_R \longrightarrow R \xrightarrow{\pi_R} B(R(1)) \longrightarrow 0
\]
in $H_H YD$. Then there is an isomorphism $\Phi : R \rightarrow F_i(R)$ in $H_H YD$ and an $\mathbb{N}_0$-graded splitting $\iota_{F_i(R)} : B(P_i(R)) \rightarrow F_i(R)$ of the exact sequence
\[
0 \longrightarrow \mathcal{F}_{F_i(R)} \longrightarrow F_i(R) \xrightarrow{\pi_{F_i(R)}} B(P_i(R)) \longrightarrow 0
\]
in $H_H YD$ with the following properties.

1. The inclusion $\Phi(R(n)) \subset \bigoplus_{k=0}^{n} F_i(R)(k)$ holds for all $n \in \mathbb{N}_0$.
2. Let $\gr \Phi \in \text{Hom}(R, F_i(R))$ such that $\gr \Phi|_R(n) = \text{pr}_n \circ \Phi|_R(n)$ for all $n \in \mathbb{N}_0$, where $\text{pr}_n : F_i(R) \rightarrow F_i(R)(n)$ is the canonical projection. Then $\gr \Phi = \lambda_i$. 
(3) The following diagram is commutative.

\[
\begin{array}{ccc}
\mathcal{B}(R(1)) & \overset{\iota_R}{\longrightarrow} & R \\
\mathcal{B}(P_i(R)) & \overset{\iota F_i(R)}{\longrightarrow} & F_i(R) \\
\mathcal{B}(P_i(R)) & \downarrow{\Phi} & \\
\end{array}
\]

\[\text{(3.1)}\]

**Proof.** First, we define the map \(\Phi|_{R(k)}\) for all \(k \in \mathbb{N}_0\) and prove that it satisfies (1) and (2). Then we show that \(\Phi\) is bijective, and that the required section \(\iota_{F_i(R)}\) of \(\pi_{F_i(R)}\), which makes diagram (3.1) commutative, exists.

For all \(k \in \mathbb{N}_0\) let \((P_i'(R))(k) = (P_i'(R)) \cap R(k)\), where \((P_i'(R)) \subset R\) is the ideal of \(R\) generated by \(P_i'(R)\). Since \(P_i'(R) \subset \bigoplus_{n=2}^{\infty} R(n)\) is an \(\mathbb{N}_0\)-graded object in \(\mathcal{H}^R\), we obtain that \((P_i'(R)) = \bigoplus_{k=2}^{\infty} (P_i'(R))(k)\) and that \((P_i'(R))(k) \in \mathcal{H}^R\) for all \(k \in \mathbb{N}_0\). Further, the sum

\[\iota_R(B^k(R(1))) + (P_i'(R))(k) \subset R(k)\]

\[\text{(3.2)}\]

is direct. Indeed, \(\pi_R\) is an algebra map, \(\pi_R(P_i'(R)) = 0\), and \(\pi_R \iota_R = \text{id}_{\mathcal{B}(R(1))}\). Hence, if \(x \in B^k(R(1))\) and \(\iota_R(x) \in (P_i'(R))(k)\), then \(\pi_R(x) \in \pi_R((P_i'(R))) = 0\) and \(\pi_R(x) = \pi_R \iota_R(x) = x\), and hence \(x = 0\).

For all \(j, k \in \mathbb{N}\) let \(V_j^{ik} \in \mathcal{H}^R\) such that

\[R(k) = (R_j \cap R(k)) \oplus V_j^{ik}\]

(see equation (2.2)). The \(V_j^{ik}\) exist, since \(R(k)\) is semisimple in \(\mathcal{H}^R\). The relation \(P_i'(R) \subset \bigoplus_{n=2}^{\infty} R(n)\) implies that

\[(P_i'(R))(k) = R_{k-1} \cap (P_i'(R))(k) = R_{k-1} \cap R(k),\]

and hence by equation (3.2) we may assume that \(\iota_R(B^k(R(1))) \subset V_j^{ik}\) if \(j < k\). Let \(k \in \mathbb{N}_0\). We define \(\Phi_k : R(k) \rightarrow F_i(R)\) by setting

\[\Phi_k := \bigoplus_{j=0}^{k} \Phi_{kj}, \quad \Phi_{kj} : R(k) \rightarrow F_i(R)(j) = R_j/R_{j-1},\]

\[\Phi_{kj}(x) := \begin{cases} 0 & \text{for } x \in V_j^{ik}, \\ x + R_{j-1} & \text{for } x \in R_j \cap R(k), \end{cases}\]

\[\text{(3.3)}\]

where \(0 \leq j \leq k\). Then it is clear that (1) holds, and since \(\text{gr} \Phi|_{R(k)} = \Phi_k\), we also get (2) by the definition of \(\lambda_i\) in equation (2.4).

Now we prove that \(\Phi\) is bijective. Suppose first that \(x = \sum_{k=0}^{n} x_k\), where \(x_k \in R(k)\) for all \(k \leq n\), and \(\Phi(x) = 0\). Then

\[\Phi(x) = \sum_{j=0}^{n} \sum_{k=j}^{n} \Phi_{kj}(x_k) \in \sum_{j=0}^{n} F_i(R)(j)\]

by definition of \(\Phi\) and by (1), and hence \(\sum_{k=j}^{n} \Phi_{kj}(x_k) = 0\) for all \(j \in \{0, 1, \ldots, n\}\). Using the definition of \(\Phi\) again and again, and the facts that \(R(k) \subset R_k\) and \(R_k\) is \(\mathbb{N}_0\)-graded for all \(k \in \mathbb{N}_0\), we conclude as follows:

\[\Phi_{nn}(x_n) = 0 \implies x_n \in R_{n-1},\]

\[\Phi_{n-1n-1}(x_{n-1}) + \Phi_{n-1n}(x_n) = 0 \implies x_{n-1} + x_n \in R_{n-2} \implies x_{n-1}, x_n \in R_{n-2},\]
Thus $x_k = 0$ for all $k \in \{0, 1, \ldots, n\}$, and hence $\Phi$ is injective. The surjectivity of $\Phi$ follows immediately from the decomposition

$$R_j/R_{j-1} = ((R(1) \oplus P'_i(R))^j + R_{j-1})/R_{j-1}$$

and the definition of $\Phi$.

Now we prove (3). Note that the canonical map $\pi_{F_i(R)} : F_i(R) \to \mathcal{B}(P'_i(R))$ is compatible with the decomposition $F_i(R) = k(R(1)) \oplus (P'_i(R))$ (see Proposition 2.3) in the sense that $\pi_{F_i(R)}(k(R(1))) \subset k(R(1))$ and $\pi_{F_i(R)}((P'_i(R))) \subset (P'_i(R))$. Let $\pi' : k(R(1)) \to k(R(1))$ and $\pi'' : (P'_i(R)) \to (P'_i(R))$ denote the components of $\pi_{F_i(R)}$, that is, $\pi_{F_i(R)} = \pi' \oplus \pi''$. Let $\iota'_{F_i(R)} : (P'_i(R)) \to (P'_i(R))$ be an arbitrary section of $\pi''$ in $\mathcal{H}YD$, and define

$$\iota'_{F_i(R)}(k(R(1))) \to k(R(1)), \quad \iota'_{F_i(R)} := \lambda_i \circ \iota_R,$$

where the domain of $\iota'_{F_i(R)}$ is a subobject of $\mathcal{B}(P'_i(R))$, and the range of $\iota'_{F_i(R)}$ is a subobject of $F_i(R)$. Let $\iota_{F_i(R)} := \iota'_{F_i(R)} \oplus \iota''_{F_i(R)}$. Note that the choice of $\iota'_{F_i(R)}$ is necessary. Indeed, in order to make diagram (3.1) commutative, we need precisely that $\iota'_{F_i(R)} = \Phi \circ \iota_R$. However, equation (3.3) and the assumption $\iota_R(\mathcal{B}(k(R(1)))) \subset \mathcal{V}_j^k$ for $j < k$ imply that $\Phi_{k,j} \iota_R(\mathcal{B}(k(R(1)))) = 0$ for $j < k$, and hence $\Phi_{k,R} = (\text{gr } \Phi)_{k,R} = \lambda_{i+k,R}$.

It remains to show that $\pi_{F_i(R)} \iota'_{F_i(R)}(k(R(1))) = \text{id}|_{k(R(1))}$. Equivalently, we have to prove that the diagram

$$\begin{array}{ccc}
\mathcal{B}(R(1)) & \xrightarrow{\iota_R} & R \\
\mathcal{B}(\iota_{F_i(R)}) & \simeq & \mathcal{B}(P'_i(R)) \supset k(R(1)) \xrightarrow{\pi_{F_i(R)}} k(R(1)) \subset F_i(R)
\end{array}$$

is commutative. Clearly, this is equivalent to the commutativity of the following diagrams:

$$\begin{array}{ccc}
\mathcal{B}(R(1)) & \xrightarrow{\iota_R} & \mathcal{B}(R(1)) \xrightarrow{\pi_{F_i(R)}} \mathcal{B}(P'_i(R)) \xrightarrow{\iota_R} \mathcal{B}(R(1)) \\
\mathcal{B}(\iota_{F_i(R)}) & \simeq & \mathcal{B}(P'_i(R)) \xrightarrow{\pi_{F_i(R)}} \mathcal{B}(P'_i(R)) \xrightarrow{\iota_R} \mathcal{B}(R(1)),
\end{array}$$

where the second one is obtained from the first one by using that $\iota_R$ is a section of $\pi_R$. The second diagram is easily seen to be commutative. Indeed, the diagram

$$\begin{array}{ccc}
\mathcal{B}(R(1)) & \xrightarrow{\pi_R} & R \\
\mathcal{B}(\iota_{F_i(R)}) & \simeq & \mathcal{B}(P'_i(R)) \xrightarrow{\pi_{F_i(R)}} F_i(R),
\end{array}$$

is a diagram in $\mathcal{H}H$, and it is commutative, since it is commutative on the generators. \qed
THEOREM 3.5. Let $R$ be a connected braided Hopf algebra that is generated as an algebra by $U \subset P(R)$, that $U \in \mathcal{H} \mathcal{Y} \mathcal{D}$, and assume that all tensor powers of $U$ are semisimple in $\mathcal{H} \mathcal{Y} \mathcal{D}$. Then there is a subobject $V \subset R$ in $\mathcal{H} \mathcal{Y} \mathcal{D}$ such that $U \subset V$, all tensor powers of $V$ are semisimple in $\mathcal{H} \mathcal{Y} \mathcal{D}$, and $R \cong B(V)$ in $\mathcal{H} \mathcal{Y} \mathcal{D}$.

Proof. It is sufficient to prove the theorem for $R \in \mathcal{H} \mathcal{H} \mathcal{Y} \mathcal{D}$ and $U = R(1)$. Indeed, define $\text{gr } R = \bigoplus_{n=0}^{\infty} R(n)$ as in Lemma 3.2. Since $U \subset P(R)$, we get

$$
\Delta : R(n) \longrightarrow \bigoplus_{k=0}^{n} R(k) \otimes R(n-k),
$$

and hence $\text{gr } R \in \mathcal{H} \mathcal{H} \mathcal{Y} \mathcal{D}$. Moreover, $\text{gr } R \cong R$ in $\mathcal{H} \mathcal{Y} \mathcal{D}$ by Lemma 3.2.

Define inductively $\Phi_{n} = \bigoplus_{n=0}^{\infty} R(n)$ for all $n \in \mathbb{N}$ by letting

$$
R(1) = R, \quad R(n) = F_{n-1}(R(n-1)) \quad \text{for all } n \in \mathbb{N}_{\geq 2},
$$

and let $P_n = P_n(R(n))$ and $P'_n = P'_n(R(n))$ for all $n \in \mathbb{N}$.

Using Corollary 3.3 and the formula $R(n)(k) = R(n)(1)^k$ for all $k, n \in \mathbb{N}$, we may apply Lemma 3.4. Hence, for each $n \in \mathbb{N}$ there exists an isomorphism

$$
\Phi_n : R(n) \longrightarrow R(n+1)
$$

which satisfies the properties in Lemma 3.4(1)–(3). In particular, the diagrams

$$
\begin{array}{ccc}
\mathcal{B}(P_{n-1}) & R(n) & \mathcal{B}(P_{n-1}) \\
\text{B}^{\Phi_{n-1}} & \Phi_n \text{B}^{\Phi_{n-1}} & \Phi_n \Phi_{n-1} \Phi_n \\
\mathcal{B}(P_n) & R(n+1) & \mathcal{B}(P_n) \\
\text{B}^{\Phi_n} & \Phi_n \text{B}^{\Phi_n} & \Phi_n \Phi_n \Phi_n
\end{array}
$$

commute for each $n \in \mathbb{N}_{\geq 2}$. Hence there is a map

$$
\Psi : \bigcup_{n=1}^{\infty} P_n = \bigcup_{n=1}^{\infty} \mathcal{B}(P_n) \longrightarrow \mathcal{B}(V)
$$

in $\mathcal{H} \mathcal{Y} \mathcal{D}$, and since $\iota_{R(n)}$ is injective for all $n \in \mathbb{N}$, the map $\Psi$ is injective. Let $V = \bigcup_{n=0}^{\infty} P_n \in \mathcal{H} \mathcal{Y} \mathcal{D}$. Since, for all $n \in \mathbb{N}$, all tensor powers of $P_n$ are semisimple in $\mathcal{H} \mathcal{Y} \mathcal{D}$, all tensor powers of $V$ are semisimple in $\mathcal{H} \mathcal{Y} \mathcal{D}$. We are left to show that $\Psi$ is surjective.

For all $n \in \mathbb{N}$ let $\varphi_n = \Phi_{n-1} \cdots \Phi_2 : R(2) = R \longrightarrow R(n)$. By Lemma 3.4, for all $x \in R$ there exist $k \in \mathbb{N}_0$ and $m \in \mathbb{N}_{\geq 2}$ such that

$$
\varphi_m(x) \in \bigoplus_{n=0}^{k} R^{(m)}(n), \quad \varphi_t(x) \notin \bigoplus_{n=0}^{k-1} R^{(m)}(n)
$$

for all $t \geq m$. We prove by induction on $k$ that $x \in \Psi(\mathcal{B}(V))$. First, let $k = 0$. Then $x \in k = \mathcal{B}(V)$, and $\Psi(x) = \Phi_t(x)$ by definition of $\Psi$.

Now let $k \geq 1$. Since

$$
\varphi_m(x) - \iota_{R(m)} \varphi_{m+1}(x) \in R^{(m)}(k) \cap \ker \pi_{R(m)} = R^{(m)}(k) \cap \mathcal{J}_{R(m)}
$$

Proposition 2.6 and Lemma 3.4(2) imply that

$$
\Phi_{m+k-2} \cdots \Phi_{m+1} \Phi_m(\varphi_m(x) - \iota_{R(m)} \varphi_{R(m)} \varphi_m(x)) \in \bigoplus_{n=0}^{k-1} R^{(m+k-1)}(n).
$$

Now $\pi_{R(m)} \varphi_m(x) \in \mathcal{B}(V)$ by definition of $\mathcal{B}(V)$, and hence

$$
\varphi_{m-1} \iota_{R(m)} \varphi_{R(m)} \varphi_m(x) \in \Psi(\mathcal{B}(V)).
$$
Further, \( x - \varphi_m^{-1} t_R(m) \pi_R(m) \varphi_m(x) \in \Psi(B(V)) \) by induction hypothesis on \( k \). Thus \( x \in \Psi(B(V)) \). \( \Box \)

Remark 3.6. Assume that \( R \in H^\mathcal{YD}_{\text{fd}}^\theta \) is a connected braided Hopf algebra and a semisimple object in \( H^\mathcal{YD}_{\text{fd}}^\theta \). Then, using the algorithm in [4, p. 24], it can be shown that \( R \cong B(V) \) in \( H^\mathcal{YD}_{\text{fd}}^\theta \) for some \( V \in H^\mathcal{YD}_{\text{fd}}^\theta \).

4. Decompositions of Nichols algebras into tensor products

Our main result in this section is Theorem 4.5 giving a decomposition of a large class of Nichols algebras. For the proof we use Theorem 3.5 and a result from [10] which is based on the ideas of Kharchenko on PBW-bases of braided Hopf algebras with diagonal braiding [17].

Let \( H^\mathcal{YD}_{\text{fd}}^\theta \) denote the category of \( \mathbb{Z}^\theta \)-graded Yetter–Drinfeld modules over \( H \) having finite-dimensional homogeneous components.

Remark 4.1. Let \( H' \) denote the Hopf algebra \( k\mathbb{Z}^\theta \otimes H \). For any \( \mathbb{Z}^\theta \)-graded object \( V \in H^\mathcal{YD}_{\text{fd}}^\theta \) let \( V' \in H^\mathcal{YD}_{\text{fd}}^\theta \) such that \( V' = V \) as an \( H' \)-module, \( \gamma v = v \) for all \( v \in V' \) and \( \gamma \in \mathbb{Z}^\theta \), and the left coaction on \( V' \) is determined by \( \delta_{V'}(v) = \gamma v(-1) \otimes v(0) \) for all \( v \in V' \), where \( v(-1) \otimes v(0) \) is the coaction of \( H \) on \( v \in V \). This way the category of \( \mathbb{Z}^\theta \)-graded objects in \( H^\mathcal{YD}_{\text{fd}}^\theta \) is equivalent to the full subcategory of \( H^\mathcal{YD}_{\text{fd}}^\theta \) consisting of those Yetter–Drinfeld modules, for which the action of \( k\mathbb{Z}^\theta \) is trivial.

Let \( A = \{\alpha_1, \ldots, \alpha_\theta\} \) be a fixed basis of \( \mathbb{Z}^\theta \). For \( \gamma = \sum_{i=1}^\theta n_i \alpha_i \in \mathbb{Z}^\theta \) let \( |\gamma|_A = \sum_{i=1}^\theta n_i \). For any \( \mathbb{Z}^\theta \)-graded object \( X \) and any \( k \in \mathbb{Z} \) let

\[
X(k) = \bigoplus_{\gamma \in \mathbb{Z}^\theta, |\gamma|_A = k} X_\gamma.
\]

Clearly, if \( X \in H^\mathcal{YD}_{\text{fd}}^\theta \) such that \( X_\gamma = 0 \) for \( \gamma \in \mathbb{Z}^\theta \setminus N_0 A \), where \( N_0 A = \sum_{i=1}^\theta \mathbb{N}_0 \alpha_i \), then \( X(k) = 0 \) for \( k < 0 \) and \( X(k) \) is finite-dimensional for all \( k \geq 0 \).

Let \( (I, \leq) \) be a totally ordered index set, and for each \( i \in I \) let \( X_i \) be a connected braided Hopf algebra in \( H^\mathcal{YD}_{\text{fd}}^\theta \). Let \( 1 \) denote the unit of (each) \( X_i \). For any finite subset \( J = \{j_1 < j_2 < \ldots < j_l\} \subseteq I \) let \( X_J = X_{j_1} \otimes X_{j_2} \otimes \ldots \otimes X_{j_l} \). The family of objects \( X_J \) forms a direct system with respect to the inclusion in the following sense. If \( J \subseteq K \subseteq I \) are finite subsets, where \( J = \{j_1 < j_2 < \ldots < j_l\} \) and \( K = \{k_1 < k_2 < \ldots < k_s\} \), then \( X_J \subseteq X_K \) via the embedding

\[
X_J \hookrightarrow X_K, \quad x_{j_1} \otimes x_{j_2} \otimes \ldots \otimes x_{j_l} \mapsto y_{k_1} \otimes y_{k_2} \otimes \ldots \otimes y_{k_s},
\]

where for all \( l \) with \( 1 \leq l \leq s \) we let

\[
y_{k_l} = \begin{cases} x_{j_m} & \text{if } k_l = j_m \text{ for some } m \leq t, \\ 1 & \text{otherwise}. \end{cases}
\]

We write \( \bigotimes_{i \in I} X_i \) for the limit of this direct system.

Remark 4.2. If \( X_i \) is \( \mathbb{Z}^\theta \)-graded for all \( i \in I \), then \( \bigotimes_{i \in I} X_i \) is \( \mathbb{Z}^\theta \)-graded. If \( (X_i)_0 = k \) and \( X_i = \bigoplus_{\gamma \in N_0 A} (X_i)_\gamma \) for all \( i \in I \), then \( \bigotimes_{i \in I} X_i \in H^\mathcal{YD}_{\text{fd}}^\theta\mathbb{Z}^\theta \) if and only if \( X_i \in H^\mathcal{YD}_{\text{fd}}^\theta\mathbb{Z}^\theta \) for all \( i \in I \) and for all \( \gamma \in N_0 A \setminus \{0\} \) the set \( \{i \in I | (X_i)_\gamma \neq 0\} \) is finite.
Theorem 4.3. Let $R$ be a connected braided Hopf algebra in $^H\mathcal{YD}$, $\kappa \in \mathbb{N}$, and let $V_1, \ldots, V_\kappa \in ^H\mathcal{YD}$ be subobjects of $P(R)$, such that $R$ is generated by $\bigoplus_{i=1}^\kappa V_i$. Let $\{e_1, \ldots, e_\kappa\}$ be the standard basis of $\mathbb{Z}^\kappa$, and assume that there is an $\mathbb{N}_0^\kappa$-grading of $R$ such that $\deg V_i = e_i$ for all $i \in \{1, \ldots, \kappa\}$. If, for all $\mathbb{N}_0^\kappa$-graded subalgebras $T \subset R$ and all $\mathbb{N}_0^\kappa$-graded ideals $T'$ of $T$, there is an $\mathbb{N}_0^\kappa$-graded splitting $T/T' \to R$ in $^H\mathcal{YD}$, then there exist a totally ordered index set $(L, \leq)$ and a family $(R_i)_{i \in L}$ of connected $\mathbb{N}_0^\kappa$-graded braided Hopf algebras $R_i \in ^H\mathcal{YD}$, such that the following hold.

1. \{1, \ldots, \kappa\} \subset L, and for all $l \in \{1, \ldots, \kappa\}$ the algebra $R_l$ is generated by an $\mathbb{N}_0^\kappa$-homogeneous subspace $E_l \subset R_l$ of degree $e_l$ such that $E_l \simeq V_l$ in $^H\mathcal{YD}$;
2. for all $l \in L \setminus \{1, \ldots, \kappa\}$ the algebra $R_l$ is generated by an $\mathbb{N}_0^\kappa$-homogeneous subspace $E_l \in H^\mathcal{YD}$ of degree $\beta_l = \sum_{j=1}^\kappa n_{lj}e_j$, where at least two of the coefficients $n_{lj}$ are non-zero;
3. $R \simeq \bigotimes_{i \in L} R_i$ as $\mathbb{N}_0^\kappa$-graded objects in $^H\mathcal{YD}$.

Proof. See [10, Theorem 4.12]. The index set $L$ corresponds to an appropriate subset of the set of Lyndon words in [10, Theorem 4.12]. By assumption, the maps $e_\nu$ in [10, Theorem 4.12] can be chosen to be $\mathbb{Z}^\kappa$-homogeneous morphisms in $^H\mathcal{YD}$. Further, following the construction of $R_l$ as subquotients of $R$ and using the assumption that $R$ is $\mathbb{N}_0^\kappa$-graded, it is clear that all objects and morphisms in the theorem, including the direct limit $\bigotimes_{i \in L}$, are $\mathbb{N}_0^\kappa$-graded.

The main technical tool in this section is the following lemma.

Lemma 4.4. Let $0 \neq V \subset H^\mathcal{YD}_{2^\theta}$ such that $V = \bigoplus_{\gamma \in \mathbb{N}_0^\kappa \setminus \{0\}} V_\gamma$ and all finite tensor powers of $V$ are semisimple in $H^\mathcal{YD}_{2^\theta}$. Let $d_V = \min\{d \in \mathbb{N} | V(d) \neq 0\}$, $\kappa \in \mathbb{N}$ such that $V(d_V) = \bigoplus_{i=1}^{\kappa} V_i$ is a decomposition into irreducible objects in $H^\mathcal{YD}_{2^\theta}$, and let $V_\kappa = \bigoplus_{n=d_V+1}^{\infty} V(n)$. Then there is a totally ordered index set $(L, \leq)$ and a family $(W_l)_{l \in L}$, $0 \neq W_l \subset H^\mathcal{YD}_{2^\theta}$ for all $l \in L$, such that the following hold.

1. $\{1, \ldots, \kappa - 1\} \subset L$ and $W_l \simeq V_l$ in $^H\mathcal{YD}_{2^\theta}$ for $l \in \{1, \ldots, \kappa - 1\}$;
2. if $l \in L \setminus \{1, 2, \ldots, \kappa - 1\}$, then $W_l = \bigoplus_{n=d_V+1}^{\infty} W_l(n) \subset H^\mathcal{YD}_{2^\theta}$;
3. $B(V) \simeq \bigotimes_{l \in L} B(W_l)$ as objects in $H^\mathcal{YD}_{2^\theta}$.

Proof. Let $H'$ be as in Remark 4.1. We will apply Theorem 4.3 to objects in $H'H^\mathcal{YD}$ instead of $H^\mathcal{YD}$.

Recall that $B(V)$ is a connected braided Hopf algebra in $H'H^\mathcal{YD}_{2^\theta}$ generated by $V \subset P(B(V))$. Let $\{e_i | 1 \leq i \leq \kappa\}$ be the standard basis of $\mathbb{Z}^\kappa$. The assignment

$$\deg v = e_i, \quad \text{where } v \in V_i,$$

defines an $\mathbb{N}_0^\kappa$-grading on $V$, and this extends to an $\mathbb{N}_0^\kappa$-grading of $B(V)$ which is compatible with the $\mathbb{Z}^\kappa$-grading: $\mathbb{Z}^\theta$-homogeneous components of a $\mathbb{Z}^\kappa$-homogeneous element are $\mathbb{Z}^\kappa$-homogeneous and vice versa. Thus we may regard $B(V)$ as a $\mathbb{Z}^\kappa$-graded object in $H'H^\mathcal{YD}$. Apply Theorem 4.3 to $B(V) \subset H'H^\mathcal{YD}$. This is possible, since the existence of the splittings $T/T' \to R$ in $H'H^\mathcal{YD}$ follows from the assumption that finite tensor powers of $V$ are semisimple in $H'H^\mathcal{YD}$. We conclude that there exists a totally ordered index set $(L, \leq)$ and a family $(R_l)_{l \in L}$ of connected $\mathbb{Z}^\theta+\kappa$-graded braided Hopf algebras $R_l \subset H'H^\mathcal{YD}$, such that Theorem 4.3(1)–(3) hold. Since $E_l$ is $\mathbb{N}_0^\kappa$-homogeneous for all $l \in L$, the degrees of the $\mathbb{N}_0^\kappa$-homogeneous components of $R_l$ are of the form $n \deg E_l$, where $n \in \mathbb{N}_0$. Further, Theorem 3.5 implies (use Rem. 4.1 with $\kappa + \theta$ instead of $\kappa$) that, for all $l \in L$, there exists an $\mathbb{N}_0^\kappa$-graded object $W_l \subset H'H^\mathcal{YD}$ such that $E_l \subset W_l$ and $R_l \simeq B(W_l)$ in $H'H^\mathcal{YD}$. By the above,
the degrees of the $\mathbb{N}_0^\kappa$-homogeneous components of $W_l$ are multiples of $\deg E_l$. Then Theorem 4.3(3) and relation $B(V) \in L^H_\mathcal{YD}^{\mathbb{Z}_0}$ imply that

(**) $B(V) \simeq \bigotimes_{l \in L} B(W_l)$ in $L^H_\mathcal{YD}^{\mathbb{Z}_0}$ and in $L^H_\mathcal{YD}^{\mathbb{Z}_0}$,

and hence claim (3) of the lemma holds.

First let $l \in \{1, 2, \ldots, \kappa - 1\}$. Then

$$B(V) = B(V_1 \oplus \tilde{V}_l) = B(V_1) \oplus (\tilde{V}_l),$$

where $\tilde{V}_l = \bigoplus_{1 \leq j \leq \kappa, j \neq l} V_j$ and $(\tilde{V}_l) \subset B(V)$ is the ideal generated by $\tilde{V}_l$. Thus

$$B(V_l) = \bigoplus_{n \in \mathbb{N}_0} \{v \in B(V)| \deg v = ne_l\}.$$ 

Further, $B(W_l)$ is a connected braided Hopf algebra generated by $W_l$, and $V_l \simeq E_l \subset W_l$. By (*), (**), and since the $\mathbb{N}_0^\kappa$-homogeneous components of $B(V_l)$ are finite-dimensional, it follows that $V_l \simeq W_l$ in $L^H_\mathcal{YD}^{\mathbb{Z}_0}$. This gives (1).

Now let $l \in L \setminus \{1, 2, \ldots, \kappa - 1\}$ and let $W_{l \beta}$ be an $\mathbb{N}_0^\kappa$-homogeneous component of $W_l$. Then either $\beta = e_\kappa$ or $\beta = \sum_{j=1}^\kappa n_j e_j$ with $\sum_{j=1}^\kappa n_j \geq 2$. In the first case (***) implies that $W_{l \beta} \subset V_{e_\kappa}$ and hence $d_{W_l} \geq d_V + 1$. In the second case $d_{W_l} \geq 2d_V \geq d_V + 1$, since $d_V \geq 1$. Hence claim (2) is proved.

**Theorem 4.5.** Let $\theta \in \mathbb{N}$ and let $V_1, \ldots, V_\theta$ be finite-dimensional irreducible objects in $L^H_\mathcal{YD}$. Let $V = \bigoplus_{i=1}^\theta V_i$ and assume that finite tensor powers of $V$ are semisimple in $L^H_\mathcal{YD}$. Define a $\mathbb{Z}_0^\theta$-grading of $B(V)$ such that $\deg V_i = \alpha_i$ for all $i$. Then the following hold.

1. There exist a totally ordered index set $(L, \leq)$ and a family $(W_l)_{l \in L}$ of irreducible objects $W_l \in L^H_\mathcal{YD}^{\mathbb{Z}_0}$ with $\deg W_l \in \mathbb{N}$, for all $l$, and

$$B(V) \simeq \bigotimes_{l \in L} B(W_l) \quad \text{in } L^H_\mathcal{YD}^{\mathbb{Z}_0}.$$ 

2. If $B(V) \simeq \bigotimes_{l \in L} B(W_l)$ and $B(V) \simeq \bigotimes_{l \in L'} B(W'_l)$ for index sets $(L, \leq)$, $(L', \leq)$, and families $(W_l)_{l \in L}$, $(W'_l)_{l \in L'}$ as in (1), then there exists a bijection $\varphi : L \to L'$ such that $W_l \simeq W'_{\varphi(l)}$ in $L^H_\mathcal{YD}^{\mathbb{Z}_0}$ for all $l \in L$.

**Proof.** The uniqueness follows from Lemma 4.7 below.

For the proof of the existence of the family $(W_l)_{l \in L}$, we first construct an inverse system of totally ordered sets and corresponding families of objects in $L^H_\mathcal{YD}^{\mathbb{Z}_0}$. This inverse system gives rise to a direct system in a natural way, and the limit of this direct system will be the family $(W_l)_{l \in L}$ we are looking for.

First, we define recursively for all $k \in \mathbb{N}_0$ a totally ordered index set $(L^k, \leq)$ and a family $(W^k_l)_{l \in L^k}$, such that

(*) $0 \neq W^k_l \in L^H_\mathcal{YD}^{\mathbb{Z}_0}$, $W^k_l$ is irreducible or $W^k_l = \bigoplus_{n=1}^\infty W^k_{l_1}(n)$ for all $l \in L^k$, and

$$B(V) \simeq \bigotimes_{l \in L^k} B(W^k_l) \quad \text{in } L^H_\mathcal{YD}^{\mathbb{Z}_0}.$$ 

Then for all $k, l, n$, note that $W^k_l(n)$ is isomorphic to a direct summand of $V^{\otimes n}$, and hence all finite tensor powers of $W^k_l$ are semisimple in $L^H_\mathcal{YD}^{\mathbb{Z}_0}$.

For $k = 0$ let $L^0 = \{1\}$ and $W^0_1 = V$. If $k \geq 0$, let $l \in L^k$, and $W^k_l$ is not irreducible, then we choose a totally ordered index set $(L_{kl}, \leq)$ and a family $(W_{klm})_{m \in L_{kl}}$ of objects in $L^H_\mathcal{YD}^{\mathbb{Z}_0}$ such that

$$B(W^k_l) \simeq \bigotimes_{m \in L_{kl}} B(W_{klm}).$$
and $0 \neq W_{klm} \in \mathcal{H}_H^* \mathcal{YD}^{2^n}$ is irreducible or $W_{klm} = \bigoplus_{n=k+2}^\infty W_{klm}(n)$. This is possible by Lemma 4.4. Now we define

$$L^k = \{ l \in L^k \mid W_l^k \text{ is irreducible} \},$$
$$L^{k+1} = L^k \cup \{(l, m) \mid l \in L^k \setminus L^k, m \in L_{kl}\} \quad \text{(disjoint union)}$$

$$W_{l}^{k+1} = \begin{cases} W_l^k & \text{if } j \in L^k, \\
W_{klm} & \text{if } j = (l, m), l \in L^k \setminus L^k, m \in L_{kl}. \end{cases} \quad (4.1)$$

Let $\leq$ be the natural order on $L^{k+1}$; that is, if $l_1, l_2 \in L^k$; then $l_1 \leq l_2$ if and only if this relation holds in $L^k$. If $l_1 \in L^k$ and $l_2 = (\tilde{l}, m) \in L^{k+1} \setminus L^k$, then $l_1 \leq l_2$ if and only if $l_1 \leq \tilde{l}$ in $L^k$, and $l_2 \leq l_1$ if and only if $\tilde{l} \leq l_1$ in $L^k$. Finally, if $l_1 = (l_1, m_1), l_2 = (l_2, m_2) \in L^{k+1} \setminus L^k$, then $l_1 \leq l_2$ if and only if $l_1 < l_2$ in $L^k$ or $l_1 = l_2$ and $m_1 \leq m_2$ in $L_{kl}$. Clearly, $(L^{k+1}, \leq)$ is totally ordered, and the family $(W_l^{k+1})_{l \in L^{k+1}}$ satisfies the properties in $(*)$.

For all $k \in \mathbb{N}_0$ define $L^k$ as in equation $(4.1)$, and let $L = \bigcup_{k \in \mathbb{N}} L^k$ with the total order $\leq$ induced by the orders defined on the sets $L^k$. Note that $L^k \subset L^{k+1}$ and $W_l^{k+1} = W_l^k$ for all $k \in \mathbb{N}_0$ and $l \in L^k$. Define $W_l = W_l^0$ for all $l \in L^k \subset L$. Then $0 \neq W_l \in \mathcal{H}_H^* \mathcal{YD}^{2^n}$ is irreducible and has positive degree (with respect to the basis $\{\alpha_i \mid 1 \leq i \leq \theta\}$) for all $l \in L$. We prove that $\mathcal{D}_{l \in L} B(W_l) \simeq B(V)$ in $\mathcal{H}_H^* \mathcal{YD}^{2^n}$. Since $L^k \subset L^k$ for all $k \in \mathbb{N}_0$, it follows that $\mathcal{D}_{l \in L^k} B(W_l)$ is isomorphic to a subobject of $B(V)$ for all $k \in \mathbb{N}_0$ by $(*)$. Moreover, $(*)$ also implies that

$$B^m(V) \simeq \left( \bigotimes_{l \in L^k} B(W_l) \right) (n) \quad (4.2)$$

for all $n, m \in \mathbb{N}_0$ with $n \leq k$. The construction of $L$ and $(W_l)_{l \in L}$ shows that

$$\left( \bigotimes_{l \in L^k} B(W_l) \right) (n) = \left( \bigotimes_{l \in L^{k+1}} B(W_l) \right) (n) = \left( \bigotimes_{l \in L} B(W_l) \right) (n)$$

for all $n, m \in \mathbb{N}_0$ and $n \leq k$. Hence

$$B(V) = \bigoplus_{n=0}^\infty B^n(V) \simeq \bigoplus_{k=0}^\infty \left( \bigotimes_{l \in L^k} B(W_l) \right) (k)$$

$$= \bigoplus_{k=0}^\infty \left( \bigotimes_{l \in L} B(W_l) \right) (k) = \bigotimes_{l \in L} B(W_l).$$

This finishes the proof of the theorem. \qed

**Remark 4.6.** In the setting of Theorem 4.5, assume that $H$ is the group algebra of an abelian group and $V_i$ is one-dimensional for all $i \in \{1, 2, \ldots, \theta\}$. In [17], Kharchenko gave a construction of a PBW-basis for a class of Hopf algebras, which can be applied to $B(V)$. Theorem 4.5 can be viewed as a weak form of a PBW-basis for Nichols algebras over arbitrary Hopf algebras in the sense that the PBW-generators of Kharchenko are replaced by the Yetter–Drinfeld modules $W_l$.

**Lemma 4.7.** Let $\theta \in \mathbb{N}_0$, $(L, \leq)$, $(L', \leq)$ be totally ordered index sets and let $(W_l)_{l \in L}$, $(W'_l)_{l \in L'}$ be families of irreducible objects in $\mathcal{H}_H^* \mathcal{YD}^{2^n}$. Assume that

$$\deg W_l \in \sum_{i=1}^\theta N_0 \alpha_i \setminus \{0\} \quad \text{for all } l \in L.$$
If $\bigotimes_{l \in L} B(W_l) \simeq \bigotimes_{l \in L} B(W'_l)$, and all $\mathbb{Z}^\theta$-homogeneous components of $\bigotimes_{l \in L} B(W_l)$ are finite-dimensional, then there is a bijection $\varphi : L \to L'$ such that $W_l \simeq W'_{\varphi(l)}$ in $H^\theta YD_{\mathcal{A}}$ for all $l \in L$.

Proof. Let $l' \in L'$. We first note that $\deg W'_l \in \sum_{i=1}^\theta \mathbb{N}_0 \alpha_i \setminus \{0\}$, since $W'_l$ is a direct summand of $\bigotimes_{l' \in L'} B(W'_l) \simeq \bigotimes_{l \in L} B(W_l)$ and the degree zero component of $\bigotimes_{l \in L} B(W_l)$ is $k1$.

For $k \in \mathbb{N}_0$ let $L_k = \{l \in L \mid |\deg W_l| = k\}$. Then $L_0 = \emptyset$, $L = \bigcup_{k \in \mathbb{N}} L_k$, and $L_k$ is a finite set for all $k \in \mathbb{N}$, since the $\mathbb{Z}^\theta$-homogeneous components of $\bigotimes_{l \in L} B(W_l)$ are finite-dimensional. Similarly $L' = \bigcup_{k \in \mathbb{N}} L'_k$. It suffices to show that

(*) for all $k \in \mathbb{N}_0$ there is a bijection $\varphi_k : L_k \to L'_k$ such that $W_l \simeq W'_{\varphi_k(l)}$ in $H^\theta YD_{\mathcal{A}}$ for all $k \in \mathbb{N}_0$, $l \in L_k$.

This is clear for $k = 0$, since $L_0 = L'_0 = \emptyset$.

Let $k \in \mathbb{N}$. Then

$$\left( \bigotimes_{l \in L} B(W_l) \right) (k) \simeq \bigoplus_{l \in L_k} W_l \oplus \left( \bigotimes_{l \in L_1 \cup L_2 \cup \ldots \cup L_{k-1}} B(W_l) \right) (k),$$

$$\left( \bigotimes_{l' \in L'} B(W'_l) \right) (k) \simeq \bigoplus_{l' \in L'_k} W'_l \oplus \left( \bigotimes_{l' \in L'_1 \cup L'_2 \cup \ldots \cup L'_{k-1}} B(W'_l) \right) (k)$$

in $H^\theta YD_{\mathcal{A}}$. Hence (*) follows by induction on $k$ and by Krull–Remak–Schmidt.

The following similar results will be used later on.

Lemma 4.8. Let $K, K'$, and $B$ be $\mathbb{Z}^\theta$-graded objects in $H^\theta YD$ such that all homogeneous components of $K$ and $B$ are finite-dimensional, $B_0$ is isomorphic to the trivial object $\mathbb{k} \in H^\theta YD$, and $K_\gamma = B_\gamma = 0$ for all $\gamma \in \mathbb{Z}^\theta \setminus \sum_{i=1}^\theta \mathbb{N}_0 \alpha_i$. If $K \otimes B \simeq K' \otimes B$ in $H^\theta YD_{\mathcal{A}}$, then $K \simeq K'$ in $H^\theta YD_{\mathcal{A}}$.

Proof. Since $B_0 \simeq \mathbb{k}$, it follows that $K'_\gamma$ (isomorphic to) a direct summand of $(K' \otimes B)_\gamma$ for all $\gamma \in \mathbb{Z}^\theta$. Hence the isomorphism $K \otimes B \simeq K' \otimes B$ implies that $K'_\gamma = 0$ for all $\gamma \in \mathbb{Z}^\theta \setminus \sum_{i=1}^\theta \mathbb{N}_0 \alpha_i$.

Using a proof similar to that of the previous lemma one can establish that

$$(K \otimes B)(k) \simeq K(k) \oplus \bigoplus_{1 \leq k' \leq k} K(k - k') \otimes B(k'),$$

$$(K' \otimes B)(k) \simeq K'(k) \oplus \bigoplus_{1 \leq k' \leq k} K'(k - k') \otimes B(k')$$

in $H^\theta YD_{\mathcal{A}}$ for all $k \in \mathbb{N}_0$. Since the $\mathbb{N}_0$-homogeneous components of $K \otimes B$ are finite-dimensional, the claim follows by induction on $k$ and by Krull–Remak–Schmidt.

Lemma 4.9. Let $\theta \in \mathbb{N}$ and let $V_1, \ldots, V_\theta$ be finite-dimensional irreducible objects in $H^\theta YD$. Let $V = \bigoplus_{i=1}^\theta V_i$, and define a $\mathbb{Z}^\theta$-grading of $B(V)$ such that $\deg V_i = \alpha_i$ for all $i$. Assume that there exists a totally ordered index set $(L, \leq)$ and a family $(W_l)_{l \in L}$ of irreducible objects
$W_l \in \mathcal{H}^Y \mathcal{D}^{2\mathbb{Z}}$ such that $\deg W_l \in \mathbb{N}_0 A$ for all $l$, and

$$B(V) \simeq \bigotimes_{l \in L} B(W_l)$$

in $\mathcal{H}^Y \mathcal{D}^{2\mathbb{Z}}$. Then for each $i \in \{1, \ldots, \theta\}$ there exists a unique $l(i) \in L$ such that $\deg W_{l(i)} \in \mathbb{N} \alpha_i$. Moreover, $\deg W_{l(i)} = \alpha_i$ and $W_{l(i)} \simeq V_i$.

Proof. Let $i \in \{1, \ldots, \theta\}$. The subspace $\bigoplus_{n \in \mathbb{Z}} B(V)_{n \alpha_i} \subset B(V)$ is the subalgebra $B(V)_{\alpha_i}$, and $B(V)_{\alpha_i} = V_i$. Since $\deg W_l \in \sum_{j=1}^{\theta} \mathbb{N}_0 \alpha_j \setminus 0$, we obtain that

$$B(V_i) \simeq \bigoplus_{n \in \mathbb{Z}} \left( \bigotimes_{l \in L} B(W_l) \right)_{n \alpha_i} = \bigotimes_{l \in L_i} B(W_l),$$

(4.3)

where $L_i = \{l \in L \mid \deg W_l \in \mathbb{N} \alpha_i\}$. Hence

$V_i = B(V_i)_{\alpha_i} \simeq \left( \bigotimes_{l \in L} B(W_l) \right)_{\alpha_i} = \bigoplus_{l \in L_i, \deg W_l = \alpha_i} W_l,$

and since $V_i$ is irreducible, it follows that there is a unique $l = l(i) \in L$ such that $V_i \simeq W_l$ and $\deg W_l = \alpha_i$. Then equation (4.3) implies that $L_i = \{l(i)\}$, which proves the lemma.

5. Weyl groupoids and root systems

In this section we give the definition of generalized root systems in [15] in terms of category theory (see [9]) and recall some results which will be needed in the following sections.

Let $I$ be a non-empty finite set and let $(\alpha_i)_{i \in I}$ be the standard basis of $\mathbb{Z}^I$.

Recall from [16, §1.1] that a generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$ is a matrix in $\mathbb{Z}^{I \times I}$ such that we have the following:

(M1) $a_{ii} = 2$ and $a_{jk} \leq 0$ for all $i, j, k \in I$ with $j \neq k$;

(M2) if $i, j \in I$ and $a_{ij} = 0$, then $a_{ji} = 0$.

Let $\mathcal{X}$ be a non-empty set; and for all $i \in I$ and $X \in \mathcal{X}$, let $r_i : \mathcal{X} \to \mathcal{X}$ be a map and let

$A^X = (a_{X,k}^j)_{j,k \in I}$

be a generalized Cartan matrix. The quadruple

$\mathcal{C} = \mathcal{C}(I, \mathcal{X}, (r_i)_{i \in I}, (A^X)_{X \in \mathcal{X}}),$

is called a Cartan scheme if we have the following:

(C1) $r_i^2 = \text{id}$ for all $i \in I$;

(C2) $a_{ij}^X = a_{ij}^X(X)$ for all $X \in \mathcal{X}$ and $i, j \in I$.

Let $\mathcal{C} = \mathcal{C}(I, \mathcal{X}, (r_i)_{i \in I}, (A^X)_{X \in \mathcal{X}})$ be a Cartan scheme. For all $i \in I$ and $X \in \mathcal{X}$ define $s_i^X \in \text{Aut}(\mathbb{Z}^I)$ by

$s_i^X(\alpha_j) = \alpha_j - a_{ij}^X \alpha_i$ for all $j \in I$.

Recall that a groupoid is a category where all morphisms are isomorphisms. The Weyl groupoid of $\mathcal{C}$ is the groupoid $\mathcal{W}(\mathcal{C})$ with $\text{Ob}(\mathcal{W}(\mathcal{C})) = \mathcal{X}$, where the morphisms are generated by all $s_i^X : X \to r_i(X)$ with $i \in I$ and $X \in \mathcal{X}$. Formally, for $X, Y \in \mathcal{X}$, the set $\text{Hom}(X, Y)$ consists of the triples $(Y, s, X)$ such that

$s = s_{i_{n-1}} \cdots s_{i_2} \cdots s_{i_1} s_{i_1} X$

and $r_{i_n} \cdots r_{i_2} r_{i_1} X = Y$ for some $n \in \mathbb{N}_0$ and $i_1, \ldots, i_n \in I$. The composition of morphisms is induced by the following group structure of $\text{Aut}(\mathbb{Z}^I)$:

$(Z, g, Y) \circ (Y, f, X) = (Z, gf, X)$
for all $(Z, g, Y), (Y, f, X) \in \text{Hom}(W(C))$. If $w = (Y, f, X) \in \text{Hom}(W(C))$ and $\alpha \in \mathbb{Z}^6$, then we define $w(\alpha) = f(\alpha)$.

Note that the inverse of the morphism $(r_i(X), s^X_i, X)$ is $(X, s^X_i r_i(X), r_i(X))$, since $s^X_i$ is a reflection, and $s^X_i = s^X_i r_i(X)$ and $r_i^2(X) = X$ by definition.

We say that

$$R = R(C, (\Delta^X)_{X \in X})$$

is a root system of type $C$ if $C = C(I, X, \{r_i\}_{i \in I}, (A^X)_{X \in X})$ is a Cartan scheme and $\Delta^X \subset \mathbb{Z}^I$, where $X \in X$, are subsets such that the following hold.

(R1) $\Delta^X = (\Delta^X \cap \mathbb{N}_0^I) \cup -(\Delta^X \cap \mathbb{N}_0^I)$ for all $X \in X$;

(R2) $\Delta^X \cap \mathbb{Z} \alpha_i = \{\alpha_i, -\alpha_i\}$ for all $i \in I, X \in X$;

(R3) $s^X_i(\Delta^X) = \Delta^X r_i(X)$ for all $i \in I, X \in X$;

(R4) $(r_i r_j)^{m^X_{i,j}}(X) = X$ for all $i, j \in I$ and $X \in X$ such that $i \neq j$ and $m^X_{i,j} := |\Delta^X \cap \{\mathbb{N}_0 \alpha_i + \mathbb{N}_0 \alpha_j\}|$ is finite.

We note that axiom (M2) is redundant for root systems by [9, Lemma 2.5].

If $R(C, (\Delta^X)_{X \in X})$ is a root system of type $C$, then $W(R) := W(C)$ is called the Weyl groupoid of $R$. The elements of $\Delta^X_+ := \Delta^X \cap \mathbb{N}_0^I$ and $\Delta^X_- := -\Delta^X_+$ are called positive and negative roots, respectively. Following [16, §5.1] we say that the roots $w(\alpha_i) \in \Delta^Y$, where $w \in \text{Hom}(X, Y), X, Y \in X$, and $i \in I$, are real roots. The set of real roots and positive real roots is denoted by $\Delta^X_{\text{re}}$ and $\Delta^X_{\text{re}}$, respectively. Note that (R3) implies that $w(\Delta^X) = \Delta Y$ for all $X, Y \in \text{Ob}(W(R))$ and $w \in \text{Hom}(X, Y)$.

Recall that a groupoid $\mathcal{G}$ is connected if, for all $X, Y \in \text{Ob}(G)$, the set $\text{Hom}(X, Y)$ is non-empty. It is finite, if $\text{Hom}(G)$ is finite.

**Lemma 5.1** [9, Lemma 2.11]. Let $C$ be a Cartan scheme and let $R$ be a root system of type $C$. Assume that $W(R)$ is connected. Then the following are equivalent:

1. $\Delta^X$ is finite for all $X \in \text{Ob}(W(R))$;
2. $\Delta^X$ is finite for at least one $X \in \text{Ob}(W(R))$;
3. $\Delta^X_{\text{re}}$ is finite for all $X \in \text{Ob}(W(R))$;
4. $W(R)$ is finite.

**Proposition 5.2** [9, Proposition 2.12]. Let $C$ be a Cartan scheme and let $R$ be a root system of type $C$. If $W(R)$ is finite, then all roots are real.

**Lemma 5.3** [15, Lemma 8(iii)]. Let $C$ be a Cartan scheme and let $R$ be a root system of type $C$. Let $X, Y \in \text{Ob}(W(R))$ and $w \in \text{Hom}(X, Y) \subset \text{Hom}(W(C))$ such that $w(\Delta^X_+) \subset \Delta^Y$. Then $\Delta^X$ is finite.

In general the Cartan matrices $A^X$ are not of finite type if the Weyl groupoid $W(R)$ is finite [9, Proposition 5.1(2)]. However, in the special case when the Cartan scheme $C$ is standard, that is, $A^X = A^Y$ for all $X, Y \in \text{Ob}(W(C))$, the following corollary holds (see [9, Theorem 3.3] for a slightly different proof).

**Corollary 5.4.** Let $C$ be a standard Cartan scheme with generalized Cartan matrix $A = A^X$ for all $X \in \text{Ob}(W(C))$. Let $R$ be a root system of type $C$. Then the following are equivalent:

1. $W(R)$ is finite;
2. $A$ is a Cartan matrix of finite type.
Proof. Let $W(A)$ be the Weyl group of the Kac–Moody Lie algebra of $A$. In particular, $W(A)$ is isomorphic to the subgroup of $\text{Aut}(\mathbb{Z}^\theta)$ generated by the reflections $s_i \in \text{Aut}(\mathbb{Z}^\theta)$, where

$$s_i(\alpha_j) = \alpha_j - a_{ij} \alpha_i \quad \text{for all } i, j \in I.$$ 

(1) $\Rightarrow$ (2). Since $\mathcal{C}$ is standard, the map $\text{Hom}(W(R)) \to W(A)$, $(Y, s, X) \mapsto s$, is well-defined and surjective. Hence $W(A)$ is finite by (1), and $A$ is of finite type.

(2) $\Rightarrow$ (1). Since $A$ is of finite type, the Weyl group $W(A)$ acts transitively on the bases of the root system. Hence there exists a permutation $\tau$ of $I$ and an element $w \in W(A)$ such that $w(\alpha_i) = -\alpha_{\tau(i)}$ for all $i \in I$. Let $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in I$ such that $w = s_{i_n} \ldots s_{i_2} s_{i_1}$. Then $w = s_{r_{i_n-1} \ldots r_1(X)} s_{i_1}^X$ for all $X \in \mathcal{X}$, since $\mathcal{C}$ is standard. Hence $w(\Delta_X^X) \subset \Delta_{r_{i_n} \ldots r_2 r_1}^X$ for all $X \in \mathcal{X}$. This proves (1) by Lemmas 5.1 and 5.3. \hfill \qed

Remark 5.5. Let $\mathcal{C} = \mathcal{C}(I, \mathcal{X}, (r_i)_{i \in I}, (A^X)_{X \in \mathcal{X}})$ be a standard Cartan scheme and let $R = R(\mathcal{C}, (\Delta_X^X)_{X \in \mathcal{X}})$ be a root system of type $\mathcal{C}$. Define $A = A^X$ for all $X \in \mathcal{X}$, and let $W(A)$ denote the Weyl group of the Kac–Moody Lie algebra of $A$. Then the maps $s_i^X \in \text{Aut}(\mathbb{Z}^\theta)$, where $i \in I$ and $X \in \mathcal{X}$, are independent of $X$. Assume that $\Delta_X^X = \Delta_X^X \tau\epsilon$ for all $X \in \mathcal{X}$. Then $\Delta_X^X = \{w(\alpha_i) \mid w \in W(A), i \in I\}$ for all $X \in \mathcal{X}$, and hence it is independent of $X$. In this case $R$ is equivalent to an action of $W(A)$ on $\mathcal{X}$.

There is an important necessary condition for the finiteness of the Weyl groupoid of a Cartan scheme. For the proof we need a lemma, which is a special case of [12, Lemma 9].

Lemma 5.6. Let $T \subset \text{SL}(2, \mathbb{Z})$ be a non-empty subsemigroup generated by matrices of the form $\begin{pmatrix} a & -b \\ c & -d \end{pmatrix}$ with $0 < d < b < a$. Then all elements of $T$ are of this form. In particular, id $\notin T$.

Proposition 5.7. Let $\mathcal{C} = \mathcal{C}(I, \mathcal{X}, (r_i)_{i \in I}, (A^X)_{X \in \mathcal{X}})$ be a Cartan scheme of rank 2. If $\text{Hom}(W(\mathcal{C}))$ is a finite set, then there exist $X \in \mathcal{X}$ and $i, j \in I$ such that $a_{ij}^X \in \{0, -1\}$.

Proof. Without loss of generality assume that $I = \{1, 2\}$. For all $a \in \mathbb{Z}$ let

$$\eta_1(a) = \begin{pmatrix} -1 & a \\ 0 & 1 \end{pmatrix}, \quad \eta_2(a) = \begin{pmatrix} 1 & 0 \\ a & -1 \end{pmatrix}.$$ 

For any $X \in \mathcal{X}$, we have

$$s_{r_2 \ldots r_1 r_2(X)} s_{r_2(X)} \ldots s_{r_2(X)} s_{r_2(X)} = \eta_1(-a_{12}^{r_2 \ldots r_1 r_2(X)}) \ldots \eta_2(-a_{21}^{r_2(X)}) \eta_1(-a_{12}^{r_2(X)}) \eta_2(-a_{21}^X).$$

(5.1)

Assume that $a_{12}^Y, a_{21}^Y < -1$ for all $Y \in \mathcal{X}$. Then the matrices

$$\eta_1(a') \eta_2(a'') = \begin{pmatrix} a' a'' - 1 & -a' \\ a'' & -1 \end{pmatrix},$$

where $a' = -a_{12}^Y$ and $a'' = -a_{21}^Y$ for some $Y \in \mathcal{X}$, satisfy the assumption of Lemma 5.6. Lemma 5.6 implies that the set of products given in equation (5.1) is infinite. This is a contradiction to the finiteness of $\text{Hom}(W(\mathcal{C}))$. \hfill \qed
6. Weyl groupoids and root systems for Nichols algebras

Let \( \theta \in \mathbb{N} \) and \( \mathbb{I} = \{1, 2, \ldots, \theta\} \). The standard basis of \( \mathbb{Z}^\theta \) is denoted by \( \{ \alpha_1, \ldots, \alpha_\theta \} \). Recall the definition of \( \mathcal{F}_\theta \) and \( \mathcal{X}_\theta \) from the introduction. For all \( N = (N_1, \ldots, N_\theta) \in \mathcal{F}_\theta \) we write \( \mathcal{B}(N) = \mathcal{B}(N_1 \oplus \ldots \oplus N_\theta) \). Let \( \mathcal{HYD}^N \) denote the set of isomorphism classes of finite-dimensional irreducible Yetter–Drinfeld modules over \( H \).

**Definition 6.1.** Let \( N = (N_1, \ldots, N_\theta) \in \mathcal{F}_\theta \). Assume that there exists a totally ordered index set \( (L, \leq) \) and a family \( (W_l)_{l \in L} \) of finite-dimensional irreducible \( \mathbb{N}_0^\theta \)-graded objects in \( \mathcal{HYD}^N \) such that

\[
\mathcal{B}(N) \cong \bigotimes_{l \in L} \mathcal{B}(W_l)
\]

as \( \mathbb{N}_0^\theta \)-graded objects in \( \mathcal{HYD}^N \), where \( \deg N_i = \alpha_i \) for all \( i \in \mathbb{I} \). In this case we say that \( N \) is decomposable. Define

\[
\tilde{\Delta}^{[N]} = \{(C, \alpha) \mid C \in \mathcal{HYD}^N, \alpha \in \mathbb{N}_0^\theta \setminus \{0\}, \exists l \in L : C = [W_l], \alpha = \deg W_l\},
\]

(6.1)

\[
\tilde{\Delta}_0^{[N]} = \{(C^*, -\alpha) \mid (C, \alpha) \in \tilde{\Delta}^{[N]}\},
\]

(6.2)

\[
\tilde{\Delta}^{[N]} = \tilde{\Delta}_0^{[N]} \cup \tilde{\Delta}^{-[N]},
\]

(6.3)

and for all \( (C, \alpha) \in \tilde{\Delta}^{[N]} \) let

\[
\text{mult}_{[N]}(C, \alpha) = \text{mult}_{[N]}(C^*, -\alpha) = |\{ l \in L \mid C = [W_l], \alpha = \deg W_l\}|.
\]

(6.4)

The sets

\[
\Delta_0^{[N]} = \{ \deg W_l \mid l \in L\}, \quad \Delta^{-[N]} = -\Delta_0^{[N]},
\]

(6.5)

\[
\Delta^{[N]} = \Delta_0^{[N]} \cup \Delta^{-[N]},
\]

(6.6)

are called the set of positive roots of \( N \), the set of negative roots of \( N \), and the set of roots of \( N \), respectively.

**Remark 6.2.** In [11], the root system of a Nichols algebra of diagonal type was defined using Kharchenko’s PBW-basis. Viewing the Yetter–Drinfeld modules \( W_l \) as generalized PBW-generators (see Remark 4.6) the definition of \( \Delta^{[N]} \) in equation (6.6) generalizes the root system given in [11].

Let \( N = (N_1, \ldots, N_\theta) \in \mathcal{F}_\theta \). By Theorem 4.5(1), \( \mathcal{B}(N) \) is decomposable, if all tensor powers of \( N_1 \oplus \ldots \oplus N_\theta \) are semisimple.

By Lemma 4.7, the definitions of \( \tilde{\Delta}_0^{[N]} \), \( \tilde{\Delta}^{[N]} \), \( \Delta_0^{[N]} \), and \( \Delta^{[N]} \) only depend on \( [N] \), but not on the choice of \( (L, \leq) \) and the family \( (W_l)_{l \in L} \).

**Lemma 6.3.** Let \( N, P \in \mathcal{F}_\theta \) such that \( \mathcal{B}(N) \) and \( \mathcal{B}(P) \) are decomposable. If \( \tilde{\Delta}^{[N]} = \tilde{\Delta}^{[P]} \), then \( [N] = [P] \).

**Proof.** See Lemma 4.7. \( \square \)

**Definition 6.4.** Let \( i \in \mathbb{I} \) and \( N = (N_1, \ldots, N_\theta) \in \mathcal{F}_\theta \). We say that \( N \) is \( i \)-finite if, for all \( j \in \mathbb{I} \setminus \{i\} \), we have \( (\text{ad}_c N_i)^h(N_j) = 0 \) for some \( h \in \mathbb{N} \).
If $N$ is not $i$-finite, then let $R_i(N) = N$ and $r_i([N]) = [N]$. Assume now that $N$ is $i$-finite. Let $a_{ij}^N = 2$ and, for all $j \in \mathbb{I} \setminus \{i\}$, let

$$-a_{ij}^N = \sup \{h \in \mathbb{N}_0 | (\text{ad}_c N_i)^h(N_j) \neq 0 \text{ in } B(N) \}.$$  \hfill (6.7)

Let $R_i(N) = (N_1', \ldots, N_0')$ and $r_i([N]) = ([N_1'], \ldots, [N_0'])$, where $N_0' = N_i^*$ and $N_j' = (\text{ad}_c N_i)^{-a_{ij}^N}(N_j)$ for all $j \in \mathbb{I} \setminus \{i\}$. This way we obtain the maps $R_i: \mathcal{F}_\theta \to \mathcal{F}_\theta$ and $r_i: \mathcal{X}_\theta \to \mathcal{X}_\theta$. Define $s_i^N \in \text{Aut}(\mathbb{Z}^\theta)$ by

$$s_i^N(\alpha_j) = \alpha_j - a_{ij}^N \alpha_i \quad \text{for all } j \in \mathbb{I},$$  \hfill (6.8)

$$s_i^N(C, \gamma) = (C, s_i^N(\gamma)) \quad \text{for all } C \in H^\theta_1 \mathcal{YD}, \gamma \in \mathbb{Z}^\theta. \hfill (6.9)$$

If $N$ is $i$-finite, then the map $s_i^N$ is a reflection in the sense of [8, Chapter V, \S 2.2], since $a_{ii}^N = 2$. In this case the definition of $R_i$ coincides with the definition of $R_i$ in [2, Equation (3.16)]. Further, $s_i^N$ only depends on $i$ and the isomorphism class $[N]$, and we also write $s_i^{[N]} = s_i^N$.

The following proposition shows how to compute the Yetter-Drinfeld module $(\text{ad}_c V)^n(W)$, where $n \in \mathbb{N}$ and $V, W \in H^\theta_1 \mathcal{YD}$. Recall the definition of $S_n$ from equation (1.3).

**Proposition 6.5.** Let $n \in \mathbb{N}$ and $V, W \in H^\theta_1 \mathcal{YD}$. Then the image of the linear map $(S_n \otimes \text{id})T_n \in \text{End}(V \otimes \cdots \otimes V \otimes W)$, where

$$T_n = (\text{id} - c_{n,n+1}^2 c_{n-1,n} \cdots c_{12}) \cdots (\text{id} - c_{n,n+1}^2 c_{n-1,n}) (\text{id} - c_{n,n+1}^2),$$

is isomorphic to $(\text{ad}_c V)^n(W) \subset B(V \oplus W)$ in $H^\theta_1 \mathcal{YD}$.

**Proof.** The kernel of $S_{n+1}$ is isomorphic to $B(V \oplus W)^{n+1}$; see Section 1. Hence the image of $S_{n+1}$ is isomorphic to $B(V \oplus W)^{n+1}$ in $H^\theta_1 \mathcal{YD}$. The lemma follows from

$$(S_n \otimes \text{id})T_n(v_1 \otimes \cdots \otimes v_n \otimes w) = S_{n+1}((\text{ad}_c v_1) \cdots (\text{ad}_c v_n)(w))$$  \hfill (6.10)

for all $v_1, \ldots, v_n \in V$ and $w \in W$, where

$$\text{ad}_c v : (V \oplus W)^{\otimes k} \longrightarrow (V \oplus W)^{\otimes k+1}, \quad x \longmapsto v \otimes x - (v_{(-1)} \cdot x) \otimes v_{(0)}$$

for all $k \in \mathbb{N}_0$, $v \in V$, and $x \in (V \oplus W)^{\otimes k}$. By equation (1.3) it suffices to show that

$$T_n(v_1 \otimes \cdots \otimes v_n \otimes w) = S_{n+1}((\text{ad}_c v_1) \cdots (\text{ad}_c v_n)(w))$$

for all $v_1, \ldots, v_n \in V$ and $w \in W$. This follows by induction on $n$ from the braid equation for $c$.

Let $N \in \mathcal{F}_\theta$ and $i \in \mathbb{I}$. Let $\mathcal{K}_i^N = B(N)^c \mathcal{B}(N_i)$ be the algebra of right coinvariant elements with respect to the coaction $(\text{id} \otimes \pi) \Delta_B(N)$, where $\pi : B(N) \to B(N_i)$ is the canonical projection; see [2, Section 3.2]. Further, $B(N), \mathcal{K}_i^N,$ and $B(N_i)$ are $\mathbb{Z}^\theta$-graded objects in $H^\theta_1 \mathcal{YD}$ with degree $N_j = \alpha_j$ for all $j \in \mathbb{I}$. Then

$$B(N) \simeq \mathcal{K}_i^N \otimes B(N_i)$$  \hfill (6.11)

as $\mathbb{N}_0^\theta$-graded objects in $H^\theta_1 \mathcal{YD}$; see [2, Lemma 3.2(ii)] and the discussion in [2, Section 3.4].

**Lemma 6.6.** Let $N \in \mathcal{F}_\theta$ and $i \in \mathbb{I}$. Assume that $B(N)$ is decomposable.

(1) Let $L$ and $(W_i)_{i \in L}$ be as in Definition 6.1. Then $\mathcal{K}_i^N \simeq \bigotimes_{i \in L, i \neq \theta} B(W_i)$ as $\mathbb{N}_0^\theta$-graded objects in $H^\theta_1 \mathcal{YD}$.

(2) Let $i, j \in \mathbb{I}, i \neq j$, and $n \in \mathbb{N}_0$ such that $(\text{ad}_c N_i)^n(N_j) \neq 0$. Then $\alpha_j + m \alpha_i \in \Delta^{[N]}$ for all $0 \leq m \leq n$. 


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Proof. (1) Since $B(N)$ is decomposable,

$$B(N) \simeq \bigotimes_{l \in L, l \neq i(i)} B(W_l) \otimes B(N_i) \otimes \bigotimes_{l \in L, l \neq i(i)} B(W_l)$$

by Lemma 4.9. Since the category $H_Y^rYD$ is braided, it follows that

$$B(N) \simeq \bigotimes_{l \in L, l \neq i(i)} B(W_l) \otimes B(N_i).$$

(6.12)

Thus equations (6.11) and (6.12), and Lemma 4.8 imply claim (1).

Now we prove (2). By [2, Proposition 3.6] the algebra $K_N^N$ is generated by the $N_0$-homogeneous subspaces $(ad_i N_i)^m(N_p)$ of degree $\alpha_p + m\alpha_i$, where $p \in \mathbb{N} \setminus \{i\}$ and $m \geq 0$, and hence

$$(K_i^N)_{\alpha_j + m\alpha_i} = (ad_i N_i)^m(N_j).$$

(6.13)

On the other hand, since $\deg W_l \in (\sum_{j=1}^N N_0\alpha_j) \setminus N_0\alpha_i$ for all $l \in L \setminus \{l(i)\}$ by Lemma 4.9, and all tensor factors are $N_0$-graded, we obtain that

$$\left( \bigotimes_{l \in L, l \neq i(i)} B(W_l) \right)_{\alpha_j + m\alpha_i} \simeq \bigoplus_{l \in L, \deg W_l = \alpha_j + m\alpha_i} W_l.$$

(6.14)

Then part (1) of the lemma and equations (6.13) and (6.14) give claim (2).

The following theorem is one of the main results of [2].

**Theorem 6.7** [2, Theorem 3.12, Lemma 3.21, Corollary 3.17]. Let $i \in \mathbb{N}$ and $N = (N_1, \ldots, N_0) \in \mathcal{F}_b$. Assume that $N$ is $i$-finite, and let $R_i(N) = (N_{i1}, \ldots, N_{i0})$. Then the following hold.

1. The family $R_i(N)$ is $i$-finite, $[R_i^2(N)] = [N]$, and $a_{ij}^N = a_{ij}^{R_i(N)}$ for all $j \in \mathbb{N}$. Moreover, if $N$ is $j$-finite for all $j \in \mathbb{N}$, then $A^N = (a^N_{jk})_{j,k \in \mathbb{N}}$ is a generalized Cartan matrix.
2. Let $\deg N_j = \alpha_j$ for all $j \in \mathbb{N}$, $\deg N_j = s_i^N(\alpha_j)$ for all $j \in \mathbb{N} \setminus \{i\}$, and $\deg N_i^* = -\alpha_i$. Then

$$B(R_i(N)) \simeq K_i^N \otimes B(N_i^*)$$

as $F^0$-graded objects in $H_Y^rYD$.
3. The Nichols algebras $B(N)$ and $B(R_i(N))$ have the same dimension.

**Lemma 6.8.** Let $N \in \mathcal{F}_b$ and $i \in \mathbb{N}$. Assume that $B(N)$ is decomposable and $N$ is $i$-finite. Then $B(R_i(N))$ is decomposable, and the map

$$s_i^N : \Delta^N \rightarrow \Delta^{R_i(N)}, \quad (C, \gamma) \mapsto (C, s_i^N(\gamma)),$$

is bijective and preserves multiplicities, that is

$$\text{mult}[N](C, \gamma) = \text{mult}[R_i(N)](C, s_i^N(\gamma))$$

for all $(C, \gamma) \in \Delta^N$.

Proof. Since $B(N)$ is decomposable, there is an index set $(L, \leq)$ and a family $(W_l)_{l \in L}$ as in Definition 6.1 such that $B(N) \simeq \bigotimes_{l \in L} B(W_l)$. 


Let $R_i(N) = (N_1', \ldots, N_n') \in \mathcal{F}_\theta$ as in Definition 6.4. By Lemma 6.6 and Theorem 6.7(2), we obtain the decomposition

$$\mathcal{B}(R_i(N)) \simeq \bigotimes_{l \in L, l \neq l(i)} \mathcal{B}(W_l) \otimes \mathcal{B}(N_i^*)$$

(6.15)
as $\mathbb{Z}^\theta$-graded objects in $\mathcal{H}_\mathcal{Y} \mathcal{D}$. We now use the group automorphism $s_i^N : \mathbb{Z}^\theta \to \mathbb{Z}^\theta$ to change the gradation. For all $l \in L$ and $l \neq l(i)$, let $W_l' = W_l$ as an object in $\mathcal{H}_\mathcal{Y} \mathcal{D}$ and let $\deg(W_l') = s_i^N(\deg(W_l))$, and $W_l' = N_i^*$ with $\deg(N_i^*) = \alpha_i$. The isomorphism in (6.15) gives an isomorphism

$$\mathcal{B}(R_i(N)) \simeq \bigotimes_{l \in L} \mathcal{B}(W_l')$$

(6.16)
of $\mathbb{Z}^\theta$-graded objects in $\mathcal{H}_\mathcal{Y} \mathcal{D}$, where $\deg N_j = \alpha_j$ for all $j \in \mathbb{I}$. Thus we have

$$\bar{\Delta}^{|R_i(N)|} = \{([W_l], s_i^N(\deg(W_l))) | l \in L, l \neq l(i)\} \cup \{([N_i^*], \alpha_i)\}$$

$$= s_i^N(\bar{\Delta}^{|N_i^*|} \setminus \{([N_i], \alpha_i)\}) \cup \{([N_i^*], -\alpha_i)\}.$$ 

Since $\bar{\Delta}^{|P\Gamma|} = \bar{\Delta}^{|P\Gamma|} \cup \bar{\Delta}^{|P\Gamma|}$ for all $P \in \mathcal{F}_\theta$, it follows that $\bar{\Delta}^{|R_i(N)|} = s_i^N(\bar{\Delta}^{|N|})$. Moreover, for all $\gamma \in \mathbb{N}_0^\mathbb{I}$, $\gamma \neq \alpha_i$, and $C \in \mathcal{H}_\mathcal{Y} \mathcal{D}$, we have

$$\text{mult}_{|N|}(C, \gamma) = \{|l \in L | C = [W_l], \gamma = \deg(W_l)|\}$$

$$= \{|l \in L | C = [W_l'], s_i^N(\gamma) = \deg(W_l')|\}$$

$$= \text{mult}_{|R_i(N)|}(C, s_i^N(\gamma)),$$

and

$$\text{mult}_{|N|}(C, \alpha_i) = 1 = \text{mult}_{|R_i(N)|}(C^*, \alpha_i) = \text{mult}_{|R_i(N)|}(C, s_i^N(\alpha_i))$$

with $C = [N_i]$. This proves the lemma.

**Definition 6.9.** For all $M \in \mathcal{F}_\theta$ let

$$\mathcal{F}_\theta(M) = \{R_{i_1} \ldots R_{i_n}(M) \in \mathcal{F}_\theta | n \in \mathbb{N}_0, i_1, \ldots, i_n \in \mathbb{I}\},$$

$$\mathcal{X}_\theta(M) = \{r_{i_1} \ldots r_{i_n}(M) \in \mathcal{X}_\theta | n \in \mathbb{N}_0, i_1, \ldots, i_n \in \mathbb{I}\}.$$ 

We say that $M$ admits all reflections, if $N$ is $i$-finite for all $i \in \mathbb{I}$ and $N \in \mathcal{F}_\theta(M)$. In this case let $A^{[N]} = (a_{ij}^{[N]}, i,j \in \mathbb{I})$ for all $N \in \mathcal{F}_\theta(M)$, and define

$$\mathcal{C}(M) = (\mathbb{I}, \mathcal{X}_\theta(M), (r_i|_{\mathcal{X}_\theta(M)}), (A^X)_{X \in \mathcal{X}_\theta(M)}).$$

**Theorem 6.10.** Let $M \in \mathcal{F}_\theta$. Assume that $M$ admits all reflections. Then $\mathcal{C}(M)$ is a Cartan scheme.

**Proof.** This follows from Theorem 6.7(1).

In the situation of Theorem 6.10, $\mathcal{C}(M)$ is called the Cartan scheme of $M$, and $\mathcal{W}(M) = \mathcal{W}(\mathcal{C}(M))$ is called the Weyl groupoid of $M$. It is clear by construction that $\mathcal{W}(M)$ is connected.

**Theorem 6.11.** Let $M \in \mathcal{F}_\theta$. Assume that $M$ admits all reflections and that $\mathcal{B}(M)$ is decomposable. Then

$$\mathcal{R}(M) = (\mathcal{C}(M), (\Delta^X)_{X \in \mathcal{X}_\theta(M)})$$

is a root system of type $\mathcal{C}(M)$, called the root system of $M$. 


Proof. We have to prove (R1)–(R4) from Section 5. 
(R1) Follows from equation (6.6) and (R2) holds by Lemma 4.9; (R3) is a direct consequence of Lemma 6.8.
(R4) Let $X \in \mathcal{X}_\theta(M)$, and fix $i, j \in I$ with $i \neq j$. Let $w_0 = id \in \text{Aut}(\mathbb{Z}^\theta)$, and for all $n \in \mathbb{N}_0$ let

$$w_{n+1} = s_i(r_i, r_j)^n(X) s_j(r_j, r_i)^n(X) w_n \in \text{Aut}(\mathbb{Z}^\theta).$$

By [15, Lemma 5] we obtain that

$$w_{m_{i,j}^X}(\alpha_i) = \alpha_i, \quad w_{m_{i,j}^X}(\alpha_j) = \alpha_j.$$ 

Note that the proof of [15, Lemma 5] applies, since in our case $\pi_a$ in [15] is always the standard basis.

Let $k \in I \setminus \{i, j\}$. By (R3) we have

$$w_{m_{i,j}^X}(\alpha_k) \in \alpha_k + \mathbb{Z}\alpha_i + \mathbb{Z}\alpha_j \cap \Delta(m_{i,j}^X)(X),$$

and hence $w_{m_X}(\alpha_k) = \alpha_k + n_i \alpha_i + n_j \alpha_j$ for some $n_i, n_j \in \mathbb{N}_0$. Since $\alpha_k \in \Delta(m_{i, j}^X)(X)$, it follows from (R3) that

$$w_{m_{i,j}^X}^{-1}(\alpha_k) = w_{m_{i,j}^X}^{-1}(w_{m_{i,j}^X}(\alpha_k) - n_i \alpha_i - n_j \alpha_j) = \alpha_k - n_i \alpha_i - n_j \alpha_j \in \Delta^X.$$

Hence $n_i = n_j = 0$ by (R1), that is, $w_{m_{i,j}^X} = id$. Then

$$\Delta^X = w_{m_{i,j}^X}(\Delta^X) = \Delta(r_i, r_j)(X)$$

by Lemma 6.8. Thus $(r_i, r_j)(m_{i,j}^X)(X) = X$ by Lemma 6.3. \hfill $\Box$

**Corollary 6.12.** Let $M \in \mathcal{F}_\theta$. Assume that $\mathcal{B}(M)$ is decomposable and that $\Delta^{[M]}$ is finite. Then $M$ admits all reflections and $\mathcal{R}(M)$ is a root system of type $C(M)$.

Proof. By Theorem 6.11 it suffices to prove that $R_{i_1} \ldots R_{i_1}(M)$ is $i$-finite for all $n \in \mathbb{N}_0$ and $i, i_1, \ldots, i_n \in I$. We proceed by induction on $n$. The case $n = 0$ is trivial. Let $n \in \mathbb{N}_0$, $i_1, \ldots, i_n \in I$, and $N = R_{i_1} \ldots R_{i_1}(M)$. Since $\mathcal{B}(M)$ is decomposable and $\Delta^{[M]}$ is finite, Lemma 6.8 implies that $\mathcal{B}(N)$ is decomposable and $\Delta^{[N]}$ is finite. Then from Lemma 6.6(2) we obtain that, for all $i, j \in I$, with $i \neq j$, there exists $n_{ij} \in \mathbb{N}$ such that $(\text{ad}_N)^{n_{ij}}(N_j) = 0$ in $\mathcal{B}(N)$, where $N = (N_1, \ldots, N_\theta)$. Hence $N$ is $i$-finite for all $i \in I$. \hfill $\Box$

The growth of the root system gives information about the growth of the corresponding Nichols algebra. The following lemma is based on an idea of Rosso [23, Lemma 19]. For any $\alpha = \sum_{i=1}^\theta m_i \alpha_i \in \mathbb{N}_0^\theta$ let $|\alpha| = \sum_{i=1}^\theta m_i$.

**Lemma 6.13.** Let $M \in \mathcal{F}_\theta$ such that $\mathcal{B}(M)$ is decomposable and $\Delta^{[M]}$ is finite. Let $(\beta_l)_{l \in \mathbb{N}}$ be a sequence of pairwise distinct elements of $\Delta^{[M]}$ with $|\beta_l| \leq |\beta_{l'}|$ whenever $l \leq l'$. Assume that there exist $p \in \mathbb{R}[x]$ and $n_0 \in \mathbb{N}_0$ such that $|\beta_n| \leq p(n)$ for all $n > n_0$. Then $\text{GK dim } \mathcal{B}(M) = \infty$.

Proof. By weakening the bounds $p(n)$, it suffices to consider the case $n_0 = 0$ and $p(x) = x^{a'} + b'$, where $a', b' \in \mathbb{N}$. Let $a, b \in \mathbb{N}$ such that $\sum_{l=1}^n b(l) \leq n^{a} + b$ for all $n \in \mathbb{N}$, and let $g(n) = n^a + b$ for all $n \in \mathbb{N}$. 
Let $k \in \mathbb{N}$. Since $B(M)$ is decomposable, Definition 6.1 implies that there exist $W_1, W_2, \ldots, W_k \in \widehat{\mathcal{YD}}$, a permutation $\sigma_k$ of $\{1, 2, \ldots, k\}$, and an injective map

$$W' := W'_1 \otimes W'_2 \otimes \ldots \otimes W'_k \to B(M)$$

of $\mathbb{N}_0^k$-graded objects in $\mathbb{YD}$, where $W'_i = k \oplus W_i$, $\deg W_i = \beta_{\sigma_k(i)}$ for all $i \in \{1, 2, \ldots, k\}$, and $\deg k = 0$. Note that $\dim W' \geq 2^k$ and that

$$|\beta| \leq \sum_{l=1}^k |\beta_l| \leq \sum_{l=1}^k p(l) \leq q(k)$$

for each homogeneous component of $W'$ of degree $\beta$, by the assumption on $p$ and the definition of $q$. Hence

$$\dim \bigoplus_{i=0}^{q(k)} B^i(M) \geq 2^k.$$ 

Let $d(n) = \dim \oplus_{i=0}^n B^i(M)$ for all $n \in \mathbb{N}$. Then we have

$$\text{GK dim } B(M) = \limsup_{n \to \infty} \frac{\log d(n)}{\log n} \geq \limsup_{n \to \infty} \frac{\log d(q(n))}{\log 2^n} \geq \limsup_{n \to \infty} \frac{\log 2^n}{\log(n^a + b)} = \infty. \tag{6.17}$$

This proves the lemma. \hfill \Box

As an application we obtain a generalization of [23, Lemma 20].

**Proposition 6.14.** Let $M \in \mathcal{F}_\vartheta$. If $B(M)$ is decomposable and $\text{GK dim } B(M)$ is finite, then $M$ is $i$-finite for all $i \in \mathbb{I}$.

**Proof.** Assume that $B(M)$ is decomposable. Let $i, j \in \mathbb{I}$, with $i \neq j$, and assume that $(\text{ad}_{M_i}^n(M_j)) \neq 0$ for all $n \in \mathbb{N}_0$. Then $\alpha_j + n\alpha_i \in \Delta^M_+$ for all $n \in \mathbb{N}_0$ by Lemma 6.6(ii). Thus Lemma 6.13 with $(\beta_l)_{l \in \mathbb{N}} = (\alpha_j + (l - 1)\alpha_i)_{l \in \mathbb{N}}$, $n_0 = 0$, and $p(x) = x$ yields the claim. \hfill \Box

The following theorem and its corollaries are new even for Nichols algebras of diagonal type.

**Theorem 6.15.** Let $M \in \mathcal{F}_\vartheta$. If $B(M)$ is decomposable, then $\text{GK dim } B(M) = \text{GK dim } B(R_i(M))$ for all $i \in \mathbb{I}$.

**Proof.** Let $i \in \mathbb{I}$. If $M$ is not $i$-finite, then $R_i(M) = M$ and the claim holds tautologically. Assume now that $M$ is $i$-finite, that is,

$$(K^M_i)_{1} := k \bigoplus_{j \neq i} (\text{ad } B(M_j))(M_j)$$

is finite-dimensional. By equation (6.11), we have $B(M) \simeq K^M_i # B(M_i)$. Further, $K^M_i$ is generated by $(K^M_i)_{1}$ by [2, Proposition 3.6]. By definition of $(K^M_i)_{1}$ and by [2, Lemmas 3.3 and 3.10], we have

$$M_i(K^M_i)_{1} \subset (K^M_i)_{1}M_i + (K^M_i)_{1} \text{ in } K^M_i # B(M_i),$$

$$M_i^*(K^M_i)_{1} \subset (K^M_i)_{1}M_i^* + (K^M_i)_{1} \text{ in } K^M_i # B(M^*_i).$$
Thus \((K^M_i)_{i=1} + M_i\) generates \(\mathcal{B}(M)\) and \((K^M_i)_{i=1} + M_i^*\) generates \(K^M_i \# \mathcal{B}(M_i^*)\), and
\[
((K^M_i)_{i=1} + M_i)^n = \bigoplus_{k=0}^{n} (K^M_i)_{i=1}^{n-k} \# M_i^k \quad \text{in} \quad K^M_i \# \mathcal{B}(M_i),
\]
(6.18)
for all \(n\), where \(M_i^k\) means \(k\)-fold product of \(M_i\) in \(\mathcal{B}(M_i)\). Recall that both \(\mathcal{B}(M_i)\) and \(\mathcal{B}(M_i^*)\) are finitely generated \(\mathbb{N}_0\)-graded algebras, and there is a non-degenerate dual pairing between them which is compatible with the grading. Thus \(\dim M_i^l = \dim(M_i^*)^l < \infty\) for all \(l \in \mathbb{N}_0\). Therefore the definition of \(\text{GK dim}\) and equation (6.18) imply that
\[
\text{GK dim} \mathcal{B}(M) = \limsup_{n \to \infty} \frac{\log \dim((K^M_i)_{i=1} + M_i)^n}{\log n} = \text{GK dim} K^M_i \# \mathcal{B}(M_i^*).
\]
Hence the theorem holds since \(\mathcal{B}(R_i(M)) \simeq K^M_i \# \mathcal{B}(M_i^*)\) as algebras by [2, Theorem 3.12].

From Theorem 6.15, Lemma 6.8, and the definition of \(\mathcal{F}_\theta(M)\) we conclude the following.

**Corollary 6.16.** Let \(M \in \mathcal{F}_\theta\). If \(\mathcal{B}(M)\) is decomposable, then \(\text{GK dim} \mathcal{B}(M) = \text{GK dim} \mathcal{B}(N)\) for all \(N \in \mathcal{F}_\theta(M)\).

By Proposition 6.14 we further obtain the following corollary.

**Corollary 6.17.** Let \(M \in \mathcal{F}_\theta\). If \(\mathcal{B}(M)\) is decomposable and \(\text{GK dim} \mathcal{B}(M)\) is finite, then \(M\) admits all reflections and \(\text{GK dim} \mathcal{B}(N) = \text{GK dim} \mathcal{B}(M)\) for all \(N \in \mathcal{F}_\theta(M)\).

7. Finite root systems for Nichols algebras

**Lemma 7.1.** Let \(M = (M_1, \ldots, M_0) \in \mathcal{F}_\theta\) such that \(M\) admits all reflections and \(\mathcal{B}(M)\) is decomposable. Let \((L, \leq)\) be a totally ordered index set and let \((W_l)_{l \in L}\) be a family of finite-dimensional irreducible \(\mathbb{Z}^0\)-graded objects in \(\mathcal{H}_\gamma \mathcal{YD}\), such that
\[
\mathcal{B}(M) \simeq \bigotimes_{l \in L} \mathcal{B}(W_l)
\]
(7.1)
as \(\mathbb{Z}^0\)-graded objects in \(\mathcal{H}_\gamma \mathcal{YD}\), where \(\deg(M_i) = \alpha_i\) for all \(i \in \mathbb{I}\). Then the following hold.

(1) For all \(\gamma \in \Delta^+[M]\) there is exactly one \(l(\gamma) \in L\) such that \(\deg W_l(\gamma) = \gamma\).

(2) If \(N = (N_1, \ldots, N_0) \in \mathcal{F}_\theta(M), w \in \text{Hom}([N], [M]),\) and \(i \in \mathbb{I}\) such that \(w(\alpha_i) \in \Delta^+[M]\), then \(W_l(w(\alpha_i)) \simeq N_i\) in \(\mathcal{H}_\gamma \mathcal{YD}\).

**Proof.** The decomposition (7.1) exists since \(\mathcal{B}(M)\) is decomposable.

Let \(\gamma \in \Delta^+[M]\). There exist \(N = (N_1, \ldots, N_0) \in \mathcal{F}_\theta(M), w \in \text{Hom}([N], [M]),\) and \(i \in \mathbb{I}\) such that \(w(\alpha_i) = \gamma\). Since \([N_i], \alpha_i \in \Delta^+[N]\), it follows that Lemma 6.8 implies that \([N_i], \gamma) \in \Delta^+[M]\), that is, there exists \(l \in L\) such that \(\deg W_l = \gamma\) and \(W_l \simeq N_i\).

Let \(l, l' \in L\) with \(\gamma = \deg(W_l) = \deg(W_{l'}) \in \Delta^+[M]\). Let \(i \in \mathbb{I}, N \in \mathcal{F}_\theta(M),\) and \(w \in \text{Hom}([N], [M]) \subset \text{Hom}(\mathcal{W}(M))\) such that \(w(\alpha_i) = \gamma\). Lemma 6.8 implies that \(w^{-1}([W_l], \gamma) = ([W_l], \alpha_i) \in \Delta^+[N]\), and similarly \(([W_{l'}], \alpha_i) \in \Delta^+[N]\). By Lemma 4.9 we obtain that \([W_l] = [N_i] = [W_{l'}]\). The second part of Lemma 6.8 implies that \(\text{mult}_{[M]}([W_l], \gamma) = \text{mult}_{[N]}([N_i], \alpha_i) = 1\). Hence \(l = l'\). This proves (1) and (2).
THEOREM 7.2. Let $M \in \mathcal{F}_0$. Assume that $M$ admits all reflections, $B(M)$ is decomposable, and $W(M)$ is finite. Let $(L, \leqslant)$ be a totally ordered index set and let $(W_i)_{i \in L}$ be a family of finite-dimensional irreducible $\mathbb{Z}^\theta$-graded objects in $\mathcal{H}_D \mathcal{YD}$, such that

$$B(M) \simeq \bigotimes_{i \in L} B(W_i)$$

(7.2)
as $\mathbb{Z}^\theta$-graded objects in $\mathcal{H}_D \mathcal{YD}$, where $\deg(M_i) = \alpha_i$ for all $i \in \mathbb{I}$. Then the following hold.

1. The map $L \to \Delta^\text{re}_+, l \mapsto \deg(W_i)$, is bijective and

$$\{ [W_i], [W_i^*] \mid l \in L \} = \{ [N_i] \mid i \in \mathbb{I}, N = (N_1, \ldots, N_\theta) \in \mathcal{F}_0(M) \}.$$  

(7.3)

2. If $\gamma \in \Delta^\text{re}_+$, then there exist $N \in \mathcal{F}_0(M)$, $w \in \text{Hom}([N], [M])$, and $i \in \mathbb{I}$ such that $\gamma = w(\alpha_i)$. In this case

$$W_{l(\gamma)} \simeq N_i$$
in $\mathcal{H}_D \mathcal{YD}$, where $l(\gamma)$ is the unique element in $L$ with $\deg W_{l(\gamma)} = \gamma$.

3. Let $i, j \in \mathbb{I}$, $i \neq j$, and $0 \leqslant m \leqslant -1_{\gamma_j}^M$. Then there is an index $l \in L$ such that $(\text{ad}_cM_i)^m(M_j) \simeq W_l$ and $\deg W_l = \alpha_j + m\alpha_i$. In particular $(\text{ad}_cM_i)^m(M_j)$ is irreducible in $\mathcal{H}_D \mathcal{YD}$.

Proof. Since $W(M)$ is finite, it follows that $\Delta^\text{re}_+ = \Delta^M$ by Proposition 5.2.

We first prove (1) and (2). The map $L \to \Delta^\text{re}_+$ is surjective by definition of $\Delta^\text{re}_+$. Hence it is bijective by Lemma 7.1(1) and the equality $\Delta^\text{re}_+ = \Delta^M$. Then (2) follows from Lemma 7.1(2), since $\Delta^\text{re}_+ = \Delta^M$. It remains to prove equation (7.3).

Let $l \in L$ and $\gamma = \deg W_l$. Then $l = l(\gamma)$ by (1). By (2) there exist $i \in \mathbb{I}$ and $N \in \mathcal{F}_0(M)$ such that $W_l \simeq N_i$. Further, if $N \in \mathcal{F}_0(M)$ and $i \in \mathbb{I}$, then $R_l(N) \in \mathcal{F}_0(M)$. Since $[R_l(N)i] = [N_i^*]$, the right-hand side of equation (7.3) is stable under passing to dual objects. Thus the inclusion $\subset$ holds in (3). Conversely, let $N \in \mathcal{F}_0(M)$ and $i \in \mathbb{I}$. Since $W(M)$ is a connected groupoid, there exists $w \in \text{Hom}([N], [M])$. Then Lemma 6.8 gives that $([N], w(\alpha_i)) \in \Delta^M$, and hence $[N_i]$ is contained in the left-hand side of equation (7.3).

Now we prove (3). Using Lemma 4.8, the isomorphism (6.11) for $N = M$ of $\mathbb{N}_0^\theta$-graded objects in $\mathcal{H}_D \mathcal{YD}$ implies that

$$K_i^M \simeq \bigotimes_{l \in L, l \neq l(i),} B(W_i)$$

(7.4)
as $\mathbb{N}_0^\theta$-graded objects in $\mathcal{H}_D \mathcal{YD}$. By [2, Proposition 3.6] the algebra $K_i^M$ is generated by the $\mathbb{N}_0^\theta$-homogeneous subspaces $(\text{ad}_cM_i)^n(M_p)$ of degree $\alpha_p + n\alpha_i$, where $n \geqslant 0$ and $p \in \mathbb{I} \setminus \{i\}$, and hence

$$(K_i^M)_{\alpha_j + m\alpha_i} = (\text{ad}_cM_i)^m(M_j).$$

(7.5)

On the other hand, since $\deg W_l \in (\sum_{j=1}^\theta \mathbb{N}_0\alpha_j) \setminus \mathbb{N}_0\alpha_i$ and all tensor factors are $\mathbb{N}_0^\theta$-graded, we obtain that

$$\bigotimes_{l \in L, l \neq l(i),} B(W_l)_{\alpha_j + m\alpha_i} \simeq W_{l(\alpha_j + m\alpha_i)}.$$  

(7.6)

Then equations (6.13), (6.14), and (7.4) give the claim of part (3).

THEOREM 7.3. Let $M \in \mathcal{F}_0$. Assume that $B(M)$ is decomposable. Then the following are equivalent:

1. $B(M)$ is finite-dimensional;
(2) (a) $M$ admits all reflections and $W(M)$ is finite, 
(b) $B(N_i)$ is finite-dimensional for all $N \in \mathcal{F}_\theta(M)$ and $i \in I$.

**Proof.** (1) $\Rightarrow$ (2). Since $B(M)$ is finite-dimensional, $N$ admits all reflections by Theorem 6.7(3). Further, $\Delta^{[M]}$ is finite, and hence $W(M)$ is finite by Lemma 5.1. Finally, $\dim B(N) = \dim B(M)$ by Theorem 6.7(3), and hence $B(N_i)$ is finite-dimensional for all $i \in I$ and $N \in \mathcal{F}_\theta(M)$.

(2) $\Rightarrow$ (1). From (2(a)) and Lemma 5.1 we obtain that $\Delta^{[M]}$ is finite. Hence $B(M)$ is finite-dimensional by (2(b)) and Theorem 7.2(1) and (2). 

Recall that $B(M)$ is decomposable, in particular, if all tensor powers of the Yetter–Drinfeld module $M_1 \oplus \ldots \oplus M_\theta$ are semisimple.

Let $M \in \mathcal{F}_\theta$. Then $M$ is called standard (see [2, Definition 3.23]) if $M$ admits all reflections and $A^N = A^M$ for all $N \in \mathcal{F}_\theta(M)$.

**Corollary 7.4.** Let $M \in \mathcal{F}_\theta$. Assume that $B(M)$ is decomposable, and that $M$ is standard. The following are equivalent:

(1) $B(M)$ is finite-dimensional;

(2) (a) $A^M$ is a Cartan matrix of finite type,
(b) $B(N_i)$ is finite-dimensional for all $N \in \mathcal{F}_\theta(M)$ and $i \in I$.

**Proof.** This follows from Theorem 7.3 and Corollary 5.4.

8. Applications for finite groups

In this section let $k$ be an algebraically closed field of characteristic 0, and let $H = kG$ be the group algebra of a finite group $G$. For any $g \in G$ let $O_g$ denote the conjugacy class of $g$ and let $G^g$ denote the centralizer of $g$ in $G$. Let $\triangleright$ denote the adjoint action in $G$, that is, $g \triangleright h = ghg^{-1}$ for all $g, h \in G$. The category $\underline{H}/YD$ will be denoted by $\underline{G}/YD$.

Let $g \in G$ and let $M$ be an irreducible $G^g$-module. Then $kG \otimes_{kG^g} M$ is an irreducible object in $\underline{G}/YD$, where

$$h \cdot (h' \otimes m) = hh' \otimes m, \quad \delta(h \otimes m) = ghg^{-1} \otimes (h \otimes m)$$

for all $h, h' \in G$ and $m \in M$. Any irreducible object in $\underline{G}/YD$ arises in this way. The Yetter–Drinfeld module $V = kG \otimes_{kG^g} M$ can be written as

$$V = \bigoplus_{s \in O_g} V_s, \quad V_s = \{v \in V | \delta(v) = s \otimes v\},$$

and $V_g = 1 \otimes M$. For all $s \in O_g$, the vector space $V_s$ is an irreducible $G^s$-module and $h \cdot V_s = V_{h \triangleright s}$ for all $h \in G$. There exists $q_v \in k^*$ such that $s \cdot v = q_v v$ for all $v \in V_s$ and $s \in O_g$.

Let $g, h \in G$. We say that $O_g$ and $O_h$ commute, if $st = ts$ for all $s \in O_g$ and $t \in O_h$.

**Proposition 8.1.** Let $g, h \in G$, and let $V = \bigoplus_{s \in O_g} V_s$ and $W = \bigoplus_{t \in O_h} W_t$ be irreducible objects in $\underline{G}/YD$. Then the following hold.

(1) If $(\text{ad}_V)(W) = 0$ in $B(V \oplus W)$, then $O_g$ and $O_h$ commute.

(2) If $(\text{ad}_V)^2(W) = 0$ in $B(V \oplus W)$, then $O_g$ commutes with itself or with $O_h$. 
Proof. (1) By Proposition 6.5, \((\text{ad}_V)(W)\) is isomorphic to \((\text{id} - c^2)(V \otimes W)\) in \(\mathcal{G}_G\mathcal{YD}\). If \(s \in \mathcal{O}_g, t \in \mathcal{O}_h,\) and \(st \neq ts\), then
\[
c^2(V_s \otimes W_t) = V_{st} \otimes W_{st},
\]
and hence \((\text{id} - c^2)(V_s \otimes W_t) \neq 0\).

(2) It suffices to show that if \((\text{ad}_V)^2(W) = 0\) and \(gh \neq hg\), then \(rg = gr\) for all \(r \in \mathcal{O}_g \setminus \{g\}\).

We note that if \(r \in \mathcal{O}_g, v \in V_r, v' \in V_g, w \in W_h,\) and \(v \otimes v' \otimes w \neq 0\), then \((S_2 \otimes \text{id})T_2(v \otimes v' \otimes w)\) is the sum of eight non-zero homogeneous terms of degrees
\[
(r, g, h), (r \triangleright g, r, h), (r, gh \triangleright g, g \triangleright h), (rgh \triangleright g, r, g \triangleright h),
(r \triangleright g, rh \triangleright r, r \triangleright h), (rgh \triangleright r, r \triangleright g, r \triangleright h),
(rgh \triangleright g, rgh^{-1} \triangleright r, r \triangleright h), (rgh \triangleright r, rgh \triangleright r, r \triangleright g, r \triangleright h).
\]

On the other hand, \((S_2 \otimes \text{id})T_2(v \otimes v' \otimes w) = 0\) by Proposition 6.5 and the assumption \((\text{ad}_V)(\text{ad}_V^2)(W_h) = 0\). The vanishing of the sum of these eight terms gives information about the corresponding degrees.

First let \(r \in \mathcal{O}_g\) such that \(rh \neq hr\) and \(r \neq g\). By comparing degrees we get \(rg \triangleright h = h\) and \(rgh \triangleright g = r\). Hence \(r = gh \triangleright g\) is uniquely determined. Since \(h \triangleright g\) and \(h^{-1} \triangleright g\) do not commute with \(h\) and are different from \(g\), we get
\[
r = h^{-1} \triangleright g = h \triangleright g = gh \triangleright g.
\]

Thus \(g = h^{-1} gh \triangleright g = r \triangleright g\) by the last and first equations in equation (8.1), that is, \(rg = gr\).

Now let \(r \in \mathcal{O}_g\) with \(rh = hr\). Then \(r \neq g\), and by comparing degrees one gets \(rg \triangleright h = g \triangleright h\) and \(r \triangleright g = g\). This proves the claim. \(\square\)

Let \(\mathcal{E}(G)\) be the set of all conjugacy classes \(\mathcal{O}\) of \(G\) such that \(\dim \mathcal{B}(V) < \infty\) for some \(V = \bigoplus_{s \in \mathcal{O}} V_s \in \mathcal{G}_G\mathcal{YD}\).

**Theorem 8.2.** Assume that any two conjugacy classes in \(\mathcal{E}(G)\) do not commute. Let \(0 \neq U \in \mathcal{G}_G\mathcal{YD}\). If \(\mathcal{B}(U)\) is finite-dimensional, then \(U\) is irreducible in \(\mathcal{G}_G\mathcal{YD}\).

**Proof.** Recall that the category \(\mathcal{G}_G\mathcal{YD}\) is semisimple, and that embeddings of Yetter–Drinfeld modules induce embeddings of the corresponding Nichols algebras. Hence it suffices to prove that \(\mathcal{B}(V \oplus W)\) is infinite-dimensional for all irreducible objects \(V, W \in \mathcal{G}_G\mathcal{YD}\).

Let \(g, h \in G\), and let \(V = \bigoplus_{s \in \mathcal{O}_g} V_s\) and \(W = \bigoplus_{t \in \mathcal{O}_h} V_t\) be irreducible objects in \(\mathcal{G}_G\mathcal{YD}\), and \(M = (V, W) \in \mathcal{F}_2\). Assume that \(\dim \mathcal{B}(V \oplus W) < \infty\). Then \(\mathcal{B}(V)\) and \(\mathcal{B}(W)\) are finite-dimensional, and hence \(\mathcal{O}_g, \mathcal{O}_h \in \mathcal{E}(G)\).

If \(a^M_{12} = 0\), then \((\text{ad}_V^4)(W) = 0\), and hence Proposition 8.1(1) gives that \(\mathcal{O}_g\) and \(\mathcal{O}_h\) commute. If \(a^M_{12} = -1\), then \((\text{ad}_V)^2(W) = 0\), and Proposition 8.1(2) gives that \(\mathcal{O}_g\) commutes with \(\mathcal{O}_h\) or with \(\mathcal{O}_h\). Thus the assumption in the theorem yields that \(a^M_{12} < -1\). Similarly, \(a^M_{21} < -1\). By Theorem 6.7(3), we have
\[
\dim \mathcal{B}(N_1 \oplus N_2) = \dim \mathcal{B}(V \oplus W)
\]
for all \(N \in \mathcal{F}_2(M)\) and \(N = (N_1, N_2)\), and hence the above arguments give that \(a^N_{12}, a^N_{21} < -1\) for all \(N \in \mathcal{F}_2(M)\). Since \(\text{Hom}(\mathcal{W}(M))\) is finite by Theorem 7.3, the theorem follows from Proposition 5.7. \(\square\)

Recall that if \(V = V_1\), then \(\dim \mathcal{B}(V) = \infty\), since the characteristic of \(k\) is 0. Thus \(\mathcal{O}_1 \notin \mathcal{E}(G)\).
Corollary 8.3. Let $G$ be a non-abelian simple group. Let $0 \neq U \in {}^G\mathcal{YD}$. If $B(U)$ is finite-dimensional, then $U$ is irreducible in $G\mathcal{YD}$.

Proof. Let $O'$ and $O''$ be two commuting conjugacy classes of $G$. Then the subgroups $G' = \langle O' \rangle$ and $G'' = \langle O'' \rangle$ of $G$ are normal. Since $G$ is simple, it follows that $G'$ and $G''$ are either 1 or $G$. Since $O'$ and $O''$ commute, it follows that $[G', G''] = 1$. Hence $G' = 1$ or $G'' = 1$, and hence $O' = 1$ or $O'' = 1$. However $1 \notin \mathcal{E}(G)$, and the claim follows from Theorem 8.2.

Corollary 8.4. Let $n \in \mathbb{N}$, with $n \geq 3$, and assume that $G = S_n$ is the symmetric group. Let $0 \neq U \in {}^G\mathcal{YD}$. If $B(U)$ is finite-dimensional, then $U$ is irreducible in $G\mathcal{YD}$.

Proof. For $n \geq 5$ the group $S_n$ has a simple group of index two. Along the lines of the proof of Corollary 8.3 it is easy to show that $S_n$, with $n \geq 5$, does not possess commuting non-trivial (that is, different from $\{1\}$) conjugacy classes. In the usual cycle notation, the only pairs of commuting non-trivial conjugacy classes of $S_n$ are $(O_{(123)}, O_{(123)})$ for $n = 3$ and $(O_{(12)(34)}, O_{(12)(34)})$ for $n = 4$. By [7, Theorem 1], $O_{(123)} \notin \mathcal{E}(S_3)$ and $O_{(12)(34)} \notin \mathcal{E}(S_4)$. Thus the claim follows from Theorem 8.2.

For the dihedral groups $D_n$, $n$ odd, an alternative proof of [2, Theorem 4.8] can be given as another application of Theorem 8.2.

We continue with some other consequences of our theory which hold for all finite groups.

Proposition 8.5. Let $g, h \in G$, and let $V = \bigoplus_{s \in O_g} V_s$ and $W = \bigoplus_{t \in O_h} W_t$ be irreducible objects in $G\mathcal{YD}$. Assume that $(\text{ad}_cV)(W)$ is irreducible. Then the following hold.

1. We have $stst = tts$ for all $s \in O_g$ and $t \in O_h$.

2. There is at most one double coset $G^h x G^g$ in $G^h \setminus G/G^g$, with $x \in G$, such that $(x \triangleright g)h \neq h(x \triangleright g)$. If it exists, then $(\text{ad}_cV)(W) \neq 0$ and $q(\text{ad}_cV)(W) = -qVW$.

Proof. By Proposition 6.5, $(\text{ad}_cV)(W)$ is isomorphic to $(id - c^2)(V \otimes W)$ in $G\mathcal{YD}$. Let $s \in O_g$, $t \in O_h$, $v \in V_s$, and $w \in W_t$. Then

$$c^2(v \otimes w) = c((s \cdot w) \otimes v) = sts^{-1} \cdot v \otimes s \cdot w.$$ 

Since $s \cdot v = qVv$ and $t \cdot w = qWw$, we obtain that

$$(id - c^2)(v \otimes w) = v \otimes w - qW^{-1}st \cdot (v \otimes w).$$

(8.2)

Since $(\text{ad}_cV)(W)$ is irreducible, there exists $q \in k^*$ such that

$$st \cdot (id - c^2)(v \otimes w) = q(id - c^2)(v \otimes w).$$

(8.3)

Comparing degrees we obtain that $st \triangleright t$ or $(st)^2 \triangleright t = t$. Thus $(st)^2 = (ts)^2$. This proves (1).

If $X \subset G$ is a set of double coset representatives for $G^h \setminus G/G^g$, then

$$V \otimes W = \bigoplus_{x \in X} kG \cdot ((x \cdot V_g) \otimes W_h)$$

is a decomposition into Yetter–Drinfeld modules over $G$. By equation (8.2),

$$(id - c^2)(V \otimes W) = \bigoplus_{x \in X} kG \cdot (id - c^2)((x \cdot V_g) \otimes W_h)$$

is a decomposition into Yetter–Drinfeld modules over $G$. Let $x \in X$. Assume that $(x \triangleright g)h \neq h(x \triangleright g)$. Then $(id - c^2)((x \cdot V_g) \otimes W_h) \neq 0$ by equation (8.2). Since $(\text{ad}_cV)(W)$ is
irreducible by the assumption, Proposition 6.5 implies that \((\text{id} - c^2)(V \otimes W)\) is irreducible. Thus \((\text{id} - c^2)(y \cdot V_g) \otimes W_h = 0\) and hence \((y \triangleright g)h = h(y \triangleright g)\) for all \(y \in X \setminus \{x\}\). The claim \(q_{(\text{id}, V)}(W) = -qv q_w\) follows from equation (8.3) for \(s = x \triangleright g\) and \(t = h\).

**Theorem 8.6.** Let \(g, h \in G\), and let \(V = \bigoplus_{s \in \mathcal{O}_g} V_s\) and \(W = \bigoplus_{t \in \mathcal{O}_h} W_t\) be irreducible objects in \(\mathcal{G} \mathcal{Y} \mathcal{D}\). If \(B(V \oplus W)\) is finite-dimensional, then the following hold.

1. For all \(s \in \mathcal{O}_g\) and \(t \in \mathcal{O}_h\), we have \((st)^2 = (ts)^2\).
2. There is at most one double coset \(G^h x G^g\) in \(G^h \backslash G/G^g\), with \(x \in G\), such that \((x \triangleright g)h \neq h(x \triangleright g)\).
3. For all \(s \in \mathcal{O}_g\) and \(t \in \mathcal{O}_h\) with \(st \neq ts\) there is an irreducible object \(U = \bigoplus_{r \in \mathcal{O}_{st}} U_r \in \mathcal{G} \mathcal{Y} \mathcal{D}\) satisfying \(q_U = -qv q_w\) and \(\dim B(U) < \infty\).

**Proof.** By Theorems 7.2(3) and 7.3, \((\text{id}, V)(W)\) is either 0 or irreducible. Then (1) and (2) follow from Proposition 8.5(2).

(3) Assume that \(s \in \mathcal{O}_g\) and \(t \in \mathcal{O}_h\) such that \(st \neq ts\). Then \(U = (\text{id}, V)(W) \neq 0\) by equation (8.2), and hence \(U = \bigoplus_{r \in \mathcal{O}_{st}} U_r\) is irreducible. Therefore \(q_U = -qv q_w\) by Proposition 8.5(2), and \(B(U)\) is finite-dimensional by Theorem 7.2 and since \(B(V \oplus W)\) is finite-dimensional.

**Acknowledgement.** We would like to express our thanks to G. Malle for providing us with information on commuting conjugacy classes of finite groups.

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