Remark on the ill-posedness for the multidimensional chemotaxis equations in critical Besov spaces

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Abstract: In this paper, we aim to solving the open question left in [Nie, Yuan: Nonlinear Anal 196 (2020); J. Math. Anal. Appl 505 (2022)] and Xiao, Fei: J. Math. Anal. Appl 514 (2022)]. We prove that a multidimensional chemotaxis system is ill-posedness in $\dot{B}_{2d,r}^{-\frac{3}{2}} \times (\dot{B}_{2d,r}^{-\frac{1}{2}})^d$ when $1 \leq r < d$ due to the lack of continuity of the solution.

Keywords: Multidimensional chemotaxis equations, Ill-posedness, Besov spaces

MSC (2010): 35G55; 35Q92; 92C17

1 Introduction

In this paper, we consider the multidimensional chemotaxis equations

\begin{equation}
\begin{aligned}
\partial_t u - \Delta u &= \text{div}(uv), \\
\partial_t v - \nabla u &= 0, \\
(u, v)(0, x) &= (u_0(x), v_0(x)),
\end{aligned}
\end{equation}

where $u(t, x)$ represents a scalar unknown function and $v(t, x)$ denotes a vector unknown function. This system is a limiting case of the mathematical model for describing chemotaxis, which was proposed by Keller and Segel [6] to represent the aggregation of certain type of bacteria from macroscopic perspective.

From the PDE’s point of view, it is crucial to know if an equation which models a physical phenomenon is well-posed in the Hadamard’s sense: existence, uniqueness, and continuous dependence of the solutions with respect to the initial data. In particular, the lack of continuous dependence would

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cause incorrect solutions or non meaningful solutions. Indeed, this means that the corresponding equation is ill-posed. There is a huge literature on the studies of the well-posedness problem and long-time behaviors of solutions for system (1.1), for example [4, 7–10]. For more background of this model and related results, we refer to [12, 13]. Many results with regard to the ill-posedness have been obtained for some important nonlinear PDEs including the incompressible Navier–Stokes equations [2, 15, 17], the stationary Navier-Stokes equations [11, 14], the compressible Navier–Stokes equations [3, 5] and so on.

In this paper, we are mainly focused on ill-posedness of solutions to system (1.1) in some critical Besov spaces. Nie-Yuan [12] proved that (1.1) is well-posed in $\dot{B}^{d-2}_{p,1} (\mathbb{R}^d) \times \left( \dot{B}^{d-1}_{p,1} (\mathbb{R}^d) \right)^d$ when $1 \leq p < 2d$ and is ill-posed when $p > 2d$. Later on, for the critical case $p = 2d$, Nie-Yuan [13] further proved that (1.1) is ill-posed in $\dot{B}^{-\frac{1}{2}}_{2d,1} (\mathbb{R}^d) \times \left( \dot{B}^{-\frac{1}{2}}_{2d,1} (\mathbb{R}^d) \right)^d$ by fully exploiting nonlinear structure of the cross term. Subsequently, Xiao-Fei [16] proved the ill-posedness of (1.1) in $\dot{B}^{-\frac{1}{2}}_{2d,r} (\mathbb{R}^d) \times \left( \dot{B}^{-\frac{1}{2}}_{2d,r} (\mathbb{R}^d) \right)^d$ for $r > 2$ by a different framework which give a special initial data. Obviously, there are still gaps on the index $r$ between Nie-Yuan and Xiao-Fei’s ill-posedness results. Precisely speaking, for the case $1 < r \leq 2$, it is still unknown whether (1.1) in $\dot{B}^{-\frac{1}{2}}_{2d,r} (\mathbb{R}^d) \times \left( \dot{B}^{-\frac{1}{2}}_{2d,r} (\mathbb{R}^d) \right)^d$ is well-posed or ill-posed. In this paper, we shall answer this question. Our main result now reads as follows:

### 1.1 Main Result

**Theorem 1.1** Let $d \geq 2$ and $1 \leq r < d$. (1.1) is ill-posed in $\dot{B}^{-\frac{1}{2}}_{2d,r} (\mathbb{R}^d) \times \left( \dot{B}^{-\frac{1}{2}}_{2d,r} (\mathbb{R}^d) \right)^d$ in the following sense: There exists an initial data $(u_0, v_0)$ satisfying

$$\lim_{n \to \infty} \left( \|u_0\|_{\dot{B}^{-\frac{1}{2}}_{2d,r}} + \|v_0\|_{\dot{B}^{-\frac{1}{2}}_{2d,r}} \right) = 0,$$

such that the corresponding solution $(u, v)$ satisfies for some positive constants $c$ and $\varepsilon$ small enough

$$\|u(t_n, \cdot)\|_{\dot{B}^{-\frac{1}{2}}_{2d,r}} + \|v(t_n, \cdot)\|_{\dot{B}^{-\frac{1}{2}}_{2d,r}} \geq c \quad \text{with} \quad t_n = \varepsilon 2^{-2n}.$$

**Remark 1.1** Theorem 1.1 demonstrates that if $d \geq 2$ and $1 \leq r < d$, there exists a sequence of initial data which converges to zero in $\dot{B}^{-\frac{1}{2}}_{2d,r} (\mathbb{R}^d) \times \left( \dot{B}^{-\frac{1}{2}}_{2d,r} (\mathbb{R}^d) \right)^d$ and yields a sequence of solutions to (1.1) which does not converge to zero in $\dot{B}^{-\frac{1}{2}}_{2d,r} (\mathbb{R}^d) \times \left( \dot{B}^{-\frac{1}{2}}_{2d,r} (\mathbb{R}^d) \right)^d$. In other words, (1.1) is ill-posed in $\dot{B}^{-\frac{1}{2}}_{2d,r} (\mathbb{R}^d) \times \left( \dot{B}^{-\frac{1}{2}}_{2d,r} (\mathbb{R}^d) \right)^d$ due to the discontinuity of the solution map at zero.

**Remark 1.2** Theorem 1.1 enriches the ill-posedness theories of the system (1.1) although our discontinuity of the solution map in Theorem 1.1 is weaker than “Norm Inflation” of [12, 13, 16].

**Remark 1.3** We should emphasize that the only question left is the well/ill-posedness of the system (1.1) in $\dot{B}^{-\frac{1}{2}}_{4,2} (\mathbb{R}^2) \times \left( \dot{B}^{-\frac{1}{2}}_{4,2} (\mathbb{R}^2) \right)^2$.  

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1.2 Main Idea

From (1.1)_{2}, one has

\[ v(t, x) = v_0 + \int_0^t \nabla u \, dt. \]  

(1.2)

By the Duhamel formula, we obtain from (1.1)_{1} that

\[ u(t, x) = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} \text{div} (uv) \, ds \]

\[ = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} \text{div} \left\{ U_1 \left( v_0 + \int_0^s \nabla U_1 \, dt \right) \right\} \, ds + \text{Remainder term}. \]  

(1.3)

Then, we decompose \( u \) into three terms, namely, \( u = U_1 + U_2 + U_3 \). Thus

\[ v(t, x) = v_0 + \int_0^t \nabla U_1 \, dt + \int_0^t \nabla U_2 \, dt + \int_0^t \nabla U_3 \, dt. \]  

(1.4)

For the convenience of using regularity estimate of heat equations later, we deduce that \( U_i \) \((i = 1, 2, 3)\) solves the following three equations respectively,

\[ \begin{cases} 
\partial_t U_1 - \Delta U_1 = 0, \\
U_{1|t=0} = u_0,
\end{cases} \]  

(1.5)

\[ \begin{cases} 
\partial_t U_2 - \Delta U_2 = \text{div}(U_1 V_1), \\
U_{2|t=0} = 0,
\end{cases} \]  

(1.6)

and

\[ \begin{cases} 
\partial_t U_3 - \Delta U_3 = \text{div} F, \\
U_{3|t=0} = 0,
\end{cases} \]  

(1.7)

where

\[ F := U_3 V_3 + U_3 (V_2 + V_1) + V_3 (U_1 + U_2) + U_1 V_2 + U_2 (V_1 + V_2). \]

To make sure that \( U_2 \) leads to the discontinuity, we decompose it as

\[ U_2 = \int_0^t e^{(t-s)\Delta} \text{div}(u_0 v_0) \, ds + \int_0^t e^{(t-s)\Delta} \text{div} ((U_1 - u_0) v_0) \, ds + \int_0^t e^{(t-s)\Delta} \text{div} \left\{ U_1 \left( \int_0^s \nabla U_1 \, dt \right) \right\} \, ds. \]

We try to extract the worst term \( U_{2,1} \). Our key argument is that, by constructing suitable initial data \((u_0, v_0)\), the other terms in \( U_2 \) can be absorbed by \( U_{2,1} \) and the terms \( U_1, U_3 \) can be small.
2 Littlewood-Paley analysis

Next, we will recall some facts about the Littlewood-Paley (L-P) decomposition, the homogeneous Besov spaces and their some useful properties.

Proposition 2.1 (L-P decomposition, See [1]) Let $\mathcal{B} := \{\xi \in \mathbb{R}^d : |\xi| \leq 4/3\}$ and $\mathcal{C} := \{\xi \in \mathbb{R}^d : 3/4 \leq |\xi| \leq 8/3\}$. Choose a radial, non-negative, smooth function $\chi : \mathbb{R}^d \mapsto [0,1]$ such that it is supported in $\mathcal{B}$ and $\chi \equiv 1$ for $|\xi| \leq 3/4$. Setting $\varphi(\xi) := \chi(\xi/2) - \chi(\xi)$, then we deduce that $\varphi$ is supported in $\mathcal{C}$ and $\varphi(\xi) \equiv 1$ for $4/3 \leq |\xi| \leq 3/2$. In particular, it holds that $\varphi(\xi) \equiv 1$ for $4/3 \leq |\xi| \leq 3/2$ which will be used in the sequel.

For every $u \in S'(\mathbb{R}^d)$, the homogeneous dyadic blocks $\hat{\Lambda}_j$ is defined as follows

$$\hat{\Lambda}_j u = \varphi(2^{-j}D)u = \mathcal{F}^{-1}(\varphi(2^{-j})\mathcal{F}u) = 2^{dj} \int_{\mathbb{R}^d} \hat{\varphi}(2^j(x-y))u(y)dy, \ \forall j \in \mathbb{Z}.$$ 

In the homogeneous case, the following Littlewood-Paley decomposition makes sense

$$u = \sum_{j \in \mathbb{Z}} \hat{\Lambda}_j u \quad \text{for any } u \in \dot{S}_h'(\mathbb{R}^d),$$

where $\dot{S}_h'$ is given by

$$\dot{S}_h' := \{u \in S'(\mathbb{R}^d) : \lim_{j \to -\infty} \|\varphi(2^{-j}D)u\|_{L^\infty} = 0\}.$$

We turn to the definition of the Besov Spaces and norms which will come into play in our paper.

Definition 2.1 ([1]) Let $s \in \mathbb{R}$ and $(p,r) \in [1,\infty]^2$. The homogeneous Besov space $\dot{B}^s_{p,r}(\mathbb{R}^d)$ consists of all tempered distribution $f$ such that

$$\dot{B}^s_{p,r} = \{f \in \dot{S}_h'(\mathbb{R}^d) : \|f\|_{\dot{B}^s_{p,r}(\mathbb{R}^d)} < \infty\},$$

where

$$\|f\|_{\dot{B}^s_{p,r}(\mathbb{R}^d)} := \begin{cases} \left( \sum_{j \in \mathbb{Z}} 2^{sjr} \|\hat{\Lambda}_j f\|_{L^p(\mathbb{R}^d)}^r \right)^{1/q}, & \text{if } 1 \leq r < \infty, \\ \sup_{j \in \mathbb{Z}} 2^{sjr} \|\hat{\Lambda}_j f\|_{L^p(\mathbb{R}^d)}, & \text{if } r = \infty. \end{cases}$$

For $0 < T \leq \infty$, $s \in \mathbb{R}$ and $1 \leq p, r, \rho \leq \infty$, we set (with the usual convention if $r = \infty$)

$$\|f\|_{\dot{L}^p_T(\mathbb{R}^d)} := \left( \sum_{j \in \mathbb{Z}} 2^{jsr} \|\hat{\Lambda}_j f\|_{L^p(0,T;\mathbb{R}^d)}^r \right)^{1/s}.$$ 

The following Bernstein’s inequalities will be used in the sequel.
Lemma 2.1 (see [1]) Let $\mathcal{B}$ be a Ball and $\mathcal{C}$ be an annulus. There exist constants $C > 0$ such that for all $k \in \mathbb{N} \cup \{0\}$, any positive real number $\lambda$ and any function $f \in L^p$ with $1 \leq p \leq q \leq \infty$, we have
\[
\text{supp} \hat{f} \subseteq \lambda \mathcal{B} \Rightarrow \|\nabla^k f\|_{L^q} \leq C^{k+1} \lambda^{k+\frac{d}{p}-\frac{d}{q}}\|f\|_{L^p},
\]
\[
\text{supp} \hat{f} \subseteq \lambda \mathcal{C} \Rightarrow C^{-k-1} \lambda^k \|f\|_{L^p} \leq \|\nabla^k f\|_{L^q} \leq C^{k+1} \lambda^k \|f\|_{L^p}.
\]
As a direct result of Bernstein’s inequalities, we have the following continuous embedding:

Lemma 2.2 (see [1]) Let $s \in \mathbb{R}$, $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq r_1 \leq r_2 \leq \infty$. Then
\[
\dot{B}^s_{p_1,r_1}(\mathbb{R}^d) \hookrightarrow \dot{B}^s_{p_2,r_2}(\mathbb{R}^d) \quad \text{with} \quad t = s - \left(\frac{d}{p_1} - \frac{d}{p_2}\right).
\]

Lemma 2.3 Let $s > 0$, $1 \leq p \leq \infty$ and $1 \leq \rho, \rho_1, \rho_2, \rho_3, \rho_4 \leq \infty$. Then
\[
\|fg\|_{L^p(B^s_{\rho})} \leq C\left(\|f\|_{L^p_t(B^s_{\rho})}\|g\|_{L^p_t(B^s_{\rho})} + \|g\|_{L^p_t(B^s_{\rho})}\|f\|_{L^p_t(B^s_{\rho})}\right),
\]
where
\[
\frac{1}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_2} = \frac{1}{\rho_3} + \frac{1}{\rho_4}.
\]

Lemma 2.4 (See [12]) Let $1 \leq \rho, \rho_1, \rho_2 \leq \infty$ with $\frac{1}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_2}$ and $1 \leq p < 2d$. Then, we have
\[
\|fg\|_{L^p_t(B^s_{\rho})} \leq C\|f\|_{L^p_t(B^s_{\rho})} \|g\|_{L^p_t(B^s_{\rho})}.
\]

Lemma 2.5 (See [12]) Let $1 \leq \rho, \rho_1, \rho_2 \leq \infty$ with $\frac{1}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_2}$ and $d < p < 2d \leq q < \infty$ with $\frac{d}{p} + \frac{d}{q} > 1$. Then, we have
\[
\|fg\|_{L^q_t(B^s_{\rho})} \leq C\|f\|_{L^q_t(B^s_{\rho})} \|g\|_{L^q_t(B^s_{\rho})}.
\]

Finally, we recall the regularity estimates for the heat equations.

Lemma 2.6 (See [1]) Let $s \in \mathbb{R}$, $1 \leq p, r \leq \infty$ and $1 \leq q_1 \leq q_2 \leq \infty$. Assume that $u_0 \in \dot{B}^s_{p,r}$ and $f \in \dot{L}^{q_1}_{\tau}(\dot{B}^{s+\frac{d}{q_1}-2}_{p,r})$. Then the heat equations
\[
\begin{cases}
\partial_t u - \Delta u = f, \\
u(0, x) = u_0(x),
\end{cases}
\]
has a unique solution $u \in \dot{L}^{q_2}_{\tau}(\dot{B}^{s+\frac{d}{q_2}-2}_{p,r})$ satisfying for all $T > 0$
\[
\|u\|_{L^{q_2}_{\tau}(\dot{B}^{s+\frac{d}{q_2}-2}_{p,r})} \leq C\left(\|u_0\|_{\dot{B}^s_{p,r}} + \|f\|_{\dot{L}^{q_1}_{\tau}(\dot{B}^{s+\frac{d}{q_1}-2}_{p,r})}\right).
\]
3 Proof of Theorem 1.1

3.1 Construction of initial data

Letting \( n \gg 1 \), we write \( n \in 16\mathbb{N} = \{16, 32, 48, \cdots \} \) and \( \mathbb{N}(n) = \left\{ k \in 8\mathbb{N} : \frac{n}{4} \leq k \leq \frac{n}{2} \right\} \).

We define a scalar function \( \hat{\theta} \in C^\infty_0(\mathbb{R}) \) with values in \([0, 1]\) which satisfies

\[
\hat{\theta}(\xi) = \begin{cases} 1, & \text{if } |\xi| \leq \frac{1}{200d}, \\ 0, & \text{if } |\xi| \geq \frac{1}{100d}. \end{cases}
\]

Let

\[
\phi(x_1, x_2, \cdots, x_d) = \theta(x_1)\theta(x_2)\cdots\theta(x_d) \sin\left(\frac{17}{24}x_d\right).
\]

Motivated by the construction in [5, 11, 15], we define

\[
f_n = n^{-\frac{d}{2}} \sum_{k \in \mathbb{N}(n)} 2^k \phi(2^k(x - 2^{n+k}\vec{e})) \sin\left(\frac{17}{12}2^n x_1\right). \tag{3.8}
\]

It is straightforward to verify that

\[
\text{supp } \hat{f}_n(\xi) \subset \left\{ \xi \in \mathbb{R}^d : \frac{33}{24}2^n \leq |\xi| \leq \frac{35}{24}2^n \right\}. \tag{3.9}
\]

We construct initial data \((u_0, v_0)\) as follows

\[
u_0 = 2^{\frac{1}{2}n}f_n \quad \text{and} \quad v_0 = 2^{\frac{1}{2}n}f_n\vec{e} \quad \text{with} \quad \vec{e} = (1, 0, \cdots, 0). \tag{3.10}
\]

3.2 Estimation of initial data

Lemma 3.1 Let \( f_n \) be defined by (3.8). Then for \( p \in [1, 2d] \) and \( r \in [1, d) \) there holds

\[
\|f_n\|_{L^p} \leq Cn^{\frac{1}{p} - \frac{1}{2} + \frac{2n}{2p} - n}. \tag{3.11}
\]

In particular, by \( L^{3.5} = [L^2, L^4]_{(1/7, 6/7)} \), then (3.11) holds for \( d = 2 \) and \( p = 3.5 \).

Proof. The proof is postponed to A.1 in Section 4.

As a direct consequence of Lemma 3.1, we have

Proposition 3.1 Let \((u_0, v_0)\) be defined by (3.8). Then for \( p \in [1, 2d] \) and \( r \in [1, d) \) there holds

\[
\|u_0\|_{B^r_{p, r}} + 2^n\|v_0\|_{B^r_{p, r}} \leq C2^{n(r+\frac{d}{2})}\frac{n^{\frac{1}{2}p}}{n^{\frac{1}{2}p} + n^{\frac{1}{2}r}}. \tag{3.12}
\]

In particular, it holds that

\[
\|u_0\|_{B^{\frac{d}{2d}r}_{2d, r}} + \|v_0\|_{B^{\frac{d}{2d}r}_{2d, r}} \leq Cn^{\frac{1}{2} - \frac{d}{2}}. \tag{3.13}
\]
Proof. Notice that \( \hat{\Delta}_j f_n = \varphi(2^{-j} \cdot) \hat{f}_n \) for all \( j \in \mathbb{Z} \) and \( \varphi(2^{-j} \xi) \equiv 1 \) in \( \{ \xi \in \mathbb{R}^d : \frac{3}{2} 2^j \leq |\xi| \leq \frac{3}{2} 2^{j+1} \} \), then we have \( \hat{\Delta}_j f_n = 0 \) for \( j \neq n \), and thus,
\[
\hat{\Delta}_j (f_n) = \begin{cases} f_n, & \text{if } j = n, \\ 0, & \text{otherwise.} \end{cases}
\]
Using the definition of Besov space, (3.9) and Lemma 3.1 yields (3.12). This completes the proof of Proposition 3.1.

3.3 Ill-posedness

In this subsection, we divide the proof into three steps. From now on, we choose “certain time” as \( t_n = T = \varepsilon 2^{-2n} \).

Step 1: The upper bound estimation of \( \| U_1(T, \cdot) \|_{B^s_{2,1}(\mathbb{R}^d)} \).

As a direct consequence of Proposition 3.1, one has
\[
\| U_1(t_n) \|_{B^s_{2,1}(\mathbb{R}^d)} \leq \| U_1 \|_{L^2_t(B^s_{2,1})} \leq C \| u_0 \|_{B^s_{2,1}} \leq C n^{\frac{3}{4} - \frac{d}{2}}. \tag{3.13}
\]

Step 2: The lower bound estimation of \( \| U_2(T, \cdot) \|_{B^s_{2,1}(\mathbb{R}^d)} \).

In view of \( V_1 = v_0 + \int_0^t \nabla U_1 \, d\tau \), we decompose \( U_2 \) as \( U_2 = U_{2,1} + U_{2,2} \), where \( U_{2,1} \) and \( U_{2,2} \) satisfy respectively
\[
\begin{align*}
\frac{\partial}{\partial t} U_{2,1} - \Delta U_{2,1} &= \text{div}(u_0 v_0) = 2^{2n} \partial_{x_1} (f_n^2), \\
U_{2,1}|_{t=0} &= 0.
\end{align*}
\tag{3.14}
\]
and
\[
\begin{align*}
\frac{\partial}{\partial t} U_{2,2} - \Delta U_{2,2} &= \text{div} \left( U_1 \int_0^t \nabla U_1 \, d\tau \right) + \text{div}((U_1 - u_0) v_0), \\
U_{2,2}|_{t=0} &= 0.
\end{align*}
\tag{3.15}
\]
Using Lemma 2.6 and noticing that \( U_1 - u_0 = \int_0^t \Delta U_1 \, d\tau \) yields
\[
\| U_{2,2} \|_{L^2_t(B^s_{2,1}(\mathbb{R}^d))} \leq \| U_{2,2} \|_{L^2_t(B^s_{2,1}(\mathbb{R}^d))} \leq C n^{\frac{3}{4} - \frac{d}{2}} \| U_{2,2} \|_{L^2_t(B^s_{2,1}(\mathbb{R}^d))} \leq C n^{\frac{3}{4} - \frac{d}{2}} \| U_{2,2} \|_{L^2_t(B^s_{2,1}(\mathbb{R}^d))} \leq CT n^{\frac{3}{4} - \frac{d}{2}} \left( \| V_1 \|_{L^2_t(B^s_{2,1}(\mathbb{R}^d))} + \| \Delta U_1 \|_{L^2_t(B^s_{2,1}(\mathbb{R}^d))} \right) \leq C T^2 n^{\frac{3}{4} - \frac{d}{2}} \left( 2^n \| u_0 \|_{L^\infty_t(L^{2d})}^2 + 2^{2n} \| u_0 \|_{L^\infty_t(L^{2d})} \| v_0 \|_{L^\infty_t(L^{2d})} \right) \leq C \varepsilon^2.
\]
Taking advantage of the Duhamel formula, then we have from (3.14)
\[
U_{2,1} = 2^{2n} \int_0^t e^{(t-\tau)\Delta} \partial_{x_1} (f_n^2) \, d\tau.
\]
Direct computations gives that for $\ell \in \mathbb{N}(n)$

$$
\Delta_t U_{2,1} = 2^n \int_0^N \mathcal{F}^{-1} \left( \varphi_\ell(\xi) e^{-|\ell_\alpha| |\xi|^2} \mathcal{F}(\partial_{\xi_1}(f_n^2)) \right) \, d\tau \\
= 2^n \mathcal{F}^{-1} \left( \varphi_\ell(\xi) \frac{1 - e^{-|\ell_\alpha| |\xi|^2}}{|\xi|^2} \mathcal{F}(\partial_{\xi_1}(f_n^2)) \right) \\
= \varepsilon \left( \Delta_t(\partial_{\xi_1}(f_n^2)) + \sum_{k \geq 1} \frac{t_n}{(k + 1)!} \frac{t_n^k}{(k_1 + 1)!} \Delta_t(\partial_{\xi_1} \Delta^k(f_n^2)) \right),
$$

where we have used Taylor’s formula

$$
\frac{1 - e^{-|\ell_\alpha| |\xi|^2}}{|\xi|^2} = t_n + t_n \sum_{k \geq 1} \frac{t_n^k}{(k + 1)!} (-|\xi|^2)^k.
$$

By Bernstein’s inequality, we deduce that

$$
\left\| \sum_{k \geq 1} \frac{t_n^k}{(k + 1)!} \Delta_t(\partial_{\xi_1} \Delta^k(f_n^2)) \right\|_{L^{2d}} \leq C \sum_{k \geq 1} \frac{t_n^k}{(k + 1)!} \left\| \Delta_t(\partial_{\xi_1} \Delta^k(f_n^2)) \right\|_{L^{2d}} \\
\leq C \sum_{k \geq 1} \frac{\varepsilon^{k}}{(k + 1)!} \left\| \Delta_t \partial_{\xi_1}(f_n^2) \right\|_{L^{2d}} \\
\leq C \varepsilon \left\| \Delta_t \partial_{\xi_1}(f_n^2) \right\|_{L^{2d}}.
$$

By the inverse triangle inequality, we have

$$
\left\| U_{2,1}(t, \cdot) \right\|_{L^2_{\beta_2, n}^{2,\ell}(\mathbb{R}^d)} \geq \varepsilon(1 - C\varepsilon) \sum_{\ell \in \mathbb{N}(n)} 2^{-\frac{d+\ell}{2}} \left\| \Delta_t \partial_{\xi_1}(f_n^2) \right\|_{L^{2d}(\mathbb{R}^d)}.
$$

For $k \in \mathbb{N}(n)$, we decompose the term $n^\ell f_n^{2}$ as

$$
n^\ell f_n^{2} = \sum_{k \in \mathbb{N}(n)} 2^k \phi^2 \left( 2^k (x - 2^{2n+\ell} \varphi) \right) \sin^2 \left( \frac{17}{12} 2^n x_1 \right) \\
+ \sum_{k \not\in \mathbb{N}(n)} 2^k 2^\ell \phi \left( 2^k (x - 2^{2n+\ell} \varphi) \right) \phi \left( 2^\ell (x - 2^{2n+\ell} \varphi) \right) \sin^2 \left( \frac{17}{12} 2^n x_1 \right) =: \mathbf{H}
$$

using the simple fact $\sin^2 \alpha = (1 - \cos 2\alpha)/2$ and noticing that (for more details see A.2 in the Appendix)

$$
\Delta_t \mathbf{H} = 0 \quad \text{for} \quad \ell \in \mathbb{N}(n),
$$

then we have

$$
n^\ell \Delta_t f_n^{2} = \frac{1}{2} \Delta_t \left( 2^\ell \phi^2 (2^\ell (x - 2^{2n+\ell} \varphi)) \right) + \frac{1}{2} \Delta_t \left( \sum_{k \in \mathbb{N}(n), k \not\in \ell} 2^k \phi^2 (2^k (x - 2^{2n+k} \varphi)) \right),
$$

where we have used Taylor’s formula

$$
\frac{1 - e^{-|\ell_\alpha| |\xi|^2}}{|\xi|^2} = t_n + t_n \sum_{k \geq 1} \frac{t_n^k}{(k + 1)!} (-|\xi|^2)^k.
$$
which in turn gives
\[ n^\frac{1}{2} \partial_{x_i} \Delta f_n^2 = 2^{2\ell-1} [h](2^\ell(x - 2^{2n+\ell} \vartheta)) + \frac{1}{2} \partial_{x_i} \Delta \left( \sum_{k \in \mathbb{N}(n), k \neq \ell} 2^k \theta^2(2^k(x - 2^{2n+k} \vartheta)) \right) = K_1 + K_2, \]
where
\[ h(x) = -\theta(x_1) \theta'(x_1) \theta(x_2) \theta^2(x_3) \cdots \theta^2(x_{d-1}) \theta^2(x_d) \cos \left( \frac{17}{12} x_d \right). \]

Let us introduce the set \( B_\ell \) defined by
\[ B_\ell \equiv \{ x : |2^\ell(x - 2^{2n+\ell} \vartheta)| \leq 1 \}, \]
then by change of variables, we have
\[ \|K_1\|_{L^2(B_\ell)} = 2^{2\ell-1} \left\| [h](2^\ell(x - 2^{2n+\ell} \vartheta)) \right\|_{L^2(B_\ell)} = 2^{\frac{3}{2}\ell-1} \|h(y)\|_{L^2(B_\ell)} = \sqrt{2} 2^{\frac{3}{2} \ell}, \]
Combining the estimate whose proof is relegated to A.3 in Section 4
\[ \|K_2\|_{L^2(B_\ell)} \leq C 2^{-\ell}, \tag{3.19} \]
thus for \( \ell \in \mathbb{N}(n) \)
\[ \left\| \Delta_{x_i} (f_n^2) \right\|_{L^2(B_\ell)} \geq n^{-\frac{1}{2}} \left( \|K_1\|_{L^2(B_\ell)} - \|K_2\|_{L^2(B_\ell)} \right) \geq n^{-\frac{1}{2}} (\sqrt{2} 2^{\frac{3}{2} \ell} - C 2^{-\ell}) \geq cn^{-\frac{1}{2}} 2^{\frac{3}{2} \ell}. \tag{3.20} \]
Inserting (3.20) into (3.16), we conclude that
\[ \left\| U_2(t_n, \cdot) \right\|_{B_{2/3}^d(3^\ell(n))} \geq \varepsilon (c - C \varepsilon). \]

**Step 3: The upper bound estimation of \( \|U_3(T, \cdot)\|_{B_{2/3}^d}. \)**

We choose the index \((d, p_0, q_0)\) to satisfy that \( q_0 = 2d \) and
\[ p_0 = \begin{cases} 3.5, & d = 2, \\ 2d - 1, & d \geq 3. \end{cases} \]
Obviously, it holds that
\[ 1 \leq r < d < p_0 < 2d = q_0 \quad \text{and} \quad \frac{3d}{2p_0} - 1 < 0. \tag{3.21} \]
For the sake of convenience, we denote
\[ X_T = \left\| U_3(t, \cdot) \right\|_{L^\infty_t(B_{p_0}^{d/2})} + \left\| U_3(t, \cdot) \right\|_{L^1_t(B_{p_0}^d)} \quad \text{and} \quad Y_T = \left\| V_3(t, \cdot) \right\|_{L^\infty_t(B_{p_0}^{d/2})} \quad \text{and} \quad Y_T \leq C \left\| U_3(t, \cdot) \right\|_{L^1_t(B_{p_0}^d)}. \]
Obviously,
\[ Y_T \leq C \left\| U_3(t, \cdot) \right\|_{L^1_t(B_{p_0}^d)} \leq C X_T. \]
Utilizing Lemma 2.6, we have
\[
X_T \leq C\|U_3 V_3 + U_3(V_1 + V_2) + V_3(U_1 + U_2) + U_1 V_2 + U_2(V_1 + V_2)\|_{L_T^1(B_{p_0,1}^{d\frac{d-1}{2}})}.
\]
(3.22)

Utilizing Lemma 2.4, one has
\[
\|U_3 V_3\|_{L_T^1(B_{p_0,1}^{d\frac{d-1}{2}})} \leq C\|U_3\|_{L_T^1(B_{p_0,1}^{d\frac{d-1}{2}})} \|V_3\|_{L_T^1(B_{p_0,1}^{d\frac{d-1}{2}})} \leq C X_T^2,
\]
(3.23)
\[
\|U_2(V_1 + V_2)\|_{L_T^1(B_{p_0,1}^{d\frac{d-1}{2}})} \leq C T\|U_2\|_{L_T^1(B_{p_0,1}^{d\frac{d-1}{2}})} \|V_1, V_2\|_{L_T^1(B_{p_0,1}^{d\frac{d-1}{2}})}.
\]
(3.24)
\[
\|U_1 V_2\|_{L_T^1(B_{p_0,1}^{d\frac{d-1}{2}})} \leq C T\|U_1\|_{L_T^1(B_{p_0,1}^{d\frac{d-1}{2}})} \|V_2\|_{L_T^1(B_{p_0,1}^{d\frac{d-1}{2}})}.
\]
(3.25)

Utilizing Lemma 2.5, one has
\[
\|U_3(V_1 + V_2)\|_{L_T^1(B_{p_0,1}^{d\frac{d-1}{2}})} \leq C\|U_3\|_{L_T^1(B_{p_0,1}^{d\frac{d-1}{2}})} \|V_1 + V_2\|_{L_T^1(B_{p_0,1}^{d\frac{d-1}{2}})}
\leq C T^\frac{1}{2}\|U_3\|_{L_T^1(B_{p_0,1}^{d\frac{d-1}{2}})} \|V_1, V_2\|_{L_T^{\infty}(B_{p_0,1}^{d\frac{d-1}{2}})}.
\]
\[
\leq C T^\frac{1}{2}\|U_3\|_{L_T^{\infty}(B_{p_0,1}^{d\frac{d-2}{2}})} \|U_3\|_{L_T(B_{p_0,1}^{d\frac{d-1}{2}})} \|V_1, V_2\|_{L_T^1(B_{p_0,1}^{d\frac{d-1}{2}})}.
\]
\[
\leq C T^\frac{1}{2} X_T\|V_1, V_2\|_{L_T^1(B_{p_0,1}^{d\frac{d-1}{2}})}.
\]
(3.26)
\[
\|V_3(U_1 + U_2)\|_{L_T^1(B_{p_0,1}^{d\frac{d-1}{2}})} \leq C\|V_3\|_{L_T^1(B_{p_0,1}^{d\frac{d-1}{2}})} \|U_1 + U_2\|_{L_T^1(B_{p_0,1}^{d\frac{d-1}{2}})} \leq C T X_T\|U_1, U_2\|_{L_T^1(B_{p_0,1}^{d\frac{d-1}{2}})}.
\]
(3.27)

Inserting (3.23)-(3.27) into (3.22) yields
\[
X_T \leq C X_T^2 + C T X_T\|U_1, U_2\|_{L_T^1(B_{p_0,1}^{d\frac{d-1}{2}})} + C T^\frac{1}{2} X_T\|V_1, V_2\|_{L_T^1(B_{p_0,1}^{d\frac{d-1}{2}})}
\leq C T\|U_2\|_{L_T^1(B_{p_0,1}^{d\frac{d-1}{2}})} \|V_1, V_2\|_{L_T^{\infty}(B_{p_0,1}^{d\frac{d-1}{2}})} + C T\|U_1\|_{L_T^1(B_{p_0,1}^{d\frac{d-1}{2}})} \|V_2\|_{L_T^{\infty}(B_{p_0,1}^{d\frac{d-1}{2}})}.
\]
(3.28)

On the one hand, using Lemma 3.1, we have
\[
\|V_1\|_{L_T^1(B_{p_0,1}^{d\frac{d-1}{2}})} \leq C\|V_1\|_{L_T^1(B_{p_0,1}^{d\frac{d-1}{2}})} + \|v_0\|_{L_T^1(B_{p_0,1}^{d\frac{d-1}{2}})} \leq C 2^{\|\nabla v_0\|_{L_T^1(B_{p_0,1}^{d\frac{d-1}{2}})}} n\frac{1}{p_0} - \frac{1}{2},
\]
and for \(q_0 = 2d\)
\[
\|U_1\|_{L_T^1(B_{p_0,1}^{d\frac{d-1}{2}})} \leq C\|U_1\|_{L_T^1(B_{p_0,1}^{d\frac{d-1}{2}})} \leq C T\|U_1\|_{L_T^1(B_{p_0,1}^{d\frac{d-1}{2}})} + \|v_0\|_{L_T^1(B_{p_0,1}^{d\frac{d-1}{2}})} \leq C 2^{\|\nabla v_0\|_{L_T^1(B_{p_0,1}^{d\frac{d-1}{2}})}} n\frac{1}{p_0} - \frac{1}{2}.
\]

\[
\|U_1\|_{L_T^1(B_{p_0,1}^{d\frac{d-1}{2}})} \leq C T\|U_0\|_{L_T^1(B_{p_0,1}^{d\frac{d-1}{2}})} \leq C N\frac{1}{p_0} - \frac{1}{2},
\]
\[
\|V_1\|_{L_T^1(B_{p_0,1}^{d\frac{d-1}{2}})} \leq C T\|U_1\|_{L_T^1(B_{p_0,1}^{d\frac{d-1}{2}})} + \|v_0\|_{L_T^1(B_{p_0,1}^{d\frac{d-1}{2}})} \leq C T\|U_1\|_{L_T^1(B_{p_0,1}^{d\frac{d-1}{2}})} + T\|v_0\|_{L_T^1(B_{p_0,1}^{d\frac{d-1}{2}})} \leq C N\frac{1}{p_0} - \frac{1}{2}.
\]
On the other hand, by the product law (Lemma 2.3) and the fact $e^{t\Delta} : L^\infty \to L^\infty$, we have

\[ T\|U_2\|_{L_T^p(B_{p,0}^{\frac{d}{2}})} + \|V_2\|_{L_T^p(B_{p,0}^{\frac{d}{2}+1})} \leq CT\|U_2\|_{L_T^p(B_{p,0}^{\frac{d}{2}})} + CT^2\|U_1V_1\|_{L_T^p(B_{p,0}^{\frac{d}{2}+1})} \]

\[ \leq CT^2(\|U_1\|_{L_T^p(L^\infty)}\|V_1\|_{L_T^p(B_{p,0}^{\frac{d}{2}+1})} + \|V_1\|_{L_T^p(L^\infty)}\|U_1\|_{L_T^p(B_{p,0}^{\frac{d}{2}+1})}) \]

\[ \leq C2^{n(\frac{d-2}{8}+\overline{\beta})}n^{\frac{1}{70} - \frac{1}{2}}, \]

where we have used the estimates

\[ \|U_1\|_{L_T^p(L^\infty)} \leq C\|u_0\|_{L^\infty} \leq C2^{\frac{d}{2}}2^{\frac{d}{n}}n^{-\frac{1}{2}} \leq C2^{\frac{d}{2}}n^{-\frac{1}{2}}, \]

\[ \|V_1\|_{L_T^p(L^\infty)} \leq C(\|v_0\|_{L^\infty} + T\|\nabla u_0\|_{L^\infty}) \leq C(2^{\frac{d}{n}}2^{\frac{d}{2}} + 2^{-2n}2^{\frac{d}{2}}2^{\frac{d}{2}}n^{-\frac{1}{2}}) \leq C2^{\frac{d}{2}}n^{-\frac{1}{2}}, \]

\[ \|V_1\|_{L_T^p(B_{p,0}^{\frac{d}{2}+1})} \leq C(\|v_0\|_{B_{p,0}^{\frac{d}{2}+1}} + T\|u_0\|_{B_{p,0}^{\frac{d}{2}+1}}) \leq C2^{n(\frac{d}{70} + \frac{d}{8})}2^{\frac{d}{4}}n^{\frac{1}{70} - \frac{1}{2}}. \]

Similarly,

\[ T\|U_2\|_{L_T^p(B_{p,0}^{\frac{d}{2}})} + T^{\frac{d}{2}}\|V_2\|_{L_T^p(B_{p,0}^{\frac{d}{2}+1})} \leq T\|U_2\|_{L_T^p(B_{p,0}^{\frac{d}{2}})} + T^{\frac{d}{2}}\|U_2\|_{L_T^p(B_{p,0}^{\frac{d}{2}+1})} \leq C2^{-\frac{d}{2}}n^{\frac{1}{70} - \frac{1}{2}}, \]

Then, we have

\[ CTX_T\|U_1, U_2\|_{L_T^p(B_{p,0}^{\frac{d}{2}})} + CT^{\frac{d}{2}}X_T\|V_1, V_2\|_{L_T^p(B_{p,0}^{\frac{d}{2}+1})} \leq CN^{\frac{1}{2m} - \frac{1}{2}}X_T, \]

\[ CT\|U_2\|_{L_T^p(B_{p,0}^{\frac{d}{2}})}\|V_1, V_2\|_{L_T^p(B_{p,0}^{\frac{d}{2}+1})} + CT\|U_1\|_{L_T^p(B_{p,0}^{\frac{d}{2}})}\|V_2\|_{L_T^p(B_{p,0}^{\frac{d}{2}+1})} \leq C2^{n(\frac{d}{70} - 1)}n^{\frac{1}{70}}. \]

Putting the above inequalities together with (3.28) yields

\[ X_T \leq CX_T^2 + CN^{\frac{1}{2m} - \frac{1}{2}}X_T + C2^{n(\frac{d}{70} - 1)}n^{\frac{1}{70}}. \]

By using the continuity argument and condition (3.21), we can take $n$ large enough such that for $T = \varepsilon 2^{-2n}$

\[ X_T \leq C2^{n(\frac{d}{70} - 1)}n^{\frac{1}{70}}. \]

Then, by the embedding $B_{p,0}^{\frac{d}{2}}(\mathbb{R}^d) \hookrightarrow B_{2d,r}^{\frac{d}{2}}(\mathbb{R}^d)$, we have

\[ \|U_3(T, \cdot)\|_{B_{2d,r}^{\frac{d}{2}}} \leq CX_T \leq C2^{n(\frac{d}{70} - 1)}n^{\frac{1}{70}}. \]

Combining Step1-Step3, we obtain that for large $n$ enough and $\varepsilon$ small enough

\[ \|u_0\|_{B_{2d,r}^{\frac{d}{2}}} + \|v_0\|_{B_{2d,r}^{\frac{d}{2}}} \leq CN^{\frac{1}{2m} - \frac{1}{2}} \to 0, \quad n \to \infty \]

and

\[ \|u\|_{B_{2d,r}^{\frac{d}{2}}} \geq \|U_3\|_{B_{2d,r}(\Omega(n))} - \|U_1\|_{B_{2d,r}^{\frac{d}{2}}} - \|U_2\|_{B_{2d,r}^{\frac{d}{2}}} \]

\[ \geq C\varepsilon - C2^{n(\frac{d}{70} - 1)}n^{\frac{1}{70}} - CN^{\frac{1}{2m} - \frac{1}{2}} \]

\[ \geq \varepsilon_0. \]

Thus, we have completed the proof of Theorem 1.1.
4 Appendix

For the sake of convenience, here we present more details in the computations.

A.1 Proof of Lemma 3.1. We assume that \( p \in \mathbb{Z}^+ \) without loss of generality. Since \( \phi \) is a Schwartz function, we have

\[
|\phi(x)| \leq C(1 + |x|)^{-M}, \quad M \geq 100. \tag{4.1}
\]

It is easy to show that

\[
\|f_n\|_{L^p} \leq n^{-\frac{d}{p}} \sum_{\ell_1, \ell_2, \ldots, \ell_p \in \mathbb{N}(n)} \frac{2^{\frac{Q}{p}(\ell_1 + \ell_2 + \cdots + \ell_p)}}{(1 + 2^{\ell_1}|x - 2^{2n+\ell_1}e|)^M \cdots (1 + 2^{\ell_p}|x - 2^{2n+\ell_p}e|)^M} \, dx
\]

\[
\leq n^{-\frac{d}{p}} \sum_{\ell \in \mathbb{N}(n)} \int_{\mathbb{R}^d} \frac{2^{\frac{Q}{p}\ell}}{(1 + 2^\ell|x - 2^{2n+\ell}e|)^{pM}} \, dx
\]

\[
+ n^{-\frac{d}{p}} \sum_{(\ell_1, \ell_2, \ldots, \ell_p) \in \Lambda} \int_{\mathbb{R}^d} \frac{2^{\frac{Q}{p}(\ell_1 + \ell_2 + \cdots + \ell_p)}}{(1 + 2^{\ell_1}|x - 2^{2n+\ell_1}e|)^M \cdots (1 + 2^{\ell_p}|x - 2^{2n+\ell_p}e|)^M} \, dx
\]

\[
\equiv n^{-\frac{d}{p}} I_1 + n^{-\frac{d}{p}} I_2, \tag{4.2}
\]

where the set \( \Lambda \) is defined by

\[
\Lambda = \{ (\ell_1, \ldots, \ell_p) \in \mathbb{N}^p(n) \mid \exists 1 \leq k \leq p \text{ s.t. } \ell_k \neq \ell_1 \}.
\]

For the term \( I_1 \), by direct computations, one has

\[
I_1 = \sum_{\ell \in \mathbb{N}(n)} 2^{\frac{Q}{p}\ell - d\ell} \int_{\mathbb{R}^d} \frac{1}{(1 + |x|)^{pM}} \, dx \leq C n^{2^{\frac{Q}{p}d} n}. \tag{4.3}
\]

For the term \( I_2 \), we assume that \( \ell_1 < \ell_2 \) without loss of generality, then \( \ell_2 - \ell_1 \geq 4 \).

\[
\int_{\mathbb{R}^d} \frac{1}{(1 + 2^{\ell_1}|x - 2^{2n+\ell_1}e|)^M (1 + 2^{\ell_2}|x - 2^{2n+\ell_2}e|)^M} \, dx
\]

\[
= \left( \int_{A_{\ell_1}} + \int_{\mathbb{R}^d \setminus A_{\ell_1}} \right) \frac{1}{(1 + 2^{\ell_1}|x - 2^{2n+\ell_1}e|)^M (1 + 2^{\ell_2}|x - 2^{2n+\ell_2}e|)^M} \, dx,
\]

where we defined the set \( A_{\ell_1} \) by

\[
A_{\ell_1} := \{ x : |x - 2^{2n+\ell_1}e| \leq 2^n \}.
\]

Thus

\[
\int_{A_{\ell_1}} \frac{1}{(1 + 2^{\ell_1}|x - 2^{2n+\ell_1}e|)^M (1 + 2^{\ell_2}|x - 2^{2n+\ell_2}e|)^M} \, dx
\]

\[
\leq C (2^{\ell_1} 2^{2n})^{-M} \int_{A_{\ell_1}} \frac{1}{(1 + 2^{\ell_2}|x - 2^{2n+\ell_2}e|)^M} \, dx
\]

\[
\leq C (2^{\ell_1} 2^{2n})^{-M} 2^{-df_2}. \tag{4.4}
\]
It is easy to deduce that for \( x \in A_{\ell_1} \)
\[
2^{\ell_2} \| x - 2^{2n+\ell_2} \| \geq 2^{\ell_2} \| 2^{2n+\ell_2} - 2^{2n+\ell_1} \| - 2^{\ell_2} 2^n \geq 2^{\ell_2} 2^n.
\]

Similarly,
\[
\int_{A_{\ell_1}} \frac{1}{(1 + 2^{\ell_1} |x - 2^{2n+\ell_1}|)^M (1 + 2^{\ell_2} |x - 2^{2n+\ell_2}|)^M} dx
\leq (2^{\ell_2} 2^n)^{-M} \int_{A_{\ell_1}} \frac{1}{(1 + 2^{\ell_1} |x - 2^{2n+\ell_1}|)^M} dx
\leq C (2^{\ell_2} 2^n)^{-M} 2^{-d\ell_1}.
\]

We infer from (4.4) and (4.5) that
\[
I_2 \leq C 2^{-M n} \sum_{(\ell_1, \ell_2, \ldots, \ell_p) \in A} (2^{-M \ell_1} 2^{-d\ell_2} + 2^{-M \ell_2} 2^{-d\ell_1}) 2^{\frac{1}{2} (\ell_1 + \ell_2 + \ldots + \ell_p)} \leq C 2^{-M n}.
\]

Inserting (4.3) and (4.6) into (4.2), we have for large enough \( n \)
\[
\| f_n \|_{L^p} \leq C n^{\frac{1}{p} - \frac{1}{2} + \frac{2}{2\pi d^2 n}}.
\]

This completes the proof of Lemma 3.1.

**A.2 Proof of (3.18).** Notice that
\[
H = \sum_{k \in \mathbb{N}(n), k \neq \ell} 2^{\frac{d}{2}} 2^{\frac{d}{2}} \Phi_{k,j}(x) \sin^2 \left( \frac{17}{12} 2^n x_1 \right) \quad \text{with} \quad \Phi_{k,j}(x) := \phi(2^k(x - 2^{2n+k} \hat{e})) \phi(2^j(x - 2^{2n+j} \hat{e})),
\]
and the definition of \( \phi \), we deduce that for \( j < k \)
\[
\text{supp } \Phi_{k,j} \subset \left\{ \xi \in \mathbb{R}^d : \frac{33}{48} 2^k \leq |\xi| \leq \frac{35}{48} 2^k \right\},
\]
\[
\Rightarrow \quad \text{supp } \Phi_{k,j} \cos \left( \frac{17}{12} 2^{n+1} \hat{e} \cdot x \right) \subset \left\{ \xi \in \mathbb{R}^d : \frac{33}{24} 2^{n+1} \leq |\xi| \leq \frac{35}{24} 2^{n+1} \right\}.
\]

Then, for \( j < k \), (3.18) holds. Similarly, (3.18) also holds for \( j > k \). Thus, we finish the proof of (3.18).

**A.3 Proof of (3.19).** Noting the fact that for \( 100d < N \in \mathbb{Z}^+ \)
\[
|\tilde{\varphi}(x)| \leq C (1 + |x|)^{-N},
\]
then we have
\[
\| K_2 \|_{L^2(B_{r})} \leq \sum_{k \in \mathbb{N}(n), k \neq \ell} 2^{\frac{d}{2}} 2^{\ell} \left\| \int_{\mathbb{R}^d} \tilde{\varphi}(2^k(x - y)) \phi^2(2^k(y - 2^{2n+k} \hat{e})) dy \right\|_{L^2(B_{r})}
\leq \sum_{k \in \mathbb{N}(n), k \neq \ell} 2^{\frac{d}{2}} 2^{\ell} \left\| \int_{\mathbb{R}^d} \left( 1 + 2^k |x - y| \right)^{-N} \left( 1 + 2^k |y - 2^{2n+k} \hat{e}| \right)^{-2N} dy \right\|_{L^2(B_{r})}.
\]
Dividing the integral region in terms of $y$ into the following two parts to estimate:

$$\mathbb{R}^d = \{y : |y - 2^{\ell+2n} \varepsilon_\ell| \leq 2^n\} \cup \{y : |y - 2^{\ell+2n} \varepsilon_\ell| \geq 2^n\} = A_1 \cup A_2,$$

For $x \in B_\ell$ and $y \in A_1$, we conclude that

$$|y - 2^{k+2n} \varepsilon_\ell| = |(y - 2^{\ell+2n} \varepsilon_\ell) + (2^{k+2n} \varepsilon_\ell - 2^{k+2n} \varepsilon_\ell)| \geq 2^{\ell+2n} - 2^{k+2n} \varepsilon_\ell| - |y - 2^{\ell+2n} \varepsilon_\ell| \geq 2^n.$$

For $x \in B_\ell$ and $y \in A_2$, it is easy to check that

$$|x - y| \geq |y - 2^{\ell+2n} \varepsilon_\ell| - |x - 2^{\ell+2n} \varepsilon_\ell| \geq 2^n - 2^{-\ell} \geq 2^{n-1}.$$

Then, we have

$$\left\| \int_{\mathbb{R}^d} \left( 1 + 2^\ell |x - y| \right)^{-N} \left( 1 + 2^k |y - 2^{n+k} \varepsilon_\ell| \right)^{-2N} dy \right\|_{L^{2d}(B_\ell)} \leq C 2^{-2(k+2n)N} \left\| \int_{A_1} \left( 1 + 2^\ell |x - y| \right)^{-N} dy \right\|_{L^{2d}(B_\ell)} + C 2^{-(\ell+2n)N} \left\| \int_{A_2} \left( 1 + 2^k |y - 2^{n+k} \varepsilon_\ell| \right)^{-2N} dy \right\|_{L^{2d}(B_\ell)} \leq C \left( 2^{-2d\ell} 2^{-2(k+2n)N} + 2^{-(\ell+2n)N} 2^{-2d\ell} \right) 2^{-\frac{\ell}{n}}.$$

Plugging the above into (4.7) yields

$$\|K_2\|_{L^{2d}(B_\ell)} \leq C \sum_{k \in \mathbb{N}(n), k \neq \ell} 2^{k+\ell} 2^{d\ell} \left( 2^{-2d\ell} 2^{-2(k+2n)N} + 2^{-(\ell+2n)N} 2^{-2d\ell} \right) 2^{-\frac{\ell}{n}} \leq C 2^{-n}.$$

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**Conflict of interest**

The authors declare that they have no conflict of interest.

**References**

[1] H. Bahouri, J.-Y. Chemin, R. Danchin, Fourier analysis and nonlinear partial differential equations, Grundlehren der Mathematischen Wissenschaften, 343, Springer-Verlag, Berlin, Heidelberg, 2011.
[2] J. Bourgain, N. Pavlović, Ill-posedness of the Navier–Stokes equations in a critical space in 3D, J. Funct. Anal., 255 (2008) 2233–2247.

[3] Q. Chen, C. Miao, Z. Zhang, On the ill-posedness of the compressible Navier–Stokes equations in the critical Besov spaces, Rev. Mat. Iberoam. 31 (4) (2015) 1375–1402.

[4] C. Hao, Global well-posedness for a multidimensional chemotaxis model in critical Besov spaces, Z. Angew. Math. Phys. 63 (5) (2012) 825–834.

[5] T. Iwabuchi, T. Ogawa, Ill-posedness for the compressible Navier-Stokes equations under the barotropic condition in the limmiting Besov spaces. J. Math. Soc. Jpn. (2021) doi: 10.2969/jmsj/81598159

[6] E.F. Keller, L.A. Segel, Initiation of slime mold aggregation viewed as an instability, J. Theoret. Biol. 26 (3) (1970) 399–415.

[7] D. Li, T. Li, K. Zhao, On a hyperbolic-parabolic system modeling chemotaxis, Math. Models Methods Appl. Sci. 21 (8) (2011) 1631–1650.

[8] T. Li, R. Pan, K. Zhao, Global dynamics of a hyperbolic-parabolic model arising from chemotaxis, SIAM J. Appl. Math. 72 (1) (2012) 417–443.

[9] T. Li, Z.A. Wang, Nonlinear stability of traveling waves to a hyperbolic-parabolic system modeling chemotaxis, SIAM J. Appl. Math. 70 (5) (2009) 1522–1541.

[10] T. Li, Z.A. Wang, Nonlinear stability of large amplitude viscous shock waves of a generalized hyperbolic-parabolic system arising in chemotaxis, Math. Models Methods Appl. Sci. 20 (11) (2010) 1967–1998.

[11] J. Li, Y. Yu, W. Zhu, Ill-posedness for the stationary Navier-Stokes equations in critical Besov spaces, arXiv:2204.08295v3.

[12] Y. Nie, J. Yuan, Well-posedness and ill-posedness of a multidimensional chemotaxis system in the critical Besov spaces, Nonlinear Anal. 196 (2020) 111782.

[13] Y. Nie, J. Yuan, Ill-posedness issue for a multidimensional hyperbolic-parabolic model of chemotaxis in critical Besov spaces $\dot{B}^{-\frac{1}{2}}_{2d,1} \times (\dot{B}^{-\frac{1}{2}}_{2d,1})^d$, J. Math. Anal. Appl. 505 (2) (2022).

[14] H. Tsurumi, Well-posedness and ill-posedness problems of the stationary Navier-Stokes equations in scaling invariant Besov spaces, Arch. Ration. Mech. Anal. 234:2 (2019) 911-923.

[15] B. Wang, Ill-posedness for the Navier-Stokes equations in critical Besov spaces $\dot{B}^{-1}_{\infty,q}$, Adv. Math. 268 (2015) 350-372.

[16] W. Xiao, X. Fei, Ill-posedness of a multidimensional chemotaxis system in the critical Besov spaces, J. Math. Anal. Appl. 514 (2022) 126302.

[17] T. Yoneda, Ill-posedness of the 3D-Navier-Stokes equations in a generalized Besov space near $\text{BMO}^{-1}$, J. Funct. Anal. 258 (2010) 3376-3387.