Oscillations of a Bose-Einstein condensate rotating in a harmonic plus quartic trap

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We study the normal modes of a two-dimensional rotating Bose-Einstein condensate confined in a quadratic plus quartic trap. Hydrodynamic theory and sum rules are used to derive analytical predictions for the collective frequencies in the limit of high angular velocities, \( \Omega \), where the vortex lattice produced by the rotation exhibits an annular structure. We predict a class of excitations with frequency \( \sqrt{\Omega} \) in the rotating frame, irrespective of the mode multipolarity \( m \), as well as a class of low energy modes with frequency proportional to \(|m|/\Omega\). The predictions are in good agreement with results of numerical simulations based on the 2D Gross-Pitaevskii equation. The same analysis is also carried out at even higher angular velocities, where the system enters the giant vortex regime.

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The availability of traps with stronger than harmonic confinement opens up new scenarios in rotating ultracold gases. In principle such traps permit the realization of configurations rotating with arbitrarily high angular velocities, since the confining potential is always stronger than the repulsive centrifugal term. The first experiments in this direction are reported in Ref. [1].

The stationary configurations of rotating Bose-Einstein condensates in the presence of a harmonic plus quartic trap have already been the subject of several theoretical papers [2, 3]. These calculations predict novel vortex structures reflecting the interplay between the centrifugal and confining forces. In particular, if the angular velocity, \( \Omega \), exceeds a critical value, the centrifugal force overcomes the harmonic confinement giving rise to a hole in the center of the condensate. For large angular velocities the radius of the resulting annulus increases linearly with \( \Omega \), while the width of the annulus decreases like \( 1/\Omega \). For such geometries the dynamical behaviour of the gas exhibits new features, whose investigation is the main purpose of this work. In particular, along with excitations involving radial deformations of the density, one expects the occurrence of low frequency sound waves propagating around the annulus.

In this work we will calculate the frequencies of the lowest modes by developing an analytical description using hydrodynamic theory and sum rules, as well as carrying out simulations based upon the numerical solution of the Gross-Pitaevskii equation. For simplicity we will restrict our discussion to 2D configurations, valid for fast rotating condensates strongly confined in the axial direction.

The expression for the trapping potential is given by the sum of quadratic and quartic components

\[
V_{\text{ext}} = \frac{\hbar \omega_{\perp}}{2} \left( \frac{r^2}{d_{\perp}^2} + \frac{r^4}{d_{\perp}^4} \right).
\]

Here \( \omega_{\perp} \) is the harmonic oscillator frequency, \( d_{\perp} = \sqrt{\hbar/M\omega_{\perp}} \) is the characteristic harmonic oscillator length where \( M \) is the atomic mass, \( r = \sqrt{x^2 + y^2} \) is the two-dimensional radial coordinate and \( \lambda \) is the dimensionless parameter characterizing the strength of the quartic term.

In the rotating frame the linearized rotational hydrodynamic equations take the form

\[
\frac{\partial}{\partial t} \delta n + \nabla \cdot (n_0 \delta \mathbf{v}) = 0 ,
\]

\[
\frac{\partial}{\partial t} \delta \mathbf{v} + g \nabla \delta n + 2 \Omega \times \delta \mathbf{v} = 0 ,
\]

where \( n_0 \) is the equilibrium density, \( g \) is the coupling constant, and \( \delta n \) and \( \delta \mathbf{v} \) are the density and velocity varia-
tions respectively. For an effectively 2D system, uniform in the axial direction over a length $Z$, the coupling constant can be written as $g = 4\pi Na/Z$ where $N$ is the number of particles and $a$ is the 3D s-wave scattering length. The integrated density is normalized to unity.

For $\Omega > \Omega_h$ the equilibrium density in the presence of the potential \( \mathbf{1} \) is given by \( \mathbf{3} \)

$$n_0 = \frac{\lambda}{2g} (R_2^2 - r^2)(r^2 - R_1^2) ,$$

where $R_{1,2}$ are the inner and the outer radius of the annulus, respectively. The mean square radius is hence $\langle r^2 \rangle = \int r^2 n_0 \, dr = (R_2^2 + R_1^2)/2$. It is useful to introduce the variable $\zeta = (r^2 - R_2^2)/(R_2^2 - R_1^2)$, where $R_2^2 = R_2^2 \pm R_1^2$. Hence $\zeta$ varies from $-1$ to $1$ and is zero at the mean square radius of the cloud. We also recall that $R_2^2 = (\Omega^2 - 1)/\lambda$ and $R_1^2 = (\Omega_h^2 - 1)/\lambda$, where the angular velocity for the formation of the hole, related to the healing length $\xi$ is given by $\Omega_h = (1 + 2\sqrt{\lambda/\Omega})^{1/2} = \sqrt{1 + (12\lambda^2g/\pi)^{1/3}}$. For large angular velocities, $R_2^2$ increases quadratically with $\Omega$ while $R_1^2$ (proportional to the area) remains constant. Hence the radius of the annulus $R_j/\sqrt{\lambda}$ increases linearly with $\Omega$ whereas the width of the annulus, $d = R_2 - R_1$, decreases like $1/\Omega$.

The hydrodynamic equations (2) and (3) can be solved by expressing the radial and azimuthal components of the velocity field $\delta \mathbf{w}$ in terms of $\delta n$, and looking for solutions of the form $\delta n = \delta n(\zeta)e^{im\phi}e^{-i\omega t}$, where $m$ is the azimuthal quantum number, $\phi$ is the azimuthal angle and $\omega$ is the excitation frequency in the rotating frame. For $\Omega > \Omega_h$, the equation for the density becomes

$$\omega^2 - 4\Omega^2 \delta n + 2m\Omega\lambda R_2^2 \zeta \delta n + \omega \lambda \frac{\partial}{\partial \zeta} \left[ (R_2^2 + R_1^2) (1 - \zeta^2) \frac{\partial}{\partial \zeta} \delta n \right] = 0 .$$

(5)

Eq. (5) can be significantly simplified in the large angular velocity limit $\Omega^2 \gg 1$ where $R_2^2 \sim \Omega^2/\lambda$, by neglecting the terms of order of $R_2^2/R_1^2 \ll 1/\Omega^2$. This case leads to a class of solutions with $\omega \propto \Omega$, obeying the equation

$$\omega^2 - 4\Omega^2 \delta n + \Omega^2 \frac{\partial}{\partial \zeta} \left[ (1 - \zeta^2) \frac{\partial}{\partial \zeta} \delta n \right] = 0 ,$$

(6)

and having the form of Legendre polynomials $P_j(\zeta)$, with $j = 1, 2, \ldots$. The corresponding eigenfrequencies are

$$\omega^2 = [4 + j(j + 1)]\Omega^2 ,$$

(7)

yielding, for the most relevant $j = 1$ mode, the prediction $\omega = \sqrt{6}\Omega$. Remarkably, result (6) is independent of both the oscillator frequency $\omega$ and the strength $\lambda$ of the quartic potential. Furthermore, it is independent of the value and the sign of $m$. The linear dependence of $\omega$ on $\Omega$ can be simply understood using the macroscopic result $\omega = cq$ for the sound wave dispersion. The sound velocity is given by the dilute gas expression $M c^2 = gn$ with $gn \propto \lambda R_2^4$ independent of $\Omega$ while $q \propto 1/d \propto (R_2^2/R_1^2)^{-1}$. Recalling that $R_2^2 \sim \Omega^2/\lambda$ one immediately finds $\omega \propto \Omega$.

The result (7) has been derived in the large $\Omega$ limit. Solutions of Eq. (6) holding for all $\Omega > \Omega_h$ can be found for $\lambda \to 0$, where $\Omega_h \sim 1$ and the terms in $R_2^2/R_1^2 \propto \lambda^{2/3}$ are negligible. For the $j = 1$ mode we find the result $\omega^2 = 6\Omega^2 - 2$. When $\Omega < 1$ the solutions for $\lambda \to 0$ tend to those obtained in Ref. (4) by solving the problem with a rotating harmonic potential.

The collective oscillations can also be investigated using a more microscopic approach based on sum rules. Let us introduce the $p$-energy weighted moments

$$m_p(F) = \sum_n \langle |n| F |0 \rangle^2 E_n^{p} ,$$

(8)

relative to a generic excitation operator $F = \sum_{k=1}^{N} f(r)_k$, where $E_{n0}$ is the energy difference between the excited state $|n\rangle$ and the ground state $|0\rangle$.

A useful estimate of the frequency of the monopole compression mode ($M$), excited by the operator $f(r) = r^2$, can be obtained using the ratio between the energy weighted ($m_1$) and inverse energy weighted ($m_{-1}$) moments. The former can be expressed in terms of the $1$ mode, the prediction

$$\omega^2 = \frac{m_1(M)}{m_{-1}(M)} = 6\lambda R_2^2 + 4 .$$

(9)

For $\Omega < \Omega_h$, $R_+ \equiv \lambda R_2^2$ is the Thomas-Fermi radius, which can be found by solving the cubic equation $R_+^4 (4\lambda R_2^4 - 3\Omega^2 + 3) = 12g/\pi$. For $\Omega \gg \Omega_h$, since $R_2^2 = (R_1^2 + R_2^2) = (\Omega^2 - 1)/\lambda$, one finds the simple result $\omega = \sqrt{6}\Omega^2 - 2$, which is consistent with the hydrodynamic prediction for $\Omega \gg \Omega_h$.

The result of estimate (8), as a function of $\Omega$, is reported in Fig. We compare to the numerical results obtained by solving the 2D time dependent Gross-Pitaevskii equation, where the numerical methods are detailed in Ref. (4). Starting from the stationary solution, the mode is excited by a sudden change in the confining $r^2$ potential, which, after some short time, is reset to its original form. The subsequent changes in the radius are then analyzed to extract the frequencies of oscillation. We have performed simulations at $g = 1000$ for $\lambda = 0.5$ and $\lambda = 10^{-3}$. The latter value of $\lambda$ is similar to that used in
of additional modes not described by Eq. (6). Indeed, as-

tent with a one-mode assumption, reveals the existence

diction of the monopole frequency. Fig. 1 also reveals

shows that the sum rule approach provides an excellent

results from the use of the Thomas-Fermi approximation

\[ \frac{m_2^3(D)}{m_1^3(D)} = 5\Omega^2 - 1 \]  

\[ \frac{m_2^3(Q)}{m_1^3(Q)} = \frac{5\Omega^2 - 1}{5\Omega^2 - 1} \]  

In the large \( \Omega \) limit Eqs. (10) and (11) both yield \( \sqrt{5}\Omega \) 

for the excitation frequency, which does not coincide with 

the prediction of Eq. (7). This result, which is inconsis-
tent with a one-mode assumption, reveals the existence 
of additional modes not described by Eq. (6). Indeed, as-

suming that \( m_3^3 \) is exhausted by the \( j = 1 \) modes, the fact 

that the ratio \( m_2^3 / m_1^3 \) is smaller than the corresponding 

frequency \( \sqrt{6}\Omega \) implies that \( m_1^3 \) includes contributions 

from lower frequency modes. In particular one concludes 

that the latter account for 1/6 of the total \( m_1^3 \) moment.

The \( \Omega \) dependence of these lowest frequency modes can 
be simply inferred from Eq. (6), where, neglecting higher 
order corrections, one finds that the frequency should 
be proportional to \( |m|\lambda R^2 / \Omega \). These modes can be 
interpreted as describing a sound wave directed along the 
azimuthal direction, in contrast to the high-lying modes 
which correspond to a radial shape oscillation of the an-

nulus. The coefficient of proportionality can be estimated 
from the ratio between the energy and inverse energy 
weighted sum rules. The low-lying modes contribute only 
1/6 of the \( m_1^3 \) moment; the \( m_2^3 \) sum rule, which is expected 
to be exhausted by the low lying modes, is given 
by the static response \( \chi \). In the large \( \Omega \) limit, the 
linearized hydronamic equations with a multipole pertur-

bation give \( m_{2,1}^3 = -\lambda = (N\pi/g)R^2 (\Omega^2 / 2m)^{|m|} \). Since 

\( m_3^3 \sim 2N m_2^3 (\Omega^2 / 2\lambda)^{|m|-1} \) in the same limit, we find the 

frequency \( \omega = (m_3^3 / 6m_1^3)^{1/2} = (\sqrt{2}/6)|m|\lambda R^2 / \Omega \). The 
same result can also be derived using a variational analy-

sis of Eq. (6). It is also worth noticing that \( \omega \propto \lambda^{2/3} \) 
tends to zero in the \( \lambda \to 0 \) limit.

Fig. 2 shows a comparison between the analytical and 
numerical results for the high-lying and low-lying dipole 
and quadrupole modes. One sees good agreement 
between the two datasets at high \( \Omega \), validating the sum 
rule approach used here. In the numerical simulations, 
the high-lying modes depart from the \( \sqrt{6}\Omega \) dependence 
for \( \Omega < \Omega_h \). The behavior at small \( \Omega \) is qualitatively 
similar to the one exhibited in a rotating harmonic trap, 
where only one mode per branch is present. In particu-
lar for \( \Omega < 1 \) the equations of rotational hydronamics 
in the rotating frame give the result 

\[ \omega(m = \pm 2) = \sqrt{2 - \Omega^2} + \Omega \]  

for the two quadrupole frequencies, while for the dipole one has \( \omega(m = \pm 1) = 1 + \Omega \). At large \( \Omega \), the numerical results also show that the low-
lying quadrupole mode frequency is larger than that of the 
dipole mode by a factor of two, in agreement with the 
arguments presented above.

The excitation energies in the laboratory frame are re-
lated to those in the rotating frame by \( E_{lab} = E_{rot} + m\Omega \). 

For a proper identification of the modes in the lab frame, 
it is crucial to consider the sign of the azimuthal quantum 
number \( m \) associated with each excitation. For this pur-

pose it is useful to evaluate the strengths \( \sigma^+ = |\langle n|F|^0\rangle|^2 \) 
and \( \sigma^- = |\langle n|F|^1\rangle|^2 \), relative to the operators \( F \) and \( F^\dagger \) 
exciting states with angular momentum \( \pm m \). A careful 
analysis of the response function reveals that the upper 
quadrupole level corresponds to an \( m = -2 \) mode, the 
\( m = +2 \) strength relative to this level being extremely 
small. A different situation takes place for the low-lying 
level. When \( \Omega < 1 \) this level has mainly an \( m = +2 \) char-
acter, as in the case of Ref. [6]. For \( \Omega > \Omega_h \), instead, both

FIG. 1: Frequency of the lowest \( m = 0 \) mode as a function of 
the angular velocity \( \Omega \) (in units of the trap frequency) for \( g = 1000 \). 
The sum rule estimates (4) for \( \lambda = 0.5 \) and \( \lambda = 10^{-3} \) 
are plotted as solid black and dotted lines respectively, while 
the results of solving the GP equation numerically are plott-
ed as solid and open circles. The gray line is the asymptotic 
prediction \( \omega = \sqrt{6}\Omega \) and the dashed lines represent the critical 
frequencies for hole formation, \( \Omega_h \), for both values of \( \lambda \).
the $m = \pm 2$ strengths significantly differ from zero. For example, at $\Omega = 5$ and for the parameters of Fig. 2, the numerical simulation shows that $\sigma_0^+ \simeq 0$, $\sigma_0^- = 12.0 N$, $\sigma_L^+ = 19.9 N$, $\sigma_L^- = 7.8 N$, where $\sigma_0^\pm$ and $\sigma_L^\pm$ are the strengths associated with the high and low-lying $m = \pm 2$ modes. In conclusion, we predict that in the lab frame only two levels (high and low-lying) are expected to hold at large $\Omega$. The vertical dashed line denotes the value of $\Omega_h$.

We finally discuss the case of the giant vortex equilibrium configuration, where the velocity field of the condensate is irrotational. In this case, linearizing the Gross-Pitaevskii equation in the rotating frame gives two coupled equations for the density and the phase variations $\delta n$ and $\delta S$:

$$\frac{\partial}{\partial t} \delta n + \left( \frac{v_{\text{irr}}}{r} - \Omega \right) \frac{\partial}{\partial \phi} \delta n + \nabla \cdot (n_0 \nabla \delta S) = 0,$$

$$\frac{\partial}{\partial t} \delta S + \left( \frac{v_{\text{irr}}}{r} - \Omega \right) \frac{\partial}{\partial \phi} \delta S + g \delta n = 0,$$

where $v_{\text{irr}} = \nu/r$ for a giant vortex with circulation $\nu$. From these equations one can derive an equation similar to Eq. (5), but for the phase rather than the density. For large $\Omega$, the solutions are again Legendre polynomials, but with eigenfrequencies $\omega^2 = 3j(j+1)\Omega^2$ where $j \geq 1$. Hence the $j = 1$ mode has the same frequency for both the irrotational and solid body cases, but the frequencies for the $j > 1$ modes are different. In the case of the low-lying modes for $m \neq 0$, using the sum rule or hydrodynamic methods discussed earlier, we find a frequency that has the same $1/\Omega$ dependence as in the vortex lattice case, but is larger by a factor $3^{1/6}$.

In summary, we have studied normal modes of a Bose condensate in a harmonic plus quartic potential using analytic methods (hydrodynamic equations and sum rule) and numerical solution of the Gross-Pitaevskii equation. At large angular velocities $\Omega$ we find a radial mode with a frequency $\sqrt{6}\Omega$ independent of the mode multipolarity and value of $\lambda$, as well as low-lying modes corresponding to waves around the annular condensate.

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[1] V. Bretin, S. Stock, Y. Seurin, and J. Dalibard, Phys. Rev. Lett. 92, 050403 (2004); S. Stock, V. Bretin, F. Chevy, and J. Dalibard, Europhys. Lett. 65, 594 (2004).
[2] A.L. Fetter, Phys. Rev. A 64, 063608 (2001); U.R. Fischer and G. Baym, Phys. Rev. Lett. 90, 140402 (2003); G.M. Kavoulakis and G. Baym, New J. Phys. 5, 51.1 (2003); E. Lundh, Phys. Rev. A 65, 043604 (2002); A.D. Jackson, G.M. Kavoulakis, and E. Lundh, Phys. Rev. A 69, 053619 (2004); A.D. Jackson and G.M. Kavoulakis, Phys. Rev. A 70, 023601 (2004).
[3] A.L. Fetter, B. Jackson, and S. Stringari, cond-mat/0407119.
[4] M. Cozzini and S. Stringari, Phys. Rev. A 67, 041602(R) (2003).
[5] The $j = 0$ solution is rejected due to its unphysical nature.
[6] The result $\omega^2 = 6\Omega^2 - 2$ can also be found within the hydrodynamic formalism by solving Eq. (5) perturbatively with respect to $1/\Omega$.
[7] For the dipole operator one finds $m_y^+(D) = 2N$ and $m_y^+(Q) = 2N(1 + 3\mu^2 + 4\lambda(r^2))$, while for the quadrupole instead has $m_y^+(Q) = 8N(r^2)$ and $m_y^+(Q) = 16N(6\Omega^2 + 1)(r^2) + (p^2 + 3\lambda(r^4) - 6\Omega(\ell)_z)$, where $p^2 = -(\ell^2_x/x^2 + \ell^2_y/y^2)$ and $\ell_z = -i\partial/\partial \phi$. In the Thomas-Fermi diffused vorticity approach the equilibrium velocity is $v_0 = \Omega \wedge r$, so that $\langle \rho \rangle = \Omega^2 (r^2)$ and $\langle \ell_z \rangle = \Omega (r^2)$.
[8] The result for the quadrupole frequencies has been confirmed in the experiment of P.C. Haljan, I. Coddington, P. Engels, and E.A. Cornell, Phys. Rev. Lett. 87, 210403 (2001), where a harmonic trap was employed.
[9] This scenario is based on the assumption that in the rotating frame only two levels (high and low-lying) are excited by the operators $F$ and $F^\dagger$. Numerical integration of Eq. (6) actually shows the occurrence of a very small splitting between the low-lying $m = \pm 2$ modes, which however barely affects the main results of the present analysis.