Low-Field Phase Diagram of Layered Superconductors: The Role of Electromagnetic Coupling

Gianni Blatter\textsuperscript{a}, Vadim Geshkenbein\textsuperscript{a,b}, Anatoli Larkin\textsuperscript{a,b}, and Henrik Nordborg\textsuperscript{a}

\textsuperscript{a} Theoretische Physik, ETH-Hönggerberg, CH-8093 Zürich, Switzerland
\textsuperscript{b} L. D. Landau Institute for Theoretical Physics, 117940 Moscow, Russia

(April 1, 2022)

Abstract

We determine the position and shape of the melting line in a layered superconductor taking the electromagnetic coupling between layers into account. In the limit of vanishing Josephson coupling we obtain a new generic reentrant low-field melting line. Finite Josephson coupling pushes the melting line to higher temperatures and fields and a new line shape $B_m \propto (1 - T/T_c)^{3/2}$ is found. We construct the low-field phase diagram including melting and decoupling lines and discuss various experiments in the light of our new results.
Since its proposal in 1988 [1], vortex-lattice melting in bulk type II material has become a central topic in the phenomenology of high temperature superconductors. The order, position, and shape of the transition have been investigated theoretically [2] as well as experimentally [3] by a large number of authors. Most recently, the main interest is concentrating on the phase diagram of the strongly layered Bi$_2$Sr$_2$Ca$_1$Cu$_2$O$_8$ (BiSCCO) superconductor which is being investigated by means of $\mu$SR [4], neutron scattering [5], SQUID magnetometry [3], and Hall-sensor arrays [7], probing the melting- and/or decoupling transition in these materials. It turns out that the most interesting regime is the low-field part of the phase diagram with $B < 1$ kG, where the electromagnetic interactions between the layers becomes relevant, and it is the purpose of this letter to derive and analyze the vortex-lattice melting transition in this regime, taking full account of electromagnetic coupling.

The importance of electromagnetic interactions, contributing to the stiffness of individual vortex lines, has been realized before within the context of vortex-lattice melting in the dilute limit [8], where the transition line exhibits a reentrant behavior (lower branch of the melting line). As we will show below, the electromagnetic interaction also influences the behavior of the upper branch of the low-field melting line and even may change its shape from the usual $B_m^1(T) \propto (1 - T/T_c)^2$ behavior to a new power-law $B_m^{em,1}(T) \propto (1 - T/T_c)^{3/2}$ within a large part of the phase diagram — this is the new and central result of this paper.

Our analysis below is based on the continuum elastic description of the vortex lattice combined with the Lindemann criterion, stating that the lattice will undergo a melting transition once the mean thermal displacement $\langle u^2 \rangle_{th}^{1/2}$ becomes comparable to the lattice spacing $a_o \approx (\Phi_o/B)^{1/2}$, $\langle u^2 \rangle_{th}^{1/2}/a_o |_{T_m,B_m} \approx c_L$. The Lindemann number $c_L$ is usually chosen to be a constant of order $c_L \approx 0.1 - 0.3$. Though not rigorous, the Lindemann-type melting scenario has proven very useful and reasonably accurate in predicting the positions of first-order melting transitions in general and the line shape of the vortex-lattice melting transition in particular.

A well known limiting case, where strong fluctuations due to dimensional reduction drive a vortex-lattice melting transition, is the superconducting film (2D dislocation-mediated
Kosterlitz-Thouless melting scenario, see [9]) and we will begin our analysis with this elementary building block of a layered superconductor. Next, we consider a layered system with electromagnetic coupling and derive the shape of the reentrant melting line in this limit. Finally, we account for the Josephson interaction between the layers producing a finite anisotropy parameter \( \varepsilon^2 = m/M < 1 \), where \( m \) and \( M \) denote the effective in-plane and \( c \)-axis masses. Our results are illustrated in Fig. 1, where we show the shape of the vortex-lattice melting line as it evolves from the 2D isolated layer, to the electromagnetically coupled system of layers, to the Josephson coupled bulk anisotropic superconductor.

Our main task is the calculation of the mean-squared thermal displacement [10]

\[
\langle u^2 \rangle_{\text{th}} \approx \int \frac{d^3k}{(2\pi)^3} \frac{T}{c_{66}K^2 + c_{44}(k)k_z^2},
\]

with the shear modulus \( c_{66} \) given by

\[
c_{66} = \begin{cases} 
\sqrt{\pi \frac{\lambda}{6} \frac{\varepsilon_o}{a_o^2} e^{-a_o/\lambda}}, & \lambda < a_o, \\
\frac{\varepsilon_o}{4a_o^2}, & a_o < \lambda,
\end{cases}
\]

and the dispersive tilt modulus \( c_{44}(k) \) consisting of a bulk term \( c^o_{44}(k) \) and a single vortex contribution \( c^v_{44}(k_z) \), \( c_{44}(k) = c^o_{44}(k) + c^v_{44}(k_z) \), with [11]

\[
c^o_{44}(k) = \frac{\varepsilon_o}{a_o^2} \frac{4\pi \lambda^2}{1 + (\lambda^2/\varepsilon^2)K_{BZ}^2 + \lambda^2k_z^2},
\]

\[
c^v_{44}(k_z) \approx \frac{\varepsilon_o}{2a_o^2} \left[ \varepsilon^2 \ln \left( \frac{\lambda^2/\varepsilon^2 \xi^2}{1 + (\lambda^2/\varepsilon^2)K_{BZ}^2 + \lambda^2k_z^2} \right) 
+ \frac{1}{\lambda^2k_z^2} \ln \left( 1 + \frac{\lambda^2k_z^2}{1 + \lambda^2K_{BZ}^2} \right) \right]
\]

(in [11] we neglect a second contribution to \( \langle u^2 \rangle_{\text{th}} \) involving lattice compression and keep only the main term). Here, \( \varepsilon_o = (\Phi_o/4\pi\lambda)^2 \) denotes the basic energy scale of the continuum elastic theory, \( \Phi_o = hc/2e \) is the flux quantum, \( \lambda \) denotes the planar London penetration depth, and \( \xi \) is the planar coherence length. The second term in the single vortex tilt \( c^v_{44} \) is due to the electromagnetic coupling between the layers and is the only term in \( c_{44} \) surviving the limit \( \varepsilon \to 0 \) (layer decoupling). The electromagnetic contribution to the tilt
modulus is strongly dispersive and produces the large stiffness \( \varepsilon_l \approx \varepsilon_o/2 \) of the vortex lines in the long-wave-length limit \( k_z < 1/\lambda \). With increasing \( k_z \) the electromagnetic stiffness decays \( \propto 1/\lambda^2 k_z^2 \) and the line tension crosses over to the well known result \( \varepsilon_l \approx \varepsilon^2 \varepsilon_o \) for the anisotropic superconductor as \( k_z \) increases beyond \( 1/\varepsilon \lambda \) (note that this residual tension is due to the Josephson coupling and is relevant only for \( \varepsilon \lambda > d \), where \( d \) denotes the layer separation). The expression given in (4) is valid for small displacements, in the elastic regime. For large displacements \( uk_z > 1 \) the logarithm in the second term of (4) should be cut on \( 2/\lambda u \) rather than \( \lambda k_z \) \cite{12}. In our analysis below we then replace the logarithm by the factor \( \lambda^2 k_z^2/(1 + \beta \lambda^2 k_z^2) \) with \( \beta = 1/\ln(1 + 4\lambda^2/c_L^2d_o^2) \) producing a smooth interpolation between the hard and soft tilt modes at large and small wave-lengths, respectively.

We start with the analysis of an individual layer (we use the definition \( \lambda^2/d = \lambda_s^2/d_s \) with \( \lambda_s \) and \( d_s \) the penetration depth and thickness of the superconducting layer). Dropping the tilt energy in (4), the integral over \( k_z \) provides a factor \( 2\pi/d \) and cutting the \( K \)-integration on a few lattice spacings we obtain the ratio \( \langle u^2 \rangle_{th}/a_o^2 \approx T/2\pi c_{66} d_o^2 \). Within our simple Lindemmann approach we then can reproduce the correct result \( T_m^{2D} \approx a_o^2 dc_{66}/4\pi \) for the dislocation-mediated melting temperature if we choose a Lindemann number \( c_L = 1/2\sqrt{2}\pi \approx 0.1 \). The high-field part \( (a_o < \lambda) \) of the melting line is field-independent,

\[
T_m^{2D} \approx \frac{\varepsilon_o d}{16\pi},
\]

and using parameters typical for the layered high-\( T_c \) superconductors, \( T_c \approx 100 \) K, \( \lambda^2(T) \approx \lambda_0^2/(1 - T^2/T_c^2) \) with \( \lambda_0 \approx 1800 \) \AA, and \( d = 15 \) \AA, we obtain \( \varepsilon_o(T = 0)d \approx 10^3 \) K and \( T_m^{2D} \approx 20 \) K. The low-field part \( (a_o > \lambda) \) of the melting line is dominated by the exponential decay of the shear modulus and we obtain the result

\[
B_m^{2D} \approx \frac{\Phi_o}{\lambda^2} \ln\left(\frac{(2\pi^3/3)^{1/2} \varepsilon_o d \varepsilon_o}{\lambda T} \right)^{-2}.
\]

The result for the melting line of an isolated layer is illustrated in Fig. 1. Note that the result (3) is a somewhat artificial construct, as we have used the low-field expression in (2) for our analysis. In this way we illustrate the behavior of the melting line in the absence
of any interlayer coupling (neither electromagnetic nor Josephson) while keeping the shear modulus of a translation invariant system along the field axis. On the other hand, the analysis of a 2D film involves a different shear modulus \[13\] which softens only at very low fields \((a_o > \lambda_{\text{eff}} = 2\lambda^2/d)\) following the power-law behavior \(c_{66} \approx 0.46\varepsilon_o\lambda_{\text{eff}}/a_o^3 \propto B^{3/2}\) rather than the exponential behavior used in our analysis.

Next we consider a finite electromagnetic coupling between the layers while keeping \(\varepsilon = 0\) (no Josephson coupling). In the high-field regime \((a_o < \lambda)\) the shear term in (1) dominates over the tilt energy and we recover the field independent 2D-result (5). For small fields with \(a_o > \lambda\) the tilt energy becomes relevant and the Lindemann criterion reads

\[
c_L^2 \approx \frac{2T}{\varepsilon_o d a_o^2} \left[ \frac{1}{4\pi\delta} \ln(1 + 4\pi\delta\beta) + \frac{d}{\lambda(4\pi\delta)^{1/2}} \right],
\]

with \(\delta = 2c_{66}\lambda^2/\varepsilon_o\). Here, the first term originates from the soft tilt modes with \(k_z\lambda > 1\), whereas the second term involves the long wave-length modes hardened by the electromagnetic coupling. This second term becomes relevant only at very small fields \(a_o/\lambda \gg 1\), where the shear modulus is exponentially small, \(\delta \propto \exp(-a_o/\lambda)\). The result (7) provides a lower branch of the melting line which is limited by soft shear and hard tilt,

\[
B_m^{\text{em,l}}(T) \approx \frac{\Phi_o}{\lambda^2} \frac{1}{4} \left[ \ln \left( \frac{4\pi c_L^2 \varepsilon_o \lambda}{(3\pi)^{1/4} T} \right) \right]^{-2},
\]

as well as a tilt limited upper branch

\[
B_m^{\text{em,u}}(T) \approx \frac{\Phi_o}{\lambda^2} \frac{c_L^2 \varepsilon_o d}{2\beta T} \sim \left( 1 - \frac{T^2}{T_c^2} \right)^2.
\]

The two branches merge near \(T_c\),

\[
1 - \frac{T_x}{T_c} \approx \frac{\beta G^{2D}}{4c_L^2} \left[ \ln \left( \frac{2\pi (2\beta)^{1/2}}{(3\pi)^{1/4}} \frac{c_L}{\sqrt{G^{2D}} d} \right) \right]^{-2},
\]

and no solid phase can exist at high temperatures beyond \(T_x\). Using typical parameters for the layered high-\(T_c\) materials and adopting a value \(c_L \approx 0.1\) for the Lindemann number we find \(T_x\) close to \(T_c\), \(1 - T_x/T_c \approx 0.05\) (in (10) we have introduced the 2D Ginzburg number \(G^{2D} \approx T_c/\varepsilon_o(T = 0)d \approx 0.1\); the logarithms in (8) and (10) take typical values
around 5 – 6). The reentrant melting line defined by (9) and (10) is illustrated in Fig. 1: The electromagnetic coupling of the layers favors the solid phase and the low-field melting line develops the characteristic “nose-like” shape of a 3D system. Note that the point of reentrance ends up in the critical region close to $T_c$. Since our approach accounts for the fluctuations of the phase-field via the thermal motion of vortices but neglects amplitude fluctuations of the order parameter our analysis breaks down in this regime.

In the final step we account for the Josephson coupling between the layers producing a finite anisotropy parameter $\varepsilon > 0$. This additional coupling becomes relevant whenever $a_o, \lambda > d/\varepsilon$ and again favors the solid phase, hence pushing the melting line further towards higher temperatures and fields. Evaluating the Lindemann criterion in the low-field regime ($a_o > \lambda$) we recover the previous result (9) with the modification that the soft tilt modes are cut off on $1/\varepsilon\sqrt{\beta}\lambda$ instead of $\pi/d$, leading to the replacement of $\ln(\ldots)/4\pi\delta$ by $(d\sqrt{\beta}/\pi\varepsilon\lambda)[\ln(\ldots)/4\pi\beta\delta + 1]$ in the first term of (9). The lower branch of the melting line remains unaffected, whereas the upper branch of the low-field melting line takes the new form

$$B_{m,J}^{em}(T) \approx \frac{\Phi_o}{\lambda^2} \frac{\pi c_L^2 \varepsilon \varepsilon_o \lambda}{4\sqrt{\beta}} \frac{T}{T_c} \propto \left(1 - \frac{T^2}{T_c^2}\right)^{3/2}. \quad (11)$$

The crossing point of the lower and upper branch of the melting line is shifted towards higher temperatures,

$$1 - \frac{T_x}{T_c} \approx \frac{1}{2} \left[ \frac{\sqrt{\beta}G^{2D}}{\varepsilon \lambda_o d} \frac{\ln\left(\frac{4\sqrt{\beta}}{(3\pi)^{1/4}\varepsilon}\right)}{\varepsilon \lambda_o} \right]^{-2}. \quad (12)$$

For $\varepsilon\lambda_0 < d$ the line $B_{m,J}^{em}$ goes over into the generic melting line $B_{m,u}^{em}$ as the temperature drops below $T^{em} \approx T_c[1 - \beta(\varepsilon\pi\lambda_0/d)^2]^{1/2}$. For the opposite case where $\varepsilon\lambda_0 > d$ the generic line $B_{m,u}^{em}$ is completely hidden and $B_{m,J}^{em}$ merges into the well known bulk anisotropic melting line $B_{m}^{J}$ as the field grows beyond $\Phi_o/\lambda^2$: At these fields the tilt energies are dominated by the dispersive bulk term $c_{44}^o \approx 4\pi\varepsilon^2\varepsilon_o/a_o^4K^2$ (see Eq. (3)) and the Lindemann criterion provides the well known result

$$B_{m}^{J}(T) \approx \frac{\Phi_o}{\lambda^2} \frac{4\pi c_L^4 \varepsilon^2 \varepsilon_o \lambda^2}{T^2} \propto \left(1 - \frac{T^2}{T_c^2}\right)^2. \quad (13)$$
At large fields \((a_o < d/\varepsilon < \lambda_o)\) the 2D result (5) is recovered.

The most interesting result is the new line shape \(B_{m}^{\text{em},J} \sim (1 - T/T_c)^{3/2}\), Eq. (11), describing the low-field/high-temperature melting in a Josephson-coupled layered or highly anisotropic superconductor (small parameter \(\varepsilon < d/\lambda_o\)). This new result is due to the electromagnetic coupling which dominates over the bulk-dispersive tilt modulus \(c_{44}^2\) as well as over the single-vortex line tension \(\varepsilon^2 \varepsilon_o\) due to Josephson-coupling in this regime. The substitution of the old result (13) by the new expression (11) is particularly relevant in the strongly layered superconductors such as BiSCCO: The \((1 - T/T_c)^{3/2}\) power-law is valid provided that \(d/\pi \sqrt{\beta} \varepsilon < \lambda < a_o\). Assuming \(\varepsilon \sim 1/150\), the second restriction implies \(T > 0.4 T_c\). In less anisotropic materials, such as YBCO with \(\varepsilon \approx 1/5\), this condition is much more stringent and the upper branch of the melting line is always described by the old result, Eq. (13). Note, however, that in YBCO the suppression of the order parameter close to the upper critical field \(H_{c_2}\) becomes relevant and the melting line cannot be described in terms of a simple power law \(\propto (1 - T/T_c)^2\) any longer, see Ref. [2], Blatter and Ivlev, for a detailed discussion (in BiSCCO the melting line is far below \(H_{c_2}\) and there is no suppression of the order parameter in this regime).

It is instructive to compare the different low-field melting lines as given by Eqs. (9), (11), and (13). A quick inspection gives the ratio \(B_{m}^{\text{em},u}/B_{m}^{J} = (d/\lambda \varepsilon c_L)^2 T/8 \pi \beta \varepsilon_o d\), which is of order unity taking the above parameters for BiSCCO and using \(\varepsilon = 1/150\), a value often quoted in the literature [4,5]. Similarly, \(B_{m}^{\text{em},J}/B_{m}^{J} = (d/\lambda \varepsilon c_L) T/16 \sqrt{\beta} c_L \varepsilon_o d \approx \alpha [T^2/T_c(T_c - T)]^{1/2}\), were again \(\alpha \sim 1\) if we use the above parameters for BiSCCO. The comparison of experimental data for the irreversibility or melting line with the theoretical prediction is often used to extract an estimate of the anisotropy parameter \(\varepsilon\), particularly in the strongly layered materials [4,5]. Following up the above discussion we draw attention to an important problem with this procedure: If the anisotropy parameter is very small, say \(\varepsilon < 1/500\), the (upper branch of the) low-field melting line (where the comparison theory/experiment is carried out) is dominated by the electromagnetic coupling and no anisotropy parameter can be extracted. The analysis of the melting line can provide a reliable estimate for the
anisotropy parameter only if \( \varepsilon \) is large enough such that either the bulk result (13) is valid or the mixed electromagnetic/Josephson result (11) can be identified via its particular line shape.

It is generally believed that the low-field melting transition takes the vortex lattice into a liquid of vortex lines. It then has been proposed that this line-liquid transforms into a pancake-liquid in a second transition where the layers decouple, see Refs. [11,10]. A simple estimate for the position and shape of this decoupling line is obtained in the following way: thermal wandering of the vortex line over a distance \( L \) produces a displacement amplitude \( \langle u^2 \rangle_{\text{th}} \approx LT/\varepsilon_l \). The layers decouple when the mean thermal displacement between line segments in neighboring layers becomes of the order of the lattice spacing, \( \langle u^2 \rangle_{\text{th}}^{1/2} (L = d) \approx a_\circ \). For a Josephson-coupled system the line tension is \( \varepsilon_l \approx \varepsilon^2 \varepsilon_\circ \) and we obtain the well known result \( B_{\text{dc}} \approx \Phi_\circ/(d/\varepsilon)^2 \varepsilon_\circ d/T \propto 1 - T/T_c \). However, for small anisotropy the electromagnetic coupling dominates at low fields and using the short wave-length elasticity \( \varepsilon_l \approx \varepsilon_\circ (d/\lambda)^2 \) we find that the decoupling line follows the melting line \( B_{\text{em}} \). We then obtain a phase diagram where the decoupling and melting lines are separate transitions at low \((T < T_m^{2D})\) and high \((T > T_{\text{em}})\) temperatures but close up in between.

Recently, a first-order phase transition has been observed in the low-field regime of a strongly layered BiSCCO superconductor [7]. The jump in the magnetization observed at the transition can be associated either with a vortex-lattice melting- or with a layer-decoupling transition. Fits using a \((1 - T/T_c)^{1.55}\) (melting) or a \((T_c/T - 1)\) (decoupling) power-law behavior produce a satisfactory agreement with the data over most of the measured temperature interval [7]. Our new result (11) then is in good agreement with the measured power-law behavior based on the melting scenario. Whether the observed transition indeed can be attributed to a first-order melting transition remains to be shown, however.

In conclusion, we have determined the position and shape of the melting line in a layered superconductor taking the electromagnetic interaction between the layers into account. Whereas the electromagnetic coupling is irrelevant at fields \( B > \Phi_\circ/\lambda^2 \), new results for the melting line have been obtained in the low-field regime \( B < \Phi_\circ/\lambda^2 \). In this regime,
the electromagnetic coupling produces a stiffening of the vortex line at long wave lengths 
\( k_z < 1/\lambda \). Both, the lower and the upper branch of the reentrant melting line are effected 
by this stiffening and a characteristic “nose”-shaped 3D melting line is found even in the 
absence of Josephson coupling between the layers. Accounting for an additional Josephson 
coupling, the upper branch of the melting line is pushed out to higher temperatures and 
fields and takes on a new characteristic line shape \( \propto (1 - T/T_c)^{3/2} \), as observed recently in 
a BiSCCO superconductor [7]. The results are crucial for an accurate understanding of the 
low-field phase diagram of layered superconductors.

We thank E. H. Brandt for helpful discussions and the Swiss National Foundation for 
financial support.
REFERENCES

[1] D. R. Nelson, Phys. Rev. Lett. 60, 1973 (1988).

[2] E. Brézin, D. Nelson, and A. Thiaville, Phys. Rev. B 31, 7124 (1985); A. Houghton, R. Pelcovits, and A. Sudbø, Phys. Rev. B 40, 6763 (1989); E. H. Brandt, Phys. Rev. Lett. 63, 1106 (1989); S. Sengupta, C. Dasgupta, H. Krishnamurthy, G. Menon, and T. Ramakrishnan, Phys. Rev. Lett. 67, 3444 (1991); S. Ryu, S. Doniach, G. Deutscher, and A. Kapitulnik, Phys. Rev. Lett. 68, 710 (1992); G. Blatter and B. Ivlev, Phys. Rev. B 50, 10272 (1994).

[3] D. Farrell, J. Rice, D. Ginsberg, Phys. Rev. Lett. 67, 1165 (1991); A. Schilling, H.-R. Ott, and Th. Wolf, Phys. Rev. B 46, 14253 (1992); H. Safar, P. Gammel, D. Huse, D. Bishop, J. Rice, and D. Ginsberg, Phys. Rev. Lett. 69, 824 (1992); W. Kwok, J. Fendrich, S. Fleshler, U. Welp, J. Downey, and G. Crabtree, Phys. Rev. Lett. 72, 1092 (1994).

[4] S. Lee et al., Phys. Rev. Lett. 71, 3862 (1993).

[5] R. Cubitt et al., Nature (London) 365, 407 (1993).

[6] H. Pastoriza, M. Goffman, A. Arribére, and F. de la Cruz, Phys. Rev. Lett. 72, 2951 (1994).

[7] E. Zeldov, D. Majer, M. Konczykowski, V. Geshkenbein, V. Vinokur, and H. Shtrikman, Nature (London) 375, 373 (1995).

[8] D. Fisher, M. P. A. Fisher, and D. Huse, Phys. Rev. B 43, 130 (1991).

[9] B. Halperin and D. Nelson, Phys. Rev. Lett. 41, 121 (1978); B. Huberman and S. Doniach, Phys. Rev. Lett. 43, 950 (1979); D. Fisher, Phys. Rev. B 22, 1190 (1980).

[10] G. Blatter, M. Feigel’man, V. Geshkenbein, A. Larkin, and V. Vinokur, Rev. Mod. Phys. 66, 1125 (1994).
[11] L. Glazman and A. Koshelev, Phys. Rev. B 43, 2835 (1991).

[12] J. Clem, Phys. Rev. B 43, 7837 (1991).

[13] E. Conen and A. Schmid, J. Low Temp. Phys. 17, 331 (1974).
FIGURES

FIG. 1. Low-field phase diagram of a strongly layered superconductor. Reduced units $b = B/(\Phi_0/\lambda^2)$ and $t = T/T_c$ have been used and the Lindemann number $c_L = 0.1$ has been chosen (else parameters appropriate for BiSCCO have been adopted, see text). The thick lines show the results for the isolated 2D layer and for the electromagnetically coupled system. The thin lines incorporate the effect of a finite Josephson coupling between the layers for anisotropy parameters $\varepsilon = 1/500$, $1/150$, and $1/50$. The inset shows the same results on a logarithmic field scale, where the reentrant behavior of the melting line becomes more visible. The dotted line traces $b(t) = B/[\Phi_0/\lambda^2(t)]$. 
interlayer coupling $\rightarrow$ 3D