MUTIPLE POSITIVE SOLUTIONS FOR SCHRÖDINGER-POISSON SYSTEMS INVOLVING CRITICAL NONLOCAL TERM

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Abstract. The present study is concerned with the following Schrödinger-Poisson system involving critical nonlocal term
\[
\begin{align*}
-\Delta u + K(x)u + |u|^3u &= \lambda f(x)|u|^{q-2}u, & x \in \mathbb{R}^3, \\
-\Delta \phi &= K(x)|u|^5, & x \in \mathbb{R}^3,
\end{align*}
\]
where \(1 < q < 2\) and \(\lambda > 0\) is a parameter. Under suitable assumptions on \(K(x)\) and \(f(x)\), there exists \(\lambda_0 = \lambda_0(q, S, f, K) > 0\) such that for any \(\lambda \in (0, \lambda_0)\), the above Schrödinger-Poisson system possesses at least two positive solutions by standard variational method, where a positive least energy solution will also be obtained.

1. Introduction and Main Results

Due to the real physical meaning, the following Schrödinger-Poisson system
\[
\begin{align*}
-\Delta u + V(x)u + \phi u &= f(x,u), & x \in \mathbb{R}^3, \\
-\Delta \phi &= u^2, & x \in \mathbb{R}^3,
\end{align*}
\]
(1.1)
has been studied extensively by many scholars in the last several decades. The system like (1.1) firstly introduced by Benci and Fortunato \cite{9} was used to describe solitary waves for nonlinear Schrödinger type equations and look for the existence of standing waves interacting with an unknown electrostatic field. We refer the readers to \cite{9, 10, 33, 36} and the references therein to get a more physical background of the system (1.1).

In recent years, by classical variational methods, there are many interesting works about the existence and non-existence of positive solutions, positive ground states, multiple solutions, sign-changing solutions and semiclassical states to the system (1.1) with different assumptions on the potential \(V(x)\) and the nonlinearity \(f(x,u)\) were established. If \(V(x) \equiv 1\) and \(f(x,u) = |u|^{p-1}u\), T. d’Aprile and D. Mugnai \cite{19} showed that the system (1.1) has no nontrivial solutions when \(p \leq 1\) or \(p \geq 5\). For the case \(4 \leq p < 6\), the existence of radial and non-radial solutions was studied in \cite{17, 18, 20}. D. Ruiz \cite{39} proved the existence and nonexistence of nontrivial solutions when \(1 < p < 5\). When
under the conditions $g \in C(\mathbb{R})$ and

$$(H_1) \quad -\infty < \liminf_{t \to 0} \frac{g(t)}{t} \leq \limsup_{t \to 0} \frac{g(t)}{t} = -m < 0;$$

$$(H_2) \quad -\infty \leq \limsup_{t \to +\infty} \frac{g(t)}{t^s} \leq 0;$$

$$(H_3) \quad \text{there exists } \xi > 0 \text{ such that } \int_0^\xi g(t)dt > 0.$$
where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with $\mu > 0$ and $1 < q < 2 < p \leq 2^* = \frac{2N}{N-2}$.

By variational method, they have obtained the existence and multiplicity of positive solutions to the problem (1.5). Subsequently, an increasing number of researchers have paid attention to semilinear elliptic equations with critical exponent and concave-convex nonlinearities, for example, see [8, 13, 16, 24, 26, 29, 38, 44] and the references therein.

To the best of our knowledge, the Schrödinger-Poisson system with critical nonlocal term was only studied in [6, 30, 34]. Meanwhile there are very few papers on existence of multiple results for Schrödinger-Poisson system with concave-convex nonlinearities. Inspired by the works mentioned above, this paper will fill the gap and prove the existence of multiple positive solutions for the following Schrödinger-Poisson system:

\[
\begin{aligned}
-\Delta u + u - K(x)\phi |u|^3u &= \lambda f(x)|u|^{q-2}u, \quad x \in \mathbb{R}^3, \\
-\Delta \phi &= K(x)|u|^3, \quad x \in \mathbb{R}^3,
\end{aligned}
\tag{1.6}
\]

where $1 < q < 2$ and $\lambda > 0$ is a parameter. The assumptions on $K(x)$ and $f(x)$ are as follows:

(K) $K(x) \in C(\mathbb{R}^3, \mathbb{R}^+)$, there exist some constants $C > 0$, $\delta > 0$ and $\beta \in [1, 3]$ such that

$|K(x) - K(x_0)| \leq C|x - x_0|^\beta$ if $|x - x_0| < \delta$,

and $x_0 \in \mathbb{R}^3$ satisfies $K(x_0) = \max_{x \in \mathbb{R}^3} K(x) := |K|_\infty < +\infty$.

(F) $f \in L^\frac{2}{q-2}(\mathbb{R}^3)$ with $f(x) \geq 0$ and $f(x) \not\equiv 0$.

Now we state our main result:

**Theorem 1.1.** Assume (K) and (F), for any $1 < q < 2$ there exists $\lambda_0 = \lambda_0(q, S, f, K) > 0$ such that the system (1.6) admits at least two positive solutions when $\lambda \in (0, \lambda_0)$. In addition, a positive least energy solution can also be established.

**Remark 1.2.**

(1) There are a lot of functions to meet the assumption (K) such as $K \equiv 1$. Without doubt that $K(x_0) > 0$ is necessary and otherwise $K(x) \equiv 0$ implies that the critical term disappears and the system (1.6) degenerates to a semilinear Schrödinger equation. This kind of assumption (K) was firstly introduced by F. Gazzola and M. Lazzarino [22] to consider a semilinear Schrödinger equation.

(2) The assumption of non-negativity for $f(x)$ is not essential to the analysis of Theorem 1.1. In a word, the method used in Theorem 1.1 can deal with the case when $f(x)$ is sign-changing.

**Remark 1.3.** It is worth to point out here that we not only show the existence of $\lambda_0$ obtained in Theorem 1.1, but also give the concrete expression:

\[
\lambda_0 = \lambda_0(q, S, f, K) := \frac{4q}{(10 - q)|f|_\infty} \left( \frac{5S^6(2 - q)}{|K|_{10^3}(10 - q)} \right)^{\frac{2}{2-q}} > 0,
\]

where $S > 0$ is the best Sobolev constant (see (2.1)).
The nonlocal critical term in (1.6) makes the problem complicated because of the lack of compactness imbedding of $H^1(\mathbb{R}^3)$ into $L^r(\mathbb{R}^3)$ for $r \in [2, 6]$. Moreover we do not assume that the functions $K(x)$ and $f(x)$ are radial symmetric, so it is impossible to work in the radial symmetric space. To overcome it, the assumption on $f(x)$ plays an vital role. However if we replace $\mathbb{R}^3$ by a bounded domain $\Omega$, the above difficult disappears. Of course, the assumption on $K(x)$ can never make a contribution to recovering the compactness. What we want to emphasize is that either $K(x) \equiv 1$ or $K(x)$ satisfies $(K)$ in our problem, the proof does not have an essential difficult, but this difference seems to cause some special obstacles in [30] with this case. Meanwhile, by means of a totally same idea but some simpler calculations employed in Theorem 1.1, one immediately has the following result which will not be proved in detail.

**Theorem 1.4.** Assume $\Omega \subset \mathbb{R}^3$ is a bounded domain with a smooth boundary $\partial \Omega$, then for any $1 < q < 2$ there exists $\lambda_0 = \tilde{\lambda}_0(q, S) > 0$ such that the conclusions obtained in Theorem 1.1 still holds for the following system

$$
\begin{align*}
-\Delta u - \phi |u|^3 u &= \lambda |u|^{q-2} u, \quad x \in \Omega, \\
-\Delta \phi &= |u|^5, \quad x \in \Omega, \\
u = \phi = 0, \quad \text{on } \partial \Omega,
\end{align*}
$$

when $\lambda \in (0, \tilde{\lambda}_0)$.

The outline of this paper is as follows. In Section 2, we present some preliminary results for Theorem 1.1. In Section 3, we will prove Theorem 1.1.

**Notations.** Throughout this paper we shall denote by $C$ and $C_i$ ($i = 1, 2, \cdots$) for various positive constants whose exact value may change from lines to lines but are not essential to the analysis of problem. We use “$\to$” and “$\rightharpoonup$” to denote the strong and weak convergence in the related function space, respectively. For any $\rho > 0$ and any $x \in \mathbb{R}^3$, $B_\rho(x)$ denotes the ball of radius $\rho$ centered at $x$, that is, $B_\rho(x) := \{y \in \mathbb{R}^3 : |y - x| < \rho\}$.

Let $(X, \| \cdot \|)$ be a Banach space with its dual space $(X^*, \| \|_*)$, and $\Psi$ be its functional on $X$. The Palais-Smith sequence at level $d \in \mathbb{R}$, $(PS)_d$ sequence in short) corresponding to $\Psi$ satisfies that $\Psi(x_n) \to d$ and $\Psi'(x_n) \to 0$ as $n \to \infty$, where $\{x_n\} \subset X$.

**2. Some Preliminaries**

In this section, we will give some lemmas which are useful for the main results. To solve the system (1.6), we introduce some function spaces. Throughout the paper, we consider the Hilbert space $H^1(\mathbb{R}^3)$ with the usual inner product

$$
(u, v) = \int_{\mathbb{R}^3} \nabla u \nabla v + uv \, dx, \quad \forall u, v \in H^1(\mathbb{R}^3)
$$

and the norm

$$
\|u\| = \left( \int_{\mathbb{R}^3} |\nabla u|^2 + u^2 \, dx \right)^{1/2}.
$$
$L^r(\mathbb{R}^3)$ ($1 \leq r < +\infty$) is the Lebesgue space, $| \cdot |_r$ means its usual $L^r$-norm and the space

$$D^{1,2}(\mathbb{R}^3) = \{ u \in L^6(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3) \}$$

equips with its usual norm and inner product

$$\| u \|_{D^{1,2}(\mathbb{R}^3)} = \left( \int_{\mathbb{R}^3} | \nabla u |^2 \, dx \right)^{\frac{1}{2}} \quad \text{and} \quad (u, v)_{D^{1,2}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \nabla u \nabla v \, dx,$$

respectively. The positive constant $S$ denotes the best Sobolev constant:

$$S := \inf_{u \in D^{1,2}(\mathbb{R}^3) \setminus \{0\}} \frac{\int_{\mathbb{R}^3} | \nabla u |^2 \, dx}{\left( \int_{\mathbb{R}^3} | u |^6 \, dx \right)^{\frac{5}{3}}}.$$  \hfill (2.1)

In the following, one can use the Lax-Milgram theorem, for every $u \in H^1(\mathbb{R}^3)$, there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ such that

$$- \Delta \phi = K(x)|u|^5$$  \hfill (2.2)

and $\phi_u$ can be written as

$$\phi_u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} K(y)|u(y)|^5 |x - y| \, dy.$$  \hfill (2.3)

Substituting (2.3) into (1.6), we get a single elliptic equation with a nonlocal term:

$$- \Delta u + u - K(x)|u|^5 u = \lambda f(x)|u|^{q-2}u$$

whose corresponding functional $J : H^1(\mathbb{R}^3) \to \mathbb{R}$ is defined by

$$J(u) = \frac{1}{2} \| u \|^2 - \frac{1}{10} \int_{\mathbb{R}^3} K(x)|\phi_u|^5 |u|^5 \, dx - \frac{\lambda}{q} \int_{\mathbb{R}^3} f(x)|u|^q \, dx.$$  \hfill (2.4)

We mention here that the idea of this reduction method was proposed by Benci and Fortunato [9] and it is a basic strategy for studying Schrödinger-Poisson system today.

For simplicity, the conditions in Theorem 1.1 always hold true thought this paper and we don’t assume them any longer unless specially needed. To know more about the solution $\phi$ of the Poisson equation in (1.6) which can leads to a critical nonlocal term, we have the following key lemma:

**Lemma 2.1.** For every $u \in H^1(\mathbb{R}^3)$, we have the following conclusions:

1. $\phi_u(x) \geq 0$ for every $x \in \mathbb{R}^3$;
2. $\| \phi_u \|^2_{D^{1,2}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} K(x)|\phi_u|^5 \, dx$;
3. For any $t > 0$, $\phi_{tu} = t^5 \phi_u$;
4. If $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, then $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{1,2}(\mathbb{R}^3)$. 
Proof. As a direct consequence of (2.2) and (2.3), one can derive (1), (2) and (3) at once.

For any $v \in D^{1,2}(\mathbb{R}^3)$, then $v \in L^6(\mathbb{R}^3)$ by (2.1) and hence $K(x)v \in L^6(\mathbb{R}^3)$ because $K(x)$ is bounded. Since $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, then $u_n \to u$ in $L^6(\mathbb{R}^3)$ and $|u_n|^5 \to |u|^5$ in $L^{5/2}(\mathbb{R}^3)$. Therefore
\[
(\phi_{u_n}, v)_{D^{1,2}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} K(x)|u_n|^5 v dx \to \int_{\mathbb{R}^3} K(x)|u|^5 v dx = (\phi_u, v)_{D^{1,2}(\mathbb{R}^3)}
\]
which implies that (4) is true. \qed

Furthermore, by (2) of Lemma 2.1, H"older’s inequality and (2.1), one has
\[
||\phi_u||_{D^{1,2}}^2 = \int_{\mathbb{R}^3} K(x)|\phi_u||u|^5 dx \leq |K|_{\infty}||\phi_u||_{6}^5 \leq |K|_{\infty}S^{-\frac{5}{2}}||\phi_u||_{D^{1,2}}^5
\]
which implies that
\[
||\phi_u||_{D^{1,2}} \leq |K|_{\infty}S^{-\frac{5}{2}}|u|^5 \tag{2.5}
\]
and
\[
\int_{\mathbb{R}^3} K(x)|\phi_u||u|^5 dx \leq |K|_{\infty}^2S^{-6}||u||^{10}. \tag{2.6}
\]

Then from $f(x) \in L^{\frac{6}{5}}(\mathbb{R}^3)$ we have that the functional $J$ given by (2.4) is well-defined on $H^1(\mathbb{R}^3)$ and is of $C^1(H^1(\mathbb{R}^3), \mathbb{R})$ class (see [43]), and for any $v \in H^1(\mathbb{R}^3)$ one has
\[
\langle J'(u), v \rangle = \int_{\mathbb{R}^3} \nabla u \nabla v + uv dx - \int_{\mathbb{R}^3} K(x)|\phi_u||u|^3 uv dx - \lambda \int_{\mathbb{R}^3} f(x)|u|^{q-2} uv dx.
\]

It is standard to verify that a critical point $u \in H^1(\mathbb{R}^3)$ of the functional $J$ corresponds to a weak solution $(u, \phi_u) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ of (1.6). In other words, if we can seek a critical point of the functional $J$, then the system (1.6) is solvable. In the following, we call $(u, \phi_u)$ a positive solution of (1.6) if $u$ is a positive critical of the functional $J$. And $(u, \phi_u)$ is a least energy solution of (1.6) if the critical point $u$ of the functional $J$ verifies
\[
J(u) = \min_{v \in \mathcal{S}} J(v),
\]
where $\mathcal{S} := \{u \in H^1(\mathbb{R}^3) \setminus \{0\} : J'(u) = 0\}$.

Motivated by the well-known Brézis-Lieb lemma [15], we have the following important lemma to prove the convergence of Schrödinger-Poisson system (1.6) involving a critical nonlocal term.

Lemma 2.2. Let $r \geq 1$ and $\Omega$ be an open subset of $\mathbb{R}^N$. Suppose that $u_n \rightharpoonup u$ in $L^r(\Omega)$, and $u_n \to u$ a.e. in $\Omega$ as $n \to \infty$, then
\[
|u_n|^p - |u_n - u|^p - |u|^p \to 0 \text{ in } L^{\frac{r}{p}}(\Omega)
\]
as $n \to \infty$ for any $p \in [1, r]$.

Proof. The proof is standard, so we omit it and the reader can refer in [30, Lemma 2.2] for the detail proof. \qed
Lemma 2.3. If $u_n \to u$ in $H^1(\mathbb{R}^3)$, then going to a subsequence if necessary, we derive

$$|u_n|^5 - |u_n - u|^5 - |u|^5 \to 0 \text{ in } L^\infty(\mathbb{R}^3),$$

(2.7)

$$\phi_{u_n} - \phi_{u_n - u} - \phi_u \to 0 \text{ in } D^{1,2}(\mathbb{R}^3),$$

(2.8)

$$\int_{\mathbb{R}^3} K(x)\phi_{u_n}|u_n|^5 \, dx - \int_{\mathbb{R}^3} K(x)\phi_{u_n - u}|u_n - u|^5 \, dx - \int_{\mathbb{R}^3} K(x)\phi_u|u|^5 \, dx \to 0,$$

and

$$\int_{\mathbb{R}^3} K(x)\phi_{u_n}|u_n|^3 u_n \varphi \, dx - \int_{\mathbb{R}^3} K(x)\phi_u|u|^3 u \varphi \, dx \to 0$$

(2.9)

for any $\varphi \in C_0^\infty(\mathbb{R}^3)$.

Proof. Since $u_n \to u$ in $H^1(\mathbb{R}^3)$, then $u_n \to u$ in $L^6(\mathbb{R}^3)$. And $u_n \to u$ a.e. in $\mathbb{R}^3$ because $u_n \to u$ in $L_{loc}^s(\mathbb{R}^3)$ with $1 \leq s < 6$ in the sense of a subsequence. If we take $r = 6$ and $p = 5$ in Lemma 2.2, one has (2.7) immediately.

It follows from (2) of Lemma 2.1 and Hölder’s inequality that

$$|\left(\phi_{u_n} - \phi_{u_n - u} - \phi_u, w\right)_{D^{1,2}(\mathbb{R}^3)}| = \left|\int_{\mathbb{R}^3} \nabla(\phi_{u_n} - \phi_{u_n - u} - \phi_u) \nabla w \, dx\right|$$

$$= \left|\int_{\mathbb{R}^3} K(x)\left(|u_n|^5 - |u_n - u|^5 - |u|^5\right) w \, dx\right|$$

$$\leq |K|_{\infty} |w|_6 \left(|u_n|^5 - |u_n - u|^5 - |u|^5\right)_{\frac{5}{6}} \to 0,$$

which implies that

$$\sup_{w \in D^{1,2}(\mathbb{R}^3), \|w\|_{D^{1,2}(\mathbb{R}^3)} = 1} \left|\left(\phi_{u_n} - \phi_{u_n - u} - \phi_u, w\right)_{D^{1,2}(\mathbb{R}^3)}\right|$$

$$\leq |K|_{\infty} |u_n|^5 - |u_n - u|^5 - |u|^5 \to 0,$$

(2.7)

hence (2.8) holds.

Using (2.8), one has $\phi_{u_n} - \phi_{u_n - u} - \phi_u \to 0$ in $L^6(\mathbb{R}^3)$. Since $\{u_n\}$ is bounded in $L^6(\mathbb{R}^3)$, then by Hölder’s inequality,

$$|A_1| := \left|\int_{\mathbb{R}^3} K(x)(\phi_{u_n} - \phi_{u_n - u} - \phi_u)|u_n|^5 \, dx\right|$$

$$\leq |K|_{\infty} |u_n|^5|\phi_{u_n} - \phi_{u_n - u} - \phi_u|_{L^6} \to 0.$$

Similarly, one can deduce that

$$A_2 := \int_{\mathbb{R}^3} K(x)(\phi_{u_n} - \phi_{u_n - u} - \phi_u)|u|^5 \, dx \to 0.$$

In view of (2.5), $\{\phi_{u_n - u}\}$ is bounded in $L^6(\mathbb{R}^3)$, then using Hölder’s inequality again,

$$|A_3| := \left|\int_{\mathbb{R}^3} K(x)\phi_{u_n - u}(u_n - u)|u_n|^5 - |u|^5 \, dx\right|$$

$$\leq |K|_{\infty} |\phi_{u_n - u}|_6 \left(|u_n|^5 - |u_n - u|^5 - |u|^5\right)_{\frac{5}{6}} \to 0.$$

(2.7)
As $u_n \to u$ in $H^1(\mathbb{R}^3)$, then one has $\phi_{u_n} \to \phi_u$ in $D^{1,2}(\mathbb{R}^3)$ by (4) of Lemma 2.1 and thus $\phi_{u_n} \to \phi_u$ in $L^6(\mathbb{R}^3)$. Clearly, $K(x)|u|^5 \in L^\frac{6}{5}(\mathbb{R}^3)$, thus

$$A_4 := \int_{\mathbb{R}^3} K(x)(\phi_{u_n} - \phi_u)|u|^5 \, dx \to 0.$$  

By $u_n \to u$ in $H^1(\mathbb{R}^3)$, one has $|u_n|^5 \to |u|^5$ in $L^\frac{6}{5}(\mathbb{R}^3)$. Since $K(x)|u|^6 \in L^6(\mathbb{R}^3)$, then

$$A_5 := \int_{\mathbb{R}^3} K(x)\phi_u(|u_n|^5 - |u|^5) \, dx \to 0.$$

Consequently,

$$\int_{\mathbb{R}^3} K(x)\phi_{u_n}|u_n|^5 \, dx - \int_{\mathbb{R}^3} K(x)\phi_{u_n} - u_n|u_n|^5 \, dx - \int_{\mathbb{R}^3} K(x)\phi_u|u|^5 \, dx$$

$$= A_1 - A_2 + A_3 + A_4 + A_5 \to 0,$$

which shows that (2.9) is true.

Since $u_n \to u$ in $H^1(\mathbb{R}^3)$, then one can deduce again that $\phi_{u_n} \to \phi_u$ in $L^6(\mathbb{R}^3)$. By $K(x)|u|^3 \varphi \in L^\frac{6}{5}(\mathbb{R}^3)$ because $\varphi \in C^\infty_0(\mathbb{R}^3)$, one has

$$\int_{\mathbb{R}^3} K(x)\phi_{u_n}|u|^3 \varphi \, dx - \int_{\mathbb{R}^3} K(x)\phi_{u_n} - u_n|u|^3 \varphi \, dx \to 0.$$  

On the other hand, by means of Hölder’s inequality and $\{\phi_{u_n}\}$ is bounded in $L^6(\mathbb{R}^3)$,

$$\left| \int_{\mathbb{R}^3} K(x)\phi_{u_n}|u_n|^3 \varphi \, dx - \int_{\mathbb{R}^3} K(x)\phi_{u_n}|u|^3 \varphi \, dx \right|$$

$$\leq |K|_\infty \int_{\text{supp } \varphi} |\phi_{u_n}||\varphi||u_n|^3 u_n - |u|^3 u| \, dx$$

$$\leq |K|_\infty |\phi_{u_n}|_0 |\varphi|_\infty \left( \int_{\text{supp } \varphi} \|u_n|^3 u_n - |u|^3 u|_\frac{6}{5} \, dx \right)^\frac{5}{6} \to 0,$$

where we have used $u_n \to u$ in $L^s_{\text{loc}}(\mathbb{R}^3)$ with $1 \leq s < 6$ in the sense of a subsequence. As a consequence of the above two facts, one has

$$\int_{\mathbb{R}^3} K(x)\phi_{u_n}|u_n|^3 \varphi \, dx - \int_{\mathbb{R}^3} K(x)\phi_u|u|^3 \varphi \, dx \to 0.$$  

The proof is complete. \hfill \Box

**Lemma 2.4.** There exists $\lambda_0 = \lambda_0(\varrho, S, f, K) > 0$ such that the functional $J(u)$ satisfies the Mountain-pass geometry around $0 \in H^1(\mathbb{R}^3)$ for any $\lambda \in (0, \lambda_0)$, that is,

1. there exist $\alpha, \rho > 0$ such that $J(u) \geq \alpha > 0$ when $\|u\| = \rho$ and $\lambda \in (0, \lambda_0)$;
2. there exists $e \in H^1(\mathbb{R}^3)$ with $\|e\| > \rho$ such that $J(e) < 0$. 


Proof.  (i) It follows from (2.6) and Hölder’s inequality that
\[
J(u) = \frac{1}{2} ||u||^2 - \frac{1}{10} \int_{\mathbb{R}^3} K(x) \phi_u |u|^5 \, dx - \frac{\lambda}{q} \int_{\mathbb{R}^3} f(x) |u|^q \, dx
\]
\[
\geq \frac{1}{2} ||u||^2 - \frac{|K|_\infty^2}{10S^6} ||u||^{10} - \frac{\lambda}{q} ||f||_{\frac{2}{2-q}} ||u||^q
\]
\[
= ||u||^q \left( \frac{1}{2} ||u||^{2-q} - \frac{|K|_\infty^2}{10S^6} ||u||^{10-q} - \frac{\lambda}{q} ||f||_{\frac{2}{2-q}} \right)
\]
\[
\geq \left( \frac{5S^6(2-q)}{|K|_\infty^2} \right)^\frac{2}{2-q} \left( \frac{4q}{10 - q} \right) \left( \frac{5S^6(2-q)}{|K|_\infty^2(10-q)} \right)^\frac{2}{2-q} > 0,
\]
(2.11)

Therefore if we set
\[
\rho = \left( \frac{5S^6(2-q)}{|K|_\infty^2} \right)^\frac{2}{2-q} > 0 \quad \text{and} \quad \lambda_0 = \frac{4q}{10 - q} \left( \frac{5S^6(2-q)}{|K|_\infty^2(10-q)} \right)^\frac{2}{2-q} > 0,
\]
(2.11)

then there exists \( \alpha > 0 \) such that \( J(u) \geq \alpha > 0 \) when \( ||u|| = \rho > 0 \) for any \( \lambda \in (0, \lambda_0) \).

(ii) Choosing \( u_0 \in H^1(\mathbb{R}^3) \setminus \{0\} \), then since \( f(x) \) is nonnegative, one has
\[
J(tu_0) = \frac{t^2}{2} ||u_0||^2 - \frac{t^{10}}{10} \int_{\mathbb{R}^3} K(x) \phi_{u_0} |u_0|^5 \, dx - \frac{\lambda t^q}{q} \int_{\mathbb{R}^3} f(x) |u_0|^q \, dx
\]
\[
\leq \frac{t^2}{2} ||u_0||^2 - \frac{t^{10}}{10} \int_{\mathbb{R}^3} K(x) \phi_{u_0} |u_0|^5 \, dx = -\infty
\]
as \( t \to +\infty \). Hence letting \( e = t_0 u_0 \in H^1(\mathbb{R}^3) \setminus \{0\} \) with \( t_0 \) sufficiently large, one has \( ||e|| > \rho \) and \( J(e) < 0 \). \qed

By Lemma 2.4, a (PS) sequence of the functional \( \Phi(u) \) at the level
\[
c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)) > 0,
\]
(2.12)
can be constructed, where the set of paths is defined as
\[
\Gamma := \{ \gamma \in C([0,1],H^1(\mathbb{R}^3)) : \gamma(0) = 0, J(\gamma(1)) < 0 \}.
\]
(2.13)
In other words, there exists a sequence \( \{u_n\} \subset H^1(\mathbb{R}^3) \) such that
\[
J(u_n) \to c, \quad J'(u_n) \to 0 \quad \text{as} \quad n \to \infty.
\]
(2.14)

Remark 2.5. It is easy to see that
\[
c \leq \inf_{u \in H^1(\mathbb{R}^3) \setminus \{0\}} \max_{t \geq 0} J(tu).
\]
(2.15)
Indeed, for any \( u \in H^1(\mathbb{R}^3) \setminus \{0\} \), similar to Lemma 2.4 (ii) there exists a sufficiently large \( t_0 > 0 \) such that \( J(t_0 u) < 0 \). Let us choose \( \gamma_0(t) = t_0 u \), therefore \( \gamma_0 \in C([0,1],H^1(\mathbb{R}^3)) \) and moreover \( \gamma_0 \in \Gamma \), thus
\[
c \leq \max_{t \in [0,1]} J(\gamma_0(t)) = \max_{t \in [0,1]} J(t_0 u) = \max_{t \in [0,t_0]} J(tu) \leq \max_{t \in [0,\infty]} J(tu).
\]
Since \( u \in H^1(\mathbb{R}^3) \setminus \{0\} \) in arbitrary, then (2.15) holds.
Because of the appearance of the critical nonlocal term, we have to estimate the Mountain-pass value given by (2.12) carefully. To do it, we choose the extremal function

$$U_{\epsilon,x_0}(x) = \frac{(3\epsilon^2)^{\frac{2}{3}}}{(\epsilon^2 + |x-x_0|^2)^{\frac{2}{3}}}$$

to solve $-\Delta u = u^5$ in $\mathbb{R}^3$, where $x_0$ is given in condition (K). Let $\varphi \in C_0^\infty(\mathbb{R}^3)$ be a cut-off function verifying that $0 \leq \varphi(x) \leq 1$ for all $x \in \mathbb{R}^3$, $\text{supp} \varphi \subset B_2(x_0)$, and $\varphi(x) \equiv 1$ on $B_1(x_0)$. Set $v_{\epsilon,x_0} = \varphi U_{\epsilon,x_0}$, then thanks to the asymptotic estimates from [14], we have

$$|\nabla v_{\epsilon,x_0}|^2 = S^{\frac{2}{3}} + O(\epsilon), \quad |v_{\epsilon,x_0}|^2 = S^{\frac{2}{3}} + O(\epsilon) \quad (2.16)$$

and

$$|v_{\epsilon,x_0}|^2 = O(\epsilon). \quad (2.17)$$

**Lemma 2.6.** Assume $1 < q < 2$, then the Mountain-pass value given by (2.12)

$$c < \frac{2}{5} K^{\frac{q}{2}}_{\infty} S^{\frac{2}{3}} - C_0 \lambda x_0^{\frac{2}{3}}, \quad \text{where} \quad C_0 = \frac{2(2-q)}{5q} \left( \frac{10-q}{8} \right)^{\frac{2}{3}} > 0$$

for any $\lambda \in (0, \lambda_0)$ and $S$ is the best Sobolev constant given in (2.1).

**Proof.** Firstly, we claim that there exist $t_1, t_2 \in (0, +\infty)$ independent of $\epsilon, \lambda$ such that

$$0 < t_1 < t_\epsilon < t_2 < +\infty. \quad (2.18)$$

Indeed, by the fact that $\lim_{t \to +\infty} J(tv_{\epsilon,x_0}) = -\infty$ and (i) of Lemma 2.4, there exists $t_\epsilon > 0$ such that

$$\max_{t \geq 0} J(tv_{\epsilon,x_0}) = J(t_\epsilon v_{\epsilon,x_0}), \quad \frac{d}{dt} J(tv_{\epsilon,x_0}) = 0, \quad \frac{d^2}{dt^2} J(tv_{\epsilon,x_0}) < 0$$

which imply that

$$\|v_{\epsilon,x_0}\|^2 - \lambda t_\epsilon^{q-2} \int_{\mathbb{R}^3} f(x)|v_{\epsilon,x_0}|^q dx = t_\epsilon^8 \int_{\mathbb{R}^3} K(x)\phi_{v_{\epsilon,x_0}} |v_{\epsilon,x_0}|^5 dx \quad (2.19)$$

and

$$\|v_{\epsilon,x_0}\|^2 - 9t_\epsilon^8 \int_{\mathbb{R}^3} K(x)\phi_{v_{\epsilon,x_0}} |v_{\epsilon,x_0}|^5 dx - \lambda (q-1)t_\epsilon^{q-2} \int_{\mathbb{R}^3} f(x)|v_{\epsilon,x_0}|^q dx < 0. \quad (2.20)$$

It follows from (2.19) that $t_\epsilon$ is bounded from above. On the other hand, combing with (2.19) and (2.20), one has

$$-8 \int_{\mathbb{R}^3} K(x)\phi_{v_{\epsilon,x_0}} |v_{\epsilon,x_0}|^5 dx < \lambda (q-2)t_\epsilon^{q-10} \int_{\mathbb{R}^3} f(x)|v_{\epsilon,x_0}|^q dx$$

which yields that $t_\epsilon$ is bounded from below because $q \in (1, 2)$. Hence (2.18) is true.

Let us define

$$g(t) := \frac{t^2}{2} \int_{\mathbb{R}^3} |\nabla v_{\epsilon,x_0}|^2 dx - \frac{t^{10}}{10} \int_{\mathbb{R}^3} K(x)\phi_{v_{\epsilon,x_0}} |v_{\epsilon,x_0}|^5 dx$$

$$:= C_1 t^2 - C_2 t^{10},$$
where

\[ C_1 = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \nu_{\epsilon, x_0}|^2 \, dx, \quad C_2 = \frac{1}{10} \int_{\mathbb{R}^3} K(x) \phi_{\nu_{\epsilon, x_0}} |\nu_{\epsilon, x_0}|^5 \, dx. \]

By some elementary calculations, we have

\[ \max_{t \geq 0} g(t) = \frac{4(C_1)^{\frac{\beta}{2}}}{5(5C_2)^{\frac{\beta}{2}}} = \frac{2}{5} \left( \frac{\int_{\mathbb{R}^3} |\nabla \nu_{\epsilon, x_0}|^2 \, dx}{\int_{\mathbb{R}^3} K(x) \phi_{\nu_{\epsilon, x_0}} |\nu_{\epsilon, x_0}|^5 \, dx} \right)^{\frac{\beta}{2}}. \quad (2.21) \]

In order to further estimate the formula (2.21), we first get the following estimate:

\[
\begin{align*}
\int_{\mathbb{R}^3} [K(x_0) - K(x)] |\nu_{\epsilon, x_0}|^6 \, dx & \quad \overset{(K)}{\leq} \quad 3^+ C \int_{|x-x_0|<\delta} \frac{|x-x_0|^\beta \epsilon^3}{(\epsilon^2 + |x-x_0|^2)^{3}} \, dx + 3^+ 2|K|_\infty \int_{|x-x_0|\geq \delta} \frac{\epsilon^3}{(\epsilon^2 + |x-x_0|^2)^{3}} \, dx \\
& \leq \quad C \epsilon^3 \int_{0}^{\delta} \frac{r^{2+\beta}}{(\epsilon^2 + r^2)^{3}} \, dr + C \epsilon^3 \int_{\delta}^{+\infty} r^{-4} \, dr \\
& \leq \quad C \epsilon^3 + C \epsilon^3 \leq C \epsilon^\beta, \quad (2.22)
\end{align*}
\]

where we use the fact that \( \beta \in [1, 3) \) in the last two inequalities. Next the Poisson equation \( -\Delta \phi_{\nu_{\epsilon, x_0}} = K(x) |\nu_{\epsilon, x_0}|^6 \) and Cauchy’s inequality give

\[
\begin{align*}
\int_{\mathbb{R}^3} K(x) |\nu_{\epsilon, x_0}|^6 \, dx &= \int_{\mathbb{R}^3} \nabla \phi_{\nu_{\epsilon, x_0}} \nabla |\nu_{\epsilon, x_0}| \, dx \\
& \leq \quad \frac{1}{2|K|_\infty} \int_{\mathbb{R}^3} |\nabla \phi_{\nu_{\epsilon, x_0}}|^2 \, dx + \frac{|K|_\infty}{2} \int_{\mathbb{R}^3} |\nabla \nu_{\epsilon, x_0}|^2 \, dx \\
& = \quad \frac{1}{2|K|_\infty} \int_{\mathbb{R}^3} K(x) \phi_{\nu_{\epsilon, x_0}} |\nu_{\epsilon, x_0}|^5 \, dx + \frac{|K|_\infty}{2} \int_{\mathbb{R}^3} |\nabla \nu_{\epsilon, x_0}|^2 \, dx
\end{align*}
\]

which implies that

\[
\begin{align*}
\int_{\mathbb{R}^3} K(x) \phi_{\nu_{\epsilon, x_0}} |\nu_{\epsilon, x_0}|^5 \, dx & \geq \quad 2|K|_\infty \int_{\mathbb{R}^3} K(x) |\nu_{\epsilon, x_0}|^6 \, dx - |K|_\infty^2 \int_{\mathbb{R}^3} |\nabla \nu_{\epsilon, x_0}|^2 \, dx \\
& \overset{(2.22)}{\geq} \quad 2|K|_\infty^2 \int_{\mathbb{R}^3} |\nu_{\epsilon, x_0}|^6 \, dx - C \epsilon^\beta - |K|_\infty^2 \int_{\mathbb{R}^3} |\nabla \nu_{\epsilon, x_0}|^2 \, dx \\
& \overset{(2.16)}{=} \quad |K|_\infty^2 S^{\frac{\beta}{2}} + O(\epsilon). \quad (\beta \in [1, 3))
\end{align*}
\]

As a consequence of the above fact, one has

\[
\max_{t \geq 0} g(t) \overset{(2.21)}{\leq} \quad \frac{2}{5} \left( \frac{S^{\frac{\beta}{2}} + O(\epsilon)}{|K|_\infty^2 S^{\frac{\beta}{2}} + O(\epsilon)} \right)^{\frac{\beta}{2}} = \frac{2}{5} |K|_\infty^{-\frac{\beta}{2}} S^{\frac{\beta}{2}} + O(\epsilon). \quad (2.23)
\]
On the other hand, for $\epsilon > 0$ with $\epsilon < 1$ we have
\[ A_{\delta e} \int_{\mathbb{R}^3} f(x)|v_{\epsilon,x,0}|^q dx = A_{\delta e} \int_{B_2(x_0)} f(x)|v_{\epsilon,x,0}|^q dx \]
\[ \geq C_{\lambda} \int_{B_1(x_0)} f(x) \frac{e^{\frac{\epsilon}{2} x}}{(\epsilon^2 + |x-x_0|^2)^{\frac{q}{2}}} dx \]
\[ \geq C_{\lambda} (\frac{\epsilon}{2})^q \int_{B_1(x_0)} f(x) dx := C_3 \lambda e^{\frac{q}{2}}, \quad (2.24) \]
where $C_3 \in (0, +\infty)$ is a constant since $f(x) \in L^{\frac{q}{2-q}}(\mathbb{R}^3)$ and then $f(x) \in L^1_{\text{loc}}(\mathbb{R}^3)$.

We have proved $\max_{t \geq 0} J(t\nu_x) = J(t\nu_x)$ at the beginning, that is,
\[ \max_{t \geq 0} J(t\nu_x) = \frac{t^2}{2} \int_{\mathbb{R}^3} |\nabla v_{\epsilon,x,0}|^2 + |v_{\epsilon,x,0}|^2 dx - \frac{t^2}{10} \int_{\mathbb{R}^3} K(x)\phi_{\epsilon,x,0} |v_{\epsilon,x,0}|^5 dx \]
\[ - \frac{A_{\delta e}^q}{q} \int_{\mathbb{R}^3} f(x)|v_{\epsilon,x,0}|^q dx \]
\[ = g(t\epsilon) + \frac{t^2}{2} |v_{\epsilon,x,0}|^2 - \frac{A_{\delta e}^q}{q} \int_{\mathbb{R}^3} f(x)|v_{\epsilon,x,0}|^q dx \]
\[ \leq \frac{2}{5} |K|^{\frac{1}{2}} S^{\frac{3}{2}} + CO(\epsilon) - C_3 \lambda e^{\frac{q}{2}}, \quad (2.25) \]
where we have used (2.17), (2.18), (2.23) and (2.24) in the last inequality.

Since $1 < q < 2$, then there exists sufficiently small $\epsilon > 0$ such that
\[ CO(\epsilon) - C_3 \lambda e^{\frac{q}{2}} < -C_0 \lambda e^{\frac{q}{2}}, \]
which indicates that $c < \frac{2}{5} |K|^{\frac{1}{2}} S^{\frac{3}{2}} - C_0 \lambda e^{\frac{q}{2}}$ by (2.15) and (2.25). \hfill \Box

**Lemma 2.7.** (see [43, Theorem A.2]) Let $\Omega$ be an open subset of $\mathbb{R}^3$ and assume that $|\Omega| < +\infty$, $1 \leq p, r < +\infty$, $g \in C(\overline{\Omega} \times \mathbb{R})$ and
\[ |g(x, u)| \leq c(1 + |u|^r). \]

Then, for every $u \in L^p(\Omega)$, $g(\cdot, u) \in L^r(\Omega)$ and the operator $A : L^p(\Omega) \to L^r(\Omega)$ defined by
\[ Au = g(x, u) \]
is continuous.

**Lemma 2.8.** Assume $f \in L^{\frac{q}{2-q}}(\mathbb{R}^3)$ and $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, then going to a subsequence if necessary, one has
\[ \int_{\mathbb{R}^3} f(x)|u_n|^q dx \to \int_{\mathbb{R}^3} f(x)|u|^q dx \quad (2.26) \]
and
\[ \int_{\mathbb{R}^3} f(x)|u_n|^{q-2} u_n \varphi dx \to \int_{\mathbb{R}^3} f(x)|u|^{q-2} u \varphi dx \quad (2.27) \]
for any $\varphi \in C_0^\infty(\mathbb{R}^3)$. 

Proof: Since \( u_n \to u \) in \( H^1(\mathbb{R}^3) \), then \( u_n \to u \) in \( L^s_{\text{loc}}(\mathbb{R}^3) \) with \( 1 \leq s < 6 \) and \( u_n \to u \) a.e. in \( \mathbb{R}^3 \) in the sense of a subsequence. Since \( f \in L^{\frac{2}{s-1}}(\mathbb{R}^3) \), for any \( \epsilon > 0 \) there exists \( R = R(\epsilon) > 0 \) such that
\[
\int_{B_R} |f(x)|^{\frac{2}{s-1}} dx \leq \epsilon.
\]
As \( \{u_n\} \) is uniformly bounded in \( H^1(\mathbb{R}^3) \), \( \{u_n\} \) and \( u \) are uniformly bounded in \( L^2(\mathbb{R}^3) \). Therefore by using Hölder’s inequality and Minkowski’s inequality, one has
\[
\left| \int_{B_R} f(x)|u_n|^q dx - \int_{B_R} f(x)|u|^q dx \right| \leq \left( \int_{B_R} |f(x)|^{\frac{2}{s-1}} dx \right)^{\frac{2-q}{2}} \left( \int_{B_R} |u_n|^q - |u|^q \right)^{\frac{2}{q}}
\leq \left( \int_{B_R} |f(x)|^{\frac{2}{s-1}} dx \right)^{\frac{2-q}{2}} (|u_n|^q_2 + |u|^q_2) \leq C \epsilon.
\]
Let \( g(x, u) = |u|^q \), then \( p := 2 \) and \( r := \frac{2}{q} > 1 \) as in Lemma 2.7. Since \( u_n \to u \) in \( L^2(B_R) \), then \( g(x, u_n) \to g(x, u) \) in \( L^{\frac{2}{q}}(B_R) \) by Lemma 2.7. Thus
\[
\left| \int_{B_R} f(x)|u_n|^q dx - \int_{B_R} f(x)|u|^q dx \right| \leq \left| f \right|_{L^{\frac{2}{s-1}}(B_R)} \left( \int_{B_R} |u_n|^q - |u|^q \right)^{\frac{2}{q}}
\leq \left| f \right|_{L^{\frac{2}{s-1}}(B_R)} \left( \int_{B_R} g(x, u_n) - g(x, u) \right)^{\frac{2}{q}} \to 0
\]
which reveals (2.26) holds together the above fact. The proof of (2.27) is similar to that of (2.26), we omit the details. \( \square \)

3. The proof of Theorem 1.1

In this section, we will prove the Theorem 1.1 in detail.

3.1. Existence of a first positive solution for (1.6).

Proof: Let \( \lambda_0 > 0 \) be given as in (2.11), then for any \( \lambda \in (0, \lambda_0) \), by Lemma 2.4, there exists a sequence \( \{u_n\} \subset H^1(\mathbb{R}^3) \) verifying (2.14). We can show that the sequence \( \{u_n\} \) is bounded in \( H^1(\mathbb{R}^3) \). Indeed,
\[
c + 1 + o(1)||u_n|| \geq J(u_n) - \frac{1}{10} \langle J'(u_n), u_n \rangle
\]
\[
= \frac{2}{5}||u_n||^2 - \lambda \left( \frac{1}{q} - \frac{1}{10} \right) \int_{\mathbb{R}^3} f(x)|u_n|^q dx
\geq \frac{2}{5}||u_n||^2 - \frac{10 - q}{10q} \lambda |f|_{L^q} \frac{2}{q}||u_n||^q,
\]
hence \( \{u_n\} \) is bounded in \( H^1(\mathbb{R}^3) \) by the fact that \( 1 < q < 2 \). It is therefore that there exists \( u_1 \in H^1(\mathbb{R}^3) \) such that \( u_n \rightharpoonup u_1 \) in \( H^1(\mathbb{R}^3) \). To end the proof, we will split it into several steps:
Step 1: \( u_1 \neq 0 \).

In fact, we will argue it indirectly and just suppose that \( u_1 \equiv 0 \). Hence it follows from (2.14) and (2.26) that

\[
J(u_n) = \frac{1}{2} ||u_n||^2 - \frac{1}{10} \int_{\mathbb{R}^3} K(x)\phi_{u_n} |u_n|^5 dx = c + o(1)
\]

and

\[
\langle J'(u_n), u_n \rangle = ||u_n||^2 - \int_{\mathbb{R}^3} K(x)\phi_{u_n} |u_n|^5 dx = o(1).
\]

Thus without loss of generality, we may assume

\[
\lim_{n \to \infty} ||u_n||^2 = \lim_{n \to \infty} \int_{\mathbb{R}^3} K(x)\phi_{u_n} |u_n|^5 dx = l, \quad \text{and} \quad c = \frac{2}{5} l.
\]

On the other hand, by (2.6) we can deduce that

\[
\int_{\mathbb{R}^3} K(x)\phi_{u_n} |u_n|^5 dx \leq |K|_{\infty}^{\frac{2}{5}} S^{-5} ||u_n||^{10}
\]

which implies that \( l \leq |K|_{\infty}^{\frac{2}{5}} S^{-5} \frac{2}{5} \). Hence either \( l = 0 \) or \( l \geq |K|_{\infty}^{\frac{2}{5}} S^{-5} \). But \( l = 0 \) yields that \( c = 0 \) which is a contradiction to (2.12), hence \( l \geq |K|_{\infty}^{\frac{2}{5}} S^{-5} \). However

\[
c = \frac{2}{5} l \geq \frac{2}{5} |K|_{\infty}^{\frac{2}{5}} S^{-5}
\]

which also yields a contradiction to Lemma 2.6. Therefore \( u_1 \neq 0 \) holds.

Step 2: \( J'(u_1) = 0 \).

To see this, since \( C_0^\infty(\mathbb{R}^3) \) is dense in \( H^1(\mathbb{R}^3) \), then it suffices to show

\[
\langle J'(u_1), \varphi \rangle = 0 \quad \text{for any} \quad \varphi \in C_0^\infty(\mathbb{R}^3).
\]

Indeed, as a direct consequence of (2.10), (2.14), (2.27), one has

\[
\langle J'(u_1), \varphi \rangle = \lim_{n \to \infty} \langle J'(u_n), \varphi \rangle = 0
\]

for any \( \varphi \in C_0^\infty(\mathbb{R}^3) \).

Step 3: \( J(u_1) = c > 0 \) and \( u_1(x) > 0 \) in \( \mathbb{R}^3 \).

We first show that

\[
J(u_1) \geq -C_0 l^{\frac{2}{5-q}}, \quad \text{where} \quad C_0 = \frac{2(2-q)}{5q} \left( \frac{(10-q)|f|}{8} \right)^{\frac{2}{1-q}} > 0.
\]

Indeed, by means of \( J'(u_1) = 0 \) and Hölder’s inequality, we derive

\[
J(u_1) = J(u_1) - \frac{1}{10} \langle J'(u_1), u_1 \rangle
\]
follows from (2.9), and (3.1), one has
\[ o(1) = \langle J'(u_n), u_n \rangle = \langle J'(u_n), u_n \rangle - \langle J'(u_1), u_1 \rangle = \|v_n\|^2 - \int_{\mathbb{R}^3} K(x)\phi_{v_n}|v_n|^5dx + o(1) \] (3.2)
and
\[ c = J(u_n) - J(u_1) + J(u_1) + o(1) \]
\[ = \frac{1}{2}\|v_n\|^2 - \frac{1}{10} \int_{\mathbb{R}^3} K(x)\phi_{v_n}|v_n|^5dx + J(u_1) + o(1) \]
\[ \geq \frac{1}{2}\|v_n\|^2 - \frac{1}{10} \int_{\mathbb{R}^3} K(x)\phi_{v_n}|v_n|^5dx - C_0 A^{\frac{2}{3-q}} + o(1) \] (3.3)

Just suppose that \( v_n \not\to 0 \) in \( H^1(\mathbb{R}^3) \), and we may assume that \( \lim_{n \to \infty} \|v_n\|^2 = l_1 > 0 \). It follows from (3.2) and
\[ \int_{\mathbb{R}^3} K(x)\phi_{v_n}|v_n|^5dx \overset{(2.6)}{=} |K|^\frac{1}{2}S^\frac{5}{2} |v_n|^10 \]
that we can derive \( l_1 \geq |K|^\frac{1}{2}S^\frac{5}{2} \). Hence as a consequence of (3.2) and (3.3), one has
\[ c = \frac{1}{2}l_1 - \frac{1}{10}l_1 - C_0 A^{\frac{2}{3-q}} + o(1) \]
\[ \geq \frac{2}{5}|K|^\frac{1}{2}S^\frac{5}{2} - C_0 A^{\frac{2}{3-q}} + o(1) \]
which yields a contradiction to Lemma 2.6. Therefore \( \|v_n\| \to 0 \) in \( H^1(\mathbb{R}^3) \) holds, or equivalently, \( u_n \to u_1 \) in \( H^1(\mathbb{R}^3) \) as \( n \to \infty \). Then \( J(u_1) = \lim_{n \to \infty} J(u_n) = c > 0 \).

On the other hand, it is obvious that \( |u_1| \) is also a nontrivial solution of problem (1.6) since the functional \( J \) is symmetric and invariant, hence we may assume that such a critical point does not change sign, i.e. \( u_1 \geq 0 \). By means of the strong maximum principle and standard arguments, see e.g. \([2, 11, 32, 37, 42]\), we obtain that \( u_1(x) > 0 \) for all \( x \in \mathbb{R}^3 \). Thus, \( (u_1, \phi_{u_1}) \) is a positive solution for the system (1.6) and the proof is complete. \( \square \)

3.2. **Existence of a second positive solution for (1.6).**

Before we obtain the second positive solution, we introduce the following well-known proposition:
Proposition 3.1. (Ekeland’s variational principle [21], Theorem 1.1) Let \( V \) be a complete metric space and \( F : V \to \mathbb{R} \cup \{+\infty\} \) be lower semicontinuous, bounded from below. Then for any \( \epsilon > 0 \), there exists some point \( v \in V \) with
\[
F(v) \leq \inf_v F + \epsilon, \quad F(w) \geq F(v) - \epsilon d(v, w) \quad \text{for all } w \in V.
\]

We are in a position to show the existence of a second positive solution for (1.6):

Proof. The main idea of this proof comes from [40], we will show it for reader’s convenience. For \( \rho > 0 \) given by Lemma 2.4(i), define
\[
\overline{B}_\rho = \{ u \in H^1(\mathbb{R}^3), \|u\| \leq \rho \}, \quad \partial B_\rho = \{ u \in H^1(\mathbb{R}^3), \|u\| = \rho \}
\]
and clearly \( \overline{B}_\rho \) is a complete metric space with the distance \( d(u, v) = \|u - v\|, \forall u, v \in \overline{B}_\rho \).

Lemma 2.4 tells us that \( J|_{\partial B_\rho} \geq \alpha > 0 \). (3.4)

It’s obvious that the functional \( J \) is lower semicontinuous and bounded from below on \( \overline{B}_\rho \). We claim that
\[
\bar{c} := \inf_{u \in \overline{B}_\rho} J(u) < 0.
\]
Indeed, we chose a nonnegative function \( \psi \in C_c^\infty(\mathbb{R}^3) \), and clearly \( \psi \in H^1(\mathbb{R}^3) \). Since \( 1 < q < 2 \), we have
\[
\lim_{t \to 0} \frac{J(t\psi)}{t^q} = \lim_{t \to 0} \frac{\frac{2}{q} \|\psi\|^2 - \frac{m_0}{10} \int_{\mathbb{R}^3} K(x) \phi \psi^5 dx - \frac{m_0}{q} \int_{\mathbb{R}^3} f(x) |\psi|^q dx}{t^q} = -\frac{\lambda}{q} \int_{\mathbb{R}^3} f(x) |\psi|^q dx < 0.
\]

Therefore there exists a sufficiently small \( t_0 > 0 \) such that \( \|t_0\psi\| \leq \rho \) and \( J(t_0\psi) < 0 \), which imply that (3.5) holds.

By Proposition 3.1, for any \( n \in N \) there exists \( \overline{u}_n \) such that
\[
\bar{c} \leq J(\overline{u}_n) \leq \bar{c} + \frac{1}{n}, \quad (3.6)
\]
and
\[
J(v) \geq J(\overline{u}_n) - \frac{1}{n} \|\overline{u}_n - v\|, \quad \forall v \in \overline{B}_\rho. \quad (3.7)
\]

Firstly, we claim that \( \|\overline{u}_n\| < \rho \) for \( n \in N \) sufficiently large. In fact, we will argue it by contradiction and just suppose that \( \|\overline{u}_n\| = \rho \) for infinitely many \( n \), without loss of generality, we may assume that \( \|\overline{u}_n\| = \rho \) for any \( n \in N \). It follows from (3.4) that
\[
J(\overline{u}_n) \geq \alpha > 0,
\]
then combing it with (3.6), we have \( c_1 \geq \alpha > 0 \) which is a contradiction to (3.5).
Next, we will show that $J'(\overline{u}_n) \to 0$ in $(H^1(\mathbb{R}^3))^\ast$. Indeed, set

$$v_n = \overline{u}_n + tu, \quad \forall u \in B_1 = \{u \in H^1(\mathbb{R}^3), ||u|| = 1\},$$

where $t > 0$ small enough such that $2t + t^2 \leq \rho^2 - ||\overline{u}_n||^2$ for fixed $n$ large, then

$$||v_n||^2 = ||\overline{u}_n||^2 + 2t(\overline{u}_n, u) + t^2 \leq ||\overline{u}_n||^2 + 2t + t^2 \leq \rho^2$$

which imply that $v_n \in \overline{B}_\rho$. So it follows from (3.7) that

$$J(v_n) \geq J(\overline{u}_n) - \frac{t}{n}||\overline{u}_n - v_n||,$$

that is,

$$\frac{J(\overline{u}_n + tu) - J(\overline{u}_n)}{t} \geq -\frac{1}{n}.$$ 

Letting $t \to 0$, then we have $\langle J'(\overline{u}_n), u \rangle \geq -\frac{1}{n}$ for any fixed $n$ large. Similarly, chose $t < 0$ and $|t|$ small enough, repeating the process above we have $\langle J'(\overline{u}_n), u \rangle \leq -\frac{1}{n}$ for any fixed $n$ large. Therefore the conclusion

$$\langle J'(\overline{u}_n), u \rangle \to 0 \text{ as } n \to \infty, \quad \forall u \in B_1$$

implies that $J'(\overline{u}_n) \to 0$ in $(H^1(\mathbb{R}^3))^\ast$.

Finally, we know that $[\overline{u}_n]$ is a $(PS)_{c_1}$ sequence for the functional $J$ with $c_1 < 0$. Since $||\overline{u}_n|| < \rho$, there exists $u_2 \in H^1(\mathbb{R}^3)$ such that $\overline{u}_n \to u_2$ in $H^1(\mathbb{R}^3)$. Hence as the Step 1, Step 2 and Step 3 in Section 3.1, $J'(u_2) = 0$ and $u_2 > 0$. In other words, $(u_2, \phi_{u_2})$ is a positive solution for (1.6). \hfill \Box

### 3.3. Existence of a positive least energy solution for (1.6).

To establish a positive least energy solution for problem (1.6), we define

$$m := \inf_{u \in S} J(u),$$

where $S := \{u \in H^1(\mathbb{R}^3) \setminus \{0\} : J'(u) = 0\}$. Firstly we have the following claims:

**Claim 1:** $J(u_2) = \overline{c} < 0$, where $u_2$ is obtained in Section 3.2.

**Proof of the Claim 1:** On one hand, it follows from Fatou’s lemma that $||u_2|| \leq \liminf_{n \to \infty} ||\overline{u}_n|| \leq \rho$ and then $J(u_2) \geq \overline{c}$ by (3.5).

On the other hand, since $J'(u_2) = 0$, then using Fatou’s lemma and (2.26) one has

$$\overline{c}_2 + o(1) = J(\overline{u}_n) - \frac{1}{10} \langle J'(\overline{u}_n), \overline{u}_n \rangle = \frac{2}{5} ||\overline{u}_n||^2 - \lambda \left( \frac{1}{q} - \frac{1}{10} \right) \int_{\mathbb{R}^3} f(x)|\overline{u}_n|^q dx$$
It's obvious that the solutions $S_1, S_2 \in S$ obtained in Section 3.1 and Section 3.2, hence $S \neq \emptyset$ and $m \leq \min\{J(u_1), J(u_2)\} \leq J(u_2) < 0$ by Claim 1.

On the other hand, $\forall u \in S$, one has

$$c = J(u) - \frac{1}{10} \langle J'(u), u \rangle$$

$$= \frac{2}{5} ||u||^2 - \lambda\left(\frac{1}{q} - \frac{1}{10}\right) \int_{\mathbb{R}^3} f(x)|u|^q dx$$

$$\geq \frac{2}{5} ||u||^2 - \lambda\left(\frac{1}{q} - \frac{1}{10}\right) \int_{\mathbb{R}^3} f(x)|u|^q dx + o(1)$$

Thus $\tilde{c} \geq J(u_2)$ and then $J(u_2) = \tilde{c} < 0$ by (3.5).

**Claim 2:** $S \neq \emptyset$ and $m \in (-\infty, 0)$.

**Proof of the Claim 2:** It's obvious that the solutions $u_1, u_2 \in S$ obtained in Section 3.1 and Section 3.2, hence $S \neq \emptyset$ and $m \leq \min\{J(u_1), J(u_2)\} \leq J(u_2) < 0$ by Claim 1.

Now let us prove the existence of a least energy solution for (1.6):

**Proof:** By means of Claim 1, we can choose a minimizing sequence of $m$, that is, a sequence $\{w_n\} \subset S$ satisfying

$$J(w_n) \to m \text{ as } n \to \infty \text{ and } J'(w_n) = 0.$$ 

Thus $\{w_n\}$ is a $(PS)_m$ sequence of the functional $J$ with $-\infty < m < 0$. It is clear that $\{w_n\}$ is bounded in $H^1(\mathbb{R}^3)$ and there exists $w \in H^1(\mathbb{R}^3)$ such that $w_n \rightharpoonup w$ in $H^1(\mathbb{R}^3)$. It is totally similar to Steps 1-3 in Section 3.1 that $J'(w) = 0$ and $w > 0$. Hence $w \in S$ and then $J(w) \geq m$. Now, we prove that $m \geq J(w)$.

In fact, since $J'(w_n) = J'(w) = 0$, then using Fatou's lemma and (2.26) one has

$$m + o(1) = J(w_n) - \frac{1}{10} \langle J'(w_n), w_n \rangle$$

$$= \frac{2}{5} ||w_n||^2 - \lambda\left(\frac{1}{q} - \frac{1}{10}\right) \int_{\mathbb{R}^3} f(x)|w_n|^q dx$$

$$= \frac{2}{5} ||w_n||^2 - \lambda\left(\frac{1}{q} - \frac{1}{10}\right) \int_{\mathbb{R}^3} f(x)|w|^q dx + o(1)$$

$$\geq \frac{2}{5} ||w||^2 - \lambda\left(\frac{1}{q} - \frac{1}{10}\right) \int_{\mathbb{R}^3} f(x)|w|^q dx + o(1)$$

$$= J(u_2) - \frac{1}{10} \langle J'(u_2), u_2 \rangle + o(1) = J(u_2) + o(1).$$
\[ J(w) - \frac{1}{10} \langle J'(w), w \rangle + o(1) = J(w) + o(1). \]

It is therefore that \( J'(w) = 0 \) with \( J(w) = m \), and \( w > 0 \). Consequently, \( (w, \phi_w) \) is a positive least energy solution for (1.6).

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