POLYNOMIAL AND RATIONAL WAVE SOLUTIONS OF KUDRYASHOV-SINELSHCHIKOV EQUATION AND NUMERICAL SIMULATIONS FOR ITS DYNAMIC MOTIONS

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Abstract   Polynomial and rational wave solutions of Kudryashov-Sinelshchikov equation and numerical simulations for its dynamic motions are investigated. Conservation flows of the dynamic motion are obtained utilizing multiplier approach. Using the unified method, a collection of exact solitary and soliton solutions of Kudryashov-Sinelshchikov equation is presented. Collocation finite element method based on quintic B-spline functions is implemented to the equation to evidence the accuracy of the proposed method by test problems. Stability analysis of the numerical scheme is studied by employing von Neumann theory. The obtained analytical and numerical results are in good agreement.

Keywords   Kudryashov-Sinelshchikov equation, unified method, finite element method, collocation, solitary waves.

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1. Introduction

Recently, the concept of soliton has become a widely-studied topic because of its applicability in the modeling of many natural phenomena arising in different fields such as optics, physics, dynamics, fluid and biology [1, 2, 5, 7, 13, 23]. The theory of soliton has been put forward in the study of nonlinear phenomena, which makes the study of the nonlinear evolution equations (NLEEs) become an essential tool in nonlinear science.

Various analytical and numerical techniques are used to investigate the NLEEs such as the inverse scattering method [3, 4, 29], the generalized unified method [16, 17], Painlevé analysis method [24], variable separation method [30], Hirota bilinear method [26], Bäcklund transformation [27], Runge-Kutta fourth-order and
Fourier spectral scheme [6], finite difference method [11], and the variational iteration method [25].

In this paper, we investigate and analyze the solutions of the Kudryashov-Sinelshchikov equation [21, 28] analytically and numerically by using the unified method (UM) [18, 19] and collocation method, respectively.

The Kudryashov-Sinelshchikov equation, denoted by KSE, is given by

$$u_t + \alpha uu_x + \beta u_{xxx} + \gamma (u u_{xx})_x + \sigma u_x u_{xx} = 0,$$  \hspace{1cm} (1.1)

where \( \alpha, \beta, \gamma \) and \( \sigma \) are real parameters. The KSE can be used to describe the nonlinear waves in gas-liquid mixtures in the absence of the viscosity [21]. The KSE is the generalization of the KdV when \( \gamma = \sigma = 0 \). For more details see [10].

Our work aims firstly to search for the solitary, soliton, and soliton rational wave solutions for the KSE by making use of the UM. Secondly, the conservation law is applied to study the invariance and the multiplier approach for the KSE. Finally, numerical simulation of the obtained results are performed on the behavior of the obtained solutions to the KSE using the quintic B-splines and the modulation instability is discussed and introduced in some figures to clarify the physical meaning for explicit and approximate solutions.

This paper is organized as follows. Section 2 constructs different types of wave solution for the KSE using the UM. Section 3 recalls some necessary preliminaries of the conservation flows for the KSE. Section 4 presents collocation finite element method for the KSE by using quintic B-splines and studies stability of the scheme. To demonstrate accuracy of the scheme, some test problems are studied and the obtained results are discussed in Section 5. Finally, conclusions are given in Section 6.

2. Exact solutions

In this section, we use the UM [18, 19] to derive exact wave solutions for Eq. (1.1). The UM classifies the solutions into two categories namely polynomial and rational wave solutions (for more details see [18, 19]).

First, we use the transformation

$$u(x, t) = u(\xi), \quad \xi = \kappa x + \nu t,$$  \hspace{1cm} (2.1)

in Eq. (1.1) and integrating both sides with respect to \( \xi \), we get

$$\kappa^3(\beta + \gamma u) u'' + \frac{1}{2} \sigma \kappa^3 u'^2 + \frac{1}{2} \alpha \kappa u^2 + \nu u = 0, \quad u' = \frac{du}{d\xi},$$  \hspace{1cm} (2.2)

where the constant of integration is set to be equal zero and \( \kappa \) and \( \nu \) are arbitrary constants.

2.1. Polynomial wave solutions

The UM asserts that the polynomial wave solutions are given by

$$u(\xi) = \sum_{j=0}^{n} p_j \Gamma^j(\xi),$$  \hspace{1cm} (2.3)
and the auxiliary function $\Gamma(\xi)$ satisfies the auxiliary equation

$$(\Gamma'(\xi))^p = \sum_{j=0}^{k} b_j \Gamma^j(\xi), \quad p = 1, 2,$$  \hspace{1cm} (2.4)$$

where $p_j$ and $b_j$ are real constants.

By considering the homogeneous balance between $u''$ and $u^2$ in Eq. (2.2), we get $n = 2(k - 1), k \geq 1$. Here, we confine ourselves to find these solutions when $k = 2$ and $p = 1$ or $p = 2$ to obtain solitary and soliton wave solutions respectively.

2.1.1. Solitary wave solutions \textbf{(k = 2 and p = 1)}

Here, we find the solitary wave solutions of Eq. (2.2). From Eqs. (2.3) and (2.4) we have

$$u(\xi) = p_0 + p_1 \Gamma(\xi) + p_2 \Gamma^2(\xi),$$  \hspace{1cm} (2.5)$$

$$\Gamma'(\xi) = b_0 + b_1 \Gamma(\xi) + b_2 \Gamma^2(\xi).$$  \hspace{1cm} (2.6)$$

Substituting Eqs. (2.5) and (2.6) into Eq. (2.2) and equating the coefficients of $\Gamma(\xi)$ to zero (by utilizing the Mathematical software), solutions for the parameters $p_i$ and $b_i, i = 0, 1, 2$ are presented as follows

$$p_0 = -\frac{12b_2b_0\kappa^2\beta}{\alpha - \mu^2\kappa^2\gamma}, \quad p_1 = -\frac{12b_2b_1\kappa^2\beta}{\alpha - \mu^2\kappa^2\gamma}, \quad p_2 = -\frac{12b_2^2\kappa^2\beta}{\alpha - \mu^2\kappa^2\gamma},$$  \hspace{1cm} (2.7)$$

$$\sigma = -3\gamma, \quad \nu = -\mu^2\kappa^2\beta, \quad \mu = \sqrt{b_1^2 - 4b_2}. $$

Using Eq. (2.7) in Eq. (2.5) and by solving the auxiliary equation given by Eq. (2.6), we get the solution of Eq. (1.1) namely

$$u(x, t) = A \text{sech}^2 \left[ B (x - ct) \right],$$  \hspace{1cm} (2.8)$$

where $A = \frac{3\kappa^2\mu^2\beta}{\alpha - \kappa^2\mu^2\gamma}, B = \frac{\kappa\mu}{2}, c = \kappa^2\mu^2\beta$ and $\alpha, \beta, \gamma, \kappa$ and $\mu > 0$ are arbitrary constants. Solitary wave solution of Eq. (1.1) is shown in Figure 1.
2.1.2. Soliton wave solutions \( (k = 2 \text{ and } p = 2) \)

To obtain the soliton wave solutions of Eq. (2.2), we put \( k = 2 \) and \( p = 2 \) in Eqs. (2.3) and (2.4) namely

\[
\begin{align*}
    u(\xi) &= p_0 + p_1 \Gamma(\xi) + p_2 \Gamma(\xi), \\
    \Gamma'(\xi) &= \Gamma(\xi) \sqrt{b_0 + b_1 \Gamma(\xi) + b_2 \Gamma^2(\xi)}.
\end{align*}
\]

Using Eqs. (2.9) and (2.10) in Eq. (2.2) and by solving the obtained set of algebraic equations, we get

\[
p_0 = p_1 = 0, \quad p_2 = -\frac{12b_2 \kappa^2 \beta}{\alpha - 4b_0 \kappa^2 \gamma}, \quad \nu = -4b_0 \kappa^3 \beta, \quad \sigma = -3 \gamma, \quad b_1 = 0.
\]

Using Eq. (2.11) into Eq. (2.9) and by solving the auxiliary equation given by (2.10), we get the solution of Eq. (1.1) namely

\[
u(x, t) = -\frac{48b_2 b_0^2 \exp \left[2\sqrt{b_0 \kappa(x - 4b_0 \kappa^2 \beta t)}\right] \kappa^2 \beta}{(\alpha - 4b_0 \kappa^2 \gamma) \left[-1 + b_0 b_2 \exp(2\sqrt{b_0 \kappa(x - 4b_0 \kappa^2 \beta t)})\right]^2},
\]

where \( \alpha, \beta, \gamma, \kappa, b_2 \) and \( b_0 > 0 \) are arbitrary constants. Soliton wave solution of Eq. (1.1) is illustrated in Figure 2.

![Figure 2](image)

Figure 2. Soliton wave solution of Eq. (1.1) for \( \alpha = -0.3, \beta = -0.1, \gamma = -0.7, \kappa = 0.5, b_0 = 0.5, \) and \( b_2 = -0.1. \)

2.2. Rational wave solutions

To find the rational wave solutions of Eq. (1.1), we assume the solution of Eq. (2.2) as

\[
u(\xi) = \sum_{i=0}^{n} p_i \Gamma^i(\xi) / \sum_{j=0}^{m} q_j \Gamma^j(\xi), \quad n \geq l,
\]

\[
(\Gamma'(\xi))^p = \sum_{i=0}^{k} b_i \Gamma^i(\xi), \quad p = 1, 2
\]

where \( p_i, q_i, \) and \( b_i \) are arbitrary constants. By considering the homogeneous balance between \( U^n \) and \( U^2 \) in Eq. (2.2), we get \( n - m = 2(k - 1) \) and \( k = 1, 2, \ldots. \)

Here, we find those solutions when \( k = 1, n = m, \) and \( p = 1 \) (soliton rational solutions).
2.2.1. Soliton rational solution \((n = m = 1 \text{ and } p = 1)\)

From Eqs. (2.13) and (2.14), we have

\[
\begin{align*}
  u(\xi) &= \frac{p_0 + p_1 \Gamma(\xi)}{q_0 + q_1 \Gamma(\xi)}, \\
  \Gamma'(\xi) &= b_0 + b_1 \Gamma(\xi).
\end{align*}
\]

(2.15)

Substituting Eq. (2.15) into Eq. (2.2) and by solving the obtained set of algebraic equations, we get

\[
\begin{align*}
  p_0 &= -\frac{2b_0 q_1 \beta}{b_1 \gamma}, \quad p_1 = -\frac{2q_1 \beta}{\gamma}, \\
  \alpha &= -b_1^2 \kappa^2 \gamma, \quad \nu = -b_0^3 \kappa^3 \beta, \quad \sigma = -4\gamma.
\end{align*}
\]

(2.16)

Using Eq. (2.16) into Eq. (2.15), we get the solution of Eq. (1.1) namely

\[
u(x,t) = -\frac{2b_1 q_1 \beta \exp(b_1 \kappa x)}{b_1 q_1 \gamma \exp(b_1 \kappa x) - (b_0 q_1 - b_1 q_0) \gamma \exp(b_1^3 \kappa^3 \beta t)},
\]

(2.17)

where \(\alpha, \beta, \gamma, \kappa, q_j\) and \(b_j, j = 0, 1\) are arbitrary constants. Soliton rational solution of Eq. (1.1) is drawn in Figure 3.

![Figure 3](attachment:image.png)

**Figure 3.** Soliton rational solution of Eq. (1.1) for \(\alpha = -0.075, \beta = -0.1, \gamma = 0.3, \kappa = 1, b_0 = -1, b_1 = 0.5, q_0 = 0.4\) and \(q_1 = -1\).

3. Conservation laws

We present some preliminaries on conservation laws that will be used in the analysis that follow.

The invariance and multiplier approach are resorted based on the well known result that the Euler-Lagrange operator annihilates a total divergence to determine conserved densities and fluxes [8]. Firstly, if \((T^x, T^x)\) is a conserved vector corresponding to a conservation law, then

\[
D_t T^x + D_x T^x = 0,
\]

(3.1)
along the solutions of the differential equation $E(x, t, u, u_x, u_t, \ldots) = 0$.

Moreover, if there exists a non-trivial differential function $Q$, called a “multiplier”, such that
\[ E_u E = 0, \]  
then $QE$ is a total divergence,
\[ QE = D_t T^t + D_x T^x, \]  
for some (conserved) vector $(T^t, T^x)$, where $E_u$ is the respective Euler-Lagrange operator. Thus, a knowledge of each multiplier $Q$ leads to a conserved vector determined by, inter alia, a homotopy operator [9].

For a system $E_1(x, t, u, v, u_x, v_x, \ldots) = 0, E_2(x, t, u, v, u_x, v_x, \ldots) = 0$, $Q = (q^1, q^2)$, say, so that
\[ q^1 E_1 + q^2 E_2 = D_t T^t + D_x T^x, \]  
and
\[ E_{(u,v)}[D_t T^t + D_x T^x] = 0. \]  
Also, the above is trivially extendable to the multi space situation so that, in the $(1 + 2)$ scenario with independent variables $(t, x, y)$, the conserved flow has the form $(T^t, T^x, T^y)$ and the total divergence is $D_t T^t + D_x T^x + D_y T^y$ (equal to $QE(x, t, u, u_x, u_y, u_t, \ldots)$).

Note. In this case, using the the language of forms, the conserved form (as opposed to the conserved flow) would be $T^t x - T^y x + T^x$. In each case, $T^t$ is the conserved density.

It is well known that the vector fields that leave the system of differential equations invariant (generators of Lie point symmetries) contain the algebra of variational symmetries, if the latter exists [12, 14, 15, 22].

The multiplier approach, for $\sigma, \gamma \neq 0$, leads to multipliers
\[ Q_1 = 1, \quad Q_2 = \left( u + \frac{\beta}{\gamma} \right)^{\sigma\gamma}. \]  
The corresponding conserved flow with $Q_1$ has components
\[ T^t = u, \]  
\[ T^x = \gamma u_{xx} u + \frac{1}{2} \sigma u_x^2 + \frac{1}{2} \alpha u^2 + \beta u_{xx}. \]  
The conserved density with $Q_2$ is
\[ T^t = \frac{1}{\gamma + \sigma} \left[ (\gamma u + \beta)^{\sigma+1} - \beta^{\sigma+1} \gamma^{-\sigma} \right]. \]  
For $\sigma = 0$, the multiplier is $Q_3 = \ln (\gamma u + \beta)$ with conserved density
\[ T^t = \frac{1}{\gamma} \left[ (\gamma u + \beta) \ln (\gamma u + \beta) - \beta \ln \beta - \gamma u \ln \gamma - \gamma u \right]. \]
4. Numerical simulation

In this section, firstly, quintic B-spline interpolation function and its some properties are defined. The collocation finite element method is implemented to Kudryashov-Sinelshchikov equation. Quintic B-spline functions are chosen as interpolation functions. Then, stability of the applied method is analyzed using von Neumann theory.

4.1. Quintic B-splines and properties

Here, Kudryashov-Sinelshchikov equation (1.1) is studied with the physical boundary conditions \( u \to 0 \) as \( x \to \pm \infty \), where \( \alpha, \beta, \gamma \) and \( \sigma \) are constants and the subscripts \( t \) and \( x \) denote the temporal and spatial differentiations, respectively.

Firstly, the solution domain is limited over an interval \( a \leq x \leq b \) to apply the numerical method. Space interval \( [a, b] \) is separated into uniformly sized finite elements of length \( h = \frac{b-a}{N} = (x_{m+1} - x_m) \) by the knots \( x_m \) like that \( a = x_0 < x_1 < \ldots < x_N = b \) for \( m = 1, 2, \ldots, N \).

To solve Eq. (1.1) homogeneous boundary conditions

\[
\begin{align*}
  u_N(a, t) &= 0, & u_N(b, t) &= 0, \\
  (u_N)_x(a, t) &= 0, & (u_N)_x(b, t) &= 0, & t > 0
\end{align*}
\]

and

\[
  u(x, 0) = f(x), \quad a \leq x \leq b.
\]

The initial condition are chosen as above.

\( \phi_m(x) \) quintic B-spline functions at the knots \( x_m \) are defined over the interval \( [a, b] \) by the following relationships for \( m = -2(1)N + 2 \):

\[
\phi_m(x) = \frac{1}{15} \begin{cases} 
  (x - x_{m-3})^5, & [x_{m-3}, x_{m-2}] \\
  (x - x_{m-3})^5 - 6 (x - x_{m-2})^5, & [x_{m-2}, x_{m-1}] \\
  (x - x_{m-3})^5 - 6 (x - x_{m-2})^5 + 15 (x - x_{m-1})^5, & [x_{m-1}, x_m] \\
  (x - x_{m-3})^5 - 6 (x - x_{m-2})^5 + 15 (x - x_{m-1})^5 - 20 (x - x_m)^5, & [x_m, x_{m+1}] \\
  + 15 (x - x_{m+1})^5, & [x_{m+1}, x_{m+2}] \\
  (x - x_{m-3})^5 - 6 (x - x_{m-2})^5 + 15 (x - x_{m-1})^5 - 20 (x - x_m)^5 & [x_{m+2}, x_{m+3}] \\
  + 15 (x - x_{m+1})^5 - 6 (x - x_{m+2})^5, & \text{elsewhere} \\
  0.
\end{cases}
\]

The values of \( \phi_m(x) \) and its derivative are tabulated as in Table 1. The splines \( \phi_m(x) \) and its four principle derivatives vanish outside the interval \( [x_{m-3}, x_{m+3}] \).

For functions defined over \( [a, b] \), the set of functions \( \{ \phi_{-2}(x), \phi_{-1}(x), \phi_0(x), \ldots, \phi_{N+1}(x), \phi_{N+2}(x) \} \) forms a basis. The approximate solution \( u_N(x, t) \) is given by

\[
u N(x, t) = \sum_{j=-2}^{N+2} \phi_j(x) \delta_j(t), \quad (4.4)\]
where $\delta_j(t)$ is time dependent parameters to be determined from the boundary and collocation conditions.

Using trial function (4.4) and quintic B-splines (4.3), the values of $u$, $u'$, $u''$, $u'''$ and $u^{iv}$ at the knots are determined in terms of the element parameters $\delta_m$ by

$$u_m = u(x_m) = \delta_{m-2} + 26\delta_{m-1} + 66\delta_m + 26\delta_{m+1} + \delta_{m+2},$$
$$u'_m = u'(x_m) = \frac{\delta}{h}(-\delta_{m-2} - 10\delta_{m-1} + 10\delta_{m+1} + \delta_{m+2}),$$
$$u''_m = u''(x_m) = \frac{30}{h^2}(\delta_{m-2} + 2\delta_{m-1} - 6\delta_m + 2\delta_{m+1} + \delta_{m+2}),$$
$$u'''_m = u'''(x_m) = \frac{60}{h^3}(-\delta_{m-2} + 2\delta_{m-1} - 2\delta_{m+1} + \delta_{m+2}),$$
$$u^{iv}_m = u^{iv}(x_m) = \frac{120}{h^4}(\delta_{m-2} - 4\delta_{m-1} + 6\delta_m - 4\delta_{m+1} + \delta_{m+2}),$$

where the symbols $'$, $''$, $'''$ and $^{iv}$ denote first, second, third and fourth differentiation with respect to $x$, respectively.

### 4.2. Implementation of the method

Substituting Eq. (4.5) into Eq. (1.1), general form of the solution approach is obtained as below.

$$\left(\delta_{m-2} + 26\delta_{m-1} + 66\delta_m + 26\delta_{m+1} + \delta_{m+2}\right) + \frac{5m\omega_m}{h}(-\delta_{m-2} - 10\delta_{m-1} + 10\delta_{m+1} + \delta_{m+2})$$
$$+ \frac{60\beta}{h^2}(-\delta_{m-2} + 2\delta_{m-1} - 2\delta_{m+1} + \delta_{m+2}) + \frac{100\gamma_0\omega_m}{h^3}(-\delta_{m-2} + 2\delta_{m-1} - 2\delta_{m+1} + \delta_{m+2})$$
$$+ \frac{100(\gamma + \sigma)\omega_m}{h^4}(-\delta_{m-2} - 10\delta_{m-1} + 10\delta_{m+1} + \delta_{m+2}) = 0,$$

where

$$\omega_m = u_m = \left(\delta_{m-2} + 26\delta_{m-1} + 66\delta_m + 26\delta_{m+1} + \delta_{m+2}\right),$$

and

$$\omega_m = \frac{h^2}{20}u_{xx} = \left(\delta_{m-2} + 2\delta_{m-1} - 6\delta_m + 2\delta_{m+1} + \delta_{m+2}\right),$$

and $\cdot$ indicates derivative with respect to $t$. Here, the term $u$ in non-linear terms $uu_x$ and $uu_{xx}$, the term $\frac{h^2}{20}u_{xx}$ in non-linear term $u_xu_{xx}$ are taken as Eqs. (4.7) and (4.8) by assuming that the quantity $u$ and $\frac{h^2}{20}u_{xx}$ are locally constants for the linearization technique.

If time parameters $\delta_i$’s and its time derivatives $\dot{\delta}_i$’s in Eq. (4.6) are discretized by using the Crank-Nicolson formula and usual finite difference approximation,
respectively:
\[
\begin{align*}
\delta_i &= \frac{\delta_i^{n+1} + \delta_i^n}{2}, & \hat{\delta}_i &= \frac{\delta_i^{n+1} - \delta_i^n}{\Delta t}.
\end{align*}
\] (4.9)

A recurrence relationship between two time levels \(n\) and \(n+1\) relating two unknown parameters \(\delta_i^{n+1}, \delta_i^n\) are obtained for \(i = m-2, m-1, ..., m+1, m+2\):
\[
\lambda_1\delta_{m-2}^{n+1} + \lambda_2\delta_{m-1}^{n+1} + \lambda_3\delta_m^{n+1} + \lambda_4\delta_{m+1}^{n+1} + \lambda_5\delta_{m+2}^{n+1} = \lambda_0\delta_{m-2}^n + \lambda_4\delta_{m-1}^n + \lambda_5\delta_m^n + \lambda_2\delta_{m+1}^n + \lambda_3\delta_{m+2}^n,
\] (4.10)

where
\[
\begin{align*}
\lambda_1 &= [1 - A\omega_m - (B + C\omega_m) - D\varpi_m], \\
\lambda_2 &= [26 - 10A\omega_m + 2(B + C\omega_m) - 10D\varpi_m], \\
\lambda_3 &= [66], \\
\lambda_4 &= [26 + 10A\omega_m - 2(B + C\omega_m) + 10D\varpi_m], \\
\lambda_5 &= [1 + A\omega_m + (B + C\omega_m) + D\varpi_m], \\
m &= 0, 1, ..., N,
\end{align*}
\]
\[
\begin{align*}
A &= \frac{5}{2h}\alpha\Delta t, & B &= \frac{30}{h^2}\beta\Delta t, & C &= \frac{30}{h^2}\gamma\Delta t, & D &= \frac{50}{h^2} (\gamma + \sigma) \Delta t.
\end{align*}
\] (4.11)

The system (4.10) includes \((N+1)\) linear equations and \((N+5)\) unknown parameters \((\delta_{-2}, \delta_{-1}, ..., \delta_{N+1}, \delta_{N+2})^T\). To obtain a unique solution from this system, four additional constraints are required. These are obtained from the boundary conditions by eliminating \(\delta_{-2}, \delta_{-1}\) and \(\delta_{N+1}, \delta_{N+2}\) in the system (4.10). In this case, a matrix equation is obtained \((N+1)\) linear equations and \(N+1\) unknowns \(d = (\delta_0, \delta_1, ..., \delta_N)^T\) in the following form
\[
Pd^{n+1} = Qd^n.
\] (4.12)

The matrices \(P\) and \(Q\) are \((N+1) \times (N+1)\) pentagonal matrices and this matrix equation (4.12) is solved by using the pentagonal algorithm. To cope with the non-linearity caused by \(\omega_m\) and \(\varpi_m\), two inner iterations are applied to the term \(\delta^{n^*} = \delta^n + \frac{1}{2}(\delta^n - \delta^{n-1})\) at each time step. After the initial vector \(d^0 = (\delta_0, \delta_1, ..., \delta_{N-1}, \delta_N)\) is determined by using the initial condition and the following derivatives at the boundary conditions,
\[
\begin{align*}
u_N(x, 0) &= u(x, 0), & m &= 0, 1, 2, ..., N, \\
(u_N)_x(a, 0) &= 0, & (u_N)_x(b, 0) &= 0, \\
(u_N)_{xx}(a, 0) &= 0, & (u_N)_{xx}(b, 0) &= 0
\end{align*}
\] (4.13)

the following matrix form of the initial vector \(d^0\) is obtained
\[
Wd^0 = R
\] (4.14)
where

\[
\begin{bmatrix}
54 & 60 & 6 \\
25.25 & 67.50 & 26.25 & 1 \\
1 & 26 & 66 & 26 & 1 \\
1 & 26 & 66 & 26 & 1 \\
\vdots \\
1 & 26 & 66 & 26 & 1 \\
1 & 26.25 & 67.50 & 25.25 & 60 & 54
\end{bmatrix}
\begin{bmatrix}
\delta_0 \\
\delta_1 \\
\vdots \\
\delta_{N-1} \\
\delta_N
\end{bmatrix}
= \begin{bmatrix}
u (x_0, 0) \\
u (x_1, 0) \\
\vdots \\
u (x_{N-1}, 0) \\
u (x_N, 0)
\end{bmatrix}
\] (4.15)

Once and for all, this matrix system is solved efficiently by using a variant of Thomas algorithm.

### 4.3. Stability of the scheme

The stability of the linearized numerical scheme is analyzed based on the von Neumann theory. To investigate stability, we use Fourier mode

\[\delta^n_m = \xi^n e^{i m k h},\] (4.16)

where \(\xi\) is growth factor of the error in a typical mode of amplitude \(\xi^n\), \(h\) is element size and \(k\) is mode number. Substituting the Fourier mode (4.16) into the system (4.10) gives the following equality

\[
\lambda_1 \xi^{n+1} e^{i (m-2) k h} + \lambda_2 \xi^{n+1} e^{i (m-1) k h} + \lambda_3 \xi^{n+1} e^{i m k h} + \lambda_4 \xi^{n+1} e^{i (m+1) k h} + \lambda_5 \xi^{n+1} e^{i (m+2) k h} = \lambda_1 \xi^n e^{i (m-2) k h} + \lambda_2 \xi^n e^{i (m-1) k h} + \lambda_3 \xi^n e^{i m k h} + \lambda_2 \xi^n e^{i (m+1) k h} + \lambda_1 \xi^n e^{i (m+2) k h}.
\] (4.17)

Then, if Euler’s formula

\[e^{i k h} = \cos (k h) + i \sin (k h),\] (4.18)

is used in Eq. (4.17) and this equation is simplified, we obtain growth factor in the following

\[\xi = \frac{a - i b}{a + i b},\] (4.19)

where

\[a = 56 + 52 \cos (k h) + 2 \cos (2 k h),\]
\[b = 2 [10 A \omega_m - 2 (B + C \omega_m) + 10 D \omega_m] \sin (k h) + 2 [A \omega_m + (B + C \omega_m) + D \omega_m] \sin (2 k h).
\] (4.20)

Because of \(|\xi| = 1\), the linearized scheme is unconditionally stable.
5. Results and discussion

In this section, numerical solutions of the Kudryashov-Sinelshchikov equation are considered for three problems which are including the motion of single solitary wave, interaction of two solitary waves, evolution of solitary waves with Gaussian and undular bore initial conditions.

To calculate the difference between analytical and numerical solutions and to show accuracy of the applied numerical scheme, the error norm $L_2$

$$L_2 = \|u^{\text{exact}} - u_N\|_2 \simeq \sqrt{\frac{1}{h} \sum_{j=1}^{N} |u_j^{\text{exact}} - (u_N)_j|^2},$$

and the error norm $L_\infty$

$$L_\infty = \|u^{\text{exact}} - u_N\|_\infty \simeq \max_j |u_j^{\text{exact}} - (u_N)_j|, \quad j = 1, 2, ..., N,$$

are used. The Kudryashov-Sinelshchikov equation (1.1) possesses two conserved quantities as follows:

$$C_1 = \int_a^b u \, dx \simeq h \sum_{j=1}^{N} u_j,$$

$$C_2 = \int_a^b \left[ \left( (\gamma u + \beta)\gamma + \beta - \beta^\gamma + 1 \right) \right] \, dx \simeq h \sum_{j=1}^{N} \left[ \left( (\gamma u_j + \beta)\gamma + 1 \right) \right].$$

The conserved quantities $C_1$ and $C_2$ are calculated to check the conservation of some quantities during the wave motion.

5.1. The motion of single solitary wave

The single solitary wave solution of the Kudryashov-Sinelshchikov equation (1.1) is given by

$$u(x, t) = \text{Asech}^2 \left[ B \, (x - ct) \right],$$

where $A = \frac{3\beta \sigma^2}{\alpha - \gamma^2 \mu^2}$, $B = \frac{\mu}{2}$, and $c = \beta \kappa^2 / \mu^2$. Note that, $\alpha$, $\beta$, $\gamma$, $\sigma = -3\gamma$, $\kappa$ and $\mu$ are arbitrary constants. During the calculation, we get the initial condition as

$$u(x, 0) = \text{Asech}^2 \left[ Bx \right].$$

with the boundary conditions $u \to 0$ as $x \to \pm \infty$.

To show the motion of the single solitary wave solution numerically over the interval $x \in [-60, 40]$, the parameters are chosen as $\alpha = 1.5$, $\beta = 1.7$, $\gamma = 0.8$, $\sigma = -2.4$, $c = 0.425$, $\kappa = 1$, $\mu = 0.5$ for $h = \Delta t = 0.1$ and $h = \Delta t = 0.025$, respectively. The solitary wave has amplitude $0.98$ for these parameters. The error norms and conserved quantities are calculated up to time $t = 20$ for different values of $h$ and $\Delta t$. The obtained results are tabulated in Table 2. It can be seen from the Table 2 that $L_2$ and $L_\infty$ error norms are found to be satisfactorily small and the conserved quantities are pretty much unchanged as the time progresses. The percentage of relative changes of $C_1$ and $C_2$ are found to be $5.920 \times 10^{-5}$ and $6.488 \times 10^{-6}$ for $h = \Delta t = 0.1$, $4.689 \times 10^{-5}$ and $4.720 \times 10^{-6}$ for $h = \Delta t = 0.025$, respectively. It is observed from the Table 2 that numerical results are better and more accurate for small values of $h$ and $\Delta t$. If we observe Figure 4(a), we find
bell shaped solitary wave solutions produced. Also, contour line for motion of the single solitary wave can be seen in Figure 4(b). As it can be clearly seen from the figures, the solitary wave moves to the right at a constant speed. As expected, its amplitude and shape are preserved as time passes. On the other hand, error graphs are plotted for different values of $h$ and $\Delta t$ in Figure 5.

Table 2. Conserved quantities and error norms for motion of the single solitary wave

| $t$   | $C_1$       | $C_2$       | $L_2 - \text{Error}$ | $L_\infty - \text{Error}$ |
|-------|-------------|-------------|----------------------|---------------------------|
| 0.0   | 7.8461558037 | -12.3622499053 | 0.000000000000       | 0.000000000000           |
| 2.5   | 7.8461557523  | -12.3622499493 | 1.365976 \times 10^{-5} | 5.729885 \times 10^{-6}   |
| 5.0   | 7.8461557779  | -12.3622499622 | 2.196199 \times 10^{-5} | 1.012597 \times 10^{-5}   |
| 7.5   | 7.8461553075  | -12.3622500428 | 3.096934 \times 10^{-5} | 1.417086 \times 10^{-5}   |
| $h = \Delta t = 0.1$ | 10.0 | 7.8461554338  | -12.3622500188 | 4.00045 \times 10^{-5} | 1.803005 \times 10^{-5} |
| 12.5  | 7.8461551619  | -12.3622503907 | 4.983849 \times 10^{-5} | 2.192837 \times 10^{-5}   |
| 15.0  | 7.8461553181  | -12.3622496956 | 5.860834 \times 10^{-5} | 2.505833 \times 10^{-5}   |
| 17.5  | 7.8461558903  | -12.3622505934 | 6.581260 \times 10^{-5} | 2.693506 \times 10^{-5}   |
| 20.0  | 7.8461551158  | -12.3622507074 | 7.438000 \times 10^{-5} | 2.959083 \times 10^{-5}   |

| $t$   | $C_1$       | $C_2$       | $L_2 - \text{Error}$ | $L_\infty - \text{Error}$ |
|-------|-------------|-------------|----------------------|---------------------------|
| 0.0   | 7.8461455100 | -12.3522259633 | 0.000000000000       | 0.000000000000           |
| 2.5   | 7.8461454628  | -12.3522259731 | 2.412955 \times 10^{-6} | 1.076032 \times 10^{-6}   |
| 5.0   | 7.8461455773  | -12.3522259455 | 3.485312 \times 10^{-6} | 1.460035 \times 10^{-6}   |
| 7.5   | 7.8461453965  | -12.3522259743 | 4.727010 \times 10^{-6} | 1.857019 \times 10^{-6}   |
| $h = \Delta t = 0.025$ | 10.0 | 7.8461452507  | -12.3522259868 | 5.791756 \times 10^{-6} | 2.159832 \times 10^{-6} |
| 12.5  | 7.8461444968  | -12.3522261185 | 8.413219 \times 10^{-6} | 3.713176 \times 10^{-6}   |
| 15.0  | 7.8461449845  | -12.3522260257 | 1.231469 \times 10^{-5} | 5.834287 \times 10^{-6}   |
| 17.5  | 7.8461430957  | -12.3522263379 | 1.707483 \times 10^{-5} | 8.178540 \times 10^{-6}   |
| 20.0  | 7.8461418312  | -12.3522265463 | 2.265791 \times 10^{-5} | 1.063514 \times 10^{-5}   |

Figure 4. Motion of single solitary wave and its contour

5.2. Interaction of two solitary waves

Secondly, the interaction of two solitary waves advancing in the same direction is considered by using the initial condition given by the linear sum of two well separated solitary waves having different amplitudes

$$u(x, 0) = \sum_{i=1}^{2} A_i \text{sech}^2 \left[ B_i (x - x_i) \right],$$  \hspace{1cm} (5.6)
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\[ h = \Delta t = 0.1 \]

\[ h = \Delta t = 0.025 \]

**Figure 5.** Error norms for motion of the single solitary wave

where \( A_i = \frac{3\beta^2 \mu_i^2}{\alpha - \kappa^2 \mu_i^2} \), \( B_i = \frac{2\mu_i}{\kappa} \) and \( c_i = \beta \kappa^2 \mu_i^2 \) for \( i = 1, 2 \). Also, \( x_i \) is arbitrary constant.

To show simulation over the interval \( x \in [-100, 100] \), the parameters are taken to be \( \alpha = 1.5, \beta = 1.7, \gamma = 0.8, \sigma = -2.4, \kappa = 1, \mu_1 = 0.7, \mu_2 = 0.5, c_1 = 0.833, c_2 = 0.425, x_1 = -35, x_2 = 10 \) for \( h = \Delta t = 0.1 \) and \( h = \Delta t = 0.025, \Delta t = 0.01 \), respectively. During the interaction of two solitary waves, the values of the two conserved quantities \( C_1 \) and \( C_2 \) can be seen in Table 3 for different values of \( h \) and \( \Delta t \). It is seen that the obtained values of the conserved quantities remain more constant sensibly for small values of \( h \) and \( \Delta t \). While interacting, the travelling wave profiles can be seen in Figure 6(a). Contour line for interaction of two solitary waves is depicted in Figure 6(b). It is clear from the figures that, the large solitary wave is located to the left of the small solitary wave at the beginning of the calculation. With increasing time, the large solitary wave catches up the small wave until \( t = 30 \), then the small solitary wave is absorbed. The overlap process continues until \( t = 60 \), then the large solitary wave has overtaken the small solitary wave and the separation process begins. At time \( t = 70 \), the interaction is completed and the large solitary wave has separated completely. At the end of this process, solitary waves preserve their original amplitudes and shapes.

| Table 3. Conserved quantities for the interaction of two solitary waves |
|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
|                            | \( h = 0.1 \)               | \( \Delta t = 0.1 \)        | \( h = 0.025 \)              | \( \Delta t = 0.01 \)       |
| \( t \)                    | \( C_1 \)                   | \( C_2 \)                   | \( C_1 \)                   | \( C_2 \)                   |
| 0                          | 20.7343903136               | -24.4006865491              | 0.00003587567               | -6.6868057075               |
| 10                         | 20.7344125274               | -24.4005864401              | 0.0001414754               | -6.6868417960               |
| 20                         | 20.7344271657               | -24.3998924146              | 0.0000488112               | -6.6868571878               |
| 30                         | 20.7344310727               | -24.3957076573              | 0.0000875714               | -6.6868507448               |
| 40                         | 20.7344325231               | -24.3832055971              | 0.0002023407               | -6.6868317004               |
| 50                         | 20.7344385138               | -24.3896941137              | 0.0001342899               | -6.6868429859               |
| 60                         | 20.7343773107               | -24.3989882874              | 0.0000622223               | -6.6868549558               |
| 70                         | 20.7344018243               | -24.4006154476              | 0.0000759852               | -6.6868526692               |
| 80                         | 20.7343755930               | -24.4007677831              | 0.0000338496               | -6.6868596667               |
5.3. Evolution of solitary waves

As a last test problem, Gaussian and undular bore initial conditions are studied to show birth of solitary waves.

5.3.1. Gaussian initial condition

Evolution of a train of solitary waves is studied on Kudryashov-Sinelshchikov equation using the Gaussian initial condition

\[ u(x,0) = \exp\left(-x^2\right), \]  

and boundary condition

\[ u(-200, t) = u(100, t) = 0, \quad t > 0. \]  

Parameters are taken as \( \alpha = 1.5, \beta = 1.7, \gamma = 0.8, \sigma = -2.4, c = 0.425, \kappa = 1, \mu = 0.5 \) for \( h = \Delta t = 0.1 \) and \( h = 0.08, \Delta t = 0.05 \) respectively. The numerical calculations are done until time \( t = 60 \). The values of the two conserved quantities of motion according to space and time steps are presented in Table 4. Here, especially the conserved quantity \( C_2 \) is preserved better for small values of \( h \) and \( \Delta t \). Also, Figure 7(a) illustrates the development of the Gaussian initial condition into solitary waves. As it is seen from the Figure 7(a), a solitary wave plus and oscillating tail are drawn. As seen from the figures, two solitary waves moving is observed. Contour line for evolution of solitary waves with Gaussian initial condition is depicted in Figure 7(b).

5.3.2. Undular bore initial condition

Production of a train of solitary waves is studied on Kudryashov-Sinelshchikov equation using the undular bore initial condition

\[ u(x,0) = \frac{1}{2}u_0 \left[ 1 - \tanh \left( \frac{|x| - x_0}{d} \right) \right], \]  

and boundary condition

\[ u(-200, t) = u(100, t) = 0, \quad t > 0. \]
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Table 4. Conserved quantities for evolution of solitary waves with Gaussian initial condition

|   | $h = 0.1$ | $\Delta t = 0.1$ | $h = 0.08$ | $\Delta t = 0.05$ |
|---|-----------|------------------|-------------|------------------|
| $t$ | $C_1$     | $C_2$            | $C_1$       | $C_2$            |
| 0  | 17.7245287586 | -51.2261857478 | 17.7245503786 | -51.2235091657 |
| 10 | 17.7245287586 | -51.2301483544 | 17.7245503794 | -51.2346847042 |
| 20 | 17.7245287628 | -51.2373589096 | 17.7245503686 | -51.2376138049 |
| 30 | 17.7245286775 | -51.2402881706 | 17.7245502309 | -51.2384401312 |
| 40 | 17.7245330256 | -51.2414655930 | 17.7245544406 | -51.2387910582 |

The undular bore refers the elevation of the water above the equilibrium surface at time $t = 0$. The changing in amplitude is centered on $x = x_0$ and the steepness of the change is measured by $d$. The values of $d$ is inversely proportional to the steepness. The parameters are choosen as $u_0 = 1$, $x_0 = 25$ and $d = 5$ in Eq. (5.9).

Choosing the parameters $\alpha = 1.5$, $\beta = 1.7$, $\gamma = 0.8$, $\sigma = -2.4$, $c = 0.425$, $\kappa = 1$, $\mu = 0.5$ for $h = \Delta t = 0.1$ and $h = \Delta t = 0.025$, the computational work is done up to $t = 60$. Variation of the conserved quantities $C_1$ and $C_2$ position is recorded in Table 5. Especially, the conserved quantity $C_1$ is preserved better for small values of $h$ and $\Delta t$. Undulation bore keeps steady-state during running the which can be observed at specified times in Figure 8(a) and contour line for evolution of solitary waves with undular bore initial condition is shown in Figure 8(b). As it is seen from these figures, the initial perturbation evolves into a good developed train of solitary waves. As the time progresses, five solitary waves moving is observed.

6. Conclusion

Polynomial and rational wave solutions of Kudryashov-Sinelshchikov equation and numerical simulations for its dynamic motions have been studied. Conservation quantities of the dynamic motion have been calculated using multiplier approach. A collection of exact solitary and soliton solutions of Kudryashov-Sinelshchikov equation has been derived. Collocation finite element method based on quintic B-spline functions has been applied to the equation. The numerical scheme has been
Table 5. Conserved quantities for evolution of solitary waves with undular bore initial condition

|   | $h = 0.1$ | $\Delta t = 0.1$ | $h = 0.25$ | $\Delta t = 0.25$ |
|---|----------|------------------|-----------|------------------|
| $t$ | $C_1$    | $C_2$            | $C_1$    | $C_2$            |
| 0  | 50.0001143360 | -47.5406405155 | 50.0002271836 | -47.5606776333 |
| 10 | 50.0001143358 | -47.5472542687 | 50.0002271800 | -47.5672722389 |
| 20 | 50.0001136922 | -47.5669315179 | 50.0002283411 | -47.5868910697 |
| 30 | 50.0001005086 | -47.5859019717 | 50.0002273206 | -47.5868910697 |
| 40 | 50.0001424769 | -47.5972898726 | 50.0002113808 | -47.6172077980 |
| 50 | 50.0001847283 | -47.6029921696 | 50.0001054225 | -47.6229362420 |
| 60 | 49.9999339684 | -47.6057580275 | 50.0000515661 | -47.6256782478 |

Figure 8. Generated waves and its contour for undular bore initial condition

shown to be unconditionally stable. To demonstrate the accuracy of the proposed method, we have considered three test problems including the motion of single solitary wave, interaction of two solitary waves and evolution of waves with Gaussian and undular bore initial conditions. Then, analytical and numerical results have been compared by aid of error norms. The obtained analytical and numerical results are in good agreement. The results of this paper come with a lot of encouragement for further future investigations.

References

[1] H. I. Abdel-Gawad, M. Tantawy and M.S. Osman, Dynamic of DNA’s possible impact on its damage, Mathematical Methods in the Applied Sciences, 2016, 39(2), 168–176.

[2] M. N. Ali, S. Ali, S. M. Husnine and T. Ak, Nonlinear self-adjointness and conservation laws of KdV equation with linear damping force, Applied Mathematics & Information Sciences Letters, 2017, 5(3), 89–94.

[3] M. N. Ali, A. R. Seadawy, S. M. Husnine and K.U. Tariq, Optical pulse propagation in monomode fibers with higher order nonlinear Schrödinger equation, Optik, 2018, 156, 356–364.

[4] M. N. Ali, S. M. Husnine, S. Noor and A. Tuna, Exact solutions of $(n + 1)$-dimensional space-time fractional Zakharov-Kuznetsov equation, Hittite Journal of Science and Engineering, 2018, 5(3), 179–183.
M. N. Ali, S. M. Husnine, T. Ak and A. Atangana, *Solitary wave solution and conservation laws of higher dimensional Zakharov-Kuznetsov equation with non-linear self-adjointness*, Mathematical Methods in the Applied Sciences, 2018, 41, 6611–6624.

M. N. Ali, S. M. Husnine, A. Saha, S. K. Bhowmik, S. Dhawan and T. Ak, *Exact solutions, conservation laws, bifurcation of nonlinear and supernonlinear traveling waves for Sharma-Tasso-Olver equation*, Nonlinear Dynamics, 2018, 94, 1791–1801.

M. N. Ali, A. R. Seadawy and S. M. Husnine, *Lie point symmetries, conservation laws and exact solutions of (1 + n)-dimensional modified Zakharov-Kuznetsov equation describing the waves in plasma physics*, Pramana-Journal of Physics, 2018, 8, 1054–1060.

S. C. Anco and G. Bluman, *Direct construction method for conservation laws of partial differential equations Part I: Examples of conservation law classifications*, European Journal of Applied Mathematics, 2002, 13(5), 545–566.

I. M. Anderson and J. Pohjanpelto, *The cohomology of invariant variational bicomplexes*, Acta Applicandae Mathematicae, 1995, 41, 3–19.

M. S. Bruzón, E. Recio, R. de la Rosa and M. L. Gandarias, *Local conservation laws, symmetries, and exact solutions for a Kudryashov-Sinelshchikov equation*, Mathematical Methods in the Applied Sciences, 2018, 41(4), 1631–1641.

B. Feng and T. Mitsui, *A finite difference method for the Korteweg-de Vries and the Kadomtsev-Petviashwili equations*, Journal of Computational and Applied Mathematics, 1998, 90(1), 95–116.

N. H. Ibragimov, *CRC Handbook of Lie Group Analysis of Differential Equations, Volume I: Symmetries, Exact Solutions, and Conservation Laws*, CRC Press, Boca Raton, 1993.

M. Inc, A. I. Aliyu, A. Yusuf and D. Baleanu, *Optical solitons for Biswas-Milovic model in nonlinear optics by Sine-Gordon equation method*, Optik-International Journal for Light and Electron Optics, 2018, 157, 267–274.

E. Noether, *Invariant and variation problems*, Transport Theory and Statistical Physics, 1971, 1(3), 186–207.

P. Olver, *Application of Lie Groups to Differential Equations*, Springer-Verlag, New York, 1986.

M. S. Osman, *Analytical study of rational and double-soliton rational solutions governed by the KdV-Sawada-Kotera-Ramani equation with variable coefficients*, Nonlinear Dynamics, 2017, 89(3), 2283–2289.

M. S. Osman and J.A. Machado, *The dynamical behavior of mixed-type soliton solutions described by (2 + 1)-dimensional Bogoyavlensky-Konopelchenko equation with variable coefficients*, Journal of Electromagnetic Waves and Applications, 2018, 32(11), 1457–1464.

M. S. Osman, J.A.T Machado and D. Baleanu, *On nonautonomous complex wave solutions described by the coupled Schrödinger-Boussinesq equation with variable coefficients*, Optical and Quantum Electronics, 2018, 50(2), 73.

M. S. Osman, A. Korkmaz, H. Rezazadeh, M. Mirzazadeh, M. Eslami and Q. Zhou, *The unified method for conformable time fractional Schrödinger equation with perturbation terms*, Chinese Journal of Physics, 2018, 56(5), 2500–2506.
[20] P. M. Prenter, *Splines and Variational Methods*, John Wiley, New York, 1975.

[21] M. Randrüüt, *On the Kudryashov-Sinelshchikov equation for waves in bubbly liquids*, Physics Letters A, 2011, 375(42), 3687–3692.

[22] H. Stephani, *Differential Equations: Their Solution Using Symmetries*, Cambridge University Press, Cambridge, 1989.

[23] H. Triki, T. Ak, S. P. Moshokoa and A. Biswas, *Soliton solutions to KdV equation with spatio-temporal dispersion*, Ocean Engineering, 2018, 91, 48.

[24] M. Vlieg-Hulstman, *The Painlevé analysis and exact travelling wave solutions to nonlinear partial differential equations*, Mathematical and Computer Modelling, 1993, 18(10), 151–156.

[25] A. M. Wazwaz, *The variational iteration method: A reliable analytic tool for solving linear and nonlinear wave equations*, Computers & Mathematics with Applications, 2007, 54(7–8), 926–932.

[26] A. M. Wazwaz, *The Hirota’s bilinear method and the tanh-coth method for multiple-soliton solutions of the Sawada-Kotera-Kadomtsev-Petviashvili equation*, Applied Mathematics and Computation, 2008, 200(1), 160–166.

[27] H. Wu, *On Bäcklund transformations for nonlinear partial differential equations*, Journal of Mathematical Analysis and Applications, 1995, 192(1), 151–179.

[28] H. Yang, *Symmetry reductions and exact solutions to the Kudryashov-Sinelshchikov equation*, Zeitschrift für Naturforschung A, 2016, 71(11)a, 1059–1065.

[29] V. E. Zakharov and A.B. Shabat, *Interaction between solitons in a stable medium*, Soviet Physics-Journal of Experimental and Theoretical Physics, 1973, 37(5), 823–828.

[30] S. Zhang and S. Hong, *Variable separation method for a nonlinear time fractional partial differential equation with forcing term*, Journal of Computational and Applied Mathematics, 2018, 339, 297–305.