HIGHER SUPPORT TILTING I: HIGHER AUSLANDER ALGEBRAS OF LINEARLY ORIENTED TYPE $A$

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Abstract. For a path algebra $A$ over a quiver $Q$, there are bijections between the support-tilting modules of $A$, torsion classes in $\text{mod}(A)$ and wide subcategories in $\text{mod}(A)$; these are part of the Ingalls-Thomas bijections. As a blueprint for further study, we show how these bijections manifest themselves for higher Auslander algebras of linearly oriented type $A$. In particular, we introduce a higher analogue of torsion classes in $d$-representation-finite algebras.

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1. Introduction

Support-tilting modules were first studied by Ringel [28]. Their importance was highlighted by Ingalls-Thomas [12], who showed the following result.

Theorem 1.1. [12, Theorem 1.1] For a finite-dimensional algebra $A = kQ$ that is hereditary and representation finite, there are bijections between the following objects:

- Isomorphism classes of basic support-tilting $A$-modules.
- Torsion classes in $\text{mod}(A)$.
- Wide subcategories in $\text{mod}(A)$.

This correspondence forms part of the Ingalls-Thomas bijections and was generalised to all representation-finite finite-dimensional algebras by Marks-Šťovíček [24]. Included in the Ingalls-Thomas bijections were also clusters in the acyclic
cluster algebra whose initial seed is given by \( Q \). A consequence was that support-tilting modules were able to capture the behaviour of clusters, and this led to further study. Significantly, support \( \tau \)-tilting theory \cite{2}, \cite{17} was able capture this behavior more generally, and has seen much activity in recent years, see for example \cite{1}, \cite{3}, \cite{6}, \cite{7}, \cite{8}, \cite{16}, \cite{18}, \cite{19}, \cite{27}.

A natural question to ask is whether similar results are true in the context of higher Auslander-Reiten theory, as introduced by Iyama in \cite{13}, \cite{14}. The limiting factor is how the support of a module behaves: for an algebra \( A \) with \( d \)-cluster-tilting subcategory \( C \subseteq \text{mod}(A) \) we would like to be able to transfer information about \( C \) to this support. So to most easily replicate the theory of support tilting, we will only consider modules whose support is determined by an idempotent ideal \( I \) such that \( C \cap \text{mod}(A/I) \subseteq \text{mod}(A/I) \) is a \( d \)-cluster-tilting subcategory. Any such module that is in addition \( d \)-tilting as an \( A/I \)-module will be said to be a proper \( d \)-support-tilting.

Another concept that needs to be generalised is that of a torsion class. Suppose that \( T \subseteq C \) is an additive subcategory such that for any \( d \)-exact sequence

\[
0 \to M_0 \to M_1 \to \cdots \to M_{d+1} \to M_{d+2} = 0,
\]

and for any \( 1 \leq i \leq d+1 \), if both \( M_{i-1} \in T \) and \( M_{i+1} \in T \) then so is \( M_i \). In this case \( T \) is a \( d \)-strong torsion class. These concepts are able to extend the classical behaviour of torsion classes to the class of \( d \)-representation-finite algebras whose \( d \)-cluster-tilting subcategories are almost directed (see Definition \cite{2}). This class of algebras includes the higher Auslander algebras of linearly oriented type \( A \).

**Theorem 1.2** (Theorem \cite{53}). Let \( A \) be a finite-dimensional algebra with a \( d \)-cluster-tilting subcategory \( C \subseteq \text{mod}(A) \) such that \( \text{gl.dim}(A) \leq d \) and \( C \) is almost directed. For a given proper support-\( d \)-tilting \( A \)-module \( T \), let \( \mathcal{T} := \text{Fac}(T) \cap C \). Then the following hold:

1. \( \mathcal{T} \) is a \( d \)-strong torsion class.
2. For all \( M \in C \), there is a module \( F_M \in \text{mod}(A) \) and an exact sequence

\[
0 \to T_1 \to T_2 \to \cdots \to T_d \to M \to F_M \to 0
\]

such that \( T_i \in \mathcal{T} \) for all \( 1 \leq i \leq d \) and

\[\mathcal{T} = \{ T \in C \mid \text{Hom}_A(T, F_M) = 0 \ \forall M \in C \}.\]

A \( d \)-strong torsion class is **standard** if it can be described as \( \text{Fac}(T) \cap C \) for a proper support-\( d \)-tilting \( A \)-module \( T \). Recently, wide subcategories were defined for \( d \)-abelian categories by Herschend-Jørgensen-Vaso \cite{11}. One might hope that there is a bijection between wide subcategories and proper support-\( d \)-tilting modules. However in higher dimensions we must instead consider certain combinations of wide subcategories called **resonant collections** (see Definition \cite{5}). This allows us to generalise Theorem \cite{11} to the following extent.
Theorem 1.3 (Theorem 6.3). Let $A$ be a $d$-Auslander algebra of linearly oriented type $A_n$ with unique $d$-cluster-tilting subcategory $C$. Then there are bijections between the following:

- Proper support-$d$-tilting $A$-modules.
- Standard $d$-strong torsion classes in $C$.
- Resonant collections of wide subcategories in $C$.
- Standard $d$-strong torsion-free classes in $C$.
- Coresonant collections of wide subcategories in $C$.

2. Background and notation

Consider a finite-dimensional algebra $A$ over a field $k$, and fix a positive integer $d$. An $A$-module will mean a finitely-generated left $A$-module; by $\text{mod}(A)$ we denote the category of finite-dimensional left $A$-modules. The functor $D = \text{Hom}_k(\text{---}, k)$ defines a duality, and we set $\tau_d = \tau \circ \Omega^{d-1}$ to be the $d$-Auslander-Reiten translation. For an $A$-module $M$, let $\text{add}(M)$ be the full subcategory of $\text{mod}(A)$ composed of all $A$-modules isomorphic to direct summands of finite direct sums of copies of $M$.

Define the dominant dimension of $A$ $\text{dom.dim}(A)$ to be the number $n$ such that for a minimal injective resolution of $A$:

$$0 \rightarrow A \rightarrow I_0 \rightarrow \cdots \rightarrow I_{n-1} \rightarrow I_n \rightarrow \cdots$$

the modules $I_0, \cdots, I_{n-1}$ are projective-injective and $I_n$ is not projective. A subcategory $C$ of $\text{mod}(A)$ is precovering if for any $M \in \text{mod}(A)$ there is an object $C_M \in C$ and a morphism $f : C_M \rightarrow M$ such that for any morphism $X \rightarrow M$ with $X \in C$ factors through $f$; that there is a commutative diagram:

$$
\begin{array}{ccc}
X & \xrightarrow{f} & M \\
\downarrow \exists & & \downarrow \\
C_M & \rightarrow & 
\end{array}
$$

The dual notion of precovering is preenveloping. A subcategory $C$ that is both precovering and preenveloping is called functorially finite. For a finite-dimensional algebra $A$, a functorially-finite subcategory $C$ of $\text{mod}(A)$ is a $d$-cluster-tilting subcategory if it satisfies the following conditions:

- $C = \{ M \in \text{mod}(A) | \text{Ext}_A^i(C, M) = 0 \ \forall \ 0 < i < d \}$.
- $C = \{ M \in \text{mod}(A) | \text{Ext}_A^j(M, C) = 0 \ \forall \ 0 < i < d \}$.

If there exists a $d$-cluster-tilting subcategory $C \subseteq \text{mod}(A)$, then $(A, C)$ is a $d$-homological pair in the sense of [11]. If $A$ is a finite-dimensional algebra such that $(A, C)$ is a $d$-homological pair and $\text{gl.dim}(A) \leq d$, then $A$ is $d$-representation finite in the sense of [15]. The class of $d$-representation-finite algebras were characterised by Iyama as follows.
Theorem 2.1. [4, Proposition 1.3, Theorem 1.10] Let $A$ be a finite-dimensional algebra such that $\text{gl.dim}(A) \leq d$. Then there is a unique $d$-cluster-tilting subcategory $\mathcal{C} \subseteq \text{mod}(A)$ if and only if

$$\text{dom.dim}(\text{End}(M)^{\text{op}}) \geq d + 1 \geq \text{gl.dim}(\text{End}(M)^{\text{op}})$$

where $M$ is an additive generator of the subcategory

$$\mathcal{C} = \text{add}(\{r_i^i(DA)|i \geq 0\}) \subseteq \text{mod}(A).$$

For an $A$-module $M$, the annihilator of $M$ is the two-sided ideal

$$\text{ann}(M) := \{a \in A|M_0 = 0 = aM\}.$$

The support of $M$, denoted $\text{supp}(M)$, is defined to be the set of vertices such that for each vertex $i$, the module $S_i$ is contained in the composition series of $M$. For a module $M$, let

$$e_M := \sum_{i \in \text{supp}(M)} e_i,$$

and for a class of modules $\mathcal{T} \subseteq \mathcal{C}$ let

$$e_\mathcal{T} := \sum_{\{M \in \mathcal{T} | i \in \text{supp}(M)\}} e_i.$$

An idempotent ideal $\langle e \rangle$ is $d$-idempotent [5, Section 1] if there are isomorphisms $\text{Ext}^i_A(M, N) \cong \text{Ext}^i_{A/\langle e \rangle}(M, N)$ for all $0 \leq i \leq d$. Within higher Auslander-Reiten theory, $d$-idempotent ideals were heavily used in the construction of higher Nakayama algebras [21, Lemma 1.20]. The singularity categories of higher Nakayama algebras were also described using idempotent ideals in [25]. For the following result to make sense, we note that it is well-known that for a finite-dimensional algebra $A$ and injective $A$-module $I$, the $A/\langle e \rangle$-module $\text{Hom}_A(A/\langle e \rangle, I)$ is injective. Dually, for an injective $A$-module $P$, the $A/\langle e \rangle$-module $P \otimes_A A/\langle e \rangle$ is projective.

Proposition 2.2. [5, Proposition 1.1] Let $A$ be a finite-dimensional algebra and let $e$ be an idempotent of $A$. Define the functor $F := \text{Hom}_A(A/\langle e \rangle, -)$ from $\text{mod}(A)$ to $\text{mod}(A/\langle e \rangle)$. Let $M$ be an $A$-module with minimal injective coresolution

$$0 \to M \to I_0 \to \cdots \to I_d.$$

Then the following are equivalent:

1. The beginning of a minimal injective coresolution of $F(M)$ in $\text{mod}(A/\langle e \rangle)$ is

$$0 \to F(M) \to F(I_0) \to \cdots \to F(I_d).$$

2. $\text{Ext}^i_A(A/\langle e \rangle, M) = 0$ for all $1 \leq i \leq d$.

3. For any $N \in \text{mod}(A/\langle e \rangle)$ and for all $1 \leq i \leq d$ there are isomorphisms

$$\text{Ext}^i_A(N, M) \cong \text{Ext}^i_{A/\langle e \rangle}(N, F(M)).$$

This leads to the following result.
Definition 1. The construction of higher Nakayama algebras relies on this property (see the usage of [21, Proposition 1.3]).

Proposition 2.3. Let $A$ be a finite-dimensional algebra and let $e$ be an idempotent of $A$. Let $d \geq 1$ be a positive integer. Then the following conditions are equivalent:

1. The ideal $\langle e \rangle$ is $d$-idempotent.
2. $\text{Ext}^i_A(\langle e \rangle, M) = 0$ for all $A/\langle e \rangle$-modules $M$ and all $1 \leq i \leq d$.
3. $\text{Ext}^i_A(\langle e \rangle, I) = 0$ for all injective $A/\langle e \rangle$-modules $I$ and all $1 \leq i \leq d$.

This has the following implication for $d$-homological pairs.

Corollary 2.4. Let $(A, \mathcal{C})$ be a $d$-homological pair such that $\text{gl.dim}(A) \leq d$. Suppose that $A/\langle e \rangle \in \mathcal{C}$, and that for any injective $A/\langle e \rangle$-module $I$, both $I \in \mathcal{C}$ and $\text{Ext}^i_A(\langle e \rangle, I) = 0$. Then the following hold:

1. The ideal $\langle e \rangle$ is $(d+1)$-idempotent.
2. $\text{gl.dim}(A/\langle e \rangle) \leq d$.

Proof. For any injective $A/\langle e \rangle$-module $I$, since both $A/\langle e \rangle \in \mathcal{C}$ and $I \in \mathcal{C}$, we have that $\text{Ext}^i_A(\langle e \rangle, I) = 0$ for all $0 < i < d$. By assumption we also have that $\text{Ext}^d_A(\langle e \rangle, I) = 0$. Finally $\text{gl.dim}(A) \leq d$ implies $\text{Ext}^i_A(\langle e \rangle, I) = 0$ for all $i > d$. By Proposition 2.3, this means the ideal $\langle e \rangle$ is $(d+1)$-idempotent. A consequence is that $\text{Ext}^{d+1}_A(M, N) = \text{Ext}^{d+1}_A(\langle e \rangle, I) = 0$ for all $M, N \in \text{mod}(A/\langle e \rangle)$. Therefore $\text{gl.dim}(A/\langle e \rangle) \leq d$.

Since $\langle e \rangle$ is $(d+1)$-idempotent, it would not be unreasonable to expect that $\mathcal{C} \cap \text{mod}(A/\langle e \rangle)$ is a $d$-cluster-tilting subcategory of $\text{mod}(A/\langle e \rangle)$. In fact the construction of higher Nakayama algebras relies on this property (see the usage of [21, Lemma 1.20]). More generally, the following is true.

Definition 1. Let $(A, \mathcal{C})$ be a $d$-homological pair such that $\text{gl.dim}(A) \leq d$. Then a subcategory $\mathcal{T} \subseteq \mathcal{C}$ is properly supported and the idempotent $e_\mathcal{T}$ properly supporting (for the idempotent $e_\mathcal{T}$ defined in equation (1)) if:

1. $\text{Ext}^d_A(A/\langle e_\mathcal{T} \rangle, I) = 0$ for all injective $A/\langle e_\mathcal{T} \rangle$-modules.
2. $\mathcal{C} \cap \text{mod}(A/\langle e_\mathcal{T} \rangle)$ is a $d$-cluster-tilting subcategory of $\text{mod}(A/\langle e_\mathcal{T} \rangle)$ such that

$$\mathcal{C} \cap \text{mod}(A/\langle e_\mathcal{T} \rangle) \cong \{ \text{Hom}_A(A/\langle e \rangle, M) | M \in \mathcal{C} \}$$
$$\cong \{ M \otimes_A A/\langle e \rangle | M \in \mathcal{C} \}.$$
Recall that for a finite-dimensional algebra $A$ and any two morphisms $f : X \to M$, $g : X \to N$ between objects $M, N, X \in \text{mod}(A)$, there is a pushout of $f$ and $g$ consisting of an object $P$ and morphisms $f' : M \to P$, $g' : N \to P$ such that $f' \circ f \cong g' \circ g$ and $P$ is universal with this property: for any $P_1$ and morphisms $f_1 : M \to P_1$, $g_1 : N \to P_1$ such that $f_1 \circ f \cong g_1 \circ g$, then $f_1$ factors through $f'$ and $g_1$ factors through $g'$. One property of the pushout is that there is an exact sequence

$$X \to M \oplus N \to P.$$  

More generally, in a functorially-finite subcategory $C \subseteq \text{mod}(A)$ and any two morphisms $f : X \to M$, $g : X \to N$ between objects $M, N, X \in C$, the property of being preenveloping implies that there is an object $P \in C$ and morphisms $f' : M \to P$, $g' : N \to P$ such that $f' \circ f \cong g' \circ g$, that $P$ is universal with this property and there is an exact sequence

$$X \to M \oplus N \to P.$$  

Returning to the category $\text{mod}(A)$, if $P$ is the pushout of two morphisms $f$ and $g$ such that $f$ is an injective morphism, then there is a commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & X \\
\downarrow & & \downarrow f \\
0 & \longrightarrow & M \\
\downarrow & & \downarrow g \\
0 & \longrightarrow & N \\
\downarrow & & \downarrow \\
0 & \longrightarrow & P \\
\downarrow & & \downarrow \\
0 & \longrightarrow & C \\
\end{array}
$$

for some $C \in \text{mod}(A)$. This concept (and the dual notion of a pullback) was generalised to $d$-pushout and $d$-pullback diagrams in [20], see also the survey article [22]. First, an exact sequence is $d$-exact if it can be written in the form

$$0 \to M_0 \to M_1 \to \cdots \to M_{d+1} \to 0.$$  

The result we need is the following:

**Proposition 2.5.** [20, Proposition 3.8] *Let $(A, C)$ be a $d$-homological pair. For any $d$-exact sequence in $C$

$$0 \to X_0 \to X_1 \to \cdots \to X_{d+1} \to 0$$

and any morphism $f : X_0 \to Y_0$ there exists a commutative diagram in $C$:

$$
\begin{array}{cccc}
0 & \longrightarrow & X_0 & \longrightarrow & X_1 & \longrightarrow & \cdots & \longrightarrow & X_d & \longrightarrow & X_{d+1} & \longrightarrow & 0 \\
\downarrow & & f & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & Y_0 & \longrightarrow & Y_1 & \longrightarrow & \cdots & \longrightarrow & Y_d & \longrightarrow & X_{d+1} & \longrightarrow & 0 \\
\end{array}
$$

such that there is an induced $d$-exact sequence

$$0 \to X_0 \to X_1 \oplus Y_0 \to X_2 \oplus Y_1 \to \cdots \to X_d \oplus Y_{d-1} \to Y_d \to 0.$$  

The commutative diagram

\[
\begin{array}{c}
X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_d \\
\downarrow f \downarrow \downarrow \downarrow \\
Y_0 \rightarrow Y_1 \rightarrow \cdots \rightarrow Y_d
\end{array}
\]

is a \textit{d-pushout diagram}. Dually, there is the notion of a \textit{d-pullback diagram}. For an algebra \(A\) with global dimension \(d\) any two \(A\)-modules \(M\) and \(N\), then as in [13, Theorem 2.3.1] there is an isomorphism

\[
\text{Hom}_A(M, \tau_d(N)) \cong \text{Ext}_A^d(N, M).
\]

For a \(d\)-representation-finite algebra \(A\) with \(d\)-cluster-tilting subcategory \(C\), there is an alternative way of phrasing this isomorphism. For any \(M, N \in C\) and any non-zero morphism \(f \in \text{Hom}(M, \tau_d(N))\), then a \(d\)-pushout diagram induces the following commutative diagram:

\[
\begin{array}{c}
0 \rightarrow M \rightarrow X_1 \rightarrow \cdots \rightarrow X_d \rightarrow N \rightarrow 0 \\
\downarrow f \downarrow \downarrow \downarrow \\
0 \rightarrow \tau_d(N) \rightarrow Y_1 \rightarrow \cdots \rightarrow Y_d \rightarrow N \rightarrow 0
\end{array}
\]

So Proposition 2.5 implies that for any modules \(M, N \in C\), then any \(d\)-exact sequence

\[
0 \rightarrow M \rightarrow E_1 \rightarrow \cdots \rightarrow E_d \rightarrow N \rightarrow 0
\]

is Yoneda equivalent to a \(d\)-exact sequence

\[
0 \rightarrow N \rightarrow X_1 \rightarrow \cdots \rightarrow X_d \rightarrow M \rightarrow 0
\]

such that \(X_1, \ldots, X_d \in C\).

### 3. Almost directed subcategories and resonance diagrams

We desire a version of directedness in order to be able to generalise torsion classes.

\textbf{Definition 2.} A \(d\)-homological pair \((A, C)\) is \textit{almost directed} if the following conditions are satisfied.

\begin{enumerate}
\item The algebra \(A\) is given by a quiver with relations whose commutativity relations and zero relations are generated by paths of length two.
\item For any indecomposable modules \(M, N \in C\) and any integer \(i \geq 1\), then \(\dim(\text{Ext}_A^i(M, N)) \leq 1\).
\item For any \(M \in C\), there exists a left properly-supporting idempotent \(e\) such that either \(M\) is projective as an \(A/\langle e\rangle\)-module, or for any \(N \in C \cap \text{mod}(A/\langle e\rangle)\) then \(\text{Hom}_A(M, N) \neq 0\) implies that \(N\) is injective as an \(A\)-module.
\end{enumerate}
(4) For any $M \in C$, there exists a right properly-supporting idempotent $e$ such that either $M$ is injective as an $A/\langle e \rangle$-module, or for any $N \in C \cap \text{mod}(A/\langle e \rangle)$ then $\text{Hom}_A(N, M) \neq 0$ implies that $N$ is projective as an $A$-module.

Observe that if a $d$-homological pair $(A, C)$ is almost directed, then for any properly-supporting idempotent $e$, then the $d$-homological pair $(A/\langle e \rangle, C \cap \text{mod}(A/\langle e \rangle))$ is almost directed. A wide range of algebras with $d$-cluster-tilting subcategories that do not satisfy property (1) can be found in [29].

**Proposition 3.1.** Let $(A, C)$ be an almost-directed $d$-homological pair such that $\text{gl.dim}(A) \leq d$. Suppose that

$$\phi_M : 0 \to M_0 \to M_1 \to \cdots \to M_{d+1} \to 0$$

$$\phi_N : 0 \to N_0 \to N_1 \to \cdots \to N_{d+1} \to 0$$

are $d$-exact sequences in $C$ such that $M_i = N_j$ for some $0 \leq i \leq j \leq d + 1$. Then there exists a commutative diagram containing $\phi_M$, $\phi_N$ and $(d - 1)$ many other $d$-exact sequences

$$\phi_k : X_{0,k} \to X_{1,k} \to \cdots \to X_{d+1,k} \to 0$$

where $1 \leq k \leq d - 1$ and such that each module $X_{i,k}$ is a term in either $\phi_M$, $\phi_N$ or $\phi_m$ for some $1 \leq m \leq d - 1$ (where $l \neq m$) and this is distinct for each $0 \leq l \leq d + 1$.

**Proof.** Because $C$ is functorially finite, we may take the “pullback” $P_{i-1}$ of $M_i = N_j$ along the morphisms $M_{i-1} \to M_i$ and $h : N_{j-1} \to N_j$ to obtain morphisms $f_{i-1} : P_{i-1} \to M_{i-1}$, $g_{i-1} : P_{i-1} \to N_{j-1}$. Now take the “pullback” $P_{i-2}$ of $M_{i-1} \to M_i$ and $f_{i-1}$ to obtain morphisms $f_{i-2} : P_{i-2} \to M_{i-2}$, $g_{i-2} : P_{i-2} \to P_{i-1}$. Since the composition of morphisms $h \circ g_{i-1} \circ g_{i-2}$ factors through $f_{i-1} \circ f_{i-2}$, we must have that $h \circ g_{i-1} \circ g_{i-2} = 0$. Since all zero relations are of length at most two, this means the composition $g_{i-1} \circ g_{i-2} = 0$. Now, as the sequences

$$P_{i-2} \to P_{i-1} \oplus M_{i-2} \to M_{i-1}$$

$$P_{i-1} \to N_{j-1} \oplus M_{i-1} \to M_i$$

are exact and $g_{i-1} \circ g_{i-2} = 0$ by uniqueness of $P_{i-1}$, we must have that

$$P_{i-2} \to P_{i-1} \to N_{j-1}$$

is exact. Continue this process through $\phi_M$, to $\phi_N$ and dually. Inductively we obtain more $d$-exact sequences, and hence every induced module must be contained in two of the induced $d$-exact sequences.

A diagram as in Proposition [3.1] will be called a *resonance diagram*. Fix an integer $n$ and consider the $d$-Auslander algebra $A$ of linearly oriented type $A_n$. There is a $d$-cluster-tilting subcategory $C \subseteq \text{mod}(A)$ that can be described by Theorem [2.1] and that has also been described further through results on higher
Nakayama algebra in [21]. It follows from these descriptions that $C$ is almost directed.

**Example 1.** Let $A$ be the Auslander algebra of linearly oriented type $A_3$

\[\begin{array}{c}
1 \\
2 \\
3
\end{array} \xrightarrow{2} \begin{array}{c}
3 \xrightarrow{5}
\end{array} \xrightarrow{4} \begin{array}{c}
4 \xrightarrow{5}
\end{array} \xrightarrow{5} \begin{array}{c}
5 \xrightarrow{6}
\end{array} \xrightarrow{6}
\]

and let $C$ be the cluster-tilting subcategory

\[\begin{array}{c}
1 \xrightarrow{2}
\end{array} \xrightarrow{3} \begin{array}{c}
2 \xrightarrow{4}
\end{array} \xrightarrow{5} \begin{array}{c}
3 \xrightarrow{4}
\end{array} \xrightarrow{4} \begin{array}{c}
4 \xrightarrow{5}
\end{array} \xrightarrow{5} \begin{array}{c}
5 \xrightarrow{6}
\end{array} \xrightarrow{6}
\]

It may be easily seen that $(A,C)$ is an almost-directed $d$-homological pair.

For algebras of linearly oriented type $A_n$, we may give a much more explicit description of their resonance diagrams.

**Example 2.** Let $A$ be a $d$-Auslander algebra of linearly oriented type $A_n$ with unique $d$-cluster-tilting subcategory $C \subseteq \text{mod}(A)$. Suppose that

\[\begin{array}{c}
\phi_M : 0 \rightarrow M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_{d+1} \rightarrow 0
\end{array} \]

\[\begin{array}{c}
\phi_N : 0 \rightarrow N_0 \rightarrow N_1 \rightarrow \cdots \rightarrow N_{d+1} \rightarrow 0
\end{array} \]

are $d$-exact sequences in $C$ such that $M_i = N_j$ for some $0 \leq i \leq j \leq d + 1$. Then there is a commutative diagram as in Figure 1 such that

- For each $0 \leq k \leq d + 1$ the sequence
  \[\begin{array}{c}
  \phi_k : 0 \rightarrow X_{0,k} \rightarrow X_{1,k} \rightarrow \cdots \rightarrow X_{k,k} \rightarrow X_{k+1,k+1} \rightarrow X_{k+1,k+2} \rightarrow \cdots \rightarrow X_{k+1,d+1} \rightarrow 0
  \end{array} \]
  is $d$-exact.
- Either $\phi_M = \phi_{i-1}$ and $\phi_N = \phi_j$ or $\phi_N = \phi_{i-1}$ and $\phi_M = \phi_i$.

**Remark 1.** Observe that for each $0 \leq k \leq d + 1$, by construction the sequence

\[\begin{array}{c}
\phi'_k : 0 \rightarrow X_{0,k} \rightarrow X_{1,k} \oplus X_{0,k+1} \rightarrow \cdots \rightarrow X_{k-1,d+1} \oplus X_{k,d} \rightarrow X_{k,d+1} \rightarrow 0
\end{array} \]

is $d$-exact. In fact this $d$-exact sequence is obtained via a $d$-pushout diagram of $\phi_{k-1}$ along the morphism $X_{0,k-1} \rightarrow X_{0,k}$. Likewise, it may also be obtained via a $d$-pullback diagram of $\phi_k$ along the morphism $X_{k,d+1} \rightarrow X_{k+1,d+1}$.

For an arbitrary resonance diagram we call $d$-exact sequences that arise from either a $d$-pushout diagram or a $d$-pullback diagram, or both, as *intermediate* $d$-exact sequences.
Figure 1. Resonance diagram for a $d$-Auslander algebra of linearly oriented type $A_n$

\[
\begin{array}{c}
X_{0,0} \\
\downarrow \\
X_{0,1} & \longrightarrow & X_{1,1} \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
X_{0,i-1} & \rightarrow & X_{1,i-1} \rightarrow \cdots \rightarrow X_{i-1,i-1} \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
X_{0,j} & \rightarrow & X_{1,j} \rightarrow \cdots \rightarrow X_{i,j} \rightarrow \cdots \rightarrow X_{j,j} \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
X_{0,d+1} & \rightarrow & X_{1,d+1} \rightarrow \cdots \rightarrow X_{i,d+1} \rightarrow \cdots \rightarrow X_{j+1,d+1} \rightarrow \cdots \rightarrow X_{d+1,d+1}
\end{array}
\]

4. Support $d$-tilting modules

The main objective of this paper is to understand support $d$-tilting theory. Recall that an $A$-module $T$ is a pre-$d$-tilting module \cite{10, 26} if:

- $\text{proj.dim}(T) \leq d$.
- $\text{Ext}^i_A(T, T) = 0$ for all $0 < i \leq d$.

Then $T$ is in addition $d$-tilting if there exists an exact sequence

$$0 \rightarrow A \rightarrow T_0 \rightarrow T_1 \rightarrow \cdots \rightarrow T_d \rightarrow 0$$

where $T_0, \ldots, T_d \in \text{add}(T)$. Dually, an $A$-module $C$ is a $d$-cotilting module if:

- $\text{inj.dim}(C) \leq d$.
- $\text{Ext}^i_A(C, C) = 0$ for all $0 < i \leq d$.
- There exists an exact sequence

$$0 \rightarrow C_d \rightarrow \cdots \rightarrow C_1 \rightarrow C_0 \rightarrow DA \rightarrow 0$$
such that $C_0, \ldots, C_d \in \text{add}(C)$.

The notion of support-$d$-tilting follows naturally, but we will be primarily interested in modules that are properly supported.

**Definition 3.** Let $(A, C)$ be a $d$-homological pair such that $\text{gl.dim}(A) \leq d$ and that $C$ is almost directed. Then an $A$-module $T \in C$ is support-$d$-tilting if

1. $T$ is $d$-tilting as an $A/\text{ann}(T)$-module.
2. $\text{ann}(T) = \langle e_T \rangle$,

where $e_T$ is the idempotent defined in equation (11). If $T$ is properly supported, then we say that $T$ is a proper support-$d$-tilting module. Dually an $A$-module $C \in C$ is proper support-$d$-cotilting if

- $C$ is $d$-cotilting as an $A/\text{ann}(C)$-module.
- $\text{ann}(C) = \langle e_C \rangle$,
- $C$ is properly supported.

then $C$ is a proper support-$d$-cotilting module.

**Lemma 4.1** (Happel [10], as found in Lemma 3.5 of [15]). Let $A$ be a finite-dimensional algebra satisfying $\text{gl.dim}(A) \leq d$. Let $T$ be a $d$-tilting $A$-module. Assume that $M \in \text{mod}(A)$ satisfies $\text{Ext}_A^i(T, M) = 0$ for all $i > 0$. Then there exists an exact sequence

$$0 \to T_d \to \cdots \to T_1 \to T_0 \to M \to 0$$

such that $T_j \in \text{add}(T)$ for all $0 \leq j \leq d$.

Consider a $d$-tilting $A$-module $T$. Since every injective module $I$ satisfies $\text{Ext}_A^i(T, I) = 0$ for all $i > 0$ we have the following corollary.

**Corollary 4.2.** Let $(A, C)$ be a $d$-homological pair such that $\text{gl.dim}(A) \leq d$. If $T$ is a support-$d$-tilting module, then every (proper) support-$d$-tilting module is also (proper) support-$d$-cotilting.

It is not true in general that every pre-$d$-tilting module is a summand of a $d$-tilting module - in other words Bongartz’ Lemma cannot be fully realised in this context. For a pre-$d$-tilting $A$-module $T$ such that there is no indecomposable $A$-module $M$ such that $T \oplus M$ is also pre-$d$-tilting, we say that $T$ is maximal pre-$d$-tilting. The following result is useful for determining support-$d$-tilting modules.

The notion of maximal support-pre-$d$-tilting can be defined similarly.

**Theorem 4.3.** [26, Theorem 1.19] For a finite-dimensional algebra $A$, if $M$ is a $d$-tilting module, then $|M| = |A|$, where $|M|$ is the number of indecomposable summands of $M$.

From this result, we would hope that an alternative definition of a support-$d$-tilting $A$-module to be a module $T$ such that $T$ is a tilting $A/\langle e_T \rangle$-module and $|T| = |A/\langle e_T \rangle|$.
5. Strong torsion classes

**Definition 4.** Let \((A,C)\) be a \(d\)-homological pair. Then a full subcategory \(T \subseteq C\) is a \(d\)-strong torsion class in \(C\) if

1. \((T1)\) For any \(T \in T\), \(M \in C\) and surjective morphism \(T \twoheadrightarrow M\), then \(M \in T\).
2. \((T2)\) For each \(d\)-exact sequence in \(C\)
   \[0 \rightarrow M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow \cdots \rightarrow M_d \rightarrow M_{d+1} \rightarrow 0\]
   such that \(M_{i-1}, M_{i+1} \in T\) for some \(1 \leq i \leq d\), then also \(M_i \in T\).

Dually, a full subcategory \(F \subseteq C\) is a \(d\)-strong torsion-free class in \(C\) if

1. \((C1)\) For any \(F \in F\), \(M \in C\) and injective morphism \(M \hookrightarrow F\), then \(M \in F\).
2. \((C2)\) For each \(d\)-exact sequence in \(C\)
   \[0 \rightarrow M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow \cdots \rightarrow M_d \rightarrow M_{d+1} \rightarrow 0\]
   such that \(M_{i-1}, M_{i+1} \in F\) for some \(1 \leq i \leq d\), then also \(M_i \in F\).

An alternative version of a torsion class for higher homological algebra was introduced in [23]. It would be interesting to find out how closely the two notions are related in general. We require two preliminary results.

**Lemma 5.1.** Let \((A,C)\) be an almost-directed \(d\)-homological pair such that \(\text{gl.dim}(A) \leq d\). Let \(T\) be a \(d\)-tilting \(A\)-module and suppose that

\[P_0 \rightarrow P_1 \rightarrow \cdots \rightarrow P_l\]

be an exact sequence such that \(P_i\) is an indecomposable projective for each \(0 \leq i \leq l\). If \(P_0, P_l \in \text{add}(T)\), then \(P_i \in \text{add}(T)\) for all \(0 \leq i \leq l\).

*Proof.* We may assume that for each \(1 \leq i \leq l - 1\), then \(P_i \notin \text{add}(T)\). Then, since \(T\) is \(d\)-tilting, for each \(1 \leq i \leq l - 1\) there exists a \(d\)-exact sequence

\[\phi_i : 0 \rightarrow P_i \rightarrow T_{0,i} \rightarrow T_{1,i} \rightarrow \cdots \rightarrow T_{d,i} \rightarrow 0\]

such that \(T_{j,i} \in \text{add}(T)\) for all \(0 \leq j \leq d\). As \(\text{Ext}^d_A(T, P_i) = 0\), we must have that \(P_{l-1} \rightarrow P_l\) factors through \(T_{0,l-1}\), as well as that \(\text{Ext}^d_A(T_{d,l-1}, P_{l-2}) \neq 0\). So we may assume that \(T_{d,l-1} \cong T_{d,l-2}\) and that the \(d\)-exact sequences \(\phi_i\) and \(\phi_{l-1}\) are contained in some resonance diagram. However, by taking an intermediate \(d\)-exact sequence we must have that \(\text{Ext}^d_A(T_{d-1,l-1}, P_{l-2}) \neq 0\). This further means that \(\text{Ext}^d_A(T_{d-1,l-1}, P_{l-3}) \neq 0\), in other words that \(T_{d,l-3} \cong T_{d-1,l-1}\). Continuing this argument further implies that \(P_0 \notin \text{add}(T)\), a contradiction. \(\square\)

**Lemma 5.2.** Let \((A,C)\) be an almost-directed \(d\)-homological pair such that \(\text{gl.dim}(A) \leq d\). Let \(T\) be a \(d\)-tilting \(A\)-module. Then for any left properly-supporting idempotent \(e\), the \(A/\langle e \rangle\)-module \(T \otimes_A A/\langle e \rangle\) is \(d\)-tilting. Dually for any right properly-supporting idempotent \(e\), the \(A/\langle e \rangle\)-module \(\text{Hom}_A(A/\langle e \rangle, T)\) is \(d\)-tilting.
Proof. Suppose that $e$ is left properly-supporting. Then, by assumption, any injective $A$-module is of the form $G(I)$ for some $I \in \text{add}(DA)$, where we use the notation $G : - \otimes_A A/\langle e \rangle$. For any injective $A$-module $I$ there is a $d$-exact sequence

$$\phi : 0 \to T_d \to \cdots \to T_0 \to I \to 0$$

such that $T_i \in \text{add}(T)$ for all $0 \leq i \leq d$. Suppose there exists a minimal $0 \leq j \leq d$ such that $T_j$ is not projective. Else apply the functor $G$ to $\phi$ and this gives a $d$-exact sequence by Proposition 2.2. Consider the morphism $T_j \to T_{j-1}$. Then there is a non-zero morphism between the projective cover of $T_j$ and $T_{j-1}$. This must factor through the projective cover of $T_{j-1}$ and form a commutative square of the form found in a resonance diagram. So complete this square to a resonance diagram between $\phi$ and the projective resolution of $T_j$:

$$0 \to P_d \to \cdots \to P_0 \to T_j \to 0.$$

This resonance diagram contains the projective resolutions of all $T_i$ for $0 \leq i \leq d$. The functor $G$ must be exact for any of these projective resolutions by Proposition 2.2, and thus in this case also $G(\phi)$ must be $d$-exact. Finally, since $e$ is properly supporting, we have that $\langle e \rangle$ is $(d+1)$-idempotent by Corollary 2.4, and hence $\text{Ext}^d_{A/\langle e \rangle}(G(T), G(T)) = 0$. So $G(T)$ is a $d$-tilting $A/\langle e \rangle$-module. Dually, if $e$ is right properly supporting then $\text{Hom}_A(A/\langle e \rangle, T)$ is a $d$-tilting $A/\langle e \rangle$-module. □

The following theorem generalises the classical behaviour of torsion classes. Specifically, we generalise Propositions VI.1.4 and VI.1.5 from the book [4].

**Theorem 5.3.** Let $(A, C)$ be an almost-directed $d$-homological pair such that $\text{gl.dim}(A) \leq d$. For any proper support-$d$-tilting $A$-module $T$, let $T := \text{Fac}(T) \cap C$. Then the following hold:

1. $T$ is a $d$-strong torsion class in $C$.
2. For all $M \in C$, there is a module $F_M \in \text{mod}(A)$ and an exact sequence

$$0 \to T_1 \to T_2 \to \cdots \to T_d \to M \to F_M \to 0$$

such that $T_i \in T$ for all $1 \leq i \leq d$ and

$$T = \{ T \in C | \text{Hom}_A(T, F_M) = 0 \ \forall M \in C \}.$$

**Proof.** To prove part (2), suppose that $T$ is a support-$d$-tilting algebra. For any indecomposable module $M \in C$, let $t(M)$ be the trace of $T$ in $M$; the sum of the images of all $A$-homomorphisms from modules in $T$ to $M$. So take $f : T' \to M$ to be the morphism from a module $T' \in \text{add}(T)$ to $M$ such that $\text{im}(f) = t(M)$.

By Lemma 4.1 every injective $A$-module is in $T$. So it follows by assumption (3) in Definition 2 that there exists a left properly-supporting idempotent $e$ such that
is projective as a $A/\langle e \rangle$-module. Let $T' := P_0$ and $G := - \otimes_A A/\langle e \rangle$. By Lemma 5.2, $G(T)$ is a $d$-tilting $A/\langle e \rangle$-module, and we may obtain an exact sequence

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M$$

such that $P_0, P_1, \ldots, P_n$ are projective as $A/\langle e \rangle$-modules. Let $M'$ be the preenvelope of $\text{coker}(P_1 \to P_0)$ in $\text{mod}(A/\langle e \rangle) \cap C$, so $M'$ is a submodule of $M$. We claim that there exists some $1 \leq i \leq d$ and a $d$-exact sequence

$$0 \to P_n \to \cdots \to P_0 \oplus N_{i-1} \to M' \oplus N_i \to \cdots \to N_{d+1} \to 0.$$ 

This is clear if $P_{n-1}$ is indecomposable. Otherwise there must be some decomposition $P_{n-1} = Q \oplus Q'$ such that cokernels of $P_n \to Q$ and $P_n \to Q'$ form a resonance diagram where there is a cokernel of $P_n \to P_{n-1}$ as an intermediate $d$-exact sequence. Therefore such a $d$-exact sequence as claimed exists. Since $M \notin T$, this means that $n < d$.

Now assume that $P_1$ is not indecomposable, and choose a resonance diagram so that there is a $d$-exact sequence

$$0 \to X_0 \to X_1 \to \cdots \to X_{d+1} \to 0$$

such that there is some $0 \leq j \leq d - 1$ where $X_j$ is an indecomposable summand of $P_1$ and $X_{j+1}$ is an indecomposable summand of $G(T)$. We also have that $X_1, \ldots, X_{j+1}$ are indecomposable projective $A/\langle e \rangle$-modules. It follows that the morphism $X_{j+1} \to M$ factors through $X_{j+1} \oplus Y$ for some $Y \in C$. Now assume that there is some $0 \leq k \leq j$ such that $X_k \notin \text{add}(G(T))$ and $X_{k+1} \in \text{add}(G(T))$. By Lemma 5.1 this means that $X_{k+1}, X_{k+2}, \ldots, X_{j+1} \in \text{add}(G(T))$. Moreover, in the $d$-exact sequence

$$\phi : 0 \to X_k \to T_0 \to \cdots \to T_d \to 0,$$

at least one of $Y \in \text{add}(G(T))$ or $X_{j+2} \in \text{add}(G(T))$. Since the functor $G$ is exact on $\phi$ (as in the proof of Lemma 5.2), this implies that $\text{Im}(P_0 \to M) \neq t(M)$, a contradiction.

So either $P_0, P_1, \ldots, P_n \in \text{add}(G(T))$ or they are all indecomposable projective $A/\langle e \rangle$-modules. As in the proof of Lemma 5.1 there is a resonance diagram containing the $\text{add}(G(T))$-coresolutions of each of these indecomposable projective modules. From this diagram, we may obtain a sequence

$$0 \to G(T_1) \to G(T_2) \to \cdots \to G(T_d) \to M \to M/t(M) \to 0.$$ 

Since $G(T_d) \cong T_d$, and $G(T_i)$ is a factor module of $T_i$ for all $1 \leq i \leq d$, this completes the proof.

Note that $\text{Hom}_A(T, F_M) = 0$ for any $T \in T$ and $M \in C$ by definition, and $\text{Hom}_A(M, F_M) = 0$ if any only if $F_M = 0$, which means $M \in T$. Part (1) follows from the exactness properties of $\text{Ext}_A^1(-, F_M)$ for any $M \in C$. \qed

For completeness’ sake, we state the dual result.
Theorem 5.4. Let \((A, C)\) be an almost-directed \(d\)-homological pair such that \(\text{gl.dim}(A) \leq d\). For any proper support-\(d\)-cotilting \(A\)-module \(F\), let 
\(\mathcal{F} := \text{Sub}(F) \cap C\). Then the following hold:

1. \(\mathcal{F}\) is a \(d\)-strong torsion-free class.
2. For all \(M \in C\), there is a module \(T_M \in \text{mod}(A)\) and an exact sequence
   \[0 \to T_M \to M \to F_0 \to F_1 \to \cdots \to F_d \to 0\]
   such that \(F_i \in \mathcal{F}\) for all \(1 \leq i \leq d\) and
   \[\mathcal{F} = \{F \in C | \text{Hom}_A(T_M, F) = 0 \forall M \in C\}\].

Theorem 5.3 does not provide a classification of strong \(d\)-torsion classes. However, we conjecture that the following is true:

Conjecture 5.5. Let \((A, C)\) be an almost-directed \(d\)-homological pair such that \(\text{gl.dim}(A) \leq d\). Then there is a bijection between \(d\)-strong torsion classes in \(C\) and maximal support \(d\)-tilting \(A\)-modules.

If \(T\) is a proper support-\(d\)-tilting module, then we call the \(d\)-strong torsion class \(T = \text{Fac}(T) \cap C\) a standard \(d\)-strong torsion class. Likewise, we call the \(d\)-strong torsion-free class \(F = \text{Sub}(T) \cap C\) a standard \(d\)-strong torsion-free class.

6. Wide subcategories

Let \((A, C)\) be a \(d\)-homological pair. As introduced in [11, Definition 2.11], an additive subcategory \(W \subseteq C\) is a wide subcategory if it satisfies the following conditions:

(W1) For each \(W_1, W_2 \in W\) and morphism \(W_1 \to W_2\) there exist \(d\)-exact sequences
   \[0 \to M_1 \to M_2 \to \cdots \to M_d \to W_1 \to W_2\]
   \[W_1 \to W_2 \to N_1 \to \cdots \to N_{d-1} \to N_d \to 0\]
   such that \(M_i, N_i \in W\) for all \(1 \leq i \leq d\).

(W2) For any \(W, W' \in W\), then any \(d\)-exact sequence
   \[0 \to W \to U_1 \to U_2 \to \cdots \to U_d \to W' \to 0\]
   is Yoneda equivalent to a \(d\)-exact sequence
   \[0 \to W \to W_1 \to W_2 \to \cdots \to W_d \to W' \to 0\]
   such that \(W_i \in W\) for all \(1 \leq i \leq d\).

We need to supplement this with the following definition.

Definition 5. Let \((A, C)\) be a \(d\)-homological pair. For two wide subcategories \(W_1 \subseteq C\) and \(W_2 \subseteq C\) such that \(W_1 \neq W_2\), we say \(W_1 \prec W_2\) if for all \(M \in W_1\) and \(N \in W_2\) such that \(M \not\cong N\), then \(\text{Hom}_A(N, M) = 0\).

A collection \((W_1, W_2, \ldots, W_n)\) of wide subcategories of \(C\) is a directed collection of wide subcategories in \(C\) if \(W_i \prec W_j\) for all \(1 \leq i < j \leq n\).
A directed collection of wide subcategories \((\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_n)\) is *resonant* if the union \(\mathcal{W}_1 \cup \mathcal{W}_2 \cup \cdots \cup \mathcal{W}_n\) has an additive generator \(M\) such that \(\text{Fac}(M) \cap \mathcal{C}\) is a standard \(d\)-strong torsion class.

Dually, a directed collection of wide subcategories \((\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_n)\) is *coresonant* if the union \(\mathcal{W}_1 \cup \mathcal{W}_2 \cup \cdots \cup \mathcal{W}_n\) has an additive generator \(M\) such that \(\text{Sub}(M) \cap \mathcal{C}\) is a standard \(d\)-strong torsion-free class.

Now fix a positive integer \(n\), and let \((A, \mathcal{C})\) be a \(d\)-homological pair such that \(A\) is a \(d\)-Auslander algebra of linearly oriented type \(A_n\). Observe that for any \(d\)-exact sequence in \(\mathcal{C}\),

\[
0 \to M_0 \to M_1 \to \cdots \to M_{d+1} \to 0
\]

and module \(N \in \mathcal{C}\), there is a \(d\)-exact sequence for every \(0 \leq i \leq d\)

\[
0 \to M_0 \to M_1 \to \cdots \to M_i \oplus N \to M_{i+1} \oplus N \to \cdots \to M_{d+1} \to 0.
\]

Unless otherwise stated, we will consider every \(d\)-exact sequence to have no such summand \(N\). The following result is implicit in the proof of [21, Theorem 2.18] using iterated application of Lemma 1.20 [21].

**Lemma 6.1.** Let \(A\) be a \(d\)-Auslander algebra of linearly oriented type \(A_n\) with unique \(d\)-cluster-tilting subcategory \(\mathcal{C}\). Then for any \(d\)-exact sequence in \(\mathcal{C}\) consisting of only indecomposable modules:

\[
0 \to X_0 \to X_1 \to \cdots \to X_{d+1} \to 0
\]

there exists a properly-supporting idempotent \(e\), such that for all \(0 \leq i \leq d\) each \(X_i\) is injective as an \(A/\langle e \rangle\)-module, and for all \(1 \leq i \leq d+1\) each \(X_i\) is projective as an \(A/\langle e \rangle\)-module.

**Definition 6.** Let \(A\) be a \(d\)-Auslander algebra of linearly oriented type \(A_n\) with unique \(d\)-cluster-tilting subcategory \(\mathcal{C}\), and let \(\mathcal{T} \subseteq \mathcal{C}\) be \(d\)-strong torsion class. Then define \(M \in \alpha(\mathcal{T})\) if for any submodule \(M'\) of \(M\) such that \(M' \in \mathcal{C}\) and indecomposable \(K_0 \in \mathcal{T}\) with surjective morphism \(f : K_0 \to M'\), then there exists a kernel of \(f\) in \(\mathcal{C}\):

\[
0 \to K_d \to \cdots \to K_2 \to K_1 \to K_0 \to M' \to 0
\]

such that \(K_i \in \mathcal{T}\) for all \(0 \leq i \leq d\).

**Lemma 6.2.** Let \(A\) be a \(d\)-Auslander algebra of linearly oriented type \(A_n\) with unique \(d\)-cluster-tilting subcategory \(\mathcal{C}\). Let \(T\) be a \(d\)-tilting \(A\)-module and \(\mathcal{T}\) the associated standard \(d\)-strong torsion class. Suppose that

\[
0 \to M_0 \to M_1 \to \cdots \to M_{d+1} \to 0
\]

is an exact sequence such that \(M_i \in \mathcal{C}\) is an indecomposable module for each \(0 \leq i \leq d+1\). If \(M_0, M_{d+1} \in \alpha(\mathcal{T})\), then \(M_i \in \alpha(\mathcal{T})\) for all \(0 \leq i \leq d+1\).
Proof. Observe that there is always some idempotent $e$ such that every projective $A/\langle e \rangle$-module is also projective as an $A$-module, as well as an idempotent $f$ such that every injective $A/\langle f \rangle$-module is also injective as an $A$-module. In the second case, for any standard $d$-strong torsion class $\mathcal{T}$ in $\mathcal{C}$, then $\mathcal{T} \cap \text{mod}(A/\langle f \rangle)$ is a standard $d$-strong torsion class in $\mathcal{C} \cap \text{mod}(A/\langle f \rangle)$.

By Lemma 6.1 there exists some properly-supporting idempotent $e$ where $M_1, \ldots, M_d$ are projective-injective $A/\langle e \rangle$-modules. By the above argument we may assume that each $M_i$ is projective as an $A$-module for $0 \leq i \leq d$. As $M_0, M_{d+1} \in \mathcal{T}$ and $\text{Ext}_A^d(M_{d+1}, M_0) \neq 0$ there must be some $X \in \text{add}(T)$ such that $M_{d+1} \not\cong X$ and there is a surjection $X \twoheadrightarrow M_{d+1}$. There is also a surjection $M_d \twoheadrightarrow M_{d+1}$. So there must either be a surjection $M_d \twoheadrightarrow X$ or $X \twoheadrightarrow M_d$. Now if there is a surjection $M_d \twoheadrightarrow X$, then we may form a resonance diagram and see that $\text{Ext}_A^d(X, M_0) \neq 0$, a contradiction. Therefore there must be a surjection $X \twoheadrightarrow M_d$ and hence $M_d \in \mathcal{T}$. Lemma 5.1 then implies that $M_i \in \mathcal{T}$ for each $0 \leq i \leq d+1$.

Now suppose there is a surjection from $T_i' \twoheadrightarrow M_i$ for some $0 < i < d+1$ and $T_j' \in \mathcal{T}$. Then we may form a resonance diagram such that there are surjections $T_j' \twoheadrightarrow M_j$ where $T_j' \in \mathcal{T}$ for all $0 \leq j \leq d+1$. Since both $M_0, M_{d+1} \in \alpha(\mathcal{T})$, applying the definition as well as Lemma 5.1 we must have that $M_i \in \alpha(\mathcal{T})$ for all $0 \leq i \leq d+1$. The proof is similar for a surjection from $T_i'$ onto a submodule of $M_i$. □

We are now ready to show our main result, a generalisation of Theorem 1.1.

**Theorem 6.3.** Let $A$ be a $d$-Auslander algebra of linearly oriented type $A_n$ with unique $d$-cluster-tilting subcategory $\mathcal{C}$. Then there are bijections between the following:

1. Proper support-$d$-tilting modules.
2. Standard $d$-strong torsion classes in $\mathcal{C}$.
3. Resonant collections of wide subcategories in $\mathcal{C}$.
4. Standard $d$-strong torsion-free classes in $\mathcal{C}$.
5. Coresonant collections of wide subcategories in $\mathcal{C}$.

**Proof.** \(1 \iff 2\) By Theorem 5.3, for any proper support-$d$-tilting module $T$, the class $\mathcal{T} := \text{Fac}(T) \cap \mathcal{C}$ is a standard $d$-strong torsion class in $\mathcal{C}$. Conversely, given a standard $d$-strong torsion class $\mathcal{T}$, consider the Ext-projectives in $\mathcal{T}$; that is the class of modules

$$\{M \in \mathcal{T} \mid \text{Ext}_A^d(M, \mathcal{T}) = 0\}.$$ 

It can be seen that $T$ is an additive generator of this class, and hence each standard $d$-strong torsion class determines a unique proper support-$d$-tilting module. 

\(2 \iff 3\)

For a proper support-$d$-tilting module, we wish to find a directed collection of wide subcategories that comprise $\alpha(T)$. For two indecomposable modules
\[ M, N \in \alpha(T), \text{ define } M \prec N \text{ if there exists a } d\text{-exact sequence in } C \]
\[ 0 \to X_0 \to X_1 \to \cdots \to M \to N \to \cdots \to X_{d+1} \to 0 \]
where at least one of \( X_0 \) and \( X_{d+1} \) is not in \( \alpha(T) \). This definition may be extended transitively.

Now divide \( \alpha(T) \) into classes such that any module \( M_1, M_2 \in \alpha(T) \) are in different classes whenever \( M_1 \prec M_2 \) and extend additively. Note that we allow there to exist modules \( M_1, M_2, N \in \alpha(T) \) such that \( M_1 \not\prec N \not\prec M_1, M_2 \not\prec N, N \not\prec M_1 \) and \( M_1 \prec M_2 \). In this case, we allow \( N \) to be in the same class as \( M_1 \) as well as the same class as \( M_2 \).

By construction, each of these classes is a wide subcategory of \( C \): by definition condition (W1) holds. In addition, Lemma 6.2 implies that condition (W2) holds: any sequence
\[ \phi : 0 \to M_0 \to M_1 \to \cdots \to M_{d+1} \to 0 \]
such that \( M_0, M_{d+1} \in \alpha(T) \) and \( M_i \not\in \alpha(T) \) must satisfy \( M_{d+1} \prec M_0 \). Assume that this is not true: there must be a summand \( N \) of \( M_2 \) such that there is a surjection \( N \to M_{d+1} \). There is some \( T_0 \in \text{add}(T) \) and there are surjections
\[ N \to T_0 \to M_{d+1}, \]
where we allow \( N \cong T_0 \) (but not \( M_{d+1} \cong T_0 \) as this would contradict the Ext-projectivity of \( T \) in \( T \)). Since \( M_{d+1} \in \alpha(T) \), there is a \( d \)-exact sequence
\[ 0 \to T_d \to \cdots \to T_1 \to T_0 \to M_{d+1} \to 0 \]
such that \( T_i \in \alpha(T) \) for all \( 0 \leq i \leq d \). This induces an exact sequence
\[ \text{Hom}_A(T_d, M_0) \to \text{Ext}^d_A(M_{d+1}, M_0) \to \text{Ext}^d_A(T_0, M_0) = 0. \]
Therefore \( \text{Hom}_A(T_d, M_0) \neq 0 \) and this morphism together with \( \phi \) induces a resonance diagram wherein \( M_{d+1} \prec M_0 \). We claim in addition that \( \prec \) is a total order: suppose there is a \( d \)-exact sequence
\[ \phi_N : 0 \to N_0 \to N_1 \to \cdots \to N_{d+1} \to 0 \]
such that \( N_i \in \alpha(T) \) for all \( 0 \leq i \leq d+1 \) apart from some \( 0 \leq l \leq d+1 \). Suppose further that there is a module \( M_j \in \alpha(T) \) such that \( M_j \prec N_i \) for all \( 1 \leq i \leq d+1 \): there is a \( d \)-exact sequence
\[ \phi_M : 0 \to M_0 \to M_1 \to \cdots \to M_{d+1} \to 0 \]
such that \( M_k \cong N_i \) for some \( k > j \). Now form a resonance diagram - by Lemma 6.1 there exists a properly-supporting idempotent \( e \) such that \( M_0, M_1, \ldots, M_d \) and \( N_0, N_1, \ldots, N_d \) are projective as \( A/\langle e \rangle \)-modules and \( M_1, M_2, \ldots, M_{d+1} \) as well as \( N_1, N_2, \ldots, N_{d+1} \) are projective as \( A/\langle e \rangle \)-modules. Also Lemma 5.2 implies that there is some functor \( F \) such that \( F(T) \) is a tilting \( A/\langle e \rangle \)-module. Assume that
$M_j$ is projective-injective as an $A/\langle e \rangle$-module and that there is a sequence in the chosen resonance diagram

$$\phi_X: 0 \to X_0 \to \cdots \to M_j \to \cdots \to N_i \to \cdots \to X_{d+1} \to 0$$

for some $0 < l < i$. We have that $M_0$ is a projective $A/\langle e \rangle$-module, and so there must be a $F(T)$-coresolution of $X_0$. If $X_0 \not\in \text{add}(F(T))$, then the $F(T)$-coresolution must factor through $M_j$ - a contradiction. We still have to check in case modules that are in $\text{add}(F(T))$ might not be in $\alpha(T)$. So assume that $M_0 \in \alpha(T)$ and that every projective $A/\langle e \rangle$-module is projective as an $A$-module. The module $M_1$ is projective as an $A$-module. Furthermore there is a morphism $M_1 \to N_1$. If $M_1 \not\in \text{add}(T)$ then there is some $T' \in \text{add}(T)$ such that $\text{Ext}_A^d(T', M_1) \neq 0$. Within the resonance diagram containing $\phi_M$ and $\phi_N$, there is a further $d$-exact sequence

$$\phi_L: 0 \to L_0 \to L_1 \to \cdots \to L_{d+1} \to 0$$

and exact sequences $L_i \to M_i \to N_i$ for $0 \leq i \leq k - 2$. We must have isomorphisms $\text{Ext}_A^d(T', L_1) \cong \text{Ext}_A^d(T', M_1) \cong \text{Ext}_A^d(T', M_2)$. Since $M_0, N_1 \in \text{add}(T)$, this means the $d$-exact sequence from $L_1$ to $T'$ factors through $N_1$. Then either the terms in a cokernel of the induced morphism $X \to N_2$ (contained in the resonance diagram) are included in the $\text{add}(T)$ coresolution of $N_2$, or there is another $T'' \in \text{add}(T)$ such that $\text{Ext}_A^d(T'', M_2) \neq 0$. Apply this argument along $\phi_M$ - at some point $M_k \cong N_i \in \alpha(T)$. Therefore $\text{Ext}_A^d(T, M_0) \neq 0$, as the terms in the cokernel of $L_{k-2} \to M_{k-1}$ must be contained in the $\text{add}(T)$-coresolution of $M_{k-1}$. This is a contradiction.

Conversely assume that every injective $A/\langle e \rangle$-module is injective as an $A$-module. Form a resonance diagram between $\phi_N$ and the projective covers of $N_i$ $(0 \leq i \leq d + 1)$, which are also injective modules for all $1 \leq i \leq d + 1$ and hence in $T$. This induces a $d$-exact sequence in $\alpha(T)$:

$$\phi'_N: 0 \to \Omega^d(N_1) \to \cdots \to \Omega^d(N_{d+1}) \to N_0 \to 0.$$ 

Form a resonance diagram with $\phi'_N$ and the morphism $M_0 \to N_0$ and apply the previous argument. This implies that $N_i \in \alpha(T)$ for all $0 \leq i \leq d + 1$.

Now let $T'$ be the minimal summand of $T$ such that $\text{Fac}(T') = \text{Fac}(T)$. By definition, $T' \in \alpha(T)$. Now let $(\mathcal{W}_1, \ldots, \mathcal{W}_n)$ be the directed collection of wide subcategories defined above. Let $M$ be an additive generator of $\mathcal{W}_1 \cup \cdots \cup \mathcal{W}_n$ - $M$ is in fact an additive generator of $\alpha(T)$. Therefore $\text{Fac}(M) = \text{Fac}(T')$, and $(\mathcal{W}_1, \ldots, \mathcal{W}_n)$ is a resonant collection of wide subcategories.

The bijections $\{1\} \leftrightarrow \{4\}$ and $\{1\} \leftrightarrow \{5\}$ are dual. \hfill $\square$
7. Examples and Further Directions

Example 3. As in Example 1 let $A$ be the algebra

\[
\begin{array}{ccc}
1 & 2 & 3 \\
\downarrow & \downarrow & \downarrow \\
4 & 5 & 6
\end{array}
\]

The support-2-tilting modules form a lattice under inclusion of standard $d$-strong torsion classes, as depicted in Figure 2. The module $T = P_1 \oplus P_2 \oplus P_3 \oplus P_4 \oplus P_5 \oplus S_4$ is a proper support-2-tilting $A$-module, and its associated 2-strong torsion class consists of

\[
\left\langle \frac{1}{2} \oplus \frac{1}{2} \oplus 1 \oplus 3 \oplus 2 \oplus 4 \oplus 5 \oplus 3 \oplus 4 \oplus 5 \oplus 6 \right\rangle.
\]

The resonant collection of wide subcategories of $C$ consists of

\[
W_1 = \left\langle \frac{1}{2} \oplus \frac{1}{2} \oplus 5 \oplus 6 \right\rangle.
\]

\[
W_2 = \left\langle 1 \oplus 2 \oplus 3 \oplus 4 \oplus 5 \right\rangle.
\]

In fact, this collection of wide subcategories is also coresonant, under the support-2-cotilting module

\[
C = 2 \oplus 1 \oplus 1 \oplus 3 \oplus 2 \oplus 4 \oplus 5 \oplus 6
\]

Example 4. For example, take $A$ to be the algebra with quiver

\[
1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7
\]

and with relations given by all paths of length 3. This is an example from a class of $d$-representation-finite algebras was introduced in [30, Section 5]. Furthermore, the wide subcategories of these algebras have been classified in [11], see also [9] for further study of wide subcategories of such algebras. Then

\[
C = \left\langle P_1 \oplus P_2 \oplus P_3 \oplus P_4 \oplus P_5 \oplus P_6 \oplus P_7 \oplus I_1 \oplus I_2 \right\rangle
\]

is a 2-cluster-tilting subcategory of $\text{mod}(A)$. There are five proper support-2-tilting $A$-modules: $S_1$, $S_7$, $A$ and the following two modules

\[
T_1 := P_1 \oplus P_2 \oplus P_3 \oplus P_4 \oplus P_5 \oplus P_6 \oplus I_2;
\]

\[
T_2 := P_1 \oplus P_2 \oplus P_3 \oplus P_4 \oplus P_5 \oplus I_1 \oplus I_2.
\]
A similar process to Theorem 6.3 shows that $T_1$ determines a resonating collection of wide subcategories: $(P_6, P_5, P_4, P_3, P_2, P_1)$ and $T_2$ determines the resonating collection of wide subcategories $(P_5, P_4, P_3, P_2, P_1)$.

Observe that for the algebra $A$ in Example 4, any wide subcategory that is contained in a resonating collection of wide subcategories generated by either the whole of $\mathcal{C}$, or by precisely one module. The classification of wide subcategories for this algebra that given in [11] tells us that not all wide subcategories are of such a form. Restricting our focus, we ask the following question:

**Question 1.** Let $(A, \mathcal{C})$ be an almost-directed $d$-homological pair. Can all wide subcategories be obtained from a resonant collection of wide subcategories associated with a proper support-$d$-tilting module?

Another question we may ask is the following:

**Question 2.** Let $(A, \mathcal{C})$ be an almost-directed $d$-homological pair such that $\text{gl.dim}(A) \leq d$. Is it possible to classify the directed collections of wide subcategories that are both resonant and coresonant?

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Figure 2. The support-2-tilting lattice for the 2-Auslander algebra of linearly-oriented type $A_3$.

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