Null weak values in multi-level systems

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Received 16 April 2012
Accepted for publication 27 June 2012
Published 30 November 2012
Online at stacks.iop.org/PhysScr/T151/014014

Abstract

A two-step measurement protocol of a quantum system, known as the weak value (WV), was introduced more than two decades ago by Aharonov et al (1988 Phys. Rev. Lett. 60 1351) and has since then been studied in various contexts. In this paper, we discuss another two-step measurement protocol that we dub the null weak value (NWV). The protocol consists of a partial-collapse measurement followed by quantum manipulation on the system and finally a strong measurement. The first step is a strong measurement which takes place with a small probability. The second strong measurement is used as a postselection on the outcome of the earlier step. Not being measured in the partial-collapse stage (null outcome) leads to an amplified signal. The NVW protocol, first defined for a two-level system (Zilberberg et al 2012 arXiv:1205.3877), is then generalized to a multi-level system and compared with the standard-WV protocol.

PACS numbers: 73.21.La, 03.65.Ta, 76.30.v, 85.35.Gv, 05.60.–k

((Some figures may appear in color only in the online journal)

1. Introduction

A measurement in quantum mechanics is a probabilistic process which, in the simplest situation, is described by the projection postulate [1]. Conditional quantum measurements, however, can lead to results that cannot be interpreted in terms of classical probabilities, due to the quantum correlations between measurements. An intriguing example of the correlated quantum measurements outcome is the so-called weak values (WVs). It is the outcome of a measurement scheme originally developed by Aharonov et al [2]. The WV measurement protocol consists of (i) initializing the system in a certain state $|i\rangle$ (preselection), (ii) weakly measuring the observable $\hat{A}$ of the system and (iii) retaining the detector output only if the system is eventually measured to be in a chosen final state $|f\rangle$ (postselection). The average signal monitored by the detector will then be proportional to the real (or imaginary) part of the complex WV, $\langle f | \hat{A} | i \rangle = \langle f | \hat{A} | i \rangle / \langle f | i \rangle$, rather than to the standard average value, $\langle f | \hat{A} | i \rangle$. Further discussion of the context in which WV should be understood has been provided [3–5].

Going beyond the peculiarities of WV protocols, a recent series of works explored the potential of WVs in quantum optics [6–12] and solid-state physics [13–16], ranging from experimental observation to their application in hypersensitive measurements. In the latter, a measurement carried out by a detector entangled with a system whose states can be preselected and postselected leads to an amplified signal in the detector that enables sensing of small quantities [8–12, 16]. Quite generally, within a WV-amplification protocol, only a subset of the detector’s readings, associated with the tail of the signal’s distribution, is accounted for. Notwithstanding the scarcity of data points, the large value of $\langle f | \hat{A} | i \rangle$ leads to an amplification [10, 16] of the signal-to-external-noise ratio $\text{SNR}_e$ for systems where the noise is dominated by an external (technical) component.

The amplification originating from WV protocols is nonuniversal. The specifics of such an amplification are diverse and system dependent. In fact, for statistical (inherent) noise, the SNR amplification resulting from large WVs is generally suppressed due to a reduction in the statistics of the collected data: postselection restricts us to a small subset of the readings at the detector. The upside of the WV procedure has several facets: if we try to enhance the statistics by increasing the intensity of the input signal through the system (e.g. the intensity of the impinging photon beam), possibly entering a nonlinear response regime, postselection
will effectively reduce this intensity back to a level accessible to the detector sensitivity [9]. Alternatively, amplification may originate from the imaginary component of the WV [8] or from the different effect of the noise and the measured variable on the detector’s signal [16]. However, as long as quantum fluctuations (leading to inherent statistical noise) dominate, the large WV is outweighed by the scarcity of data points, failing to amplify the signal-to-statistical-noise ratio [16, 17].

We recently presented an alternative measurement protocol dubbed the null weak value (NWV), which leads to high-fidelity discrimination between qubit states on the background of quantum fluctuations [18]. The main features of this protocol are: (i) it is distinctly different from the standard WV in making use of a partial-collapse measurement (a strong measurement that occurs with a small probability realized by finely tuned, time-resolved detector), in which the system experiences weak backaction only for a subset of all possible measurement outcomes (see, e.g., [19, 20]); (ii) unlike standard WV-amplification procedures [8–12, 16], where one needs to employ two degrees of freedom (related, respectively, to the ‘system’ and ‘detector’) that are entangled by the weak measurement, in NWVs there is a single quantum degree of freedom (‘system’) since the detector is classical; (iii) the SNR amplification is versus inherent quantum and statistical fluctuations, and not only against external detector noise.

Here we present a general formalism for NWVs. After reviewing briefly the derivation of standard-WV (section 2), in section 3 we generalize the definition of an NWV to a multi-level system. The main results are summarized in section 4.

2. Weak values

WVs describe the outcome read in a detector when the measured system is subsequently found to be in a specific state. The weak coupling between a system and a detector is performed by an ideal von Neumann measurement [1], described by the Hamiltonian

$$H = H_S + H_D + H_{int}, \quad H_{int} = \lambda g(t) \hat{P} \hat{A},$$

(1)

where $H_{S(D)}$ is the Hamiltonian of the system (detector), and $H_{int}$ is the interaction Hamiltonian between the two. Here $\hat{P}$ is the momentum canonically conjugate to the position of the detector’s pointer, $\hat{A}$, and $\lambda g(t)$ ($\lambda \ll 1$) is a time-dependent coupling constant. $\hat{A} = \sum_i a_i |a_i\rangle \langle a_i|$ is the measured observable. We assume, for simplicity, that the free Hamiltonians of the system and the detector vanish and that $g(t) = \hat{A}(t - t_0)$.

The system is initially prepared in the state $|i\rangle$ and the detector in the state $|\phi_0\rangle$. The latter is assumed to be a Gaussian wave packet centered at $q = q_0$, $|\phi_0\rangle = C e^{-i(q - q_0)^2/4\Delta^2}$. After the interaction with the detector, the entangled state of the two is

$$|\psi\rangle = e^{-i\hat{P} \hat{A}} |i\rangle |\phi_0\rangle.$$  

(2)

In a regular measurement the signal in the detector, i.e. the pointer’s position $\langle q \rangle = q_0 + \lambda \langle \hat{A} \rangle$, is read. From the classical signal, $\langle q \rangle$, one can infer the average value of the observable $\hat{A}$.

In a WV protocol the signal in the detector is kept provided that the system is successfully postselected to be in a state $|f\rangle$. Hence, the detector ends up in the state

$$|\psi\rangle = |\phi_0\rangle - i\lambda [\langle f | \hat{A} |i\rangle]/\langle f |i\rangle \hat{P} |\phi_0\rangle \approx e^{-i\lambda/\langle f |i\rangle} \hat{P} |\phi_0\rangle,$$

(3)

which corresponds to a shift in the position of the pointer proportional to $\text{Re} \langle f | \hat{A} |i\rangle$. Hence, the expectation value of the coordinate of the pointer is given by

$$\langle f | \hat{A} |i\rangle = \langle f |i\rangle \lambda \text{Re} \langle f | \Lambda |i\rangle \equiv \text{NWV}(A),$$

(5)

Here, too, the conditional average value of $\hat{A}$ is inferred from the detector’s reading.

We note that the approximation in equation (3) is valid when $\Delta \gg \max_i \langle |a_i - a_i| \rangle$. This means that the initial detector’s wave function and the shifted one due to the weak interaction with the system are strongly overlapping. In turn, this means that for any outcome of the detector the state of the system is weakly changed. This corresponds to a weak measurement. As long as the measurement of the observable, $\hat{A}$, is weak, the WV is a general result and does not depend on the details of the coupling or the specific choice of the detector.

3. NWVs in multi-level systems

A different measurement protocol based on a postselected readout has recently been proposed as an efficient tool for qubit state discrimination [18]. The NWV protocol consists, like the standard-WV protocol, of a two-step procedure. Consider first a two-level system. It is initially prepared in a state $|i\rangle$. The first measurement, $M_{\text{post}}$, is a strong (projective) measurement which is carried out on the system with a small probability $p$. Specifically, during the time of this measurement we allow the system to decay, thus inducing a signal in the detector. The detector then ‘clicks’ (the measurement outcome is positive) and the qubit system is destroyed. Very importantly, having a ‘null outcome’ (no click) still results in a weak backaction on the system. We refer to this stage of the measurement process as the ‘weak partial collapse’. Subsequently, the qubit state is (strongly) measured a second time (postselected), $M_\text{fin}$, to be in the state $|f\rangle$ (click), $|\bar{f}\rangle$ (no click). The conditional probability of a ‘click’ in the weak partial-collapse measurement conditional to ‘no click’ in the postselection, $P(M_{\text{fin}}|M_{\text{post}})$, defines the NWV with $\text{M}_{\text{fin}}$ and $\text{M}_{\text{post}}$ representing ‘click’ and ‘no click’, respectively. Events in which the qubit is measured strongly (in the second measurement), $M_\text{fin}$, are discarded.

This protocol substantially differs from the standard-WV in the employment of the weak partial-collapse measurement. As long as a single measurement is concerned, the partial-collapse measurement will give the same results as a standard von Neumann measurement. However, in the NWV
observable measurement would result in the measurement of the signal. Hence, the average outcome of the partial-collapse states (the ‘detector’) through a tunnel barrier. Tunneling is collapse measurement. One can think of a partial-collapse states, allows us to discriminate between two possible initial qubit cases and comparing the respective conditional outcomes, the system is destroyed; hence there are no clicks upon further evolution, into $|\tilde{\psi}\rangle$. For example, if the tunneling probability of a specific state, probabilities and is therefore not a pure observable of the multi-level system. For the sake of being pedagogical, let define, for the sake of brevity, the operator of the system.

We present here a general formalism for NWVs of a protocol, involving a postselection leads to results which are qualitatively different as compared with the application of a standard WV protocol. In fact, the NWV protocol is comprised of two consecutive measurements subsequent to the tunneling (measurement of the ‘system’ by the ‘detector’) the system undergoes a general unitary evolution, $\hat{U}(\tau)$. Finally, the state of the particle is (strongly) measured a second time (postselected to be, for example, in state $|n\rangle$, $M_{s} = |n\rangle\langle n|$). One may now define an expectation value of the operator $\hat{A}$, conditioned on the reading of the second measurement being negative ($M_{s}$),

\begin{equation}
\hat{A} \equiv \frac{\hat{M}_{w}}{f(p)}, \quad (7)
\end{equation}

where $f(p)$ is a calibration function that translates the probabilistic outcome of the partial-collapse procedure (characterized by $p = p_{0}, \ldots, p_{n-1}$) to an observable of the system.

In order to include postselection, we assume that subsequent to the tunneling (measurement of the ‘system’ by the ‘detector’) the system undergoes a general unitary evolution, $\hat{U}(\tau)$. Finally, the state of the particle is (strongly) measured a second time (postselected to be, for example, in state $|n\rangle$, $M_{s} = |n\rangle\langle n|$). One may now define an expectation value of the operator $\hat{A}$, conditioned on the reading of the second measurement being negative ($M_{s}$),

\begin{equation}
\hat{A} \equiv \frac{\hat{M}_{w}}{f(p)}, \quad (8)
\end{equation}

where $P\{M_{w}|\hat{M}_{s}\}$ is the conditional probability of having a click in the first measurement conditional to not having a click the second time. This is the NWV. Note that, similarly to a standard-WV [2, 14], $\hat{A}_{i}\langle \hat{A}\rangle_{i}$ may give rise to large values, beyond the interval $[0,1]$.

To shed some light on this expression, we calculate explicitly the conditional probabilities following the measurement procedure sketched in figure 1. As a first step, we note that when the detector clicks during the first measurement (positive outcome of the detector) the state of the system is destroyed. Very importantly, having a ‘null outcome’ (no click) still results in a weak backaction (partial collapse) on the system. In such a case the state can be written as

\begin{equation}
|\tilde{\psi}\rangle = \frac{1}{\sqrt{P(M_{w})}} \sum_{m=0}^{n} \alpha_{m} \sqrt{1 - p_{m}} e^{i\phi_{m}} |m\rangle, \quad (9)
\end{equation}

where $P(M_{w}) = 1 - P(\tilde{\psi}) = \sum_{m=1}^{n} p_{m} |\alpha_{m}|^{2}$ is the probability of the state to be positively measured. We include general relative phases that such a coupling could induce in the form of $\phi_{m}$.

Rewriting the conditional probability on the right-hand side of equation (8) using the Bayes theorem, we obtain

\begin{equation}
P(M_{w}|\hat{M}_{s}) = \frac{P(M_{w})P(M_{s}|M_{w})}{P(M_{w})P(M_{s}|M_{w}) + P(M_{w})P(M_{s}|M_{w})}. \quad (10)
\end{equation}

The conditional probability $P(M_{w}|M_{s}) = 1$ represents classical correlation between the two measurements: namely, given that the detector has clicked in the first measurement (the particle has tunneled out to the detector), the result of the second measurement will be negative with probability 1. By contrast, $P(M_{w}|M_{s})$ embeds nontrivial quantum correlations. If no tunneling takes place (with probability $P(M_{w}|\hat{M}_{s})$) the state evolves into $|\tilde{\psi}\rangle$ of equation (9).
During the time $\tau$ between the two measurements, the state is evolved in a controlled fashion to $|\tilde{\psi}_p\rangle = \hat{U}(\tau)|\psi_p\rangle$. If the state was measured in the first measurement, this evolution is truncated. The second strong measurement $M_s$ gives a positive outcome (‘click’ of the detector) with probability $P(M_s|\tilde{\psi}_w) = |\langle \tilde{m} | \tilde{\psi}_w \rangle|^2$ and a negative one with $P(\tilde{M}_I|\tilde{\psi}_w) = \sum_{m \neq \tilde{m}} |\langle m | \tilde{\psi}_w \rangle|^2$. Collecting this into equation (10), we obtain

$$P(M_s|\tilde{\psi}_w) = \frac{P(M_s)}{P(M_s) + P(\tilde{M}_I|\tilde{\psi}_w)},$$

and after calibration the NWV of equation (8) can be written as

$$\hat{n}_t(\hat{A}) = \frac{\langle \hat{A} \rangle}{P(M_s)}.$$  \hspace{1cm} (12)

An interesting case is that of two-level systems, where only two tunneling probabilities exist and the NWV protocol uniquely defines the measured observable $\hat{n}_1 = |1\rangle \langle 1|$ = $(\tilde{\psi}_w - p_0)/(p_1 - p_0)$, i.e. the population of the state $|1\rangle$. The postselection is arbitrarily chosen to consist of a projection on the state $|1\rangle$. Taking a weak partial-collapse limit $p_0, p_1 \ll 1$ such that $P(M_s) \ll P(\tilde{M}_I|\tilde{\psi}_w)$, equation (12) yields

$$f(\hat{n}_1) \approx \frac{\langle \langle \hat{n}_1 \rangle \rangle}{|\langle f | i \rangle|^2} \equiv \text{NWV}(\hat{n}_1),$$

where we have defined the final state to include a general rotation between the partial-collapse measurement and the postselection, $|f\rangle = \hat{U}^\dagger(\tau)|0\rangle$. The expression for the NWV result in equation (13) is indeed different from the standard-WV expression (see equation (5)). This is a manifestation of the partial-collapse measurement, to be contrasted with a von Neumann-type [1] weak measurement underlying the latter. The differences between the two values are shown in figure 2. It appears that the NWV diverges with a wider envelope than the standard-WV as the pre- and postselection states become more orthogonal.

4. Conclusions

We have presented here a novel measurement protocol applicable to a general system spanning an $n$-dimensional Hilbert space. Similar to standard WV's, the outcome of this protocol—NWV—is the result of a first (weaker) measurement correlated with a strong postselection. Ostensibly, as long as a single measurement is concerned, the first measurement in both protocols yields the same outcome. However, the substantial difference between the standard and null weak values shows that its backaction on the system is profoundly different. Hence, involving a postselection leads to qualitatively different correlated results.

Acknowledgments

This work was supported by the German Israeli Foundation (GIF), the Israeli Science Foundation (ISF), the Minerva Foundation, an Israel–Korea MOST grant and the EU GEOMDISS.

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