A Sublinear Algorithm of Sparse Fourier Transform for Nonequispaced Data

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Abstract

We present a sublinear randomized algorithm to compute a sparse Fourier transform for nonequispaced data. Suppose a signal $S$ is known to consist of $N$ equispaced samples, of which only $L < N$ are available. If the ratio $p = L/N$ is not close to 1, the available data are typically non-equispaced samples. Then our algorithm reconstructs a near-optimal $B$-term representation $R$ with high probability $1 - \delta$, in time and space $\text{poly}(B, \log(L), \log(1/\delta), \epsilon^{-1})$, such that $\|S - R\|_2 \leq (1 + \epsilon)\|S - R_{\text{opt}}^B\|_2$, where $R_{\text{opt}}^B$ is the optimal $B$-term Fourier representation of signal $S$. The sublinear $\text{poly}(\log(L))$ time is compared to the superlinear $O(N \log N + L)$ time requirement of the present best known Inverse Nonequispaced Fast Fourier Transform (INFFT) algorithms. Numerical experiments support the advantage in speed of our algorithm over other methods for sparse signals: it already outperforms INFFT for large but realistic size $N$ and works well even in the situation of a large percentage of missing data and in the presence of noise.

1 Introduction

We consider the problem in which the recovery of a discrete time signal $S$ of length $N$ is sought when only $L$ signal values are known. In general, this is of course an insoluble problem; we consider it here under the additional assumption that the signal has a sparse Fourier transform. Let us fix the notations: the signal is denoted by $S = (S(t))_{t=0, \ldots, N-1}$, but we have at our disposal only the $(S(i))_{i \in T}$, where the set $T$ is a subset of $\{0, \ldots, N-1\}$ and $|T| = L$. The Fourier transform of signal $S$ is $\hat{S} = (\hat{S}(0), \ldots, \hat{S}(N-1))$, defined by $\hat{S}(\omega) = \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} S(t) e^{-2\pi i \omega t/N}$. In terms of the Fourier basis functions $\phi_\omega(t) = \frac{1}{\sqrt{N}} e^{2\pi i \omega t/N}$, $S$ can be written as $S = \sum_{\omega=0}^{N-1} \hat{S}(\omega) \phi_\omega(t)$; this is the (discrete) Fourier representation of $S$. A signal $S$ is said to have a $B$-sparse Fourier representation, if there exists a subset $\Omega \subset \{0, \ldots, N-1\}$ with $|\Omega| = B$, and values $c(\omega) \neq 0$ for $\omega \in \Gamma$, such that $S(t) = \sum_{\omega \in \Omega} c(\omega) \phi_\omega$. For a signal that does not have a $B$-sparse Fourier representation, we denote by $R_{\text{opt}}^B(S)$ the optimal $B$-term Sparse Fourier representation of $S$.

∗This work was partially supported by NSF grant DMS-03168875 and AFOSR grant 109-6047.
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This paper presents a sublinear algorithm to recover a $B$-sparse Fourier representation of a signal $S$ from incomplete data. Our algorithm also extends to the case where the Fourier transform $\hat{S}$ is not $B$-sparse, where we aim to find a near-optimal $B$-term Fourier representation, i.e. $R = \sum_{\omega \in \Gamma} c(\omega) \phi_\omega$, such that
\[
\|S - R\| = \|S - \sum_{\omega \in \Gamma} c(\omega) \phi_\omega\| \leq (1 + \epsilon)\|S - R^B_{opt}(S)\|.
\] (1)

A typical situation where our study applies is the observation of non-equispaced data, where the samples are nevertheless all elements of $\tau \mathbb{Z}$ for some $\tau > 0$. For a signal with evenly spaced data, the famous Fast Fourier Transform (FFT) computes all the Fourier coefficients in time $O(N \log N)$. However, the requirement of equally distributed data by FFT raises challenges for many important applications. For instance, because of the occurrence of instrumental drop-outs, the data may be available only on a set of non-consecutive integers. Another example occurs in astronomy, where the observers cannot completely control the availability of observational data: a telescope can only see the universe on nights when skies are not cloudy. In fact, computing the Fourier representation from irregularly spaced data has wide applications [19] in processing astrophysical and seismic data, the spectral method on adaptive grids, the tracking of Lagrangian particles, and the implementation of semi-Lagrangian methods.

In many of these applications, a few large Fourier coefficients already capture the major time-invariant wave-like information of the signal, and we can thus ignore very small Fourier coefficients. To find a small set of the largest Fourier coefficients and hence a (near) optimal $B$-sparse Fourier representation of a signal that describes most of the signal characteristics is a fundamental task in applied Fourier Analysis.

An equivalent version of this problem is as follows: define the matrix $A := (e^{2\pi i k t_j})_{k=0,\ldots,N; j=0,\ldots,L-1}$, where the $t_j$ are the locations of the available samples. Given $S(t_j)$, we want to reconstruct the signal $S$, or equivalently, its Fourier coefficients $\hat{S}_k$, so that $A\hat{S} = S$. This linear system is over-determined. Several algorithms [2] [11] [12] have provided efficient approaches to solve this problem. Among all NFFT algorithms, the iterative CGNE approach of [6] in the benchmark software NFFT 2.0 is one of the fastest methods; it takes time $O(L^{1+(d-1)/\beta} \log L)$, where $L$ is the number of available points, $d$ is the number of dimensions, and $\beta > 1$ is the smoothness for the original signal. The super-linearity relationship between the running time and $N$ (recall $L = pN$, where $p$ is the percentage of available data) poses difficulties in processing large dimensional signals, which have nothing to do with the unequal spacing. It follows that identifying a sparse number of significant modes and amplitudes is expensive for even fairly modest $N$. Our goal in this paper is to discuss much faster (sublinear) algorithms that can identify the sparse representation or approximation with coefficients $a_1, \ldots, a_B$ and modes $\omega_1, \ldots, \omega_B$ for unevenly spaced data. These algorithms will not use all the samples $S(0), \ldots, S(N-1)$, but only a very sparse subset of them.

Our approach is based on the paper [8] that shows how to construct the Fourier representation for a signal $S$ with $B$-sparse Fourier representation in time and space $\text{poly}(B, \log N, 1/\epsilon, \log(1/\delta))$ on equal spacing data. The algorithm contains some random elements (which do not depend on the signal); their approach guarantees that the error of estimation is of order $\epsilon \|S\|^2$ with probability exceeding $1 - \delta$. The ideas in [8] have also been applied by its authors to sparse wavelet,
wavelet packet representation, and histograms [7]. We have dubbed the whole family of algorithms RA/STA (for Randomized Algorithm for Sparse Transform Approximation); when dealing only with Fourier Transforms, as is the case here, we specialize it to RA/SFA (F for Fourier). Zou, Gilbert, Strauss and Daubechies [20] improved and implemented the algorithm greatly. It convincingly beats FFT when the number of grid points \( N \) is reasonably large. The crossover point lies at \( N \approx 25,000 \) in one dimension, and at \( N \approx 460 \) for data on a \( N \times N \) grid in two dimensions for a two-mode signal. When \( B = 13 \), RA/SFA surpasses FFT at \( N \geq 300,000 \) for one dimensional signals and 1100 for two dimensional signals.

In this paper, we modify RA/SFA to solve the irregularly spaced data problem. The new NERA/SFA (Nonequispaced RA/SFA) uses sublinear time and space \( \text{poly}(B, \log L, \epsilon, \log(1/\delta), \log p) \) to find a near-optimal \( B \)-term Fourier representation, such that \( \|S - R\|_2 \leq (1 + \epsilon) \|S - R_{\text{opt}}\|_2 \) with high probability \( 1 - \delta \). Similar to the RA/SFA algorithm, it outperforms existing INFFT algorithms in processing sparse signals of large size.

**Notation and Terminology** Denote by \( \chi_T \) a signal that equals 1 on a set \( T \) and zero elsewhere in the time domain. We say a signal \( H \) is \( q \) percent pure, if there exists a frequency \( \omega \) and a signal \( \rho \), such that \( H = ae^{2\pi i \omega t/N} + \rho \), with \( |a|^2 \geq (q\%) \|H\|^2 \). To quantify the unevenness of the data, introduce a parameter \( p = L/N \) to be the percentage of the available data over all the data, where \( L \) is the number of available data. Obviously a larger \( p \) corresponds to more information about the signal. We use \( L^2 \)-norm throughout the paper, which is denoted by \( \| \cdot \| \). The convolution \( F \ast G \) is defined as \( F \ast G(t) = \sum_s F(s)G(t-s) \). It follows that \( \hat{F} \ast \hat{G}(\omega) = \sqrt{N} \hat{F}(\omega) \hat{G}(\omega) \).

A Box-car filter with width \( 2k + 1 \) is defined as follows:

\[
\chi_k(t) = \begin{cases} 
\frac{\sqrt{N}}{2k+1} & \text{if } -k \leq t \leq k, \\
0 & \text{if } t > k \text{ or } t < -k 
\end{cases}
\]

In the frequency domain, this filter is in the form of

\[
\hat{\chi}_k(\omega) = \begin{cases} 
\frac{\sin((2k+1)\pi\omega/N)}{(2k+1)\sin(\pi\omega/N)} & \text{if } \omega \neq 0 \\
1 & \text{if } \omega = 0 
\end{cases}
\]  

A dilation operation on signal \( H \) with a dilation factor \( \sigma \) is defined as \( H^{(\sigma)}(t) = H(\sigma t) \) for every points \( t \).

**Organization** The paper is organized as follows. In Section 2, we give the outline of the RA/SFA algorithm. Section 3 presents the modification of RA/SFA that deals with the unavailability of some samples by a greedy method. In Section 4, an interpolation technique is introduced for better performance. Finally, we compare numerical results with existing algorithms in Section 5.

### 2 Set-up of RA/SFA

Given a signal \( S \) of length \( N \), the optimal \( B \)-term Fourier representation \( R_{\text{opt}}^B(S) \) uses only \( B \) frequencies; it is simply a truncated version of the Fourier representation of \( S \), retaining only the \( B \) largest coefficients. The following theorem is the main result of [8].
Theorem 2.1. Let an accuracy factor $\epsilon$, a failure probability $\delta$, and a sparsity target $B \in \mathbb{N}, B \ll N$ be given. Then for an arbitrary signal $S$ of length $N$, RA/$SFA$ will find a $B$-term approximation $R$ to $S$, at a cost in time and space of order $\text{poly}(B, \log(N), 1/\epsilon, \log(1/\delta))$ and with probability exceeding $1 - \delta$, so that $\|S - R\|_2^2 \leq (1 + \epsilon)\|S - R_{\text{opt}}^B(S)\|_2^2$.

The striking fact is that RA/$SFA$ can build a near-optimal representation $R$ in sublinear time $\text{poly}(\log N)$ instead of the $O(N \log N)$ time requirement of other algorithms. Its speed surpasses FFT as long as the length of a signal is sufficiently large. If a signal is composed of only $B$ modes, RA/$SFA$ constructs $S$ without any error.

The main procedure is a Greedy Pursuit with the following steps:

Algorithm 2.2. Total Scheme \[20\]

1. Initialize the representation signal $R$ to 0. Set the maximum number of iterations $\text{ITER} = B \log(N) \log(1/\delta)/\epsilon^2$.

2. Test whether $\|S - R\|$ appears to be less than some user threshold, $\iota$. If yes, return the representation signal $R$ and the whole algorithm ends; else go to step 3.

3. Locate Fourier Modes $\omega$ for the signal $S - R$ by isolation and group test procedures.

4. Estimate Fourier Coefficients at $\omega$: $(\widehat{S - R})(\omega)$.

5. Update the representation signal $R \leftarrow R + (\widehat{S - R})(\omega)\phi_\omega(t)$.

6. If the total number of iterations is less than $\text{ITER}$, go to 2; else return the representation $R$.

The basic idea of Algorithm 2.2 is to identify significant frequencies and then estimate their corresponding coefficients. In order to locate those nonzero frequencies, we first construct a new signal where a previous significant frequency becomes predominant. Then a recursive approach called group test finds the exact label of this predominant mode, by splitting intervals, comparing energies, and keeping only intervals with large energies. After the frequency is located, coefficient estimation procedures give a good estimation by taking means and medians of random samples.

3 NERA/$SFA$ with Greedy Technique

RA/$SFA$ samples from a signal, implicitly assuming that uniform and random sampling is possible, with a fixed cost per sample. This raises challenges for processing unevenly spaced data. Specifically speaking, Fourier coefficients and norms can not be estimated properly. Thus one has to modify steps 3 and 4 accordingly. In this section, NERA/$SFA$, a modified version of RA/$SFA$ with greedy technique, is introduced to overcome these problems.

The basic idea is a greedy pursuit for an available data point. Whenever the algorithm samples at a missing data point, it searches some other random indices $t$ until it finds one available data
point $S(t)$ as the substitute. This technique is used in estimating both Fourier coefficients and norms.

A good data structure is important to save running time cost. We denote the availability of a data point by a label, say +1 for available and 0 for unavailable. Hence, the label is tested to see if its corresponding sample is valid. An alternative solution is to store all the sorted labels of available data in a long list. However, each search takes time $O (\log(N))$, which introduces a $O (\log N)^2$ factor into the whole computation. As the empirical results show, the running time of NERA/SFA algorithm is linear to $\log N$. For this reason, we selected the first method.

We now give a more detailed discussion of the different procedures used in steps 3 and 4 of Algorithm 2.2.

### 3.1 Estimating Fourier Coefficients

First, we give the procedure for estimating Fourier coefficients for unevenly spaced data as follows.

**Algorithm 3.1. ESTIMATING INDIVIDUAL FOURIER COEFFICIENTS**

*Input a signal $S$, a frequency $\omega$, $n = 2 \log(1/\delta)$, $m = 8/\epsilon^2$.*

1. For $i = 1, \ldots, n$

2. For $j = 1, \ldots, m$
   
   Randomly generate the index $t$ until $S(t)$ is available.

   Then let $t_{ij} = t$. Evaluate $k(t_{ij}) = \langle S(t_{ij}), \phi_\omega(t_{ij}) \rangle$.

3. Take the means of $m$ samples $k(t_{ij})$, i.e. $p(i) = \sum_{j=1}^{m} k(t_{ij})$, where $i = 1, \ldots, n$.

4. Take the median of $n$ samples $c = \text{median}(p(i))$, where $i = 1, \ldots, n$.

5. Return $c$ as the estimation of the Fourier coefficient $\hat{S}(\omega)$.

Next, we show that using unevenly spaced data leads to a very good approximation to the true coefficient. The first lemma is one of most fundamental theorems in randomized algorithms. It essentially states that by repeating an experiment enough times, a small probability event will happen eventually.

**Lemma 3.2.** If an event happens with probability $p$, then in the first $k > \log \delta / \log(1-p)$ iterations, it happens at least once with success probability $1 - \delta$.

In our case, only $p = L/N$ percentage of the data is available, so that $k > \log \delta / \log(1 - L/N)$ trials are needed to generate one available data point with success probability at least $1 - \delta$.

In fact, most of the Fourier coefficients of a characteristic function on a typical set $T$ are small, under some conditions. The following lemma makes this more explicit.
Lemma 3.3. Suppose the components $X_j$ of a discrete random variable $X = (X_j)_{j=0}^{N-1}$ are identically and independently distributed in $\{0, 1\}$, with $p = \text{Prob}(X_j = 1)$. Define the random set $T = \{ j \in \{0, \ldots, N-1\} | X_j = 1 \}$ to be the set of all available data; $\hat{\chi}_T(\omega)$ is the Fourier transform of $\chi_T(t) = \sum_{j=0}^{N-1} X_j$. If $p \geq \frac{1}{1+(N-1)\lambda \tau^2}$, then

$$\text{Prob}(|\hat{\chi}_T(\omega)|^2 \geq \lambda) \leq \tau^2.$$  
(3)

Proof. First, we claim that $E(|\hat{\chi}_T(\omega)|^2) \leq \frac{(1-p)}{p(N-1)}$.

Since $\hat{\chi}_T(\omega) = \frac{1}{pN} \sum_{j \in T} e^{2\pi i \omega j/N}$, we have

$$|\hat{\chi}_T(\omega)|^2 = \frac{1}{p^2 N^2} \sum_{j, k \in T} e^{2\pi i (j-k)/N}$$

$$= \frac{1}{p^2 N^2} \sum_{j \in T} 1 + \frac{1}{p^2 N^2} \sum_{j, k \in T, j \neq k} e^{2\pi i (j-k)/N}.$$  
(4)

It follows that

$$E(|\hat{\chi}_T(\omega)|^2) = \frac{1}{pN} + \frac{1}{p^2 N^2} p \frac{N-1}{N-1} \sum_{j, k = 0, j \neq k}^{N-1} e^{2\pi i (j-k)/N}.$$  

Observe that $\sum_{j, k = 0, j \neq k}^{N-1} e^{2\pi i (j-k)/N} = |\sum_{j=0}^{N-1} e^{2\pi i j/N}|^2 - \sum_{j=0}^{N-1} 1 = (N\delta_{\omega,0})^2 - N$, hence

$$E(|\hat{\chi}_T(\omega)|^2) = \frac{1}{pN} + \frac{1}{p^2 N^2} p \frac{N-1}{N-1} (N^2 \delta_{\omega,0} - N) = \frac{1}{pN} \left\{ 1 + \frac{p(N-1)}{N-1} (N\delta_{\omega,0} - 1) \right\}$$

$$= \frac{1}{pN(N-1)} \left\{ N - 1 + (pN-1)(N\delta_{\omega,0} - 1) \right\}.$$  

By Markov’s Inequality, when $\omega \neq 0$, we have

$$\text{Prob}(|\hat{\chi}_T(\omega)|^2 \geq \lambda) \leq \frac{E(|\hat{\chi}_T(\omega)|^2)}{\lambda} = \frac{1-p}{p(N-1)\lambda}.$$  

Since $p \geq \frac{1}{1+(N-1)\lambda \tau^2}$, it follows that

$$\text{Prob}(|\hat{\chi}_T(\omega)|^2 \geq \lambda) \leq \tau^2.$$  

That is, for any $\omega \neq 0$, with probability at least $1 - \tau^2$

$$|\hat{\chi}_T(\omega)| \leq \sqrt{\lambda}.$$  
(5)
In particular, we want both \( \lambda \) and \( \tau \) to be small, meaning that \( p \) cannot be too small itself.

Next, we consider the conditions for the two coefficients \( \hat{S} \) and \( \hat{S}_1 = \hat{S} \cdot \hat{\chi}_T(\omega) \) to be close.

**Lemma 3.4.** Suppose the parameters \( T, S, \chi_T(t), \lambda, \tau, p \) are as stated in Lemma 3.3 and define \( S_1(t) = S(t) \chi_T(t) \). If \( p \geq \frac{1}{1+(N-1)\lambda^2}, \) and \( \tau \leq \sqrt{1 - (1 - \delta)^{\frac{4}{3}}} \), then, for any \( \omega \),

\[
| \hat{S}(\omega) - \hat{S}_1(\omega) | \leq \sqrt{B \lambda \| S \|_2}.
\]

with probability exceeding \( 1 - \delta \).

**Proof.** Suppose the significant terms of signal \( S \) are \( \omega_i \), where \( i = 1, \ldots, B \).

Since \( S_1(t) = S(t) \chi_T(t) \) and thus \( \hat{S}_1(\omega) = \hat{S}(\omega) \ast \hat{\chi}_T(\omega) \), then

\[
\hat{S}_1(\omega_j) = \sum_{i=1}^B \hat{S}(\omega_i) \hat{\chi}_T(\omega_j - \omega_i) = \hat{S}(\omega_j) \hat{\chi}_T(0) + \sum_{i=1, \omega_i \neq \omega_j}^B \hat{S}(\omega_i) \hat{\chi}_T(\omega_j - \omega_i)
\]

\[
= \hat{S}(\omega_j) + \sum_{i=1, \omega_i \neq \omega_j}^B \hat{S}(\omega_i) \hat{\chi}_T(\omega_j - \omega_i).
\]

Therefore

\[
| \hat{S}_1(\omega_j) - \hat{S}(\omega_j) | = | \sum_{i=1, \omega_i \neq \omega_j}^B \hat{S}(\omega_i) \hat{\chi}_T(\omega_j - \omega_i) | \tag{7}
\]

\[
\leq \sqrt{ \sum_{i=1, \omega_i \neq \omega_j}^B |\hat{S}(\omega_i)|^2 } \sqrt{ \sum_{i=1, \omega_i \neq \omega_j}^B |\hat{\chi}_T(\omega_j - \omega_i)|^2 } \leq ||S||_2 \sqrt{ \sum_{i=1, \omega_i \neq \omega_j}^B |\hat{\chi}_T(\omega_j - \omega_i)|^2 }.
\]

Because \( p \geq \frac{1}{1+(N-1)\lambda^2} \), we have \( |\hat{\chi}_T(\omega)|^2 \leq \lambda \) with probability at least \( 1 - \tau^2 \) for any \( \omega \neq 0 \).

This implies that \( |\hat{S}_1(\omega_j) - \hat{S}(\omega_j) | \leq ||S||_2 \sqrt{B \lambda} \) with probability at least \( (1 - \tau^2)^B \geq (1 - \delta) \).

Then

\[
| \hat{S}_1(\omega_j) - \hat{S}(\omega_j) | \leq \sqrt{B \lambda ||S||_2} \tag{8}
\]

For those \( \omega \notin \{ \omega_i, i = 1, \ldots, B \} \),

\[
\hat{S}_1(\omega) = \sum_{i=1}^B \hat{S}(\omega) \hat{\chi}_T(\omega - \omega_i),
\]

and we conclude similarly that \( | \hat{S}_1(\omega) - \hat{S}(\omega) | \leq \sqrt{B \lambda ||S||_2} \). with probability at least \( 1 - \delta \). \( \square \)

We shall use Algorithm 3.1 to estimate \( \hat{S}_1(\omega) \); we now look at how close the approximation \( \hat{S} \) (i.e. the output of Algorithm 3.1) of \( \hat{S}_1(\omega) \) is to the true coefficient \( \hat{S}(\omega) \).
Lemma 3.5. For a set of parameters $T, S, \chi_T(t), \lambda, \tau, p$ as stated in Lemma 3.3, if $p \geq \frac{1}{1+(N-1)\lambda \tau^2}$, and $\tau \leq \sqrt{1-(1-\delta)^{1/B}}$, then Algorithm 3.7 for signal $S_1(t) = S(t)\chi_T(t)$ gives a good estimation $A$ of $\hat{S}(\omega)$, such that

$$|A - \hat{S}(\omega)| \leq (\sqrt{\lambda} + \sqrt{B\lambda})\|S\|_2.$$  \hspace{1cm} (10)

with high probability.

Proof. Lemma 4.2 in [20] says that the coefficient estimation algorithm returns $A$, such that

$$|A - \hat{S}_1(\omega)| \leq \sqrt{\lambda}\|S\|_2.$$ \hspace{1cm} (11)

By Lemma 3.4

$$|\hat{S}_1(\omega) - \hat{S}(\omega)| \leq \sqrt{B\lambda}\|S\|_2.$$ \hspace{1cm} (12)

Thus

$$|A - \hat{S}(\omega)| \leq |A - \hat{S}_1(\omega)| + |\hat{S}_1(\omega) - \hat{S}(\omega)| \leq (\sqrt{\lambda} + \sqrt{B\lambda})\|S\|_2.$$ \hspace{1cm} (13)

Finally, we derive the conclusion about estimating coefficients.

Theorem 3.6. For a set of parameters $T, S, \chi_T(t), \lambda, \tau, p$ as stated in Lemma 3.3, if $\lambda \leq \epsilon$ and $p \geq \frac{1}{1+(N-1)\lambda \tau^2}$, then every application of Algorithm 3.7 produces, for each frequency $\omega$ and each signal $S$, and each $\lambda > 0$, with high probability, an output $A$ (after inputting $(S, \omega, \epsilon)$), such that $|A - \hat{S}(\omega)|^2 \leq \epsilon\|S\|_2^2$.

Proof. By Lemma 3.5

$$|A - \hat{S}(\omega)| \leq (\sqrt{\lambda} + \sqrt{B\lambda})\|S\|_2.$$ \hspace{1cm} (14)

Thus we have

$$|A - \hat{S}(\omega)|^2 \leq 2(\lambda + B\lambda)\|S\|_2^2.$$ \hspace{1cm} (15)

From the conditions $2(\lambda + B\lambda) \leq \epsilon$, it follows that

$$|A - \hat{S}(\omega)|^2 \leq \epsilon\|S\|_2^2.$$ \hspace{1cm} (16)

When we are able to get most of the data, the computational cost for estimating Fourier coefficients on unevenly spaced data is only slightly more than for the evenly spaced data case. The time to compute the signal value remains almost the same as for the evenly spaced data case. The extra time, in the worst case $O\left(\frac{\log \delta}{\epsilon^2 p \log(1-p)}\right)$, comes from visiting unavailable data. Fortunately, the visit operation is very fast and therefore contributes little to the total time, especially when most of the data are available.

Moreover, as in [20], one can speed up the algorithm by using multi-step coarse-to-fine coefficient estimation procedures, which turns out to be more efficient than single-step accurate estimation; the proof is entirely analogous to Lemma 4.3 in [20].
3.2 Estimating Norms

The basic idea for locating the label of a significant frequency is to compare the energies (i.e. the $L^2$ norm) of signals restricted in different frequency intervals. If the energy of some interval is relatively large, the significant mode is in that region with higher probability. We construct the following new signals to focus on certain intervals relatively large, the significant mode is in that region with higher probability. We construct the following new signals to focus on certain intervals

$$H_j(t) = \chi_1(t)e^{\frac{2\pijt}{q_1}} * \chi_{[-q_1,q_1]}(\sigma t)e^{\frac{2\pi\theta t}{q_1}} * S$$

where $2q_1 + 1$ is the filter width, $j = 0, \ldots, 15$, $\sigma$ and $\theta$ are random dilation and modulation factors. (Please see [20] for an explanation of the role of $\sigma$ and $\theta$). For convenience, we denote $H_j(t)$ by $H(t)$.

We need to evaluate values $H(t)$ for random indices $t \in \{0, \ldots, N-1\}$. Note that the signal $H$ results from the convolutions of two finite bandwidth Box-car filters with the original signal $S$. Therefore, any missing point needed by the two convolutions would lead to a failure of computing $F(t)$. The total number of signal points involved depends on the number of nonzero taps in these two filters. Moreover, random dilation and modulation factors of the second Box-car filter make computation more tricky.

One naive way is to dive into the two convolutions and sample each signal point. If it is not available, stop evaluating this $F(t)$ and start with a new index $t$. This definitely increases time cost by wasting abundant computation. For example, suppose five data are needed and only one of them is missing, then the algorithm may compute four data in vain in the worst case, where the missing data point is visited last in the sequence of 5.

To avoid the above situation, we first compute the locations of all the points that will be needed for the convolution; only if they are all available will we start the computation. The locations related to the convolution are given in the following lemma.

**Lemma 3.7.** Suppose we have a signal $H(t) = (\chi_1^{(\sigma_1)} * (\chi_{\theta_1}^{(\sigma_2)} * S)^{(\sigma_3)}(\sigma_4))(t)$, where $\sigma_1$, $\sigma_2$, $\sigma_3$, and $\sigma_4$ are dilation factors. From the definition of Box car filter, the taps for $\chi_1$ lies in the interval $[-1,1]$, the taps for $\chi_{\theta_1}$ in $[-q_1,q_1]$, then in order to evaluate $H(t)$, we need values of $S$ with indices at $\sigma_3\sigma_4 t - \sigma_3\sigma_1 i - j\sigma_2$, where integers $i = -1, \ldots, 1$, $j = -q_1, \ldots, q_1$.

**Proof.** To evaluate $H(t)$, first let signal $r = (\chi_{\theta_1}^{(\sigma_2)} * S)^{(\sigma_3)}$, then

$$H(t) = (\chi_1^{(\sigma_1)} * r)^{(\sigma_4)}(t) = \sum_{i=-1}^{1} \chi_1(\sigma_1 i) r(\sigma_4 t - \sigma_1 i)$$

$$r(\sigma_4 t - \sigma_1 i) = (\chi_{\theta_1}^{(\sigma_2)} * S)^{(\sigma_3)}(\sigma_4 t - \sigma_1 i) = (\chi_{\theta_1}^{(\sigma_2)} * S)(\sigma_3\sigma_4 t - \sigma_3\sigma_1 i)$$

$$= \sum_{j=-q_1}^{q_1} \chi_{\theta_1}(\sigma_2 j) S(\sigma_3\sigma_4 t - \sigma_3\sigma_1 i - \sigma_2 j).$$

Thus, in order to get the value of $H(t)$, we need values of all $S(t')$, where $t' = \sigma_3\sigma_4 t - \sigma_3\sigma_1 i - \sigma_2 j$, with $i = -1, \ldots, 1$ and $j = -q_1, \ldots, q_1$. □
The scheme of the norm estimation algorithm is as follows.

**Algorithm 3.8. NORM ESTIMATION**

**Input:** signal $H$, $k = 0$, the number of iterations $M = 1.2 \ln(1/\delta)$.

**While** $k < M$:

1. Randomly generate the index $t_k$.

2. Compute all indices needed by the two convolutions: $\Upsilon = \{t’, t’ = \sigma_3 \sigma_4 t - \sigma_3 \sigma_1 i - \sigma_2 j\}$, where $i = -1, \ldots, 1$ and $j = -q_1, \ldots, q_1$.

3. If all the points $t’ \in \Upsilon$ are available, then compute $H(t_k)$ else go to step 1 and generate another index $t_k$.

4. Estimate $= 60$-th percentile of the sequence $\{|H(t_k)|^2N\}$, where $k = 0, \ldots, M - 1$.

If there exist satisfactory data groups, although maybe very few, the norm estimation will eventually find them. However, when most data are unavailable, the program may struggle in a long loop and take a huge amount of time. We introduce some tricks to avoid this. For example, set an upper bound MAX on the number of the loops. If it is reached, just use the sample points generated so far to estimate the norms. This technique may lead to a larger error, and thus hamper our frequency identification. However, by repeating the calculation, as stipulated by Lemma 3.2, we reduce the inaccuracy. Anyway we cannot hope to recover the signal, if $p$ is too small.

The following lemma investigates the number of repetitions to get a satisfactory data group for estimating norms.

**Lemma 3.9.** Suppose $\chi_{q_1}$ and $\chi_{q_2}$ are two Box-car filters with numbers of taps $2q_1 + 1$ and $2q_2 + 1$ respectively. Define $D_{q_1,q_2} = \chi_{q_1} \ast \chi_{q_2}$. Then $D_{q_1,q_2}$ has $2q_1 + 2q_2 + 1$ nonzero taps in the time domain.

**Lemma 3.10.** Randomly choose an index for signal $H(t)$, then after $k > \log \delta / \log(1 - (1 - p)^{(2q_1+2q_2+1)})$ iterations, we can get at least one satisfactory index with high probability $1 - \delta$.

**Proof.** It is easy to prove by Lemma 3.2.

Here is a new scheme for estimating norms, which uses much fewer samples than the original one and still achieves good estimation. In [20], we propose a lemma that enabled us to achieve a good norm estimation by only a few samples. The following lemma is its adaption to the case of unevenly spaced data.

**Lemma 3.11.** If a signal $H$ is 95% pure and if $r > 1.2 \ln(1/\delta)$, the output of Algorithm 3.8 gives an estimation of its energy which exceeds $\|H\|^2 / 3$ with probability exceeding $1 - \delta$. 

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Proof. The proof is very similar to that of Lemma 4.5 in [20]. We shall present only the difference of these two proofs. Suppose we sample \( r \) times for the signal \( H \). Let \( \kappa = \{ t : N|H(t)|^2 < \|H\|^2/3 \} \), with \( \kappa^c \) as its complement, we have

\[
\left| \sum_{t \in \kappa} H(t) \right|^2 \leq |\kappa| \sum_{t \in \kappa} |H(t)|^2 \leq |\kappa|^2 \frac{1}{N} \|H\|^2. \tag{20}
\]

On the other hand, we know that the signal is 95% pure, i.e. \( |\hat{H}(\omega_0)|^2 \geq 0.95 \|H\|^2 \) for some \( \omega_0 \). By modulating, \( \omega_0 \) can be moved to 0; therefore, we can, without loss of generality, suppose most of the energy concentrates at the frequency 0; then

\[
\left| \sum_{t \in \kappa^C} H(t) \right|^2 = |\hat{H}(0)|^2 \geq 0.95 \|H\|^2. \tag{21}
\]

So we have

\[
\left| \sum_{t \in \kappa^C} H(t) \right| \geq \sqrt{0.95N\|H\|} - |\kappa| \frac{1}{\sqrt{3N}} \|H\|. \tag{22}
\]

On the other hand, \( |\sum_{t \in \kappa} H(t)| \leq |\kappa^C||H| = (N - |\kappa|)||H|| \), so that

\[
N - |\kappa| \geq \left( \sqrt{0.95N} - \frac{|\kappa|}{\sqrt{3N}} \right)^2. \tag{23}
\]

Let \( \alpha = \frac{|\kappa|}{N} \); the above inequality becomes

\[
\alpha^2 + \left( 3 - 2\sqrt{0.95 \times 3} \right) \alpha - 0.15 \leq 0. \tag{24}
\]

Thus \( 0 \leq \alpha \leq 0.075 \). Define now a random variable \( X_\kappa = \left( \sum_{i=1}^{N} \chi_\kappa(i) \right) \); it will be useful to estimate

\[
E(X_\kappa) = \frac{|\kappa|}{N} \leq 0.075, \tag{25}
\]

and the expectation of the random variable \( e^{zX_\kappa} \),

\[
E(e^{zX_\kappa}) = e^0 \text{Prob}(\chi_\kappa(i) = 0) + e^z \text{Prob}(\chi_\kappa(i) = 1) = 1 - \alpha + \alpha e^z. \tag{26}
\]

Suppose now we sample the signal \( H \) \( r \) times, and take the 60-th percentile of the numbers \( N|H(t_1)|^2, \ldots, N|H(t_r)|^2 \). By Chernoff’s standard argument and similar procedure of Lemma 4.5 in [20], we have for \( z > 0 \),

\[
\text{Prob} \left( \text{60-th percentile} < \frac{1}{3}\|H\|^2 \right) = \left[ (1 - \alpha)e^{-0.6z} + \alpha e^{0.4z} \right]^r.
\]

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Take $z = \ln(1.25(1 - \alpha)/\alpha)$, then
\[
(1 - \alpha)e^{-0.6z} + \alpha e^{0.4z} = 1.97\alpha^{0.6}(1 - \alpha)^{0.4}. \tag{27}
\]
The right hand side of (35) is increasing in $\alpha$ on the interval $[0, 0.075]$; since $\alpha \leq 0.075$, we obtain an upper bound by substituting 0.075 for $\alpha$:
\[
[(1 - \alpha)e^{-0.6z} + \alpha e^{0.4z}]^r = [1.97\alpha^{0.6}(1 - \alpha)^{0.4}]^r \leq e^{-0.90r}. \tag{28}
\]
For $\text{Prob}(60-\text{th percentile} < \frac{1}{3}||H||^2) \leq \delta$, we need $r \geq 1.2 \ln(1/\delta)$, we have
\[
\text{Prob}(\text{Output} \geq ||H||^2/3) = \text{Prob}(60-\text{th percentile of } N||H(t)||^2 \geq ||H||^2/3) \geq 1 - \delta. \tag{29}
\]

This norm estimation procedure will be used repeatedly in the group testing step below.

### 3.3 Isolation

For a significant frequency in signal $S$, isolation aims to construct a series of new signals, such that this significant frequency becomes predominant in at least one of the new isolation signals.

**Lemma 3.12.** Given signals $S$, $S_1$, and the parameters as stated in Lemma 3.3, suppose $F_1(t) = S_1(t) * \chi_1(t) = (\chi_1(t)S(t)) * \chi_1(t)$, $F(t) = S(t) * \chi_1(t)$. If $p \geq \frac{1}{1+(N-1)\lambda r^2}$, then for each $\omega$ with $|\hat{S}(\omega)|^2 > B\lambda||S||^2$, isolation algorithm can create a signal $F_1^*$, such that
\[
|\hat{F}_1^*(\omega)|^2 \geq 0.98||F_1^*||^2. \tag{30}
\]

**Proof.** Since $|\hat{S}(\omega)|^2 > B\lambda||S||^2$, we have $|\hat{S}(\omega)| > \sqrt{B\lambda||S||}$. Then there exists some $\eta > 0$, such that $|\hat{S}(\omega)| \geq (\sqrt{\eta} + \sqrt{B\lambda})||S||$. Lemma 3.4 states that $|\hat{S}_1(\omega) - \hat{S}(\omega)| \leq \sqrt{B\lambda}||S||$. Therefore
\[
|\hat{S}_1(\omega)| \geq \sqrt{\eta}||S|| \geq \sqrt{\eta}||S_1||. \tag{31}
\]

Isolation algorithm returns $F_1^{(0)}, \ldots, F_1^{(2k)}$ with $k < O(\frac{1}{\eta})$, as described in [8]. For any $\omega$ with $|\hat{S}_1(\omega)|^2 \geq \eta||S_1||^2$, there exists some $j$, such that
\[
|\hat{F}_1^{(j)}(\omega)|^2 \geq 0.98||F_1^{(j)}||^2. \tag{32}
\]
Let $F_1^* = F_1^{(j)}$, then
\[
|\hat{F}_1^*(\omega)|^2 \geq 0.98||F_1^*||^2. \tag{33}
\]

Theoretically, in order to capture a significant mode, we need $O(1/\eta)$ signals. However, in practice, much fewer signals is enough to achieve this goal.
3.4 Group Testing

Isolation has produced several signals, one of which contains the most significant frequency. Group testing uses repeated zoom-ins on one of the signals, and norm testing to select where to zoom in, in order to determine the frequency. The goal of group testing is thus to find the most significant mode of the signal $F^*_1$ from isolation. It uses recursive procedures MSB (Most Significant Bit) to approach this mode gradually.

**Definition:** Denote a set $\{\omega : (2l - 1)N/32 \leq \omega \leq (2l + 1)N/32\}$ by $\text{interval}_l$.

Group test algorithm is given as follows.

**Algorithm 3.13. Group Testing**

*Input isolation signal $F^*_1$ to $F^{(0)}_1$, $i = 0$, $q = 1$*

*While $q < N$, in the $i$-th iteration,*

1. Find the most significant bit $v$ and the number of significant intervals $c$ by the procedure MSB.
2. Update $i = i + 1$, modulate the signal $F^{(i)}_1$ by $\lfloor (v + 0.5)N/16 \rfloor$ and dilate it by a factor of $\lceil 16/c \rceil$. Store it in $F^{(i+1)}_1$.
3. Call Group Test again with the new signal $F^{(i)}_1$, denote its output by $g$.
4. Update the accumulation factor $q = q \times \lfloor 16/c \rfloor$.
5. If $g > N/2$, then $g = g - N$.
6. return $\lfloor g/\lfloor 16/c \rfloor \rfloor + (v + 1/2)N/16 + 0.5](mod N)$;

The MSB procedure is as follows.

**Algorithm 3.14. MSB (Most Significant Bit)**

*Input: signal $F^{(i)}_1$ with length $N$, a threshold $0 < \eta < 1$.*

1. Get a series of new signals $H_j(t) = F^{(i)}_1(t) \ast (e^{2\pi ij/16} \chi_1), j = 0, \ldots, 15$.
2. Estimate the energies $e_j$ of $H_j$, $j = 0, \ldots, 15$.
3. for $l = 0, \ldots, 15$, compare the energies $e_l$ with all other energies $e_j$, where $j = (l + 4)\mod 16, (l + 5)\mod 16, \ldots, (l + 12)\mod 16$. If $e_l > e_j$ for all these $j$, label it as an interval with large energy.
4. Find the longest consecutive intervals of large energies. Take their center as $v$, and the number of those intervals as $c$.
5. If $c < 8$, then do the original MSB in [8] to get $v$ and set $c = 8$;
6. Return the dilation-related factor $c$ and the most significant bit $v$. 
Lemma 3.15. Given a 98% pure signal $F_1$, suppose $G_j(t) = e^{2\pi ijt/16} \chi_1(t)$. Then Algorithm 3.14 with Algorithm 3.13 as its subroutine, can find the significant frequency $\omega_1$ of the signal $F_1$ with high probability.

Proof. The proof is similar to that of Lemma 5 in [8], with some changes:

Since the signal $F_1$ is 98% pure, there exist a frequency mode $\omega_1$ and a signal $\rho$, such that $F_1 = a \phi_{\omega_1} + \rho$, where $|a|^2 \geq 0.98 \|F_1\|^2$ and $\|\rho\|^2 \leq 0.02 \|F_1\|^2$. Without loss of generality, assume $\omega_1 \in [-N/32, N/32]$. The whole region is divided into 16 subintervals $[jN/16 - N/32, jN/16 + N/32]$, where $j = 0, \ldots, 15$. To estimate $F_1^* G_0(\omega_1)$ for $|\omega_1| \leq N/32$, we use that $|\hat{G}_0(\omega_1)| = |\hat{\chi}_1(\omega_1)| \geq 0.987$ for $|\omega_1| \leq N/32$. It follows that

$$
|\hat{F}_1^* \hat{G}_0(\omega_1)|^2 = N \left| \hat{F}_1(\omega_1) \hat{G}_0(\omega_1) \right|^2 \geq N 0.987^2 |\hat{F}_1(\omega_1)|^2 \geq N 0.987^2 0.98 \|F_1\|^2 \\
\geq 0.954 N \|\hat{F}_1\|^2 \geq 0.954 N \|\hat{F}_1 \hat{G}_0\|^2 = 0.954 \|F_1^* G_0\|^2.
$$

Therefore the estimation $X$ of $\|F_1^* G_0\|$ satisfies:

$$
X \geq \|F_1^* G_0\|^2 / 3 = \|\hat{F}_1^* \hat{G}_0\|^2 / 3 = \sum_{\omega} |\hat{F}_1^* \hat{G}_0(\omega)|^2 / 3 \geq |\hat{F}_1^* \hat{G}_0(\omega_1)|^2 / 3 \\
\geq 0.954 N \|\hat{F}_1\|^2 / 3 \geq 0.318 N \|F_1\|^2.
$$

Next consider the energy of $F_1^* G_4$.

$$
\|\hat{\rho} \hat{G}_4\|^2 = \sum_{\omega} |\hat{\rho}(\omega) \hat{G}_4(\omega)|^2 \\
\leq \sum_{\omega} |\hat{\rho}(\omega)|^2 = \|\rho\|^2 \leq 0.02 \|F_1\|^2.
$$

Since $|\hat{G}_4(\omega_1)| < 0.464$, we have

$$
|\hat{F}_1(\omega_1) \hat{G}_4(\omega_1)| \leq |\hat{F}_1(\omega_1)||\hat{G}_4(\omega_1)| \leq |\hat{F}_1(\omega_1)| 0.464 \leq 0.464 \|F_1\|
$$

Also $\|\hat{F}_1 \hat{G}_4\|^2 - |\hat{F}_1(\omega_1) \hat{G}_4(\omega_1)|^2 \leq 0.02 \|F_1\|^2$. Thus

$$
\|\hat{F}_1 \hat{G}_4\|^2 \leq 0.464^2 \|F_1\|^2 + 0.02 \|F_1\|^2 = 0.24 \|F_1\|^2.
$$

It follows that

$$
\|F_1^* G_4\|^2 = \|\hat{F}_1^* \hat{G}_4\|^2 = N \|\hat{F}_1 \hat{G}_4\| \leq 0.24 N \|F_1\|^2.
$$

Then we compare $\|F_1^* G_4\|^2$ with the lower bound of the estimation of $\|F_1^* G_0\|^2$, which is

$$
0.24 N \|F_1\|^2 \leq 0.318 N \|F_1\|^2,
$$

which is less than the estimation for $\|F_1^* G_0\|^2$. In general, $\omega \in \text{interval}_j$, for $j$ not necessarily 0. Therefore we compare $\|F_1^* G_j^\prime\|^2$ with $\|F_1^* G_j\|^2$, where $|j - j^\prime| \geq 4$. If there is some $j$ with
\[ \|F_1 \ast G_j\|^2 \text{ apparently larger than } \|F_1 \ast G_{j'}\|^2, \] then we conclude \( \omega_1 \notin \text{ interval}_{j'} \). Otherwise, possibly \( \omega_1 \in \text{ interval}_{j'} \). By the above argument, we can always eliminate 9 consecutive interval regions out of 16, leaving a cyclic interval of length at most \( 7N/16 \). The remaining proof is exactly the same as Lemma 8 in paper [8].

Remark: In [20], we showed that group testing works for a Box-car filter with width more than 21, i.e. \( k > 10 \). In that case, \( 2k + 1 \) intervals are sufficient. A similar conclusion still holds in the unevenly spaced data case. However, the lemma above proves the success of group testing under different conditions. In our proof, we use a Box-car filter with much shorter width, namely 3 in time domain; this works well if 16 intervals are taken. In practice, we use these shorter filters; we can usually (if \( B \) is small) get away with using much fewer intervals as well (e.g. 3 instead of 16).

3.5 Adaptive Greedy Pursuit

In summary, given a signal \( S \), for an accuracy \( \epsilon \) and for \( B \) modes, we can find a very good approximation of the signal \( S \) by using Algorithm 2.2.

**Theorem 3.16.** Given a signal \( S \), an accuracy \( \epsilon \), success probability \( 1 - \delta \), Algorithm 2.2 can output a \( B \)-term representation \( R \) with sum-square-error \( \|S - R\|^2 \leq (1 + \epsilon)\|S - R_{opt}\|^2 \), where \( R_{opt} \) is the \( B \)-term representation for \( S \) with the least sum-square-error, with time and space cost \( \text{poly}(B, \log(N), \frac{1}{\epsilon}, \log(1/\delta)) \) for computing and \( \frac{B \log M \log N \log \delta}{\lambda \log(1 - (1 - p)^2t_1 + 2q_2 + 1)} + \frac{\log(1/\delta) \log M}{\lambda \log p} \) for just visiting samples.

**Proof.** We omit the proof since it is very similar to Theorem 9 in [8].

4 NERA\textsuperscript{SFA} with Interpolation Technique

The greedy algorithm described above is fast. When \( p \) is sufficiently large (e.g. \( p > 0.7 \)), the approach proposed and discussed in the previous section works well. For smaller \( p \), the amount of time wasted to find available sample groups becomes unacceptably long. For example, when \( B = 2, N = 100, p = 0.4 \), the algorithm couldn’t find the signal within 200 greedy pursuit iterations. For this reason, we introduced an interpolation technique to get an approximate value of the missing point in the norm estimation procedure. This algorithm is efficient even in smaller \( p \) cases.

4.1 Lagrange Interpolation Technique

The task of interpolation is to estimate \( S(t) \) for arbitrary \( t \) by drawing a smooth curve through all the known points [17]. It is called interpolation when the desired \( t \) is between the largest and smallest of these \( t_i \)'s. We use Lagrange Polynomial Interpolation, one of the simplest and most popular interpolation techniques.

Generally, the number of interpolation points determines the degree of a polynomial. A polynomial of higher degree is smoother with smaller approximation errors at the expense of more
computation. Thus we choose a second degree polynomial, as a balance between computational complexity and accuracy. It is given explicitly by Lagrange’s classical formula. If the three nearest neighbors are \((t_1, S(t_1)), (t_2, S(t_2)), (t_3, S(t_3))\), the polynomial is

\[
P(t) = \frac{(t-t_2)(t-t_3)}{(t_1-t_2)(t_1-t_3)}S(t_1) + \frac{(t-t_1)(t-t_3)}{(t_2-t_1)(t_2-t_3)}S(t_2) + \frac{(t-t_2)(t-t_1)}{(t_3-t_2)(t_3-t_1)}S(t_3)
\]  

(34)

If \(S(t)\) is three times differentiable in an interval \([a, b]\), and the points \(t_1, t_2, t_3 \in [a, b]\) are different, then there exists some \(v \in [a, b]\), such that the approximation error is \(S(t) - P(t) = \frac{S^{(3)}(v)}{3!}(t-t_1)(t-t_2)(t-t_3)\).

### 4.2 Estimate Norms with Interpolation

We introduce the interpolation scheme into estimating norms. The idea is to estimate the value of a missing point by the Lagrange interpolation. The detailed algorithm for estimating norms is as follows.

**Algorithm 4.1. Estimate Norm with Interpolation Technique**

*Input: signal \(H\), \(k = 0\), the maximum number of samples \(M\).*

1. Randomly generate the index \(t_k\), where \(k = 0, \ldots, M-1\).

2. For each \(k\), if \(H(t_k)\) is not available, estimate \(H(t_k)\) by Lagrange interpolation; else compute \(H(t_k)\) directly.

3. Estimation = 60-th percentile of the sequence \(\{|H(t_k)|^2N\}\), where \(k = 0, \ldots, M-1\).

Note that we use interpolation only in norm estimation steps, where precision is less critical. With less precise norm estimation, the localization of important modes could still work well when iterated. For coefficient estimation, which needs to be more precise, we always search for available samples.

### 5 Numerical Results

In this section, we present striking numerical results of NERA/SFA, comparing to the Inverse Non-equispaced Fast Fourier Transform (INFFT) algorithms. The popular benchmark software NFFT version 2.0 is used to give performance of INFFT, with default CGNE_R method and Dirichlet kernel. Its time cost excludes the precomputation of samples values, which takes \(O(L)\). Numerical experiments show the advantage of our NERA/SFA algorithm in processing large amount of data. We begin in Section 5.1 with comparing NERA/SFA with INFFT for some one and two dimensional examples with different length. In Section 5.2, the performance for different number of modes is shown. Finally, we test the capability of NERA/SFA to recover the signal in the situation with a large amount of missing data and in presence of large noise.

All the experiments were run on an AMD Athlon(TM) XP1900+ machine with Cache size 256KB, total memory 512 MB, Linux kernel version 2.4.20-20.9 and compiler gcc version 3.2.2. The numerical data is an average of 10 runs of the code; errors are given in the \(L^2\) norm.
Table 1: Experiments with fixed $B = 8$, $p = 0.7$, $d = 1$ (one dimension), and varying length $N$ of signals; an i.i.d. white noise is added with $\sigma = 0.5$, or $SNR \simeq 30dB$ (see text). For each length of the signal, 10 different runs were carried out; the average result is shown. We did all the tests for NERA/SFA with Lagrange interpolation, as explained in the text. Two kinds of time costs for NERA/SFA are provided. One is the total running time and another is the running time excluding the sampling time. The time of INFFT does not include the precomputation time for samples.

5.1 Experiments with Different Length of Signals

We ran the comparison for a 8-mode superposition signal $S(t) = \sum_{i=1}^{B} \phi_{\omega_i}$, plus white noise $\nu$ with the standard deviation $\sigma = 0.5$, damped by a factor of $1/\sqrt{N}$, (so that $\|\nu\|^2 = \sigma^2 = 0.25$; since $\|S\|^2 = 8$, this implies $SNR = 20 \log_{10} 32 \approx 30.1dB$). Other parameters are $B = 8$, $\epsilon = 0.02$, $\delta = 0.01$, and $p = 70\%$. The missing data are randomly and uniformly distributed. NERA/SFA outperforms INFFT in speed when $N$ is large; see Table 1 and Figure 2. The corresponding crossover point is $N \geq 2^{15} = 32768$. For example, to process $2^{19} = 524,288$ data, more than nineteen minutes (estimated) are needed for INFFT versus approximately one second for NERA/SFA. Experiments support the theoretical conclusion that NERA/SFA would be faster than INFFT after some $N$ for a sparse signal; whatever the sparsity, i.e. whatever the value of $B$, there always exists some crossover $N$.

In two dimensions, we test a noisy 6-mode superposition signal $S(t) = \sum_{i=1}^{B} \phi_{\omega_{x_i}\omega_{y_i}} + \nu$, with $B = 6$, $\epsilon = 0.02$, $\delta = 0.01$, $p = 80\%$, and $\sigma = 0.1$. Missing data are randomly and uniformly distributed. As the number of grid points $N$ in each dimension grows, two dimensional NERA/SFA outperforms two dimensional INFFT at $N \geq 512$, as Table 1 and Figure 2 show. The crossover point becomes much smaller in high dimensions situation. It would not be surprising that for recovering a 6-mode three dimensional signal, NERA/SFA surpasses INFFT at a hundred sampling grid points in each dimension.

5.2 Experiments with Different Number of Modes

The number of modes has an important influence on the running time since the crossover point varies for signals with different $B$. To investigate this, we did the experiments with fixed $N = 2^{18} = 262144$, $p = 0.6$ and varying $B$. As before, we take $S$ to be a superposition of exactly $B$ modes with white noise, i.e. $S(t) = \sum_{i=1}^{B} c_i \phi_{\omega_i} + \nu$, with standard deviation of noise $\sigma = 0.05$. 

Figure 2: Time Comparison between INFFT and NERALSFA for different $N$ with $B = 8$, $p = 0.7$, $d = 1$. The result in Table 1 is shown in the form of a graph here. The $x$ coordinate is the $\log_2(N)$, the $y$ coordinate presents the running time for each algorithm. NERALSFA without sampling surpasses INFFT at $N = 2^{14} = 16384$.

| $N$  | INFFT | NERALSFA (+sampling) | NERALSFA (w/o sampling) |
|------|-------|----------------------|-------------------------|
| 128  | 0.13  | 2.86                 | 1.57                    |
| 256  | 0.73  | 2.60                 | 1.46                    |
| 512  | 3.00  | 3.70                 | 2.13                    |
| 1024 | 11.59 | 4.31                 | 2.94                    |
| 2048 | 54.94 | 6.56                 | 4.90                    |

Table 3: Experiments with fixed $B = 6$, $p = 0.8$, $d = 2$ (two dimensions), and varying length $N$ of signals; an i.i.d white noise is added with $\sigma = 0.1$, or $SNR \simeq 56dB$ (see text). For each length of the signal, 10 different runs were carried out; the average result is shown. We did all the tests for NERALSFA with two dimensional interpolation techniques as shown in the appendix. Again, two kinds of time costs for NERALSFA, the one with and without sampling time is provided. The time of INFFT excludes the sampling time.
Figure 4: Time comparison between INFFT and NERA/ SFA for different $N$ with fixed $B = 6$, $p = 0.8$, $d = 2$. The $x$ coordinate is the logarithm of length $N$ of signal in each dimension. INFFT is very fast when $N$ is relatively small and slows down quickly as $N$ increases. On the contrary, it takes NERA/ SFA similar time to process small and large $N$ problem. NERA/ SFA without sampling outperforms INFFT at $N = 2^{8.5}=362$.

Available data are uniformly and randomly distributed. Table 5 and Figure 6 compare the running time for different $B$ using INFFT and NERA/ SFA. At first, NERA/ SFA takes less time because $N$ is so large. However, the execution time of INFFT keeps constant for different number of modes $B$, while that of modified RA/ SFA is polynomial of higher order. INFFT is faster than NERA/ SFA when $B \geq 10$. The regression techniques shows empirically that the order of $B$ in NERA/ SFA is greater than quadratic. This is one of the characteristics of this version of the RA/ SFA algorithms and irrelevant to the nonequispaceness of the data. (A different version of RA/ SFA in [9] is linear in $B$, but maybe less easily used when not all equispaced data are available.)

### 5.3 Experiments for Different Percentage of Missing Data

The advantage of interpolation techniques is to recover a signal even when a large percentage of data is missing. Table 7 shows the recovery effect for a two-mode pure signal $c_1 \phi_{\omega_1} + c_2 \phi_{\omega_2}$, $N = 10^6$ with all the other parameters $\epsilon$ and $\delta$ the same as before. When the percentage of available data is large, both algorithms recover the signal well with similar running time.

We tried another example of signal when $N = 100$. NERA/ SFA without interpolation techniques fails to recover the signal with high probability if more than 45% data are unavailable. In contrast, with the help of interpolation technique, the NERA/ SFA can always recover the signal with only 25% available data.

Experiments also show that for NERA/ SFA with interpolation technique, the total number of
Table 5: Experiments with fixed $N = 2^{18}$, $p = 0.6$, $d = 1$ (one dimension), $\sigma = 0.05$, and varying number of modes $B$ of signals. For each length of the signal, 10 different runs were carried out; the average result is shown. We did all the tests for NERA/SFA with interpolation techniques. We present two different time costs of NERA/SFA, with and without sampling.

| Number of Modes $B$ | SNR (dB) | NERA/SFA (+sampling) | NERA/SFA (w/o sampling) | INFFT |
|---------------------|----------|----------------------|-------------------------|-------|
| 2                   | 58       | 0.06                 | 0.01                    | 1.35  |
| 4                   | 64       | 0.24                 | 0.06                    | 1.35  |
| 6                   | 68       | 0.61                 | 0.23                    | 1.35  |
| 8                   | 70       | 1.44                 | 0.69                    | 1.35  |
| 10                  | 72       | 2.45                 | 1.39                    | 1.35  |
| 13                  | 74       | 5.78                 | 3.64                    | 1.35  |
| 16                  | 76       | 10.03                | 7.17                    | 1.35  |

Figure 6: Time Comparison between INFFT and NERA/SFA for different $B$ with fixed $N = 2^{18}$, $p = 0.6$, $d = 1$ (one dimension), $\sigma = 0.05$, a graph of the result in Table 5. The $x$ coordinate is the number of modes $B$, the $y$ coordinate presents running time. The running time of NERA/SFA is polynomial to $B$. In contrast, the time of INFFT keeps constant for different $B$, excluding precomputation for the samples. NERA/SFA without sampling begins to be slower than INFFT at $B = 10$ for $N = 2^{18}$.
Table 7: Experiments with fixed $B = 2, N = 10^6$, no noise, and varying percentage of available data. Each entry is based on the average of 10 different runs. In each run, the number of iterations is limited to 200; (this also corresponds to a fixed limit to the number of samples taken.) the success probability indicates the number of runs in which all 6 modes were found. When only 30% of data is available, the NERA/$\ell$SFA without interpolation cannot find all two significant modes within 200 iterations.

available data, instead of the percentage of available data determines the success probability. On the contrary, The success of NERA/$\ell$SFA without interpolation is determined by the percentage.

5.4 Experiments to Recover Noisy Signals

To recover a signal from very noisy data is a challenging problem. The following tests are done for $S(t) = \sum_{i=1}^{B} c_i \phi_{\omega_i} + \nu$, $B = 6$, $\epsilon = 0.02$, $N = 2^{17}$, $p = 0.6$, and different standard deviation $\sigma$ for noise. The amplitude of noise is still multiplied by a factor of $1/\sqrt{N}$. As Table X shows, NERA/$\ell$SFA excels at extracting information from noisy data even in the case of small signal to noise ratio.

6 Conclusion

We provide a sublinear sampling algorithm that recovers, with high probability, a $B$-term Fourier representation for an unevenly spaced signal. It is faster than any existed methods for processing sparse signals of large size. Moreover, it recovers the signal in the situation of large percentage of missing data or small signal to noise ratio.
Table 8: Experiments with fixed $B = 6$, $N = 2^{17}$, $p = 0.6$, and varying noise levels. For each noise level, 10 different runs were carried out; the average result is shown. In each run, the number of iterations is limited to 200; (this also corresponds to a fixed limit to the number of samples taken.) the success probability indicates the number of runs in which all 6 modes were found. The average relative error is the error of reconstructed signal with respect to the original signal.

| $\sigma$ (dB) | SNR (dB) | Time of NERA/SFA (+sampling) | Time of NERA/SFA (w/o sampling) | Relative Error (%) | Success probability |
|---------------|----------|-----------------------------|-------------------------------|-------------------|-------------------|
| 0             | -        | 0.48                        | 0.21                          | 0.02              | 100%              |
| 0.5           | 27.60    | 0.56                        | 0.22                          | 2.00              | 100%              |
| 1.0           | 15.56    | 0.87                        | 0.32                          | 4.50              | 90%               |
| 1.5           | 8.53     | 3.94                        | 1.59                          | 5.83              | 80%               |
| 2.0           | 3.52     | 4.78                        | 1.86                          | 7.67              | 50%               |
| 2.5           | -0.35    | 7.96                        | 2.14                          | 8.50              | 30%               |

Acknowledgments

For many helpful suggestions and discussions, I would thank my adviser Ingrid Daubechies. In addition, I thank Weinan E, Anna Gilbert, Martin Strauss for their suggestions.

Appendix

How to interpolate the two dimensional data to get values for missing points

In one dimension, values of missing points can be interpolated by its few nearest left and right available neighbors. The idea can be extended to higher dimensional cases with more techniques.

For instance, in two dimensions, we first find four nearest available neighbors of a missing point in each quadrant. Suppose a missing point is $(x, y)$, its four neighbors are $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$, $(x_4, y_4)$. The weights of neighbors can be derived by solving the following linear system of equations.

$$
\begin{pmatrix}
x_1 & x_2 & x_3 & x_4 \\
y_1 & y_2 & y_3 & y_4 \\
x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 \\
1 & 1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
w_1 \\
w_2 \\
w_3 \\
w_4
\end{pmatrix} =
\begin{pmatrix}
x \\
y \\
x y \\
1
\end{pmatrix}
$$

(35)

However, the matrix in (35) could be singular. In this case we choose the three nearest neighbors in different quadrants and use the following equations:

$$
\begin{pmatrix}
x_1 & x_2 & x_3 \\
y_1 & y_2 & y_3 \\
1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
w_1 \\
w_2 \\
w_3
\end{pmatrix} =
\begin{pmatrix}
x \\
y \\
1
\end{pmatrix}
$$

(36)
Figure 9: Some geometrical shapes of available neighboring points that occur most often. A missing point (denoted by a small cross) is at the center of the cross. Available points are denoted by dots. Left: the four available neighbors are located in the shape of cross. The distances of each neighbor to the missing point are equal. Right: almost the same as configuration in the left side, except one point moved off to the diagonal.

The time to locate those nearest neighbors and compute corresponding weights is considered a part of precomputation and excluded from total running time.

Note that we can use geometrical arguments to simplify the pre-computation of the weights. One easily sees that the system of equations (35) is translation invariant: the two linear system of equations

\[
\begin{pmatrix}
  x_1 + l \\
  y_1 + p \\
  (x_1 + l)(y_1 + p) \\
  1
\end{pmatrix}
\begin{pmatrix}
  x_2 + l \\
  y_2 + p \\
  (x_2 + l)(y_2 + p) \\
  1
\end{pmatrix}
\begin{pmatrix}
  x_3 + l \\
  y_3 + p \\
  (x_3 + l)(y_3 + p) \\
  1
\end{pmatrix}
\begin{pmatrix}
  x_4 + l \\
  y_4 + p \\
  (x_4 + l)(y_4 + p) \\
  1
\end{pmatrix}
\begin{pmatrix}
  w_1 \\
  w_2 \\
  w_3 \\
  w_4
\end{pmatrix}
= 
\begin{pmatrix}
  l \\
  p \\
  lp \\
  1
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
  x_1 & x_2 & x_3 & x_4 \\
  y_1 & y_2 & y_3 & y_4 \\
  1 & 1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
  w_1 \\
  w_2 \\
  w_3 \\
  w_4
\end{pmatrix}
= 
\begin{pmatrix}
  0 \\
  0 \\
  0 \\
  1
\end{pmatrix}
\]

have the same solutions for any \( l \) and \( p \). That means the location of the missing points does not influence the weights. Only the geometrical shape and relative distance of the available neighbors of a missing point matters.

Thus, we compute weights for the geometrical shapes of available neighboring points which occur most often. As we go through every missing point, we check if the shape of its neighboring available points matches those popular ones; if it does, we can directly get the weights without computation. This saves a huge amount of work, especially when \( p \) is large.
Table 10: Two possibilities corresponding to the geometrical shapes in Figure 9. The parameter \( p \) is the percentage of available data. The left side of Figure 9 happens with probability \( p^4 \); the right side appears with probability \( 4p^4(1-p)(2-p) \).

| \( p \) | \( p^4 \) | \( 4p^4(1-p)(2-p) \) | sum: \( p^4 + 4p^4(1-p)(2-p) \) |
|---|---|---|---|
| 1 | 100% | 0 | 100% |
| 0.9 | 65% | 29% | 94% |
| 0.8 | 41% | 39% | 80% |
| 0.7 | 24% | 37% | 61% |
| 0.6 | 13% | 29% | 42% |
| 0.5 | 6% | 19% | 25% |

For example, if the four neighboring points are located in the shape of a cross with the missing point as their center, as the left side of Figure 9 shows, then all of the weights are equal to one quarter. This situation happens with probability \( p^4 \), which is almost \( 2/3 \) when \( p = 0.9 \). Another often occurring case typically has one of the four neighbors of the previous configuration moved off to the diagonal (see the right side of Figure 9), which happens with probability \( 4p^4(1-p)(2-p) \), i.e., about 28% when \( p = 0.9 \). In this case, the two neighbors on the same line as the mirroring points have a weight 0.5 respectively; the other two points have weight zero. Table 10 shows the probabilities of these two situations as \( p \) varies.

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