Replica Analysis for Portfolio Optimization with Single-Factor Model

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In this paper, we use replica analysis to investigate the influence of correlation among the return rates of assets on the solution of the portfolio optimization problem. We consider the behavior of the optimal solution for the case where the return rate is described with a single-factor model and compare the findings obtained from our proposed methods with correlated return rates with those obtained with independent return rates. We then analytically assess the increase in the investment risk when correlation is included. Furthermore, we also compare our approach with analytical procedures for minimizing the investment risk from operations research.

KEYWORDS: mean-variance model, single-factor model, investment risk, investment concentration, replica analysis

In recent decades, investment strategies for the portfolio optimization problem have been considered extensively using a combination of analytical approaches from different research fields, including econophysics and statistical mechanical informatics.1–15) Recently, the mean-variance model, which is one of the most popular portfolio optimization problems, has been the subject of renewed interest in a variety of cross-disciplinary studies.5–15) In particular, the objective function for the investment risk in the mean-variance model is mathematically similar to the Hamiltonian of the Hopfield model, which has been widely used in studies on the associative memory problem, as both objective functions are described by using the quadratic form with respect to thermodynamic variables, and Hebb’s rule is related to the variance-covariance matrix of the return rate.6) The optimal portfolio which minimizes the investment risk is also interpreted as corresponding to the ground state in the spin glass model, and consequently, several previous studies have applied techniques that were developed in spin glass theory such as replica analysis, belief propagation, and random matrix theory to investigate the optimal portfolio.

Although in5–15) it is usually assumed that the return rates are independent, the return rates of assets in actual investment portfolios may be correlated, meaning that the models developed in these studies may underestimate the risk of loss (negative return rates) and should be used with caution. To analyze the portfolio optimization problem analytically with correlated return rates, we need to utilize and extend existing methods from a variety of fields. As a first step for characterizing the correlation among return rates, we consider a single-factor model that is widely used in mathematical finance and discuss whether the optimal portfolio which minimizes the investment risk with budget constraints is affected by correlation among the return rates using replica analysis.

Following previous work, we begin by considering the situation where rational investors invest into \( N \) assets over \( p \) periods in a steady investment market with no short-selling. The portfolio of asset \( i (i=1,2,\ldots, N) \) is denoted by \( w_i \in \mathbb{R} \), and \( \bar{w} = (w_1, w_2, \ldots, w_N)^T \in \mathbb{R}^N \) is the entire portfolio, where \( T \) denotes its transpose. Since there is no short-selling, we note that \( w_i \) is not always positive. Furthermore, \( \bar{x}_{ij} \) indicates the return rate of asset \( i \) at period \( \mu (\mu = 1, 2, \ldots, p) \) and its expectation is \( E[\bar{x}_{ij}] \). Then, in investing periods, the investment risk of portfolio \( \bar{w} \), \( \mathcal{H}(\bar{w}|X) \), is defined as follows:

\[
\mathcal{H}(\bar{w}|X) = \frac{1}{2N} \sum_{\mu=1}^{p} \left( \sum_{i=1}^{N} \bar{x}_{ij}w_i - \sum_{i=1}^{N} E[\bar{x}_{ij}]w_i \right)^2 
\]

\[
= \frac{1}{2} \bar{w}^T J \bar{w}, \quad (1)
\]

where \( x_{ij} = \bar{x}_{ij} - E[\bar{x}_{ij}] \) is the modified return rate and the return rate matrix \( X = \{x_{ij}\} \in \mathbb{R}^{N \times p} \) is defined using the modified return rates, and entry \( i, j \) of the variance-covariance (or Wishart) matrix \( J = \{J_{ij}\} = XX^T \in \mathbb{R}^{N \times N} \) in \( J_{ij} = \frac{1}{N} \sum_{\mu=1}^{p} \bar{x}_{ij} \bar{x}_{j\mu} = (XX^T)_{ij} \). Here, the budget constraint

\[
\sum_{i=1}^{N} w_i = N \quad (2)
\]

is used. From this, we need to determine the optimal portfolio which minimizes the investment risk \( \mathcal{H}(\bar{w}|X) \) in Eq. (1) from the set of portfolios that satisfy the budget constraint in Eq. (2). With respect to the optimal portfolio \( \bar{w}^* = \arg \min_{\bar{w} \in W} \mathcal{H}(\bar{w}|X) \), determining analytically the minimal investment risk per asset \( \varepsilon = \frac{1}{N} \mathcal{H}(\bar{w}^*|X) \) and its investment concentration \( q_w = \frac{1}{N} (\bar{w}^*)^T \bar{w}^* \) is one of the most active issues being researched for the portfolio optimization problem, and a variety of cross-disciplinary approaches have been developed. Here, \( W = \)
\{ \vec{w} \in \mathbb{R} \mid \sum_{i=1}^{N} w_i = N \} \) is the feasible subset of portfolios satisfying Eq. (2). Our previous work\(^6\) discussed the case where \( x_{i\mu} \) is independently and identically distributed with mean 0 and variance 1, and the minimal investment risk per asset \( \varepsilon \) and its investment concentration \( q_w \) were determined as follows:

\[
\varepsilon = \frac{\alpha - 1}{2}, \quad q_w = \frac{\alpha - 1}{\alpha - 1} \tag{3}
\]

For the case where \( x_{i\mu} \) is independently distributed with mean 0 and variance \( v_i \), that is, the variance of each asset is distinct, the minimal investment risk per asset \( \varepsilon \) and its investment concentration \( q_w \) were also determined as follows:

\[
\varepsilon = \frac{\alpha - 1}{2(v_i)}, \quad q_w = \frac{\alpha - 1}{\alpha - 1} \tag{4}
\]

where \( \alpha = p/N \sim O(1) \). In order to determine uniquely the optimal portfolio \( \vec{w}^* \), the squared matrix \( J \) should be regularized, and then the above-mentioned results hold for \( \alpha > 1 \). Similarly, in the present work, we assume for \( \alpha > 1 \), the optimal portfolio is uniquely determined. Moreover, the notation \( \langle g(v) \rangle = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} g(v_i) \) is used.

Namely, in previous work, the return rates \( x_{i\mu} \) were assumed to be independently and identically distributed with mean 0 and variance 1, or independently (but not identically) distributed with mean 0 and variance \( v_i \). However, the return rates of assets in many practical situations are correlated, and the findings in previous work which assumed independent rates may be unsuitable for practical applications, as they will underestimate the investment risk. Thus, as a first step for characterizing the correlations among return rates, we should analyze the minimal investment risk per asset \( \varepsilon \) and its investment concentration \( q_w \) for the portfolio minimizing the investment risk for the case where the return rate of each asset is determined with a single-factor model. Here, using a single-factor model, the return rate \( x_{i\mu} \) is defined as follows:

\[
x_{i\mu} = \frac{1}{\sqrt{N}} b_i f_\mu + y_{i\mu}, \tag{7}
\]

where \( \frac{1}{\sqrt{N}} \) is the scaling parameter which can be adjusted to simplify the analytical results. Moreover, \( f_\mu \) is the macroeconomic indicator at period \( \mu \) (the probability of \( f_\mu \) is already known and its mean is assumed to be 0, and we do not require the indicator to be normally distributed), and \( b_i \) denotes the level of influence of the macroeconomic indicator \( f_\mu \) on asset \( i \). (Hereafter we call this the factor loading. The probability of \( b_i \) is also assumed to be known and does not need to be normally distributed.) Further, the (independent) return rate \( y_{i\mu} \) is independent of the other return rates and is not correlated with macroeconomic indicator \( f_\mu \) and factor loading \( b_i \), and the mean and the variance are 0 and \( v_i \), respectively. That is, \( x_{i\mu} \) in Eq. (7) is regarded as a linear regression equation with noise term \( y_{i\mu} \). In general, since macroeconomic indicators may include temporal trends, we do not assume independence among macroeconomic indicators. Similarly, there may exist correlation among factor loadings, and the assumption of independence among factor loadings is not required in this work.

Let us reformulate the above optimization problem in the framework of statistical mechanical informatics and analyze the minimal investment risk per asset \( \varepsilon \) and its investment concentration \( q_w \) using replica analysis. First, from the framework of statistical mechanical informatics, the partition function \( Z(\beta, X) \) at inverse temperature \( \beta(> 0) \) is defined as follows:

\[
Z(\beta, X) = \int_{\vec{w} \in \mathcal{W}} d\vec{w} e^{-\beta H(\vec{w}||X)}. \tag{8}
\]

From this, we can determine the average of the logarithm of the partition function per asset as follows:

\[
\phi = \lim_{N \to \infty} \frac{1}{N} E \log Z(\beta, X) = \lim_{N \to \infty} \frac{1}{N} \frac{\partial}{\partial \alpha} \log [Z^\alpha(\beta, X)]. \tag{9}
\]

From the formula

\[
\varepsilon = - \frac{\partial \phi}{\partial \beta}, \tag{10}
\]

we can evaluate the minimal investment risk per asset analytically, where the notation \( E[g(X)] \) means the expectation of \( g(X) \) with respect to the return rate. From replica analysis,

\[
\phi = \text{Extr}_{\Theta} \left\{ -k - \beta \chi + \frac{1}{2}(\chi w + q w)(\tilde{\chi} w - \tilde{q} w) \right. \\
+ \frac{1}{2} q w \tilde{q} w + \frac{1}{2}(\chi w + q w)(\tilde{\chi} w - \tilde{q} w) + \frac{1}{2} q w \tilde{q} w \\
- \alpha \frac{1}{2}(1 + \beta \chi w) - \frac{\alpha \beta (q w + F m^2)}{2(1 + \beta \chi w)} \\
- \frac{1}{2}(\chi w + v X w) \\
+ \left. \frac{1}{2} \left( \tilde{q} w + v \tilde{q} w + k + bh \right)^2 \right\}, \tag{11}
\]

is obtained, where \( \text{Extr}_r g(r) \) is the extremum of \( g(r) \) with respect to the parameter \( r \) and \( \Theta = \{ k, m, h, \chi w, q w, w w, \chi w, q w, \chi w, q w \} \) represents the set of order parameters,

\[
F = \lim_{p \to \infty} \frac{1}{N} \sum_{\mu=1}^{p} f^2 \mu, \tag{12}
\]

\[
\langle g(b, v) \rangle = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} g(b_i, v_i), \tag{13}
\]

and \( \alpha = p/N \sim O(1) \) (see Appendix for further details). Note that since \( F \) in Eq. (12) is the average of the square of the macroeconomic indicators, we can determine \( F \) easily, regardless of the presence or absence of correlation among the macroeconomic indicators. In addition, from
Eq. (13), it is also easy to assess \((g(b, v))\) regardless of the presence or absence of correlation among factor loadings. From the above, 
\[
\chi_w = \langle v^{-1} \rangle \beta(\alpha - 1),
\]
\[
q_w = \frac{1}{\alpha - 1} \left( 1 + Fmm(\langle v^{-1} \rangle) + C \right),
\]
\[
\chi_s = \frac{1}{\beta(\alpha - 1)},
\]
\[
q_s = \frac{1}{\langle v^{-1} \rangle} + F^2m^2V_1\langle v^{-1} \rangle + \frac{1}{\alpha - 1} \left( \frac{1}{\langle v^{-1} \rangle} + Fmm \right),
\]
are determined, where 
\[
m = \frac{m_1}{1 + FV_1\langle v^{-1} \rangle}.
\]
Furthermore, 
\[
m_1 = \frac{(1 - b_1^2)}{(1 - v_1)}, \quad V_1 = \frac{(1 - b_1^2)}{(1 - v_1)} - \left( \frac{(1 - b_1^2)}{(1 - v_1)} \right)^2,
\]
\[
m_2 = \frac{(1 - b_2^2)}{(1 - v_2)}, \quad V_2 = \frac{(1 - b_2^2)}{(1 - v_2)} - \left( \frac{(1 - b_2^2)}{(1 - v_2)} \right)^2,
\]
and 
\[
C = F^2m^2V_2\langle v^{-2} \rangle + \frac{(1 - b_2^2)}{(1 - v_2)} \left( 1 + Fm(m_1 - m_2) \langle v^{-1} \rangle \right)^2.
\]
From this, the minimal investment risk per asset \(\varepsilon\) is derived using Eq. (10), 
\[
\varepsilon = -\lim_{\beta \to \infty} \frac{\partial \varepsilon}{\partial \beta} = \lim_{\beta \to \infty} \left\{ \frac{\alpha q}{2(1 + \beta \chi_v)} + \frac{q q_m}{2(1 + \beta \chi_v)} \right\}
\]
as follows:
\[
\varepsilon = \frac{\alpha - 1}{2\langle v^{-1} \rangle} + \frac{\alpha - 1}{2} Fmm. \tag{19}
\]
We note that \(mm_1 \geq 0\) is determined from Eq. (18), so that this findings is not smaller than the one obtained in our previous work, (or see Eq. (5)).

We now consider whether the models obtained in the present work include the results obtained in previous work as special cases. First, from the assumption of independent return rates that are not influenced by macroeconomic indicators, that is, when \(b_i = 0\), \(m = m_1 = m_2 = 0\), and \(V_1 = V_2 = 0\), Eqs. (15) and (19) become
\[
\varepsilon = \frac{\alpha - 1}{2\langle v^{-1} \rangle}, \tag{20}
\]
\[
q_w = \frac{1}{\alpha - 1} + \frac{\langle v^{-2} \rangle}{\langle v^{-1} \rangle^2}, \tag{21}
\]
where \(C = \frac{(1 - b_2^2)}{(1 - v_2)}\). These equations are consistent with the results of previous work (Eqs. (5) and (6)). Further, the second term in Eq. (19), \(\frac{\alpha - 1}{2} Fmm\), quantifies the influence from common factors \(f_1, f_2, \cdots, f_p\) in a single-factor model. Note that \(Fmm\) is a monotonic nondecreasing function with respect to \(F\), \(\lim_{F \to \infty} Fmm = 0\) and \(\lim_{\beta \to \infty} Fmm = \frac{m^2}{v_1}\). In addition, although it is a superfluous consideration, for the case where the variance of the return rate of each asset is unique, that is, when \(v_i = 1\), by substituting \(\langle v^{-1} \rangle = \langle v^{-2} \rangle = 1\) into Eqs. (20) and (21), Eqs. (3) and (4) are obtained. Since our results include the findings obtained in previous work as special cases, it is confirmed that our model is a natural extension which can handle the case of correlated return rates.

Finally, we compare the minimum expected investment risk which is obtained with an analytical procedure that is well known in operations research. First, from the portfolio which minimizes the expected investment risk \(E[\mathcal{H}(\bar{w} | X)]\), that is, \(\bar{w}^{\text{OR}} = \arg \min_{\bar{w} \in W} E[\mathcal{H}(\bar{w} | X)]\), the minimum expected investment risk per asset \(\varepsilon^{\text{OR}} = \lim_{N \to \infty} \frac{1}{N} E[\mathcal{H}(\bar{w}^{\text{OR}} | X)]\) can easily be obtained as follows:
\[
\varepsilon^{\text{OR}} = \frac{\alpha}{2 \langle v^{-1} \rangle} + \frac{\alpha - 1}{2} Fmm. \tag{22}
\]
From this, the opportunity loss \(\kappa = \frac{\varepsilon^{\text{OR}}}{\varepsilon}\) is computed as follows:
\[
\kappa = \frac{\alpha}{\alpha - 1}. \tag{23}
\]
Using a similar argument as in our previous work,\(^{15}\) we note that although the opportunity loss \(\kappa\) depends on the period ratio \(\alpha\), it does not depend on the statistical properties of \(v_i, b_i, f_i\). Moreover, from the investment concentration of the portfolio \(\bar{w}^{\text{OR}}\) which is derived analytically using a procedure from operations research, \(q_w = \frac{1}{N}(\bar{w}^{\text{OR}})^{T} \bar{w}^{\text{OR}}\) is calculated as follows:
\[
q_w^{\text{OR}} = F^2m^2V_2 \langle v^{-2} \rangle + \frac{\langle v^{-2} \rangle}{\langle v^{-1} \rangle^2} \left( 1 + Fm(m_1 - m_2) \langle v^{-1} \rangle \right)^2, \tag{24}
\]
where it is found that \(q_w^{\text{OR}}\) corresponds to the last term \(C\) in the investment concentration \(q_w\) of the optimal portfolio \(\bar{w}^{\text{OR}}\) in Eq. (15). As noted in,\(^6\) since rational investors prefer to invest in assets whose risks are comparatively low, in the investing periods when \(\alpha\) is close to 1 the risks of the assets vary greatly and the investment concentration of the optimal portfolio increases. In contrast, when \(\alpha\) is sufficiently large, the risks of the assets are almost indistinguishable in terms of the return rates, and rational investors will invest equally across all assets; therefore, the investment concentration will tend to be low. This behavior is reflected in our proposed approach; however, the investment concentration of the portfolio \(\bar{w}^{\text{OR}}\) derived with the approach from operations research, \(q_w^{\text{OR}}\), is always constant with period ratio \(\alpha\), and this is inconsistent with the optimal investment behavior of the rational investors. We have also verified that the analysis of the annealed disordered systems (related to the ordinary operations research approach) is distinct from the analysis of quenched disordered systems (the analytical procedure based on our proposed method which corresponds to the analysis of the optimal investment strategy).

In the present work, we have analyzed the minimization of the investment risk with budget constraints for the case of correlated return rates using a cross-disciplinary replica analysis approach from econophysics and statistical mechanical informatics. As there are many different types of dependence among the return rates of assets in an actual investment market, we used the single-factor model, as it is one of the most fundamental mod-
els for correlation among return rates in mathematical finance. We discussed whether the correlation among return rates characterized by a single-factor model would influence the optimal solution. Further, we compared our approach with the findings obtained from previous work and verified the effectiveness of the methodology proposed here for determining analytically and explicitly the investment risk in the presence of correlated assets.

In actual investment markets, there are a myriad of macroeconomic indicators, and thus, in future work we will try to adapt the techniques developed for the associative memory problem,\textsuperscript{16–18} to apply them to the optimization problem when the number of factors is $O(1)$ and $O(N)$.

This paper discussed the investment risk minimization problem with budget constraints only, while the investment risk minimization problem in practice involves several constraints, for instance, the expected return and investment concentration constraints, and we need to analyze how these additional constraints influence the optimal solution for minimizing the investment risk with and without correlated return rates.

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Appendix

In this appendix, we derive $\phi$ using replica analysis. Following the discussion in our previous work,\textsuperscript{15} $E[Z^n(\beta, X)]$, $(n \in Z)$ is described as follows:

\begin{equation}
E[Z^n(\beta, X)] = \text{Extr}_{\beta N} \frac{1}{(2\pi i)^m} \int_{C_N} \prod_a d\tilde{a}_a d\tilde{a}_a d\tilde{z}_a \exp \left( -\frac{\beta}{2} \sum_{\mu,a} \tilde{z}_{\mu,a}^2 + \sum_{a} k_a \left( \sum_i w_{ia} - N \right) + \sum_{\mu,a} u_{\mu,a} \left( z_{\mu,a} - \frac{1}{\sqrt{N}} \sum_i w_{ia} x_{i\mu} \right) \right),
\end{equation}

where $\Pi_a$ displays $\Pi_{i=1}^n$, $\sum_i$ is $\sum_{i=1}^{n_k}$, $\sum_{\mu}$ means $\sum_{\mu=1}^{n_\mu}$, and $\sum_a$ means $\sum_{a=1}^{n_a}$. Moreover, $\tilde{w}_a = (w_{1a}, w_{2a}, \ldots, w_{Na})^T \in \mathbb{R}^N$, $\tilde{a}_a = (a_{1a}, a_{2a}, \ldots, a_{pa})^T \in \mathbb{R}^p$, $\tilde{z}_a = (z_{1a}, z_{2a}, \ldots, z_{pa})^T \in \mathbb{R}^p$, $(a = 1, \ldots, n)$, $\tilde{k} = (k_1, k_2, \ldots, k_n)^T \in \mathbb{R}^n$, and $\tilde{k}_a$ is the auxiliary parameter related to the budget constraint in Eq. (2). Next, the order parameters are defined by

\begin{align}
    m_a &= \frac{1}{N} \sum_{i=1}^N b_i w_{ia}, \quad (A-2) \\
    q_{wab} &= \frac{1}{N} \sum_{i=1}^N w_{ia} w_{ib}, \quad (A-3) \\
    q_{sab} &= \frac{1}{N} \sum_{i=1}^N v_i w_{ia} w_{ib}, \quad (A-4)
\end{align}

and $h_a, \tilde{q}_{wab}, \tilde{q}_{sab}$ are the conjugate parameters, where $a, b = 1, 2, \ldots, n$. From the ansatz of the replica symmetry solution, which comprises $k_a = k, m_a = m, h_a = h, q_{waa} = \chi_w + \tilde{q}_w, q_{wab} = q_{w}, (a \neq b), q_{wab} = \tilde{q}_w, (a \neq b), q_{saa} = \chi_s + \tilde{q}_s, q_{sab} = q_{s}, (a \neq b),$ and $\tilde{q}_{saa} = \chi_s - \tilde{q}_s, \tilde{q}_{sab} = -\tilde{q}_s, (a \neq b)$, the following is obtained:

\begin{align}
    \lim_{N \to \infty} \frac{1}{N} \log E[Z^n(\beta, X)] &= \text{Extr}_{\beta N} \left\{ -nk - nhm + \frac{n}{2} (\chi_w + \tilde{q}_w)(\tilde{\chi}_w - \tilde{q}_w) - \frac{n(n-1)}{2} q_{wab} (\tilde{\chi}_w + \tilde{q}_w)(\tilde{\chi}_s - \tilde{q}_s) - \frac{n(n-1)}{2} q_{sab} (\tilde{\chi}_w - \tilde{q}_w) \left( \tilde{\chi}_s - \tilde{q}_s \right) \right\}.
\end{align}

We substitute this result into Eq. (9) and obtain Eq. (11).

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