ON AN ANALYTIC ESTIMATE IN THE THEORY OF THE RIEHMANN ZETA FUNCTION AND A THEOREM OF BÁEZ-DUARTE

JEAN-FRANÇOIS BURNOL

Abstract. We establish a uniform upper estimate for the values of \(\zeta(s)/\zeta(s + A)\), \(0 \leq A\), on the critical line (conditionally on the Riemann Hypothesis). We use this to give a variant, purely complex analytic, to Báez-Duarte’s proof of a strengthened Nyman-Beurling criterion for the validity of the Riemann Hypothesis.

1. Introduction

The following theorem has been established by Báez-Duarte \[\{x\} = x - \lfloor x \rfloor\] is the fractional part of the real number \(x\):

**Theorem 1.1** (Báez-Duarte \[\{x\} = x - \lfloor x \rfloor\]). If the Riemann Hypothesis holds then the function \(1_{0 < t \leq 1}\) belongs to the closure in \(L^2((0, \infty), dt)\) of the finite linear combinations of the functions \(t \rightarrow \{1/nt\}\), \(n \geq 1\), \(n \in \mathbb{N}\).

That the converse holds is (a special case of) the easy half of the classical Nyman-Beurling criterion \[\{x\} = x - \lfloor x \rfloor\] (in a minor variant as the original formulation is entirely inside \(L^2((0, 1), dt)\).) The not-so-easy other half of the original version of this criterion states that the Riemann Hypothesis implies that \(1_{0 < t \leq 1}\) belongs to the span of the functions \(t \rightarrow \{1/\Lambda t\}\) where \(\Lambda\) varies in the continuous interval \([1, \infty)\). The connection with the Riemann zeta function is established with the help of the classical formula

\[
\int_0^\infty \left\{ \frac{1}{t} \right\} t^{s-1} dt = -\frac{\zeta(s)}{s},
\]

where the integral is absolutely convergent for \(0 < \text{Re}(s) < 1\). Generally speaking the Mellin transform \(f(t) \mapsto \hat{f}(s) = \int_0^\infty f(t)t^{s-1} dt\) establishes an isometry between \(L^2((0, \infty), dt)\) and the Hilbert space of square-integrable functions on the critical line for the measure \(|ds|/2\pi\). So the Nyman-Beurling criterion is that the Riemann Hypothesis holds if and only if one can approximate in the square-mean sense the function \(1/s\) on the critical line by expressions \(\zeta(s)P(s)/s\) where \(P(s) = \sum_k c_k \Lambda_k^{-s}\), \(\Lambda_k \geq 1\).

This topic has attracted some interest in recent years in a number of publications, among them \[\{1, 2, 3, 5, 6, 7, 12, 13, 14, 17, 18\}\.

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and it was expected from the expression $1/\zeta(s) = \sum_{n=1}^{\infty} \mu(n)n^{-s}$ that the values $\Lambda_k = n$, $n \in \mathbb{N} \setminus \{0\}$ are enough. This is what has been established by Báez-Duarte. We state again here for future reference his theorem in complex analytic form:

**Theorem 1.2** (Báez-Duarte [4]). If the Riemann Hypothesis holds then $1/s$ can be arbitrarily well approximated in square-mean on the critical line by functions $\zeta(s)P(s)/s$ with $P(s) = \sum_{n=1}^{N} c_{n,N} n^{-s}$ a Dirichlet polynomial.

**Remark 1.1.** It remains an open problem to exhibit for $N \to \infty$ an explicit sequence of “natural approximations” $\sum_{n=1}^{N} c_{n,N} n^{-s}$. It is known [5] that the Hilbert space distance from $\zeta(s)P(s)/s$ to $1/s$ is bounded below asymptotically by $C/\sqrt{\log(N)}$ for a $C > 0$ (and from [9] it is known that $C \geq 2 + \gamma - \log(4\pi)$.) This applies even to Dirichlet polynomials $\sum_{k} c_k \Lambda_k^{-s}$ allowing non-integer $\Lambda_k$’s ($1 \leq \Lambda_k \leq N$). It is not known whether the restriction to integer $\Lambda$’s may result in a worse rate of convergence. It is naturally expected that this is not the case. But no explicit upper bound to the Hilbert space distance to the optimal natural approximation of degree $N$ is currently known.

It is a quite notable feature of Báez-Duarte’s proof that it frees completely the Nyman-Beurling criterion from any appeal to the theory of invariant subspaces of Hardy spaces. In this manner the topic of the Nyman-Beurling criterion becomes less foreign to the more classical analytic number theoretical topics as they are treated in Titchmarsh’s book [16].

The purpose of this paper is to reinforce this with the help of a purely complex analytical proof of 1.2. While inspired by the original proof and quite directly related to it, our approach relies on a novel analytic estimate on the Riemann zeta function, conditional on the Riemann Hypothesis:

**Theorem 1.3.** Let $\epsilon > 0$. Conditional on the Riemann Hypothesis one has:

$$\left| \frac{\zeta(s)}{\zeta(s + A)} \right| = O(\epsilon^{|s|^{\inf(\epsilon,A/2)}})$$

on the critical line, uniformly with respect to $0 \leq A < \infty$.

The other main component of the proof, as in Báez-Duarte’s, is the use of a Theorem of Balazard and Saias [6, Lemme 2], which is a version with a fine error estimate of the convergence of $\sum_{n=1}^{\infty} \mu(n)/n^s$ to $1/\zeta(s)$ for Re$(s) > 1/2$. We state a slightly weakened form of this Theorem which is directly suitable for our later use:

**Theorem 1.4** (Balazard-Saias [6]). Let $\frac{1}{2} > \epsilon > 0$ and $\theta > 0$. Conditional on the Riemann Hypothesis one has:

$$\sum_{n=1}^{N} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)} + O_{\epsilon,\theta}(\frac{|s|^\theta}{N^{\epsilon/3}})$$

on Re$(s) = 1/2 + \epsilon$. 
It will be apparent that the proof of the strengthened Nyman-Beurling criterion in fact would go through equally well with much coarser error estimates than the ones provided by \[1.3\] and \[1.4\]. But the future developments could possibly use the finer estimates.

This paper is organized as follows: in the first section we prove the Báez-Duarte strengthened “only if” Nyman-Beurling criterion using \[1.3\] and \[1.4\]. This takes only a few lines. The second section proves \[1.3\]. In a third section we show that the Riemann Hypothesis holds if and only if certain functions considered by Báez-Duarte are square-integrable, and we conclude with some related comments.

2. Proof of \[1.2\]

From \[1.3\] we have in particular on the critical line a uniform upper estimate for \(0 \leq A \leq 1/2\):
\[
\left| \frac{\zeta(s)}{\zeta(s + A)} \right| \leq C|s|^{1/4}
\]
which implies that the functions \(\zeta(s)/s\zeta(s + A)\) converge in square-mean on the critical line to \(1/s\) as \(A \to 0\). If we now pick one such fixed small \(A = \epsilon\) we have from \[1.4\] that
\[
\frac{\zeta(s)}{s} \sum_{n=1}^{N} \frac{\mu(n)}{n^{s+\theta}} = \zeta(s) s^\alpha \left( \frac{|s|^\theta}{N^{\epsilon/3}} \left| \frac{\zeta(s)}{s} \right| \right) + O_{\epsilon, \theta}
\]
We then only need to invoke (as \(\theta > 0\) may be chosen arbitrarily) a weak bound like \(|\zeta(s)| = O(|s|^{1/4})\) (\[16\], 5.1.8) on the critical line to conclude that the left hand side converge in square-mean sense as \(N \to \infty\) to \(\zeta(s)/\zeta(s + \epsilon)s\). This concludes the proof of \[1.2\].

It is apparent from this proof that any \(\theta < 1/4\) in \[1.4\] will do, even any \(\theta < 1/2\) if we are ready to use the Lindelöf Hypothesis, which is a known corollary to the Riemann Hypothesis (\[16\], XIV). Similarly we used only a coarse version of \[1.3\]. What is essential nevertheless is the uniformity as \(A \to 0\) which is granted by \[1.3\].

3. Proof of \[1.3\]

We restate \[1.3\]:

**Theorem 3.1.** If the Riemann Hypothesis holds one has for \(\epsilon > 0\)
\[
\left| \frac{\zeta(s)}{\zeta(s + A)} \right| = O_E(|s|^\inf(E, A/2))
\]
on the critical line, uniformly for \(0 \leq A < \infty\).

**Proof.** We fix \(\epsilon > 0\) and we may assume \(\epsilon < 1/4\). The range \(2 \leq A\) obviously reduces to the known Lindelöf Hypothesis bound \(|\zeta(s)| = O(|s|^\epsilon)\). For the range \(2\epsilon \leq A \leq 2\) it is enough to combine \(|\zeta(s)| = O(|s|^\epsilon/2)\) on the critical line with \(|1/\zeta(s)| = O(|s|^\epsilon/2)\) on \(\text{Re}(s) \geq 1/2 + 2\epsilon\) which is known to be true
under the Riemann Hypothesis ([16, XIV]). Finally for the range $0 \leq A \leq 2\varepsilon$ we invoke the next lemma.

**Lemma 3.1.** If the Riemann Hypothesis holds one has on the critical line

$$\left| \frac{\zeta(s)}{\zeta(s + A)} \right| = O(|s|^{A/2})$$

uniformly for $0 \leq A \leq 1/2$.

**Proof.** Assuming the validity of the Riemann Hypothesis we consider the function

$$\frac{\zeta(s - A/2)}{\zeta(s + A/2)}$$

for $\text{Re}(s) = 1/2$. Let $\gamma_+(s)$ be the function appearing in the functional equation of the zeta function:

$$\gamma_+(s) = \frac{\pi^{-s/2}\Gamma(s/2)}{\pi^{-(1-s)/2}\Gamma((1-s)/2)}$$

The following uniform estimate

$$|\gamma_+(w)| = O(|w|^\sigma - \frac{1}{2})$$

on $\text{Re}(w) = \sigma$, $1/4 \leq \sigma \leq 3/4$, is known ([16, 4.12.3]).

So on the critical line and uniformly for $0 \leq A \leq 1/2$:

$$\left| \frac{\zeta(s - A/2)}{\zeta(s + A/2)} \right| = |\gamma_+(s + A/2)| = O(|s|^{A/2})$$

Next we consider for a fixed $A > 0$ the holomorphic function of $s$, for $\text{Re}(s) \geq 1/2$, given by

$$\frac{(s - 1 - A/2)\zeta(s - A/2)}{(s - 1 + A/2)\zeta(s + A/2)} \frac{1}{s^{A/2}}$$

On a closed vertical strip $\frac{1}{2} \leq \text{Re}(s) \leq \Lambda$ we are in a position to apply the Phragmen-Lindelöf principle. Modest growth information is necessary inside the strip for $|\zeta(s - A/2)|$ and for $|1/\zeta(s + A/2)|$ and in fact as in the previous proof both functions are $O(|s|^C)$ for a suitable $C$. We then let $\Lambda \to \infty$ and the conclusion is that

$$\left| \frac{(s - 1 - A/2)\zeta(s - A/2)}{(s - 1 + A/2)\zeta(s + A/2)} \right| = O(|s|^{A/2})$$

on the closed half-plane $\text{Re}(s) \geq 1/2$, with an implied constant which is uniform with respect to $0 < A \leq 1/2$. We may of course include $A = 0$ now. The Lemma then follows from looking at the line $\text{Re}(s) = \frac{1}{2} + \frac{A}{2}$.

**Remark 3.1.** A slightly more direct method (in place of the Phragmen-Lindelöf principle) is to invoke the theory of Nevanlinna functions and a result such as [10, Lemma 2.2]. This or the simpler Phragmen-Lindelöf principle will also extend the bound [16] from the critical line to the closed half-plane $\text{Re}(s) \geq 1/2$ (avoiding a neighborhood of $s = 1$ of course.)
Remark 3.2. The bound $|\zeta(s-\epsilon)/\zeta(s+\epsilon)| = |\gamma_+(s+\epsilon)| = O(|s|^\epsilon)$ on the critical line also plays an important rôle in Báez-Duarte’s proof [4]. We comment more on this proof in the next section.

4. ON THE FUNCTIONS $f_\epsilon$ CONSIDERED BY BÁEZ-DUARTE AND THEIR RELATION WITH THE RIEMANN HYPOTHESIS

Báez-Duarte considers for $\epsilon > 0$ the function of $t > 0$, defined pointwise as:

$$f_\epsilon(t) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^\epsilon \{nt\}}$$

for $0 < \epsilon$. From the bound $\{1/nt\} \leq 1/nt$ one has absolute convergence and $|f_\epsilon(t)| \leq \zeta(1+\epsilon)/t$. Another expression is given by:

$$f_\epsilon(t) = \frac{1}{\zeta(1+\epsilon)t} - \sum_{n=1}^{\infty} \frac{\mu(n)}{n^\epsilon \{nt\}}$$

where the second term is pointwise a finite sum.

Let $C_n$ be the unitary operator $\phi(t) \to \sqrt{n}\phi(nt)$. One has pointwise $f_\epsilon(t) = \sum_{n\geq1} \mu(n)n^{-\epsilon-1/2}C_n(\{1/t\})$, so for $\epsilon > 1/2$ the series is absolutely convergent in $L^2$ and the $L^2$-Mellin transform $\hat{f}_\epsilon(s)$ is

$$\sum_{n\geq1} \frac{\mu(n)}{n^{\epsilon+1/2}n^{s-1/2}} \frac{-\zeta(s)}{s} = \frac{-\zeta(s)}{\zeta(s+\epsilon)s}$$

It is known (from [16, 3.6.5]) that $1/\zeta(s)$ is $O(|s|^\eta)$ in Re$(s) \geq 1$ ($\eta > 0$ arbitrary) and also that $\zeta(s) = O(|s|^{1/4})$ on Re$(s) = 1/2$ ([14, 5.1.8]). So as $\epsilon \to \frac{1}{2}^+$ the functions $f_\epsilon(s)$ on the critical line converge in square mean. Clearly $f_\epsilon(t) \to f_{1/2}(t)$ pointwise so this implies that $f_{1/2}(t)$ is square-integrable. Let us recapitulate these simple results:

Lemma 4.1. For $\epsilon \geq \frac{1}{2}$ the functions $f_\epsilon(t)$ are square-integrable and their $L^2$-Mellin transforms are:

$$\hat{f}_\epsilon(s) = \frac{-1}{\zeta(s+\epsilon)} \frac{\zeta(s)}{s}$$

It is apparent from this expression that the square-integrability of $f_\epsilon(t)$ for $\epsilon < 1/2$ should be related with the Riemann Hypothesis. Indeed Báez-Duarte proves, conditionally on the Riemann Hypothesis, that these functions are square integrable (at least for small $\epsilon$). He actually proves the stronger result that $t^{-\epsilon/2}f_\epsilon(t)$ is square-integrable and that its $(L^2)$-Mellin Transform is $-\zeta(s-\epsilon/2)/(s-\epsilon/2)\zeta(s+\epsilon/2)$. This is done so that the limit $\epsilon \to 0$ is easy to deal with. But we can copy his argument and apply it directly to $f_{1/2}(t)$: under the Riemann Hypothesis, and using the Theorem of Balazard-Saias [13,4], the $(L^2)$-Mellin transforms of the finite partial sums converge in square-mean to the function $-\zeta(s)/\zeta(s+\epsilon)s$. So the function
$f_{\epsilon}(t)$ which is their pointwise limit must be square-integrable. We sum this up as a Lemma, essentially contained in [4]:

**Lemma 4.2.** If the Riemann Hypothesis holds the functions $f_{\epsilon}(t)$ for $0 < \epsilon < 1/2$ are square-integrable and their $L^2$-Mellin transforms are:

$$\widehat{f}_{\epsilon}(s) = \frac{-1}{\zeta(s + \epsilon)} \zeta(s)$$

What we wish to point out here is the simple observation that the square-integrability of the function $f_{\epsilon}(t)$ in itself allows to say things about the Riemann Hypothesis quite independently of its connection with the Nyman-Beurling criterion.

Let $\epsilon > 0$. We have $f_{\epsilon}(t) = O(1/t)$ and in particular the function $\int_0^1 f_{\epsilon}(u)u^{s-1} du$ is analytic for $\text{Re}(s) > 1$. We first identify this function unconditionally:

**Lemma 4.3.** One has unconditionally for $\text{Re}(s) > 1$:

$$\int_0^1 f_{\epsilon}(u)u^{s-1} du = \frac{1}{\zeta(1 + \epsilon)} \frac{1}{s - 1} - \frac{1}{\zeta(s + \epsilon)} \frac{\zeta(s)}{s}$$

**Proof.**

$$\int_0^1 f_{\epsilon}(u)u^{s-1} du = \int_0^1 \left( \frac{1}{u} \frac{1}{\zeta(1 + \epsilon)} - \sum_{a=1}^{\infty} \frac{\mu(a)}{a^s} \frac{1}{au} \right) u^{s-1} du$$

$$= \frac{1}{\zeta(1 + \epsilon)} \frac{1}{s - 1} - \sum_{a=1}^{\infty} \frac{\mu(a)}{a^s} \int_0^1 \frac{1}{au} u^{s-1} du$$

$$= \frac{1}{\zeta(1 + \epsilon)} \frac{1}{s - 1} - \sum_{a=1}^{\infty} \frac{\mu(a)}{a^s} \int_0^{1/a} \frac{1}{v} v^{s-1} dv$$

$$= \frac{1}{\zeta(1 + \epsilon)} \frac{1}{s - 1} - \sum_{a=1}^{\infty} \frac{\mu(a)}{a^{s+\epsilon}} \int_0^{1/a} \frac{1}{v} v^{s-1} dv$$

$$= \frac{1}{\zeta(1 + \epsilon)} \frac{1}{s - 1} - \sum_{a=1}^{\infty} \frac{\mu(a)}{a^{s+\epsilon}} \frac{\zeta(s)}{s}$$

\[ \square \]

**Theorem 4.1.** If $f_{\epsilon} \in L^2$ then its $L^2$-Mellin transform on the critical line is the function

$$\frac{-1}{\zeta(s + \epsilon)} \frac{\zeta(s)}{s}$$

Furthermore this function is analytic in $\text{Re}(s) \geq \frac{1}{2}$ except for a simple pole at $s = 1$. 
Proof. Let us assume that $f_\epsilon \in L^2$. Its restriction to $t > 1$ is a non-zero multiple of $1/t$ and the corresponding Mellin transform is $-(1+\epsilon)/(1+\epsilon)(s-1)$. Let us write $k_\epsilon$ for the restriction of $f_\epsilon$ to the interval $0 < t < 1$. Its Mellin transform defines an element of the Hardy space of $\Re(s) > 1/2$, in particular it has to be analytic there. We have obviously for $\Re(s) = 1/2$: 

$$\hat{f}_\epsilon(s) = \hat{k}_\epsilon(s) + \frac{1}{\zeta(1+\epsilon)} \frac{1}{s-1}$$

and we have computed explicitly $\hat{k}_\epsilon(s)$ for $\Re(s) > 1$ hence also for $\Re(s) \geq 1/2$. This gives the formula

$$\frac{-1}{\zeta(s+\epsilon)} \frac{\zeta(s)}{s}$$

and that it cannot have any pole in $\Re(s) \geq 1/2$ apart from $s = 1$. 

So the square-integrability of $f_\epsilon(t)$ is already a strong statement; we will be content with a trivial observation:

**Corollary 4.1.** If $f_\epsilon \in L^2$ and $\rho$ is a zero with $\Re(\rho) \geq 1/2 + \epsilon$ then $\rho - \epsilon$ is again a zero.

This leads to:

**Theorem 4.2.** If $f_\epsilon \in L^2$ for a sequence of $\epsilon > 0$ going to 0 then the Riemann Hypothesis holds.

**Proof.** The $\rho - \epsilon$ have to be among the zeros for an infinite sequence of $\epsilon \to 0$, but they accumulate at $\rho$. 

We have considered the functions $f_\epsilon(t)$ but we could as well consider the functions $t^{-\epsilon/2}f_\epsilon(t)$ ($0 < \epsilon < 1/2$). But here arises the situation that under the hypothesis that $t^{-\epsilon/2}f_\epsilon(t)$ is square-integrable its Fourier-Mellin transform on the critical line has to be the function

$$\frac{-\zeta(s-\epsilon/2)}{\zeta(s+\epsilon/2)} \frac{1}{s-\epsilon/2}$$

which is already known to be square-integrable!

Let us define the function $g_\epsilon(t)$ to be the inverse $(L^2)$-Mellin transform of this function from the critical line to the half-axis $0 < t \leq \infty$. The Riemann Hypothesis then becomes equivalent to the statement that $g_\epsilon(t) = t^{-\epsilon/2}f_\epsilon(t)$ (for small $\epsilon$’s) or even seemingly weaker statements like $g_\epsilon(t) = C_{\epsilon}/t^{1+\epsilon/2}$ for $t > 1$. We state here just one such strange reformulation of the Riemann Hypothesis:

**Theorem 4.3.** The Riemann hypothesis holds if and only if for small $\epsilon > 0$ (a sequence going to 0 is enough), and for all $\Re(z) < 1/2$ one has:

$$\frac{1}{2\pi i} \int_{\Re(s)=\frac{1}{2}} \frac{\zeta(s-\frac{\epsilon}{2})}{\zeta(s+\frac{\epsilon}{2})} \frac{1}{s-\frac{\epsilon}{2}} \frac{|ds|}{s-z} = \frac{1}{\zeta(1+\epsilon)} \frac{1}{z-\frac{\epsilon}{2}-1}$$
Proof. Let us assume that the integral formula is true. As is known this kind of integral corresponds to orthogonal projection to the Hardy space of the left half-plane \( \text{Re}(z) < \frac{1}{2} \). Indeed the right hand side is up to a constant the Mellin transform of the function \( 1_{x>1}(x)x^{-1-\epsilon/2} \). But this then implies that

\[
\frac{\zeta(s - \frac{\epsilon}{2})}{\zeta(s + \frac{\epsilon}{2})} \frac{1}{s - \frac{\epsilon}{2} - 1} - \frac{1}{\zeta(1 + \epsilon)} \frac{1}{s - \frac{\epsilon}{2} - 1}
\]

belongs to the Hardy space of the right half-plane \( \text{Re}(s) > \frac{1}{2} \). From the absence of poles one then sees using a sequence of \( \epsilon \to 0 \) that this implies the Riemann Hypothesis.

If the Riemann Hypothesis holds then

\[
\frac{\zeta(s - \frac{\epsilon}{2})}{\zeta(s + \frac{\epsilon}{2})} \frac{1}{s - \frac{\epsilon}{2}}
\]

is the Mellin transform of \( -x^{-\epsilon/2}f_{\epsilon}(x) \) which restricts to \( -x^{-1-\epsilon/2}/\zeta(1+\epsilon) \) for \( x > 1 \) and this gives its orthogonal projection to the Hardy space of \( \text{Re}(z) < \frac{1}{2} \):

\[
\frac{-1}{\zeta(1+\epsilon)} \int_{1}^{\infty} x^{-1-\epsilon/2}x^{-1} dx = \frac{1}{\zeta(1+\epsilon)} \frac{1}{z - \frac{\epsilon}{2} - 1}
\]

\( \square \)

We may envision this formulation as a kind of causality statement about the Riemann Zeta function: the ratio \( \zeta(s - \epsilon/2)/\zeta(s + \epsilon/2) \) has the same modulus (\( \text{Re}(s) = 1/2 \)) but not the same phase as the spectral function \( \gamma_+(s+\epsilon/2) \) of the Fourier cosine transform. In [10] we formulated, generally speaking, the Riemann Hypothesis for all abelian \( L \)-functions as a statement of causality. And we expressed elsewhere [11] the wish to see this implemented in a novel framework, more truly arithmetical in nature than the ones we have been working in in this and other papers.

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E-mail address: burnol@math.unice.fr