Mutations of group species with potentials and their representations. Applications to cluster algebras.
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MUTATIONS OF GROUP SPECIES WITH POTENTIALS
AND THEIR REPRESENTATIONS.
APPLICATIONS TO CLUSTER ALGEBRAS.

LAURENT DEMONET

Abstract. This article tries to generalize former works of Derksen, Weyman and Zelevinsky about skew-symmetric cluster algebras to the skew-symmetrizable case. We introduce the notion of group species with potentials and their decorated representations. In good cases, we can define mutations of these objects in such a way that these mutations mimic the mutations of seeds defined by Fomin and Zelevinsky for a skew-symmetrizable exchange matrix defined from the group species. These good cases are called non-degenerate. Thus, when an exchange matrix can be associated to a non-degenerate group species with potential, we give an interpretation of the $F$-polynomials and the $g$-vectors of Fomin and Zelevinsky in terms of the mutation of group species with potentials and their decorated representations. Hence, we can deduce a proof of a series of combinatorial conjectures of Fomin and Zelevinsky in these cases. Moreover, we give, for certain skew-symmetrizable matrices a proof of the existence of a non-degenerate group species with potential realizing this matrix. On the other hand, we prove that certain skew-symmetrizable matrices can not be realized in this way.

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1. Introduction

The aim of this paper is to extend the results of [DWZ2] and [DWZ1] to the case of skew-symmetrizable exchange matrices. Unfortunately, the
techniques presented here do not work in any situation, but nevertheless in some important cases.

For this, we introduce group species with potential (GSP), which can be seen as quivers with potential with more than one idempotent at each vertex. Thus, we can also define a Jacobian ideal and a Jacobian algebra and study their representations. More precisely, we define the notion of a group species with potential with a decorated representation (GSPDR) and the notion of the mutation of a GSPDR at a vertex $k$ (which is called the direction of the mutation). In good cases, we can mutate a GSPDR as many times as we want in any direction. In this case, the underlying GSP is called non-degenerate. Moreover, we can associate to certain GSPs, called locally free, a skew-symmetrizable matrix in such a way that the mutation we introduce projects, when it exists, to the mutation of matrix introduced by Fomin and Zelevinsky [FZ1]. Any skew-symmetrizable matrix can be reached in this way using a locally free GSP. The hard problem is to find which skew-symmetrizable matrix can be reached using a non-degenerate GSP. It is the case of matrices of the form $DS$ where $D$ is diagonal with positive integer coefficients and $S$ is skew-symmetric with integer coefficients. It is also the case for the skew-symmetrizable matrices which occur in the situation of [Der1] in particular in all acyclic cases. Nevertheless, it is not always true, as shown by the counterexample at the end of section 12. The techniques presented in [DWZ2] work here almost in the same way. The only problem is that it is not always the case that for any 2-cycle, there exists a potential canceling it (this fact is very easy in the context of [DWZ2]).

We now explain the content of this article in more details. Let $K$ be an algebraically closed field. Let $I$ be a finite set and $E$ be an $(E,E)$-bimodule. This data is called a group species and its complete path algebra is

$$E^A = \prod_{n \in \mathbb{N}} E^n.$$

A potential $S$ on this group species can be seen as a (maybe infinite) linear combination of cyclic path, up to rotation. It permits to construct a two sided ideal $J(S)$, called the Jacobian ideal and a quotient algebra $\mathcal{P}(A,S) = E^A/J(S)$ called the Jacobian algebra. A decorated representation of the GSP is a pair consisting of a $\mathcal{P}(A,S)$-module $X$ and an $E$-module $V$. In sections 6 and 7 we define the mutation of a GSP with a decorated representation (GSPDR). This mutation is well defined if the group species has no loop and is 2-acyclic (that is, for any $i \in I$, $E_i(A \otimes A \otimes E)A_i = 0$, where $E_i = K[\Gamma_i] \subset E$).

In what follows, we suppose that the $\Gamma_i$ are commutative and that the GSP is locally free, that is, for any $i,j \in I$, $E_iAE_j$ is a free $E_i$-left module and a free $E_j$-right module. In section 6, we define the exchange matrix $B$ of the group species by

$$b_{ij} = \dim_{E_j} A_{ji} - \dim_{E_i} A_{ij}^*.$$

Thus, the mutation of GSPDRs descends to the mutation of matrices defined by Fomin and Zelevinsky [FZ1]. In section 7, we discuss a class of matrices,
namely those of the form $DS$, for which there is always a non-degenerate GSP. Moreover, we remark that there exists also non-degenerate GSP in all cases which are categorified in \cite{Dem} (because the endomorphisms rings of cluster-tilting objects constructed in \cite{Dem} are Jacobian algebras). Remark also that there is no chance, with definitions given here, to construct non-degenerate GSPs for any skew-symmetrizable matrix, as shown by the counterexample ending section 12.

Following the ideas of \cite{DWZ1}, we explain in section 9 how to reinterpret the $F$-polynomials and $g$-vectors defined in \cite{FZ2} in terms of GSPDRs and their mutations. We deduce in section 11 that, when a skew-symmetrizable matrix can be obtained from a non-degenerate GSP, then the following conjectures are true:

Conjecture (\cite{FZ2}, conjecture 5.4). For any $i \in I^n$ and $k \in I$, $F^B_{k;1}$ has constant term 1.

Conjecture (\cite{FZ2}, conjecture 5.5). For any $i \in I^n$ and $k \in I$, $F^B_{k;1}$ has a maximum monomial for divisibility order with coefficient 1.

Conjecture (\cite{FZ2}, conjecture 7.12). For any $i \in I^n$, $k \in I$, we denote by $k_i$ the concatenation of $(k)$ and $i$. Let $j \in I$ and $(g_i)_{i \in I} = g^B_{j;1}$ and $(g'_i)_{i \in I} = h^B_{j;k;1}$. Then we have, for any $i \in I$,

$$g'_i = \begin{cases} -g_i & \text{if } i = k; \\ g_i + \max(0, b_{ik})g_k - b_{jk}\min(g_k, 0) & \text{if } i \neq k. \end{cases}$$

Conjecture (\cite{FZ2}, conjecture 6.13). For any $i \in I^n$, the vectors $g^B_{i;1}$ for $i \in I$ are sign-coherent. In other terms, for $i, i', j \in I$, the $j$-th components of $g^B_{i;1}$ and $g^B_{i';1}$ have the same sign.

Conjecture (\cite{FZ2}, conjecture 7.10(2)). For any $i \in I^n$, the vectors $g^B_{i;1}$ for $i \in I$ form a $\mathbb{Z}$-basis of $\mathbb{Z}^I$.

Conjecture (\cite{FZ2}, conjecture 7.10(1)). For any $i, i' \in I^n$, if we have

$$\sum_{i \in I} a_i g^B_{i;1} = \sum_{i \in I} a'_i g^B_{i;i'}$$

for some nonnegative integers $(a_i)_{i \in I}$ and $(a'_i)_{i \in I}$, then there is a permutation $\sigma \in S_I$ such that for every $i \in I$,

$$a_i = a'_{\sigma(i)} \quad \text{and} \quad a_i \neq 0 \Rightarrow g^B_{i;1} = g^B_{\sigma(i);i'} \quad \text{and} \quad a_i \neq 0 \Rightarrow F^B_{i;1} = F^B_{\sigma(i);i'}.$$

In particular, $F^B_{i;1}$ is determined by $g^B_{i;1}$.

Thus, as stated in \cite{FZ2}, remark 7.11, if $B$ is a full rank skew-symmetrizable matrix which correspond to a non-degenerate GSP, then the cluster monomials of a cluster algebra with exchange matrix $B$ are linearly independent.

2. Group species and path algebras

Let $K$ be a field.
Definition 2.1. A group species is a triple \((I, (\Gamma_i)_{i \in I}, (A_{ij})_{(i,j) \in I^2})\) where \(I\) is a finite set, for each \(i \in I\), \(\Gamma_i\) is a finite group and for each \((i, j) \in I^2\), \(A_{ij}\) is a finite dimensional \(\{K[\Gamma_i], K[\Gamma_j]\}\)-bimodule (the first acting on the left and the second on the right).

Fix now such a group species \(Q = (I, (\Gamma_i)_{i \in I}, (A_{ij})_{(i,j) \in I^2})\)

Definition 2.2. A representation of \(Q\) is a pair \(((V_i)_{i \in I}, (x_{ij})_{(i,j) \in I^2})\) where for each \(i \in I\), \(V_i\) is a right finite dimensional \(K[\Gamma_i]\)-module and for each \((i,j) \in I^2\),

\[ x_{ij} \in \text{Hom}_{\Gamma_j}(V_i \otimes_{\Gamma_i} A_{ij}, V_j). \]

Definition 2.3. Let \(((V_i)_{i \in I}, (x_{ij})_{(i,j) \in I^2})\) and \(((V'_i)_{i \in I}, (x'_{ij})_{(i,j) \in I^2})\) be two representations of \(Q\). A morphism from the first one to the second one is a family \((f_i)_{i \in I} \in \prod_{i \in I} \text{Hom}_{\Gamma_i}(V_i, V'_i)\) such that for each \((i,j) \in I^2\) the following diagram commute:

\[
\begin{array}{ccc}
V_i \otimes \Gamma_i A_{ij} & \xrightarrow{x_{ij}} & V_j \\
f_i \otimes \text{id}_{A_{ij}} & & f_j \\
V'_i \otimes \Gamma_i A_{ij} & \xrightarrow{x'_{ij}} & V'_j
\end{array}
\]

Remarks 2.4.  
- The previous definitions give rise to an abelian category.
- If for each \(i \in I\), \(\Gamma_i\) is the trivial group, we get back the classical definition of a quiver (up to the choice of a basis of each \(A_{ij}\)) and of the category of representations of a quiver.
- If for each \(i \in I\), \(K[\Gamma_i]\) is replaced by a division algebra, we obtain the usual definition of a species (see for example [DR]).

Definition 2.5. For each \(i \in I\), denote \(E_i = K[\Gamma_i]\). Denote also \(E = \bigoplus_{i \in I} E_i\) and \(A = \bigoplus_{(i,j) \in I^2} A_{ij}\). Thus, we put the natural \((E, E)\)-bimodule structure on \(A\) and define the graded algebras

\[ E\langle A \rangle = \bigoplus_{n \in \mathbb{N}} A^{\otimes n} \quad \text{and} \quad E\langle\langle A \rangle\rangle = \prod_{n \in \mathbb{N}} A^{\otimes n} \]

the first one being called the path algebra of the group species and the second one the complete path algebra of the group species (note that every tensor product is taken over \(E\)).

Remarks 2.6.  
- As usual for quiver, the category of representations of a group species is equivalent to the category of finite dimensional right modules over its path algebra. Moreover, the category of nilpotent representations of a group species is equivalent to the category of finite dimensional right modules over its complete path algebra.
- If one denotes

\[ m = \prod_{n > 0} A^{\otimes n} \subset E\langle\langle A \rangle\rangle \]

which is clearly a two-sided ideal, then \(E\langle\langle A \rangle\rangle\) becomes a topological algebra for the \(m\)-adic topology and \(E\langle A \rangle\) is a dense subalgebra of it.
As in [DWZ2], m is the unique maximal two-sided ideal of $E\langle\langle A\rangle\rangle$ not intersecting $E$. Moreover, if we have another group species with the same vertices whose arrows are encoded in the $(E,E)$-bimodule $A'$, then, again as in [DWZ2], the morphisms $\varphi$ from $E\langle\langle A\rangle\rangle$ to $E\langle\langle A'\rangle\rangle$ such that $\varphi_E = \Id_E$ (later called $E$-morphisms) are parameterized in an obvious way by a pair $(\varphi^{(1)}, \varphi^{(2)})$ where $\varphi^{(1)} : A \to A'$ and $\varphi^{(2)} : A \to m^2$ are $(E,E)$-bimodule morphisms. Thus, $\varphi$ is an isomorphism if and only if $\varphi^{(1)}$ is an isomorphism.

Introduce now the analogous of [DWZ2, definition 2.5]:

**Definition 2.7.** An $E$-automorphism $\varphi$ of $E\langle\langle A\rangle\rangle$ will be called a change of arrows if $\varphi^{(2)} = 0$ and a unitriangular automorphism if $\varphi^{(1)} = \Id_A$.

Finally, introduce the following useful notation:

**Notation 2.8.** For all $i, j \in I$,

$E\langle\langle A\rangle\rangle_{ij} = E_i E\langle\langle A\rangle\rangle E_j$ and $E\langle\langle A\rangle\rangle_{ij} = E_i E\langle\langle A\rangle\rangle E_j$

and for $n \in \mathbb{N}$,

$A_{ij}^{\otimes n} = A_{ij}^{\otimes n} \cap E\langle\langle A\rangle\rangle_{ij} = A_{ij}^{\otimes n} \cap E\langle\langle A\rangle\rangle_{ij}$

so that

$E\langle\langle A\rangle\rangle_{ij} = \bigoplus_{n \in \mathbb{N}} A_{ij}^{\otimes n}$ and $E\langle\langle A\rangle\rangle_{ij} = \prod_{n \in \mathbb{N}} A_{ij}^{\otimes n}$.

3. Potential and their Jacobian ideals

Following [DWZ2] define:

**Definition 3.1.** Define

$E\langle\langle A\rangle\rangle_{\text{cyc}} = \frac{E\langle\langle A\rangle\rangle}{[E\langle\langle A\rangle\rangle, E\langle\langle A\rangle\rangle]}$

whose elements are called potentials (here, $[E\langle\langle A\rangle\rangle, E\langle\langle A\rangle\rangle]$ is the closure of the two-sided ideal generated by commutators). As $[E\langle\langle A\rangle\rangle, E\langle\langle A\rangle\rangle]$ is generated by its homogeneous elements, we can decompose $E\langle\langle A\rangle\rangle_{\text{cyc}} = \prod_{n \in \mathbb{N}} A_{\text{cyc}}^{\otimes n}$ where

$A_{\text{cyc}}^{\otimes n} = A_{\text{cyc}}^{\otimes n} \cap E\langle\langle A\rangle\rangle_{\text{cyc}}$

and, if $S \in E\langle\langle A\rangle\rangle_{\text{cyc}}$, we write $S^{(n)}$ its summand which lies in $A_{\text{cyc}}^{\otimes n}$.

**Definition 3.2.** Define the continuous linear map

$\partial : (E\langle\langle A\rangle\rangle)^* \otimes_k E\langle\langle A\rangle\rangle \to E\langle\langle A\rangle\rangle$

in the following way. First remark that $(E\langle\langle A\rangle\rangle)^* \cong \bigoplus_{n \in \mathbb{N}} (A^{\otimes n})^*$. Then, if $\xi \in (A^{\otimes n})^*$ and $a_1, a_2, \ldots, a_\ell \in A$ define $\partial \xi (a_1 a_2 \ldots a_\ell) = 0$ if $\ell < n$ and

$\partial \xi (a_1 a_2 \ldots a_\ell) = \sum_{j=1}^{\ell} \sum_{g,h, k \in B} \xi \left( g^{-1} a_j a_{j+1} \ldots a_{j+n-1} h \right) h^{-1} a_{j+n} a_{j+n+1} \ldots a_{j+1} g$

if $\ell \geq n$ where all indices are taken modulo $\ell$ and $B = \bigcup_{i \in I} \Gamma_i \subset E$. It is easy to see that $\partial$ is well defined and moreover that it vanishes on commutators. Thus, we can descend $\partial$ to a continuous linear map

$\partial : (E\langle\langle A\rangle\rangle)^* \otimes_k E\langle\langle A\rangle\rangle_{\text{cyc}} \to E\langle\langle A\rangle\rangle$. 

**Definition 4.4.** We say that a GSP $E$ is an equivalence if the species has no loop (for all $b$, $b$ is trivial). The quotient $E/\langle A \rangle$ is the Jacobian algebra and is denoted by $\mathcal{P}(A, S)$ (we do not keep trace of $(I, (\Gamma_i))$ in the notation because it will be fixed).

Note that every $E$-morphism $\varphi : E/\langle A \rangle \to E/\langle A' \rangle$ descends to $\varphi : E/\langle A \rangle_{\text{cyc}} \to E/\langle A' \rangle_{\text{cyc}}$.

It is now easy to adapt the proof of [DWZ2, proposition 3.7]:

**Proposition 3.5.** Let $S \in E/\langle A \rangle_{\text{cyc}}$. Every $E$-isomorphism $\varphi : E/\langle A \rangle \to E/\langle A' \rangle$ maps $J(S)$ to $J(\varphi(S))$ and therefore induces an isomorphism $\mathcal{P}(A, S) \to \mathcal{P}(A', \varphi(S))$.

4. **Group species with potentials**

For the rest of this article, the data $(I, (\Gamma_i))$ and so $E$ will be fixed. Following the ideas of [DWZ2], define:

**Definition 4.1.** As before, $A$ is an $(E, E)$-bimodule and we take $S \in E/\langle A \rangle_{\text{cyc}}$. We say that $(A, S)$ is a group species with potential (GSP for short) if the species has no loop (for all $i \in I$, $E_i A E_i = [0]$) and $S \in \prod_{n>1} A_{\text{cyc}}^{\otimes n}$.

**Definition 4.2.** Let $(A, S)$ and $(A', S')$ be two GSPs. One says that an $E$-isomorphism $\varphi : E/\langle A \rangle \to E/\langle A' \rangle$ is a right-equivalence if $\varphi(S) = S'$.

Note that this definition induces a equivalence relation. Moreover, a right equivalence $(A, S) \simeq (A', S')$ induces isomorphisms of $(E, E)$-bimodules $A \simeq A'$, $J(S) \simeq J(S')$ and $\mathcal{P}(A, S) \simeq \mathcal{P}(A', S')$ as said before.

**Notation 4.3.** If $(A, S)$ and $(A', S')$ are two GSPs, define $(A, S) \oplus (A', S') = (A \oplus A', S + S')$ so that $\mathcal{P}((A, S) \oplus (A', S'))$ is the completion of $\mathcal{P}(A, S) \oplus \mathcal{P}(A', S')$ for the product topology.

**Definition 4.4.** We say that a GSP $(A, S)$ is trivial if $S \in A_{\text{cyc}}^{\otimes 2}$ and $\{\partial_\xi(S) \mid \xi \in A^*\} = A$, or, equivalently, if $\mathcal{P}(A, S) = E$.

The following easy proposition is an adaptation of [DWZ2, proposition 4.4]:

**Proposition 4.5.** A GSP $(A, S)$ is trivial if and only if there exist an $(E, E)$-bimodule $B$ and an $(E, E)$-bimodules isomorphism $\varphi : A \to B \oplus B^*$ such that

$$\varphi(S) = \sum_{b \in B} b \oplus b^*$$

where $\varphi$ is naturally extended to an isomorphism $E/\langle A \rangle_{\text{cyc}} \to E/\langle B \oplus B^* \rangle_{\text{cyc}}$ and the right member does not depend of the choice of a basis $B$ of $B$.

One gets also this proposition, similar to [DWZ2, proposition 4.5]:
Proposition 4.6. If $(A, S)$ is a GSP and $(B, T)$ is a trivial GSP, then the canonical embedding $E\langle\langle A\rangle\rangle \hookrightarrow E\langle\langle A \oplus B\rangle\rangle$ induces an isomorphism $\mathcal{P}(A, S) \cong \mathcal{P}(A \oplus B, S + T)$.

For a GSP $(A, S)$, we define the trivial and reduced part of $A$ as the $(E, E)$-bimodules

$$A_{\text{triv}} = \{\partial_\xi S^{(2)} \mid \xi \in A^*\} \quad \text{and} \quad A_{\text{red}} = A/A_{\text{triv}}.$$

Moreover, we say that $(A, S)$ is reduced if $S^{(2)} = 0$, or, equivalently, if $A_{\text{triv}} = \{0\}$.

Again, the proof of [DWZ2, theorem 4.6] is easy to adapt:

Theorem 4.7. For any GSP $(A, S)$, there exist $S_{\text{triv}} \in E\langle\langle A_{\text{triv}}\rangle\rangle$ and $S_{\text{red}} \in E\langle\langle A_{\text{red}}\rangle\rangle$ such that $(A, S)$ is right equivalent to $(A_{\text{triv}}, S_{\text{triv}}) \oplus (A_{\text{red}}, S_{\text{red}})$.

Moreover, the right equivalence classes of $(A_{\text{triv}}, S_{\text{triv}})$ and $(A_{\text{red}}, S_{\text{red}})$ are uniquely determined by the right equivalence class of $(A, S)$.

Definition 4.8. A group species $(I, (\Gamma_i), A)$ is called 2-acyclic if, for any $i, j \in I$, $E_i A \otimes_2 E_i = \{0\}$.

We will see now how to find, as in [DWZ2], algebraic conditions guaranteeing the 2-acyclicity of the reduced part of a group species. Let $K [E\langle\langle A\rangle\rangle_{\text{cyc}}]$ be the ring of polynomial functions on $E\langle\langle A\rangle\rangle_{\text{cyc}}$ vanishing on all but a finite number of the $A^*_{\text{cyc}}$.

For each $S \in E\langle\langle A\rangle\rangle_{\text{cyc}}$ and $i, j \in I$, define the bilinear form $\alpha_{S, ij}$ by:

$$A^*_{ij} \times A^*_{ji} \to K$$

$$(f, g) \mapsto \sum_{\gamma \in \Gamma_i, \gamma' \in \Gamma_j} \left( \gamma' f \gamma'^{-1} \otimes \gamma g \gamma^{-1} \right) \left( S^{(2)} \right) + \left( \gamma g \gamma'^{-1} f \gamma^{-1} \right) \left( S^{(2)} \right) .$$

First, an easy lemma:

Lemma 4.9. Let $i, j \in I$. The followings are equivalent:

(i) there exists $S \in E\langle\langle A\rangle\rangle_{\text{cyc}}$ such that $\alpha_{S, ij}$ is of maximal rank;

(ii) either $A^*_{ij}$ is a submodule of $A_{ji}$ or $A^*_{ji}$ is a submodule of $A_{ij}$.

Proof. We clearly have $\alpha_{S, ij} = \alpha_{S, ji}$ for any $S$ and therefore, one can suppose without loss of generality that $\dim_K A_{ij} \leq \dim_K A_{ji}$. Suppose that $\alpha_{S, ij}$ is of maximal rank. In any basis, the matrix of $\alpha_{S, ij}$ is the matrix of $A^*_{ij} \to A_{ji} : \xi \mapsto \partial_\xi (S^{(2)})$ and therefore, $A^*_{ij}$ is a submodule of $A_{ji}$.

Reciprocally, suppose that $A^*_{ij}$ is a submodule of $A_{ji}$. Thus, if $B$ is a basis of $A_{ij}$, define

$$S = \sum_{a \in B} a \otimes a^*$$

where $a^* \in A^*_{ij}$ is identified with its image in $A_{ji}$. Then, it is clear that $\alpha_{S, ij}$ is of maximal rank. \(\square\)

Again, it is easy to generalize [DWZ2, proposition 4.15]:

Proposition 4.10. The reduced part of a GSP $(A, S)$ is 2-acyclic if and only if, for any $i, j \in I$, $\alpha_{S, ij}$ is of maximal rank. This condition is open. Moreover, if, for any $i, j \in I$, either $A^*_{ij}$ is a submodule of $A_{ji}$, either
A*_{ij} is a subbimodule of A_{ij}, then there is a non empty Zariski open subset U of E\langle\langle A\rangle\rangle_{cyc}, a 2-acyclic (E,E)-bimodule A' and a regular map H : U \to E\langle\langle A'\rangle\rangle_{cyc} such that for any S \in U, (A_{red}, S_{red}) is right equivalent to (A', H(S)).

Proof. The arguments are the same than in [DWZ2]. For each i, j \in I^2, choose \overline{A}_{ij} \subset A^*_{ij} such that \overline{A}_{ij} = A^*_ij \text{ or } \overline{A}_{ij} \simeq A_{ij}. Let U be the non-empty open subset of E\langle\langle A\rangle\rangle_{cyc} containing the S such that for all i, j \in I, \alpha_{S,ij} : \overline{A}_{ij} \times \overline{A}_{ij} \to A^*_{ij} \text{ is non-degenerate (it corresponds to the non-vanishing of a fixed maximal minor of } \alpha_{S,ij}). Define A' to be the intersection of the kernels of the \alpha_{S,ij}. Then the construction of H follows the proof of [DWZ2, theorem 4.6]. □

5. Mutations of group species with potential

Let (A, S) and k \in I be a vertex such that E_kA^{\otimes 2}E_k = \{0\} (we say that (A, S) is 2-acyclic at k). We suppose also that for any i \in I, the characteristic of K does not divide \#\Gamma_i. As in [DWZ2 §5], one defines \hat{\mu}_k(A, S) = (\hat{A}, \hat{S}) where, if i, j \in I,

\hat{\Lambda}_{ij} = \begin{cases} A^*_{ij} & \text{if } k \in \{i, j\}; \\ A_{ij} \oplus A_{ik} \otimes E_k A_{kj} & \text{otherwise.} \end{cases}

In other terms,

\hat{A} = \overline{E}_k A E_k \oplus AE_k A \oplus (E_k A)^* \oplus (AE_k)^*

where \overline{E}_k = \bigoplus_{i \neq k} E_i. Let now [-] : \overline{E}_k E\langle\langle A\rangle\rangle \overline{E}_k \to E\langle\langle \hat{A}\rangle\rangle be the morphism of k-algebras generated by \{a\} = a if a \in \overline{E}_k A \overline{E}_k \text{ and } [ab] = ab \in AE_k A \text{ if } a \in AE_k A \text{ and } b \in E_k A \text{ which is well defined because } (A, S) \text{ has no loop. Again, because } (A, S) \text{ has no loop, every potential } S \in E\langle\langle A\rangle\rangle_{cyc} \text{ has a representative in } \overline{E}_k E\langle\langle A\rangle\rangle \overline{E}_k \text{ and it is easy to see that } [-] \text{ descends to a map }

[-] : E\langle\langle A\rangle\rangle_{cyc} \to E\langle\langle \hat{A}\rangle\rangle_{cyc}.

Moreover, as for any i \in I the characteristic of K does not divide \#\Gamma_i, we have a canonical sequence of isomorphisms

\text{Hom}_{E_k}(AE_k A, AE_k A) \simeq (AE_k A)^* \otimes E_k AE_k A \simeq (AE_k \otimes E_k A)^* \otimes E_k AE_k A

\simeq (E_k A)^* \otimes E (AE_k)^* \otimes E AE_k A \subset E\langle\langle \hat{A}\rangle\rangle

and we define \Delta_k(A) to be the image of Id_{AE_k A} through this isomorphism. Thus, define

\hat{S} = [S] + \Delta_k(A).

The proof of [DWZ2 proposition 5.1] can be easily generalized:

**Proposition 5.1.** If (A', S') is another GSP such that E_k A' = A'E_k = \{0\}, then

\hat{\mu}_k(A \oplus A', S + S') = \mu_k(A, S) \oplus (A', S').

Now, the proof of [DWZ2 theorem 5.2] is easy to generalize:

**Theorem 5.2.** The right-equivalence class of the GSP \hat{\mu}_k(A, S) is fully determined by the right-equivalence class of (A, S).
Definition 5.3. Using theorem 5.2 together with theorem 4.7, the right-equivalence class of the reduced part of $\{\hat{\mu}_k(A, S)\}$ is fully determined by the right-equivalence class of $(A, S)$. Thus we can define the map $\mu_k$ from the set of right-equivalence classes which are 2-acyclic at $k$ to itself. It is called the \textit{mutation at vertex} $k$.

Again, the proof of [DWZ2, theorem 5.7] is easy to generalize:

\textbf{Theorem 5.4.} $\mu_k$ is an involution.

Let us also remark that [DWZ2, proposition 6.1], [DWZ2, proposition 6.4] and [DWZ2, corollary 6.6] can be generalized:

\textbf{Proposition 5.5.} The algebras $\mathbb{E}_k \mathcal{P}(A, S) \mathbb{E}_k$ and $\mathbb{E}_k \mathcal{P}(\hat{\mu}_k(A, S)) \mathbb{E}_k$ are isomorphic.

\textbf{Proposition 5.6.} The Jacobian algebra $\mathcal{P}(A, S)$ is finite-dimensional if and only if $\mathcal{P}(\hat{\mu}_k(A, S))$ is.

\textbf{Corollary 5.7.} The Jacobian algebras $\mathbb{E}_k \mathcal{P}(A, S) \mathbb{E}_k$ and $\mathbb{E}_k \mathcal{P}(\mu_k(A, S)) \mathbb{E}_k$ are isomorphic and $\mathcal{P}(A, S)$ is finite-dimensional if and only if $\mathcal{P}(\mu_k(A, S))$ is.

As stated in [DWZ2, remark 6.8], the following definition makes sense:

\textbf{Definition 5.8.} We define the \textit{deformation space} of $(A, S)$ to be

$\text{Def}(A, S) = \frac{\mathcal{P}(A, S)}{E + [\mathcal{P}(A, S), \mathcal{P}(A, S)]}$

where $[\mathcal{P}(A, S), \mathcal{P}(A, S)]$ is the closure of the two-sided ideal of $\mathcal{P}(A, S)$ generated by the commutators.

Thus, let us introduce the following extension of [DWZ2, proposition 6.9]:

\textbf{Proposition 5.9.} We have an isomorphism:

$\text{Def}(A, S) \simeq \text{Def}(\hat{\mu}_k(A, S))$.

\textit{Proof.} It is enough to prove that

$\mathbb{E}_k \mathcal{P}(A, S) \mathbb{E}_k \to \text{Def}(A, S)$

is in fact an isomorphism (which is true because $A$ has no loop) and to use proposition 5.3.

As in [DWZ2],

\textbf{Definition 5.10.} The GSP $(A, S)$ is called \textit{rigid} if $\text{Def}(A, S) = \{0\}$.

\textbf{Corollary 5.11.} The GSP $(A, S)$ is rigid if and only if $\mu_k(A, S)$ is.
6. Exchange matrices

We suppose now that $A$ has neither loop nor 2-cycle (that is $A_{cyc}^{\oplus 1} = A_{cyc}^{\oplus 2} = \{0\}$). We suppose also that for any $(i, j) \in I^2$, $A_{ij}$ is a free left $E_i$-module and a free right $E_j$-module (we will call it a locally free GSP). Define the matrix $B = B(A) = B(A, S)$ to be the matrix with rows and columns indexed by $I$ and coefficients

$$b_{ij} = \dim_{E_j} A_{ji} - \dim_{E_i} A^*_{ij}$$

(by default, dimension are taken relatively to the left module structure). This matrix is clearly skew-symmetrizable since

$$\# \Gamma_j \times b_{ij} = \dim_K A_{ji} - \dim_K A^*_{ij}.$$

**Definition 6.1.** The matrix $B$ is called the exchange matrix of $A$.

The following proposition justifies this generalization of [DWZ2]:

**Proposition 6.2.** Every skew-symmetrizable matrix $B$ can be reached in this way from a GSP.

**Proof.** Let $B$ be a skew-symmetrizable matrix and $D = (d_i)_{i \in I}$ be a diagonal matrix with positive integer coefficients such that $BD$ is skew-symmetric. Let $\Gamma_i = \mathbb{Z}/d_i \mathbb{Z}$ and for $(i, j) \in I^2$ such that $b_{ij} > 0$,

$$A_{ji} = K \left[ \mathbb{Z}/(d_j b_{ij}) \mathbb{Z} \right] = K \left[ \mathbb{Z}/(-d_i b_{ji}) \mathbb{Z} \right]$$

which is a left and right free $(\Gamma_j, \Gamma_i)$-bimodule. It is clear that $A = \bigoplus_{i, j \in I} A_{ij}$ has exchange matrix $B$. \hfill \Box

**Proposition 6.3.** Let $k \in I$.

(i) The GSP $\tilde{\mu}_k(A, S)$ is locally free.

(ii) If $\mu_k(A, S)$ is 2-acyclic then it is locally free.

(iii) If $\mu_k(A, S)$ is 2-acyclic then

$$\mu_k(B(A, S)) = B(\mu_k(A, S))$$

where the $\mu_k$ on the left hand is the one defined in [FZ1]. Namely:

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } k \in \{i, j\} \\ b_{ij} + \frac{b_{ik}\abs{b_{kj}} + b_{jk}\abs{b_{ki}}}{2} & \text{otherwise} \end{cases}$$

if $B' = \mu_k(B)$.

**Proof.**

(i) First of all, it is clear that for $i \in I$, $E^*_i \cong E_i$ as $(E_i, E_i)$-bimodules (as $E_i$ is finite dimensional). Thus, for any $i$, $A^*_k$ and $A^*_k$ are left and right free modules. Moreover, as a right module,

$$A_{ik} \otimes_{E_k} A_{kj} \cong A^*_{kj}^{\dim E_k(A^*_k)}$$

and, as a left module,

$$A_{ik} \otimes_{E_k} A_{kj} \cong A^*_{ik}^{\dim E_k(A^*_k)}$$

which ends the proof that $\tilde{\mu}_k(A, S)$ is locally free.
(ii) If one denotes \( \hat{\mathcal{A}}, \hat{\mathcal{S}} = \tilde{\mu}_k(A, S) \), one has
\[
\hat{\mathcal{A}} = \mathcal{A}_{\text{red}} \oplus \mathcal{A}_{\text{triv}}
\]
As \( \mathcal{A}_{\text{red}} \) is 2-acyclic, for any \( i, j \in I \), \( \mathcal{A}_{\text{red},ij} = 0 \) or \( \mathcal{A}_{\text{red},ji} = 0 \).
Suppose that \( \mathcal{A}_{\text{red},ij} = 0 \). Hence \( \mathcal{A}_{\text{triv},ji} \cong \mathcal{A}_{\text{triv},ij} \) is left and right free (thanks to the previous point). Moreover, \( \mathcal{A}_{ji} = \mathcal{A}_{\text{red},ji} \oplus \mathcal{A}_{\text{triv},ji} \) and, as the categories of left \( E_i \)-modules and right \( E_j \)-modules are Krull-Schmidt, \( \mathcal{A}_{\text{red},ji} \) is left and right free.

(iii) It is enough to remark that
\[
\dim_{E_i} A_{ik} \otimes_{E_k} A_{kj} = \dim_{E_i} A_{ik} \dim_{E_k} A_{kj} = \dim_{E_i} (A_{ik}) \dim_{E_k} (A_{kj})
\]
and that
\[
\dim_{E_i} (A_{jk} \otimes_{E_k} A_{kj})^* = \dim_{E_i} (A_{kj}^* \dim_{E_k} A_{jk}^*) = \dim_{E_i} (A_{kj}^*) \dim_{E_k} (A_{jk}^*)
\]
and to use the definition and the duality \( \mathcal{A}_{\text{triv},ij} \cong \mathcal{A}_{\text{triv},ji}^* \).

**Definition 6.4.** The group species is said to be **globally free** if, for any \( i, j \in I \), \( A_{ij} \) is a free \( (E_i, E_j) \)-bimodule (that is a free \( E_i \otimes_K E_j^{\text{op}} \)-module).

**Remark 6.5.** The class of globally free group species is stable under mutation.

**Proposition 6.6.** If a matrix is of the form \( DB \), where \( D \) is diagonal with positive integer coefficients and \( B \) is skew-symmetric, then the group species constructed in proposition 6.2 is globally free.

### 7. Existence of nondegenerate potentials

If \( (I, (\Gamma), A) \) is a group species without loop nor 2-cycle, a potential \( S \in E(\langle A \rangle)_{\text{cyc}} \) will be said to be **non-degenerate** if every sequence of mutation going from \( (A, S) \) yields to a 2-acyclic GSP.

We cite the following adapted result, whose proof is the same than the proof of [DWZ2, corollary 7.4]:

**Theorem 7.1.** If the group species is globally free then there is a countable number of non-constant polynomials in \( K [E(\langle A \rangle)_{\text{cyc}}] \) such that the non-vanishing of these polynomials on \( S \in E(\langle A \rangle)_{\text{cyc}} \) implies that \( S \) is non-degenerate. In particular if \( K \) is uncountable, there exist non-degenerate potentials.

**Proof.** The only thing to change is that, if the group species is globally free, then for each \( i, j \in I \), either \( A_{ij}^* \) is a subbimodule of \( A_{ji} \), or \( A_{ji}^* \) is a subbimodule of \( A_{ij} \) and, therefore, proposition 4.10 can be applied. □

**Remark 7.2.** It is also easy to prove that for any skew-symmetrizable matrix \( B \) coming from the categories with an action of a group \( \Gamma \) considered in [Dem], there is a non-degenerate GSP realizing it. More precisely, the endomorphism ring of a \( \Gamma \)-stable cluster-tilting object in the stable category of a category constructed in [Dem] can be realized by a non-degenerate GSP (it is the case because \( \Gamma \)-2-cycles do not appear after mutations). In particular, the only potential for an acyclic group species is non-degenerate.
Another proposition linking rigid and non-degenerate potentials can be adapted from [DWZ2, proposition 8.1 and corollary 8.2]:

**Proposition 7.3.** Every rigid globally free $GSP (A, S)$ is 2-acyclic and, in this case, $S$ is non-degenerate.

As in [DWZ2, §8], there exist group species without rigid potentials. The techniques of [DWZ2, §8] work also in the context of this article.

8. Decorated representations and their mutations

The aim of this section is to adapt the results of [DWZ2, §10]. We suppose here that for any $i \in I$, the characteristic of $K$ does not divide the cardinal of $\Gamma_i$.

Following [DWZ2, definition 10.1],

**Definition 8.1.** A decorated representation of a GSP $(A, S)$ is a pair $(X, V)$ where $X$ is a $P(A, S)$-module and $V$ is an $E$-module.

In the following, we will look at pairs consisting of a GSP $(A, S)$ and a decorated representation of it. We will denote this type of objects by $(A, S, X, V)$ and call them group species with potential and decorated representation (GSPDR).

Following [DWZ2, definition 10.2],

**Definition 8.2.** A right-equivalence between two GSPDRs $(A, S, X, V)$ and $(A', S', X', V')$ is a triple $(\varphi, \psi, \eta)$ such that:

- $\varphi : E\langle A \rangle \rightarrow E\langle A' \rangle$ is a right-equivalence from $(A, S)$ to $(A', S')$ (see definition 4.3);
- $\psi : X \rightarrow X'$ is a linear isomorphism such that the following diagram commutes:

$$
\begin{array}{ccc}
X & \xrightarrow{u_X} & X \\
\downarrow \psi & & \downarrow \psi \\
X' & \xrightarrow{\psi(u_X)} & X'
\end{array}
$$

for any $u \in E\langle A \rangle$;
- $\eta : V \rightarrow V'$ is an isomorphism.

Using proposition 4.6, for each GSPDR $(A, S, X, V)$, the decorated representation $(X, V)$ can be seen as a representation of $(A_{\text{red}}, S_{\text{red}})$. Thus, we can call $(A_{\text{red}}, S_{\text{red}}, X, V)$ the reduced part of $(A, S, X, V)$. As in [DWZ2, proposition 10.5], the right-equivalence class of the reduced part of a GSPDR is fully determined by the right-equivalence class of this GSPDR.

Now, we can define the mutation of a GSPDR $(A, S, X, V)$. Let $k \in I$. Our aim is to define a GSPRD $\mu_k(A, S, X, V) = (A', S', X', V')$ such that $(A', S') = \mu_k(A, S)$. Denote:

$$
X_{\text{in}} = X \otimes_E A E_k \quad \text{and} \quad X_{\text{out}} = X \otimes_E A^* E_k.
$$

Thus, we can define two right $E_k$-module morphisms. One, $\alpha$, from $X_{\text{in}}$ to $X_k = X E_k$ which is the application $(x \otimes a) \mapsto xa$ and one from $X_k$ to $X_{\text{out}}$ which is defined by

$$
\beta(x) = \sum_{b \in B} xb \otimes b^*.
$$
which does not depend on the basis $B$ of $E_kA$. Observe also that we have a canonical sequence of isomorphisms:

$$\text{Hom}_{E_k}(X_{\text{out}}, X_{\text{in}}) \simeq \text{Hom}_E(X \otimes_E A^* E_k \otimes_{E_k} E_k A^*, X)$$

$$\simeq \text{Hom}_E(X \otimes_E (AE_k A)^*, X)$$

It is not hard to see that $[x \otimes \xi \mapsto x(\partial E)] \in \text{Hom}_E(X \otimes_E (AE_k A)^*, X)$. Let $\gamma$ be the corresponding element of $\text{Hom}_{E_k}(X_{\text{out}}, X_{\text{in}})$.

So we get, as in [DWZ2] a commutative diagram of right $E_k$-modules:

$$
\begin{array}{ccc}
X_{\text{in}} & \xrightarrow{\alpha} & X_k \\
\downarrow{\gamma} & & \downarrow{\beta} \\
X_{\text{out}} & \xleftarrow{\gamma} & X_{\text{out}}
\end{array}
$$

with $\alpha \gamma = \gamma \beta = 0$ [DWZ2 lemma 10.6]. For $i \in I$, define:

$$X'_i = \begin{cases} 
X_i & \text{if } i \neq k \\
\ker \gamma \oplus \ker \alpha \oplus \ker \gamma \oplus V_i & \text{if } i = k
\end{cases}$$

and

$$V'_i = \begin{cases} 
V_i & \text{if } i \neq k \\
\ker \beta \cap \ker \alpha & \text{if } i = k
\end{cases}$$

To get the structure of an $\mathcal{P}(A', S')$-module on $X'$, we must define the way $\tilde{A}$ acts on it where $(\tilde{A}, \tilde{S}) = \tilde{\mu}_k(A, S)$ (as $\mathcal{P}(A', S') \simeq \mathcal{P}(\tilde{A}, \tilde{S})$). Recall from §5, that

$$\tilde{A} = \overline{E_k A E_k} \oplus A E_k A \oplus (E_k A^*)_* \oplus (AE_k)*.$$
define
\[
\alpha' = \begin{pmatrix}
-\pi & 0 \\
-\gamma & 0 \\
0 & 0 \\
\end{pmatrix}
\] and \[
\beta' = \begin{pmatrix}
0 & \nu & 0 \\
\iota & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}
\]

where \(\pi\) designs the canonical projection and \(\iota\) the canonical injections.

Again, [DWZ2, proposition 10.7] can be adapted:

**Proposition 8.3.** The above definition gives rise to a decorated representation of \((\tilde{\mathcal{A}}, \tilde{\mathcal{S}})\) and, therefore, through the isomorphism \(\mathcal{P}(\tilde{\mathcal{A}}, \tilde{\mathcal{S}}) \simeq \mathcal{P}(\mathcal{A}', \mathcal{S}')\), to a decorated representation of \((\mathcal{A}', \mathcal{S}')\).

**Notation 8.4.** We denote \[
\tilde{\mu}_k(A, S, X, V) = (\tilde{\mathcal{A}}, \tilde{\mathcal{S}}, X', V') \quad \text{and} \quad \mu_k(A, S, X, V) = (A', S', X', V').
\]

We can adapt [DWZ2, proposition 10.9]:

**Proposition 8.5.** The isomorphism class of the GSPDR \(\tilde{\mu}_k(A, S, X, V)\) does not depend on the choice of the splitting data.

and [DWZ2, proposition 10.10 and corollary 10.12]:

**Proposition 8.6.** The right-equivalence classes of the GSPDRs \[
\tilde{\mu}_k(A, S, X, V) \quad \text{and} \quad \mu_k(A, S, X, V)
\]
depend only on the right-equivalence class of \((A, S, X, V)\).

Now an important theorem whose proof is the same as the one of [DWZ2, theorem 10.13]:

**Theorem 8.7.** On the right-equivalence classes of GSPDRs which are 2-acyclic at \(k\), \(\mu_k\) is an involution.

It is easy to define the notion of a direct sum of two decorated representations of a GSP and, therefore, the notion of an indecomposable decorated representation of a GSP. Thus, as \(\mu_k\) clearly commutes with this type of direct sums, \(\mu_k\) acts on GSPs with indecomposable decorated representations. We call a GSPDR \((A, S, X, V)\) positive if \(V \neq 0\) and negative if \(X \neq 0\). Moreover, it is called simple at \(i \in I\) if \(X \oplus V\) is an indecomposable \(E_i\)-module. Then we adapt [DWZ2, proposition 10.15]:

**Proposition 8.8.** An indecomposable GSPDR is either positive, or negative simple. The mutation \(\mu_k\) exchange a positive simple at \(k\) with the corresponding negative simple at \(k\). Moreover, it is the only case where a mutation interchanges positive and negative indecomposable GSPDRs.

As in [DWZ1, §6], denote, for \(k \in I\) and \(X, X' \in \text{mod} \mathcal{P}(A, S)\),
\[
\text{Hom}^{[k]}_{\mathcal{P}(A,S)}(X, X') = \left\{ f \in \text{Hom}_{\mathcal{P}(A,S)}(X, X') \mid f|_{X_{\mathcal{E}_k}} = 0 \right\}.
\]

Cite now easy to adapt [DWZ1, proposition 6.1]:

**Proposition 8.9.** The mutation \(\mu_k\) induces an isomorphism
\[
\frac{\text{Hom}^{[k]}_{\mathcal{P}(A,S)}(X, X')}{\text{Hom}^{[k]}_{\mathcal{P}(A,S)}(X, X')} \simeq \frac{\text{Hom}^{[k]}_{\mathcal{P}(\mu_k(A,S))}(\mu_k(X), \mu_k(X'))}{\text{Hom}^{[k]}_{\mathcal{P}(\mu_k(A,S))}(\mu_k(X), \mu_k(X'))}.
\]
Remark 8.10. As claimed in [DWZ1, §6], the isomorphism of proposition 8.9 can be seen as a functorial isomorphism by introducing adapted quotient categories.

9. $F$-POLYNOMIALS AND $g$-VECTORS OF DECORATED REPRESENTATIONS

The aim of this section is to define the notions of the $F$-polynomial and the $g$-vector of a GSPDR and to give a link with the usual notion (see [FZ2]). It is an extension of [DWZ1]. As before, $(I, (\Gamma_i))$ and therefore $E$ are fixed. We suppose also that the characteristic of $K$ does not divide any of the cardinals of the groups $\Gamma_i$. We suppose moreover that $K$ is algebraically closed and that all the $\Gamma_i$ are commutative (as seen in section 6, this case is sufficient to realize skew-symmetrizable exchange matrices).

Notation 9.1. For any $i \in I$, denote $\text{irr}_i = \text{irr}(\Gamma_i)$ the set of isomorphism classes of irreducible representations of $\Gamma_i$. One defines $\text{irr} = \bigcup_{i \in I} \{i\} \times \text{irr}_i$ and for $i \in I$, $C_i = K_0(\Gamma_i) \simeq \mathbb{Z}^{\text{irr}_i}$. We also denote $C = K_0(E) = \bigoplus_{i \in I} C_i \simeq \mathbb{Z}^{\text{irr}}$. If $V \in \text{mod } E$ (resp. $V \in \text{mod } E_i$), $[V]$ is its class in $C$ (resp. in $C_i$). If $e \in C$ (resp. $e \in C_i$) and $(j, \rho) \in \text{irr}$ (resp. $\rho \in \text{irr}_i$) then $e_{j, \rho}$ (resp. $e_{\rho}$) is the coefficient of $(j, \rho)$ (resp. $\rho$) in $e$.

If $(Y_j)_{j \in \text{irr}}$ (resp. $(Y_j)_{j \in \text{irr}_i}$) is a family of indeterminates or of elements of a ring, and $e \in C$ (resp. $e \in C_i$), one denotes

$$Y^e = \prod_{j \in \text{irr}} Y_j^{e_{j, \rho}}.$$  

If $(A, S)$ is a GSP, $X$ a representation of it, $[X]$ is its class, seen as an $E$-module, in $C$. If $e \in C$ then $\text{Gr}_e(X)$ is the Grassmanian of the $\mathcal{P}(A, S)$-submodules $X'$ of $X$ such that $[X'] = e$.

Let $(A, S, X, V)$ be a GSPDR, we recall the diagram of section 8, by changing a little the notation:

\[
\begin{array}{ccc}
X(k) & \downarrow \beta_k & \leftarrow X_{\text{out}}(k) \\
X_{\text{in}}(k) & \leftarrow & \gamma_k
\end{array}
\]

Definition 9.2. One defines the $F$-polynomial $F_X$ of $X$ to be a polynomial in $\mathbb{Z}[\{Y_j\}_{j \in \text{irr}}]$ defined by:

$$F_X(Y) = \sum_{e \in C} \chi(\text{Gr}_e(X)) Y^e$$

where $\chi$ is the Euler characteristic. One define also the $g$-vector $g_{X,V} = (g_k)_{k \in I} \in C = \bigoplus_{k \in I} C_k$ by

$$g_k = [\ker \gamma_k] - [X(k)] + [V(k)].$$

With the same indexing, define $h_{X,V} = (h_k)_{k \in I}$ by

$$h_k = -[\ker \beta_k].$$
Notation 9.3. If $(Y)$ is a family of indeterminates, we denote by $\mathbb{Q}_+(Y)$ the free commutative semifield generated by its elements. If $(y)$ is a family of elements of a commutative semifield, we denote by $\mathbb{Q}_+(y)$ the subsemifield generated by its elements.

Then, it is easy to adapt [DWZ1, proposition 3.1], [DWZ1, proposition 3.2] and [DWZ1, proposition 3.3]:

**Proposition 9.4.** The polynomial $F_Y(Y)$ has constant term 1 and maximum term (for divisibility of monomials) $Y[I]$.

**Proposition 9.5.** If $X'$ is another $\mathcal{P}(A, S)$-module then $F_{X \oplus X'} = F_X F_{X'}$.

**Proposition 9.6.** If $F_X \in \mathbb{Q}_+(Y)$, then $F_X$ can be evaluated in the semifield $\text{Trop}(Y')$ where $(Y')_{\text{irr}}$ is a family of indeterminates. Then $h_X$ and $F_X$ are related by the following formula:

$$Y^h_X = F_X|_{\text{Trop}(Y')} \left( Y'\rho^{-1}Y'_{\rho \otimes E_i, E_i^*} \right)_{(i, \rho) \in \text{irr}}.$$

**Proof.** We follow the proof of [DWZ1]. Remark that for any $e \in C$,

$$(Y^e)|_{\text{Trop}(Y')} \left( Y'\rho^{-1}Y'_{\rho \otimes E_i, E_i^*} \right)_{(i, \rho) \in \text{irr}} = Y'\rho^{-1}Y'_{\rho \otimes E_i, E_i^*}.$$ 

For $i \in I$, the exponent of $Y'_i = (Y'_{i, \rho})_{\rho \in \text{irr}}$ can be rewritten as

$$-e_i + [e \otimes E_i^*]$$

which can be interpreted as

$$-[X'(i)] + [X_{\text{out}}(i)]$$

for any submodule $X'$ of $X$ such that $[X'] = e$. Thus, the end of the proof is the same as in [DWZ1]. □

Recall the definition of a $Y$-seed:

**Definition 9.7** ([DWZ1, §2]). A $Y$-seed is a pair $(y, B)$ where $y$ is a family of elements of a semifield indexed by $I$ and $B$ is a skew-symmetrizable matrix. For $k \in I$, we define $\mu_k(y, B) = (y', \mu_k(B))$ where, for $i \in I$,

$$y'_i = \begin{cases} y_i^{-1} & \text{if } i = k \\ y_i y_k^{\max(0, b_{ik})} (1 + y_k)^{-b_{ki}} & \text{if } i \neq k. \end{cases}$$

Now, define the notion of an extended $Y$-seed:

**Definition 9.8.** An extended $Y$-seed is a pair $(y, (A, S))$ where $y$ is a family of elements of a semifield indexed by $I$ and $(A, S)$ is a non-degenerate GSP. For $k \in I$, we define $\mu_k(y, (A, S)) = (y', \mu_k(A, S))$ where, for $(i, \rho) \in \text{irr},$

$$y'_{i, \rho} = \begin{cases} y_i^{-1} & \text{if } i = k \\ y_i y_k^{[\rho \otimes E_i^* A_{ik}]} (1 + y_k)^{[\rho \otimes E_i^* A_{ik}]} & \text{if } i \neq k. \end{cases}$$

**Remark 9.9.** The mutation of extended $Y$-seeds is involutive.

**Definition 9.10.** A $Y$-seed or an extended $Y$-seed will be called *free* if its variables $y$ are algebraically independent.
Remark 9.11. The notion of freeness for a $Y$-seed (or an extended $Y$-seed) is stable under mutations. The semifield $\mathbb{Z}_+(y)$ and the algebra $\mathbb{Z}[y]$ are also stable under mutation, as the mutation is involutive.

Definition 9.12. Let $(y, (A, S))$ be a free extended $Y$-seed and $(z, B(A))$ be a $Y$-seed (for the same $A$). The following morphism of algebra is called the specialization map:

$$
\Phi_{y \to z} : \mathbb{Z}_+(y) \to \mathbb{Z}_+(z)
$$

$$_y \mapsto z_i.$$

The analogous for $\mathbb{Z}[y]$ and $\mathbb{Z}[z]$ is also denoted by $\Phi$.

Proposition 9.13. Let $(y, (A, S))$ be a free extended $Y$-seed such that $(A, S)$ is a locally free GSP, and $(z, B(A))$ be a $Y$-seed. Let $k \in I$. Denote $y' = \mu_k(y)$, and $z' = \mu_k(z)$. Then, $\Phi_{y' \to z} = \Phi_{y \to z}$.

Proof. As $y'$ generates $\mathbb{Z}_+(y') = \mathbb{Z}_+(y)$, it is enough to look at this for the $y'_{i, \rho}$ for $(i, \rho) \in \text{irr}$. By definition,

$$
\Phi_{y' \to z} (y'_{i, \rho}) = z'_i
$$

If $i = k$, then

$$
\Phi_{y \to z} (y'_{i, \rho}) = \Phi_{y \to z} (y^{-1}_{i, \rho}) = z^{-1}_i = z'_i.
$$

If $i \neq k$, then

$$
\Phi_{y \to z} (y'_{i, \rho}) = \Phi_{y \to z} \left( y_{i, \rho} \left[ \rho_{\otimes E_i} A_{ik} \right] (1 + y_k) \left[ \rho_{\otimes E_i} A^*_i \rho - \rho_{\otimes E_i} A_{ik} \right] \right)
$$

$$
= z_i \prod_{\sigma \in C_k} \left[ \frac{1}{2^k} (1 + z_k)^{\dim_K(\rho_{\otimes E_i} A_{ik}) - \dim_K(\rho_{\otimes E_i} A^*_i \rho)} \right]
$$

$$
= z_i \left[ \frac{1}{2^k} (1 + z_k)^{\dim_K(\rho_{\otimes E_i} A^*_i \rho)} \right]
$$

$$
= z_i \left[ \frac{1}{2^k} (1 + z_k)^{\max(0, b_{ki})} \right]
$$

(here we use the fact that every considered irreducible representation is of dimension 1, as the considered groups are commutative and $K$ is algebraically closed).

To make the relation with $F$-polynomials and $g$-vectors in cluster algebras, we need the following adaptation of [DWZ1, lemma 5.2]:

Proposition 9.14. Let $(A, S, X, V)$ be a GSPDR such that $(A, S)$ is non-degenerate. Let $k \in I$. Denote $(A', S', X', V') = \mu_k(A, S, X, V)$. Suppose also that the extended $Y$-seed $(y', (A', S'))$ is obtained from $(y, (A, S))$ by the mutation at $k$. Denote $g_{X, V} = (g_i)_{i \in I}, g_{X', V'} = (g'_i)_{i \in I}, h_{X, V} = (h_i)_{i \in I}$ and $h_{X', V'} = (h'_i)_{i \in I}$. Then

(i) $g_{X, V} = h_{X, V} - h_{X', V'}$;

(ii) one has

$$(y_k + 1)^{h_k} F_X(y) = (y'_k + 1)^{h'_k} F_{X'}(y').$$
where
\[(y_k + 1)^{h_k} = \prod_{i \in \text{irr}_k} (y_{k,i} + 1)^{h_{k,i}};\]

(iii) for any \(j \in I\),
\[
g'_j = \begin{cases} 
-g_j & \text{if } j = k \\
g_j + [g_k \otimes E_k A_{kj}] - [h_k \otimes E_k A_{kj}] + [h_k \otimes E_k A_{kj}^*] & \text{if } j \neq k.
\end{cases}
\]

Proof. (i) By definition, for \(i \in I\), \(g_i = [\ker \gamma_i] - [X(i)] + [V(i)]\), \(h_i = -[\ker \beta_i]\) and \(h'_i = -[\ker \beta'_i]\) (where \(\beta'_i\) is the analogous of \(\beta\) for \((X', V')\)). So it is enough to prove that
\[
[\ker \gamma_i] + [V_i] + [\ker \beta_i] = [X(i)] + [\ker \beta'_i].
\]
From the definition of \(\beta'_i\) given in section 8, it is easy to see that \(\ker \beta'_i \simeq \ker(\gamma_i)/\text{im}(\beta_i) \otimes V_i\). And, therefore, the searched equality reduces to
\[
[\text{im } \beta_i] + [\ker \beta_i] = [X(i)]
\]
which is obvious.

(ii) We follow the proof of [DWZ1, lemma 5.2]. Let \(e \in C\) and \(e'\) its projection in \(\bigoplus_{i \neq k} C_i\). Let \(X_0 = X \overline{E}_k\) which is a \(\overline{E}_k \mathcal{P}(A, S) \overline{E}_k\)-module. For any \(\overline{E}_k \mathcal{P}(A, S) \overline{E}_k\)-submodule \(W\) of \(X_0\), one can define
\[
W_{\text{in}}(k) = W \otimes \overline{E}_k A E_k \subset X_{\text{in}}(k) \quad \text{and} \quad W_{\text{out}}(k) = W \otimes \overline{E}_k A^s E_k \subset X_{\text{out}}(k)
\]
which are well defined because \((A, S)\) has no loop (and therefore \(X_{\text{in}} = X \otimes \overline{E}_k A E_k\) and \(X_{\text{out}} = X \otimes \overline{E}_k A^s E_k\).

For \(r, s \in C_k\), define \(Z_{e', r, s}(X)\) to be the subvariety of \(\text{Gr}_e(X_0)\) consisting of the \(W\) satisfying
- \([\alpha_k (W_{\text{in}}(k))] = r;\)
- \([\beta^{-1}_k (W_{\text{out}}(k))] = s;\)
- \(\alpha_k (W_{\text{in}}(k)) \subset \beta^{-1}_k (W_{\text{out}}(k)).\)

Define also the variety
\[
\hat{Z}_{e', r, s}(X) = \{ W \in \text{Gr}_e(X) | W \overline{E}_k \in Z_{e', r, s}(X) \}
\]
so that, by an easy computation, \(\hat{Z}_{e', r, s}(X)\) is a fiber bundle over \(Z_{e', r, s}(X)\) with fiber \(\text{Gr}_{e_k - r} (s - r)\) (where, by abuse of notation, we identify \(s - r \geq 0\) with any of its representatives in \(\text{mod } E_k\), and \(\text{Gr}_{e_k - r} (s - r) = \emptyset\) if \(e_k - r\) or \(s - r\) are nonnegative). Hence, using the easy fact that \(\text{Gr}_e(X)\) is the disjoint union of the \(\hat{Z}_{e', r, s}(X)\), we obtain, as every considered irreducible representation is of dimension 1,
\[
\chi(\text{Gr}_e(X)) = \sum_{r, s \in C_k} \left( s - r \right) \chi(\hat{Z}_{e', r, s}(X)).
\]

where, for any \(r_1, r_2 \in C_k\),
\[
\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \prod_{\rho \in \text{ind}_k} \begin{pmatrix} r_{1, \rho} \\ r_{2, \rho} \end{pmatrix}.
\]
Then, substituting this expression in the definition of \(F_X\), we obtain:

\[
F_X(y) = \sum_{e \in C} \left[ \sum_{r, s \in C_k} \left( \sum_{e_k \in C_k} \left( s - r \right) \chi \left( Z_{e', r, s}(X) \right) \right) y^e \right] \\
= \sum_{e' \in \bigoplus_i \mathbb{T}_{k_i}} \chi \left( Z_{e', r, s}(X) \right) y^{e'} \sum_{e_k \in C_k} \left( s - r \right) y^e_k \\
= \sum_{e' \in \bigoplus_i \mathbb{T}_{k_i}} \chi \left( Z_{e', r, s}(X) \right) y^{e' + r}(1 + y_k)^{s - r}.
\]

Now, as in [DWZ], we have easily that

\[
Z_{e', r, s}(X) = Z_{e', r, s}(X')
\]

where

\[
\mathbb{F} = \left[ e' \otimes_{\mathbb{T}_k} A^s E_k \right] - h_k - s \quad \text{and} \quad \mathbb{S} = \left[ e' \otimes_{\mathbb{T}_k} A E_k \right] - h'_k - r.
\]

Using this, one gets

\[
(1 + y_k)^{h'_k} F_X(y') = \sum_{e' \in \bigoplus_i \mathbb{T}_{k_i}} \chi \left( Z_{e', \mathbb{F}, \mathbb{S}}(X') \right) y^{e'} (1 + y_k)^{h'_k + \mathbb{S} - \mathbb{F}}
\]

\[
= \sum_{e' \in \bigoplus_i \mathbb{T}_{k_i}} \chi \left( Z_{e', r, s}(X) \right) y^{e'} y_k^{h'_k - h_k} (1 + y_k)^{h'_k + \mathbb{S} - \mathbb{F}}
\]

\[
= \sum_{e' \in \bigoplus_i \mathbb{T}_{k_i}} \chi \left( Z_{e', r, s}(X) \right) y^{e'} (1 + y_k)^{h_k + s - r}
\]

\[
= (1 + y_k)^{h_k} F_X(y)
\]

(iii) As \(g_k = h_k - h'_k\), \(g'_k = -g_k\). If \(j \neq k\), the equality we want to prove becomes, using again \(g_k = h_k - h'_k\),

\[
[\ker \gamma_j'] - [\ker \beta_k \otimes E_k A_{kj}] = [\ker \gamma_j] - [\ker \beta_k \otimes E_k A^s_{kj}]
\]

and, up to a possible exchange of \((A, S, X, V)\) and \((A', S', X', V')\), we can suppose that \(A_{kj} = 0\) (because \(A\) is 2-acyclic) and therefore, we have to prove that

\[
[\ker \gamma_j'] = [\ker \gamma_j] - [\ker \beta_k \otimes E_k A^s_{kj}].
\]

Let

\[
(\hat{A}, \hat{S}, \hat{X}, \hat{V}) = \hat{\mu}_k(A, S, X, V)
\]

in such a way that \((A', S')\) is right-equivalent to \((\hat{A}, \hat{S})_{\text{red}}\). In this setting, one will prove that

\[
[\ker \hat{\gamma}_j] = [\ker \gamma_j] - [\ker \beta_k \otimes E_k A^s_{kj}].
\]

We can decompose

\[
X_{\text{out}}(j) = X \otimes_{E} A^s E_j = X(k) \otimes_{E_k} A^s_{kj} \oplus X \mathbb{T}_k \otimes_{\mathbb{T}_k} E_k A^s E_j
\]
Corollary 9.16. Let 

\[ X_{\text{out}}(j) = X_{\text{out}}(k) \otimes E_k A_{jk}^* \oplus X E_k \otimes E_k E_j \]

and 

\[ X_{\text{in}}(j) = X (k) \otimes E_k \hat{A}_{kj} \oplus X_{\text{in}}(j) = X'(k) \otimes E_k A_{jk}^* \oplus X_{\text{in}}(j). \]

Along these decompositions, one has: 

\[ \gamma_j = (\psi \circ (\beta_k \otimes E_k A_{jk}^*) \eta) \quad \text{and} \quad \tilde{\gamma}_j = \left( \begin{array}{cc} \alpha_k^* \otimes E_k & \Phi_k \otimes E_k \end{array} \right. \]

where \( \psi : X_{\text{out}}(k) \otimes E_k A_{jk}^* \to X_{\text{in}}(j) \) and \( \eta : X E_k \otimes E_k E_j \to X_{\text{in}}(j) \) are two \( E_j \)-modules morphisms (basically speaking, these two morphisms encode the part of \( \gamma_j \) which is not modified by the mutation at \( k \)). Using definitions of section \( \text{[DWZ]} \), we get easily that \( \ker \alpha_k^* = \text{im} \beta_k \) and we get an exact sequence of \( E_j \)-modules: 

\[ 0 \to \ker \beta_k \otimes E_k A_{jk}^* \oplus \{0\} \to \ker \gamma_i \to \ker \tilde{\gamma}_i \to 0 \]

where, along the previous decompositions 

\[ f(u, v) = (\beta_k \otimes E_k A_{jk}^*)u, v). \]

This short exact sequence implies that 

\[ [\ker \tilde{\gamma}_j] = [\ker \gamma_j] - [\ker \beta_k \otimes E_k A_{jk}^*]. \]

To finish, it remains to prove that \( [\ker \tilde{\gamma}_j] = [\ker \gamma_j] \). The proof is the same than in \( \text{[DWZ]} \). \( \Box \)

Definition 9.15. For any GSPDR \( (A, S, X, V) \), we define in the following way the reduced \( \mathfrak{g} \)-vectors, \( \mathfrak{h} \)-vectors and \( F \)-polynomials:

- for \( i \in I \), let \( \mathfrak{g}_{X, V} = (\hat{g}_i)_{i \in I} \) defined by \( \hat{g}_i = \text{dim}_K g_i \) where \( (g_i)_{i \in I} = \mathfrak{g}_{X, V}; \)
- for \( i \in I \), let \( \mathfrak{h}_{X, V} = (\hat{h}_i)_{i \in I} \) defined by \( \hat{h}_i = \text{dim}_K h_i \) where \( (h_i)_{i \in I} = \mathfrak{h}_{X, V}; \)
- \( \tilde{F}_X = \Phi_{Y \to Z}(F_X) \) where \( (Y_i)_{i \in \text{irr}} \) and \( (Z_i)_{i \in I} \) are families of indeterminates.

Corollary 9.16. Let \( (A, S, X, V) \) be a GSPDR such that \( (A, S) \) is non-degenerate and locally free. Let \( k \in I \). Denote 

\( (A', S', X', V') = \mu_k(A, S, X, V). \)

Suppose also that the \( Y \)-seed \( (z', B(A')) \) is obtained from \( (z, B(A)) \) by the mutation at \( k \). Denote \( \hat{g}_{X, V} = (\hat{g}_i)_{i \in I}, \hat{g}_{X', V'} = (\hat{g}'_i)_{i \in I}, \hat{h}_{X, V} = (\hat{h}_i)_{i \in I} \) and \( \hat{h}_{X', V'} = (\hat{h}'_i)_{i \in I}. \) We also denote by \( (b_{ij})_{i, j \in I} \) the coefficients of \( B(A). \) Then

(i) \( \forall i \in I, \hat{g}_i = \hat{g}_i - \hat{h}_i; \)

(ii) one has 

\[ (z_k + 1)^{\hat{h}_i} \tilde{F}_X(z) = (z'_k + 1)^{\hat{h}'_i} \tilde{F}_{X'}(z'); \]

(iii) for any \( j \in I, \)

\[ \hat{g}'_j = \begin{cases} -\hat{g}_j & \text{if } j = k \\ \hat{g}_j + \max(0, b_{jk})\hat{g}_k - b_{jk}\hat{h}_k & \text{if } j \neq k \end{cases} \]
(iv) if $F_X \in \mathbb{Q}_+(Y)$, then $\hat{F}_X \in \mathbb{Q}_+(Z)$. Then $\hat{h}_X$ and $\hat{F}_X$ are related by the following formula:

$$Z_0^{\hat{h}_X} = \hat{F}_X|_{\text{Trop}(Z_0)} \left( Z_{0,i}^{-1} \prod_{j \neq i} Z_{0,j}^{\max(0,-b_{ji})} \right)_{i \in I} .$$

Proof. The points (ii) and (iii) are immediate consequences of proposition 9.14. To prove (i), it is enough to apply $\Phi_{y \rightarrow z}$ to the analogous identity in proposition 9.14 (for any extended free $Y$-seed $(y, (A, S))$ and then apply proposition 9.13. For (iv), remark that for any $(i, \rho) \in \text{irr}$,

$$\Phi_{Y_0 \rightarrow Z_0} \left( Y_{0,i,\rho}^{-1} V_{0,i,\rho}^{[\rho \otimes E,E,A^*]} \right)_{(i, \rho) \in I} = Z_0^{-1} \prod_{j \neq i} Z_{0,j}^{\max(0,-b_{ji})}$$

is independent of $\rho$ and therefore, it is easy to see that

$$\hat{F}_X|_{\text{Trop}(Z_0)} \left( Z_{0,i}^{-1} \prod_{j \neq i} Z_{0,j}^{\max(0,-b_{ji})} \right)_{i \in I}.$$

is independent of $\rho$ and therefore, it is easy to see that

$$\hat{F}_X|_{\text{Trop}(Z_0)} \left( Z_{0,i}^{-1} \prod_{j \neq i} Z_{0,j}^{\max(0,-b_{ji})} \right)_{i \in I}.$$

is independent of $\rho$ and therefore, it is easy to see that

$$\hat{F}_X|_{\text{Trop}(Z_0)} \left( Z_{0,i}^{-1} \prod_{j \neq i} Z_{0,j}^{\max(0,-b_{ji})} \right)_{i \in I}.$$

using proposition 9.6.

In [FZ2], (see also [DWZ1, §2]), Fomin and Zelevinsky defined the notions of the $F$-polynomials and the $g$-vectors associated to a sequence of mutation. More precisely, for a skew-symmetrizable matrix $B$ (which will play the role of an initial seed), a sequence of indices $i = (i_1, i_2, \ldots, i_n) \in I^n$ and $k \in I$, they define a polynomial $F_{k,i}^B \in \mathbb{Z}[Z]_{\neq I}$ and a vector $g_{k,i}^B \in \mathbb{Z}^I$.

**Definition 9.17.** Let $(A, S)$ be a non-degenerate GSP and $i = (i_1, \ldots, i_n)$ be in $I^n$ and $V$ an $E$-module. We denote

$$\left( A_{V,i_1}^{A,S}, S_{V,i_1}^{A,S}, X_{V,i_1}^{A,S}, V_{V,i_1}^{A,S} \right) = (\mu_{i_1} \mu_{i_2} \ldots \mu_{i_n}, (\mu_{i_1} \ldots \mu_{i_2} \mu_{i_1}(A, S), 0, V).$$

Remark that $\left( A_{V,i_1}^{A,S}, S_{V,i_1}^{A,S} \right)$ is right-equivalent to $(A, S)$.

Thus, we can adapt theorem [DWZ1, theorem 5.1]:

**Theorem 9.18.** Let $(A, S)$ be a non-degenerate locally free GSP. Let $i = (i_1, i_2, \ldots, i_n) \in I^n$, $k \in I$ and $\rho \in \text{irr}_k$. Then

$$g_{k,i}^B = g_{X_{\rho,i}^{A,S}, V_{\rho,i}^{A,S}} \quad \text{and} \quad F_{k,i}^B = F_{X_{\rho,i}^{A,S}}.$$

Proof. With corollary 9.16, it is the same proof as in [DWZ1].

We get also this following, analogous to [DWZ1, corollary 5.3]:

**Corollary 9.19.** In the situation of theorem 9.18, suppose that $F_{k,i}^B \neq 1$, hence $X_{\rho,i}^{A,S} \neq \{0\}$ and $V_{\rho,i}^{A,S} \neq \{0\}$ (see proposition 9.13). Let $x_{k,i}^{B(A)}$ be the corresponding cluster variable in the coefficient-free cluster algebra. In other terms

$$\left( x_{k,i}^{B(A)} \right)_{i \in I} = (\mu_{i_1} \ldots \mu_{i_2} \mu_{i_1}(x_{i})_{i \in I} \cdot B(A)).$$
Theorem 10.1. We have, for any negative decorated representation $/D_4/D_8$ 

Corollary 10.2. If $B$ is the GSP is locally free and $/D_6$ 

In particular, 

Then we have the following cluster character formula:

Define the three following integer functions:

where $X = X^A.S$, $d_i = \dim_K X(i)$ and $e_i = \dim_K e_i$.

10. $\mathcal{E}$-INVARIANT

The aim of this part is analogous to [DWZ1], §7, §8. Let $(A, S, X, V)$ and $(A, S, X', V')$ be two GSPDRs with the same non-degenerate GSP. We denote:

Define the three following integer functions:

where $[X] \in C$ is the class of $X$ seen as an $E$-module, and, for $e, e' \in C$ (resp. $e, e' \in C_k$ for $k \in I$),

Then, we get, with the same proof as [DWZ1], theorem 7.1:

**Theorem 10.1.** We have, for any $k \in I$,

In particular, $\mathcal{E}^\text{sym}$ and $\mathcal{E}$ are stable under mutations.

**Proof.** The only difference with [DWZ1] is that computations have to be done in the Grothendieck groups. Moreover, we have to worry about the skew-symmetrizability: with our convention, informally speaking, all $b_{ik}$ should be replaced by $-b_{ki}$ in the proof of [DWZ1]). For example,

For $\mathcal{E}^{\text{sym}} (\mu_k (X, V); \mu_k (X', V')) - \mathcal{E}^{\text{inj}} (X, V; X', V')$

We get also the following analogous of [DWZ1], corollary 7.2:

**Corollary 10.2.** If $(X, V)$ is obtained by a sequence of mutations from a negative decorated representation $(\{0\}, V)$ then $\mathcal{E}(X, V) = 0$. 

\[ x_{k;i}^{B(A)} = \prod_{i \in I} \chi \left( \mathcal{G}_{e_i}(X) \right) \prod_{i \in I} x_i^{-d_i} \sum_{e \in C} \chi(\mathcal{G}_{e}(X)) \prod_{i \in I} x_i^{-rk_{\gamma} + \sum_{e \in I} \max(0, b_{ij} e) + \max(0, -b_{ij} (d_j - e_j))} \] 

where $X = X^A.S$, $d_i = \dim_K X(i)$ and $e_i = \dim_K e_i$. 

\[ \mathcal{E}^{\text{inj}}(X, V; X', V') = \langle X, X' \rangle + \langle X, X' \rangle - \frac{\mathcal{E}^{\text{sym}}(X, V; X', V')}{2} \] 

\[ \mathcal{E}(X, V) = \mathcal{E}^{\text{inj}}(X, V; X, V) = \] 

\[ \mathcal{E}(X, V) = \mathcal{E}^{\text{inj}}(X, V; X, V) = \frac{\mathcal{E}^{\text{sym}}(X, V; X, V)}{2} \] 

\[ (e|e') = \sum_{i \in \text{irr}} e_i e'_i. \]
Proposition 10.3. We have

\[ \mathcal{E}(X^*, V^*) = \mathcal{E}(X, V). \]

Proof. As for any \( i \in I \), the characteristic of \( K \) does not divide \( \# \Gamma_i \), we have an isomorphism of right \( E \)-modules

\[ (X \otimes_E A)^* \rightarrow X^* \otimes_E A^{op} \Rightarrow X^* \otimes_E A^{op} \]

\[ f \mapsto \sum_{a \in B} f(x \otimes a)x^* \otimes a^{op} \]

which does not depend on the bases \( B_X \) and \( B_A \) of \( X \) and \( A \). Thus, we have, as in [DWZ1,]

\[ \mathcal{E}(X, V) = \langle X, X \rangle + ([X][X \otimes_E A^*]) + \left( [X] \biggl[ V \biggl] - [X] - \sum_{i \in I} \text{im} \gamma_i \biggr) \]

\[ = \langle X, X \rangle + ([X \otimes_E A][X]) + \left( [X] \biggl[ V \biggl] - [X] - \sum_{i \in I} \text{im} \gamma_i \biggr) \]

\[ = \langle X^*, X^* \rangle + \left( ([X \otimes_E A]^*)[[X]] \right) \]

\[ + \left( [X^*] \biggl[ V^* \biggl] - [X^*] - \sum_{i \in I} \text{im} \gamma_i \biggr) \]

\[ = \mathcal{E}(X^*, V^*) \]

where we used that

\[ ([X][X \otimes_E A^*]) = \dim_K \text{Hom}_E(X, X \otimes_E A^*) \]

\[ = \dim_K \text{Hom}_E(X \otimes_E A, X) = ([X \otimes_E A][X]). \]

Hence, the following theorem has the same proof as [DWZ1, theorem 8.1] (note that all [DWZ1, §10] can be easily adapted in this case):

Theorem 10.4. The \( \mathcal{E} \)-invariant satisfies

\[ \mathcal{E}(X, V) \geq \left( \biggl( \bigoplus_{i \in I} \ker \beta_i \biggr) \biggl( \bigoplus_{i \in I} \ker \gamma_i \biggr) \right) + ([X][V]). \]
Then, we obtain the analogous of \cite[corollary 8.3]{DWZ1}:

**Corollary 10.5.** If $\mathcal{E}(X, V) = 0$ then for each $(k, \rho) \in \text{irr}$,

(i) either $[M_k]_{\rho} = 0$ or $[V_k]_{\rho} = 0$;

(ii) either $[\ker \gamma_k]_{\rho} = 0$ or $[\ker \gamma_k]_{\rho} = [\text{im} \beta_k]_{\rho}$.

11. **Applications to cluster algebras**

We conclude here that the following conjectures of \cite{FZ2} are true for skew-symmetrizable integer matrix which can be obtained from a non-degenerate GSP with abelian groups. In particular, every matrix of the form $DS$ where $D$ is diagonal with integer coefficients and $S$ is skew-symmetric with integer coefficients can be obtained in view of section 7. Every exchange matrix corresponding to the situation described in \cite{Dem} (in particular every acyclic ones) can also be raised. Let $B$ be such a skew-symmetrizable integer matrix. We suppose moreover that some $(A, S)$ is fixed satisfying the hypothesis of section 9 such that $B(A) = B$.

**Proposition 11.1** (\cite[conjecture 5.4]{FZ2}). For any $i \in \mathcal{I}$ and $k \in I$, $F^B_{k; i}$ has constant term 1.

**Proposition 11.2** (\cite[conjecture 5.5]{FZ2}). For any $i \in \mathcal{I}$ and $k \in I$, $F^B_{k; i}$ has a maximum monomial for divisibility order with coefficient 1.

These first two are immediate, as in \cite[§9]{DWZ1}.\n
**Proposition 11.3** (\cite[conjecture 7.12]{FZ2}). For any $i \in \mathcal{I}$, $k \in I$, we denote by $k_i$ the concatenation of $(k)$ and $i$. Let $j \in I$ and $(g_i)_{i \in I} = g^B_{j; i}$ and $(g'_i)_{i \in I} = g^B_{j; k_i}$. Then we have, for any $i \in I$,

$$g'_i = \begin{cases} 
-g_i + \max(0, b_k)g_k - b_j \min(g_k, 0) & \text{if } i = k; \\
g_i & \text{if } i \neq k.
\end{cases}$$

*Proof.* We need here to add some trick to the proof of \cite[§9]{DWZ1}. Indeed, we need to prove, as in \cite[§9]{DWZ1}, that

$$\min(0, g_k) = h_k.$$ 

But what we obtain by using corollary 10.1 is

$$\min(0, g_{k, \rho}) = h_{k, \rho}$$

for any $\rho \in \text{irr}_k$. Moreover, we have, as seen before,

$$g_k = \sum_{\rho \in \text{irr}_k} g_{k, \rho} \quad \text{and} \quad h_k = \sum_{\rho \in \text{irr}_k} h_{k, \rho}$$

and therefore, what we need is equivalent to the fact that the $g_{k, \rho}$ are of the same sign. We will prove this with an indirect method. Retaining the notation of definition 9.17, we get

$$X^{A,S}_{E,j;i} = \sum_{\rho \in \text{irr}_j} X^{A,S}_{\rho;i}$$

and therefore, by linearity of $g$,

$$g_{X^{A,S}_{E,j;i}} = \sum_{\rho \in \text{irr}_j} g_{X^{A,S}_{\rho;i}}.$$
Hence, we get:
\[(\#\Gamma_j)g_k = \dim_K \left[ g_{X_j}^{A_i} \right]_k \]
In the same way,
\[(\#\Gamma_j)h_k = \dim_K \left[ h_{X_j}^{A_i} \right]_k \]
Moreover, by an immediate induction using proposition 9.14, as \([E_j] = \text{the~class~of~a~free~} E_j\text{-module}\], \(g_{i,j}^{B_{i,j}}\) and \(g_{i,j}^{B_{i,j}}\) are also free and therefore, their coefficients in term of the irreducible representations of \(E_k\) are of the same sign. Hence, we obtain, by adding these components
\[\min(0,(\#\Gamma_j)g_k) = (\#\Gamma_j)h_k\]
and the rest follows as in [DWZ1]. Note that it implies also that the \(g_{k,\rho}\) are of the same sign. □

The three following propositions have the same proof than in [DWZ1, §9]:

**Proposition 11.4** ([FZ2, conjecture 6.13]). For any \(i \in I^n\), the vectors \(g_{i,j}^B\) for \(i \in I\) are sign-coherent. In other terms, for \(i, i', j \in I\), the \(j\)-th components of \(g_{i,j}^B\) and \(g_{i',j}^B\) have the same sign.

**Proposition 11.5** ([FZ2, conjecture 7.10(2)]). For any \(i \in I^n\), the vectors \(g_{i,j}^B\) for \(i \in I\) form a \(\mathbb{Z}\)-basis of \(\mathbb{Z}I\).

**Proposition 11.6** ([FZ2, conjecture 7.10(1)]). For any \(i, i' \in I^n\), if we have
\[
\sum_{i \in I} a_{i} g_{i,j}^B = \sum_{i \in I} a'_{i} g_{i,j}^B
\]
for some nonnegative integers \((a_i)_{i \in I}\) and \((a'_i)_{i \in I}\), then there is a permutation \(\sigma \in \mathfrak{S}_I\) such that for every \(i \in I\),
\[
a_i = a'_{\sigma(i)} \quad \text{and} \quad a_i \neq 0 \Rightarrow g_{i,j}^B = g_{\sigma(i),j}^B \quad \text{and} \quad a_i \neq 0 \Rightarrow F_{i,j}^B = F_{\sigma(i),j}^B.
\]
In particular, \(F_{i,j}^B\) is determined by \(g_{i,j}^B\).

12. An example and a counterexample

The aim of this part is to show an example where the technique shown in the previous sections works and a counterexample where there is no non-degenerate potential.

Suppose here that \(K = \mathbb{C}\). We fix \(\Gamma_1 = \Gamma_2\) to be the trivial group and \(\Gamma_3 = \mathbb{Z}/2\mathbb{Z}\). We take \(A_{12} = \mathbb{C}\) and \(A_{23} = \mathbb{C}[\mathbb{Z}/2\mathbb{Z}]\), the other \(A_{ij}\) vanishing. Then \(A\) is acyclic and therefore \(S = 0\) is a non-degenerate potential, in view of section 1. Moreover,
\[
B(A) = \begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & -1 \\
0 & 2 & 0
\end{pmatrix}
\]
which is of type \(C_3\). Its exchange graph is given on figure[] where the small dots (•) symbolize vertices with trivial group and big dots (○) symbolize
Figure 1. Exchange graph of type $B_3$

vertices with group $\mathbb{Z}/2\mathbb{Z}$. Simple arrows symbolize $\mathbb{C}$ and double arrows symbolize $\mathbb{C}[\mathbb{Z}/2\mathbb{Z}]$. Thus, $(A, S)$ will be symbolized by

Finally, wave lines ($\sim$) symbolize mutations composed with the exchange of vertices 1 and 2.

Now, we will compute explicitly $F_{3,213}^B$ and $g_{3,213}^B$. We will follow the construction of section 9. According to the exchange graph,

$$\mu_3\mu_1\mu_2(A, 0) = \left( \begin{array}{c} \mathcal{A} \\ \mathcal{A'} \end{array}, 0 \right) = (A', S').$$
Let $\rho$ be one of the two irreducible modules over $\mathbb{Z}/2\mathbb{Z}$. Then

$$\mu_3(A', S', 0, \rho) = \begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mu_1 \mu_3(A', S', 0, \rho) = \begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mu_2 \mu_1 \mu_3(A', S', 0, \rho) = \begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(the arrows are obvious) and therefore,

$$X_{\rho;213}^B = \begin{pmatrix} \rho \\ \rho \end{pmatrix}$$

which induces that:

$$F_{X_{\rho;213}^B} = 1 + Y_\rho + Y_2 Y_\rho + Y_1 Y_2 Y_\rho$$

and therefore

$$\tilde{F}_{X_{\rho;213}^B} = 1 + Y_3 + Y_2 Y_3 + Y_1 Y_2 Y_3$$

Moreover,

$$g_{X_{\rho;213}^B} = \begin{pmatrix} 0 \\ 0 \\ -\rho \end{pmatrix}$$

and therefore

$$\tilde{g}_{X_{\rho;213}^B} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$ 

It is easy to check by hand that these coincide with $F_{3;213}^B$ and $g_{3;213}^B$ obtained for example by formulas of [DWZ1, §2].

Let now $B$ be the matrix defined by

$$B = \begin{pmatrix} 0 & 0 & 1 & 1 & -1 & -2 \\ 0 & 0 & -1 & -1 & 1 & 2 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$ 

We will show that there is no non-degenerate locally free GSP with mutation matrix $B$. Suppose that $(I, (\Gamma_i), A, S)$ is a non-degenerate GSP with mutation matrix $B$. Then, $\Gamma_1, \ldots, \Gamma_5$ have the same cardinal which is two times the one of $\Gamma_6$. Applying $\mu_3$ followed by $\mu_5$ create 2-cycles and implies, in view of proposition 4.10, that

$$A_{23} \otimes E_3 A_{31} \simeq (A_{15} \otimes E_5 A_{52})^*.$$
In the same way, applying $\mu_4$ followed by $\mu_5$ implies that
\[ A_{24} \otimes_{E_4} A_{41} \simeq (A_{15} \otimes_{E_5} A_{32})^\ast. \]
With the same type of argument, applying $\mu_3$, $\mu_4$ and $\mu_6$ implies that
\[ (A_{23} \otimes_{E_3} A_{31})^{\oplus 2} \simeq A_{24} \otimes_{E_4} A_{41} \oplus A_{23} \otimes_{E_3} A_{31} \simeq (A_{16} \otimes_{E_6} A_{62})^\ast. \]
As all considered groups are semisimple, it is easy to see that the $(E_1, E_6)$-bimodule $A_{16}$ can be decomposed as a direct sum of the form
\[ A_{16} = \bigoplus_{i=1}^m r_i \otimes_K s_i \]
where the $r_i$ are irreducible left $E_1$-modules and the $s_i$ are irreducible right $E_6$-modules. Moreover, the $r_i \otimes_K s_i$ are irreducible bimodule and satisfy, because of $B$,
\[ \forall r \in \text{irr}_1, \sum_{i | r_i \simeq r} \dim_K s_i = \dim_K r \quad \text{and} \quad \forall s \in \text{irr}_6, \sum_{i | s_i \simeq s} \dim_K r_i = 2 \dim_K s. \]
Thus, there are exactly two indices which can be supposed to be 1 and 2 such that $s_1$, $s_2$ are trivial and $r_1$ and $r_2$ are of dimension 1 and appear only one time in the sequence $(r_i)$. In the same way,
\[ A_{62} = \bigoplus_{i=1}^n t_i \otimes_K u_i \]
with
\[ \forall t \in \text{irr}_1, \sum_{i | t_i \simeq t} \dim_K u_i = 2 \dim_K t \quad \text{and} \quad \forall u \in \text{irr}_2, \sum_{i | u_i \simeq u} \dim_K t_i = \dim_K u. \]
Thus, there are exactly two indices which can be supposed to be 1 and 2 such that $t_1$, $t_2$ are trivial and the $u_1$ and $u_2$ are of dimension 1 and appear only one time in the sequence $(u_j)$. Hence,
\[ (A_{16} \otimes A_{62})^\ast = \bigoplus_{i=1}^m \bigoplus_{j=1}^n (u_j^* \otimes_K r_i^*)^{\dim_K s_i} \]
contains $u_1^* \otimes r_1^* \oplus u_2^* \otimes r_2^* \oplus u_3^* \otimes r_1^* \oplus u_3^* \otimes r_2^*$ as the only summands containing $u_1^*$, $u_2^*$, $r_1^*$ and $r_2^*$. Finally, $(A_{16} \otimes A_{62})^\ast$ can not be decomposed as a direct sum of two times the same bimodule, which is a contradiction.

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