Local Conformal Instability and Local Non-Collapsing in Quantum Spacetime

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In this paper, we try to answer two questions, does gravitation itself suffer from conformal instability, if it is true, does the conformal instability cause the collapse of local spacetime region or even collapse the whole spacetime? The problems are studied in the framework of the Quantum Spacetime Reference Frame (QSRF) and induced spacetime Ricci flow. We find that if the lowest eigenvalue of an operator, associated with the F-functional in a local compact region, is positive, the local region is conformally unstable and will tend to volume-shrinking and curvature-pinching along the Ricci flow-time $t$; if the eigenvalue is negative, the local region is conformally stable up to a trivial rescaling. However, the local non-collapsing theorem ensures that the instability will not cause the local compact spacetime region collapse into nothing. The total effective action is also proved positive defined and bounded from below keeping the whole spacetime conformally stable, which can be considered as a generalization of the classical positive mass theorem of gravitation to the quantum level. The motivation and logical derivation of QSRF are also reviewed in the appendix of the paper.

I. CONFORMAL INSTABILITY PROBLEM IN QUANTUM GENERAL RELATIVITY

Through a standard Wick rotation, the resulting Euclidean Einstein-Hilbert action

$$S_E = -\frac{1}{16\pi G} \int d^4x \sqrt{g} R = -\frac{1}{16\pi G} \int d^4x \sqrt{\tilde{g}} \left[ \Omega^2 \tilde{R} + 6(\nabla \Omega)^2 \right]$$

of General Relativity suffers a well known “bottomless” problem [1, 2], in which $\Omega$ is a conformal factor

$$g_{\mu\nu}(x) = \Omega^2(x) \tilde{g}_{\mu\nu}(x).$$

The problem is associated with the conformal instability of the theory, a naive observation is that the kinetic term $(\nabla \Omega)^2$ of the conformal factor of gravity has a “wrong sign”, making the Euclidean action potentially become arbitrarily negative or unbounded from below. For this reason, the functional integral quantization of the corresponding theory

$$Z = \int [Dg_{\mu\nu}] e^{-S_E}$$

is divergent and ill defined. This divergence is not related to the ultraviolet divergences of quantum gravity, it must be taken care of first before one can renormalize the theory.

Various possible solutions for dealing with the conformal instability problem have been proposed. For example, it was suggested that the integration of the conformal factor should be performed by distorting the integration contour in the complex plane to avoid the unboundness action [1, 3]. However, the contour deformations in functional integral would lead to complex non-perturbative contributions to the calculations [4, 5]. Another approach has been proposed that the theory should be formulated in terms of dynamics and physical (transverse-traceless) degrees of freedom by taking account of certain Jacobian factors in the functional integral measure [6–8], leading to a non-standard Wick rotation [9, 10] of the conformal factor. In the approach, the “wrong sign” of the conformal factor can flip to be “right” depending on specific range of value of an undetermined constant in the supermetric. Further more, the approach works only at the linearized level, and beyond the linearized level the Euclidean action is rather complicated, and it is not obvious whether the “wrong sign” will also flip beyond the linearized level.

An alternative possibility is that the unboundedness of the Euclidean action may be not necessarily a real problem in defining its quantum theory. On the one hand, there are a number of bottomless theories at naive classical level which are well defined and having good properties at the quantum level. For example, electron in an attractive Coulomb well (e.g. the hydrogen atom) has a similar instability before the quantum mechanics is discovered, since the action is also unbounded from below which should cause the extranuclear electron falling into the infinitely deep

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Coulomb well and hence the atom finally collapses. But now it is known that the quantum mechanical treatment ultimately evades such instability problem: the energy eigenvalue is bounded from below and the electron has almost vanishing wavefunction and probability amplitude to fall into the infinitely deep well. On the other hand, similar type of conformal instability in gravitational system may be necessary and even crucial in understanding an inflationary universe at early epoch. For example, the conformon inflation [11] is caused by such instability of the conformal factor: the conformon fields.

II. CONFORMAL STABILITY OF LOCAL QUANTUM SPACETIME AND THE F-FUNCTIONAL

Does the local conformal instability of gravity real? Here we propose another viewing angle to the problem from the framework of quantum spacetime reference frame (QSRF) and the induced spacetime Ricci flow. The Ricci flow in $D = 3$ dimensions on compact manifolds is now well understood due to the seminal works of Perelman [12–14], and some basic theorems can be generalized to the $D = 4$ Riemannian manifolds (compact case [15, 16], and noncompact [17,19]), even though the case is much harder and still under development. Applying the Ricci flow to a $D = 4$ Lorentzian manifolds is more physical relevant, although the rigorous validity at the mathematical level is still lacking, there already exist many applications of the Ricci flow to the physical spacetime (see e.g. [20–28]). In our previous works (motivation and logical derivation will be reviewed in the appendix of the paper), we have tried to lay a solid physical foundation for the spacetime Ricci flow based on a well-defined relativistic quantum fields theory of spacetime reference frame [29–35], in which the Ricci flow arises as the RG flow of quantum spacetime at the gaussian approximation. It seems that several fundamental issues of quantum gravity are consistently treated within the frameworks, e.g. the cosmological constant problem, the trace anomaly, the thermodynamics of spacetime and the early inflationary universe. Thus we have reasons to believe that, at the physical rigorous level, it is natural to conjecture that some basic techniques of Perelman and theorems of the Ricci flow are still true for the physical spacetime.

Note that there is also a “wrong sign” in the F-functional [12] of the $D = 4$ spacetime Ricci flow

$$
\mathcal{F}(g, u) = \int d^D x \left( R + |\nabla f|^2 \right)
$$

which is similar with the Einstein-Hilbert action [11] up to a constant multiple. The conformal stability of a spacetime configuration depends on the sign of the lowest eigenvalue of an operator $-4\Delta + R$ associated with the F-functional $\mathcal{F}(g, \phi)$ [36, 37],

$$
\Lambda(g) = \inf \left\{ \mathcal{F}(g, \phi) = \int d^4 x \left( R\phi^2 + 4|\nabla \phi|^2 \right), \text{ with } \lambda \int \phi^2 d^4 x = 1 \right\}
$$

rather than naively depends on the sign of the local kinetic term, just as the stability of quantum treatment of hydrogen atom.

To probe the stability of a local compact region centered at $X$, one could choose $\phi^2(X) = u(X) = e^{-f(X)}$ an approximate delta function centered at $X$ to calculate $\Lambda(g(X))$. From the backwards heat equation [15] and (14), it is clear that if the eigenvalue $\Lambda(g(X)) > 0$, the local compact 4-volume $d^4 X \sim u^{-1}$ of the region will decay and shrink by the increasing of $u$ along the flow-time $t$ (but the physical-time) into a shrinking soliton and tend to disappear (not completely collapse, see later) or develop local neckpinch and tend to thorns in finite flow-time $t$ (finite scale) during the Ricci flow, so the local compact region around $X$ is linearly unstable. The positive eigenvalue $\Lambda$ is similar with the negative unbounded classical action $S_E$, leading to a conformally unstable region, and the function $\phi$ plays a similar role of the conformal factor $\Omega$, not only they both have similar “wrong sign” in their kinetic terms, but also $\phi = \sqrt{\gamma}$ represents the longitudinal and trace part of the degrees of freedom of metric. Actually $\phi = \sqrt{\gamma}$ is indeed a conformal factor of the gravity up to a constant multiple, a positive sign of $\Lambda$ will indeed induce a conformal instability of gravity at least in a local compact region. Further more, it is also found [34] that such local unstable shrinking region modeled by a shrinking Ricci soliton reproduces a spatial inflationary region at a local physical-time. If $\Lambda(g(X)) \leq 0$, the spacetime compact region expands its volume and hence it is stable up to a rescaling, and it is equivalent to the positive mass theorem in that region.

III. LOCAL NON-COLLAPSING OF QUANTUM SPACETIME AND THE W-FUNCTIONAL

From previous discussions, we can see that some local compact spacetime regions are possibly conformally unstable for the similar reason of the “wrong sign” in the local F-functional, the functional or the eigenvalue is also possibly negative making some other local regions stable. The next natural question is that, does the local unstable compact
region completely collapse into nothing? Or even worse, does the instability finally leads to a disastrous collapse of the whole spacetime? A no-go answer to second question is seem guaranteed by the positive mass theorem \cite{38,39} at the classical level, but what is the case at the quantum level?

The monotonicity of the F-functional ensures its boundedness, which claims that $\Lambda(g)$ is nondecreasing along the flow-time $t$ during any Ricci flow without any curvature condition, and bounded above by $\mathcal{F} \leq \frac{D}{2(t_\ast - t)}$ \cite{12}, where $D$ is the dimension of spacetime, $t$ the flow-time, $t_\ast$ is certain finite singular flow-time when the local curvature diverges. The equal sign is saturate if and only if the local spacetime region flows and finally becomes to a gradient shrinking Ricci soliton configuration eq.(30).

However, the right hand side of $\mathcal{F} \leq \frac{D}{2(t_\ast - t)}$ also diverges when $t \to t_\ast$, the bound still does not answer the question whether the local shrinking Ricci soliton region finally local collapse as $t \to t_\ast$. More precisely, to define the notion “local collapse” of a compact local spacetime region, we consider if there exists a sequence $(k = 1, 2, 3, \ldots)$ of flow-time $t_k \to t_\ast$ and radii $r_k \in (0, \infty)$ of the local compact space and time region with $r_k^2/t_k$ uniformly bounded from above, and the Riemannian curvature of the local region is bounded comparable to the radius $|Rm(g(t_k))| \leq r_k^{-2}$ in the compact spacetime ball $B_{g(t_k)}(r_k)$, here the volume of the ball in the local compact region $V(B_{g(t_k)}(r_k))$ shrinks to zero in the limit of the sequence $\lim_{k \to \infty} V(B_{g(t_k)}(r_k))/r_k^D = 0$. If the situation actually happens, the local compact spacetime region is said to be local collapse.

To probe whether the local compact region is local collapse, we need a dimensionless and scale invariant version of the F-functional. To achieve scale invariance, Perelman includes an explicit insertion of the scale parameter $\tau = t_\ast - t$ to the F-functional and defines the W-functional \cite{12},

$$W(g,u,\tau) = \int d^Dx u \left( \tau (R + |\nabla f|^2) + f - D \right)$$

in which $D = 4$ is the dimension of the spacetime, $u = \frac{1}{(4\pi \tau)^{D/2}}e^{-f}$ is the positive defined manifolds density allowing one to localize to the concern compact region, $\tau = t_\ast - t$ telling us at what distance scale to localize (i.e. $\sqrt{\tau}$). The W-functional will be essential for understanding the critical structure near a shrinking spacetime region. First, it is invariant under simultaneous rescaling of $\tau$ and $g$. Second, it is non-decreasing along the flow-time during any Ricci flow $\frac{\partial W}{\partial \tau} \geq 0$ without any assumption on curvature, for this reason, it is also often called the W-entropy.

One can also use the $u$ density to probe the local compact region of $g(t)$ where one concerns. For example, the collapse or non-collapse of the region near a point $X$ can be detected from the value of the W-functional. If one chooses $u(X,t)$ an approximate delta function centered at $X$ at flow-time $t$, then the more collapse of the region, the more negative the value of $W(g(X),u(X),t)$. However, by the monotonicity of the W-functional during the Ricci flow, since the W-functional of a certain initial metric of the region is bounded from below at certain initial scales (bounded from above and below), after a certain amount of flow-time, the W-functional must also be bounded from below at all bounded scales, so the local collapsing of the compact spacetime region corresponding to the arbitrary negative value of the W-functional must be ruled out by the monotonicity of it.

This leads to the local non-collapsing theorem of Perelman \cite{12}, which states that if a local compact spacetime region around point $X$ is unstable and hence tends to shrink, as one approaches the singular finite flow-time $t_\ast$ when the local curvature diverges, collapsing of the spacetime region cannot actually occur at the scales $r \sim O(\sqrt{t_\ast - t})$, the volume of a unit ball in the compact space and time region is bounded from below

$$\frac{V(B_{g(t)}(X,r))}{r^D} \geq \kappa(g,D) > 0. \quad (7)$$

The local non-collapsing theorem has important physical consequences. Since the inequality $\frac{\partial W}{\partial \tau} \geq 0$ saturates when the spacetime region is a gradient shrinking Ricci soliton eq.(30). Although the volume of the shrinking soliton is shrinking due to the conformal instability, the scale invariant W entropy has already maximized to be a finite constant value, the local shape of the shrinking soliton does not change, the information of its shape or topology is preserved rather than lost (when W entropy is arbitrarily negative and hence collapse, then the information is completely lost). The information of its size or volume is not preserved (relative to an observer and depends on the definition of ruler and clock of the observer), but the essential information of its structure encoded by its shape and topology can always be zoom in by the observer if it is not completely collapse into nothing. More physical speaking, the conformal instability may shrink a local compact spacetime region but will not collapse it into nothing, since the scale-independent information of its local topology is preserved during the Ricci flow. As a consequence, there may exist local conformal instability happening in the places where $\Lambda(g(X)) > 0$ for a given general initial spacetime manifolds, i.e. the Lagrangian in the functional integral is not necessarily positive defined, instead, the total effective action $S_{eff}(M^D) = - \log Z = \frac{D}{2} - \lambda N(M^D)$ should be positive defined and bounded from below, where the
effective action $\frac{\mathcal{L}}{\kappa} - \lambda \tilde{N}(M^D)$ is given by QSRF corresponding to Perelmann’s partition function $[12]$ in the Ricci flow. It is easy to verify that it is indeed the case, because the Relative Shannon entropy $\tilde{N}(M^D)$ is bounded above $\tilde{N}(M^D) \leq 0$, eq. (31).

The positive and boundedness of the total effective action $S_{\text{eff}}(M^D)$ ensures the stability of the whole spacetime, which can be seen as a quantum generalization of the classical positive mass theorem in certain sense. In the geometric point of view, starting from an arbitrary initial spacetime manifolds, the Ricci flow as a RG flow gradually deforms and smooths out local irregularities on it, after certain proper treatment of the local neckpinch (“surgeries” by hand $[13]$ or by internal mechanism) the spacetime will finally flow to a stable and non-collapse spacetime up to a rescaling.

IV. CONCLUSIONS

In this paper, the conformal instability problem is studied in the framework of the quantum spacetime reference frame and induced spacetime Ricci flow. Instead of naively observing a “wrong sign” in front of the kinetic term of the conformal factor, the conformal stability of quantum spacetime and gravitation depend on the sign of the lowest eigenvalue $\Lambda(g(X))$ of the operator $-4\Delta + R$, associated with the $F$-functional, around a local compact spacetime region. If $\Lambda(g(X)) > 0$ the local compact region is conformally unstable and will tend to shrink its volume (along the flow-time $t$ but the physical-time), if $\Lambda(g(X)) < 0$ the local compact region is conformally stable up to a trivial rescaling. Thus a given general spacetime will possibly develop local volume shrinking and local curvature pinching in some places. Although conformal instability may happen in some places of a given general spacetime, the instability will not cause the structure of the local region collapse into nothing, a finite residue information taken by the scale-invariant W-functional of that region is preserved. The residue information of the shrinking local region will not be lost, otherwise the W-entropy will become infinitely negative which is ruled out by the monotonicity of it.

Different from the possible approaches to the conformal instability of the euclidean general relativity in literature, which try to locally flip the “wrong sign” into a right one in the Lagrangian, the framework of QSRF and induced Ricci flow allow the local Lagrangian in the functional integral being negative, leading to conformally unstable in certain places of the spacetime, but the total effective action is proved positive defined and bounded below, so the whole spacetime is stable, which generalizes the classical positive mass theorem of spacetime to the quantum level.

The presence of the conformal instability of a quantum spacetime may have important physical consequence, for instance, an inflationary universe at early epoch may be driven by the mechanism of the conformal instability, in other words, the spatial inflationary region near the local physical-time origin may be modeled by a gradient shrinking Ricci soliton configuration, leading to a possible quantum theory of early universe different from the textbook inflation models driven by inflaton.

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Appendix I: Quantum Spacetime Reference Frame (QSRF)

In this appendix, we briefly review the QSRF as a possible theory of quantum spacetime. It is starting from the idea that spacetime is nothing but an ideal and standard reference that moving body is relative to. In fact “ideal” can not be realized in rigor, all things are quantum fluctuating. A quantum standard reference can be measured (relative to a fiducial lab) by quantum rods and quantum clocks which are also subject to quantum fluctuations. Surprisingly, gravitation is nothing but emergent phenomenon coming from relative measurement of the under-studied moving body being reference to the quantum standard reference, i.e. quantum reference frame. If the classical equivalence principle is generalized to the quantum level, then some universal (classical or quantum) properties (e.g. the universal acceleration of a free falling body which is independent to its mass, the Hubble redshift (universal recession velocity) and even the broadening (universal acceleration) of spectral lines are independent to theirs energies) are not merely the (classical or quantum) properties of matter, they are actually the (classical or quantum) properties of the spacetime itself. Thus in this framework, the most fundamental thing is not individual quantum state but the relations between two quantum states, i.e. the under-studied quantum body $|\psi\rangle$ and the quantum reference frame system $|X\rangle$, describing by an entangled state $|\Psi\rangle = \sum_{ij} |\psi\rangle_i \otimes |X\rangle_j$ in the whole Hilbert space $\mathcal{H}_\psi \otimes \mathcal{H}_X$. 
If the quantum spacetime reference frame $|X_\mu\rangle$ ($\mu = 0, 1, 2, \ldots D - 1$) itself is considered as the under-studied quantum body, then we have a quantum theory of the quantum spacetime. In this case the reference system could be the fiducial lab spacetime $|x_a\rangle$, $(a = 0, 1, 2, \ldots d - 1)$. The entangled state is constructed by a one-to-one correspondence between two states, i.e. $|x_a\rangle \rightarrow |X_\mu\rangle$, which is nothing but a generally non-linear differentiable mapping $X : \mathbb{R}^d \rightarrow M^D$. From the geometric point of view, the entangled state $\sum_i |X_i\rangle \otimes |x_j\rangle$ or $X(x)$ is a mapping from a local coordinate flat patch $x \in \mathbb{R}^d$ to a Riemannian (even with Lorentz signature) manifolds $X \in M^D$. From the physics point of view, the generally non-linear mapping can be realized by a kind of field theory, the non-linear sigma model (NLSM)

$$S[X] = \frac{1}{2} \lambda \int d^d x g_{\mu\nu} \frac{\partial X^\mu}{\partial x_a} \frac{\partial X^\nu}{\partial x_a}$$

(8)

where $\lambda$ is a constant with dimension of energy density $[L^{-d}]$ taking the value of the critical density of the universe $\lambda = \frac{3H_0^2}{8\pi G}$. $x_a$ having dimension of length $[L]$ is the base space of NLSM interpreted as the fiducial lab spacetime as the starting reference. Field $X_\mu$ also having dimension of length is the target space of NLSM interpreted as rods and clocks, i.e. the reference frame fields measuring the quantum spacetime coordinate. $D = 4$ is the least number of the frame fields capable to measure the coordinates of spacetime. $d$ is the dimensions of the fiducial lab spacetime, it could take $d = 4 - \epsilon$, where $0 < \epsilon \ll 1$ is for topological reason that the homotopic group $\pi_d(M^D)$ should be trivial so that the mapping $X(x) : \mathbb{R}^d \rightarrow M^D$ should be free from topological obstacles and singularities. Since a quantum fields theory is well-formulated in an inertial frame, and the local patch is also flat, so the fiducial lab spacetime $x_a$ is considered flat and rigid. It can be either Minkowskian or Euclidean, since $d^d x$ in fact $d^d x \det e$ ($\det e$ the Jacobian) is an invariant measure independent to the metric of the base space. Thus without loss of generality, we consider the base space as the Euclidean one which is better defined when one uses the functional integral quantization method since the action in this case is always real defined, and remember that the Euclidean theory will give the same result as the Minkowskian one. It is well-known that the NLSM in $d = 2$ is perturbative renormalizable, and $d = 3$ and $d = 4 - \epsilon < 4$ are non-perturbative renormalizable, so it is known as a well-defined relativistic quantum fields theory.

The quantum frame fields model of spacetime has practical physical interpretation, for instance, in the lab scale, it can be considered as the multi-wire proportional chamber system constructed relative to the wall and clock of a lab that are used to measure coordinates of events in the lab, where the frame fields can be interpreted as the spin-less electron signal. When the spacetime measuring scale is beyond the lab’s scale, for instance, to the cosmic scale, the quantum fluctuation or broadening of the frame fields (e.g. by using spin-less light signal) become unignorable, and curved metric $g_{\mu\nu}$ measured by the comparison between the frame fields and fiducial coordinates become non-trivial and important, since the quantum fluctuation $X_\mu = \langle c_\sigma^a \rangle x_a + \delta X_\mu$ have nontrivial physical consequence. The lowest 2nd order moment (variance) modifies the quadratic form distance of Riemannian geometry and hence gives correction to the quantum expectation of the metric

$$\langle g_{\mu\nu} \rangle = \left\langle \frac{\partial X_\mu}{\partial x_a} \frac{\partial X_\nu}{\partial x_a} \right\rangle = \left\langle \frac{\partial X_\mu}{\partial x_a} \right\rangle \left\langle \frac{\partial X_\nu}{\partial x_a} \right\rangle + \frac{\partial^2}{\partial x_a^2} \langle \delta X_\mu \delta X_\nu \rangle = g_{\mu\nu}^{(1)}(X) + \delta g_{\mu\nu}^{(2)}(X)$$

(9)

The 2nd order moment quantum fluctuation

$$\delta g_{\mu\nu}^{(2)}(X) = \frac{R_{\mu\nu}^{(1)}(X)}{32\pi^2\lambda} \delta k^{d-2}$$

(10)

derms the metric, where $k^{d-2}$ is the cutoff energy scale.

**Appendix II: Ricci Flow**

The deformation of the metric at the gaussian approximation is driven by the 2nd moment quantum fluctuation and hence the Ricci curvature, which gives rise to the Ricci flow equation

$$\frac{\partial g_{\mu\nu}}{\partial t} = -2R_{\mu\nu},$$

(11)

where the flow-time interval

$$\delta t = \frac{1}{64\pi^2\lambda} \delta k^{d-2}$$

(12)

has the dimension of distance squared $[L^2]$ for any dimension of the base space $d$, in our relativistic quantum frame fields setting, we have taken $d = 4 - \epsilon$. 

The Ricci flow was initially introduced in 1980s by Friedan in $d = 2 + \epsilon$ NLSM and independently by Hamilton in mathematics. The main motivation of the Ricci flow from the mathematical point of view is to classify manifolds, a specific goals is to proof the Poincare conjecture. Hamilton used it as a useful tool to gradually deform a manifolds into a more and more "simple and good" manifolds whose topology can be readily recognized for some simple cases. A general realization of the program is achieved by Perelman at around 2003, who introduced several monotonic functionals (the F-functional and the W-functional) to prove the local non-collapsing theorem for finite flow-time local singularities which may be developed in general initial manifolds. By using the local non-collapsing theorem, he ruled out the cigar-type of singularity during the Ricci flow, which removes a main stumbling block of Hamilton’s program to the Poincare conjecture. Due to their seminal works, in $D = 3$ dimensions on compact manifolds the Ricci flow is by now well understood, and many basic technics and theorems can be generalized to $D = 4$ Riemannian manifolds and Lorentzian manifolds (spacetime).

For the Ricci curvature is non-linear for the metric, the Ricci flow equation is a non-linear version of a heat equation for the metric, and flow along $t$ introduces an averaging or coarse-graining process to the intrinsic non-linear quantum spacetime which is highly non-trivial. In general, if the flow is free from local singularities there exists long flow-time solution in $t \in (-\infty, 0)$, which is often called ancient solution in mathematical literature. This range of the $t$-parameter corresponds to $k \in (0, \infty)$, that is from $t = -\infty$, i.e. the short distance (high energy) UV scale $k = \infty$ forwardly to $t = 0$ i.e. the long distance (low energy) IR scale $k = 0$. The metric at certain scale $t$ is given by being averaged out the shorter distance details which produces an effective correction to the metric at that scale. So along $t$, the manifolds loss its information in shorter distance, thus the flow is irreversible, i.e. generally having no backwards solution, which is the underlying reason for the entropy of a spacetime.

As it is shown in (10), the 2nd order moment fluctuation modifies the local (quadratic) distance of the spacetime, so the flow is non-isometry. The non-isometry is not important for its topology, so along $t$, the flow preserves the topology of the spacetime but its local metric, shape and size (volume) changes. There also exists a very special solution of the Ricci flow called Ricci soliton, which only changes the local volume while keeps its local shape. The Ricci soliton, and its generalized version, the Gradient Ricci Soliton, as the flow limits, are the generalization of the notion of fixed point in the sense of RG flow.

Although the Ricci flow is not strongly parabolic, DeTurck provide a simple way to prove the short-flow-time existence of the Ricci flow by slightly modified it, named Ricci-DeTurck flow

$$\frac{\partial g_{\mu\nu}}{\partial t} = -2 \left( R_{\mu\nu} + \nabla_\mu \nabla_\nu f \right)$$

which is equivalent to the standard Ricci flow equation up to a diffeomorphism given by $u = e^{-f}$. From the geometric point of view, $u = e^{-f}$ introduces a positive defined density bundle over the Riemannian geometric, so it is also called a density of manifolds. From the physics point of view $u$ is referred to as a conformal factor up to a constant multiple which compensates the flow of the volume (i.e. the longitudinal degrees of freedom of gravity). From the statistic physics point of view, $u$ can also be interpreted as a positive defined density matrix of the frame fields system described by NLSM. Thus $u$ is usually normalized by the condition

$$\lambda \int u d^D X = 1. \quad (14)$$

The condition, together with the Ricci-DeTurck flow, gives the flow equation of $u$,

$$\frac{\partial u}{\partial t} = (\Delta + R) u. \quad (15)$$

Note the minus sign in front of the Laplacian, it is a backwards heat-like equation. Naively speaking, the solution of the backwards heat flow will not exist. But we could also note that if one let the Ricci flow flows to certain IR scale $t_*$, and at $t_*$ one may then choose an appropriate $u(t_*) = u_0$ arbitrarily (up to a diffeomorphism gauge) and flows it backwards in $\tau = t_* - t$ to obtain a solution $u(\tau)$ of the backwards equation. In this case, $u$ satisfies the heat-like equation $\frac{\partial u}{\partial \tau} = (\Delta - R) u$, which does admit a solution along $\tau$, often called the conjugate heat equation in mathematical literature.

Appendix III: Anomaly Induced Action of QSRF

The quantum fluctuation and hence the Ricci flow does not preserve the quadratic distance of a Riemannian geometry. The non-isometry of the quantum fluctuation induces a breakdown of diffeomorphism or general coordinate...
transformation at the quantum level, namely the anomaly. More precisely, here we consider functional quantization of the pure frame fields, the partition function is

$$Z(M^D) = \int [\mathcal{D}X] \exp \left(-S[X]\right) = \int [\mathcal{D}X] \exp \left(-\frac{1}{2} \lambda \int d^4x g^{\mu\nu} \partial_\mu X_\mu \partial_\nu X_\nu\right),$$  \hspace{1cm} (16)

where $M^D$ is the target space of the NLSM, and the partition function is real independent to whether the base space is Euclidean or Minkowskian. Note that a general coordinate transformation

$$X_\mu \rightarrow \hat{X}_\mu = \frac{\partial \hat{X}_\mu}{\partial X_\nu} X_\nu = e^\nu_\mu X_\nu$$  \hspace{1cm} (17)
does not change the action $S[X] = S[\hat{X}]$, but the measure of the functional integral changes

$$\mathcal{D}\hat{X} = \prod_x \prod_{\mu=0}^D d\hat{X}_\mu(x) = \prod_x \epsilon_{\mu\nu\rho\sigma} \epsilon_{\mu}^0 \epsilon_{\nu}^1 \epsilon_{\rho}^2 \epsilon_{\sigma}^3 dX_\mu(x)dX_1(x)dX_2(x)dX_3(x)$$

= $$\prod_x \left| \det e(x) \right| \prod_x \prod_{a=0}^D dX_a(x) = \left( \prod_x \left| \det e(x) \right| \right) \mathcal{D}X,$$  \hspace{1cm} (18)

where

$$\epsilon_{\mu\nu\rho\sigma} \epsilon_{\mu}^0 \epsilon_{\nu}^1 \epsilon_{\rho}^2 \epsilon_{\sigma}^3 = \left| \det e_\mu^a \right| = \sqrt{\left| \det g_{\mu\nu} \right|}$$  \hspace{1cm} (19)
is the Jacobian of the diffeomorphism. The Jacobian is nothing but a local relative volume element $d\mu(\hat{X}_\mu)$ w.r.t. the fiducial volume $d\mu(X_\mu)$, and hence it is the inverse of the frame fields density matrix

$$u(\hat{X}) = \frac{d\mu(X_\mu)}{d\mu(\hat{X}_\mu)} = \frac{\left| \det e_\mu^a \right|}{\left| \det e_\mu^\nu \right|}.$$  \hspace{1cm} (20)

Here the absolute value symbol of the determinant is because the density $u$ and the volume element is always positive defined even in the pseudo-Riemannian spacetime with Lorentz signature.

If we parameterize a dimensionless solution $u$ of the conjugate heat equation as

$$u(\hat{X}) = \frac{1}{\lambda (4\pi T)^{D/2}} e^{-f(\hat{X})},$$  \hspace{1cm} (21)

then the partition function $Z(M^D)$ is transformed to

$$Z(\hat{M}^D) = \int [\mathcal{D}\hat{X}] \exp \left(-S[\hat{X}]\right) = \int \left( \prod_x \left| \det e \right| \right) [\mathcal{D}X] \exp \left(-S[X]\right)$$

= $$\int \left( \prod_x e^{f + \frac{D}{2} \log(4\pi T)} \right) [\mathcal{D}X] \exp \left(-S[X]\right)$$

= $$\int [\mathcal{D}X] \exp \left\{ -S[X] + \lambda \int d^4x \left[ f + \frac{D}{2} \log(4\pi T) \right] \right\}$$

= $$\int [\mathcal{D}X] \exp \left\{ -S[X] + \lambda \int_{M^D} d^D X u \left[ f + \frac{D}{2} \log(4\pi T) \right] \right\}.$$  \hspace{1cm} (22)

Note that the change of the partition function

$$Z(\hat{M}^D) = e^{\lambda N(\hat{M}^D)} Z(M^D)$$  \hspace{1cm} (23)
is nothing but a pure real Shannon entropy in terms of the density $u$

$$N(\hat{M}^D) = \int_{\hat{M}^D} d^D X u \left[ f + \frac{D}{2} \log(4\pi T) \right] = - \int_{\hat{M}^D} d^D X u \log u.$$  \hspace{1cm} (24)
The classical action $S[X]$ is invariant under the general coordinates transformation or diffeomorphism, but the quantum partition function is no longer invariant under the general coordinates transformation or diffeomorphism, which is called diffeomorphism anomaly, meaning a breaking down of the diffeomorphism at the quantum level. The diffeomorphism anomaly is purely due to the quantum fluctuation and Ricci flow of the frame fields which do not preserve the functional integral measure and change the spacetime volume at the quantum level. The diffeomorphism anomaly has many profound consequences to the theory of quantum reference frame, e.g. non-unitarity, the trace anomaly, the notion of entropy, reversibility, and the cosmological constant.

Without loss of generality, if we simply consider the under-transformed coordinates $X_\mu$ as the coordinates of the fiducial lab $x_\mu$ which can be treated as a classical parameter coordinates, in this situation the classical action of NLSM is just a topological invariant, i.e. half the dimension of the target spacetime

$$\exp (-S_{cl}) = \exp \left( -\frac{1}{2} \lambda \int d^4x g^{\mu\nu} \partial_\mu x_\alpha \partial_\nu x_\alpha \right) = \exp \left( -\frac{1}{2} \lambda \int d^4x g^{\mu\nu} g_{\mu\nu} \right) = e^{-\frac{D}{2}}. \tag{25}$$

Thus the total partition function of the frame fields takes a simple form

$$Z(M^D) = e^\lambda N(M^D) - \frac{D}{2}. \tag{26}$$

A relative density

$$u_r(X) = \frac{u}{u_*} \tag{27}$$

can be defined by a density $u(X)$ being relative to

$$u_*(X) = \frac{1}{\lambda (4\pi \tau)^{D/2}} \exp \left( -\frac{1}{4\tau} g_{\mu\nu} X^\mu X^\nu \right) \tag{28}$$

which corresponds to the maximum Shannon entropy $N_*$

$$N_* = - \int d^D X u_* \log u_* = \int d^D X u \frac{D}{2} \left[ 1 + \log(4\pi \tau) \right] = \frac{D}{2\lambda} \left[ 1 + \log(4\pi \tau) \right], \tag{29}$$

when the spacetime becomes a gradient shrinking Ricci soliton (GSRS) satisfying

$$R_{\mu\nu} + \nabla_\mu \nabla_\nu f = \frac{1}{2\tau} g_{\mu\nu}. \tag{30}$$

By using the relative density, a relative Shannon entropy $\tilde{N}$ can be defined by

$$\tilde{N}(M^D) = - \int d^D X u \log u_r = - \int d^D X u \log u + \int d^D X u_\ast \log u_\ast = N - N_* = - \log Z_P \leq 0, \tag{31}$$

where $Z_P$ is nothing but the Perelman’s partition function

$$\log Z_P = \int_{M^D} d^D X u \left( \frac{D}{2} - f \right) \geq 0, \tag{32}$$

In terms of the relative Shannon entropy, the total partition function of the frame fields is normalized by the extreme value

$$Z(M^D) = \frac{e^{\lambda N} - \frac{D}{2}}{e^{\lambda N_*}} = e^{\lambda \tilde{N} - \frac{D}{2}}. \tag{33}$$

### Appendix IV: Effective Theory of Gravity

The relative Shannon entropy $\tilde{N}$ as the anomaly vanishes at gradient shrinking Ricci soliton (GSRS) or IR scale, however, it is non-zero at ordinary lab scale up to UV where the fiducial volume of the lab is considered rigid and fixed $\lambda \int d^4x = 1$. The cancellation of the anomaly at the lab scale up to UV is physically required, which leads to the counter term $\nu(M^D_{\tau=\infty})$ or cosmological constant. The monotonicity of $\tilde{N}$ and the $W$-functional implies

$$\nu(M^D_{\tau=\infty}) = \lim_{\tau \to \infty} \lambda \tilde{N}(M^D, u, \tau) = \lim_{\tau \to \infty} \lambda W(M^D, u, \tau) = \inf_\tau \lambda W(M^D, u, \tau) < 0, \tag{34}$$
where $\mathcal{W}$, the Perelman’s W-functional, is the Legendre transformation of $\tilde{N}$ w.r.t. $\tau^{-1}$.

$$\mathcal{W} \equiv \tau \frac{\partial \tilde{N}}{\partial \tau} + \tilde{N} = \tau \tilde{F} + \tilde{N} = \frac{d}{d\tau} \left( \tau \tilde{N} \right).$$

(35)

In other words, the difference between the effective actions (relative Shannon entropies) at UV and IR is finite

$$\nu = \lambda (\tilde{N}_{UV} - \tilde{N}_{IR}) < 0.$$  

(36)

In fact $\epsilon^\nu < 1$ (usually called the Gaussian density) is a relative volume or the reduced volume $\tilde{V}(M^D)$ of the backwards limit manifolds introduced by Perelman, or the inverse of the initial condition of the manifolds density $u^{-1}_{\tau=0}$. A finite value of it makes an initial spacetime with unit volume from UV flow and converge to a finite $u_{\tau=0}$, and hence the manifolds finally converges to a finite relative volume/reduced volume instead of shrinking to a singular point at $\tau = 0$.

As an example, for a homogeneous and isotropic universe for which the sizes of space and time (with a “ball” radius $a_\tau$) are on an equal footing, i.e. $ds^2 = a_\tau^2 (-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2)$, it can be approximately given by a compact 4-ball $\nu(B^4) \approx -0.8$.

So the partition function, which is anomaly canceled at UV and having a fixed-volume fiducial lab, is

$$Z(M^D) = e^{\lambda \tilde{N} - \frac{D^2}{2} - \nu}.  

(37)$$

Since $\lim_{\tau \to 0} \tilde{N}(M^D) = 0$, so at small $\tau$, $\tilde{N}(M^D)$ can be expanded by powers of $\tau$

$$\tilde{N}(M^D) = \frac{\partial \tilde{N}}{\partial \tau} \tau + O(\tau^2) = \tau \tilde{F} + O(\tau^2)$$

$$= \int_{M^D} d^D X u \left[ \left( R_{\tau=0} + |\nabla f_{\tau=0}|^2 - \frac{D}{2\tau} \right) \right] + O(\tau^2)$$

$$= \int_{M^D} d^D X u R_{0} \tau + O(\tau^2),$$

(38)

in which $\lambda \int d^D X u(r) \sqrt{|\nabla f_{\tau=0}|^2} \approx \frac{D^2}{2}$ (at GSRS) has been used.

For $D = 4$ and small $\tau$, the effective action of $Z(M^4)$ can be given by

$$- \log Z(M^4) = S_{eff} \approx \int_{M^4} d^4 X u (2\lambda - \lambda R_{0} \tau + \lambda \nu)$$

(small $\tau$)

(39)

or by taking $u = \sqrt{|g|}$ and $\tau = -t = \frac{1}{64\pi^2} k^2$, we have

$$S_{eff} = \int_{M^4} d^4 X \sqrt{|g|} \left( 2\lambda - \frac{R_0}{64\pi^2} k^2 + \lambda \nu \right)$$

(small $k$).

(40)

The effective action can be interpreted as a low energy effective action of pure gravity. As the cutoff scale $k$ ranges from the lab scale to the solar system scale ($k > 0$), the action must recover the well-tested Einstein-Hilbert (EH) action. But at the cosmic scale ($k \to 0$), we know that the EH action deviates from observations and the cosmological constant becomes important. In this picture, as $k \to 0$, the action leaving $2\lambda + \lambda \nu$ should play the role of the standard EH action with a limit constant background scalar curvature $R_0$ plus the cosmological constant, so

$$2\lambda + \lambda \nu = \frac{R_0 - 2\Lambda}{16\pi G}.  

(41)$$

While at $k \to \infty$, $\lambda \tilde{N} \to \nu$, the action leaving only the fiducial Lagrangian $\frac{\partial^2}{\partial \tau^2} \lambda = 2\lambda$ which should be interpreted as a constant EH action without the cosmological constant

$$2\lambda = \frac{R_0}{16\pi G}.  

(42)$$

Thus we have the cosmological term

$$\lambda \nu = \frac{-2\lambda}{16\pi G} = -\rho_\Lambda. 

(43)$$
The action can be rewritten as an effective EH action plus a cosmological term

\[ S_{\text{ef f}} = \int_{M^4} d^4x \sqrt{|g|} \left( \frac{R_k}{16\pi G} + \lambda \nu \right) \quad \text{(small k),} \tag{44} \]

where

\[ \frac{R_k}{16\pi G} = 2\lambda - \frac{R_0}{64\pi^2 k^2}, \tag{45} \]

which is nothing but the flow equation of the scalar curvature

\[ R_k = \frac{R_0}{1 + \frac{1}{2\pi G k^2}}, \quad \text{or} \quad R_\tau = \frac{R_0}{1 + \frac{2}{9} R_0 \tau}. \tag{46} \]

Since at the cosmic scale \( k \to 0 \), the effective scalar curvature is bounded by \( R_0 \) which can be measured by the “Hubble’s constant” \( H_0 \) at the cosmic scale,

\[ R_0 = D(D-1)H_0^2 = 12H_0^2, \tag{47} \]

so \( \lambda \) is nothing but the critical density of the 4-spacetime Universe

\[ \lambda = \frac{3H_0^2}{8\pi G} = \rho_c, \tag{48} \]

so the cosmological constant is always of order of the critical density with a “dark energy” fraction

\[ \Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c} = -\nu \approx 0.8, \tag{49} \]

which is not far from observations. The detail discussions about the cosmological constant problem and the observational effect in the cosmology, especially the modification of the Distance-Redshift relation leading to the acceleration parameter \( q_0 \approx -0.68 \) can be found.

If we include matter into the gravity theory, consider the entangled system in \( \mathcal{H}_\psi \otimes \mathcal{H}_X \) between the under-studied quantum body and the quantum reference frame fields system. Without loss of generality, we could take a scalar field \( \psi \) as the under-studied system, which shares the base space with the frame fields, the total action of the two entangled system is a direct sum of each system

\[ S[\psi, X] = \int d^4x \left[ \frac{1}{2} \frac{\partial \psi}{\partial x^a} \frac{\partial \psi}{\partial x_a} - V(\psi) + \frac{1}{2} \lambda g_{\mu\nu} \frac{\partial X^\mu}{\partial x_a} \frac{\partial X^\nu}{\partial x_a} \right], \tag{50} \]

where \( V(\psi) \) is some potential of the \( \psi \) fields. Since both \( \psi \) field and the frame fields \( X \) share the same base space (fiducial lab), here they are formulated on the usual lab spacetime \( x \). If we interpret the frame fields as the physical spacetime coordinates, the coordinates or reference frames of \( \psi \) field must be transformed from \( x \) to \( X \). At the semi-classical level, or 1st moment approximation when the fluctuation of \( X \) can be ignored, it is simply a coordinates transformation

\[
\begin{align*}
S[\psi, X] \overset{\text{(1)}}{=}& S[\psi(X)] = \int d^4X \sqrt{|\det g^{(1)}|} \left\{ \frac{1}{4} \left( g^{(1)\mu\nu} \frac{\partial X^\mu}{\partial x_a} \frac{\partial X^\nu}{\partial x_a} \right) \left( \frac{1}{2} g^{(1)\mu\nu} \frac{\delta \psi}{\delta X^\mu} \frac{\delta \psi}{\delta X^\nu} + 2\lambda \right) - V(\psi) \right\} \\
=& \int d^4X \sqrt{|\det g^{(1)}|} \left\{ \frac{1}{4} g^{(1)\mu\nu} \frac{\delta \psi}{\delta X^\mu} \frac{\delta \psi}{\delta X^\nu} - V(\psi) + 2\lambda \right\},
\end{align*}
\]

in which \( \overset{\text{(1)}}{=} \) stands for the 1st moment approximation, and \( \frac{1}{4} \left( g^{(1)\mu\nu} \frac{\partial X^\mu}{\partial x_a} \frac{\partial X^\nu}{\partial x_a} \right) \) has been used. It is easy to see, at the semi-classical level, i.e., only consider the 1st moment of \( X \) while 2nd moment fluctuations are ignored, the (classical) coordinates transformation reproduces the scalar field action in general coordinates \( X \) up to a constant \( 2\lambda \), and the derivative \( \frac{\partial }{\partial x_a} \) is replaced by the functional derivative \( \frac{\delta }{\delta X^a} \). \( \sqrt{|\det g^{(1)}|} \) is the Jacobian determinant of the coordinate transformation, note that the determinant requires the coordinates transformation matrix a square matrix, so at semi-classical level \( d \) must be very close to \( D = 4 \), which is not necessarily true beyond the semi-classical level, when the 2nd moment quantum fluctuations are important. For instance, since \( d \) is a parameter but an observable in the theory, it could even not necessary be an integer but effectively fractal at the quantum level, and we have chosen \( d = 4 - \epsilon \).
When the gravity and quantum spacetime frame fields are normalized by the Ricci flow, the term in the Ricci flow 2λ term in eq. (61) is normalized by 2nd moment fluctuation, by using eq. (40) and eq. (45), a matter-coupled-gravity is emerged from the Ricci flow

$$S[\psi, X] \approx \int d^4X \sqrt{\left| \det g \right|} \left[ \frac{1}{2} g^{\mu \nu} \frac{\delta \psi}{\delta X^\mu} \frac{\delta \psi}{\delta X^\nu} - V(\psi) + 2\lambda - \frac{R_0}{64\pi^2} k^2 + \lambda \nu \right]$$

$$= \int d^4X \sqrt{\left| \det g \right|} \left[ \frac{1}{2} g^{\mu \nu} \frac{\delta \psi}{\delta X^\mu} \frac{\delta \psi}{\delta X^\nu} - V(\psi) + \frac{R_t}{16\pi G} + \lambda \nu \right]$$

(52)

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