ON A CLASS OF LIFTING MODULES

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Abstract. In this paper, we introduce principally $\delta$-lifting modules which are analogous to $\delta$-lifting modules and principally $\delta$-semiperfect modules as a generalization of $\delta$-semiperfect modules and investigate their properties.

1. Introduction

Throughout this paper all rings have an identity, all modules considered are unital right modules. Let $M$ be a module and $N, P$ be submodules of $M$. We call $P$ a supplement of $N$ in $M$ if $M = P + N$ and $P \cap N$ is small in $P$. A module $M$ is called supplemented if every submodule of $M$ has a supplement in $M$. A module $M$ is called lifting if, for all $N \leq M$, there exists a decomposition $M = A \oplus B$ such that $A \leq N$ and $N \cap B$ is small in $M$. Supplemented and lifting modules have been discussed by several authors (see [2, 4, 6]) and these modules are useful in characterizing semiperfect and right perfect rings (see [4, 7]).

In this note, we study and investigate principally $\delta$-lifting modules and principally $\delta$-semiperfect modules. A module $M$ is called principally $\delta$-lifting if for each cyclic submodule has the $\delta$-lifting property, i.e., for each $m \in M$, $M$ has a decomposition $M = A \oplus B$ with $A \leq mR$ and $mR \cap B$ is $\delta$-small in $B$, where $B$ is called a $\delta$-supplement of $mR$. A module $M$ is called principally $\delta$-semiperfect if, for each $m \in M$, $M/mR$ has a projective $\delta$-cover. We prove that if $M_1$ is semisimple, $M_2$ is principally $\delta$-lifting, $M_1$ and $M_2$ are relatively projective, then $M = M_1 \oplus M_2$ is a principally $\delta$-lifting module. Among others we also prove that for a principally $\delta$-semiperfect module $M$, $M$ is principally $\delta$-supplemented, each factor module of $M$ is principally $\delta$-semiperfect, hence any homomorphic image and any direct summand of $M$ is principally $\delta$-semiperfect. As an application, for a projective module $M$, it is shown that $M$ is principally $\delta$-semiperfect if and only if it is principally $\delta$-lifting, and therefore a ring $R$ is principally $\delta$-semiperfect if and only if it is principally $\delta$-lifting.

In section 2, we give some properties of $\delta$-small submodules that we use in the paper, and in section 3, principally $\delta$-lifting modules are introduced and various properties of principally $\delta$-lifting and $\delta$-supplemented modules are obtained. In section 4, principally $\delta$-semiperfect modules are defined and characterized in terms of principally $\delta$-lifting modules.

In what follows, by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_n$ and $\mathbb{Z}/\mathbb{Z}_n$ we denote, respectively, integers, rational numbers, the ring of integers and the $\mathbb{Z}$-module of integers modulo $n$. For unexplained concepts and notations, we refer the reader to [1, 4].

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2. $\delta$-Small Submodules

Following Zhou [9], a submodule $N$ of a module $M$ is called a $\delta$-small submodule if, whenever $M = N + X$ with $M/X$ singular, we have $M = X$. We begin by stating the next lemma which is contained in [9] Lemma 1.2 and 1.3].

**Lemma 2.1.** Let $M$ be a module. Then we have the following.

(1) If $N$ is $\delta$-small in $M$ and $M = X + N$, then $M = X \oplus Y$ for a projective semisimple submodule $Y$ with $Y \subseteq N$.

(2) If $K$ is $\delta$-small in $M$ and $f : M \to N$ is a homomorphism, then $f(K)$ is $\delta$-small in $N$. In particular, if $K$ is $\delta$-small in $M \subseteq N$, then $K$ is $\delta$-small in $N$.

(3) Let $K_1 \subseteq M_1 \subseteq M$, $K_2 \subseteq M_2 \subseteq M$ and $M = M_1 \oplus M_2$. Then $K_1 \oplus K_2$ is $\delta$-small in $M_1 \oplus M_2$ if and only if $K_1$ is $\delta$-small in $M_1$ and $K_2$ is $\delta$-small in $M_2$.

(4) Let $N, K$ be submodules of $M$ with $K$ is $\delta$-small in $M$ and $N \leq K$. Then $N$ is also $\delta$-small in $M$.

**Lemma 2.2.** Let $M$ be a module and $m \in M$. Then the following are equivalent.

(1) $mR$ is not $\delta$-small in $M$.

(2) There is a maximal submodule $N$ of $M$ such that $m \notin N$ and $M/N$ singular.

**Proof.** (1) $\Rightarrow$ (2) Let $\Gamma := \{B \leq M \mid B \neq M, mR + B = M, M/B$ singular$\}$. Since $mR$ is not $\delta$-small in $M$, there exists a proper submodule $B$ of $M$ such that $mR + B = M$ and $M/B$ singular. So $\Gamma$ is non empty. Let $A$ be a nonempty totally ordered subset of $\Gamma$ and $B_0 := \cup_{B \in A} B$. If $m$ is in $B_0$ then there is a $B \in A$ with $m \in B$. Then $B = mR + B = M$ which is a contraction. So we have $m \notin B_0$ and $B_0 \neq M$. Since $mR + B_0 = M$ and $M/B_0$ singular, $B_0$ is upper bound in $\Gamma$. By Zorn’s Lemma, $\Gamma$ has a maximal element, say $N$. If $N$ is a maximal submodule of $M$ there is nothing to do. Assume that there exists a submodule $K$ containing $N$ properly. Since $N$ is maximal in $\Gamma$, $K$ is not in $\Gamma$. Since $M = mR + N$ and $N \leq K$, so $M = mR + K$. $M/K$ is singular as a homomorphic image of singular module $M/N$. Hence $K$ must belong to the $\Gamma$. This is the required contradiction.

(2) $\Rightarrow$ (1) Let $N$ be a maximal submodule with $m \in M \setminus N$ and $M/N$ singular. We have $M = mR + N$. Then $mR$ is not $\delta$-small in $M$. $\square$

Let $A$ and $B$ be submodules of $M$ with $A \leq B$. $A$ is called a $\delta$-cosmall submodule of $B$ in $M$ if $B/A$ is $\delta$-small in $M/A$. Let $A$ be a submodule of $M$. $A$ is called a $\delta$-coclosed submodule in $M$ if $A$ has no proper $\delta$-cosmall submodules in $M$. A submodule $A$ is called $\delta$-coclosure of $B$ in $M$ if $A$ is $\delta$-coclosed submodule of $M$ and it is $\delta$-cosmall submodule of $B$. Equivalently, for any submodule $C \leq A$ with $A/C$ is $\delta$-small in $M/C$ implies $C = A$ and $B/A$ is $\delta$-small in $M/A$. Note that $\delta$-coclosed submodules need not always exist.
Lemma 2.3. Let $A$ and $B$ be submodules of $M$ with $A \subseteq B$. Then we have:

1. $A$ is $\delta$-cosmall submodule of $B$ in $M$ if and only if $M = A + L$ for any submodule $L$ of $M$ with $M = B + L$ and $M/L$ singular.

2. If $A$ is $\delta$-small and $B$ is $\delta$-coclosed in $M$, then $A$ is $\delta$-small in $B$.

Proof. (1) Necessity: Let $M = B + L$ and $M/L$ be singular. We have $M/A = B/A + (L + A)/A$ and $M/(L + A)$ is singular as homomorphic image of singular module $M/L$. Since $B/A$ is $\delta$-small, $M/A = (L + A)/A$ or $M = L + A$.

Sufficiency: Let $M/A = B/A + K/A$ and $M/K$ singular. Then $M = B + K$. By hypothesis, $M = A + K$ and so $M = K$. Hence $A$ is $\delta$-cosmall submodule of $B$ in $M$.

(2) Assume that $A$ is $\delta$-small submodule of $M$ and $B$ is $\delta$-coclosed in $M$. Let $B = A + K$ with $B/K$ singular. Since $B$ is $\delta$-coclosed in $M$, to complete the proof, by part (1) it suffices to show that $K$ is $\delta$-small submodule of $B$ in $M$. Let $M = B + L$ with $M/L$ singular. By assumption, $M = A + K + L = K + L$ since $M/(K + L)$ is singular. By (1), $K$ is $\delta$-small submodule of $B$ in $M$.

Lemma 2.4. Let $A$, $B$ and $C$ be submodules of $M$ with $M = A + C$ and $A \subseteq B$. If $B \cap C$ is a $\delta$-small submodule of $M$, then $A$ is a $\delta$-cosmall submodule of $B$ in $M$.

Proof. Let $M/A = B/A + L/A$ with $M/L$ singular. We have $M = B + L$ and $B = A + (B \cap C)$. Then $M = A + (B \cap C) + L = (B \cap C) + L$. Hence $M = L$ since $B \cap K$ is $\delta$-small in $M$ and $M/L$ is singular. Hence $B/A$ is $\delta$-small in $M/A$. Thus $A$ is $\delta$-cosmall submodule of $B$ in $M$.

3. Principally $\delta$-Lifting Modules

In this section, we study and investigate some properties of principally $\delta$-lifting modules. The following definition is motivated by [9, Lemma 3.4] and Lemma 3.4.

Definition 3.1. A module $M$ is called finitely $\delta$-lifting if for any finitely generated submodule $A$ of $M$ has the $\delta$-lifting property, that is, there is a decomposition $M = N \oplus S$ with $N \leq A$ and $A \cap S$ is $\delta$-small in $S$. In this case $A \cap S$ is $\delta$-small in $S$ if and only if $A \cap S$ is $\delta$-small in $M$. A module $M$ is called principally $\delta$-lifting if for each cyclic submodule has the principally $\delta$-lifting property, i.e., for each $m \in M$, $M$ has a decomposition $M = A \oplus B$ with $A \leq mR$ and $mR \cap B$ is $\delta$-small in $B$.

Example 3.2. Every submodule of any semisimple module satisfies principally $\delta$-lifting property.

Example 3.3. Let $p$ be a prime integer and $n$ any positive integer. Then the $\mathbb{Z}$-module $M = \mathbb{Z}/Zp^n$ is a principally $\delta$-lifting module.
Lemma 3.4 is proved in [7] and [9].

**Lemma 3.4.** The following are equivalent for a module $M$.

1. $M$ is finitely $\delta$-lifting.
2. $M$ is principally $\delta$-lifting.

Let $M$ be a module and $N$ a submodule of $M$. A submodule $L$ is called a $\delta$-supplement of $N$ in $M$ if $M = N + L$ and $N \cap L$ is $\delta$-small in $L$ (therefore in $M$).

**Proposition 3.5.** Let $M$ be a principally $\delta$-lifting module. The we have:

1. Every direct summand of $M$ is a principally $\delta$-lifting module.
2. Every cyclic submodule $C$ of $M$ has a $\delta$-supplement $S$ which is a direct summand, and $C$ contains a complementary summand of $S$ in $M$.

**Proof.** (1) Let $K$ be a direct summand of $M$ and $k \in K$. Then $M$ has a decomposition $M = N \oplus S$ with $N \leq kR$ and $kR \cap S$ is $\delta$-small in $M$. It follows that $K = N \oplus (K \cap S)$, and $kR \cap (K \cap S) \leq kR \cap S$ is $\delta$-small in $M$ and so $kR \cap (K \cap S)$ is $\delta$-small in $K$. Therefore $K$ is a principally $\delta$-lifting module.

(2) Assume that $M$ is a principally $\delta$-lifting module and $C$ is a cyclic submodule of $M$. Then we have $M = N \oplus S$, where $N \leq C$ and $C \cap S$ is $\delta$-small in $M$. Hence $M = N + S \leq C + S \leq M$, we have $M = C + S$. Since $S$ is direct summand and $C \cap S$ is $\delta$-small in $M$, $C \cap S$ is $\delta$-small in $S$. Therefore $S$ is a $\delta$-supplement of $C$ in $M$. $\square$

**Theorem 3.6.** The following are equivalent for a module $M$.

1. $M$ is a principally $\delta$-lifting module.
2. Every cyclic submodule $C$ of $M$ can be written as $C = N \oplus S$, where $N$ is direct summand and $S$ is $\delta$-small in $M$.
3. For every cyclic submodule $C$ of $M$, there is a direct summand $A$ of $M$ with $A \leq C$ and $C/A$ is $\delta$-small in $M/A$.
4. Every cyclic submodule $C$ of $M$ has a $\delta$-supplement $K$ in $M$ such that $C \cap K$ is a direct summand in $C$.
5. For every cyclic submodule $C$ of $M$, there is an idempotent $e \in \text{End}(M)$ with $eM \leq C$ and $(1 - e)C$ is $\delta$-small in $(1 - e)M$.
6. For each $m \in M$, there exist ideals $I$ and $J$ of $R$ such that $mR = mI \oplus mJ$, where $mI$ is direct summand of $M$ and $mJ$ is $\delta$-small in $M$.

**Proof.** (1)$\Rightarrow$(2) Let $C$ be a cyclic submodule of $M$. By hypothesis there exist $N$ and $S$ submodules of $M$ such that $N \leq C$, $C \cap S$ is $\delta$-small in $M$ and $M = N \oplus S$. Then we have $C = N \oplus (C \cap S)$. 
(2) ⇒ (3) Let $C$ be a cyclic submodule of $M$. By hypothesis, $C = N \oplus S$, where $N$ is direct summand and $S$ is $\delta$-small in $M$. Let $\pi : M \to M/N$ be the natural projection. Since $S$ is $\delta$-small in $M$, we have $\pi(S)$ is $\delta$-small in $M/N$. Since $\pi(S) \cong S \cong C/N$, $C/N$ is $\delta$-small in $M/N$.

(3) ⇒ (4) Let $C$ be a cyclic submodule of $M$. By hypothesis, there is a direct summand $A \leq M$ with $A \leq C$ and $C/A$ is $\delta$-small in $M/A$. Let $M = A \oplus A'$. Hence $C = A \oplus (A' \cap C)$. Let $\sigma : M/A \to A'$ denote the obvious isomorphism. Then $\sigma(C/A) = A' \cap C$ is $\delta$-small in $A'$.

(4) ⇒ (5) Let $C$ be any cyclic submodule of $M$ and $K \leq M$ such that $C \cap K$ is a direct summand of $C$, $M = C + K$ and $C \cap K$ is $\delta$-small in $K$. Hence $C = (C \cap K) \oplus X$ for some $X \leq C$. Then $M = X + (C \cap K) + K = X \oplus K$. Let $e : M \to X : e(x + k) = x$ and $(1 - e) : M \to K ; e(x + k) = k$ are projection maps. $e(M) \subseteq X \subseteq C$ and $(1 - e)C = C \cap (1 - e)M = C \cap K$ is $\delta$-small in $(1 - e)M$.

(5) ⇒ (6) Let $mR$ be any cyclic submodule of $M$. By hypothesis, there exists an idempotent $e \in \text{End}(M)$ such that $eM \leq mR$, $M = eM \oplus (1 - e)M$ and $(1 - e)mR$ is $\delta$-small in $(1 - e)M$. Note that $(mR) \cap ((1 - e)M) = (1 - e)mR$ (for if $m = em_1 + y$, where $em_1 \in eM$, $y \in (mR) \cap ((1 - e)M)$. Then $(1 - e)m = em_1 + (1 - e)y = y$ and so $(1 - e)mR \leq (mR) \cap ((1 - e)M)$. Let $mr = (1 - e)m' \in (mR) \cap ((1 - e)M)$. Then $mr = (1 - e)m' \in (1 - e)mR$. So $(mR) \cap ((1 - e)M) \leq (1 - e)mR$. Thus $(mR) \cap ((1 - e)M) = (1 - e)mR$. So $mR = eM \oplus (1 - e)mR$. Let $I = \{ r \in R : mr \in eM \}$ and $J = \{ t \in R : mt \in (1 - e)mR \}$. Then $mR = mI \oplus mJ$, $mI = eM$ and $mJ = (1 - e)mR$ is $\delta$-small in $(1 - e)M$.

(6) ⇒ (1) Let $m \in M$. By hypothesis, there exist ideals $I$ and $J$ of $R$ such that $mR = mI \oplus mJ$, where $mI$ is direct summand and $mJ$ is $\delta$-small in $M$. Let $M = mI \oplus K$ for some submodule $K$. Since $K \cap mR \cong mJ$ and $mJ$ is $\delta$-small in $M$, $M$ is principally $\delta$-lifting. \qed

Note that every lifting module is principally $\delta$-lifting. There are principally $\delta$-lifting modules but not lifting.

**Example 3.7.** Let $M$ be the $\mathbb{Z}$-module $\mathbb{Q}$ and $m \in M$. It is well known that every cyclic submodule $mR$ of $M$ is small, therefore $\delta$-small in $M$. Hence $M$ is a principally $\delta$-lifting $\mathbb{Z}$-module. If $N$ is a nonsmall proper submodule of $M$, then $N$ is neither direct summand nor contains a direct summand of $M$. It follows that $M$ is not a lifting $\mathbb{Z}$-module.

It is clear that every $\delta$-lifting module is principally $\delta$-lifting. However the converse is not true.

**Example 3.8.** Let $R$ and $T$ denote the rings in [9] Example 4.1, where
\[ R = \sum_{i=1}^{\infty} \bigoplus \mathbb{Z}_2 + \mathbb{Z}_2.1 = \{(f_1, f_2, \ldots, f_n, f, \ldots) \in \prod_{i=1}^{\infty} \mathbb{Z}_2\} \]
and \[ T = \left\{ \begin{bmatrix} x & y \\ o & x \end{bmatrix} : x \in R, y \in \text{Soc}(R) \right\}. \]
Then \( \text{Rad}_3(T) = \begin{bmatrix} 0 & \text{Soc}(R) \\ 0 & 0 \end{bmatrix} \)
and \( T/\text{Rad}_3(T) \) is not semisimple as isomorphic to \( R \). So \( T \) is not \( \delta \)-semiperfect by \cite{9} Theorem 3.6. Hence \( T \) is not a \( \delta \)-lifting module over \( T \). It is easy to show that \( T/\text{Rad}_3(T) \) lift to idempotents of \( T \), so \( T \) is a semiregular ring. Since \( T \) is a \( \delta \)-semiregular ring, every finitely generated right ideal \( H \) of \( T \) can be written as \( H = aT \oplus S \), where \( a^2 = a \in T \) and \( S \leq \text{Rad}_3(T) \) by \cite{9} Theorem 3.5. Hence \( T \) is a principally \( \delta \)-lifting module.

**Proposition 3.9.** Let \( M \) be a principally \( \delta \)-lifting module. If \( M = M_1 + M_2 \) such that \( M_1 \cap M_2 \) is cyclic, then \( M_2 \) contains a \( \delta \)-supplement of \( M_1 \) in \( M \).

**Proof.** Assume that \( M = M_1 + M_2 \) and \( M_1 \cap M_2 \) is cyclic. Then we have \( M_1 \cap M_2 = N \oplus S \), where \( N \) is direct summand of \( M \) and \( S \) is \( \delta \)-small in \( M \). Let \( M = N \oplus N' \) and \( M_2 = N \oplus (M_2 \cap N') \). It follows that \( M_1 \cap M_2 = N \oplus (M_1 \cap M_2 \cap N') = N \oplus S \).

Let \( \pi : M_2 = N \oplus (M_2 \cap N') \rightarrow N' \) be the natural projection. It follows that \( \pi(M_1 \cap M_2 \cap N') = M_1 \cap M_2 \cap N' = \pi(S) \). Since \( S \) is \( \delta \)-small in \( M \), it is \( \delta \)-small in \( N' \) by Lemma 2.2. Hence \( M = M_1 + (M_2 \cap N'), M_2 \cap N' \leq M_2 \) and \( M_1 \cap (M_2 \cap N') \) is \( \delta \)-small in \( M_2 \cap N' \). \( M_2 \cap N' \) is contained in \( M_2 \) and a \( \delta \)-supplement of \( M_1 \) in \( M_2 \). This completes the proof. \( \square \)

Let \( M \) be a module. A submodule \( N \) is called **fully invariant** if for each endomorphism \( f \) of \( M \), \( f(N) \leq N \). Let \( S = \text{End}(M_R) \), the ring of \( R \)-endomorphisms of \( M \). Then \( M \) is a left \( S \)-, right \( R \)-bimodule and a principal submodule \( N \) of the right \( R \)-module \( M \) is fully invariant if and only if \( N \) is a sub-bimodule of \( M \). Clearly \( 0 \) and \( M \) are fully invariant submodules of \( M \). The right \( R \)-module \( M \) is called a **duo module** provided every submodule of \( M \) is fully invariant. For the readers’ convenience we state and prove Lemma 3.10 which is proved in \cite{5}.

**Lemma 3.10.** Let a module \( M = \bigoplus_{i \in I} M_i \) be a direct sum of submodules \( M_i \) (\( i \in I \)) and let \( N \) be a fully invariant submodule of \( M \). Then \( N = \bigoplus_{i \in I} (N \cap M_i) \).

**Proof.** For each \( j \in I \), let \( p_j : M \rightarrow M_j \) denote the canonical projection and let \( i_j : M_j \rightarrow M \) denote inclusion. Then \( i_j p_j \) is an endomorphism of \( M \) and hence \( i_j p_j(N) \subseteq N \) for each \( j \in I \). It follows that \( N \subseteq \bigoplus_{j \in I} i_j p_j(N) \subseteq \bigoplus_{j \in I} (N \cap M_j) \subseteq N \), so that \( N = \bigoplus_{j \in I} (N \cap M_j) \). \( \square \)

One may suspect that if \( M_1 \) and \( M_2 \) are principally \( \delta \)-lifting modules, then \( M_1 \oplus M_2 \) is also principally \( \delta \)-lifting. But this is not the case.
Example 3.11. Consider the \( \mathbb{Z} \)-modules \( M_1 = \mathbb{Z}/2\mathbb{Z} \) and \( M_2 = \mathbb{Z}/28\mathbb{Z} \). It is clear that \( M_1 \) and \( M_2 \) are principally \( \delta \)-lifting. Let \( M = M_1 \oplus M_2 \). Then \( M \) is not a principally \( \delta \)-lifting \( \mathbb{Z} \)-module. Let \( N_1 = (1,2)\mathbb{Z} \) and \( N_2 = (1,1)\mathbb{Z} \). Then \( M = N_1 + N_2 \), \( N_1 \) is not a direct summand of \( M \) and does not contain any nonzero direct summand of \( M \). For any proper submodule \( N \) of \( M \), \( M/N \) is singular \( \mathbb{Z} \)-module. Hence the principal submodule does not satisfy \( \delta \)-lifting property. It follows that \( M \) is not principally \( \delta \)-lifting \( \mathbb{Z} \)-module. By the same reasoning, for any prime integer \( p \), the \( \mathbb{Z} \)-module \( M = (\mathbb{Z}/\mathbb{Z}p) \oplus (\mathbb{Z}/\mathbb{Z}p^3) \) is not principally \( \delta \)-lifting.

We have already observed by the preceding example that the direct sum of principally \( \delta \)-lifting modules need not be principally \( \delta \)-lifting. Note the following fact.

Proposition 3.12. Let \( M = M_1 \oplus M_2 \) be a decomposition of \( M \) with \( M_1 \) and \( M_2 \) principally \( \delta \)-lifting modules. If \( M \) is a duo module, then \( M \) is principally \( \delta \)-lifting.

Proof. Let \( M = M_1 \oplus M_2 \) be a duo module and \( mR \) be a submodule of \( M \). By Lemma 3.10, \( mR = ((mR) \cap M_1) \oplus ((mR) \cap M_2) \). Since \((mR) \cap M_1 \) and \((mR) \cap M_2 \) are principal submodules of \( M_1 \) and \( M_2 \) respectively, there exist \( A_1, B_1 \leq M_1 \) such that \( A_1 \leq (mR) \cap M_1 \leq M_1 = A_1 \oplus B_1 \), \( B_1 \cap ((mR) \cap M_1) = B_1 \cap (mR) \) is \( \delta \)-small in \( B_1 \), and \( A_2, B_2 \leq M_2 \) such that \( A_2 \leq (mR) \cap M_2 \leq M_2 = A_2 \oplus B_2 \), \( B_2 \cap ((mR) \cap M_2) = B_2 \cap (mR) \) is \( \delta \)-small in \( B_2 \). Then \( M = A_1 \oplus A_2 \oplus B_1 \oplus B_2 \), \( A_1 \oplus A_2 \leq N \) and \((mR) \cap (B_1 \oplus B_2) = ((mR) \cap B_1) \oplus ((mR) \cap B_2) \) is \( \delta \)-small in \( M_1 \oplus M_2 \). \( \square \)

Lemma 3.13. The following are equivalent for a module \( M = M' \oplus M'' \).

1. \( M' \) is \( M'' \)-projective.
2. For each submodule \( N \) of \( M \) with \( M = N + M'' \), there exists a submodule \( N' \leq N \) such that \( M = N' \oplus M'' \).

Proof. See [7, 41.14] \( \square \)

Theorem 3.14. Let \( M_1 \) be a semisimple module and \( M_2 \) a principally \( \delta \)-lifting module. Assume that \( M_1 \) and \( M_2 \) are relatively projective. Then \( M = M_1 \oplus M_2 \) is principally \( \delta \)-lifting.

Proof. Let \( 0 \neq m \in M \) and let \( K = M_1 \cap ((mR) + M_2) \). We divide the proof into two cases:

Case (i): \( K \neq 0 \). Then \( M_1 = K \oplus K_1 \) for some submodule \( K_1 \) of \( M_1 \) and so \( M = K \oplus K_1 \oplus M_2 = (mR) + (M_2 \oplus K_1) \). Hence \( K \) is \( M_2 \oplus K_1 \)-projective. By Lemma 3.13 there exists a submodule \( N \) of \( mR \) such that \( M = N \oplus (M_2 \oplus \)
We may assume $(mR) \cap (M_2 \oplus K_1) \neq 0$. Note that for any submodule $L$ of $M_2$, we have $(mR) \cap (L + K_1) = L \cap ((mR) + K_1)$. In particular $(mR) \cap (M_2 + K_1) = M_2 \cap ((mR) + K_1)$. Then $mR = N \oplus (mR) \cap (K_1 \oplus M_2)$. There exist $n \in N$ and $m' \in (mR) \cap (K_1 \oplus M_2)$ such that $m = n + m'$. Then $nR = N$ and $m'R = (mR) \cap (K_1 \oplus M_2)$. Since $(mR) \cap (M_2 + K_1) = M_2 \cap ((mR) + K_1)$, $M_2 \cap ((mR) + K_1)$ is a direct submodule of $M_2$ and $M_2$ is principally $\delta$-lifting, there exists a submodule $X$ of $M_2 \cap ((mR) + K_1) = (mR) \cap (M_2 \oplus K_1)$ such that $M_2 = X \oplus Y$ and $Y \cap M_2 \cap ((mR) + K_1) = Y \cap ((mR) + K_1)$ is $\delta$-small in $M_2 \cap ((mR) + K_1)$ and in $M_2$. Hence $M = (N \oplus X) \oplus (Y \oplus K_1)$. Since $N \oplus X \leq mR$ and $(mR) \cap (Y \oplus K_1) = Y \cap ((mR) + K_1)$, $(mR) \cap (M_2 \oplus K_1)$ is $\delta$-small in $Y \oplus K_1$. So $M$ is $\delta$-lifting.

Case (ii): $K = 0$. Then $mR \leq M_2$. Since $M_2$ is $\delta$-lifting, there exists a submodule $X$ of $mR$ such that $M_2 = X \oplus Y$ and $(mR) \cap Y$ is $\delta$-small in $Y$ for some submodule $Y$ of $M_2$. Hence $M = X \oplus (M_1 \oplus Y)$. Since $(mR) \cap (M_1 \oplus Y) = (mR) \cap Y$ and $(mR) \cap (M_1 \oplus Y) = (mR) \cap Y$ is $\delta$-small in $Y$. By Lemma 2.11(3), $(mR) \cap (M_1 \oplus Y)$ is $\delta$-small in $M_1 \oplus Y$. It follows that $M$ is $\delta$-lifting.

A module $M$ is said to be a \textit{principally semisimple} if every cyclic submodule is a direct summand of $M$. Tuganbayev calls a principally semisimple module as a regular module in $[3]$. Every semisimple module is principally semisimple. Every principally semisimple module is principally $\delta$-lifting. For a module $M$, we write $\text{Rad}_3(M) = \sum \{L \mid L$ is a $\delta$-small submodule of $M \}$.

**Lemma 3.15.** Let $M$ be a principally $\delta$-lifting module. Then $M/\text{Rad}_3(M)$ is a principally semisimple module.

**Proof.** Let $m \in M$. There exists $M_1 \leq mR$ such that $M = M_1 \oplus M_2$ and $(mR) \cap M_2$ is $\delta$-small in $M_2$. So$(mR) \cap M_2$ is $\delta$-small in $M$. Then

\[
M/\text{Rad}_3(M) = [(mR + \text{Rad}_3(M))/\text{Rad}_3(M)] \oplus [(M_2 + \text{Rad}_3(M))/\text{Rad}_3(M)]
\]

because $(mR + \text{Rad}_3(M)) \cap (M_2 + \text{Rad}_3(M)) = \text{Rad}_3(M)$. Hence every principal submodule of $M/\text{Rad}_3(M)$ is a direct summand. $\square$

**Proposition 3.16.** Let $M$ be a principally $\delta$-lifting module. Then $M = M_1 \oplus M_2$, where $M_1$ is a principally semisimple module and $M_2$ is a module with $\text{Rad}_3(M)$ essential in $M_2$.

**Proof.** Let $M_1$ be a submodule of $M$ such that $\text{Rad}_3(M) \oplus M_1$ is essential in $M$ and $m \in M_1$. Since $M$ is principally $\delta$-lifting, there exists a direct summand $M_2$ of $M$ such that $M_2 \leq mR$, $M = M_2 \oplus M_2'$ and $mR \cap M_2'$ is $\delta$-small in $M$. Hence $mR \cap M_2'$ is a submodule of $\text{Rad}_3(M)$ and so $mR \cap M_2' = 0$. Then $m \in M_2$ and $mR = M_2$. Since $M_2 \cap \text{Rad}_3(M) = 0$, $M_2$ is isomorphic to a submodule of $M/\text{Rad}_3(M)$. By
Lemma 3.15. \( M/\text{Rad}_\delta(M) \) is principally semisimple, \( M_2 \) is principally semisimple.

On the other hand, \( \text{Rad}_\delta(M) = \text{Rad}_\delta(M'_2) \) is essential in \( M_2 \) that it is clear from the construction of \( M'_2 \).

A nonzero module \( M \) is called \( \delta \)-hollow if every proper submodule is \( \delta \)-small in \( M \), and \( M \) is \emph{principally} \( \delta \)-hollow if every proper cyclic submodule is \( \delta \)-small in \( M \), and \( M \) is \emph{finitely} \( \delta \)-hollow if every proper finitely generated submodule is \( \delta \)-small in \( M \). Since finite direct sum of \( \delta \)-small submodules is \( \delta \)-small, \( M \) is principally \( \delta \)-hollow if and only if it is finitely \( \delta \)-hollow.

Lemma 3.17. The following are equivalent for an indecomposable module \( M \).

1. \( M \) is a principally \( \delta \)-lifting module.
2. \( M \) is a principally \( \delta \)-hollow module.

Proof. (1)\( \Rightarrow \) (2) Let \( m \in M \). Since \( M \) is a principally \( \delta \)-lifting module, there exist \( N \) and \( S \) submodules of \( M \) such that \( N \leq mR \), \( mR \cap S \) is \( \delta \)-small in \( M \) and \( M = N \oplus S \). By hypothesis, \( N = 0 \) and \( S = M \). So that \( mR \cap S = mR \) is \( \delta \)-small in \( M \).

(2)\( \Rightarrow \) (1) Let \( m \in M \). Then \( mR = (mR) \oplus (0) \). By (2) \( mR \) is \( \delta \)-small and \( (0) \) is direct summand in \( M \). Hence \( M \) is a principally \( \delta \)-lifting module.

Lemma 3.18. Let \( M \) be a module, then we have

1. If \( M \) is principally \( \delta \)-hollow, then every factor module is principally \( \delta \)-hollow.
2. If \( K \) is \( \delta \)-small submodule of \( M \) and \( M/K \) is principally \( \delta \)-hollow, then \( M \) is principally \( \delta \)-hollow.
3. \( M \) is principally \( \delta \)-hollow if and only if \( M \) is local or \( \text{Rad}_\delta(M) = M \).

Proof. (1) Assume that \( M \) is principally \( \delta \)-hollow and \( N \) a submodule of \( M \). Let \( m + N \in M/N \) and \( (mR + N)/N + K/N = M/N \). Suppose that \( M/K \) is singular. We have \( mR + K = M \). Since \( M/K \) is singular and \( M \) is principally \( \delta \)-hollow, \( M = K \).

(2) Let \( m \in M \). Assume that \( mR + N = M \) for some submodule \( N \) with \( M/N \) singular. Then \( (m + K)R = (mR + K)/K \) is a cyclic submodule of \( M/K \) and \( (mR + K)/K + (N + K)/K = M/K \) and \( M/(N + K) \) is singular as an homomorphic image of \( M/N \). Hence \( (N + K)/K = M/K \) or \( N + K = M \). By hypothesis \( N = M \).

(3) Suppose that \( M \) is principally \( \delta \)-hollow and it is not local. Let \( N \) and \( K \) be two distinct maximal submodules of \( M \) and \( k \in K \setminus N \). Then \( M = kR + N \) and \( M/N \) is a simple module, and so \( M/N \) is a singular or projective module. If \( M/N \) is singular, then \( M = N \) since \( kR \) is \( \delta \)-small. But this is not possible since \( N \) is maximal. So \( M/N \) is projective. Hence \( N \) is direct summand. So \( M = N \oplus N' \)
for some nonzero submodule $N'$ of $M$, that is, $N$ and $kR$ are proper submodules of $M$. Since every proper submodule of $M$ is contained in $\text{Rad}_\delta(M)$, $M = \text{Rad}_\delta(M)$. The converse is clear.

**Proposition 3.19.** Let $M$ be a module. Then the following are equivalent.

1. $M$ is principally $\delta$-hollow.
2. If $N$ is submodule with $M/N$ cyclic, then $N$ is a $\delta$-small submodule of $M$.

**Proof.** (1) $\Rightarrow$ (2) Assume that $N$ is a submodule with $M/N$ cyclic. Lemma 2.1 implies that $M/N$ is principally $\delta$-hollow since being $\delta$-small is preserved under homomorphisms. Since $M/N$ has maximal submodules, and by Lemma 3.18, $M/N$ is local. There exists a unique maximal submodule $N_1$ containing $N$. Hence $N$ is small, therefore it is $\delta$-small.

(2) $\Rightarrow$ (1) We prove that every cyclic submodule is $\delta$-small in $M$. So let $m \in M$ and $M = mR + N$ with $M/N$ singular. Then $M/N$ is cyclic. By hypothesis, $N$ is $\delta$-small submodule of $M$. By Lemma 2.1 there exists a projective semisimple submodule $Y$ of $N$ such that $M = (mR) \oplus Y$. Let $Y = \bigoplus_{i \in I} N_i$ where each $N_i$ is simple. Now we write $M = ((mR) \bigoplus_{i \neq j} N_j) \oplus N_i$. Then $M/( (mR) \bigoplus_{i \neq j} N_j)$ is cyclic module as it is isomorphic to simple module $N_i$. By hypothesis, $((mR) \bigoplus_{i \neq j} N_j)$ is $\delta$-small in $M$. Again by Lemma 2.1 there exists a projective semisimple submodule $Z$ of $((mR) \bigoplus_{i \neq j} N_j)$ such that $M = Z \oplus N_i$. Hence $M$ is projective semisimple module. So $M = N \oplus N'$ for some submodule $N'$. Then $N'$ is projective. $M/N$ is projective as it is isomorphic to $N'$. Hence $M/N$ is both singular and projective module. Thus $M = N$. \hfill $\Box$

4. Applications

In this section, we introduce and study some properties of principally $\delta$-semiperfect modules. By [9], a projective module $P$ is called a **projective $\delta$-cover** of a module $M$ if there exists an epimorphism $f : P \twoheadrightarrow M$ with $\text{Ker} f$ is $\delta$-small in $P$, and a ring is called **$\delta$-perfect** (or **$\delta$-semiperfect**) if every $R$-module (or every simple $R$-module) has a projective $\delta$-cover. For more detailed discussion on $\delta$-small submodules, $\delta$-perfect and $\delta$-semiperfect rings, we refer to [9]. A module $M$ is called **principally $\delta$-semiperfect** if every factor module of $M$ by a cyclic submodule has a projective $\delta$-cover. A ring $R$ is called **principally $\delta$-semiperfect** in case the right $R$-module $R$ is principally $\delta$-semiperfect. Every $\delta$-semiperfect module is principally $\delta$-semiperfect. In [9], a ring $R$ is called **$\delta$-semiregular** if every cyclically presented $R$-module has a projective $\delta$-cover.

**Theorem 4.1.** Let $M$ be a projective module. Then the following are equivalent.

1. $M$ is principally $\delta$-semiperfect.
(2) $M$ is principally $\delta$-lifting.

Proof. (1)⇒ (2) Let $m \in M$ and $P \xrightarrow{f} M/mR$ be a projective $\delta$-cover and $M \xrightarrow{\pi} M/mR$ the natural epimorphism.

Then there exists a map $M \xrightarrow{g} P$ such that $fg = \pi$. Then $P = g(M) + \text{Ker}(f)$. Since $\text{Ker}(f)$ is $\delta$-small, by Lemma 2.1 there exists a projective semisimple submodule $Y$ of $\text{Ker}(f)$ such that $P = g(M) \oplus Y$. So $g(M)$ is projective. Hence $M = K \oplus \text{Ker}(g)$ for some submodule $K$ of $M$. It is easy to see that $g(K \cap mR) = g(K) \cap \text{Ker}(f)$ and $\text{Ker}(g) \leq mR$. Hence $M = K + mR$. Next we prove $K \cap (mR)$ is $\delta$-small in $K$. Since $\text{Ker}(f)$ is $\delta$-small in $P$, $g(K) \cap \text{Ker}(f) = g(K \cap mR)$ is $\delta$-small in $P$ by Lemma 2.1(4). Hence $K \cap (mR)$ is $\delta$-small in $K$ since $g^{-1}$ is an isomorphism from $g(M)$ onto $K$.

(2)⇒ (1) Assume that $M$ is a principally $\delta$-lifting module. Let $m \in M$. There exist direct summands $N$ and $K$ of $M$ such that $M = N \oplus K$, $N \leq mR$ and $mR \cap K$ is $\delta$-small in $K$. Let $K \xrightarrow{\pi} M/mR$ denote the natural epimorphism defined by $\pi(k) = k + mR$ where $k \in K$, $k + mR \in M/mR$. It is obvious that $\text{Ker}(\pi) = mR \cap K$. It follows that $K$ is projective $\delta$-cover of $M/mR$. So $M$ is principally $\delta$-semiperfect.

Corollary 4.2. Let $R$ be a ring. Then the following are equivalent.

(1) $R$ is principally $\delta$-semiperfect.
(2) $R$ is principally $\delta$-lifting.
(3) $R$ is $\delta$-semiregular.

Proof. (1)⇔ (2) Clear by Theorem 4.1

(2)⇔ (3) By Theorem 3.6 (2), $R$ is principally $\delta$-lifting if and only if for every principal right ideal $I$ of $R$ can be written as $I = N \oplus S$, where $N$ is direct summand and $S$ is $\delta$-small in $R$. This is equivalent to being $R$ $\delta$-semiregular since for any ring $R$, $\text{Rad}_\delta(R)$ is $\delta$-small in $R$ and each submodule of a $\delta$-small submodule is $\delta$-small.

The module $M$ is called principally $\delta$-supplemented if every cyclic submodule of $M$ has a $\delta$-supplement in $M$. Clearly, every $\delta$-supplemented module is principally $\delta$-supplemented. Every principally $\delta$-lifting module is principally $\delta$-supplemented.
In a subsequent paper we investigate principally $\delta$-supplemented modules in detail. Now we prove:

**Theorem 4.3.** Let $M$ be a principally $\delta$-semiperfect module. Then

1. $M$ is principally $\delta$-supplemented.
2. Each factor module of $M$ is principally $\delta$-semiperfect, hence any homomorphic image and any direct summand of $M$ is principally $\delta$-semiperfect.

**Proof.** (1) Let $m \in M$. Then $M/mR$ has a projective $\delta$-cover $P \xrightarrow{\beta} M/mR$. There exists $P \xrightarrow{\alpha} M$ such that the following diagram is commutative, $\beta = \pi \alpha$, where $M \xrightarrow{\pi} M/mR$ is the natural epimorphism.

```
\begin{array}{ccc}
P & \xrightarrow{} & M/mR \\
\uparrow{\alpha} & & \downarrow{\beta} \\
M & \xrightarrow{\pi} & 0
\end{array}
```

Then $M = \alpha(P) + mR$, and $\alpha(P) \cap mR$ is $\delta$-small in $\alpha(P)$, by Lemma 2.1 (1). Hence $M$ is principally $\delta$-supplemented.

(2) Let $M \xrightarrow{f} N$ be an epimorphism and $nR$ a cyclic submodule of $N$. Let $m \in f^{-1}(nR)$ and $P \xrightarrow{g} M/(mR)$ be a projective $\delta$-cover. Define $M/(mR) \xrightarrow{h} N/nR$ by $h(m' + mR) = f(m') + nR$, where $m' + mR \in M/(mR)$. Then $\ker(g)$ is contained in $\ker(hg)$. By projectivity of $P$, there is a map $\alpha$ from $P$ to $N$ such that $hg = \pi \alpha$.

```
\begin{array}{ccc}
P & \xrightarrow{g} & M/mR \\
\downarrow{\alpha} & & \downarrow{h} \\
N & \xrightarrow{\pi} & N/nR \\
\end{array}
```

It is routine to check that $(nR) \cap \alpha(P) = \alpha(\ker(g))$. By Lemma 2.1 (2), $\alpha(\ker(g))$ is $\delta$-small in $N$ since $\ker(g)$ is $\delta$-small. Let $x \in \ker(\pi \alpha)$. Then $hg(x) = (\pi \alpha)(x) = 0$ or $\alpha(x) \in (nR) \cap \alpha(P)$. So $\ker(\pi \alpha)$ is $\delta$-small. Hence $P$ is a projective $\delta$-cover for $N/(nR)$.

**Theorem 4.4.** Let $P$ be a projective module with $\text{Rad}_\delta(P)$ is $\delta$-small in $P$. Then the following are equivalent.

1. $P$ is principally $\delta$-lifting.
2. $P/\text{Rad}_\delta(P)$ is principally semisimple and, for any cyclic submodule $\pi R$ of $P/\text{Rad}_\delta(P)$ that is a direct summand of $P/\text{Rad}_\delta(P)$, there exists a cyclic direct summand $A$ of $P$ such that $\pi R = A$. 

$\square$
Proof. (1)⇒(2) Since \( P \) is a principally \( \delta \)-lifting module, \( P/\text{Rad}_{\delta}(P) \) is principally semisimple by Lemma 3.15. Let \( \pi R \) be any cyclic submodule of \( P/\text{Rad}_{\delta}(P) \). By Theorem 3.6, there exists a direct summand \( A \) of \( P \) and a \( \delta \)-small submodule \( B \) such that \( xR = A \oplus B \). Since \( B \) is contained in \( \text{Rad}_{\delta}(R) \), \( xR + \text{Rad}_{\delta}(R) = A + \text{Rad}_{\delta}(R) \). Hence \( xR = \overline{A} \).

(2)⇒(1) Let \( xR \) be any cyclic submodule of \( P \). Then we have \( P/\text{Rad}_{\delta}(P) = [(xR+\text{Rad}_{\delta}(P))/\text{Rad}_{\delta}(P)] \oplus [U/\text{Rad}_{\delta}(P)] \) for some \( U \leq P \). By (2), there exists a direct summand \( A \) of \( P \) such that \( P = A \oplus B \) and \( U = B + \text{Rad}_{\delta}(P) \). Then \( P = A + B = A + U + \text{Rad}_{\delta}(P) \). Since \( \text{Rad}_{\delta}(P) \) is \( \delta \)-small in \( P \), there exists a projective and semisimple submodule \( Y \) of \( P \) such that \( P = A \oplus (A + U) \oplus Y \). Since \( P \) is projective, \( A + B \) is also projective and so by Lemma 3.13, we have \( A + B = V \oplus B \) for some \( V \leq A \). Hence \( P = V \oplus B \oplus Y \). On the other hand \( (xR) \cap (B \oplus Y) = (xR) \cap B \leq (xR) \cap U \leq \text{Rad}_{\delta}(R) \). Since \( \text{Rad}_{\delta}(R) \) is \( \delta \)-small in \( P \), it is \( \delta \)-small in \( B \oplus Y \) by Lemma 2.1 (3). Thus \( P \) is principally \( \delta \)-lifting. \( \square \)

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