Classical and quantum anomalous diffusion in a system of $2\delta$-kicked Quantum Rotors

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We study the dynamics of cold atoms subjected to pairs of closely time-spaced $\delta$-kicks from standing waves of light. The classical phase space of this system is partitioned into momentum cells separated by trapping regions. In a certain range of parameters it is shown that the classical motion is well described by a process of anomalous diffusion. We investigate in detail the impact of the underlying classical anomalous diffusion on the quantum dynamics with special emphasis on the phenomenon of dynamical localization. Based on the study of the quantum density of probability, its second moment and the return probability we identify a region of weak dynamical localization where the quantum diffusion is still anomalous but the diffusion rate is slower than in the classical case. Moreover we examine how other relevant time scales such as the quantum-classical breaking time or the one related to the beginning of full dynamical localization are modified by the classical anomalous diffusion. Finally we discuss the relevance of our results for the understanding of the role of classical cantori in quantum mechanics.

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I. INTRODUCTION

The effect of the underlying classical dynamics on the quantum motion has been a recurrent topic of research since the early days of the quantum theory. In recent years the use of experimental techniques based on ultracold atoms in optical lattices [1] has permitted to study in great detail the role of classical mechanics in simple quantum systems. In these experiments a very dilute almost free gas of atoms (Cs and Rb) is cooled down to temperatures of the order of tens $\mu K$ and then interacts with an optical lattice. In its simplest form, the optical lattice consists of two laser beams prepared in such a way that the resulting interference pattern is a stationary plane wave in space. The laser frequency is tuned close to a resonance of the atomic system in order to enhance the atom-laser coupling but not too close to avoid spontaneous emission. In this limit the laser-atom system can be considered as a point particle in a sine potential, namely, the quantum pendulum. If the laser is turned on and off in a series of short periodic pulses, the resulting system is very well approximated by the so called quantum kicked rotor (QKR) (see Ref. [2] for a review) extensively studied in the context of quantum chaos,

$$\mathcal{H} = \frac{p^2}{2} - K \cos(q) \sum_n \delta(t - Tn). \quad (1.1)$$

For short time scales, quantum and classical motion agrees. However quantum diffusion is eventually suppressed due to destructive interference that localize eigenstates in momentum space. This counter-intuitive feature, usually referred to as dynamical localization [3], was fully understood [4] after mapping the kicked rotor problem onto a short range one dimensional disordered system where localization is well established. The theoretical predictions of Ref. [4] were eventually confirmed experimentally [1] (see also Ref. [5]) by using the cold atoms techniques mentioned previously. The standard kicked rotor is thus an ideal candidate for the study of the quantum properties of one dimensional systems whose classical motion is diffusive. A natural question to ask is whether this analysis can be extended to other types of (anomalous) diffusive motion.

For values of the kick strength $K$ in Eq. (1.1) sufficiently small, the classical phase space is composed of chaotic and integrable parts and, for certain initial conditions, the classical motion is well described by a process of anomalous diffusion. The quantum transport properties in systems with a mixed phase space [6] depend strongly on the details of the Hamiltonian [7]. Even for a given configuration many types of anomalous diffusion are observed depending on the initial conditions or the time scales studied [8]. This lack of universality makes it difficult to precisely assess the effect of the underlying classical anomalous diffusion on the quantum dynamics.

The situation is different if the smooth sinusoidal optical potential is replaced by a potential with a logarithmic or power-law singularity [9]. The classical phase space is homogeneous but still the classical motion is superdiffusive. In the quantum realm, as a consequence of interference effects, the particle still diffuses but at a slower rate. In fact, for certain types of singularities, full dynamical localization never occurs and diffusion persists at all times. In other cases exponential localization is eventually observed but anomalous diffusion different from the classical one is still observed for shorter times. The classical density of probability $P(p,t)$ in this region is accurately described by the solution of an anomalous
Fokker-Planck equation \[^{[10]}\]. We note that, unlike the case of normal diffusion, the information obtained from the knowledge of a few moments of the density of probability may not be sufficient to fully characterize the classical motion \[^{[10]}\]. Thus for a correct understanding of these systems it is essential a detailed investigation of \(P(p,t)\).

The models studied in Ref. \[^{[9]}\] are non-KAM but classical anomalous diffusion can also exist in KAM systems. One example is the kicked rotor with a smooth potential but subjected to pairs of closely time-spaced kicks: the 2\(\delta\)-kicked rotor (2\(\delta\)-KR) \[^{[11,12]}\].

The dynamics of this model in a certain region of parameters has already been investigated in the literature: theoretically in Ref. \[^{[11]}\] and experimentally in Ref. \[^{[12]}\]. Unlike the single kicked rotor the classical dynamics is strongly correlated. The momentum space is divided into regions of fast momentum diffusion separated by porous boundaries, ie narrow trapping regions where classical trajectories ‘stick’ for relatively long periods. The 2\(\delta\)-KR trajectory spends considerable time trapped in a cell before escaping to the next.

In this paper we provide a detailed account of the type of classical and quantum anomalous diffusion associated with this motion. We shall restrict ourselves to the one cell region, namely, to time/momentum scales such that a particle initially trapped manages to escape and eventually reaches a new trapping region. We aim to better understand of how generic features of the classical anomalous diffusion affect the quantum motion in a KAM system. We restrict ourselves to a region of parameters such that the dynamics is generic, namely, the classical phase space is fully chaotic with no island of stability. However the trapping-leaking mechanism of our model still causes strong deviations from the single kicked rotor results. Other important motivation to study the motion of a fully chaotic 2\(\delta\)-KR is that it mimics the effect of cantori in chaotic systems. Thus a particle typically gets trapped in a cantorus for a long time until it escapes to the chaotic sea. We argue the findings of this paper shed light about the role of classical cantori in quantum mechanics \[^{[13,15]}\].

The organization of the paper is as follows: in the next section we introduce the model, review some of its more relevant dynamical features and discuss the region of parameters to be studied. In section III, we study the classical and quantum transport properties. Among others we analyze the classical and quantum density of probability, the classical-quantum breaking time and the return probability. We aim to describe how dynamical localization arises in systems whose classical diffusion is anomalous and how other relevant scales of the problem such as the quantum-classical breaking time are affected by the underlying classical dynamics.

In summary, our main new results are: 1) In the region of interest both classical and quantum dynamics is well characterized by a process of anomalous diffusion. 2) We identify two routes to dynamical localization as a function of \(\hbar\): for \(h > h_c\) (\(h_c\) is function of the kick strength and the separation between pairs of kicks) standard dynamical localization occurs in the trapping region and the particle does not escape; for \(h < h_c\) the central part probability density is exponential and almost time independent. However diffusion does not stop since eventually the particle escapes from the trapping region. True dynamical localization in this case occurs for time scales much longer than the typical time to escape from the trapping region. For intermediate times we find the quantum diffusion is anomalous but slower than the classical one. 3) The classical anomalous diffusion induces a fractional scaling with \(\hbar\) in different quantities of interest such as the quantum-classical breaking time. 4) The 2\(\delta\)-KR can be used as a simplified model to study the effect of cantori in classical and quantum mechanics.

II. THE MODEL

We consider a system with a Hamiltonian corresponding to a sequence of closely spaced pairs of kicks:

\[
\mathcal{H} = \frac{p^2}{2} - K \cos x \sum_n \left[ \delta(t - nT) + \delta(t - nT + \epsilon) \right],
\]

where \(\epsilon \ll T\) is a short time interval and \(K\) is the kick-strength.

A. Classical dynamics

The classical map for the 2\(\delta\)-KP is a straightforward extension of the Standard Map:

\[
\begin{align*}
p_{n+1} &= p_n + K \sin x_n; & p_{n+2} &= p_{n+1} + K \sin x_{n+1} \\
x_{n+1} &= x_n + \epsilon p_{n+1}; & x_{n+2} &= x_{n+1} + \tau p_{n+2}
\end{align*}
\]

(2.1)

where \(\epsilon\) is a very short time interval between two kicks in a pair and \(\tau = T - \epsilon\) is the (much longer) time interval between the pairs. Clearly, the limit \(\epsilon = \tau = 0\) corresponds to the Standard Map, which describes the classical dynamics of the quantum kicked rotor:

\[
\begin{align*}
p_{i+1} &= p_i + K \sin x_i, \\
x_{i+1} &= x_i + p_{i+1}. 
\end{align*}
\]

(2.2)

Particles with momenta \(p_0 \simeq (2m + 1)\pi/\epsilon\) and \(m = 0, \pm 1, \pm 2, \cdots\) (relative to the optical lattice) are confined in momentum trapping regions and absorb little energy; conversely, particles prepared near \(p_0 \simeq 2m\pi/\epsilon\) experience rapid energy growth up to localization. The basic mechanism of trapping is fairly intuitive: atoms for which \(p_0 = (2m + 1)\pi/\epsilon\) experience an impulse \(K \sin x\) followed by another one \(\simeq K \sin(x + \pi)\) which in effect cancels the first. Over time, however, there is a gradual de-phasing of this classical ‘anti-resonant’ process. A theoretical study \[^{[12]}\] of the classical diffusion in this system
found anomalous momentum diffusion for any \( p_0 \) and intermediate time scales. It was also observed long-ranged corrections to the uncorrelated diffusion rate not present in the standard kicked rotor. Inside the trapping region diffusion is normal but the coefficient of diffusion \( D \) has a dependence on \( K \) as \( D \sim K^3 \), similar to the one found in single kicked rotors in a region of the classical phase space densely populated by cantori. By contrast, if the phase space is fully chaotic it is expected \( D \sim K^2 \). This suggests that our model may reproduce to a good approximation the effect of cantori in a KAM system.

The size in momentum space of the trapping region \( \Delta p \) strongly depends on the parameters \( \epsilon, K \) defining the model.

For an accurate analysis of the trapped region is necessary that: i) the particle is initially trapped, ii) the particle dwells on average a time long enough in this region before it escapes from it. In Ref. [12] it was concluded that this implies the criterion \( K\epsilon \ll 1 \). However, if \( K\epsilon \) is too small, the phase space would be too regular. Since we are interested in generic feature of the motion, we take iii) \( \tau \gg 1 \). This implies less correlations between successive space positions of the particle. We thus restrict ourselves through the paper to the range \( K\epsilon \ll 1 \) and \( \tau \gg 1 \) with initial conditions inside the trapping region.

### B. Quantum dynamics

The time evolution operator for this system can be written as

\[
\hat{U}_\epsilon = e^{-i \frac{\epsilon \pi}{\hbar} \cos x} e^{-i \frac{\epsilon^2 \pi}{\hbar} \cos x} .
\]

In a basis of plane waves, \( \hat{U}_\epsilon \) has matrix elements

\[
U_{lm}^\tau = U_{lm}^{\text{free}}, \quad U_{lm}^{2-\text{kick}} = e^{-i \frac{\epsilon^2 \pi}{\hbar}} \delta^{l-m}
\]

\[
\sum_k J_{l-k}(K\hbar) \ J_{k-m}(K\hbar) \ e^{-i \frac{\epsilon^2 \pi}{\hbar}}
\]

where \( K\hbar = K/\hbar \) and \( J_n(K\hbar) \) is integer Bessel functions of the first kind. It is easy to see that \( U_{lm}^{2-\text{kick}} \) is invariant if the products \( K\epsilon = K\epsilon \) and \( K\hbar = K\hbar \) are kept constant; while the free propagator \( U_{lm}^{\text{free}} = e^{-i \frac{\epsilon \pi}{\hbar} \cos x} \) simply contributes a near-random phase. Thus the results are quite insensitive to the magnitude of \( \tau = T - \epsilon \) provided that \( \tau \gg 1 \). We will stick to \( K\epsilon \ll 1 \) and \( \tau \gg 1 \) in all numerical calculations.

The result in Eq. (2.3) may be compared with the one-kick map in Eq. (2.1)

\[
U_{lm}^{(0)} = e^{-i \frac{\epsilon \pi}{\hbar}} J_{l-m}(K\hbar) .
\]

The one-kick matrix for the QKR has a well-studied band-structure: since \( J_{l-m}(x) \simeq 0 \) for \( |l-m| \gg x \), we can define a bandwidth for \( U^{(0)} \), i.e. \( b = K\hbar \) (this is strictly a half-bandwidth) which is independent of the angular momenta \( l \) and \( m \). However, this is not the case for the matrix of \( U^\tau \).

Assuming \( |l-m| \) is small it was shown in Ref. [11] that

\[
U_{lm}^\tau \approx e^{-i \Phi} J_{l-m} [2K\hbar \cos (\hbar \epsilon /2)]
\]

where the phase \( \Phi = (l^2 T + e l m + e^2 T^2 \hbar /2 + \pi (l - m)/2 \). Hence we infer a momentum dependent bandwidth, \( b(p) = 2K\hbar \cos (\epsilon p /2) \).

While \( U^{(0)} \) has a constant bandwidth, the bandwidth for the matrix of \( U^\tau \) oscillates with \( l \) from a maximum value \( b_{\text{max}} = 2K\hbar \), equivalent to twice the bandwidth of \( U^{(0)} \), down to a minimum value \( b_{\text{min}} \sim 0 \). \( U^\tau \) is thus partitioned into sub-matrices of dimension \( N = 2\pi /\hbar \) corresponding precisely to the momentum cells of width \( \Delta p = N\hbar \) observed in experiments [12].

In this paper we aim to study both the evolution of an initially given wave-packet and its low order moments. The structure of the evolution operator \( \hat{U}^\tau \) allows an efficient and accurate numerical calculation of these quantities. The action of the operator \( \hat{U}^\tau \) on a quantum state can be decomposed into four steps: two associated with the kicks, which are diagonal in the space presentation; and two associated with the free rotations between neighboring kicks, which are diagonal in the momentum presentation. We thus take the space and momentum representations alternatively to facilitate the calculations. The transformation between the representations is efficiently carried out with the fast Fourier transformation (FFT) algorithm. In our numerical simulations we can set the size of the basis up to \( 2^{20} \) which is big enough to guarantee a double precision accuracy.

### III. RESULTS: ANOMALOUS DIFFUSION AND DYNAMICAL LOCALIZATION

In this section we investigate the classical and quantum dynamics of the 2\( \delta \)-KR. Our main motivation is to describe the effect on the quantum motion of generic dynamical features of the classical dynamics such as the trapping-escaping mechanism. We mainly focus on time/momentum scales such that the motion is confined inside one cell. Initial conditions are always chosen within the trapping region. For a study in the multi-cell region we refer to Ref. [11, 12].

As was mentioned previously, we restrict ourselves to the window of parameters \( K\epsilon \ll 1 \). In addition it is also imposed \( \tau \gg 1 \) in order to remove correlations that make the dynamics non generic. In the numerical calculations \( \tau, K, \) and \( \epsilon \) are fixed within the above limits. Our main conclusions do not depend on the specific value of the parameters. Quantum effects are investigated by varying \( \hbar \).
The classical dynamics in the trapping region was investigated analytically in Ref. [12]. It was found that energy diffusion $\langle p^2(t) \rangle \sim D t$ increases linearly with time. However, the dependence of $D \sim K^3$ on the kicking strength is different from the single kicked rotor prediction $D \sim K^2$ in the chaotic region but similar to the one observed if the classical phase space is populated by cantori [14]. This suggests that our model captures generic features of the effect of cantori in classical mechanics.

Eventually the particle reaches the edge of the trapping region and leaks outside. Diffusion becomes then anomalous until the particle gets trapped again. As is shown in Fig. 1, the parameters $K$, $\epsilon$ and $\tau$ were chosen such that the typical time for considerable leaking is $t_{\text{leak}} \approx 200 \gg 1$. Before this time the leakage is still possible but with a negligible probability. As the particle approaches the outer boundaries of the two cells that sandwich the trapping region (at $t \approx 6 \times 10^3$), the diffusion begins to slow down. The time unit we take in all figures is the number of the kick pairs.

In the quantum case the dynamics is richer. If we restrict ourselves to the one cell region three time scales are distinguished. In a first stage, $t < t_{c}$, the quantum averaged energy agrees with its classical counterpart. As a result of the peculiarities of the classical dynamics, scaling with $h$ of the classical-quantum breaking time $t_c$ may be different from $t_c \sim h^{-2}$, the result for a single kicked rotor. For longer times, $t_c < t < t_d$, interference effects start to be important. As a consequence we expect the particle still diffuses, $\langle p^2(t) \rangle \approx D_{\text{quan}} t^\gamma$, but at a rate which decreases as $h$ increases and it is lower than the classical one; namely $\gamma \leq 1$ and $D_{\text{quan}} \leq D_{\text{clas}}$. This is a novel regime caused by the interplay of destructive interference and classical anomalous diffusion. This stage lasts up to a time $t_d$ in which full dynamical localization occurs and diffusion is totally arrested.

Our numerical results confirm the above qualitative picture. Results for quantum and classical energy diffusion $\langle p^2(t) \rangle$ of the quantum case) as a function of time $t$ for $\tau = 10^4$ and $K = 0.05 < 1$ for different $h$’s are shown in Fig. 1. In the classical case $2 \times 10^7$ initial conditions uniformly distributed along $p_0 = \pi/\epsilon$ (located at the center of the trapping region) have been utilized for the ensemble average. The quantum initial condition is set as the momentum eigenstate $|l_0\rangle$ with $l_0$ the integer closest to $p_0/h$.

We restrict ourselves to time/momentum scales such that the motion is confined to one cell. We observe that for short times both classical and quantum results coincide up to a certain breaking time $t_c$ which has a fractional scaling with $h$, different from the single kick rotor case (see insert of Fig. 1b, where $t_c$ is evaluated as the maximum time such that the deviation between classical and quantum $\langle p^2(t) \rangle$ is below 20%). For later times the quantum particle still diffuses normally but with a smaller diffusion coefficient $D_{\text{quan}}$ (see Tab. I). This is a direct consequence of destructive interference effects. We relate this new region of weak dynamical localization to the effect of classical trapping on the quantum dynamics. This interpretation is reinforced by the observed dependence of $D_{\text{quan}}$ on $h$ in the trapping region (see Tab. I). For $h$ sufficiently small ($h < 10^{-3}$), $D_{\text{quan}}$ is close to the classical prediction. For larger $h$ we observe a growing dependence on $h$ though the exponent $\gamma \approx 1$ is not modified. It decreases as $h$ increases. This situation lasts until a maximum $h$, denoted as $h_{\text{max}}$, such that the breaking-time $t_c$ coincides with the time $t_d$ in which full dynamical localization starts. For the parameters utilized $h_{\text{max}} \approx 6 \times 10^{-3}$.

We have observed an additional $h$ scale, denoted as $h_c$, relevant for a precise characterization of the trapping-leaking mechanism. It is defined by the smallest $h$ such that $t_d < t_{\text{leak}}$. For the parameters used $h_c \approx 10^{-3} <

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**FIG. 1:** (Color online) Comparison of quantum and classical energy diffusion (from bottom to top: $h = 4 \times 10^{-3}, 1 \times 10^{-3}, 0.1 \times 10^{-3}, 0.05 \times 10^{-3}$ and classical respectively) for $K = 0.2, \epsilon = 0.25$ and $\tau = 10^3$. The two figures only differ in the scale of time/momentum represented. Fig. (a) shows the effect of leaking from the trapping region in longer time scales while Fig. (b) describes diffusion inside the trapping region. The inset of Fig. (b) shows the dependence of quantum-classical breaking time $t_c$ on $h$; the best fitting (thin solid line) suggests that $t_c \sim h^{-1.39 \pm 0.05}$ for $h < 10^{-3}$. This is in contrast with the single kicked rotor where $t_c \sim h^{-2}$.

### A. Energy diffusion


Finally we note that $\langle p^2(t) \rangle \sim t$ is only a necessary condition for normal diffusion but this by no means assures that the density of probability is Gaussian-like [10]. Indeed, in the next section we will show that the quantum and classical density of probability of our model strongly deviate from the normal diffusion prediction.

### B. Density of probability

The density of probability of finding a particle with momentum $p$ after a time $t$ from a given initial state $|\psi(0)\rangle = |\Psi_0\rangle$ is given by $P_q(p,t) = |\langle k|\psi(t)\rangle|^2$ with $p = kh$. The parameter set $K = 0.2$, $\epsilon = 0.05$ and $\tau = 10^4$ chosen permits us to study generic features of the motion. By generic we mean the trapping region is large enough to be studied and the classical phase space has not stability island; namely, $K\epsilon \ll 1$. In addition $\tau \gg 1$, so consecutive pairs of kicks are uncorrelated. In all cases our initial state is located in the trapping region.

#### 1. Classical density of probability

The classical $P(p,t)$ is obtained by evolving the classical equation of motion for $2 \times 10^7$ different initial conditions at the center of the trapping region. Positions are uniformly distributed along the interval $(-\pi, \pi)$. $P(p,t)$ is approximated as the set of the states of the whole ensemble that falls in $(p-\Delta p/2, p+\Delta p/2)$ at time $t$. In the numerical calculations we set $\Delta p = 0.02\sqrt{\langle p^2(t)\rangle}$.

We distinguish the following regions in the classical density of probability (see Fig. 2a): For short times such that the particle is well inside the trapping region the diffusion is normal and $P(p,t)$ is Gaussian (see inset of Fig. 2a). As the boundary of the trapping region is approached we observe a gradual crossover from normal to anomalous (super) diffusion. For small momentum $P(p,t)$ is still Gaussian as leaking to the outside region is weak. As time approaches $t_{\text{leak}} \approx 200$, the typical time to reach the edge of the trapping region, the central (small momentum) Gaussian region becomes smaller and smaller. Meanwhile, the outskirts bend down and a power-like behavior typical of anomalous diffusion is observed.

For longer times, $t > t_{\text{leak}}$, $P(p,t) \sim p^{-\alpha}$ with $\alpha$ a decreasing function of time (Fig. 2a). For $t \to \infty$, $\alpha \to 0$. This behavior is not surprising due to the transient nature of the trapping-leaking mechanism. The power-law decay is a direct consequence of the enhanced diffusion once the particle has escaped from the trapped region. However, precisely due to this fast diffusion, the particle soon approaches a new trapped region where diffusion is slowed down again until the particle leaks from the trap. As a consequence of this bottle neck, the exponent $\alpha$ decreases and the density of probability in the region of fast diffusion gradually increases with time. In addition a sharp jump in $P(p,t)$ is observed when the enhanced

| $h$          | $t_c$ | $t_d$ | $D_{\text{quan}}/D_{\text{clas}}$ |
|--------------|-------|-------|----------------------------------|
| $8 \times 10^{-5}$ | $95 \pm 30$ | $> 10^4$ | $0.85 \pm 0.05$ |
| $1 \times 10^{-4}$ | $64 \pm 25$ | $> 10^4$ | $0.74 \pm 0.04$ |
| $2 \times 10^{-4}$ | $22 \pm 6$ | $> 10^4$ | $0.43 \pm 0.04$ |
| $5 \times 10^{-4}$ | $7 \pm 3$ | $> 10^4$ | $0.07 \pm 0.02$ |
| $1 \times 10^{-3}$ | $3 \pm 1$ | $> 10^4$ | $0.02 \pm 0.01$ |
| $2 \times 10^{-3}$ | $3 \pm 1$ | $80 \pm 30$ | $0.02 \pm 0.01$ |
| $3 \times 10^{-3}$ | $3 \pm 1$ | $50 \pm 20$ | $0.02 \pm 0.01$ |
| $6 \times 10^{-3}$ | $3 \pm 1$ | $4 \pm 2$ | $- \pm$ |

TABLE I: Breaking time $t_c$, dynamical localization time $t_d$, and classical/quantum diffusion coefficient against $h$ for $K = 0.2$, $\epsilon = 0.25$, and $\tau = 10^4$. $D_{\text{clas}} \approx 1.83 \times 10^{-5}$, which is evaluated over $t < t_{\text{leak}} \approx 200$. $D_{\text{quan}}$ is evaluated over the time range $t_c < t < t_{\text{leak}}$ for $h \leq h_c \approx 10^{-3}$ and $t_c < t < t_d$ otherwise.
quantum probability density with power-law tails as well (see Fig. (b)). The best fitting exponent varies in time ($\alpha \approx 4.5, 2.9, 2.3, 1.3$ for $t = 500, 1000, 2000$ and 6000 respectively). In the quantum case for small $\hbar$ leaking leads to power-law tails as well (see Fig. (b)). The best fitting exponent, indicated by thin black lines, is $\alpha \approx 7.1, 4.2, 3.0, 0.94$ for $t = 500, 1000, 2000$ and 6000 respectively). For larger $\hbar$ (see Fig. (c)), dynamical localization suppresses the diffusion before leaking occurs. $K = 0.2, \epsilon = 0.25$ and $\tau = 10^3$.

The quantum density of probability $P_q(p, t)$ was calculated from an initial condition $|\psi(0)\rangle = |l_0\rangle$ with $l_0 = \pi/\epsilon$ exactly for both classical and quantum calculations. As in the classical case, $P_q(p, t)$ was evaluated by summing up the probability of falling in a bin of width $\Delta p$. We set $\Delta p = 0.0315$ for $\hbar \approx 5 \times 10^{-4}$ (Fig. 2b) and $\Delta p = 0.006$ for $\hbar \approx 3 \times 10^{-3}$ (Fig. 2c). Hence it is in fact a coarse-grained result where part of quantum fluctuations have been suppressed. It is more instructive to start our analysis of the quantum probability density with a general account of the expected effect of the classical motion on the quantum dynamics. As in the single kicked rotor we expect to observe dynamical localization for sufficiently long times $t > t_d$ [4]. However the trapping (and eventual release) mechanism changes qualitatively the route to dynamical localization in our model. In fact we distinguish two different scales for full dynamical localization. For $\hbar > \hbar_c$, full dynamical localization will occur well inside the trapping region where the classical diffusion is normal. As a consequence the particles will not have time to escape, $(t_d < t_{\text{leak}} \approx 200$ with our parameters) and the density of probability will be time independent and will decay exponentially as a function of the momentum $p$. These are typical signatures of exponential localization similar to the one observed in a single kicked rotor. The situation is different if $\hbar < \hbar_c$ is small enough such that leaking to the enhanced diffusion region is possible. In this case we may still observe typical features of dynamical localization within the trapping region. However for larger momenta the density of probability develops time-dependent power law tails typical of anomalous diffusion. Full dynamical localization will not typically take place until the next trapping region is reached or later if $\hbar$ is small enough. This revival of quantum diffusion is a typical feature of our model that it is not observed in the single kicked rotor. Below we provide a more detailed account of the relevant time and momentum scales (see Fig. 2b) for an accurate description of the quantum density of probability:

1. For $t, p$ well within the trapping region: $t < t_{\text{leak}}$...
and $|p - p_0| < p_{\text{leak}} \sim 0.1$. After a narrow region in which classical and quantum results fully agree (see inset Fig. 2a), we observe in the quantum case a cusp for $p \sim p_0$ caused by the incipient dynamical localization in the core of the trapping region. However for larger momentum the distribution is Gaussian in agreement with previous results (see Fig. 1) for the energy diffusion.

2. For $p$ within the trapping region ($|p - p_0| < p_{\text{leak}}$) but $t > t_{\text{leak}}$. The core of the quantum probability is still exponentially localized (see Fig. 2c) but the outskirts $P_\varphi(p, t) \sim 1/p^\alpha$ develops a power-law tail (see Fig. 2b) typical of anomalous diffusion. In this region the exponent $\alpha$ depends on $h$ but it is time independent.

3. For $p$ outside or close to the edge of the trapping region and $t \geq t_{\text{leak}}$. The quantum probability $P_\varphi(p, t) \sim 1/p^\alpha$ develops power-law tails but in this case the exponent $\alpha$ decreases with time and eventually tends to zero for $t \rightarrow \infty$ (see Fig. 2b). This is in principle surprising. It is in general expected that destructive interference slows down the quantum motion. As a consequence the quantum density of probability should decay faster than the classical one. However there is a simple explanation for this behavior: in a first stage, the quantum exponent $\alpha$ is larger than the classical one as a consequence of destructive interference effects. For later times, the classical probability between two trapping region become smaller than the quantum one since the classical particle leaks faster into the next cell. At the same time destructive interference slows down the quantum motion in this region. Eventually $P_\varphi(p, t)$ saturates, $\alpha \rightarrow 0$ before the next trapping region is reached (see $t = 6000$ in Fig. 2a and Fig. 2b).

4. For times sufficiently long $t > t_d$ quantum destructive interference effects dominate, standard dynamical localization takes place, and diffusion stops. $P_\varphi(p, t)$ becomes time independent and decays exponentially as a function of the momentum. As was mentioned previously the value of $t_d$ depends dramatically on whether dynamical localization occurs in the trapping region or after it has escaped from the trapping region. In the former case $t_d < t_{\text{leak}} \approx 200$ and in the latter $t_d > 10^4$ with no intermediate values.

## C. Return Probability

Having studied diffusion in momentum for a fixed time $t$ we now look at the explicit dependence in time. In order to proceed we calculate return probabilities $P(t)$ of a wave-packet as a function of time. We average over $N$ initial starting conditions close to the center of the trapping region to suppress quantum fluctuations,

$$P(t) = \frac{1}{N} \sum_{l_1} \langle P_l(t) \rangle,$$

(3.1)

where $P_l(t) = \langle |\psi(t)|\psi(t = 0)\rangle^2$. The initial condition was taken to be an angular momentum eigenstate $|\psi(t = 0)\rangle = |l\rangle$. The results were further averaged from $l_1 \approx \pi/h_0 - N/2$ to $l_2 = l_1 + N$. In order to ensure the initial conditions are within the trapping region, the value $N$ is set to be $N \approx \delta p/h$ with $\delta p$ the width of the trapping region.

The return probability provides valuable information about the degree of localization of a system. Indeed it was already utilized in the landmark paper by Anderson [17] about localization. A non zero $P(t)$ in the $t \rightarrow \infty$ limit is a signature of exponential localization. On the other hand $P(t) \propto t^{-d/2}$ with $d$ the spatial dimensionality is typical of fully delocalized eigenstates (normal diffusion). In the theory of localization a power-law decay $P(t) \propto 1/t^\gamma$ with $\gamma < d/2$ is a signature of localization typical of a disordered conductor close to a localization transition. Such slow decay has also been related to the effect of cantori in transport [18] properties.

Our results are summarized in Fig. 3. For the sake of comparison, we first study the classical counterpart of $P(t)$. Again $2 \times 10^7$ initial states uniformly located in space and with $p_0 = \pi/\epsilon$ were evolved. $P(t)$ was evaluated as the set of states that fall in the momentum region $|p - p_0| < 0.004$ at time $t$. For the parameters ($K = 5, \epsilon = 0.04$ and $\tau = 10^4$) chosen for the calculation, the width of the trapping region $\delta p \approx 4$, and $t_{\text{leak}} \approx 30$. For $t < t_{\text{leak}}$, $P(t) \sim t^{-0.5}$, in agreement with the Gaussian probability density distribution obtained in the previous section. For $t > t_{\text{leak}}$, leaking gradually takes over and after a crossover $P(t)$ undergoes a faster power-law decay, $P(t) \sim t^{-1.5}$ until the next trap is reached.

Within the trapping region the quantum $P(t)$ decays as $t^{-\gamma}$ with $\gamma < 0.5$, a decreasing function of $h$. The fact that $\gamma < d/2$ is a clear indication that destructive inter-
ference is already slowing down the quantum diffusion. Not surprisingly, $\gamma$ approaches its classical limit, 0.5, as $\hbar \to 0$.

For sufficiently large $\hbar \geq 0.1$ the particle cannot escapes from the trapping region and $P(t)$ becomes constant (see for example $P(t)$ for $\hbar = 2^{-3}$ in Fig. 3). This is fully consistent with previous results for the energy diffusion and the density of probability.

For smaller $\hbar$ the particle escapes from the trapping region and full dynamical localization occurs only for times much longer than those shown in Fig. 3. For $t > t_{\text{leak}}$ we observe a rapid decrease of $P(t)$. The decay is still power-law but the exponent is larger than for $t < t_{\text{leak}}$ though smaller than the classical result ($\gamma \approx 1.5$). This time scale can be utilized to study the interplay between quantum effects and classical anomalous diffusion. Destructive interference slows down the quantum motion however diffusion is not arrested due to the underlying classical anomalous (super)diffusion.

Finally we address the similarities of our model with those of a generic Hamiltonian in a region of the phase space dominated by cantori. Qualitatively the effect of the trapping region is similar to that of cantori [8,13]. In both cases the particle remains in a small region of the phase space for a long time until eventually is released to the chaotic sea. In addition the dependence of the classical diffusion constant $D \sim K^3$ [11] is identical in both cases. In the quantum realm cantori [16,19,20] induces slow power-law decay in quantities such as the return probability [18] or the eigenstates themselves [19]. In addition cantori have been linked to fractional scaling with $\hbar$ [15] and anomalous dynamical localization [21]. All these features have also been observed in the model studied in this paper. These similarities strongly suggest that the $2 - \delta$ KR can be utilized as a toy model to study the effect of cantori in classical and quantum mechanics.

IV. CONCLUSIONS

We have studied the effect of classical anomalous diffusion on the quantum dynamics of atoms exposed to pairs of $\delta$-kicks. Our result are generic, do not depend on the parameters $K \epsilon \ll 1$ and $\tau \gg 1$ utilized. We have identified a regime of quantum diffusion where the motion is slow down due to destructive interference but full dynamical localization has not occurred yet. This has been related to the effect of the underlying classical trapping and releasing mechanism on the quantum dynamics. We have argued that the $2\delta$-KR can be used as a simplified model to study the effect of cantori in classical and quantum mechanics.

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