Triple Intersections and Geometric Transitions

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Abstract
The local neighborhood of a triple intersection of fivebranes in type IIA string theory is shown to be equivalent to type IIB string theory on a noncompact Calabi-Yau fourfold. The phases and the effective theory of the intersection are analyzed in detail.
1. Introduction

Many supersymmetric gauge theories are obtained as a sector of string theory in a particular geometric background. Varying the moduli of the string geometry may allow passage between different backgrounds and open a window onto the strong coupling regions of the gauge theory. Some of the phases are better described by open strings while others are described by closed strings with nonzero fluxes of background fields.

The conifold is a noncompact Calabi-Yau (CY) threefold described geometrically as a cone over an $S^2 \times S^3$ base. The transitions between the geometries where an $S^2$ is blown up ($O(-1) + O(-1) \to P^1$) and an $S^3$ is blown up ($T^* S^3$) have been studied [1]. Recently, the transitions between open strings in one of the conifold geometries and fluxes in the other have been studied as a means to describe phases of $N = 1$ gauge theories [2]. The conifold has a T-dual description as a pair of orthogonally intersecting Neveu-Schwarz (NS) fivebranes on $R^3$ [3]. Our aim in the present paper is to understand similarly the triple intersection of NS fivebranes on a string. Our motivation was to provide evidence for conjectural bound states associated to this intersection [4]. After T-duality the triple intersection has a geometric description, and one can describe the phases through geometric transitions. Work is in progress to determine the spectrum of BPS states and possibly the metric.

In section two we will discuss the supergravity solution for triple intersections of fivebranes in eleven dimensional supergravity. The solution will be reduced to ten dimensions along one of the transverse directions and shown to be equivalent to the stringy cosmic string construction of [5]. Finally, a T-duality along three directions (generally, mirror symmetry for a CY threefold) transforms the solution to a CY fourfold. In section three we present a linear sigma model describing the local neighborhood of the singularity of the CY fourfold. The model turns out to be $O(-1, -1) + O(-1, -1) \to P^1 \times P^1$. We show that this model has flops to isomorphic phases as well as transitions to CY fourfolds with various other cycles. In section four we show that the perturbative modes of the supersymmetric spacetime two-dimensional theory of a Dirichlet fivebrane in the $P^1 \times P^1$ phase match with those in the dual fivebrane picture. The dual fivebrane is a manifold partly constructed from a fivebrane. Anomaly cancellation for the fivebrane guarantees consistency in the CY fourfold framework. We conjecture that a fivebrane instanton is the mechanism for resolving the singularity. A local mirror for the CY fourfold is given. The other phases are compared and also seem to agree. The analogy with the conifold leads
to various conjectures discussed in this section. Our results indicate that chiral anomalies only occur in the $\mathbb{P}^1 \times \mathbb{P}^1$ phase where the fivebranes are unseparated and that the theories are effectively four-dimensional in the other phases.

2. Supergravity Solution of Intersecting Fivebranes

We start with the following configuration of fivebranes preserving one-eighth of the supersymmetry in eleven dimensional supergravity.

The metric and equations that result from solving the supersymmetry constraints for this configuration have been determined in [6][7][8][9]. The metric can be written in the following form.

$$ds_{11}^2 = H^{-1/3}(-dx_0^2 + dx_1^2) + 2H^{-1/3}g_{m\bar{n}}dz^m dz^{\bar{n}} + H^{2/3}dx^\alpha dx^\alpha$$

$$g = Hf\bar{f}$$

(2.1)

In the above $f$ is an arbitrary holomorphic function of the $z^m$, $z^1 = x^2 + ix^3$, $z^2 = x^4 + ix^5$, $z^3 = x^6 + ix^7$, and $\alpha \in \{8, 9, 10\}$. Note that $g$ is the determinant of the $3 \times 3$ matrix $g_{m\bar{n}}$.

The threefold metric $g_{m\bar{n}}$ is required by supersymmetry to be Kahler. The field strength of the three-form gauge field takes the following form.

$$F_{\bar{n}\alpha\beta\gamma} = -i\partial_{\bar{n}} \ln H\epsilon_{\alpha\beta\gamma}$$

$$F_{m\alpha\beta\gamma} = i\partial_m \ln H\epsilon_{\alpha\beta\gamma}$$

$$F_{m\bar{n}\beta\gamma} = -i\partial_{\alpha} g_{m\bar{n}} H^{-2/3}\epsilon^\alpha_{\beta\gamma}$$

(2.2)

The equation of motion for the gauge field yields

$$2\partial_m \partial_{\bar{n}} H + \partial^2_\alpha g_{m\bar{n}} = J^{source}_{m\bar{n}}$$

(2.3)

where $J^{source}_{m\bar{n}}$ is the magnetic source for the fivebranes. These equations are interesting to analyze directly in eleven dimensional supergravity since they resemble a generalized
system of monopoles. For our present purposes, however, we need to reduce to type IIA string theory by taking $x^{10}$ to be a hidden compact dimension. The metric becomes
\[ ds_{10}^2 = -dx^{02} + dx^{12} + g_{m\bar{n}}dz^m dz^{\bar{n}} + H(dzd\bar{z}) \] (2.4)
where $z = x^8 + ix^9$, $H = e^{2\phi} = \frac{g}{f^2}$ ($\phi$ is the dilaton). The field strengths are
\[ H_{mz\bar{z}} = \frac{-1}{2} \partial_m H = -\partial_m g_{z\bar{z}} \]
\[ H_{\bar{n}z\bar{z}} = \frac{1}{2} \partial_{\bar{n}} H = \partial_{\bar{n}} g_{z\bar{z}} \]
\[ H_{m\bar{n}z} = -\partial_z g_{m\bar{n}} \]
\[ H_{m\bar{n}\bar{z}} = \partial_{\bar{z}} g_{m\bar{n}}, \] (2.5)
and the equation of motion is
\[ \partial_m \partial_{\bar{n}} H + 2\partial_z \partial_{z\bar{z}} g_{m\bar{n}} = J_{m\bar{n}}^{source}. \] (2.6)
Defining the Kahler form $J = ig_{m\bar{n}}dz^m \wedge dz^{\bar{n}}$ leads to the following equation for the Kahler parameters, $\rho = B + iJ$.
\[ \partial_{\bar{z}} \rho = 0 \] (2.7)
This is precisely the mirror of the stringy cosmic string construction where one has $\partial_{\bar{z}} \tau = 0$ for the complex structure deformation parameters $\tau$ of a Kahler threefold. Note that the complex fourfold total space fibered over the $z$ plane with fiber the Kahler threefold $X_3$ is not Kahler since $\partial_z J_{m\bar{n}} \neq 0$. The would be Kahler form $J_{total}^\text{fourfold}$ of the fourfold satisfies the equation
\[ (d^2 J_{total}^\text{fourfold})_{z\bar{z}m\bar{n}} = \frac{i}{2} J_{source}^\text{source} \] (2.8)
since the fivebranes contribute to the stress-energy tensor. The equations of motion can be solved by taking $g_{m\bar{n}}$ to be a CY metric dependent on $z$ and piecing together neighborhoods covering the CY in which $\partial_m \partial_{\bar{n}} H = 0$. By replacing $X_3$ by its mirror $\tilde{X}_3$ in type IIB, we obtain the stringy cosmic string construction of generally noncompact CY fourfolds [10]. There are constraints on the number of strings, and it remains unclear whether a Ricci flat fourfold can be constructed for all $\tilde{X}_3$ (in particular ones where the complex structure degenerates at infinity).

In this paper we will restrict the analysis to the simplest case of a $T^2 \times T^2 \times T^2$ fibration over the $z$ plane in which each fiber degenerates at one or more points in the $z$
plane. Performing a T-duality along one cycle of a torus exchanges the Kahler parameter, \( \rho = B + i\sqrt{G} \), of the torus (\( B \) is the NS two-form, \( G \) is the determinant of the \( T^2 \) metric) with the complex structure parameter \( \tau \) of the torus. T-dualizing on one cycle of each torus yields a \( T^2 \times T^2 \times T^2 \) fibration over the \( z \) plane in which the \( \tau \) parameters are holomorphic functions of \( z \), and the \( \rho \) parameters are independent of \( z \). To avoid questions of global consistency due to orbifold points on the tori, we will study the local neighborhood of the degeneration where the three fibers are noncompact. The \( T^2 \times T^2 \) case corresponds to the conifold while a \( T^2 \) fibration that degenerates at one point corresponds to a Kaluza-Klein fivebrane and is nonsingular. There is additional freedom in the above solution to add various fluxes of background fields. The supergravity analysis is not usually valid in the limit where the number of fivebranes is small and the sources are delta functions, but this analysis has provided insight into the nature of the singularity. In the next section we will determine a noncompact CY fourfold corresponding to this degeneration.

3. Geometry of the Triple Intersection

The equations describing the local neighborhood of the degenerate fibers in the triple \( T^2 \) fibration can be written in the form

\[
z = a_1 a_2 = b_1 b_2 = c_1 c_2.
\]

(3.1)

The singularity can be expressed as the intersection of two quadrics in \( C^6 \).

\[
a_1 a_2 - b_1 b_2 = 0 \quad (3.2)
\]

\[
b_1 b_2 - c_1 c_2 = 0 \quad (3.3)
\]

Each equation by itself describes a conifold so the two equations represent the intersection of two conifolds. Doing a small resolution of each conifold (blowing up a \( P^1 \)) yields four phases—each conifold has two flops. There is one large resolution describing the deformations of intersecting \( S^3 \)'s. Additionally, there are six phases in which one conifold is blown up and the other is deformed. We will present each of these phases in detail and show that there are transitions in which the holomorphic four-form is preserved.
3.1. The Small Resolution

The small resolution is described by the following four equations for eight complex variables.

\[
\begin{align*}
  a_1 &= z_1 b_1 & b_2 &= z_1 a_2 \\
  b_1 &= z_2 c_1 & c_2 &= z_2 b_2
\end{align*}
\]  

(3.4)

The flops are generated by \(a_1 \leftrightarrow a_2\) and \(c_1 \leftrightarrow c_2\). Actually, there are many possible deformations of the small resolution when one takes into account deformations of equation (3.3). Fixing (3.2) by a linear transformation, there is still an \(SO(4, \mathbb{C})\) invariance. Equation (3.3) then has \(21 - 12 - 1 = 8\) complex deformations where we have subtracted an overall scale. This counting agrees with the fivebrane side. All of these deformations are nonnormalizable since the fivebrane is noncompact. We conjecture that there are no deformations of the small resolution preserving supersymmetry. In section four we will find evidence for this conjecture by obtaining a fivebrane manifold which looks very dissimilar to this small resolution but produces an equivalent theory.

This resolution is also described by a gauged linear sigma model with two \(U(1)'s\) and the following charges for six chiral fields \(x_i\) under the two \(U(1)'s\).

\[
\begin{align*}
  l_1 &= (1, 1, -1, -1, 0, 0) \\
  l_2 &= (0, 0, -1, -1, 1, 1)
\end{align*}
\]  

(3.5)

We can partly visualize the resolution in the following “toric” diagram that projects four dimensions onto a plane. Generically, the diagram depicts a \(T^4\) fibration that shrinks to a \(T^3\) on three-dimensional boundaries enclosed by three lines, a \(T^2\) on two-plane boundaries, an \(S^1\) on lines in the toric base, and a point at the intersection of four lines.
Diag. 1. Toric Diagram of the Small Resolution
Here, $t_1$ and $t_2$ are the Kahler parameters for the two $\mathbb{P}^1$’s. Note that there are other two-planes not labeled in the diagram.

3.2. The Large Resolution

To describe the large resolution we pick a nongeneric deformation of the moduli space that can be understood fairly easily and then argue that if we deform along a path that does not pass through singular regions, topological features of the manifold should remain invariant. Because the CY fourfold is noncompact, most of the parameters of the deformation are not normalizable. The nongeneric deformation that we choose is

\[
\begin{align*}
    a_1a_2 - b_1b_2 &= \mu \\
    b_1b_2 - c_1c_2 &= \nu
\end{align*}
\]  

(3.6)

with $\mu, \nu$ real and $\mu > \nu > 0$. The base of this manifold can be depicted schematically in the following diagram as a four-torus fibration.

Diag. 2. Base of the Large Resolution

One can describe the space as $T^*\mathbb{S}^1$ fibered over $T^*\mathbb{S}^3$ where the $T^*\mathbb{S}^1$ degenerates away from the base $\mathbb{S}^3$. The two independent $\mathbb{S}^3$’s are noncontractible with sizes determined by $\mu$ and $\nu$. The manifold is simply connected. Each $\mathbb{S}^3$ intersects the other two $\mathbb{S}^3$’s along a $T^2$. There is a third $\mathbb{S}^3$ with size parameter $\mu + \nu$ that intersects $a_1 = a_2 = 0$ and $c_1 = c_2 = 0$ but not $b_1 = b_2 = 0$ or anywhere else on the above diagram. The $\mathbb{S}^3$’s are not isolated since the $T^*\mathbb{S}^1$ direction is a zero mode direction. There are also two independent noncontractible four-cycles with topology $\mathbb{S}^3 \times \mathbb{S}^1$. The $\mathbb{S}^3 \times \mathbb{S}^1$ is homologous to a four-cycle where the $\mathbb{S}^1$ shrinks over a portion of the $\mathbb{S}^3$ at loci where the $T^*\mathbb{S}^1$ fiber degenerates, but there is no flat direction preserving the area of the $\mathbb{S}^3$ and connecting the
two four-cycles. Another picture of the base as a three-torus fibration is presented below with \( \mu \) real and \( \nu \to -i\nu \). (Angles are not drawn accurately.) In the diagram \( k = \frac{\mu - i\nu}{\sqrt{\mu^2 + \nu^2}} \).

The three edges are noncontractible four-cycles of topology \( S^3 \times S^1 \) bounding an open five-chain.

**Diag. 3.** Base of the Large Resolution (\( \mu \) real, \(-i\nu \) imaginary)

To deform the moduli space in a generic way we choose coordinates such that the form of the first equation is preserved. Writing the second equation as

\[
x_i G_{ij} x_j = \nu
\]

(3.7)

with \( x_1 = a_1, x_2 = a_2, x_3 = b_1, x_4 = b_2, x_5 = c_1, x_6 = c_2 \), and \( \nu \) complex, one can analyze the condition of transversality. For every symmetric, nonzero matrix \( G \), there will be a possible nontransverse intersection of the hypersurfaces if \( \det(G_{ij} - a\delta_{ij}\epsilon_j) = 0 \) where \( \epsilon_1 = \epsilon_2 = -\epsilon_3 = -\epsilon_4 = 1 \) and \( \epsilon_5 = \epsilon_6 = 0 \). Nontransversality implies that \( a = \frac{\nu}{\mu} \) for one of the roots \( a \). Obviously, this condition will not be a generic one. One can show that transversality implies that the holomorphic four-form has no zeroes or poles. The deformation should preserve the structure of three three-cycles that are topologically \( S^3 \) (two are independent) intersecting as described above. In terms of the original fivebranes, this deformation should correspond to varying the holomorphic four cycle on which one of the fivebranes is wrapped without compromising transversality. From this point of view there are nine complex parameters determining the deformation. At a first glance there are twenty-one complex parameters deforming the second equation where we fix \( \nu \). However, there is an \( SO(4, \mathbb{C}) \) group of symmetries preserving the undeformed equation so that we are left with nine parameters in agreement with the fivebrane count. There are also the modes \( \mu \) and \( \nu \) which control the sizes of the \( S^3 \)'s and will turn out not to be normalizable.
3.3. The Mixed Resolution

We are calling the mixed resolution the CY fourfold obtained by doing a small resolution on one of the two conifolds and a large resolution on the other. There are six isomorphic phases and we will describe one. The equations are as follows.

\[ a_1 = zb_1 \quad b_2 = za_2 \]
\[ b_1b_2 - c_1c_2 = \nu \]  \hspace{1cm} (3.8)

The base of this CY fourfold is a noncontractible five-cycle containing an \( S^2 \) and an \( S^3 \). We can visualize the base of the manifold in the following diagram where we take \( \nu \) real and positive. The horizontal edges are \( S^2 \times S^1 \) with the \( S^1 \) contracting at the opposite edge while the angled edges are \( S^3 \). Since the minimal \( S^2 \) is isolated at \( b_1 = 0 \) inside the five-cycle, the \( S^2 \times S^1 \) three-cycle, although homologous to the \( S^2 \) at \( c_1 = 0 \), is not connected to it by a flat direction that preserves the radius of the \( S^2 \). This cycle is not supersymmetric since it is not Lagrangian.

![Diagram](image)

Diag. 4. Base of the Mixed Resolution

The normal directions are \( b_1 - \bar{b}_2 \) and \( c_1 + \bar{c}_2 \) constrained by \( b_1b_2 - c_1c_2 = \nu \). There is a zero mode direction for the \( S^2 \) along \( b_1 = b_2 = 0 \) and for \( S^3 \) inside the five-cycle. In this phase the modes associated with the deformation \( \nu \) are massive because the four-cycle of topology \( S^3 \times S^1 \) shrinks to an \( S^3 \) at the poles of the \( S^2 \).

3.4. Geometric Transitions

To show that there are possible transitions between all of the phases that are described above, we note that there is a common set of coordinates for the small resolution, the
nongeneric large resolution, and the mixed resolution in which the holomorphic four-form is expressed as
\[
\Omega = \frac{da_1 \wedge db_1 \wedge db_2 \wedge dc_2}{a_1 c_2}.
\]
(3.9)

The transversality of the intersections for the various resolutions implies that \( \Omega \) has no zeroes or poles. The large resolution generic \( \Omega \) takes another form in general but can be continuously deformed to the above \( \Omega \) without encountering zeroes or poles. The existence of \( \Omega \) ensures that the holonomy is no larger group than \( SU(4) \). The mixed resolution can be described as a manifold with base \( O(-1) + O(-1) \to \mathbb{P}^1 \) and fiber \( T^*S^1 \) where the fiber degenerates along \( b_1 b_2 = \nu \) (see (3.8)). The large resolution can be described as a \( T^*S^1 \) fibration over \( T^*S^3 \) that degenerates along \( b_1 = 0 \) and \( b_2 = 0 \). The base in each case has a nonvanishing and covariantly constant holomorphic three-form. The pullback of this form to the CY fourfold must have a zero or pole. Otherwise, we could obtain a covariantly constant one-form by wedging with the antiholomorphic four-form and taking the Hodge star dual. This is impossible since the manifold is simply connected. The holonomy in all cases is, thus, \( SU(4) \). The dynamics of the possible transitions will be discussed in the next section.

4. Effective Theory of the Triple Intersection

4.1. The Small Resolution

The effective two-dimensional theory for the triple intersection of fivebranes in type IIA has a chiral \((0,4)\) supersymmetry. Similarly, type IIB on a CY fourfold has \((0,4)\) supersymmetry. We require that the spatial dimension of the two-dimensional theory be a circle so that fivebrane deformations in the fivebrane theory are normalizable and to avoid supergravity anomalies in the CY fourfold theory as discussed below. To determine the zero modes of the triple intersection, we first consider the case of a smooth fivebrane wrapping a four-cycle on a six torus as discussed in [11]. This analysis is mostly inapplicable when the six-torus is decompactified, and the intersection is localized. We do not have a rigorous derivation of the zero mode spectrum but argue that the self-intersecting fivebrane can be resolved to a smooth fivebrane wrapped on a \( \mathbb{P}^2 \) face of the manifold \( \mathbb{P}^3 \). Requiring that the first Chern class of the total space vanish, we obtain \( O(-4) \to \mathbb{P}^3 \). This resolution reduces the supersymmetry to \((0,2)\). The moduli space of \( \mathbb{P}^2 \)'s inside \( \mathbb{P}^3 \) is \( \mathbb{P}^3 \). We illustrate
the fivebrane resolution of the singularity as a tetrahedron in the following diagram. The
fivebrane is wrapped on the shaded face.

![Diagram 5: Small Fivebrane Resolution-Toric Diagram of $\mathbb{P}^3$](image)

**Diag. 5.** Small Fivebrane Resolution-Toric Diagram of $\mathbb{P}^3$

If we exclude deformations of the fivebrane that intersect the point labeled $x$, the
moduli space of deformations is $\mathbb{C}^3$. When the fivebrane reaches $x$, there is a “flop”
transition but no singularity. The moduli space of the fivebrane inside $\mathbb{P}^3$ encompasses all
of the flopped phases of the CY fourfold with no singularities. If we were to remove a $\mathbb{P}^2$
face on which the fivebrane is wrapped and wrap a fivebrane on the three other faces, we
would return to the singularity of the triply intersecting fivebrane in $\mathbb{C}^3$. The resolution
is a compactification in which the fivebrane flops to the compactifying face. Since this
manifold allows for fivebrane instantons, it is natural to conjecture that this transition
is mediated by a fivebrane instanton in $\mathbb{P}^3$. The transition is depicted in the following
diagram.

![Diagram 6: Instanton Transition from Triple Intersection to Smooth Fivebrane](image)

**Diag. 6.** Instanton Transition from Triple Intersection to Smooth Fivebrane

Additional arguments for selecting the above manifold are that the broken translational
invariance due to the intersection provides six real modes on $\mathbb{R}^6$ and one translational
mode for the compactified dimension of eleven-dimensional supergravity. Translations on
the complex plane acquire a mass because of the conical deficit. As we have seen in the supergravity analysis, the manifold with the fivebrane is not Kahler so there is no scale modulus, and the $\mathbb{P}^3$ cannot shrink. From the supergravity equations we also see that the scale of the $\mathbb{P}^3$ should be set by the number of fivebranes which in our case is one. The generator of the second homology class inside the $\mathbb{P}^2$ of the fivebrane has genus zero and positive self-intersection and is therefore self-dual. All together we have the following worldsheet theory for scalar bosons ($B$) and fermions ($F$)

\begin{align}
N^B_L &= 7 & N^B_R &= 8 \\
N^F_L &= 0 & N^F_R &= 8 \\
c_L &= 7 & c_R &= 12
\end{align}

(4.1)

where $R$ is for right movers and $L$ for left movers and $c$ is the central charge.

Now compare this result with the CY fourfold theory in the $\mathcal{O}(-1, -1) + \mathcal{O}(-1, -1) \to \mathbb{P}^1 \times \mathbb{P}^1$ phase. In two dimensions all moduli of the vacua must be dynamical variables. Supersymmetry constrains the number of left moving matter scalars to be a multiple of four. Reduction of the Kahler form and the two $B$ fields on the $\mathbb{P}^1$’s yields six nonchiral scalars. The Ramond-Ramond (RR) scalar provides another nonchiral scalar. Additionally, there is one antiself-dual four-cycle which provides a left moving scalar. Since the four-cycle is isolated, its self-intersection must be negative. Supersymmetry pairs the left movers with fermions. In the compact case the number of right moving fermions is determined by index theorems to be proportional to the number of three-cycles \cite{12}. We assume that if the small resolution is a limit of a compact CY fourfold with three-cycles that these modes are nonnormalizable in this limit. The dilaton goes into the supergravity multiplet which includes four right moving gravitinos and four left moving spin one-half fermions. In order to match the supersymmetry of the effective theory of the fivebrane resolution which is broken to $(0, 2)$ by the CY fourfold, we need to wrap a Dirichlet fivebrane around the four-cycle. The fivebrane also adds a $U(1)$ field to match the $U(1)$ from the membrane potential reduced on the $\mathbb{P}^1$ of $\mathbb{P}^2$. The moduli of the four-cycle (Kahler parameters) are zero modes in the two-dimensional effective theory of the fivebrane. Additionally, the RR fields couple to the fivebrane giving the same zero modes as for the CY fourfold. The scalar from the antiself-dual four-cycle is charged under the $U(1)$ of the fivebrane. The zero mode spectrum of the matter multiplets agrees with that found for the triple intersection except that left and right movers are exchanged. These zero modes would be precisely those of the heterotic string on a circle if seventeen right movers were not missing.
The conformal anomaly of this theory is the reduction of the fivebrane anomaly which is cancelled by bulk counterterms as has been shown in great detail by many authors. On the other hand type IIB on a compact CY fourfold is not anomalous as shown in \[12\]. Accordingly, the appropriate counterterms are not present for type IIB on a compact CY fourfold. The T-duality relation ensures that these terms are present to cancel the anomaly for our noncompact case. From the discussion we see that a compact CY fourfold cannot be constructed easily out of intersecting fivebranes because the conformal anomaly of the \((0, 4)\) theory generally requires a bulk counterterm. The supergravity multiplet in type IIA is not anomalous, but there is a two-dimensional anomaly for this multiplet in type IIB theory. The T-dual counterterms will not cancel this anomaly. Consistency requires that either the supergravity multiplet be decoupled or that there are further corrections to type IIB on a noncompact CY fourfold. As we mentioned above the two-dimensional theory is compactified on a circle so there is no supergravity anomaly. Threebrane instantons may smooth the flop transitions.

Note that the case of four triply intersecting fivebranes can be resolved by fivebrane instantons to a smooth fivebrane wrapped around a \(K3\) surface embedded in \(P^3\). Counting deformations and two-cycles we obtain 72 right moving scalars and fermions and 88 left movers. The amount of supersymmetry makes this theory likely equivalent to the heterotic string on the product of a two-torus and a noncompact CY threefold.

**Local Mirror Symmetry**

The resolution of the triple intersection singularity by a fivebrane instanton reduces the supersymmetry to \((0, 2)\). There is another resolution that preserves \((0, 4)\) supersymmetry and should be equivalent to the CY fourfold in the small resolution phase by local mirror symmetry and T-duality. Our discussion here has many gaps as we are not able to fully analyze this system. Despite these gaps this presentation may be useful for further analysis of this system. The techniques of local mirror symmetry applied to a gauged linear sigma model are discussed in \[13\]. Starting with the linear sigma model \((3.5)\), the mirror CY fourfold is obtained from the following equations.

\[
\begin{align*}
    z &= 1 + e^u + e^v + e^w + e^{v-u-t_1} + e^{v-w-t_2} \\
    z &= xy
\end{align*}
\]

(4.2)

The first equation is a noncompact holomorphic surface fibered over the \(z\) plane that degenerates at two points in the \(z\) plane for finite \(u, v, w\) and at \(z = 1\) for \(u, v, w \to -\infty\). The second equation is a \(\mathbb{C}^*\) fibration that degenerates at \(z = 0\). The total space has no
singularity for generic $t_1$ and $t_2$. At each degeneration of the holomorphic surface for finite $u$, $v$, and $w$, we expect but cannot prove that an $S^2$ shrinks. We can form two four-cycles of topology $S^4$ by connecting $z = 0$ with the two points where the holomorphic surface degenerates. A linear combination of these two cycles as well as a cycle joining $z = 0$ and $z = 1$ is expected to be mirror to the four-cycle, $P^1 \times P^1$, and antiself-dual. Mirror symmetry requires that this four-cycle be rigid as there should not be a compact two-cycle. It is not clear to me how this works. The mirror of the zero-cycle is expected to be noncompact.

The two $S^4$'s are combinations of cycles with Hodge type $(1,3)$ and $(3,1)$, and their deformations correspond to complex structure deformations. Generically, the deformations of middle-dimensional Lagrangian cycles are not normalizable but are logarithmically divergent. In two dimensions massless scalars have logarithmically divergent two-point functions so that this type of divergence is acceptable here. Each $S^4$ yields three nonchiral scalars in the effective two-dimensional theory from the complex structure deformations and the reduction of the RR four-form. The $(2,2)$ cycle mirror to $P^1 \times P^1$ gives a left moving scalar, and the RR scalar gives a nonchiral scalar. Supersymmetry generates left moving fermionic partners. With a lot of assumptions, the zero mode spectrum is the same as what we previously found.

By T-dualizing on the circle of the $C^*$ fibration, we obtain a type IIA fivebrane wrapped on the noncompact surface at $z = 0$.

$$1 + e^u + e^v + e^w + e^{v-u-t_1} + e^{v-w-t_2} = 0 \quad (4.3)$$

Introducing another coordinate and taking the coordinates to be projective, we can write this equation as

$$xyz + y^2z + wyz + yz^2 + \lambda_1 wxz + \lambda_2 wxy = 0. \quad (4.4)$$

If either or both $\lambda_1$ and $\lambda_2$ diverge, the singularity of the triply intersecting fivebrane is recovered. Otherwise, there are two isolated singularities when three of the coordinates are zero which are on the boundary of (4.3). The boundary is the intersection of $wxyz = 0$ and (4.4). We will not in this paper be able to determine the deformations and two-cycles of the noncompact four-cycle to analyze the effective two dimensional theory.
4.2. The Mixed Resolution

The phase where we separate one fivebrane from the other two corresponds to the mixed resolution. The effective theory at the intersection of two fivebranes is four-dimensional, and the matter content in this phase is nonchiral. There are four nonchiral translational modes moving the intersection of the two fivebranes in $\mathbb{R}^4$ as well as a supersymmetric completion of fermions. The supersymmetry breaking by the third fivebrane occurs at a distance from the intersection and only affects fields that interact with the bulk. The zero modes restricted to the intersection are unaffected by this breaking and are nonchiral. Going to the CY fourfold, we argue that the effective theory is four-dimensional and nonchiral. The $\mathbb{S}^2$ has four flat, normal spacetime directions, $T^*\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{R}$. The $T^*\mathbb{S}^1$ fibration degenerates, and the supersymmetry is broken in half at a distance of order $|\nu|^{\frac{1}{2}}$ from the $\mathbb{S}^2$. Zero modes which come from reducing the theory on $\mathbb{S}^2$ are not affected by this breaking since they do not interact with the bulk. We obtain a left-right symmetric combination of four scalar bosons and four fermions as well as a $U(1)$ gauge field from the $\mathbb{S}^3$. The zero mode scalars come from reducing the Kahler form, two $B$ fields, and RR four-form on $\mathbb{S}^2$. In the four dimensions of spacetime normal to the $\mathbb{S}^2$ ($T^*\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{R}$), the reduced RR four-form is dual to a scalar. For the effective four-dimensional theory we expect these modes to be nondynamical. If we “compactified” this theory on $T^*\mathbb{S}^1$, we would have a two-dimensional theory with the following zero mode spectrum.

\[
N_L^B = 4 \quad N_R^B = 4 \\
N_L^F = 4 \quad N_R^F = 4 \\
c_L = 6 \quad c_R = 6 \tag{4.5}
\]

We would actually have a continuous family of these theories labeled by $\nu$ which was what we found before we realized that the effective theories are four-dimensional.

The modes related to the complex parameter $\nu$ are massive because of the four-cycle that shrinks at the poles of the $\mathbb{S}^2$. Unless there is a five-form flux on the five-cycle, the four-form gauge field is pure gauge. The situation here is analogous to the magnetic field for a Dirac monopole outside of a two-sphere of radius $r$ in $\mathbb{R}^3$. If there is a magnetic field, its flux is quantized. The total energy of the field outside of the two-sphere is $\frac{g^2}{r}$ where $g$ is the magnetic charge unit. Our case is more complicated because there are two scales, the areas of the $\mathbb{S}^2$ and the $\mathbb{S}^3$, and there are three normal directions. One of these directions is compact. There is also the zero mode direction for the minimal $\mathbb{S}^2$ which potentially causes
a divergence in the energy. The minimal $S^2$ is a set of measure zero in the five-cycle, and to estimate the energy we assume there is no divergence. A rough estimate of the energy of the five-form field is to divide the volume of a minimal eight-ball surrounding the five-cycle by the area squared of the five-cycle. Without knowing the metric and taking the square of the radius to be the mean square radius of the $S^2$ and the $S^3$, $|\nu|^3 R$, we obtain an energy scaling as $\frac{1}{|\nu|}$. Assuming that this energy should be independent of the Kahler parameter uniquely determines that the energy scale as $\frac{1}{|\nu|}$. The zero modes of the Kahler parameter, the expectation value of the RR $B$ field on the $S^2$, and that of the RR four-form form a massive supermultiplet in the presence of five-form flux. Note that the mass diverges at the origin of the mixed resolution so that dynamical transitions into this phase when the $S^3$ shrinks are not possible. On the fivebrane side this flux should correspond to a magnetic field. Another argument to support the above hand waving is the following. If $\nu \rightarrow \infty$ what remains is the small resolution of the conifold with no flux, and this hypermultiplet should be massless. On the other hand, taking the Kahler parameter to infinity leaves the $S^3$ resolution of the conifold, and this hypermultiplet should be massive. There is another massive supermultiplet with mass proportional to $|\nu|^{3/2}$ from the threebrane wrapping $S^3$ as in the conifold [13]. The four massive modes are the $S^2 \times S^1$ directions and the global $U(1)$ charge mode.

4.3. The Large Resolution

The phase where all three fivebranes are separated corresponds to the large resolution. There are no normalizable translational zero modes in this phase. The effective theory for two fivebranes joined by a fourbrane is a four-dimensional theory with a decoupled $U(1)$ parametrized by the separation of the fivebranes in ten-dimensional spacetime and their difference in position in the compactified eleventh dimension. The two independent separations of the fivebranes and differences in positions along the compactified eleventh dimension will not be normalizable modes. One way to see this is that these modes correspond to complex parameter deformations of $S^3$ inside $T^*S^3$ which are logarithmically divergent.

On the CY fourfold side we will explain why the theory is trivial at low energies. There is a subtlety concerning the deformations of the four-cycles. The four-cycles are topologically $S^3 \times S^1$'s, and the parameter governing the size of the $S^3$ is distinct from that determining the size of $S^1$. Deforming an $S^3$ inside $T^*S^3$ would yield a logarithmically divergent four-dimensional mode and integrating over the zero mode $T^*S^1$ direction would
generate an additional divergence for a two-dimensional mode. We obtain no normalizable scalar modes from the four-cycles in this phase. To understand this phase we need to remember that each $S^3$ has a cylindrical zero mode direction along $T^*S^1$. In the four dimensions of spacetime $S^1 \times \mathbb{R} \times T^*S^1$ remaining after reduction on $S^3$, the RR scalar, the two $B$ fields, and the dilaton are on an equal footing since scalars and two-forms are dual. Again, one would expect these modes to be nondynamical in four dimensions. The same argument about supersymmetry breaking in the mixed resolution applies here. We should obtain no chiral fermions in this phase. “Compactifying” the theory on $T^*S^1$ would yield a trivial two-dimensional theory at low energies with no conformal anomaly. We summarize this below.

\[
\begin{align*}
N^B_L &= 0 & N^B_R &= 0 \\
N^F_L &= 0 & N^F_R &= 0 \\
c_L &= 0 & c_R &= 0
\end{align*}
\]  

(4.6)

As in the mixed resolution the puzzle that we appear to have a family of two-dimensional vacua parametrized by $\mu$ and $\nu$ is resolved by realizing that the effective theory is really four-dimensional here. There are two massive supermultiplets from threebranes wrapping the independent $S^3$’s. The zero modes are the global $U(1)$ and $T^*S^1 \times S^1$. These supermultiplets become massless at the singularity.

The theories we have found are not strictly two-dimensional because of the bulk counterterms needed for anomaly cancellation and the zero mode directions. The mixed and large resolutions are effectively four-dimensional and nonchiral. Any dynamical transition from the small resolution into the other phases seems unlikely. Wrapping a large number of fivebranes on $\mathbb{P}^1 \times \mathbb{P}^1$ would yield a large $N$ two-dimensional gauge theory without dynamics. On the other hand the mixed and large resolutions are analogous to the $S^2$ and $S^3$ resolutions of the conifold. It is natural to conjecture that there is a duality relating fivebranes wrapped on $S^2 \times T^*S^1 \times S^1 \times \mathbb{R}$ in the mixed resolution with threeform flux on one of the $S^3$’s in the large resolution since the effective theories are four-dimensional. By taking the radius of the other $S^3$ to infinity, one retrieves the two resolutions of the conifold. It might be interesting to explore further the equations for the fivebrane following from local mirror symmetry to have a purely geometric understanding of the fivebrane manifold in the mixed and large phases. Determining whether there is a compact manifold with these transitions would be interesting. Also, can one understand a triple intersection of fivebranes in type IIB similarly as giving an effective two-dimensional theory of
two (2, 2) supersymmetric vector multiplets in type IIA? The triple intersection theory is related to string theory black holes. In type IIA the number of intersecting fivebranes is limited by the conical deficit angle, but the number in eleven dimensional supergravity is unlimited. The various phases should still be present as the eleventh dimension opens up. There are many additional questions including nonperturbative terms in these theories left for further work.

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