Counting substructures and eigenvalues II: quadrilaterals

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Abstract: Let $G$ be a graph and $\lambda(G)$ be the spectral radius of $G$. A previous result due to Nikiforov [Linear Algebra Appl., 2009] in spectral graph theory asserted that every graph $G$ on $m \geq 10$ edges contains a 4-cycle if $\lambda(G) > \sqrt{m}$. Define $f(m)$ to be the minimum number of copies of 4-cycles in such a graph. A consequence of a recent theorem due to Zhai et al. [European J. Combin., 2021] shows that $f(m) = \Omega(m)$. In this article, by somewhat different techniques, we prove that $f(m) = \Theta(m^2)$. We left the solution to $\lim_{m \to \infty} \frac{f(m)}{m^2}$ as a problem, and also mention other ones for further study.

Keywords: Quadrilaterals; Spectral radius; Counting

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1 Introduction

This is the second paper of our project [19], which aims to study the relationship between copies of a given substructure and the eigenvalues of a graph. In this article, we study the supersaturation problem of 4-cycles under the eigenvalue condition.

The study of 4-cycles plays an important role in the history of extremal graph theory. The extremal number of $C_4$ (i.e., a 4-cycle), denoted by $ex(n, C_4)$, is defined to be the maximum number of edges in a graph which contains no 4-cycle as a subgraph. The study of $ex(n, C_4)$ can be at least dated back to Erdős [7] eighty years ago. A longstanding conjecture of Erdős and Simonovits [8] (see also [5, p. 84]) states that every graph on $n$ vertices and at least $ex(n, C_4) + 1$ edges contains at least two copies of 4-cycles when $n$ is large. Very recently, He, Ma and Yang [21]
announced this conjecture does not hold for the cases $n = q^2 + q + 2$ where $q = 4^k$ is large.

The original supersaturation problem of subgraphs in graphs focuses on the following function: for a given graph $H$ and for integers $n, t \geq 1$,

$$h_H(n, t) = \min \{ \#H : |V(G)| = n, |E(G)| = ex(n, H) + t \},$$

where $ex(n, H)$ is the Turán function of $H$. Establishing a conjecture of Erdős, Lovász and Simonovits [13] proved that $h_{C_3}(n, k) \geq k \lfloor \frac{n}{2} \rfloor$ for all $1 \leq k < \lfloor \frac{n}{2} \rfloor$. But He et al.’s result tells us $h_{C_4}(n, 1) = 1$ for some positive integers $n$. This means that supersaturation phenomenon of $C_4$ is quite different from the cases of triangles [13].

On the other hand, counting the copies of 4-cycles plays a heuristic important role in measuring the quasirandom-ness of a graph (see Chap. 9 in [1]).

As an important case of spectral Zarankiewicz problem, Nikiforov [15] proved that every $n$-vertex $C_4$-free graph satisfies that $\lambda(G) \leq \frac{1}{2} + \sqrt{n - \frac{3}{4}}$ where $\lambda(G)$ is the spectral radius of $G$, and the earlier bound of Babai and Guiduli [2] gives the correct order of the main term. As the counterpart of these results, we consider sufficient eigenvalue condition (in terms of the size of a graph) for the existence of 4-cycles. A pioneer result can be found in [14].

**Theorem 1** ([14]). Let $G$ be a graph with $m$ edges, where $m \geq 10$. If $\lambda(G) \geq \sqrt{m}$ then $G$ contains a 4-cycle, unless $G$ is a star (possibly with some isolated vertices).

Recently, Theorem 1 was extended by the following.

**Theorem 2** ([22]). Let $r$ be a positive integer and $G$ be a graph with $m$ edges where $m \geq 16r^2$. If $\lambda(G) \geq \sqrt{m}$, then $G$ contains a copy of $K_{2,r+1}$, unless $G$ is a star (possibly with some isolated vertices).

Let $B_r$ be an $r$-book, that is, the graph obtained from $K_{2,r}$ by adding one edge within the partition set of two vertices. Very recently, Nikiforov [17] proved that, if $m \geq (12r)^4$ and $\lambda(G) \geq \sqrt{m}$, then $G$ contains a copy of $B_{r+1}$, unless $G$ is a complete bipartite graph (possibly with some isolated vertices). This result further extends above two theorems and solves a conjecture proposed in [22].

The central topic of this article is the following spectral radius version of supersaturation problem of 4-cycles:

**Problem 1.** Let $f(m)$ be the minimum number of copies of 4-cycles over all labelled graph $G$ on $m$ edges with $\lambda(G) > \sqrt{m}$. Give an estimate of $f(m)$.

Till now, the only counting result related to Problem 1 is a consequence of Theorem 2. Note that $K_{2,r+1}$ contains $r(r+1)$ 4-cycles for $r = \frac{\sqrt{m}}{4}$. Theorem 2 implies that $f(m) \geq \frac{m^2}{32}$, unless $G$ is a star (possibly with some isolated vertices).

One may ask for the best answer to Problem 1. In this paper, we make the first progress towards this problem.
Theorem 3. Let $m \geq 3.6 \times 10^9$ be a positive integer. Then $f(m) \geq \frac{m^2}{2000}$, unless the graph $G$ is a star (possibly with some isolated vertices).

Throughout the left part, we also define $f(G)$ to be the number of copies of 4-cycles in a graph $G$.

Proposition 1. $f(m) \leq \frac{(m-1)(m-2\sqrt{m})}{8}$.

Proof. Let $s = \sqrt{m} + 1$ and $K_s^+$ be the graph obtained from the complete graph $K_s$ by adding $m - \binom{s}{4}$ pendent edges to one vertex of $K_s$. Clearly, $\lambda(K_s^+) \geq \lambda(K_s) = \sqrt{m}$. However, observe that $K_s^+$ contains $\binom{s}{4}$ copies of $K_4$ and every $K_4$ contains three copies of 4-cycles. Consequently, $f(K_s^+) = 3\binom{s}{4} = \frac{(m-1)(m-2\sqrt{m})}{8}$.

Together with Theorem 3 and Proposition 1, one can easily find that $f(m) = \Theta(m^2)$.

Let us introduce some necessary notation and terminologies. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex $u \in V(G)$, we denote by $N_G(u)$ the set of neighbors of $u$, and by $d_G(u)$ the degree of $u$. The symbol $G - v$ denotes the subgraph induced by $V(G)\{v\}$ in $G$.

The paper is organized as follows. In Section 2, we shall give some necessary preliminaries and prove a key lemma. We present a proof of our main theorem in Section 3. We conclude this article with one corollary and some open problems for further study.

## 2 Preliminaries

In this section, we introduce some lemmas, which will be used in the subsequent proof.

The first lemma is known as Cauchy’s Interlace Theorem.

Lemma 2.1 ([4]). Let $A$ be a symmetric $n \times n$ matrix and $B$ be an $r \times r$ principal submatrix of $A$ for some $r < n$. If the eigenvalues of $A$ are $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, and the eigenvalues of $B$ are $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r$, then $\lambda_i \geq \mu_i \geq \lambda_{i+n-r}$ for all $1 \leq i \leq r$.

The following inequality is due to Hofmeister.

Lemma 2.2 ([9]). Let $G$ be a graph of order $n$ and $M(G) = \sum_{u \in V(G)} d_G^2(u)$. Then

$$\lambda(G) \geq \sqrt{\frac{1}{n} M(G)},$$

(1)

with equality if and only if $G$ is either regular or bipartite semi-regular.
Lemma 2.3 ([12]). Let $G$ be a graph of order $n$ and size $m$. Then

$$f(G) = \frac{1}{8} \sum_{i=1}^{n} \lambda_i^4 + \frac{m}{4} - \frac{1}{4} M(G),$$

(2)

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $G$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

The following result is well-known [3,20]. A short proof can also be found in [18].

Lemma 2.4 ([3,18,20]). Let $G$ be a bipartite graph with $m$ edges, where $m \geq 1$. Then $\lambda(G) \leq \sqrt{m}$, with equality if and only if $G$ is a complete bipartite graph (possibly with some isolated vertices).

We need prove the last lemma. A proof of its one special case that $m \leq n - 2$ can be found in [10, Lemma 2.4].

Lemma 2.5. Let $G$ be a graph with $m$ edges. Then $M(G) \leq m^2 + m$.

Proof. Let $G$ be an extremal graph with the maximum $M(G)$ and $n := |G|$. Let $V(G) = \{u_1, \ldots, u_n\}$, and $d_i := d_G(u_i)$ for each $u_i \in V(G)$. We may assume that $d_1 \geq \cdots \geq d_n \geq 1$.

If there exists some integer $i \geq 2$ such that $u_i u_1 \notin E(G)$, then we choose a vertex $u_j \in N_G(u_i)$ and define $G' := G - u_i u_j + u_i u_1$. Now $d_{G'}(u_1) = d_1 + 1$, $d_{G'}(u_j) = d_j - 1$ and $d_{G'}(u_k) = d_G(u_k)$ for each $k \in \{2, \ldots, n\} \setminus \{j\}$. Consequently,

$$M(G') - M(G) = (d_1 + 1)^2 + (d_j - 1)^2 - d_1^2 - d_j^2 = 2d_1 - 2d_j + 2 \geq 2,$$

a contradiction. Thus, $N_G(u_1) = V(G) \setminus \{u_1\}$, and so $d_1 = n - 1$.

Now let $e(G - u_1)$ be the number of edges in $G - u_1$. Clearly, $e(G - u_1) = m - d_1$. If $e(G - u_1) = 0$, then $G \cong K_{1,m}$, and so $\sum_{i=1}^{n} d_i^2 = m^2 + m$, as desired. In the following, we assume $e(G - u_1) \geq 1$.

If $e(G - u_1) \leq d_1 - 2$, then $d_i + d_j \leq e(G - u_1) + 3 \leq d_1 + 1$ for each $u_i u_j \in E(G - u_1)$. Now let $G' = G - u_i u_j + u_i u_0$, where $u_i u_j \in E(G - u_1)$ and $u_0$ is a new vertex adjacent only to $u_1$ in $G'$. Then

$$M(G') - M(G) = (d_1 + 1)^2 + 1 + (d_i - 1)^2 + (d_j - 1)^2 - d_1^2 - d_i^2 - d_j^2 = 2(d_1 - d_i - d_j) + 4.$$

It follows that $M(G') > M(G)$, a contradiction. Therefore, $e(G - u_1) \geq d_1 - 1$.

Now let $e(G - u_1) = k$ and define a new graph $G'' := K_{1,d_1+k}$. Then $k \geq d_1 - 1$ and $e(G'') = d_1 + k = e(G) = m$. Note that $n = d_1 + 1$ and $2k = 2e(G - u_1) = \sum_{i=2}^{n} (d_i - 1)$. Hence, $2kd_1 \geq \sum_{i=2}^{n} d_i^2 - d_1^2 = M(G) - 2d_1^2$. It follows that

$$M(G') - M(G) = (k + d_1)^2 + (k + d_1) - M(G) \geq k^2 - d_1^2 + (k + d_1) \geq 0,$$

as $k \geq d_1 - 1$. Thus, $M(G) \leq M(G') = m^2 + m$. This proves Lemma 2.5.
3 Proof of Theorem 3

In this section, we give a proof of Theorem 3. We would like to point out that the techniques used in the left part are completely different from [19].

3.1 A key lemma

We first prove a key lemma.

**Lemma 3.1.** Let \( G \) be a graph of size \( m \geq 1.8 \times 10^9 \) and \( X \) be the Perron vector of \( G \) with component \( x_u \) corresponding to \( u \in V(G) \). If \( \lambda(G) \geq \sqrt{m} \) and \( x_u x_v > \frac{1}{9\sqrt{m}} \) for any \( uv \in E(G) \), then \( f(G) \geq \frac{m^2}{900} \) unless \( G \) is a star (possibly with some isolated vertices).

**Proof.** We may assume that \( \delta(G) \geq 1 \), where \( \delta(G) \) is the minimum degree of \( G \). Then \( G \) is connected (otherwise, we can find an edge \( uv \) with \( x_u x_v = 0 \)). By Perron-Frobenius theorem, \( X \) is a positive vector. Let \( A = \{ u \in V(G) : x_u > \frac{1}{3\sqrt{m}} \} \) and \( B = V(G) \setminus A \). Clearly, \( B \) is an independent set. Now suppose that \( f(G) < \frac{m^2}{900} \) and set \( \lambda := \lambda(G) \). We will prove a series of claims.

**Claim 3.1.** We have \( \lambda(G) \geq 2 \) unless \( G \cong K_{1,m} \).

**Proof.** Assume that there exists a vertex \( u \in V(G) \) with \( d_G(u) = 1 \) and \( N_G(u) = \{ u \} \). Then \( x_u x_u = \frac{x_u^2}{\lambda} \leq \frac{x_u^2}{\sqrt{m}} \). Since \( x_u x_u > \frac{1}{9\sqrt{m}} \), we have \( x_u > \frac{1}{3} \). Let \( u^* \in V(G) \) with \( x_{u^*} = \max_{v \in V(G)} x_v \). Then \( x_{u^*} > \frac{1}{3} \).

Now let \( S := N_G(u^*) \), \( T := V(G) \setminus (S \cup \{ u^* \}) \), and \( N_S(v) = N_G(v) \cap S \) for any \( v \in V(G) \). Moreover, we partite \( S \) into three subsets \( S_1, S_2 \) and \( S_3 \), where \( S_1 = \{ v : \frac{1}{6} < x_v \leq x_{u^*} \} \), \( S_2 = \{ v : \frac{1}{6} < x_v \leq \frac{1}{2} \} \), and \( S_3 = \{ v : 0 < x_v \leq \frac{1}{6} \} \).

Choose a vertex \( u \in S_1 \) arbitrarily. By Cauchy-Schwarz inequality,

\[
(\lambda x_u)^2 = \left( \sum_{v \in N_G(u)} x_v \right)^2 \leq d_G(u) \sum_{v \in N_G(u)} x_v^2 \leq d_G(u)(1 - x_u^2). \tag{3}
\]

Since \( x_u > \frac{1}{4} \) and \( \lambda \geq \sqrt{m} \), we have \( d_G(u) \geq \frac{m}{15} \). If \( |N_T(u)| \leq \frac{m}{450} \), then \( |N_S(u)| \geq \frac{m}{15} - \frac{m}{450} - 1 \geq \frac{m}{15\lambda} \), and thus \( G \) contains a copy of \( K_{2,\lfloor \frac{m}{15\lambda} \rfloor} \). Hence, \( G \) contains at least \( \left( \frac{m}{2\sqrt{m}} \right) \) quadrilaterals, a contradiction. Therefore, \( |N_T(u)| \geq \frac{m}{450} \) and \( |N_S(u)| < \frac{m}{15\lambda} \). Now let \( S^* = \{ v \in S : x_v < \frac{1}{108} \} \), \( T^* = \{ v \in T : x_v < \frac{1}{108} \} \) and \( V' = (S \setminus S^*) \cup (T \setminus T^*) \). Since \( X \) is a unit vector, we have \( |V'| \leq 108^2 \). By Cauchy-Schwarz inequality,

\[
\sum_{v \in V'} x_v \leq \sqrt{|V'|} \sum_{v \in V'} x_v^2 \leq \sqrt{|V'|} \leq 108. \tag{4}
\]

Consequently,

\[
\sum_{v \in N_S(u)} x_v = \sum_{v \in N_{S \setminus S^*}(u)} x_v + \sum_{v \in N_{S^*}(u)} x_v \leq 108 + \frac{1}{108}|N_{S^*}(u)|.
\]
Recall that $|N_{S^*}(u)| \leq |N_S(u)| \leq \frac{m}{156}$ and $x_u^* > \frac{1}{3}$. It follows that
\[
\sum_{v \in N_S(u)} x_v \leq 324 + \frac{1}{36} |N_{S^*}(u)| x_u^* < \frac{1}{36} \cdot \frac{m}{15} x_u^*.
\]

On the other hand, note that $|N_{T^*}(u)| \geq |N_T(u)| - 108^2 \geq \frac{m}{525}$ and $x_v < \frac{1}{108} < \frac{1}{36} x_u^*$ for any $v \in T^*$. Then
\[
\sum_{v \in N_T(u)} x_v < \sum_{v \in N_{T \setminus T^*}(u)} x_u^* + \sum_{v \in N_{T^*}(u)} \frac{1}{36} x_u^* \leq |N_T(u)| x_u^* - \frac{35}{36} \cdot \frac{m}{525} x_u^* = |N_T(u)| x_u^* - \frac{1}{36} \cdot \frac{m}{15} x_u^*.
\]

It follows that $\sum_{v \in N_{S \cup T}(u)} x_v < |N_T(u)| x_u^*$. Let $e(S, T)$ be the number of edges from $S$ to $T$, and $e(S)$ be the number of edges within $S$. Then
\[
\sum_{u \in S_1} \sum_{v \in N_{S \cup T}(u)} x_v < e(S_1, T) x_u^*.
\]

Secondly, consider a vertex $u \in S_2$ arbitrarily. Note that $x_u > \frac{1}{12}$ and $\lambda \geq \sqrt{m}$. Then (3) gives $d_G(u) \geq \frac{m}{30}$. Since $S^* \subseteq S_3$ and $x_u^* - x_u > \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$, we have
\[
\sum_{v \in N_{S^*}(u)} x_v \leq \frac{1}{108} |N_{S_3}(u)| \leq \frac{1}{9} |N_{S_3}(u)| (x_u^* - x_u),
\]
and by (4) we have $\sum_{v \in N_{S^*}(u)} x_v \leq \sum_{v \in V'} x_v \leq 108$. Then
\[
\sum_{v \in N_S(u)} x_v = \sum_{v \in N_{S \setminus S^*}(u)} x_v + \sum_{v \in N_{S^*}(u)} x_v \leq 108 + \frac{1}{9} |N_{S_3}(u)| (x_u^* - x_u).
\]

If $|N_{S^*}(u)| \geq \frac{m}{72}$, then $|N_{S_3}(u)| \geq |N_{S^*}(u)| \geq \frac{m}{72}$. Since $x_u^* - x_u > \frac{1}{12}$, it follows from (6) that $\sum_{v \in N_S(u)} x_v < |N_{S_3}(u)| (x_u^* - x_u)$, and thus
\[
\sum_{v \in N_{S \cup T}(u)} x_v < |N_{S_3}(u)| (x_u^* - x_u) + |N_T(u)| x_u^*. \tag{7}
\]

If $|N_{S^*}(u)| \leq \frac{m}{72}$, then
\[
|N_{T^*}(u)| \geq d_G(u) - |N_{S^*}(u)| - 108^2 > \frac{m}{72}.
\]

Hence,
\[
\sum_{v \in N_{T^*}(u)} x_v \leq |N_{T^*}(u)| \cdot \frac{1}{108} < |N_{T^*}(u)| x_u^* - 108,
\]
as $x_u^* > \frac{1}{3}$. It follows that $\sum_{v \in N_{T^*}(u)} x_v < |N_T(u)| x_u^* - 108$. Combining with (6), we can also obtain (7). Therefore, in both cases we have
\[
\sum_{u \in S_2} \sum_{v \in N_{S \cup T}(u)} x_v < e(S_2, S_3) x_u^* + e(S_2, T) x_u^* - \sum_{u \in S_2} |N_{S_3}(u)| x_u. \tag{8}
\]
Thirdly, we consider an arbitrary vertex \( u \in S_3 \). Since \( x_u > \frac{1}{3} \), we have \( x_v \leq \frac{1}{6} < \frac{1}{2} x_u \) for each \( v \in N_{S_3}(u) \). Thus, \( \sum_{u \in S_3} \sum_{v \in N_{S_3}(u)} x_v \leq e(S_3) x_u \), with equality if and only if \( e(S_3) = 0 \). Therefore,

\[ \sum_{u \in S_3} \sum_{v \in N_{S_3,T}(u)} x_v \leq e(S_3, S_1) x_u + e(S_3) x_u + e(S_3, T) x_u + \sum_{u \in S_3} \sum_{v \in N_{S_3}(u)} x_v. \tag{9} \]

Notice that

\[ \sum_{u \in S_2} |N_{S_3}(u)| x_u = \sum_{u \in S_3} \sum_{v \in N_{S_2}(u)} x_v. \]

Combining with (5), (8) and (9), we have

\[ \sum_{u \in S} \sum_{v \in N_{S_i,T}(u)} x_v \leq (e(S) + e(S, T)) x_u, \tag{10} \]

where if equality holds then \( S_1 \cup S_2 = \emptyset \) and \( e(S_3) = 0 \), that is, \( e(S) = 0 \). Furthermore, we can see that

\[ \lambda x_u = \sum_{u \in S} \sum_{v \in N_{G}(u)} x_v = |S| x_u + \sum_{u \in S} \sum_{v \in N_{S_i,T}(u)} x_v \leq (|S| + e(S) + e(S, T)) x_u \leq m x_u. \]

Since \( \lambda \geq \sqrt{m} \), the above inequality holds in equality, that is, \( \lambda = \sqrt{m} \). Therefore, \( m = |S| + e(S) + e(S, T) \), and (10) holds in equality (hence \( e(S) = 0 \)). This implies that \( G \) is a complete bipartite graph. By Lemma 2.4, \( G \) is a complete bipartite graph. Since \( f(G) < \frac{m^2}{500} \), \( G \) can only be a star. This completes the proof. \( \square \)

In the following, we may assume that \( G \not\cong K_{1,m} \). Then by Claim 3.1, \( \delta(G) \geq 2 \).

Claim 3.2. \( |A| \leq 9 \sqrt{m}. \)

Proof. Recall that \( x_u > \frac{1}{2 \sqrt{m}} \) for each \( u \in A \). Thus \( \sum_{u \in A} x_u > \frac{|A|}{9 \sqrt{m}} \), and hence \( |A| \leq 9 \sqrt{m} \). \( \sum_{u \in A} x_u^2 \leq 9 \sqrt{m}. \) \( \square \)

Claim 3.3. Let \( |G| = \frac{m}{2} + b \). Then \(-\frac{m}{125} \leq b \leq |A| \).

Proof. Set \( \lambda := \lambda_{|G|} \). Note that \( \lambda \geq \sqrt{m} \). By Lemmas 2.2 and 2.3,

\[ f(G) \geq \frac{1}{8}(\lambda^4 + \lambda^4) - \frac{1}{4} M(G) \geq \frac{1}{8}(\lambda^4 + \lambda^4) - \left| \frac{|G|}{4} \right| \geq \frac{1}{8} \lambda^4 - \frac{b}{4} \lambda^2. \tag{11} \]

If \( b < -\frac{m}{125} \), then

\[ f(G) \geq -\frac{b}{4} \lambda^2 \geq \frac{m}{500} \lambda^2 \geq \frac{m^2}{500}, \]

a contradiction. Thus, \( b \geq -\frac{m}{125} \).

On the other hand, recall that \( e(B) = 0 \) and \( \delta(G) \geq 2 \), then

\[ m \geq e(B, A) \geq 2|B| = 2(|G| - |A|) = 2\left(\frac{m}{2} + b - |A|\right). \]

Thus, \( b \leq |A| \), as desired. \( \square \)
Claim 3.4. $\Delta(G) \leq \frac{2}{15} m$, where $\Delta(G)$ is the maximum degree of $G$.

**Proof.** We know that $\sum_{i=1}^{\lfloor |G| \rfloor} \lambda_i^2 = 2m$. Thus, $\lambda^2 = \lambda_i^2 \leq 2m$. Combining with (11) and $b \leq |A| \leq 9\sqrt{m}$, we have

$$f(G) \geq \frac{1}{8} \lambda^4 - \frac{b}{4} \lambda^2 \geq \frac{1}{8} \lambda^4 - 9m^\frac{3}{2}. \quad (12)$$

Now if there exists some $u \in V(G)$ with $d_G(u) > \frac{2}{15} m$, then

$$|N_B(u)| \geq d_G(u) - |A| > \frac{2}{15} m - 9\sqrt{m}.$$ 

Since $e(B) = 0$, $G$ contains $K_{1,|N_B(u)|}$ as an induced subgraph. By Lemma 2.1,

$$\lambda' \leq -\sqrt{|N_B(u)|} < -\sqrt{\frac{2}{15} m - 9\sqrt{m}},$$

and by (12) we have

$$f(G) \geq \frac{1}{8} \lambda^4 - 9m^\frac{3}{2} \geq \frac{1}{8} \left(\frac{2}{15} m - 9\sqrt{m}\right)^2 - 9m^\frac{3}{2} > \frac{m^2}{500},$$

for $m \geq 1.8 \times 10^9$. We have a contradiction. Therefore, $\Delta(G) \leq \frac{2}{15} m$. \(\square\)

Claim 3.5. Let $B^* = \{u \in V(G) : d_G(u) = 2\}$. Then $B^* \subseteq B$ and

$$\frac{m}{2} + 3(b - |A|) \leq |B^*| \leq \frac{m}{2}. \quad (13)$$

**Proof.** Let $u \in B^*$ and $N_G(u) = \{u_1, u_2\}$. Then $\lambda x_u = x_{u_1} + x_{u_2} \leq 2$. Since $\lambda \geq \sqrt{m}$, we have $x_u \leq \sqrt{m} < \frac{1}{3} \sqrt{m}$, and so $u \in B$.

Recall that $e(B) = 0$. Thus, $e(B^*) = 0$, and $m \geq e(B^*, A) \geq 2|B^*|$. This gives $|B^*| \leq \frac{m}{2}$. On the other hand, note that $|B| = |G| - |A| = \frac{m}{2} + b - |A|$, then

$$m \geq e(B, A) \geq 2|B^*| + 3(|B| - |B^*|) = \frac{3}{2} m + 3(b - |A|) - |B^*|.$$ 

It follows that $|B^*| \geq \frac{m}{2} + 3(b - |A|)$. \(\square\)

Claim 3.6. Let $A^* = \{v \in N_G(u) : u \in B^*\}$. Then $A^* \subseteq A$ and $|A^*| \leq 24$.

**Proof.** Since $e(B) = 0$, we have $N_G(u) \subseteq A$ for any $u \in B^*$. Thus, $A^* \subseteq A$. Furthermore, we will see that $\frac{1}{20} < x_v^2 \leq \frac{2}{17}$ for each $v \in A^*$.

Let $v$ be an arbitrary vertex in $A^*$. By Cauchy-Schwarz inequality,

$$(\lambda x_v)^2 = \left( \sum_{u \in N_G(v)} x_u \right)^2 \leq d_G(v) \sum_{u \in N_G(v)} x_u^2 \leq d_G(v)(1 - x_v^2) \leq \frac{2}{15} m(1 - x_v^2),$$

as $\Delta(G) \leq \frac{2}{15} m$. Since $\lambda \geq \sqrt{m}$, we have $x_v^2 \leq \frac{2}{15}$. 


If there exists a vertex \( v \in A^* \) with \( x_v^2 \leq \frac{1}{25} \), then by the definition of \( A^* \), we can find a vertex \( u \in N_{B^*}(v) \). Clearly,
\[
\lambda x_u \leq x_v + \sqrt{\frac{2}{17}} \leq \frac{1}{5} + \sqrt{\frac{2}{17}} < \frac{5}{9},
\]
Consequently,
\[
x_u x_v < \frac{1}{\lambda} \cdot \frac{5}{9} \cdot \frac{1}{5} \leq \frac{1}{9/\sqrt{m}},
\]
which contradicts the condition of Lemma 3.1. Therefore, \( x_v^2 > \frac{1}{25} \) for any \( v \in A^* \), and so \( |A^*| \leq 24 \).

**Claim 3.7.** Let \( V'' := (A \setminus A^*) \cup (B \setminus B^*) \). Then \( |V''| \leq \frac{m}{60} \) and \( e(V'') \leq \frac{m}{20} \).

**Proof.** Recall that \( |A \cup B| = |G| = \frac{m}{2} + b \). Combining with (13), we obtain that \( |V''| \leq |G| - |B^*| \leq 3|A| - 2b \). Moreover, by Claims 3.2 and 3.3, we have \( |A| \leq 9/\sqrt{m} \) and \( b \geq -\frac{m}{125} \). Thus, \( |V''| \leq 27/\sqrt{m} + \frac{2}{125}m \leq \frac{m}{60} \).

Now we estimate \( e(V'') \). Again by \( |A| \leq 9/\sqrt{m} \), \( b \geq -\frac{m}{125} \) and (13), we have
\[
e(A^*, B^*) = 2|B^*| \geq m + 6(b - |A|) \geq m - \frac{6}{125}m - 54/\sqrt{m}.
\]
It follows that \( e(V'') \leq m - e(A^*, B^*) \leq \frac{6}{125}m + 54/\sqrt{m} \leq \frac{m}{20} \).

Now we give the final proof of Lemma 3.1. For convenience, let \( d'(u) := |N_{V''}(u)| \) for each \( u \in V'' \). Note that \( e(V'', B^*) = 0 \). Thus by Claim 3.6,
\[
d_G(u) \leq d'(u) + |A^*| \leq d'(u) + 24
\]
for each vertex \( u \in V'' \). Consequently,
\[
\sum_{u \in V''} d_G^2(u) \leq \sum_{u \in V''} (d'(u) + 24)^2 = 96e(V'') + 24^2|V''| + \sum_{u \in V''} d'^2(u). \tag{14}
\]
Since \( e(V'') \leq \frac{m}{20} \), by Lemma 2.5 we have \( \sum_{u \in V''} d'^2(u) \leq \frac{m^2}{400} + \frac{m}{20} \). Combining this with Claim 3.7 and (14), we have
\[
\sum_{u \in V''} d_G^2(u) \leq 96 \cdot \frac{m}{20} + 24^2 \cdot \frac{m}{60} + \frac{m^2}{400} + \frac{m}{20} < \frac{m^2}{225}. \tag{15}
\]
On the other hand, by Claim 3.4, \( \Delta(G) \leq \frac{2}{15}m \), and so
\[
\sum_{u \in A^*} d_G^2(u) \leq |A^*|\Delta(G)^2 \leq \frac{96}{225}m^2
\]
(as \( |A^*| \leq 24 \)). Moreover, by Claim 3.5 \( |B^*| \leq \frac{m}{2} \), and thus
\[
\sum_{u \in B^*} d_G^2(u) = 4|B^*| \leq 2m.
\]
Combining with (15), we get
\[ M(G) = \sum_{u \in V'' \cup A^* \cup B^*} d^2_G(u) \leq \frac{1}{225}m^2 + \frac{96}{225}m^2 + 2m < \frac{100}{225}m^2 = \frac{4}{9}m^2. \]

Now by Lemma 2.3, we have
\[ f(G) \geq \frac{1}{8} \lambda^4 - \frac{1}{4} M(G) \geq \frac{1}{8}m^2 - \frac{1}{9}m^2 = \frac{1}{72}m^2 > \frac{1}{500}m^2, \]
a contradiction. This completes the proof. \( \square \)

### 3.2 Nikiforov’s deleting small eigenvalue edge method

Over the past decades, Nikiforov developed some novel tools and techniques for solving problems in spectral graph theory (see [16]). One is the method we called “deleting small eigenvalue edge method”, or “The DSEE Method”. Generally speaking, an edge \( xy \in E(G) \) is called a small eigenvalue edge, if \( x_u x_v \) is small where \( x_u, x_v \) are Perron components.

By using this method, Nikiforov [14] successfully proved the following results, of which some original ideas appeared in [16] earlier:

- Every graph on \( m \) edges contains a 4-cycle if \( \lambda(G) \geq \sqrt{m} \) and \( m \geq 10 \), unless it is a star with possibly some isolated vertices (see Claim 4 in [14, pp. 2903]);

- Every graph on \( m \) edges satisfies that the booksize \( bk(G) > \frac{\sqrt{m}}{12} \) if \( \lambda(G) \geq \sqrt{m} \), unless it is a complete bipartite graph with possibly some isolated vertices (see [17], this confirmed a conjecture in [22]).

One main ingredient in the proof of Theorem 3 is using this method.

### 3.3 Proof of Theorem 3

Now we are ready to give the proof of Theorem 3.

**Proof of Theorem 3.** Let \( G \) be a graph with \( e(G) = m \) and \( \lambda(G) \geq \sqrt{m} \). By using the Nikiforov DESS Method [17], we first construct a sequence of graphs.

(i) Set \( i := 0 \) and \( G_0 := G \).

(ii) If \( i = \lfloor \frac{m}{2} \rfloor \), stop.

(iii) Let \( X = (x_1, x_2, \ldots, x_{|G_i|})^T \) be the Perron vector of \( G_i \).

(iv) If there exists \( uv \in E(G_i) \) with \( x_u x_v \leq \frac{1}{9\sqrt{e(G_i)}} \), set \( G_{i+1} := G_i - uv \) and \( i := i + 1 \).

(v) If there is no such edge, stop.

Assume that \( G_k \) is the resulting graph of the graph sequence constructed by the above algorithm. Then \( k \leq \lfloor \frac{m}{2} \rfloor \). We can obtain the following two claims.
Claim 3.8. $\lambda(G_{i+1}) \geq \sqrt{m-i-1}$ for each $i \in \{0, 1, \ldots, k-1\}$.

Proof. Let $X$ be the Perron vector of $G_i$ with component $x_u$ corresponding to $u \in V(G_i)$. Then, there exists $uv \in E(G_i)$ with $x_u x_v \leq \frac{1}{9 \sqrt{e(G_i)}}$. Thus,

$$\lambda(G_{i+1}) \geq X^T A(G_{i+1}) X = X^T A(G_i) X - 2 x_u x_v \geq \lambda(G_i) - \frac{2}{9 \sqrt{e(G_i)}}.$$ 

Hence,

$$\lambda(G_0) \leq \lambda(G_1) + \frac{2}{9 \sqrt{m}} \leq \cdots \leq \lambda(G_{i+1}) + \sum_{j=0}^{i} \frac{2}{9 \sqrt{m-j}}.$$ 

It follows that

$$\lambda(G_{i+1}) \geq \lambda(G_0) - \frac{2(i+1)}{9 \sqrt{m-i-1}} \geq \sqrt{m} - \frac{2(i+1)}{9 \sqrt{m-i-1}} \quad (16)$$

This implies that $\lambda(G_{i+1}) \geq \sqrt{m-i-1}$, as $i+1 \leq k \leq \lfloor \frac{m}{2} \rfloor$. □

Now we may assume that all isolated vertices are removed from each $G_i$, where $i \in \{0, 1, \ldots, k\}$.

Claim 3.9. $G_k$ cannot be a star unless $G_k = G_0 \cong K_{1,m}$.

Proof. Suppose to the contrary that $k \geq 1$ while $G_k$ is a star. Since $e(G_k) = m-k$, we have $G_k \cong K_{1,m-k}$. Let $u_0$ be the central vertex of $G_k$ and $u_1, \ldots, u_{m-k}$ be the leaves. We now let $G_k = G_{k-1} - uv$ and $X$ be the Perron vector of $G_{k-1}$.

If $uv$ is a pendent edge incident to $u_0$, say $uv = u_0 u_{m-k+1}$, then

$$\lambda(G_{k-1}) = \sqrt{e(G_{k-1})} = \sqrt{m-k+1}$$

and $\lambda(G_{k-1}) x_{u_i} = x_{u_0}$ for $i \in \{1, 2, \ldots, m-k+1\}$. Hence, $\|X\|_2 = \sum_{i=0}^{m-k+1} x_{u_i}^2 = 2 x_{u_0}^2$, which gives $x_{u_0}^2 = \frac{1}{2}$. It follows that

$$x_{u_0} x_{u_{m-k+1}} = \frac{x_{u_0}^2}{\sqrt{e(G_{k-1})}} > \frac{1}{9 \sqrt{e(G_{k-1})}},$$

which contradicts the definition of $G_k$.

If $uv$ is an isolated edge or a pendent edge not incident to $u_0$, then $G_{k-1}$ is bipartite but not complete bipartite. By Lemma 2.4, $\lambda(G_{k-1}) < \sqrt{e(G_{k-1})}$, which contradicts Claim 3.8.

Now we conclude that $uv$ is an edge within $V(G_k) \setminus \{u_0\}$, say $uv = u_1 u_2$, then $x_{u_1} = x_{u_2}$ and $\lambda(G_{k-1}) x_{u_1} = x_{u_0} + x_{u_2}$. Hence, $x_{u_1} = \frac{x_{u_0}}{\lambda(G_{k-1})} < \frac{1}{2} x_{u_0}$, as $\lambda(G_{k-1}) \geq \sqrt{m-k+1}$ by Claim 3.8. Consequently,

$$\lambda^2(G_{k-1}) x_{u_0} = \sum_{i=1}^{m-k} \lambda(G_{k-1}) x_{u_i} = (m-k) x_{u_0} + (x_{u_1} + x_{u_2}) < (m-k+1) x_{u_0},$$

It follows that $\lambda(G_{k-1}) < \sqrt{m-k+1}$, which also contradicts Claim 3.8. □
Now we finish the final proof of Theorem 3. Assume that $G$ is not a star. Then $G_k$ is not a star by Claim 3.9; moreover, $\lambda(G_k) \geq \sqrt{m - k} = \sqrt{e(G_k)}$ by Claim 3.8. If $k < \lfloor \frac{m}{2} \rfloor$, then $x_u x_v > \frac{1}{9 \sqrt{e(G_k)}}$ for any edge $uv \in E(G_k)$. Since $e(G_k) = m - k > \frac{m}{2}$, by Lemma 3.1 $f(G_k) \geq \frac{(e(G_k))^2}{500} > \frac{m^2}{2000}$, and so $f(G) > \frac{m^2}{2000}$, as desired.

If $k = \lfloor \frac{m}{2} \rfloor$, then by (16) we have

$$\lambda(G_k) \geq \sqrt{m - \frac{2k}{9 \sqrt{m - k}}} \geq \sqrt{m - \frac{m}{9 \sqrt{\frac{m}{2}}}} = \left(1 - \frac{\sqrt{2}}{9}\right) \sqrt{m},$$

and so

$$\lambda^4(G_k) \geq (1 - \frac{\sqrt{2}}{9})^4 m^2 = 0.5047 m^2 > 0.504 m^2 + 4m.$$

On the other hand, by Lemma 2.5,

$$M(G_k) \leq (e(G_k))^2 + e(G_k) = \left\lfloor \frac{m}{2} \right\rfloor^2 + \left\lfloor \frac{m}{2} \right\rfloor \leq 0.25 m^2 + 2m.$$

Thus by Lemma 2.3,

$$f(G_k) \geq \frac{1}{8} \lambda^4(G_k) - \frac{1}{4} M(G_k) > \frac{1}{8} (0.504 - 0.5) m^2 = \frac{1}{2000} m^2,$$

and so $f(G) > \frac{m^2}{2000}$. This completes the proof. 

\section{Concluding remarks}

We do not try our best to optimize the constant "$\frac{1}{2000}$" in Theorem 3. So it is natural to pose the following problem:

\textbf{Problem 2.} Determine $\lim_{m \to \infty} \frac{f(m)}{m^2}$. (We think that the upper bound in Proposition 1 is close to the truth.)

By Theorem 3 and an inequality $\lambda(G) \geq \frac{2m}{n}$ due to Collatz and Sinogowitz [6], we deduce the following.

\textbf{Theorem 4.} Let $G$ be a graph on $n$ vertices and $m$ edges. If $m > \max\{\frac{n^2}{4}, 3.6 \times 10^9\}$, then $G$ contains $\frac{n^4}{32000}$ copies of 4-cycles.

On the other hand, we would like to mention the following conjecture.

\textbf{Conjecture 4.1} (Conjecture 5.1 in [22]). Let $k \geq 2$ be a fixed positive integer and $G$ be a graph of sufficiently large size $m$ without isolated vertices. If $\lambda(G) \geq k - 1 + \sqrt{\frac{4m - k^2 + 1}{2}}$, then $G$ contains a cycle of length $t$ for every $t \leq 2k + 2$, unless $G = S_{\frac{m}{k + 1}}$.

When $k = 1$, the above conjecture reduces to Nikiforov’s result (Theorem 1). Let $B_r,k$ be the join of an $r$-clique with an independent set of size $k$. We conclude this note with a new conjecture appeared in [11] which extends Theorem 1.
**Conjecture 4.2** (Conjecture 1.20 in [11]). *Let m be large enough and G be a $B_{r,k}$-free graph with m edges. Then $\lambda(G) \leq (1 - \frac{1}{r^2})2m$, with equality if and only if G is a complete bipartite graph for $r = 2$, and G is a complete regular r-partite graph for $r \geq 3$ with possibly some isolated vertices.*

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