Quantitative \((p, q)\) theorems in combinatorial geometry

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Abstract

We show quantitative versions of classic results in discrete geometry, where we require in the conclusion to find sets which contain either many points from a given discrete set or to have large volume. We give versions of this kind for the classic first selection lemma of Bárány, the existence of weak epsilon-nets for convex sets and the \((p, q)\) theorem of Alon and Kleitman.

1 Introduction

Helly’s theorem is one of the central results regarding the intersection structure of convex sets. It says that a finite family of convex sets in \(\mathbb{R}^d\) is intersecting if and only if every subfamily of size \(d+1\) is intersecting [Hel23]. Among the many generalisations and extensions of Helly’s theorem, a crowning achievement of combinatorial convexity is the proof of the \((p, q)\) theorem by Alon and Kleitman, answering positively a conjecture by Hadwiger and Debrunner [HD57].

Theorem (N. Alon and D. J. Kleitman [AK92]). Given integers \(p \geq q \geq d+1\), there is a constant \(c = c(p, q, d)\) such that the following statement holds. For every finite family \(\mathcal{F}\) of convex sets in \(\mathbb{R}^d\) such that out of every \(p\) elements of \(\mathcal{F}\) there are \(q\) which are intersecting, there is a set \(K\) of \(c\) points that intersects every element in \(\mathcal{F}\).

Recently, there have been developments regarding versions of Helly’s theorem which work if we additionally require the condition that the intersection of the convex sets is large in some sense. This could mean that it contains many points from a certain discrete subset \(S\) of \(\mathbb{R}^d\) or that it has a large volume. For example, the integer lattice admits a Helly-type theorem, as shown by Doignon.
Theorem (J.-P. Doignon [Doi73]). If $F$ is a finite family of convex sets in $\mathbb{R}^d$ such that the intersection of every subfamily of size $2^d$ contains a point with integer coordinates, then the intersection of $F$ contains a point with integer coordinates.

This result was rediscovered several times [Bel76, Sca77, Hol79]. The result above has a quantitative extension [ABDLL14]. Aliev et. al. showed that there is a constant $c(k, d)$ such that for any finite family of convex sets in $\mathbb{R}^d$ such that the intersection of every $c(k, d)$ of them contains at least $k$ points with integer coordinates, the intersection of the whole family contains at least $k$ points with integer coordinates.

Whenever we are given a discrete set $S$, it makes sense to talk about version of Helly’s theorem for $S$, and, if possible, a quantitative version as well.

Definition 1. For a discrete set $S \subset \mathbb{R}^d$, we define the $k$-quantitative Helly number $H_S(k)$ as the smallest number for which the following statement holds. For any finite family $F$ of convex sets in $\mathbb{R}^d$, if for every subfamily $F'$ of size $H_S(k)$ we know that $\cap F'$ has at least $k$ points of $S$, then $\cap F$ contains at least $k$ points of $S$. If no such number exists, we consider $H_S(k) = \infty$.

Volumetric versions of Tverberg, Helly, and Carathéodory have also appeared recently [DLLHRS15]. The first results of this kind were given by Bárány, Katchalski and Pach [BKP82, BKP84]. In the same spirit, the following version of Helly’s theorem was obtained.

Theorem (J.A. De Loera et al. [DLLHRS15]). There is a constant $n^h(d, \varepsilon)$ such that the following holds. If $F$ is a finite family of convex sets in $\mathbb{R}^d$ such that all subfamilies of size at least $n^h(d, \varepsilon)$ have an intersection of volume at least one, then $\text{vol}(\cap F) \geq 1 - \varepsilon$.

The quantity $n^h(d, \varepsilon) \sim (\frac{cd}{\varepsilon})^{(d-1)/2}$ for an absolute constant $c$ is closely related to results of approximation of convex sets by polytopes with few facets. One should note that it is impossible to obtain a result with $\text{vol}(\cap F) \geq 1$ in the conclusion of the theorem, regardless of the size of the subfamilies we are willing to check.

The aim of this paper is to show quantitative versions of the $(p, q)$ theorem. One version deals with the condition of having $k$ points of a discrete set $S$ and the other deals with having a volume of fixed size.

Theorem 1.1 (Quantitative discrete $(p, q)$ theorem). Let $S$ be a discrete subset of $\mathbb{R}^d$ and $k$ be an integer such that $H_S(k)$ is finite. Let $p \geq q \geq \text{vol}(\cap F) \geq 1 - \varepsilon$. 

\[ n^h(d, \varepsilon) \sim (\frac{cd}{\varepsilon})^{(d-1)/2} \]
be positive integers. Then, there is a constant $c = c(p, q, S)$ such that the following is true. For any finite family $F$ of convex sets in $\mathbb{R}^d$, each containing at least $k$ points of $S$ and such that for every $p$ sets in $F$ there are $q$ whose intersection contains $k$ points of $S$, we can find $c$ sets $K_1, K_2, \ldots, K_c$ of $k$ points of $S$ each such that every $F \in F$ contains at least one $K_i$.

**Theorem 1.2** (Volumetric $(p, q)$ theorem). Given $d$ a positive integer and $\varepsilon > 0$, there is a constant $n(d, \varepsilon)$ such that the following holds. Let $p \geq q \geq n(d, \varepsilon)$ be positive integers. Then, there is a constant $c_{\text{vol}} = c_{\text{vol}}(p, q, \varepsilon)$ such that the following is true. For any finite family $F$ of convex sets in $\mathbb{R}^d$, each of volume at least $1$ and such that for every $p$ sets in $F$ there are $q$ whose intersection has volume at least $1$, we can find $c_{\text{vol}}$ sets $C_1, C_2, \ldots, C_{c_{\text{vol}}}$ of volume $1 - \varepsilon$ each such that every $F \in F$ contains at least one $C_i$.

The quantity $n(d, \varepsilon)$ of Theorem 1.2 is $O\left(\varepsilon^{-(d^2-1)/4}\right)$, where the constants hidden by the $O$ notation depend on $d$. For $d = 1$, it is a simple exercise to show that the classic $(p, q)$ theorem implies Theorem 1.2, with $\text{vol}(C_i) = 1$ for all $i$.

The case $k = 1$, $S = \mathbb{Z}^d$ of Theorem 1.1 has already been proven [BM03]. Moreover, the surprising result is that the conditions needed on $p, q$ are only $p \geq q \geq d + 1$, while $\mathbb{Z}^d(1) = 2^d$. However, it seems that one of their key ingredients [BM03, Corollary 2.2] fails when $k \geq 2$. The fractional Helly theorems of Averkov and Weismantel [AW10] and the methods below show that we can relax the condition to $p \geq q \geq d + 1$ if $k = 1$ as long as $\mathbb{H}_S(1)$ is finite.

An example of a large family of sets $S$ with finite $\mathbb{H}_S(k)$ for all $k$ is the following. Let $L$ be a lattice in $\mathbb{R}^d$ and $L_1, L_2, \ldots, L_m$ be sublattices. If $S = L \setminus (L_1 \cup \ldots \cup L_m)$, then $\mathbb{H}_S(k) \leq (2^{m+1}k + 1)^{\text{rank}L}$ [DLLHS15, Thm. 1.9]. For the case when $S$ is the integer lattice, better bounds are found in [ABDLL14].

In order to prove Theorem 1.1 and 1.2, we need quantitative versions of several classic results. These are commonly known as the first selection lemma [Bár82], the fractional Helly theorem [KL79] and the existence of weak $\varepsilon$-nets for convex sets [ABFK92]. For the discrete case, these results can be deduced naturally using the quantitative discrete versions of Carathéodory and Tverberg’s theorem from [DLLHS15]. However, in order to prove a $(p, q)$ theorem, we need slightly different interpretations of the quantitative versions of these results. The latter ones also have volumetric analogues, which we use to prove Theorem 1.2. We believe that both versions of these results are interesting, so we present all in this paper.
The key difference is that the results of [DLLHRS15] need to have subsets of $S$ with many distinct points to work. However, the versions we need for Theorem 1.1 only require that the sets contain many $k$-tuples of $S$, with multiplicities allowed.

The fractional version of quantitative Helly for discrete sets $S$ is the following one.

**Theorem 1.3** (Quantitative fractional Helly theorem). For each $\alpha \in (0, 1]$ there is a $\beta = \beta(\alpha, S) > 0$ such that the following is true. For each finite family $\mathcal{F}$ of $n \geq \Pi(k)$ closed convex sets in $\mathbb{R}^d$ such that there are at least

$$\alpha \left( \frac{n}{\Pi(k)} \right)$$

$\Pi(k)$-tuples of $\mathcal{F}$ whose intersection contains $k$ points of $S$, there is a subfamily $\mathcal{F}' \subset \mathcal{F}$ of size at least $\beta n$ such that $\bigcap \mathcal{F}'$ contains at least $k$ points of $S$.

Moreover, if $\alpha \to 1$, then $\beta \to 1$.

Its volumetric analogue is the following one.

**Theorem 1.4** (Volumetric fractional Helly theorem). For each $\alpha \in (0, 1]$ there is a $\beta_{vol}(\alpha, d) > 0$ such that the following is true. For each finite family $\mathcal{F}$ of $n \geq n^h(d, \varepsilon)$ convex sets in $\mathbb{R}^d$ such that there are at least

$$\alpha \left( \frac{n}{n^h(d, \varepsilon)} \right)$$

subfamilies of size $n^h(d, \varepsilon)$ whose intersection has volume at least one, there is a subfamily $\mathcal{F}' \subset \mathcal{F}$ of size at least $\beta_{vol} n$ such that

$$\text{vol}(\bigcap \mathcal{F}') \geq 1 - c\varepsilon^{2/(d+1)}$$

for some constant $c$ depending only on the dimension.

It would be interesting if Theorem 1.3 could be proved with the condition being on the $\max\{kd, d+1\}$-tuples of $\mathcal{F}$ or even the $(d+1)$-tuples, rather than the $\Pi(k)$-tuples. If this was the case, it would immediately improve Theorem 1.1. The case $S = \mathbb{Z}^d$ and $k = 1$ of Theorem 1.3 was first noted in [AKMM02], along several other variations of the $(p, q)$ theorem.

The arguments we use to prove for Theorem 1.4 can also be used to give a different proof of the volumetric colourful Helly theorem [DLLHRS15, Thm. 1.6], which is inspired in the colourful Helly theorem by Lovász. Lovász’ proof is contained in [Bár82]. We get a slightly different presentation.
Theorem 1.5. Let $N = n^h(d, \varepsilon)$, and $F_1, F_2, \ldots, F_N$ be finite families of convex sets in $\mathbb{R}^d$, considered as colour classes. We know that the intersection of every colourful choice $F_1 \in F_1, \ldots, F_n \in F_N$ has volume at least one. Then, there is a colour class $F_i$ for which
\[
\text{vol}(\cap F_i) \geq 1 - c\varepsilon^{2/(d+1)}
\]
for some constant $c$ depending only on the dimension.

One of the advantages of this proof is that it is easier to get bounds on the number of colour classes needed to guarantee that one of the colours has an intersection with volume at least $1 - \varepsilon$. This quantity is $O\left(\varepsilon^{-(d^2-1)/4}\right)$.

For the first selection lemma, the result below is the version focusing on having many distinct points of $S$. The version focusing on $k$-tuples is Theorem 3.2, and the volumetric version is Theorem 5.2.

Theorem 1.6 (Quantitative first selection lemma). There is a constant $\lambda = \lambda(S)$ such that for any set $T \subset \mathbb{R}^d$ of $n$ distinct points of $S$, there is a subset $K' \subset S$ of $k$ points contained in the convex hull of at least
\[
\lambda\left(\frac{n}{\max\{kd, d+1\}}\right)
\]
sets of size $\max\{kd, d+1\}$

The reason why the constant $\max\{kd, d+1\}$ appears is due to the following quantitative version of quantitative Carathéodory’s theorem. The case $k = 1$ is a classic result by Bárány [Bár82].

Theorem (Theorem 1.3 in [DLLRS15]). Let $A$ be a set of $k$ points in $\mathbb{R}^d$, and $n = \max\{kd, d+1\}$. Given $n$ sets $X_1, X_2, \ldots, X_n$ such that $A \subset \text{conv}(X_i)$ for each $i$, we can find $x_1 \in X_1, \ldots, x_n \in X_n$ such that
\[
A \subset \text{conv}\{x_1, \ldots, x_n\}
\]

The monochromatic case is simply when $X_1 = X_2 = \ldots = X_n$.

Using this result, we can find quantitative versions of weak $\varepsilon$-nets. Again, we give an alternative discrete version in Theorem 5.3 and a volumetric version in Theorem 5.3.

Theorem 1.7 (Quantitative weak $\varepsilon$-nets). There is a constant $m = m(\varepsilon, S)$ such that the following statement is true. Given a finite subset $T \subset S$ with $|T| \geq k\varepsilon^{-1}$, there is a family $K_1, K_2, \ldots, K_m$ of subsets of $k$ points of $S$ each such that for any $T' \subset T$ with $|T'| \geq \varepsilon|T|$, there is at least one $K_i \subset \text{conv}(T')$. 
The condition on the size of \( T \) is necessary to avoid trivial counterexamples.

In section 2 we prove theorems 1.3, 1.6 and 1.7. In section 3 we present the alternative discrete versions of these results and their proofs. In section 4 we prove Theorem 1.1. In section 5 we state and prove all the volumetric results.

2 Proofs for direct quantitative results

Proof of Theorem 1.3. This proof uses ideas very similar to the ones in [DLRPOLH15, Sect. 5]. We may assume without loss of generality that the convex sets are bounded.

Consider a direction which is not orthogonal to any segment made by two points of \( S \). Since \( S \) is discrete and thus numerable, such a \( v \) exists. We consider a \( v \)-halfspace a set of the form \( \{ x : \langle x, v \rangle \leq \alpha \} \) for some real \( \alpha \). For each \((H_S(k) - 1)\)-tuple \( M = \{F_1, F_2, \ldots, F_{H_S(k) - 1}\} \) such that its intersection contains at least \( k \) points of \( S \), let \( H_M \) be the containment-minimal \( v \)-halfspace such that \((\cap M) \cap H_M \) contains at least \( k \) points of \( S \). Note that given the choice of \( v \), we have that this set contains exactly \( k \) points of \( S \). We denote by \( K_M \) the set of \( k \) points of \( S \) in \((\cap M) \cap H_M \).

Now consider an \( H_S(k) \)-tuple \( A \) of \( F \) whose intersection contains \( k \) points of \( S \). Among its \((H_S(k) - 1)\)-tuples, there must be one, call it \( M_0 \), such that \( H_{M_0} \) is containment-maximal. It is clear that if we add \( H_{M_0} \) to \( A \), in the resulting set the intersection of any \( H_S(k) \) contains \( k \) points of \( S \). By the definition of \( H_S(k) \) we have the intersection of this whole family contains at least \( k \) points of \( S \). However, this implies that every set in the family contains \( K_{M_0} \).

For each \( H_S(k) \)-tuple \( A \) whose intersection contains at least \( k \) points of \( S \), let \( A' \) be its \((H_S(k) - 1)\)-tuple with containment-maximal \( H_{A'} \). If a positive fraction of the \( H_S(k) \)-tuples satisfy the condition of the problem, then a simple counting argument shows that there must be an \((H_S(k) - 1)\)-tuple \( B \) which was assigned to at least \( \beta n \) different \( H_S(k) \)-tuples, for some positive \( \beta \) not depending on \( n \). Thus at least \( \beta n \) sets contain \( K_B \), as desired.

For the following proof we need the quantitative Tverberg theorem from [DLLHRS15].

Theorem. Given a set of at least \((m - 1)kd \cdot H_S(k) + k\) distinct points of
there is a partition of them into \( m \) sets \( A_1, A_2, \ldots, A_m \) such that
\[
\bigcap_{i=1}^{m} \text{conv}(A_i)
\]
contains at least \( k \) points of \( S \).

Proof of Theorem 1.6. Assume \( n \) is sufficiently large, or a small \( \lambda \) would make the result true. By the discrete version of quantitative Tverberg, \( T \) can be split into \( \left\lceil \frac{n-k+1}{kd \cdot H_S(k)} \right\rceil \) sets such that their intersection of their convex hulls has a set of \( k \) points of \( S \). Denote by \( K' \) this set. Now colour each part with a different colour. Notice that every time we take \( \max\{kd, d+1\} \) of them, by the colourful quantitative Carathéodory theorem, we can take one point from each such the convex hull of the resulting set contains \( K' \). Thus, \( K' \) is in the convex hull of at least
\[
\left( \frac{n-k+1}{kd \cdot H_S(k)} \right) \sim \left( \frac{1}{kd \cdot H_S(k)} \right)^{\max\{kd,d+1\}} \left( \frac{n}{\max\{kd,d+1\}} \right)
\]
set of size \( \max\{kd,d+1\} \).

Proof of Theorem 1.7. We construct the family \( K = \{K_1, K_2, \ldots, K_m\} \) inductively. Let \( r \) be the number of subsets \( A \subset T \) of size \( \max\{kd,d+1\} \) which do not contain any of the \( K_i \). If there is a subset \( T' \subset T \) with \( |T'| \geq \varepsilon |T| \) such that \( \text{conv}(T') \) does not contain any \( K_i \), by applying Theorem 1.6 we can find a \( K' \) that is in the convex hull of at least
\[
\lambda \left( \frac{|T'|}{\max\{kd,d+1\}} \right) \sim \lambda \varepsilon^{\max\{kd,d+1\}} \left( \frac{|T'|}{\max\{kd,d+1\}} \right)
\]
subsets of \( T' \) of size \( \max\{kd,d+1\} \). By adding \( K' \) to \( K \), we have reduced \( r \) by at least \( \lambda \varepsilon^{\max\{kd,d+1\}} \). We can repeat this process at most \( (\lambda \varepsilon^{\max\{kd,d+1\}})^{-1} \) times, which finishes the proof.

3 Discrete variations needed to prove \((p, q)\) theorems

Let us prove a slightly different version of the discrete quantitative Tverberg of [DLLHRS15].
Theorem 3.1. Let $S \subset \mathbb{R}^d$ be a discrete set such that $\mathcal{H}_S(k)$ is finite. Let $n \geq \mathcal{H}_S(k)(m-1)kd + 1$ and $\mathcal{T} = \{T_1, T_2, \ldots, T_n\}$ be a family of sets $T_i \subset S$ of size $k$ each. Then, there is a partition of $\mathcal{T}$ into $m$ families $A_1, A_2, \ldots, A_m$ so that

$$\bigcap_{i=1}^{m} \text{conv}(\bigcup A_i)$$

contains at least $k$ points of $S$.

Proof. Consider the family

$$\mathcal{F} = \{F : F = \text{conv}(\cup T') \text{ for some } T' \subset T, |T'| = (m-1)(\mathcal{H}_S(k)-1)kd + 1\}$$

Note that each $F \in \mathcal{F}$ is missing at most $(m-1)kd$ families of $\mathcal{T}$. Thus, any $\mathcal{H}_S(k)$ of them have at least one $T_i$ in common, which means they intersect in $k$ points of $S$. By the definition of $\mathcal{H}_S(k)$, there is a family $T_0$ of $k$ points of $S$ which is contained in $\cap \mathcal{F}$.

Every closed half-space that contains a point of $T_0$ also contains points of at least $kd(m-1) + 1$ sets of $\mathcal{T}$. If this was not the case, there would be a closed half-space containing points of at most $kd(m-1)$ sets $T_i$. All the other sets would be contained in the other open half-space, contradicting the fact that $T_0 \subset \cap \mathcal{F}$.

Now we construct $A_1, A_2, \ldots, A_m$ inductively. Assume $k \geq 2$. By the quantitative Carathéodory theorem, there is a set $P \subset \cup \mathcal{T}$ of at most $kd$ points such that $T_0 \subset \text{conv}(P)$. Take $A_1$ the union of all sets $T_i$ that have points in $P$. Since we have removed at most $kd$ sets, it means that every closed half-space containing a point of $T_0$ must have points of at least $(m-2)kd + 1$ different $T_i$ not in $A_1$. Thus, we can repeat this process until we have $A_1, A_2, \ldots, A_m$.

If $k = 1$, we can only guarantee that $P$ has at most $d+1$ points. However, if we take $P$ to be minimal, it either has at most $d$ points or (the point) $T_0$ is in the interior of $\text{conv}(P)$. In the latter case, every closed half-space containing $T_0$ in its boundary is only losing $d$ points. Thus we can follow the same arguments as above. \hfill \square

This result allows us to prove the following version of Theorem 1.6.

Theorem 3.2 (Second quantitative first selection lemma). Let $S \subset \mathbb{R}^d$ be a discrete set such that $\mathcal{H}_S(k)$ is finite. Then, there is a constant $\lambda_2 = \lambda_2(S)$ such that for any family $T \subset \mathbb{R}^d$ of $n$ sets $T_1, T_2, \ldots, T_n$ of $k$ points each of
there is a subset $K' \subset S$ of $k$ points contained in the convex hull of the union of at least

$$\lambda_2 \left( \frac{n}{\max\{kd, d + 1\}} \right)$$

subfamilies $\mathcal{T}' \subset \mathcal{T}$ of size $\max\{kd, d + 1\}$

Proof. We may assume that $n > kd \cdot H_S(k)$ or a small enough $\lambda_2$ would make the result trivial. By Theorem 3.1, $\mathcal{T}$ can be split into $\left\lceil \frac{n}{kd \cdot H_S(k)} \right\rceil$ families such that the intersection of the convex hulls of their unions has at least $k$ points of $S$. Denote by $K'$ this set. Now colour each part with a different colour. Notice that every time we take $\max\{kd, d + 1\}$ parts, by the colourful quantitative Carathéodory, we can take one point from each such the convex hull of the resulting set contains $K'$. Thus, $K'$ is in the convex hull of the union of at least

$$\left( \frac{\left\lceil \frac{n}{kd \cdot H_S(k)} \right\rceil}{\max\{kd, d + 1\}} \right)^{\max\{kd, d + 1\}} \sim \left( \frac{1}{kd \cdot H_S(k)} \right)^{\max\{kd, d + 1\}} \left( \frac{n}{\max\{kd, d + 1\}} \right)$$

subfamilies $\mathcal{T}' \subset \mathcal{T}$ of size $\max\{kd, d + 1\}$

Using the same method as before we get the following version of Theorem 1.7

**Theorem 3.3.** Let $S \subset \mathbb{R}^d$ be a discrete set such that $H_S(k)$ is finite. Then, there is a constant $m_2 = m_2(\varepsilon, S)$ such that the following statement is true. Given a finite family $\mathcal{T}$ of subsets of $S$ with $k$ points each, there is a family $K_1, K_2, \ldots, K_{m_2}$ of subsets of $S$ of $k$ points each such that for any $\mathcal{T}' \subset \mathcal{T}$ with $|\mathcal{T}'| \geq \varepsilon |\mathcal{T}|$, there is at least one $K_i \subset \text{conv}(\cup \mathcal{T}')$.

Proof. We construct the family $\mathcal{K} = \{K_1, K_2, \ldots, K_{m_2}\}$ inductively. Let $r$ be the number of subfamilies $A \subset \mathcal{T}$ of size $\max\{kd, d + 1\}$ such that $\text{conv}(\cup A)$ does not contain any of the $K_i$. If there is a subfamily $\mathcal{T}' \subset \mathcal{T}$ with $|\mathcal{T}'| \geq \varepsilon |\mathcal{T}|$ such that $\text{conv}(\cup \mathcal{T}')$ does not contain any $K_i$, by applying Theorem 3.2, we can find a $K'$ that is in the convex hull of the union of at least

$$\lambda_2 \left( \frac{|\mathcal{T}'|}{\max\{kd, d + 1\}} \right) \sim \lambda_2^{\varepsilon^{\max\{kd, d + 1\}}} \left( \frac{|\mathcal{T}|}{\max\{kd, d + 1\}} \right)$$

subfamilies of $\mathcal{T}'$ of size $\max\{kd, d + 1\}$. By adding $K'$ to $\mathcal{K}$, we have reduced $r$ by at least $\lambda_2^{\varepsilon^{\max\{kd, d + 1\}}} \left( \frac{|\mathcal{T}|}{\max\{kd, d + 1\}} \right)$. Thus, we can repeat this process at most $\left( \lambda_2^{\varepsilon^{\max\{kd, d + 1\}}} \right)^{-1}$ times. 

\qed
4 Proof of Theorem 1.1

In order to prove quantitative \((p, q)\)-theorems, the last step is to make quantitative versions of the transversal and packing numbers and follow the classic proof of Alon and Kleitman.

**Definition 2.** Given a finite family \(\mathcal{F}\) of convex sets in \(\mathbb{R}^d\), each containing at least \(k\) points of \(S\), we define

- the \(k\)-transversal number \(\tau_k(\mathcal{F})\) as the minimum \(\sum_{K \subseteq \binom{S}{k}} w(K)\) over all functions \(w : \binom{S}{k} \to \{0, 1\}\) such that
  \[
  \sum_{K \subseteq F, K \subseteq \binom{S}{k}} w(K) \geq 1
  \]
  for all \(F \in \mathcal{F}\),

- the fractional \(k\)-transversal number \(\tau^*_k(\mathcal{F})\) as the minimum \(\sum_{K \subseteq \binom{S}{k}} w(K)\) over all functions \(w : \binom{S}{k} \to [0, 1]\) such that
  \[
  \sum_{K \subseteq F, K \subseteq \binom{S}{k}} w(K) \geq 1
  \]
  for all \(F \in \mathcal{F}\), and

- the fractional \(k\)-packing number \(\nu^*_k(\mathcal{F})\) as the maximum \(\sum_{F \in \mathcal{F}} w(F)\) for \(w : \mathcal{F} \to [0, 1]\) such that
  \[
  \sum_{F : K \subseteq F} w(F) \leq 1
  \]
  for all \(K \in \binom{S}{k}\).

Note that a simple duality argument shows that \(\tau^*_k = \nu^*_k\). Thus, it suffices to prove that

- \(\tau_k\) is bounded by a function of \(\tau^*_k\) via the quantitative weak \(\varepsilon\)-nets

- \(\nu^*_k\) is bounded if \(\mathcal{F}\) satisfies the quantitative \((p, q)\) property, via the quantitative fractional Helly theorem.

**Lemma 4.1.** Let \(S\) be a fixed discrete subset of \(\mathbb{R}^d\) with \(\mathcal{H}_S(k)\) finite. If \(\mathcal{F}\) is a finite family of convex sets in \(\mathbb{R}^d\), then \(\tau_k(\mathcal{F})\) is bounded by a function depending only \(\tau^*_k(\mathcal{F})\).
Proof. Consider a function $w : \binom{S}{k} \rightarrow [0,1]$ which realises $\tau_k^*$. Without loss of generality, we may assume that $w$ has finite support and only has rational values. Let $T$ be the family that formed by the disjoint union of $M \cdot w(K)$ copies of $K$, for each $K \in \binom{S}{k}$. Now consider $K$ a quantitative weak $(\frac{1}{\tau_k(F)})$-net of $T$, as in Theorem 3.3. Notice that $K$ is a quantitative transversal to $F$, so $\tau_k(F) \leq |K|$ which in turn is only bounded by a function of $\tau_k^*(F)$. □

Lemma 4.2. If $p \geq q \geq \Pi_S(k)$ and $F$ is a finite family of convex sets with the quantitative $(p,q)$ condition, then $\nu_k^*(F)$ is bounded.

Proof. Let $w : F \rightarrow [0,1]$ be a function that attains $\nu_k^*(F)$. We may assume without loss of generality that $w(C)$ is rational for all $C \in F$. Let $w(C) = \frac{n_c}{m}$ where $m$ is the common denominator for all $w(C)$ for $C \in F$. Let $F'$ be the family consisting of $n_c$ copies of $C$ for each $C \in F$ and let $N = |F'|$. Note that $\frac{N}{m} = \sum_{C \in F} \frac{n_c}{m} = \nu_k^*(F)$.

The family $F'$ satisfies the quantitative $((q-1)(p-1)+1,q)$ property. This comes immediately from the fact that every $[(q-1)(p-1)+1]$-tuple from $F'$ contains either $q$ copies of the same set or $p$ different sets of $F$. In either case we have a $q$-tuple containing $k$ points of $S$. However, since $q \geq \Pi_S(k)$ this implies that there is a positive fraction of the $\Pi_S(k)$-tuples of $F'$ whose intersections contain at least $k$ points of $S$. Theorem 1.3 implies then that there is a positive fraction $\beta$ depending only on $p,q,S$ such that there is a set $K_0$ of $k$ points of $S$ contained in the intersection of at least $\beta N$ sets of $F'$. Thus

$$1 \geq \sum_{C \in F : K_0 \subset C} w(C) = \sum_{C \in F : K_0 \subset C} \frac{n_c}{m} \geq \frac{1}{m} \cdot \beta N = \beta \nu_k^*(F).$$

This implies $\nu_k^*(F) \leq \frac{1}{\beta}$, as desired. □

Proof of Theorem 1.1. Combining Lemma 4.1 and Lemma 4.2, we obtain the desired result. □

5 Volumetric theorems and proofs

The following result was used by De Loera et al. in [DLLHRS15] to find a colourful quantitative Steinitz theorem for volume.
**Theorem.** There is a constant $n^c(d,\varepsilon)$ such that the following holds. Given a bounded convex set $K$ with non-empty interior, there is a polytope $K' \subset K$ of at most $n^c(d,\varepsilon)$ vertices such that

$$\text{vol}(K') \geq (1 - \varepsilon) \text{vol}(K)$$

Moreover, $n^c(d,\varepsilon) \sim \left(\frac{d}{\varepsilon}\right)^{(d-1)/2}$ for some absolute constant $c'$.

The result above comes from approximated convex sets with polytopes contained in them via the Nikodym metric [Bro08, GMR95].

Let us prove a fractional Helly for volume. For this result we will need to use previous results on the convex floating body. For a convex set $K$ with positive volume, we define $\text{float}(K,\varepsilon)$ as the set of points $x$ such that $\text{vol}(H \cap K) \geq \text{vol}(K)(1 - \varepsilon)$ for all closed halfspaces $H \ni x$. There are several results regarding the volume of the floating body [BL10]. For sufficiently smooth bodies $K$ of unit volume, we have

$$\text{vol}(\text{float}(K,\varepsilon)) \geq 1 - ce^{2/(d+1)}$$

where $c$ is a constant depending only on the dimension. Note that if $K$ is a convex set of unit volume and $K'$ is another convex set such that $\text{vol}(K \cap K') \geq 1 - \varepsilon$, then $\text{float}(K,\varepsilon) \subset K'$.

**Proof of Theorem 1.4.** This proof is very similar to the one of Theorem 1.3. We may assume without loss of generality that the sets in $\mathcal{F}$ are bounded.

Consider $v$ a direction. We consider a $v$-halfspace to be a set of the form $\{x : \langle x, v \rangle \leq \alpha\}$ for some real $\alpha$. For each $(n^h(d,\varepsilon) - 1)$-tuple $M = \{F_1, F_2, \ldots, F_{n^h(d,\varepsilon) - 1}\}$ such that $\text{vol}(\cap M) \geq 1$, let $H_M$ be the $v$-halfspace such that $\text{vol}(\cap M \cap H_M) = 1$. We denote this intersection by $K_M$.

Now consider an $n^h(d,\varepsilon)$-tuple $A$ of $\mathcal{F}$ such that $\text{vol}(\cap A) \geq 1$. Among its $(n^h(d,\varepsilon) - 1)$-tuples, there must be one, call it $M_0$, such that $H_{M_0}$ is containment-maximal. It is clear that if we add $H_{M_0}$ to $A$, in the resulting set the intersection of any $n^h(d,\varepsilon)$ sets has volume at least 1. By the definition of $n^h(d,\varepsilon)$ we have that the intersection of this whole family has volume at least $1 - \varepsilon$. However, this implies that the set not in $M_0$ contains $\text{float}(K_{M_0},\varepsilon)$.

For each $n^h(d,\varepsilon)$-tuple $A$ such that $\text{vol}(\cap A) \geq 1$, let $A'$ be its $(n^h(d,\varepsilon) - 1)$-tuple with containment-maximal $H_{A'}$. If a positive fraction of the $n^h(d,\varepsilon)$-tuples satisfy the condition of the problem, then a simple counting argument shows that there must be an $(n^h(d,\varepsilon) - 1)$-tuple $B$ which was assigned to at least $\beta_{\text{vol}} \cdot n$ different $n^h(d,\varepsilon)$-tuples, for some positive $\beta_{\text{vol}}$ not depending on $n$. Thus at least $\beta_{\text{vol}} n$ sets contain $\text{float}(K_B,\varepsilon)$, as desired. Note that if $K_B$
is not sufficiently smooth, it can be approximated with arbitrary precision
with a body that satisfies this and is contained in it. Since \( \varepsilon \) is fixed, this
causes no loss of volume in the intersection. \( \square \)

The facts about the floating body are not absolutely necessary. By using
repeatedly the volumetric version of Helly’s theorem we can obtain a bound
on the size of the tuples with big intersection in order to guarantee the
existence of the subfamily \( \mathcal{F}' \). However, these are much worse than what
one gets by looking at the floating body.

**Proof of Theorem 1.5.** We follow the same technique as in the proof above.
We may assume without loss of generality that the sets in \( \mathcal{F} \) are bounded.

Consider \( v \) a direction. We consider a \( v \)-halfspace to be a set of the form
\( \{ x : \langle x, v \rangle \leq \alpha \} \) for some real \( \alpha \). For each colourful \((n^h(d, \varepsilon) - 1)\)-tuple \( M = \{ F_1, F_2, \ldots, F_{n^h(d, \varepsilon)} \} \) (i.e. each \( F_i \) is in a different colour class) we have
\( \text{vol}(\cap M) \geq 1 \). Let \( H_M \) be the \( v \)-halfspace such that
\( \text{vol}((\cap M) \cap H_M) = 1 \). We denote this intersection by \( K_M \).

Now consider a colourful \( n^h(d, \varepsilon) \)-tuple \( A \) of \( \mathcal{F} \). We know that \( \text{vol}(\cap A) \geq 1 \). Among its \((n^h(d, \varepsilon) - 1)\)-tuples, there must be one, call it \( M_0 \), such that
\( H_{M_0} \) is containment-maximal. It is clear that if we add \( H_{M_0} \) to \( A \), in the
resulting set the intersection of any \( n^h(d, \varepsilon) \) sets has volume at least 1. By
the definition of \( n^h(d, \varepsilon) \) we have that the intersection of this whole family
has volume at least \( 1 - \varepsilon \). However, this implies that the set not in \( M_0 \)
contains \( \text{float}(K_{M_0}, \varepsilon) \).

Now let \( B \) be a colourful \((n^h(d, \varepsilon - 1) \)-tuple with containment-maximal
\( H_B \). Let \( \mathcal{F}_i \) be the colour class that does not have a set in \( B \). The observations above imply that

\[
\text{float}(K_B, \varepsilon) \subset \bigcap \mathcal{F}_i,
\]

finishing the proof. \( \square \)

Now let us prove a volumetric version of Tverberg’s theorem.

**Theorem 5.1.** Let \( n \geq (m-1)d \cdot n^h(d, \varepsilon_1) n^c(d, \varepsilon_2) + 1 \) and \( \mathcal{T} = \{ T_1, T_2, \ldots, T_n \} \)
be a family of sets \( T_i \) of volume 1 each. Then, there is a partition of \( \mathcal{T} \) into
\( m \) families \( A_1, A_2, \ldots, A_m \) so that

\[
\text{vol} \left( \bigcap_{i=1}^{m} \text{conv}(\bigcup A_i) \right) \geq (1 - \varepsilon_1)(1 - \varepsilon_2).
\]
Proof. Consider the family

\[ \mathcal{F} = \{ F : F = \text{conv}(\cup T') \text{ for some } T' \subset T, |T'| = (m-1)(n^b(d, \varepsilon_1) - 1)n^c(d, \varepsilon_2)d + 1 \} \]

Note that each \( F \in \mathcal{F} \) is missing at most \((m-1)n^c(d, \varepsilon_2)d\) sets of \( T \). Thus, any \( n^h(d, \varepsilon_1) \) of them have at least one \( T_i \) in common, which means their intersection has volume at least 1. By the definition of \( n^h(d, \varepsilon_1) \), there is a set \( T_0 \) of volume \( 1 - \varepsilon_1 \) which is contained in \( \cap \mathcal{F} \).

Every closed half-space that contains a point of \( T_0 \) also contains points of at least \( n^c(d, \varepsilon_2)d \) sets \( T_i \). If this was not the case, there would be a closed half-space containing points of at most \( n^c(d, \varepsilon_2)d(m-1) \) sets \( T_i \). All the other sets would be contained in the other open half-space, contradicting the fact that \( T_0 \subset \cap \mathcal{F} \).

Now we construct \( A_1, A_2, \ldots, A_m \) inductively. By the result in the beginning of the section, there is a polytope \( T'_0 \subset T_0 \) of at most \( n^c(d, \varepsilon_2) \) vertices such that \( \text{vol}(T'_0) \geq (1 - \varepsilon_2) \text{vol}(T_0) \geq (1 - \varepsilon_2)(1 - \varepsilon_1) \). By the quantitative Carathéodory theorem, there is a set \( P \subset \cup T \) of at most \( n^c(d, \varepsilon_2)d \) points such that \( T'_0 \subset \text{conv}(P) \). Take \( A_1 \) the union of all sets \( T_i \) that have points in \( P \). Since we have removed at most \( n^c(d, \varepsilon_2)d \) sets, it means that every closed halfspace containing a point of \( T'_0 \) must have points of at least \((m-2)n^c(d, \varepsilon_2)d + 1\) different \( T_i \) not in \( A_1 \). Thus, we can repeat this process until we have \( A_1, A_2, \ldots, A_m \).

\[ \textbf{Theorem 5.2} \ (\text{Volumetric quantitative first selection lemma}). \text{ There is a constant } \lambda_{vol} = \lambda_{vol}(d, \delta) \text{ such that for any family } T \subset \mathbb{R}^d \text{ of } n \text{ bounded sets } T_1, T_2, \ldots, T_n \text{ of volume 1 each, there is a set } K' \text{ of volume } (1 - \delta)^3 \text{ contained in the convex hull of the union of at least } 
\]

\[ \lambda_{vol} \left( \frac{n}{d \cdot n^c(d, \delta)} \right) \]

subfamilies \( T' \subset T \) of size \( d \cdot n^c(d, \delta) \)

\[ \text{Proof. We may assume that } n \text{ is large or a small enough } \lambda_{vol} \text{ would make the result trivial. By Theorem 5.1 with } \varepsilon_1 = \varepsilon_2 = \delta, T \text{ can be split into } \left[ \frac{n}{d \cdot n^h(d, \delta)n^c(d, \delta)} \right] \text{ families such that the intersection of the convex hulls of their unions has volume at least } (1 - \delta)^2. \text{ Let } K' \text{ be a polytope contained in this set with at most } n^c(d, \delta) \text{ vertices which has a volume of at least } (1 - \delta)^3. \text{ Now colour each part with a different colour. Notice that every time we take } d \cdot n^c(d, \delta) \text{ coloured sets, by the colourful quantitative Carathéodory theorem,} \]
we can take one point from each such the convex hull of the resulting set contains $K'$. Thus, $K'$ is in the convex hull of the union of at least

$$\left(\left\lfloor \frac{n}{d \cdot n^{b(d, \delta)} n^{c(d, \delta)}} \right\rfloor \right) \sim \left(\frac{1}{d \cdot n^{b(d, \delta)} n^{c(d, \delta)}} \right)^{n^{c(d, \delta)}} \left(\frac{n}{n^{c(d, \delta)}} \right)$$

subfamilies $T' \subset T$ of size $d \cdot n^{c(d, \delta)}$.

**Theorem 5.3.** There is a constant $m_{\text{vol}} = m_{\text{vol}}(\varepsilon, \delta, d)$ such that the following statement is true. Given a finite family $T$ of volume one each, there is a family $K_1, K_2, \ldots, K_m$ of sets of volume $(1 - \delta)^3$ each such that for any $T' \subset T$ with $|T'| \geq \varepsilon|T|$, there is at least one $K_i \subset \text{conv}(\bigcup T')$.

**Proof.** We construct the family $K = \{K_1, K_2, \ldots, K_{m_{\text{vol}}}\}$ inductively. Let $r$ be the number of subfamilies $A \subset T$ of size $d \cdot n^{c(d, \delta)}$ which do not contain any of the $K_i$. If there is a subfamily $T' \subset T$ with $|T'| \geq \varepsilon|T|$ such that $\text{conv}(\bigcup T')$ does not contain any $K_i$, by applying Theorem 3.2 we can find a $K'$ of volume $(1 - \delta)^3$ that is in the convex hull of the union of at least

$$\lambda_{\text{vol}}\left(\frac{|T'|}{d \cdot n^{c(d, \delta)}}\right) \sim \lambda_{\text{vol}} \cdot \varepsilon^{d \cdot n^{c(d, \delta)}} \left(\frac{|T|}{d \cdot n^{c(d, \delta)}}\right)$$

subfamilies of $T'$ of size $d \cdot n^{c(d, \delta)}$. Thus, by adding $K'$ to $K$, we have reduced $r$ by at least $\lambda_{\text{vol}} \cdot \varepsilon^{d \cdot n^{c(d, \delta)}} \left(\frac{|T|}{n^{c(d, \delta)}}\right)$. We can repeat this process at most $\left(\lambda_{\text{vol}} \cdot \varepsilon^{d \cdot n^{c(d, \delta)}}\right)^{-1}$ times, giving the bound on $m_{\text{vol}}$.

**Proof of Theorem 1.2.** The proof follows the same steps as in Section 4. We just need to replace $\left(\begin{array}{c} n \\varepsilon \end{array}\right)$ by the family of bounded subsets of $\mathbb{R}^d$ with volume $1 - \varepsilon$, and the quantitative fractional Helly and quantitative weak $\varepsilon$-nets for their volumetric counterparts shown in this section. Let

$$C_{d, \varepsilon} = \{F \subset \mathbb{R}^d : \text{vol}(F) = 1 - \varepsilon\}$$

and define

- the volumetric transversal number $\tau_{\varepsilon, \text{vol}}(F)$ as the minimum $\sum_{C \in C_{d, \varepsilon}} w(C)$ over all functions $w : C_{d, \varepsilon} \rightarrow \{0, 1\}$ such that

$$\sum_{C : C \subset F, C \in C_{d, \varepsilon}} w(C) \geq 1$$

for all $F \in F$. 

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the fractional volumetric transversal number $\tau^*_{\mathcal{E},\text{Vol}}(\mathcal{F})$ as the minimum $\sum_{C \in \mathcal{C}_{d,\varepsilon}} w(C)$ over all functions $w : \mathcal{C}_{d,\varepsilon} \to [0,1]$ such that
\[ \sum_{C : C \subset F, \ C \in \mathcal{C}_{d,\varepsilon}} w(C) \geq 1 \]
for all $F \in \mathcal{F}$, and

the fractional volumetric packing number $\nu^*_k(\mathcal{F})$ as the maximum $\sum_{F \in \mathcal{F}} w(F)$ for $w : \mathcal{F} \to [0,1]$ such that
\[ \sum_{F : C \subset F, \ F \in \mathcal{F}} w(F) \leq 1 \]
for all $C \in \mathcal{C}_{d,\varepsilon}$.

For these quantities, we can prove the analogues of Lemma 4.1 and 4.2 using the volumetric versions of the results needed in Section 4.

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