Quantum key distribution between two groups using secret sharing

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In this paper, we investigate properties of some multi-particle entangled states and, from the properties applying the secret sharing present a new type of quantum key distribution protocols as generalization of quantum key distribution between two persons. In the protocols each group can retrieve the secure key string, only if all members in each group should cooperate with one another. We also show that the protocols are secure against an external eavesdropper using the intercept/resend strategy.

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I. INTRODUCTION

The computational power of quantum computers has threatened classical cryptosystems. For example, public key cryptosystems, such as Rivest-Shamir-Adleman public key cryptosystem [1], can be broken by quantum computers to be able to perform the fast factorization. On the other hand, quantum mechanical phenomena provide us a new kind of cryptosystems, called quantum key distribution (QKD), from which we can in principle obtain perfectly random and secure key strings.

The first quantum cryptographic protocol was presented by Bennett and Brassard [2] and their protocol bore the acronym BB84. In 1991, Ekert [3] proposed a QKD protocol using entangled particles. It was modified by Bennett, Brassard, and Mermin [4]. Let us call the modified version the Einstein-Podolsky-Rosen (EPR) protocol. The EPR protocol is a QKD between two persons using an EPR pair of spin $\frac{1}{2}$ particles in the state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

Using the Greenberger-Horne-Zeilinger (GHZ) state $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ [5] the secret sharing protocol was presented by Hillery, Bužek and Berthiaume [6]. In this protocol, Alice distributes the information on a key to Bob and Charlie. And the key can be restored only when their information are collected by them.

In this paper, applying the secret sharing protocol, we generalize the EPR protocol on noiseless channels by the properties of several cat states [7] and then obtain QKD protocols between group $A$ and group $B$. In each group the information of a secret key is distributed to all members. After the process for recovery of the key, the two groups get the secret key. And the protocols require each member’s approval and cooperation. Furthermore, when some members try to affect the shared bit adversely, if the shared key does not have the correct correlation (or anti-correlation) then it should be revealed to others in the test step. Any external eavesdropper should also be detected even if several members assist the eavesdropper.

This paper is organized as follows: In Section 2, we investigate some properties of several cat states. The QKD protocol between two groups and its modification are presented in Section 3. We analyze the security for the protocol in Section 4.

II. NONORTHOGONAL CAT STATES

Let us begin with reviewing cat states [7]. The $t$-particle cat state is defined as an entangled state of the type

$$\bigotimes_{i=1}^{t} |u_i^c\rangle \pm \bigotimes_{i=1}^{t} |u_i^f\rangle$$

whereby $u_i$ stands for the binary variable in $\{0, 1\}$, and $u_i^c = 1 - u_i$. Furthermore, Equation (1) becomes one of the Bell states when $t = 2$ and one of the GHZ states when $t = 3$.

From now on, we use the following several cat states:

$$|\Phi^\pm_i\rangle = \frac{1}{\sqrt{2}}(\bigotimes_{i=1}^{t} |0\rangle \pm \bigotimes_{i=1}^{t} |1\rangle)$$

$$|\Lambda^\pm_i\rangle = \frac{1}{\sqrt{2}}(\bigotimes_{i=1}^{t} |0\rangle \pm i\bigotimes_{i=1}^{t} |1\rangle).$$

We define $|0\rangle_x = |\Phi^+_i\rangle, |1\rangle_x = |\Phi^-_i\rangle, |0\rangle_y = |\Lambda^+_i\rangle, \text{ and } |1\rangle_y = |\Lambda^-_i\rangle$.

For $n = k + l$, we notice the states in Equation (2) and (3) have the following relations:

$$|\Phi^+_n\rangle_{AB} = \frac{1}{\sqrt{2}} (|\Phi^+_k\rangle_A |\Phi^+_l\rangle_B + |\Phi^-_k\rangle_A |\Phi^-_l\rangle_B)$$

$$|\Lambda^+_n\rangle_{AB} = \frac{1}{\sqrt{2}} (|\Lambda^+_k\rangle_A |\Lambda^+_l\rangle_B + |\Lambda^-_k\rangle_A |\Lambda^-_l\rangle_B),$$

$$|\Phi^-_n\rangle_{AB} = \frac{1}{\sqrt{2}} (|\Phi^-_k\rangle_A |\Phi^-_l\rangle_B + |\Phi^+_k\rangle_A |\Phi^+_l\rangle_B)$$

$$|\Lambda^-_n\rangle_{AB} = \frac{1}{\sqrt{2}} (|\Lambda^-_k\rangle_A |\Lambda^-_l\rangle_B + |\Lambda^+_k\rangle_A |\Lambda^+_l\rangle_B).$$

When $G$ is a group of $t$ persons, assume that, for one of the above four cat states, each person takes its one particle and measure in the $x$- or $y$-direction. Firstly we let $N^G_x$ be the number of members modulo 4 who measure
in the y-direction, $\mathcal{M}_y^G = \left\lceil \frac{N_y^G}{2} \right\rceil$, and $\mathcal{P}^G$ the sum of the measurement outcome of all members modulo 2. Then the following results are obtained.

(a) Suppose $N_y^G$ is even. Then $\mathcal{P}^G \oplus \mathcal{M}_y^G$ is 0 for $|\Phi^+_t\rangle$, and it is 1 for $|\Lambda^-_t\rangle$, where $a \oplus b = a + b \pmod{2}$ for any $a, b \in \mathbb{N}$.

(b) Suppose $N_y^G$ is odd. Then $\mathcal{P}^G \oplus \mathcal{M}_y^G$ is 0 for $|\Lambda^+_t\rangle$, and it is 1 for $|\Lambda^-_t\rangle$.

Also, it is noticed that if the above suppositions of $N_y^G$ are not satisfied, $\mathcal{P}^G \oplus \mathcal{M}_y^G$ becomes 0 or 1 with probability $\frac{1}{2}$ i.e. it has no rules.

Using an induction on $t$ the proof of such facts is given. To begin with, for $t = 1$ it is trivial. Assume that these statements are true for $t - 1$. The cat state $|\Phi^+_t\rangle$ is considered. Let $N_y^G$ be even. Equation (4) implies
\[
|\Phi^+_t\rangle = \frac{1}{\sqrt{2}} (|0\rangle_x |\Phi^+_t\rangle - |1\rangle_x |\Phi^-_t\rangle).
\] (8)

If any one member takes measurement in the $x$-direction and obtains 0 then $N_y^G = N_y^{G'}$ and $N_y^{G'}$ will be even, where $G'$ is the group of all members except that member. From Equation (5) $\mathcal{M}_y^{G'} \oplus \mathcal{P}^{G'} \equiv 0$ and $\mathcal{P}^G = \mathcal{P}^{G'}$. Thus $\mathcal{M}_y^G \oplus \mathcal{P}^G \equiv 0$. Otherwise, $\mathcal{M}_y^{G'} \oplus \mathcal{P}^{G'} \equiv 1$ by (5) and $\mathcal{P}^G \equiv \mathcal{P}^{G'} \equiv 1$. Thus $\mathcal{M}_y^G \oplus \mathcal{P}^G \equiv 0$.

On the other hand, for the case that the member takes a measurement in the $y$-direction, the proof is similar to the above case. Hence, we hold that $\mathcal{M}_y^G \oplus \mathcal{P}^G \equiv 0$. That is, the previous assumption holds for $t$. For other cat states, all of the proofs are similar.

Now, we consider two parties, A and B, that consist of $k$ members and $l$ members, respectively. Applying the previously described properties of the cat states, we obtain the Table II.

![Table II](image)

III. PROTOCOLS

By means of the properties of the cat states, we describe the QKD protocols between two groups. We first discuss how two groups proceed to share the secret key string. Next, by modifying several steps we show to be able to use the cat states efficiently.

A. Protocol

In this section, we present a QKD protocol between two groups, A and B, that consist $k \,(k > 1)$ members and $l \,(l > 2)$ members respectively. From here, with $n = k + l$ we use the $n$-particle cat states and suppose that all members are arbitrarily ordered. For each shared bit, each group requires a member who collects the information that has been distributed to all members. We call such members the ‘collectors’. We present one of the methods to collect the information after description of the protocol. We presume that a collector chooses the used cat state and its information is possessed by only the collector. However, if secure classical channels among members in A exists, all members in A may share the information on demand and then may choose the cat state together under their agreement.

1. A collector in A randomly chooses an $n$-particle cat state out of $|\Phi^+_n\rangle$ and $|\Lambda^+_n\rangle$ which is denoted by $|S\rangle$. Each particle of $|S\rangle$ is transmitted to each member of the two groups.

2. Each member of the two groups randomly performs a measurement on his own particle either in the $x$- or $y$-direction, respectively.

3. Each member in the two groups announces the basis he used through the public channel, but not the result he obtained. The two groups, A and B, obtain $N_A^y$ and $N_B^y$, respectively. We call the member who finally announces the basis in each group the ‘last member’ Here, the announcement of the last member in A should be followed by B’s.

4. Two groups, A and B, collect the outcomes to obtain $\mathcal{P}^A$ and $\mathcal{P}^B$, respectively, and then obtain the shared bit $\mathcal{M}_y^A \oplus \mathcal{P}^A$ and $\mathcal{M}_y^B \oplus \mathcal{P}^B$, respectively. The last member is never the collector and it will be discussed in Section IV.B.

In order to obtain the key bit strings, the two groups should repeat the above steps a sufficient number of times.

5. The two groups have a public discussion on a set of bits used to detect an eavesdropper’s presence. For the test bits, A reveals $\mathcal{P}^A$ and is followed by B. The reason will be treated in Section IV.B.

6. A announces the cat states $|S\rangle$ that were chosen at first. For $|S\rangle = |\Phi^+_n\rangle$, if $N_A^y + N_B^y$ is even, then the shared bit will be kept, and otherwise, it will be discarded. In case $|S\rangle = |\Lambda^+_n\rangle$, if $N_A^y + N_B^y$ is odd, it will be kept, and otherwise, it will be discarded. So the two groups keep it with probability $\frac{1}{2}$.

With a set of test bits, the two groups make independently a test to detect the presence of eavesdroppers or the faulty bit string made by some members who behave wrong.

If an error exists, all shared keys should be discarded, and the two groups should go back to Step 1. Otherwise, they go on the next step.

We suggest a method of obtaining $\mathcal{P}^A$ (or $\mathcal{P}^B$). Here, we consider the first member as a collector and all operations are module 2. The collector chooses a random
TABLE I: Relations between outcomes of $A$ and $B$.

| $|\Phi_+\rangle (|\Phi_-\rangle)$ | even | $N_y^A + N_y^B$ | $N_y^A M_y^B + P^A$ | $N_y^B M_y^A + P^B$ |
|-------------------------------|------|----------------|---------------------|---------------------|
| even                          | 0    | even 0(1)      | 1                   | 1(0)                |
| odd                           | 0    | odd 1(0)       | 1                   | 0(1)                |

We clearly remark that any member can play the chairperson if all members in $A$ have the information on $|S\rangle$.

To obtain a more efficient protocol, Step 3 and 4 in the previous protocol are modified as the followings.

3’. Except the chairperson each member in the two groups randomly performs a measurement on his own particle either in $x$- or $y$-direction.

4’. (a) Let $A'$ be the group consisting of all members in $A$ except the chairperson. All members in $A'$ and $B$ announce the measurement bases. Then the two groups get $N_y^{A'}$ and $N_y^B$, respectively. Now, the collector in $B$ never plays the last member.

(b) Using the properties of cat states, the chairperson performs the measurement on his particle depending on $N_y^{A'} + N_y^B$ and $|S\rangle$ in order to prevent the shared bit from being discarded.

We remark that the order of the basis announcements of two groups is not important in this protocol because $N_y^{A'}$ is determined by $N_y^B$ and $|S\rangle$.

IV. ANALYSIS OF SECURITY

In this section, we analyze security of our protocols. Firstly, we discuss the case that several members have some wrong behavior. The second case treat an eavesdropper who uses the intercept/resend strategy [10]. We again divide the second case into two cases according to the existence of members who give an eavesdropper some helps. Since the protocols should be secure even if all members in $A$ share the information on $|S\rangle$ before the transmission of the particles in the first step, we assume that all members in $A$ know the information.

A. Members with wrong behaviors

In this part, we discuss that when there are some members who behave wrong if two groups, particularly the
collectors, have the faulty key strings then they can notice it from the test.

To begin with, we treat a chairperson in the modified protocol. The measurement basis of the chairperson is exactly determined by the other members’ ones and the state |S⟩. Thus, he cannot change his basis arbitrarily, and can affect only his measurement result. From the above fact, we clearly obtain that he can have no more influence on the key information than the other members’ one. Therefore, it suffices to consider the investigation of the other members’ behavior.

We think over all members’ behavior except the collector’s one. Because $P^A$ (or $P^B$) is possessed by just a collector, any member except the collector cannot know it and hence cannot notice the shared bit. While some members are having behavior wrong, they cannot perceive what is the key bit made by their actions. Moreover, before the test step they cannot perceive if errors will be detected in the test step and what are the bit strings used to test. Hence, the nonexistence of errors in sufficiently many test bits implies that almost all the key bits have correct correlations (or anti-correlations). Since the collectors only possess the shared bits, if the bits have the correct correlations (or anti-correlations) the two groups can share correct keys although there exist some members to behave wrong. Therefore, if some members have a wrong effect on some of bit strings and if the two groups share the faulty bit strings, then they can find errors from sufficiently many test bits.

B. Eavesdropper and conspirators in two groups

Suppose that there are an eavesdropper and some members who assist her. Here, the eavesdropper and the members are called ‘Eve’ and ‘conspirators’, respectively. We first discuss that without assistance of conspirators Eve uses the intercept/resend strategy [11], and then discuss that with some helps of conspirators Eve uses such strategy and the l-particle entangled state to resend to B. Finally it is treated that under the same strategy Eve uses the n’-particle entangled state to resend to B ($n’ > l$).

1. No conspirator in intercept/resend strategy

We consider that Eve uses the intercept/resend strategy, i.e., Eve intercepts l particles travelling from A to B, performs a measurement on that particles, and resends an l-particle fake state instead. Even if Eve chooses a fake state according to the measurement result and resends it, the two groups will detect an error in the shared bit with probability $\frac{1}{2}$ for the original protocol, and with probability $\frac{1}{4}$ for the modified protocol, respectively. The difference of probability comes from the nonexistence of the discarded bits in the modified protocol. Hence, the two groups can find errors for sufficiently many test bits.

2. Using intercept/resend strategy: Eve and conspirators

We consider that Eve adopts the intercept/resend strategy and has some conspirators in two groups. First conspirators should try to change $N^A$, $N^B$, $P^A$ or $P^B$, to make no errors which are caused by Eve’s eavesdropping. However, as stated in Section [11] it is impossible for any member except the collector to change $P^A$ and $P^B$ into what they want. In addition, any member except the last members can never change $N^A$ and $N^B$ into what they want. Thus, it suffices to treat the case that one more conspirator plays collectors (or last members) under assumption that the collector (or the last members) is played in rotation by each member.

We assume that Eve eavesdrops with the probability $\lambda$, $0 \leq \lambda \leq 1$ using the intercept/resend strategy; $\lambda = 0$ means that Eve is not eavesdropping at all. Let $r_n$ be the number of conspirators in A and $r_b$ the number of conspirators in B. The two groups randomly select t shard bits in order to estimate the error rate. In the first protocol, Eve’s eavesdropping then causes at least the following error rates according to $r_n$ and $r_b$. In the case that $r_n = 0$ and $r_b \geq 1$, we have

$$1 - \left(\frac{7}{8}\right)^{\lambda t \left(1 - \frac{2r_b - 1}{r_b}\right)}.$$  

Applying Eve’s the measurement result on the intercepted particles, they can notice the values of $N^B$ and $P^B$ to make no errors. For example, in the case that the measurement result is $|\Phi^+_B\rangle$, if $N^B$ is even and $M^B + P^B$ is 0 then there is no error. So, only having assistance of a collector and a last member in B at once, they can forbid errors to be caused. They have the chance with ratio $\frac{2r_b - 1}{r_b}$ for one shared bit, i.e., the conspirators in B can play a collector and a last member simultaneously with the ratio. Hence Equation (9) is obtained.

We note that if a last member is a collector in B then the ratio becomes greater than $\frac{2r_b - 1}{r_b}$. Thus, in the first protocol, a last member can never be the identical person with a collector during one key agreement.

In the case $r_n \geq 1$ and $r_b = 0$, we have

$$1 - \left(\frac{7}{8}\right)^{\lambda t \left(1 - \frac{2r_n - 1}{r_n}\right)}.$$  

From the information on |S⟩ and Eve’s measurement result they can find the suitable values for $N^B$ and $P^A$ which induce no error. Thus Eve and conspirators in A can change these values into the found suitable ones, only if the conspirators play the collector and the last member in A simultaneously. Hence Equation (10) is found. It also becomes the reason that a last member in A never plays a collector.

In the case $r_n \geq 1$ and $r_b \geq 1$, we have

$$1 - \left(\frac{7}{8}\right)^{\lambda t \left(1 - \frac{2r_n - 1}{r_n}\right) \left(1 - \frac{2r_b - 1}{r_b}\right)}.$$  

(11)
In this case, it is clear that they are able to use two methods discussed in the above paragraphs. In addition, they are able to change \( N^B \) to make the shared bit be discarded, or change \( P^B \) to make no error by means of information on \( N^A, N^B, P^A \) and \(|S|\). To do so, they need assistance of any conspirator in \( A \) for \(|S|\) and either the collector or the last member in \( B \). The rate that such cases occur in \( B \) is \( \frac{r_a + r_b}{3} \). Therefore, we obtain the Equation (11).

Furthermore, we perceive that \( N^A_y \) have to be announced before \( N^B_y \). This is because if not, Eve is able to make no errors even with assistance of either the collector or the last member in \( A \) without any conspirator in \( B \).

Now, we can notice that the probability in Equations (9) and (10) are not less than in Equation (11). So it is sufficient to treat only Equation (11).

For case of \( r_b = l - 1 \), the probability in Equation (11) is 0 and then this protocol is not secure, but it is not so in the modified protocol which will be treated later. Next, we consider the case, \( r_a = k - 1 \) and \( r_b = l - 2 \).

\[
1 - \left( \frac{7}{8} \right)^{\lambda t - \left( 1 - \frac{r_a - 1}{k} \right) \left( 1 - \frac{r_b + 1}{l} \right)} \geq 0.95. \tag{12}
\]

if and only if

\[
\left( 1 - \frac{r_a - 1}{k} \right) \left( 1 - \frac{r_b + 1}{l} \right) = \frac{2}{k \cdot l} \cdot \lambda t \geq 270. \tag{13}
\]

Though \( r_a = k - 1 \) and \( r_b = l - 2 \), their existence can be detected with probability 0.95 by choosing sufficiently many test bits which satisfy \( \lambda t \geq 135 kl \). From the equations we can know that by making test bits be increased, even for extreme cases, eavesdropping can also be detected with as high probability as the two groups need. However, the more many test bits are required to detect Eve’s eavesdropping as the rate of existence of the conspirators increases.

We remark that quite many test bits should be chosen in the case that \( r_a = k - 1 \) and \( r_b = l - 2 \). Hence, upon all members’ deliberation for presumption of the number of members that can behave wrong, the number of test bits can effectively be modulated.

We now consider the case that \( r_a \) and \( r_b \) are not more than a half of the number of all members in \( A \) and \( B \), respectively. The probability in Equation (11) is larger than 0.95 if and only if

\[
\lambda t \left( 1 - \frac{r_a - 1}{k} \right) \left( 1 - \frac{r_b + 1}{l} \right) = \frac{2}{k \cdot l} \geq 22.44. \tag{14}
\]

Then if \( \frac{r_a}{k} \) and \( \frac{r_b}{l} \) are fixed, the probability increases as \( k \)’s value increases or \( l \)’s one decreases. From comparisons between Figures 2 and 4 and between Figures 3 and 4, we can certainly perceive the above facts.

If \( \frac{r_a}{k} \leq \frac{1}{2}, \frac{r_b}{l} \leq \frac{1}{3} \) and \( \lambda = 1 \), it follows from the condition \( l > 2 \) that the number of required test bits is not less than 270. However, for fixed \( k \) and \( l \) the fewer number of test bits are required. The change of the error rate according to the number of test bits is exemplified in Tables II for the case that \( \frac{r_a}{k} \leq \frac{1}{2} \) and \( \frac{r_b}{l} \leq \frac{1}{3} \).

The probability in the modified protocol has a little difference from the primary protocol because there are no discarded bits and \( N^A_y \) is determined by \( N^B_y \) and \(|S|\). By removing the strategy to use \( N^A \), we can easily get the following error rates of the modified protocol.

In the case \( r_a = 0 \),

\[
1 - \left( \frac{3}{4} \right)^{\left( 1 - \frac{r_b - 1}{l} \right) \lambda t}. \tag{15}
\]

In the case \( r_a \geq 1 \),

\[
1 - \left( \frac{3}{7} \right)^{\left( 1 - \frac{r_b}{l} \right) \lambda t}. \tag{16}
\]

As in the case of the first protocol, we analyze just the case \( r_a \geq 1 \). The error rate in Equation (16) has no connections with the values of \( r_a \) and \( k \), and depends just on \( r_b \) and \( l \). For \( r_b = l - 1 \), we require only \( l \) that satisfies \( \lambda t \geq 10.4 l \). For \( \frac{r_b}{l} = \frac{1}{2} \), it becomes \( \lambda t \geq 21.4 \). In the modified protocol, we can notice that two groups require remarkably smaller test bits than the first protocol, and furthermore errors can be detected from the test step.
FIG. 4: The probability in Equation (11) when \( k = 6, l = 6 \) and \( r_a = 3 \).

TABLE II: The probability in Equation (11) when \( k = 6, l = 6 \) and \( r_a = 3 \)

| \( r_b \) | 20 | 40 | 60 | 80 | 100 | 120 | 140 |
|-------|----|----|----|----|-----|-----|-----|
| 1     | 0.6948 | 0.9069 | 0.9716 | 0.9913 | 0.9974 | 0.9992 | 0.9998 |
| 2     | 0.5894 | 0.8314 | 0.9308 | 0.9716 | 0.9883 | 0.9952 | 0.998 |
| 3     | 0.4476 | 0.6948 | 0.8314 | 0.9069 | 0.9486 | 0.9716 | 0.9843 |

even in the cases \( l = 2 \) or, \( r_a = k - 1 \) and \( r_b = l - 1 \), while errors cannot be detected for the case in the first protocol.

We remark that if just a collector can take the information on \( |S\rangle \) or only one member plays the collector for all shared bits then a fewer test bits would be required.

3. Intercept/resend strategy using entangled states

We assume that Eve intercepts \( l \) particles travelling from \( A \) to \( B \), and call this state ‘the intercepted state’. She chooses an \( n' \)-particle cat state and resends \( l \) particles of this cat state to \( B \) (\( n' > l \)). We refer to the remainder \( (n' - l) \)-particle state as ‘the remainder state’.

Before announcement of \( N_y^A \) (or \( N_y^B \)) the measurement of the intercepted state (or the remainder state) cannot give her the information on \( \mathcal{P}^A \) (or \( \mathcal{P}^B \)). So she should measure on the intercepted state and the remainder state, after \( N_y^A \) and \( N_y^B \) are announced. Even though she measures in the way, she should have the information on \( S \) to obtain \( \mathcal{P}^A \), since \( \mathcal{P}^A \) is completely determined by \( |S\rangle \) and \( N_y^A \). In order to take information on \( |S\rangle \), she needs any conspirator in \( A \).

On the other hand, she wants to change \( \mathcal{P}^A \) or \( \mathcal{P}^B \) into the values she desires to prevent errors from occurring. Thus she needs collectors’ assistance in \( A \) or \( B \). Without any conspirators the test induces errors with probability \( \frac{1}{2} \) in the first protocol and probability \( \frac{1}{2} \) in the modified protocol, respectively. For these facts Eve’s strategy makes at least the following error rate in the first protocol.

In the case \( r_a = 0 \),

\[
1 - \left( \frac{3}{4} \right)^{\lambda t}. \tag{17}
\]

In the case \( r_a \geq 1 \),

\[
1 - \left( \frac{3}{4} \right)^{\lambda t(1 - \frac{r_a}{3})(1 - \frac{r_a}{3})}. \tag{18}
\]

In the modified protocol the error rates are similar to the first protocol, because Eve cannot have a different strategy. From these equations we can know that this strategy is not optimal to Eve.

V. Summary

Applying the properties of cat states and the secret sharing \( \tilde{c} \), we proposed two generalized QKD protocols between two groups and showed that the protocols are secure against an external eavesdropper using the intercept/resend strategy. The importance of these protocols is that any member in the two groups cannot obtain the secret key strings without cooperation, that is, the secret key strings can be obtained only under all member’s approval.

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