BIORTHOGONAL ENSEMBLES

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ABSTRACT. One object of interest in random matrix theory is a family of point ensembles (random point configurations) related to various systems of classical orthogonal polynomials. The paper deals with a one-parametric deformation of these ensembles, which is defined in terms of the biorthogonal polynomials of Jacobi, Laguerre and Hermite type.

Our main result is a series of explicit expressions for the correlation functions in the scaling limit (as the number of points goes to infinity). As in the classical case, the correlation functions have determinantal form. They are given by certain new kernels which are described in terms of the Wright’s generalized Bessel function and can be viewed as a generalization of the well-known sine and Bessel kernels.

In contrast to the conventional kernels, the new kernels are non-symmetric. However, they possess other, rather surprising, symmetry properties.

Our approach to finding the limit kernel also differs from the conventional one, because of lack of a simple explicit Christoffel–Darboux formula for the biorthogonal polynomials.

1. Introduction

Orthogonal polynomial ensembles are widely known. They are characterized by the property that the joint probability density of an $N$-point ensemble has the form

$$p(x_1, \ldots, x_N) = \text{const} \cdot \prod_{i=1}^{N} \omega(x_i) \prod_{i<j} (x_i - x_j)^2$$

for some positive weight function $\omega(x)$. The phase space $I$ of an orthogonal ensemble is a finite or infinite interval of the real line. Usually the couple $(\omega(x), I)$ corresponds to one of the classical systems of orthogonal polynomials.

These ensembles play a very important role in the random matrix theory, see [Me]. They also serve as a rich source of various mathematical problems which include Selberg integrals, differential equations for Fredholm determinants and many others, see, for example, [Me], [TW].

One of the main properties of orthogonal ensembles is the existence of simple formulas for all correlation functions. Namely, the $n$th correlation function has the form

$$\rho_n(x_1, \ldots, x_n) = \prod_{i=1}^{n} \omega(x_i) \cdot \det [K_N(x_i, x_j)]_{i,j=1}^{n}$$

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where \( K_N(x, y) \) is the \( N \)th Christoffel-Darboux kernel:

\[
K_N(x, y) = \frac{p_N(x)p_{N-1}(y) - p_{N-1}(x)p_N(y)}{x - y};
\]

\( \{p_k(x)\} \) is the system of orthogonal polynomials on \( I \) with weight \( \omega(x) \) (see [D], [Me], [NW1]). Determinantal formulas (*) often allow to study the asymptotic behaviour of an \( N \)-point biorthogonal ensemble when the number of points \( N \) goes to infinity. It turns out that after the appropriate scaling the correlation functions tend to a limit. The limit correlation functions also have determinantal form with a certain limit kernel.

The well–known sine kernel

\[
\frac{\sin \pi(x - y)}{\pi(x - y)}
\]

arises in the scaling limit of the classical polynomial ensembles in the bulk of spectrum. In particular, the sine kernel arises in the case of the Hermite weight function, see, e.g., [Me, section 5.2].

At the left edge of the spectrum, both the Jacobi and Laguerre weights produce the same Bessel kernel ([F], [NW2])

\[
\frac{\varphi_1(x)\varphi_2(y) - \varphi_1(y)\varphi_2(x)}{x - y}
\]

where

\[
\varphi_1(x) = J_\alpha(2\sqrt{x}), \quad \varphi_2(x) = x\varphi'_1(x);
\]

\( \alpha > -1 \), \( J_\alpha(z) \) is the Bessel function.

Both the sine and the Bessel kernels are symmetric and represent positive self-adjoint integral operators in \( L^2 \) on \( \mathbb{R}_+ \) and on \( \mathbb{R} \) respectively.

In this paper we study a one parameter generalization of the orthogonal polynomial ensembles, the joint probability densities of our ensembles have the form

\[
p^b(x_1, \ldots, x_N) = \text{const} \cdot \prod_{i=1}^{N} \omega(x_i) \prod_{i<j} \left[ (x_i - x_j)(x_i^\theta - x_j^\theta) \right]
\]

where \( \theta \) is a fixed positive number. We call these ensembles biorthogonal. Orthogonal polynomial ensembles correspond to \( \theta = 1 \).

Biorthogonal ensembles inherit several nice properties from the orthogonal ones. For example, their correlation functions also have determinantal form, and corresponding formulas can be expressed via so-called biorthogonal polynomials, see [K1] for general definitions. This fact was proved by K. A. Muttalib in [Mu]. He also argued that biorthogonal ensembles are of certain interest in physics.

We consider three different cases:

1. \( I = (0, 1) \), \( \omega(x) = x^\alpha \),
2. \( I = (0, +\infty) \), \( \omega(x) = x^\alpha e^{-x} \),
3. \( I = (-\infty, +\infty) \), \( \omega(x) = |x|^\alpha e^{-x^2} \).
Naturally, we shall call the second ensemble \textit{biorthogonal Laguerre ensemble}, it depends on one real parameter \( \alpha > -1 \).

Ensemble (1) is given by a special case of the Jacobi weight function. Namely, the factor \((1 - x)^\beta\) of the general Jacobi weight is absent in our formula for \(\omega(x)\). Our techniques does not allow to handle the case of arbitrary \(\beta\), so we shall set \(\beta = 0\) but we shall still use the words \textit{biorthogonal Jacobi ensemble} in this case. Thus, our Jacobi biorthogonal ensemble also depends on one real parameter \(\alpha > -1\).

The weight function in (3) is, in contrary, more general than the Hermite weight function, which corresponds to \(\alpha = 0\). However, we shall deal with this more general case and use the words \textit{biorthogonal Hermite ensemble}. This ensemble depends on one real parameter \(\alpha > -1\).

We prove that in all these cases after an appropriate scaling there exists a limit of the correlation functions, and we explicitly compute the limit kernels thus obtained.

It turns out that the limit kernel in Jacobi and Laguerre cases is the same and it equals

\[
K^{(\alpha, \theta)}(x, y) = \sum_{k, l=0}^{\infty} \frac{(-1)^k x^k}{k! \Gamma(\frac{\alpha + 1 + k}{\theta})} \frac{(-1)^l y^l}{l! \Gamma(\alpha + 1 + \theta l)} \theta \frac{\theta}{\alpha + 1 + k + \theta l}.
\]

This kernel can be represented as the integral

\[
K^{(\alpha, \theta)}(x, y) = \theta \int_0^1 J_{\alpha + 1, \theta}(xt) \cdot J_{\alpha + 1, \theta}(yt) t^\alpha dt
\]

where

\[
J_{a,b}(x) = \sum_{m=0}^{\infty} \frac{(-x)^m}{m! \Gamma(a + bm)}
\]

is the Wright’s generalized Bessel function, see [Wr], [E2, 18.1].

The limit correlation functions have the form

\[
\rho_n^{Jac(\alpha, \theta)}(x_1, \ldots, x_n) = \rho_n^{Lag(\alpha, \theta)}(x_1, \ldots, x_n) = \prod_{i=1}^{n} x_i^\alpha \cdot \det \left[ K^{(\alpha, \theta)}(x_i, x_j) \right]_{i,j=1}^{n}.
\]

These functions are defined on \(\mathbb{R}_+\).

The limit kernel in the Hermite case is expressed via \(K^{(\alpha, \theta)}(x, y)\) and has the form

\[
K^{Her(\alpha, \theta)}(x, y) = K^{\left(\frac{\alpha + 1}{2}, \theta\right)}(x^2, y^2) + x^\theta y \cdot K^{\left(\frac{\alpha - 1}{2}, \theta\right)}(x^2, y^2).
\]

The limit correlation functions are

\[
\rho_n^{Her(\alpha, \theta)}(x_1, \ldots, x_n) = \prod_{i=1}^{n} |x_i|^{\alpha} \cdot \det \left[ K^{Her(\alpha, \theta)}(x_i, x_j) \right]_{i,j=1}^{n}.
\]

They are defined on the whole real axis.

Two limit kernels \(|xy|^\alpha/2 K^{Her(\alpha, \theta)}(x, y)\) and \((xy)^{\alpha/2} K^{(\alpha, \theta)}(x, y)\) can be considered as biorthogonal generalizations of sine and Bessel kernels, see below Examples 5.5 and 3.5, respectively.
Though these kernels are not symmetric anymore, the transposition $x \leftrightarrow y$ leads to a non-trivial symmetry in the asymptotics of Laguerre and Hermite ensembles. Namely, the change of parameters

$$\alpha \mapsto \frac{\alpha + 1}{\theta} - 1, \quad \theta \mapsto \frac{1}{\theta}$$

turns out to be equivalent to the transformation $x \mapsto x^\theta$ of the phase space, see Corollaries 4.7 and 5.6 below. Finite point ensembles do not possess this symmetry, it appears only in the asymptotics.

The method that we use in this paper also provides a new approach to the asymptotics of classic orthogonal polynomial ensembles.

The paper is organized as follows. Section 1 is the introduction. In Section 2 we prove some general statements for further purposes. Section 3 deals with the biorthogonal Jacobi ensemble. In Section 4 we work with the biorthogonal Laguerre ensemble. The trick that we use there to obtain a formula for Christoffel-Darboux kernels originates from our work on stochastic point processes arising in the representation theory of the infinite symmetric group (see [O], [B], [BO]). We hope to explain this connection in subsequent publications. In Section 5 we compute the asymptotics of the biorthogonal Hermite ensemble. Section 6 is an appendix, it contains rigorous proofs of two main theorems. Heuristic proofs of these theorems can be found at the appropriate places of the main text.

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2. Generalities

We start with a discussion of well-known orthogonal polynomial ensembles, see [NW1], [Me]. Consider an ensemble of $N$ points on a (possibly infinite) interval $(a, b)$ of the real line with the joint probability density of the form

$$p(x_1, \ldots, x_N) = \text{const} \cdot \prod_{i=1}^{N} \omega(x_i) \prod_{i<j} (x_i - x_j)^2$$

where $\omega(x)$ is some positive weight function on $(a, b)$. We can explicitly compute all correlation functions of this ensemble using orthogonal polynomials

$$\{p_k(x)\}_{k=0}^{\infty}, \quad \deg p_k = k;$$

$$\int_{a}^{b} p_k(x)p_l(x)\omega(x)dx = \delta_{kl}.$$  

Namely, we consider the $N$th Christoffel-Darboux kernel

$$K_N(x, y) = \sum_{i=0}^{N-1} p_i(x)p_i(y).$$

The integral operator in $L^2([a, b], \omega(x)dx)$ with this kernel is the orthogonal projection on the $N$–dimensional subspace $\text{Span}\{1, x, \ldots, x^{N-1}\}$. 
The correlation functions

\begin{equation}
\rho_n(x_1, \ldots, x_n) = \frac{N!}{(N-n)!} \int_a^b \cdots \int_a^b p(x_1, \ldots, x_N) dx_{n+1} \cdots dx_N
\end{equation}

have the following form (see \cite{D}, \cite{Me}, \cite{NW1})

\begin{equation}
\rho_n(x_1, \ldots, x_n) = \prod_{i=1}^n \omega(x_i) \cdot \det[K_N(x_i, x_j)]_{i,j=1}^n.
\end{equation}

The Christoffel-Darboux formula (see \cite{S}, \cite{E1})

\begin{equation}
K_N(x, y) = \frac{p_N(x)p_{N-1}(y) - p_{N-1}(x)p_N(y)}{x-y}
\end{equation}

provides a convenient way of analyzing the asymptotic behaviour of $K_N(x, y)$ when $N \to \infty$.

Let us look at $K_N(x, y)$ from another point of view. Clearly, the kernel has the form

\begin{equation}
K_N(x, y) = \sum_{k,l=0}^{N-1} c_{kl} x^k y^l,
\end{equation}

where $c_{kl}$ are some constants. There is a simple way of saying what these constants are. Set

\[ g_{ij} = \int_a^b x^{i+j} \omega(x) dx. \]

Let

\[ C = (c_{kl})_{k,l=0}^{N-1}, \quad G = (g_{ij})_{i,j=0}^{N-1}. \]

**Proposition 2.1.** With the preceding notation

\begin{equation}
C = G^{-1}.
\end{equation}

**Proof.** By definition of the Christoffel-Darboux kernel, for all $j = 0, 1, \ldots, N-1$

\[ \int_a^b K_N(x, y) y^j dy = x^j. \]

On the other hand

\[ \int_a^b K_N(x, y) y^j dy = \int_a^b \sum_{k,l=0}^{N-1} c_{kl} x^k y^l \cdot y^j dy = \sum_{k,l=0}^{N-1} c_{kl} g_{lj} x^k. \]
By equating these two expressions, we get
\[ \sum_{l=0}^{N-1} c_{kl} g_{lj} = \delta_{kj}. \]

Proposition 2.1 and (2.6) provide another way of analyzing the kernel $K_N(x, y)$. As we shall see, this will also work for biorthogonal ensembles, where Christoffel-Darboux formula becomes rather complicated.

Let us introduce the biorthogonal ensembles in the following very general way. Consider an ensemble of $N$ points on $(a, b) \subset \mathbb{R}$ with the joint probability density of the form

\begin{equation}
(2.8) \quad p_b(x_1, \ldots, x_N) = \text{const} \cdot \prod_{i=1}^{N} \omega(x_i) \cdot \det[\xi_i(x_j)]_{i,j=1}^{N} \cdot \det[\eta_i(x_j)]_{i,j=1}^{N},
\end{equation}

where $\xi_i(x), \eta_i(x)$: $i = 1, 2, \ldots$ are some functions defined on $(a, b)$. The formula (2.1) is clearly a special case of (2.8), take $\xi_i(x) = \eta_i(x) = x^{i-1}$.

Suppose, we managed to biorthogonalize \( \{\xi_i\} \) and \( \{\eta_i\} \) with respect to the pairing

\begin{equation}
(2.9) \quad \langle \xi, \eta \rangle = \int_{a}^{b} \xi(x) \eta(x) \omega(x) dx.
\end{equation}

That is, we have two other systems of functions, say \( \{\zeta_i(x)\}_{i=1}^{N} \) and \( \{\psi_i(x)\}_{i=1}^{N} \), such that

\[ \zeta_i \in \text{Span}\{\xi_1, \ldots, \xi_i\}, \quad \psi_j \in \text{Span}\{\eta_1, \ldots, \eta_j\}; \]
\[ \langle \zeta_i, \psi_j \rangle = \delta_{ij} \]

for all possible $i$ and $j$.

Exactly the same argument as for orthogonal ensembles proves that the correlation functions (defined as in (2.3)) of a biorthogonal ensemble have the form

\begin{equation}
(2.10) \quad \rho^b_n(x_1, \ldots, x_n) = \prod_{i=1}^{n} \omega(x_i) \cdot \det[K^b_N(x_i, x_j)]_{i,j=1}^{n},
\end{equation}

where

\begin{equation}
(2.11) \quad K^b_N(x, y) = \sum_{i=1}^{N} \zeta_i(x) \psi_i(y).
\end{equation}

is an analog of the Christoffel-Darboux kernel (2.2).

In the next section we shall restrict ourselves to the case

\[ \xi_i(x) = x^{i-1}, \quad \eta_i(x) = x^{\theta(i-1)}; \theta > 0. \]

Then \( \{\zeta_i\} \) and \( \{\psi_i\} \) are so-called biorthogonal polynomials, see [K1]. In this case determinantal formula (2.10) was proved in [Mu].
(In Section 5, however, it will be more convenient to take
\[ \xi_i(x) = x^{\theta(i-1)}, \quad \eta_i(x) = x^{i-1}. \]
This transposition, clearly, does not change the ensemble (2.8) and its correlation functions (2.10), but it interchanges the variables \( x \) and \( y \) in the Christoffel–Darboux kernel (2.11).

Unfortunately, there is no simple analog of Christoffel-Darboux formula in the biorthogonal case. For example, for integer \( \theta = k \in \mathbb{N} \), the following formula is proved in [I] (cf. (2.5))

\[
K_N^b(x,y) = \frac{1}{x^k - y^k} \left( \sum_{r,s \geq 0, r + s \leq k-1} \alpha_{rs} \zeta_{N+s+1}(x) \psi_{N-r}(y) + \beta \zeta_N(x) \psi_{N+1}(y) \right),
\]
where \( \alpha_{rs}, \beta \) are some constants. But even in the simplest case \( \theta = 2 \) it requires a lot of technical work to compute these constants for known biorthogonal polynomials.

We shall use another approach. The kernel \( K_N^b(x,y) \) has the following form (cf. (2.6))

\[
(2.12) \quad K_N^b(x,y) = \sum_{k,l=1}^{N} c_{kl}^b \xi_k(x) \eta_l(y)
\]
for some constants \( c_{kl}^b \). Set

\[
g_{ij}^b = \int_a^b \xi_j(x) \eta_i(x) \omega(x) dx.
\]
Let

\[
C^b = (c_{kl}^b)_{k,l=1}^N, \quad G^b = (g_{ij}^b)_{i,j=1}^N.
\]

It turns out that we have the exact analog of Proposition 2.1.

**Proposition 2.2.** With the preceding notation

\[
(2.13) \quad C^b = (G^b)^{-1}.
\]

The proof is just the same.

To conclude this section, we shall consider even more general situation, where an analog of Propositions 2.1 and 2.2 holds.

Let us fix two (possibly infinite) intervals \((a_1, b_1)\) and \((a_2, b_2)\) of the real line. We consider a distribution \( p(x_1, \ldots, x_N; y_1, \ldots, y_N) \) defined on \((a_1, b_1)^N \times (a_2, b_2)^N\) of the form

\[
p(x_1, \ldots, x_N; y_1, \ldots, y_N) = \text{const} \cdot \det w(x_i, y_j) \cdot \det \xi_i(x_j) \cdot \det \eta_i(y_j),
\]
all subscripts vary from 1 to \( N \). Here \( w(x, y) \) is a (generalized) function on \((a_1, b_1) \times (a_2, b_2); \xi_i(x), \eta_i(x)\) are some (generalized) functions defined on \((a_1, b_1)\) and \((a_2, b_2)\), respectively.
We choose the constant so that
\[ \int_{(a_1,b_1)^N \times (a_2,b_2)^N} p(x_1, \ldots, x_N; y_1, \ldots, y_N) dx_1 \cdots dx_N dy_1 \cdots dy_N = 1. \]

Suppose again that we managed to biorthogonalize the systems \( \{\xi_i\} \) and \( \{\eta_i\} \) with respect to the pairing
\[ \langle \xi, \eta \rangle = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \xi(x) \eta(y) w(x, y) dx dy. \]

Thus, we have somehow constructed two systems \( \{\zeta_i(x)\}_{i=1}^N \) and \( \{\psi_i(y)\}_{i=1}^N \), such that
\[ \zeta_i \in \text{Span}\{\xi_1, \ldots, \xi_i\}, \quad \psi_j \in \text{Span}\{\eta_1, \ldots, \eta_j\}; \]
\[ \langle \zeta_i, \psi_j \rangle = \delta_{ij} \]
for all possible \( i \) and \( j \).

If \( a_1 = a_2 = a, b_1 = b_2 = b \), and \( w(x, y) = \omega(x) \delta(x - y) \), then we return to the situation described above: (2.15) and (2.9) will coincide.

It turns out that if we compute the “correlation functions” of the “measure” \( p \) only in \( (a_1, b_1) \) (i.e., we integrate \( p \) over all \( N \) \( y \)'s and over some \( x \)'s), then these correlation functions also have determinantal form. (In fact, as was recently proved in [EM], all correlation functions of measures of the type (2.14) have determinantal form, but this statement is much harder.)

The following statement is proved in [MS].

**Proposition 2.3.** With the preceding notation
\[ \frac{N!}{(N-n)!} \int_{(a_1,b_1)^{N-n} \times (a_2,b_2)^N} p(x_1, \ldots, x_N; y_1, \ldots, y_N) dx_{n+1} \cdots dx_N dy_1 \cdots dy_N \]
\[ = \det[\tilde{K}_N(x_i, x_j)]_{i,j=1}^n, \]
where
\[ \tilde{K}_N(x, t) = \sum_{i=1}^N \zeta_i(x) \int_{a_2}^{b_2} \psi_i(y) w(t, y) dy. \]

We shall use Proposition 2.3 in Section 4.

Note that if \( a_1 = a_2 = a, b_1 = b_2 = b \), and \( w(x, y) = \omega(x) \delta(x - y) \), then
\[ \tilde{K}_N(x, t) = \omega(t) K^b_N(x, t). \]

As above, we can write
\[ \sum_{i=1}^N \zeta_i(x) \psi_i(y) = \sum_{k,l=1}^N \tilde{c}_{kl} \xi_k(x) \eta_l(y) \]
for some constants \( \tilde{c}_{kl} \).

Set
\[ \tilde{g}_{ij} = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \xi_j(x) \eta_i(y) w(x, y) dx dy; \]
\[ \tilde{C} = (\tilde{c}_{kl})_{k,l=1}^N, \quad \tilde{G} = (\tilde{g}_{ij})_{i,j=1}^N. \]

Quite similarly to Propositions 2.1 and 2.2 we get the following assertion.
Proposition 2.4. With the preceding notation

\( C = \tilde{G}^{-1} \).

By (2.17) and (2.18) we get

\[
K_N(x,t) = \sum_{k,l=1}^{N} c_{kl} \xi_k(x) \int_{a_2}^{b_2} \eta_l(y) w(t,y) dy.
\]

Note that everywhere above we supposed that the corresponding biorthogonal systems exist. The following simple statement will guarantee the existence in all our further examples.

Proposition 2.5. If all principal minors of \( \tilde{G} \) are not zero, then there exist biorthogonal systems \( \{ \zeta_i(x) \}_{i=1}^{N} \) and \( \{ \psi_i(y) \}_{i=1}^{N} \).

Proof. The hypothesis implies that \( \tilde{G} \) possesses a Gauss decomposition: it can be represented as the product of a lower triangular and an upper triangular matrices. Thus, there exist a lower triangular matrix \( L = (l_{ij}) \) and an upper triangular matrix \( U = (u_{ij}) \) such that

\[
\tilde{L} \tilde{G} U = \text{Id}.
\]

Set

\[
\zeta_i = \sum_{j=1}^{N} u_{ji} \xi_j; \quad \psi_j = \sum_{i=1}^{N} l_{ji} \eta_i.
\]

A straightforward check shows that these systems are biorthogonal. \( \square \)

3. Biorthogonal Jacobi ensemble

Our goal in this section is to study the asymptotic behaviour of the \( N \)-point ensemble on \((0,1)\) with the joint probability density (cf. (2.8))

\[
p_{Jac}^N(x_1, \ldots, x_N) = \text{const} \prod_{i=1}^{N} x_i^\alpha \prod_{i<j} (x_i^\theta - x_j^\theta) = \text{const} \prod_{i=1}^{N} x_i^\alpha \cdot \det x_i^{j-1} \cdot \det x_i^{\theta(j-1)}
\]

where \( \theta > 0 \) and \( \alpha > -1 \). We call this ensemble the biorthogonal Jacobi ensemble. Let us compute the matrix of pairwise scalar products \( g_{ij}^{Jac} \). We have

\[
g_{ij}^{Jac} = \int_0^1 x^{j-1+\theta(i-1)+\alpha} dx = \frac{1}{j + \theta(i-1) + \alpha}.
\]

To invert this matrix we shall use the following lemma.
Lemma 3.1. Let $A = \{A_1, \ldots, A_N\}$ and $B = \{B_1, \ldots, B_N\}$ be two sequences of complex numbers such that $A_i + B_j \neq 0$ for all $i, j = 1, \ldots, N$, and $A_i \neq A_j$, $B_i \neq B_j$ for $i \neq j$. Then

$$\left( \frac{1}{A_i + B_j} \right)^{-1} = (C_{kl})$$

where

$$C_{kl} = \frac{\prod_{i=1}^{N} [(B_i + A_l)(A_i + B_k)]}{\prod_{i \neq j} (A_i - A_j) \prod_{j \neq k} (B_k - B_j) \prod_{i=1}^{N} (A_i + B_j)}.$$  \hspace{1cm} (3.3)

Proof. As is known, the elements of the inverse matrix are the ratios of the cofactors of corresponding elements of the initial matrix and the determinant of the initial matrix. The determinant of our matrix

$M = \left( \frac{1}{A_i + B_j} \right)$

is well-known Cauchy determinant, see [W]:

$$\det M = \prod_{i<j} [(A_i - A_j)(B_i - B_j)] \prod_{i=1}^{N} \prod_{j=1}^{N} (A_i + B_j).$$

Every submatrix of $M$ has the same form as $M$ itself for some other sets $A$ and $B$. Then we can use the formula for the Cauchy determinant for computing any minor of $M$. In particular, we can compute all cofactors, and, thus, the inverse matrix. The result is exactly (3.3). \Box

Proposition 3.2. The inverse of the Gram matrix $(g_{ij}^{Jac})$ has the form

$$\left( g_{ij}^{Jac} \right)^{-1} = \theta \left( \frac{(k+\alpha)}{(k-1)!} \right)^N \cdot \frac{(\theta(l-1) + \alpha + 1)}{(l-1)!} \cdot \frac{(-1)^{k+l}}{k + \theta(l-1) + \alpha},$$

where $\theta(a) = a(a+1)\cdots(a+m-1)$ stands for the Pohhammer symbol.

Proof. Direct application of Lemma 3.1 for

$$A_i = \theta(i-1), \quad B_i = i + \alpha; \quad i = 1, \ldots, N. \-box$$

Proposition 3.3. The correlation functions of the biorthogonal $N$-point Jacobi ensemble have the form

$$\rho_{nN}(x_1, \ldots, x_n) = \prod_{i=1}^{n} x_i^\alpha \cdot \det[K_n^{Jac}(x_i, x_j)]_{i,j=1}^{n}$$

where

$$K_n^{Jac}(x, y) = \theta \sum_{k,l=1}^{N} \frac{(k+\alpha)}{(k-1)!} x^{k-1} \cdot \frac{(\theta(l-1) + \alpha + 1)}{(l-1)!} \cdot \frac{(-1)^{k+l}}{k + \theta(l-1) + \alpha},$$

$$\hspace{1cm} (3.5)$$
Proof. The claim follows from Proposition 2.2 and Proposition 3.2. The existence of the Christoffel-Darboux kernel $K_N^{ac}(x, y)$ is guaranteed by Proposition 2.5 because all minors of the matrix $(g_{ij}^{ac})$ are nonzero. □

Now we are in a position to compute the asymptotics of our ensemble as $N \to \infty$. We shall employ the following entire function introduced by E. M. Wright, see [Wr], [E2, 18.1(27)]

$$J_{a, b}(x) = \sum_{m=0}^{\infty} \frac{(-x)^m}{m! \Gamma(a + bm)}$$

(our notation differs from that used in [Wr], [E2]). It is closely related to Mittag–Leffler type functions, see [E2, 18.1].

Note that

$$x^{\frac{a}{2}}J_{a+1, 1}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{m+\frac{a}{2}}}{m! (a + m + 1)} = J_a(2\sqrt{x})$$

where $J_a(x)$ is the Bessel function.

**Theorem 3.4.** For all $n = 1, 2, \ldots$ there exists the limit

$$\lim_{N \to \infty} \frac{1}{N^n(1+\frac{\theta}{\alpha})^r} \rho^{ac}_{nN} \left( \frac{x_1}{N^{1+\frac{\theta}{\alpha}}}, \ldots, \frac{x_n}{N^{1+\frac{\theta}{\alpha}}} \right) = \prod_{i=1}^{n} x_i^{\alpha} \cdot \det \left[ K^{(\alpha, \theta)}(x_i, x_j) \right]_{i,j=1}^{n}$$

where

$$K^{(\alpha, \theta)}(x, y) = \sum_{k,l=0}^{\infty} \frac{(-1)^k x^k}{k! \Gamma\left(\frac{\alpha + 1 + k}{\theta}\right)} \frac{(-1)^l y^l}{l! \Gamma\left(\alpha + 1 + \theta l\right)} \frac{\theta}{\alpha + 1 + k + \theta l}$$

(3.6)

$$= \theta \int_{0}^{1} J_{\alpha+\frac{\theta}{\alpha}}(xt) \cdot J_{\alpha+1, \theta}(yt) t^{\alpha} dt.$$

Heuristic proof. If we substitute the asymptotic relations

$$\left(\frac{k+\alpha}{\theta}\right)_N \sim \frac{N^{(k+\alpha)/\theta} + 1}{\Gamma\left(\frac{k+\alpha}{\theta}\right)}$$

$$\left(\frac{\theta(l-1)+\alpha}{\theta}\right)_N \sim \frac{N^{(\theta(l-1)+\alpha)} + 1}{\Gamma(\alpha + 1 + \theta(l-1))}$$

in (3.5) and shift the summation indices $k$ and $l$ by one, then we get (3.6). □

A rigorous proof of Theorem 3.4 (which is nothing more than a formalization of the argument above) can be found in Section 6.

**Example 3.5.** For $\theta = 1$, Theorem 3.4 (and Theorem 4.5, see below) is well known, see [NW2], [F]. In this case it is easy to check that

$$(xy)^{\alpha/2} K^{(\alpha, \theta)}(x, y) = \frac{\varphi_1(x)\varphi_2(y) - \varphi_1(y)\varphi_2(x)}{x - y}$$

where

$$\varphi_1(x) = J_{\alpha}(2\sqrt{x}), \quad \varphi_2(x) = x\varphi_1'(x),$$
is the Bessel kernel.

Let us draw one more interesting corollary from Lemma 3.1.

Note that in the notation of Section 2 if we know the Christoffel-Darboux kernels (2.11) then we can easily obtain explicit formulas for biorthogonal functions \( \{ \zeta_i(x) \} \) and \( \{ \psi_i(x) \} \). Namely, we have the relation

\[
\zeta_m(x) \psi_m(y) = K^b_m(x,y) - K^b_{m-1}(x,y).
\]

By Lemma 3.1 we can compute the kernel \( K^b_m(x,y) \) for the systems

\[
\xi_i(x) = x^{a_i}, \quad \eta_i(x) = x^{b_i};
\]

\[ a_i + b_j > -1, \quad i, j = 1, 2, \ldots \]

in \( L^2([0,1],dx) \). The result of computation of the corresponding biorthogonal systems is expressed in the next statement.

**Proposition 3.6.** Let \( a_1, a_2, \ldots \) and \( b_1, b_2, \ldots \) be two sequences of complex numbers such that

\[ a_i + b_j > -1, \quad i, j = 1, 2, \ldots \]

and \( a_i \neq a_j, \ b_i \neq b_j \) for \( i \neq j \). Then two systems of functions

\[
\zeta_n(x) = \sqrt{a_n + b_n + 1} \sum_{i=1}^{n} \prod_{k=1,k \neq i}^{n-1} \frac{(a_i + b_k + 1)}{(a_i - a_k)} x^{a_i}, \quad n = 1, 2, \ldots
\]

and

\[
\psi_n(x) = \sqrt{a_n + b_n + 1} \sum_{i=1}^{n} \prod_{k=1,k \neq i}^{n-1} \frac{(b_i + a_k + 1)}{(b_i - b_k)} x^{b_i}, \quad n = 1, 2, \ldots
\]

are biorthonormal in \( L^2([0,1],dx) \). In other words

\[
\int_0^1 \zeta_m(x) \psi_n(x) dx = \delta_{mn}.
\]

**Example 3.7.** For

\[ a_i = b_i = i - 1 \]

our biorthogonal systems are classic Jacobi polynomials. If

\[ a_i = i - 1, \quad b_i = \theta(i - 1) \]

for some \( \theta > 0 \) we get explicit formulas for biorthogonal Jacobi polynomials, cf. [MT1].
4. Biorthogonal Laguerre ensemble

In this section we study the asymptotic behaviour of the \( N \)-point ensemble on \((0, +\infty)\) with the joint probability density

\[
p_{\text{Lag}}^N(x_1, \ldots, x_N) = \text{const} \cdot \prod_{i<j} [(x_i - x_j)(x_i^\theta - x_j^\theta)] \cdot \prod_{i=1}^N x_i^\alpha \cdot e^{-x_1 - \cdots - x_N} \]  

where \( \alpha > -1 \) and \( \theta > 0 \).

We call it the \( N \)-point biorthogonal Laguerre ensemble. It turns out that the asymptotics of this ensemble is governed by the same kernel as that of biorthogonal Jacobi ensemble, see the previous section. However, in this case it is much more difficult to show. Here we do not know an analog of Lemma 3.1 for inverting the matrix of scalar products. However, the following trick allows to reduce the computation of the correlation functions to Lemma 3.1.

Let us introduce \( N \) new variables \( y_1, \ldots, y_N \) and a new distribution in \( 2N \) variables

\[
p_N(x_1, \ldots, x_N; y_1, \ldots, y_N) = p_{\text{Lag}}^N(x_1, \ldots, x_N) \prod_{i=1}^N \delta(y_i).
\]

It is quite clear that

\[
\rho_{N}^{\text{Lag}}(x_1, \ldots, x_n) = \frac{N!}{(N-n)!} \int p_N(x_1, \ldots, x_N) dx_{n+1} \cdots dx_N
\]

Thus, it suffices to compute the correlation functions of \( p_N \). To do this we shall use Proposition 2.3, but, first, we need to show that \( p_N \) has the form (2.14).

**Proposition 4.1.** With the preceding notation

\[
p_N(x_1, \ldots, x_N; y_1, \ldots, y_N) = \text{const} \cdot \det w(x_i, y_j) \cdot \det \xi_i(x_j) \cdot \det \eta_i(y_j),
\]

all indices vary from 1 to \( N \), and

\[
\xi_i(x) = x^\alpha + N + \theta(i-1), \quad \eta_i(y) = \delta^{(i-1)}(y)
\]

for all \( i = 1, \ldots, N \);

\[
w(x, y) = \frac{e^{-x-y}}{x+y}.
\]

**Proof.** Using the formula for Cauchy determinant mentioned in the proof of Lemma 3.1, we get

\[
\det w(x_i, y_j) = \frac{\prod_{i<j} [(x_i - x_j)(y_i - y_j)]}{\prod_{i,j} (x_i + y_j)} \cdot e^{-\sum_{i=1}^N (x_i + y_i)}
\]
Next,
\[ \prod_{i<j} (y_i - y_j) \cdot \det \delta^{(i-1)}(y_j) = (-1)^{\frac{n(n-1)}{2}} N! \prod_{i=1}^{N} \delta(y_i) \]
because \( y^m \cdot \delta^{(n)}(y) = \delta_{mn} \) for \( m \geq n \). Finally,
\[
p_N(x_1, \ldots, x_N; y_1, \ldots, y_N) = \text{const} \cdot \prod_{i<j} \left( x_i - x_j \right) \cdot e^{-x_1 - \cdots - x_N} \cdot N! \prod_{i=1}^{N} \delta(y_i). \]

Now, according to Section 2, we need to compute
\[
\tilde{g}_{ij} = \int_{0}^{+\infty} \int_{0}^{+\infty} \xi_j(x) \eta_i(y) w(x, y) \, dx \, dy.
\]

Proposition 4.2.
\[
\tilde{g}_{ij} = \int_{0}^{+\infty} \int_{0}^{+\infty} x^\alpha x^\theta (x_i - x_j) \cdot N! \prod_{i=1}^{N} \delta(y_i) = \text{const} \cdot \prod_{i<j} \left( x_i - x_j \right) \cdot e^{-x_1 - \cdots - x_N} \cdot N! \prod_{i=1}^{N} \delta(y_i). \]

Before proving Proposition 4.2, let us explain our achievement. By Lemma 3.1 we can now easily compute the inverse matrix \( \tilde{C} = (\tilde{c}_{kl}) = (\tilde{g}_{ij})^{-1} \).

Corollary 4.3. With the preceding notation
\[
(\tilde{c}_{kl}) = \theta \left( \frac{\Gamma(1 + \alpha + \theta(k-1))}{(k-1)!(N-k)!} \right)^{-1} \left( \frac{\Gamma(1 + \alpha + \theta(l-1))}{(l-1)!(N-l)!} \right)^{-1} (-1)^{N+k+l+1} \]

Proof of Corollary 4.3. Apply Lemma 3.1 for
\[
A_i = -(i-1), \quad B_i = \alpha + N + \theta(i-1), \quad i = 1, \ldots, N. \]

Proof of Proposition 4.2. It is easy to check that
\[
\int_{0}^{+\infty} \int_{0}^{+\infty} \frac{x^a y^b}{\Gamma(a+1) \Gamma(b+1)} e^{-x-y} \, dx \, dy = \frac{1}{a+b+1}
\]
if, say, \( \Re a \) and \( \Re b > 0 \). (Change of variables \( r = x+y \), \( s = \frac{x}{x+y} \) reduces the integral to the product of Euler’s gamma and beta integrals.)
As is well known, there exists a distribution
\[ \phi_c(u) = \frac{u^c}{\Gamma(c+1)} \]
which depends on \( c \) analytically, such that for \( c > -1 \) it is just an integrable function \( u^c/\Gamma(c+1) \) for \( u > 0 \) and 0 for \( u \leq 0 \). For \( c \leq -1 \) it is defined via analytic continuation. In particular, we always have the relation \( \phi'_c = \phi_{c-1} \) and, thus, for any positive integer \( k \)
\[ \phi_{-k}(u) = \delta^{(k-1)}(u). \]
Analytic continuation of (4.3) gives the relation
\[ \int_0^\infty \int_0^\infty \frac{x^a}{\Gamma(a+1)} \phi_b(y) e^{-x-y} dx dy = \frac{1}{a+b+1} \]
for, say, \( \Re a > 0 \) and \( \Re(a+b+1) > 0 \). Our claim is the special case of this formula for
\[ a = \alpha + N + \theta(i-1), \quad b = -j. \]
Thus, we have inverted the matrix \((\tilde{g}_{ij})\), and as the result we get a formula for the correlation functions.

**Theorem 4.4.** The correlation functions of the \( N \)-point biorthogonal Laguerre ensemble have the form
\[ \rho_{\alpha N}(x_1, \ldots, x_n) = \prod_{i=1}^n x_i^\alpha e^{-x_i} \cdot \det[K^L_{\alpha N}(x_i, x_j)]_{i,j=1}^n \]
where
\[ K^L_{\alpha N}(x, y) = \theta \sum_{k, i=0}^{N-1} \sum_{r=1}^{N-1} \frac{\Gamma(N + i + n + 1)}{\Gamma(\alpha + \theta k + 1)\Gamma(\frac{i+k+1}{\theta})} \]
\[ \times \frac{(-1)^{i+k}}{k!(N-k-1)!!(r-i)!} x^{\theta k} y^r. \]

**Proof.** By the definition of \( p_N \),
\[ \rho^L_{\alpha N}(x_1, \ldots, x_n) = \frac{N!}{(N-n)!} \int p^L_{\alpha N}(x_1, \ldots, x_N) dx_{n+1} \cdots dx_N \]
\[ = \frac{N!}{(N-n)!} \int p_N(x_1, \ldots, x_N; y_1, \ldots, y_N) dx_{n+1} \cdots dx_N dy_1 \cdots dy_N. \]
Thus, we can apply Propositions 2.3 and 2.4. We have
\[ \int \psi_1(y)w(t, y)dy = \int \delta^{(l-1)}(y) \frac{e^{-t-y}}{t+y} dy \]
\[ = (-1)^{l-1} \left. \frac{\partial^{l-1}}{\partial y^{l-1}} \frac{e^{-t-y}}{t+y} \right|_{y=0} = \sum_{s=0}^{l-1} \frac{(l-1)!}{(l-1-s)!} t^{l-s-1} e^{-t}. \]
Then, by Proposition 2.3, we get the determinantal formula
\[ \rho_{nN}^{\text{Lag}}(x_1, \ldots, x_n) = \det \tilde{K}_N(x_i, x_j) \]
where, see (4.2),
\[
\tilde{K}_N(x, t) = \sum_{k,l=1}^N \tilde{c}_{kl} \sum_{s=0}^{l-1} \frac{(l-1)!}{(l-1-s)!} t^{l-s-1} e^{-t} \frac{x^N}{t^N} \\
\times \sum_{k,l=1}^N \sum_{s=0}^{l-1} \frac{\Gamma(1 + \alpha + \theta(k-1))}{(k-1)!(N-k)!} \left( \frac{\alpha+N-(l-1)}{\theta} \right)_N \\
\times \frac{(-1)^{N+k+l+1} x^{\theta(k-1)} t^{N-s-1}}{\alpha + N + \theta(k-1) - (l-1)}. 
\]
Note that the factor \((x/t)^N\) disappears, when we take the determinant, and the factor \(x^\alpha e^{-t}\) produces the factor \(\prod_{i=1}^n x^\alpha e^{-x_i}\) outside the determinant. Introducing new summation indices
\[ i = N - l, \quad r = N - s - 1 \]
and shifting index \(k\) by one, we arrive at our claim. □

One may ask whether the kernel \(K_N^{\text{Lag}}(x, y)\) defined above is of the Christoffel–Darboux type in the sense of Section 2 (see (2.11)). The answer is positive: if we take (using the notation of Section 2)
\[ \xi_i(x) = x^{\theta(i-1)}, \quad \eta_i(x) = x^{i-1} \]
then
\[ K_N^{\text{Lag}}(x, y) = \sum_{i=1}^N \xi_i(x) \psi_i(y), \]
see Proposition 5.1.

Now we are ready to pass to the limit \(N \to \infty\).

**Theorem 4.5.** For all \(n = 1, 2, \ldots\) there exists the limit
\[
\lim_{N \to \infty} \frac{1}{N^n} \rho_{nN}^{\text{Lag}} \left( \frac{x_1}{N^{\frac{1}{\theta}}}, \ldots, \frac{x_n}{N^{\frac{1}{\theta}}} \right) = \prod_{i=1}^n x_i^\alpha \cdot \det \left[ K^{(\alpha, \theta)}(x_i, x_j) \right]_{i,j=1}^n 
\]
where \(K^{(\alpha, \theta)}(x, y)\) is defined in (3.6).

**Heuristic proof.** Indeed, if we substitute in (4.4) the asymptotic relation
\[
\frac{\Gamma \left( n + \frac{\alpha + 1}{\theta} \right)}{(N-k-1)!} \sim N^{\frac{n+\alpha+1}{\theta}+k} 
\]
then we get
\[
\frac{1}{N^\frac{1}{\theta}} \left( \frac{x}{N^\frac{1}{\theta}} \right)^\alpha e^{-x/N^\frac{1}{\theta}} K_N^{\text{Log}} \left( \frac{x}{N^\frac{1}{\theta}}, \frac{y}{N^\frac{1}{\theta}} \right) \sim x^\alpha e^{-x/N^\frac{1}{\theta} \theta} \\
\times \sum_{k,i=0}^{n-1} \frac{(-1)^{i+k} x^{\theta k} y^i}{k! \Gamma(\alpha + \theta k + 1) \Gamma\left(\frac{\alpha + \theta k + 1}{\theta}\right) (\alpha + \theta k + i + 1)} \sum_{r=0}^{n-1} \frac{(y/N^\frac{1}{\theta})^r}{(r-i)!}.
\]
But
\[
\sum_{r=0}^{n-1} \frac{(y/N^\frac{1}{\theta})^r}{(r-i)!} = 1 + o(1), \quad e^{-x/N^\frac{1}{\theta}} = 1 + o(1),
\]
and we arrive at our asymptotic kernel. □

A justification of the heuristic argument above is given in the appendix (Section 6).

Remark 4.6. Asymptotic behaviour of biorthogonal Jacobi and Laguerre ensembles is the same, while the scaling factors are different.

Now let us point out a certain symmetry in the kernel \( K^{(\alpha, \theta)}(x, y) \) as defined by (3.6):
\[
K^{(\alpha, \theta)}(y^\frac{1}{\theta}, x^\frac{1}{\theta}) = \frac{1}{\theta} K^{\left(\frac{\alpha + 1}{\theta} - 1, \frac{1}{\theta}\right)}(x, y).
\]
It follows that the correlation functions (defined by the determinantal formula of Theorem 4.5), or rather the correlation measures
\[
\prod_{i=1}^{n} x_i^\alpha \cdot \det \left[ K^{(\alpha, \theta)}(x_i, x_j) \right]_{i,j=1}^{n} \cdot dx_1 \cdots dx_n
\]
are stable under the change of parameters
\[
(4.5) \quad \alpha \mapsto \frac{\alpha + 1}{\theta} - 1, \quad \theta \mapsto \frac{1}{\theta}
\]
combined with the transformation of the phase space \( x \mapsto x^\frac{1}{\theta}, \ y \mapsto y^\frac{1}{\theta} \).

For the Jacobi ensemble this is perfectly understandable, the transformation of the phase space is equivalent to the change of parameters, because if \( x_i \mapsto x_i^\frac{1}{\theta} \) then
\[
\text{const} \cdot \prod_{i=1}^{N} x_i^\alpha \prod_{i<j} \left[ (x_i^\theta - x_j^\theta) \right] dx_1 \cdots dx_N \\
\quad \mapsto \text{const} \cdot \prod_{i=1}^{N} x_i^{\frac{\alpha + 1}{\theta} - 1} \prod_{i<j} \left[ (x_i^\frac{1}{\theta} - x_j^\frac{1}{\theta}) \right] dx_1 \cdots dx_N
\]
However, for the Laguerre ensemble this symmetry is rather surprising, the change of parameters is not equivalent to a transformation of the phase space, because the factor \( e^{-x_1^\frac{1}{\theta} - \cdots - x_N^\frac{1}{\theta}} \) does not behave properly. Thus, we get a non-trivial conclusion, which deserves a separate statement.

Corollary 4.7. The asymptotic behaviour of the Laguerre biorthogonal ensemble at the left edge of spectrum is invariant with respect to the change of parameters (4.5) and the transformation \( x \mapsto x^\frac{1}{\theta} \) of the phase space \((0, +\infty)\).
5. **Biorthogonal Hermite ensemble**

Everywhere below the symbol $x^\theta$ stands for $\text{sign}(x)|x|^\theta$, and for all integers $k$

$$x^{\theta k} = (x^\theta)^k.$$ 

In this section we are dealing with ensembles on $(-\infty, +\infty)$ with the joint probability densities

$$p_N^{\text{Her}}(x_1, \ldots, x_N) = \text{const} \cdot \prod_{i=1}^N |x_i|^\alpha \prod_{i<j} \left[ (x_i^\theta - x_j^\theta) e^{-x_i^2 \cdots - x_N^2} \right]$$

(5.1)

$$= \text{const} \cdot \prod_{i=1}^N |x_i|^\alpha e^{-x_i^2} \cdot \det x_i^{j-1} \cdot \det x_i^{\theta(j-1)},$$

where $\theta > 0$ and $\alpha > -1$. We call these ensembles the **biorthogonal Hermite ensembles**.

We shall reduce the study of their asymptotics to that of the Laguerre ensembles, see the previous section. To do this we need to introduce a notation for biorthogonal Laguerre and Hermite polynomials.

Let us denote two sequences of biorthogonal Laguerre polynomials by $\{Z_{n}^{\alpha}(x, \theta)\}$ and $\{Y_{n}^{\alpha}(y, \theta)\}$. That is

$$\deg Z_{n}^{\alpha}(x, \theta) = \deg Y_{n}^{\alpha}(x, \theta) = n$$

and

$$\int_{0}^{\infty} Z_{m}^{\alpha}(x^\theta, \theta)Y_{n}^{\alpha}(x, \theta)|x|^\alpha e^{-x} dx = \delta_{mn}.$$ 

Such polynomials were explicitly constructed in [K2], [C]:

$$Z_{n}^{\alpha}(x, \theta) = \sum_{j=0}^{n} \binom{n}{j} (-1)^j x^j \frac{\Gamma(\theta j + \alpha + 1)}{n!},$$

$$Y_{n}^{\alpha}(x, \theta) = \frac{1}{n!} \sum_{r=0}^{n} x^r \sum_{i=0}^{r} (-1)^i \binom{r}{i} \left( \frac{i + \alpha + 1}{\theta} \right)^n.$$ 

Note that our notation is slightly different from the conventional one, usually $Z_{n}^{\alpha}(x, \theta)$ is multiplied by $\frac{\Gamma(\theta n + \alpha + 1)}{n!}$, but we want our polynomials to be orthonormal.

Following [MT2] we can construct biorthogonal Hermite polynomials $\{S_{n}^{\alpha}(x, \theta)\}$ and $\{T_{n}^{\alpha}(y, \theta)\}$ satisfying

$$\deg S_{n}^{\alpha}(x, \theta) = \deg T_{n}^{\alpha}(y, \theta) = n$$

$$\int_{-\infty}^{+\infty} S_{m}^{\alpha}(x^\theta, \theta)T_{n}^{\alpha}(x, \theta)|x|^\alpha e^{-x^2} dx = \delta_{mn}.$$
Namely,
\[
S_{2n}^\alpha(x, \theta) = Z_n^{(\alpha-1)/2}(x^2, \theta);
\]
\[
T_{2n}^\alpha(x, \theta) = Y_n^{(\alpha-1)/2}(x^2, \theta);
\]
\[
S_{2n+1}^\alpha(x, \theta) = xZ_n^{(\alpha+\theta)/2}(x^2, \theta);
\]
\[
T_{2n+1}^\alpha(x, \theta) = xY_n^{(\alpha+\theta)/2}(x^2, \theta).
\]
Again, our notation differs from the usual one by scalar factors.

It is quite clear that the Christoffel-Darboux kernel for Hermite case can be expressed via that for Laguerre case, we shall do this in Proposition 5.3.

**Proposition 5.1.** With the preceding notation
\[
K_{N}\text{Lag}(x, y) = \sum_{i=0}^{N-1} Z_i^\alpha(x, \theta)Y_i^\alpha(y, \theta).
\]
where \(K_{N}\text{Lag}(x, y)\) is defined by (4.4).

*Idea of the proof.* One can prove this statement by direct verifying that
\[
K_{N+1}\text{Lag}(x, y) - K_{N}\text{Lag}(x, y)
\]
is equal to
\[
Z_N^\alpha(x, \theta)Y_N^\alpha(y, \theta),
\]
we have explicit formulas for both these expressions. However, this check is rather tedious. \(\square\)

**Remark 5.2.** Proposition 5.1 proves that \(K_{N}\text{Lag}(x, y)\) is the Christoffel–Darboux kernel (2.11) for
\[
\xi_i(x) = x^{0(i-1)}, \quad \eta_i(x) = x^{i-1}.
\]
Note that this fact and (2.10) provide another proof of Theorem 4.4. However, it is just a *checking* proof, the explicit formulas for biorthogonal Laguerre polynomials above do not prompt a suitable expression for the Christoffel–Darboux kernel (4.4).

In Section 4, using the trick with \(p_N\), we, actually, managed to derive the formula (4.4), and its relative simplicity allowed us to analyze the asymptotic behaviour of the biorthogonal Laguerre ensemble (Theorem 4.5).

Moreover, we have a conceptual proof of the following statement which generalizes Proposition 5.1: let \(\xi_i(x) = x^{a_i}\) for arbitrary complex numbers \(a_i, \Re a_i > 0\), and \(\eta_i(x) = x^{i-1}\); then the trick with \(p_N\) described in Section 4 always produces the Christoffel–Darboux type kernel. But the proof is based on a certain formalism which exceeds the limits of the present paper.

Set
\[
K_{N}\text{Her}(x, y) = \sum_{i=0}^{N-1} S_i^\alpha(x, \theta)T_i^\alpha(y, \theta).
\]
By (2.10),
\[
\rho_{nN}^{\text{Her}}(x_1, \ldots, x_n) = \prod_{i=1}^{n} |x_i|^\alpha e^{-x_i^2} \cdot \det \left[ K_{N}\text{Her}(x_i, x_j) \right]_{i,j=1}^{n}.
\]
Let us use a more detailed notation for Christoffel-Darboux kernels and write
\[ K_{\text{Lag}}^N(x,y), \quad K_{\text{Her}}^N(x,y) \]
instead of
\[ K_N^\text{Lag}(x,y), \quad K_N^\text{Her}(x,y) \]
indicaing the dependence on \( \alpha \).

**Proposition 5.3.** The Christoffel-Darboux kernel for the \( N \)-point biorthogonal Hermite ensemble has the form
\[
K_{\text{Her}}^N(x,y) = K_{\text{Lag}}^{(\alpha-1/2)}(x^2, y^2) + x^\theta y K_{\text{Lag}}^{(\alpha+\theta/2)}(x^2, y^2) \quad \text{for } N = 2M
\]

\[
K_{\text{Her}}^N(x,y) = K_{\text{Lag}}^{(\alpha-1/2)}(x^2, y^2) + x^\theta y K_{\text{Lag}}^{(\alpha+\theta/2)}(x^2, y^2) \quad \text{for } N = 2M + 1
\]

where \( K_N^\text{Lag}(\alpha)(x,y) \) is defined by (4.4).

**Proof.** Immediately follows from explicit expressions for biorthogonal Hermite polynomials via biorthogonal Laguerre polynomials. \( \square \)

Now we can express the asymptotics of the Hermite ensemble via that of the Laguerre ensemble.

**Theorem 5.4.** For all \( n = 1, 2, \ldots \) there exists the limit (set \( M = \frac{N}{2} \))
\[
\lim_{N \to \infty} \frac{1}{M^\frac{n}{2}} \rho_{nN}^\text{Her} \left( \frac{x_1}{M^\frac{1}{2}}, \ldots, \frac{x_n}{M^\frac{1}{2}} \right) = \prod_{i=1}^n |x_i|^\alpha \cdot \det \left[ K_{\text{Her}}^{(\alpha,\theta)}(x_i, x_j) \right]_{i,j=1}^n
\]

where
\[
K_{\text{Her}}^{(\alpha,\theta)}(x,y) = K^{(\alpha-1/2,\theta)}(x^2, y^2) + x^\theta y \cdot K^{(\alpha+\theta/2,\theta)}(x^2, y^2)
\]

and \( K^{(\alpha,\theta)}(x,y) \) is defined in (3.6).

The proof is straightforward.

**Example 5.5.** The asymptotic kernel of the classic Hermite ensemble in the bulk of spectrum is the *sine-kernel*
\[
\frac{\sin \pi (\xi - \eta)}{\pi (\xi - \eta)}.
\]

Let us obtain it from our formulas. For \( \theta = 1 \) we have, see (3.6),
\[
K^{(\alpha,1)}(x,y) = \int_0^1 J_{\alpha+1,1}(xt) \cdot J_{\alpha+1,1}(yt)t^\alpha dt
\]

But
\[
K_{\text{Her}}^{(0,1)}(x,y) = K^{(-\frac{1}{2},1)}(x^2, y^2) + xy \cdot K^{(-\frac{1}{2},1)}(x^2, y^2)
\]

and
\[
J_{\frac{1}{2},1}(x) = \sum_{k=0}^\infty \frac{(-x)^k}{k! \Gamma(k + 1/2)} = \frac{1}{\sqrt{\pi}} \cos(2\sqrt{x}),
\]
$$J_{3,1}(x) = \sum_{k=0}^{\infty} \frac{(-x)^k}{k! \Gamma(k + 3/2)} = \frac{1}{\sqrt{\pi}} \frac{\sin(2\sqrt{x})}{\sqrt{x}}.$$ Thus,

$$K^{(-\frac{3}{2}, 1)}(x^2, y^2) = \frac{1}{\pi} \int \cos(2x\sqrt{t}) \cos(2y\sqrt{t}) \frac{dt}{\sqrt{t}} = \frac{1}{2\pi} \left( \frac{\sin(2x - y)}{x - y} + \frac{\sin(2x + y)}{x + y} \right),$$

$$xy \cdot K^{(\frac{3}{2}, 1)}(x^2, y^2) = \frac{1}{\pi} \int \sin(2x\sqrt{t}) \sin(2y\sqrt{t}) \frac{dt}{\sqrt{t}} = \frac{1}{2\pi} \left( \frac{\sin(2x - y)}{x - y} - \frac{\sin(2x + y)}{x + y} \right).$$

Then (5.3) brings us to the sine-kernel for $\xi = 2x/\pi$, $\eta = 2y/\pi$.

Similarly to the Laguerre case, the biorthogonal Hermite ensemble also possesses a strange symmetry, cf. Corollary 4.7.

**Corollary 5.6.** The asymptotic behaviour of the Hermite biorthogonal ensemble in the bulk of spectrum is invariant with respect to the change of parameters (4.5) and the transformation $x \rightarrow \text{sign}(x) \cdot |x|^{\frac{1}{\theta}}$ of the phase space $(-\infty, +\infty)$.

**Proof.** The claim easily follows from Corollary 4.7 and (5.2). If we set, see (4.5),

$$\tilde{\alpha} = \frac{\alpha + 1}{\theta} - 1, \quad \tilde{\theta} = \frac{1}{\theta},$$

then

$$\frac{\frac{\alpha + 1}{\theta} - 1}{2} = \frac{\tilde{\alpha} - 1}{2}, \quad \frac{\frac{\alpha + \theta}{\theta} + 1}{2} - 1 = \frac{\tilde{\alpha} + \tilde{\theta}}{2}.$$

These identities show that each summand of (5.2) is invariant under changes from the hypothesis. \(\square\)

It would be very interesting to find some kind of natural explanation for Corollaries 4.7 and 5.6.

6. **Appendix**

**Proof of Theorem 3.4.** The formula (3.5) for the Christoffel-Darboux kernel $K_N^{Jac}(x, y)$ implies that

$$K_N^{Jac}(x, y) = \theta \int_0^1 \frac{1}{A_N(\theta t)B_N((\theta t)^\theta) t^\theta} dt$$

where

$$A_N(x) = \sum_{k=1}^{N} \frac{(-x)^{k-1}}{(k - 1)! (N - k)!}.$$
(6.3) \[ B_N(y) = \sum_{l=1}^{N} \frac{(\theta(l-1) + \alpha + 1)_{N-l}(-y)^{l-1}}{(l-1)!(N-l)!} \]

Comparing (6.1) to (3.6) we see that it suffices to prove the following

(6.4) \[ \lim_{N \to \infty} \frac{1}{N^{\alpha+1}} A_N \left( \frac{x}{N^{1+\frac{1}{\theta}}} \right) = J_{\alpha+1,\theta}^{\alpha+1}(x), \]

(6.5) \[ \lim_{N \to \infty} \frac{1}{N^{\alpha+1}} B_N \left( \frac{y}{N^{1+\frac{1}{\theta}}} \right) = J_{\alpha+1,\theta}(y) \]

where the convergence is uniform on every compact subset of \( \mathbb{R}_+ \). Indeed, these relations imply

\[ \lim_{N \to \infty} \frac{1}{N^{1+\frac{1}{\theta}}} \left( \frac{x}{N^{1+\frac{1}{\theta}}} \right)^{\alpha} K_N^{\alpha} \left( \frac{x}{N^{1+\frac{1}{\theta}}}, \frac{y}{N^{1+\frac{1}{\theta}}} \right) = x^{\alpha} K^{(\alpha,\theta)}(x, y), \]

which is what we want to prove.

We shall prove only (6.4); the proof of (6.5) is quite similar.

Let us split the sum (6.2) into two parts: in the first part the summation index \( k \) runs from 1 to some \( M < N \) which will be chosen later, and the second part is the remainder. We shall denote these parts by \( A_N'(x) \) and \( A_N''(x) \), respectively. Thus,

\[ A_N(x) = A_N'(x) + A_N''(x). \]

It will be sufficient to prove the following (uniform) estimates

(6.6) \[ \frac{1}{N^{\alpha+1}} A_N' \left( \frac{x}{N^{1+\frac{1}{\theta}}} \right) = \sum_{k=0}^{M-1} \frac{(-1)^k x^k}{k! \Gamma(\frac{\alpha+1+k}{\theta})} + o(1); \]

(6.7) \[ \frac{1}{N^{\alpha+1}} A_N'' \left( \frac{x}{N^{1+\frac{1}{\theta}}} \right) = o(1) \]

as \( N \to \infty \).

To verify (6.6) we shall use Stirling formula

(6.8) \[ \ln \Gamma(z) = \left( z - \frac{1}{2} \right) \ln z - z + \frac{1}{2} \ln(2\pi) + o(z^{-1}). \]

Applying (6.8) to \( z = N + \frac{k+\alpha}{\theta} \) and to \( z = N - k + 1 \) we get (we shall choose \( M \) so that it will be \( o(N) \), that is why the last term in (6.8) produces \( O\left(\frac{1}{N}\right) \) in the next formula)

\[ \ln \frac{\Gamma \left( N + \frac{k+\alpha}{\theta} \right)}{\Gamma(N-k+1)N^{\frac{k+\alpha}{\theta}+k-1}} = \left( N + \frac{k+\alpha}{\theta} - \frac{1}{2} \right) \ln \left( N + \frac{k+\alpha}{\theta} \right) \]

\[ - \left( N - k + \frac{1}{2} \right) \ln(N-k+1) - \left( \frac{k+\alpha}{\theta} + k - 1 \right) (\ln N + 1) + O \left( \frac{1}{N} \right). \]
Using asymptotic expansions

\[ \ln \left( N + \frac{k + \alpha}{\theta} \right) = \ln N + \frac{k + \alpha}{N\theta} + O \left( \frac{(k + \alpha)^2}{N} \right) \]

\[ \ln(N - k + 1) = \ln N - \frac{k - 1}{N} + O \left( \left( \frac{k - 1}{N} \right)^2 \right) \]

we arrive at the following estimate

\[ \ln \frac{\Gamma \left( N + \frac{k + \alpha}{\theta} \right)}{\Gamma(N - k + 1)N^{k+\alpha+k^{-1}}} = \frac{k + \alpha}{N\theta} \left( \frac{k + \alpha}{\theta} - \frac{1}{2} \right) \]

\[ + \frac{k - 1}{N} \left( -k + \frac{1}{2} \right) + O \left( \frac{k^2}{N} \right) + O \left( \frac{1}{N} \right) = O \left( \frac{k^2}{N} \right). \]

Now we want this expression to converge to 0 as \( N \to \infty \). Since \( k \leq M \), we may set, for example, \( M = \lceil N^{1/3} \rceil \). Then

\[ O \left( \frac{k^2}{N} \right) = O(N^{-\frac{2}{3}}), \]

and hence we get

\[ \frac{\Gamma \left( N + \frac{k + \alpha}{\theta} \right)}{\Gamma(N - k + 1)N^{k+\alpha+k^{-1}}} = 1 + O(N^{-\frac{2}{3}}) \]

and

\[ \frac{1}{N^{x+1}} A^* \left( \frac{x}{N^{1+\frac{1}{3}}} \right) = \frac{1}{N^{x+1}} \sum_{k=1}^{M} \frac{(k + \alpha)}{\theta} \left( \frac{k + \alpha}{\theta} \right)_N \left( \frac{-x}{N^{1+\frac{1}{3}}} \right)^{k-1} \]

\[ = \sum_{k=1}^{M} \frac{\Gamma \left( N + \frac{k + \alpha}{\theta} \right)}{\Gamma(N - k + 1)N^{k+\alpha+k^{-1}} (k - 1)!} \left( \frac{-x}{N^{1+\frac{1}{3}}} \right)^{k-1} \]

\[ = (1 + O(N^{-\frac{2}{3}})) \sum_{k=1}^{M} \left( \frac{-x}{N^{1+\frac{1}{3}}} \right)^{k-1} \frac{1}{(k - 1)!} \frac{\Gamma \left( \frac{k + \alpha}{\theta} \right)}{\Gamma \left( \frac{k + \alpha}{\theta} \right)} = \sum_{k=0}^{M-1} \frac{(-1)^k x^k}{k! \Gamma \left( \frac{a+1+k}{\theta} \right)} + o(1). \]

Thus, (6.6) is proved.

To prove (6.7) we notice that for any \( a, b > 0 \), \( b \) is an integer < \( N \), we have the following simple estimate

\[ \frac{\Gamma(N + a)}{\Gamma(N - b)N^{a+b}} \leq \frac{(N + a)^{a+b+1}}{N^{ab+1}} \leq N \left( 1 + \frac{a}{N} \right)^{a+b+1}, \]

which can be obtained by applying the identity \( \Gamma(z + 1) = z\Gamma(z) \) to the numerator of the left-hand side \([a] + b + 1\) times if \( N - b > 1 \), and \([a] + b\) times if \( N - b = 1 \).
Proof of Theorem 4.5.

Let us rewrite the formula (4.4) for the kernel $K_N^Lag(x, y)$ in the following form

$$K_N^Lag(x, y) = \theta \sum_{k,i=0}^{N-1} \frac{\Gamma(N)}{\Gamma(\alpha + \theta k + 1)k!(N - k - 1)! \Gamma(N)\Gamma(i + \frac{\alpha + 1}{\theta})!} \times (-x^\theta)^k(-y)^i \alpha + \theta k + i + 1 \sum_{r=i}^{N-1} \frac{y^{r-i}}{(r-i)!}$$

(6.8)

Note now, that if we substitute $yN^{-\frac{1}{\theta}}$ instead of $y$ into the last sum, it will be close to 1:

$$\sum_{r=i}^{N-1} \frac{1}{(r-i)!} \left( \frac{y}{N^{\theta}} \right)^{r-i} = 1 + O(N^{-\frac{1}{\theta}})$$

where $O(N^{-\frac{1}{\theta}})$ does not depend on $i$. Thus, we can neglect this sum while computing the limit.

The rest of (6.8) can be written in the form (cf. (6.1))

$$\theta \sum_{k,i=0}^{N-1} \frac{\Gamma(N)}{\Gamma(\alpha + \theta k + 1)k!(N - k - 1)! \Gamma(N)\Gamma(i + \frac{\alpha + 1}{\theta})!} \times (-x^\theta)^k(-y)^i \alpha + \theta k + i + 1 \sum_{r=i}^{N-1} \frac{y^{r-i}}{(r-i)!}$$

(6.9)

$$= \theta \int_{0}^{1} C_N((xt)^\theta)D_N(yt)t^\alpha dt$$

Since $1 + \frac{N + \alpha}{N\theta}$ is bounded, say, by some constant $c$, and $N \leq (M + 1)^3 \leq k^3$, the last sum does not exceed

$$\sum_{k=M+1}^{N} k^3 c^{\frac{N + \alpha}{N\theta} + k} \frac{|x|^{k-1}}{(k-1)! \Gamma\left(\frac{k + \alpha}{\theta}\right)}$$

which is the difference of two partial sums $S_N$ and $S_M$ of the series

$$\sum_{k=1}^{\infty} k^3 c^{\frac{k + \alpha}{N\theta} + k} \frac{|x|^{k-1}}{(k-1)! \Gamma\left(\frac{k + \alpha}{\theta}\right)}.$$

This series converges for all $x$ uniformly on every compact set, and, consequently,

$$\frac{1}{N^{\frac{\alpha + 1}{\theta}}} A^*_N \left( \frac{x}{N^{1+\frac{1}{\theta}}} \right)$$

converges to 0 uniformly on every compact set, as was to be proved. □

Proof of Theorem 4.5. Let us rewrite the formula (4.4) for the kernel $K_N^Lag(x, y)$ in the following form

$$K_N^Lag(x, y) = \theta \sum_{k,i=0}^{N-1} \frac{\Gamma(N)}{\Gamma(\alpha + \theta k + 1)k!(N - k - 1)! \Gamma(N)\Gamma(i + \frac{\alpha + 1}{\theta})!} \times (-x^\theta)^k(-y)^i \alpha + \theta k + i + 1 \sum_{r=i}^{N-1} \frac{y^{r-i}}{(r-i)!}$$

(6.8)

Note now, that if we substitute $yN^{-\frac{1}{\theta}}$ instead of $y$ into the last sum, it will be close to 1:

$$\sum_{r=i}^{N-1} \frac{1}{(r-i)!} \left( \frac{y}{N^{\theta}} \right)^{r-i} = 1 + O(N^{-\frac{1}{\theta}})$$

where $O(N^{-\frac{1}{\theta}})$ does not depend on $i$. Thus, we can neglect this sum while computing the limit.

The rest of (6.8) can be written in the form (cf. (6.1))

$$\theta \sum_{k,i=0}^{N-1} \frac{\Gamma(N)}{\Gamma(\alpha + \theta k + 1)k!(N - k - 1)! \Gamma(N)\Gamma(i + \frac{\alpha + 1}{\theta})!} \times (-x^\theta)^k(-y)^i \alpha + \theta k + i + 1 \sum_{r=i}^{N-1} \frac{y^{r-i}}{(r-i)!}$$

(6.9)

$$= \theta \int_{0}^{1} C_N((xt)^\theta)D_N(yt)t^\alpha dt$$
where
\[ C_N(x) = \sum_{k=0}^{N-1} \frac{\Gamma(N)(-x)^k}{\Gamma(\alpha + \theta k + 1)k!(N-k-1)!} \]
\[ D_N(y) = \sum_{i=0}^{N-1} \frac{\Gamma(N + \frac{i+\alpha+1}{\theta})(-y)^i}{\Gamma(N)\Gamma\left(\frac{i+\alpha+1}{\theta}\right)i!}. \]

Comparing (6.9) to (3.6) we see that it suffices to prove the following

\[ \lim_{N \to \infty} C_N \left( \frac{x}{N^{\frac{1}{\theta}}} \right) = J_{\alpha+1,1}(x), \]

\[ \lim_{N \to \infty} \frac{1}{N^{\frac{1}{\theta}}} D_N \left( \frac{y}{N^{\frac{1}{\theta}}} \right) = J_{\alpha+1,1} \left( \frac{y}{N^{\frac{1}{\theta}}} \right) \]

as \( N \to \infty \), because these relations imply the desired one

\[ \lim_{N \to \infty} \frac{1}{N^{\frac{1}{\theta}}} \left( \frac{x}{N^{\frac{1}{\theta}}} \right)^\alpha \exp \left( \frac{x}{N^{\frac{1}{\theta}}} \right) K_L^{\text{Log}} \left( \frac{x}{N^{\frac{1}{\theta}}}, \frac{y}{N^{\frac{1}{\theta}}} \right) = x^\alpha K^{(\alpha,\theta)}(y,x). \]

(The interchange \( x \leftrightarrow y \) in the last expression does not change the determinants of the type \( \det[K^{(\alpha,\theta)}(x_i, x_j)] \).)

The proofs of (6.10) and (6.11) are very similar to the proof of (6.4) which we carried out above, and we shall not give them here. \( \square \)

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