Hopf algebroids with balancing subalgebra

Zoran Škoda\textsuperscript{a,*}, Martina Stojić\textsuperscript{b}

\textsuperscript{a}Faculty of Science, University of Hradec Králové, Rokitanského 62, Hradec Králové, Czech Republic
\textsuperscript{b}Department of Mathematics, University of Zagreb, Bijenička cesta 3, HR-10000 Zagreb, Croatia

Abstract

Recently, S. Meljanac proposed a construction of a class of examples of an algebraic structure with properties very close to the Hopf algebroids $H$ over a noncommutative base $A$ of other authors. His examples come along with a subalgebra $B$ of $H \otimes H$, here called the balancing subalgebra, which contains the image of the coproduct and such that the intersection of $B$ with the kernel of the projection $H \otimes H \to H \otimes_A H$ is a two-sided ideal in $B$ which is moreover well behaved with respect to the antipode. We propose a set of abstract axioms covering this construction and make a detailed comparison to the Hopf algebroids of Lu. We prove that every scalar extension Hopf algebroid can be cast into this new set of axioms. We present an observation by G. Böhm that the Hopf algebroids constructed from weak Hopf algebras fit into our framework as well. At the end we discuss the change of balancing subalgebra under Drinfeld-Xu procedure of twisting of associative bialgebroids by invertible 2-cocycles.

Keywords: Hopf algebroid, balancing subalgebra, Takeuchi product

1. Introduction

Hopf algebroids \cite{Lu1, Lu2, Lu3} are generalizations of Hopf algebras, which are roughly in the same relation to groupoids as Hopf algebras are to groups. They are bialgebroids possessing a version of an antipode, where an (associative) bialgebroid is the appropriate generalization of a bialgebra. Hopf

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*Corresponding author

Email addresses: zoran.skoda@uhk.cz (Zoran Škoda), stojic@math.hr (Martina Stojić)
algebroids comprise several structure maps defined on a pair of associative unital algebras, the **total algebra** $H$ (generalization of a function algebra on the space of morphisms of a groupoid), and the **base algebra** $A$ (generalization of a function algebra on the space of objects (equivalently, units) of a groupoid). The main structure on the total algebra of a bialgebroid is an $A$-bimodule structure on $H$ and a coproduct $\Delta : H \to H \otimes_A H$. The commutative Hopf algebroids (where both $H$ and $A$ are commutative) are easy to define by a categorical dualization of the groupoid concept. They are used as a classical tool in stable homotopy theory [13, 24]. Noncommutative Hopf algebroids over a commutative base ($H$ noncommutative, $A$ commutative) are also rather straightforward to introduce; this theory has been studied from late 1980-s, under the influence of the quantum group theory [21]. The most obvious examples are the convolution algebras of finite groupoids. Bialgebroids and Hopf algebroids over a noncommutative base are much more complicated to define; several versions were developed in early 1990s by Lu [18], Xu [32], Böhm [1], Böhm-Szlachányi [6], Day-Street (in [9] and a different abstract notion in [11]) and others, including an earlier notion of $\times_A$-bialgebra [31] and its Hopf version by Schauenburg [25]; for comparisons see [2, 7]. Böhm has also introduced an internalization of a bialgebroid in any symmetric monoidal category with coequalizers commuting with tensor product [3]; this has been extended to an internalization of Hopf algebroids in [28]. Many examples of Hopf algebroids over noncommutative bases have been studied in the contexts of inclusions of von Neumann algebra factors [14, 2], dynamical Yang-Baxter equation [10], weak Hopf algebras [2], deformation quantization [32], noncommutative torsors, noncommutative differential calculus and cyclic homology [15] etc.

In 2012, S. Meljanac devised a new approach to some examples of (topological) Hopf algebroids over a noncommutative base restricting the codomain of the coproduct map in a useful, but somewhat *ad hoc* way. To construct that codomain, he chooses a subalgebra $\mathcal{B}$ in the tensor square $H \otimes H$ of the total algebra $H$, such that the intersection of $\mathcal{B}$ with the kernel $I_A$ of the projection $\pi : H \otimes H \to H \otimes_A H$ to the tensor square over the noncommutative base algebra $A$ is a two-sided ideal $I_A \cap \mathcal{B} \subset \mathcal{B}$ (with an appropriate behaviour under the antipode map). The appearance of the two-sided ideal is a novel and somewhat unexpected feature reminding of the classical case where the base algebra $A$ is commutative and $I_A$ is two-sided itself. The approach is developed in collaborative works [17], with more details in [16].

Papers [16] and [17] neglect two mathematical issues. Firstly, no care
is taken about the implicit use of completions: the values of the coproduct involve infinite sums, hence its codomain should be a completed tensor square. Secondly, at the algebraic level, they do not state a complete axiomatic framework for their version of Hopf algebroid, nor state its precise relation to other definitions. Instead, they construct an interesting class of examples and give a partial list of essential properties. In our article [22] with Meljanac, using the explicit formulas from [12], we treat a somewhat wider class of examples in a mathematically rigorous way, using G. Böhm’s definition of a symmetric Hopf algebroid, partly adapted to a formally completed tensor product. For a better adaptation, which gives rise to an internal Hopf algebroid in a symmetric monoidal category of filtered-cofiltered vector spaces, entailing a more sensible completion, see [28, 29]. These works took care of completions, but instead of the two-sided ideal approach they relied on (an internalization of) symmetric Hopf algebroid axiomatics [2, 6]. To return closer to the original idea, we here propose a new set of axioms expressing the essence of the two-sided ideal approach and discuss it in the context. The subalgebra $B \subset H \otimes H$ in new axioms is named the balancing subalgebra and our new version of Hopf algebroid over noncommutative base algebra $A$ is named a Hopf $A$-algebroid with balancing subalgebra $B$.

In Theorem 3.4 we compare Hopf $A$-algebroid with a balancing subalgebra to the Hopf algebroids of Lu instead to symmetric Hopf algebroids from [2, 6, 22]. This is because Lu’s axioms for the antipode map involve a choice of certain map (section $\gamma$ below) which is close in spirit to the choice of balancing subalgebra in our axiomatics and in the informal approach of Meljanac. Our main result is Theorem 4.12 (based on nontrivial Lemmas 4.10, 4.11) stating that every scalar extension Hopf algebroid can be cast into a Hopf algebroid with a suitable choice of balancing subalgebra $B$.

In Section 5 we worked out the observation of Böhm that each weak Hopf algebra gives rise not only to Hopf algebroids in the sense of Lu [18] and Böhm [1, 6], but also to a Hopf algebroid with a balancing subalgebra.

Throughout the paper, $k$ is a commutative ground field, and the unadorned tensor symbol $\otimes$ between symbols for $k$-vector spaces, is meant over $k$, however we often use $\otimes_k$ for emphasis. When used among elements in calculations, symbol $\otimes$ is interpreted from the context.

3
2. Bialgebroids over noncommutative base

The axioms of bialgebroids and Hopf algebroids over a noncommutative base algebra are far less obvious to formulate \([2, 7, 18, 25, 32]\). Let us detect a problem naively. For a commutative \(k\)-algebra \(A\), an \(A\)-bialgebra is a monoid (algebra) and a comonoid (coalgebra) in the same symmetric monoidal category, namely that of \(A\)-modules, with a compatibility condition utilizing the symmetry of the tensor product \(\otimes_A\). For a noncommutative base algebra \(A\) over \(k\), the category of \(A\)-modules is not monoidal, so it is natural to try replacing it with the monoidal category of \(A\)-bimodules. However, the latter is neither symmetric nor braided monoidal in general, so the usual compatibility condition between the comonoids and monoids makes no sense. Instead, it appears that the monoid and the comonoid part of the left \(A\)-bialgebroid structure live in different monoidal categories \([2]\).

The monoid structure \((H, \mu, \eta)\) on \(H\) is in the monoidal category of \(A \otimes A^{\text{op}}\)-bimodules; equivalently \([2]\), Lemma 2.2), \((H, \mu)\) is an associative \(k\)-algebra and the unit map \(\eta\) is a morphism of \(k\)-algebras \(\eta : A \otimes_k A^{\text{op}} \to H\) (we say that \((H, \mu, \eta)\) is an \(A \otimes A^{\text{op}}\)-ring). The unit \(\eta : A \otimes A^{\text{op}} \to H\) is usually described in terms of its left leg \(\alpha := \eta(- \otimes 1_{A^{\text{op}}}) : A \to H\) and its right leg \(\beta := \eta(1_A \otimes -) : A^{\text{op}} \to H\), also called the source and target maps respectively; then, their images commute because

\[
\alpha(a)\beta(b) = \eta(a \otimes 1)\eta(1 \otimes b) = \eta(a \otimes b) = \eta(1 \otimes b)\eta(a \otimes 1) = \beta(b)\alpha(a). \quad (1)
\]

An \(A \otimes A^{\text{op}}\)-ring \((H, \mu, \eta)\) is described below as the equivalent datum \((H, \mu, \alpha, \beta)\).

On the other hand, the comonoid structure \((H, \Delta, \epsilon)\) is in the monoidal category of \(A\)-bimodules (we say that \(H\) is an \(A\)-coring, \([8]\)).

**Definition 2.1.** An \(A \otimes A^{\text{op}}\)-ring \((H, \mu, \alpha, \beta)\) and an \(A\)-coring \((H, \Delta, \epsilon)\) with underlying \(A\)-bimodule \(H\) form a **left associative \(A\)-bialgebroid** \((H, \mu, \alpha, \beta, \Delta, \epsilon)\) if they satisfy the following compatibility conditions:

(C1) The underlying \(A\)-bimodule structure of \(A\)-coring \(H\) is determined by the source and target maps (part of the \(A \otimes A^{\text{op}}\)-ring structure): \(a.h.b = \alpha(a)\beta(b)h\) for \(a, b \in A\) and \(h \in H\). This is indeed a bimodule by \([1]\).

(C2) Formula \(\triangleright:\ \sum_\lambda h_\lambda \otimes a_\lambda \mapsto \epsilon(\sum_\lambda h_\lambda \alpha(a_\lambda))\) defines an action \(H \otimes A \xrightarrow{\delta} A\).

(C3) The map \(H \otimes_k (H \otimes_k H) \to H \otimes_A H\) given by the rule \(h \otimes (g \otimes f) \mapsto \Delta(h)(g \otimes f)\), factorizes through a map \(H \otimes_k (H \otimes_A H) \to H \otimes_A H\) which
is moreover a unital action. Expression \( \Delta(h)(g \otimes_k f) \) is understood by taking any representative of \( \Delta(h) \) in \( H \otimes_k H \), then multiplying in each tensor factor separately with \( g \otimes f \in H \otimes_k H \); the result is well defined in \( H \otimes_A H \). Unitality of the action implies \( \Delta(1) = 1 \otimes_A 1 \).

By (C1), \( \varepsilon \) being a bimodule map means \( \varepsilon(\alpha(a)h) = a\varepsilon(h) \) and \( \varepsilon(\beta(b)h) = \varepsilon(h)b \). In particular, \( \varepsilon \circ \alpha = \varepsilon \circ \beta = \text{id}_A \) and \( \varepsilon(1_H) = \varepsilon(\alpha(1_A)) = 1_A \). Action in (C2) is unital by its defining formula and it extends the left regular action \( A \otimes A \to A \) along the inclusion \( A \otimes A \xrightarrow{\alpha \otimes \alpha} H \otimes A \) by direct calculation, \( \varepsilon(\alpha(a)\alpha(b)) = \varepsilon(\alpha(ab)) = ab \). Action axiom (C2), for acting on \( 1_A \), implies \( \varepsilon(hg) = \varepsilon(hg\alpha(1_A)) = (hg) \triangleright 1_A = h \triangleright (g \triangleright 1_A) = \varepsilon(h\alpha(\varepsilon(g))) \). In particular, \( \varepsilon(h\beta(b)) = \varepsilon(h\alpha((\varepsilon \circ \beta)(b))) = \varepsilon(h\alpha(b)) = h \triangleright b \). Action axiom on \( 1_A \) together with \( a = \varepsilon(\alpha(a)) = \alpha(a) \triangleright 1 \), implies the general case, \( h \triangleright (g \triangleright a) = h \triangleright (g \triangleright (\alpha(a) \triangleright 1)) = h \triangleright ((g\alpha(a)) \triangleright 1) = (h\alpha(g\alpha(a))) \triangleright 1 = (h\alpha(g)) \triangleright 1 = (h \triangleright a) \triangleright 1 = (hg) \triangleright a \).

From (C1) and \( \Delta(1_H) = 1_H \otimes_A 1_H, \Delta \) being an \( A \)-bimodule map implies \( \Delta(\alpha(a)) = \Delta(\alpha(a)1_A) = \alpha(\alpha(a) \otimes A 1_H) \) and \( \Delta(\beta(b)) = 1_H \otimes_A \beta(b) \). It follows that \( \Delta(h\alpha(a)) = \Delta(h)(\alpha(a) \otimes_A 1) = h(1)\alpha(a) \otimes_A h(2) \) and \( \Delta(h\beta(a)) = h(1) \otimes_A h(2)\beta(a) \). Applying the counit axiom we obtain, for all \( h \in H \) and \( a \in A \),

\[
\begin{align*}
  h\alpha(a) &= \alpha(\varepsilon(h(1)\alpha(a)))h(2) = \alpha(h(1) \triangleright a)h(2), \quad (2) \\
  h\beta(a) &= \beta(\varepsilon(h(2)\beta(a)))h(2) = \beta(h(2) \triangleright a)h(1). \quad (3)
\end{align*}
\]

The condition (C1) implies that the kernel \( I_A = \text{Ker} \pi \) of the projection map

\[
\pi : H \otimes_k H \to H \otimes_A H
\]

of \( H \)-bimodules is a right ideal in the algebra \( H \otimes_k H \), generated by the set of elements of the form \( \beta(a) \otimes 1 - 1 \otimes \alpha(a) \):

\[
I_A = \{ \beta(a) \otimes_k 1 - 1 \otimes_k \alpha(a) \mid a \in A \} \cdot (H \otimes_k H) \quad (4)
\]

Regarding that \( I_A \) is a right ideal, and \( \Delta(h) \) is defined up to \( I_A \), the map \( H \otimes_k (H \otimes_k H) \to H \otimes_A H \) in (C3) is well defined. Its factorization through a map \( H \otimes_k (H \otimes_A H) \to H \otimes_A H \) exists iff for every \( h \), \( \Delta(h)I_A \subset I_A \), which is clearly equivalent to \( \Delta(h)(\beta(a) \otimes 1 - 1 \otimes \alpha(a)) \in I_A \) for all \( a \in A \). Hence \( \Delta(h) \) must belong to a set

\[
H \times_A H = \left\{ \sum_i b_i \otimes b'_i \in H \otimes_A H \mid \sum_i b_i \otimes b'_i \alpha(a) = \sum_i b_i \beta(a) \otimes b'_i, \ \forall a \in A \right\},
\]

5
which is a \(A\)-subbimodule of \(H \otimes_A H\) \([30, 31]\), called the Takeuchi product \([7, 2]\). In these terms, the factorization property from (C3) is equivalent to the property \(\text{Im} \, \Delta \subset H \times_A H\). Another part of (C3), stating that the induced map is an action, may also be expressed in terms of Takeuchi product. A direct check shows that the preimage \(\pi^{-1}(H \times_A H) = \{ \sum_i b_i \otimes k b'_i, \sum_i b_i \otimes b'_i \alpha(a) - b_i \beta(a) \otimes b'_i \in I_A \}\) is closed under multiplication; in fact a unital subalgebra. The right ideal \(I_A\) is spanned by the elements of the form \(\beta(a)d \otimes d' - d \otimes \alpha(a)d'\) and if \(\sum_i b_i \otimes b'_i \in H \times_A H\) then
\[
\left( \sum_i b_i \otimes b'_i \right) (\beta(a)d \otimes d' - d \otimes \alpha(a)d') = \left( \sum_i b_i \beta(a) \otimes b'_i - b_i \otimes b'_i \alpha(a) \right) (d \otimes d')
\]
and the right hand side clearly belongs to \(I_A\). Thus, \(I_A \cap \pi^{-1}(H \times_A H)\) is not only a right ideal but a two sided ideal of \(\pi^{-1}(H \times_A H)\) showing that \(H \times_A H \sim = \pi^{-1}(H \times_A H) / (I_A \cap \pi^{-1}(H \times_A H))\) is, unlike \(H \otimes_A H\), an associative algebra with respect to the componentwise product. The componentwise rule is not well defined in \(H \otimes_A H\) because it may depend on the chosen representatives in \(H \otimes_k H\); this is because \(I_A\) is only a right ideal in general.

This discussion shows that (C3) is equivalent to the joint assertion of the following two requirements:

(C3a) \(\text{Im} \, \Delta \subset H \times_A H\),

(C3b) \(\Delta\) as a map from \(H\) to \(H \times_A H\) is a homomorphism of algebras.

Of course, (C3b) makes sense only because of (C3a). Observe now a commutative diagram of \(A\)-bimodules:

\[
\begin{array}{ccc}
\pi^{-1}(H \times_A H) & \longrightarrow & H \otimes_k H \\
\downarrow \pi|_{\pi^{-1}(H \times_A H)} & & \downarrow \pi \\
H & \underset{\Delta}{\longrightarrow} & H \times_A H & \longrightarrow & H \otimes_A H
\end{array}
\]

(5)

All arrows except those into \(H \otimes_A H\) are also homomorphisms of algebras.

The equation \(\sum_i b_i \otimes b'_i \alpha(a) = \sum_i b_i \beta(a) \otimes b'_i\) for elements in \(H \otimes_A H\) is demanded in the quotient, hence it holds only up to elements in \(I_A\); if we take the same equation strictly in \(H \otimes_k H\) to cut some subalgebra (actually a left ideal) \(H \times h \subset H \otimes_k H\), then the projection \(\pi|_{H \times_A H}\) maps this subalgebra within \(H \times_A H\), but is not necessarily onto. In a categorical language, \(H \times_A H\)
is an end (kind of a categorical limit) of a coend (kind of a colimit), not the other way around. However, Meljanac in his examples takes some other subalgebra $\mathcal{B} \subset H \otimes_k H$ (not a universal construction) first and then passes to the quotient by $\pi|_\mathcal{B}$ (hence a colimit), with a result which is still an algebra (different from $H \times_A H$). To achieve this, he needs that

(C3MI) $I_A \cap \mathcal{B}$ is a two-sided ideal in $\mathcal{B}$.

In addition, he (implicitly) requires

(C3Ma) $\text{Im } \Delta \subset \mathcal{B}/(I_A \cap \mathcal{B})$,

(C3Mb) $\Delta$ as a map from $H$ to $\mathcal{B}/(I_A \cap \mathcal{B})$ is a homomorphism of algebras.

**Definition 2.2.** A left $A$-bialgebroid with balancing subalgebra $\mathcal{B}$ comprises an $A \otimes_k A^{op}$-ring $(H, m, \alpha, \beta)$ and an $A$-coring $(H, \Delta, \epsilon)$ with the same underlying $A$-bimodule $H$ and satisfying (C1) and (C2), and a (not necessarily unital) subalgebra $\mathcal{B} \subset H \otimes_k H$ satisfying (C3MI), (C3Ma) and (C3Mb). $\mathcal{B}$ is called the **balancing subalgebra**.

A left $A$-bialgebroid with balancing subalgebra $\mathcal{B}$ is not necessarily a left associative $A$-bialgebroid in the standard sense, because (C3) does not always hold. However, if $\mathcal{B}$ is the preimage $\pi^{-1}(H \times_A H)$ of the Takeuchi product under the natural projection $\pi$ then (C3) follows. Conversely, given a left associative $A$-bialgebroid $H = (H, m, \alpha, \beta, \Delta, \epsilon)$, we have presented above that $\pi^{-1}(H \times_A H) \cap I_A$ is a two sided ideal of the subalgebra $\pi^{-1}(H \times_A H)$. Therefore, (C3) implies that it is a balancing subalgebra called the **trivial balancing subalgebra** of $H \otimes_k H$. It follows that in a left associative $A$-bialgebroid any subalgebra of $\pi^{-1}(H \times_A H)$ containing $\pi^{-1}(\text{Im } \Delta)$ is also balancing. Therefore, on the level of bialgebroids, balancing algebras are interesting only when either we can not determine whether $\text{Im } \Delta \subset H \times_A H$, or when it does not hold but there is a balancing subalgebra, which is in the latter case automatically not a subalgebra of Takeuchi product $H \times_A H$. However, for Hopf algebroids, as we shall see below, balancing subalgebras provide more flexible approach to introducing the antipode than using Lu’s section, while it is technically more compact (less structure and axioms) than Böhm’s symmetric Hopf algebroids.

Observe a commutative diagram of $A$-bimodules where all arrows except those into $H \otimes_A H$ are homomorphisms of algebras:
Proposition 2.3. Let \((H, \mu, \alpha, \beta, \Delta, \epsilon)\) be the data defining an \(A \otimes A^\text{op}\)-
ring and \(A\)-coring satisfying (C1), (C2) and (C3a). Suppose there exist a
subalgebra \(B \subset H \otimes_k H\) such that (C3MI) and (C3Ma) hold. Then (C3b) iff
(CM3b). In other words, these data define a left \(A\)-bialgebroid with balancing
subalgebra \(B\) iff they (without \(B\)) define a left associative \(A\)-bialgebroid.

Proof. This is a rather simple observation: (C3a) and (C3Ma) together imply
that \(\text{Im} \Delta \subset \frac{B}{I_A \cap B} \cap H \times_A H\) which has the structure of a subalgebra of
\(B/(I_A \cap B)\) and also of \(H \times_A H\); the multiplications in \(B/(I_A \cap B)\) and in \(H \times_A H\)
are both defined componentwise, hence they are equal on the intersection. If
we assume (C3b), then algebra map \(\Delta : H \to H \times_A H\) corestricts to algebra
map \(H \to \frac{B}{I_A \cap B} \cap H \times_A H\) which postcomposed with inclusion of algebras
into \(B/(I_A \cap B)\) is again an algebra map, hence \(\Delta : H \to B/(I_A \cap B)\) is
also an algebra map, hence (C3Mb) holds. Likewise we infer (C3b) from
(C3Mb).

3. Hopf algebroids: antipode

Definition 3.1. A Hopf \(A\)-algebroid in the sense of J-L. Lu [18] (or a
Lu-Hopf algebroid) is a left associative \(A\)-bialgebroid \((H, \mu, \alpha, \beta, \Delta, \epsilon)\) with
an antipode map \(\tau : H \to H\), which is a linear antiautomorphism satisfying

\[
\tau \beta = \alpha \quad (7)
\]

\[
\mu(\text{id} \otimes_k \tau) \gamma \Delta = \alpha \epsilon \quad (8)
\]

\[
\mu(\tau \otimes_A \text{id}) \Delta = \beta \epsilon \tau \quad (9)
\]

for some linear section \(\gamma : H \otimes_A H \to H \otimes H\) of the projection \(\pi : H \otimes H \to
H \otimes_A H\).

The reason for introducing \(\gamma\) in (8) is the fact that \(\mu(\text{id} \otimes_A \tau) \Delta\) is not a
well defined map because \(\mu(\text{id} \otimes_k \tau)(I_A) \neq 0\) in general. Indeed, \(I_A\) is a linear
span of the set of all elements of the form \(\beta(a)h \otimes k - h \otimes \alpha(a)k\), where \(a \in A\)
and \( h, k \in H \), and \( \mu(\text{id} \otimes \tau)(\beta(a)h \otimes k - h \otimes \alpha(a)k) = \beta(a)h\tau(k) - h\tau(k)\tau(\alpha(a)) \)
which can be nonzero in general. No such problems occur with (9) because

\[
\mu(\tau \otimes \text{id})(\beta(a)h \otimes k - h \otimes \alpha(a)k) = \tau(h)\tau(\beta(a))k - \tau(h)\alpha(a)k = 0.
\]

**Definition 3.2.** A Hopf \( A \)-algebroid with balancing subalgebra \( B \) is a left \( A \)-bialgebroid \((H, \mu, \alpha, \beta, \Delta, \varepsilon)\) with balancing subalgebra \( B \) together with an algebra antihomomorphism \( \tau : H \to H \), called the antipode, such that

\[
\mu(\text{id} \otimes_k \tau)(I_A \cap B) = 0 \tag{10}
\]
\[
\tau\beta = \alpha \tag{11}
\]
\[
\mu(\text{id} \otimes_A \tau)\Delta = \alpha\varepsilon \tag{12}
\]
\[
\mu(\tau \otimes_A \text{id})\Delta = \beta\varepsilon\tau \tag{13}
\]

Two equations are the same as in Definition 3.1: (11) is identical to (7) and (13) to (9). Equation (12) now makes sense because of (10). There is no more need for a choice of a section \( \gamma \). Choice of the subalgebra \( B \) which accomplishes the same.

**Remark 3.3.** The map \( \mu(\text{id} \otimes_k \tau) : h \otimes h' \mapsto h\tau(h') \) is \( k \)-linear, but neither a homomorphism nor an antihomomorphism of algebras. Hence, it is not sufficient to check (10) on the algebra generators of \( I_A \cap B \), and a fortiori, on its generators as an ideal in \( B \). This will be the central difficulty in Section 4.

**Theorem 3.4.** If a Hopf algebroid with a balancing subalgebra satisfies (C3a) then it admits a (possibly nonunique) structure of Lu-Hopf algebroid.

**Proof.** Choose a vector space splitting of \( H \otimes_A H \) into \( \text{Im} \Delta \) and a linear complement; for \( \gamma \) take any linear section of the projection \( \pi : H \otimes_k H \to H \otimes_A H \) such that values \( \gamma(p) \) over all \( p \in \text{Im} \Delta \) are in \( B \) (this can be done by (C3Ma)) and on the linear complement prescribe any linear choice for \( \gamma \), for instance 0. Condition (C3b) holds by (CM3b) and Proposition 2.3. Then \( \mu(\text{id} \otimes_k \tau)\gamma(\Delta(h)) = \mu(\text{id} \otimes_A \tau)\Delta(h) \) as the right hand side is defined by choosing any representative of \( \Delta(h) \) in \( H \otimes_k H \) and evaluating \( \mu(\text{id} \otimes_k \tau) \). Thus (8) holds, and other conditions on the antipode become identities. \( \Box \)

Böhm and Szlachányi exhibited Example 4.9 in [6] of a symmetric Hopf algebroid which does not carry a structure of a Lu-Hopf algebroid. An application of Theorem 3.4 implies that this example is not a Hopf algebroid with
a balancing subalgebra either. We do not know if for any Lu-Hopf algebroid there is a balancing subalgebra (containing the image of $\gamma$). However, we exhibit recipes for a balancing subalgebra for several most prominent classes of Lu-Hopf algebroids. Notably, in Section 4 for any scalar extension $H$ of a Hopf algebra $T$ (with bijective antipode) by a braided commutative Yetter-Drinfeld module algebra $A$ we exhibit a Hopf $A$-algebroid with total algebra $H$ and with a balancing subalgebra given by a specified set of generators. Clearly, every Hopf algebroid over a commutative base is both a Lu-Hopf algebroid and a Hopf algebroid with a balancing subalgebra, namely $B = H \otimes_k H$.

Lu [18] exhibits an example which she calls a coarse Hopf algebroid, nowadays often called a minimal Hopf algebroid. Given a unital associative algebra, the tensor product algebra $A \otimes A^{\text{op}}$ carries the structure of Hopf algebroid with source map $\alpha(a) = a \otimes 1$, target map $\beta(b) = 1 \otimes b$, comultiplication $\Delta(a \otimes b) = (a \otimes 1) \otimes_A (1 \otimes b)$, counit $\epsilon(a \otimes b) = ab$ and antipode $\tau(a \otimes b) = b \otimes a$. It has the balancing subalgebra $B = (A \otimes k) \otimes_A (k \otimes A^{\text{op}})$.

4. Scalar extension Hopf algebroids

4.1. Scalar extensions, elements $R(a)$ and section $\gamma$

Given any associative $k$-algebra $A$ equipped with a left Hopf action $\triangleright$ of a bialgebra $T$, vector space $A \otimes T$ carries a structure of a unital associative $k$-algebra with multiplication bilinearly extending formula $(a \otimes t)(a' \otimes t') = \sum a(t(1) \triangleright a') \otimes t(2)t'$ and with unit $1_A \otimes 1_T$. This algebra is called the smash product algebra [23] and denoted $A \sharp T$. It comes along with canonical algebra monomorphisms $A \cong A \otimes k \hookrightarrow A \sharp T$ and $T \cong k \otimes T \hookrightarrow A \sharp T$. The images of these two embeddings will be denoted $A \sharp 1$ and $1 \sharp T$.

Let $T$ be a Hopf $k$-algebra with a comultiplication $\Delta_T : T \rightarrow T \otimes_k T$ and a bijective antipode $S$. A braided-commutative left-right Yetter-Drinfeld $T$-module algebra $A$ is a unital associative algebra with a left $T$-action $\triangleright : T \otimes A \rightarrow A$ which is Hopf in the sense that

$$t \triangleright (ab) = (t(1) \triangleright a)(t(2) \triangleright b), \quad t \triangleright 1_A = \epsilon(t) 1_A,$$

and a right $T$-coaction $a \mapsto a_{[0]} \otimes a_{[1]}$ which is morphism of algebras $A \rightarrow A \otimes T^{\text{op}}$ (see [7]), satisfying the left-right Yetter-Drinfeld condition

$$(t(1) \triangleright a_{[0]}) \otimes (t(2) a_{[1]}) = (t(2) \triangleright a)_{[0]} \otimes (t(1) \triangleright a)_{[1]} t(1), \quad \forall t \in T, \forall a \in A \quad (14)$$
and the **braided commutativity**

\[ x_0(x_1 \triangleright a) = ax, \quad \forall a, x \in A. \quad (15) \]

**Lemma 4.1.** Braided commutativity condition is equivalent to the condition

\[ (Sd_{[1]}) \triangleright a)d_{[0]} = da, \quad \forall d, a \in A. \quad (16) \]

**Proof.** This is rather standard. Assuming braided commutativity (15),

\[
\begin{align*}
da &= d_{[0]}((d_{[1]}Sd_{[2]}) \triangleright a) \\
&= d_{[0][0]}(d_{[0][1]} \triangleright ((Sd_{[1]}) \triangleright a)) \\
&= ((Sd_{[1]}) \triangleright a)d_{[0]}.
\end{align*}
\]

In other direction, assuming (16)

\[
\begin{align*}
x_0(x_1 \triangleright a) &= (Sx_0[1] \triangleright (x_1 \triangleright a))x_0[0] \\
&= (Sx_1[1] \triangleright (x_1[2] \triangleright a))x_0 \\
&= (\varepsilon(x_1[1])1_T \triangleright a)x_0[0] \\
&= ax.
\end{align*}
\]

\[ \square \]

If \( A \) is in fact a braided commutative Yetter-Drinfeld algebra over \( T \) then the smash product \( H = A \sharp T \) is a total algebra of a Hopf \( A \)-algebroid called a scalar extension Hopf algebroid. For a Lu-Hopf algebroid this is proven in [7], modifying slightly an earlier construction of Lu [18], Section 5, where instead of Yetter-Drinfeld modules closely related modules over Drinfeld double \( D(H) \) are considered. Both works entail a circular argument in the proof that the antipode of the algebroid is an antihomomorphism, checking the property on rather trivial case of binary products of generators of the form \( a \sharp 1 \) and \( 1 \sharp t \) only. Antihomomorphism property for products of general elements is checked in [28], assuming that \( S \) is bijective. Lu-Brzeziński-Militaru construction has been adapted to the symmetric Hopf algebroids of Böhm [1, 2, 6, 28].

The \( A \)-bimodule structure of \( A \sharp H \) is determined by the source and target maps

\[ \alpha(a) = a \sharp 1, \quad \beta(a) = a_{[0]} \sharp a_{[1]}, \quad (17) \]

and the comonoid structure of \( A \sharp H \) is given by

\[ \Delta_{A \sharp T}(a \sharp t) = (a \sharp t_{(1)}) \otimes_A (1 \sharp t_{(2)}), \quad \varepsilon_{A \sharp T}(a \sharp t) = ae_T(t). \quad (18) \]

11
Finally, the antipode $\tau$ for the Lu-Hopf algebroid is (cf. [7, 18])
\[
\tau(a \# t) = S(t)S^2(a_{[1]}) \cdot a_{[0]}.
\] (19)

We often identify $A_\# 1 = \text{Im} \, \alpha$ with $A$ and $1\# T$ with $T$. By Definition 4 the right ideal $I_A \subset H \otimes A H$ is generated by the set of all elements of the form
\[
I(a) := \beta(a) \otimes 1 - 1 \otimes \alpha(a) = a_{[0]} \# a_{[1]} \otimes 1 - 1 \otimes a, \quad a \in A.
\] (20)

There is also another set of generators $R(a)$ of $I_A$, more convenient for our analysis below.

**Proposition 4.2.** In the case of scalar extension $H = A_\# T$, right ideal $I_A \subset H \otimes A H$ is generated by the set of all elements of the form
\[
R(a) := a \otimes 1 - S a_{[1]} \otimes a_{[0]}, \quad a \in A.
\] (21)

**Proof.** In the notation (20),
\[
I(a) = (a_{[0]} \# 1 - S a_{[0][1]} \otimes a_{[0][0]})(a_{[1]} \otimes 1) = R(a_{[0]})(a_{[1]} \otimes 1).
\]

Notice that $a_{[0]} \in A$. On the other hand,
\[
R(a) = (a_{[0][0]} \# a_{[0][1]} \otimes 1 - 1 \otimes a_{[0]})(S a_{[1]} \otimes 1) = I(a_{[0]})(S a_{[1]} \otimes 1).
\]

Therefore, the right ideal generated by $\{I(a) \mid a \in A\}$ and the right ideal generated by all $R(a)$ coincide. \qed

Given a linear basis $\hat{x}_1, \ldots, \hat{x}_{\dim g}$ of a finite dimensional Lie algebra $g$, references [16, 17] introduce elements $R_\mu (\mu = 1, \ldots, \dim g)$ in a related example of a formally completed version of a scalar extension Hopf $U(g)$-algebroid. In Subsection 4.4 we observe that $R_\mu = R(\hat{x}_\mu)$.

**Lu’s section for scalar extension bialgebroids.** For any scalar extension $H = A_\# T$, J-H. Lu [18] exhibits a section $\gamma : H \otimes_A H \to H \otimes_k H$ by the unique $k$-linear extension of the formula
\[
\gamma : h \otimes_A (a \# t) \mapsto \beta(a) h \otimes_k (1 \# t), \quad h \in H, a \in A, t \in T.
\] (22)

Section $\gamma$ is well defined by (22), namely on the generators
\[
\beta(b) h \otimes (c \# t) - h \otimes \alpha(b)(c \# t)
\]
of the ideal $I_A$ the formula evaluates to $\beta(c)\beta(b) h \otimes (1 \# t) - \beta(bc) h \otimes (1 \# t) = 0$.

Linear map $\gamma$ is a section of the projection $\pi : H \otimes_k H \to H \otimes_A H$ because $h \otimes_A (a \# t) = h \otimes_A \alpha(a)(1 \# t) = \beta(a) h \otimes_A (1 \# t) = (\pi \circ \gamma)(h \otimes_A (a \# t))$.

In particular, formula (22) gives
\[
(\gamma \circ \Delta)(a \# t) = (a \# t_{(1)}) \otimes_k (1 \# t_{(2)}).
\] (23)
4.2. Subalgebra $W \subset H \otimes H$ where $H = A^\natural T$ is a scalar extension Hopf $A$-algebroid

**Notation 4.3.** Let $T$ be a Hopf algebra and $A$ a braided commutative algebra in the category of left-right Yetter-Drinfeld $T$-modules. For a scalar extension $A^\natural T$ let $W \subset (A^\natural T) \otimes (A^\natural T)$ be the smallest unital subalgebra such that all elements of the form $a \otimes 1$ and all elements of the form $Sa[1] \otimes a[0]$ (where $a \in A \cong A^\natural 1 \subset A^\natural T$) are in $W$. Let $W^+$ be the two sided ideal in $W$ generated by all elements of the form $R(a) = a \otimes 1 - Sa[1] \otimes a[0]$ where $a \in A$ (compare (21)).

Let $W^+_0 \subset W$ be the linear subspace of $W$ spanned by all elements of the form $(x \otimes 1 - Sx[1] \otimes x[0])(x' \otimes 1)$ where $x, x' \in A \cong A^\natural 1$.

We now formulate two lemmas which together imply $W^+_0 = W^+$.

**Lemma 4.4.** For $x, z \in A$ we have $(x \otimes 1)(z \otimes 1 - Sx[1] \otimes z[0]) \in W^+_0$.

*Proof.* Multiplying out, and using $xS(t) = S(t[1])t[2]xS(t[3]) = S(t[1])t[2] \triangleright x$ for $x \in A, t \in T$, we obtain

$$x z \otimes 1 - x S(z[1]) \otimes z[0] = x z \otimes 1 - S(z[1])(z[2] \triangleright x) \otimes z[0]$$

= by braided commutativity

$$= z[0](z[1] \triangleright x) \otimes 1 - S(z[1])(z[2] \triangleright x) \otimes z[0]$$

$$= (z[0] \otimes 1 - S(z[0][1])) \otimes z[0][0](z[2] \triangleright x \otimes 1)$$

and the right hand side is clearly in $W^+_0$ as claimed. \qed

**Lemma 4.5.** $R(x)R(z) = (x \otimes 1 - Sx[1] \otimes x[0])(z \otimes 1 - Sx[1] \otimes x[0]) \in W^+_0$.

*Proof.* Since $x \mapsto x[1] \otimes x[0]$ is a morphism of algebras $A \to T^{op} \otimes_k A$ and $T \otimes_k A \hookrightarrow A^\natural T \otimes_k A^\natural T = H \otimes_k H$ inclusion of algebras, we conclude that $x \mapsto Sx[1] \otimes x[0]$ is a morphism of algebras $A \to H \otimes_k H$ (with respect to the componentwise multiplication in $H \otimes_k H$). Therefore,

$$(x \otimes 1 - Sx[1] \otimes x[0])(z \otimes 1 - Sw[1] \otimes w[0]) =$$

$$= (x \otimes 1 - Sx[1] \otimes x[0])(z \otimes 1) + xSw[1] \otimes z[0] - xz \otimes 1 + xz \otimes 1 - S(xz)[1] \otimes (xz)[0]$$

$$= (x \otimes 1 - Sx[1] \otimes x[0])(z \otimes 1) + (-x \otimes 1)(z \otimes 1 - Sw[1] \otimes w[0]) + (xz \otimes 1 - S(xz)[1] \otimes (xz)[0]).$$

The first and the third summands on the right hand side are manifestly in $W^+_0$ while for the second summand we apply Lemma 4.4. \qed
Corollary 4.6. \( \forall x, z \in A, (Sx_{[1]} \otimes x_{[0]})(z \otimes 1 - Sx_{[1]} \otimes z_{[0]}) \in W_0^+ \),

(i) \( W_0^+ \) is a two-sided ideal in \( W \),

(ii) \( W^+ = W_0^+ \).

Proof. (i) follows by subtracting the expression \( (x \otimes 1 - Sx_{[1]} \otimes x_{[0]})(z \otimes 1 - Sx_{[1]} \otimes z_{[0]}) \) which is in \( W_0^+ \) from the expression \( (x \otimes 1)(z \otimes 1 - Sx_{[1]} \otimes z_{[0]}) \) which is in \( W_0^+ \) by Lemma \( \ref{lemma1} \).

(ii) \( W_0^+ \) is a right ideal: by its definition, we can multiply by \( z \otimes 1 \) from the right; this together with the assertion of Lemma \( \ref{lemma1} \) implies that we can also multiply by \( Sx_{[1]} \otimes z_{[0]} \) from the right.

(ii) \( W_0^+ \) is a left ideal: using Lemma \( \ref{lemma1} \), \( (x \otimes 1)R(z)(x' \otimes 1) \in W_0^+(x' \otimes 1) \) which is in \( W_0^+ \) because it is a right ideal. Combining with Lemma \( \ref{lemma1} \) we also conclude that \( (Sx_{[1]} \otimes x_{[0]})(z \otimes 1) \in W_0^+ \).

For (iii) notice first that, trivially, \( W_0^+ \subset W^+ \). For the converse inclusion, \( W^+ \subset W_0^+ \), it is sufficient to observe that \( R(a) \in W_0^+ \), apply (ii) and the definition of \( W^+ \).

\[ \square \]

Theorem 4.7. \( \mu(id \otimes_k \tau)W^+ = \{0\} \).

Proof. By Corollary (iii) \( W^+ = W_0^+ \), which is the span of the elements of the form

\( (x \otimes 1 - Sx_{[1]} \otimes x_{[0]})(z \otimes 1), \) where \( x, z \in A \).

Taking the standard formula for the antipode for the scalar extensions \( \tau(a^*t) = S(t)S^2(a_{[1]} \cdot a_{[0]}) \), we can now compute \( \mu(id \otimes \tau) \) on such an element as

\[ xz - S(x_{[2]})zS^2(x_{[1]})(x_{[0]} = xz - ((Sx_{[1]}) \triangleright z)x_{[0]} = 0, \]

by the braided commutativity.

\[ \square \]

4.3. Subalgebra \( B \) and two-sided ideal \( B^+ \subset B \)

In this section, we want to show that every scalar extension Lu-Hopf algebroid \( H = A\sharp T \) is also a Hopf algebroid with a (carefully chosen) balancing subalgebra \( B \).

Using the inclusion \( T \otimes_k T \hookrightarrow A\sharp T \otimes_k A\sharp T \), we identify the image of the coproduct \( \Delta_T : T \rightarrow T \otimes_k T \) of the Hopf algebra \( T \) with a subalgebra of \( H \otimes_k H \) which will be denoted by \( \Delta_T(T) \).
Definition 4.8. Subalgebra \( \mathcal{B} \subset A \sharp T \otimes_k A \sharp T \) is the subalgebra generated by \( W \) and \( \Delta_T(T) \) or, equivalently, by the set
\[
\{ a \otimes 1, \quad Sa_{[1]} \otimes a_{[0]} \mid a \in A \} \cup \Delta_T(T).
\]
The elements of this set are called the distinguished generators of \( \mathcal{B} \).

Recall now elements \( R(a) \in I_A \cap \mathcal{B} \) defined by formula (21). Let \( \mathcal{B}^+ \) be the two-sided ideal in \( \mathcal{B} \) generated by the subset
\[
\{ R(a) \mid a \in A \} = \{ a \otimes 1 - Sa_{[1]} \otimes a_{[0]} \mid a \in A \} \subset \mathcal{B},
\]
whose elements \( R(a) \) are called the distinguished generators of \( \mathcal{B}^+ \).

Theorem 4.9. Suppose \( H = A \sharp T \) is a smash product, where \( T \) is a Hopf algebra with bijective antipode and \( A \) a braided-commutative Yetter-Drinfeld algebra over \( A \); in other words, \( A \sharp T \) is the underlying algebra of usual scalar extension bialgebroid. Let \( \mathcal{B} \) and \( \mathcal{B}^+ \) be as in Definition 4.8.

(i) \( \mathcal{B}^+ = I_A \cap \mathcal{B} \).
(ii) (C3MI) holds: \( I_A \cap \mathcal{B} \) is a two-sided ideal in \( \mathcal{B} \).
(iii) (C3Ma) holds: \( \text{Im} \Delta \subset \mathcal{B}/(I_A \cap \mathcal{B}) \).
(iv) The scalar extension \( A \sharp T \) is a bialgebroid with the balancing subalgebra \( \mathcal{B} \).
(v) \( \mathcal{B} \subset \pi^{-1}(H \times_A H) \).
(vi) Inclusion from (v) induces an inclusion of algebras \( \mathcal{B}/\mathcal{B}_+ \hookrightarrow H \times_A H \) whose image is \( \Delta_L(A \sharp T) \subset A \sharp T \otimes_A A \sharp T \). On generators this isomorphism onto the image is given by \( k \)-linear extension of the correspondence \( a \otimes 1 \mapsto a \sharp 1 \otimes_A 1 \), \( Sa_{[1]} \otimes a_{[0]} \mapsto a \sharp 1 \otimes_A 1 \) for \( a \in A \), and \( \Delta_T(t) = t_{(1)} \otimes t_{(2)} \mapsto 1 \sharp t_{(1)} \otimes_A 1 \sharp t_{(2)} \) for \( t \in T \).

Proof. (i) follows immediately from Definition 4.8 for \( \mathcal{B}^+ \) and Proposition 4.2.
(ii) follows from (i) and the definition of \( \mathcal{B}^+ \). (iii) is an immediate check knowing the generators and (i).

For (iv) use (ii), (iii), Proposition 2.3 and the fact that the (C3b) is known for scalar extensions [7, 13].

For (v) we have two proofs. One is to check for each distinguished generator separately that it belongs to \( \pi^{-1}(H \times_A H) \). Another is to write \( I(a) \) in terms of \( R(a) \), use that \( R(a) \in I_A \cap \mathcal{B} \) and hence \( bR(a) \in b\mathcal{B}^+ \subset \mathcal{B}^+ \) because \( R(a) \in \mathcal{B}^+ \) and the latter is a two-sided ideal.

(vi) Clearly, \( a \otimes 1 - Sa_{[1]} \otimes a_{[0]} = R(a) \in \mathcal{B} \cap I_A \mapsto I_A \), hence the values on \( a \otimes 1 \) and \( Sa_{[1]} \otimes a_{[0]} \) are the same. \( \square \)
Lemma 4.10. Let $\sum_{\rho_i} K_{i}^{\rho_i} \otimes L_{i}^{\rho_i} \in \{x \otimes 1, \; Sx_{[1]} \otimes x_{[0]} \mid x \in A\} \cup \{f_{(1)} \otimes f_{(2)} \mid f \in T\}$ be a distinguished generator. Then for any $a \in A$,

$$\sum_{\rho_i} K_{i}^{\rho_i} \cdot (a \sharp 1) \cdot \tau(L_{i}^{\rho_i}) \in A \sharp 1.$$  

Proof. We inspect the claim case by case as follows.

(a) If $\sum_{\rho_i} K_{i}^{\rho_i} \otimes L_{i}^{\rho_i} \in \{x \otimes 1, \; Sx_{[1]} \otimes x_{[0]} \mid x \in A\}$, then by (19)

$$\sum_{\rho_i} K_{i}^{\rho_i} \cdot (a \sharp 1) \cdot \tau(L_{i}^{\rho_i}) = \sum_{\rho_i} K_{i}^{\rho_i} \cdot a \sharp 1 \cdot S^{2}(L_{i[1]}^{\rho_i}) \cdot L_{i[0]}^{\rho_i}. \quad (24)$$

The dot product sign $\cdot$ denotes here the multiplication in the smash product $A \sharp T$; $A$ is identified there with $A \sharp 1$ and $T$ with $1 \sharp T$. There are now two subcases, (a1) and (a2).

(a1) For $\sum_{\rho_i} K_{i}^{\rho_i} \otimes L_{i}^{\rho_i} = x \otimes 1$, (24) equals $xa = xa \sharp 1$ which is in $A \sharp 1$.

(a2) For $\sum_{\rho_i} K_{i}^{\rho_i} \otimes L_{i}^{\rho_i} = Sx_{[1]} \otimes x_{[0]}$, (24) equals $Sx_{[1]} \cdot a \cdot S^{2}(x_{[1]}) \cdot x_{[0]} = ((Sx)[1] \triangleright a) \cdot (Sx)[2] \cdot S((Sx)[3]) \cdot x_{[0]} = ((Sx)[1] \triangleright a) \cdot x_{[0]} = xa$ which is in $A \sharp 1$.

(b) If $\sum_{\rho_i} K_{i}^{\rho_i} \otimes L_{i}^{\rho_i} = f_{(1)} \otimes f_{(2)}$, $f \in T$, then $\sum_{\rho_i} K_{i}^{\rho_i} \cdot a \sharp 1 \cdot \tau(L_{i}^{\rho_i}) = f_{(1)} \cdot a \sharp 1 \cdot S(f_{(2)}) = (f \triangleright a) \sharp 1$ which is again in $A \sharp 1$.

Therefore the claim $\sum_{\rho_i} K_{i}^{\rho_i} \cdot a \sharp 1 \cdot \tau(L_{i}^{\rho_i}) \in A \sharp 1$ follows. \qed

Lemma 4.11. Let $U$ be a product of finitely many distinguished generators of $B$. Then

$$\mu(\text{id} \otimes \tau)(U) \in A \sharp 1.$$  

Proof. Let $U = (\sum_{\rho_1} K_{1}^{\rho_1} \otimes L_{1}^{\rho_1}) \cdots (\sum_{\rho_n} K_{n}^{\rho_n} \otimes L_{n}^{\rho_n})$. The antipode $\tau$ is an algebra antihomomorphism, hence

$$\mu(\text{id} \otimes \tau)(U) = \sum_{\rho_{1, \ldots, \rho_{n}}} K_{1}^{\rho_{1}} K_{2}^{\rho_{2}} \cdots K_{n}^{\rho_{n}} \tau(L_{n}^{\rho_{n}}) \cdots \tau(L_{1}^{\rho_{1}}). \quad (25)$$

We prove that

$$\sum_{\rho_{n-p, \ldots, \rho_{n}}} K_{n-p}^{\rho_{n-p}} \cdots K_{n}^{\rho_{n}} \tau(L_{n}^{\rho_{n}}) \cdots \tau(L_{n-p}^{\rho_{n-p}}) \in A \sharp 1,$$
by induction on $p$ where $0 \leq p \leq n - 1$, the assertion of the lemma is then the case $p = n - 1$. For the base of induction, $p = 0$, this is the identity $\sum_{\rho} K_{n}^{\rho} \cdot \tau(L_{n}^{\rho}) \in \mathbb{A}^{\#}1$ which follows from Lemma 4.10 when $a = 1$. The step of the induction on $p$ is clearly also a special case of Lemma 4.10.

By (25) it follows that $\mu(id \otimes \kappa \tau)(U) \in \mathbb{A}^{\#}1$; in other words, $\mu(id \otimes \kappa \tau)(U)$ is of the form $d^{\#}1$ for some $d \in \mathbb{A}$.

\textbf{Theorem 4.12.} Let $\tau : \mathbb{A}^{\#}T \to \mathbb{A}^{\#}T$ given by the formula (19) be the antipode of the scalar extension as a Lu-Hopf algebroid. Then

(i) $\mu(id \otimes \tau)B^{+} = \{0\}$.

(ii) $\tau$ makes the corresponding $\mathbb{A}$-bialgebroid with a balancing subalgebra from Theorem 4.12 into a Hopf $\mathbb{A}$-algebroid with a balancing subalgebra.

\textbf{Proof.} (i) A general element of $B^{+}$ is a linear combination of the elements of the form

$$\prod_{j=1}^{m} \sum_{\sigma_{j}} M_{j}^{\sigma_{j}} \otimes N_{j}^{\sigma_{j}} \cdot (x \otimes 1 - Sx_{[1]} \otimes x_{[0]}) \cdot \prod_{k=1}^{n} K_{k}^{\rho_{k}} \otimes L_{k}^{\rho_{k}},$$

(26)

where $\sum_{\sigma_{j}} M_{j}^{\sigma_{j}} \otimes N_{j}^{\sigma_{j}}$, $\sum_{\rho_{k}} K_{k}^{\rho_{k}} \otimes L_{k}^{\rho_{k}}$ are some distinguished generators of $B$, and the middle factor $x \otimes 1 - Sx_{[1]} \otimes x_{[0]}$ is some distinguished generator in $B^{+}$. Notice that $M_{j}^{\sigma_{j}}$, $N_{j}^{\sigma_{j}}$, $K_{k}^{\rho_{k}}$, $L_{k}^{\rho_{k}} \in \mathbb{A}^{\#}1 \cup 1^{\#}T$ and $x \in \mathbb{A}$.

By the linearity of $\mu(id \otimes \tau)$ it is sufficient to prove the assertion for one element of the form above. Rewrite (26) as,

$$\prod_{j,k} \sum_{\sigma_{j},\phi_{j}} (M_{j}^{\sigma_{j}} \cdot x_{1}^{\#} \cdot K_{k}^{\rho_{k}}) \otimes (N_{j}^{\sigma_{j}} \cdot L_{k}^{\rho_{k}}) - (M_{j}^{\sigma_{j}} \cdot 1_{x[1]} \cdot x_{[0]}^{\#} \cdot x_{1}^{\#} \cdot d^{\#}1 \cdot \tau(L_{k}^{\rho_{k}}) \otimes (N_{j}^{\sigma_{j}} \cdot x_{[0]}^{\#} \cdot 1_{x[1]} \cdot \tau(L_{k}^{\rho_{k}})).$$

(27)

By Lemma 4.11 we can define $d \in \mathbb{A}$ by

$$d^{\#}1 := K_{1}^{\rho_{1}} \cdots K_{n}^{\rho_{n}} \cdot \tau(L_{n}^{\rho_{n}}) \cdots \tau(L_{1}^{\rho_{1}}) \in \mathbb{A}^{\#}1.$$ 

(28)

We apply map $\mu(id \otimes \tau)$ to (27) and substitute (28). Notice that $\tau$, being an antihomomorphism, reverses the order. Thus we need the vanishing of

$$\sum_{\sigma} M_{1}^{\sigma_{1}} \cdots M_{m}^{\sigma_{m}} (x_{1}^{\#} \cdot d^{\#}1 - 1_{x[1]}^{\#} \cdot Sx_{[1]} \cdot d^{\#}1 \cdot \tau(x_{[0]}^{\#} \cdot 1_{x[1]} \cdot \tau(N_{m}^{\sigma_{m}}) \cdots \tau(N_{1}^{\sigma_{1}})).$$

(29)
Therefore to finish the proof of the assertion (i) it is sufficient to show that for all \( x, d \in A \) we have

\[
x^{\ast}1 \cdot d^{\ast}1 = 1^{\ast}Sx_{[1]} \cdot d^{\ast}1 \cdot \tau(x_{[0]}^{\ast}1),
\]

where, by the formula for the antipode (19), \( \tau(x_{[0]}^{\ast}1) = S^{2}(x_{[1]}) \cdot x_{[0]}. \)

This amounts to showing

\[
xd^{\ast}1 = ((Sx)_{[1]} \triangleright d) \triangleright (Sx)_{[2]} S((Sx)_{[3]}) \cdot x_{[0]}^{\ast}1,
\]

that is,

\[
xd^{\ast}1 = ((Sx)_{[1]} \triangleright d) x_{[0]}^{\ast}1,
\]

which is by Lemma 4.1 an expression of the braided commutativity (15) for \( A \). Therefore, (i) is proven.

For the part (ii), according to Theorem 4.9 part (iv), it remains only to check the axioms for the antipode. The antipode requirements (11) and (13) have the same content as in the case of Lu-Hopf algebroid definition hence they are true. Now, thanks to (ii) the left-hand side of the equation (12), that is, \( \mu(id \otimes A \tau) \Delta \), does not depend on the representatives of \( \Delta(at) = (a^{\ast}t(1)) \otimes (1^{\ast}t(2)) \) in \( H \otimes k H \) where \( a \in A \) and \( t \in T \). So we need to show that

\[
(a^{\ast}t(1))(S^{2}(t(2))_{[1]} \cdot (t(2))_{[0]}) = a^{\ast}t,
\]

which boils down to the same computation as for the Lu’s choice of \( \gamma \), see (23). Our result is stronger only in the sense that we allow for an additional freedom in \( B_{+} \) and that \( B \) is a balancing subalgebra in the bialgebroid sense.

4.4. Comparison with the examples of Meljanac

S. Meljanac has devised his method [16, 17] to the study topological Hopf algebroids related to a Lie algebra \( g_{\kappa} \) with the universal enveloping algebra \( U(g_{\kappa}) \) in physics literature called the \( \kappa \)-Minkowski space. Some extensions of this Hopf algebroid (including some symmetries into the algebra) from the point of view of Lu-Hopf algebroid have been studied in [19] in an informal style of mathematical physics and, in just slightly more mathematical treatment, in [20]. Works [16, 17] made it clear that their construction applies to any finite dimensional Lie algebra \( g \) in characteristic zero. We comment below on how our construction of \( B \) relates to theirs for general \( g \). As stated in the introduction, we neglect here the issues related to the adaptation of the notion of Hopf algebroid to the completed tensor products [22, 28].
We use the notation from [22]. Generators of the Lie algebra $\mathfrak{g}$ are denoted $\hat{x}_1, \ldots, \hat{x}_n$ with commutators $[\hat{x}_\mu, \hat{x}_\nu] = C^\lambda_{\mu\nu} \hat{x}_\lambda$ and the generators of the symmetric algebra of the dual $S(\mathfrak{g}^*)$ by $\partial^1, \ldots, \partial^n$. The completed dual $T = \hat{S}(\mathfrak{g}^*)$ is a topological Hopf algebra, namely the coproduct $\Delta_T : \hat{S}(\mathfrak{g}^*) \to \hat{S}(\mathfrak{g}^*) \otimes \hat{S}(\mathfrak{g}^*)$ may be identified with the dual (transpose) map to the multiplication $\mu(\hat{g}) \to \hat{U}(\hat{g}) \otimes \hat{U}(\hat{g})$. The identification is made with help of the symmetrization map $\hat{S}(\hat{g}) \cong \hat{U}(\hat{g})$, which is an isomorphism of coalgebras [12, 22] and its dual isomorphism of algebras $\hat{S}(\hat{g}) \cong \hat{U}(\hat{g})^*$. Now $A = U(\hat{g})$ becomes a braided-commutative Yetter-Drinfeld module algebra over $T$ (internally in a symmetric category of filtered-cofiltered vector spaces [28]). Regarding $H$ generated by $\mathfrak{g}$, we observe that $R(a)$ is defined by (21), then $\hat{x}_{[0]} \otimes \hat{x}_{[1]} = \hat{x}_\sigma \otimes (\mathcal{O}^{-1})^\sigma_\tau$ where $\mathcal{O}^\sigma_\tau, (\mathcal{O}^{-1})^\sigma_\tau$ are certain elements in $T$ (see [22] for the definition and properties), $\mathcal{O}^{-1}$ is a matrix inverse of $\mathcal{O}$, $\Delta_T \mathcal{O}^\sigma_\nu = \mathcal{O}^\sigma_\nu \otimes \mathcal{O}^\nu_\sigma$ and $S(\mathcal{O}^{-1})^\mu_\nu = \mathcal{O}^\mu_\nu$. Thus we obtain,

$$R(\hat{x}_\mu) = \hat{x}_\mu \otimes 1 - \hat{S}_{[1]} \otimes \hat{x}_{[0]} = \hat{x}_\mu \otimes 1 - \mathcal{O}^\sigma_\mu \otimes \hat{x}_\sigma.$$  

(30)

We observe that $R(\hat{x}_\mu)$ is identical to $R_\mu$ of [17, 16]. Using identities $[\mathcal{O}^\mu_\sigma, \hat{x}_\mu] = C^\lambda_{\mu\sigma} \mathcal{O}^\lambda_\sigma$ (formula (17) in [22]) and $C^\sigma_\nu \mathcal{O}^\lambda_\tau = C^\lambda_\rho \mathcal{O}^\rho_\nu \mathcal{O}^\sigma_\rho$ (formula (20) in [22]), we obtain

$$[R(\hat{x}_\mu), R(\hat{x}_\nu)] = C^\sigma_{\mu\nu} R(\hat{x}_\sigma),$$

(31)

$$[\hat{x}_\mu \otimes 1, R(\hat{x}_\nu)] = C^\lambda_{\mu\nu} R(\hat{x}_\lambda)$$

(32)

(generalizing Eq. (32),(33) in arXiv version of [16], (3.2),(3.3) in journal v.). Moreover, (31) is the only relation among $R_\mu$-s, hence the subalgebra in $H \otimes_k H$ generated by $\{R_\mu \mid \mu = 1, \ldots, n\}$, is isomorphic to the universal enveloping algebra $U(\mathfrak{g})$, but with generators $R_\mu$ in place of $\hat{x}_\mu$. Following [16], denote this subalgebra by $U(R)$. The relation [16] (32) shows that the products of the form $r(u_r^1 \otimes 1)$ where $r \in U(R)$ and $u_r^1 \otimes 1 \in U(\mathfrak{g}) \otimes_k k \subset U(\mathfrak{g}) \otimes_k T \otimes_k U(\mathfrak{g})$ span a subalgebra in $H \otimes_k H$. This is precisely our subalgebra $W$ in this case. However, the relations (3.3),(3.4) in [16] for $\mathfrak{g}_k$ case, and the
generalizations (31), (32) for general $g$, used to show that $W$ is a subalgebra of $H \otimes_k H$, do not have a simple analogue for general scalar extension $A\sharp T$ (not of enveloping algebra type). It is also not clear what is the precise structure of the subalgebra generated by $R(a)$-s for all $a \in A$, in general. On the other hand, our Hopf algebraic definition (21) of $R(a)$ and the corresponding definition of $W$ in the subsection 4.2 along with the lemmas therein guarantee that such general $W$ is a subalgebra in $H \otimes_k H$ in full generality.

The issues are more complicated when we pass from $W$ to $B$. In the enveloping algebra case, the subalgebra $B$ (denoted $\hat{B}$ in [16]) is defined in [16] rather simply as the subalgebra of all elements of the form $\sum_i w_i \Delta \hat{S}(g^\ast)(t_i)$ where $w_i \in W$, $t_i \in T = \hat{S}(g^\ast)$ are arbitrary (the sums may be infinite, in an appropriate completion). Equations (3.3),(3.4) in [16], can be abstracted and generalized to an arbitrary finite dimensional Lie algebra as the following proposition.

**Proposition 4.13.** For general $g$,

\[ [\Delta \partial^\mu, R(\hat{x}_\nu)] = 0 \]
\[ [\Delta \partial^\mu, \hat{x}_\nu \sharp 1 \otimes 1] \in \Delta_T(T). \]  

Regarding that $\partial^\mu$ generate a dense subalgebra of $T$, this implies immediately that $\{ \sum_i w_i \Delta \hat{S}(g^\ast)(t_i) \mid w_i \in W, t_i \in T \}$ is a subalgebra of $H \otimes_k H$, and that $B$ has a very simple structure of all sums of products of the form: an element in $U(R)$ times an element in $A\sharp 1 \otimes_k k \subset H \otimes_k H$ times an element of the form $\Delta_T(t_i)$ with $t_i \in T$. We have exhibited above a similar structure – as a sum of products of elements from three subalgebras in this fixed order – for general scalar extension Hopf algebroids. In this generality, $P_i$ do not commute with elements in $W$ and multiple products (e.g. of the form $w t w' t' w''$) of elements in $W$ and elements in $\Delta_T(T)$ may appear, as analysed in the subsection 4.3. Regarding that $\mu(\text{id} \otimes_k \tau)$ is not an antihomomorphism, the multiple products bring the main difficulty in our proof that the antipode $\tau$ is well defined (see Theorem 4.12 (i)).

Analogous comparisons may be made for the ideal $B_+$ which is in [16] not defined as the intersection $I_A \cap B$, but an equivalent description is given, constructing it in analogy to $B$, but with the enveloping algebra $U(R)$ replaced by its ideal $U_+(R) \subset U(R)$ of elements which are not degree 0 in the standard filtration of the universal enveloping algebra. This explains the notation $B_+$. The commutation relations (3.1)-(3.4) in [16] imply that such $B_+$ is indeed a two-sided ideal in $B$. 

20
Our approach also differs from [16] in insisting that the coproduct is still defined as taking values in $H \otimes_A H$ (rather than in $B/(I_A \cap B)$ as an abstract algebra); the two-sided ideal trick is used only to make sense of the requirement and to check that the induced map into $B/(I_A \cap B)$ is a morphism of algebras. Moreover, they view $B$ as an abstract algebra constructed from its pieces $U(R)$, $A_1 \otimes_k k$ and $\Delta_T(T) \cong T$. In our approach, the coherently associative tensor product of bimodules $\otimes_A$ is used to formulate the coassociativity of the coproduct as in the standard definition of a bialgebroid [2, 6, 7, 18, 32]. In [16] the coproduct is taking values in $B/(I_A \cap B)$ by definition and, in the spirit of their viewpoint, the higher iterations of the coproduct in subalgebras $B^{(j)} \subset H \otimes_k H \otimes_k \cdots \otimes_k H$ ($j$ tensor factors),

which define higher analogues of the subalgebra $B^{(2)} := B \subset H \otimes_k H$. One also considers the higher analogues $B^{(j)}_* = I_A^{(j)} \cap B^{(j)}$ of $I_A \cap B$ in order to deal with the (co)associativity issues. For example, $B^{(3)}$ is generated by all ordered products of the form $r \cdot (a \otimes 1 \otimes 1) \cdot (\Delta \otimes \text{id})(\Delta(t))$ where $a \in A$, $t \in T$ and $r$ belongs to the subalgebra generated by $\{ R(a) \otimes 1 \mid a \in A \} \cup \{ 1 \otimes R(b) \mid b \in A \}$. The right ideal $I_A^{(j)}$ is the smallest right ideal in $H \otimes^k$ containing right ideals $I_A \otimes H^{(j-2)}), H \otimes I_A \otimes H^{(j-3)}, \ldots, H^{(j-2)} \otimes I_A$. These are interesting structures, but in our view more cumbersome than the familiar usage of the bimodule tensor product $\otimes_A$.

5. Weak Hopf algebras

It is well-known that from the data of any weak Hopf algebra one can construct a corresponding Lu-Hopf algebroid. Upon looking at our axiomatics, G. Böhm has observed and sketched to us how to construct a Hopf algebroid with balancing subalgebra from a weak Hopf algebra. We present her results in this section, starting with a short review of weak Hopf algebras.

5.1. Weak bialgebras, standard definitions

A weak $k$-bialgebra $\mathbb{H}$ (see [3]) is a tuple $(\mathbb{H}, \mu, \eta, \Delta, \epsilon)$ where $(A, \mu, \eta)$ is an associative unital $k$-algebra, $(\mathbb{H}, \Delta, \epsilon)$ is a coassociative counital $k$-coalgebra, and the following compatibilities hold:

(i) $\Delta$ is multiplicative, $\Delta(ab) = \Delta(a)\Delta(b)$ for all $a, b \in \mathbb{H}$.
(ii) Weak multiplicativity of the counit: for all \( x, y, z \in H \),

\[
\epsilon(xyz) = \epsilon(xy_{(1)})\epsilon(y_{(2)}z), \quad (34)
\]

\[
\epsilon(xyz) = \epsilon(xy_{(2)})\epsilon(y_{(1)}z). \quad (35)
\]

If we assume (i), then it is elementary (see formulas (4) and (1) in [4]) that (34) and (35) are respectively equivalent to the conditions

\[
g\epsilon(1_{(2)}h)1_{(1)} = \epsilon(g_{(2)}h)g_{(1)}, \quad \forall g, h \in H, \quad (36)
\]

\[
g\epsilon(1_{(1)}h)1_{(2)} = \epsilon(g_{(1)}h)g_{(2)}, \quad \forall g, h \in H. \quad (37)
\]

(iii) Weak comultiplicativity of the unit:

\[
\Delta^{(2)}(1) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1))
\]

\[
\Delta^{(2)}(1) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1)
\]

where we denoted \( \Delta^{(2)} := (\text{id} \otimes \Delta)\Delta = (\Delta \otimes \text{id})\Delta \). In Sweedler notation,

\[
1_{(1)} \otimes 1_{(2)} \otimes 1_{(3)} = 1_{(1)} \otimes 1_{(2)}1_{(1)' \otimes 1_{(2)'}, \quad (38)
\]

\[
1_{(1)} \otimes 1_{(2)} \otimes 1_{(3)} = 1_{(1)' \otimes 1_{(1)}1_{(2)' \otimes 1_{(2)}}. \quad (39)
\]

For every weak \( k \)-bialgebra there are \( k \)-linear maps \( \Pi^L, \Pi^R : H \rightarrow H \) with properties \( \Pi^R\Pi^R = \Pi^R \) and \( \Pi^L\Pi^L = \Pi^L \) and defined by

\[\Pi^L(x) := \epsilon(1_{(1)}x)1_{(2)}), \quad \Pi^R(x) := 1_{(1)}\epsilon(x_{1(2)}).\]

These expressions are met below in two of the axioms for the antipode of a weak Hopf algebra. Less frequently, one also encounters idempotents \( \bar{\Pi}^L, \bar{\Pi}^R \) given by

\[\Pi^L(x) := \epsilon(1_{(2)}x)1_{(1)}, \quad \Pi^R(x) := 1_{(2)}\epsilon(x_{1(1)}).\]

Now

\[
\epsilon(xz) = \epsilon(x_{1z}) = \epsilon(x_{1(2)})\epsilon(1_{(1)}z) = \epsilon(x_{1(2)})1_{(1)}\epsilon(z) = \epsilon(\Pi^R(x)z),
\]

\[
= \epsilon(1_{(1)}z)1_{(2)} = \epsilon(x_{\Pi^L(z)}).
\]

The images of the idempotents \( \Pi^R \) and \( \Pi^L \),

\[
\mathbb{H}^R := \Pi^R(\mathbb{H}), \quad \mathbb{H}^L := \Pi^L(\mathbb{H}),
\]

are mutually dual as \( k \)-linear spaces via the canonical nondegenerate pairing \( \mathbb{H}^L \otimes \mathbb{H}^R \rightarrow k \) given by \( (x, y) \mapsto \epsilon(xy) \).

The identities \( \Pi^L(x_{\Pi^L(y)}) = \Pi^L(xy) \) and \( \Pi^R(\Pi^R(x)y) = \Pi^R(xy) \) hold. Dually also \( \Delta(\mathbb{H}^L) \subset \mathbb{H} \otimes \mathbb{H}^L, \Delta(\mathbb{H}^R) \subset \mathbb{H}^R \otimes \mathbb{H} \), and \( \Delta(1) \in \mathbb{H}^R \otimes \mathbb{H}^L \).
5.2. Böhm’s recipes

It is known that a weak Hopf algebra $H$ can be regarded as a Lu-Hopf algebroid over $A := \Pi^L(\mathbb{H})$ where the source map $\alpha$ being the inclusion $\Pi^L(\mathbb{H}) \subset \mathbb{H}$ and where the target map is given by

$$\beta(a) = \tilde{\Pi}^L(a) = \epsilon(1_{(2)}a)1_{(1)} \quad \text{for} \quad a \in A,$$

and the comultiplication $\Delta' = \pi \circ \Delta$ of the bialgebroid is the comultiplication $\Delta : \mathbb{H} \rightarrow \mathbb{H} \otimes_k \mathbb{H}$ of the weak Hopf algebra followed by the canonical projection $\pi : \mathbb{H} \otimes_k \mathbb{H} \rightarrow \mathbb{H} \otimes_{\Pi^L(\mathbb{H})} \mathbb{H}$.

**Lemma 5.1.** $1 \otimes 1 - \Delta(1) \in I_A$.

*Proof.* By definition, $I_A$ is generated by all expressions of the form

$$I(h) := \beta(\Pi^L(h)) \otimes 1 - 1 \otimes \alpha(\Pi^L(h)), \quad h \in \mathbb{H}.$$

Now $\Pi^L(h) = \epsilon(1_{(1)}h)1_{(2)}$ hence by (40)

$$\beta(\Pi^L(h)) = \epsilon(1_{(2)})\epsilon(1_{(2')})1_{(1)}$$

$$= \epsilon(1_{(2)}1_{(2')})\epsilon(1_{(2')})1_{(1)}$$

$$= \epsilon(1_{(2)})1_{(1)},$$

where $1_{(1')} \otimes 1_{(2')}$ denotes another copy of $\Delta(1)$.

$$I(h) = \epsilon(1_{(2)}h)1_{(1)} \otimes 1 - 1 \otimes \epsilon(1_{(1)}h)1_{(2)}$$

$$= \tilde{\Pi}^L(h) \otimes 1 - 1 \otimes \Pi^L(h).$$

It is sufficient to prove that $1 \otimes 1 - \Delta(1) = I(1_{(2)})(1_{(1)} \otimes 1)$ because the right hand side manifestly belongs to $I_A$. From (42) we calculate

$$I(1_{(2')})(1_{(1')} \otimes 1) = \epsilon(1_{(2')}1_{(2')})1_{(1)}1_{(1')} \otimes 1 - 1_{(1')} \otimes \epsilon(1_{(1)}1_{(2')})1_{(2)}$$

$$= \epsilon((1 \cdot 1_{(2)})(1 \cdot 1_{(2)}))1_{(1)} \otimes 1 - 1_{(1)} \otimes \epsilon(1_{(2)})1_{(3)}$$

$$= 1 \otimes 1 - \Delta(1),$$

where in the middle line the axioms on $\Delta^{(2)}$ were used for the second summand.

**Lemma 5.2.** $\Delta(1)I(h) = 0$. 

23
Proof. By (42), \( \Delta(1)I(h) = 1(1)\epsilon(1(2)'h)1(1') \otimes 1(2) - 1(1) \otimes 1(2)'\epsilon(1(1)'h)1(2) \).

\[ 1(1)\epsilon(1(2)'h)1(1') \otimes 1(2) = 1(1) \otimes 1(2)'\epsilon(1(1)'h)1(2) \]

\[ = 1(1) \otimes 1(2)\epsilon(1(1)'h)1(2) \]

Here we used (36) with \( g = 1(1) \) and (37) with \( g = 1(2) \).

\[ \square \]

Corollary 5.3. The right ideal \( I_A \) coincides with the principal right ideal generated by \( 1 \otimes 1 - \Delta(1) \).

Proof. By Lemma 5.1 element \( 1 \otimes 1 - \Delta(1) \in I_A \) and by Lemma 5.2 for every \( h \in H, I(h) = (1 \otimes 1 - \Delta(1))I(h) \), hence also \( I_A \subset (1 \otimes 1 - \Delta(1))H \).

\[ \square \]

Theorem 5.4. For a weak bialgebra \( (H, \mu, \eta, \Delta, \epsilon) \), define the subalgebra

\[ B := \Delta(1)(H \otimes H)\Delta(1) \subset H \otimes H. \]

Then \( (H, \mu, \Pi^L(HH), \Pi^L, \pi|_B \circ \Delta, \Pi^L) \) is a left \( \Pi^L(HH) \)-bialgebroid with balancing subalgebra \( B \).

Proof. It is clear that \( B \) is a subalgebra with unit \( \Delta(1) \) and that \( \text{Im} \Delta \subset B \).

By Lemma 5.2 the intersection \( B \cap I_A \subset \Delta(1)I_A = 0 \) is the zero ideal of \( B \) hence (C3MI) holds and \( \Delta' : H \rightarrow HH \otimes \Pi^L(HH) H \) factorizes, indeed, through an algebra homomorphism \( H \rightarrow B/(B \cap I_A) \). It is clear that \( \Delta(h) = \Delta(1)\Delta(h)\Delta(1) \in B \) for every \( h \in H \), hence (C3Ma) holds. Then \( \pi|_B : B \cong B/(B \cap I_A) \), hence as \( \Delta \) is homomorphism, its corestriction \( \Delta|_B \) to \( B \) is homomorphism and, finally, the corestriction followed by the restriction of the projection \( \pi|_B \circ \Delta|_B \) is a homomorphism. Other properties (e.g. that \( (H, \Delta, \Pi^L) \) is a \( \Pi^L(HH) \)-coring) are well known as they coincide with the axioms of a left associative \( A \)-bialgebroid.

\[ \square \]

5.3. Antipode

A weak \( k \)-bialgebra \( H \) is a weak Hopf algebra if there is a \( k \)-linear map \( S : H \rightarrow H \) (which is then called an antipode) such that for all \( x \in H \)

\[ x(1)S(x(2)) = \epsilon(1(1)x)1(2), \]  

\[ S(x(1))x(2) = 1(1)\epsilon(x1(2)), \]  

\[ 24 \]
Notice that the right hand side of (43) equals $\Pi^L(x)$ and the right hand side of (44) equals $\Pi^R(x)$. Suppose the antipode $S$ is bijective. Set the antipode of the corresponding Hopf algebroid with a balancing subalgebra to be $\tau = S$. Since $I_A \cap B = \{0\}$ any $k$-linear map vanishes on it; hence so does the map $\mu \circ (\text{id} \otimes_k \tau)$ of (11). Axiom (11) can be restated as $(S \circ \beta \circ \Pi^L)(h) = \Pi^L(h)$). To show this identity, notice that $(S \circ \beta \circ \Pi^L)(h) = S(\epsilon(1(2)h)1(1))$ by (41) and then it is enough to quote $S(1(1)\epsilon(1(2)x)) = \Pi^L(x)$, which is the identity (2.24a) in [5]. Axiom (12) reads $h(1)\beta(S(h(2))) = 1(1)\epsilon(h(2))$, where the last equality is (2.23b) in [5], proven using axiom (45).

6. Twisting by invertible 2-cocycles

Ping Xu [32] has generalized Drinfeld’s procedure of twisting of bialgebras by invertible counital 2-cocycles to associative bialgebroids. Basic treatment involves several subtle points [32] not appearing in the bialgebra case. These do not readily generalize to arbitrary bialgebroids with balancing subalgebra. Thus we consider usual 2-cocycles for bialgebroids, but consider the effect of twisting on the balancing subalgebra.

**Definition 6.1.** Let $H$ be a left associative $A$-bialgebroid with balancing subalgebra $B \subset H \otimes_k H$ such that $\pi(B) \subset H \times_A H$. An element $\mathcal{F} \in H \times_A H$ is called a **2-cocycle** if the equation

$$[(\Delta \otimes_A \text{id})(\mathcal{F})](\mathcal{F} \otimes_k 1) = [(\text{id} \otimes_A \Delta)(\mathcal{F})](1 \otimes_k \mathcal{F})$$

holds in $H \otimes_A H \otimes_A H$. 2-cocycle $\mathcal{F}$ is **counital** if $(\text{id} \otimes_A \epsilon)\mathcal{F} = 1 = (\epsilon \otimes_A \text{id})\mathcal{F}$. If we write $\mathcal{F} = \sum_i F^1_i \otimes F^2_i := F^1 \otimes F^2$, then the counitality can be rewritten as $\beta(\epsilon(F^2)F^1) = 1 = \alpha(\epsilon(F^1))F^2$.

This equation (46) makes sense by $\mathcal{F} \in H \times_A H$. The case $\pi(B) \subset H \times_A H$ is by Proposition 2.3 not quite a novel case of a bialgebroid. Still, we are now interested in a recipe for the change of a concrete balancing subalgebra under twisting. Following Xu, for $a \in A$ we define

$$\beta_{\mathcal{F}}(a) := \beta(F^2 \triangleright a)F^1, \quad \alpha_{\mathcal{F}}(a) := \alpha(F^1 \triangleright a)F^2.$$
Xu has proved [32] that the twisted product $\star_{\mathcal{F}}$ on $A$ defined by $a \star_{\mathcal{F}} b = \alpha_{\mathcal{F}}(a) \beta_{\mathcal{F}}(b)$ is associative and unital. For the $\mathcal{F}$-twisted base algebra $A_{\mathcal{F}} = (A, \star_{\mathcal{F}})$ maps $\alpha_{\mathcal{F}} : A_{\mathcal{F}} \rightarrow H$ and $\beta_{\mathcal{F}} : A_{\mathcal{F}}^{op} \rightarrow H$ are morphisms of $k$-algebras with mutually commuting images. In particular, $H$ becomes an $A_{\mathcal{F}}$-bimodule and an $A_{\mathcal{F}} \otimes A_{\mathcal{F}}^{op}$-ring; use $H^{\mathcal{F}}$ to emphasize the twisted structures. Xu has further shown that

$$\mathcal{F}(\beta_{\mathcal{F}}(a) \otimes 1 - 1 \otimes \alpha_{\mathcal{F}}(a)) \in I_{A}.$$  \hspace{1cm} (48)

Define $I_{\mathcal{F}}$ as the right ideal in $H \otimes_{k} H$ generated by all elements of the form $\beta_{\mathcal{F}}(a) \otimes 1 - 1 \otimes \alpha_{\mathcal{F}}(a)$. Then (48) implies $\mathcal{F} I_{\mathcal{F}} \subseteq I_{A}$. One says that $\mathcal{F}$ is \textbf{invertible} if there is an element $\tilde{\mathcal{F}}^{-1} \in H \otimes_{k} H$ such that $\tilde{\mathcal{F}}^{-1} I_{A} \subseteq I_{\mathcal{F}}$ and for $\mathcal{F}^{-1} := \tilde{\mathcal{F}}^{-1} + I_{\mathcal{F}}$ the identities $\mathcal{F} \mathcal{F}^{-1} = 1 \otimes_{k} 1 + I_{A}$ and $\mathcal{F}^{-1} \mathcal{F} = 1 \otimes_{k} 1 + I_{\mathcal{F}}$ hold. Denote also by $\tilde{\mathcal{F}} \in H \otimes_{k} H$ any representative of $\mathcal{F}$. This is not the original definition of invertibility, but it is equivalent to it [27]. It follows that $\mathcal{F} I_{\mathcal{F}} = I_{A}$ (and $\tilde{\mathcal{F}} I_{\mathcal{F}} = I_{A}$). Clearly, $H^{\mathcal{F}} \otimes_{A_{\mathcal{F}}} H^{\mathcal{F}} = H \otimes_{k} H/ I_{\mathcal{F}}$. If we define $\Delta_{\mathcal{F}}(h) := \mathcal{F}^{-1} \Delta(h) \mathcal{F} : H^{\mathcal{F}} \rightarrow H^{\mathcal{F}} \otimes_{A_{\mathcal{F}}} H^{\mathcal{F}}$, this map of $A_{\mathcal{F}}$-bimodules is coassociative due the 2-cocycle property, with counit $\epsilon_{\mathcal{F}} = \epsilon$. Notice that $\Delta_{\mathcal{F}}(h) I_{\mathcal{F}} = \mathcal{F}^{-1} \Delta(h) \mathcal{F} I_{\mathcal{F}} \subseteq \mathcal{F}^{-1} \Delta(h) I_{A} \subseteq \mathcal{F}^{-1} I_{A} = I_{\mathcal{F}}$. In other words, $\text{Im} \Delta_{\mathcal{F}} \subseteq H^{\mathcal{F}} \times_{A_{\mathcal{F}}} H^{\mathcal{F}}$, where the factorwise multiplication is well defined. Conjugation with $\mathcal{F}$ (in the sense up to corresponding ideals) can easily be checked to preserve the multiplicativity property of $\Delta_{\mathcal{F}}$. Thus Xu obtains a new twisted $A_{\mathcal{F}}$-bialgebroid $H^{\mathcal{F}}$ from the old $A$-bialgebroid $H$. We want to modify this a bit to allow for a balancing subalgebra. For this we first describe twisted Takeuchi product in terms of the original one. Suppose

$$\sum_{i} b_{i} \otimes_{A} b_{i}' \in H \times_{A} H,$$

that is $\sum_{i} (b_{i} \beta_{\mathcal{F}}(a) \otimes_{k} b_{i}' - b_{i} \otimes_{A} b_{i}' \alpha(a)) \in I_{A}$. Then

$$\sum_{i} b_{i} F^{1} \beta_{\mathcal{F}}(a) \otimes_{A} b_{i}' F^{2} \overset{47}{=} \sum_{i} b_{i} \beta(F^{1}_{2} F^{r2} \triangleright a) F^{1}_{1} \otimes_{A} b_{i}' F^{2} \overset{46}{=} \sum_{i} b_{i} \beta(F^{2}_{1} F^{r1} \triangleright a) F^{1} \otimes_{A} b_{i}' F^{2} \overset{2}{=} \sum_{i} b_{i} F^{1} \otimes_{A} b_{i}' \alpha(F^{1}_{2} F^{r1} \triangleright a) F^{2}_{2} \overset{2}{=} \sum_{i} b_{i} F^{1} \otimes_{A} b_{i}' F^{2} \alpha_{\mathcal{F}}(a)\,$$

Transformations in this calculations are allowed because elements in Takeuchi product multiply elements in $H \otimes_{A} H$ from the left, and the maps $(h \otimes_{A} g) \mapsto \beta(g \triangleright a) h$ and $(h \otimes_{A} g) \mapsto \alpha(h \triangleright a) g$ are well defined. We obtain

$$\sum_{i} b_{i} F^{1} \beta_{\mathcal{F}}(a) \otimes_{k} b_{i}' F^{2} - \sum_{i} b_{i} F^{1} \otimes_{k} b_{i}' F^{2} \alpha_{\mathcal{F}}(a) \in I_{A}.$$  Multiplying this by $\mathcal{F}^{-1}$
from the left, we obtain

$$\mathcal{F}^{-1}\left(\sum b_i \otimes A b'_i\right) \mathcal{F} \in \mathcal{F}^{-1} I_A = I_{\mathcal{F}}.$$

(49)

Therefore, $\mathcal{F}^{-1}(H \times_A H) \mathcal{F} \subset H^\mathcal{F} \times_{A^\mathcal{F}} H^\mathcal{F}$, and similarly for the converse inclusion, obtaining

$$\mathcal{F}^{-1}(H \times_A H) \mathcal{F} = H^\mathcal{F} \times_{A^\mathcal{F}} H^\mathcal{F}$$

Since $I_A$ is right ideal and $\mathcal{F} \mathcal{F}^{-1} = 1 \otimes_k 1 + I_A$, there are inclusions $I_A \mathcal{F}^{-1} \subset I_A = I_A \mathcal{F} \mathcal{F}^{-1} \subset I_A \mathcal{F}^{-1}$, hence $I_A \mathcal{F}^{-1} = I_A$. Denote the new projection $\pi_\mathcal{F} : H \otimes_k H \rightarrow (H \otimes_k H)/I_{\mathcal{F}}$. Then define the twisted balancing subalgebra by $\mathcal{B}_\mathcal{F} := \pi^{-1}(\mathcal{F}^{-1} \pi(\mathcal{B}) \mathcal{F})$. Then $I_{\mathcal{F}} \cap \mathcal{B}_\mathcal{F} = (\mathcal{F}^{-1} I_A) \cap \mathcal{B}_\mathcal{F} = \mathcal{F}^{-1}(I_A \cap \mathcal{B}) \mathcal{F}$. It is a conjugate of a two-sided ideal within algebra $H \otimes_k H$, hence itself a two-sided ideal. If $\Delta : H \rightarrow \mathcal{B}/(I_A \cap \mathcal{B})$ is a morphism of $k$-algebras, then clearly $\mathcal{F}^{-1} \Delta(-) \mathcal{F} : H^\mathcal{F} \rightarrow \mathcal{B}/(I_{\mathcal{F}} \cap \mathcal{B})$ is. Thus we obtain a twisted bialgebroid with a balancing subalgebra.

Xu [32] does not consider the antipode. A general proof that Drinfeld-Xu twist can be used to twist the antipode by using a canonical formula has been missing for 20 years due technical difficulties resolved only in [27], for Hopf algebroids with an invertible antipode in the sense of Böhm and Szlachányi [6]. We leave the extension of twisting to antipode in the setting of balancing subalgebras to a future treatment.

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