On the symmetries of the 1+1 dimensional relativistic fluid dynamics

C Alexa
I.F.I.N. - High Energy Dept. Magurele-Bucharest 76900

D Vrinceanu
University of Bucharest, Dept. of Theoretical Physics, Bucharest

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Abstract

A very intuitive description of nucleus-nucleus collision phenomena is provided by the relativistic fluid dynamics. We consider a 1+1 dimensional relativistic imperfect fluid flow to approximate the high energy heavy ion collision. The article investigates the application of the continuous symmetry group on the relativistic fluid energy-momentum tensor conservation equations in the ultrarelativistic limit $\gamma \to \infty$.

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1 Introduction

There are three important theoretical approaches of the heavy ion collisions: fluid dynamics, Boltzman equation and statistical models.

A simple but intuitive description of the general nuclear phenomena is given by fluid dynamics and it has been successfully applied to the ultrarelativistic central heavy ion collisions [1].

The Boltzmann equation has already been applied to heavy ion collisions for relative low energies [2] and there is also a relativistic Boltzmann equation
using Jüttner distribution $f_{\text{Jüttner}} = 1/(2\pi\hbar)^3 \exp((\mu - p^\alpha u_\alpha)/T)$ where $\mu$ is the chemical potential, $u_\alpha$ and $p^\alpha$ are the four-velocity and four-momentum of the particle, $T$ is the temperature parameter [3].

In statistical models, one use the equilibrium thermodynamics to get the properties of the system. One of the main interest is the study of the phase transition from hadrons to quarks and gluons. At this point we can mention that the lattice quantum chromodynamics and subsequent use of Monte Carlo techniques enabled us to study physical observables over the entire temperature range from 0 to $\infty$ [4] and the very existence of a phase transition.

A consistent way of investigation of the high energy heavy ion collisions is the manifest covariant formulation of relativistic fluid dynamics. An excellent review about the status of fluid dynamics models for relativistic heavy ion reactions is the article written by D.Strottman [5]. One of the arguments to justify relativistic fluid dynamics is that the mean free path $\Lambda$ is smaller than the size $L$ of the nucleus. A simple estimate of the order of magnitude of $\Lambda$ is about 1 fm, which is to be compared to the size of heavy nucleus 15 fm (for uranium). On the other hand, it was showed [6] that a single particle quantum mechanics is formally analogous to the Euler hydrodynamics.

Solving the hydrodynamics equations is hardly discouraged by the puzzle of choosing the rest frame. This situation leads us to acausal and instable solutions [5]. Furthermore, the choose of the initial conditions is rather unclear. In the same time the numerical approaches do not give us satisfactory quantitative results.

Complementary information may be achieved by exploiting the Lie symmetry group of the covariant relativistic hydrodynamics equations. The power of this technique consists in the possibility to explore the properties of physical systems, like the symmetry structure and the invariants, without solving the corresponding differential equations. The group structure and the invariants can help us to reduce the order of the equation and even to integrate them. Complex systems were successfully studied using this approach [7]. In [8] we study a simple form of energy-momentum tensor conservation, where we already obtained symmetries and invariants.
2 Energy-momentum tensor conservation equation

Relativistic fluid dynamics is well described by the number of particles $N$ and energy-momentum tensor $T_{\alpha\beta}$ conservation equations [5]. In the ideal case we have:

$$
T_{\alpha\beta} = p\eta_{\alpha\beta} + (p + \rho)U_{\alpha}U_{\beta}
$$

$$
N_{\alpha} = nU_{\alpha}
$$

where $p$ is the pressure, $\rho$ is the energy density, $n$ is the number of particles density and $U_{\alpha} : (\gamma \vec{\beta}, \gamma)$ is the 4-velocity field.

There are two ways of choosing the rest frame: in the Landau way $U_{\alpha}$ is the energy transport velocity where $T_{i0} = 0$ in the rest frame, while in the Eckart way $U_{\alpha}$ is the particle transport velocity where $N_{i} = 0$ in the rest frame. The dissipation contribution is introduced by redefining the energy-momentum and number of particle tensor by adding correction terms:

$$
T_{\alpha\beta} = p\eta_{\alpha\beta} + (p + \rho)U_{\alpha}U_{\beta} + \Delta T_{\alpha\beta},
$$

$$
N_{\alpha} = nU_{\alpha} + \Delta N_{\alpha}
$$

In the Eckart frame $\Delta N_{\alpha} = 0$, so the dissipation contribution is present only in the energy-momentum terms. In the following we choose the Eckart approach. The construction of the most general dissipation term $\Delta T_{\alpha\beta}$ is coming up from the positivity of the entropy production:

$$
\Delta T^{\alpha\beta} = -\eta H^{\alpha\gamma}H^{\beta\delta}W_{\gamma\delta} - \chi \left( H^{\alpha\gamma}U^{\beta} - H^{\beta\gamma}U^{\alpha} \right) Q_{\gamma} - \zeta H^{\alpha\beta}\partial_{\gamma}U^{\gamma}
$$

where we have shear tensor:

$$
W_{\alpha\beta} = \partial_{\beta}U_{\alpha} + \partial_{\alpha}U_{\beta} - \frac{2}{3}\eta_{\alpha\beta}\partial_{\gamma}U^{\gamma},
$$

heat-flow vector:

$$
Q_{\alpha} = \partial_{\alpha}T + T U^{\beta}\partial_{\beta}U_{\alpha},
$$

$T$ is the temperature and projection tensor on the hyperplane normal to $U_{\alpha}$

$$
H_{\alpha\beta} = \eta_{\alpha\beta} + U_{\alpha}U_{\beta}.
$$
We may identify \( \chi, \eta, \zeta \) as the coefficients of heat conduction, shear viscosity and bulk viscosity.

Making some calculations we can write the energy-momentum tensor in the following form:

\[
T_{\alpha\beta} = p\eta_{\alpha\beta} + (p + \rho)U_\alpha U_\beta - \eta \left[ \partial_\alpha U_\beta + \partial_\beta U_\alpha - \frac{2}{3}(\eta_{\alpha\beta} + U_\alpha U_\beta) \partial_\gamma U_\gamma + U_\gamma \partial_\gamma (U_\alpha U_\beta) \right] - \chi \left[ U_\alpha \partial_\beta T + U_\beta \partial_\alpha T + 2U_\alpha U_\beta U_\gamma \partial_\gamma T + TU_\gamma \partial_\gamma (U_\alpha U_\beta) \right] - \zeta \left( \eta_{\alpha\beta} + U_\alpha U_\beta \right) \partial_\gamma U_\gamma
\]

With this form we rewrite the energy-momentum conservation:

\[
\partial^\alpha T_{\alpha\beta} = \partial^\beta \left[ p + \left( \frac{2}{3} \eta - \zeta \right) \partial^\alpha U_\alpha \right] + \partial^\alpha \left\{ U_\alpha U_\beta \left[ p + \rho + \left( \frac{2}{3} \eta - \zeta \right) \partial_\gamma U_\gamma + 2\chi U_\gamma \partial_\gamma T \right] \right\} - \partial^\alpha \left[ (\eta + \chi T) U_\gamma \partial_\gamma (U_\alpha U_\beta) - \eta (\partial_\alpha U_\beta + \partial_\beta U_\alpha) - \chi (U_\alpha \partial_\beta T + U_\beta \partial_\alpha T) \right]
\]

We can see this equation as a polynomial in \( U \), with the power 3, 2, 1, 0. From this expression we select only the highest power of \( U \) and the free terms. Because \( U \) is proportional with \( \gamma \) Lorentz \( U^3 \gg U^2 \gg U \gg 1 \) in the ultrarelativistic limit, or we can make a boost to a system were \( \gamma \) is big enough to obtain to above inequality. This approximation is convenable for us because the form of the equation is much more simpler. In the same time we don’t have any physical contradictions - as we will see later on Lorentz transformation will be one of the symmetry group transformation - and we don’t loose physical information about the fluid.

Doing this one can find the following form of the equation:

\[
\partial^\alpha T_{\alpha\beta} = \partial_\beta \left[ p + \left( \frac{2}{3} \eta - \zeta \right) \partial^\alpha U_\alpha \right] - 2\chi \cdot \partial^\alpha (U_\alpha U_\beta U_\gamma \partial_\gamma T)
\]

In the final form we have in the 1+1 dimensional approximation two equations:

\[
\begin{align*}
p_x + \left( \frac{2}{3} \eta - \zeta \right) (u_{xx} - v_{xt}) - 2\chi (u^3 T_{xx} + uv^2 T_{tt} - 2u^2 v T_{xt}) &= 0 \\
p_t + \left( \frac{2}{3} \eta - \zeta \right) (u_{xt} - v_{tt}) - 2\chi (u^2 v T_{xx} + v^3 T_{tt} - 2uv^2 T_{xt}) &= 0
\end{align*}
\]
where $p_x, p_t, \text{etc.}$ means the partial derivative of $p$ with respect to $x, t$, etc. The $u$ and $v$ are the spatial and temporal components of the velocity field $U(\gamma\vec{v}, \gamma) = U(u, v)$.

3 Symmetry group of differential equations

The symmetry group of a system of differential equations is the largest local group of transformation acting on the independent and dependent variables of the system with the property that it transform solutions of the system to other solutions.

We restrict our attention to local Lie group of symmetries, leaving aside problems involving discrete symmetries such as reflections.

Let $\mathcal{S}$ be a system of differential equations. A symmetry-group of the system $\mathcal{S}$ is a local group of transformations $G$ acting on an open subset $M$ of the space of independent and dependent variables for the system with the property that whenever $u=f(x)$ is a solution of $\mathcal{S}$, and whenever $g \cdot f$ is defined for $g \in G$, then $u = g \cdot f(x)$ is also a solution of the system.

Applying the standard procedure \[7\] we solve the defining equations for the symmetry group of the given system of differential equations. We find a nine parameter symmetry group for the ultrarelativistic fluid dynamics equations. The basis of the corresponding solvable Lie algebra of this group is:

$$
\begin{align*}
V_1 &= \partial_x \quad \text{(spatial translation)} \\
V_2 &= \partial_t \quad \text{(temporal translation)} \\
V_3 &= \partial_T \quad \text{(temperature translation)} \\
V_4 &= \partial_p \quad \text{(pressure translation)} \\
V_5 &= x\partial_T \\
V_6 &= t\partial_T \\
V_7 &= u \partial_u + v \partial_v - 2T \partial_T - p \partial_p \quad \text{(dilatations)} \\
V_8 &= t \partial_x + x \partial_t - u \partial_v - v \partial_u \quad \text{(hyperbolic rotations)} \\
V_9 &= 4x \partial_x + 4t \partial_t - u \partial_u - v \partial_v + 2T \partial_T + 5p \partial_p \quad \text{(dilatations)}
\end{align*}
$$
The symmetry group infinitesimal generator is defined by:

\[ \tilde{V} = \xi \partial_x + \tau \partial_t + \Phi \partial_u + \Psi \partial_v + \Gamma \partial_T + \Omega \partial_p \]

where

\[ \begin{align*}
\xi &= c_1 + c_8 \cdot t + 4c_9 \cdot x \\
\tau &= c_2 + c_8 \cdot x + 4c_9 \cdot t \\
\Phi &= (c_7 - c_9)u - c_8v \\
\Psi &= (c_7 - c_9)v - c_8u \\
\Gamma &= c_3 + c_5x + c_6t - 2(c_7 - c_9)T \\
\Omega &= c_4 - (c_7 - 5c_9)p
\end{align*} \]

and \( a_i \) are arbitrary group parameters.

We calculate the second order prolongation of \( \tilde{V} \):

\[ \text{pr}(2)\tilde{V} = \xi \partial_x + \tau \partial_t + \Phi \partial_u + \Psi \partial_v + \Gamma \partial_T + \Omega \partial_p + \Phi_x \partial_u + \Phi_t \partial_v + \Psi_x \partial_v + \Psi_t \partial_u + \Gamma_x \partial_T + \Gamma_t \partial_T + \Omega_x \partial_p + \Omega_t \partial_p + \Phi_{xx} \partial_u + \Phi_{xt} \partial_v + \Psi_{xx} \partial_v + \Psi_{xt} \partial_u + \Gamma_{xx} \partial_T + \Gamma_{xt} \partial_T + \Omega_{xx} \partial_p + \Omega_{xt} \partial_p \]

The coefficient functions of the prolongation of \( \text{pr}^{(n)}\tilde{V} \) are given by the following formula \[^{[7]}\]:

\[ \Phi^{J,k}_{\alpha} = D_k \Phi^{J}_{\alpha} - \sum_{i=1}^{p} D_k \xi^i u^\alpha_{J,i} \]

where \( p \) is the number of the independent variables, \( \xi^i \) are the coefficients of the partial derivative of the independent variables \((x, t)\) - here \( \xi^i = (\xi, \tau) \), \( q \) is the number of dependent variables \( u = (u, v, T, p) \) - in this case we have \( q = 4 \) so \( \Phi_{\alpha} = (\Phi, \Psi, \Gamma, \Omega) \) and \( D_i \) is the total derivative

\[ D_i f = \frac{\partial f}{\partial x^i} + \sum_{a=1}^{q} \sum_{j} u^a_{J,i} \frac{\partial f}{\partial u^a_j} \]

6
\[ u^\alpha_{j,i} = \frac{\partial u^\alpha_j}{\partial x^i} = \frac{\partial^{k+1} u^\alpha}{\partial x^i \partial x^{j_1} \ldots \partial x^{j_k}} \]

\( J = (j_1, \ldots, j_k), 0 \leq \# J \leq n \), \( n \) is the highest order derivative appearing in \( f \). For example \( \Phi^x = D_x(\Phi - \xi u_x - \tau u_t) + \xi u_{xx} + \tau u_{xt} \), \( \Psi^{xx} = D_x^2(\Psi - \xi v_x - \tau v_t) + \xi v_{xxx} + \tau v_{xxt} \) and \( D_x \Gamma = \Gamma_x + \Gamma_u u_x + \Gamma_v v_x + \Gamma_T T_x + \Gamma_T p_x \).

An n-th order differential invariant of a group \( G \) is a smooth function depending on the independent and dependent variables and their derivatives, invariant on the action of the corresponding n-th prolongation of \( G \) [7].

We use the method of characteristics for computing the invariants for the prolongation of the every element of the Lie algebra; then we re-express the next vector in terms of first vector’s invariants; we repet this procedure until we obtaine all global invariants of the vectors. In the end we obtained seven global independent invariants of the symmetry group:

\[ I_1 = (u + v)(p_x + p_t) \exp \left(-\frac{2v_x}{u_x + v_t}\right) \]

\[ I_2 = \sqrt{(u^2 - v^2)(p_x^2 - p_t^2)} \frac{u_t - v_x}{u_x - v_t} \]

\[ I_3 = \sqrt{p_x^2 - p_t^2} \frac{(p_x + p_t)(u_t - v_x)}{p_{xx} - p_{tt}} \exp \left(-\frac{2v_x}{u_x + v_t}\right) \]

\[ I_4 = \frac{(p_{xx} - p_{tt})^2 (u + v)^2}{(p_x + p_t)^2 (u_x v_t - u_t v_x)} \]

\[ I_5 = \frac{(u_x v_t - u_t v_x)(p_x + p_t)^2}{(u + v)^2 (T_{xx} - T_{tt})} \]

\[ I_6 = \frac{(u_x v_t - u_t v_x)^3 (T_{xx} - T_{tt})^3}{(p_{xt}^2 - \frac{1}{2} p_{xx}^2 - p_{xx} p_{tt})^2} \left[ u_{xx}^2 + u_{xx} v_{xx} - \frac{1}{2} u_{xx}^2 - u_{xx} (u_{tt} + v_{xx}) \right]^{-2} \]

\[ I_7 = \frac{(p_{xt}^2 - \frac{1}{2} p_{xx}^2 - p_{xx} p_{tt}) (T_{xt}^2 - \frac{1}{2} T_{xx}^2 - T_{xx} T_{tt})}{[v_{xt}^2 + v_{xt} u_{xx} - \frac{1}{2} v_{xx}^2 - v_{xx} (u_{xt} + v_t)]^2} \left[ u_{xx}^2 + u_{xx} v_{xx} - \frac{1}{2} u_{xx}^2 - u_{xx} (u_{tt} + v_{xx}) \right] \]
4 Conclusions

Relativistic imperfect fluid flow seems to be a very good approach of the ultrarelativistic heavy ion collision, because we can very well suppose a zero mean free path and a instantaneous local equilibrium.

In spite of the simplification of the equations, our results indicate a non-trivial structure of the symmetry group for the 1+1 dimensional relativistic fluid dynamics equations.

As expected due to the covariant formulation of the theory, the symmetry group contain the Poincare generators. And of course, other system specific transformations mentioned above are present.

Beside the physical meaning that can be associated to the invariants, they can be used for reducing the order of the original equations. Doing this one can hope to find simpler equations that can be integrated; unfortunately only for special type of groups one can find the general solutions of the equations by quadratures alone.

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