A GENUINE ANALOGUE OF WIENER TAUBERIAN THEOREM FOR SL(2, R)

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Abstract. We prove a genuine analogue of Wiener Tauberian theorem for integrable functions on SL(2, R).

1. Introduction

Let \( f \in L^1(\mathbb{R}) \) and \( \hat{f} \) be its Fourier transform. The celebrated Wiener-Tauberian (W-T) theorem says that the ideal generated by \( f \) in \( L^1(\mathbb{R}) \) is dense in \( L^1(\mathbb{R}) \) if and only if \( \hat{f} \) is nowhere vanishing on \( \mathbb{R} \). This theorem has been extended to abelian groups. In 1955, Ehrenpreis and Mautner observed that the exact analogue of the theorem above fails for the commutative algebra of the integrable \( K \)-biinvariant functions on the group \( SL(2, \mathbb{R}) \), where \( K = SO(2) \) is a maximal compact subgroup. Nonetheless the authors proved that if a \( K \)-biinvariant integrable function \( f \) on \( G \) satisfies a “not-to-rapid decay” condition and nonvanishing condition on an extended strip \( S_{1, \delta} = \{ \lambda \in \mathbb{C} \mid |\Re \lambda| \leq 1 + \delta \} \) for \( \delta > 0 \), etc, that is,

\[
\hat{f}(\lambda) \neq 0 \quad \text{for all } \lambda \in S_{1, \delta},
\]

and “not-to-rapid decay” condition

\[
\limsup_{|t| \to \infty} |\hat{f}(it)|e^{Ke^{|t|}} > 0 \quad \text{for all } K > 0
\]

together with some other conditions then the ideal generated by \( f \) in \( L^1(G//K) \) is dense in \( L^1(G//K) \) (see [EM] for the precise statements). Using the extended strip condition the results has been generalised to the full group \( SL(2, \mathbb{R}) \) (see [Rs1]) and to the real rank one semi simple Lie groups (see [BBH], [BWH], [Rs2], [As]). We also refer [En], and [NS] for an analogue of W-T theorem for semisimple Lie groups of arbitrary real rank.

Y. Ben Natan, Y. Benyamini, H. Hedenmalm and Y. Weit (in [BBH] [BWH] proved a genuine analogue of the W-T theorem without the extended strip condition for \( L^1(SL(2, \mathbb{R})//SO(2)) \). In [PS] the authors extended this result to real rank one semisimple Lie group in the \( K \)-biinvariant setting. In this article we generalize the result to the full group \( SL(2, \mathbb{R}) \) and therefore this improves the corresponding result of [Rs1].

Let \( G \) be the group \( SL(2, \mathbb{R}) \) and \( K \) be its maximal compact subgroup \( SO(2) \). A complex valued function \( f \) on \( G \) is said to be of left (resp. right) \( K \)-type \( n \) if

\[
f(kx) = e_n(k)f(x) \quad \text{(resp. } f(xk) = e_{-n}(k)f(x) \text{)}
\]

for all \( k \in K \) and \( x \in G \), \hspace{1em} (1.1)

where \( e_n(k) = e^{in\theta} \). For a class of functions \( \mathcal{F} \) on \( G \) (e.g. \( L^1(G) \)), \( \mathcal{F}_n \) denotes the corresponding subclass of functions of right \( n \) type and \( \mathcal{F}_{m,n} \) will denote the subclass of \( \mathcal{F}_n \) which are also of left type \( m \). We denote the subclass of \( \mathcal{F} \) consisting of functions with integral zero by \( \mathcal{F}^0 \).

The main result (Theorem [1.1]) of this article is an analogue of W-T theorem to the the full group without the redundant extended strip condition. We first prove the W-T theorem for \( L^1(G)_{n,n} \) (Theorem [1.3]) for all \( n \in \mathbb{Z} \). This is the most crucial step in the direction of proving the W-T theorem to the full group. Before stating our main result we introduce some notation. For a

2010 Mathematics Subject Classification. Primary 43A85; Secondary 22E30.

Key words and phrases. Wiener Tauberian theorem, estimate of hypergeometric functions, resolvent transform.
function \( f \in L^1(G) \) its principal and discrete parts of the Fourier transform will be denoted by \( \hat{f}_H \) and \( \hat{f}_B \) respectively. Let \( M = \{ \pm i \} \) and \( \hat{M} = \{ \sigma^+, \sigma^- \} \) which consists of the trivial \((\sigma^+)\) and the non-trivial \((\sigma^-)\) irreducible representations of \( M \). The representation \( \pi_{\sigma^-,0} \) has two irreducible subrepresentations, so called mock series. We will denote them by \( D_+ \) and \( D_- \). The representation spaces of \( D_+ \) and \( D_- \) contain \( e_n \in L^2(K) \) respectively for positive odd \( n \)'s and negative odd \( n \)'s. For each \( \sigma \in \hat{M}, \mathbb{Z}^* \) stands for the set of even integers for \( \sigma = \sigma^+ \), and the set of odd integers for \( \sigma = \sigma^- \). Moreover, we express \(-\sigma\) by \(-\sigma^+ = \sigma^- \) and \(-\sigma^- = \sigma^+ \). We define,

\[
S_1 = \{ \lambda \in \mathbb{C} \mid |\Re \lambda| \leq 1 \} \quad \text{and} \quad \Gamma_n = \begin{cases} 
\{ k \mid 0 < k < n \; \text{and} \; k \in \mathbb{Z}^- \} & \text{if } n > 0 \\
\{ k \mid n < k < 0 \; \text{and} \; k \in \mathbb{Z}^- \} & \text{if } n < 0 
\end{cases}.
\]

For any function \( F \) on \( i\mathbb{R} \), we let

\[
\delta_{\infty}^+(F) = -\lim_{t \to \infty} e^{-\frac{\pi}{t}} \log |F(\pm it)|. \tag{1.2}
\]

We now state our main theorem.

**Theorem 1.1.** Let \( \{ f^\alpha \mid \alpha \in \Lambda \} \) be a collection of functions in \( L^1(G) \) such that the collections \( \{ \hat{f}_H^\alpha \mid \alpha \in \Lambda \} \) and \( \{ \hat{f}_B^\alpha \mid \alpha \in \Lambda \} \) have no common zero in \( \hat{M} \times S_1 \cup \{ D_+, D_- \} \) and \( \mathbb{Z}^* \) respectively. If \( \inf_{\alpha \in \Lambda, m,n \in \mathbb{Z}} \delta_{\infty}^+(\hat{f}_H^\alpha)_{m,n} = 0 \), then the \( L^1(G) \)-bimodule generated by \( \{ f^\alpha \mid \alpha \in \Lambda \} \) is dense in \( L^1(G) \).

Moreover, if the integral of \( f^\alpha \) is zero for all \( \alpha \), then the ideal is dense in \( L^1(G) \).

For \( f \in L^1(G)_n \), the natural domain of the principal part \( \hat{f}_H \) and the discrete part \( \hat{f}_B \) of the Fourier transform is \( S_1 \) and \( \Gamma_n \) respectively. To prove Theorem 1.1 we will prove the following theorem.

**Theorem 1.2.** Let \( \{ f^\alpha \mid \alpha \in \Lambda \} \) be a collection of functions in \( L^1(G)_n \) such that the collection \( \{ \hat{f}_H^\alpha \mid \alpha \in \Lambda \} \) and \( \{ \hat{f}_B^\alpha \mid \alpha \in \Lambda \} \) have no common zero in \( S_1 \) and \( \Gamma_n \) respectively. Moreover, if \( \inf_{\alpha \in \Lambda, m,n \in \mathbb{Z}} \delta_{\infty}^+(\hat{f}_H^\alpha)_{m,n} = 0 \), then the \( L^1(G) \) module generated by \( \{ f^\alpha \mid \alpha \in \Lambda \} \) is dense in \( L^1(G)_n \).

Theorem 1.2 will follow from the theorem below.

**Theorem 1.3.** Let \( \{ f^\alpha \mid \alpha \in \Lambda \} \) be a collection of functions in \( L^1(G)_{n,n} \) and \( I \) be the smallest closed ideal in \( L^1(G)_{n,n} \) containing \( \{ f^\alpha \mid \alpha \in \Lambda \} \) such that the collection \( \{ (\hat{f}_H^\alpha)_{n,n} \mid \alpha \in \Lambda \} \) and \( \{ (\hat{f}_B^\alpha)_{n,n} \mid \alpha \in \Lambda \} \) have no common zero in \( S_1 \) and \( \Gamma_n \) respectively. Moreover, if \( \inf_{\alpha \in \Lambda} \delta_{\infty}^+(\hat{f}_H^\alpha)_{n,n} = 0 \), then \( I = L^1(G)_{n,n} \).

Proof of the theorem above borrows heavily from the ideas and methods of [BBH], [PS] which uses the method of the resolvent transform. In the following we give a sketch of our proof.

1. We will begin by showing that for all \( \lambda \in \mathbb{C}^+ = \{ \lambda \in \mathbb{C} \mid \Re \lambda > 0 \} \) except for a finite set \( \mathcal{B} \) there is a family \( b_\lambda \) such that \( \hat{b}_\lambda H(i\xi) = \frac{1}{\lambda^2+i\xi^2} \) for all \( \xi \in \mathbb{R} \) and \( \hat{b}_\lambda B(k) = \frac{1}{\lambda^2+k^2} \) for all \( k \in \Gamma_n \). For \( \Re \lambda > 1 \), \( b_\lambda \in L^1(G)_{n,n} \) and \( \{ b_\lambda \mid \Re \lambda > 1 \; \text{and} \; \lambda \notin \mathcal{B} \} \) spans a dense subset of \( L^1(G)_{n,n} \). We will show \( ||b_\lambda||_1 \to 0 \) if \( \lambda \to \infty \) along the positive real axis.

2. By the Banach algebra theory (using the fact that principal part and discrete part of the Fourier transforms of the elements of \( I \) have no common zero), we define \( \lambda \mapsto B_\lambda \) as a \( L^1(G)_{n,n}/I \) valued even entire function.

3. Let \( g \in L^\infty(G)_{n,n} \) such that \( g \) annihilates \( I \). We define the resolvent transform \( \mathcal{R}[g] \) by

\[
\mathcal{R}[g](\lambda) = (B_\lambda, g).
\]

Considering \( g \) as a bounded linear functional on \( L^1(G)_{n,n}/I \), we write

\[
\mathcal{R}[g](\lambda) = (B_\lambda, g).
\]
where $B_\lambda = b_\lambda + I \in L^1(G)_{h,n}/I$, for all $\lambda$ with $\Re \lambda > 1$ and $\lambda \notin B$.

4. We need an explicit formula of the function $R[g](\lambda)$. For this we find a representative $T_\lambda f$ in $L^1(G)_{h,n}$ of the cosets $B_\lambda$ for $0 < \Re \lambda < 1$ where $f \in I$ such that $(\hat{f}_H(\sigma, \lambda))_{h,n} \neq 0$.

5. By the estimates of $||b_\lambda||_1$, $||T_\lambda f||_1$ and using a continuity argument we get the necessary estimate of $R[g](\lambda)$. Then using a log-log type theorem $[\text{PS}, \text{Theorem 6.3}]$ we show $R[g] = 0$.

6. By denseness of $\{b_\lambda \ | \ Re \lambda > 1$ and $\lambda \notin B\}$, it follows that $g = 0$.

Let $\Omega$ be the Casimir element of $G$. In $[\text{PS}]$ the solutions $\phi_{\sigma,\lambda}^{0,0}$ and $\Phi_{\sigma,\lambda}^{0,0}$ of

$$\Omega f = \left(\frac{\lambda^2 - 1}{4}\right) f$$

played a crucial role and they are given in terms of hypergeometric functions in $[\text{Ee}]$. From $[\text{AKS}, \text{p.31}]$ observing a formula of $\phi_{\sigma,\lambda}^{0,0}$ we found a way to derive the second solution of (1.3) in terms of hypergeometric functions. We are also able to write $\Phi_{\sigma,\lambda}^{n,n}$ as a linear combination of $\Phi_{\sigma,\lambda}^{n,n}$ and $\Phi_{\sigma,-\lambda}^{n,n}$ that is,

$$\phi_{\sigma,\lambda}^{n,n} = c_\sigma^{n,n}(\lambda)\Phi_{\sigma,\lambda}^{n,n} + c_\sigma^{n,n}(-\lambda)\Phi_{\sigma,-\lambda}^{n,n}. \quad (1.4)$$

It is an essential tool to find the Fourier transforms of $b_\lambda$'s.

As in $[\text{PS}, \text{Lemma 8.1}]$, using asymptotic behaviour $\Phi_{\sigma,\lambda}^{n,n}(a_t)$ for $\lambda \in \mathbb{C}$ near $t = \infty$ we show,

$$\lim_{t \to \infty} \frac{\phi_{\sigma,\lambda}(a_t)}{\Phi_{\sigma,\lambda}^{n,n}(a_t)} = 0. \quad (1.5)$$

This directly gives $\tilde{b}_{\lambda H}(i\xi) = \frac{1}{\lambda^2 + \xi^2}$, for all $\xi \in \mathbb{R}$. But for a general $n$, we also need to find $\tilde{b}_{\lambda B}(k)$ for all $k \in \Gamma_n$ and for that we need to show

$$\lim_{t \to \infty} \frac{\phi_{\sigma,|k|}(a_t)}{\Phi_{\sigma,|k|}^{n,n}(a_t)} = 0. \quad (1.6)$$

Here asymptotic behaviour of $\Phi_{\sigma,\lambda}^{n,n}$ is not enough. We have to use the full potential of decay of discrete series matrix coefficient $\phi_{\sigma,|k|}^{n,n}$. From $[\text{Ba}, \text{Theorem 8.1}]$ we get $\phi_{\sigma,|k|}^{n,n}$ has sufficient decay as $t \to \infty$. Using this we proved (1.6) and consequently $\tilde{b}_{\lambda B}(k) = \frac{1}{\lambda^2 - k^2}$ for all $k \in \Gamma_n$.

By inverse Fourier transform we find the representative of $B_\lambda$. For all but finitely many $\lambda$ with $\Re \lambda > 1$ we show $B_\lambda = b_\lambda + I$. In general there are some zeros of $c_\sigma^{n,n}(-\lambda)$ in $\mathbb{C}^+$ for $n \in \mathbb{Z}$. For this reason we have to remove a neighbourhood $B_1$ to find estimate of $||b_\lambda||_1$ on $|\Re \lambda| > 1$. Using estimate of $||b_\lambda||_1$ we find the estimate of $R[g](\lambda)$ on $\{\lambda \in \mathbb{C} \ | \ |\Re \lambda| > 1\} \setminus B_1$. We use continuity to get estimate of $R[g](\lambda)$ on $0 < \Re \lambda < 1$ and consequently we find the necessary estimate of $R[g](\lambda)$. Finally using a log-log type theorem it will follow that $R[g] = 0$.

Next we extend W-T theorem for $L^1(G)_{h,n}$ to $L^1(G)_n$. From the given collection $\{f^\alpha | \alpha \in \Lambda \} \subset L^1(G)_n$ and using the isomorphism between $L^1$-Schwartz space and its image under Fourier transform we construct a new collection of $L^1(G)_{h,n}$ functions $\{g_m * f^\alpha | m \in \mathbb{Z}^\sigma\}$. We show the new collection satisfy the hypothesis of Theorem (1.3). Therefore the collection $\{g_m * f^\alpha | m \in \mathbb{Z}^\sigma\}$ is dense in $L^1(G)_{h,n}$ and consequently Theorem (1.2) will follow. Following similar idea as above we prove W-T theorem for $L^1(G)$. 

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2. Preliminaries

In this article most of our notations are standard can be found in [Ba], [AKS], [PS] and [Rs1]. We will denote $C$ as constant and its value can change from one line to another. For any two positive expressions $f_1$ and $f_2$, $f_1 \sim f_2$ stands for that there are positive constants $C_1$, $C_2$ such that $C_1f_1 \leq f_2 \leq C_2f_1$. For $z \in \mathbb{C}$ we will use $\Re z$ and $\Im z$ to denote real and imaginary parts of $z$ respectively.

For $k \in \mathbb{Z}^*$ and $\sigma \in \widehat{M}$ be determined by $k \in \mathbb{Z}^{-\sigma}$ and we define,

$$Z(k) = \begin{cases} 1 \quad \text{if } k \geq 1 \\ \{m \in \mathbb{Z}^{-\sigma} : m \leq -k \} \quad \text{if } k \leq -1. \end{cases} \quad (2.1)$$

The Iwasawa decomposition for $G$ gives a diffeomorphism of $K \times A \times N$ onto $G$ where $A = \{a_t \mid t \in \mathbb{R}\}$ and $N = \{n_\xi \mid \xi \in \mathbb{R}\}$. That is, by Iwasawa decomposition $x \in G$ can be uniquely written as $x = k_0a_tn_\xi$ and using this we define $K(x) = k_0$ and $H(x) = t$. Let $A^+ = \{a_t \mid t > 0\}$. The Cartan decomposition for $G$ gives $G = K\mathbb{A}K$. Let $dg$, $dn$, $dk$ and $dm$ be the Haar measures of $G$, $N$, $K$ and $M$ respectively where $\int_K dk = 1$ and $\int_M dm = 1$. We have the following integral formulæ corresponding to the Cartan decomposition, which holds for any integrable function:

$$\int_G f(x)dx = \int_K \int_{\mathbb{R}^+} \int_K f(k_1a_tk_2)\Delta(t)dk_1dtdk_2, \quad (2.2)$$

where $\Delta(t) = 2\sinh 2t$.

For all $\lambda \in \mathbb{C}$ let us define,

$$\phi_{n,n}^{x,\lambda}(x) = \int_K e^{(\lambda-1)H(x)}e_n(k^{-1})e_n(K(xk)^{-1})dk,$$

for all $x \in G$.

Then we have for all $\lambda \in \mathbb{C}$, $\phi_{n,n}^{x,\lambda}$ is a smooth eigenfunctions of the Casimir element $\Omega$ that is,

$$\Omega \phi_{n,n}^{x,\lambda} = \frac{\lambda^2 - 1}{4} \phi_{n,n}^{x,\lambda}. \quad (2.3)$$

Let $\Pi_{n,n}(\Omega)$ be the differential operator on $A \setminus \{1\}$ defined by

$$\Pi_{n,n}(\Omega)f = \frac{d^2}{dt^2}f(a_t) + 2\coth 2t \frac{d}{dt}f(a_t) + \frac{1}{4 \cosh^2 t}f(a_t), t > 0. \quad (2.4)$$

Then from [Ba] p.62, eqm. (13.2)) we get that $\phi_{n,n}^{x,\lambda}$ is a solution of the following equation,

$$\Pi_{n,n}(\Omega)f = (\lambda^2 - 1)f. \quad (2.5)$$

We also have the following properties of $\phi_{n,n}^{x,\lambda}$:

1. $\phi_{n,n}^{x,\lambda}$ is a $(n,n)$ type function.
2. $\phi_{n,n}^{x,\lambda} = \phi_{n,-\lambda}^{x,-\lambda}$, $\phi_{n,n}^{x,\lambda}(a_t) = \phi_{n,n}^{x,\lambda}(a_{-t})$.
3. For any fixed $x \in G$, $\lambda \mapsto \phi_{n,n}^{x,\lambda}(x)$ is an entire function.
4. $|\phi_{n,n}^{x,\lambda}(x)| \leq 1 \text{ for } x \in G \text{ if } \lambda \in S_1$.

For $f \in L^1(G)_{n,n}$ the principal and discrete parts of the Fourier transform are defined by,

$$\widehat{f}_H(\sigma, \lambda)_{n,n} = \int_G f(x)\phi_{n,n}^{x,\lambda}(x^{-1})dx \quad \text{for all } \lambda \in S_1, \quad (2.6)$$

$$\widehat{f}_B(k)_{n,n} = \int_G f(x)\psi_k^{x,n}(x^{-1})dx \quad \text{for all } k \in \Gamma_n. \quad (2.7)$$
It follows from Riemann-Lebesgue lemma that if \( f \in L^1(G)_{n,n} \) then \( \hat{f}(\sigma, \lambda)_{n,n} \to 0 \) as \( |\Im \lambda| \to \infty \) in \( S_1 \). We also have from [Ba p.30 propn 7.3]
\[
\psi_{k}^{n,m} = \phi_{\sigma,|k|}^{n,m} \quad \text{for all } k \in \Gamma_n . \tag{2.6}
\]

We denote \( C^1(G) \) the \( L^1 \)- Schwartz space of \( G \). Suppose \( \sigma \in \hat{M}, m, n \in \mathbb{Z}^\sigma \) then the space \( C^1_H(\hat{G})_{m,n} \) denotes the collection of functions \( F : S_1 \to \mathbb{C} \) such that

1. \( F \) is continuous on \( S_1 \) and homomorphic on \( \text{Int} \ S_1 \),
2. \( F(\lambda) = \varphi_{\lambda}^{m,n} F(-\lambda) \) for all \( \lambda \in S_1 \), where
\[
\varphi_{\lambda}^{m,n} = P_{m,n}(\lambda)/P_{m,n}(-\lambda) \tag{2.7}
\]
is the rational function defined in [Ba Prop. 7.1],
3. \( \hat{\rho}_{H,l,r}(F) < \infty \) for all \( l \in \mathbb{N}, r \in \mathbb{R}^+ \), where
\[
\hat{\rho}_{H,l,r}(F) = \sup_{\lambda \in S_1} \left| \left( \frac{d}{d\lambda} \right)^l F(\lambda) \right| (1 + |\lambda|)^r ,
\]
4. \( F(k) = 0 \) if \( nm < 0, \ k \in \mathbb{Z}^- \) and \( |k| \leq \min\{|m|, |n|, 1\} \).

We note that for the particular case \( m = n, P_{n,n} = 1 \) so the property 2 in the definition of \( C^1_H(\hat{G})_{m,n} \) reduces to \( F(\lambda) = F(-\lambda) \) and property 4 becomes irrelevant. Let,
\[
\mathbb{Z}^1_{m,n} = \begin{cases} 
\{ k \mid 1 < k < \min\{m,n\} \text{ and } k \in \mathbb{Z}^\sigma \} & \text{if } mn > 0, m > 0 \\
\{ k \mid \max\{m,n\} < k < 0 \text{ and } k \in \mathbb{Z}^\sigma \} & \text{if } mn > 0, m < 0 \\
\phi & \text{if } mn \leq 0
\end{cases}
\]
and \( C^1_B(\hat{G})_{m,n} \) is the set of all functions \( F : \mathbb{Z}^1_{m,n} \to \mathbb{C} \). Then from [Ba Theorem 18.2] we have the following result:

**Lemma 2.1.** The Fourier transform, \( f \mapsto (\hat{f}_H, \hat{f}_B) \) is a topological isomorphism between \( C^1(G)_{m,n} \) and \( C^1(\hat{G})_{m,n} = C^1_H(\hat{G})_{m,n} \times C^1_B(\hat{G})_{m,n} \). Moreover, the restriction of that isomorphism gives,

1. \( C^1_H(G)_{m,n} \) isomorphic to \( C^1_H(\hat{G})_{m,n} \),
2. \( C^1_B(G)_{m,n} \) isomorphic to \( C^1_B(\hat{G})_{m,n} \).

**Hypergeometric function:** We are going to use the following properties of hypergeometric function,

(a) The hypergeometric function has the following integral representation for \( \Re c > \Re b > 0 \),
\[
\begin{align*}
_2F_1(a, b; c; z) &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 s^{b-1}(1-s)^{c-b-1}(1-sz)^{-a} ds, \quad |z| < 1. \tag{2.8}
\end{align*}
\]
(see [Ln p. 239, eqn 9.1.4])

(b) \[
\begin{align*}
c(c+1)_2F_1(a, b; c; z) &= c(c-a+1)_2F_1(a, b+1; c+2; z) \\
&\quad + a \left[ c -(c-b)z \right] _2F_1(a+1, b+1; c+2; z), \quad z \in \mathbb{C} \setminus [1, \infty). \tag{2.9}
\end{align*}
\]
(see [Ln p. 240, eqn. (9.1.7)])

(c) \[
\int_0^1 x^{d-1}(1-x)^{b-d-1} _2F_1(a, b; c; x) dx = \frac{\Gamma(c)\Gamma(d)\Gamma(b-d)\Gamma(c-a-d)}{\Gamma(b)\Gamma(c-a)\Gamma(c-d)}, \quad \text{if } \Re d > 0, \Re(b-d) > 0, \Re(c-a-d) > 0. \tag{2.10}
\]
The functions \( b_\lambda : \) Representatives of \( B_\lambda, \Re \lambda > 1 \)

In this section first we find the expression of \( \phi^{n,n}_{\sigma,\lambda} \) and the second solutions of (2.3) in terms of hypergeometric functions. We substitute

\[ f(t) = g(t) \cosh^n t, \]

in the equation (2.3). Then we get the following ODE,

\[ \frac{d^2 g}{dt^2} + ((2n + 1) \tanh t + \coth t) \frac{dg}{dt} + ((n + 1)^2 - \lambda^2)g = 0, \quad t > 0. \]

By the change of variable \( z := - \sinh^2 t \) the equation reduces to the following hypergeometric differential equation

\[ z(1 - z) \frac{d^2 g}{dz^2} + (c - (1 + a + b)z) \frac{dg}{dz} - \frac{1}{4} abg = 0, \quad (2.11) \]

with \( a = \frac{n+1}{2} + \frac{\lambda}{2}, b = \frac{n+1}{2} - \frac{\lambda}{2}, c = 1. \) Therefore

\[ g_1(t) = 2F_1 \left( \frac{n + 1}{2} + \frac{\lambda}{2}, \frac{n + 1}{2} - \frac{\lambda}{2}; 1; - \sinh^2 t \right), \]

is a solution of (2.11) which is regular at origin, so by the uniqueness of regular solution,

\[ \phi^{n,n}_{\sigma,\lambda}(a_t) = (\cosh t)^n 2F_1 \left( \frac{n + 1}{2} + \frac{\lambda}{2}, \frac{n + 1}{2} - \frac{\lambda}{2}; 1; - \sinh^2 t \right), \quad t > 0. \]

Also from [Er, p.105, 2.9 (11)] second solutions of (2.11) on \((0, \infty)\) are,

\[ g_2(t) = (\cosh t)^{-(1+\lambda+n)} 2F_1 \left( \frac{1+\lambda}{2}, \frac{n}{2}, \frac{1+\lambda}{2} - \frac{n}{2}; 1 + \lambda; \cosh^{-2} t \right), \]

\[ g_3(t) = (\cosh t)^{-(1-\lambda+n)} 2F_1 \left( \frac{1-\lambda}{2}, \frac{n}{2}, \frac{1-\lambda}{2} - \frac{n}{2}; 1 - \lambda; \cosh^{-2} t \right). \]

We now define,

\[ \Phi^{n,n}_{\sigma,\lambda}(a_t) = (2 \cosh t)^{-(1+\lambda)} 2F_1 \left( \frac{1+\lambda}{2}, \frac{|n|}{2}, \frac{1+\lambda}{2} - \frac{|n|}{2}; 1 + \lambda; \cosh^{-2} t \right). \quad (2.12) \]

Then \( \Phi^{n,n}_{\sigma,\lambda} \) and \( \Phi^{n,n}_{\sigma,-\lambda} \) both are solutions of (2.3), both has singularity at \( t = 0 \) and they are linearly independent. For \( \lambda \in \mathbb{C} \setminus \mathbb{Z} \), from [Er, p.110, 2.10(2,3 and 5)] we have,

\[ \phi^{n,n}_{\sigma,\lambda} = c^{n,n}_{\sigma}(\lambda) \Phi^{n,n}_{\sigma,\lambda} + c^{n,n}_{\sigma}(-\lambda) \Phi^{n,n}_{\sigma,-\lambda}, \quad (2.13) \]

where \( \sigma \) is determined by \( n \in \mathbb{Z}^\sigma \) and

\[ c^{n,n}_{\sigma}(\lambda) = \frac{2^{1+\lambda} \Gamma(-\lambda)}{\Gamma(\frac{1-\lambda-|n|}{2}) \Gamma(\frac{1-\lambda+|n|}{2})}. \quad (2.14) \]

We have for \( t \to \infty \),

\[ \Phi^{n,n}_{\sigma,\lambda}(a_t) = e^{-(\lambda+1)t} (1 + O(1)). \quad (2.15) \]

Hence for \( \Re \lambda < 0 \) and as \( t \to \infty \),

\[ \phi^{n,n}_{\sigma,\lambda}(a_t) = c^{n,n}_{\sigma}(\lambda) e^{-(\lambda+1)t} (1 + O(1)). \quad (2.16) \]

For simplicity if \( f \in L^1(G)_{n,n} \) then we denote the principal and discrete parts of the Fourier transform by \( \widehat{f}_H \) and \( \widehat{f}_B \) respectively. Since \( n \in \mathbb{Z}^\sigma \) determines \( \sigma \) we will use \( c^{n,n}_{\sigma}(\lambda) \) instead of \( c^{n,n}_{\sigma}(\lambda) \).
Lemma 2.2. Let \( B_1 = \bigcup_{i=0}^{k_0-1} B(|n| - 2i - 1; 1) \) where \( B(z; 1) \) denotes a Euclidean ball of radius 1 centered at \( z \) and \( k_0 = \left\lfloor \frac{|n|}{2} \right\rfloor + 1 \). Then for \( \lambda \in \mathbb{C}_+ \setminus B_1 \), we have

(i) There is a positive constant \( C \) independent of \( \lambda \) such that for all \( t \in (0, 1/2] \),

\[
|b_\lambda(a_t)| \leq C \log \frac{1}{t}.
\]

(ii) There is a positive constant \( C \) independent of \( \lambda \) such that for all \( t \in [1/2, \infty) \),

\[
|b_\lambda(a_t)| \leq Ce^{-(\Re \lambda + 1)t}.
\]

Proof. (i) Since

\[
\Phi_{\sigma, \lambda}^{n,n}(a_t) = (2 \cosh t)^{-(1+\lambda)} 2F_1 \left( \frac{1 + \lambda + |n|}{2}, \frac{1 + \lambda - |n|}{2}; 1 + \lambda; \cosh^{-2} t \right),
\]

we first find the estimate of the hypergeometric function near \( t = 0 \) and then by polynomial approximation of gamma functions we will finally find the estimate of \( b_\lambda \). Now by (2.9),

\[
\begin{align*}
2F_1 \left( \frac{1 + \lambda + |n|}{2}, \frac{1 + \lambda - |n|}{2}; 1 + \lambda; \cosh^{-2} t \right) &= \frac{1}{(1 + \lambda)(2 + \lambda)} \left[ (1 + \lambda) \left( \frac{1 + \lambda - |n|}{2} + 1 \right) 2F_1 \left( \frac{1 + \lambda + |n|}{2}, \frac{1 + \lambda - |n|}{2}; 1 + \lambda + 2; \cosh^{-2} t \right) \\
&\quad + \frac{1 + \lambda + |n|}{2} \left( 1 + \lambda - \frac{1 + \lambda + |n|}{2} \cosh^{-2} t \right) 2F_1 \left( \frac{1 + \lambda + |n|}{2}, \frac{1 + \lambda - |n|}{2}; 1 + \lambda + 2; \cosh^{-2} t \right) \right]
\end{align*}
\]

Since \((1 + \lambda)(2 + \lambda)\) and \( \cosh t \times 1 \) near \( t = 0 \), so for all \( t \in (0, \frac{1}{2}] \) we have

\[
\frac{(1 + \lambda)(1 + \lambda - |n|)}{(1 + \lambda)(2 + \lambda)} \asymp C \quad \text{and} \quad \frac{\left( \frac{1 + \lambda - |n|}{2} \right) (1 + \lambda)}{(1 + \lambda)(2 + \lambda)} \cosh^{-2} t \]

\( \asymp C \) for all \( \lambda \in \mathbb{C}_+ \).

By the same argument and applying the formula (2.9) \( k_0 = \left\lfloor \frac{|n|}{2} \right\rfloor + 1 \) times we can write,

\[
\begin{align*}
2F_1 \left( \frac{1 + \lambda + |n|}{2}, \frac{1 + \lambda - |n|}{2}; 1 + \lambda; \cosh^{-2} t \right) &= \sum_{i=0}^{k_0} P_i(\lambda, \cosh^{-2} t) \frac{Q_i(\lambda)}{Q_0(\lambda)} 2F_1 \left( \frac{1 + \lambda + |n|}{2}, \frac{1 + \lambda - |n|}{2}; 1 + \lambda + 2k_0; \cosh^{-2} t \right)
\end{align*}
\]

where \( P_i \)'s are polynomials in \( \lambda \) and \( \cosh^{-2} t \) and \( Q_i \)'s are polynomials in \( \lambda \) which has no zero in \( \mathbb{C}_+ \) such that for all \( t \in (0, \frac{1}{2}] \),

\[
\frac{P_i(\lambda, \cosh^{-2} t)}{Q_i(\lambda)} \asymp C.
\]
for all $\lambda \in \mathbb{C}_+$. Now since $\Re(1 + \lambda + 2k_0) > \Re\left(\frac{1 + |\lambda|}{2} + k_0\right)$ from (2.8),

$$2F_1\left(\frac{1 + \lambda + |n|}{2}, \frac{1 + \lambda - |n|}{2}, 1 + \lambda + 2k_0; \cosh^2 t\right)$$

$$= C(\lambda; n)\int_0^1 s^{\frac{1 + |n|}{2} + k_0 - 1} (1 - s)^{\frac{1 + |n|}{2} + k_0 - 1} (1 - s \cosh^2 t)^{-\frac{1 + |n|}{2} - i} ds$$

where $C(\lambda; n) = \frac{\Gamma(1 + \lambda + 2k_0)}{\Gamma\left(\frac{1 + |n|}{2} + k_0\right) \Gamma\left(\frac{1 + |n|}{2} + k_0\right)}$

$$= C(\lambda; n)(\cosh t)^{1 + |n| + 2i} \int_0^1 s^{\frac{1 + |n|}{2} + k_0 - 1} (1 - s)^{\frac{1 + |n|}{2} + k_0 - 1} (\cosh^2 t - s)^{-\frac{1 + |n|}{2} - i} ds$$

( writing $\cosh^2 t = 1 + x$ and making the change of variable $s \to 1 - s$ we get)

$$= C(\lambda; n)(\cosh t)^{1 + |n| + 2i} \int_0^1 (1 - s)^{\frac{1 + |n|}{2} + k_0 - 1} s^{\frac{1 + |n|}{2} + k_0 - 1} (x + s)^{-\frac{1 + |n|}{2} - i} ds. \quad (2.19)$$

Let $I$ be the integration above. Then,

$$|I| \leq \int_0^1 (1 - s)^{\frac{1 + |n|}{2} + k_0 - 1} s^{\frac{1 + |n|}{2} + k_0 - 1} (x + s)^{-\frac{1 + |n|}{2} - i} ds.$$

Now we let $I_1$ be the integration on $(0, \frac{1}{2})$ and $I_2$ on $(\frac{1}{2}, 1)$. Then,

$$I_2 \leq C \int_{\frac{1}{2}}^1 (1 - s)^{-\frac{1 - |n|}{2} + k_0} 2^{-\frac{1 + |n|}{2} + i} ds \leq C(\text{independent of } \lambda),$$

and

$$I_1 \leq C \int_0^{\frac{1}{2}} s^{\frac{1 + |n|}{2} + k_0 - 1} (x + s)^{-\frac{1 + |n|}{2} - i} ds$$

$$\leq C 2^{-\frac{1 + |n|}{2} + k_0} (x + \frac{1}{2})^{\frac{1 + |n|}{2} - i} + C \int_0^{\frac{1}{2}} s^{\frac{1 + |n|}{2} + k_0 - 1} (x + s)^{-\frac{1 + |n|}{2} - i} ds$$

$$\leq C + C \int_0^{\frac{1}{2}} \left( \frac{s}{x + s} \right)^{\frac{1 + |n|}{2} + i} \frac{1}{x + s} ds$$

$$\leq C + C \log \left( 1 + \frac{1}{2x} \right).$$

Since $x = \sinh^2 t$ and log is an increasing function we have, $I_1 \leq C \log \frac{1}{t}$, hence it follows $|I| \leq C \log \frac{1}{t}$ for all $t \in (0, \frac{1}{2}]$. 

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We now turn to the estimate of $b_\lambda$. Using the inequality above of $|I|$ and applying the expression of $c^{n,n}(\lambda)$ in \((2.17)\) we get,

$$|b_\lambda(a_t)| \leq C \left| \frac{\Gamma\left(\frac{1+\lambda-|n|}{2}\right)\Gamma\left(\frac{1+\lambda+|n|}{2}\right)}{\Gamma(1+\lambda)\Gamma\left(\frac{1+\lambda-|n|}{2} + k_0\right)\Gamma\left(\frac{1+\lambda+|n|}{2} + k_0\right)} \right| \log \frac{1}{t}$$

\[
\leq C \left| \frac{\Gamma\left(\frac{1+\lambda-|n|}{2}\right)\Gamma\left(\frac{1+\lambda+|n|}{2}\right)}{(1+\lambda-|n|)(1+\lambda+|n|) + 1} \right| \log \frac{1}{t}
\]

The last line of the inequality above follows from \([PS\text{ Appendix, Lemma A.3}]\) and the fact that $k_0 - \frac{|n|}{2} \geq \frac{1}{2}$. Therefore for all $\lambda \in \mathbb{C}_+ \setminus B_1$, (where $B_1 = \bigcup_{i=0}^{k_0-1} B(|n| - 2i - 1; 1)$)

$$|b_\lambda(a_t)| \leq C \log \frac{1}{t}, \quad (2.20)$$

for all $t \in (0, \frac{1}{2}]$ and $C$ is independent of $\lambda$.

(ii) Since $\Phi_{r,\lambda}^{n,n}(a_t) \asymp e^{-(\Re \lambda + 1)t}$ near $\infty$ and by the definition of $b_\lambda$ and $c^{n,n}(\lambda)$ we get for all $t \in [\frac{1}{2}, \infty)$,

$$|b_\lambda(a_t)| \leq C \left| \frac{\Gamma\left(\frac{1+\lambda-|n|}{2}\right)\Gamma\left(\frac{1+\lambda+|n|}{2}\right)}{\Gamma(1+\lambda)\Gamma\left(\frac{1+\lambda-|n|}{2} + k_0\right)\Gamma\left(\frac{1+\lambda+|n|}{2} + k_0\right)} \right| |\Phi_{r,\lambda}^{n,n}(a_t)|$$

\[
\leq C \left| \frac{\Gamma\left(\frac{1+\lambda-|n|}{2}\right)\Gamma\left(\frac{1+\lambda+|n|}{2}\right)}{(1+\lambda-|n|)(1+\lambda+|n|) + 1} \right| e^{-(\Re \lambda + 1)t}
\]

\[
\leq C \left| \frac{\Gamma\left(\frac{1+\lambda-|n|}{2}\right)\Gamma\left(\frac{1+\lambda+|n|}{2}\right)}{(1+\lambda-|n|)(1+\lambda+|n|) + 1} \right| e^{-(\Re \lambda + 1)t} \quad \text{(using \([PS\text{ Appendix, Lemma A.3}])}
\]

\[
\leq C e^{-(\Re \lambda + 1)t} \quad \text{(Since $\lambda \notin B_1$)}.
\]

Therefore for all $t \in [\frac{1}{2}, \infty)$, $|b_\lambda(a_t)| \leq C e^{-(\Re \lambda + 1)t}$ where $C$ is independent of $\lambda$.

\[\square\]

**Remark 2.3.** The proof of the lemma above shows that to get the estimate of $b_\lambda$ near $0$ and $\infty$ we only need to remove a neighbourhood of the zeros of $c^{n,n}(-\lambda)$ and origin (when $n$ is odd). If we only remove the zeros of $c^{n,n}(-\lambda)$ and origin but not the neighbourhoods then on both cases the constant on the right hand side will depend on $\lambda$, for example for $t \in (0, \frac{1}{2}]$, $|b_\lambda(a_t)| \leq C_\lambda \log \frac{1}{t}$.

**Lemma 2.4.**

(a) For all $\lambda \in \mathbb{C}_+ \setminus B$, $b_\lambda$ is locally integrable at $e$.

(b) For $\Re \lambda > 1$ and $\lambda \notin B$, $b_\lambda \in L^1(G)_{n,n}$.

(c) For all $\lambda \in \mathbb{C}_+$ and $\lambda \notin B$, $b_\lambda$ is in $L^2$ outside neighbourhood of $e$.

(d) For each $\lambda \in \mathbb{C}_+ \setminus B$, there exists $p < 2$ (depending on $\lambda$) such that $b_\lambda$ is in $L^p$ outside neighbourhood of $e$.

**Proof.** Proof of this lemma follows directly from previous Lemma 2.3 and the asymptotic behaviour of $\Delta(t)$ near 0 and $\infty$. \[\square\]
**Remark 2.5.** By the lemma above $b_\lambda$ can be written as a sum of $L^1$ and $L^p \ (p < 2)$ functions on $G$. Therefore its principal part of the Fourier transform is a continuous function on $\mathbb{C}$, vanishing at infinity in $\mathbb{C}$. In fact in the next lemma we are going to find the Fourier transforms of $b_\lambda$.

**Lemma 2.6.** Let $\lambda \in \mathbb{C}_+ \setminus B$. Then,

$$\hat{b}_{\lambda H}(i\xi) = \frac{1}{\lambda^2 + \xi^2}, \text{ for all } \xi \in \mathbb{R} \text{ and}$$

$$\hat{b}_{\lambda B}(k) = \frac{1}{\lambda^2 - k^2}, \text{ for all } k \in \Gamma_n.$$

**Proof.** For two smooth functions $f$ and $g$ on $(0, \infty)$, we define

$$[f, g](t) = \Delta(t) \left[ f'(t)g(t) - f(t)g'(t) \right], \quad t > 0.$$ 

An easy calculation shows that $[f, g]'(t) = [\Pi_{n,n}(\Omega)f \cdot g - f \cdot \Pi_{n,n}(\Omega)g](t)\Delta(t)$. Therefore, for any $b > a > 0$, we have

$$\int_a^b (\Pi_{n,n}(\Omega)f \cdot g - f \cdot \Pi_{n,n}(\Omega)g)(t)\Delta(t) = \int_a^b [f, g](b) - [f, g](a). \quad (2.21)$$

Then by similar calculations in [PS, Lemma 8.1] we have the following two results,

$$[\phi_{\sigma,\lambda}', \Phi_{\sigma,\lambda}^n](\cdot) = 2\lambda c^{n,n}(-\lambda)$$

and if $f$ is an even smooth function on $\mathbb{R}$ then

$$\lim_{t \to 0^+} [f, \Phi_{\sigma,\lambda}^n](t) = 2\lambda c^{n,n}(-\lambda)f(0). \quad (2.22)$$

**CASE 1 :** $\hat{b}_{\lambda H}(i\xi) = \frac{1}{\lambda^2 + \xi^2}$, for all $\xi \in \mathbb{R}$.

For $\xi \in \mathbb{R}$, we put $f = \phi_{\sigma,\lambda}^{n,n}, g = \Phi_{\sigma,\lambda}^n$ in equation (2.21) we get,

$$\int_a^b \Phi_{\sigma,\lambda}^n(t)\phi_{\sigma,\lambda}^{n,n}(t)\Delta(t)dt = \frac{1}{-\lambda^2 - \xi^2} \left( [\phi_{\sigma,\lambda}^{n,n}(b) - \phi_{\sigma,\lambda}^{n,n}(a)] - [\phi_{\sigma,\lambda}^{n,n}(b) - \phi_{\sigma,\lambda}^{n,n}(a)] \right).$$

Taking $a \to 0^+$, we get from (2.22)

$$\int_0^b \Phi_{\sigma,\lambda}^n(t)\phi_{\sigma,\lambda}^{n,n}(t)\Delta(t)dt = \frac{2\lambda c^{n,n}(-\lambda)}{\lambda^2 + \xi^2} - \frac{[\phi_{\sigma,\lambda}^{n,n}, \Phi_{\sigma,\lambda}^n](b)}{\lambda^2 + \xi^2}.$$ 

Therefore if we could show $[\phi_{\sigma,\lambda}^{n,n}, \Phi_{\sigma,\lambda}^n](b) \to 0$ as $b \to \infty$ then we will be done.

We note that the existence of limit is guaranteed by the equation above. As like before we can write,

$$\lim_{b \to \infty} [\phi_{\sigma,\lambda}^{n,n}, \Phi_{\sigma,\lambda}^n](b) = \lim_{b \to \infty} e^{-2\lambda b} \left( \frac{\phi_{\sigma,\lambda}^{n,n}}{\Phi_{\sigma,\lambda}^n} \right)'(b).$$

By the asymptotic behavior of $\phi_{\sigma,\lambda}^{n,n}$ and $\Phi_{\sigma,\lambda}^n$

$$\lim_{b \to \infty} \frac{\phi_{\sigma,\lambda}^{n,n}(b)}{\Phi_{\sigma,\lambda}^n} = 0.$$

Finally for all $\lambda \in \mathbb{C}_+ \setminus B$,

$$\hat{b}_{\lambda H}(i\xi) = \frac{1}{\lambda^2 - (i\xi)^2}, \text{ for all } \xi \in \mathbb{R}.$$
CASE 2: \( \tilde{b}_{\lambda H}(k) = \frac{1}{\lambda^2 - k^2} \) for all \( k \in \Gamma_n \).

We note that from \([Ba\text{ p.30 propn 7.3}]\) we have \( \psi_k^{n, n} = \phi_{n, |k|}^{n, n} \) for all \( k \in \Gamma_n \).

Let \( k \in \Gamma_n \), we put \( f = \psi_k^{n, n}, g = \Phi_{\sigma, \lambda}^{n, n} \) in equation (2.21) to get,

\[
\int_a^b \Phi_{\sigma, \lambda}^{n, n}(t)\psi_k^{n, n}(t)\Delta(t)dt = \frac{1}{\lambda^2 + k^2} \left( [\psi_k^{n, n}, \Phi_{\sigma, \lambda}^{n, n}] (b) - [\psi_k^{n, n}, \Phi_{\sigma, \lambda}^{n, n}] (a) \right).
\]

Taking \( a \to 0^+ \), we get from (2.22)

\[
\int_0^b \Phi_{\sigma, \lambda}^{n, n}(t)\psi_k^{n, n}(t)\Delta(t)dt = \frac{2\lambda c^{\sigma, n}(-\lambda)}{\lambda^2 - k^2} - \frac{[\psi_k^{n, n}, \Phi_{\sigma, \lambda}^{n, n}] (b)}{\lambda^2 - k^2}.
\]

Therefore if we could show \( [\psi_k^{n, n}, \Phi_{\sigma, \lambda}^{n, n}] (b) \to 0 \) as \( b \to \infty \) then we will be done.

From \([Ba\text{ p.33 Theorem 8.1}]\) we get that there exist constants \( C, r_1, r_2, r_3 \geq 0 \) such that

\[
|\psi_k^{n, n}(t)| \leq C(1 + |n|)^{r_1}(1 + |k|)^{r_2}(1 + t)^{r_3}e^{-2t}
\]

for all \( k \in \mathbb{Z}^* \) for which \( |k| \geq 1 \) and for all \( n \in \mathbb{Z}(k) \).

Now by the asymptotic behaviour of \( \Phi_{\sigma, \lambda}^{n, n} \) we get,

\[
\frac{|\psi_k^{n, n}(b)|}{e^{2\lambda b \Phi_{\sigma, \lambda}^{n, n}(b)}} \leq \frac{C}{e^{2\lambda b}e^{-\lambda b}b} \leq \frac{C}{e^{(\lambda - 1)b}}
\]

for a fixed \( n \). Therefore \( \lim_{b \to \infty} \frac{|\psi_k^{n, n}(b)|}{e^{2\lambda b \Phi_{\sigma, \lambda}^{n, n}(b)}} = 0 \). This completes the proof.

\[\square\]

Remark 2.7. Since for \( \Re \lambda > 1 \) and \( \lambda \notin \mathbb{B} \), \( \lambda \) is in \( L^1(G)_{\sigma, n} \) and its principal Fourier transform is a well defined continuous function on the strip \( S_1 \), which is also holomorphic in \( S_1^0 \). Therefore by analytic continuation we can write for \( \Re \lambda > 1 \) and \( \lambda \notin \mathbb{B} \),

\[
\hat{b}_{\lambda H}(z) = \frac{1}{\lambda^2 - z^2}, \text{ for all } z \in S_1.
\]

We now turn to the estimates of \( \|b_{\lambda}\|_1 \) which is essential in \([3]\).

Lemma 2.8. (i) If \( \Re \lambda > 1 \) and \( \lambda \notin \mathbb{B}_1 \), \( \|b_{\lambda}\|_1 \leq C \frac{(1 + |\lambda|)}{\Re \lambda - 1} \) for some \( C > 0 \).

(ii) \( \|b_{\lambda}\|_1 \to 0 \) if \( \lambda \to \infty \) along the positive real axis.

Proof. (i) Since \( \Delta(t) \simeq t \) near 0 and \( \Delta(t) \simeq e^{2t} \) near \( \infty \), from Lemma 2.2 we can write,

\[
\|b_{\lambda}\|_1 = \int_0^{1/2} |b_{\lambda}(a_t)|\Delta(t)dt + \int_{1/2}^{\infty} |b_{\lambda}(a_t)|\Delta(t)dt
\]

\[
\leq \int_0^{1/2} t\log \frac{1}{t} + C \int_{1/2}^{\infty} e^{(1 - \Re \lambda)t}
\]

\[
\leq C + \frac{C}{\Re \lambda - 1}
\]

\[
\leq C \frac{1 + |\lambda|}{\Re \lambda - 1}.
\]
(ii) If \( \lambda = \xi \in \mathbb{R} \) and \( \xi > n + 1 \) then \( b_\xi(a_t) \) is nonnegative. Hence
\[
||b_\xi||_1 = \int_{\mathbb{R}_+} b_\xi(a_t) \Delta(t) dt \leq \int_{\mathbb{R}_+} (\cosh t)^n ||b_\xi(a_t)\Delta(t) dt \]
\[
= \frac{1}{\xi^2 - (n + 1)^2}.
\]
The last line of the inequalities follows from similar calculation of \( \text{[PS] Lemma 3.3} \), which uses (2.10). Hence the proof follows. \( \square \)

**Lemma 2.9.** The functions \( \{b_\lambda \mid \Re \lambda > 1 \text{ and } \lambda \notin \mathcal{B} \} \) span a dense subset of \( L^1(G)_{n,n} \).

**Proof.** We will show that \( \text{span}\{b_\lambda \mid \Re \lambda > 1 \text{ and } \lambda \notin \mathcal{B} \} \) contains \( C_c^\infty(G)_{n,n} \) and since \( C_c^\infty(G)_{n,n} \) is dense in \( L^1(G)_{n,n} \), the lemma will follow.

Let \( f \in C_c^\infty(G)_{n,n} \). Since \( \hat{f}_H \) is entire and it has polynomial decay on any bounded vertical strip (by Paley-Wiener theorem) Cauchy’s formula implies that
\[
\hat{f}_H(w) = \frac{1}{2\pi i} \int_{\Gamma_1} \hat{f}_H(z) - w \ dz + \frac{1}{2\pi i} \int_{\Gamma_2} \hat{f}_H(z) - w \ dz, \quad \text{for } w \in \mathbb{C},
\]
where \( \Gamma_1 = (|n| + 2) + i\mathbb{R} \) downward and \( \Gamma_2 = -(|n| + 2) + i\mathbb{R} \) upward. Next by the change of variable \( z \to -z \) in the second integral,
\[
\hat{f}_H(w) = \frac{1}{2\pi i} \int_{\Gamma_1} \hat{f}_H(z) d(-z) + \frac{1}{2\pi i} \int_{\Gamma_1} \hat{f}_H(z) d(-z).
\]
We know \( \hat{f}_H(z) \) is an even function, therefore for all \( w \in \mathbb{C} \)
\[
\hat{f}_H(w) = \frac{1}{2\pi i} \int_{\Gamma_1} 2z \hat{f}_H(z) dz. \tag{2.23}
\]
Since \( \hat{f}_B(k) = \hat{f}_H(|k|) \) for all \( k \in \Gamma_n \), so from (2.23) and together with Lemma 2.6 we get,
\[
\hat{f}_H(w) = \frac{1}{2\pi i} \int_{\Gamma_1} 2z \hat{f}_H(z) b_z H(w) dz, \quad \text{for all } w \in S_1, \tag{2.24}
\]
\[
\hat{f}_B(k) = \frac{1}{2\pi i} \int_{\Gamma_1} 2z \hat{f}_H(z) b_z B(k) dz, \quad \text{for all } k \in \Gamma_n. \tag{2.25}
\]

The decay condition on \( \hat{f}_H \) and Lemma 2.8 imply that the \( L^1(G)_{n,n} \) valued integral
\[
\frac{1}{2\pi i} \int_{\Gamma_1} 2z \hat{f}_H(z) b_z(\cdot) dz
\]
converges and (2.24), (2.25) implies that it must converge to \( f \). Thus the Riemann sums which are nothing but finite linear combinations of \( b_\xi \’s \) converge to \( f \). So we can conclude that \( f \) is in the closed subspace spanned by \( \{b_\lambda \mid \Re \lambda > 1 \text{ and } \lambda \notin \mathcal{B} \} \). The lemma follows. \( \square \)

### 3. Resolvent transform

Let \( L^1_\delta(G)_{n,n} \) be the unitization of \( L^1(G)_{n,n} \) and \( \delta \), where \( \delta \) is the \( (n,n) \) type distribution defined by \( \delta(\phi) = \phi(e) \) for all \( \phi \in C_c^\infty(G)_{n,n} \). Maximal ideal space of \( L^1_\delta(G)_{n,n} \) is \( \{L_z : z \in S_1 \cup \{\infty\} \} \) and \( \{L^\prime_k : k \in \Gamma_n \} \), where \( L_z \) and \( L^\prime_k \) are the complex homomorphism on \( L^1_\delta(G)_{n,n} \) defined by
\[
L_z(f) = \hat{f}_H(z) \text{ and } L^\prime_k(f) = \hat{f}_B(k) \text{ for all } f \in L^1_\delta(G)_{n,n}.
\]
From now on we will denote $I$ as a closed ideal of $L^1(G)_{n,n}$ such that $\{\hat{f}_H : f \in I\}$ and $\{\hat{f}_B : f \in I\}$ does not have common zero on $S_1$ and $\Gamma_n$ respectively. Since $\delta \ast f = f$ for all $f \in L^1(G)_{n,n}$ so $I$ is also an ideal of $L^1_\delta(G)_{n,n}$ and $L^1_\delta(G)_{n,n}/I$ makes sense.

In Banach algebra theory if $J$ is a closed ideal of a commutative Banach algebra $\mathcal{A}$ then the maximal ideal space of $\mathcal{A}/J$ is

$$\Sigma(\mathcal{A}/J) = \{h \in \Sigma(\mathcal{A}) : h = 0 \text{ on } J\},$$

where $\Sigma(\mathcal{A})$ denotes the maximal ideal space of $\mathcal{A}$.

From the theory above the maximal ideal space of $L^1_\delta(G)_{n,n}/I$ is the complex homomorphism $\hat{L}_\infty$ and it is defined by

$$\hat{L}_\infty(f + I) = \hat{f}_H(\infty) \text{ for all } f \in L^1_\delta(G)_{n,n}/I.$$ 

It also follows that an element $f + I$ in $L^1_\delta(G)_{n,n}/I$ is invertible if and only if $\hat{f}_H(\infty) \neq 0$.

Let $\lambda_0$ be a fixed complex number with $\Re \lambda_0 > n + 1$. Then by Lemma 2.6 $\hat{b}_{\lambda_0}$ is in $L^1(G)_{n,n}$. For $\lambda \in \mathbb{C}$ the function,

$$\lambda \mapsto \hat{\delta} - (\lambda_0^2 - \lambda^2) \hat{b}_{\lambda_0},$$

does not vanish at $\infty$ and hence $\delta - (\lambda_0^2 - \lambda^2) b_{\lambda_0} + I$ is invertible in the quotient algebra $L^1_\delta(G)_{n,n}/I$.

We put

$$B_{\lambda} = (\delta - (\lambda_0^2 - \lambda^2) b_{\lambda_0} + I)^{-1} \ast (b_{\lambda_0} + I), \quad \text{for } \lambda \in \mathbb{C}. \quad (3.1)$$

Now let $g \in L^\infty(G)_{n,n}$ annihilates $I$, so we can take $g$ as a bounded linear functional on $L^1(G)_{n,n}/I$.

We define the resolvent transform $\mathcal{R}[g]$ of $g$ by

$$\mathcal{R}[g](\lambda) = \langle B_{\lambda}, g \rangle. \quad (3.2)$$

From (3.1), $\lambda \mapsto B_{\lambda}$ is a Banach space valued even holomorphic function on $\mathbb{C}$. So $\mathcal{R}[g]$ is an even holomorphic function on $\mathbb{C}$.

We need an explicit formula of the function $\mathcal{R}[g]$ almost everywhere in $\mathbb{C}$. We will show for $\Re \lambda > 1$ and $\lambda \notin B$, $B_{\lambda} = b_{\lambda} + I$. Also, for $0 < \Re \lambda < 1$ we find a representative of the cosets $B_{\lambda}$ in the next section.

4. **Representatives of $B_{\lambda}, 0 < \Re \lambda < 1$ and properties of $\mathcal{R}[g]$**

Let $\lambda$ be such that $0 < \Re \lambda < 1$. For $f \in L^1(G)_{n,n}$ we define

$$T_{\lambda}f := \hat{f}_H(\lambda)b_{\lambda} - f * b_{\lambda}. \quad (4.1)$$

Since $b_{\lambda}$ is a sum of $L^1$ and $L^p$ functions (by Lemma 2.4) $T_{\lambda}f$ is well defined and the principal and discrete part of Fourier transforms exist on $i\mathbb{R}$ and $\Gamma_n$ respectively. The proof follows directly from Lemma 2.6.

**Lemma 4.1.** Let $0 < \Re \lambda < 1$ and $f$ be a $L^1(G)_{n,n}$ function on $G$. Then

$$\hat{T}_{\lambda}f_H(i\xi) = \frac{\hat{f}_H(\lambda) - \hat{f}_H(i\xi)}{\lambda^2 + \xi^2}, \quad \text{for all } \xi \in \mathbb{R},$$

$$\hat{T}_{\lambda}f_B(k) = \frac{\hat{f}_B(\lambda) - \hat{f}_B(k)}{\lambda^2 - k^2}, \quad \text{for all } k \in \Gamma_n.$$ 

**Lemma 4.2.** Let $\lambda \in \mathbb{C}_+ \setminus B$. Then,

$$\int_K b_{\lambda}(a_s k a_t) e_n(k^{-1})dk = \begin{cases} 
    b_{\lambda}(a_s) \phi_{\sigma,\lambda}^{n,n}(a_t) & \text{if } s > t \geq 0, \\
    b_{\lambda}(a_t) \phi_{\sigma,\lambda}^{n,n}(a_s) & \text{if } t > s \geq 0.
\end{cases}$$
Proof. Since $b_\lambda$ is smooth outside $K$ and $a_s k_a t \notin K$ as $s \neq t$, the integral is well defined. Fix $s > 0$, as $b_\lambda$ is a $(n, n)$ type eigenfunction of $\Omega$ on $G \setminus K$ with eigenvalue $\frac{\sigma}{4}$, the function

$$g \mapsto \int_K b_\lambda(a_s k g) e_n(k^{-1})dk$$

is smooth $(n, n)$ type eigenfunction of $\Omega$ on the open ball $B_s = \{k_1 a_s k_2 \in KA^+ K \mid r < s\}$. Hence the function $t \mapsto \int_K b_\lambda(a_s k_a t) e_n(k^{-1})dk$ is a solution of (4.3) on $(0, s)$ which is regular at 0. Therefore,

$$\int_K b_\lambda(a_s k_a t) e_n(k^{-1})dk = C\phi_{\sigma, \lambda}^n(a_t) \quad \text{for all } 0 \leq t < s$$

Putting $t = 0$ in the equation above we get $C = b_\lambda(a_s)$. Therefore for $s > t \geq 0$ we have,

$$\int_K b_\lambda(a_s k a t) e_n(k^{-1})dk = b_\lambda(a_s)\phi_{\sigma, \lambda}^n(a_t).$$

Similarly the second case follows. \hfill \Box

Next we will show $T_\lambda f$ is in $L^1(G)_{n,n}$ and to do that we will use the following representation of $T_\lambda f$.

**Lemma 4.3.** Let $0 < \Re \lambda < 1$ and $f \in L^1(G)_{n,n}$. Then for all $t > 0$,

$$T_\lambda f(a_t) = b_\lambda(a_t) \int_t^\infty f(a_s)\phi_{\sigma, \lambda}^n(a_s)\Delta(s)ds - \phi_{\sigma, \lambda}^n(a_t) \int_t^\infty f(a_s)b_\lambda(a_s)\Delta(s)ds. $$

Proof. Here we are going to use the fact that there exists $k_0 \in K$ such that $k_0 a_s k_0^{-1} = a_{-s}$ for all $s \geq 0$. Now

$$f * b_\lambda(a_t) = \int_K \int_0^\infty \int_K f(k_1 a_s k_2) b_\lambda(k_2^{-1} a_{-k_1^{-1}} a_t) \Delta(s)dk_1 ds dk_2$$

$$= \int_0^\infty f(a_s) \int_K b_\lambda(a_{-s} k_1 a_t) e_n(k_1^{-1}) \Delta(s)dk_1 ds \quad \text{(change of variable } k_1 \to k_1^{-1})$$

$$= \int_0^\infty f(a_s) \int_K b_\lambda(k_0 a_s k_0^{-1} k_1 a_t) e_n(k_1^{-1}) \Delta(s)dk_1 ds$$

$$= \int_0^\infty f(a_s) \int_K b_\lambda(a_s k_1 a_t) e_n(k_1^{-1}) \Delta(s)dk_1 ds \quad \text{(change of variable } k_1 \to k_0 k_1)$$

$$= \int_0^t f(a_s)b_\lambda(a_t)\phi_{\sigma, \lambda}^n(a_s)\Delta(s)ds + \int_t^\infty f(a_s)b_\lambda(a_s)\phi_{\sigma, \lambda}^n(a_t)\Delta(s)ds \quad \text{(4.2)}$$

The last line follows from Lemma 4.2. Next,

$$\tilde{f}_H(\lambda)b_\lambda(a_t) = b_\lambda(a_t) \int_K \int_0^\infty \int_K f(k_1 a_s k_2) \phi_{\sigma, \lambda}^n((k_2^{-1} a_{-k_1^{-1}}) \Delta(s)dk_1 ds dk_2$$

$$= b_\lambda(a_t) \int_0^\infty f(a_s)\phi_{\sigma, \lambda}^n(a_s)\Delta(s)ds \quad \text{(Since } \phi_{\sigma, \lambda}^n(a_s) = \phi_{\sigma, \lambda}^n(a_{-s})). \quad \text{(4.3)}$$

Putting the expressions above (4.2) and (4.3) in the definition of $T_\lambda f$ the result follows. \hfill \Box

Next we show $T_\lambda f$ is in $L^1(G)_{n,n}$ for $0 < \Re \lambda < 1$ and find the estimates of $\|T_\lambda f\|_1$.

**Lemma 4.4.** Let $0 < \Re \lambda < 1$ and $f$ be a $(n, n)$ type integrable function on $G$. Then $T_\lambda f \in L^1(G)_{n,n}$ and moreover if $\lambda \notin B(0; 1) \cup B(1; 1)$, its $L^1$ norm satisfies,

$$\|T_\lambda f\|_1 \leq C \|f\|_1(1 + |\lambda|)d(\lambda, \partial S_1)^{-1},$$
where $d(\lambda, \partial S_1)$ denotes the Euclidean distance of $\lambda$ from the boundary $\partial S_1$ of the strip $S_1$.

**Proof.** Proof of the Lemma above follows exactly in the same line as [PS, Lemma 4.4]. □

Now we summarize the necessary properties of the resolvent transform.

**Lemma 4.5.** Assume $g \in L^\infty(G)_{n,n}$ annihilates $I$ and fix a function $f \in I$. Let $Z(\hat{f}_H) := \{z \in S_1 : \hat{f}_H(z) = 0\}$. Then

(a) $\mathcal{R}[g](\lambda)$ is an even holomorphic function on $\mathbb{C}$. It is given by the following formula:

\[
\mathcal{R}[g](\lambda) = \begin{cases} 
(b_\lambda, g), & \Re \lambda > 1, \lambda \notin B \\
\left(\frac{T_\lambda f}{f_H(\lambda)}\right), & 0 < \Re \lambda < 1, \lambda \notin Z(\hat{f}_H).
\end{cases}
\]

(b) For $|\Re \lambda| > 1$, $|\mathcal{R}[g](\lambda)| \leq C||g||_\infty \frac{(1+|\lambda|)}{d(\lambda, \partial S_1)}$.

(c) For $|\Re \lambda| < 1$, $|\hat{f}_H(\lambda)\mathcal{R}[g](\lambda)| \leq C||f||_1 ||g||_\infty \frac{(1+|\lambda|)}{d(\lambda, \partial S_1)}$, where the constant $C$ is independent of $f \in I$.

**Proof.** (a) **CASE-1** : Let $\Re \lambda > 1$ and $\lambda \notin B$ then by (2.4) $b_\lambda$ is in $L^1(G)_{n,n}$. For $z \in S_1$ we have from Lemma 2.6 and 2.7

\[
\frac{1}{b_{\lambda H}(z)} - \frac{1}{b_{H}(z)} = \lambda_0^2 - \lambda^2
\]

so,

\[
\left(1 - (\lambda_0^2 - \lambda^2)b_{\lambda H}(z)\right)\hat{b}_{\lambda H}(z) = \hat{b}_{\lambda H}(z).
\]

Similarly for $k \in \Gamma_n$ we have,

\[
\left(1 - (\lambda_0^2 - \lambda^2)^2\hat{b}_{\lambda B}(k)\right)\hat{b}_{\lambda B}(k) = \hat{b}_{\lambda B}(k).
\]

So

\[
(\delta - (\lambda_0^2 - \lambda^2)b_{\lambda B}(\cdot)) b_\lambda(\cdot) = b_{\lambda B}(\cdot)
\]

as $L^1(\hat{G})_{n,n}$ functions. Hence in the quotient algebra $L^1(\hat{G})_{n,n}/I$,

\[
(\delta - (\lambda_0^2 - \lambda^2)b_{\lambda} + I) \ast (b_\lambda + I) = b_\lambda + I,
\]

(4.4)

Now $(\delta - (\lambda_0^2 - \lambda^2)b_{\lambda} + I)$ is invertible in $L^1(\hat{G})_{n,n}/I$ so from (3.1) and (4.4) we get $B_\lambda = b_\lambda + I$

Therefore by the definition of $\mathcal{R}[g](\lambda)$,

\[
\mathcal{R}[g](\lambda) = \langle b_\lambda, g \rangle.
\]

**CASE-2** : Let $0 < \Re \lambda < 1$, $\lambda \notin Z(\hat{f}_H)$. Then by Lemma 4.4 $T_\lambda f$ is in $L^1(G)_{n,n}$. Similarly as in previous case we have from Lemma 4.1

\[
\left(1 - (\lambda_0^2 - \lambda^2)b_{\lambda H}(z)\right)\frac{\hat{T}_\lambda f_H(z)}{\hat{f}_H(\lambda)} = \hat{b}_{\lambda H}(z) - \frac{\hat{f}_H(z)b_{\lambda H}(z)}{\hat{f}_H(\lambda)} \quad \text{for all } z \in S_1
\]

and

\[
\left(1 - (\lambda_0^2 - \lambda^2)b_{\lambda B}(k)\right)\frac{\hat{T}_\lambda f_B(k)}{\hat{f}_H(\lambda)} = \hat{b}_{\lambda B}(k) - \frac{\hat{f}_B(k)b_{\lambda B}(k)}{\hat{f}_H(\lambda)} \quad \text{for all } k \in \Gamma_n.
\]

Therefore

\[
(\delta - (\lambda_0^2 - \lambda^2)b_{\lambda B}(\cdot)) \left(\frac{T_\lambda f(\cdot)}{\hat{f}_H(\lambda)}\right) = b_{\lambda B}(\cdot) - \frac{f(\cdot)b_{\lambda B}(\cdot)}{\hat{f}_H(\lambda)}
\]
in $L^1(G)_{n,n}$. Since $f \in I$

$$
(\delta - (\lambda^2 - \lambda^2)b_{\lambda_0} + I) \ast \left(\frac{T_{\lambda}f}{f(\lambda)} + I\right) = b_{\lambda_0} + I.
$$

(4.5)

Again from (3.1) and the equation above

$$
B_{\lambda} = \frac{T_{\lambda}f}{f(\lambda)} + I,
$$

which implies

$$
\mathcal{R}[g](\lambda) = \frac{\langle T_{\lambda}f, g \rangle}{f(\lambda)}.
$$

(b) Since $\mathcal{R}[g](\lambda)$ is even we only need to consider the case $\Re \lambda > 1$. For $\Re \lambda > 1$ and $\lambda \not\in B_1$ we have from Lemma 2.8

$$
||b_{\lambda}||_1 \leq C \frac{(1 + |\lambda|)}{d(\lambda, \partial S_1)}
$$

for some $C > 0$.

Now from (3.2) it follows that $\mathcal{R}[g](\lambda)$ is bounded on $B_1$. Hence

$$
|\mathcal{R}[g](\lambda)| \leq C ||g||_{\infty} \frac{(1 + |\lambda|)}{d(\lambda, \partial S_1)}.
$$

(c) From Lemma 4.4 we get for $0 < \Re \lambda < 1$ and $\lambda \not\in B(0; 1) \cup B(1; 1)$,

$$
\left| f_H(\lambda)\mathcal{R}[g](\lambda) \right| \leq C ||f||_1 ||g||_{\infty} \frac{(1 + |\lambda|)}{d(\lambda, \partial S_1)}.
$$

Since $\hat{f}_H(\lambda)\mathcal{R}[g](\lambda)$ is an even continuous function on $S_1$, the same estimate is true for $0 < |\Re \lambda| < 1$, $\lambda \not\in B(0; 1) \cup B(1; 1)$. Now from (3.2) it follows that $\mathcal{R}[g](\lambda)$ is bounded on $B(0; 1) \cup B(1; 1)$ with bound independent of $f$, Therefore for $0 < |\Re \lambda| < 1$ and $\lambda \in B(0; 1) \cup B(1; 1)$,

$$
\left| \hat{f}(\lambda)\mathcal{R}[g](\lambda) \right| \leq C ||f||_1,
$$

where $C$ is independent of $f$ and $\lambda$. So we have for $0 < |\Re \lambda| < 1$,

$$
\left| \hat{f}_H(\lambda)\mathcal{R}[g](\lambda) \right| \leq C ||f||_1 ||g||_{\infty} \frac{(1 + |\lambda|)}{d(\lambda, \partial S_1)}.
$$

Finally the constant in the inequality above is independent of $f$ so by continuity of $\mathcal{R}[g]$ and $\hat{f}$ the lemma follows.

\[ \square \]

5. Results from complex analysis

For any function $F$ on $i\mathbb{R}$, we let

$$
\delta^+_\infty(F) = \limsup_{t \to \infty} e^{-\frac{\pi}{2t}} \log |F(it)| \quad \text{and} \quad \delta^-\infty(F) = -\limsup_{t \to \infty} e^{-\frac{\pi}{2t}} \log |F(-it)|.
$$

Next from [PS, Theorem 6.3] we have the following theorem.

**Theorem 5.1.** Let $M : (0, \infty) \to (e, \infty)$ be a continuously differentiable decreasing function with

$$
\lim_{t \to 0^+} t \log \log M(t) < \infty, \quad \int_0^\infty \log \log M(t) dt < \infty.
$$

Let $\Omega$ be a collection of bounded holomorphic functions on $S_1^0$ such that

$$
\inf_{F \in \Omega} \delta^+_\infty(F) = \inf_{F \in \Omega} \delta^-\infty(F) = 0.
$$
Suppose $H$ satisfies the following estimates for some nonnegative integer $N$:

$$|H(z)| \leq (1 + |z|)^N M(d(z, \partial S_1)), \quad z \in \mathbb{C} \setminus S_1,$$

$$|F(z)H(z)| \leq (1 + |z|)^N M(d(z, \partial S_1)), \quad z \in S_1^0, \text{ for all } F \in \Omega.$$

1. If in addition, $H$ is a holomorphic function on $S_1 \setminus \{ \pm 1 \}$ then $H$ is dominated by a polynomial outside a bounded neighbourhood of $\{ \pm 1 \}$.

2. If $H$ is an entire function, then it is a polynomial.

### 6. Proof of W-T Theorem for $L^1(G)_{n,n}$

**Proof of Theorem 6.3.** Since the ideal generated by $\{f^\alpha \mid \alpha \in \Lambda \}$ is same as the ideal generated by the elements $\left\{ \frac{f^\alpha}{||f^\alpha||} \mid \alpha \in \Lambda \right\}$ and $\delta^\pm_\infty(\hat{f}_H) = \delta^\pm_\infty\left(\frac{\hat{f}_H}{||\hat{f}_H||}\right)$, we can assume that the functions $f^\alpha$ are of unit $L^1$ norm. Let $g \in L^\infty(G)_{n,n}$ annihilates the closed ideal $I$ generated by $\{f^\alpha \mid \alpha \in \Lambda \}$. We will show that $g = 0$. Then by an application of Hahn Banach theorem it will follow that $I = L^1(G)_{n,n}$. From the hypothesis we have,

$$\inf_{\alpha \in \Lambda} \delta^\pm_\infty(\hat{f}_H) = \inf_{\alpha \in \Lambda} \delta^\pm_\infty(\frac{\hat{f}_H}{||\hat{f}_H||}) = 0.$$

By Lemma 6.3, the entire function $R[g]$ satisfies the following estimates

$$|R[g](z)| \leq C(1 + |z|)(d(z, \partial S_1))^{-1}, \quad z \in \mathbb{C} \setminus S_1,$$

$$|\hat{f}_H(z)R[g](z)| \leq C(1 + |z|)(d(z, \partial S_1))^{-1}, \quad z \in S_1^0,$$

for all $\alpha \in \Lambda$, where $C$ is a constant and we choose it is greater than $e$. We can define $M : (0, \infty) \to (e, \infty)$ to be a continuously differentiable decreasing function such that $M(t) = \frac{C}{t}$ for $0 < t < 1$, and $\int_1^\infty \log \log M(t) dt < \infty$. With this definition of $M$, we have

$$|R[g](z)| \leq (1 + |z|)M(d(z, \partial S_1)) \quad z \in \mathbb{C} \setminus S_1,$$

$$|\hat{f}_H(z)R[g](z)| \leq (1 + |z|)M(d(z, \partial S_1)) \quad z \in S_1^0, \text{ for all } \alpha \in \Lambda.$$

Therefore, by Theorem 5.1 $R[g](z)$ is a polynomial. From Lemma 4.3

$$R[g](z) \leq ||b_2||_1 ||g||_\infty.$$

Then Lemma 2.3 implies $R[g](z) \to 0$ when $z \to \infty$ along the positive real axis. Therefore $R[g]$ must be the zero polynomial. Hence $\langle b_\Lambda, g \rangle = 0$ whenever $\Re \lambda > 1$ and $\lambda \not\in B$ but the collection $\{b_\Lambda : \Re \lambda > 1 \text{ and } \lambda \not\in B\}$ spans a dense subset of $L^1(G)_{m,n}$ by Lemma 2.9. So $g = 0$ and the proof follows.

Finally we like to mention here that we first started to prove a W-T theorem for $L^1(G)_{m,n}$ but our method fails in this general setting as $L^1(G)_{m,n}$ is not necessarily a commutative Banach algebra.

### 7. Final Results

Now we prove Wiener Tauberian theorem for $L^1(G)_{n}$ using Theorem 6.3. Here we will follow similar technique as in [RS1].

For $f \in L^1(G)$ we have from [Ba, p. 30, prop 7.3],

$$(\hat{f}_B(k))_{m,n} = \eta^{m,n}(k)(\hat{f}_H(k))_{m,n} \text{ for all } k \in \{ \pm 1 \} \text{ and } m,n \in \mathbb{Z}(k),$$

where $\eta^{m,n}(k)$ is a positive number. Therefore

$$(\hat{f}_B(k))_{m,n} \neq 0 \iff (\hat{f}_H(k))_{m,n} \neq 0.$$
Suppose \( \hat{f}_B(k) \neq 0 \) for all \( k \in \Gamma_n \), then it implies the following:

(a) If \( n \) is positive then for every \( m < n \), \( (\hat{f}_B(n - 1))_{m,n} = 0 \), so \( f \) has at least one non zero component of left type \( m \) such that \( m \geq n \). Similarly when \( n \) is negative \( f \) has at least one left type \( m \) for some \( m \leq n \).

(b) Let \( f \in L^1(G)_n \) and \( n \) is even. If \( n > 0 \) then by the hypothesis above \( \hat{f}_B(1) \neq 0 \) and so there is an \( m \) such that \( m \in \mathbb{Z}(1) \) and \( (\hat{f}_B(1))_{m,n} \neq 0 \). Therefore \( (\hat{f}_B(1))_{m,n} = 0 \). For \( n < 0 \) one can have a similar statement.

Proof of Theorem 1.2. We first consider the case when the collection indexed by \( \Omega \) contains exactly one function, \( f \in L^1(G)_n \). Let \( f \) be an \((m,n)\) type function and \((m,n)\)-th matrix coefficient \( \hat{f}_H \), \( \hat{f}_H = \hat{f}_mH \).

Now we will construct a family of functions in \( \{G_m(\cdot) \mid m \in \mathbb{Z}^n\} \) such that \( G \in C^1_H(\hat{G})_{n,m} \).

When \( mn \geq 0 \) let us define \( G_m(\lambda) = e^{-\lambda^4}Q_{n,m}(\lambda) \) where \( Q_{n,m} = P_{n,m} \) which is the numerator of the rational function \( \varphi_{\lambda}^{n,m} \) from [2.7]. Hence \( e^{-\lambda^4}Q_{n,m}(\lambda) = \varphi_{\lambda}^{n,m}e^{-\lambda^4}Q_{n,m}(-\lambda) \) which shows that

\[
G_m(\lambda) = e^{-\lambda^4}Q_{n,m}(\lambda) \in C^1_H(\hat{G})_{n,m}
\tag{7.2}
\]

for the case \( mn \geq 0 \). Here we note that \( Q_{n,m}(0) \neq 0 \).

If \( mn < 0 \) then we will have to choose the polynomial in a slightly different way because we want \( G_m(\lambda) \) to satisfy all the properties of \( C^1_H(\hat{G})_{n,m} \).

Case 1. Let \( n \) be odd. Then we take the polynomial \( Q'_{n,m}(\lambda) = P_{n,m}(\lambda) \cdot \lambda^2 \). Now \( Q'_{n,m}(0) = 0 \) and \( e^{-\lambda^4}Q'_{n,m}(\lambda) = \varphi_{\lambda}^{n,m}e^{-\lambda^4}Q'_{n,m}(-\lambda) \). Therefore, in this case

\[
G_m(\lambda) = e^{-\lambda^4}Q'_{n,m}(\lambda) \in C^1_H(\hat{G})_{n,m}.
\tag{7.3}
\]

Case 2. Let \( n \) be even (hence \( |n|, |m| \geq 2 \) as \( nm < 0 \)). Then the required polynomial is \( Q''_{n,m}(\lambda) = P_{n,m}(\lambda)(1 - \lambda^2) \). So \( Q''_{n,m}(\pm 1) = 0 \) and \( e^{-\lambda^4}Q''_{n,m}(\lambda) = \varphi_{\lambda}^{n,m}e^{-\lambda^4}Q''_{n,m}(-\lambda) \). Therefore in this case also,

\[
G_m(\lambda) = e^{-\lambda^4}Q''_{n,m}(\lambda) \in C^1_H(\hat{G})_{n,m}.
\tag{7.4}
\]

Now for all \( m, n \)

\[
G_m(\lambda)\hat{f}_mH(\lambda) = e^{-\lambda^4}Q_{n,m}(\lambda)\hat{f}_mH(\lambda)
\]

\[
= e^{-\lambda^4}Q_{n,m}(-\lambda)\varphi_{\lambda}^{n,m}\varphi_{\lambda}^{m,n}\hat{f}_mH(-\lambda)
\]

\[
= G_m(-\lambda)\hat{f}_mH(-\lambda).
\]

Since \( f_m \) is an \((m,n)\) type function on \( G \) so \( \hat{f}_mH(\lambda) = \varphi_{\lambda}^{m,n}\hat{f}_mH(-\lambda) \) and \( \varphi_{\lambda}^{m,n} = (\varphi_{\lambda}^{m,n})^{-1} \).

This shows that for all \( m, G_m(\lambda)\hat{f}_mH(\lambda) \) is the Fourier transform of an \((n,n)\) type function with respect to principal series representation. Now we claim that \( \lambda \in S_1 \) there is an \( m \) such that \( G_m(\lambda)\hat{f}_mH(\lambda) \neq 0 \). The only possible zeros of the polynomials \( Q_{n,m}, Q'_{n,m} \) and \( Q''_{n,m} \) in \( S_1 \) are \( \{0, \pm 1\} \) and everywhere else it is non-zero. Given \( \hat{f}_H(\lambda) \neq 0 \) for all \( \lambda \in S_1 \). If we could show that for each \( \lambda \in \{0, \pm 1\} \) there is an \( m \) such that \( G_m(\lambda)\hat{f}_mH(\lambda) \neq 0 \) then we will be done.

Before proving our claim we find out exactly when \( \{0, \pm 1\} \) are zeros of the polynomials above.

(i) \( P_{n,m}(-1) = 0 \) if and only if \( n = 0 \) and \( m \neq 0 \), \( P_{n,m}(+1) \neq 0 \) for all \( m \neq 0 \) and \( P_{n,0}(+1) \neq 0 \) when \( n \neq 0 \), therefore

\[
Q_{n,m}(\pm 1) \neq 0 \text{ when } nm \neq 0 \text{ and } Q'_{n,m}(\pm 1) \neq 0 \text{ for all } m \neq 0.
\tag{7.5}
\]
Case 2. Let $Q_n(0) \neq 0$ so $Q''_{n,m}(0) \neq 0$. Therefore $\hat{f}_{mH}(0) \neq 0$ as $Q_{m,n}(0) \neq 0$.

First we consider the case for $\lambda = 0$. By hypothesis there is an $m$ such that $\hat{f}_{mH}(0) \neq 0$. If $n$ is odd then $mn > 0$, otherwise $\phi_{n,m}^{m,n} \equiv 0$ which implies $\hat{f}_{mH}(0) = 0$. Therefore $G_m(0)\hat{f}_{mH}(0) \neq 0$ as $Q_{m,n}(0) \neq 0$.

Next suppose $n$ is even. Now If $nm \geq 0$ then $G_m(0)\hat{f}_{mH}(0) \neq 0$ as $Q_{m,n}(0) \neq 0$. When $nm < 0$ then also $G_m(0)\hat{f}_{mH}(0) \neq 0$ because from (7.6) $Q''_{m,n}(0) \neq 0$.

Now we prove our claim for $\lambda = \pm 1$. Here we will consider several case for $n$.

Case 1. Let $n = 0$, then $\hat{f}_{mH}(1) = 0$ for all $m \neq 0$ as $\phi_{m,n}^{m,0} \equiv 0$. Therefore $\hat{f}_{0H}(1) \neq 0$ and also $\hat{f}_{0H}(1) = \hat{f}_{0H}(-1)$.

Case 2. Let $n(\neq 0)$ be an even no. If $n > 0$ then by discussion(b) preceding this proof, there exists an $r \in \mathbb{Z}(1)$ such that $\hat{f}_{rH}(1) \neq 0$ and so $G_r(1)\hat{f}_{rH}(1) \neq 0$ (since $Q_{m,n}(\pm 1) \neq 0$ for $nm > 0$ see (7.5)). But $\hat{f}_{rH}(-1) = \phi_{n,r}^{1,r}\hat{f}_{rH}(1)$ and $\phi_{n,r}^{1,r}$ has no zero at $\lambda = 1$ (see [Ba], prop 7.2). This shows that $\hat{f}_{rH}(-1) \neq 0$ and so $G_m(-1)\hat{f}_{mH}(-1) \neq 0$.

When $n < 0$ we will give similar arguments. By the same discussion(b) there exists an $s \in \mathbb{Z}(-1)$ such that $\hat{f}_{sH}(-1) \neq 0$ and so $G_s(-1)\hat{f}_{sH}(-1) \neq 0$. But $\hat{f}_{sH}(-1) = \phi_{n,s}^{n,s}\hat{f}_{sH}(1)$ and $\phi_{n,s}^{n,s}$ has no pole at $\lambda = 1$. This implies $\hat{f}_{sH}(1) \neq 0$ hence $G_m(-1)\hat{f}_{mH}(-1) \neq 0$ (since $Q_{m,n}(\pm 1) \neq 0$ for $nm > 0$ see (7.5)). This concludes our claim when $n$ is an even no.

Case 3. Let $n$ be an odd no. Then by the hypothesis there exist $m \in \mathbb{Z}^{\sigma}$ such that $\hat{f}_{mH}(1) \neq 0$.

Then $mn \neq 0$ so from (7.5) it follows $G_m(1)\hat{f}_{mH}(1) \neq 0$. Proof for $\lambda = -1$ is exactly similar.

Let $G'_m(k) = e^{-k^2}Q_{n,m}(k)$ for all $k \in \Gamma_n$ where $Q_{n,m}$ is chosen in the same way as before. Now let for $k_0 \in \Gamma_n$, $f_{m0}(k_0) \neq 0$ then $m_0 \in \mathbb{Z}(k_0)$. Therefore $Q_{m_0,m}(k_0) \neq 0$ as all the zeros of the polynomial $P_{n,m}$ are either between $m_0$ and $n$ or between $-m_0$ and $-n$ (see [Ba] prop. 7.1). Now from Lemma 2.1 an isomorphism between $C^1(G)_{n,m}$ and $C^1(\hat{G})_{n,m}$ for every $m$, there exists $g_m \in C^1(G)_{n,m}$ such that $\hat{g}_{mH}(\lambda) = G_m(\lambda)$ for all $\lambda \in \mathcal{S}_1$ and $\hat{g}_{mH}(k) = G_m(k)$ for all $k \in \Gamma_n$.

Now we show the set of $L^1(G)_{n,n}$ functions $\{g_m * f_m \mid m \in \mathbb{Z}^{\sigma}\}$ satisfies all the conditions of Theorem 4.3. Since $Q_{m,n}$’s are always polynomial in $\lambda$, by a simple argument of analysis shows that

$$\lim_{t \to \infty} e^{-\frac{t}{2}} \log | G_m(it) | = 0. \quad (7.8)$$

Hence,

$$\limsup_{t \to \infty} e^{-\frac{t}{2}} \log | G_m(it) \hat{f}_{mH}(it) | = \lim_{t \to \infty} e^{-\frac{t}{2}} \log | (G_m(it)) + \limsup_{t \to \infty} e^{-\frac{t}{2}} \log | \hat{f}_{mH}(it) |$$

$$= \limsup_{t \to \infty} e^{-\frac{t}{2}} \log | \hat{f}_{mH}(it) |. \quad (7.9)$$

Therefore by the given hypothesis,

$$\inf_{m \in \mathbb{Z}^{\sigma}} \delta^\pm (G_m \hat{f}_{mH}) = 0. \quad (7.10)$$

So we have established that the ideal generated by $\{g_m * f_m \mid m \in \mathbb{Z}^{\sigma}\}$ is dense in $L^1(G)_{n,m}$. But $g_m * f_m = g_m * f$; so the result follows from the fact that the left $L^1(G)$ module generated by $L^1(G)_{n,m}$ is all of $L^1(G)_n$.

Now suppose $\Lambda$ is an arbitrary index set. Then out of each $f^\alpha$ by projections we get $f^{\alpha}_{jH}$ for all $j \in \mathbb{Z}$ which are functions of type $(j,n)$. We apply previous arguments to the collection $\{f^{\alpha}_{jH} \mid \alpha \in \Lambda\}$.
\[ \Lambda, j \in \mathbb{Z} \} \] of functions in \( L^1(\hat{G})_n \) and the theorem follows.

\[ \square \]

**Proof of Theorem 1.1.** As we have seen in the proof of previous theorem, it is enough to consider the case when the collection contains a single function, namely \( f \). Let \( f_j \) be the projection of \( f \) to \( L^1(G)_j \), for every \( j \in \mathbb{Z} \). For each \( j, m \in \mathbb{Z} \), we choose a polynomial \( Q_{j,m} \) in \( \lambda \) involving \( j \) and \( m \) so that 
\[ e^{-\lambda^4} Q_{j,m}(\lambda) \in C^1_H(\hat{G})_{j,m}. \]

When \( jm \geq 0 \), \( Q_{j,m} = P_{j,m} \) is the numerator of the rational function \( \varphi_j^m \). Now suppose \( jm < 0 \), then whenever \( j, m \) are odd integers we take \( Q'_{j,m} = \lambda^2 P_{j,m} \) and if \( j, m \) are even integers then we choose \( Q''_{j,m} = (1-\lambda^2)P_{j,m} \), where \( P_{j,m} \) is as above. Then for \( m \in \mathbb{Z} \), 
\[ e^{-\lambda^4} Q_{j,m}(\lambda) \in C^1_H(\hat{G})_{j,m}. \]

By the isomorphism of \( L^1 \) Schwartz space \( C^1(G)_{j,m} \) and \( C^1(\hat{G})_{j,m} \) (see Lemma 2.1), there exists \( g_{j,m} \in C^1(G)_{j,m} \) such that \( \hat{g}_{j,m}(\lambda) = e^{-\lambda^4} Q_{j,m}(\lambda) \) for all \( \lambda \in S_1 \) and \( \hat{g}_{j,m_B}(k) = e^{-k^4} Q_{j,m}(k) \) for all \( k \in \Gamma_j \). Now for all \( m \in \mathbb{Z} \) we consider the following collection of functions,
\[ \mathcal{F}_m = \{ f_j * g_{j,m} \mid j \in \mathbb{Z} \} \]
contained in \( L^1(G)_m \).

As in (7.8) and (7.9) we have for each \( m \in \mathbb{Z} \),
\[ \lim_{t \to \infty} e^{-\frac{t^4}{4}} \log \left| \hat{g}_{j,m}(it) \right| = \lim_{t \to \infty} e^{-\frac{t^4}{4}} \log \left| \hat{f}_{i,j}(it) \right| \]
for all \( i, j \in \mathbb{Z} \). So,
\[ \inf_{i,j \in \mathbb{Z}} \delta^+ (\hat{g}_{j,m} \hat{f}_{i,j}) = 0. \]

Now for all \( m \in \mathbb{Z} \), Fourier transforms of the elements of \( \mathcal{F}_m \) do not have common zeros, follows from [Rs1], Theorem 1.2. Therefore together with (7.12) it follows that for every \( m \), elements of \( \mathcal{F}_m \) satisfies all the conditions of Theorem 1.2 and so \( \mathcal{F}_m \) generates \( L^1(G)_m \) under left convolution. Now \( f_j * g_{j,m} = f * g_{j,m} \), for every \( m \). So the two sided closed ideal generated by \( f \) contains \( L^1(G)_m \) for all \( m \). The smallest closed right \( G \)-invariant subspace of \( L^1(G)_m \) containing \( L^1(G)_m \), for all \( m \in \mathbb{Z} \), is \( L^1(G)_m \). Hence the first part of the Theorem 1.2 follows. The second part of the theorem follows similarly as in [Rs1], Theorem 1.2.

\[ \square \]

**Acknowledgement.** The author would like to thank to his supervisor Prof. Sanjoy Pusti for introducing him to the problem and for the many useful discussions during the course of this work. I am grateful to him for encouraging me in research and his guidance.

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