A CFT description of the BTZ black hole: topology versus geometry (or thermodynamics versus statistical mechanics)

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Abstract

In this paper we review the properties of the black hole entropy in the light of a general conformal field theory treatment. We find that the properties of horizons of the BTZ black holes in $\text{AdS}_3$, can be described in terms of an effective unitary CFT$_2$ with central charge $c = 1$ realized in terms of the Fubini-Veneziano vertex operators.

It is found a relationship between the topological properties of the black hole solution and the infinite algebra extension of the conformal group in 2D, $\text{SU}(2,2)$, i.e. the Virasoro Algebra, and its subgroup $\text{SL}(2,\mathbb{Z})$ which generates the modular symmetry. Such a symmetry induces a duality for the black hole solution with angular momentum $J \neq 0$. On the light of such a global symmetry we reanalyze the Cardy formula for CFT$_2$ and its possible generalization to $D > 2$ proposed by E. Verlinde.
1 Introduction

Gravity in 2+1 dimensions is a very simple model where it is possible to derive exactly the correspondence between quantum black holes properties and thermodynamical quantities.

In presence of a negative cosmological constant $\Lambda = -1/\ell^2$, the General Relativity in 2 + 1 dimensions admits the analogues of the 3 + 1 Schwarzschild and the Kerr black hole known as the BTZ black hole \cite{1}.

In fact these solutions exhibit all the usual thermodynamic properties of black holes \cite{2}. Their entropy is found to be 1/4 of the horizon area divided by $G_3$ the gravitational constant in three dimensions

$$S = \frac{2\pi r_+}{4G_3}. \quad (1.1)$$

It satisfies the first law of thermodynamics, as it should,

$$dU = TdS + \Omega dJ \quad (1.2)$$

where $U$ is identified with the ADM energy, $T$ is the Hawking “temperature”, $J$ is the angular momentum and $\Omega$ is the angular potential \cite{3}.

It is well known that for the 2 + 1 gravity there are no bulk dynamical degrees of freedom, or, in other words, all the dynamics is in the boundary. But there is a clear separation between the gravitational sector and the matter sector.

In the following we analyze the black hole physical properties by using the powerful tools of 2D conformal quantum field theory, CFT$_2$; indeed, it is will be argued that the boundary degrees of freedom of ADS$_3$ are described by an unitary representation of the “dual” CFT$_2$, i.e. of the infinite Virasoro algebra centrally extended with the central charge $c$ given by the Brown-Henneaux relation \cite{4}

$$c = \frac{3\ell}{2G_3}. \quad (1.3)$$

The paper is organized as follows. In sect. 2 we discuss the geometrical diffeomorphisms which preserve the conformal boundary and its possible description by a Chern-Simons topological field theory. The classical black hole general solution (which we shall call in the following as BTZ) is given in section 3 together with its properties. The explicit relation of the mass $M$ and the angular momentum $J$ in terms of the generators of the classical Virasoro algebra is found. In section 4 the quantum version of the BTZ solution in the Euclidean space is briefly analyzed. Moreover it is argued that the relevant unitary representation of CFT$_2$ is the $c = 1$ Fubini-Veneziano vertex operator one \cite{5}, following the Cardy argument \cite{6}. In section 5 we
give a short summary of the mathematical properties of the unitary representation of CFT\(_2\) which are used in this paper. In particular the Hilbert space of the quantized “momenta” \(\hat{p}\) and the winding “numbers” \(\hat{w}\) for the Fubini-Veneziano scalar field compactified on a circle \(S_1\) is analyzed using the SL(2,Z) modular symmetry. That is reflected in a “duality” relation between \((\ell, J)\) or \((r_+, r_-)\). In section 6 we derive the Brown-Henneaux relation by evaluating the quantum “anomaly” for the locally conformal (Weyl) transformations which is a classical symmetry in both cases, for the gravitational equations in the bulk and for CFT\(_2\) defined on the boundary and identifying the results \([7]\). That is also a proof of the validity of the ADS\(_3\)-CFT\(_2\) duality guessed by Maldacena \([8]\).

The analogue of the Hawking temperature and entropy \(S_H\) is derived in section 7 from the topology of spacetime, identified to be \(\mathbb{R}^2 \times S^1\). Then as for the duality between \((r_+, r_-)\), the topology of spacetime is strictly related to the physical quantities of BTZ. In section 8 we stress how the modular invariance, SL(2,Z), of CFT\(_2\), which implies the Cardy equation (see sect. 4) is also crucial for the validity of the cosmological Bekenstein bound \([9]\) in 2D. We present a derivation of the Casimir energy \(E_C\) and entropy \(S_C\) which should be relevant for the possible extension to higher dimensions.

Some hints in such a direction are briefly discussed in section 9 where general comments and conclusions are also given.

2 The 2 + 1 anti-de Sitter spacetime (ADS\(_3\))

Before discussing the BTZ black hole, i.e. the black hole in 2+1 dimensions, let us introduce the ADS\(_3\) spacetime. The reason is that a black hole solution in 2+1 dimension, differently from the Schwarzschild solution, has not a Newtonian asymptotic limit but an ADS\(_3\) one.

The ADS\(_3\) is a vacuum solution of Einstein equations in 3 dimensions with a negative cosmological constant \(\Lambda\).

Its metric in polar coordinates is

\[
ds^2 = \left[ \left( \frac{r}{\ell} \right)^2 + 1 \right] dt^2 + \left[ \left( \frac{r}{\ell} \right)^2 + 1 \right]^{-1} dr^2 + r^2 d\varphi^2.
\]

where \(\ell = |\Lambda|^{-1/2}\).

An ADS\(_3\) is characterized by being a manifold with a (conformal) boundary. In this case not all the diffeomorphisms are allowed, but only those which preserve the conformal boundary.

These can be identified with the conformal group SL(2,C), which is the covering group of O(2,2) \([10]\).
From the geometrical point of view, these diffeomorphisms are generated by the vector fields

\[ \xi^+(t) = \ell T^+ + \frac{\ell^3}{2r^2} \partial_u T^+ + O \left( \frac{1}{r^4} \right) \]

(2.2)

\[ \xi^-(t) = \ell T^- + \frac{\ell^3}{2r^2} \partial_v T^- + O \left( \frac{1}{r^4} \right) \]

(2.3)

\[ \xi^+(\phi) = \ell T^+ - \frac{\ell^3}{2r^2} \partial_u T^+ + O \left( \frac{1}{r^4} \right) \]

(2.4)

\[ \xi^-(\phi) = -\ell T^- + \frac{\ell^3}{2r^2} \partial_v T^- + O \left( \frac{1}{r^4} \right) \]

where \( T^\pm \) are functions of \( u = t/\ell + \phi \) and \( v = t/\ell - \phi \).

The commutators \([\xi^+_1, \xi^-_2] = \xi^-_3\) define new vector fields of the form given above and induce two Virasoro algebras with vanishing central charges on the functions \( T^\pm \).

It is possible to show that the presence of a boundary modifies these algebras by introducing a central charge \( c \) related to \( \ell \) as in eq. (1.3).

Some properties of the \( \text{ADS}_3 \) spacetime are better described by observing that the General Relativity action in \( 2 + 1 \) dimensions is equivalent to the Chern-Simons theory \[3\] \[11\].

Then by doing appropriate identifications the general relativistic action with a negative cosmological constant can be expressed by the following action

\[ I_{CS} = \frac{k}{4\pi} \int_M \text{Tr} \left\{ A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right\} \]

(2.5)

which can be split in the spatial and the time parts

\[ I_{CS} = \frac{k}{4\pi} \int dt \int_\Sigma \text{Tr} \left\{ A \wedge \dot{A} - A_0 F \right\} \]

(2.6)

where \( A_0 \) is a multiplier and the resulting field equation

\[ F \equiv dA + A = 0 \]

(2.7)

is a constraint. The solutions in the bulk take the trivial form \( A_\mu \sim G^{-1} \partial_\mu G \) and \( \bar{A}_\mu \sim (\partial_\mu \bar{G}) \bar{G}^{-1} \); where

\[ G = \left( \begin{array}{cc} \sqrt{r} & 0 \\ 0 & \frac{1}{\sqrt{r}} \end{array} \right) g(u) \]

(2.8)
\[
\tilde{G} = \tilde{g}(v) \begin{pmatrix}
\frac{1}{\sqrt{r}} & 0 \\
0 & \frac{1}{\sqrt{r}} 
\end{pmatrix}
\]

(2.9)

are diffeomorphisms in the bulk and the connections are just trivial gauge conditions.

But these transformations are not anymore diffeomorphisms on the boundary and the connections \(A\) and \(\tilde{A}\), are the dynamical variables of the theory. \(g(u)\) and \(\tilde{g}(v)\) are restricted (see ref.) so that the “affine” algebra (Kac-Moody algebra) currents \(g^{-1} \partial_u g\) and \(\hat{\partial}_u g \cdot g^{-1}\) (i.e. the left and right currents of the algebra) induce the two sectors of the Virasoro algebra, whose generators are

\[
T_{uu} = \sum_n L_n e^{-i n u} = \frac{k}{2} T r (\sigma_3 \partial_u A_u - A_u A_u),
\]

(2.10)

\[
T_{vv} = \sum_n L_n e^{-i n v} = \frac{k}{2} T r (\sigma_3 \partial_v \tilde{A}_v + \tilde{A}_v \tilde{A}_v).
\]

(2.11)

3 Classical BTZ black hole

The black hole solution in \(\text{ADS}_3\) was found by Bañados, Henneaux, Teitelboim and Zanelli in \[1\][12]. The corresponding quantum mechanical and thermodynamic properties are discussed in \[13\].

In \[12\] it is shown that an axially symmetric stationary metric in a \(2 + 1\) dimensional lorentzian spacetime (labelled with the subscript \(\text{Lor}\)) with cosmological constant \(\Lambda = -1/\ell^2\) is

\[
ds^2 = - \left( -M_{\text{Lor}} + \frac{r^2}{\ell^2} + \frac{J_{\text{Lor}}^2}{4r^2} \right) dt^2 + \left( -M_{\text{Lor}} + \frac{r^2}{\ell^2} + \frac{J_{\text{Lor}}^2}{4r^2} \right)^{-1} dr^2 + r^2 \left( d\phi - \frac{J_{\text{Lor}}}{2r} dt \right)^2
\]

(3.1)

We can note the following features in this solution.

Differently from the \(3 + 1\) solution the length scale is given by the “curvature” radius \(\ell\), because the mass \(M_{\text{Lor}}\) is a dimensionless quantity.

The lapse function

\[
N = -M_{\text{Lor}} + \frac{r^2}{\ell^2} + \frac{J_{\text{Lor}}^2}{4r^2}
\]

(3.2)

vanishes for

\[
r_{\pm}^2 = \frac{\ell^2 M_{\text{Lor}}}{2} \left( 1 \pm \sqrt{1 - \frac{J_{\text{Lor}}^2}{\ell^2 M_{\text{Lor}}^2}} \right).
\]

(3.3)
It follows

\[ M_{\text{Lor}} = \frac{r_+^2 + r_-^2}{\ell^2} \] (3.4)

and

\[ J_{\text{Lor}} = \frac{2r_+ r_-}{\ell}. \] (3.5)

g\_0\_0 vanishes at

\[ r = r_{\text{erg}} \equiv \ell M_{\text{Lor}}^{1/2}, \] (3.6)

which as for Kerr solution called ergodic radius, which defines the infinite red-shift surface of the black hole.

Finally at \( r = 0 \) there is a singularity on the causal structure but not in the curvature, because the curvature is everywhere finite and constant.

For large \( r \) the BTZ metric (3.1) approaches the \( \text{ADS}_3 \) metric (2.1), then the asymptotic symmetry group for this metric is the conformal group in two dimensions \( \text{SO}(2, 2) \) or its covering \( \text{SL}(2, \mathbb{C}) \).

It is noteworthy that one can find some dual relations in the BTZ black hole in presence of an angular momentum. Indeed the second degree algebraic equation \( N = 0 \) can be solved in terms of the unknown \( r^2 \) or its inverse \( 1/r^2 \). The solutions in terms of \( r^2 \) are related to the solutions of their inverse through (3.5) according to which

\[ r_-^2 = \frac{J_{\text{Lor}}^2 \ell^2}{4r_+^2}, \] (3.7)

moreover in the equation \( N = 0 \) there is a dual relation between \( J^2 \) and \( 1/\ell^2 \). This relation, as of the well-known duality relation in CFT\_2, is a consequence of the \( \text{SL}(2, \mathbb{Z}) \) invariance of the unitary representations of Virasoro algebra in CFT\_2. We will discuss such fact in sect. 5.

The Lie algebra of the conformal group consists in two copies of the Virasoro algebra with a central charge proportional to \( \ell \), the radius of curvature. It has generators \( L_n \) and \( \bar{L}_n \) for any \( n \in \mathbb{Z} \). The two non zero charges for metric (4.1) are

\[ M_{\text{Lor}} = \bar{L}_0 + L_0 \] (3.8)

and

\[ J_{\text{Lor}} = L_0 - \bar{L}_0, \] (3.9)

which are consistent with the fact that the eigenvalues of the dilatation operators \( L_0(\bar{L}_0) \) are \( r_+^2/\ell^2 \left( r_+^2/\ell^2 \right) \). An explicit derivation is given in sect. 5. The \( \text{ADS}_3 \) is a maximally symmetric space, it has 6 Killing vectors. A spacetime with a black hole shares with the AdS spacetime such symmetries
only asymptotically, but it generally has a lesser number of Killing vectors. Therefore a black hole can be defined by its symmetries.

Given the Killing vectors $\xi$, one can construct a parameter subgroup such that to a given point $P$ we have $P \to e^{t \xi} P$. When $t$ is an integral multiple of a step (conventionally one can take it as $2\pi$) we define an identification subgroup of $SO(2,2)$.

Correspondingly we can take the quotient space, which preserves the properties of the ADS$^3$. This quotient space is still a solution of Einstein’s equations.

If we label the coordinates by $x^a = (u,v,x,y)$, then the six Killing vectors of the ADS$_3$ are

$$J_{ab} = x_b \frac{\partial}{\partial x^a} - x_a \frac{\partial}{\partial x^b}$$  \hspace{1cm} (3.10)

In [12] it is proved that the black hole solutions are obtained by making the identifications defined previously by the discrete group generated by the Killing vector

$$\xi = \frac{1}{2} \omega^{ab} J_{ab} = \frac{r_+}{\ell} J_{12} - \frac{r_-}{\ell} J_{03} - J_{13} + J_{23},$$  \hspace{1cm} (3.11)

where $\omega^{ab}$ is an antisymmetric tensor, defined by the preceding relation, with eigenvalues $\pm r_+/\ell$, and $\pm r_-/\ell$.

The Casimir invariants are

$$I_1 = \omega_{ab} \omega^{ab} = -\frac{2}{\ell^2} (r_+^2 + r_-^2) = -2M$$  \hspace{1cm} (3.12)

$$I_2 = \frac{1}{2} \epsilon_{abcd} \omega^{ab} \omega^{cd} = -\frac{4}{\ell^2} = -2\frac{|J|}{\ell}.$$  \hspace{1cm} (3.13)

which are respectively proportional to the quantities defined in (3.8) and (3.9).

4 BTZ black hole quantum mechanics: from Minkowski to the Euclidean description

Sometimes it is more convenient to work in an Euclidean frame, where the Euclidean “time” is $t \to \tau = it$, and we put $M_{\text{Lor}} = M$ and $J_{\text{Lor}} = iJ$

$$ds^2 = - \left( -M + \frac{r^2}{\ell^2} - \frac{J^2}{4r^2} \right) d\tau^2 + \left( -M + \frac{r^2}{\ell^2} - \frac{J^2}{4r^2} \right)^{-1} dr^2$$

$$+ r^2 \left( d\phi - \frac{J}{2r} d\tau \right)^2.$$  \hspace{1cm} (4.1)
The singularities of this metric are in $r = 0$, $r = \infty$ and in

$$r_{\pm}^2 = \frac{\ell^2 M}{2} \left( 1 \pm \sqrt{1 + \frac{J^2}{\ell^2 M^2}} \right).$$

(4.2)

In the limit $J \to 0$, the metric reduces to

$$ds^2 = \left( -M + \frac{r^2}{\ell^2} \right) d\tau^2 + \left( -M + \frac{r^2}{\ell^2} \right)^{-1} dr^2 + r^2 d\phi^2.$$

(4.3)

the metric is not singular at $r = 0$ and the singularities (4.2) reduce to $r_+ = \sqrt{M\ell}$ and $r_- = 0$, which is the equivalent in $2 + 1$ dimensions of the Schwarzschild radius in $3 + 1$ dimensions [12].

By appropriate changes of coordinates we can put in evidence the properties of the metric (4.3) in this limiting case, it is possible to show the existing of the periodicity condition $\theta \sim \theta + 2\pi$ and the identifications of the coordinates lead to quotient the upper semispace by identifying $R \sim e^{2\pi \sqrt{M} R}$, one can make a further change of metric and work in the region delimited between the upper hemisphere $R = 1$ and $R = e^{2\pi \sqrt{M}}$, where the singular points are identified through the radial lines, the manifold so defined is a solid torus. The extension of this procedure to the case $J \neq 0$ is straightforward.

Going back to the CFT$_2$ for the boundary dynamics, we stress a very relevant result. It has been proven in [6] that the density of states is given by

$$S = 2\pi \left[ \frac{cL_0}{6} \right]^{1/2} + \left[ \frac{c\bar{L}_0}{6} \right]^{1/2}.$$

(4.4)

In [14] the Cardy equation was used to derive an effective CFT$_2$ with $c = 1$ described by the Coulomb gas (vertex) operator (for details see section 5) as a consequence of modular invariance, $SL(2, Z)$, and other very general properties of CFT$_2$. Then the “highest weight states” of such CFT$_2$ can be considered as the microstates of the gravity in $2 + 1$ dimensions, while the global states are described by the boundary dynamics. More precisely the gravity is determined only by the global geometric data and does not have “local excitations”, however eq. (4.4) sets for ADS$_3$ spacetime the correspondence with local states of the (conformal) field theory which are the “microscopic” excitations for $2 + 1$ gravity.

This point of view resembles the one advocated by Martinec in [14] and it will be made precise in the following section.
5 Short summary of unitary representations of CFT$_2$

To analyze the properties of the unitary representations of CFT$_2$, it is customary to use the Euclidean spacetime.

In complex coordinates the metric is

\[ ds_E^2 = dzd\bar{z} \]  

where \( z = x + iy \) and \( \bar{z} = x - iy \). Being the field theory conformal invariant one can split all the field in an analytic and an antianalytic part, i.e. for \( \Phi(x, y) \) one can write

\[ \Phi(x, y) = \Phi_L(z) + \Phi_R(\bar{z}). \]  

Henceforth we discuss only the analytic part, i.e. the left sector.

Relevant conformal fields called highest weight states are

a ) The energy-momentum tensor \( T_{ab}(x, y) \) which is an operator of conformal dimension 2 and it is written as

\[ T(x, y) = T_{zz}(z) + \bar{T}_{\bar{z}\bar{z}}(\bar{z}) = T_L(z) + \bar{T}_R(\bar{z}). \]  

b ) The currents \( J_a(z) \) which have conformal dimension 1 and are generators of symmetries.

They are a necessary ingredient for the solvability of CFT$_2$, for details see [10] and [16].

Then one can define the Virasoro algebra for the left sector as follows

\[ [L_n, L_m] = (n - m)L_{n+m} + cn(n^2 - 1)\delta_{n+m;0} \]  

where

\[ L_n = \frac{1}{2\pi i} \oint z^n T_L(z). \]  

Naturally \( \bar{L}_n \) defined for the antianalytic part defines the same algebra.

Notice that the central extension, i.e. the second term in eq. (5.4) is absent for \( n = 0, \pm 1 \) as it should be. In fact \( L_0, L_{\pm 1} \) generate the 2-D conformal group \( SO(2, 2) \) (whose covering group is \( SL(2, C) \)). But the relevance of the entire algebra is a milestone of theoretical physics (see [10]).

The Operator Product Expansion (OPE) is another crucial technique of conformal field theory in any dimension. For example from simple scaling arguments one derives the

\[ T(z)J(w) \xrightarrow{(z-w)\to0} \frac{J(w)}{(z-w)^2} + \frac{\partial_w J}{(z-w)} + \text{regular terms} \]  

where

\[ J(z) = \frac{1}{2\pi i} \oint z^n T_L(z). \]
A CFT\(_2\) is completely and exactly known if one can derive all the highest weight states and the associated operators \(O_a(z)\) \(a = 1, \ldots k\) and their 3 point functions exactly

\[
O_a(z)O_b(w) = \lim_{z \to w} \frac{O_c(w)}{(z - w)^{d_a + d_b + d_c}}
\]

(5.7)

where \(d_a\) are the conformal dimensions.

We refer for the complete analysis of such beautiful results to [10] [15].

For our purpose we recall only the results concerning the representation of the \(c = 1\) CFT\(_2\).

The highest weight states can be described by the vertex operators [5],

\[
V(z) = e^{i\alpha \Phi(z)}
\]

We will see that for rational CFT\(_2\) the values of \(\alpha^2\) are rational numbers. The scalar real field [5] is represented in Fock Space as follows

\[
\Phi(z) = a_0 + \rho_0 \ln z + \sum_{n=-\infty; n \neq 0}^{\infty} a_n z^n
\]

(5.8)

where \(a_n^\dagger = a_{-n}\) due to the reality of \(\Phi\). The commutator relations are given by

\[
[x_0, p_0] = 1; \quad [a_n, a_{-m}] = \delta_{n+m;0}
\]

(5.9)

The Green function \(G(z)\) is evaluated as

\[
[\Phi(z), \Phi(w)] = \ln \left(\frac{z-w}{u}\right)
\]

(5.10)

\(u\) being an infrared cut-off.

A very interesting case is when the scalar field \(\Phi(z)\) is compactified on a circle \(S_1\) with radius \(R\) Then the highest weight states are defined by the Hilbert space as

\[
\hat{p} |l > = \frac{l}{R} |l >
\]

(5.11)

and for the “dual” state by the winding number \(\hat{w}\)

\[
\hat{w}|k > = k R |k >
\]

(5.12)

So we get a \(U(1)\) symmetry for the highest weight states enhanced to \(U(1) \times U(1)\). In fact it can be shown that the conformal weights \(\Delta\) and \(\bar{\Delta}\) for the \(R, L\) sector are

\[
\Delta + \bar{\Delta} = \frac{\hat{p}^2}{R^2} + R^2 \hat{w}^2
\]

(5.13)
\[ \Delta - \tilde{\Delta} = 2\hat{p} \cdot \hat{w}. \quad (5.14) \]

from which
\[ \Delta = \left( \frac{\hat{p}}{R} + R\hat{w} \right)^2; \quad \tilde{\Delta} = \left( \frac{\hat{p}}{R} - R\hat{w} \right)^2 \quad (5.15) \]

If \( R^2 = m \) and \( m \in \mathbb{Z}_+ \) then \( (l, k) \leq m \). In other terms the highest weight states are finite and all the correlation functions are given by products of binomials as \((z - w)^{\alpha^2}\) with \( \alpha^2 \) a positive integer. Furthermore \( \hat{p} \) and \( \hat{w} \) can be interpreted as charged in the so-called 2D Coulomb gas interpretation of the vertex operators \[\text{(16)}\ [\text{17}]. \] Eqs. \((5.13)\) and \((5.14)\) for \( R^2 = 1 \) reproduce respectively eqs. \((3.4)\) and \((3.5)\) if we identify
\[ \hat{p} = \frac{r_+}{\ell}; \quad \hat{w} = \frac{r_-}{\ell} \quad (5.16) \]

from which
\[ \Delta + \tilde{\Delta} = \hat{p}^2 + \hat{w}^2 = \frac{r_+^2 + r_-^2}{\ell^2} = M \quad (5.17) \]
\[ \Delta - \tilde{\Delta} = 2\hat{p} \cdot \hat{w} = 2r_+r_- = J\ell. \quad (5.18) \]

then
\[ \Delta = (\hat{p} + \hat{w})^2 = \frac{(r_+ + r_-)^2}{\ell^2} \quad (5.19) \]

and
\[ \tilde{\Delta} = (\hat{p} - \hat{w})^2 = \frac{(r_+ - r_-)^2}{\ell^2} \quad (5.20) \]

For \( J = 0 \Leftrightarrow r_- = 0 \) one gets \( \Delta = \tilde{\Delta} \). In the particular case of maximal \( J \) i.e. \( r_+ = r_- \) one finds
\[ \Delta = 4 \frac{r_+^2}{\ell^2}, \quad \tilde{\Delta} = 0. \quad (5.21) \]

We notice that for such extremal case the CFT\(_2\) contains one sector or in other terms it is a chiral theory. In such a case it looks similar to the one used in \([\text{17}]\) to describe the Laughlin anyons for a Quantum Hall fluid \([\text{18}]\). Such a connection has previously been noticed, in a different context, in \([\text{19}]\).

6 Cosmological constant and central charge

As it is well known there have been different ways to analyze the gravitational properties of the anti de Sitter space in \(2 + 1\) dimensions.

Here we shall start from a very interesting fact derived in the work by Brown and Henneaux, i.e. the relation between the cosmological constant

\[ \Delta = \frac{4}{\ell^2} r_+^2, \quad \tilde{\Delta} = 0. \quad (5.21) \]
\[ \Lambda = -1/\ell^2 \text{ in } AdS_3 \] and the central charge of the boundary CFT$_2$, equation (1.3). We shall briefly summarize the derivation as given in [7]. In fact starting from the action

\[
S = -\frac{1}{16\pi G} \int_M d^3x \sqrt{g} \left[ R - \frac{d(d-1)}{\ell^2} \right] - \frac{1}{8\pi G} \int_{\partial M} \sqrt{-\gamma} \Theta + \frac{1}{8\pi G} S_{ct}(\gamma_{\mu\nu})
\]

where \( \Theta \) is the trace of the extrinsic curvature of the boundary. \( M \) is AdS and \( \partial M \) is its boundary. From (6.1) one gets

\[
T^{\mu\nu} = \frac{1}{8\pi G} \left[ \Theta^{\mu\nu} - \Theta \gamma^{\mu\nu} + \frac{2}{\sqrt{-\gamma}} S_{ct} \delta\gamma_{\mu\nu} \right]
\]

(6.2)

where \( S_{ct} \) has the role of canceling the divergences when \( \delta M \) goes to the AdS boundary \( \delta M \).

By a careful analysis one finds

\[
T^{\mu\nu} = \frac{1}{8\pi G} \left[ \Theta^{\mu\nu} - \Theta \gamma^{\mu\nu} - \frac{1}{\ell} \gamma^{\mu\nu} \right]
\]

(6.3)

where all the quantities refer to the boundary metric and

\[
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}.
\]

(6.4)

Remind that in 2d \( R_{\mu\nu} = g_{\mu\nu} R \). For AdS$_3$ one can compute the mass and angular momentum of BTZ black holes.

Let us introduce the AdS$_3$ metric

\[
ds^2 = \left( \frac{\ell}{r} \right)^2 dr^2 + \left( \frac{r}{\ell} \right)^2 (dx^2 - dt^2).
\]

(6.5)

At a fixed \( r \) we have a boundary conformal to \( R^{1,1} \) i.e.

\[-g_{tt} = g_{xx} = \left( \frac{r^2}{\ell^2} \right).
\]

(6.6)

Following such line of thought one reproduces general results usually derived with conventional technique see ref [7] for details.

In particular when \( M = -1/8\pi G; J = 0 \) the BTZ metric approaches the global AdS$_3$, while \( M = 0 \) and \( J = 0 \) it is similar to the Poincaré metric.

Let us derive the Weyl anomaly, in a covariant way. As it is well-known for Euclidean CFT$_2$ with metric is \( ds^2 = dz \bar{z} \) the diffeomorphisms are defined as by

\[
z \rightarrow z - f(z)
\]

\[
\bar{z} \rightarrow \bar{z} - g(\bar{z})
\]

(6.7)
and
\[ T_{zz} \rightarrow T_{zz} + (2\partial_z f(z)T_{zz} + z\partial_z T_{zz}) - \frac{c}{24\pi} \partial_z^+ f \]
\[ T\bar{z}\bar{z} \rightarrow T\bar{z}\bar{z} + (2\partial_{\bar{z}} g(\bar{z})T\bar{z}\bar{z} + \bar{z}\partial_{\bar{z}} T\bar{z}\bar{z}) - \frac{c}{24\pi} \partial_{\bar{z}}^+ g \]  \hspace{1cm} (6.8)

where \( z = x + iy \) and \( \bar{z} = x - iy \).

Really equation (6.7) is a symmetry of the CFT\(_2\) only at a classical level.

At the quantum level. In fact we have to introduce an ultraviolet cut-off \( u \) in quantum time-like convergent loops and show that a field theory in ADS\(_3\) remains invariant if we rescale \( z \rightarrow z' = e^\lambda z \) \( (\lambda > 0) \). Equivalently the metric should be Weyl rescaled to preserve \( ds^2 = -dzd\bar{z} \). Starting from eq. (6.5), if we consider the diffeomorphism eq. (6.7) there is a Weyl scaling of the boundary metric

Then we require that the asymptotic form (for \( r^2 \rightarrow \infty \)) remains invariant. One can prove that it is so if for \( r^2 \rightarrow \infty \)

\[ g_{zz} = -\frac{r^2}{2} \quad g_{++} = g_{--} = O(1) \]  \hspace{1cm} (6.9)
\[ g_{rr} = \frac{\ell^2}{r^2} + O\left(\frac{1}{r^4}\right) \quad g_{+r} = g_{-r} = O\left(\frac{1}{r^3}\right) \]  \hspace{1cm} (6.10)

From now on the boundary CFT\(_2\) is analyzed in the Euclidean metric as usual one does for the relativistic quantum field theory.

With these diffeomorphisms the metric changes as

\[ ds^2 \rightarrow \frac{\ell^2}{r^2}dr^2 - r^2dzd\bar{z} - \frac{\ell^2}{2}(\partial^3_\xi^+)dz^2 - \frac{\ell^2}{2}(\partial^3_\xi^+)d\bar{z}^2 \]  \hspace{1cm} (6.11)

and we can compute the stress-energy tensor which is

\[ T_{zz} = -\frac{\ell}{16\pi G} \partial_\xi^z \quad T_{\bar{z}\bar{z}} = -\frac{\ell}{16\pi G} \partial_{\bar{z}}^\xi \]  \hspace{1cm} (6.12)

Equations (6.8) do agree with (6.12) if

\[ c = \frac{3\ell}{2G} \]  \hspace{1cm} (6.13)

according to (1.3).

Naturally the analysis above is in agreement with the ADS\(_3\)/CFT\(_2\) duality on which we have not much to say here. We stress that \( r \) plays the rôle of the ultraviolet cut-off in the general relativity analysis of the BTZ metric as the usual cut-off \( a \) does on the quantum field theory (CFT\(_2\)) side.
7 Thermodynamics and topology of a BTZ black hole

We will show that the BTZ black hole is a thermodynamic object with “effective” temperature

\[ T_0 = \frac{1}{\beta_0} = \frac{r_+^2 - r_-^2}{2\pi\ell r_+}. \]  

(7.1)

as one would guess by analogy to the Schwarzschild case.

To do so we can apply the Euclidean path integral method [20]. In complete (and straightforward) analogy to the $3+1$ dimensional case one finds the Euclidean metric (4.1) (with $t = i\tau$) introduced in section 4. Such metric is singular at

\[ r_+^2 = \left\{ \frac{M\ell^2}{2} \left[ 1 + \left(1 + \frac{J^2}{M\ell^2}\right)^{1/2} \right] \right\}, \quad (7.2) \]

and

\[ r_-^2 \equiv [-i|r-|]^2 = \left\{ \frac{M\ell^2}{2} \left[ 1 - \left(1 + \frac{J^2}{M\ell^2}\right)^{1/2} \right] \right\}. \quad (7.3) \]

As shown in [13] the metric (4.1) is positive definite of constant negative curvature; then it is isometric to the hyperbolic three-space $H^3$. With a coordinate change one obtains the metric of the standard half-space of $H^3$, i.e.

\[ ds^2 = \frac{\ell^2}{z^2}(dx^2 + dy^2 + dz^2), \quad z > 0 \]

\[ = \frac{\ell^2}{\sin^2 \chi} \left( \frac{dR^2}{R^2} + d\chi^2 + \cos^2 \chi d\vartheta^2 \right); \quad z > 0 \]

(7.4)

with the identifications

\[ (R, \theta, \chi) \sim (Re^{2\pi r+}/\ell, \theta + 2\pi|r-|/\ell, \chi), \]

(7.5)

the resulting topology is $R^2 \times S^1$ as expected.

The requirements of smoothness lead to

\[ (\Phi, \tau) \sim (\Phi + m\Phi, \tau + \beta_0) \]

(7.6)

\[ \tilde{\Phi} = \frac{2\pi r_- \ell}{r_+ - r_-} = \frac{2\pi r_+ \ell^2}{r_+^2 - r_-^2}, \quad \beta_0 = \frac{2\pi r_+ \ell^2}{r_+^2 - r_-^2}. \]

(7.7)

\[ \tilde{\Phi}(r_- = 0) = 0 \quad \text{and} \quad \beta_0(r_- = 0) = \frac{2\pi \ell^2}{r_+}. \]

(7.8)
For a temperature $T_0 = \beta_0^{-1}$ and a rotational chemical potential $\Omega$ then

$$I_E = 4\pi r_+ - \beta_0 (M - \Omega J) \quad (7.9)$$

then one has

$$S \equiv S_E = \frac{2\pi r_+}{4\ell} \quad (7.10)$$

which is the analogous of the Bekenstein-Hawking entropy, in $2 + 1$ dimensions. Therefore we can identify $T_0$ with $T_{BH}$ and $S$ with $S_{BH}$.

8 Cardy formula and the Verlinde proposal

We have seen in sect. 4 that the entropy of a CFT$_2$ (the Cardy formula) depends crucially on the central charge $c$. We shall write it in a simplified way as

$$S = 2\pi \sqrt{\frac{c}{6}} \left( L_0 - \frac{c}{12} \right). \quad (8.1)$$

The term $c/12$ is the vacuum energy of the system.

Moreover the central charge $c$, for a system of finite volume, is strictly related to the boundary (surface) energy, i.e. the Casimir energy $E_c$. Such a quantity is not an extensive term, i.e. proportional to the volume $V$, but a subextensive one.

All the previous statements have been proved to be true for a large class of CFT$_2$ [6]. Finally it has been argued recently by Verlinde [21] that eq. (8.1) can be generalized for any dimension $D$ if the central charge $c/12$ is replaced by the Casimir energy [see also ref 3].

The main support of such an assumption consists in relating the Cardy formula to the cosmological Beckenstein bound [9]

$$S \leq S_{BH}. \quad (8.2)$$

Here $S$ is the entropy of the entire system, while $S_{BH}$ is the Beckenstein entropy, which for any $D$ can be defined as

$$S_{BH} = \frac{2\pi}{D - 1} E R \quad (8.3)$$

Such a bound is quite restrictive for low energy density, see [22]. To clarify the origin of equation (8.3) we can apply a simple scaling argument [21] by observing that, for $V_D \rightarrow \lambda V_D$, $E \rightarrow \lambda E$ and $S_{BH} \rightarrow \lambda^{1 + \frac{2}{D}} S_{BH}$.

By looking more closely to the role of Casimir energy in the cosmological bounds (see [21] and [23] for details) we will see that eq. (8.2) has a very surprising physical interpretation.
To this aim we shall repeat a general argument. The entropy of a thermodynamical system $S_V$ and the associated energy $E_V$ are extensive quantities, i.e. proportional to the volume $V$ in $D$ dimensions.

That simply implies the relation

$$\rho + p = Ts. \tag{8.4}$$

where $\rho$, $p$ and $s$ are respectively the energy density, the pressure and the entropy density.

Now the extensiveness of $E$ means that $E(\lambda V, \lambda E) = \lambda E(S, V)$. By differentiating one finds

$$E = V \left( \frac{\partial E}{\partial V} \right)_S + S \left( \frac{\partial E}{\partial S} \right)_V \tag{8.5}$$

for $\lambda = 1$.

The first law of thermodynamics tells that

$$\left( \frac{\partial E}{\partial V} \right)_S = -p \tag{8.6}$$

i.e.

$$TS = E + pV. \tag{8.7}$$

But for a system with a boundary there is a surface energy, the Casimir energy, which implies a non-extensive contribution, $S_C$, to the entropy $S$, as it was already stated.

At this point we stress that $S_C$ is a finite size effect due to quantum fluctuations, then its contribution is proportional to the area $A$ of the boundary is not zero at any temperature, included $T = 0$. It is appropriate to notice here the strong resemblance with the physics of quantum phase transitions recently so much studied [24] and [25]. The origin and physical relevance of the area law and/or of the Black Hole entropy seems very surprising in these papers.

This fact is crucial for understanding the relation between the topology of a $2+1$ dimensional spacetime and the thermodynamics of the BTZ black hole. Now we will analyze how to generalize some of the previous results for $D > 2$.

A possible definition of Casimir energy in any spatial dimension $D$ can be the following

$$E_C = (D - 1)(E + pV - TS) \tag{8.8}$$

i.e the term which violates the Euler identity eq. (8.5). This definition is quite useful in our context.

A simple dimensional analysis tells us that $E_C$ scales as

$$E_C(\lambda S, \lambda V) = \lambda^{1-2/(D-1)}E_C(S, V) \tag{8.9}$$
Notice that the exponent of $\lambda$ depends on the dimensions $D$ in contrast to the extensive quantities.

Finally we can separate the two terms in $E$

$$E = E_{\text{ext.}} + \frac{1}{2} E_C$$  \hspace{1cm} (8.10)

where the extensive term is $E_{\text{ext.}}$ and the non extensive one is $E_C$, the factor $1/2$ is introduced for convenience.

Now we come back to the bold assumption about $S_c$ and the definition of $S_B$ in eq. (8.3). We notice that $ER$ is independent of $V$ as a consequence of conformal invariance. Therefore it is only a function of $S$. Naturally both terms $E_{\text{ext.}}$ and $E_C$ are only functions of $S$, more precisely by usual scaling arguments one can write

$$E_{\text{ext.}} = \frac{a}{4\pi R} S^{1+1/(D-1)}; \quad E_C = \frac{b}{2\pi R} S^{1-1/(D-1)}$$  \hspace{1cm} (8.11)

with $(a, b)$ positive constants.

From that the entropy $S$ is given by

$$S = \frac{2\pi R}{\sqrt{ab}} \sqrt{E_C(2E - E_C)} = \frac{2\pi R}{\sqrt{ab}} \sqrt{2E_CE_{\text{ext.}}}. \hspace{1cm} (8.12)$$

That is exactly the Cardy formula (up to normalization) if one makes the identifications

1) $L_0 \rightarrow ER$

2) $c \rightarrow E_C R$  \hspace{1cm} (8.13)

Notice that we have used only conformal invariance and scaling arguments to derive eq. (8.12) then it is very tempting to assume that it is true for any $D > 2$.

For fixed $E$ eq. (8.12) has a maximum when $E = E_C$, i.e.

$$S \leq \frac{2\pi}{\sqrt{ab}} ER.$$  \hspace{1cm} (8.14)

That is the Beckenstein bound up to a constant term.

Many of our results are nice exemplifications of Maldacena ADS$_k$/CFT$_{k-1}$ correspondence for ADS$_3$. More precisely the thermodynamics of CFT$_2$ is identified with the thermodynamics of the BTZ black hole as argued in [26].
9 Comments and conclusions

In this paper we have emphasized the role and the strict relations between the different properties,

1) the origin of the Brown-Henneaux relation \(1.3\) is found to be the Weyl (trace) anomaly which for CFT\(_2\) is proportional to the central charge of the Virasoro algebra (sect. 6).

A related (dual) derivation of the Weyl anomaly \(6.9\) gives a contribution proportional to \(\ell/G\). The eq. \(1.3\) relates the (quantum) vacuum energy in CFT\(_2\) with the (classical) gravity vacuum energy in ADS\(_3\) parameterized by \(\ell\).

2) The metric of the space-time ADS\(_3\) and its symmetries (i.e. the geometry of space-time) implies the symmetries (diffeomorphisms) i.e. the classical conformal group \(SO(2,2)\) at the boundary

3) The quantum extension of this symmetry, the Virasoro algebra, is assumed to be true for the boundary of ADS\(_3\) (assumption needed in order to reproduce the Weyl anomaly).

Therefore the role of the global symmetry \(SL(2,Z)\) (the modular invariance of CFT\(_2\)) is to generate a “duality” relation between \(r_+\) and \(r_-\), eq. \(5.16\) or equivalently between \(\ell\) and \(J\) very reminiscent of the electromagnetic duality in the description of the Quantum Hall effect by CFT\(_2\) (see ref. [17]). This fact needs a deeper understanding.

4) The thermodynamics of CFT\(_2\) now gives the results of sect. 7 where the Bekenstein-Hawking temperature \(T_{BH}\), and the related “entropy” eq. \(8.3\) are evaluated to be the correct ones.

Surprisingly those results are consequence of the topology of the space-time metric when a BTZ black hole of mass \(M\) and angular momentum \(J\) is present. Then topology implies thermodynamical properties of CFT\(_2\) and viceversa. Moreover the black-hole physical quantities eqs. \(3.4\) and \(3.5\) are all expressed in terms of \(\ell\) and become zero when \(c \to 0\) (\(E_C \to 0\)) or equivalently when \(\ell \to 0\). That seems to be a further support for the validity of the Cardy formula for \(D > 2\) or of a related more general formula.

These results emphasize the relevance of the Casimir energy \(E_C\) for the cosmological implication of the Cardy formula eq. \(8.1\) analyzed in papers [21] and [23]. Our result shows that in 2D the ADS\(_3\) entropy \(S = S_{BH}\) as derived in section 4 does saturate the Bekenstein bound \(8.2\) implying that \(E = E_C\) which, of course, is the maximum value of the Casimir energy.

Then for \(D > 2\) one should find black hole solutions which do not saturate the bound. Therefore it seems quinte important to deepen the study of black hole phase transitions as the Hawking-Page [27], started in [23] [28] and [29], to understand the evolution of its physical quantities as \((r_+, r_-)\) or \((M, J)\) as functions of \(E_c\) or \(\ell\).

It is our opinion that there is a strong interplay between the physics of
Quantum Phase Transitions, the Hawking-Page phase transition and the thermodynamics of Black Holes as one can infer from the work done by the community of cosmologists [30][31], the community of statistical mechanics and field theorists [24][25]. In this contest one of the main problems is to clarify the relation between the various definitions of entropy as pointed out in [31]. The problem of “information loss” can be also carefully analyzed in this framework. There are already clear indications in favor of the recent work by Hawking [32] in which the unitarity of the black hole theory is preserved. See also the paper of [33].

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