Multiplicity of a zero of an analytic function
on a trajectory of a vector field

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Dedicated to Vladimir Igorevich Arnol’d on his 60th birthday

Let $P(x)$ be a germ at the origin of an analytic function in $\mathbb{C}^n$, where $x = (x_1, \ldots, x_n)$, and let $\xi = \xi_1(x)\partial/\partial x_1 + \ldots + \xi_n(x)\partial/\partial x_n$ be a germ at the origin of an analytic vector field. Suppose that $\xi(0) \neq 0$, and let $\gamma$ be a trajectory of $\xi$ through the origin. Suppose that $P|_\gamma \neq 0$, and let $\mu(P|_\gamma)$ be the multiplicity of a zero of $P|_\gamma$ at the origin. Let $\xi P = \xi_1 \partial P/\partial x_1 + \ldots + \xi_n \partial P/\partial x_n$ be derivative of $P$ in the direction of $\xi$, and let $\xi^k P$ be the $k$th iteration of this derivative.

We give a formula (Theorem 1) for $\mu(P|_\gamma)$ in terms of the Euler characteristic of the Milnor fibers defined by a deformation of $P, \xi P, \ldots, \xi^{n-1} P$. For a polynomial $P$ of degree $p$ and a vector field $\xi$ with polynomial coefficients of degree $q$, this allows one to compute $\mu(P|_\gamma)$ in purely algebraic terms (Theorem 2), and to give an estimate (Theorem 3) for $\mu(P|_\gamma)$ in terms of $n, p, q$, single exponential in $n$ and polynomial in $p$ and $q$. This estimate improves previous results [7, 1] which were double exponential in $n$.

For a system $\Xi = \{\xi_i\}$ of vector fields in $\mathbb{R}^n$ with polynomial coefficients of degree not exceeding $q$, this implies a single exponential in $n$ and polynomial in $q$ estimate for the degree of nonholonomy of $\Xi$, i.e., the minimal order of brackets of $\xi_i$ necessary to generate the maximal possible subspace at each point of $\mathbb{R}^n$.

For $n = 2$, our estimate coincides with the estimate for the multiplicity of a Pfaffian intersection [4, 2]. In case $n = 3$, a similar estimate was obtained in [3].

The main result of this paper can be reformulated as follows. Let $x(t) : \mathbb{C}_{t,0} \to \mathbb{C}_{x,0}^n$ be a germ of an analytic vector-function satisfying a system of nonlinear algebraic differential equations $S_i(x(t), t)dx_i/dt = Q_i(x(t), t)$ where $S_i$ and $Q_i$ are polynomial in $(x, t)$ of degree $q$, and $S_i(0, 0) \neq 0$. Let $p(t) = P(x(t), t)$ where $P$ is a polynomial in $(x, t)$ of degree $p$. Suppose that $p(t) \neq 0$. Then the multiplicity of a zero of $p(t)$ at $t = 0$ can be computed in purely algebraic terms, and there is an estimate for this multiplicity in terms of $n, p, q$, single exponential in $n$ and polynomial in $p$ and $q$. 
Definition. Let \( \epsilon \in \mathbb{C} \). A germ \( \tilde{P}(x, \epsilon) \) of an analytic function at the origin in \( \mathbb{C}^{n+1} \) is called a deformation of \( P \) if \( \tilde{P}(x, 0) = P(x) \). For a fixed \( \epsilon \), we write \( \tilde{P}^\epsilon(x) \) for the function \( \tilde{P}(x, \epsilon) \) considered as a function of \( x \).

Proposition 1. [7]. Let \( \tilde{P}(x, \epsilon) = (\tilde{P}_1(x, \epsilon), \ldots, \tilde{P}_k(x, \epsilon)) \) be a deformation of \( P(x) = (P_1(x), \ldots, P_k(x)) \). For a positive number \( \delta \), let \( B_\delta \) be an open ball in \( \mathbb{C}^n \) of radius \( \delta \) centered at the origin. Then, for a small positive \( \delta \) and for a nonzero \( \epsilon \in \mathbb{C} \) much smaller than \( \delta \), the homotopy type of the set \( \{ \tilde{P}^\epsilon = 0 \} \cap B_\delta \) does not depend on \( \delta \) and \( \epsilon \). This set is called the Milnor fiber of the deformation \( \tilde{P} \).

Proposition 2. [7]. Let \( \tilde{P}(x, \epsilon) = (\tilde{P}_1(x, \epsilon), \ldots, \tilde{P}_k(x, \epsilon)) \) be a deformation of \( P(x) = (P_1(x), \ldots, P_k(x)) \). Suppose that, for small \( \epsilon \neq 0 \), the Milnor fiber \( X^\epsilon \) of \( \tilde{P} \) is nonsingular, i.e., \( dP_1^\epsilon \wedge \ldots \wedge dP_k^\epsilon \neq 0 \) at each point of \( X^\epsilon \).

Let \( X \) be the closure of \( \bigcup_{\epsilon \neq 0} X^\epsilon \), and let \( Z = X \cap \{ \epsilon = 0 \} \). Let \( \{Z_\alpha\} \) be a good stratification (Thom’s \( A_\epsilon \) stratification) of \( Z \setminus 0 \), i.e., a Whitney stratification such that, for any sequence \( (x_\nu, \epsilon_\nu) \) converging to \( (x^0, 0) \in Z_\alpha \), the limit of the tangent spaces to \( X^\epsilon \) at \( (x_\nu, \epsilon_\nu) \), if exists, contains the tangent space to \( Z_\alpha \) at \( (x^0, 0) \).

Let \( l(x) \) be a linear function in \( \mathbb{C}^n \) such that \( \{l(x) = 0\} \) is transversal to all \( Z_\alpha \). Then the set of critical points of \( l|_{X^\epsilon} \), for small \( \epsilon \neq 0 \), is zero-dimensional. Let \( \nu \) be the number of these points, counted with their multiplicities, converging to the origin as \( \epsilon \to 0 \). Then the Milnor fiber of \( P \) can be obtained from the Milnor fiber of \( (P, l) \) by attaching \( \nu \) cells of dimension \( n - k \).

Theorem 1. Let \( P(x) \) be a germ of an analytic function in \( \mathbb{C}^n \), and let \( \tilde{P}(x, \epsilon) \) be a deformation of \( P(x) \). Suppose that, for \( k = 1, \ldots, n \), the Milnor fiber \( V_k \) of \( \tilde{P}_k = (\tilde{P}, \xi \tilde{P}, \ldots, \xi^{k-1} \tilde{P}) \) is nonsingular, and let \( \chi(\tilde{P}_k) \) be the Euler characteristic of \( V_k \). Then

\[
\mu(P|_\gamma) = \chi(\tilde{P}_1) + \ldots + \chi(\tilde{P}_n). 
\]

Remark. A. Khovanskii suggested an alternative proof of this theorem, valid also when the Milnor fibers \( V_k \) are singular, as long as \( V_{n+1} \) is empty. In fact, he proved the following:

Theorem 1’. Let \( P(x) \) be a germ of an analytic function in \( \mathbb{C}^n \), and let \( \tilde{P}(x, \epsilon) \) be any deformation of \( P(x) \). Let \( V_k \) be the Milnor fiber of \( \tilde{P}_k = (\tilde{P}, \xi \tilde{P}, \ldots, \xi^{k-1} \tilde{P}) \), and let \( \chi(\tilde{P}_k) \) be the Euler characteristic of \( V_k \). Then

\[
\mu(P|_\gamma) = \chi(\tilde{P}_1) + \ldots + \chi(\tilde{P}_\mu). 
\]

Proof. Let \( (z, y_2, \ldots, y_n) \) be a system of coordinates in \( \mathbb{C}^n \) where \( \xi = \partial/\partial z \), and let \( \pi \) be projection \( \mathbb{C}^n \to \mathbb{C}^{n-1}_y \). For each \( y \), define \( \zeta_k(y) = \chi(\pi^{-1} y \cap V_k) \). Because each
set \( \pi^{-1}y \cap V_k \) is finite, its Euler characteristic equals to the number of points in it (not counting multiplicities). Then
\[
\sum_{k=1}^{\mu} \zeta_k(y) \equiv \mu.
\]
Standard “integration over Euler characteristic” arguments [9] show that
\[
\int \zeta_k(y)d\chi = \chi(\tilde{P}_k), \quad \text{and} \quad \int \sum_{k=1}^{\mu} \zeta_k(y)d\chi = \int \mu d\chi = \mu.
\]

**Proof of Theorem 1.** Let us choose a coordinate system \((z,y_2,\ldots,y_n)\) in a neighborhood of the origin in \(\mathbb{C}^n\) so that \(\xi = \partial/\partial z\) in this coordinate system. In particular, \(\gamma\) becomes \(z\)-axis, and \(\mu(P|_{\gamma})\) equals to the multiplicity of a zero of \(P(z,0,\ldots,0)\) at the origin.

We proceed by induction on \(n\). For \(n = 1\) the statement is obvious. Suppose that it holds for \(n-1\), so we can apply it to the subspace \(\left\{y_n = 0\right\}\) of \(\mathbb{C}^n\). For a generic coordinate \(y_n\), the Milnor fibers of \(\tilde{P}_k' = \tilde{P}_k|_{y_n = 0}\) are nonsingular, and the condition of Theorem 1 is satisfied. We have then
\[
\mu(P|_{\gamma}) = \chi(\tilde{P}_1') + \cdots + \chi(\tilde{P}_{n-1}').
\]  
Let \(1 \leq k \leq n - 1\). Let us fix a small nonzero \(\epsilon\). According to Proposition 2, \(V_k\) can be obtained from \(V_k'\) by attaching \(\nu_k\) cells of dimension \(n - k\), where \(\nu_k\) is the number of points of \(V_k\) where \(dy_n|_{V_k} = 0\), counted with the proper multiplicities. In particular,
\[
\chi(\tilde{P}_k') = \chi(\tilde{P}_k) - (-1)^{n-k} \nu_k.
\]
The necessary transversality conditions in Proposition 2 are satisfied for a generic coordinate \(y_n\), because \(P(z,0,\ldots,0) \not\equiv 0\).

The critical points of \(y_n|_{V_k}\) are defined by the linear dependence at the points of \(V_k\) of the following 1-forms:
\[
d(\tilde{P}_\epsilon'), \; d(\xi \tilde{P}_\epsilon'), \; \ldots \; d(\xi^{k-1} \tilde{P}_\epsilon'), \; dy_n.
\]
In other words, the rank of the following \(k \times (n-1)\)-matrix \(A_k\) is less than \(k\):
\[
A_k = \left( \begin{array}{cccc}
\frac{\partial}{\partial z} \tilde{P}_\epsilon & \frac{\partial}{\partial y_2} \tilde{P}_\epsilon & \cdots & \frac{\partial}{\partial y_{n-1}} \tilde{P}_\epsilon \\
\frac{\partial}{\partial z} \xi \tilde{P}_\epsilon & \frac{\partial}{\partial y_2} \xi \tilde{P}_\epsilon & \cdots & \frac{\partial}{\partial y_{n-1}} \xi \tilde{P}_\epsilon \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial}{\partial z} \xi^{k-1} \tilde{P}_\epsilon & \frac{\partial}{\partial y_2} \xi^{k-1} \tilde{P}_\epsilon & \cdots & \frac{\partial}{\partial y_{n-1}} \xi^{k-1} \tilde{P}_\epsilon
\end{array} \right)
\]
Taking into account that \( \xi = \partial/\partial z \), we find that, at the points of \( X_k \), all the entries in the first column of the matrix \( A_k \) are zero, except for the last entry which is \( \xi^k \bar{P}^\epsilon \).

Let \( B_k \) be the matrix \( A_k \) with the first column removed, and let \( C_k \) be the matrix \( B_k \) with the last row removed. For \( k = 1, \ldots, n - 2 \), we have

\[
B_k = C_{k+1} = \begin{pmatrix}
\frac{\partial}{\partial y_2} \bar{P}^\epsilon & \cdots & \frac{\partial}{\partial y_{n-1}} \bar{P}^\epsilon \\
\frac{\partial}{\partial y_2} \xi \bar{P}^\epsilon & \cdots & \frac{\partial}{\partial y_{n-1}} \xi \bar{P}^\epsilon \\
\vdots & \ddots & \vdots \\
\frac{\partial}{\partial y_2} \xi^{k-1} \bar{P}^\epsilon & \cdots & \frac{\partial}{\partial y_{n-1}} \xi^{k-1} \bar{P}^\epsilon
\end{pmatrix}.
\]

For \( k = 1, \ldots, n - 1 \), the rank of \( A_k \) is less than \( k \) if and only if either \( \xi^k \bar{P}^\epsilon = 0 \) and the rank of \( B_k \) is less than \( k \) or \( \xi^k \bar{P}^\epsilon \neq 0 \) and the rank of \( C_k \) is less than \( k - 1 \). Modifying, if necessary, \( \bar{P} \) (adding \( z^k \) with a small coefficient) we can always suppose that \( \xi^k \bar{P}^\epsilon \neq 0 \) at the points of \( V_k \) where the rank of \( C_k \) is less than \( k - 1 \).

This means that the set of the critical points of \( y_n|_{V_k} \) is a union of two disjoint sets, \( X_k \cap \{ \xi^k \bar{P}^\epsilon = 0 \} \cap \{ \text{rank} \, B_k < k \} \) and \( X_k \cap \{ \text{rank} \, C_k < k - 1 \} \). Hence \( \nu_k = \nu'_k + \nu''_k \), where \( \nu'_k \) and \( \nu''_k \) are the numbers of critical points of \( y_n|_{V_k} \) in these two sets, counted with the proper multiplicities.

Taking into account that \( B_k = C_{k+1} \) and \( X_k \cap \{ \xi^k \bar{P}^\epsilon = 0 \} = X_{k+1} \), we have \( \nu'_k = \nu''_{k+1} \), for \( k = 1, \ldots, n - 2 \). For \( k = 1 \), we have \( \nu_1 = \nu'_1 \). For \( k = n - 1 \), we have \( \nu'_{n-1} = \chi(\bar{P}_n) \), the number of points in the set \( V_n \).

Replacing \( \nu_k \) in (4) by \( \nu'_k + \nu''_k \) and substituting (4) into (3), we see that all the values \( \nu'_k \) and \( \nu''_k \) cancel out, except \( \nu'_{n-1} \), and (3) implies (1).

**Theorem 2.** Let \( \bar{P}(x, \epsilon) \) be a deformation of \( P(x) \) such that, for \( k = 1, \ldots, n \), the Milnor fiber \( V_k \) of \( (\bar{P}, \xi \bar{P}, \ldots, \xi^k \bar{P}) \) is nonsingular. Let \( l_1(x), \ldots, l_{n-1}(x) \) be generic linear forms in \( \mathbb{C}^n \). For \( k = 1, \ldots, n \) and \( i = 1, \ldots, n - k \), let

\[
\omega_{i,k} = d\bar{P}^\epsilon \wedge d(\xi \bar{P}^\epsilon) \wedge \ldots \wedge d(\xi^{k-1} \bar{P}^\epsilon) \wedge dl_1 \wedge \ldots \wedge dl_{n-k-i+1} = 0.
\]

Let \( \nu_{0,k} \) be the number of isolated zeroes of the system

\[
\bar{P}^\epsilon = \xi \bar{P}^\epsilon = \ldots = \xi^{k-1} \bar{P}^\epsilon = l_1 = \ldots = l_{n-k} = 0
\]

converging to the origin as \( \epsilon \to 0 \), and let \( \nu_{i,k} \), for \( i = 1, \ldots, n - k \), be the number of isolated zeroes of the system

\[
\bar{P}^\epsilon = \xi \bar{P}^\epsilon = \ldots = \xi^{k-1} \bar{P}^\epsilon = l_1 = \ldots = l_{n-k-i} = 0, \quad \omega_{i,k} = 0
\]
converging to the origin as $\epsilon \to 0$. Then
\[
\mu(P|_\gamma) = \sum_{k=1}^{n} \sum_{i=0}^{n-k} (-1)^i \nu_{i,k}.
\] (6)

For a polynomial $\tilde{P}$ and a vector field $\xi$ with polynomial coefficients, equations (5) are algebraic.

**Proof.** This theorem follows from Theorem 1 and the standard relations for the Euler characteristics of hyperplane sections (Proposition 2):
\[
\chi(\tilde{P}, \xi_l \tilde{P}, \ldots, \xi_{k-1} \tilde{P}, l_1, \ldots, l_{n-k}) = \nu_{0,k},
\]
\[
\chi(\tilde{P}, \xi_l \tilde{P}, \ldots, \xi_{k-1} \tilde{P}, l_1, \ldots, l_{n-k-i+1}) - \chi(\tilde{P}, \xi_l \tilde{P}, \ldots, \xi_{k-1} \tilde{P}, l_1, \ldots, l_{n-k-i}) = (-1)^i \nu_{i,k}.
\]

**Lemma 1.** Let $l(x)$ be a germ of an analytic function such that $\xi_l(0) \neq 0$. Let us choose $\delta > 0$ so that there exists a representative of $P$ in $B_\delta$. For $c = (c_1, \ldots, c_n) \in \mathbb{C}^n$, let $P_c(x) = P(x) + c_1 + c_2 l(x) + \ldots + c_n l^{n-1}(x)$.

(i) For a generic $c$, the set $X_{k,c} = \{P_c = \xi P_c = \ldots = \xi_{k-1} P_c = 0\} \cap B_\delta$ is nonsingular, for $k = 1, \ldots, n$.

(ii) For a generic $c$, the deformation $\tilde{P}(x, \epsilon) = P(x) + \epsilon (c_1 + c_2 l(x) + \ldots + c_n l^{n-1}(x))$ satisfies conditions of Theorem 1, i.e., the Milnor fiber of $(\tilde{P}, \xi \tilde{P}, \ldots, \xi_{k-1} \tilde{P})$ is nonsingular, for $k = 1, \ldots, n$.

**Proof.** Consider coordinate system $(z, y_2, \ldots, y_n)$ where $\xi = \partial / \partial z$. Let us note first that the set $\{P_c = \xi P_c = \ldots = \xi_{k-1} P_c = 0\}$ coincides with the set
\[
P_c = \frac{\partial}{\partial l} P_c = \ldots = \frac{\partial^{k-1}}{\partial l^{k-1}} P_c = 0,
\]
where $\partial / \partial l$ is the partial derivative in a coordinate system $(l, y_2, \ldots, y_n)$. This follows from the chain rule. Geometrically, both sets represent the points where restriction of $P_c$ to a line parallel to $z$-axis has a zero of multiplicity at least $k$.

Now our statement is a special case of Thom’s transversality theorem. Let $Z_k \subset \mathbb{C}_{x,c}^{2n}$ be defined as $\{x, c : x \in B_\delta, P_c(x) = \xi P_c(x) = \ldots = \xi_{k-1} P_c(x) = 0\}$, for $k = 1, \ldots, n$. Each set $Z_k$ is nonsingular. Let $\pi : \mathbb{C}_{x,c}^{2n} \to \mathbb{C}_c^n$ be a natural projection. The set $X_{k,c}$ is nonsingular if and only if $c$ is not a critical value of the restriction of $\pi$ to $Z_k$. Due to Sard’s theorem, this holds for a general $c$.

To prove (ii), note that the set of those values of $c$ for which $X_{k,c}$ is nonsingular is a real subanalytic set which admits a complex-analytic stratification. From (i), this set does
not contain open subsets, hence its stratification does not have any strata of the complex dimension \( n \). Hence the intersection of this set with a generic complex line through the origin is zero-dimensional.

**Theorem 3.** Let \( P \) be a polynomial of degree not exceeding \( p \geq n - 1 \), and let \( \xi \) be a vector field with polynomial coefficients of degree not exceeding \( q \geq 1 \). Then \( \mu(P|_\gamma) \) does not exceed

\[
2^{2n-1} \sum_{k=1}^{n} [p + (k - 1)(q - 1)]^{2n}.
\]

(7)

**Proof.** From Lemma 1, there exists a deformation \( \tilde{P} \) of \( P \) satisfying conditions of Theorem 1, such that \( P^e \) is a polynomial of degree not exceeding \( p \). Hence degree of \( \xi^k \tilde{P}^e \) does not exceed \( p + k(q - 1) \). From [6], the sum of Betti numbers of the Milnor fiber \( V_k \) of \((\tilde{P}, \xi \tilde{P}, \ldots, \xi^{k-1} \tilde{P})\) does not exceed \((p + (k - 1)(q - 1))(2p + 2(k - 1)(q - 1) - 1)2^{n-1}\), which does not exceed \( 2^{2n-1} [p + (k - 1)(q - 1)]^{2n} \). The estimate (7) follows now from Theorem 1.

**Theorem 4.** Let \( P \) be a polynomial of degree not exceeding \( p \geq n - 1 \), and let \( \xi \) be a vector field with polynomial coefficients of degree not exceeding \( q \geq 1 \). Suppose that \( P|_\gamma \equiv 0 \), and let \( P_\nu \) be any sequence of polynomials of degree not exceeding \( p \) converging to \( P \). Then the number of isolated zeros of \( P_\nu|_\gamma \) converging to the origin as \( \nu \to \infty \) does not exceed (7).

**Proof.** This follows from Theorem 3 and the results of [10]. An alternative argument was suggested by Khovanskii. Let \( L \) denote the linear space of all polynomials of degree not exceeding \( p \) modulo polynomials identically vanishing on \( \gamma \). Let \( P_\nu \) be a sequence of polynomials \( P_\nu \) converging to \( P \) such that \( M \) zeroes of \( P_\nu|_\gamma \) converge to the origin as \( \nu \to \infty \). These polynomials define a sequence of points \( Q_\nu \) in \( L \). Note that the zeros of \( P_\nu|_\gamma \) depend only on \( Q_\nu \), and do not change when we multiply \( Q_\nu \) by a constant. If we define any norm in \( L \), we obtain a sequence of points \( Q_\nu/|Q_\nu| \) in \( L \) that has a non-zero limit point \( Q_0 \). Let \( P_0 \) be a polynomial of degree not exceeding \( p \) such that its image in \( L \) is \( Q_0 \). Obviously, \( P_0|_\gamma \) has a zero of the multiplicity \( M \) at 0. Hence \( M \) does not exceed (7).

**Theorem 5.** Let \( \Xi = \{\xi_i\} \) be a system of vector fields in \( \mathbb{C}^n \) or \( \mathbb{R}^n \) with polynomial coefficients of degree not exceeding \( q \geq 1 \). Let \( d = d(\Xi, 0) \) be dimension of the vector space \( L \) spanned by the values at the origin of the vector fields \( \xi_i \) and their Lie brackets of all orders. The degree of nonholonomy \( N = N(\Xi, 0) \), i.e., the minimal number \( N \) such
that the values at the origin of the vector fields $\xi_i$ and their Lie brackets of the order not exceeding $N$ generate $L$, does not exceed

$$2^{d-2} \left( 1 + 2^{2n(d-2)-2} q^{2n} \sum_{k=4}^{n+3} k^{2n} \right), \quad \text{for } d > 2,$$

$$1 + 2^{2n-1} q^{2n} \sum_{k=2}^{n+1} k^{2n}, \quad \text{for } d = 2.$$  \hfill (8)

\textbf{Proof.} According to [1], there exists a vector field $\xi$ with polynomial coefficients of degree not exceeding $2^d - 2q$ (for $d = 2$, not exceeding $q$) and a polynomial $P$ of degree not exceeding $2^{d-1}q$ such that:

(1) $\xi(0) \neq 0$;

(2) $P|_\gamma \neq 0$, where $\gamma$ is the trajectory of $\xi$ through 0;

(3) $N$ does not exceed $2^{d-2} + 2^{d-3}\mu$ (for $d = 2$, does not exceed $1 + \mu$) where $\mu$ is the multiplicity of a zero of $P|_\gamma$ at 0.

Applying this to the estimate (7) for $\mu$, we obtain (8) and (9).

\textbf{Definition 2.} (Khovanskii, unpublished; see [8].) A \textit{Noetherian chain} of order $m$ and degree $\alpha$ is a system $f(x) = (f_1(x), \ldots, f_m(x))$ of germs of analytic functions at the origin 0 of a complex or real $n$-dimensional space, satisfying

$$\frac{\partial f_i}{\partial x_j} = g_{ij}(x, f(x)), \quad \text{for } i = 1, \ldots, m \text{ and } j = 1, \ldots, n,$$  \hfill (10)

where $g_{ij}$ are polynomials in $x$ and $f$ of degree not exceeding $\alpha \geq 1$. A function $\phi(x) = P(x, f(x))$, where $P$ is a polynomial in $x$ and $f$ of degree not exceeding $p$, is called a \textit{Noetherian function} of degree $p$, with the Noetherian chain $f$.

The following two theorems can be reduced to Theorems 3 and 5 by adding $m$ new variables corresponding to the functions of the Noetherian chain (see [1]).

\textbf{Theorem 6.} Let $f = (f_1, \ldots, f_m)$ be a Noetherian chain of order $m$ and degree $\alpha$, and let $\xi = \sum_j \phi_j(x) \partial / \partial x_j$ be a vector field with the coefficients $\phi_j$ Noetherian of degree $q$, with the Noetherian chain $f$. Let $\psi$ be a Noetherian function of degree $p$, with the Noetherian chain $f$. Suppose that $\xi(0) \neq 0$ and that $\psi$ does not vanish identically on the trajectory $\gamma$ of $\xi$ through 0. Then the multiplicity of the zero of $\psi|_\gamma$ at 0 does not exceed

$$2^{2(n+m)-1} \sum_{k=1}^{n+m} [p + (k-1)(q+\alpha-1)]^{2(n+m)}.$$  \hfill (11)
Theorem 7. Let \( f = (f_1, \ldots, f_m) \) be a Noetherian chain in \( \mathbb{C}^n \) or \( \mathbb{R}^n \) of order \( m \) and degree \( \alpha \geq 1 \). Let \( \Xi = \{\xi_i\} \) be a set of vector fields with Noetherian coefficients:

\[
\xi_i = \sum_j Q_{ij}(x, f(x)) \frac{\partial}{\partial x_j}
\]

with \( Q_{ij} \) polynomial in \( x \) and \( f \) of degree not exceeding \( q \geq 1 \). Let \( d = d(\Xi, 0) \) be dimension of the vector space spanned by the values at the origin of the vector fields \( \xi_i \) and their Lie brackets of all orders. The degree of nonholonomy \( N(\Xi, 0) \) does not exceed

\[
2^{d-2} \left( 1 + 2^{2(n+m)(d-2)-2}(q + \alpha)^{2(n+m)} \sum_{k=4}^{n+m+3} k^{2(n+m)} \right), \quad \text{for } d > 2, \quad (12)
\]

\[
1 + 2^{2(n+m)-1}(q + \alpha)^{2(n+m)} \sum_{k=2}^{n+m+1} k^{2(n+m)}, \quad \text{for } d = 2. \quad (13)
\]

Remark. The “integration over Euler characteristic” arguments allow one to obtain an effective estimate on the multiplicity of an isolated intersection defined by Noetherian functions of degree \( p \) in \( n \) variables, with a Noetherian chain of order \( m \) and degree \( \alpha \), in terms of \( n, m, \alpha, \) and \( p \). The proof will appear in a joint paper of A. Khovanskii and the author.

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