On a remarkable semigroup of homomorphisms with respect to free multiplicative convolution

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Abstract

Let $\mathcal{M}$ denote the space of Borel probability measures on $\mathbb{R}$. For every $t \geq 0$ we consider the transformation $B_t : \mathcal{M} \to \mathcal{M}$ defined by

$$B_t(\mu) = \left( \mu \boxtimes (1+t) \right)^{1/(1+t)}, \quad \mu \in \mathcal{M},$$

where $\boxtimes$ and $\uplus$ are the operations of free additive convolution and respectively of Boolean convolution on $\mathcal{M}$, and where the convolution powers with respect to $\boxtimes$ and $\uplus$ are defined in the natural way. We show that $B_s \circ B_t = B_{s+t}$, $\forall s, t \geq 0$ and that, quite surprisingly, every $B_t$ is a homomorphism for the operation of free multiplicative convolution $\boxdot$ (that is, $B_t(\mu \boxdot \nu) = B_t(\mu) \boxdot B_t(\nu)$ for all $\mu, \nu \in \mathcal{M}$ such that at least one of $\mu, \nu$ is supported on $[0, \infty)$).

We prove that for $t = 1$ the transformation $B_1$ coincides with the canonical bijection $B : \mathcal{M} \to \mathcal{M}_{inf-div}$ discovered by Bercovici and Pata in their study of the relations between infinite divisibility in free and in Boolean probability. Here $\mathcal{M}_{inf-div}$ stands for the set of probability distributions in $\mathcal{M}$ which are infinitely divisible with respect to the operation $\boxtimes$. As a consequence, we have that $B_t(\mu)$ is $\boxtimes$-infinitely divisible for every $\mu \in \mathcal{M}$ and every $t \geq 1$.

On the other hand we put into evidence a relation between the transformations $B_t$ and the free Brownian motion; indeed, Theorem 4 of the paper gives an interpretation of the transformations $B_t$ as a way of re-casting the free Brownian motion, where the resulting process becomes multiplicative with respect to $\boxdot$, and always reaches $\boxtimes$-infinite divisibility by the time $t = 1$.

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1 Introduction

1.1 The transformations $B_t$.

In this paper we put into evidence a remarkable semigroup of homomorphisms of $(\mathcal{M}_+, \boxtimes)$, where $\mathcal{M}_+$ is the space of probability measures on $[0, \infty)$, and $\boxtimes$ is free multiplicative convolution – the operation with measures from $\mathcal{M}_+$ which corresponds to the multiplication of free positive random variables (for a basic introduction to the ideas of free harmonic analysis, see for instance Chapter 3 of [23]).

For $\mu \in \mathcal{M}_+$ and $n \in \mathbb{N}$, the $n$-fold convolution $\mu \boxtimes \mu \boxtimes \cdots \boxtimes \mu$ is denoted by $\mu^{\boxtimes n}$. It turns out ([5], Section 2) that for every $\mu \in \mathcal{M}_+$, the measures $\mu^{\boxtimes n}$ can be incorporated in a family $\{\mu^{\boxtimes t} | t \in [1, \infty)\}$ such that $(\mu^{\boxtimes s}) \boxtimes (\mu^{\boxtimes t}) = \mu^{\boxtimes (s+t)}$, for all $s, t \geq 1$. In particular, the extended family of $\boxtimes$-powers provides us with a continuous semigroup of $\boxtimes$-homomorphisms for $\mathcal{M}_+$, consisting of the maps $\mathcal{M}_+ \ni \mu \mapsto \mu^{\boxtimes t} \in \mathcal{M}_+$, $t \geq 1$.

In this paper we show that, quite surprisingly, there is a simple formula which defines another semigroup of homomorphisms for $\boxtimes$, by using powers of two additive convolutions on $\mathcal{M}_+$: the free additive convolution $\boxplus$, and the Boolean convolution $\uplus$. (A brief review of $\boxplus$ and of $\uplus$ is made in Section 2 below.) This other semigroup of $\boxtimes$-homomorphisms consists of the maps

$$\mathcal{M}_+ \ni \mu \mapsto \left( \mu^{\boxplus (1+t)} \right)^{\omega(1/(1+t))} \in \mathcal{M}_+, \quad t \geq 0. \quad (1.1)$$

The formula shown in (1.1) actually makes sense when $\mu$ belongs to the larger space $\mathcal{M}$ of all probability measures on $\mathbb{R}$ (without requiring that $\mu$ is supported on $[0, \infty)$). When moving to $\mathcal{M}$, one gets the issue that the convolution $\mu \boxtimes \nu$ isn’t generally defined for arbitrary $\mu, \nu \in \mathcal{M}$. However, it is still possible to define $\mu \boxtimes \nu$ when $\mu \in \mathcal{M}$ and $\nu \in \mathcal{M}_+$, and we have the following theorem, which contains in particular the fact that the maps in (1.1) form a semigroup of $\boxtimes$-homomorphisms of $\mathcal{M}_+$.

**Theorem 1.** For every $t \geq 0$ one can define a one-to-one map $B_t : \mathcal{M} \to \mathcal{M}$ by

$$B_t(\mu) = \left( \mu^{\boxplus (1+t)} \right)^{\omega(1/(1+t))}, \quad \mu \in \mathcal{M}. \quad (1.2)$$

Every $B_t$ is continuous with respect to the weak topology on $\mathcal{M}$, and carries $\mathcal{M}_+$ into itself. Moreover, the maps $\{B_t | t \geq 0\}$ satisfy:

$$B_s \circ B_t = B_{s+t}, \quad \forall s, t \geq 0, \quad (1.3)$$

and for every $t \geq 0$ we have that

$$B_t(\mu \boxtimes \nu) = B_t(\mu) \boxtimes B_t(\nu), \quad \forall \mu \in \mathcal{M}, \nu \in \mathcal{M}_+. \quad (1.4)$$

The semigroup of transformations $\{B_t | t \geq 0\}$ has two other interesting features, which are discussed in the next subsections 1.2 and 1.3.
1.2 Relation to ⊠-infinite divisibility.

A probability measure \( \mu \in \mathcal{M} \) is said to be \( \text{infinitely divisible} \) with respect to \( \boxplus \) if for every \( n \geq 1 \) there exists \( \mu_n \in \mathcal{M} \) such that \( \mu \boxplus \mu_n = \mu \); we will denote the set of probability measures which have this property by \( \mathcal{M}_{\text{inf-div}} \). In terms of convolution powers with respect to \( \boxplus \), the fact that \( \mu \in \mathcal{M}_{\text{inf-div}} \) amounts to saying that \( \mu \boxplus t \) can be defined for every \( t > 0 \) (in contrast with the situation of an arbitrary probability measure \( \mu \in \mathcal{M} \), for which the powers \( \mu \boxplus t \) can in general be defined only for \( t \geq 1 \)). Infinite divisibility in free sense has a well-developed theory – see section 2.11 of the survey [22]. An aspect of this theory which is of particular relevance for this paper is a special bijection

\[
\mathbb{B} : \mathcal{M} \to \mathcal{M}_{\text{inf-div}},
\]

found by Bercovici and Pata ([9], Section 6), in connection to their parallel study of infinite divisibility with respect to \( \boxplus \) and to \( \boxdot \). We will refer to \( \mathbb{B} \) as the \textit{Boolean Bercovici-Pata bijection}. The transformations \( \mathbb{B}_t \) from our Theorem 1 connect to this as follows.

**Theorem 2.** We have

\[
\mathbb{B}_1(\mu) = \mathbb{B}(\mu), \quad \forall \mu \in \mathcal{M},
\]

where the transformation \( \mathbb{B}_1 : \mathcal{M} \to \mathcal{M} \) is defined as in Theorem 1 (by setting \( t = 1 \) there), and \( \mathbb{B} \) is the Boolean Bercovici-Pata bijection.

A consequence of Theorem 2 and of Equation (1.5) from Theorem 1 is that the Boolean Bercovici-Pata bijection is multiplicative with respect to \( \boxdot \). This phenomenon was observed, via combinatorial methods, to also hold for a multi-variable generalization of the Boolean Bercovici-Pata bijection which was recently studied in [7].

Another consequence of Theorem 2, in conjunction with the semigroup property (1.3) from Theorem 1 is that for \( t \geq 1 \) we have \( \mathbb{B}_t(\mathcal{M}) = \mathbb{B}(\mathbb{B}_{t-1}(\mathcal{M})) \subseteq \mathcal{M}_{\text{inf-div}} \). That is, we have the following corollary.

**Corollary.** The probability measure \( \mathbb{B}_t(\mu) \) is infinitely divisible with respect to \( \boxplus \), for every \( t \geq 1 \) and every \( \mu \in \mathcal{M} \).

The statement of the above corollary can be sharpened by introducing a numerical quantity, defined as follows.

**Definition.** For \( \mu \in \mathcal{M} \) we denote

\[
\phi(\mu) := \sup \{ t \in [0, \infty) \mid \mu \in \mathbb{B}_t(\mathcal{M}) \} \in [0, \infty].
\]

We will call \( \phi(\mu) \) the \( \boxplus \)-divisibility indicator of \( \mu \).

In terms of the \( \boxdot \)-divisibility indicator, the statement of the preceding corollary gets translated into the fact that for \( \mu \in \mathcal{M} \) we have the equivalence

\[
\mu \in \mathcal{M}_{\text{inf-div}} \iff \phi(\mu) \geq 1
\]

(see Proposition 5.3 below). Thus if \( \mu \) has \( \phi(\mu) \geq 1 \), then we are sure we can consider the \( \boxplus \)-power \( \mu^{\boxplus t} \) for every \( t > 0 \). On the other hand, if \( \mu \in \mathcal{M} \) has \( 0 < \phi(\mu) < 1 \) (and thus does
not belong to $\mathcal{M}_{\text{inf-div}}$), then we can still take some subunitary $\boxplus$-powers of $\mu$ — namely $\mu \boxplus t$ for every $t \geq 1 - \phi(\mu)$; see Remark 5.5.3 below.

The values of $\phi(\mu)$ for a few distributions of importance in free probability are listed in the next table.

| Distribution $\mu$                                      | $\phi(\mu)$ | Distribution $\mu$                                      | $\phi(\mu)$ |
|--------------------------------------------------------|--------------|--------------------------------------------------------|--------------|
| Symmetric Bernoulli distribution, $\mu = \frac{1}{2}(\delta_{-1} + \delta_1)$ | 0            | Marchenko-Pastur distribution of parameter 1,           | 1            |
|                                                         |              | $d\mu(x) = \frac{1}{2\pi} \sqrt{(4-x)/x} \, dx$ on $[0,4]$ |              |
| Arcsine law of variance 1, $d\mu(x) = \frac{1}{\pi\sqrt{2-x^2}} \, dx$ on $[-\sqrt{2},\sqrt{2}]$ | $1/2$        | Cauchy distribution, $d\mu(x) = \frac{1}{\pi(x^2+1)} \, dx$ | $\infty$     |
| Standard semicircular distribution, $d\mu(x) = \frac{1}{2\pi} \sqrt{4-x^2} \, dx$ on $[-2,2]$ | 1            |                                                        |              |

Table 1. $\phi(\mu)$ for a few important distributions $\mu$.

1.3 Relation to free Brownian motion, and to the complex Burgers equation.

For a probability measure $\mu \in \mathcal{M}$, we will use the notation $F_\mu$ for the reciprocal Cauchy transform of $\mu$. That is, $F_\mu$ is the analytic self-map of the upper half-plane $\mathbb{C}^+$ defined as

$$F_\mu(z) := \frac{1}{G_\mu(z)}, \quad \forall z \in \mathbb{C}^+,$$

where $G_\mu$ is the Cauchy transform of $\mu$ (a brief review of $F_\mu$ and of some of its basic properties appears in Section 2.2 below).

**Theorem 3.** Let $\mu$ be in $\mathcal{M}$, and consider the function $h : (0, \infty) \times \mathbb{C}^+ \rightarrow \mathbb{C}$ defined by

$$h(t, z) = F_{B_t(\mu)}(z) - z, \quad \forall t > 0, \quad \forall z \in \mathbb{C}^+. \quad (1.9)$$

Then $h$ satisfies the complex Burgers equation,

$$\frac{\partial h}{\partial t}(t, z) = h(t, z) \frac{\partial h}{\partial z}(t, z), \quad t > 0, \quad z \in \mathbb{C}^+. \quad (1.10)$$

The complex Burgers equation has previously appeared in free harmonic analysis in the work of Voiculescu [21], in connection to the free Brownian motion started at a probability
measure \( \nu \in \mathcal{M} \) – that is, in connection to the family of measures \( \{ \nu \boxplus \gamma_t \mid t > 0 \} \), where \( \gamma_t \) is the semicircular distribution of variance \( t \). More precisely, when one considers the Cauchy transforms of the measures in this family, it turns out that the function

\[
(0, \infty) \times \mathbb{C}^+ \ni (t, z) \mapsto -G_{\nu \boxplus \gamma_t}(z) \in \mathbb{C}
\]

satisfies the complex Burgers equation (in exactly the form stated above in Equation (1.10)). These two occurrences of the complex Burgers equation (for the functions in (1.9) and (1.11)) are in fact connected to each other, in the way described as follows.

**Theorem 4.** Let \( \nu \) be a probability measure in \( \mathcal{M} \), and consider the analytic function \( -G_\nu \) from \( \mathbb{C}^+ \) to itself. Then there exists a unique probability measure \( \mu \in \mathcal{M} \) such that

\[
- G_\nu(z) = F_\mu(z) - z, \quad z \in \mathbb{C}^+.
\]

Moreover, the relation (1.12) between \( \mu \) and \( \nu \) is not affected when \( \nu \) evolves via the free Brownian motion, while \( \mu \) evolves under the action of the transformations \( \mathbb{B}_t \); that is, we have that

\[
- G_{\nu \boxplus \gamma_t}(z) = F_{\mathbb{B}_t(\mu)}(z) - z, \quad \forall t > 0, \quad \forall z \in \mathbb{C}^+.
\]

Clearly, Theorem 3 would follow from Theorem 4 and the corresponding result for the free Brownian motion, if it were true that every analytic function from the collection \( \{ F_\mu(z) - z \mid \mu \in \mathcal{M} \} \) can be put in the form \( -G_\nu \) for some \( \nu \in \mathcal{M} \); but this is not the case – see Remark 4.3 below. Nevertheless, Theorem 4 gives an interesting interpretation of the transformations \( \mathbb{B}_t \), as a way of re-casting the free Brownian motion where the resulting process becomes multiplicative with respect to \( \boxplus \).

### 1.4 Further remarks, and organization of the paper.

In [8], using combinatorial methods, we have found multi-variable analogues to several results we present in this paper. Most notably, [8] provides a multi-variable analogue of Theorem 4, and an operatorial model for the correspondence described in this theorem. However, the methods of [8] have the important draw-back that they only apply to distributions with compact support. All the results of the present paper use complex analytic tools, and apply to arbitrary Borel probability measures on the real line.

Let us now give a brief outline of the paper’s organization: After a review of background material in Section 2, we give the proofs of Theorems 1 and 2 in Section 3. What stands behind Theorem 1 are two interesting connections which the Boolean convolution \( \boxplus \) turns out to have with the operations \( \boxtimes \) and \( \boxdot \) from free probability. The first connection is a kind of *commutation relation* between the convolution powers with respect to \( \boxplus \) and to \( \boxtimes \): a probability measure of the form \((\mu \boxplus \nu)^{p \boxtimes q}\) can also be written as \((\mu^{p \boxtimes q}) \boxdot \nu'\) for some new exponents \( p', q' \), where \( p', q' \) are given explicitly in terms of \( p \) and \( q \) – see Proposition 3.1 below. The second connection is a kind of “distributivity” which involves \( \boxdot \) and a fixed convolution power with respect to \( \boxplus \):

\[
(\mu^{\nu_{\boxplus t}}) \boxdot (\nu'_{\boxplus t}) = (\mu \boxtimes \nu)^{\nu_{\boxplus t}} \circ D_{1/t}, \quad \forall t > 0, \quad \forall \mu, \nu \in \mathcal{M}.
\]

In (1.14), “\( D_{1/t} \)” stands for the natural operation of dilation (by a factor of \( 1/t \)) for probability measures on \( \mathbb{R} \). Together with an analogous distributivity formula involving \( \boxdot \) and
a convolution power of $⊞$, this relation explains why each of the transformations $B_t$ from Theorem 1 is a homomorphism with respect to $⊞$. See Proposition 3.5 and Remark 3.7 below.

In Section 3 we also point out that the transformations $B_t$ can be very nicely described by using Voiculescu’s $S$-transform (see Remark 3.9). Theorem 2 can be easily proved as an application of this (see Remark 3.10).

Section 4 is devoted to the relation with the free Brownian motion, and to the proofs of Theorems 3 and 4. A very nice example of process $\{B_t(\mu) \mid t \geq 0\}$ which can be described explicitly by using Theorem 4 is the one started at the symmetric Bernoulli distribution; this process turns out to go both through the arcsine law and through the standard semicircle law (see Example 4.5).

Finally, in Section 5 we discuss a few miscellaneous facts related to the transformations $B_t$. In Proposition 5.1 and Corollary 5.2 we describe atoms and regularity for the measures $B_t(\mu)$ and the convergence of $B_t(\mu)$ when $t$ tends to zero. In Remarks 5.5 and 5.6 we prove some basic properties of the $⊞$-divisibility indicator $\phi(\mu)$ introduced in Section 1.2, including the verification for the values of $\phi(\mu)$ that were listed in Table 1.

### 2 Background and notations

**2.1 The convolution operations $⊞, ⊠, ⊎$.**

Same as in the introduction, we use the notation $M$ for the set of Borel probability measures on $\mathbb{R}$, and the notation $M_+$ for the set of all probability measures $\mu \in M$ with the property that $\mu([0, +\infty)) = 1$.

In the literature on non-commutative probability one encounters several “convolution” operations for probability distributions in $M$, which are defined to reflect operations with non-commutative random variables.

The operations $⊞$ and $⊛$ are from free probability theory. They are defined in order to reflect the addition and respectively the multiplication of free random variables; we refer to Chapter 3 of [23] or Chapters 2, 3 in [22] for a precise description of how this goes. In this paper we will not pursue the approach to $⊞$ and $⊛$ in terms of free random variables, but we will rather use the analytic function theory developed in order to deal (and do computations) with these operations. In particular, in the next subsection 2.2 we will review the “transforms” that are mainly used to study these operations: the $R$-transform for $⊞$, and the $S$-transform for $⊛$.

There is a third convolution operation which appears in this paper, the Boolean additive convolution $\oplus$. This comes from the world of random variables that are “Boolean independent”, and reflects the addition of such variables. One of the points emphasized by this paper is that $\oplus$ has nice relations with $⊞$ and $⊛$, and, surprisingly, has a role to play in free probability, in connection to infinite divisibility with respect to $⊞$. In the next subsection we will also review how $\oplus$ is handled by using complex function theory – the analytic function theory for $\oplus$ is in fact simpler than the one required for $⊞$ and $⊛$.

**2.2 A glossary of transforms.**

Here we collect some basic formulas concerning the various kinds of transforms used in non-commutative probability, and which appear in this paper. We are not aiming to
a self-contained presentation of the transforms, for the most part we will only state the formulas and properties that we need, and indicate references for them.

1° Cauchy transform and reciprocal Cauchy transform.

The Cauchy transform of a probability measure \( \mu \in \mathcal{M} \) is the analytic function \( G_\mu \) defined by

\[
G_\mu(z) = \int_{\mathbb{R}} \frac{d\mu(s)}{z-s}, \quad z \in \mathbb{C} \setminus \mathbb{R}.
\]  

(2.1)

The reciprocal Cauchy transform \( F_\mu \) is defined by

\[
F_\mu(z) = \frac{1}{G_\mu(z)}, \quad z \in \mathbb{C} \setminus \mathbb{R}.
\]  

(2.2)

It can be easily checked that \( G_\mu \) maps \( \mathbb{C}^+ \) to \( \mathbb{C}^- \); as a consequence of this, \( F_\mu \) can be viewed as an analytic self-map of \( \mathbb{C}^+ \).

We will denote by \( \mathcal{F} \) the set of analytic self-maps of \( \mathbb{C}^+ \) which can appear as \( F_\mu \) for some \( \mu \in \mathcal{M} \). One has a very nice intrinsic description for the functions in \( \mathcal{F} \), namely

\[
\mathcal{F} = \left\{ F : \mathbb{C}^+ \to \mathbb{C}^+ \mid F \text{ is analytic and } \lim_{y \to \infty} \frac{F(iy)}{iy} = 1 \right\}.
\]  

(2.3)

For a function \( F \in \mathcal{F} \), the limit \( \lim F(z)/z = 1 \) holds in fact under the weaker condition that \( z \) converges non-tangentially to \( \infty \) (i.e. \( z \to \infty \) in an angular domain of the form \( \{ z \in \mathbb{C}^+ \mid |\Re(z)| < c \cdot |\Im(z)| \} \), for some \( c > 0 \).

Another fact worth recording is that a function \( F \in \mathcal{F} \) always increases the imaginary part:

\[ \Im F_\rho(z) \geq \Im z \text{ for all } z \in \mathbb{C}^+. \]

Moreover, if there exists \( z_0 \in \mathbb{C}^+ \) such that \( \Im F_\rho(z_0) = \Im z_0 \), then the equality holds for all \( z \in \mathbb{C}^+ \) and \( \rho \) is a point mass. Proofs of all these facts can be found in \([1]\). For a nice review, one can also consult Section 2 in Maassen’s paper \([16]\) or Section 5 of \([10]\).

2° R-transform.

Let \( \mu \) be a probability measure in \( \mathcal{M} \), and consider its Cauchy transform \( G_\mu \). It can be proved \(([19],[23],[10])\) that the composition inverse \( G_\mu^{-1} \) of \( G_\mu \) is defined on a truncated angular domain of the form

\[
\left\{ z \in \mathbb{C}^+ \mid |\Re(z)| < c \cdot |\Im(z)|, \ |z| > M \right\},
\]

for some \( c, M > 0 \). For \( z \) in this domain we define

\[
R_\mu(z) = G_\mu^{-1}(z) - \frac{1}{z}, \quad \text{and} \quad R_\mu(z) = zR_\mu(z).
\]

The function \( R_\mu \) is called the R-transform of \( \mu \). (Note that in the free probability literature the function \( R_\mu \) also goes under the same name, of R-transform of \( \mu \); in this paper we will only work with \( R_\mu \).

The R-transform is the linearizing transform for the operation \( \boxplus \). That is, for every \( \mu, \nu \in \mathcal{M} \) we have

\[
R_{\mu \boxplus \nu}(z) = R_\mu(z) + R_\nu(z),
\]  

(2.4)

with \( z \) running in truncated angular domain where all of \( R_\mu(z), R_\nu(z) \) and \( R_{\mu \boxplus \nu}(z) \) are defined.
3° $S$-transform.

Let $\mu \in \mathcal{M}$ be a probability measure in $\mathcal{M}$ which has compact support and has the first moment $\int t \, d\mu(t)$ different from 0. Consider the moment generating series of $\mu$,

$$\psi_\mu(z) = \sum_{n=1}^{\infty} m_n z^n,$$

deﬁned on a neighborhood of 0, and where for every $n \geq 1$ we denote $m_n = \int t^n \, d\mu(t)$ (the moment of order $n$ of $\mu$). Then $\psi_\mu$ is invertible under composition on a suﬃciently small disc centered at 0, and it makes sense to deﬁne

$$S_\mu(z) = \frac{z + 1}{z} \psi_\mu^{-1}(z), \quad \text{for } |z| \text{ sufﬁciently small.} \quad (2.5)$$

$S_\mu$ is called the $S$-transform of $\mu$.

An equivalent way of deﬁning the $S$-transform is by relating it directly to the $R$-transform $R_\mu$. Indeed, for $\mu$ as above it is easily seen that $R_\mu$ is deﬁned and invertible under composition on a small disc centered at 0, and it turns out that we have

$$S_\mu(z) = \frac{1}{z} R_\mu^{-1}(z), \quad (2.6)$$

again holding for $z \in \mathbb{C}$ such that $|z|$ is suﬃciently small.

It can be proved ([20]) that the $S$-transform is multiplicative in for $\boxplus$, in the sense that if $\mu \in \mathcal{M}$ and $\nu \in \mathcal{M}_+$ have compact support and the ﬁrst moment diﬀerent from zero, then

$$S_{\mu \boxplus \nu}(z) = S_\mu(z)S_\nu(z). \quad (2.7)$$

4° $\eta$-transform.

For a probability measure $\mu \in \mathcal{M}$ we denote

$$\psi_\mu(z) = \int_{\mathbb{R}} \frac{s z}{1 - s z} d\mu(s), \quad z \in \mathbb{C} \setminus \mathbb{R}. \quad (2.8)$$

We will refer to $\psi_\mu(z)$ as the moment generating function of $\mu$. (In the case when $\mu$ has compact support, it is easily seen that the integral formula given in (2.8) matches the series expansion, also denoted by $\psi_\mu$, which appeared in the above discussion about the $S$-transform.)

The $\eta$-transform of $\mu$ is then deﬁned in terms of the moment generating function $\psi_\mu$ by the formula

$$\eta_\mu(z) = \frac{\psi_\mu(z)}{1 + \psi_\mu(z)}, \quad z \in \mathbb{C} \setminus \mathbb{R}. \quad (2.9)$$

Note that the denominator of the fraction on the right-hand side of Equation (2.9) is always diﬀerent from zero, due to the fact (immediately seen from the deﬁnitions) that we have

$$1 + \psi_\mu(z) = \frac{1}{z} G_\mu(1/z) \neq 0, \quad \forall z \in \mathbb{C} \setminus \mathbb{R}. \quad (2.10)$$

Let us also record here that $\eta_\mu$ can be directly obtained from the reciprocal Cauchy transform $F_\mu$ via the formula

$$\eta_\mu(z) = 1 - z F_\mu(1/z), \quad z \in \mathbb{C} \setminus \mathbb{R}. \quad (2.11)$$
(which is obtained by combining the preceding two Equations (2.9) and (2.10)).

The $\eta$-transform plays a role in the study of the Boolean convolution $\triangledown$. Indeed, it is known [18] that $\triangledown$ can be neatly described by using reciprocal Cauchy transforms: for $\mu, \nu \in \mathcal{M}$ we have

$$F_{\mu \triangledown \nu}(z) = F_\mu(z) + F_\nu(z) - z, \quad \forall z \in \mathbb{C} \setminus \mathbb{R}. \quad (2.12)$$

This amounts to saying that "$F_\mu(z) - z$ is a linearizing transform for $\triangledown$" (in the sense that the function "$F(z) - z$" calculated for $\mu \triangledown \nu$ is the sum of the corresponding functions calculated for $\mu$ and $\nu$). But now, thanks to the direct connection between $\eta$-transform and reciprocal Cauchy transform which was recorded in (2.11), one immediately sees that the latter linearization formula (2.12) is equivalent to

$$\eta_{\mu \triangledown \nu}(z) = \eta_\mu(z) + \eta_\nu(z), \quad \forall \mu, \nu \in \mathcal{M}, \forall z \in \mathbb{C}^+. \quad (2.13)$$

Thus we can conclude that the $\eta$-transform also is "a linearizing transform for $\triangledown$".

5° $\Sigma$-transform.

By comparing the above Equations (2.14) and (2.13), one could say that the $\eta$-transform is an analogue of the $R$-transform, living in the parallel world of Boolean probability. But the $\eta$-transform also has a direct connection (which is more than a "Boolean vs. free" analogy) with the $R$-transform. This connection is in terms of the composition inverses for $R$ and $\eta$, and is best put into evidence by introducing the $\Sigma$-transform of $\mu$.

Let us assume that $\mu \in \mathcal{M}$ has compact support and has the first moment different from 0. It is easily seen that in this case the $\eta$-transform $\eta_\mu$ is defined and invertible under composition on a small disc centered at 0, and it thus makes sense to define the $\Sigma$-transform $\Sigma_\mu$ of $\mu$ by the formula

$$\Sigma_\mu(z) = \frac{1}{z} \eta_\mu^{-1}(z), \text{ for } |z| \text{ sufficiently small.} \quad (2.14)$$

Note that this is very similar to the formula used when one defines the $S$-transform in terms of the $R$-transform (see Equation (2.6) above). But more than having an analogy between how $S_\mu$ and $\Sigma_\mu$ are defined, it turns out that we have a direct relation between them, namely:

$$\Sigma_\mu(z) = S_\mu \left( \frac{z}{1 - z} \right), \text{ for } |z| \text{ sufficiently small.} \quad (2.15)$$

Equation (2.15) is obtained by taking inverses under composition in Equation (2.9) (this gives us that $\eta_\mu^{-1}(z) = \psi^{-1}_\mu(z/(1-z))$, for $|z|$ sufficiently small), and then by invoking the formulas (2.3) and (2.14) which were used to define $S_\mu$ and respectively $\Sigma_\mu$.

From Equation (2.15) it is immediate that the $\Sigma$-transform has a multiplicativity property with respect to $\triangledown$, analogous to the one enjoyed by the $S$-transform:

$$\Sigma_{\mu \triangledown \nu}(z) = \Sigma_\mu(z) \Sigma_\nu(z), \quad (2.16)$$

whenever $\mu \in \mathcal{M}$ and $\nu \in \mathcal{M}_+$ have compact support and have first moment different from zero.

2.3 Remark. One should keep in mind that a probability measure $\mu \in \mathcal{M}$ is uniquely determined by any of the transforms ($G_\mu, F_\mu, R_\mu, \ldots$) reviewed in the above glossary, and
which are defined for \( \mu \). So, for instance, if \( \mu, \nu \in \mathcal{M} \) are such that \( F_\mu = F_\nu \), then it follows that \( \mu = \nu \). Or: if \( \mu, \nu \in \mathcal{M} \) have compact support and first moment different from 0, and if we know that \( S_\mu \) and \( S_\nu \) coincide on a neighborhood of 0, then we can infer that \( \mu = \nu \). The reason for this is that each of the transforms considered in the above glossary determines the Cauchy transform of the measure; and a measure \( \mu \in \mathcal{M} \) can always be retrieved from its Cauchy transform \( G_\mu \), by a procedure called “the Stieltjes inversion formula” (see e.g. [1]).

In the remaining part of this section we review some facts about convolution powers and infinite divisibility that are used later on in the paper.

### 2.4 Convolution powers with respect to \( \oplus, \odot \).

1° For \( \mu \in \mathcal{M} \) and a positive integer \( n \), one denotes the \( n \)-fold convolution \( \mu \odot \cdots \odot \mu \) by \( \mu^{\otimes n} \). The probability measure \( \mu^{\otimes n} \) is very nicely characterized in terms of \( R \)-transforms, via the formula \( R_{\mu^{\otimes n}}(z) = n \cdot R_\mu(z) \). It turns out that the latter formula can be extended to the case when \( n \) is not an integer. More precisely, for every \( \mu \in \mathcal{M} \) and \( t \in [1, \infty) \) there exists a probability measure \( \mu_t \in \mathcal{M} \) so that

\[
R_{\mu_t}(z) = t R_\mu(z),
\]

with \( z \) running in a truncated angular domain where both sides of the equation are defined. The existence of \( \mu_t \) was first observed in [17] in the case when \( \mu \) has compact support, and then extended to arbitrary \( \mu \in \mathcal{M} \) in [4].

The measure \( \mu_t \) appearing in (2.17) (which is uniquely determined by the prescription for the \( R \)-transform \( R_{\mu_t} \)) is called the \( t \)-th convolution power of \( \mu \) with respect to \( \odot \), and denoted by \( \mu^{\odot t} \). It is immediate that we have

\[
\mu^{\odot t} \odot \mu^{\odot s} = \mu^{\odot t+s}, \quad \forall s, t \in [1, +\infty).
\]

In the following sections we shall use some other properties of the \( \bigodot \)-convolution powers. It is an immediate consequence of the operatorial realization of \( \mu^{\odot t} \) given in [17] that if \( \mu \in \mathcal{M} \) has compact support, then so does \( \mu^{\odot t} \) for any \( t \geq 1 \), and if the support of \( \mu \) is included in \([0, +\infty)\), then so is the support of \( \mu^{\odot t} \) for any \( t \geq 1 \). The operation of taking convolution powers with respect to \( \bigodot \) is well behaved with respect to the usual topologies on \( \mathcal{M} \) and the real line: it follows from equation (2.17) and the characterization of continuity in terms of the \( R \)-transform given in [10] that the correspondence \( \mu \mapsto \mu^{\odot t} \) is continuous in the weak topology for any \( t \), and the correspondence \([1, +\infty) \ni t \mapsto \mu^{\odot t} \in \mathcal{M} \) is continuous for every \( \mu \in \mathcal{M} \), where \([1, +\infty) \) is considered with the usual topology and \( \mathcal{M} \) is endowed with the weak topology. As a consequence of the two properties above, we easily see that if \( \mu \in \mathcal{M}_+ \) then \( \mu^{\odot t} \in \mathcal{M}_+, \forall t \geq 1 \).

2° We now do the same kind of discussion as above, but in connection to the operation of Boolean convolution \( \forall \). For \( \mu \in \mathcal{M} \) and a positive integer \( n \), one denotes the \( n \)-fold convolution \( \mu \forall \cdots \forall \mu \) by \( \mu^{\forall n} \). From the discussion about the linearizing of \( \forall \) (see Equation (2.12) in the glossary of transforms) we see that \( \mu^{\forall n} \) is nicely characterized in terms of its reciprocal Cauchy transform, via the formula

\[
F_{\mu^{\forall n}}(z) = nF_\mu(z) + (1 - n)z, \quad z \in \mathbb{C}^+.
\]

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This formula can be extended to the case when the exponent \( n \) is not an integer; in fact, it turns out that the convolution power \( \mu^{\otimes t} \) can be defined for every \( \mu \in \mathcal{M} \) and every \( t > 0 \) (we no longer have the restriction “\( t \geq 1 \)” which we had in the above discussion about \( \boxplus \)). That is, for every \( \mu \in \mathcal{M} \) and every \( t > 0 \), the \( \otimes \)-convolution power \( \mu^{\otimes t} \) is defined [18] as the unique probability measure in \( \mathcal{M} \) whose reciprocal Cauchy transform satisfies

\[
F_{\mu^{\otimes t}}(z) = tF_{\mu}(z) + (1 - t)z, \quad z \in \mathbb{C}^+.
\] (2.18)

It follows easily from properties of the functions \( F_{\mu} \) that convolution powers with respect to \( \boxplus \) enjoy the same properties as the ones mentioned for \( \boxplus \) at the end of part 1° of this remark.

2.5 Infinite divisibility with respect to \( \boxplus \), \( \otimes \).

1° A probability measure \( \mu \in \mathcal{M} \) is said to be infinitely divisible with respect to \( \boxplus \) if for every \( n \geq 1 \) there exists \( \mu_n \in \mathcal{M} \) such that \( \mu_n^{\boxplus n} = \mu \). The set of probability measures \( \mu \in \mathcal{M} \) which are \( \boxplus \)-infinitely divisible will be denoted in this paper by \( \mathcal{M}_{\text{inf-div}} \). It is easily seen that for a measure \( \mu \in \mathcal{M}_{\text{inf-div}} \) one can define the \( \boxplus \)-convolution powers \( \mu^{\boxplus t} \) in exactly the same way as described in Equation (2.17), and where now \( t \) can be an arbitrary number in \((0, \infty)\). (And conversely, if \( \mu \in \mathcal{M} \) has the property that \( \mu^{\boxplus t} \) is defined for all \( t > 0 \), then it is immediate that \( \mu \in \mathcal{M}_{\text{inf-div}} \).

Infinite divisibility with respect to \( \boxplus \) was first studied in [19], then in [10]. It was observed there that \( \boxplus \)-infinite divisibility is very nicely described in terms of the \( R \)-transform: given \( \mu \in \mathcal{M} \), one has that \( \mu \) is \( \boxplus \)-infinitely divisible if and only if the \( R \)-transform \( R_{\mu} \) can be extended analytically to all of \( \mathbb{C} \setminus \mathbb{R} \).

2° In the same vein as above, one could consider the parallel concept of infinite divisibility with respect to \( \otimes \). But here the situation turns out to be much simpler. Indeed, as already pointed out in part 2° of the preceding subsection, we have that every \( \mu \in \mathcal{M} \) is infinitely divisible with respect to \( \otimes \). (In particular, no new notation is needed for the set of measures in \( \mathcal{M} \) which are \( \otimes \)-infinitely divisible.)

2.6 The Boolean Bercovici-Pata bijection.

In the paper [9], Bercovici and Pata have proved the existence of a strong connection between free, Boolean, and classical infinite divisibility. We reproduce here the result from [9] which is relevant for the present paper. Let \( (\mu_n)_{n=1}^{\infty} \) be a sequence in \( \mathcal{M} \), and let \( k_1 < k_2 < \cdots < k_n < \cdots \) be a sequence of positive integers. Then (as proved in [9], Theorem 6.3) the following statements are equivalent:

1. The sequence \( \mu_n \boxplus \mu_n \boxplus \cdots \boxplus \mu_n \) converges weakly to a probability measure \( \nu \in \mathcal{M} \).

2. The sequence \( \mu_n \otimes \mu_n \otimes \cdots \otimes \mu_n \) converges weakly to a probability measure \( \mu \in \mathcal{M} \).

Moreover, suppose that the statements (1) and (2) are both true. Then the limit \( \nu \) from (1) is \( \boxplus \)-infinitely divisible, and we have the following relation between \( \mu \) and \( \nu \):

\[
z - F_{\nu}(z) = zR_{\nu}(1/z), \quad z \in \mathbb{C} \setminus \mathbb{R}.
\] (2.19)
Note that the right-hand side of the above equation makes indeed sense because, as mentioned in the preceding subsection, the $R$-transform of a $⊞$-infinitely divisible distribution extends analytically to all of $\mathbb{C} \setminus \mathbb{R}$.

And finally (this is also part of Theorem 6.3 in [9]), the correspondence $\mu \mapsto \nu$ with $\mu, \nu$ as in Equation (2.19) is a bijection between $\mathcal{M}$ and $\mathcal{M}_{\text{inf-div}}$. This correspondence will be called the **Boolean Bercovici-Pata bijection**, and will be denoted by $\mathbb{B}$.

Let us record here that an alternative form of the Equation (2.19) describing the Boolean Bercovici-Pata bijection is

$$R_{\mathbb{B}(\mu)}(z) = \eta_{\mu}(z), \quad z \in \mathbb{C} \setminus \mathbb{R}.$$  \hspace{1cm} (2.20)

This is obtained by replacing $z$ with $1/z$ in (2.19), and by invoking the formula (2.11) which connects the $\eta$-transform to the reciprocal Cauchy transform.

Another useful reformulation of the Equation (2.19) describing the bijection $\mathbb{B}$ is obtained by using the $S$-transform and the $\Sigma$-transform. Let us suppose that $\mu \in \mathcal{M}$ has compact support and has first moment different from 0. Then the equality from (2.20) can be extended to a small disc centered at 0, where we can also perform inversion under composition for the functions on its two sides. Due to the analogy between how $S$ and $\Sigma$ were defined in terms of $R$ and respectively $\eta$ (see Equations (2.6) and (2.14) in the above glossary of transforms), we thus arrive to the fact that

$$S_{\mathbb{B}(\mu)}(z) = \Sigma_{\mu}(z), \quad \text{for } |z| \text{ sufficiently small.}$$  \hspace{1cm} (2.21)

We conclude this subsection by recording a few other properties of the bijection $\mathbb{B}$, which were established in [9] or follow easily from the arguments presented there. The bijection $\mathbb{B}$ turns out to be a weakly continuous isomorphism from $(\mathcal{M}, \oplus)$ onto $(\mathcal{M}_{\text{inf-div}}, \ominus)$. Moreover, as an easy consequence of relation (2.20), if $\mu$ has compact support, then so does $\mathbb{B}(\mu)$.

### 2.7 Subordination.

Let $\mu$ and $\sigma$ be two probability measures in $\mathcal{M}$. One says that $F_\sigma$ is subordinated to $F_\mu$ (abbreviated in this paper as “$\sigma$ is subordinated to $\mu$”) if there exists a function $\omega \in \mathcal{F}$ (with $\mathcal{F}$ as described in Equation (2.3) above), such that

$$F_\sigma = F_\mu \circ \omega.$$  

If it exists, this function $\omega \in \mathcal{F}$ is uniquely determined, and is called the **subordination function** of $\sigma$ with respect to $\mu$. An important phenomenon in the study of the operation of free additive convolution $\oplus$ is that for every $\mu, \nu \in \mathcal{M}$, the convolution $\mu \oplus \nu$ is subordinated with respect to $\mu$ and with respect to $\nu$. This phenomenon has been first observed in [21] and then proved in full generality in [13].

In the same vein, one has that a $\oplus$-convolution power $\mu^{\oplus t}$ is always subordinated with respect to $\mu$, for every $\mu \in \mathcal{M}$ and every $t \geq 1$. This fact appears in [4]. In the same paper it is also pointed out that the subordination function $\omega$ of $\mu^{\oplus t}$ with respect to $\mu$ can be given by a “direct” formula (not involving composition) in terms of the reciprocal Cauchy transform of $\mu^{\oplus t}$, namely

$$\omega(z) = \frac{1}{t}z + \left(1 - \frac{1}{t}\right)F_{\mu^{\oplus t}}(z), \quad z \in \mathbb{C}^+.$$  \hspace{1cm} (2.22)
We will repeatedly use this formula in what follows, rewritten in order to express $F_{\mu \Box t}$ in terms of $\omega$, when it thus says that

$$F_{\mu \Box t}(z) = \frac{t \omega(z) - z}{t - 1}, \quad z \in \mathbb{C}^+$$

(2.23)

(where $\mu$, $t$ and $\omega$ are the same as in (2.22)).

On the other hand let us note that the above Equation (2.22) can be also put in the form

$$\omega(z) = \frac{1}{t} z + \left(1 - \frac{1}{t}\right) F_\mu(\omega(z)).$$

(2.24)

The latter formula can be viewed as a functional equation, for which it is known that $\omega$ is the only solution belonging to the set of analytic maps $\mathcal{F}$ from Equation (2.3).

Another benefit of Equation (2.24) is that we can use it in order to write $z$ in terms of $\omega(z)$:

$$z = t \omega(z) + (1 - t) F_\mu(\omega(z)), \quad z \in \mathbb{C}^+,$$

(2.25)

and the latter means that the function

$$H : \mathbb{C}^+ \to \mathbb{C}, \quad H(w) = tw + (1 - t) F_\mu(w),$$

(2.26)

is a left-inverse for $\omega$. It was in fact observed in [4] (and will be used in this paper too) that an equality of subordination functions of the kind discussed above is equivalent to the equality of their left-inverses defined as in (2.26). More precisely: let $\mu, t, \omega, H$ be as above, and let $\tilde{\mu}, \tilde{t}, \tilde{\omega}, \tilde{H}$ be another set of data given in the same way; then we have that

$$\omega = \tilde{\omega} \iff H = \tilde{H}.$$  

(2.27)

Finally, let us record here one more formula concerning subordination functions, which will be invoked in Section 4 below. This formula is not about $\Box$-convolution powers, but rather concerns the free Brownian motion, i.e. the free additive convolution with a semicircular distribution. For $t > 0$, let us denote by $\gamma_t$ the centered semicircular distribution of variance $t$, $d\gamma_t(x) = \frac{1}{2\pi t^2 \sqrt{4t - x^2}} dx$ on $[-2\sqrt{t}, 2\sqrt{t}]$. Let $\mu$ be a probability measure in $\mathcal{M}$, let $t$ be in $(0, \infty)$, and let $\omega$ be the subordination function of $\mu \boxplus \gamma_t$ with respect to $\mu$. Then one has a “direct” formula (not involving compositions) which relates $\omega$ to the Cauchy transform of $\mu \boxplus \gamma_t$, namely

$$\omega(z) = z - t G_{\mu \boxplus \gamma_t}(z), \quad z \in \mathbb{C}^+.$$  

(2.28)

This formula was observed in [12] (it is obtained by putting together the statements of Lemma 4 and Proposition 2 of that paper).

3 Proof of Theorems 1 and 2

The fact that the transformations $\{B_t \mid t \geq 0\}$ form a semigroup under composition will follow from a “commutation relation”, stated in the next proposition, concerning the convolution powers with respect to $\boxplus$ and to $\boxdot$. 

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Proposition 3.1 Let \( p, q \) be two real numbers such that \( p \geq 1 \) and \( q > (p - 1)/p \). We have
\[
\left( \mu^{\boxplus p} \right)^{\boxtimes q} = \left( \mu^{\boxtimes q'} \right)^{\boxplus p'}, \quad \forall \mu \in \mathcal{M},
\] (3.1)
where the new exponents \( p' \) and \( q' \) are defined by
\[
p' := pq/(1 - p + pq), \quad q' := 1 - p + pq
\] (3.2)
(note that \( p' \geq 1 \) and \( q' > 0 \), thus the convolution powers appearing on the right-hand side of (3.1) do indeed make sense).

Proof. If \( p = 1 \), then we have \( p' = 1 \) and \( q' = q \), and both sides of (3.1) are equal to \( \mu^{\boxtimes q} \).
Throughout the remaining of the proof we will assume that \( p > 1 \) (which also implies that \( p' > 1 \), thus allowing divisions by \( p - 1 \) and \( p' - 1 \) in our calculations).

We will prove the equality (3.1) by showing that the probability measures appearing on its two sides have the same reciprocal Cauchy transform. In order to do this we will take advantage of the specific formulas we have for reciprocal Cauchy transforms, when we look at convolution powers for \( \boxplus \) and for \( \boxtimes \). For \( \boxplus \) we will simply use the formula (2.18) which defined \( \nu^{\boxplus p} \) in the above subsection 2.4. The convolution powers with respect to \( \boxtimes \) are a bit more complicated: we will handle their reciprocal Cauchy transforms via subordination functions, by invoking Equation (2.23) from subsection 2.7.

Let \( F_{\text{lhs}} \) denote the reciprocal Cauchy transform of the probability measure on the left-hand side of (3.1). We have:
\[
F_{\text{lhs}}(z) = (1 - q)z + qF_{\mu^{\boxplus p}}(z) = (1 - q)z + q \cdot \frac{p \cdot \omega_{\text{lhs}}(z) - z}{p - 1}
\] (by Equation (2.18))
\[
= \left( (1 - q) - \frac{q}{p - 1} \right) z + \frac{pq}{p - 1} \omega_{\text{lhs}}(z)
\] (by Equation (2.23))
\[
= -\frac{1}{p' - 1} z + \frac{p'}{p' - 1} \omega_{\text{lhs}}(z),
\]
where \( \omega_{\text{lhs}} \) denotes the subordination function of \( \mu^{\boxplus p} \) with respect to \( \mu \). At the last equality sign in the above calculation we used the fact that, due to how \( p' \) is defined, we have \((1 - q) - q/(p - 1) = -1/(p' - 1) \) and \( pq/(p - 1) = p'/(p' - 1) \).

On the other hand, let \( F_{\text{rhs}} \) be the reciprocal Cauchy transform of the probability measure on the right-hand side of (3.1). If we also make the notation \( \nu := \mu^{\boxtimes q'} \), then the formula (2.23) gives us that
\[
F_{\text{rhs}}(z) = \frac{p' \cdot \omega_{\text{rhs}}(z) - z}{p' - 1} = -\frac{1}{p' - 1} z + \frac{p'}{p' - 1} \omega_{\text{rhs}}(z),
\]
where \( \omega_{\text{rhs}} \) is the subordination function of \( \nu^{\boxplus p'} \) with respect to \( \nu \). By comparing the latter expression with the one obtained in the preceding paragraph, we see that the desired equality \( F_{\text{lhs}} = F_{\text{rhs}} \) is tantamount to the equality of subordination functions \( \omega_{\text{lhs}} = \omega_{\text{rhs}} \).

Now, in order to prove that \( \omega_{\text{lhs}} = \omega_{\text{rhs}} \) we invoke the equivalence (2.24) from subsection 2.7, which tells us that it suffices to check the equality of the left inverses of these
subordination functions. Let us denote these inverses by $H_{\text{lhs}}$ and $H_{\text{rhs}}$, respectively. From Equation (2.26) we know that

$$H_{\text{lhs}}(w) = pw + (1 - p)F_\mu(w), \quad H_{\text{rhs}}(w) = p'w + (1 - p')F_\nu(w),$$

(3.3)

and the equality $H_{\text{lhs}} = H_{\text{rhs}}$ thus amounts to the fact that

$$F_\nu(w) = \frac{(p - p')w + (1 - p)F_\mu(w)}{1 - p'},$$

(3.4)

But on the other hand, since $\nu$ is defined as $\mu \bowtie q'$, formula (2.18) tells us that

$$F_\nu(w) = (1 - q')w + q'F_\mu(w);$$

(3.5)

and it is straightforward to check that (3.4) reduces to (3.5) — that is, we have $(p - p')/(1 - p') = 1 - q'$ and $(1 - p)/(1 - p') = q'$, due to how $p'$ and $q'$ were defined in terms of $p$ and $q$. □

**Remark 3.2** For a measure $\mu \in \mathcal{M}$ which has compact support, the above proposition can be also proved by using combinatorial methods, on the line developed in [7] (which has the merit that they extend to a multi-variable framework - see Proposition 4.2 in [8]). Here we preferred the treatment via analytic methods, which apply directly to a general measure $\mu \in \mathcal{M}$.

We now turn to examine why every transformation $\mathbb{B}_t$ is a homomorphism with respect to free multiplicative convolution. We will prove that, in fact, each of the two kinds of convolution powers involved in the definition of $\mathbb{B}_t$ is “only a dilation away” from being a homomorphism with respect to $\boxtimes$. In order to do that, it will be convenient to start by recording how the various transforms considered in the paper behave under dilation.

**Notation 3.3** For $\mu \in \mathcal{M}$ and $r > 0$ we denote by $\mu \circ D_r$ the probability measure on $\mathbb{R}$ defined by

$$(\mu \circ D_r)(A) := \mu(rA), \quad \forall A \subseteq \mathbb{R}, \text{ Borel set.}$$

**Remark 3.4** Let $\mu$ be a probability measure in $\mathcal{M}$ and let $r$ be a positive real number. Then the Cauchy transform of the dilated measure $\mu \circ D_r$ is given by the formula

$$G_{\mu \circ D_r}(z) = rG_\mu(rz), \quad z \in \mathbb{C}^+.$$  

(3.6)

This follows directly from the definition of the Cauchy transform:

$$G_{\mu \circ D_r}(z) = \int_\mathbb{R} \frac{d\mu \circ D_r(x)}{z - x} = \int_\mathbb{R} \frac{d\mu(x)}{z - x/r} = r \int_\mathbb{R} \frac{d\mu(x)}{rz - x} = rG_\mu(rz).$$

From Equation (3.6) one easily obtains formulas for how various other transforms considered in this paper change under dilations. For the $R$-transform and the $S$-transform it turns out that we have

$$R_{\mu \circ D_r}(z) = R_\mu(z/r), \quad S_{\mu \circ D_r}(z) = rS_\mu(z);$$

(3.7)
while for the $\eta$-transform and the $\Sigma$-transform we have
\[ \eta_{\mu \circ D_r}(z) = \eta_{\mu}(z/r), \quad \Sigma_{\mu \circ D_r}(z) = r \Sigma_{\mu}(z). \] (3.8)

Each of the formulas listed in (3.7) and (3.8) holds for $z$ running in the appropriate domain where the corresponding transform is defined. The verification of these formulas is immediate, and left as exercise to the reader (one just has to start from (3.6) and then move through the definitions of the other four transforms appearing in the formulas).

**Proposition 3.5** For $\mu \in \mathcal{M}$ and $\nu \in \mathcal{M}_+$ we have
\[ (\mu \boxplus t) \boxdot (\nu \boxplus t) = (\mu \boxdot \nu)^{\text{id}} \circ D_{1/t}, \quad \forall \ t \geq 1, \] (3.9)

and
\[ (\mu \lozenge t) \boxdot (\nu \lozenge t) = (\mu \boxdot \nu)^{\text{id}} \circ D_{1/t}, \quad \forall \ t > 0. \] (3.10)

**Proof.** By using the continuity of the operations $\boxplus$, $\lozenge$, and $\boxdot$ with respect to the weak topology, and by doing suitable approximations of $\mu$, $\nu$ in this topology, we see that it suffices to verify the required formulas in the case when $\mu$ and $\nu$ have compact support and have first moment different from 0. We fix for the whole proof two such measures $\mu$ and $\nu$. For these $\mu$ and $\nu$ we will verify (3.9) and (3.10) by using the $S$-transform and respectively the $\Sigma$-transform.

In order to start on the proof of (3.9), it is useful to record how the $S$-transform behaves under $\boxplus$-convolution powers: if $\rho \in \mathcal{M}$ has compact support and non-vanishing first moment, then we have
\[ S_{\rho^{\boxplus t}}(z) = \frac{1}{t} S_{\rho}(z/t), \quad \forall \ t \geq 1. \] (3.11)

This formula follows immediately from how the $S$-transform is expressed in terms of the $R$-transform (that is, $S(z) = \frac{1}{z} R^{-1}(z)$), combined with the fact that $R_{\rho^{\boxplus t}} = t R_\rho$.

Now, let us verify that the probability measures appearing on the two sides of Equation (3.9) have indeed the same $S$-transform. We compute:
\[ S_{(\mu \boxdot \nu)^{\text{id}}}(z) = S_{\mu^{\boxdot t}}(z) \cdot S_{\nu^{\boxdot t}}(z) \quad \text{(by multiplicativity of $S$-transform)} \]
\[ = \frac{1}{t^2} S_\mu(z/t) \cdot S_\nu(z/t) \quad \text{(by Equation (3.11))}, \]

and on the other hand
\[ S_{(\mu \boxdot \nu)^{\text{id} \circ D_{1/t}}}(z) = \frac{1}{t} S_{(\mu \boxdot \nu)^{\text{id}}}(z) \quad \text{(by the $S$-transform formula in Equation (3.7))} \]
\[ = \frac{1}{t^2} S_\mu(z/t) \quad \text{(by Equation (3.11))} \]
\[ = \frac{1}{t^2} S_\mu(z/t) \cdot S_\nu(z/t) \quad \text{(by multiplicativity of $S$-transform)}. \]

This completes the verification of (3.9).

The proof of Equation (3.10) goes on the same lines as above, with the difference that we now use $\eta$ and $\Sigma$ instead of $R$ and $S$. We first note the analogue of Equation (3.11):
\[ \Sigma_{\rho^{\boxplus t}}(z) = \frac{1}{t} \Sigma_{\rho}(z/t), \quad \forall \ t > 0. \] (3.12)
holding for same kind of $\rho$ as in (3.11). Formula (3.12) follows immediately from how the $\Sigma$-transform is expressed in terms of the $\eta$-function (that is, $\Sigma_{\rho}(z) = \frac{1}{z} \eta^{-1}(z)$), combined with the fact that $\eta_{\rho \circ t} = t \eta_{\rho}$. By using (3.12) and the multiplicativity of the $\Sigma$-transform with respect to $\circ$, we obtain (by calculations which are virtually identical to those shown in the preceding paragraph) that the probability measures on both sides of (3.10) have the same $\Sigma$-transform, equal to $\frac{1}{t} \Sigma_{\mu}(z/t) \cdot \Sigma_{\nu}(z/t)$. □

Remark 3.6 It is immediately seen that Equation (3.10) from the above proposition can also be put in the alternative form

$$\left( \left( \mu \boxprod \nu \right) \boxprod (\nu \boxplus t) \right)^{\frac{1}{t}} = (\mu \boxprod \nu) \circ D_{1/t}, \quad \forall t > 0 \quad (3.13)$$

(where $\mu \in \mathcal{M}$ and $\nu \in \mathcal{M}_+$, as in Proposition 3.5). We note here that in the terminology of [14], the left-hand side of Equation (3.13) would be said to define “the $t$-transform of free multiplicative convolution”; so in this terminology, (3.13) shows that the $t$-transform of $\boxprod$ is simply obtained by dilating $\boxprod$ by a factor of $1/t$.

Remark 3.7 (Proof of Theorem 1). At this moment it has become quite easy to verify all the properties of the transformations $\mathbb{B}_t$ that were stated in Theorem 1 of the introduction. Indeed, every $\mathbb{B}_t$ is injective, continuous and carries $\mathcal{M}_+$ itself because each of the maps

$$\mathcal{M} \ni \mu \mapsto \mu \boxprod (1 + t) \in \mathcal{M}$$

has these properties. The fact that $\mathbb{B}_t$ is a homomorphism with respect to $\boxprod$ is a straightforward consequence of Proposition 3.5: the dilation factors which appear when we take successively the powers “$\boxprod (t+1)$” and “$\boxprod 1/(t+1)$” cancel each other, and we are left with the plain multiplicativity stated in Equation (1.4). Finally, the formula $\mathbb{B}_t \circ \mathbb{B}_s = \mathbb{B}_{t+s}$ follows from a direct application of Proposition 3.1

$$\mathbb{B}_t(\mathbb{B}_s(\mu)) = \mathbb{B}_t \left( (\mu \boxprod (s+1))^{\frac{1}{t+s}} \right)$$

$$= \left[ \left( (\mu \boxprod (s+1))^{\frac{1}{t+s}} \right)^{\frac{1}{t+s+1}} \right]^{\frac{1}{t+s+1}}$$

$$= \left[ \left( \mu \boxprod (s+1) \right)^{\frac{1}{t+s+1}} \right] \cdot \left( \mu \boxprod 1/(t+s+1) \right)^{\frac{1}{t+s+1}}$$

$$= (\mu \boxprod (s+1))^{\frac{1}{t+s+1}} = \mathbb{B}_{t+s}(\mu),$$

where at the third equality sign we used Proposition 3.1 with $p = (t+s+1)/(s+1)$ and $q = (t+1)/(t+s+1)$. □

Remark 3.8 As stated in Theorem 1, every $\mathbb{B}_t$ maps $\mathcal{M}_+$ into itself; but let us point out here that the converse implication “$\mathbb{B}_t(\mu) \in \mathcal{M}_+ \Rightarrow \mu \in \mathcal{M}_+$” is not true in general. Indeed, it may happen that a measure $\mu \in \mathcal{M} \setminus \mathcal{M}_+$ has $\mu \boxprod \mu \in \mathcal{M}_+$. It is true on the other hand that convolution powers with respect to $\boxprod$ always preserve $\mathcal{M}_+$ (that is, $\nu^{\boxprod t} \in \mathcal{M}_+$ for every $\nu \in \mathcal{M}_+$ and every $t > 0$). So if $\mu \boxprod \mu \in \mathcal{M}_+$ then it follows that

$$\mathbb{B}_1(\mu) = (\mu \boxprod \mu)^{1/2} \in \mathcal{M}_+,$$
even though $\mu$ itself might not belong to $\mathcal{M}_+$.

In order to give a concrete example of probability measure $\mu \in \mathcal{M} \setminus \mathcal{M}_+$ such that $\mu \boxplus \mu \in \mathcal{M}_+$, one can take for instance a suitable $\boxplus$-power of the non-symmetric Bernoulli distribution $\mu_o = \frac{1}{4}\delta_{-1} + \frac{3}{4}\delta_1$. We leave it as an exercise to the reader to check (by computing explicitly the necessary Cauchy transforms and $R$-transforms) that $\mu^\boxplus_{10}$ is in $\mathcal{M}_+$ for sufficiently large $t > 1$. The set $\{n \in \mathbb{N} \mid n \geq 1, \mu^{\boxplus_{2n}}_o \in \mathcal{M}_+\}$ is therefore non-empty. This set has a minimal element $m$, and the probability measure $\mu := \mu^\boxplus_{2m-1}$ has the property that $\mu \notin \mathcal{M}_+$, but $\mu \boxplus \mu \in \mathcal{M}_+$. (The explicit calculations of transforms that were left as exercise can in fact be pursued to yield the explicit value for $m$ – one has $m = 4$, thus the measure $\mu$ of this example is $\mu^\boxplus_{18}$.)

**Remark 3.9** The Equations (3.11) and (3.12) shown during the proof of Proposition 3.5 can be used in order to obtain a formula which expresses directly the $S$-transform of $\mathbb{B}_t(\mu)$ in terms of the $S$-transform of $\mu$.

More precisely, let $\mu$ be a probability measure in $\mathcal{M}$ which has compact support and has first moment different from 0. Then for every $t \geq 0$, the measure $\mathbb{B}_t(\mu)$ also has these two properties (this happens because taking convolution powers with respect to $\boxplus$ or to $\boxplus_1$ preserves the compactness of the support, and rescales the first moment by a factor equal to the exponent used in the convolution power). Thus, for every $t \geq 0$, it makes sense to consider the $S$-transform of $\mathbb{B}_t(\mu)$. We claim that for $z$ in a sufficiently small disc centered at 0, this $S$-transform is given by the formula:

$$S_{\mathbb{B}_t(\mu)}(z) = S_\mu\left(\frac{z}{1-tz}\right). \tag{3.14}$$

Indeed, for $|z|$ small enough we can write:

$$S_{\mathbb{B}_t(\mu)}(z) = S_{(\mu^{\boxplus(1+t)})^{1/(1+t)}}(z)$$

$$= \Sigma_{(\mu^{\boxplus(1+t)})^{1/(1+t)}}\left(\frac{z}{1+z}\right) \quad \text{(by relation between $S$ and $\Sigma$, Eqn. (2.15))}$$

$$= (1+t) \cdot \Sigma_{\mu^{\boxplus(1+t)}}\left(1+t \cdot \frac{z}{1+z}\right) \quad \text{(by Equation (3.12))}$$

$$= (1+t) \cdot \Sigma_{\mu^{\boxplus(1+t)}}\left(\frac{(1+t) \cdot z/(1+z)}{1 - (1+t) \cdot z/(1+z)}\right) \quad \text{(by relation between $S$ and $\Sigma$)}$$

$$= (1+t) \cdot S_\mu^{\boxplus(1+t)}\left(\frac{z}{1-tz}\right) \quad \text{(by Equation (3.11))}.$$
But on the other hand, for $|z|$ small enough we also have that

$$S_{B_1(\mu)}(z) = S_{\mu} \left( \frac{z}{1-z} \right) \quad \text{(by Equation (3.14) in Remark 3.9)}$$

$$= \Sigma_{\mu}(z) \quad \text{(by the relation between $S$ and $\Sigma$, Equation (2.15)).}$$

Thus the measures $B(\mu)$ and $B_1(\mu)$ must indeed be equal to each other, since they have the same $S$-transform. □

4 Relation to free Brownian motion and to complex Burgers equation

We first record, in the following lemma, an analytic description of $B_t$.

**Lemma 4.1** Consider $\mu \in M$ and $t \geq 0$, and consider the measure $B_t(\mu) \in M$. We have

$$F_{B_t(\mu)}(z) = \left( 1 - \frac{1}{t} \right) z + \frac{1}{t} \omega(z), \quad z \in \mathbb{C}^+,$$

where $\omega$ is the subordination function of $\mu^{\Xi(t+1)}$ with respect to $\mu$.

**Proof.** Since $B_t(\mu) = (\mu^{\Xi(t+1)})^{\omega_1/(t+1)}$, Equation (2.18) from subsection 2.4 gives us that

$$F_{B_t(\mu)}(z) = \left( 1 - \frac{1}{1+t} \right) z + \frac{1}{1+t} F_{\mu^{\Xi(t+1)}}(z), \quad z \in \mathbb{C}^+. \quad (4.2)$$

On the other hand, Equation (2.23) from subsection 2.7 (with $t$ replaced by $t+1$) tells us that

$$F_{\mu^{\Xi(t+1)}}(z) = \frac{(t+1)\omega(z) - z}{t}, \quad z \in \mathbb{C}^+. \quad (4.3)$$

Substituting (4.3) into (4.2) leads to the required formula (4.1). □

**Remark 4.2** (Proof of Theorem 3). Throughout this proof it will come in handy to use the notation $h_\nu(z) := F_\nu(z) - z$, for $\nu \in M$ and $z \in \mathbb{C}^+$.

Let $\mu$ and $h$ be as in the statement of Theorem 3. For every $t \geq 1$, let $\omega_t$ denote the subordination function of $\mu^{\Xi_t}$ with respect to $\mu$, and let $H_t$ be the left-inverse for $\omega_t$ defined as in Equation (2.26) from Section 2. Let us observe that, by Lemma 4.1 and the definition of $h_\mu$, we have that $\omega_{t+1}(z) = z + th_\mu(\omega_{t+1}(z))$, for all $z \in \mathbb{C}^+, t \geq 0$. Thus, using again Lemma 4.1 we find that:

$$h(t, z) = h_{B_t(\mu)}(z) = F_{B_t(\mu)}(z) - z = \frac{1}{t} (\omega_{t+1}(z) - z) = h_\mu(\omega_{t+1}(z)), \quad z \in \mathbb{C}^+, t \geq 0. \quad (4.4)$$

We observe that the function $\omega_{t+1}(z)$ is indeed differentiable in both variables; this follows immediately from the equation $H_{t+1}(\omega_{t+1}(z)) = z$ and the definition of $H_{t+1}(z)$. Let us denote $\frac{\partial}{\partial z}$ by $\partial_x$. Differentiating with respect to $z$ gives
∂_t \omega_{t+1}(z) = \frac{1}{(\partial_z H_{t+1})(\omega_{t+1}(z))},

and differentiating with respect to \( t \) gives

\[ \partial_t \omega_{t+1}(z) = -\frac{(\partial_1 H_{t+1})(\omega_{t+1}(z))}{(\partial_2 H_{t+1})(\omega_{t+1}(z))} = h_\mu(\omega_{t+1}(z))\partial_z \omega_{t+1}(z), \]

where we have used the definitions of \( H \) and \( h \) in the last equality.

Then

\[ \partial_t h(t, z) = \partial_t h_{\mathbb{B}_t(\mu)}(z) = h'_\mu(\omega_{t+1}(z))\partial_1 \omega_{t+1}(z) + h'_\mu(\omega_{t+1}(z))h_\mu(\omega_{t+1}(z))\partial_z \omega_{t+1}(z), \]

and

\[ \partial_z h(t, z) = \partial_z h_{\mathbb{B}_t(\mu)}(z) = h'_\mu(\omega_{t+1}(z))\partial_z \omega_{t+1}(z). \]

The two relations above prove equation \([1.9]\). \( \Box \)

**Remark 4.3** We now move to the framework of Theorem 4. Let \( \nu \) be in \( \mathcal{M} \), and let us consider the Cauchy transform \( G_\nu \). It is easily verified that the map \( \mathbb{C}^+ \ni z \mapsto z - G_\nu(z) \in \mathbb{C} \) belongs to the set \( \mathcal{F} \) of analytic self-maps of \( \mathbb{C}^+ \) considered in the above Equation \([2.2]\); thus there exists a unique \( \mu \in \mathcal{M} \) such that \( z - G_\nu(z) = F_\mu(z), z \in \mathbb{C}^+ \). Clearly, this \( \mu \) is related to the given \( \nu \) in exactly the way described by Equation \([1.12]\) of Theorem 4. It can be shown that \( \mu \) has variance equal to 1 and is centered, that is, it satisfies

\[ \int_{-\infty}^{\infty} t^2 \, d\mu(t) = 1, \quad \int_{-\infty}^{\infty} t \, d\mu(t) = 0. \quad (4.5) \]

Moreover, it can be shown that the correspondence \( \nu \mapsto \mu \) (with \( \nu \) and \( \mu \) as above) is a bijection between \( \nu \in \mathcal{M} \) and \( \mu \) running in the set of probability measures in \( \mathcal{M} \) which satisfy the conditions in \([4.5]\). A detailed presentation of these facts appears in Section 2 of the paper \([16]\) of Maassen (see Proposition 2.2 of that paper).

**Remark 4.4** *(Proof of Theorem 4).* Let us fix two probability measures \( \mu, \nu \in \mathcal{M} \) which are connected to each other as in Remark 4.3, via the relation

\[ -G_\nu(z) = F_\mu(z) - z, \quad z \in \mathbb{C}^+. \]

Let us also fix a real number \( t > 0 \). Our goal in this proof is to show that we have the formula

\[ -G_{\nu \boxplus \gamma_t}(z) = F_{\mathbb{B}_t(\mu)}(z) - z, \quad z \in \mathbb{C}^+, \quad (4.6) \]

where \( \gamma_t \) is the centered semicircular distribution of variance \( t \).

Let \( \theta: \mathbb{C}^+ \to \mathbb{C}^+ \) be the subordination function of \( \nu \boxplus \gamma_t \) with respect to \( \nu \). By the definition of \( \theta \), we thus have

\[ G_{\nu \boxplus \gamma_t} = G_\nu \circ \theta \quad (4.7) \]

(composition of self-maps of \( \mathbb{C}^+ \)). But, as noted at the end of the above subsection 2.7, the maps \( \theta \) and \( G_{\nu \boxplus \gamma_t} \) are also related via the formula

\[ \theta(z) = z - tG_{\nu \boxplus \gamma_t}(z), \quad z \in \mathbb{C}^+. \quad (4.8) \]
Now let \( \omega : \mathbb{C}^+ \to \mathbb{C}^+ \) be the subordination function of \( \mu \boxplus (t+1) \) with respect to \( \mu \). We claim that \( \omega \) is in fact equal to the subordination function \( \theta \) from the preceding paragraph. In order to prove this claim, we write the Cauchy transform of \( \nu \boxplus \gamma_t \) in two different ways.

On one hand, for every \( z \in \mathbb{C}^+ \) we have
\[
G_{\nu \boxplus \gamma_t}(z) = G_\nu(\theta(z)) \quad \text{(by Equation (4.7))},
\]
(4.9)

where we used the formula which connects \( G_\nu \) to \( F_\mu \), applied to the complex number \( \theta(z) \in \mathbb{C}^+ \). On the other hand, from Equation (4.8) we have that
\[
G_{\nu \boxplus \gamma_t}(z) = 1 - t \left( z - \theta(z) \right),
\]
(4.10)

By eliminating \( G_{\nu \boxplus \gamma_t}(z) \) between Equations (4.9) and (4.10), we find that
\[
-F(\theta(z)) + \theta(z) = \frac{1}{t} \left( z - \theta(z) \right),
\]
which implies that
\[
\theta(z) = \frac{1}{t+1} z + \left( 1 - \frac{1}{t+1} \right) F_\mu(\theta(z)), \quad z \in \mathbb{C}^+.
\]
(4.11)

We obtained that \( \theta \) satisfies the functional equation which determines uniquely the subordination function \( \omega \) (cf. discussion in subsection 2.7 above); the equality \( \theta = \omega \) follows.

Finally, let us observe that for every \( z \in \mathbb{C}^+ \) we have
\[
F_{B_t(\mu)}(z) = (1 - \frac{1}{t}) z + \frac{1}{t} \omega(z) - z \quad \text{(by Lemma 4.1)}
\]
\[
= \frac{1}{t} \left( \omega(z) - z \right)
\]
\[
= \frac{1}{t} \left( \theta(z) - z \right) \quad \text{(since \( \omega = \theta \)}
\]
\[
= -G_{\nu \boxplus \gamma_t}(z) \quad \text{(by Equation (4.10))},
\]
and the desired Equation (4.6) is obtained. \(\square\)

**Example 4.5** Let \( \mu \) be the symmetric Bernoulli distribution, \( \mu = \frac{1}{2} (\delta_{-1} + \delta_1) \). The measure \( \nu \) corresponding to this \( \mu \) via the bijection \( \nu \leftrightarrow \mu \) from Remark 4.3 is the Dirac measure \( \delta_0 \); indeed, with \( \nu = \delta_0 \), we clearly have \( -G_\nu(z) = F_\mu(z) - z = -1/z, z \in \mathbb{C}_+ \). But then Theorem 4 implies that for every \( t \geq 0 \) we have
\[
F_{B_t(\mu)}(z) = -G_{\delta_0 \boxplus \gamma_t}(z) = -G_{\gamma_t}(z) = -z - \sqrt{z^2 - 4t}, \quad z \in \mathbb{C}_+.
\]

We thus get an explicit formula for \( F_{B_t(\mu)} \), leading to a formula for the Cauchy transform of \( B_t(\mu) \):
\[
G_{B_t(\mu)}(z) = \frac{(2t - 1)z - \sqrt{z^2 - 4t}}{2(1 - (1-t)z^2)} \quad \text{z \in \mathbb{C}_+}. \quad \text{(4.12)}
\]
By a straightforward application of the Stieltjes inversion formula, one can then determine exactly what $B_t(\mu)$ is. For $t \geq 1/2$ the result of the calculation is that $B_t(\mu)$ is absolutely continuous, with density

$$x \mapsto \frac{\sqrt{4t - x^2}}{2\pi \cdot (1 - (1 - t)x^2)}, \quad |x| \leq 2\sqrt{t}. \quad (4.13)$$

For $0 < t < 1/2$, one finds that $B_t(\mu)$ has an absolutely continuous part with density described exactly as in the above Equation (4.13); and in addition to that, $B_t(\mu)$ has two atoms at $\pm 1/\sqrt{1-t}$, each of them of mass equal to $(1 - 2t)/(2 - 2t)$. (The details of the calculation are left as an exercise to the reader.)

Note that for $t = 1/2$ the density from (4.13) simplifies to

$$x \mapsto \frac{1}{\pi \sqrt{2 - x^2}}, \quad |x| \leq \sqrt{2}.$$  

Thus $B_{1/2}(\mu)$ is the arcsine law on the interval $[-\sqrt{2}, \sqrt{2}]$.

On the other hand, for $t = 1$ the density from (4.13) becomes

$$x \mapsto \frac{\sqrt{4 - x^2}}{2\pi}, \quad |x| \leq 2,$$

which shows that $B_1(\mu)$ is the standard semicircle law $\gamma_1$. (The fact that $B_1(\mu) = \gamma_1$ could also be obtained directly from Theorem 2. Indeed, $B_1(\mu)$ is equal to $B(\mu)$, and is hence determined by the fact that it has $R$-transform equal to $\eta_\mu(z)$. But it is immediate that $\mu$ of this example has $\eta_\mu(z) = z^2$, which is exactly the $R$-transform of $\gamma_1$.)

**Remark 4.6** The above Example 4.5 can be placed within the framework of a family of probability measures on $\mathbb{R}$ which are called *free Meixner states*. (We thank Michael Anshelevich for bringing this observation to our attention.) We briefly outline here how the connection to free Meixner states appears; for more details, see Section 4.2 in the recent paper [3].

Let $b$ and $c$ be two real parameters such that $c \geq -1$. The free Meixner state of mean 0 and variance 1 which is indexed by the parameters $b, c$ is a probability measure $\mu_{b,c} \in \mathcal{M}$ that can be defined as follows: if $(P_n)_{n=1}^\infty$ is the sequence of monic orthogonal polynomials for $\mu_{b,c}$, then we have $P_0(t) = 1$, $P_1(t) = t$, and the recurrence

$$\begin{cases} tP_1(t) = P_2(t) + bP_1(t) + P_0(t), \\
 tP_n(t) = P_{n+1}(t) + bP_n(t) + (c + 1)P_{n-1}(t), \quad \forall n \geq 2. \quad (4.14)\end{cases}$$

For the explanation of why the probability measures $\mu_{b,c}$ are called “free Meixner”, we refer to the paper [2].

Now, it is not hard to obtain an explicit formula for the Cauchy transform of $\mu_{b,c}$ – see the displayed equations in Theorem 4 of [2] (the parameters “$a, t$” from those equations have to be suitably set, in order to match the $b, c$ from the above Equation (4.14)). This leads to an explicit formula for the analytic function $F_{\mu_{b,c}}(z) - z$; it turns out that what one gets is exactly the relation

$$F_{\mu_{b,c}}(z) - z = -G_{\gamma_{b,c+1}}(z), \quad z \in \mathbb{C}^+, \quad (4.15)$$
where $\gamma_{b,t}$ stands for the semicircular distribution of mean $b$ and variance $t$. Or in other words, the correspondence $\nu \mapsto \mu$ from Remark 4.3 has the property that

$$\gamma_{b,t} \mapsto \mu_{b,t-1}, \quad \forall \, b \in \mathbb{R} \text{ and } t \geq 0. \quad (4.16)$$

(Thus the set of all free Meixner states with mean 0 and variance 1 is precisely what one obtains when applying the correspondence from Remark 4.3 to the set of all – not necessarily centered – semicircular distributions!)

It is of course clear that we have

$$\gamma_{b,t} = \delta_b \boxplus \gamma_t, \quad \forall \, b \in \mathbb{R}, \, \forall \, t \geq 0,$$

where $\gamma_t$ is the centered semicircular distribution of variance $t$. Thus Theorem 4 of this paper can be applied in conjunction to the above Equation (4.16), and it gives us that

$$B_t(\mu_{b,c}) = \mu_{b,c+t}, \quad \forall \, b,c,t \in \mathbb{R} \text{ such that } c \geq -1 \text{ and } t \geq 0. \quad (4.17)$$

If we fix a value $b \in \mathbb{R}$, we hence see that the family of probability measures $\{\mu_{b,c} \mid c \geq -1\}$ is exactly what one obtains by starting with the measure $\mu_{b,-1}$ and by letting it evolve under the semigroup of transformations $B_t$. It is easily computed that $\mu_{b,-1}$ is actually a Bernoulli measure,

$$\mu_{b,-1} = \frac{q}{q-p} \delta_p + \frac{p}{p-q} \delta_q, \quad (4.18)$$

with $p,q$ determined from the equations $p+q = b$, $pq = -1$. The situation from Example 4.5 is obtained by setting $b = 0$ in this discussion, when $\mu_{b,-1}$ becomes the symmetric Bernoulli distribution $\frac{1}{2}(\delta_{-1} + \delta_1)$.

## 5 Miscellaneous facts about $\mathbb{B}_t(\mu)$

We start this section with a proposition discussing atoms and regularity for a measure $\mathbb{B}_t(\mu)$, $t > 0$. We show that densities of such probability measures tend to be quite smooth, while singular continuous parts cannot appear.

**Proposition 5.1** Let $\mu \in \mathcal{M}$ and any $t > 0$. Then:

1. The singular continuous part of $\mathbb{B}_t(\mu)$ with respect to the Lebesgue measure on the real line is zero.

2. The probability $\mathbb{B}_t(\mu)$ has at most $[1/t]$ atoms for $t \leq 1$ and at most one atom for $t > 1$. The point $x \in \mathbb{R}$ is an atom of $\mathbb{B}_t(\mu)$ if and only if

$$\lim_{y \to 0} F'_\mu((1-t)x+iy) = -tx \quad \text{and} \quad \lim_{y \to 0} F''_\mu((1-t)x+iy) = \frac{1 + t(1 - \mathbb{B}_t(\mu)(\{x\}))}{\mathbb{B}_t(\mu)(\{x\}) + t(1 - \mathbb{B}_t(\mu)(\{x\})}. \tag{We have denoted by $[a]$ the largest integer less than or equal to $a$.}$$

3. The absolutely continuous part of $\mathbb{B}_t(\mu)$ with respect to the Lebesgue measure is zero if and only if $\mu = \delta_c$ for some $c \in \mathbb{R}$. Moreover, its density is analytic wherever positive and finite.
Proof. We shall prove first item 2. As known from equation (5.7) in [5], if \( \lim_{y \to 0} F_{\mathcal{B}_t(\mu)}(x + iy) = 0 \), then we have

\[
\mathbb{B}_t(\mu)\{x\}^{-1} = \lim_{y \to 0} \frac{F_{\mathcal{B}_t(\mu)}(x + iy)}{iy} = \lim_{y \to 0} F'_{\mathcal{B}_t(\mu)}(x + iy),
\]

where we use the convention \( 1/0 = \infty \); conversely, for \( x \) to be an atom of \( \mathbb{B}_t(\mu) \) it is required that \( \lim_{y \to 0} F_{\mathcal{B}_t(\mu)}(x + iy) = 0 \), and then (5.1) holds. From Lemma 4.1 it follows that this is equivalent to

\[
\lim_{y \to 0} \omega_{t+1}(x + iy) = (1-t)x
\]

and

\[
\lim_{y \to 0} \omega_{t+1}(x + iy) - (1-t)x = \lim_{y \to 0} \omega'(x + iy) = (1-t) + \frac{t}{\mathbb{B}_t(\mu)\{x\}}.
\]

These statements can be seen to be equivalent to \( \lim_{y \to 0} H_{t+1}((1-t)x + iy) = x \) and \( \lim_{y \to 0} H'_{t+1}((1-t)x + iy) = (1-t) + \frac{t}{\mathbb{B}_t(\mu)\{x\}} \), where \( H_{t+1}(z) = (t+1)z - tF_{\mu}(z) \) (See also Proposition 4.7 of [5]). Using the definition of \( H_{t+1} \) and [5 (5.7)], we obtain

\[
\lim_{y \to 0} F_{\mu}((1-t)x + iy) = -tx
\]

and

\[
\lim_{y \to 0} F'_{\mu}((1-t)x + iy) = \frac{t + 1}{t} \cdot \frac{\mathbb{B}_t(\mu)\{x\}}{t + \mathbb{B}_t(\mu)\{x\}(1-t)} = \frac{1 + t(1 - \mathbb{B}_t(\mu)\{x\})}{\mathbb{B}_t(\mu)\{x\} + t(1 - \mathbb{B}_t(\mu)\{x\})}.
\]

Assume first that \( t < 1 \). Then the function \( F_{\mu}((1-t)z + tz, z \in \mathbb{C}^+ \) maps the upper half-plane into itself, since \( \exists F_{\mu}((1-t)z) \geq (1-t)\exists z \), so if \( 1-t > 0 \) we must have \( \exists F_{\mu}((1-t)z + tz) > 0 \) for all \( z \in \mathbb{C}^+ \). Moreover, the function \( G_{\lambda}(z) = 1/(F_{\mu}((1-t)z) - tz) \) is the Cauchy transform of a probability measure \( \lambda \) and \( F_{\lambda}(z) = 1/G_{\lambda}(z) \) satisfies

\[
\lim_{y \to 0} F_{\lambda}(x + iy) = 0,
\]

so for any \( x \) we have

\[
\lim_{y \to 0} F'_{\lambda}(x + iy) = \frac{t + 1}{t} \cdot \frac{\mathbb{B}_t(\mu)\{x\}}{t + \mathbb{B}_t(\mu)\{x\}(1-t)} - \frac{1}{1 + \frac{t}{\mathbb{B}_t(\mu)\{x\} + t(1 - \mathbb{B}_t(\mu)\{x\})}}.
\]

This holds for any \( x \) so that \( \lim_{y \to 0} F_{\mathcal{B}_t(\mu)}(x + iy) = 0 \). Thus, any such point \( x \) must be an atom of \( \lambda \) and \( \lambda\{x\} = \frac{\mathbb{B}_t(\mu)\{x\}}{1 + \mathbb{B}_t(\mu)\{x\}} \). Thus, \( \lim_{y \to 0} F_{\mathcal{B}_t(\mu)}(x + iy) = 0 \) and \( 0 < t < 1 \) implies \( \lambda\{x\} = 0 \). Since the total mass of all atoms of \( \lambda \) cannot exceed one, we conclude that there exist at least \( 1/t \) points \( x \) where \( \lim_{y \to 0} F_{\mathcal{B}_t(\mu)}(x + iy) = 0 \), and in particular at most \( 1/t \) atoms of \( \mathbb{B}_t(\mu) \).

If \( t \geq 1 \), then we have proved in Theorem 2 that \( \mathbb{B}_t(\mu) \) is infinitely divisible, so, as observed in [11], it can have at most one atom. This proves item 2.

We shall now show that \( \mathbb{B}_t(\mu) \) cannot have a nonzero singular continuous part. Observe that since \( \mathbb{B}_t(\mu) \) is \( \mathbb{B} \)-infinitely divisible for all \( t \geq 1 \), according to Theorem 2, the statement
is a trivial consequence of Proposition 5.1 in [3] whenever \( t \geq 1 \). Thus we will focus again on the case \( 0 < t < 1 \). Assume that \( \mathbb{B}_t(\mu) \) has a nontrivial singular continuous part. As noted in [3], for uncountably many points \( x \) in the support of the singular continuous part of \( \mathbb{B}_t(\mu) \), we must have

\[
\lim_{y \to 0} F_{\mathbb{B}_t(\mu)}(x + iy) = 0 \quad \text{and} \quad \lim_{y \to 0} \frac{F_{\mathbb{B}_t(\mu)}(x + iy)}{iy} = \lim_{y \to 0} F'_{\mathbb{B}_t(\mu)}(x + iy) = \infty.
\]

As shown in the proof of item 2, this implies that there exists a probability \( \lambda \) that has uncountably many atoms of mass at least \( t \), an obvious contradiction. This proves item 1.

We prove next item 3. As observed in [3], for any \( t > 0 \), the function \( F_{\mu^{\mathbb{H}+1}} \) extends analytically through any point \( x \) where \( F_{\mu^{\mathbb{H}+1}} \) has finite nontangential limit with strictly positive imaginary part, so, by Lemma 4.1 the function \( F_{\mathbb{B}_t(\mu)} \) extends analytically through any point \( x \) where \( F_{\mu^{\mathbb{H}+1}}(x) > 0 \) has finite nontangential limit with strictly positive imaginary part. Since the density of \( \mathbb{B}_t(\mu) \) is just \( -\pi^{-1}\Im(1/F_{\mathbb{B}_t(\mu)}) \), its analyticity follows immediately.

It has been observed in Proposition 4.7 of [3] that \( \mu^{\mathbb{H}+1} \) has a nonzero absolutely continuous part for any \( t > 0 \) if and only if \( \mu \) is not concentrated in one point. Thus, by Lemma 4.1 and the remarks above, we conclude that for any \( t > 0, \mathbb{B}_t(\mu) \) has a nonzero absolutely continuous part if and only if \( \mu \) is not concentrated in one point. This proves item 3.

**Corollary 5.2** Assume that \( \mu \in \mathcal{M} \) is absolutely continuous with respect to the Lebesgue measure on the real line, with density \( f_0 \). Denote by \( f_t \) the density with respect to the Lebesgue measure of the absolutely continuous part of \( \mathbb{B}_t(\mu) \). Then \( f_t \) converge to \( f_0 \) almost everywhere when \( t \) tends to zero.

**Proof.** It is known [15, Theorem 1.3] that any bounded function which is analytic in the unit disk has nontangential limit at almost all points of the unit circle. Since \( G_\mu \) maps the upper into the lower half-plane, it can be conjugated to a self-map of the unit disk by using the map \( z \mapsto \frac{z - i}{z + i} \), and thus has nontangential limit at almost all points \( x \in \mathbb{R} \) (denote the limit function by \( G_\mu(x) \)). Also, by the upper half-plane version of [15, Theorem 1.2], \( \Im G_\mu(x) = -\pi f_0(x) \) for Lebesgue-almost all \( x \in \mathbb{R} \). Since the same holds when \( \mu \) is replaced by \( \mathbb{B}_t(\mu) \) and \( f_0 \) by \( f_t \), it is enough to prove that \( \lim_{t \to 0} G_{\mathbb{B}_t(\mu)}(x) = G_\mu(x) \) for all \( x \in \mathbb{R} \), except possibly for a set of zero Lebesgue measure. The statement of the corollary follows from this limit by taking the imaginary parts.

Now, from the relation between \( G_\mu \) and \( h_\mu \) it is immediate that it will be enough to show that for any \( x \in \mathbb{R} \) with the property that the nontangential limit of \( h_\mu \) at \( x \) belongs to the upper half-plane, we have \( \lim_{t \to 0} h_{\mathbb{B}_t}(x) = h_\mu(x) \). We observe that for any such \( x \in \mathbb{R} \), the function \( h_x(z) := h_\mu(x + z), z \in \mathbb{C}^+ \), has nontangential limit at zero belonging to the upper half-plane, and the function \( \gamma(t) := \theta h_{\mathbb{B}_t}(x) \) satisfies the condition \( \gamma(t) = \theta h_x(\gamma(t)) \). Indeed, the first statement is just a reformulation of the fact that the nontangential limit of \( h_\mu \) at \( x \) belongs to the upper half-plane. For the second observation, note that by Lemma 4.1 we have \( \omega(z) = z + \theta h_{\mathbb{B}_t}(\mu)(z) \), and by Theorem 4.6 in [3], this relation extends by continuity to the real line. Thus, relation (4.4) together with the above implies that \( \gamma(t) = \theta h_x(\gamma(t)) \). These conditions have been proved in Lemma 2.13 in [3] to imply that \( \lim_{t \to 0} h_x(\gamma(t)) \) exists and equals the nontangential limit at zero of \( h_x \). But this is equivalent to saying that \( \lim_{t \to 0} h_{\mathbb{B}_t}(x) = h_\mu(x) \). □
Remark 5.3 The proposition above gives a full description of the atoms of $B_t(\mu)$ in terms of the reciprocal Cauchy transform of $\mu$, and a quite strict bound on the possible number of such atoms. However, it does not guarantee that the lack of atoms for $\mu$ must translate into a lack of atoms for $B_t(\mu)$. We will provide an example showing that this situation in fact can occur. Consider the probability measure $\mu$ whose reciprocal Cauchy transform is given by the formula

$$F_{\mu}(z) = z - 2 + \frac{z - i}{z + i}, \quad z \in \mathbb{C}^+ \cup \mathbb{R}.$$ 

It is easy to see that $F_{\mu}$ is in fact a rational transformation of the whole complex plane, with one pole at $-i$. We observe that $F_{\mu}(1) = -1$ and $F_{\mu}(x) \in \mathbb{C}^+$ for all $x \in \mathbb{R} \setminus \{1\}$ so $\mu$ has no atoms.

Let $x = 2$, $t = 1/2$. Then

$$F_{\mu} \left( \left( 1 - \frac{1}{2} \right) \cdot 2 \right) = F_{\mu}(1) = -1 = -\frac{1}{2} \cdot 2,$$

and since $F'_{\mu}(z) = 1 + \frac{2i}{z + i}$, we have

$$F'_{\mu} \left( \left( 1 - \frac{1}{2} \right) \cdot 2 \right) = F'_{\mu}(1) = 2.$$

According to the proposition above, $B_{1/2}(\mu)(\{2\}) = 1/3$.

The remaining part of this section is about the $\boxplus$-divisibility indicator $\phi(\mu)$.

Proposition 5.4 Let $\mu$ be in $\mathcal{M}$. We have that $\mu$ is infinitely divisible with respect to $\boxplus$ if and only if $\phi(\mu) \geq 1$.

Proof. We have already seen that if $\mu$ is infinitely divisible, then $\mu \in \mathbb{B}(\mathcal{M})$, so that $\phi(\mu) \geq 1$. Assume now that $\phi(\mu) \geq 1$. We shall prove that $\mu$ is infinitely divisible. This is clearly true if $\phi(\mu) > 1$, so we may assume without loss of generality that $\phi(\mu) = 1$. Observe that $\lim_{t \downarrow 0} B_t(\mu) = \mu$ in the weak topology, and (by the definition of $\phi$) $\phi(B_t(\mu)) > 1$ for all $t > 0$, so that by Theorem 2, $B_t(\mu)$ is $\boxplus$-infinitely divisible. Thus, the statement of our proposition will be proved once we show that the set of $\boxplus$-infinitely divisible probability measures is closed in the topology of weak convergence.

As shown in [10], Proposition 5.7, weak convergence of probability measures translates into convergence on compact subsets of the corresponding $R$-transforms: there exists a cone $\Gamma$ with vertex at zero so that $R_\mu$ and $R_{B_t(\mu)}$ are all defined on $\Gamma$ for $t > 0$ small enough and $\lim_{t \downarrow 0} R_{B_t(\mu)}(z) = R_\mu(z)$ uniformly on compact subsets of $\Gamma$. But by Theorem 5.10 in the same [10], $\mu$ is infinitely divisible if and only if $R_\mu$ has an analytic extension to the upper half-plane. As the $R$-transform $R$ maps points from $\mathbb{C}^+$ into $\mathbb{C}^+ \cup \mathbb{R}$, the family $\{R_{B_t(\mu)} : t \geq 0\}$ is normal, so, by the classical theorem of Montel, there exists a subsequence $t_n \to 0$ so that $R_{B_{t_n}(\mu)}$ is convergent on $\mathbb{C}^+$. The existence of the limit on $\Gamma$ together with the uniqueness of analytic continuation guarantees that $R_\mu$ has an analytic extension to the upper half-plane, and thus $\mu$ is $\boxplus$-infinitely divisible. □
Remark 5.5 (Some basic properties of $\phi(\mu).$)

1° A useful formula for deriving explicit values of $\phi(\mu)$ is

$$\phi(\mathbb{B}_t(\mu)) = t + \phi(\mu), \quad \forall \mu \in \mathcal{M}, \forall t \in [0, +\infty).$$

(5.2)

Indeed, if $\phi(\mu) = 0$, then this is obvious from the injectivity of $\mathbb{B}_t$ and the semigroup property. If $\phi(\mu) > 0$, then consider a sequence $q_n$ that increases to $\phi(\mu)$ and define $\nu_n$ by the equation $\mathbb{B}_{q_n}(\nu_n) = \mu$. Clearly

$$\phi(\mathbb{B}_t(\mu)) = \phi(\mathbb{B}_{q_n+t}(\nu_n)) \geq q_n + t \to t + \phi(\mu)$$

as $n \to \infty$. Thus, $\phi(\mathbb{B}_t(\mu)) \geq t + \phi(\mu)$. We claim that the inequality cannot be strict. Indeed, if this were not the case, we would find an $\epsilon$ so that $\phi(\mathbb{B}_t(\mu)) - (t + \phi(\mu)) > \epsilon > 0$ and a measure $\nu_\epsilon$ so that $\mathbb{B}_t(\mu) = \mathbb{B}_{t+\phi(\mu)+\epsilon}(\nu_\epsilon)$, and thus, by injectivity of $\mathbb{B}_t$ together with the semigroup property, $\mu = \mathbb{B}_{\phi(\mu)+\epsilon}(\nu_\epsilon)$, so that $\phi(\mu) \geq \phi(\mu) + \epsilon$, an obvious contradiction.

2° Let $\mu \in \mathcal{M}$ be such that $\phi(\mu) =: p > 0$. From the definition of $\phi(\mu)$ and the semigroup property of the transformations $\mathbb{B}_t$ it is immediate that for every $0 < q < p$ one can find a probability measure $\nu_q \in \mathcal{M}$ such that $\mathbb{B}_q(\nu_q) = \mu$. However, as an immediate consequence of 1° above and of Proposition 5.4, we observe that a stronger result holds:

$$\exists \nu \in \mathcal{M} \text{ such that } \mathbb{B}_p(\nu) = \mu.$$  

(5.3)

The proof is as follows: assume first that $\phi(\mu) = 1$. Then $\mu$ is $\mathbb{H}$-infinitely divisible by Proposition 5.4 so $\mu \in \mathbb{B}_1(\mathcal{M})$, by Theorem 2, and (5.3) follows. If $\phi(\mu) < 1$, consider $\nu = \mathbb{B}_{1-\phi(\mu)}(\mu)$. By 1° above, $\phi(\nu) = 1$, so there exists $\nu_0 \in \mathcal{M}$ so that $\nu = \mathbb{B}_1(\nu_0)$. Thus, $\mathbb{B}_1(\mu) = \mathbb{B}_{\phi(\mu)}(\nu) = \mathbb{B}_{1+\phi(\mu)}(\nu_0)$, and, from injectivity of $\mathbb{B}_1$ and the semigroup property, $\mu = \mathbb{B}_{\phi(\mu)}(\nu_0)$, proving our statement for $\phi(\mu) < 1$. The similar (and easier) case $\phi(\mu) > 1$ is left to the reader.

3° Let $\mu \in \mathcal{M}$ be such that $0 < \phi(\mu) < 1$. Then we can take convolution powers $\mu^{\mathbb{H}t}$ for any $1 - \phi(\mu) \leq t \leq 1$. Indeed, let us fix such a $t \in [1 - \phi(\mu), 1]$. Then $0 \leq 1 - t \leq \phi(\mu)$ and hence there exists $\nu \in \mathcal{M}$ so that $\mathbb{B}_{1-t}(\nu) = \mu$. Now,

$$\mu = \mathbb{B}_{1-t}(\nu) = (\mu^{\mathbb{H}(2-t)})^{\Phi(2-t)} = (\nu^{\mathbb{H}t})^{\mathbb{H}/t} \quad (\text{by Proposition 3.1}).$$

Since $t \leq 1$, the last term in the above equalities is defined, and thus $\mu^{\mathbb{H}t}$ is well defined by the above and equals $\nu^{\mathbb{H}t}$.

4° We have

$$\phi(\mu \boxtimes \nu) \geq \min\{\phi(\mu), \phi(\nu)\}, \quad \forall \mu, \nu \in \mathcal{M}. \quad (5.4)$$

This follows immediately from the homomorphism property of $\mathbb{B}_t$ proved in Theorem 1, and from 2° above. An immediate consequence of this fact is that if both $\mu$ and $\nu$ are infinitely divisible with respect to $\mathbb{H}$, then so is $\mu \boxtimes \nu$. (This statement is not obvious from the definitions, but can also be derived directly from Equation (3.9) in Proposition 3.5 without resorting to $\mathbb{H}$-divisibility indicators.)
Remark 5.6 It is now easy to derive the concrete values \( \phi(\mu) \) that were listed in Table 1 of the introduction section. Indeed, from Proposition 5.1 it is immediate that \( \mathbb{B}_t(\mu) \) can never have finite support when \( t > 0 \), and this implies in particular that the symmetric Bernoulli distribution \( \mu_0 = \frac{1}{2}(\delta_1 + \delta_{-1}) \) has \( \phi(\mu_0) = 0 \). But for this \( \mu_0 \) we know that (as observed in the above Example 4.5) the measure \( \mathbb{B}_{1/2}(\mu_0) \) is the arcsine law, while \( \mathbb{B}_1(\mu_0) \) is the standard semicircle law. The values of \( \phi(\mu) \) listed on the left column of Table 1 then follow from Equation (5.2) of Remark 5.5.

On the other hand, let us observe that the Marchenko-Pastur distribution of parameter 1 can be written as \( \mathbb{B}_1(\tilde{\mu}_o) \), where \( \tilde{\mu}_o = \frac{1}{2}(\delta_0 + \delta_2) \). Indeed, it is immediately seen by direct calculation that \( \eta_{\tilde{\mu}_o}(z) = z/(1 - z) \), hence (in view of the above Equation (2.20)) the measure \( \mathbb{B}_1(\tilde{\mu}_o) \) is determined by the fact that its \( R \)-transform is \( z/(1 - z) \); but it is well-known that the latter function is exactly the \( R \)-transform of the Marchenko-Pastur (also called free Poisson) distribution of parameter 1 – see for instance Section 2.7 of [22]. The corresponding entry in Table 1 is then obtained by just writing that

\[
\phi(\mathbb{B}(\tilde{\mu}_o)) = 1 + \phi(\tilde{\mu}_o) = 1,
\]

where (same as in the preceding paragraph) the fact that \( \phi(\tilde{\mu}_o) = 0 \) follows from Proposition 5.1 and we used Equation (5.2) of Remark 5.5.

Finally, it remains to look at the case when \( \mu \) is the Cauchy distribution. This special \( \mu \) has the remarkable property that

\[
\mu^{\text{ext}} = \mu^{\text{int}}, \quad \forall \, t \geq 0.
\]  

(In order to verify Equation (5.5) one checks that the measures on its both sides have the same reciprocal Cauchy transform, which is just \( F(z) = z + it \).) From (5.5) it follows that \( \mu \) is fixed by \( \mathbb{B}_t \), for every \( t \geq 0 \), and this in turn makes clear that \( \mu \in \mathbb{B}_t(\mathcal{M}) \) for all \( t \geq 0 \) (hence that \( \phi(\mu) = \infty \), as stated in Table 1).

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