ROOK PLACEMENTS IN YOUNG DIAGRAMS AND PERMUTATION ENUMERATION

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ABSTRACT. Given two operators $\hat{D}$ and $\hat{E}$ subject to the relation $\hat{D}\hat{E} - q\hat{E}\hat{D} = p$, and a word $w$ in $M$ and $N$, the rewriting of $w$ in normal form is combinatorially described by rook placements in a Young diagram. We give enumerative results about these rook placements, particularly in the case where $p = (1-q)/q^2$. This case naturally arises in the context of the PASEP, a random process whose partition function and stationary distribution are expressed using two operators $D$ and $E$ subject to the relation $DE - qED = D + E$ (matrix Ansatz). Using the link obtained by Corteel and Williams between the PASEP, permutation tableaux and permutations, we prove a conjecture of Corteel and Rubey about permutation enumeration. This result gives the generating function for permutations of given size with respect to the number of ascents and occurrences of the pattern 13-2, this is also the moments of the $q$-Laguerre orthogonal polynomials.

1. Introduction

In the recent work of Postnikov [12], permutations appear with a new description, as pattern-avoiding fillings of Young diagrams. More precisely, he made a correspondence between positive Grassmann cells, these pattern-avoiding fillings called J-diagrams, and decorated permutations (which are permutations with a weight 2 on each fixed point). In particular, the usual permutations are in bijection with permutation tableaux, a subset of J-diagrams. Permutation tableaux have then been studied by Steingrimsson, Williams, Burstein, Corteel, Nadeau [1, 6, 7, 14], and revealed themselves very useful in the combinatorics of permutations.

Corteel and Williams observed, and explained, a rather surprising link between these permutation tableaux and the stationary distribution of a classical process of statistical physics, the Partially Asymmetric Self-Exclusion Process (PASEP). This model is described in [7, 8]. More precisely, the stationary probability of a given state in the process is proportional to the sum of weights of permutation tableaux of a given shape. The factor behind this proportionality is the partition function, which is the sum of weights of permutation tableaux of a given half-perimeter.

Another way of finding the stationary distribution of the PASEP is the matrix Ansatz of Derrida, Evans, Hakim, and Pasquier [8]. Suppose that we have operators $D$ and $E$, a row vector $\langle W |$ and a column vector $| V \rangle$ such that

$$DE - qED = D + E, \quad \langle W | E = \langle W |, \quad D | V \rangle = | V \rangle, \quad \text{and} \quad \langle W | | V \rangle = 1.$$

Then, coding any state of the process by a word $w$ of length $n$ in $D$ and $E$, the stationary probability of the state $w$ is given by $\langle W | w | V \rangle (\langle W | (D + E)^n | V \rangle)^{-1}$. This denominator $\langle W | (D + E)^n | V \rangle$ is the partition function.

We briefly describe how the matrix Ansatz is related to permutation tableaux [7]. First, notice that there are unique polynomials $n_{i,j} \in \mathbb{Z}[q]$ such that

$$(D + E)^n = \sum_{i,j\geq 0} n_{i,j} E^i D^j.$$

This sum is called the normal form of $(D + E)^n$. It is particularly useful, since for example the sum of the coefficients $n_{i,j}$ give an evaluation of $\langle W | (D + E)^n | V \rangle$. If $D$ and $E$ would commute, the expansion of $(D + E)^n$ would be described by binomial coefficients. But in this non-commutative context, the process of
operators. They derive the eigenvalues and eigenvectors of \( \hat{S} \). Sasamoto and Wadati use the new relation between a linear operation to a generalization of the inversion number for permutations. Indeed, if the number of crosses is an important statistic on rook placements. This statistic was introduced in [9], as same column) or to the left (in the same row) of a rook (see Figures 5, 6 and 7 further). We will see that \( \circ \) with rooks (denoted by a circle \( (\circ) \)) each cell of the Young diagram that is not below (in the same row) in normal form is combinatorially described by permutation tableaux.

One of the ideas at the origin of this article is the following. From \( D \) and \( E \) of the matrix Ansatz, we define new operators

\[
\hat{D} = \frac{q-1}{q}D + \frac{1}{q} \quad \text{and} \quad \hat{E} = \frac{q-1}{q}E + \frac{1}{q}.
\]

Some immediate consequences are

\[
\hat{D}\hat{E} - q\hat{E}\hat{D} = \frac{1-q}{q^2}, \quad \langle W|\hat{E} = \langle W|, \quad \text{and} \quad \hat{D}|V\rangle = |V\rangle.
\]

This new commutation relation is in a way much more simple than the one satisfied by \( D \) and \( E \). It is close to the relation between creation and annihilation operators classicaly studied in quantum physics. Moreover, from these definitions we have \( q(y\hat{D} + \hat{E}) + (1-q)(y\hat{D} + E) = 1+y \) for some parameter \( y \). By isolating one term of the left-hand side and raising to the \( n \) with the binomial rule, we get the following inversion formulas between \((y\hat{D} + E)^n\) and \((y\hat{D} + \hat{E})^n\):

\[
(1-q)^n(y\hat{D} + E)^n = \sum_{k=0}^{n} \binom{n}{k}(1+y)^{n-k}(-1)^k q^k(y\hat{D} + \hat{E})^k, \quad \text{and} \quad q^n(y\hat{D} + \hat{E})^n = \sum_{k=0}^{n} \binom{n}{k}(1+y)^{n-k}(-1)^k(1-q)^k(y\hat{D} + E)^k.
\]

In particular, the first formula means that if we want to compute the coefficients of the normal form of \((y\hat{D} + E)^n\), it is enough to compute the ones of \((y\hat{D} + \hat{E})^n\) for all \( n \). Notice that taking the normal form is a linear operation.

Up to a factor \(-q\), these operators \( \hat{D} \) and \( \hat{E} \) are also defined in [19] and [2]. In the first reference, Uchiyama, Sasamoto and Wadati use the new relation between \( \hat{D} \) and \( \hat{E} \) to find explicit matrix representations of these operators. They derive the eigenvalues and eigenvectors of \( D + \hat{E} \), and consequently the ones of \( D + E \), in terms of orthogonal polynomials. In the second reference, Blythe, Evans, Colaiori and Essler also use these eigenvalues and obtain an integral form for \( \langle W|(D + E)^n|V\rangle \). They also provide an exact integral-free formula of this quantity, although quite complicated since it contains three sum signs and several \( q \)-binomial coefficients.

In this article, instead of working on representations of \( \hat{D} \) and \( \hat{E} \) and their eigenvalues, we study the combinatorics of the rewriting in normal form of \((\hat{D} + \hat{E})^n\), and more generally \((y\hat{D} + \hat{E})^n\) for some parameter \( y \). In the case of \( \hat{D} \) and \( \hat{E} \), the objects that appear are the rook placements in Young diagrams, long-known since the results of Kaplansky, Riordan, Goldman, Foata and Schützenberger (see [13] and references therein). This method is described in [24], and is the same as the one leading to permutation tableaux or alternative tableaux in the case of \( D \) and \( E \).

**Definition 1.** Let \( \lambda \) be a Young diagram. A rook placement of shape \( \lambda \) is a partial filling of the cells of \( \lambda \) with rooks (denoted by a circle \( \circ \)), such that there is at most one rook per row (resp. per column).

For convenience, we distinguish with a cross (\( \times \)) each cell of the Young diagram that is not below (in the same column) or to the left (in the same row) of a rook (see Figures 5, 6 and 7 further). We will see that the number of crosses is an important statistic on rook placements. This statistic was introduced in [9], as a generalization of the inversion number for permutations. Indeed, if \( \lambda \) is a square of side length \( n \), a rook placements \( R \) with \( n \) rooks may be seen as the graph of a permutation \( \sigma \in S_n \), and then the number of crosses in \( R \) is the inversion number of \( \sigma \).

**Definition 2.** The weight of a rook placement \( R \) with \( r \) rooks and \( s \) crosses is \( w(R) = p^r q^s \).
Theorem 1. For any \( n > 0 \), we have

\[
\langle W | (yD + E)^{n-1} | V \rangle = \frac{1}{(1-q)^n} \sum_{k=0}^{n} (-1)^{k} \sum_{j=0}^{n-k} y^j \left( \binom{n}{j+k} - \binom{n}{j-1} \binom{n}{j+k+1} \right) \left( \sum_{i=0}^{k} q^i \right) (k+1-i).
\]

The combinatorial interpretation of this polynomial, in terms of permutations, is given in Proposition 15.

When \( y = 1 \), this can be specialized to:

Theorem 2. For any \( n > 0 \), we have

\[
\langle W | (D + E)^{n-1} | V \rangle = \frac{1}{(1-q)^n} \sum_{k=0}^{n} (-1)^{k} \left( \binom{2n}{n-k} - \binom{2n}{n-k-2} \right) \left( \sum_{i=0}^{k} q^i \right) (k+1-i).
\]

We can see Theorem 2 as a variation of the Touchard-Riordan formula \([18]\). This classical formula gives the \( q \)-enumeration of fixed-point-free involutions of size \( 2n \) with respect to the number of crossings, and it is also the \( 2n \)th moment of the \( q \)-Hermite polynomials. This formula is:

\[
\sum_{I \in \text{Inv}(2n,0)} q^{\text{cr}(I)} = \frac{1}{(1-q)^n} \sum_{k=0}^{n} (-1)^{k} \left( \binom{2n}{n-k} - \binom{2n}{n-k-2} \right) \left( \sum_{i=0}^{k} q^i \right) \frac{i(k+1)}{2},
\]

where \( \text{Inv}(2n,0) \) is the set of fixed-point-free involutions on \( 2n \) elements, and where the number of crossings \( \text{cr}(I) \) is given in Definition 4.

These two theorems were conjectured by Corteel and Rubey. The earliest conjecture, when \( y = 1 \) and here stated as Theorem 2, was first proved by Rubey and Prellberg \([17]\) in May 2008. The same method also can be used to give an alternative proof of our Theorem 1. This alternative proof, as well as the material of this article, are summarized in the extended abstract \([5]\).

This alternative proof relies on a decomposition of weighted Motzkin paths, which gives a combinatorial explanation of the factor \( \sum_{i=0}^{k} y^i \binom{n}{j+k} - \binom{n}{j-1} \binom{n}{j+k+1} \). But on the other hand, the factor \( \sum_{i=0}^{k} q^i \) is obtained by solving a functional equation and this is a completely non-combinatorial step. It may be possible to use the involution principle instead of a functional equation to obtain \( \sum_{i=0}^{k} y^i q^{i(k+1-i)} \) but this is still an open problem at the time of writing.

Besides references earlier mentioned, we have to point out the previous results of Williams \([24]\), where Corollary 6.3 gives the coefficients of \( y^m \) in \( \langle W | (yD + E)^n | V \rangle \). It was obtained by a more direct approach, via the enumeration of \( J \)-diagrams, and was the only known polynomial formula for the distribution of a permutation pattern of length greater than 2 (see Proposition 15). Whereas Williams’s work is rather focused on \( J \)-diagrams, our results give more simple formulas in the case of permutation tableaux and permutations. Moreover Williams’s formulas have also been obtained by Kasraoui, Stanton and Zeng in their work on orthogonal polynomials \([19]\).

This article is organised as follows. In Section 2, we describe the link between rook placements and the rewriting of \( (D + E)^n \) in normal form. In Sections 3, 4, 5, 6, we obtain enumerative results about rook placements, in particular Section 4 contains the bijective step of this enumeration. In Section 6, we use these results to prove Theorem 1, give the combinatorial interpretation of \( \langle W | (yD + E)^n | V \rangle \) and some applications of the main theorem. In an appendix we give a combinatorial proof of Proposition 12 which gives a generalization of the Touchard-Riordan formula.
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Notations and conventions

We denote by $\text{Par}(n-k,k)$ the set of Young diagrams with exactly $k$ rows and $n-k$ columns, allowing empty rows and columns. The integer $n$ is the half-perimeter of the diagram $\lambda \in \text{Par}(n-k,k)$, and we can see $\lambda$ as an integer partition $(\lambda_1, \ldots, \lambda_k)$ with $n-k \geq \lambda_1 \geq \ldots \geq \lambda_k \geq 0$. We use the French convention.

We denote by $|\lambda|$ the number of cells in $\lambda$, which is also $\sum \lambda_i$.

The North-East boundary of $\lambda \in \text{Par}(n-k,k)$ is a path of $n$ steps, $k$ of them being vertical and $n-k$ horizontal. Reciprocally, for any word $w$ of length $n$ in $\hat{D}$ and $\hat{E}$, with $k$ occurrences of $\hat{E}$, we define $\lambda(w) \in \text{Par}(n-k,k)$ by the following rule: we read $w$ from left to right, and draw one step East for each factor $\hat{D}$, and one step South for each factor $\hat{E}$.

We denote by $\text{Inv}(n,k)$ the set of involutions on $\{1, \ldots, n\}$ with $k$ fixed points.

We use the classical $q$-analogs of integers, factorials, and binomial coefficients:

$$[n]_q = \frac{1-q^n}{1-q}, \quad [n]_q! = \prod_{i=1}^{n} [i]_q, \quad \text{and} \quad \binom{n}{k}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$ 

Proposition 1. \cite{13} The $q$-binomial coefficient has the following combinatorial interpretation:

$$\binom{n}{k}_q = \sum_{\lambda \in \text{Par}(n-k,k)} q^{|\lambda|}, \quad \text{and} \quad q^{k(k+1)/2} \binom{n}{k}_q = \sum_{\lambda \in \text{Par}(n,k), \lambda \text{ has distinct non-zero parts}} q^{|\lambda|}.$$ 

Definition 3. For any $k,n \geq 0$, the Delannoy numbers are defined by

$$\binom{n}{k} = \binom{n}{k} - \binom{n}{k-1}.$$ 

Proposition 2. When $2k \leq n$, the number $\binom{n}{k}$ counts the left factors of Dyck paths of $n$ steps ending at height $n-2k$. In particular, $\binom{2n}{n}$ is the $n$th Catalan number. They satisfy the relations:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}, \quad \binom{n}{n-k} = \binom{n}{k}, \quad \binom{0}{0} = 1, \quad \text{and} \quad \binom{n}{k} = 0 \text{ if } k \notin \{0, \ldots, n+1\}.$$ 

Proof. The number of left factors of Dyck paths of $n$ steps ending at height $n-2k$ is easily seen to satisfy the same relations as $\binom{n}{k}$: we just have to distinguish two cases whether the last step is going up or down. \qed
2. FROM OPERATOR RELATIONS TO ROOK PLACEMENTS

In this section, we make the link between the coefficients of the normal form of \((\hat{D} + \hat{E})^n\), and rook placements in Young diagrams. This is done via a combinatorial description of the rewriting in normal form. When \(q = 1\), we can view it as a combinatorial statement of a classical result in statistical physics, called Wick’s theorem. The principle of this method is the same as the one described in the introduction, making the link between \(D\) and \(E\) and permutation tableaux. Moreover the results of these section are presented in \([20]\) in a slightly different form.

From now on we assume that \(\hat{D}\) and \(\hat{E}\) are such that \(\hat{D}\hat{E} - q\hat{E}\hat{D} = p\) for some parameter \(p\), which is a slight generalization of the relation \((\Pi)\). As in the case of \(D\) and \(E\), any word \(w\) in \(\hat{D}\) and \(\hat{E}\) can be uniquely written in normal form:

\[
w = \sum_{i,j \geq 0} c_{i,j}(w)\hat{E}^i\hat{D}^j,
\]

where \(c_{i,j}(w) \in \mathbb{Z}[p,q]\). We have:

\[
\langle W|w|V \rangle = \sum_{i,j \geq 0} c_{i,j}(w).
\]

The combinatorial interpretation of this polynomial is given by the following proposition:

**Proposition 3.** Let \(w\) be a word in \(\hat{E}\) and \(\hat{D}\). Then \(\langle W|w|V \rangle\) is the sum of weights of rook placements of shape \(\lambda(w)\).

**Proof.** Let us denote by \(T_w\) the sum of weights of rook placements of shape \(\lambda(w)\). We prove with a recurrence on \(|\lambda(w)|\), that \(T_w = \langle W|w|V \rangle\).

The base case, \(|\lambda(w)| = 0\), is the situation where the word \(w\) is already in normal form: \(w = \hat{E}^i\hat{D}^j\) for some \(i\) and \(j\). So we directly have \(\langle W|w|V \rangle = 1\) from the properties of \(\langle W|\) and \(|V|\) given in \((\Pi)\). This 1 corresponds to the unique rook placement of shape \(\lambda(w)\), which contains no rook and no cross since there is no cell in this diagram.

Now we assume that \(|\lambda(w)| > 0\). It is possible to factorize \(w\) into \(w = w_1\hat{D}\hat{E}w_2\). Indeed, this factor \(\hat{D}\hat{E}\) corresponds to a corner of \(\lambda(w)\), and there is at least one corner since \(|\lambda(w)| > 0\). The commutation relation of \(\hat{D}\) and \(\hat{E}\) gives:

\[
w = qw_1\hat{E}\hat{D}w_2 + pw_1w_2, \quad \text{hence} \quad \langle W|w|V \rangle = q\langle W|w_1\hat{E}\hat{D}w_2|V \rangle + p\langle W|w_1w_2|V \rangle.
\]

So it remains to show that the same relation holds for \(T_w\):

\[
T_w = qT_{w_1\hat{E}\hat{D}w_2} + pT_{w_1w_2}.
\]

To this end, we distinguish two kinds of rook placements of shape \(\lambda(w)\), whether the corner corresponding to the factor \(\hat{D}\hat{E}\) contains a cross or a rook. It cannot be empty, since being a corner there is no cell above it or to its right that may contain a rook.

The sum of weights of rook placements of the first type is \(qT_{w_1\hat{E}\hat{D}w_2}\). Indeed, when removing the corner the weight is divided by \(q\) and we can get any rook placement of shape \(\lambda(w_1\hat{D}\hat{E}w_2)\).

Similarly, the sum of weights of rook placements of the second type is \(pT_{w_1w_2}\). Indeed, when removing the corner, its row and its column, the weight is divided by \(p\) and we can get any rook placement of shape \(\lambda(w_1w_2)\).

Since \((\hat{D} + \hat{E})^n\) expands into the sum of all words of length \(n\) in \(\hat{D}\) and \(\hat{E}\), we also obtain:

**Proposition 4.** For any \(n\), \(\langle W|(\hat{D} + \hat{E})^n|V \rangle\) is the sum of weights of rook placements of half-perimeter \(n\).

We can also expand \((y\hat{D} + \hat{E})^n\) and get the sum over all words of length \(n\) in \(\hat{D}\) and \(\hat{E}\). But in this case each word \(w\) has a coefficient \(y^n\), where \(m\) is the number of occurrences of \(\hat{D}\) in \(w\). Via the correspondence between words and Young diagrams, the number of occurrences of \(\hat{D}\) in \(w\) is the number of columns in \(\lambda(w)\). This leads to a refined version of the previous proposition.
Proposition 5. For any $n$, $\langle W | (y\hat{D} + \hat{E})^n | V \rangle$ is the generating function for rook placements of half-perimeter $n$, the parameter $y$ counting the number of columns.

3. Basic results about rook placements

In this section we introduce the recurrence relation which will be used in the enumeration of rook placements, and we present two simple examples of enumeration. These two examples involve the $q$-binomial coefficients and the Delannoy numbers defined at the end of the introduction, and they introduce the more general formulas we will show later.

Definition 4. Let $T_{j,k,n}(p,q)$ be the sum of weights of rook placements of half-perimeter $n$, with $k$ rows, and with $j$ rows containing no rook (or equivalently, with $k-j$ rooks). We also define:

$$T_{k,n}(p,q) = \sum_{j=0}^{k} T_{j,k,n}(p,q), \quad \text{and} \quad T_n(p,q,y) = \sum_{k=0}^{n} y^k T_{k,n}(p,q).$$

So $T_{k,n}(p,q)$ is the sum of weights of rook placements of half-perimeter $n$ with $k$ rows, and $T_n(p,q,y)$ is the generating function of rook placements of half-perimeter $n$, the parameter $y$ counting the number of rows.

Since there is an obvious transposition-symmetry, we can also view the parameter $y$ as counting the number of columns. These are polynomials in the variables $p$, $q$ and $y$, so we will sometimes omit the arguments. From Proposition 5 we know that $T_n(p,q,y)$ is equal to $\langle W | (y\hat{D} + \hat{E})^n | V \rangle$. In Figure 1 we give some examples of these polynomials.

$$T_{0,1,3} = pq + p + p, \quad T_{1,1,3} = 1 + q + q^2, \quad T_{2} = 1 + (1 + q + p)y + y^2.$$

Figure 1. Some small values of $T_{j,k,n}$ and $T_n$, together with the rook placements corresponding to each term.

Proposition 6. We have the following recurrence relation:

$$(5) \quad T_{j,k,n} = T_{j-1,k-1,n-1} + q^j T_{j,k,n-1} + p[j+1]q T_{j+1,k,n-1}.$$ 

Proof. We can distinguish three kinds of rook placements enumerated by $T_{j,k,n}$ (see Figure 2):

- the first column is of size strictly less than $k$,
- the first column is of size $k$ and contains no rook,
- or the first column is of size $k$ and contains exactly one rook.

We show that these three types respectively lead to the three terms of the recurrence relation.

Figure 2. The three kinds of rook placements we distinguish for proving Proposition 6.

The first case is the situation where the first step of the North-East boundary is a step down, or equivalently the first row is of size 0. Removing this step (or row) is a bijection between these first-type rook placements, and the ones enumerated by $T_{j-1,k-1,n-1}$, the first term of (5).
In the second case, the first column contains exactly \( j \) crosses, one per row without rook. So removing the first column is a bijection between the second-type rook placements, and the ones enumerated by \( T_{j,k,n-1} \), and this bijection changes the weight by a factor \( q^j \). This explains the second term of (5).

In the third case, removing the first column is not a bijection since there are several possibilities for the position of the rook in this column. But this map has the property that for any \( R \) enumerated by \( T_{j+1,k,n-1} \), the preimage set of \( R \) contains \( j + 1 \) elements, and their weights are \( pw(R), pqw(R), \ldots, pq^jw(R) \). See Figure 3 for an example. This shows that the sum of weights of the third-type rook placements is the third term of (5), and completes the proof. □

![Figure 3](image)

**Figure 3.** We have here four rook placements of the third type, which are equal when we remove the first column. Here we have \( n = 10, k = 6, j = 3 \), and the sum of their weights is \((p + pq + pq^2 + pq^3)p^2q^3 = p[j + 1]q(p^2q^3)\). This illustrates the third term of (5).

**Proposition 7.** For any \( k, n \) we have:

\[
T_{k,k,n} = \left\{ \begin{array}{ll}
\binom{n}{k} & \\
q & 
\end{array} \right.
\]

**Proof.** We are counting rook placements without any rook, \( i.e. \) such that all cells contain a cross. So this is a direct application of Proposition 1. □

This proposition is illustrated for example in Figure 1 where we see that \( T_{1,1,3} = 1 + q + q^2 = [3]_q \). The second example of this section is more subtle and we begin with the following lemma.

**Lemma 1.** Given a Young diagram \( \lambda \), the number of rook placements of shape \( \lambda \) having no cross and exactly one rook per row is either 0 or 1. It is 1 in the case where the North-East boundary is a Dyck path (which means that the \( i \)th row of \( \lambda \) starting from the top contains at least \( i \) cells, for any \( i \) between 1 and the number of rows).

**Proof.** Suppose that \( R \) is a rook placement with no cross and exactly one rook per row. Then the \( i \) first rows contain \( i \) rooks, which are necessarily in \( i \) different columns. So the \( i \)th row contain at least \( i \) cells. this is true for any \( i \), so the North-East boundary is a Dyck path.

It remains to prove that there is a unique such rook placement in the case where the North-East boundary of a Young diagram \( \lambda \) is a Dyck path. We show that there is only one way to build this rook placement starting from an empty diagram \( \lambda \). First, notice that each corner of the diagram must contain a rook (as we saw in previous section, the general statement is that each corner contains either a rook or a cross). Then, if we consider the subdiagram of cells that are not in the same row or column of these rooks (see Figure 3), again all corners of this subdiagram must contain a rook by the same argument. We can even say that his North-East boundary is also a Dyck path: indeed, the boundary of the subdiagram is obtained from the boundary of the diagram by removing each occurrence of a step right followed by a step down. So we can conclude by recurrence. □

**Proposition 8.** If \( 2k < n \), we have:

\[
T_{0,k,n}(p,0) = p^k \binom{n}{k}.
\]

**Proof.** We are counting rook placements with no cross (since \( q = 0 \) here) and exactly \( k \) rooks. Each of these rook placements has weight \( p^k \), so we just have to prove that there are \( \binom{n}{k} \) such rook placements. Knowing that \( \binom{n}{k} \) is the number of left factors of Dyck paths of \( n \) steps ending at height \( n - 2k \), this is a consequence of the previous lemma. □
4. Rook placements and involutions

In this section we present the bijective step of the enumeration of rook placements. Indeed, the recurrence (5) is rather complicated to be solved directly. But thanks to this bijective step, we show that there is a simple relation between $T_{j,k,n}$ and $T_{0,k-j,n}$, and also that there is a simple recurrence relation satisfied by $T_{0,k,n}$.

Given a rook placement $R$ of half-perimeter $n$, we define an involution $\alpha(R)$ by the following construction. We label the North-East boundary of $R$ with integers from 1 to $n$, as shown in the left part of Figure 5. This way, each column and each row has a label between 1 and $n$. If a column, or a row, is labelled with $i$ and does not contain a rook, then $i$ is a fixed point of $\alpha(R)$. If there is a rook at the intersection of column $i$ and row $j$, then the involution $\alpha(R)$ sends $i$ to $j$ (and $j$ to $i$).

Given a rook placement $R$ of half-perimeter $n$, we also define a Young diagram $\beta(R)$ by the following construction. The North-East boundary of $R$ is a sequence of East steps and South steps, respectively denoted by $S$ and $E$. In this sequence, we overline the letter $E$ if it corresponds to a East step on top of a column containing a rook. Similarly, we overline the letter $S$ if it corresponds to a South step on the right of a row containing a rook. The non-overlined letters give a sub word of this sequence, and $\beta(R)$ is the Young diagram corresponding to this sequence.

For example, with $R$ given in Figure 5, the North-East boundary is $EEEESEES$. With the overlined letters we obtain $\overline{EEEESEES}$. The non-overlined letters give the word $EES$. So in this case $\beta(R)$ is the rectangular Young diagram with 1 row and 2 columns. See Figure 6 for another example.

Figure 6. Example of a rook placement and its image by the map $\beta$. The grey cells are the ones that are in the same row or the same column as a rook. The shape of the rook placement is given by the sequence $\overline{EEEESEES}$. So $\beta(R)$ is defined by the sequence $ESE$. 

Remark: We can see in Figure 8 that the cells of \( \beta(R) \) are obtained from \( R \) by removing all gray cells. This is a general fact, and \( |\beta(R)| \) is the number of cells in \( R \) with no rook in the same row and no rook in the same column.

We also define \( \phi(R) = (\alpha(R), \beta(R)) \). See Figure 7 for another example of a rook placement and its images by the maps \( \alpha \) and \( \beta \).

\[
R = \begin{array}{cccc}
\times & \times & \times & \\
\circ & \times & \times & \\
\circ & \times & \times & \\
\times & \times & \times & \\
\end{array}
\quad \quad \phi(R) = (\phi_1, \phi_2) = \left( \begin{array}{cc}
\cdots & \cdots \\
\circ & \times \\
\times & \circ \\
\end{array}, \begin{array}{c}
\circ \\
\circ \\
\end{array} \right)
\]

Figure 7. Example of a rook placement and its image by the map \( \phi \).

**Proposition 9.** The map \( \phi \) is a bijection between rook placements in Young diagrams of half-perimeter \( n \), and couples \( (I, \lambda) \) where \( I \) is an involution on \( \{1, \ldots, n\} \) and \( \lambda \) a Young diagram of half-perimeter \( |\text{Fix}(I)| \). If \( \phi(R) = (I, \lambda) \), the number of rows (resp. columns) of \( \lambda \) is equal to the number of rows (resp. columns) without a rook in \( R \).

This is a classical argument so we don’t give a complete proof. This bijection was already defined in [11], in terms of partial involutions, i.e. involutions on subsets of \( \{1, \ldots, n\} \). These partial involutions are equivalent to the couples \( (I, \lambda) \) in the sense that they are involutions with a weight 2 for each fixed point.

Indeed, a couple \( (I, \lambda) \) may be seen as an involution with two kinds of fixed points: those corresponding to vertical steps in \( \lambda \) and those corresponding to horizontal steps. Similarly, a partial involution \( I \) on \( \{1, \ldots, n\} \) may also be seen as an involution on \( \{1, \ldots, n\} \) with two kinds of fixed points: the ones that are not in the domain of \( I \), and the ones that are in the domain and fixed by \( I \).

Now that we have built a bijection, it remains to describe how the weight of a rook placement reads in the couple \( (I, \lambda) \). We need the following definitions.

**Definition 5.** For any involution \( I \), we call
- an arch of \( I \), a couple \( (i,j) \) such that \( i < j \) and \( I(i) = j \),
- a crossing of \( I \), a pair of arches \( ((i,j),(k,l)) \) such that \( i < k < j < l \),
- the height of a fixed point \( k \in \text{Fix}(I) \) the number of arches \( (i,j) \) such that \( i < k < j \).

We denote by \( \text{cr}(I) \) the number of crossings of \( I \), and by \( \text{ht}(k) \) the height of the fixed point \( k \).

For example, let us consider the involution \( \alpha(R) \) in Figure 4. There are two crossings, \(((1,6),(4,9))\) and \(((1,6),(5,8))\). The fixed points are \( \text{Fix}(I) = \{2,3,7,10\} \) and their respective heights are 1, 1, 2 and 0.

**Proposition 10.** Let \( (I, \lambda) = \phi(R) \). Then:
- each crossing of \( I \) corresponds to a cell of \( R \) containing a cross, having a rook to its left (in the same row) and a rook below (in the same column).
- each triple \( (i,k,j) \) such that \( i < k < j \), \( k \in \text{Fix}(I) \) and \( (i,j) \) is an arch of \( I \) corresponds to a cell of \( R \) containing a cross, having either a rook to its left (in the same row) or a rook below (in the same column).

**Proof.** These two statements are respectively illustrated in the left part and the right part of Figure 8.

- Let \( ((i,j),(k,l)) \) be a crossing of \( I \). Since \( k < j \), column \( k \) intersects row \( j \) in some cell \( c \). Then, \( (i,j) \) is an arch of \( I \), which means that there is a rook at the intersection of column \( i \) and row \( j \), to the left of the cell \( c \). Similarly, \( (k,l) \) is an arch so there is a rook at the intersection of column \( k \) and row \( l \), below the cell \( c \). So to this crossing \( ((i,j),(k,l)) \) we associate the cell \( c \).
Let \((i, k, j)\) be such that \(i < k < j\), \(k \in \text{Fix}(I)\) and \((i, j)\) is an arch of \(I\). We suppose for example that \(k\) is the label of a column. Since \(k < j\), row \(j\) intersects column \(k\) in some cell \(c\). There is no rook below the cell \(c\) because \(k\) is a fixed point of \(I\). But there is a rook in row \(k\), to the left of \(c\). So to this triple \((i, k, j)\) we associate the cell \(c\).

\[
\begin{array}{c}
\text{Figure 8. Interpretation of crossings, and sum of heights of fixed points, in terms of rook placements.}
\end{array}
\]

**Proposition 11.** If \(\phi(R) = (I, \lambda)\) then the number of crosses in \(R\) is \(|\lambda| + \mu(I)\), where \(\mu\) is the statistic on involutions defined by

\[
\mu(I) = \text{cr}(I) + \sum_{x \in \text{Fix}(I)} \text{ht}(x).
\]

**Proof.** From the definition of the map \(\beta\), we directly see that \(|\lambda| = |\beta(R)|\) is the number of crosses in \(R\) with no rook in the same row and no rook in the same column. Besides, from Proposition 10 we know that the number of crossings \(\text{cr}(I)\) counts the crosses of \(R\) with one rook to its left and one rook below. From the same proposition, we also know that the sum of heights of fixed points counts all remaining crosses. \(\Box\)

The previous proposition means that the number of crosses in rook placements is an additive parameter with respect to the decomposition \(R \mapsto (I, \lambda)\). This situation naturally leads to a factorisation of the corresponding generating functions, so we get the following corollary:

**Corollary 1.** For any \(j,k,n\), we have

\[
T_{j,k,n} = \left[ n - 2k + 2j \right]_q T_{0,k−j,n}.
\]

**Proof.** We assume that \(n - 2k + 2j \geq 0\), since otherwise both sides are 0. Indeed, a rook placement enumerated by \(T_{j,k,n}\) contains \(k - j\) rooks, so it has at least \(k - j\) different rows and \(k - j\) different columns, so its half-perimeter is at least \(2k - 2j\). Using the bijection \(\phi\), we can compute \(T_{j,k,n}\) by summing the weights of couples \((I, \lambda)\) where \(I \in \text{Inv}(n, n - 2k + 2j)\) and \(\lambda \in \text{Par}(n - 2k + j, j)\). Hence:

\[
T_{j,k,n} = p^{k-j} \sum_{(\lambda, I)} q^{|\lambda| + \mu(I)} \left( \sum_{\lambda} q^{\lambda} \right) \left( p^{k-j} \sum_{I} q^{\mu(I)} \right).
\]

The first factor of the right-hand side is \(\left[n - 2k + 2j\right]_q\) by a direct application of Proposition 11. The second factor can be seen as a sum over couples \((I, \lambda)\) where \(\lambda\) has 0 rows and \(n - 2k + 2j\) columns. So using again the bijection \(\phi\), this second factor is \(T_{0,k−j,n}\). \(\Box\)

Thanks to this factorisation property of \(T_{j,k,n}\), the problem is reduced to the evaluation of \(T_{0,k,n}\). But this factorisation property also gives a recurrence relation satisfied by \(T_{0,k,n}\).

**Corollary 2.** We have the following recurrence relation:

\[
T_{0,k,n} = T_{0,k,n−1} + p[n+1−2k]_q T_{0,k−1,n−1}.
\]

**Proof.** When \(j = 0\), the relation 6 gives

\[
T_{0,k,n} = T_{0,k,n−1} + p T_{1,k,n−1}.
\]

Applying the previous corollary to the second term of this sum gives the desired equality. \(\Box\)
5. Enumeration of rook placements

In this section we solve the recurrence (11), and we obtain an expression for $T_{0,k,n}$ involving both $q$-binomials and Delannoy numbers, generalizing the two examples of Section 3. Using the factorisation property of $T_{j,k,n}$ and summing over $j$, we obtain an expression for

$$T_{k,n} = \sum_{j=0}^{k} T_{j,k,n},$$

i.e. for the sum of weights of rook placements of half-perimeter $n$ with $k$ rows. This expression is rather lengthy, with a sum over three indices, but for certain values of $p$ we can simplify it with the $q$-binomial identities of Lemma 2. So in these particular specializations we get expressions for $T_{k,n}$ and $T_n$ without $q$-binomials.

**Proposition 12.** When $p = 1 - q$, we have

$$(10) \quad T_{0,k,n}(1-q,q) = \sum_{i=0}^{k} (-1)^i q^{\frac{i(i+1)}{2}} \binom{n-2k+i}{i} q^{\binom{n}{k-i}}.$$

**Proof.** We give here a recursive proof. In the appendix we give an alternative proof, which is much more combinatorial.

Let us denote by $f(k,n)$ the right-hand side of (10). The initial condition is $f(k,0) = T_{0,k,0} = \delta_{0k}$ so it remains to check relation (11) when $p = 1 - q$. Let us define

$$A = \binom{n-1-2k+i}{k-i}_q, \quad B = q^{n-2k} \binom{n-1-2k+i}{i-1}_q,$$

$$C = \binom{n-1}{k-i}_q, \quad D = \binom{n-1}{k-i-1}_q,$$

so that we have

$$f(k,n) = \sum_{i=0}^{k} (-1)^i q^{\frac{i(i+1)}{2}} \binom{n-2k+i}{i} q^{\binom{n}{k-i}}.$$ 

After expanding this sum, the second term gives

$$\sum_{i=0}^{k} (-1)^i q^{\frac{i(i+1)}{2}} BC = -\sum_{i=0}^{k-1} (-1)^i q^{\frac{i(i+1)}{2}} q^{n-2k} \binom{n-2k+i}{i} q^{\binom{n}{k-i}} \binom{n-1}{k-i}_q.$$

(The sum is reindexed such that $i$ becomes $i+1$. And the third term gives

$$\sum_{i=0}^{k} (-1)^i q^{\frac{i(i+1)}{2}} (A + B)D = \sum_{i=0}^{k-1} (-1)^i q^{\frac{i(i+1)}{2}} \binom{n-2k+i}{i} q^{\binom{n}{k-i}} \binom{n-1}{k-i-1}_q.$$  

(after noticing that the term where $i = k$ is 0). Adding the previous two identities yields

$$\sum_{i=0}^{k} (-1)^i q^{\frac{i(i+1)}{2}} (BC + AD + BD) = \sum_{i=0}^{k-1} (-1)^i q^{\frac{i(i+1)}{2}} \binom{n-2k+i}{i} q^{\binom{n}{k-i}} (1 - q^{n-2k+i+1}).$$

But we have $[n-2k+i]_q [n-2k+i]_q = [n-2k+1]_q [n-2k+i+1]_q$, hence

$$\sum_{i=0}^{k} (-1)^i q^{\frac{i(i+1)}{2}} (BC + AD + BD) = \sum_{i=0}^{k-1} (-1)^i q^{\frac{i(i+1)}{2}} \binom{n-2k+i+1}{i} q^{\binom{n}{k-i}} (1 - q^{n-2k+1})$$

$$= (1 - q^{n-2k+1}) f(k-1,n-1).$$
Since $\sum_{i=0}^{k}(-1)^i q^{\binom{i+1}{2}} A C$ readily gives $f(k, n - 1)$, we get the relation
\[ f(k, n) = f(k, n - 1) + (1 - q^n - 2k) f(k - 1, n - 1), \]
which is precisely (9) when $p = 1 - q$.

Remark: The rook placements enumerated by $T_{0,k,n}$ contain exactly $k$ rooks, so $T_{0,k,n}(p, q) = p^k T_{0,k,n}(1, q)$. This shows that there is no loss of generality in the assumption $p = 1 - q$ of the previous proposition. Moreover, the Touchard-Riordan formula (4) mentioned in the introduction is a particular case of (10). Indeed, via the bijection of the previous section, involutions without fixed points correspond to rook placements with exactly one rook per row and one rook per column (therefore with as many rows as columns). So knowing (10), we directly obtain (4):
\[
\sum_{t \in \text{Inv}(2n, 0)} q^{|t|} = T_{0, n, 2n}(1, q) = (1-q)^n T_{0, n, 2n}(1 - q, q) = \frac{1}{(1-q)^n} \sum_{i=0}^{n} (-1)^i \left\{ \begin{array}{c} 2n \\ n-i \end{array} \right\} q^{i+i+1}.
\]

Now using (8) and (10), we have the following equality:
\[ T_{j,k,n}(1 - q, q) = \left[ \begin{array}{c} n - 2k + 2j \\ j \end{array} \right] \sum_{i=0}^{k-j} (-1)^i q^{i+i+1} \left[ \begin{array}{c} n - 2k + 2j + i \\ i \end{array} \right] \left[ \begin{array}{c} n \\ k - j - i \end{array} \right].
\]

And as in the previous remark, $T_{j,k,n}(p, q) = p^{k-j} T_{j,k,n}(1, q)$ so that we have a similar expression for any value of $p$. Summing it over $j$ will give an expression for $T_{k,n}(p, q)$. For certain values of $p$, it is possible to simplify greatly this sum. To perform this simplification we need the following lemma:

Lemma 2. For any $k, n \geq 0$ we have the following $q$-binomial identities:
\[ \sum_{j=0}^{k} (-1)^j q^{\binom{j+1}{2}} \left[ \begin{array}{c} n - j \\ j - k \end{array} \right] q^{n-k} = 1, \]
\[ \sum_{j=0}^{k} (-1)^j q^{\binom{j+1}{2}} \left[ \begin{array}{c} n - j \\ j - k \end{array} \right] q^{n-k} = q^{k(n-k)}, \]
\[ \sum_{j=0}^{k} (-1)^j q^{\binom{j+1}{2}} \left[ \begin{array}{c} n - j \\ j - k \end{array} \right] q^{n-k} = q^{k+1}(1-q)^{\binom{j+1}{2}} q^{k(n-k) - q^{k(n-k)} + q^{k(n+1-k)} - q^{k+1}(n+1-k)} \]
\[ q^{n-1}(1-q). \]

Proof. The first two are proved combinatorially using Proposition 1. We first prove (13), because it is slightly more simple. It seems that there is no simple combinatorial proof of (14) so we prove it with a recurrence, which is quite similar to the one of Proposition 12.

- The left-hand side of (13) counts the pairs $(\lambda, \mu) \in \text{Par}(n - k, k - j) \times \text{Par}(n - k - 1, j)$, for some $j$ between 0 and $k$, signed by $(-1)^j$ and such that $\mu$ has distinct parts. More precisely, $\lambda$ is such that $n - k \geq \lambda_1 \geq \ldots \geq \lambda_{k-j} \geq 0$ and $\mu$ is such that $n - k > \mu_1 > \ldots > \mu_j \geq 0$. When $k - j > 0$, such a couple $(\lambda, \mu)$ satisfying $\lambda_{k-j} < \mu_j$ or $\mu = (\emptyset)$ can be paired with the couple $(\lambda', \mu')$ such that:
\[ \lambda' = (\lambda_1, \ldots, \lambda_{k-j-1}), \quad \mu' = (\mu_1, \ldots, \mu_j, \lambda_{k-j}). \]

This couple satisfies $|\lambda| + |\mu| = |\lambda'| + |\mu'|$ but it has opposite sign. The only couple which is not paired with any other is such that $\lambda_1 = \ldots = \lambda_k = n - k$ and $\mu = (\emptyset)$, it contributes to the sum with a $q^{k(n-k)}$.

- The proof of (12) is quite similar. Here the factor $q^{(j+1)/2}$ means that we count pairs $(\lambda, \mu)$ as before but such that $n - k \geq \mu_1 > \ldots > \mu_j > 0$, (because $(j+1)/2 = 1 + \ldots + j$). Now the pairing is done by comparing the smallest non-zero part of $\lambda$ with the smallest part of $\mu$. Depending on the situation, one of these parts is moved from $\lambda$ to $\mu$, or from $\mu$ to $\lambda$. The only couple $(\lambda, \mu)$ which is not paired with any other is such that $\lambda_1 = \ldots = \lambda_k = 0$ and $\mu = (\emptyset)$, and it contributes to the sum with a 1.
When \( k = 0 \), both sides of (14) are equal to \( q \). Let us denote by \( g(n, k) \) the left-hand side of (14). We define:

\[
A = q^{n-k} {n-k \choose j}, \quad B = {n-j \choose n-k-1}, \quad C = {n-k-1 \choose j}, \quad D = q^{n-k-j} {n-k-1 \choose j-1},
\]

so that \( g(n+1, k+1) = \sum_{j=0}^{k+1} (-1)^{j} q^{(j-1)(j-2)/2} (A + B)(C + D) \). After expanding this product, we get the recurrence relation

\[
g(n+1, k+1) = q^{n-k} g(n, k) + g(n, k+1) - q^{n-k} g(n-1, k).
\]

In view of the simple expression of the right-hand side of (14), it is straightforward to check that it satisfies the same relation.

\[\Box\]

**Proposition 13.**

\[
T_{k,n}(1-q, q) = {n \choose k}, \quad T_{k,n} \left( \frac{1-q}{q} \right, q \right) = \sum_{j=0}^{k} \binom{n}{j} q^{(k-j)(n-k-j)},
\]

(15)

\[
T_{k,n} \left( \frac{1-q}{q} \right, q \right) = \sum_{j=0}^{k} \binom{n}{j} \left( \frac{q^{(k+1-j)(n-k-j)-q^{k-j}(n-k-j)}+q^{k-j}(n+1-k-j)-q^{(k+1-j)(n+1-k-j)}}{(1-q)q^{n-j}} \right).
\]

(16)

**Proof.** The three identities of this proposition respectively come from (12), (13) and (14). We prove only the last one, because it is the most important case. The two others are proved similarly but more simply. Multiplying (11) by \( q^{2j-2k} \) and summing over \( j \) gives

\[
T_{k,n} \left( \frac{1-q}{q} \right, q \right) = \sum_{j=0}^{k} q^{2j-2k} \binom{n-2k+2j}{j} \sum_{i=0}^{k-j} (-1)^{i} q^{\frac{i(i+1)}{2}} \binom{n-2k+2j+i}{i} \binom{n}{k-j-i}.
\]

Introducing \( l = k - j - i \), we get:

\[
T_{k,n} \left( \frac{1-q}{q} \right, q \right) = \sum_{l=0}^{k} \binom{n}{l} \sum_{j=0}^{k-l} q^{2j-2l} \binom{n-2k+2j}{j} (-1)^{j-l} q^{\frac{(k-j-l)(k-j-l+1)}{2}} \binom{n-k-j-l}{k-j-l}.
\]

and after replacing \( j \) with \( k - l - j \) we also have:

\[
T_{k,n} \left( \frac{1-q}{q} \right, q \right) = \sum_{l=0}^{k} \binom{n}{l} \sum_{j=0}^{k-l} q^{-2l-2j} \binom{n-2l-j}{j} (-1)^{j-l} q^{\frac{(j+1)(j)}{2}} \binom{n-2l-j}{j}.
\]

At this point we can apply (13) with \( n' = n - 2l \) and \( k' = k - l \), and get (16).

\[\Box\]

**Remark:** By an obvious argument of symmetry by transposition, we have \( T_{k,n} = T_{n-k,n} \), and this can be directly seen in (10). The summand \( q^{(k+1-j)(n-k-j)} - q^{k-j)(n-k-j)} + q^{k-j)(n+1-k-j)} - q^{(k+1-j)(n+1-k-j)} \) is unchanged when \( j \) is replaced with \( n+1-j \). Besides, we have \( \binom{n}{j} = - \binom{n}{n+1-j} \), so

\[
\sum_{j=k+1}^{n-k} \binom{n}{j} \left( \frac{q^{(k+1-j)(n-k-j)} - q^{(k-j)(n-k-j)} + q^{(k-j)(n+1-k-j)} - q^{(k+1-j)(n+1-k-j)}}{(1-q)q^{n-j}} \right) = 0.
\]
A consequence is that in (16) instead of summing over \( j \) between 0 and \( k \), we can sum over \( j \) between 0 and \( \min(k, n-k) \). This is also true for the second identity of (15). In this form, it is clear that \( T_{k,n} = T_{n-k,n} \).

The last step of this section is the summing over \( k \) to get an expression for \( T_n(\frac{1}{q^2}, q, y) \).

**Proposition 14.**

\[
(1 - q)q^n T_n \left( \frac{1}{q^2}, q, y \right) = (1 + y)G(n) - G(n + 1),
\]

where \( G(n) = \sum_{j=0}^n \binom{n}{j} \sum_{i=0}^{n-2j} y^{i+j-1} q^{i(n+1-2j-i)} \).

**Proof.** First we define

\[
P_k = \sum_{i=0}^k y^i q^{(k+1-i)}.\]

We have to multiply (16) by \( y^k \), and sum over \( k \) between 0 and \( n \). This gives:

\[
(1 - q)q^n T_n \left( \frac{1}{q^2}, q, y \right) = \sum_{0 \leq j \leq k \leq n} y^k \binom{n}{j} \left( q^{(k+1-j)(n-k-j)} - q^{(k-j)(n-k-j)} + q^{(k-j)(n+1-k-j)} - q^{(k+1-j)(n+1-k-j)} \right)
\]

\[
= \sum_{j=0}^n \binom{n}{j} \left( \sum_{k=j}^n y^k q^{(k+1-j)(n-k-j)} - \sum_{k=j}^n y^k q^{(k-j)(n-k-j)} + \sum_{k=j}^n y^k q^{(k-j)(n+1-k-j)} - \sum_{k=j}^n y^k q^{(k+1-j)(n+1-k-j)} \right)
\]

\[
= \sum_{j=0}^n \binom{n}{j} \left( \sum_{i=1}^{n-j} y^{i+j-1} q^{i(n+1-2j-i)} - \sum_{i=0}^{n-j} y^{i+j} q^{i(n-2j-i)} + \sum_{i=0}^{n-j} y^{i+j} q^{i(n+1-2j-i)} - \sum_{i=1}^{n+1-j} y^{i+j-1} q^{i(n+2-2j-i)} \right),
\]

after a reindexing of the second and third sums with \( i = k - j \), and of the first and fourth sums with \( i = k + 1 - j \). Since \( (1 - q)q^n T_n \) is a polynomial, we can discard all negative powers of \( q \) appearing in these sums. Modulo non-positive powers of \( q \), these four sums are respectively equal to \( y^{j-1} P_{n-2j} \), \( y^j P_{n-1-2j} \), \( y^j P_{n-2j} \), \( y^{j-1} P_{n+1-2j} \). But we have to be careful when it comes to the constant terms in \( q \). These constant terms are respectively:

\[
[q^0] \sum_{i=1}^{n+1-j} y^{i+j-1} q^{i(n+1-2j-i)} = y^{n-j} \chi_{\{1 \leq n+1-2j \leq n-j+1\}},
\]

\[
[q^0] \sum_{i=0}^{n-j} y^{i+j} q^{i(n-2j-i)} = 1 + y^{n-j} \chi_{\{0 \leq n-2j \leq n-j\}},
\]

\[
[q^0] \sum_{i=0}^{n-j} y^{i+j} q^{i(n+1-2j-i)} = 1 + y^{n+1-j} \chi_{\{0 \leq n+1-2j \leq n-j\}},
\]

\[
[q^0] \sum_{i=1}^{n+1-j} y^{i+j-1} q^{i(n+2-2j-i)} = y^{n+1-j} \chi_{\{1 \leq n+2-2j \leq n+1-j\}},
\]

14
where $\chi_P$ is either 0 or 1 whether the property $P$ is false or true. We see that these constant terms in $q$ actually cancel two-by-two, so that it remains:

\[
(1 - q)q^m T_n \left( \frac{1 - q}{q^2}, q, y \right) = \sum_{j=0}^{n} \binom{n}{j} \left( (y^j + y^j^{1-1})P_{n-2j} - y^j P_{n-1-2j} - y^j^{1-1}P_{n+1-2j} \right) \\
= (1 + y) \sum_{j=0}^{n} \binom{n}{j} y^{j-1}P_{n-2j} - \sum_{j=0}^{n+1} \left( \binom{n}{j-1} + \binom{n}{j} \right) y^{j-1}P_{n+1-2j} \\
= (1 + y)G(n) - G(n + 1), \quad \text{where} \quad G(n) = \sum_{j=0}^{n} \binom{n}{j} y^{j-1}P_{n-2j}.
\]

Since the polynomial $P_{n-2j}$ is zero when $n - 2j < 0$, we can sum over $j$ between 0 and $|n/2|$ in the definition of $G(n)$, so that we get (17).

\[ \square \]

6. Application to permutation enumeration

In the previous section we have computed $T_n$, which is also equal to $\langle W | (yD + E)^n | V \rangle$ thanks to the results of Section 2. Now, using the inversion formula (2), we can compute $\langle W | (yD + E)^n | V \rangle$ and prove Theorem 1. At the beginning of this section we describe the combinatorial interpretation of this polynomial in terms of permutations and permutation tableaux. Then we prove Theorem 1 and Theorem 2 and give some applications.

**Proposition 15.** \([1, 6, 7, 10, 14, 22]\) For any $n \geq 1$ the following polynomials are equal:

- the generating function for permutation tableaux of size $n$, the number of lines counted by $y$ and the number of superfluous 1’s counted by $q$,
- the generating function for permutations of size $n$, the number of ascents plus 1 counted by $y$ and the occurrences of the pattern 13-2 counted by $q$,
- the generating function for permutations of size $n$, the number of weak excedances counted by $y$ and the number of crossings counted by $q$,
- the $n$th moment of the $q$-Laguerre polynomials.

**Proof.** All this material is present in the references. See also the references for definitions. In particular there are several possible definitions for the $q$-Laguerre polynomials: the one we mention is defined as a rescaled version of the Al-Salam-Chihara polynomials as in [10]. We recall that the $n$th moment of these $q$-Laguerre polynomials is the sum of weights of histoires de Laguerre of $n$ steps. This is also present in [4].

**Definition 6.** An histoire de Laguerre is a weighted Motzkin path such that:

- the weight of an horizontal step at height $h$ is $q^i$ for some $i \in \{0, \ldots, h-1\}$ or $yq^i$ for some $i \in \{0, \ldots, h\}$,
- the weight of a North-East step starting at height $h$ is $q^i$ for some $i \in \{0, \ldots, h\}$,
- the weight of a South-East step starting at height $h$ is $yq^i$ for some $i \in \{0, \ldots, h-1\}$.

The classical bijections between permutations and histoires de Laguerre, namely the Françon-Viennot and Foata-Zeilberger bijections, give the equality of the last three items in the list of Proposition 15.

As said in the introduction, the link between the operators $D$ and $E$ of the matrix Ansatz and the permutation tableaux was first exposed by Corteel and Williams in [7]. This shows the equality of the first two items in the list. See also [22].

To end this proof we can use the bijection between permutation tableaux and permutations exposed in [6]: the number of columns in permutation tableaux corresponds to the number of ascents in permutations, and the number of superfluous 1’s corresponds to the number of occurrences of the pattern 13-2. We also have to mention the previous results of Postnikov, who made the link between J-diagrams, which generalize the permutation tableaux, and alignments in decorated permutations [12, 22].
We now give the formula for the polynomials of Proposition 15. This is the Theorem 1 stated in the introduction.

**Theorem 1.** For any \( n \geq 1 \), we have

\[
(W|(yD + E)^{n-1}|V) = \frac{1}{y(1-q)^n} \sum_{k=0}^{n} (-1)^k \left( \sum_{j=0}^{n-k} y^j \left( \begin{pmatrix} n_j \end{pmatrix} - \begin{pmatrix} n_{j+k} \end{pmatrix} \right) \right) \left( \sum_{i=0}^{k} y^i q^{k+1-i} \right).
\]

**Proof.** Using the main result of the previous section and the inversion formula, we obtain:

\[
(W|(1-q)^n(yD + E)^{n-1}|V) = (1-q) \sum_{k=0}^{n-1} \binom{n-1}{k}(1+y)^{n-1-k}(-q)^k (W|(yD + E)^{k}|V)
\]

\[
= \sum_{k=0}^{n-1} \binom{n-1}{k}(1+y)^{n-1-k}(-1)^k \left( (1+y)G(k) - G(k+1) \right) = \sum_{k=0}^{n} \binom{n}{k}(1+y)^{n-k}(-1)^kG(k)
\]

\[
= \sum_{0 \leq k \leq n \atop i \equiv k \mod 2} \binom{n}{k}(1+y)^{n-k}(-1)^k \left\{ \begin{pmatrix} k-j \end{pmatrix} \right\} y^{k-i/2-1} P_t = \frac{1}{y} \sum_{i=0}^{n} (-1)^i \left( \sum_{k=0}^{n-i} \binom{n-i}{k} (1+y)^{n-2k-i} \left\{ \begin{pmatrix} 2k+i \end{pmatrix} \right\} r_k \right) P_t,
\]

after a reindexing such that \( k \) becomes \( 2k + i \). It remains to simplify the sum between parentheses. After expanding the power of \( 1+y \), this sum is:

\[
= \sum_{0 \leq k \leq m} \binom{m-k}{n-2k-i} \sum_{j=0}^{n-k-2i} \binom{n}{2k+i} \binom{n-2k-i}{j} \binom{2k+i}{k} y^{k+j}
\]

\[
= \sum_{0 \leq k \leq m} \frac{n!}{j!(n-2k-i-j)!} \left( \frac{1}{k!(k+i)!} - \frac{1}{(k-1)!(k+i+1)!} \right) y^{k+j}
\]

\[
= \sum_{0 \leq k \leq m} \frac{n!}{(m-k)!(n-m-k-i)!} \left( \frac{1}{k!(k+i)!} - \frac{1}{(k-1)!(k+i+1)!} \right) y^{m}
\]

\[
= \sum_{m=0}^{n-i} y^m \left( \binom{n}{m} \sum_{k=0}^{m} \binom{m}{k} \binom{n-m}{k+i} - \binom{n}{m-1} \sum_{k=0}^{m-1} \binom{m-1}{k} \binom{n-m+1}{k+i+1} \right).
\]

But thanks to the Vandermonde identity, the two sums over \( k \) may be simplified:

\[
\sum_{k=0}^{m} \binom{m}{k} \binom{n-m}{k+i} = \binom{n}{m+i}, \quad \sum_{k=0}^{m} \binom{m-1}{k} \binom{n-m+1}{k+i+1} = \binom{n}{m+i+1},
\]

and this completes the proof. \( \square \)

**Remark:** The number \( \binom{n}{j} \binom{n}{j+k} - \binom{n}{j+k+1} \) may be seen as the determinant of a \( 2 \times 2 \)-matrix of binomial coefficients. The Lindström-Gessel-Viennot lemma gives a combinatorial interpretation of this quantity in terms of lattice paths: it is the number of pairs of non-intersecting paths with starting points \((1,0)\) and \((0,1)\), with end points \((j,n-j+1)\) and \((j+k+1,n-k+j)\), and only with unit steps going North or East, as in Figure 9. In particular when \( k = 0 \), this is the Narayana number \( N(n,j+1) \).

**Proposition 16.** The coefficient of \( y^m \) in \( (W|(yD + E)^{n-1}|V) \) is given by:

\[
[y^m](W|(yD + E)^{n-1}|V) = \frac{1}{(1-q)^n} \sum_{k=0}^{n} \sum_{j=m-k}^{m} (-1)^k q^{(m-j)(k+j+1-m)} \left( \binom{n}{j} \binom{n}{j+k} - \binom{n}{j-1} \binom{n}{j+k+1} \right).
\]

**Proof.** We just have to expand the products in the equality of Theorem 1 since each of the factors between parentheses is a polynomial in \( y \) and their coefficients are explicit. \( \square \)
illustration we give the formulas for 13-2. By taking the Taylor series of (2), we obtain direct and quick proofs for these previous results. As an example, see for example [3, 6, 14, 15]. In particular in [3, 15] we can find methods for obtaining, as a function of \( n \) for a given \( k \), the number of permutations of size \( n \) with exactly \( k \) occurrences of the pattern 13-2. By taking the Taylor series of (2), we obtain direct and quick proofs for these previous results. As an illustration we give the formulas for \( k \leq 3 \) in the following proposition.

Among the several objects of the list in Proposition 15, the most studied are probably permutations and the pattern 13-2, see for example [3, 6, 14, 15]. In particular in [3, 15] we can find methods for obtaining, as a function of \( n \) for a given \( k \), the number of permutations of size \( n \) with exactly \( k \) occurrences of the pattern 13-2. By taking the Taylor series of (2), we obtain direct and quick proofs for these previous results. As an illustration we give the formulas for \( k \leq 3 \) in the following proposition.

We now give the specialization when \( y = 1 \). This is the Theorem 2 stated in the introduction.

**Theorem 2.** For any \( n \geq 1 \), we have

\[
(W | (D + E)^{n-1} | V) = \frac{1}{(1-q)^n} \sum_{k=0}^{n} (-1)^k \left( \binom{2n}{n-k} - \binom{2n}{n-k-2} \right) \sum_{i=0}^{k} q^{i(k+1-i)}.
\]

**Proof.** We just have to substitute \( y = 1 \) into the equality of Theorem 1. We can simplify the resulting expression using again the Vandermonde identity, indeed we have

\[
\sum_{j=0}^{n-k} \binom{n}{j} \binom{n}{j+k} = \sum_{j=0}^{n-k} \binom{n}{j} \binom{n}{n-j} = \binom{2n}{n-k},
\]

\[
\sum_{j=0}^{n-k} \binom{n}{j} \binom{n}{j+k+1} = \sum_{j=0}^{n-k} \binom{n}{j} \binom{n}{n-j-1} = \binom{2n}{n-k-2},
\]

and the result follows.\[\square\]
Proposition 17. The order 3 Taylor series of \( \langle W \rangle (D + E)^{n-1}|V \) is
\[
\langle W \rangle (D + E)^{n-1}|V \rangle = C_n + \left( \frac{2n}{n-3} \right) q + \left( \frac{2n}{n-4} \right) q^2 + \left( \frac{(n+1)(n+2)}{6} \right) q^3 + O(q^4),
\]
where \( C_n \) is the \( n \)th Catalan number.

Proof. On one side, we have \((1-q)^{-n} = 1 + nq + \left( \frac{n+1}{2} \right) q^2 + \left( \frac{n+2}{3} \right) q^3 + O(q^4)\). On the other side, we have \( \sum_{k=0}^{n} q^{k(k+1-i)} = 1 + q6\delta_k + 2q^2\delta_{2k} + 2q^3\delta_{3k} + O(q^4)\). The constant term is:
\[
\sum_{k=0}^{n} \left( \frac{2n}{n-k} - \left( \frac{2n}{n-k-2} \right) \right) = \left( \frac{2n}{n} \right) - \left( \frac{2n}{n-1} \right) = C_n.
\]
So this Taylor series is
\[
(1 + nq + \left( \frac{n+1}{2} \right) q^2 + \left( \frac{n+2}{3} \right) q^3)^{-1} \left( C_n - \left( \frac{2n}{n-1} \right) q + \left( \frac{2n}{n-2} \right) q^2 - \left( \frac{2n}{n-3} \right) q^3 \right).
\]
After expanding the product, all coefficients can be seen as the product of \( \binom{2n}{n} \) and a rational fraction of \( n \). So the simplification is just a matter of simplifying rational fractions of \( n \), which is straightforward. \( \square \)

More generally, a computer algebra system can provide higher order terms, for example it takes no more than a few seconds to obtain the following closed formula for \([q^{10}]{\langle W \rangle (D + E)^{n-1}|V} \):
\[
\frac{\binom{2n}{n}!}{10!(n+12)!/(n-8)!} \left( n^{13} + 70n^{12} + 2093n^{11} + 32354n^{10} + 228543n^9 - 318990n^8 - 17493961n^7 - 104051458n^6 - 6828164n^5 + 2022876520n^4 + 6310831968n^3 + 5832578304n^2 + 14397419520n + 574801920 \right),
\]
which is quite an improvement when compared to the methods of [13]. Besides these exact formulas, the following proposition gives the asymptotic for permutations with a given fixed number of occurrences of the pattern 13-2.

Theorem 3. For any \( m \geq 0 \) we have the following asymptotic when \( n \) goes to infinity:
\[
[q^m]{\langle W \rangle (D + E)^{n-1}|V} \sim \frac{4^n n^m - \frac{1}{2}}{\sqrt{\pi m!}}.
\]

Proof. When \( n \) goes to infinity, the numbers \( \binom{2n}{n-k} - \binom{2n}{n-k-2} \) are dominated by the Catalan number \( \frac{1}{n+1} \binom{2n}{n} \). It implies that in \( (1-q)^n \langle W \rangle (D + E)^{n-1}|V \rangle \), each higher order term grows at most as fast as the constant term \( C_n \). On the other side, the coefficient of \( q^n \) in \( (1-q)^{-n} \) is equivalent to \( n^m/m! \). So we have the asymptotic
\[
[q^m]{\langle W \rangle (D + E)^{n-1}|V} \sim \frac{C_n n^m}{m!}.
\]
Knowing the asymptotic of the Catalan numbers, we can conclude the proof. \( \square \)

Since any occurrence of the pattern 13-2 in a permutation is also an occurrence of the pattern 1-3-2, a permutation with \( k \) occurrences of the pattern 1-3-2 has at most \( k \) occurrences of the pattern 13-2. So we get the following corollary.

Corollary 3. Let \( \psi_k(n) \) be the number of permutations in \( \mathfrak{S}_n \) with at most \( k \) occurrences of the pattern 1-3-2. For any constant \( C > 1 \) and \( k \geq 0 \), we have
\[
\psi_k(n) \leq C \frac{4^n n^k - \frac{1}{2}}{\sqrt{\pi k!}}
\]
when \( n \) is sufficiently large.
Proof. By the previous remark we have
\[ \psi_k(n) \leq \sum_{i=0}^{k} |q^i|(W|(D + E)^n - 1|V), \]
so this is a consequence of Theorem \[3\] which gives the asymptotics of each of these terms. \[\square\]

So far we have mainly used Theorem \[2\]. Now we illustrate what we can do with the refined formula given in Theorem \[1\]. We already mentioned that we get Narayana numbers when \( q = 0 \), but we can also get the coefficients of higher degree in \( q \). For example it is conjectured in \[23\] that the coefficient of \( q^y \) in \( \langle W|y(yD + E)^n - 1|V \rangle \) is equal to \( \left( \begin{array}{c} n \\ m \end{array} \right) \left( \begin{array}{c} n - 1 \\ m - 2 \end{array} \right) \). With our results we can prove:

**Proposition 18.** The coefficients of \( q^y \) and \( q^2y \) in \( \langle W|y(yD + E)^n - 1|V \rangle \) are respectively
\[ \left( \begin{array}{c} n \\ m + 1 \end{array} \right) \left( \begin{array}{c} n \\ m - 2 \end{array} \right) \text{ and } \left( \begin{array}{c} n + 1 \\ m - 2 \end{array} \right) \left( \begin{array}{c} n + 1 \\ m + 2 \end{array} \right) \frac{nm + m - m^2 - 4}{2(n + 1)}. \]

**Proof.** A naive expansion of the Taylor series in \( q \) gives a lengthy formula, which is simplified straightforwardly after noticing it is the product of \( \left( \begin{array}{c} n \\ m \end{array} \right)^2 \) and a rational fraction of \( n \) and \( m \). \[\square\]

## Appendix

We give here a combinatorial proof of Proposition \[12\]. As noticed earlier, this result is a generalization of the Touchard-Riordan formula \[4\], and this combinatorial proof is a generalization of Penaud’s combinatorial proof \[10\] of \[4\]. We follow very closely this reference, even in some notations. Moreover the ideas of this proof were inspired by the alternative proof of Theorem \[1\] mentioned in the introduction (see \[3\]).

**Proposition 19.** There is a bijection between involutions on \( \{1, \ldots, n\} \) and weighted Motzkin paths of \( n \) steps with the following properties:

- The weight of an East step at height \( h \) is \( q^h \).
- The weight of a South-East step starting at height \( h \) is \( q^i \) for some \( i \in \{0, \ldots, h - 1\} \).

Moreover the image of an involution \( I \) on \( \{1, \ldots, n\} \) is a weighted Motzkin path with total weight \( q^{\mu(I)} \).

**Proof.** This is obtained via the same methods as the bijection between involution without fixed points and Hermite histories, see \[16\]. It is also very similar to the Foata-Zeilberger bijection. See Figure 10 for an example. \[\square\]

![Figure 10. An involution and the corresponding weighted Motzkin path.](image)

To compute \( T_{0,k,n}(1, q) \), we have to sum the weights of the weighted Motzkin paths having \( n \) steps, and \( n - 2k \) East steps. When we multiply by \( (1 - q)^k \), there are many cancellations in this sum. Indeed we easily see that to compute \( T_{0,k,n}(1 - q, q) \), we have to sum the weights of Motzkin paths of \( n \) steps satisfying conditions (C2):

- the weight of an East step at height \( h \) is \( q^h \).
- the weight of a South-East step starting at height \( h \) is either 1 or \( -q^h \).

Now, we give a decomposition of these weighted Motzkin paths.

**Proposition 20.** There is a weight-preserving bijection between weighted Motzkin paths satisfying (C2) and couples \((H_1, H_2)\) such that for some \( i \in \{0, \ldots, k\} \),

- \( H_1 \) is a left factor of a Dyck path, with \( n \) steps and ending at height \( n - 2k + 2i \),
• $H_2$ is a weighted Motzkin path of $n-2k+2i$ steps, with $n-2k$ East steps, statifying conditions (C2) above, and also that any South-East step following a North-East step has weight $-q^h$ (i.e. not 1).

**Proof.** This is similar to Lemma 1 in [5].

A weighted Motzkin path as $H_2$ above is called a core. The enumeration of left factors of Dyck path is given by Delannoy numbers. On the other hand, to compute the sum of weights of cores we need two other lemmas.

**Lemma 3.** There is an involution $\gamma_i$ on cores of length $n-2k+2i$ with $n-2k$ East steps, with the following properties:

• if a core and its image are different they have opposite weights,
• the fixed points of $\gamma_i$ are the cores such that:
  • the $i$ first steps are North-East, and all following steps are East or South-East,
  • a South-East step starting at height $h$ has weight $-q^h$ (i.e. not 1).

**Proof.** In this proof we use a word notation for cores: the letters $x$, $z$, and $\bar{y}$ respectively correspond to North-East steps, East steps, South-East steps weighted by 1, and South-East steps weighted by $-q^h$. For a core $c$, let $u(c)$ be the length the last sequence of consecutive $x$’s. Let $v(c)$ be the height of the last $y$ if there is no $x$ after this $y$, and $i$ otherwise. The fixed points of $\gamma_i$ are the cores such that $u(c) = v(c) = i$.

From now on we assume that $c$ does not satisfy $u(c) = v(c) = i$. The involution $\gamma_i$ is such that $u(c) \geq v(c)$ if and only if $u(\gamma_i(c)) < v(\gamma_i(c))$. Suppose that $u(c) \geq v(c)$. Let $\tilde{c}$ be the word obtained from $c$ when we replace the last $y$ with a $\bar{y}$. There is a unique factorization $\tilde{c} = f_1x^{u(c)}ay^jf_2$ such that:

• $a$ is either $z$ or $\bar{y}$,
• $f_2$ begins with a $\bar{y}$ and contains no $x$.

We set

$$\gamma_i(c) = f_1x^{u(c)}-v(c)ay^jx^{v(c)}f_2.$$ 

See Figure 11 for an example.

![Figure 11. A core $c$ and its image by $\gamma_i$. The thick lines indicate the $\bar{y}$, i.e. the South-East steps weighted by $-q^h$. In this example we have $n-2k = 3$, $i = 9$, $u = 4$, $v = 2$. We can check that $w(c) = -q^{17} = -w(\gamma_i(c))$.](image)

Simple arguments of word combinatorics show that:

• $c$ and its image have opposite weights,
• any core $c'$ such that $u(c') < v(c')$ is obtained as a $\gamma_i(c)$ for some $c$ satisfying $u(c) \geq v(c)$. Indeed, let $\tilde{c}'$ be the word obtained from $c'$ by replacing the last $y$ at height $u(c')$ with a $y$. There is unique factorization $\tilde{c}' = f_1ay^jx^{u(c')}f_2$, where $a$ is $z$ or $\bar{y}$ and $f_2$ contains no $x$. Then $c = f_1x^{u(c')}ay^jf_2$ has the required properties.

These arguments, put together, show that $\gamma_i$ has the claimed properties. □

**Lemma 4.** The sum of weights of the fixed points of $\gamma_i$ is

$$\sum_{H_2 \in Fix(\gamma_i)} w(H_2) = (-1)^i q^{\frac{2n+2i}{2}} \binom{n-2k+i}{i}.$$ 

**Proof.** A fixed point of $\gamma_i$ is fully characterized by the heights $h_1, \ldots, h_{n-2k}$ of the $n-2k$ East steps, and these heights can take any values such that $i \geq h_1 \geq \cdots \geq h_{n-2k} \geq 0$. Such a fixed point of $\gamma_i$ has weight

$$(-1)^i q^{\frac{2n+2i}{2}} q^{\sum_{h_i}}.$$
indeed the South-East steps have weights \(-q^{i}, \ldots, -q^{2}, -q\) and they correspond to the factor \((-1)^{i}q^{\frac{i(i+1)}{2}}\). It remains to sum over \(h_{i}\) and we can conclude thanks to Proposition 11.

Now we can prove:

**Proposition 12.**

\[
T_{0,k,n}(1-q,q) = \sum_{i=0}^{k} (-1)^{i}q^{\frac{i(i+1)}{2}} \left[ \begin{array}{c} n-2k+i \\ i \end{array} \right] q^{\begin{array}{c} n \\ k-i \end{array}}.
\]

**Proof.** The decomposition of weighted Motzkin paths stated in Proposition 20 gives

\[
T_{0,k,n}(1-q,q) = \sum_{i=0}^{k} \left\{ \begin{array}{c} n \\ k-i \end{array} \right\} \sum_{H_{2}} w(H_{2}),
\]

where the second sum is over cores \(H_{2}\) of \(n-2k+2i\) steps with \(n-2k\) East steps. Thanks to Lemma 3 we can restrict the second sum to the fixed points of the involution \(\gamma_{i}\). And thanks to Lemma 4 this sum is

\[
\sum_{H_{2}} w(H_{2}) = (-1)^{i}q^{\frac{i(i+1)}{2}} \left[ \begin{array}{c} n-2k+i \\ i \end{array} \right] q.
\]

This completes the proof. □

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