STATISTICAL PROPERTIES OF INTERVAL MAPS WITH CRITICAL POINTS AND DISCONTINUITIES

HONGFEI CUI

Abstract. We consider dynamical systems given by interval maps with a finite number of turning points (including critical points, discontinuities) possibly of different critical orders from two sides. If such a map $f$ is continuous and piecewise $C^2$, satisfying negative Schwarzian derivative and some summability conditions on the growth of derivatives and recurrence along the turning orbits, then $f$ has finitely many attractors whose union of basins of attraction has total probability, and each attractor supports an absolutely continuous invariant probability measure $\mu$. Over each attractor there exists a renormalization $(f^m, \mu)$ that is exact, and the rates of mixing (decay of correlations) are strongly related to the rates of growth of the derivatives and recurrence along the turning orbits in the attractors. We also give a sufficient condition for $(f^m, \mu)$ to satisfy the Central Limit Theorem. In some sense, we give a fairly complete global picture of the dynamics of such maps. Similarly, we can get similar statistical properties for interval maps with critical points and discontinuities under some more assumptions.

1. Introduction and statement of results

1.1. Introduction. In the last three decades, many results on statistical properties were obtained for iterations of one dimensional maps, for example the existence of absolutely continuous invariant probability measures (acip for short), the decay of correlations and the Central Limit Theorem. Various conditions have been shown to guarantee the existence of acip and corresponding statistical properties. In the area of interval maps with turning points (including critical points, discontinuities), these results generally require more restriction on the turning points, for example the critical orders of each turning point from two sides are equal in the continuous cases, and positive Lyapunov exponents at the critical values and the recurrence of turning point are sub-exponential in the discontinuous cases.

Our aim in this paper is to obtain the same conclusion but relax as much as possible the conditions on the orbit of the critical points, to include in particular cases in which the growth of derivatives may be sub-exponential and/or the recurrence of the turning points exponential, or the critical orders of turning points from both sides do not equal. More precisely, we will show the existence and finiteness of the number of acip for a general map under two summability conditions on the growth of derivatives and recurrence along the turning points, and study its statistical properties such as decay.
of correlations and the Central Limit Theorem. In these processes, we give a complete global picture of the dynamics from a probabilistic perspective.

1.2. **Dynamical assumption.** We now give the precise statement of our assumption. Let \( \mathcal{A} \) denote the class of the interval map satisfying the conditions formulated in Subsections 1.2.1-1.2.2 below. Denote \( \mathcal{A}_1 \) as continuous maps in \( \mathcal{A} \) and \( \mathcal{A}_2 \) as discontinuous maps in \( \mathcal{A} \) respectively.

1.2.1. **Critical set.** Let \( M \) be a compact interval \([0,1]\) and \( f : M \to M \) be a piecewise \( C^2 \) map. This means that there exists a finite set \( C \) such that \( f \) is \( C^2 \) and a diffeomorphism on each component of \( M \setminus C \), and \( f \) admits a continuous extension to the boundary so that both the left and the right limits \( f(c\pm) = \lim_{x \to c\pm} f(x) \) exist. We regard each \( c \in C \) as two points: \( c^+, c^- \), the concrete values depend on the corresponding one-side neighborhoods. We assume that each \( c \in C \) has a one-side critical order \( l(c\pm) \in [1,\infty) \), this means that

\[
|Df(x)| \approx |x - c|^{l(c\pm) - 1}, \quad |f(x) - f(c\pm)| \approx |x - c|^{l(c\pm)} \quad \text{and} \quad |D^2f(x)| \approx |x - c|^{l(c\pm)^2}
\]

for \( x \) in the corresponding one-side neighborhood of \( c \), where we say \( f \approx g \) if the ratio \( f/g \) is bounded above and below uniformly in its domain. When we use the notion \( l(c) \), it may be either \( l(c+) \) or \( l(c-) \). If \( l(c) = 1, c \) is a bound derivative point, and if \( l(c) > 1 \) we say that \( c \) is a critical point. Note that \( c \) may be a critical point on one side and is a bound derivative point on the other side. When there is no possibility of confusion, each point \( c \in C \) will be called a critical point without distinguishing whether \( c \) is really a critical point with \( l(c) > 1 \), or \( c \) is a bounded derivative point with \( l(c) = 1 \).

We also assume that \( f \) is with negative Schwarzian derivative outside of \( C \), i.e., \( |Df|^{-\frac{1}{2}} \) is a convex function on each component of \( M \setminus C \). In particular, if \( f \) is continuous, and the critical orders with \( l(c) > 1 \) are equal from two sides for each critical point, we can get rid of this assumption (but need to add a natural topological assumption that all period points are hyperbolic repelling) by a result of Kozlovski [15] (generalized to the multimodal setting by van Strien and Vargas [26]).

1.2.2. **Summability conditions.** We suppose \( f \) satisfies the following summability conditions along the critical orbits. The first summability condition is

\[
\sum_{n=1}^{\infty} \left( \frac{|f^n(c) - \tilde{c}|^{l(\tilde{c})}}{|f^n(c) - \tilde{c}|^{l(c)}|Df^n(f(c))|} \right)^{1/(2l(c)-1)} < \infty, \quad \forall c \in C,
\]

where \( \tilde{c} \) is the critical point closest to \( f^n(c) \), and \( l(c), l(\tilde{c}) \) depend on the corresponding one-side neighborhoods, and the second summability condition is

\[
\sum_{n=1}^{\infty} \left( \frac{1}{|Df^n(f(c))|} \right)^{1/l(c)} < \infty, \quad \forall c \in C.
\]

One of the most simplest example satisfying the above conditions is the contracting Lorenz maps considered in [20] and [24], which motivated by
the study of the return map of the Lorenz equations near classical parameter values. Notice that above summability conditions are satisfied if the derivative is growing exponentially fast and the recurrence is not faster than exponential in the sense that for each critical point \( c \in \mathcal{C} \), \( |Df^n(f(c))| > \lambda^n \), for some \( \lambda > 1 \), and \( |f^{n-1}(f(c)) - \mathcal{C}| > e^{-\alpha n} \) for some \( \alpha \) small enough, for all \( n \geq 1 \).

**Remark 1.** According to the first summability condition, critical points of \( f \in A \) are not on the forward orbits of the critical set, i.e., \( \mathcal{C} \cap \bigcup_{n \geq 1} f^n(\mathcal{C}) = \emptyset \). It is easy to see that if all of the critical orders are equal, one can get rid of the recurrence condition containing in the first summability condition.

### 1.3. Statement of results.

In a previous paper, we have shown the following theorem under some weaker summability conditions,

**Theorem 1.** [10] If \( f \) satisfies assumption in Subsection 1.2.1 and summability condition (4), and the following summability condition

\[
\sum_{n=1}^{\infty} \left( \frac{|f^n(c) - \tilde{c}|^{l(c)}}{|f^n(c) - \tilde{c}|^{l(c)}|Df^n(f(c))|} \right)^{1/l(c)} < \infty, \forall c \in \mathcal{C},
\]

then \( f \) admits an acip. Furthermore, if \( l_{\max} > 1 \), then its density is in \( L^p \) for all \( 1 \leq p < \frac{l_{\max}}{l_{\max} - 1} \), where \( l_{\max} \) is the maximum of the orders of the critical points.

In this paper, we will consider the general properties of acip, including the finiteness of acip, the support of each acip and its properties (decay of correlations, Central Limit theorem). In general, if \( f \) has many turning points, the acip \( \mu \) need not to be unique and not Lebesgue ergodic (note that unimodal maps with negative Schwarzian derivative and equal critical orders from both sides are ergodic with respect to Lebesgue measure). If \( f \in A_1 \), then we can choose the minimal cycle (see Section 3) of \( f \), show that there exists a unique acip \( \mu \) support on the minimal cycle and the renormalization \( f^m \) of \( f \) corresponding the minimal cycle is exact, hence \( (f^m, \mu) \) is mixing and ergodic. So it is natural to estimate the rates of mixing, quantified through the correlation functions

\[
C_n(f^m, \mu) := |\int (\varphi \circ f^m)\psi d\mu - \int \varphi d\mu \int \psi d\mu|,
\]

where \( \varphi \) and \( \psi \) are respectively bounded and Hölder continuous functions on the minimal cycle.

Another important characterization of \( \mu \) is by Central Limit Theorem, which describes the oscillations of finite averages \( \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) \) around the expect value \( \int \varphi d\mu \). We say that \( (f^m, \mu) \) satisfies the Central Limit Theorem if given a Hölder continuous functions \( \varphi \) which is not coboundary (\( \varphi \neq \psi \circ f - \psi \) for any \( \psi \)), then there exists \( \sigma > 0 \) such that for any interval \( I \subset R \),

\[
\mu \left\{ x \in M; \left( \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \varphi(f^j(x)) - \int \varphi d\mu \right) \in I \right\} \rightarrow \frac{1}{\sigma \sqrt{2\pi}} \int_I e^{-t^2/2\sigma^2} dt, \text{ as } n \rightarrow \infty.
\]
We shall state the results in this paper. Write

\[ \gamma_n(c) := \min \left\{ \left( \frac{|f^n(c) - \tilde{c}|^{l(c)}}{|f^n(c) - \tilde{c}|^l(c) |Df^n(f(c))|} \right)^{1/(2l(c)-1)}, \frac{1}{2} \right\} \quad \forall c \in \mathcal{C}, \]

\[ d_n(c) := \min_{i<n} \left( \frac{\gamma_i(c)}{|Df^i(f(c))|} \right)^{1/(2l(c)-1)} |f^i(c) - \tilde{c}|^{l(c)/l(c)}, \quad \forall c \in \mathcal{C}. \]

By the first summability condition and elementary calculations, we have \( d_n(c) \leq \gamma_{n-1}(c). \)

**Theorem 2.** Let \( f \in \mathcal{A}_1 \), then there exists at least one and at most \( \sharp \mathcal{C} \) different ergodic acips \( \mu_i, 1 \leq i \leq N \). These measures are support on minimal cycles of intervals with pairwise disjoint interiors. Over each such cycles there exists a renormalization \( (f^{m_i}, \mu_i) \) that is exact. For a fixed minimal cycle with critical point \( C_k \), the corresponding renormalization \( (f^m, \mu) \) is mixing with the following rates:

**Polynomial case:** If \( d_n(c) \leq Cn^{-\alpha}, \alpha > 1, C > 0 \) for all \( c \in \mathcal{C}_c \) and \( n \geq 1 \), then for each \( \bar{\alpha} < \alpha - 1 \), there exists \( \bar{C} > 0 \) such that

\[ C_n(f^m, \mu) \leq \bar{C}n^{-\bar{\alpha}}. \]

**Stretched exponential case:** If \( \gamma_n(c) \leq Ce^{-\beta n^\alpha}, \beta > 0, \alpha \in (0,1), C > 0 \) for all \( c \in \mathcal{C}_c \) and \( n \geq 1 \), then for each \( \bar{\alpha} \in (0, \alpha) \) there exist \( \bar{\beta}, \bar{C} > 0 \) such that

\[ C_n(f^m, \mu) \leq \bar{C}e^{-\bar{\beta} n^{\bar{\alpha}}}. \]

**Exponential case:** If \( \gamma_n(c) \leq Ce^{-\beta n}, \beta > 0 \) for all \( c \in \mathcal{C}_c \) and \( n \geq 1 \), then there exist \( \bar{\beta}, \bar{C} > 0 \) such that

\[ C_n(f^m, \mu) \leq \bar{C}e^{-\bar{\beta} n}. \]

If \( d_n(c) \leq Cn^{-\alpha}, \alpha > 2 \) for \( c \in \mathcal{C}_c \) and \( n \geq 1 \), then \( (f^m, \mu) \) satisfies the Central Limit theorem. If \( l_{\max} > 1 \) for critical point in the minimal cycle, then the density of \( \mu \) is \( L^p \) for all \( 1 \leq p < \frac{l_{\max}}{l_{\max} - 1} \). The union of the basin \( B(\mu_i) \) has full probability measure in the interval \( M \).

For a map \( f \in \mathcal{A}_2 \), we suppose \( f \) satisfies the following

**Density of preimages:** There exists \( c \in \mathcal{C} \) whose preimages are dense in some \( J^* \in M \), where \( J^* \) is a union of intervals and satisfies \( f(J^*) = J^* \).

**Remark 2.** The interval \( J^* \) plays the same role to the notation of the minimal cycle in the continuous case.

**Theorem 3.** Let \( f \in \mathcal{A}_2 \) and satisfies the assumption of density of preimages. Then \( f \) has an acip \( \mu \), and there exists an integer \( k > 1 \) such that \( (f^k, \mu) \) is exact. For Hölder continuous functions \( \varphi, \psi, (f^k, \mu) \) is mixing with the following rates:

**Polynomial case:** If \( d_n(c) \leq Cn^{-\alpha}, \alpha > 1, C > 0 \) for \( c \in J^* \) and \( n \geq 1 \), then for each \( \bar{\alpha} < \alpha - 1 \), there exists \( \bar{C} > 0 \) such that

\[ C_n(f^k, \mu) \leq \bar{C}n^{-\bar{\alpha}}. \]
Then the density of \( \hat{\mu} \) then there exist the Central Limit Theorem. For the contracting Lorenz map \( f \) maps have finite number of acips, and exponential decay of correlations and condition and subexponential recurrence conditions, and proved that such point are equal from two sides under the assumption the Coll et-Eckmann critical orders may not equal, but need the critical orders of each critical point are equal from two sides in general.

On the other hand, the results on the existence of acip in smooth case need the assumption that the recurrence of the critical point is not increasing rapidly for interval maps with critical points and discontinuities. An observation is that we only consider the derivatives of the critical value on the smooth case \([6, 5]\), while we consider the smooth case use weaker assumptions than the case of interval maps with critical points and discontinuities. An observation is that we only consider the derivatives of the critical value on the smooth case \([6, 5]\), while we should need the assumption that the recurrence of the critical point is not increasing rapidly for interval maps with critical points and discontinuities.

1.4. Comments on results. The story about the existence of acip of interval maps has a long history, see \([17]\) and reference therein for a comprehensive survey. Quite general conditions are known to guarantee the existence of acip for uniformly expanding maps, for smooth maps with critical points (for S-unimodal maps satisfying Collect-Eckmann condition in \([9]\) and Nowicki-van Strien condition in \([23]\), and for multimodal maps under the most general condition in this setting \([5]\) recently), for interval maps with critical points and singularities under summability conditions in \([11]\) recently, for smooth maps with a countable number of critical point in \([2]\). Notice that the results on the smooth case use weaker assumptions than the case of interval maps with critical points and discontinuities, an observation is that we only consider the derivatives of the critical value on the smooth case \([6, 5]\), while we should need the assumption that the recurrence of the critical point is not increasing rapidly for interval maps with critical points and discontinuities.

On the other hand, the results on the existence of acip in smooth case need an assumption that the critical orders of each critical point should equal from two sides in general.

The results regarding decay of correlations and the Central Limit Theorem for the unimodal maps with same critical orders were proved in \([27, 14]\). In \([27]\), Young considered quadratic maps satisfying Collect-Eckmann condition and critical recurrence at a sufficiently slow exponential rate, and proved that such maps have exponential decay of correlations and satisfy the Central Limit Theorem. Independently, Keller and Nowicki \([14]\) obtained the same results for S-unimodal maps (with same critical orders) satisfying only the Collet-Eckmann condition. Later, in \([4]\), Bruin et.al considered multimodal maps (with the same critical orders for all critical points and \( l(c) > 1, \forall c \in C, \) note not only from both sides), obtained the same results to Theorem 2 under the summability condition \( \sum_{n=0}^{\infty} |Df^n(f(c))|^{-\frac{1}{2n+1}} < \infty \). Moreover, the construction in that paper made it possible to show a direct link between the rate of decay of correlations and the rate of growth of \( |Df^n(f(c))| \). Cederwall \([8]\) considered the interval maps with critical points ( \( l(c) > 2 \), critical orders may not equal, but need the critical orders of each critical point are equal from two sides) under the assumption the Collet-Eckmann condition and subexponential recurrence conditions, and proved that such maps have finite number of acips, and exponential decay of correlations and the Central Limit Theorem. For the contracting Lorenz map \( f \) satisfying \( |Df^n(f(c\pm))| > \lambda^n \), for each \( n \geq 1 \) and some \( \lambda > 1 \), and \( |f^{n-1}(f(c\pm)) - c| > \)
$e^{-\alpha n}$ for some $\alpha$ small enough, and for all $n \geq 1$, it was shown that $f$ admits an acip which has exponential decay of correlations in [20].

For interval maps with critical points and singularities, it was obtained exponential decay of correlations and Central Limit Theorem, but need more additional assumptions in [12] [11] [18]. Observe that the decay of correlations and Central Limit Theorem hold for non-Hölder observables in [11] [18].

Note that in the unimodal cases (continuous) with same critical orders from two sides, $|Df^n(f(c))| \geq Ce^{\lambda n}, C > 0, \lambda > 0$ if and only if there is a renormalization $f^n$ with exponential decay of correlations [22]. We are not sure whether it holds for unimodal maps with different orders from two sides, but according Theorem 2, we can get that if the unique critical point satisfies $\gamma_n(c) \leq Ce^{-\beta n}$ for $C > 0$ and $\beta > 0$, then there is a renormalization $f^n$ with exponential decay of correlations.

If $0 < l(c) < 1$, $c$ is called a singular point. Since the negative Schwarzian derivative condition rules out the existence of singularities, once one can get rid of the negative Schwarzian derivative condition, the results in this paper may easily generalized to interval maps with critical points and singularities.

2. Ideas and organization of the proof

To obtain decay of correlations and the Central Limit Theorem of interval maps, a useful technique is based on Frobenius-Perron operator or transfer operator. Exponential decay corresponds to a gap in the spectrum of this operator, various technique has been developed for proving the existence of this gap. For example, in [16], it was shown that the Frobenius-Perron operator is contracting with respect to Hilbert metric on defined cones of density functions for an expanding map. Another powerful tool was proposed by Young in [28] [29]. She has shown that for an induced Markov map, the upper bound of the decay of correlations are strongly related to the tail estimates of the inducing time. Notice the method in [28] [29] could capture mixing rates that are slower than exponential rates.

Our strategy is to apply the results of Young in [29], so it is crucial for us to construct an induced Markov map based on the general maps we considered. Induced Markov map has been constructed before for unimodal maps with equal critical orders and satisfying Collet-Eckmann condition, for multimodal maps with equal critical orders (not only from two sides) and the summability condition $\sum_n |Df^n(f(c))|^{-1/[2l(c)-1]} < \infty$ for any $c \in C$ [4], for multimodal maps with critical points ( $l(c) > 2$ critical orders may not equal, but need the critical orders of each critical point from two sides are equal) under the assumption the Collet-Eckmann condition and subexponential recurrence conditions [8]. Observed that all the critical orders are equal (from both sides) is necessary in the construction in [4] [8].

In order to carry out almost the same construction [4] of an induced Markov map with corresponding estimates when the critical points are allowed to have different orders from two sides, we need do some modifications. Firstly, we need a new definition of binding period, which involves the recurrence of the critical points, see the details of the proof on Lemma [5] in Section 3. Secondly, we shall prove the nonexistence of wandering intervals and backward bound contraction property (BBC) for maps in $\mathcal{A}$, but we
have shown these results under a weaker condition in [10] for interval maps without singularities, recently.

The structure of the paper is as follows. In Section 3, for $f \in \mathcal{A}_1$, we identify the topological attractor and metric attractor, and consider the intervals in the attractor. We use a binding argument to obtain some estimations of growth in terms of the derivative and recurrence along the appropriate critical orbits. Similarly, we derived the same estimates for $f \in \mathcal{A}_2$ on $J^*$. In Section 4, for $f \in \mathcal{A}_1$, we choose any interval $J$ in the minimal cycle, and construct an induced Markov map on $J$ which has uniformly distortion and of the image bounded below on each element of the partition of $J$. Then we give the induced time estimates. In Section 5, we construct a full Markov map for an appropriate interval based on the induced map constructed in Section 4. We also state some estimates about this full Markov map. In Section 6, we apply Young’s result and present the proof of Theorem 2 and Theorem 3. For readers who are familiar to the construction in [4], they can skip Sections 4,5, except the proof of Lemma 9.

3. Notations and some estimates

For $f \in \mathcal{A}_1$, we will identify the attractors of $f$ which be both topological and metric in this Section, then we shall restrict $f$ to the attractors. For $f \in \mathcal{A}_2$, we restrict $f$ to the interval $J^*$ directly. Taking a small interval in the each attractor (or $J^*$), we use a binding argument to obtain estimates in terms of the derivatives and recurrence along the appropriate critical orbit. Similar argument have been applied before by Jakobson [13] and Benedicks and Carleson [3] under the strong conditions on $Df^n(f(c))$ and on the recurrence along the critical orbit. The way we defined here is imitated on [4], but [4] only consider the growth of $Df^n(f(c))$ in the definition of the binding period, without the assumption on the rate of recurrence. However, our definition of binding period relates the recurrence of $f^n(c)$, then it is useful to tackle the case that the critical points have different orders (including different orders from both sides). This is the main point of this work.

A point is called a period point if $f^n(x) = x$ for integer $n > 0$, it is attracting if its basin include a interior. A general notation of period orbit is the cycle of interval, if $J \subset M$ is a nontrivial closed interval for which there exists a positive integer $n$ such that $f^n(J) \subset J$ and $n$ is the least such integer, we call the set $\bigcup_{i=0}^{n} f^i(J)$ a cycle of intervals for $f$ with period $n$. If the interiors of intervals in the cycle are pairwise disjoint we say that the cycle is proper. If a cycle contains no small cycle, we say it is minimal. If $f$ has a proper cycle $\bigcup_{i=0}^{n} f^i(J)$, define $g : M \to M$ by $g = \Lambda^{-1} \circ f^n \circ \Lambda$, where $\Lambda$ is an affine transformation from $M$ onto $J$. We say $g$ is a renormalization of $f$.

The following Lemma collects some basic properties of proper cycle and minimal cycle for continuous maps on $M$.

Lemma 1. Let $f$ be a continuous map on $M$, then we have the following statements:
(1) the minimal cycle is a proper cycle,
(2) minimal cycles ether coincide or have disjoint interiors.
Proof. These follow from the definition of the proper cycle and the minimal cycle immediately. \hfill \Box

Proposition 1. If \( f \) is a map in \( A_1 \), then \( f \) has finite renormalizations, and \( f \) has at least one and at most \( 2^C \) minimal cycles. Moreover, \( \bigcup_{i=0}^n f^{-i}(c) \) for each \( c \) in the minimal cycle is dense in the corresponding minimal cycle.

Proof. At first, since \( f \) satisfies some summability conditions and negative Schwarzian derivative, \( f \) has no attracting or neutral periodic orbits by Singer’s Theorem \cite{25}. On the other hand, \( f \) has no wandering interval by \cite{10}, thus from the contraction principle, there exists \( \delta > 0 \) such that for any interval \( J \in M \) with \( |J| > 0 \), there exists \( N_J > 0 \) such that we have \( f^n(J) > \delta \) for \( n > N_J \). Hence all intervals constituting a cycle have the length greater than \( \delta \), this implies that the period of any proper cycle is bound by \( \delta^{-1} \). Thus \( f \) has a finite number of renormalizations.

Next, we suppose that all intervals in a cycle \( \bigcup_{i=0}^n f^i(J) \) don’t contain a turning point of \( f \) in its interior, then \( f^n \) or \( f^{2n} \) is monotone increasing and continuous on \( J \), and \( f^n(J) \subseteq J \) or \( f^{2n}(J) \subseteq J \), this contradicts \( f \) has no attracting or neutral period orbits, then \( f \) has at most \( 2^C \) minimal cycles. On the other hand, if \( M \) is not a minimal cycle of period one, by the finiteness of proper cycles, then there is a cycle could contain no small cycle, i.e., this is a minimal cycle.

Finally, we consider a minimal cycle \( \bigcup_{i=0}^n f^i(J) \), \( c \) is a critical point in \( \bigcup_{i=0}^n f^i(J) \). Because \( f \) has no wandering interval, and has no attracting and neutral periodic orbits, \( f \) has no homterval. Therefore, any small interval in the cycle will eventually visit the critical point in the cycle. If the preimages of \( c \) isn’t dense in the minimal cycle, we can find a cycle smaller the minimal cycle, this would induce a contradiction.

\hfill \Box

For \( f \in A_1 \) and each minimal cycle \( X := \bigcup_{i=0}^n f^i(J) \) with period \( m \), denote \( C_c \in \mathcal{C} \) as critical set in \( X \), we will consider the subsystem \((X, f^m)\). For \( f \in A_2 \), we consider the subsystem \((J^*, f)\) directly, denote \( C_c \in \mathcal{C} \) as critical set in \( J^* \) too.

For \( x \in X \) or \( x \in J^* \), let \( c \) be the critical point closest to \( x \). Given a critical neighborhood \( \Delta \) of \( C_c \), we define the binding period as follows,

\[
(5) \quad p(x) := \begin{cases} \max \left\{ p; |f^k(x) - f^k(c)| \leq \gamma_k(c)|f^k(c) - c|, \ 1 \leq k \leq p - 1 \right\} & \text{if } x \in \Delta, \\ 0 & \text{if } x \notin \Delta. \end{cases}
\]

The size of the critical neighborhood \( \Delta \) depends on the following Lemmas.

Lemma 2. (BBC property) Let \( f \in A_1 \), then there exists \( K > 0 \) such that for all \( \delta_0 > 0 \) there exist \( 0 < \delta < \delta_0 \), \( \Delta_\delta = \cup_{c \in \mathcal{C}}(c - \delta, c + \delta) \), for each \( x \in M \) we have

\[
(6) \quad |Df^n(x)| > K, \quad \text{where } n = \min\{i \geq 0; f^i(x) \in \Delta_\delta\}.
\]

Proof. See Theorem C in \cite{10}.

\hfill \Box

Remark 3. In fact, BBC holds for maps satisfying a weaker condition than \( A_1 \). For symmetric unimodal maps with negative Schwarzian derivative,
BBC holds [21]. For the multimodal case which be with the same critical orders of all critical points and an increasing condition
\[ \lim_{n \to \infty} |Df^n(f(c))| = \infty, \quad \forall c \in C, \]
it was shown in [4].

**Lemma 3.** (Uniformly expanding outside of \( \triangle \)) Let \( f \in A \), there exist \( C(\delta) > 0 \) and \( \lambda_\delta > 0 \) such that for \( x, f(x), \ldots, f^{n-1}(x) \notin \triangle_\delta \), then
\[ |Df^n(x)| \geq C(\delta)e^{\lambda_\delta n}. \]

**Proof.** Since \( f \in A \), all the periodic orbits of \( f \) are repelling [25]. We can define a new map \( \tilde{f} \) such that \( \tilde{f} \) is \( C^2 \) in \( \triangle_\delta \) and has no change with \( f \) outside of \( \triangle_{\delta_1} \), then the above is a consequence of Mañé theorem [19]. \( \square \)

**Lemma 4.** Suppose \( G_p \geq 0 \) and \( \sum_p G_p < \infty \), then for any \( \zeta > 1 \) there exists \( p_0 > 0 \) such that
\[ \sum_{s \geq 1} \sum_{(p_1, \ldots, p_s) \text{and } \sum p_i \geq p_0} \zeta G_{p_i} \leq 1. \]

**Proof.** See [4]. \( \square \)

For each \( c \in C_c \) or \( c \in J^* \), let \( \Delta := \cup_{c \in C_c} \text{ or } J^*(c - \delta, c + \delta) \), \( p_i := p(c + \delta) \) or \( p(c - \delta) \) depending the neighborhood we consider. Note that \( p_\delta \to \infty \) as \( \delta \to 0 \). Using Lemmas [21, 2, 4], and the summability of \( \gamma_n(c) \) for each \( c \), we can fix at this moment and for the rest of the paper \( \delta \) so small that
\[ \text{(1) BBC holds and uniformly expanding outside of } \Delta, \]
\[ \text{(2) } \sum_{s \geq 1} \sum_{(p_1, \ldots, p_s) \text{ and } \sum p_i \geq p_0} \zeta \gamma_{p_i}(c) \leq 1 \text{ for all } c \in C_c \text{ or } J^*, \]
where \( \zeta \) is a constant which depends only the map itself and will be specified explicitly in the proof, see the proof of Lemma [9].

The following Lemma gives an estimation of derivative growth for points in \( \Delta \).

**Lemma 5.** Let \( f \in A \), denote \( I_p := \{ x; \ p(x) = p \} \), and for each critical point \( c \in C_c \) or \( J^* \), put \( DF_p(c) := \min\{ |DF^p(x)| ; x \in I_p \cap \triangle \} \), then there exists constant \( K_1 > 0 \) such that
\[ DF_p(c) \geq K_1 \gamma_p^{-1}(c). \]

**Proof.** For any interval \( B \in M \) and integer \( n \geq 1 \), let \( B_j = f^j(B) \) for \( j = 0, \ldots, n \) and define the generalized distortion:
\[ D(f^n, B) = \prod_{j=0}^{n-1} \sup_{x_j, y_j \in B_j} \left| \frac{Df(x_j)}{Df(y_j)} \right|. \]

This definition and the mean value theorem imply
\[ \frac{Df(x_j)}{Df(y_j)} = 1 + \frac{Df(x_j) - Df(y_j)}{Df(y_j)} = 1 + \frac{Df^2(\xi_j)}{Df(y_j)}(x_j - y_j) \]
where \( \xi_j \in (x_j, y_j) \), and
\[ D(f^n, B) \leq \prod_{j=0}^{n-1} \left( 1 + \frac{\sup_{B_j} |D^2f|}{\inf_{B_j} |Df|} |B_j| \right). \]
We start by considering the points in \( \Delta \), for any \( x \in I_p \cap \Delta \), \( c \) is the closest critical point to \( x \).

**Claim:** There exists a positive constant \( K \) independent of \( x \) such that for any \( 1 \leq k \leq p(x) - 1 \), then

\[
D(f^k, [f(x), f(c)]) \leq K.
\]

Indeed, put \( B_0 = [x, c] \) and \( B_j = f^j(B_0) \), and denote \( d(B_j) = \text{dist}(B_j, C) \), from the definition of binding period, and \( \gamma_j(c) < \frac{1}{2} \) for every \( 1 \leq j \leq p(x) - 1 \), we have

\[
\frac{|B_j|}{d(B_j)} \leq \frac{|f^j(x) - f^j(c)|}{|f^j(c) - C| - |f^j(c) - f^j(x)|} \leq \frac{\gamma_j(c)|f^j(c) - C|}{(1 - \gamma_j(c))|f^j(c) - C|} = \frac{\gamma_j(c)}{1 - \gamma_j(c)} \leq 2\gamma_j(c).
\]

(11)

By the orders of the critical points, we obtain that

\[
\sup_{x_j, y_j \in B_j} \frac{|D^2 f(x_j)|}{|D^2 f(y_j)|} = \sup_{x_j, y_j \in B_j} \frac{|D^2 f|}{\inf_{B_j} |D^2 f|} \leq \frac{K^2}{d(B_j)}.
\]

where \( K_i \) is a constant from the orders of the critical point. Combining (10) and (11) we have

\[
D(f^k, B_1) \leq \prod_{j=1}^{k-1} \left(1 + \sup_{x_j, y_j \in B_j} \frac{|D^2 f(x_j)|}{|D^2 f(y_j)|} |B_j| \right)
\]

\[
\leq \prod_{j=1}^{k-1} \left(1 + K^2 \frac{|B_j|}{d(B_j)} \right) \leq \prod_{j=1}^{k} \left(1 + 2K^2 \gamma_j(c) \right).
\]

Using the inequality \( \ln(1 + x) \leq x \), and the summability of \( \gamma_j(c) \), we obtain the uniform bound of the general distortion \( D(f^k, B_1) \). The Claim follows.

Using the above Claim, we have

\[
|D f^p(x)| = |D f^{p-1}(f(x))| |D f(x)| \geq \frac{K}{K_i} |D f^{p-1}(f(c))| |x - c|^{(l(c)-1)}
\]

\[
\geq K K_i \frac{1}{l(c)} |D f^{p-1}(f(c))| |f(x) - f(c)|^{(l(c)-1)/l(c)}.
\]

On the other hand, by the mean value theorem and the definition of the binding period, we obtain

\[
K |D f^{p-1}(f(c))| |f(x) - f(c)| \geq |f^p(x) - f^p(c)| \geq \gamma_p(c) |f^p(c) - C|.
\]

So it concludes that \( |f(x) - f(c)| \geq \frac{\gamma_p(c)|f^p(c) - C|}{K |D f^{p-1}(f(c))|} |f^p(c) - C|. \)

Therefore,

\[
|D f^p(x)| \geq K K_i \frac{1}{l(c)} \left(\frac{\gamma_p(c)|f^p(c) - C|}{K |D f^{p-1}(f(c))|} \right)^{(l(c)-1)/l(c)} |D f^{p-1}(f(c))|.
\]

From the orders of the critical points, we can assume that \( |D f(f^p(c))| \leq K_i |f^p(c) - \hat{c}|^{(l(\hat{c})-1)} \), where \( \hat{c} \) is the closest critical point to \( f^n(c) \). Put \( K_1 = K K_i^{1/l(c)} \), we have
There exists $\delta$.

The proof of Lemma is complete. \(\square\)

(12) 
\[
|D^p(x)| \geq K_1 \gamma_p(c) \frac{\|c\|^{-1}}{|\gamma_p(f(c))| \gamma_p(f(c))^{1/\beta}} |f^p(c)| - \frac{1}{\beta} |x|^{-\alpha} \gamma_p(f(c))^{1/\beta} \\
\geq K_1 \gamma_p(c) \frac{\|c\|^{-1}}{|\gamma_p(f(c))| \gamma_p(f(c))^{1/\beta}} |f^p(c)| - \frac{1}{\beta} |x|^{-\alpha} \gamma_p(f(c))^{1/\beta} = K_1 \gamma_p^{-1}(c).
\]

The proof of Lemma is complete.

4. The construction of the induce map

Our aim in this Section is to construct a countable partition $\hat{P}$ of an interval $J$ into open intervals, define an inducing time function $\tau: \hat{P} \to N$ which is constant on elements of $\hat{P}$, and let $\tilde{F}: \hat{P} \to M$, denote the induce map by 
\[
\tilde{F}(x) = f^{\tau(x)}(x).
\]

We will show this induce map has uniformly bounded distortion and its image has bounded below on each element of $\hat{P}$. We will give the corresponding estimates about the induce time in the last Subsection. This construction is essentially indented to that of [4], we will use the estimates without proof if these estimates will not effect by the difference of the critical orders from two sides. More precisely, it is the following,

**Proposition 2.** Let $f \in A_1$, then there exists $\delta' > 0$ such that for all $\delta'' > 0$ the following properties hold. For an arbitrary interval $J \subset X$ with $|J| > \delta''$, there exists a countable partition $\hat{P}$ of $J$ and an induced time function $\hat{\rho}: \hat{P} \to N$ constant on each element $w$ of $\hat{P}$, such that the induced map $\tilde{F} = f^{\hat{\rho}(x)}(x)$ has uniformly bounded distortion and $|\tilde{F}(w)| = |f^{\hat{\rho}(w)}(w)| \geq \delta'$. Moreover, it satisfies the following estimates.

1. (Summability induced time)
\[
\sum_n |\hat{\rho} > n| J | < \infty.
\]

2. (Polynomial inducing time) If $d_n(c) < Cn^{-\alpha}$, $C > 0$, $\alpha > 1$ for all $c \in \mathcal{C}$ and $n \geq 1$, then there exists $\hat{C} > 0$ such that
\[
|\hat{\rho} > n| J | < \hat{C}n^{-\alpha}.
\]

3. (Stretched exponential case) If $\gamma_n(c) < Ce^{-\beta n^\alpha}$, $C > 0$, $\alpha \in (0,1)$, $\beta > 0$ for all $c \in \mathcal{C}$ and $n \geq 1$, then for each $\hat{\alpha} \in (0,\alpha)$, there exist $\hat{C}, \hat{\beta} > 0$ such that
\[
|\hat{\rho} > n| J | < \hat{C}e^{-\hat{\beta}n^{\hat{\alpha}}}.
\]

4. (Exponential case) If $\gamma_n(c) < Ce^{-\beta n}$, $C > 0$, $\beta > 0$ for all $c \in \mathcal{C}$ and $n \geq 1$, then there exist $\hat{\beta}, \hat{C} > 0$ such that
\[
|\hat{\rho} > n| J | < \hat{C}e^{-\hat{\beta}n}.
\]

At first, we clarify the role of the constant and some notations in the above Proposition. Let $\Delta_1 = \cup_{c \in \mathcal{C}} (c - \frac{\delta}{2}, c + \frac{\delta}{2})$, using a result in [10], i.e., for any Borel set $A$, there exists $C > 0$ such that $|f^{-n}(A)| \leq C|A|^{1/\max}$, then there exists $\delta' > 0$ such that for each component $w \in \Delta_1 \setminus \mathcal{C}$ and each $n \geq 0$,
\[ |f^n(w)| \geq \delta' \] \footnote{In fact, it suffices to choose \( \delta' \) so that \( f'(w) \geq \delta' \) for each \( w \in \triangle_1 \setminus \mathcal{C} \) and \( 0 \leq i \leq n = \min\{k; \text{Int}(f^k(w)) \cap \mathcal{C} \neq \emptyset\} \), where \( \text{Int}(f^k(w)) \) is the interior point of \( f^k(w) \).} We also suppose \( \delta' \leq \frac{\delta}{2} \), where \( \delta \) is a constant we have fixed in previous Section. \( \delta'' \) is a constant to be fixed in next section. We denote \(|\{\hat{p} > n|J\}||f^n|\) \footnote{H. CUI} by the conditional probability
\[ \frac{|\{x \in J; \hat{p}(x) > n\}|}{|J|}. \]

4.1. The construction of the induced map \( \hat{F} \). Denote \( \triangle_1 = \cup_{c \in C_\epsilon}(c - \frac{\epsilon}{2}, c + \frac{\epsilon}{2}) \) as above. For any interval \( J \) with \(|J| \geq \delta''\), we subdivide \( J \) by the
\[ I_p, p \geq 0. \]
For each subinterval \( w = I_p \cap J \), let
\[ \nu_1 = \min\{n \geq 0; f^n(w) \cap \triangle_1 \neq \emptyset\}, \]
be the first visit of \( w \) to \( \triangle_1 \), denote \( \hat{w} = f^{\nu_1}(w) \), we distinguish two cases:

1. \( |\hat{w}| < \delta' \), we subdivide \( \hat{w} \) with the elements \( \{I_p\} \). Each interval \( I_p \cap \hat{w} \)
for \( p > 0 \) is labeled as deep return. Notice that by the choice of \( \delta' \), each return in this case is a deep return.

2. \( |\hat{w}| \geq \delta' \), we cut off two side intervals of length \( \epsilon \) from \( \hat{w} \), where \( \epsilon \)
is a small parameter to be fixed (seeLemma \[ \ref{lem} \]). The middle part is called the large scale, add the sub interval \( w_0 \subset w \) to the partition that \( f^{\nu_1}(w_0) \)
is equals to this middle part of \( \hat{w} \), and stop to work on this middle part.
Suppose that \( \hat{w} \pm \) are the two pieces of length \( \epsilon \) that are cut off, we subdivide \( \hat{w} \pm \) further by the elements in \( \{I_p\} \). Each interval \( I_p \cap \hat{w} \) for \( p > 0 \) is labeled as deep return, \( I_0 \cap \hat{w} \) is labeled as shallow return.

Let \( w' \) be a partition interval by \( \{I_p\} \) in \( \hat{w} \) which has not reached the large scale, denote the binding period by \( p(w') \), let
\[ \nu_2 = \nu_1 + \min\{n \geq p(w'); f^n(w') \cap \triangle_1 \neq \emptyset\}. \]
Then we subdivide \( f^{\nu_2-\nu_1}(w') \) according above rules, stop until some parts reach the large scale. Notice that we have applied binding period (if exist) time iteration to guarantee expansion after each return time if it has not reach large scale.

We then construct inductively a sequence of partitions \( \hat{P}_n \) by only considering at most \( n \) iterates, denote \( \hat{P} \) by the partition of \( J \) by considering all iterates of \( f \).

Let \( x \in w \) where the above algorithm eventually stops, then there exists \( n > 0 \) such that \( x \in \hat{P}_n \), denote the stopping time by \( \hat{P}_j(x) = n \), otherwise, set \( \hat{P}_j(x) = \infty \).

Let \( \hat{J} = \{x \in J; \hat{P}_j(x) < \infty\} \), we have defined the induced map
\[ \hat{F}_j : \hat{J} \rightarrow M, \quad \hat{F}_j(x) = f^{\hat{P}_j(x)}(x). \]

In next subsection we will show \( \hat{J} \) is a partition of \( J \) up to a set of zero Lebesgue measure, and give the corresponding estimates of \( \hat{F}_j \), but now we first want to know the structure of \( \hat{J} \). Given an interval \( w \) in some partition \( \hat{P}_n, n > 0 \), we associated it with a sequence
\[ (\nu_1, p_1), (\nu_2, p_2), \ldots, (\nu_n, p_n), \]
where \( \nu_i, 1 \leq i \leq s \) is the return times to \( \triangle_1 \), \( p_i \) is the corresponding periods, and \( s \) is the maximum integer such that \( \nu_s \leq n \). If \( w \) is an interval on which \( (\nu_1, p_1), (\nu_2, p_2), \ldots, (\nu_{j-1}, p_{j-1}) \), is fixed and \( \nu_j \) is the next return, then \( \{ x \in w; p(f^{\nu_j}x) = p \} \) has at most 4 components. This maximum is attained when \( |f^{\nu_j}(w)| \geq \delta \), and the outmost intervals of length of \( \epsilon \) contain a critical point and \( p_j \) is big enough.

Notice that the corresponding sequence \( (\nu_1, p_1), (\nu_2, p_2), \ldots, (\nu_s, p_s) \) contains information about the iterations of \( f \) on \( w \), although many sequences don’t correspond to partition intervals. For a given sequence \( (\nu_1, p_1), (\nu_2, p_2), \ldots, (\nu_s, p_s) \), let

\[
\mathcal{S}_d := \{ i \leq s; \nu_i \text{ is a deep return} \} = \{ i \leq s; p_i > 0 \},
\]

\[
\mathcal{S}_s := \{ i \leq s; \nu_i \text{ is a shallow return} \} = \{ i \leq s; p_i = 0 \},
\]

\[
\mathcal{S}_{s,s} := \{ i < s; p_i = 0 \text{ and } p_{i+1} = 1 \}.
\]

### 4.2. Important estimates

**Lemma 6.** There exists \( K_\iota > 0 \) such that for all \( w \in \hat{P} \), the distortion of \( \hat{F}_\lambda|_w \) is bounded by \( K_\iota \).

**Proof.** Let \( w \in \hat{P}_n \), then \( f^n(w) \) has reached large scale. By the construction of the induced map, there is an interval \( T \supset w \) such that \( f^n(T) \) is a \( \epsilon \)-scaled neighborhood of \( f^n(w) \), i.e., \( |f^n(T) \setminus f^n(w)| \geq \epsilon |f^n(w)| \). On the other hand, it is easy to see that \( f^n \) is a diffeomorphism on \( T \). Using the Koebe principle, we can obtain the result. \( \square \)

For a given sequence \( (\nu_1, p_1), (\nu_2, p_2), \ldots, (\nu_s, p_s) \), the following Lemma contains an important metric estimation of the length of the corresponding interval.

**Lemma 7.** Let \( C = C(\frac{\delta}{2}) \) and \( \lambda = \lambda_{\delta/2} \) be as in Lemma 3, and \( K \) be constant in Lemma 3. There exist \( K_0 > 0 \) independent of \( \epsilon \) and \( \rho \in (0, 1) \) with the following properties. For a given sequence \( (\nu_1, p_1), (\nu_2, p_2), \ldots, (\nu_s, p_s) \) with \( \nu_s \leq n \), the corresponding interval \( w_{p_1p_2\ldots p_s} \in \hat{P}_n \), we have

\[
\frac{|w_{p_1p_2\ldots p_s}|}{|f^m(w_{p_1p_2\ldots p_s})|} \leq \min \left\{ C^{-\delta s_d} e^{-\lambda (m-s_0 p_0)}, \left( \frac{K_0}{K} \right) \rho^{s_{s,s}} \right\} \prod_{i \in \mathcal{S}_d} (DF_{p_i})^{-1}
\]

for \( m = \max \{ n, \nu_s + p_s \} \). And \( \rho \to 0 \) as \( \epsilon \to 0 \). Moreover there exists \( T > 0 \) which can be chosen arbitrarily large if \( \epsilon \) is small, such that \( \nu_{i+1} - \nu_i \geq T \) whenever \( p_i = p_{i+1} = 1 \).

**Proof.** See the proof of Lemma 3.2 in [4]. \( \square \)

### 4.3. Induced time estimates

The aim of this subsection is to estimate the tail behaviors of the induced time function \( \hat{P} \), i.e., the estimation of \( \{ x \in J; \hat{p}(x) > n \} \).

We fixed \( n \) for the rest of this subsection. Let \( \eta > 0 \) be a small constant to be determined in Lemma 8, we can divide the elements in \( \hat{P}_n \) with \( \hat{p}|_w > n \) into two parts

\[
\hat{P}_n' = \{ w \in \hat{P}_n; \hat{p}|_w > n, \sum_{i=1}^{s} p_i \leq \eta n \}.
\]
Given a sequence \( \nu \) return term such that has length \( s \).

Then we have
\[
|\{\hat{p} > n\}| = \sum_{w \in \hat{P}_n'} |w| + \sum_{w \in \hat{P}_n''} |w|.
\]

To treat the exponential and stretched exponential cases, we subdivide \( \hat{P}_n'' \) further into
\[
\hat{P}_n'' = \{w \in \hat{P}_n'' \mid s \leq \rho n^{\hat{\alpha}}\}, \quad \hat{P}_n'' = \{w \in \hat{P}_n'' \mid s > \rho n^{\hat{\alpha}}\},
\]
where \( \hat{\alpha} \in (0, 1] \) and \( \rho > 0 \) are constants to be fixed below.

Intuitively, \( p_1, p_2, \ldots, p_s \) in \( \hat{P}_n'' \) is small, then the elements in \( \hat{P}_n'' \) spend much time in \( X \setminus \triangle \). Thus we can use Lemma 3 to get exponential rate of decay of \( \hat{P}_n'' \). For elements in \( \hat{P}_n'' \), they spend much time in \( \triangle \), then the estimations of \( \hat{P}_n'' \) relate closely to the property of the critical point. So we obtain the estimations of \( \hat{P}_n'' \) by using Lemma 2 and Lemma 7 under the assumption of the rates of decay of \( \gamma_n(c) \).

**Lemma 8.** For any \( \theta > 0 \) there exists \( \eta_0 > 0 \) such that for all \( 0 < \eta < \eta_0 \) and for \( n \) sufficiently large,
\[
\sum_{w \in \hat{P}_n''} |w| \leq e^{- (\lambda - \theta) n}.
\]

See the proof of Lemma 3.5 in [4].

**Lemma 9.** Fix \( L \in \{1, \ldots, n\} \) arbitrary and let
\[
d_{n,s}(c) := d_i(c) \text{ for } i = \max\{\frac{\eta m}{2s^2}, L\}.
\]
Write \( s(w) = s \) if the associate sequence \((\nu_1, p_1), (\nu_2, p_2), \ldots, (\nu_s, p_s)\) of \( w \) has length \( s \). For any \( \eta > 0 \) there exists \( C_1 > 0 \) such that
\[
\sum_{w \in \hat{P}_n'' : s(w) \geq L} |w| \leq C_1 \max_{c \in C} \sum_{s=L}^{n} 2^{-s} \hat{d}_{n,s}(c).
\]

See the proof of Lemma 4.6 in [4]. Given a sequence \((\nu_1, p_1), (\nu_2, p_2), \ldots, (\nu_{j'}, p_{j'})\), let \( p_{j'} \) be the first return term such that \( p_{j'} \geq \frac{\eta m}{2j'^2} \). Because \( p_1 + p_2 + \cdots + p_s \geq \eta n \), such \( j' \) exists. Take \( j = \max\{L, j'\} \).

Let \( \bar{w}_{p_1 p_2 \ldots p_s} \) be the union of adjacent intervals \( w_{p_1 p_2 \ldots p_s} \) with common return times \( \nu_1, \ldots, \nu_{j'} \) and \( p \geq p_{j'} \). Then \( f^{p_j} \) maps \( \bar{w}_{p_1 p_2 \ldots p_j} \) diffeomorphically onto a interval \((x, y)\) such that \( p(x), p(y) \geq p_j \). Assume without loss of generality that \(|x - c| \geq |y - c|\) and \( \hat{c} \) is the closest critical point to \( f^i(c) \), therefore for \( i < p_j \),
\[
\gamma_i(c)|f^i(c) - \hat{c}| \geq |f^i(x) - f^i(c)| \geq K|Df^{i-1}(f(c))||f(x) - f(c)|
\]
(13)
\[
\geq K|Df^{i-1}(f(c))||x - c|^{l(c)}
\]
\[
\geq KK_{i'} |Df^{i}(f(c))||x - c|^{l(c)}
\]
\[
|f^{i}(c) - \hat{c}|^{l(c)-1},
\]
where $K$ is the constant in the proof of Lemma \[5\]. This gives that
\[
\gamma(t)|f^i(c) - \tilde{c}|^{l(c)} \geq KK_l|Df^i(f(c))||x - c|^{l(c)}.
\]

Thus
\[
|x - y| \leq 2|x - c| \leq 2KK_l\left(\frac{\gamma(t)|f^i(c) - \tilde{c}|^{l(c)}}{|Df^i(f(c))|}\right)^{1/l(c)}|f^i(c) - \tilde{c}|^{l(c)/l(c)}.
\]

Since the above inequality holds for all $i \leq p_j$, and by the definition of $d_{p_j}(c)$, we can obtain that
\[
|x - y| \leq 2KK_l\max_{p \geq \frac{\alpha}{2}} d_{p}(c) = 2KK_l\hat{d}_{n,j}(c).
\]

Let $S_s$ and $S_d$ be shallow returns times and deep return times before the indices smaller than $j$, $S_d' := S_d\setminus\{j\}$, since $f^{\rho}$ maps $\bar{w}_{p_1p_2...,p_j}$ diffeomorphically onto an interval $(x, y)$, and by Lemma \[7\] we have
\[
\sum_{w \in \bar{P}_{n_s'}(s(w) \geq L)} |w| \leq \sum_{j=L}^{n} \sum_{p_1,p_2,...,p_j} |\bar{w}_{p_1,...,p_j}|
\]
\[
\leq \sum_{j=L}^{n} 2KK_l\max_{i \in C_c} \hat{d}_{n,j}(c) \sum_{p_1,p_2,...,p_{j-1}} 4^j \frac{K_1}{K} \rho \#S_{s,s} \prod_{i \in S_d'} \frac{1}{DF_{p_i}}.
\]

On the other hand, by an element calculation,
\[
\sum_{p_1,p_2,...,p_{j-1}} 4^j \frac{K_1}{K} \rho \#S_{s,s} \prod_{i \in S_d'} \frac{1}{DF_{p_i}} \leq 2^{-j} \frac{512K_1}{K} \sum_{p_1,p_2,...,p_{j-1}} (8\rho) \#S_{s,s} \prod_{i \in S_d'} \frac{64K_1}{KDF_{p_i}}.
\]

Notice that $p_i \geq p_\delta$ for $i \in S_d$, and we can take $\epsilon$ so small that $\rho \leq 1/8$. Thus by Lemma \[4\] and the choose of $\delta$, we have
\[
\sum_{p_1,p_2,...,p_{j-1} \geq p_\delta} (8\rho) \#S_{s,s} \prod_{i \in S_d'} \frac{64K_1}{KDF_{p_i}} \leq 1.
\]

Therefore, take $C_1 = 1024K_1K_l$, the Lemma follows. \hfill \Box

To distinguish the exponential and stretched exponential case, we need the following Lemma.

**Lemma 10.** Assume that there exist $C, \beta > 0$ and $\alpha \in (0, 1]$ such that
\[
\gamma_0(c) \leq Ce^{-\beta n^\alpha}
\]
for all positive integer $n$ and $c \in C_c$. Then for each $\tilde{\alpha} \in (0, \alpha)$ (or $\tilde{\alpha} = 1$ if $\alpha = 1$) there exist $\rho, C', \beta'$ such that
\[
\sum_{w \in \tilde{P}_{n_{\tilde{\alpha}}}} |w| \leq C'e^{-\beta' n^\alpha}.
\]

**Proof.** See the proof of Lemma 3.7 in \[4\]. \hfill \Box

**Proof of Proposition 3.1** We first show that $\hat{J}$ has full measure in $J$.

Because $f$ has no wandering intervals \[10\], almost all $x \in J$, $f^{n_k}(x) \to C_c$ as $n_k \to \infty$. Hence $x$ has infinitely many deep returns (if it has not reach large
scale), assume that \( x \) is contained in \( w_{p_1, \ldots, p_s} \) with arbitrary \( s \). By Lemma \ref{lem:bounded_divergence} and Lemma \ref{lem:extension}, we have
\[
|\{x \in J; \hat{p}(x) = \infty\}| \leq \lim_{s \to \infty} \sum_{(p_1, \ldots, p_s)} \left| w_{(p_1, \ldots, p_s)} \right| = 0.
\]

By the construction of the induce map \( F \); \( \hat{F} \) is a diffeomorphism with uniformly bounded distortion, and \( |\hat{F}(w)| \geq \delta' \) on any component \( w \) of \( \hat{J} \).

Next we will show the estimates mentioned in the Proposition. Notice that we have exponential bounds for \( \hat{P}_n \) by Lemma \ref{lem:bounded_divergence}, so we only consider \( \hat{P}_n' \) here. First observe that for each \( k \) there are at most
\[
\# \{ n; k - 1 \leq \eta m/2 s^2 \leq k \} \leq 2 s^2 / \eta
\]

numbers \( n \) such that \( k = [\eta m/2 s^2] \). Therefore, using Lemma \ref{lem:approximation} with \( L = 1 \),
\[
\sum_{n \geq 1} \sum_{s=1}^n \hat{d}_{n,s}(c) \leq \sum_{s \geq 1} 2 s^2 / \eta \sum_{k \geq 1} (\gamma_k(c) / |Df^k(f(c))|^{1/l(c)}) (f^k(c) - \hat{c}^{l(c)/l(c)})
\]
\[
\leq \frac{12}{\eta} \sum_{k \geq 1} \gamma_k(c) \leq \infty.
\]

Hence
\[
\sum_n |\hat{p} > n| J | = \sum_n |\hat{P}_n'| + \sum_n |\hat{P}_n''| \leq \infty.
\]

For polynomial case, take \( L = 1 \) in Lemma \ref{cor:exponential} if \( \gamma_n(c) < C n^{-\alpha} \), \( \alpha > 1 \) for all \( c \in \mathcal{C}_c \), and \( n \geq 1 \), then
\[
\sum_{w \in \hat{P}_n'} \leq C_1 \max_{c \in \mathcal{C}_c} \sum_{s=1}^n 2^{-s} \hat{d}_{n,s}(c) \leq C_1 \max_{c \in \mathcal{C}_c} \sum_{s=1}^n 2^{-s} d_{\{m/2 s^2\}}(c)
\]
\[
\leq C_1 \max_{c \in \mathcal{C}_c} \sum_{s=1}^n 2^{-s} \frac{\eta m}{2 s^2} \leq \frac{12 C_1}{(\eta m)^{\alpha}}.
\]

For the stretched exponential case, if \( \gamma_n(c) \leq C e^{-\beta n^\alpha} \) for all \( n \) and \( c \in \mathcal{C}_c \), using Lemma \ref{cor:exponential} and take \( L = \rho n^{\hat{\alpha}} \), we obtain
\[
\sum_{w \in \hat{P}_n'} \leq C_1 \max_{c \in \mathcal{C}_c} \sum_{s=\rho n^{\hat{\alpha}}}^n 2^{-s} \hat{d}_{n,s}(c) \leq C_1 \max_{c \in \mathcal{C}_c} \sum_{s=\rho n^{\hat{\alpha}}}^n 2^{-s} d_{\{m/2 s^2\}}(c)
\]
\[
\leq C_1 2^{-\rho n^{\hat{\alpha}}} \max_{c \in \mathcal{C}_c} d_{\rho n^{\hat{\alpha}}}(c) \leq C_1 e^{-\rho' n^{\hat{\alpha}}},
\]
where the third inequality follows from that \( d_n(c) \) is decreasing as \( n \). On the other hand, by Lemma \ref{cor:exponential}, we have for each \( \hat{\alpha} \in (0, 1) \),
\[
\sum_{w \in \hat{P}_n'} |u| \leq C' e^{-\beta n^\hat{\alpha}}.
\]

Since \( \hat{\alpha} < \alpha \), we obtain result in the stretched case.

For the exponential case, it suffices to take \( \alpha = 1 \) in the stretched case. So we obtain all results claimed in Proposition.
5. The construction of the full Markov map

In this Section we will construct a full Markov map \( \hat{f} \) and give its estimates based on the map \( \hat{F} : \hat{J} \to X \) constructed in previous section. Let \( J \in X \) be an arbitrary interval, \( \hat{P} \) be the corresponding partition constructed in Section 4.

**Lemma 11.** Let \( f \in A_1 \). There exists an open interval \( \Omega \) around each critical point \( c \) that is contained in a minimal cycle \( X \), an integer \( t_0 \) and a constant \( \zeta > 0 \) such that for every \( w \in \hat{P} \), there exists an interval \( \tilde{w} \subset w \) with the following properties:

1. there exists an integer \( t \leq t_0 \) such that \( f^t(\tilde{w}) \) maps \( \tilde{w} \) diffeomorphically onto \( \Omega \),
2. \( |\tilde{w}| \geq \zeta |w| \),
3. both components of \( f^{t}(w \setminus \tilde{w}) \) have length \( \geq \delta' / 3 \).

**Proof.** By Theorem 1, the preimages of each critical point \( c \in C_\zeta \) are dense in \( X \). Then there exists a finite integer \( t_0 > 1 \) such that \( X \setminus \bigcup_{i=0}^{t_0} f^{-i}(c) \) contains no interval of length greater than \( \delta' \). Let \( x := f^{-t}(c) \) for some \( 0 < t \leq t_0 \), we can choose \( w_x \) be a maximal interval containing \( x \) such that \( w_x \) diffeomorphically onto a neighborhood \( \Omega \) of \( c \) and \( x \) in \( w_x \) middle fifth. By adjusting the size of \( w_x \), we can make sure that they all map onto exactly the same critical neighborhood \( \Omega \) and \( w_x \leq \delta'/10 \). Let \( \tilde{w} \subset w \in \hat{P} \) be the interval that is mapped onto \( w_x \) by \( f^t(\tilde{w}) \), such a \( \tilde{w} \) exists by Proposition 2. Then the first and the third statement follow. Because \( f^t(\tilde{w}) \) has bounded distortion on \( w \) by Lemma 6 and the diffeomorphism \( f^t (t \leq t_0) \) doesn’t effect the distortion significantly, the second statement follows immediately. \( \square \)

We have fixed \( \Omega \) and let \( \delta'' = \min \{ \delta'/3, |\Omega| \} \). We shall define a full Markov map by the construction in Proposition 2 and Lemma 11.

Let \( \hat{f} : \Omega \to \Omega \) be defined as follows, \( Q \) be the associated partition of \( \Omega \) and \( R \) be the induced time function. For each element \( w \) in the partition \( \hat{P} \) of \( \Omega \), let \( \tilde{w} \) denote the subinterval given in Lemma 11. We put \( \hat{w} \) into \( Q \) and \( R(\hat{w}) = \hat{p}(w) + t \). Then \( \hat{f}(\hat{w}) = f^R(\hat{w}) = \Omega \) by Lemma 11. On the other hand, both components of \( f^{R(\hat{w})}(w) \setminus f^{R(\hat{w})}(\hat{w}) \) has size at least \( \delta'/3 \) (\( \geq \delta'' \)), we consider them as new starting intervals and carry out the construction in Section 4 and repeat the procedure as above. In this way, for each \( w \in Q \), we can define an associated sequence before it has reached full return (i.e., \( \exists n > 0, f^n(w) = \Omega \)), write \( \hat{p}_1 = \hat{p}(x) \) and \( \hat{p}_{i+1}(x) = \hat{p}(f^{\hat{r}_i(x)}(x)) \), \( 1 < i < s - 1 \), where \( s \) is the integer such that \( f^s \) maps corresponding element containing \( x \) to reach full return. Then we have \( R(w) = \hat{p}_s(w) + t \) for \( s \geq 1, t \leq t_0 \).

Next we collect some important results on the induced full Markov map and the return time estimates, they are important bounds used in Young [29].

**Lemma 12.** For each \( n \leq 0 \) and each interval \( w \in Q \) on which \( \hat{f}^n \) is continuous, the distortion of \( \hat{f}^n \) is uniformly bounded.

**Proof.** It follows from the construction of \( \hat{f} \) immediately. \( \square \)
Lemma 13. Let \( f \in A_1 \), \( \hat{f} = f^R \) be the induced full Markov map defined above. Then we have following statements:

Summable case: Under no conditions on \( d_n(c) \),
\[
\sum_n |\{R > n\}| < \infty.
\]

Polynomial case: If \( d_n(c) \leq Cn^{-\alpha}, \ C > 0, \ \alpha > 1 \) for all \( c \in C_c \) and \( n \geq 1 \), then there exists \( \tilde{C} > 0 \) such that
\[
|\{R > n\}| \leq \tilde{C}n^{-\alpha}.
\]

Stretched exponential case: If \( \gamma_n(c) \leq Ce^{-\beta n^\alpha}, \ C > 0, \ \alpha \in (0,1), \ \beta > 0 \) for all \( c \in C_c \) and \( n \geq 1 \), then for each \( \tilde{\alpha} \in (0,\alpha) \) there exist \( \tilde{\beta}, \tilde{C} > 0 \) such that
\[
|\{R > n\}| \leq \tilde{C}e^{-\tilde{\beta}n^\tilde{\alpha}}.
\]

Exponential case: If \( \gamma_n(c) \leq Ce^{-\beta n}, \ C > 0, \ \beta > 0 \) for all \( c \in C_c \) and \( n \geq 1 \), then there exist \( \tilde{\beta}, \tilde{C} > 0 \) such that
\[
|\{R > n\}| \leq \tilde{C}e^{-\tilde{\beta}n}.
\]

Proof. See the details of the proof of Proposition 4.1 in [4]. □

For \( x, y \in Q \), we let separation time \( s(x,y) \geq 0 \) be the least integer \( k \) such that \( \hat{f}^k(x) \) and \( \hat{f}^k(y) \) belong to different elements of \( Q \). So \( s(x,y) = 0 \) if \( x, y \) are in different component of \( Q \), and \( s(x,y) \geq 1 \) if they belongs to the same component of \( Q \).

Lemma 14. There exist \( \beta \in (0,1) \) and \( C > 0 \) such that for all \( w \in Q \) and all \( x, y \in w \), we have
\[
\left| \frac{D\hat{f}(x)}{D\hat{f}(y)} - 1 \right| \leq C\beta^{s(x,y)}.
\]

Proof. See [4]. □

6. Proof of Theorem 2 and Theorem 3

For \( f \in A_1 \), we focus on a minimal cycle of \( f \). Let it be \( X = \bigcup_{i=0}^{m-1} f^i(J) \), where \( m \) is the period of the minimal cycle \( X \). In previous Section, we have constructed an induced full Markov map \( f^R(w) \) on a neighborhood \( \Omega \) of a critical point \( c \in J \). Then denote \( g = \Lambda^{-1} \circ f^m \circ \Lambda \) be a renormalization of \( f \), where \( \Lambda \) is an affine transformation from interval \( M \) to \( J \). We will apply results of Young [29] to study the statistical properties of \( g \) stated in Theorem 2.

From the definition of \( g \), \( g \) induces a Markov map \( \hat{G} := \Lambda^{-1} \circ \hat{f} \circ \Lambda \) on the interval \( \Delta_0 = \Lambda^{-1}(\Omega) \). Assume that the induced time of \( \hat{G} \) about \( g \) is \( T(w) \), clearly \( T(w) = R(w)/m \).

To treat the statistical properties of \( g \), we define a tower about \( g \)
\[
\hat{\Delta} = \{(x,k) \in \Delta_0 \times Z; 0 \leq k \leq T(x)\}
\]
consists of levels \( \Delta_n = \{(x,k) \in \hat{\Delta}; k = n\} \). We then define the map \( \hat{g} : \Delta \to \hat{\Delta} \) by
\[ \hat{g}(x, k) := \begin{cases} (x, k + 1) & \text{if } k + 1 \leq T(x), \ x \in w, \\ (\hat{G}(x), 0) & \text{if } k + 1 = T(x), \ x \in w. \end{cases} \]

Let \( \sigma : \hat{\Delta} \to M \) by \( \sigma(x, k) = g^k(x) \), which is a semi-conjugacy between \( \hat{g} \) and \( g \). So we can get the statistical properties of \( g \) from the properties of \( \hat{g} \).

To study the structure of \( \hat{g} \), we define \( \pi \) be a projection between \( \hat{\Delta} \) and \( \Delta_0 \) by \( \pi(x, l) = x \), clearly \( \pi \hat{g} = \hat{G} \pi \). The powerful result of [29] relates the tower map \( \hat{g} \) and the full induced Markov map \( \hat{G} \).

We summarize Young’s result from [29] as we need. Recall some notations from above, for a fixed \( \beta \in (0, 1) \), let

\[ C_\beta = \{ \hat{\varphi} : \hat{\Delta} \to R; \ \exists C > 0, \forall x, y \in \hat{\Delta}, \ |\hat{\varphi}(x) - \hat{\varphi}(y)| \leq C \beta^{\alpha(x,y)} \}, \]

and

\[ C_\beta^+ = \{ \hat{\varphi} \in C_\beta; \hat{\varphi} > 0 \}. \]

**Theorem 4.** [29] Suppose that \( \hat{G} : \Delta_0 \to \Delta_0 \) and \( \hat{g} : \Delta \to \Delta \) defined as above. Let \( \rho_n \) be a sequence of positive numbers related to the tail behavior as follows. If \( |\{ T > n \}| \leq n^{-\alpha} \), then \( \rho_n = n^{1-\alpha} \). If \( |\{ T > n \}| \leq \exp(-\beta n) \), then \( \rho_n = \exp(-\beta n) \) for some \( \beta < 1 \). If \( |\{ T > n \}| \leq n^{-\alpha} \) for some \( \alpha \in (0, 1) \), then \( \rho_n = \exp(-n^\alpha) \) for some \( \alpha' < \alpha \). If

\[ \left| \frac{D \hat{G}(x)}{D \hat{G}(y)} \right| - 1 \leq C \beta^{\alpha(x,y)} \]

for any \( x, y \in \Delta_0 \), some \( \beta \in (0, 1) \) and \( C > 0 \), and the induce time function \( T \) satisfies \( \sum_n |T > n| < \infty \), then we have

1. \( \hat{\Delta} \) carries a \( \hat{g} \) acip \( \hat{\nu} \) and \( \frac{d\nu}{dm_\hat{\Delta}} \in C_\beta^+ \), where \( m_\hat{\Delta} \) be Lebesgue measure on the tower. \( (\hat{g}, \hat{\nu}) \) is exact, hence ergodic and mixing.

2. For any pair of functions \( \hat{\varphi} \in L^\infty(\hat{\Delta}, m_\hat{\Delta}) \) and \( \hat{\psi} \in C_\beta \), there exists \( C_{\hat{\varphi}, \hat{\psi}} \) such that

\[ |\int (\hat{\varphi} \circ \hat{g}) \hat{\psi} d\hat{\nu} - \int \hat{\varphi} d\hat{\nu} \int \hat{\psi} d\hat{\nu}| \leq C_{\hat{\varphi}, \hat{\psi}} \rho_n. \]

3. If \( |\{ T > n \}| \leq O(n^{-\alpha}) \) for some \( \alpha > 2 \), then for any \( \hat{\varphi} \in C_\beta \) which is not a coboundary \( (\hat{\varphi} \neq \int \hat{\psi} d\hat{\nu} - \hat{\psi}) \), the Central Limit Theorem holds, i.e., there exists \( \sigma > 0 \) such that \( \frac{1}{\sqrt{n}} \sum_{i=1}^{n-1} \hat{\varphi} \circ \hat{g}^i \) converges to the normal distribution \( \mathcal{N}(\int \hat{\varphi} d\hat{\nu}, \sigma) \).

At first, Lemma [13] and Lemma [14] imply that conditions in Young’s Theorem hold. If we define a measure on \( M \) by \( \mu = \sigma_* \hat{\nu} \), i.e., \( \mu(B) = \hat{\nu}(\sigma^{-1}(B)) \) for all measurable set \( B \), we have

\[ \mu(g^{-1}(B)) = \hat{\nu}(\sigma^{-1}(g^{-1}(B))) = \hat{\nu}(\sigma^{-1}(g^{-1}(B))) = \hat{\nu}(\sigma^{-1}(B)) = \mu(B). \]

So \( \mu \) is an invariant measure of \( g \). Note that the reference measure \( m_\hat{\Delta} \) defined by \( m_\hat{\Delta}(A) = \sum_{k \geq 0} m(\pi(A \cap \Delta_k)) \) for all measurable set \( A \in \hat{\Delta} \). Let
Thus the decay of correlations of $g$, Young's Theorem.

We have for $0 < k$ for all $x$ be two H"older continuous functions so that there exist $C > 0$ from above theorem. Then the support is equal to the entire interval $M$, which in turn implies that $\mu$ is the unique acip for $g$ on $M$. According to Theorem 1, if $l_{\text{max}} > 1$ for all $c$ in the cycle $X$, we obtains that $\frac{d\mu}{dm} \in L^\infty$ for all $0 < \tau < \frac{l_{\text{max}}}{l_{\text{max}} - 1}$.

We also obtain that $(g, \mu)$ is exact. Indeed, if there are sets $B_0, B_1, \ldots, B_i \subset M$ such that $\mu(B_j) \in (0, 1)$ and $B_0 = g^{-j}(B_j)$ for all integer $j \geq 0$, then let $A_j = \sigma^{-1}(B_j)$, $\hat{\nu}(A_j) \in (0, 1)$, therefore

$$\hat{g}^{-j}(A_j) = g^{-j}(\sigma^{-1}(B_j)) = \sigma^{-1}g^{-j}(B_j) = \sigma^{-1}(B_0) = A_0,$$

which contradicts to the first statement of Young's Theorem.

We shall consider the decay of correlations of $(g, \mu)$. Let $\varphi, \psi : M \to R$ be two H"older continuous functions so that there exist $C > 0$ and $\alpha \in (0, 1)$

$$|\varphi(x_1) - \varphi(x_2)| \leq C|x_1 - x_2|^\alpha,$$

for all $x_1, x_2 \in M$. Let $\hat{\varphi} = \varphi \circ \sigma$, then if $x, y \in \Delta_0$ with $s(x, y) \geq 1$, we have for $0 < k \leq T(x) = T(y)$

$$|\hat{\varphi}(x, k) - \hat{\varphi}(y, k)| = |\varphi \circ g^k(x) - \varphi \circ g^k(y)| \leq C|g^k(x) - g^k(y)|^\alpha \leq C|f(x) - f(y)|^\alpha \leq C(\beta^\alpha)^{s(x, y)},$$

thus $\alpha$-H"older continuous observables on $M$ correspond to observables in $C_{\beta_1}$ for $\beta_1 = \beta^\alpha$, where $\beta$ is the constant in Lemma 14.

Since $\mu = \sigma_\ast \hat{\nu}$, $\hat{\varphi} = \varphi \circ \sigma$ and $\hat{\psi} = \psi \circ \sigma$, we have the following relations

$$\int \hat{\varphi}d\hat{\nu} = \int \varphi d\mu, \int \hat{\psi}d\hat{\nu} = \int \psi d\mu$$

and

$$\int (\varphi \circ g^k) \psi d\mu = \int \varphi d\mu \int \psi d\nu = \int \hat{\varphi}d\hat{\nu} \int \hat{\psi}d\hat{\nu}.$$

Thus the decay of correlations of $g$ follows from the second statement of the Young's Theorem.

Similarly, because

$$\mu\{x \in M; \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \varphi(g^j(x)) - \int \varphi d\mu \in I\} = \hat{\nu}\{y \in \hat{\Delta}; \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \hat{\varphi}(g^j(y)) - \int \hat{\varphi}d\hat{\nu} \in I\},$$

applying the third statement of Young's Theorem, we obtain the Central Limit Theorem for the H"older continuous observable $\varphi$. Therefore, we have finished the proof of Theorem 2.
For $f \in \mathcal{A}_2$, we concentrated on the dynamics of $(f, J^*)$. In previous sections, we have constructed an induced full Markov map $f^{R(w)}$ on a neighborhood $\Omega$ of a critical point $c \in J^*$. Let $k$ be the greatest common divisor of all values taken by the function $R(w)$, then we can obtain Theorem 3 by using Young’s results and the above argument similarly.

References

[1] V. Araújo, S. Luzzatto, and M. Viana. Invariant measures for interval maps with critical points and singularities. *Adv. Math.*, 221(5):1428–1444, 2009.

[2] V. Araújo and M.J. Pacifico. Physical measures for infinite-modal maps. 2007.

[3] M. Benedicks and L. Carleson. On iterations of $1 – ax^2$ on $(-1, 1)$. *Ann. of Math. (2)*, 122(1):1–25, 1985.

[4] H. Bruin, S. Luzzatto, and S. van Strien. Decay of correlations in one-dimensional dynamics. *Ann. Sci. École Norm. Sup. (4)*, 36(4):621–646, 2003.

[5] H. Bruin, J. Rivera-Letelier, W. Shen, and S. van Strien. Large derivatives, backward contraction and invariant densities for interval maps. *Invent. Math.*, 172(3):509–533, 2008.

[6] H. Bruin, W. Shen, and S. van Strien. Invariant measures exist without a growth condition. *Comm. Math. Phys.*, 241(2-3):287–306, 2003.

[7] H. Bruin and S. van Strien. Expansion of derivatives in one-dimensional dynamics. *Israel J. Math.*, 137:223–263, 2003.

[8] S. Cedervall. Invariant measures and correlation decay for s-multimodal interval maps. *Ph.D. thesis*, page 111, 2006.

[9] P. Collet and J.-P. Eckmann. Positive Liapunov exponents and absolute continuity for maps of the interval. *Ergodic Theory Dynam. Systems*, 3(1):13–46, 1983.

[10] H. Cui and Y. Ding. Wandering intervals and absolutely continuous invariant probability measures of interval maps. *arXiv:0912.1469v2*, 2009.

[11] K. Díaz-Ordaz. Decay of correlations for non-Hölder observables for one-dimensional expanding Lorenz-like maps. *Discrete Contin. Dyn. Syst.*, 15(1):159–176, 2006.

[12] K. Díaz-Ordaz, M. P. Holland, and S. Luzzatto. Statistical properties of one-dimensional maps with critical points and singularities. *Stoch. Dyn.*, 6(4):423–458, 2006.

[13] M. V. Jakobson. Absolutely continuous invariant measures for one-parameter families of one-dimensional maps. *Comm. Math. Phys.*, 81(1):39–88, 1981.

[14] G. Keller and T. Nowicki. Spectral theory, zeta functions and the distribution of periodic points for Collet-Eckmann maps. *Comm. Math. Phys.*, 149(1):31–69, 1992.

[15] O. S. Kozlovski. Getting rid of the negative Schwarzian derivative condition. *Ann. of Math. (2)*, 152(3):743–762, 2000.

[16] C. Liverani. Decay of correlations. *Ann. of Math. (2)*, 142(2):239–301, 1995.

[17] S. Luzzatto. Stochastic-like behaviour in nonuniformly expanding maps. In *Handbook of dynamical systems. Vol. 1B*, pages 265–326. Elsevier B. V., Amsterdam, 2006.

[18] V. Lynch. Decay of correlations for non-Hölder observables. *Discrete Contin. Dyn. Syst.*, 16(1):19–46, 2006.

[19] R. Mañé. Hyperbolicity, sinks and measure in one-dimensional dynamics. *Comm. Math. Phys.*, 100(4):495–524, 1985.

[20] R. J. Metzger. Sinai-Ruelle-Bowen measures for contracting Lorenz maps and flows. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 17(2):247–276, 2000.

[21] M. Misiurewicz. Absolutely continuous measures for certain maps of an interval. *Inst. Hautes Études Sci. Publ. Math.*, (53):17–51, 1981.

[22] T. Nowicki and D. Sands. Non-uniform hyperbolicity and universal bounds for $S$-unimodal maps. *Invent. Math.*, 132(3):633–680, 1998.

[23] T. Nowicki and S. van Strien. Invariant measures exist under a summability condition for unimodal maps. *Invent. Math.*, 105(1):123–136, 1991.

[24] A. Rovella. The dynamics of perturbations of the contracting Lorenz attractor. *Bol. Soc. Brasil. Mat. (N.S.)*, 24(2):233–259, 1993.
22 H. CUI

[25] D. Singer. Stable orbits and bifurcation of maps of the interval. *SIAM J. Appl. Math.*, 35(2):260–267, 1978.

[26] S. van Strien and E. Vargas. Real bounds, ergodicity and negative Schwarzian for multimodal maps. *J. Amer. Math. Soc.*, 17(4):749–782 (electronic), 2004.

[27] Lai-sang Young. Decay of correlations for certain quadratic maps. *Comm. Math. Phys.*, 146:123–138, 1992.

[28] Lai-Sang Young. Statistical properties of dynamical systems with some hyperbolicity. *Ann. of Math. (2)*, 147(3):585–650, 1998.

[29] Lai-Sang Young. Recurrence times and rates of mixing. *Israel J. Math.*, 110:153–188, 1999.

Wuhan Institute of Physics and Mathematics, The Chinese Academy of Sciences, P.O. Box 71010, Wuhan 430071, China

E-mail address: cuihongfei05@mails.gucas.ac.cn