A New Polar Coding Scheme for the Interference Channel

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Abstract

We consider the problem of polar coding for the 2-user discrete memoryless interference channel (DM-IC). Existing schemes are designed according to the original description of the Han-Kobayashi region, which involves two deterministic functions that map auxiliary random variables to channel inputs. Inspired by Chong et al.’s compact description of the Han-Kobayashi region, we propose a novel, simplified design of polar codes to achieve the Han-Kobayashi region which does not require the aforementioned mapping functions. We design two types of polar coding schemes and show that every point on the dominant faces of the Han-Kobayashi region can be achieved. We further extend our proposed schemes to discrete memoryless interference networks (DM-IN) and show that they in fact represent a heterogeneous superposition coding scheme with rate splitting, which can simplify the code design for DM-INs with private messages while still achieve the optimal rate region.

I. INTRODUCTION

Polar codes, proposed by Arikan [1], are the first class of channel codes that can provably achieve the capacity of any memoryless binary-input output-symmetric channels with low encoding and decoding complexity. Since its invention, polar codes have been widely adopted to many other scenarios, such as source compression [2]–[5], wiretap channels [6]–[11], relay channels [6], [12], [13], multiple access channels (MAC) [5], [14]–[17], broadcast channels [18], [19], broadcast channels with confidential messages [10], [20], and bidirectional broadcast channels with common and confidential messages [21]. In these scenarios, polar codes have also shown capacity-achieving capabilities.

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The interference channel (IC), first initiated by Shannon [22] and further studied by Ahlswede [23], models the situation where \( m \) sender-receiver pairs try to communicate simultaneous through a common channel. In this model, it is assumed that there is no cooperation between any of the senders or receivers, and the signal of each sender is seen as interference by the unintended receivers. Although the 2-user discrete memoryless IC (DM-IC) is rather simple in appearance, except for some special cases [24]–[31], determining the capacity region of a general IC remains an open problem. Reference [23] gave simple but fundamental inner and outer bounds on the capacity region of the IC. In [32], Carleial determined an improved achievable rate region for the IC by applying the superposition coding technique of Cover [33], which was originally designed for the broadcast channel. Later, Han and Kobayashi established the best achievable rate region for the general IC to date [34]. A more compact description of the Han-Kobayashi region was given in [35]. The idea of the Han-Kobayashi coding strategy is to split each sender’s message into a private part and a common part, and allow the unintended receiver to decode the common part so as to enhance the total transmission rates. To achieve the whole Han-Kobayashi region, it is required that each receiver decodes its intended private message and both senders’ common messages jointly.

There are limited studies on the design of specific coding schemes that can achieve the Han-Kobayashi rate region. A low-density parity-check (LDPC) code-based Han-Kobayashi scheme was proposed for the Gaussian IC in [36], which has close-to-capacity performance in the case of strong interference. In [37], a specific coding scheme was designed for the binary-input binary-output Z IC using LDPC codes, and an example was shown to outperform time sharing of single user codes. For polar codes, reference [38] pointed out how alignment of polarized bit-channels can be of use for designing coding schemes for interference networks, and presented an example of the one-sided discrete memoryless 3-user IC with a degraded receiver structure. A polar coding scheme that achieves the Han-Kobayashi inner bound for the 2-user IC was proposed in [39], and [40] used a similar scheme to achieve the Han-Kobayashi region in the 2-user classical-quantum IC. The idea of [39] is to transform the original IC into two 3-user MACs from the two receivers’ perspectives, and design a compound MAC polar coding scheme for them. The achievable rate region of the compound MAC equals the Han-Kobayashi region, and can be achieved by polar codes. This design is based on the original description of the Han-Kobayashi region in [34] (also shown in Theorem 1 in this paper), in which five auxiliary random variables are used, and two deterministic functions that map auxiliary random variables to the
actual channel inputs are assumed to be known. By ranging over all possible choices of these functions and distributions of auxiliary random variables, the whole Han-Kobayashi region can be achieved. However, such an approach could be problematic in practice since finding proper such functions may be a complex task, especially for interference networks (IN), in which the number of random variables involved in the deterministic functions can be large.

Our work is inspired by the compact description of the Han-Kobayashi region in [35] (also shown in Theorem 2 in this paper), in which no deterministic functions are required, and only three auxiliary random variables are needed. The Han-Kobayashi coding strategy requires a fully joint decoder at each receiver that jointly decodes all its three messages, which is not directly implementable using polar codes from the compact description since there are no random variables standing for private messages in it. By analyzing points on the dominant faces of the Han-Kobayashi region, we find that it is possible to loosen the fully joint decoding requirement and propose to use two types of partially joint decoders. Each receiver can either jointly decode both senders’ common messages first and then the intended sender’s private message, or solely decode the intended sender's common message first and then jointly decode the rest two. Based on this finding and enlightened by Goela et al.’s superposition polar coding scheme for the broadcast channel [18], we design two types of superposition coding schemes and show that every point on the dominant faces of the Han-Kobayashi region can be achieved. The deterministic functions used in [39] can be seen as implicit in our code design, which makes our scheme a simpler approach. The two types of schemes can be seen as a rate splitting scheme. Unlike conventional rate splitting which transforms the original 3-user MAC into three point-to-point channels, we now create a point-to-point channel and a 2-user MAC. By extending our proposed two types of schemes to discrete memoryless interference networks (DM-IN), we show that they in fact represent a simple heterogeneous superposition coding scheme [41] for DM-INs with private messages, and the proposed rate splitting method can still achieve the optimal rate region. This result shows that the auxiliary random variables for private messages in DM-INs are also unnecessary for the code design, not just their cardinality bounds [35].

In our proposed schemes, joint decoders and the corresponding code constructions are implemented using the 2-user MAC polarization method based on Arıkan’s monotone chain rule expansions [5], whose encoding and decoding complexity is similar to the single-user case. We use Şaşoğlu’s result on polarization for arbitrary discrete alphabet [42] to extend it to arbitrary prime input alphabet case. To deal with non-uniform input distribution, one may apply Gallager’s
alphabet extension method \[43, p. 208\] as in \[39\], or a more direct approach by invoking results on polar coding for lossless compression \[18\], \[20\], \[44\]–\[46\]. In this paper, we take Chou and Bloch’s low-complexity approach \[20\], \[45\], which only requires a vanishing rate of shared randomness between communicators. One crucial point in designing capacity-achieving polar codes for a general multi-user channel is how to properly align the polar indices. One solution for this problem is the chaining method, which has already been used in several areas \[9\]–\[11\], \[19\], \[47\]. Another way is to add additional stages of polarization to align the incompatible indices, as shown in \[48\] and used in \[39\]. In this paper, we adopt the chaining method as it does not change the original polar transformation and may be easier to understand.

The rest of this paper is organized as follows. In Section II we introduce the 2-user DM-IC model and the Han-Kobayashi region. In Section III we review some background on polarization and polar codes necessary for our code design. In Section IV we provide an overview of our scheme and analyze its feasibility. Details of our proposed schemes are presented in Section V. In Section VI we extend our proposed schemes to arbitrary DM-INs. Section VII concludes this paper with some discussions.

Notations: \([N]\) is the abbreviation of an index set \(\{1, 2, ..., N\}\). Vectors are denoted as \(X^N \triangleq \{X^1, X^2, ..., X^N\}\) or \(X^{a:b} \triangleq \{X^a, X^{a+1}, ..., X^b\}\) for \(a \leq b\). For a subset \(A \subset [N]\), \(X^A\) denotes the subvector \(\{X^i : i \in A\}\) of \(X^{1:N}\). \(G_N = B_N F^\otimes n\) is the generator matrix of polar codes \[1\], where \(N = 2^n\) with \(n\) being an arbitrary integer, \(B_N\) is the bit-reversal matrix, and \(F = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\). \(H_q(X)\) stands for the entropy of \(X\) with \(q\)-based logarithm. The Bhattacharyya parameter of a random variable pair \((X, Y) \in \mathcal{X} \times \mathcal{Y}\) with joint distribution \(P_{XY}\) and \(|\mathcal{X}| = q_X\) is defined as

**Definition 1** (Bhattacharyya parameter).

\[
Z(X|Y) \triangleq \frac{1}{q_X - 1} \sum_{x, x' \in \mathcal{X}, y \in \mathcal{Y}} \sqrt{P_{XY}(x, y) P_{XY}(x', y)}. \tag{1}
\]

**II. Problem Statement**

**A. Channel Model**

**Definition 2.** A 2-user DM-IC consists of two input alphabets \(\mathcal{X}_1\) and \(\mathcal{X}_2\), two output alphabets \(\mathcal{Y}_1\) and \(\mathcal{Y}_2\), and a probability transition function \(P_{Y_1Y_2|X_1X_2}(y_1, y_2|x_1, x_2)\). The conditional joint
probability distribution of the 2-user DM-IC over \(N\) channel uses can be factored as
\[
P_{Y_1^N Y_2^N | X_1^N X_2^N}(y_1^N, y_2^N | x_1^N, x_2^N) = \prod_{i=1}^{N} P_{Y_1 Y_2 | X_1 X_2}(y_i^1, y_i^2 | x_i^1, x_i^2). \tag{2}
\]

**Definition 3.** A \((2^{NR_1}, 2^{NR_2}, N)\) code for the 2-user DM-IC consists of two message sets \(\mathcal{M}_1 = \{1, 2, ..., [2^{NR_1}]\}\) and \(\mathcal{M}_2 = \{1, 2, ..., [2^{NR_2}]\}\), two encoding functions
\[
x_1^N(m_1) : \mathcal{M}_1 \mapsto \mathcal{X}_1^N \quad \text{and} \quad x_2^N(m_2) : \mathcal{M}_2 \mapsto \mathcal{X}_2^N, \tag{3}
\]
and two decoding functions
\[
\hat{m}_1(y_1^N) : \mathcal{Y}_1^N \mapsto \mathcal{M}_1 \quad \text{and} \quad \hat{m}_2(y_2^N) : \mathcal{Y}_2^N \mapsto \mathcal{M}_2. \tag{4}
\]

**Definition 4.** The average probability of error \(P_e^{(N)}\) of a \((2^{NR_1}, 2^{NR_2}, N)\) code for the 2-user DM-IC is defined as the probability that the decoded message pair is not the same as the transmitted one averaged over all possible message pairs,
\[
P_e^{(N)} = \frac{1}{2^{N(R_1+R_2)}} \sum_{(M_1, M_2) \in \mathcal{M}_1 \times \mathcal{M}_2} \Pr\left\{ (\hat{m}_1(Y_1^N), \hat{m}_2(Y_2^N)) \neq (M_1, M_2) | (M_1, M_2) \text{ sent} \right\}, \tag{5}
\]
where \((M_1, M_2)\) are assumed to be uniformly distributed over \(\mathcal{M}_1 \times \mathcal{M}_2\).

**B. The Han-Kobayashi Rate Region**

In the Han-Kobayashi coding strategy, each sender’s message is split into two parts: a private message, which only needs to be decoded by the intended receiver, and a common message, which is allowed to be decoded by the unintended receiver. Each receiver decodes its intended private message and two common messages jointly so that a higher transmission rate can be achieved. In the rest of this paper, we will refer to the two senders and two receivers as Sender 1, Sender 2, Receiver 1 and Receiver 2 respectively. Sender 1’s message, denoted as \(M_1\), is split into \((M_{1p}, M_{1c})\), where \(M_{1p} \in \mathcal{M}_{1p} \triangleq \{1, 2, ..., [2^{NS_1}]\}\) denotes its private message and \(M_{1c} \in \mathcal{M}_{1c} \triangleq \{1, 2, ..., [2^{NT_1}]\}\) the common message. Similarly, Sender 2’s message \(M_2\) is split into \((M_{2p}, M_{2c})\) with \(M_{2p} \in \mathcal{M}_{2p} \triangleq \{1, 2, ..., [2^{NS_2}]\}\) and \(M_{2c} \in \mathcal{M}_{2c} \triangleq \{1, 2, ..., [2^{NT_2}]\}\). Define \(W_1, W_2, V_1\) and \(V_2\) as the random variables for messages \(M_{1c}, M_{2c}, M_{1p}\) and \(M_{2p}\) respectively, with \(\mathcal{W}_1, \mathcal{W}_2, \mathcal{V}_1\) and \(\mathcal{V}_2\) being their alphabets. Then each encoding function can be decomposed into three functions. For \(x_1^N(m_1)\), the three functions are
\[
\begin{align*}
w_1^N(M_{1c}) : \mathcal{M}_{1c} &\mapsto \mathcal{W}_1^N, \\
v_1^N(M_{1p}) : \mathcal{M}_{1p} &\mapsto \mathcal{V}_1^N
\end{align*}
\]
and \(x_1^N(W_1^N, V_1^N) : \mathcal{W}_1^N \times \mathcal{V}_1^N \mapsto \mathcal{X}_1^N.\ \tag{6}
\]
Similarly, for $x_2^N(m_2)$, the three functions are

$$w_2^N(M_{2c}) : M_{2c} \mapsto W_2^N, \quad v_2^N(M_{2p}) : M_{2p} \mapsto Y_2^N$$

and

$$x_2^N(W_2^N, V_2^N) : W_2^N \times V_2^N \mapsto X_2^N. \quad (7)$$

The coding strategy described in (6) and (7) is also known as superposition coding, which is proposed by Cover for the broadcast channel [33], and first used in the interference channel by Carleial [32]. Han and Kobayashi proposed the best achievable rate region for the general interference channel to date in [34]. The result is summarized in Theorem 1.

**Theorem 1** ([34], [49]). Let $\mathcal{P}^*$ be the set of probability distributions $P^*(\cdot)$ that factor as

$$P^*(q, v_1, v_2, w_1, w_2, x_1, x_2) = P_Q(q)P_{V_1|Q}(v_1|q)P_{V_2|Q}(v_2|q)P_{W_1|Q}(w_1|q)P_{W_2|Q}(w_2|q)$$

$$\times P_{X_1|V_1,W_1}(x_1|v_1, w_1, q)P_{X_2|V_2,W_2}(x_2|v_2, w_2, q),$$

where $Q \in \mathcal{Q}$ is the time-sharing parameter, and $P_{X_1|V_1,W_1}(\cdot)$ and $P_{X_2|V_2,W_2}(\cdot)$ equal either 0 or 1, i.e., they are deterministic functions. For a fix $P^*(\cdot) \in \mathcal{P}^*$, Consider Receiver 1 and the set of non-negative rate-tuples $(S_1, T_1, S_2, T_2)$ denoted by $\mathcal{R}_{HK}^{\alpha_1}(P^*)$ that satisfy

$$0 \leq S_1 \leq I(V_1; Y_1|W_1W_2Q), \quad (9)$$

$$0 \leq T_1 \leq I(W_1; Y_1|V_1W_2Q), \quad (10)$$

$$0 \leq T_2 \leq I(W_2; Y_1|V_1W_1Q), \quad (11)$$

$$S_1 + T_1 \leq I(V_1W_1; Y_1|W_2Q), \quad (12)$$

$$S_1 + T_2 \leq I(V_1W_2; Y_1|W_1Q), \quad (13)$$

$$T_1 + T_2 \leq I(W_1W_2; Y_1|V_1Q), \quad (14)$$

$$S_1 + T_1 + T_2 \leq I(V_1W_1W_2; Y_1|Q). \quad (15)$$

Similarly, let $\mathcal{R}_{HK}^{\alpha_2}(P^*)$ be the set of non-negative rate-tuples $(S_1, T_1, S_2, T_2)$ that satisfy (9)–(15) with indices 1 and 2 swapped everywhere. For a set $S$ of 4-tuples $(S_1, T_1, S_2, T_2)$, let $\mathcal{R}(S)$ be the set of $(R_1, R_2)$ such that $0 \leq R_1 \leq S_1 + T_1$ and $0 \leq R_2 \leq S_2 + T_2$ for some $(S_1, T_1, S_2, T_2) \in S.$ Then we have that

$$\mathcal{R}_{HK}^\circ = \mathcal{R} \left( \bigcup_{P^* \in \mathcal{P}^*} \mathcal{R}_{HK}^{\alpha_1}(P^*) \cap \mathcal{R}_{HK}^{\alpha_2}(P^*) \right) \quad (16)$$

is an achievable rate region for the DM-IC.
The original description for the Han-Kobayashi region in Theorem 1 needs five auxiliary random variables and two deterministic functions to describe. This poses difficulties in the code design in practice, since finding such functions and distributions of auxiliary random variables that can achieve a desired rate region may not be easy. Reference [35] presented a simplified description for the region, in which only three auxiliary random variables are used and no deterministic functions are needed. Their result is summarized in Theorem 2.

**Theorem 2 ([35], [49])**. Let \( P_{1}^\ast \) be the set of probability distributions \( P_{1}^\ast (\cdot) \) that factor as
\[
P_{1}^\ast (q, w_{1}, w_{2}, x_{1}, x_{2}) = P_{Q}(q)P_{X_{1}W_{1}|Q}(x_{1}, w_{1}|q)P_{X_{2}W_{2}|Q}(x_{2}, w_{2}|q),
\]
where \( |W_{j}| \leq |X_{j}| + 4 \) for \( j = 1, 2 \), and \( |Q| \leq 6 \). For a fixed \( P_{1}^\ast (\cdot) \in P_{1}^\ast \), let \( \mathcal{R}_{HK}(P_{1}^\ast) \) be the set of \( (R_{1}, R_{2}) \) satisfying
\[
0 \leq R_{1} \leq I(X_{1}; Y_{1}|W_{2}Q) \overset{\Delta}{=} a,
\]
\[
0 \leq R_{2} \leq I(X_{2}; Y_{2}|W_{1}Q) \overset{\Delta}{=} b,
\]
\[
R_{1} + R_{2} \leq I(X_{1}W_{2}; Y_{1}|Q) + I(X_{2}; Y_{2}|W_{1}W_{2}Q) \overset{\Delta}{=} c,
\]
\[
R_{1} + R_{2} \leq I(X_{1}; Y_{1}|W_{1}W_{2}Q) + I(X_{2}W_{1}; Y_{2}|Q) \overset{\Delta}{=} d,
\]
\[
R_{1} + R_{2} \leq I(X_{1}W_{2}; Y_{1}|W_{1}Q) + I(X_{2}W_{1}; Y_{2}|W_{2}Q) \overset{\Delta}{=} e,
\]
\[
2R_{1} + R_{2} \leq I(X_{1}W_{2}; Y_{1}|Q) + I(X_{1}; Y_{1}|W_{1}W_{2}Q) + I(X_{2}W_{1}; Y_{2}|W_{2}Q) \overset{\Delta}{=} f,
\]
\[
R_{1} + 2R_{2} \leq I(X_{2}; Y_{2}|W_{1}W_{2}Q) + I(X_{2}W_{1}; Y_{2}|Q) + I(X_{1}W_{2}; Y_{1}|W_{1}Q) \overset{\Delta}{=} g.
\]

Then we have that
\[
\mathcal{R}_{HK} = \bigcup_{P_{1}^\ast \in P_{1}^\ast} \mathcal{R}_{HK}(P_{1}^\ast)
\]
is an achievable rate region for the DM-IC.

It is shown in [35] that the regions described in Theorem 1 and 2 are equivalent, and constraints (10), (11) and (14) and their counterparts for the second receiver are unnecessary. It is straightforward to see that \( \mathcal{R}_{HK}^{o} \subseteq \mathcal{R}_{HK} \) by using Fourier-Motzkin elimination [35], but to
prove the converse, we will need [35, Lemma 2], which states that \( R_{HK}(P^*_1) \subseteq R^o_{HK}(P^*) \cup R^o_{HK}(P^{**}) \cup R^o_{HK}(P^{***}) \), where

\[
P^*_1(q, w_1, w_2, x_1, x_2) = \sum_{v_1 \in V_1, v_2 \in V_2} P^*(q, v_1, v_2, w_1, w_2, x_1, x_2),
\]

\[
P^{**} = \sum_{w_1 \in W_1} P^*, \quad P^{***} = \sum_{w_2 \in W_2} P^*.
\]

This indicates that to achieve \( R_{HK}(P^*_1) \) for a given \( P^*_1 \) with the scheme of [39], one generally will need to use three codes designed for different joint distributions. In this paper, we aim to design a polar coding scheme to achieve \( R_{HK}(P^*_1) \) directly. The deterministic functions used in the original description can be seen as included in the joint distributions of \( P_{X_1W_1|Q}(x_1, w_1|q) \) and \( P_{X_2W_2|Q}(x_2, w_2|q) \). Since the purpose of introducing the time-sharing parameter \( Q \) in Theorem 1 and 2 is just to replace the convex-hull operation, in the rest of this paper, we will consider a fixed \( Q = q \) and drop the condition \( Q = q \) in expressions for simplicity, including \( a, b, \ldots, g \) in (18)–(24). If for any \( q \in Q \), the corresponding Han-Kobayashi region is achievable, then the whole region defined in Theorem 2 can be achieved by time sharing.

III. POLAR CODING PRELIMINARIES

A. Polar Coding for Lossless Source Compression

First, let us recap the lossless source polarization scheme introduced in [2] and generalized to arbitrary alphabet in [42]. Let \( X^{1:N} \in \mathcal{X}^N \) be \( N \) independent copies of a random variable \( X \) with \( |\mathcal{X}| = q_X \) being a prime number\(^1\) and \( U^{1:N} = X^{1:N}G_N \). It is shown that the components of \( U^{1:N} \) polarize in the sense that \( U^j (j \in [N]) \) are either almost independent of \( U^{1:j-1} \) and uniformly distributed, or almost determined by \( U^{1:j-1} \). Therefore, we can define the following two sets of polarized indices:

\[
\mathcal{H}^{(N)}_X = \{ j \in [N] : Z(U^j|U^{1:j-1}) \geq 1 - \delta_N \}, \quad (26)
\]

\[
\mathcal{L}^{(N)}_X = \{ j \in [N] : Z(U^j|U^{1:j-1}) \leq \delta_N \}, \quad (27)
\]

where \( \delta_N = 2^{-N^\beta} \) and \( \beta \in (0, 1/2) \). It is shown that [42]

\[
\lim_{N \to \infty} \frac{1}{N} |\mathcal{H}^{(N)}_X| = H_{q_X}(X),
\]

\[
\lim_{N \to \infty} \frac{1}{N} |\mathcal{L}^{(N)}_X| = 1 - H_{q_X}(X). \quad (28)
\]

\(^1\)Although for composite \( q_X \), polarization can also happen if we use some special type of quasigroup operation instead of group operation [42], we only consider the prime number case in this paper for simplicity.
Next, consider compression of two correlated sources. Let \((X, Y) \sim p_{X,Y}\) be a pair of random variables over \((\mathcal{X} \times \mathcal{Y})\) with \(|\mathcal{X}| = q_X\) being a prime number. Consider \(X\) as the memoryless source to be compressed and \(Y\) as side information of \(X\). Similar to the single source case, let \(U^{1:N} = X^{1:N}G_N\). As \(N\) goes to infinity, \(U^j (j \in [N])\) becomes either almost independent of \((Y^{1:N}, U^{1:j-1})\) and uniformly distributed, or almost determined by \((Y^{1:N}, U^{1:j-1})\) \[2\]. For \(\delta_N = 2^{-N^\beta}\) with \(\beta \in (0, 1/2)\), define the following sets of polarized indices:

\[
H_X^{(N)} = \{j \in [N] : Z(U^j|Y^{1:j-1}) \geq 1 - \delta_N\},
\]

\[
L_X^{(N)} = \{j \in [N] : Z(U^j|Y^{1:j-1}) \leq \delta_N\}.
\]

Also, we have \[42\]

\[
\lim_{N \to \infty} \frac{1}{N} |H_X^{(N)}| = H_{q_X}(X|Y),
\]

\[
\lim_{N \to \infty} \frac{1}{N} |L_X^{(N)}| = 1 - H_{q_X}(X|Y).
\]

\[B. Polar Coding for Arbitrary Discrete Memoryless Channels\]

Polar codes were originally developed for symmetric channels. By invoking results in source polarization, one can construct polar codes for asymmetric channels without alphabet extension, as introduced in \[44\]. However, the scheme of \[44\] requires the encoder and the decoder to share a large amount of random mappings, which raises a practical concern of not being explicit. In \[18\], \[20\], \[45\], \[46\], deterministic mappings are used to replace (part of) the random mappings so as to reduce the amount of shared randomness needed. Next, we briefly review the method of \[20\], \[46\], which only requires a vanishing rate of shared randomness.

Let \(W(Y|X)\) be a discrete memoryless channel (DMC) with a \(q_X\)-ary input alphabet \(\mathcal{X}\), where \(q_X\) is a prime number. Let \(U^{1:N} = X^{1:N}G_N\) and define \(H_X^{(N)}\) and \(L_X^{(N)}\) as in \[26\] and \[27\]. Consider \(Y\) as side information about \(X\) and define \(H_{X|Y}^{(N)}\) and \(L_{X|Y}^{(N)}\) as in \[29\] and \[30\]. Define the information set, frozen set and deterministic set respectively as follows:

\[
I \triangleq H_X^{(N)} \cap L_X^{(N)},
\]

\[
F_r \triangleq H_X^{(N)} \cap (L_X^{(N)})^c,
\]

\[
F_d \triangleq (H_X^{(N)})^c.
\]

\[^2\text{Strictly speaking, not all of } \{u^j\}_{j \in F_d} \text{ are deterministic. More detailed discussions about this set can be found in } [45].\]
The encoding procedure goes as follows: \( \{ u^j \}_{j \in I} \) carry information, \( \{ u^j \}_{j \in F_r} \) are filled with uniformly distributed frozen symbols (shared between the sender and the receiver), and \( \{ u^j \}_{j \in F_d} \) will be assigned by random mappings \( \lambda_j(u^{1:j-1}) \), which randomly generate an output \( u \in \mathcal{X} \) with probability \( P_{U|U^{1:j-1}}(u|u^{1:j-1}) \).

In order for the receiver to decode successfully, \( \{ u^j \}_{j \in (\mathcal{H}_X^{(N)})^C \cap (\mathcal{L}_{X|Y}^{(N)})^C} \) is separately transmitted to the receiver with some reliable error-correcting code, the rate of which vanishes as \( N \) goes large \([20], [46]\).

After receiving \( y^{1:N} \) and recovered \( \{ u^j \}_{j \in (\mathcal{H}_X^{(N)})^C \cap (\mathcal{L}_{X|Y}^{(N)})^C} \), the receiver computes the estimate \( \tilde{u}^{1:N} \) of \( u^{1:N} \) as

\[
\tilde{u}^j = \begin{cases} 
  u^j, & \text{if } j \in (\mathcal{L}_{X|Y}^{(N)})^C, \\
  \arg \max_{u \in \{0,1\}} P_{U|U^{1:N}U^{1:j-1}}(u|y^{1:N}, u^{1:j-1}), & \text{if } j \in \mathcal{L}_{X|Y}^{(N)}.
\end{cases}
\]  

(35)

The rate of this scheme, \( R = |I|/N \), satisfies \([44]\)

\[
\lim_{N \to \infty} R = I(X;Y).
\]  

(36)

The block error probability of this scheme can be upper bounded by \([42], \text{Proposition 3.2}\]

\[
P_e \leq (q_X - 1) \sum_{j \in \mathcal{L}_{X|Y}^{(N)}} Z(U^j|Y^{1:N}, U^{1:j-1}) = O(N2^{-N^3}).
\]  

(37)

C. Polar Coding for Multiple Access Channels

Let \( P_{Y|X_1X_2}(y|x_1,x_2) \) be the transition probability of a discrete memoryless 2-user MAC, where \( x_1 \in \mathcal{X}_1 \) with \( |\mathcal{X}_1| = q_{X_1} \) and \( x_2 \in \mathcal{X}_2 \) with \( |\mathcal{X}_2| = q_{X_2} \). For a fixed product distribution of \( P_{X_1}(x_1)P_{X_2}(x_2) \), the achievable rate region of \( P_{Y|X_1X_2} \) is given by \([50]\)

\[
\mathcal{R}(P_{Y|X_1X_2}) \triangleq \left\{ \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \right\} \begin{cases} 
  0 \leq R_1 \leq I(X_1;Y|X_2) \\
  0 \leq R_2 \leq I(X_2;Y|X_1) \\
  R_1 + R_2 \leq I(X_1, X_2;Y)
\end{cases}
\]  

(38)

Polar coding for MACs has been studied in \([5], [14]–[17]\). Although \([17]\) provides a more general scheme that can achieve the whole uniform rate region of a \( k \)-user MAC, in our scheme, we adopt the monotone chain rule expansion method in \([5]\) because it has simple structure and possesses similar complexity to the single-user polar codes. Reference \([5]\) mainly deals with the Slepian-Wolf problem in source coding, but the method can be readily applied to the problem of coding for the 2-user MAC since they are dual problems, which has been studied in \([14]\) and
used in [39]. However, both [14] and [39] consider uniform channel inputs. Here we generalize it to arbitrary input case. Note that although the input alphabets of the two users can be different, the extension is ready since there is no polarization operation between the two channel inputs.

For simplicity, we assume \( q_{X_1} \) and \( q_{X_2} \) are prime numbers. Define

\[
U_{1}^{1:N} = X_{1}^{1:N}G_{N}, \quad U_{2}^{1:N} = X_{2}^{1:N}G_{N}.
\]

(39)

Let \( S^{1:2N} \) be a permutation of \( U_{1}^{1:N}U_{2}^{1:N} \) such that it preserves the relative order of the elements of both \( U_{1}^{1:N} \) and \( U_{2}^{1:N} \), called a monotone chain rule expansion. Such an expansion can be represented by a string \( b_{2N} = b_{1}b_{2}...b_{2N} \), called the path of the expansion, where \( b_{j} = 0 \) \((j \in [2N])\) represents that \( S^{j} \in U_{1}^{1:N} \), and \( b_{j} = 1 \) represents that \( S^{j} \in U_{2}^{1:N} \). Then we have

\[
I(Y^{1:N};U_{1}^{1:N},U_{2}^{1:N}) = H(U_{1}^{1:N},U_{2}^{1:N}) - H(U_{1}^{1:N},U_{2}^{1:N}|Y^{1:N})
\]

\[
= H(U_{1}^{1:N}) + H(U_{2}^{1:N}) - \sum_{j=1}^{2N} H(S^{j}|Y^{1:N},S^{1:j-1})
\]

\[
= NH(X_{1}) + NH(X_{2}) - \sum_{j=1}^{2N} H(S^{j}|Y^{1:N},S^{1:j-1}).
\]

It is shown in [5] that \( H(S^{j}|Y^{1:N},S^{1:j-1}) \) \((j \in [2N])\) polarizes to 0 or 1 as \( N \) goes to infinity.

Because the input alphabets of two users can be different, to show the fraction of information a transmitted symbol can carry, we define user \( k \)'s \((k = 1, 2)\) rate with \( q_{X_k} \)-based logarithm:

\[
R_{U_{1}} = H_{q_{X_{1}}}(X_{1}) - \frac{1}{N} \sum_{j \in S_{U_{1}}} H_{q_{X_{1}}}(S^{j}|Y^{1:N},S^{1:j-1}),
\]

\[
R_{U_{2}} = H_{q_{X_{2}}}(X_{2}) - \frac{1}{N} \sum_{j \in S_{U_{2}}} H_{q_{X_{2}}}(S^{j}|Y^{1:N},S^{1:j-1}),
\]

(40)

where \( S_{U_{1}} \triangleq \{j \in [2N] : b_{j} = 0\} \) and \( S_{U_{2}} \triangleq \{j \in [2N] : b_{j} = 1\} \).

**Proposition 1** (45). Let \((R_{1}, R_{2})\) be a rate pair on the dominant face of \( \mathcal{R}(P_{Y|X_{1}X_{2}}) \). For any given \( \epsilon > 0 \), there exists \( N \) and a chain rule \( b_{2N} \) on \( U_{1}^{1:N}U_{2}^{1:N} \) such that \( b_{2N} \) is of the form \( 0^{1}1^{N}0^{N-i} \) \((0 \leq i \leq N)\) and has a rate pair \((R_{U_{1}}, R_{U_{2}})\) satisfying

\[
|R_{1} - R_{U_{1}}| \leq \epsilon \text{ and } |R_{2} - R_{U_{2}}| \leq \epsilon.
\]

(41)

Although the permutations can have lots of variants, even non-monotone [17], Proposition 1 shows that expansions of type \( 0^{1}1^{N}0^{N-i} \) \((0 \leq i \leq N)\) are sufficient to achieve every point on the

\(^{3}\text{The entropy here is calculated adaptively. If } j \in S_{U_k} \text{ } (k = 1, 2), \text{ then entropy is calculated with } q_{X_k} \text{-based logarithm.} \)
dominant face of $R(P_{Y|X_1,X_2})$ given sufficiently large $N$, which can make our code design and construction simpler. To polarize a MAC sufficiently while keeping the above rate approximation intact, we need to scale the path. For any integer $l = 2^n$, let $l b_{2N}$ denote
\[
\underbrace{b_1 \cdots b_1}_l \underbrace{b_2 \cdots b_2}_l \cdots \underbrace{b_{2N} \cdots b_{2N}}_l,
\]
which is a monotone chain rule for $U_1^{1:2N} U_2^{1:2N}$. It is shown in \cite{5} that the rate pair for $b_{2N}$ is also the rate pair for $l b_{2N}$.

Now we can construct a polar code for the 2-user MAC with arbitrary inputs. Let $f_k(i) : [N] \to S_{U_k}$ ($k = 1, 2$) be the mapping from indices of $U_k^{1:2N}$ to those of $S_{U_k}$. For $\delta_N = 2^{-N^\alpha}$ with $\beta \in (0, 1/2)$, define
\[
\mathcal{H}_{S_{U_k}}^{(N)} \triangleq \{ j \in [N] : Z(S_{f_k(j)} | S^{1:f_k(j)-1}) \geq 1 - \delta_N \},
\]
\[
\mathcal{L}_{S_{U_k}}^{(N)} \triangleq \{ j \in [N] : Z(S_{f_k(j)} | Y^{1:N}, S^{1:f_k(j)-1}) \leq \delta_N \}.
\]  
(42)

Since $X_1$ and $X_2$ are independent, we have
\[
\mathcal{H}_{S_{U_k}}^{(N)} = \mathcal{H}_{X_k}^{(N)} \triangleq \{ j \in [N] : Z(U_{k}^j | U_{k}^{1:j-1}) \geq 1 - \delta_N \}.
\]  
(43)

Then we can partition user $k$’s ($k = 1, 2$) indices as
\[
\mathcal{I}_k \triangleq \mathcal{H}_{S_{U_k}}^{(N)} \cap \mathcal{L}_{S_{U_k}}^{(N)} \cap \mathcal{L}_{S_{U_k}}^{(N)} \cap \mathcal{L}_{S_{U_k}}^{(N)} C,
\]
\[
\mathcal{F}_{kr} \triangleq \mathcal{H}_{S_{U_k}}^{(N)} \cap \mathcal{L}_{S_{U_k}}^{(N)} \cap \mathcal{L}_{S_{U_k}}^{(N)} \cap \mathcal{L}_{S_{U_k}}^{(N)} C,
\]
\[
\mathcal{F}_{kd} \triangleq (\mathcal{H}_{S_{U_k}}^{(N)})^C.
\]  
(44)

The polarization result can be summarized as the following proposition.

**Proposition 2** [5]. Let $P_{Y|X_1,X_2}(y|x_1,x_2)$ be the transition probability of a discrete memoryless 2-user MAC. Consider the transformation defined in \cite{39}. Let $N_0 = 2^{n_0}$ for some $n_0 \geq 1$ and fix a path $b_{2N_0}$ for $U_1^{1:N_0} U_2^{1:N_0}$. The rate pair for $b_{2N_0}$ is denoted by $(R_{U_1}, R_{U_2})$. Let $N = 2^l N_0$ for $l \geq 1$ and let $S^{1:2N}$ be the expansion represented by $2^l b_{2N_0}$. Then, for any given $\delta > 0$, as $l$ goes to infinity, we have
\[
\frac{1}{2N} \left| \{ 1 \leq j \leq 2N : \delta < Z(S_j | Y^{1:N}, S^{1:j-1}) < 1 - \delta \} \right| \to 0,
\]
\[
\frac{|\mathcal{I}_1|}{N} \to R_{U_1} \quad \text{and} \quad \frac{|\mathcal{I}_2|}{N} \to R_{U_2}.
\]  
(45)

Proposition 1 and 2 can be readily extended from Theorem 1 and Theorem 2 in \cite{5} by considering $Y$ as side information of source pair $(X_1, X_2)$ and performing the same analysis. Similar to
the single-user channel case, the 2-user MAC polar coding scheme is for user \( k \) \((k = 1, 2)\) to put information symbols to \( \{u^k_j \}_{j \in I_k} \), fill \( \{u^k_\ell \}_{\ell \in F_{kd}} \) with frozen symbols, and assign \( \{u^k_j \}_{j \in F_{kr}} \) by random mappings according to the conditional distributions. \( \{u^k_j \}_{j \in F_{kr}} \) is separately transmitted to the receiver with some reliable error-correcting code. The receiver uses a successive cancellation decoder to decode two users’ information jointly according to the expansion order. The error performance of such a scheme is upper bounded by

\[
P_e \leq \sum_{k=1,2} \left( q_{X_k} - 1 \right) \sum_{j \in \mathcal{L}^{(N)}_{SU_k} \cap \mathcal{L}^{(N)}_{SU_k} | Y } Z(S^{f_k(j)} | Y^{1:N}, S^{1:f_k(j)}^{-1} ) = O(N^{-2N^2}). \tag{46}
\]

**IV. AN OVERVIEW OF OUR NEW APPROACH**

In this section, we introduce the main idea of our scheme. We follow the Han-Kobayashi coding strategy to design polar codes for the interference channel. Sender \( k \)’s \((k = 1, 2)\) splits its message \( M_k \) into a private message, denoted by \( M_{kp} \), and a common message, denoted by \( M_{kc} \), and encodes them into one codeword using superposition coding. For a target rate pair \( P \), let \( R^p_k \) and \( R^c_k \) respectively denote the corresponding private and common message rates of Sender \( k \), and define \( P^1 \triangleq (R^p_1 + R^c_1, R^c_2) \) and \( P^2 \triangleq (R^c_1, R^p_2 + R^c_2) \) as Receiver 1’s and Receiver 2’s receiving rate pairs respectively. Furthermore, define \( P^c \triangleq (R^c_1, R^c_2) \) as the common message rate pair. In the rest of this paper, we refer \((R^p_1, R^c_1, R^p_2, R^c_2)\) to a rate decomposition of \( P \).

**A. Synthesized MACs for Receivers**

According to the characteristics of random variables used in Theorem 2, we design our superposition polar coding scheme as illustrated in Fig. 1. For Sender 1, encoder \( \mathcal{E}_{1b} \) maps its common message \( M_{1c} \) into a sequence \( U_1^{1:N} \) of length \( N \), which goes into a polar encoder to generate an intermediate codeword \( W_1^{1:N} \) (corresponding to auxiliary random variables \( W_1 \) in Theorem 2). Encoder \( \mathcal{E}_{1a} \) then maps its private message \( M_{1p} \) together with \( W_1^{1:N} \) into \( U_1^{1:N} \), which goes into another polar encoder to generate the final codeword \( X_1^{1:N} \). \( \mathcal{E}_{2b} \) and \( \mathcal{E}_{2a} \) serve the same purposes for Sender 2.

Our proposed scheme is inspired by Goela et al.’s superposition polar coding scheme for the broadcast channel [18]. The main challenge in applying such a scheme to DM-ICs is the design of joint decoders. With polar codes, \( W_k^{1:N} \) \((k = 1, 2)\) will be treated as side information when we encode \( M_{kp} \). Thus, the intended common message needs be recovered before decoding the private message at each receiver, and therefore they can not be jointly decoded. Each receiver
can either jointly decode two common messages first and then its private message, or solely decode its intended common message first and then jointly decode the rest two. We will show that these two types of decoding orders are sufficient to achieve every point on the dominant faces of the Han-Kobayashi region. Since only two of the three messages for each receiver are jointly decoded in these schemes, we refer to them as the partially joint decoders in this paper.

For the purpose of decomposing a target rate pair into a private and common message rate tuples for our scheme, we first define the effective channel of each receiver. For Receiver 1, its effective channel, $P_{Y_1|X_1,W_2}$, is defined as

$$P_{Y_1|X_1,W_2}(y_1|x_1,w_2) \triangleq \sum_{x_2} P_{Y_1|X_1,X_2}(y_1|x_1,x_2) P_{X_2|W_2}(x_2|w_2,q).$$

(47)

Similarly, the effective channel of Receiver 2 is defined as

$$P_{Y_2|W_1,X_2}(y_2|w_1,x_2) \triangleq \sum_{x_1} P_{Y_2|X_1,X_2}(y_2|x_1,x_2) P_{X_1|W_1}(x_1|w_1,q).$$

(48)

The achievable rate regions for these two MACs are

$$\mathcal{R}(P_{Y_1|X_1,W_2}) = \left\{ \begin{array}{ll} (R_1) & 0 \leq R_1 \leq I(X_1;Y_1|W_2) \\ (R_2) & 0 \leq R_2 \leq I(W_2;Y_1|X_1) \\ & R_1 + R_2 \leq I(X_1W_2;Y_1) \end{array} \right\},$$

(49)

$$\mathcal{R}(P_{Y_2|W_1,X_2}) = \left\{ \begin{array}{ll} (R_1) & 0 \leq R_1 \leq I(W_1;Y_2|X_2) \\ (R_2) & 0 \leq R_2 \leq I(X_2;Y_2|W_1) \\ & R_1 + R_2 \leq I(X_2W_1;Y_2) \end{array} \right\}.$$

(50)

Now we can study the Han-Kobayashi coding problem in $P_{Y_1|X_1,W_2}$ and $P_{Y_2|W_1,X_2}$. In these two MACs, the rate of $X_k$ ($k = 1, 2$) equals the overall rate of Sender $k$, while the rate of $W_k$ equals
the common message rate of Sender $k$. Obviously, $\mathbf{P}^1$ and $\mathbf{P}^2$ must lie inside $\mathcal{R}(P_{Y_1|X_1W_2})$ and $\mathcal{R}(P_{Y_2|W_1X_2})$ respectively in order to make reliable communication possible.

Giving only two effective channels is insufficient to determine the suitable decoding order for a target point. If we hope to use a partially joint decoder, the following two MACs, $P_{Y_1|W_1W_2}$ and $P_{Y_2|W_1W_2}$, will be useful. For $k = 1, 2$, define

$$P_{Y_k|W_1W_2}(y_k|w_1, w_2) \triangleq \sum_{x_1} \sum_{x_2} P_{Y_k|X_1X_2}(y_k|x_1, x_2)P_{X_1|W_1Q}(x_1|w_1, q)P_{X_2|W_2Q}(x_2|w_2, q), \quad (51)$$

the achievable rate region of which is

$$\mathcal{R}(P_{Y_k|W_1W_2}) = \left\{ \begin{array}{l} R_1 \\ R_2 \end{array} \right\} \begin{array}{l} 0 \leq R_1 \leq I(W_1; Y_k|W_2) \\ 0 \leq R_2 \leq I(W_2; Y_k|W_1) \\ R_1 + R_2 \leq I(W_1W_2; Y_k) \end{array}. \quad (52)$$

The relations between the above four achievable rate regions are shown in Fig. 2. For a target rate pair $\mathbf{P}$, if there exists a possible common message rate pair $\mathbf{P}^c \in \mathcal{R}(P_{Y_k|W_1W_2})$ ($k = 1, 2$), then Receiver $k$ can decode two common messages separate from its private message while still being able to achieve $\mathbf{P}^k$. Otherwise, it will have to jointly decode its private message and the common messages. In the next subsection, we will discuss how to design decoding orders with these four synthesized MACs.
B. The General Idea of Our Scheme

Definition 5 (Type I points). A Type I point $P$ in $\mathcal{R}_{HK}(P^*_1)$ is a rate pair which can be decomposed into a private and common message rate tuple that satisfies:

\[
(R^c_1, R^c_2) \in \mathcal{R}(P_{Y_1|W_1W_2}) \cap \mathcal{R}(P_{Y_2|W_1W_2}),
\]

\[
R^p_1 = I(X_1; Y_1|W_1W_2),
\]

\[
R^p_2 = I(X_2; Y_2|W_1W_2).
\]

Definition 6 (Type II points). A Type II point $P$ in $\mathcal{R}_{HK}(P^*_1)$ is a rate pair which can be decomposed into a private and common message rate tuple that satisfies:

\[
(R^c_1, R^c_2) \in \mathcal{R}(P_{Y_k|W_1W_2}),
\]

\[
R^p_k \leq I(W_k'; Y_k'),
\]

\[
R^p_k = I(X_k; Y_k|W_1W_2),
\]

\[
R^p_{k'} = I(X_{k'}W_k; Y_{k'}|W_{k'}) - R^c_k,
\]

where $k, k' \in \{1, 2\}$ and $k \neq k'$.

For a Type I points $P$ with common message rate pair $P^c \in \mathcal{R}(P_{Y_1|W_1W_2}) \cap \mathcal{R}(P_{Y_2|W_1W_2})$, we can first design a MAC polar code for two common messages that achieves $P^c$ in the compound MAC composed of $P_{Y_1|W_1W_2}$ and $P_{Y_2|W_1W_2}$, and then design a point-to-point polar code for each senders’ private message with the common messages being side information.

For a Type II points $P$, let us consider $P^c \in \mathcal{R}(P_{Y_2|W_1W_2})$ as an example. The code constructions for two common messages $(M_{1c}, M_{2c})$ and Sender 1’s private message $M_{1p}$ are jointly designed in such a way that, Receiver 1 can first decode $M_{1c}$ (equivalently $W_1^{1:N}$) with $Y_1^{1:N}$, and then jointly decode $(M_{1p}, M_{2c})$ with the decoding result of $W_1^{1:N}$ being side information, while Receiver 2 can jointly decode $(M_{1c}, M_{2c})$ with $Y_2^{1:N}$. The code construction for Sender 2’s private message $M_{2p}$ is simply a point-to-point polar code with $W_2^{1:N}$ being side information.

To show that these two types of schemes can achieve the whole Han-Kobayashi region, we have the following lemma.

Lemma 1. Every point on the dominant faces of $\mathcal{R}_{HK}(P^*_1)$ can be classified into either Type I or Type II.

Proof. See Appendix A.
V. PROPOSED POLAR CODING SCHEMES

In this section, we describe details of our proposed polar coding scheme for the 2-user DM-IC. We consider the case when \( q_{X_1} = |X_1| \) and \( q_{X_2} = |X_2| \) are two prime numbers, \( q_{W_1} = |W_1| \) is the smallest prime number larger than \( q_{X_1} + 4 \), and \( q_{W_2} = |W_2| \) is the smallest prime number larger than \( q_{X_2} + 4 \). For a rate pair \( \mathbf{P} \), let \( \mathbf{P}(1) \) and \( \mathbf{P}(2) \) respectively denote its first and second component.

A. Polar Coding Scheme for Type I Points

Let \( \mathbf{P}^c = (R_1^c, R_2^c) \) be the common message rate pair for a Type I point \( \mathbf{P} \). Obviously, \( \mathbf{P}^c \) must lie on the dominant face of either \( \mathcal{R}(P_{Y_1|W_1}W_2) \) or \( \mathcal{R}(P_{Y_2|W_1}W_2) \), otherwise we can choose a larger common message rate pair to achieve higher rates. Without loss of generality, we assume that \( \mathbf{P}^c \) is on the dominant face of \( \mathcal{R}(P_{Y_1|W_1}W_2) \) in this subsection as an example.

1) Common Message Encoding: First, choose a point \( \tilde{\mathbf{P}}^c \) on the dominant face of \( \mathcal{R}(P_{Y_2|W_1}W_2) \), which is larger than \( \mathbf{P}^c \) in the sense that \( \tilde{\mathbf{P}}^c(1) \geq \mathbf{P}^c(1) \) and \( \tilde{\mathbf{P}}^c(2) \geq \mathbf{P}^c(2) \), as the target point for conducting the monotone chain rule expansion in our code design. Let \( S^{1:2N} \) be the monotone chain rule expansion for achieving \( \mathbf{P}^c \) in \( \mathcal{R}(P_{Y_1|W_1}W_2) \), and \( T^{1:2N} \) be that for achieving \( \tilde{\mathbf{P}}^c \) in \( \mathcal{R}(P_{Y_2|W_1}W_2) \). Denote the sets of indices in \( S^{1:2N} \) with \( S^j \in U_1^{1:N} \) and \( S^j \in U_2^{1:N} \) by \( S_{U_1} \) and \( S_{U_2} \) respectively, and those in \( T^{1:2N} \) with \( T^j \in U_1^{1:N} \) and \( T^j \in U_2^{1:N} \) by \( T_{U_1} \) and \( T_{U_2} \) respectively. For \( k = 1, 2 \), let \( f_k(j) : [N] \rightarrow S_{U_k} \) be the mapping from indices of \( U_k^{1:N} \) to those of \( S_{U_k} \), and \( g_k(j) : [N] \rightarrow T_{U_k} \) the mapping from indices of \( U_k^{1:N} \) to those of \( T_{U_k} \). For \( \delta_N = 2^{-N^\beta} \) with \( 0 < \beta < 1/2 \), define the following polarized sets

\[
\mathcal{H}_{S_{U_k}}^{(N)} = \left\{ j \in [N] : Z(S_{f_k(j)}|S^{1:1(N)}_{f_k(j)}-1) \geq 1 - \delta_N \right\},
\]

\[
\mathcal{L}_{S_{U_k}}^{(N)}|Y_1 = \left\{ j \in [N] : Z(S_{f_k(j)}|Y_1^{1:N}, S^{1:1(N)}_{f_k(j)}-1) \leq \delta_N \right\},
\]

\[
\mathcal{H}_{T_{U_k}}^{(N)} = \left\{ j \in [N] : Z(T_{g_k(j)}|T^{1:1(N)}_{g_k(j)}-1) \geq 1 - \delta_N \right\},
\]

\[
\mathcal{L}_{T_{U_k}}^{(N)}|Y_2 = \left\{ j \in [N] : Z(T_{g_k(j)}|Y_2^{1:N}, T^{1:1(N)}_{g_k(j)}-1) \leq \delta_N \right\}.
\]

Since two senders’ common messages are independent from each other, we have

\[
\mathcal{H}_{S_{U_k}}^{(N)} = \mathcal{H}_{T_{U_k}}^{(N)} = \mathcal{H}_{W_k}^{(N)},
\]

where \( \mathcal{H}_{W_k}^{(N)} = \left\{ j \in [N] : Z(U_k^{1:N}|U_k^{1:1(N)}-1) \geq 1 - \delta_N \right\} \).
Define the following sets of indices for Sender 1,

\[ C_1 \triangleq \mathcal{H}_{S_{U_1'}}^{(N)} \cap \mathcal{L}_{S_{U_1'}|Y_1}^{(N)}, \quad C_2 \triangleq \mathcal{H}_{T_{U_1'}}^{(N)} \cap \mathcal{L}_{T_{U_1'}|Y_2}^{(N)}, \]  

and similarly define \( C_{12} \) and \( C_{22} \) for Sender 2. From (45) we have

\[ \lim_{N \to \infty} \frac{1}{N} |C_1| = P^c(1), \quad \lim_{N \to \infty} \frac{1}{N} |C_2| = \tilde{P}^c(1) \geq P^c(1), \]  

\[ \lim_{N \to \infty} \frac{1}{N} |C_{12}| = P^c(2), \quad \lim_{N \to \infty} \frac{1}{N} |C_{22}| = \tilde{P}^c(2) \geq P^c(2). \]  

(57)

Now we can use the chaining method introduced in [47] to achieve \( P^c \). Suppose the block number is \( K \). Choose an arbitrary subset of \( C_{12} \setminus C_1 \), denoted as \( C_{21} \), such that \( |C_{21}| = |C_1 \setminus C_{12}| \), and an arbitrary subset of \( C_{22} \setminus C_2 \), denoted as \( C_{22} \), such that \( |C_{22}| = |C_2 \setminus C_{22}| \). Define the following sets of indices for Sender 1:

\[ I_{1c} = C_1 \cap C_{12}, \quad I_{1r} = C_1 \setminus C_1, \quad I_{1e} = C_{12}, \]  

\[ F'_{1r} = \mathcal{H}_{W_1}^{(N)} \setminus (I_{1e} \cup I_{1c} \cup I_{1r}), \quad F'_{1d} = (\mathcal{H}_{W_1}^{(N)})^C, \]  

and similarly define \( I_{2c}, I_{2r}, I_{2e}, F'_{2r}, \) and \( F'_{2d} \) for Sender 2. Fig. 3 shows the code construction for Sender 1’s common message. Sender 1 encodes its common messages as follows.

1. In Block 1,
   - \( \{u_{1}^{i,j}\}_{j \in I_{1c} \cup I_{1e}} \) store common message symbols.
   - \( \{u_{1}^{i,j}\}_{j \in F'_{1r} \cup I_{1r}} \) carry uniformly distributed frozen symbols.
   - \( \{u_{1}^{i,j}\}_{j \in F'_{1d}} \) are assigned by random mappings \( \lambda_j(u_{1}^{i,j-1}) \) that generate an output \( u \in W_1 \) according to conditional probability \( P_{U_{1}^{i,j}|U_{1}^{i,j-1}}(u|U_{1}^{i,j-1}) \).

2. In Block \( i \) (\( 1 < i < K \)),
   - \( \{u_{1}^{i,j}\}_{j \in I_{1c} \cup I_{1e}}, \{u_{1}^{i,j}\}_{j \in F'_{1r}}, \) and \( \{u_{1}^{i,j}\}_{j \in F'_{1d}} \) are determined in the same ways as in Block 1.
• \{u_{ij}^i\}_{j \in \mathcal{II}_c^i} are assigned to the same value as \{u_{1j}^i\}_{j \in \mathcal{II}_c^1} in Block \(i - 1\).

(3) In Block \(K\),

• \{u_{ij}^i\}_{j \in \mathcal{II}_c^i}, \{u_{1j}^i\}_{j \in \mathcal{II}_c^1}, \{u_{1j}^i\}_{j \in \mathcal{II}_d^1} and \{u_{ij}^i\}_{j \in \mathcal{II}_d^i} are determined in the same ways as in Block \(i\) \((1 < i < K)\).

• \{u_{ij}^i\}_{j \in \mathcal{II}_c^i} also carry uniformly distributed frozen symbols.

To guarantee that both receivers can decode the common messages reliably, a vanishing fraction of deterministic symbols in each block, \{u_{ij}^i\}_{j \in \mathcal{II}_c(N) \cap (\mathcal{L}_c(N))} \text{ and } \{u_{1j}^i\}_{j \in \mathcal{II}_c(N) \cap (\mathcal{L}_c(N))}^c, are separately transmitted to Receiver 1 and 2 respectively with some reliable error-correcting code.

Sender 2 encodes its common messages similarly.

The common message rates of two senders in this scheme are

\[
R_1^c = \frac{K|\mathcal{I}_c| + (K - 1)|\mathcal{I}_c^1|}{KN} - \frac{|C_1|}{KN},
\]
\[
R_2^c = \frac{K|\mathcal{I}_c| + (K - 1)|\mathcal{I}_c^2|}{KN} - \frac{|C_2|}{KN}.
\]

From (57) we have

\[
\lim_{N \to \infty, K \to \infty} R_1^c = P^c(1), \quad \lim_{N \to \infty, K \to \infty} R_2^c = P^c(2).
\]

2) Private Message Encoding: Let \(\delta_N = 2^{-N^{\beta}}\) for \(0 < \beta < 1/2\). Define the following polarization sets:

\[
\mathcal{H}_{X_1|W_1W_2}^{(N)} \triangleq \{ j \in [N] : Z(U_1^j|U_1^{1:N}, U_2^{1:j-1}) \geq 1 - \delta_N \},
\]
\[
\mathcal{L}_{X_1|Y_1W_1W_2}^{(N)} \triangleq \{ j \in [N] : Z(U_1^j|Y_1^{1:j}, U_2^{1:j-1}, U_1^{1:j-1}) \leq \delta_N \},
\]

which satisfy (42)

\[
\lim_{N \to \infty} \frac{1}{N} |\mathcal{H}_{X_1|W_1W_2}^{(N)}| = H_{q_{X_1}}(X_1|W_1W_2),
\]
\[
\lim_{N \to \infty} \frac{1}{N} |\mathcal{L}_{X_1|Y_1W_1W_2}^{(N)}| = 1 - H_{q_{X_1}}(X_1|Y_1W_1W_2).
\]

Similarly we can define \(\mathcal{H}_{X_2|W_1W_2}^{(N)}\) and \(\mathcal{L}_{X_2|Y_2W_1W_2}^{(N)}\). Due to the independence between two senders’ messages, we have

\[
\mathcal{H}_{X_1|W_1W_2}^{(N)} = \mathcal{H}_{X_1|W_1}, \quad \mathcal{H}_{X_2|W_1W_2}^{(N)} = \mathcal{H}_{X_2|W_2},
\]

where \(\mathcal{H}_{X_k|W_k}^{(N)} \triangleq \{ j \in [N] : Z(U_1^j|U_1^{1:N}, U_2^{1:j-1}) \geq 1 - \delta_N \}\) for \(k = 1, 2\).
The private message encoding is just standard point-to-point polar coding, we have
\[ \mathcal{I}_{1p} \triangleq \mathcal{H}_{X_1|W_1W_2} \cap \mathcal{L}_{X_1|Y_1W_1W_2}, \quad \mathcal{I}_{2p} \triangleq \mathcal{H}_{X_2|W_1W_2} \cap \mathcal{L}_{X_2|Y_2W_1W_2}. \]  
(63)

The private message rates are then
\[ R_1^p = \frac{1}{N} | \mathcal{I}_{1p} |, \quad R_2^p = \frac{1}{N} | \mathcal{I}_{2p} |. \]  
(64)

Since the private message encoding is just standard point-to-point polar coding, we have
\[ \lim_{N \to \infty} R_1^p = I(X_1; Y_1|W_1W_2), \quad \lim_{N \to \infty} R_2^p = I(X_2; Y_2|W_1W_2). \]  
(65)

The remaining symbols are partitioned into two subsets. For Sender 1, the two subsets are the deterministic set of \( \mathcal{F}_{1d} = (\mathcal{H}_{X_1|W_1W_2}^C)^C \), and the frozen set of \( \mathcal{F}_{1r} = \mathcal{H}_{X_1|W_1W_2} \cap (\mathcal{L}_{X_1|Y_1W_1W_2}^C)^C \).

For Sender 2, \( \mathcal{F}_{2d} \) and \( \mathcal{F}_{2r} \) are similarly defined. In each block, Sender 1 generates its final codewords as follows.

- \( \{ \bar{u}_j^i \}_{j \in \mathcal{I}_{1p}} \) store private message symbols.
- \( \{ u_j^i \}_{j \in \mathcal{F}_{1r}} \) carry uniformly distributed frozen symbols.
- \( \{ u_j^i \}_{j \in \mathcal{F}_{1d}} \) are assigned by random mappings \( \lambda_j(u_1^i, \ldots, u_j^i) \) that generate an output \( u \in \mathcal{X} \) according to conditional probability \( P_{U^i|U_1^i}\ldots U_{j-1}^i}(u|u_1^i, \ldots, u_j^i) \).
- \( \{ u_j^i \}_{j \in (\mathcal{H}_{X_1|W_1W_2}^C \cap (\mathcal{L}_{X_1|Y_1W_1W_2}^C)^C} \) is separately transmitted to Receiver 1 with some reliable error-correcting code.

Sender 2 generates its final codewords similarly.

3) Decoding: Receiver 1 decodes two senders’ common messages from Block 1 to Block \( K \).

- In Block 1, for \( k = 1, 2 \),
  \[ \bar{u}_k^i = \begin{cases} 
  u_k^i, & \text{if } j \in (\mathcal{L}_{S_{U_k}^i|Y_1}^C) \\
  \arg \max_{u \in (0,1)} P_{S_{U_k}^i|Y_1^N S_{U_k}^i}(u|y_k^i, s_{U_k}^i), & \text{if } j \in \mathcal{L}_{S_{U_2}^i|Y_1}^C 
  \end{cases} \]  
(66)

- In Block \( i \) (\( 1 < i < K \)), \( \{ \bar{u}_1^j \}_{j \in \mathcal{I}_{1c}} \) and \( \{ \bar{u}_2^j \}_{j \in \mathcal{I}_{2c}} \) are deduced from \( \{ \bar{u}_1^j \}_{j \in \mathcal{I}_{1c}} \) and \( \{ \bar{u}_2^j \}_{j \in \mathcal{I}_{2c}} \) in Block \( i - 1 \) respectively, and the rest are decoded in the same ways as in Block 1.

- In Block \( K \), \( \{ \bar{u}_1^j \}_{j \in \mathcal{I}_{1c}} \) and \( \{ \bar{u}_2^j \}_{j \in \mathcal{I}_{2c}} \) are assigned to the pre-shared value between Sender 1 and the two receivers, and the rest are decoded in the same ways as in Block \( i \) (\( 1 < i < K \)).

Having recovered the common messages in a block, Receiver 1 decodes its private message in that block as
\[ \bar{u}_1^j = \begin{cases} 
  u_1^j, & \text{if } j \in (\mathcal{L}_{X_1|Y_1W_1W_2}^C) \\
  \arg \max_{u \in (0,1)} P_{U_1^i|Y_1^N U_1^i} U_2^N U_1^i}(u|y_1^i, \bar{u}_1^i, \bar{u}_2^i, u_1^j), & \text{if } j \in \mathcal{L}_{X_1|Y_1W_1W_2}^C 
  \end{cases} \]  
(67)
According to our encoding rules, the induced joint distribution by our encoding scheme is decomposed as

\[ P_{U_1^{1:N}U_1'^{1:N}U_2^{1:N}U_2'^{1:N}}(u_{1:1:N}^1, u_{1:1:N}^2, u_{2:1:N}^1, u_{2:1:N}^2) \]

\[ = P_{U_1'^{1:N}}(u_{1:1:N}^1)P_{U_1^{1:N}U_1'^{1:N}}(u_{1:1:N}^1)P_{U_2'^{1:N}}(u_{2:1:N}^2)P_{U_2^{1:N}U_2'^{1:N}}(u_{2:1:N}^1) \]

\[ = \prod_{j=1}^{N} P(u_1^j|u_{1:j-1}^1)P(u_1^j|u_{1:j-1}^1)P(u_2^j|u_{2:j-1}^2)P(u_2^j|u_{2:j-1}^2). \]

According to our encoding rules, the induced joint distribution by our encoding scheme is

\[ Q_{U_1^{1:N}U_1'^{1:N}U_2^{1:N}U_2'^{1:N}}(u_{1:1:N}^1, u_{1:1:N}^2, u_{2:1:N}^1, u_{2:1:N}^2) \]

\[ = \prod_{j=1}^{N} Q(u_1^j|u_{1:j-1}^1)Q(u_1^j|u_{1:j-1}^1)Q(u_2^j|u_{2:j-1}^2)Q(u_2^j|u_{2:j-1}^2), \]

where

\[ Q(u_k^j|u_k^{1:j-1}) = \begin{cases} \frac{1}{q_{W_k}}, & \text{if } j \in \mathcal{H}^{(N)}_{W_k}, \\ P(u_k^j|u_k^{1:j-1}), & \text{otherwise}. \end{cases} \]

and

\[ Q(u_k^j|u_k^{1:j-1}) = \begin{cases} \frac{1}{q_{X_k}}, & \text{if } i \in \mathcal{H}^{(N)}_{X_k|W_k}, \\ P(u_k^j|u_k^{1:j-1}), & \text{otherwise}. \end{cases} \]

for \( k = 1, 2. \) From [18] Lemma 5\(^4\) we have

\[ \| P_{U_1^{1:N}U_1'^{1:N}U_2^{1:N}U_2'^{1:N}}(u_{1:1:N}^1, u_{1:1:N}^2, u_{2:1:N}^1, u_{2:1:N}^2) - Q_{U_1^{1:N}U_1'^{1:N}U_2^{1:N}U_2'^{1:N}}(u_{1:1:N}^1, u_{1:1:N}^2, u_{2:1:N}^1, u_{2:1:N}^2) \| \]

\[ = O(2^{-N^3}), \]

where \( \| P - Q \| \) denotes the total variation distance between distributions \( P \) and \( Q \).

5) Achievable Rates: From (60) and (65) we can see that our proposed scheme achieves the target Type I point \( P \).

\(^4\) Although the number of random variables and alphabet size are different in [18] Lemma 5, the proof method is the same.
6) Error Performance: For Receiver 1, the error probability in decoding two senders’ common messages in the overall $K$ blocks can be upper bounded by

$$P_{e1}^c \leq (q_{w1} - 1) \left( K \sum_{j \in \mathcal{C}_1} Z(S^{f_1(j)}|Y_1^{1:N}, S^{1:f_1(j)-1}) - \sum_{j \in \mathcal{I}_{1c}} Z(S^{f_1(j)}|Y_1^{1:N}, S^{1:f_1(j)-1}) \right) + \left( q_{w2} - 1 \right) \left( K \sum_{j \in \mathcal{C}_2} Z(S^{f_2(j)}|Y_1^{1:N}, S^{1:f_2(j)-1}) - \sum_{j \in \mathcal{I}_{2c}} Z(S^{f_2(j)}|Y_1^{1:N}, S^{1:f_2(j)-1}) \right)$$

(68)

$$= O(KN2^{-N^\beta}).$$

The error probability in decoding its private message in the overall $K$ blocks assuming it has successfully decoded the common messages can be upper bounded by

$$P_{e1}^p \leq K(q_{x1} - 1) \sum_{j \in \mathcal{I}_{1p}} Z(U_1^j|Y_1^{1:N}, U_1^{1:j-1}, U_2^{1:j-1}, U_1^{1:j-1})$$

(69)

$$= O(KN2^{-N^\beta}).$$

Thus, the overall error probability can be upper bounded by

$$P_{e1} \leq P_{e1}^c + P_{e1}^p = O(KN2^{-N^\beta}).$$

(70)

Similarly, the overall error probability of Receiver 2 can be upper bounded by

$$P_{e2} \leq O(KN2^{-N^\beta}).$$

(71)

B. Polar Coding Scheme for Type II Points

Let $P$ be a point of Type II, $P^c$ be the corresponding common message rate pair, and $P^1$ and $P^2$ be Receiver 1’s and Receiver 2’s rate pairs respectively. Without loss of generality, we consider the case when $P^c \in \mathcal{R}(P_{Y_2|W_1W_2}) \setminus \mathcal{R}(P_{Y_1|W_1W_2})$ and $P^c(1) \leq I(W_1; Y_1)$ in this subsection as an example.

Choose $\tilde{P}^1 = \left(I(X_1W_2; Y_1) - P^1(2), P^1(2)\right)$, which is on the dominant face of $\mathcal{R}(P_{Y_1|X_1W_2})$ and larger than $P^1$, and $\tilde{P}^c = \left(I(W_1W_2; Y_2) - P^c(2), P^c(2)\right)$, which is on the dominant face of $\mathcal{R}(P_{Y_2|W_1W_2})$ and larger than $P^c$, as the target points for conducting monotone chain rule expansions in our code design. From Definition 6 we know that

$$R_1^p = \tilde{P}^1(1) - I(W_1; Y_1),$$

(72)

$$R_2^p = I(X_2; Y_2|W_1W_2).$$

(73)

Let $S^{1:2N}$ be the monotone chain rule expansion for achieving $\tilde{P}^1$ in $\mathcal{R}(P_{Y_1|X_1W_2})$, and $T^{1:2N}$ be the monotone chain rule expansion for achieving $\tilde{P}^c$ in $\mathcal{R}(P_{Y_2|W_1W_2})$. Denote the sets of
indices in $S^{1:2N}$ with $S^j \in U^{1:N}_1$ and $S^j \in U^{1:N}_2$ by $S_{U_1}$ and $S_{U_2}$ respectively, and those in $T^{1:2N}$ with $T^j \in U^{1:N}_1$ and $T^j \in U^{1:N}_2$ by $T_{U_1}$ and $T_{U_2}$ respectively. Let $f_1(j) : [N] \rightarrow S_{U_1}$ be the mapping from indices of $U^{1:N}_1$ to those of $S^{S_{U_1}}$, $f_2(j) : [N] \rightarrow S_{U_2}$ the mapping from indices of $U^{1:N}_2$ to those of $S^{S_{U_2}}$, and $g_k(j) : [N] \rightarrow T_{U_k}$ the mapping from indices of $U^{1:N}_k$ to those of $T^{U_k}$. For $k = 1, 2$. For $\delta_N = 2^{-\alpha/\beta}$ with $0 < \beta < 1/2$, define $H^{(N)}_{W_1}$, $H^{(N)}_{W_2}$, $H^{(N)}_{S_{U_2}}$, $H^{(N)}_{T_{U_1}}$, $H^{(N)}_{T_{U_2}}$, $L_{S_{U_2}}(N)$, $L_{T_{U_1}}(N)$, $L_{T_{U_2}}(N)$, $L_{S_{U_2}}(N)$, $L_{T_{U_1}}(N)$, $L_{T_{U_2}}(N)$, and $L_{S_{U_2}}(N)$ in the same forms as in the previous subsection, and define the following extra polarized sets

\[
L^{(N)}_{W_1|Y_1} \triangleq \{ j \in [N] : Z(U^{j}|Y_1^{1:N}, U_1^{1:j-1}) \leq \delta_N \},
\]

\[
L^{(N)}_{S_{U_1}|Y_1W_1} \triangleq \{ j \in [N] : Z(S^{f_1(j)}|Y_1^{1:N}, U_1^{1:N}, S^{1:f_1(j)-1}) \leq \delta_N \}. \tag{74}
\]

1) Common Message Encoding: Define the following sets of indices for two senders:

\[
C'_1 \triangleq H^{(N)}_{W_1} \cap L^{(N)}_{W_1|Y_1}, \quad C''_1 \triangleq H^{(N)}_{W_1} \cap L^{(N)}_{T_{U_1}|Y_2},
\]

\[
C'_2 \triangleq H^{(N)}_{W_2} \cap L^{(N)}_{S_{U_2}|Y_1}, \quad C''_2 \triangleq H^{(N)}_{W_2} \cap L^{(N)}_{T_{U_2}|Y_2}, \tag{75}
\]

which satisfy

\[
\lim_{N \rightarrow \infty} \frac{1}{N} |C'_1| = I(W_1; Y_1) \geq P^c(1), \quad \lim_{N \rightarrow \infty} \frac{1}{N} |C''_1| = \tilde{P}^c(1) \geq P^c(1),
\]

\[
\lim_{N \rightarrow \infty} \frac{1}{N} |C'_2| = \tilde{P}^1(2) = P^c(2), \quad \lim_{N \rightarrow \infty} \frac{1}{N} |C''_2| = \tilde{P}^c(2) = P^c(2).
\]

For Sender 1, if $P^c(1) = I(W_1; Y_1)$, let $C'_1 = C'_1$. Otherwise choose $C'_1 \subset C'_1$ such that $|C'_1| = N P^c(1)$. Since $\lim_{N \rightarrow \infty} \frac{1}{N} |C'_1| > P^c(1)$ in the latter case, we can always find such a subset given sufficiently large $N$. Similarly, if $\tilde{P}^c = P^c$, let $C''_1 = C''_1$. Otherwise choose $C''_1 \subset C''_1$ such that $|C''_1| = N P^c(1)$.

Now we can use the chaining method to achieve $P^c$. Although $\lim_{N \rightarrow \infty} \frac{1}{N} |C'_1| = \lim_{N \rightarrow \infty} \frac{1}{N} |C''_1|$, for a finite $N$, their sizes may be different. If $|C''_1| \geq |C'_1|$, we choose an arbitrary subset of $C''_1 \setminus C'_1$, denoted as $C''_{11}$, such that $|C''_{11}| = |C'_1 \setminus C''_1|$. Otherwise we choose an arbitrary subset of $C'_1 \setminus C''_1$, denoted as $C'_1$. Then, we proceed with the process as follows:

\[\ldots\]
denoted as $C_1^{12}$, such that $|C_1^{12}| = |C_1^2 \setminus C_1^1|$. Partition the indices of $U'_1:N$ as follows:

$$I_{1c} = C_1^1 \cap C_1^2,$$

$$I_{1c}^1 = \begin{cases} C_1^1 \setminus C_1^2, & \text{if } |C_1^2| \geq |C_1^1| \\ C_{12}, & \text{otherwise} \end{cases}$$

$$I_{1c}^2 = \begin{cases} C_1^{21}, & \text{if } |C_1^2| \geq |C_1^1| \\ C_1^2 \setminus C_1^1, & \text{otherwise} \end{cases}$$

$$F'_{1r} = \mathcal{H}_{W_1}^{(N)} \setminus (I_{1c} \cup I_{1c}^1 \cup I_{1c}^2),$$

$$F'_{1d} = (\mathcal{H}_{W_1}^{(N)})^c.$$ (80)

The code construction of $U'_1:N$ in this case is illustrated in Fig. 4. For Sender 2, similarly, if $|C_2^2| \geq |C_2^1|$, choose an arbitrary subset of $C_2^2 \setminus C_2^1$, denoted as $C_{21}^2$, such that $|C_{21}^2| = |C_2^1 \setminus C_2^2|$. Otherwise choose an arbitrary subset of $C_2^1 \setminus C_2^2$, denoted as $C_{22}^1$, such that $|C_{22}^1| = |C_2^2 \setminus C_2^1|$. Then we can define $I_{2c}, I_{2c}^1, I_{2c}^2, F'_{2r}$ and $F'_{2d}$ similarly.

Let $K$ be the number of blocks used. The encoding procedures for two senders’ common messages can be expressed in the same form as the Type I scheme, and the common message rates are also

$$R_c^1 = \frac{|C_1^1|}{N} - \frac{|I_{1c}^1|}{KN}, \quad R_c^2 = \frac{|C_2^1|}{N} - \frac{|I_{2c}^1|}{KN},$$ (81)

with

$$\lim_{N \to \infty, K \to \infty} R_c^1 = P^c(1), \quad \lim_{N \to \infty, K \to \infty} R_c^2 = P^c(2).$$
2) Encoding for Sender 1’s Private Message: Define the information set, frozen set and deterministic set for $U_1^{1:N}$ as

\[ I_{1p} = \mathcal{H}_{X_1|W_1}^{(N)} \cap \mathcal{L}_{S_{U_1}|Y_1W_1}^{(N)}, \]
\[ F_{1r} = \mathcal{H}_{X_1|W_1}^{(N)} \cap (\mathcal{L}_{S_{U_1}|Y_1W_1}^{(N)})^C, \]
\[ F_{1d} = (\mathcal{H}_{X_1|W_1}^{(N)})^C. \] (82)

Then sender 1 can generate its final codewords in the same way as in the Type I scheme. Note that the information set and frozen set in this scheme are different from the conventional ones described in Section III-C. This is because $W_1^{1:N}$ is superimposed on $X_1^{1:N}$, which is considered as side information in encoding and decoding. The permutation $S_{1:2}^{1:2}$ is chosen to achieve $P_{Y_1}$ in Receiver 1’s effective channel $P_{Y_1|X_1W_2}$, but the code construction for $U_1^{1:N}$ is determined jointly by this permutation and the knowledge of $W_1^{1:N}$.

3) Encoding for Sender 2’s Private Message: The same as the Type I scheme.

4) Decoding: Receiver 1 decodes from Block 1 to Block $K$.

- In Block 1, Sender 1’s common message is decoded as

\[ \bar{u}_1^j = \begin{cases} 
  u_1^j, & \text{if } j \in (\mathcal{L}_{W_1|Y_1}^{(N)})^C \\
  \arg \max_{u \in \{0,1\}} P_{U_1^j|Y_1^1:1:2}^j(u|y_1^1:1:2, \bar{u}_1^1:j-1), & \text{if } j \in \mathcal{L}_{W_1|Y_1}^{(N)}
\end{cases} \] (83)

Sender 1’s private messages and Sender 2’s common message are jointly decoded as

\[ \bar{u}_1^j = \begin{cases} 
  u_1^j, & \text{if } j \in (\mathcal{L}_{S_{U_1}|Y_1}^{(N)})^C \\
  \arg \max_{u \in \{0,1\}} P_{S_{U_1}(j)|Y_1^1:1:2}^j(u|y_1^1:1:2, \bar{u}_1^1:j-1), & \text{if } j \in \mathcal{L}_{S_{U_1}|Y_1}^{(N)}
\end{cases} \] (84)

\[ \bar{u}_2^j = \begin{cases} 
  u_2^j, & \text{if } j \in (\mathcal{L}_{S_{U_2}|Y_1}^{(N)})^C \\
  \arg \max_{u \in \{0,1\}} P_{S_{U_2}(j)|Y_1^1:1:2}^j(u|y_1^1:1:2, s_{1:2}^j), & \text{if } j \in \mathcal{L}_{S_{U_2}|Y_1}^{(N)}
\end{cases} \] (85)

- In Block $i$ ($1 < i < K$), $\{\bar{u}_1^j\}_{j \in T_{1e}^i}$ and $\{\bar{u}_2^j\}_{j \in T_{2e}^i}$ are deduced from $\{\bar{u}_1^j\}_{j \in T_{1e}^{i-1}}$ and $\{\bar{u}_2^j\}_{j \in T_{2e}^{i-1}}$ in Block $i - 1$ respectively, and the rest are decoded in the same way as in Block 1.

- In Block $K$, $\{\bar{u}_1^j\}_{j \in T_{1e}^K}$ and $\{\bar{u}_2^j\}_{j \in T_{2e}^K}$ are assigned to the pre-shared value between Sender 1 and the two receivers, and the rest are decoded in the same way as in Block $i$ ($1 < i < K$).

Receiver 2 decodes from Block $K$ to Block 1 in the same way as the Type I scheme.

5) Total Variation Distance: With a similar analysis to the Type I scheme, one can show that the induced joint distribution by our encoding rules is asymptotically indistinguishable from the target one.
6) Achievable Rates: We have shown that our proposed scheme achieves the target common message rate pair and the private message rate of Sender 2. For Sender 1’s private message rate, the following lemma shows that our proposed scheme satisfies (72).

**Lemma 2.** \( \lim_{N \to \infty} \frac{1}{N}|I_1| = \mathbb{P}^1(1) - I(W_1; Y_1) \).

**Proof.** See Appendix B.

7) Error Performance: For Receiver 1, the error probability in decoding Sender 1’s common messages in the overall \( K \) blocks can be upper bounded by

\[
P_{e_1}^c \leq K(qw_1 - 1) \sum_{j \in C_1^c} Z(U_1^{j} | Y_{1}^{1:N}, U_1^{1:j-1}) - (qw_1 - 1) \sum_{j \in T_1^c} Z(U_1^{j} | Y_{1}^{1:N}, U_1^{1:j-1}) = O(KN2^{-N^\alpha}).
\]

(86)

The error probability in decoding its private messages and Sender 2’s common messages in the overall \( K \) blocks assuming it has successfully decoded Sender 1’s common messages can be upper bounded by

\[
P_{e_1}^p \leq K(qx_1 - 1) \sum_{j \in I_1^p} Z(S_{f_1}^{(j)} | Y_{1}^{1:N}, U_1^{1:j-1}, S_{1}^{1:j-1}) + K(qw_2 - 1) \sum_{j \in C_2^p} Z(S_{f_2}^{(j)} | Y_{1}^{1:N}, S_{1}^{1:j-1}) - (qw_2 - 1) \sum_{i \in T_2^c} Z(S_{f_2}^{(j)} | Y_{1}^{1:N}, S_{1}^{1:j-1}) = O(KN2^{-N^\alpha}).
\]

(87)

Thus, the overall error probability can be upper bounded by

\[
P_{e_1} \leq P_{e_1}^c + P_{e_1}^p = O(KN2^{-N^\alpha}).
\]

(88)

The error probability of Receiver 2 can be similarly analyzed as the Type I scheme.

VI. EXTENSION TO INTERFERENCE NETWORKS

So far we have shown that our proposed two types of schemes can achieve the Han-Kobayashi rate region for the 2-user DM-IC. A natural question is whether they can be extended to arbitrary DM-INs to simplify the code design. Although the feasibility of the two schemes are proved by decomposing rate pairs under the constraints in Theorem 1 and 2 in this section, we will show that they in fact represent a heterogeneous superposition coding scheme with rate splitting that can be applied to DM-INs with private messages.
Homogeneous superposition coding and heterogeneous superposition coding are two variants of superposition coding in the literature [41]. In the homogeneous variant, component messages of each sender are independently encoded into auxiliary sequences, which are then mapped into the channel input sequence by some symbol-by-symbol deterministic mapping. This variant was first proposed by Cover [51]. In the heterogeneous variant, the coarse messages are encoded into auxiliary sequences first, and then a satellite codebook for the fine message is generated around it conditionally independently. This approach was introduced by Bergmans [52]. We can see that the scheme of [39] belongs to the homogeneous type while ours belongs to the heterogeneous type. The private messages in our scheme play the role of fine messages. Usually the heterogeneous scheme is simpler than the homogeneous one since it requires fewer auxiliary random variables, as our proposed scheme has shown. To achieve the optimal rate region, both variants require joint decoding, which can not be implemented using polar codes with our proposed encoding structure. This is why we design two types of decoding orders, which in fact represent a rate splitting method. Unlike conventional rate splitting which transforms the original channel into several point-to-point channels, our method creates a point-to-point channel and a MAC. We will show that with regard to the overall rate of each user pair rather than each component message rate, our heterogeneous approach can achieve the same rate region as the homogeneous one in DM-INs with private messages, and our rate splitting scheme suffers no rate loss compared to joint decoding.

A $K$-sender $L$-receiver DM-IN, denoted by $(K, L)$-DM-IN, consists of $K$ senders and $L$ receivers. Each sender $k \in [K]$ transmits an independent message $M_k$ at rate $R_k$, while each receiver $l \in [L]$ wishes to recover a subset $D_l \subset [K]$ of the messages. Similar to the Han-Kobayashi strategy in the 2-user DM-IC, $M_k$ can be split into several component messages, each intended for a group of receivers. If a message is intended for only one receiver, we refer to it as a private message. Otherwise we refer to it as a common message. We only consider the case when each sender has only one private message intended for some receiver and (possibly) multiple common messages intended also for this receiver. More complicated cases can be resolved by decomposing a sender with multiple private and common messages into a certain number of virtual senders of this type.

Fig. 5 shows Sender 1’s part of the equivalent channel of the $(K, L)$-DM-IN with a private message $M_{11}$ intended for Receiver 1, and common messages $M_{1C_1}$, $M_{1C_2}$, ..., $M_{1C_{a_1}}$ ($a_1 \geq 1$) intended for Receiver 1 and some other receiver groups. It is shown in [53] that the optimal
achievable rate region when the encoding is restricted to random coding ensembles is the intersection of rate regions for its component multiple access channels in which each receiver recovers its private messages as well as its common messages. Thus, one can design a code for the compound MAC to achieve the optimal rate region, as [39] has presented. Here we discuss using the idea of our proposed schemes to simplify the code design.

Firstly, consider the case when only Sender 1 uses our proposed approach. Instead of generating a codeword for each message and then merging them with some mapping function as in Fig. 5, now we only generate codewords for the common messages and then superimpose them with the private message $M_{11}$ using coding techniques, as shown in Fig. 6. Let $P^*(X_1|U_{11}, U_{1C_1}, ..., U_{1C_{a_1}})$ be the deterministic mapping from auxiliary random variables $U_{11}, U_{1C_1}, ..., U_{1C_{a_1}}$ to the channel input $X_1$ in Fig. 5, and let $P^*(X_1|U_{1C_1}, ..., U_{1C_{a_1}}) = \sum_{U_{11}} P^*(U_{11})P(U_{11})$ be the conditional distribution of random variables $X_1, U_{1C_1}, ..., U_{1C_{a_1}}$ in Fig. 6. We can see that synthesized MACs for other receivers are not affected with this setting since $U_{11}$ plays no part in them. Thus, the achievable rate regions of other receivers’ synthesized MACs remain the same. Note that although $P^*$ is a deterministic mapping as in the Han-Kobayashi scheme, the resulting $P_1^*$ becomes random, which yields a more direct polar code design. Thus, like the 2-user case, our proposed schemes
can also reduce the use of deterministic mappings in designing codes for DM-INs. Also note that the auxiliary random variable $U_{11}$ for private message $M_{11}$ is no longer needed in this design.

Now let us discuss the achievable rates from Receiver 1’s point of view. Denote Sender 1’s common messages as a whole by $U_{1c_1}$ with rate $R_{1}^{c_1}$, and other senders’ common messages which are intended for Receiver 1 by $U_{1c_o}$ with rate $R_{1}^{c_o}$. The private message rate is denoted by $R_{1}^{p}$.

With the original approach, the achievable rate region $\mathcal{R}_{MAC}^{1}(P^*)$ of Receiver 1’s synthesized MAC $P(Y_1|U_{11}, U_{1c_1}, U_{1c_o})$ is

\[ R_{1}^{p} \leq I(U_{11}; Y_1|U_{1c_1}, U_{1c_o}), \quad (89) \]
\[ R_{1}^{c_1} \leq I(U_{1c_1}; Y_1|U_{11}, U_{1c_o}), \quad (90) \]
\[ R_{1}^{c_o} \leq I(U_{1c_o}; Y_1|U_{11}, U_{1c_1}), \quad (91) \]
\[ R_{1}^{p} + R_{1}^{c_1} \leq I(U_{11}, U_{1c_1}; Y_1|U_{1c_o}), \quad (92) \]
\[ R_{1}^{p} + R_{1}^{c_o} \leq I(U_{11}, U_{1c_o}; Y_1|U_{1c_1}), \quad (93) \]
\[ R_{1}^{c_1} + R_{1}^{c_o} \leq I(U_{1c_o}, U_{1c_1}; Y_1|U_{11}), \quad (94) \]
\[ R_{1}^{p} + R_{1}^{c_1} + R_{1}^{c_o} \leq I(U_{11}, U_{1c_1}, U_{1c_o}; Y_1). \quad (95) \]

With our proposed approach, the achievable rate region $\mathcal{R}_{MAC}^{1'}(P^*)$ of Receiver 1’s synthesized MAC $P(Y_1|X_1, U_{1c_o})$ becomes

\[ R_{1}^{c_o} \leq I(U_{1c_o}; Y_1|X_1), \quad (96) \]
\[ R_{1}^{p} + R_{1}^{c_1} \leq I(X_1; Y_1|U_{1c_o}), \quad (97) \]
\[ R_{1}^{p} + R_{1}^{c_1} + R_{1}^{c_o} \leq I(X_1, U_{1c_o}; Y_1). \quad (98) \]

Since $(U_{11}, U_{1c_1}) \rightarrow X_1$ is a deterministic mapping, we can readily see that upper bounds for $R_{1}^{c_o}$, $R_{1} = R_{1}^{p} + R_{1}^{c_1}$ and $R_{1}^{III} = R_{1}^{p} + R_{1}^{c_1} + R_{1}^{c_o}$ are invariant with our proposed approach. Thus, if we are interested in the overall rate between the user pair of Sender 1 and Receiver 1 rather than each component message rate, our proposed approach can achieve the same or even a larger\(^5\) rate region than the original approach for a given joint distribution.

Similar to the 2-user DM-IC case, when we apply polar codes to this encoding structure, the common messages of a sender must be decoded before its private message. Thus, fully joint

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\(^5\)As shown in [35], in the 2-user DM-IC case, for a given joint distribution, the compact description of the Han-Kobayashi region may be larger than the original one. However, when maximized over all possible distributions, they are equivalent.
decoding can not be performed. Our proposed two types of partially joint decoding orders for the 2-user DM-IC can also be applied here. With the first type, all common messages intended for Receiver 1 are jointly decoded before the private message, and the achievable rate region is

$$R_{p1} \leq I(X_1; Y_1 | U_{1c1}, U_{1co}),$$ \hspace{1cm} (99)

$$R_{c1}^{ci} \leq I(U_{1c1}; Y_1 | U_{1co}),$$ \hspace{1cm} (100)

$$R_{c1}^{co} \leq I(U_{1co}; Y_1 | U_{1c1}),$$ \hspace{1cm} (101)

$$R_{c1}^{ci} + R_{c1}^{co} \leq I(U_{1c1}, U_{1co}; Y_1).$$ \hspace{1cm} (102)

With the second type, Sender 1’s common messages are decoded first, then the private message and other senders’ common messages are jointly decoded. Thus, the achievable rate region is

$$R_{c1}^{ci} \leq I(U_{1c1}; Y_1),$$ \hspace{1cm} (103)

$$R_{p1} \leq I(X_1; Y_1 | U_{1c1}, U_{1co}),$$ \hspace{1cm} (104)

$$R_{c1}^{co} \leq I(U_{1co}; Y_1 | X_1),$$ \hspace{1cm} (105)

$$R_{p1} + R_{c1}^{co} \leq I(X_1, U_{1co}; Y_1 | U_{1c1}).$$ \hspace{1cm} (106)

We can see that these two decoding orders together can achieve every rate pair in \(\{(R_1, R_{c1}^{co}) : R_1 = R_{p1} + R_{c1}^{ci}, (R_{p1}, R_{c1}^{ci}, R_{c1}^{co}) \in \mathcal{R}_{MAC}(P^*)\}\). Thus, such a rate-splitting method can achieve the same rate region as the original fully joint scheme. To apply polar coding, one simply needs to adopt MAC polarization with more than 2 users and follow our proposed schemes for the 2-user DM-IC.

In the above discussion we have only considered the case when only one sender uses our proposed scheme. However, the case when multiple senders use such a scheme can be readily derived from this simple case by adding one sender with the proposed scheme at a time. To conclude, we have the following proposition.

**Proposition 3.** The proposed heterogeneous superposition polar coding scheme with the two types of partially joint decoding orders achieves the optimal rate region of DM-INs when the encoding is restricted to random coding ensembles.

**VII. CONCLUSION REMARKS**

Based on the compact description of the Han-Kobayashi rate region and the coding strategy lying behind [35], we have shown that every point on the dominant faces of the Han-Kobayashi
region can be achieved by polar codes in a simpler way compared to the scheme of [39]. Extending our proposed scheme to DM-INs with private messages, we have shown that they in fact represent a heterogeneous superposition coding scheme which can reduce the use of deterministic mappings and the number of auxiliary random variables in the code design, and the two types of partially joint decoders stand for a rate splitting method capable of achieving the optimal achievable rate region. This result reveals more insights on the roles of auxiliary random variables and coding strategy for DM-INs.

The chaining method we use in this paper and the polar alignment technique used in [39] both make polar coding schemes lengthy, since both methods result in increasing of block length. It is shown in [42] that the non-universality of polar codes is a property of the successive cancellation decoding algorithm. Under maximum likelihood (ML) decoding, polar codes are universal. This makes us wonder if there exist decoding algorithms which maintain universality while still enjoy low complexity. If the answer is yes, our proposed scheme may be further simplified as well as polar coding schemes for some other multi-user channels.

APPENDIX A

PROOF OF Lemma 1

Since \( R_1 + R_2 = c, R_1 + R_2 = d, R_1 + R_2 = e, 2R_1 + R_2 = f \) and \( R_1 + 2R_2 = g \) are the possible dominant faces of the Han-Kobayashi region, we prove Lemma 1 by deriving value ranges of common message rates for points on each of them.

A. Points on \( R_1 + R_2 = c \) and \( R_1 + R_2 = d \)

Suppose \( P \in \mathcal{R}_{HK}(P_{1}^{*}) \) is a point on line

\[
R_1 + R_2 = c. \tag{107}
\]

Let \( (R_1^p, R_1^c, R_2^p, R_2^c) \) be a rate decomposition of \( P \). From (9) and (15) we have

\[
R_2^p = I(X_2; Y_2 | W_1 W_2),
\]

\[
R_1^p + R_1^c + R_2^c = I(X_1 W_2; Y_1).
\]

From (23), (24) and (107) we have

\[
I(X_2; Y_2 | W_1 W_2) + I(W_1 W_2; Y_1) - I(W_1; Y_2 | W_2) \leq R_2 \leq I(X_2 W_1; Y_2) - I(W_1; Y_1).
\]
Thus,
\[ I(W_1W_2; Y_1) - I(W_1; Y_2|W_2) \leq R_2^c \leq I(W_1W_2; Y_2) - I(W_1; Y_1). \]

If \( R_1 + R_2 = c \) is a dominant face of the Han-Kobayashi region, \( c \leq d \) and \( c \leq e \) must hold.

Then we have
\[ I(W_1; Y_2|W_2) \geq I(W_1; Y_1), \]
\[ I(W_1W_2; Y_2) \geq I(W_1W_2; Y_1). \]

For \( I(W_1W_2; Y_1) - I(W_1; Y_2|W_2) \leq R_2^c \leq I(W_2; Y_1|W_1) \), let
\[ R_1^c = I(W_1W_2; Y_1) - R_2^c, \]
\[ R_1^p = I(X_1; Y_1|W_1W_2). \]

We can see that \((R_1^c, R_2^c) \in \mathcal{R}(P_{1|W_1W_2}) \cap \mathcal{R}(P_{2|W_1W_2})\). Thus, \( P \) is of Type I.

For \( I(W_2; Y_1|W_1) \leq R_2^c \leq I(W_1W_2; Y_2) - I(W_1; Y_1) \), let
\[ R_1^c = I(W_1; Y_1), \]
\[ R_1^p = I(X_1W_2; Y_1|W_1) - R_2^c. \]

In this case, \( P \) belongs to Type II.

For a point \( P \in \mathcal{R}_{HK}(P_1^*) \) on line \( R_1 + R_2 = d \), the analysis is similar.

**B. Points on \( R_1 + R_2 = e \)**

Suppose \( P \in \mathcal{R}_{HK}(P_1^*) \) is a point on line
\[ R_1 + R_2 = e. \quad (108) \]

Let \((R_1^p, R_1^c, R_2^p, R_2^c)\) be a rate decomposition of \( P \). From (13) and its counterpart for Receiver 2, we have
\[ R_1^p + R_2^c = I(X_1W_2; Y_1|W_1), \]
\[ R_2^p + R_1^c = I(X_2W_1; Y_2|W_2). \]

Then from (9), (15) and their counterparts for Receiver 2, we have
\[ I(W_1; Y_2|W_2) \leq R_1^c \leq I(W_1; Y_1), \]
\[ I(W_2; Y_1|W_1) \leq R_2^c \leq I(W_2; Y_2). \]
From (23), (24) and (108) we have

\[ I(X_1W_2; Y_1|W_1) + I(W_1; Y_2|W_2) - I(W_2; Y_2) \leq R^c_1 + R^c_2 \leq I(X_1; Y_1|W_1W_2) + I(W_1; Y_1), \]
\[ I(X_2W_1; Y_2|W_2) + I(W_2; Y_1|W_1) - I(W_1; Y_1) \leq R^c_2 + R^p_2 \leq I(X_2; Y_2|W_1W_2) + I(W_2; Y_2). \]

1) If \( I(X_2; Y_1|W_1W_2) + I(W_2; Y_1|W_1) \leq P(2) \leq I(X_2; Y_2|W_1W_2) + I(W_2; Y_2) \), let \( R^c_2 = I(X_2; Y_2|W_1W_2) \). Then

\[ R^c_2 = P(2) - I(X_2; Y_2|W_1W_2), \]
\[ R^p_2 = I(Y_1; W_2), \]
\[ R^c_2 = I(X_1W_2; Y_1|W_1) - R^c_2. \]

2) If \( I(X_2W_1; Y_2|W_2) + I(W_2; Y_1|W_1) - I(W_1; Y_1) \leq P(2) < I(X_2; Y_2|W_1W_2) + I(W_2; Y_1|W_1) \), let \( R^p_1 = I(X_1; Y_1|W_1W_2) \). Then

\[ R^c_2 = I(W_2; Y_1|W_1), \]
\[ R^p_2 = P(2) - I(W_2; Y_1|W_1), \]
\[ R^c_2 = I(X_2W_1; Y_2|W_2) + I(W_2; Y_1|W_1) - P(2). \]

In both cases, \( P \) belongs to Type II.

C. Points on \( 2R_1 + R_2 = f \) and \( R_1 + 2R_2 = g \)

Suppose \( P \in \mathcal{R}_{HK}(P^*) \) is a point on line

\[ 2R_1 + R_2 = f. \] (109)

Let \((R^p_0, R^c_1, R^p_2, R^c_2)\) be a rate decomposition of \( P \). From (109), (9), (15) and the counterpart of (13) for Receiver 2, we have

\[ R^p_1 = I(X_1; Y_1|W_1W_2), \] (110)
\[ R^c_1 + R^p_2 = I(X_2W_1; Y_2|W_2), \] (111)
\[ R^c_1 + R^c_2 = I(W_1W_2; Y_1). \] (112)

Then we obtain from (12) that

\[ R^c_1 \leq I(W_1; Y_1|W_2). \] (113)
From (20)–(22), (109) and (110) we have
\[
R^c_1 \geq \max \left\{ I(W_1; Y_2|W_2), \right. \\
I(W_1W_2; Y_1) - I(W_2; Y_2), \\
I(W_1; Y_1) \left. \right\}.
\]

Thus,
\[
R^c_2 \leq \min \left\{ I(W_1W_2; Y_1) - I(W_1; Y_2|W_2), \\
I(W_2; Y_2), \\
I(W_2; Y_1|W_1) \right\}.
\]

We can see that \((R^c_1, R^c_2) \in \mathcal{R}(P_{Y_1|W_1W_2})\) and \(R^c_2 \leq I(W_2; Y_2)\). Thus, \(P\) belongs to Type II.

For a point \(P \in \mathcal{R}_{HK}(P_1^*)\) on line \(R_1 + 2R_2 = g\), the analysis is similar.

Now we have completed the proof.

**APPENDIX B**

**PROOF OF LEMMA 2**

For \(\delta_N = 2^{-N^\beta}\) with \(0 < \beta < 1/2\), define
\[
\mathcal{H}^{(N)}_{S_U|Y_1W_1} = \left\{ j \in [N] : Z(Sf(j)|Y_1^{1:N}, U_1^{1:N}, S^{1:f(j)-1}) \geq 1 - \delta_N \right\},
\]
and let \(\mathcal{B}^{(N)}_{S_U|Y_1W_1} = (\mathcal{H}^{(N)}_{S_U|Y_1W_1} \cup \mathcal{C}^{(N)}_{S_U|Y_1W_1})^C\). Then we have
\[
\frac{1}{N}|\mathcal{I}_{1p}| = \frac{1}{N}|\mathcal{H}^{(N)}_{X_1|W_1} \cap (\mathcal{H}^{(N)}_{S_U|Y_1W_1} \cup \mathcal{B}^{(N)}_{S_U|Y_1W_1})^C| \\
= \frac{1}{N}|\mathcal{H}^{(N)}_{X_1|W_1} \cap (\mathcal{H}^{(N)}_{S_U|Y_1W_1})^C \cap (\mathcal{B}^{(N)}_{S_U|Y_1W_1})^C| \\
\geq \frac{1}{N}|\mathcal{H}^{(N)}_{X_1|W_1} \cap (\mathcal{H}^{(N)}_{S_U|Y_1W_1})^C| - \frac{1}{N}|\mathcal{B}^{(N)}_{S_U|Y_1W_1}| \\
= \frac{1}{N}|\mathcal{H}^{(N)}_{X_1|W_1}| - \frac{1}{N}|\mathcal{H}^{(N)}_{S_U|Y_1W_1}| - \frac{1}{N}|\mathcal{B}^{(N)}_{S_U|Y_1W_1}|.
\]
Since
\[
\lim_{N \to \infty} \frac{1}{N} |\mathcal{H}^{(N)}_{S_{U_1}, Y_1W_1}| = \lim_{N \to \infty} \frac{1}{N} \sum_{j \in S_{U_1}} H(S^j|Y_1^{1:N}, W_1^{1:N}, S_1^{1:j-1})
\]
\[
= \lim_{N \to \infty} \left( \frac{1}{N} H(S^{1:2N}|Y_1^{1:N}, W_1^{1:N}) - \frac{1}{N} \sum_{j \in S_{U_2}} H(S^j|Y_1^{1:N}, W_1^{1:N}, S_1^{1:j-1}) \right)
\]
\[
= \lim_{N \to \infty} \frac{1}{N} H(S^{1:2N}|Y_1^{1:N}) + \lim_{N \to \infty} \frac{1}{N} H(W_1^{1:N}|Y_1^{1:N}, S_1^{1:2N}) - H(W_1|Y_1)
\]
\[
- \lim_{N \to \infty} \frac{1}{N} \sum_{j \in S_{U_2}} H(S^j|Y_1^{1:N}, W_1^{1:N}, S_1^{1:j-1})
\]
\[
= H(X_1W_2|Y_1) - H(W_1|Y_1) - \lim_{N \to \infty} \frac{1}{N} \sum_{j \in S_{U_2}} H(S^j|Y_1^{1:N}, S_1^{1:j-1}) \quad (115)
\]
\[
= H(X_1W_2|Y_1) - H(W_1|Y_1) - (H(W_2) - I(X_1W_2; Y_1) + \bar{P}_1(1)) \quad (116)
\]
\[
= H(X_1) - H(W_1|Y_1) - \bar{P}_1(1),
\]
and
\[
\lim_{N \to \infty} \frac{1}{N} |\mathcal{H}^{(N)}_{X_1|W_1}| = H(X_1|W_1), \quad \lim_{N \to \infty} \frac{1}{N} |\mathcal{B}^{(N)}_{S_{U_1}, Y_1W_1}| = 0,
\]
we have
\[
\lim_{N \to \infty} \frac{1}{N} |\mathcal{I}_{1p}| = H(X_1|W_1) - (H(X_1) - H(W_1|Y_1) - \bar{P}_1(1))
\]
\[
= H(X_1W_1) - H(W_1) - H(X_1) + H(W_1|Y_1) + \bar{P}_1(1)
\]
\[
= \bar{P}_1(1) - I(W_1; Y_1), \quad (117)
\]
where (115) holds because \(H(W_1^{1:N}|Y_1^{1:N}, S_1^{1:2N}) = 0\), (116) holds because
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{j \in S_{U_2}} H(S^j|Y_1^{1:N}, S_1^{1:j-1}) = H(W_2) - \bar{P}_1(2)
\]
\[
= H(W_2) - (I(X_1W_2; Y_1) - \bar{P}_1(1)),
\]
and (117) holds because \(H(X_1W_1) = H(X_1)\).
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