A CYCLAGE POSET STRUCTURE
FOR LITTLEWOOD-RICHARDSON TABLEAUX

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Abstract. A graded poset structure is defined for the sets of Littlewood-Richardson (LR) tableaux that count the multiplicity of an irreducible $gl(n)$-module in the tensor product of irreducible $gl(n)$-modules corresponding to rectangular partitions. This poset generalizes the cyclage poset on column-strict tableaux defined by Lascoux and Schützenberger, and its grading function generalizes the charge statistic. It is shown that the polynomials obtained by enumerating LR tableaux by shape and the generalized charge, are none other than the Poincaré polynomials of isotypic components of the certain modules supported in the closure of a nilpotent conjugacy class. In particular explicit tableau formulas are obtained for the special cases of these Poincaré polynomials given by Kostka-Foulkes polynomials, the coefficient polynomials of two-column Macdonald-Kostka polynomials, and the Poincaré polynomials of isotypic components of coordinate rings of closures of conjugacy classes of nilpotent matrices. These $q$-analogs conjecturally coincide with $q$-analogs of the number of certain sets of rigged configurations and the $q$-analogs of LR coefficients defined by the spin-weight generating functions of ribbon tableaux of Lascoux, Leclerc, and Thibon.

1. Introduction

In a series of papers [8] [11] [12] Lascoux and Schützenberger developed the deep theory of the cyclage poset on column-strict tableaux, to give combinatorial explanations of properties of the Kostka-Foulkes polynomials, which are $q$-analogues of weight multiplicities in type $A$. In particular they assign to each tableau a nonnegative integer called the charge and show that the Kostka-Foulkes polynomials are the $q$-enumeration of column-strict tableaux by the charge [1] [11].

These polynomials occur as instances of Poincaré polynomials $K_{\lambda;R}(q)$ of isotypic components of coordinate rings of closures of nilpotent conjugacy classes of matrices twisted by line bundles. The polynomials $K_{\lambda;R}(q)$ are $q$-analogues of the tensor product multiplicities given by Littlewood-Richardson coefficients of the form

$$c_{\lambda}^{R} = \langle s_{\lambda}, s_{R_1} s_{R_2} \cdots s_{R_t} \rangle$$

(1.1)

where $s_{\lambda}$ is the Schur polynomial and $R = (R_1, R_2, \ldots, R_t)$ is a sequence of rectangular partitions. This multiplicity has a well-known description as the cardinality of a set of Littlewood-Richardson tableaux $\text{LRT}(\lambda; R)$. Many properties of Kostka-Foulkes polynomials have generalizations for the Poincaré polynomials $K_{\lambda;R}(q)$ [20]. In [1] [20] many combinatorial conjectures were proposed to explain these properties.

The key construction of this paper is a direct definition of a poset structure on Littlewood-Richardson tableaux that generalizes the cyclage poset on column-strict tableaux. This new poset is graded by a function $\text{charge}_{R}$ that generalizes the
charge. The most important consequence of this construction is a proof that the Poincaré polynomial \( K_{\lambda; R}(q) \) is given by the \( q \)-enumeration of the set \( \text{LRT}(\lambda; R) \) of LR tableaux by charge \( R \). Other consequences include proofs of monotonicity and symmetry properties of the polynomials \( K_{\lambda; R}(q) \) \[7\]; these proofs will appear elsewhere.

Conjecturally the polynomials \( K_{\lambda; R}(q) \) coincide with the \( q \)-enumeration of rigged configurations \[7\]. A. N. Kirillov has given a bijection from LR tableaux to rigged configurations, but the conjecture that it preserves the appropriate statistics, remains open.

The polynomials \( K_{\lambda; R}(q) \) also conjecturally coincide with a subfamily of the \( q \)-analogues of LR coefficients arising from the spin-weight generating function over ribbon tableaux \[9\]. At this time, little is known about this larger family of polynomials.

While preparing this paper the author discovered that the polynomials \( K_{\lambda; R}(q) \) appear as multiplicities of Schur functions in certain Demazure characters of affine type \( A \), generalizing a level-one formula of \[4\] to arbitrary level. This and the many connections between the affine type \( A \) crystal theory and the tableau combinatorics in this paper, will appear elsewhere.

The first section reviews the definitions of the Poincaré polynomials, the LR tableaux, the action of the symmetric group on LR tableaux by the generalized automorphisms of conjugation, the generalized charge statistic, and the main result. The main construction, the \( R \)-cocyclage poset structure on LR tableaux, is introduced in Section 3, where the important features of this graded poset and its grading function charge \( R \) are described. Section 4 sketches the proof of the main theorem. The proofs appear in the last two sections.

Thanks to M. Okado for pointing out the reference \[15\] which has considerable overlap with this paper and \[19\].

2. Definitions

This section reviews the definitions necessary to state the main result.

2.1. The Poincaré polynomials \( K_{\lambda; R} \). We recall the definition of the Poincaré polynomials \[2\].

Let \( \eta = (\eta_1, \eta_2, \ldots, \eta_t) \) be a sequence of positive integers summing to \( n \) and \( \gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}^n \) an integral weight. Given the pair \( (\eta, \gamma) \) we define a generating function \( H_{\eta, \gamma}(x; q) \) for the polynomials \( K_{\lambda; R}(q) \). This generating function is the graded Euler characteristic character of the coordinate ring of a nilpotent conjugacy class closure, twisted by a line bundle.

Let \( \text{Roots}_\eta \) be the set of positions in an \( n \times n \) matrix above the block diagonal given by the parts of \( \eta \), that is,

\[
\text{Roots}_\eta = \{(i, j) \mid 1 \leq i \leq \eta_1 + \cdots + \eta_r \text{ and } j < \eta_r \text{ for some } 1 \leq r < t\}.
\]

Define

\[
B_{\eta, \gamma}(x; q) = x^\gamma \prod_{(i, j) \in \text{Roots}_\eta} (1 - qx_i/x_j)^{-1},
\]

which is a formal power series in \( q \) whose coefficients are Laurent polynomials in the set of variables \( x = (x_1, x_2, \ldots, x_n) \). Let \( W \) be the symmetric group that permutes the \( x \) variables, \( J = \sum_{w \in W} (-1)^w w \) the antisymmetrization operator,
and \( \rho = (n - 1, n - 2, \ldots, 1, 0) \). Then define the formal power series \( H_{\eta, \gamma}(x; q) \) and \( K_{\lambda, \eta, \gamma}(q) \) by

\[
H_{\eta, \gamma}(x; q) = J(x^\rho)^{-1} J(x^\rho B_{\eta, \gamma}(x; q))
\]

\[
H_{\eta, \gamma}(x; q) = \sum_{\lambda} K_{\lambda, \eta, \gamma}(q) s_\lambda(x)
\]

where \( \lambda \) runs over the dominant integral weights and \( s_\lambda(x) = J(x^\rho)^{-1} J(x^\rho x^\lambda) \) is the irreducible character of highest weight \( \lambda \). It can be shown that the formal power series \( K_{\lambda, \eta, \gamma}(q) \) are polynomials with integer coefficients. These can be calculated explicitly using the \( q \)-Kostant partition formula that follows immediately from this definition.

For the rest of the paper we shall only be concerned with the following special case. Let \( R = (R_1, R_2, \ldots, R_t) \) be a sequence of rectangular partitions where \( R_i \) has \( \eta_i \) rows and \( \mu_i \) columns. Let \( n = \sum_i \eta_i \) and

\[
\gamma(R) = (\mu_1^{\eta_1}, \mu_2^{\eta_2}, \ldots, \mu_t^{\eta_t}),
\]

the weight obtained by juxtaposing the parts of the rectangular partitions \( R_1 \) through \( R_t \). Say that \( R \) is dominant if \( \gamma(R) \) is, that is, the number of columns of the rectangles in \( R \) weakly decrease. Let

\[
K_{\lambda; R}(q) = K_{\lambda, \eta, \gamma(R)}(q).
\]

2.2. Notation for rectangle sequences. Let us fix notation associated with the sequence of rectangular partitions \( R = (R_1, R_2, \ldots, R_t) \). Let \( R_i \) have \( \eta_i \) rows and \( \mu_i \) columns for \( 1 \leq i \leq t \) and let

\[
N = \sum_{i=1}^t |R_i| = \sum_{i=1}^t \mu_i \eta_i
\]

\[
n = \sum_{i=1}^t \eta_i.
\]

Let \( A_1 \) be the first \( \eta_1 \) elements of \( [n] = \{1, 2, \ldots, n\} \), \( A_2 \) the next \( \eta_2 \) elements, etc.

Let \( Y_i \) be the unique column-strict tableau of shape \( R_i \) and content \( R_i \) in the alphabet \( A_i \), meaning that the \( j \)-th row of \( Y_i \) consists of \( \mu_i \) copies of the \( j \)-th largest letter (namely \( \eta_1 + \eta_2 + \cdots + \eta_{i-1} + j \)) of \( A_i \).

Example 1. Let \( R = ((3, 3), (2, 2), (1, 1, 1)) \), so that \( \mu = (3, 2, 1), \eta = (2, 2, 3), n = 7, N = 13, A_1 = \{1, 2\}, A_2 = \{3, 4\}, A_3 = \{5, 7\} \), and

\[
Y_1 = \begin{array}{ccc} 1 & 1 & 1 \\ 2 & 2 & 2 \end{array}, \quad Y_2 = \begin{array}{cc} 3 & 3 \\ 4 & 4 \end{array}, \quad Y_3 = \begin{array}{c} 5 \\ 6 \\ 7 \end{array}.
\]

Remark 2. Suppose \( \eta_i = 1 \) for all \( i \) and \( \mu \) is a partition of \( N \); this shall be referred to as the Kostka case. Then \( A_i = \{i\} \) for all \( i \) and \( Y_i \) is the one-row tableau consisting of \( \mu_i \) copies of the letter \( i \).
2.3. \textit{R-LR words.} Let \( w \in \mathbb{[n]}^* \) be a word in the alphabet \([n]\). For \( B \subseteq \mathbb{[n]} \), denote by \( w|_B \) the restriction of \( w \) to the subalphabet \( B \), obtained from \( w \) by erasing all letters not in \( B \). The Knuth equivalence on words will be denoted \( \sim_K w \). For a (possibly skew) tableau \( T \), let \( \text{word}(T) \) denote the row-reading word of \( T \), given by \( \cdots u^2 u^1 \) where \( u^i \) is the \( i \)-th row of \( T \) read from left to right. Often we write \( T \) instead of \( \text{word}(T) \) when only the Knuth equivalence class of \( T \) is needed.

Say that \( w \) is \textit{R-LR} (short for \textit{R-Littlewood-Richardson}) if

\[
\text{word}(P(w)) \sim_K w
\]

for all \( i \). Denote by \( W(R) \) the set of \textit{R-LR} words. In the Kostka case, \( w \) is \textit{R-LR} if and only if it has content \( \mu \).

\begin{example}
In the running example, the word \( w = 7442632512131 \) is \textit{R-LR} since \( w|_{A_1} = 221211 \sim_K 221211, w|_{A_2} = 4433 \) and \( w|_{A_3} = 765 \).
\end{example}

\begin{remark}
\( W(R) \) is the set of shuffles of the Knuth classes of \{\text{word}(Y_i)\}. Since each \( \text{word}(Y_i) \) consists of letters in the alphabet \( A_i \) and the intervals \( A_i \) are non-overlapping, it follows that \( W(R) \) is closed under Knuth equivalence.

Let \( 1 \leq i \leq j \leq t \). Write \( R_{i,j} = (R_i, R_{i+1}, \ldots, R_j) \) and \( B := \cup_{r=i}^j A_r \). As an immediate consequence of the definitions, if \( w \in W(R) \) then \( w|_B \in W(R_{i,j}) \).
\end{remark}

2.4. \textit{Littlewood-Richardson tableaux.} Let \( \text{LRT}(R) \) be the set of column-strict tableaux \( T \) of arbitrary partition shape such that \( \text{word}(T) \) is \textit{R-LR}. For a skew shape \( D \), write \( \text{LRT}(D; R) \) for the set of column-strict tableaux \( T \) of shape \( D \) such that \( \text{word}(T) \) is \textit{R-LR}.

\begin{remark}
For a word \( w \), denote by \( P(w) \) its Schensted \( P \) tableau, the unique column-strict tableau of partition shape such that \( \text{word}(P(w)) \sim_K w \). By Remark \( \Box \) a word \( w \) is \textit{R-LR} if and only if \( P(w) \in \text{LRT}(R) \).

In the Kostka case, \( \text{LRT}(R) \) is the set of column-strict tableaux of content \( \mu \) and arbitrary partition shape, and \( \text{LRT}(D; R) \) is the set of column-strict tableaux of shape \( D \) and content \( \mu \).
\end{remark}

\begin{proposition}
The multiplicity

\[
c^\lambda_R := \langle s_\lambda, s_{R_1} s_{R_2} \cdots s_{R_t} \rangle
\]

is given by the cardinality of the set \( \text{LRT}(\lambda; R) \).
\end{proposition}

This is an easy consequence of the well-known rule of Littlewood and Richardson \( \Box \) (see section 3.1).

2.5. \textit{Generalization of automorphisms of conjugation.} In the Kostka case, there is a content-permuting action of the symmetric group \( S_t \) on words in the alphabet \( \mathbb{[t]} \) induced by the automorphisms of conjugation \( \Box \) (see section 3.1). We recall from \( \Box \) a construction that generalizes these bijections for \textit{R-LR} words and LR tableaux. It should be mentioned that in \( \Box \) many of the properties of the generalized automorphisms of conjugation were conjectural but are proven here using the direct definition of cyclage on LR tableaux.

The symmetric group \( S_t \) has an obvious action on the set of all sequences of rectangles of length \( t \). For any two sequences \( R \) and \( R' \) in the same \( S_t \)-orbit, Proposition \( \Box \) implies that

\[
|\text{LRT}(\lambda; R)| = |\text{LRT}(\lambda; R')|.
\]
We wish to define a family of bijections
\[ \tau_R^{R'} : \text{LRT}(\lambda; R) \to \text{LRT}(\lambda; R') \]
that is functorial in the sense that \( \tau_R^{R'} \) is the identity on \( \text{LRT}(\lambda; R) \) and
\[ \tau_R^{R'} = \tau_R^{R''} \circ \tau_R^{R'} \]
for \( R, R', \) and \( R'' \) in the same \( S_t \) orbit. Let \( R \) be an \( S_t \) orbit of a sequence of \( t \) rectangles. This functoriality then makes it possible to define an action of the permutation \( \sigma \in S_t \) on the union \( \bigcup_{R \in \mathcal{R}} \text{LRT}(\lambda; R) \) such that the action of the permutation \( \sigma \in S_t \) is defined by
\[ \sigma|_{\text{LRT}(\lambda; R)} = \tau_R^{\sigma} \]
for every \( R \in \mathcal{R} \). This action of \( S_t \) generalizes the automorphisms of conjugation.

It is helpful to describe the above situation rather carefully, since each sequence of rectangles \( R \in \mathcal{R} \) has a different set of subalphabets \( \{A_1, \ldots, A_t\} \) of \([n]\). When the context is clear we will abuse notation by writing \( \sigma \) instead of \( \tau_R^{\sigma} \).

Recall that the (row-insertion) Robinson-Schensted (RS) correspondence assigns to each word \( w = w_1 w_2 \cdots w_N \) (where \( w_i \) is a letter) the pair of tableaux \( (P(w), Q(w)) \) of the same partition shape, where \( Q(w) \) is the unique standard tableau such that the shape of the tableaux \( Q(w)[i] \) and \( P(w_1 \cdots w_i) \) coincide for all \( 0 \leq i \leq N \).

As with the automorphisms of conjugation, the action of \( S_t \) is defined on words. By definition, the RS correspondence restricts to a bijection
\[ W(R) \to \bigcup_{\lambda} \text{LRT}(\lambda; R) \times \text{ST}(\lambda) \]
sending \( w \) to the tableau pair \( (P(w), Q(w)) \), where \( \text{ST}(\lambda) \) is the set of standard tableaux of shape \( \lambda \). We wish to define a functorial family of bijections \( \tau_R^{R'} : W(R) \to W(R') \) (for \( R \) and \( R' \) in \( \mathcal{R} \)) with the property that the following diagram commutes:
\[ \begin{array}{ccc}
W(R) & \xrightarrow{\text{RS}} & \bigcup_{\lambda} \text{LRT}(\lambda; R) \times \text{ST}(\lambda) \\
\tau_R^{R'} & \downarrow & \downarrow_{\bigcup_{\lambda} \tau_R^{R'} \times \text{id}_{\text{ST}(\lambda)}} \\
W(R') & \xrightarrow{\text{RS}} & \bigcup_{\lambda} \text{LRT}(\lambda; R') \times \text{ST}(\lambda)
\end{array} \]
that is, \( P(\tau_R^{R'} w) = \tau_R^{R'} P(w) \) and \( Q(\tau_R^{R'} w) = Q(w) \) for all \( w \in W(R) \). Define the action of \( S_t \) on \( \bigcup_{R \in \mathcal{R}} W(R) \) by
\[ \sigma|_{W(R)} = \tau_R^{\sigma} \]
The commutation of the above diagram can then be rephrased as
\[ P(\sigma w) = \sigma P(w) \]
\[ Q(\sigma w) = Q(w) \]
for all \( w \in W(R) \) and \( \sigma \in S_t \).

As in the definition of the automorphisms of conjugation, we define the action of \( S_t \) in terms of the action of the adjacent transpositions. The simple reflections in the symmetric group \( S_t \) shall be denoted \( \tau_p \) for \( 1 \leq p \leq t - 1 \). The corresponding bijection on LR words and tableaux shall also be denoted \( \tau_p \). When we have occasion to use the original automorphisms of conjugation acting in the symmetric
group $S_n$, they will be denoted $s_r$ for $1 \leq r \leq n - 1$. We call the bijections that give the action of $\tau_p$ “rectangle-switching bijections”.

The definition of $\tau_p$ proceeds in a sequence of cases. Let

$$R' = \tau_p R = (R_1, \ldots, R_{p-1}, R_p, R_{p+1}, \ldots, R_t)$$

with rectangles $(R'_1, R'_2, \ldots, R'_t)$, subalphabets $A'_1$, $A'_2$, etc., and tableaux $Y'_1$, $Y'_2$, etc.

(1) Suppose $t = 2$ and $p = 1$. Then define $\tau_1 : \text{LRT}(\lambda; R) \to \text{LRT}(\lambda; R')$ to be the unique map between the two sets. It is necessarily a bijection since both the domain and range are either both empty or singleton sets (see Proposition 33).

(2) Suppose $t = 2$ and $p = 1$ with the notation of the previous case. Define the map $\tau_1 : W(R) \to W(R')$ by $P(\tau_1 w) = \tau_1 P(w)$ and $Q(\tau_1 w) = Q(w)$, which is well-defined by the bijectivity of the RS correspondence and case (1).

(3) Let $t$ be arbitrary and $1 \leq p \leq t - 1$. Let $B = A_p \cup A_{p+1}$. Given $w \in W(R)$, let $\tau_p w$ be the word obtained from $w$ by replacing the letters of the subword $w|_B$ (in the same positions) by those of $\tau_p(w|_B)$, which is defined by case (2), since $w|_B \in W((R_p, R_{p+1}))$ in the alphabet $B$ by Remark 4.

(4) With the same hypotheses as case (3), let $T$ be a (possibly skew) tableau such that $\text{word}(T) \in W(R)$. Define $\tau_p T$ to be the unique tableau of the same shape as $T$ such that

$$\text{word}(\tau_p T) = \tau_p \text{word}(T).$$

This tableau is column-strict (Theorem 9 (A1)).

**Example 7.** With $R$ as before, consider the tableau $S \in \text{LRT}(R)$ and some of its images under operators $\tau_p$.

$$S = \begin{array}{cccc}
1 & 1 & 1 & 3 \\
2 & 2 & 2 & 4 \\
4 & 6 & 7 \\
\end{array}$$

$$\tau_1 S = \begin{array}{cccc}
1 & 1 & 3 & 3 \\
2 & 2 & 4 & 4 \\
4 & 6 & 7 \\
\end{array}$$

$$\tau_2 S = \begin{array}{cccc}
1 & 1 & 1 & 6 \\
2 & 2 & 2 & 6 \\
5 & 7 & 7 \\
\end{array}$$

$$\tau_2 \tau_1 S = \begin{array}{cccc}
1 & 1 & 3 & 6 \\
2 & 2 & 4 & 7 \\
5 & 7 & 7 \\
\end{array}$$

$$\tau_1 \tau_2 S = \begin{array}{cccc}
1 & 4 & 4 & 6 \\
2 & 5 & 5 & 7 \\
3 & 7 & 7 \\
\end{array}$$

$$\tau_1 \tau_2 \tau_1 S = \begin{array}{cccc}
1 & 4 & 4 & 6 \\
2 & 5 & 5 & 7 \\
3 & 7 & 7 \\
\end{array}$$

**Remark 8.** In the Kostka case the bijection $\tau_p$ is precisely the automorphism of conjugation $s_p$.

Say that a word $w$ fits the skew shape $D$ if there is a column-strict tableau $T$ of shape $D$ such that $w = \text{word}(T)$.

**Theorem 9.** The maps $\tau_p$ are well-defined bijections

$$W(R) \to W(\tau_p R)$$

$$\text{LRT}(\lambda; R) \to \text{LRT}(\lambda; \tau_p R)$$
If $D$ is a skew shape, then $w \in W(R)$ fits $D$ if and only if $\tau_p w$ fits $D$.

(A2) If $\nu =_K w$ are in $W(R)$, then $\tau_p \nu =_K \tau_p w$ in $W(\tau_p R)$.

(A3) $P(\tau_p w) = \tau_p P(w)$ for all $w \in W(R)$.

(A4) $Q(\tau_p w) = Q(w)$ for all $w \in W(R)$.

(A5) The bijections $\tau_p$ (1 ≤ $p$ ≤ $t$ − 1) satisfy the Moore-Coxeter relations for $S_t$, defining an action of the symmetric group on LR tableaux and words.

Suppose $\sigma \in S_t$ stabilizes the subintervals $[1, i−1]$, $I = [i, j]$, and $[j + 1, t]$. Let $w \in W(R)$ and let $\sigma w$ denote the action of $\sigma \in S_t$ on $w$ given in (A5). Then the positions of the letters of $J = \bigcup_{i \in I} A_j$ in $w$ coincide with the positions of the letters of $J$ in $\sigma w$.

Suppose $R_p = R_{p+1}$. Then $\tau_p(w) = w$ for all $w \in W(R)$.

2.6. Statistic on LR tableaux. Lascoux, Leclerc, and Thibon gave a formula for the charge that averages a simpler statistic over the orbit of a word by the symmetric group acting by automorphisms of conjugation [14]. We recall from [14] a statistic $c_R$ on LR tableaux that generalizes the charge. In [14] even the well-definedness of $c_R$ was based on a conjecture (namely, Theorem 9 (A5)); here these conjectures regarding $c_R$ are proven.

First let $R = (R_1, R_2)$ and $w \in W(R)$. Then $P(w) \in \text{LRT}(R)$. Define

$$d_{R_1, R_2}(w) = d_{R_1, R_2}(P(w))$$

to be the number of cells of the shape of $P(w)$ that lie in columns strictly to the right of the max($\mu_1, \mu_2$)-th column. In the notation of Proposition 33, this is the size of the partition $\lambda_{nc}$ where $\lambda$ is the shape of $P(w)$.

Next, for arbitrary $R$, 1 ≤ $i$ ≤ $t$ − 1, and $w \in W(R)$, recall that by definition $w|_{A_i \cup A_{i+1}} \in W((R_i, R_{i+1})).$ Define $d_{i, R}(w) = d_{(R_i, R_{i+1})}(w|_{A_i \cup A_{i+1}})$.

Finally, define the statistic $c_R : W(R) \to \mathbb{N}$ as the following average over the symmetric group $S_t$:

$$(2.1) \quad c_R(w) = \frac{1}{t!} \sum_{\tau \in S_t} \sum_{i=1}^{t-1} (t - i) d_{i, \tau R}(\tau w).$$

This makes sense if $R$ has two or more rectangles. Let $c_R(w) = 0$ if $R$ has less than two rectangles. For $T \in \text{LRT}(R)$, define $c_R(T) = c_R(\text{word}(T))$.

Example 10. For the tableaux in the $S_3$-orbit of the tableau $S$ in the previous example, in order one has the following values for $d_1$ and $d_2$: (3, 1), (3, 1), (2, 1), (2, 1), (2, 2), (2, 2). So $c_R(T) = 1/6(7 + 7 + 5 + 5 + 6 + 6) = 6$.

2.7. Main result.

Theorem 11. Let $R$ be a dominant sequence of rectangles and $\lambda$ a partition. Then

$$(2.2) \quad K_{\lambda; R}(q) = \sum_{T \in \text{LRT}(\lambda; R)} q^{n(T)}.$$ 

The proof relies on the cyclage poset structure on LR tableaux introduced below.

3. The cyclage poset for LR tableaux

Section 3.1 reviews the definitions of the crystal operators for type $A_{n-1}$, including the action of the symmetric group $S_n$ given by the automorphisms of conjugation. This is necessary to define the action of the cyclic group $\mathbb{Z}/N\mathbb{Z}$ on $R$-LR
words in section 3.2, which in turn makes possible the definition in section 3.3 of the R-cocyclage relation on R-LR tableaux. Section 3.3 gives the main theorems on the structure of the R-cocyclage poset on R-LR tableaux and its grading function chargeR.

3.1. Crystal operators and automorphisms of conjugation. We recall the definitions of the crystal raising, lowering, and reflection operators e_r, f_r, and s_r associated with the crystal graph of type A_{n-1}. The combinatorial constructions as given here appear in [12] (see also [14]), and are equivalent to those coming from the computation of this crystal graph by Kashiwara and Nakashima [5].

Fix 1 ≤ r ≤ n - 1. For a word u, regard the letters r as right parentheses, the letters r+1 as left parentheses, and ignore other letters. Perform the usual matching of parentheses, leaving a subword of unpaired letters of the form r^a(r + 1)^b. Then the words e_r u, f_r u, and s_r u are defined by replacing this subword of u (in the same positions) by r^{a+1}(r + 1)^{b-1}, r^{a-1}(r + 1)^{b+1}, and r^b(r + 1)^a respectively, where e_r u and f_r u are defined only if b > 0 and a > 0 respectively. Say that two words are in the same r-string if one is obtained from the other by a power of f_r.

Theorem 12. [12] There is an action of S_n on the words in the alphabet [n], where the simple reflection s_r ∈ S_n acts by the above operator s_r on words.

The operators on words corresponding to permutations of S_n are called automorphisms of conjugation in [12].

3.2. Action of the cyclic group Z/NZ on W(R). In the Kostka case, the cyclic group Z/NZ acts on words of content µ by cyclic rotation of positions, leading to the definition of the cyclage poset on column-strict tableaux of content µ [12]. This simple action of Z/NZ is extended to the set of R-LR words.

Example 13. The naive action that merely rotates positions does not preserve the set of R-LR words. It is illustrative to consider the case R = R_1). Write a = µ_1 and Y = Y_1. Consider w = word(Y) and its right circular rotation v, given explicitly by

\[ w = n^a(n - 1)^a \cdots 2^a1^a \]
\[ v = 1 n^a(n - 1)^a \cdots 2^a1^{a-1} \]

Clearly v is not R-LR unless n = 1.

This right circular rotation must be modified to preserve R-LRness. Let w ∈ [n]^*. Write w = ux where x is a letter. Let w^R_0 be the longest permutation in the Young subgroup of S_n that stabilizes all the intervals A_i. Define

(3.1) \[ \chi_R(w) = (w^R_0 x) (w^R_0 u) \]

where w^R_0 acts by the automorphism of conjugation (see Remark 3).

In the Kostka case, the interval A_1 = {i} so w^R_0 is the identity permutation and \chi_R(w) is merely the right circular rotation of w.

Example 14. In the previous example, w^R_0 is the longest permutation in S_n, x = 1, xw^R_0 = n,

\[ u = n^a(n - 1)^a \cdots 2^a1^{a-1} \]
\[ uw^R_0 = (n-1)n^{a-1}(n-2)(n-1)^{a-1} \cdots 23^{a-1}12^{a-1}1^{a-1}. \]

Observe that \chi_R(w) ∈ W(R).
Proposition 15. Let $k$ be a nonnegative integer.
1. $w \in W(R)$ if and only if $\chi^k_R(w) \in W(R)$.
2. Let $w = uv \in W(R)$ with $v$ of length $k$, respectively. Then
   \[ \chi^k_R(uv) = (w_0^R v)(w_0^R u) \]
   where $w_0^R \in S_n$ acts by an automorphism of conjugation.

The following key result generalizes [12, Theorem 4.6].

Theorem 16. Let $R' = \tau_p R$. The following diagram commutes:
\[
\begin{array}{ccc}
W(R) & \xrightarrow{\chi_R} & W(R) \\
\downarrow{\tau_p} & & \downarrow{\tau_p} \\
W(R') & \xrightarrow{\chi_{R'}} & W(R')
\end{array}
\]

3.3. $R$-cocyclage. Let $S, T \in \text{LRT}(R)$. Let $\leq_R$ be the transitive closure of the relation $T <_R S$, which holds if there is a word $w = ux \in W(R)$ with $x$ a letter, such that $S = P(w)$, $T = P(\chi_R(w))$, and the cell given by the difference of the shapes of $P(w)$ and $P(u)$, is in a column strictly east of the $a$-th where $a = \max_i \mu_i$.

Remark 17. All covering relations $T <_R S$ can be realized as follows. Let $S \in \text{LRT}(R)$ and $s$ a corner cell of $S$ in a column strictly east of the $a$-th. Perform the reverse row insertion on $S$ starting at $s$, producing the tableau $U$ and ejecting the letter $x$. Then letting $u = \text{word}(U)$ and $w = ux$, one has $T <_R S$ where $T = P(\chi_R(w))$.

Example 18. Using the tableau $S$ from the running example and the corner cell $(2, 4)$, we have $w_0^R s = s_1 s_3 s_5 s_6 s_8$.

\[
S = \begin{array}{cccc}
1 & 1 & 1 & 3 \\
2 & 2 & 2 & 4 \\
4 & 6 & & \\
7 & & & \\
\end{array}
\quad U = \begin{array}{cccc}
1 & 1 & 1 & 3 \\
2 & 2 & 2 & \\
4 & 6 & & \\
7 & & & \\
\end{array}
\]

and $x = 3$. Now $w_0^R x = 4$ and

\[
w_0^R U = \begin{array}{cccc}
1 & 1 & 1 & 3 \\
2 & 2 & 2 & \\
4 & 6 & & \\
7 & & & \\
\end{array}
\quad T = \begin{array}{cccc}
1 & 1 & 1 & 3 \\
2 & 2 & 2 & \\
4 & 4 & 6 & \\
7 & & & \\
\end{array}
\]

Theorem 19. 1. $(\text{LRT}(R), \leq_R)$ is a graded poset.
2. An element of $\text{LRT}(R)$ is $\leq_R$-minimal if and only if it has exactly $a = \max_i \mu_i$ columns. In particular, if all of the rectangles in $R$ have the same number of columns then there is a unique minimal tableau.
3. Suppose $\mu_1 = a$ and $T \in \text{LRT}(R)$ is $\leq_R$-minimal. Then $T$ consists of the tableau $Y_1$ sitting atop a tableau $\widehat{T} \in \text{LRT}(\widehat{R})$ in the alphabet $[\eta_1 + 1, n]$ where $\widehat{R} = (R_2, R_3, \ldots, R_t)$.

Theorem 20. $\tau_p$ is an isomorphism of graded posets
\[(\text{LRT}(R), \leq_R) \rightarrow (\text{LRT}(\tau_p R), \leq_{\tau_p R})\]
Theorem 21. There is a unique function \( \text{charge}_R : \text{LRT}(R) \to \mathbb{N} \) such that:

(C1) If \( R = \emptyset \), then \( \text{charge}_R(\emptyset) = 0 \).
(C2) If \( T' <_R T \) is a covering relation for \( T \) and \( T' \) in \( \text{LRT}(R) \), then \( \text{charge}_R(T) = \text{charge}_R(T') + 1 \).
(C3) With the assumptions and notation of Theorem 19, \( \text{charge}_R(T) = \text{charge}_{\hat{R}}(\hat{T}) \).
(C4) \( \text{charge}_{\tau_p}(\tau_p T) = \text{charge}_R(T) \) for \( T \in \text{LRT}(R) \).

Theorem 22. The function \( c_R \) is constant on Knuth equivalence classes and satisfies conditions (C1) through (C4) of Theorem 21.

Proposition 23. Let \( w \in W(R) \). For each \( \tau \in S_t \), let \( f(\tau) \in [t] \) be such that the last letter of \( \tau w \) is in the alphabet \( A_{f(\tau)} \), and \( N \) the set of indices \( i \in [t] \) such that \( f(\tau_1\tau_2\cdots\tau_{i-1}) = 1 \). Then

\[
(c_R(w) = c_R(\chi_R(w)) + 1 - |N|.
\]

4. Proof of main theorem

The proof follows the suggested outline in [20]. The important difference is that here we work with LR tableaux, as opposed to the \( R \)-catabolizable tableaux used in [20]. As in the approach of Lascoux and Schützenberger for the Kostka-Foulkes polynomials, Theorem 11 is proven by showing that the right hand side of (2.2) satisfies (a form of) the generalized Morris recurrence for \( K_{\lambda;R}(q) \):}

\[
K_{\lambda;R}(q) = \sum_{w \in W} (-1)^w q^{\alpha(w) - |R_1|} \sum_{\nu} K_{\nu,\alpha(w)-R_1,\beta(w)} K_{\nu;\hat{R}}(q)
\]

where \( \alpha(w) \) and \( \beta(w) \) are the first \( m := n_1 \) and last \( n - m \) parts of the weight

\[
\xi(w) := w^{-1}(\lambda + \rho) - \rho
\]

and \( K_{\lambda,\alpha} \) is the Kostka number [14, I.6], the multiplicity of the weight \( \alpha \) in the \( \lambda \)-th irreducible \( gl(n) \)-module, or equivalently the number of column strict tableaux of shape \( \lambda \) and content \( \alpha \).

Let \( S \) be the set of triples \((w, T, U)\) where \( w \in W \), and \( T \) and \( U \) are column-strict tableaux of the same partition shape, where the word of \( T \) is \( R \)-LR in the alphabet \([m+1,n]\) and \( U \) has content \((\beta(w),\alpha(w) - R_1)\). The reordering of parts of the content of \( U \) is justified since the Kostka number is symmetric in its second index.

Define a sign and weight on \( S \) by

\[
\text{sign}(w, T, U) = (-1)^w
\]

\[
\text{weight}(w, T, U) = q^{\alpha(w) - |R_1| + \text{charge}_{\hat{R}}(T)}
\]

By induction the right hand side of (4.1) is given by

\[
\sum_{(w, T, U) \in S} \text{sign}(w, T, U) \text{weight}(w, T, U).
\]

We wish to write the sum (4.2) as the generating function of another signed weighted set that is more amenable to cancellation. Let \( T \) be the set of triples
The words \( u \) and \( \lambda \) of the sequence of words \( \xi \).

Define

\[
\text{sign}(w, P, Q) = (-1)^w \\
\text{weight}(w, P, Q) = q^{\text{charge}_n(P)}.
\]

We define a sign- and weight-preserving map \( \Phi : S \to T \) as follows.

The column insertion version of the Robinson-Schensted-Knuth correspondence (column RSK) sends a sequence of weakly increasing words \( (u^n, u^{n-1}, \ldots, u^1) \) to a pair of column-strict tableaux \((P, Q)\) where \( P = P(u^n, u^{n-1}, \ldots, u^1) \) is the usual Schensted \( P \)-tableau and \( Q \) is the column-strict tableau in the alphabet \([n]\) such that the shape of \( Q \) coincides with the shape of \( P(u^i u^{i-1} \cdots u^1) \) for all \( 0 \leq i \leq n \).

Given the triple \((w, T, U)\) \( \in S \), consider the sequence of words given by the inverse image of the tableau pair \((T, U)\) under column RSK. Here we use a non-standard indexing for the weakly increasing words, writing

\[
P(u^m u^{m-1} \cdots u^1 u^n u^{n-1} \cdots u^{m+1}) = T.
\]

Since \( U \) has content \( (\beta(w), \alpha(w) - R_1) \), this nonstandard indexing allows us to say that the length of \( u^i \) is the \( i \)-th part of the weight \( \xi(w) - (R_1, 0^{n-m}) = (\alpha(w) - R_1, \beta(w)) \).

Let \( y^i \) be the word of the same length as \( u^i \), such that

\[
\chi_{\hat{R}}^{\beta(w) - \lfloor R_1 \rfloor}(y^n y^{n-1} \cdots y^1) = u^m u^{m-1} \cdots u^1 u^n u^{n-1} \cdots u^{m+1}.
\]

This is well-defined since \( \chi_{\hat{R}} \) is invertible. By Proposition 17 and Theorem 9 for the automorphism of conjugation \( w^R \), it follows that each \( y^i \) is a weakly increasing word with letters greater than \( m \). Let

\[
(4.3) \quad y^i = \begin{cases} 
    i^\mu y^i & \text{if } 1 \leq i \leq m \\
    y^i & \text{if } m < i \leq n.
\end{cases}
\]

The word \( y^i \) is weakly increasing since \( y^i \) is. Note that \( |y^i| = \xi(w)_i \).

Finally, let \((P, Q)\) be the pair of tableaux given by the image under column RSK, of the sequence of words \( \{v^i\} \) so that \( P(v^n v^{n-1} \cdots v^1) = P \). Note that \( Q \) has content \( \xi(w) \). Then define \( \Phi(w, T, U) := (w, P, Q) \).

**Example 24.** Let \( n = 8 \), \( \eta = (2, 2, 2, 1, 1) \), and \( \mu = (3, 3, 3, 3, 3) \), so that \( m = 2 \) and

\[
R = ((3, 3), (3, 3), (3, 3), (3, 3)), \\
\hat{R} = ((3, 3), (3, 3), (3, 3), (3, 3)).
\]

Let \( w = 32154678 \) and \( \lambda = (6, 5, 5, 5, 2, 1, 0, 0) \) so that \( \xi(w) = (3, 5, 8, 1, 6, 1, 0, 0) \), \( \alpha(w) = (3, 5) \) and \( \beta(w) = (8, 1, 6, 1, 0, 0) \). Let \( T \) and \( U \) be given by

\[
T = \begin{pmatrix}
    3 & 3 & 3 & 5 & 5 & 7 & 7 & 1 \\
    4 & 4 & 6 & 8 & 8 & 2 & 3 & 3 \\
    5 & 6 & 8 & & & & & 3 \\
    6 & & & & & & & 4
\end{pmatrix}, \\
U = \begin{pmatrix}
    1 & 1 & 1 & 1 & 1 & 1 & 1 \\
    2 & 3 & 3 & 3 & 3 & 3 & 3 \\
    3 & 8 & 8 & & & & & 3 \\
    4 & & & & & & & 4
\end{pmatrix}
\]

The words \( u^i \) are given by \( u^2 = 66 \), \( u^1 = u^8 = u^7 = \emptyset \), \( u^6 = 8 \), \( u^5 = 445688 \), \( u^4 = 4 \), \( u^3 = 33355777 \). Since

\[
\chi_{\hat{R}}^{-2}(66//8/445688/4/33355777) = //8/446688/4/33356777/55/,
\]
we have $v^1 = 111$, $v^2 = 22255$, $v^3 = 33356777$, $v^4 = 4$, $v^5 = 446688$, $v^6 = 8$, and $v^7 = v^8 = \emptyset$.

The tableaux $P$ and $Q$ are given by

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 5 & 5 & 5 & 7 & 7 \\
2 & 2 & 2 & 6 & 6 & 7 \\
3 & 3 & 3 & 8 & 8 \\
4 & 4 & 4 & 8 \\
\end{array}
\quad
\begin{array}{cccccccc}
1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 \\
2 & 2 & 2 & 3 & 3 & 5 \\
3 & 3 & 3 & 5 & 5 \\
4 & 5 & 5 \\
\end{array}
\]

Lemma 25. The map $\Phi: S \rightarrow T$ is a sign- and weight-preserving injection.

By Lemma 25 and (4.2), the right hand side of (4.1) is equal to

\[
\sum_{(w, P, Q) \in \Phi(S)} \text{sign}(w, P, Q) \text{weight}(w, P, Q).
\]

We define a sign-reversing, weight-preserving involution $\theta$ on the set $T$. Let $(w, P, Q) \in T$.

1. If $Q$ is lattice (see section 5.1), it follows that $Q$ has partition content, $w$ is the identity, and the content of $Q$ is $\lambda$. In this case define $\theta(w, P, Q) = (w, P, Q)$.

2. If $Q$ is not lattice, let $r+1$ be the rightmost letter in word($Q$) that violates the lattice property. Define $\theta(w, P, Q) = (w', P', Q')$ where $w' = ws_r$, $P' = P$, and $Q' = s_r e_r Q$.

$\theta$ is an involution on all of $T$, due to the following easy lemma.

Lemma 26. Consider the following map $\Psi$ on the set of non-lattice words. Suppose the word $u$ is not lattice. Let $r+1$ be the rightmost letter where latticeness fails, and let $\Psi(u) = s_r e_r u$. Then $\Psi$ is an involution.

Example 27. In computing $\theta'(w, P, Q) = (w', P', Q')$, the first violation of latticeness in the row-reading word of $Q$ occurs at the cell $(1,8)$ so $r = 2$. We have $w' = 31254678$, $P' = P$ and

\[
\begin{array}{cccc}
1 & 1 & 1 & 2 \\
2 & 2 & 2 & 3 \\
3 & 3 & 3 & 5 \\
4 & 5 & 5 \\
\end{array}
\quad
\begin{array}{cccc}
2 & 2 & 3 & 3 \\
5 & 5 \\
\end{array}
\]

Lemma 28. $\theta$ is a sign-reversing, weight-preserving involution on the subset $\Phi(S)$ of $T$.

By definition the fixed points of the restriction of $\theta$ to $\Phi(S)$ are the triples $(w, P, Q)$ where $w$ is the identity, $Q = \text{key}(\lambda)$ (see section 5.1), and $P \in \text{LRT}(\lambda; R)$. Cancelling down from (4.4) using $\theta$, we have

\[
K_{\lambda; R}(q) = \text{right hand side of (4.4)}
\]

\[
= \sum_{P \in \text{LRT}(\lambda; R)} q^{\text{charge}_R(P)}
\]

proving Theorem 11.
5. Proofs

5.1. $R$-LR property and proof of Proposition 3. Say that a word is lattice if the content of every final subword is a partition.

More generally, if $\mu$ is a partition, say that a word is $\mu$-lattice if, for every final subword, the sum of $\mu$ and its content, is a partition. For the skew shape $\lambda/\mu$ and the partitions $\alpha$ and $\beta$, let $\text{LRT}(\lambda/\mu; \alpha, \beta)$ be the set of column-strict tableaux of shape $\lambda/\mu$ and content $\alpha - \beta$ whose row-reading words are $\beta$-lattice.

For every composition $\alpha$ (sequence of nonnegative integers, almost zero) let $\alpha^+$ be the partition obtained by sorting $\alpha$ into weakly decreasing order. There is a unique column-strict tableau of shape $\alpha^+$ and content $\alpha$, called the key tableau of content $\alpha$, denoted $\text{key}(\alpha)$. Explicitly, the $j$-th column of $\text{key}(\alpha)$ consists of the letters $i$ such that $\alpha_i \geq j$. When $\alpha$ is a partition $\text{key}(\alpha)$ is called a Yamanouchi tableau.

The following result has a straightforward proof.

**Proposition 29.**
1. A word is $\mu$-lattice if and only if the number of $r$-unpaired letters $r + 1$ is at most $\mu_r - \mu_{r+1}$ for all $r$.
2. The property of $\mu$-latticeness is preserved under Knuth equivalence.
3. A word $u$ is lattice of content $\lambda$ if and only if $u \sim_K \text{key}(\lambda)$.

Write $\text{LRT}(D; \alpha; \beta)$ for the set of column-strict tableaux of shape $D$, content $\alpha$, whose row-reading words are $\beta$-lattice.

**Theorem 30.** [13] (Littlewood-Richardson rule) The coefficient
\[ c_{\lambda/\mu}^\nu := \langle s_{\lambda/\mu}, s_{\nu} \rangle \]

is equal to the cardinality of the set $\text{LRT}(\lambda/\mu; \nu; ())$.

The LR rule has an immediate corollary that gives the skew-skew LR coefficient.

Given the skew shapes $D$ and $E$, let $D \otimes E$ denote any skew shape given by the union of a translate of $D$ and a translate of $E$ such that every cell of $D$ is strictly south and strictly west of every cell of $E$. Clearly $s_{D \otimes E} = s_D s_E$ for any skew shapes $D$ and $E$.

**Corollary 31.** The coefficient
\[ c_{\alpha/\beta}^{\lambda/\mu} := \langle s_{\lambda/\mu}, s_{\alpha/\beta} \rangle \]

is equal to the cardinality of the set $\text{LRT}(\lambda/\mu; \alpha, \beta)$.

**Proof.** Applying the LR rule to the skew shape $\lambda/\mu \otimes \beta$ and the partition $\alpha$, one has
\[
\langle s_{\lambda/\mu}, s_{\alpha/\beta} \rangle = \langle s_{\lambda/\mu} s_{\beta}, s_{\alpha} \rangle = \langle s_{\lambda/\mu} \otimes \beta, s_{\alpha} \rangle = |\text{LRT}(\lambda/\mu \otimes \beta; \alpha, ())|
\]

Let $T' \in \text{LRT}(\lambda/\mu \otimes \beta; \alpha, ())$. Write $T' = T \otimes U$ where $T$ and $U$ are column-strict tableaux of shape $\lambda/\mu$ and $\beta$ respectively. Clearly $\text{word}(T') = \text{word}(T) \text{word}(U)$.

By the definition of latticeness, any final subword of $\text{word}(T')$ must itself be lattice. In particular $\text{word}(U)$ is lattice; it is also the row-reading word of the column-strict tableau $U$ of shape $\beta$. It follows from Proposition 29 that $U = \text{key}(\beta)$. This given, a direct translation of the definitions shows that there is a bijection
\[
\text{LRT}(\lambda/\mu; \alpha, \beta) \to \text{LRT}(\lambda/\mu \otimes \beta; \alpha, ())
\]
given by $T \mapsto T \otimes \text{key}(\beta)$. \hfill \Box

Proof of Proposition 33

Proof. The skew shape $R_i \otimes R_{i-1} \otimes \cdots \otimes R_1$ can be explicitly written in the form $\alpha/\beta$, where

$$\alpha = ((\mu_1 + \cdots + \mu_i)^{n_i}, (\mu_2 + \cdots + \mu_i)^{n_i}, \ldots, \mu_1^{n_i})$$
$$\beta = ((\mu_2 + \cdots + \mu_i)^{n_i}, (\mu_3 + \cdots + \mu_i)^{n_i}, \ldots, \mu_1^{n_i-1}, 0^{n_i}).$$

Note that the row indices corresponding to the translate of $R_i$ are given by the subinterval $A_i$ defined in section 2.2. By Corollary 31.

$c_R^\lambda = \{s_\lambda, s_{R_1} s_{R_2} \cdots s_{R_i}\} = \{s_\lambda, s_{R_1} \otimes R_{i-1} \otimes \cdots \otimes R_1\} = \{s_\lambda, s_{\alpha/\beta}\} = |\text{LRT}(\lambda; \alpha, \beta)|.$

The $\beta$-lattice condition is vacuous for indices $r$ of the form $\eta_1 + \eta_2 + \cdots + \eta_i$, since for such $r$, $\beta_r - \beta_{r+1} = \mu_{r+1}$, which is the total number of letters $r+1$ in a word of content $\alpha - \beta$. For $r$ not of this form, say $\eta_1 + \cdots + \eta_{i-1} + 1 \leq r < \eta_1 + \cdots + \eta_i$, we have $\beta_r - \beta_{r+1} = 0$. From Proposition 29 and the definitions, by restricting to the subalphabets $A_i$, a word is $\beta$-lattice of content $\alpha - \beta$ if and only if it is $R$-LR. \hfill \Box

5.2. The one-rectangle case. For later proofs it is necessary to conduct detailed analyses of the cases where $R$ consists of one or two rectangles.

Suppose $R = (R_1)$. Then $n = \eta_1$. Write $a = \mu_1$ and $Y = Y_1 = \text{key}((a^n))$.

The following result is easy to show by direct computation and Proposition 29.

Proposition 32. Suppose $u$ and $v$ words of content $\alpha$ and $\beta$ respectively. The following are equivalent.

1. $vu \sim_K Y$.
2. $\alpha$ is a partition, $\alpha + \beta = (a^n)$, $u \sim_K \text{key}(\alpha)$ and $v \sim_K \text{key}(\beta)$.

5.3. The two rectangle case. An important feature of this case is that the set of tableaux $\text{LRT}(\lambda; (R_1, R_2))$ is either empty or a singleton. If it is the latter it can be described explicitly. In the Kostka case these are column-strict tableaux of partition shape with letters in the alphabet $\{1, 2\}$.

The pair of rectangles $R = (R_1, R_2)$ defines three disjoint regions in $\mathbb{N}^2$: the partition diagram $R_1 \cup R_2$, the northeast region $\text{NE}(R)$ consisting of the cells $(r, c)$ such that $r \leq l' := \min(\eta_1, \eta_2)$ and $c > a := \max(\mu_1, \mu_2)$, and the southwest region $\text{SW}(R)$, consisting of the cells $(r, c)$ such that $r > l := \max(\eta_1, \eta_2)$ and $c \leq a' := \min(\mu_1, \mu_2)$. Clearly for the reversed sequence $R' = (R_2, R_1)$ one has $\text{NE}(R') = \text{NE}(R)$ and $\text{SW}(R') = \text{SW}(R)$.

Proposition 33. The set $\text{LRT}(\lambda; (R_1, R_2))$ is either empty or a singleton. The following are necessary and sufficient conditions that the set $\text{LRT}(\lambda; (R_1, R_2))$ be nonempty.

1. $R_1 \cup R_2 \subset \lambda \subset R_1 \cup R_2 \cup \text{NE}(R) \cup \text{SW}(R)$.
2. The shapes $\lambda_{ne} := \lambda \cap \text{NE}(R)$ and $\lambda_{sw} := \lambda \cap \text{SW}(R)$ are complementary inside the rectangle $R_1 \cap R_2$ in the sense that both are contained in $R_1 \cap R_2$, and if $R_1 \cap R_2$ is skewed by the removal of one, then the resulting shape is the 180 degree rotation of the other.
Proof. Suppose LRT(\(\lambda; (R_1, R_2)\)) is nonempty; let \(T\) be a member. Let \(R' = (R_2, R_1)\), with alphabets \(A'\) and \(A'\) and Yamanouchi tableaux \(Y_1'\) and \(Y_2'\). Due to the symmetry of tensor product multiplicities and Proposition 3, LRT(\(\lambda; (R_2, R_1)\)) is also nonempty, containing the tableau \(T'\) say. By definition \(T_{|A_1} = Y_1\) which has shape \(R_1\), so \(R_1 \subseteq \lambda\). A similar argument applied to \(T'\) shows that \(R_2 \subseteq \lambda\).

By switching the rectangles \(R_1\) and \(R_2\) if necessary, we may assume that either \(\eta_1 > \eta_2\), or \(\eta_1 = \eta_2\) and \(\mu_1 \geq \mu_2\). In particular \(R_1\) is not properly contained in \(R_2\).

Suppose \(\lambda \notin (R_1 \cup R_2 \cup \text{NE}(R) \cup \text{SW}(R))\). We consider cases. Suppose first that \(R_2\) is not contained in \(R_1\), that is, \(\eta_1 > \eta_2\) and \(\mu_1 < \mu_2\). By assumption \((\eta_2 + 1, \mu_1 + 1) \in \lambda\). This cell and those due north of it, are in \(\lambda\) but not in \(R_1\). Thus the skew tableau \(T - Y_1\) contains a column of length \(\eta_2 + 1\), which means it contains at least that many distinct letters, which is impossible since it only contains the letters of \(A_w\), which has cardinality \(\eta_2\). Otherwise suppose that \(R_2\) is contained in \(R_1\). Then by assumption \(\lambda\) either contains the cell \((l + 1, a' + 1)\) or the cell \((l' + 1, a + 1)\). If \((l + 1, a' + 1) \in \lambda\) then one arrives at a contradiction in a manner similar to the previous case. Suppose \((l' + 1, a + 1) \in \lambda\). It and all the cells due west of it, are contained in \(\lambda\) but are not in \(R_2\). This means that the skew tableau \(T' - Y'_1\) contains a row of length \(a + 1\). But then \(\text{word}(T' - Y'_1)\) contains a contiguous weakly increasing subword of length \(a + 1\), so by Theorem 14 the recording tableau of word \((T' - Y'_1)\) contains cells in at least \(a + 1\) different columns. It follows that the shape of \(P(T' - Y'_1) = Y'_2\) has at least \(a + 1\) columns, which is a contradiction, since \(Y'_2\) has \(a = \mu_1\) columns. Thus \(\lambda\) is contained in \(R_1 \cup R_2 \cup \text{NE}(R) \cup \text{SW}(R)\).

To show the second condition is necessary, recall that the column-reading word of a (possibly skew) tableau \(S\) is the word \(w^1 w^2 \cdots\) where \(w^i\) is the word comprising the \(i\)-th column of \(S\), read from bottom to top. Let us consider the column-reading word \(\text{cword}(T - Y_1)\). By definition it satisfies \(\text{cword}(T - Y_1) \sim_K Y_2\). Consider another dissection of \(\lambda\) into three pieces: \(R_1\), the part \(\lambda_w\) of \(\lambda - R_1\) in the first \(\mu_1\) columns, and the part \(\lambda_e\) of \(\lambda - R_1\) not in the first \(\mu_1\) columns. Label the corresponding parts of the tableau \(T\) by \(Y_1, T_w,\) and \(T_e\). Clearly \(\text{cword}(T - Y_1) = \text{cword}(T_w) \circ \text{cword}(T_e)\). By Proposition 12 \(T_w = \text{key}(\beta)\) and \(T_e = \text{key}(\alpha)\) (both in the alphabet \(A_2\)) where \(\alpha = \lambda_e, \alpha + \beta = R_2,\) and \(\beta^+ = \lambda_w\). In particular, \(\lambda_w\) and \(\lambda_e\) are complementary in \(R_2\). It is not hard to see that this implies that the shapes \(\lambda_{nc}\) and \(\lambda_{sw}\) are complementary in \(R_1 \cap R_2\).

The above argument shows that if LRT(\(\lambda; (R_1, R_2)\)) is nonempty then it is a singleton, since the entire tableau \(T \in \text{LRT}(\lambda; (R_1, R_2))\) was specified.

For the converse, suppose \(\lambda\) satisfies the two properties given above. Let \(\lambda_w\) (resp. \(\lambda_{\mu}\)) be the part of \(\lambda - R_1\) in (resp. not in) the first \(\mu_1\) columns. Let \(\alpha = \lambda_e, \beta = R_2 - \alpha,\) and \(T\) the (not necessarily column-strict) tableau of shape \(\lambda\) whose restrictions to the subshapes \(R_1, \lambda_e,\) and \(\lambda_w\) of \(\lambda\) are given by \(Y_1, \text{key}(\alpha),\) and \(\text{key}(\beta)\) where the key tableaux are taken with respect to the alphabet \(A_2\). The column-reading word of \(T\) satisfies \(\text{cword}(T)\mid_{A_1} \sim_K Y_1\) and \(\text{cword}(T)\mid_{A_2} \sim_K Y_2\), so it only remains to show that \(T\) is indeed column-strict. Since all the letters of \(A_1\) are strictly smaller than those of \(A_2\), the only possible violations of column-strictness in \(T\) are of the form \((r, \mu_1) > (T(r, \mu_1 + 1)\) where the cell \((r, \mu_1)\) is in \(\lambda_w\) and the cell \((r, \mu_1 + 1)\) is in \(\lambda_e\). Now \((r, \mu_1) \in \lambda_w\) (and \(\lambda_w \subseteq R_2\) as partitions) implies that \(r > \eta_1\) and \(\mu_2 \geq \mu_1\), while \((r, \mu_1 + 1) \in \lambda_e\) (and \(\lambda_e \subseteq R_2\) as partitions) implies that
\( \eta_2 \geq r \). Now

\[
T(r, \mu_1) = \text{key}(\beta)(r - \eta_1, \mu_1)
\]
is the \((r - \eta_1)\)-th smallest letter in the last column of \( \text{key}(\beta) \), which consists of the smallest \( \eta_2 - m \) letters in \( A_2 \), where \( m \) is the number of parts of \( \alpha \) equal to \( \mu_2 \). It follows that \( T(r, \mu_1) = m + r \). On the other hand,

\[
T(r, \mu_1 + 1) = \text{key}(\alpha)(r, 1) = \eta_1 + r,
\]
being the \( r \)-th smallest letter in \( A_2 \). By assumption \( m + r = T(r, \mu_1) > T(r, \mu_1 + 1) = \eta_1 + r \), that is, \( m > \eta_1 \). Now the partition \( \alpha \) contains the cell \((m, \mu_2)\), and \( \lambda_e = \alpha \) as partitions, so it follows that \( \lambda \) contains the cell \((m, \mu_1 + \mu_2)\). But this cell lies outside \( R_1 \cup R_2 \cup \text{NE}(R) \cup \text{SW}(R) \) since \( m > \eta_1 = \min(\eta_1, \eta_2) \) and \( \mu_1 + \mu_2 \geq \max(\mu_1, \mu_2) \).

**Example 34.** Let \( \mu_1 = 3, \mu_2 = 5, \eta_1 = 2, \eta_2 = 3 \), and \( \lambda = (76521) \). Then in the notation of Proposition 33, \( R_1 \cup R_2 = R_2 = (555), \lambda_{ne} = (210), \lambda_{sw} = (210), \alpha = \lambda_e = (432) \) and \( \beta = (123) \) so that \( \lambda_{w} = (321) \). The unique tableaux in \( \text{LRT}(\lambda; (R_1, R_2)) \) and \( \text{LRT}(\lambda; (R_2, R_1)) \) are given respectively by

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 3 & 3 & 3 & 3 & 3 \\
2 & 2 & 2 & 4 & 4 & 4 & 2 & 2 \\
3 & 4 & 5 & 5 & 5 & 3 & 3 & 3 \\
4 & 5 & & & & 4 & 5 \\
5 & & & & & & & 5 \\
\end{array}
\]

5.4. Proof of Theorem 3.

**Proof.** All will be proven here except the part of (A5) given by the braid relation \( \tau_p\tau_{p+1}\tau_p = \tau_{p+1}\tau_p\tau_{p+1} \), which is proven later using the induction coming from \( R \)-coeycage.

For the well-definedness of the map \( \tau_p \), the only step requiring proof is (4), and this follows from (A1).

(A1): Since \( w \) and \( \tau_p w \) agree at all positions except those occupied by letters of \( B = A_p \cup A_{p+1} \), without loss of generality it may be assumed (by restricting to \( B \)) that \( R = (R_1, R_2) \) and \( p = 1 \). In this case, \( Q(w) = Q(\tau_p w) \) by definition. But a theorem of D. White [21] asserts that a word fits a skew shape if and only if its recording tableau satisfies a condition that depends only on \( D \). Thus \( w \) fits \( D \) if and only if \( \tau_p w \) does, proving (A1).

(A2): Without loss of generality suppose \( v \sim_k w \) is an elementary Knuth equivalence of words in \( W(R) \). Suppose first that \( v = t_y x z u \) and \( w = t_y x z u \) where \( t \) and \( u \) are words of length \( l \) and \( m \) respectively and \( x \leq y < z \) are letters. This case is settled by the following lemma.

**Lemma 35.** There exist words \( t' \) and \( u' \) of lengths \( l \) and \( m \) respectively, and letters \( x' \leq y' < z' \) such that

\[
\tau_p v = t'y'x'z'u' \quad \text{and} \quad \tau_p w = t'y'z'x'u'
\]

**Proof.** Let \( B = A_p \cup A_{p+1} = A'_p \cup A'_{p+1} \). Suppose \( x \notin B \). Then the removal of \( x \) makes \( v \) and \( w \) identical; call this common subword \( v' \). By the definition of \( \tau_p, \tau_p v \) and \( \tau_p w \) are identical if the letter \( x \) is removed; this common subword is \( \tau_p v' \). It follows that \( \tau_p v \) and \( \tau_p w \) have the desired form.
The same argument works if \( z \notin B \). So it may be assumed that \( x \in B \) and \( z \in B \). Since \( B \) is an interval and \( x \leq y < z \), \( y \in B \) as well. Let \( I \) be the set of positions of letters of \( v \) that are not in \( B \). By the definition of \( \tau_p \), the positions of the letters not in \( B \) are equal to \( I \) for all of the words \( v, w, \tau_p v \) and \( \tau_p w \), and all of these words agree at those positions. Thus by restricting to \( B \), it may be assumed that \( R = (R_p, R_{p+1}) \). In this case we have \( Q(\tau_p v) = Q(v) \) and \( Q(\tau_p w) = Q(w) \), so two applications of Lemma 45 show that the words \( \tau_p v \) and \( \tau_p w \) have the desired form.

The other kind of elementary Knuth equivalence takes the form \( v \sim_K w \) where \( v = t x z y u \) and \( w = t z x y u \) with \( t \) and \( u \) as above and \( x < y \leq z \). Let \( R\# = (R_t, R_{t-1}, \ldots, R_1) \) and \( A_1\# \) through \( A_t\# \) the corresponding subalphabets for \( R\# \).

Let \( v\# \) be the reverse of the complement of the word \( v \) in the alphabet \( \lbrack n \rbrack \). Then \( v\# = u'y'z'x't' \) and \( w\# = u'y'x'z't' \) where \( t' \) and \( u' \) are words of the same length as \( t \) and \( u \) respectively, and \( z' < x' \). By Lemma 12, \( v\# \) and \( w\# \) are in \( W(R\#) \). This puts us in the previous case, so by an application of the Lemma we have \( \tau_{1-t}\#v\# \sim_K \tau_{1-t}\#w\# \). By Lemma 12 \( (\tau_p v)\# \sim_K (\tau_p w)\# \), which implies that \( \tau_p v \sim_K \tau_p w \) as desired. This proves (A2).

(A3) follows immediately from (A2).

It remains to show (A4). Suppose first that \( p < t-1 \). Let \( A' = A_1 \cup A_2 \cup \cdots \cup A_{t-1} \). By definition the positions of the letters of \( A_i \) are the same in \( w \) and \( \tau_p w \). By \( \eta_i \) applications of Lemma 13 it is enough to show that

\[
Q(w|_{A'}) = Q((\tau_p w)|_{A'})
\]

since one uses the same process to pass from these tableaux to \( Q(w) \) and \( Q(\tau_p w) \) respectively. But this holds by induction on \( t \) since the latter tableau is equal to \( Q(\tau_p(w|_{A'})]) \).

Thus it may be assumed that \( p = t - 1 \). For the case \( t = 2 \), (A4) holds by definition. So it may be assumed that \( t > 2 \). Let \( R\# = (R_t, R_{t-1}, \ldots, R_1) \). Then \( Q(w) = Q(w\#) \) and

\[
Q(\tau_p w) = Q((\tau_p w)\#) = Q(\tau_1(w\#))
\]

by Lemma 12. Since \( \tau_1 \) switches the first two rectangles and \( t > 2 \), \( Q(\tau_1(w\#)) = Q(w\#) \) by a previous argument. Thus \( Q(w) = Q(\tau_p w) = Q(w) = Q(\tau_p w) \), proving (A4).

(A5): The fact that \( \tau_p \) is an involution, and that \( \tau_p \tau_q = \tau_q \tau_p \) if \( |p - q| > 1 \), follow easily from the definitions.

(A6): This follows from the special case of the operator \( \tau_p \) and interval \( I = [p, p + 1] \).

(A7): This immediately reduces to the case (1) where \( R = (R_1, R_1) \). But \( \tau_1 : \text{LRT}(\lambda;(R_1, R_1)) \rightarrow \text{LRT}(\lambda;(R_1, R_1)) \) is the identity since \( \text{LRT}(\lambda;(R_1, R_1)) \) is empty or a singleton.

5.5. Proof of Proposition 13.

Remark 36. Let \( w \) be a word of content \( \chi(R) = (\mu^R_1, \ldots, \mu^R_t) \), the content of any word in \( W(R) \).

1. For each \( 1 \leq j \leq t \), the set of positions occupied by the letters of \( A_j \) in \( \chi^R_k(w) \) are rotated cyclically to the right \( k \) positions from the corresponding set of positions in \( w \) (by Theorem 3 (A6) applied to the automorphism of conjugation \( w^R_0 \)).
2. If \( j \neq i \), then the letters of \( A_j \) just move to the right by one position (and are otherwise unchanged) in passing from \( w \) to \( \chi_R(w) \) (by 1 and Theorem (A7) applied to the automorphism of conjugation \( w_0^R \) and the word \( w \), which has \( \mu_j \) copies of each letter in the interval \( A_j \)).

3. Let \( B = A_i \cup A_{i+1} \cup \cdots \cup A_j \) and \( w = uv \) where \( v \) has length \( k \). Say \( v|_B \) has length \( k' \). Then

\[
\chi_R^k(uv)|_B = \chi_R^{k'}(u|_B v|_B).
\]

This follows from the case \( k = 1 \), where 2 applies.

**Proof.** Let \( w \in W(R) \). For 1, it suffices to prove that \( \chi_R(w) \in W(R) \), since 2 implies that \( \chi_R^N \) is the identity, where \( N \) is the length of the word \( w \) (making \( \chi_R \) invertible). Write \( w = ux \) with \( x \in A_i \), say. For \( j \neq i \), \( \chi_R(w)|_{A_j} = w|_{A_j} \sim_K Y_j \) by Remark 3 and the fact that \( w \in W(R) \). For \( j = i \), \( \chi_R(w)|_{A_i} = \chi_R(w|_{A_i}) \). Thus it may be assumed that \( R = (R_1) \). Let \( R_1 = (a^n) \), \( Y = Y_1 \), and \( w_0 = w_0^R \). For this it certainly suffices to let \( uv \in W((R_1)) \) and show that \( (w_0v)(w_0u) \in W(R) \), for we only need the case that \( v \) is a letter. Let \( \nu \) be the content of \( v \). By Lemma 32 \( \nu \) is a partition, \( P(\nu) = \text{key}(\nu) \), and \( u \sim_K \text{key}((a^n) - \nu) \). Then by Lemma 32 and the definition of key tableau,

\[
P((w_0v)(w_0u)) = P((w_0\text{key}(\nu))(w_0\text{key}((a^n) - \nu)))
\]

\[
= P(\text{key}(\text{rev}(\nu))(\text{key}(a^n) - \text{rev}(\nu)))
\]

\[
= \text{key}(a^n)
\]

where \( \text{rev}(a_1, \ldots, a_n) = (a_n, \ldots, a_1) \).

Note that 2 holds by definition when \( k = 1 \). In light of Remark 36 1, it is enough to show that

\[
\chi_R^k(uv)|_{A_i} = ((u_0^R v)(u_0^R w)|_{A_i},
\]

for all \( i \). Fix \( i \). Let \( k' \) be the length of \( v|_{A_i} \). Then by Remark 36 3,

\[
\chi_R^{k'}(uv)|_{A_i} = \chi_R^{k'}((u|_{A_i} v|_{A_i})).
\]

Thus we have reduced to the case that \( R = (R_1) \). Let \( R_1 = (a^n) \), \( Y = Y_1 \), and \( w_0 = w_0^R \). By the bijectivity of the RS correspondence it is enough to show that \( \chi_R^k(uv) \) and \( (w_0v)(w_0u) \) have the same \( P \) tableaux and the same \( Q \) tableaux. By 1 and its proof, both words have \( P \) tableau equal to \( Y_1 \).

Let \( Q = Q(w) = Q(uv) \), \( Q' = Q((w_0v)(w_0u)) \) and \( Q'' = Q(\chi_R^k(uv)) \). It only remains to show that \( Q' = Q'' \). Recall that all of these tableaux have shape equal to that of \( Y \), which is rectangular with \( n \) columns and \( a \) rows. Let \( N = k + l \) be the number of cells in \( Q \). In light of Lemma 37 it is enough to show that \( Q' = \text{pr}_k(Q) \) and \( Q'' = \text{pr}_l(Q) \). Let \( T + j \) denote the tableau whose entries are obtained from those of \( T \) by adding the integer \( j \). We have

\[
P(Q|_{i+1,i+k}) - l = Q(v) = Q(w_0v) = Q'|_{[k]}
\]

by Lemma 41, Theorem 3 for \( w_0 \), and the definition of recording tableau. Also

\[
P(Q'|_{[k+1,k+l]}) - k = Q(w_0u) = Q(u) = Q|_{[l]}.
\]

It follows that \( Q' = \text{pr}_k(Q) \). To show that \( Q'' = \text{pr}_l(Q) \), it suffices to show that \( Q(\chi_R(w)) = \text{pr}_1(Q) \). But the above argument with \( k = 1 \) proves this. \( \square \)
5.6. Proof of Theorem 16. The proof requires a few preliminary results.

Lemma 37. Suppose $uyxz \in W(R)$ with $x < y \leq z$ letters, so that $uyxz \sim_K uyxx$ is an elementary Knuth equivalence. Then $\chi^3_R(uyxz) \sim_K \chi^3_R(uyzx)$.

Proof. We have

$$\chi^3_R(uyxz) = (w_0R yxz)(w_0R u) \sim_K (w_0R yxxz)(w_0R u) = \chi^3_R(uyzx)$$

by two applications of Proposition 15 and Theorem 9. \boxed{}

Before giving the proof of Theorem 16 it is useful to state a more detailed version in the two rectangle case. Let $R = (R_1, R_2)$ and $R' = s_1 R = (R_2, R_1)$. Suppose $w \in W(R)$, written $w = ux$ with $x$ a letter. Let $\lambda$ (resp. $\rho$) be the shape of $P(w) = P(ux)$ (resp. $P(\chi_R(w)) = P((w_0R x)(w_0R u))$). Let $s$ (resp. $s'$) be the cell giving the difference of $\lambda$ (resp. $\rho$) and the shape of $P(u)$, which is the same as the shape as $P(w_0R u)$.

Proposition 38. With the above notation and that of Proposition 32:

1. If $s \in \text{NE}(R)$ then $s' \in \text{SW}(R)$, $\rho_{ne} = \lambda_{ne} - \{s\}$, and $x \in A_2$.
2. If $s \in \text{SW}(R)$ then $s' \in \text{NE}(R)$, $\rho_{se} = \lambda_{se} - \{s\}$, and $x \in A_1$.
3. If $s \in R_1 \cup R_2$ then $s' = s$ and $\rho = \lambda$. If $s \in R_1$ and $s \notin R_2$ then $x \in A_1$. If $s \in R_2$ and $s \notin R_1$ then $x \in A_2$. (The corner cell $s$ cannot lie in $R_1 \cap R_2$).

In particular, $\chi_R$ and $\tau_1$ commute for $R = (R_1, R_2)$.

The proof, which relies on explicit computations on two-rectangle LR tableaux, is straightforward but tedious and is omitted. In each of the three cases there are two subcases depending on whether one rectangle contains the other or not.

Proof of Theorem 16:

Proof. Let $A'_1, A'_2, \ldots, A'_a$, etc. be the subalphabets for $R'$ and $Y'_1, Y'_2, \ldots, Y'_b$, etc. the Yamanouchi tableaux. Let $w = ux \in W(R)$ with $x$ a letter. Say $x \in A_i$.

Suppose $i \notin \{p, p + 1\}$. We have

$$\chi^R(\tau_p w) = \chi^R(\tau_p u)x = (w_0R x)(w_0R \tau_p u)$$

$$\tau_p \chi^R w = \tau_p (w_0R x)(w_0R u) = (w_0R x)(\tau_p w_0R u)$$

since $x \in A_i$ and $w_0R x \in A_i$ are unchanged by $\tau_p$. But $w_0R' x = w_0R x$ so it enough to show that $w_0R' \tau_p u = \tau_p w_0R u$. It is clear that these two words agree at positions of letters not in $B = A_p \cup A_{p+1}$, so it is enough to show that $w_0R' \tau_p (u|B) = \tau_p w_0R (u|B)$. But $w_0R' u|B = u|B$ and $w_0R' \tau_p (u|B) = \tau_p u|B$ by Theorem A7 for the automorphisms of conjugation $w_0R'$ and $w_0R$. Thus both words are equal to $\tau_p u|B$.

The other case is that $i \in \{p, p + 1\}$. For each $j \notin \{p, p + 1\}$ it follows from the definitions and Remark 32 that $\chi^R(\tau_j w)$ and $\tau_p (\chi_R(w))$ agree at the positions containing letters of $A_j = A'_j$. Again it is enough to show that

$$\chi^R(\tau_p w)|B = ((\chi_R(w))\tau_p)|B$$

where $B = A_p \cup A_{p+1} = A'_p \cup A'_{p+1}$. Let $\tau_p w = u'x'$ where $x'$ is a letter. Since $w = ux$ with $x \in B$, $x' \in B$ by the definition of $\tau_p$. We have

$$\chi^R(\tau_p w)|B = \chi^R((\tau_p w)|B) = \chi^R(\tau_p (w|B))$$

$$(\tau_p (\chi_R(w)))|_B = \tau_p ((\chi_R(w))|B)\tau_p (\chi_R(w)|B)$$

by Remark 32 and the definition of $\tau_p$. 





Thus we have reduced to the two rectangle case \((R_p, R_{p+1})\). For simplicity of notation let \(t = 2, p = 1, \text{ and } \tau = \tau_1\). By the bijectivity of the RS correspondence it suffices to show that the two words \(\chi_R(\tau w)\) and \(\tau \chi_R(w)\) have the same \(P\) and \(Q\) tableaux. For this we claim that it suffices to show that their \(P\) tableaux have the same shape. Indeed, since we are in the two rectangle case and both \(P\) tableaux are in the set \(\text{LRT}(R)\) and are assumed to have the same shape, they must be equal by Proposition \(\text{A3}\). For the equality of the \(Q\) tableaux, it suffices to show that they agree after applying the invertible operator \(\text{pr}_1\) (see section \(\text{A2}\)). But since it is assumed that these \(Q\) tableaux have the same shape, it is enough to show that

\[
P(Q(\chi_R(\tau w))_{[2,N]}) = P(Q(\tau \chi_R(w))_{[2,N]}).
\]

Let \(w = ux\) and \(\tau w = u'x'\) where \(x\) and \(x'\) are letters. Then \(\chi_R(w) = (w_0^R x)(w_0^R u)\) and \(\chi_R(\tau w) = (w_0^R x')(w_0^R u')\). Let \(N\) be the length of \(w\). We have

\[
P(Q(\tau \chi_R(w))_{[2,N]}) - 1 = P(Q(\chi_R(w))_{[2,N]}) - 1
= P(Q((w_0^R x)(w_0^R u))_{[2,N]}) - 1
= Q(w_0^R u) = Q(u) = Q(w)_{[N-1]}
\]

by Theorem \(\text{A4}\), definition of \(\chi_R(w)\), Lemma \(\text{A1}\), Theorem \(\text{A3}\), and the fact that \(w = ux\). On the other hand,

\[
P(Q(\chi_R(\tau w))_{[2,N]}) - 1 = P(Q((w_0^R x')(w_0^R u'))_{[2,N]}) - 1
= Q(w_0^R u') = Q(u') = Q(\tau w)_{[N-1]} = Q(w)_{[N-1]}
\]

for similar reasons. But this establishes \((5.1)\).

So it only remains to show that the shapes of \(W := P(\chi_R(w))\) and \(W' := P(\chi_R(\tau w))\) coincide, since \(W\) has the same shape as \(\tau W = P(\tau \chi_R(w))\) by Theorem \(\text{A4}\). Let \(s\) be the cell giving the difference of the shapes of \(P(w)\) and \(P(u)\); this is also the difference of the shapes of \(P(w \tau)\) and \(P(u')\), by Theorem \(\text{A3}\) and the previously proven fact that \(Q(u) = Q(u')\). Propositions \(\text{A3}\) and \(\text{A5}\) explicitly show how the shape of \(P(w)\) (resp. \(P(\tau w)\)), together with the cell \(s\), determines the shape of \(W\) (resp. \(W'\)). But \(P(w)\) and \(P(\tau w)\) have the same shape, so \(W\) and \(W'\) do as well.

5.7. Proof of Theorem \(\text{A4}\)

Proof: First it is shown that \(\text{LRT}(R)\) under \(\leq_R\) is a partial order. It is enough to show that \(\leq_R\) is an extension by a partial order \(\leq\). Let \(n_j(T)\) be the number of letters in \(T|_{A_j}\) in the first \(\mu_j\) columns. Define the partial order \(\leq\) on \(\text{LRT}(R)\) by \(T < S\) if there is an index \(j\) such that \(n_j(T) = n_j(S)\) for all \(j < i\) but \(n_j(T) > n_j(S)\). Suppose \(T < R < S\) is a covering relation coming from the word \(w = ux \in W(R)\) where \(x\) is a letter. Let \(x \in A_i\) say. \(x\) is the smallest letter in \(A_i\) since \(x\) is the last letter of \(w|_{A_i}\) and \(P(w|_{A_i}) = Y_i\), by Lemma \(\text{A3}\).

Let \(s\) be the cell giving the difference of the shapes of \(U := P(u)\) and \(S = P(w)\). Write \(\chi_R(ux) = x'u'\) so that \(T = P(x'u')\). Write \(U' := P(u') = w_0^R U\). Now \(x' = w_0^R x\) is the largest letter of \(A_i\) and \(u' = w_0^R u\). For \(j < i\) we have

\[
n_j(T) = n_j(U) = n_j(U') = n_j(S).
\]

The first equality holds, for by a property of Schensted insertion, since the cell \(s\) is not in the first \(a\) columns, the tableaux \(S\) and \(U\) must agree in the first \(a\) (and hence \(\mu_i\)) columns. The second equality comes from Theorem \(\text{A3}\) (A7) for the
The first two equalities hold for the above reasons. Recall that $S$ is obtained from $U'$ by the column insertion of the letter $x'$. But $x'$, being the maximum letter of $A_i$, only bumps other copies of $x'$ in $U'$. There are $\mu_i - 1$ copies of $x'$ in $U'$, hence at most that many in the first $\mu_i$ columns of $U'$. Let $B = A_1 \cup \cdots \cup A_i$. Then by explicit calculation $P((x'U')|B)$ is obtained from $U'|B$ by adjoining the letter $x'$ to the bottom of the first column of $U'|B$ that does not contain the letter $x'$. This puts another letter of $A_i$ into the first $\mu_i$ columns. Thus $S < T$.

This proves that $\text{LRT}(R)$ is a poset under $\leq_R$.

Before showing that $\text{LRT}(R)$ is graded, let us prove 2 and 3. Let $T \in \text{LRT}(R)$. If $T$ has more than $a$ columns, then it has a corner cell in a column strictly east of the $a$-th, and therefore admits a covering relation $T' <_R T$. Continuing this process one produces a saturated chain in $\leq_R T$ down to an element $\min(T) \in \text{LRT}(R)$ that has at most $a$ columns. Since $\min(T)|_{A_i} \sim_K Y_i$ and $a = \mu_i$ for some $i$, $\min(T)$ must have exactly $a$ columns. For 3, note that for any $T \in \text{LRT}(R)$, $T|_{A_i} = Y_i$. For $T$ minimal $T$ must consist of $Y_1$ atop a tableau $\hat{T}$. By Remark 1, $\hat{T} \in \text{LRT}(\hat{R})$.

To show $\text{LRT}(R)$ is graded, it is enough to show that for any $T \in \text{LRT}(R)$, there is only one such tableau $\min(T)$ (that is, a tableau in $\text{LRT}(R)$ that has $a$ columns and satisfies $\min(T) \leq_R T$), and any saturated chain from $T$ down to $\min(T)$ has the same length. Let $T \in \text{LRT}(R)$ and $S_1$ and $S_2$ in $\text{LRT}(R)$ with $a$ columns and suppose that there are saturated chains from $T$ down to $S_1$ and $S_2$. Let $T_1 <_R T$ and $T_2 <_R T$ be the first steps in these saturated chains from $T$ down to $S_1$ and $S_2$ respectively. We construct a tableau $T_3$ that is two steps below both $T_1$ and $T_2$. To see that this suffices, by induction and definition one has $\min(T_3) = \min(T_1) = S_1$ and $\min(T_3) = \min(T_2) = S_2$, so $S_1 = S_2$ and $\min(T)$ is the common min of $T$, $T_1$, $T_2$, and $T_3$. Furthermore the distance from $T_1$ and $T_2$ down to the common minimum is well-defined and equal, since both distances are two more than the distance from $T_3$ down to the common minimum. So it suffices to construct $T_3$. Its construction uses a rectangular analogue of [2, Lemma 2.13].

Let $s_1$ and $s_2$ be the two corner cells of the shape of $T \in \text{LRT}(\lambda; R)$, both strictly east of the $a$-th column, which induce the covering relations $T_1 <_R T$ and $T_2 <_R T$. Without loss of generality assume that $s_1$ is strictly north and strictly east of $s_2$. Let $(U_i, x_i)$ be the pair consisting of a column-strict tableau $U_i$ of shape of shape $\lambda - \{s_i\}$ and a letter $x_i$ for $1 \leq i \leq 2$, computed by reverse row insertion on $T$ at $s_i$. Clearly there is a corner cell $s_3$ of the shape $\lambda - \{s_1, s_2\}$ that is strictly east of the $a$-th column, such that $s_2$ is strictly south and weakly west of $s_3$ and $s_1$ is strictly east and weakly north of $s_3$. Performing reverse row insertions on $T$ at the cells $s_3$, then $s_2$, then $s_1$, let $V_1$ be the resulting column-strict tableau and $y'x$ the three ejected letters. Doing the same thing except using the order $s_3$, then $s_2$, then $s_1$, let $V_2$ be the resulting tableau and $y'z'x'$ the three ejected letters. By Lemma 15 we have $V_1 = V_2$ and $y' = y$, $z' = z$ and $x' = x$ with $x \leq y < z$, and $T \sim_K U'yxz \sim_K U'yxx$. Let $T_3 = P(\chi_R(U'yxz)) = P(\chi_R(U'yxx))$; the latter automorphism of conjugation $w_0^R$. The third follows from the fact that $x' \in A_i$ is strictly greater than any letter in $A_j$, so the column insertion of $x'$ into $U'$ doesn’t move any letters in $A_j$. Moreover,

$$n_i(T) = n_i(U) = n_i(U') = n_i(S) - 1.$$
equality holds by Lemma 37. There are saturated chains
\[ P((\chi_R^h(U'yxz)) <_R P((\chi_R^h(U'yxz)) <_R P((\chi_R(U'yxz)) = T_1 <_R P(U'yxz) \]
\[ P((\chi_R^h(U'yxz)) <_R P((\chi_R^h(U'yxz)) <_R P((\chi_R(U'yxz)) = T_2 <_R P(U'yxz), \]
where both left hand tableaux are \( T_1 \) and both right hand tableaux are \( T_2 \). The fact that these covering relations are produced by corner cells that are strictly east of the \( a \)-th column, is a consequence of Lemma 38.

5.8. Proof of Proposition 24

Proof. \( \tau_p \) is a bijection so it is enough to show that if \( T <_R S \) is a covering relation, then \( \tau_p T <_R' \tau_p S \) is a covering relation.

Let \( T \in \text{LRT}(\lambda; R) \), \( s \) a corner cell of \( \lambda \) that is strictly east of the \( a \)-th column where \( a := \max_i \mu_i \), \( U \) the column-strict tableau of shape \( \lambda - \{s\} \) and \( x \) the letter such that \( T = P(Ux) \), and \( W = P(\chi_R(Ux)) \). Then \( W <_R T \), and all covering relations have this form.

Let \( \tau = \tau_p \) and \( N \) be the number of cells in \( T \). Let \( U'x' = \tau(Ux) \) and \( T' = \tau T \). By Theorem 16, \( T \) and \( T' \) have the same shape and \( Q(U') = Q(U'x')|_{[N-1]} = Q(U)|_{[N-1]} = Q(U) \), so that \( U' \) is a column-strict tableau of the same shape as \( U \). Define \( W' := P(\chi_R(U'x')) \). By Theorem 16, \( W' = \tau W \). But by construction, \( W' <_R T' \).

With this poset structure in place, we now show that \( \tau_p \tau_{p+1} \tau_p = \tau_{p+1} \tau_p \tau_{p+1} \), the only part of Theorem 23(A5) that was not proven.

Proof. Let \( B = A_p \cup A_{p+1} \cup A_{p+2} \). Consider the operators \( \tau_p \tau_{p+1} \tau_p \) and \( \tau_{p+1} \tau_p \tau_{p+1} \). Both do not disturb the letters outside the interval \( B \), so by restriction to \( B \), we may assume that \( R = (R_1, R_2, R_3) \) and \( p = 1 \). By Theorem 16(A3) and (A4) and the bijectivity of the RS correspondence, it is enough to check the equality of the two operators on \( T \in \text{LRT}(\lambda; R) \). Using the fact that \( \tau_1 \) and \( \tau_2 \) are involutions, we may reorder \( R \) so that \( \mu_1 \) is the largest among the \( \mu_i \). Proceeding by induction on height in the poset \( \text{LRT}(R) \) with order \( \leq_R \), by Theorem 16 it may be assumed that \( T \) is minimal, that is, \( T \) has exactly \( \mu_1 \) columns. Using Remark 39 below, it is easy to see that \( \tau_1 \) and \( \tau_2 \) satisfy the braid relation on \( T \).

Remark 39. Let \( T \in \text{LRT}(\lambda; R) \) where \( \lambda \) has \( a = \max_i \mu_i \) columns with \( \mu_j = a \). Let \( A_j \) be the \( i \)-th subalphabet for \( \tau_j R \). Then \( \tau_j T \) is obtained from \( T \) by a rather trivial vertical exchanging process. In each column of \( T \), first replace the letters of \( A_j \) (resp. \( A_{j+1} \)) by their counterparts in \( A'_{j+1} \) (resp. \( A'_{j} \)) and then sort the resulting column; only the letters in \( A_j \cup A_{j+1} = A'_j \cup A'_{j+1} \) need to be moved. The key fact is that each column of \( T|_{A_j} \) must contain each of the numbers in \( A_j \).

5.9. Proof of Proposition 23

Proof. Observe that
\[ (5.2) \quad d_{i,R}(w) = d_{i,\tau_i R}(\tau_i w) \]
since \( \tau_i \) preserves shape by Theorem 16. Consider first the case that \( t = 2 \), that is, \( R = (R_1, R_2) \). Let \( R' = \tau_1 R \). Then
\[ 2!(c_R(w) - c_R(\chi_R(w))) = d_R(w) - d_R(\chi_R(w)) + d_R(\tau_1 w) - d_R(\chi_R(\tau_1 w)) \]
\[ = 2(d_R(w) - d_R(\chi_R(w))) \]
by definition, the fact that \( \chi \) and \( \tau_1 \) commute (Theorem \[16\]), and (5.2). The values of \( d_R(w) \) and \( d_R(\chi_R(w)) \) are determined by the shapes of the two-rectangle LR tableaux \( P(w) \) and \( P(\chi_R(w)) \). By Lemma \[18\] it follows that

\[
(5.3) \quad d_R(w) - d_R(\chi_R(w)) = \begin{cases} 
1 & \text{if } f(id) = f(\tau_1) = 2 \\
-1 & \text{if } f(id) = f(\tau_1) = 1 \\
0 & \text{otherwise.}
\end{cases}
\]

In general we have

\[
(5.4) \quad t!(c_R(w) - c_R(\chi_R(w))) = \sum_{\tau \in S_t} \sum_{i=1}^{t-1} (t - i)(d_{i,\tau R}(\tau w) - d_{i,\tau w}(\chi_R(\tau w))).
\]

Next it is shown that

\[
(5.5) \quad d_{i,\tau R}(\tau w) - d_{i,\tau w}(\chi_R(\tau w)) = \begin{cases} 
1 & \text{if } f(\tau) = f(\tau \tau_i) = i + 1 \\
-1 & \text{if } f(\tau) = f(\tau \tau_i) = i \\
0 & \text{otherwise.}
\end{cases}
\]

Suppose first that \( f(\tau) \not\in \{i, i+1\} \), that is, the last letter of \( \tau w \) is not in the union \( B \) of the \( i \)-th or \( (i+1) \)-st subalphabets for \( \tau R \). Then \( (\chi_R(\tau w))|_B = (\tau w)|_B \) and the difference on the left-hand side of \( (5.5) \) is zero. If \( f(\tau) \in \{i, i+1\} \), then one can deduce \( (5.3) \) using the formula \( (5.3) \) for the two-rectangle case. This proves \( (5.3) \).

As by-product of the above calculation we obtain

\[
(5.6) \quad f(\tau_i \tau) \in \{f(\tau), \tau_i f(\tau)\}
\]

for any \( \tau \in S_t \) and \( 1 \leq i \leq t - 1 \).

For a statement \( P \) let \( \chi(P) \) be 1 if \( P \) holds and 0 otherwise. Define

\[
\Delta c(f):= \sum_{\tau \in S_t} \sum_{i=1}^{t-1} (t - i)(\chi(f(\tau) = f(\tau \tau_i) = i + 1) - \chi(f(\tau) = f(\tau \tau_i) = i)).
\]

By \( (5.3) \) we have

\[
\Delta c(f) = t!(c_R(w) - c_R(\chi_R(w))).
\]

To compute \( \Delta c(f) \), it is convenient to replace \( f \) by a simpler function \( f' \) such that \( \Delta c(f') = \Delta c(f) \). Suppose \( f \) has maximum value \( M + 1 \) for \( 1 < M < t \). Define

\[
f'(\tau) = \begin{cases} 
f(\tau) & \text{if } f(\tau) \neq M + 1 \\
M & \text{if } f(\tau) = M + 1.
\end{cases}
\]

By definition one has the equality of inverse images

\[
(5.7) \quad f^{-1}(i) = (f')^{-1}(i) \quad \text{if } i \not\in \{M, M + 1\}.
\]

We have

\[
(5.8) \quad \Delta c(f) - \Delta c(f') = \sum_{i=1}^{t-1} (t - i) \sum_{\tau \in S_t} (\chi(f(\tau) = f(\tau_i \tau) = i + 1) - \chi(f'(\tau) = f'(\tau_i \tau) = i + 1) - \chi(f(\tau) = f(\tau_i \tau) = i) + \chi(f'(\tau) = f'(\tau_i \tau) = i)).
\]
The summand is zero unless \( \{i, i+1\} \cap \{M, M + 1\} \neq \emptyset \) or equivalently \( i \in \{M - 1, M, M+1\} \). For \( i \in [t - 1] \), let \( Y_{i, \tau} \) be the inner sum of (5.8). We have

\[
Y_{M-1, \tau} = \chi(f(\tau) = f(\tau_{M-1} \tau) = M) - \chi(f'(\tau) = f'(\tau_{M-1} \tau) = M)
\]

by (5.7), the definition of \( f \) by (5.7), the definition of \( f \) by the fact that \( f \) has maximum value \( M \) and (5.6), and (5.6) again. We have

\[
Y_{M, \tau} = \chi(f(\tau) = f(\tau_M \tau) = M + 1) - \chi(f(\tau) = f(\tau_M \tau) = M)
\]

\[
+ \chi(f'(\tau) = f'(\tau_M \tau) = M)
\]

\[
= \chi(f(\tau) = f(\tau_M \tau) = M + 1) - \chi(f(\tau) = f(\tau_M \tau) = M)
\]

\[
+ \chi(\{f(\tau), f(\tau_M \tau)\} \subseteq \{M, M + 1\})
\]

\[
= 2\chi(f(\tau) = f(\tau_M \tau) = M + 1) + \chi(\{f(\tau), f(\tau_M \tau)\} = \{M, M + 1\})
\]

by the fact that \( f' \) has maximum value \( M \), the definition of \( f' \), and considering the four cases for values of \( f(\tau) \) and \( f(\tau_M \tau) \) in the set \( \{M, M + 1\} \). We have

\[
Y_{M+1, \tau} = -\chi(f(\tau) = f(\tau_{M+1} \tau) = M + 1)
\]

\[
= -\chi(f(\tau) = M + 1)
\]

since \( f \) has maximum value \( M + 1 \) and \( f' \) has maximum value \( M \), and (5.6). Define

\[
S_0 = \{\tau \in S_i | f(\tau) = f(\tau_M \tau) = M + 1\}
\]

\[
S_1 = \{\tau \in S_i | f(\tau) = M + 1 \text{ and } f(\tau_M \tau) = M\}.
\]

Note that \( f^{-1}(M + 1) \) is the disjoint union of \( S_0 \) and \( S_1 \), and there is a bijection between \( S_1 \) and the set

\[
\{\tau \in S_i | f(\tau) = M \text{ and } f(\tau_M \tau) = M + 1\}.
\]

Putting the above calculations together we have

\[
\Delta c(f) - \Delta c(f') = -(t - (M - 1))|f^{-1}(M + 1)| + (t - M)(2|S_0| + 2|S_1|)
\]

\[
- (t - (M + 1))|f^{-1}(M + 1)| = 0.
\]

In this calculation, the only property of \( f \) that was used was (5.6). Furthermore it is easy to check that \( f' \) satisfies (5.4) and \( (f')^{-1}(1) = f^{-1}(1) \). Repeating this construction, we obtain a function \( h : S_i \to [t] \) that has maximum value \( 2 \) and satisfies \( \Delta c(h) = \Delta c(f) \), (5.4), and \( h^{-1}(1) = f^{-1}(1) \). Let us explicitly calculate \( \Delta c(h) \). Let \( n_{ij} \) be the number of \( \tau \in S_i \) such that \( h(\tau) = i \) and \( h(\tau_1 \tau) = j \). Then we have

\[
t! = n_{11} + n_{12} + n_{21} + n_{22}
\]

\[
n_{21} = n_{12}
\]

\[
n_{11} + n_{12} = |h^{-1}(1)| = |f^{-1}(1)|
\]

\[
n_{21} + n_{22} = |h^{-1}(2)| = t! - |f^{-1}(1)|
\]
Since $h$ has maximum value 2, the condition $h(\tau) = h(\tau_2 \tau) = 2$ is equivalent to $h(\tau) = 2$. Then

$$\Delta c(f) = \Delta c(h)$$
$$= -(t-1)n_{11} + (t-1)n_{22} - (t-2)(n_{21} + n_{22})$$
$$= -(t-1)n_{11} - (t-2)n_{12} + n_{22}$$
$$= -(t-1)n_{11} - (t-2)n_{12} + (t! - n_{11} - 2n_{12})$$
$$= t! - t n_{11} - t n_{12}$$
$$= t! - t |f^{-1}(1)|$$
$$= t! - t! |N|.$$  

The last equality follows from the fact that for $1 \leq i \leq t$ the elements $\tau_1 \tau_2 \cdots \tau_{i-1}$ form a system of coset representatives of $S_i \times S_{i-1} \setminus S_t$, $f^{-1}(1)$ is a union of the corresponding cosets, each of which has cardinality $(t-1)!$. 

5.10. Proof of Theorem 22.

Proof. To show that $c_R$ is constant on Knuth classes it is enough to show that $w \mapsto d_{i,\sigma R}(\sigma w)$ is, where $\sigma$ acts on $w$ by a composite of $\tau_i$'s. Write $R'_i$ and $R''_i$ for $(\sigma R)_i$ and $(\sigma R)_{i+1}$ and $B$ the union of the $i$-th and $(i+1)$-st alphabets for $\sigma R$. Recall that

$$d_{i,\sigma R}(\sigma w) = d_{R'_i, R''_i}((\sigma w)|_{A'_i \cup A''_i}).$$

Now $v \sim_K w$ implies $\sigma v \sim_K \sigma w$ which implies $(\sigma v)|_B \sim_K (\sigma w)|_B$ by Theorem 3 (A2) and the fact that $B$ is an interval. But then $P((\sigma v)|_B) = P((\sigma w)|_B)$, and the map $d_{i,\sigma R}$ depends only on this $P$ tableau.

(C1) certainly holds for $c_R$. For (C2), in light of Proposition 24, it is enough to show that the set $\mathcal{N}$ is empty. Write $T' <_{K} T$ where $w = ux \in W(R)$, $T = P(w)$, $T' = P(\chi_R(u))$, and $s$ is the cell where the shapes of $T = P(w)$ and $U = P(u)$ differ. Since $s$ lies in a column strictly east of the $a$-th (where $a = \max_i \mu_i$), it follows that $x$ came from a cell of $T$ in a column strictly east of the $a$-th. This means $x \notin A_1$, since $T|_{A_1} = Y_1$ lies entirely in the first $\mu_1$ columns, and $\mu_1 \leq a$. The same argument shows that for any permutation $\sigma \in S_t$, the reverse row insertion on $\sigma T$ at $s$ ejects a number that cannot be in the first subalphabet of $\sigma R$. This shows $\mathcal{N}$ is empty, so that (C2) holds.

(C3): For $\tau \in S_t$, define $\sigma \in S_{t-1}$ by

$$\sigma(i) = \begin{cases} 
\tau(i+1) & \text{if } \tau(i+1) < \tau(1) \\
\tau(i+1) - 1 & \text{if } \tau(i+1) > \tau(1).
\end{cases}$$

There is a bijection $\tau \leftrightarrow (\sigma, \tau(1))$. By Remark 39 and Theorem 33,

$$d_{i,\tau R}(\tau T) = \begin{cases} 
d_{i,\sigma \hat{R}}(\sigma \hat{T}) & \text{if } 1 \leq i < \tau(1) \\
0 & \text{if } i \in \{\tau(1), \tau(1)+1\} \\
d_{i-1,\sigma \hat{R}}(\sigma \hat{T}) & \text{if } \tau(1)+1 < i \leq t-1.
\end{cases}$$

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Then letting \( j = \tau(1) \) we have

\[
\begin{align*}
\tau! c_R(T) &= \sum_{\tau \in S_t} \sum_{i=1}^{t-1} (t-i) d_{i,\tau R}(\tau T) \\
&= \sum_{j=1}^{t} \sum_{\sigma \in S_{t-1}} (\sum_{i=1}^{j-1} (t-i) d_{i,\sigma \hat{R}}(\sigma \hat{T})) \\
&\quad + \sum_{j=j+2}^{t-1} (t-i) d_{i-1,\sigma \hat{R}}(\sigma \hat{T})) \\
&= \sum_{\sigma \in S_{t-1}} (\sum_{j=1}^{t} \sum_{i=1}^{j-1} (t-i) d_{i,\sigma \hat{R}}(\sigma \hat{T})) \\
&\quad + \sum_{j=j+1}^{t-2} (t-i-1) d_{i,\sigma \hat{R}}(\sigma \hat{T})) \\
&= \sum_{\sigma \in S_{t-1}} (\sum_{j=1}^{t} \sum_{i=1}^{j-1} d_{i,\sigma \hat{R}}(\sigma \hat{T})) \\
&\quad - (t-j-1) d_{j,\sigma \hat{R}}(\sigma \hat{T}) + \sum_{i=1}^{t-2} (t-i-1) d_{i,\sigma \hat{R}}(\sigma \hat{T})) \\
&= \tau! c_R(\hat{T}) + \sum_{\sigma \in S_{t-1}} (\sum_{j=1}^{t} \sum_{i=1}^{j-1} d_{i,\sigma \hat{R}}(\sigma \hat{T})) \\
&\quad + \sum_{1 \leq i < j \leq t-1} d_{i,\sigma \hat{R}}(\sigma \hat{T})) \\
&= \tau! c_R(\hat{T}) + \sum_{\sigma \in S_{t-1}} (\sum_{j=1}^{t} \sum_{i=1}^{j-1} d_{i,\sigma \hat{R}}(\sigma \hat{T})) \\
&\quad + \sum_{1 \leq i \leq t-2} (t-i-1) d_{i,\sigma \hat{R}}(\sigma \hat{T})) \\
&= \tau! c_R(\hat{T}).
\end{align*}
\]

(C4) holds by the definition of \( c_R \).

5.11. **Proof of Theorem 21.**

**Proof.** For existence, one has the explicit function \( c_R \) by Theorem 22.

For uniqueness, the proof proceeds by induction on \( c_R \), then on the number of inversions of \( R \) (the number of pairs \( 1 \leq i < j \leq t \) such that either \( \mu_i < \mu_j \) or \( \mu_i = \mu_j \) and \( \eta_i < \eta_j \)), then on the number \( t \) of rectangles in \( R \).

Suppose \( T \) is not \( <_R \)-minimal. Then by Remark 17 (C2) applies and drops \( c_R \). Otherwise suppose \( T \) is \( <_R \)-minimal.

Now suppose \( R \) has an inversion. Then it has an adjacent inversion, say \( (p,p+1) \). Then (C4) applies and \( \tau_p R \) has fewer inversions than \( R \) does. Otherwise suppose \( R \) has no inversions.

In this case \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_t \), so (C3) applies and drops \( t \) by one.
5.12. Proofs of Lemmas for main theorem. Proof of Lemma 29

Proof. Let \((w, T, U) \in \mathcal{S}\) and \(\Phi(w, T, U) = (w, P, Q)\). First it must be shown that \((w, P, Q) \in \mathcal{T}\), that is, the word of \(P\) is \(R\)-LR. By direct computation we have

\[
y^n \cdots y^1 Y_1 \sim_K y^n \cdots y^{m+1} m^{\mu_1} y^m (m-1)^{\mu_1} y^{m-1} \cdots 1^{\mu_1} y^1.
\]

Let \(d = |\alpha(w)| - |R_1|\). Since none of the cycled letters are in the first alphabet \(A_1 = \lfloor m \rfloor\), we have

\[
\text{word}(T)\text{word}(Y_1) \sim_K u^n \cdots u^1 y^n \cdots y^{m+1}\text{word}(Y_1)
\]

\[
= \chi^d_R(y^n \cdots y^{m+1}\text{word}(Y_1)y^m \cdots y^1)
\]

\[
\sim_K \chi^d_R(y^n \cdots y^{m+1} m^{\mu_1} y^m (m-1)^{\mu_1} y^{m-1} \cdots 1^{\mu_1} y^1)
\]

\[
= \chi^d_R(v^n \cdots v^1) \sim_K \chi^d_R(P),
\]

which holds by explicit Knuth equivalences and Lemma 37.

By assumption \(\text{word}(T)\) is \(\bar{R}\)-LR, so by definition \(\text{word}(T)\text{word}(Y_1)\) is \(R\)-LR. But both Knuth equivalence and \(\chi_R\) preserve \(R\)-\(LR\)ness, so \(\text{word}(P)\) is \(R\)-LR and \(\Phi\) is well-defined.

By definition \(\Phi\) is injective, being a composition of injective maps. \(\Phi\) is sign-preserving by definition.

To show that \(\Phi\) is weight-preserving, it suffices to show that each of the \(d\) instances of the operator \(\chi_R\) in the above computation, induce a \(\leq_R\)-covering relation. That is, if

\[
W_i := P(\chi^d_R(y^n \cdots y^{m+1}\text{word}(Y_1)y^m \cdots y^1))
\]

then it must be shown that \(W_{i+1} <_R W_i\). Let us say that a cell is sufficiently east if it lies in a column strictly east of the \(\mu_1\)-st.

Fix \(0 \leq i \leq d - 1\). Write

\[
uxv = y^n \cdots y^{m+1}\text{word}(Y_1)y^m \cdots y^1
\]

where \(v\) has length \(i\) and \(x\) is a letter. Also write \(w^0_0ux = u'x'\) and \(w^0_0v = v'\). By Theorem 35, \(W_i = \chi^d_R(uxv) = v'u'x'\). It is enough to show that the cell \(s\) given by the difference of the shapes of \(P(u'v'x')\) and \(P(u'v')\) is sufficiently east. By Proposition 18 it is enough to show that the cell \(s'\) given by the difference of the shapes of \(P(u'x')\) and \(P(u')\) is sufficiently east. The cell \(s'\) is also the difference of the shapes of \(Q(u'x')\) and \(Q(u')\). Now \(Q(u'x') = Q(w^0_0ux) = Q(ux)\) by Theorem 9(A4). By the definition of recording tableau it follows that \(s'\) is the difference of the shapes of \(Q(ux)\) and \(Q(ux)\), or equivalently, of \(P(ux)\) and \(P(u)\).

Now \(ux\) is an initial subword of \(\text{word}(Y_1)y^n \cdots y^1\) and \(ux\) contains \(\text{word}(Y_1)\). Each of the words \(y_i\) is a weakly increasing word consisting of letters that are strictly greater than those in the tableau \(Y_1\). By Theorem 44 it follows that the tableau \(P(Y_1y^n \cdots y^1)\) has at most \(m\) rows and consists of the tableaux \(Y_1\) and \(P(y^n \cdots y^1)\) sitting side by side. In particular the cells of the difference of the shapes \(P(Y_1y^n \cdots y^1)\) and \(Y_1\) is sufficiently east, which implies that \(s'\) is sufficiently east.

Proof of Lemma 28.

Proof. By definition \(\theta\) is sign-reversing and weight-preserving on all of \(\mathcal{T}\). By Lemma 29 \(\theta\) is an involution. It remains to show that \(\theta\) stabilizes the set \(\Phi(\mathcal{S})\). Let \((w, P, Q) \in \Phi(\mathcal{S})\) and \(\theta(w, P, Q) = (w', P', Q')\). It may be assumed that \((w, P, Q)\)
is not a fixed point of \( \theta \). Let \( v' \) be to \((P',Q')\) as \( v \) is to \((P,Q)\) in the definition of \( \Phi \). It is enough to show that \( (v')^i \) starts with the subword \( i^{\mu_1} \) for every \( 1 \leq i \leq m \), since the other steps in the map \( \Phi \) are invertible by definition.

Let us apply Lemma 49 (see section 6.6) to \( P \) and \( Q' = s_r P \). There is nothing to prove unless \( r \leq m \). Suppose first that \( r < m \). We need only check that \( v'^r \) starts with \( r^{\mu_1} \) and \( v'^{r+1} \) starts with \( (r+1)^{\mu_1} \). Since \( v'^r = v'^{\mu_1} y'^r \) and \( v'^{r+1} = (r+1)^{\mu_1} y'^{r+1} \) where all the letters of \( y'^r \) and \( y'^{r+1} \) are strictly greater than \( m \), it follows that

\[
P(v'^{r+1} v'^r)_{[r,r+1]} = P(v'^{r+1} v'^r)_{[r,r+1]} = P((r+1)^{\mu_1} r^{\mu_1}).
\]

This, together with the fact that \( v'^{r+1} \) and \( v'^r \) are weakly increasing words, implies that all of the letters \( r + 1 \) must precede all of the letters \( r \) in the word \( v'^{r+1} v'^r \), that is, \( v'^i \) starts with \( i^{\mu_1} \) for \( i \in [r,r+1] \).

The remaining case is \( r = m \). Then \( v'^r = v'^{\mu_1} y'^r \) and \( v'^{r+1} = y'^{r+1} \). Let us calculate \( v'^{r+1} \) and \( v'^r \) using a two-row jeu-de-taquin. Let \( V \) (resp. \( V' \)) be the (skew) two row tableau with first row \( v'^r \) (resp. \( v'^r \)) and second row \( v'^{r+1} \) (resp. \( v'^{r+1} \)) in which the two rows achieve the maximum overlap. The overlaps of \( V \) and \( V' \) are equal by Lemma 49 and the fact that \( Q \) and \( Q' \) are in the same \( r \)-string and hence have the same \( r \)-paired letters. Furthermore this common overlap is at least \( \mu_1 \). To see this, note that the overlap weakly exceeds the minimum of \( \mu_1 \) and \( |y'|^{r+1} \) since all of the letters in \( y'^{r+1} \) have values in the alphabet \([m+1,n] = [r+1,n] \) and there are \( \mu_1 \) copies of \( r \) in \( v'^r \). On the other hand, \( |y'^{r+1}| > \mu_1 \), for otherwise by Lemma 49 all of the letters \( r+1 \) in \( Q \) would be \( r \)-paired, contradicting the choice of \( r \).

We calculate \( V' \) from \( V \) in two stages. Let \( V'' \) be the two row skew tableau (whose rows have maximum overlap) such that \( P(V'') = P(V) \), where the first row of \( V'' \) is one cell longer than that of \( V \). By Lemma 44 this tableau exists since \( Q \) has an \( r \)-unpaired letter \( r+1 \); the corresponding recording tableau is \( e_r P \). Furthermore \( V'' \) is obtained by sliding the “hole” in the cell just to the left of the first letter in the first row of \( V' \), into the second row. By the same reasoning as above, \( V'' \) has the same overlap that \( V \) does. Finally we calculate \( V' \) from \( V'' \) by another two row jeu-de-taquin. If the first row of \( V'' \) is shorter than the second, we are done, for in this case the first row \( v''_1 \) of \( V' \) contains the first row of \( V'' \), which in turn contains the first row of \( V \), which contains \( r^{\mu_1} \). So suppose the second row of \( V'' \) is shorter than the first, by \( p \) cells, say. Now \( p \) is less than or equal to the number of cells on the right end of the first column of \( V'' \) that have no cell of \( V'' \) below them. Since \( V'' \) has maximum overlap it follows that when \( p \) holes are slid from the second row of \( V'' \) to the first, they all exchange with numbers lying in the portion of the first row of \( V'' \) that extends properly to the right of the second. Thus the subword \( r^{\tau r} \) remains in the first row of \( V' \), and we are done.

\[\Box\]

**Example 40.** In \( v' \) all the subwords are the same as in \( v \) except that \( v'^2 = 2225677 \) and \( v'^3 = 333567 \). In this example \( r = m \). The tableaux \( V, V'', \) and \( V' \) are given
6. Schensted miscellany

This section is the repository for some well-known (or should be well-known) facts regarding the Robinson-Schensted-Knuth correspondence.

6.1. Evacuation. Given a word $w = w_1 \cdots w_N$ in the alphabet $[n]$, let $w\#_n$ be the reverse of the complement of $w$ with respect to the alphabet $[n]$, that is, the $i$-th letter of $w\#_n$ is $n + 1 - w_{N+1-i}$ for $1 \leq i \leq N$.

Given a column-strict tableau $T$ of partition shape in the alphabet $[n]$, define $T^{ev_n}$ to be the unique column-strict tableau in the alphabet $[n]$ such that the shape of $(T^{ev_n})_{[i]}$ is equal to that of $P(T)_{[n+1-i,n]}$ for all $1 \leq i \leq n$.

**Theorem 41.** Let $w$ be a word of length $N$ in the alphabet $[n]$, $P = P(w)$ and $Q = Q(w)$. Then $P(w\#_n) = P^{ev_n}$ and $Q(w\#_n) = Q^{ev_n}$.

Applying evacuation to LR tableaux produces other LR tableaux.

**Lemma 42.** Let $R = (R_1, \ldots, R_t)$ and $R\# = (R_t, R_{t-1}, \ldots, R_1)$. Then $\#_n$ and $ev_n$ restrict to bijections such that the following diagram commutes:

\[
\begin{array}{ccc}
W(R) & \xrightarrow{RS} & \bigcup_{\lambda} \text{LRT}(\lambda; R) \times \text{ST}(\lambda) \\
\downarrow \#_n & & \downarrow \uparrow_{\lambda(\#_n \times ev_n)} \\
W(R\#) & \xrightarrow{RS} & \bigcup_{\lambda} \text{LRT}(\lambda; R\#) \times \text{ST}(\lambda)
\end{array}
\]

**Proof.** Let $A'_1$ through $A'_t$ be the subalphabets for $R\#$ and $Y'_1$ through $Y'_t$ the corresponding Yamanouchi tableaux. Let $w \in W(R)$, $\# = \#_n$ and $ev = ev_n$. In light of Theorem 41 it is enough to check that $w\#$ is in $W(R\#)$.

\[P(w\#|A'_p) = P((w|A_p)\#) = P(w|A_p)^{ev} = (Y_p)^{ev} = Y'_p.\]

Thus $w\# \in W(R\#)$. \qed

6.2. Removal of large letters.

**Lemma 43.** Suppose $w = w_1 w_2 \cdots w_N$ has maximum letter $n$, occurring in positions $1 \leq i_1 < i_2 < \cdots < i_r \leq N$, and let $\tilde{w}$ be the word obtained by removing these letters $n$ from $w$. Let $P = P(w)$, $Q = Q(w)$, $\tilde{P} = P(\tilde{w})$ and $\tilde{Q} = Q(\tilde{w})$ where $\tilde{w}$ is recorded in $\tilde{Q}$ by the letters in $[N] - \{i_1, \ldots, i_r\}$. Then $\tilde{P} = P(w)_{[n-1]}$ and $Q$ is obtained from $\tilde{Q}$ by the row insertion of the letters $i_1$ through $i_r$ in that order.

6.3. Pieri’s rules.

**Theorem 44.** Let $w = w_1 w_2 \cdots w_N$ and $Q = Q(w)$ its row-insertion recording tableau. Then

1. If $w_i \leq w_{i+1}$ then $i+1$ is strictly east and weakly north of $i$ in $Q$.
2. If $w_i > w_{i+1}$ then $i+1$ is strictly south and weakly west of $i$ in $Q$. 

6.4. Knuth equivalence and recording tableaux.

**Lemma 45.** Let \( v \) and \( w \) be words with \( P(v) = P(w) \). The following are equivalent.
1. \( v \sim_K w \) is an elementary Knuth equivalence of the form \( v = tyxzv \) and \( w = tyzvx \) where \( t \) and \( u \) are words \( x \leq y < z \) are letters with \( x \) and \( z \) in positions \( i \) and \( i + 1 \) in \( v \).
2. \( Q(v) \) and \( Q(w) \) differ by the transposition of \( i \) and \( i + 1 \), and in \( Q(v) \), \( i \) is strictly south and weakly west of \( i - 1 \) and \( i + 1 \) is strictly east and weakly north of \( i - 1 \).

6.5. Recording tableaux. The following result is not hard to prove and appears as [18, Lemma 2.4.4].

**Lemma 46.** Let \( w = uv \) be a word where \( u \) and \( v \) have lengths \( N - k \) and \( k \) respectively. Let \( Q = Q(uv) \). Then
\[
P(Q[v][N-k+1,N]) = Q(v) + N - k
\]
where \( Q(v) + N - k \) is the tableau formed by adding the number \( N - k \) to every entry in \( Q(v) \).

Let \( Q \) be a standard tableau with \( N \) letters. For the integer \( i \), define \( \text{pr}_i(Q) \) to be the standard tableau such that
\[
\text{pr}_i(Q)[|N-i|] = P(Q[i+1,N]) - i
\]
\[
P(\text{pr}_i(Q)[|N-i+1,N|]) = Q[i] + N - i
\]
\( \text{pr}_i(Q) \) can be defined in terms of jeu-de-taquin or exchanging tableaux [2]; it is obtained by exchanging the subtableaux \( Q[i,i] \) and \( Q[i+1,N] \) (and then relabeling). For standard tableaux \( \text{pr}_1 \) is Schützenberger’s promotion operator [17].

The following lemma is not hard to prove using the jeu-de-taquin techniques of [3]. It is crucial that the shape of \( Q \) be both normal (having a unique northwestmost cell) and antinormal (having a unique southwestmost cell), that is, the shape of \( Q \) must be rectangular.

**Lemma 47.** Let \( Q \) be a standard tableau of rectangular shape. Then \( \text{pr}_k(Q) = \text{pr}_1^k(Q) \).

**Lemma 48.** Let \( U \) be a column-strict tableau of partition shape, \( x \) a letter, and \( v \) a word. Let \( s \) (resp. \( s' \)) be the cell given by the difference of the shapes of \( P(Ux) \) (resp. \( P(vUx) \)) and \( U \) (resp. \( P(vU) \)). Then \( s' \) is weakly south and weakly east of \( s \).

**Proof.** Let \( Q \) be the skew standard tableau that records the column insertion of \( v \) into \( P(Ux) \). Then \( s' \) is the vacated cell after a jeu-de-taquin that slides \( Q \) to the northwest into the cell \( s \). This precise statement follows from [18, Lemma 21]. More crudely, the tableau \( P(vU) \) is entrywise smaller than \( U \), viewing empty cells as containing the letter \( \infty \). So the row insertion of \( x \) into \( P(vU) \) must necessarily end at a cell \( s' \) that is weakly south and weakly west of the cell \( s \) where the row insertion of \( x \) into \( U \) ends. \( \square \)
6.6. **Two row jeux-de-taquin.** There is a duality between the crystal operators and jeux-de-taquin on two-row skew column strict tableaux. This is described below.

Define the overlap of the pair \((v, u)\) of weakly increasing words to be the length of the second row in the tableau \(P(vu)\), or equivalently, the maximum number of columns of size two among the skew column strict tableaux with first row \(u\) and second row \(v\).

The following results appear in [20].

**Lemma 49.** Let \((P, Q)\) be the tableau pair obtained by column RSK from the sequence of words \(\{v^i\}\). Then the overlap of the pair of words \((v^{r+1}, v^r)\) is equal to the number of \(r\)-pairs in \(Q\).

**Lemma 50.** Let \(\{v^i\}\) and \((P, Q)\) be as in Lemma 49 and let \(\{v'^i\}\) be another sequence of weakly increasing words with corresponding tableau pair \((P', Q')\). The following are equivalent.

1. \(P = P'\), and \(Q\) and \(Q'\) are in the same \(r\)-string.
2. \(v'^i = v^i\) for \(i \not\in [r, r+1]\) and \(P(v'^{r+1}v^r) = P(v^{r+1}v^r)\).

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