ON SCATTERING FOR GENERALIZED NLS ON WAVEGUIDE MANIFOLDS

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Abstract. In this paper, we prove the large data scattering for fractional nonlinear Schrödinger equations (FNLS) on waveguide manifolds \( \mathbb{R}^d \times \mathbb{T} \), \( d \geq 3 \). This result can be regarded as the fractional analogue of \([43, 44]\) and the waveguide analogue of \([16]\). A key ingredient of the proof is a Morawetz-type estimate for the setting of this model. This result also extends the recent result \([35]\) by proving the scattering behavior.

Keywords: Fractional Schrödinger equation, Strichartz estimate, waveguide manifold, scattering, global well-posedness, Morawetz estimate

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1. INTRODUCTION

We consider the defocusing fractional nonlinear Schrödinger equations (FNLS) on waveguide manifolds \( \mathbb{R}^d \times \mathbb{T} \) in energy space as follows

\[
(i \partial_t + (-\Delta_x)^\sigma - \partial_y^2) u = \mu |u|^p u, \quad u(0, x, y) = u_0 \in H^\sigma(\mathbb{R}^d \times \mathbb{T}),
\]

where \( \mu = 1 \), \( d \geq 3 \) and \( \frac{d\sigma}{d} < p < \frac{d\sigma}{d+1-2\sigma} \).

For FNLS (1.1), like the classical NLS, the following quantities are conserved:

Mass: \( M(u(t)) = \int_{\mathbb{R}^d \times \mathbb{T}} |u(t, x, y)|^2 \, dx \, dy \),

Energy: \( E(u(t)) = \int_{\mathbb{R}^d \times \mathbb{T}} \left( \frac{1}{2} |(\Delta_x)^\sigma u(t, x, y)|^2 + \frac{1}{p+2} |u(t, x, y)|^{p+2} \right) \, dx \, dy \).

The well-posedness theory for model (1.1) has been established recently in \([35]\). However, the long time behavior of this model is still unclear. The purpose of this paper is to show scattering behavior for (1.1), i.e. there exist \( f^\pm \in H^\sigma_{x,y}(\mathbb{R}^d \times \mathbb{T}) \) such that

\[
\lim_{t \to \pm \infty} \|u(t, x, y) - e^{it\Delta_x^\sigma} f^\pm\|_{H^\sigma_{x,y}(\mathbb{R}^d \times \mathbb{T})} = 0.
\]

Scattering behavior is an important property of dispersive equations such as Schrödinger-type equations, which indicates that the nonlinear solutions will resemble the linear solutions in the long run.
1.1. **Statements of the main result.** The main theorem of this paper reads,

**Theorem 1.1.** Let \( \sigma > \frac{1}{2} \) in (1.1) and assume the spatial variable is symmetric in \( \mathbb{R}^d \). Then we have: for any initial datum \( u_0 \in H^\sigma_{x,y} \), the Initial value problem (IVP) (1.1) has a unique global solution \( u(t,x,y) \in C((\infty, \infty); H^\sigma_{x,y}) \); Moreover, the solution scatters in the sense of (1.2).

**Remark 1.2.** We consider the exponent \( p \) in (1.1) is in the subcritical range for some technical reasons. The left endpoint indicates mass-critical if we ignore the torus-direction; the right endpoint indicates energy-critical if we regard the torus-direction as Euclidean-direction. So essentially, the problem is energy-subcritical and mass-supercritical. We will make a few remarks on the critical cases in Section 5. It is also interesting to consider NLNS on more general waveguide manifolds \( \mathbb{R}^d \times T^m \).

**Remark 1.3.** The assumption \( \sigma > \frac{1}{2} \) is also due to some technical reasons if one wants to modify the method in Tzvetkov-Visciglia [43]. Roughly speaking, it is because the Sobolev embedding exponent in 1D is \( \frac{1}{2}+ \).

**Remark 1.4.** Since NLS with a harmonic potential has similar properties/behaviors with NLS on tori, heuristically one may compare NLS with a partial harmonic potential with NLS on waveguides. (See [1, 5, 18] and the references therein. Thus one may conjecture that, it is possible to obtain the analogue of Theorem 1.1 for NLNS with a partial harmonic potential i.e. the following model

\[
(i\partial_t + (-\Delta_x)^\sigma - \partial^2_x + |x|^2)^2)u = \mu|u|^p u, \quad u(0, x, y) = u_0 \in H^\sigma(\mathbb{R}^{d+1}),
\]

where \( \mu = \pm 1 \) and \( \frac{4\sigma}{d+1} < p < \frac{4\sigma}{d+1-2\sigma} \). We leave the research for interested readers.

1.2. **Background.** The FNLS (1.1) is a fundamental equation of fractional quantum mechanics, which was derived by Laskin [29] as a result of extending the Feynman path integral, from the Brownian-like to Lévy-like quantum mechanical paths. The corresponding physical realizations of the FNLS were made in condensed matter physics [36] and in nonlinear optics [30].

The long time behaviors (such as global well-posedness, scattering, blow up and the existence of invariant measures) of the solutions of the FNLS are interesting and widely studied. In [3], the blow up with radial data in certain regime was constructed by deriving a localized virial estimate for the fractional Schrödinger equation. In [16], the first author of this paper together with Guo, Wang and Zhao performs the Kenig-Merle’s concentration-compactness-rigidity method [25] and obtains global well-posedness and scattering of (1.1) in the energy space in the defocusing case, and in the focusing case with energy below the ground state. We also refer the reader to [9, 11, 15, 21, 24, 28, 34, 37, 38, 39, 40, 41] and references therein for many other results on the long time behavior for FNLS. Product spaces \( \mathbb{R}^d \times T^m \) are known as ‘waveguide manifolds’ and are of particular interest in nonlinear optics. We refer to [7, 8, 17, 19, 20, 22, 23, 26, 45, 47, 48, 49] with regard to the torus and waveguide settings.

The well-posedness theory and the long time dynamics for FNLS on waveguide manifolds are understudied. To the authors’ best knowledge, the current work is the first scattering result towards understanding long time dynamics for the FNLS within the context of waveguides.

Now we briefly discuss the outlines of the proof for the main result of this paper (large data scattering). First, we recall the important known results for model (1.1) such as Strichartz-type estimates and the well-posedness theory. Next, we will establish a Morawetz-type estimate for the setting of (1.1). Then, based on the Morawetz theory, we use a contradiction argument to obtain the decay property of (1.1); At last, we use the decay property to control a few spacetime norms of the nonlinear solution and then obtain the scattering result (Theorem 1.1).

1.3. **Notations.** We write \( A \lesssim B \) to say that there is a constant \( C \) such that \( A \leq CB \). We use \( A \simeq B \) when \( A \lesssim B \lesssim A \). Particularly, we write \( A \lesssim u B \) to express that \( A \leq C(u)B \) for some constant \( C(u) \) depending on \( u \).

Then we give some more preliminaries in the setting of waveguide manifold. The tori case can be defined similarly. In fact, it is included since it is a special case. Throughout this paper, we regularly refer to the
spacetime norms
\[ \|u\|_{L^p_t L^q_x(\mathbb{R}^m \times \mathbb{T}^n)} = \left( \int_0^1 \left( \int_{\mathbb{R}^m \times \mathbb{T}^n} |u(t,z)|^q \, dz \right)^{\frac{p}{q}} \, dt \right)^{\frac{1}{p}}. \]

Similarly we can define the composition of three \(L^p\)-type norms like \(L^p_t L^q_x L^r_y\).

1.4. Organizations of the rest of this paper. In Section 2, we give an overview for the well-posedness theory which is covered by [35]; in Section 3, we establish a Morawetz-type estimate for the model (1.1) which is a crucial step for obtaining the decay property of (1.1); in Section 4, we give the proof for the main theorem (large data scattering); in Section 5, we give a few more remarks on the research line of ‘dispersive equations on waveguide manifolds’.

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2. An overview for the well-posedness theory

We recall Strichartz estimates and the local well-posedness theory for ‘FNLS on waveguides’ as follows. These results are discussed and proved in [35].

We note that we say \((p, q)\) is \(\sigma\)-admissible with \(\gamma\) regularity if the previous gap condition is satisfied, i.e.
\[ \frac{2\sigma}{p} + \frac{d}{q} = \frac{d}{2} - \gamma, \quad \frac{2\sigma}{\tilde{p}} + \frac{d}{\tilde{q}} = \frac{d}{2} + \gamma. \]
We say \((p, q)\) is \(\sigma\)-admissible if \(\gamma = 0\).

Following [43] (Proposition 2.1) for the standard case we consider mixed-type norms of Strichartz type. Consider
\[ (i\partial_t + (-\Delta_x)^\sigma - \partial_y^{2\sigma})u = F, \quad u(0, x, y) = u_0 \in H^\sigma(\mathbb{R}^d \times \mathbb{T}), \]
we have

\textbf{Lemma 2.1.} Consider a solution \(u\) of (2.2) (with symmetric-in-\(\mathbb{R}^d\) spatial variable) and admissible pairs \((p, q), (\tilde{p}, \tilde{q})\) satisfying exponent relations (2.1). Then the following estimates hold
\[ \|u\|_{L^p_t L^q_x L^r_y} \lesssim \|u_0\|_{H^\gamma} + \|F\|_{L_x^{\tilde{p}} L_y^{\tilde{q}}}, \]
and
\[ \|u\|_{L^p_t L^q_x H^r_y} \lesssim \|u_0\|_{H^\gamma} + \|F\|_{L_x^{\tilde{p}} L_y^{\tilde{q}}}. \]

Then using the similar method in [43], one can establish the well-posedness theory for (1.1) via a contraction mapping argument based on the Strichartz estimate above,

\textbf{Theorem 2.2.} Let \(\sigma > \frac{1}{2}\) in (1.1) and assume the spatial variable is symmetric in \(\mathbb{R}^d\). Then we have: for any initial datum \(u_0 \in H^\sigma_{x,y}\), the IVP (1.1) has a unique local solution \(u(t, x, y) \in C((-T; T); H^\sigma_{x,y})\) where \(T = T(\|u_0\|_{H^\sigma_{x,y}}) > 0\); Moreover, when \(\mu = 1\) (defocusing case), the solution \(u(t, x, y)\) can be extended globally in time.

After having the global well-posedness theory, it is natural to study the long time behavior for (1.1). To obtain scattering, investigating the decay property of the nonlinear is needed in general (for example, a Morawetz-type estimate). We will discuss the proof of scattering behavior for (1.1) (Theorem 1.1) in Section 3 and Section 4.
3. Morawetz estimates on $\mathbb{R}^d \times T$

In this section, we establish a Morawetz estimate for solutions to (1.1) on $\mathbb{R}^d \times T$. This step is crucial to obtain the decay property of (1.1).

We first define the following Morawetz action on the waveguide $\mathbb{R}^d \times T$:

$$M_\phi[u(t)] := 2 \Im \int_{\mathbb{R}^d \times T} \overline{u(t, x, y)} \nabla_x \phi(x) \cdot \nabla_x u(t, x, y) \, dx \, dy.$$ 

Note that employing the similar idea in [33], the weight function $\nabla_x \phi(x)$ that we chose here depends on only $x$, not on $y$.

Then we present the main result in this section. For a ready-to-use Morawetz estimate, see Corollary 3.2.

**Lemma 3.1.** If $u$ solves (1.1), then the Morawetz action satisfies the identity

$$\frac{d}{dt} M_\phi[u(t)] = \int_0^\infty m^s \int_{\mathbb{R}^d \times T} (4 \partial_x u_m(\partial_x x_1 \phi) \partial_x x_m - \Delta_x^2 \phi |u_m|^2) \, dx \, dy \, dm + \frac{2p}{p + 2} \int_{\mathbb{R}^d \times T} \Delta_x \phi |u|^{p+1} \, dx \, dy.$$ 

**Proof of Lemma 3.1.** Following the strategy in [3], we define

$$\Gamma_\phi := i(\nabla_x \cdot \nabla_x \phi + \nabla_x \phi \cdot \nabla_x),$$

that is

$$\Gamma_\phi f := i[\nabla_x \cdot ((\nabla_x \phi) f) + \nabla_x \phi \cdot \nabla_x f].$$ 

Under this notation, we claim that

$$\langle u(t), \Gamma_\phi(t) \rangle = M_\phi[u(t)].$$ 

Note that $\langle f, g \rangle = \Re \int_{\mathbb{R}^d \times T} \overline{f} \, g \, dx \, dy$.

In fact,

$$\langle u(t), \Gamma_\phi(t) \rangle = \langle u(t), i[\nabla_x \cdot ((\nabla_x \phi) u) + \nabla_x \phi \cdot \nabla_x u] \rangle = \langle u, i\nabla_x \cdot ((\nabla_x \phi) u) \rangle + \langle u, i\nabla_x \phi \cdot \nabla_x u \rangle.$$ 

We then compute the two inner products separately:

$$\langle u, i\nabla_x \phi \cdot u \rangle = \Re \int_{\mathbb{R}^d \times T} \overline{\nabla_x \phi \cdot \nabla_x u} \, dx \, dy = i \Re \int_{\mathbb{R}^d \times T} \nabla_x \phi \cdot \nabla_x u \, dx \, dy = \Im \int_{\mathbb{R}^d \times T} \nabla_x \phi \cdot \nabla_x u \, dx \, dy = \frac{1}{2} M_\phi;$$

and

$$\langle u, i\nabla_x \cdot ((\nabla_x \phi) u) \rangle = \Re \int_{\mathbb{R}^d \times T} \overline{\nabla_x \cdot ((\nabla_x \phi) u)} \, dx \, dy = i \Re \int_{\mathbb{R}^d \times T} \nabla_x \phi \cdot \nabla_x u \, dx \, dy = \Im \int_{\mathbb{R}^d \times T} \nabla_x \phi \cdot \nabla_x u \, dx \, dy = \frac{1}{2} M_\phi.$$ 

Therefore, combining these two terms, we conclude the claim

$$\langle u(t), \Gamma_\phi(t) \rangle = M_\phi[u(t)].$$
Next, we compute the derivative of $M_\phi[u(t)]$ with respect to time $t$. Using (1.1) and Plancherel theorem, we write
\[
\frac{d}{dt} M_\phi[u(t)] = \left\langle \frac{d}{dt} u(t), \Gamma_\phi u(t) \right\rangle + \left\langle u(t), \frac{d}{dt} \Gamma_\phi u(t) \right\rangle
\]
\[
= \left\langle -i(\Delta)^\sigma u - |u|^2 u, \Gamma_\phi u(t) \right\rangle
\]
\[
= \left\langle -i(\Delta)^\sigma u, \Gamma_\phi u(t) \right\rangle + \left\langle -i|u|^2 u, \Gamma_\phi u(t) \right\rangle
\]
\[
= \left\langle u(t), -i(\Delta)^\sigma \Gamma_\phi u(t) \right\rangle + \left\langle u(t), i\Gamma_\phi (u(t)) \right\rangle
\]
\[
= \left\langle u(t), (\Delta)^\sigma \Gamma_\phi u(t) \right\rangle + \left\langle u(t), [u]^p \Gamma_\phi u(t) \right\rangle
\]
\[
= \langle u(t), [(\Delta)^\sigma, \Gamma_\phi] u(t) \rangle + \langle u(t), [u]^p, \Gamma_\phi \rangle u(t) \rangle
\]
where we used the commutator notation $[A, B] = AB - BA$.

Therefore,
\[
\frac{d}{dt} M_\phi[u(t)] = \langle u(t), [(\Delta)^\sigma, \Gamma_\phi] u(t) \rangle + \langle u(t), [u]^p, \Gamma_\phi \rangle u(t) \rangle = I + II.
\]

In the rest of the proof, we will work on the linear term $I$ and the nonlinear term $II$ separately.

First, we consider the linear term $I$. In order to deal with the $[(\Delta)^\sigma, \Gamma_\phi]$ term inside $I$, we will employ the following Balakrishnan’s representation formula for $(-\Delta)^\sigma$ introduced in [2],
\[
(\Delta)^\sigma = \frac{\sin(\pi \sigma)}{\pi} \int_0^\infty m^{\sigma-1} \frac{-\Delta}{-\Delta + m} \, dm,
\]
where $\Delta = \Delta_{\mathbb{R}^d \times T} = \Delta_{x,y}$.

In general, for $A \geq 0$, $m > 0$, the following commutator has the form of
\[
[A + m, B] = m \frac{1}{A + m} [A, B] - \frac{1}{A + m}.
\]
In particular, if taking $A = -\Delta$ and combining with (3.1), we write
\[
[(\Delta)^\sigma, B] = \frac{\sin(\pi \sigma)}{\pi} \int_0^\infty m^{\sigma-1} \frac{1}{-\Delta + m} \frac{1}{-\Delta + m} \, dm.
\]

Then taking $B = i\Gamma_\phi$ in (3.2), we have
\[
[(\Delta)^\sigma, i\Gamma_\phi] = \frac{\sin(\pi \sigma)}{\pi} \int_0^\infty m^{\sigma-1} \frac{1}{-\Delta + m} \frac{1}{-\Delta + m} \, dm.
\]

Now we claim that
\[
(3.3) = -\Delta_{x,y}, i\Gamma_\phi = -4\partial_{x} (\partial_{x} \phi) \partial_{x} - \Delta_x^2 \phi,
\]
where to indicate that $\Delta$ takes derivatives in full space directions, we write $\Delta = \Delta_{x,y}$; and to emphasize that $\Delta_x$ takes derivative only in the $\mathbb{R}^d$ direction, we put $x$ in its subscript. Similarly, in the following calculations, $\partial_{x}$ and $\partial_{x}$ are differential operators in $\mathbb{R}^d$ directions, while $\partial_{y}$ is the $T$ direction derivative.

In fact,
\[
[-\Delta_{x,y}, i\Gamma_\phi] f = -\Delta_{x,y} (i\Gamma_\phi f) + i\Gamma_\phi (\Delta_{x,y} f)
\]
\[
= -\Delta_{x,y} \nabla_x \cdot (\nabla_x \phi f) - \Delta_{x,y} (\nabla_x \phi \cdot \nabla_x f) + \nabla_x ((\nabla_x \phi) \Delta_{x,y} f) + \nabla_x f \cdot \nabla_x (\Delta_{x,y} f)
\]
\[
= -\partial_{x} \partial_{x} \phi (\partial_{x} f) - \partial_{x} \partial_{x} \phi (\partial_{x} f) + \partial_{x} \partial_{x} \phi (\partial_{x} f) + \partial_{x} \partial_{x} \phi (\partial_{x} f)
\]
\[
(3.5) \]
Note that the last line in (3.5) is in fact 0 (derivatives along mixing $\mathbb{R}^d$ and $\mathbb{T}$ directions are 0). A direct computation gives
\[
- \partial_{yy} \partial_x (\partial_x \phi f) - \partial_{yy} (\partial_x \phi \partial_{xx} f) + \partial_x (\partial_x \phi) \partial_{yy} f + \partial_x \phi \partial_{xx} f = - \partial_{yy} \partial_x (\partial_x \phi f) - 2 \partial_{yy} (\partial_x \phi \partial_{xx} f) + \partial_x \phi \partial_{yy} f + \partial_x \phi \partial_{xx} f = - \partial_x \phi \partial_{yy} f - 2 \partial_x \phi \partial_{xx} f + \partial_x \phi \partial_{yy} f + 2 \partial_x \phi \partial_{xx} f = 0.
\]

Hence, combining this, we continue from (3.5) using the product rule
\[
[-\Delta_{xy}, i\Gamma_\phi] f = - \partial_{x_k x_k} (\partial_{x_k x_k} \phi f) - 2 \partial_{x_k x_k} (\partial_{x_k} \phi \partial_{x_k} f) + \partial_{x_k} \phi \partial_{x_k x_k} f + \partial_{x_k} \phi \partial_{x_k x_k} f - \partial_{x_k} \phi \partial_{x_k x_k} f - 2 \partial_{x_k} \phi \partial_{x_k x_k} f - 2 \partial_{x_k} \phi \partial_{x_k x_k} f + \partial_{x_k} \phi \partial_{x_k x_k} f + \partial_{x_k} \phi \partial_{x_k x_k} f = - \Delta_{xy}^2 \phi f - 4 \partial_{x_k} \phi \partial_{x_k} f - 4 \partial_{x_k} \phi \partial_{x_k} f = - 4 \partial_{x_k} (\partial_{x_k} \phi \partial_{x_k} f - \Delta_{xy}^2 \phi f).
\]

This proves the claim (3.4).

At this point, we are in a good position to compute term $I$, and claim:
\[
\langle u(t), [(\Delta)^\sigma, i\Gamma_\phi] u \rangle = \int_{\mathbb{R}^d \times \mathbb{T}} m^\sigma \int_0^\infty (4 \partial_{x_k} u_m (\partial_{x_k} \phi) \partial_{x_k} u_m - \Delta_{xy}^2 \phi |u_m|^2) \, dxdydm.
\]

In fact, combining (3.4) and (3.3), we write
\[
[(\Delta)^\sigma, i\Gamma_\phi] = \frac{\sin(\pi \sigma)}{\pi} \int_0^\infty m^\sigma \int_{-\Delta + m}^\infty \frac{1}{\Delta + m} [-\Delta, i\Gamma_\phi] \frac{1}{\Delta + m} \, dm
\]
(3.6)

Therefore
\[
I = \langle u(t), [(\Delta)^\sigma, i\Gamma_\phi] u \rangle = \int_{\mathbb{R}^d \times \mathbb{T}} \frac{\sin(\pi \sigma)}{\pi} \int_0^\infty \frac{1}{\Delta + m} [-4 \partial_{x_k} (\partial_{x_k} \phi) \partial_{x_k} - \Delta_{xy}^2 \phi] \frac{1}{\Delta + m} \, dm.
\]

For $m > 0$, we define
\[
u_m(t) := c_\sigma \frac{1}{-\Delta_{\mathbb{R}^d \times \mathbb{T}} + m} u(t) = c_\sigma \mathcal{F}^{-1} \left( \frac{\hat{u}(t, \xi)}{|\xi|^2 + m} \right),
\]
where
\[
c_\sigma := \sqrt{\frac{\sin(\pi \sigma)}{\pi}}.
\]

Under such change of variables and with Fubini’s theorem, Plancherel theorem and integration by parts, we obtain
\[
(3.6) = \int_0^\infty \int_{\mathbb{R}^d \times \mathbb{T}} m^\sigma \int_{-\Delta + m}^\infty \frac{1}{\Delta + m} [-4 \partial_{x_k} (\partial_{x_k} \phi) \partial_{x_k} - \Delta_{xy}^2 \phi] u_m dxdydm
\]
\[
= \int_0^\infty m^\sigma \int_{\mathbb{R}^d \times \mathbb{T}} \frac{c_\sigma}{\Delta + m} [-4 \partial_{x_k} (\partial_{x_k} \phi) \partial_{x_k} - \Delta_{xy}^2 \phi] u_m dxdydm
\]
\[
= \int_0^\infty m^\sigma \int_{\mathbb{R}^d \times \mathbb{T}} \frac{\nu_m}{\Delta + m} [-4 \partial_{x_k} (\partial_{x_k} \phi) \partial_{x_k} - \Delta_{xy}^2 \phi] u_m dxdydm
\]
\[
= \int_0^\infty m^\sigma \int_{\mathbb{R}^d \times \mathbb{T}} 4 \partial_{x_k} u_m (\partial_{x_k} \phi) \partial_{x_k} u_m - \Delta_{xy}^2 \phi |u_m|^2 dxdydm.
\]
Then we consider the nonlinear term $II$. Noticing that
\[ \nabla(|u|^{p+2}) = \frac{p+2}{p} \nabla(|u|^p)|u|^2, \]
we obtain
\[ II = \langle u(t), |u|^p, i\Gamma \rangle |u(t)\rangle = \langle u(t), |u|^p, \nabla_x \phi \nabla_x \phi + \nabla_x \phi \nabla_x \phi \rangle u(t) \]
\[ = \langle u(t), |u|^p \nabla_x \phi \nabla_x ((|u|^2)u) \rangle + \langle u(t), |u|^p \nabla_x \phi \nabla_x u \rangle \]
\[ = \frac{2p}{p+2} \int_{\mathbb{R}^d \times T} (\Delta_x \phi)|u|^{p+2} \, dx \, dy. \]

Therefore, together with the computation on terms $I$ and $II$, we get
\[ \frac{d}{dt} M_\phi[u(t)] = \int_0^\infty m^s \int_{\mathbb{R}^d \times T} (4\partial_{x_k} u_m(\partial_{x_k} \phi) \partial_{x_k} u_m \Delta^2 \phi|u_m|^2) \, dx \, dy \, dm + \frac{2p}{p+2} \int_{\mathbb{R}^d \times T} \Delta_x \phi|u|^{p+2} \, dx \, dy, \]
which implies Lemma 3.1.

**Corollary 3.2.** Assume $u$ to be a smooth solution to the initial value problem (1.1) with $d \geq 3$, then we have the following Morawetz inequality
\[ \int_0^\infty \int_{\mathbb{R}^d \times T} \frac{|u(t, x, y)|^{p+2}}{|x|} \, dx \, dy \, dt \lesssim \sup_{t \in \mathbb{R}} \|u(t)\|_{H^\frac{p}{2}}^2 \lesssim \sup_{t \in \mathbb{R}} \|u(t)\|_{H^s}^2. \]

**Proof of Corollary 3.2.** We take $\phi(x) = |x|$ (independent on $y$) in Lemma 3.1. Hence
\[ \nabla_x \phi = \frac{x}{|x|}, \]
\[ \Delta_x \phi = \frac{d-1}{|x|}, \]
\[ \partial_{x_k} \phi = \frac{\delta_{x_k x_l}}{|x|} \frac{x_k x_l}{|x|^3}, \]
\[ \Delta^2 \phi = \begin{cases} -\pi \delta(x), & d = 3, \\ -(d-1)(d-3)|x|^{-3}, & d \geq 4. \end{cases} \]

Under such choice of $\phi$, we claim that
\[ \int_0^\infty m^s \int_{\mathbb{R}^d \times T} (4\partial_{x_k} u_m(\partial_{x_k} \phi) \partial_{x_k} u_m \Delta^2 \phi|u_m|^2) \, dx \, dy \, dm \geq 0. \]
Assuming (3.7), we obtain
\[ \frac{d}{dt} M_{|x|}[u(t)] \geq \frac{2p}{p+2} \int_{\mathbb{R}^d \times T} \Delta_x (|x|)|u(t, x, y)|^{p+2} \, dx \, dy \]
\[ = \frac{2p}{p+2} \int_{\mathbb{R}^d \times T} \frac{d-1}{|x|}|u(t, x, y)|^{p+2} \, dx \, dy, \]
which gives Corollary 3.2 by combining the following upper bound of $M_{|x|}$ in [10]
\[ |M_{|x|}[u(t)]| \lesssim \sup_t \|u(t)\|^2_{H^\frac{p}{2}} \lesssim \sup_t \|u(t)\|^2_{H^s}, \]
and integrating in $t$ using the fundamental theorem of calculus.

To see (3.7), we write (when $d \geq 4$)
\[ \int_{\mathbb{R}^d} \left(4\partial_{x_k} u_m(\partial_{x_k} \phi) \partial_{x_k} u_m \Delta^2 \phi|u_m|^2\right) \, dx \]
\[ = \int_{\mathbb{R}^d} \left(4\partial_{x_k} u_m \frac{\delta_{x_k x_l}}{|x|} \frac{x_k x_l}{|x|^3} \partial_{x_k} u_m - \frac{(d-1)(d-3)}{|x|^3}|u_m|^2\right) \, dx. \]
The $d = 3$ case can be handled similarly, hence omitted.
Using the notation $\nabla_{\vec{e}} u = (\vec{e} \cdot u) \frac{\vec{e}}{|\vec{e}|}$ and $\nabla_{\vec{e}}^2 u = \nabla u - \nabla_{\vec{e}} u$ with $\vec{e} = \vec{x}$, we can decompose $\nabla u$ orthogonally. Then we have

$$
\partial_{x_k} u_m x_l \partial_{x_l} u_m \leq \frac{1}{2} |\partial_{x_k} u_m x_k|^2 + \frac{1}{2} |\partial_{x_l} u_m x_l|^2
$$

$$
\leq \frac{1}{2} |\nabla_{\vec{e}} u_m|^2 |x|^2 + \frac{1}{2} |\nabla_{\vec{e}} u_m|^2 |x|^2
$$

$$
= |\nabla_{\vec{e}} u_m|^2 |x|^2.
$$

Then continuing from (3.9), we obtain

$$(3.9) \geq \int_{\mathbb{R}^d} \left( 4 \frac{|\nabla_{\vec{e}} u_m|^2}{|x|} - 4 \frac{|\nabla_{\vec{e}} u_m|^2 |x|^3}{|x|^3} + \frac{(d-1)(d-3)}{|x|^3} |u_m| \right) dx
$$

$$
= \int_{\mathbb{R}^d} \left( 4 \frac{|\nabla_{\vec{e}} u_m|^2}{|x|} + \frac{(d-1)(d-3)}{|x|^3} |u_m|^2 \right) dx \geq 0,
$$

then (3.7) follows by integrating (3.8) in both $y$ and $m$. This completes the proof of Corollary 3.2.

4. Proof of the scattering result

In this section, we give the proof for the scattering result Theorem 1.1. There are four steps and we will discuss them step by step. The strategy has similar spirit with [43, 44].

4.1. Step 1: the Morawetz bound. Recall the Morawetz estimate (Corollary 3.2) established in Section 3, we have

$$(4.1) \int_\mathbb{R} \int_{\mathbb{R}^d \times T} \frac{|u(t, x, y)|^{2+p}}{|x|} dtdxdy \lesssim_{\text{data}} 1.
$$

4.2. Step 2: Proof of the decay property. Based on the Morawetz bound, we aim to show the decay property of (1.1), i.e.

$$
\lim_{t \to \infty} \|u(t, x, y)\|_{L^q(x \times \mathbb{T})} = 0,
$$

where $2 < q \leq 2 + r$ (for any $r < \frac{2 + 4r + dp + 2p^2}{2d}$).

Remark 4.1. One may also consider the (stronger) pointwise type decay which describes the decay rate of nonlinear solutions quantitatively. See [14, 46] for recent results and the references therein.

In viewing of interpolation with mass conservation law, it suffices to show the endpoint case, that is,

$$
(4.2) \lim_{t \to \infty} \|u(t, x, y)\|_{L^2(x \times \mathbb{T})} = 0.
$$

We will prove it by contradiction. Before starting with the proof, we recall a radial Sobolev embedding as follows,

Lemma 4.2 (Radial Sobolev Embeddings in $\mathbb{R}^d$ in [42]). Let $d \geq 1$, $1 \leq q < \infty$, $0 < s < d$ and $\beta \in \mathbb{R}$ obey the conditions

$$
\beta > -\frac{d}{q}, \quad 0 \leq \frac{1}{p} - \frac{1}{q} \leq s
$$

and the scaling condition

$$
\beta + s = \frac{d}{p} - \frac{d}{q}
$$

We will prove it by contradiction. Before starting with the proof, we recall a radial Sobolev embedding as follows,
with at most one of the equalities

\[ p = 1, \quad p = \infty, \quad q = 1, \quad q = \infty, \quad \frac{1}{p} - \frac{1}{q} = s \]

holding. Then for any spherically symmetric function \( f \in \dot{W}^{s,p}(\mathbb{R}^d) \), we have

\[ \| |x|^\beta f \|_{L^q(\mathbb{R}^d)} \lesssim \| \nabla |x|^s f \|_{L^p(\mathbb{R}^d)}. \]

Let \( \beta \) satisfies \((2 + p)\beta + 1 = 2 + r\).

Via Lemma 4.2, H"older inequality and Sobolev embedding, we have

\[
\| u(t, x, y) \|_{L_\infty^{\frac{2+\tau}{r}}(\mathbb{R}^d \times T)} = \left( \int |u(t, x, y)|^{2+r} \, dx \, dy \right)^{\frac{1}{2+r}} \\
= \left( \int \frac{|u(t, x, y)|^{(2+p)\beta}}{|x|^\beta} \cdot |x|^\beta |u| \right)^{\frac{1}{2+r}} \\
\lesssim \left( \int_{\mathbb{R}^d \times T} \frac{|u(t, x, y)|^{2+p}}{|x|} \, dx \, dy \right)^{\frac{\beta}{2+r}} \cdot \left( \int (|x|^\beta |u|)^{\frac{1}{2+r}} \, dx \, dy \right)^{\frac{1}{2+r}} \\
= \left( \int_{\mathbb{R}^d \times T} \frac{|u(t, x, y)|^{2+p}}{|x|} \, dx \, dy \right)^{\frac{\beta}{2+r}} \cdot \left( \| |x|^\beta |u| \|_{L^1_{x,y}} \right)^{\frac{1}{2+r}} \\
\lesssim \left( \int_{\mathbb{R}^d \times T} \frac{|u(t, x, y)|^{2+p}}{|x|} \, dx \, dy \right)^{\frac{\beta}{2+r}} \cdot \left( \| |\nabla|^s u \|_{L^2_{x,y}} \right)^{\frac{1}{2+r}} \\
\lesssim \left( \int_{\mathbb{R}^d \times T} \frac{|u(t, x, y)|^{2+p}}{|x|} \, dx \, dy \right)^{\frac{\beta}{2+r}} \\
\lesssim \left( \int_{\mathbb{R}^d \times T} \frac{|u(t, x, y)|^{2+p}}{|x|} \, dx \, dy \right)^{\frac{\beta}{2+r}}.
\]

We require the indices satisfy:

\[ s + \tau \leq \sigma \quad \text{(regularity requirement from the energy conservation)}, \]
\[ (\beta - \frac{1}{2})+ = \tau \quad \text{(Sobolev embedding in 1D)}, \]
\[ \beta + s = \frac{d}{2} - d(1 - \beta) \quad \text{(radial Sobolev embedding)}, \]
\[ (2 + p)\beta + 1 = 2 + r \quad \text{(the relation between \( \beta \) and \( r \))}. \]

We need to choose \( \beta \) satisfies \( \beta < \frac{1}{2} + \frac{d+\sigma}{d} \). Correspondingly, \( r < \frac{2+4\sigma+d\sigma+p+2p\sigma}{2d} \). That is the exponent requirement in the decay estimate (4.2).

We are now ready to prove (4.2) by contradiction argument. If (4.2) does not hold, using the estimate above, we deduce the existence of a sequence \( \{t_n\}_n \to \infty \) and \( \epsilon_0 > 0 \) such that

\[
\left( \int_{\mathbb{R}^d \times T} \frac{|u(t_n, x, y)|^{2+p}}{|x|} \, dx \, dy \right)^{\frac{1}{2+r}} \geq \epsilon_0 > 0.
\]

Without loss of generality, we consider \( \{t_n\}_n \to +\infty \). Similar as in [43], we get the existence of \( T > 0 \) such that

\[
\inf_{n} \inf_{t \in [t_n, t_n+T]} \left( \int_{\mathbb{R}^d \times T} \frac{|u(t, x, y)|^{2+p}}{|x|} \, dx \, dy \right)^{\frac{1}{2+r}} \geq \frac{\epsilon_0}{2}.
\]
Notice that since \( \{t_n\} \rightarrow +\infty \) then we can assume (modulo subsequence) that the intervals \((t_n, t_n + T)\) are disjoint. In particular we have
\[
\sum_n T(\frac{\epsilon_0}{2})^{2+p} \leq \sum_n \int_{t_n}^{t_n+T} \int_{\mathbb{R}^d \times T} \frac{|u(t, x, y)|^{2+p}}{|x|} \, dx \, dy \, dt
\]
\[
\leq \int \left( \int_{\mathbb{R}^d \times T} \frac{|u(t, x, y)|^{2+p}}{|x|} \, dx \, dy \right) \, dt
\]
and hence we get a contradiction since the left hand side is divergent and the right hand side is bounded by (4.1).

4.3. Step 3: Proof of the spacetime bound. We aim to show,
\[
(4.3) \quad u \in L^q_t L^p_x H^{1+\delta}_y (\mathbb{R}^t \times \mathbb{R}^d \times T)
\]
and
\[
\lim_{t_1, t_2 \rightarrow \infty} \left( \| u \|_{L^q_t L^p_x L^{2}_y ([t_1, t_2] \times \mathbb{R}^d \times T)} + \| \nabla_x \|_{L^q_t L^p_x L^{2}_y ([t_1, t_2] \times \mathbb{R}^d \times T)} + \| \partial_y \|_{L^q_t L^p_x L^{2}_y ([t_1, t_2] \times \mathbb{R}^d \times T)} \right) < \infty.
\]

The above spacetime bounds are sufficient to show the scattering for (1.1). In this step, all spacetime norms are over \( \mathbb{R}^t \times \mathbb{R}^d \times T \) unless indicated otherwise. For example, we define
\[
\| f(t) \|_{L^q_t} := \left( \int_{t_0}^{\infty} |f(t)|^q \, dt \right)^{\frac{1}{q}}
\]
for any given time-dependent function \( f(t) \), and similarly we can define \( \| f(t) \|_{L^q_t} \). We note that we will apply an \( H^{1+\delta}_y \) valued version of the critical analysis of [4].

**Proof.** Using Strichartz estimates and the Hölder inequality,
\[
\| u \|_{L^q_t L^p_x L^{2}_y H^{1+\delta}_y} \lesssim \| u_0 \|_{H^s(\mathbb{R}^d \times T)} + \| |u|_t^p \|_{L^q_t L^p_x L^{2}_y H^{1+\delta}_y}
\]
\[
\lesssim \| u_0 \|_{H^s(\mathbb{R}^d \times T)} + \| u \|_{L^{1+p}_t L^{(1+p)}_x}^{1+p} \| L^{(1+p)}_x \|^{1-\delta} H^{1+\delta}_y
\]
\[
\lesssim \| u_0 \|_{H^s(\mathbb{R}^d \times T)} + \| u \|_{L^{q_0}_t L^{q_0}_x L^{2}_y} \| u \|^{(1+p)(1-\delta)}_{L^{q_0}_t L^{q_0}_x L^{2}_y} H^{1+\delta}_y.
\]
Similar to Lemma 2.5 in [43] (this lemma is an analysis result which does not involve the nonlinear PDE structure so we can use it directly), based on the decay property (4.2), we can further obtain
\[
(4.4) \quad \| u \|_{L^q_t L^p_x H^{1+\delta}_y} = o(1).
\]

Using the decay property (4.4), we see for every \( \epsilon > 0 \) there exists \( t_0 = t_0(\epsilon) > 0 \) such that
\[
\| u \|_{L^q_t L^p_x L^{2}_y H^{1+\delta}_y} \leq C \| u_0 \|_{H^2(\mathbb{R}^d \times T)} + \epsilon \| u \|_{L^q_t L^p_x L^{2}_y H^{1+\delta}_y}.
\]

We can now use the continuity argument to obtain
\[
\| u \|_{L^q_t L^p_x L^{2}_y H^{1+\delta}_y} < \infty.
\]
Similarly, we obtain \( \| u \|_{L^q_t L^p_x L^{2}_y H^{1+\delta}_y} < \infty \).

Now we consider the second estimate. We show \( \| \partial_y \|_{L^q_t L^p_x L^{2}_y} \), the other estimates are similar. Using Strichartz estimate and the Hölder inequality,
\[
\| \partial_y \|_{L^q_t L^p_x L^{2}_y} \lesssim \| u_0 \|_{H^s(\mathbb{R}^d \times T)} + \| \partial_y \|_{L^q_t L^p_x L^{2}_y} \| u \|_{L^q_t L^p_x L^{2}_y} \| u \|_{L^q_t L^p_x L^{2}_y}.
\]
We conclude by choosing $t_0$ large enough and by recalling (4.3).

For FNLS, due to the Strichartz estimates and Sobolev embedding, we choose the indices satisfying,

$$s + \frac{1}{2} + \delta \leq \sigma, \quad \text{(regularity requirement from the energy conservation)}$$

$$\frac{2\sigma}{q_0} + \frac{d}{r_0} = \frac{d}{2} - s, \quad \frac{2\sigma}{q_0} + \frac{d}{r_0} + \frac{2\sigma}{p_0} + \frac{d}{r_0} = d, \quad \text{(Strichartz exponent relations)}$$

$$\frac{1}{(p + 1)q_0} = \frac{\theta}{q_0}, \quad \frac{1}{(p + 1)r_0} = \frac{\theta}{r_0} + \frac{2(1 - \theta)}{pd}, \quad \text{(the Hölder inequality, or say, interpolation)}$$

and

$$\frac{2\sigma}{l} + \frac{d}{m} = \frac{d}{2}, \quad \frac{1}{m'} = \frac{1}{m} + \frac{p}{r_0}, \quad \frac{1}{l'} = \frac{1}{l} + \frac{p}{q_0} \quad \text{(Strichartz exponent relations and the Hölder inequality).}$$

\[\square\]

4.4. **Step 4: Proof of the scattering asymptotics.** In fact by using the integral equation, it is sufficient to prove that

$$\lim_{t, t_2 \to \infty} \left\| \int_{t_1}^{t_2} e^{-its((-\Delta_x)^{\sigma} - \partial_y^{2\sigma})} (|u|^p u) \, ds \right\|_{H_x^s(y_0(R^d \times \mathbb{T}))} = 0.$$ 

Moreover, using Strichartz estimates, we only need to show,

$$\lim_{t, t_2 \to \infty} \left( \left\| |u|^p u \right\|_{L_t^1 L_y^p \cap L_t^2 L_y^2((t_1, t_2) \times \mathbb{R}^d \times \mathbb{T})} + \left\| \nabla_x |u|^p u \right\|_{L_t^1 L_y^p \cap L_t^2 L_y^2((t_1, t_2) \times \mathbb{R}^d \times \mathbb{T})} + \left\| \partial_y |u|^p u \right\|_{L_t^1 L_y^p \cap L_t^2 L_y^2((t_1, t_2) \times \mathbb{R}^d \times \mathbb{T})} \right) = 0.$$ 

Noticing the two established estimates, the above limit follows in a straightforward way. Thus we proved scattering in the energy space.

5. **Further remarks**

In this section, we make a few more remarks on this research line, i.e. *long time dynamics for dispersive equations on waveguide manifolds*. As mentioned in the introduction, this area has been developed a lot in recent decades. The authors are interested in this research line for several years. Though many theories/tools/results have been established, there are still many interesting open questions left. We list some interesting related problems in this line for interested readers.

1. **The critical regime.** The cases we are considering in this paper are of ‘double subcritical’ nature (see (1.1)). In fact, it is also quite interesting to consider the scattering theory for the critical regime. For example,

$$(i\partial_t + (-\Delta_x)^{\sigma} - \partial_y^{2\sigma})u = |u|^pu, \quad u(0) = u_0(x, y) \in H_x^s(\mathbb{R}^d \times \mathbb{T}),$$

and

$$(i\partial_t + (-\Delta_x)^{\sigma} - \partial_y^{2\sigma})u = |u|^pu, \quad u(0, x, y) = u_0 \in H_x^s(\mathbb{R}^d \times \mathbb{T}).$$

The first one is of mass-critical nature and the second one is of energy critical nature. New techniques are needed including function spaces, profile decomposition, profile approximations and even resonant systems. See [7, 17, 47, 48] for the NLS case.

2. **Improvements/generalizations for Theorem 1.1.** One may also try to remove the radial assumption in Theorem 1.1 or consider FNLS on more general waveguide manifolds $\mathbb{R}^n \times \mathbb{T}^m$. However, when $m \geq 2$, one can at most show the global well-posedness since the scattering is not expected to hold. Another point is to consider the cases $d = 1, 2$ for model (1.1). Some more techniques are required, such as the Morawetz-type estimate in 1D and 2D (see Section 3).
One may consider other problems for model (1.1) such as growth of Sobolev norms (weak turbulence) or low regularity type results (see [49] and the references therein).

3. Scattering for focusing NLS/4NLS/FNLS on waveguide manifolds. The model discussed in this paper concerns the defocusing case. In general, large data scattering for the focusing NLS (or other dispersive equations) on waveguides are comparably less understood than the defocusing case. Threshold assumptions are necessary and new ingredients are needed to handle this type of problems. See [45] for a recent global well-posedness result, see [12, 13, 25, 27] for the Euclidean results and see [6, 31] for some very recent scattering result.

4. Critical NLS on higher dimensional waveguide manifolds. For critical NLS (or other dispersive models) on waveguide manifolds, most of the models are lower dimensional (with no higher than four whole dimensions), which leads quintic or cubic nonlinearity. This gives one advantages to apply function spaces to deal with the nonlinearity. In general, the difficulty of the critical NLS problem on $\mathbb{R}^m \times \mathbb{T}^n$ increases if the dimension $m + n$ is increased or if the number $m$ of copies of $\mathbb{R}$ is decreased (which is concluded in [23]). There is no large data global results for critical NLS on waveguide manifolds with at least 5 whole dimensions, to the best knowledge of the authors. Moreover, the Hartree analogues are also less understood.

5. NLS on other product spaces. Instead of waveguide manifolds, one may consider dispersive equations on other types of product spaces, for example, $\mathbb{R}^d \times S^n$ where $S^n$ are n-dimensional spheres ($S^n$ can be replaced by other manifolds). See [32] for a global well-posedness result of NLS on pure spheres. In this regime, NLS may be a good model to start with. One can also replace $S^n$ by other manifolds.

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