A short study of generalized Dirichlet Integrals via tempered distributions.

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Abstract. In the present article, the author uses Fourier theory of tempered distributions (generalized functions) in deriving a formula for Dirichlet-like integrals. The applied method is remarkably efficient and allows a solution in a few calculational steps. The interest of the present article lies not only in the derivation of an apparently unknown formula but also in the original methodology of its derivation.

1. Introduction

In the present article we are concerned with integrals of the following type:

\[ I(m, n) := \int_0^\infty \frac{\sin^n x}{x^m} \, dx \]

where \( n, m \) are positive integers.

The improper integral \( I(m, n) \) is absolut convergent when \( n \geq m \geq 2 \).

The case \( m = 1 \) has to be treated separately since the integrals \( I(1, n) \) are not absolut convergent. It turns out that these integrals are divergent for even \( n \) and convergent for odd \( n \). A corresponding formula for the case of equal exponents \( n = m \) is known \([1, 3]\). The simplest case \( n = m = 1 \) is the well-known Dirichlet Integral:

\[ I(1, 1) = \int_0^\infty \frac{\sin x}{x} \, dx = \frac{1}{2\pi} \]

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The main purpose of this article is to derive the following formulae.

In the case $n \geq m \geq 2$:

\begin{equation}
I(m, n) = \frac{i^{m-n+1}}{2^n(m-1)!} \sum_{l=0}^{n} (-1)^l \binom{n}{l} (n-2l)^m \ln[i(2l-n)]
\end{equation}

In the case $m = 1$:

\begin{equation}
I(1, n) = \begin{cases} 
\text{divergent} & n \text{ even} \\
\frac{1}{2} \sum_{l=0}^{n} (-1)^l \binom{n}{l} \ln[i(2l-n)] & n \text{ odd}
\end{cases}
\end{equation}

where in formula (1.3) the term $2l = n$ is omitted in the summation. Note that the complex logarithm in the above formulae may be written as follows:

\[\ln[i(2l - n)] = \ln|2l - n| + \frac{i\pi}{2} \text{sign}(2l - n)\]

**Example 1.1.**

\begin{align*}
I(1, 1) &= \frac{1}{2} \pi & I(1, 2) &= \text{divergent} & I(1, 3) &= \frac{1}{4} \pi \\
I(2, 2) &= \frac{3}{2} \pi & I(2, 3) &= \frac{3}{4} \ln 3 & I(2, 4) &= \frac{1}{4} \pi \\
I(3, 3) &= \frac{3}{8} \pi & I(3, 4) &= \ln 2 & I(3, 5) &= \frac{5}{32} \pi \\
I(4, 4) &= \frac{1}{3} \pi & I(4, 5) &= \frac{125}{96} \ln 5 - \frac{45}{32} \ln 3 & I(4, 6) &= \frac{1}{8} \pi \\
I(5, 5) &= \frac{115}{384} \pi & I(5, 6) &= \frac{27}{16} \ln 3 - 2 \ln 2 & I(5, 7) &= \frac{77}{768} \pi
\end{align*}

**1.1. Dirichlet Integral.** In the present section we introduce the method by calculating the well-known Dirichlet Integral. We rewrite this integral in the following suggestive form:

\[\int_{0}^{\infty} \frac{\sin kx}{x} dx = \frac{1}{2i} \int_{-\infty}^{\infty} \frac{\Theta(x) e^{-i(-k)x}}{x} - \frac{\Theta(x) e^{-ikx}}{x} dx\]

Where $\Theta(\cdot)$ denotes Heaviside’s unit step function defined by:

\[\Theta(x) = \begin{cases} 
1 & \text{if } x > 0 \\
\frac{1}{2} & \text{if } x = 0 \\
0 & \text{if } x < 0
\end{cases}\]

Obviously, splitting of the latter integral is not possible. However, if one reinterprets the quantity $\frac{\Theta(x)}{x}$ as a tempered distribution rather than a function (See Appendix A) one may write:

\begin{equation}
\int_{0}^{\infty} \frac{\sin kx}{x} dx = \frac{1}{2i} \mathcal{F} \left[ \frac{\Theta(x)}{x} \right]_{-k} - \frac{1}{2i} \mathcal{F} \left[ \frac{\Theta(x)}{x} \right]_{k}
\end{equation}
Note that the Fourier transform \( \mathcal{F} \left[ \frac{\Theta(x)}{x} \right] \) is well-defined in the context of tempered distributions. The explicit formula reads as follows (See Appendix A):

\[
\mathcal{F} \left[ \frac{\Theta(x)}{x} \right] = -\gamma - \ln |k| - i \frac{\pi}{2} \text{sign}(k) = -\gamma - \ln i k
\]

where \( \gamma = 0.5772156 \ldots \) denotes Euler’s constant. By inserting formula (1.6) into Eq. (1.5) we obtain the correct result:

\[
\int_{0}^{\infty} \frac{\sin kx}{x} \, dx = \frac{\pi}{2} \text{sign}(k)
\]

2. Derivation of the general formula

In the present section we derive a general formula for the integrals \( \int_{1}^{\infty} \frac{\sin kx}{x} \, dx \). For this purpose we need the following Lemma.

**Lemma 2.1.**

\[
\left( \frac{\Theta(x)}{x} \right)^{(n)} = -\delta^{(n)}(x) \left[ 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \right] + (-1)^n n! \left( \frac{\Theta(x)}{x^{n+1}} \right)
\]

where \( \delta^{(n)}(x) \) denotes the \( n \)th derivative of Delta-Dirac-distribution.

**Proof.** We give a proof by induction on \( n \). Evidently, the Lemma holds for \( n = 1 \):

\[
\left( \frac{\Theta(x)}{x} \right)' = \delta(x) \cdot \frac{1}{x} - \frac{\Theta(x)}{x^2} = -\delta'(x) - \frac{\Theta(x)}{x^2}
\]

Where the following well-known identities have been used.

\[
\Theta'(x) = \delta(x)
\]

\[
\delta'(x) = -\frac{\delta(x)}{x}
\]

Inductive step \( n \Rightarrow n + 1 \):

\[
\left( \frac{\Theta(x)}{x} \right)^{(n+1)} = -\delta^{(n+1)}(x) \left[ 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \right] + (-1)^n n! \left[ \delta(x) - \frac{\Theta(x)}{x^{n+1}} \right] \left[ 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1} \right] + (-1)^{n+1} (n+1)! \left( \frac{\Theta(x)}{x^{n+2}} \right)
\]

Where the following equally well-known formula for the \( n \)th derivative of delta function has been used:

\[
\delta^{(n)}(x) = (-1)^n n! \frac{\delta(x)}{x^n}
\]

Lemma 2.1 leads to the following Corollary:

**Corollary 2.1.**

\[
\mathcal{F} \left[ \frac{\Theta(x)}{x^{n+1}} \right] = \left( \frac{\Theta(x)}{x} \right)^{(n)} + \sum_{l=1}^{n} \frac{1}{l}
\]

\[
\mathcal{F} \left[ \frac{\Theta(x)}{x} \right] = (-i x)^n \frac{\Theta(x)}{n!} \left( \mathcal{F} \left[ \frac{\Theta(x)}{x} \right] + \sum_{l=1}^{n} \frac{1}{l} \right)
\]
Proof. On one hand, $\Theta(x)/x$ fulfills the following identity:

$$F\left(\frac{\Theta(x)}{x}\right)^{(n)} = (ix)^n F\left(\frac{\Theta(x)}{x}\right)$$

On the other hand, according to Lemma 2.1, we have:

$$F\left(\frac{\Theta(x)}{x}\right)^{(n)} = -(ix)^n \sum_{l=1}^{n} \frac{1}{l} + (-1)^n n! F\left(\frac{\Theta(x)}{x^n+1}\right)$$

We obtain formula of Corollary 2.1 by equating Eqs. 2.2, 2.3.

It is now a straightforward task to derive a formula for the integrals 1.1. By virtue of the Eulerian formula we may use:

$$\sin^n x = \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^n$$

Expanding the latter expression yields:

$$\sin^n x = \frac{1}{(2i)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l e^{-i2l-n}$$

By inserting expansion 2.4 into 1.1 and by adding an exponential we obtain formally:

$$\int_{0}^{\infty} \frac{\sin^n x}{x^m} e^{-\epsilon x} \, dx = \frac{1}{(2i)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l F\left(\frac{\Theta(x)}{x^m}\right)_{k=2l-n-i\epsilon}$$

We now take advantage of Corollary 2.1 and formula 1.6 which leads to:

$$\int_{0}^{\infty} \frac{\sin^n x}{x^m} e^{-\epsilon x} \, dx = \frac{i^{m-n+1}}{2^n (m-1)!} \sum_{l=0}^{n} (-1)^l \binom{n}{l} (n-2l + i\epsilon)^{m-1} \ln[i(2l - n - i\epsilon)]$$

$$+ (C - \gamma) \frac{i^{m-1-n}}{2^n (m-1)!} \sum_{l=0}^{n} (-1)^l \binom{n}{l} (n-2l + i\epsilon)^{m-1}$$

Where $C := 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m-1}$

In the next step we take the limit $\epsilon \to 0$ on both sides of Eq. 2.6.

In the case $n \geq m > 1$ we obtain the formula:

$$\int_{0}^{\infty} \frac{\sin^n x}{x^m} \, dx = \frac{i^{m-n+1}}{2^n (m-1)!} \sum_{l=0}^{n} (-1)^l \binom{n}{l} (n-2l + i\epsilon)^{m-1} \ln[i(2l - n)]$$

Where the following identity has been used (See Appendix B)

$$\sum_{l=0}^{n} (-1)^l \binom{n}{l} (n-2l)^{m-1} = 0, \quad n \geq m \geq 2$$
In the case \( m = 1 \) we get:

\[
\int_0^\infty \frac{\sin^n x}{x} \, dx = \begin{cases} 
\text{divergent} & \text{n even} \\
-\frac{1}{12} \sum_{l=0}^{n} (-1)^l \binom{n}{l} \ln[i(2l - n)] & \text{n odd}
\end{cases}
\]

where the following elementary identity has been used:

\[
\sum_{l=0}^{n} (-1)^l \binom{n}{l} = (1 - 1)^n = 0
\]

**Remark 2.1.** By using the identity

\[
\ln[i(2l - n)] = \ln|2l - n| + i \frac{\pi}{2} \text{sign}(2l - n)
\]

one derives easily the following formula for \( n > m > 1 \):

\[
\int_0^\infty \frac{\sin^n x}{x^m} \, dx = \begin{cases} 
\frac{\pi}{2} \sum_{l=0}^{m-1} (-1)^l \binom{n}{l} (n - 2l)^{m-1} \ln[|2l - n|] & \text{if n even} \\
\frac{\pi}{2} \sum_{l=0}^{m-1} (-1)^l \binom{n}{l} (n - 2l)^{m-1} \ln|2l - n| & \text{if n odd}
\end{cases}
\]

A formula without case distinction is possible by using the following definitions:

\[
0^{m-1} \ln(i \cdot 0) := 0 \quad m = 2, 3, \ldots \\
0^{m-1} \ln(i \cdot 0) := +\infty \quad m = 1
\]

Presuming this we may write down a formula valid in the entire range \( n \geq m > 1 \):

\[
I(m, n) = \begin{cases} 
\frac{\pi}{2} \sum_{l=0}^{m-1} (-1)^l \binom{n}{l} (n - 2l)^{m-1} \ln[|2l - n|] & \text{if n even} \\
\frac{\pi}{2} \sum_{l=0}^{m-1} (-1)^l \binom{n}{l} (n - 2l)^{m-1} \ln|2l - n| & \text{if n odd}
\end{cases}
\]

Evidently, these definitions get their justification due to the following limits:

\[
\lim_{x \to 0} x^{m-1} \ln ix = \begin{cases} 
0 & \text{if } m = 2, 3, \ldots \\
\infty & \text{if } m = 1
\end{cases}
\]

**Appendix A.**

We rewrite the function \( \Theta(x)/x \) in the form:

\[
\Theta(x) = \frac{1}{2|x|} + \frac{1}{2x}
\]

It is well-known that the functions \( 1/|x| \) and \( 1/x \) can be turned into tempered distributions via the following operations [4, 5, 2]:

\[
\text{PV} \frac{1}{x} := \lim_{\epsilon \to 0^+} \int_{|x| > \epsilon} \frac{\varphi(x)}{x} \, dx
\]

\[
\text{Pf} \frac{1}{|x|} := \text{f. p. of } \lim_{\epsilon \to 0^+} \int_{|x| > \epsilon} \frac{\varphi(x)}{|x|} \, dx
\]
Where $PV$ refers to Cauchy’s Principal Value and $Pf$ to Hadamard’s finite part. The function $\varphi$ denotes a test function in the Schwartz space $\mathcal{S}(\mathbb{R})$ of smooth and rapidly decaying functions.

As a tempered distribution $\Theta(x)/x$ has a well-defined Fourier transform. In order to derive it we take advantage of the following formula [2]:

\[(A.1)\quad \mathcal{F}\left[\frac{1}{|x|}\right] = -2\gamma - 2 \ln |x|\]

By inserting the identity

\[\frac{1}{|x|} = \frac{2 \Theta(x) - 1}{x}\]

into formula \[(A.1)\] we obtain

\[\mathcal{F}\left[\frac{\Theta(x)}{x}\right] = -\gamma - \ln |x| - i \pi \frac{\text{sign}(x)}{2} = -\gamma - \ln ix\]

**Appendix B.**

\[\sum_{l=0}^{n} (-1)^l \binom{n}{l} (n - 2l)^{m-1} = 0, \quad n \geq m \geq 2\]

**PROOF.** It is enough to prove the following identity:

\[\sum_{l=0}^{n} (-1)^l \binom{n}{l} l^{m-1} = 0, \quad n \geq m \geq 2\]

We proceed by induction on $m$. As shown in the following, the identity holds for $m = 2$ and arbitrary $n \geq m$. By using the binomial theorem we obtain:

\[\frac{d}{dx} (1 - x)^n = -n (1 - x)^{n-1} = \sum_{l=0}^{n} (-1)^l \binom{n}{l} l x^{l-1}, \quad x \neq 0\]

Inserting the value $x = 1$ leads to the statement for $m = 2$.

Inductive step $m \Rightarrow m + 1$:

\[(B.1)\quad \frac{d^{m+1}}{dx^{m+1}} (1 - x)^n = (-1)^{m+1} n (n - 1) \cdots (n - m) (1 - x)^{n-m-1}\]

On the other hand:

\[(B.2)\quad \frac{d^{m+1}}{dx^{m+1}} (1 - x)^n = \sum_{l=0}^{n} (-1)^l \binom{n}{l} \frac{l (l - 1) \cdots (l - m)}{=l^{m+1} + \Theta_{m}(l)} x^{l-m-1}, \quad x \neq 0\]

Where $P_m(l)$ denotes a polynom in $l$ of order $m$. Once again, equating Eq. \[(B.1)\] and Eq. \[(B.2)\] and inserting the value $x = 1$ leads to the statement. □
A STUDY OF GENERALIZED DIRICHLET INTEGRALS.

References

[1] T.M. Apostol. *Math. Magazine* 53, page 183, 1980.
[2] G. E. Shilov I. M. Gel’fand. *Generalized Functions, Volume 1: Properties and Operations*, volume 1. AMS Chelsea Publishing: An Imprint of the American Mathematical Society, 1964.
[3] G. Schumacher R. Remmert. *Funktionentheorie, pages 366-367*, volume 1. Springer-Verlag Berlin, 5th ed., 2001.
[4] L. Schwartz. *Théorie Des Distributions. Tome I*. Hermann Paris, 1950.
[5] L. Schwartz. *Théorie Des Distributions. Tome II*. Hermann Paris, 1951.

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