A NEW UPPER BOUND ON THE CHROMATIC NUMBER OF GRAPHS WITH NO ODD $K_t$ MINOR

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Gerards and Seymour conjectured that every graph with no odd $K_t$ minor is $(t-1)$-colorable. This is a strengthening of the famous Hadwiger’s Conjecture. Geelen et al. proved that every graph with no odd $K_t$ minor is $O(t\sqrt{\log t})$-colorable. Using the methods the present authors and Postle recently developed for coloring graphs with no $K_t$ minor, we make the first improvement on this bound by showing that every graph with no odd $K_t$ minor is $O(t(\log t)^\beta)$-colorable for every $\beta > 1/4$.

1. Introduction

All graphs in this paper are finite, and have no loops or parallel edges. Given graphs $G$ and $H$, we say that $G$ has an $H$ minor if a graph isomorphic to $H$ can be obtained from a subgraph $G'$ of $G$ by contracting edges; and $G$ has an odd $H$ minor if, in addition, the set of contracted edges forms a cut in $G'$. (Note that the empty set $\emptyset$ is a cut.) We denote the complete graph on $t$ vertices by $K_t$.

Gerards and Seymour in 1993 (see [7, Section 6.5]) made the following conjecture.

Conjecture 1.1 (Odd Hadwiger’s Conjecture). For every integer $t \geq 1$, every graph with no odd $K_t$ minor is $(t-1)$-colorable.

Conjecture 1.1 substantially strengthens the famous Hadwiger’s Conjecture [6] which states that every graph with no $K_t$ minor is $(t-1)$-colorable.
Conjecture 1.1 is trivially true for $t \leq 3$. The case $t = 4$ was proved by Catlin [2]. Guenin (see [16]) announced the proof for the case $t = 5$, but it has not yet been published. Conjecture 1.1 remains open for all $t \geq 6$. We refer the reader to a recent survey by Seymour [16] for further background.

The general upper bound on the number of colors sufficient to properly color graphs with no odd $K_t$ minor was established by Geelen, Gerards, Reed, Seymour and Vetta [5], who proved the following.

**Theorem 1.2 ([5]).** Every graph with no odd $K_t$ minor is $O(t^{\sqrt{\log t}})$-colorable.

Kawarabayashi [8] gave a simpler proof of Theorem 1.2. Both proofs rely on the following celebrated result, obtained independently by Kostochka [9,10] and Thomason [17].

**Theorem 1.3 ([9,10,17]).** Every graph with no $K_t$ minor is $O(t^{\sqrt{\log t}})$-degenerate.

Note that Theorem 1.3 directly implies that every graph with no $K_t$ minor is $O(t^{\sqrt{\log t}})$-colorable. Very recently, the present authors [14] made the first improvement on the order of magnitude of this bound. Shortly after, Postle [15] further improved a major part of the argument from [14] showing the following.

**Theorem 1.4 ([15]).** For every $\beta > \frac{1}{4}$, every graph with no $K_t$ minor is $O(t(\log t)^\beta)$-colorable.

In this paper we combine the results and ideas of [5,14,15] to extend Theorem 1.4 to odd minors, as follows.

**Theorem 1.5.** For every $\beta > \frac{1}{4}$, every graph with no odd $K_t$ minor is $O(t(\log t)^\beta)$-colorable.

The proof of Theorem 1.5 occupies the rest of the paper.

**Notation** We use largely standard graph-theoretical notation. We denote by $v(G)$, $e(G)$, $\chi(G)$, and $\kappa(G)$ the number of vertices, number of edges, chromatic number, and (vertex) connectivity of a graph $G$, respectively. We use $d(G) = e(G)/v(G)$ to denote the density of a non-null graph $G$, and $G[X]$ to denote the subgraph of a graph $G$ induced by a set $X \subseteq V(G)$. For a positive integer $n$, let $[n]$ denote $\{1, 2, \ldots, n\}$. The logarithms in the paper are natural unless specified otherwise.

Let $H$ and $G$ be graphs. An $H$-expansion in $G$ is a function $\eta$ with domain $V(H) \cup E(H)$ such that
• for every vertex \( v \in V(H) \), \( \eta(v) \) is a subgraph of \( G \) which is a tree, and the trees \( \{ \eta(v) \}_{v \in V(H)} \) are pairwise vertex-disjoint, and
• for every edge \( e = uv \in E(H) \), \( \eta(e) \) is an edge of \( G \) with one end in \( V(\eta(u)) \) and the other in \( V(\eta(v)) \).

We call the trees \( \{ \eta(v): v \in V(H) \} \) the nodes of the expansion, and denote by \( \bigcup \eta \) the subgraph of \( G \) with vertex set \( \bigcup_{v \in V(H)} V(\eta(v)) \) and edge set \( \{ E(\eta(v)): v \in V(H) \} \cup \{ \eta(e): e \in E(H) \} \). An \( H \)-expansion \( \eta \) is bipartite if \( \bigcup \eta \) is bipartite. Moreover, we say that an \( H \)-expansion \( \eta \) in \( G \) is \( S \)-rooted for \( S \subseteq V(G) \) if \( |S| = v(H) \) and \( |V(\eta(v)) \cap S| = 1 \) for every \( v \in V(H) \).

It is well-known and easy to see that \( G \) has an \( H \) minor if and only if there exists a bipartite \( H \)-expansion in \( G \). We say that \( G \) has a bipartite \( H \)-minor if there exists a bipartite \( H \)-expansion in \( G \).

2. Proof of Theorem 1.5

In this section we prove Theorem 1.5, pending the proof of a key technical result, Theorem 2.6, which is proved in Section 3.

We use the same strategy as in [14], where we established a new upper bound on the chromatic number of graphs with no \( K_t \) minor, and we refer the reader to [14, Section 2] for the outline of the argument. Several parts of the proof, however, become much more involved. In particular, in [14] we could easily reduce the proof to the case when the graph has high connectivity. Here we shall introduce a non-standard technical notion of connectivity which we now present.

Recall that the pair \((A, B)\) is a separation of a graph \( G \) if \( A \cup B = V(G) \) and every edge of \( G \) has both ends in \( A \) or both in \( B \). A separation \((A, B)\) of \( G \) is proper if \( A - B \neq \emptyset \) and \( B - A \neq \emptyset \). The order of a separation \((A, B)\) is \( |A \cap B| \). We say that a graph \( G \) is weakly \( k \)-connected if for every proper separation \((A, B)\) of \( G \) of order at most \( k \), we have

\[
\min\{|A - B|, |B - A|\} < |A \cap B|.
\]

Our first lemma ensures that every graph \( G \) with large chromatic number contains a subgraph \( H \) with high weak connectivity such that the chromatic number of \( H \) is still fairly large. All colorings in the proof of Lemma 2.1 are proper vertex-colorings.

**Lemma 2.1.** Let \( k, l \) be positive integers with \( k \geq 3l \). Then every graph \( G \) with \( \chi(G) > k \) contains a weakly \( l \)-connected subgraph \( H \) such that

\[
\chi(H) > k - 2l.
\]
Theorem 2.2 ([5]). If a graph $H$ has a bipartite $K_{12t}$ minor, then either $H$ contains an odd $K_t$ minor, or there exists $X \subseteq V(H)$ with $|X| \leq 8t-2$ such that some component of $H \setminus X$ is bipartite and contains at least $8t+2$ vertices.

Corollary 2.3. Let $k,t$ be positive integers with $k \geq 16t$. Assume that every graph $H$ with $\chi(H) > k$ has either a bipartite $K_{12t}$ minor or an odd $K_t$ minor. Then every graph $G$ with $\chi(G) > k + 16t$ has an odd $K_t$ minor.

Proof. Let $G$ be a graph with $\chi(G) > k + 16t$. Suppose for a contradiction that $G$ has no odd $K_t$ minor. By Lemma 2.1, $G$ contains a weakly $8t$-connected subgraph $H$ with $\chi(H) > k$. Note that $H$ has no odd $K_t$ minor.

Proof. Let $H$ be a subgraph of $G$ with $v(H)$ minimum such that for some $Z \subseteq V(H)$ with $|Z| \leq 2l$ there exists a $k$-coloring $\phi_Z: Z \to [k]$ of $H[Z]$ which cannot be extended to a $k$-coloring of $H$. It is easy to see that such a subgraph $H$ exists, and satisfies $\chi(H) > k - 2l$.

It remains to show that $H$ is weakly $l$-connected. Suppose $H$ is not weakly $l$-connected. Then there exists a proper separation $(A,B)$ of $H$ of order at most $l$ such that

$$\min\{|A-B|, |B-A|\} \geq |A \cap B|.$$ 

We may assume that $|Z \cap (B-A)| \leq l$. Suppose first that $B-A \neq Z \cap (B-A)$. By the choice of $H$, there exists a $k$-coloring $\phi_A: Z \cup A \to [k]$ of $H[Z \cup A]$ such that $\phi_A$ extends $\phi_Z$. Let $Z' = (Z \cup A) \cap B$. Then

$$|Z'| = |Z \cap (B-A)| + |A \cap B| \leq 2l.$$ 

By the choice of $H$, $\phi_A|_{Z'}$ can be extended to a $k$-coloring of $H[B]$. Thus $\phi_A$ (and so $\phi_Z$) can be extended to a $k$-coloring of $H$, a contradiction. This proves that $B-A=Z \cap (B-A)$. Let $Z'' = (Z \cup B) \cap A$. Then

$$|Z'' \cup Z| \leq |Z| + |A \cap B| \leq 3l \leq k.$$ 

It follows that $A-(Z'' \cup Z) \neq \emptyset$ because $\chi(H) > k$. By the choice of $H$, there exists a $k$-coloring $\phi$ of $H[Z'' \cup Z]$ such that $\phi$ extends $\phi_Z$. Note that $|B-A| \geq |A \cap B|$ and so $|Z''| \leq |Z| - |B-A| + |A \cap B| \leq |Z| \leq 2l$. By the minimality of $H$, the coloring $\phi|_{Z''}$ can be further extended to a $k$-coloring of $H[A]$. This yields a $k$-coloring of $H$ extending $\phi_Z$, a contradiction.

Our first application of Lemma 2.1, combined with the following result of Geelen et al. [5], allows us to convert bipartite clique minors to odd clique minors.
By our assumption, \( H \) has a bipartite \( K_{12t} \) minor. Furthermore, by Theorem 2.2, there exists a proper separation \((A, B)\) of \( H \) of order at most \( 8t - 2 \) such that \( H[B - A] \) is bipartite, and \(|B - A| \geq 8t + 2 > |A \cap B|\). Since \( H \) is weakly \( 8t \)-connected, we see that \(|A - B| < |A \cap B|\).

\[
\chi(H) \leq \chi(H[B - A]) + \chi(H[A \cap B]) + \chi(H[A - B]) < 16t - 2 < k,
\]
a contradiction.

The second ingredient of the proof of Theorem 1.5 allows us to deal with the case when \( \nu(G) \) is small. It is based on the following bound due to Kawarabayashi and the second author [12] on the independence number of graphs with no odd \( K_t \) minor.

**Theorem 2.4 ([12]).** Let \( G \) be a graph with no odd \( K_t \) minor. Then \( \alpha(G) \geq \frac{\nu(G)}{2t} \).

Theorem 2.4 implies the following bound on the chromatic number of graphs with no odd \( K_t \) minor.

**Corollary 2.5.** Let \( G \) be a graph with no odd \( K_t \) minor. Then

\[
(1) \quad \chi(G) \leq 2t \left( 1 + \log \left( \frac{\nu(G)}{t} \right) \right).
\]

**Proof.** By Theorem 2.4, for every integer \( s \geq 1 \), there exist pairwise disjoint, independent subsets \( X_1, X_2, \ldots, X_s \subseteq V(G) \) such that

\[
|V(G) - \bigcup_{i=1}^{s} X_i| \leq (1 - 1/(2t))^s \cdot \nu(G).
\]

Let \( s = \lceil 2t \cdot \log(\nu(G)/t) \rceil \). Then \((1 - 1/(2t))^s \cdot \nu(G) \leq t\). It follows that \( \chi(G \setminus \bigcup_{i=1}^{s} X_i) \leq t \) and so

\[
\chi(G) \leq \chi(G[\bigcup_{i=1}^{s} X_i]) + \chi(G \setminus \bigcup_{i=1}^{s} X_i) \leq s + t,
\]
as desired.

The third ingredient is Theorem 2.6 on the existence of bipartite \( K_t \) minors in weakly \( l \)-connected graphs. The proof of Theorem 2.6 is more involved and will be given in Section 3.

**Theorem 2.6.** There exists a constant \( C = C_{2,6} \) satisfying the following. Let \( t, l, r \geq 2 \) be integers such that \( r \geq \sqrt{\log t/2} \) and \( l = \lceil Ct(\log t)^{1/4} \rceil \). Let \( G \) be a weakly \( l \)-connected graph. If there exist pairwise disjoint sets \( X_1, X_2, \ldots, X_r \subseteq V(G) \) such that \( d(G[X_i]) \geq l \) for every \( i \in [r] \), and \( \chi(G \setminus \bigcup_{i \in [r]} X_i) \geq l \), then \( G \) has a bipartite \( K_t \) minor.
Our fourth tool is a bound from [5] on the density sufficient to force a bipartite $K_t$ minor. Note that Kostochka [9] proved that every graph $G$ with $d(G) \geq 3.2s\sqrt{\log s}$ has a $K_s$ minor.

**Theorem 2.7 ([5]).** Every graph $G$ with $d(G) \geq 7t\sqrt{\log t}$ contains a bipartite $K_t$ minor.

Finally, we use the result of Postle [15], which is an improvement of a similar theorem from [14].

**Theorem 2.8 ([15]).** For every $\delta > 0$ there exists $C = C_{2.8}(\delta) > 0$ such that for every $D > 0$ the following holds. Let $G$ be a graph with $d(G) \geq C$, and let $s = D/d(G)$. Then $G$ contains either

(i) a minor $J$ with $d(J) \geq D$, or

(ii) a subgraph $H$ with $\nu(H) \leq s^{1+\delta}CD$ and $d(H) \geq s^{-\delta}d(G)/C$.

In the remainder of this section we deduce Theorem 1.5. It follows immediately from Corollary 2.3 and the following Theorem 2.9. The proof of Theorem 2.9 utilizes all the tools presented above.

**Theorem 2.9.** For every $\delta > 0$ there exists $t_0 = t_0(\delta)$ such that for all positive integers $t \geq t_0$, every graph $G$ with neither a bipartite $K_t$ minor nor an odd $K_t$ minor satisfies

$$\chi(G) < t(\log t)^{1/4+\delta}.$$

**Proof.** We may assume that $\delta < 1/4$. Let $C_1 = C_{2.8}(\delta)$ and $C_2 = C_{2.6}$. We choose $t_0 \gg \max\{C_1, C_2, 1/\delta\}$ implicitly to satisfy the inequalities appearing throughout the proof.

Let $t \geq t_0$ be an integer and let $k = t(\log t)^{1/4+\delta}/6$. Suppose for a contradiction that there exists a graph $G$ with neither a bipartite $K_t$ minor nor an odd $K_t$ minor such that $\chi(G) \geq 6k$. By Lemma 2.1, $G$ contains a weakly $2k$-connected subgraph $H$ with $\chi(H) \geq 2k$. Our goal is to apply Theorem 2.6 and Theorem 2.8 to $H$ to obtain a contradiction.

Choose a maximal collection $\{X_1, X_2, \ldots, X_{r'}\}$ of pairwise disjoint subsets of $V(H)$ such that $d(H[X_i]) \geq C_2t(\log t)^{1/4}$ and $|X_i| \leq t(\log t)^{3/4}$ for all $i \in [r']$. Let $r = \lceil \sqrt{\log t}/2 \rceil$, $r^* = \min\{r', r\}$, and $X = \bigcup_{i \in [r^*]} X_i$. Then $|X| \leq t(\log t)^{5/4}$. By Corollary 2.5, for sufficiently large $t$,

$$\chi(H[X]) \leq 3t \cdot \log(\log t) \leq k - 1.$$

Thus $\chi(H \setminus X) \geq k + 1$. If $r' \geq r$, then $\chi(H \setminus \bigcup_{i \in [r]} X_i) \geq k + 1 \geq C_2 \cdot t(\log t)^{1/4}$. By Theorem 2.6 applied to $H$ and $\{X_1, X_2, \ldots, X_r\}$, we see that $H$ has a
bipartite $K_t$ minor, contrary to the choice of $G$. Thus $r' < r$. Then $X = \bigcup_{i \in [r']} X_i$.

Let $H'$ be a minimal subgraph of $H \setminus X$ with $\chi(H') \geq k + 1$. Then the minimum degree of $H'$ is at least $k$, and so $d(H') \geq k/2 = t(\log t)^{1/4 + \delta}/12 \geq C_1$. Let $D = 7t \sqrt{\log t}$. We next apply Theorem 2.8 to $D$ and $H'$. Note that $H'$ has no bipartite $K_t$ minor by the choice of $G$. Thus by Theorem 2.7, $H'$ has no minor $J$ with $d(J) \geq D$. By Theorem 2.8, there must exist $Z \subseteq V(H')$ such that $|Z| \leq s^{1+\delta} D$ and $d(H[Z]) \geq s^{-\delta} d(H')/C_1$, where $s = D/d(H') \leq 100(\log t)^{1/4 - \delta}$. It is easy to check that, for sufficiently large $t$, the above conditions yield that $d(H[Z]) \geq C_2 t (\log t)^{1/4}$ and $|Z| \leq t(\log t)^{3/4}$. But then the collection $\{X_1, X_2, \ldots, X_{r'}, Z\}$ contradicts the maximality of $\{X_1, X_2, \ldots, X_{r'}\}$.

3. Proof of Theorem 2.6

The proof of Theorem 2.6 is an adaptation of the proof of [14, Lemma 3.3], which in turn is based on the ideas of Thomason [18]. In addition to some of the lemmas from the previous section we use an array of extra tools from the literature, which we now present.

We first use a classical result of Mader, which ensures a highly-connected subgraph in a dense graph, to deduce a highly-connected bipartite subgraph in a graph with either high density or large chromatic number.

**Lemma 3.1 ([13]).** Every graph $G$ contains a subgraph $H$ such that $\kappa(H) \geq d(G)/2$.

**Corollary 3.2.** Every graph $G$ contains a bipartite subgraph $H$ such that

$$\kappa(H) \geq \max\{d(G)/4, (\chi(G) - 1)/8\}.$$

**Proof.** By a well-known result of Erdős [4], $G$ contains a bipartite subgraph $G'$ with $d(G') \geq d(G)/2$. By Lemma 3.1, $G'$ contains a subgraph $H$ with $\kappa(H) \geq d(G)/4$.

Next, let $k = \chi(G)$, and let $G''$ be a minimal subgraph of $G$ such that $\chi(G'') = k$. Then the minimum degree of $G''$ is at least $k - 1$ and so $d(G'') \geq (k - 1)/2$. As shown in the previous paragraph $G''$ contains a bipartite subgraph $H$ such that $\kappa(H) \geq (k - 1)/8$, as desired.

A large part of the proof of Theorem 2.6 involves linking together small bipartite clique-expansions, which we find in each $G[X_i]$ given in the theorem. We now present the terminology and tools needed to accomplish this.
Let $l$ be a positive integer and let $S = (\{s_i, t_i\})_{i \in [l]}$ be a sequence of pairwise disjoint pairs of vertices of a graph $G$, except, for each $i \in [l]$, it is possible that $s_i = t_i$. An $S$-linkage $P$ in $G$ is a sequence $(P_i)_{i \in [l]}$ of vertex-disjoint paths in $G$ such that $P_i$ has ends $s_i$ and $t_i$ for every $i \in [l]$. For an $S$-linkage $P$, let $I$ be the set of all $i \in [l]$ such that $P_i$ has an odd number of edges. Then we say that $P$ is an $(S, I)$-parity linkage. Let $S \subseteq V(G)$ with $|S| = 2l$. We say that $S$ is linked in $G$ if for every ordered partition $S = (\{s_i, t_i\})_{i \in [l]}$ of $S$ into pairs there exists an $S$-linkage in $G$; and $S$ is parity-linked if, in addition, for every $I \subseteq [l]$ there exists an $(S, I)$-parity linkage in $G$. Finally, a graph $G$ with $|V(G)| \geq 2l$ is $l$-linked if every set $S \subseteq V(G)$ with $|S| = 2l$ is linked in $G$.

Our second tool is the following theorem of Thomas and Wollan [19], which improves an earlier result of Bollobás and Thomason [1].

**Theorem 3.3 ([19]).** For every integer $l \geq 1$, every graph $G$ with $\kappa(G) \geq 10l$ is $l$-linked.

Kawarabayashi and Reed [11] extended Theorem 3.3 to parity linkages. They proved that for every graph $G$ with $\kappa(G) \geq 50l$, either there exists $X \subseteq V(G)$ such that $|X| < 4l - 3$ and $G \setminus X$ is bipartite, or every set $S \subseteq V(G)$ with $|S| = 2l$ is parity-linked in $G$. We need a variant of their result for weakly connected graphs. Fortunately, we are able to reuse most of the ingredients of the proof from [11]. One of these ingredients is the “Erdös–Pósa property” for odd $S$-paths.

**Theorem 3.4 ([3,5]).** Let $k \geq 1$ be an integer. For any set $S$ of vertices of a graph $G$, either

(i) there are $k$ vertex-disjoint paths each of which has an odd number of edges and both its ends in $S$, or

(ii) there exists $X \subseteq V(G)$ with $|X| \leq 2k - 2$ such that $G \setminus X$ contains no such path.

Let $G$ be a graph, and let $H$ be a bipartite subgraph of $G$. We say that a path $P$ in $G$ is a parity-breaking path for $H$, if the ends of $P$ are in $V(H)$, $P$ is otherwise vertex-disjoint from $H$, and $H \cup P$ contains an odd cycle. Note that such a parity-breaking path may have only one edge. For a graph $G$ and $X, Y \subseteq V(G)$, we say that $X$ is joined to $Y$ in $G$ if there exist $|X|$ vertex-disjoint paths in $G$ each of which has one end in $X$ and the other in $Y$. We need the following lemma which is a consequence of the result of Kawarabayashi and Reed [11, Theorem 1.2].

**Lemma 3.5 ([11]).** Let $H$ be a $2k$-linked bipartite subgraph of a graph $G$. Suppose that $G$ contains $2k$ vertex-disjoint parity-breaking paths for $H$. 
Then every set \( X \subseteq V(G) \) with \( |X| = 2k \) that is joined to \( V(H) \) in \( G \) is parity-linked in \( G \).

We say that a set \( X \) of vertices of a graph \( G \) is \textit{parity-knitted} if for every pair of partitions \((A,B)\) and \((X_1,X_2,\ldots,X_r)\) of \( X \), there exist pairwise vertex-disjoint trees \( T_1,T_2,\ldots,T_r \) in \( G \) such that \( X_i \subseteq V(T_i) \) and \( (A \cap X_i,B \cap X_i) \) extends to the bipartition of \( T_i \) for every \( i \in [r] \). A \textit{linkage} in a graph \( G \) is a collection of pairwise vertex-disjoint paths.

**Corollary 3.6.** Let \( H \) be a \( 2k \)-linked bipartite subgraph of a graph \( G \). Suppose that \( G \) contains \( 2k \) vertex-disjoint parity-breaking paths for \( H \). Then every set \( X \subseteq V(G) \) with \( |X| = k \) that is joined to \( V(H) \) in \( G \) is parity-knitted in \( G \).

**Proof.** Let \((A,B)\) and \((X_1,X_2,\ldots,X_r)\) be two partitions of \( X \). As \( X \) is joined to \( V(H) \) in \( G \), there exists a linkage \( \mathcal{P} \) in \( G \) such that \( |\mathcal{P}| = k \), and every path in \( \mathcal{P} \) has one end in \( X \), the other in \( V(H) \), and is otherwise vertex-disjoint from \( H \). Let \( Y = \{V(P) \cap V(H) | P \in \mathcal{P}\} \). Then \( |Y| = k \). Note that the minimum degree of \( H \) is at least \( 4k-1 \) because \( H \) is \( 2k \)-linked. It follows that we can greedily find pairwise vertex-disjoint trees \( T'_1,T'_2,\ldots,T'_r \) in \( H \setminus Y \) such that \( v(T'_i) = |X_i| \) for each \( i \in [r] \). Let \( X' = X \cup (\bigcup_{i \in [r]} V(T'_i)) \). Then \( |X'| = 2|X| = 2k \) and \( X' \) is joined to \( V(H) \) in \( G \). By Lemma 3.5, \( X' \) is parity-linked in \( G \). Thus we can find pairwise vertex-disjoint linkages \( \mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_r \) in \( G \) such that for each \( i \in [r], |\mathcal{P}_i| = |X_i| \); every path in \( \mathcal{P}_i \) has one end in \( X_i \) and the other in \( V(T'_i) \), and is otherwise disjoint from \( X' \); and in addition, by choosing the desired parity of each path in \( \mathcal{P}_i \), the partition \( (A \cap X_i,B \cap X_i) \) extends to the bipartition of the tree \( T_i = T'_i \cup (\bigcup_{P \in \mathcal{P}_i} P) \), as desired.

Finally, we need a lemma from [14].

**Lemma 3.7 ([14]).** There exists a constant \( C = C_{3,7} > 0 \) satisfying the following. Let \( G \) be a graph, let \( m,s \geq 2 \) be positive integers. Let \( s_1,\ldots,s_m \), \( t_1,\ldots,t_m, r_1,\ldots,r_s \in V(G) \) be distinct, except, for each \( i \in [m] \), it is possible that \( s_i = t_i \). If \( \kappa(G) \geq C \cdot \max\{m,s\sqrt{\log s}\} \), then there exists a \( K_s \)-expansion \( \eta \) in \( G \) rooted at \( \{r_1,\ldots,r_s\} \) and an \( (\{s_i,t_i\}_{i \in [m]} \)-linkage \( \mathcal{P} \) in \( G \) such that \( \bigcup \eta \) and \( \mathcal{P} \) are vertex-disjoint.

We are now ready to prove Theorem 2.6 by building a bipartite \( K_t \) minor from the pieces constructed in each \( G[X_i] \).
Proof of Theorem 2.6. We show that the theorem holds for

\[ C_{2.6} = \max\{2000, 48C_{3.7}\}. \]

Let \( k = \lceil t(\log t)^{1/4} \rceil \). Then \( l \geq \max\{2000, 48C_{3.7}\} \cdot k \). By Lemma 2.1, \( G \setminus \cup_{i \in [r]} X_i \) contains a weakly \((l/3)\)-connected subgraph \( G_0 \) with \( \chi(G_0) \geq l/3 \). By Corollary 3.2 and the choice of \( C \), \( G_0 \) contains a bipartite subgraph \( H_0 \) with \( \kappa(H_0) \geq (l-3)/24 > 80k \). We next show that

(*) \( G_0 \) contains at least \( 8k \) vertex-disjoint parity-breaking paths for \( H_0 \).

Suppose (*) is not true. Let \( (A_0, B_0) \) be a bipartition of \( H_0 \). Then \( |A_0| \geq \kappa(H_0) > 80k \). Note that every path in \( G_0 \) with an odd number of edges and both its ends in \( A_0 \) contains a parity-breaking path for \( H_0 \). Thus \( G_0 \) does not contain \( 8k \) vertex-disjoint paths each of which has an odd number of edges and both its ends in \( A_0 \). By Theorem 3.4 applied to \( G_0 \) and \( A_0 \), there exists \( X \subseteq V(G_0) \) with \( |X| \leq 16k - 2 \) such that \( G_0 \setminus X \) contains no such path. As \( \kappa(H_0) > 80k \) and \( \chi(G_0 \setminus X) \geq \chi(G_0) - |X| \geq l/4 \), it follows that the block of \( G_0 \setminus X \) containing \( H_0 \setminus X \) is bipartite, and \( G_0 \setminus X \) contains a block \( J \) with \( \chi(J) \geq l/4 \). Now consider a proper separation \( (A_1, B_1) \) of \( G_0 \) with \( V(H_0) \subseteq A_1 \) and \( V(J) \subseteq B_1 \) such that \( |A_1 \cap B_1| \leq |X| + 1 < 16k < l/3 \). Then \( |A_1 - B_1| \geq \nu(H_0) - |A_1 \cap B_1| > 16k \). Since \( G_0 \) is weakly \((l/3)\)-connected, we have \( |B_1 - A_1| < |A_1 \cap B_1| \) and so \( \nu(J) = |B_1| = |A_1 \cap B_1| + |B_1 - A_1| < 32k \). But then \( \chi(J) < 32k < l/4 \), a contradiction. This proves (*)&n

Let \( y = [(\log t)^{1/4}] \) and \( x = \lfloor t/y \rfloor \). Assume first that \( y \leq 1 \). Then \( G[X_i] \) contains a bipartite \( K_r \) minor by Theorem 2.7, as desired. Assume next that \( y \geq 2 \). Then \( r \geq \left( \frac{y}{2} \right) \), \( xy \geq t \), and \( xy(y-1) \leq 4k \). It suffices to show that \( G \) contains a bipartite \( K_{xy} \)-expansion.

By Corollary 3.2, \( G[X_i] \) contains a bipartite subgraph \( H_i \) with \( \kappa(H_i) \geq l/4 \) for each \( i \in [r] \). Let \( \mathcal{H} = \{H_1, H_2, \ldots, H_{\binom{y}{2}}\} \). We relabel the graphs in \( \mathcal{H} \) to \( \{H_{\{i,j\}}\}_{\{i,j\} \subseteq [y]} \). We claim that there exist pairwise vertex-disjoint linkages \( Q_{\{i,j\}} \) for all \( i, j \in [y] \) with \( i \neq j \), such that each \( Q_{\{i,j\}} \) consists of \( x \) vertex-disjoint paths \( Q_{\{i,j\}}^1, \ldots, Q_{\{i,j\}}^x \) each of which starts in \( V(H_{\{i,j\}}) \), ends in \( V(H_0) \), and is otherwise vertex-disjoint from \( H_{\{i,j\}} \). Suppose not. By Menger’s theorem there exists a proper separation \( (A, B) \) of \( G \) with \( |A \cap B| < xy(y-1) \leq 4k < l \) such that \( V(H_0) \subseteq A \) and \( V(H) \subseteq B \) for some \( H \in \mathcal{H} \). But then

\[ \min\{|A - B|, |B - A|\} > |A \cap B|, \]

contrary to the fact that \( G \) is weakly \( l \)-connected.
Let $Q=\cup_{i,j \in [y], i \neq j} Q_{(i,j)}$. We now apply Lemma 3.7 consecutively to each subgraph $H_{(i,j)}$ with $s = 2x$ and $m \leq xy(y-1) - 2x$ equal to the number of paths in $Q - (Q_{(i,j)} \cup Q_{(j,i)})$ which are not vertex-disjoint from $H_{(i,j)}$. The vertices $\{(s_i,t_i)\}_{i \in [m]}$ are then chosen to be the first and last vertices of these paths in $H_{(i,j)}$, while the vertices $r^1_{(i,j)}, \ldots, r^x_{(i,j)}$ and $r^1_{(j,i)}, \ldots, r^x_{(j,i)}$ are the ends of the paths in $Q_{(i,j)}$ and $Q_{(j,i)}$ in $H_{(i,j)}$, respectively. Note that $\kappa(H_{(i,j)}) \geq 1/4 > C_{3.7}\cdot\max\{m, s \sqrt{\log s}\}$ by the choice of $C$. By using the linkage $P$ given by Lemma 3.7 to reroute the paths in $Q - (Q_{(i,j)} \cup Q_{(j,i)})$ within $H_{(i,j)}$, we may assume that $H_{(i,j)}$ contains a bipartite $K_{2x}$-expansion $\eta_{(i,j)}$ rooted at $\{r^1_{(i,j)}, \ldots, r^x_{(i,j)}, r^1_{(j,i)}, \ldots, r^x_{(j,i)}\}$, which is otherwise vertex-disjoint from $Q$. We may assume that $\eta_{(i,j)}$ has domain $\{i,j\} \times [x]$ such that for each $z \in [x]$, we have $r^z_{(i,j)} \in V(\eta_{(i,j)}(i,z))$ and $r^z_{(j,i)} \in V(\eta_{(i,j)}(j,z))$.

For $i,j \in [y]$ with $i \neq j$ and $z \in [x]$, let $s^z_{(i,j)}$ be the first vertex of $H_0$ encountered as we traverse $Q^z_{(i,j)}$ from the end $r^z_{(i,j)}$ in $H_{(i,j)}$; let $R^z_{(i,j)}$ be the subpath of $Q^z_{(i,j)}$ with ends $r^z_{(i,j)}$ and $s^z_{(i,j)}$; and let $S^z_i = \{s^z_{(i,j)} : j \in [y] - \{i\}\}$. Finally, let

$$S = \bigcup_{i \in [y], z \in [x]} S^z_i \quad \text{and} \quad H^* = \bigcup_{\{i,j\} \subseteq [y]} \left( \left( \cup_{\{i,j\}} \cup \{R^z_{(i,j)} \cup R^z_{(j,i)} : z \in [x]\} \right) \right).$$

It is easy to see that $(S^z_i)_{i \in [y], z \in [x]}$ partitions $S$, and $H^*$ is a bipartite subgraph of $G$ with $V(H^*) \cap V(H_0) = S$. Let $(A^*, B^*)$ be a bipartition of $H^*$. Note that $|S| = xy(y-1) \leq 4k$ and $S$ is joined to $V(H_0)$ in $G_0$. By Theorem 3.3, $H_0$ is $8k$-linked because $\kappa(H_0) > 80k$. Furthermore, by $(*)$ and Corollary 3.6, $S$ is parity-knitted in $G_0$. Thus for the pair of partitions $(A^* \cap S, B^* \cap S)$ and $(S_i^z)_{i \in [y], z \in [x]}$ of $S$, there exists a collection $(T^z_i)_{i \in [y], z \in [x]}$ of pairwise vertex-disjoint trees in $G_0$ such that $S^z_i \subseteq V(T^z_i)$, and $G^* = H^* \cup (\cup_{i \in [y], z \in [x]} T^z_i)$ is a bipartite subgraph of $G$.

It remains to show that $G^*$ contains a $K_{xy}$-expansion. We construct such an expansion $\eta$ with domain $[y] \times [x]$. For $i \in [y]$ and $z \in [x]$, let

$$\eta(i,z) = T^z_i \cup \bigcup_{j \in [y] - \{i\}} \left( \eta_{(i,j)}(i,z) \cup R^z_{(i,j)} \right).$$

It is not hard to see that $\{\eta(i,z)\}_{i \in [y], z \in [x]}$ is a collection of pairwise vertex-disjoint trees in $G^*$. Moreover, for each pair of distinct elements $(i, z)$ and $(i', z')$ in the domain of $\eta$, $G^*$ contains an edge with one end in $V(\eta(i,z))$ and the other in $V(\eta(i',z'))$. Indeed, if $i = i'$, then $z \neq z'$ and for each $j \in [y] - \{i\}$, the edge in $\eta_{(i,j)}$ with one end in $V(\eta_{(i,j)}(i,z))$ and the other in
\(V(\eta_{(i,j)}(i,z'))\) is such an edge; and, if \(i \neq i'\), then the edge in \(\eta_{(i,i')}\) with one end in \(V(\eta_{(i,i')}((i,z))\) and the other in \(V(\eta_{(i,i')}((i',z'))\) is the desired one.

This completes the proof of Theorem 2.6.

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