Storage and conversion of quantum-statistical properties of light in the resonant quantum memory on tripod atomic configuration

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We have considered theoretically the feasibility of the broadband quantum memory based on the resonant tripod-type atomic configuration. In this case, the writing of a signal field is carried out simultaneously into two channels, and characterized by an excitation of two spin waves of the atomic ensemble. With simultaneous read out from both channels quantum properties of the original signal are mapped on the retrieval pulse no worse than in the case of memory based on Λ-type atomic configuration. At the same time new possibilities are opened up for manipulation of quantum states associated with sequential reading out (and/or sequential writing) of signal pulses. For example, the pulse in squeezed state is converted into two partially entangled pulses with partially squeezed quadratures. Alternatively, two independent signal pulses with orthogonal squeezed quadratures can be converted into two entangled pulses.

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I. INTRODUCTION

Under an optical quantum memory is most often understood the process of light-matter interaction, which may be divided into three stages: "writing" — the mapping of quantum state of light on the state of matter, "storage" of the state and "read out" that is the retrieving of light with quantum state close to the original. Thus, the protocols of quantum memory have one main goal: to store and then to retrieve the quantum state with high efficiency and fidelity. Since the quantum memory is considered as a resource for quantum information and communication networks, it is no less important aspect of the multimode quantum memory as possibility to manipulate with a few qubits simultaneously [1–10]. In addition to the passive task of information storage [11, 12], quantum memory can also be used to convert quantum states directly in the same cell. This approach requires not one but several (at least two) degrees of freedom of the cell memory, allowing to store different quantum states of light. In this regard, ensembles with multilevel atoms are interesting. Typically, multilevel factor is considered as deleterious, leading to losses during storage of signals [13, 14]. We demonstrate here that basing on a more complicated atomic configuration and choosing the proper way to retrieve the signal, we not only do not earn additional losses in the system, but also get the opportunity to transform initial quantum states straight in the memory cell. In this article we consider the quantum memory based on the four-level tripod atomic configuration as a source of pairs of entangled light pulses. Classical aspects of interaction of fields with atoms in tripod configuration were discussed in [15].

There are several options for implementation of quantum optical memory which differ in duration of interaction, as well as the configuration and geometry of the fields involved in the process of storage (e.g. see [11]). For us in this work the protocol of "high-speed (broadband) resonant quantum memory" [16–18] will be of the most interest as one of the closest to real requirements. Here as in many other approaches the Λ-type atomic ensemble (in appropriate basis of states) is used as a storage system. Due to collective properties of atomic ensemble the scheme is suitable for the writing of short light pulses even when one suppose that their duration (writing time) is much shorter then a life time of the excited state. This is because the absorption band of light from the ground state is not determined by the absorption band of individual atom but by the band of all the collective (where optical depth can achieve 100 or even higher).

In the framework of high-speed resonant quantum memory protocol we will consider the interaction of the ensemble of atoms with light pulses. Herewith the read out process of the signal will be performed in two different variants allowing to obtain different (but controllable) quantum states of the retrieved light.

We believe it is important to demonstrate that an additional noise source (associated with the inclusion of an additional energy sublevel and with a random distribution of atoms between sublevels in the writing process) does not lead to loss of efficiency and quantum correlations in the system with adequate choice of the read out procedure.

The article is structured as follows. In Section II main equations describing processes of light-matter interaction in tripod configuration are obtained. In Section III a transition to Λ atomic configuration and an appropriate conversion of the system of equations are described. Here we also obtained a solution for spin subsystem during the writing process of the signal pulse in a resonant medium. In Section IV dynamic and quantum-statistical properties of the input signal are defined. We considered the radiation of sub-Poissonian laser as the signal. In Sections V and VI quantum-statistical properties of spin waves, which arise due to the mapping of this light into the medium, are...
the z and their carrier frequencies coincide with the frequencies of atomic transitions:

The consideration is limited by the rotating wave approximation where all fields are treated as quasi-monochromatic.

![Figure 1: Tripod type atomic configuration, $\Omega_1$ and $\Omega_2$ are Rabi frequencies of the driving fields, $\hat{a}$ is the slowly varying amplitude of the signal field.](image)

discussed. In Section VIII different variants for observing of quantum correlations of spin waves are discussed and the necessity to pass to the Schmidt modes (see Section IX) for the correct analysis of the storage of quantum features is substantiated. Finally, Sections X and XI are devoted to the discussion of two variants of signal retrieval from the tripod-type memory and the analysis of quantum-statistical properties of the output light in depending on the readout process.

II. MAIN EQUATIONS OF HIGH-SPEED RESONANT MEMORY

In this article we consider the tripod-type atomic ensemble as a storage system. In accordance with Fig. 1, each atom can be represented by four actual stationary states. Two driving fields with Rabi frequencies $\Omega_1$ and $\Omega_2$ interact resonantly with optical transitions $|1\rangle - |4\rangle$ and $|2\rangle - |4\rangle$, respectively. In our consideration we assume that the transitions differ in frequency so that each of driving field interacts with only one of them. Initially all atoms are pumped into the state $|3\rangle$. It can be achieved, for example, by the use of an optical pumping [10]. We will aim at storing the quantum state of the signal field which operates on the transition $|3\rangle - |4\rangle$ resonantly.

A thermal motion will be ignored under the assumption that the medium is prepared inside the effective atomic trap. We will not take into account any relaxation processes supposing the light-matter interaction much faster than the typical times of spontaneous relaxation. We consider three lower states as long-lived and believe that their excitation is preserved long enough. It provides the complete emptying of the upper state $|4\rangle$ during storage.

These requirements can be satisfied, for example, on a basis of the $D2$-transition of a hyperfine structure of atoms of Rubidium $87: 5^2S_{1/2}(F = 1) \leftrightarrow 5^2P_{3/2}(F = 0)$ when a degeneracy of the lower sub-level $5^2S_{1/2}(F = 1)$ is lifted by a stationary magnetic field. Sub-levels $5^2P_{3/2} F = 0$ and $F = 1$ are close to each other ($\approx 0.3 \mu eV$, $72.2 \, MHz$), therefore one should apply the magnetic field about $1G$ [20].

Physical conditions formulated above are mostly coincide with the conditions in the article [18]. For that reason the light-matter interaction Hamiltonian in the dipole approximation obtained there can be easily generalized on the case of tripod configuration:

$$\hat{V} = i\hbar \int d^3r [\hat{g}_4(\vec{r}, t)\hat{\sigma}_{43}(\vec{r}, t)e^{ik_zz} + \Omega_1\hat{\sigma}_{41}(\vec{r}, t)e^{ik_1z} + \Omega_2\hat{\sigma}_{42}(\vec{r}, t)e^{ik_2z*}] + h.c.$$  \hspace{1cm} (1)

Here $k_s, k_1$ and $k_2$ are respectively wave numbers of signal and two driving fields. The coupling constant $g$ defines the force of the dipole interaction of the signal field with a single atom on the transition $|4\rangle - |3\rangle$ with a dipole moment $d_{43}$. It can be written in form

$$g = \left(\frac{\omega_s}{2\varepsilon_0 c}\right)^{1/2} d_{43}.$$  \hspace{1cm} (2)

The consideration is limited by the rotating wave approximation where all fields are treated as quasi-monochromatic and their carrier frequencies coincide with the frequencies of atomic transitions: $\omega_s = \omega_{43}$, $\omega_1 = \omega_{41}$ and $\omega_2 = \omega_{42}$.

The positive-frequency operator of the electric field of a quasiplane and a quasimonochromatic wave travelling in the $+z$ direction can be written in terms of space- and time-dependent photon annihilation operator $\hat{a}(\vec{r}, t)$ of the signal as:

$$\hat{E}_s(\vec{r}, t) = i \left(\frac{\hbar \omega_s}{2\varepsilon_0 c}\right)^{1/2} e^{i(k_z z - \omega_s t)}\hat{a}(z, \vec{p}, t) + h.c., \quad \vec{r} = (z, \vec{p}), \quad \vec{p} = (x, y).$$  \hspace{1cm} (3)
Creation and annihilation operators \( \hat{a}^\dagger(\vec{r}, t) \) and \( \hat{a}(\vec{r}, t) \) obey the following commutation relations \[21\]:

\[
[\hat{a}(\vec{r}, t), \hat{a}^\dagger(\vec{r}', t)] = c \left( 1 - \frac{i}{k_s} \partial_z - \frac{1}{2k_s^2} \Delta_\perp \right) \delta^3(\vec{r} - \vec{r}'), \quad \Delta_\perp = \partial_x^2 + \partial_y^2
\]

(4)

and are normalized so that the mean value \( \langle \hat{a}^\dagger(\vec{r}, t) \hat{a}(\vec{r}, t) \rangle \) determines the mean number of photons passing through the cross section per unit time with a dimension of sec\(^{-1}\)cm\(^{-2}\).

According to \[18\], collective atomic variables (coherences and populations) are introduced as a linear superpositions of individual variables:

\[
\hat{\sigma}_{i\neq k}(\vec{r}, t) = \sum_a \hat{\sigma}_{i\neq k}^a(t) \delta^3(\vec{r} - \vec{r}_a) \quad \hat{N}_i(\vec{r}, t) = \sum_a \hat{\sigma}_{i}^a(t) \delta^3(\vec{r} - \vec{r}_a), \quad i, k = 1, 2, 3, 4.
\]

(6)

Here vector \( \vec{r}_a \) indicates a position of the individual atom with an index \( a \). The set of operators \( \hat{\sigma}_{i\neq k}^a(t) \) corresponds to individual atoms, and there are well known relations \( \hat{\sigma}_{i\neq k}^a(t) = \hat{\sigma}_{i}^a(t) \hat{\sigma}_{k}^a(t) \). Taking into account that variables of different atoms commute with each other it is easy to get commutation relations for collective atomic variables:

\[
[\hat{\sigma}_{ik}(\vec{r}, t), \hat{\sigma}_{kj}(\vec{r}', t)] = (\hat{N}_i(\vec{r}, t) - \hat{N}_k(\vec{r}, t)) \delta^3(\vec{r} - \vec{r}'),
\]

(7)

Now we have all required to derive a system of differential equations in partial derivatives for variables of the field and the matter in Heisenberg representation on the basis of Hamiltonian \[1\]. We omit details of this procedure, they are described quite well in previous discussions. As a result, we get the following set of equations:

\[
\left( \frac{1}{c} \partial_t + \partial_z - \frac{i}{2k_s} \Delta_\perp \right) \hat{a} = -g \hat{\sigma}_{34},
\]

\[
\partial_t \hat{\sigma}_{12} = -\Omega_1 \hat{\sigma}_{42} - \Omega_2 \hat{\sigma}_{14},
\]

\[
\partial_t \hat{\sigma}_{31} = -\Omega_1 \hat{\sigma}_{34} - g\hat{a} \hat{\sigma}_{41},
\]

\[
\partial_t \hat{\sigma}_{32} = -\Omega_2 \hat{\sigma}_{34} - g\hat{a} \hat{\sigma}_{42},
\]

\[
\partial_t \hat{\sigma}_{34} = g\hat{a} (\hat{N}_3 - \hat{N}_4) + \Omega_1 \hat{\sigma}_{31} + \Omega_2 \hat{\sigma}_{32},
\]

\[
\partial_t \hat{\sigma}_{41} = \Omega_1 (\hat{N}_1 - \hat{N}_4) + g\hat{a}^\dagger \hat{\sigma}_{31} + \Omega_2 \hat{\sigma}_{41},
\]

\[
\partial_t \hat{\sigma}_{42} = \Omega_2 (\hat{N}_2 - \hat{N}_4) + g\hat{a}^\dagger \hat{\sigma}_{32} + \Omega_1 \hat{\sigma}_{12},
\]

\[
\partial_t \hat{N}_1 = -\Omega_1 (\hat{\sigma}_{14} + \hat{\sigma}_{41}),
\]

\[
\partial_t \hat{N}_2 = -\Omega_2 (\hat{\sigma}_{24} + \hat{\sigma}_{42}),
\]

\[
\partial_t \hat{N}_3 = -g\hat{a}^\dagger \hat{\sigma}_{34} - g\hat{a} \hat{\sigma}_{43}.
\]

(8)

To close this system of equations we have to complete it by the equality \( \hat{N}_1 + \hat{N}_2 + \hat{N}_3 + \hat{N}_4 = N, \) where \( N \) is time independent concentration of atoms involved in the interaction process. Let us remind that according to definitions all of collective variables (and also concentration \( N \)) have a fine-grained spatial structure with a characteristic scale of the order of an average distance between immovable atoms.

We omit here well known assumptions \[18\] \[22\], which allow us to pass from Eqs. \[5\] to simplified ones in the form

\[
\partial_z \hat{a} = -g\sqrt{N} \hat{c},
\]

\[
\partial_t \hat{c} = g\sqrt{N} \hat{a} + \Omega_1 \hat{b}_1 + \Omega_2 \hat{b}_2,
\]

\[
\partial_t \hat{b}_1 = -\Omega_1 \hat{c},
\]

\[
\partial_t \hat{b}_2 = -\Omega_2 \hat{c}.
\]

(9)\(10\)\(11\)\(12\)

Under the deriving of these equations, the averaging on the above-mentioned fine-grained spatial structure is applied. Hereafter we treat \( N \) and other atomic variables as average values.

In set \(9\)\(10\)\(11\)\(12\) we introduced renormalized operator amplitudes:

\[
\hat{c} = \hat{\sigma}_{34}/\sqrt{N}, \quad \hat{b}_1 = \hat{\sigma}_{31}/\sqrt{N}, \quad \hat{b}_2 = \hat{\sigma}_{32}/\sqrt{N},
\]

(13)

for which canonical commutation relations obey:

\[
[\hat{c}(z, t), \hat{c}^\dagger(z', t)] = \delta(z - z'), \quad [\hat{b}_1(z, t), \hat{b}_1^\dagger(z', t)] = \delta_{ij}\delta(z - z'), \quad i, j = 1, 2.
\]

(14)
Now the considered problem can be treated as an interaction of four quantum oscillators with Heisenberg amplitudes $\hat{a}, \hat{b}_1, \hat{b}_2, \hat{c}$.

Further we will assume that initially all atoms are pumped on the level $|3\rangle$ and will neglect changing of the population of this state caused by the interaction with actual fields. To satisfy this it is enough to suppose that the number of photons in the signal pulse (which is absorbed on the transition $|3\rangle \rightarrow |4\rangle$) is much smaller than the population of the level $|3\rangle$.

Also in this article further we will not take into account the transverse structure of the field and the atomic ensemble in particular neglecting the diffraction of light under its propagation through the medium. We assume that one-dimensional approximation is applicable. It means that actual fields can be treated as the set of plane waves travelling in the $z$-direction through the atomic layer with thickness $L$. Then in the original field-matter interaction Hamiltonian the integration over the volume is replaced by one-dimensional integration along the axis $z$ ranging from zero to the thickness of medium $L$. In equality $(6)$ for collective atomic variables, three-dimensional delta-function $\delta^3(\vec{r} - \vec{r}_a)$ should be replaced by one-dimensional function $\delta(z - z_a)$. Due to this change the dimensions of collective variables and the concentration $N$ turn out to be equal $cm^{-1}$.

\section{III. Excitation of Spin Waves in Atomic Ensemble}

In future we will assume that the Rabi frequencies of both driving fields are equal to each other: $\Omega_1 = \Omega_2 \equiv \Omega/\sqrt{2}$. Let us introduce instead of spin amplitudes $\hat{b}_i$ their linear combinations:

$$\hat{b}_\pm = (\hat{b}_1 \pm \hat{b}_2)/\sqrt{2}.$$  

Then instead of Eqs. (9) - (12) we get

$$\begin{align*}
\partial_z \hat{a} &= -g\sqrt{N}\hat{c}, \\
\partial_t \hat{c} &= g\sqrt{N}\hat{a} + \Omega \hat{b}_+,
\end{align*}$$

$$\begin{align*}
\partial_t \hat{b}_+ &= -\Omega \hat{c}, \\
\partial_t \hat{b}_- &= 0.
\end{align*}$$

The first three equations represent the closed system. They are exactly the same as for the three level atoms in $\Lambda$-configurations interacting with relevant fields \[13\]. Let us remind, in that case driving fields are defined by Rabi frequency $\Omega = \sqrt{2}\Omega_1 = \sqrt{2}\Omega_2$. Thus we can use the solutions obtained in \[17 \, 18\], and derive the expression for the amplitude $\hat{b}_+$ in the explicit form:

$$\hat{b}_+(z) = -\frac{1}{\sqrt{2}} \int_0^{T_W} dt \, \hat{a}_{in}(T_W - t) G_{ab}(z, t) + \hat{v}_+(z, T_W), \quad \hat{b}_+(z) \equiv \hat{b}_+(z, T_W).$$

This formula represents the quantum state of the medium at the end of the writing process $t = T_W$. The kernel $G_{ab}(z, t)$ has a following form:

$$G_{ab}(z, t) = \int_0^t dt' e^{-it'} J_0 \left(\sqrt{z t'}\right) \Theta(t') e^{i(t-t')/2} J_0 \left(\sqrt{z(t-t')}\right) \Theta(t-t').$$

Here $J_0 (\sqrt{z t})$ is Bessel function of the first kind and zero order, and window-function $\Theta(t)$ is different from zero (equal to 1) in the time interval $0 \leq t \leq T_W$.

The operator $\hat{v}_+$ in the right side of the Eq. \[20\] determines the contribution of all kinds of vacuum channels to the spin coherence. The solution \[20\] is formed not only by the input signal pulse $\hat{a}_{in}(t)$ (the first term on the right) but also due to the initial vacuum excitations of spin waves. In quantum theory we have to consider these processes that form the explicit expression of the operator $\hat{v}_+$. However, strictly speaking, we do not need to know this expression explicitly, because vacuum channels do not introduce any contributions in the normal ordering averages of the operators. Of course, for physical analysis, we have to calculate not only the normal ordering, but also ordinary averages of the operators. However, they can be easily expressed via the normal ordering values.

In the formulas \[20\], \[21\] and everywhere after we use dimensionless coordinates and time, which are introduced according to the relations

$$\Omega t \rightarrow t, \quad 2g^2 N z/\Omega \rightarrow z.$$
Respectively, there are dimensionless times of writing and reading of the signal pulses and the dimensionless length of the resonant medium

\[ \Omega T_W \rightarrow T_W, \quad \Omega T_R \rightarrow T_R, \quad 2g^2 NL/\Omega \rightarrow L. \]  

(23)

As for the operator \( \hat{b}_- \), according to (19) corresponding spin wave keeps its vacuum state during all the time:

\[ \hat{b}_-(z, t) = \hat{b}_-(z, 0). \]  

(24)

In further discussion apart from formulas for \( \hat{b}_\pm \), we will need the explicit expressions for initial spin amplitudes \( \hat{b}_1 \) and \( \hat{b}_2 \). Since

\[ \hat{b}_1 = (\hat{b}_+ + \hat{b}_-)/\sqrt{2}, \quad \hat{b}_2 = (\hat{b}_+ - \hat{b}_-)/\sqrt{2}, \]  

(25) then it is easy to get that at the end of the writing process \( t = T_W \) amplitudes of spin waves are given by the expressions

\[ \hat{b}_{1,2}(z) = \frac{1}{2} \int_0^{T_W} dt \, \hat{a}_{in}(T_W - t) \, G_{ab}(z, t) + \hat{v}_{1,2}(z, T_W), \quad \hat{b}_{1,2}(z) \equiv \hat{b}_{1,2}(z, T_W). \]  

(26)

The meaning of operators \( \hat{v}_{1,2} \) remains the same, namely, they correspond to the contributions of the various subsystems which are initially in a vacuum state. We imply the average over these states for the resulting values.

**IV. INPUT SIGNAL PULSE**

Equations (26) allow us to describe the quantum-statistical properties of spin waves (the quantum state of the atomic ensemble) with the help of known properties of the input signal pulse. The latter, for example, can be postulated formally by defining various moments on the input edge of the cell memory. However, we believe that more consistent is to define a particular source of light, the quantum statistical properties of which are well described theoretically. For example, it could be a synchronized sub-Poissonian laser or optical-parametric generator. Statistical properties of these sources were described in detail in articles [23, 24].

In what follows the sub-Poissonian laser will be discussed as the source of light carried quantum information to the cell. Choosing phase conditions, we can achieve that \( X \)-quadrature is squeezed and normally ordered correlation function can be written for its fluctuations [23, 24]:

\[ \langle \delta \hat{X}_{in}(t) \delta \hat{X}_{in}(t') \rangle = \frac{p \kappa (1 - \mu)}{8 (1 - \mu/2)} e^{-\kappa (1 - \mu/2)|t - t'|} \Theta^W(t)\Theta^W(t'). \]  

(27)

Let us recall the notations in this formula. Hermitian quadratures \( \hat{X}_{in}(t) \) and \( \hat{Y}_{in}(t) \) are introduced in the usual way as the real and imaginary parts of Heisenberg amplitude:

\[ \hat{a}_{in}(t) = \hat{X}_{in}(t) + i\hat{Y}_{in}(t). \]  

(28)

The value \( \kappa \) is a spectral width of laser mode; the parameter \( p \) determines a degree of ordering of the excitation of the laser medium \(-1 < p < \infty\); \( p = 0 \) corresponds to the quite random Poissonian statistics; when \( p < 0 \) the sub-Poissonian statistics takes place, moreover, when \( p = -1 \) there is a strictly regular pumping; \( p > 0 \) corresponds to the super-Poissonian statistics. Synchronization of the laser is ensured by the external weak field in the coherent state. Quantitativelly it is described by the parameter \( \mu \), equal to the ratio of the power of this field inside the cavity to the power of generation. We choose \( \mu \ll 1 \) to ensure the safety of the quantum properties of generation under synchronization by the coherent external field.

The formula (27) is preserved if we turn to the description in terms of the dimensionless time (23). Herewith values \( \kappa \) and \( \delta \hat{X}_{in} \) also become to be dimensionless according to \( \kappa \rightarrow R/\Omega \) and \( \delta \hat{X}_{in}/\sqrt{R} \).

Factors \( \Theta^W(t) \) on the right of (27) are defined as above in (21). They convert the stationary solution into the pulse one.

It is well known that light squeezing occurs when the normal ordering correlation function (27) is negative. As one can see, it is possible only when pumping of the laser medium is sub-Poissonian: \( p < 0 \).

Let us rewrite (27) in Fourier domain performing a Fourier transform in the form:

\[ \delta \hat{X}_{in,\omega} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dt \, \delta \hat{X}_{in}(t)e^{i\omega t}. \]  

(29)
A similar expression holds for the second quadrature $\delta \hat{Y}_{in,\omega}$. One can obtain a nonzero commutation relation for the spectral components:

$$[\delta \hat{X}_{in,\omega}, \delta \hat{Y}_{in,\omega'}] = i/2 \delta(\omega + \omega').$$ (30)

In further calculations, it is convenient to turn the continuous frequency scale into a discrete one. The "grain" of this scale we determine as $2\pi/T_W$, then the whole set of frequencies is given by

$$\omega \rightarrow \omega_n = (2\pi/T_W)n, \quad \omega' \rightarrow \omega_m = (2\pi/T_W)m, \quad n, m = 0, \pm 1, \pm 2, \ldots$$ (31)

Now we can do the following substitutions in the Eq. (30): transform the delta function into the Kronecker symbol

As a result, instead of (30) we obtain

$$[\delta \hat{X}_{in,\omega_n}, \delta \hat{Y}_{in,\omega_m}] = i/2 \delta_{\omega_n, -\omega_m}. \quad (32)$$

We should note that the discretization of the frequency scale is associated with the ability to find appropriate grain for this. Such a grain is not always obviously determined but in our case we can well justify the choice made above. Indeed, according to our requirements the distance between adjacent spectral components in the discretized scale is $2\pi/T_W$. This value is much smaller than the spectral width $\kappa$ that allows us to follow the specific spectral behaviour of the system with a good accuracy.

Now we can get from (37)

$$\langle \delta \hat{X}_{in,\omega} \rangle^2 = 1 + 4\langle \delta \hat{Y}_{in,\omega} \rangle^2 = 1 + \frac{p\kappa^2(1 - \mu)}{\kappa^2(1 - \mu/2)^2 + \omega^2}. \quad (34)$$

One can see, when $\mu = -1$ (the regular laser pumping) the maximum possible two-frequency quadrature squeezing takes place, so that at zero frequency

$$4 \langle \delta \hat{X}_{in,\omega=0} \rangle^2 = \mu^2/4 \ll 1. \quad (35)$$

V. SQUEEZING OF QUADRATURES OF SPIN WAVES

Let us now consider the signal pulse in squeezed state according to (34) when $\mu < 0$ at the input of the memory cell. Spin waves of the resonant medium are formed due to the absorption of the signal pulse, and described by Heisenberg amplitudes:

$$\hat{b}_j(z) = \hat{X}_j(z) + i\hat{Y}_j(z), \quad j = 1, 2. \quad (36)$$

It is interesting to find out how squeezing of the signal maps on the frequency waves after the writing process. We are interested only in X-quadratures, since they are directly dependent on the squeezed quadrature of the signal. According to (26), normal ordering averages of spin quadratures are derived in the form:

$$\langle \delta \hat{X}_j(z) \delta \hat{X}_{j'}(z') \rangle = \frac{1}{4} \int_0^{T_W} dt \int_0^{T_W} dt' \langle \delta \hat{X}_{in}(T_W-t) \delta \hat{X}_{in}(T_W-t') \rangle \gamma_{ab}(z, t) \gamma_{ab}(z', t') =$$

$$= \frac{p\kappa}{32} \frac{1 - \mu}{1 - \mu/2} \int_0^{T_W} dt \int_0^{T_W} dt' e^{-\kappa(1 - \mu/2)(t - t')} \gamma_{ab}(z, t) \gamma_{ab}(z', t') + \{ z \rightleftharpoons z' \}. \quad (37)$$

The second equality is written taking into account Eq. (27). Here the exponential function can be replaced by $\delta$-function under the condition $\kappa T_W \gg 1$:

$$\kappa(1 - \mu/2) e^{-\kappa(1 - \mu/2)(t - t')} \rightarrow \delta(t - t'). \quad (38)$$
We rewrite this equation in the Fourier domain according to transformation
\[
\delta \hat{X}_{j,k} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dz \delta \hat{X}_j(z) e^{-ikz}.
\] (39)

Then we discretize scale \( k \) with the "grain" \( 2\pi/L \) by analogy with derivation of the Eq. (33). As a result we get
\[
4\langle |\delta \hat{X}_{j,k}|^2 \rangle = 1 + \frac{\mu}{4} \frac{(1 - \mu)(1 - \mu/2)}{(1 - \mu/2)^2} \int_0^{Tw} dt |G_{ab}(k,t)|^2,
\] (40)
where \( G_{ab}(k,t) \) and \( G_{ab}(z,t) \) are related to each other as
\[
G_{ab}(k,t) = \frac{1}{\sqrt{L}} \int_0^L dz G_{ab}(z,t) e^{-ikz}.
\] (41)

As one can see from (40), when \( p < 0 \) quadratures of spin waves are squeezed. Let us remind that in the case when \( p = -1 \) the initial pulse was in the perfectly squeezed state. Its excitation has been randomly distributed between two spin waves. The behaviour of quantum oscillators here is similar to the light passed through the symmetrical beamsplitter, therefore we can expect that the squeezing of each of spin waves can not exceed 50%. Below we demonstrate this numerically.

VI. ENTANGLEMENT OF SPIN WAVES

We apply the Duan criterion (see Appendix A) to estimate the degree of the entanglement of two quantum subsystems. To formulate this correctly one should specify the canonical pair (generalized coordinate and momentum). It is easy to do on the basis of spectral amplitudes \( \hat{b}_{i,k} \) (i=1,2) (33) separating their real and imaginary parts:
\[
\hat{b}_{1,k} = \hat{Q}_{1,k} + i\hat{P}_{1,k}, \quad \hat{b}_{2,k} = \hat{Q}_{2,k} + i\hat{P}_{2,k}
\] (42)

\[
[\hat{Q}_{i,k}, \hat{P}_{j,k}] = i/2 \delta_{ij} \delta(k-k')
\] (43)

Operators \( \hat{Q}_{i,k} \) and \( \hat{P}_{i,k} \) can play the role of the canonical variables since they are Hermitian and obey the canonical commutation relations.

We can consider the issue of the entanglement of any two oscillators with any wave numbers \( k \) and \( k' \), which belong to different spin waves (or even the same wave). However, here we consider only the case of two oscillators from different spin waves with wave numbers \( k \) and \( k' = -k \). For this case, the Duan-criterion can be written in the form of inequality:
\[
D_{k,-k} = \langle (\delta \hat{Q}_{1,k} + \delta \hat{Q}_{2,-k})^2 \rangle + \langle (\delta \hat{P}_{1,k} - \delta \hat{P}_{2,-k})^2 \rangle < 1.
\] (44)

The positive value \( D_{k,-k} \) is often called as the spectral parameter of the entanglement.

According to (A8)-(A9), canonical operators \( \hat{Q}_{i,k} \) and \( \hat{P}_{i,k} \) can be rewritten in terms of non-Hermitian quadrature spectral components \( \hat{X}_{i,k} \) and \( \hat{Y}_{i,k} \). Then the Duan-criterion (41) will be as follows:
\[
0 < D_{k,-k} = \langle |\delta \hat{X}_{1,k} + \delta \hat{X}_{2,k}|^2 \rangle + \langle |\delta \hat{Y}_{1,k} - \delta \hat{Y}_{2,k}|^2 \rangle < 1.
\] (45)

When we derived this expression, we required that in accordance with (26) \( X - \) and \( Y - \)quadratures are statistically independent. This fact is satisfied since the kernel \( G_{ab} \) is real. Of course, the connection between quadratures could occur due to the properties of the signal source. However we consider a theoretical model of laser, where the statistical correlations between quadratures are absent.

Let us rewrite inequality (45) in terms of the normal ordering averages of operators:
\[
-1 < \langle |\delta \hat{X}_{1,k} + \delta \hat{X}_{2,k}|^2 \rangle + \langle |\delta \hat{Y}_{1,k} - \delta \hat{Y}_{2,k}|^2 \rangle < 0,
\] (46)

Due to the normal order we can ignore the vacuum contributions and in accordance with (26) equate operators \( \delta \hat{X}_{1,k} = \delta \hat{X}_{2,k} \) and \( \delta \hat{Y}_{1,k} = \delta \hat{Y}_{2,k} \). Then, returning back to the inequality (45) for ordinary averages, we get
\[
D_{k,-k} = 4\langle |\delta \hat{X}_{1,k}|^2 \rangle < 1.
\] (47)

As one can see, spin waves are entangled in the same extent in which spectral quadratures of each spin wave are squeezed.
VII. OPTIMAL OBSERVATION OF SQUEEZED SPIN QUADRATURES

Quantum properties such as squeezing or entanglement of light (or spins) can be discussed either in spectral representation (temporal or spatial) or in any other mode representation depending on the procedure of measurement. In the previous sections we studied the fluctuations of the spectral quadratures $\hat{X}_{i, k}$ of the $i$-th ($i = 1, 2$) spin wave. Alternatively, one could use, for example, a complete orthonormal set of eigenfunctions of the memory (Schmidt modes).

Let us analyze the situation when we follow the spectral components of quadratures. According to (34)-(35) the signal pulse at the input of the memory cell, which was formed from the stationary radiation of the synchronized sub-Poissonian laser ($p = -1$), is in a multi-frequency squeezed state. Herewith the maximum squeezing of the quadrature $\langle |\delta \hat{X}_{\text{in}, \omega}|^2 \rangle \ll 1$ achieves almost 100% at $\omega = 0$. After the writing of the signal pulse on the tripod atomic ensemble, two spin waves $\hat{b}_1$ and $\hat{b}_2$ (or $\hat{b}_\pm$) arise instead of one wave $\hat{a}_\text{in}$.

To estimate the effects of the squeezing and the entanglement of spin waves, we should find numerically the value $\langle |\delta \hat{X}_{i, \omega}|^2 \rangle$. For the single-mode memory protocols the light-matter interaction is formally similar to the transmission of the light through a beamsplitter. If this conclusion remains correct also in our (multimode) case then the quantum statistics of two spin waves would be the same as for mixing of two waves in squeezed and vacuum states on the glass plate. It is easy to see that quadratures of spin waves would be entangled and simultaneously squeezed by 50%. Let us discuss this conclusion on the basis of the numerical analysis of Eq. (40).

In our case of the high-speed resonant memory, losses of the signal are determined by two factors. One of them is associated with photon leakage, when a part of signal photons are not absorbed during the writing process. Another factor is associated with a population of the upper atomic level. These losses can be minimized by matching of two parameters: the dimensionless thickness $L$ of the memory cell and the dimensionless time $T_W$ of the writing process. The set of parameters which provides minimum losses for given optical depth we will refer as "matched" or "well-matched" set. For example, for $L = 10$ the requirement of minimum losses satisfied at $T_W = 5.5$. Curve 1 in Figs. 2 and 3 corresponds to this choice. As one can see, the squeezing (and the entanglement) of spin waves reaches almost 50% at zero wave number $k = 0$. (The difference from 50% is determined by the choice of the relatively small value of $L$, achievable in experiments.) Thus, in this case the analogy with the mixing of squeezed and vacuum light on the beamsplitter is quite justified.

Now let us slightly increase the dimensionless time of the signal writing $T_W$, keeping the same dimensionless thickness of layer $L$. For example, instead of the matched set ($L = 10, T_W = 5.5$) we choose a set ($L = 10, T_W = 7.25$).
A two-pulse excitation in the medium results in the excitation of two spin waves. We will now consider the specific outcomes of such an excitation in the context of quantum memory and signal processing.

**Signal Readout and Quantum Properties**

The readout of the signal involves the retrieval of the quantum state of the spin waves. This can be achieved using measurements on the spin waves, which are described by the operators $\hat{X}_{\text{out}}(t)$ and $\hat{Y}_{\text{out}}(t)$. The output signal is given by the equation:

$$\hat{a}_{\text{out}}(t) = \hat{X}_{\text{out}}(t) + i\hat{Y}_{\text{out}}(t) = \int_0^{T_W} dt' \hat{a}_{\text{in}}(T_W - t')G(t, t') + \hat{v}(t).$$

Here, $\hat{v}(t)$ represents the noise or the vacuum field that is mixed with the signal. The kernel $G(t, t')$ encodes the interaction between the input and output signals. It is determined by the medium properties and the geometry of the system.

**VIII. FULL MEMORY CYCLE (WRITING, STORAGE AND READING): SCHMIDT MODES**

In our model of quantum memory, the squeezed signal pulse is mapped onto the medium, and the medium evolves according to the Hamiltonian $H(T)$. The memory process is characterized by the spectral properties of the medium, which can lead to the entanglement of spin waves.

**Entanglement and Signal Degradation**

The entanglement of spin waves is a critical aspect of quantum memory, and it depends on the spectral properties of the medium. As discussed in previous sections, the spectral squeezing of spin waves is influenced by the geometry of the memory cell and the parameters of the medium.

The spectral squeezing of the readout signal is compared to the corresponding parameters of the initial pulse. Curve 1 in Figs. 3a and 3b (blue curve) shows the spectral squeezing for the matched set of parameters. Curves 2 (red curves) correspond to unmatched sets. As one can see from the figures, the deviation from the matched set of parameters influences slightly on the squeezing of the readout signal, whereas the spectral squeezing of spin waves changes essentially.

Thus, when we discuss the quantum state of the atomic ensemble, which arises due to the mapping of the signal, we should take into account that the spectral basis can be unsuitable for this purpose. Alternatively, it is reasonable to use the complete orthonormal set associated with light-matter interaction in the memory process (Schmidt modes).
Let us apply the Schmidt decomposition (52) for the kernels $G$ retrieved light $\langle \hat{\delta} \hat{X}_{\text{out}} (t) \hat{X}_{\text{out}} (t') \rangle$ corresponding Rabi frequency $\Omega$. To calculate the normally ordered average $\langle \hat{\delta} \hat{X}_{\text{out}} (t) \rangle$ we arrive at the equation

\[
\langle \hat{\delta} \hat{X}_{\text{out}} (t) \hat{X}_{\text{out}} (t') \rangle = \int_0^{T_W} dt_1 dt_2 \langle \hat{\delta} \hat{X}_{\text{in}} (T_W - t_1) \hat{\delta} \hat{X}_{\text{in}} (T_W - t_2) \rangle G(t, t_1) G(t', t_2).
\]  

(56)

This section, we will assume that both the control pulses act simultaneously during the reading time. The corresponding Rabi frequency $\Omega_1$ and $\Omega_2$, in general, can be chosen arbitrarily, but here we will assume that $\Omega_1 = \Omega_2 = \Omega$. In this case, the analytical equations are the most simple.

Taking into account Eq. (48), it is easy to get the equality for the normally ordered correlation function for the retrieved light $\langle \hat{\delta} \hat{X}_{\text{out}} (t) \hat{X}_{\text{out}} (t') \rangle$ expressed through the input correlation function:

\[
\langle \hat{\delta} \hat{X}_{\text{out}} (t) \hat{\delta} \hat{X}_{\text{out}} (t') \rangle = \int_0^{T_W} dt \langle \hat{\delta} \hat{X}_{\text{in}} (T_W - t) \hat{\delta} \hat{X}_{\text{in}} (T_W - t') \rangle G(t, t) G(t', t).
\]  

(56)

Let us apply the Schmidt decomposition (52) for the kernels $G$ under the sign of integration as well as decompose the input quadrature:

\[
\hat{\delta} \hat{X}_{\text{in}} (T_W - t) = \sum_i \hat{\delta} \hat{x}_{i,1} \varphi_i (t), \quad \hat{\delta} \hat{x}_{i,1} = x_{i,1} + i y_{i,1}, \quad [\hat{\delta} x_{i,1}, \hat{\delta} x_{j,1}] = \delta_{ij}.
\]  

(57)

where $\hat{\delta} x_{i,1}$ — operator coefficients of the decomposition. Then we arrive at the equation

\[
\langle \hat{\delta} \hat{X}_{\text{out}} (t) \hat{\delta} \hat{X}_{\text{out}} (t') \rangle = \sum_{i,j} \langle \hat{\delta} \hat{x}_{i,1} \hat{\delta} \hat{x}_{j,1} \rangle \sqrt{\lambda_i \lambda_j} \varphi_i (t) \varphi_j (t').
\]  

(58)

To calculate the normally ordered average $\langle \hat{\delta} \hat{x}_{i,1} \hat{\delta} \hat{x}_{j,1} \rangle$ let us use inverse transformation with respect to the decomposition (57), which is given by

\[
\hat{\delta} \hat{x}_{i,1} = \int_0^{T_W} dt \hat{\delta} \hat{X}_{\text{in}} (T_W - t) \varphi_i (t).
\]  

(59)

Then one can derive

\[
\langle \hat{\delta} \hat{x}_{i,1} \hat{\delta} \hat{x}_{j,1} \rangle = \int_0^{T_W} dt \int_0^{T_W} dt' \langle \hat{\delta} \hat{X}_{\text{in}} (T_W - t) \hat{\delta} \hat{X}_{\text{in}} (T_W - t') \rangle \varphi_i (t) \varphi_j (t').
\]  

(60)
Now we again take into account Eq. (22) under the condition \( \kappa T_W \gg 1 \), and we get
\[
\langle \delta \hat{x}_{in,i} \delta \hat{x}_{in,j} \rangle = \frac{p}{4} \frac{1 - \mu}{(1 - \mu/2)^2} \delta_{ij}.
\] (61)
Substituting this equality to (23) and passing to the Fourier domain, we obtain for \( \omega = -\omega' \):
\[
4\langle |\delta \hat{X}_{out,\omega}|^2 \rangle = 1 + \frac{p}{4} \frac{(1 - \mu)}{(1 - \mu/2)^2} \sum_{i} \lambda_i |\varphi_{i,\omega}|^2.
\] (62)
Let us recall that this equation was calculated under the condition \( T_W = T_R \), but it can be generalized to an arbitrary relation between the writing and reading times, which we will not consider here. As before we implied the discretization of the frequency scale, that results in
\[
\delta \hat{X}_{out,\omega} = \frac{1}{\sqrt{T}} \int_{0}^{T} dt \delta \hat{X}_{out}(t)e^{i\omega t} dt, \quad \varphi_{i,\omega} = \frac{1}{\sqrt{T}} \int_{0}^{T} dt \varphi_{i}(t)e^{i\omega t} dt.
\] (63)
Let us consider the numerical evaluations of squeezing of the spectral components of the retrieved signal after the full cycle of memory. We will use above mentioned well-matched set of the dimensionless parameters \( L = 10 \), \( T_W = 5.5 \). Under this choice of parameters only the first two modes play a significant role in the writing/reading process as one can see from the estimation of eigenvalues \( \lambda_i \) and eigenfunctions \( \varphi_{i,\omega} \) (see [1]). For these modes \( \lambda_1 = 1.0 \), \( |\varphi_{1,0}| = 0.69 \) and \( \lambda_2 = 0.9 \), \( |\varphi_{2,0}| = 0.66 \). These parameters are negligibly small for all the other modes and their contributions in the series (22) can be omitted.
Substituting these numerical values in Eq. (23) under the condition \( p = -1 \) we get \( 4\langle |\delta \hat{X}_{out,\omega=0}|^2 \rangle \approx 0.13 \) that means the zero spectral component of the retrieved light quadrature is squeezed by 87%, herewith we assumed the perfect squeezing of the input light.
We can conclude that although the initial squeezed pulse was written in the tripod medium into two channels with a random distribution of photons between the channels, but the initial quantum features are almost completely retrieved in the output signal.

**X. SUCCESSIVE READ OUT FROM TWO CHANNELS**

In the previous section, we have studied the simultaneous reading from two spin waves and we have shown that the single signal pulse is squeezed as far as the efficiency of the memory. Now we will discuss the situation when only one control pulse (for example, with the Rabi frequency \( \Omega_1 \)) acts in the time interval from 0 to \( T_R \), and then the other one acts in the time interval from \( T_0 \) (\( \gg T_R \)) to \( T_0 + T_R \). The time interval between pulses must be chosen sufficiently large to avoid all possible transient processes associated with the first pulse before the arrival of the second pulse.

For such a procedure of signal retrieval, it is follow from Eqs. (64-65) and clear from physics, the reading is carried out first from one channel and then from the other one. Both these processes are analogous to ones at \( \Lambda \)-configuration of atoms. This allows us to express quadratures of both read out signals through the quadrature of spin waves in the form:

\[
\delta \hat{X}_{out}^{(1)}(t) = -\frac{1}{\sqrt{2}} \int_{0}^{L} dz \delta \hat{X}_1(z)G_{ba}(z,t)\Theta(t) + \hat{v}_1(t),
\] (64)

\[
\delta \hat{X}_{out}^{(2)}(t) = -\frac{1}{\sqrt{2}} \int_{0}^{L} dz \delta \hat{X}_2(z)G_{ba}(z,t-T_0)\Theta(t-T_0) + \hat{v}_2(t).
\] (65)
It is important to note that the kernel $G_{ba}$ coincides with the kernel $G_{ab}$ (see Eq. (21)), which we have analyzed above. This means that for the readout of the first pulse we have chosen $\Omega_1 = \Omega$ and $\Omega_2 = 0$, and for the second $\Omega_2 = \Omega$ and $\Omega_1 = 0$. Let us remind that on the writing stage we assumed $\Omega_1 = \Omega_2 = \Omega/\sqrt{2}$.

Taking into account Eqs. (54)–(55), Schmidt decomposition and Fourier transform for the discrete frequency scale (63), we arrive at the expressions for the spectral squeezing in each of pulses:

$$4|\langle \delta \hat{X}_{out,\omega}^{(1)} \rangle^2| = 4|\langle \delta \hat{X}_{out,\omega}^{(2)} \rangle^2| = 1 + \frac{p}{2}(1 - \mu) \sum_i \lambda_i |\varphi_{i,\omega}|^2.$$  \hspace{1cm} (66)

Let us estimate the value of the second term in (66) on the basis of data shown in Fig. 4. One can be convinced that under the previous matched set of parameters $L = 10$, $T_W = 5.5$ in the case of the sub-Poissonian pumping of laser ($p = -1$) the squeezing in each of pulses reaches $43.5\%$ (this value tends to $50\%$ with increasing the optical depth of the cell).

Now let us check the degree of the entanglement of the read out pulses. As we discussed above (see Appendix A), the Duan-criterion of entanglement for two canonical oscillators can be given by

$$D_{\omega,-\omega} = \langle \; \langle \delta \hat{X}_{out,\omega}^{(1)} + \delta \hat{X}_{out,\omega}^{(2)} \; \rangle \rangle + \langle \; \langle \delta \hat{Y}_{out,\omega}^{(1)} - \delta \hat{Y}_{out,\omega}^{(2)} \; \rangle \rangle < 0.$$  \hspace{1cm} (67)

In accordance with the previous statement, the second term (with a difference of quadratures) is equal zero. That is only the first term (with a sum of quadratures) contributes into this value. As a result, we get inequality:

$$D_{\omega,-\omega} = 4\langle |\delta \hat{X}_{out,\omega}^{(1,2)}|^2 \rangle = \frac{p}{2}(1 - \mu) \sum_i \lambda_i |\varphi_{i,\omega}|^2 < 0.$$  \hspace{1cm} (68)

One can see from the Eq. (68) that similar to spin waves there exists not only the squeezing of light quadratures but also the entanglement of signal pulses which reaches $43\%$ at $p = -1$ (and tends to $50\%$ with increasing of $L$). Thus, under the successive reading, the quantum states of the signal pulses well repeat the quantum states of the spin waves.

**XI. CONCLUSION**

In this work we considered the protocol of quantum memory based on tripod atomic configuration in the simplest case, when there is only one signal pulse in squeezed state at the input of the system. In this case the writing of the signal is carried out in two channels and two spin waves are excited in the resonance medium. It would seem that under this process there is an additional source of noise as compared with the memory, based on the $\Lambda$-configuration of atoms, which associated with the random distribution of signal photons between spin waves. However, the simultaneous read out from two channels (with the correct choice of the characteristic parameters for efficient operation) demonstrates the preservation of quantum light features: the squeezing of the input signal is almost completely retrieved in the output light. The situation is similar to the mixture of squeezed and vacuum fields on the Mach–Zehnder interferometer.

The presence of an additional degree of freedom (the second channel) in the system creates a possibility for various manipulations with the quantum states. Particularly, under successive reading, the retrieved light reproduce the state of the spin waves, as opposed to the traditional approaches, where the original state of the signal pulse is retrieved. This spin wave state can be different in dependence on the quantum state and the temporal structure of the input signal.

Let us discuss qualitatively what happens when not one, but two successive signal pulses in orthogonally squeezed states are incident at the cell input. As has been obtained above, choosing $\Omega_1 = \Omega_2$ for the writing of the first pulse, we provide the excitation of spin waves $\hat{b}_+$ with squeezed $X$-quadrature. Herewith the spin wave $\hat{b}_-$ is not affected and keeps in the vacuum state after the writing of the first pulse. Then, the second pulse can be written in the memory with the control field $\Omega_1 = -\Omega_2$ that provides the excitation of the wave $\hat{b}_-$ with squeezed $Y$-quadrature. At the same time the spin wave $\hat{b}_+$ keeps its $X$-quadrature in the squeezed state, which caused by the first pulse. It leads to the entanglement of the spin waves $\hat{b}_{1,2}$ that can be converted into the retrieval signal pulses under successive read out.
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Appendix A: Duan criterion of entanglement

In our task we do not define formally some pure quantum state for the input signal pulse, but determine the source of light, which gives the field in the mixed state. Due to this fact, the Duan criterion [27] looks like the most suitable for the analysis of the entanglement. This criterion can be easily formulated for two oscillators, which are described by canonical variables $\hat{q}_i = \hat{q}_i^\dagger$ and $\hat{p}_i = \hat{p}_i^\dagger$ ($i = 1, 2$). They obey the following commutation relations:

$$[\hat{q}_i, \hat{p}_j] = i/2 \delta_{ij}, \quad i, j = 1, 2, \quad (A1)$$

and are introduced as a real and an imaginary parts of the annihilation operator $\hat{a}_i = \hat{q}_i + i\hat{p}_i$.

The Duan criterion is introduced as follows. If the fluctuations of canonical variables $\delta \hat{q}_i = \langle \hat{q}_i \rangle - \langle \hat{q}_i \rangle$ and $\delta \hat{p}_i = \langle \hat{p}_i \rangle - \langle \hat{p}_i \rangle$ obey the inequality

$$D = \langle (\delta \hat{q}_1 + \delta \hat{q}_2)^2 \rangle + \langle (\delta \hat{p}_1 - \delta \hat{p}_2)^2 \rangle < 1, \quad (A2)$$

then these two oscillators are in the entangled state.

Let us now apply this for the electromagnetic field. We consider two beams, which propagate along the $z$-axis and can be formally described by two Heisenberg amplitudes $\hat{E}_i(t)$ ($i = 1, 2$) in some cross-section (for example, on the output edge of some device). Balanced homodyne detection allows us to select the quadrature components of the fields as its real and imaginary parts:

$$\hat{E}_i(t) = \hat{X}_i(t) + i\hat{Y}_i(t), \quad \hat{X}_i^\dagger(t) = \hat{X}_i(t), \quad \hat{Y}_i^\dagger(t) = \hat{Y}_i(t). \quad (A3)$$
Next we pass to the Fourier domain and show how the Duan criterion can be formulated in terms of spectral fluctuations of quadrature components $\delta \hat{X}_{i,\omega}$ and $\delta \hat{Y}_{i,\omega}$. Thereby we demonstrate how to measure the entanglement in the scheme of balanced homodyne detection of spectral field components.

Let us rewrite Eqs. (A3) in the Fourier domain:

$$\hat{E}_{i,\omega} = \hat{X}_{i,\omega} + i\hat{Y}_{i,\omega}, \quad \hat{X}_{i,\omega} = \hat{X}_{i,-\omega}, \quad \hat{Y}_{i,\omega} = \hat{Y}_{i,-\omega},$$

(A4)

Non-Hermitian amplitudes $\hat{E}_{i,\omega}$ describe the behavior of the field oscillators with frequency $\omega_0 + \omega$ and physically are similar to the amplitudes $\hat{a}_i$ introduced in the beginning of this section. However, the spectral quadratures $\hat{X}_{i,\omega}, \hat{Y}_{i,\omega}$ are not canonical variables, because they are not Hermitian as well as the amplitudes $\hat{E}_{i,\omega}$. By analogy with the previous construction, let us introduce the real and imaginary parts of the amplitudes $\hat{E}_{i,\omega}$:

$$\hat{E}_{i,\omega} = \hat{Q}_{i,\omega} + i\hat{P}_{i,\omega}, \quad \hat{Q}_{i,\omega} = \hat{Q}_{i,-\omega}, \quad \hat{P}_{i,\omega} = \hat{P}_{i,-\omega}.$$  

(A5)

Now the operators $\hat{Q}_{i,\omega}$ and $\hat{P}_{i,\omega}$ can play the role of canonical variables, since they are Hermitian and obey the commutation relation

$$\left[\hat{Q}_{i,\omega}, \hat{P}_{i,\omega}\right] = i/2 \delta(\omega - \omega').$$

(A6)

First, we estimate the degree of entanglement of two oscillators with the same frequency, but from different light beams. In this case the Duan criterion can be written in the form

$$D_1 = \langle (\delta \hat{Q}_{1,\omega} + \delta \hat{Q}_{2,\omega})^2 \rangle + \langle (\delta \hat{P}_{1,\omega} - \delta \hat{P}_{2,\omega})^2 \rangle < 1.$$  

(A7)

The measuring procedure can be constructed so that $\hat{Q}$ and $\hat{P}$ are observables. This allows us to apply the Duan criterion in form (35) directly, to evaluate the entanglement of the state. However, if the experiment is based on the balanced homodyne detection, we should rewrite this equation via the quadratures. This can be done, taking into account the following equalities:

$$\hat{Q}_{i,\omega} = \frac{1}{2} \left( \hat{X}_{i,\omega} + \hat{X}_{i,-\omega} \right) - \frac{1}{2i} \left( \hat{Y}_{i,\omega} - \hat{Y}_{i,-\omega} \right),$$  

(A8)

$$\hat{P}_{i,\omega} = \frac{1}{2} \left( \hat{X}_{i,\omega} - \hat{X}_{i,-\omega} \right) + \frac{1}{2} \left( \hat{Y}_{i,\omega} + \hat{Y}_{i,-\omega} \right).$$  

(A9)

Then the inequality (35) is given by

$$D_1 = \langle (\delta \hat{X}_{1,\omega} + \delta \hat{X}_{2,-\omega})^2 \rangle + \langle (\delta \hat{Y}_{1,\omega} - \delta \hat{Y}_{2,-\omega})^2 \rangle < 1.$$  

(A10)

In the left side of this inequality we have omitted

$$-i\langle \delta \hat{X}_{1,\omega} + \delta \hat{X}_{2,-\omega} \rangle \langle \delta \hat{Y}_{1,-\omega} - \delta \hat{Y}_{2,-\omega} \rangle + \text{h.c.}$$  

(A11)

that is correct when different quadratures are statistically independent.

Similarly, we can calculate the correlation between oscillators in different beams with arbitrary frequencies $\omega$ and $\omega'$. Let us derive Duan criterion in terms of the canonical variables for the case when $\omega' = -\omega$:

$$D_2 = \langle (\delta \hat{Q}_{1,\omega} + \delta \hat{Q}_{2,\omega})^2 \rangle + \langle (\delta \hat{P}_{1,\omega} - \delta \hat{P}_{2,-\omega})^2 \rangle < 1.$$  

(A12)

Then, substituting Eqs. (A8)-(A9), we obtain

$$D_2 = \langle (\delta \hat{X}_{1,\omega} + \delta \hat{X}_{2,\omega})^2 \rangle + \langle (\delta \hat{Y}_{1,\omega} - \delta \hat{Y}_{2,\omega})^2 \rangle < 1.$$  

(A13)

Here we have again assumed that the different quadratures are statistically independent, and have omitted

$$-i\langle \delta \hat{X}_{1,\omega} + \delta \hat{X}_{2,\omega} \rangle \langle \delta \hat{Y}_{1,-\omega} - \delta \hat{Y}_{2,-\omega} \rangle + \text{h.c.}$$  

(A14)

Finally, let us investigate the entanglement of two field oscillators with frequencies $\omega_0 \pm \omega$ in the same beam. In this case, the Duan criterion is as follows

$$D = \langle (\delta \hat{Q}_0 + \delta \hat{Q}_-)^2 \rangle + \langle (\delta \hat{P}_0 - \delta \hat{P}_-)^2 \rangle < 1.$$  

(A15)

Substituting (A8)-(A9), we get

$$D = 4\langle (\delta \hat{X}_0)^2 \rangle < 1.$$  

(A16)

As one can see, the Duan criterion is satisfied for multimode squeezed light beam.