Equivalent norms for the Morrey spaces with non-doubling measures

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Abstract

In this paper under some growth condition we investigate the connection between RBMO and the Morrey spaces. We do not assume the doubling condition which has been a key property of harmonic analysis. We also obtain another type of equivalent norms.

KEYWORDS : Morrey space, Campanato space, equivalent norms

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1 Introduction

In this paper we discuss equivalent norms for the (vector-valued) Morrey spaces with non-doubling measures. We consider the connection between the Morrey spaces and the Campanato spaces with underlying measure \( \mu \) non-doubling. The Morrey spaces appeared in \([5]\) originally in connection with the partial differential equations and the Campanato spaces in \([1]\) and \([2]\). We refer to \([6]\) for

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the result of Morrey spaces coming with the doubling measures. Before we state our main theorem, let us make a brief view of the terminology of measures on \( \mathbb{R}^d \).

We say that a (positive) Radon measure \( \mu \) on \( \mathbb{R}^d \) satisfies the growth condition if

\[
\mu(Q(x, l)) \leq C_0 l^n \quad \text{for all } x \in \text{supp}(\mu) \text{ and } l > 0,
\]

where \( C_0 \) and \( n \in (0, d] \) are some fixed numbers. A measure \( \mu \) is said to satisfy the doubling condition if

\[
\mu(Q(x, 2l)) \leq C \mu(Q(x, l)) \quad \text{for all } x \in \mathbb{R}^d \text{ and } l > 0
\]

for some constant \( C > 0 \). A measure \( \mu \) which satisfies the growth condition will be called growth measure while a measure \( \mu \) with the doubling condition will be called the doubling measure.

By a “cube” \( Q \subset \mathbb{R}^d \) we mean a closed cube having sides parallel to the axes. Its center will be denoted by \( z_Q \) and its side length by \( \ell(Q) \). By \( Q(x, l) \) we will also denote the cube centered at \( x \) of sidelength \( l \). For \( \rho > 0 \), \( \rho Q \) means a cube concentric to \( Q \) with its sidelength \( \rho \ell(Q) \). Let \( Q(\mu) \) denote the set of all cubes \( Q \subset \mathbb{R}^d \) with positive \( \mu \)-measures. If \( \mu \) is finite, we include \( \mathbb{R}^d \) in \( Q(\mu) \) as well. In [7], the authors defined the Morrey spaces \( \mathcal{M}_q^p(k, \mu) \) for non-doubling measures normed by

\[
\| f : \mathcal{M}_q^p(k, \mu) \| := \sup_{Q \in Q(\mu)} \mu(kQ)^{\frac{1}{q} - \frac{1}{p}} \left( \int_Q |f|^q d\mu \right)^{\frac{1}{q}}, \quad 1 \leq q \leq p < \infty, \ k > 1.
\]

The fundamental property of this norm is

\[
\| f : \mathcal{M}_q^p(k_1, \mu) \| \leq \| f : \mathcal{M}_q^p(k_2, \mu) \| \leq C_d \left( \frac{k_1 - 1}{k_2 - 1} \right)^d \| f : \mathcal{M}_q^p(k_1, \mu) \|
\]

for \( 1 < k_1 < k_2 < \infty \). With this relation in mind, we will denote \( \mathcal{M}_q^p(\mu) = \mathcal{M}_q^p(2, \mu) \). The aim of this paper is to find some norms equivalent to this Morrey norm.

## 2 Equivalent norm of doubling type

In this section we investigate an equivalent norm related to the doubling cubes. Although we now envisage the non-homogeneous setting, we are still able to place ourselves in the setting of the doubling cubes. In [10], Tolsa defined the notion of doubling cubes. Let \( k, \beta > 1 \). We say that \( Q \in Q(\mu) \) is a \( (k, \beta) \)-doubling cube, if \( \mu(kQ) \leq \beta \mu(Q) \). It is well-known that, if \( \beta > k^d \), then for \( \mu \)-almost all \( x \in \mathbb{R}^d \) and for all \( Q \in Q(\mu) \) centered at \( x \), we can find a \( (k, \beta) \)-doubling
cube from $k^{-1}Q, k^{-2}Q, \ldots$. In what follows we denote by $Q(\mu; k, \beta)$ the set of all $(k, \beta)$-doubling cubes in $Q(\mu)$. We fix $k, \beta > 1$ with $\beta > k^d$. Let $1 \leq q \leq p < \infty$. For $f \in L^1_{loc}(\mu)$ define

$$
\|f : M^p_q(\mu)\|_d := \sup_{Q \in Q(\mu; k, \beta)} \mu(Q)^{\frac{1}{p} - \frac{1}{q}} \left( \int_Q |f(y)|^q \, d\mu(y) \right)^{\frac{1}{q}}.
$$

Now we present the main theorem in this section.

**Theorem 2.1.** Let $\mu$ be a Radon measure which does not necessarily satisfy the growth condition nor the doubling condition and let $1 \leq q < p < \infty$. If $\beta > k^{\frac{dpq}{p-q}}$, then

$$
C^{-1} \|f : M^p_q(\mu)\|_d \leq \|f : M^p_q(k, \mu)\| \leq C \|f : M^p_q(\mu)\|_d, \quad f \in M^p_q(\mu),
$$

for some constant $C > 0$.

Before we come to the proof of Theorem 2.1, two clarifying remarks may be in order.

**Remark 2.2.** If $p = q$, this theorem fails in general. However, if we assume the growth condition or the doubling condition, the theorem is still available for $p = q$. In fact, under the growth condition or the doubling condition for any cube $Q \in Q(\mu)$ we can find a large integer $j \gg 1$ such that $2^jQ \in Q(\mu; k, \beta)$.

**Remark 2.3.** This theorem readily extends to the vector-valued version. Let $1 \leq q \leq p < \infty$ and $r \in (1, \infty)$. We define the vector-valued Morrey spaces $M^p_q(l^r, \mu)$ by the set of sequences of $\mu$-measurable functions $\{f_j\}_{j \in \mathbb{N}}$ for which

$$
\|f_j : M^p_q(l^r, \mu)\| := \sup_{Q \in Q(\mu)} \mu(2^jQ)^{\frac{1}{r} - \frac{1}{q}} \left( \int_Q \|f_j : l^r\|_q \, d\mu(y) \right)^{\frac{1}{q}} < \infty.
$$

The theorem can be extended to the vector valued version. Let

$$
\|f_j : M^p_q(l^r, \mu)\|_d := \sup_{Q \in Q(\mu; k, \beta)} \mu(Q)^{\frac{1}{p} - \frac{1}{q}} \left( \int_Q \|f_j(y) : l^r\|_q \, d\mu(y) \right)^{\frac{1}{q}}.
$$

Then $C^{-1} \|f_j : M^p_q(l^r, \mu)\|_d \leq \|f_j : M^p_q(l^r, \mu)\| \leq C \|f_j : M^p_q(l^r, \mu)\|_d$. The same proof as the scalar-valued spaces works for the vector-valued spaces, so in the actual proof we concentrate on the scalar-valued cases.

**Proof.** Given $k > 1$, we shall prove

$$
C^{-1} \|f : M^p_q(\mu)\|_d \leq \|f : M^p_q(k, \mu)\|, \quad \|f : M^p_q(\mu)\| \leq C \|f : M^p_q(\mu)\|_d
$$

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for large $\beta > 0$. The left inequality is obvious, so let us prove the right inequality. We have only to show that, for every cube $Q \in \mathcal{Q}(\mu)$,

$$\mu(2Q)^{\frac{1}{p} - \frac{1}{q}} \left( \int_Q |f(y)|^q d\mu(y) \right)^{\frac{1}{q}} \leq C \|f : \mathcal{M}^p_q(\mu)\|_d.$$ 

Let $x \in Q \cap \text{supp } (\mu)$ and $Q(x)$ the largest doubling cube centered at $x$ and having sidelength $k^{-j}\ell(Q)$ for some $j \in \mathbb{N}$. Existence of $Q(x)$ can be ensured for $\mu$-almost all $x \in \mathbb{R}^d$. Set

$$Q_0(j) := \{Q(x) : \ell(Q(x)) = k^{-j}\ell(Q)\}, j \in \mathbb{N}.$$ 

By Besicovitch’s covering lemma we can take $Q(j) \subset Q_0(j)$ so that $\sum_{R \in Q(j)} \chi_R \leq 4^d\chi_{2Q}$ and that $x \in \bigcup_{R \in Q(j)} R$ for $\mu$-almost all $x \in Q$ with $\ell(Q(x)) = k^{-j}\ell(Q)$.

Volume argument gives us that $\#(Q(j)) \leq 8^d k^j$. Since

$$\left( \int_Q |f(y)|^q d\mu(y) \right)^{\frac{1}{q}} \leq \sum_{j=1}^{\infty} \sum_{R \in Q(j)} \left( \int_R |f(y)|^q d\mu(y) \right)^{\frac{1}{q}} \mu(R)^{\frac{1}{p} - \frac{1}{q}} \left( \int_R |f(y)|^q d\mu(y) \right)^{\frac{1}{q}}$$

$$\leq \sum_{j=1}^{\infty} 8^d k^j \beta^{j\left(\frac{1}{p} - \frac{1}{q}\right)} \|f : \mathcal{M}^p_q(\mu)\|_d$$

$$= \sum_{j=1}^{\infty} 8^d \exp \left\{ j \left( d \log k + \left( \frac{1}{p} - \frac{1}{q} \right) \log \beta \right) \right\} \|f : \mathcal{M}^p_q(\mu)\|_d \leq C \|f : \mathcal{M}^p_q(\mu)\|_d,$$

where the constant $C$ is finite, provided $\beta > k^{\frac{d\mu}{p-2q}}$.

## 3 Equivalent norms of Campanato type

Throughout the rest of this paper we assume that $\mu$ satisfy the growth condition (1). We do not assume that $\mu$ is doubling. Before we formulate our theorems, let us recall the definition of the RBMO spaces due to Tolsa [10]. Given two cubes $Q \subset R$ with $Q \in \mathcal{Q}(\mu)$, we denote

$$\delta(Q, R) := \int_{\ell(Q)}^{\ell(Q_R)} \frac{\mu(Q(zQ, l))}{l^n} \frac{dl}{l}, \quad K_{Q,R} = 1 + \delta(Q, R),$$

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where $Q_R$ denotes the smallest cube concentric to $Q$ containing $R$. Here and below we abbreviate the $(2, 2^{d+1})$-doubling cube to the doubling cube and $Q(\mu; 2, 2^{d+1})$ to $Q(\mu, 2)$. Given $Q \in Q(\mu)$, we set $Q^*$ as the smallest doubling cube $R$ of the form $R = 2^jQ$ with $j = 0, 1, \ldots.$

Tolsa defined a new BMO for the growth measures, which is suitable for the Calderón-Zygmund theory. We say that $f \in L^1_{loc}(\mu)$ is an element of $\text{RBMO}$ if it satisfies

$$\|f\|_* := \sup_{Q \in Q(\mu)} \frac{1}{\mu\left(\frac{3}{2}Q\right)} \int_Q |f(x) - m_Q^*(f)| \, d\mu(x) + \sup_{Q \subset R \in Q(\mu, 2)} \frac{|m_Q(f) - m_R(f)|}{K_{Q,R}} < \infty,$$

where $m_Q(f) := \frac{1}{\mu(Q)} \int_Q f(y) \, d\mu(y)$. Further details may be found in [10, Section 2]. The following lemma is due to Tolsa.

**Lemma 3.1.** [10, Corollary 3.5] Let $f \in \text{RBMO}$.

1. There exist positive constants $C$ and $C'$ independent of $f$ so that, for every $\lambda > 0$ and every cube $Q \in Q(\mu)$,

$$\mu\{x \in Q : |f(x) - m_Q^*(f)| > \lambda\} \leq C \mu\left(\frac{3}{2}Q\right) \exp\left(-\frac{C'\lambda}{\|f\|_*}\right).$$

2. Let $1 \leq q < \infty$. Then there exists a constant $C$ independent of $f$, so that, for every cube $Q \in Q(\mu)$,

$$\left(\frac{1}{\mu\left(\frac{3}{2}Q\right)} \int_Q |f(x) - m_Q^*(f)|^q \, d\mu(x)\right)^{\frac{1}{q}} \leq C \|f\|_*.$$

**Elementary property of $\delta(\cdot, \cdot)$** Below we list elementary properties of $\delta(\cdot, \cdot)$ used in this paper.

**Lemma 3.2.** Let $Q \in Q(\mu)$. Then the following properties hold:

1. For $\rho > 1$, we have $\delta(Q, \rho Q) \leq C_0 \log \rho$.
2. $\delta(Q, Q^*) \leq C_0 2^{n+1} \log 2$.
3. Let $k_0 \in \mathbb{N}$ and $\alpha > 0$. Assume, for some $\theta > 0$, $\alpha \leq \mu(Q) \leq \mu(2^{k_0}Q) \leq \theta \alpha$. Then $\delta(Q, 2^{k_0}Q) \leq 2^n \log 2 \cdot \theta C_0 c_n$, where $c_n := \sum_{k=0}^{\infty} 2^{-nk}$.

\[\text{By the growth condition [11] there are a lot of big doubling cubes. Precisely speaking, given a cube } Q \in Q(\mu), \text{ we can find } j \in \mathbb{N} \text{ with } 2^jQ \in Q(\mu, 2) \text{ (see [10]).}\]
(4) Given the cubes \( P \subset Q \subset R \) with \( P \in \mathcal{Q}(\mu) \), then
\[
|\delta(P, R) - (\delta(P, Q) + \delta(Q, R))| \leq C,
\]
where \( C \) is a constant depending only on \( C_0, n, d \).

(5) Let \( Q, R \in \mathcal{Q}(\mu) \). Suppose, for some constant \( c_1 > 1 \), \( Q \subset R \) and \( \ell(R) \leq c_1 \ell(Q) \). Then there exists a doubling cube \( S \in \mathcal{Q}(\mu, 2) \) such that \( Q^*, R^* \subset S \) and \( \delta(Q^*, S), \delta(R^*, S) \leq C \), where \( C \) is a constant depending only on \( c_1, C_0, n, d \).

**Proof.** In \([7]\), we have proved (1)–(4). For reader’s convenience the full proof is given here. (1) is obvious. To prove (2) we set \( Q^* = 2^{k_0}Q_0 \). We may assume that \( k_0 \geq 1 \). The dyadic argument yields that
\[
\delta(Q, 2^{k_0}Q) = \int_{\ell(Q)}^{\ell(2^{k_0}Q)} \frac{\mu(Q(z, l))}{l^n} \, dl \leq 2^n \log 2 \sum_{k=1}^{k_0} \frac{\mu(2^kQ)}{(2^{k_0}Q)^n}.
\]
Note that \( 2^{d+1} \mu(2^k-1Q) \leq \mu(2^kQ) \) for \( k = 1, 2, \ldots, k_0 \), since \( 2^k-1Q \) is not doubling, which yields, together with the fact that \( d \geq n \),
\[
\delta(Q, 2^{k_0}Q) \leq 2^n \log 2 \frac{\mu(2^{k_0}Q)}{\ell(2^{k_0}Q)^n} \sum_{k=1}^{k_0} (2^{n-k}) \leq C_0 2^{n+1} \log 2.
\]

We prove (3). It follows by the dyadic argument and the assumption that
\[
\delta(Q, 2^{k_0}Q) \leq 2^n \log 2 \sum_{k=1}^{k_0} \frac{\mu(2^kQ)}{\ell(2^{k_0}Q)^n} \leq 2^n \log 2 \cdot \frac{\theta \alpha}{\ell(Q)^n} \sum_{k=0}^{k_0} 2^{-nk} \leq 2^n \log 2 \cdot \theta C_0 c_n.
\]

Now we prove (4). It suffices to prove that
\[
A := |\delta(P, Q) - \delta(Q, R)| \leq C.
\]

We decompose \( A \) as
\[
A = \left| \int_{\ell(P_R)}^{\ell(P_Q)} \frac{\mu(Q(z, l))}{l^n} \, dl - \int_{\ell(Q)}^{\ell(Q_R)} \frac{\mu(Q(z, l))}{l^n} \, dl \right|
\]
\[
\leq \left| \int_{\ell(P_Q)}^{\ell(P_Q)} \frac{\mu(Q(z, l))}{l^n} \, dl + \int_{\ell(Q)}^{\min \{\ell(P_R), \ell(Q_R)\}} (\mu(Q(z, l)) - \mu(Q(z, l))) \, dl \right|
\]
\[
+ \int_{\min \{\ell(P_R), \ell(Q_R)\}}^{\max \{\ell(P_R), \ell(Q_R)\}} \left( \frac{\mu(Q(z, l))}{l^n} + \frac{\mu(Q(z, l))}{l^n} \right) \, dl =: A_1 + A_2 + A_3.
\]
By (1) the integrals $A_1$ and $A_3$ are easily estimated above by some constant $C$. So we estimate $A_2$. Bound $A_2$ from above by

$$A_2 \leq \int_{\ell(P_Q)}^{\infty} \mu(Q(z_p,l)\Delta Q(z_Q,l)) \frac{dl}{l^{n+1}} = \int_{\ell(P_Q)}^{\infty} \int_{\mathbb{R}^d} \chi_{Q(z_p,l)\Delta Q(z_Q,l)}(y) d\mu(y) \frac{dl}{l^{n+1}}.$$

A simple geometric observation tells us that $\chi_{Q(z_p,l)\Delta Q(z_Q,l)}(y) = 0$ if

$$l \notin \min\{|y - z_p|_\infty, |y - z_Q|_\infty\}, \max\{|y - z_p|_\infty, |y - z_Q|_\infty\},$$

where $|y|_\infty := \max\{|y_1|, \ldots, |y_d|\}$. This observation and Fubini’s theorem yield

$$A_2 \leq C \int_{\mathbb{R}^d \setminus P_Q} \frac{1}{|y - z_p|_\infty^n - |y - z_Q|_\infty^n} d\mu(y)
\leq C \int_{|y - z_p|_\infty \geq \ell(P_Q)/2} \frac{|z_p - z_Q|_\infty}{|y - z_p|_\infty^n + 1} d\mu(y) \leq C \frac{|z_p - z_Q|_\infty}{\ell(P_Q)} \leq C.$$

This proves (2).

Finally we establish (4). Let $Q^* = 2^l Q$. Then we claim $\delta(R, 2^l R) \leq C$. Indeed, by virtue of the fact that $Q \subset R$ we see that if $l \geq \ell(R)$ then $Q(z_R,l) \subset Q(z_Q,2l)$. As a consequence we obtain

$$\delta(R, 2^l R) = \int_{\ell(R)}^{2^l \ell(R)} \frac{\mu(Q(z_R,l))}{l^n} dl
\leq \int_{\ell(R)}^{2^l \ell(R)} \frac{\mu(Q(z_Q,2l))}{l^n} dl
\leq \int_{\ell(Q)}^{c_1 2^{l+1} \ell(Q)} \frac{\mu(Q(z_Q,l))}{l^n} dl \leq C.$$

If we put $S := (2^l+1)R^*$, then $\delta(R^*, S) \leq C$. (1) and (4) finally give us

$$\delta(Q^*, S) \leq \delta(Q^*, 2^l+1 R) + \delta(2^l+1 R, S) + C \leq C.$$

This is the desired result. \[\blacksquare\]

**Scalar-valued Campanato space** Having cleared up the definition of RBMO, we will find a relationship between RBMO and the Morrey spaces. With the definition of RBMO in mind, we shall define the Campanato spaces.

Let $f \in L^1_{loc}(\mu)$. We define the Campanato spaces $C^p_q(k, \mu)$ normed by

$$\|f : C^p_q(k, \mu)\| := \sup_{Q \subset \mathcal{Q}(\mu)} \mu(kQ)^{\frac{1}{q} - \frac{1}{p}} \left( \int_Q |f(x) - m_Q(f)|^q \mu(dx) \right)^{\frac{1}{q}} + \sup_{Q \subset \mathcal{R} \quad Q,R \in \mathcal{Q}(\mu,2)} \mu(Q)^{\frac{1}{p}} \left| m_Q(f) - m_R(f) \right| K_{Q,R}, 1 \leq q \leq p \leq \infty, k > 1.$$
Let $k_1, k_2 > 1$. Then $C^p_q(k_1, \mu)$ and $C^p_q(k_2, \mu)$ coincide as a set and their norms are mutually equivalent. Speaking more precisely, we have the norm equivalence

$$\| f : C^p_q(k_1, \mu) \| \sim \| f : C^p_q(k_2, \mu) \|.$$  \hspace{1cm} (3)

To prove (3) we may assume that $k_2 = 2k_1 - 1$ because of the monotonicity of $C^p_q(k, \mu)$ with respect to $k$. Then all we have to prove is

$$\mu(k_1 Q) \frac{1}{p} - \frac{1}{q} \left( \int_Q |f(x) - m_{Q^*}(f)|^q d\mu(x) \right)^{\frac{1}{q}} \leq C \| f : C^p_q(k_2, \mu) \|$$

for fixed cube $Q \in Q(\mu)$. Divide equally $Q$ into $2^d$ cubes and collect those in $Q(\mu)$. Let us name them $Q_1, Q_2, \ldots, Q_N, N \leq 2^d$. The triangle inequality reduces the matter to showing

$$\mu(k_1 Q) \frac{1}{p} - \frac{1}{q} \left( \int_{Q_l} |f(x) - m_{Q^*}(f)|^q d\mu(x) \right)^{\frac{1}{q}} \leq C \| f : C^p_q(k_2, \mu) \|, \quad 1 \leq l \leq N.$$ 

Note that $k_2Q_l \subset k_1Q$. We apply Lemma 3.2 (5) to obtain an auxiliary doubling cube $R$ which contains $(Q_l)^*, Q^*$ and satisfies $K(Q_l)^*, R, K_{Q^*, R} \leq C$. Thus, we obtain

$$\mu(k_1 Q) \frac{1}{p} - \frac{1}{q} \left( \int_{Q_l} |f(x) - m_{Q^*}(f)|^q d\mu(x) \right)^{\frac{1}{q}} \leq \mu(k_1 Q) \frac{1}{p} - \frac{1}{q} \left( \int_{Q_l} |f(x) - m_{Q_l^*}(f)|^q d\mu(x) \right)^{\frac{1}{q}}$$

$$+ \mu(Q_l) \frac{1}{p} |m_{Q_l^*}(f) - m_R(f)| + \mu(Q_l) \frac{1}{p} |m_R(f) - m_{Q^*}(f)|$$

$$\leq C \| f : C^p_q(k_2, \mu) \|.$$ 

As a result (3) is proved.

Since $C^p_q(k_1, \mu)$ and $C^p_q(k_2, \mu)$ are isomorphic to each other as Banach spaces, no confusion can occur if we denote $C^p_q(\mu) = C^p_q(2, \mu)$.

Note that $C^\infty_q(\mu) = \text{RBMO}$, if $1 \leq q < \infty$. This is an immediate consequence of Lemma 3.1. Thus we can say that RBMO is a limit function space of $C^p_q(\mu)$ as $p \to \infty$ with $q \in [1, \infty)$ fixed.

Next, we observe $Q(\mu, 2)$ can be seen as a net whose order is induced by natural inclusion. With the aid of the following proposition, we shall cope with the ambiguity of constant functions in the semi-norm of the Campanato spaces.

**Proposition 3.3.** Let $1 \leq q \leq p < \infty$. Then the limit $M(f) := \lim_{Q \in Q(\mu, 2)} m_Q(f)$ exists for every $f \in C^p_q(\mu)$. That is, given $\varepsilon > 0$, we can find a doubling cube $Q \in Q(\mu, 2)$ such that

$$|m_R(f) - m_Q(f)| \leq \varepsilon$$
for all \( R \in \mathcal{Q}(\mu, 2) \) engulfing \( Q \). In particular there exists an increasing sequence of concentric doubling cubes \( I_0 \subset I_1 \subset \ldots \subset I_k \subset \ldots \) such that

\[
\{ m_{I_k}(f) \}_{k \in \mathbb{N}_0} \text{ is Cauchy and } \bigcup_k I_k = \mathbb{R}^d.
\] (4)

We remark that the condition like (4) appears in [4]. We are mainly interested in the function \( f \in C_p^q(\mu) \) such that \( M(f) = 0 \).

**Proof.** Before we come to the proof of Proposition 3.3, we note that

\[
\| | \cdot | : C_1^p(\mu) \| \leq C \| f : C_1^p(\mu) \|. \tag{5}
\]

Indeed, we have

\[
\mu \left( \frac{3}{2}Q \right)^{\frac{1}{p-1}} \int_Q |f(x)| - m_{Q^*}(|f|) \, d\mu(x) \\
= \mu \left( \frac{3}{2}Q \right)^{\frac{1}{p-1}} \int_Q \left| \int_{Q^*} |f(x)| - |f(y)| \, d\mu(y) \right| \, d\mu(x) \\
\leq \mu \left( \frac{3}{2}Q \right)^{\frac{1}{p-1}} \frac{1}{\mu(Q^*)} \int_Q \int_{Q^*} |f(x)| - |f(y)| \, d\mu(y) \, d\mu(x) \\
\leq \mu \left( \frac{3}{2}Q \right)^{\frac{1}{p-1}} \frac{1}{\mu(Q^*)} \int_Q \int_{Q^*} |f(x)| - f(y) \, d\mu(y) \, d\mu(x) \\
\leq \mu \left( \frac{3}{2}Q \right)^{\frac{1}{p-1}} \int_Q |f(x)| - m_{Q^*}(f) \, d\mu(x) + \mu(Q^*)^{\frac{1}{p-1}} \int_{Q^*} |m_{Q^*}(f) - f(y)| \, d\mu(y) \\
\leq C \| f : C_1^p(\mu) \|.
\]

In the same way we can prove

\[
\sup_{Q \subset R, Q, R \in \mathcal{Q}(\mu, 2)} \mu(Q)^{\frac{1}{p}} \frac{|m_Q(|f|) - m_R(|f|)|}{K_{Q,R}} \leq C \| f : C_1^p(\mu) \|.
\]

As a consequence (5) is justified.

We now turn to the proof of Proposition 3.3. By the monotonicity of \( C_p^q(\mu) \) with respect to \( q \), we may assume \( q = 1 \).

**Case 1 \( \mu \) is infinite.** Take a sequence of concentric doubling cubes \( \{ Q_j \}_{j \in \mathbb{N}} \) such that for all \( j \in \mathbb{N} \)

\[
\mu(Q_1) \geq 1, \ \mu(Q_{j+1}) \geq 2\mu(Q_j), \ \delta(Q_j, Q_{j+1}) \leq C
\]
for some $C > 0$ depending only on $C_0$. Then by the definition of $\mathcal{C}_p^q(\mu)$ it holds that

$$|m_{Q_j}(f) - m_{Q_{j+1}}(f)| \leq C 2^{-\frac{j}{p}}\|f : \mathcal{C}_p^q(\mu)\|,$$

$j \in \mathbb{N}$.

Thus we establish at least the existence of $M(f) := \lim_{j \to \infty} m_{Q_j}(f)$. Let $Q \in \mathcal{Q}(\mu, 2)$ which contains $Q_j$ and does not contain $Q_{j+1}$. Set $Q' = (Q_j Q)^\ast$. Then by using Lemma 3.2 it is easy to see that $\delta(Q, Q') \leq C$ for some absolute constant $C > 0$. Then we have

$$|m_{Q}(f) - m_{Q}(f)|, |m_{Q'}(f) - m_{Q}(f)| \leq C 2^{-\frac{j}{p}}\|f : \mathcal{C}_1^p(\mu)\|,$$

which implies

$$|m_{Q}(f) - M(f)| \leq C 2^{-\frac{j}{p}}\|f : \mathcal{C}_1^p(\mu)\|.$$

Thus we finally establish $M(f) = \lim_{Q \in \mathcal{Q}(\mu, 2)} m_{Q}(f)$.

**Case 2** $\mu$ is finite. In this case, we have only to prove

Claim 3.4. If $\mu$ is finite and $\|f : \mathcal{C}_1^p(\mu)\| < \infty$, then $f \in L^1(\mu)$.

In proving Claim 3.4 allows us to assume $f$ is positive.

We take an increasing sequence of concentric doubling cubes $\{Q_j\}_{j \in \mathbb{N}}$ such that $\delta(Q_1, Q_k) \leq C$ for all $k \in \mathbb{N}$. Then we have

$$m_{Q_k}(f) \leq m_{Q_1}(f) + \mu(Q_1)^{-\frac{1}{p}}(1 + C)\|f : \mathcal{C}_1^p(\mu)\|.$$

Passage to the limit then gives

$$\int_{\mathbb{R}^d} f \, d\mu \leq \mu(\mathbb{R}^d) \left( m_{Q_1}(f) + \mu(Q_1)^{-\frac{1}{p}}(1 + C)\|f : \mathcal{C}_1^p(\mu)\| \right).$$

This establishes $f \in L^1(\mu)$.

The main theorem in this section is the following.

**Theorem 3.5.** Let $1 \leq q \leq p < \infty$. Assume $f \in \mathcal{C}_q^p(\mu)$ satisfies $M(f) = \lim_{Q \in \mathcal{Q}(\mu, 2)} m_{Q}(f) = 0$. Then

$$C^{-1}\|f : \mathcal{C}_1^p(\mu)\| \leq \|f : \mathcal{M}_q^p(\mu)\| \leq C \|f : \mathcal{C}_q^p(\mu)\|$$

for some constant $C > 0$.

The left inequality is obvious. To prove the right inequality we need a lemma.

**Lemma 3.6.** Under the assumption of Theorem 3.5, given $R \in \mathcal{Q}(\mu, 2)$, there exists a sequence of increasing doubling cubes $\{R_k\}_{k=1}^K$ such that
The triangle inequality enables us to majorize the above integral by 

\[ C \] majorized by some constants dependent only on \( \mu \).

Consequently we can reduce the matters to the estimate of \( \mu^3 \). Since by the properties of the lemma, \( \mu(\Omega) \leq C \).

Proof. Suppose we have defined \( R_k \). If \( \mu(\mathbb{R}^d) \leq 2^k \mu(R) \), then we set \( R_{k+1} = \mathbb{R}^d \) and we stop. Suppose otherwise. We define \( R_{k+1} \) as the smallest doubling cube of the form \( 2^l R_k \) with \( l \geq 3 \) whose \( \mu \)-measure exceeds \( 2^k \mu(R) \). By virtue of Lemma 3.2 (3) it is easy to verify that \( \{ R_k \}_{k=1}^K \) obtained in this way satisfies the property of the lemma. \[ \square \]

Let us return to the proof of Theorem 3.5. Let \( R \in \mathcal{Q}(\mu) \). We shall estimate

\[ \mu(2R) \left( \int_R |f(x)|^q \, d\mu(x) \right)^{\frac{1}{q}}. \]

The triangle inequality enables us to majorize the above integral by

\[ \mu \left( \frac{3}{2} R \right) \left( \int_R |f(x) - m_{R^*}(f)|^q \, d\mu(x) \right)^{\frac{1}{q}} + \mu(R) \frac{1}{2} |m_{R^*}(f)|. \]

Consequently we can reduce the matters to the estimate of \( \mu(R^*) \frac{1}{2} |m_{R^*}(f)|. \)

Now we invoke Lemma 3.6 for \( K_0 \) taken so that \( \mu(R) \frac{1}{2} |m_{I_{K_0}}(f)| \leq \|f : C_q^p(\mu)\|. \) Using the sequence \( \{ R_k \}_{k=1}^K \), we obtain

\[ \mu(R^*) \frac{1}{2} |m_{R_k}(f) - m_{R_{k+1}}(f)| \]

\[ \leq C \frac{2}{2^k \mu(R_k)} \left( \frac{|m_{R_k}(f) - m_{R_{k+1}}(f)|}{2^k \mu(R_k)} \right) \leq C \frac{2}{2^k \mu(R_k)} \|f : C_q^p(\mu)\|. \]

We also have \( \mu(R^*) \frac{1}{2} |m_{R_{K_0}}(f) - m_{I_{K_0}}(f)| \leq C \frac{2}{2^k \mu(R_k)} \|f : C_q^p(\mu)\|, \) since by the properties 3 and 4 of Lemma 3.6 we see that the \( \delta(R_{K_0}, R_{K_0} + 1), \delta(I_{K_0}, R_{K_0} + 1) \) are majorized by some constants dependent only on \( C_0 \). The triangle inequality gives us

\[ \mu(R^*) \frac{1}{2} |m_{R^*}(f)| \]

\[ \leq \mu(R) \sum_{k=1}^{k_0} \frac{1}{2^k \mu(R_k)} \left( |m_{R_k}(f) - m_{I_{K_0}}(f)| + |m_{I_{K_0}}(f)| \right) \]

\[ \leq C \left( \sum_{k=1}^{\infty} 2^{-k \frac{1}{2}} \right) \|f : C_q^p(\mu)\| + \mu(R^*) \frac{1}{2} |m_{I_{K_1}}(f)| \leq C \|f : C_q^p(\mu)\|. \]

The proof of Theorem 3.5 is therefore complete.
Vector-valued extension  Finally we consider the vector-valued extensions of Theorem 3.5. Let $\|a_j : l^r\|$ denote the $l^r$-norm of $a = \{a_j\}_{j \in \mathbb{N}}$. If possible confusion can occur, then we write $\|\{a_j\}_{j \in \mathbb{N}} : l^r\|$. For $f \in L^1_{\text{loc}}(\mu)$, we define the sharp maximal operator due to Tolsa by

$$M^s f(x) := \sup_{x \in Q \in \mathcal{Q}(\mu)} \frac{1}{\mu(\frac{3}{2}Q)} \int_Q |f(y) - m_Q^*(f)| \, d\mu(y) + \sup_{\substack{x \in Q \subset R \subset \mathbb{R}^d \mu \in \mathcal{Q}(\mu,2)}} \frac{|m_Q(f) - m_R(f)|}{K_{Q,R}}.$$ 

Lemma 3.7 can be extended to the following vector-valued version.

**Lemma 3.7.** [9] Let $f_j \in \text{RBMO}$ for $j = 1, 2, \ldots$. For any cube $Q \in \mathcal{Q}(\mu)$ and $q, r \in (1, \infty)$, there exists a constant $C$ independent of $f_j$ such that

$$\left( \frac{1}{\mu(\frac{3}{2}Q)} \int_Q \|f_j(x) - m_{Q^*}(f_j) : l^r\|^q d\mu(x) \right)^\frac{1}{q} \leq C \sup_{x \in \mathbb{R}^d} \|M^s f_j(x) : l^r\|. \quad (6)$$

We now define the vector-valued Campanato spaces. Let $1 \leq q \leq p \leq \infty$ and $r \in (1, \infty)$. We say that $\{f_j\}_{j \in \mathbb{N}}$ belongs to the vector-valued Campanato spaces $\mathcal{C}_q^p(l^r, \mu)$ if each $f_j$ is $\mu$-measurable and

$$\|f_j : \mathcal{C}_q^p(l^r, \mu)\| := \sup_{Q \in \mathcal{Q}(\mu)} \mu(2Q)^{\frac{1}{p} - \frac{1}{q}} \left( \int_Q \|f_j(x) - m_{Q^*}(f_j) : l^r\|^q d\mu(x) \right)^\frac{1}{q} + \sup_{\substack{Q \subset R \mu \in \mathcal{Q}(\mu,2)}} \mu(Q)^{\frac{1}{p}} \frac{\|m_Q(f_j) - m_R(f_j) : l^r\|}{K_{Q,R}} < \infty.$$

As for the vector-valued spaces, the norm equivalence of the Campanato type still holds.

**Theorem 3.8.** Let $1 \leq q \leq p < \infty$ and let $\{f_j\}_{j \in \mathbb{N}}$ be a sequence in $\mathcal{C}_q^p(\mu)$. Assume that there exists an increasing sequence of concentric doubling cubes $I_0 \subset I_1 \subset \ldots \subset I_k \subset \ldots$ such that

$$\lim_{k \to \infty} m_{I_k}(f_j) = 0 \text{ for all } j \text{ and } \bigcup_k I_k = \mathbb{R}^d.$$ 

Then there exists a constant $C > 0$ independent of $\{f_j\}_{j \in \mathbb{N}}$ such that

$$C^{-1} \|f_j : \mathcal{C}_q^p(l^r, \mu)\| \leq \|f_j : \mathcal{M}_q^p(l^r, \mu)\| \leq C \|f_j : \mathcal{C}_q^p(l^r, \mu)\|.$$

Using Lemma 3.7, we can say more about $\mathcal{C}_q^\infty(l^r, \mu)$, which gives us a partial clue to the definition of the vector-valued RBMO spaces. Speaking precisely, we obtain the following proposition.
Proposition 3.9. Let \( \{f_j\}_{j \in \mathbb{N}} \) be a sequence of \( L^1_{\text{loc}}(\mu) \) functions. Then

\[
\sup_{Q \subset R} \frac{\|m_Q(f_j) - m_R(f_j) : l'\|}{K_{Q,R}} \leq c \sup_{x \in \mathbb{R}^d} \|M^2 f_j(x) : l'\|. \tag{7}
\]

In particular, we have

\[
\|f_j : C^\infty_q(l', \mu)\| \leq c \sup_{x \in \mathbb{R}^d} \|M^2 f_j(x) : l'\|. \tag{8}
\]

Proof. Fix \( Q \subset R \) such that \( Q \in Q(\mu) \). Then \( \frac{|m_Q(f_j) - m_R(f_j)|}{K_{Q,R}} \leq c M^2 f_j(x) \) for all \( x \in Q \). By taking the \( l' \)-norm of both sides we obtain

\[
\frac{\|m_Q(f_j) - m_R(f_j) : l'\|}{K_{Q,R}} \leq c \sup_{x \in Q} \|M^2 f_j(x) : l'\| \leq c \sup_{x \in \mathbb{R}^d} \|M^2 f_j(x) : l'\|.
\]

Now since \( Q \) and \( R \) are taken arbitrarily, (7) is proved. (8) can be obtained with the help of (6) and (7).

Before we conclude this section, a remark may be in order.

Remark 3.10. Let \( 0 < \alpha < n \). For \( Q, R \in Q(\mu) \) with \( Q \subset R \) we define

\[
K_{Q,R}^{(\alpha)} = 1 + \sum_{k=1}^{N_{Q,R}} \left( \frac{\mu(2^k Q)}{\ell(2^k Q)^n} \right)^{\frac{n-\alpha}{n}},
\]

where \( N_{Q,R} \) is the least integer \( j \) with \( 2^j Q \supset R \). For the definition of this constant we refer to [3]. Theorems in this section still hold, if we replace \( K_{Q,R} \) by \( K_{Q,R}^{(\alpha)} \) whenever \( 1 \leq q \leq p < \infty \).

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