FINITE RANK ISOPAIRS
UDENI WIJESOORIYA

Abstract. An algebraic isopair is a commuting pair of pure isometries that is annihilated by a polynomial. The notion of the rank of a pure algebraic isopair with finite bimultiplicity is introduced as an s-tuple $\alpha = (\alpha_1, ..., \alpha_s)$ of natural numbers. A pure algebraic isopair of finite bimultiplicity with rank $\alpha$, acting on a Hilbert space is nearly max{$\alpha_1, ..., \alpha_s$}-cyclic and there is a finite codimensional invariant subspace such that the restriction to that subspace is max{$\alpha_1, ..., \alpha_s$}-cyclic.

1. INTRODUCTION

Given a polynomial $p \in \mathbb{C}[z, w]$ (or in $\mathbb{C}[z]$) let $Z(p)$ denote its zero set. We say $p$ is square free if $q^2$ does not divide $p$ for every non-constant polynomial $q(z, w) \in \mathbb{C}[z, w]$. We say $q \in \mathbb{C}[z, w]$ is the square free version of $p$ if $q$ is the polynomial with smallest degree such that $q$ divides $p$ and $Z(p) = Z(q)$. The square free version is unique up to multiplication by a nonzero constant.

Let $D$, $T$ and $E$ denote the open unit disk, the boundary of the unit disk and complement of the closed unit disk in $\mathbb{C}$ respectively. In [AKM12] the notion of an inner toral polynomial is introduced. (See also [AMS06, AM06, K10, PS14].) A polynomial $q \in \mathbb{C}[z, w]$ is inner toral if

$$Z(q) \subset D^2 \cup T^2 \cup E^2.$$ 

In other words, if $(z, w) \in Z(q)$ then either $|z|, |w| < 1$ or $|z| = 1 = |w|$ or $|z|, |w| > 1$. A distinguished variety in $\mathbb{C}^2$ is the zero set of an inner toral polynomial.

Let $V$ be an isometry defined on a Hilbert space $H$. By the Wold Decomposition, there exist two reducing subspaces for $V$, say $K$ and $L$, such that $H = K \oplus L$ and $S = V|_K$ is a shift operator and $U = V|_L$ is a unitary operator. We say $V$ is pure, if there is no unitary part. An isometry $V$ is pure if and only if $\bigcap_{j=1}^{\infty} V^j(H) = \{0\}$. A subspace $\mathcal{W}$ of $H$ is called
a wandering subspace for $V$ if $V^n(W) \perp V^m(W)$ for $n \neq m$ and $H = \bigoplus_{n=0}^{\infty} V^n(W)$. If $V$ is a pure isometry and $W = H \ominus V(H) = \ker(V^*)$, then $\ker(V^*)$ is a wandering subspace for $V$. Moreover, if $V$ is a pure isometry then $V \cong M_z$ on the Hilbert Hardy space $H^2_W$ of $W$-valued functions for a Hilbert space $W$ with dimension of $\dim(\ker(V^*))$. The \textit{multiplicity} of a pure isometry $V$ is defined as $\text{mult}(V) = \dim(\ker(V^*))$.

A \textit{pure isopair} is a pair of commuting pure isometries. A pure isopair $V = (S, T)$ is a \textit{pure algebraic isopair} if there is a nonzero polynomial $q \in \mathbb{C}[z, w]$ such that $q(S, T) = 0$ and is also referred to as \textit{pure $q$-isopair}. The study of pure algebraic isopairs was initiated in [AKM12] and also discussed in [M12]. Among the many results in [AKM12] it is shown (see Theorem 1.20) if $V = (S, T)$ is a pure algebraic isopair, then there is a square free inner toral polynomial $p$ such that $p(S, T) = 0$ that is minimal in the sense if $q(S, T) = 0$, then $p$ divides $q$. We call this polynomial $p$ the \textit{minimal polynomial} of $V$. The minimal polynomial of $V$ is unique up to multiplication by a nonzero constant. Moreover, in [AKM12] the notion of a \textit{nearly cyclic} pure isopair is introduced. Here we fix a square free inner toral polynomial $p$ and consider nearly multi-cyclic pure isopairs with the minimal polynomial $p$.

An isopair $V = (S, T)$ acting on a Hilbert space $H$ is called \textit{at most nearly $k$-cyclic} if there exist distinct $f_1, ..., f_k \in H$ such that the closure of

$$\{ \sum_{j=1}^{k} q_j(S, T)f_j : q_j \in \mathbb{C}[z, w] \text{ for } j = 1, 2, ..., k \}$$

is of finite codimension in $H$. It is called \textit{at least nearly $k$-cyclic} if the closure of

$$\{ \sum_{j=1}^{l} q_j(S, T)f_j : q_j \in \mathbb{C}[z, w] \text{ for } j = 1, 2, ..., l \}$$

is not of finite codimension in $H$ for any $l < k$ and for any set of $f_1, ..., f_l \in H$. We say $V = (S, T)$ is \textit{nearly $k$-cyclic} if it is both at most nearly $k$-cyclic and at least nearly $k$-cyclic. Moreover, $V = (S, T)$ is called \textit{$k$-cyclic} if it is nearly $k$-cyclic and the span given in (1) is dense in $H$.

Given a pair of isometries $V = (S, T)$, define the \textit{bimultiplicity} of $V$ by

$$\text{bimult}(V) = (\text{mult}(S), \text{mult}(T)).$$

It is a well known fact that we can view pure isopaires as pairs of multiplication operators. In particular, if $V = (S, T)$ is a pure $p$-isopair of finite multiplicity $(M, N)$, then there exists an $M \times M$ matrix-valued rational inner function $\Phi$ with its poles in $\mathbb{E}$, such that $V$ is unitarily equivalent to $(M_z, M_\Phi)$ on $H^2_{CM}$ and $p(M_z, M_\Phi) = 0$ (see [AKM12]). Moreover

$$p(\lambda, \Phi(\lambda)) = 0 \text{ for } \lambda \in \overline{D}. $$
**Definition 1.1.** We say a point \((\lambda, \mu) \in \mathbb{C}^2\) is a regular point for \(p\) if \((\lambda, \mu) \in Z(p)\), but
\[
\nabla p(\lambda, \mu) = \left( \frac{\partial p}{\partial z}, \frac{\partial p}{\partial w} \right) |(\lambda, \mu) \neq 0.
\]

Let \(p\) be a square free inner toral polynomial. Write \(p = p_1 p_2 \cdots p_s\) as a product of (distinct) irreducible factors. Then each \(p_j\) is inner toral. In other words, each \(Z(p_j)\) is a distinguished variety. The zero set of \(p\) is the union of the zero sets of \(p_j\). Let
\[
\mathfrak{V}(p_j) = Z(p_j) \cap \mathbb{D}^2, \quad \mathfrak{V}(p) = Z(p) \cap \mathbb{D}^2 = \bigcup_{j=1}^s \mathfrak{V}(p_j).
\]

Let \(\mathbb{N}\) denote the non negative integers and \(\mathbb{N}_+\) denote the positive integers.

**Proposition 1.2.** Let \(V = (S, T)\) be a pure \(p\)-isopair of finite bimultiplicity with minimal polynomial \(p\) and suppose \(p = p_1 p_2 \cdots p_s\), a product of distinct irreducible factors. For each \(j\) and \((\lambda, \mu) \in \mathfrak{V}(p_j)\) that is a regular point for \(p\), the dimension of the intersection of \(\ker(S - \lambda)^* \cap \ker(T - \mu)^*\) is a nonzero constant.

**Definition 1.3.** Let \(V = (S, T)\) be a pure \(p\)-isopair of finite bimultiplicity with minimal polynomial \(p\) and suppose \(p = p_1 p_2 \cdots p_s\), a product of distinct irreducible factors. The rank of \(V\) is a \(s\)-tuple, \(\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{N}_+^s\), denoted by \(\text{rank}(V)\), where
\[
\alpha_j = \dim (\ker(S - \lambda)^* \cap \ker(T - \mu)^*),
\]
and \((\lambda, \mu) \in \mathfrak{V}(p_j)\) and a regular point for \(p\).

**Theorem 1.4.** Suppose \(V = (S, T)\) is a pure \(p\)-isopair of finite bimultiplicity with minimal polynomial \(p\) and write \(p = p_1 p_2 \cdots p_s\) as a product of distinct irreducible factors. If \(V\) has rank \((\alpha_1, \alpha_2, ..., \alpha_s)\), then \(V\) is nearly \(\max\{\alpha_1, ..., \alpha_s\}\)-cyclic.

**Remark 1.5.** Compare Theorem 1.4 with the results in [AKM12].

We prove Theorem 1.4 in section 5. An important ingredient in the proof of Theorem 1.4 is a representation for a pure \(p\)-isopair as a pair of multiplication operators on a reproducing kernel Hilbert space over \(\mathfrak{V}(p)\) in the case \(p\) is irreducible. Representations of this type already appear in the literature, [C97, Theorem D.14] for instance. Here we provide additional information. See Theorems 4.1 and 4.9.

**Remark 1.6.** The concept of nearly multi-cyclic isopairs was introduced in [AKM12]. A discussion on multicyclicity of a bundle shift given in terms of its multiplicities can be found in [AD76]. In [R69], the article presents a way to realize a Riemann surface with a distinguished variety.
2. PRELIMINARIES

Proposition 2.1. Suppose \( p, q \in \mathbb{C}[z, w] \).

(i) \( Z(p) \cap Z(q) \) is a finite set if and only if \( p \) and \( q \) are relatively prime.
(ii) If \( p \) and \( q \) are relatively prime, then the ideal \( I \subset \mathbb{C}[z, w] \) generated by \( p \) and \( q \) has finite codimension in \( \mathbb{C}[z, w] \); i.e. there is a finite dimensional subspace \( \mathcal{R} \) of \( \mathbb{C}[z, w] \) such that for each \( \psi \in \mathbb{C}[z, w] \) there exist polynomials \( s, t \in \mathbb{C}[z, w] \) and \( r \in \mathcal{R} \) such that

\[
\psi = sp + tq + r.
\]

Bezout’s Theorem says that if two algebraic curves, say described by \( p = 0 \) and \( q = 0 \), do not have any common factors, then they have only finitely many points in common. In particular if \( p \) and \( q \) do not have any common factors, then \( Z(p) \) and \( Z(q) \) have only finitely many points in common. In particular, for the ideal \( I \) generated by \( p \) and \( q \), the affine variety \( V(I) = Z(p) \cap Z(q) \) is finite. The Finiteness Theorem of [C97, page 13], says that if \( V(I) \) is finite then the quotient ring \( \mathbb{C}[z, w]/I \) has a finite dimension. Hence the ideal \( I \) has finite codimension in \( \mathbb{C}[z, w] \).

For \( p \in \mathbb{C}[z, w] \) and \( \lambda \in \mathbb{D} \), let \( p_\lambda(w) = p(\lambda, w) \).

Lemma 2.2. Suppose \( p \) is square free and inner toral and write \( p = p_1 p_2 \cdots p_s \) as a product of irreducible factors. Let \( q \) be a nonzero polynomial.

(i) If \( q \) vanishes on a countably infinite subset of \( \mathcal{V}(p_j) \), then \( p_j \) divides \( q \).
(ii) If \( q \) vanishes on a cofinite subset of \( \mathcal{V}(p) \), then \( p \) divides \( q \).
(iii) If \( Z(q) \cap Z(p) \cap \mathbb{D}^2 \) is finite, then \( q \) and \( p \) are relatively prime.
(iv) The polynomial \( \frac{\partial p}{\partial w} \) has only finitely many zeros in \( \mathcal{V}(p) \).
(v) If \( q \frac{\partial p}{\partial w} \) is zero on a cofinite subset of \( \mathcal{V}(p) \), then \( p \) divides \( q \).
(vi) If \( \Lambda \) is the set of all \( \lambda \in \mathbb{D} \) for which \( p_\lambda(w) \) has distinct zeros, then \( \Lambda \subset \mathbb{D} \) is cofinite.

Proof. The proof of item (i) follows from Proposition 2.1 item (i) and by the fact that \( p_j \) is irreducible. By item (i), each \( p_j \) divides \( q \). Since the \( p_j \)’s are distinct, their product divides \( q \), proving item (ii). If \( q \) and \( p \) have a common factor, then because \( p \) is inner toral, \( Z(q) \) and \( Z(p) \) have infinitely many common points in \( \mathbb{D}^2 \), proving (iii).

Let \( q = \frac{\partial p}{\partial w} \) and suppose \( q \) has infinitely many zeros in \( \mathcal{V}(p) \). In this case there is a \( j \) such that \( q \) has infinitely many zeros in \( \mathcal{V}(p_j) \). Hence by (i), \( q \) vanishes on \( \mathcal{V}(p_j) \). Therefore, either \( \frac{\partial p}{\partial w} \) has infinitely many zeros in \( \mathcal{V}(p_j) \) or there is an \( \ell \) such that \( p_\ell \) has infinitely many zeros in \( \mathcal{V}(p_j) \) and thus, by part (i), \( p_j \) divides \( \frac{\partial p}{\partial w} \) or \( p_j \) divides \( p_\ell \), a contradiction. Item (v) follows from item (ii). To prove item (vi), if \( \Lambda \) is not cofinite, then \( \frac{\partial p}{\partial w} \) has infinitely many
zeros in $Z(p)$. Since $p$ is inner toral, $\frac{\partial p}{\partial w}$ has infinitely many zeros in $\mathcal{W}(p)$, a contradiction to item (iv) and hence $\Lambda$ is cofinite. \hfill \Box

**Proposition 2.3.** Suppose $p \in \mathbb{C}[z,w]$ is a square free polynomial and write $p = p_1 p_2 \cdots p_s$ as a product of irreducible factors $p_j \in \mathbb{C}[z,w]$. If $q \in \mathbb{C}[z,w]$ and $Z(p) \subseteq Z(q)$, then there exist $\gamma = (\gamma_1, \ldots, \gamma_s) \in \mathbb{N}_+^s$ and an $r \in \mathbb{C}[z,w]$ such that $p_j$ and $r$ are relatively prime and

$$q = p_1^{\gamma_1} p_2^{\gamma_2} \cdots p_s^{\gamma_s} r.$$

**Proof.** The proof is an application of Bezout’s Theorem. \hfill \Box

**Remark 2.4.** If $p$ and $q$ are inner toral polynomial, then we may replace the condition $Z(p) \subseteq Z(q)$ with $\mathcal{W}(p) \subseteq Z(q)$.

3. RESULTS FOR GENERAL $p$

In this section $p = p_1 p_2 \cdots p_s$ is a general square free inner toral polynomial with (distinct) irreducible factors $p_j$. Let $(n_j, m_j)$ be the bidegree of $p_j(z,w)$.

In [AKM12] it is proven that any nearly cyclic pure $p$-isopair is unitarily equivalent to a cyclic pure $p$-isopair restricted to a finite codimensional invariant subspace (See Proposition 3.6 in [AKM12]). Next Proposition is a more generalized version of this result.

**Proposition 3.1.** Suppose $V = (S,T)$ is a pure $p$-isopair of finite bimultiplicity $(M,N)$ acting on the Hilbert space $K$. If $H$ is a finite codimension $V$-invariant subspace of $K$ and $W$ is the restriction of $V$ to $H$, then there exists a finite codimension subspace $L$ of $H$ such that $V$ is unitarily equivalent to the restriction of $W$ to $L$.

**Remark 3.2.** In case the codimension of $H$ is one, the codimension of $L$ (in $H$) can be chosen as $N - 1$ (or as $M - 1$). In general, the proof yields a relation between the codimensions of $H$ in $K$ and $L$ in $H$ (or in $K$).

**Corollary 3.3.** Suppose $V = (S,T)$ is a pure $p$-isopair of finite bimultiplicity $(M,N)$ acting on the Hilbert space $K$. If there exists a finite codimension $V$-invariant subspace $H$ of $K$ such that the restriction of $V$ to $H$ is $\beta$-cyclic, then there exists a $\beta$-cyclic pure isopair $W$ acting a Hilbert space $L$ and a finite codimension $W$-invariant subspace $F$ of $L$ such that $W|_F$ is unitarily equivalent to $V$.

**Proof of Theorem 3.1.** Following the argument in [AKM12, Proposition 3.6], let $F = K \ominus H$ and write, with respect to the decomposition $K = H \oplus F$,

$$V = (S,T) = \begin{pmatrix} W \mid H & (X,Y) \\ 0 & (A,B) \end{pmatrix}. $$
In particular $A$ (and likewise $B$) is a contraction on a finite dimensional Hilbert space. Because $V$ is pure and $A$ is a contraction, $A$ has spectrum in the open disc $\mathbb{D}$. Choose a (finite) Blaschke $u$ such that $u(A) = 0$. Note that $u(S)$ is an isometry on $K$ and moreover the codimension of the range of $u(S)$ (equal to the dimension of the kernel of $u(S)^*$) in $K$ is (at most) $dM$, where $d$ is the degree (number of zeros) of $u$. Further, since

$$u(S) = \begin{pmatrix} u(S|_H) & X' \\ 0 & u(A) = 0 \end{pmatrix},$$

the range $L = u(S)K$ of $u(S)$ is a subspace of $H$ of finite codimension. Since $u(S)V = Wu(S)$ it follows that $L$ is invariant for $W$ and $V$ is unitarily equivalent to $W$ restricted to $L$.

To prove the remark, note that if $A$ is a scalar (equivalently $H$ has codimension one in $K$), then $u$ can be chosen to be a single Blaschke factor. In which case the codimension of $L$ is $N$ in $K$ and hence $N - 1$ in $H$. In general, if $d$ is the degree of the Blaschke $u$, then the codimension of $L$ in $K$ is $dN$. By reversing the roles of $S$ and $T$ one can replace $N$ with $M$, the multiplicity of the shift $T$. \[\square\]

**Proposition 3.4.** Let $(M_z, M_\Phi)$ be a pure isopair of finite bimultiplicity $(M, N)$ with minimal polynomial $p$, where $\Phi(z)$ is an $M \times M$ matrix-valued rational inner function. There exists an $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{N}^s_+$ such that

(i) for $\lambda \in \mathbb{D}$, the characteristic polynomial $f_\lambda(w)$ of $\Phi(\lambda)$ satisfies

$$f_\lambda(w) = \det(w - \Phi(\lambda)) = c(\lambda)p_{1,\lambda}^{\alpha_1}(w) \cdots p_{s,\lambda}^{\alpha_s}(w),$$

for a constant (in $w$) $c(\lambda)$.

(ii) for each $\lambda$ such that $p_\lambda$ has $m$ distinct zeros, $\Phi(\lambda)$ is diagonalizable and similar to

$$\bigoplus_{j=1}^{s} \bigoplus_{\mu_j \in Z(p_{j,\lambda})} \mu_j I_{\alpha_j},$$

(iii) if $(\lambda, \mu) \in Z(p_{j})$ and $\frac{\partial p}{\partial w}|(\lambda, \mu) \neq 0$, then

$$\dim \ker(\Phi(\lambda) - \mu) = \alpha_j.$$

**Proof.** First note that, by equation (2), for all $\lambda \in \overline{\mathbb{D}}$

$$p_{\lambda}(\Phi(\lambda)) = p(\lambda, \Phi(\lambda)) = 0.$$

In particular, the spectrum, $\sigma(\Phi(\lambda))$, is a subset of $Z(p_{\lambda})$.

Note that $\det(wI_m - \Phi(z))$ is a rational function whose denominator $d(z)$ (a polynomial in $z$ alone) doesn’t vanish in $\overline{\mathbb{D}}$. Let $q(z, w) = d(z)\det(wI_m - \Phi(z))$, the numerator of
det(wI_m - \Phi(z)). For fixed z \in \mathbb{D}, let
\begin{equation}
q_z(w) = d(z) \det(wI_m - \Phi(z)) = \sum_{j=0}^{M} q_j(z)w^j.
\end{equation}

By Cayley-Hamilton Theorem, \(q_z(\Phi(z)) = \sum_{j=0}^{M} q_j(z)\Phi(z)^j = 0\) and therefore \(q(z, \Phi(z)) = 0\) for all \(z \in \mathbb{D}\). Now for \(\gamma \in \mathbb{C}^M\) and \(\lambda \in \mathbb{D}\),
\begin{equation}
q(M_z, M_{\Phi}(z))^{*}\gamma s_{\lambda} = q(\lambda, \Phi(\lambda))\gamma s_{\lambda} = 0.
\end{equation}

Therefore, \(q(M_z, M_{\Phi}) = 0\). Since \(p\) is the minimal polynomial for \((M_z, M_{\Phi}), \mathfrak{M}(p)\) is a subset of \(Z(q)\). Hence there exist an \(\alpha = (\alpha_1, ..., \alpha_s) \in \mathbb{N}_+^s\) and a polynomial \(r\) such that \(p_j\) does not divide \(r\) for each \(j\) and
\begin{equation}
d(z) \det(w - \Phi(z)) = q(z, w) = p_1^{\alpha_1}(z, w) \cdots p_s^{\alpha_s}(z, w) r(z, w).
\end{equation}

For \((\lambda, \mu) \in \mathbb{D} \times \mathbb{D}\), \(\mu\) is in the spectrum of \(\Phi(\lambda)\) if and only if \(q(\lambda, \mu) = 0\). In particular, \(q(z, w)\) is a polynomial whose zero set in \(\mathbb{D} \times \mathbb{C}\) is the set \(\{(z, w) : z \in \mathbb{D}, w \in \sigma(\Phi(z))\} \subseteq \mathfrak{M}(p)\). Observe \(Z(r) \cap [\mathbb{D} \times \mathbb{C}] \subseteq Z(q) \cap [\mathbb{D} \times \mathbb{C}] \subseteq \mathfrak{M}(p)\). On the other hand, \(r\) can have only finitely many zeros in \(\mathfrak{M}(p)\) as otherwise \(r\) has infinitely many zeros on some \(\mathfrak{M}(p_j)\) and, by Lemma 2.2 item (i) \(p_j\) divides \(r\). Hence \(r(z, w)\) has only finitely many zeros in \(\mathbb{H} = \mathbb{D} \times \mathbb{C}\).
We conclude there are only finitely many \(z \in \mathbb{D}\) such that \(r_z(w) = r(z, w)\) has a zero and consequently \(r\) depends on \(z\) only so that \(r_z(w) = r(z)\). Thus, for \(\lambda \in \mathbb{D}\), the characteristic polynomial \(f_\lambda(w)\) of \(\Phi(\lambda)\) satisfies
\begin{equation}
f_\lambda(w) = \det(w - \Phi(\lambda)) = c(\lambda)p_1^{\alpha_1}(w) \cdots p_s^{\alpha_s}(w),
\end{equation}
for a constant \((w) c(\lambda)\).

Let \(\Lambda\) be the set of all \(\lambda \in \mathbb{D}\) for which \(p_\lambda\) has \(\sum_{j=1}^{s} m_j\) distinct zeros. By Lemma 2.2 item (vi), \(\Lambda \subseteq \mathbb{D}\) is cofinite. For \(\lambda \in \Lambda\), the polynomial \(p_\lambda\) has distinct zeros and by (5), \(p_\lambda(\Phi(\lambda)) = 0\). Hence, \(\Phi(\lambda)\) is diagonalizable and, for given \(\mu_j \in Z(p_{j,\lambda})\), the dimension of the eigenspace of \(\Phi(\lambda)\) at \(\mu_j = \alpha_j\). Thus \(\Phi(\lambda)\) is similar to
\begin{equation}
\bigoplus_{j=1}^{s} \bigoplus_{\mu_j \in Z(p_{j,\lambda})} \mu_j I_{\alpha_j}.
\end{equation}

Let \((\lambda, \mu) \in Z(p_j)\) be such that \(\frac{\partial \Phi}{\partial w}(\lambda, \mu) \neq 0\). The minimal polynomial for \(\Phi(\lambda)\) has a zero of multiplicity 1 at \(\mu\), since it divides \(p_\lambda\). Hence \(\Phi(\lambda)\) is similar to \(\mu I_{\alpha_j} \oplus J\) where the spectrum of \(J\) does not contain \(\mu\). Therefore, the kernel of \(\Phi(\lambda) - \mu\) has dimension \(\alpha_j\).

**Proposition 3.5.** Let \(V = (S, T)\) be a pure \(p\)-isopair of finite bimultiplicity and suppose \(p = p_1 p_2 \cdots p_s\) a product of distinct irreducible factors. For each \(j\) and \((\lambda, \mu) \in \mathfrak{M}(p_j)\)
such that $\frac{\partial p}{\partial w}|_{(\lambda,\mu)} \neq 0$, the dimension of the intersection of $\ker(S - \lambda)^* \cap \ker(T - \mu)^*$ is a nonzero constant.

Proof. By the standard model theory for pure isopairs with finite bimultiplicity, there exists an $M \times M$ matrix-valued rational inner function $\Phi$ such that $V = (S, T)$ is unitarily equivalent to $(M, M)$ on $H^2_{\C^M}$ and $p(M, M) = 0$. Let $(\lambda, \mu) \in \Psi(p_j)$ be a regular point for $p$. Observe that for any $\gamma \in \ker(\Phi(\lambda) - \mu)^*$, both $(S - \lambda)^*s_\lambda \gamma = 0$ and $(T - \mu)^*s_\lambda \gamma = 0$. Hence $s_\lambda \gamma \in \ker(S - \lambda)^* \cap \ker(T - \mu)^*$. Now suppose $f \in \ker(S - \lambda)^* \cap \ker(T - \mu)^*$. Since $(S - \lambda)^*f = 0$, there is a vector $\gamma \in \mathbb{C}^N$ such that $f = s_\lambda \gamma$. Thus, $0 = (T - \mu)^*s_\lambda \gamma = s_\lambda(\Phi(\lambda)^* - \mu^*)\gamma$.

Hence

$$s_\lambda \ker(\Phi(\lambda) - \mu)^* = \ker(S - \lambda)^* \cap \ker(T - \mu)^*.$$ 

Since $\dim \ker(\Phi(\lambda) - \mu)^* = \dim \ker(\Phi(\lambda) - \mu)$, we have

$$(8) \quad \dim [\ker(S - \lambda)^* \cap \ker(T - \mu)^*] = \dim \ker(\Phi(\lambda) - \mu),$$

and hence by Proposition 3.4 item (iii), $\dim [\ker(S - \lambda)^* \cap \ker(T - \mu)^*] = \alpha_j$. \qed

Corollary 3.6. Let $V = (S, T)$ be a pure $p$-isopair of finite bimultiplicity and suppose $p = p_1 p_2 \cdots p_s$ a product of distinct irreducible factors. For each $\lambda$ and $(\lambda, \mu) \in \Psi(p_j)$ such that $\frac{\partial p}{\partial z}|_{(\lambda,\mu)} \neq 0$, dimension of the intersection of $\ker(S - \lambda)^*$ and $\ker(T - \mu)^*$ is a nonzero constant.

Proof. The proof is immediate from the symmetry of $S$ and $T$ and Proposition 3.5. \qed

Proof of Proposition 1.2. Let $(\lambda, \mu) \in \Psi(p_j)$. If $\frac{\partial p}{\partial w}|_{(\lambda,\mu)} \neq 0$, then by Proposition 3.5, there exists a non zero constant $\alpha_j \in \mathbb{N}^+$ such that

$$\dim(\ker(S - \lambda)^* \cap \ker(T - \mu)^*) = \alpha_j.$$ 

If $\frac{\partial p}{\partial z}|_{(\lambda,\mu)} \neq 0$, then by Corollary 3.6, there exists a non zero constant $\beta_j \in \mathbb{N}^+$ such that

$$\dim(\ker(S - \lambda)^* \cap \ker(T - \mu)^*) = \beta_j.$$ 

Note that, since $p$ is square free, so is $p_j$ and hence there are infinitely many points in $\Psi(p_j)$ such that both partial derivatives $\frac{\partial p}{\partial z}|_{(\lambda,\mu)}$ and $\frac{\partial p}{\partial w}|_{(\lambda,\mu)}$ do not vanish. If $(\lambda, \mu)$ is a regular point for $p$ such that $\frac{\partial p}{\partial z}|_{(\lambda,\mu)} \neq 0$ and $\frac{\partial p}{\partial w}|_{(\lambda,\mu)} \neq 0$, then $\alpha_j = \beta_j$. Therefore, if $(\lambda, \mu) \in \Psi(p_j)$ is a regular point for $p$, then dimension of the intersection of $\ker(S - \lambda)^*$ and $\ker(T - \mu)^*$ is a nonzero constant. \qed

Corollary 3.7. If $(S, T)$ is a pure $p$-isopair of finite bimultiplicity $(M, N)$ with rank $\alpha = (\alpha_1, ..., \alpha_s) \in \mathbb{N}^s$, then

$$(9) \quad M = \sum_{j=1}^s m_j \alpha_j \text{ and } N = \sum_{j=1}^s n_j \alpha_j.$$
Proof. First, view \((S, T)\) as \((M_z, M_\Phi)\) where \(\Phi(z)\) is an \(M \times M\) matrix-valued rational inner function. By Proposition 3.4 item (i), for \(\lambda \in \mathbb{D}\),
\[
\det(w - \Phi(\lambda)) = c(\lambda)p_{1,\lambda}^{\alpha_1}(w) \cdots p_{s,\lambda}^{\alpha_s}(w)
\]
for a constant (in \(w\)) \(c(\lambda)\). Comparing the degree in \(w\) on the left and the right, for all but finitely many \(\lambda\), we have
\[
M = \sum_{j=1}^{s} \alpha_j m_j.
\]

To see the relation on \(N\), view \(p\) as \(p(w, z)\) a polynomial of bidegree \((m, n)\). Note that each factor \(p_j = p_j(w, z)\) has bidegree \((m_j, n_j)\). Moreover \(p(T, S) = 0\) and \((T, S)\) has bimultiplicity \((N, M)\). Model \((T, S)\) as \((M_w, M_\Psi(w))\), where \(\Psi(w)\) is an \(N \times N\) matrix valued ration inner function. By Proposition 3.4, item (i), there exists \((\beta_1, \beta_2, ..., \beta_s) \in \mathbb{N}_s^s\) such that for \(\mu \in \mathbb{D}\),
\[
\det(z - \Psi(\mu)) = c'(\mu)p_{1,\mu}^{\beta_1}(z) \cdots p_{s,\mu}^{\beta_s}(z)
\]
for a constant (in \(z\)) \(c'(\mu)\). By Proposition 3.4 item (iii), for \((\mu, \lambda) \in Z(p_j)\) that is a regular point for \(p\),
\[
\dim \ker(\Psi(\mu) - \lambda) = \beta_j.
\]

Now by equation (8),
\[
\dim [\ker(S - \lambda)^* \cap \ker(T - \mu)^*] = \beta_j.
\]
Since \((S, T)\) has rank \(\alpha\), we get \(\beta_j = \alpha_j\) for \(j = 1, ..., s\) and by comparing the degree in \(z\) on the left and the right of (10), for all but finitely many \(\mu\), we have
\[
N = \sum_{j=1}^{s} \alpha_j n_j. \quad \Box
\]

Proposition 3.8. If \(V = (S, T)\) is a finite bimultiplicity \(k\)-cyclic pure \(p\)-isopair acting on the Hilbert space \(K\), then for each \((\lambda, \mu) \in \mathcal{V}(p)\),
\[
\dim (\ker(S - \lambda)^* \cap \ker(T - \mu)^*) \leq k.
\]
In particular, if \(p\) is the minimal polynomial for \(V\) and if \(V\) has rank \(\alpha\), then \(k \geq \max\{\alpha_1, ..., \alpha_s\}\).

Proof. Let \(\{f_1, ..., f_k\}\) be a cyclic set for \((S, T)\). For any \(q(z, w) \in \mathbb{C}[z, w], f \in \ker(S - \lambda)^* \cap \ker(T - \mu)^*\) and \(1 \leq j \leq k\),
\[
\langle q(S, T)f_j, f \rangle = \langle f_j, q(S, T)^* f \rangle = \langle f_j, q(\lambda)^* f \rangle = q(\lambda, \mu) \langle f_j, f \rangle.
\]
If \( \dim (\ker(S - \lambda)^* \cap \ker(T - \mu)^*) > k \), then there exists a non zero vector \( f \in \ker(S - \lambda)^* \cap \ker(T - \mu)^* \) perpendicular to \( f_j \) for all \( j \). Thus \( \langle q(S,T)f_j, f \rangle = 0 \) for all \( j \) and for any \( q \), and hence \( \langle g, f \rangle = 0 \) for any \( g \in \{ \sum_{j=1}^k q_j(S,T) f_j : q_j \in \mathbb{C}[z,w] \} \), a contradiction. Therefore, \( \dim (\ker(S - \lambda)^* \cap \ker(T - \mu)^*) \leq k \). The last statement of the proposition follows from the definition of the rank. \( \square \)

**Proposition 3.9.** Suppose \( V = (S,T) \) is a finite bimultiplicity pure \( p \)-isopair with minimal polynomial \( p \) and with rank \( \alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{N}_+^s \) acting on a Hilbert space \( K \). If \( H \) is a finite codimension \( V \)-invariant subspace of \( K \), then \( W = V|_H \) has rank \( \alpha \) too.

**Proof.** Write \( W = V|_H = (S_0,T_0) \). Let \( F = K \ominus H \). Thus \( F \) has finite dimension and \( K = H \oplus F \). With respect to this decomposition, write

\[
S^* = \begin{pmatrix} S_0^* & 0 \\ X^* & A^* \end{pmatrix}, \quad T^* = \begin{pmatrix} T_0^* & 0 \\ Y^* & B^* \end{pmatrix}.
\]

Observe that \( \sigma(A) \times \sigma(B) \) is a finite set since \( A \) and \( B \) act on a finite dimensional space. Fix \( 1 \leq j \leq s \). Let \( \Gamma \) be the set of all \((\lambda, \mu) \in \mathfrak{V}(p_j)\) such that the dimension of \( \ker(S - \lambda)^* \cap \ker(T - \mu)^* \) is \( \alpha_j \) and \((\lambda, \mu) \notin \sigma(A) \times \sigma(B)\). Hence by Proposition 1.2, \( \Gamma \) contains the cofinite set of all regular points. Since also the set \( \sigma(A) \times \sigma(B) \) is finite, \( \Gamma \) is a cofinite subset of \( \mathfrak{V}(p_j) \). Fix \((\lambda, \mu) \in \Gamma \) and let

\[
L = \ker(S - \lambda)^* \cap \ker(T - \mu)^* \quad \text{and} \quad L_0 = \ker(S_0 - \lambda)^* \cap \ker(T_0 - \mu)^*.
\]

Let \( \mathcal{P} \subseteq H \) be the projection of \( L \) onto \( H \). Given \( f \in L \), write \( f = f_1 \oplus f_2 \), where \( f_1 \in H \) and \( f_2 \in F \). Since \( f \in L \), the kernel of \( (S_0 - \lambda)^* \) contains \( f_1 \). Likewise the kernel of \( (T_0 - \lambda)^* \) contains \( f_1 \). Therefore, \( \mathcal{P} \subseteq L_0 \). If \( \dim(L_0) < \alpha_j \), then, since \( \dim(L) = \alpha_j \), there exists a non zero vector of the form \( 0 \oplus v \) in \( L \) and hence \( \ker(A - \lambda)^* \cap \ker(B - \mu)^* \) is non-empty. But, \( \ker(A - \lambda)^* \cap \ker(B - \mu)^* \) is empty by the choice of \((\lambda, \mu)\). Thus \( \dim(L_0) = \alpha_j \) for almost all \((\lambda, \mu) \in \mathfrak{V}(p_j)\). Therefore \( W \) also has rank \( \alpha \). \( \square \)

**Corollary 3.10.** Suppose \( V = (S,T) \) is a finite bimultiplicity pure \( p \)-isopair with minimal polynomial \( p \) and with rank \( \alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{N}_+^s \) acting on a Hilbert space \( K \). If \( H \) is a finite codimension \( V \)-invariant subspace of \( K \), then \( W = V|_H \) is at least \( \beta = \max\{\alpha_1, \ldots, \alpha_s\} \)-cyclic. Hence \( V \) is at least nearly \( \beta \)-cyclic.

**Proof.** By Proposition 3.9, \( W \) has rank \( \alpha \). By Proposition 3.8, \( W \) is at least \( \beta \)-cyclic. Thus, each restriction of \( V \) to a finite codimension invariant subspace is at least \( \beta \)-cyclic and hence \( V \) is at least nearly \( \beta \)-cyclic. \( \square \)
4. THE CASE $p$ IS IRREDUCIBLE

In this section $p$ is an irreducible square free inner toral polynomial of bidegree $(n, m)$.

A rank $\alpha$-admissible kernel $\mathcal{K}$ over $\mathfrak{V}(p)$ consists of an $\alpha \times m\alpha$ matrix polynomial $Q$ and an $\alpha \times n\alpha$ matrix polynomial $P$ such that

$$
\frac{Q(z, w)Q(\zeta, \eta)^*}{1 - z\zeta^*} = \mathcal{K}((z, w), (\zeta, \eta)) = \frac{P(z, w)P(\zeta, \eta)^*}{1 - w\eta^*}, \quad (z, w), (\zeta, \eta) \in \mathfrak{V}(p)
$$

where $Q$ and $P$ have full rank $\alpha$ at some point in $\mathfrak{V}(p)$. In particular, at some point $x \in \mathfrak{V}(p)$ the matrix $\mathcal{K}(x, x)$ has full rank $\alpha$ [JKM12]. An $\alpha \times \alpha$ matrix-valued kernel on a set $\Omega$ has full rank at $x \in \Omega$, if $\mathcal{K}(x, x)$ has full rank $\alpha$. We refer to $(\mathcal{K}, P, Q)$ as an $\alpha$-admissible triple.

Let $H^2(\mathcal{K})$ denote the Hilbert space associated to the rank $\alpha$ admissible kernel $\mathcal{K}$. For a point $y \in \mathfrak{V}(p)$, denote by $\mathcal{K}_y$ the $\alpha \times \alpha$ matrix function on $\mathfrak{V}(p)$ defined by $\mathcal{K}_y(x) = \mathcal{K}(x, y)$. Elements of $H^2(\mathcal{K})$ are $\mathbb{C}^\alpha$ vector-valued functions on $\mathfrak{V}(p)$ and the linear span of $\{\mathcal{K}_y \gamma : y \in \mathfrak{V}(p), \gamma \in \mathbb{C}^\alpha\}$ is dense in $H^2(\mathcal{K})$. Note that the operators $X$ and $Y$ determined densely on $H^2(\mathcal{K})$ by $X\mathcal{K}_{(\lambda, \mu)\gamma} = \lambda^*\mathcal{K}_{(\lambda, \mu)}\gamma$ and $Y\mathcal{K}_{(\lambda, \mu)\gamma} = \mu^*\mathcal{K}_{(\lambda, \mu)}\gamma$ are contractions. By Theorem (4.1) item (i) below, $X^*$ is a bounded operator on $H^2(\mathcal{K})$. Further for $f \in H^2(\mathcal{K})$,

$$
\langle X^* f, \mathcal{K}_{(\lambda, \mu)\gamma} \rangle = \lambda\langle f(\lambda, \mu), \gamma \rangle.
$$

Hence $X^*$ is the operator of multiplication by $z$ on $H^2(\mathcal{K})$. Likewise, $Y^*$ is a bounded operator on $H^2(\mathcal{K})$ and it is the multiplication by $w$ on $H^2(\mathcal{K})$.

**Theorem 4.1.** If $\mathcal{K}$ is a rank $\alpha$-admissible kernel over $\mathfrak{V}(p)$, then

(i) $X$ is bounded on the linear span of $\{\mathcal{K}_y \gamma : y \in \mathfrak{V}(p), \gamma \in \mathbb{C}^\alpha\}$;

(ii) for each $1 \leq j \leq m\alpha$ and each positive integer $n$, the vector $z^nQe_j$ ($Qe_j$ is the $j$-th column of $Q$) lies in $H^2(\mathcal{K})$;

(iii) the span of $\{s_\lambda Q(\lambda, \mu)^*\gamma : (\lambda, \mu) \in \mathfrak{V}(p), \gamma \in \mathbb{C}^\alpha\}$ is dense in $H^2_{C^{m\alpha}}$;

(iv) the set $\mathcal{B} = \{z^nQe_j : n \in \mathbb{N}, 1 \leq j \leq m\alpha\}$ is an orthonormal basis for $H^2(\mathcal{K})$; and

(v) operators $S$ and $T$ densely defined on $\mathcal{B}$ by $Sf = zf$ and $Tf = wf$ extend to a pair of pure isometries on $H^2(\mathcal{K})$. 
Proof. For a finite set of points $(\lambda_1, \mu_1), \ldots, (\lambda_n, \mu_n) \in \mathcal{W}(p)$, and $\gamma_1, \ldots, \gamma_n \in \mathbb{C}^a$, observe that

$$
\langle (I - X^*X) \sum_{j=1}^{n} \mathcal{K}_{(\lambda_j, \mu_j)} \gamma_j, \sum_{k=1}^{n} \mathcal{K}_{(\lambda_k, \mu_k)} \gamma_k \rangle = \sum_{j,k=1}^{n} \langle (1 - \lambda_k \bar{\lambda_j}) \mathcal{K}_{(\lambda_j, \mu_j)}(\lambda_k, \mu_k) \gamma_j, \gamma_k \rangle
$$

$$
= \sum_{j,k=1}^{n} \langle Q(\lambda_k, \mu_k)Q^*(\lambda_j, \mu_j) \gamma_j, \gamma_k \rangle
$$

$$
= \left( \sum_{j=1}^{n} Q^*(\lambda_j, \mu_j) \gamma_j, \sum_{k=1}^{n} Q^*(\lambda_k, \mu_k) \gamma_k \right)
$$

$$
\geq 0.
$$

Therefore, $X$ is bounded on the linear span of $\{\mathcal{K}_{\gamma} \gamma : y \in \mathcal{W}(p), \gamma \in \mathbb{C}^a\}$.

To prove item (ii), note that by [PR16, Theorem 4.15], if $f$ is a $\mathbb{C}^a$ valued function defined on $\mathcal{W}(p)$ and if $\mathcal{K}((z, w), (\zeta, \eta)) - f(z, w)q(\zeta, \eta)^*$ is a (positive semidefinite) kernel function then $f \in H^2(\mathcal{K})$. Since

$$
\mathcal{K}((z, w), (\zeta, \eta)) = (z\zeta^*)^n Q(z, w) Q^*(\zeta, \eta) = \sum_{j=1}^{n-1} (z\zeta^*)^j Q(z, w) Q^*(\zeta, \eta)
$$

$$
+ (z\zeta^*)^n Q\mathcal{K}((z, w), (\zeta, \eta))
$$

is positive semidefinite, it follows that $z^n Qe_j \in H^2(\mathcal{K})$.

By a result in [JKM12, Lemma 4.1], there exists a cofinite subset $\Lambda \subset \mathbb{D}$ such that for each $\lambda \in \Lambda$ there exist distinct points $\mu_1, \ldots, \mu_m \in \mathbb{D}$ such that $(\lambda, \mu_j) \in \mathcal{W}(p)$ and the $ma \times ma$ matrix,

$$
R(\lambda) := \begin{pmatrix} Q(\lambda, \mu_1)^* & \cdots & Q(\lambda, \mu_m)^* \end{pmatrix}
$$

has full rank. Define a map $U$ from $H^2(\mathcal{K})$ to $H^2_{\mathbb{C}^a}$ by

$$
U\mathcal{K}_{(\lambda, \mu)}(z, w)^{\gamma} = s_\lambda(z)Q(\lambda, \mu)^* \gamma.
$$

Observe that for $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in \mathbb{D}^2$ and $\gamma, \delta \in \mathbb{C}^a$,

$$
\langle U\mathcal{K}_{(\lambda_1, \mu_1)}(z, w)_{\gamma}, U\mathcal{K}_{(\lambda_2, \mu_2)}(z, w)^{\delta} \rangle = \langle s_{\lambda_1}(z)Q(\lambda_2, \mu_2)Q^*(\lambda_1, \mu_1)^{\gamma}, s_{\lambda_2}(z) \delta \rangle
$$

$$
= \delta^* Q(\lambda_2, \mu_2)Q^*(\lambda_1, \mu_1) \gamma \langle s_{\lambda_1}(z), s_{\lambda_2}(z) \rangle
$$

$$
= \delta^* \mathcal{K}((\lambda_2, \mu_2), (\lambda_1, \mu_1)) \gamma
$$

$$
= \langle \mathcal{K}_{(\lambda_1, \mu_1)}(z, w)_{\gamma}, \mathcal{K}_{(\lambda_2, \mu_2)}(z, w)^{\delta} \rangle.
$$

Therefore, $U$ is an isometry and hence a unitary, onto its range. Given $\lambda \in \mathbb{D}$, the span of

$$
\{U\mathcal{K}_{(\lambda, \mu)} \gamma : \mu_j \in Z(p_\lambda), \gamma \in \mathbb{C}^a\}
$$
is equal to $s_\lambda$ times the span of
\[ \{Q(\lambda, \mu_j)^*e_k : 1 \leq j \leq m, 1 \leq k \leq \alpha \} \subseteq \mathbb{C}^{m_\alpha}. \]
If $\lambda \in \Lambda$, then $R(\lambda)$ has full rank. Thus for such $\lambda$, the span of $\{Q(\lambda, \mu)^*\gamma : \mu$ such that $(\lambda, \mu) \in \Gamma, \gamma \in \mathbb{C}^\alpha \}$ is all of $\mathbb{C}^{m_\alpha}$. Since $\Lambda \subseteq \mathbb{D}$ is cofinite, $\{s_\lambda\mathbb{C}^{m_\alpha} : \lambda \in \Lambda \}$ is dense in $H^2_{\mathbb{C}^{m_\alpha}}$. Since,
\[ \{s_\lambda\mathbb{C}^{m_\alpha} : \lambda \in \Lambda \} \subseteq \text{span}\{s_\lambda Q(\lambda, \mu)^*\gamma : (\lambda, \mu) \in \mathcal{U}(p), \gamma \in \mathbb{C}^\alpha \}, \]
the span of $\{s_\lambda Q(\lambda, \mu)^*\gamma : (\lambda, \mu) \in \mathcal{U}(p), \gamma \in \mathbb{C}^\alpha \}$ is also dense in $H^2_{\mathbb{C}^{m_\alpha}}$, proving item (iii). Moreover, it proves that $U$ is onto and hence unitary.

Let $q_k$ denote the $k$-th column of $Q$. Thus $q_k = Qe_k$. Note that, for any $a \in \mathbb{N}$ and $1 \leq j \leq m_\alpha$,
\[
\langle U^*z^a e_j(\zeta, \eta), e_k \rangle = \langle U^*z^a e_j, K_{(\zeta, \eta)} e_k \rangle \\
= \langle z^a e_j, U K_{(\zeta, \eta)} e_k \rangle \\
= \sum_{i=1}^{m_\alpha} \langle z^a e_j, (s_\xi q_i^*(\zeta, \eta) e_k) e_i \rangle \\
= \langle q_j(\zeta, \eta) \zeta^a, e_k \rangle \\
= \langle (z^a q_j)(\zeta, \eta), e_k \rangle
\]
and hence it follows that $U^*z^a e_j = z^a q_j$ and $Uz^a q_j = z^a e_j$. In particular, $\{z^a q_j : a \in \mathbb{N}, 1 \leq j \leq m_\alpha \}$ is an orthonormal basis for $H^2(K)$ completing the proof of item (iv).

To prove item (v), observe that $M_\gamma U = US$ on $\mathcal{B}$ and then extending to $H^2(K)$, it is true on $H^2(K)$ too. It is now evident that $S$ is a pure isometry of multiplicity $m_\alpha$ with wandering subspace $\{Q_\gamma : \gamma \in \mathbb{C}^{m_\alpha} \}$ (the span of the columns of $Q$). Likewise for $T$ by symmetry.

**Proposition 4.2** ([AM02]). *Suppose $\Phi$ is an $M \times M$ matrix-valued rational inner function and the pair $(M_\gamma, M_\Phi)$ of multiplication operators on $H^2_{\mathbb{C}^M}$. If the rank of the projection $I - M_\Phi M_\Phi^*$ is $N$, then there exists a unitary matrix $U$ of size $(M + N) \times (M + N)$,*

\[ U = \begin{pmatrix} M & N \\ A & B \\ C & D \end{pmatrix} \begin{pmatrix} M \\ N \end{pmatrix}, \]

*such that*

\[ \Phi(z) = A + zB(I - zD)^{-1}C. \]

**Proposition 4.3.** *If $V = (S, T)$ is a finite bimultiplicity $(M, N)$ pure $p$-isopair of rank $\alpha$, modeled as $(M_\gamma, M_\Phi)$ on $H^2_{\mathbb{C}^M}$, where $\Phi$ is an $M \times M$ matrix-valued rational inner function, then $M = m_\alpha$ and*
(i) there exists an $\alpha \times m\alpha$ matrix polynomial $Q$ such that $Q(z, w)$ has full rank at almost all points of $\mathcal{V}(p)$;
(ii) for $(z, w) \in \mathcal{V}(p)$
\[ Q(z, w)(\Phi(z) - w) = 0; \]
(iii) there exists an $\alpha \times n\alpha$ matrix polynomial $P$ such that $P(z, w)$ has full rank at almost all points of $\mathcal{V}(p)$ and an $\alpha$-admissible kernel $K$ such that
\[ \frac{Q(z, w)Q(\zeta, \eta)}{1 - z\zeta^*} = K((z, w), (\zeta, \eta)) = \frac{P(z, w)P(\zeta, \eta)}{1 - w\eta^*} \text{ on } \mathcal{V}(p) \times \mathcal{V}(p). \]

**Remark 4.4.** The triple $(K, P, Q)$ in Proposition 4.3 is a rank $\alpha$-admissible triple.

**Proof.** Applying Corollary 3.7 to irreducible $p$ gives $M = m\alpha$. Let $\Lambda$ denote the set of $\lambda \in \mathbb{D}$ such that $p\lambda$ has $m$ distinct zeros. By Lemma (2.2) item (vi) $\Lambda$ is cofinite. Let
\[ \Gamma = \{ (\lambda, \mu) : \lambda \in \Lambda, \mu \in Z(p_\lambda) \}. \]
By Proposition 3.4 item (ii), for each $(\lambda, \mu) \in \Gamma$, the matrix $\Phi(\lambda)$ is diagonalizable and $\Phi(\lambda) - \mu$ has an $\alpha$ dimensional kernel. Now fix $(\lambda_0, \mu_0) \in \Gamma$. Hence there exist unitary matrices $\Pi$ and $\Pi_*$ such that
\[ \Pi_*(\Phi(\lambda_0) - \mu_0)\Pi = \begin{pmatrix} 0_\alpha & 0 \\ 0 & A \end{pmatrix}, \]
where $A$ is $(m - 1)\alpha \times (m - 1)\alpha$ and invertible. Let
\[ \Sigma(z, w) = \Pi_*(\Phi(z) - w)\Pi. \]
For $(\lambda, \mu) \in \Gamma$, the matrix $\Sigma(z, w)$ has an $\alpha$ dimensional kernel. Write,
\[ \Sigma(z, w) = \begin{pmatrix} E(z) - w & G(z) \\ H(z) & L(z) - w \end{pmatrix}, \]
where $E$ is $\alpha \times \alpha$ and $L$ is of size $(m - 1)\alpha \times (m - 1)\alpha$. By construction $L(z) - w$ is invertible at $(\lambda_0, \mu_0)$ and the other entries are 0 there. In particular, $L(\lambda) - \mu$ is invertible for almost all points $(\lambda, \mu) \in \mathcal{V}(p)$. Moreover, if $L(z) - w$ is invertible, then
\[ \Sigma(z, w) = \begin{pmatrix} I & G(z) \\ 0 & L(z) - w \end{pmatrix} \begin{pmatrix} \Psi(z, w) & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I \\ (L(z) - w)^{-1}H(z) \\ 0 \end{pmatrix}, \]
where
\[ \Psi(z, w) = E(z) - w - G(z)(L(z) - w)^{-1}H(z). \]
Thus, on the cofinite subset of $\mathcal{V}(p)$ where $L(\lambda) - \mu$ is invertible and $\Sigma(\lambda, \mu)$ has an $\alpha$ dimensional kernel, $\Psi(\lambda, \mu) = 0$ and moreover,
\[ \begin{pmatrix} I_\alpha & -G(\lambda)(L(\lambda) - \mu)^{-1} \end{pmatrix} \Pi_*(\Phi(\lambda) - \mu) = 0. \]
Let
\[ Q(z, w) = \left( I_\alpha - G(z)(L(z) - w)^{-1} \right) \Pi. \]
It follows that
\[ Q(z, w)(\Phi(z) - w) = 0 \]
for almost all points in \( \mathfrak{V}(p) \). After multiplying \( Q \) by an appropriate scalar polynomial we obtain an \( \alpha \times m\alpha \) matrix polynomial \( Q(z, w) \) that has full rank at almost all points of \( \mathfrak{V}(p) \) and satisfies
\[ Q(z, w)(\Phi(z) - w) = 0 \]
for all \((z, w)\in \mathfrak{V}(p)\).

Since \( T \) has multiplicity \( N \), the operator \( M_\Phi \) also has multiplicity \( N \) and hence the projection \( I - M_\Phi M_\Phi^* \) has rank \( N \). By Theorem 4.2, there exists a unitary matrix \( U \) of size \((M+N)\times(M+N)\),
\[ U = \begin{pmatrix} M & N \\ A & B \\ C & D \end{pmatrix} \]
such that
\[ \Phi(z) = A + zB(I - zD)^{-1}C. \]
Define \( P \) by \( P(z, w) = Q(z, w)B(I - zD)^{-1} \) and verify, for \((z, w)\in \mathfrak{V}(p)\),
\[ \begin{pmatrix} Q & zP \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} wQ & P \end{pmatrix} \text{ on } \mathfrak{V}(p). \]
It follows that, for \((\zeta, \eta)\in \mathfrak{V}(p)\),
\[ Q(z, w)Q(\zeta, \eta)^* + z\zeta^*P(z, w)P(\zeta, \eta)^* = w\eta^*Q(z, w)Q(\zeta, \eta)^* + P(z, w)P(\zeta, \eta)^*. \]
Rearranging gives,
\[ \frac{Q(z, w)Q(\zeta, \eta)^*}{1 - z\zeta^*} = K((z, w), (\zeta, \eta)) = \frac{P(z, w)P(\zeta, \eta)^*}{1 - w\eta^*} \text{ on } \mathfrak{V}(p) \times \mathfrak{V}(p). \]
Finally, if \((\zeta, \eta)\in \mathfrak{V}(p)\) is such that \( Q(\zeta, \eta) \) has full rank \( \alpha \), then \( P(\zeta, \eta)P(\zeta, \eta)^* \) also has full rank \( \alpha \). Therefore, \( P(\zeta, \eta) \) also has full rank \( \alpha \) and hence \( K \) is a rank \( \alpha \)-admissible kernel. \( \square \)

**Theorem 4.5.** If \( V = (S, T) \) is a finite bimultiplicity \((M, N)\) pure \( p \)-isopair with rank \( \alpha \), then there exists a rank \( \alpha \)-admissible triple \((K, P, Q)\) such that \( V \) is unitarily equivalent to the operators of multiplication by \( z \) and \( w \) on \( H^2(K) \).
Proof. Note that \((S, T)\) is unitarily equivalent to \((M_z, M_\Phi)\) on \(H^2_{\mathcal{C}M}\), where \(\Phi\) is an \(M \times M\) matrix-valued rational inner function. By Proposition 4.3, there exists a rank \(\alpha\)-admissible triple \((\mathcal{K}, P, Q)\) such that

\[
Q(z, w)(\Phi(z) - w) = 0
\]

for all \((z, w) \in \mathfrak{M}(p)\). Define

\[
U : H^2_{\mathcal{C}M} \to H^2(\mathcal{K})
\]

on the span of

\[
\mathcal{B} = \{s_\zeta Q^*(\zeta, \eta)\gamma : (\zeta, \eta) \in \mathfrak{M}(p), \gamma \in \mathbb{C}^\alpha\} \subseteq H^2_{\mathcal{C}M}
\]

by

\[
Us_\zeta(z)Q^*(\zeta, \eta)\gamma = \mathcal{K}_{(\zeta, \eta)}(z, w)\gamma.
\]

For \((\zeta, \eta) \in \mathfrak{M}(p)\) and \(\gamma_j \in \mathbb{C}^\alpha\) for \(1 \leq j \leq 2\),

\[
\langle Us_{\zeta_1}(z)Q^*(\zeta_1, \eta_1)\gamma_1, Us_{\zeta_2}(z)Q^*(\zeta_2, \eta_2)\gamma_2 \rangle = \langle \mathcal{K}_{(\zeta_1, \eta_1)}(z, w)\gamma_1, \mathcal{K}_{(\zeta_2, \eta_2)}(z, w)\gamma_2 \rangle
\]

\[
= \langle \mathcal{K}_{(\zeta_1, \eta_1)}(\zeta_2, \eta_2)\gamma_1, \gamma_2 \rangle
\]

\[
= \langle s_{\zeta_1}(\zeta_2)Q(\zeta_2, \eta_2)Q^*(\zeta_1, \eta_1)\gamma_1, \gamma_2 \rangle
\]

\[
= \langle s_{\zeta_1}(z)Q^*(\zeta_1, \eta_1)\gamma_1, s_{\zeta_2}(z)Q^*(\zeta_2, \eta_2)\gamma_2 \rangle.
\]

Hence \(U\) is an isometry. By Theorem 4.1 item (iii) the span of \(\mathcal{B}\) is dense in \(H^2_{\mathcal{C}M}\). Moreover, the range of \(U\) is dense in \(H^2(\mathcal{K})\). Thus, \(U\) is a unitary. Rewrite (11) as,

\[
w^*Q^*(z, w) = \Phi^*(z)Q^*(z, w).
\]

Let \(\tilde{M}_z\) and \(\tilde{M}_w\) be the operators of multiplication by \(z\) and \(w\) on \(H^2(\mathcal{K})\) respectively. For \((\zeta, \eta) \in \mathfrak{M}(p)\) and \(\gamma \in \mathbb{C}^\alpha\), using (12), observe that,

\[
\tilde{M}_w^*Us_{\zeta}(z)Q^*(\zeta, \eta)\gamma = \tilde{M}_w^*(\mathcal{K}_{(\zeta, \eta)})(z, w)\gamma
\]

\[
= \tilde{\eta}\mathcal{K}_{(\zeta, \eta)}(z, w)\gamma
\]

\[
= \tilde{\eta}(s_{\zeta}Q^*(\zeta, \eta)\gamma
\]

\[
= U(s_{\zeta}(z)\tilde{\eta}Q^*(\zeta, \eta)\gamma
\]

\[
= U(s_{\zeta}(z)\Phi(\zeta)Q^*(\zeta, \eta)\gamma
\]

\[
= UM_\Phi^*(s_{\zeta}(z)Q^*(\zeta, \eta)\gamma).
\]

Similarly,

\[
\tilde{M}_z^*Us_{\zeta}(z)Q^*(\zeta, \eta)\gamma = UM_z^*(s_{\zeta}(z)Q^*(\zeta, \eta)\gamma).
\]

Therefore, \(UM_z^* = \tilde{M}_z^*U\) and \(UM_\Phi^* = \tilde{M}_w^*U\) on the span of \(\mathcal{B}\), and hence on \(H^2_{\mathcal{C}M}\). Thus our original \((S, T)\) is unitarily equivalent to \((\tilde{M}_w, \tilde{M}_w)\) on \(H^2(\mathcal{K})\). \(\square\)
**Definition 4.6.** If $B$ is a subspace of vector space $X$, then the **codimension** of $B$ in $X$ is the dimension of the quotient space $X/B$.

**Lemma 4.7.** Suppose $X$ is a vector space (over $\mathbb{C}$) and $Q$ and $B$ are subspaces of $X$. If $Q \subset B$ and $Q$ has finite codimension in $X$, then $Q$ has finite codimension in $B$.

**Lemma 4.8.** Suppose $K$ is a Hilbert space and $Q \subset B \subset K$ are linear subspaces (thus not necessarily closed) and let $\overline{Q}$ denote the closure of $Q$. If $Q$ has finite codimension in $B$ and if $B$ is dense in $K$, then there exists a finite dimensional subspace $D$ of $K$ such that $K = Q \oplus D$.

**Theorem 4.9.** If $K$ is a rank $\alpha$ admissible kernel function defined on $\mathcal{V}(p)$ and $S = M_z$, $T = M_w$ are the operators of multiplication by $z$ and $w$ respectively on $H^2(K)$, then the pair $(S, T)$ is nearly $\alpha$-cyclic.

**Proof.** Since $K$ is a rank $\alpha$ admissible kernel, there exist matrix polynomials $Q$ and $P$ of size $\alpha \times m\alpha$ and $\alpha \times n\alpha$ respectively, such that

$$K((z,w),(\zeta,\eta)) = \frac{Q(z,w)Q^*(\zeta,\eta)}{1-z\bar{\zeta}} = \frac{P(z,w)P^*(\zeta,\eta)}{1-w\bar{\eta}}, \quad (z,w),(\zeta,\eta) \in \mathcal{V}(p)$$

and $Q$ and $P$ have full rank $\alpha$ at some point in $\mathcal{V}(p)$. Fix $(\zeta,\eta) \in \mathcal{V}(p)$ so that $Q(\zeta,\eta)$ has full rank $\alpha$. By the definition of $K$ and [JKM12, Lemma 3.3], $K((z,w),(\zeta,\eta))$ has full rank $\alpha$ at almost all points in $\mathcal{V}(p)$. Let

$$Q_0 = Q_0(z,w) = Q(z,w)Q^*(\zeta,\eta).$$

Then $Q_0e_j = (1 - S\bar{\zeta})K(\zeta,\eta)e_j$. By Theorem (4.1) item (ii), $Q_0e_j$, the $j^{th}$ column of $Q_0$, is also in $H^2(K)$. Letting $\tilde{q} = \tilde{q}(z,w)$ be the determinant of $Q_0$, since $K((z,w),(\zeta,\eta))$ has full rank $\alpha$ at almost all points in $\mathcal{V}(p)$, $\tilde{q}$ is nonzero except for finitely many points in $\mathcal{V}(p)$. Thus, $p$ and $\tilde{q}$ have only finitely many common zeros in $\mathcal{V}(p)$. By Lemma 2.2 item (iii), $p$ and $\tilde{q}$ are relatively prime. Let $I$ be the ideal generated by $p$ and $\tilde{q}$. By Proposition 2.1 item (ii), $\mathbb{C}[z,w]/I$ is finite dimensional. Observe that

$$\tilde{q}_j = \tilde{q}e_j = Q_0\text{Adj}(Q_0)e_j = \sum_{k=1}^{\alpha} b_{kj}Q_0e_k \in H^2(K),$$

where $b_{kj}$ is the $(k,j)$-entry of $\text{Adj}(Q_0)$. If $\tilde{r}$ is an $\alpha \times 1$ matrix polynomial with entries $r_j$, then

$$\tilde{r}\tilde{q} = \sum_{j=1}^{\alpha} r_j \text{Adj}(Q_0)Q_0e_j \in H^2(K).$$
Since $\mathbb{C}[z, w]/I$ is finite dimensional, there is a finite dimensional subspace $\mathscr{S} \subseteq \mathbb{C}[z, w]$ such that
$$\{r\tilde{q} + sp + t \mid r, s \in \mathbb{C}[z, w], t \in \mathscr{S}\} = \mathbb{C}[z, w].$$

Therefore
$$\left\{\tilde{r}\tilde{q} + \tilde{s}p + \tilde{t} : \tilde{r}, \tilde{s} \text{ are vector polynomials}, \tilde{t} \in \bigoplus_{i}^{\alpha} \mathscr{S}\right\} = \bigoplus_{i}^{\alpha} \mathbb{C}[z, w],$$
and hence the span $Q$ of $\{r_1\tilde{q}_1, ..., r_\alpha\tilde{q}_\alpha : r_1, ..., r_\alpha \in \mathbb{C}[z, w]\}$ is of finite codimension in $\bigoplus_{i}^{\alpha} \mathbb{C}[z, w]$.

Let $B = \bigvee\{z^nQe_j : n \in \mathbb{N}, 1 \leq j \leq m\alpha\} \subseteq \bigoplus_{i}^{\infty} \mathbb{C}[z, w]$. By equation (13) $Q \subset B$. By Lemma 4.7, $Q$ has finite codimension in $\bigoplus_{i}^{\infty} \mathbb{C}[z, w]$. Moreover, $B$ is dense in $H^2(K)$ by Theorem 4.1 item (iv). Hence by Lemma 4.8, the closure of $Q$ in $H^2(K)$ has finite codimension in $H^2(K)$. Equivalently, the closure of $\{\sum_{j=1}^{n} r_j(S, T)\tilde{q}_j : r_j \in \mathbb{C}[z, w]\}$ is of finite codimension in $H^2(K)$. Thus $(S, T)$ is $\alpha$-cyclic on $Q$ and hence at most nearly $\alpha$-cyclic in $H^2(K)$.

Moreover, by Corollary 3.10, $(S, T)$ has rank at most $\alpha$. For $(\zeta, \eta) \in \mathfrak{Z}(p)$ and for $\gamma \in \mathbb{C}^{\alpha}$, note that
$$\mathcal{K}_{(\zeta, \eta)}\gamma \in \ker(M_z - \zeta)^* \cap \ker(M_w - \eta)^*.$$ 
Hence, if $(\zeta, \eta) \in \mathfrak{Z}(p)$ is such that $\mathcal{K}_{(\zeta, \eta)}$ has full rank $\alpha$, then $\ker(M_z - \zeta)^* \cap \ker(M_w - \eta)^*$ has dimension at least $\alpha$. Therefore, $(S, T)$ has rank at least $\alpha$. Thus $(S, T)$ has rank $\alpha$. By Corollary 3.10, $(S, T)$ is at least nearly $\alpha$-cyclic and hence $(S, T)$ is nearly $\alpha$-cyclic on $H^2(K)$.

**Proposition 4.10.** If $V = (S, T)$ is a finite bimultiplicity pure $p$-isopair of rank $\alpha$ acting on the Hilbert space $K$, then there exists a finite codimension $V$ invariant subspace $H$ of $K$ such that the restriction of $V$ to $H$ is $\alpha$-cyclic.

**Proof.** Combine Theorems 4.5 and 4.9. ☐

### 5. DECOMPOSITION OF FINITE RANK ISOPAIRS

**Proposition 5.1.** Suppose $p_1, p_2 \in \mathbb{C}[z, w]$ are relatively prime square free inner toral polynomials, but not necessarily irreducible. If $V_j = (S_j, T_j)$ are $\beta_j$-cyclic $p_j$-pure isopairs, then $V = V_1 \oplus V_2$ is a $p_1p_2$-isopair and is at most nearly $\max\{\beta_1, \beta_2\}$-cyclic.

**Proof.** Clearly,
$$p_1p_2(V) = (p_1(V_1) \oplus p_1(V_2))(p_2(V_1) \oplus p_2(V_2)) = (0 \oplus p_1(V_2))(p_2(V_1) \oplus 0) = 0.$$
Let $I$ be the ideal generated by $p_1$ and $p_2$. By Proposition 2.1 item (ii), $I$ has finite codimension in $\mathbb{C}[z, w]$. Hence there exists a finite dimension subspace $\mathcal{R}$ of $\mathbb{C}[z, w]$ such that, for each $\psi \in \mathbb{C}[z, w]$, there exist $s_1, s_2 \in \mathbb{C}[z, w]$ and $r \in \mathcal{R}$ such that

$$\psi = s_1p_1 + s_2p_2 + r.$$ 

Let $K$ denote the Hilbert space that $V$ acts upon. Let $\beta = \max\{\beta_1, \beta_2\}$ and suppose without loss of generality $\beta_1 = \beta_2 = \beta$. For $j = 1, 2$, choose cyclic sets $\Gamma_j = \{\gamma_{j,1}, \ldots, \gamma_{j,\beta}\}$ for $V_j$. (In the case where $\beta_1 < \beta_2$ we can set $\Gamma_1$ to be $\{\gamma_{1,1}, \ldots, \gamma_{1,\beta_1}, 0, 0, \ldots\}$, so that this new $\Gamma_1$ has $\beta = \beta_2$ vectors.) Let $K_0 = \{\psi_1(V_1)\gamma_{1,k} + \psi_2(V_2)\gamma_{2,k} : 1 \leq k \leq \beta, \psi_j \in \mathbb{C}[z, w]\}$. By the hypothesis, $K_0$ is dense in $K$. For given polynomials $\psi_1, \psi_2 \in \mathbb{C}[z, w]$, there exist $s_1, s_2 \in \mathbb{C}[z, w]$ and $r \in \mathcal{R}$ such that

$$\psi_1 - \psi_2 = -s_1p_1 + s_2p_2 + r.$$ 

Rearranging gives,

$$\psi_1 + s_1p_1 = \psi_2 + s_2p_2 + r.$$ 

Let $\varphi = \psi_1 + s_1p_1$. It follows that

$$\varphi = \psi_2 + s_2p_2 + r.$$ 

Consequently,

$$\varphi(V) [\gamma_{1,k} \oplus \gamma_{2,k}] = \varphi(V_1)\gamma_{1,k} \oplus \varphi(V_2)\gamma_{2,k}$$

$$= \psi_1(V_1)\gamma_{1,k} \oplus (\psi_2(V_2)\gamma_{2,k} + r(V_2)\gamma_{2,k}).$$

Let $H_0$ denote the span of $\{\varphi(V) [\gamma_{1,k} \oplus \gamma_{2,k}] : 1 \leq k \leq \beta, \psi \in \mathbb{C}[z, w]\}$ and $H$ be the closure of $H_0$. Let $\mathcal{L}$ denote the span of $\{0 \oplus r(V_2)\gamma_{2,k} : 1 \leq k \leq \beta, r \in \mathcal{R}\}$. Note that $\mathcal{L}$ is finite dimensional since $\mathcal{R}$ is and hence $\mathcal{L}$ is closed. Moreover,

$$K_0 = H_0 + \mathcal{L}.$$ 

Hence $H_0$ has finite codimension in $K_0$. By Lemma 4.8, $H$ has finite codimension in $K$. Evidently $H$ is $V$ invariant and the restriction of $V$ to $H$ is at most $\beta$-cyclic. Therefore, $V$ is at most nearly $\beta$-cyclic.

\[\square\]

**Proposition 5.2.** If $V_j = (S_j, T_j)$ are finite bimultiplicity pure $p_j$-isopairs with rank $\alpha_j$ acting on Hilbert spaces $K_j$, where $p_j$ are irreducible and relatively prime inner toral polynomials for $1 \leq j \leq s$, then $\bigoplus_{j=1}^s V_j$ is nearly max\{$\alpha_1, \alpha_2, \ldots, \alpha_s$\}-cyclic on $\bigoplus_{j=1}^s K_j$.

**Proof.** First suppose $s = 2$. By Proposition 4.10, each $V_j$ is $\alpha_j$-cyclic on some finite codimensional invariant subspace $H_j$ of $K_j$. By Proposition 5.1, $V_j|_{H_j} \oplus V_j|_{H_j}$ is at most nearly max\{$\alpha_1, \alpha_2$\}-cyclic on $H_1 \oplus H_2$. Since each $H_j$ has finite codimension in $K_j$, it follows that $V = V_1 \oplus V_2$ is at most nearly max\{$\alpha_1, \alpha_2$\}-cyclic on $K_1 \oplus K_2$. On the other hand, $V$ has
rank \((\alpha_1, \alpha_2)\) and hence, by Corollary 3.10, is at least \(\max\{\alpha_1, \alpha_2\}\)-cyclic. Thus \(V\) is nearly \(\max\{\alpha_1, \alpha_2\}\)-cyclic.

Arguing by induction, suppose the result is true for \(0 \leq j - 1 < s\). Thus \(V' = V_1 \oplus \cdots \oplus V_{j-1}\) is nearly \(\beta = \max\{\alpha_1, \alpha_2, \ldots, \alpha_{j-1}\}\)-cyclic on \(K' = K_1 \oplus K_2 \oplus \cdots \oplus K_{j-1}\). Hence there exists a finite codimensional invariant subspace \(H'\) of \(K'\) such that the restriction of \(V'\) to \(H'\) is \(\beta\)-cyclic. Since \(V_j\) is a finite bimultiplicity \(p\)-isopair with rank \(\alpha_j\), by Proposition 4.10, there exists a finite codimensional invariant subspace \(H_j\) of \(K_j\) such that \(V_j|_{H_j}\) is \(\alpha_j\)-cyclic. Note that \(p_{j-1}\) and \(p_j\) are relatively prime. Applying Proposition 5.1 to \(V'|_{H'}\) and \(V_j|_{H_j}\), it follows that \(V'|_{H'} \oplus V_j|_{H_j}\) is at most nearly \(\gamma = \max\{\beta, \alpha_j\}\)-cyclic on \(H' \oplus H_j\). Since \(H'\) and \(H_j\) have finite codimension in \(K'\) and \(K_j\) respectively, \(W = V_1 \oplus V_2 \oplus \cdots \oplus V_j\) is at most nearly \(\gamma\)-cyclic on \(K_1 \oplus K_2 \oplus \cdots \oplus K_j\). On the other hand, \(W\) has rank \((\alpha_1, \ldots, \alpha_j)\) and is therefore at least nearly \(\gamma\)-cyclic by Corollary 3.10. Thus \(W\) is nearly \(\gamma = \max\{\alpha_1, \ldots, \alpha_j\}\)-cyclic. \(\square\)

**Proof of Theorem 1.4.** By [AKM12, Theorem 2.1], there exist a finite codimension subspace \(H\) of \(K\) that is invariant for \(V\) and pure \(p\)-isopairs \(V_j\) such that

\[
W = V|_{H} = V_1 \oplus V_2 \oplus \cdots \oplus V_s.
\]

By Proposition 3.9, \(W\) has rank \(\alpha\). Hence \(V_j\) has rank \(\alpha_j\). By Proposition 5.2, there is a finite codimension invariant subspace \(L\) of \(H\) such that the restriction of \(W\) to \(L\) is \(\beta = \max\{\alpha_1, \alpha_2, \ldots, \alpha_s\}\)-cyclic. Thus \(L\) is a finite codimensional subspace of \(K\) such that \(V|_{L}\) is \(\beta\)-cyclic. Hence \(V\) is at most nearly \(\beta\)-cyclic. By Corollary 3.10, \(V\) is at least nearly \(\beta\)-cyclic. Therefore, \(V\) is nearly \(\max\{\alpha_1, \alpha_2, \ldots, \alpha_s\}\) cyclic. \(\square\)

**Corollary 5.3.** Suppose \(V = (S, T)\) is a pure \(p\)-isopair of finite bimultiplicity with minimal polynomial \(p\) and write \(p = p_1 p_2 \cdots p_s\) as a product of distinct irreducible factors. If \(V\) has rank \(\alpha\) and \(\beta = \max\{\alpha_1, \ldots, \alpha_s\}\), then

(i) there exists a finite codimension invariant subspace \(H\) of \(V\) such that the restriction of \(V\) to \(H\) is \(\beta\)-cyclic;

(ii) \(V\) is not \(k\)-cyclic for any \(k < \beta\); and

(iii) there exists a \(\beta\)-cyclic pure \(p\)-isopair \(V'\) and an invariant subspace \(K\) for \(V'\) such that \(V\) is the restriction of \(V'\) to \(K\).

**Proof.** Proofs of items (i) and (ii) follow from Theorem 1.4 and the definition of nearly \(k\)-cyclic isopairs. The proof of item (iii) is an application of item (i) and Corollary 3.3. \(\square\)

5.1. **Example.** In this section we discuss an example on pure \(p\)-isopairs of finite rank to illustrate the connection of the rank of a pure \(p\)-isopair to nearly cyclicity and to the representation as direct sums.

Consider the irreducible, square free inner toral polynomial, \( p = z^3 - w^2 \). The distinguished variety, \( \mathcal{V} \), defines by \( p \) is called Neil parabola [JKM12]. The triple \((K_1, Q_1, P_1)\) given by

\[
Q_1(z, w) = \begin{pmatrix} 1 & w \end{pmatrix}
\]

\[
P_1(z, w) = \begin{pmatrix} 1 & z & z^2 \end{pmatrix}
\]

and the corresponding kernel function

\[
\frac{1 + w\eta}{1 - z\zeta} = \mathcal{K}_1((z, w), (\zeta, \eta)) = \frac{1 + z\zeta + z^2\zeta^2}{1 - w\eta},
\]

is a 1-admissible triple. Likewise for the choice of

\[
Q_2(z, w) = \begin{pmatrix} z & w \end{pmatrix}
\]

\[
P_2(z, w) = \begin{pmatrix} w & z & z^2 \end{pmatrix}
\]

and the corresponding kernel function

\[
\frac{z\zeta + w\eta}{1 - z\zeta} = \mathcal{K}_2((z, w), (\zeta, \eta)) = \frac{w\eta + z\zeta + z^2\zeta^2}{1 - w\eta},
\]

the triple \((K_2, Q_2, P_2)\) is also a 1-admissible triple. For \( j = 1, 2 \), let \( V_j \) be the pair \((M_z, M_w)\) defines on \( H^2(K_j) \). Now each \( V_j \) is a pure \( p \)-isopair or rank 1 and each \( V_j \) is nearly 1-cyclic.

Let \( Q = Q_1 \oplus Q_2 \), \( P = P_1 \oplus P_2 \) and \( \mathcal{K} = K_1 \oplus K_2 \). Observe that the triple \((\mathcal{K}, Q, P)\) is a 2-admissible triple and \( V = (M_z, M_w) \) define on \( H^2(\mathcal{K}) \) is a pure \( p \)-isopair and nearly 2-cyclic. In fact \( V \) is a pure \( p \)-isopair of rank 2 that can be written as direct sum of two pure \( p \)-isopairs, \( V_1 \) and \( V_2 \).

However this is not in true general. In other words, there exist pure \( p \)-isopairs of finite rank (say \( \alpha \in \mathbb{N} \)), that cannot be expressed as a direct sum of \( \alpha \) number of pure \( p \)-isopairs. For instance, let

\[
H' = \{ f \in H^2(\mathcal{K}) : \langle f, (1 - z)^\top \rangle = 0 \}
\]

and \( V' = V|_{H'} \). Observe that \( H' \) is a finite codimensional subspace of \( H^2(\mathcal{K}) \) and \( H' \) is invariant under \( V \). By the stability of the rank \( V' \) has rank 2 and hence nearly 2-cyclic.

Moreover, the collection of vectors of the form:

\[
\begin{pmatrix} 1/\sqrt{2} \\ z/\sqrt{2} \end{pmatrix}, \begin{pmatrix} z^n \end{pmatrix}_{n \geq 1}, \begin{pmatrix} wz^n \end{pmatrix}_{n \geq 0}, \begin{pmatrix} 0 \end{pmatrix}_{n \geq 2}, \begin{pmatrix} 0 \\ wz^n \end{pmatrix}_{n \geq 0}
\]
forms an orthonormal basis for $H'$. Hence the reproducing kernel, $\tilde{K}$, for $H'$ has the form

$$
\tilde{K}((z, w), (\zeta, \eta)) = 
\begin{pmatrix}
\frac{1}{2} + \frac{z\zeta + w\eta}{1 - z\zeta} & \frac{\zeta}{2} \\
\frac{z}{2} & z\zeta \left(\frac{1}{2} + \frac{z\zeta + w\eta}{1 - z\zeta}\right)
\end{pmatrix}.
$$

Since $\tilde{K}((z, w), (0, 0))$ is not diagonalizable, $\tilde{K}$ is not diagonalizable. Consequently, $H'$ and $V'$ are not direct sums. In other words, $V'$ is a pure $p$-isopair of rank 2 that cannot be expressed as a direct sum of two other pure $p$-isopairs.

6. ACKNOWLEDGEMENTS

I am very grateful to my adviser, Scott McCullough, for his valuable guidance and insights that greatly improved the content of this paper.

REFERENCES

[AD76] M. B. Abrahamse and R. G. Douglas, A Class of Subnormal Operators Related to Multiply-Connected Domains, in Advances in Mathematics, 106–148, 1976.

[AKM12] Jim Agler, Greg Knese and John E. McCarthy, Algebraic pairs of isometries, J. Operator Theory 67 (2012), no. 1, 215–236.

[AM05] Jim Agler and John E. McCarthy, Distinguished Varieties, Acta Mathematica 194 (2005), no. 2, 133–153.

[AM06] Jim Agler and John E. McCarthy, Parametrizing distinguished varieties, Recent advances in operator-related function theory, 2934, Contemp. Math., 393, Amer. Math. Soc., Providence, RI, 2006.

[AM02] J. Agler and J.E. McCarthy, Pick interpolation and Hilbert Function Spaces, American Mathematical Society, Providence, 2002.

[AMS06] Jim Agler, John McCarthy and Mark Stankus, Toral algebraic sets and function theory on polydisks, Journal of Geometric Analysis 16 (2006), no. 4, 551–562.

[C97] David A. Cox, Introduction to Grobner Bases, Applications of computational algebraic geometry (San Diego, CA, 1997), 124, Proc. Sympos. Appl. Math., 53, Amer. Math. Soc., Providence, RI, 3, 4.

[JKM12] Michael T. Jury, Greg Knese and Scott McCullough, Nevalinna-Pick interpolation on distinguished varieties in the bidisk, Journal of Functional Analysis, 262 (2012) 3812–3838.

[K10] Greg Knese, Polynomials defining distinguished varieties, Transactions of the American Mathematical Society 362 (2010), no. 11, 5635–5655.

[M12] John E. McCarthy, Shining a Hilbertian lamp on the bidisk, in Topics in complex analysis and operator theory, 49–65, Contemp. Math., 561, American Mathematical Society, Providence, RI, 2012.

[PR16] Vern Paulsen and Mrinal Raghupathi, An Introduction to the Theory of Reproducing Kernel Hilbert Spaces, Cambridge Studies in Advanced Mathematics, 2016.

[PS14] Sourav Pal and Orr Moshe Shalit, Spectral sets and distinguished varieties in the symmetrized bidisc, Journal of Functional Analysis 266 (2014), no. 9, 5779–5800.
[R69] Walter Rudin, *Pairs of Inner Functions on Finite Riemann surfaces*, Trans. Amer. Math. Soc., 140:423–434, 1969. 3