Generalized Gibbons-Hawking-York term for $f(R)$ gravity

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Abstract. A derivation of the Gibbons-Hawking-York boundary term for modified gravity metric is presented. The GHY term is generalized for the case of $f(R)$ modified gravity. It is shown that a stable set of field equations can be formed with aid of the new GHY term for the case of $f(R)$ gravity action. It is proven that the new term demands no particular value of $\delta g_a$ or $\partial_a \delta g_a$ on the boundary. Finally, the surface terms are shown to vanish in the derivation of the new equations.

1. Introduction

There are three popular formulations for $f(R)$ gravity theories: metric $f(R)$ gravity, metric-affine $f(R)$ gravity and Palatini $f(R)$ gravity. Metric $f(R)$ has been the most popular of the three, mainly because it passes all the observational and theoretical constrains [1]. Recently, metric $f(R)$ gravity has proven quite useful in providing a toy model for studying the problem of dark energy. One problem with metric $f(R)$ that has received less attention is the problem of fixing the surface terms on the boundary when deriving the field equations. Unlike General Relativity[5,6], the boundary terms in modified $f(R)$ gravity do not vanish in the addition of the total divergence to the action [2]. In this paper, we will show that in fact the boundary terms can be expressed as a total variation of some quantity. To achieve stable field equations, the GHY boundary term will be modified such that the surface equations vanish. While it has been demonstrated that the addition of GHY term to the gravitational action is attractive because of the removal of the equations involving $\delta g_a$ , and so setting $\delta g_a = 0$ achieves the aim of making the action stationary. In order to see the role Gibbons-Hawking-York boundary term we must first review its role in general relativity. If we consider the Einstein-Hilbert action [3]

$$S_E = \frac{1}{2\kappa} \int d^4x \sqrt{-g} R,$$  \hspace{2cm} (1)

where $R = R_a g^a$ is the Ricci scalar.

The variation of (1) with respect to $\delta g_a$ gives
\[ \delta S_E = \frac{1}{2\kappa} \int d^4x (\delta \sqrt{-g} \mathcal{R} + \sqrt{-g} \delta \mathcal{L}), \]  

(2)

using \( \delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g} g_a \delta g^a \) and \( \delta \mathcal{R} = \nabla_a \delta \Gamma^c_{ab} - \nabla_b \delta \Gamma^c_{ab} \), the variation \( \delta S_E \) becomes

\[ \delta S_E = \int d^4x \left( -\frac{1}{2} \delta \mathcal{R} + \mathcal{L} \right) + \sqrt{-g} g^a \left( \nabla_a \delta \mathcal{L} - \nabla_b \delta \mathcal{L} \right) \]  

(3)

Finally, using the fact that the covariant derivative of the metric tensor vanishes in general relativity, we can rewrite the third term as the divergence of a vector field \( \mathcal{J}^c \), where \( \mathcal{J}^c = g^a \delta \Gamma^c_{ab} - g^a \delta \Gamma^b_{ab} \). Using Gauss-Stokes theorem [7], the variation (3) becomes

\[ \delta S_E = \int d^4x \mathcal{J} \delta \mathcal{L} + \oint d^3 \mathcal{S} \mathcal{J} n_c, \]  

(4)

where \( h \) is the induced metric tensor associated with hypersurface and \( n_c \) is the normal unit vector on hypersurface \( \partial \). The first term in (4) is the familiar Einstein field equations multiplied by \( \delta g^a \), however in order to derive the field equations, the action must be stationary and consequently the second term must vanish. We can cancel the second term by adding a boundary term to the total action in (1) that cancels this surface term.

Using the definition of the christoffel symbol (i.e. \( \Gamma^a_{bc} = \frac{1}{2} g_{ab} \left( \partial_a g_{bc} + \partial_b g_{ac} - \partial_c g_{ab} \right) \)), \( \mathcal{J}_c \) becomes

\[ \mathcal{J}^c = g^a \left( V_b \delta \mathcal{L}^c - V_c \delta \mathcal{L}^b \right), \]  

(5)

substituting the completeness relation \( g^a = h^a - n^a n^b \), so that

\[ n_c \mathcal{J}_c = n_c (h^a - n^a n^b) \left( V_b \delta \mathcal{L}^c - V_c \delta \mathcal{L}^b \right) \]

\[ = n^c h^a V_b \delta \mathcal{L}^c - n^c h^a V_c \delta \mathcal{L}^b \]

\[ - n^a n^b n^c V_b \delta \mathcal{L}^c + n^a n^b n^c V_c \delta \mathcal{L}^b. \]  

(6)

The third and fourth terms vanish because of the antisymmetry of \( (V_b \delta \mathcal{L}^c - V_c \delta \mathcal{L}^b) \). Consequently we obtain
\[ n^c f_c |_{\partial} = n^c h^a (\partial_b \delta_{\varepsilon}^c - \partial_c \delta_{\varepsilon}^b ). \]  

(7)

Since \( \delta g_{\alpha} \) vanishes everywhere on \( \partial \), its tangential derivative must also vanishes. It follows that \( \alpha^a \partial_b \delta_{\varepsilon}^c = 0 \) and we obtain

\[ n^c f_c |_{\partial} = -n^c h^a \partial_c \delta_{\varepsilon}^b , \]  

(8)

substituting (8) in (4), the last term becomes

\[ -\oint_{\partial} d^3 x \sqrt{|h|} n^c h^a \partial_c \delta_{\varepsilon}^b . \]  

(9)

Now, the trick is to construct a term that cancels this term. The GHY term was originally proposed in the form of an additional trace of the extrinsic curvature of the boundary to the action. If we consider the Gibbons-Hawking-York boundary term

\[ S_G = 2 \oint_{\partial} d^3 x \sqrt{|h|} K, \]  

(10)

where \( K = \nabla_c h^c \) is intrinsic curvature, using \( \delta |_{\partial} = n^c h^a \partial_c \delta_{\varepsilon}^b \), the variation of this term gives

\[ \delta S_G = \oint_{\partial} d^3 x \sqrt{|h|} n^c h^a \partial_c \delta_{\varepsilon}^b . \]  

(11)

Notice that this boundary term cancels (9) totally. It must be noted that the addition of boundary term is only needed when the manifold has a boundary, otherwise no additional term is needed such as the cases for a closed space time manifold.

2. Redefining the function of Gibbons-Hawking-York term

Generally, the Gibbons-Hawking-York term is defined as the term added to the total action to cancel the surface terms if the action is to be stationary. However, if we redefine the Gibbons-Hawking-York term as the term add to the Ricci scalar, such that

\[ \int d^4 x \sqrt{-\tilde{g}} R + 2 \oint_{\partial} d^3 x \sqrt{|\tilde{h}|} K \]

\[ = \int d^4 x \sqrt{-\tilde{g}} R, \]  

(12)

where \( \tilde{R} = R - \phi \) and \( \phi = 2 \nabla_c (h^a \Gamma^c_{\alpha} \). The variation of (12) gives

\[ \delta \int d^4 x \sqrt{-\tilde{g}} R = \int d^4 x \sqrt{-\tilde{g}} (\delta - \delta - \frac{1}{2} \delta_{\alpha} \delta^\alpha R) \]
\[ \int d^4x \sqrt{-g}(K_a - \frac{1}{2} g_a \ K) \delta \ a \]

\[ + \oint d^3x \sqrt{|h|}(f^c n_c - 2 h^a \ \delta l^c_{a} n_c), \quad (13) \]

where we have used

\[ \int d^4x \sqrt{-g} \delta - \int d^4x \sqrt{-g} \frac{1}{2} g_a \ \delta \ a \ \phi \]

\[ = \delta \int d^4x \sqrt{-g} \delta = \int d^4x \sqrt{-g} \frac{1}{2} R g_a \ ) \delta \ a, \quad (15) \]

Using the fact that the quantity \( f^c n_c - h^a \ \delta l^c_{a} n_c \) vanishes at the boundary, the variation of (12) gives the familiar field equations without boundary terms, i.e.

\[ \delta \int d^4x \sqrt{-g} R = \int d^4x \sqrt{-g} (K_a - \frac{1}{2} R g_a ) \delta \ a, \]

It’s clear that the modification considered in (12) does the same job as the Gibbons-Hawking-York term without the need to consider the vanishing of \( \delta g_a \) at the boundary of the hypersurface. Although this may seem simple, in the next section we show that this new definition is crucial in deriving the field equations for \( f(R) \) metric gravity without the need to worry about any surface terms that need be to canceled.

3. Variational principle in metric \( f(R) \) gravity

If we consider the modified \( f(R) \) gravity action

\[ S_R = \int d^4x \sqrt{-g} f(R), \quad (16) \]

the variation of this action

\[ \delta S_R = \int d^4x (\delta \sqrt{-g} f(R) + \sqrt{-g} f'(R) \delta), \quad (17) \]

using \( \delta = \delta \ a \ K_a + g_a \ V_c V^c(\delta \ a) \) and \( \delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g} g_a \ \delta g^a \), (17) becomes

\[ \delta S_R = \int d^4x \sqrt{-g} (f'(R) \delta \ a \ K_a + f'(R) g_a \ V_c V^c(\delta \ a)) \]

\[ = \int d^4x \sqrt{-g} f'(R) \delta \ a \ K_a + f'(R) g_a \ V_c V^c(\delta \ a) \]
\[-f'(R)V_aV_b(\delta^a) - \frac{1}{2}f(R)g_{ab}\delta^a.\]  

To derive the field equations we must reexpress the second and third terms as \(\delta^a\) multiplied by some quantity. To do that we define

\[W^c = f'(R)[g_{a} g^{c} V_a \delta^a - V_a \delta^c]\]

\[-\delta^a [g_{a} g^{c} V_{af'}(R) - \delta^{c}_b V_{af'}(R)].\]  

Rewriting the second and third terms in (18) in terms of \(V_a W^c\) and using Gauss-Stokes theorem on the last term, (18) becomes

\[\delta S_R = \int d^4 x \sqrt{-g} (f''(R) K_a + g_a V_c V^c(f'(R))
\]

\[-V_a V_b (f'(R)) - \frac{1}{2}f(R)g_{ab}) \delta^a + \oint d^3 x \sqrt{|K|} n_c W^c.\]  

Evaluating the quantity \(W^c\) at the boundary yields

\[W^c|_\partial = -f'(R) n^c h^a \partial_c \delta^a.\]  

If the action is to be stationary, \(W^c\) must be canceled with boundary term. One way to do that is to add the Gibbons-Hawking-York term multiplied by \(f'(R)\) to action in (16), such that

\[S_R + S_G = \int d^4 x \sqrt{-g} f(R)
\]

\[+2 \oint d^3 x \sqrt{|K|} f'(R) K.\]  

However, the variation of this term requires setting \(\delta^a = 0\) on the boundary of the hypersurface [4], consequently this creates a strong condition on how \(\delta^a\) is varied near the hypersurface since \(\partial_c \delta^a\) is no longer arbitrary on the boundary. To overcome this restriction we use our new definition of the Gibbons-Hawking-York and replace \(f(R)\) by \(f(R) = f(R - \Phi) = \sum_{n=0}^{\infty} (-\Phi)^n \frac{d^n}{d\Phi^n}f(R),\) where \(\Phi\) is now defined such that \(\delta^a = V_c Y^c_a \delta^a - \delta^a V^c_a ,\) with the operator \(Y^c_a = g_{ca} V^c - \delta^c_a V_b.\) The variation of (16) gives
\[
\delta S_R = \delta \int d^3 x \sqrt{-g} \sum_{n=1}^{\infty} (-\phi)^n \frac{d^n}{dR^n} f(R)
= \delta \int d^3 x \sqrt{-g}(\delta - \delta )f'(\bar{R})
= \delta \int d^3 x \sqrt{-g}(\delta^a R_a + \xi^a \delta_a - V_e \nabla_e^a \delta^a
+ \delta^a V^e \nabla_e^a )f'(\bar{R}) - \frac{1}{2} f(\bar{R}) g_a \delta^a .
\]

Using the definition of \( \gamma_a^e \), this variation becomes
\[
\delta S_R = \int d^3 x \sqrt{-g}(f'(\bar{R}) R_a + \xi^a V^e(f'(\bar{R}))
- V^a V_b(f'(\bar{R})) - \frac{1}{2} f(\bar{R}) g_a )\delta^a ,
\]

for an arbitrary variation \( \delta^a \), we get the field equation
\[
f'(\bar{R}) R_a + \xi^a V^e(f'(\bar{R}))
- V^a V_b(f'(\bar{R})) - \frac{1}{2} f(\bar{R}) g_a = 0
\]

### 4. Conclusions

We have redefined the Gibbons-Hawking-York term as the term add to the Ricci scalar and have shown that this definition works just as well as the original Gibbons-Hawking-York in illuminating any possible boundary terms so that we can have a well defined stationary action. This derivation requires no restrictions on the value of \( \delta^a \) or \( \delta \delta^a \) on the boundary of the manifold. Although, unlike the familiar metric \( f(R) \) field equation [2], (25) is a differential equation for \( f(R - \phi) \). Rewriting the variation (23) in terms of \( f(R) \) yields
\[
\delta S_R = \int d^3 x \sqrt{-g}(\delta f'(R) - \frac{1}{2} f(R) g_a \delta^a
- \delta f'(R) - \delta f''(R) + \cdots),
\]

the variation of \( S_R + S_G \) gives
\[
\delta S_R + \delta S_G = \int d^4x \sqrt{g} \left( \delta f'(R) - \frac{1}{2} f(R) g_{\alpha \beta} \delta g^{\alpha \beta} \right) + \int d^3x \sqrt{|H|} \left( \delta f'(R) + K f''(R) \right) + \delta S_G
\]

If the action \( S_R + S_G \) is to be stationary, the last term must vanish (i.e. setting \( \delta = 0 \) on the boundary). However, in (26) there is no need to set such a condition on the boundary since the second and higher derivatives of \( f(R) \) give us a series of field equations for \( f(R) \) and its derivatives that can be gathered to give one field equation for \( f(R) \). In conclusion, the Gibbons-Hawking-York term must be an infinite number of boundary integrals in terms of the first and higher derivatives of \( f(R) \) that are added to the total action to cancel the surface terms. The reason that there is only one boundary integral for the Einstein-Hilbert action is because the second and higher derivatives of \( f(R) \) vanish. \( S_G \) require setting \( \delta = 0 \) because we have not considered the rest of the boundary integrals.

5. References

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