Speed Scaling On Parallel Servers with MapReduce Type Precedence Constraints

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Abstract—A multiple server setting is considered, where each server has tunable speed, and increasing the speed incurs an energy cost. Jobs arrive to a single queue, and each job has two types of sub-tasks, map and reduce, and a precedence constraint among them: any reduce task of a job can only be processed once all the map tasks of the job have been completed. In addition to the scheduling problem, i.e., which task to execute on which server, with tunable speed, an additional decision variable is the choice of speed for each server, so as to minimize a linear combination of the sum of the flow times of jobs/tasks and the total energy cost. The precedence constraints present new challenges for the speed scaling problem with multiple servers, namely that the number of tasks that can be executed at any time may be small but the total number of outstanding tasks might be quite large. We present simple speed scaling algorithms that are shown to have competitive ratios, that depend on the power cost function, and/or the ratio of the size of the largest task and the shortest reduce task, but not on the number of jobs, or the number of servers.

I. INTRODUCTION

In distributed/parallel processing systems such as MapReduce [9] or Hadoop [29], Dryad [18], jobs/flows alternate between computation and communication stages, where a new stage cannot start until all the required tasks/flows have been processed in the preceding stage. Essentially, there are precedence constraints between different tasks, e.g. until all the map tasks of a job are not completed, no reduce task of that job can be started.

A typical figure of merit in these systems is the delay seen by a job/task, where a job consists of multiple map and reduce tasks, and the precedence constraints present a new set of challenges for deriving optimal routing and scheduling policies that minimize the sum of job/task delays. There are typically three different metrics that are considered with multiple jobs, makespan, i.e., the finish time of the last job, completion time, i.e., the sum of finish times (counted from time 0) of all jobs, and flow time, i.e., the sum of response times (finish minus the arrival time) of all jobs. Solving the makespan problem is typically easier than the completion time problem and solving the flow time problem is the hardest.

The MapReduce scheduling problem has been considered in prior work quite extensively for both single and multiple server systems, however, under the typical assumption that the server speeds are fixed. In particular, for the offline setting, where non-causal job arrival information is assumed, [25], [6], [30], [35], [31], [34], [32] considered either the makespan or the completion time minimization problem, and derived approximation algorithms mostly assuming that any map task can be split arbitrarily to allow parallel execution on multiple servers. The more challenging online setting has been considered in [25], [6], [3], [22], where very recently [21] found an online algorithm with competitive ratio of 3 for the makespan minimization problem assuming as before that any map task can be split and executed in parallel on multiple servers. There is some work on energy efficient MapReduce scheduling [33], [23] with heuristic algorithms.

Makespan and flow time minimization has also been considered for general precedence constraints represented by an acyclic directed graphs (DAG) with fixed speed multiple servers [12], [5], [8], [27]. Most recent work in this model does not even assume knowing the size of the jobs when they arrive [11] (called the non-clairvoyant model).

In this paper, we consider a parallel server setting, where each server has a tunable speed. Corresponding to the speed of operation, there is a power/energy cost for each server identified by function $P$ which is typically assumed to have the form $P(s) = s^\alpha$ for $\alpha > 1$. Clearly, increasing the speed of the server reduces the flow time but incurs a larger energy cost. Thus, there is a natural tradeoff between the the flow time and the total energy cost, and a natural objective with tunable servers is to minimize a linear combination of the flow time and total energy, called flow time plus energy.

Jobs arrive over time where each job has a fixed number of map and reduce tasks, and any reduce task of a job can be executed only after all the map tasks belonging to the job are completed. Both preemption and migration are allowed, i.e., a task can be preempted on one server and restarted on another server later, which is a natural requirement with MapReduce constraints, since map tasks have inherently higher priority. Moreover, no task can be split, and thus cannot be processed on multiple servers at the same time.

To keep the problem most general, we assume an arbitrary input setting, where both the arrival times of jobs, and sizes (of tasks) are arbitrary (can be chosen by an adversary), and the objective is to find optimal online algorithms (that use only causal information) in terms of the competitive ratio. Competitive ratio is defined to be ratio of the cost incurred by the online algorithm to the cost of the optimal offline algorithm OPT that knows the entire input in advance, maximized over all possible inputs.

In prior work, there is a large body of work on online algorithms for the flow time plus energy problem, however, to the best of our knowledge except [27], [3], does not consider any precedence constraints. Without precedence constraints,
online algorithms for multiple servers to minimize flow time and energy with constant competitive ratios have been derived in [19], [13], [4], [2], [16], [20], [15], [1], [10], under both the homogenous server model, i.e., \( P(.) \) is identical for all servers, and the heterogeneous model.

The most relevant work for our considered problem is [27], [24], [3], that considers multiple servers with tuneable speed, and the objective is to find an optimal algorithm that minimizes the completion time under a total energy constraint for arbitrary precedence constraints defined by a directed acyclic graph (DAG) [27] and for MapReduce type constraints [24], [3]. However, the results of [27] are limited for the case when all jobs are available at time 0, and its approximation ratio scales as square of the logarithm of the number of servers. In [24], [3], an offline problem (where the exact arrival sequence of jobs is known a priori) to minimize the completion time with MapReduce precedence constraints is considered. Moreover, [3] requires that for each task there is a preassigned server that only is allowed to process it. A power function dependent approximation ratio has been derived in [3] with energy augmentation, improving upon [24] that considered only a single server setting.

In this paper, our endeavour is to only consider the MapReduce type of precedence constraints and find online algorithms for the flow time plus energy problem with competitive ratios that do not scale with either the number of jobs or the number of servers. The main difficulty with multiple servers under precedence constraints is that at certain time, there might be a large number of outstanding tasks (few map but large number of reduce tasks), but very small (less than the number of servers) number of tasks that can be executed (e.g. only map tasks), making some servers idle. Essentially, all the non-triviality stems in controlling this event, since we are looking for a sample path result against the optimal offline algorithm (OPT) that might keep all servers busy making the comparison between any proposed online algorithm and OPT difficult.

With precedence constraints, flow time can be counted in two ways, job-wise (departure time of the last reduce task - arrival time of job) or task-wise. We consider both these settings in this paper. For the job-wise setting, our results are limited for the case when all jobs arrive at time 0 similar to [27]. For the task-wise flow time, we consider the general online setting, where jobs arrive over time and the algorithm has only causal information.

Throughout this paper, we consider the shortest remaining processing time (SRPT) algorithm for executing the outstanding executable tasks on multiple servers. In particular, let \( n \) be the total number of outstanding tasks, and \( k \leq K \) be the number of tasks that can be executed at time \( t \). Then with \( K > 1 \) servers, the \( \min\{k, K\} \) tasks with the shortest remaining size are executed on the \( \min\{k, K\} \) servers.

With \( K \) homogenous servers with power function \( P(s) = s^\alpha \), our contributions are as follows:

1. For the job-wise flow time + energy problem, the proposed algorithm achieves a competitive ratio of at most \( 4(2 - 1/K)^{\alpha} + o(1) \).
2. For the task-wise flow time + energy problem, we first propose an algorithm that achieves the competitive ratio of at most
   \[
   P(2 - 1/K)(2\beta + 2(\alpha - 1)) + o(1),
   \]
   where
   \[
   \beta = \frac{w^t_{\max}}{w_{\text{reduce, min}}},
   \]
   and \( w^t_{\max} \) and \( w_{\text{reduce, min}} \) is the maximum size of any map/reduce task and the minimum size of any reduce task, respectively. Ideally, the competitive ratio should have no dependence on the size of the map or reduce tasks, but the competitive ratio of the proposed algorithm is a linear function of \( \beta \) (the ratio of the size of the largest map/reduce task and the shortest reduce task). For practical MapReduce applications, \( \beta \) typically small (since map tasks corresponding to any job are designed to have nearly equal size, as are reduce tasks [17]), and hence the derived guarantee is still meaningful. Moreover, from a theoretical point of view, as far as we know, no competitive ratio result is known in prior work even when \( \beta = 1 \), i.e. all tasks have equal size.
3. For the task-wise flow time + energy problem, for the power function \( P(s) = s^\alpha \), we remove the dependence of \( \beta \) on the competitive ratio next, where we show that the proposed algorithm achieves a competitive ratio of at most \( 8 + \frac{8}{\alpha} + 3^\alpha \), however, the result holds for only \( 1 < \alpha < 2 \).

Since we adapt the SRPT algorithm to work with MapReduce constraints in this paper, we next review and contrast this work with our recent prior work [31] that considers the online SRPT algorithm without precedence constraints. In [31], the competitive ratio of the SRPT algorithm with multiple servers without any precedence constraints has been shown to be upper bounded by \( P(2 - 1/m) \left( 2 + \frac{2}{\alpha} \max(1, P(s)) \right) \), where \( s \) is a constant associated with the function \( P(.) \). To compare the results derived in this paper to that of [31], we note that for the job based flow time + energy (offline) problem, adapting SRPT algorithm to work with MapReduce constraints only changes the competitive ratio by an additive \( o(1) \) term. Thus, SRPT is adaptable with MapReduce constraints and incurs a very small penalty in this case.

For the task based flow time + energy problem, compared to [31], using SRPT algorithm in the presence of MapReduce constraints there is an additional penalty of \( \beta = \frac{w^t_{\max}}{w_{\text{reduce, min}}} \) and \( w^t_{\max} \) in the competitive ratio, \( w_{\text{reduce, min}} \) is the maximum size of any map/reduce task and the minimum size of any reduce task. Typically, \( \beta = \frac{w^t_{\max}}{w_{\text{reduce, min}}} \) is small [17] for MapReduce applications, the derived competitive ratio guarantee is still meaningful and adapting SRPT algorithm with MapReduce algorithm is efficient.

In terms of technical novelty compared to [31], enforcing MapReduce constraints bring in new challenges while using the SRPT algorithm. In particular, for the job based flow time...
+ energy problem, we introduce a new job-SRPT algorithm that is clearly sub-optimal but helps in analytical tractability. In particular, we prove Lemma 1 which allows us to prove the same competitive ratio bound (upto $o(1)$ term) as in [31] even when there are arbitrary precedence constraints between different tasks of any one job and not just MapReduce constraints.

For the task based flow time + energy problem, as long as there are at least $K$ (number of servers) map tasks, one can directly use results from [31]. The main technical difficulty arises when there are less than $K$ map tasks but a large number of reduce tasks (which cannot be processed because of the precedence constraint). This presents a unique challenge in multi-server systems, that there are a large number of outstanding tasks but some of the servers are idling. Since we are considering worst case input, we have to analyze every single sample path. To deal with this case, we construct a novel potential function [9], and derive its drift in Lemma 9 and show that the competitive ratio of the SRPT algorithm in presence of the MapReduce constraints for the task based flow time + energy problem, increases by a factor of $\beta = \frac{K}{\max w_{reduce}}$ compared to the SRPT algorithm without precedence constraints [31].

To remove the dependence of the competitive ratio on $\beta$, we next consider a small tweak to the SRPT algorithm that allows multiple tasks to be processed simultaneously by a single server, and show that we can get a competitive ratio independent of the instance of the problem $\beta$ that only depends on the value of $\alpha$ where $P(s) = s^\alpha$ as long as $\alpha < 2$. The main technical contribution is the construction of potential function [19] and the analysis of its derivative $(d\Phi_t(t)/dt)$ in Appendix XIV similar to Lemma 9.

### II. System Model

We consider a MapReduce profile of job arrivals, where each job $i$ has $m_i$ map tasks and $r_i$ reduce tasks, where any of the reduce tasks of a job can be executed only after the completion of all the $m_i$ map tasks. Let the input consist of a set of jobs $\mathcal{J}$ with cardinality $J$, where job $i \in \mathcal{J}$ arrives (is released) at time $a_i$, with the work:size of its $k^{th}$ map task being $m_{ik}, k = 1, \ldots, m_i$, and that of its $\ell^{th}$ reduce task being $r_{i\ell}, \ell = 1, \ldots, r_i$. The set of tasks belonging to job $i$ is denoted by $\text{Job}_i$. The total number of map (respectively, reduce) tasks summed across all jobs is denoted by $m_m$ (respectively, $n_r$). Note that we are not assuming that the size of the map and reduce tasks are identical. The model is general, where a map or reduce task can have an arbitrary size, and derive competitive ratio guarantees that are either dependent or independent of the map/reduce task sizes.

There are $K$ homogenous parallel servers, each with the same power function $P(s)$, where $P(s)$ denotes the power consumed while running at speed $s$. Any task can be processed by any of the $K$ servers, and both preemption and task migration are allowed, which is essential with precedence constraints (since map tasks have inherently higher priority, and reduce tasks need to be migrated).

**Definition 1.** A task is defined to be free at time $t$, if it can be processed at time $t$. Thus, a map task is always free, while a reduce task is free at time $t$ if all the map tasks belonging to the same job have been completed at time $t$. A reduce task that is not free is called caged.

Let $c_{ik}$ be the time at which the task (map/reduce) $k$ of job $i$ is completed. Then, the flow time $t_{ik}$ for a map/reduce task $k$ of job $i$ is defined as $t_{ik} = c_{ik} - a_i$ (completion time minus the arrival time). We consider the two possible scenarios of counting flow times that correspond to counting the delay seen by a job, or the sum of the delays seen by all tasks within a job. For the job-by-job case, we define the flow time of job $i$ as $f_i = T_i - a_i$, where $T_i = \max_{k \in \text{Job}_i} c_{ik}$ is the time at which the last reduce task of job $i$ finishes, and total flow time is defined as $F_{job} = \sum_{i \in \mathcal{J}} f_i$. For the task-by-task case, $f_i = \sum_{k \in \text{Job}_i} t_{ik}$ and the total flow time is $F_{task} = \sum_{i \in \mathcal{J}} f_i$.

$F_{job}$ corresponds to counting per-job delay cost, while the $F_{task}$ reflects per-bit delay cost or average flow time. With both these definitions, the following expressions for the flow time are useful.

$$F_{job} = \int n(t) dt, \quad F_{task} = \int (n_f(t) + n_c(t)) dt,$$

where at time $t$, $n(t)$ is the number of outstanding jobs, while $n_f(t)$ and $n_c(t)$ is the number of outstanding free and caged tasks, respectively.

Let server $k$ run at speed $s_k(t)$ at time $t$. The energy cost is defined as $\sum_{k=1}^{K} P(s_k(t))$ integrated over the flow time. Choosing larger speeds reduces the flow time, however, increases the energy cost, and the natural objective function that has been considered extensively in the literature is the sum of flow time and energy cost, which we define as

$$C_{job} = \int n(t) dt + \int_{k=1}^{K} P(s_k(t)) dt, \quad (1)$$

$$C_{task} = \int (n_f(t) + n_c(t)) dt + \int_{k=1}^{K} P(s_k(t)) dt. \quad (2)$$

Note that we have added the two costs without weighting them, since the weight can be absorbed in the power function $P$. Any online algorithm only has causal information, i.e., it becomes aware of job $i$ only at time $a_i$. Using only this causal information, any online algorithm has to decide at what speed each server should be run at each time. Let the job arrival sequence be $\sigma = \{(a_i), (m_{ik}, r_{i\ell}), i \in \mathcal{J}\}$. For the online algorithm $\sigma$ is revealed causally, while for an offline algorithm $\sigma$ is assumed to be known in advance. Let the cost [11] of an online algorithm $A$ be $C_A^A$, and the cost for the offline optimal algorithm $OPT$ be $C_{OPT}^A$. Then the worst case competitive ratio of the online algorithm $A$ is defined as $c_A = \max_{\sigma} \frac{C_A^A(\sigma)}{C_{OPT}^A(\sigma)}$, and the objective function considered in this paper is to find an online algorithm that minimizes the worst case competitive ratio $c^* = \min_{A} c_A$.

A typical approach in speed scaling literature to upper bound the competitive ratio $c_A$ for an algorithm $A$ is via the
construction of a potential function $\Phi(t)$ and show that for any input sequence $\sigma$,

$$n(t) + \sum_{k=1}^{K} P(s_k(t)) + \frac{d\Phi(t)}{dt} \leq c_A \left( n^0(t) + \sum_{k \in \mathcal{OPT}} P(\tilde{s}_k(t)) \right),$$

whenever $\frac{d\Phi(t)}{dt}$ exists, and $\tilde{s}_k$ is the speed of server $k$ with the OPT and $n^0$ stands for the number of outstanding jobs with the OPT, and that $\Phi(t)$ satisfies the following boundary conditions.

1) Before any job arrives and after all jobs are finished, $\Phi(t) = 0$, and
2) $\Phi(t)$ does not have a positive jump discontinuity at any point of non-differentiability.

For the task based flow time, $n(t)$ is replaced by $n_f(t) + n_r(t)$ in (3). Then, integrating (3) with respect to $t$, we get that

$$\int \left( n(t) + \sum_{k=1}^{K} P(s_k(t)) \right) \leq c_A \int \left( n^0(t) + \sum_{k \in \mathcal{OPT}} P(\tilde{s}_k(t)) \right),$$

which is equivalent to showing that $C^A(\sigma) \leq c_A C^{OPT}(\sigma)$ for any input $\sigma$ as required.

**Remark 2.** Suppose that the second boundary condition is not satisfied, and let $\Phi(t)$ increase by amount $D_j$ at the $j^{th}$ discontinuous point, such that the total increase is $\sum_j D_j \leq \mathcal{D} C^{OPT}$. If (3) holds at points where $\Phi(t)$ is differentiable, then we get that $C^A(\sigma) \leq (c_A + D) C^{OPT}(\sigma)$ for any input $\sigma$, making the upper bound on the competitive ratio $c_A + D$.

One easy lower bound on the OPT’s cost can be obtained by removing the precedence constraints, and considering that the arrival sequence is such that no job/task has to wait behind any other job/task. Thus, the cost of OPT for any job/task with size $w = \min_j w_j/s + (w/s) P(s)$.

**Proposition 1.** For the OPT, $C^*_{OPT}(\sigma) \geq P^*(s^*) (\sum_{j \in \mathcal{J}} w_j)$ and $C_{OPT}(\sigma) \geq P^*(s^*) (\sum_{j \in \mathcal{J}} (m_j + r_j))$, where $1 + P^*(s^*) = s^* P^*(s^*)$.

Throughout this paper, we make the following assumptions about the power function $P(\cdot)$.

**Assumption 3.** $P: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $P(0) = 0$ is a differentiable, strictly increasing, and strictly convex function, which implies $\lim_{s \to \infty} P(s) = \infty$, and $\bar{s} := \inf\{ s > 0 \mid P(s) > s \} < \infty$. Moreover, for $x, y > 0$, $P(xy) \leq P(x)P(y)$. We also assume that $\Delta(x) = P'(P^{-1}(x)) = o(x)$ as $x \to \infty$.

**Assumption 4.** In Section III, where we consider the job-based flow time metric, we assume a natural many jobs scaling, where the total number of jobs $J$ is large, while the maximum size of any job $w_{max}$ and the numbers of servers $K$ satisfy $K w_{max} \Delta(1/K) = o(\sum_j w_j)$, which is quite a mild requirement and easily satisfied in practice. For $P(s) = s^\alpha$, this means $K^{1/\alpha} w_{max}^{1 - 1/\alpha} = o(\sum_j w_j)$. In Section IV where we consider task-based flow time metric, we assume a natural many tasks scaling where the total number of map plus reduce tasks $n_m + n_r$ is large, and $O(J w_{max}^{K^{1 - 1/\alpha}}) = o(\sum_{j \in J} (m_j + r_j))$, where $w_{max}$ denotes the size of the largest map/reduce task, and $r_{max}$ is the maximum number of reduce tasks any job can have.

We essentially need Assumption 4 to couple the increase in the potential functions at the points of discontinuities, and the total cost of the OPT.

### III. JOB BASED FLOW TIME

In this section, we consider the job-by-job flow time metric and the cost metric is $C_{job}(\cdot)$. We restrict ourselves to the case that all jobs are available at time $0$. The objective is to propose an algorithm with a constant approximation (competitive) ratio. This is typically a difficult task since there are intra-job precedence constraints, and the flow time is counted as the departure time of the last reduce task of a job minus its arrival time, coupling the processing of different tasks of the system.

To overcome this difficulty, we propose a job-by-job SRPT algorithm, called job-SRPT, as follows. For any task (map or reduce) $k$ of job $j$, let $w_{kj}(t)$ be its total remaining work at time $t$. Then the total remaining cumulative work of job $j$ at time $t$ is defined as $w_j(t) = \sum_{k \in job_j} w_{kj}(t)$ (sum of the sizes of all its maps and reduce tasks). Index the $n(t)$ outstanding jobs at time $t$ in increasing order of their remaining cumulative work. If $n(t) \geq K$, the job-SRPT algorithm processes the $K$ shortest jobs on the $K$ servers, and on each server the shortest free task of each job is executed. Thus, each server is executing a task corresponding to a different job at each time. Otherwise, if $n(t) < K$, then the $n(t)$ distinct jobs (shortest free task of that job) are executed on the $n(t)$ servers, and for the rest of $K - n(t)$ servers, the $K - n(t)$ shortest free tasks (other than the ones already executing) across all outstanding $n(t)$ jobs are executed. For the job-SRPT algorithm, we propose the following speed. For server $k$,

$$s_k(t) = \begin{cases} 
P^{-1} \left( \frac{n(t)}{K} \right) & \text{if } n(t) \geq K, \\
P^{-1}(1), & \text{otherwise} \end{cases}$$

The main result of this section for the job-based flow time + energy problem is as follows.

**Theorem 5.** Under Assumptions 3 and 4 the job-SRPT algorithm with speed scaling 4 for job based flow time problem has competitive ratio $P(2 - 1/K) \left( 2 + \frac{2}{\pi^2} \max(1, P(\bar{s})) \right) + o(1)$, where $\bar{s}$ is defined in Assumption 2.

Taking $P(s) = s^\alpha$ for $\alpha > 1$, the competitive ratio equals $4(2 - 1/K)^{\alpha} + o(1)$. It is worth noting that this competitive
ratio is same as with the no precedence constraints case in [31] with multi-server SRPT algorithm up to a $o(1)$ term.

Clearly, the job-SRPT algorithm is not optimal since one can easily construct an example where executing multiple tasks from the same job on different servers is better, however, the job-SRPT algorithm allows analytical tractability because of the following lemma.

**Lemma 1.** With MapReduce precedence constraints, when all jobs are available at time $0$, for $K$ fixed speed servers, OPT that follows the job-SRPT algorithm with server speed $(2 - 1/K)$ has a sum of job flow times that is at most that under OPT with $K$ servers having speed 1 for any job input sequence $\sigma$.

This lemma is similar to the claim for classical SRPT [26], where each job is a single autonomous entity without any precedence constraints, and follows by proving Lemma [1] and Lemma [2]. Importantly, the claim in [26] is true for classical SRPT even in the online case, when jobs are released over time. However, with precedence constraints, it is not true for the job-SRPT algorithm (Lemma [2] no longer holds) in the online case. The main reason is that, if at any time there is a single job with multiple tasks that are being processed simultaneously, an arrival of a new job does not guarantee that the number of jobs departed by any future time is as small as many as the ones would have departed if the new job would not have arrived.

How Lemma [1] helps is as follows. In light of Lemma [1], we can let OPT execute the job-SRPT algorithm with an arbitrary unknown speed and claim that it is $P(2 - 1/K)$-approximate with respect to OPT (executes arbitrary scheduling and speed choice). Note that OPT will process all tasks of a job at the same speed because of convexity of $P$. Thus, for a job $j$, let the speed OPT uses be $s_j$. Then for OPT following job-SRPT, the speed of job $j$ (whenever any task of job $j$ is scheduled) is chosen to be $(2 - 1/K)s_j$. Thus, OPT following job-SRPT will have at most the same flow-time as the OPT from Lemma [1] with an additional energy cost of $P(2 - 1/K)$. Thus, OPT following job-SRPT is $P(2 - 1/K)$-approximate with respect to OPT.

For the rest of the section, we call OPT following job-SRPT as OPT. The main advantage of considering job-SRPT for both the algorithm and the OPT, is that we can get rid of all the precedence constraints within a job, since at any time only the $K$ distinct shortest (cumulative) jobs are being processed if the number of outstanding jobs is at least $K$. Since all jobs are available at time 0, the case that multiple tasks belonging to the same job are processed simultaneously occurs only when the number of outstanding jobs is less than $K$, which can be handled separately.

Using job-SRPT algorithm for scheduling, we use the same potential function that has been proposed in [31] to analyze multi-server SRPT algorithm. Recall that the remaining cumulative size of any job be the sum of the remaining sizes of its map and reduce tasks. At time $t$, let $n^o(t, q)$ and $n(t, q)$ denote the number of unfinished jobs with the OPT and the algorithm, respectively, with remaining cumulative size at least $q$. In particular, $n^o(t, 0) = n^o(t)$ and $n(t, 0) = n(t)$. Let

$$d(t, q) = \max \{0, n(t, q) - n^o(t, q)\}. $$

Let $f : \{i/K : i \in \mathbb{Z}_+\} \to \mathbb{R}_+$ is defined as follows: $f(0) = 0$, and for $i \geq 1$, $f(i/K) = f((i-1)/K) + \Delta(i/K)$, where $\Delta(x) := P(P^{-1}(x))$. We consider the potential function

$$\Phi(t) = \Phi_1(t) + \Phi_2(t),$$

$$\Phi_1(t) = c_1 \int_0^t f \left( \frac{d(t, q)}{K} \right) dq,$$

$$\Phi_2(t) = c_2 \int_0^\infty (n(t, q) - n^o(t, q))dq,$$

where $c_1, c_2$ are positive constants whose value will be specified later. $\Phi_1$ part is sufficient for the case when the $n(t) \geq K$, while $\Phi_2$ is needed to handle the case when $n(t) < K$.

**Remark 6.** As long as there are $K$ distinct jobs with both the algorithm and the OPT, because of processing by the algorithm, $d(t+dt, q) = d(t, q) - 1$ for $q \in [q_k - s_k(t)dt, q_k]$, where $q_k, k = 1, \ldots, K$ is the remaining (cumulative) size of the shortest $K$ jobs. Similarly, for the OPT, $d(t, q) = d(t, q) + 1$ for $q \in [q_k^* - s_k(t)dt, q_k^*]$, for $q_k^*, k = 1, \ldots, K$. Thus, one can use the results from [31] (Lemma [3]) directly when both the algorithm and the OPT have $K$ distinct jobs, to bound the drift $d\Phi_1/dt$, that are valid for the case when each job has a single task, and no job can be split and processed parallely on multiple servers. When either the algorithm or the OPT has less than $K$ jobs available, the job-SRPT algorithm, may process more than one task belonging to the same job simultaneously on multiple servers and [31] (Lemma [3]) is not directly applicable. Since the algorithm’s contribution to the drift $d\Phi_1/dt$ is negative, we can just count one term $d(q) = d(q) - 1$ for $q \in [q_k - s_k(t)dt, q_k]$ for each of the distinct jobs and disregard the contribution in drift $d\Phi_1/dt$ from multiple tasks of a same job that are being processed simultaneously. This, however, cannot be done for the OPT, since OPT’s contribution is positive in drift $d\Phi_1/dt$, and to handle this case, we introduce deletion of tasks for the OPT (defined next) which will make $\Phi_1$ and $\Phi_2$ increase discontinuously, and we will count the total increase in $\Phi_1(t)$ and $\Phi_2(t)$ at all points of discontinuities.

**Deleting tasks with the OPT:** Let the time instant at which OPT has $k$ outstanding jobs be $t_k$, where $k = K - 1, K - 2, \ldots, 1$. Note that $t_k$’s need not be distinct. Let at time $t_k$, the set of jobs $j$ for which the OPT is about to process more than one task parall ely be $S_k$. Then at $t_k$, for all jobs $j \in S_k$, we delete all its reduce tasks and all the map tasks other than its shortest map task. This will ensure that there is at most one task of each job being processed by any of the servers throughout, allowing the applicability of Lemma 3 [31]. The deletion of tasks will result in a total upward jump in $\Phi_1(t)$ and $\Phi_2(t)$ which can be bounded as follows (proof in Appendix [X]).

**Lemma 2.** The total upward jump in the potential function $\Phi_1(t)$ and $\Phi_2(t)$ because of deletion of tasks with the OPT is $\sum_{j=1}^{K-1} c_1 w_j \Delta(j/K) + \sum_{j=1}^{K-1} c_2 w_j$, where $w_j$ are the sizes of
potentially the $K - 1$ largest (cumulative) size jobs, and $J$ is the total number of jobs in the input.

Next, we state Lemma 3 and Lemma 4 from [31] that allows us to bound the derivative of $\Phi_1(t)$ and $\Phi_2(t)$ as long as $\text{OPT}$ does not process more than one task belonging to the same job on multiple servers, and we count contribution of only distinct jobs of the algorithm in $d\Phi_1(t)/dt$ and $d\Phi_2(t)/dt$.

**Lemma 3.** When $\text{OPT}$ does not process more than one task belonging to the same job simultaneously on multiple servers, for $n \geq K$, $d\Phi_1(t)/dt \leq c_1 n^\epsilon - c_1 + c_1 \left( \frac{K}{2} \right) + c_1 \sum_{k \in \text{OPT}} P(\bar{s}_k)$, while for $n < K$, $d\Phi_1(t)/dt \leq c_1 n^\epsilon - c_1 \frac{n \ln(n)}{2K} + c_1 \sum_{k \in \text{OPT}} P(\bar{s}_k)$.

**Lemma 4.** $d\Phi_2(t)/dt \leq -c_2 \min(K, n) P^{-1}(1) + c_2 \sum_{k \in \text{OPT}} \max\{P(\bar{s}), P(\bar{s}_k)\}$.

Now, we prove Theorem 5 from Proposition 1 note that the minimum cost $C_{\text{job}}$ of $\text{OPT}$ is at least $P(s^*) \sum_{j=1} w_j$. From Lemma 2, we have that the total jump size at all points of discontinuities is $\sum_{j=1} c_1 w_j \Delta(\frac{1}{x}) + \sum_{j=1} c_2 w_j$. From Assumption 3, $\Delta(\frac{1}{x}) = o(J/K)$, e.g., for $P(s) = s^\epsilon$, $\Delta(x) = O(x^{\epsilon - 1/2})$. Thus, $D = o(1)$ from Assumptions 4 and Remark 2. Thus, if we can show that 3 holds for some $c$, we will have that the total competitive ratio is upper bounded by $c + o(1)$.

From Remark 6, we ensure that the $\text{OPT}$ never processes more than one task belonging to the same job, thus applying, Lemma 3 and 4 (similar to [31]) we get (3)

$$n + \sum_{k \in A} P(s_k) + d\Phi(t)/dt = n + n + 1 + d\Phi(t)/dt,$$

$$\leq c(n^\epsilon + \sum_{k \in \text{OPT}} P(\bar{s}_k)),$$

is true for $c = \left(2 + \frac{2}{\epsilon} \max(1, P(\bar{s}))\right)$, where $c_1 = 2$, $c_2 = 2/P^{-1}(1)$.

Discussion: By introducing the job-SRPT algorithm, we have effectively removed the precedence constraints within each job. By processing as many distinct jobs, the job-SRPT policy ensures that there is sufficient drift available for the choice of the potential function, and allows leveraging of the recent analysis of the multi-server SRPT algorithm [31]. It is worth noting that the analysis of this section (Theorem 5) holds even when the precedence constraints are arbitrary within each job.

**IV. TASK BASED FLOW TIME**

In this section, we consider the task-by-task flow time where the cost metric is $C_{\text{task}}$ [2]. We consider the online setting, where jobs arrive over time arbitrarily, and the objective is to propose an online algorithm and bound its competitive ratio. The competitive ratio guarantee of our proposed online algorithm will be a function of $\beta = \frac{w_{\text{max}}}{w_{\text{reduce}, \text{min}}}$, where $w_{\text{max}}$ and $w_{\text{reduce}, \text{min}}$ is the maximum size of any map/reduce task and the minimum size of any reduce task, respectively. Since for the MapReduce framework, this ratio is typically small, the guarantees are meaningful [17]. Moreover, even when $\beta = 1$, the problem is non-trivial, since how to construct potential function for the case when the number of map tasks is smaller than the number of servers but there are large number of caged reduce jobs is not clear, and to the best of our knowledge no competitive ratio guarantees are known in the online case.

$\text{OPT}$: Throughout this section, we assume that the $\text{OPT}$ has no precedence constraints across tasks, and can process any of its outstanding tasks at any time. This can only decrease the $\text{OPT}$‘s cost. Moreover, we assume that without precedence constraints, $\text{OPT}$ follows the multi-server SRPT algorithm, i.e., among its all outstanding tasks $n^\epsilon$, it processes the $\min\{n^\epsilon, K\}$ shortest tasks (map/reduce since there are no precedence constraints for $\text{OPT}$) on the $\min\{n^\epsilon, K\}$ servers at any time. Since the multi-server SRPT algorithm that has access to future arrivals and can choose arbitrary speed depending on it is $P(2 - 1/K)$-competitive with respect to $\text{OPT}$ with no precedence constraints [31], we will get an additional factor of $P(2 - 1/K)$ in the final competitive ratio.

We need the following definition to describe our algorithm.

**Definition 7.** For a map task $j$ belonging to job $i$, define its load as $l_j = r_j + 1$, the number of reduce tasks in the job $i$ that it precedes plus 1 for itself. For a map task $j$, $b_j(t)$ be the number of (brother map tasks) outstanding map tasks belonging to the same job as task $j$ with the algorithm ($\text{OPT}$) at time $t$. Moreover, let $b(t) = \min\{b(t), K\}$ and $z_j(b_j(t)) = t_j + b_j(t)$.

**Algorithm:** Let the total number of outstanding tasks at time $t$ be $n(t)$ out of which $n_j(t)$ are free, i.e., $n_j(t)$ is the sum of all the map tasks and all the free reduce tasks. The algorithm processes the $\min\{n_j(t), K\}$ shortest free tasks on the $\min\{n_j(t), K\}$ servers. If $n_j(t) \geq K$, then all servers process at speed $P^{-1}(\frac{n(t) + 1}{K})$. Otherwise, server $k$ has speed $s_k(t) = \left(\begin{array}{l} P^{-1}(t_j + b_j(t) + 1) \text{ if } k \text{ is processing map task } j, \\ P^{-1}(1) \text{ if } k \text{ is processing a free reduce task}. \end{array}\right)$ (8)

The main result of this section is as follows.

**Theorem 8.** The competitive ratio of the proposed online algorithm is $P(2 - 1/K)(2\beta + 2(2\alpha - 1)) + o(1)$, where $\beta = \frac{w_{\text{max}}}{w_{\text{reduce}, \text{min}}}$. Next, we work towards proving Theorem 8. At time $t$, let $n^\epsilon(t, q)$ and $n(t, q)$ denote the number of unfinished tasks (map + reduce) under the $\text{OPT}$ and the algorithm, respectively, with remaining size at least $q$. In particular, $n^\epsilon(t, 0) = n^\epsilon(t)$ and $n(t, 0) = n(t)$. Let $d(t, q) = \max\{0, n(t, q) - n^\epsilon(t, q)\}$. We consider the potential function

$$\Phi(t) = \Phi_1(t) + \Phi_2(t) + \Phi_3(t),$$

$$\Phi_1(t) = c_1 \int_0^t f(\frac{d(t, q)}{K}) dq,$$

$$\Phi_2(t) = c_2 \int_0^t (n(t, q) - n^\epsilon(t, q))dq,$$

where $c_1, c_2$ are positive constants and function $\Phi_3$ which is specially designed for handling the precedence constraints
Lemma 8. [Lemma 3.1 in [4]] Let $P$ satisfy Assumption 3. Then for $s_k, \tilde{s}_k, x \geq 0$,

$$
\Delta(x)(-s_k + \tilde{s}_k) \leq (-s_k + P^{-1}(x)) \Delta(x) + P(\tilde{s}_k) - x.
$$

Let $n_{f}(t)$ be the free reduce tasks at time $t$, and let $\tilde{n}(t) = n(t) - n_{f}(t) (∩ n^o(t))$ be the number of outstanding tasks other than the free reduce tasks at time $t$ with the algorithm (OPT). The main new result we need with precedence constraints is as follows, where $1_{\{e\}} = 1$ if event $e$ is true and zero otherwise.

Lemma 9. $d\Phi_3/dt \leq -c_3 \mathbf{1}_{\{n_f < K\}} \tilde{n}(t) + c_3(\alpha - 1)(\tilde{n}^o(t)) + c_3 \sum_{k=1}^m P(\tilde{s}_k)$.

Proof. For notational simplicity, we suppress the time index, since we are considering a fixed time $t$. Consider the change in $\Phi_3$ due to $OPT (h^o_j(q) - h^o_j(q) - 1)$ for at most $m^o$ map tasks in the interval $[q^o_j, q^o_j - \tilde{s}_j dt]$ where $q^o_j$ is the remaining size of the $j^{th}$ map task with the OPT:

$$
d\Phi_3 \leq -c_3 \sum_{j=1}^m \left[ \int z_j^{o}(\hat{b}^{o}, \hat{t}) h_j^{o}(q^o_j) \right] dt \cdot \tilde{s}_j dt,
$$

$$
= c_3 \sum_{j=1}^m \Delta \left( \frac{\hat{\ell}_j + \hat{b}_j}{\hat{b}_j^2} h_j^{o}(q^o_j) \right) \tilde{s}_j dt,
$$

$$
\leq c_3(\alpha - 1)(\tilde{n}o(t)) + c_3 \sum_{k=1}^m P(\tilde{s}_k),
$$

since $h^o_j(q^o_j) = n^o_j = 1$. Using Lemma 8 individually on the $m^o$ terms of (10) with $s_k = 0$, and noting that $P^{-1}(x) \Delta(x) = (\alpha - 1)x$, we get

$$
d\Phi_3 \leq -c_3(\alpha - 1) \sum_{j=1}^m \left( \frac{\hat{\ell}_j + \hat{b}_j}{\hat{b}_j^2} \right) + c_3 \sum_{j=1}^m P(\tilde{s}_k),
$$

Since $\tilde{s}_k$ is the speed of server $k$ with the OPT.

Lemma 6. [Lemma 10] For $n_f \geq K, d\Phi_1/dt \leq c_3 n^o - \frac{c_3}{2} n + c_3(\hat{\ell}_j + \hat{b}_j) + c_1 \sum_{k=OPT} P(\tilde{s}_k)$, while for $n_f < K$,

$$
d\Phi_1/dt \leq c_3 n^o - \frac{c_3 n^o}{2} K + c_1 \sum_{k=OPT} P(\tilde{s}_k).
$$

Lemma 7. [Lemma 11] $d\Phi_2/dt \leq -c_2 \min(K, n) P^{-1}(1)$, where $P$ is the optimal policy.

Next, we quantify the more non-trivial drift of $\Phi_3(t)$ because of the processing by the algorithm and the OPT. We will need the following result from [4].
algorithm. Thus,

$$
\frac{d\Phi_3}{dt} = c_3 \sum_{j=1}^{n_m} \left[ g_{z_j(b)} \left( z_j(b) (h_j(q_j) - 1) \right) s_j dt, \right.
$$

$$
- c_3 \sum_{j=1}^{n_m} \Delta \left( \ell_j + \hat{b}_j \right) h_j(q_j) b_j,
$$

$$
= - c_3 \sum_{j=1}^{n_m} \Delta \left( \ell_j + \hat{b}_j \right) b_j,
$$

$$
= - c_3 \sum_{j=1}^{n_m} \Delta \left( \ell_j + \hat{b}_j \right) b_j,
$$

$$
\text{since } h_j(t, q_j) = 1. \text{ Using Lemma 8 individually on each of the } n_m \text{ terms with } \hat{s}_k = 0, \text{ we get}
$$

$$
\frac{d\Phi_3}{dt} \leq c_3 \sum_{j=1}^{n_m} \left( -s_j + P^{-1} \left( \ell_j + \hat{b}_j \right) \right) \Delta \left( \ell_j + \hat{b}_j \right)
$$

$$
= - c_3 \sum_{j=1}^{n_m} \left( \ell_j + \hat{b}_j \right) b_j,
$$

$$
\text{(a)} = - c_3 \sum_{j=1}^{n_m} \left( \ell_j + \hat{b}_j \right) b_j,
$$

$$
\text{(b)} = - c_3 \sum_{k=1}^{\lvert J \rvert} \sum_{j=1, j \in J_k}^{n_m} \left( \ell_j + \hat{b}_j \right) b_j,
$$

$$
= c_3 \hat{n}(t),
$$

(14)

where (a) follows since the speed of map task \( j \) is \( s_j = P^{-1} \left( \ell_j + \hat{b}_j(t) + 1 \right) \) making the first term \( \leq 0 \), and in (b) we have separated the contributions from the \( \lvert J \rvert \) outstanding tasks, and \( J_k \) represents the \( k^{th} \) job which has \( b_j + \ell_j \) tasks for \( j \in J_k \), and the final inequality follows since the total number of map tasks is \( \sum_{j=1}^{n_m} b_j \leq K - 1 \). Combining (11) and (14), we get the result. \[\]
A. Single Server Case

First we consider the single server case, since this itself poses new challenges with the precedence constraints. The cumulative (remaining) map size of a job is defined as the sum of the sizes of all the (remaining) map tasks in a job.

Algorithm: We propose the following algorithm, that defines which tasks should be executed and at what speed. At time $t$, let the number of outstanding map tasks (summed across different jobs) be $m(t)$, the number of free reduce tasks be $r_f(t)$, and the number of jobs with at least one unfinished map task be $J(t)$. The algorithm will process (up to) three tasks simultaneously.

- $M_1$: The shortest remaining map task among all the outstanding map tasks with the algorithm at speed $s_m(t) = P^{-1}(m(t) + 1)$.
- $M_2$: The shortest remaining map task of the job that has the smallest remaining cumulative map size among all the outstanding jobs at speed $s_j(t) = P^{-1}(rJ(t) + 1)$.
- $R_f$: The shortest remaining free reduce task among all the outstanding free reduce tasks at speed $s_f(t) = P^{-1}(r_f(t) + 1)$.

Thus, the total speed is $s(t) = s_m(t) + s_j(t) + s_f(t)$. Note that $M_1$ could be equal to $M_2$ in which case only two tasks are processed simultaneously.

For the algorithm, let $m(q)$, $J(q)$, and $r(q)$ denote the number of unfinished map tasks, jobs with at least one unfinished map task, and the number of unfinished reduce tasks (free or caged) with remaining cumulative size of at least $q$, respectively. The quantities for the OPT are denoted with a superscript $o$. Let $d_m(q) = \max \{0, m(q) - m^o(q)\}$, $d_J(q) = \max \{0, r(J(q) - J^o(q))\}$, and $d_r(q) = \max \{0, r(q) - r^o(q)\}$. We consider the potential function

$$\Phi(t) = \Phi_1(t) + \Phi_2(t) + \Phi_3(t),$$

$$\Phi_1(t) = c_1 \int_0^\infty f(d_m(q)) dq,$$

$$\Phi_2(t) = c_2 \int_0^\infty f(d_J(q)) dq,$$

$$\Phi_3(t) = c_3 \int_0^\infty f(d_r(q)) dq,$$

and $\Phi_3(t) = c_3 \int_0^\infty f(d_r(q)) dq$, where $c_1$, $c_2$, $c_3$ are positive constants to be determined later.

The motivation to consider processing three tasks simultaneously is closely tied with the choice of potential function $\Phi(t)$, where we need the total drift (derivative of the potential function $\Phi$) to decrease at a rate $-cn(t)$, where $n(t)$ is the total number of outstanding tasks with the algorithm. Because of the precedence constraints, the algorithm is not necessarily allowed to process the shortest task (which can be a caged reduce task), in contrast to the no-precedence constraint case as in [4]. Thus, to get sufficient drift, we combine the drifts from processing three jobs, $M_1$ gives $-m(t)$, $M_2$ gives $-rJ(t)$ and $R$ gives $-r_f(t)$. Since $m(t) + rJ(t) + r_f(t) \geq n(t)$, we get sufficient drift.

Following this approach, the main result of this section is as follows.

Theorem 9. For the single server case, for $\alpha < 2$, the proposed algorithm has a competitive ratio of $8 + \frac{1}{2\alpha} + 3^\alpha$. The main ingredients in proving Theorem 2 are the following three Lemmas (proofs in Appendix), where $\tilde{s}(t)$ is the speed of the server with the OPT at time $t$.

Lemma 10. $d\Phi_1/dt \leq c_1 P(\tilde{s}(t) - c_1(m(t) - m^o(t))$.

Lemma 11. $d\Phi_2/dt \leq c_2 P(\tilde{s}(t) - c_2(\alpha - 2)rJ(t) + c_2(2 - \alpha)r^o(t))$.

Lemma 12. $d\Phi_3/dt \leq c_3 P(\tilde{s}(t) - c_3(r_f(t) - (\alpha - 1)r(t) - (2 - \alpha)r^o(t))$.

Next, we prove that (3) holds for some $c$ to prove Theorem 9 using Lemma 10, 11, 12.

Proof of Theorem 9. We only describe the computation when $m(t) \geq m^o(t)$, $J(t) \geq J^o(t)$ and $r(t) \geq r^o(t)$, since otherwise result follows easily. Recall that the total number of outstanding jobs with the algorithm is $m(t) + rJ(t) + r_f(t)$, and the total power used by the algorithm is $P(\tilde{s}(t)) = P(P^{-1}(m(t) + 1) + P^{-1}(rJ(t) + 1) + P^{-1}(r_f(t) + 1)) \leq 3^{\alpha - 1}(m(t) + rJ(t) + r_f(t) + 3)$, using Lemma 14 since $P^{-1}(x) = x^{\alpha}/\alpha$ with $n = 3$ and $\alpha \geq 1$. Thus, the running cost $n(t) + P(\tilde{s}(t)) + d\Phi/dt \leq (3^{\alpha - 1} + 1)(m(t) + rJ(t) + r_f(t)) + 3^\alpha + (c_1 + c_2 + c_3)P(\tilde{s}(t)) - c_1(m(t) - m^o(t)) - c_2(\alpha - 2)rJ(t) + c_2(2 - \alpha)r^o(t) - c_3(r_f(t) - (\alpha - 1)r(t) - (2 - \alpha)r^o(t)) \leq c(m^o(t) + rJ^o(t) + r^o(t) + P(\tilde{s}(t))) \leq c(n_o(t) + P(\tilde{s}(t)))$.

B. Multiple Server Case

In this section, we consider the multi-server version of the problem, where there are $K$ homogenous servers each with identical power function $P$. Except this change, everything else remains the same about the model as described in Section 2.4A. In the presence of precedence constraints, speed scaling with multiple servers pose the following new challenge. Let the number of map plus free reduce jobs be less than $K$, while the number of caged reduce jobs be large. In this case, even though there are a large number of outstanding jobs, some of the servers have to idle, making the construction of the potential function a difficult task.

Algorithm: With multiple servers, we propose the following algorithm, that defines which tasks should be executed and at what speed. At time $t$, let the number of outstanding map tasks (summed across different jobs) be $m(t)$, the number of free reduce tasks be $r_f(t)$, and the number of jobs with at least one unfinished map task be $J(t)$.

- $M_1$'s: The $\min\{m(t), K\}$ shortest remaining map tasks among all the outstanding map tasks available with the algorithm at speed $s_m(t) = P^{-1}\left(\min\left\{\frac{m(t) + 1}{K}, 1\right\}\right)$ are processed on any $\min\{m(t), K\}$ servers.
\[ M_2' \text{ jobs: The } \min\{J(t), K\} \text{ jobs that have the smallest remaining cumulative size of all the outstanding map tasks at speed } s_{cm}(t) = P^{-1} \left( \min \left\{ \frac{r(t)+1}{K}, r \right\} \right) \text{ are processed on any } \min\{m(t), K\} \text{ servers, where the shortest map task of each job is processed.} \]

\[ R_1' \text{ jobs: The } \min\{m(t), K\} \text{ shortest remaining free reduce tasks among all the free reduce tasks at speed } s_f(t) = P^{-1} \left( \min \left\{ \frac{r_f(t)+1}{K}, 1 \right\} \right) \text{ are processed on any } \min\{m(t), K\} \text{ servers.} \]

Thus, the algorithm will process (upto) three tasks on each of the \( K \) servers simultaneously, where the speed of any server is \( s(t) = s_m(t) + s_{cm}(t) + s_f(t) \). For any server, \( M_1 \) could be equal to \( M_2 \) for some jobs in which case only two tasks are processed on each server simultaneously.

**Theorem 10.** For the multi-server case, for \( \alpha < 2 \), the proposed algorithm has a competitive ratio of \( 8 + \frac{4}{2-\alpha} + 3^\alpha \).

Proof is provided in the Appendix XIV where the novelty over the single server case is the drift \( d \Phi_1(t)/dt \) that is needed only when \( K > 1 \). This distinction is required since when number of jobs is less than \( K \), but there are a large number of caged reduce jobs, all jobs of \( J(t) \) are being processed by the algorithm, thus giving the negative drift of \( -\sum_{j \in J(t)} \hat{A}(r) s_j(t) \) where \( s_j(t) \) is the speed at which each of the jobs’ shortest map task is being executed.

**VI. SIMULATIONS**

In this section, we present simulation results for both the job-based flow time + energy and task-based flow time + energy problem. Since the OPT is unknown, for benchmarking the performance, we remove the MapReduce constraints, and use the same speed scaling choice as that of the algorithms presented in the paper. For the task-based flow time + energy problem, without the MapReduce constraints the algorithm is the same the multi-server SRPT algorithm with speed choice prescribed in [31], that is constant competitive with respect to the OPT. Thus, our benchmark is meaningful. For the job-based flow time + energy problem, without the MapReduce constraints, there is no guarantee known on the Job-SRPT algorithm as far as we know, or for that matter any algorithm.

For all the plots, we use \( \alpha = 2.5 \), and number of servers \( K = 5 \). In the simulation setup, we consider a slotted time where in each slot, the number of jobs arriving is Poisson distributed with mean \( \lambda \). For each job, the number of map and reduce tasks are Poisson distributed with mean 2 and 2 (reasonable with \( K = 5 \)), and the size of a map job and a reduce job is exponentially distributed with mean 3 and 5, respectively. For each iteration, we generate jobs for 1000 slots, and count its flow time + energy, and iterate over 1000 iterations. The performance of the two algorithms is compared for the same realizations of the random variables, and then averaged over iterations.

In Fig. 1 we plot the comparison of the job-based flow time + energy with and without MapReduce constraints as a function of \( \lambda \), where even though in theory we consider that all jobs are available at time 0, in simulation we let jobs arrive over time. Clearly, the ratio of the average costs is at most 5. Moreover, from simulations we observed that even the maximum ratio of the two costs remain below 5. Similarly, in Fig. 2 we plot the comparison of the task-based flow time + energy with and without MapReduce constraints as a function of \( \lambda \), and observe similar behaviour.

Recall that the competitive ratio bound for the proposed algorithm for the task-based flow time + energy problem is a function of the ratio of the largest map/reduce task and the smallest reduce task \( \beta \). To illustrate that, in Fig. 3 we plot the task-based flow time + energy with and without MapReduce constraints with increasing sizes of map tasks while holding the size of the reduce tasks constant. In particular, we let the size of the reduce tasks to be exponentially distributed with mean 1, and the size of the map tasks to be exponentially distributed with variable mean \( \mu_m \). For Fig. 4 the number of jobs arriving is Poisson distributed with mean .5, and for each job, the number of map and reduce slots are Poisson distributed with mean 2 and 5, respectively. We see that unlike the theoretical results, the competitive ratio between the algorithm with and without enforcing the MapReduce constraints, does not increase with the ratio of the largest map/reduce task and the smallest reduce task.

**VII. CONCLUSIONS**

In this paper, we have made progress in deriving online algorithms for scheduling and speed scaling jobs with MapReduce precedence constraints. Without the precedence constraints there is a large body of literature on efficiently solving the flow time plus energy problem, however, very little is known with precedence constraints, that is inherently a hard problem. The algorithms we proposed followed a simple SRPT policy, (process as many shortest jobs/tasks as possible) and the total power used by all the servers is equal to the total
number of outstanding tasks, which has an intuitive appeal in the sense that it balances the energy and the delay cost effectively. The derived competitive ratio guarantee depends on the power function $P$, and/or the ratio of the size of the largest map/reduce task and the smallest reduce task, that is typically small for MapReduce applications.

Fig. 2. Comparison of task-based flow time + energy with and without precedence constraints.

Fig. 3. Comparison of task-based flow time + energy with and without precedence constraints as a function of size of map task being exponentially distributed with mean $\mu_m$, while the size of map task is exponentially distributed with mean $\mu_r = 1$.

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Let $A(j, t)$ denote the amount of work (total volume of jobs processed) completed by a job-work conserving algorithm $A$ for job $j$ by time $t$. For set of jobs $J$, $A(J, t) = \sum_{j \in J} A(j, t)$.

**Lemma 13.** For a job arrival sequence $\sigma$, at any time $t$, let $1 \leq \rho \leq (2 - 1/K)$, and $\gamma = \frac{2 - 1/K}{\rho}$. For any job-work conserving algorithm $A$ with $K$ servers having speed $\gamma$, $A(\sigma, t) \geq A'(\sigma, t)$ for any algorithm $A'$ (including OPT) using $K$ speed 1 servers.

Thus, the amount of work completed by any job-work conserving algorithm by time $(2 - 1/K)t$ is at least as much as work done by OPT by time $t$. The proof is identical to [26], that proves an identical result when each job has a single task.

The key idea that makes this work is that the algorithm is job-work conserving, to ensure that the algorithm processes as many distinct jobs as possible at any given time. Clearly, it does not hold if an algorithm processes more than one task of a job on multiple servers simultaneously and provides no processing for any task of an outstanding job.

With all jobs being available at time $0$, the input sequence $\sigma$ is essentially the set of jobs (that consists of map and reduce tasks). For the job-SRPT algorithm the following property is easy to prove.

**Proposition 2.** If the number of jobs finished completely by job-SRPT algorithm is $k$ by time $t$ with input sequence $\sigma'$. Then the number of jobs finished completely by job-SRPT algorithm is at least $k$ by time $t$ with input sequence $\sigma''$, where $\sigma' \subseteq \sigma''$.

**Proof of Lemma 7** First note that the job-SRPT algorithm is a job-work conserving algorithm. Let $\sigma' \subseteq \sigma''$ be the set of jobs that the OPT completely finishes by time $t$ when the input sequence is $\sigma$. Therefore from Lemma [13] job-SRPT algorithm finishes all $|\sigma_t|$ jobs by time $(2 - 1/K)t/\gamma$ given speed $\gamma$ for each server if the input sequence is $\sigma_t$. Now using Lemma [2] we let $\sigma_t = \sigma'$ and $\sigma = \sigma''$ to conclude that for any time $t$, at least $|\sigma_t|$ jobs will be completed by time $(2 - 1/K)t/\gamma$ with the job-SRPT algorithm with input sequence $\sigma$. This implies that for any $k$, the completion time of the $k^{th}$ job with job-SRPT algorithm is no later than the $(2 - 1/K)/\gamma$ times the completion time of the $k^{th}$ job with the OPT for any job arrival sequence $\sigma$. Recall that the flow time of job $j$ is $F_j = T_j - a_j$, where $T_j$ is the departure time of the last reduce task of job $j$ and $a_j$ is the arrival time of the job $j$. Note that the order of departure of jobs with the job-SRPT algorithm and OPT might be different, but since flow time is $\sum_{j \in \sigma} (T_j - a_j)$ it is sufficient to show that $\sum_{j \in \sigma} F_j(\text{OPT}) = \sum_{j \in \sigma} F_j(\text{job-SRPT})$ to claim that $F_j(\text{OPT}) = F_j(\text{job-SRPT})$. Choosing $\gamma = 2 - 1/K$, we get that $\sum_{j \in \sigma} F_j(\text{OPT}) = \sum_{j \in \sigma} F_j(\text{job-SRPT})$ and the claim follows. □

**IX. Proof of Lemma 2**

Proof. Let the total size of a job $j$ be $w_j$ which is the sum of the sizes of all its map and reduce tasks. Recall that map/reduce tasks of at most the last $K - 1$ jobs among the total $J$ are deleted with the OPT that is executing job-SRPT algorithm. Thus, on any map/reduce tasks deletion of job $j$,
the increase in $d(t, q) \leq 1$ for $q \leq w_j$. Thus, the increase in $\Phi_1(t)$ is at most $c_1 \int_0^{w_j} f(d(t, q)) dq - c_1 \int_0^{w_j} d(t, q) dq \leq c_1 \int_0^{w_j} \Delta \left( \frac{1}{K} \right) dq \leq c_1 \Delta \left( \frac{1}{K} \right) w_j$. Since this happens for at most $K - 1$ jobs, the total increase in $\Phi_1(t)$ at points of discontinuities is $\sum_{j=1}^{K-1} c_1 w_j \Delta \left( \frac{1}{K} \right)$. The increase in $\Phi_2(t)$ because of deletion of map and reduce task of the $j$th among the last $K - 1$ jobs is simply $c_2 \int_0^{w_j} 1 dq \leq c_2 w_j$. Since this happens for at most $K - 1$ jobs, the total increase in $\Phi_2(t)$ at points of discontinuities is $\sum_{j=1}^{K-1} c_2 w_j$.}

\[ \Box \]

X. PROOF OF LEMMA 5

Proof. On an arrival of a new job, the new addition to the two terms of $\Phi_3(t)$ corresponding to the new map tasks is identical, thus keeping $\Phi_3(t)$ unchanged.

Whenever a map task $j$ is completed by the algorithm or OPT, $n(t, q)$ or $n_j(t, q)$ is changed for only a single point of $q = 0$, keeping the integral unchanged, and the term corresponding to map task $j$ vanishes continuously in $\Phi_3$, and there is no discontinuity. Map task $j$’s departure with the algorithm (OPT), however, reduces $b_k(t)$ by 1 for map task $k$ that belong to the same job as map task $j$ if $b_k(t) \leq K$. If a map task departs with the OPT, the change in $\Phi_3$ is negative, and we disregard this change, and only consider the increase because of a map task departure with the algorithm. For a job $i$, departure of its at most $K$ map tasks with the algorithm result in discontinuities (since $b_k(t) \leq K$ for any discontinuity to arise), where for the $k$th discontinuity, there are at most $k$ map tasks of job $i$ remaining in the system. Thus, the increase in $\Phi_3(t)$ for any such discontinuity is bounded as follows. Thus, only when $b_k(t) \leq K$,

\[ \Phi_3(t^+) - \Phi_3(t) = \sum_{k=1}^{b_k(t)} c_3 \delta_k, \]

where

\[ \delta_k = \int g_{z_k(t, q)} (z_k(t, q)) dq, \]

\[ - c_3 \int g_{z_k(t, q)} (z_k(t, q)) dq, \]

\[ \leq \Delta \left( \frac{\ell_j + b_k(t)}{b_k(t)} \right) \int_{w_j}^{w_j^t} 1 dq, \]

\[ = \Delta \left( \frac{\ell_j + b_k(t)}{b_k(t)} \right) w_j^t, \]

where (a) follows by dropping the second term, and using the definition of $f$ and fact that $h_k(t, q) \leq 1$ for all $q \leq w_j$, where $w_j^t$ is an upper bound on the size of any map task.

Thus, for any one map task departure for a fixed value of $b_k(t)$,

\[ \Phi_3(t^+) - \Phi_3(t) = c_3 w_j^t b_k(t) \Delta \left( \frac{\ell_j + b_k(t)}{b_k(t)} \right), \]

where $\Delta(x) = o(x)$ by Assumption 3. In particular, $\Delta(x) = ax^{1-1/\alpha}$ for $P(s) = s^{\alpha}$. Since at most $K$ such map tasks departures corresponding to $b_k(t)$ give rise to discontinuities, counting for all $K$ such events we have that for any one job, the maximum increase in $\Phi_3(t)$ is $c_3 w_j^t \sum_{j=1}^{K} \frac{b_k(t)}{b_k(t)} = O(c_3 w_j^t K^2 \Delta(\ell_j + 1)).$

\[ \Box \]

XI. PROOF FOR REMAINING CASE OF THEOREM 8

When the algorithm has $n_f < K$ and $n_m = 0$, i.e., there less than $K$ free tasks and all of them are of reduce type, i.e., $n = n_f$, we only count the positive terms (because of OPT) of $d\Phi_3/dt$ of Lemma 9 and ignore the negative contribution because of the execution of the algorithm. Combining Lemma 6, Lemma 7 and Lemma 9

\[ n + \sum_{k=1}^{K} P(\tilde{s}_k) + d\Phi_1/dt + d\Phi_2/dt + d\Phi_3/dt \]

\[ \leq n_f + n_f P(P^{-1}(1)) + c_1 n - c_1 \frac{n(n+1)}{2K}, \]

\[ + c_1 \sum_{k \in OPT} P(\tilde{s}_k) - c_2 n_f P(P^{-1}(1)) + c_2 \sum_{k \in OPT} P(\tilde{s}_k), \]

\[ + c_3 (\alpha - 1) n^\alpha + c_3 \sum_{k \in OPT} P(\tilde{s}_k), \]

for $c = c_1 + c_2 + (\alpha - 1) c_3$, where $c_2 \geq 2$. Thus, $c$ for which 3 holds for all the three cases is $c = 2\beta + 2(\alpha - 1)$, completing the proof of Theorem 8.

XII. PROOFS OF LEMMA 10, 11 AND 12

Proof of Lemma 10. Since there are only map tasks in $\Phi_1(t)$, this proof is identical to that of Theorem 1.1 [4] for single server speed scaling with no precedence constraints, since map tasks are always free.

Proof of Lemma 11. We consider the case when $J(t) \geq J^o(t)$ since otherwise we can show that $d\Phi_2/dt \leq 0$ similar to 4.

With $J(t) \geq J^o(t)$, for the algorithm, let $q_c$ be the sum of the sizes of all the map tasks for the job with the smallest cumulative map size. The OPT might be processing a map or a reduce task. If it is a reduce task, then it does not affect $d\Phi_2/dt$. Otherwise, let OPT process a map task $k$ of job $j$, where the sum of the sizes of the map tasks of job $j$ is $q_c$. Note that job $j$ may not be the job with the smallest sum of the map tasks, i.e., $J^o(q_c) \neq J^o(t)$. Now we need to separate, the three cases $q_c < q_c^0$, $q_c > q_c^0$, $q_c = q^0_c$.

Case 1: $q_c < q_c^0$. The algorithm is processing the shortest (cumulative map task size) job, thus making $J(q) = J(q) - 1$ for $q \in [q_c - s_j(t)/dt, q]$. Thus, $d\Phi_2(t)/dt$ because of processing of the algorithm is $f(J(q_c) - J^o(q_c) - 1 - f(J(q_c) - J^o(q_c)))s_j = -\Delta(r(J(q_c) - J^o(q_c)))s_j$. Now note that $J(q_c) = J(t)$ since the algorithm is processing the shortest (cumulative map task size) job, while $J^o(q_c) \leq J^o(t)$ always. Hence

\[ d\Phi_3(t)/dt \leq \Delta(r(J(q_c) - J^o(q_c)))s_j = -\Delta(r(J(t) - J^o(t)))s_j. \]
Since OPT is not necessarily processing the map task for a job that has the smallest cumulative map task size of all the outstanding map jobs, \( J^0(q') \neq J^0(q) \) and the best bound we can get is \( J^0(q') \geq 1 \). Moreover, since \( J(q') \leq J(t) \) always, we get the OPT’s contribution as \( d\Phi_2(t)/dt \leq \Delta(r(t) - 1)(-\dot{s}_J + \dot{s}) \).

Hence, the argument inside \( \Delta \) function for the algorithms’ and the OPT’s contribution are not identical and we have to apply Lemma 8 separately on the algorithms’ and the OPT’s contribution for \( d\Phi_2(t)/dt \) to get \( d\Phi_2(t)/dt \leq c_2 P(\dot{s}(t)) - c_2(\alpha - 2) rJ(t) + c_2(2 - \alpha)rJ^0(t) \) by using the fact that the speed of the algorithm is \( s_J(t) = P^{-1}(r(t) + 1) \).

The case of \( q_c > q_c^* \) and \( q_c = q_c^* \) follows similarly.

\[ \square \]

**Proof of Lemma 14.** We consider the case when \( r(t) \geq r^0(t) \) since otherwise we can show that \( d\Phi_3(t)/dt \leq 0 \) similar to [4].

With \( r(t) \geq r^0(t) \), let \( q_r(q_r') \) be the size of the smallest free reduce task remaining with the algorithm (OPT). Note that with OPT, all reduce jobs are always free. Now we need to separate, the three cases \( q_r < q_r^* \), \( q_r > q_r^* \), \( q_r = q_r^* \).

**Case I:** \( q_r < q_r^* \). The algorithm is processing the shortest free reduce task, thus making \( r(q) = r(q) - 1 \) for \( q \in [q_r - s_r(t)/dt, q_r] \) while the processing by OPT makes \( r^0(q) = r^0(q) - 1 \) for \( q \in [q_r^* - \dot{s}(t)/dt, q_r^*] \) if the OPT is processing a reduce task. If OPT is processing a map task, then it does not affect \( d\Phi_3(t)/dt \). Thus, \( d\Phi_3(t)/dt \) because of processing by the algorithm is \((f(r(q_r) - r^0(q_r) - 1) - f(r(q_r) - r^0(q_r))) s_r = -\Delta(r(q_r) - r^0(q_r)) s_r \) always. Note that \( q_r \geq q_r^* \) since the algorithm is processing the shortest free reduce task, while \( r^0(q_r) \leq r^0(t) \) always. Hence:

\[ d\Phi_3(t)/dt \leq \Delta(r(q_r) - r^0(q_r)) s_r = -\Delta(r(t) - r^0(t)) s_r. \]

Similarly for the OPT’s contribution,

\[ d\Phi_3(t)/dt \leq (f(r(q_r^*) - r^0(q_r^*)) + 1) - (f(r(q_r^*) - r^0(q_r^*))) \dot{s}, \]

\[ = \Delta(r(q_r^*) - r^0(q_r^*)) \dot{s}, \]

\[ \leq \Delta(r(t) - r^0(t)) \dot{s}, \]

as \( r(q_r^*) \leq r(t) - 1 \) since \( q_r < q_r^* \). Therefore, combining the algorithm’s and OPT’s contribution, we get \( d\Phi_3(t)/dt \leq \Delta(r(t) - r^0(t))(-s_r + \dot{s}) \). Using Lemma 8 and the fact that \( s_r = P^{-1}(r(t) + 1) \), we get \( d\Phi_3(t)/dt \leq c_3 P(\dot{s}(t) - c_3(r(t) - r^0(t))^\alpha \dot{s} - c_3(r(t) - r^0(t))^\alpha \dot{s} \).

**Case II:** \( q_r > q_r^* \): The proof for this case is more non-trivial, and essentially reflects why the competitive ratio guarantee holds only for \( \alpha < 2 \). In this case, the algorithm’s contribution is

\[ d\Phi_3(t)/dt \leq \Delta(r(q_r) - r^0(q_r)) s_r = -\Delta(r(t) - r^0(t) + 1) s_r, \]

since \( r^0(q_r) \leq r^0(t) - 1 \) as \( q_r > q_r^* \). The OPT’s contribution

\[ d\Phi_3(t)/dt \leq (f(r(q_r^*) - r^0(q_r^*)) + 1) - (f(r(q_r^*) - r^0(q_r^*))) \dot{s}, \]

\[ = \Delta(r(q_r^*) - r^0(t) + 1) \dot{s}. \]

**Case III:** \( q_r = q_r^* \): We would want \( r(q_r^*) = r(t) \), however, that need not be true, since \( q_r > q_r^* \) and there can be caged reduce tasks that are smaller in size than the free reduce jobs in which case \( r(q_r^*) = r(t) \), where \( r(t) \) is the total number of reduce jobs with the algorithm. Thus, the OPT’s contribution is \( d\Phi_3(t)/dt \leq \Delta(r(t) - r^0(t)) \dot{s} \). Hence, the argument inside \( \Delta \) function for the algorithms’ and the OPT’s contribution are not identical and we have to apply Lemma 8 separately on the algorithms’ and the OPT’s contribution for \( d\Phi_3(t)/dt \) to get \( d\Phi_3(t)/dt \leq c_3 P(\dot{s}(t)) - c_3(r(t) - (\alpha - 1)r(t) - (2 - \alpha)r^0(t)) \dot{s} \), where we have used the fact that the speed of the algorithm for processing the shortest free reduce task is \( P^{-1}(r(t) + 1) \).

The case for \( q_c = q_c^* \) follows similarly.

**XIII. Lemma 14.** For \( \alpha \geq 1 \), and \( x_i \geq 0 \), \( \left( \sum_{i=1}^{n} x_i^{1/\alpha} \right) \leq n^{\alpha - 1} \sum_{i=1}^{n} x_i. \)

**Proof.** Let \( X \) be a random variable with distribution \( D \) over the support \( x_1, \ldots, x_n \). Let \( \eta > \nu > 0 \). Then \( E(|X|^\eta)^{1/\eta} = E(|X|^\nu)^{1/\nu} \geq E(|X|^\nu)^{1/\nu} \geq E(|X|^\nu)^{1/\nu} \), where the inequality follows from Jensen’s inequality since \( x^\eta/\nu \) is convex. Choosing \( D \) to be the uniform distribution, and \( \nu = 1/\alpha \) with \( \alpha \geq 1 \), we get the inequality.

**XIV. Proof of Theorem 10.**

The potential function we construct as follows is essentially the multi-server version of (19) as proposed in [31] for analyzing the multi-server SRPT algorithm, except for the \( \Phi_4 \) term that is needed to handle the unique challenge posed by the precedence constraints described in the previous paragraph. Let \( d_m(q) = \max\{0, m(q) - m^0(q)\} \), \( d_r(q) = \max\{0, r(q) - r^0(q)\} \). We consider the potential function

\[ \Phi(t) = \Phi_1(t) + \Phi_2(t) + \Phi_3(t) + \Phi_4(t), \]

where \( \Phi_i(\cdot) \) are defined as follows.

\[ \Phi_1(t) = c_{11} \int_0^t f(d_m(q)) \, dq + c_{12} \int_0^t (m(q) - m^0(q)) \, dq, \]

\[ \Phi_2(t) = c_{21} \int_0^t f(d_r(q)) \, dq, \]

\[ \Phi_3(t) = c_{31} \int_0^t f(d_r(q)) \, dq + c_{32} \int_0^t (r(q) - r^0(q)) \, dq. \]

The sub-potential function \( \Phi_4(t) \) is special and needed to handle the case when the number of jobs with the algorithm is less than \( K \). For the \( j^{th} \), \( j = 1, \ldots, J(t,J^0(t)) \) job, let \( q_j(q_j') \) be the sum of the remaining size of all its map tasks with the algorithm and the OPT, respectively. Then for the \( j^{th} \) job \( \eta_j(t, q) = 1/(n_j^0(t, q)) \) for \( q \leq q_j \) and \( q_j < q_j^* \) and zero otherwise.
Let $\Phi_4(t)$

$$= c_4 \left( \sum_{j \in J(t)} \int_0^\infty f \left( m_j(q) \right) dq - \sum_{j \in J(t)} \int_0^\infty f \left( m_j'(q) \right) dq \right)$$

c_{11}, c_{12}, c_{21}, c_{22}, c_{31}, c_{32}, c_4$ are positive constants to be determined later.

The drift of $\Phi_1, \Phi_2, \Phi_3$ can be computed similar to Lemma 10 and 11 [51], when OPT has no precedence constraints but is not restricted to perform multi-server SRPT scheduling.

**Lemma 15.** $d\Phi_1/dt$ For $m(t) \geq K$,

$$\leq c_{11} m^o - c_{11}(2-\alpha)m + c_{12}(2-\alpha) \left( \frac{K-1}{2} \right)$$

$$+ c_{11} \sum_{k \in \text{OPT}} P(\tilde{s}_k) - c_{12} \min(K, m) + c_{12} \sum_{k \in \text{OPT}} P(\tilde{s}_k)$$

while for $m(t) < K$,

$$d\Phi_1/dt \leq c_{11} m^o - c_{12} \min(K, m) + c_{12} \sum_{k \in \text{OPT}} P(\tilde{s}_k)$$

**Lemma 16.** For $J(t) \geq K$,

$$d\Phi_2/dt \leq c_{21} r^o - c_{21}(2-\alpha)rJ + c_{21}(2-\alpha) \left( \frac{K-1}{2} \right)$$

$$+ c_{21} \sum_{k \in \text{OPT}} P(\tilde{s}_k)$$

while for $J(t) < K$, (by disregarding the algorithm’s contribution)

$$d\Phi_2/dt \leq c_{21} r^o + c_{21}(2-\alpha) \left( \frac{K-1}{2} \right) + c_{21} \sum_{k \in \text{OPT}} P(\tilde{s}_k)$$

**Lemma 17.** For $r_J(t) \geq K$, $d\Phi_3/dt$

$$\leq c_{31} r^o - c_{31}(2-\alpha)r_J + c_{31}(2-\alpha) \left( \frac{K-1}{2} \right)$$

$$+ c_{31} \sum_{k \in \text{OPT}} P(\tilde{s}_k) - c_{32} \min(K, m) + c_{32} \sum_{k \in \text{OPT}} P(\tilde{s}_k)$$

while for $r_J(t) < K$,

$$d\Phi_3/dt \leq c_{31} r^o + c_{31}(2-\alpha)r_J + c_{31} \sum_{k \in \text{OPT}} P(\tilde{s}_k),$$

$$- c_{32} \min(K, m) + c_{32} \sum_{k \in \text{OPT}} P(\tilde{s}_k)$$

The only non-trivial drift is for $\Phi_4(t)$ which is needed when $J(t) < K$ is described as follows.

**Lemma 18.** When

$$d\Phi_4(t)/dt \leq c_4(-1 \mathbf{1}_{J(t)<K} r_J(t) + r \min\{J^o(t), K\}).$$

Proof is similar to Lemma 9 and hence omitted.

**Proof of Theorem 10** To prove the Theorem we will show that (3) holds for an appropriate choice of $c$ using Lemma 15 [16, 17] and 18.