Integrable mixed vertex models from braid-monoid algebra

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Abstract

We use the braid-monoid algebra to construct integrable mixed vertex models. The transfer matrix of a mixed $SU(N)$ model is diagonalized by nested Bethe ansatz approach.

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1 Introduction

The quantum version of the inverse scattering method paved the way for the discovery and solution of new exactly solvable two-dimensional lattice models. Of particular interest are non-homogeneous vertex models whose transfer matrix are constructed by mixing different local vertex operators. These operators, known as Lax $\mathcal{L}$-operators, define the local structure of Boltzmann weights of the system. A sufficient condition for integrability is that all the $\mathcal{L}$-operators should satisfy the Yang-Baxter equation with the same invertible $R$-matrix. More precisely we have \cite{1},

$$R(\lambda - \mu)\mathcal{L}_{Ai}(\lambda) \otimes \mathcal{L}_{Ai}(\mu) = \mathcal{L}_{Ai}(\mu) \otimes \mathcal{L}_{Ai}(\lambda)R(\lambda - \mu) \quad (1)$$

The auxiliary space $\mathcal{A}$ corresponds to the horizontal degrees of freedom of the vertex model in the square lattice. The operator $\mathcal{L}_{Ai}(\lambda)$ is a matrix in the auxiliary space $\mathcal{A}$ and its matrix elements are operators on the quantum space $\prod_{i=1}^{L} \otimes V_i$, where $V_i$ represents the vertical space of states and $i$ the sites of a one-dimensional lattice of size $L$. The tensor product in formula (1) is taken with respect the auxiliary space and $\lambda$ is a spectral parameter. One way of producing a mixed vertex model is by choosing $\mathcal{L}$-operators intertwining between different representations $V_i$ of a given underlying algebra. This approach has first been used by Andrei and Johannesson \cite{2} to study the Heisenberg model in the presence of an impurity of spin-$S$ and by De Vega and Woynarovich \cite{3} to construct alternating Heisenberg spin chains. Subsequently, several papers have discussed physical properties of the latter models \cite{4, 5, 6} as well as considered generalizations to include other Lie algebras \cite{7, 8} and superalgebras \cite{9, 10, 11, 12}.

In this paper, which we are pleased to dedicate to James McGuire, we construct and solve mixed vertex models whose $\mathcal{L}$-operators can be expressed in terms of the generators of the braid-monoid algebra \cite{13}. In particular, we show that the recent results by Abad and Rios \cite{8, 10} and by Links and Foerster \cite{11} for the $SU(3)$ and $gl(2|1)$ algebras can be reobtained from such algebraic approach. The novel feature as compared to these works is that we are able to diagonalize the corresponding transfer matrix by using a standard variant of the nested Bethe ansatz approach. Furthermore, this algebraic approach allows us to derive extensions
to the $SU(N)$ and $Sl(N|M)$ algebras in a more direct way.

This paper is organized as follows. In section 2 we recall the basics of the braid-monoid algebra and show how it produces two different $L$-operators satisfying the Yang-Baxter equation (1). In section 3 a mixed vertex model based on the $SU(N)$ algebra is diagonalized by the algebraic Bethe ansatz method.

## 2 Braid-monoid L-operators

It is well known that the braid algebra produces the simplest rational $R$-matrix solution of the Yang-Baxter equation. In this case the braid operator becomes a generator of the symmetric group and the $R$-matrix is given by

$$R(\lambda) = I_i + \lambda b_i$$  \hfill (2)

where $I_i$ is the identity and $b_i$ is the braid operator acting on the sites $i$ and $i + 1$ of a one-dimensional chain. Here we choose the braid operator as the graded permutation between $N$ bosonic and $M$ fermionic degrees of freedom [14]

$$b_i = \sum_{a,b=1}^{N+M} (-1)^{p(a)p(b)} e_{ab}^i \otimes e_{ba}^{i+1}$$  \hfill (3)

where $p(a)$ is the Grassmann parity of the $a$-th degree of freedom, assuming values $p(a) = 0$ for bosons and $p(a) = 1$ for fermions.

The $R$-matrix (2,3) has a null Grassmann parity, and consequently can produce a vertex operator $L_{\lambda_i}(\lambda)$ satisfying either the Yang-Baxter equation or its graded version [14]. In the latter case, the tensor product in formula (1) should be taken in the graded sense(supertensor product) [14]. In the latter case the associated $L$-operator is

$$L_{\lambda_i}(\lambda) = \lambda I_i + b_i$$  \hfill (4)

The next step is to search for extra $L$-operators which should satisfy equation (1) with the $R$-matrix (2,3). As we shall see below, this is possible when we enlarge the braid algebra by
including a Temperley-Lieb operator $E_i$. This operator satisfies the following relations

$$E_i^2 = qE_i, \ E_iE_{i+1}E_i = E_i, \ E_iE_j = E_jE_i \quad |i - j| \geq 2$$  \hspace{1cm} (5)

where $q$ is a $c$-number. It turns out that the braid operator (3) together with the monoid $E_i$ close the braid-monoid algebra [13] at its degenerated point ($b_i^2 = I_i$). The extra relations between $b_i$ and $E_i$ closing the degenerated braid-monoid algebra are (see e.g [15])

$$b_iE_i = E_ib_i = \hat{t}E_i$$  
$$E_ib_{i\pm 1}b_i = b_{i\pm 1}b_iE_{i\pm 1} = E_iE_{i\pm 1} \hspace{1cm} (6)$$

where the constant $\hat{t}$ assumes the values $\pm 1$.

Now we have to solve the Yang-Baxter equation (1) with the $R$-matrix (2,3) assuming the following general ansatz for the $L$-operator

$$L_{Ai}(\lambda) = f(\lambda)I_i + g(\lambda)b_i + h(\lambda)E_i \hspace{1cm} (7)$$

where $f(\lambda)$, $g(\lambda)$ and $h(\lambda)$ are functions to be determined as follows. Substituting this ansatz in equation (1), and taking into account the braid-monoid relations, we find two classes of solutions. The first one has $h(\lambda) = 0$ and clearly corresponds to the standard solution already given in equation (4). The second one is new and is giving by $g(\lambda) = 0$. We find that the new $L$-operator, after normalizing the solution by function $h(\lambda)$, is given by

$$\tilde{L}_{Ai}(\lambda) = \hat{t}(\lambda + \eta)I_i - E_i \hspace{1cm} (8)$$

where $\eta$ is an arbitrary constant. This constant can be fixed imposing unitary property, i.e $\tilde{L}_{Ai}(\lambda)\tilde{L}_{Ai}(-\lambda) \sim I_i$. By using this property and the first equation (5) we find

$$\eta = \frac{q}{2\hat{t}} \hspace{1cm} (9)$$

After having found two distincts $L$-operators which satisfy the Yang-Baxter algebra with the same $R$-matrix, the construction of an integrable mixed vertex model becomes standard [2 3]. The monodromy matrix of a vertex model mixing $L_1$ operators of type $L_{Ai}(\lambda)$ and $L_2$ operators of type $\tilde{L}_{Ai}(\lambda)$ is written as

$$\mathcal{T}^L_{L_1L_2}(\lambda) = \tilde{L}_{AL_1+L_2}(\lambda)\tilde{L}_{AL_1+L_2-1}(\lambda) \cdots \tilde{L}_{A1}(\lambda) \hspace{1cm} (10)$$
where $\tilde{L}_{\lambda i}(\lambda)$ is defined by

$$\tilde{L}_{\lambda i}(\lambda) = \begin{cases} L_{\lambda i}(\lambda) & \text{if } i \in \{\beta_1, \cdots, \beta_{L_1}\} \\ \tilde{L}_{\lambda i}(\lambda) & \text{otherwise} \end{cases}$$

and the partition $\{\beta_1, \cdots, \beta_{L_1}\}$ denotes a set of integer indices assuming values in the interval $1 \leq \alpha_i \leq L_1 + L_2$. Although the integrability does not depend on how we choose such partition, the construction of local conserved charges commuting with the respective transfer matrix does.

One interesting case is when the number of operators $L_{\lambda i}(\lambda)$ and $\tilde{L}_{\lambda i}(\lambda)$ are equally distributed ($L_1 = L_2 = L$) in an alternating way in the monodromy matrix \cite{3}. In this case, the first non-trivial charge, known as Hamiltonian, is given in terms of nearest neighbor and next-to-nearest neighbor interactions. More specifically, the expression for the Hamiltonian in the absence of fermionic degrees of freedom ($M = 0$) is

$$H = \sum_{m \in \text{odd}}^{2L} \tilde{L}_{m-1,m}(0) + \sum_{i \in \text{even}}^{2L} \tilde{L}_{n-2,n-1}(0)\tilde{L}_{n,n-1}(0)P_{n-2,n}$$

where in the computations it was essential to use the unitary property $\tilde{L}_{\lambda i}(0)\tilde{L}_{\lambda i}(0) = \frac{q^2}{4}I_i$ at the regular point $\lambda = 0$. We also recall that $P_{ij}$ denotes permutation between sites $i$ and $j$ (equation (3) with $M = 0$).

We close this section discussing explicit representations for the monoid $E_i$. Such representations can be found in terms of the invariants of the superalgebra $Osp(N|2M)$, where $N$ and $2M$ are the number of bosonic and fermionic degrees of freedom, respectively. Here we shall consider a representation that respects the $U(1)$ invariance, which will be very useful in Bethe ansatz analysis. Following ref. \cite{15} the monoid is written as

$$E_i = \sum_{a,b,c,d=1}^{N+2M} \alpha_{ab}\alpha_{cd}^{-1}e_{ac}^i \otimes e_{bd}^{i+1}$$

and the matrix $\alpha$ has the following block anti-diagonal structure

$$\alpha = \begin{pmatrix} O_{N \times M} & O_{N \times M} & \mathcal{I}_{N \times N} \\ O_{M \times M} & \mathcal{I}_{M \times M} & O_{M \times N} \\ -\mathcal{I}_{M \times M} & O_{M \times M} & O_{M \times N} \end{pmatrix}$$
where $I_{k_1 \times k_2}$ and $O_{k_1 \times k_2}$ are the anti-diagonal and the null $k_1 \times k_2$ matrices, respectively. We also recall that for $\hat{t} = 1$ the sequence of grading is $f_1 \cdots f_M b_1 \cdots b_N f_{M+1} \cdots f_{2M}$ and the Temperley-Lieb parameter $q$ is the difference between the number of bosonic and fermionic degrees of freedom.

$$q = N - 2M$$  \hfill (15)

Finally, we remark that new $\tilde{L}_A^i(\lambda)$ operators are obtained only when $N + 2M \geq 3$. Indeed, for the special cases $N = 2$, $M = 0$ and $N = 0$, $M = 1$ it is possible to verify that such operator has the structure of the 6-vertex model, which is precisely the same of $L_A^i(\lambda)$, modulo trivial phases and scaling. In the cases $N = 3, M = 0$ and $N = 1, M = 1$ we reproduce, after a canonical transformation, the $L$-operators used recently in the literature \[8, 10, 11\] to construct mixed $SU(3)$ and $t-J$ models. As we shall see in next section, however, an important advantage of our approach is that representation (14) is the appropriate one to allows us to perform standard nested Bethe Ansatz diagonalization.

### 3 Bethe ansatz diagonalization

In this section we look at the problem of diagonalization of the transfer matrix $T^{L_1, L_2}(\lambda) = Tr_A[T^{L_1, L_2}(\lambda)]$, namely

$$T^{L_1, L_2}(\lambda) \mid \Phi \rangle = \Lambda(\lambda) \mid \Phi \rangle$$  \hfill (16)

by means of the quantum inverse scattering method.

For sake of simplicity we restrict ourselves to the case of mixed vertex models in the absence of fermionic degrees of freedom. In this case the $L$-operators $L_A^i(\lambda)$ and $\tilde{L}_A^i(\lambda)$ are given by formulae (4) and (8) with $M = 0$, $\hat{t} = 1$ and $\eta = N/2$. An important object in this framework is the reference state $\mid 0 \rangle$ we should start with in order to construct the full Hilbert space $\mid \Phi \rangle$. The structure of the $L$-operators suggests us to take the standard ferromagnetic pseudovacuum
as our reference state, i.e

\[ |0\rangle = \prod_{i=1}^{L_1+L_2} |0\rangle_i, \quad |0\rangle_i = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{N} \]  

where the index \( N \) represents the length of the vectors \( |0\rangle_i \). It turns out that this state is an exact eigenvector of the transfer matrix, since both operators \( L_{Ai}(\lambda) \) and \( \tilde{L}_{Ai}(\lambda) \) satisfy the following important triangular properties

\[ L_{Ai}(\lambda) |0\rangle_i = \begin{pmatrix} a(\lambda) |0\rangle_i \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{N \times N} \]  

and

\[ \tilde{L}_{Ai}(\lambda) |0\rangle_i = \begin{pmatrix} \tilde{b}(\lambda) |0\rangle_i \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{N \times N} \]

where the symbol \( * \) stands for non-null values that are not necessary to evaluate in this algebraic approach. The functions \( a(\lambda) \), \( b(\lambda) \), \( \tilde{a}(\lambda) \) and \( \tilde{b}(\lambda) \) are obtained directly from expressions (8,9), and they are given by

\[ a(\lambda) = \lambda + 1, \quad b(\lambda) = \lambda, \quad \tilde{a}(\lambda) = \lambda + \frac{N}{2} - 1, \quad \tilde{b}(\lambda) = \lambda + \frac{N}{2} \]  

To make further progress we have to write an appropriate ansatz for the monodromy matrix \( \mathcal{T}^{L_1,L_2}(\lambda) \) in the auxiliary space \( \mathcal{A} \). The triangular properties (18,19) suggest us to seek for standard structure used in nested Bethe ansatz diagonalization of \( SU(N) \) vertex models \[16\],

\[ \mathcal{T}^{L_1,L_2}(\lambda) = \begin{pmatrix} A(\lambda) & B_i(\lambda) \\ C_i(\lambda) & D_{ij}(\lambda) \end{pmatrix}_{N \times N} \]

where \( i, j = 1, \cdots, N-1 \). As a consequence of properties (18,19) we derive how the monodromy matrix elements act on the reference state. The fields \( B_i(\lambda) \) play the role of creation operators.
while $C_i(\lambda)$ are annihilation fields, i.e $C_i(\lambda) |0\rangle = 0$. Furthermore, the action of the “diagonal” operators $A(\lambda)$ and $D_{ij}(\lambda)$ are given by

$$A(\lambda) |0\rangle = [a(\lambda)]^{L_1} [\tilde{b}(\lambda)]^{L_2} |0\rangle$$

$$D_{ii}(\lambda) |0\rangle = [b(\lambda)]^{L_1} [\tilde{b}(\lambda)]^{L_2} |0\rangle \quad \text{for } i \neq N - 1, \quad D_{N-1,N-1}(\lambda) |0\rangle = [b(\lambda)]^{L_1} [\tilde{a}(\lambda)]^{L_2} |0\rangle$$

$$D_{i,j}(\lambda) |0\rangle = 0 \quad \text{for } i \neq j \quad \text{and} \quad j \neq N - 1, \quad D_{i,N-1}(\lambda) |0\rangle \neq 0 \quad \text{for } i \neq N - 1$$

We observe that although matrix $D_{ij}(\lambda) |0\rangle$ is non-diagonal it is up triangular, which is an important property to carry on higher level Bethe ansatz analysis. In order to construct other eigenvectors we need to use the commutation relations between the monodromy matrix elements which are obtained by extending the Yang-Baxter relation to the monodromy matrix ansatz (21). Due to the structure of the $R$-matrix, the commutation rules are the same of that already known for isotropic $SU(N)$ models [16], and the most useful relations for subsequent derivations are

$$A(\lambda) B_i(\mu) = \frac{a(\mu - \lambda)}{b(\mu - \lambda)} B_i(\mu) A(\lambda) - \frac{1}{b(\mu - \lambda)} B_i(\lambda) A(\mu)$$

$$D_{ij}(\lambda) B_k(\mu) = \frac{1}{b(\lambda - \mu)} B_p(\mu) D_{ij}(\lambda) r^{(1)}(\lambda - \mu)^{jk}_{pq} - \frac{1}{b(\mu - \lambda)} B_j(\lambda) D_{ik}(\mu)$$

$$B_i(\lambda) B_j(\mu) = B_p(\mu) r^{(1)}(\lambda - \mu)^{ij}_{pq}$$

where $r^{(1)}(\lambda)^{ij}_{pq}$ are the elements of the $R$-matrix $I_i + \lambda b_i$ on the subspace $(N - 1) \times (N - 1)$. The eigenvectors are given in terms of the following linear combination [16]

$$\Phi_{m_1}(\lambda^{(1)}_1, \ldots, \lambda^{(1)}_{m_1}) = B_{a_1}(\lambda^{(1)}_1) \cdots B_{a_{m_1}}(\lambda^{(1)}_{m_1}) \mathcal{F}^{a_{m_1} \cdots a_1}$$

where the components $\mathcal{F}^{a_{m_1} \cdots a_1}$ are going to be determined a posteriori.

By carrying on the diagonal fields $A(\lambda)$ and $D_{ii}(\lambda)$ over the above $m_1$-particle state we generate the so-called wanted and unwanted terms. The wanted terms are those proportional to $\Phi_{m_1}(\lambda^{(1)}_1, \ldots, \lambda^{(1)}_{m_1})$ and they contribute directly to the eigenvalue $\Lambda^{L_1,L_2}(\lambda, \{\lambda^{(1)}_i\})$. These terms are easily obtained by keeping only the first term of the commutation rules (25,26)
each time we turn $A(\lambda)$ and $D_{ii}(\lambda)$ over one of the $B_{ai}(\lambda^{(1)}_i)$ component. The result of this computations leads us to the following expression

$$T^{L_1,L_2}(\lambda) \left| \Phi_{m_1}(\lambda^{(1)}_1, \ldots, \lambda^{(1)}_{m_1}) \right> = [a(\lambda)]^{L_1} [\bar{b}(\lambda)]^{L_2} \prod_{i=1}^{m_1} \frac{a(\lambda^{(1)}_i - \lambda)}{b(\lambda^{(1)}_i - \lambda)} \left| \Phi_{m_1}(\lambda^{(1)}_1, \ldots, \lambda^{(1)}_{m_1}) \right> + \prod_{i=1}^{m_1} \frac{1}{b(\lambda - \lambda^{(1)}_i)} B_{b_1}(\lambda^{(1)}_1) \cdots B_{b_{m_1}}(\lambda^{(1)}_{m_1}) T^{(1)}(\lambda, \{\lambda^{(1)}_j\}) |a^{1 \cdots a_{m_1}} F^{a_{m_1} \cdots a_1} + \text{unwanted terms} \right>$$

(29)

where $T^{(1)}(\lambda, \{\lambda^{(1)}_j\})$ is the transfer matrix of the following inhomogeneous auxiliary vertex model

$$T^{(1)}(\lambda, \{\lambda^{(1)}_i\})_{b_1 \cdots b_{m_1}} = r^{(1)}(\lambda - \lambda^{(1)}_1)_{b_1 d_1} r^{(1)}(\lambda - \lambda^{(1)}_2)_{b_2 d_2} \cdots r^{(1)}(\lambda - \lambda^{(1)}_{m_1})_{b_{m_1} d_{m_1}} D_{a d_{m_1}}(\lambda) |0\rangle$$

(30)

The unwanted terms arise when one of the variables $\lambda^{(1)}_i$ of the $m_1$-particle state is exchanged with the spectral parameter $\lambda$. It is known [16] how to collect these in a close form, thanks to the commutation rule (27) which makes possible to relate different ordered multiparticle states. We find that the unwanted terms of kind $B_{a_1}(\lambda^{(1)}_1) \cdots B_{a_1}(\lambda) \cdots B_{a_{m_1}}(\lambda^{(1)}_{m_1})$ are cancelled out provided we impose further restriction to the $m_1$-particle state rapidities $\lambda^{(1)}_i$, namely

$$[a(\lambda^{(1)}_i)]^{L_1} [\bar{b}(\lambda^{(1)}_i)]^{L_2} \prod_{j=1, j \neq i}^{m_1} b(\lambda^{(1)}_i - \lambda^{(1)}_j) \frac{a(\lambda^{(1)}_j - \lambda^{(1)}_i)}{b(\lambda^{(1)}_j - \lambda^{(1)}_i)} F^{a_{m_1} \cdots a_1} = T^{(1)}(\lambda = \lambda^{(1)}_i, \{\lambda^{(1)}_j\})_{a_1 \cdots a_{m_1}} F^{b_{m_1} \cdots b_1}, i = 1, \ldots, m_1$$

(31)

Now it becomes necessary to introduce a second Bethe ansatz in order to diagonalize the auxiliary transfer matrix $T^{(1)}(\lambda, \{\lambda^{(1)}_i\})$. The only difference as compare to standard cases [16] is the presence of the “gauge” matrix $g_{ab} = D_{ab}(\lambda) |0\rangle$. It turns out that this problem is still integrable since the tensor product $g \otimes g$ commutes with the auxiliary $R$-matrix $r^{(1)}(\lambda)$ (see e.g ref. [17]). From equations (23,24) we also note that this gauge does not spoil the triangular form of the monodromy matrix associated to $T^{(1)}(\lambda, \{\lambda^{(1)}_i\})$ when it acts on the

1 This occurs because the off-diagonal elements $D_{i,N-1}(\lambda)$ belongs to a commutative ring.
usual ferromagnetic state,

\[ |0^{(1)}\rangle = \prod_{i=1}^{m_1} \otimes \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{N-1} \]

By defining \( \Lambda^{(1)}(\lambda, \{\lambda_i^{(1)}\}) \) as the eigenvalue of the auxiliary transfer matrix \( T^{(1)}(\lambda, \{\lambda_i^{(1)}\}) \), i.e.

\[ T^{(1)}(\lambda, \{\lambda_i^{(1)}\})^{b_1 \ldots b_m}_{a_1 \ldots a_m} \mathcal{F}^{b_m \ldots b_1} = \Lambda^{(1)}(\lambda, \{\lambda_i^{(1)}\}) \mathcal{F}^{a_m \ldots a_1} \]  

we derive from equation (29) that the eigenvalue of \( T^{L_1,L_2}(\lambda) \) is given by

\[ \Lambda^{L_1,L_2}(\lambda, \{\lambda_i^{(1)}\}) = [a(\lambda)]^{L_1} [\tilde{b}(\lambda)]^{L_2} \prod_{j=1, j \neq i}^{m_1} b(\lambda_i^{(1)} - \lambda_j^{(1)}) \frac{a(\lambda_j^{(1)} - \lambda_i^{(1)})}{b(\lambda_j^{(1)} - \lambda_i^{(1)})} = \Lambda^{(1)}(\lambda = \lambda_i^{(1)}, \{\lambda_j^{(1)}\}), \quad i = 1, \ldots, m_1 \]

In order to find the auxiliary eigenvalue \( \Lambda^{(1)}(\lambda = \lambda_i^{(1)}, \{\lambda_j^{(1)}\}) \) we have to introduce a new set of variables \( \{\lambda_i^{(2)}, \ldots, \lambda_{m_2}^{(2)}\} \) which parametrize the eigenvectors of \( T^{(1)}(\lambda, \{\lambda_i^{(1)}\}) \). The structure of the commutations rules as well as the eigenvector ansatz (28) remains basically the same, and the expression for \( \Lambda^{(1)}(\lambda = \lambda_i^{(1)}, \{\lambda_j^{(1)}\}) \) will again depend on another auxiliary inhomogeneous vertex model having \((N-2)\) states per link. We repeat this procedure until we reach the \((N-2)\)th step, where the auxiliary problem becomes of 6-vertex type. Since this nesting approach is well known in the literature \[13\], we here only present our final results.

The eigenvalue of the transfer matrix \( T^{L_1,L_2}(\lambda) \) is given by

\[ \Lambda^{L_1,L_2} \left( \lambda; \{\lambda_j^{(1)}\}, \ldots, \{\lambda_j^{(N-1)}\} \right) = [a(\lambda)]^{L_1} [\tilde{b}(\lambda)]^{L_2} \prod_{j=1}^{m_1} a(\lambda_i^{(1)} - \lambda_j^{(1)}) \frac{a(\lambda_j^{(1)} - \lambda_i^{(1)})}{b(\lambda_j^{(1)} - \lambda_i^{(1)})} \]

\[ + [b(\lambda)]^{L_1} [\tilde{b}(\lambda)]^{L_2} \sum_{i=1}^{m_1} \prod_{j=1}^{m_{i+1}} b(\lambda - \lambda_j^{(l)}) \frac{a(\lambda - \lambda_j^{(l)})}{b(\lambda - \lambda_j^{(l)})} \prod_{j=1}^{m_{i+1}} a(\lambda_j^{(l+1)} - \lambda_j^{(l)}/b(\lambda_j^{(l+1)} - \lambda_j^{(l)})} \]

\[ + [b(\lambda)]^{L_1} [\tilde{a}(\lambda)]^{L_2} \prod_{j=1}^{m_{N-1}} a(\lambda - \lambda_j^{(N-1)}) \frac{a(\lambda - \lambda_j^{(N-1)})}{b(\lambda - \lambda_j^{(N-1)})} \]
while the nested Bethe ansatz equations are given by

\[
\begin{align*}
\left[ a(\lambda_i^{(1)}) \frac{1}{b(\lambda_i^{(1)})} \right]^{L_1} &= \prod_{j=1, j \neq i}^{m_i} \frac{a(\lambda_i^{(1)} - \lambda_j^{(1)})}{a(\lambda_j^{(1)} - \lambda_i^{(1)})} \prod_{j=1}^{m_2} \frac{a(\lambda_j^{(2)} - \lambda_i^{(1)})}{a(\lambda_j^{(2)} - \lambda_i^{(1)})} \\
\prod_{j=1, j \neq i}^{m_i} \frac{a(\lambda_i^{(1)} - \lambda_j^{(1)})}{a(\lambda_j^{(1)} - \lambda_i^{(1)})} &= \prod_{j=1}^{m_i-1} \frac{a(\lambda_j^{(l)} - \lambda_i^{(l)})}{a(\lambda_i^{(l)} - \lambda_j^{(l)})} \prod_{j=1}^{m_{i+1}} \frac{b(\lambda_j^{(l+1)} - \lambda_i^{(l)})}{b(\lambda_i^{(l+1)} - \lambda_j^{(l)})}, \ l = 2, \ldots, N - 2
\end{align*}
\]

(37)

\[
\begin{align*}
\left[ \tilde{a}(\lambda_i^{(N-1)}) \frac{1}{b(\lambda_i^{(N-1)})} \right]^{L_2} &= \prod_{j=1, j \neq i}^{m_{N-1}} \frac{\tilde{a}(\lambda_i^{(N-1)} - \lambda_j^{(N-1)})}{\tilde{a}(\lambda_j^{(N-1)} - \lambda_i^{(N-1)})} \prod_{j=1}^{m_{N-2}} \frac{\tilde{a}(\lambda_i^{(N-1)} - \lambda_j^{(N-2)})}{\tilde{a}(\lambda_j^{(N-1)} - \lambda_i^{(N-2)})} \\
\prod_{j=1, j \neq i}^{m_{N-1}} \frac{\tilde{a}(\lambda_i^{(N-1)} - \lambda_j^{(N-1)})}{\tilde{a}(\lambda_j^{(N-1)} - \lambda_i^{(N-1)})} &= \prod_{j=1}^{m_{N-1}-1} \frac{\tilde{a}(\lambda_j^{(l)} - \lambda_i^{(l)})}{\tilde{a}(\lambda_i^{(l)} - \lambda_j^{(l)})} \prod_{j=1}^{m_{l+1}} \frac{\tilde{a}(\lambda_j^{(l+1)} - \lambda_i^{(l)})}{\tilde{a}(\lambda_i^{(l+1)} - \lambda_j^{(l)})}, \ l = 2, \ldots, N - 2
\end{align*}
\]

(38)

(39)

We would like to close this paper with the following remarks. First we note that it is possible to perform convenient shifts in the Bethe ansatz rapidities, \( \{\lambda_i^{(p)}\} \rightarrow \{\lambda_i^{(p)}\} - \frac{\pi}{2} \), in order to present the results (36-39) in a more symmetrical form. For instance, after these shifts, the nested Bethe ansatz equations can be compactly written as

\[
\begin{align*}
\left[ \frac{\lambda_i^{(a)} - \delta_{a,w}}{\lambda_i^{(a)} + \delta_{a,w}} \right]^{L_w} &= \prod_{b=1}^{r} \prod_{k=1, k \neq i}^{m_b} \frac{\lambda_i^{(a)} - \lambda_k^{(b)}}{\lambda_i^{(a)} - \lambda_k^{(b)}} - \frac{C_{a,b}}{2}, \ i = 1, \ldots, m_a; \ a = 1, \ldots, N - 1
\end{align*}
\]

(40)

where \( C_{a,b} \) is the Cartan matrix of the \( A_N \) Lie algebra and \( w = 1, N - 1 \). We note that for \( N = 3 \) we recover the results by Abad and Rios [8]. It is also straightforward to extend the above Bethe ansatz results to the superalgebra \( SL(N|M) \). The \( N = 0 \) case is the simplest one, since the only modification is the addition of minus signs in functions \( \tilde{a}(\lambda) \) and \( \tilde{b}(\lambda) \).

Next remark concerns the physical meaning of the \( \mathcal{L} \)-operators as scattering \( S \)-matrices. It is known that \( \mathcal{L}_{\mathcal{A} i}(\lambda) \) might represent the scattering matrix of particles belonging to the fundamental representation of \( SU(N) \). The extra solution \( \tilde{\mathcal{L}}_{\mathcal{A} i}(\lambda) \), however, should be seen as the forward scattering amplitude between a particle and an antiparticle. In fact, it is possible to show that the whole particle-antiparticle scattering (even backward amplitudes) can be closed in terms of the braid-monoid algebra. We leave a detailed analysis of this possibility, their Bethe ansatz properties as well as generalizations to include trigonometric solutions for a forthcoming paper [18].
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