EMBEDDINGS BETWEEN WEIGHTED CESÀRO FUNCTION SPACES

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Abstract. In this paper, we give the characterization of the embeddings between weighted Cesàro function spaces. The proof is based on the duality technique, which reduces this problem to the characterizations of some direct and reverse Hardy-type inequalities and iterated Hardy-type inequalities.

1. Introduction

Our principle goal in this paper is to obtain two-sided estimates of the best constant $c$ in the inequality

$$
\left( \int_0^\infty \left( \int_0^t f(s)^{p_2} v_2(s)^{p_2} ds \right)^{\frac{q_2}{p_2}} u_2(t)^{q_2} dt \right)^{\frac{1}{q_2}} \leq c \left( \int_0^\infty \left( \int_0^t f(s)^{p_1} v_1(s)^{p_1} ds \right)^{\frac{q_1}{p_1}} u_1(t)^{q_1} dt \right)^{\frac{1}{q_1}},
$$

where $0 < p_1, p_2, q_1, q_2 < \infty$ and $u_1, u_2, v_1, v_2$ are non-negative measurable functions.

Let $X$ and $Y$ be quasi normed vector spaces. If $X \subset Y$ and the identity operator is continuous from $X$ to $Y$, that is, there exists a positive constant $c$ such that $\| I(z) \|_Y \leq c \| z \|_X$ for all $z \in X$, we say that $X$ is embedded into $Y$ and write $X \hookrightarrow Y$. We denote by $\mathcal{M}^+$, the set of all non-negative measurable functions on $(0, \infty)$. A weight is a function such that measurable, positive and finite a.e on $(0, \infty)$ and we will denote the set of weights by $\mathcal{W}$.

We denote by $\text{Ces}_{p,q}(u,v)$, the weighted Cesàro function spaces and $\text{Cop}_{p,q}(u,v)$, the weighted Copson function spaces, the collection of all functions on $\mathcal{M}^+$ such that

$$
\| f \|_{\text{Ces}_{p,q}(u,v)} = \left( \int_0^\infty \left( \int_0^t f(s)^p v(s)^p ds \right)^{\frac{q}{p}} u(t)^q dt \right)^{\frac{1}{q}} < \infty,
$$

and

$$
\| f \|_{\text{Cop}_{p,q}(u,v)} = \left( \int_0^\infty \left( \int_t^\infty f(s)^p v(s)^p ds \right)^{\frac{q}{p}} u(t)^q dt \right)^{\frac{1}{q}} < \infty,
$$

respectively, where $p, q \in (0, \infty)$, $u \in \mathcal{M}^+$ and $v \in \mathcal{W}$. Then with this denotation, we can formulate the main aim of this paper as the characterization of the embeddings between weighted Cesàro function spaces, that is,

$$
\text{Ces}_{p_1,q_1}(u_1,v_1) \hookrightarrow \text{Ces}_{p_2,q_2}(u_2,v_2).
$$
The classical Cesàro function spaces \( \text{Ces}_{1,p}(x^{-1}, 1) \) have been defined by Shiue in [32] and it was shown in [22] that these spaces are Banach spaces when \( p > 1 \).

In [11], it was shown that \( \text{Ces}_{1,p}(x^{-1}, 1) \) and \( \text{Cop}_{1,p}(1, x^{-1}) \) coincide when \( 1 < p < \infty \) and the dual of the \( \text{Ces}_{1,p}(x^{-1}, 1) \) function spaces is given with a simpler description than in [33] as a remark.

During the past decade, these spaces have not been studied to a high degree but recently Astashkin and Maligranda began to examine the properties of classical Cesàro and Copson spaces in various aspects ([7, 8, 11, 6, 2, 3, 4, 5]), for the detailed information see the survey paper [9]. In [3], they gave the proof of the characterization of dual spaces of classical Cesàro function spaces.

Later, in [23] authors computed the dual norm of the spaces \( \text{Ces}_{1,p}(w, 1) \) generated by an arbitrary positive weight \( w \), where \( 1 < p < \infty \). In [10], factorizations of spaces \( \text{Ces}_{1,p}(1, x^{-1}, v) \) and \( \text{Cop}_{1,p}(x^{-1}, v) \) are presented.

In their newly papers, Leóšnik and Maligranda ([26, 27, 28, 29]) started the study of these spaces in an abstract setting and they replaced the role of \( L_p \) spaces with a more general function space \( X \).

A Banach ideal space \( X \) on \((0, \infty)\) is a Banach space contained in \( M^+ \) which satisfies the monotonicity property, that is, \( f, g \in M^+, f \leq g \) a.e on \((0, \infty)\) and \( g \in X \) then \( f \in X \) and \( \|f\| \leq \|g\| \).

For a Banach ideal space \( X \) on \((0, \infty)\), Leóšnik and Maligranda defined an abstract Cesàro space \( CX \) as
\[
CX = \{ f \in M^+, Cf \in X \}
\]
with the norm \( \|f\|_{CX} = \|Cf\|_X \) and an abstract Copson space \( C^*X \) as
\[
C^*X = \{ f \in M^+, C^*f \in X \}
\]
with the norm \( \|f\|_{C^*X} = \|C^*f\|_X \), where
\[
Cf(x) = \frac{1}{x} \int_0^x f(t)dt \quad \text{and} \quad C^*f(x) = \int_x^\infty \frac{f(t)}{t}dt, \quad x \in (0, \infty).
\]

Moreover, in [12] abstract Cesàro spaces were considered for rearrangement invariant spaces.

Note that taking \( X = L_p \), the definition of abstract spaces is related to our definition in the following way: \( CL_p = \text{Ces}_{1,p}(x^{-1}, 1) \).

Let \( X \) and \( Y \) be (quasi-) Banach spaces of measurable functions on \((0, \infty)\). Denote by \( M(X, Y) \), the space of all multipliers, that is,
\[
M(X, Y) := \{ f : f \cdot g \in Y \quad \text{for all} \quad g \in X \}.
\]
The Köthe dual \( X' \) of \( X \) is defined as the space \( M(X, L_1) \) of multipliers into \( L_1 \).

The space of all multipliers from \( X \) into \( Y \) is a quasi normed space with the quantity
\[
\|f\|_{M(X,Y)} := \sup_{g \neq 0} \frac{\|fg\|_Y}{\|g\|_X}.
\]
Now, define a weighted space \( Y_f = \{ g : f \cdot g \in Y, \quad f \in W \} \). Then
\[
\|f\|_{M(X,Y)} = \sup_{g \neq 0} \frac{\|g\|_{Y_f}}{\|g\|_X} = \|I\|_{X \rightarrow Y_f}.
\]

We should mention that, in [21], it is stated that the characterizing the multipliers between Cesàro and Copson spaces are difficult and note that the weighted Cesàro and Copson spaces...
are related to the spaces $C$ and $D$ defined in [21] as follows:

$$\text{Ces}_{p,q}(u,v) = C(p,q,u)_v \quad \text{and} \quad \text{Cop}_{p,q}(u,v) = D(p,q,u)_v.$$ 

Among all, recently in [29], multipliers between $\text{Ces}_{1,p}(x^{-1},1)$ and $\text{Cop}_{1,q}(1,x^{-1})$ is given when $1 < q \leq p \leq \infty$.

With this motivation in [17], the embeddings between weighted Cesàro and Copson function spaces, that is,

$$\text{Cop}_{p_1,q_1}(u_1,v_1) \hookrightarrow \text{Ces}_{p_2,q_2}(u_2,v_2)$$

have been characterized under the restriction $p_2 \leq q_2$ arises from duality approach. Also, using these results pointwise multipliers of weighted Cesàro and Copson spaces is given in [18]. We want to extend these results. In [17] Sawyer duality principle reduced the problem of embeddings to solutions of iterated Hardy-type inequalities of the forms

$$\left(\int_{0}^{\infty} \left( \sup_{s \in (0,t)} u(s) \int_{s}^{\infty} f(t) \, dw(t) dt \right)^{\frac{1}{q}} \right)^{\frac{1}{p}} \leq \left( \int_{0}^{\infty} w(t) dt \right)^{\frac{1}{q}} \left( \int_{0}^{\infty} f(t)^p w(t) dt \right)^{\frac{1}{p}},$$

and

$$\left(\int_{0}^{\infty} \left( \int_{0}^{t} \left( \int_{0}^{\infty} u(s) ds \right)^{\frac{m}{q}} \, dw(t) dt \right)^{\frac{1}{q}} \right)^{\frac{1}{p}} \leq \left( \int_{0}^{\infty} w(t) dt \right)^{\frac{1}{q}} \left( \int_{0}^{\infty} f(t)^p w(t) dt \right)^{\frac{1}{p}},$$

where $0 < m, q \leq 1$ and $1 < p < \infty$ (see, for instance [14, 15, 16, 20]).

Using the same approach as in [17], inequality (1.1) reduces to the characterization of iterated inequalities (containing iterated Copson-type operators) of the following type,

$$\left(\int_{0}^{\infty} \left( \sup_{s \in (t,\infty)} u(s) \int_{s}^{\infty} f(t) \, dw(t) dt \right)^{\frac{1}{q}} \right)^{\frac{1}{p}} \leq \left( \int_{0}^{\infty} w(t) dt \right)^{\frac{1}{q}} \left( \int_{0}^{\infty} f(t)^p w(t) dt \right)^{\frac{1}{p}},$$

and

$$\left(\int_{0}^{\infty} \left( \int_{t}^{\infty} \left( \int_{s}^{\infty} u(s) ds \right)^{\frac{m}{q}} \, dw(t) dt \right)^{\frac{1}{q}} \right)^{\frac{1}{p}} \leq \left( \int_{0}^{\infty} w(t) dt \right)^{\frac{1}{q}} \left( \int_{0}^{\infty} f(t)^p w(t) dt \right)^{\frac{1}{p}},$$

where $1 < p < \infty$ and $0 < q, m < \infty$. Until recently the solutions of these problems were not known but not long ago different characterizations have been given for these inequalities, see [24, 15, 16, 30, 25], therefore now we are able to continue this study. We will use characterizations from [24] and [25].

In order to shorten the formulas and simplify the notation, we will characterize the following inequality:

$$\left(\int_{0}^{\infty} \left( \int_{0}^{t} f(s)^p v(s) ds \right)^{\frac{m}{q}} \, u(t) dt \right)^{\frac{1}{q}} \leq \left( \int_{0}^{\infty} \left( \int_{0}^{t} f(s) ds \right)^{\theta} \, w(t) dt \right)^{\frac{1}{p}}. \quad (1.2)$$

It is easy to see that taking parameters $p = \frac{q}{p_2}$, $q = \frac{q}{p_1}$, $\theta = \frac{q}{p_1}$ and weights $v = v_1^{-p_2/p_2}$, $u = u_2^{q_2}$, $w = u_1^{q_1}$, we can obtain the characterization of inequality (1.1).

When $p = q$ or $\theta = 1$, (1.2) has been characterized in [17] by using direct and reverse Hardy-type inequalities. Unfortunately in this paper we will solve this problem under the restriction $p < q$ arising from the techniques we used, we will deal with the case when $q < p$ in the future paper with a different approach. On the other hand we always assume that $p < 1$, since otherwise inequality (1.2) holds only for trivial functions (see Lemma 3.1).

We adopt the following usual conventions. Throughout the paper we put $0/0 = 0$, $0 \cdot (\pm \infty) = 0$ and $1/(\pm \infty) = 0$. For $p \in (1, \infty)$, we define $p' = \frac{p}{p-1}$. We always denote by $c$ and
Theorem 1.2. Assume that \( u, v \in M^+ \) and \( w \in W \) such that \( \int_t^\infty w < \infty \) for all \( t \in (0, \infty) \).

(i) If \( 1 \leq q < \infty \), then inequality (1.2) holds for all \( f \in M^+ \) if and only if

\[
A_1 := \sup_{x \in (0, \infty)} \left( \int_x^\infty w \right)^{-\frac{1}{q}} \sup_{t \in (x, \infty)} \left( \int_t^\infty \frac{1}{v^{1-p}} \right)^{\frac{1-p}{p}} \left( \int_t^\infty u \right)^{-\frac{1}{q}} < \infty.
\]

Moreover, the best constant in (1.2) satisfies \( C \approx A_1 \).

(ii) If \( q < 1 \), then inequality (1.2) holds for all \( f \in M^+ \) if and only if

\[
A_2 := \sup_{x \in (0, \infty)} \left( \int_x^\infty w \right)^{-\frac{1}{q}} \left( \int_x^\infty \left( \int_x^t \frac{1}{v^{1-p}} \right)^{\frac{q(1-p)}{p(1-q)}} \left( \int_t^\infty u \right)^{\frac{1}{q}} \right)^{\frac{1-q}{q}} < \infty.
\]

Moreover, the best constant in (1.2) satisfies \( C \approx A_2 \).

Theorem 1.3. Let \( p = 1 \) and \( 0 < \theta \leq 1 < q < \infty \). Assume that \( u, v \in M^+ \) and \( w \in W \) such that \( \int_t^\infty w < \infty \) for all \( t \in (0, \infty) \). Then inequality (1.2) holds for all \( f \in M^+ \) if and only if

\[
A_3 := \sup_{t \in (0, \infty)} \left( \int_t^\infty u \right)^{-\frac{1}{q}} \text{ess sup}_{s \in (0, t)} v(s) \left( \int_s^\infty w \right)^{-\frac{1}{q}} < \infty.
\]

Moreover, the best constant in (1.2) satisfies \( C \approx A_3 \).

Theorem 1.4. Let \( 0 < p < \min\{1, q, \theta\} \). Assume that \( u, v \in M^+ \) and \( w \in W \) such that \( \int_t^\infty w < \infty \) for all \( t \in (0, \infty) \). Suppose that

\[
0 < \left( \int_0^t \left( \int_s^t \frac{1}{v^{1-p}} \right)^{\theta(1-p)} \left( \int_s^\infty w \right)^{-\frac{\theta}{p(p-\theta)}} w(s) ds \right)^{\frac{p}{p-\theta}} < \infty
\]

holds for all \( t \in (0, \infty) \).

(i) If \( \max\{1, \theta\} \leq q < \infty \), then inequality (1.2) holds for all \( f \in M^+ \) if and only if

\[
A_4 := \left( \int_0^\infty w \right)^{-\frac{1}{q}} \sup_{t \in (0, \infty)} \left( \int_0^t \frac{1}{v^{1-p}} \right)^{\frac{1-p}{p}} \left( \int_t^\infty u \right)^{-\frac{1}{q}} < \infty,
\]

and

\[
A_5 := \sup_{t \in (0, \infty)} \left( \int_0^t \left( \int_s^\infty w \right)^{-\frac{\theta}{p(p-\theta)}} w(s) \left( \int_0^t \frac{1}{v^{1-p}} \right)^{\theta(1-p)} ds \right)^{\frac{p}{p-\theta}} \left( \int_t^\infty u \right)^{-\frac{1}{q}} < \infty.
\]

Moreover, the best constant in (1.2) satisfies \( C \approx A_4 + A_5 \).
(ii) If $1 \leq q < \theta < \infty$, then inequality (1.2) holds for all $f \in M^+$ if and only if $A_4 < \infty$,
\[
A_6 := \left( \int_0^\infty \left( \int_0^t \left( \int_s^\infty w \right)^{\frac{\theta}{\theta-p}} w(s) \, ds \right)^{\frac{\theta(q-p)}{\theta(q-p)-(\theta-q)}} \left( \int_t^\infty w \right)^{\frac{\theta-p}{\theta-q}} w(t) \right) \times \sup_{z \in (t, \infty)} \left( \int_t^z \frac{\theta(q-1-p)}{p(q-p)} \left( \int_v^\infty u \right)^{\frac{\theta}{\theta-q}} \, dt \right) < \infty,
\]
and
\[
A_7 := \left( \int_0^\infty \left( \int_0^t \left( \int_s^\infty w \right)^{\frac{\theta}{\theta-p}} w(s) \left( \int_s^t v^{\frac{1}{1-p}} \right)^{\frac{\theta(q-p)}{\theta(q-p)-(\theta-q)}} \left( \int_t^\infty w \right)^{\frac{\theta-p}{\theta-q}} w(t) \right) \times \sup_{z \in (t, \infty)} \left( \int_t^z \frac{\theta(q-1-p)}{p(q-p)} \left( \int_v^\infty u \right)^{\frac{\theta}{\theta-q}} \, dt \right) \right) < \infty,
\]
where $A_4$ is defined in (1.3). Moreover, the best constant in (1.2) satisfies $C \approx A_4 + A_6 + A_7$.

(iii) If $\theta \leq q < 1$, then inequality (1.2) holds for all $f \in M^+$ if and only if $A_5 < \infty$,
\[
A_8 := \left( \int_0^\infty \left( \int_0^t \left( \int_s^\infty w \right)^{\frac{1}{p(q-n)}} \left( \int_t^\infty u \right)^{\frac{\theta}{\theta-q}} u(t) \, dt \right)^{1-q} \right) < \infty,
\]
and
\[
A_9 := \sup_{t \in (0, \infty)} \left( \int_t^\infty \left( \int_s^\infty w \right)^{\frac{1}{p(q-n)}} \left( \int_t^s v^{\frac{1}{1-p}} \right)^{\frac{1}{p(q-p)}} \left( \int_t^\infty u \right)^{\frac{\theta}{\theta-q}} u(s) \, ds \right)^{1-q} < \infty,
\]
where $A_5$ is defined in (1.4). Moreover, the best constant in (1.2) satisfies $C \approx A_5 + A_8 + A_9$.

(iv) If $\theta < \infty$ and $q < \min\{1, \theta\}$, then inequality (1.2) holds for all $f \in M^+$ if and only if $A_7 < \infty$, $A_8 < \infty$ and
\[
A_{10} := \left( \int_0^\infty \left( \int_0^t \left( \int_s^\infty w \right)^{\frac{1}{p(q-n)}} \left( \int_t^\infty w \right)^{\frac{\theta}{\theta-p}} w(t) \right) \times \left( \int_t^\infty \left( \int_t^s v^{\frac{1}{1-p}} \right)^{\frac{1}{p(q-p)}} \left( \int_t^\infty u \right)^{\frac{\theta}{\theta-q}} u(s) \, ds \right)^{1-q} \right) < \infty,
\]
where $A_7$ and $A_8$ are defined in (1.5) and (1.6), respectively. Moreover, the best constant in (1.2) satisfies $C \approx A_7 + A_8 + A_{10}$.

**Theorem 1.4.** Let $1 < \min\{q, \theta\}$, $q, \theta < \infty$, and $p = 1$. Assume that $u, v \in M^+$ and $w$ is a weight such that $\int_t^\infty w < \infty$ for all $t \in (0, \infty)$. Suppose that $v$ is continuous and
\[
0 < \int_0^t \frac{1}{\theta-p} < \infty, \quad 0 < \int_0^t \left( \int_s^\infty w \right)^{-\frac{\theta}{\theta-p}} w(t) \, dt < \infty, \quad 0 < \int_0^t u^{-\frac{1}{q}} \, dt < \infty
\]
holds for all $t \in (0, \infty)$.

(i) If $\theta \leq q$, then inequality (1.2) holds for all $f \in \mathcal{M}^+$ if and only if

$$A_{11} := \left( \int_0^\infty w \right)^\frac{1}{q} \sup_{t \in (0, \infty)} \left( \int_t^\infty u \right)^\frac{1}{q} \sup_{s \in (0, t)} v(s) < \infty,$$

and

$$A_{12} := \sup_{t \in (0, \infty)} \left( \int_0^t \left( \int_x^\infty w \right)^{-\frac{\theta}{\theta - 1}} w(x) \sup_{z \in (x, t)} v(z)^{\frac{\theta}{\theta - 1}} dx \right)^{\theta - 1 \theta q} \left( \int_t^\infty u \right)^{\frac{1}{q}} < \infty.$$

Moreover, the best constant in (1.2) satisfies $C \approx A_{11} + A_{12}$.

(ii) If $q < \theta$, then inequality (1.2) holds for all $f \in \mathcal{M}^+$ if and only if $A_{11} < \infty$,

$$A_{13} := \left( \int_0^\infty \left( \int_0^t \left( \int_x^\infty w \right)^{-\frac{\theta}{\theta - 1}} w(x) dx \right)^{\frac{\theta - 1}{\theta q}} \left( \int_t^\infty w \right)^{-\frac{\theta}{\theta - 1}} w(t) \right) \times \sup_{z \in (t, \infty)} v(z)^{\frac{\theta}{\theta - q}} \left( \int_z^\infty u \right)^{\frac{\theta}{\theta - q}} dt < \infty,$$

and

$$A_{14} := \left( \int_0^\infty \left( \int_0^t \left( \int_x^\infty w \right)^{-\frac{\theta}{\theta - 1}} w(x) \sup_{z \in (x, t)} v(z)^{\frac{\theta}{\theta - 1}} dx \right)^{\frac{\theta - 1}{\theta q}} \left( \int_t^\infty w \right)^{-\frac{\theta}{\theta - 1}} w(t) \right) \times \sup_{z \in (t, \infty)} v(z)^{\frac{\theta}{\theta - 1}} \left( \int_z^\infty u \right)^{\frac{\theta}{\theta - q}} dt < \infty,$$

where $A_{11}$ is defined in (1.7). Moreover, the best constant in (1.2) satisfies $C \approx A_{11} + A_{13} + A_{14}$.

It should be noted that, using the characterization of the embedding between weighted Cesàro function spaces, one can obtain the characterization of the embedding between weighted Copson function spaces. Indeed, using change of variables $x = 1/t$, it is easy to see that the embedding

$$\text{Cop}_{p_1,q_1}(u_1, v_1) \hookrightarrow \text{Cop}_{p_2,q_2}(u_2, v_2)$$

is equivalent to the embedding

$$\text{Ces}_{p_1,q}(\tilde{u}_1, \tilde{v}_1) \hookrightarrow \text{Cop}_{p_2,q_2}(\tilde{u}_2, \tilde{v}_2),$$

where $\tilde{u}_i(t) = t^{-2/q_i}u_i(1/t)$ and $\tilde{v}_i(t) = t^{-2/p_i}v_i(1/t)$, $i = 1, 2$, $t > 0$. We will not formulate the results here.

The paper is organized as follows. In the second section we present necessary background materials. In the third section we prove the main results of this paper.

2. Definitions and Preliminaries

Now, we will present some background information we need to prove our main results. Let us begin with the characterization of the well known Hardy-type inequalities (see, for instance, [31], Section 1.)
**Theorem 2.1.** Assume that $1 \leq p < \infty$, $0 < q < \infty$ and $v, w \in \mathcal{M}^+$. Let

$$H = \sup_{f \in \mathcal{M}^+} \left( \frac{\left( \int_0^\infty \left( \int_t^\infty f(s)ds \right)^q w(t)dt \right)^\frac{1}{q}}{\left( \int_0^\infty f(t)^p v(t)dt \right)^\frac{1}{p}} \right).$$

(i) If $1 < p \leq q$, then $H \approx H_1$, where

$$H_1 = \sup_{t \in (0, \infty)} \left( \frac{\left( \int_0^t w(s)ds \right)^\frac{1}{q}}{\left( \int_t^\infty v(s)^{1-p'}ds \right)^\frac{1}{p'}} \right).$$

(ii) If $1 < p$ and $q < p$, then $H \approx H_2$, where

$$H_2 = \left( \int_0^\infty \left( \int_0^t w(s)ds \right)^{\frac{p}{p-q}} \left( \int_t^\infty v(s)^{1-p'}ds \right)^{\frac{p(q-1)}{p-q}} v(t)^{1-p'}dt \right)^{\frac{p-q}{p}}.$$

**Theorem 2.2.** Assume that $1 < p < \infty$ and $v, w \in \mathcal{M}^+$. Let

$$H = \sup_{f \in \mathcal{M}^+} \left( \frac{\left( \int_0^\infty f(s)ds \right) w(t)}{\left( \int_0^\infty f(t)^p v(t)dt \right)^\frac{1}{p}} \right).$$

Then $H \approx H_5$, where

$$H_3 = \sup_{t \in (0, \infty)} \left( \text{ess sup}_{s \in (0,t)} w(s) \left( \int_t^\infty v(s)^{1-p'}ds \right)^{\frac{1}{p'}} \right).$$

Let us now recall the characterizations of reverse Hardy-type inequalities.

**Theorem 2.3.** [13, Theorem 5.1] Assume that $0 < q \leq p \leq 1$. Suppose that $v, w \in \mathcal{M}^+$ such that $w$ satisfies $\int_t^\infty w < \infty$ for all $t \in (0, \infty)$. Let

$$R = \sup_{f \in \mathcal{M}^+} \left( \frac{\left( \int_0^\infty f(t)^p v(t)dt \right)^\frac{1}{p}}{\left( \int_0^\infty \left( \int_0^t f(s)ds \right)^{\frac{q}{q-1}} w(t)dt \right)^\frac{1}{q}} \right).$$

(2.1)

(i) If $p < 1$, then $R \approx R_1$, where

$$R_1 = \sup_{t \in (0, \infty)} \left( \int_t^\infty w(s)ds \right)^{-\frac{1}{q}} \left( \int_t^\infty v(s)^{\frac{1-p}{p}}ds \right)^{\frac{1-p}{p}}.$$

(ii) If $p = 1$, then $R \approx R_2$, where

$$R_2 = \sup_{t \in (0, \infty)} \left( \int_t^\infty w(s)ds \right)^{-\frac{1}{q}} \left( \text{ess sup}_{s \in (t, \infty)} v(s) \right).$$
Theorem 2.4. [13, Theorem 5.4] Assume that $0 < p \leq 1$ and $p < q < \infty$. Suppose that $v, w \in \mathcal{M}^+$ such that $w$ satisfies $\int_t^\infty w < \infty$ for all $t \in (0, \infty)$ and $w \neq 0$ a.e. on $(0, \infty)$. Let $R$ be defined by (2.1).

(i) If $p < 1$, then $R \approx R_3$, where

$$R_3 = \left( \int_0^\infty \left( \int_t^\infty v(s)^{\frac{1}{1-p}} ds \right)^{\frac{q(1-p)}{q}} \left( \int_t^\infty w(s) ds \right)^{-\frac{q}{q-p}} w(t) dt \right)^{\frac{1}{q-p}} + \left( \int_0^\infty v(s)^{\frac{1}{1-p}} ds \right)^{\frac{1}{p}} \left( \int_0^\infty w(s) ds \right)^{-\frac{1}{q}}.$$

(ii) If $p = 1$, then $R \approx R_4$, where

$$R_4 = \left( \int_0^\infty \left( \sup_{s \in (0, \infty)} v(s)^{\frac{1}{q-1}} \right) \left( \int_t^\infty w(s) ds \right)^{-\frac{q-1}{q}} w(t) dt \right)^{\frac{1}{q-1}} + \left( \sup_{s \in (0, \infty)} v(s) \right) \left( \int_0^\infty w(s) ds \right)^{-\frac{1}{q}}.$$

Theorem 2.5. [25, Theorem 1.1] Let $1 < p < \infty$ and $0 < q, m < \infty$ and define $r := \frac{pq}{p-q}$. Assume that $u, v, w \in \mathcal{M}^+$ such that

$$0 < \left( \int_0^t \left( \int_s^t u \right)^{\frac{m}{q-m}} w(s) ds \right)^{\frac{1}{q}} < \infty$$

for all $t \in (0, \infty)$. Let

$$I = \sup_{f \in \mathcal{M}^+} \left( \int_0^\infty \left( \int_t^\infty \left( \int_s^t f \right)^m u(s) ds \right)^{\frac{q}{m}} w(t) dt \right)^{\frac{1}{q}} \left( \int_0^\infty f(t)^p v(t) dt \right)^{\frac{1}{p}}.$$

(i) If $p \leq \min\{m, q\}$, then $I \approx I_1$, where

$$I_1 := \sup_{t \in (0, \infty)} \left( \int_0^t w(s) \left( \int_s^t u \right)^{\frac{m}{q}} ds \right)^{\frac{1}{q}} \left( \int_t^\infty v^{1-p'} \right)^{\frac{1}{p'}}.$$

(ii) If $q < p \leq m$, then $I \approx I_2 + I_3$, where

$$I_2 := \left( \int_0^\infty \left( \int_0^t w \right)^{\frac{p}{q-p}} w(t) \sup_{z \in (t, \infty)} \left( \int_z^\infty u \right)^{\frac{m}{q-m}} \left( \int_x^\infty v^{1-p'} \right)^{\frac{p}{p'}} dt \right)^{\frac{1}{r}},$$

and

$$I_3 := \left( \int_0^\infty \sup_{z \in (t, \infty)} \left( \int_t^z u \right)^{\frac{m}{q-m}} \left( \int_z^\infty v^{1-p'} \right)^{\frac{p}{p'}} \times \left( \int_0^t w(s) \left( \int_s^t u \right)^{\frac{m}{q}} ds \right)^{\frac{1}{q}} w(t) dt \right)^{\frac{1}{r}}.$$
(iii) If $m < p \leq q$, then $I \approx I_1 + I_4$, where $I_1$ is defined in (2.2) and

$$I_4 := \sup_{t \in (0, \infty)} \left( \int_0^t w \left( \int_s^\infty \left( \int_t^s u \right)^{\frac{p}{p-m}} \left( \int_s^\infty v^{1-p'} \right)^{\frac{p(m-1)}{p-m}} v(s)^{1-p'} ds \right)^\frac{p-m}{p} \right).$$

(iv) If $\max\{m, q\} < p$ then $I \approx I_3 + I_5$, where $I_3$ is defined in (2.3) and

$$I_5 := \left( \int_0^\infty \left( \int_0^t w \right)^\frac{p}{m} w(t) \right) \times \left( \int_0^\infty \left( \int_t^s u \right)^{\frac{p}{p-m}} \left( \int_s^\infty v^{1-p'} \right)^{\frac{p(m-1)}{p-m}} v(s)^{1-p'} ds \right)^\frac{p-m}{p}.$$ 

**Theorem 2.6.** [24] Theorem 6] Let $1 < p < \infty$ and $0 < q < \infty$ and set $r := \frac{pq}{p-q}$. Assume that $u, v, w \in M^+$ such that $u$ is continuous and

$$0 < \int_0^t u < \infty, \quad 0 < \int_0^t v < \infty, \quad 0 < \int_0^t w < \infty$$

hold for all $t \in (0, \infty)$. Let

$$I = \sup_{f \in M^+} \left( \int_0^\infty \left( \sup_{s \in (t, \infty)} u(s) \int_s^\infty f^q w(t) dt \right)^\frac{1}{q} \right) \left( \int_0^\infty f(t)^p v(t) dt \right)^\frac{1}{p}.$$

(i) If $p \leq q$ then $I \approx I_6$, where

$$I_6 := \sup_{t \in (0, \infty)} \left( \int_0^t w(s) \sup_{z \in (s, t)} u(z)^q ds \right)^\frac{1}{q} \left( \int_t^\infty v^{1-p'} \right)^\frac{1}{p'}.$$

(ii) If $q < p$, then $I \approx I_7 + I_8$, where

$$I_7 := \left( \int_0^\infty \left( \int_0^t w \right)^\frac{p}{m} w(t) \sup_{s \in (t, \infty)} u(s)^r \left( \int_s^\infty v^{1-p'} \right)^\frac{1}{p'} dt \right)^\frac{1}{r},$$

and

$$I_8 := \left( \int_0^\infty \left( \int_0^t w(s) \sup_{z \in (s, t)} u(z)^q ds \right)^\frac{r}{m} w(t) \sup_{z \in (t, \infty)} u(z)^q \left( \int_z^\infty v^{1-p'} ds \right)^\frac{1}{p'} dt \right)^\frac{1}{q}.$$ 

### 3. Proofs of the Main Results

In this section we will prove our main results. Following lemma explains the assumption $p < 1$, when characterizing our main inequality.

**Lemma 3.1.** Let $0 < p, q, \theta < \infty$. Assume that $u, v, w \in M^+$ such that

$$0 < \int_t^\infty u < \infty, \quad 0 < \int_t^\infty w < \infty,$$

for all $t \in (0, \infty)$. If $p > 1$, then inequality (1.2) holds only for trivial functions.
Proof. Suppose that there exists a positive constant \( C \) such that inequality (1.2) holds for all non-negative measurable functions on \((0, \infty)\).

Let \( 0 < \tau_1 < \tau_2 < \infty \) and assume that \( h \) is a non-negative measurable function such that \( \text{supp} \, h \subset [\tau_1, \tau_2] \). Testing inequality (1.2) with \( h \), one can see that

\[
\left( \int_0^{\infty} \left( \int_0^t h^p v \right)^{\frac{q}{p}} u(t) dt \right)^{\frac{1}{q}} \geq \left( \int_{\tau_1}^{\tau_2} \left( \int_0^t h^p v \right)^{\frac{q}{p}} u(t) dt \right)^{\frac{1}{q}}
\]

and

\[
\left( \int_0^{\infty} \left( \int_0^t f(s) ds \right)^{\theta} w(t) dt \right)^{\frac{1}{\theta}} = \left( \int_{\tau_1}^{\tau_2} \left( \int_0^t f(s) ds \right)^{\theta} w(t) dt \right)^{\frac{1}{\theta}} \leq \left( \int_{\tau_1}^{\tau_2} h \left( \int_{\tau_1}^{\infty} w \right)^{\frac{1}{\theta}} \right)
\]

hold. Hence the validity of inequality (1.2) implies that

\[
\left( \int_{\tau_1}^{\tau_2} h^p v \right)^{\frac{q}{p}} \left( \int_{\tau_1}^{\infty} u(t) dt \right)^{\frac{1}{q}} \leq \left( \int_{\tau_1}^{\tau_2} h \right) \left( \int_{\tau_1}^{\infty} w \right)^{\frac{1}{\theta}}.
\]

Since \( 0 < \int_t^\infty u, \int_t^\infty w < \infty \) for all \( t \in (0, \infty) \), we arrive at \( L_1(1) \hookrightarrow L_p(v) \) when \( p > 1 \), which is a contradiction.

\[\square\]

Proof of Theorem 1.1. We begin with the well-known duality principle in weighted Lebesgue spaces. Recall that \( p \in (1, \infty) \), \( f \in M^+ \) and \( v \) is a weight on \((0, \infty)\), then

\[
\left( \int_0^{\infty} f(t)^p v(t) dt \right)^{\frac{1}{p}} = \sup_{h \in M^+} \frac{\int_0^{\infty} f(t) h(t) dt}{\left( \int_0^{\infty} h(t)^p v(t)^{1-p'} dt \right)^{\frac{1}{p'}}}.
\]

It is clear that since in our case \( q/p > 1 \), using Sawyer duality, the best constant of inequality (1.2) satisfies

\[
C = \sup_{f \in M^+} \frac{1}{\left( \int_0^{\infty} \left( \int_0^t f \right)^{\theta} w(t) dt \right)^{\frac{1}{\theta}}} \sup_{h \in M^+} \left( \int_0^{\infty} h(t) \left( \int_0^t f(s)^p v(s) ds dt \right)^{\frac{1}{p}} \right)^{\frac{q}{p-q}}.
\]

Interchanging suprema and applying Fubini, we get that

\[
C = \sup_{h \in M^+} \frac{1}{\left( \int_0^{\infty} h(t)^{\frac{q}{p-q}} u(t)^{\frac{q}{q-p} dt} \right)^{\frac{q}{q-p}}} \sup_{f \in M^+} \left( \int_0^{\infty} f(t)^p v(t) \int_t^{\infty} h(s) ds dt \right)^{\frac{1}{p}} \left( \int_0^{\infty} \left( \int_0^t f \right)^{\theta} w(t) dt \right)^{\frac{1}{\theta}}.
\]
Denote by

\[ D := \sup_{f \in \mathcal{M}^+} \left( \int_0^\infty f(t)^p v(t) \int_1^\infty h(s) ds dt \right)^{\frac{1}{p}}. \]  

(3.1)

Since \( \theta \leq p < 1 \), we have by applying [Theorem 2.3 (i)] that

\[ D \approx \sup_{x \in (0,\infty)} \left( \int_x^\infty v(s)^{\frac{1}{1-p}} \left( \int_s^\infty \frac{1}{1-p} \, ds \right)^{\frac{1-p}{p}} \left( \int_x^\infty w \right)^{-\frac{\theta}{p}} \right). \]

Therefore,

\[ C \approx \sup_{x \in (0,\infty)} \left( \int_x^\infty w \right)^{-\frac{1}{q'}} \sup_{h \in \mathcal{M}^+} \left( \int_0^\infty h(t)^{\frac{q}{q-p}} u(t)^{-\frac{p}{q-p}} \, dt \right)^{\frac{q-p}{q'}}. \]

Interchanging suprema yields that

\[ C \approx \sup_{x \in (0,\infty)} \left( \int_x^\infty w \right)^{-\frac{1}{q'}} \sup_{h \in \mathcal{M}^+} \left( \int_0^\infty h(t)^{\frac{q}{q-p}} u(t)^{-\frac{p}{q-p}} \, dt \right)^{\frac{q-p}{q'}}. \]

Then, it remains to apply Theorem 2.1. To this end, we need to split into two cases.

(i) If \( 1 \leq q \), in this case \( \frac{1}{1-p} \geq \frac{q}{q-p} \), then applying [Theorem 2.1 (i)], we obtain that

\[ C \approx \sup_{x \in (0,\infty)} \left( \int_x^\infty w \right)^{-\frac{1}{q'}} \max \left\{ \sup_{t \in (0,\infty)} \left( \int_t^\infty v(s)^{\frac{1}{1-p}} \chi_{(x,\infty)}(s) ds \right)^{\frac{1-p}{p}} \left( \int_t^\infty u \right)^{\frac{1}{q}}, \right. \]

\[ \sup_{t \in (x,\infty)} \left( \int_0^t v(s)^{\frac{1}{1-p}} \chi_{(x,\infty)}(s) ds \right)^{\frac{1-p}{p}} \left( \int_t^\infty u \right)^{\frac{1}{q}} \left) \right. \]

(ii) If \( q < 1 \), in this case \( \frac{1}{1-p} < \frac{q}{q-p} \), then applying [Theorem 2.1 (ii)], we arrive at

\[ C \approx \sup_{x \in (0,\infty)} \left( \int_x^\infty w \right)^{-\frac{1}{q'}} \left( \int_0^\infty \left( \int_0^t v(s)^{\frac{1}{1-p}} \chi_{(x,\infty)}(s) ds \right)^{\frac{q(1-p)}{p(1-q)}} \left( \int_t^\infty u \right)^{\frac{q}{1-q}} u(t) dt \right)^{\frac{1}{q'}}. \]
Proof of Theorem 1.2. As in the previous proof since \( q/p > 1 \), duality approach yields that,

\[
C = \sup_{h \in \mathcal{M}^+} \frac{D}{\left( \int_0^\infty h(t)^{\frac{q}{q-p}} u(t)^{-\frac{p}{q-p}} dt \right)^{\frac{q-p}{q}}} ,
\]

where \( D \) is defined in (3.1). Since, in this case \( \theta \leq p = 1 \), we have by applying [Theorem 2.3 (ii)], that

\[
C \approx \sup_{h \in \mathcal{M}^+} \frac{\left( \int_0^\infty h(t)^{\frac{q}{q-p}} u(t)^{-\frac{p}{q-p}} dt \right)^{\frac{q-p}{q-p}}}{\left( \int_0^\infty h(t)^{\frac{q}{q-p}} u(t)^{-\frac{p}{q-p}} dt \right)^{\frac{q-p}{q}}}.
\]

Recall that if \( F \) is a non-negative, non-decreasing measurable function on \((0, \infty)\), then

\[
\text{ess sup}_{t \in (0, \infty)} F(t) G(t) = \text{ess sup}_{t \in (0, \infty)} F(t) \text{ ess sup}_{\tau \in (t, \infty)} G(\tau),
\]

(3.2)

holds (see, for instance, page 85 in [19]). On using (3.2), we obtain that

\[
C \approx \sup_{h \in \mathcal{M}^+} \frac{\left( \int_0^\infty h(t)^{\frac{q}{q-p}} u(t)^{-\frac{p}{q-p}} dt \right)^{\frac{q-p}{q-p}}}{\left( \int_0^\infty h(t)^{\frac{q}{q-p}} u(t)^{-\frac{p}{q-p}} dt \right)^{\frac{q-p}{q}}}.
\]

Finally, applying Theorem 2.2, we arrive at

\[
C \approx \sup_{t \in (0, \infty)} \left( \int_t^\infty u \right)^{\frac{1}{q}} \text{ ess sup}_{s \in (0, t)} \left( \int_s^\infty w \right)^{-\frac{1}{q}}.
\]

Proof of Theorem 1.3. Similar to the previous proofs, we have that

\[
C = \sup_{h \in \mathcal{M}^+} \frac{D}{\left( \int_0^\infty h(t)^{\frac{q}{q-p}} u(t)^{-\frac{p}{q-p}} dt \right)^{\frac{q-p}{q}}} ,
\]

where \( D \) is defined in (3.1). Since in this case \( p < 1 \) and \( p < \theta \), we have, by applying [Theorem 2.4 (i)], that

\[
C \approx \left( \int_0^\infty w \right)^{-\frac{1}{q}} \sup_{h \in \mathcal{M}^+} \frac{\left( \int_0^\infty h (\int_s^\infty \frac{1}{1-p} v(s)^{-\frac{1}{1-p}} ds)^{\frac{1-p}{1}} \left( \int_0^\infty h(t)^{\frac{q}{q-p}} u(t)^{-\frac{p}{q-p}} dt \right)^{\frac{q-p}{q}} \right)}{\left( \int_0^\infty h(t)^{\frac{q}{q-p}} u(t)^{-\frac{p}{q-p}} dt \right)^{\frac{q-p}{q}}} + \sup_{h \in \mathcal{M}^+} \frac{\left( \int_0^\infty h \left( \int_t^\infty \frac{1}{1-p} v(t)^{-\frac{1}{1-p}} dt \right)^{\frac{p(1-p)}{q-p}} \left( \int_t^\infty w \right)^{-\frac{p}{q-p}} w(x) dx \right)^{\frac{q-p}{q}}}{\left( \int_0^\infty h(t)^{\frac{q}{q-p}} u(t)^{-\frac{p}{q-p}} dt \right)^{\frac{q-p}{q}}}.
\]
Thus, we need to consider the conditions on parameters in four cases.

We begin with the condition \( p < 1 \) and \( \theta < q < \theta \approx 2 \approx 2 \).

On the other hand, if \( p < q < 1 \), using [Theorem 2.1, (ii)], we get that

\[
C_1 \approx \left( \int_0^{\infty} w \right)^{-\frac{1}{p}} \left( \int_0^{t} \frac{1}{v^{1-p}} \right)^{\frac{1-p}{\theta}} \left( \int_t^{\infty} u \right)^{\frac{1}{q}} =: A_4. \tag{3.3}
\]

Let us now evaluate \( C_2 \). We will apply Theorem 2.5 with parameters

\[
m = \frac{1}{1-p}, \quad q = \frac{\theta}{\theta - p}, \quad p = \frac{q}{q - p}. \]

Thus, we need to consider the conditions on parameters in four cases.

(i) If \( p < \min\{1, q, \theta\} \) and \( \max\{1, \theta\} \leq q \), then applying [Theorem 2.5, (i)], we get that

\[
C_2 \approx I_1^{\frac{1}{p}}, \text{ where }
\]

\[
I_1^{\frac{1}{p}} = \sup_{t \in (0, \infty)} \left( \int_0^{t} \left( \int_0^{\infty} w \right)^{-\frac{\theta}{\theta - p}} w(s) \left( \int_s^{\infty} v^{\frac{1}{\theta - p}} \right)^{\frac{q(1-p)}{\theta - p}} ds \left( \int_t^{\infty} u \right)^{\frac{1}{q}} u(t) \right)^{\frac{1}{q} - \frac{1}{p}} =: A_5. \tag{3.5}
\]

Then, since \( 1 < q \) in this case, we have that \( C_1 \approx A_4 \). Therefore \( C = C_1 + C_2 \approx A_4 + A_5 \).

(ii) If \( p < \min\{1, q, \theta\} \) and \( 1 \leq q < \theta \), then applying [Theorem 2.5, (ii)], we get that

\[
C_2 \approx I_2^{\frac{1}{p}} + I_3^{\frac{1}{p}}, \text{ where }
\]

\[
I_2^{\frac{1}{p}} = \left( \int_0^{\infty} \left( \int_0^{t} \left( \int_0^{\infty} w \right)^{-\frac{\theta}{\theta - p}} w(s) \left( \int_s^{\infty} v^{\frac{1}{\theta - p}} \right)^{\frac{q(1-p)}{\theta - p}} ds \left( \int_t^{\infty} u \right)^{\frac{1}{q}} w(t) \right)^{-\frac{\theta}{\theta - p}} \right) \times \sup_{s \in (t, \infty)} \left( \int_s^{\infty} v^{\frac{1}{\theta - p}} \right)^{\frac{\theta q}{\theta - p}} \left( \int_s^{\infty} u \left( \frac{\theta}{\theta - q} \right) \right)^{\frac{\theta - q}{\theta - p}} =: A_6. \tag{3.6}
\]

and

\[
I_3^{\frac{1}{p}} = \left( \int_0^{\infty} \left( \int_0^{t} \left( \int_0^{\infty} w \right)^{-\frac{\theta}{\theta - p}} w(s) \left( \int_s^{\infty} v^{\frac{1}{\theta - p}} \right)^{\frac{q(1-p)}{\theta - p}} ds \left( \int_t^{\infty} u \right)^{\frac{1}{q}} w(t) \right)^{-\frac{\theta}{\theta - p}} \right) \times \sup_{s \in (t, \infty)} \left( \int_s^{\infty} v^{\frac{1}{\theta - p}} \right)^{\frac{\theta q}{\theta - p}} \left( \int_s^{\infty} u \left( \frac{\theta}{\theta - q} \right) \right)^{\frac{\theta - q}{\theta - p}} =: A_7. \tag{3.5}
\]

Since, \( 1 < q \) in this case, we have that \( C_1 \approx A_4 \). Therefore \( C = C_1 + C_2 \approx A_4 + A_6 + A_7 \).

(iii) If \( p < \min\{1, q, \theta\} \) and \( \theta \leq q < 1 \), then applying [Theorem 2.5, (iii)], we get that

\[
C_2 \approx I_4^{\frac{1}{p}} + I_5^{\frac{1}{p}}, \text{ where } I_1 \text{ is given in } (3.5) \text{ and }
\]

\[
I_4^{\frac{1}{p}} := \sup_{t \in (0, \infty)} \left( \int_0^{t} \left( \int_0^{\infty} w \right)^{-\frac{\theta}{\theta - p}} w(s) ds \right)^{\frac{\theta - q}{\theta - p}} \left( \int_t^{\infty} u \right)^{\frac{1}{q}} u(t) dt =: A_8. \tag{3.4}
\]
Since \( q < 1 \), we have that \( C_1 \approx A_8 \). Thus, \( C = C_1 + C_2 \approx A_8 + A_5 + A_9 \).

(iv) If \( p < \min\{1, q, \theta\} \) and \( q < \min\{1, \theta\} \), then applying [Theorem 2.5 (iv)], we get that

\[
C_2 \approx I_3^\frac{1}{p} + I_5^\frac{1}{p},
\]

where \( I_3 \) is given in (3.11) and

\[
I_5^\frac{1}{p} := \left( \int_0^\infty \left( \int_t^\infty \left( \int_s^\infty w \frac{\theta}{q-p} w(s) ds \right)^\frac{(1-p)}{q-p} \right)^\frac{q-p}{\theta} \right)^\frac{\theta}{q-p} w(t) \times \left( \int_t^\infty \left( \int_s^\infty u \frac{q}{1-q} u(s) ds \right)^\frac{1-q}{q} \right)^\frac{q-1}{1-q} dt =: A_{10}.
\]

Since, \( q < 1 \), again we have that \( C_1 \approx A_8 \), which yields that \( C = C_1 + C_2 \approx A_8 + A_7 + A_{10} \).

**Proof of Theorem 1.4.** We have already shown that

\[
C = \sup_{h \in \mathcal{M}^+} \frac{D}{\left( \int_0^\infty h(t) \frac{q}{q-p} u(t) \frac{1}{q-p} dt \right)^\frac{q-p}{q}}
\]

where \( D \) is defined in (3.11). Since in this case \( p = 1 \) and \( p < \theta \), we have, by applying [Theorem 2.4 (ii)], that

\[
C \approx \left( \int_0^\infty w \right)^{-\frac{1}{\theta}} \sup_{h \in \mathcal{M}^+} \frac{\text{ess sup } v(x) \int_x^\infty \frac{1}{h}}{\left( \int_0^\infty h(t) \frac{q}{q-p} u(t) \frac{1}{q-p} dt \right)^\frac{q-p}{q}} \times \left( \int_0^\infty \left( \text{ess sup } v(s) \int_s^\infty h \right)^\frac{\theta}{q-1} \left( \int_x^\infty \frac{1}{w} w(s) ds \right)^\frac{q-1}{q} \right)^\frac{q-1}{q} \times \left( \int_0^\infty h(t) \frac{q}{q-1} u(t) \frac{1}{q-1} dt \right)^\frac{q-1}{q} dt
\]

\[=: C_3 + C_4.\]

Since \( q/q - 1 > 1 \), applying Theorem 2.2, we have that

\[
C_3 \approx \left( \int_0^\infty w \right)^{-\frac{1}{\theta}} \sup_{r \in (0, \infty)} \left( \text{ess sup } v(s) \right) \left( \int_t^\infty u \right)^\frac{1}{q} =: A_{11}.
\]

On the other hand, in order to calculate \( C_4 \), we will apply Theorem 2.6 with parameters

\[
q = \frac{\theta}{\theta - 1} \quad \text{and} \quad p = \frac{q}{q - 1}.
\]

We need to apply this theorem to the cases \( \theta \leq q \) and \( q < \theta \) separately.
(i) If $\theta \leq q$, then applying [Theorem 2.6 (i)], we have that $C_4 \approx I_6$, where

$$I_6 = \sup_{t \in (0, \infty)} \left( \int_0^t \left( \int_x^\infty w(x) \sup_{z \in (x,t)} v(z) \frac{\theta}{\theta - q} dx \right) \left( \int_t^\infty u \right) \right) \approx: A_{12}.$$ 

Therefore, $C = C_3 + C_4 \approx A_{11} + A_{12}$.

(ii) If $q < \theta$, then applying [Theorem 2.6 (ii)], we have that $C_4 \approx I_7 + I_8$, where

$$I_7 = \left( \int_0^\infty \left( \int_0^t \left( \int_x^\infty w(x) \left( \int_t^\infty u \right) \right) \right) \left( \int_t^\infty w(t) \right) \right) \times \sup_{z \in (t, \infty)} v(z) \frac{\theta q}{\theta - q} \left( \int_z^\infty u \right) \approx: A_{13},$$

and

$$I_8 = \left( \int_0^\infty \left( \int_0^t \left( \int_x^\infty w(x) \sup_{z \in (x,t)} v(z) \frac{\theta(q - 1)}{\theta - q} dx \right) \left( \int_t^\infty w(t) \right) \right) \times \sup_{z \in (t, \infty)} v(z) \frac{\theta q}{\theta - q} \left( \int_z^\infty u \right) \right) \approx: A_{14}.$$

Then, we arrive at $C = C_3 + C_4 \approx A_{11} + A_{13} + A_{14}$.

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