A Cosmological Mass Function with Broken Hierarchy

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ABSTRACT

We construct an analytic formalism for the mass function of cold dark matter halos, assuming that there is a break in the hierarchical merging process. According to this broken-hierarchy scenario, due to the inherent nature of the gravitational tidal field the formation of massive pancakes precedes that of dark halos of low-mass. In the framework of the Zel’dovich approximation which generically predicts the presence of pancakes, we first derive analytically the conditional probability that a low-mass halo observed at present epoch was embedded in an isolated pancake at some earlier epoch. Then, we follow the standard Press-Schechter approach to count analytically the number density of low-mass halos that formed through anti-hierarchical fragmentation of the massive pancakes. Our mass function is well approximated by a power-law $dN/dM = M^{-l}$ in the mass range $10^6h^{-1}M\odot \leq M \leq 10^{10}h^{-1}M\odot$ with the slope $l = 1.86$ shallower than that of the currently popular Sheth-Tormen mass function $l = 2.1$. It is expected that our mass function will provide a useful analytic tool for investigating the effect of broken hierarchy on the structure formation.

Subject headings: cosmology:theory — large-scale structure of universe

1. INTRODUCTION

In the cold dark matter (CDM) paradigm the gravitationally bound objects made of dark matter particles (dark halos) are supposed to form through hierarchical merging process. Press & Schechter (1974, hereafter PS) devised for the first time an analytic formalism to evaluate the mass distribution function of CDM halos that formed hierarchically. Later Sheth & Tormen (1999, hereafter ST) refined the PS mass function by taking into account the non-spherical collapse. The ST mass function has been tested against many N-body simulations, showing good agreements (e.g., Reed et al. 2003). This success of the ST mass function
implies the validity of the hierarchical merging scenario since it was originally derived under the assumption that the CDM halos form through hierarchical merging.

Yet it is premature to assert that the formation of CDM halos is always hierarchical over the entire mass range given the fact that the validity of the ST mass function was rather limited to the relatively high-mass section (> $10^{10} h^{-1} M_\odot$). Results from N-body simulations still suffer from the lack of information on the low-mass section ($\leq 10^{10} h^{-1} M_\odot$) due to the resolution limit.

Very recently, Mo et al. (2005) came up with a new halo-formation scenario where the formation of low-mass halos is somewhat anti-hierarchical, preceded by that of massive pancakes. According to their model, the preheated medium caused by the gravitational pancaking effect suppresses the star-formation sufficiently in the low-mass halos, which can explain the observed low HI-mass as well as the faint-end slope of the galaxy luminosity function.

Although the work of Mo et al. (2005) was focused on explaining the suppression of star-formation in the low-mass halos, we note that if the formation of low-mass halos was indeed preceded by the formation of massive pancakes, what is suppressed should not be only the star-formation but also the formation of low-mass halos itself. To take into account the break in the hierarchical process and to predict the abundance of low-mass halos more accurately, it is desirable to have an analytic model for it derived from first principles.

Our goal here is to construct an analytic model for the low-mass halos that form through anti-hierarchical fragmentation of massive pancakes. To achieve this goal, we adopt the Zel’dovich approximation as a simplest footstep which generically predicts the formation of pancakes, and we follow the standard PS approach to count the abundance of dark halos as a function of mass. The hypotheses, the mathematical layout and the predictions of our model are presented in §2, and the summary and discussion of the final results are provided in §3.

2. ANALYTIC FORMALISM

In the Press-Schechter formalism (Press & Schechter 1974, hereafter, PS), an isolated dark halo (a halo just collapsed) of mass $M$ forms from the regions in the density field whose average density contrast $\delta \equiv \Delta \rho/\bar{\rho}$ ($\bar{\rho}$: the mean mass density of the universe) smoothed on the mass scale of $M$ reaches some critical value, $\delta_c$. The value of the critical density $\delta_c$ is approximately 1.68, which depends very weakly on the background cosmology and the redshift (Kitayama & Suto 1996).
The Gaussian probability distribution of the linear density contrast smoothed with the sharp k-space filter on the mass scale \( M \) is given as

\[
p(\delta) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-\delta^2}{2\sigma^2}\right), \quad \sigma^2(M) \equiv \int_{-\infty}^{\ln k_c} \Delta^2(k) d\ln k, \quad M = 6\pi^2 \bar{\rho} k_c^{-3},
\]

where \( \sigma(M) \) is the rms density fluctuation, and \( \Delta^2(k) \) is the dimensionless power spectrum. Throughout this Letter, we use the power spectrum of the concordance \( \Lambda \)CDM cosmology with \( \Omega_\Lambda = 0.7, \Omega_m = 0.3, \Omega_b = 0.044, h = 0.7 \) (Bardeen et al. 1986).

In the Zel’ dovich approximation (Zeldovich 1970, hereafter, ZEL), the mass density is given as

\[
\rho = \frac{\bar{\rho}}{(1-\lambda_1)(1-\lambda_2)(1-\lambda_3)},
\]

where \( \lambda_1, \lambda_2, \lambda_3 \) (with \( \lambda_1 \geq \lambda_2 \geq \lambda_3 \)) are the three eigenvalues of the deformation tensor \( d_{ij} \) defined as the second derivative of the perturbation potential \( \Phi \): \( d_{ij} \equiv \partial_i \partial_j \Phi \). Doroshkevich (1970) derived the joint distribution of the three eigenvalues:

\[
p(\lambda_1, \lambda_2, \lambda_3) = \frac{3375}{8\sqrt{5\pi}\sigma^6} \exp\left(-\frac{3I_1^2}{\sigma^2} + \frac{15I_2}{2\sigma^2}\right) \frac{1}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_1-\lambda_3)},
\]

with \( I_1 \equiv \lambda_1 + \lambda_2 + \lambda_3 \) and \( I_2 \equiv \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_2\lambda_3 \).

Equation (2) implies that the mass-density diverges along the direction of the major principal axis of the deformation tensor if the largest eigenvalue reaches unity and the intermediate and the smallest eigenvalues are less than zero. In other words, an isolated pancake (a two-dimensional object just collapsed only along the first principal axis) of mass \( M \) forms if the following condition is satisfied: \( \lambda_1 = \lambda_c, \lambda_2 < 0, \lambda_3 < 0 \) with \( \lambda_c = 1 \) on the mass scale \( M \).

Now that the conditions for the formation of isolated halos and pancakes are specified, we would like to find the probability that a halo at present epoch was embedded in a pancake at some earlier epoch before it formed. For this, it is required to have the joint distribution of the linear density and the three eigenvalues of the deformation tensor on two different scales on two different epochs. Let \( \delta \) be defined at present epoch on the galactic mass scale \( M_g \), and let \( \lambda_1, \lambda_2, \lambda_3 \) be defined at some earlier epoch of redshift \( z > 0 \) on the larger mass scale \( M_p > M_g \). The rms density fluctuation at redshift \( z \) on mass scale \( M_p \) is given as \( \sigma(M_p, z) = b(z)\sigma(M_p) \) where \( b(z) \) is the growth factor of the linear density that is normalized to satisfy \( b(0) = 1 \). Since the growth factor as well as the rms density fluctuation is a decreasing function of \( z \), we have \( \sigma(M_p, z) < \sigma(M_g) \). From here on, we use the notations of \( \sigma', d'_{ij}, \lambda'_1, \lambda'_2, \lambda'_3 \) to represent the rms density fluctuation, the deformation tensor and its three eigenvalues at redshift \( z \) on the mass scale \( M_p \).
To derive the joint distribution of $\delta$ and $\lambda_1', \lambda_2', \lambda_3'$, we first derive the multivariate Gaussian distribution of $\delta$ and the 6 independent components of the symmetric tensor, $d'_{ij}$. Rotating the frame into the principal axis of $d'_{ij}$ and using the fact that $\delta$ is invariant under the axis-rotation, we derive analytically the joint distribution of $\delta$ and $\lambda_1', \lambda_2', \lambda_3'$

$$p(\delta, \lambda_1', \lambda_2', \lambda_3') = \frac{1}{\sqrt{2\pi}\delta\Delta} \frac{3375}{8\sqrt{5\pi}\sigma^6} \exp \left[ -\frac{(\delta - I_1')^2}{2\sigma^2_\Delta} \right] \times \exp \left( -\frac{3I_1'^2}{\sigma^2} + \frac{15I_2'}{2\sigma^2} \right) \left( \lambda_1' - \lambda_2' \right) \left( \lambda_2' - \lambda_3' \right) \left( \lambda_1' - \lambda_3' \right),$$

where $\sigma^2_\Delta \equiv \sigma^2 - \sigma'^2$, $I_1' = \lambda_1' + \lambda_2' + \lambda_3'$, and $I_1 = \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1$.

Through equations and (1) and (4) we find the probability that a halo of mass $M_g$ observed at present epoch was embedded in an isolated pancake of larger mass $M_p$ at redshift $z$ with the help of the Bayes theorem:

$$p(\lambda_1' = \lambda_c, \lambda_2' < 0, \lambda_3' < 0 | \delta \geq \delta_c) = \frac{p(\delta \geq \delta_c, \lambda_1' = \lambda_c, \lambda_2' < 0, \lambda_3' < 0)}{P(\delta \geq \delta_c)},$$

$$= \frac{\int_{\lambda_1'}^{0} d\lambda_2' \int_{-\infty}^{0} d\lambda_3' \int_{-\infty}^{0} d\delta p(\delta, \lambda_1', \lambda_2', \lambda_3')} {\int_{\delta_c}^{\infty} d\delta p(\delta)}. \quad (5)$$

In equation (5), the integration over $\delta$ can be readily evaluated

$$\int_{\lambda_1'}^{0} d\lambda_2' \int_{-\infty}^{0} d\lambda_3' \int_{-\infty}^{0} d\delta p(\delta, \lambda_c, \lambda_2', \lambda_3') = \frac{1}{2} \int_{\lambda_1'}^{0} d\lambda_2' \int_{-\infty}^{0} d\lambda_3' \text{erfc} \left( \frac{\delta_c - I_1'}{\sqrt{2}\sigma_\Delta} \right) \times p(\lambda_1' = \lambda_c, \lambda_2', \lambda_3'), \quad (6)$$

$$\int_{\delta_c}^{\infty} d\delta p(\delta) = \frac{1}{2} \text{erfc} \left( \frac{\delta_c}{\sqrt{2}\sigma} \right). \quad (7)$$

This probability (eq. [5]) will allow us to determine the most-likely epoch when the formation of pancakes precedes that of low-mass halos, and the typical mass scale for the formation of pancakes as well.

For comparison, we consider the probability that a halo of mass $M_g$ observed at present epoch just formed hierarchically at redshift $z$ which is approximately given as (Bower 1991; Lacey & Cole 1994):

$$p(\delta'' = \delta_c | \delta \geq \delta_c) = \frac{p(\delta \geq \delta_c, \delta'' = \delta_c)}{P(\delta \geq \delta_c)},$$

$$= \frac{1}{\sqrt{2\pi}\sigma''} \left[ \text{erfc} \left( \frac{\delta_c}{\sqrt{2}\sigma} \right) \right]^{-1} \exp \left( -\frac{\delta_c^2}{2\sigma''^2} \right), \quad (8)$$
where $\delta''$ represents the linear density on the mass scale $M_g$ at redshift $z$. The relative difference between $p(\lambda'_1 = \lambda_c, \lambda'_2 < 0, \lambda'_3 < 0 | \delta \geq \delta_c)$ and $p(\delta' = \delta_c | \delta \geq \delta_c)$ indicates how probable the anti-hierarchical formation of low-mass halos is at given epoch.

The direct comparison of the two conditional probabilities (eqs. [5] and [8]) is shown in Fig. 1. The halo mass $M_g$ observed at present epoch is set at the dwarf galactic scale $M_g = 10^6 h^{-1} M_\odot$, and three different cases of the pancake’s mass $M_p$ are considered: $M_p = 10^8 h^{-1} M_\odot$ (dashed); $M_p = 10^{10} h^{-1} M_\odot$ (solid); $M_p = 10^{12} h^{-1} M_\odot$ (long dashed). As can be seen, the value of $p(\lambda'_1 = \lambda_c, \lambda'_2 < 0, \lambda'_3 < 0 | \delta \geq \delta_c)$ is twice higher than that of $p(\delta' = \delta_c | \delta \geq \delta_c)$ at $z \sim 2$ for the case of $10^{10} h^{-1} M_\odot \leq M_p \leq 10^{12} h^{-1} M_\odot$.

Figure 2 also plots the two probabilities as solid and dashed lines. In this Fig. 2 the pancake’s mass is set at $M_p = 10^{11} h^{-1} M_\odot$, and the four different cases of the halo mass $M_g$ are considered in separate panels: $M_g = 10^6 h^{-1} M_\odot$ (upper left); $M_g = 10^7 h^{-1} M_\odot$ (upper right); $M_g = 10^8 h^{-1} M_\odot$ (lower left); $M_g = 10^9 h^{-1} M_\odot$ (lower right). As shown, the probability distribution $p(\lambda'_1 = \lambda_c, \lambda'_2 < 0, \lambda'_3 < 0 | \delta \geq \delta_c)$ has a maximum value around $z = 2$, position of which shifts to the low-redshift section as the halo mass $M_g$ increases. For all four cases of $M_g$ at $z \sim 2$, the value of $p(\lambda'_1 = \lambda_c, \lambda'_2 < 0, \lambda'_3 < 0 | \delta \geq \delta_c)$ is consistently higher than that of $p(\delta' = \delta_c | \delta \geq \delta_c)$. The results shown in Figs. 1 and 2 imply that the halo of mass $M_g \leq 10^{10} h^{-1} M_\odot$ observed at present epoch are more likely to have been embedded in massive pancakes of mass $M_p \approx 10^{11} h^{-1} M_\odot$ around $z = 2$ rather than formed through hierarchical merging.

Setting the typical mass scale and redshift for the formation of pancakes at $10^{11} h^{-1} M_\odot$ and $z = 2$, respectively, we follow the standard PS approach to evaluate the mass distribution function of the low-mass halos that formed anti-hierarchically. According to the theory the differential number density of the dark halos in the mass range $[M, M + dM]$ is related to the volume fraction $F$ occupied by the proto-halo regions in the linear density field that satisfy a specified collapse condition:

$$\frac{dN}{dM} \equiv A \frac{\bar{\rho}}{M} \left| \frac{dF}{dM} \right| = A \frac{\bar{\rho}}{M^2} \left| \frac{d\sigma}{d\ln M} \right| \left| \frac{\partial F}{\partial \sigma} \right|, \quad (9)$$

where $A$ is the normalization factor, which is exactly 2 in the original PS theory (Peacock & Heavens 1990; Bond et al. 1991; Jedamzik 1995). If the halos observed at present epoch were embedded in massive pancakes at redshift $z$, the volume fraction $F$ should be written as

$$F(\sigma) = \int_{\delta_c}^{\infty} d\delta \ p(\delta | \lambda'_1 = \lambda_c, \lambda'_2 < 0, \lambda'_3 < 0). \quad (10)$$

The conditional probability $p(\delta | \lambda'_1 = \lambda_c, \lambda'_2 < 0, \lambda'_3 < 0)$ in this equation (10) can be found
from equations (3) and (4) by using the Bayes theorem again:

\[
p(\delta|\lambda_1' = \lambda_c, \lambda_2' < 0, \lambda_3' < 0) = \frac{p(\delta, \lambda_1' = \lambda_c, \lambda_2' < 0, \lambda_3' < 0)}{P(\lambda_1' = 1, \lambda_2' < 0, \lambda_3' < 0)} = \frac{\int_{X_3'}^0 d\lambda_2' \int_{-\infty}^0 d\lambda_3' p(\delta, \lambda_1' = \lambda_c, \lambda_2', \lambda_3')}{\int_{X_3'}^0 d\lambda_2' \int_{-\infty}^0 d\lambda_3' p(\lambda_1' = \lambda_c, \lambda_2', \lambda_3')}
\]

(11)

Now, the differential volume fraction \( \partial F/\partial \sigma \) in equation (9) is found to be

\[
\frac{\partial F}{\partial \sigma} = \frac{\partial}{\partial \sigma} \int_{\delta_c}^\infty d\delta' p(\delta|\lambda_c, \lambda_2' < 0, \lambda_3' < 0),
\]

\[
= -\left( \frac{\sigma}{\sigma} \right) \left[ \int_{X_3'}^0 d\lambda_2' \int_{-\infty}^0 d\lambda_3' p(\lambda_c, \lambda_2', \lambda_3') \right]^{-1} \times \int_{X_3'}^0 d\lambda_2' \int_{-\infty}^0 d\lambda_3' (\delta_c - \lambda_1' - \lambda_2' - \lambda_3') p(\delta', \lambda_c, \lambda_2', \lambda_3').
\]

(12)

The logarithmic derivative of the rms density fluctuation \( d\sigma/d\ln M \) in equation (9) for the case of the sharp k-space filter is also found to be

\[
\frac{d\sigma}{d\ln M} = -\frac{1}{6\sigma} \Delta^2(\ln k_c),
\]

(13)

where \( k_c \) is given in equation (1).

By equations (9)-(13), we evaluate the mass function of the low-mass halos in the mass range \( 10^6 h^{-1} M_\odot \leq M \leq 10^{10} h^{-1} M_\odot \), assuming that all halos in this mass range were embedded in pancakes of mass \( 10^{11} h^{-1} M_\odot \) at redshift \( z = 2 \). The normalization factor \( A \) in equation (9) is determined from the constraint that our mass function on the mass scale \( M = 10^{10} h^{-1} M_\odot \) should give the same value as that of the ST formula which is known to agree very well with N-body simulation in the mass range \( M \geq 10^{10} h^{-1} M_\odot \).

Figure 3 plots our result (solid), and compares it with the original PS (dotted) and the ST (dashed) mass functions. As can be seen, our model predicts less number of low-mass halos when compared with the PS and the ST mass functions. That is, the formation of low-mass halos is suppressed by the earlier formation of massive pancakes. Our mass function is found to be well fitted by a power-law, \( dN/dM \approx M^{-1.86} \), which is shallower than the PS and the ST ones, \( dN/dM \approx M^{-2.1} \). This shallow shape of our mass function in the low-mass tail is consistent with the recent high-resolution simulation (Yahagi et al. 2004).
3. SUMMARY AND DISCUSSION

By taking into account the possibility that the low-mass CDM halos form through anti-hierarchical fragmentation of the massive pancakes, we have derived a new analytic mass function for the low-mass halos in the ΛCDM cosmology with the help of the Zel’dovich approximation and the Press-Schechter mass function theory. It has been shown that our mass function has a shallower slope in the low-mass tail and predicts maximum five times less abundance of dwarf galactic halos of mass $10^6h^{-1}M_\odot$ than the currently popular Sheth-Tormen formula.

The concept of broken-hierarchy should modify not only the mass function but also the other halo statistics from the previous models that were constructed under the assumption that the halo formation is perfectly hierarchical. For instance, the two-point correlation of dwarf galactic halos would be different in the broken-hierarchy scenario, which in turn implies the mass-to-light bias on the dwarf galactic scale would be altered in accordance. Our future work will be in the direction of investigating the effect of broken-hierarchy on the halo n-point correlations and the mass-to-light bias as well.

Since our mass function has been derived analytically from first principles without introducing any fitting parameters, one may not expect it to be very realistic. The formation of low-mass halos should be dominated by complicated non-linear processes which cannot be described by using first principles alone. However, as it is the first attempt to model the broken hierarchy which can accommodate future refinements, it is concluded that our model will provide a useful guideline for the theoretical study of the effects of broken hierarchy on the structure formation.

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Fig. 1.— Probability that a galactic halo observed on mass scale, $M_g = 10^6 h^{-1} M_\odot$ at present epoch was embedded in a pancake at redshift $z$ for the three cases of the pancake’s mass: $M_p = 10^8 h^{-1} M_\odot$ (dashed); $M_p = 10^{10} h^{-1} M_\odot$ (solid); $M_p = 10^{12} h^{-1} M_\odot$ (long dashed). For comparison, the probability that a galactic halo formed hierarchically at redshift $z$ is also plotted (dotted).
Fig. 2.— Comparison of the probability that a galactic halo forms in the anti-hierarchically (solid) with the probability that it forms in the purely hierarchical way for four different cases of the halo mass $M_g$: $M_g = 10^6 h^{-1} M_\odot$ (upper left); $M_g = 10^7 h^{-1} M_\odot$ (upper right); $M_g = 10^8 h^{-1} M_\odot$ (lower right); $M_g = 10^9 h^{-1} M_\odot$ (lower left). The pancake’s mass is set at $M_p = 10^{11} h^{-1} M_\odot$. 

\begin{align*}
M_g &= 10^6 h^{-1} M_\odot \\
M_p &= 10^{11} h^{-1} M_\odot \\
M_g &= 10^7 h^{-1} M_\odot \\
M_g &= 10^8 h^{-1} M_\odot \\
M_g &= 10^9 h^{-1} M_\odot \\
M_g &= 10^{10} h^{-1} M_\odot
\end{align*}
Fig. 3.— Predictions of our (solid), the Press-Schechter (dashed) and the Sheth-Tormen (dotted) models for the number density of dark halos as a function of logarithmic mass.