Exact Multiplicities in the Three-Anyon Spectrum

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Abstract

Using the symmetry properties of the three-anyon spectrum, we obtain exactly the multiplicities of states with given energy and angular momentum. The results are shown to be in agreement with the proper quantum mechanical and semiclassical considerations, and the unexplained points are indicated.
It is well known that the quantum mechanical spectrum of \(N\) non-interacting bosons or fermions can be obtained given just the single-particle spectrum, but for anyons this does not hold because the \(N\)-anyon problem is essentially many-particle. For arbitrary \(N\) there are two classes of exact solutions but their relative number decreases rapidly with \(N\) increasing. The only case in which one can proceed with the exact analysis more or less far ahead is that of \(N = 3\). In the previous work we have shown that it is possible to calculate exactly all the degeneracies in the three-anyon spectrum using certain symmetry properties of the latter. Here we will carry out an analogous calculation, taking into account in addition the angular momentum. Our present consideration will allow us to shed at least some light on the problem of quantum mechanical description of anyonic spectra, which at the moment is far from being closed.

As it was done earlier, we consider the problem of three non-interacting anyons in a harmonic potential, with the particle mass and the frequency set to unity. The Hamiltonian is \(\hat{H} = \sum_{j=1}^{3} \hat{H}_j\) with the one-particle Hamiltonian \(\hat{H}_j = \frac{1}{2}(-\Delta_j + \mathbf{r}_j^2)\). The single-particle state is uniquely determined by the two quantum numbers – energy \(E\) which may equal \(1, 2, \ldots\) and angular momentum \(L\). Formally, the number of single-particle states with energy \(E\) and momentum \(L\) is given by

\[
g_1(E, L) = \begin{cases} r(E - L, 2) & \text{if } |L| \leq E - 1 \\ 0 & \text{otherwise} \end{cases}
\]  

(1)

where \(r(a, b)\) is the remainder of the division of \(a\) by \(b\). (Here and further \(E\) is implicit to be a positive integer and \(L\) an integer.) In what follows it will be also convenient to use, along with \(E\) and \(L\), the numbers \(\ell^+ = (E + L - 1)/2\) and \(\ell^- = (E - L - 1)/2\). Obviously, any pair of non-negative integers \((\ell^+, \ell^-)\) corresponds to exactly one state.

To calculate the multiplicities \(\tilde{g}_{3B}(E, L)\) and \(\tilde{g}_{3F}(E, L)\) in the relative motion spectrum of three bosons and three fermions, respectively, we start from the multiplicities \(g_3(E, L)\) corresponding to Boltzmann statistics, then come to \(g_{3B}(E, L)\) and \(g_{3F}(E, L)\) for the full spectrum, and afterwards separate the center-of-mass motion.

The calculation is simplified by using

\[
\mathcal{L}^\pm = \frac{1}{2}(E \pm L - 3);
\]  

(2)

since \(\mathcal{L}^\pm = \ell_1^\pm + \ell_2^\pm + \ell_3^\pm\) with the \(\ell_i^\pm\)'s as defined earlier, the Boltzmann degeneracy is \(g_3(E, L) = S_3(\mathcal{L}^+)S_3(\mathcal{L}^-)\) where \(S_3(N)\) is the number of ordered triples of integers the sum of which is \(N\). One has \(S_3(N) = \frac{1}{2}(N + 1)(N + 2)\) and consequently

\[
g_3(E, L) = \frac{1}{64} \left[ (E^2 - 1)^2 - 2(E^2 + 1)L^2 + L^4 \right].
\]  

(3)
Now, in Boltzmann count one bosonic state is taken six times if all the particles occupy different single-particle states, three times if two are in the same state but the third is in another one and one time if all are in one and the same state. Therefore

\[
g_{3B}(E, L) = \frac{1}{6} \left[ g_3(E, L) + 3d_3(E, L) + 2t_3(E, L) \right]. \tag{4}
\]

where \(d_3(E, L)\) is the number of states with two particles in the same state (the third one may be or not be in that state) and \(t_3(E, L)\) is the number of states with all three particles in the same state. For fermions, correspondingly,

\[
g_{3F}(E, L) = \frac{1}{6} \left[ g_3(E, L) - 3d_3(E, L) + 2t_3(E, L) \right]. \tag{5}
\]

By virtue of the aforesaid,

\[
d_3(E, L) = Q_3(L^+)Q_3(L^-) \tag{6}
\]

with \(Q_3(N)\) the number of ordered pairs \((\ell_1, \ell_3)\) such that \(2\ell_1 + \ell_3 = N\) \((\ell_2 = \ell_1)\),

\[
Q_3(N) = \left\lfloor \frac{N}{2} \right\rfloor + 1 \equiv \frac{1}{2} [N - r(N, 2)] + 1, \tag{7}
\]

where \([n]\) stands for the entire part of \(n\). Finally,

\[
t_3(E, L) = d(E, 3)d(L, 3), \tag{8}
\]

where we have introduced the "multiplicity function"

\[
d(a, b) = \begin{cases} 
1 & \text{if } r(a, b) = 0 \\
0 & \text{otherwise}
\end{cases}. \tag{9}
\]

Explicit expressions for \(g_{3B}\) and \(g_{3F}\) read

\[
g_{3B}(E, L) = \frac{1}{384} \left[ (E^2 - L^2)^2 + 10E^2 - 14L^2 + 24E + 13 \right] \\
+ \frac{1}{16} \left[ 2\mathcal{M}^+\mathcal{M}^- + \mathcal{M}^+(L - E - 1) - \mathcal{M}^-(L + E + 1) \right] + \frac{1}{3}d(E, 3)d(L, 3), \tag{10}
\]

\[
g_{3F}(E, L) = \frac{1}{384} \left[ (E^2 - L^2)^2 - 14E^2 + 10L^2 - 24E - 11 \right] \\
- \frac{1}{16} \left[ 2\mathcal{M}^+\mathcal{M}^- + \mathcal{M}^+(L - E - 1) - \mathcal{M}^-(L + E + 1) \right] + \frac{1}{3}d(E, 3)d(L, 3), \tag{11}
\]

where \(\mathcal{M}^\pm = r(L^\pm, 2)\), \(L^\pm\) are defined by \((2)\), and \(E\) and \(L\) should be of different parity.
In order to separate the center-of-mass motion, note that
\[ g_{3S}(E, L) = \sum_{e=1}^{E} \sum_{l=-e+1}^{-1} \tilde{g}_{3S}(E - e, L - l) \] (12)
for \( S = B, F \) (\( e \) and \( l \) are the center-of-mass energy and angular momentum, respectively). Eq.(12) implies
\[ \tilde{g}_{3S}(E, L) = g_{3S}(E + 1, L) + g_{3S}(E - 1, L) - g_{3S}(E, L - 1) - g_{3S}(E, L + 1). \] (13)
Substituting (10) and (11), one gets
\[ \tilde{g}_{3(B/F)}(E, L) = \frac{1}{24}(E^2 - L^2) + \frac{1}{3}[d(E + 1, 3)d(L, 3) + d(E - 1, 3)d(L, 3) \]
\[ -d(E, 3)d(L - 1, 3) - d(E, 3)d(L + 1, 3)] \pm \frac{1}{8}(E - L, 4)r(E + L, 4), \] (14)
the upper/lower sign referring to \( B/F \); here, \( E \) and \( L \) have to be of the same parity.
In the table below, the values of \( \tilde{g}_{3B}(E, L) \) and \( \tilde{g}_{3F}(E, L) \) for all \( E \leq 12 \) are listed.

Now, our main goal is to investigate how the spectrum interpolates between the bosonic and fermionic ones. Anyonic wave functions satisfy the interchange conditions \( \mathcal{P}_{jk}\Psi = \exp(i\pi\delta)\Psi \) for each pair \( j, k \), where \( \mathcal{P}_{jk} \) is the operator of anticlockwise interchange of particles \( j \) and \( k \) and \( \delta \) is the statistical parameter; there is a continuous transition from bosons to fermions as \( \delta \) goes from 0 to 1. Having turned an \( N \)-anyon system by one complete revolution in an anticlockwise direction, one will have the wave function multiplied by \( \exp[i\pi N(N - 1)\delta] \). Consequently, the possible values of angular momentum for anyons are \( L = \frac{N(N - 1)}{2}\delta + \text{integer} \). In our case \( N = 3 \) one gets \( L = 3\delta + \text{integer} \). Thus, a state with angular momentum \( L \) at Bose statistics interpolates to that with angular momentum \( L + 3 \) at Fermi statistics. As for energy, it is known [6, 7, 8] that possible values of the difference \( E_{\text{Fermi}} - E_{\text{Bose}} \) are \( +3, +1, -1, -3 \). In the first and the last cases the \( \delta \) dependence of \( E \) is linear and the states may be found exactly (in Ref.[8] we called them "good" states), in the other two ones ("bad" states) this dependence is non-linear and not found exactly (see Ref.[9] for an interesting hypothesis concerning the mentioned dependence). To summarize, all bosonic states with the quantum numbers \( (E, L) \) fall into four classes according to the quantum numbers of fermionic states to which they interpolate; those can equal \( (E + 3, L + 3), (E + 1, L + 3), (E - 1, L + 3), \) or \( (E - 3, L + 3) \). (It is implicit of course that in the subspace of states with the same \( (E, L) \) one chooses the "correct" ones in the same way as in perturbation theory with degenerate states; it can always be done since \( E \) and \( L \) themselves are good quantum numbers for anyons.)

We will use \( \tilde{r}^n(E, L) \) to denote the number of states which come from \( (E, L) \) at Bose statistics to \( (E + n, L + 3) \) at Fermi statistics. (Since it has been shown by numerical calculations [6, 7] that the energies of different "bad" states interpolating between the
same bosonic and fermionic ones are in general different for fractional $\delta$, one has for anyons, strictly speaking, “numbers” of such states rather than “multiplicities.”) As $n$ can take four values, one needs four equations for each $(E, L)$ to determine those numbers. The “law of conservation of states” at bosonic and fermionic points reads

$$\tilde{r}^3(E, L) + \tilde{r}^{-1}(E, L) + \tilde{r}^{-1}(E, L) + \tilde{r}^{-3}(E, L) = \tilde{g}_{3B}(E, L), \quad (15)$$

$$\tilde{r}^3(E - 3, L - 3) + \tilde{r}^{-1}(E - 1, L - 3) + \tilde{r}^{-1}(E + 1, L - 3) + \tilde{r}^{-3}(E + 3, L - 3) = \tilde{g}_{3F}(E, L), \quad (16)$$

respectively. Two more equations follow from the known symmetry properties of the spectrum. First, in perturbation theory it is easy to establish \[10, 11\] that at Fermi statistics, two states with opposite angular momenta have opposite values of slopes, that is, derivatives $(dE/d\delta)_{\delta=1}$. Since the linear behavior of the ”good” states is exact, this implies the equality

$$\tilde{r}^3(E - 3, L - 3) = \tilde{r}^{-3}(E + 3, -L - 3) \quad (17)$$

(see Fig.1). Second, there is the supersymmetry property. Namely, it was pointed out by Sen \[12\] that there exists an operator $\hat{Q}$ which annihilates some of the ”good” states but acting on a ”bad” state with statistical parameter $\delta$ always produces another ”bad” state with same energy and with statistical parameter $1+\delta$. It is straightforward to show that $[\hat{L}, \hat{Q}] = 2\hat{Q}$ so that a state coming from $(E, L)$ at $\delta = 0$ to $(E+1, L+3)$ at $\delta = 1$ turns under $\hat{Q}$ into that coming from $(E, L+2)$ at $\delta = 1$ to $(E+1, L+5)$ at $\delta = 2$.

Now, parity transformation turns the latter into a state coming from $(E+1, -L - 5)$ at $\delta = 0$ to $(E, -L - 2)$ at $\delta = 1$ (Fig.2). Therefore the last equation is

$$\tilde{r}^{-1}(E, L) = \tilde{r}^{-1}(E + 1, -L - 5). \quad (18)$$

The four equations (15)-(18) and the expression (14) for $\tilde{g}_{3B}$ and $\tilde{g}_{3F}$ would suffice to calculate $\tilde{r}^n(E, L)$. It seems, however, hardly possible to obtain closed-form expressions for them in a straightforward manner as it is not clear in which form they should be searched for. Instead, it is easy to solve (15)-(18) numerically for low-lying levels and it turns out to be possible to pick up the regularity in the numbers.

The results for $\tilde{r}^n(E, L)$ for $E \leq 12$ are summarized in Tab.2. The general tendency is clear and can be said to coincide with what one could more or less expect, but to move further it is useful to involve the concept of towers \[6, 11, 12\]. Again following Ref.\[12\], we recall that there exists an operator $\hat{K}_-$ such that $[\hat{H}, \hat{K}_-] = -2\hat{K}_-$ and $[\hat{L}, \hat{K}_-] = 0$ — therefore it either annihilates a common eigenstate of $\hat{H}$ and $\hat{L}$ or lowers its energy by two units without changing its angular momentum. This means in turn that all the states fall into ”towers” descending along which is realized by $\hat{K}_-$ until one reaches a bottom state for which $\hat{K}_- \Psi = 0$. Therefore any $(E, L)$ state either is a bottom state or has its correspondent $(E - 2, L)$ state obtained from it by action of
Denoting by \( \tilde{b}^n(E, L) \) the number of bottom states with the same characteristics as in \( \tilde{r}^n(E, L) \), we have

\[
\tilde{b}^n(E, L) = \tilde{r}^n(E, L) - \tilde{r}^n(E - 2, L),
\]

and correspondingly

\[
\tilde{r}^n(E, L) = \sum_{k=0}^{[E/2]} \tilde{b}^n(E - 2k, L).
\]

Thus, instead of counting (and finding) all states it is sufficient to count (and find) only bottom states. The values of \( \tilde{b}^{+3}(E, L) \) and \( \tilde{b}^{+1}(E, L) \) are displayed in Tabs.3-4.

Here at last the regularities are obvious. The formal expressions read

\[
\tilde{b}^{+3}(E, L) = \left[ \frac{E + 1}{6} \right] - \left[ \frac{E + L}{4} \right] + \left[ \frac{L}{2} \right] + 1 - d(E, 6)
\]

for \( \frac{E - 2}{3} \leq L \leq E - 2; \)

\[
\tilde{b}^{+1}(E, L) = \left[ \frac{E^*}{6} \right] + \left[ \frac{L - L^*}{4} \right] + \frac{1 - r(E, 2)}{2} + r(E^*, 6)
\]

\[
+ [1 - 2r(E, 2)] \frac{d(E^* - 2, 6)[r(L - L^*, 4) - 1]}{2}
\]

for \( \frac{-E - 2}{3} \leq L \leq E - 6,
\]

where

\[
E^* = E - 3r(E, 2), \quad L^* = 2 \left[ \frac{E^*}{6} \right] - 2 + r(E, 2).
\]

From these, one obtains the formulas for \( \tilde{r}^n(E, L) \) immediately by applying (20):

\[
\tilde{r}^{+3}(E, L) = \left\{ \begin{array}{ll}
\frac{L^2 + 6L + 5}{12} + \frac{1 - r(L, 2)}{4} + \frac{d(L, 3)}{3} & \text{for } 0 \leq L \leq \frac{E - 2}{3}, \\
\frac{-E^2 + 6EL - 5L^2 + 12E - 12L}{48} + \frac{d(L, 3) - d(E, 3)}{3} & \text{for } \frac{E - 2}{3} \leq L \leq E - 2;
\end{array} \right.
\]

\[
+ [1 - 2r(E, 2)] \frac{d(E - L - 2, 4)}{4}
\]

for \( \frac{E - 2}{3} \leq L \leq E - 2; \)
\[
\tilde{r}^+ (E, L) = \begin{cases} 
\frac{E^2 + 6EL + 9L^2 + 16E + 48L + 48}{48} + \frac{r(E - L, 4)}{8} + \frac{d(E - 1, 3)}{3} 
& \text{for } \frac{-E-2}{3} \leq L < -1, \\
\frac{E^2 + 6EL - 15L^2 + 16E - 72L - 96}{48} + \frac{r(E - L, 4)}{8} + \frac{d(E - 1, 3)}{3} 
& \text{for } -1 \leq L \leq \frac{E-8}{3}, \\
\frac{E^2 - 2EL + L^2 - 4E + 4L}{16} + \frac{r(E - L, 4)}{8} 
& \text{for } \frac{E-8}{3} < L \leq E - 6.
\end{cases}
\] (25)

Formulas (21), (22), (24), (25) are exact and are the main result of this paper. In these formulas, \(E\) and \(L\) should be of the same parity, i.e. \(r(E-L, 2) = 0\); otherwise, as well as if \(L\) does not fall in any of the ranges specified in the formulas, the corresponding multiplicities vanish. Formulas for \(\tilde{b}^{-1}, \tilde{r}^{-1}\) and \(\tilde{b}^{-3}, \tilde{r}^{-3}\) follow immediately from the obtained ones upon applying (17) and (18).

Finally, summation over \(L\) yields the total numbers of states with given energy and slope:

\[
\tilde{r}^{+3} (E) = \frac{1}{216} \left\{ E^3 + 9E^2 + [42 - 27r(E, 2) - 24d(E, 3)] E + [f_r(E^*, 6) - 81r(E, 2)] \right\}, \\
\tilde{r}^{+1} (E) = \frac{1}{216} \left\{ 2E^3 + 3E^2 + [24d(E - 1, 3) - 18] E + [4r(E^*, 6) - 27r(E, 2)] \right\},
\] (26) (27)

where \(f_0 = 0, f_2 = 88, f_4 = 56,\) and \(E^*\) has been defined by (23); these are exactly the expressions obtained in Ref.[5] (where slightly different notations were used).

Let us now discuss the obtained results. For clarity, the plots of \(\tilde{r}^n (E, L)\) for \(E = 100\) are displayed in Fig.3. We have learned that for a given \(E\), the states of each of the four classes exist for not all values of \(L\) allowed for that \(E\). Tab.5 shows the lowest and the highest values of \(L\) for which the corresponding states exist, as well as the values for which the numbers of such states are maximal.

Since all the states are solutions of the Schrödinger equation, all these values should in principle be deduced directly from this equation. Producing them in this way is an interesting and apparently difficult problem, and perhaps at least some points of its solution, if available, could be used also to understand the structure of the \(N\)-anyon spectrum. What we can do at the moment is to explain why for \(L = E - 2\) and \(E - 4\), as the table shows, there exist only the (+3) states. In the context of our
previous considerations, this follows immediately from \((17)\) and \((13)\) by noticing that 
\[ \tilde{r}^{+1}(E, -k) = \tilde{r}^{-1}(E + 1, -E + k - 5) \]
does not vanish only for \(k \geq 6\), because there are no \((E, L)\) states with \(L < -E + 2\), and analogously for \(\tilde{r}^{-1}\) and \(\tilde{r}^{-3}\). However, a purely quantum mechanical proof of this fact can be given \([1, 13]\), which we will describe here.

Choosing as independent coordinates the one of the center of mass \(Z = (z_1 + z_2 + z_3)/\sqrt{3}\) (\(z_j = x_j + iy_j\) is the complex coordinate of \(j\)-th particle) and two relative ones \(z_{12} = (z_1 - z_2)/\sqrt{2}\), \(z_{23} = (z_2 - z_3)/\sqrt{2}\) and searching for the wave function in the form 
\[ \Psi = \phi_{cm} \chi \exp \left\{ -\frac{i}{2} \left( z_1 z_1^* + z_2 z_2^* + z_3 z_3^* \right) \right\}, \]
one has for the relative Hamiltonian 
\( \tilde{H} = \tilde{H}_1 + \tilde{H}_2 \) where 
\[ \tilde{H}_1 = z_{12} \partial_{12} + z_{12}^* \partial_{12}^* + z_{23} \partial_{23} + z_{23}^* \partial_{23}^* + 2, \]
\[ \tilde{H}_2 = -2(\partial_{12} \partial_{12}^* + \partial_{23} \partial_{23}^*), \]
\(\partial_{jk} \equiv \partial/\partial z_{jk}\) \((\tilde{H} \text{ acts on } \chi)\), and for the relative angular momentum 
\(L = z_{12} \partial_{12} - z_{12}^* \partial_{12}^* + z_{23} \partial_{23} - z_{23}^* \partial_{23}^*\). The function \(\chi\) is to be searched for as a linear combination of functions of the form

\[ \left| l_{12} \bar{l}_{12} ; l_{23} \bar{l}_{23} ; l_{31} \bar{l}_{31} \right> _+ \equiv \left\{ (z_{12})^{l_{12}} (z_{12}^*)^{l_{12}} (z_{23})^{l_{23}} (z_{23}^*)^{l_{23}} (z_{31})^{l_{31}} (z_{31}^*)^{l_{31}} \right\} _+ \]

(\(z_{31} = -z_{12} - z_{23}\)) where \(\{ \ldots \} _+\) means symmetrization over \(z_j\)'s and the equalities \(l_{jk} - \bar{l}_{jk} = \delta + \text{integer}\) should fulfill. The function \(\chi\) then satisfies the anyonic interchange conditions \(\mathcal{P}_{jk} \chi = \exp(\text{i} \delta) \chi\), and the coefficients in the linear combination should be chosen so that \(\chi\) be non-singular and satisfy the equation \(\tilde{H} \chi = E \chi\). Obviously,

\[ \tilde{L} \left| l_{12} \bar{l}_{12} ; l_{23} \bar{l}_{23} ; l_{31} \bar{l}_{31} \right> _+ \]

\[ = \left( l_{12} - \bar{l}_{12} + l_{23} - \bar{l}_{23} + l_{31} - \bar{l}_{31} \right) \left| l_{12} \bar{l}_{12} ; l_{23} \bar{l}_{23} ; l_{31} \bar{l}_{31} \right> _+, \]

(29)

\[ \tilde{H}_1 \left| l_{12} \bar{l}_{12} ; l_{23} \bar{l}_{23} ; l_{31} \bar{l}_{31} \right> _+ \]

\[ = \left( l_{12} + \bar{l}_{12} + l_{23} + \bar{l}_{23} + l_{31} + \bar{l}_{31} + 2 \right) \left| l_{12} \bar{l}_{12} ; l_{23} \bar{l}_{23} ; l_{31} \bar{l}_{31} \right> _+, \]

(30)

and \(\tilde{H}_1 \left| l_{12} \bar{l}_{12} ; l_{23} \bar{l}_{23} ; l_{31} \bar{l}_{31} \right> _+\) consists of pieces of the form

\[ l_{jk} \bar{l}_{mn} | \ldots l_{jk} - 1 \ldots \bar{l}_{mn} - 1 \ldots > _+. \]

The states which at \(\delta = 0\) have the maximal angular momentum \(L = E - 2\) are those with all \(\bar{l}_{jk}\) vanishing, i.e. of the form \(| l \ 0; m \ 0; n \ 0 > _+\) (with integral \(l, m, n\)); the corresponding anyonic states \(| l + \delta \ 0; m + \delta \ 0; n + \delta \ 0 >\) always are stationary because \(\tilde{H}_2\) annihilates them, and their energy is \(E = E_{\text{Bose}} + 3\delta\) (where \(E_{\text{Bose}} = l + m + n + 2\)). Further, to construct a state which has \(L = E - 4\) at \(\delta = 0\) one should start from \(| l + \delta \ 1; m + \delta \ 0; n + \delta \ 0 > _+ \equiv | 1 >\). Acting on this with \(\tilde{H}_2\) yields a sum of the terms of the form \(| p + \delta \ 0; q + \delta \ 0; r + \delta \ 0 > _+\), which are annihilated by \(\tilde{H}_2\). If all of them are non-singular, then a stationary state can be built. (One has \(\tilde{H}_1 | 1 > = E | 1 >\), \(\tilde{H}_2 | 1 > = | 2 >\), \(\tilde{H}_1 | 2 > = (E - 2) | 2 >\), and \(\tilde{H}_2 | 2 > = 0\), consequently
\[ \hat{H} \left[ |1\rangle + \frac{1}{2} |2\rangle \right] = E \left[ |1\rangle + \frac{1}{2} |2\rangle \right]. \] Now, the only potentially singular term in $|2\rangle$ could be $|-1 + \delta 0; q + \delta 0; r + \delta 0\rangle_\ast$ with non-negative integral $q$ and $r$. However, such function, which equals $(z_{12} z_{23} z_{31})^4 |-1 0; q 0; r 0\rangle$, can easily be shown to be non-singular: the singularities cancel out due to symmetrization \[4, 13]. This completes the proof of the fact that all bosonic states with $E = L - 4$, as well as those with $E = L - 2$, have their correspondent anyonic states with the slope $+3$.

At present it is unknown how the other features of the obtained results could be derived from such quantum mechanical considerations. However, the qualitative picture can be justified using semiclassical arguments. In the two-anyon problem, there is only one relative coordinate $z_{12}$, the change of which is governed by the oscillator equation. The corresponding classical trajectories are ellipses, and the anyonic interchange conditions demand that the relative angular momentum be $L = \delta + 2l$, where the integer $l$ determines the sign of $L$ at $\delta = 0$. Thus, as one starts increasing $\delta$ from zero, one should expect linear increase of the energy of those states in which the relative vector, in classical terms, rotates anticlockwise ($l > 0$) or does not rotate ($l = 0$), and linear decrease for those in which it rotates clockwise. This is indeed the case: The semiclassical description of the two-anyon spectrum turns out to yield the exact values of levels \[14, 8\]. For $N$ anyons, there are $\frac{N(N-1)}{2}$ relative vectors, and the analogous consideration would yield linear dependences with the slopes $E(1) - E(0) = \frac{N(N-1)}{2} - 2s$ with $s$ the number of the relative vectors rotating clockwise. The set of the slopes is correct \[17\], which encourages us to use this picture for a qualitative analysis (although we do not argue that it is good in other senses, for example that it gives the correct multiplicities). So, for a state to have the slope $+3$, neither of the relative vectors should rotate clockwise. Certainly it cannot be so if $L < 0$. For $L = 0$ a (+3) state can be only realized as a radial excitation of the bosonic ground state; indeed, \[\tilde{r}^{+3}(E,0) = 1\] (for even $E$). Further, if we choose a state with a given $L$ "at random", then the more is $L$, the more vectors "in average" rotate anticlockwise. Therefore the states with $L$ close to maximal $L = E - 2$ have at most the slope $+3$, and with $L$ decreasing, states with successively decreasing slopes emerge, pass the point of their maximal number, and vanish, just as one observes in Fig.3. Apparently the picture is analogous for arbitrary $N$.

To summarize, we have determined exactly the multiplicities of states with given energies and angular momenta in the spectrum of three non-interacting anyons in a harmonic well and shown that the results are in conformity with certain quantum mechanical and semiclassical considerations. Generally speaking, these results should follow directly from the Schrödinger equation, but the concrete procedure of deriving them this way is still to be found.

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| $E$ | $L$ | $\tilde{g}_{3B}$ | $\tilde{g}_{3F}$ | $E$ | $L$ | $\tilde{g}_{3B}$ | $\tilde{g}_{3F}$ | $E$ | $L$ | $\tilde{g}_{3B}$ | $\tilde{g}_{3F}$ | $E$ | $L$ | $\tilde{g}_{3B}$ | $\tilde{g}_{3F}$ |
|-----|-----|-----------------|-----------------|-----|-----|-----------------|-----------------|-----|-----|-----------------|-----------------|-----|-----|-----------------|-----------------|
| 2   | 0   | 1               | 0               | 7   | 1   | 2               | 2               | 9   | -5  | 2               | 2               | 11  | -1  | 5               | 5               |
| 3   | 1   | 0               | 0               | 7   | -1  | 2               | 2               | 9   | -7  | 1               | 1               | 11  | -3  | 5               | 5               |
| 3   | -1  | 0               | 0               | 7   | -3  | 2               | 2               | 10  | 8   | 2               | 1               | 11  | -5  | 4               | 4               |
| 4   | 2   | 1               | 0               | 7   | -5  | 1               | 1               | 10  | 6   | 3               | 3               | 11  | -7  | 3               | 3               |
| 4   | 0   | 1               | 1               | 8   | 6   | 2               | 1               | 10  | 4   | 4               | 3               | 11  | -9  | 2               | 2               |
| 4   | -2  | 1               | 0               | 8   | 4   | 2               | 2               | 10  | 2   | 4               | 4               | 12  | 10  | 2               | 1               |
| 5   | 3   | 1               | 1               | 8   | 2   | 3               | 2               | 10  | 0   | 5               | 4               | 12  | 8   | 3               | 3               |
| 5   | 1   | 1               | 1               | 8   | 0   | 3               | 3               | 10  | -2  | 4               | 4               | 12  | 6   | 5               | 4               |
| 5   | -1  | 1               | 1               | 8   | -2  | 3               | 2               | 10  | -4  | 4               | 3               | 12  | 4   | 5               | 5               |
| 5   | -3  | 1               | 1               | 8   | -4  | 2               | 2               | 10  | -6  | 3               | 3               | 12  | 2   | 6               | 5               |
| 6   | 4   | 1               | 0               | 8   | -6  | 2               | 1               | 10  | -8  | 2               | 1               | 12  | 0   | 6               | 6               |
| 6   | 2   | 1               | 1               | 9   | 7   | 1               | 1               | 11  | 9   | 2               | 2               | 12  | -2  | 6               | 5               |
| 6   | 0   | 2               | 1               | 9   | 5   | 2               | 2               | 11  | 7   | 3               | 3               | 12  | -4  | 5               | 5               |
| 6   | -2  | 1               | 1               | 9   | 3   | 3               | 3               | 11  | 5   | 4               | 4               | 12  | -6  | 5               | 4               |
| 6   | -4  | 1               | 0               | 9   | 1   | 3               | 3               | 11  | 3   | 5               | 5               | 12  | -8  | 3               | 3               |
| 7   | 5   | 1               | 1               | 9   | -1  | 3               | 3               | 11  | 1   | 5               | 5               | 12  | -10 | 2               | 1               |
| 7   | 3   | 2               | 2               | 9   | -3  | 3               | 3               |                  |                  |                  |                  |                  |                  |                  |

Tab.1. The values of $\tilde{g}_{3B}(E, L)$ and $\tilde{g}_{3F}(E, L)$. 

10
| $E$ | $L$ | $n$ | $E$ | $L$ | $n$ | $E$ | $L$ | $n$ |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 2   | 0   | 1   | 8   | 4   | 2   | 10  | -8  | 1   |
| 3   | 1   |     | 8   | 2   | 1   | 11  | 9   | 2   |
| 3   | -1  |     | 8   | 0   | 1   | 11  | 7   | 3   |
| 4   | 2   | 1   | 8   | -2  | 2   | 11  | 5   | 3   |
| 4   | 0   | 1   | 8   | -4  | 2   | 11  | 3   | 3   |
| 4   | -2  | 1   | 8   | -6  | 1   | 11  | 1   | 4   |
| 5   | 3   | 1   | 9   | 7   | 1   | 11  | -1  | 4   |
| 5   | 1   | 1   | 9   | 5   | 2   | 11  | -3  | 2   |
| 5   | -1  | 1   | 9   | 3   | 2   | 11  | -3  | 4   |
| 5   | -3  | 1   | 9   | 1   | 1   | 11  | -7  | 2   |
| 6   | 4   | 1   | 9   | -1  | 3   | 11  | -9  | 1   |
| 6   | 2   | 1   | 9   | -3  | 1   | 12  | 10  | 2   |
| 6   | 0   | 1   | 9   | -5  | 2   | 12  | 8   | 3   |
| 6   | -2  | 1   | 9   | -7  | 1   | 12  | 6   | 4   |
| 6   | -4  | 1   | 10  | 8   | 2   | 12  | 4   | 3   |
| 7   | 5   | 1   | 10  | 6   | 3   | 12  | 2   | 2   |
| 7   | 3   | 2   | 10  | 4   | 3   | 12  | 0   | 1   |
| 7   | 1   | 1   | 10  | 2   | 2   | 12  | -2  | 4   |
| 7   | -1  | 2   | 10  | 0   | 1   | 12  | -4  | 1   |
| 7   | -3  | 1   | 10  | -2  | 3   | 12  | -6  | 4   |
| 7   | -5  | 1   | 10  | -4  | 1   | 12  | -8  | 2   |
| 8   | 6   | 2   | 10  | -6  | 2   | 12  | 10  | 1   |

Tab.2. The values of $\tilde{r}^n(E,L)$. Zeros are not written down for clarity.
| $E$ | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
|-----|---|---|---|---|----|----|----|----|----|----|
| 0   | 1 | 0 | 0 | 0 | 0  | 0  | 0  | 0  | 0  | 0  |
| 2   | 1 | 0 | 1 | 0 | 0  | 0  | 0  | 0  | 0  | 0  |
| 4   | 1 | 1 | 1 | 0 | 1  | 0  | 0  | 0  | 0  | 0  |
| 6   | 2 | 1 | 1 | 1 | 1  | 1  | 0  | 1  | 1  |    |
| 8   | 2 | 1 | 2 | 1 | 1  | 1  | 1  | 1  | 1  | 1  |
| 10  | 2 | 2 | 2 | 1 | 2  |    |    |    |    |    |
| 12  | 3 | 2 | 2 | 2 | 2  |    |    |    |    |    |
| 14  | 3 | 2 | 3 |    |    |    |    |    |    |    |
| 16  | 3 | 3 |    |    |    |    |    |    |    |    |
| 18  | 4 |    |    |    |    |    |    |    |    |    |

| $L$ | 3 | 5 | 7 | 9 | 11 | 13 | 15 | 17 | 19 | 21 |
|-----|---|---|---|---|----|----|----|----|----|----|
| 1   | 0 | 1 | 0 | 0 | 0  | 0  | 0  | 0  | 0  | 0  |
| 3   | 1 | 1 | 0 | 1 | 0  | 0  | 0  | 0  | 0  | 0  |
| 5   | 1 | 1 | 1 | 1 | 1  | 0  | 1  | 0  | 0  |    |
| 7   | 1 | 2 | 1 | 1 | 1  | 1  | 0  |    |    |    |
| 9   | 2 | 2 | 1 | 2 | 2  | 1  | 1  | 1  | 1  | 1  |
| 11  | 2 | 2 | 2 | 2 | 2  | 2  |    |    |    |    |
| 13  | 2 | 3 | 2 | 2 |    |    |    |    |    |    |
| 15  | 3 | 3 | 2 |    |    |    |    |    |    |    |
| 17  | 3 | 3 |    |    |    |    |    |    |    |    |
| 19  | 3 |    |    |    |    |    |    |    |    |    |

Tab.3. The values of $\tilde{b}^3(E, L)$. 
Tab. 4. The values of $\tilde{b}^{+1}(E, L)$. 

| $E$ | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
|-----|---|---|---|---|----|----|----|----|----|----|
| -6  | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 |
| -4  | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 1 |
| -2  | 1 | 0 | 1 | 1 | 1 | 1 | 2 | 1 | 2 | 1 |
| 0   | 1 | 1 | 2 | 1 | 2 | 2 | 2 | 2 | 2 | 2 |
| 2   | 1 | 1 | 2 | 2 | 3 | 2 | 2 | 2 | 2 | 2 |
| 4   | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 3 |
| 6   | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 3 |
| 8   | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 3 |
| 10  | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 3 |
| 12  | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 3 |
| 14  | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 3 |
| 16  | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 3 |

| $E$ | 3 | 5 | 7 | 9 | 11 | 13 | 15 | 17 | 19 | 21 |
|-----|---|---|---|---|----|----|----|----|----|----|
| -7  | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| -5  | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 1 |
| -3  | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 1 | 2 | 1 |
| -1  | 1 | 1 | 1 | 1 | 2 | 1 | 2 | 2 | 2 | 2 |
| 1   | 1 | 1 | 2 | 2 | 2 | 2 | 3 | 2 | 2 | 2 |
| 3   | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 3 |
| 5   | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 3 |
| 7   | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 3 |
| 9   | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 3 |
| 11  | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 3 |
| 13  | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 15  | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
Tab. 5. The lowest and highest values of $L$ for which the states belonging to $\tilde{r}^a(E, L)$ exist, and the value for which the number of such states is maximal.

|    | Low | Max.     | High  |
|----|-----|----------|-------|
| +3 | 0   | $\frac{3E-6}{5}$ | $E-2$ |
| +1 | $\frac{-E-2}{3}$ | $\frac{E-12}{5}$ | $E-6$ |
| −1 | $-E+2$ | $\frac{-E-12}{5}$ | $E-14$ |
| −3 | $-E+2$ | $\frac{-3E-6}{5}$ | −6  |
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Figure Captions

Fig.1. Two "good" states with opposite angular momenta in the fermion limit.

Fig.2. Two "bad" states related by (supersymmetry+parity) transformation.

Fig.3. The plots of $\tilde{r}^n(E, L)$ for $E = 100$. 