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On the Solvability of Nonlinear Third-Order Two-Point Boundary Value Problems

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Abstract: Under barrier strips type assumptions we study the existence of $C^3[0,1]$—solutions to various two-point boundary value problems for the equation $x''' = f(t, x, x', x'')$. We give also some results guaranteeing positive or non-negative, monotone, convex or concave solutions.

Keywords: third-order differential equation; boundary value problem; existence; sign conditions

MSC: 34B15; 34B18

1. Introduction

In this paper, we study the solvability of boundary value problems (BVPs) for the differential equation

$$x''' = f(t, x, x', x''), t \in (0, 1),$$

with some of the boundary conditions

$$x(0) = A, x'(1) = B, x''(1) = C, \quad (2)$$

$$x(0) = A, x'(0) = B, x''(1) = C, \quad (3)$$

$$x(0) = A, x(1) = B, x''(1) = C, \quad (4)$$

$$x(0) = A, x'(0) = B, x'(1) = C, \quad (5)$$

$$x(1) = A, x'(0) = B, x'(1) = C, \quad (6)$$

where $f : [0, 1] \times D_x \times D_p \times D_q \rightarrow \mathbb{R}$, $D_x, D_p, D_q \subseteq \mathbb{R}$, and $A, B, C \in \mathbb{R}$.

The solvability of BVPs for third-order differential equations has been investigated by many authors. Here, we will cite papers devoted to two-point BVPs which are mostly with some of the above boundary conditions; in each of these works $A, B, C = 0$. Such problems for equations of the form

$$x''' = f(t, x), t \in (0, 1),$$

have been studied by H. Li et al. [1], S. Li [2] (the problem may be singular at $t = 0$ and/or $t = 1$), Z. Liu et al. [3,4], X. Lin and Z. Zhao [5], S. Smirnov [6], Q. Yao and Y. Feng [7]. Moreover, the boundary conditions in References [2,3] are (3), in Reference [4] they are (4), in References [1,5,7] they are (5), and in Reference [6] are

$$x(0) = x(1) = 0, x'(0) = C.$$
Y. Feng [8] and Y. Feng and S. Liu [9] have considered the equation
\[ x''' = f(t,x,x'), t \in (0,1), \]
with (6) and (5), respectively. Y. Feng [10] and R. Ma and Y. Lu [11] have studied the equations
\[ f(t,x,x',x'') = 0 \text{ and } x''' + Mx'' + f(t,x) = 0, t \in (0,1). \]
with (5). BVPs for the equation
\[ x''' = f(t,x,x',x''), t \in (0,1), \]
have been investigated by A. Granas et al. [12], B. Hopkins and N. Kosmatov [13], Y. Li and Y. Li [14]; the boundary conditions in [12] are (5), these in Reference [13] are (2) and (3), and in Reference [14]—(2).

Results guaranteeing positive or non-negative solutions can be found in References [2–4,7–11,13,14], and results that guarantee negative or non-positive ones in References [7,9,10]. The existence of monotone solutions has been studied in References [3,7,9].

As a rule, the main nonlinearity is defined and continuous on a left- and/or a right-unbounded set; in Reference [13] it is a Carathéodory function on an unbounded set. Besides, the main nonlinearity is monotone with respect to some of the variables in References [1,5], does not change its sign in References [2–4,14] and satisfies Nagumo type growth conditions in Reference [14]. Maximum principles have been used in References [8,10], Green’s functions in References [1,2,4,5], and upper and lower solutions in References [1,7–11].

Here, we use a different tool—barrier strips which allow the right side of the equation to be defined and continuous on a bounded subset of its domain and to change its sign.

To prove our existence results we apply a basic existence theorem whose formulation requires the introduction of the BVP
\[
\begin{align*}
x''' + a(t)x'' + b(t)x' + c(t)x &= f(t,x,x',x''), \\ V_i(x) &= r_i, i = 1,2,3 (i = 1,3 \text{ for short}),
\end{align*}
\]
with constants \(a_{ij}\) and \(b_{ij}\) such that \(\sum_{j=0}^{2}(a^2_{ij} + b^2_{ij}) > 0, i = 1,3,\) and \(r_i \in \mathbb{R}, i = 1,3.\) Next, consider the family of BVPs for
\[ x''' + a(t)x'' + b(t)x' + c(t)x = g(t,x,x',x'',\lambda), t \in (0,1), \lambda \in [0,1] \]
with boundary conditions (8), where \(g\) is a scalar function defined \([0,1] \times D_x \times D_y \times [0,1]\), and \(a, b, c\) are as above. Finally, \(BC\) denotes the set of functions satisfying boundary conditions (8), and \(BC_0\) denotes the set of functions satisfying the homogeneous boundary conditions \(V_i(x) = 0, i = 1,3.\) Besides, let \(C^3_{BC}[0,1] = C^3[0,1] \cap BC\) and \(C^3_{BC_0}[0,1] = C^3[0,1] \cap BC_0.\)

The proofs of our existence results are based on the following theorem. It is a variant of Reference [12] (Chapter I, Theorem 5.1 and Chapter V, Theorem 1.2). Its proof can be found in Reference [15]; see also the similar result in Reference [16] (Theorem 4).

**Lemma 1.** Suppose:
(i) Problem (7), (8) has a unique solution \(x_0 \in C^3[0,1].\)
(ii) Problems (7), (8) and (7), (8) are equivalent.
(iii) The map $L_i : C^3_{bc0}[0, 1] \rightarrow C[0, 1]$ is one-to-one: here,

$$L_i x = x''' + a(t)x'' + b(t)x' + c(t)x.$$  

(iv) Each solution $x \in C^3[0, 1]$ to family $(7)_{\lambda}$, $(8)$ satisfies the bounds

$$m_i \leq x^{(i)} \leq M_i \text{ for } t \in [0, 1], i = 0, 2, 3,$$

where the constants $-\infty < m_i, M_i < \infty, i = 0, 2, 3$, are independent of $\lambda$ and $x$.

(v) There is a sufficiently small $\sigma > 0$ such that

$$m_0 - \sigma, M_0 + \sigma \subseteq D_x, [m_1 - \sigma, M_1 + \sigma] \subseteq D_p, [m_2 - \sigma, M_2 + \sigma] \subseteq D_q,$$

and $g(t, x, p, q, \lambda)$ is continuous for $(t, x, p, q, \lambda) \in [0, 1] \times J \times [0, 1]$ where $J = \left[ m_0 - \sigma, M_0 + \sigma \right] \times \left[ m_1 - \sigma, M_1 + \sigma \right] \times \left[ m_2 - \sigma, M_2 + \sigma \right]$; $m_i, M_i, i = 0, 2, 3$, are as in (iv).

Then boundary value problem $(7), (8)$ has at least one solution in $C^3[0, 1]$.

For us, the equation from $(7)_{\lambda}$ has the form

$$x''' = \lambda f(t, x, x', x'').$$  

Preparing the application of Lemma 1, we impose conditions which ensure the a priori bounds from (iv) for the eventual $C^3[0, 1]$-solutions of the families of BVPs for $(7)_{\lambda}, \lambda \in [0, 1]$, with one of the boundary conditions $(k), k = \bar{a}, \bar{b}$.

So, we will say that for some of the BVPs $(1), (k), k = \bar{a}, \bar{b}$, the conditions $(H_1)$ and $(H_2)$ hold for a $K \in \mathbb{R}$ (it will be specified later for each problem) if:

$(H_1)$ There are constants $F'_i, L'_i, i = 1, 2$, such that

$$F'_2 < F'_1 \leq K \leq L'_1 < L'_2, [F'_2, L'_2] \subseteq D_q,$$

\begin{align*}
    f(t, x, p, q) &\geq 0 \text{ for } (t, x, p, q) \in [0, 1] \times D_x \times D_p \times [L'_1, L'_2], & (9) \\
    f(t, x, p, q) &\leq 0 \text{ for } (t, x, p, q) \in [0, 1] \times D_x \times D_p \times [F'_2, F'_1]. & (10)
\end{align*}

$(H_2)$ There are constants $F_i, L_i, i = 1, 2$, such that

$$F_2 < F_1 \leq K \leq L_1 < L_2, [F_2, L_2] \subseteq D_q,$$

\begin{align*}
    f(t, x, p, q) &\leq 0 \text{ for } (t, x, p, q) \in [0, 1] \times D_x \times D_p \times [L_1, L_2], \\
    f(t, x, p, q) &\geq 0 \text{ for } (t, x, p, q) \in [0, 1] \times D_x \times D_p \times [F_2, F_1].
\end{align*}

Besides, we will say that for some of the BVPs $(1), (k), k = \bar{a}, \bar{b}$, the condition $(H_3)$ holds for constants $m_i \leq M_i, i = 0, 2, 3$ (they also will be specified later for each problem) if:

$(H_3)$ $[m_0 - \sigma, M_0 + \sigma] \subseteq D_x, [m_1 - \sigma, M_1 + \sigma] \subseteq D_p, [m_2 - \sigma, M_2 + \sigma] \subseteq D_q$ and $f(t, x, p, q)$ is continuous on the set $[0, 1] \times J$, where $J$ is as in (v) of Lemma 1, and $\sigma > 0$ is sufficiently small.

In fact, the present paper supplements P. Kelevedjiev and T. Todorov [15] where only conditions $(H_2)$ and $(H_3)$ have been used for studying the solvability of various BVPs for $(1)$ with other boundary conditions. Here, $(H_1)$ is also needed. Now, only $(H_1)$ guarantees the a priori bounds for $x'''(t), x''(t)$ and $x(t)$, in this order, for each eventual solution $x \in C^3[0, 1]$ to the families $(1)_{\lambda}, (k), k = \bar{a}, \bar{b}$, and $(H_1)$ and $(H_2)$ together guarantee these bounds for the families $(1)_{\lambda}, (k), k = 5, 6$. As in Reference [15], $(H_3)$ gives the bounds for $x'''(t)$. 

The auxiliary results which guarantee a priori bounds are given in Section 2, and the existence theorems are in Section 3. The ability to use (H1) and (H2) for studying the existence of solutions with important properties is shown in Appendix A. Examples are given in Section 4.

2. Auxiliary Results

This part ensures a priori bounds for the eventual C^3[0, 1]-solutions of each family (1)\Lambda. (k), k = \frac{\Xi}{2}, b, that is, it ensures the constants m_i, M_i, i = 0, 2, from (iv) of Lemma 1 and (H3).

Lemma 2. Let x ∈ C^3[a, b] be a solution to (1)\Lambda. Suppose (H1) holds with [0, 1] replaced by [a, b] and K = x''(b). Then

F_1 ≤ x''(t) ≤ L_1 on [a, b].

Proof. By contradiction, assume that x''(t) > L_1 for some t ∈ [a, b]. This means that the set

S_+ = \{ t ∈ [a, b] : L_1 < x''(t) ≤ L_2 \}

is not empty because x''(t) is continuous on [a, b] and x''(b) ≤ L_1. Besides, there is a γ ∈ S_+ such that

x'''(γ) < 0.

As x(t) is a C^3[a, b]—solution to (1)\Lambda,

x'''(γ) = λf(γ, x(γ), x'(γ), x''(γ)).

But, (γ, x(γ), x'(γ), x''(γ)) ∈ S_+ × D_x × D_y × (L_1, L_2) and (9) imply

x'''(γ) ≥ 0,

a contradiction. Consequently,

x''(t) ≤ L_1 for t ∈ [a, b].

Along similar lines, assuming on the contrary that the set

S_- = \{ t ∈ [a, b] : F_1 ≤ x''(t) < F_2 \}

is not empty and using (10), we achieve a contradiction which implies that

F_1 ≤ x''(t) for t ∈ [a, b].

The proof of the next assertion is virtually the same as that of Lemma 2 and is omitted; it can be found in [15].

Lemma 3. Let x ∈ C^3[a, b] be a solution to (1)\Lambda. Suppose (H2) holds with [0, 1] replaced by [a, b] and K = x''(a). Then

F_1 ≤ x''(t) ≤ L_1 on [a, b].

Let us recall, conditions of type (H1) and (H2) are called barrier strips, see P. Kelevedjiev [17]. As can we see from Lemmas 2 and 3 they control the behavior of x''(t) on [a, b], depending on the sign of f(t, x, x', x'') the curve of x''(t) on [a, b] crosses the strips [a, b] × [L_1, L_2], [a, b] × [L_1, L_2], [a, b] × [F_2, F_3] and [a, b] × [F_2, F_3] not more than once. This property ensures the a priori bounds for x''(t).
Lemma 4. Let (H₁) hold for \( K = C \). Then every solution \( x \in C^3[0,1] \) to (1)ₜ, (2) or (1)ₜₜ, (3) satisfies the bounds

\[
|x(t)| \leq |A| + |B| + \max\{|F'_1|, |L'_1|\}, \quad t \in [0,1],
\]
\[
|x'(t)| \leq |B| + \max\{|F'_1|, |L'_1|\}, \quad t \in [0,1],
\]
\[
F'_1 \leq x''(t) \leq L'_1, \quad t \in [0,1]. \quad (11)
\]

Proof. Let first \( x(t) \) be a solution to (1)ₜₜ, (2). Using Lemma 2 we conclude that (11) is true. Then, according to the mean value theorem, for each \( t \in [0,1] \) there is an \( \xi \in (t,1) \) such that

\[
x'(1) - x'(t) = x''(\xi)(1 - t),
\]

which together with (11) gives the bound for \( |x'(t)| \). Again from the mean value theorem for each \( t \in (0,1] \) there is an \( \eta \in (0,t) \) with the property

\[
x(t) - x(0) = x'(\eta)t,
\]

which yields the bound for \( |x(t)| \). The assertion follows similarly for (1)ₜₜ, (3). \( \square \)

Lemma 5. Let (H₁) hold for \( K = C \). Then every solution \( x \in C^3[0,1] \) to (1)ₜₜ, (4) satisfies the bounds

\[
|x(t)| \leq |A| + |B - A| + \max\{|F'_1|, |L'_1|\}, \quad t \in [0,1],
\]
\[
|x'(t)| \leq |B - A| + \max\{|F'_1|, |L'_1|\}, \quad t \in [0,1],
\]
\[
F'_1 \leq x''(t) \leq L'_1, \quad t \in [0,1].
\]

Proof. By Lemma 2, \( F'_1 \leq x''(t) \leq L'_1 \) on \([0,1]\). Clearly, there is a \( \mu \in (0,1) \) for which \( x'(\mu) = B - A \). Further, for each \( t \in [0,\mu) \) there is an \( \xi \in (t,\mu) \) such that

\[
x'(\mu) - x'(t) = x''(\xi)(\mu - t),
\]

from where, using the obtained bounds for \( x''(t) \), we get

\[
|x'(t)| \leq |B - A| + \max\{|F'_1|, |L'_1|\}, \quad t \in [0,\mu].
\]

We can proceed analogously to see that the same bound is valid for \( t \in [\mu,1] \). Finally, for each \( t \in (0,1] \) there is an \( \eta \in (0,t) \) such that

\[
x(t) - x(0) = x'(\eta)t,
\]

which together with the obtained bound for \( |x'(t)| \) yields the bound for \( |x(t)| \). \( \square \)

Lemma 6. Let (H₁) and (H₂) hold for \( K = C - B \). Then every solution \( x \in C^3[0,1] \) to (1)ₜₜ, (5) or (1)ₜₜₜ, (6) satisfies the bounds

\[
|x(t)| \leq |A| + |B| + \max\{|F_1|, |L_1|, |F'_1|, |L'_1|\}, \quad t \in [0,1],
\]
\[
|x'(t)| \leq |B| + \max\{|F_1|, |L_1|, |F'_1|, |L'_1|\}, \quad t \in [0,1],
\]
\[
\min\{F_1, F'_1\} \leq x''(t) \leq \max\{L_1, L'_1\}, \quad t \in [0,1].
\]
Proof. Let $x(t)$ be a solution to (1), (5); the proof is similar for (1), (6). We know there is a $v \in (0, 1)$ for which $x''(v) = C - B$. Then, applying Lemmas 2 and 3 on the intervals $[0, v]$ and $[v, 1]$, respectively, we get

$$F_1' \leq x''(t) \leq L_1'$$

and so the bounds for $x''(t)$ follow. Further, as in the proof of Lemma 4 we establish consecutively the bounds for $|x'(t)|$ and $|x(t)|$. □

3. Existence Results

Theorem 1. Let $(H_1)$ hold for $K = C$ and $(H_3)$ hold for

$$M_0 = |A| + |B| + \max \{|F_1^I, |L_1^I| \}, m_0 = -M_0,$$

$$M_1 = |B| + \max \{|F_1^I, |L_1^I| \}, m_1 = -M_1, m_2 = F_1', M_2 = L_1'.$$

Then each of BVPs (1), (2) and (1), (3) has at least one solution in $C^3[0, 1]$.

Proof. We will establish that the assertion is true for problem (1), (2) after checking that the hypotheses of Lemma 1 are fulfilled; it follows similarly and for (1), (3). We easily check that (ii) holds for (1), (2). Clearly, BVP (1), (2) is equivalent to BVP (1), (2) and so (ii) is satisfied. Since now $L_0 = x''$, (iii) also holds. Next, according to Lemma 4, for each solution $x \in C^3[0, 1]$ to (1), (2) we have

$$m_i \leq x^{(i)}(t) \leq M_i, t \in [0, 1], i = 0, 1, 2.$$

Now use that $f$ is continuous on $[0, 1] \times J$ to conclude that there are constants $m_3$ and $M_3$ such that

$$m_3 \leq \lambda f(t, x, p, q) \leq M_3$$

for $\lambda \in [0, 1]$ and $(t, x, p, q) \in [0, 1] \times J$, which together with $(x(t), x'(t), x''(t)) \in J$ for $t \in [0, 1]$ and Equation (1), (ii) implies

$$m_3 \leq x''(t) \leq M_3, t \in [0, 1].$$

These observations imply that (iv) holds, too. Finally, the continuity of $f$ on the set $J$ gives (v) and so the assertion is true by Lemma 1. □

Theorem 2. Let $(H_1)$ hold for $K = C$ and $(H_3)$ hold for

$$M_0 = |A| + |B - A| + \max \{|F_1^I, |L_1^I| \}, m_0 = -M_0,$$

$$M_1 = |B - A| + \max \{|F_1^I, |L_1^I| \}, m_1 = -M_1, m_2 = F_1', M_2 = L_1'.$$

Then BVP (1), (4) has at least one solution in $C^3[0, 1]$.

Proof. It follows the lines of the proof of Theorem 1. Now the bounds

$$m_i \leq x^{(i)}(t) \leq M_i, t \in [0, 1], i = 0, 1, 2,$$

for each solution $x \in C^3[0, 1]$ to a $(1), (4)$ follow from Lemma 5. □

Theorem 3. Let $(H_1)$ and $(H_2)$ hold for $K = C - B$ and $(H_3)$ hold for

$$M_0 = |A| + |B| + \max \{|F_1^I, |L_1^I|, |F_1^I, |L_1^I| \}, m_0 = -M_0,$$

$$M_1 = |B| + \max \{|F_1^I, |L_1^I|, |F_1^I, |L_1^I| \}, m_1 = -M_1,$$
\[ m_2 = \min \{F_1, F'_1\}, M_2 = \max \{L_1, L'_1\}. \]

Then each of BVPs (1), (5) and (1), (6) has at least one solution in \( C^3[0,1] \).

**Proof.** Arguments similar to those in the proof of Theorem 1 yield the assertion. Now the bounds
\[ m_i \leq x^{(i)}(t) \leq M_i, t \in [0,1], i = 0,1,2, \]
for each solution \( x \in C^3[0,1] \) to \((1)_x\), \((5)\) and \((1)_x\), \((6)\) follow from Lemma 6. \( \square \)

4. Examples

Through several examples we will illustrate the application of the obtained results.

**Example 1.** Consider the BVPs for the equation
\[ x'''(t) = \exp(x'') - 3 + 5x'''(x^2 + 1) - t \sin x, t \in (0,1), \]
with boundary conditions (2) or (3).

For \( F'_2 = -|C| - 2, F''_1 = -|C| - 1, L'_1 = \max \{|C|, 3\} + 1, L''_1 = \max \{|C|, 3\} + 2 \) and \( \sigma = 0.1 \), for example, each of these problems has a solution in \( C^3[0,1] \) by Theorem 1.

**Example 2.** Consider the BVP
\[ x'''(t) = (\log((x'' + 50)(60 - x'')) - 3), t \in (0,1), \]
\[ x(0) = 5, x'(0) = 10, x'(1) = 40, \]
where \( \varphi : [0,1] \times \mathbb{R}^2 \to \mathbb{R} \) is continuous and does not change its sign.

If \( \varphi(t,x,p) \geq 0 \) on \([0,1] \times \mathbb{R}^2\), the assumptions of Theorem 3 are satisfied for \( F_2 = -36, F_1 = -35, F'_2 = -46, F'_1 = -45, L'_1 = 40, L''_1 = 41, L_1 = 55, L_2 = 56 \) and \( \sigma = 0.01 \), for example, and if \( \varphi(t,x,p) \leq 0 \) on \([0,1] \times \mathbb{R}^2\), they are satisfied for \( F'_2 = -36, F'_1 = -35, F_2 = -46, F_1 = -45, L_1 = 40, L_2 = 41, L'_1 = 55, L''_1 = 56 \) and \( \sigma = 0.01 \), for example; it is clear, \( K = 30 \). Thus, the considered problem has at least one solution in \( C^3[0,1] \). Let us note, here \( D_q = (-50,60) \).

**Example 3.** Consider the BVP
\[ x'''(t) = \frac{t(x'' + 8)(x'' + 3)\sqrt{625 - x'^2}}{\sqrt{900 - x^2}\sqrt{100 - x'^2}}, t \in (0,1), \]
\[ x(0) = 9, x(1) = 1, x''(1) = -4. \]

For \( F_2 = -6, F_1 = -5, L'_1 = -3, L''_1 = -2 \) and \( \sigma = 0.1 \), for example, this problem has a positive, decreasing, concave solution in \( C^3[0,1] \) by Theorem A1; notice, here \( D_x, D_p \) and \( D_q \) are bounded.

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Appendix A

In this part we show how the barrier strips can be used for studying the existence of positive or non-negative, monotone, convex or concave $C^3[0,1]$ solutions. Here, we demonstrate this on problem (1), (4) but it can be done for the rest of the BVPs considered in this paper. Similar results for various other two-point boundary conditions can be found in R. Agarwal and P. Kelevedjiev [16] and P. Kelevedjiev and T. Todorov [15].

**Lemma A1.** Let $A,B \geq 0, C \leq 0$. Suppose (H$_1$) holds for $K = C$ with $L'_1 \leq 0$. Then each solution $x \in C^3[0,1]$ to (1), (4) satisfies the bounds

$$\begin{align*}
\min\{A,B\} \leq x(t) &\leq A + |B - A| + |F'_1|, \quad t \in [0,1], \\
B - A + F'_1 \leq x'(t) &\leq B - A - F'_1, \quad t \in [0,1].
\end{align*}$$

**Proof.** From Lemma 2 we know that $F'_1 \leq x''(t) \leq L'_1$ for $t \in [0,1]$. Besides, for some $\mu \in (0,1)$ we have $x'(\mu) = B - A$. Then,

$$\int_{t}^{\mu} F'_1 ds \leq \int_{t}^{\mu} x''(s) ds \leq \int_{t}^{\mu} L'_1 ds, \quad t \in [0,\mu),$$

gives

$$B - A \leq x'(t) \leq B - A - F'_1, \quad t \in [0,\mu],$$

and

$$\int_{\mu}^{t} F'_1 ds \leq \int_{\mu}^{t} x''(s) ds \leq \int_{\mu}^{t} L'_1 ds, \quad t \in (\mu,1],$$

implies

$$B - A + F'_1 \leq x'(t) \leq B - A - F'_1, \quad t \in [\mu,1].$$

As a result,

$$B - A + F'_1 \leq x'(t) \leq B - A - F'_1, \quad t \in [0,1].$$

Using Lemma 5, conclude

$$|x(t)| \leq A + |B - A| + |F'_1|$$

for $t \in [0,1]$.

From $x''(t) \leq L'_1 \leq 0$ for $t \in [0,1]$ it follows that $x(t)$ is concave on $[0,1]$ and so, in view of $A,B \geq 0, x(t) \geq \min\{A,B\}$ on $[0,1]$, which completes the proof.  \hfill $\Box$

**Theorem A1.** Let $A \geq B \geq 0$ and $C \leq 0$ ($A \geq B > 0$ and $C < 0$). Suppose (H$_1$) holds for $K = C$ with $B - A \leq F'_1$ ($B - A < F'_1$) and $L'_1 \leq 0$, and (H$_3$) holds for

$$m_0 = B, M_0 = 2A - B + |F'_1|,$$

$$m_1 = B - A + F'_1, M_1 = B - A - F'_1, m_2 = F'_1, M_2 = L'_1.$$

Then BVP (1), (4) has at least one non-negative, non-increasing (positive, decreasing), concave solution in $C^3[0,1]$.

**Proof.** By Lemma 5, for every solution $x \in C^3[0,1]$ to (1), (4) we have $F'_1 \leq x''(t) \leq L'_1$ on $[0,1]$, and Lemma A1 yields

$$B - A + F'_1 \leq x'(t) \leq B - A - F'_1, \quad t \in [0,1]$$

$$\min\{A,B\} \leq x(t) \leq A + |B - A| + |F'_1|, \quad t \in [0,1].$$
Because of $A \geq B$, the last inequality gets the form

$$B \leq x(t) \leq 2A - B + |F'_1|, \ t \in [0,1].$$

So, $x(t)$ satisfies the bounds

$$m_0 \leq x^{(i)}(t) \leq M_0, t \in [0,1], i = 0, 1, 2.$$

Essentially the same reasoning as in the proof of Theorem 1 establishes that (1), (4) has a solution in $C^3[0,1]$. Since $m_0 = B \geq 0 (m_0 > 0), M_1 = B - A - F'_1 \leq 0 (M_1 < 0)$ and $M_2 = L'_1 \leq 0$, this solution has the desired properties. \(\square\)

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