An efficient arc-search interior-point algorithm for convex quadratic programming with box constraints

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Abstract

This paper proposes an arc-search interior-point algorithm for convex quadratic programming with box constraints. The problem has many applications, such as optimal control with actuator saturation. It is shown that an explicit feasible starting point exists for this problem. Therefore, an efficient feasible interior-point algorithm is proposed to tackle the problem. It is proved that the proposed algorithm is polynomial and has the best known complexity bound $O(\sqrt{n} \log(1/\epsilon))$. Moreover, the search neighborhood for this special problem is wider than an algorithm for general convex quadratic programming problems, which implies that longer steps and faster convergence are expected. Finally, an engineering design problem is considered and the algorithm is applied to solve the engineering problem.

Keywords Arc-search · Interior-point method · Convex quadratic programming · Polynomial algorithm

1 Introduction

For three decades, the most efficient interior-point algorithms use high-order derivatives, wide search neighborhood, and infeasible starting point (when feasible starting point is not available) because all of these strategies have demonstrated their computational efficiency and effectiveness [14, 15, 17]. However, widely known theoretical analyses for these algorithms gave different conclusions because all these strategies lead to poorer polynomial bounds than the ones of the algorithms that do not use these strategies [19, 20, 23]. Even worse, Monhrotra’s algorithm, which
demonstrated the efficiency of these strategies and was widely regarded as the most efficient interior-point algorithm for linear programming for decades, does not have any convergence result [3, 29].

In a series of recent papers, we proposed several arc-search algorithms for linear and convex quadratic programming which solved the long-standing dilemma for the interior-point method in linear and convex quadratic programming. In [26, 27], higher-order arc-search algorithms were devised and proved to have the best polynomial bound for linear and convex quadratic programming problems. However, the algorithms proposed in [26, 27] required the initial point to be an interior point which may not be trivial to obtain [4] (finding an initial interior point may be a major obstacle for the use of feasible interior-point methods). To demonstrate the merit of the arc-search strategy, an infeasible interior-point algorithm, which mimics Monhrotra’s algorithm but replaces the line search by the arc-search strategy, was devised in [28]. The extensive computational comparison between Monhrotra’s algorithm and arc-search algorithm against Netlib problems was performed and results favor clearly the arc-search method. Like Monhrotra’s algorithm, there is no convergent result for the algorithm discussed in [28]. To find convergent infeasible interior-point algorithms, several authors [25, 32] made progress. In [29], we reported a significant result that a new arc-search infeasible interior-point algorithm is not only computationally more attractive than Monhrotra’s algorithm, but also converges in polynomial iterations with the best polynomial bound for all interior-point algorithms, infeasible or feasible. The merit of arc-search strategy has also been demonstrated in [10–13, 16, 22, 24, 33] where symmetric optimization, symmetric cones optimization, and semidefinite optimization problems were solved. A state-of-the-art treatment of the arc-search techniques is given in [31].

Since it is difficult to find a feasible initial point for general inequality constrained optimization problems, an infeasible interior-point method becomes very popular. However, if a feasible initial point is available, intuitively, the feasible interior-point method is more attractive than the infeasible interior-point method, because the effort to bring the iterates gradually from the infeasible region to the feasible region is not needed anymore. In this paper, we show that the convex quadratic programming problem subject only to bound (box) constraints does have an explicit feasible initial point. This problem has been studied in [8] where a first-order interior-point algorithm is considered. In this paper, we will consider a higher-order interior-point algorithm, which starts at a feasible initial point. We believe that these two features will improve the computational efficiency. We propose an algorithm that is specially designed for this problem and show that the algorithm is also more efficient than the algorithm in [27] because of two improvements due to the special structure of the constraints: (1) the enlarged search neighborhood and (2) an explicit feasible initial interior point. Moreover, we show that this algorithm has a very desirable theoretical property, the best polynomial complexity. Although the idea of the proof of polynomiality is somewhat similar to that used in [27], we provided the proof in the Appendix for several reasons: (i) it shows that the search neighborhood is larger than the general convex quadratic programming with linear constraints, which suggests a superior efficiency.
of the newly proposed algorithm, (ii) it is fairly different from the [27] because of some special structure of the box constraints and it carefully takes care of these differences, and (iii) it provides complete results. We have implemented the algorithm in MATLAB. We provide some numerical test to demonstrate the effectiveness and efficiency of the proposed algorithm. Finally, we want to point out that the problem under the consideration has many applications, such as optimal control with actuator saturation [30].

The remainder of the paper is organized as follows. Section 2 describes the problem and provides the notations and some simple technical results to be used in the late sections. Section 3 describes the central path of quadratic programming with box constraints. Section 4 proposes an arc-search algorithm for the convex quadratic programming with box constraints. Section 5 gives convergence analysis. Section 6 addresses implementation issues. Section 7 presents some numerical test example. Section 8 summarizes the conclusions. Some technical proofs are in the Appendix to enhance the readability of the paper.

2 Some notations and technical lemmas

Throughout the paper, we will use notations adopted in [27]. We denote n-dimensional vector space by \( \mathbb{R}^n \), \( n \times m \)-dimensional matrix space by \( \mathbb{R}^{n \times m} \), Hadamard (element-wise) product of two vectors \( y \in \mathbb{R}^n \) and \( \lambda \in \mathbb{R}^n \) by \( y \circ \lambda \), the \( i \)th component of \( y \) by \( y_i \), element-wise division of the two vectors by \( y \div \lambda \) if \( \min |\lambda_i| > 0 \), the Euclidean norm of \( y \) by \( \| y \| \), the identity matrix of any dimension by \( I \), the vector of all ones with appropriate dimension by \( e \), and element-wise absolute value vector by \( |y| = [|y_1|, \ldots, |y_n|]^T \). To simplify the notation for block column vectors, we will denote, for example, \( [y^T, \lambda^T]^T \) by \( (y, \lambda) \). For vectors \( x \in \mathbb{R}^n \), \( y \in \mathbb{R}^n \), \( z \in \mathbb{R}^n \), \( \lambda \in \mathbb{R}^n \), and \( \gamma \in \mathbb{R}^n \), we will use capital letters \( X \), \( Y \), \( Z \), \( \Lambda \), and \( \Gamma \) for some related diagonal matrices whose diagonal elements are the components of the corresponding vectors. For example, we will use \( \Lambda = \text{diag}(\lambda) \) and \( \Gamma = \text{diag}(\gamma) \) for the diagonal matrices. For a matrix \( H \in \mathbb{R}^{n \times n} \), we use \( H \geq 0 \) if \( H \) is positive semidefinite, and \( H > 0 \) if \( H \) is positive definite. Finally, we define an initial point of any algorithm by \( x^0 \), the point at the \( k \)th iteration by \( x^k \).

We will consider the convex quadratic problem with box constraints in a standard form:

\[
(QP) \min \frac{1}{2} x^T H x + c^T x, \text{ subject to } -e \leq x \leq e, \tag{1}
\]

where \( 0 < H \in \mathbb{R}^{n \times n} \) is a positive definite matrix, \( c \in \mathbb{R}^n \) is given, and \( x \in \mathbb{R}^n \) is the variable vector to be optimized. The remaining discussion of this paper is focused on the solution to the convex quadratic programming problem with box constraints described by (1). We will use some technical lemmas which are independent of the problem. The first two simple lemmas are given in [27].
Lemma 2.1 Let $p > 0$, $q > 0$, and $r > 0$ be some constants. If $p + q \leq r$, then $pq \leq \frac{r^2}{4}$.

Lemma 2.2 For $\alpha \in [0, \frac{\pi}{2}]$,
\[ \sin(\alpha) \geq \sin^2(\alpha) = 1 - \cos^2(\alpha) \geq 1 - \cos(\alpha). \]

The following lemma is proved in [19].

Lemma 2.3 Let $u$, $v$, and $w$ be real vectors of the same size satisfying $u + v = w$ and $u^T v \geq 0$. Then,
\[ 2\|u\| \cdot \|v\| \leq \|u\|^2 + \|v\|^2 \leq \|u\|^2 + \|v\|^2 + 2u^T v = \|u + v\|^2 = \|w\|^2. \tag{2} \]

The next technical lemma is from [23, page 88].

Lemma 2.4 Let $u$ and $v$ be the vectors of the same dimension, and $u^T v \geq 0$. Then,
\[ \|u \circ v\| \leq 2^{-\frac{3}{2}} \|u + v\|^2. \]

We will use the famous Cardano’s formula which can be found in [21].

Lemma 2.5 Let $p$ and $q$ be the real numbers that are related to the following cubic algebra equation
\[ x^3 + px + q = 0. \]
If
\[ \Delta = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 > 0, \]
then the cubic equation has one real root that is given by
\[ x = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}. \]

For quartic polynomials, the roots can also be represented by analytic formulas, and we do not list all the possible cases and solutions but refer to [9] for the detailed discussion. The last technical lemma in this section is as follows.

Lemma 2.6 Let $u$ and $v$ be the $n$-dimensional vectors. Then,
\[ \|u \circ v - \frac{1}{n} (u^T v) e\| \leq \|u \circ v\|. \]
Proof Simple calculation gives

\[ \left\| u \circ v - \frac{1}{n} \left( u^T v \right) e \right\|^2 = \sum_{i=1}^{n} \left( u_i v_i - \frac{1}{n} \sum_{i=1}^{n} u_i v_i \right)^2 \]

\[ = \sum_{i=1}^{n} \left( \frac{u_i^2 v_i^2}{n} - \frac{2 u_i v_i}{n} \sum_{i=1}^{n} u_i v_i + \frac{1}{n^2} \left( \sum_{i=1}^{n} u_i v_i \right)^2 \right) \]

\[ = \sum_{i=1}^{n} \left( \frac{u_i^2 v_i^2}{n} - \frac{2}{n} \left( \sum_{i=1}^{n} u_i v_i \right)^2 + \frac{1}{n} \left( \sum_{i=1}^{n} u_i v_i \right)^2 \right) \]

\[ = \sum_{i=1}^{n} \left( \frac{u_i^2 v_i^2}{n} - \frac{1}{n} \left( \sum_{i=1}^{n} u_i v_i \right)^2 \right) \leq \left\| u \circ v \right\|^2. \]

\[ \square \]

3 Central path of convex QP with box constraints

It is well known that \( x \) is an optimal solution of (1) if and only if \( x, \lambda, \text{ and } \gamma \) meet the following KKT conditions

\[ -\lambda + \gamma - Hx = c, \]

\[ -e \leq x \leq e, \quad (3a) \]

\[ (\lambda, \gamma) \geq 0, \quad (3b) \]

\[ \lambda_i(e_i - x_i) = 0, \quad \gamma_i(e_i + x_i) = 0, \quad i = 1, \ldots, n. \quad (3c) \]

Denote \( y = e - x \geq 0, z = e + x \geq 0 \). The KKT condition can be rewritten as

\[ Hx + c + \lambda - \gamma = 0, \quad (4a) \]

\[ x + y = e, \quad x - z = -e, \quad (4b) \]

\[ (y, z, \lambda, \gamma) \geq 0, \quad (4c) \]

\[ \lambda_i y_i = 0, \quad \gamma_i z_i = 0, \quad i = 1, \ldots, n. \quad (4d) \]

For the convex (QP) problem, the KKT condition is also sufficient for \( x \) to be a global optimal solution. Denote the feasible set \( \mathcal{F} \) as a collection of all points that meet the constraints (4a), (4b), and (4c)

\[ \mathcal{F} = \{ (x, y, z, \lambda, \gamma) : Hx + c + \lambda - \gamma = 0, \quad (y, z, \lambda, \gamma) \geq 0, \quad x + y = e, \quad x - z = -e \}, \quad (5) \]
and the strictly feasible set \( F^o \) as a collection of all points that meet the constraints (4a) and (4b), and are strictly positive in (4c)

\[
F^o = \{(x, y, z, \lambda, \gamma) : Hx + c + \lambda - \gamma = 0, (y, z, \lambda, \gamma) > 0, x + y = e, x - z = -e\}. \tag{6}
\]

Similar to the linear programming, we define the central path \( C \in F^o \subset F \), as a curve in finite dimensional space parameterized by a scalar \( \tau > 0 \) as follows. For each interior point \( (x, y, z, \lambda, \gamma) \in F^o \) on the central path, there is a \( \tau > 0 \) such that

\[
\begin{align*}
Hx + c + \lambda - \gamma &= 0, \tag{7a} \\
x + y &= e, x - z = -e, \tag{7b} \\
(y, z, \lambda, \gamma) &> 0, \tag{7c} \\
\lambda_i y_i = \tau, \gamma_i z_i = \tau, &i = 1, \ldots, n. \tag{7d}
\end{align*}
\]

Therefore, the central path is an arc that is parameterized as a function of \( \tau \) and is denoted as

\[
C = \{(x(\tau), y(\tau), z(\tau), \lambda(\tau), \gamma(\tau)) : \tau > 0\}. \tag{8}
\]

As \( \tau \to 0 \), the moving point \( (x(\tau), y(\tau), z(\tau), \lambda(\tau), \gamma(\tau)) \) on the central path represented by (7) approaches the solution of (QP) represented by (1). Throughout the paper, we make the following assumption.

**Assumption**

1. \( F^o \) is not empty.

Assumption 1 implies the existence of a central path. This assumption is always true for (1), and we will provide an explicit initial interior point in Section 5.

Let \( 1 > \theta > 0 \), denote \( p = (y, z) \), \( \omega = (\lambda, \gamma) \), and the duality gap

\[
\mu = \frac{\lambda^T y + \gamma^T z}{2n} = \frac{p^T \omega}{2n}. \tag{9}
\]

We define a set of neighborhood of the central path as

\[
N_2(\theta) = \{(x, y, z, \lambda, \gamma) \in F^o : \|p \circ \omega - \mu e\| \leq \theta \mu, \} \subset F^o. \tag{10}
\]

As we reduce the duality gap to zero, the neighborhood of \( N_2(\theta) \) will be a neighborhood of the central path that approaches the optimizer(s) of the QP problem; therefore, all points inside \( N_2(\theta) \) will approach the optimizer(s) of the QP problem. For \( (x, y, z, \lambda, \gamma) \in N_2(\theta) \), since \( (1 - \theta) \mu \leq \omega_i p_i \leq (1 + \theta) \mu \), where \( \omega_i \) is either \( \lambda_i \) or \( \gamma_i \), and \( p_i \) is either \( y_i \) or \( z_i \), we have

\[
\frac{\omega_i p_i}{1 + \theta} \leq \frac{\max i \omega_i p_i}{1 + \theta} \leq \mu \leq \frac{\min i \omega_i p_i}{1 - \theta} \leq \frac{\omega_i p_i}{1 - \theta}. \tag{11}
\]
4 An arc-search algorithm for convex QP with box constraints

The idea of the arc-search technique proposed in this paper is very simple. The algorithm starts from a feasible point in $N_2(\theta)$ close to the central path, constructs an arc that passes through the point and approximates the central path, searches along the arc to a new point in a larger area $N_2(2\theta)$ that reduces the duality gap $p^T\omega$, and meets (3a), (3b), and (3c). The process is repeated by finding a better point close to the central path or on the central path in $N_2(\theta)$ that simultaneously meets (7a), (7b), and (7c).

Following the idea used in [27], we will use an ellipse $\mathcal{E}$ [5] in an appropriate dimensional space to approximate the central path $\mathcal{C}$ described by (7), where

$$\mathcal{E} = \left\{ (x(\alpha), y(\alpha), z(\alpha), \lambda(\alpha), \gamma(\alpha)) : (x(\alpha), y(\alpha), z(\alpha), \lambda(\alpha), \gamma(\alpha)) = \tilde{a} \cos(\alpha) + \tilde{b} \sin(\alpha) + \tilde{c}, \right\},$$  \hspace{1cm} (12)

$\tilde{a} \in \mathbb{R}^{5n}$ and $\tilde{b} \in \mathbb{R}^{5n}$ are the axes of the ellipse, and $\tilde{c} \in \mathbb{R}^{5n}$ is the center of the ellipse. Given a point $(x, y, z, \lambda, \gamma) = (x(\alpha_0), y(\alpha_0), z(\alpha_0), \lambda(\alpha_0), \gamma(\alpha_0)) \in \mathcal{E}$ which is close to or on the central path, $\tilde{a}$, $\tilde{b}$, and $\tilde{c}$ are functions of $\alpha$, $(x, \lambda, \gamma, y, z)$, $(\dot{x}, \dot{y}, \dot{z}, \dot{\lambda}, \dot{\gamma})$, and $(\ddot{x}, \ddot{y}, \ddot{z}, \ddot{\lambda}, \ddot{\gamma})$, where $(x, \dot{x}, \ddot{x}, \lambda, \dot{\lambda}, \ddot{\lambda}, \gamma, \dot{\gamma}, \ddot{\gamma}, \bar{x}, \bar{y}, \bar{z}, \bar{\lambda}, \bar{\gamma})$ are defined as

$$\begin{bmatrix}
H & 0 & 0 & I & -I \\
I & I & 0 & 0 & 0 \\
I & 0 & -I & 0 & 0 \\
0 & \Lambda & 0 & Y & 0 \\
0 & 0 & \Gamma & 0 & Z
\end{bmatrix} \begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{z} \\
\dot{\lambda} \\
\dot{\gamma}
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
\lambda \circ y \\
\gamma \circ z
\end{bmatrix},$$ \hspace{1cm} (13)

$$\begin{bmatrix}
H & 0 & 0 & I & -I \\
I & I & 0 & 0 & 0 \\
I & 0 & -I & 0 & 0 \\
0 & \Lambda & 0 & Y & 0 \\
0 & 0 & \Gamma & 0 & Z
\end{bmatrix} \begin{bmatrix}
\ddot{x} \\
\ddot{y} \\
\ddot{z} \\
\ddot{\lambda} \\
\ddot{\gamma}
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
-2\dot{\lambda} \circ \dot{\gamma} \\
-2\dot{\gamma} \circ \dot{z}
\end{bmatrix}. \hspace{1cm} (14)

The first rows of (13) and (14) are equivalent to

$$H\dot{x} = \dot{\gamma} - \dot{\lambda}, \quad H\ddot{x} = \ddot{\gamma} - \ddot{\lambda}. \hspace{1cm} (15)$$

The next 2 rows of (13) and (14) are equivalent to

$$\dot{x} = -\dot{y}, \quad \ddot{x} = \ddot{z}, \quad \ddot{\bar{x}} = -\ddot{\bar{y}}, \quad \ddot{\bar{x}} = \ddot{\bar{z}}. \hspace{1cm} (16)$$

The last 2 rows of (13) and (14) are equivalent to

$$p \circ \dot{\omega} + \ddot{p} \circ \omega = p \circ \omega, \hspace{1cm} (17)$$

$$p \circ \ddot{\omega} + \ddot{p} \circ \omega = -2\ddot{p} \circ \omega. \hspace{1cm} (18)$$

It has been shown in [27] that one can avoid the calculation of $\tilde{a}$, $\tilde{b}$, and $\tilde{c}$ in the expression of the ellipse. The following formulas are used instead.
Theorem 4.1 Let \((x(\alpha), y(\alpha), z(\alpha), \lambda(\alpha), \gamma(\alpha))\) be an arc defined by (12) passing through a point \((x, y, z, \lambda, \gamma) \in \mathcal{E}\), and its first and second derivatives at \((x, y, z, \lambda, \gamma)\) be \((\dot{x}, \dot{y}, \dot{z}, \dot{\lambda}, \dot{\gamma})\) and \((\ddot{x}, \ddot{y}, \ddot{z}, \ddot{\lambda}, \ddot{\gamma})\) which are defined by (13) and (14). Then, an ellipse approximation of the central path is given by

\[
x(\alpha) = x - \dot{x} \sin(\alpha) + \ddot{x}(1 - \cos(\alpha)),
\]

\[
y(\alpha) = y - \dot{y} \sin(\alpha) + \ddot{y}(1 - \cos(\alpha)),
\]

\[
z(\alpha) = z - \dot{z} \sin(\alpha) + \ddot{z}(1 - \cos(\alpha)),
\]

\[
\lambda(\alpha) = \lambda - \dot{\lambda} \sin(\alpha) + \ddot{\lambda}(1 - \cos(\alpha)),
\]

\[
\gamma(\alpha) = \gamma - \dot{\gamma} \sin(\alpha) + \ddot{\gamma}(1 - \cos(\alpha)).
\]

We will also use a compact format for \(p(\alpha) = (y(\alpha), z(\alpha))\) and \(\omega(\alpha) = (\lambda(\alpha), \gamma(\alpha))\), which are given by

\[
p(\alpha) = p - \dot{p} \sin(\alpha) + \ddot{p}(1 - \cos(\alpha)),
\]

\[
\omega(\alpha) = \omega - \dot{\omega} \sin(\alpha) + \ddot{\omega}(1 - \cos(\alpha)).
\]

We denote the duality gap at point \((x(\alpha), p(\alpha), \omega(\alpha))\) as

\[
\mu(\alpha) = \frac{\lambda(\alpha)^T y(\alpha) + \gamma(\alpha)^T z(\alpha)}{2n} = \frac{p(\alpha)^T \omega(\alpha)}{2n}.
\]

Assuming \((y, z, \lambda, \gamma) > 0\), one can easily see that if \(\dot{y}, \dot{z}, \dot{\lambda}, \dot{\gamma}\) are bounded (we will show that this is true), and if \(\alpha\) is small enough, then \(y(\alpha) > 0, z(\alpha) > 0, \lambda(\alpha) > 0, \) and \(\gamma(\alpha) > 0\). We will also show that searching along this ellipse will reduce the duality gap, i.e., \(\mu(\alpha) < \mu\).

Lemma 4.1 Let \((x, y, z, \lambda, \gamma)\) be a strictly feasible point of (QP), \((\dot{x}, \dot{y}, \dot{z}, \dot{\lambda}, \dot{\gamma})\) and \((\ddot{x}, \ddot{y}, \ddot{z}, \ddot{\lambda}, \ddot{\gamma})\) meet (13) and (14), and \((x(\alpha), y(\alpha), z(\alpha), \lambda(\alpha), \gamma(\alpha))\) be calculated using (19), (20), (21), (22), and (23), then the following conditions hold.

\[x(\alpha) + y(\alpha) = e, x(\alpha) - z(\alpha) = -e, Hx(\alpha) + c + \lambda(\alpha) + \gamma(\alpha) = 0.\]

Proof Since \((x, y, z, \lambda, \gamma)\) is a strictly feasible point, the result follows from direct calculation by using (6), (13), (14), and Theorem 4.1.

Lemma 4.2 Let \((\dot{x}, \dot{p}, \dot{\omega})\) be defined by (13), \((\ddot{x}, \ddot{p}, \ddot{\omega})\) be defined by (14), and \(H\) be positive definite matrix. Then, the following relations hold.

\[
p^T \dot{\omega} = \dot{x}^T (\dot{y} - \dot{\lambda}) = \dot{x}^T H \dot{x} \geq 0,
\]

The equality holds if and only if \(\|\dot{x}\| = 0\).

\[
p^T \ddot{\omega} = \ddot{x}^T (\ddot{y} - \ddot{\lambda}) = \ddot{x}^T H \ddot{x} \geq 0,
\]
The equality holds if and only if \( \| \ddot{x} \| = 0 \).

\[
\ddot{p}^T \dot{\omega} = \ddot{x}^T (\dot{y} - \dot{\lambda}) = \dot{x}^T (\dot{y} - \dot{\lambda}) = \ddot{p}^T \dot{\omega} = \dot{x}^T \dddot{x}. \tag{29}
\]

\[
-(\ddot{x}^T \dddot{x})(1 - \cos(\alpha))^2 - (\dddot{x}^T \dddot{x} \sin^2(\alpha) \leq (\ddot{x}^T (\dot{y} - \dot{\lambda}) + \dddot{x}^T (\dot{y} - \dot{\lambda})) \sin(\alpha)(1 - \cos(\alpha)) \leq (\dddot{x}^T \dddot{x})(1 - \cos(\alpha))^2 + (\ddot{x}^T \dddot{x} \sin^2(\alpha). \tag{30}
\]

\[
-(\ddot{x}^T \dddot{x} \sin^2(\alpha) - (\ddot{x}^T \dddot{x})(1 - \cos(\alpha))^2 \leq (\ddot{x}^T (\dot{y} - \dot{\lambda}) + \dddot{x}^T (\dot{y} - \dot{\lambda})) \sin(\alpha)(1 - \cos(\alpha)) \leq (\dddot{x}^T \dddot{x} \sin^2(\alpha) + (\ddot{x}^T \dddot{x})(1 - \cos(\alpha))^2 \tag{31}
\]

For \( \alpha = \frac{\pi}{2} \), (30) and (31) reduce to

\[
-\left( \ddot{x}^T \dddot{x} + \dddot{x}^T \dddot{x} \right) \leq (\dddot{x}^T \dddot{x} + \dddot{x}^T \dddot{x} \leq \dddot{x}^T \dddot{x} + \dddot{x}^T \dddot{x}. \tag{32}
\]

Proof See Appendix.

Using Lemmas 4.2, 4.1, and 4.3, we can show that \( \ddot{p} := \left( \dot{y}, \dot{y}, \dot{z}, \dot{z}, \dot{\lambda}, \dot{\lambda} \right) \), \( \ddot{\omega} := \left( \dot{\lambda}, \dot{\lambda}, \dot{\gamma}, \dot{\gamma} \right) \) are all bounded as claimed in the following two Lemmas.

**Lemma 4.3** Let \( (x, p, \omega) = (x, y, z, \lambda, \gamma) \in N_2(\theta) \) and \( (\dot{x}, \dot{p}, \dot{\omega}) = (\dot{x}, \dot{y}, \dot{z}, \dot{\lambda}, \dot{\gamma}) \) meet (13). Then,

\[
\left\| \ddot{p} \right\|^2 + \left\| \dot{\omega} \right\|^2 \leq \frac{2n}{1 - \theta}, \tag{33}
\]

\[
\left\| \ddot{p} \right\|^2 \left\| \dot{\omega} \right\|^2 \leq \left( \frac{n}{1 - \theta} \right)^2, \tag{34}
\]

\[
0 \leq \frac{\ddot{p}^T \dot{\omega}}{\mu} \leq \frac{1 + \theta}{1 - \theta} n := \delta_1 n. \tag{35}
\]

Proof See Appendix.

**Lemma 4.4** Let \( (x, p, \omega) = (x, y, z, \lambda, \gamma) \in N_2(\theta) \), \( (\dot{x}, \dot{y}, \dot{z}, \dot{\lambda}, \dot{\gamma}) \) and \( (\ddot{x}, \ddot{y}, \ddot{z}, \ddot{\lambda}, \ddot{\gamma}) \) meet (13) and (14). Then,

\[
\left\| \ddot{p} \right\|^2 + \left\| \dot{\omega} \right\|^2 \leq \frac{4(1 + \theta)n^2}{(1 - \theta)^3}, \tag{36}
\]

\[
\left\| \ddot{p} \right\|^2 \left\| \dot{\omega} \right\|^2 \leq \left( \frac{2(1 + \theta)n^2}{(1 - \theta)^3} \right)^2. \tag{37}
\]
\[
0 \leq \frac{\tilde{p}^T \dot{\omega}}{\mu} \leq \frac{2(1 + \theta)^2}{(1 - \theta)^3} n^2 := \delta_2 n^2, \\
\left| \frac{\ddot{p}^T \dot{\omega}}{\mu} \right| \leq \frac{(2n(1 + \theta))^\frac{3}{2}}{(1 - \theta)^2} := \delta_3 n^{\frac{3}{2}}, \\
\left| \frac{\dddot{p}^T \dot{\omega}}{\mu} \right| \leq \frac{(2n(1 + \theta))^\frac{3}{2}}{(1 - \theta)^2} := \delta_3 n^{\frac{3}{2}}.
\]  

(38) \hspace{10cm} (39)

**Proof** See Appendix. \(\square\)

Using the bounds established in Lemmas 4.2, 4.3, 4.4, and 2.2, we can obtain the lower bound and upper bound for \(\mu(\alpha)\).

**Lemma 4.5** Let \((x, p, \omega) = (x, y, z, \lambda, \gamma) \in \mathcal{N}_2(\theta)\), \((\dot{x}, \dot{y}, \dot{z}, \dot{\lambda}, \dot{\gamma})\) and \((\ddot{x}, \ddot{y}, \ddot{z}, \ddot{\lambda}, \ddot{\gamma})\) meet (13) and (14). Let \(x(\alpha), y(\alpha), z(\alpha), \lambda(\alpha), \gamma(\alpha)\) be defined by (19), (20), (21), (22), and (23). Then,

\[
\mu(1 - \sin(\alpha)) - \frac{1}{2n} \dot{x}^T H \dot{x} (1 - \cos(\alpha))^2 + \sin^2(\alpha)
\leq \mu(\alpha) = \mu(1 - \sin(\alpha)) + \frac{1}{2n} \left( \dot{x}^T (\dot{\gamma} - \dot{\lambda}) - \ddot{x}^T (\dot{\gamma} - \dot{\lambda}) \right)(1 - \cos(\alpha))^2
\leq \mu(1 - \sin(\alpha)) + \frac{1}{2n} \dot{x}^T H \ddot{x} (1 - \cos(\alpha))^2 + \sin^2(\alpha).
\]

(40)

**Proof** See Appendix. \(\square\)

To keep all the iterates of the algorithm inside the strictly feasible set, we need \((p(\alpha), \omega(\alpha)) > 0\) for all iterations. We will prove that this is guaranteed if \(\mu(\alpha) > 0\) holds. The following corollary states the condition for \(\mu(\alpha) > 0\) to hold.

**Corollary 4.1** If \(\mu > 0\), then for any fixed \(\theta \in (0, 1)\), there is an \(\bar{\alpha} > 0\) depending on \(\theta\), such that for any \(\sin(\alpha) \leq \sin(\bar{\alpha}), \mu(\alpha) > 0\). In particular, if \(\theta = 0.19\), \(\sin(\bar{\alpha}) \geq 0.6158\).

**Proof** From Lemmas 4.2 and 2.2, we have \(\dot{x}^T H \dot{x} = \dot{x}^T (\dot{\gamma} - \dot{\lambda}) = p^T \dot{\omega}\) and \((1 - \cos(\alpha))^2 \leq \sin^4(\alpha)\). Therefore, from Lemmas 4.5 and 4.3, we have

\[
\mu(\alpha) \geq \mu \left( 1 - \sin(\alpha) - \frac{1}{2n} \frac{\dot{p}^T \dot{\omega}}{\mu} \left( \sin^4(\alpha) + \sin^2(\alpha) \right) \right) \\
\geq \mu \left( 1 - \sin(\alpha) - \frac{(1 + \theta)}{2(1 - \theta)} \left( \sin^4(\alpha) + \sin^2(\alpha) \right) \right) := \mu r(\alpha).
\]

Since \(\mu > 0\), and \(r(\alpha)\) is a monotonic decreasing function of \(\alpha\) in \([0, \frac{\pi}{2}]\) with \(r(0) > 0\), \(r(\frac{\pi}{2}) < 0\), there is a unique real solution \(\sin(\bar{\alpha}) \in (0, 1)\) of \(r(\alpha) = 0\) such that for
all \( \sin(\alpha) < \sin(\bar{\alpha}) \), \( r(\alpha) > 0 \), or \( \mu(\alpha) > 0 \). It is easy to check that if \( \theta = 0.19 \), \( \sin(\bar{\alpha}) = 0.6158 \) is the solution of \( r(\alpha) = 0 \). \( \square \)

**Remark 4.1** Corollary 4.1 indicates that for any \( \theta \in (0, 1) \), there is a positive \( \tilde{\alpha} \) such that for \( \alpha \leq \tilde{\alpha} \), \( \mu(\alpha) > 0 \). Intuitively, to search in a wider region will generate a longer step. Therefore, the larger the \( \theta \) is, the better. But to derive the convergence result, \( \theta \leq 0.22 \) is imposed in Lemma 4.9 and \( \theta \leq 0.19 \) is imposed in Lemma 4.13.

To reduce the duality gap in any iteration, we need to have \( \mu(\alpha) \leq \mu \). For linear programming, it is known \([26]\) that \( \mu(\alpha) \leq \mu \) for \( \alpha \in [0, \hat{\alpha}] \) with \( \hat{\alpha} = \frac{\pi}{2} \), and the larger the \( \alpha \) in the interval is, the smaller the \( \mu(\alpha) \) will be. This claim is not true for the convex quadratic programming with box constraints and it needs to be modified as follows.

**Lemma 4.6** Let \((x, p, \omega) = (x, y, z, \lambda, \gamma) \in \mathcal{N}_2(\theta), (\dot{x}, \dot{y}, \dot{\tilde{x}}, \dot{\tilde{y}})\) and \((\ddot{x}, \ddot{y}, \ddot{\tilde{x}}, \ddot{\tilde{y}}, \ddot{\lambda}, \ddot{\gamma})\) be calculated by solving (13) and (14). Let \( x(\alpha), y(\alpha), z(\alpha), \lambda(\alpha), \) and \( \gamma(\alpha) \) be defined by (19), (20), (21), (22), and (23). Then, there exists

\[
\hat{\alpha} = \begin{cases} 
\frac{\pi}{2}, & \text{if } \frac{\dot{\tilde{x}}^TH\dot{\tilde{x}}}{n\mu} \leq 1 \\
\sin^{-1}(g), & \text{if } \frac{\dot{\tilde{x}}^TH\dot{\tilde{x}}}{n\mu} > 1 
\end{cases}
\]

where

\[
g = 3\sqrt{\frac{n\mu}{\dot{x}^TH\dot{x}}} + \sqrt{\left(\frac{n\mu}{\dot{x}^TH\dot{x}}\right)^2 + \left(\frac{1}{3}\right)^3} + 3\sqrt{\frac{n\mu}{\dot{x}^TH\dot{x}}} - \sqrt{\left(\frac{n\mu}{\dot{x}^TH\dot{x}}\right)^2 + \left(\frac{1}{3}\right)^3},
\]

such that for every \( \alpha \in [0, \hat{\alpha}] \), \( \mu(\alpha) \leq \mu \).

**Proof** See Appendix. \( \square \)

According to Theorem 4.1 and Lemmas 4.1, 4.3, 4.4, and 4.6, if \( \alpha \) is small enough, then \((p(\alpha), \omega(\alpha)) > 0\), and \( \mu(\alpha) < \mu \), i.e., the search along the ellipse defined by Theorem 4.1 will generate a strictly feasible point with a smaller duality gap. Since \((p, \omega) > 0\) holds in all iterations, reducing the duality gap to zero means approaching the solution of the convex quadratic programming. We will apply a similar idea used in \([18, 27]\), i.e., starting with an iterate in \(\mathcal{N}_2(\theta)\), searching along the approximated central path to reduce the duality gap and to keep the iterate in \(\mathcal{N}_2(2\theta)\), and then making a correction to move the iterate back to \(\mathcal{N}_2(\theta)\). First, we will introduce the following notations.

\[
a_0 = -\theta \mu < 0, \\
a_1 = \theta \mu > 0, \\
a_2 = 2\theta \frac{\dot{p}^T\omega}{2n} = 2\theta \frac{\dot{x}^T(\dot{y} - \dot{\lambda})}{2n} = 2\theta \frac{\dot{x}^TH\dot{\lambda}}{2n} \geq 0, 
\]

\(\square\ Springer\)
\[
a_3 = \| \dot{p} \circ \ddot{\omega} + \ddot{\omega} \circ \ddot{p} - \frac{1}{2n}(\ddot{p}^T \dot{\omega} + \dot{\omega}^T \ddot{p})e \| \geq 0,
\]

\[
a_4 = \| \ddot{p} \circ \ddot{\omega} - \dot{\omega} \circ \ddot{p} - \frac{1}{2n}(\ddot{p}^T \dot{\omega} - \dot{\omega}^T \ddot{p})e \| + 2\theta \frac{\dot{\theta}^T \dot{\theta}}{2n} \geq 0.
\]

We also define a quartic polynomial in terms of \( \sin(\alpha) \) as follows

\[
q(\alpha) = a_4 \sin^4(\alpha) + a_3 \sin^3(\alpha) + a_2 \sin^2(\alpha) + a_1 \sin(\alpha) + a_0 = 0. \tag{42}
\]

Since \( q(\alpha) \) is a monotonic increasing function of \( \alpha \in [0, \frac{\pi}{2}] \), \( q(0) = -\theta \mu < 0 \) and \( q\left(\frac{\pi}{2}\right) = a_2 + a_3 + a_4 > 0 \) if \( \dot{x} \neq 0 \), the polynomial has exactly one positive root in \( [0, \frac{\pi}{2}] \). Moreover, since (42) is a quartic equation, all the solutions are analytical and the computational cost is independent of the size of \( H \) and therefore negligible [9].

**Lemma 4.7** Let \((x, p, \omega) = (x, y, z, \lambda, \omega) \in \mathcal{N}_2(\theta), (\dot{x}, \dot{y}, \dot{z}, \dot{\lambda}, \dot{\omega}) \) and \((\ddot{x}, \ddot{y}, \ddot{z}, \ddot{\lambda}, \ddot{\omega}) \) be calculated from (13) and (14). Denote \( \sin(\tilde{\alpha}) \) be the only positive real solution of (42) in \([0, 1]\). Assume \( \sin(\alpha) \leq \min\{\sin(\tilde{\alpha}), \sin(\tilde{\alpha})\} \), let \((x(\alpha), y(\alpha), z(\alpha), \lambda(\alpha), \gamma(\alpha))\) and \(\mu(\alpha)\) be updated as follows

\[
(x(\alpha), y(\alpha), z(\alpha), \lambda(\alpha), \gamma(\alpha)) = (x, y, z, \lambda, \gamma) - (\ddot{x}, \ddot{y}, \ddot{z}, \ddot{\lambda}, \ddot{\gamma}) \sin(\alpha) + (\ddot{x}, \ddot{y}, \ddot{z}, \ddot{\lambda}, \ddot{\gamma})(1 - \cos(\alpha)), \quad (43)
\]

\[
\mu(\alpha) = \mu(1 - \sin(\alpha)) + \frac{1}{2n} \left( (\ddot{p} \circ \ddot{\omega} - \ddot{\omega} \circ \ddot{p})(1 - \cos(\alpha))^2 - (\ddot{p} \circ \ddot{\omega} + \ddot{\omega} \circ \ddot{p}) \sin(\alpha)(1 - \cos(\alpha)) \right). \quad (44)
\]

Then, \((x(\alpha), y(\alpha), z(\alpha), \lambda(\alpha), \gamma(\alpha)) \in \mathcal{N}_2(2\theta)\).

**Proof** See Appendix.

**Remark 4.2** It is worthwhile to note, by examining the proof of Lemma 4.7, that \( \sin(\tilde{\alpha}) \) is selected for the proximity condition (89) to hold, and \( \sin(\tilde{\alpha}) \) is selected for \( \mu(\alpha) > 0 \), thereby assuring the positivity condition (90) to hold.

The lower bound of \( \sin(\tilde{\alpha}) \) is estimated in Corollary 4.1. To estimate the lower bound of \( \sin(\tilde{\alpha}) \), we need the following lemma.

**Lemma 4.8** Let \((x, p, \omega) \in \mathcal{N}_2(\theta), (\ddot{x}, \ddot{p}, \ddot{\omega}) \) and \((\dddot{x}, \dddot{p}, \dddot{\omega}) \) be calculated by solving (13) and (14). Then,

\[
\| \dot{p} \circ \ddot{\omega} \| \leq \frac{(1 + \theta)}{(1 - \theta)} n\mu, \quad (45)
\]

\[
\| \dddot{p} \circ \dddot{\omega} \| \leq \frac{2(1 + \theta)^2}{(1 - \theta)^3} n^2 \mu, \quad (46)
\]

\[
\| \dddot{p} \circ \dddot{\omega} \| \leq \frac{2\sqrt{2}(1 + \theta)^{3}}{(1 - \theta)^2} n^\frac{3}{2} \mu, \quad (47)
\]
\[
\|\dot{\rho} \circ \dot{\omega}\| \leq \frac{2\sqrt{2}(1 + \theta)^{\frac{3}{2}}}{(1 - \theta)^2} n^{\frac{3}{2}} \mu. \tag{48}
\]

**Proof** See Appendix.

**Lemma 4.9** Let \( \theta \leq 0.22 \). Then, \( \sin(\tilde{\alpha}) \geq \frac{\theta}{\sqrt{n}} \).

**Proof** See Appendix.

Corollary 4.1 and Lemmas 4.7 and 4.9 prove the feasibility of searching optimizer along the ellipse while keeping the iterate in \( \mathcal{N}_2(2\theta) \). To move the iterate back to \( \mathcal{N}_2(\theta) \), we use the direction \((\Delta x, \Delta y, \Delta z, \Delta \lambda, \Delta \gamma)\) defined by

\[
\begin{bmatrix}
H & 0 & 0 & I & -I \\
I & I & 0 & 0 & 0 \\
I & 0 & -I & 0 & 0 \\
0 & \Lambda(\alpha) & 0 & Y(\alpha) & 0 \\
0 & 0 & \Gamma(\alpha) & 0 & Z(\alpha)
\end{bmatrix}
\begin{bmatrix}
\Delta x \\
\Delta y \\
\Delta z \\
\Delta \lambda \\
\Delta \gamma
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
\mu(\alpha)e - \lambda(\alpha) \circ y(\alpha) \\
\mu(\alpha)e - \gamma(\alpha) \circ z(\alpha)
\end{bmatrix}, \tag{49}
\]

and we update \((x^{k+1}, p^{k+1}, \omega^{k+1})\) and \(\mu^{k+1}\) by

\[
(x^{k+1}, p^{k+1}, \omega^{k+1}) = (x(\alpha), p(\alpha), \omega(\alpha)) + (\Delta x, \Delta p, \Delta \omega), \tag{50}
\]

\[
\mu^{k+1} = \frac{p^{k+1T} \omega^{k+1}}{2n}, \tag{51}
\]

where \(\Delta p = (\Delta y, \Delta z)\) and \(\Delta \omega = (\Delta \lambda, \Delta \gamma)\). Denote \(P(\alpha) = \begin{bmatrix} Y(\alpha) & 0 \\ 0 & Z(\alpha) \end{bmatrix}\), \(\Omega(\alpha) = \begin{bmatrix} \Lambda(\alpha) & 0 \\ 0 & \Gamma(\alpha) \end{bmatrix}\), and \(D = P^{\frac{1}{2}}(\alpha) \Omega^{-\frac{1}{2}}(\alpha)\). Then, the last 2 rows of (49) can be rewritten as

\[
P \Delta \omega + \Omega \Delta p = u(\alpha)e - P(\alpha) \Omega(\alpha)e. \tag{52}
\]

Now, we show that the correction step brings the iterate from \(\mathcal{N}_2(2\theta)\) back to \(\mathcal{N}_2(\theta)\). 

**Lemma 4.10** Let \((x(\alpha), p(\alpha), \omega(\alpha)) \in \mathcal{N}_2(2\theta)\) and \((\Delta x, \Delta p, \Delta \omega)\) be calculated by solving (49). Let \((x^{k+1}, p^{k+1}, \omega^{k+1})\) and \(\mu^{k+1}\) be updated by using (50) and (51) respectively. Then, for \(\theta \leq 0.29\) and \(\sin(\alpha) \leq \sin(\tilde{\alpha})\), \((x^{k+1}, p^{k+1}, \omega^{k+1}) \in \mathcal{N}_2(\theta)\).

**Proof** See Appendix.

Next, we show that the combined step (searching along the arc in \(\mathcal{N}_2(2\theta)\) and moving back to \(\mathcal{N}_2(\theta)\)) will reduce the duality gap of the iterate, i.e., \(\mu^{k+1} < \mu^k\), if we select some appropriate \(\theta\) and \(\alpha\). We introduce the following two lemmas before we prove this result.
Lemma 4.11 Let \((x(\alpha), p(\alpha), \omega(\alpha)) \in N_2(2\theta)\) and \((\Delta x, \Delta p, \Delta \omega)\) be calculated by solving (49). Then,

\[
0 \leq \frac{\Delta p^T \Delta \omega}{2n} \leq \frac{\theta^2 (1 + 2\theta)}{n(1 - 2\theta)^2} \mu(\alpha) := \frac{\delta_0}{n} \mu(\alpha).
\]

(53)

Proof See Appendix.

Lemma 4.12 Let \((x(\alpha), p(\alpha), \omega(\alpha)) \in N_2(2\theta)\) and \((\Delta_1x, \Delta_1p, \Delta_1\omega)\) be calculated by solving (49). Let \((x^{k+1}, p^{k+1}, \omega^{k+1})\) be updated as in (50). Then,

\[
\mu(\alpha) \leq \mu^{k+1} := \frac{p^{k+1}^T \omega^{k+1}}{2n} \leq \mu(\alpha) \left(1 + \frac{\theta^2 (1 + 2\theta)}{n(1 - 2\theta)^2}\right) = \mu(\alpha) \left(1 + \frac{\delta_0}{n}\right)
\]

Proof Using the fact that \(p(\alpha)^T \Delta \omega + \omega(\alpha)^T \Delta p = 0\) established in (96) in the proof of Lemma 4.10, and Lemma 4.11, it is therefore straightforward to obtain

\[
\mu(\alpha) \leq \frac{p(\alpha)^T \omega(\alpha)}{2n} + \frac{1}{2n} \Delta p^T \Delta \omega = \frac{(p(\alpha) + \Delta p)^T (\omega(\alpha) + \Delta \omega)}{2n} = \mu^{k+1} \leq \mu(\alpha) + \frac{\theta^2 (1 + 2\theta)}{n(1 - 2\theta)^2} \mu(\alpha).
\]

This proves the lemma.

For linear programming, it is known [18, 26] that \(\mu^{k+1} = \mu(\alpha)\). This claim is not always true for the convex quadratic programming as is pointed out in Lemma 4.12. Therefore, some extra work is needed to make sure that the duality gap will be reduced in every iteration.

Lemma 4.13 For \(\theta \leq 0.19\), if

\[
\sin(\alpha) = \frac{\theta}{\sqrt{n}},
\]

then \(\mu^{k+1} < \mu^k\). Moreover, for \(\sin(\alpha) = \frac{\theta}{\sqrt{n}} = \frac{0.19}{\sqrt{n}}\),

\[
\mu^{k+1} \leq \mu^k \left(1 - \frac{0.0185}{\sqrt{n}}\right).
\]

(55)

Proof See Appendix.

Remark 4.3 As we have seen in this section that starting with \((x^0, p^0, \omega^0)\), the interior-point algorithm proceeds with finding \((x(\alpha), p(\alpha), \omega(\alpha)) \in N_2(2\theta)\) and \((x^{k+1}, p^{k+1}, \omega^{k+1}) \in N_2(\theta)\) such that \(\mu^{k+1} < \mu^k\). In view of the proofs of Lemmas 4.7, 4.10, and 4.13, the positivity of \((x(\alpha), p(\alpha), \omega(\alpha)) > 0\) and \((x^{k+1}, p^{k+1}, \omega^{k+1}) > 0\) relies on \(\mu(\alpha) > 0\) which, according to Corollary 4.1, is
achievable for any $\theta$ and is given by a bound in terms of $\bar{\alpha}$. The proximity condition for $(x(\alpha), p(\alpha), \omega(\alpha))$ relies on the real positive root of $q(\sin(\alpha))$, denoted by $\sin(\tilde{\alpha})$, which is conservatively estimated in Lemma 4.9 under the condition that $\theta \leq 0.22$; the proximity condition for $(x^{k+1}, p^{k+1}, \omega^{k+1})$ is established in Lemma 4.10 under the condition that $\theta \leq 0.29$. Finally, duality gap reduction $\mu^{k+1} < \mu^k$ is established in Lemma 4.13 under the condition that $\theta \leq 0.19$. For all these results to hold, we just need to take the smallest bound $\theta = 0.19$.

We summarize all the results in this section as the following theorem.

\textbf{Theorem 4.2} Let $\theta = 0.19$ and $(x^k, p^k, \omega^k) \in N_2(\theta)$. Then, $(x(\alpha), p(\alpha), \omega(\alpha)) \in N_2(2\theta)$. Moreover, $(x^{k+1}, p^{k+1}, \omega^{k+1}) \in N_2(\theta)$, and $\mu^{k+1} \leq \mu^k \left(1 - \frac{0.0185}{\sqrt{n}}\right)$.

\textbf{Proof} From Corollary 4.1 and Lemma 4.9, we can select $\sin(\alpha) \leq \min\{\sin(\bar{\alpha}), \sin(\tilde{\alpha})\}$. Therefore, Lemma 4.7 holds, i.e., $(x(\alpha), p(\alpha), \omega(\alpha)) \in N_2(2\theta)$. Since $\sin(\alpha) \leq \sin(\bar{\alpha})$ and $(x(\alpha), p(\alpha), \omega(\alpha)) \in N_2(2\theta)$, Lemma 4.10 states $(x^{k+1}, p^{k+1}, \omega^{k+1}) \in N_2(\theta)$. For $\theta = 0.19$ and $\sin(\alpha) = \frac{\theta}{\sqrt{n}}$, Lemma 4.13 states $\mu^{k+1} \leq \mu^k \left(1 - \frac{0.0185}{\sqrt{n}}\right)$. This finishes the proof. \hfill $\Box$

\textbf{Remark 4.4} It is worthwhile to point out that $\theta = 0.19$ for the box constrained quadratic optimization problem is larger than the $\theta = 0.148$ for linearly constrained quadratic optimization problem established in [27]. This makes the searching neighborhood larger and the following algorithm more efficient than the algorithm in [27].

We present the proposed method as the following

\textbf{Algorithm 4.1 (Arc-search path-following)}

\textbf{Data:} $H \geq 0$, $c$, $n$, $\theta = 0.19$, $\epsilon > 0$, initial point $(x^0, p^0, \omega^0) \in N_2(\theta)$, and $\mu^0 = \frac{p^0\omega^0}{2n}$.

\textbf{for} iterations $k = 1, 2, \ldots$

Step 1: Solve the linear systems of equations (13) and (14) to get $(\dot{x}, \dot{p}, \dot{\omega})$ and $(\ddot{x}, \ddot{p}, \ddot{\omega})$.

Step 2: Let $\sin(\alpha) = \frac{\theta}{\sqrt{n}}$. Update $(x(\alpha), p(\alpha), \omega(\alpha))$ and $\mu(\alpha)$ by using (43) and (44).

Step 3: Solve (49) to get $(\Delta x, \Delta p, \Delta \omega)$, update $(x^{k+1}, p^{k+1}, \omega^{k+1})$ and $\mu^{k+1}$ by using (50) and (51).

Step 4: Set $k + 1 \rightarrow k$. Go back to Step 1.

\textbf{end (for)}

\section{5 Convergence analysis}

The first result in this section extends a result of linear programming (c.f. [23]) to convex quadratic programming subject to box constraints.
Lemma 5.1 Suppose $\mathcal{F}^o \neq \emptyset$. Then for each $K \geq 0$, the set

$$\{(x, p, \omega) \mid (x, p, \omega) \in \mathcal{F}, p^T \omega \leq K\}$$

is bounded.

Proof First, $x$ is bounded because $-e \leq x \leq e$. Since $x + y = e$ and $-e \leq x \leq e$, we have $0 \leq y = e - x \leq 2e$. Since $x - z = -e$, we have $0 \leq z = x + e \leq 2e$. Therefore, $y$ and $z$ are also bounded. Let $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{\gamma})$ be any fixed point in $\mathcal{F}^o$, and $(x, y, z, \lambda, \gamma)$ be any point in $\mathcal{F}$ with $y^T \lambda + z^T \gamma \leq K$. Then,

$$H(\bar{x} - x) + (\bar{\lambda} - \lambda) - (\bar{\gamma} - \gamma) = 0.$$

Therefore,

$$(\bar{x} - x)^T H(\bar{x} - x) + (\bar{x} - x)^T (\bar{\lambda} - \lambda) - (\bar{x} - x)^T (\bar{\gamma} - \gamma) = 0,$$

or equivalently

$$(\bar{x} - x)^T (\bar{y} - y) - (\bar{x} - x)^T (\bar{\lambda} - \lambda) = (\bar{x} - x)^T H(\bar{x} - x) \geq 0.$$

Using the relations $x - e = -y$ and $x + e = z$, we have

$$((\bar{x} + e) - (x + e))^T (\bar{y} - y) - ((\bar{x} - e) - (x - e))^T (\bar{\lambda} - \lambda) \geq 0,$$

or equivalently

$$(\bar{z} - z)^T (\bar{y} - y) + (\bar{y} - y)^T (\bar{\lambda} - \lambda) \geq 0.$$

This leads to

$$z^T \bar{y} + z^T \gamma - z^T \bar{y} - z^T \bar{\gamma} + \bar{y}^T \bar{\lambda} + y^T \lambda - y^T \bar{\lambda} - \bar{y}^T \lambda \geq 0,$$

or in a compact form

$$\bar{p}^T \bar{\omega} + p^T \omega - p^T \bar{\omega} - \bar{p}^T \omega \geq 0.$$

Sine $(\bar{p}, \bar{\omega}) > 0$ is fixed, let

$$\xi = \min_{i=1, \ldots, n} \min \{\bar{p}_i, \bar{\omega}_i\}.$$

Then, using $p^T \omega \leq K$,

$$\bar{p}^T \bar{\omega} + K \geq \xi e^T (p + \omega) \geq \max_{i=1, \ldots, n} \max \{\xi p_i, \xi \omega_i\},$$

i.e., for $i \in \{1, \ldots, n\}$,

$$0 \leq p_i \leq \frac{1}{\xi} (K + \bar{p}^T \bar{\omega}), 0 \leq \omega_i \leq \frac{1}{\xi} (K + \bar{p}^T \bar{\omega}).$$

This proves the lemma. \qed
The following theorem is a direct result of Lemmas 5.1 and 4.1, Theorem 4.2, KKT conditions, and Theorem A.2 in [23].

**Theorem 5.1** Suppose that Assumption 1 holds, then the sequence generated by Algorithm 4.1 converges to a set of accumulation points, and all these accumulation points are global optimal solutions of the convex quadratic programming subject to box constraints.

Let \((x^*, p^*, \omega^*)\) be any solution of (3d), following the notation of [1], we denote index sets \(B, S, \text{ and } T\) as

\[
B = \{ j \in \{1, \ldots, 2n\} \mid p^*_j \neq 0 \}. \tag{56}
\]

\[
S = \{ j \in \{1, \ldots, 2n\} \mid \omega^*_j \neq 0 \}. \tag{57}
\]

\[
T = \{ j \in \{1, \ldots, 2n\} \mid p^*_j = \omega^*_j = 0 \}. \tag{58}
\]

According to the Goldman-Tucker theorem [6], for the linear programming, \(B \cap S = \emptyset = T\) and \(B \cup S = \{1, \ldots, 2n\}\). A solution with this property is called strictly complementary. This property has been used in many papers to prove the locally super-linear convergence of interior-point algorithms in linear programming. However, it is pointed out in [7] that this partition does not hold for general quadratic programming problems. We will show that as long as a convex quadratic programming subject to box constraints has strictly complementary solution(s), an interior-point algorithm will generate a sequence to approach strict complementary solution(s). As a matter of fact, from Lemma 5.1, we can extend the result of [23, Lemma 5.13] to the case of convex quadratic programming subject to box constraints, and obtain the following lemma which is independent of any algorithm.

**Lemma 5.2** Let \(\mu^0 > 0, \text{ and } \rho \in (0, 1)\). Assume that the convex QP (1) has strictly complementary solution(s). Then for all points \((x, p, \omega)\) with \((x, p, \omega) \in \mathcal{F}^o, p_i \omega_i > \rho \mu, \text{ and } \mu < \mu^0\), there are constants \(M, C_1, \text{ and } C_2\) such that

\[
\| (p, \omega) \| \leq M, \tag{59}
\]

\[
0 < p_i \leq \mu/C_1 (i \in S), 0 < \omega_i \leq \mu/C_1 (i \in B). \tag{60}
\]

\[
\omega_i \geq C_2 \rho (i \in S), p_i \geq C_2 \rho (i \in B). \tag{61}
\]

**Proof** The first result (59) follows immediately from Lemma 5.1 by setting \(K = 2n \mu^0\). Let \((x^*, p^*, \omega^*)\) be any strictly complementary solution. Since \((x^*, p^*, \omega^*)\) and \((x, p, \omega)\) are both feasible, we have

\[
(y - y^*) = -(x - x^*) = -(z - z^*), H(x - x^*) + (\lambda - \lambda^*) - (\gamma - \gamma^*) = 0.
\]

Therefore,

\[
(y - y^*)^T (\lambda - \lambda^*) + (z - z^*)^T (\gamma - \gamma^*) = (x - x^*)^T H(x - x^*) \geq 0. \tag{62}
\]
Since \((x^*, y^*, z^*, \lambda^*, \gamma^*) = (x^*, p^*, \omega^*)\) is strictly complementary solution, \(T = \emptyset\), 
\(p_i^* = 0\) for \(i \in S\), and \(\omega_i^* = 0\) for \(i \in B\). Since \(p^T \omega = 2n \mu\), \((p^*)^T \omega^* = 0\), from (62), we have

\[
p^T \omega = y^T \lambda + z^T \gamma + ((y^*)^T \lambda^* + (z^*)^T \gamma^*) \geq y^T \lambda^* + z^T \gamma^* + ((y^*)^T \lambda^* + (z^*)^T \gamma^*) = p^T \omega^* + \omega^T p^*
\]

\[
\iff 2n \mu \geq p^T \omega^* + \omega^T p^* = \sum_{i \in S} p_i^* \omega_i^* + \sum_{i \in B} p_i^* \omega_i.
\]

(63)

Since each term in the summations is positive and bounded above by \(2n \mu\), we have for any \(i \in S\), \(\omega_i^* > 0\); therefore,

\[0 < p_i \leq \frac{2n \mu}{\omega_i^*}.
\]

Denote \(\Omega_D = \{(p^*, \omega^*) | \omega_i^* > 0\}\) and \(\Omega_P = \{(p^*, \omega^*) | p_i^* > 0\}\), we have

\[
0 < p_i \leq \frac{2n \mu}{\sup_{(p^*, \omega^*) \in \Omega_D} \omega_i^*}.
\]

This leads to

\[
\max_{i \in S} p_i \leq \frac{2n \mu}{\min_{i \in S} \sup_{(p^*, \omega^*) \in \Omega_D} \omega_i^*}.
\]

Similarly,

\[
\max_{i \in B} \omega_i \leq \frac{2n \mu}{\min_{i \in B} \sup_{(p^*, \omega^*) \in \Omega_P} p_i^*}.
\]

Combining these 2 inequalities gives

\[
\max \{ \max_{i \in S} p_i, \max_{i \in B} \omega_i \} \leq \frac{2n \mu}{\min \{ \min_{i \in S} \sup_{(p^*, \omega^*) \in \Omega_D} \omega_i^*, \min_{i \in B} \sup_{(p^*, \omega^*) \in \Omega_P} p_i^* \}} = \frac{\mu}{C_1}.
\]

This proves (60). Finally, \(p_i \omega_i \geq \rho \mu\); hence for any \(i \in S\),

\[
\omega_i \geq \frac{\rho \mu}{p_i} \geq \frac{\rho \mu}{\mu / C_1} = C_2 \rho.
\]

Similarly, for any \(i \in B\),

\[
p_i \geq \frac{\rho \mu}{\omega_i} \geq \frac{\rho \mu}{\mu / C_1} = C_2 \rho.
\]

\[\blacksquare\]

**Lemma 5.2** leads to the following

**Theorem 5.2** Let \((x^k, p^k, \omega^k) \in N_2(\theta)\) be generated by Algorithms 4.1. Assume that the convex QP with box constraints has strictly complementary solution(s). Then, every limit point of the sequence is a strictly complementary solution of the convex quadratic programming with box constraints, i.e.,

\[
\omega_i^* \geq C_2 \rho (i \in S),\ p_i^* \geq C_2 \rho (i \in B).
\]

(64)
Proof From Lemma 5.2, \((p^k, \omega^k)\) is bounded; therefore, there is at least one limit point \((p^*, \omega^*)\). Since \((p_i^k, \omega_i^k)\) is in the neighborhood of the central path, i.e., \(p_i^k \omega_i^k > \rho\mu^k := (1 - 3\theta)\mu^k\),
\[
\omega_i^k \geq C_2\rho \quad (i \in S), \quad p_i^k \geq C_2\rho \quad (i \in B),
\]
every limit point will meet (64) due to the fact that \(C_2\rho\) is a constant. \(\square\)

We now show that the complexity bound of Algorithm 4.1 is \(O(\sqrt{n} \log(1/\epsilon))\). We need the following theorem from [23] for this purpose.

**Theorem 5.3** Let \(\epsilon \in (0, 1)\) be given. Suppose that an algorithm for solving (3d) generates a sequence of iterations that satisfies
\[
\mu^{k+1} \leq \left(1 - \frac{\delta}{n^x}\right)\mu^k, \quad k = 0, 1, 2, \ldots, \quad (65)
\]
for some positive constants \(\delta\) and \(\chi\). Suppose that the starting point \((x^0, p^0, \omega^0)\) satisfies \(\mu^0 \leq 1/\epsilon\). Then, there exists an index \(K\) with
\[
K = O(n^x \log(1/\epsilon))
\]
such that
\[
\mu^k \leq \epsilon, \quad \text{for } \forall k \geq K.
\]

Combining Lemma 4.13 and Theorem 5.3 gives

**Theorem 5.4** The complexity of Algorithm 4.1 is bounded by \(O(\sqrt{n} \log(1/\epsilon))\).

6 Implementation issues

Algorithm 4.1 is presented in a form that is convenient for the convergence analysis. Some implementation details that make the algorithm more efficient are discussed in this section.

6.1 Termination criterion

Algorithm 4.1 needs a termination criterion in real implementation. One can use
\[
\mu^k \leq \epsilon, \quad (66a)
\]
\[
\|r_X\| = \|Hx^k + \lambda^k - \gamma^k + c\| \leq \epsilon, \quad (66b)
\]
\[
\|r_Y\| = \|x^k + y^k - e\| \leq \epsilon, \quad (66c)
\]
\[
\|r_Z\| = \|x^k - z^k + e\| \leq \epsilon, \quad (66d)
\]
\[
\|r_r\| = \|p^k \Xi^k e - \mu e\| \leq \epsilon, \quad (66e)
\]
\[
(p^k, \omega^k) > 0. \quad (66f)
\]
An alternate criterion is similar to the one used in \texttt{linprog} [34]

$$\kappa := \frac{\|r_Y\| + \|r_Z\|}{2n} + \frac{\|r_X\|}{\max\{1, \|c\|\}} + \frac{\mu_k}{\max\{1, \|x^T H x^k + c^T x^k\|\}} \leq \epsilon. \quad (67)$$

### 6.2 Initial \((x^0, \lambda^0, s^0) \in \mathcal{N}_2(\theta)\)

For feasible interior-point algorithms, an important prerequisite is to start with a feasible interior point. While finding an initial feasible point may not be a simple task for even linear programming with equality constraints [4], for the quadratic programming subject to box constraints, finding the initial point is not an issue. We show that the following initial point \((x^0, y^0, z^0, \lambda^0, \gamma^0)\) is an interior point; moreover, \((x^0, y^0, z^0, \lambda^0, \gamma^0) \in \mathcal{N}_2(\theta)\).

\begin{align*}
  x^0 &= 0, \quad y^0 = z^0 = e > 0, \quad (68a) \\
  \lambda^0_i &= 4(1 + \|c\|^2) - \frac{c_i}{2} > 0, \quad (68b) \\
  \gamma^0_i &= 4(1 + \|c\|^2) + \frac{c_i}{2} > 0. \quad (68c)
\end{align*}

It is easy to see that this selected point meets (6). Therefore, we will show that it meets (10). Since

$$\mu^0 = \sum_{i=1}^n (\lambda^0_i + \gamma^0_i) = \sum_{i=1}^n \frac{8(1 + \|c\|^2)}{2n} = 4(1 + \|c\|^2), \quad (69)$$

we have, for \(\theta = 0.19\),

$$\|p^0 \circ \omega^0 - \mu^0 e\|^2 = \sum_{i=1}^n (\lambda^0_i - \mu^0)^2 + \sum_{i=1}^n (\gamma^0_i - \mu^0)^2 = \frac{\|c\|^2}{2} \leq 16\theta^2(1 + \|c\|^2)^2 = \theta^2(\mu^0)^2.$$

### 6.3 Step size

Directly using \(\sin(\alpha) = \frac{\theta}{\sqrt{n}}\) in Algorithm 4.1 provides an effective formula to prove the polynomiality. However, this choice of \(\sin(\alpha)\) is too conservative in practice because this search step in \(\mathcal{N}_2(2\theta)\) is too small and the speed of duality gap reduction is slow. A better choice of \(\sin(\alpha)\) should have a larger step in every iteration so that the polynomiality is reserved and fast convergence is achieved. In view of Remark 4.3, conditions that restrict step size are positivity conditions, proximity conditions, and duality reduction condition. We examine how to enlarge the step size under these restrictions.

First, from (90) and (99), \(\mu(\alpha) > 0\) is required for positivity conditions \((p(\alpha), \omega(\alpha)) > 0\) and \((p^{k+1}, \omega^{k+1}) > 0\) to hold. Since \(\sin(\tilde{\alpha})\) estimated in Corollary 4.1 is conservative, we find a better \(\tilde{\alpha}\) directly from (40).

$$\mu(\alpha) \geq \mu(1 - \sin(\alpha)) - \frac{1}{2n} (\dot{p}^T \dot{\omega}) \left( \sin^4(\alpha) + \sin^2(\alpha) \right) := f(\sin(\alpha)) = \sigma, \quad (70)$$
where $\sigma > 0$ is a small number, and $f(\sin(\alpha))$ is a monotonic decreasing function of $\sin(\alpha)$ with $f(\sin(0)) = 1$ and $f(\sin(\frac{\pi}{2})) < 0$. Therefore, (70) has a unique positive real solution for $\alpha \in (0, \frac{\pi}{2})$. Since (70) is a quartic function of $\sin(\alpha)$, the cost of finding the smallest positive solution is negligible [9].

Second, for $\theta \leq 0.19$, from (98), the proximity condition for $(x^{k+1}, y^{k+1}, z^{k+1}, \lambda^{k+1}, y^{k+1})$ holds without further restriction. The proximity condition (89) is met for $\sin(\alpha) \in [0, \sin(\tilde{\alpha})]$, where $\sin(\tilde{\alpha})$ is the smallest positive solution of (42) and it is estimated very conservatively in Lemma 4.9. An efficient implementation should use $\sin(\tilde{\alpha})$, the smallest positive solution of (42). Actually, there exist a $\tilde{\alpha}$ which is normally larger than $\tilde{\alpha}$ such that the proximity condition (89) is met for $\sin(\alpha) \in [0, \sin(\tilde{\alpha})]$. Let

$$
b_0 = -\theta \mu < 0, \quad b_1 = \theta \mu > 0, \quad b_3 = \left\| \tilde{p} \circ \tilde{\omega} + \dot{\omega} \circ \tilde{p} - \frac{1}{2n}(\tilde{p}^T \tilde{\omega} + \dot{\omega}^T \tilde{p})e \right\| + \frac{\theta}{n} \left( \tilde{p}^T \tilde{\omega} + \dot{\omega}^T \tilde{p} \right), \quad b_4 = \left\| \tilde{p} \circ \tilde{\omega} - \dot{\omega} \circ \tilde{p} - \frac{1}{2n}(\tilde{p}^T \tilde{\omega} - \dot{\omega}^T \tilde{p})e \right\| - \frac{\theta}{n} \left( \tilde{p}^T \tilde{\omega} - \dot{\omega}^T \tilde{p} \right),
$$

and

$$
p(\alpha) := b_4(1 - \cos(\alpha))^2 + b_3 \sin(\alpha)(1 - \cos(\alpha)) + b_1 \sin(\alpha) + b_0. \quad (71)
$$

Applying the second inequality of (31) to $\frac{\theta}{n} \left( \tilde{p}^T \tilde{\omega} + \dot{\omega}^T \tilde{p} \right) \sin(\alpha)(1 - \cos(\alpha))$, we can easily show that

$$
p(\alpha) \leq q(\alpha),
$$

where $q(\alpha)$ is defined in (42). Therefore, the smallest positive solution $\dot{\alpha}$ of $p(\alpha)$ is larger than the smallest positive solution $\tilde{\alpha}$ of $q(\alpha)$. We will show that for $\sin(\alpha) \in [0, \sin(\tilde{\alpha})]$, the proximity condition (89) holds. Since for $\sin(\alpha) \in [0, \sin(\tilde{\alpha})]$, $p(\alpha) \leq 0$, we have

$$
\left\| \tilde{p} \circ \tilde{\omega} - \dot{\omega} \circ \tilde{p} - \frac{1}{2n}(\tilde{p}^T \tilde{\omega} - \dot{\omega}^T \tilde{p})e \right\| (1 - \cos(\alpha))^2 + \left\| \tilde{p} \circ \tilde{\omega} + \dot{\omega} \circ \tilde{p} - \frac{1}{2n}(\tilde{p}^T \tilde{\omega} + \dot{\omega}^T \tilde{p})e \right\| \sin(\alpha)(1 - \cos(\alpha)) \leq (2\theta) \left( \frac{1}{2n}(\tilde{p}^T \tilde{\omega} - \dot{\omega}^T \tilde{p}) (1 - \cos(\alpha))^2 - \frac{1}{2n}(\tilde{p}^T \tilde{\omega} + \dot{\omega}^T \tilde{p}) \sin(\alpha)(1 - \cos(\alpha)) \right) - \theta \mu(1 - \sin(\alpha)). \quad (72)
$$

Substituting this inequality into (88) gives

$$
\left\| p(\alpha) \circ \omega(\alpha) - \mu(\alpha)e \right\| \leq 2\theta \left( \mu(1 - \sin(\alpha)) + \frac{1}{2n} \left( \tilde{x}^T (\dot{\gamma} - \tilde{\lambda}) - \tilde{x}^T (\dot{\gamma} - \tilde{\lambda}) \right) (1 - \cos(\alpha))^2 - \frac{1}{2n} \left( \tilde{x}^T (\dot{\gamma} - \tilde{\lambda}) + \tilde{x}^T (\dot{\gamma} - \tilde{\lambda}) \right) \sin(\alpha)(1 - \cos(\alpha)) \right) = 2\theta \mu(\alpha). \quad (73)
$$
This is the proximity condition for \((x(\alpha), y(\alpha), z(\alpha), \lambda(\alpha), \gamma(\alpha))\). Denote \(\hat{b}_0 = b_0, \hat{b}_1 = b_1\),

\[
\hat{b}_3 = \begin{cases} 
  b_3 & \text{if } b_3 \geq 0, \\
  0 & \text{if } b_3 < 0,
\end{cases}
\quad
\hat{b}_4 = \begin{cases} 
  b_4 & \text{if } b_4 \geq 0, \\
  0 & \text{if } b_4 < 0,
\end{cases}
\]

and

\[
\hat{p}(\alpha) := \hat{b}_4(1 - \cos(\alpha))^2 + \hat{b}_3 \sin(\alpha)(1 - \cos(\alpha)) + \hat{b}_1 \sin(\alpha) + \hat{b}_0. \tag{74}
\]

Since \(\hat{p}(\alpha) \geq p(\alpha)\), the smallest positive solution \(\hat{\alpha}\) of \(\hat{p}(\alpha)\) is smaller than smallest positive solution \(\tilde{\alpha}\) of \(p(\alpha)\). To estimate the smallest solution of \(\hat{\alpha}\), by noticing that \(\hat{p}(\alpha)\) is a monotonic increasing function of \(\alpha\) and \(\hat{p}(0) = -\theta \mu < 0\), we can simply use the bisection method. The computational cost is impendent of the problem size \(n\) and is negligible. Since both estimated step sizes \(\hat{\alpha}\) and \(\tilde{\alpha}\) guarantee the proximity condition for \((x(\alpha), y(\alpha), z(\alpha), \lambda(\alpha), \gamma(\alpha))\) to hold, we select \(\hat{\alpha} = \max\{\hat{\alpha}, \tilde{\alpha}\} \geq \tilde{\alpha}\) which guarantees the polynomiality claim to hold.

Third, from (107a) and Lemma 4.5, we have

\[
\mu^{k+1} \leq \mu^k \left(1 + \frac{\theta^2(1 + 2\theta)}{n(1 - 2\theta)^2} - \left(1 + \frac{\theta^2(1 + 2\theta)}{n(1 - 2\theta)^2}\right) \sin(\alpha) + \left(1 + \frac{\theta^2(1 + 2\theta)}{n(1 - 2\theta)^2}\right) \frac{\bar{p}^T \bar{\omega}}{2n\mu} \left(\sin^2(\alpha) + \sin^4(\alpha)\right)\right).
\]

For \(\mu^{k+1} \leq \mu^k\) to hold, we need

\[
\frac{\theta^2(1 + 2\theta)}{n(1 - 2\theta)^2} = \left(1 + \frac{\theta^2(1 + 2\theta)}{n(1 - 2\theta)^2}\right) \sin(\alpha) + \left(1 + \frac{\theta^2(1 + 2\theta)}{n(1 - 2\theta)^2}\right) \frac{\bar{p}^T \bar{\omega}}{2n\mu} \left(\sin^2(\alpha) + \sin^4(\alpha)\right) \leq 0.
\]

For the sake of convenience in convergence analysis, a conservative estimate is used in Lemma 4.13. For efficient implementation, the following solution should be adopted. Denote \(u = \frac{\theta^2(1 + 2\theta)}{n(1 - 2\theta)^2} > 0\), \(v = \frac{\bar{p}^T \bar{\omega}}{2n\mu} > 0\), \(z = \sin(\alpha) \in [0, 1]\), and

\[
F(z) = (1 + u) vz^4 + (1 + u) vz^2 - (1 + u)z + u.
\]

For \(z \in [0, 1]\) and \(v \leq \frac{1}{6}\), \(F'(z) = (1 + u)(4vz^3 + 2vz - 1) \leq 0\); therefore, the upper bound of the duality gap is a monotonic decreasing function of \(\sin(\alpha)\) for \(\alpha \in [0, \frac{\pi}{2}]\). The larger \(\alpha\) is, the smaller the upper bound of the duality gap will be. For \(v > \frac{1}{6}\), to minimize the upper bound of the duality gap, we can find the solution of \(F'(z) = 0\).

It is easy to check from discriminator [21] that the cubic polynomial \(F'(z)\) has only one real solution which is given by (see Lemma 2.5)

\[
\sin(\tilde{\alpha}) = \sqrt[3]{\frac{n\mu}{4\bar{p}^T \bar{\omega}}} + \sqrt[3]{\left(\frac{n\mu}{4\bar{p}^T \bar{\omega}}\right)^2 + \left(\frac{1}{6}\right)^3} + \sqrt[3]{\frac{n\mu}{4\bar{p}^T \bar{\omega}}} - \sqrt[3]{\left(\frac{n\mu}{4\bar{p}^T \bar{\omega}}\right)^2 + \left(\frac{1}{6}\right)^3}.
\]
Since \( F''(\sin(\alpha)) = (1 + u)(12v \sin^2(\alpha) + 2v) > 0 \), at \( \sin(\alpha) \in [0, 1) \), the upper bound of the duality gap is minimized. Therefore, we can define

\[
\tilde{\alpha} = \begin{cases} 
\frac{\pi}{2}, & \text{if } \frac{\hat{p}_t \hat{\omega}}{2n\mu} \leq \frac{1}{6} \\
\sin^{-1} \left( \sqrt{\frac{n\mu}{4p^2\omega}} + \sqrt{\left( \frac{n\mu}{4p^2\omega} \right)^2 + \left( \frac{1}{6} \right)^3} \right) - \sin^{-1} \left( \sqrt{\left( \frac{n\mu}{4p^2\omega} \right)^2 + \left( \frac{1}{6} \right)^3} \right), & \text{if } \frac{\hat{p}_t \hat{\omega}}{2n\mu} > \frac{1}{6}.
\end{cases}
\]

(75)

It is worthwhile to note that for \( \alpha < \tilde{\alpha}, F'(\sin(\alpha)) < 0 \), i.e., \( F(\sin(\alpha)) \) is a monotonic decreasing function of \( \alpha \in [0, \tilde{\alpha}] \).

We summarize the step size selection process as a simple algorithm as follows.

**Algorithm 6.1 (Step Size Selection)**

**Data:** \( \sigma > 0 \).

**Step 1:** Find the positive real solution of (70) to get \( \sin(\tilde{\alpha}) \)

**Step 2:** Find the smallest positive real solution of (74) to get \( \sin(\hat{\alpha}) \), the smallest positive real solution of (42) to get \( \sin(\check{\alpha}) \), and set \( \sin(\bar{\alpha}) = \max\{\sin(\tilde{\alpha}), \sin(\hat{\alpha}), \sin(\check{\alpha})\} \).

**Step 3:** Calculate \( \tilde{\alpha} \) given by (75)

**Step 4:** The step size is obtained as \( \sin(\alpha) = \min\{\sin(\bar{\alpha}), \sin(\check{\alpha}), \sin(\tilde{\alpha})\} \).

### 6.4 The practical implementation

Therefore, Algorithm 4.1 can be implemented as follows.

**Algorithm 6.2 (Arc-search path-following)**

**Data:** \( H \geq 0, c, n, \theta = 0.19, \epsilon > \sigma > 0 \).

**Step 0:** Find initial point \((x^0, p^0, \omega^0) \in N_2(\theta)\) using (68c), \( \kappa \) using (67), and \( \mu^0 \) using (69).

\[ \textbf{while} \ \kappa > \epsilon \]

**Step 1:** Compute \((\dot{x}, \dot{p}, \dot{\omega})\) and \((\ddot{x}, \ddot{p}, \ddot{\omega})\) using (13) and (14).

**Step 2:** Select \( \sin(\alpha) \) using Algorithm 6.1. Update \((x(\alpha), p(\alpha), \omega(\alpha))\) and \( \mu(\alpha) \) using (43) and (44).

**Step 3:** Compute \((\Delta x, \Delta p, \Delta \omega)\) using (49), update \((x^{k+1}, p^{k+1}, \omega^{k+1})\) and \( \mu^{k+1} \) using (50) and (51).

**Step 4:** Compute \( \kappa \) using (67).

**Step 5:** Set \( k + 1 \rightarrow k \). Go back to Step 1.

**end (while)**

**Remark 6.1** The condition \( \mu > \sigma \) guarantees that the equation (70) has a positive solution before terminate criterion is met.
In this section, we use the design example of [2] to demonstrate the effectiveness and efficiency of the proposed algorithm. The linear time-invariant system under consideration is given by

\[
x_{s+1} = \begin{bmatrix} 1 & h \\ -h & 1 \end{bmatrix} x_s + \begin{bmatrix} 0 \\ h \end{bmatrix} u_s, \quad s = 0, \ldots, N - 1,
\]

(76)

with the initial state given by \( x_0 = [15, 5]^T \). The control constraints are

\[-1 \leq u_s \leq 1, \quad s = 0, \ldots, N - 1.\]

(77)

The problem is to minimize

\[
J = \min_{u_0, u_1, \ldots, u_{N-1}} \frac{1}{2} x_N^T P x_N + \frac{h}{2} \sum_{k=0}^{N-1} \left( x_k^T Q x_k + R u_k^2 \right)
\]

(78)

where the matrices \( P \), \( Q \), and scalar \( R \) are given by

\[
P = Q = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = 6.
\]

This problem arises from discretization of the continuous-time problem of minimizing

\[
\frac{1}{2} x_T^T P x_T + \int_0^T \left( x(t)^T Q x(t) + R u(t)^2 \right) dt
\]

subject to

\[
\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) = Ax(t) + B u(t),
\]

Fig. 1 Optimal control with saturation constraint
and
\[ 1 \leq u(t) \leq 1, \, t \in [0, T], \]
where the interval \([0, T]\) is discretized into \(N\) intervals of length \(h = \frac{T}{N}\).

In our implementation of Algorithm 6.2, \(\epsilon = 10^{-8}\) and \(\sigma = 10^{-10}\) are selected. For the simple design example, \(T = 50\) and \(N = 500\) are used. After 27 iterations, the algorithm converges. Using the optimal control inputs, we can calculate the state space response from (76). The control inputs and state space response are displayed in Fig. 1.

8 Conclusions

This paper proposes an arc-search interior-point algorithm for convex quadratic programming subject to box constraints that searches the optimizers along ellipses that approximate the central path. The algorithm is proved to be polynomial with the complexity bound \(O(\sqrt{n} \log(1/\epsilon))\). A constrained LQR design example from [2] is used to demonstrate how the algorithm works. A preliminary test on this simple design problem shows that the proposed algorithm is promising.

Appendix: Proofs of technical lemmas

Proof of Lemma 4.2 From (16), we have
\[
\dot{x}^T (\dot{\gamma} - \dot{\lambda}) = \ddot{\gamma}^T \gamma + \dot{\gamma}^T \lambda = \ddot{p}^T \omega, \quad \dddot{x}^T (\dddot{\gamma} - \dddot{\lambda}) = \dddot{\gamma}^T \gamma + \dot{\gamma}^T \lambda = \dddot{p}^T \omega, \quad \text{and} \quad \dddot{x}^T (\dddot{\gamma} - \dddot{\lambda}) = \dddot{p}^T \omega. \]

Pre-multiplying \(\dot{x}^T\) and \(\dddot{x}^T\) to (15) gives
\[
\dot{x}^T (\dot{\gamma} - \dot{\lambda}) = \dot{x}^T \dot{H} \dddot{x}, \quad \dddot{x}^T (\dddot{\gamma} - \dddot{\lambda}) = \dddot{x}^T \dddot{H} \dddot{x}, \quad \dddot{x}^T (\dddot{\gamma} - \dddot{\lambda}) = \dddot{x}^T \dddot{H} \dddot{x} = \dddot{x}^T (\dddot{\gamma} - \dddot{\lambda}).
\]

Equations (27) and (28) follow from the first two equations and the fact that \(H\) is positive definite. The last equation gives (29). Using (27), (28), and (29) gives
\[
(\dot{x}(1 - \cos(\alpha)) + \dddot{x} \sin(\alpha))^T H(\dot{x}(1 - \cos(\alpha)) + \dddot{x} \sin(\alpha)) = (\dot{x}^T \dot{H} \dddot{x})(1 - \cos(\alpha))^2 + 2(\dot{x}^T \dot{H} \dddot{x}) \sin(\alpha)(1 - \cos(\alpha)) + (\dddot{x}^T \dddot{H} \dddot{x}) \sin^2(\alpha) = (\dot{x}^T \dot{H} \dddot{x})(1 - \cos(\alpha))^2 + (\dddot{x}^T \dddot{H} \dddot{x}) \sin^2(\alpha) + \dddot{x}^T (\dddot{\gamma} - \dddot{\lambda}) + \dddot{x}^T (\dddot{\gamma} - \dddot{\lambda}) \sin(\alpha)(1 - \cos(\alpha)) \geq 0,
\]
which is the first inequality of (30). Using (27), (28), and (29) also gives
\[
(\dot{x}(1 - \cos(\alpha)) - \dddot{x} \sin(\alpha))^T H(\dot{x}(1 - \cos(\alpha)) - \dddot{x} \sin(\alpha)) = (\dot{x}^T \dot{H} \dddot{x})(1 - \cos(\alpha))^2 - 2(\dot{x}^T \dot{H} \dddot{x}) \sin(\alpha)(1 - \cos(\alpha)) + (\dddot{x}^T \dddot{H} \dddot{x}) \sin^2(\alpha) = (\dot{x}^T \dot{H} \dddot{x})(1 - \cos(\alpha))^2 + (\dddot{x}^T \dddot{H} \dddot{x}) \sin^2(\alpha) - \dddot{x}^T (\dddot{\gamma} - \dddot{\lambda}) + \dddot{x}^T (\dddot{\gamma} - \dddot{\lambda}) \sin(\alpha)(1 - \cos(\alpha)) \geq 0,
\]
which is the second inequality of (30). Replacing \(\dot{x}(1 - \cos(\alpha))\) and \(\dddot{x} \sin(\alpha)\) by \(\dddot{x} \sin(\alpha)\) and \(\dddot{x}(1 - \cos(\alpha))\), and following the same method, we can obtain (31).
Proof of Lemma 4.3 From the last two rows of (13) or equivalently (17), we have
\begin{align*}
\Lambda \dot{y} + Y \dot{\lambda} &= \Lambda Ye \\
\Gamma \dot{z} + Z \dot{\gamma} &= \Gamma Ze.
\end{align*}

Pre-multiplying \( Y^{-\frac{1}{2}} \Lambda^{-\frac{1}{2}} \) on both sides of the first equality gives
\begin{align*}
Y^{-\frac{1}{2}} \Lambda^{-\frac{1}{2}} \dot{y} + Y^{-\frac{1}{2}} \Lambda^{-\frac{1}{2}} \dot{\lambda} &= Y^{-\frac{1}{2}} \Lambda^{-\frac{1}{2}} e.
\end{align*}

Pre-multiplying \( Z^{-\frac{1}{2}} \Gamma^{-\frac{1}{2}} \) on both sides of the second equality gives
\begin{align*}
Z^{-\frac{1}{2}} \Gamma^{-\frac{1}{2}} \dot{z} + Z^{-\frac{1}{2}} \Gamma^{-\frac{1}{2}} \dot{\gamma} &= Z^{-\frac{1}{2}} \Gamma^{-\frac{1}{2}} e. \quad (79)
\end{align*}

Let \( u = \begin{bmatrix} Y^{-\frac{1}{2}} \Lambda^{-\frac{1}{2}} \dot{y} \\ Z^{-\frac{1}{2}} \Gamma^{-\frac{1}{2}} \dot{z} \end{bmatrix}, v = \begin{bmatrix} Y^{-\frac{1}{2}} \Lambda^{-\frac{1}{2}} \dot{\lambda} \\ Z^{-\frac{1}{2}} \Gamma^{-\frac{1}{2}} \dot{\gamma} \end{bmatrix} \), and \( w = \begin{bmatrix} Y^{-\frac{1}{2}} \Lambda^{-\frac{1}{2}} e \\ Z^{-\frac{1}{2}} \Gamma^{-\frac{1}{2}} e \end{bmatrix} \), and using (16) and Lemma 4.2, we have
\begin{align*}
\dot{p}^T \dot{\omega} &= \dot{x}^T (\dot{\gamma} - \dot{\lambda}) = \dot{x}^T H \dot{x} \geq 0, \text{ we have the first inequality of (35).}
\end{align*}
**Proof of Lemma 4.4** Similar to the proof of Lemma 4.3, from (18), we have
\[
\Lambda \ddot{y} + Y \ddot{x} = -2 (\dot{y} \circ \dot{\lambda})
\]
\[
\iff Y^{-\frac{1}{2}} \Lambda^{\frac{1}{2}} \ddot{y} + Y^{-\frac{1}{2}} \Lambda^{-\frac{1}{2}} \ddot{x} = -2 Y^{-\frac{1}{2}} \Lambda^{-\frac{1}{2}} (\dot{y} \circ \dot{\lambda}),
\]
and

\[
\Gamma \ddot{x} + Z \ddot{y} = -2 (\dot{z} \circ \dot{y})
\]
\[
\iff Z^{-\frac{1}{2}} \Gamma^{\frac{1}{2}} \ddot{x} + Z^{-\frac{1}{2}} \Gamma^{-\frac{1}{2}} \ddot{y} = -2 Z^{-\frac{1}{2}} \Gamma^{-\frac{1}{2}} (\dot{z} \circ \dot{y}).
\]

Let \( u = \begin{bmatrix} Y^{-\frac{1}{2}} \Lambda^{\frac{1}{2}} \ddot{y} \\ Z^{-\frac{1}{2}} \Gamma^{\frac{1}{2}} \ddot{x} \end{bmatrix}, v = \begin{bmatrix} Y^{\frac{1}{2}} \Lambda^{-\frac{1}{2}} \ddot{x} \\ Z^{\frac{1}{2}} \Gamma^{-\frac{1}{2}} \ddot{y} \end{bmatrix}, \) and \( w = \begin{bmatrix} -2 Y^{-\frac{1}{2}} \Lambda^{-\frac{1}{2}} (\dot{y} \circ \dot{\lambda}) \\ -2 Z^{-\frac{1}{2}} \Gamma^{-\frac{1}{2}} (\dot{z} \circ \dot{y}) \end{bmatrix}, \) using (16) and Lemma 4.2, we have \( u^T v = \ddot{y}^T \ddot{x} + \ddot{z}^T \ddot{y} = \ddot{x}^T (\ddot{y} - \ddot{x}) \leq 0. \) Using Lemma 4.3, we have
\[
\|u\|^2 + \|v\|^2 = \sum_{i=1}^{n} \left( \frac{\ddot{y}_i^2 \lambda_i}{y_i} + \frac{\ddot{z}_i^2 \gamma_i}{z_i} \right) + \sum_{i=1}^{n} \left( \frac{\dot{\lambda}_i^2 \gamma_i}{\lambda_i} + \frac{\dot{\gamma}_i^2 \lambda_i}{\gamma_i} \right)
\]
\[
\leq \left\| -2 Y^{-\frac{1}{2}} \Lambda^{-\frac{1}{2}} (\dot{y} \circ \dot{\lambda}) \right\|^2 + \left\| -2 Z^{-\frac{1}{2}} \Gamma^{-\frac{1}{2}} (\dot{z} \circ \dot{y}) \right\|^2
\]
\[
= 4 \sum_{i=1}^{n} \left( \frac{\ddot{y}_i^2 \lambda_i^2}{y_i} + \frac{\ddot{z}_i^2 \gamma_i^2}{z_i} \right).
\]

Dividing both sides of the inequality by \( \mu \) and using (11) gives
\[
(1 - \theta) \left( \sum_{i=1}^{n} \left( \frac{\ddot{y}_i^2 \lambda_i}{y_i} + \frac{\ddot{z}_i^2 \gamma_i}{z_i} \right) + \sum_{i=1}^{n} \left( \frac{\dot{\lambda}_i^2 \gamma_i}{\lambda_i} + \frac{\dot{\gamma}_i^2 \lambda_i}{\gamma_i} \right) \right)
\]
\[
= (1 - \theta) \left( \frac{\ddot{p}}{p} \right)^2 + \frac{\ddot{\omega}}{\omega} \right)^2
\]
\[
\leq 4(1 + \theta) \left( \sum_{i=1}^{n} \left( \frac{\ddot{y}_i^2 \lambda_i^2}{y_i^2} \lambda_i^2 + \frac{\ddot{z}_i^2 \gamma_i^2}{z_i^2} \gamma_i^2 \right) \right),
\]
in view of Lemma 4.3, this leads to
\[
\frac{\ddot{p}}{p} \right)^2 + \frac{\ddot{\omega}}{\omega} \right)^2 \leq 4 \frac{1 + \theta}{1 - \theta} \left( \frac{\ddot{p}}{p} \right)^2 \leq 4 \frac{1 + \theta}{1 - \theta} \left( \frac{\ddot{\omega}}{\omega} \right)^2 \leq 4(1 + \theta)n^2 \left( \frac{2(1 + \theta)n^2}{(1 - \theta)^3} \right)^2.
\]
This proves (36). Combining (36) and Lemma 4.1 yields
\[
\left( \frac{\ddot{p}}{p} \right)^2 \frac{\ddot{\omega}}{\omega} \right)^2 \leq \left( \frac{2(1 + \theta)n^2}{(1 - \theta)^3} \right)^2.
\]
Using (11) and Cauchy-Schwarz inequality yields
\[
\frac{\ddot{p}^T \ddot{\omega}}{\mu} \leq \frac{\ddot{p}^T \ddot{\omega}}{\mu} \leq (1 + \theta) \max_i \frac{\ddot{p}_i \ddot{\omega}_i}{\frac{\ddot{p}}{p} \ddot{\omega}} \leq (1 + \theta) \left( \frac{\ddot{p}}{p} \right)^2 \left( \frac{\ddot{\omega}}{\omega} \right)^2 \leq \frac{2n^2(1 + \theta)^2}{(1 - \theta)^3}.
\]
which is the second inequality of (38). Using (16) and Lemma 4.2, we have $\tilde{p}^T\tilde{\omega} = \tilde{y}^T\tilde{x} + \tilde{z}^T\tilde{y} = \tilde{x}^T(\tilde{\omega} - \tilde{\lambda}) = \tilde{x}^TH\tilde{x} \geq 0$. This proves the first inequality of (38). Finally, using (11), Cauchy-Schwarz inequality, (33), and (36) yields

$$\frac{|\hat{p}^T\tilde{\omega}|}{\mu} \leq \frac{|\hat{p}^T|\tilde{\omega}|}{\mu} \leq (1 + \theta) \frac{|\hat{p}^T|\tilde{\omega}|}{\max_ip_i\omega_i} \leq (1 + \theta) \left(\frac{|\hat{p}|}{p}\right)^T \left(\frac{|\tilde{\omega}|}{\omega}\right) \leq (1 + \theta) \left(\frac{2n}{1 - \theta}\right)^{\frac{1}{2}} \left(\frac{4(1 + \theta)n^2}{(1 + \theta)^3}\right)^{\frac{1}{2}} \leq \frac{2n(1 + \theta)^3}{(1 - \theta)^2}.$$  

This proves the first inequality of (39). Replacing $\hat{p}$ by $\tilde{p}$ and $\tilde{\omega}$ by $\tilde{\omega}$, then using the same reasoning, we can prove the second inequality of (39).

**Proof of Lemma 4.5** Using (20), (22), (17), and (18), we have

$$y^T(\alpha)\lambda(\alpha) = \left(y^T - \tilde{y}^T\sin(\alpha) + \tilde{y}^T(1 - \cos(\alpha))\right)\left(\lambda - \tilde{\lambda}\sin(\alpha) + \tilde{\lambda}(1 - \cos(\alpha))\right)$$

$$= y^T\lambda - y^T\tilde{\lambda}\sin(\alpha) + y^T\tilde{\lambda}(1 - \cos(\alpha))$$

$$- \tilde{y}^T\lambda\sin(\alpha) + \tilde{y}^T\tilde{\lambda}\sin^2(\alpha) - \tilde{y}^T\tilde{\lambda}\sin(\alpha)(1 - \cos(\alpha))$$

$$+ \tilde{y}^T\tilde{\lambda}(1 - \cos(\alpha)) - \tilde{y}^T\tilde{\lambda}\sin(\alpha)(1 - \cos(\alpha)) + \tilde{y}^T\tilde{\lambda}(1 - \cos(\alpha))^2$$

$$= y^T\lambda - (y^T\tilde{\lambda} + \lambda^T\tilde{y})\sin(\alpha) + (y^T\tilde{\lambda} + \lambda^T\tilde{y})(1 - \cos(\alpha))$$

$$- (y^T\tilde{\lambda} + \tilde{\lambda}^T\tilde{y})\sin(\alpha)(1 - \cos(\alpha)) + y^T\tilde{\lambda}\sin^2(\alpha) + y^T\tilde{\lambda}(1 - \cos(\alpha))^2$$

$$= y^T\lambda(1 - \sin(\alpha)) - 2y^T\tilde{\lambda}(1 - \cos(\alpha))$$

$$- (y^T\tilde{\lambda} + \tilde{\lambda}^T\tilde{y})\sin(\alpha)(1 - \cos(\alpha))$$

$$+ y^T\tilde{\lambda}(1 - \cos^2(\alpha)) + y^T\tilde{\lambda}(1 - \cos(\alpha))^2$$

$$= y^T\lambda(1 - \sin(\alpha)) + (y^T\tilde{\lambda} - y^T\tilde{\lambda})(1 - \cos(\alpha))^2 - (y^T\tilde{\lambda} + \tilde{\lambda}^T\tilde{y})\sin(\alpha)(1 - \cos(\alpha)). \quad (84)$$

Using (21), (23), (17), (18), and a similar derivation of (84), we have

$$z^T(\alpha)\gamma(\alpha) = z^T(1 - \sin(\alpha)) + (\tilde{z}^T\tilde{y} - \tilde{z}^T\tilde{y})(1 - \cos(\alpha))^2 - (\tilde{z}^T\tilde{y} + \tilde{y}^T\tilde{z})\sin(\alpha)(1 - \cos(\alpha)). \quad (85)$$

Combining (84) and (85) gives

$$2n\mu(\alpha) = p^T(\alpha)\omega(\alpha)$$

$$= y^T(\alpha)\lambda(\alpha) + z^T(\alpha)\gamma(\alpha)$$

$$= (y^T\tilde{\lambda} + z^T\gamma)(1 - \sin(\alpha)) + (y^T\tilde{\lambda} + z^T\gamma - y^T\tilde{\lambda} - \tilde{z}^T\gamma)(1 - \cos(\alpha))^2$$

$$- (y^T\tilde{\lambda} + z^T\gamma + \tilde{y}^T\tilde{\lambda} + \tilde{z}^T\gamma)\sin(\alpha)(1 - \cos(\alpha))$$

$$= (y^T\tilde{\lambda} + z^T\gamma)(1 - \sin(\alpha)) + (\tilde{y}^T(\tilde{\omega} - \tilde{\lambda}) - \tilde{x}^T(\tilde{\omega} - \tilde{\lambda}))(1 - \cos(\alpha))^2 \quad \text{use (16)}$$

$$- (\tilde{x}^T(\tilde{\omega} - \tilde{\lambda}) + \tilde{x}^T(\tilde{\omega} - \tilde{\lambda}))\sin(\alpha)(1 - \cos(\alpha)) \quad \text{(86)}$$

$$\leq (y^T\tilde{\lambda} + z^T\gamma)(1 - \sin(\alpha)) + (\tilde{x}^T H\tilde{x} - \tilde{x}^T H\tilde{x})(1 - \cos(\alpha))^2 \quad \text{use (30) in Lemma 4.2}$$

$$+ \tilde{x}^T H\tilde{x}(1 - \cos(\alpha))^2 + \tilde{x}^T H\tilde{x}\sin^2(\alpha)$$

$$= (y^T\tilde{\lambda} + z^T\gamma)(1 - \sin(\alpha)) + \tilde{x}^T H\tilde{x}(1 - \cos(\alpha))^2 + \tilde{x}^T H\tilde{x}\sin^2(\alpha).$$

Dividing both sides by $2n$ proves the second inequality of the lemma. Combining (86) and (31) proves the first inequality of the lemma. \qed
Proof of Lemma 4.6 From the second inequality of (40), we have

\[ \mu(\alpha) - \mu \leq \mu \sin(\alpha) \left( -1 + \frac{\dot{x}^T H \ddot{x}}{2n\mu} \sin(\alpha) + \frac{\dot{x}^T H \ddot{x}}{2n\mu} \sin^3(\alpha) \right). \]

Clearly, if \( \frac{\dot{x}^T H \ddot{x}}{2n\mu} \leq \frac{1}{2} \), for any \( \alpha \in [0, \frac{\pi}{2}] \), the function

\[ f(\alpha) := \left( -1 + \frac{\dot{x}^T H \ddot{x}}{2n\mu} \sin(\alpha) + \frac{\dot{x}^T H \ddot{x}}{2n\mu} \sin^3(\alpha) \right) \leq 0, \]

and \( \mu(\alpha) \leq \mu \). If \( \frac{\dot{x}^T H \ddot{x}}{2n\mu} > \frac{1}{2} \), using Lemma 2.5, the function \( f \) has one real solution \( \sin(\alpha) \in (0, 1) \). The solution is given as

\[ \sin(\hat{\alpha}) = \sqrt[3]{\frac{n\mu}{\dot{x}^T H \ddot{x}}} + \sqrt[3]{\left( \frac{n\mu}{\dot{x}^T H \ddot{x}} \right)^2 + \left( \frac{1}{3} \right)^3} + \sqrt[3]{\frac{n\mu}{\dot{x}^T H \ddot{x}}} - \sqrt[3]{\left( \frac{n\mu}{\dot{x}^T H \ddot{x}} \right)^2 + \left( \frac{1}{3} \right)^3}. \]

This proves the Lemma.

Proof of Lemma 4.7 Since \( \sin(\hat{\alpha}) \) is the only positive real solution of (42) in \([0, 1]\) and \( q(0) < 0 \), substituting \( a_0, a_1, a_2, a_3, \) and \( a_4 \) into (42), we have, for all \( \sin(\alpha) \leq \sin(\hat{\alpha}) \),

\[ \left( \left\| \hat{p} \circ \hat{\omega} - \Omega - \frac{1}{2n}(\hat{p}^T \hat{\omega} - \hat{\omega}^T \hat{p}) e \right\| \right) \sin^4(\alpha) + \left( \left\| \hat{p} \circ \hat{\omega} + \hat{\omega} \circ \hat{p} - \frac{1}{2n}(\hat{p}^T \hat{\omega} + \hat{\omega}^T \hat{p}) e \right\| \right) \sin^3(\alpha) \]

\[ \leq - \left( \frac{2\theta}{2n} \right) \sin^4(\alpha) - \left( \frac{2\theta}{2n} \right) \sin^2(\alpha) + \theta \mu (1 - \sin(\alpha)). \] (87)

Using (24), (25), (17), (18), (44), Lemma 2.2, (87), and the first inequality of (40), we have

\[ \left\| p(\alpha) \circ \omega(\alpha) - \mu(\alpha) e \right\| \]

\[ = \left\| \left( p - \hat{p} \sin(\alpha) + \hat{p}(1 - \cos(\alpha)) \right) \circ \left( \omega - \hat{\omega} \sin(\alpha) + \hat{\omega}(1 - \cos(\alpha)) \right) - \mu(\alpha) e \right\| \]

\[ = \left\| (p \circ \omega - \mu e)(1 - \sin(\alpha)) + \left( \hat{p} \circ \hat{\omega} - \hat{p} \circ \omega - \frac{1}{2n}(\hat{p}^T \hat{\omega} - \hat{\omega}^T \hat{p}) e \right) (1 - \cos(\alpha))^2 \right. \]

\[ - \left. \left( \hat{p} \circ \hat{\omega} + \hat{\omega} \circ \hat{p} - \frac{1}{2n}(\hat{p}^T \hat{\omega} + \hat{\omega}^T \hat{p}) e \right) \sin(\alpha) (1 - \cos(\alpha)) \right\| \]

\[ \leq (1 - \sin(\alpha)) \left\| p \circ \omega - \mu e \right\| + \left\| (\hat{p} \circ \hat{\omega} - \hat{p} \circ \omega - \frac{1}{2n}(\hat{p}^T \hat{\omega} - \hat{\omega}^T \hat{p}) e) \right\| (1 - \cos(\alpha))^2 \]

\[ + \left\| (\hat{p} \circ \hat{\omega} + \hat{\omega} \circ \hat{p} - \frac{1}{2n}(\hat{p}^T \hat{\omega} + \hat{\omega}^T \hat{p}) e) \right\| \sin(\alpha) (1 - \cos(\alpha)) \] (88)
\[
\leq \theta \mu (1 - \sin(\alpha)) + \left\| (\hat{p} \circ \hat{\omega} - \hat{p} \circ \hat{\omega} - \frac{1}{2n} (\hat{p}^T \hat{s} - \hat{p}^T \hat{\omega})) e \right\| \sin^4(\alpha) + a_3 \sin^3(\alpha)
\]
\[
\leq 2\theta \mu (1 - \sin(\alpha)) - \left( 2\theta \frac{\hat{p}^T \hat{\omega}}{2n} \right) (\sin^4(\alpha) + \sin^2(\alpha))
\]
\[
\leq 2\theta \left( \mu (1 - \sin(\alpha)) - \left( \frac{\hat{x}^T \hat{H} \hat{x}}{2n} \right) \left( (1 - \cos(\alpha))^2 + \sin^2(\alpha) \right) \right)
\]
\[
\leq 2\theta \mu (\alpha).
\]  
(89)

Hence, the point \((x(\alpha), p(\alpha), \omega(\alpha))\) satisfies the proximity condition for \(N_2^2(2\theta)\). To check the positivity condition \((p(\alpha), \omega(\alpha)) > 0\), note that the initial condition \((p, \omega) > 0\). It follows from (89) and Corollary 4.1 that, for \(\sin(\alpha) \leq \sin(\bar{\alpha})\) and \(\theta < 0.5\),

\[
p_i(\alpha)\omega_i(\alpha) \geq (1 - 2\theta) \mu(\alpha) > 0.
\]  
(90)

Therefore, we cannot have \(p_i(\alpha) = 0\) or \(\omega_i(\alpha) = 0\) for any index \(i\) when \(\alpha \in [0, \sin^{-1}(\bar{\alpha})]\). This proves \((p(\alpha), \omega(\alpha)) > 0\). \qed

**Proof of Lemma 4.6** Since

\[
\left\| \frac{\hat{p}}{p} \right\|^2 = \sum_{i=1}^{2n} \left( \frac{\dot{p}_i}{p_i} \right)^2, \quad \left\| \frac{\hat{\omega}}{\omega} \right\|^2 = \sum_{i=1}^{2n} \left( \frac{\dot{\omega}_i}{\omega_i} \right)^2,
\]

from Lemma 4.3 and (11), we have

\[
\left( \frac{n}{1 - \theta} \right)^2 \geq \left\| \frac{\hat{p}}{p} \right\| \left\| \frac{\hat{\omega}}{\omega} \right\|^2 = \left( \sum_{i=1}^{2n} \left( \frac{\dot{p}_i}{p_i} \right)^2 \right) \left( \sum_{i=1}^{2n} \left( \frac{\dot{\omega}_i}{\omega_i} \right)^2 \right)
\]

\[
\geq \sum_{i=1}^{2n} \left( \frac{\dot{p}_i}{p_i} \frac{\dot{\omega}_i}{\omega_i} \right)^2 = \left\| \frac{\hat{p} \circ \hat{\omega}}{p \circ \omega} \right\|^2
\]

\[
\geq \sum_{i=1}^{2n} \left( \frac{\dot{p}_i}{(1 + \theta) \mu} \frac{\dot{\omega}_i}{(1 + \theta) \mu} \right)^2 \geq \frac{1}{(1 + \theta)^2 \mu^2} \left\| \hat{p} \circ \hat{\omega} \right\|^2,
\]

i.e.,

\[
\left\| \hat{p} \circ \hat{\omega} \right\|^2 \leq \left( \frac{1 + \theta}{1 - \theta n \mu} \right)^2
\]

This proves (45). Using

\[
\left\| \frac{\hat{p}}{p} \right\|^2 = \sum_{i=1}^{2n} \left( \frac{\dot{p}_i}{p_i} \right)^2, \quad \left\| \frac{\hat{\omega}}{\omega} \right\|^2 = \sum_{i=1}^{2n} \left( \frac{\dot{\omega}_i}{\omega_i} \right)^2,
\]
and Lemma 4.4, then following the same procedure, it is easy to verify (46). From (33) and (36), we have

\[
\left( \frac{2n}{1 - \theta} \right) \left( \frac{4(1 + \theta)n^2}{(1 - \theta)^3} \right) \geq \left( \| \ddot{p} \| + \| \dot{\omega} \| \right) \left( \| \ddot{p} \| + \| \dot{\omega} \| \right)
\]

\[
\geq \left( \sum_{i=1}^{2n} \left( \frac{\ddot{p}_i \dot{\omega}_i}{p_i \omega_i} \right)^2 \right) + \left( \sum_{i=1}^{2n} \left( \frac{\dot{p}_i \ddot{\omega}_i}{p_i \omega_i} \right)^2 \right)
\]

\[
\geq \left( \sum_{i=1}^{2n} \left( \frac{\ddot{p}_i \dot{\omega}_i}{(1 + \theta) \mu} \right)^2 \right) + \left( \sum_{i=1}^{2n} \left( \frac{\dot{p}_i \ddot{\omega}_i}{(1 + \theta) \mu} \right)^2 \right)
\]

\[
= \frac{1}{(1 + \theta)^2 \mu^2} \left( \| \ddot{p} \circ \dot{\omega} \|^2 + \| \dot{p} \circ \ddot{\omega} \|^2 \right).
\]

(91)

i.e.,

\[
\| \ddot{p} \circ \dot{\omega} \|^2 + \| \dot{p} \circ \ddot{\omega} \|^2 \leq \frac{(2n)^3(1 + \theta)^3}{(1 - \theta)^4} \mu^2.
\]

This proves the lemma. \(\square\)

**Proof of Lemma 4.9** First notice that \(q(\sin(\alpha))\) is a monotonic increasing function of \(\sin(\alpha)\) for \(\alpha \in [0, \frac{\pi}{2}]\) and \(q(\sin(0)) < 0\); therefore, we need only to show that \(q\left(\frac{\theta}{\sqrt{n}}\right) < 0\) for \(\theta \leq 0.22\). Using Lemma 2.6, we have

\[
\| \ddot{p} \circ \dot{\omega} + \dot{\omega} \circ \ddot{p} - \frac{1}{2n} (\ddot{p}^T \dot{\omega} + \omega^T \ddot{p}) e \| \leq \| \ddot{p} \circ \dot{\omega} \| + \| \dot{\omega} \circ \ddot{p} \|.
\]

\[
\| \ddot{p} \circ \dot{\omega} - \dot{\omega} \circ \ddot{p} - \frac{1}{2n} (\ddot{p}^T \dot{\omega} - \omega^T \ddot{p}) e \| \leq \| \ddot{p} \circ \dot{\omega} \| + \| \dot{\omega} \circ \ddot{p} \|.
\]

In view of Lemmas 4.8, 4.3, and 4.4, from (42), we have, for \(\alpha \in [0, \frac{\pi}{2}]\),

\[
q(\sin(\alpha)) \leq \left( \| \ddot{p} \circ \dot{\omega} \| + \| \dot{\omega} \circ \ddot{p} \| + 2\theta \frac{\ddot{p}^T \dot{\omega}}{2n} \right) \sin^4(\alpha) + \left( \| \ddot{p} \circ \dot{\omega} \| + \| \dot{\omega} \circ \ddot{p} \| \right) \sin^3(\alpha)
\]

\[
+ 2\theta \frac{\ddot{p}^T \dot{\omega}}{2n} \sin^2(\alpha) + \theta \mu \sin(\alpha) - \theta \mu
\]

\[
\leq \mu \left( \left( \frac{2(1 + \theta)^2}{(1 - \theta)^3} n^2 + \frac{n(1 + \theta)}{(1 - \theta)} + \frac{\theta(1 + \theta)}{(1 - \theta)} \right) \sin^4(\alpha) + 4\sqrt{2} \left( \frac{1 + \theta}{(1 - \theta)^2} n^2 \right) \sin^3(\alpha)
\]

\[
+ \theta(1 + \theta) \sin^2(\alpha) + \theta \sin(\alpha) - \theta \right).
\]
Since \( n \geq 1 \) and \( \theta > 0 \), substituting \( \sin(\alpha) = \frac{\theta}{\sqrt{n}} \) gives

\[
q\left(\frac{\theta}{\sqrt{n}}\right) \leq \mu\left(\frac{2(1 + \theta)^2}{(1 - \theta)^3} n^2 + \frac{n(1 + \theta)}{(1 - \theta)} + \frac{\theta(1 + \theta)}{(1 - \theta)} \right) \frac{\theta^4}{n^2} + 4\sqrt{2} \frac{(1 + \theta)^3 n^3 \theta^3}{(1 - \theta)^2 n^2} \\
+ \frac{\theta(1 + \theta) \theta^2}{(1 - \theta) n} + \frac{\theta \theta}{\sqrt{n}} - \theta \right) \\
= \theta \mu\left(\frac{2\theta^3(1 + \theta)^2}{(1 - \theta)^3} + \frac{\theta^3(1 + \theta)}{(1 - \theta)} + \frac{\theta^4(1 + \theta)}{(1 - \theta) n^2} \\
+ \frac{4\sqrt{2}\theta^2(1 + \theta)^2}{(1 - \theta)^2} + \frac{\theta^2(1 + \theta)}{(1 - \theta) n} + \frac{\theta \theta}{\sqrt{n}} - 1 \right) \\
\leq \theta \mu\left(\frac{2\theta^3(1 + \theta)^2}{(1 - \theta)^3} + \frac{\theta^3(1 + \theta)}{(1 - \theta)} + \frac{\theta^4(1 + \theta)}{(1 - \theta) n^2} \\
+ \frac{4\sqrt{2}\theta^2(1 + \theta)^2}{(1 - \theta)^2} + \frac{\theta^2(1 + \theta)}{(1 - \theta) n} + \theta - 1 \right) := \theta \mu p(\theta). \tag{92}
\]

Since \( p(\theta) \) is a monotonic increasing function of \( \theta \in [0, 1) \), \( p(0) < 0 \), it is easy to verify that \( p(0.22) < 0 \). This proves the lemma. \( \square \)

**Proof of Lemma 4.10** Using Lemma 2.6, we have

\[
0 \leq \bigg\| \Delta p \circ \Delta \omega - \frac{1}{2n} (\Delta p^T \Delta \omega) e \bigg\|^2 \leq \| \Delta p \circ \Delta \omega \|^2. \tag{93}
\]

Pre-multiplying \( \left( P(\alpha)\Omega(\alpha) \right)^{-\frac{1}{2}} \) on both sides of (52) yields

\[
D \Delta \omega + D^{-1} \Delta p = \left( P(\alpha)\Omega(\alpha) \right)^{-\frac{1}{2}} \left( \mu(\alpha)e - P(\alpha)\Omega(\alpha)e \right).
\]

Let \( u = D \Delta \omega, v = D^{-1} \Delta p, \) from (49), we have

\[
u^T v = D p^T \Delta \omega = D y^T \Delta \lambda + D z^T \Delta \gamma = D x^T (\Delta \gamma - \Delta \lambda) = D x^T H D x \geq 0. \tag{94}
\]

Use Lemma 2.4 and the assumption of \( (x(\alpha), p(\alpha), \omega(\alpha)) \in \mathcal{N}_2(2\theta) \), we have

\[
\| \Delta p \circ \Delta \omega \| = \| u \circ v \| \leq 2^{-\frac{3}{2}} \left\| \left( P(\alpha)\Omega(\alpha) \right)^{-\frac{1}{2}} \left( \mu(\alpha)e - P(\alpha)\Omega(\alpha)e \right) \right\|^2 \\
= 2^{-\frac{3}{2}} \sum_{i=1}^{2n} \frac{(\mu(\alpha) - p_i(\alpha)\omega_i(\alpha))^2}{p_i(\alpha)\omega_i(\alpha)} \\
\leq 2^{-\frac{3}{2}} \| \mu(\alpha)e - p(\alpha) \circ \omega(\alpha) \|^2 \min_i p_i(\alpha)\omega_i(\alpha) \\
\leq 2^{-\frac{3}{2}} \frac{(2\theta)^2 \mu(\alpha)^2}{(1 - 2\theta) \mu(\alpha)} = 2^{\frac{1}{2}} \frac{\theta^2 \mu(\alpha)}{1 - 2\theta}. \tag{95}
\]
Define \((p^{k+1}(t), \omega^{k+1}(t)) = (p(\alpha), \omega(\alpha)) + t(\Delta p, \Delta \omega)\). From (52) and (26), we have
\[
p(\alpha)^T \Delta \omega + \omega(\alpha)^T \Delta p = 2n \mu - \sum_{i=1}^{2n} p_i(\alpha) \omega_i(\alpha) = 0. \tag{96}
\]
Therefore,
\[
\mu^{k+1}(t) = \left(\frac{p(\alpha) + t \Delta p}{2n}\right)^T \left(\frac{\omega(\alpha) + t \Delta \omega}{2n}\right) = \frac{p(\alpha)^T \omega(\alpha) + t^2 \Delta p^T \Delta \omega}{2n} = \mu(\alpha) + \frac{t^2 \Delta p^T \Delta \omega}{2n}. \tag{97}
\]
Since \(\Delta p^T \Delta \omega = \Delta x^T H \Delta x \geq 0\), we conclude that \(\mu^{k+1}(t) \geq \mu(\alpha)\). Using (97), (52), (93), and (95), we have
\[
\left\| p^{k+1}(t) \circ \omega^{k+1}(t) - \mu^{k+1}(t)e \right\|
= \left\| (p(\alpha) + t \Delta p) \circ (\omega(\alpha) + t \Delta \omega) - \mu(\alpha)e - \frac{t^2}{2n} \left(\Delta p^T \Delta \omega\right) e\right\|
\leq (1 - t)(2\theta) \mu(\alpha) + t^2 \frac{2 \theta^2}{(1 - 2\theta)} \mu(\alpha)
\leq \left(1 - t)(2\theta) + t^2 \frac{2 \theta^2}{(1 - 2\theta)} \right) \mu^{k+1} := f(t, \theta) \mu^{k+1}. \tag{98}
\]
Therefore, taking \(t = 1\) gives
\[
\left\| p^{k+1} \circ \omega^{k+1} - \mu^{k+1}e \right\| \leq \frac{2 \theta^2}{(1 - 2\theta)} \mu^{k+1}. \] It is easy to see that, for \(\theta \leq 0.29\),
\[
\frac{2 \theta^2}{(1 - 2\theta)} = 0.2832 < \theta.
\]
For \(\theta \leq 0.29\) and \(t \in [0, 1]\), noticing \(0 \leq f(t, \theta) \leq f(t, 0.29) \leq 0.58(1 - t) + 0.2832t^2 < 1\), and using Corollary 4.1, we have, for an additional condition \(\sin(\alpha) \leq \sin^{-1}(\hat{\alpha})\),
\[
p_i^{k+1}(t) \omega_i^{k+1}(t) \geq (1 - f(t, \theta)) \mu^{k+1}(t)
\geq (1 - f(t, \theta)) \left(\mu(\alpha) + \frac{t^2}{n} \Delta p^T \Delta \omega\right)
\geq (1 - f(t, \theta)) \mu(\alpha)
> 0. \tag{99}
\]
Therefore, \((p^{k+1}(t), \omega^{k+1}(t)) > 0\) for \(t \in [0, 1]\), i.e., \((p^{k+1}, \omega^{k+1}) > 0\). This finishes the proof. \(\square\)
Proof of Lemma 4.11  The first inequality of (53) follows from (94). Pre-multiplying both sides of (52) by $P^{-\frac{1}{2}}(\alpha)\Omega^{-\frac{1}{2}}(\alpha)$ gives

$$
P^{-\frac{1}{2}}(\alpha)\Omega^{-\frac{1}{2}}(\alpha)\Delta p + P^{\frac{1}{2}}(\alpha)\Omega^{-\frac{1}{2}}(\alpha)\Delta \omega = P^{-\frac{1}{2}}(\alpha)\Omega^{-\frac{1}{2}}(\alpha)\left(\mu(\alpha)e - P(\alpha)\Omega(\alpha)e\right).
$$

Let $u = P^{-\frac{1}{2}}(\alpha)\Omega^{-\frac{1}{2}}(\alpha)\Delta p$, $v = P^{\frac{1}{2}}(\alpha)\Omega^{-\frac{1}{2}}(\alpha)\Delta \omega$, and $w = P^{-\frac{1}{2}}(\alpha)\Omega^{-\frac{1}{2}}(\alpha)\left(\mu(\alpha)e - P(\alpha)\Omega(\alpha)e\right)$, from (94), we have $u^Tv = \Delta p^T\Delta \omega \geq 0$. Using Lemma 4.3 and the assumption of $(x(\alpha), p(\alpha), \omega(\alpha)) \in \mathcal{N}_2(2\theta)$, we have

$$
\|u\|^2 + \|v\|^2 = \sum_{i=1}^{2n} \left(\frac{(\Delta p_i)^2 \omega_i(\alpha)}{p_i(\alpha)} + \frac{(\Delta \omega_i)^2 p_i(\alpha)}{\omega_i(\alpha)}\right) \\
\leq \|w\|^2 = \sum_{i=1}^{2n} \frac{(\mu(\alpha) - p_i(\alpha)\omega_i(\alpha))^2}{p_i(\alpha)\omega_i(\alpha)} \\
\leq \frac{\sum_{i=1}^{2n} (\mu(\alpha) - p_i(\alpha)\omega_i(\alpha))^2}{\min_i p_i(\alpha)\omega_i(\alpha)} \\
\leq \frac{(2\theta)^2 \mu^2(\alpha)}{(1 - 2\theta)\mu(\alpha)} = \frac{(2\theta)^2 \mu(\alpha)}{(1 - 2\theta)}.
$$

(100)

Dividing both sides by $\mu(\alpha)$ and using $p_i(\alpha)\omega_i(\alpha) \geq \mu(\alpha)(1 - 2\theta)$ yield

$$
\sum_{i=1}^{2n} (1 - 2\theta) \left(\frac{(\Delta p_i)^2}{p_i^2(\alpha)} + \frac{(\Delta \omega_i)^2}{\omega_i^2(\alpha)}\right) \\
= (1 - 2\theta) \left(\left\|\frac{\Delta p}{p(\alpha)}\right\|^2 + \left\|\frac{\Delta \omega}{\omega(\alpha)}\right\|^2\right) \\
\leq \frac{(2\theta)^2}{(1 - 2\theta)},
$$

(101)

i.e.,

$$
\left\|\frac{\Delta p}{p(\alpha)}\right\|^2 + \left\|\frac{\Delta \omega}{\omega(\alpha)}\right\|^2 \leq \left(\frac{2\theta}{1 - 2\theta}\right)^2.
$$

(102)

Invoking Lemma 4.1, we have

$$
\left\|\frac{\Delta p}{p(\alpha)}\right\|^2 \cdot \left\|\frac{\Delta \omega}{\omega(\alpha)}\right\|^2 \leq \frac{1}{4} \left(\frac{2\theta}{1 - 2\theta}\right)^4.
$$

(103)
This gives
\[
\left\| \frac{\Delta p}{p(\alpha)} \right\| \cdot \left\| \frac{\Delta \omega}{\omega(\alpha)} \right\| \leq \frac{2\theta^2}{(1 - 2\theta)^2}.
\] (104)

Using Cauchy-Schwarz inequality, we have
\[
\frac{(\Delta p)^T (\Delta \omega)}{\mu(\alpha)} \leq \sum_{i=1}^{2n} \frac{|\Delta p_i| |\Delta \omega_i|}{\mu(\alpha)}
\]
\[
\leq (1 + 2\theta) \sum_{i=1}^{2n} \frac{|\Delta p_i| |\Delta \omega_i|}{p_i(\alpha) \omega_i(\alpha)}
\]
\[
= (1 + 2\theta) \left( \frac{\Delta p}{p(\alpha)} \right)^T \left( \frac{\Delta \omega}{\omega(\alpha)} \right)
\]
\[
\leq (1 + 2\theta) \left\| \frac{\Delta p}{p(\alpha)} \right\| \cdot \left\| \frac{\Delta \omega}{\omega(\alpha)} \right\|
\]
\[
\leq \frac{2\theta^2 (1 + 2\theta)}{(1 - 2\theta)^2}.
\] (105)

Therefore,
\[
\frac{(\Delta p)^T (\Delta \omega)}{2n} \leq \frac{\theta^2 (1 + 2\theta)}{n(1 - 2\theta)^2} \mu(\alpha).
\] (106)

This proves the lemma. \(\square\)

**Proof of Lemma 4.13** Using Lemmas 4.12, 4.5, 2.2, 4.2, 4.3, and 4.4, and noticing \(\hat{p}^T \hat{\omega} \geq 0\) and \(\hat{p}^T \hat{\omega} \geq 0\), we have

\[
\mu^{k+1} \leq \mu(\alpha) \left( 1 + \frac{\theta^2 (1 + 2\theta)}{n(1 - 2\theta)^2} \right) = \mu(\alpha) \left( 1 + \frac{\delta_0}{n} \right)
\] (107a)

\[
= \mu^k \left( 1 - \sin(\alpha) + \left( \frac{\hat{p}^T \hat{\omega}}{2n\mu} - \frac{\hat{\omega}^T \hat{p}}{2n\mu} \right)(1 - \cos(\alpha))^2 \right.
\]
\[
- \left( \frac{\hat{p}^T \hat{\omega}}{2n\mu} + \frac{\hat{\omega}^T \hat{p}}{2n\mu} \right) \sin(\alpha)(1 - \cos(\alpha)) \left( 1 + \frac{\delta_0}{n} \right)
\]
\[
\leq \mu^k \left( 1 - \sin(\alpha) + \frac{\hat{p}^T \hat{\omega}}{2n\mu} \sin^4(\alpha) + \left( \frac{\hat{p}^T \hat{\omega}}{2n\mu} + \frac{\hat{\omega}^T \hat{p}}{2n\mu} \right) \sin^3(\alpha) \left( 1 + \frac{\delta_0}{n} \right) \right)
\]
\[
\leq \mu^k \left( 1 - \sin(\alpha) + \frac{n(1 + \theta)^2}{(1 - \theta)^3} \sin^4(\alpha) + \frac{2(2n)^2 (1 + \theta)^3}{(1 - \theta)^2} \sin^3(\alpha) \left( 1 + \frac{\delta_0}{n} \right) \right)
\] (107b)
Substituting $\sin(\alpha) = \frac{\theta}{\sqrt{n}}$ into (107b) gives

$$
\mu^{k+1} \leq \mu^k \left( 1 - \frac{\theta}{\sqrt{n}} + \frac{n(1+\theta)^2 \theta^4}{(1-\theta)^3 n^2} + \frac{2(2n)\frac{1}{2}(1+\theta)^2 \theta^3}{(1-\theta)^2 n^2} \right) \left( 1 + \frac{\delta_0}{n} \right)
$$

$$
= \mu^k \left( 1 - \frac{\theta}{\sqrt{n}} + \frac{\theta^4(1+\theta)^2}{n(1-\theta)^3} + \frac{2\theta^3(1+\theta)^2}{n(1-\theta)^2} \right) \left( 1 + \frac{\delta_0}{n} \right)
$$

$$
= \mu^k \left( 1 - \frac{\theta}{\sqrt{n}} + \frac{\theta^4(1+\theta)^2}{n(1-\theta)^3} + \frac{2\theta^3(1+\theta)^2}{n(1-\theta)^2} \right) \left( 1 + \frac{\delta_0}{n} \right)
$$

$$
= \mu^k \left( 1 - \frac{\theta}{\sqrt{n}} \left[ 1 - \frac{\delta_0}{\sqrt{n}} - \frac{\theta^4(1+\theta)^2}{\sqrt{n}(1-\theta)^3} - \frac{2\theta^3(1+\theta)^2}{\sqrt{n}(1-\theta)^2} \right] \right)
$$

Since

$$
1 - \frac{\theta^3(1+\theta)^2}{\sqrt{n}(1-\theta)^3} - \frac{2\theta^2(1+\theta)^3}{\sqrt{n}(1-\theta)^2} \geq 1 - \frac{\theta^3(1+\theta)^2}{(1-\theta)^3} - \frac{2\theta^2(1+\theta)^3}{(1-\theta)^2} := f(\theta),
$$

where $f(\theta)$ is a monotonic decreasing function of $\theta$, it follows that for $\theta \leq 0.37$, $f(\theta) > 0$. Therefore, for $\theta \leq 0.37$,

$$
\mu^{k+1} \leq \mu^k \left( 1 - \frac{\theta}{\sqrt{n}} \left[ 1 - \frac{\delta_0}{\sqrt{n}} - \frac{\theta^4(1+\theta)^2}{\sqrt{n}(1-\theta)^3} - \frac{2\theta^3(1+\theta)^2}{\sqrt{n}(1-\theta)^2} \right] \right)
$$

$$
= \mu^k \left( 1 - \frac{\theta}{\sqrt{n}} \left[ 1 - \frac{\theta(1+2\theta)}{\sqrt{n}(1-2\theta)^2} - \frac{\theta^3(1+\theta)^2}{\sqrt{n}(1-\theta)^3} - \frac{2\theta^2(1+\theta)^3}{\sqrt{n}(1-\theta)^2} \right] \right)
$$

(108)

Since

$$
1 - \frac{\theta(1+2\theta)}{\sqrt{n}(1-2\theta)^2} - \frac{\theta^3(1+\theta)^2}{\sqrt{n}(1-\theta)^3} - \frac{2\theta^2(1+\theta)^3}{\sqrt{n}(1-\theta)^2} \geq 1 - \frac{\theta(1+2\theta)}{(1-2\theta)^2} - \frac{\theta^3(1+\theta)^2}{(1-\theta)^3} - \frac{2\theta^2(1+\theta)^3}{(1-\theta)^2} := g(\theta),
$$

where $g(\theta)$ is a monotonic decreasing function of $\theta$, it follows that for $\theta \leq 0.19$, $g(\theta) > 0.0976 > 0$. For $\theta = 0.19$, $\theta g(\theta) > 0.0185$ and

$$
\mu^{k+1} \leq \mu^k \left( 1 - \frac{0.0185}{\sqrt{n}} \right).
$$

This proves (55).

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**Data availability** The datasets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

**Declarations**

**Conflict of interest** The author declares no competing interests.
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