Abstract
We prove the asymptotic of the logarithmic Bergman kernel. And as an application, we calculate the conditional expectation of density of zeros of Gaussian random sections of powers of a positive line bundle that vanish along a fixed smooth subvariety.

1 Introduction
Let \((M,L)\) be a polarized Kähler manifold of dimension \(m\). We endow \(L\) with a Hermitian metric \(h\) with positive curvature. And we use \(\omega = \frac{i}{2} \Theta_h\) as the Kähler form. By abuse of notation, we still use \(h\) to denote the induced metric on the \(k\)-th power \(L^k\). Then we have a Hermitian inner product on \(H^0(M,L^k)\), defined by

\[
\langle s_1, s_2 \rangle = \int_M h(s_1, s_2) \frac{\omega^m}{m!}
\]

Let \(\{s_i\}\) be an orthonormal basis of \(H^0(M,L^k)\). Then the on-diagonal Bergman kernel

\[
\rho_k(z) = \sum |s_i(z)|^2_h
\]

has very nice asymptotic expansion by the results of Tian, Zelditch, Lu, etc. \([5,16,17,34,35]\). Recall that

\[
\rho_k(z) = \frac{k^m}{\pi^m} \left[ 1 + \frac{S(z)}{2k} + O \left( \frac{1}{k^2} \right) \right].
\]
where $S(z)$ is the scalar curvature of the Riemannian metric associated to $\omega$, and the other coefficients are all functions of the derivatives of the Riemannian curvature tensor. Let $V$ be a subvariety(subscheme) of $M$, and we denote by $H_{k,V}$ the subspace of $H^0(M, L^k)$ consisting of sections that vanish along $V$. We will call the Bergman kernel of $H_{k,V}$ the $k$-th logarithmic Bergman kernel of $(M, L, V)$, denoted by $\rho_{k,V}$. It is necessary to compare our logarithmic Bergman kernel to the partial Bergman kernel studied by Zelditch-Zhou [36,37], Ross-Singer [21], Coman-Marinescu [8], etc. The partial Bergman kernel they studied is for the space of sections that vanish along a subset to an order that grows with the power $k$. While it is natural to call $\rho_{k,V}$ the partial Bergman kernel, we call it logarithmic Bergman kernel to distinguish the two situations. It is also reasonable to call $\rho_{k,V}$ the conditional density of states.

The asymptotics of on- and off-diagonal Bergman kernel have been extensively used in the value distribution theory of sections of line bundles by Shiffman–Zelditch and others [1–4,10–14,22–28]. Of closest relation to this article are [1–4,13,27]. In [1–4], the $n$-point correlation functions of Gaussian random holomorphic sections of $H^0(M, L^k)$ were calculated, together with the scaling limits of these correlation functions, which very interestingly have universality among manifolds of the same dimension. In particular, in the case of Riemann surfaces, they showed that the scaling limit of pair correlation depends only on the distance of the pair of points. In [27], Shiffman–Zelditch–Zhong calculated the asymptotics of the conditional expectation of zero currents of sections conditioned on the vanishing at several points, both unscaled and scaled. As a direct application, they proved the scaled limits of the conditional expectation. In [13], in the case of Riemann surfaces, Feng calculated the conditional expectation of density of critical points given a fixed zero and the conditional expectation of zeros given a fixed critical point of Gaussian random holomorphic sections of $H^0(M, L^k)$. He also calculated the rescaling limits, which also exhibit universality. In particular, the universal scaling limit of conditional expectation of density of zeros with a fixed critical point depends also only on the distance between a point and the given critical point. It was first shown in [23] by Shiffman and Zeldtich that the mean of the zero currents $[Z_s]$ for $s \in H^0(M, L)$ endowed with the Gaussian measure is the pull back of the Fubini–Study form $\omega_{FS}$, whose difference from $\omega$ is just the $i \partial \bar{\partial} \log$ of the Bergman kernel. So it is not surprising that the conditional expectation of zeros is very closely related to the logarithmic Bergman kernel.

In this article, we will study the asymptotics of logarithmic Bergman kernels. And as an application, we will calculate the conditional expectation of density of zeros of Gaussian random holomorphic sections of $H^0(M, L^k)$ which vanish along a fixed smooth subvariety $V$ of $M$, generalizing the results of [27]. The logarithmic Bergman kernel behaves like Bergman kernel for the singular metric [6,7,30,32,33] in the sense that it behaves very much like the smooth case “away” from the subvariety $V$, while it exhibits a very different nature “around” $V$. We use the notation $\varepsilon(k)$ to mean a term that is bounded by $Ck^{-r}$ for all $r$, which becomes invisible in any asymptotic expansion in inverse powers of $k$.

Our main result is about the asymptotics near $V$. We have

**Theorem 1.1** For $z \in M$, let $r = d(z, V)$ be the distance. Then
when \( r \leq \sqrt{\frac{\log k}{2k}} \), we have

\[
\frac{\rho_{k,V}(z)}{\rho_k}(z) = (1 - e^{-kr^2})(1 + R_k(z)),
\]

where \( |R_k(z)| \leq C \varepsilon \frac{kr^2}{k^{1/2}} \) for any \( \varepsilon > 0 \).

- When \( r \) satisfies

\[
\sqrt{\frac{\log k}{2k}} < r < \frac{\log k}{\sqrt{k}},
\]

then we have

\[
\frac{\rho_{k,V}(z)}{\rho_k}(z) = 1 - O(e^{-kr^2}).
\]

"Away" from \( V \), the asymptotics is same as the smooth case as expected:

**Theorem 1.2** For \( z \in M \), let \( r = d(z, V) \) be the distance. Then when \( r \geq \frac{\log k}{\sqrt{k}} \), we have

\[
\rho_{k,V}(z) = \rho_k(z) - \varepsilon(k).
\]

In particular, \( \rho_{k,V}(z) \) has the same asymptotic expansion as \( \rho_k(z) \).

We would like to comment that like the asymptotic of the Bergman kernel in the smooth case, the asymptotic around \( V \) also depends on mainly on the geometry of the manifolds \( M \) and \( V \). But unlike the smooth case where the geometry only kicks in from the second term, the dependence on the geometry appears from the first term in the asymptotic of the logarithmic Bergman kernel near \( V \). When \( V \) has singularities, the asymptotics should be more interesting near the singularities. Also, the main difference between \( \rho_k - \rho_{k,V} \) and the restricted Bergman kernel studied in [15] is that the \( L^2 \)-norm is by integration over \( X \) in our case compared to integration over \( V \) in [15]. Our proof depends on two important tools: the Ohsawa–Takegoshi–Manivel extension theorem and the asymptotic of the off-diagonal Bergman kernel developed by Shiffman–Zelditch, etc.

Recall that the expectation of density of zeros of sections of a line bundle is defined as a \((1, 1)\)-current. More precisely, given a line bundle \( L \to M \), with \( \dim M = m \), and a Hermitian inner product \( H \) on \( H^0(M, L) \), we have a complex Gaussian measure \( d\mu \) on \( H^0(M, L) \). Let \( Z_s \) denote the zero variety of \( s \in H^0(M, L) \). Then the expectation is defined as

\[
E(Z_s)(f) = \int_{H^0(M,L)} d\mu(s) \int_{Z_s} f
\]

for any smooth \((m - 1, m - 1)\)-form \( f \) with compact support. Given a subset \( V \) of \( M \), the conditional expectation of density of zeros of sections of \( L^k \) is denoted by
$Z_k(z|V)$, defined by

$$Z_k(z|V)(f) = E_{(H^0(M,L^k),d\mu_k)} \left( \int_{Z_s} f \mid s|V = 0 \right).$$

for any smooth $(m-1, m-1)$-form $f$ with compact support, where $d\mu_k$ is the complex Gaussian measure on $H^0(M, L^k)$ corresponding to the Hermitian inner product on this space. The inner product on $\mathcal{H}_{k, V}$ is inherited from that on $H^0(M, L^k)$. We denote by $d\gamma_k$ the induced complex Gaussian measure on $H^k, V$, and then we have the following:

$$E_{(H^0(M,L^k),d\mu_k)} \left( \int_{Z_s} f \mid s|V = 0 \right) = E_{(\mathcal{H}_{k, V}, d\gamma_k)} \left( \int_{Z_s} f \right).$$

Then it follows from [24] and [31] that

**Proposition 1.3**

$$Z_k(z|V) = \frac{i}{2\pi} \partial \bar{\partial} \log \rho_{k, V} + k\omega.$$ 

Therefore, as a corollary of Theorem 1.2, we have

**Corollary 1.4** For $z \in M \setminus V$, we have

$$Z_k(z|V) = k\omega \left( 1 + O \left( \frac{1}{k^2} \right) \right).$$

And as a Corollary of Theorem 1.1, we have

**Corollary 1.5** For $z_0 \in V$, we fix a normal coordinates $(w_1, \ldots, w_m)$ so that $V$ is given by $w_{n+1} = \cdots = w_m$ and $\omega(z_0) = \sqrt{-1} \sum_{i=1}^{m} dw_i \wedge d\bar{w}_i$. Then for $z \in \mathbb{C}^m$ we have the scaling limit

$$Z_{\infty}(z|V) = \lim_{k \to \infty} Z_k \left( z_0 + \frac{z}{\sqrt{k}} \mid V \right) = \sqrt{-1} \sum_{i=1}^{n} dz_i \wedge d\bar{z}_i + \sqrt{-1} \partial \bar{\partial} \log(e^{\sum_{i=n+1}^{m} |z_i|^2 - 1}).$$

The structure of this article is as follows. We will first do the calculations in the complex projective space, which gives us important insight for the general picture. Then we use the Ohsawa–Takegoshi–Manivel extension theorem to prove Lemma 3.3, which is very important for this article, and Theorem 1.2. Then we use the asymptotic of the off-diagonal Bergman kernel to prove the two parts of Theorem 1.1. Then we quickly prove Corollaries 1.4 and 1.5.
2 Complex Projective Space and Its Geometry

Let \([Z_0, \ldots, Z_N]\) be the homogeneous coordinates of \(\mathbb{CP}^N\). \(U_0 = \{[1, z], z \in \mathbb{C}^N\}\) is a coordinate patch with \(z_i = \frac{Z_i}{Z_0}\). The \(Z_i\)'s can be identified as generating sections in \(H^0(\mathbb{CP}^N, \mathcal{O}(1))\). In particular, \(Z_0\) is a local frame in \(U_0\). Then on \(U_0\), the Fubini–Study form \(\omega = \frac{i}{2} \overline{\partial} \overline{\partial} \log(1 + |z|^2)\) has the following explicit form:

\[
\omega = \frac{i}{2} \left(1 + |z|^2\right) \sum dz^i \wedge d\bar{z}^i - \left(\sum \bar{z}_i dz_i\right) \left(\sum z_i d\bar{z}_i\right) \left(1 + |z|^2\right)^2
\]

and the point-wise norm of \(Z_0\) is given by

\[
|Z_0|^2_{FS}(z) = \frac{1}{1 + |z|^2} = e^{-\varphi}.
\]

For simplicity, we use the volume form \(\omega^N_N\) instead of \(\omega^N_N\). Then the total volume of \(M = \mathbb{CP}^N\) is 1. With the Riemannian metric associated to \(\omega\), the distance between two points \([Z]\) and \([W]\) in \(M\) is given by \(\arccos \frac{|\langle Z, W \rangle|}{|Z||W|}\).

Sections in \(H^0(\mathbb{CP}^N, \mathcal{O}(k))\) are represented by homogeneous polynomials of variables \(Z_0, \ldots, Z_N\). Endowed with the inner product

\[
\langle s_1, s_2 \rangle = \int_{\mathbb{CP}^N} s_1 \overline{s_2} e^{-k\varphi} \omega^N
\]

\(H^0(\mathbb{CP}^N, \mathcal{O}(k))\) is then a Bergman space, denoted by \(\mathcal{H}_k\). And the Bergman kernel \(\rho_k(z)\), by \(U(N+1)\)-invariance, is constant. So we have \(\rho_k(z) = \dim H^0(\mathbb{CP}^N, \mathcal{O}(k)) = N_k\). Therefore, we can read out an orthonormal basis of \(\mathcal{H}_k\) from the binomial expansion of \(\rho_k(z)e^{k\varphi} = N_k(1 + |z|^2)^k\).

Let \(V \subset \mathbb{CP}^N\) be a linear subspace of codimension \(m\). But for a \(U(N+1)\)-change of coordinates, we can assume that in the coordinate patch \(U_0 = \{[1, z], z \in \mathbb{C}^N\}\), \(V\) is defined by \(z_1 = \cdots = z_m = 0\). Then the sections of \(H^0(\mathbb{CP}^N, \mathcal{O}(1))\) that vanish along \(V\) are generated by \(Z_i, i = 1, \ldots, m\), where \(Z_i\) is represented by \(z_i\) in \(U_0\). More generally, we consider the space \(\mathcal{H}_{k,V}\), consisting of sections of \(H^0(\mathbb{CP}^N, \mathcal{O}(k))\) that vanish along \(V\). An orthonormal basis of \(\mathcal{H}_{k,V}\) consists of sections in the binomial expansion of \(N_k(1 + |z|^2)^k\). Therefore, the Bergman kernel of \(\mathcal{H}_{k,V}\) on \(U_0\) is

\[
\rho_{k,V} = N_k \frac{(1 + |z|^2)^k - (1 + \sum_{i=m+1}^{N} |z_i|^2)^k}{(1 + |z|^2)^k}.
\]

In particular

\[
\rho_{1,V} = (N + 1) \frac{\sum_{i=1}^{m} |z_i|^2}{1 + |z|^2}.
\]
We will need the following lemma in the proof of Theorem 4.4:

**Lemma 2.1** For each point \([Z] \in M\), the number \(\arcsin \sqrt{\rho_{1, V}(Z)}\) is just the distance of \([Z]\) to \(V\) under the Fubini–Study distance.

**Proof** This is straightforward, since the distance of \([1, z] \in U_0\) to \([1, 0]\) is just \(\arccos \frac{1}{1+|z|^2}\). \(\qed\)

We clearly have

\[
\lim_{k \to \infty} \sqrt{-1} \partial \bar{\partial} \log \left[ (1 + |z|^2/k)^k - \left( 1 + \sum_{i=m+1}^{N} |z_i|^2/k \right)^k \right]
\]

\[
= \sqrt{-1} \partial \bar{\partial} \log (e^{|z|^2} - e^{\sum_{i=m+1}^{N} |z_i|^2})
\]

\[
= \sqrt{-1} \sum_{i=m+1}^{N} dz_i \wedge d\bar{z}_i + \sqrt{-1} \partial \bar{\partial} \log (e^{\sum_{i=1}^{m} |z_i|^2} - 1)
\]

which gives us a hint on why Corollary 1.5 should be true.

### 3 Setting Up for the General Case

Recall that the on-diagonal Bergman kernel can also be defined as

\[
\rho_k(z) = \sup_{\|s\| = 1} |s(z)|^2.
\]

And at each point \(p \in M\), the supremum is achieved by a unique (up to a multiple of \(e^{i\theta}\)) unit section, denoted by \(s_p\), called the peak section at \(p\). \(s_p\) can also be characterized as the unit section that is orthogonal to all holomorphic sections that vanish at \(p\). The techniques of peak sections have been very useful in the calculation of asymptotics of Bergman kernel, for example [16]. For reader’s convenience and for later use, let us quickly copy some details of the construction of the peak sections.

First of all, we can choose local holomorphic coordinates \(\{w_a\}\) centered at a given point \(p\) and local Kähler potential \(\varphi\) for the Kähler form \(\omega\) so that

\[
\varphi(w) = \sum_a w_a \bar{w}_a + O(w^3).
\]

Then by a careful change of coordinates we can make \(\varphi\) be of the form

\[
\varphi(w) = \sum_a w_a \bar{w}_a + \sum P_{abcd} w_a w_b \bar{w}_c \bar{w}_d + \text{higher order terms}.
\]
Then by the rescaling of coordinates $z_a = \sqrt{k} w_a$, $\varphi$ becomes

$$\Phi(z) = |z|^2 + \frac{1}{k} P(z) + k^{-3/2} Q(z) + O(k^{-2}).$$

And the volume form $(i \partial \overline{\partial} \Phi)^m$ is of the form

$$J = 1 + k^{-1} p(z) + k^{-3/2} q(z) + O(k^{-2}).$$

Then one can choose local frame $\sigma_0$ for the line bundle $L^k$ on the ball $|z| \leq k^{1/4}$ for example so that $|\sigma_0|_h^2 = e^{-\Phi}$. We will mention $\sigma_0$ as a “normal frame”. One can then modify $\sigma_0$ to get the peak section. More precisely, $\sigma_0$, when regarded as a global discontinuous section of $L^k$, extending by zero outside our ball, is almost orthogonal to all holomorphic sections vanishing at $p$. In fact, by the symmetry of $e^{-|z|^2}$, we have

$$|\langle \sigma_0, \tau \rangle| \leq C k^{-1} \| \tau \|$$

and

$$\| \sigma_0 \|^2 = \pi^m (1 + a k^{-1} + O(k^{-2})).$$

Then by Hörmander’s $L^2$-techniques, we can modify $\sigma_0$ to get a global section, by introducing an error of the size $\varepsilon(k)$, due to the fact that $e^{-|z|^2}$ decays very fast near the boundary of our ball. Therefore, under the local frame $\sigma_0$, the peak section $s_p$ is represented by a holomorphic function of the form $\sqrt{\frac{k}{\pi^m}} (1 + O(k^{-1}))$. We will be using the important property of $s_p$ that

$$|s_p(z)|_h = \varepsilon(k),$$

for $z$ whose distance to $p$ is $\geq \frac{\log k}{\sqrt{k}}$, since $e^{(\log k)^2} = \varepsilon(k)$.

Let $V$ be a subvariety of $M$, and then for $k$ large enough, we have the exact sequence

$$0 \to H^0(M, L^k \otimes I_V) \to H^0(M, L^k) \to H^0(V, L^k) \to 0,$$

where $I_V$ is the ideal sheaf of $V$. So $\mathcal{H}_{k,V} = H^0(M, L^k \otimes I_V)$, and the orthogonal complement $\mathcal{H}_{k,V}^\perp$ is isomorphic to $H^0(V, L^k)$. We have the following lemma.

**Lemma 3.1** $\mathcal{H}_{k,V}^\perp$ is spanned by the peak sections $s_p$ for $p \in V$.

**Proof** We denote by $W$ the linear space spanned by the peak sections $\{s_p\}_{p \in V}$. Then for any $s \in \mathcal{H}_{k,V}$, and each $p \in V$, $s$, vanishing at $p$, is orthogonal to $s_p$. Therefore $s_p \in \mathcal{H}_{k,V}^\perp$, namely $W \subseteq \mathcal{H}_{k,V}^\perp$. On the other hand, if $s \in \mathcal{H}_{k}$ is orthogonal to $W$, then $s$ has to vanish at each $p \in V$, so $s \in \mathcal{H}_{k,V}$. $\square$
We denote by $\pi$ the restriction map $H^0(M, L^k) \to H^0(V, L^k)$. The restriction of $\pi$ on $H^0_k(M, V)$ is an isomorphism, and is denoted by $R$.

When $V$ is of pure dimension $n$, $H^0(V, L^k)$ is also endowed with a Hermitian inner product by integrating over the smooth part of $V$. We want to show that $R\sqrt{km-n}$ is a quasi-isometry, meaning $\|R\sqrt{km-n}\| = O(1)$. For this purpose, we need to use the Ohsawa–Takegoshi–Manivel extension theorem. There are several versions of this theorem, for example, [9,18,19]. The version that is most useful for our purpose is the one from [20]. In order to state the theorem, we copy the setting up from [20].

Let $M$ be a complex manifold of dimension $m$ with continuous measure $d\mu_M$ and let $(E, h)$ be a holomorphic Hermitian vector bundle over $M$. Let $S$ be a closed complex submanifold of dimension $n$. Consider a class of continuous functions $/Psi1 : M \to [−\infty, 0)$ such that

(1) $S \subset /Psi1(−\infty)$
(2) If $S$ is $n$-dimensional around a point $x$, there exists a local coordinate $(z_1, \ldots, z_m)$ on a neighborhood $U$ of $x$ such that $z_{n+1} = \cdots = z_m = 0$ on $S \cap U$ and

$$\sup_{U \setminus S} |/Psi1(z) - (m - n) \log \sum_{n+1}^m |z_j|^2| < \infty.$$ 

The set of such functions $/Psi1$ will be denoted by $\sharp(S)$. Clearly, the condition (2) does not depend on the choice of local coordinate. For each $/Psi1 \in \sharp(S)$, one can associate a positive measure $dV_M[/Psi1]$ on $S$ as the minimum element of the partially ordered set of positive measure $d\mu$ satisfying

$$\int_S f \, d\mu \geq \lim_{t \to \infty} \frac{2(m - n)}{\sigma_{2m-2n-1}} \int_M f e^{-/Psi1} \chi_{R(/Psi1, t)} \, dV_M$$

for any nonnegative continuous function $f$ with $\text{supp} \, f \subset M$. Here $\sigma_m$ denotes the volume of the unit sphere in $\mathbb{R}^{m+1}$, and $\chi_{R(/Psi1, t)}$ denotes the characteristic function of the set

$$R(/Psi1, t) = \{x \in M | -t - 1 < /Psi1(x) < -t\}.$$ 

Let $/Theta_h$ be the curvature form of the fiber metric $h$. Let $\Delta_h(S)$ be the set of functions $/Psi$ in $\sharp(S)$ such that, for any point $x \in M$, $e^{-/Psi} h = e^{-/Psi} \hat{h}$ around $x$ for some plurisubharmonic function $/Psi$ and some fiber metric $\hat{h}$ whose curvature form is semipositive in the sense of Nakano.

**Theorem 3.2** (Ohsawa–Takegoshi–Manivel, [20]) Let $M$ be a complex manifold with a continuous volume form $dV_M$, let $E$ be a holomorphic vector bundle over $M$ with a $C^\infty$ fiber metric $h$, let $S$ be a closed complex submanifold of $M$, let $/Psi \in \sharp(S)$ and let $K_M$ be the canonical line bundle of $M$. If the following are satisfied,

(1) There exists a closed subset $X \subset M$ such that
(a) \( X \) is locally negligible with respect to \( L^2 \) holomorphic functions, i.e., for any local coordinate neighborhood \( U \subset M \) and for any \( L^2 \) holomorphic function \( f \) on \( U \setminus X \), there exists a holomorphic function \( \tilde{f} \) on \( U \) such that \( \tilde{f}|_{U \setminus X} = f \).

(b) \( M \setminus X \) is a Stein manifold which intersects with every component of \( S \).

(2) \( \Theta_h \geq 0 \) in the sense of Nakano.

(3) \( (1 + \delta) \Psi \in \Delta_h(S) \cap C^\infty(M \setminus S) \) for some \( \delta > 0 \).

then there exists a constant \( C \) such that for any \( f \in H^0(S, E \otimes K_M|_S) \) such that

\[
\int_S |f|_{h_{\otimes}(dV_M)}^2 dV_M[\Psi] < \infty,
\]

there exists \( F \in H^0(M, E \otimes K_M) \) such that

\[
\int_M |F|_{h_{\otimes}(dV_M)}^2 dV_M \leq (C + \delta^{-3/2})^2 \int_S |f|_{h_{\otimes}(dV_M)}^2 dV_M[\Psi].
\]

If \( \Psi \) is plurisubharmonic, the constant \( (C + \delta^{-3/2})^2 \) can be chosen to be less than \( 256\pi \).

In our situation, \( V \) is the \( S \) in the theorem. The volume form is \( dV_M = \omega^m |_M \). The vector bundle \( E \) is the line bundle \( L^k - K_M \) with a twisting of the metric \( e^{-k\Psi} \otimes dV_M = e^{-k\Psi + \kappa} \). Since our manifold \( M \) is projective, the \( X \) in the theorem exists. Let \( N_V \) denote the normal bundle of \( V \), with the metric induced by \( \omega \) on \( T_M \). Let \( r \) denote the length of vectors in \( N_V \). Denote by \( N_V(\rho) \) the subset of vectors with length \( r < \rho \), and for \( \rho \) small enough, the exponential map

\[
\exp : N_V(\rho) \to M
\]

is a diffeomorphism of \( N_V(\rho) \) with its image. Then \( r \) is a function in a neighborhood of \( V \) in \( M \); we then choose a nonnegative smooth function \( \chi \) on \( [0, \infty) \), which is concave and satisfies the following conditions:

(1) \( \chi(x) = x \) for \( x \leq \frac{(\log k)^2}{k} \);

(2) \( \chi(x) \) is constant for \( x \geq \frac{(10\log k)^2}{k} \).

So \( \chi(r^2) \) can be seen as a smooth function on \( M \), which is constant away from \( V \). Then we twist the metric on \( L^k - K_M \) by \( e^{\beta_k \chi(r^2)} \), for \( \beta_k \) to be determined. Then we let \( \Psi = (m - n) \log r^2 \) and extend it smoothly to be defined on \( M \). Clearly this function \( \Psi \) satisfies the two conditions in the definition of \( \varpi(V) \). Also \( (1 + 1)\Psi \in \Delta_h(V) \cap C^\infty(M \setminus S) \) for \( h = e^{-k\Psi + \kappa + \beta_k \chi(r^2)} \) when \( \beta_k \) is not too big. We want to make \( \beta_k \) as large as possible. So we calculate

\[
\partial \bar{\partial} \chi(r^2) = \chi' \partial \bar{\partial} r^2 + \chi'' \partial r^2 \wedge \bar{\partial} r^2.
\]

By our construction of \( \chi \), we have \( 0 \leq \chi' \leq 1 \) and \( \chi'' \leq 0 \). So one can see that for \( k \) large, we can allow \( \beta_k \) to be of the size \( k - O(\log k) \). Finally, we calculate the measure

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dV_M[\Psi]. By integrating along fibers, one sees easily that for our \Psi, the measure dV_M[\Psi] is just the smooth measure \frac{\omega^n}{n!}.

Now we can apply the Ohsawa–Takegoshi–Manivel extension theorem to get that for any f \in H^0(V, L^k), one can find F \in H^0(M, L^k) satisfying the inequality:

\[ \int_M |F|^2 e^{-k\varphi + \beta_k \chi(r^2)} \frac{\omega^m}{m!} \leq C \int_V |f|^2 e^{-k\varphi} \frac{\omega^n}{n!} \]  

(1)

for some constant C independent of k. For simplicity, we assume that \int_V |f|^2 e^{-k\varphi} \frac{\omega^n}{n!} = 1.

First of all, since \epsilon = \frac{\log k}{\sqrt{k}} for r \geq \rho = \log \frac{k}{\sqrt{k}}, we see that

\[ \int_{M \setminus N_V(\rho)} |F|^2 e^{-k\varphi} dV_M,\omega = \epsilon(k). \]

So we can see that the mass of F is concentrated within a small tubular neighborhood of V with radius \rho = \frac{\log k}{\sqrt{k}}. To estimate the integral within the small neighborhood, we only need to notice that

\[ \int_C |z|^{2a} e^{-|z|^2} dV = \frac{\pi a!}{ka^{a+1}}. \]

Since \beta_k = k - O(\frac{\log k}{\sqrt{k}}), by integrating along fibers of N_V first, one can see that

\[ \int_{N_V(\rho)} |F|^2 e^{-k\varphi} dV_M,\omega \approx \frac{1}{k^{m-n}} \int_{N_V(\rho)} |F|^2 e^{-k\varphi + \beta_k \chi(r^2)} \frac{\omega^m}{m!}. \]

Therefore, we have the following:

**Theorem 3.3** The restriction map R : \mathcal{H}_{k, V}^+ \rightarrow H^0(V, L^k) has norm satisfying

\[ \| R \|^2 = O \left( \frac{1}{k^{m-n}} \right). \]

**Remark** This theorem is very important for our subsequent arguments. If one wishes to be more precise than the big O, one needs to have optimal constant in the Ohsawa–Takegoshi–Manivel extension theorem [38,39].

As a direct application, we have Theorem 1.2.

**Proof of Theorem 1.2** Consider the peak section s_z. We know that for any point w with distance d(z, w) \geq \log \frac{k}{\sqrt{k}}, the length |s_z(w)|_h = \epsilon(k). So, the L^2 norm of s_z|_V is \epsilon(k).

So by the theorem above, if we write s_z = s_1 + s_2 with s_1 \in \mathcal{H}_{k, V} and s_2 \in \mathcal{H}_{k, V}^+, then \| s_2 \|^2 = \epsilon(k). Therefore \| s_1 \|^2 = 1 - \epsilon(k) and

\[ |s_1(z)|_h = |\langle s_1, s_z \rangle| |s_z(z)|_h = |\langle s_1, s_1 \rangle| |s_z(z)|_h = (1 - \epsilon(k)) |s_z(z)|_h. \]
So \( \rho_{k,V}(z) \geq \rho_k(z) - \varepsilon(k) \); on the other hand, we clearly have \( \rho_{k,V}(z) < \rho_k(z) \), so the theorem is proved.

\[ \square \]

4 Calculations Near \( V \)

Next, we study the asymptotic of \( \rho_{k,V} \) around \( V \). For this, we need the off-diagonal asymptotics for the Bergman kernel, about which we now recall some details from [23–25].

Let \( \pi : X \to M \) be the circle bundle of unit vectors in the dual bundle \( L^* \to M \) with respect to \( h \). Sections of \( L^k \) lift to equivariant functions on \( X \). Then \( s \in H^0(M, L^k) \) lifts to a CR holomorphic function on \( X \) satisfying \( \hat{\delta}(e^{i\theta}x) = e^{ik\theta}\hat{s}(x) \). We denote the space of such functions by \( \mathcal{H}^2_k(X) \). The Szegö projector is the orthogonal projector

\[ \Pi_k : L^2(X) \to \mathcal{H}^2_k(X), \]

which is given by the Szegö kernel (Bergman kernel)

\[ \Pi_k(x, y) = \sum \hat{s}_j(x)\overline{\hat{s}_j(y)} \quad (x, y \in X). \]

(Here the functions \( \hat{s}_j \) are the lifts to \( \mathcal{H}^2_k(X) \) of the orthonormal sections \( \{s_j\} \); they provide an orthonormal basis for \( \mathcal{H}^2_k(X) \).)

The covariant derivative \( \nabla s \) of a section \( s \) lifts to the horizontal derivative \( \nabla_h \hat{s} \) of its equivariant lift \( \hat{s} \) to \( X \); the horizontal derivative is of the form

\[ \nabla_h \hat{s} = \sum_{j=1}^m \left( \frac{\partial \hat{s}}{\partial z_j} - A_j \frac{\partial \hat{s}}{\partial \theta} \, dz_j \right). \]

For \( z = \pi(x), w = \pi(y) \in M \), we will write

\[ |\Pi_k(z, w)| = |\Pi_k(x, y)|. \]

In particular, on the diagonal \( \Pi_k(z, z) > 0 \) is the same as our previous notation \( \rho_k(z) \). For each point \( z_0 \in M \), we choose a neighborhood \( U \), a local coordinate chart \( \rho : (U, z_0) \to (\mathbb{C}^m, 0) \), and a preferred local frame at \( z_0 \), which is a local frame \( e_L \) such that

\[ \| e_L(z) \|_h = 1 - \frac{1}{2} |\rho(z)|^2 + \cdots \]

For \( u = (u_1, \ldots, u_m) \in \rho(U), \theta \in (-\pi, \pi) \), let

\[ \tilde{\rho}(u_1, \ldots, u_m, \theta) = \frac{e^{i\theta}}{|e^*_L(\rho^{-1}(u))|_h} e^*_L(\rho^{-1}(u)) \in X \]
so that \((u_1, \ldots, u_m, \theta) \in \mathbb{C}^m \times \mathbb{R}\) give local coordinates on \(X\). We then write

\[
\Pi_k^0(u, \theta; v, \varphi) = \Pi_k(\tilde{\rho}(u, \theta), \tilde{\rho}(v, \varphi)).
\]

**Theorem 4.1** ([23–25]) Let \((L, h)\) be a positive Hermitian holomorphic line bundle over a compact \(m\)-dimensional Kähler manifold \(M\). \(\omega = \frac{1}{2} \Theta_h\) is the Kähler form. Let \(z_0 \in M\), then

(i) \[
\pi^m k^{-m} \Pi_k^0 \left( \frac{u}{\sqrt{k}}, \frac{\theta}{\sqrt{k}} ; \frac{v}{\sqrt{k}}, \frac{\varphi}{\sqrt{k}} \right) = e^{i(\theta - \varphi) + u \cdot v - \frac{1}{2}(|u|^2 + |v|^2)} \left[ 1 + \sum_{r=1}^{l} k^{-\frac{r}{2}} p_r(u, v) + k^{-\frac{l+1}{2}} R_{kl}(u, v) \right],
\]

where the \(p_r\) are polynomials in \((u, v)\) of degree \(\leq 5r\) (of the same parity as \(r\)), and

\[
|\nabla^j R_{kl}(u, v)| \leq C_{jkl} k^\varepsilon \quad \text{for} \quad |u| + |v| < b\sqrt{\log k},
\]

for \(\varepsilon, b \in \mathbb{R}^+\), \(j, l \geq 0\). Furthermore, the constant \(C_{jkl}\) can be chosen independent of \(z_0\).

(ii) For \(b > \sqrt{j + 2l + 2m}\), \(j, l \geq 0\), we have

\[
|\nabla^j \Pi_k(z, w)| = O(k^{-l})
\]

uniformly for \(\text{dist}(z, w) \geq b\sqrt{\log k}/k\).

The so-called normalized Bergman kernel \(P_k(v, z_0)\) was also defined in [23] as

\[
P_k(v, z_0) = \frac{|\Pi_k(v, z_0)|}{\sqrt{\Pi_k(v, v)} \sqrt{\Pi_k(z_0, z_0)}}
\]

which contains plenty of information of the geometry of the image of \(M\) under the Kodaira embedding [29]. Recall that it was proved in [23] the following estimations.

**Theorem 4.2**

\[
P_k \left( z_0 + \frac{u}{\sqrt{k}}, z_0 + \frac{v}{\sqrt{k}} \right) = e^{-\frac{1}{2} |u - v|^2} (1 + R_N(u, v)),
\]

where the remainder satisfies the following:

\[
|R_k(u, v)| \leq C_1 |u - v|^2 k^{-1/2 + \varepsilon}
\]

\[
|\nabla R_k(u, v)| \leq C_1 |u - v| k^{-1/2 + \varepsilon}
\]

\[
|\nabla^j R_k(u, v)| \leq C_j k^{-1/2 + \varepsilon}
\]

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for $|u| + |v| < b \sqrt{\log k}$, where the $C_i$'s all depend on $b$.

Locally, the orthonormal basis $\{s_i\}$ are represented by holomorphic functions $\{f_i\}$, so the Kodaira embedding $\Phi_k$ is given by

$$\Phi_k(z) = [f_0(z), \ldots, f_N(z)].$$

We denote by

$$Q_k(z, w) = \sum f_i(z) \bar{f}_i(w).$$

So we have

$$Q_k(z, z) = \frac{N^m}{\pi^m} \left( 1 + O \left( \frac{1}{k} \right) \right) e^{k \phi}.$$ 

For simplicity, we first work on the case of codimension 1 to illustrate the idea of calculations. One immediately realizes that this idea works for codimension $\geq 1$.

Now we assume $V$ is a smooth divisor $D$. Notice that the property required for the choice of $e_L$ by the theorem above is only that $\phi = |z|^2 + \text{higher order terms}$. So we are allowed to choose local coordinates $z$ so that $D$ is defined by $\{z_m = 0\}$.

We use the notation $f_{i,m} = \frac{\partial}{\partial z_m} f_i$. Then by taking the derivatives $\frac{\partial}{\partial z_m} P(z, z)$ and $\frac{\partial^2}{\partial z_m \partial \bar{z}_m} P(z, z)$, and since $\partial \phi(z_0) = 0$ and $\partial \partial \phi(z_0) = \sum d z_i \wedge d \bar{z}_i$, we get the following estimations:

$$\sum f_i(z_0) \bar{f}_{i,m}(z_0) = O(k^{m-1}) \quad (2)$$

$$\sum f_{i,m}(z_0) \bar{f}_{i,m}(z_0) = \frac{k^{m+1}}{\pi^m} (1 + O(k^{-1})). \quad (3)$$

We denote by $f = (f_0, \ldots, f_N)$ and $f_m = (f_{1,m}, \ldots, f_{N,m})$. Then we define a unit section of $L^k$ as

$$\alpha(z_0) = \frac{\sum \bar{f}_{i,m}(z_0) s_i(z)}{|f_m(z_0)|}.$$ 

The estimations above imply that

$$|\alpha(z_0)|^2 = O(k^{m-3}).$$

We need also to estimate the norm of $\alpha(z_0)$ for points $z \neq z_0$. For $z, w \in U$, by the asymptotics for the off-diagonal Bergman kernel, we have the following:

$$\sum f_i(z/\sqrt{k}) f_i(w/\sqrt{k}) = \frac{k^m}{\pi^m} e^{-z \cdot w} \left[ 1 + \sum_{r=1}^l \frac{k^{-r/2} p_r(z, w) + k^{-l/2}}{r} R_{kl}(z, w) \right],$$
where $p_r$ and $R_{kl}$ are slightly different from those in the theorem, but enjoy similar estimations. Therefore, we have

$$\sum_i f_i(z) \tilde{f}_{i,m}(z_0) = O(k^m)$$

for $z \in D$ satisfying $|z| < \sqrt{2m + 3\frac{\log k}{k}}$, since $z_m = 0$. This implies that

$$|\alpha_{z_0}(z)|_h^2 = O(k^{m-1})e^{-k|z|^2},$$

for these $z$. Finally, when $\text{dist}(z, z_0) \geq \sqrt{2m + 3\frac{\log k}{k}}$, we can use part (ii) of Shiffman–Zelditch’s theorem to get that

$$|\alpha_{z_0}(z)|_h^2 = O(k^{-1}).$$

Therefore we can estimate the $L^2$ norm of $\alpha_{z_0}$ on $D$ to get

$$\int_D |\alpha_{z_0}(z)|_h^{2m-1} \frac{\omega^{m-1}}{(m-1)!} = O\left(1 + \frac{1}{k}\right) = O(1)$$

as $\int_{C_{m-1}} e^{-k|z|^2} dV = O(k^{1-m})$. So we have proved the following:

**Lemma 4.3** $\alpha_{z_0}$ is almost orthogonal to the space $\mathcal{H}_{k,D}$. More precisely, if we write

$$\alpha_{z_0} = s_1 + s_2,$$

with $s_1 \in \mathcal{H}_{k,D}$ and $s_2 \in \mathcal{H}_{k,D}^\perp$, then $\|s_2\|^2 = O(k^{-1})$.

Before proceeding, we want to explain the meaning of this lemma in the sense of complex projective geometry. A complex vector space equipped with a Hermitian inner product can be identified with its dual space by a conjugate linear map. In our case, let $W = H^0(M, L^k)$, then the Kodaira map $\Phi_k$ maps $M$ to $\mathbb{P}W^*$. Fixing an orthonormal basis $(s_0, \ldots, s_N)$, and local frame $e_L$, then $\Phi_k(z) = (f_0(z), \ldots, f_N(z))$, where $s_i = f_i e_L^k$. Then by taking the complex conjugate $(\bar{f}_0(z), \ldots, \bar{f}_N(z))$, we get a conjugate-holomorphic embedding $\Phi_k : M \to \mathbb{P} W$, with $\Phi_k(z) = \{\bar{f}_i(z)s_i\}$. What interesting is that $\bar{\Phi_k}(p)$ is just the complex line in $W$ containing the peak section $s_p$. And Lemma 3.1 implies that $\bar{\Phi_k}(D)$ linearly span $\mathbb{P}\mathcal{H}^\perp_{k,D}$. So the preceding lemma, in this setting, says that the image of $\frac{\partial}{\partial z_m}(z_0)$ under the tangent map of $\bar{\Phi_k}$ is almost orthogonal to the linear subspace $\mathbb{P}\mathcal{H}^\perp_{k,D}$.

We can apply similar calculation for the point in the $z_m$-disk passing through $z_0$, which have coordinates $(0, \ldots, 0, z_m)$ with $z_m$ small. We use $v$ to denote the points on this disk. We again define

$$\alpha_v = \sum_i \frac{\tilde{f}_{i,m}(v)s_i}{|f_{m}(v)|}.$$
as a unit section of $L^k$. Then we estimate the point-wise norm of $\alpha_v$ along $D$ by differentiating the function $Q_k(z, w)$. We have

$$
\sum f_i(z)\overline{f_i,m}(v) = O(k^m) |e^{k^2v^2}| = O(k^m)
$$

(4)

$$
\sum f_i,m(v)\overline{f_i,m}(v) = \frac{k^{m+1}}{\pi^m}(1 + O(k^{-1}))(1 + k|v|^2)e^{k|v|^2}
$$

(5)

for $z \in D$ satisfying $|z| < \sqrt{2m + 3\log k \sqrt{k}}$. Therefore, for these $z$, we have

$$
|\alpha_v(z)|^2 = O(k^{m-1})e^{-k|z|^2 - k|z|^2}.
$$

So the integral over this patch of $z$ is $O(1)$ as $\alpha_{z_0}$. And for the remaining $z \in D$, we still have $|\alpha_v(z)|^2 = O(1/k)$, so in conclusion, we have

$$
\int_D |\alpha_v|^2 \frac{\omega^{m-1}}{(m-1)!} = O(1).
$$

Again, this implies that $\alpha_v$ is almost orthogonal to $H_{k,D}^\perp$ with the same estimation as in Lemma 4.3. Now we put all these ideas together to get

**Theorem 4.4** We have the following estimation for the logarithmic Bergman kernel along the disk:

$$
1 - P_k^2(v, z_0) \leq \frac{\rho_k,D(v)}{\rho_k} \leq 1 - P_k^2(v, z_0),
$$

where $r = |v| = |v_m|$ and $\beta(r) = C \int_0^r \sqrt{1 + kx^2}e^{kx^2/2}dx$ with $C$ independent of $k$.

**Proof** We decompose $f(v) - \tilde{f}(z_0) = b + c$, with $b \in H_{k,D}$ and $c \in H_{k,D}^\perp$. Let $d$ denote the distance in the Fubini–Study metric. Then

$$
\cos d(\tilde{f}(v), \tilde{f}(z_0)) = \frac{|(\tilde{f}(v), \tilde{f}(z_0))|}{|f(v)||f(z_0)|} = \frac{\langle \tilde{f}(v) + c, \tilde{f}(z_0) \rangle}{|f(v)||f(z_0)|}
$$

$$
\cos d(\tilde{f}(v), H_{k,D}^\perp) = \frac{|(\tilde{f}(v), \tilde{f}(z_0) + c)|}{|f(v)||f(z_0) + c|} = \frac{|\tilde{f}(z_0) + c|}{|f(v)|}
$$

So

$$
\frac{\cos d(\tilde{f}(v), \tilde{f}(z_0))}{\cos d(\tilde{f}(v), H_{k,D}^\perp)} = \frac{\langle \tilde{f}(z_0) + c, \tilde{f}(z_0) \rangle}{|\tilde{f}(z_0) + c||f(z_0)|}.
$$

Then

$$
|c(r)| \leq \int_0^r O \left( \frac{1}{\sqrt{k}} \right) \sqrt{\frac{k^{m+1}}{\pi^m}(1 + kx^2)e^{kx^2/2}}dx.
$$
So

\[ \frac{|c(r)|}{|f(z_0)|} \leq C \int_0^r \sqrt{1 + kx^2} e^{kx^2/2} \, dx = \beta(r). \]

When \( \frac{|c(r)|}{|f(z_0)|} < 1 \), we have

\[ 1 \leq \frac{\cos d(\bar{f}(v), \mathcal{H}_{k,D}^\perp)}{\cos d(\bar{f}(v), \bar{f}(z_0))} \leq \left( 1 - \left( \frac{|c(r)|}{|f(z_0)|} \right)^2 \right)^{-1}. \]

By Lemma 2.1, we are interested in \( \lambda_v = \sin^2 d(\bar{f}(v), \mathcal{H}_{k,D}^\perp) \), which satisfies

\[ 1 - \left( 1 - \left( \frac{|c(r)|}{|f(z_0)|} \right)^2 \right)^{-1} \cos^2 d(\bar{f}(v), \bar{f}(z_0)) \leq \lambda_v \leq 1 - \cos^2 d(\bar{f}(v), \bar{f}(z_0)). \]

We have that \( \frac{|(\bar{f}(v), \bar{f}(z_0))|}{|f(v)||f(z_0)|} = \frac{|\Pi_k(v, z_0)|}{\sqrt{\Pi_k(v, v)\sqrt{\Pi_k(z_0, z_0)}}} \) is just the normalized Bergman kernel \( P_k(v, z_0) \). So we have proved the theorem. \( \square \)

Notice that the term \( \beta(r) \) is small when \( r \) is small. With the substitution \( R = \sqrt{k}r \), it becomes

\[ C \int_0^R \sqrt{1 + x^2} e^{x^2/2} \, dx. \]

So for fixed \( R \), \( \beta = O(\frac{1}{\sqrt{k}}) \).

More generally, when \( V \) is a smooth subvariety, we can choose local coordinates \((z_1, \ldots, z_m)\) so that \( V \) is defined as \( z_m = z_{m-1} = \cdots = z_{m-r+1} = 0 \). Then we can repeat our calculations for the divisor case without any difficulties. More precisely, in each normal direction at a point of \( V \), we can apply a unitary change of coordinates, so that direction is contained in the space spanned by \( \frac{\partial}{\partial z_m} \). So the conclusions for the divisor case hold for this more general case.

Notice that when \( |z| \) is small, \( |z| \) is about the distance of \( z \) to \( V \), since \( \omega(z) = \sum \delta_{ij} dz_i \wedge d\bar{z}_i + O(|z|) \). More precisely, we have

\[ d(z, D) = |z|(1 + O(|z|)), \]

so we can talk about the asymptotics without going local. In particular, when \( |z| \leq \frac{\log k}{\sqrt{k}} \),

\[ e^{-k|z|^2 + k|z|^2} = \left( 1 + k|z|^2 \frac{\log k}{\sqrt{k}} \right). \]
When \( r \leq \frac{\sqrt{\log k}}{\sqrt{2k}} \), we have

\[
\beta^2(r) = O\left(\frac{\log k}{\sqrt{k}}\right),
\]

so we have the first part of Theorem 1.1.

**Remark** If the reader is careful enough, he/she must have noticed there is a gap area between our estimations around \( V \) and away from \( V \), namely when

\[
\frac{\sqrt{\log k}}{\sqrt{2k}} < r < \frac{\log k}{\sqrt{k}},
\]

which, very interestingly, have also been seen in the case of Bergman kernel for Poincaré type metrics [30,32,33], where it was called the “neck”. Luckily, unlike the Poincaré type metrics, the “neck” in our case is not very difficult.

Let \( z_0 \in M \) with distance \( r \) satisfying

\[
\frac{\sqrt{\log k}}{\sqrt{2k}} < r < \frac{\log k}{\sqrt{k}}.
\]

We consider the peak section \( s_{z_0} \). Using the normal coordinates centered at \( z_0 \), and the normal frame, we know that

\[
|s_{z_0}(z)|_h^2 = \left(1 + O\left(\frac{1}{k}\right)\right) \frac{k^m}{\pi^m} e^{-k|z|^2}.
\]

Let \( w \in V \) be the point that is closest to \( z_0 \). Then for \( z \in V \) in this coordinates patch, we have

\[
|z|^2 \approx |w|^2 + |z - w|^2,
\]

since we are looking at geodesics of very small scales, meaning things work like Euclidean spaces. Therefore

\[
\int_V |s_{z_0}(z)|_h^2 \frac{\omega^n}{n!} = O(k^{m-n})e^{-kr^2}.
\]

Therefore, the image of the orthogonal projection of \( s_{z_0} \) onto \( \mathcal{H}_{k,V}^1 \) has \( L^2 \) norm \( O(e^{-kr^2}) \). If we decompose \( s_{z_0} = s_1 + s_2 \), with \( s_1 \in \mathcal{H}_{k,V} \) and \( s_2 \in \mathcal{H}_{k,V}^1 \), then

\[
||s_1||^2 = 1 - O(e^{-kr^2}) \quad \text{and} \quad |s_1(z_0)|_h = |\langle s_1, s_{z_0} \rangle||s_{z_0}|_h = (s_1, s_1)|s_{z_0}|_h = (1 - O(e^{-kr^2}))|s_{z_0}|_h.
\]

Therefore

\[
\rho_{k,V}(z_0) > |s_1(z_0)|_h^2 = (1 - O(e^{-kr^2}))\rho_k(z_0).
\]
So we have proved the second part of Theorem 1.1.

**Proof of Corollary 1.4** We have $\rho_{k,V} = \frac{k^m}{\pi^m} (1 + O\left(\frac{1}{k}\right))$, so $i \partial \bar{\partial} \log \rho_{k,V} = O\left(\frac{1}{k}\right) \omega$. \(\square\)

**Proof of Corollary 1.5** In this normal coordinates, let $r = d(w, V)$, then we have $r^2 = \sum_{i=n+1}^{m} |w_i|^2 (1 + o(\frac{1}{k^{1/4}}))$ for $w$ satisfying $|w|^2 \leq \frac{\log k}{k}$, by Toponogov's comparison theorem. We also choose local frame so that $\varphi = |w|^2 + \text{higher order terms}$. Therefore by Theorem 1.1, we have

$$
(\rho_{k,V} e^{k\varphi}(\frac{z}{\sqrt{k}})) = \frac{k^m}{\pi^m} (e^{\sum_{i=1}^{n} |z_i|^2} (1 + o \left(\frac{1}{k^{1/4}}\right))).
$$

Then by taking the limit of $\sqrt{-1} \partial \bar{\partial} \log(\rho_{k,V} e^{k\varphi}(\frac{z}{\sqrt{k}}))$, we get the conclusion. \(\square\)

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