Detecting Bifurcations in Numerical Simulation of Fluid Flow

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Abstract. We present a general method for detecting bifurcations in numerical simulations of dissipative fluid flow whose solutions possess attracting sets for certain values of the parameters of the system. We describe the method as ‘fast-slow’ steering. The parameters governing system behaviour are slowly steered within a chosen range of interest, as the system evolves in time. The parameter changes are continuous and are ‘slow’ relative to the ‘fast’ rate of the time stepping of the system. This allows the simulated system to evolve continuously through a series of states sufficiently close to a trajectory of attractors of the system held fixed at each successive value on the trajectory. We present the method in its general form which can be applied to any suitable dynamical system. We then present a detailed unfolding of a bifurcation sequence in the numerical solutions of magnetohydrodynamic equations modelling dynamo acting in a toroidal volume. Torus-shaped discs have been observed in astrophysical contexts around collapsed objects (e.g. black holes). Our results conform to those obtained using the traditional computational approach of ‘quasi-static’ steering, and are also compatible with the mathematical analysis.

1. Introduction
When moving through a bifurcation point, a dynamical system may undergo a transition from one stable state to another, via a series of intermediate states which may be unstable. From a geometric viewpoint these stable states are represented by geometric structures in state space. Such structures can be simple, e.g. points or limit cycles, or more complex, such as tori or strange attractors. Currently most software for following solution branches and detecting bifurcations is designed for bifurcations involving the simpler zero or one dimensional structures in phase space. For example, continuation software can follow certain types of solutions (fixed points, periodic orbits, homo/heteroclinic connections, etc) and indicate their bifurcations, but they are generally only capable of dealing with zero or one dimensional equilibria. The use of scientific visualisation can be employed to gain a qualitative understanding of changes in geometry (and hence topology) involving attractors of higher and/or fractal dimension. Scenarios such as the Ruelle-Takens-Newhouse [1, 2] route to chaos, and symmetry breaking due to on-off intermittency [3, 4, 5], involve interesting changes in the topologies of the solutions in phase space which can be detected via visualisation. Computational steering has developed as a means of investigating the changes in behaviour of numerical simulations as key control parameters are steered. Generally, the visualisation is of the development of the solutions in three dimensions of space, for example in studying development of different structures in multi-phase fluids [6, 7]. However, we are
concerned here with visualisation and steering of structures in the state space of the solutions, since the theory of dynamical systems is related to behaviour in state space. Visual methods for studying behaviour are very useful when an analytic or purely numerical approach is difficult or impossible. However, a major challenge that we face is that the state space of the solutions of partial differential is infinite dimensional, and, even when some form of discretization reduces this to a finite dimension, it is very large involving thousands or even millions of dimensions that represent the degrees of freedom of the system.

In this paper we describe a novel method for studying bifurcations and illustrate it by examining the solutions of a magnetohydrodynamic dynamo where physical effects of turbulence and rotation are parameterised, giving the control parameter of the system, and a nonlinearity is introduced which enables the existence of a variety of stable solutions which evolve as the control parameter is varied. The structure of the paper is as follows. In section 2 we present various methods for following changes of state in dynamical systems and discuss their different areas of application. In section 3 we present a novel method for studying bifurcations, which we call fast-slow steering. In section 4 we present the system we choose to illustrate the use of this method. In section 5 we present our results and in section 6 we present conclusions and suggestions for future work.

2. Methods for detecting bifurcations

2.1. Numerical continuation

A widely used class of tools for bifurcation analysis are based on numerical continuation [8, 9], which itself relies fundamentally on the knowledge of the model equations of the system. To detect bifurcations during the continuation of stable or unstable branches of solutions, the characteristic values of the system, such as eigenvalues or Floquet multipliers, are monitored. Bifurcations happen when these values cross certain manifolds on the complex plane. In this way, continuation software [8, 10, 11, 12] such as AUTO [9] and MATCONT [13] offer a software solution for scientists and engineers who need a systematic way for bifurcation analysis.

Continuation has thus far been used to deal with mathematical problems defined as a system of differential equations. Sometimes however, the same global behaviour can be reproduced using local microscopic rules applied to a set of nodes, agents or particles which constitute the model, such as predator-prey models approximated by cellular automata modelling techniques [14, 15]. Conclusions about the mathematical model can not be easily extended to the computational one, and vice versa, due to the complexity of nonlinear phenomena and the discrepancies between the modelling approaches. The loss of mathematical rigour in the computational approach is compensated for through the richer insight that this experimental framework provides, because it captures aspects which are not easily mathematically defined.

Besides its dependence on the existence of the model equations, continuation tools use a large number of numerical methods, and require an advanced understanding of numerical stability analysis and bifurcation theory. Consequently, most existing continuation software are hard to use by non-specialists. Continuation algorithms do not rely on visualisation, and as such can be used with systems of high order without the necessity of model reduction, nonetheless, they do become prohibitively expensive for such systems. Furthermore, continuation software provide close to no functionality to deal with more complex topologies, such as tori, fractal or chaotic attractors, which has to be dealt with using visual approaches.

2.2. Computational steering

There has been an increasing use of non-conventional modelling techniques, where the governing equations of the system are not represented as partial differential equations. Examples of such systems, which lack a macroscopic mathematical model, are mesoscale simulations [16, 17], cellular automata models [18] and design optimisation [19, 20].

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In such cases the use of computational approaches to match parameter configurations to system behaviour, became an indispensable scientific tool. Traditionally, such maps have been constructed by task farming; that is by performing large numbers of simulations concurrently, each corresponding to a set of different values of the control parameters. However, task farming is not explorative, as it does not allow any intuition and is not user driven, because it relies on post-processing of the data. Consequently, using this brute-force approach may lead to wasted computational power, and inefficient use of valuable resources. Many investigative scenarios involve a process of exploration of the behaviour of solutions which can be made more efficient if human know-how is utilised. Task farming methods have been used in various domains, for example in identifying interesting features in large-scale molecular dynamics simulations [21] or design steering and optimisation [22].

An alternative to task farming is the exploration of parameter space by a human agent, who intuitively steers the parameter values towards regions where the solutions display some desired behaviour, or simply for exploration. Such methods are termed computational steering [23, 24]. Like task farming, steering can be used to build maps between points in parameter space and the corresponding dynamical behaviour in solution space [25]. However, steering utilises on-line visualisation as a basis for user interaction and steering decisions support [26, 27, 28]. Steering proved well suited to high-order systems where theory alone falls short of describing the system behaviour, and systems which lack a macroscopic mathematical model, such as mesoscale simulations [16, 17], cellular automata models [18] and design optimisation [19, 20].

Steering starts with a point in state space (often, but not necessarily, a point on a stable attractor), and an initial direction in parameter space. A subsequent simulation is started at an adjacent point in parameter space, using the previous convergent solution for initialisation, until eventually a numerically unstable regime (or a regime with a qualitatively different behaviour) is found. Successive simulations can then be launched to home in on the location of the boundary between stable and unstable behaviour, using an interval bisection algorithm. This method provides an advantage over task farming, as it allows an interactive investigation of the system by allowing choice of the direction in parameter space. However, this conventional way of performing parameter steering — that we shall term ‘quasi-static’ steering — still suffers from relying on multiple simulation runs or, at least, multiple isolated parameter evaluations and the corresponding induced transient delays. This is because quasi-static steering does not take advantage of system continuability with respect to the parameter vector at hyperbolic neighbourhoods, because it deals with solutions whose asymptotic behaviour approximates the actual attractors, but which are nevertheless different by definition (they are not subject to continuation). Consequently, the adoption of computational steering for stability analysis and bifurcation detection has been limited. Beside the limited application of bifurcation theory outside the realm of mathematical models, which confines the problems of interest to those amenable to continuation, this meagre uptake of computational steering has been partly due to the lack of a sound mathematical basis for its use as a stability analysis tool. It is for this reason that numerical continuation is more widely deployed in the study of bifurcations of dynamical systems, even if this limits the types of dynamical systems that may be studied.

2.3. Topology-based visualisation

In steering, as a human agent must be involved, the representation of the data in the solutions is of paramount importance. Unlike with a machine, humans get overwhelmed with large amounts of data, therefore the data must be presented in a form which emphasises the important aspects. One such aspect is the evolution of structural stability which can be conveyed using topology tracking techniques. The topology of a system can be seen via a visual representation, for example as a change of topology of an attractor from a limit cycle (one dimension) to a torus (at least two dimensions). The roots of this approach can be traced back to the works of Smale
who introduced concepts of topological equivalence and their relation to structural
stability in dynamical systems, and inaugurated the application of topology to the study of
dynamical systems. Topology-based visualisation for tracking singularities in vector fields was
practically initiated by Helman and Hesselink 1989 [31]. The approach of topology tracking is
now used in a number of fields, most notably in fluid flow studies.

Current topology-based techniques are, again, based on analytical tools, such as space
interpolation [32, 33], time interpolation [34, 35], streamline and pathline integration [35], or
Poincaré-Hopf index tracking [33]. Furthermore, many are designed to deal with data sets
composed of consecutive discrete snapshots of the whole vector field, and in particular two-
dimensional vector fields. In these low-order fields, the time development can be tracked in
a third dimension. The vector field is computed for discrete steps, and then reduced into
topological skeletons. Interpolation techniques are subsequently used to deduce the topological
changes that took place in the missing time intervals. The more challenging problem, at least
in terms of visualisation, lies in developing such methods for higher-order topologies, whether
for the vector field as a whole or for individual attractors.

3. Fast-slow steering

To reconcile the experimental approach of computational steering with the mathematical theory
of dynamical systems, when dealing with simulations which exhibit rich nonlinear behaviour,
extensive analysis has to be carried out, to match results to theory. However, for certain
simulated models, which exhibit behaviour characteristic of dynamical systems defined as
differential equations, such as continuability and rules governing structural stability, conventional
quasi-static steering can be augmented to provide an in-line analysis tool by taking this into
consideration. By exploiting the concept of parameter continuation, steering can further
distinguish itself from the traditional approach of task farming. In this paper we introduce
fast-slow steering as a form of parameter continuation, where parameter updates take place as
small steps, while the system evolves in time. Slow time functions can be used to guide and
automate the parameter stepping in control space, while the system state is recorded for post-
processing, or transformed into some visual output for online monitoring. Evolving the system
in this manner produces solutions which do not represent a single isolated fixed configuration,
but which travel through a series of consecutive configurations. It is demonstrated that, due to
the slow evolution of the parameter vector trajectory, these solutions represent the continuation
of the consecutive frozen configurations along the created parameter trajectory. This way, fast-
slow steering makes the investigated systems natural candidates for applying topology-based
visualisation. Tracking dynamical behaviour would then amount to tracking the evolution of
the system in a suitable observation space, which can be chosen to geometrically convey and
emphasise the topological information about the system. The novelty in this approach lies in
incorporating a combination of aspects from steering, continuation, topology-based visualisation,
and system control, and as such, it provides a powerful alternative to these traditional approaches
for studying the dynamical behaviour of a simulated model. Another major distinction of this
method, is that it does not rely on the availability of a mathematical model defined as a system
of differential equations, nor does it use the analytical information derived thereof, and thus can
be applied to a wider range of problems in computational science.

For the purpose of validating fast-slow steering, we demonstrate its effectiveness for a high-
order dissipative system (described in section 4). Such systems have, what can be informally
described as ‘well-behaved’ dynamics, as they have a set number of attractors to which all
neighbouring solutions converge. Furthermore, the system chosen should preferably have basins
of attraction which divide the parameter space ‘nicely’, rather than being locally intermingled
basins of co-existing attractors, such as in the case of the Hénon map [36, 37]. The system will
then have solutions which, when initialised in a particular basin of attraction, would exhibit
an easily discernible, deterministic succession of dynamical behaviour in a range of parameter values. This produces results which can be easily validated, through using analytical tools, or using quasi-static steering for some sample parameter evaluations.

4. A bifurcation sequence in the solutions of a mean field dynamo

4.1. Physical processes to be modelled

The study of the process by which the magnetic field is regenerated and sustained within a conductive body through its motion is called dynamo theory. This sustainability of magnetic fields depends on the presence of a highly conducting fluid, such as the earth’s molten iron or the ionised gas of the sun, where the flow of these fluids in the interior of the object sustains the magnetic field. Earth’s magnetic field for example, is due to the convection of liquid iron within the outer core, which is in turn due to earth’s rotation that organises the flow in rolls around the north-south polar axis. Electric currents are induced, which again produces a magnetic field that reinforces the original, and a dynamo that sustains itself is created. Such dynamo action is also believed to be responsible for the large scale fields in stars, including the sun.

We illustrate the method of fast-slow steering by investigating a model derived from mean field dynamo theory, in which the key processes driving the dynamo appear as parameters in the model. We restrict the spatial dimensions from three to two by assuming axisymmetry of the system. The control parameters represent processes on much smaller length scale compared with the mean field. More realistic equations and parameterisations can be derived by combining this approach with direct numerical simulation of the turbulent flow. Our model is solved with the electrically conducting fluid confined to torus surrounded by a vacuum, which is known to exhibit a very rich sequence of bifurcations, ending in chaotic behaviour within a range of parameter values close to the onset of dynamo action. Such bifurcation sequences were observed in dynamos operating in spheres and spherical shells but at much more supercritical parameter values.

The method of ‘fast-slow’ steering to unfold sequences of bifurcations has not been previously applied to solutions of partial differential equations, therefore it is sensible to begin with a test case whose solutions are known but which is rich enough in its behaviour to demonstrate the full power of the method. The model equations are much simpler than those being used in current dynamo theory which are fully three dimensional and which have much more sophisticated representations of the interaction of the turbulent fluid motions and the growing magnetic field. We use these equations because they are sufficiently rich to allow us to demonstrate the exploration of nonlinear behaviour in fluids, whilst being simple enough to explore a sufficiently large region of parameter space in a computationally feasible time.

4.2. The model equations

Our model is represented by the following equation which has been nondimensionalised by an appropriate choice of length and time scales so that there is a single dimensionless parameter $\alpha$:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B} + \alpha f(\theta, |\mathbf{B}|) \mathbf{B} - \nabla \times \mathbf{B}).$$

(1)

- $\mathbf{B}(s, \theta)$ is the magnetic field vector.
- $\mathbf{u}(s, \theta)$ is the fluid velocity.
- $\alpha$ is the dimensionless parameter representing the effect of turbulence and rotation as a term for field generation. This effect is necessary for a dynamo to operate.
- $f(\theta, |\mathbf{B}|)$ is a scalar function that represents both the antisymmetry of the parameterisation of the turbulence in the rotating torus and also the back reaction of the magnetic field to quench the turbulence hence to give the nonlinear feedback. In the results presented here $f(\theta, |\mathbf{B}|) = \sin(\theta)/(1 + |\mathbf{B}|^2)$. This is just about the simplest form for these effects.
Further details of the model and the exact mathematical representation of the differential operators in toroidal geometry is described in [38, 39]. The expressions are given in terms of a set of coordinates $s, \theta, \phi$ where $s$ is a coordinate that determines (but is not equivalent to) the minor radius of the torus. Surfaces of constant $s$ represent tori when the other coordinates range through their full set of values. $\theta$ represents the angle around the torus starting from the equator, thus $\theta = 0, \theta = \pi$ represent the points where the cross section of the torus cuts the equator. $\phi$ is the azimuthal angle along the major axis of the torus, since we assume the system is axisymmetric about the central axis of the torus, this coordinate does not appear in the equations and is therefore ignored.

We choose $\alpha$ as the parameter which will be slowly varied as the model (1) is solved via an explicit time-stepping scheme (Dufort-Frankel). Thus, $\alpha$ changes its value at each time step but sufficiently slowly that the system appears to evolve through the trajectory of attractors. Bifurcations along this trajectory are detected as major changes in the topology of the attractor. In accordance to the language of dynamical systems theory we call a solution where the fields are time invariant a fixed point solution (often called a steady dynamo). A single period solution is a limit cycle. A doubly periodic solution is called a solution on a torus being quasi-periodic if the two periods are incommensurate.

4.3. Projection method

![Diagram](a) ![Diagram](b)

**Figure 1.** Poincaré section with control and sampling points, 1(a) one hemisphere only, 1(b) north and south hemipheres (symmetrical sampling).

When we solve equation (1) via discretization of time and space we represent the solutions as values at points on a grid, essentially a matrix. In the current case we are using a finite difference scheme for discretization with a grid of typically 100 values for $s$ and 400 values for $\theta$. This gives a state space of order $10^5$ dimensions. To visualise we must find a method of projecting from this space onto something that can be visualised, i.e. using three spatial dimensions and some other visual clues such as colour for extra dimensions. We now describe this method.

We sample the solutions as they progress in time via the Poincaré map. This is obtained by choosing a suitable hyperplane in phase space and plotting the points at which the orbit intersects this hyperplane. To provide graphical insight this hyperplane is projected onto two-dimensional plane. The hyperplane is better chosen not tangential or near tangential to the solution, as it might give results that are difficult to interpret. We do this via a physically motivated method. The dynamo solutions are known to propagate as waves travelling clockwise
in the northern and anticlockwise in the southern hemisphere. We choose a small number of points on the solution grid as in figure 1. The Poincaré section is produced by monitoring values of the field at a two chosen control points $A$ and $C$ to check when these values pass through a zero in the north and south hemisphere. Each zero represents the passing of the wave, since they are alternately opposite in magnetic sign. Strictly, for a Poincaré section one should only record crossings in one direction. Here we record both, negative to positive or vice versa. This is essentially to aid the visualisation as will be shown later, we treat positive and negative regions of the field as individual waves whereas a strict interpretation would be that they represent together a single wave. To project the Poincaré section to a subspace low enough for visualisation we only record data values at two data points, $B$ and $D$, which are recorded as tuples of three elements. Since the magnetic field $B$ is a vector in axisymmetric geometry it has two components. The first is the poloidal component, the magnitude of the field in the meridional plane, the second is the toroidal representing the field in the azimuthal direction. The third value is the time recorded when the values of the field at $A$ (for $B$) and $C$ (for $D$) are zero. Note that because the zero crossing was monitored in both directions, the attractors obtained are symmetric with respect to the origin whenever symmetry is preserved. Therefore, we observe that whenever there is a periodic solution, the period is an even number, whenever the solution is doubly periodic (torus) the two loops are commensurate, etc. It is important to note that both control and data points are symmetric with respect to the equator. Just after the value of $\alpha$ at which the dynamo waves start to appear, solutions are symmetric about the equator, this continues until symmetry is broken in a bifurcation. Our choice of points allows our visualisation to be sensitive to this symmetry breaking effect.

5. Results using $\alpha$ as the steering parameter

5.1. Bifurcation sequence

For the investigated model, fast-slow steering proved a powerful method for investigating the way bifurcations unfold as the parameter passes through the critical values. The parameter $\alpha$ was steered in the interval $[2,6]$ where most of the dynamics of the system related to this parameter are observed. Figure 2 represents the time development of the toroidal component of the Poincaré projection. The figure conveys the bifurcation sequence, through which the solution travels as the parameter $\alpha$ is steered in the range of interest for ascending then descending values. One immediately interesting result is the existence of hysteresis; the bifurcation sequence is different in the ascending and descending ranges of $\alpha$. The area where the red (north hemisphere) data points are observed, indicates the presence of symmetry-breaking and chaos.

**Periodic Behaviour (Limit Cycles and Tori):** The solution starts as a limit cycle represented by a one dimensional loop, and we observe the evolution of two point solutions representing the intersection of its orbit with the applied hyperplane. This limit cycle develops into a torus, or a two dimensional loop, through a Neimark-Sacker bifurcation (equivalent to a secondary Hopf in state space). The intersection of the torus with the Poincaré projection corresponds to a commensurate pair of closed invariant curves. This bifurcation is observed better for decreasing $\alpha$ values, as for the latter case, there is no hidden unstable solutions — i.e. the limit cycle in the onset of the torus behaviour — because they become stable at the bifurcation point. On the other hand, when the parameter is being increased, the solution keeps following the still existing but now unstable limit cycle, before it suddenly jumps to the emerging torus attractor.

It is possible for periodic orbits to emerge on a torus via saddle-node bifurcations, as the parameter crosses the Arnold tongues bounded by them. This is how we observe the temporary disappearance of the growing torus, and its replacement by a limit cycle, manifested by $2 \times 3$ branches on the Poincaré map. Saddle-node bifurcations are dual by nature, as they create a saddle and a node. The steered solution follows the node behaviour as it is attracting. It is
Figure 2. Toroidal component time series for $\alpha \in [2 - 6]$ using $\alpha = |4\sin(\epsilon t)| + 2$ and $\epsilon = 10^{-6}$ (red curve). The north data points are in red and the south in blue. Where the solutions are symmetric the points coincide and only the south values can be displayed. The symmetry-breaking can be clearly seen in the chaos region where the red (north) data points do not coincide with the blue (south) ones. A hysteresis cycle can also be seen.

We observed that the saddle and the node switch roles depending on the direction from which the solution approaches them, i.e. depending on the steering direction. This takes place as part of the global hysteresis taking place throughout the investigated parameter range. Therefore, in actual fact, there is another hidden $2 \times 3$ periodic solution in this $\alpha$ range. This fact is backed by two observations. First, the $2 \times 3$ node limit cycle disappears for decreasing $\alpha$ values, as the steered solution follows the previously hidden saddle limit cycle, which is now attracting. Second, the examination of the torus attractor near the saddle-node bifurcation, shows that it is denser around 12 points (figure 7(b)), which means that the field is approaching the transition to two $2 \times 3$ periodic solutions, rather than just one. In both cases, the observed attracting limit cycle is the image of the hidden unstable one by symmetry with respect to the origin. Therefore, despite the ‘apparent’ breaking of origin symmetry (each of the two observed limit cycles do not show origin symmetry), this symmetry is still present in the set of all existing solutions. North-South Symmetries are discussed in section 5.3.

Chaos and 11 fold Periodic Windows: After the recovery of the torus behaviour, we observe the breaking of the torus, as the solution enters the chaotic regime. There is however, a window of periodic solutions, characterised by 11 fold branches (the case shown in the figure shows $4 \times 11$ branches), which appears shortly after the emergence of chaotic behaviour. For other configurations of this model, various 11 fold branches were observed, (e.g. a $3 \times 5 \times 11$ periodic solution is reported in [38]). These orbits can display very delicate structure in phase space. This window of limit cycles does not appear for the decreasing $\alpha$ values, due to the previously mentioned global hysteresis.
5.2. Bifurcation tracking in state space

(a) $\alpha \in (2.0, 3.5)$: One-dimensional limit cycle. The loop looks rather like a spiral (growing diameter), due to the continuous steering.

(b) $\alpha \in (3.5, 4.8)$: Solution evolving on a torus topology, this solution is 1M pts in size.

(c) subset of 3(b): Building block pattern of the torus solution, a subset of the previous attractor (100K pts).

(d) $\alpha \in (4.8, 5.0)$: $2 \times 3$ periodic solution forming on top of the torus. window inside the chaos onset. This solution retains a torus topology.

(e) $\alpha \in (5.8, 5.9)$: $4 \times 11$ periodic solution inside the chaos onset. The ghost of a torus can still be observed.

(f) $\alpha \in (5.9, 6.0)$: Chaotic attractor.

Figure 3. We are using four data points, two toroidal values and two poloidal. We are visualising the data in a four-dimensional projection of phase space. We used colour to show the second toroidal value. These solutions correspond to the evolution of the system as the parameter is steered in the indicated ranges.

To actually visualise what is happening to the topology of solutions as the system travels through the described bifurcations, the Poincaré section, which served as a dimension reduction technique to show the continuation of the solutions, was removed. The recorded toroidal and poloidal components of the two data points are coupled in a four-dimensional space, with the fourth dimension represented as colour. This enables a more detailed topology tracking, and provides added information about the global dynamical behaviour of the system, which cannot be induced from tracking a single data point. The generic choice of the pair of sampling points allows us to draw conclusions about the global dynamics in the whole spacial domain. In other words, this enables us to investigate whether the dynamics of the system at each grid point are consistent for a given parameter configuration. Results showed that the data points are indeed dynamically synchronous (as will be demonstrated in section 5.3 the grid points exhibit spacio-temporal synchronisation as well).

Figure 3(a) shows the first observed attractor; a limit cycle. Due to the continuous steering of
the parameter $\alpha$ in the range $[2.0-3.5]$, the cycle is observed to be slowly growing in diameter as a spiral. For a static configuration in this range, a single line loop that closes on itself is observed. Figure 3(b) shows the appearance of a torus via a supercritical secondary Hopf bifurcation at $\alpha \simeq 3.5$. This behaviour persists through the range $\alpha \in [3.5 - 4.8]$. Two thin edges, where the folding of the torus takes place, are also observed. Figure 3(c) represents the building pattern of the torus. This pattern is obtained by taking a subset of the evolution of the solution in the range $\alpha \in [3.5 - 4.8]$, which is 100K points in size instead of the 1M points sampled to demonstrate the torus behaviour in the whole range. The torus is built by the slow rotation of this pattern. Despite the apparent overall origin symmetry in the torus, the solution develops this symmetry as this pattern rotates. Theoretically, the full symmetry is revealed when this block pattern is dense on the torus. Unfortunately, this requires an infinite integration time. Figure 3(d) represents the saddle-node bifurcation due to the crossing of an Arnold tongue of toroidal/poloidal synchronisation. This bifurcation leads to phase locking and, consequently, the emergence of a limit cycle on the torus. As mentioned before, the loss of symmetry here is only ‘apparent’, because of the existence of another hidden unstable solution, which becomes stable and hence detectable for the opposite steering direction. The difference between this limit cycle and the pattern in figure 3(c) is a denser edge on the stripe, where the limit cycle is reached. Figures 3(f) (1M pts) represents the chaotic attractor over the range $\alpha \in [5.9 - 6.0]$. The shape of the torus (ghost torus) can still be seen but now the trajectories do not close and we have sensitive dependence on initial conditions. Figure 3(e) (300K pts) illustrates the $4 \times 11$ periodic window discussed above. It can be observed that this periodic solution develops on a topology equivalent to the previously broken torus.

It is worth emphasising that to obtain these numerical approximations of the attractors, the solution behaviour was tracked for ranges of parameter values, rather than monitoring the evolution of static configurations of the model. This was possible thanks to the topological equivalence of successive attractors in ranges where there are no stability transitions. Another interesting result, for this particular model, is the fact that the geometrical properties of the attractors were also preserved throughout the explored range, which made easy the task of separating them into sub-solutions of distinct behaviour. This variant behaviour however, developed around a unified topological skeleton; the torus.

5.3. North-south symmetries:

![Figure 4](image_url)

**Figure 4.** The solutions are travelling waves that migrate from the side nearest the rotation axis to the opposite side. In the case of (anti)symmetric solutions the files are ($-$) equal about the equator. The solution on the left is symmetrical, while the one on the right is taken after a bifurcation which breaks north-south symmetry.

Figure 5(a) represents solution behaviour throughout the range $\alpha \in [2.0 - 6.0]$. The torus, acting as a topological skeleton, can be seen within the chaotic attractor and within the torus.
regions of higher density. Regions of higher density outside the torus correspond to the $4 \times 11$ periodic window, which evolve on a similar but ‘bigger’ torus. An overall origin symmetry is also conveyed. Figure 5(b) is the result of superimposing the northern and southern attractors. For this figure, the sampling points were chosen symmetrical with respect to the equator. Solutions are demonstrated to exhibit an equator anti-symmetry, i.e. data points in the northern hemisphere, with the same toroidal component as their counterpart in the southern hemisphere, have a sign-inversed poloidal component.

The use of fast-slow steering, in which parameters are coupled with time, enabled us to investigate the development of these spatial symmetries in more detail by looking at the time series of the solution. Comparison between the two hemispheres time series data, revealed a temporal relationship between the two solutions. The component signals exhibited a constant phase difference between north and south. In particular, the toroidal components were always in phase, and the poloidal ones had a phase difference of $1 \pi$. This phase coupling persisted through the whole explored steering interval, even in the chaotic region where local symmetries are broken. Amplitudes match in the periodic regions resulting in an exact symmetry (figures 6(a) and 6(b)), and differ in the chaotic region where the local symmetries are broken (figures 6(c) and 6(d)), and only a global symmetry is prevalent. Consequently, solutions in the chaotic range do not show spatial symmetry about the equator at all times. However hemisphere coupling is maintained in phase. This is an interesting result, because even if these symmetries existed at discrete parameter values (e.g. using quasi-static steering), their persistence as the system evolves through a range of parameter values cannot be easily induced, in particular, if that range contains chaotic attractors. Through allowing attractor evolution, fast-slow steering allows the separation of the spatial symmetries into their one-dimensional time components, and hence a simpler way of tracking symmetries and the mechanisms governing symmetry-breaking. This might not be of much benefit when the phase space is of low dimension, however, when phase space is four-dimensional or higher, visualising the spatial symmetries can be more of a challenge. Figures 6(e) and 6(f) convey the correlation between solutions of the two hemispheres in a more compact way. For the toroidal component, the data points representing the north-south correlation function, lie on the line $x = y$ (periodic range) or scattered around it (chaotic range). Similarly, for the poloidal data, the correlation function fits on the curve $x = -y$ due to the phase difference of $1 \pi$. 

**Figure 5.** Attractors superimposed
(a) North/south toroidal time series in periodic range: phase and amplitude periodic range: constant phase difference synchronised, solution has spatial symmetry at all times (complete synchronisation).

(b) North/south poloidal time series in periodic range: constant phase difference of $\pi$, and matching amplitudes (complete synchronisation).

(c) North/south toroidal time series in chaotic range: synchronised in phase but not in amplitude (weak synchronisation).

(d) North/south poloidal time series in chaotic range: constant phase difference of $\pi$, and mismatching amplitudes (weak synchronisation).

(e) Toroidal: complete (red) / phase-only (blue) synchronisation.

(f) Poloidal: complete (red) / phase-only (blue) synchronisation.

Figure 6. North-south symmetries

5.4. Fast-slow steering advantage

Fast-slow steering can offer a number of advantages over quasi-static steering. Fast-slow steering automates the process of parameter updating, where the user updates parameter trajectories rather than individual parameter values, hence reducing checkpointing frequencies. The user can monitor the system evolution as the bifurcations unfold, which provides more insight on the transition mechanisms during bifurcation occurrence. Furthermore, the slow steering of the
(a) Saddle-node bifurcation giving rise to two periodic solutions: a saddle (unstable) and a node (stable) $2 \times 3$ limit cycles. This is observed on the torus before the bifurcation information cannot be directly extracted from (dense). The appearance of only one of the possible period-3 orbits seems to be linked to hysteresis. One orbit is selected on the increasing $\alpha$ and the other on the decreasing.

(b) The parameter $\alpha$ is linked to time, thus we unfold the bifurcation. The hidden saddle is observed on the torus before the bifurcation (dense). The appearance of only one of the possible period-3 orbits seems to be linked to hysteresis. One orbit is selected on the increasing $\alpha$ and the other on the decreasing.

(c) The appearance of a $4 \times 11$ periodic window after the breaking of the torus.

(d) Three-dimensional view describing how the $11$ fold periodic solution emerges from the broken torus.

Figure 7. Using fast-slow steering to unfold the bifurcation

parameter vector through consecutive attractors cancels the effect of transient delays, caused by repeated initialisation. Comparing it to continuation, fast-slow steering provides a more intuitive tool for studying dynamical behaviour, because it conveys this information in a more conceivable visual format, and offers the user more experimental freedom during the simulation run-time. Fast-slow steering can be easily implemented for already available simulations lacking continuation codes which would otherwise be developed from scratch for the specific simulation. Typically, with small coding effort, simulations can be augmented with steering routines in the context of many available steering environments.

Most importantly, fast-slow steering allows the user to unfold the bifurcation, and investigate the mechanisms by which emerging attractors replace the pre-bifurcation ones. This information is not always easily induced from just looking at static snapshots of the system (figure 7). Furthermore, when the behaviour of orbits is affected by the previous evolution of the system, such as in the case of the hysteresis, steered solutions can be made to reveal results of accumulative nature.
6. Conclusions
We introduced a novel approach to computational steering, which is shown to take advantage of system continuability with respect to the parameter vector, and can be used to unfold bifurcating solutions across a non-hyperbolic points. This steering technique is best suited for dissipative systems, where the notion of an attracting solution is used to track its topological development in a suitable observation space. The emergence of structures at bifurcation points, such as the appearance of strange attractors after the breaking of tori, can be observed as an evolutionary process to gain insight into the transition mechanisms. The fast-slow steering method reveals the dynamic details of the bifurcation sequence that are difficult to see when building up solutions at fixed parameter values. Therefore, we believe that it is a good basis for parameter space exploration in dynamical systems.

The use of controllable trajectories allows an efficient exploration of parameter space, and enables the automation of the steering process. It was also demonstrated how, using this method, spacial symmetries can be decomposed into their time components and hence easily monitored. The concept of fast-slow steering is simple and can be easily implemented in existing simulation codes. Computational scientists who do not have a detailed understanding of the mathematical theory of bifurcations, can use this method for the investigation of dynamical systems using their domain-specific knowledge the dynamics of their systems, and without the need for a deep mathematical grasp of bifurcation theory or continuation methods.

In terms of the actual example studied in this paper, the axisymmetric dynamo in torus geometry, our method reproduces the results obtained with quasistatic steering given in [38] and in [39]. However our method reveals details of the transitions between different solutions that are very difficult, if not impossible, to see via quasistatic steering. This may be due to the ability of the fast-slow method to detect the effects of unstable solution branches on the evolution of stable solutions. In time-stepping methods, unstable solutions are very difficult to detect because the slightest numerical noise destroys them. Our method therefore permits comparison of high-dimensional models with low-dimensional models where methods of analysis exist which can track unstable solutions. This is very important in the study of fluid dynamics and magnetohydrodynamics, where nearly all our solution methods involve some dimensional reduction, the most drastic being from the infinite dimensional state space of partial differential equations to the finite dimensional spaces of numerical discretisations or lattice methods.

In future work we intend to apply more rigorous methods of dimensional reduction from the data produced by the numerical solutions. We have used in our current work, methods based on a knowledge of the behaviour of the dynamo solutions as travelling waves but we would like to use more general methods of model reduction. We are also investigating the use of fast-slow steering in systems where we do not know the governing equations, this is a key area for our method as the conventional continuation methods cannot work with these systems. We will also investigate the extension of our method to fully three-dimensional fluid flows by linking it to simulations on parallel computers. In these cases, efficient methods of dimensionality reduction will be essential for the visualisation used in our methods.

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