Stability and Convergence of a Randomized Model Predictive Control Strategy

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Abstract—RBM-MPC is a computationally efficient variant of model predictive control (MPC) in which the random batch method (RBM) is used to speed up the finite-horizon optimal control problems at each iteration. In this article, stability and convergence estimates are derived for RBM-MPC of unconstrained linear systems. The obtained estimates are validated in a numerical example that also shows a clear computational advantage of RBM-MPC.

Index Terms—Error estimates, model predictive control (MPC), random batch method (RBM), receding horizon control, stability.

I. INTRODUCTION

Model predictive control (MPC) is a well-established and widely used method to control complex dynamical systems, see, e.g., [1], [2], and [3] for an overview of the large body of research in this area. MPC requires the real-time solution of a sequence of optimal control problems (OCPs) on a finite time horizon, which can be computationally demanding. This is, for example, the case when the model is the result of the (spatial) discretization of a partial differential equation (PDE) or in the simulation of interaction particle systems.

One recently-proposed numerically-efficient approximation method is the random batch method (RBM) [4], which is closely related to the stochastic algorithms, such as stochastic gradient descent (SGD). In the RBM-dynamics, a random subset/batch of interconnections between degrees of freedom (DOFs) is considered during small time intervals.

This can reduce the computational cost significantly and leads to a good approximation of the original dynamics when these time intervals are chosen sufficiently small, see, e.g., [4]. Recently, this idea has been extended to infinite-dimensional systems [5]. RBM-constrained OCPs have been analyzed in [6].

The RBM can be used to speed up the solution of the finite-horizon OCPs in MPC. The feedback nature of MPC also makes more robustness against the accumulating error in the RBM approximation (see, e.g., [6]). The effectiveness of this combination of model predictive control with random batch method (RBM-MPC) for nonlinear interacting particle systems has been demonstrated in [7], but a rigorous stability and convergence analysis is still missing.

The RBM in RBM-MPC fulfills a similar role as the reduced-order models (ROM) in MPC based on ROMs. There has been research on stability guarantees for MPC based on ROMs in constrained linear systems, see, e.g., [8] and [9]. The RBM is typically easier to apply than ROM techniques, but the analysis of RBM-MPC is nonetheless involved due to the stochasticity introduced by the RBM.

In this article, we provide the first rigorous analysis of the RBM-MPC algorithm. Our analysis is limited to the unconstrained linear quadratic setting and, thus, extends the open-loop analysis from [6] to a closed-loop setting. The obtained error estimates demonstrate the influence of the different parameters in RBM-MPC on the expected performance, and the obtained convergence rates are validated in a numerical example.

The rest of this article is structured as follows. The RBM-MPC algorithm is presented in Section II. After the introduction of preliminary estimates and notation in Section III, the stability and convergence of RBM-MPC are proven in Sections IV and V, respectively. The convergence rates are validated in a numerical example in Section VI. Finally, the discussions in Section VII conclude this article.

We will use the following notation. The (Euclidean) norm of a vector $x \in \mathbb{R}^n$ is $|x| = \sqrt{x^\top x}$. For a matrix $M \in \mathbb{R}^{n \times m}$, we write $\|M\| = \sup_{x \neq 0} |Mx|$. For symmetric $M \in \mathbb{R}^{n \times n}$, $M \succeq 0$ or $M \preceq 0$ indicates that $M$ is positive semidefinite or positive-definite, respectively. For $M \succ 0$, $|x|_M = \sqrt{x^\top Mx}$.

II. RBM-MPC ALGORITHM

The RBM-MPC algorithm analyzed in this article is a way to approximate the control $u_\infty(t)$ that minimizes

$$J_\infty(u) = \int_0^\infty \left( |x(t)|_Q^2 + |u(t)|_W^2 \right) dt$$

subject to the dynamics

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

where the state $x(t)$ evolves in $\mathbb{R}^n$ starting from the initial condition $x_0$, the control $u(t)$ evolves in $\mathbb{R}^m$, $0 \leq t < \bar{t}$, and $W \in \mathbb{R}^{m \times m}$, $A \in \mathbb{R}^{n \times n}$, and $B \in \mathbb{R}^{n \times m}$. It is assumed that $(A, B)$ is stabilizable and $(A, Q)$ is detectable.
The OCP (1) and (2) can be approximated using MPC. In MPC, two parameters arise: The prediction horizon $T$ and the shorter control horizon $\tau$. Set $\tau_i := i\tau$ (with $i \in \mathbb{N}$) and let $u_T^J(t; x_{i-1}, \tau_i)$ and $x_T^J(t; x_{i-1}, \tau_i)$ denote the control and state trajectory that minimize
\[ J_T(u_T; x_{i-1}, \tau_i) = \left| x_T^J(\tau_i + T) \right|^2 + \int_{\tau_i}^{\tau_i+T} \left| x_T^J(t) \right|_Q^2 + \left| u_T(t) \right|_W^2 \, dt \] (3)
where $F > 0$ and $x_T(t)$ fulfills $t \in [\tau_i-1, \tau_i-1 + T]$.

When $n$ is large and $A$ not sparse, finding $u_T^J(t; x_{i-1}, \tau_i)$ and $x_T^J(t; x_{i-1}, \tau_i)$ is computationally demanding. We, therefore, replace $A$ by a randomized sparser matrix $A_R$.

The randomized matrix $A_R(\omega, t)$ is constructed as follows. First, $A$ is written as the sum of sparse submatrices $A_m$
\[ A = \sum_{m=1}^{M} A_m. \] (5)

Next, the subsets of $\{1, 2, \ldots, M\}$ are enumerated as $S_1, S_2, \ldots, S_M$ and a probability $p_m \in [0, 1]$ is assigned to each subset $S_m$ ($\omega \in \{1, 2, \ldots, 2^M\}$), such that $\sum_m p_m = 1$.

The time interval $[0, T]$ is divided into $K$ time intervals of equal length $h$. For each of the $K$ time intervals, an element $\omega_{i,k} \in \{1, 2, \ldots, 2^M\}$ of the vector $\omega$ is selected according to the probabilities $p_m$. The matrix $A_R$ is now defined as follows:
\[ A_R(\omega, t) = \sum_{m \in \omega_{i,k}} A_m, \quad t \in [(k-1)h, kh). \] (6)
The scaling factors $\pi_m$ are defined, such that the expected value of $A_R(\omega, t)$ is equal to $A$. In particular, $\pi_m$ denotes the probability of having the index $m$ in the selected subset
\[ \pi_m := \sum_{\omega \in \{1, 2, \ldots, 2^M\}} p_m. \] (7)

The definition of $A_R$, thus, requires that the probabilities $p_m$ are selected, such that $\pi_m > 0$ for all $m \in \{1, 2, \ldots, M\}$.

The dynamics generated by $A_R(\omega, t)$ is in expectation close to the dynamics generated by $A$ for $h$ sufficiently small (see [6]) and replacing $A$ by $A_R(\omega, t)$ reduces the computational cost when $A_R(\omega, t)$ is much sparser than $A$. Consider, therefore, the control $u_R^J(\omega, t; x_{i-1}, \tau_i)$ and state trajectory $x_R^J(\omega, t; x_{i-1}, \tau_i)$ that minimize
\[ J_R(u_R; \omega, x_{i-1}, \tau_i) = \left| x_R^J(\omega, \tau_i + T) \right|^2 + \int_{\tau_i}^{\tau_i+T} \left| x_R^J(\omega, t) \right|_Q^2 + \left| u_R(t) \right|_W^2 \, dt \] (8)
where $x_R(t)$ fulfills $t \in [\tau_i-1, \tau_i-1 + T]$.
\[ x_R^J(\omega, t) = A_R(\omega, t - \tau_i) x_R(\omega, t) + B u_R(t) \]
\[ x_R(\omega, \tau_i - 1) = x_{i-1}. \] (9)

It has been proven in [6] that $u_R^J(\omega, t; x_{i-1}, \tau_i)$ is (in expectation) close to $u_T^J(t; x_{i-1}, \tau_i)$ for $h$ small enough, see also Section III. Because $u_R^J(\omega, t; x_{i-1}, \tau_i)$ is used to control the dynamics generated

by $A$, consider also the solution $y_R^J(\omega, t; x_{i-1}, \tau_i)$ of
\[ y_R^J(\omega, t) = A y_R^J(\omega, t) + B u_R^J(t; x_{i-1}, \tau_i) \]
\[ y_R(\omega, \tau_i - 1) = x_{i-1}. \] (10)

where $y_R^J(\omega, t)$ denotes $y_R^J(\omega, t; x_{i-1}, \tau_i)$ for brevity.

The RBM-MPC algorithm now computes the control $u_{R,M}(t)$ and state trajectory $x_{R,M}(t)$ on $[0, \infty)$ as follows:
1) Initialize $x_{R,M}(0) = x_0$ and $i = 1$;
2) Select a random vector $\omega_i \in \{1, 2, \ldots, 2^M\}$;
3) Compute $u_R^j(\omega_i, t; x_{R,M}(\tau_i - 1), \tau_i)$ and $y_R^j(\omega_i, t; x_{R,M}(\tau_i - 1), \tau_i)$ on $[\tau_i - 1, \tau_i - 1 + T]$;
4) Set $u_{R,M}(t) = u_R^j(\omega_i, t; x_{R,M}(\tau_i - 1), \tau_i)$ and $x_{R,M}(t) = y_R^j(\omega_i, t; x_{R,M}(\tau_i - 1), \tau_i)$ on $[\tau_i - 1, \tau_i]$; and
5) Set $i = i + 1$ and go to Step 2.

Note that RBM-MPC reduces to standard MPC when $A_R(\omega, t) = A$ and that $x_{R,M}(\tau_i)$ depends on the previously selected sequences $\omega_j$ with $j \leq i$, which are denoted by
\[ \Omega := (\omega_1, \omega_2, \ldots, \omega_i) \in \{1, 2, \ldots, 2^M\}^i. \] (11)

The construction of the matrix $A_R(\omega, t)$ leaves freedom in the choice of the submatrices $A_m$, the probabilities $p_m$, and the grid spacing $h$. As for the submatrices $A_m$, splittings of the form (5) are standard in operator-splitting methods, which are well-established in numerical analysis, see, e.g., [10]. The specific choice of the $A_m$’s is often guided by physical insight. In many finite-dimensional examples, each $A_m$ represents an interaction between two degrees of freedom (DOFs), so that $M \leq n(n-1)/2$. Regarding the grid spacing $h$, note that the estimates in Theorems 1 and 2 below are proportional to $\sqrt{\text{Var}[A_R]}$, where
\[ \text{Var}[A_R] := \sum_{\omega=1}^{2^M} \left| A - \sum_{m \in S_\omega} A_m \right| \sum_{\omega=1}^{2^M} \left| A - \sum_{m \in S_\omega} A_m \right|^2 p_\omega. \] (12)

Reducing $\text{Var}[A_R]$, thus, enables us to use a larger step size $h$. Finally, note that assigning nonzero probabilities $p_m$ to larger subsets $S_\omega$ reduces $\text{Var}[A_R]$, but will also make $A_R(\omega, t)$ less sparse and, thus, potentially increases the computational cost, see [6, Sec. 2.3] for further discussions and examples.

Error estimates for the RBM, as in Section III and in Theorems 1 and 2, require a uniform quasidissipativity bound on $A_R$ in the tradition of [11], i.e., we fix a $\mu_R \geq 0$ such that
\[ x^T A_R(\omega, t)x \leq \mu_R |x|^2 \] (13)
for all $x \in \mathbb{R}^n$, $\omega \in \{1, 2, \ldots, 2^M\}^K$, and $t \in [0, T]$. Note that this condition implies that the eigenvalues of the symmetric part of $A_R(\omega, t)$ do not exceed $\mu_R$.

Remark 1: Note that $\mu_R = 0$ when $x^T A_R x \leq 0$ for all $x \in \mathbb{R}^n$ and all $m \in \{1, 2, \ldots, M\}$, i.e., when all $A_m$ are dissipative. The latter condition can be achieved in many examples, see, e.g., Section VI and [6, Sec. 4].

Remark 2: Condition (13) readily extends to a setting in which the $A_m$ are quasidissipative bounded operators on a Banach space, see, e.g., [11]. However, the appearance of the operator norm $\| \cdot \|$ in (12) is an indication that extending the RBM to such setting is not trivial, see, e.g., [5].

III. Preliminary Estimates

In the following, $C$ denotes a constant depending only on $A$, $B$, $Q$, $W$, and $F$. The notation $C_T$ indicates that the constant also depends on $T$. The constants $C$ and $C_T$ may vary from expression to expression,
e.g., $(\|A\| + T)C_T \leq C_T$. Because we are interested in the limit $h\text{Var}[A_R] \to 0$, we will only consider the lowest power of $h\text{Var}[A_R]$ in our estimates. The following lemma now directly follows from (13).

**Lemma 1:** The solution $x_R(\omega_i, t)$ of (9) satisfies for all $\tau_{i-1} \leq t \leq \tau_{i-1} + T$ and all $\omega_i \in \{1, 2, \ldots, 2^M\}^K$

$$|x_R(\omega_i, t)| \leq C_T e^{hR(t - \tau_{i-1})} (|\omega_i|_1 + |u_R(\omega_i)|_{L^2(\tau_{i-1}, t; \mathbb{R}^p)}).$$

(14)

**Proof:** Differentiate $|x_R(\omega_i, t)|^2$ using (9), use (13), integrate from $\tau_{i-1}$ to $t$, apply Cauchy–Schwarz in $L^2(\tau_{i-1}, t; \mathbb{R}^p)$ and then Gronwall’s lemma.

Our analysis will use Riccati theory. For the infinite-horizon OCP (1) and (2), let $P_\infty$ denote the (unique) symmetric positive-definite solution of the algebraic Riccati equation

$$A^T P_\infty + P_\infty A - P_\infty B W^{-1} B^T P_\infty + Q = 0. \quad (15)$$

It is, then, well-known that, e.g., [12, Section 5.1]

$$u^*_R(t) = -W^{-1} B^T P_\infty x^*_R(t). \quad (16)$$

Therefore, $x^*_R(t)$ follows the dynamics generated by

$$A_{\infty} = A - B W^{-1} B^T P_\infty \quad (17)$$

which is stable, i.e., there exist $M_{\infty} \geq 1$ and $\mu_{\infty} > 0$

$$\|e^{A x}\| \leq M_{\infty} e^{-\mu_{\infty} t}. \quad (18)$$

For the finite-horizon OCP (3) and (4), let $P_2(t)$ solve the Riccati differential equation

$$-\dot{P}_2(t) = A^T P_2(t) + P_2(t) A^T - P_2(t) B W^{-1} B^T P_2(t) + Q, \quad P_2(T) = F \quad (19)$$

on $t \in [0, T]$. It is well-known that, see, e.g., [12, Section 5.2]

$$u^*_2(t; x_{i-1}, \tau_{i-1}) = - W^{-1} B^T P_2(t - \tau_{i-1}) x^*_2(t; x_{i-1}, \tau_{i-1}). \quad (20)$$

For the randomized OCP (8) and (9), let $P_R(\omega_i, t)$ solve the randomized Riccati differential equation on $[0, T]$

$$\dot{P}_R(\omega_i, t) = A_R(\omega_i, t)^T P_R(\omega_i, t) + P_R(\omega_i, t) A_R(\omega_i, t) - P_R(\omega_i, t) B W^{-1} B^T P_R(\omega_i, t) + Q, \quad P_R(\omega_i, t) = F. \quad (21)$$

Similarly as in (20), it holds that

$$u^*_R(\omega_i, t; x_{i-1}, \tau_{i-1}) = - W^{-1} B^T P_R(\omega_i, t - \tau_{i-1}) x^*_R(\omega_i, t; x_{i-1}, \tau_{i-1}). \quad (22)$$

The following lemma shows that $P_2(t) \to P_\infty$ for $T \to \infty$.

**Lemma 2:** If $(A, B)$ is stabilizable, $(A, Q)$ is detectable, and $\mu_{\infty}$ is as in (18), then, for all $t \in [0, T]$

$$\|P_2(t) - P_\infty\| \leq C \|F - P_\infty\| e^{-2\mu_{\infty}(T - t)}. \quad (23)$$

**Proof:** See [13]. A shorter proof for the case that $(A, B)$ is controllable and $(A, Q)$ is observable is given in [14].

**Remark 3:** Because $\|P_\infty\| \leq C$, Lemma 2 implies that $\|P_2(t)\| \leq C$ for all $t$.

Let $V$ be a vector space. The expected value of a random variable $X : \{1, 2, \ldots, 2^M\}^K \to V$ depending on $\omega_i$ is

$$\mathbb{E}_i[X] = \sum_{\omega_i \in \{1, 2, \ldots, 2^M\}^K} X(\omega_i) p(\omega_i). \quad (24)$$

where $p(\omega_i) = p_{i1} p_{i2} \cdots p_{ik}$. The expected value of a random variable $X : \{1, 2, \ldots, 2^M\}^K \to V$ is denoted by

$$\mathbb{E}[X] = \sum_{\omega_i \in \{1, 2, \ldots, 2^M\}^K} X(\omega_i) p(\omega_i). \quad (25)$$

For the expected value of a random variable $X(\Omega_i)$ w.r.t. the last $\omega_i$, we write $\mathbb{E}_i[X(\Omega_{i-1})]$ to indicate that the result depends on $\Omega_{i-1}$. For random variables $X(\Omega_i)$ and $Y(\Omega_i)$,

$$\mathbb{E}[XY] \leq \sqrt{\mathbb{E}[X^2] \mathbb{E}[Y^2]} \quad (26)$$

$$\sqrt{\mathbb{E}[(X + Y)^2]} \leq \sqrt{\mathbb{E}[X^2]} + \sqrt{\mathbb{E}[Y^2]} \quad (27)$$

Similar expressions hold for the expectation $\mathbb{E}_i$.

For $0 \leq s < t \leq T$, let $S_R(\omega_i, t, s)$ be the evolution operator generated by $A_R(\omega_i, t)$, i.e., $S_R(\omega_i, t, s)x = x(\omega_i, t)$ where

$$\dot{x}(\omega_i, t) = A_R(\omega_i, t)x(\omega_i, t), \quad x(\omega_i, s) = x. \quad (28)$$

The following lemma from [6], then, shows that $S_R(\omega_i, t, s)$ is (in expectation) close to $e^{-A(t-s)}$ when $h\text{Var}[A_R]$ is small.

**Lemma 3:** Let $\text{Var}[A_R]$ and $\mu_R$ be as in (12) and (13) and $0 \leq s \leq t \leq T$, then

$$\mathbb{E}_i[\|S_R(\omega_i, t, s) - e^{-A(t-s)}\|^2] \leq C_T e^{\mu_R(t-s)} h\text{Var}[A_R]. \quad (29)$$

**Proof:** See [6, Theorem 1 and Corollary 1].

With Lemma 3, it is also possible to bound the difference between controlled state trajectories.

**Lemma 4:** Let $u_R : \{1, 2, \ldots, 2^M\}^K \times [\tau_{i-1}, \tau_{i-1} + T] \to \mathbb{R}^n$ be a random control and $x_{i-1} : \{1, 2, \ldots, 2^M\}^K \to \mathbb{R}^n$ a random initial condition. If $x_R(\omega_i, t)$ and $y_R(\omega_i, t)$ satisfy

$$\dot{x}_R(\omega_i, t) = A x_R(\omega_i, t) + B u_R(\omega_i, t) \quad (30)$$

$$\dot{y}_R(\omega_i, t) = A_R(\omega_i, t - \tau_{i-1}) y_R(\omega_i, t) + B u_R(\omega_i, t) \quad (31)$$

then

$$\mathbb{E}[\|u_R(t; x_i, \tau_i) - x_R(t; x_i, \tau_i)\|^2] \leq C_T e^{2\mu_R(t-\tau_i)} \times h\text{Var}[A_R] \left( \max_{\omega_i} \|x_{i-1}(\omega_i)\| + \max_{\omega_i} \|u_R(\omega_i)\|_{L^2(\tau_{i-1}, t; \mathbb{R}^p)} \right)^2. \quad (33)$$

**Proof:** By a slight modification of [6, Th. 2] in which the random initial condition was not considered.

Although Riccati theory will be used in the analysis, the OCPs in Section II are typically more efficiently solved by a gradient-based algorithm, especially when $n$ is large.

**IV. STABILITY ANALYSIS**

For the stability result in Theorem 1 at the end of this section, we first establish two lemmas. Consider $i \in \mathbb{N}$ and $t \in [\tau_{i-1}, \tau_i]$. Because $x_{R-M}(\Omega_i, t)$ satisfies (10)

$$\dot{x}_{R-M}(\Omega_i, t) = A_{x-M} x_{R-M}(\Omega_i, t) + r(\Omega_i, t) \quad (34)$$

with $A_{x-M}$ as in (17) and

$$r(\Omega_i, t) = BW^{-1} B^T P_{x-R-M}(\Omega_i, t) + Bu_{x-M}(\omega_i, t) \leq BW^{-1} B^T (P_{x-M} - P_R(t - \tau_{i-1})) x_{R-M}(\Omega_i, t) \quad (35)$$

For the expected value of a random variable $X(\Omega_i)$ w.r.t. the last $\omega_i$, we write $\mathbb{E}_i[X(\Omega_{i-1})]$ to indicate that the result depends on $\Omega_{i-1}$. For random variables $X(\Omega_i)$ and $Y(\Omega_i)$,
+ B \left( \begin{bmatrix} W^{-1} B^T P_T (t - \tau_{i-1}) x_{R-M} (\Omega_1, t) + u_{R}^i (\omega_1, t) \end{bmatrix} \right) \right] \}

\left. \right\rangle_{\omega_1}^{g(\Omega_1, t)} \quad \text{(35)}

where \( u_{R}^i (\omega_1, t) \) denotes \( u_{R}^i (\omega_1, t; x_{R-M} (\Omega_1, \tau_{i-1}, \tau_{i-1})) \) for brevity. The first auxiliary lemma is now as follows.

**Lemma 5:** Let \( P_R (\omega_1, t) \) and \( P_T (t) \) satisfy (21) and (19), then, for \( t \in [0, T] \)

\[
E_i \| P_R (t) - P_T (t) \| \leq C_T e^{2 \mu_r T} \sqrt{\operatorname{Var}[\varphi R]}.
\]

**Proof:** We will only prove (36) for \( t = 0 \). The result for \( t > 0 \) can be obtained similarly. By definition,

\[
\| P_R (\omega_1, 0) - P_T (0) \| = | \bar{\omega}(\omega_1) \|^T (P_R (\omega_1, 0) - P_T (0)) \bar{\omega}(\omega_1)
\]

\[
= \left| J_R (\omega_1, u^R_1 (\omega_1); \bar{\omega}(\omega_1), 0) - J_T (u^T_2 (\omega_1); \bar{\omega}(\omega_1), 0) \right|
\]

where \( \bar{\omega}(\omega_1) = \arg \max_{\omega_1} \omega_1 \right)\right\| P_R (\omega_1, 0) - P_T (0) \| = \left| J_R (\omega, u^R_1 (\omega_1); \bar{\omega}(\omega_1), 0) - J_T (u^T_2 (\omega_1); \bar{\omega}(\omega_1), 0) \right| \quad \text{(37)}

because \( u_{R}^i (\omega_1, t) = u_{R}^i (\omega_1) \) and \( u_{R}^i (\omega_1) \) minimizes \( J_R (\omega_1, \omega_1) \). Similarly, when \( J_R (\omega_1, u^R_1 (\omega_1)) < J_T (u^T_2 (\omega_1)) \)

\[
\| P_R (\omega_1, 0) - P_T (0) \| = \left| J_T (u^T_2 (\omega_1); \bar{\omega}(\omega_1), 0) - J_T (u^T_2 (\omega_1)) \right| \quad \text{(38)}
\]

because \( u_{R}^i (\omega_1, t) = u_{R}^i (\omega_1) \) and \( u_{R}^i (\omega_1) \) minimizes \( J_T (\omega_1, \omega_1) \). Combining (38) and (39), thus, shows that

\[
\| P_R (\omega_1, 0) - P_T (0) \| \leq \left| J_R (\omega_1, u^R_1 (\omega_1); \bar{\omega}(\omega_1), 0) - J_T (u^T_2 (\omega_1)) \right|
\]

\[
\leq \left| \left( y^R (\omega_1, T) \right) \right|^2 \quad \text{(40)}
\]

where \( \langle \cdot, \cdot \rangle_L \) denotes the \( L^2 \)-inner product on \([0, T] \) and \( e_{R} (\omega, t) \) is \( y^R (\omega_1, t) - x_{R} (\omega_1, t) \). Furthermore, when \( J_T (u^T_2 (\omega_1)) \leq J_R (\omega_1, u^R_1 (\omega_1)) \)

\[
\| x_{R} (\omega, T) \| \left( J_T (u^T_2 (\omega_1)) \right) \leq J_T (u^T_2 (\omega_1)) \leq C | \bar{\omega}(\omega_1) |^2.
\]

(41)
where $x_R(\omega, t; x_{i-1}, \tau_i)$ satisfies (9) with $u_R(t) = 0$ and $x_{i-1} = x_{R-M}(\Omega_{i-1}, \tau_{i-1})$ and the last inequality follows from Lemma 1. Lemma 4 now shows that

$$E[|x_{R-M}(\Omega_{i-1}, t) - x_R^*(t)|] \leq C\sqrt{\text{Var}[A_R]} e^{\mu_R t} \leq \frac{\sqrt{\text{Var}[A_R]}}{e^{\mu_R t}} |x_{R-M}(\Omega_{i-1}, \tau_{i-1})|.$$

To bound $|x_R^*(\omega, t)|$, note that $x_R^*(\omega, t)$ satisfies (9) with $u_R(\omega, t) = u_R(\omega, t)$ and $x_{R-M}(\Omega_{i-1}, \tau_{i-1})$. Inserting (48) into (14), thus, shows that for $t \in [\tau_{i-1}, \tau_i]$

$$|x_R^*(\omega, t)| \leq C\sqrt{\text{Var}[A_R]} e^{\mu_R(T+\tau)} |x_{R-M}(\Omega_{i-1}, \tau_{i-1})|.$$

Now insert (49), (36), and (50) into (47) to find (46).

We are now ready to prove the main stability result.

**Theorem 1:** If $(A, B)$ is stabilizable, $(A, Q)$ is detectable, and $M_\infty$ and $\mu_\infty$ are as in (18), then

$$E[|x_{R-M}(t)|] \leq M_\infty e^{-\mu_M t} |x_0|$$

where

$$\mu_M = \mu_\infty - C\|F + P_\infty e^{-2\mu_\infty(T+\tau)}\|$$

$$- C_T e^{\mu_R(2T+\tau)} e^{\mu_\infty \tau} \sqrt{\text{Var}[A_R]}.$$

**Proof:** Applying the variation of constants formula to (34), taking the norm and the expectation yields

$$E[|x_{R-M}(t)|] \leq M_\infty e^{-\mu_\infty t} |x_0| + M_\infty \int_0^t e^{-\mu_\infty (t-s)} E[|r(s)|] ds$$

where (18) has been used. Taking the norm and the expectation (first w.r.t. $\omega_{i-s/\tau+1}$ and, then, w.r.t. to the other $\omega_j$’s in (35) using Lemmas 2 and 6, it follows that:

$$E[|r(s)|] \leq C_1 E[|x_{R-M}(s)|] + C_2 E[|x_{R-M}(\tau_{i-1})|]$$

where we have introduced $C_1 = C\|F + P_\infty e^{-2\mu_\infty(T+\tau)}\|$ and $C_2 = C_T e^{\mu_R(2T+\tau)} \sqrt{\text{Var}[A_R]}$. By inserting (54) into (53) and writing $f(t) = E[|x_{R-M}(t)|]$, we obtain

$$f(t) \leq M_\infty e^{-\mu_\infty t} |x_0| + \int_0^t e^{-\mu_\infty(t-s)} \left(C_1 f(s) + C_2 f(\tau_{i-1})\right) ds.$$

Setting $\tilde{f}(t) = e^{\mu_\infty t} f(t)$, it follows that $\tilde{f}(t) \leq \tilde{F}(t)$ where

$$\tilde{F}(t) = M_\infty |x_0| + \int_0^t \left(C_1 \tilde{f}(s) + C_2 e^{\mu_\infty \tau} \tilde{f}(\tau_{i-1})\right) ds.$$

Because $\tilde{F}(t)$ is monotonically increasing and $\tilde{f}(t) \leq \tilde{F}(t)$

$$\tilde{F}(t) \leq M_\infty |x_0| + (C_1 + C_2 e^{\mu_\infty \tau}) \int_0^t \tilde{F}(s) ds.$$

By Gronwall’s lemma, we, thus, obtain that $e^{\mu_\infty t} f(t) = \tilde{f}(t) \leq \tilde{F}(t) \leq M_\infty |x_0| e^{(C_1 + C_2 e^{\mu_\infty \tau}) t}$

and the result follows.

**Remark 4:** Note that $\mu_M > 0$ will be positive for $h\text{Var}[A_R]$ and $\|F + P_\infty e^{-2\mu_\infty(T+\tau)}\|$ sufficiently small.

**Remark 5:** When $\mu_M > 0$, the RBM-MPC scheme is stabilizing with probability 1. To see this, note that Markov’s inequality and

$$\mathbb{P}[|x_{R-M}(t)| \geq \epsilon] \leq \frac{E[|x_{R-M}(t)|]}{\epsilon} \leq \frac{M_\infty e^{-\mu_M t} |x_0|}{\epsilon}.$$

Because $\mu_M > 0$, the probability that $x_{R-M}(t)$ is outside any $\epsilon$-neighborhood of the origin approaches zero for $t \to \infty$.

**V. CONVERGENCE**

We first consider the convergence of MPC. Note that $x_M(t)$ follows the dynamics generated by the $\tau$-periodic matrix

$$A_T(t) = A - BW^{-1} B^T P_T(t \mod \tau).$$

We then, have the following lemma.

**Lemma 7:** If $(A, B)$ is stabilizable, $(A, Q)$ is detectable and $M_\infty$ and $\mu_\infty$ are as in (18), then, for all $0 \leq s \leq t$,

$$\|e^{\int_0^t A_T(s) \, ds \, \sigma}\| \leq M_\infty e^{-\mu_M t}.$$

Furthermore, if $\mu_M > 0$, then

$$|x_M(t) - x_\infty(t)| + |u_M(t) - u_\infty(t)| \leq C\|F + P_\infty e^{-2\mu_\infty(T+\tau)}|x_0|.$$

**Remark 6:** Lemma 7 shows that the dynamics generated by $A_T(t)$ is stable for $T - \tau$ sufficiently large or $\|F + P_\infty\|$ sufficiently small and that $(x_M(t), u_M(t)) \to (x_\infty(t), u_\infty(t))$ for $T - \tau \to \infty$ or $\|F - P_\infty\| \to 0$.

**Proof:** Let $x(t)$ denote the solution to $\dot{x}(t) = A_T(t)x(t)$ with initial condition $x(s) = x_s$. By (17),

$$\dot{x}(t) = (A_\infty + BW^{-1} B^T (P_\infty - P_T(t \mod \tau))) x(t).$$

The variation of constants formula, thus, shows that $x(t) = e^{A_\infty(t-s)} x_s$

$$+ \int_s^t e^{A_\infty(t-\sigma)} BW^{-1} B^T (P_\infty - P_T(\sigma \mod \tau)) x(\sigma) \, d\sigma.$$

Taking norms using (18) and Lemma 2, it follows that:

$$\|x(t)\| = M_\infty e^{-\mu_\infty t} |x_0| + C\|F - P_\infty\| e^{-2\mu_\infty(T+\tau)} \int_s^t \|x(\sigma)\| \, d\sigma.$$

Applying Gronwall’s lemma and noting that the initial condition $x_s$ is arbitrary now yields (59). For the bound on $e_M(t) := x_M(t) - x_\infty(t)$ in (61), note that $\dot{x}_\infty(t) = A_\infty x_\infty(t)$ and that $x_M(t)$ satisfies (62), so that $e_M(t) = A_\infty e_M(t) - BW^{-1} B^T (P_T(t \mod \tau) - P_\infty) x_M(t)$.

Applying the variation of constants formula and taking the norm using (18), Lemma 2, and the inequality $|x_M(t)| \leq M_\infty |x_0| \leq C |x_0|$ when $\mu_M \geq 0$ by (59), it follows that:

$$\|e_M(t)\| \leq C\|F - P_\infty\| e^{-2\mu_\infty(T+\tau)} \int_0^t e^{-\mu_\infty(t-s)} ds |x_0|.$$

The bound on $e_M(t)$ follows because the remaining integral is bounded by $1/\mu_\infty \leq C$. For $u_M(t) - u_\infty(t)$, note that (20) implies that

$$\|u_M(t) - W^{-1} B^T P_T(t \mod \tau) x_M(t)\| \leq M_\infty e^{-\mu_M t} |x_0|.$$
Fig. 1. RBM-MPC control and state trajectory $u_{R-M}(\Omega_i, t)$ and $x_{R-M}(\Omega_i, t)$ for 20 realizations of $\Omega_i$ compared with $u_M(t)$, $x_M(t)$, $u_\infty(t)$, and $x_\infty(t)$ for $n=100$, $h=1$, $\tau=10$, and $T=15$. Lines for $|x_{R-M}(\Omega_i, t)|$ and $|x_M(t)|$ in Fig. 1(b) almost overlap. (a) Controls $u_{R-M}(\Omega_i, t)$, $u_M(t)$, and $u_\infty(t)$. (b) Norm of the state trajectories $x_{R-M}(\Omega_i, t)$, $x_M(t)$, and $x_\infty(t)$.

Fig. 2. Differences between the RBM-MPC state trajectory $x_{R-M}(\Omega_i, t)$, the MPC state trajectory $x_M(t)$, and the infinite horizon state trajectory $x_\infty(t)$ for $n=100$. The error bars indicate the $2\sigma$ confidence intervals estimated based on 20 realizations of $\Omega_i$. (a) Varying $h$, $T=15$, $\tau=10$. (b) $h=1$, varying $T$, $\tau=10$. (c) $h=1$, $T=40$, varying $\tau$.

Fig. 3. Differences between the RBM-MPC control $u_{R-M}(\Omega_i, t)$, the MPC control $u_M(t)$, and the infinite horizon control $u_\infty(t)$ for $n=100$. The error bars indicate the $2\sigma$ confidence intervals estimated based on 20 realizations of $\Omega_i$. (a) Varying $h$, $T=15$, $\tau=10$. (b) $h=1$, varying $T$, $\tau=10$. (c) $h=1$, $T=40$, varying $\tau$.

so that subtracting (16) shows that

$$u_M(t) - u_\infty(t) = W^{-1}B^\top(P_\infty - P_\tau(t \mod \tau))x_M(t) - W^{-1}B^\top P_\infty e_M(t).$$

Using Lemma 2 and that $|x_M(t)| \leq M_\infty|x_0| \leq C|x_0|$ for the first term, and the previously derived bound for $|e_M(t)|$ for the second, the result follows.

Now the convergence of RBM-MPC can be established.

**Theorem 2**: If $(A, B)$ is stabilizable, $(A, Q)$ is detectable, and $\mu_{R-M}$ in (52) is positive, then

$$\mathbb{E}[|x_{R-M}(t) - x_M(t)|] + \mathbb{E}[|u_{R-M}(t) - u_M(t)|]$$

$$\leq \frac{C_T}{\mu M} e^{\mu R(2T+\tau)} \sqrt{h \text{Var}[A_R|x_0|]}.$$  

(67)
Proof: Consider \( i \in \mathbb{N} \) and \( t \in [\tau_{i-1}, \tau_i) \). For the bound on \( e_{R-M}(\Omega, t) = x_{R-M}(\Omega, t) - x_M(t) \), note that \( x_M(t) \) satisfies (62), so that (35) into (34) and subtracting (62) yields
\[
\dot{e}_{R-M}(\Omega, t) = A_c(t)e_{R-M}(\Omega, t) +Bg(t),
\]
and \( e_{R-M}(\Omega, 0) = 0 \). Applying the variation of constants formula, taking the norm and the expectation, thus, shows that
\[
\mathbb{E}[|e_{R-M}(t)|] = C \int_0^t \left( e^{\int_s^t A_c(r) dr} \right) \mathbb{E}[|g(s)|] s d s.
\]
By Lemma 6, it follows that:
\[
\mathbb{E}[|g(t)|] \leq C_T e^{C_T |t-T|} \sqrt{\text{Var}[A_R] \mathbb{E}[|x_{R-M}(\tau_{i/\varepsilon})|]} + \sqrt{\text{Var}[A_R] \mathbb{E}[|x_0|]}
\]
which has it been used that \( |x_{R-M}(\tau_{i/\varepsilon})| \leq M_\infty \mathbb{E}[x_0] \leq C_0 \mathbb{E}[x_0] \) by Theorem 1 because \( \mu_{R-M} \geq 0 \). Using (59) and (70) in (69), the bound for \( e_{R-M}(\Omega, t) \) follows because the integral of \( e^{C_T |t-T|} \) is bounded by \( 1/\mu_M \).

To bound \( u_{R-M}(\Omega, t) - u_M(t) \), note that for \( t \in [\tau_{i-1}, \tau_i) \)
\[
u_{R-M}(\Omega, t) = u_R^{\tau_{i}}(\omega_i, t; x_{R-M}(\Omega_{i-1}, \tau_{i-1}), \tau_{i-1}).
\]
Subtracting (65) using the definition of \( g(\Omega, t) \) in (35) yields
\[
\nu_{R-M}(\Omega, t) - u_M(t) = g(\Omega, t) - W^{-1}B^TP(t \bmod \tau)e_{R-M}(\Omega_{i-1}, t)
\]
The bound now follows after taking the norm and the expected value, and then, using Lemma 6 to bound \( \mathbb{E}[|g(t)|] \) and the previously derived estimate for \( \mathbb{E}[|e_{R-M}(t)|] \).

Remark 7: Combining Theorem 2 and (61), one obtains estimates for \( \mathbb{E}[|x_{R-M}(\tau_{i/\varepsilon})|] + \mathbb{E}[|u_{R-M}(\Omega, t) - u_M(t)|] \).

The estimates also indicate a natural approach to tuning the parameters in RMB-MPC. First, \( T - \tau \) should be chosen such that the MPC strategy is stabilizing with sufficient margin, i.e., such that \( C^T [F - P]\mathbb{E}[|u_{R-M}(\tau_{i/\varepsilon})|] < \mu_{\infty} \). After that, \( h \) can be chosen such that \( \mu_{R-M} > 0 \) and \( h \) such that RMB-MPC leads to a sufficiently good approximation of MPC.

VI. NUMERICAL EXAMPLE

We consider a problem of the form (1) and (2) with \( n \in \{11, 101, 1001\} \) states and \( m = 1 \) input, \( A \in \mathbb{R}^{n \times n} \) is
\[
A = (n-1)^2\begin{bmatrix}
-2 & 1 & 0 & \cdots & 0 & 1 \\
1 & -2 & 1 & 0 & \cdots & 0 \\
0 & 1 & -2 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & -2 & 0 & \cdots \\
1 & 0 & 0 & \cdots & 1 & -2
\end{bmatrix}
\]
when the relative change in the control is below 10^{-5} or after 1000 iterations.

To construct the randomized matrix \( A_R(\omega_i, t) \), note that \( A \) can be written as the sum of \( M = n \) interconnection matrices as in (5), where the first \( n - 1 \) interconnection matrices \( A_{in} \) are zero except for a diagonal block of the form
\[
\begin{bmatrix}
-1 & 1 \\
1 & -1
\end{bmatrix}
\]
and the last interconnection matrix has only nonzero entries in its four corners. The sum of the first \( n - 1 \) interconnection matrices leads to a tridiagonal matrix, which reduces the computational cost for each time step to \( O(n) \), see, e.g., [10, Section 2.1.1]. In fact, the symmetry of the problem implies that omitting any one of the \( n \) submatrices \( A_{in} \) reduces the computational cost for one time step to \( O(n) \). A probability \( 1/n \) is assigned to each subset of \( \{1, 2, \ldots, n\} \) of size \( n - 1 \). The probabilities \( \pi_{in} \) in (74) are, thus, \( \pi_{in} = 1/n - 1/n \). The grid spacing \( h \) is chosen as small as possible, so \( h = \Delta t \). All \( A_{in} \) are dissipative, so \( \mu_R = 0 \) by Remark 1.

Table I compares 20 realizations of the RMB-MPC control \( u_{R-M}(\Omega, t) \) to the MPC control \( u_M(t) \) and the infinite horizon control \( u_M^{\infty}(t) \) for \( n = 100 \) spatial grid points. As can be seen, \( u_M^{\infty}(t) \) is smooth, \( u_M(t) \) jumps when \( t \) is a multiple of \( \tau = 10 \), and the realizations of \( u_{R-M}(\Omega, t) \) contain high-frequent oscillations related to the grid spacing \( \Delta t = h = 1 \). Fig. 1(b) shows that despite the relatively large deviations of \( u_{R-M}(\Omega, t) \) from \( u_M(t) \), \( |x_{R-M}(\Omega, t)| \) is very close to \( |x_M(t)| \) for all 20 considered realizations \( \Omega_i \). RMB-MPC, thus, leads to almost the same decay rate as the MPC here. Note that \( T = 15 \) is not much larger than \( \tau = 10 \), but the simulations indicate that MPC and RMB-MPC are stabilizing.

Table I shows that the running times for RMB-MPC are smaller than those for MPC, which are again smaller than those for solving the OCP on [0, 200] directly. The numbers between round brackets in Table I indicate the estimated standard deviation of the running times based on 20 runs. For \( n = 100 \), MPC is almost three times faster than a classical optimal control approach, and RMB-MPC is again almost three times faster than MPC. For \( n = 1000 \), MPC is still approximately three times faster than solving the OCP directly, but RMB-MPC is five times faster than MPC. Note that the relative speed-up of RMB-MPC compared with MPC may not always match theoretical estimates due to overhead and potential additional iterations in the RMB-constrained OCP compared with the original OCP.

These observations are particularly interesting because Table II shows that the errors do not increase significantly when \( n \) is increased. The numbers between round brackets in Table II indicate the estimated standard deviation based on 20 realizations of \( \Omega_i \). Here, \( \|x\|_{L^\infty} := \max_i \sqrt{\mathbb{E}[(x(t))]^2} \).

The convergence rates from Lemma 7 and Theorem 2 are validated in Figs. 2 and 3. Figs. 2(a) and 3(a) show that \( \|x_{R-M}(\Omega_i) - x_M\|_{L^\infty} \) and \( |u_{R-M}(\Omega_i) - u_M|_{L^\infty} \) decay as \( \sqrt{n} \) for \( h \to 0 \) and that \( x_{R-M}(\Omega_i) \) and \( u_{R-M}(\Omega_i) \) do not converge to \( x_M^* \) and \( u_M^* \) for \( h \to 0 \), as the
estimates from Section V indicate. Figs. 2(b) and 3(b) show that \( \| x_M - x^*_M \|_{L^\infty} \) and \( \| u_M - u^*_M \|_{L^2} \) are proportional to \( e^{-2\mu_{\infty} T} \), as Lemma 7 indicates. Increasing \( T \) increases \( \| x_{R-M}(\Omega_t) - x_M \|_{L^\infty} \) and \( \| u_{R-M}(\Omega_t) - u_M \|_{L^2} \), which confirms that the constant \( C_T \) in Theorem 2 increases with \( T \). Figs. 2(c) and 3(c) show that varying \( \tau \) does not affect \( \| x_{R-M}(\Omega_t) - x_M \|_{L^\infty} \) and \( \| u_{R-M}(\Omega_t) - u_M \|_{L^2} \) strongly and \( \| x_M - x^*_M \|_{L^\infty} \) and \( \| u_M - u^*_M \|_{L^2} \) increase with \( \tau \).

The code used to generate the results in this section can be found online.

VII. CONCLUSION

This article considers a randomized MPC strategy called RBM-MPC to efficiently approximate the solution of a large-scale infinite-horizon linear-quadratic OCP. In RBM-MPC, the finite-horizon OCPs in each MPC-iteration are simplified by replacing the system matrix \( A \) by a randomized one. The estimates in this article demonstrate that 1) RBM-MPC is stabilizing for \( h \text{Var}[A_R] \) sufficiently small and either \( T - \tau \) sufficiently large or \( \| F - P_k \| \) sufficiently small, and 2) RBM-MPC states and controls converge in expectation to their MPC counterparts for \( h \text{Var}[A_R] \to 0 \). In an example with \( n = 100 \) states, RBM-MPC is nine times faster than solving the OCP directly and three times faster than classical MPC. The estimates in this note form a natural starting point for the analysis of RBM-MPC in nonlinear and/or constrained settings in future works. The computational advantage of RBM-MPC has already been demonstrated in a nonlinear setting, see [7]. Because the training of residual deep neural networks (DNNs) can be seen viewed as a nonlinear OCP (see, e.g., [16] and [17]), RBM-MPC may also be applied to speed up the training of DNNs. RBM-MPC may also be used for the control of (networks of) PDEs, that, for example, appear in the modeling of gas transport, see, e.g., [18].

Finally, other variations of RBM-MPC could be considered. One variation would be to first fix a RBM approximation over the whole time axis \( [0, \infty) \) and use this as the plant model for MPC. Another interesting variation would be to consider a new (independent) RBM approximation in each step of the gradient descent algorithm used to solve the (finite horizon) OCPs in MPC.