SKEW GROUP ALGEBRAS, INVARIANTS
AND WEYL ALGEBRAS

ROBERTO MARTÍNEZ-VILLA AND JERONIMO MONDRAGÓN

ABSTRACT. The aim of this paper is two fold:
First to study finite groups $G$ of automorphisms of the homogenized Weyl algebra $B_n$, the skew group algebra $B_n \ast G$, the ring of invariants $B_n^G$, and the relations of these algebras with the Weyl algebra $A_n$, with the skew group algebra $A_n \ast G$, and with the ring of invariants $A_n^G$. Of particular interest is the case $n = 1$.

In the on the other hand, we consider the invariant ring $C[X]^G$ of the polynomial ring $K[X]$ in $n$ generators, where $G$ is a finite subgroup of $GL(n, C)$ such that any element in $G$ different from the identity does not have one as an eigenvalue. We study the relations between the category of finitely generated modules over $C[X]^G$ and the corresponding category over the skew group algebra $C[X] \ast G$. We obtain a generalization of known results for $n = 2$ and $G$ a finite subgroup of $SL(2, C)$. In the last part of the paper we extend the results for the polynomial algebra $C[X]$ to the homogenized Weyl algebra $B_n$.

1. AUTOMORPHISMS OF THE HOMOGENIZED WEYL ALGEBRA.

In this section we will assume the reader is familiar with basic results on Weyl algebras as in [Co]. For results on the homogenized Weyl algebra we refer to [MMo].

Let $K$ be a field of zero characteristic. In this section we consider the homogenized Weyl algebra $B_n$ defined by generators and relations as:

$$B_n = K < X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n, Z > / \langle [X_i, \delta_j] - \delta_{ij} Z^2, [X_i, X_j], [Y_i, Y_j], [X_i, Z^i], [Y_i, Z^j] \rangle,$$

where $K < X_1, X_2, ..., X_n, Y_1, Y_2, ..., Y_n, Z >$ the free algebra in $2n+1$ generators, $[u, v]$ the commutator $uv - vu$ and $\delta_{ij}$, Kronecker's delta.

It is known $B_n$ has a Poincaré-Birkhoff basis and it is quadratic, hence by [Li], [GH] it is Koszul. Let $B_n^!$ be its Yoneda algebra [GM1],[GM2], $B_n^! = \bigoplus_{k \geq 0} \text{Ext}^k_B(K, K)$.

The algebra $B_n^!$ has the same quiver as $B_n$ and relations orthogonal with respect to the canonical bilinear form, it is easy to see that $B_n^!$ has the following form:

$$B_n^! = K_q \{ X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n, Z \} / \{ X_i^2, Y_i^2, \sum_{i=0}^n X_i Y_i + Z^2 \},$$

where $K_q \{ X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n, Z \}$ denote the quantum polynomial ring.

$K < X_1, X_2, ..., X_n, Y_1, Y_2, ..., Y_n, Z > / \langle (X_i, X_j), (Y_i, Y_j), (X_i, Z), (Y_i, Z) \rangle$. Here $(u, v)$ denotes the anti commutators $uv + vu$.

The polynomial algebra $C_n = K[X_1, X_2, ..., X_n, Y_1, Y_2, ..., Y_n, Z]$ is a Koszul algebra with Yoneda algebra $C_n^! = K_q \{ X_1, X_2, ..., X_n, Y_1, Y_2, ..., Y_n, Z \} / \{ X_i^2, Y_i^2 \}$. The
Weyl algebra is defined by \( A_n = K < X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots Y_n, > /\{[X_i, Y_j] = - \delta_{ij}, [X_i, X_j], [Y_i, Y_j], [X_i, Z], [Y_i, Z]\} \).

We denote by \( \sigma \) a homogeneous element of degree one, hence \( \eta \).

\[ \mbox{These algebras are related as follows: } B_n / \mathbb{Z} B_n \cong C_n \mbox{ and } B_n / (Z - 1) B_n \cong A_n \]

The algebra \( B_n \) is a free \( C_n \)-module of rank two.

In fact we have:

**Proposition 1.** There exists a \( C_n \)-module decomposition: \( B_n = C_n \oplus Z C_n \).

We consider now a finite group \( G \) of grade preserving automorphisms of \( B_n \).

It was proved in [MMo] that the center of \( B_n \) is \( K[Z] \) and any automorphism \( \sigma \in G \) takes the center to the center, and since \( \sigma \) is grade preserving \( \sigma(Z) \) is an homogeneous element of degree one, hence \( \sigma(Z) = \lambda \sigma Z \). We assume \( G \) has finite order \( m \), it follows \( \sigma^m(Z) = Z = \lambda^n Z \) and \( \lambda \) is an \( m \)-th root of unity. If we denote by \( \sqrt[m]{T} \) the group of \( m \) roots of unity, there is a group homomorphism \( \eta : G \rightarrow \sqrt[m]{T} \) given by \( \eta(\sigma) = \lambda \), since \( \sqrt[m]{T} \) is a cyclic group the image of \( \eta \) is cyclic and we have a group extension: \( 1 \rightarrow \mathbb{N} \rightarrow G \rightarrow Z_k \rightarrow 0 \) with \( Z_k \) the cyclic group of order \( k \) and \( \mathbb{N} \) is the subgroup of \( G \) such that for every \( \sigma \) in \( G \), \( \sigma(Z) = Z \).

Since \( B_n \) is generated in degree one, the action of \( G \) on \( B_n \) is determined by the action on \( M = (B_n)_1 = \bigoplus_{i=1}^{n} K X_i \oplus (\bigoplus_{i=1}^{n} K Y_i) \oplus K Z \).

To determine the structure of \( G \) we need to look for automorphisms of \( M \) which leave \( K Z \) invariant and preserve the relations: \([X_i, \delta_j] = \partial_{ij} Z^2, [X_i, X_j] = [Y_i, Y_j] = 0\).

For any element \( \sigma \) of \( G \) we have equations:

\[
\sigma(X_j) = \sum_{i=1}^{n} A_{2i-1,2j-1} X_i + \sum_{i=1}^{n} A_{2i,2j-1} Y_i + \mu_j Z
\]

\[
\sigma(Y_k) = \sum_{\ell=1}^{n} A_{2\ell-1,2k} X_\ell + \sum_{\ell=1}^{n} A_{2\ell,2k} Y_\ell + \nu_k Z
\]

\[
\sigma(Z) = \lambda Z.
\]

We must have equalities:

\[
\sigma(X_j X_k - X_k X_j) = 0 = \sigma(X_j) \sigma(X_k) - \sigma(X_k) \sigma(X_j).
\]

Using the relations that define \( B_n \), we obtain after cancellation the following equation involving \( 2 \times 2 \) determinants:

\[
\sum_{i=1}^{n} \begin{vmatrix}
A_{2i-1,2j-1} & A_{2i,2j-1} \\
A_{2i-1,2k-1} & A_{2i,2k-1} \\
\end{vmatrix}
(X_i Y_j - Y_i X_j) = 0.
\]

Since for every \( i \) we have \( X_i Y_i - Y_i X_i = Z^2 \), it follows:

\[
\sum_{i=1}^{n} \begin{vmatrix}
A_{2i-1,2j-1} & A_{2i,2j-1} \\
A_{2i-1,2k-1} & A_{2i,2k-1} \\
\end{vmatrix} = 0.
\]

We may assume \( j < k \). Using that a matrix and its transpose have the same determinant we obtain the equivalent relations:

\[
1) \sum_{i=1}^{n} \begin{vmatrix}
A_{2i-1,2j-1} & A_{2i-1,2k-1} \\
A_{2i,2j-1} & A_{2i,2k-1} \\
\end{vmatrix} = 0, \text{ for all } j, k \mbox{ with } j < k.
\]

In a similar way, for we obtain from the relation:
Theorem 1. Let $\sigma(Y_iY_j - Y_j Y_i) = 0 = \sigma(Y_j)\sigma(Y_k) - \sigma(Y_k)\sigma(Y_j)$, the following equation:

$$2) \sum_{i=1}^{n} \begin{vmatrix} A_{2i-1,2j}^{1} & A_{2i-1,2k}^{1} \\ A_{2i,2j} & A_{2i,2k} \end{vmatrix} = 0,$$

for all $j, k$ with $j < k$.

From the relation:

$$\sigma(X_j Y_j - Y_j X_j) = \sigma(X_j)\sigma(Y_j) - \sigma(Y_j)\sigma(X_j) = \sigma(Z^2) = \lambda^2 Z^2,$$

the following equation:

$$\sum_{i=1}^{n} \begin{vmatrix} A_{2i-1,2j-1} & A_{2i-1,2j} \\ A_{2i,2j-1} & A_{2i,2j} \end{vmatrix} (X_i Y_i - Y_i X_i) = \lambda^2 Z^2.$$

Since for every $i$ we have $X_i Y_i - Y_i X_i = Z^2$, it follows:

$$3) \sum_{i=1}^{n} \begin{vmatrix} A_{2i-1,2j-1} & A_{2i-1,2j} \\ A_{2i,2j-1} & A_{2i,2j} \end{vmatrix} = \lambda^2$$

With similar calculations, from the equation:

$$\sigma(X_j Y_k - Y_k X_j) = 0 = \sigma(X_j)\sigma(Y_k) - \sigma(Y_k)\sigma(X_j),$$

for $k \neq 0$, we get the equation:

$$\sum_{i=1}^{n} \begin{vmatrix} A_{2i-1,2j-1} & A_{2i-1,2k} \\ A_{2i,2j-1} & A_{2i,2k} \end{vmatrix} (X_i Y_i - Y_i X_i) = 0.$$ 

which implies:

$$4) \sum_{i=1}^{n} \begin{vmatrix} A_{2i-1,2j-1} & A_{2i-1,2k} \\ A_{2i,2j-1} & A_{2i,2k} \end{vmatrix} = 0,$$

for $i \neq k$.

We have proved the following:

**Theorem 1.** Let $B_n$ be the homogenized Weyl algebra in $n+1$ generators. Then an automorphism $\sigma$ of $M = \bigoplus_{i=1}^{n} KX_i \oplus \bigoplus_{i=1}^{n} KY_i \oplus KZ$, with matrix in block form:

$$\begin{bmatrix} A & 0 \\ \rho & \lambda \end{bmatrix}$$

where $\rho$ is the vector: $\rho = (\mu_1, \mu_2, ..., \mu_n, \nu_1, \nu_2, ..., \nu_n)$ and $\lambda \neq 0$, extends to an automorphism of $B_n$ if and only if $A$ satisfies the following equations:

$$1) \sum_{i=1}^{n} \begin{vmatrix} A_{2i-1,2j-1} & A_{2i-1,2k-1} \\ A_{2i,2j-1} & A_{2i,2k-1} \end{vmatrix} = 0,$$

for all $j, k$ with $j < k$,

$$2) \sum_{i=1}^{n} \begin{vmatrix} A_{2i-1,2j} & A_{2i-1,2k} \\ A_{2i,2j} & A_{2i,2k} \end{vmatrix} = 0,$$

for all $j, k$ with $j < k$,

$$3) \sum_{i=1}^{n} \begin{vmatrix} A_{2i-1,2j-1}^{1} & A_{2i-1,2j}^{1} \\ A_{2i,2j-1} & A_{2i,2j} \end{vmatrix} = \lambda^2,$$

$$4) \sum_{i=1}^{n} \begin{vmatrix} A_{2i-1,2j-1} & A_{2i-1,2k} \\ A_{2i,2j-1} & A_{2i,2k} \end{vmatrix} = 0,$$

for $i \neq k$.

A particular case is obtained when $\rho = 0$, $\lambda = 1$ and the matrix $A$ satisfies

$$\begin{vmatrix} A_{2i-1,2j-1} & A_{2i-1,2k} \\ A_{2i,2j-1} & A_{2i,2k} \end{vmatrix} = 1$$

for all $i$ and all remaining $2 \times 2$ minors of $A$ are zero. This is the product of $n$ matrices of size $2 \times 2$ and determinant one.

**Corollary 1.** Let $G_1, G_2, ..., G_n$ be finite subgroups of $SL(2, K)$, then the product $G = G_1 \times G_2 \times ... \times G_n$ acts as automorphism group of the homogenized algebra in $n+1$ variables, $B_n$. 

2. Structure of the homogenized Weyl algebra $B_n$
and its skew group algebra $B_n \ast G$.

In this section we study the structure of the homogenized Weyl algebras $B_n$. We will see that they can be obtained from the homogenized algebras $B_i$, $B_j$ with $i + j = n$, as follows: $B_n = B_i \otimes_K B_j / (Z \otimes 1 - 1 \otimes Z) B_i \otimes_K B_j$. This result is very similar to the situation of the Weyl algebras for which it is well known [Co] that for $i + j = n$, there is an isomorphism of $K$-algebras: $A_n = A_i \otimes_K A_j$, or the polynomial algebras $C_i = K[X_1, X_2, ..., X_i]$, $C_j = K[X_{i+1}, X_{i+2}, ..., X_n]$ for which the isomorphism $C_n \cong C_i \otimes C_{n-1}$ is well known.

**Theorem 2.** For integers $n, m > 0$ let $B_n$, $B_m$ and $B_{n+m}$ be the corresponding homogenized algebras. Then there exists an isomorphism of graded $K$-algebras:

$$B_{n+m} = B_n \otimes_K B_m / (Z \otimes 1 - 1 \otimes Z) B_n \otimes_K B_m.$$  

**Proof:** The algebras $B_n$, $B_m$ and $B_{n+m}$ can be written as:

$$B_n = K < X_1, X_2, ..., X_n, Y_1, Y_2, ..., Y_n, Z > / \{ [X_i, \delta_j] - \delta_{ij} Z^2, [X_i, X_j], [Y_i, Y_j], [X_i, Z], [Y_i, Z] \}$$

$$B_m = K < X_{n+1}, ..., X_{n+m}, Y_{n+1}, ..., Y_{n+m}, Z > / \{ [X_i, \delta_j] - \delta_{ij} Z^2, [X_i, X_j], [Y_i, Y_j], [X_i, Z], [Y_i, Z] \}$$

$$B_{n+m} = K < X_1, X_2, ..., X_{n+m}, Y_1, Y_2, ..., Y_{n+m}, Z > / \{ [X_i, \delta_j] - \delta_{ij} Z^2, [X_i, X_j], [Y_i, Y_j], [X_i, Z], [Y_i, Z] \}.$$  

Then there are natural inclusions: $\varphi_1 : B_n \to B_{n+m}$ and $\varphi_2 : B_m \to B_{n+m}$ given by: $\varphi_i(X_j) = X_j$, $\varphi_i(Y_j) = Y_j$, $\varphi_i(Z) = Z$ and $i = 1, 2$.

Let $j_1 : B_n \to B_{n+m}$ and $j_2 : B_m \to B_{n+m}$, be the inclusions $j_1(b_1) = b_1 \otimes 1$, and $j_2(b_2) = 1 \otimes b_2$. Since multiplication is bilinear we have a vector space map: $\varphi : B_n \otimes_K B_m \to B_{n+m}$ given by $\varphi(b_1 \otimes b_2) = \varphi_1(b_1) \varphi_2(b_2)$, such that the diagram:

$$\begin{array}{ccc}
B_n & \xrightarrow{j_1} & B_n \otimes_K B_m \\
\varphi_1 & \downarrow \varphi & \downarrow \varphi_2 \\
B_{n+m} & \to & B_{n+m}
\end{array}$$

commutes.

Since $\varphi_1(b_1) \varphi_2(b_2) = \varphi_2(b_2) \varphi_1(b_1)$ with $b_1 \in B_n$, $b_2 \in B_m$, $\varphi$ is an algebra homomorphism. It is clear $\varphi$ is surjective and $\varphi(Z \otimes 1 - 1 \otimes Z) = 0$, hence $(Z \otimes 1 - 1 \otimes Z) B_n \otimes_K B_m$ is contained in the kernel of $\varphi$. We will prove that they are actually equal.

Let $b$ be an element of degree $t$ in the kernel of $\varphi$. The element $b$ has the following form:

$$\sum_{a+b+k+\ell+\alpha+\beta=0} C_{\alpha, \beta, k} X^\alpha Y^\beta Z^k \otimes B_{\alpha', \beta', \ell, \alpha''} Y_m^\alpha Y_m^\beta Z'$$

with $X^\alpha = X_1^{\alpha_1} X_2^{\alpha_2} ... X_n^{\alpha_n}$, $Y^\beta = Y_1^{\beta_1} Y_2^{\beta_2} ... Y_n^{\beta_n}$ and $X_m^\alpha = X_{n+1}^{\alpha_1} X_{n+2}^{\alpha_2} ... X_{n+m}^{\alpha_m}$, $Y_m^\beta = Y_{n+1}^{\beta_1} Y_{n+2}^{\beta_2} ... Y_{n+m}^{\beta_m}$.

Assume $k \geq 0$.

Then $X^\alpha Y^\beta Z^k \otimes Z_m^\alpha Y_m^\beta Z' = X^\alpha Y^\beta Z^{k-1} \otimes X_m^\alpha Y_m^\beta Z' (Z \otimes 1) = X^\alpha Y^\beta Z^{k-1} \otimes X_m^\alpha Y_m^\beta Z' (Z \otimes 1) + X^\alpha Y^\beta Z^k \otimes X_m^\alpha Y_m^\beta Z'$.

By induction,

$$C_{\alpha, \beta, k} X^\alpha Y^\beta Z^k \otimes B_{\alpha', \beta', \ell, \alpha''} Y_m^\alpha Y_m^\beta Z' = g(X, Y, Z) (Z \otimes 1 - 1 \otimes Z) + C_{\alpha, \beta, k} X^\alpha Y^\beta \otimes B_{\alpha', \beta', \ell, \alpha''} Y_m^\alpha Y_m^\beta Z^{k+\ell}$$

and $g(X, Y, Z)$ an expression in $X, Y, Z$. Then
\[ \sum_{a+\beta+k+\ell+a'+\beta'=t} C_{\alpha,\beta,k} X^{\alpha \beta} Z^k \otimes B_{\alpha',\beta',\ell} X_m^{\alpha'} Y_m^{\beta'} Z^\ell = G(X,Y,Z) \text{ (} Z \otimes 1 - 1 \otimes Z \text{)} + \sum_{r} \sum_{a+\beta+a'+\beta'=t-r} \sum_{r=k+\ell} C_{\alpha,\beta,k} B_{\alpha',\beta',\ell} X^{\alpha \beta} Z^k \otimes X_m^{\alpha'} Y_m^{\beta'} (1 \otimes Z^r). \]

Applying the above expression, we get:

\[ \varphi(b) = 0 = \sum_{r} \sum_{a+\beta+a'+\beta'=t-r} \sum_{r=k+\ell} C_{\alpha,\beta,k} B_{\alpha',\beta',\ell} X^{\alpha \beta} X_1^{\alpha_1} X_2^{\alpha_2} \ldots X_n^{\alpha_n} Y_1^{\beta_1} Y_2^{\beta_2} \ldots Y_n^{\beta_n} Z^r. \]

Using the fact \( X_{n+i} Y_i = Y_i X_{n+i} \) for \( 1 \leq i \leq m \) we obtain:

\[ \sum_{r} \sum_{a+\beta+a'+\beta'=t-r} \sum_{r=k+\ell} C_{\alpha,\beta,k} B_{\alpha',\beta',\ell} X^{\alpha \beta} X_1^{\alpha_1} X_2^{\alpha_2} \ldots X_n^{\alpha_n} Y_1^{\beta_1} Y_2^{\beta_2} \ldots Y_n^{\beta_n} Z^r = 0. \]

It follows:

\[ \sum_{a+\beta+a'+\beta'=t-r} \sum_{r=k+\ell} C_{\alpha,\beta,k} B_{\alpha',\beta',\ell} X^{\alpha \beta} X_m^{\alpha'} Y_m^{\beta'} = 0. \]

Therefore:

\[ \sum_{a+\beta+k+\ell+a'+\beta'=t} C_{\alpha,\beta,k} X^{\alpha \beta} Z^k \otimes B_{\alpha',\beta',\ell} X_m^{\alpha'} Y_m^{\beta'} Z^\ell = G(X,Y,Z) \text{ (} Z \otimes 1 - 1 \otimes Z \text{)} \]

is an element of \( (Z \otimes 1 - 1 \otimes Z) B_n \otimes_K B_m \), as claimed. \( \square \)

Let \( \Lambda, \Gamma \) be \( K \)-algebras and \( G, H \) finite group of automorphisms of \( \Lambda, \Gamma \), respectively. Assume the characteristic of \( K \) does not divide neither the order of \( G \) nor the order of \( H \). By the universal property of the coproduct, given \( \sigma \in G, \tau \in H \), there is a commutative diagram:

\[ \Lambda \xrightarrow{\downarrow \sigma} \Lambda \otimes_K \Gamma \xrightarrow{\downarrow \tau} \Gamma \]

with \( (\sigma,\tau)(\lambda \otimes \gamma) = \sigma(\lambda) \otimes \tau(\gamma) \), the homomorphism \( (\sigma,\tau) \) is an automorphism, such that \( (\sigma,\tau) = 1 \) implies \( \sigma = 1 \) and \( \tau = 1 \). Hence \( G \times H \) embeds faithfully in the automorphism group of \( \Lambda \otimes_K \Gamma \).

The following proposition is well known: (see for example [CR]).

**Proposition 2.** Let \( K \) be a field and \( G, H \), finite groups. Then there is a natural isomorphism of finite dimensional algebras: \( K(G \times H) \cong KG \otimes_K K(H) \).

**Proof.** We define a map \( \varphi : K(G \times H) \rightarrow KG \otimes_K K(H) \) as \( \varphi(g,h) = g \otimes h \) and extend it linearly. The map sends basis to basis, hence it is a vector space isomorphism. It is easy to see it is also a ring isomorphism. \( \square \)

We use this result to prove the following:

**Theorem 3.** Let \( \Lambda, \Gamma \) be graded \( K \)-algebras and \( G, H \) finite group of (grade preserving) automorphisms of \( \Lambda, \Gamma \), respectively. Assume the characteristic of \( K \) does not divide neither the order of \( G \) nor the order of \( H \). Denote by \( \Lambda \ast G \) and \( \Gamma \ast H \) the corresponding skew group algebras. Then there is a natural isomorphism of (graded) \( K \)-algebras: \( \Lambda \otimes_K \Gamma \ast (G \times H) \cong \Lambda \ast G \otimes_K \Gamma \ast H \).
Proof. We have vector space isomorphisms: $\Lambda \otimes_K \Gamma \ast (G \times H) \cong \Lambda \otimes_K \Gamma \otimes_K K(G \times H) \cong \Lambda \otimes_K \Gamma \otimes_K K(G \times H) \cong \Lambda \otimes_K \Gamma \otimes_K K(H) \cong \Lambda \ast G \otimes_K \Gamma \ast H$.

We look to the action of $G \times H$ on $\Lambda \otimes_K \Gamma$.

Let $(\sigma, \tau)$ be an element of $G \times H$ and $\lambda \otimes \gamma \in \Lambda \otimes_K \Gamma$. Then $(\sigma, \tau)\lambda \otimes \gamma = \lambda^\sigma \otimes \gamma^\tau = \lambda^\sigma \otimes \gamma^\tau \otimes \sigma \otimes \tau = \lambda^\sigma \otimes \sigma \otimes \gamma^\tau \otimes \tau = (\sigma \otimes \tau)(\lambda \otimes \gamma)$.

It follows $\Lambda \otimes_K \Gamma \ast (G \times H) \cong \Lambda \ast G \otimes_K \Gamma \ast H$ as (graded) algebras. \hfill $\square$

Corollary 2. Let $K$ be a field of zero characteristic, $B_n$, $B_m$, homogenized Weyl algebras and $G$, $H$, finite groups of grade preserving automorphisms of $B_n$ and $B_m$, respectively. Assume for all $\sigma \in G$ and $\tau \in H$, $\sigma(Z) = Z$ and $\tau(Z) = Z$. Then $G \times H$ acts on $B_{n+m}$ and there is an isomorphism of graded algebras: $B_{n+m} \ast (G \times H) \cong B_{n} \ast G \otimes_K B_{m} \ast H/(Z \otimes 1 - 1 \otimes Z)B_n \ast G \otimes_K B_m \ast H$.

If we denote by $B_{G \times H}^G$, $B_{G}^G$, $B_{H}^H$ the rings of invariants, then we have an isomorphism of algebras: $B_{G \times H}^G \cong B_{n} \otimes_K B_{m}^H / (Z \otimes 1 - 1 \otimes Z)B_n \otimes_K B_m^H$.

Proof. Given $\sigma \in G$, $\tau \in H$, we have a commutative diagram:

\[
\begin{array}{cccc}
B_n & \xrightarrow{\hat{\phi}_1} & B_n \otimes_K B_m & \xrightarrow{\hat{\phi}_2} & B_m \\
\downarrow{\sigma} & & \downarrow{\sigma \otimes \tau} & & \downarrow{\tau} \\
B_n & \xrightarrow{\hat{\phi}_1} & B_n \otimes_K B_m & \xrightarrow{\hat{\phi}_2} & B_m \\
\end{array}
\]

Let $\sum_{i=1}^{n} b_i \otimes b'_i$ be an element of the kernel of $\varphi \sigma \otimes \tau$. Then $\varphi(\sum_{i=1}^{n} b_i \otimes b'_i) = 0$.

By that above description of $\text{Ker}\varphi$, $\sum_{i=1}^{n} b_i \otimes b'_i = g(X, Y, Z)(Z \otimes 1 - 1 \otimes Z)$. Therefore:

$\sum_{i=1}^{n} b_i \otimes b'_i = ((\sigma^{-1}, \tau^{-1})(g(X, Y, Z))(\sigma^{-1}Z \otimes 1 - 1 \otimes \tau^{-1}Z)) = g(X, Y, Z)(Z \otimes 1 - 1 \otimes Z)$.

It follows $\text{Ker}\varphi \sigma \otimes \tau = (Z \otimes 1 - 1 \otimes Z)B_n \otimes_K B_m$, and the map $\varphi \sigma \otimes \tau$ factors through $B_{n+m}$, denote by $\sigma \otimes \tau$ the induced map, which is clearly an automorphism of $B_{n+m}$ such that the diagram:

\[
\begin{array}{cccc}
B_n & \xrightarrow{\hat{\phi}_1} & B_{n+m} & \xrightarrow{\hat{\phi}_2} & B_m \\
\downarrow{\sigma} & & \downarrow{\sigma \otimes \tau} & & \downarrow{\tau} \\
B_n & \xrightarrow{\hat{\phi}_1} & B_{n+m} & \xrightarrow{\hat{\phi}_2} & B_m \\
\end{array}
\]

The above diagram *) induces a commutative diagram of graded algebras and homomorphisms:

\[
\begin{array}{cccc}
B_n \ast G & \xrightarrow{\phi_1} & B_n \otimes_K B_m(G \times H) & \xrightarrow{\phi_2} & B_m \ast H \\
\downarrow{\phi_1} & & \downarrow{\phi} & & \downarrow{\phi_2} \\
B_{n+m} \ast (G \times H) & & & & \\
\end{array}
\]

There is an exact sequence:

$0 \rightarrow (Z \otimes 1 - 1 \otimes Z)B_n \otimes_K B_m(G \times H) \rightarrow B_n \otimes_K B_m(G \times H) \rightarrow B_{n+m}(G \times H) \rightarrow 0$

From the isomorphism $B_n \otimes_K B_m \ast (G \times H) \cong B_n \ast G \otimes_K B_m \ast H$, it follows $B_{n+m} \ast (G \times H) \cong B_n \ast G \otimes_K B_m \ast H/(Z \otimes 1 - 1 \otimes Z)B_n \ast G \otimes_K B_m \ast H$.

Now let $e = 1/|G| \sum_{g \in G} g$, and $f = 1/|H| \sum_{h \in H} h$ be elements of $KG$ and $KH$, respectively. The elements $e$ and $f$ are idempotents, it is well known and easy to prove that the rings $B_n^G$ and $B_m^H$ are isomorphic $e(B_n \ast G)e$ and $f(B_m \ast H)f$ respectively.
Under the isomorphism $K(G \times H) \to KG \otimes_K KH$ the idempotent $(e,f) = 1/(G \times H) \sum (g,h)$ correspond to $e \otimes f$.

Then from the isomorphism: $(B_n \otimes_K B_m) \ast (G \times H) \cong B_n \ast G \otimes_K B_m \ast H$ we get an isomorphism:

$$(e,f)(B_n \otimes_K B_m) \ast (G \times H)(e,f) \cong e \otimes f(B_n \ast G \otimes_K B_m \ast H)e \otimes f \cong e(B_n \ast G)e \otimes_K f(B_m \ast H)f$$

Therefore $(B_n \otimes_K B_m)^{G \times H} \cong B_n^G \otimes_K B_m^H$.

It follows $(e,f)(B_{n+m}) \ast (G \times H)(e,f) \cong B_{n+m}^{G \times H} \cong (e \otimes f)(B_n \ast G \otimes_K B_m \ast H)(e \otimes f)/(Z \otimes 1 - 1 \otimes Z) \cong e(B_n \ast G)e \otimes_K f(B_m \ast H)f/(Z \otimes 1 - 1 \otimes Z)$.

From this we have the isomorphism of algebras:

$B_{n+m}^{G \times H} \cong B_n^G \otimes_K B_m^H / (Z \otimes 1 - 1 \otimes Z) B_n^G \otimes_K B_m^H$. 

For Weyl algebras we have the following analogous of the previous theorem.

**Theorem 4.** Given finite groups of automorphisms $G$, $H$ of the Weyl algebras $A_n$ and $A_m$, $G \times H$ acts as a group of automorphisms of $A_{n+m}$ and there is an isomorphism of $K$-algebras, $A_{n+m}^{G \times H} \cong A_n^G \otimes_K A_m^H$, where $A_n^G$, $A_m^H$.

3. **The Weyl algebras $B_1$, and $A_1$ and their skew group algebras $B_1 \ast G$ and $A_1 \ast G$, with $G$ a finite subgroup of $SL(2,C)$.**

In this section we describe the basic algebras Morita equivalent to $B_1 \ast G$ and $A_1 \ast G$ by quivers and relations. To achieve this we will make use of the following result, [AR], [L], [C-B]:

**Theorem 5.** For any subgroup $G$ of $SL(2,C)$ the skew group algebra $C[X,Y] \ast G$ is Morita equivalent to the preprojective algebra of an Euclidean diagram.

We will end the section sketching the situation for a product $G = G_1 \times G_2 \times ... G_n$ of finite subgroups $G_i$ of $SL(2,C)$ acting on $B_n$ and on $A_n$.

We start by recalling the situation of $C[X,Y] \ast G$, following the approach of [GuM] and take the opportunity to correct some inaccuracies.

Recall the construction of a McKay quiver of a finite subgroup $G$ of the linear group $GL(n,C)$. [Mc]

Let $S_1$, $S_2$, ..., $S_n$ be the non isomorphic irreducible representations of $G$ and $M$ the representation corresponding to the inclusion of $G$ in $GL(n,C)$. Tensoring $M$ with some $S_j$ we obtain a decomposition in irreducible representations: $M \otimes_K S_j \cong \bigoplus a_{ij} S_i$.

The McKay quiver of $G$ has vertices $v_1$, $v_2$, ..., $v_n$, with each vertex $v_i$ corresponding to an irreducible representation $S_i$ and we put $a_{ij}$ arrows from $v_i$ to $v_j$.

For the proof we will make use of the following well known result from ordinary group representations:

**Theorem 6.** [Mu] Let $G$ be a finite group and $L$ a complex irreducible representation. Then the dimension of $L$ as $C$-vector space divides the order of $G$.

We reproduce here the proof given in [Ste] of the following:

**Theorem 7.** Let $M$ be a $C$-vector space of dimension 2 and $G$ a finite subgroup of the special linear group $SL(2,M)$. Then the McKay quiver has no loops.
Proof. For the proof of the theorem we have two cases:

1) $M$ is an irreducible representation.

According to the previous theorem, 2 divides the order of $G$ and by Sylow theorems, there exists an element $g \in G$ of order 2. Since $g \in sl(2, M)$, by a change of basis $PgP^{-1} = \begin{bmatrix} \lambda & 0 \\ \mu & \lambda' \end{bmatrix}$, with $\lambda\lambda' = 1$, and

$$
\begin{bmatrix}
\lambda^2 \\
\lambda\mu + \lambda'\mu \\
\lambda^2
\end{bmatrix} = 1
$$

Therefore: $\lambda^2 = 1 = (\lambda')^2$ and either $\lambda = \lambda' = 1$ or $\lambda = \lambda' = -1$, in any case $\mu = 0$ and $g$ is in the center of $G$.

Let $S$ be another irreducible representation of $G$ and $\varphi : CG \rightarrow S$ an epimorphism.

Let’s suppose $g - 1$ is not in the kernel of $\varphi$. Then there exists $s' \in S$ with $(g - 1)s' \neq 0$. Since $S$ is simple, for any $s \in S$, $s \neq 0$, there exists $\rho \in CG$ such that $\rho(g - 1)s' = s$ and $\rho(g - 1) = (g - 1)\rho$. It follows $(g + 1)s = (g + 1)(g - 1)\rho s' = (g^2 - 1)\rho s' = 0$ and $gs = -s$.

This means that $g$ acts on $S$ either as the identity or as $-1$.

If $g$ acts as $-1$ on $M$, and as $1$ on $S$, then $g$ acts as $-1$ on $M \otimes_K S$ and $S$ can not appear as a summand of $M \otimes_K S$ and if $g$ acts as $-1$ on $M$ and as $-1$ on $S$, then it acts as $1$ on $M \otimes_K S$ and again $S$ can not be a summand of $M \otimes_K S$.

We have proved that in this case the McKay quiver has no loops.

2) The representation $M$ is reducible, this is: $M = M_1 \oplus M_2$ and $\dim_C M_1 = \dim_C M_2 = 1$. Let’s say that $M_1$ is generated by $m_1$ and $M_2$ is generated by $m_2$. If the order of $G$ is $n$, then for any $g \in G$, $g(m_1) = \lambda_1 m_1$ and $g(m_2) = \lambda_2 m_2$ with $\lambda_1$ and $\lambda_2$ n-th roots of unity.

The element $g$ has form: $g = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, with $\lambda_1, \lambda_2 = 1$.

There exists an injective homomorphism from $G$ to the group of nth roots of unity, $\sqrt[n]{1}$ given by $g \rightarrow \lambda_1$, hence $G$ is cyclic and all irreducible representations have dimension one.

Let $Cs = S$ be an irreducible representation $G = < g >$ and $gs = ts$.

Suppose $M \otimes_K S = S_1 \oplus S_2$. Then $M_1 \otimes S = S_1$ and $M_2 \otimes S = S_2$.

Assume $S = S_1$, then $m_1 \otimes s = s'$, $s' \in S$ and $s' = rs$. It follows $g(m_1 \otimes s) = g(m_1) \otimes g(s) = \lambda_1 m_1 \otimes ts = g(rs) = rt\lambda_1(m_1 \otimes s) = t\lambda_1 rs$. Therefore $\lambda_1 = 1$ and $g$ is the identity.

We have proved the McKay quiver has no loops. \(\square\)
Consider now an arbitrary Koszul $C$-algebra $\Lambda$ and $G$ a finite group of automorphisms of $\Lambda$. Given a complete set of primitive orthogonal idempotents $e_1, e_2, \ldots, e_n$ of $CG$, they are also a complete set of orthogonal idempotents of $\Lambda * G$ and taking $e = \sum_{i=1}^{n} e_i$, the algebra $e(\Lambda * G)e$ is basic and Morita equivalent to $\Lambda * G$.

There is a natural isomorphism:

$$\text{Ext}^k_{\Lambda * G}(CGe, CGe) \cong e\text{Ext}^k_{\Lambda * G}(CG, CG)e$$

By the Morita theorems applied to graded algebras [MV4], the functor $\text{Hom}_{\Lambda * G}(\Lambda * Ge, -)$ induces an equivalence of categories $Gr_{\Lambda * G} \cong Gr_{\Lambda * Ge}$. It follows that for each $k \geq 0$, there is an isomorphism:

$$\text{Ext}^k_{\Lambda * G}(CGe, CGe) \cong \text{Ext}^k_{\Lambda * Ge}(eCGe, eCGe).$$

Using this two isomorphisms and adding up, we get an isomorphism of graded $K$-algebras.

$$e(\bigoplus \text{Ext}^k_{\Lambda * G}(CG, CG))e \cong \bigoplus \text{Ext}^k_{\Lambda * Ge}(eCGe, eCGe)$$

In particular when $\Lambda$ is the exterior algebra in two generators we have:

$$C[X, Y] * G \cong \bigoplus \text{Ext}^k_{\Lambda * G}(CG, CG)$$

and

$$e(C[X, Y] * G)e \cong \bigoplus \text{Ext}^k_{\Lambda * Ge}(eCGe, eCGe).$$

The algebra $e(\Lambda * G)e$ is basic Koszul selfinjective of radical cube zero with Yoneda algebra the basic noetherian algebra $e(C[X, Y] * G)e$.

It follows from [GMT] that the separated quiver of $e(\Lambda * G)e$ is an Euclidean diagram $Q$.

It is easy to check that the quiver of $e(C[X, Y] * G)e$ is the McKay quiver of $G$, by Theorem ?, this quiver does not have loops. By the properties of Koszul algebras $e(C[X, Y] * G)e$ and $e(\Lambda * G)e$ have the same quiver $\overset{\wedge}{Q}$.

We know by [GMT] $\overset{\wedge}{Q}$ is a translation quiver with translation $\tau$ the Nakayama permutation, but $\text{soc}(\Lambda * G)e_i = J^2 * Ge_i$, and since $\Lambda$ has simple socle $J^2 = K$. Then $\text{soc}(\Lambda * G)e_i = KG_e_i = \text{top}(\Lambda * G)e_i$ and $\tau$ is the identity.

The quiver $\overset{\wedge}{Q}$ is the complete quiver of an Euclidean diagram, this means $\overset{\wedge}{Q}_0 = Q_0$ and $\overset{\wedge}{Q}_1 = Q_i \cup Q_1^{op}$, with $Q$ an Euclidean diagram. For each arrow $\alpha : i \to j$ in $\overset{\wedge}{Q}$ we have an arrow $\alpha^{-1} : j \to i$ in $\overset{\wedge}{Q}:

\begin{array}{ccc}
\overset{\wedge}{Q} & \overset{\wedge}{Q} \\
\alpha & \alpha^{-1} \\
\beta & \beta^{-1} \\
\end{array}

Since $(\Lambda * G)e_i$ has simple socle, for any pair of arrows $\alpha : i \to k$, $\beta : i \to k$, there is a non zero $c \in K$, such that $\alpha^{-1}\alpha = c\beta^{-1}\beta$. If we assume $Q$ is a tree then we can change the maps $b : (\Lambda * G)e_i \to (\Lambda * G)e_j$, corresponding to the arrow $\beta$ by $cb$ and we get an arrow which we denote again by $\beta$ such that $\alpha^{-1}\alpha = \beta^{-1}\beta$ and we obtain an isomorphism $e(\Lambda * G)e \cong KQ/I$, where $I$ is the ideal generated by relations: $\alpha^{-1}\alpha - \beta^{-1}\beta$, $\alpha\alpha^{-1} - \beta\beta^{-1}$ and $\alpha\delta$ if $\delta \neq \alpha^{-1}$, $\delta\alpha$ if $\delta \neq \alpha^{-1}$.
From this is clear that \( e(C[X,Y] \ast G)e \cong K \tilde{Q}/I^\perp \), where \( I^\perp \) is the ideal generated by mesh relations: \[
abla \begin{array}{cc}
abla & \alpha_1 \nearrow \\
abla & j_1 \swarrow \\
abla & \alpha_k \nearrow \\
abla & j_k \swarrow \\
abla & \end{array}
abla \] \[i \rightarrow j_2 \rightarrow \ldots \rightarrow i \), this is \[\sum_{i=1}^{k} \alpha_i^{-1} \alpha_i \in I^\perp.\]

We have proved \( e(C[X,Y] \ast G)e \) is isomorphic to the preprojective algebra.

The case \( A_n \) is the skew group algebra corresponding to the cyclic group \( Z_n \) and has to be considered separately, since for \( A_n \) it is not clear that we could choose the arrows in the algebra \( e(\Lambda \ast G)e \) in such a way that for any pair of arrows \( \alpha, \beta \) with the same origin, \( \alpha^{-1} = \beta^{-1} \).

We also need to prove that all preprojective algebras appear in this way.

A full description of the quiver of a preprojective algebra and its relations with the McKay graph for finite subgroups of \( SL(2, \mathbb{C}) \) has appeared in several papers by authors like: Crawley-Boevey or Lenzing [L], [C-B].

We now consider finite groups of automorphisms \( G \) of the homogenized Weyl algebra \( B_n \) such that for any \( \sigma \in G \), \( \sigma(Z) = Z \) and for all \( 1 \leq i \leq n \), \( \sigma(X_i) \), \( \sigma(Y_j) \in \bigoplus_i C X_i \oplus \bigoplus_i C Y_i \).

Given a Koszul algebra \( \Lambda \) with Yoneda algebra \( \Gamma \) and a finite group of automorphisms \( G \) of \( \Lambda \), we recall from [MV1] , how the action transfers to a group of automorphisms of \( \Gamma \):

Let \( \Lambda \) be a module and \( \sigma \in G \), we define \( \sigma^\sigma = \sigma M \) as vector space and multiplication given as follows: for \( \lambda \in \Lambda \), \( m \in \sigma^\sigma \) we define \( \lambda \ast m = \sigma^n m \).

In case \( M \) is a \( G \)-module, there is an isomorphism: \( \varphi_\sigma : \sigma M \rightarrow \sigma^\sigma \) given by \( \varphi_\sigma(\lambda m) = \sigma(\lambda m) = \lambda^\sigma \sigma m = \lambda \ast \varphi_\sigma(m) \).

Now given an extension \( \delta \in \operatorname{Ext}_\Lambda^1(S_i, S_j) \), with \( S_i, S_j \) graded simple modules:

\[
\delta : 0 \rightarrow S_j \rightarrow E_1 \rightarrow E_2 \rightarrow \ldots \rightarrow E_k \rightarrow S_i \rightarrow 0
\]

we define \( \sigma(\delta) : 0 \rightarrow S_j^\sigma \rightarrow E_1^\sigma \rightarrow E_2^\sigma \rightarrow \ldots \rightarrow E_k^\sigma \rightarrow S_i^\sigma \rightarrow 0 \).

It is clear that the modules \( S_j^\sigma, S_i^\sigma \) are again graded simple and \( \sigma(\delta) \in \operatorname{Ext}_\Lambda^1(S_i^\sigma, S_j^\sigma) \).

In this way we have an isomorphism of \( C \)-algebras: \[ \sigma : \bigoplus_{k \geq 0} \operatorname{Ext}_\Lambda^k(\Lambda/J, \Lambda/J) \rightarrow \bigoplus_{k \geq 0} \operatorname{Ext}_\Lambda^k(\Lambda/J, \Lambda/J) \] given by \( \sigma(\delta_1, \delta_2, \ldots, \delta_n) = (\sigma(\delta_1), \sigma(\delta_2), \ldots, \sigma(\delta_n)) \).

In the particular case \( \Lambda = B_n \) and \( \Gamma = B_n^1 \), we want to see that for any group of automorphisms \( G \) of \( B_n \) such that any element \( \sigma \) of \( G \) has matrix form \( \sigma = \begin{bmatrix} A & 0 \\
0 & 1 \end{bmatrix} \), with \( A \) a \( n-1 \times n-1 \) matrix, the action of \( G \) on \( (B_n^1)_1 \) is such that every \( \sigma \in G \) has matrix form \( \sigma = \begin{bmatrix} * & 0 \\
0 & 1 \end{bmatrix} \), with \( * \) a \( n-1 \times n-1 \) matrix.

The element \( Z \) of \( B_n^1 \) corresponds to the extension:
We have an isomorphism: $\text{Ext}^1_{B_n}(C,C) \cong \text{Hom}_{B_n}(J,C) \cong \text{Hom}_{B_n}(J/J^2,C)$, where $J/J^2 \cong \bigoplus_{i=1}^n C \tilde{X}_i \oplus \bigoplus_{i=1}^n C \tilde{Y}_i \oplus CZ$.

The element $X_i$ of $B^*_n$ corresponds to the map: $f : J \to J/J^2 \to C \tilde{X}_i \cong K$ applying the homomorphism $\sigma$ we get a map: $f^\sigma : J/J^2 \to C \tilde{X}_i \cong K$.

But by hypothesis $(\tilde{X}_i)^\sigma = \sum A_{ij} X_j + \sum B_{ij} Y_j$ hence in the expansion of $f^\sigma$ does not appear $Z$.

For $Y_i \in B^*_n$, the situation is similar and the automorphism $\sigma$ of $B^*_n$ has form

$\sigma = \left[ \begin{array}{cc} [1] & 0 \\ 0 & 1 \end{array} \right]$, with $[1]$ a $n-1 \times n-1$ matrix.

We want to use the previous remarks to describe $B^*_n \ast G$ for $G$ a finite subgroup of $Sl(2,C)$ with action on $B^*_n$ as above.

Let $e_1, e_2, \ldots, e_k$ be a complete set of orthogonal idempotents of $CG$ and $e = \sum_{i=1}^k e_i$, $B_1/ZB_1 \cong C_1$, the polynomial algebra in two variables.

We saw in previous section $eC_1 \ast Ge$ is the preprojective algebra and its Yoneda algebra is the selfinjective radical cube zero algebra $eC_1 \ast Ge$, with $C_1$ the exterior algebra in two variables. and for any pair of arrows $\alpha : i \to j$, $\beta : j \to k$ in the quiver of $eC_1 \ast Ge$ there exists arrows $\alpha^{-1} : j \to i$ and $\beta^{-1} : k \to i$ such that $\alpha^{-1} \beta = \beta^{-1} \beta$ and $\delta \alpha = 0$ if $\delta$ is an arrow different from $\alpha^{-1}$.

We also saw in Section 1, that there is an isomorphism of $C^*_n$-bimodules: $B^*_n \cong C^*_n \oplus CZC^*_n$.

If we assume $G$ acts on $B_n$ in such a way that for $\sigma \in G$, $\sigma(Z) = Z$ and $\sigma(X_i)$, $\sigma(Y_i)$ is contained in the vector space generated by the $X$'s and the $Y$'s, then $G$ acts on the same way on $B^*_n$ and we have an isomorphism: $B^*_n \ast G \cong C^*_n \ast G \oplus CZC^*_n \ast G$ of $C^*_n \ast G$-bimodules.

which induces an isomorphism: $e(B^*_n \ast G)e \cong e(C^*_n \ast G)e \oplus Ze(C^*_n \ast G)e$ of $e(C^*_n \ast G)e$-bimodules. In particular, for $n = 1$, $e(B^*_1 \ast G)e \cong e(C^*_1 \ast G)e \oplus Ze(C^*_1 \ast G)e$.

The Jacobson radical of $B_1$ is of the form: $J = J_{C^*_1} \oplus ZC^*_1$, with $J_{C^*_1}$ the Jacobson radical of $C^*_1$. It follows the Jacobson radical of $B^*_n \ast G$ is $J \ast G = J_{C^*_1} \ast G \oplus ZC^*_1 \ast G$ and $(J \ast G)^2 = J_{C^*_1} \ast G + Z(J_{C^*_1} \ast G) + Z^2C^*_1 \ast G$. But $Z^2 = -XY \in J^2_{C^*_1}$, hence $(J \ast G)^2 = J^2_{C^*_1} \ast G + Z(J_{C^*_1} \ast G)$.

Therefore: $J \ast G/J^2 \ast G \cong J/J^2 \ast G \cong J_{C^*_1}/J^2_{C^*_1} \ast G \oplus Z(C^*_1J_{C^*_1} \ast G) \cong J_{C^*_1}/J^2_{C^*_1} \ast G \oplus Z(CG)$. Then $e(J \ast G/J^2 \ast G)e \cong e(J_{C^*_1}/J^2_{C^*_1} \ast G)e \oplus Ze(CG)e$. Hence for $1 \leq i \leq n$, $e_i(J \ast G/J^2 \ast G)e_i \cong e_i(J_{C^*_1}/J^2_{C^*_1} \ast G)e_i \oplus Ze(CG)e_j$.  

$0 \to \quad J \to \quad B_n \to \quad C \to \quad 0$

$0 \to \quad J/J^2 \to \quad E \to \quad C \to \quad 0$

$0 \to \quad C \to \quad C[Z]/(Z^2) \to \quad C \to \quad 0$

$0 \to \quad C \to \quad C[Z]/(Z^2) \to \quad C \to \quad 0$
Since the preprojective algebra has no loops, \( e_i(Jc_1^*/Jc_1^2 * G) e_i = 0 \). In the other hand, for \( i \neq j \), \( e_i(CG)e_j = 0 \) and \( e_i( J * G/J^2 * G)e_i = ZCe_i \).

We have proved that the quiver of \( e(B_1^* G) e \) has the same vertices and arrows as a preprojective algebra corresponding to an Euclidean diagram, and in addition a loop for each vertex.

We shall next find the relations. Given an idempotent \( e_j \notin \{e_1, e_2, \ldots, e_k \} \), there exists one of the idempotents \( e_j \) and an isomorphism \( \varphi : CGe_j \rightarrow CGe_j \), in particular \( \varphi(e_j) = e_j^\prime \rho e_j \) and \( \varphi^{-1}(e_j) = e_j \gamma e_j^\prime \) with \( \rho, \gamma \in CG \). Then \( e_j = e_j \gamma e_j^\prime \rho e_j \) and \( e_j^\prime = e_j^\prime \rho e_j \gamma e_j^\prime \).

Note that \( CX_\alpha Y = J_{c_1^2}^2 \), and \( G \) acts trivially on \( XY \).

Let \( g \in G \), \( g(X) = aX + bY \), \( g(Y) = cX + dY \). Hence, \( g(XY) = g(X)g(Y) = (aX + bY)(cX + dY) = acX^2 + adXY + bdYX + bdY^2 = adX - bcY = XY \), since \( det(g) = 1 \). In consequence, \( XY e_i = e_i XY = -e_i Z^2 \).

It follows \( XY e_i = e_i XY e_i = \sum_{i=1}^n e_i X e_j^\prime Y e_i \).

Consider paths of the following form:

\[
\begin{array}{ccc}
\alpha & k & \alpha^{-1} \\
\downarrow & \nearrow & \downarrow \\
i & i & j \\
e_j \gamma e_j^\prime Y e_i & e_i X e_j^\prime \rho e_j
\end{array}
\]

Where \( e_i X e_j^\prime Y e_i = e_i X e_j^\prime \rho e_j \gamma e_j^\prime Y e_i \).

Since each indecomposable projective of \( e(B_1^* G) e \) has simple socle and \( e(B_1^* G) e \) is a radical cube zero algebra, there are constants \( e_j \in C \) such that \( e_j \alpha^{-1} \alpha = e_i X e_j^\prime Y e_i = e_i X e_j^\prime \rho e_j \gamma e_j^\prime Y e_i \).

Therefore: \( e_i XY = (\sum_{j=1}^n c_j) \alpha^{-1} \alpha = t_i \alpha^{-1} \alpha = -e_i Z^2 \), with \( t_i \in C - \{0 \} \). This is \( t_i \alpha^{-1} \alpha + Z_i = 0 \), with \( Z_i = e_i e_i \).

In a similar way we obtain relations \( t_k \alpha^{-1} \alpha + Z_k = 0 \), \( \alpha^{-1} \alpha \) and \( \alpha \alpha^{-1} \) are paths in \( e(C_1^* G) e \).

The element \( Z \) of \( B_1^* \) anticommutes with \( X, Y \) and commutes with all the elements \( g \) of \( G \). Then \( Z \) anticommutes with \( \alpha \) and \( \alpha^{-1} \). This is \( Z \alpha = -Z \alpha \).

It follows: \( 0 = t_k \alpha \alpha^{-1} \alpha + Z_k \alpha = t_i \alpha \alpha^{-1} \alpha + Z_i \alpha \), then \( \alpha \alpha^{-1} \alpha + (Z_k^2/t_k) \alpha = \alpha \alpha^{-1} \alpha + \alpha Z_i^2/t_i \).

We have the following equalities: \( \alpha Z_i^2/t_i = (\alpha Z) Z_i/t_i = -Z \alpha Z/t_i = (Z^2 \alpha)/t_i = (Z_k^2/t_k) \alpha = (Z^2 \alpha)/t_k \). Therefore \( t_k = t_i \) for any pair of vertices \( i, k \) connected by an arrow. By connectivity, \( t_k = t_i \) for all \( k, i \).

Assume every element of \( K \) has a square root and let \( c = \sqrt{t_c} \), we can make a change of variables, \( z_k = Z/c \) to get relations: \( \{z_k \alpha + \alpha z_i \) for every arrow \( \alpha \), and \( \alpha \alpha^{-1} + z_k^2, \alpha^{-1} \alpha + z_i^2 \) = \( R \).

We have proved the algebra \( e(B_1^* G) e \) is isomorphic to the quiver algebra \( \hat{K}^\wedge Q / R \), with \( \hat{Q}_0 = Q_0 \) and \( \hat{Q}_1 = Q_1 \cup Q_0^\perp \cup \{z_i \mid i \in Q_0 \} \) and \( Q \) an Euclidean diagram.

Using Koszulity, \( e(B_1^* G) e \) is isomorphic to the quiver algebra \( \hat{K}^\wedge Q / R^\perp \), with \( \hat{Q}_0 = Q_0 \) and \( \hat{Q}_1 = Q_1 \cup Q_0^\perp \cup \{z_i \mid i \in Q_0 \} \) and \( Q \) an Euclidean diagram and \( R^\perp \) the orthogonal relations: \( R^\perp = \{z_k \alpha - \alpha z_i, \sum \alpha_j \alpha_j^{-1} - z_i^2, \sum \alpha_j^{-1} \alpha_j - z_k^2 \} \).
For the first Weyl algebra we have: \( e(A_1 \ast G)e = e(B_1 \ast G)e / (z - 1)e(B_1 \ast G)e \).

It follows \( A_1 \ast G \) is Morita equivalent to an algebra with quiver \( Q \), such that \( \hat{Q}_0 = Q_0 \) and \( \hat{Q}_1 = Q_1 \cup Q_1^{op} \) and \( Q \) an Euclidean diagram and relations \( \{ \sum \alpha_j \alpha_j^{-1} - e_k, \sum \alpha_j^{-1} \alpha_j - e_i \} \). This means, \( e(A_1 \ast G)e \) is the deformed preprojective algebra.

We state the previous results as a theorem:

**Theorem 8.** Let \( G \) be a finite subgroup of \( SL(2, K) \), with \( K \) a field of zero characteristic and containing the square root of each element. Let \( A_1 \) be the first Weyl algebra and \( B_1 \) its homogenized algebra. The group \( G \) acts as a grade preserving automorphism group of \( B_1 \), fixing \( Z \) and sending \( X \) and \( Y \) to a linear combination of \( X \) and \( Y \). Then \( G \) acts in the same way on \( B_1^\Gamma \), the Yoneda algebra of \( B_1 \), and \( G \) acts as a group of automorphisms of \( A_1 \). The skew group algebras \( B_1 \ast G, B_1^\Gamma \ast G \) and \( A_1 \ast G \) are Morita equivalent to the algebras defined by quivers and relations as follows:

i) The skew group algebra \( B_1 \ast G \) is Morita equivalent to the quiver algebra \( \hat{K}Q \)

\( / R \), with \( \hat{Q}_0 = Q_0 \) and \( \hat{Q}_1 = Q_1 \cup Q_1^{op} \) \( \{ z_i : v_i \in Q_0 \} \) and \( Q \) an Euclidean diagram and \( R = \{ z_k \alpha + \alpha z_i, \text{for every arrow } \alpha \text{, and } \alpha \alpha^{-1} + z_i^2, \alpha^{-1} \alpha + z_k^2 \} \).

ii) The skew group algebra \( B_1 \ast G \) is Morita equivalent to the quiver algebra \( \hat{K}Q \)

\( / R^\perp \), with \( \hat{Q}_0 = Q_0 \) and \( \hat{Q}_1 = Q_1 \cup Q_1^{op} \) \( \{ z_i : v_i \in Q_0 \} \) and \( Q \) an Euclidean diagram and \( R^\perp = \{ z_k \alpha - \alpha z_i, \sum \alpha_j \alpha_j^{-1} - z_i^2, \sum \alpha_j^{-1} \alpha_j - z_k^2 \} \).

iii) The skew group algebra \( A_1 \ast G \) is Morita equivalent to the algebra \( \hat{K}Q/ I \), such that \( \hat{Q}_0 = Q_0 \) and \( \hat{Q}_1 = Q_1 \cup Q_1^{op} \) and \( Q \) an Euclidean diagram and relations

\( I = \{ \sum \alpha_j \alpha_j^{-1} - e_k, \sum \alpha_j^{-1} \alpha_j - e_i \} \).

We want to sketch the situation for groups of the form \( G = G_1 \times G_2 \times \ldots \times G_n \), such that \( G_i \) is a finite subgroup of \( SL(2, C) \) acting as subgroups of the automorphism group of the homogenized Weyl algebra \( B_n \).

We will recall first the description by quivers and relations of the tensor product of two quiver algebras.

Let \( K \) be a field of characteristic zero and let \( \Lambda = KQ_1 / I_1 \) and \( \Gamma = KQ_2 / I_2 \) be two graded quiver algebras, \( Q \) the quiver of \( \Lambda \otimes K \Gamma \) and \( I \) an admissible ideal of \( KQ \) such that \( KQ / I \cong \Lambda \otimes K \Gamma \). We next describe \( Q \) and \( I \).

Let \( \{ e_1, e_2, \ldots, e_n \} \) and \( \{ f_1, f_2, \ldots, f_m \} \) be complete sets of orthogonal primitive idempotents of \( \Lambda \) and \( \Gamma \), respectively. Then \( \{ e_i \otimes f_j \mid 1 \leq i \leq n, 1 \leq j \leq m \} \) is a complete set of primitive orthogonal idempotents of \( \Lambda \otimes K \Gamma \). This means that the quiver \( Q \) has vertices \( Q_0 = (Q_1)_0 \times (Q_2)_0 \), where the pair \( (v_i, w_j) \) corresponds to the idempotent \( e_i \otimes f_j \) and \( v_i \) is the vertex corresponding to \( e_i \) and \( w_j \) the vertex corresponding to \( f_j \).

The arrows \( \alpha_k : v_i \to v_j \) in \( (Q_1)_1 \) correspond to elements \( a_k \in e_j r_\Delta e_i - e_j r_\Delta^2 e_i \) and the arrows \( \beta_k : v_i \to w_j \) of \( (Q_2)_1 \) correspond to elements \( b_k \in f_j r_\Delta f_i - f_j r_\Delta^2 f_i \). We have in \( Q \) arrows \( Q_1 = (Q_1)_1 \times (Q_2)_0 \cup (Q_1)_0 \cup (Q_2)_1 \). This is an arrow \( \alpha : v_i \to v_j \) in \( (Q_1)_1 \) and a vertex \( w_k \) of \( (Q_2)_0 \) corresponds an arrow \( (\alpha, w_k) : (v_i, w_k) \to (v_j, w_k) \) of \( Q_1 \) associated to \( a \otimes f_k \in e_i \otimes f_k(r_\Delta \otimes K \Gamma) e_j \otimes f_k - e_j \otimes f_k(r_\Delta \otimes K \Gamma) e_i \otimes f_k \). Similarly, to an arrow \( \beta : v_i \to w_j \) of \( (Q_2)_1 \) and a vertex \( v_k \) of \( (Q_1)_0 \) corresponds the arrow \( (v_k, \beta) : (v_k, w_i) \to (v_k, w_j) \) of \( Q_1 \) associated to \( e_k \otimes b \in e_k \otimes f_i(r_\Delta \otimes K \Gamma) e_k \otimes f_j - e_k \otimes f_i(r_\Delta \otimes K \Gamma) e_k \otimes f_j \).
Given paths \( \alpha_1 \alpha_2 \ldots \alpha_r : v_i \rightarrow v_j \) and \( \beta_1 \beta_2 \ldots \beta_s : w_k \rightarrow w_l \) in \( Q_1 \) and \( Q_2 \), respectively, there are paths \((\alpha_1 \alpha_2 \ldots \alpha_r, w) : (v_i, w) \rightarrow (v_j, w)\) and \((v, \beta_1 \beta_2 \ldots \beta_s) : (v, w_k) \rightarrow (v, w_l)\). Now it is clear that given readable relations \( \rho = \sum c_k \gamma_k \) in \( I_1 \), and \( \rho' = \sum b_k \gamma'_k \) in \( I_2 \) there are induced readable relations \( (\rho, w) = \sum c_k (\gamma_k, w) \) and \( (v, \rho) = \sum b_k (v, \gamma'_k) \) in \( KQ \).

If we have two arrows \( \alpha : v_2 \rightarrow v_1 \) and \( \beta : w_2 \rightarrow w_1 \) in \( Q_1 \) and \( Q_2 \), respectively, the we have maps:

\[
\begin{align*}
\Lambda \otimes K \Gamma e_1 \otimes f_1 & \overset{a \otimes f_1}{\longrightarrow} \Lambda \otimes K \Gamma e_2 \otimes f_1 \\
\downarrow e_1 \otimes b & \quad \downarrow e_2 \otimes b \\
\Lambda \otimes K \Gamma e_2 \otimes f_1 & \overset{a \otimes f_2}{\longrightarrow} \Lambda \otimes K \Gamma e_2 \otimes f_2
\end{align*}
\]

making the diagram commute.

Hence we have in \( I \) a commutative relation \( \zeta_{\alpha, \beta} = (v_2, \beta)(\alpha_1, w_1) - (\alpha_1, w_2)(v_1, \beta) \).

Then \( I \) is the ideal generated by \( \{ \rho \times (Q_2)_0 \mid \rho \) a readable relation in \( I_1 \} \cup \{ (Q_1)_0 \times \rho' \mid \rho' \) a readable relation in \( I_2 \} \cup \{ \zeta_{\alpha, \beta} \mid \alpha \in (Q_1)_1, \beta \in (Q_2)_1 \} \).

We obtained the description of \( Q \) and \( I \) such that \( KQ/I \cong \Lambda \otimes K \Gamma \).

We return to the case \( G = G_1 \times G_2 \) with \( G_1, G_2 \) finite subgroups of \( \text{SL}(2, \mathbb{C}) \) acting in the way above described as an automorphism group of \( B_2 \).

Taking complete sets of primitive orthogonal idempotents \( \{ e_1, e_2, \ldots, e_k \} \) and \( \{ f_1, f_2, \ldots, f_i \} \) of \( KG_1 \) and \( KG_2 \), respectively. Letting \( e, f \) be the idempotents \( e = \sum_{i=1}^k e_i \) and \( f = \sum_{i=1}^k f_i \), we saw above \( e (B_1 \ast G_1) e \) and \( f (B_1 \ast G_2) f \) are preprojective algebras of Euclidean diagrams, \( e (B_1 \ast G_1) e \cong KQ_1/I_1 \) and \( f (B_1 \ast G_2) f \cong KQ_2/I_2 \).

The quiver \( \hat{Q}_1 \) is of the form \( \hat{Q}_1 = (Q_1)_0, \hat{Q}_1 = (Q_1)_1 \cup (Q_1)^{op}_1 \cup \{ z_i \mid z_i \in (Q_1)_0 \} \) and \( \hat{Q}_2 = (Q_2)_0, \hat{Q}_2 = (Q_2)_1 \cup (Q_2)^{op}_1 \cup \{ z_i \mid z_i \in (Q_2)_0 \} \) with \( Q_1 \) and \( Q_2 \) Euclidean diagrams. and \( I_1, I_2 \) the ideals described above.

Then \( e \otimes f (B_1 \ast G_1) e \otimes K \hat{Q}_1/I_1 \otimes K \hat{Q}_2/I_2 \) is the tensor product of two algebras related to preprojective algebras of Euclidean diagrams, as described above.

We have \( \sum_{i,j} e_i \otimes f_j \) and \( \sum_{i,j} Z e_i \otimes f_j \), hence \( (Z \otimes 1 - 1 \otimes Z) \) is the ideal generated by readable relations \( e_i \otimes Z f_j - Z e_i \otimes f_j \).

Therefore in \( (e, f) (B_2 \ast (G_1 \times G_2)) (e, f) \cong (e \otimes f (B_1 \ast G_1) B_1 \ast G_2) e \otimes f / (Z \otimes 1 - 1 \otimes Z) \) we are identifying the two loops \( e_i \otimes Z f_j \) and \( Z e_i \otimes f_j \).

Then \( B_2 \ast (G_1 \times G_2) \) can be completely described by quiver and relations: it has the same quiver as the tensor product of two preprojective algebras of Euclidean diagrams and in addition a loop in each vertex, the relations are naturally induced from those in each preprojective algebra as studied in detail above. We leave to the reader to write explicitly the quiver and relations as well as the induced relations in \( (e, f) A_2 (G_1 \times G_2) (e, f) \cong ((e, f) (B_2 \ast (G_1 \times G_2)) (e, f)) / (Z - 1) \). Observe that from this could also give a full description of the relations in \( B_2 \ast (G_1 \times G_2) \) and \( A_2 (G_1 \times G_2) \).

By induction a full description by quivers and relations of the basic algebras Morita equivalent to \( B_n \ast (G_1 \times G_2 \times \ldots \times G_n) \) can be given, the algebras come from iterated tensor products of preprojective algebras of Euclidean diagrams and additional loops for each vertex and naturally induced relations.
4. The algebras $C[X] * G$ and $C[X]^G$, with $G$ a finite subgroup of $Gl(n, C)$, such that no $\sigma \in G$, $\sigma \neq 1$, has a fixed point.

In this section we study the relations between the categories of finitely generated modules mod$_{C[X] * G}$ and mod$_{C[X]^G}$, where $C[X] * G$ is the skew group algebra and $C[X]^G$ the invariant ring of a finite subgroup of $Gl(n, C)$, such that no $\sigma \in G$, $\sigma \neq 1$, has a fixed point.

More precisely, we prove that after killing the modules of finite length, the categories mod$_{C[X] * G}$ and mod$_{C[X]^G}$ are equivalent. These results generalize known results for $C[X, Y]$ and finite subgroups of $Sl(2, C)$, [C-B].

**Lemma 1.** Assume $G$ is a finite subgroup of $Gl(n, C)$. Then $G$ acts naturally on the polynomial ring in $n$ variables $C[X]$ and the invariant ring $C[X]^G$ is the center of $C[X] * G$.

**Proof.** It is clear $C[X]^G$ is contained in the center of $C[X] * G$. Let $v \in Z(C[X] * G)$ be an element in the center, $v = \sum_{g \in G} c_g g$ and $c_g \in C[X]$.

For any $X_i v = v X_i$ for any $X_i$. Then $X_i v = \sum_{g \in G} c_g X_i g = \sum_{g \in G} c_g g X_i = \sum_{g \in G} c_g X_i^g g$. If $c_g \neq 0$, then $X_i^g = X_i$. Since $X_i$ is arbitrary, $g = 1$ and $v \in C[X]$.

Now, $g v = v^g g = v g$ implies $v = v^g$ for all $g \in G$ and $v \in C[X]^G$. $\square$

It is well known [Be], [Stu], $C[X]^G$ is an affine algebra and $C[X]^G \rightarrow C[X]$ an integral extension, in particular, dim $C[X]^G = \dim C[X]$ and the maximal spectrums $\text{max spec} C[X]^G$ and $\text{max spec} C[X]$ are isomorphic [Ku].

Moreover, it is known [Be] that given a prime $p$ of $C[X]^G$ and primes $P$, $Q$ of $K[X]$ above $p$, this is: $P \cap C[X]^G = Q \cap C[X]^G = p$, there exists $\sigma \in G$ such that $P^\sigma = Q$ and for any $\sigma \in G$, $P^\sigma \cap C[X]^G = P \cap C[X]^G = p$.

In particular for $m$ a maximal ideal of $C[X]^G$ and maximal ideals $n_1$, $n_2$ of $C[X]$ with $n_1 \cap C[X]^G = n_2 \cap C[X]^G = m$, there exists $\sigma \in G$ with $n_2 = n_1^\sigma$ and for any $\sigma \in G$, $n_1^\sigma$ is a maximal ideal with $n_1^\sigma \cap C[X]^G = n_1 \cap C[X]^G = m$.

The points $v$ of $V = \bigoplus_{i=1}^n C X_i$ correspond to maximal ideals $n_v$ of $C[X]$ . In particular, $0 \in V$ corresponds to the irrelevant maximal ideal $n_0 = (X_1, X_2, ... X_n)$ of $C[X]$.

By hypothesis, for $v \neq 0$, the orbit of $v$ under $G$, $O(v) = \{ \sigma(v) \mid \sigma \in G \}$ has order $|O(v)| = |G|$. The point $v$ corresponds to a maximal ideal $n$ such that $n \cap C[X]^G = m$ and there are $|G|$ ideals above $m$ and they are precisely $\{ n^\sigma \mid \sigma \in G \}$.

This is $m$ is an unramified prime.

In the case $v = 0$, for any $\sigma \in G$, $\sigma(v) = v$ and the maximal ideal $n_0$ satisfies $n_0^\sigma = n_0$ for all $\sigma \in G$.

The maximal ideal $n_0 \cap C[X]^G = m_0$ has a unique maximal ideal, $n_0$, above it.

We consider the two cases separately.

a) The ideal $m$ of $C[X]^G$ is maximal and different of $m_0$.

Let $n^\sigma$ with $\sigma \in G$ be the set of all maximal ideals of $C[X]$ above $m_0$, hence $mC[X] \subseteq \bigcap_{\sigma \in G} n^\sigma$. It follows $\sqrt{mC[X]} = \bigcap_{\sigma \in G} n^\sigma$, the radical of $mC[X]$.

It is clear that for any $\tau \in G$, $(\bigcap_{\sigma \in G} n^\sigma)^\tau = \bigcap_{\sigma \in G} n^{\sigma \tau} = \bigcap_{\sigma \in G} n^\sigma$ and $\sqrt{mC[X]}$ is $G$-invariant.

b) The ideal $m$ of $C[X]^G$ is not maximal and $m_0$.

In this case, it is easy to see that $\sqrt{mC[X]} = C[X]^G$. Hence $mC[X] = \bigcap_{\sigma \in G} n^\sigma$. It follows $\sqrt{mC[X]} = \bigcap_{\sigma \in G} n^\sigma$, the radical of $mC[X]$.
By the Chinese Remainder Theorem, $C[X]/\sqrt{mC[X]} = \bigcap_{\sigma \in G} C[X]/n^\sigma = \prod_{\sigma \in G} C[X]/n^\sigma$.

Since $C[X]/n \cong C$$V$, $\prod_{\sigma \in G} C[X]/n^\sigma \cong \prod_{\sigma \in G} C\sigma$, with $1 = \sum_{\sigma \in G} v^\sigma$, and $\{ v^\sigma \}_{\sigma \in G}$ is a complete set of primitive orthogonal idempotents.

For the skew group algebra we have the following isomorphism:

$$C[X] \ast G/\sqrt{mC[X]} \ast G \cong (C[X]/\sqrt{mC[X]}) \ast G \cong (\prod_{\sigma \in G} C\sigma) \ast G.$$  

Since $\prod_{\sigma \in G} C\sigma$ semisimple, it follows by $[,]$, $C[X] \ast G/\sqrt{mC[X]} \ast G$ is semisimple.

Let $S$ be $S = \prod_{\sigma \in G} C\sigma$, the group $G$ acts transitively on the basis $\{ v^\sigma \}_{\sigma \in G}$ of $S$.

We claim $S \ast G$ is simple, it will be enough to prove that the center of $S \ast G$ is $C$.

It is clear that $C$ is contained in the center. Let $z \in Z(S \ast G)$ be an element of the center, $z = \sum s_\sigma \sigma$. Each $s_\sigma$ has form $s_\sigma = \sum_{\tau \in G} c_{\sigma,\tau} v^\tau$, then $v^\sigma = \sum_{\tau \in G} c_{\sigma,\tau} v^\tau = c_{\sigma,\sigma} v^\sigma$.

It follows $v^\sigma z = \sum_{\sigma \in G} v^\sigma s_\sigma = \sum_{\sigma \in G} c_{\sigma,\rho} v^\rho = z v^\sigma = \sum_{\sigma \in G} s_\sigma v^\rho = \sum_{\sigma \in G} s_\sigma v^\sigma = \sum_{\sigma \in G} c_{\rho,\sigma} v^\rho$. Therefore: $c_{\rho,\sigma} v^\sigma = c_{\sigma,\rho} v^\rho$ for all $\rho, \sigma$, and $c_{\sigma,\sigma} = 1$ for $\sigma \neq 1$.

It follows $z \in S$, this is $z = \sum c_\rho v^\rho$ and $c_\rho \in C$. Hence, $\tau z = \sum c_\rho v^\rho = \sum_{\rho \in G} c_\rho v^\rho = \tau z$ and $\sum_{\rho \in G} c_\rho v^\rho = \sum c_\rho v^\rho = \sum_{\rho \in G} c_\rho v^\rho = c_{\rho,\tau} v^\rho$. It follows $c_{\rho,\tau} = c_\rho$ for all $\rho, \tau$, in particular, $c_1 = c$ and $z = c \sum v^\rho = c_1 1$.

Since $S^G \subset Z(S \ast G)$ we have proved $S^G = Z(S \ast G) = C$.

The element $e = 1/|G| \sum g$ is an idempotent of $KG$, hence a non zero idempotent of $(C[X]/\sqrt{mC[X]}) \ast G$ and $e((C[X]/\sqrt{mC[X]}) \ast G) \neq 0$.

Being the algebra $(C[X]/\sqrt{mC[X]}) \ast G$ simple and the ideal:

$$((C[X]/\sqrt{mC[X]}) \ast G)e((C[X]/\sqrt{mC[X]}) \ast G)$$

non zero

$$(C[X]/\sqrt{mC[X]}) \ast G = e((C[X]/\sqrt{mC[X]}) \ast G) = e(C[X]/\sqrt{mC[X]}) \ast G).$$

Case 2.

In this case we have: $C[X] \ast G/\sqrt{m_0C[X]} \ast G \cong (C[X]/n_0) \ast G \cong CG$.

Let $L_1, L_2, \ldots, L_s$ the two sided maximal ideals of $CG$ and $L_1, L_2, \ldots, L_s$ maximal two sided ideals $C[X] \ast G$ containing $\sqrt{m_0C[X]} \ast G$. In particular, $C[X] \ast G/L_i \cong CG/L_i \cong C$ and $\bigcap_{i=1}^s L_i = n_0 \ast G$.

Let $\{ m_i \}_{i \in I}$ be the set of maximal ideals of $C[X]^G$ different from $m_0$, we have a natural homomorphism:

$$\Psi: C[X] \ast G \to \prod_{i \in I} C[X]/\sqrt{m_iC[X]} \ast G \times C[X] \ast G/m_0 \ast G,$$

given by $\Psi(\sum g_i) = (\sum_{g_i \in G} r_ig_i + \sqrt{m_iC[X]} \ast G), \sum_{g_i \in G} r_ig_i + n_0 \ast G)$, which has kernel $\Psi = \cap \bigcap_{i \in I} n_0^G = \cap \bigcap_{i \in I} n_0^G = 0$, since $\cap \bigcap_{i \in I} n_0^G = 0$, the intersection of all maximal ideals of $C[X]$. The map $\Psi$ is an injective ring homomorphism.

Denote by $\Sigma$ the product $\prod S_i \ast G = \Sigma$ and $\Sigma = \Sigma \times CG$. 

The map $\Psi$ sends the idempotent $e = 1/|G| \sum g$, into $\hat{e} = ((\tau, e))$, where $\tau = e + \sqrt{m_iC[X]eG}$.

Then $\sum \hat{e} \sum = (\prod_{i \in I} S_i \ast GeS_i \ast G) \times CGeCG = \prod_{i \in I} S_i \ast G \times Ce = \Sigma \times Ce$.

It is clear that $\Psi(C[X] \ast GeC[X] \ast G)$ is contained in $\sum \hat{e} \sum \Psi(C[X] \ast G) = (\Sigma \ast Ce) \cap \Psi(C[X] \ast G)$.

We want to prove they are equal.

Let $((\tau_i), ce)$ be an element of $(\Sigma \times Ce) \cap \Psi(C[X] \ast G)$. This means that there exists an element $r = \sum_{g \in G} r_g g \in C[X] \ast G$ such that $\tau = \tau_i$ for all $i \in I$ and $\tau = e\tau$ in $C[X] \ast Ge/n_0 \ast Ge \cong CGe$. Then $\sum_{g \in G} r_g g = ce + he$ with $h$ a polynomial without constant term. This implies for all $g \in G$, $r_g = c + h$ and $\sum_{g \in G} r_g g = (c + h)e \in C[X] \ast GeC[X] \ast G$. It follows $(\Sigma \times Ce) \cap \Psi(C[X] \ast G) = \Psi(C[X] \ast GeC[X] \ast G)$.

We have a commutative exact diagram:

\[
\begin{array}{ccc}
0 & \rightarrow & C[X] \ast GeC[X] \ast G \\
\downarrow & & \downarrow \\
0 & \rightarrow & \prod_{i \in I} S_i \ast G \times Ce \\
\downarrow & & \downarrow \\
0 & \rightarrow & C[X] \ast G \\
\downarrow & & \downarrow \\
0 & \rightarrow & C[X] \ast G/\mathbb{C}[X] \ast GeC[X] \ast G \\
\downarrow & & \downarrow \\
0 & \rightarrow & CG(1 - e)
\end{array}
\]

It follows $C[X] \ast G/\mathbb{C}[X] \ast GeC[X] \ast G$ is a subalgebra of a semisimple algebra, hence it is a finite dimensional $C$-algebra.

We can identify the category of $C[X] \ast G/\mathbb{C}[X] \ast GeC[X] \ast G$-modules with the category of $C[X] \ast G$-modules $M$ with $eM = 0$.

Let’s assume more generally that $R$ is a ring and $e$ an idempotent of $R$. Then there is a functor $\text{Hom}_R(Re, -) : \text{Mod}_R \rightarrow \text{Mod}_{Re}$, which is exact and it has a left adjoint $Re \otimes_{Re} e\text{-}Re$ such that $\text{Hom}_R(Re, Re \otimes_{Re} e\text{-}Re M) = M$.

It follows $\text{Hom}_R(Re, -)$ is a dense functor with kernel $\text{Mod}_{R/Re}$. The category $A = \text{Mod}_{R/Re} = \{ M \in \text{Mod}_R \mid eM = 0 \}$ is a dense (Serre) subcategory of $\text{Mod}_R$. [P]

Given an exact sequence of $R$-modules: $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ applying the exact functor $\text{Hom}_R(Re, -)$ we obtain an exact sequence of $e\text{-}Re$-modules: $0 \rightarrow eL \rightarrow eM \rightarrow eN \rightarrow 0$, hence $eM = 0$ if and only if $eN = eL = 0$.

Given a dense subcategory $A$ of $\text{Mod}_R$ we define the multiplicative system of all maps $f : M \rightarrow N$ such that the kernel and the cokernel of $f$ are in $A$.

We define a quotient category $\text{Mod}_R/A = (\text{Mod}_R)_\Sigma$ (see [Ga], [P], [Mi]).

The category $(\text{Mod}_R)_\Sigma$ is abelian and the quotient functor $\pi : \text{Mod}_R \rightarrow (\text{Mod}_R)_\Sigma$ is exact an it has the following universal property: Given an abelian category $C$ and an exact functor $F : \text{Mod}_R \rightarrow C$ such that $F(X) = 0$ for all $X \in A$, there is a unique exact functor $H : (\text{Mod}_R)_\Sigma \rightarrow C$ such that $H\pi = F$. 

Theorem 9. Let $R$ be a ring and $e$ an idempotent of $R$ and $A$ the category $A = \{ M \in \text{Mod}_R \mid eM = 0 \}$. Then there is a commutative diagram of categories and functors:

\[
\begin{array}{ccc}
\text{Mod}_R & \xrightarrow{\text{Hom}_R(Re,-)} & \text{Mod}_{eR} \\
\downarrow \pi & = & \downarrow \cong \\
\text{Mod}_{R/A} & \xrightarrow{H} & \text{Mod}_{eR}
\end{array}
\]

with $H$ an equivalence.

Proof. 1) The functor $H$ is dense.

Since $\text{Hom}_R(Re,-)$ is dense, it follows $H$ is dense.

2) The functor $H$ is full.

Let $M$ and $N$ be $R$-modules and $\text{Re}_e eM \xrightarrow{\mu} M$ the map given by multiplication, taking the kernel and the cokernel of $\mu$ we get an exact sequence:

$0 \rightarrow L \rightarrow \text{Re}_e eM \xrightarrow{\mu} M \rightarrow M/\text{Re} M \rightarrow 0$.

From the exact sequence:

$0 \rightarrow eL \rightarrow e(\text{Re}_e eM) \xrightarrow{\mu} eM \rightarrow e(M/\text{Re} M) \rightarrow 0$ and the fact $e\mu$ is an isomorphisms $eL = 0 = e(M/\text{Re} M)$.

Hence the multiplication maps $\text{Re}_e eM \xrightarrow{\mu} M$ and $\text{Re}_e eN \xrightarrow{\mu'} N$ are isomorphisms in $\text{Mod}_{R/A}$.

A map in $\text{Mod}_{R/A}$ such that when we apply the functor $\text{Hom}_R(Re,-)$ we obtain $\text{Hom}_R(\text{Re}_e eM)$ maps:

$\text{Hom}_R(Re,\mu) \xrightarrow{\mu} \text{Hom}_R(Re,\mu' \otimes f)$ is an isomorphism with $\text{Hom}_R(Re,\mu), \text{Hom}_R(Re,\mu')$ isomorphisms.

Hence $H(\mu' \otimes f\mu^{-1}) = f$.

3) The functor $H$ is faithful.

Let $s \xrightarrow{\mu} f$ be a map in $\text{Mod}_{R/A}$. Then the kernel and the cokernel of $s$ are in $A$. Assume $H(fs^{-1}) = 0$.

Hence, $H(fs^{-1}) = \text{Hom}_R(Re,f)\text{Hom}_R(Re,s)^{-1} = 0$ and $\text{Hom}_R(Re,s)$ an isomorphism implies $\text{Hom}_R(Re,f) = ef = 0$.

From the commutative diagram:

$0 \rightarrow eW \xrightarrow{j} W \xrightarrow{f} \text{Mod}_{R/A}$

we get $fj = 0$. But $j : eW \rightarrow W$ is an isomorphism in $\text{Mod}_{R/A}$. Therefore: $f = 0$ and $fs^{-1} = 0$ in $\text{Mod}_{R/A}$. \qed

We come back now to the case of a finite subgroup $G$ of $\text{Gl}(n,C)$ such that no $\sigma \in G$ different from the identity has a fixed point, $G$ acting as a group of automorphism of the polynomial ring $C[X]$ in $n$ variables the idempotent $e = 1/|G| \sum g \in G$ is an idempotent of the skew group algebra $C[X]^G$ such that $eC[X]^G Ge$ is isomorphic to the group of invariants $C[X]^G$. The algebra $C[X]^G$ is a sub algebra of $C[X]$ and $C[X]$ is finitely generated as $C[X]^G$-module. Let $f_1, f_2, ..., f_m$ be the generators. The epimorphism $\rho : \oplus C[X]/G \rightarrow C[X]$ given
by \( \rho(\gamma_1,\gamma_2,...,\gamma_m) = \sum_{i=1}^{m} \gamma_if_i \), extends to a map: \( \hat{\rho} : \oplus C[X]^G e \to C[X]e \) of left\( C[X]^G \)-modules \( \hat{\rho}(\gamma_1e,\gamma_2e,...,\gamma_me) = \sum_{i=1}^{m} \gamma_if_ie \). Let \( e\lambda = \lambda e = e\lambda e \in C[X]^Ge \).

Then \( \hat{\rho}(\gamma_1\lambda,\gamma_2\lambda,\gamma_me) = \hat{\rho}(\gamma_1\lambda e,\gamma_2\lambda e,\gamma_me) = \sum_{i=1}^{m} \gamma_i\lambda f_ie = (\sum_{i=1}^{m} \gamma_i f_i)e\lambda = (\sum_{i=1}^{m} \gamma_i f_i)\lambda\) and \( \hat{\rho} \) is a map of \( C[X]^G - eC[X]^G e \) bimodules. If \( M \) is a \( eC[X]^G e \)-module of finite dimensional over \( C \), from the epimorphism: \( \hat{\rho}\otimes 1 : \oplus C[X]^G e\otimes eC[X]^G e \) \( M \to C[X]e \otimes eC[X]^G e \) and the isomorphism: \( C[X]^G e \otimes eC[X]^G e \) \( M \cong eC[X]^G e \otimes eC[X]^G e \) \( M \) we obtain that \( C[X]e \otimes eC[X]^G e = C[X]e \otimes eC[X]^G e \) \( M \) is finite dimensional over \( C \).

It is also clear that if \( M \) is a \( C[X]^G e \) \( C[X]^G e \) \( M \) is finite dimensional over \( C \).

To simplify the notation we will write \( R = C[X]^G e \) and \( T = eC[X]^G e \) \( C[X]^G e \) \( M \) and denote by \( S_R \) and \( S_T \) the categories of finite dimensional \( R \) and \( T \)-modules, respectively.

The categories \( S_R \) and \( S_T \) are dense subcategories of the categories of finitely generated, \( \text{mod}_R, \text{mod}_T, R \) and \( T \)-modules, respectively.

Then we have:

**Theorem 10.** Let \( G \) be a finite subgroup of \( \text{Gl}(n,C) \) such that no \( \sigma \in G \) different from the identity has a fixed point, \( G \) acting as a group of automorphism of the polynomial ring \( C[X] \) in \( n \) variables and \( e \) the idempotent \( e = 1/|G| \sum g \) of the skew group algebra \( C[X]^G \). Writing \( R = C[X]^G e \) and \( T = eC[X]^G e \equiv C[X]^G e \) and denoting by \( S_R \) and \( S_T \) the categories of finite dimensional \( R \) and \( T \)-modules, respectively and by \( \text{Hom}_R, \text{mod}_R, \text{mod}_T, \) the categories of finitely generated, \( R \) and \( T \)-modules, respectively. Then there is a commutative diagram of categories and functors:

\[
\begin{array}{ccc}
\text{mod}_R & \xrightarrow{\text{Hom}_R(\text{Re},-)} & \text{mod}_T \\
\downarrow \pi_R & & \downarrow \pi_T \\
\text{mod}_R/S_R & \xrightarrow{H} & \text{mod}_T/S_T
\end{array}
\]

Before proving the theorem, observe that there is a graded version.

The ring \( e(C[X]^G)e \) is a positively graded \( C \)-algebra with grading \( (e(C[X]^G)e)_k = e((C[X]^G)e)_k \), hence the isomorphism of \( C \)-algebras \( e(C[X]^G)e \cong C[X]^G \) induces a positive grading on \( C[X]^G \). In what follows we will assume \( C[X]^G \) has this grading and denote by \( R \) and \( T \) the positively graded \( C \)-algebras \( C[X]^G \) and \( C[X]^G \), respectively. If we denote by \( g_R \) and \( g_T \) the categories of finitely generated graded \( R \) and \( T \) modules, respectively, and degree zero maps. Following [1], [2], we call the quotient categories \( g_R/S_R \) and \( g_T/S_T \) the categories of tails \( \text{tails}_{SR} \) and \( \text{tails}_{ST} \), another notation used in [MV2] is \( Qgr_R \) and \( Qgr_T \).

**Theorem 11.** Let \( G \) be a finite subgroup of \( \text{Gl}(n,C) \) such that no \( \sigma \in G \) different from the identity has a fixed point, \( G \) acting as a group of automorphism of the polynomial ring \( C[X] \) in \( n \) variables and \( e \) the idempotent \( e = 1/|G| \sum g \) of the skew group algebra \( C[X]^G \). Writing \( R = C[X]^G e \) and \( T = eC[X]^G e \equiv C[X]^G e \) and considering both as positively graded algebras, denoting by \( S_R \) and \( S_T \) the categories of finite dimensional graded \( R \) and \( T \)-modules, respectively and by \( g_R, g_T \),
the categories of finitely generated, $R$ and $T$-modules, respectively and by tails$_R$ and tails$_T$ the quotient categories $\text{gr}_R/S_R$ and $\text{gr}_T/S_T$. Then there is a commutative diagram of categories and functors:

$$
\begin{array}{ccc}
\text{gr}_R & \xrightarrow{\text{Hom}_R(Re,-)} & \text{gr}_T \\
\downarrow \pi_R & & \downarrow \pi_T \\
tails_R & \xrightarrow{H} & tails_T
\end{array}
$$

with $H$ an equivalence.

We will prove only the ungraded case, the other follows with the same line of arguments.

Proof. 1) The functor $H$ is dense.

It follows as above from the fact $\text{Hom}_R(Re,-)$ is dense.

2) The functor $H$ is full.

Let $X$, $Y$ be objects in $\text{mod}_R/S_R$ and consider a map $\varphi : H(X) \to H(Y)$ in $\text{mod}_T/S_T$, with $H(X) = H\text{om}_R(Re,X)$, $H(Y) = H\text{om}_R(Re,Y)$ and $\varphi$ a "roof"

$W \xrightarrow{\varphi} Y$, where the kernel $L$, and the cokernel $N$, of $\varphi$ are of finite length.

Applying the tensor functor $Re \otimes eRe -$ and composing with multiplication we obtain the maps:

$$
\begin{array}{ccc}
\mu' & & \\
\mu & & \\
& Re \otimes eRe \sigma & \\
& \sigma & \\
X & \xrightarrow{\sigma} & Re \otimes eRe X
\end{array}
\quad
\begin{array}{ccc}
\mu' & & \\
\mu & & \\
& Re \otimes eRe \sigma & \\
& \sigma & \\
Y & \xrightarrow{\sigma} & Re \otimes eRe Y
\end{array}
$$

We also have exact sequences:

$$
\begin{array}{ccc}
Re \otimes eRe L & \xrightarrow{Re \otimes eRe j} & Re \otimes eRe W \\
& & Re \otimes eRe s \\
& & Re \otimes eRe Im s \to 0
\end{array}
\quad
\begin{array}{ccc}
Re \otimes eRe Im s & \to & Re \otimes eRe \sigma X \\
& & Re \otimes eRe N \to 0
\end{array}
$$

By the above observation, $Re \otimes eRe L$ and $Re \otimes eRe N$ are finite dimensional $C$-vector spaces, then $U = \text{Im} Re \otimes eRe j = Ker Re \otimes eRe s$ is finite dimensional.

We have a commutative exact diagram:

$$
\begin{array}{cccccccc}
0 & & & & & & & 0 \\
& & & & & & & \\
0 & & & & & & & Z \\
& & & & & & & \\
0 & \to & U & \to & Re \otimes W & \to & Re \otimes \text{Im} s & \to & 0 \\
\downarrow & & & & & & & \downarrow \downarrow \\
0 & \to & V & \to & Re \otimes W & \to & Re \otimes eX & \to & Re \otimes N & \to 0 \\
\downarrow & & & & & & & \downarrow \downarrow \downarrow \\
0 & & & & & & & Z & \to & Re \otimes N & Re \otimes N & \to 0 \\
\downarrow & & & & & & & \downarrow & & & & \downarrow \\
0 & & & & & & & 0 & & 0 & 0
\end{array}
$$

Applying the functor $\text{Hom}_R(Re,-)$ to the diagram we obtain the commutative exact diagram:
Since the map $i$ is a monomorphism $eZ=0$ and by Theorem 9, $Z$ is finite dimensional, but both $U$ and $Z$ finite dimensional implies $V$ is finite dimensional.

By the above remark, $N$ of finite dimension implies $\text{Re} \otimes N$ is of finite dimension and both the kernel and cokernel of $\text{Re} \otimes e \text{Re}s$ are in $S_R$.

It is clear that $H(\mu(\text{Re} \otimes f)(\text{Re} \otimes s)^{-1}) = \text{Hom}_R \text{Re}, \mu(\text{Re} \otimes f))\text{Hom}_R(\text{Re}, \text{Re} \otimes s)^{-1} = fs^{-1}$.

3) The functor $H$ is faithful.

Let $s$ ↗ $\downarrow g$ be a map in $\text{mod}_{R/S_R}$, where $s$ has kernel $L$ and cokernel $N$, both finite dimensional. Assume $H(gs^{-1}) = 0$.

There exist an exact sequence: $0 \to eL \to eW \to eX \to N \to 0$ and $L, N \in S_R$ implies $eL, eN \in S_T$. Then we have in $\text{mod}_{T/S_T}$ the map:

$$\begin{array}{ccc}
&eW & eG \\
&\downarrow eS & \downarrow eG \\
eX & eY \\
\end{array}$$

which by assumption satisfies $(eg)(es)^{-1} = H(gs^{-1}) = 0$.

Then we have a commutative diagram:

$$\begin{array}{ccc}
&eW & eG \\
&\downarrow es & \downarrow es \\
eX & eY \\
\end{array}$$

with $esr_1 = t'r_2$ having kernel and cokernel in $S_T$ and $egr_1 = 0$.

We want to see first that $r_1$ has kernel and cokernel in $S_T$.

We have a commutative exact diagram:

$$\begin{array}{ccc}
0 & \downarrow eL & 0 \\
\downarrow & \downarrow & \downarrow \\
0 \to B' & \to Z & eW \to D' \to 0 \\
\downarrow & 1 & 1 \downarrow \\
0 \to B & \to Z \to eX \to D \to 0 \\
\downarrow & \downarrow & \downarrow \\
& eN & \\
\downarrow & \\
0 & \\
\end{array}$$
with $B$, $D$, $eL$, $eN$ in $S_T$. We need to check that $B'$ and $D'$ are also in $S_T$.

From the commutative exact diagram:

\[
\begin{array}{cccccc}
0 & \rightarrow & B' & \rightarrow & Z & \rightarrow & E' & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & B & \rightarrow & Z & \rightarrow & E & \rightarrow & 0 \\
\end{array}
\]

We obtain the following commutative exact diagram:

\[
\begin{array}{cccccc}
0 & \rightarrow & E'' & \rightarrow & eL & \rightarrow & D'' & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & E' & \rightarrow & eW & \rightarrow & D' & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & E & \rightarrow & eX & \rightarrow & D & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & eN & \rightarrow & eN & \rightarrow & 0 \\
\end{array}
\]

Since $D''$ is a quotient of $eL$ and $\text{Im } u$ a submodule of $D$ it follows both $D''$ and $\text{Im } u$ are in $S_T$. Therefore: $D'$ is in $S_T$.

From the equality $eg\gamma = 0$ it follows $eg$ factors through the finite dimensional module $D'$. Tensoring with $R$, the map: $R \otimes e : R \otimes eReW \rightarrow R \otimes eReY$ factors through a finite dimensional $R$-module.

We have a commutative exact diagram:

\[
\begin{array}{cccccc}
\text{Re} \otimes eReW & \rightarrow & W & \rightarrow & Y & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{Re} \otimes eg & \rightarrow & \mu & \rightarrow & Y & \rightarrow & 0 \\
\end{array}
\]

where the multiplication maps $\mu$ and $\mu'$ have kernels and cokernels in $S_R$. It follows that a the nap $g\mu$ factors through a module of finite length. This implies $g\mu = 0$ in $\text{mod}_R/S_R$ and $\mu$ an isomorphism in $\text{mod}_R/S_R$ implies $g = 0$ and $s^{-1}g = 0$. \hfill \square

For the benefit of the reader we prove the last claim in more detail.

**Lemma 2.** Assume $H : L \rightarrow M$ is a map in $\text{mod}_R$ which factors through a finite dimensional module. Then $h = 0$ in $\text{mod}_R/S_R$.

**Proof.** $h = vu$ with $u : L \rightarrow U$, $v : U \rightarrow M$ homomorphisms and $U$ a $R$-module of finite dimension. Changing $U$ by $\text{Im } u$, we can assume $u$ is an epimorphism.

We have a commutative exact diagram:
Since $u$ is an epimorphism, then $\text{Im} \, v = \text{Im} \, h$ and $U'$ are of finite dimension. Therefore the injective map $j : B \to L$ has kernel and cokernel in $S_R$.

We have a commutative diagram:

\[
\begin{array}{ccc}
L & \xrightarrow{h} & M \\
1 & \searrow & \downarrow \quad j \\
B & \searrow & 0 \\
& U' & \downarrow \\
& B' & \xrightarrow{v} \\
& 0 \\
\end{array}
\]

We have proved $h = 0$ in $\text{mod}_R/S_R$.

We comeback to the last claim of the theorem

Let $s : Z \to L$ be a map in $\text{mod}_R$ whose kernel and cokernel are finite dimensional, $h : L \to M$ a map such that $hs$ factors through a module of finite dimension and let $j : B \to Z$ be the kernel of $hs$. Then $j$ has kernel and cokernel of finite dimension and $sj$ has kernel and cokernel of finite dimension. We have a commutative diagram:

\[
\begin{array}{ccc}
L & \xrightarrow{h} & M \\
1 & \searrow & \downarrow \quad j \\
B & \searrow & 0 \\
& Z & \downarrow \\
& s & \downarrow \\
\end{array}
\]

It follows $h = 0$ in $\text{mod}_R/S_R$.

**Remark 1.** In our case $R = C[X] * G$, $T = e(C[X] * G)e \cong C[X]^G$, the algebra $C[X]^G$ is affine this is, the is a polynomial ring $C[Y_1, Y_2, \ldots, Y_m] = C[Y]$ and an ideal $I$ of $C[Y]$ such that $C[Y]/I \cong C[X]^G$. The ring $C[X]^G$ has the grading induced by the isomorphism $e(C[X] * G)e \cong C[X]^G$, but in general $I$ is not an homogeneous ideal in the natural grading of $C[Y]$.

Changing notation, from the graded version of the theorem we have an equivalence of categories: $\text{gr} C[X]_{*G} \cong Cgr C[X]^G$. This equivalence induces at the level of bounded derived equivalences: $D^b(\text{Qgr} C[X]_{*G}) \cong D^b(\text{Qgr} C[X]^G)$.

As remarked above, the exterior algebra in $n$ variables, $\Lambda_n$ is the Yoneda algebra of $C[X]$ , $G$ acts as an automorphism group of $\Lambda_n$ and the Yoneda algebra of $C[X] * G$ is $\Lambda_n * G$.

By a theorem of [MS] and by [MM] there is an equivalence of triangulated categories: $\text{gr} \Lambda_n * G \cong D^b(\text{Qgr} C[X]^G)$, where $\text{gr} \Lambda_n * G$ denotes the stable category of finitely generated graded modules. Therefore: there is an equivalence of triangulated categories: $\text{gr} \Lambda_n * G \cong D^b(\text{Qgr} C[X]^G)$. 

We have proved the following:

**Corollary 3.** Let $G$ be a finite subgroup of $\text{Gl}(n, C)$ such that no $\sigma \in G$ different from the identity has a fixed point, $G$ acting as a group of automorphism of the polynomial ring $C[X]$ in $n$ variables and $e$ the idempotent $e = 1/|G| \sum g$ of the skew group algebra $C[X] \rtimes G$ and let $\Lambda_n$ be the exterior algebra in $n$ variables. Then there are isomorphisms of triangulated categories: $\text{gr}_{\Lambda_n \rtimes G} \cong \text{D}^b(\text{gr}_{C[X]} \rtimes G)$. In particular, the categories $\text{D}^b(\text{gr}_{C[X]} \rtimes G)$ and $\text{D}^b(\text{Qgr}_{C[X]} \rtimes G)$ have Auslander-Reiten triangles and they are of the form $ZA_n$.

**Proof.** For the proof we use the fact $\Lambda_n \rtimes G$ is selfinjective Koszul and results from [MZ].

4.1. Invariants for the Weyl algebra $A_n$ and the homogenized Weyl algebra $B_n$. In this subsection we study the ring of invariants of the Weyl algebra $A_n^G$ with $G$ a finite subgroup of the automorphism group of $A_n$. We prove $A_n^G$ is a simple algebra Morita equivalent to the skew group algebra $A_n \rtimes G$. We then consider the homogenized Weyl algebra $B_n$ and a subgroup $G$ of the group of automorphisms of $B_n$ satisfying some mild conditions, and prove for the invariant group $B_n^G$ and the skew group algebra $B_n \rtimes G$ there is a theorem relating the category of finitely generated left $B_n^G$ modules and the category of finitely generated $B_n \rtimes G$ modules, similar to the last theorem of the previous section.

**Theorem 12.** Let $\Lambda$ be a simple algebra over a field $K$ and, $G$ a finite group of automorphisms of $\Lambda$. Then the skew group algebra $\Lambda \rtimes G$ is simple.

**Proof.** Let $I$ be a non zero two sided ideal of $\Lambda \rtimes G$ and $r = a_0 + a_1 \sigma_1 + \ldots + a_i \sigma_i$ with $a_i \in \Lambda$ and $\sigma_i \in G$, if some $a_i \neq 0$ the element $r \sigma_i^{-1} \neq 0$ is in $I$ and the coefficient of the identity is no zero, hence we can assume all $a_i$ in the expression of $r$ are non zero and call $r \sigma_i^{-1} \neq 0$ the length of $r$. We can choose $r$ to be of minimal length among the non zero elements of $I$. Consider the set $L_0$ defined by $L_0 = \{ b_0 \mid b_0 + b_1 \sigma_1 + \ldots + b_i \sigma_i \in I \} \cup \{ 0 \}$. By definition $L_0$ is non zero, we prove it is a two sided ideal of $\Lambda$.

Let $r_1 = b_0 + b_1 \sigma_1 + \ldots + b_i \sigma_i$ and $r_2 = c_0 + c_1 \sigma_1 + \ldots c_i \sigma_i$ be two elements of $I$. Then $r_1 + r_2 = (b_0 + c_0) + (b_1 + c_1) \sigma_1 + \ldots + (b_i + c_i) \sigma_i$ is in $I$ and $\ell(r_1 + r_2) \leq \ell(r_1)$, by minimality either $r_1 + r_2 = 0$ or $\ell(r_1 + r_2) = \ell(r_1)$ in any case $b_0 + c_0 \in L_0$.

Let $a$ be a non zero element of $\Lambda$. Then $ar_1 = ab_0 + ab_1 \sigma_1 + \ldots + ab_i \sigma_i$ and $r_1 a = b_0 a + b_1 a \sigma_1 + \ldots b_i a \sigma_i$ are in $I$ with $ab_0$ and $b_0 a$ in $L_0$. Hence $L_0$ is a two sided ideal.

It follows $L_0 = \Lambda$ and there is an expression of minimal length $1 + b_1 \sigma_1 + \ldots + b_i \sigma_i$ in $I$.

If $a_0 + a_1 \sigma_1 + \ldots a_i \sigma_i$ is another non zero expression in $I$, then $(a_0 b_1 - a_1 b_0) \sigma_i \in I$ and by minimality $a_0 b_i - a_i = 0$ for $i \neq 0$. This is: $a_0 + a_1 \sigma_1 + \ldots a_i \sigma_i = a_0(1 + b_1 \sigma_1 + \ldots b_i \sigma_i)$.

Multiplying by $\sigma_i^{-1}$ we obtain the expression $b_0 + b_1 \sigma_i^{-1} + \ldots b_{i-1} \sigma_i^{-1} + b_i^{-1} + b_{i+1} \sigma_i^{-1} + \ldots b_n \sigma_i^{-1}$ belongs to $I$ is non zero and of minimal length. We define as above the set $L_1 = \{ a_0 \mid a_0 + a_1 \sigma_i^{-1} + \ldots a_{i-1} \sigma_i^{-1} + a_i + a_{i+1} \sigma_i^{-1} + \ldots a_n \sigma_i^{-1} \in I \}$. As before $L_1$ is a non zero two sided ideal of $A_n$. Then there is an expression $1 + c_1 \sigma_i^{-1} + \ldots c_{i-1} \sigma_i^{-1} + c_i + c_{i+1} \sigma_i^{-1} + \ldots c_n \sigma_i^{-1}$ in $I$ and $b_1 + b_1 \sigma_i^{-1} + \ldots b_{i-1} \sigma_i^{-1} + b_i^{-1} + b_{i+1} \sigma_i^{-1} + \ldots b_n \sigma_i^{-1}$
Consider the idempotent of \( \Lambda \).

In particular, \( b_i c_i = 1 \) and \( b_i \in C - \{0\} \) is a unity. Since the element was arbitrary all coefficients in \( 1 + b_1 \sigma_1 + \ldots + b_t \sigma_t \) are units, this is they are non zero complex numbers.

Assume the length \( t > 0 \).

Let \( \{ X_i \}_{i \in \Phi} \) be a set of algebra generators of \( \Lambda \) as \( K \)-algebra. Then \( X_i + X_i b_i \sigma_1 + \ldots + X_i b_i \sigma_t \) and \( X_i + b_i X_i^{\sigma_1} \sigma_1 + \ldots + b_i X_i^{\sigma_t} \sigma_t \) are elements of \( I \) and \( b_i (X_i - X_i^{\sigma_i}) \sigma_i \) is in \( I \). By minimality \( X_i = X_i^{\sigma_i} \) for an arbitrary \( X_i \).

But if \( \sigma_1 \) fixes all the generators of \( \Lambda \), then \( \sigma_1 = 1 \), a contradiction. It follows \( t = 0 \) and \( \Lambda = \{ a \in \Lambda \mid a1 \in I \} \). Therefore \( 1 \in I \) and \( I = \Lambda \ast G \).

As a corollary we obtain the following:

**Corollary 4.** Let \( A_n = C < X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n > / \{ [X_i, X_j], [Y_i, Y_j], [X_i, Y_j] - \delta_{ij} \} \) be a Weyl algebra and, \( G \) a finite group of automorphisms of \( A_n \). Then the skew group algebra \( \Lambda \ast G \) is simple.

Another consequence of the theorem is the following:

**Theorem 13.** Let \( \Lambda \) be a simple algebra over a field \( K \) and, \( G \) a finite group of automorphisms of \( \Lambda \). Then the algebra of invariants \( \Lambda^G \) is simple.

**Proof.** Let \( I \) be a non zero ideal of \( \Lambda^G \). Then \( I e = e I e \) is a non zero ideal of \( \Lambda^e = e \Lambda \ast Ge \).

The ideal \( \Lambda \ast Ge \Lambda \ast G \) of \( \Lambda \ast G \) is non zero otherwise, \( e \Lambda \ast Ge \Lambda \ast Ge = e I e = 0 \). Therefore \( \Lambda \ast Ge \Lambda \ast G = \Lambda \ast G \) and \( e I e = e \Lambda \ast Ge \Lambda \ast Ge = e \Lambda \ast Ge \). \( \square \)

**Corollary 5.** Let \( A_n = C < X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n > / \{ [X_i, X_j], [Y_i, Y_j], [X_i, Y_j] - \delta_{ij} \} \) be a Weyl algebra and, \( G \) a finite group of automorphisms of \( A_n \). Then the algebra of invariants \( A^G_n \) is simple.

We can prove now the following:

**Theorem 14.** Let \( \Lambda \) be a simple algebra over a field \( K \) and, \( G \) a finite group of automorphisms of \( \Lambda \). Then the algebra of invariants \( \Lambda^G \) and the skew group algebra \( \Lambda \ast G \) are Morita equivalent.

**Proof.** Consider the idempotent of \( \Lambda \ast G \), \( e = 1 \mid | G | \sum_{\sigma \in G} \sigma \). We prove first that the functor \( F = Hom_{\Lambda \ast G}(\Lambda \ast Ge, -) : Mod_{\Lambda \ast G} \rightarrow Mod_{\Lambda \ast Ge} \) has zero kernel.

A \( \Lambda \ast G \)-module \( M \) is in the kernel of \( F \) if and only if \( e M = 0 \), this is: if and only if \( \Lambda \ast Ge \ast G M = 0 \). But \( \Lambda \ast G \) simple implies \( \Lambda \ast G = \Lambda \ast Ge \ast G \) and \( M \) is in the kernel of \( F \) if and only if \( M = 0 \).

The functor \( F \) is always dense.

Let \( f : M \rightarrow N \) be a map of \( \Lambda \ast G \)-modules and \( ef : e M \rightarrow e N \) the restriction.

There is a commutative diagram:

\[
\begin{array}{ccc}
\Lambda \ast Ge \otimes e_{\Lambda \ast Ge} e M & \xrightarrow{\mu} & M \\
\downarrow & & \downarrow f \\
\Lambda \ast Ge \otimes e_{\Lambda \ast Ge} e N & \xrightarrow{\mu'} & N
\end{array}
\]

with \( \mu \) and \( \mu' \) multiplication. The kernels of \( \mu \) and \( \mu' \) are annihilated by \( e \), hence they are zero and the cokernel of \( \mu \) is \( M/\Lambda \ast Ge \ast G M = 0 \.

Similarly for \( \mu' \). It follows both \( \mu \) and \( \mu' \) are isomorphisms.

Therefore \( ef = 0 \) implies \( f = 0 \) and \( F \) is faithful.
Let $g : M' \to N'$ be a map of $e\Lambda*Ge$-modules. Since $F$ is dense, there exists $\Lambda*G$-modules $M$ and $N$ such that $eM \cong M'$ and $eN \cong N'$ and $g$ can be identified with a map $g : eM \to eN$. There are maps:

\[ \Lambda * Ge \otimes e\Lambda * Ge eM \xrightarrow{\mu} M \]
\[ \downarrow \Lambda * Ge \otimes g \]
\[ \Lambda * Ge \otimes e\Lambda * Ge eN \xrightarrow{\mu'} N \]

with $\mu$, $\mu'$ isomorphisms and the map $f = \mu'\Lambda*Ge \otimes g\mu^{-1}$ is such that $\text{Hom}_{\Lambda*G}(\Lambda*Ge, f) = ef = g$.

Therefore $\text{Hom}_{\Lambda*G}(\Lambda*Ge, \cdot)$ is full. □

**Corollary 6.** Let $A_n = C \langle X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n \rangle / \langle [X_i, X_j], [Y_i, Y_j], [X_i, Y_j] - \delta_{ij} \rangle$ be a Weyl algebra and, $G$ a finite group of automorphisms of $A_n$. Then the algebra of invariants $A_n^G$ and the skew group algebra $A_n \ast G$ are Morita equivalent.

**Corollary 7.** Let $G$ be a finite subgroup of $\text{SL}(2, C)$ acting as automorphism group of $A_1$. Then $A_1 \ast G$ and $A_1^G$ are simple and Morita equivalent to the deformed preprojective algebra. In particular the deformed preprojective algebra is simple.

For more results on the deformed preprojective algebra we refer to [C-BH].

We analyze next the relations between $\text{mod}_{B_n*G}$ and $\text{mod}_{B_G}$ for a special class of subgroups of automorphisms of $B_n$, which include finite products of finite subgroups of $\text{SL}(2, C)$.

We will need the following:

**Theorem 15.** Let $\Lambda$ be a noetherian algebra over a field $K$ and, $G$ a finite group of automorphisms of $\Lambda$ such that $\Lambda$ is finitely generated as left module over the ring of invariants $\Lambda^G$. Then $\Lambda^G$ is a noetherian algebra.

**Proof.** Let $I$ be a left ideal of $\Lambda^G$. Then $\Lambda I$ is a left ideal and by hypothesis, $\Lambda I$ is finitely generated as $\Lambda$-module and $\Lambda$ finitely generated over $\Lambda^G$ implies $\Lambda I$ is a finitely generated $\Lambda^G$-module. Let $\{\sum_{i=1}^m b_i^j u_i^j \mid 1 \leq j \leq m \mid b_i^j \in \Lambda, u_i^j \in I\}$ be a set of generators of $\Lambda I$. Then $\{b_i^j u_i^j \mid 1 \leq j \leq m \mid b_i^j \in \Lambda, u_i^j \in I\}$ is also a set of generators of $\Lambda I$, after re-indexing we have a set of generators $\{b_i u_i, 1 \leq i \leq n \mid b_i \in \Lambda, u_i \in I\}$.

Let $x$ be an element of $I$. Then for $1 \leq i \leq n$, there exists $c_i \in \Lambda^G$ such that $x = \sum_{i=1}^n c_i b_i u_i$. Then for $\sigma \in G$, $x = x^\sigma = \sum_{i=1}^m c_i^\sigma b_i^\sigma u_i^\sigma = \sum_{i=1}^m c_i b_i u_i$ and $x = 1/|G| \sum_{\sigma \in G} \sum_{i=1}^m c_i b_i^\sigma u_i = 1/|G| \sum_{i=1}^m c_i b_i u_i = \sum_{i=1}^m c_i (1/|G| \sum_{\sigma \in G} b_i^\sigma) u_i = \sum_{i=1}^m c_i t_i u_i$ with $c_i, t_i \in \Lambda^G$.

Therefore: $I$ is finitely generated as $\Lambda^G$-module. □

**Proposition 3.** Let $G$ be a finite group of grade preserving automorphisms of $B_n$ fixing $Z$. Then $B_n$ is a finitely generated left (right) $B_n^G$-module.

**Proof.** Since $G$ is a group of grade preserving automorphisms $B_n^G$ is a positively graded ring with $(B_n^G)_i = \{b \in (B_n)_i \mid \sigma(b) = b \text{ for all } \sigma \in G\}$ and the inclusion $j_1 : B_n^G \to B_n$ is a homomorphism of graded $C$-algebras.

Let $C_n$ be the polynomial ring in $2n$ variables, $G$ acts as an automorphism group of $C_n$, let $C_n^G$ be the ring of invariants and $j_0 : C_n^G \to C_n$ the inclusion as a graded subring.
We have a commutative exact diagram:

\[
\begin{array}{cccc}
0 & \to & ZB_G^n & \to & B_G^n & \to & C_G^n & \to & 0 \\
\downarrow j_2 & & \downarrow j_1 & & \downarrow j_0 & & & & \\
0 & \to & ZB_n & \to & B_n & \to & C_n & \to & 0
\end{array}
\]

We know \(C_n\) is a finitely generated \(C_G^n\)-module. \([\text{ ]}\) and we can choose homogeneous generators \(c_1, c_2, \ldots, c_t\) of \(C_n\) as \(C_G^n\)-module. Let \(b_1, b_2, \ldots, b_t\) be homogeneous elements of \(B_n\) such that \(b_i + ZB_n = c_i\) for \(1 \leq i \leq t\).

Let \(b\) be an element of \(B_n\) of degree \(d\). Then \(b + ZB_n = \sum_{i=1}^{t} r_i^0 b_i + ZB_n\), with \(r_i^0 \in B_G^n\) and \(b = \sum_{i=1}^{t} r_i^0 b_i + Z\mu_1\).

Since \(b, b_1, b_2, \ldots, b_t\) are homogeneous we can choose \(r_i^0\) and \(\mu_1\) homogeneous elements with degree \(r_i^0 + \deg b_i\) and \(\deg \mu_1 = d - 1\).

Then \(\mu_1 = \sum_{i=1}^{t} r_i^1 b_i + Z\mu_2\) with \(\mu_2\) homogeneous of degree \(d - 2\). Continuing by induction we obtain \(\mu_i\) homogeneous of degree \(d - i\), in particular \(\mu_d\) has degree zero, which means it is a constant and we get:

\[b = \sum_{i=1}^{t} r_i^0 b_i + \sum_{j=1}^{d-1} \sum_{i=1}^{t} Z^j r_i^1 b_i + Z^d k,\] \(k\) a complex number. Then \(b = \sum_{i=1}^{t} (\sum_{j=0}^{d-1} Z^j r_i^1) b_i + Z^d k\), with \(Z^j r_i^1 \in B_G^n\) and 1, \(b_1, b_2, \ldots, b_t\) generate \(B_n\) as left \(B_G^n\)-module. \(\square\)

**Corollary 8.** Let \(G\) be a finite group of grade preserving automorphisms of \(B_n\) fixing \(Z\). Then \(B_G^n\) is a noetherian algebra.

**Lemma 3.** Let \(B_n\) be the homogenized Weyl \(K\)-algebra over an infinite field \(H\). Then \(\cap_{c \in K} (Z - c) B_n = 0\).

**Proof.** Since \(B_n\) has a Poincare-Birkhoff basis with \(Z\) in the center, any element of \(B_n\) is a polynomial \(g/X,Y,Z\) in \(X\)'s, \(Y\)'s and \(Z\). Given \(d \in K\) \(Z - d\) divides \(g\) if and only if \(g(X,Y,d) = 0\).

It is clear that \(g(X,Y,Z) = q(X,Y,Z)(Z - d)\) implies \(g(X,Y,d) = 0\). We can write \(g\) as a polynomial in \(Z\), \(g(X,Y,Z) = g_0(X,Y) + g_1(X,Y)Z + g_2(X,Y)Z^2 + \ldots g_l(X,Y)Z^l\) with \(g_i(X,Y)\) polynomials in \(X\)'s and \(Y\)'s.

Then \(g_i(X,Y)Z^i = g_i(X,Y)((Z - d) + d)^i = g_i'(X,Y,Z)(Z - d) + g_i(X,Y)d^i\) and \(g(X,Y,Z) = q(X,Y,Z)(Z - d) + g(X,Y,d)\). Therefore \(g(X,Y,d) = 0\) implies \(Z - d\) divides \(g\).

Assume now \(0 \neq h \in \cap_{c \in K} (Z - c) B_n\). Then \(h = (Z - c_1)q_1 = (Z - c_2)f\) with \(c_1 \neq c_2\). Then \((c_2 - c_1)q_1(X,Y,c_2) = (c_2 - c_2)f = 0\) implies \(Z - c_2\) divides \(q_1(X,Y,Z)\) and \(h = (Z - c_1)(Z - c_2)f\).

Assume \(h\) is a polynomial in \(Z\) of degree \(m\). Continuing by induction we obtain \(h = (Z - c_1)(Z - c_2)\ldots(Z - c_{m+1})q_{m+1}\) a contradiction.

Therefore: \(\cap_{c \in K} (Z - c) B_n = 0\). \(\square\)

In what remains of the section we want to extend the theorems of the previous subsection to the homogenized Weyl algebras, the following result will be crucial:

**Proposition 4.** Let \(B_n = C\langle X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n, Z \rangle/[\langle X_i X_j, Y_i, Y_j, X_i, Y_j \rangle, \delta_{ij} Z^2, [X_i, Z], [Y_i, Z]]\) be the homogenized Weyl algebra in \(2n+1\) variables, \(G\) a finite
group of grade preserving automorphisms of $B_n$ and $e$ the idempotent of $B_n \ast G$, $e = 1/|G| \sum_{\sigma \in G} \sigma$. Moreover, assume $B_n$ satisfies the following conditions:

i) For all $\sigma \in G$, $\sigma(Z) = Z$.

ii) The $C$-subspace $V = \bigoplus_{i=1}^{n} CX_i \oplus \bigoplus_{j=1}^{n} CY_j$ of $B_n$ is $G$- invariant.

iii) If $v \in V$, $v \neq 0$ and $\sigma(v) = v$, then $\sigma = 1$.

Then $B_n \ast G/B_n \ast GeB_n \ast G$ is a finite dimensional $C$-algebra.

Proof. We will use the commutative exact diagram:
\[
\begin{array}{ccc}
0 & \rightarrow & ZB_n \\
\downarrow j_2 & & \downarrow j_1 \\
0 & \rightarrow & ZB_n \\
\end{array}
\]

\[
\begin{array}{ccc}
\downarrow j_2 & & \downarrow j_0 \\
ZB_n & \rightarrow & B_n \\
\end{array}
\]

\[
\begin{array}{ccc}
0 & \rightarrow & \ast \\
\downarrow j_2 & & \downarrow j_0 \\
0 & \rightarrow & \ast \\
\end{array}
\]

Let $\{\overline{m}_i\}_{i \in I}$ be the set of maximal ideals of $C \ast G$, for each $\overline{m}_i$ an ideal $m_i$ of $C_n$ such that $C_n \cap m_i = \overline{m}_i$. Let $\overline{m}_0 = (X_1, X_2, ..., X_n, Y_1, Y_2, ..., Y_n)$ be the maximal irrelevant ideal of $C_n$ and $\overline{m}_0 = C_n \cap \overline{m}_0$.

We saw in Proposition ?, that given any ideal $\overline{m}_i \neq \overline{m}_0$, $\{\overline{m}_i\}_{\sigma \in G}$, such that $\overline{m}_i = m_i$ implies $\sigma = \tau$, is the set of maximal ideals of $C_n$ above $\overline{m}_i$. In particular, the radical $\sqrt{m_i C_n} = \cap m_i$ is a $G$-invariant ideal of $C_n$, and $\sqrt{m_0 C_n} = \overline{m}_0$.

By the isomorphism theorems, there exists maximal ideals $\{m_i\}_{i \in I}$ of $B_n \ast G$ containing $ZB_n$, such that $m_i/ZB_n \ast G \cong m_i$ and maximal ideals $\{n_i\}_{i \in I}$ of $B_n$ containing $ZB_n$ such that $n_i/ZB_n \cong \overline{m}_i$.

From the isomorphisms: $n_i/ZB_n \ast G/ZB_n = n_i \cap B_n \ast G/ZB_n = m_i/ZB_n$, we get $n_i \cap B_n = m_i$ and $n_i \cap B_n = m_i$ for all $\sigma \in G$. It follows $L_i = \cap_{\sigma \in G} m_i$ is a $G$-invariant ideal of $B_n$, with $L_0 = n_0$.

We have $ZB_n \leq \cap_{\sigma \in G} n_i = \cap_{i \in I} L_i$ and $\cap_{i \in I} L_i/ZB_n = \cap_{i \in I} \cap_{\sigma \in G} m_i = 0$. Therefore:
\[
\cap_{i \in I} n_i = ZB_n.
\]

It was proved in [], that for any $c \in C - \{0\}$ there is an isomorphism $B_n/(Z - c)B_n \cong A_n$, hence each $(Z - c)B_n$ is a maximal ideal of $B_n$.

Then we have: $\cap_{i \in I} \cap_{\sigma \in G} n_i = \cap_{c \in C - \{0\}} (Z - c)B = \cap_{c \in C} (Z - c)B = 0$.

We have a ring homomorphism:
\[
\psi : B_n \ast G \rightarrow \prod_{i \in I - \{0\}} B_n \ast G/L_i \ast G \times B_n \ast G/n_0 \ast G \times \prod_{c \in C - \{0\}} B_n \ast G/(Z - c)B_n \ast G
\]

whose kernel is zero.

By the isomorphism theorems: $B_n/L_i \cong B_n/ZB_n/L_i/ZB_n \cong C_n/\sqrt{m_i C_n}$ and $B_n/n_0 \cong B_n/ZB_n/n_0/ZB_n \cong C_n/\sqrt{m_i C_n}$.

Therefore: $B_n \ast G/n_0 \ast G \cong CG$ and for each $i \neq 0$, $S_i \ast G = B_n \ast G/L_i \ast G \cong C_n/\sqrt{m_i C_n} \ast G$ is a simple finite dimensional algebra, as we proved in Proposition ?.

Then we have an injective ring homomorphism:
\[
\psi : B_n \ast G \rightarrow \prod_{i \in I - \{0\}} S_i \ast G \times CG \times \prod_{c \in C - \{0\}} A_n \ast G
\]

By the simplicity of $S_i \ast G$ and $A_n \ast G$ for the idempotent $e$ we have: $S_i \ast GeS_i \ast G = S_i \ast G$ and $A_n \ast GeA_n \ast G = A_n \ast G$.

$\psi(e) = (\hat{e}, (e), (e))$ is an idempotent of $\prod_{i \in I - \{0\}} S_i \ast G \times CG \times \prod_{c \in C - \{0\}} A_n \ast G$

such that:
the fact of the group of grade preserving automorphisms of $B$

ring homomorphism:

$B_n * G / B_n * GeB_n * G \rightarrow CG / CGe$.

It follows $B_n * G / B_n * GeB_n * G$ is a finite dimensional $C$-algebra. □

We have all the ingredients to prove for the homogenized Weyl algebras, theorem analogous to Theorem 9 and Theorem 10, what was essential in the proof of those theorems was the fact $C_n * G / C_n * GeC_n * G$ is a finite dimensional $C$-algebra and the fact $C_n$ is a finitely generated $C_n^G$-module. Then the proof of the next two theorems follows by similar arguments to those used in the proof of Theorem 9 and Theorem 10 and we will skip it.

**Theorem 16.** Let $B_n = \langle X_1, X_2, ..., X_n, Y_1, Y_2, ..., Y_n, Z \rangle / \langle [X_i, X_j], [Y_i, Y_j], [X_i, Y_j], \delta_{ij}Z^2, [X_i, Z], [Y_i, Z] \rangle$ be the homogenized Weyl algebra in $2n+1$ variables, $G$ a finite group of grade preserving automorphisms of $B_n$ and $e$ the idempotent of $B_n * G$, $e = 1/|G| \sum_{\sigma \in G} \sigma$. Moreover, assume $B_n$ satisfies the following conditions:

i) For all $\sigma \in G$, $\sigma(Z) = Z$.

ii) The $C$-subspace $V = \oplus_{i=1}^n CX_i \oplus \oplus_{j=1}^n CY_j$ of $B_n$ is $G$- invariant.

iii) If $v \in V$, $v \neq 0$ and $\sigma(v) = v$, then $\sigma = 1$.

Then: writing $R = B_n * G$ and $T = eB_n * Ge \cong B_n^G$ and denoting by $S_R$ and $S_T$ the categories of finite dimensional $R$ and $T$-modules, respectively and by $mod_R$, $mod_T$, the categories of finitely generated, $R$ and $T$-modules, respectively. Then there is a commutative diagram of categories and functors:

\[
\begin{array}{ccc}
mod_R & \xrightarrow{\text{Hom}(\text{Re}, -)} & mod_T \\
\downarrow \pi_R & & \downarrow \pi_T, \hbox{ with } H \hbox{ an equivalence.} \\
\text{mod}_R / S_R & \xrightarrow{H} & \text{mod}_T / S_T 
\end{array}
\]

We have also the graded version:

**Theorem 17.** Let $B_n = \langle X_1, X_2, ..., X_n, Y_1, Y_2, ..., Y_n, Z \rangle / \langle [X_i, X_j], [Y_i, Y_j], [X_i, Y_j], \delta_{ij}Z^2, [X_i, Z], [Y_i, Z] \rangle$ be the homogenized Weyl algebra in $2n+1$ variables, $G$ a finite group of grade preserving automorphisms of $B_n$ and $e$ the idempotent of $B_n * G$, $e = 1/|G| \sum_{\sigma \in G} \sigma$. Moreover, assume $B_n$ satisfies the following conditions:

i) For all $\sigma \in G$, $\sigma(Z) = Z$.

ii) The $C$-subspace $V = \oplus_{i=1}^n CX_i \oplus \oplus_{j=1}^n CY_j$ of $B_n$ is $G$- invariant.

iii) If $v \in V$, $v \neq 0$ and $\sigma(v) = v$, then $\sigma = 1$. Then: writing $R = B_n * G$ and $T = eB_n * Ge \cong B_n^G$ and considering both as positively graded algebras, denoting by $S_R$ and $S_T$ the categories of finite dimensional graded $R$ and $T$-modules, respectively and by $gr_R$, $gr_T$, the categories of finitely generated, $R$ and $T$-modules, respectively and by $\text{tails}_R$ and $\text{tails}_T$ the quotient categories $gr_R / S_R$ and $gr_T / S_T$. Then there is a commutative diagram of categories and functors:

\[
\begin{array}{ccc}
gr_R & \xrightarrow{\text{Hom}(\text{Re}, -)} & gr_T \\
\downarrow \pi_R & & \downarrow \pi_T, \hbox{ with } H \hbox{ an equivalence.} \\
\text{tails}_R & \xrightarrow{H} & \text{tails}_T 
\end{array}
\]
We also have as in Theorem 10: 

Changing notation, from the graded version of the theorem we have an equivalence of categories: $\text{gr}_{B_n \ast G} \cong \text{Qgr}_{B_n}$. This equivalence induces at the level of bounded derived equivalences: $D^b\left(\text{Qgr}_{B_n \ast G}\right) \cong D^b\left(\text{Qgr}_{B_n}\right)$.

As remarked above, $G$ acts as an automorphism group of $B_n$, the Yoneda algebra of $B_n$ and the Yoneda algebra of $B_n \ast G$ is $B_n' \ast G$.

The algebra $B_n' \ast G$ is Koszul selfinjective. By a theorem of [MS] and by [MM] there is an equivalence of triangulated categories: $\text{gr}_{B_n' \ast G} \cong D^b\left(\text{Qgr}_{B_n \ast G}\right)$, where $\text{Qgr}_{B_n \ast G}$ denotes the stable category of finitely generated graded modules. Therefore: there is an equivalence of triangulated categories: $\text{gr}_{B_n' \ast G} \cong D^b\left(\text{Qgr}_{B_n}\right)$.

We have proved the following:

**Corollary 9.** Let $B_n = C < X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n, Z > / \langle [X_i, X_j], [Y_i, Y_j], [X_i, Y_j], \delta_{ij}, Z, [X_i, Z], [Y_i, Z] \rangle$ be the homogenized Weyl algebra in $2n+1$ variables, $G$ a finite group of grade preserving automorphisms of $B_n$ and $e$ the idempotent of $B_n \ast G$, $e = 1/|G| \sum \sigma$. Moreover, assume $B_n$ satisfies the following conditions:

i) For all $\sigma \in G$, $\sigma(Z) = Z$.

ii) The C-subspace $V = \bigoplus_{i=1}^{n} CX_i \oplus \bigoplus_{j=1}^{n} CY_j$ of $B_n$ is $G$-invariant.

iii) If $v \in V$, $v \neq 0$ and $\sigma(v) = v$, then $\sigma = 1$.

Let $B_n'$ be the Yoneda algebra of $B_n$. Then there are isomorphisms of triangulated categories: $\text{gr}_{B_n' \ast G} \cong D^b\left(\text{Qgr}_{B_n}\right) \cong D^b\left(\text{Qgr}_{B_n \ast G}\right)$. In particular, the categories $D^b\left(\text{Qgr}_{B_n}\right)$ and $D^b\left(\text{Qgr}_{B_n \ast G}\right)$ have Auslander-Reiten triangles and they are of the form $ZA_n$.

**Proof.** As before, the proof uses the fact $B_n' \ast G$ is selfinjective Koszul and results from [MZ].

---

**References**

[AR] Auslander, M.; Reiten, I.; McKay quivers and extended Dynkin diagrams. Trans. Amer. Math. Soc. 293 (1986), no. 1, 293–301

[Be] Benson D.; Polynomial Invariants of Finite Groups. London Mathematical Society Lecture Notes Series 190, Cambridge University Press, 1993.

[CR] Curtis Ch. W. Reiner I. Representation Theory of Finite Groups and Associative Algebras, Pure and Applied Mathematics Vol XI, Interscience 1962

[C-BH] Crawley-Boevey, W.; Holland, M. P. Noncommutative deformations of Kleinian singularities. Duke Math. J. 92 (1998), no. 3, 605–635.

[C-B] Crawley-Boevey W. DMV Lectures on Representations of quivers, preprojective algebras and deformation of quotient singularities.(preprint Leeds)

[Co] Coutinho S.C. A Primer of Algebraic D-modules, London Mathematical Society, Students Texts 33, 1995

[Ga] Gabriel, Pierre Des catégories abéliennes. Bull. Soc. Math. France 90 1962 323–448.

[GH] Green, E.; Huang, R. Q. Projective resolutions of straightening closed algebras generated by minors. Adv. Math. 110 (1995), no. 2, 314–333.

[GM1] Green, E. L.; Martínez Villa, R.; Koszul and Yoneda algebras. Representation theory of algebras (Cocoyoc, 1994), 247–297, CMS Conf. Proc., 18, Amer. Math. Soc., Providence, RI, 1996.
[GM2] Green, E. L.; Martínez-Villa, R.: Koszul and Yoneda algebras. II. Algebras and modules, II (Geiranger, 1996), 227–244, CMS Conf. Proc., 24, Amer. Math. Soc., Providence, RI, 1998.

[GMT] Guo, Jin Yun; Martínez-Villa, R.; Takane, M.; Koszul generalized Auslander regular algebras. Algebras and modules, II (Geiranger, 1996), 263–283, CMS Conf. Proc., 24, Amer. Math. Soc., Providence, RI, 1998.

[GuM] Guo, Jin Yun; Martínez-Villa, R.; Algebra pairs associated to McKay quivers. Comm. Algebra 30 (2002), no. 2, 1017–1032.

[Ku] Kunz E.; Introduction to Commutative Algebra and Algebraic Geometry, Birkhäuser, 1985.

[Le] Lenzing H.; Polyhedral groups and the geometric study of tame hereditary algebras (preprint Paderborn).

[Li] Li Huishi. Noncommutative Groebner Bases and Filtered-Graded Transfer, Lecture Notes in Mathematics 1795, Springer, 2002.

[MM] Martínez-Villa, R., Martsinkovsky; A. Stable Projective Homotopy Theory of Modules, Tails, and Koszul Duality, (aceptado 11 de septiembre 2009), Comm. Algebra 38 (2010), no. 10, 3941–3973.

[MS] Martínez Villa, Roberto; Saorín, M.; Koszul equivalences and dualities. Pacific J. Math. 214 (2004), no. 2, 359–378.

[MZ] Martínez-Villa, Roberto; Zacharia, Dan; Approximations with modules having linear resolutions. J. Algebra 266 (2003), no. 2, 671–697.

[MV1] Martínez-Villa, R.; Skew group algebras and their Yoneda algebras. Math. J. Okayama Univ. 43 (2001), 1–16.

[MV2] Martínez-Villa, R.; Serre duality for generalized Auslander regular algebras. Trends in the representation theory of finite-dimensional algebras (Seattle, WA, 1997), 237–263, Contemp. Math., 229, Amer. Math. Soc., Providence, RI, 1998.

[MV3] Martínez-Villa, R.; Applications of Koszul algebras: the preprojective algebra. Representation theory of algebras (Cocoyoc, 1994), 487–504, CMS Conf. Proc., 18, Amer. Math. Soc., Providence, RI, 1996.

[MV4] Martinez-Villa, R. Graded, Selfinjective, and Koszul Algebras, J. Algebra 215, 34-72 1999.

[MMo] Martinez-Villa, R. Mondragon J. On the homogeneized Weyl Algebra. (preprint 2011).

[Mi] Miyachi, Jun-Ichi; Derived Categories with Applications to Representations of Algebras, Chiba Lectures, 2002.

[Mu] Müller W. Darstellungstheorie von eindlichen Gruppen, Teubner Studienbücher, 1980.

[Mc] McKay J.;, Graphs, singularities and finite groups, Proc. Sympos. Pure Math., vol. 37, Amer. Math. Soc., Providence, R. I., 1980, pp. 183-186.

[P] Popescu N.; Abelian Categories with Applications to Rings and Modules;, L.M.S. Monographs 3, Academic Press 1973.

[RR] Reiten, I.; Riedtmann, Ch. Skew group algebras in the representation theory of Artin algebras. J. Algebra 92 (1985), no. 1, 224–282.

[Sm] Smith, P.S.: Some finite dimensional algebras related to elliptic curves, Rep. Theory of Algebras and Related Topics, CMS Conference Proceedings, Vol. 19, 315-348, Amer. Math. Soc 1996.
[Ste] Steinberg R.; Finite subgroups of SU$_2$, Dynkin Diagrams and Affine Coxeter elements. Pacific Journal of Mathematics, Vol. 118, No. 2, 1985.

[Stu] Sturmfels B.; Algorithms in Invariant Theory, Texts and Monographs in Symbolic Computation, Springer-Verlag, 1993

(A. One and A. Two) Centro de Ciencias Matemáticas, UNAM, Morelia, Mich. México

E-mail address, A. One: mvilla@matmor.unam.mx

URL: http://www.matmor.unam.mx

Current address, A. Two: Centro de Ciencias Matemáticas, UNAM, Morelia, Mich. México

E-mail address, A. Two: jeronimo@matmor.unam.mx

URL: http://www.matmor.unam.mx