SUPER TRIPLE SYSTEMS AND APPLICATIONS TO
PARA-STATISTICS AND YANG-BAXTER EQUATION

S. OKUBO
Department of Physics and Astronomy, University of Rochester
Rochester, NY 14627, USA

ABSTRACT
We introduce the notion of ortho-symplectic super triple system, and apply it to find solutions of super Yang-Baxter equation. Also, the para-statistics are formulated as a Lie-super triple system.

1. Triple Product

1.1 Quaternionic and Octonionic Triple Systems
Let $V$ be a $N$-dimensional vector space, which we write hereafter as

$$ N = \text{Dim} \ V \ . $$

A triple product $w = [x, y, z]$ in $V$ is an assignment of $w \in V$ for any three elements $x, y, z \in V$, which is linear in each variables $x, y, \text{and } z$. In other words, we may write

$$ [x, y, z] ; \ V \otimes V \otimes V \rightarrow V \ . $$

Perhaps the simplest non-trivial example is the case of $N = 4$, which can be constructed below. Let $e_1, e_2, e_3, \text{and } e_4$ be a basis of $V$ and assign $[e_j, e_k, e_\ell] \in V$ by

$$ [e_j, e_k, e_\ell] = \sum_{m=1}^{4} \epsilon_{j k \ell m} e_m $$

for any $j, k, \ell = 1, 2, 3, 4$, where $\epsilon_{j k \ell m}$ is the totally antisymmetric Levi-Civita symbol with $\epsilon_{1234} = 1$ in 4-dimensional space. We then extend the definition of $[x, y, z]$ for any $x, y, z \in V$ from Eq. (1.3) by the linearity. Further, we can introduce a bi-linear form $<x|y>$ in $V$ by

$$ < e_j | e_k > = \delta_{jk} \ (j, k = 1, 2, 3, 4) \ . $$
Then, \(< x|y >\) defines clearly a symmetric bi-linear non-degenerate form in \(V\).

The triple product \([x, y, z]\) with the bi-linear form \(< x|y >\) can be easily verified to satisfy the following properties:

(i) \(< x|y > = < y|x >\) is non-degenerate,
(ii) \([x, y, z]\) is totally antisymmetric in \(x, y, z\),
(iii) \(< w|[x, y, z]\) is totally antisymmetric in \(w, x, y, z\),
(iv) \(< [x, y, z]|u, v, w >\)

\[ = < x|u > < y|v > < z|w > + < x|v > < y|w > < z|u > + < x|w > < y|u > < z|v > - < x|u > < y|v > < z|w > \]

\[ - < x|v > < y|u > < z|w > - < x|u > < y|w > < z|v > . \] (1.5d)

(v) \([u, v, [x, y, z]]\)

\[ = ( < y|v > < z|u > - < y|u > < z|v > )x + ( < z|v > < x|u > - < z|u > < x|v > )y + ( < x|v > < y|u > - < x|u > < y|v > )z . \] (1.5e)

For examples, Eq. (1.5c) is an immediate consequence of

\(< e_m|[e_j, e_k, e_\ell]\) = \(\epsilon_{jkm}\)

being totally antisymmetric in \(j, k, \ell,\) and \(m\), while Eqs. (1.5d) and (1.5e) is equivalent to the validity of the identity,

\[
\sum_{m=1}^{4} \epsilon_{jkm} \epsilon_{abcm} = \delta_{ja} (\delta_{kb}\delta_{\ell c} - \delta_{kc}\delta_{\ell b}) + \delta_{jb} (\delta_{kc}\delta_{\ell a} - \delta_{ka}\delta_{\ell c}) + \delta_{jc} (\delta_{ka}\delta_{\ell b} - \delta_{kb}\delta_{\ell a}) .
\]

Conversely, we can prove that any system satisfying Eqs. (1.5) is possible only for Dim \(V = 4\), and moreover that we can find a basis \(e_1, e_2, e_3\) and \(e_4\) which satisfy Eqs. (1.3) and (1.4). We call the system to be a quaternionic triple (or ternary) system\(^1\).

We can generalize Eqs. (1.5) by adding extra terms

\[ \beta\{ < x|u > < y|z, v, w > \]

\[ + < y|u > < z|[x, v, w] > + < z|u > < x|[y, v, w] > + < x|v > < y|[z, w, u] > + < y|v > < z|[x, w, u] > + < z|v > < x|[y, w, u] > + < y|w > < z|[x, u, v] > + < z|w > < x|[y, u, v] > \} \] (1.5d)'
to the right side of Eq. (1.5d) and
\[
-\beta \{ <u|[v,y,z]>x + <u|[v,z,x]>y + <u|[v,x,y]>z \}
-\beta \{ <x|v>[u,y,z] + <y|v>[u,z,x] + <z|v>[u,x,y] \\
+ <x|u>[v,z,y] + <y|u>[v,x,z] + <z|u>[v,y,x] \},
\]
to the right side of Eq. (1.5e) for a constant \( \beta \), while other relations Eqs. (1.5a), (1.5b), and (1.5c) remain unchanged. We have shown elsewhere\(^1\) that the new system is possible only for two cases of
\[
\begin{align*}
(a) & \quad N = 4 , \quad \beta = 0 \quad & (1.6a) \\
(b) & \quad N = 8 , \quad \beta = \pm 1 \quad . \quad (1.6b)
\end{align*}
\]
We call the new case of \( N = \text{Dim} \ V = 8 \) to be an octonionic triple system. Let us normalize \( \beta \) to be \( \beta = -1 \) for the octonionic triple system by changing the sign of \([x, y, z]\) if necessary. Let \( e \in V \) be any fixed element satisfying
\[
<e|e> = 1 \quad , \quad (1.7a)
\]
and introduce a bi-linear product \( xy \) in \( V \) by
\[
xy = [x, y, e] + <x|e>y + <y|e>x - <x|y>e \quad . \quad (1.7b)
\]
Then, it is easy to show the validity of
\[
xe = ex = x \quad , \quad (1.8a)
\]
\[
<x|xy> = <x|x><y|y> \quad . \quad (1.8b)
\]
so that the bi-linear product \( xy \) defines the 8-dimensional octonion algebra\(^2\).
Conversely, we can determine the original triple product \([x, y, z]\) in terms of the bi-linear octonionic product, although we will not go into detail.

1.2 Orthogonal Triple System

Hereafter in this note, we assume that \( <x|y> \) is a bi-linear non-degenerate form in \( V \) which is, however, not necessarily symmetric. Also, let \( e_1, e_2, \ldots, e_N \) be a basis of \( V \) and set
\[
g_{jk} = <e_j|e_k> \quad . \quad (1.9)
\]
Because of the non-degeneracy of \( <x|y> \), \( g_{jk} \) possesses its inverse \( g^{jk} \) satisfying
\[
\sum_{\ell=1}^{N} g_{jt}g^{t\ell} = \sum_{\ell=1}^{N} g^{kt}g_{t\ell} = \delta^k_j \quad . \quad (1.10)
\]
We set
\[ e^j = \sum_{k=1}^{N} g^{jk} e_k, \quad e_j = \sum_{k=1}^{N} g_{jk} e^k. \] (1.11)

Then, we can expand any \( x \in V \) by
\[ x = \sum_{j=1}^{N} e_j < e^j | x > = \sum_{j=1}^{N} < x | e_j > e^j. \] (1.12)

We can now introduce the notion of orthogonal triple system as follows. Consider a vector space \( V \) with a triple linear product \( x y z \) and with a bi-linear non-degenerate form \( < x | y > \), satisfying the axioms
\[
\begin{align*}
    (i) \quad & < y | x > = < x | y > \quad (1.13a) \\
    (ii) \quad & y x z + x y z = 0 \quad (1.13b) \\
    (iii) \quad & x z y + x y z = 2\lambda < y | z > x - \lambda < x | y > z - \lambda < z | x > y \quad (1.13c) \\
    (iv) \quad & < uvx | y > = - < x | uvy > \quad , \quad (1.13d) \\
    (v) \quad & uv(xyz) = (uvx)yz + x(uvy)z + xy(uvz) \quad , \quad (1.13e)
\end{align*}
\]

where \( \lambda \) is a constant. We call any such a \( V \) to be an orthogonal triple system$^1$.

For any orthogonal triple system, we can introduce the second triple product \( [x, y, z] \) by
\[ x y z = [x, y, z] + \lambda < y | z > x - \lambda < z | x > y \] (1.14)

Then, Eqs. (1.13a)–(1.13d) imply that both \( [x, y, z] \) and \( < w | [x, y, z] > \) are totally antisymmetric in \( x, y, \) and \( z \), as well as in \( w, x, y, \) and \( z \), respectively. However, the last relation Eq. (1.13e) will become a rather complicated equation in terms of \( [x, y, z] \)’s. The notion of the orthogonal triple system is a generalization of both quaternionic and octonionic triple systems. Indeed for the totally antisymmetric product \( [x, y, z] \) of both quaternionic and octonionic triple systems, we conversely introduce the triple product \( x y z \) by Eq. (1.14) where the constant \( \lambda \) is assumed to be
\[
\begin{align*}
    (a) \quad & \lambda = \text{arbitrary for } N = 4 \quad (1.15a) \\
    (b) \quad & \lambda = -3 \beta \text{ for } N = 8 \quad . \quad (1.15b)
\end{align*}
\]

We can then verify the validity of Eqs. (1.13), so that they become orthogonal triple systems. The reason why we can choose \( \lambda \) to be arbitrary for the case of \( N = 4 \) is validities of special identities, as has been explained elsewhere$^1$. 

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2. Lie-Super Triple System

2.1 Super Space

Let the underlying vector space \( V \) admit a \( Z_2 \)-grading, so that

\[
V = V_B \oplus V_F
\]

(2.1)

is a direct sum of the bosonic vector space \( V_B \) and of the fermionic vector space \( V_F \). We set their dimensions as

\[
\text{Dim } V_B = N_B, \quad \text{Dim } V_F = N_F
\]

(2.2)

so that

\[
N = N_B + N_F.
\]

(2.3)

We define the signature function \( \sigma(x) \) in \( V \) by

\[
\sigma(x) = \begin{cases} 
0, & \text{if } x \in V_B \\
1, & \text{if } x \in V_F
\end{cases}
\]

(2.4a)

\[
\sigma(x) = \begin{cases} 
0, & \text{if } x \in V_B \\
1, & \text{if } x \in V_F
\end{cases}
\]

(2.4b)

We consider only homogeneous elements \( x \) of \( V \), i.e. either \( x \in V_B \) or \( x \in V_F \). Any triple product \( x y z \) (or \([x, y, z]\)) defined in \( V \) is hereafter assumed to satisfy

\[
\sigma(xyz) = \{\sigma(x) + \sigma(y) + \sigma(z)\} \pmod{2}.
\]

(2.5)

Also, any bi-linear non-degenerate form \( <x|y> \) in \( V \) is called super-symmetric, if we have

\[
\begin{align*}
&\text{(a) } <x|y> = 0, \text{ unless } \sigma(x) = \sigma(y) \pmod{2}, \quad (2.6b) \\
&\text{(b) } <y|x> = (-1)^{\sigma(x)\sigma(y)} <x|y>, \quad (2.6c)
\end{align*}
\]

which we assume throughout in this note. In what follows, we often write

\[
(-1)^{xy} = (-1)^{\sigma(x)\sigma(y)}
\]

(2.7)

whenever there is no confusion.

2.2 Lie-Super Triple System

Suppose that a triple product \( x \, y \, z \) satisfies the following axioms.

\[
\begin{align*}
\text{(i) } & y \, x \, z = -(-1)^{xy} x \, y \, z, \quad (2.8a) \\
\text{(ii) } & (-1)^{xz} x \, y \, z + (-1)^{yx} y \, z \, x + (-1)^{zy} z \, x \, y = 0, \quad (2.8b)
\end{align*}
\]
(iii) \[ uv(xyz) = (uvx)yz + (-1)^{(u+v)x}x(uvy)z \]
\[ + (-1)^{(u+v)(x+y)}xy(uvz) \]  
(2.8c)

where we have set for simplicity

\[ (-1)^{(u+v)x} = (-1)^{ux+vx} = (-1)^{[\sigma(u)+\sigma(v)]\sigma(x)} \]

etc. in accordance with Eq. (2.7). If \( V = V_B \) with \( V_F = 0 \), then this reduces to the well-known Lie triple system\(^3\(^,\)\(^4\), so that we call the system satisfying Eqs. (2.8) to be a Lie-super triple system.

The Lie-super triple system is intimately related to Lie-super algebra\(^5\). Let \( L \) be a Lie-super algebra with the Lie-product \([x, y]\), so that we have

(i) \( \sigma([x, y]) = \{\sigma(x) + \sigma(y)\} \mod 2 \)  
(ii) \([y, x] = -(-1)^{xy}[x, y]\),  
(iii) \((-1)^{xz}[[x, y], z] + (-1)^{yx}[[y, z], x] + (-1)^{zy}[[z, x], y] = 0 \).

(2.9c)

If we set

\[ x \ y \ z \equiv [[x, y], z] \]  
(2.10)

then it is not difficult to see that the triple product \( x \ y \ z \) defined by Eq. (2.10) satisfies Eqs. (2.8). The converse statement is also true in a sense to be specified below.

For this, we introduce first the linear multiplication operator \( L_{x,y} \) in \( V \) by

\[ L_{x,y}z = x \ y \ z \]  
(2.11)

and set

\[ M = \text{vector space spanned by } L_{x,y}'s, \ (x, \ y \in V) \]  
(2.12)

Calculating the commutators

\[ [L_{x,y}, L_{u,v}] = L_{x,y}L_{u,v} - (-1)^{(x+y)(u+v)}L_{u,v}L_{x,y} \]  
(2.13)

from Eq. (2.8c), we find

\[ [L_{x,y}, L_{u,v}] = L_{xyu,v} + (-1)^{(x+y)u}L_{u,xyv} \]  
(2.14)

so that this defines a Lie-super algebra with

\[ \sigma(L_{x,y}) = [\sigma(x) + \sigma(y)] \mod 2 \]  
(2.15)
Actually, we can construct a larger Lie-super algebra in a larger space

\[ V_0 = V \oplus M \]  \hspace{1cm} (2.16)

as follows. The commutators \([M, M]\) are still defined by Eq. (2.13), while we set

\[ [L_{x,y}, z] = -(-1)^{(x+y)z} [z, L_{x,y}] \equiv x \ y \ z \]  \hspace{1cm} (2.17)

for \([M, V]\) and

\[ [x, y] = -(-1)^{yx} [y, x] \equiv L_{x,y} \]  \hspace{1cm} (2.18)

for \([V, V]\). We note especially

\[ [M, M] \subset M \ , \ [M, V] \subset V \ , \ [V, V] \subset M \ , \]  \hspace{1cm} (2.19)

which is familiar in the theory of symmetric homogeneous spaces. It is not hard to verify that these define a Lie-super algebra in \(V_0\). If \(V = V_B\) with \(V_F = 0\), this construction reduces of course to the familiar canonical construction\(^3\),\(^4\).

Then, we calculate

\[ [[x, y], z] = [L_{x,y}, z] = x \ y \ z \]  \hspace{1cm} (2.20)

from Eqs. (2.18) and (2.17), so that the relation Eq. (2.10) is formally valid, although the meaning of \([x, y]\) is different, since it is an element of \(M\), but not of \(V\) itself.

2.3 Para-statistics as a Lie-Super Triple System

Let \(< x_1, x_2, \ldots, x_N >\) be a vector space spanned by \(x_1, x_2, \ldots, x_N\), and set

\[ V_B = < a_j, a_j^+ ; j = 1, 2, \ldots, n > \]  \hspace{1cm} (2.21a)

\[ V_F = < b_\alpha, b_\alpha^+ ; \alpha = 1, 2, \ldots, m > \]  \hspace{1cm} (2.21b)

Moreover, we introduce \(< x | y >\) in \(V = V_B \oplus V_F\) by

\[ < a_j | a_k^+ > = < a_k^+ | a_j > = \delta_{jk} \ , \ (j, k = 1, 2, \ldots, n) \]  \hspace{1cm} (2.22a)

\[ < b_\alpha | b_\beta^+ > = - < b_\beta^+ | b_\alpha > = \delta_{\alpha\beta} \ , \ (\alpha, \beta = 1, 2, \ldots, m) \]  \hspace{1cm} (2.22b)

while all other combinations such as \(< a_j | a_k >\, , \, < a_j | b_\alpha >\, , \, < a_j | b_\alpha^+ >\) etc. are assumed to be identically zero. Then, \(< x | y >\) defines a bi-linear symmetric non-degenerate form in the sense of Eqs. (2.6).
For any vector space $V$ with $< x|y >$ satisfying Eq. (2.6), the triple product defined by

$$x y z = \lambda < y|z > x - \lambda(-1)^{yz} < x|z > y$$

(2.23)

for a non-zero constant $\lambda$ can be readily verified to give a Lie-super triple system. Applying this fact to the system $V = V_B \oplus V_F$ given by Eqs. (2.21) and (2.22), we see that $V$ is a Lie-super triple system, while $V_B$ itself defines a Lie triple system. Especially, for $V = V_B$, we see

$$[[a_j, a_k^+, a_\ell]] = 2 \delta_{k\ell} a_j$$

(2.23a)

$$[[a_j^+, a_k^+, a_\ell]] = 2 \delta_{k\ell} a_j^+ - 2 \delta_{j\ell} a_k^+$$

(2.23b)

where we have set $\lambda = 2$. These relations are essentially equivalent to the para-Fermi statistics$^6$. Similarly, $V_F$ defines the para-boson statistics$^6$. However, for $V = V_B \oplus V_F$, we may have rather peculiar relations such as

$$[[a_j, b_\alpha^+, a_k^+] = -2 \delta_{jk} b_\alpha^+$$

(2.23c)

so that bosonic and fermionic operators are no longer independent of each other in contrast to the standard theory. This possibility will be explored, however, elsewhere. At any rate, the para-statistic relations may be rewritten as

$$[[x, y], z] = 2 < y|z > x - 2 (-1)^{yz} < x|z > y$$

(2.24a)

$$[x, y] = xy - (-1)^{xy}yx$$

(2.24b)

where boson variables and fermion variables are now regarded to possess opposite signature values as in Eqs. (2.21) and (2.22), i.e.

$$\sigma(x) = \begin{cases} 0 & \text{for fermions} \\ 1 & \text{for bosons} \end{cases}$$

(2.25)

Also, this realizes a ortho-symplectic Lie-super algebra$^5$.

3. Ortho-Symplectic Super-Triple System

3.1 Definition

Let $< x|y >$ be a bi-linear non-degenerate super-symmetric form in the sense of Eqs. (2.6). Suppose now that a triple linear product $x y z$ satisfies the following axioms:
(i)  \[ < y|x > = (-1)^{xy} < x|y > , \]  
(3.1a)  
(ii)  \[ x y z + (-1)^{xy} x z y = 0 , \]  
(3.1b)  
(iii)  \[ x y z + (-1)^{yz} y z x = 2 \lambda < y|z > x - \lambda < x|y > z - \lambda (-1)^{yz} < x|z > y , \]  
(3.1c)  
(iv)  \[ < uvx|y > = -(-1)^{(u+v)x} < x|uvy > , \]  
(3.1d)  
(v)  \[ uv(xy) = (uvx)y z + (-1)^{(u+v)x} x(uvy)z \]  
\[ + (-1)^{(u+v)(x+y)}xy(uvz) , \]  
(3.1e)  

for a constant \( \lambda \). We call the system to be ortho-symplectic super-triple system. If \( V = V_B \) with \( V_F = 0 \), then it reduces to the orthogonal triple system, while the other case of \( V = V_F \) with \( V_B = 0 \) is equivalent to the symplectic triple system of Yamaguchi and Asano\(^7\). Although the present super triple system is still a special case of more general super systems considered by many authors, it is still of some intrinsic interest as we will see below.

Here, we will construct a class of ortho-symplectic super triple system as follows. Let \(< x|y > \) be as before, and let  

\[ J_\alpha : V \to V \quad (\alpha = 1, 2, \ldots, n) \]  
(3.2)  

be a signature-preserving linear mapping in \( V \) so that  

\[ \sigma(J_\alpha x) = \sigma(x) \quad . \]  
(3.3)  

Moreover, we assume the validity of  

\[ < x|J_\alpha y > = -< J_\alpha x|y > \]  
(3.4)  

as well as  

\[ J_\alpha J_\beta = \lambda \delta_{\alpha\beta} 1 + \sum_{\gamma=1}^{n} f_{\alpha\beta\gamma} J_\gamma \]  
(3.5)  

for a constant \( \lambda \), where \( 1 \) stands for the identity linear mapping in \( V \). We consider two cases of  

(i)  \[ n = 1 \quad , \quad f_{\alpha\beta\gamma} = 0 , \]  
(3.6a)  
(ii)  \[ n = 3 \quad , \quad f_{\alpha\beta\gamma} = \epsilon_{\alpha\beta\gamma} = \text{Levi-Civita symbol} \]  
(3.6b)
so that the case of $n = 3$ corresponds to the quaternion algebra for $J_\alpha$ ($\alpha = 1, 2, 3$). Now, we introduce a triple product by

$$x \cdot y \cdot z = \sum_{\alpha=1}^{n} \left\{ J_\alpha x < y | J_\alpha z > + (-1)^{z(y+x)} J_\alpha y < z | J_\alpha x > + J_\alpha z < x | J_\alpha y > \right\} + \lambda \left\{ < y | z > x - (-1)^{z(x+y)} < z | x > \right\}.$$  \hspace{1cm} (3.7)

We can verify then that it satisfies Eqs. (3.1).

### 3.2 Its Relationship to Lie-super Triple System

Then, from a given ortho-symplectic super-triple system $V$, we can construct Lie-super triple systems in two different ways. The first one is essentially a straight-forward generalization of the method by Freudenthal-Yamaguchi-Asano in the space $V \oplus V$, which will be reported elsewhere. Here, we will consider a generalization of the earlier result given in ref. 8 which will be referred hereafter to as II. Let $e_1, e_2, \ldots, e_N$ be a basis of $V$ and define its dual $e^j$ by Eq. (1.11). We introduce a new triple product in the same vector space by

$$x \cdot y \cdot z = - \sum_{j=1}^{N} (xye^j)e^j z - \frac{1}{3} \lambda (N_0 - 16) x \cdot y \cdot z,$$  \hspace{1cm} (3.8)

where we have set

$$N_0 = N_B - N_F.$$  \hspace{1cm} (3.9)

Then, we can show as in II that $x \cdot y \cdot z$ defines a Lie-super triple system. However, it could happen that we have $x \cdot y \cdot z = 0$ identically as in octonionic triple system as well as in other cases studied in II. As a matter of fact, the condition $x \cdot y \cdot z = 0$ is crucial to obtain some class of solutions of Yang-Baxter equation as we already noted in II.

We now calculate $x \cdot y \cdot z$ for the ortho-symplectic super-triple system given by Eq. (3.7) to find

$$x \cdot y \cdot z = c \cdot x \cdot y \cdot z,$$  \hspace{1cm} (3.10a)

$$c = \begin{cases} -\frac{1}{3} \lambda (N_0 - 4), & \text{for } n = 1 \\ -\frac{1}{3} \lambda (N_0 + 8), & \text{for } n = 3 \end{cases}$$  \hspace{1cm} (3.10b)

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where \( x * y * z \) is a Lie-super triple product defined by

\[
x * y * z = \sum_{\alpha=1}^{n} \left\{ J_{\alpha} x < y | J_{\alpha} z > \right. \\
+ (-1)^{z(x+y)} J_{\alpha} y < z | J_{\alpha} x > - 2 J_{\alpha} z < x | J_{\alpha} y > \right\} \\
+ \lambda \left\{ < y | z > x - (-1)^{z(x+y)} < z | x > y \right\} .
\]

(3.11)

Especially, we see

\[
x \cdot y \cdot z = 0
\]

(3.12)

for two cases of

(i) \( n = 1 \), \( N_{0} = 4 \) \hspace{1cm} (3.13a)

(ii) \( n = 3 \), \( N_{0} = -8 \) . \hspace{1cm} (3.13b)

We also note that a special case of

\[
x y z = \lambda \left\{ < y | z > x - (-1)^{z(x+y)} < z | x > y \right\}
\]

(3.14a)

gives

\[
x \cdot y \cdot z = -\frac{1}{3} \lambda (N_{0} - 10) x y z
\]

(3.14b)

Especially \( x y z \) defined by Eq. (3.14a) describes both ortho-symplectic and Lie super triple systems at the same time.

4. Yang-Baxter Equation

Let \( R_{ab}^{cd}(\theta) \) \( (a, b, c, d = 1, 2, \ldots, N) \) be the scattering matrix for \( a + b \rightarrow c + d \) in one-dimensional line with the rapidity difference \( \theta \). The (super) Yang-Baxter equation\(^9\) is the relation.

\[
\sum_{a',b',c'=1}^{N} (-1)^{a'b' + a_2 c'} + b_2 c_2 R_{a_1 b_1}^{b_1 a'} (\theta) R_{a' c_1}^{c_1 a_2} (\theta') R_{b' c_2}^{c_2 b_2} (\theta'')
\]

(4.1)

\[
= \sum_{a',b',c'=1}^{N} (-1)^{b' c' + a' c_2 + a_2 b_2} R_{b_1 c_1}^{b_1 c'} (\theta'') R_{a_1 c'}^{a_1 a'} (\theta') R_{a' b_2}^{b_2 a_2} (\theta)
\]

with the energy-momentum conservation law

\[
\theta' = \theta + \theta''
\]

(4.2)
This relation can be graphically depicted as in Fig. 1, below.

Figure 1

Here, \((-1)^{ab} = (-1)^{\sigma(a)\sigma(b)}\) is the signature factor as before.

If we set further

\[ S_{ab}^c(\theta) = (-1)^{cd} R_{ab}^c(\theta) \]  

then we can eliminate all sign factors from Eq. (4.1) in terms of \(S_{ab}^c(\theta)\). However, it will lose then some interesting features of the super-space.

We assume hereafter that \(R_{ab}^c(\theta)\) satisfies a condition

\[ R_{ab}^c(\theta) = 0 \quad \text{if} \quad \sigma(c) + \sigma(d) \neq \{\sigma(a) + \sigma(b)\} \mod 2 \]  

We now introduce two \(\theta\)-dependent triple linear products by

\[ [e^b, e_c, e_d]_\theta = \sum_{a=1}^N e_a R_{ab}^{cd}(\theta) \]  

or equivalently by

\[ R_{ab}^{cd}(\theta) = \langle e^a | [e^b, e_c, e_d]_\theta \rangle \]

\[ = (-1)^{ab+cd} \langle e^b | [e^a, e_d, e_c]_\theta \rangle \].
Note that Eqs. (4.4) and (4.5) are consistent with Eq. (2.5).

Now, the Yang-Baxter equation Eq. (4.1) can be rewritten as a triple product equation

\[ \sum_{j=1}^{N} (-1)^{\sigma(e_j)\sigma(z)} [v, [u, e_j, z]_{\theta''}, [e^j, x, y]_{\theta'\theta''}] = \sum_{j=1}^{N} (-1)^{\sigma(u)\sigma(v) + \sigma(x)\sigma(z)} \sum_{j=1}^{N} (-1)^{\sigma(e_j)\sigma(x)} [u, [v, e_j, x]^{*}_{\theta''}, [e^j, z, y]^{*}_{\theta''}] . \] (4.7)

Here, we will present a class of solutions of Eq. (4.7). Let \( x \ y \ z \) be the ortho-symplectic super-triple product defined as in section 3. We seek a solution with ansatz of

\[ [z, x, y]_{\theta} = P(\theta) (-1)^{(x+y)z} x \ y \ z + Q(\theta) < x|y > z + R(\theta) < z|x > y + S(\theta) (-1)^{xy} < z|y > x \] (4.8)

for some functions \( P(\theta), Q(\theta), R(\theta), \) and \( S(\theta) \) of \( \theta \) to be determined. We note that Eq. (4.8) implies

\[ [z, x, y]^{*}_{\theta} = [z, x, y]_{\theta} . \] (4.9)

Moreover, we assume that the triple product \( x \cdot y \cdot z \) defined by Eq. (3.8) vanishes identically, i.e.

\[ x \cdot y \cdot z = 0 . \] (4.10)

Then, the solution for \( P(\theta) \neq 0 \) can be found to be

\[ R(\theta)/P(\theta) = a + k\theta , \] (4.11a)

\[ Q(\theta)/P(\theta) = \lambda - \frac{2a\lambda}{2(a - \lambda) + k\theta} , \] (4.11b)

\[ S(\theta)/P(\theta) = -2 \lambda - \frac{2\lambda a}{k\theta} , \] (4.11c)

where we have set

\[ a = -\frac{1}{6} \lambda (N_0 - 4) = -\frac{1}{6} \lambda (N_B - N_F - 4) , \] (4.12)

and \( k \) is an arbitrary constant. Evidently, the present solution generalizes that given in II. Especially, this reproduces the solution of de Vega and Nicolai\(^{10}\), which is based upon the octonionic triple product corresponding to \( N_B = 8 \).
and $N_F = 0$. Also, $x \cdot y \cdot z$ given by Eq. (3.7) with $n = 1$ satisfies $x \cdot y \cdot z = 0$ for $N_0 = 4$ with $a = 0$, so that the solution admits existence of non-zero $P(0)$, $Q(0)$, $R(0)$, and $S(0)$ for $\theta = 0$ which may be of some interest to the knot theory.

For other applications of super-triple systems, see also ref. 11.

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