Observation of a localized flat-band state in a photonic Lieb lattice

Seababrata Mukherjee,1 Alexander Spracklen,1 Debaditya Choudhury,1 Nathan Goldman,2,3 Patrik Öhberg,1 Erika Andersson,1 and Robert R. Thomson1

1SUPA, Institute of Photonics and Quantum Sciences, Heriot-Watt University, Edinburgh, EH14 4AS, United Kingdom
2Center for Nonlinear Phenomena and Complex Systems, Université Libre de Bruxelles, CP 231, Campus Plaine, B-1050 Brussels, Belgium
3Laboratoire Kastler Brossel, Collège de France, 11 place Marcelin Berthelot, 75005, Paris, France

We demonstrate experimentally that a non-diffracting state can be excited in a photonic Lieb lattice. This lattice supports three energy bands, including a perfectly flat middle band which corresponds to an infinite effective mass with zero dispersion. We show that a suitable optical input state can be prepared so as to only excite the flat band. We analyse, both experimentally and theoretically, the evolution of such photonic flat-band states, and show their remarkable robustness, even in the presence of disorder.

PACS numbers: 63.20.Pw, 42.82.Et, 78.67.Pt

Introduction. Crystal structures are at the heart of our understanding of solid state matter and its properties. It is intriguing that by slightly changing rules for transport between the different lattice sites in the crystal, we can obtain dramatically different transport properties. The Lieb lattice (Fig 1(a)) illustrates this concept in a pertinent way. It is an edge-centered square two-dimensional lattice, exhibiting three bands; the middle one is perfectly flat with no momentum dependence in energy. The Lieb lattice is perhaps the simplest lattice which exhibits two-dimensional flat bands, see Refs. [1–7] for other lattices sharing the same property. Originally, flat-band Hubbard models were introduced in the context of magnetism, where it was shown that unusual ferromagnetic ground states were due to electrons populating the flat band [5]. Lieb lattices and flat-band effects have also been studied using cold atoms, where exotic geometries can be achieved using suitable optical-lattice configurations [8–10]. Very recently, the possibility of diffraction-free optical propagation, by exciting the flat-band eigenstates of a photonic Kagome lattice, was proposed by Vicencio et. al. [11]. In this context, Guzman-Silva et. al. [12] have reported bulk and edge transport phenomena in a photonic Lieb lattice. However, up to now, the diffraction-free propagation of a flat-band state has not been observed.

Here, we present a purely optical version of the Lieb lattice, where light propagation through an array of optical waveguides is shown to be non-diffractive due to the effective infinite mass associated with the flat-band. We present the first experimental observation of a stationary and localized flat-band state in a photonic Lieb lattice, which we fabricated using femtosecond laser writing (Fig. 1(b)) [13]. A superposition of eigenstates in the non-dispersive flat band is excited at the input to the lattice, and no significant tunneling of light into the other waveguides is observed at the output.

FIG. 1: (a) Edge-centered square Lieb lattice. The basis consists of three sites (A, B and C). To avoid edge effects and effects due to lattice inhomogeneity with depth, all measurements were performed near the circled A-site (A7−7), see also (b). The lattice constant is \(a = 44\mu m\). (b) White light transmission optical micrograph of the facet of a finite Lieb lattice with 323 waveguides fabricated by femtosecond laser writing. Each waveguide supports only a single fundamental mode at 780 nm. The next-nearest neighbor coupling for a 7 cm long glass chip was observed to be negligible. To minimise the difference in the next-nearest neighbor coupling constants along the x- and y-axes, the lattice was rotated by 45° around the z axis. (c) Representation of the three energy bands, including the flat band in the middle, for \(\{k_x, a, k_y, a\} = [0, \pi]\).

The transport of light across an array of evanescently coupled optical waveguides, i.e. a photonic lattice, is in the paraxial approximation described by a Schrödinger
The role of the wavefunction is played by the envelope where the refractive index profile across the lattice is modeled by a tight-binding Hamiltonian. In this paper it describes a lattice structure, it is possible to use photonic lattices to observe and probe phenomena known in solid-state physics, such as Bloch oscillations, dynamic localization, Bloch-Zener oscillations, and Landau-Zener tunneling.

In the case of weak evanescent coupling, Eq. (1) can be modeled by a tight-binding Hamiltonian. In this paper we consider an edge-centered square (Lieb) lattice, as shown in Fig. 1 (a). This lattice is composed of a three-site basis. When Fourier transforming the Hamiltonian into $k$-space, an energy spectrum with three bands is obtained, with

\begin{align}
\Omega_\pm(k) &= \pm 2 \sqrt{\kappa_x^2 \cos^2(k_x a) + \kappa_y^2 \cos^2(k_y a)} \\
\Omega_0(k) &= 0,
\end{align}

where $\kappa_x$ and $\kappa_y$ are the hopping amplitudes (or coupling constants) for nearest-neighbor sites along the $x$- and $y$-axes, and where $a$ is the lattice constant. $\Omega_\pm$ are the energies of the upper and the lower bands, respectively, and $\Omega_0$ represents the non-dispersive flat band. The Brillouin zone spans $0 < k_x, k_y < \pi/a$. The three bands intersect at $k_x = k_y = \pi/2a$, known as the M point, see Fig. 1(c).

The corresponding Hamiltonian for the lattice dynamics displays particle-hole symmetry, and so the energies of two of these bands can be determined by this symmetry.

Particle-hole symmetry means that for a given quasi-momentum, if $E$ is an eigenenergy, then $-E$ must also be an eigenenergy. This symmetry, combined with the statement that at each $k$ there are three energy states, automatically implies a flat band, as for each $k$, one of these energies must be zero. This argument breaks down in presence of disorder, as $k_{x,y}$ are no longer good quantum numbers. However, as shown in the supplementary material, the flat band persists in the case of off-diagonal disorder (i.e., disordered tunneling matrix elements) of arbitrary strength. This form of disorder is present in our lattices, and is due to small random variations in the waveguide-to-waveguide separations, both across and along, the lattice. In contrast, diagonal disorder would occur if different waveguides exhibited random variations in their propagation constants. As we show later, this form of disorder is not significant in our lattices.

The dotted square in Fig. 1 (a) shows a primitive cell of the lattice. There are four A-sites at the corners of each cell; two B-sites and two C-sites lie on the edges. If the Lieb lattice is isotropic, with $\kappa_x = \kappa_y$ everywhere, then a superposition of states in the flat band can be excited under the following requirements: (a) the lattice has insignificant next-nearest-neighbor coupling; and (b) the two B-sites and two C-sites of a primitive cell are excited with equal intensities ($I_B = I_C$) and alternating phases ($\phi_B = \phi_C \pm \pi$). In this letter, we demonstrate experimentally and theoretically that the flat-band state excited at the input of a photonic Lieb lattice remains localized and does not diffract.

**Fabrication of the photonic Lieb lattice.** Photonic Lieb lattices were fabricated using the well established technique of femtosecond laser writing. Accordingly, the substrate material (Corning Eagle) was mounted on air-bearing Aerotech $x$-$y$-$z$ translation stages (ABL1000), and each lattice waveguide was fabricated by translating the substrate once through the focus of a 500 kHz train of circularly polarized sub-picosecond (~400 fs) laser pulses (generated from a Menlo BlueCut fibre laser system). The various laser writing parameters were optimized to produce low propagation loss, single-mode waveguides for operation at a wavelength of 780 nm. The refractive index profiles of the waveguides was controlled using the slit-beam shaping method, by placing a slit directly in front of the 0.4 numerical aperture (NA) lens used to focus the laser pulses inside the material. The effective NA’s of the laser focus were calculated to be $\approx 0.2$ and 0.3, along the axis perpendicular and parallel to the waveguide axis, respectively. The final Lieb lattices were inscribed in a 7 cm long glass chip, and the individual waveguides exhibited a propagation loss of $\approx 1\,\text{dB/cm}$ at 780 nm.

Thirteen complete Lieb lattices (lattice constant $a = 24$ to 48 $\mu m$ in steps of $2\,\mu m$) were fabricated, with each containing 323 single-mode waveguides. However, as discussed earlier, it was not possible to observe the non-
diffraction state in lattices with lattice constants of \( \leq 42 \mu m \). This is because the next-nearest neighbor coupling in these lattices is non-negligible over the 7 cm length of the chip, destroying the flat band. The remainder of the paper shall, therefore, focus only on the lattice fabricated with \( a = 44 \mu m \) – the most compact Lieb lattice fabricated where the flat band could be excited. A white light transmission optical micrograph of the facet of this lattice is shown in Fig. 1 (b), where it can be seen that the \( x \)- and \( y \)-axes of the lattice were rotated by 45° relative to the vertical axis. The individual waveguide modes were known to be slightly elliptical in shape, exhibiting major and minor mode-field diameters of 8.6 and 7.4 \( \mu m \) along the vertical and horizontal axis respectively, this rotation was therefore chosen so as to minimize any differences in the nearest-neighbor coupling coefficients for
of the Lieb lattice. It is well known that when fabricating waveguides using femtosecond laser writing, depth-dependent aberrations imparted on the laser beam by the air-glass interface can significantly affect the properties of the written waveguides [24] – a phenomenon that will clearly result in depth-dependent lattice properties. To assess the potential impact of this on the homogeneity of the fabricated Lieb lattice, we investigated how the coupling constant between two evanescently coupled waveguides varied as a function of depth. We inscribed arrays of two-waveguide evanescent field couplers at six different depths, from 100 to 600 µm in steps of 100 µm. Each coupler was fabricated using the same waveguide-to-waveguide separation in the interaction region (22 µm), and same waveguide-to-waveguide angle (relative to the vertical axis in Fig. 1 (a)) as the waveguides in the Lieb lattice. At each depth, 5 sets of 17 couplers were fabricated, with each set consisting of couplers with interaction lengths between 1 and 65 mm, in steps of 4 mm. Each coupler was characterized by injecting 780 nm light into one of the waveguides and measuring the output coupling ratio. Using this data, and the procedure outlined in [25], the mean and standard deviation of the evaluated coupling constant at each depth was evaluated. In this manner, the mean coupling constant, and variance in coupling constant, were measured to be \( \approx 0.01 \text{ mm}^{-1} \) and \( \approx 0.002 \text{ mm}^{-1} \) respectively for couplers fabricated up to a maximum depth of 300 µm, after which both the coupling constant and variance become a function of depth, with deeper structures exhibiting a progressively higher variance.

To investigate whether the observed variance in coupling constant was due to random variations in the waveguide-to-waveguide separation (off-diagonal disorder), or random variations in waveguide propagation constants (diagonal disorder), we performed a second set of experiments, in which an array of couplers with different interaction lengths was fabricated at a single depth inside the substrate. For these couplers, the waveguide-to-waveguide angle and separation in the interaction region were set to 45° and 15 µm respectively – the reduced interaction separation was used to reduce the coupling length. It was observed that the coupling characteristic for these couplers was close to ideal, and that near-complete transfer of power from the input waveguide to the other waveguide was achieved after one coupling length. Because the complete transfer of energy from one waveguide to the another in an evanescent field coupler is only possible if the waveguides support modes with identical propagation constants, we conclude that local variations in the waveguide propagation constants are negligible in our Lieb lattices, and that diagonal disorder is not significant.

Given the results outlined above, all optical measurements of the Lieb lattice were performed by injecting light into the primitive sites surrounding the \( A_{7-7} \) site (circled in Fig. 1 (b)), which is at a depth of \( \approx 150 \text{ µm} \). First, 780 nm light was individually coupled to the nine \( A \)-, six \( B \)- and six \( C \)-sites surrounding the \( A_{7-7} \) site, and the output diffraction patterns were measured. For each type of injection site, the obtained diffraction patterns were normalized and averaged. The results of these measurements are presented in Fig. 2. The white circle in each of the images in Fig. 2 highlights the site excited at the input. It can clearly be seen that the circled \( A \)-site in Fig. 2 (a) contains less light than the circled \( B \)-site in Fig. 2 (b), or circled \( C \)-site in Fig. 2 (c). This confirms that light injected into \( A \)-sites diffracts more than light injected into \( B \)- or \( C \)-sites, a phenomenon that has been recently shown by Guzman-Silva et. al. [12].

A schematic of the experimental set-up used to excite the flat-band state is presented in Fig. 3 (a). As shown in this schematic, 780 nm laser light emerging from a single-mode fibre was collimated using lens L1. This light was then focused through a zero-order nulled (for 780 nm) diffractive optical element (DOE) (binary-phase, square-checker-board pattern) using lens L2, to generate a square array of diffracted orders at the focus of L2. Using lenses L3 and L4, these diffraction orders were relay imaged to the spatial filter, the transmission aperture of which was adjusted to block all orders, except for the four first-order “spots”. Using lenses \( L_5 \) and \( L_6 \), the four spots were relay imaged to the input-facet of the Lieb lattice (Fig. 3 (b)). The physical size of each spot on the lattice facet could be controlled via the diameter of the beam entering lens \( L_6 \) and its focal length, and the spacing between the spots could be controlled via the distance of the DOE from lens \( L_2 \). The relative optical phases between the spots could be inferred by viewing the four-spot interference pattern in the Fraunhofer regime (using a camera not shown in Fig. 3 (a)), and as shown in Figs. 3 (c) and (d), the relative phases between the spots could be controlled by translating the DOE in the \( x/y \)-plane (the \( z \) axis is the beam propagation axis). The independent control over the spacing and size of the four spots on the facet of the Lieb lattice enabled the simultaneous excitation of two \( B \)- and two \( C \)-site waveguides for a given primitive cell. To couple the spots to the lattice in a controllable manner, 780±10 nm light from a filtered broad-band source was used to flood-illuminate the output end of the lattice and excite all the guided modes. Using Camera-1, it was thus possible to simultaneously view both the lattice modes and the excitation spots, enabling us to launch light specifically to the modes of the lattice.

To excite the flat-band state, the DOE position was set to produce four equal intensity spots (relative standard deviation (RSD): 5.7 %), with alternating 0 and \( \pi \) phases, Fig. 3 (c). These four spots were then coupled to the \( B \)- and \( C \)-sites for a given primitive cell of interest, in order
to excite a superposition of eigenstates in the flat band – a proof that such a state will excite the flat band is given in the Supplementary material. The (7-7), (6-7), (7-6) and (6-6) primitive cells were each excited independently, and the output diffraction patterns observed in each case using Camera-2. As shown in Fig. 3 (a-c), when the flat-band state was excited, no significant tunneling of light into the surrounding lattice sites could be observed after 7 cm of propagation. This non-diffracting state remains localized. The non-diffracting state was not observed in lattices fabricated using lattice constants of $\leq 42 \mu m$, an observation that we attribute to non-negligible nearest-neighbor coupling in these lattices over the 7 cm length of the chip.

To confirm that the observed non-diffracting phenomenon really is unique to the phase and intensity distribution of the injected state, we coupled an orthogonal phase state to the lattice, where all the B- and C-sites of a primitive cell are excited with close to equal intensity (RSD: 5.7 %) and equal phase (the equal-phase state), Fig. 3 (d). As presented in Fig. 4 (f-j), this input state is not localized when injected into the (7-7), (6-7), (7-6) and (6-6) primitive cells – a direct result of its orthogonality to the flat-band-state. Interestingly, the diffraction patterns shown in Figs. 4 (f-i) exhibit significant differences, depending on which cell is excited. These differences are due to off-diagonal disorder, as discussed earlier, yet we are still able to successfully excite the flat band (see also Supplementary material [21] for a theoretical description).

**Conclusions.** We have experimentally shown how a flat-band state can be excited in a photonic Lieb lattice, where the propagating light is found to be non-diffractive. Such states may provide useful applications in, for instance, image processing and precision measurements. It is certainly intriguing to extend this study to the case where nonlinearities are present in the dynamics, which would provide a platform to simulate and investigate the behavior of interacting particles with flat dispersion relations. This suggests an interesting route towards strongly-correlated states of matter in photonic systems [26, 27].

While in the final stage of preparing the current manuscript we became aware of similar work by Vicencio et al. [28].

R.R.T. gratefully acknowledges funding from the UK Science and Technology Facilities Council (STFC) in the form of an STFC Advanced Fellowship (ST/H005595/1) and through the STFC Project Research and Development (STFC-PRD) scheme (ST/K00235X/1). RRT also thanks the European Union for funding via the OPTICON Research Infrastructure for Optical/IR astronomy (EU-FP7 226604). S.M. and R.R.T. thank Andrew Wad-die and Neil Ross for designing and fabricating the DOE respectively. A.S. acknowledges support from the EPSRC CM-DTC. S.M. thanks Heriot Watt University for a James-Watt Ph.D Scholarship. N.G. is financed by the FRS-FNRS Belgium.

---

* Electronic address: snm32@hw.ac.uk

[1] A. Mielke, Journal of Physics A: Mathematical and General 25, 4335 (1992).
[2] H. Aoki, M. Ando, and H. Matsumura, Phys. Rev. B 54, 1517930 (1996).
[3] S. Deng, A. Simon, and J. Köhler, Journal of Solid State Chemistry 176, 412 (2003).
[4] C. Wu, D. Bergman, L. Balents, and S. Das Sarma, Phys. Rev. Lett. 99, 70401 (2007).
[5] H. Tasaki, Eur. Phys. J. B 64, 365 (2008).
[6] Z. Lan, N. Goldman, and P. Öhberg, Phys. Rev. B 85, 155451 (2012).
[7] T. Jaqmin, I. Carusotto, I. Sagnes, M. Abbarchi, D. D. Solnyshkov, G. Malpuech, E. Galopin, A. Lemaître, J. Bloch, and A. Amo, Phys. Rev. Lett. 112, 116402 (2014).
[8] R. Shen, L. B. Shao, B. Wang, and D. Y. Xing, Phys. Rev. B 81, 041402 (2010).
[9] V. Apaja, M. Hylkäs, and M. Manninen, Phys. Rev. A 83, 063601 (2011).
[10] N. Goldman, D. F. Urban, and D. Bercioux, Phys. Rev. A 83, 063601 (2011).
[11] R. A. Vicencio and C. Mejía-Cortés, Journal of Optics 16, 015706 (2014).
[12] D. Guzmán-Silva, C. Mejía-Cortés, M. A. Bandres, M. C. Rechtsman, S. Weimann, S. Nolte, M. Segev, A. Szameit, and R. A. Vicençio, New Journal of Physics 16, 063061 (2014).
[13] K. M. Davis, K. Miura, N. Sugimoto, and K. Hirao, Opt. Lett. 21, 1729 (1996).
[14] T. Pertsch, P. Dammberg, W. Elßlein, A. Bräuer, and F. Lederer, Phys. Rev. Lett. 83, 4752 (1999).
[15] R. Morandotti, U. Peschel, J. S. Aitchison, H. S. Eisenberg, and Y. Silberberg, Phys. Rev. Lett. 83, 4756 (1999).
[16] G. Lenz, I. Talanina, and C. M. de Sterke, Phys. Rev. Lett. 83, 963 (1999).
[17] N. Chiodo, T. Talanina, and C. M. de Sterke, Phys. Rev. Lett. 83, 963 (1999).
[18] F. Dreisow, M. Heinrich, A. Szameit, S. Doering, S. Nolte, A. Tünnemann, S. Fahr, and F. Lederer, Opt. Express 16, 3474 (2008).
[19] F. Dreisow, A. Szameit, M. Heinrich, T. Pertsch, S. Nolte, A. Tünnemann, and S. Longhi, Phys. Rev. Lett. 102, 076802 (2009).
[20] F. Dreisow, A. Szameit, M. Heinrich, S. Nolte, A. Tünnemann, M. Ormigotti, and S. Longhi, Phys. Rev. A 79, 055802 (2009).
[21] URL, to be provided.
[22] M. Ams, G. Marshall, D. Spence, and M. Withford, Opt. Express 13, 5676 (2005).
[23] Y. Cheng, K. Sugio, K. Midorikawa, M. Masuda, K. Toyoda, M. Kawachi, and K. Shiroyama, Opt. Lett. 28, 55 (2003).
[24] P. S. Salter, M. Baum, I. Alexeiev, M. Schmidt, and M. J. Booth, Opt. Express 22, 17644 (2014).
SUPPLEMENTARY MATERIAL

In this supplementary material, we first show that the non-diffracting input state consists of only flat-band eigenstates when the horizontal and vertical couplings are equal. Next we consider the effect of disorder, and show that the flat band persists also in the presence of off-diagonal disorder. This type of disorder is known to be present in the experiment, as the coupling strengths between waveguides is highly sensitive to their spacing.

1. THE NON-DIFFRACTING INPUT STATE LIES IN THE FLAT BAND

The tight-binding Hamiltonian for the Lieb Lattice can be written as
\[
\hat{H} = \sum_{(n,m)} (\kappa_y \hat{b}_{n,m}^\dagger \hat{a}_{n,m} + \kappa_y \hat{b}_{n,m}^\dagger \hat{a}_{n,m-1} + \kappa_x \hat{c}_{n,m}^\dagger \hat{a}_{n,m} + \kappa_x \hat{c}_{n,m}^\dagger \hat{a}_{n,m-1}) + \text{h.c.,}
\]
where \(\hat{a}_{n,m}, \hat{b}_{n,m}\) and \(\hat{c}_{n,m}\) are the destruction operators for the A, B and C sites in lattice unit \((n,m)\), respectively. Fourier transforming the real-space tight-binding Hamiltonian results in a k-space Hamiltonian which is a 3x3 matrix due to the three inequivalent lattice sites per unit cell,
\[
\hat{H} = \sum_k \begin{pmatrix} \hat{a}_k^\dagger & \hat{b}_k & \hat{c}_k \end{pmatrix} H_k \begin{pmatrix} \hat{a}_k \\ \hat{b}_k \\ \hat{c}_k \end{pmatrix},
\]
where
\[
H_k = \begin{pmatrix} 0 & 2\kappa_y \cos(k_y) & 2\kappa_x \cos(k_x) \\ 2\kappa_y \cos(k_y) & 0 & 0 \\ 2\kappa_x \cos(k_x) & 0 & 0 \end{pmatrix}.
\]
The energy spectrum consists of three bands, two dispersive and one flat band. The eigenvectors have the form
\[
|\psi_{\pm,0}(k)\rangle = \frac{1}{\sqrt{N}} \left( e^{-ik_y} B_k^\pm \hat{b}_k^\dagger + e^{-ik_x} C_k^\pm \hat{c}_k^\dagger \right) |0\rangle,
\]
where \(N\) is the number of unit cells and \(R_a, R_b, R_c\) are the lattice positions of the different A, B and C sites. These eigenvectors form a complete orthonormal basis for real space. It can be shown that the \(A_k^\pm, B_k^\pm\) and \(C_k^\pm\) coefficients for the two dispersive bands are
\[
A_k^\pm = \pm \frac{\cos(k_x)}{\sqrt{1 + \cos(2k_y)}},
\]
\[
B_k^\pm = \frac{\kappa_y \cos(k_y) \sec(k_x)}{\sqrt{(\kappa_x^2 + \kappa_y^2 \cos(2k_x) + \kappa_x^2 \cos(2k_y)) \sec(k_x)^2}},
\]
\[
C_k^\pm = \frac{\kappa_x}{f(k_x, k_y)},
\]
The two dispersive bands differ only in their \(A_k\) coefficients. We denote the non-diffracting input state in the experiment by \(|\phi_0\rangle\), with
\[
|\psi_{k,0}^{\pm,0}\rangle = \langle \psi_{\pm,0}(k) | \phi_0 \rangle.
\]
The non-diffracting input state in the experiment has +1 on two C sites and -1 on two B sites. We will now show that this state has zero overlap with the dispersive-band states. Setting the origin on any A site gives, up to an unimportant global shift in phase,
\[
\psi_{k,0}^{\pm,0} = \frac{1}{\sqrt{N}} (e^{-ik_x} C_k^\pm + e^{-ik_x} - 2ik_x C_k^\pm - e^{-ik_y} B_k^\pm - e^{-2ik_x - ik_y} B_k^\pm) - \frac{1}{\sqrt{N}} f(k_x, k_y) (e^{-ik_y} C_k^\pm + e^{-ik_y} - 2ik_y C_k^\pm - e^{-ik_x - ik_y} B_k^\pm - e^{-2ik_x - ik_y} B_k^\pm).
\]
Equation (5) applies to coefficients for all three bands. For the coefficients for states in the dispersive bands, we obtain
\[
|\psi_{k}^{\pm}\rangle = \frac{1}{\sqrt{N}} (e^{-ik_x} C_k^\pm + e^{-ik_y} - 2ik_y C_k^\pm - e^{-2ik_x - ik_y} B_k^\pm - e^{-2ik_x - ik_y} B_k^\pm)
\]
where
\[
b_k = \frac{2\kappa_y \cos(k_y)}{e^{ik_x} (1 + e^{-2ik_x})}.
\]
Therefore

\[
\psi^\pm_k = \frac{1}{\sqrt{N}} \left[ e^{-ik_x c_k(1 + e^{-2ik_y})} - b_ke^{-ik_y (1 + e^{-2ik_x})} \right] \\
= \frac{1}{\sqrt{N}} \left[ e^{-ik_x - ik_y \kappa_x} \left( e^{ik_x + e^{-ik_y}} - 2\kappa_y \cos(k_y) e^{-ik_y - ik_x} (1 + e^{-2ik_x}) \right) \right] \\
= 2e^{-ik_x - ik_y \kappa_y} \cos(k_y) (\kappa_x - \kappa_y).
\]

When \( \kappa_x = \kappa_y \) there is therefore no overlap between the non-diffracting input state in the experiment and states in either of the dispersive bands. Consequently, the non-diffracting state must be composed of only flat-band eigenvectors.

2. DISORDER AND THE FLAT BAND

In this section we show that in the case of a Lieb Hamiltonian \( \hat{H}_1 \) with random couplings, that is, off-diagonal disorder, there are \( N \) eigenvectors which satisfy the equation \( \hat{H}_1 | \eta_i \rangle = 0 \). Hence these \( N \) eigenvectors form a flat band. \( N \) is the number of unit cells. Off-diagonal disorder is known to be present in the experiment. Definitions: Let \( | \psi^0(k) \rangle \) and \( | \psi^\pm(k) \rangle \) be eigenvectors of the disorder-free Hamiltonian \( \hat{H}_0 \). The superscript refers to the band. Let \( \hat{H}_1 \) be the disordered Hamiltonian. In a finite but periodic lattice, there are only a discrete number of allowed quasimomenta. These are labelled \( k_i \), and this label is often summed over.

Outline

The mechanism at the heart of the derivation is that a zero-energy eigenstate must satisfy

\[
\hat{H}_1 | \eta_i \rangle = \hat{H}_1 \sum_k \left[ c^\dagger(k)|\psi^0(k)\rangle + c^+(k)|\psi^+(k)\rangle + c^-(k)|\psi^-(k)\rangle \right] = 0. \tag{6}
\]

The key steps are

1. We prove that particle-hole symmetry holds also for the disordered lattice. One can then show that \( \hat{H}_1 | \psi^0(k) \rangle \) can only contain population on A sites.

2. The constraint in 1. means that \( \sum_k [c^+(k)|\psi^+(k)\rangle + c^-(k)|\psi^-(k)\rangle] \) can only contain population in A sites if \( | \eta_i \rangle \) is to satisfy equation (6).

3. The constraint in 2. implies that

\[
\sum_k [c^+(k)|\psi^+(k)\rangle + c^-(k)|\psi^-(k)\rangle] = \sum_k c(k)[|\psi^+(k)\rangle + |\psi^-(k)\rangle]. \tag{7}
\]

4. The vectors \( \hat{H}_1[|\psi^+(k)\rangle + |\psi^-(k)\rangle] \) form a basis for the A sites.

5. Points 1.-4. show that there exist zero-energy eigenfunctions of the form \( | \eta_i \rangle = | \psi^0(k_i) \rangle + \sum_k c_i(k)(|\psi^+(k)\rangle + |\psi^-(k)\rangle) \).

More specifically, point 5 can be seen in the following way. It follows from point 1. that acting with \( \hat{H}_1 \) on the first term on the rhs of \( | \eta_i \rangle \), \( \sum_k c^\dagger(k)|\psi^0(k)\rangle \), gives a vector that only has A site population. By points 2 and 3, acting with \( \hat{H}_1 \) on the remaining part of \( | \eta_i \rangle \) gives \( \sum_k c(k)[|\psi^+(k)\rangle + |\psi^-(k)\rangle] \). Item 4 says that \( \hat{H}_1[|\psi^+(k)\rangle + |\psi^-(k)\rangle] \) form a basis for the A sites. Therefore, any state with A site population can be written in the form \( \sum_k c(k)[|\psi^+(k)\rangle + |\psi^-(k)\rangle] \). In particular, by a suitable choice of \( c(k) \), the A site population coming from \( \hat{H}_1 | \psi^0(k) \rangle \) can be cancelled, giving \( \hat{H}_1 | \eta_i \rangle = 0 \) as desired.

Disorder and particle-hole symmetry

Claim. In the case of off-diagonal disorder, particle-hole symmetry is still maintained.

Proof. The disordered Hamiltonian can be written

\[
\hat{H}_1 = \sum_{nm} \left( J_{x,(n,m)}^+ c_{n,m}^\dagger \hat{a}_n \hat{a}_m + (J_{x,(n,m)}^-) c_{n-1,m}^\dagger \hat{a}_{n-1,m} \hat{a}_m \right) \\
+ \left( J_{y,(n,m)}^+ \hat{b}_{n,m}^\dagger \hat{a}_n \hat{a}_m + (J_{y,(n,m)}^-) \hat{b}_{n,m}^\dagger \hat{a}_{n+1,m} + \hat{a}_{n,m} \right) + \text{h.c.}
\]

where \( J^+ \) refers to hopping within the same unit cell whilst \( J^- \) refers to hopping to a different unit cell. The subscripts are needed on the hoppings due to the disorder. The Schrödinger equation for the system is given by

\[
-\frac{i}{\partial t} | \xi \rangle = \hat{H}_1 | \xi \rangle \Leftrightarrow \\
- i \sum_{n,m} \left( \hat{a}_{n,m} \hat{a}_{n,m}^\dagger \hat{b}_{n,m}^\dagger + \hat{b}_{n,m} \hat{b}_{n,m}^\dagger + \hat{c}_{n,m} \hat{c}_{n,m}^\dagger \right) | 0 \rangle = \hat{H}_1 | \xi \rangle
\]
The rhs of the above equation is

\[ \hat{H}_1|\xi\rangle = \hat{H}_1 \sum_{n,m} (a_{n,m}^\dagger b_{n,m} + b_{n,m}^\dagger c_{n,m} + c_{n,m}^\dagger a_{n,m})|0\rangle \]

\[ = \sum_{n,m,p,q} \left[ (J^{+}_{x,(p,q)})_{p,q}^\dagger \hat{a}_{p,q} + (J^{-}_{x,(p,q)})_{p,q}^\dagger \hat{a}_{p,q}^{-1} \hat{a}_{p,q} \right] \]

\[ + (J^{+}_{y,(p,q)})_{p,q}^\dagger \hat{b}_{p,q} \hat{a}_{p,q} + (J^{-}_{y,(p,q)})_{p,q}^\dagger \hat{b}_{p,q}^{-1} \hat{a}_{p,q} \]

\[ + (J^{+}_{y,(p,q)})_{p,q} \hat{a}_{p,q}^\dagger \hat{b}_{p,q} + (J^{-}_{y,(p,q)})_{p,q} \hat{a}_{p,q}^{-1} \hat{b}_{p,q} \]

\[ + (J^{+}_{y,(p,q)})_{p,q} \hat{b}_{p,q}^\dagger \hat{b}_{p,q} + (J^{-}_{y,(p,q)})_{p,q} \hat{b}_{p,q}^{-1} \hat{b}_{p,q} \]

\[ \times (a_{n,m}^\dagger b_{n,m} + b_{n,m}^\dagger c_{n,m} + c_{n,m}^\dagger a_{n,m})|0\rangle \]

\[ = \sum_{n,m} (J^{+}_{x,(n,m)} a_{n,m}^\dagger c_{n,m} + J^{-}_{x,(n,m)} a_{n,m})|a_{n,m}\rangle \]

\[ + (J^{+}_{y,(n,m)} a_{n,m}^\dagger b_{n,m} + J^{-}_{y,(n,m)} a_{n,m})|b_{n,m}\rangle \]

\[ + (J^{+}_{y,(n,m)} c_{n,m}^\dagger a_{n,m} + J^{-}_{y,(n,m)} c_{n,m})|c_{n,m}\rangle \]

\[ + (J^{+}_{y,(n,m)} b_{n,m}^\dagger a_{n,m} + J^{-}_{y,(n,m)} b_{n,m})|a_{n,m}\rangle \]

\[ \times (a_{n,m}^\dagger b_{n,m} + b_{n,m}^\dagger c_{n,m} + c_{n,m}^\dagger a_{n,m})|0\rangle \]

Equation (11) gives

\[ - i\hat{a}_{n,m} = J^{+}_{x,(n,m)} a_{n,m} + J^{-}_{x,(n,m)} a_{n,m+1} \]

\[ - i\hat{b}_{n,m} = J^{+}_{y,(n,m)} b_{n,m} + J^{-}_{y,(n,m)} b_{n,m-1} \]

\[ - i\hat{c}_{n,m} = J^{+}_{y,(n,m)} c_{n,m} + J^{-}_{y,(n,m)} c_{n,m+1} \]

For a state to be an eigenvector of \( H_1 \) its probability amplitudes must satisfy the equations

\[ Eq. (11) \]

\[ E_{a_{n,m}} = J^{+}_{x,(n,m)} a_{n,m} + J^{-}_{x,(n,m)} a_{n,m+1} \]

\[ E_{b_{n,m}} = J^{+}_{y,(n,m)} b_{n,m} + J^{-}_{y,(n,m)} b_{n,m-1} \]

\[ E_{c_{n,m}} = J^{+}_{y,(n,m)} c_{n,m} + J^{-}_{y,(n,m)} c_{n,m+1} \]

Consider a new vector with the elements \( a'_{n,m}, b'_{n,m} \) and \( c'_{n,m} \), where \( b'_{n,m} \) and \( c'_{n,m} \) are the same as \( b_{n,m} \) and \( c_{n,m} \) for the eigenvector but \( a'_{n,m} = -a_{n,m} \). Equation (9) gives

\[ - i\hat{a}'_{n,m} = J^{+}_{x,(n,m)} c_{n,m} + J^{-}_{x,(n,m)} c_{n,m+1} \]

\[ = E_{a_{n,m}} \]

\[ = -E_{a_{n,m}} \]

A similar equation as for \( c_{n,m} \) holds for \( b_{n,m} \). Therefore, the new vector with elements \( a'_{n,m}, b'_{n,m} \) and \( c'_{n,m} \) is also an eigenvector with energy \(-E\). For an eigenvector of the disordered Hamiltonian with energy \(-E\) there exists another eigenvector with energy \(-E\), which is particle-hole symmetry. This new eigenvector is obtained from the original one by sending \( a_{n,m} \rightarrow -a_{n,m} \) for all the the A sites. The operator \( \hat{U} \) that performs this operation therefore leaves B and C sites unchanged. This operator anticommutes with the Hamiltonian since

\[ \hat{U} \hat{H}_1 \hat{U}^\dagger = E \hat{U} \psi, \]

\[ \hat{H}_1 \hat{U} \psi = -E \hat{U} \psi. \]

In equation (16), we used particle-hole symmetry. Adding equations (15) and (16) gives

\[ \hat{U} \hat{H}_1 + \hat{H}_1 \hat{U} \psi = 0. \]

**Probability on A-sites in \( \hat{H}_1|\psi^0(k)\rangle \)**

In this section it is shown that \( \hat{H}_1|\psi^0(k)\rangle \) only has occupation on A-sites.

**Claim.** \( \hat{H}_1|\psi^0(k)\rangle \) only has probability amplitude on A-sites.

**Proof.** Consider the effect \( \hat{H}_1 \) has on a flat-band eigenvector, \( |\psi^0(k)\rangle \):

\[ |\phi\rangle = \hat{H}_1|\psi^0(k)\rangle \]

\[ = \hat{H}_1 \hat{U} |\psi^0(k)\rangle \]

\[ = -\hat{U} \hat{H}_1 |\psi^0(k)\rangle \]

\[ = -\hat{U} |\phi\rangle. \]

Here we used the fact that \( \hat{U} \) only effects A-sites, and that \( |\psi^0(k)\rangle \) has zero probability amplitude for all A sites. The operator \( \hat{U} \) leaves B and C sites unchanged. Therefore, for an arbitrary B-site component, (18) gives \( b_{n,m} = -b_{n,m} \rightarrow b_{n,m} = 0 \) and similarly for the C-site components. This means that \( |\phi\rangle \) can only have A site population.

**Consequences.** Any eigenvector of the disordered Hamiltonian can be written in terms of the eigenvectors
for the disorder-free Hamiltonian as

\[
|\eta^0\rangle = \sum_k c^0(k)|\psi^0(k)\rangle + \sum_k c^+(k)|\psi^+(k)\rangle
+ \sum_k c^-(k)|\psi^-(k)\rangle.
\]

For this eigenvector to be a zero-energy eigenvector of the disordered Hamiltonian, it is required that

\[
0 = \hat{H}_1 \left[ \sum_k c^0(k)|\psi^0(k)\rangle + \sum_k c^+(k)|\psi^+(k)\rangle 
+ \sum_k c^-(k)|\psi^-(k)\rangle \right].
\] (19)

Therefore, since \(\hat{H}_1|\psi^0(k)\rangle\) can only have A-site population, for equation (19) to hold, it is required that \(\hat{H}_1[\sum_k c^+(k)|\psi^+(k)\rangle + \sum_k c^-(k)|\psi^-(k)\rangle]\) should only have A-site population.

**Form of dispersive band component**

In the previous section it was shown that for \(|\eta^0\rangle\), \(\hat{H}_1[\sum_k c^+(k)|\psi^+(k)\rangle + c^-(k)|\psi^-(k)\rangle]\) should only have A-site population. In this section we show that this forces the \(\sum_k c^+(k)|\psi^+(k)\rangle + c^-(k)|\psi^-(k)\rangle\) component of \(|\eta^0\rangle\) to be equal to \(\sum_k c(k)|\psi^+(k)\rangle + |\psi^-(k)\rangle\). The proof of this fact relies upon the following claim.

**Claim.** \([c^+(k)|\psi^+(k)\rangle + c^-(k)|\psi^-(k)\rangle\) having no A-site population is a solution to the requirement that \(\hat{H}_1[|\psi^+(k)\rangle + c^-(k)|\psi^-(k)\rangle]\) only have A-site population.

**Proof.** \(\sum_k [c^+(k)|\psi^+(k)\rangle + c^-(k)|\psi^-(k)\rangle]\) has a general form given by \(\langle \xi | = \sum_{n,m} (a_{n,m}a_{n,m}^\dagger + b_{n,m}b_{n,m}^\dagger + c_{n,m}c_{n,m}^\dagger) |0\rangle\). It was shown in equation (3) that \(H_1\) acting upon this state gives

\[
\hat{H}_1|\xi\rangle = \sum_{n,m} [(J_{x,(n,m)}^+\lambda_{n,m}e_{n-1,m}^\dagger + (J_{x,(n,m)}^-\lambda_{n,m}e_{n-1,m}^\dagger)
+ (J_{y,(n,m)}^+\lambda_{n,m}b_{n,m}^\dagger + (J_{y,(n,m)}^-\lambda_{n,m}b_{n,m-1}^\dagger)
+ (J_{x,(n,m)}^+c_{n,m}a_{n,m}^\dagger + (J_{x,(n,m)}^-c_{n,m}a_{n,m+1}^\dagger)
+ (J_{y,(n,m)}^+b_{n,m}a_{n,m}^\dagger + (J_{y,(n,m)}^-b_{n,m+1}a_{n,m+1}^\dagger) |0\rangle.
\] (20)

Lines 1 and 2 of Eq. (20) show that the probability amplitude on the B and C sites of the vector \(H_1|\xi\rangle\) is proportional to \(a_{n,m}\), which is \(|\xi\rangle\)’s A-site probability amplitude. Therefore, if \(\sum_k [c^+(k)|\psi^+(k)\rangle + c^-(k)|\psi^-(k)\rangle]\) contains no A-site probability then this satisfies the requirement that \(\hat{H}_1[\sum_k c^+(k)|\psi^+(k)\rangle + c^-(k)|\psi^-(k)\rangle]\) contain only A-site probability.

We are now in a position to show \(c^+(k) = c^-(k)\).

**Claim.** For \(\sum_k [c^+(k)|\psi^+(k)\rangle + c^-(k)|\psi^-(k)\rangle]\) to have no A-site population it is required that \(c^+(k) = c^-(k)\).

**Proof.** In the disorder-free case the dispersive bands have eigenvectors of the form

\[
|\psi^\pm(k)\rangle = \sum_{n,m} (\pm A_k e^{ikR_{nm}^A} a_{n,m}^\dagger + B_k e^{ikR_{nm}^B} b_{n,m}^\dagger
+ C_k e^{ikR_{nm}^C} c_{n,m}^\dagger) |0\rangle.
\]

It is required that there is no A-site population. Therefore, for arbitrary \(p\) and \(q\) it must hold that

\[
0 = \langle 0| \partial_{pq} \sum_k (c^+(k)|\psi^+(k)\rangle + c^-(k)|\psi^-(k)\rangle
= \sum_k [c^+(k)A_k e^{ikR_{pq}^A} - c^-(k)A_k e^{ikR_{pq}^A}]
= \sum_k A_k e^{ikR_{pq}^A} [c^+(k) - c^-(k)] .
\]

This has to hold for all \(R^A\) and so \(c^+(k) = c^-(k)\). The consequence of \(c^+(k) = c^-(k)\) is that the expression for the zero-energy eigenstate becomes

\[
|\eta^0\rangle = \sum_k c^0(k)|\psi^0(k)\rangle + \sum_k c(k) [|\psi^+(k)\rangle + |\psi^-(k)\rangle] .
\]

**Basis for A sites**

In the previous section it was shown that if a zero-energy state exists it must have the form

\[
|\eta^0(k)\rangle = \sum_k c^0(k)|\psi^0(k)\rangle + \sum_k c(k) [|\psi^+(k)\rangle + |\psi^-(k)\rangle] .
\]

Consider acting with \(\hat{H}_1\) upon this state. It has been shown that \(\hat{H}_1|\psi^0(k)\rangle\) produces a vector with only A-site population, and in the previous section it was shown that \(\hat{H}_1[|\psi^+(k)\rangle + |\psi^-(k)\rangle]\) also has only A-site population. This suggests that it may be possible through some appropriate choice of \(c(k)\) to cancel the terms coming from \(\hat{H}_1|\psi^0(k)\rangle\). If this were the case then \(\hat{H}_1|\eta^0(k)\rangle\) would be zero and we would have a zero-energy eigenstate.

To show that this can happen we must show that the set of vectors \(\hat{H}_1[|\psi^+(k)\rangle + |\psi^-(k)\rangle]\), which we now call \(|\mu^k\rangle\), are a basis for the A sites. In the disorder-free case there are as many values of \(k\) as there are unit cells, \(N\), and so there are \(N\) different \(|\mu^k\rangle\) states. There are \(N\) A sites in the lattice, and so if the \(|\mu^k\rangle\) are linearly independent then \(|\mu^k\rangle\) will form a basis for the A sites.

**Claim.** \(|\mu^k\rangle\) are linearly independent.

**Proof.** To test for linear independence it must be shown that the only solution to \(\sum_i \alpha_i |\mu^k_i\rangle = 0\) is \(\alpha_i = 0\) \(\forall i\). Therefore the equation under consideration is

\[
\sum_i \alpha_i \hat{H}_1[|\psi^+(k)\rangle + |\psi^-(k)\rangle] = 0.
\]
If a solution would exist which does not have \( \alpha_k \neq 0 \ \forall i \) then this solution, \( |\phi\rangle = \sum \alpha_k (|\psi^+ (k)\rangle + |\psi^- (k)\rangle) \), satisfies \( \hat{H}_1 |\phi\rangle = 0 \). Therefore \( |\phi\rangle \) would represent a zero-energy eigenstate. If it can be shown that \( \sum_i \alpha_k (|\psi^+ (k)\rangle + |\psi^- (k)\rangle) \) cannot be a zero-energy eigenvector, then this would prove that the \( |\mu^k\rangle \) are linearly independent and therefore a basis for the A sites. As has been discussed previously, \( |\mu^k\rangle \) only has A-site population. For \( |\mu^k\rangle \) to be a zero-energy eigenvector we obtain from equations (14) and (13) that

\[
0 = - J_x^{+, \mu} a_{n,m} - J_x^{+, (n+1,m)} a_{n+1,m}, \quad (21) \\
0 = - J_y^{+, \mu} a_{n,m} - J_y^{+, (n,m+1)} a_{n,m+1}. \quad (22)
\]

Let us assume that these equations hold and let the probability amplitude on the \((n,m)^{th}\) A-site be \( x \). Then Eqn. (22) gives

\[
a_{n,m+1} = - \frac{J_y^{+, (n,m)}}{J_y^{+, (n,m+1)}} x.
\]

This means that \( a_{n,m+1} \) is now determined by \( x \). In turn, \( a_{n,m+1} \) determines \( a_{n+1,m+1} \) via Eqn. (21),

\[
a_{n+1,m+1} = - \frac{J_x^{+, (n+1,m)} a_{n,m+1}}{J_x^{+, (n+1,m+1)}},
\]

\[
= - \frac{J_x^{+, (n+1,m)} a_{n+1,m+1}}{J_x^{+, (n+1,m+1)}},
\]

\[
= - \frac{J_y^{+, (n,m+1)} a_{n,m+1}}{J_y^{+, (n,m+1)}}, \quad (23)
\]

The probability amplitude \( x \) also determines \( a_{n+1,m} \) via equation (21),

\[
a_{n+1,m} = - \frac{J_x^{+, (n+1,m)} x}{J_x^{+, (n+1,m+1)}}
\]

and \( a_{n+1,m} \) determines \( a_{n+1,m+1} \) via equation (22),

\[
a_{n+1,m+1} = - \frac{J_y^{+, (n,m+1)} a_{n+1,m}}{J_y^{+, (n,m+1+1)}} = - \frac{J_y^{+, (n,m+1)} a_{n+1,m+1}}{J_y^{+, (n,m+1+1)}}, \quad (24)
\]

Therefore, this gives us two equations for what \( a_{n+1,m+1} \) must be if \( a_{n,m} = x \). Equating (21) and (23) gives

\[
\frac{J_x^{+, (n+1,m)} a_{n,m+1}}{J_x^{+, (n+1,m+1)}}, \quad \frac{J_y^{+, (n,m+1)} a_{n,m+1}}{J_y^{+, (n,m+1+1)}} = \frac{J_x^{+, (n+1,m)} a_{n+1,m+1}}{J_x^{+, (n+1,m+1)}}, \quad \frac{J_y^{+, (n,m+1)} a_{n+1,m+1}}{J_y^{+, (n,m+1+1)}}.
\]

Equation (25) says that for the solution \( a_{n,m} = x \) to be consistent with the eigenvector equations, it is required that the expression of the bonds in the \( x \) direction in the lhs and the expression of the bonds in the \( y \) direction in the rhs be equal. For a randomly disordered Hamiltonian, this will in general not hold. If equation (25) does not hold then the only solution that avoids a contradiction is \( x = 0 \). This means that for a disordered Hamiltonian, a state with only A-site population cannot be a zero-energy state. Therefore, the \( |\mu^k\rangle \) are linearly independent.

### Zero-energy eigenstates

The consequence of the \( |\mu^k\rangle \) being linearly independent is that the \( |\mu^k\rangle \) form a basis for the A sites. This is crucial as it allows the creation of zero-energy eigenstates of the disordered lattice. A zero-energy eigenstate has the form

\[
|\eta^0\rangle = |\psi^0(k)\rangle + \sum_k c(k)(|\psi^+(k)\rangle + |\psi^-(k)\rangle). \quad (26)
\]

A zero-energy eigenvector must satisfy \( \hat{H}_1 |\eta^0\rangle = 0 \), and this holds for the state of equation (26) as shown now. Acting with \( \hat{H}_1 \) on the first term on the rhs of (26) gives a vector that only has A site population. Acting with \( \hat{H}_1 \) on the second term gives \( \sum_k c(k)|\mu^k\rangle \) where \( |\mu^k\rangle \) form a basis for the A sites. Therefore, by a suitable choice of \( c(k) \), the A site population coming from \( \hat{H}_1 |\psi^0(k)\rangle \) can be cancelled giving \( \hat{H}_1 |\eta^0\rangle = 0 \) as desired.

### Flat band

A whole family of such zero-energy eigenvectors can be created by changing the value of \( k \) that is used for \( |\psi^0(k)\rangle \). A general member of this family has the form

\[
|\eta_i^0\rangle = |\psi^0(k_i)\rangle + \sum_k c_i(k)(|\psi^+(k)\rangle + |\psi^-(k)\rangle).
\]

**Claim.** The \( |\eta_i^0\rangle \) are linearly independent.

**Proof.**

\[
0 = \sum_i \alpha_i |\eta_i^0\rangle = \sum_i \alpha_i (|\psi^0(k_i)\rangle + \sum_k c_i(k)(|\psi^+(k)\rangle + |\psi^-(k)\rangle)).
\]

The term in the brackets is a linear combination of disorder-free eigenstates, and therefore it cannot be zero as this would mean that the disorder-free eigenvectors were not linearly independent. The other possibility is that the brackets corresponding to two different \( k \) values cancel. Therefore assume that there is a solution with \( \alpha_i \neq 0 \) for some \( i \). Let one of the non-zero \( \alpha_i \) be \( \alpha_0 \). Then

\[
0 = \alpha_0 |\psi^0(k_0)\rangle + \sum_k c_0(k)(|\psi^+(k)\rangle + |\psi^-(k)\rangle) + \sum_{i,i \neq 0} \alpha_i (|\psi^0(k_i)\rangle + \sum_k c_i(k)(|\psi^+(k)\rangle + |\psi^-(k)\rangle))
\]

and

\[
|\psi^0(k_0)\rangle = \beta_0 \sum_k c_0(k)(|\psi^+(k)\rangle + |\psi^-(k)\rangle) + \sum_{i,i \neq 0} \beta_i (|\psi^0(k_i)\rangle + \sum_k c_i(k)(|\psi^+(k)\rangle + |\psi^-(k)\rangle)).
\]
This expression violates the linear independence of the disorder-free eigenvectors, which are a basis. Therefore, the only solution is $\alpha_i = 0 \ \forall i$, and therefore the $|\eta_i\rangle$ are linearly independent. These linearly independent eigenvectors when combined with other linearly independent non-zero energy state can be converted into an orthonormal basis via the Gram-Schmidt process. This procedure does not change the eigenvalues, and so the disordered Hamiltonian always possesses $N$ zero-energy orthonormal eigenvectors, where $N$ is the number of unit cells. Hence the flat band persists even in the presence of disorder, which was what we wanted to show.

**Comment on diagonal disorder**

Finally we would like to note that in the case of diagonal disorder it can be seen numerically that the flat band is destroyed by this type of disorder. By numerically calculating the eigenvalues of the disordered lattice, and ensemble averaging, a distribution function for the energy spacing between adjacent eigenvalues was obtained for the eigenvalues close to zero. This distribution function goes to zero as the spacing goes to zero. Therefore there is only a small probability of degenerate eigenvalues and hence the flat band is broken. This repulsion of neighbouring eigenvalues is a well known phenomena in random matrix theory and is described by the Wigner-Dyson distribution. Furthermore, numerically, it has been observed that the experimental flat band state only weakly disperses in the presence of diagonal disorder. This dispersive behaviour is due to the breaking of the flat band. The breaking of this flat band can be directly related to the diagonal disorder using the Bauer-Fike theorem [F.L. Bauer and C.F. Fike, Numer. Math. 2, 137 (1960)]. The theorem states that the eigenvalues of the perturbed matrix, i.e. the Hamiltonian with disorder, cannot differ from the eigenvalues of the disorder free Hamiltonian by more than the largest eigenvalue of the perturbing Hamiltonian. As the perturbation matrix is diagonal the eigenvalues are easily obtained. Therefore, for weak disorder the flat band is almost maintained. In the very long time limit the experimental state would disperse throughout the lattice for diagonal disorder. The same is not true for off-diagonal disorder. In this case the flat band persists, as shown above, and therefore the projection of the original state onto this flat band will remain localised.