Quantum chaos of a kicked particle in a 1D infinite square potential well

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We study quantum chaos in a non-KAM system, i.e. a kicked particle in a one-dimensional infinite square potential well. Within the perturbative regime the classical phase space displays stochastic web structures, and the diffusion coefficient \( D \) in the regime increases with the perturbative strength \( K \) giving a scaling \( D \propto K^{2.5} \), and in the large \( K \) regime \( D \) goes as \( K^2 \). Quantum mechanically, we observe that the level spacing statistics of the quasi eigenenergies changes from Poisson to Wigner distribution as the kick strength increases. The quasi eigenstates show power-law localization in the small \( K \) region, which become extended one at large \( K \). Possible experimental realization of this model is also discussed.

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In the study of quantum chaos, most works are concentrated on those quantum systems whose classical counterparts obey the Kolmogorov-Arnold-Moser (KAM) theorem. That is to say, changing the external or driven parameter, the invariant curves gradually break up and local chaos becomes global chaos, and the classical motion becomes diffusive. The widely studied models are the kicked rotator model (KRM) \( [0] \) and quantum billiards \( [5] \). In these models, an ostensible phenomenon is the dynamical localization, namely, the quantum suppression of classical diffusion. This phenomenon was first discovered numerically by Casati et al. \( [1] \) in the KRM, and later on confirmed by several experiments such as the Rydberg atom in microwave field \( [4] \) and an atom moving in a modulated standing wave etc. \( [5] \). This phenomenon has been found to be generic not only in the kicked quantum systems but also in the conservative Hamiltonian systems such as the quantum billiards \( [6] \), the Wigner band random matrix model \( [7] \), and a single ion confined in a Paul trap \( [8] \) and so on.

However, in addition to the systems mentioned above which have been studied extensively in the past two decades, there exists another class of systems which are out of the KAM frame. In these systems, the invariant curves do not exist at all for any small external/driven parameters. Compared with the KAM systems, much less is known about quantum chaos in such systems. For instance, we have only limited knowledge of the kicked harmonic oscillator introduced by Zaslavsky et al. \( [9,10] \) to describe a charged particle moving in a magnetic field, and under the disturbance of a wave packet. This model can also be used to describe a single ion trapped in a harmonic potential \( [11] \). This system is a degenerate one and does not satisfy the KAM theorem. The quasi eigenenergies, the quasi eigenstates, and the long time diffusion of this model cannot be studied by using nowadays computer facilities \( [12] \), because its phase space is unbounded and cannot be reduced to a cylinder, as in the case of the KRM.

The purpose of this letter is two-fold: (1) to construct a simple non-KAM system which could be investigated numerically both classically and quantum mechanically; (2) to study quantum chaos in such a system. As we shall see, in spite of its simplicity, our model shows stochastic webs in classical phase space which is the essential property of a non-KAM system. Unlike that of the kicked harmonic oscillator, the quasi eigenenergies and quasi eigenstates of this model can be computed easily. Furthermore, like the KRM, our model might be realized experimentally. The study of this model aims to enrich our understanding of quantum chaos in the non-KAM quantum systems.

The model we are considering in this letter is a particle moving inside a one-dimensional (1D) infinite square potential well, and under the influence of a kicked periodic external potential. The difference of this model from the kicked harmonic oscillator lies in its phase configuration. As mentioned before, the phase space of the kicked harmonic oscillator is unbounded both in momentum and in space, whereas it is bounded on a cylinder with flattened end in our model.

The Hamiltonian of our system is

\[
H = \frac{p^2}{2} + V_0(q) + k \cos(q + \alpha) \sum_{n=-\infty}^{\infty} \delta(t - nT),
\]

\[
V_0(q) = \begin{cases} 
\infty, & \text{for } q = 0, \text{ and } \pi \\
0, & \text{elsewhere} 
\end{cases}
\]

\( \alpha \) is a phase shift, in general case \( \alpha \neq 0 \). It is readily seen from this Hamiltonian that our model is a modification of the KRM. There are two minor changes: (1) two hard walls are set up at \( q = 0 \) and \( \pi \), respectively; (2) a phase shift \( \alpha \) for potential is made. The two hard walls destroy
the analyticity of the potential, and make the model out of the KAM system. The phase shift destroys the parity symmetry.

**Classical Dynamics** - The main characteristic of this system is the stochastic webs in the classical phase space. Thus the diffusion can take place along the stochastic webs for any small perturbation \( K(= kT) \), see e.g. Fig. 1 for \( K = 0.01, \alpha = 1 \). This is the fundamental difference from the KAM system e.g. the KRM. In the KRM, for any \( K < K_c = 0.971635 \ldots \), no global classical diffusion occurs due to the invariant curves. In this letter, we restrict our calculations to \( \alpha = 1 \). It has been checked that changing \( \alpha \) does not change any results quantitatively; it just shifts the stochastic webs in phase space left or right. The properties of the stochastic webs such as the thickness and symmetry etc. are also of great interest. We leave this part work over for the future study. As the \( K \) increases, the stochastic layer becomes wider and wider, and eventually covers the whole phase space. In calculating the diffusion coefficient for a given \( K \), we have taken 10,000 points starting from stochastic regions, and all the initial trajectories evolve for one million periods. Averages are taken over 10,000 trajectories for each time period. It is found that the energy diffusion is asymptotically linear for all values of \( K \).

The diffusion coefficient \( D(= \langle E_n \rangle / n) \) \( (n \) is the time in unit of \( T ) \) versus \( K \) is plotted in Fig. 2. It is evident that there exist two different diffusion regions. For \( K \gg 1 \) the diffusion coefficient is \( D \sim K^2 \), whereas for \( K \ll 1 \), the diffusion behavior is \( D \sim K^{2.5} \) which is similar to that of the discontinuous twist map. However, the underlying mechanism is different. In the case of the discontinuous twist map, the super slow diffusion is caused by the stickiness to the cantori, whereas in our model it is due to the stable islands.

Now we turn to quantum behaviors of this model. One may ask: how do these kinds of characteristics, the stochastic webs and the super slow diffusion, manifest themselves in quantum mechanics in terms of the statistics of quasi eigenenergies and quasi eigenstates? These are most interesting problems in the study of quantum chaos. Since our model is a periodically driven system, the evolution operator over the period \( T \) of the kick is given by

\[
\hat{U}(T) = \exp(-\frac{i\hat{p}^2 T}{4\hbar}) \exp(-\frac{iV(\hat{q})}{\hbar}) \exp(-\frac{i\hat{p}^2 T}{4\hbar}),
\]

where \( V(q) = k\cos(q+\alpha) \). The operator \( \hat{U}(T) \) is also called the Floquet operator, it is time-reversal invariant. Moreover, it is unitary and satisfies the following eigenvalue equation, \( \hat{U}(T)|\Psi_{\lambda}\rangle = e^{-\frac{i\pi}{T}}|\Psi_{\lambda}\rangle \) where the eigenphase \( \lambda \) is real, \( \lambda/T \) is the so called quasi eigenenergy, and \( \Psi_{\lambda} \) is the quasi eigenstate (or the Floquet state).

The quasi eigenenergies can be obtained by diagonalizing \( \hat{U}(T) \) within a large number of bases \( |n\rangle \), which we chose as the eigenstates of the non-perturbative Hamiltonian system:

\[
\langle q|n\rangle = \sqrt{\frac{2}{\pi}} \sin(nq), \; q \in [0,\pi]; \; n = 1, 2, \ldots, N. \tag{3}
\]

In our calculations \( N \) is kept at 1024. (The calculation is also performed with 512 bases, but no quantitative difference is found.) The elements of matrix are \( U_{nm} = \langle n|\hat{U}(T)|m\rangle \). As \( \hat{U}(T) \) is a unitary operator, we construct \( C = \langle U(T) + \hat{U}^+(T) \rangle / 2 \) as a Hermitian operator. Then the elements of \( C_{nm} = -A_{nm} + iB_{nm} \). The matrix \( A \) and \( B \) satisfy the condition \( A^T = A \) and \( B^T = -B \), respectively. \( \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \) is a \( 2N \times 2N \) symmetric matrix. The standard algorithm \( \mathbb{E} \) is used to diagonalize the above matrix and to obtain eigenvalues and eigenvectors \( [u, v] \). Then we project the \( N \) dimensional vector \( n + iv \) on the basis of a plane wave to obtain the eigenstates of \( \hat{U}(T) \). The fast Fourier transform (FFT) of sinusoidal form \( \mathbb{E} \) is employed to transform the wave function between the position representation and energy representation in our calculations.

**Quasi eigenstates** - The quasi eigenstates show behaviors quite different from that of the KRM. In the KRM, the quasi eigenstates are exponentially localized in the momentum space. In our model, however, the quasi eigenstates are power-law localized, as is shown in Fig. 3. In this figure, we demonstrate a few typical states at different values of \( K \) for localized, intermediate and extended ones. It is clearly seen that the localized states gradually transit to extended ones as we increase the strength of the kicked potential. This transition, as we shall see later, will manifest itself in the statistics of quasi eigenenergies.

In fact, the power-law localization of the eigenstates can be traced back to the structure of the matrix \( U \). In the KRM, the values of matrix elements \( U_{nm+n} \) decay faster than exponential when \( n \) exceeds the band width \( b \) which is proportional to \( K \), thus the elements outside this band can be regarded as zero. Within the band of width \( b \), the elements are proved to be pseudorandom \( \mathbb{E} \). Such kind of band random matrix attracted much attention in the past years. However, in our model the situation is different. Careful analysis yields that the elements outside the band decay as a power law with \( |U_{nm+n}| \approx 1/n^2 \). We calculate \( \langle U^2 \rangle_n \approx \langle U^2_{nm+n} \rangle \) (the average is done over \( n \)) for four different \( K \)'s, and plot them in Fig. 4. The typical slope of the curves over a large range is approximately minus 4. And the band width \( b \) in our model is found also to be approximately proportional to the perturbation strength \( K \). Inside this band the magnitudes of the matrix elements are almost constant. This kind of band random matrix, describes a new class of physical systems e.g. systems with non-analytic singular boundary, has not yet been fully studied \( \mathbb{E} \). More works are expect to be done.
Statistics of the quasi eigenenergies – The structure of the quasi eigenstates determines the energy level statistics. As is well known that the level repulsion can occur between the Floquet eigenvalues when the Floquet eigenfunctions overlap. In the KRM, the quasi eigenfunctions are exponentially localized in angular momentum. Since the angular momentum has a finite range, the Floquet states with very close eigenvalues may lie so far apart that they don’t overlap. Thus, we don’t have any level repulsion for these two eigenvalues. This is the reason why Poisson-like spectral statistics persists in the KRM even though the system is classically chaotic. To observe the transition from Poisson to Wigner distribution, one has to consider a KRM defined on a torus [2].

In our model, however, the Floquet states do overlap in momentum space as is shown above. Therefore, we are expecting to observe the transition of the quasi eigenstates determines the energy level statistics. As is well known that the level repulsion can occur between the Floquet eigenvalues when the Floquet eigenfunctions overlap.

To quantify this transition, the Brody distribution [10] is used to best fit the above four distributions. (In fact we use the cumulative distribution function $I(s) = \int_0^s P(s’)ds’$). The best fitting gives rise to the Brody parameter $\beta = 0.03, 0.08, 0.46$ and 0.82, for the four distributions in Fig. 5, respectively. To check the approach to Poisson distribution as $K$ goes down to zero, the $P(s)$ is also calculated for $K = 10^{-4}$, as expected, we obtain a good Poisson distribution, the best fitting gives rise to $\beta = 0.01$.

It must be stressed that the difference between our model and the KRM shown in level spacing statistics of the quasi eigenenergies and the quasi eigenstates comes from non-analyticity of the potential, which makes the phase space in our model a half cylinder with the end flattened. This non-analyticity also leads to a different structure of the evolution matrix $U$. Moreover, we would like to point out that since the KRM has already been realized in laboratory by putting the cold (sodium/cesium) atoms in a periodically pulsed standing wave of light [3], our model could be also realized experimentally. One possible way is to put the cold atoms in a quasi 1D quantum dot. The atoms are then driven by a periodically pulsed standing wave of light. The quasi 1D quantum dot might be realized by formulating a 2D quantum dot elongated in one direction, namely, the size in one direction is much larger than another. This experiment would allow us to study quantum chaos in a non KAM system.

In summary, we have studied the classical dynamics and quantum behaviors of a kicked particle in a 1D infinite square potential well. In spite of its simplicity, our model exhibits stochastic webs which is one of the basic features of the non-KAM systems. The classical dynamics is diffusive for any infinitesimal perturbation, and the classical diffusion rate is found to be $D \propto K^{2.5}$ for small values of $K$, and $D \propto K^2$ for large values of $K$. The level statistics of quasi eigenenergies shows a smooth transition from Poisson to Wigner distribution for a fixed dimension of the Floquet matrix. The quasi eigenstates are found to be power-law localized with exponent equal to two. Our model provides a new paradigm in investigating classical quantum correspondence of stochastic motion exhibited in Hamiltonian systems with non-analytic boundary conditions.

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FIG. 1. A typical classical phase space of our model at very small perturbation strength. The stochastic webs is clearly seen. Here we have $K = 0.01$, $\alpha = 1$. One trajectory starts from $(q_0 = 0.1, p_0 = 0.012)$ and evolves for 100,000 periods.

FIG. 2. Classical diffusion coefficient $D$ versus perturbation strength $K$. The best fitting by using the data $K > 1$ gives rise to a slope 1.97, whereas that by using the data $K < 0.1$ gives rise to a slope 2.47. A clear turning point of the slope can be seen at $K$ about 1.

FIG. 3. Typical quasi eigenstates in different regimes, localized ($K = 0.1$ and 5), intermediate ($K = 25$), and extended ($K = 50$). The corresponding values of $K$ are shown in the figure.
FIG. 4. The averaged matrix element $\langle U^2 \rangle$ versus $n$ for different values of $K$. See the text for its definition. From left to right, the dashed curve for $K = 0.1$, dotted curve for 5, thick solid curve for 25, and thin solid curve for 50. The bandwidth is about the order of $K$, which is clearly seen from the figure. The slopes of these four curves are about minus four.

FIG. 5. The distribution of the nearest neighbor level spacing $P(s)$ for different $K$. The corresponding values of $K$ are given in the figures. The dotted curve is Poisson and the thin curve is Wigner distribution. The histograms are numerical results. $P(s)$ at $K = 0.1$ and 5 are close to Poisson distribution and at $K = 25$ is intermediate, and $K = 50$ is close to Wigner distribution. The corresponding best fitting Brody parameter $\beta$ are 0.03, 0.08, 0.46, and 0.83, respectively.
