Stabilization of two-dimensional solitons and vortices against supercritical collapse by lattice potentials

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It is known that optical-lattice (OL) potentials can stabilize solitons and solitary vortices against the critical collapse, generated by the cubic attractive nonlinearity in the 2D geometry. We demonstrate that OLs can also stabilize various species of fundamental and vortical solitons against the supercritical collapse, driven by the double-attractive cubic-quintic nonlinearity (however, solitons remain unstable in the case of the pure quintic nonlinearity). Two types of OLs are considered, producing similar results: the 2D Kronig-Penney “checkerboard”, and the sinusoidal potential. Soliton families are obtained by means of a variational approximation, and as numerical solutions. The stability of the families, which include fundamental and multi-humped solitons, vortices of oblique and straight types, vortices built of quadrupoles, and supervortices, strictly obeys the Vakhitov-Kolokolov criterion. The model applies to optical media and BEC in “pancake” traps.

PACS numbers: 03.75.Lm, 05.45.Yv, 42.65.Tg, 42.70.Nq

I. INTRODUCTION AND THE MODEL

Formation of multidimensional solitons and solitary vortices (solitons with embedded vorticity) has drawn a great deal of attention in studies of nonlinear optics and Bose-Einstein condensation (BEC), see review [1]. (2+1)-dimensional spatial solitons and quasi-2D spatiotemporal ones were created in crystals with the quadratic ($\chi^{(2)}$) nonlinearity [2]. Also reported were spatial solitons, vortices, and dipole-mode states in photorefractive crystals with a photo-induced lattice, where the nonlinearity is saturable [3]. However, truly 2D or 3D solitons have not yet been observed in media with the generic cubic ($\chi^{(3)}$) nonlinearity, the problem being their instability against the collapse [4, 5]. Vortex solitons are subject (in the uniform space) to a still stronger azimuthal instability, which occurs even in the absence of the collapse [6].

A relevant ingredient of both optical and BEC settings which may stabilize multidimensional solitons and vortices is an effective periodic potential. In optics, it implies a periodic modulation of the refractive index in the transverse plane, as in photonic-crystal fibers), and in BEC it is induced by optical lattices (OLs). In the medium with the self-focusing (SF) $\chi^{(3)}$ nonlinearity, the stabilization of fundamental and vortical 2D solitons (those with topological charges $S=0$ and 1, respectively) under the action of the square-lattice potential was predicted in works [7] and [8] (see also works [9]; stable solitons were found too in photonic-crystal-fiber models [10]). The action of the stabilization mechanism can be summarized as follows. In the free 2D space, the $\chi^{(3)}$ nonlinearity supports a family of Townes solitons (TSs), which have the same norm, $Q=Q_{TS}$, at all values of the propagation constants, $k$ [5, 11]. Although there are no unstable eigenvalues in the spectrum of small perturbations around Townes solitons, a specific zero eigenvalue accounts for the instability against sub-exponentially growing perturbations, which is a manifestation of the critical character of the collapse induced by the $\chi^{(3)}$ nonlinearity in the 2D space. As demonstrated by numerical findings and the variational approximation (VA), the action of the OL potential with small strength $\epsilon$ lifts the degeneracy of the Townes-soliton family, stretching the single point, $Q=Q_{TS}$, into interval $Q_{TS}-\Delta Q<Q<Q_{TS}$ of width $\Delta Q \sim \epsilon$, which is filled by stable solitons, that obey the Vakhitov-Kolokolov (VK) criterion, $dQ/dk>0$. This criterion is well known to be a necessary condition for the stability of solitons in media with SF nonlinearities [5, 12].

As concerns vortex solitons supported by the square OL, the most compact “crater-shaped” ones, represented by a single density peak with a vorticity-induced hole in the center, are unstable [13]. Stable vortices with $S=1$ can be built as sets of four (or eight) peaks, with the phase shift $\pi/2$ (or $\pi/4$, respectively) between adjacent ones [7, 8]. Stable vortices of higher orders, up to $S=6$, built of up to 12 peaks, were found too, as well as stable “supervortices”, i.e., ring chains of 12 (or more) compacts vortices carrying local spins $s=1$, with global vorticity $S=\pm 1$ imprinted onto the entire ring. These patterns were found in models with cubic and saturable SF nonlinearities [15]. Stable vortices of the gap-soliton type were also found in the model combining the OL and self-repulsive nonlinearity [16].

A general mathematical classification of various localized states with an intrinsic phase structure (in particular, vortices) in the 2D model with the square-lattice cos potential and cubic nonlinearity was developed in [17].

In the 2D geometry, supercritical collapse is generated by the SF quintic ($\chi^{(5)}$) nonlinearity, which may occur in a combination with $\chi^{(3)}$ terms. In optics, the cubic-quintic (CQ) nonlinearity with the SF $\chi^{(5)}$ part was predicted [18] and recently observed [19] in aqueous colloids. It was also observed in dye solutions [20], and recently predicted,
through the cascaded mechanism, in two-level media \[21\]. It is relevant to mention that the $\chi^{(5)}$ nonlinearity with
the self-defocusing sign is observed in various uniform optical media \[22\].

Besides the context of nonlinear optics, the CQ nonlinearity of the SF type appears in the description of BEC with
attraction between atoms trapped in a “pancake” configuration. Strictly speaking, the reduction of the underlying 3D
Gross-Pitaevskii equation (GPE) to the 2D form produces a more complex nonpolynomial nonlinearity \[23\]. However,
for 1D configurations corresponding to a cigar-shaped trap, it was demonstrated that the respective GPE with the
SF quintic term, which represents a “vestige” of the underlying multi-dimensionality, provides for an appropriate
description of the soliton dynamics, unless one is interested in the collapse per se \[24\]. Another straightforward
interpretation of the quintic term in the GPE is the contribution of three-body collisions, provided that the lossy part
of this interaction (kicking out atoms from the condensate) may be neglected \[25\].

The OL potential is not necessary for the stability of 1D solitons in the SF CQ model, as they are stable in the
free 1D space, despite the possibility of the collapse \[26\]. On the contrary, in the 2D model with the SF $\chi^{(3)}$ and
$\chi^{(5)}$ nonlinearities, the OL is a crucial factor for the stabilization of solitons against the supercritical collapse. In the
normalized form, the corresponding model is based on the equation for the mean-field wave function, $u(x, y, t)$ (or the
local amplitude of an electromagnetic wave, in terms of optics):

$$iu_t + u_{xx} + u_{yy} - V(x, y)u + 2|u|^2u + \gamma|u|^4u = 0.$$  \[1\]

The derivation of Eq. \[1\] from the full GPE yields $\gamma \sim a_z^2/D^2$, where $a_z$ is the width of the transverse confinement,
and $D$ the half-period of the OL potential. We will consider two different potentials, viz., the cosinusoidal (cos)
one, $V = -V_0 [\cos (\pi x/D) + \cos (\pi y/D)]$, and the 2D Kronig-Penney (KP) potential, in the form of a “checkerboard”
composed of cells of size $D$ with potential difference $V_0$ between adjacent ones \[13\] (2D states in the checkerboard
potential combined with the CQ nonlinearity in which the $\chi^{(5)}$ term is self-defocusing, i.e., $\gamma < 0$, hence the collapse
does not occur, were recently investigated in Ref. \[13\]). Equation \[1\] conserves the norm,

$$Q \equiv \int \int |u(x, y)|^2 dx dy.$$  \[2\]

Undoing scalings used in the derivation of Eq. \[1\] from the underlying GPE, one concludes that the actual number
of atoms in the BEC is $N \sim (a_z/|a_z|) Q$, where $a_z$ is the scattering length. In terms of nonlinear optics, evolution
variable $t$ in Eq. \[1\] is the propagation distance, rather than time.

For the purpose of comparison, we will also consider a modification of Eq. \[1\] with pure quintic nonlinearity,

$$iu_t + u_{xx} + u_{yy} - V(x, y)u + |u|^4u = 0.$$  \[3\]

Stationary solutions to Eqs. \[1\] and \[3\] for 2D fundamental (single-humped) solitons were found by means of the
VA and in a numerical form, as reported below in Section II. Higher-order (multi-humped) nontopological solitons,
as well as vortices (including those built of quadrupoles, rather than of fundamental solitons) and supervortices, were
constructed by means of numerical methods, see Sections II and III, respectively. The stability of all these localized
patterns was inferred from the VK criterion and verified by direct simulations. To test the stability numerically,
the initial perturbation was, typically, imposed by the multiplication of a numerically exact stationary solution by
perturbing factor $(1 + \varepsilon)$, typically with $|\varepsilon| \lesssim 0.03$ (this was quite sufficient to identify stable and unstable solitons,
monitoring their evolution in the course of sufficiently long simulations). As a result, a border between stable and unstabe solutions has been found in each soliton family in the CQ model, while all solitons of Eq. \[3\] are unstable.
Approaching the stability border, the width of the stability margin, i.e., the maximum value of perturbation amplitude
$\varepsilon$ in the above expression, which does not trigger the onset of instability, shrinks to zero.

II. FUNDAMENTAL AND MULTI-HUMPED SOLITONS

Solutions to Eq. \[1\] with real chemical potential $-k$ (in optical models, $k$ is the propagation constant) are sought
for as $u(x, z) = \exp (ikt) U(x, y)$, with $U(x, y)$ obeying the stationary equation,

$$U_{xx} + U_{yy} - V(x, y)U + 2|U|^2U + \gamma|U|^4U = kU,$$  \[4\]

which is associated with Lagrangian $L = \int \int L(x, y)dxdy$, whose density is

$$L = |U_x|^2 + |U_y|^2 + [k + V(x, y)] |U|^2 - |U|^4 - (\gamma/3) |U|^6.$$  \[5\]
FIG. 1: (Color online) A typical example of numerically found stable fundamental soliton in the model with the KP (checkerboard) potential, for \( D = 3, V_0 = 2, \) and \( \gamma = 1. \) The soliton is pertains to \( k = 1.53, \) its norm being \( Q = 2.05. \)

Fundamental solitons with amplitude \( A \) and width \( W \) may be approximated by ansatz \( U(x, y) = A \exp\left(-\frac{x^2 + y^2}{2W^2}\right), \) whose norm \( (2) \) is \( Q = \pi A^2 W^2, \) hence the ansatz can be written as

\[
U(x, y) = \sqrt{\frac{Q}{\pi W}} \exp\left(-\frac{x^2 + y^2}{2W^2}\right),
\]

(6)

where \( Q \) and \( W \) may be treated as free variational parameters. The use of the isotropic ansatz is suggested by numerically found shapes of the fundamental solitons, which feature an approximate axial symmetry, except for “tails”, where the local amplitude of the wave field is very small, see Fig. 1.

Subsequent derivation of the VA from Lagrangian (5) and ansatz (6) is straightforward for the cos potential. In the KP model, the checkerboard potential may be replaced, for this purpose, by its two lowest spatial harmonics [13]. Thus, the calculation of the respective effective Lagrangian, \( L = L(Q, W), \) leads to the variational equations, \( \partial L/\partial Q = \partial L/\partial W = 0, \) which take the form of

\[
k + 1 - Q = 0,
\]

(7)

\[
1 - \frac{Q}{2\pi} - \frac{2\gamma Q^2}{9\pi^2 W^2} - \frac{\pi^2 V_0 W^4}{8D^2} \exp\left(-\frac{\pi^2 W^2}{4D^2}\right) = 0.
\]

The modification of the VA for the quintic-only equation [3] is obvious: in Eqs. (7), the terms linear in \( Q \) should be dropped, and \( \gamma = 1 \) should be substituted.

Families of fundamental solitons, as predicted by the VA, i.e., obtained from a numerical solution of Eqs. (7), and found from a numerical solution of Eq. (4), that was performed by means of a modification of the relaxation method, are represented in Fig. 2 by a set of curves \( Q(k), \) for different values of \( \chi(5) \) coefficient \( \gamma. \) The case of the pure quintic nonlinearity, which corresponds to Eq. (3), is included too.

It is seen that solitons in the CQ model do not exist with the norm below a threshold value, \( Q_{\text{min}}, \) which is explained by the delocalization transition [14] in the region dominated by the cubic nonlinearity (therefore \( Q_{\text{min}} \) weakly depends on \( \gamma \)). Further, the VK criterion suggests that, in the presence of the cubic term and for any \( \gamma > 0, \) the solitons are stable only up to a point at which \( dQ/dk \) changes the signs. In accordance with works [7], the stability-change point does not exist in the model with \( \gamma = 0. \) Note that the norm of stable solitons attains the largest value, which is \( Q = Q_{\text{TS}}, \) for \( \gamma = 0; \) with the growth of \( \gamma, \) the largest norm of the stable solitons decreases, which demonstrates the increasing difficulty in the stabilization of the solitons with the transition from the critical (cubic) to supercritical (quintic) nonlinearity in the 2D geometry. In accordance with this trend, all solitons in the pure quintic model (the
FIG. 2: The norm of the fundamental solitons versus the propagation constant, $k$, as predicted by the VA (dotted curves) and found from numerical solution of Eq. (4) with the cos potential (solid curves). Fixed parameters are $V_0 = 2$ and $D = 3$. In addition, the dashed-dotted curve displays the same dependence as predicted by the VA for the model with the pure quintic nonlinearity, i.e., Eq. (3).

one without the cubic term) are unstable (as per the VK criterion), as the $Q(k)$ curve for the quintic model (the dashed-dotted curve in Fig. 2) features solely the negative slope, $dQ/dk < 0$. Direct simulations exactly corroborate all the predictions of the VK criterion. While Fig. 2 displays the results for the model with the cos potential, the situation in the KP model is the same, the respective $Q(k)$ curves being very close to those shown in Fig. 2.

In the CQ model, the decrease of $V_0$ leads to shrinkage of the portions of the $Q(k)$ curves with the positive slope, and they disappear at some $(V_0)_{min}$, leaving only unstable states, with $dQ/dk < 0$. The respective stability regions for the fundamental solitons in the $(V_0,Q)$ plane are displayed in Fig. 3 for the KP model (the situation for its counterpart with the cos potential is very similar). The lower stability boundary in this figure is the delocalization border, below which solitons do not exist, while the upper border is exactly predicted by the VK criterion (i.e., solitons are unstable above it). The evident trend to the shrinkage of the stability region with the decrease of $D$ is explained by the exponential smallness of the force of the interaction of a broad soliton with a short-period OL.

Values of $k$ for all solutions reported in this work belong to the semi-infinite gap in the OL-induced spectrum. Solitons can also be found in finite bandgaps; however, as well as in the 2D KP model with the self-repulsive $\chi^{(3)}$ term [13], all gap solitons turn out to be unstable.

As shown above, the supercritical collapse imposes an upper bound on the norm of stable fundamental solitons [undoing rescalings leading to Eq. (1)], one can conclude that the number of atoms in the respective matter-wave soliton is $\lesssim 10^4$. Stable localized states with a larger norm can be built as multi-humped solitons. Due to the symmetry imposed by the OL in two dimensions, the first species of that type following fundamental solitons features five peaks, cf. Ref. [13]. Figure 4 displays examples of stable five-peaked solitons found in the KP and cos models, for a common value of the chemical potential, $k = 1.8$ (the solitons are rotated relative to each other by angle $\pi/4$ due to the difference in the definition of the periodic potential in the models). As well as the fundamental solitons, families of these solutions feature the stability-change point, separating portions of the respective $Q(k)$ curves with $dQ/dk > 0$ and $< 0$ (not shown here). Actually, dependences $Q(k)$ and stability regions in the plane of $(V_0,Q)$ for the five-peak solitons are similar to those for the fundamental solitons, which are displayed above in Figs. 2 and 3 with a difference that $Q$ is larger by a factor of $\approx 4$. Injection of more norm gives rise to higher-order solitons. For $Q$ still larger than the maximum value admitted by the five-peak solitons, their nine-peak counterparts were obtained.

Besides the solutions built as complexes of in-phase peaks, the model also gives rise to stable dipole, quadrupole, and multi-pole localized states. In particular, examples of a quadrupole can be seen below in Fig. 8 as building blocks used to compose a new type of vortices (quadrupole vortices).
FIG. 3: Stability regions (between the upper and lower borders) for fundamental solitons in the plane of the OL strength \( V_0 \) and soliton's norm \( Q \) in the KP model for different values of OL half-period \( D \). The coefficient in front of the quintic term is \( \gamma = 1 \).

FIG. 4: (Color online) Typical examples of the five-humped soliton found in models with the KP (b) and cosinusoidal (b) potential, respectively. Both solitons pertain to propagation constant \( k = 1.8 \) and differ in the total norm: \( Q = 12.42 \) for panel (a), and \( Q = 14.44 \) for (b). Lattice parameters are the same as those corresponding to Fig. [1].

III. VORTEX SOLITONS, QUADRUPOLE VORTICES, AND SUPERVORTICES

Solitary vortices with topological charge \( S \) are found as complex solutions to Eq. (4) with the phase circulation of \( 2\pi S \). Compact (“crater-shaped”) vortex solitons, with the vorticity nested in a single peak, are unstable (the “crater” splits into a set of nonsteady pulses resembling fundamental solitons, with a single one surviving the subsequent evolution). Two species of stable vortices with \( S = 1 \) have been found in the present models (with the KP and cos potentials alike), either one being arranged as a set of four peaks with phase shifts \( \pi/2 \) between them. Referring to the orientation of diagonals connecting the opposite peaks, which compose the vortices, relative to the KP “checkerboard”, the species may be called oblique and straight, examples of which are displayed together in Fig. [5]. While oblique
vortices include a nearly empty site at the center, the straight vortex places its center at a local potential maximum, without any vacancy.

Families of the solitary vortices of both types are presented in Fig. 6 by the respective $Q(k)$ curves, which summarize numerically obtained results. The VK criterion is only a necessary condition for the stability of vortices, as it does not detect azimuthal instabilities. Nevertheless, systematic simulations have demonstrated that the stability of the vortex solitons in the present model precisely obeys the VK criterion, i.e., they are stable up to the point where $dQ/dk$ changes its sign, see Fig. 6. This is explained by the fact that the lattice potential is strong enough to suppress the azimuthal instability of the vortices, similar to the situation in the cubic model. Note that, as well as in the case of fundamental solitons, cf. 2, the stability-change point, $dQ/dk = 0$, does not exist in the limit of the cubic equation, i.e., $\gamma = 0$, and the norm of the solitary vortices attains its maximum just in this limit.

The model also supports more complex stable vortex structures, such as supervortices, see an example of the oblique type in Fig. 7. Although each compact crater-shaped vortex, of which the structure is built, is unstable in isolation, the ring formed by four of them, with the global vorticity imprinted onto it, is stable, as verified by direct simulations. $Q(k)$ curves for families of the supervortices are similar to those shown in Fig. 6, their stability also precisely obeying the VK criterion. In particular, for the same values of parameters as in Fig. 7, the stability border (which coincides with the point where $qQ/dt$ vanishes) is found at $k \approx 12$. At this point, the norm of the supervortex attains its maximum, $Q_{\text{max}} \approx 27$.

Families of higher-order vortices with $S > 1$ have been found too. A novel type of stable solitary vortices can be constructed using, as building blocks, a set of four quadrupoles (rather than simple peaks, cf. Fig. 5), with the phase shift of $\pi/2$ between adjacent ones, which corresponds to $S = 1$. An example of a corresponding quadrupole vortex of the straight type is displayed in Fig. 8. These findings will be reported in a detailed form elsewhere.

IV. CONCLUSION

We have demonstrated that various species of solitons and vortices can be stabilized by periodic potentials in the 2D geometry against the supercritical collapse, driven by the self-attractive cubic-quintic nonlinearity. For the Kronig-Penney and cos potentials, soliton families were obtained by means of the variational approximation and in a numerical form. The stability of all the families, including fundamental and multi-humped solitons, solitary vortices of the oblique and straight types and supervortices precisely obeys the VK criterion. A novel species of quadrupole vortices has been demonstrated. The model can be realized in composite (colloidal) optical media, and
FIG. 6: $Q(k)$ curves for families of oblique and straight vortex solitons (dotted and solid curves, respectively), with $S = 1$, in the KP model with $V_0 = 10, D = 3, \gamma = 1$.

FIG. 7: (Color online) The distribution of the local density (a) and phase (b) in a stable supervortex supported by the KP potential with $V_0 = 10, D = 2$, and $\gamma = 1$, for $k = 9$ and $Q = 24.26$. The global vorticity of the pattern is $S = +1$, while spins of four individual crater-shaped vortices, with centers placed at points $(x, y) = (\pm 4, \pm 4)$, are $s = -1$.

in self-attractive BEC in “pancake”-shaped traps.

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FIG. 8: (Color online) An example of a stable quadrupole vortex (of the straight type), built as a chain of four quadrupoles, with the global phase circulation of $2\pi$. Panels (a) and (b) display the distribution of the local density and phase in the pattern. Parameters are $V_0 = 10$, $D = 3$, $\gamma = 1$, and $k = 8.5$, $Q = 1.17$.

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