WEAK FACTORIZATION OF HARDY SPACES IN THE BESSEL SETTING

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Abstract. We provide the weak factorization of the Hardy spaces $H^p(\mathbb{R}_+, dm_\lambda)$ in the Bessel setting, for $p \in \left(\frac{2\lambda+1}{2\lambda+2}, 1\right]$. As a corollary we obtain a characterization of the boundedness of the commutator $[b, R_{\Delta\lambda}]$ from $L^q(\mathbb{R}_+, dm_\lambda)$ to $L^r(\mathbb{R}_+, dm_\lambda)$ when $b \in \text{Lip}_\alpha(\mathbb{R}_+, dm_\lambda)$ provided that $\alpha = \frac{1}{q} - \frac{1}{r}$. The results are a slight generalization and modification of the work of Duong, Li, Yang, and the second named author, which in turn are based on modifications and adaptations of work by Uchiyama.

1. Introduction

The theory of Hardy spaces has been studied and developed extensively in harmonic analysis and more precisely the theory of Hardy spaces on the Euclidean setting has been shown to have many applications, see [6–8,15] for an instance of general references.

The real-variable Hardy space theory on $n$-dimensional Euclidean space $\mathbb{R}^n$, $n \geq 1$, plays an important role in harmonic analysis and has been systematically developed [6,8]. There are many equivalent definitions of the Hardy spaces $H^p(\mathbb{R}^n)$, $0 < p < \infty$. It is well-known that when $p > 1$, the actual definition of $H^p(\mathbb{R}^n)$ makes it equivalent to $L^p(\mathbb{R}^n)$, but when $p \in (0,1]$, these spaces are much better suited to ask questions about harmonic analysis than are the $L^p(\mathbb{R}^n)$ spaces, see [9,15] for an account of all of this.

In the case of the real-variable Hardy space $H^1(\mathbb{R}^n)$, Coifman, Rochberg and Weiss [5] provided a factorization that works in studying function theory and operator theory of $H^1(\mathbb{R}^n)$ which was called the weak factorization. This weak factorization for $H^1(\mathbb{R}^n)$ consist of the following: every $f \in H^1(\mathbb{R}^n)$ can be written as

$$f = \sum_{j=1}^\infty \sum_{i=1}^n (g^j_i R_i h^j_i + h^j_i R_i g^j_i),$$

where $\{g^j_i\}_{i,j}, \{h^j_i\}_{i,j} \in H^2(\mathbb{R}^n)$ and $R_i$ are the Riesz transforms on $\mathbb{R}^n$ and $\|f\|_{H^1(\mathbb{R}^n)} \simeq \inf \left\{ \sum_{j=1}^\infty \sum_{i=1}^n \|g^j_i\|_{L^2(\mathbb{R}^n)} \|g^j_i\|_{L^2(\mathbb{R}^n)} \right\}$, with the infimum taken over all possible representations of $f$ as above. Later, Uchiyama [17] found an algorithmic way to generalize this weak factorization for Hardy spaces $H^p(\mathbb{R}^n)$ with values of $p \in (0,1]$, but close to 1. In fact, the algorithm that Uchiyama provides works in spaces of homogeneous type for Calderón–Zygmund operators that satisfy certain lower bounds on their kernels. On the other hand, it is also well-known, as pointed in [5], that this weak factorization is closely related with the boundedness of some commutator on $L^p$ spaces, to be defined
later. Since then, many authors generalized the boundedness of this commutator between different $L^p$ spaces [10, 11, 13, 14].

The theory of the classical Hardy space $H^p(\mathbb{R}^n)$ is intimately connected to the Laplacian $\Delta$. Changing the differential operator introduces new challenge and directions to explore. In 1965, Muckenhoupt and Stein in [12] introduced the notion of conjugacy associated with this Bessel operator $\Delta_\lambda$, $\lambda > 0$, which is defined by

$$\Delta_\lambda f(x) := -\frac{d^2}{dx^2}f(x) - \frac{2\lambda}{x} \frac{d}{dx}f(x), \quad x > 0.$$ 

They developed a theory in the setting of $\Delta_\lambda$. Results on $L^p(\mathbb{R}^+, dm_\lambda)$-boundedness of conjugate functions and fractional integrals associated with $\Delta_\lambda$ were obtained, where $p \in (1, \infty)$, $\mathbb{R}^+ := (0, \infty)$ and $dm_\lambda(x) := x^{2\lambda}dx$. Since then, many problems based on the Bessel context were studied; see, for example, [1, 3, 16, 18, 20]. In particular, the properties and $L^p$ boundedness $(1 < p < \infty)$ of Riesz transforms

$$R_{\Delta_\lambda} f := \partial_x (\Delta_\lambda)^{-1/2}f$$

related to $\Delta_\lambda$ have been studied in [12, 18]. The related Hardy space

$$H^1(\mathbb{R}^+, dm_\lambda) := \{ f \in L^1(\mathbb{R}^+, dm_\lambda) : R_{\Delta_\lambda} f \in L^1(\mathbb{R}^+, dm_\lambda) \}$$

with norm $\|f\|_{H^1(\mathbb{R}^+, dm_\lambda)} := \|f\|_{L^1(\mathbb{R}^+, dm_\lambda)} + \|R_{\Delta_\lambda} f\|_{L^1(\mathbb{R}^+, dm_\lambda)}$ has been studied by Betancor et al. in [2] where they established the characterizations of the atomic Hardy space $H^1(\mathbb{R}^+, dm_\lambda)$ associated with $\Delta_\lambda$ in terms of the Riesz transform and the radial maximal function associated with the Hankel convolution of a class $Z^{[\lambda]}$ of functions, which includes the Poisson semigroup and the heat semigroup as special cases. Duong, Li, Wick and Yang [16] used Uchiyama’s algorithm to prove the weak factorization on the Bessel setting of Hardy space $H^1(\mathbb{R}^+, dm_\lambda)$ in terms of the Riesz transform $R_{\Delta_\lambda}$. One can not appeal to Uchiyama’s results directly since the kernel of the Bessel Riesz transforms do not satisfy the hypotheses of his results in [17]; however, with appropriate modifications one can carry out his approach.

For the general case of $p \in (0, 1]$, Yang and Yang [20] characterized the Hardy spaces $H^p(\mathbb{R}^+, dm_\lambda)$ via atomic decomposition for values of $p \in (\frac{2\lambda+1}{2\lambda+2}, 1]$. They also showed a counterpart of the characterization of Hardy spaces in terms of the Riesz transforms. More concretely, in [20, Theorem 1.2], they proved that for $p \in (\frac{2\lambda+1}{2\lambda+2}, 1]$, we have that $f \in H^p(\mathbb{R}^+, dm_\lambda)$ if and only if there exist $C > 0$ such that

$$\|f \ast_\lambda \phi_\delta\|_p + \|R_{\Delta_\lambda} (f \ast_\lambda \phi_\delta)\|_p \leq C,$$

where $\phi_\delta(x) = \alpha^{-2\lambda-1} \phi(x/\delta)$, $\phi \in Z^{[\lambda]}$ and

$$f \ast_\lambda g(x) := \int_0^\infty f(y) \tau_y^{[\lambda]} g(y) dm_\lambda(y),$$

where for $x \in (0, \infty)$, $\tau_y^{[\lambda]} g(y)$ denotes the Hankel translation of $g(y)$, that is,

$$\tau_y^{[\lambda]} g(y) := \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\lambda) \sqrt{\pi}} \int_0^\pi (\sin \theta y)^{2\lambda-1} d\theta.$$

Then the aim of this paper is the following. We first build up a weak factorization for the Hardy space $H^p(\mathbb{R}^+, dm_\lambda)$, for $p \in (\frac{2\lambda+1}{2\lambda+2}, 1]$, in terms of a bilinear form related to $R_{\Delta_\lambda}$. Secondly, as a consequence, we further prove that this weak factorization implies
the characterization of the commutator with a symbol in \( \text{Lip}_\alpha(\mathbb{R}_+, dm_\lambda) \), the space of Lipschitz (or Hölder) continuous of order \( \alpha > 0 \) functions.

Throughout this paper, for any \( x, r \in \mathbb{R}_+ \), \( I(x, r) := (x - r, x + r) \cap \mathbb{R}_+ \). From the definition of the measure \( m_\lambda \), i.e., \( dm_\lambda(x) := x^{2\lambda}dx \), it is obvious that there exists a positive constant \( C \in (1, \infty) \) such that for all \( x, r \in \mathbb{R}_+ \),

\[
(1.1) \quad C^{-1}m_\lambda(I(x, r)) \leq x^{2\lambda}r + r^{2\lambda+1} \leq Cm_\lambda(I(x, r)).
\]

Thus \( (\mathbb{R}_+, \rho, dm_\lambda) \) is a space of homogeneous type in the sense of Coifman and Weiss [6], where \( \rho(x, y) := |x - y| \) for all \( x, y \in \mathbb{R}_+ \). We denote by \( \| \cdot \|_p \) the norm of \( L^p(\mathbb{R}_+, dm_\lambda) \), for any \( 0 < p \leq \infty \).

We now state our main result on the weak factorization of the Hardy spaces \( H^p(\mathbb{R}_+, dm_\lambda) \), for \( p \in (\frac{2\lambda+1}{2\lambda+2}, 1] \).

**Theorem 1.1.** Let \( p \in (\frac{2\lambda+1}{2\lambda+2}, 1] \) and \( q, r > 1 \) such that

\[
(1.2) \quad \frac{1}{p} = \frac{1}{q} + \frac{1}{r}.
\]

For any \( f \in H^p(\mathbb{R}_+, dm_\lambda) \), there exists numbers \( \{\alpha_j^k\}_{j,k} \), functions \( \{g_j^k\}_{j,k} \subset L^q(\mathbb{R}_+, dm_\lambda) \) and \( \{h_j^k\}_{j,k} \subset L^r(\mathbb{R}_+, dm_\lambda) \) such that

\[
(1.3) \quad f = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \alpha_j^k \Pi(g_j^k, h_j^k)
\]
in \( H^p(\mathbb{R}_+, dm_\lambda) \) where \( \Pi \) is defined as

\[
\Pi(g, h) := gR_{\Delta \lambda}h - h\widetilde{R_{\Delta \lambda}}g,
\]

where \( \widetilde{R_{\Delta \lambda}} \) is the adjoint operator of \( R_{\Delta \lambda} \). Moreover, there exists a positive constant \( C \) independent of \( f \) such that

\[
C^{-1} \|f\|_{H^p(\mathbb{R}_+, dm_\lambda)} \leq \inf \left\{ \left( \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\alpha_j^k|^p \|g_j^k\|_q^p \|h_j^k\|_r^p \right)^{\frac{1}{p}} : f = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \alpha_j^k \Pi(g_j^k, h_j^k) \right\} \leq C \|f\|_{H^p(\mathbb{R}_+, dm_\lambda)} .
\]

Note that the case \( p = 1 \) is exactly as Duong, Li, Wick and Yang [16] did, so our contribution here is the cases of \( p \in (\frac{2\lambda+1}{2\lambda+2}, 1) \).

As a corollary we get the following second main result that provides a characterization of the \( \text{Lip}_\alpha(\mathbb{R}_+, dm_\lambda) \) space in terms of the boundedness of the commutators adapted to the Riesz transform \( R_{\Delta \lambda} \). Recall the definition of the \( \text{Lip}_\alpha(\mathbb{R}_+, dm_\lambda) \) space associated with the Bessel operator, which is the dual space of \( H^p(\mathbb{R}_+, dm_\lambda) \), for \( \alpha = \frac{1}{p} - 1 \), see [6, Theorem B].

**Definition 1.2** ([6, p.591]). Let \( \alpha > 0 \). The space \( \text{Lip}_\alpha(\mathbb{R}_+, dm_\lambda) \) is defined as the set of functions \( f \) measurable on \( (\mathbb{R}_+, dm_\lambda) \) which satisfy

\[
|f(x) - f(y)| \leq Cm_\lambda(I)^{\alpha},
\]

where \( \lambda \in (\mathbb{R}_+, dm_\lambda) \).
where $I$ is any interval containing both $x$ and $y$ and $C > 0$ is a constant independent of $x$ and $y$. The greatest lower bound of these constants is denoted by $\|f\|_{\text{Lip}_a(\mathbb{R}^+, dm)}$.

Suppose $b \in L^1_{\text{loc}}(\mathbb{R}^+, dm)$ and $f \in L^p(\mathbb{R}^+, dm)$, for $p \in (1, \infty)$. Let $[b, R_{\Delta \lambda}]$ be the commutator defined by

$$[b, R_{\Delta \lambda}]f(x) := b(x)R_{\Delta \lambda}f(x) - R_{\Delta \lambda}(bf)(x).$$

**Theorem 1.3.** Let $b \in L^1_{\text{loc}}(\mathbb{R}^+, dm)$ and $1 < p < q < \infty$ and $0 < \alpha < \frac{1}{2\lambda + 1}$ such that

$$\alpha = \frac{1}{p} - \frac{1}{q},$$

(1) If $b \in \text{Lip}_a(\mathbb{R}^+, dm)$, then the commutator $[b, R_{\Delta \lambda}]$ is bounded from $L^p(\mathbb{R}^+, dm)$ to $L^q(\mathbb{R}^+, dm)$ with the operator norm

$$\|[b, R_{\Delta \lambda}]\|_{L^p(\mathbb{R}^+, dm) \to L^q(\mathbb{R}^+, dm)} \leq C\|b\|_{\text{Lip}_a(\mathbb{R}^+, dm)},$$

where $C > 0$ is a constant independent of $b$.

(2) If $[b, R_{\Delta \lambda}]$ is bounded from $L^p(\mathbb{R}^+, dm)$ to $L^q(\mathbb{R}^+, dm)$, then $b \in \text{Lip}_a(\mathbb{R}^+, dm)$ and

$$\|b\|_{\text{Lip}_a(\mathbb{R}^+, dm)} \leq C\|[b, R_{\Delta \lambda}]\|_{L^p(\mathbb{R}^+, dm) \to L^q(\mathbb{R}^+, dm)},$$

where $C > 0$ is another constant independent of $b$.

The proofs of Theorems 1.1 and 1.3 are shown via the following arguments. We first provide some preliminary results that we need for the proof of Theorem 1.1, and the case (1) of Theorem 1.3 is also proved beforehand using similar ideas as [16]. Once we have Theorem 1.1, the second part (2) of Theorem 1.3 follows from that.

The structure of the paper is as follows. In Section 2 we recall all the preliminary results from the literature that we will need later on. For example, we recall the Hardy spaces associated with $\Delta \lambda$ and also we collect some fundamental estimates of the kernel of the Riesz transform $R_{\Delta \lambda}$. In Section 3 we prove Theorems 1.1 and 1.3. In addition we also show some new lemmas that are the key to prove our main results. Finally, in Section 4 we explain some difficulties we have encountered and open problems related to that.

Throughout the paper, if $f \lesssim g$ or $g \gtrsim f$ denote that there exists a constant $C$ independent of $f$ and $g$ such that $f \leq Cg$. When $f \lesssim g$ and $g \lesssim f$ we write $f \simeq g$.

## 2. Preliminaries

In this section we recall the preliminary notions that we need. More concretely, we recall the Hardy spaces and Riesz transform related to the Bessel operator $\Delta \lambda$ from [2, 12].

We now recall the atomic characterization of the Hardy spaces $H^p(\mathbb{R}^+, dm)$, for $p \in \left(\frac{2\lambda + 1}{2\lambda + 2}, 1\right]$ in [20].

**Definition 2.1.** Let $p \in \left(\frac{2\lambda + 1}{2\lambda + 2}, 1\right]$. A function $a$ is called an $H^p(\mathbb{R}^+, dm)$-atom (or simply a $p$-atom) if there exists an open bounded interval $I \subset \mathbb{R}^+$ such that $\text{supp} \ a \subset I$, $\|a\|_\infty \leq m_\lambda(I)^{-1/p}$ and $\int_0^\infty a(x)dm_\lambda(x) = 0$. 
Let $p \in \left( \frac{2\lambda + 1}{2\lambda + 2}, 1 \right]$. We point out that from [20], the Hardy space $H^p(\mathbb{R}_+, dm_\lambda)$ can be characterized via an atomic decomposition. That is, an $L^p(\mathbb{R}_+, dm_\lambda)$ function $f \in H^p(\mathbb{R}_+, dm_\lambda)$ if and only if

$$f = \sum_{k=1}^{\infty} \alpha_k a_k \text{ in } L^p(\mathbb{R}_+, dm_\lambda),$$

where for every $k$, $a_k$ is a $p$-atom and $\alpha_k \in \mathbb{R}$ satisfying that $\sum_{k=1}^{\infty} |\alpha_k|^p < \infty$. Moreover,

$$\|f\|_{H^p(\mathbb{R}_+, dm_\lambda)} \simeq \inf \left\{ \left( \sum_{k=1}^{\infty} |\alpha_k|^p \right)^{\frac{1}{p}} \right\},$$

where the infimum is taken over all the decompositions of $f$ as above.

We also note that $H^p(\mathbb{R}_+, dm_\lambda)$ can also be characterized in terms of the radial maximal function, the nontangential maximal function, the grand maximal function, the Littlewood-Paley $g$-function and the Lusin-area function. More interesting in our case, we have also the Riesz transform characterization of these Hardy spaces; see [20, Theorem 1.2].

Next we recall the Poisson integral, the conjugate Poisson integral and the properties of the Riesz transforms. As in [2, 16, 20], let $\{P^\lambda_t\}_{t>0}$ be the Poisson semigroup $\{e^{-t\sqrt{\Delta_\lambda}}\}_{t>0}$ defined by

$$P^\lambda_t f(x) := \int_0^\infty P^\lambda_t(x, y) f(y) y^{2\lambda} dy,$$

where

$$P^\lambda_t(x, y) = \int_0^\infty e^{-tz} (xz)^{-\lambda+\frac{1}{2}} J_{\lambda-\frac{1}{2}}(z)(yz)^{-\lambda+\frac{1}{2}} J_{\lambda-\frac{1}{2}}(yz) z^{2\lambda} dz$$

and $J_\nu$ is the Bessel function of the first kind and order $\nu$. Weinstein [19] established the following formula for $P^\lambda_t(x, y)$: for $t, x, y \in \mathbb{R}_+$,

$$P^\lambda_t(x, y) = \frac{2\lambda}{\pi} \int_0^\pi \frac{(\sin \theta)^{2\lambda-1}}{(x^2 + y^2 + t^2 - 2xy \cos \theta)^{\lambda+1}} d\theta.$$

The $\Delta_\lambda$-conjugate of the Poisson integral of $f$ is defined by

$$Q^\lambda_t f(x) := \int_0^\infty Q^\lambda_t(x, y) f(y) y^{2\lambda} dy,$$

where

$$Q^\lambda_t(x, y) = \frac{2\lambda}{\pi} \int_0^\pi \frac{(x - y \cos \theta)(\sin \theta)^{2\lambda-1}}{(x^2 + y^2 + t^2 - 2xy \cos \theta)^{\lambda+1}} d\theta.$$

From this, we deduce that for any $x, y \in \mathbb{R}_+$,

$$R_{\Delta_\lambda}(x, y) = \partial_x \int_0^\infty P^\lambda_t(x, y) dt = -\frac{2\lambda}{\pi} \int_0^\pi \frac{(x - y \cos \theta)(\sin \theta)^{2\lambda-1}}{(x^2 + y^2 + t^2 - 2xy \cos \theta)^{\lambda+1}} d\theta = \lim_{t \to 0} Q^\lambda_t(x, y).$$

We note that, as indicated in [2], this Riesz transform $R_{\Delta_\lambda}$ is a standard Calderón-Zygmund operator. We recall that if $R_{\Delta_\lambda}(x, y)$ is the kernel of the Riesz transform $R_{\Delta_\lambda}$ then for any measurable function $f$ in $(\mathbb{R}_+, dm_\lambda)$,

$$R_{\Delta_\lambda} f(x) = \int_{\mathbb{R}_+} f(y) R_{\Delta_\lambda}(x, y) dm_\lambda(y), \quad x \in \mathbb{R}_+. $$
Lemma 3.1. Let $f$ be a function satisfying the following conditions:

(i) $\int_0^\infty f(x)x^{2\lambda} \, dx = 0$;

(ii) there exists $I(x_1, r)$ and $I(x_2, r)$ for some $x_1, x_2, r \in \mathbb{R}_+$ and $C_1, C_2 > 0$ such that

$$|f(x)| \leq C_1 \chi_{I(x_1, r)}(x) + C_2 \chi_{I(x_2, r)}(x);$$

(iii) $|x_1 - x_2| \geq 4r$.

Then there exists $C > 0$ (independent of $x_1, x_2, r, C_1, C_2$) such that

$$\|f\|_{H^p(\mathbb{R}_+, dm_\lambda)} \leq C \left[ C_1 \chi_{I(x_1, r)}(x) + C_2 \chi_{I(x_2, r)}(x) \right].$$

for any $p \in \left(\frac{2\lambda+1}{2\lambda+2}, 1\right].$
Proof. Assume that \( f := f_1 + f_2 \), where \( \text{supp} \ f_i \subset I(x_i, r) \) for \( i = 1, 2 \) (we can suppose that thanks to (ii)). We will show that \( f \) has a particular \( p \)-atomic decomposition. To this end, we write

\[
f = \sum_{i=1}^{2} (f_i - \tilde{\alpha_i}^1 \chi_{I(x_i, 2r)}) + \sum_{i=1}^{2} \tilde{\alpha_i}^1 \chi_{I(x_i, 2r)} = f_1 + f_2 + \sum_{i=1}^{2} \tilde{\alpha_i}^1 \chi_{I(x_i, 2r)}
\]

where

\[
\tilde{\alpha_i}^1 := \frac{1}{m_\lambda(I(x_i, 2r))} \int_{I(x_i, r)} f_i(x) \, dm_\lambda(x)
\]

and define \( a_i^1 := f_i^1 / \alpha_i^1 \), where \( \alpha_i^1 := \|f_i^1\|_\infty m_\lambda(I(x_i, 2r))^{1/p} \), for \( i = 1, 2 \). Then we can see that \( a_i^1 \) is a \( p \)-atom supported on \( I(x_i, 2r) \). Indeed, clearly \( \text{supp} \ a_i^1 \subset I(x_i, 2r) \) and

\[
\|a_i^1\|_\infty = \frac{1}{\alpha_i^1} \|f_i^1\|_\infty = m_\lambda(I(x_i, 2r))^{-1/p}.
\]

Finally, since

\[
\int_{0}^{\infty} \tilde{\alpha_i}^1 \chi_{I(x_i, 2r)}(x) \, dm_\lambda(x) = \int_{I(x_i, r)} f_i(x) \, dm_\lambda(x),
\]

we have that \( \int_{0}^{\infty} a_i^1(x) \, dm_\lambda(x) = 0 \) and so \( a_i^1 \) is a \( p \)-atom. Moreover, since \( \|f_i\|_\infty \leq C_i \) (for (ii)) and \( m_\lambda \) is doubling, we have that

\[
|a_i^1| \lesssim 2^{1/p - 1} C_i m_\lambda(I(x_i, r))^{1/p}.
\]

For \( i = 1, 2 \), we further write

\[
\tilde{\alpha_i}^1 \chi_{I(x_i, 2r)} = \tilde{\alpha_i}^1 \chi_{I(x_i, 2r)} - \tilde{\alpha_i}^2 \chi_{I(x_i, 4r)} + \tilde{\alpha_i}^2 \chi_{I(x_i, 4r)} =: f_i^2 + \tilde{\alpha_i}^2 \chi_{I(x_i, 4r)},
\]

where

\[
\tilde{\alpha_i}^2 := \frac{1}{m_\lambda(I(x_i, 4r))} \int_{I(x_i, r)} f_i(x) \, dm_\lambda(x).
\]

Let

\[
\alpha_i^2 := \|f_i^2\|_\infty m_\lambda(I(x_i, 4r))^{1/p}
\]

and \( a_i^2 := f_i^2 / \alpha_i^2 \). Then \( a_i^2 \) is a \( p \)-atom supported on \( I(x_i, 4r) \). Indeed, it is obvious that \( \text{supp} \ a_i^2 \subset I(x_i, 4r) \) and

\[
\|a_i^2\|_\infty = \frac{1}{\alpha_i^2} \|f_i^2\|_\infty = m_\lambda(I(x_i, 4r))^{-1/p}.
\]

Finally, since

\[
\int_{0}^{\infty} \tilde{\alpha_i}^2 \chi_{I(x_i, 4r)}(x) \, dm_\lambda(x) = \int_{I(x_i, r)} f_i(x) \, dm_\lambda(x),
\]

we have that \( \int_{0}^{\infty} a_i^2(x) \, dm_\lambda(x) = 0 \) and so \( a_i^2 \) is a \( p \)-atom. Moreover, using (1.1) and the fact that \( m_\lambda \) is doubling, we obtain that

\[
|a_i^2| \leq \|\tilde{\alpha_i}^1\| m_\lambda(I(x_i, 4r))^{1/p} \leq \frac{m_\lambda(I(x_i, r))}{m_\lambda(I(x_i, 2r))} \|f_i\|_\infty m_\lambda(I(x_i, 4r))^{1/p}
\]

\[
\lesssim 2^{(1/p - 1)} C_i m_\lambda(I(x_i, r))^{1/p}.
\]
Continuing this fashion we see that
\[
f = \sum_{i=1}^{2} \left[ \sum_{j=1}^{J_0} f_{i}^{j} \right] + 2 \sum_{i=1}^{2} \alpha_i^j \chi_{I(x_i,2^{j}r)} = \sum_{i=1}^{2} \left[ \sum_{j=1}^{J_0} \alpha_i^j a_i^{j} \right] + 2 \sum_{i=1}^{2} \tilde{\alpha}_i^j \chi_{I(x_i,2^{j}r)},
\]
where \( J_0 \) is the smallest integer larger than \( \log_2 \frac{|x_1-x_2|}{r} \) and for \( j \in \{2, 3, \ldots, J_0\} \),
\[
\alpha_i^j := \frac{1}{m_\lambda(I(x_i,2^{j}r))} \int_{I(x_i,r)} f_i(x) \, dm_\lambda(x),
\]
\[
f_i^j := \tilde{\alpha}_i^{j-1} \chi_{I(x_i,2^{j-1}r)} - \tilde{\alpha}_i^{j} \chi_{I(x_i,2^{j}r)},
\]
\[
\alpha_i^j := \| f_i^j \|_\infty m_\lambda(I(x_i,2^{j}r))^{1/p} \text{ and } a_i^j := f_i^j / \alpha_i^j.
\]
By condition (iii) we guarantee that \( J_0 \geq 2 \). Moreover, for each \( i \) and \( j \), \( a_i^j \) is a \( p \)-atom supported on \( I(x_i,2^{j}r) \). Indeed, for \( j \in \{2, 3, \ldots, J_0\} \), it is obvious that \( \text{supp} \, a_i^j \subset I(x_i,2^{j}r) \) and
\[
\| a_i^j \|_\infty = \frac{1}{\alpha_i^j} \| f_i^j \|_\infty = m_\lambda(I(x_i,2^{j}r))^{-1/p}.
\]
Finally, since
\[
\int_0^\infty \tilde{\alpha}_i^j \chi_{I(x_i,2^{j}r)}(x) \, dm_\lambda(x) = \int_{I(x_i,r)} f_i(x) \, dm_\lambda(x),
\]
we have that \( \int_0^\infty a_i^j(x) \, dm_\lambda(x) = 0 \) and so \( a_i^j \) is a \( p \)-atom. In addition, using again (1.1) and the fact that \( m_\lambda \) is doubling we also have that
\[
|a_i^j| \leq |\tilde{\alpha}_i^{j-1}| m_\lambda(I(x_i,2^{j-1}r))^{1/p} \leq \frac{m_\lambda(I(x_i,r))}{m_\lambda(I(x_i,2^{j-1}r))} \| f_i \|_\infty m_\lambda(I(x_i,2^{j}r))^{1/p} \leq 2^{j(1/p-1)} C_i m_\lambda(I(x_i,r))^{1/p}.
\]
For the last part, set
\[
\sum_{i=1}^{2} \tilde{\alpha}_i^j \chi_{I(x_i,2^{j}r)} = \left( \tilde{\alpha}_1^j \chi_{I(x_1,2^{j}r)} - \tilde{\alpha}_2^j \chi_{I(x_2,2^{j}r)} \right) + \left( \tilde{\alpha}_1^j \chi_{I(x_1,2^{j+1}r)} + \tilde{\alpha}_2^j \chi_{I(x_2,2^{j+1}r)} \right) := \sum_{i=1}^{2} f_i^{j+1},
\]
where
\[
\tilde{\alpha}_i^j := \frac{1}{m_\lambda(I(x_i,2^{j+1}r))} \int_{I(x_i,r)} f_i(x) \, dm_\lambda(x)
\]
\[
= - \frac{1}{m_\lambda(I(x_2,2^{j+1}r))} \int_{I(x_2,r)} f_2(x) \, dm_\lambda(x),
\]
using (i) in the last equality. Now, for \( i = 1, 2 \), let
\[
\alpha_i^{j_0+1} := \| f_i^{j_0+1} \|_\infty m_\lambda(I(x_i,\frac{x_1+x_2}{2},2^{j_0+1}r))^{1/p}
\]
and
\[
\alpha_i^{j_0+1} := f_i^{j_0+1} / \alpha_i^{j_0+1}.
\]
We see that $a_i^{j_0+1}$ is a $p$-atom: it is not difficult to see that $\text{supp} a_i^{j_0+1} \subset I((x_1 + x_2)/2, 2^{j_0+1}r)$ because $J_0$ is the smallest integer larger than $\log_2 |x_1 - x_2|/r$ and
\[
\|a_i^{j_0+1}\|_\infty = \frac{1}{\alpha_i^{j_0+1}} \|f_i^{j_0+1}\|_\infty = m_\lambda(I((x_1 + x_2)/2, 2^{j_0+1}r))^{-1/p}.
\]

Finally, since
\[
\int_0^\infty \tilde{\alpha}_J \chi_I((x_1 - x_2)/2, 2^{j_0+1}r) \, dx \, d\lambda(x) = \int_{I(x_1, r)} f_1(x) \, d\lambda(x) = -\int_{I(x_2, r)} f_2(x) \, d\lambda(x),
\]
by (i), we have that $\int_0^\infty a_i^{j_0+1}(x) \, d\lambda(x) = 0$ and so $a_i^{j_0+1}$ is a $p$-atom. Moreover, by (1.1),
\[
|a_i^{j_0+1}| \leq |\tilde{\alpha}_J| m_\lambda(I((x_1 + x_2)/2, 2^{j_0+1}r))^{1/p} \leq \frac{m_\lambda(I((x_1 + x_2)/2, 2^{j_0+1}r))^{1/p-1}}{m_\lambda(I(x_i, r))^{1/p-1}} \|f_i\|_\infty m_\lambda(I(x_i, r))^{1/p} \lesssim 2^{(j_0+1)(1/p-1)} C_i m_\lambda(I(x_i, r))^{1/p}.
\]

In conclusion, we get the following $p$-atomic decomposition
\[
f = \sum_{i=1}^{J_0+1} \sum_{j=1}^{2} \alpha_i^j a_i^j,
\]
with, for $i \in \{1, 2\}$,
\[
|\alpha_i^j| \lesssim 2^{(1/p-1)} C_i m_\lambda(I(x_i, r))^{1/p}, \quad j \in \{1, \ldots, J_0 + 1\}.
\]
By [20] we have that $f \in H^p(\mathbb{R}_+, d\lambda)$ and
\[
\|f\|_{H^p(\mathbb{R}_+, d\lambda)} \leq \left( \sum_{i=1}^{J_0+1} \sum_{j=1}^{2} |\alpha_i^j|^p \right)^{1/p} \lesssim \left( \sum_{j=1}^{2^{(1/p-1)}} \left( \sum_{i=1}^{J_0+1} 2^{j-1/p} \right)^{1/p} \left( \sum_{i=1}^{2} C_i^p m_\lambda(I(x_i, r)) \right)^{1/p} \right.
\]
\[
\leq (J_0 + 1)^{1/p} (2^{(j_0+1)(1/p-1)})^{1/p} \left( \sum_{i=1}^{2} C_i^p m_\lambda(I(x_i, r)) \right)^{1/p} \lesssim \left( \log_2 \frac{|x_1 - x_2|}{r} \right)^{1/p} \left( \frac{|x_1 - x_2|}{r} \right)^{1/p-1} \left( \sum_{i=1}^{2} C_i^p m_\lambda(I(x_i, r)) \right)^{1/p}.
\]
This finishes the proof of Lemma 3.1. \qed

**Proposition 3.2.** Let $p \in \left(\frac{2k+1}{2k+2}, 1\right]$ and $q, r > 0$ such that (1.2) holds. For every $\varepsilon > 0$, there exist $M > 0$ and $C > 0$ such that for all $p$-atom $a$, exists $g \in L^q(\mathbb{R}_+, d\lambda)$ and $h \in L^r(\mathbb{R}_+, d\lambda)$ satisfying that
\[
\|a - \Pi(g, h)\|_{H^p(\mathbb{R}_+, d\lambda)} < \varepsilon
\]
and $\|g\|_q \|h\|_r \leq CM\frac{r^{k+1}}{r^{k+1}}$. 

Proof. Let \( a \) be a \( p \)-atom with \( \text{supp} \, a \subset I(x_0, r) \) where \( x_0, r > 0 \). Observe that if \( r > x_0 \), then \( I(x_0, r) = (x_0 - r, x_0 + r) \cap \mathbb{R}_+ = I(x_0, 2r) \). Therefore, without loss of generality, we may assume that

\[
(3.1) \quad r \leq x_0.
\]

Let \( K_1 \) and \( K_2 \) be the constants appeared in (i) and (ii) of Proposition 2.3 respectively, and \( K_0 > \max \left\{ \frac{1}{K_1}, \frac{1}{K_2} \right\} + 1 > 1 \) large enough. For any \( \varepsilon > 0 \), let \( M \) be a positive constant large enough such that \( M \geq 100K_0 \) and \( \frac{\log_2 M}{M^{1/p}} < \varepsilon^p \) (possible since \( p > 1/2 \) when \( p \in \left( \frac{2\lambda + 1}{2\lambda + 2}, 1 \right) \)).

We now consider the following two cases.

**Case (a):** Assume that \( x_0 \leq 2Mr \). In this case, let \( y_0 := x_0 + 2MK_0r \). Then, by (3.1),

\[
(3.2) \quad (1 + K_0)x_0 \leq y_0 \leq (1 + 2MK_0)x_0
\]

and

\[
(3.3) \quad 2MK_0r \leq y_0 \leq (1 + K_0)2Mr.
\]

Define

\[
(3.4) \quad g(x) := \chi_{I(y_0, r)}(x) \quad \text{and} \quad h(x) := -\frac{a(x)}{R_{\Delta \lambda}g(x_0)}.
\]

By (3.1) and (3.2) we know that \( y/x_0 > K_0 > K_1^{-1} \) for any \( y \in I(y_0, r) \). Using this fact, Proposition 2.3 (i) and (3.3) we see that

\[
(3.5) \quad \left| \tilde{R}_{\Delta \lambda}g(x_0) \right| = \left| \int_{y_0-r}^{y_0+r} R_{\Delta \lambda}(y, x_0) \, dm_\lambda(y) \right| \gtrsim \int_{y_0-r}^{y_0+r} \frac{dy}{y} \sim \frac{r}{y_0} \sim \frac{1}{M}.
\]

Moreover, from the definition of \( g \) and \( h \), and using (3.5) and (3.2), it follows that

\[
\|g\|_q \|h\|_r \leq \frac{1}{\left| \tilde{R}_{\Delta \lambda}g(x_0) \right|} [m_\lambda(I(y_0, r))]^{1/q} [m_\lambda(I(x_0, r))]^{1/r-1/p} \lesssim M \left( \frac{2^{\lambda}}{y_0} \right)^{1/q} \left( \frac{2^{\lambda}}{x_0} \right)^{-1/q} \lesssim M^{2k+1}.
\]

By the definition of the operator \( \Pi \), we write

\[
a(x) - \Pi(g, h)(x) = a(x) \frac{\tilde{R}_{\Delta \lambda}g(x_0) - \tilde{R}_{\Delta \lambda}g(x)}{\tilde{R}_{\Delta \lambda}g(x_0)} - g(x)R_{\Delta \lambda}h(x) =: W_1(x) + W_2(x).
\]

Then it is obvious that \( \text{supp} \, W_1 \subset I(x_0, r) \) and \( \text{supp} \, W_2 \subset I(y_0, r) \). From the cancellation property \( \int_0^\infty a(y) \, dm_\lambda(y) = 0 \), the Hölder’s regularity of the Riesz kernel \( R_{\Delta \lambda}(x, y) \) in Proposition 2.2 (ii), (3.5) and the fact that \( |x - y| \simeq |x_0 - y_0| \) for \( y \in I(x_0, r) \) and
\( x \in I(y_0, r) \), we have that
\[
|W_2(x)| = \chi_{I(y_0, r)}(x) |R_{\Delta, h}(x)|
\lesssim MX_{I(y_0, r)}(x) \left| \int_{I(x_0, r)} [R_{\Delta, h}(x, y) - R_{\Delta, h}(x, x_0)]a(y) \, dm_\lambda(y) \right|
\lesssim MX_{I(y_0, r)}(x) \int_{I(x_0, r)} \frac{|y - x_0|}{|y - x|} \frac{|a(y)|}{m_\lambda(I(y, |y - x|))} \, dm_\lambda(y)
\lesssim MX_{I(y_0, r)}(x) \frac{r}{|y_0 - x_0|} \|a\|_\infty \int_{I(x_0, r)} \frac{dm_\lambda(y)}{m_\lambda(I(y, |y - x|))}
\lesssim C_2 \chi_{I(y_0, r)}(x),
\]
where
\[
C_2 := \frac{m_\lambda(I(x_0, r))^{1 - 1/p}}{m_\lambda(I(y_0, |y_0 - x_0|))}.
\]

On the other hand, using again (3.5), Proposition 2.2 (ii) and the fact that \(|y - x_0| \lesssim |y_0 - x_0|\) for \(y \in I(y_0, r)\),
\[
|W_1(x)| \lesssim MX_{I(x_0, r)}(x) \|a\|_\infty \int_{I(y_0, r)} |R_{\Delta, h}(y, x_0) - R_{\Delta, h}(y, x)| \, dm_\lambda(y)
\lesssim MX_{I(x_0, r)}(x) \frac{1}{m_\lambda(I(x_0, r))^{1/p}} \int_{I(y_0, r)} \frac{|x_0 - x|}{|x_0 - y|} \frac{dm_\lambda(y)}{m_\lambda(I(x_0, |x_0 - y|))}
\lesssim C_1 \chi_{I(x_0, r)}(x),
\]
where
\[
C_1 := \frac{m_\lambda(I(y_0, r)) m_\lambda(I(x_0, r))^{-1/p}}{m_\lambda(I(x_0, |x_0 - y_0|))}.
\]
Moreover, using Fubini’s theorem we can see that \(\int_0^\infty \Pi(g, h) \, dm_\lambda = 0\), then
\[
\int_0^\infty [a - \Pi(g, h)] \, dm_\lambda = 0.
\]
Hence, the function \(f(x) := a(x) - \Pi(g, h)(x)\) satisfies all conditions in Lemma 3.1. Now from Lemma 3.1, we have that
\[
\|a - \Pi(g, h)\|_{H^p(R^n, dm_\lambda)}^p \leq \left( \frac{x_0 - y_0}{r} \right)^{1-p} \log_2 \left( \frac{x_0 - y_0}{r} \right) \times
\left[ C_1^p m_\lambda(I(x_0, r)) + C_2^p m_\lambda(I(y_0, r)) \right]
= \left( \frac{x_0 - y_0}{r} \right)^{1-p} \log_2 \left( \frac{x_0 - y_0}{r} \right) \left[ \frac{m_\lambda(I(x_0, r))}{m_\lambda(I(x_0, |x_0 - y_0|))^{p^p}} + \frac{m_\lambda(I(x_0, r))^{p^p - 1} m_\lambda(I(y_0, r))}{m_\lambda(I(y_0, |y_0 - x_0|))^{p^p}} \right]
\approx \left( \frac{x_0 - y_0}{r} \right)^{1-p} \log_2 \left( \frac{x_0 - y_0}{r} \right) \frac{r^p}{|x_0 - y_0|^p}
\approx \log_2 M \frac{M^{2p - 1}}{M^{2p - 1}} < \varepsilon^p.
\]
Case (b): In this case we assume that \( x_0 > 2Mr \) and let \( y_0 := x_0 - Mr/K_0 \). Then it is clear that

\[
\frac{2K_0 - 1}{2K_0} x_0 < y_0 < x_0.
\]

Let \( g \) and \( h \) be as in Case (a) in (3.4). For every \( y \in I(y_0, r) \) and the fact that \( K_0 > \max\{1/K_1, 1/K_2\} + 1 \) and \( M \geq 100K_0 \) we have

\[
0 < \frac{x_0}{y} - 1 < K_2.
\]

By Proposition 2.3 (ii) and the fact that \( y \simeq y_0 \simeq x_0 \) for \( y \in I(y_0, r) \), we conclude that

\[
\left| \tilde{R}_{\Delta \lambda} g(x_0) \right| \gtrsim \int_{y_0-r}^{y_0+r} \frac{1}{x_0 y_0^\lambda} \frac{d\lambda}{x_0 - y - 1} \simeq \int_{y_0-r}^{y_0+r} \frac{dy}{x_0 - y} \simeq \frac{1}{M}.
\]

Moreover, using the same operations as Case (a) and (3.6),

\[
\| g \|_q \| h \|_r \lesssim M \left( y_0^{2\lambda} r \right)^{1/q} \left( x_0^{2\lambda} r \right)^{-1/q} \lesssim M.
\]

Let \( W_1, W_2, C_1 \) and \( C_2 \) be the same as in Case (a). Then similarly, we obtain the same estimates \(|W_1(x)| \lesssim C_1 \chi_{I(\log r)}(x)\) and \(|W_2(x)| \lesssim C_2 \chi_{I(y_0, r)}(x)\) using (3.7) instead of (3.5). Then by Lemma 3.1 we obtain that

\[
\| a - \Pi(g, h) \|_{H^p(\mathbb{R}_+, dm_\lambda)} \lesssim \left( \frac{x_0 - y_0}{r} \right)^{1-p} \log_2 \left( \frac{x_0 - y_0}{r} \right) \times \\
\left[ C_1^p m_\lambda(I(\log r)) + C_2^p m_\lambda(I(y_0, r)) \right] \\
\lesssim \left( \frac{x_0 - y_0}{r} \right)^{1-p} \log_2 \left( \frac{x_0 - y_0}{r} \right) \left[ \frac{m_\lambda(I(y_0, r))}{m_\lambda(I(x_0, |x_0 - y_0|))} \right]^{p-1} \left[ m_\lambda(I(y_0, |y_0 - x_0|)) \right]^{p-1} \\
\lesssim \frac{\log_2 M}{M^{2p-1}} < \varepsilon^p.
\]

which together with Case (a) completes the proof of Proposition 3.2.

We also need the following estimate of the bilinear operator \( \Pi \).

**Proposition 3.3.** Let \( p \in \left( \frac{1}{2}, 1 \right) \) and \( q, r > 1 \) such that (1.2) holds. There exists \( C > 0 \) such that for any \( g \in L^q(\mathbb{R}_+, dm_\lambda) \) and \( h \in L^r(\mathbb{R}_+, dm_\lambda) \),

\[
\| \Pi(g, h) \|_{H^p(\mathbb{R}_+, dm_\lambda)} \leq C \| g \|_q \| h \|_r.
\]
Proof. Let \( \alpha := \frac{1}{p} - 1 \). It is known [6, Theorem B] that the dual of \( H^p(\mathbb{R}_+, dm_\lambda) \) is \( \text{Lip}_\alpha(\mathbb{R}_+, dm_\lambda) \). Then, by Hölder's inequality,

\[
\|\Pi(g, h)\|_{H^p(\mathbb{R}_+, dm_\lambda)} = \sup_{b \in \text{Lip}_\alpha(\mathbb{R}_+, dm_\lambda), \|b\|_{\text{Lip}_\alpha(\mathbb{R}_+, dm_\lambda)} = 1} \|\langle \Pi(g, h), b \rangle_{L^2(\mathbb{R}_+, dm_\lambda)}\| \\
= \sup_{b \in \text{Lip}_\alpha(\mathbb{R}_+, dm_\lambda), \|b\|_{\text{Lip}_\alpha(\mathbb{R}_+, dm_\lambda)} = 1} \|\langle [b, R_{\Delta_\lambda}]h, g \rangle_{L^2(\mathbb{R}_+, dm_\lambda)}\| \\
\leq \sup_{b \in \text{Lip}_\alpha(\mathbb{R}_+, dm_\lambda), \|b\|_{\text{Lip}_\alpha(\mathbb{R}_+, dm_\lambda)} = 1} \|b, R_{\Delta_\lambda}\|_q \|g\|_q,
\]

where \( q' \) is the conjugate exponent of \( q \). Now, since \( b \in \text{Lip}_\alpha(\mathbb{R}_+, dm_\lambda) \) and using the usual kernel estimate in Proposition 2.2 (i) and (1.1) we obtain that

\[
\|b, R_{\Delta_\lambda}\|_q h(x) \leq \int_0^\infty |h(y)| |R_{\Delta_\lambda}(x, y)| |b(x) - b(y)| dm_\lambda(y) \\
\leq \|b\|_{\text{Lip}_\alpha(\mathbb{R}_+, dm_\lambda)} \int_0^\infty \frac{|h(y)| dm_\lambda(y)}{m_\lambda(I(x, |x - y|))^{1-\alpha}} \\
= \|b\|_{\text{Lip}_\alpha(\mathbb{R}_+, dm_\lambda)} I^+_\alpha h(x),
\]

where \( I^+_\alpha \) is the positive fractional integral operator defined by

\[
I^+_\alpha f(x) := \int_0^\infty \frac{|f(y)| dm_\lambda(y)}{m_\lambda(I(x, |x - y|))^{1-\alpha}}.
\]

It follows then that

\[
\|b, R_{\Delta_\lambda}\|_q \lesssim \|b\|_{\text{Lip}_\alpha(\mathbb{R}_+, dm_\lambda)} \|I^+_\alpha h\|_q.
\]

By [4, Theorem 4.1], \( I^+_{\alpha} \) is bounded on \( L^r(\mathbb{R}_+, dm_\lambda) \rightarrow L^{q'}(\mathbb{R}_+, dm_\lambda) \) provided that \( 1 < r < 1/\alpha \) with the condition \( \frac{1}{q'} = \frac{1}{r} - \alpha \), which is true in our case by (1.2). Note that in their proof, they actually prove that

\[
\| I^+_{\alpha} f \|_{q'} \leq \| I^+_{\alpha} f \|_{q'} \leq C \| f \|_r, \quad f \in L^r(\mathbb{R}_+, dm_\lambda).
\]

Therefore,

\[
\sup_{b \in \text{Lip}_\alpha(\mathbb{R}_+, dm_\lambda), \|b\|_{\text{Lip}_\alpha(\mathbb{R}_+, dm_\lambda)} = 1} \|b, R_{\Delta_\lambda}\|_q \lesssim \|h\|_r
\]

and the proposition is proved. \( \square \)

Now we are in the situation to prove our main result.

Proof of Theorem 1.1. By Proposition 3.3 we have that for any \( g \in L^q(\mathbb{R}_+, dm_\lambda) \) and \( h \in L^r(\mathbb{R}_+, dm_\lambda) \),

\[
\|\Pi(g, h)\|_{H^p(\mathbb{R}_+, dm_\lambda)} \lesssim \|g\|_q \|h\|_r.
\]

From this, for any \( f \in H^p(\mathbb{R}_+, dm_\lambda) \) having the representation (1.3) with

\[
\left( \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \left( |\alpha_j^k| \|g_j^k\|_q \|h_j^k\|_r \right)^p \right)^{1/p} < \infty
\]
it follows that

\[
\|f\|_{H^p(\mathbb{R}, dm_\lambda)}^p = \left\| \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \alpha_j^k \Pi(g_j^k, h_j^k) \right\|_{H^p(\mathbb{R}, dm_\lambda)}^p \\
\leq \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\alpha_j^k|^p \|\Pi(g_j^k, h_j^k)\|_{H^p(\mathbb{R}, dm_\lambda)}^p \\
\lesssim \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\alpha_j^k|^p \|g_j^k\|_{q}^p \|h_j^k\|_{r}^p.
\]

Then, we have

\[
\|f\|_{H^p(\mathbb{R}, dm_\lambda)} \lesssim \inf \left\{ \left( \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\alpha_j^k|^p \|g_j^k\|_{q}^p \|h_j^k\|_{r}^p \right)^{\frac{1}{p}} : f = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \alpha_j^k \Pi(g_j^k, h_j^k) \right\}.
\]

To see the converse, let \( f \in H^p(\mathbb{R}, dm_\lambda) \). We will show that \( f \) has a representation as in (1.3) with

\[
(3.8) \quad \inf \left\{ \left( \sum_{k,j=1}^{\infty} \left( |\alpha_j^k| \|g_j^k\|_{q} \|h_j^k\|_{r} \right)^p \right)^{\frac{1}{p}} : f = \sum_{k,j=1}^{\infty} \alpha_j^k \Pi(g_j^k, h_j^k) \right\} \lesssim \|f\|_{H^p(\mathbb{R}, dm_\lambda)}.
\]

To this end, assume that \( f \) has the following atomic representation \( f = \sum_{j=1}^{\infty} \alpha_j^1 a_j^1 \) with

\[
\left( \sum_{j} |\alpha_j^1|^p \right)^{1/p} \leq C \|f\|_{H^p(\mathbb{R}, dm_\lambda)}
\]

for certain constant \( C \in (1, \infty) \) (see [20, Theorem 1.1]), where \( \{\alpha_j^1\}_j \) are numbers and \( \{a_j^1\}_j \) are \( p \)-atoms.

First of all, for given \( \varepsilon > 0 \), such that \( \varepsilon C < 1 \), and \( a_j^1 \), by Proposition 3.2, there exist \( g_j^1 \in L^q(\mathbb{R}, dm_\lambda) \) and \( h_j^1 \in L^r(\mathbb{R}, dm_\lambda) \) with

\[
\|g_j^1\|_q \|h_j^1\|_r \lesssim M^{\frac{2k}{r}+1}
\]

and

\[
\|a_j^1 - \Pi(g_j^1, h_j^1)\|_{H^p(\mathbb{R}, dm_\lambda)} \lesssim \varepsilon.
\]

Now we write

\[
f = \sum_{j=1}^{\infty} \alpha_j^1 a_j^1 = \sum_{j=1}^{\infty} \alpha_j^1 \Pi(g_j^1, h_j^1) + \sum_{j=1}^{\infty} \alpha_j^1 \left[ a_j^1 - \Pi(g_j^1, h_j^1) \right] =: M_1 + E_1.
\]

Observe that

\[
\|E_1\|_{H^p(\mathbb{R}, dm_\lambda)}^p \leq \sum_{j=1}^{\infty} |\alpha_j^1|^p \|a_j^1 - \Pi(g_j^1, h_j^1)\|_{H^p(\mathbb{R}, dm_\lambda)}^p \leq \varepsilon^p C^p \|f\|_{H^p(\mathbb{R}, dm_\lambda)}^p.
\]

Since \( E_1 \in H^p(\mathbb{R}, dm_\lambda) \), by [20, Theorem 1.1] again, there exist a sequence of \( p \)-atoms \( \{a_j^2\}_j \) and numbers \( \{\alpha_j^2\}_j \) such that \( E_1 = \sum_{j=1}^{\infty} \alpha_j^2 a_j^2 \) and

\[
\left( \sum_{j=1}^{\infty} |\alpha_j^2|^p \right)^{1/p} \leq C \|E_1\|_{H^p(\mathbb{R}, dm_\lambda)} \leq \varepsilon C^2 \|f\|_{H^p(\mathbb{R}, dm_\lambda)}.
\]
Another application of Proposition 3.2 with $\varepsilon$ and $a_j^2$ implies that there exist functions $g_j^2 \in L^q(\mathbb{R}_+, dm_\lambda)$ and $h_j^2 \in L^r(\mathbb{R}_+, dm_\lambda)$ with
\[
\|g_j^2\|_q \|h_j^2\|_r \lesssim M_{\lambda}^{\frac{2p}{q} + 1}
\]
and
\[
\|a_j^2 - \Pi(g_j^2, h_j^2)\|_{H^p(\mathbb{R}_+, dm_\lambda)} < \varepsilon.
\]
Thus, we have
\[
E_1 = \sum_{j=1}^{\infty} \alpha_j^2 a_j^2 = \sum_{j=1}^{\infty} \alpha_j^2 \Pi(g_j^2, h_j^2) + \sum_{j=1}^{\infty} \alpha_j^2[a_j^2 - \Pi(g_j^2, h_j^2)] =: M_2 + E_2.
\]
Moreover,
\[
\|E_2\|_{H^p(\mathbb{R}_+, dm_\lambda)}^p \leq \sum_{j=1}^{\infty} \|\alpha_j^2\|^p \|a_j^2 - \Pi(g_j^2, h_j^2)\|_{H^p(\mathbb{R}_+, dm_\lambda)}^p \leq \varepsilon^p C^2 \|f\|_{H^p(\mathbb{R}_+, dm_\lambda)}^p.
\]
Then, we conclude that
\[
f = \sum_{j=1}^{\infty} \alpha_j^1 a_j^1 = \sum_{k=1}^{2} \sum_{j=1}^{\infty} \alpha_k^j \Pi(g_j^k, h_j^k) + E_2.
\]
Continuing in this way, we deduce that for any $K \in \mathbb{N}$, $f$ has the following representation
\[
f = \sum_{k=1}^{K} \sum_{j=1}^{\infty} \alpha_k^j \Pi(g_j^k, h_j^k) + E_K
\]
satisfying, for any $k \in \{1, \ldots, K\}$,
\[
\|g_j^k\|_q \|h_j^k\|_r \lesssim M_{\lambda}^{\frac{2k}{q} + 1},
\]
\[
\left(\sum_{j=1}^{\infty} |\alpha_j^k|^p\right)^{1/p} \leq \varepsilon^{k-1} C_k \|f\|_{H^p(\mathbb{R}_+, dm_\lambda)} \]
and
\[
\|E_K\|_{H^p(\mathbb{R}_+, dm_\lambda)} \leq (\varepsilon C)^K \|f\|_{H^p(\mathbb{R}_+, dm_\lambda)}.
\]
Thus, letting $K \to \infty$, we see that (1.3) holds. Moreover, since $\varepsilon C < 1$, we have that
\[
\sum_{j=1}^{\infty} |\alpha_j^k|^p \leq \sum_{k=1}^{\infty} \varepsilon^{-p}(\varepsilon C)^k \|f\|_{H^p(\mathbb{R}_+, dm_\lambda)}^p \lesssim \|f\|_{H^p(\mathbb{R}_+, dm_\lambda)}^p
\]
which implies (3.8) and hence, completes the proof of Theorem 1.1. \hfill \Box

Finally, we can prove Theorem 1.3 that follows from Theorem 1.1.

**Proof of Theorem 1.3.** (1) This is already proved in the proof of Proposition 3.3.
(2) Assume that \([b, R_{\lambda}]\) is bounded from \(L^p(\mathbb{R}_+, dm_\lambda)\) to \(L^q(\mathbb{R}_+, dm_\lambda)\) with norm \(\|b, R_{\lambda}\|_{p \to q} := \|b, R_{\lambda}\|_{L^p(\mathbb{R}_+, dm_\lambda) \to L^q(\mathbb{R}_+, dm_\lambda)}\). Let \(f \in H^t(\mathbb{R}_+, dm_\lambda)\) such that
\[
\frac{1}{t} = \frac{1}{p} + \frac{1}{q'},
\]
where \(q'\) is the conjugate exponent of \(q\). By hypothesis and Theorem 1.1, there exists numbers \(\{\lambda_j^k\}_{j,k}\), functions \(\{g_j^k\}_{j,k} \subset L^q'(\mathbb{R}_+, dm_\lambda)\) and \(\{h_j^k\}_{j,k} \subset L^p(\mathbb{R}_+, dm_\lambda)\) such that
\[
\langle b, f \rangle_{L^2(\mathbb{R}_+, dm_\lambda)} = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \lambda_j^k \langle b, \Pi(g_j^k, h_j^k) \rangle_{L^2(\mathbb{R}_+, dm_\lambda)} \]
\[
= \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \lambda_j^k \langle g_j^k, [b, R_{\lambda}]h_j^k \rangle_{L^2(\mathbb{R}_+, dm_\lambda)}.
\]

By Hölder’s inequality, the hypothesis, the fact that \(t < 1\) and Theorem 1.1 again, it implies that
\[
\|\langle b, f \rangle_{L^2(\mathbb{R}_+, dm_\lambda)}\| \leq \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \|\lambda_j^k\| \|g_j^k\|_{q'} \|b, R_{\lambda}]h_j^k\|_q
\]
\[
\leq \|b, R_{\lambda}]\|_{p \to q} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \|\lambda_j^k\| \|g_j^k\|_{q'} \|h_j^k\|_p
\]
\[
\leq \|b, R_{\lambda}]\|_{p \to q} \left( \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \left( \|\lambda_j^k\| \|g_j^k\|_{q'} \|h_j^k\|_p \right)^t \right)^{\frac{1}{t}}
\]
\[
\leq \|b, R_{\lambda}]\|_{p \to q} \|f\|_{H^t(\mathbb{R}_+, dm_\lambda)}.
\]

Then, if \(\alpha = \frac{1}{t} - 1 = \frac{1}{p} - \frac{1}{q}\), which is true by hypothesis, by duality [6, Theorem B] between \(H^t(\mathbb{R}_+, dm_\lambda)\) and \(\text{Lip}_\alpha(\mathbb{R}_+, dm_\lambda)\), we finish the proof of Theorem 1.3. ☐

4. Comments

We want to point out that the proofs here only work for \(p\) values that are large enough. Namely, we have proved the weak factorization of Hardy spaces \(H^p(\mathbb{R}_+, dm_\lambda)\) for values of \(p \in (\frac{2k+1}{2k+2}, 1]\). A reasonable question would be what happens when \(p\) is small. More concretely for values of \(p \in (0, \frac{1}{2}\] \). This is an open problem and we do not know exactly how to proceed to solve this question. In fact, in order to solve that we need a crucial result that characterizes the Hardy spaces \(H^p(\mathbb{R}_+, dm_\lambda)\) and the atomic ones for values of \(p \in (0, \frac{1}{2}\) and we think this is not trivial at all. We need more conditions on the atoms, that is, we need to impose more vanishing moment conditions as [6] suggested.

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