Two-parametric hyperbolic octagons and reduced Teichmüller space in genus two

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Abstract

It is explored a model of compact Riemann surfaces in genus two, represented geometrically by two-parametric hyperbolic octagons with an order four automorphism. We compute the generators of associated isometry group and give a real-analytic description of corresponding Teichmüller space, parametrized by the Fenchel-Nielsen variables, in terms of geometric data. We state the structure of parameter space by computing the Weil-Petersson symplectic two-form and analyzing the isoperimetric orbits. The results of the paper may be interesting due to their applications to the quantum geometry, chaotic systems and low-dimensional gravity.

1 Introduction

The Riemann surface of genus two serves as geometry carrier in a great number of the models of string theory \(^1\), statistical physics \([2,3]\), chaology \([4,5,6]\), low-dimensional gravity \([7,8]\). Problems, in which surface geometry is not fixed and is developing in time, have a special interest. The changes of underlying geometry can be described in different ways, for instance, by evolution equations, by averaging over surface moduli or parameters, etc. Then it is naturally to require the surface deformation to be represented by continuous and smooth trajectory in a some space with the properties which should be carefully studied.

Although the case of genus two gives us access to quite explicit calculations, most of problems cannot be solved in general. This fact forces us to concentrate on a family of the surfaces with the reduced number of geometric degrees of freedom. Using the more convenient geometric approach we consider the surfaces represented by two-parametric hyperbolic octagons embedded into the unit disk. Assuming that an octagon form remains the same under rotation by \(\pi/2\), we first construct the fundamental domain with opposite sides identified and the associated Fuchsian group, using as “input” the two real parameters: length and angle, determining the position of vertices, i.e. the octagon geometry. Although the general formalism linking the geometric data and the Fuchsian group is known \([9]\), we pay great attention to manifest dependence of octagon boundary segments and isometry group generators on these parameters in order to make the functions straightforwardly applicable in forthcoming calculations.

We aim to investigate a real-analytic structure of parameter space, that is dictated by isometry group of Teichmüller space, usually called as the mapping class group, and essentially determines an initial octagon evolution in various physical problems. Thus, to realise this, we introduce Teichmüller space for the surfaces under consideration as a subset of total Teichmüller space for all surfaces in genus two and compute the Fenchel-Nielsen variables regarding as the global coordinates on it.

We perform main analysis (in Section 3) within the Weil-Petersson geometry allowing us to endow the parameter space with the symplectic two-form which is invariant by definition under action of the

\(^{1}\)Here we refer to few works but directly related to a given topic.
mapping class group. Key tool is the Wolpert’s formula [10] permitting us to express this form in terms of the Fenchel-Nielsen variables. As the result, we shall see that the accessibility domain of geometric data used is the symplectic orbifold. Furthermore, the symmetry group of the reduced Teichmüller space is expected to be wide than the mapping class group because of geometric constraints imposed. Note that the involution of the surfaces with an order four automorphism and the associated generators are discussed in details in [11].

We supplement our results by description of the isoperimetric orbits in the parameter space (Section 4), what gives us additional information about the structure of this space and reflects a particular diffeomorphism of octagon. On the other hand, the dense set of isoperimetric orbits can serve as a tool for further geometric quantization, independent of the octagon automorphisms and pants decompositions. Physically, such an approach leads to a study of the quantum spectra and quantum fluctuations of geometric quantities of the objects with the same topology/geometry but of different nature.

2 Model octagons and their symmetries

We will deal with the Poincaré model of two-dimensional hyperbolic space, namely, with the open unit disk centered at the origin,

\[ \mathbb{D} = \{ z = x + iy \in \mathbb{C} | |z| < 1 \}, \]

endowed with the metric

\[ ds^2 = \frac{4 \, dx^2 + dy^2}{(1 - x^2 - y^2)^2} \]

of the Gaussian curvature \( K = -1 \).

Hyperbolic distance between complex coordinates \( z \) and \( w \) in space \((\mathbb{D}, ds^2)\) is denoted by \( \text{dist}_\mathbb{D}(z, w) \) and determined from relation:

\[ \cosh \text{dist}_\mathbb{D}(z, w) = 1 + \frac{2|z - w|^2}{(1 - |z|^2)(1 - |w|^2)}, \]

where \(|z - w|\) is the Euclidean distance.

Solution to the geodesic equation in \((\mathbb{D}, ds^2)\) is the function:

\[ z(s) = \frac{\cosh s + iR \sinh s}{\sqrt{1 + R^2} \cosh s + R} \exp(-i\phi), \]

which is defined in the interval \( s \in (-\infty, +\infty) \) and describes an arc inside \( \mathbb{D} \) with radius \( R \) and center at the point \( z_0 = \sqrt{1 + R^2} \exp(i\phi) \) lying beyond the unit disk. In particular case, the geodesics emanating from the origin are the Euclidean straight lines (diameters). All geodesics intersect the boundary \( \partial \mathbb{D} \) orthogonally.

The group of all orientation-preserving isometries of \((\mathbb{D}, ds^2)\), denoted by \( \text{Isom}^+(\mathbb{D}) \), acts via the Möbius transformation:

\[ z \mapsto \gamma(z) = \frac{uz + v}{\bar{v}z + \bar{u}, \quad z \in \mathbb{D},} \]

where \( u \) and \( v \) are complex numbers satisfying relation \(|u|^2 - |v|^2 = 1\), and \( \bar{u}, \bar{v} \) are the complex conjugates. Thus, it is convenient to identify a generator \( \gamma \) with an element of group \( \text{SU}(1, 1)/\{\pm 1\} \),

\[ \text{SU}(1, 1) = \left\{ \begin{pmatrix} u & v \\ \bar{v} & \bar{u} \end{pmatrix} | u, v \in \mathbb{C}, |u|^2 - |v|^2 = 1 \right\}. \]
We shall concentrate on the properties of Riemann surface $S$ of genus $g = 2$, which is understood here as a compact two-dimensional orientable manifold with the Riemannian metric of the constant negative curvature. Such a surface is obtained from a hyperbolic simply connected octagon $F$ in $\mathbb{D}$ via gluing opposite sides formed by eight geodesic arcs, whose intersections serve as vertices.

In our model, which has been already declared and geometrically described in [3], we assume that the vertices are at the points $a \exp (i k \pi/2)$, $b \exp [i(\alpha + k \pi/2)]$, where $0 < \alpha < \pi/2$, $0 < a, b < 1$, $k = 0, 3$. We also require the sum of the inner angles of $F$ to be equal to $2\pi$. This is the same as requiring $\text{area}(F) = 2\pi (2g - 2) = 4\pi$ in accordance with the Gauss-Bonnet theorem [12].

Such an octagon (sketched in figure 1, left panel) is two-parametric: we choose a pair $(a, \alpha)$ as independent real variables while parameter $b$ together with the parameters of geodesics (sides) are functions of those. Although the surface of genus $g$ is generally determined by $6g-6$ real parameters, we reduce 6-parametric object to 2-parametric one by imposing constraints that simplifies considerations. Therefore, the model octagon $F$ can be viewed as a “minimal deformation” of the regular hyperbolic octagon with $b = a = 2^{-1/4}$, $\alpha = \pi/4$, well studied in the context of the chaology (see, for example, [5] and references therein).

The octagon we have obtained is stable under rotation by $\pi/2$. It means that the surface has an order four automorphism. Note that the connection between this geometric model and algebraic curves was intensively explored in the works of P. Buser and R. Silhol (for example, see [11], [13] and references therein).

In this Section we review the connection between the model octagon $F$, the corresponding Riemann surface $S$ and the Fuchsian group $\Gamma \subset \text{Isom}^+(\mathbb{D})$ isomorphic to fundamental group $\pi_1(S)$ of surface. For more details we refer to [9].

In the case at hand, the octagon boundary $\partial F$ is formed by geodesics of two kinds (labeled by “$\pm$” below). These geodesics are completely determined by the radii $R_\pm$ and the angles $\phi_\pm + k \pi/2$, $k = 0, 3$ (see [4]), defining the positions of the circle centers. Satisfying the conditions imposed above (and collected in manifest form in Appendix of [3]), we obtain that

$$R_\pm = \frac{1}{2a} \sqrt{T_\pm^2 + (1 - a^2)^2}, \quad \phi_\pm = \arctan \left[ \left( \frac{T_\pm}{1 + a^2} \right)^{\pm1} \right], \quad (7)$$

where $0 < \phi_+ < \alpha < \phi_- < \pi/2$, and

$$T_\pm = a^2 \pm \tan \tilde{\alpha}, \quad \tilde{\alpha} = \alpha - \pi/4. \quad (8)$$
Moreover, introducing the inner angle $\beta$ by vertices $a \exp (ik\pi/2)$ (the angle by vertices $b \exp [i(\alpha + k\pi/2)]$ is then equal to $\pi/2 - \beta$) such that

$$\tan \beta = (1 - a^2) \frac{2a^2 \cos^2 \tilde{\alpha}}{2a^2 \cos^2 \tilde{\alpha} - 1},$$

we should control the condition $0 < \beta < \pi/2$, determining the region of variety of parameters $(a, \alpha)$:

$$-\pi/4 < \tilde{\alpha} < \pi/4, \quad (\sqrt{2} \cos \tilde{\alpha})^{-1} < a < 1,$$

which is sketched in figure 4 below.

The last formulae define a domain, which we denote by $\mathcal{A}$, whose points completely determine the geometry of octagon $\mathcal{F}$. Our aim is to investigate the structure of $\mathcal{A}$.

To complete geometry description, the parameter $b$, pointed out in figure 1, is calculated as

$$b = (\sqrt{2}a \cos \tilde{\alpha})^{-1}.$$

Note that the manifest dependence of the octagon parameters on the pair $(a, \alpha)$ is necessary in the different problems where geometry is not fixed. For instance, $(a, \alpha)$ would be dynamical variables in topological field theory and gravity; it is able to average over $(a, \alpha)$ in statistical physics, etc.

In order to define the Riemann surface $S$ based on the geometrical model, let us recall that the opposite sides of $\mathcal{F}$ have the same lengths by construction. We therefore have a uniquely defined isometry $g_k \in \text{Isom}^+(\mathbb{D})$ mapping geodesic boundary segment $s_{k+4}$ onto $s_k$ for all $k = 0, 3$ (see figure 1, right panel). For these isometries we get $g_k[F] \cap F = s_k$, where $g_k[F]$ means the set $\{g_k[z] \mid z \in F\}$. Pasting sides $s_{k+4}$ and $s_k$ together by identifying any $z \in s_{k+4}$ with $g_k[z] \in s_k$, we obtain a closed surface of genus two that carries the hyperbolic metric inherited from $\mathcal{F}$.

After introduction of four isometries $g_k$ and their inverses $g_k^{-1}$ generating Fuchsian group $\Gamma$ with a single relation,

$$g_0 g_1^{-1} g_2 g_3^{-1} g_6^{-1} g_1 g_2^{-1} g_3 = \text{id},$$

it is purely to define surface $S$ as a quotient $\mathbb{D}/\Gamma$ for which $\pi : \mathbb{D} \to S$ is the natural covering map. This is a Fuchsian model $\Gamma$ of the Riemann surface $S$.

In general, we can equip $S$ with a structure $(U_p, f_p)_{p \in S}$ by specifying a chart $U_p$ around each point $p \in S$ and a homeomorphism $f_p$ identifying $U_p$ with any of the open sets in $\mathbb{D}$ covering $U_p$.

It turns out that the different octagons may lead to the same surface. For this reason, we mark a surface by generators of $\Gamma$. Two marked surfaces $(S, \Gamma)$ and $(S', \Gamma')$ are called marking equivalent if there exist an isometry $\gamma : S \to S'$ satisfying $g_k^\gamma = \gamma g_k \gamma^{-1}$ ($k = 0, 3$). Then all marking equivalent surfaces form a marking equivalence class $[S, \Gamma]$ representing the Riemann surface $S$ together with a structure defined on it.

It is useful sometimes to mark a surface by selecting a curve system $\Sigma$ of simple closed geodesics on it. Then the marking equivalence also means an existence of isometry $\gamma : S \to S'$ sending $\Sigma \to \Sigma'$. In this case an equivalence class is formed by a pair $[S, \Sigma]$.

The set of all marking equivalence classes of Riemann surfaces is called the Teichmüller space and is simply denoted by $\mathcal{T}_g$ in the case of the closed and compact Riemann surfaces of genus $g$. The definition of $\mathcal{T}_g$ depends in general on choice of a marking of Riemann surfaces. In any case, the real dimension of $\mathcal{T}_g$ like vector space equals $6g - 6$ in accordance with the Riemann–Roch theorem. We immediately note that the Riemann surfaces, constructed with geometrical constraints imposed above, result only in the subset of the total $\mathcal{T}_2$ of dimension six. In this sense we call such a space as reduced Teichmüller one.
Coming back to our model, octagon $\mathcal{F}$ is a fundamental domain of the Fuchsian group $\Gamma$, elements of which are hyperbolic: $|\text{Tr} \gamma| > 2$ for all $\gamma \in \Gamma$ and depend on the form of octagon. In order to find generators $g_k$, it is reasonable to introduce the set of auxiliary variables: $\omega_0 = \omega_+, \omega_1 = \omega_-, \omega_2 = i\omega_+, \omega_3 = i\omega_-, \omega_5 = 0$, where

$$\omega_{\pm} = \frac{be^{ia(1-a^2)} + ae^{i\pi(1+\gamma)/4}(1-b^2)}{1-a^2b^2}$$

are functions of parameters $(a, \alpha)$.

Let us also define the matrices $M_k = M(\omega_k)$, where

$$M(\omega) = \frac{i}{\sqrt{1-|\omega|^2}} \begin{pmatrix} 1 & -\omega \\ \omega & -1 \end{pmatrix}.$$  

The generators of $\Gamma$ are directly expressed in these terms as

$$g_k = M_kM_5, \quad k = 0, 3.$$  

This form is general and can be applied for arbitrary admissible octagon [9].

Our calculations give us the manifest dependence of $g_0$ and $g_1$ on $(a, \alpha)$:

$$g_0 = N(a, \tilde{\alpha}) \begin{pmatrix} a(1 - \tan \tilde{\alpha}) & (a^2 - \tan \tilde{\alpha}) + i(1 - a^2) \\ (a^2 - \tan \tilde{\alpha}) - i(1 - a^2) & a(1 - \tan \tilde{\alpha}) \end{pmatrix},$$

$$g_1 = N(a, \tilde{\alpha}) \begin{pmatrix} a(1 + \tan \tilde{\alpha}) & (1 - a^2) + i(a^2 + \tan \tilde{\alpha}) \\ (1 - a^2) - i(a^2 + \tan \tilde{\alpha}) & a(1 + \tan \tilde{\alpha}) \end{pmatrix},$$

where

$$N(a, \tilde{\alpha}) = \frac{-\cos \tilde{\alpha}}{\sqrt{(1-a^2)(2a^2\cos^2 \tilde{\alpha} - 1)}}, \quad \tilde{\alpha} = \alpha - \frac{\pi}{4}.$$  

The remaining generators are simply obtained by rotations:

$$g_{2,3} = R_{\frac{\pi}{2}}g_0, g_2^{-1}, \quad g_{k}^{-1} = R_{\pi}g_kR_{\pi}^{-1}, \quad R_{\varphi} = \begin{pmatrix} \exp(i\varphi/2) & 0 \\ 0 & \exp(-i\varphi/2) \end{pmatrix}.$$  

It is convenient sometimes to represent the generators of $\Gamma$ by half turns as follows. Let $p_k$ be the mid-point of $k$-th side, $k = 0, 3$. Since the opposite sides of octagon have the same lengths, the generators are then written as $g_k = H(p_k)$ (see [9]), where

$$H(p) = \frac{-1}{1 - |p|^2} \begin{pmatrix} 1 + |p|^2 & 2p \\ 2\overline{p} & 1 + |p|^2 \end{pmatrix}.$$  

The operation of matrices $H(p)$ consists of composition of the half turn (rotation with angle $\pi$) of geodesic segment around the origin $z = 0$ and the half turn around point $p$.

Due to symmetry of our model, $p_0 = p_+, p_1 = p_-, p_2 = ip_+, p_3 = ip_-$, where

$$p_\pm = \frac{\omega_\pm}{1 + \sqrt{1 - |\omega_\pm|^2}}.$$  

Having got the Fuchsian group, it is possible to build the octagonal lattice with a given fundamental octagon: action of the group elements $\gamma \in \Gamma$ on $\mathcal{F}$ produces the daughter cells tiling unit disk, $\mathbb{D} = \bigcup_{\gamma \in \Gamma} \gamma[\mathcal{F}]$. 

5
Figure 2: Pants decomposition of the octagon with \( a = 0.8, \alpha = \pi/3 \).

3 Fenchel–Nielsen parameters and symplectic form

A hyperbolic Riemann surface of genus \( g \) without boundary always contains a system of \( 3g - 3 \) simple closed geodesics that are neither homotopic to each other nor homotopically trivial. Regardless of which curve system we choose, the cut along these geodesics always decomposes surface into \( 2g - 2 \) pairs of pants (three-holed spheres), playing a role of natural building blocks for Riemann surfaces (for instance, see [13]).

In the case at hand, surface \( S \) constructed is a two-holed torus which can be decomposed into two pairs of pants by a system of three closed geodesics. Such a surgery permits to calculate the global Fenchel–Nielsen (FN) parameters: lengths of these geodesics and twists, needed for further investigation and defined as follows.

Let us consider the geodesic arcs from \( p_0 \) to \( p_1 \) and from \( p_5 \) to \( p_4 \) on the octagon \( F \) (see figure 2). On the surface \( S \) obtained by gluing the sides of the octagon, these two arcs together form a smooth closed geodesic \( \gamma_1 \). Similarly, a closed geodesic \( \gamma_2 \) is obtained from the arcs running from \( p_2 \) to \( p_3 \) and from \( p_7 \) to \( p_6 \), respectively. The line \( p_8p_9 \) results in a closed geodesic \( \gamma_3 \).

The triple \( \gamma_1, \gamma_2, \gamma_3 \) dissects \( S \) into two pairs of pants determined up to isometry by the hyperbolic lengths \( \ell_k, k = 1, 3 \). Immediate calculations yield

\[
\ell_1,2 \equiv 2 \text{ dist}_D(p_+,p_-) = 2 \arccosh \frac{a^2}{1 - a^2},
\]

\[
\ell_3 \equiv 2 \text{ dist}_D(0,a) = 2 \ln \frac{1 + a}{1 - a},
\]

where \( \text{dist}_D(p_{n-1},p_n) = \text{dist}_D(p_+,p_-) \) for \( n = 1, 3, 5, 7 \).

When the pairs of pants are pasted together again to recover \( S \), there arise additional degrees of freedom at each \( \gamma_k \), named the twist parameters \( \tau_k \) and defined as follows. On each pair of pants, one takes disjoint orthogonal geodesic arcs between each pair of boundary geodesics. It is known that the feet of two perpendiculars on each geodesic are diametrically opposite. Let us paste together two tubular neighborhoods of pair(s) of pants with the boundaries of closed geodesics \( \gamma_k' \) and \( \gamma_k'' \) of the same orientation and hyperbolic length and let us denote the weld by \( \gamma_k \). In principle, the feet of perpendiculars, arriving at the previously separated \( \gamma_k' \) and \( \gamma_k'' \), do not coincide on \( \gamma_k \). The twist parameter \( \tau_k \) is then the hyperbolic distance (shift) along \( \gamma_k \) between the feet of perpendiculars on opposite sides of the weld. Globally, the surfaces arising from different \( \tau_k \) are not in general isometric. This fact is often used for investigation of Riemann surface deformations [14]–[16].

Let us now concentrate on computational aspects. One of convenient methods of geodesic lengths computation is the matrix formalism. Here we follow the algorithms from [9] based on it.

We have already used the matrices $M_i$ ($i = 0, 3, 5$) for finding the generators of Fuchsian group in previous Section. However, in order to calculate the twists, it is necessary to complete the set of generators by introducing matrix $M_4$, determined by auxiliary parameter $\omega_4 = 2a/(1 + a^2)$ (see [13]).

Introducing the set of parameters:

\begin{align*}
  c_1 &= -\frac{1}{2} \text{Tr}(M_0M_1), \quad c_2 = -\frac{1}{2} \text{Tr}(M_2M_3), \quad c_3 = -\frac{1}{2} \text{Tr}(M_4M_5), \\
  d_1 &= \frac{1}{2} \text{Tr}^2(M_0M_4M_5) - 1, \quad d_2 = \frac{1}{2} \text{Tr}^2(M_2M_1M_3) - 1, \quad d_3 = \frac{1}{2} \text{Tr}^2(M_5M_3M_2) - 1,
\end{align*}

we can immediately check that $c_k = \cosh(\ell_k/2)$ ($k = 1, 3$) and

\begin{align*}
  d_{1,2} + 1 &= \frac{4}{(1 - a^2)(1 - b^2)}, \quad d_3 + 1 = \frac{2}{(1 - a^2)^2}.
\end{align*}

On the other hand, parameters $d_k$ are geometrically related to the twists as

\begin{align*}
  d_k &= \frac{p}{c_k^2 - 1} (1 + \cosh \tau_k) - 1,
\end{align*}

where $p = c_1^2 + c_2^2 + c_3^2 + 2c_1c_2c_3 - 1$.

Equating (25) and (26), we obtain the twists in terms of model parameters:

\begin{align*}
  \tau_{1,2} &= \text{arccosh} \left[ \frac{2a^2 - 1}{a^2(1 - b^2)} - 1 \right], \quad \tau_3 = \ln \frac{1 + a}{1 - a}.
\end{align*}

It is known that the Teichmüller space of marked Riemann surfaces of genus two forms a manifold homeomorphic to $\mathbb{R}^6$. This fact allows one to identify the FN variables with global coordinates on it. However, the Teichmüller space carries additional structure, namely, the Weil–Petersson (WP) symplectic two-form. Actually, it is imaginary part of a natural Kählerian metric. Due to a theorem of Wolpert [10, 17] (see also Thm. 3 in [18]), WP two-form for compact closed Riemann surfaces of genus $g$ takes on a particularly simple form in terms of FN variables,

\begin{align*}
  \omega_{\text{WP}} &= \frac{1}{2} \sum_{k=1}^{3g-3} d\ell_k \wedge d\tau_k,
\end{align*}

with respect to any pants decomposition. It says in the sense of theoretical mechanics that $\ell_k$ play the role of the action variables while $\theta_k = 2\pi\tau_k/\ell_k$ are the angle variables. Indeed, the so-called simple Dehn twist $\theta_k \to \theta_k + 2\pi$ gives us isometrically the same surface.

Using the pants decomposition presented in figure 2 and substituting the functions $\ell_k$ and $\tau_k$ of $(a, \alpha)$ into (28), the WP symplectic form becomes

\begin{align*}
  \omega_{\text{WP}} &= \frac{8a}{(1 - a^2)(2a^2 \cos^2 \tilde{\alpha} - 1)} da \wedge d\tilde{\alpha}, \quad \tilde{\alpha} = \alpha - \frac{\pi}{4}.
\end{align*}
To verify the uniqueness of the last formula, let us consider another pants decomposition by changing connections between arc mid-points and main diagonal of octagon, which give us new $\gamma'_1, \gamma'_2$ and $\gamma'_3$, respectively. It is easily seen that a performed decomposition simply leads to the replacements,

$$a \leftrightarrow b, \quad \tilde{\alpha} \leftrightarrow -\tilde{\alpha},$$

in the length and twist functions of the previous decomposition. We have

$$\ell'_{1,2} = 2 \text{dist}_D(ip_+ , p_-) = 2 \text{arccosh} \frac{b^2}{1 - b^2},$$

$$\ell'_3 = 2 \text{dist}_D(0,b) = 2 \ln \frac{1+b}{1-b},$$

$$\tau'_{1,2} = \text{arccosh} \left[ \frac{2b^2 - 1}{b^2(1-a^2)} - 1 \right], \quad \tau'_3 = \ln \frac{1+b}{1-b},$$

where $\text{dist}_D(p_{n-1}, p_n) = \text{dist}_D(ip_+, p_-)$ for $n = 2, 4, 6, 8$.

Although we have obtained the set of new functions, the resulting two-form remains the same, that is, $\omega'_{WP} = \omega_{WP}$ due to the fact $\text{sgn} \tau_k = -\text{sgn} \tau'_k$. Moreover, analysis shows

$$d\ell'_{1,2} \wedge d\tau_{1,2} = d\ell'_{1,2} \wedge d\tau'_{1,2}, \quad d\ell_3 \wedge d\tau_3 = d\ell'_3 \wedge d\tau'_3 = 0.$$  

Thus, we can conclude that i) the permission domain $A$ of parameters $(a, \alpha)$ is non-trivial symplectic manifold $(A, \omega_{WP})$; ii) the Weil–Petersson symplectic two-form (29) is a closed and invariant under action of the modular (sub)group $\mathbb{Z}_2$ represented by transformation (30). Formally, we can treat the form (29) as an area element of manifold $A$, associated with the moduli space of Riemann surfaces under consideration.

Furthermore, introducing the quantities,

$$T_{1,2} \equiv \cosh \frac{\tau_{1,2}}{2} = \sqrt{\frac{2a^2 - 1}{2a^2(1 - b^2)}}, \quad L_{1,2} \equiv \cosh \frac{\ell_{1,2}}{2} = \frac{a^2}{1 - a^2},$$

$$T'_{1,2} \equiv \cosh \frac{\tau'_{1,2}}{2} = \sqrt{\frac{2b^2 - 1}{2b^2(1 - a^2)}}, \quad L'_{1,2} \equiv \cosh \frac{\ell'_{1,2}}{2} = \frac{b^2}{1 - b^2},$$

we can establish the following relations among them,

$$L_3^{(t)} \equiv \cosh \frac{\ell_3^{(t)}}{2} = 2L_1^{(t)} + 1, \quad \tau_3^{(t)} = \ell_3^{(t)}/2,$$

$$L'_1 = T_1^2 \frac{2L_1}{L_1 - 1} - 1, \quad T'_1 = \sqrt{\frac{L_1^2 T_1^2 + L_1 T_1^2 - L_1^2 + 1}{2L_1 T_1^2 - L_1 + 1}}.$$  

These formulae reflect the symmetry of the model in terms of geometric constraints and correspond to a special case of the surface with an order four automorphism, previously studied in the literature ([11], Lm. 3.5).
4 Isoperimetric curves in $A$

We can also obtain additional information about structure of $A$ by means of analysis of principal geometric characteristics. One of those is an area, fixed by the Gauss–Bonnet theorem and equal to $4\pi$ for genus two. Therefore, the area cannot obviously be a measure of octagon deformation (evolution) preserving genus.

Simplest way to describe changes globally consists in consideration of a perimeter of hyperbolic octagon. Within the present model, the perimeter is given by the formula:

$$P = 8 \arccosh \left( \frac{1 - a^2 b^2 + \sqrt{(1 - a^2)^2 + (1 - b^2)^2}}{(1 - a^2)(1 - b^2)} \right).$$  \hspace{1cm} (39)

In a some sense, this characteristic is a good candidate due to invariance of $P$ under the octagon automorphisms and pants decomposition. It means that $P$ can take on the same value for various values of $(a, \alpha)$. In this Section, we are aiming to describe the corresponding orbits.

Evidence of this fact came from numerical investigation of information entropy within the model of directed random walk on the Cayley tree generated by the Fuchsian group $[16]-[18]$, where the perimeter and the entropy are connected $[3]$. Moreover, the entropy had a maximum for octagonal lattice with $(a = 2^{-1/4}, \alpha = \pi/4)$, what says about uniqueness of this configuration which should be proved here.

For further investigation, it is useful to introduce an auxiliary quantity,

$$E = 2 \left( \cosh \left( \frac{P}{8} \right) + 1 \right).$$  \hspace{1cm} (40)

Note that $E \to \exp \left( \frac{P}{8} \right)$ at $P \to \infty$.

For a given $E$, maximal and minimal values of parameter $a$ are found at $\tilde{\alpha} = 0$ from equation

$$E = \frac{4a^2}{(1 - a^2)(2a^2 - 1)}. \hspace{1cm} (41)$$

We get

$$a_{\pm}(E) = \left( 2\sqrt{E} \right)^{-1} \sqrt{3E - 4 \pm \sqrt{E^2 - 24E + 16}}. \hspace{1cm} (42)$$

It means that one can parametrize $a$ as follows

$$a(E, \varphi) = \left( 2\sqrt{E} \right)^{-1} \sqrt{3E - 4 + \cos \varphi \sqrt{E^2 - 24E + 16}}, \hspace{1cm} (43)$$

where periodic variable $\varphi \in [0, 2\pi)$ is used.

Let us now solve algebraic equation

$$E^2 - 24E + 16 = 0. \hspace{1cm} (44)$$

We immediately obtain

$$E_{\text{reg}} = 12 + 8\sqrt{2}, \quad P_{\text{reg}} = 8 \arccosh \left( \frac{5 + 4\sqrt{2}}{1} \right), \quad a_{\text{reg}} = 2^{-1/4}. \hspace{1cm} (45)$$

At $\tilde{\alpha} = 0$, these quantities correspond to the regular hyperbolic octagon as it must be. Thus, trajectory in $A$ at $P_{\text{reg}}$ is contracted to a point. Moreover, $P_{\text{reg}}$ is a minimal value of $P$ among possible ones.
Therefore, the maximal symmetry of the regular octagon explains an extremum of information entropy observed previously. This fact could be important in description of the physical systems, in which geometry carrier (two-holed torus) changes.

Substituting (43) in (39) and resolving the equation obtained with respect to \( \tilde{\alpha} \), we deduce that

\[
\tilde{\alpha}(E, \varphi) = \arctan \left( \sqrt{2E} \right)^{-1} \frac{\sqrt{(E - 4)(E^2 - 24E + 16)} \sin \varphi}{\sqrt{E - 12 - \cos \varphi \sqrt{E^2 - 24E + 16}}}.
\] (46)

Equations (43), (46) allow us to see the orbits \( P = \text{const} \), presented in figure 3. The point corresponds to the parameters of the regular octagon \( P_{\text{reg}} \approx 24.45713 \); cyclic curves are orbits for \( P \) from \( P = 25 \) to \( P = 41 \) with step 2.

At \( P \to \infty \), one obtains the asymptotics:

\[
a_\infty(\varphi) = \frac{1}{2} \sqrt{3 + \cos \varphi}, \quad \tilde{a}_\infty(\varphi) = \arctan \frac{\sin \varphi}{\sqrt{2(1 - \cos \varphi)}}.
\] (47)

We may assume that the functions \( a(E, t), \alpha(E, t) = \pi/4 + \tilde{\alpha}(E, t) \) describe some kind of octagon evolution, generated by translation operator \( \partial/\partial t \) and determined by the initial coordinates \( (a_0, \alpha_0) \) only. This diffeomorphism produces continuous and smooth trajectory in \( A \) preserving the constant hyperbolic length of the closed geodesics forming an octagon perimeter. Since the set of orbits is dense in \( A \) there arises a possibility to geometrically quantize the symplectic orbifold \( A \) in a spirit of [2]. In order to realise it, it is necessary to consider a Weil-Petersson (WP) area \( \text{Area}(P_s) \) of the domain in \( A \), bounded by isoperimetric orbit for some fixed \( P_s \). Physically, \( \text{Area}(P_s) \) can be treated as an action variable, that is, the only integral of motion \( \{a(E_s, t), \alpha(E_s, t) | t \in \mathbb{R} \} \), where \( E_s \) is related to \( P_s \) by (10). Canonical quantization in terms of \( \text{Area}(P_s) \) and conjugate angle variable has to give us the number of quantum states inside of the domain in \( A \). However, detailed development of quantum geometry of \( A \) and the reduced Teichmüller space is a subject of another investigation which will be published elsewhere.

Here, using the WP symplectic form (29) and equations (43), (46), we are limiting ourselves by...
Figure 4: WP area of domain bounded by isoperimetric curve $P = \text{const.}$

introduction of WP area:

$$\text{Area}(P) = \int_{P_{\text{reg}} < P < P_*} \frac{8a}{(1 - a^2)(2a^2 \cos^2 \tilde{\alpha} - 1)} \text{dad} \tilde{\alpha}. \quad (48)$$

This double integral is reduced to a single one:

$$\text{Area}(P) = \int_{a_{+}(E_*)}^{a_{-}(E_*)} \frac{16a}{(1 - a^2)\sqrt{2a^2 - 1}} \text{arctanh} f(E_*, a) da, \quad (49)$$

where functions $a_{\pm}(E_*)$ are determined by (42), and

$$f(E_*, a) = \sqrt{\frac{(E_* - 4)(1 - a^2)}{E_*(1 - a^2) - 4}} \sqrt{1 - \frac{E}{E_*}} \quad (50)$$

is $\tan \tilde{\alpha}/\sqrt{2a^2 - 1}$ expressed in terms of $E_*$ and $a$; $E$ is the function of $a$ given by (11).

Further calculations are performed numerically, and the result is demonstrated in figure 4. Semi-analytical analysis shows that this curve can be approximated by a parabola, $c_1(P - P_{\text{reg}})^2 + c_2(P - P_{\text{reg}})$, with accuracy of the order $O(\exp(-P/8))$. Best fit in the presented range of $P$ gives $c_1 = 0.05622$, $c_2 = 2.62132$.

5 Conclusions

We studied the real-analytic structure of two-dimensional domain of geometric parameters variety of hyperbolic octagon which is stable under an order four automorphism. Carrying the Fenchel-Nielsen surgery and exploiting the Wolpert’s theorem, it was shown that this manifold is endowed with the fundamental symplectic two-form coming from the Weil-Petersson (WP) geometry in the case of the closed compact surfaces in genus two. In principal, the WP geometry also allows us to introduce the Riemannian (Kählerian) metric. However, this important problem remains unsolved and requires in general the use of quasi conformal mappings [16], theory of which we have not touched.
in this paper. Perspective of these investigations consists in the classical and quantum description of free geometrodynamics of the surface in genus two. Attempts to do it have been already performed in [7, 8].

Additionally, the structure of parameter space has been considered from the point of view of isoperimetric curves. The obtained trajectories might be treated as some kind of hyperbolic octagon evolution and can be reformulated as quasi conformal mapping. Furthermore, combining the formulae derived, we have evaluated the WP area of domain with the boundary formed by the orbit of fixed perimeter. In principal, it gives a tool for quantization of parameter space, associated with the moduli.

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