RANDOM WALKS IN NEGATIVE CURVATURE AND
BOUNDARY REPRESENTATIONS.

KEVIN BOUCHER

ABSTRACT. In this paper we establish a version of the Margulis Roblin
equidistribution theorem’s [28, 32] for harmonic measures. As a conse-
quency a von Neumann type theorem is obtained for boundary actions
and the irreducibility of the associated quasi-regular representations [8]
is deduced.

1. Introduction

Given a non-elementary discrete group, $\Gamma$, acting properly by isome-
tries on a proper roughly geodesic hyperbolic space $(X, d)$, two main
classes of measures can be constructed on the Gromov boundary $\partial X$ of
$(X, d)$: the harmonic measures and quasiconformal measures. Let us briefly recall these
constructions.

A measure $\nu$ on $\partial X$ is called $\Gamma$-quasiconformal if $\Gamma$ preserves the class of
$\nu$ and one can find $C \geq 1$, for all $g \in \Gamma$ and $\nu$-almost every $\xi \in \partial X$:

$$C^{-1} e^{\delta_\Gamma \beta(a, g^{-1} \cdot a \xi)} \leq \frac{d(\cdot, g \cdot \cdot)}{d\nu}(\xi) \leq C e^{\delta_\Gamma \beta(a, g^{-1} \cdot a \xi)}$$

where $\beta$ denote the Buseman cocycle on $\partial X$.

Given a point $o \in X$ in $(X, d)$, the critical exponent of $\Gamma$ is defined as:

$$\delta_\Gamma = \limsup_r \frac{1}{r} \ln |B_X(o, r) \cap \Gamma \cdot o|$$

It is strictly positive when $\Gamma$ is non-elementary.

The Patterson-Sullivan procedure consists to take a weak limit in $\hat{\text{Prob}}(X \cup \partial X)$ when $s$ goes to $\delta_\Gamma$ of the sequence of probabilities:

$$\frac{1}{H(s)} \sum_{g \in \Gamma} e^{-sd(g \cdot o, o)} \delta_{g \cdot o}$$

for $s > \delta_\Gamma$ and $H(s) = \sum_{g \in \Gamma} e^{-sd(g \cdot o, o)}$. These Patterson-Sullivan measures
are quasiconformal. Moreover these are Hausdorff measures for a visual dis-
tance on $\partial X$ when $\Gamma$ acts cocompactly on $(X, d)$ [15].
One the other hand given a probability measure, \( \mu \), with 1st moment and support not contained in an elementary group. If \((X_n)\) denote the random walk on \( \Gamma \) starting at the identity \( e \in \Gamma \), the sequence of \( X \)-valued random variables \((X_n,0)\) converges almost surely to a random variable \( Z \) with values in \( \partial X \). The law of \( Z \) is by definition the harmonic measure on \( \partial X \) for the random walk generated by \( \mu \).

The dynamical systems associated to these measures was extensively investigated these last years. Initiated by U. Bader and R. Muchnik in \cite{Bader-Muchnik} a particular interest grown around the quasi-regular representations associated to these systems called \textit{boundary representations}. These representations generalized the well known parabolic inductions \( L^2(G/P) \) for semisimple Lie groups \( G \) and \( P \) minimal parabolic subgroups of \( G \). In their work the authors proved the irreducibility of boundary representations arising from Patterson-Sullivan measures and conjectured the following:

\textbf{Conjecture [Bader-Muchnik]} Let \( G \) be a locally compact group and \( \mu \) a admissible probability measure on \( G \), the boundary representation of \( G \) on the Poisson boundary \( \mathcal{P}(G,\mu) \) is irreducible.

Using the interplay between quasiconformal and harmonic measures in negative curvature \cite{Benjamini-Schramm} \cite{Croke} we propose another approach of these questions that morally attempt to understand boundary dynamics in terms of spectral properties instead of geometric ones. This led us to investigate probabilistic analogues of certain geometric phenomenons.

1.1. \textbf{Harmonic equidistribution and its consequences}. Let \( \mu \) be a symmetric probability measure on \( \Gamma \). The Markov operator \( P_\mu \) associated to a continuous action \( \alpha : \Gamma \times K \to K \) of \( \Gamma \) on a compact space \( K \) is the positive contraction defined on the space of continuous functions \( (C^0(K), \| \cdot \|_\infty) \) given by:

\[
P_\mu \varphi(\xi) = \sum_{g \in \Gamma} \mu(g) \varphi(g \xi)
\]

for \( \varphi \in C^0(K) \). A probability measure, \( \nu \), on \( K \) is \( \mu \)-stationary if \( \mu \ast \nu = \alpha_\ast(\mu \otimes \nu) = \nu \). Moreover the measure \( \mu \) is called \( \nu \)-ergodic if \( \nu \) is stationary and every \( P_\mu \)-invariant function is \( \nu \)-almost everywhere constant.

The principal result of this paper is the following probabilistic version of the Margulis Roblin equidistribution theorem’s \cite{Margulis} \cite{Roblin} for stationary measures on the Gromov boundary of \( (X,d) \):
**Theorem 1** Let \( \nu \) be a probability measure on the Gromov boundary, \( \partial X \), of \( (X,d) \). Assume \( \mu \in \text{Prob}(\Gamma) \) is a 1st moment, non-elementary \( \nu \)-ergodic probability measure on \( \Gamma \).

Then the following equidistribution holds:

\[
\mathbb{E}^\mu[\varphi(X_n,\xi)\psi(X_n^{-1} \cdot \eta)] = \sum_{g \in \Gamma} \varphi(g\xi)\psi(g^{-1} \cdot \eta)\mu^*\nu(g) \xrightarrow{n \to +\infty} \int_{\partial X} \varphi \, d\nu \int_{\partial X} \psi \, d\nu
\]

for all \( \varphi, \psi \in C^0(\partial X) \) continuous functions and \( \xi, \eta \in \partial X \).

Let \( X \) be a Hadamard manifold with pinched curvature. This means that \( X \) is a complete simply connected Riemannian manifold of dimension greater than 2 and curvature, \( K_X \), which satisfies: \( -b^2 \leq K_X \leq -a^2 \). The Brownian motion \( (B_t)_t \) on \( X \), that is the diffusion process generated by the Laplace-Beltrami operator on \( X \), converges almost surely to a random variable \( B_\infty \) with values in the geometric boundary \( \partial X \). When the action \( \Gamma \) on \( X \) has finite covolume \( W \), Ballman and F. Ledrappier in [5] refined the discretization procedure introduced by T. Lyons and D. Sullivan in [27] and earlier discussed by H. Furstenberg in [19] which allows to interpret the distribution of \( B_\infty \), also called harmonic, has a stationary measure for a random walk performed on \( \Gamma \):

**Theorem [Ballman, Ledrappier [5]]** There exists an admissible symmetric probability measure on \( \Gamma \) with 1st moment such that the harmonic measure of the random walk generated by \( \mu \) and the distribution of the limit random variable \( B_\infty \) coincide.

This result guarantees the ergodicity of the measure \( \mu \) and as a direct consequence of Theorem 1 one has:

**Corollary** Let \( X \) be a Hadamard manifold with pinched negative curvature and \( \Gamma \) a discrete group of isometries with finite covolume. Let \( \nu \) be the unique harmonic measure on the visual boundary \( \partial X \) of \( X \) for the Brownian motion \( (B_t)_t \).

Then \( \Gamma \) carries an admissible probability measure with 1st moment, \( \mu \), such that:

\[
\mathbb{E}^\mu[\varphi(X_n,\xi)\psi(X_n^{-1} \cdot \eta)] = \sum_{g \in \Gamma} \varphi(g\xi)\psi(g^{-1} \cdot \eta)\mu^*\nu(g) \xrightarrow{n \to +\infty} \int_{\partial X} \varphi \, d\nu \int_{\partial X} \psi \, d\nu
\]

for all continuous functions \( \varphi, \psi \in C^0(\partial X) \) and \( \xi, \eta \in \partial X \).

Back to a more general situation where \( (X,d) \) is a proper roughly geodesic hyperbolic space. Assuming the action of \( \Gamma \) on \( (X,d) \) is cocompact and \( \mu \)
is finitely supported symmetric and admissible it was proved in [8] that the harmonic measure, \( \nu \), can be interpreted as a quasiconformal measure coming for the Green distance on \( \Gamma \) that satisfies hyperbolic properties. Conversely when \( (X, d) \) is a CAT\((-1)\) space [11] every quasiconformal measure for \( \Gamma \) can be interpreted as a stationary measure for a symmetric measure on \( \Gamma \) [14].

This interplay motivate the investigation on quasiconformal measures that can be regarded as stationary measures.

A measure \( \nu \) on a compact metric space \( (K, d) \) is called Ahlfors-regular if there exist \( \alpha > 0 \) and \( C \geq 1 \) such that:
\[
C^{-1}r^\alpha \leq \nu(B(x, r)) \leq Cr^\alpha
\]
for \( 0 < r < \text{Diam}(K) \).

Examples of such measures in our setting are given by quasiconformal measures on \( \partial X \) associated to discrete groups which act geometrically on \( (X, d) \).

Considering Ahlfors-regular quasiconformal measures on \( \partial X \) for the action of \( \Gamma \) on \( (X, d) \), Theorem 1 together with techniques inspired by [14] [3] led us to the following von Neumann type result for measure class preserving actions:

**Theorem 2** Let \( \mu \in \text{Prob}(\Gamma) \) be a admissible finitely supported symmetric probability on \( \Gamma \) and \( \nu \) its harmonic measure on \( \partial X \). Denote \( P_n \), with \( n \geq 0 \), the operator defined as:
\[
P_n = \mathbb{E}_\mu \left[ \frac{\pi_o(X_n)}{\Xi_o(X_n)} \right] = \sum_{g \in \Gamma} \frac{\pi_o(g)}{\Xi_o(g)} \mu^{sn}(g)
\]
where \( \Xi_o = (\pi_o 1_{\partial X}|1_{\partial X})_{L^2} \) denotes the Harish-Chandra function of the boundary representation \( \pi_o \) on \( \Gamma \).

Then for all \( \varphi, \psi \in L^2[\partial X, \nu] \), square integrable functions on \( \partial X \), the following von Neumann type convergence holds:
\[
(P_n \varphi | \psi)_{L^2} \, \xrightarrow{n \to \infty} \, \int_{\partial X} \varphi d\nu \int_{\partial X} \psi d\nu
\]

In other words if \( P_{1_{\partial X}} \) denote the one dimensional projector on the constant function \( 1_{\partial X} \) given by:
\[
P_{1_{\partial X}}(\varphi) = (\varphi | 1_{\partial X}) 1_{\partial X}
\]
for \( \varphi \in L^2[\partial X, \nu] \). The Theorem 2 corresponds to the weak convergence of the sequence of operators \( (P_n)_n \) to \( P_{1_{\partial X}} \).
In opposite to the measure preserving situation, the vector \(1_{\partial X}\) appears to be cyclic for the boundary representation \((\Gamma, \pi_o, L^2[\partial X, \nu])\) and a standard argument leads to the following:

**Corollary 4.1** Let \(\mu \in \text{Prob}(\Gamma)\) be a admissible finitely supported symmetric probability on \(\Gamma\) and \(\nu\) its harmonic measure on \(\partial X\). Then the associated boundary representation \((\Gamma, \pi_o, L^2[\partial X, \nu])\) is irreducible.

1.2. **Outlines.**

In Section 2 we remind the reader standard facts about random walk on groups as well as some part of the theory of Patterson-Sullivan on quasiconformal measures. Assuming no prior familiarity with this topic we propose a relatively self-contained account of the relevant ingredients needed for the understanding of the rest. In Section 3 we prove Theorem 1. The Section 4 is devoted to the proof of Theorem 2 which is splitted in two parts: the first part is a reduction from matrix coefficients analysis to equidistribution, cf. Proposition 4.5, and the second part is dedicated to the uniform boundedness of the sequence of operators \((\mathcal{P}_n)_n\) introduced in Theorem 2 statement.

1.3. **Notations and terminologies.**

Usually \((X, d)\) stands for a locally finite Gromov hyperbolic space pointed at \(o \in X\) upon which a discrete group \(\Gamma\) acts geometrically. An action of \(\Gamma\) on \((X, d)\) is called *geometric* if it is properly discontinuous and cocompact.

Except mention groups considered are assumed to be non elementary, i.e. non-virtually cyclic.

A probability measure \(\mu \in \text{Prob}(\Gamma)\) on a group \(\Gamma\) is called admissible if its support generates \(\Gamma\) as a semi-group and symmetric if \(\mu(g) = \mu(g^{-1})\) for all \(g \in \Gamma\). Except mention in the rest every measures considered on \(\Gamma\) are assumed to have a 1st moment, i.e. the quantity \(\sum_{g \in \Gamma} d(g.o, o)\mu(g)\) is finite.

We will use the notations: \(|x|_z = d(x, z)\) and \(|g|_z = d(gz, z)\) for \(x, z \in X\) and \(g \in \Gamma\).

In order to avoid the escalation of constants coming from estimates up to controlled additive or multiplication error terms we will use the following conventions. Given two real valued functions, \(a, b\), of a set \(Z\), we write \(a \leq b\) if there exists \(C > 0\) such that \(a(z) \leq Cb(z)\) for all \(z \in Z\) and \(a = b\) if \(a \leq b\) and \(b \leq a\). Analogously we write \(a \preceq b\) if there exists \(c\) such that \(a(z) \preceq b(z) + c\) and \(a \simeq b\) if \(a \preceq b\) and \(b \preceq a\).
2. Preliminaries

2.1. Hyperbolic spaces and compactifications.
Given \((X, d)\) a discrete locally finite metric space, the Gromov product on \(X\) at the basepoint \(o \in X\) is defined as:

\[
(x|y)_o = \frac{1}{2}(|x|_o + |y|_o - d(x, y))
\]

for \(x, y \in X\). The space \((X, d)\) is called \((\delta)\)-hyperbolic, with \(\delta \geq 0\), if it satisfies:

\[
(x|y)_w \geq \min\{(x|z)_w, (z|y)_w\} - \delta
\]

for all \(x, y, z, w \in X\).

**Definition 2.1.** A discrete metric space \((X, d)\) is called roughly geodesic if there exists a constant \(C\) such that for all pair of points \(x, y \in X\), there exists a map \(p : [0, T] \to X\) such that \(p(0) = x\), \(p(T) = y\) and

\[
|t - t'| - C \leq d(p(t), p(t')) \leq |t - t'| + C
\]

for all \(t, t' \in [0, T]\).

Note that hyperbolicity is preserved by quasi-isometries between proper roughly geodesic spaces [9].

Given such a metric space, \((X, d)\), one can associate a compact space, \(\partial X\), called Gromov boundary. Let us recall briefly a construction of this object and its properties.

2.1.1. Hyperbolic boundaries viewed as equivalence classes of sequences.
A sequence \((x_n)_n\) in \(X\) goes to infinity if \(\lim_{n,m} (x_n|x_m)_o \to +\infty\).

The boundary \(\partial X\) of \(X\) can be defined as the equivalence classes on the set of sequences which go to infinity, \(X^\infty \subset X^\mathbb{N}\), endowed with the equivalence relation:

\[
(a_n) \sim (b_n) \text{ if and only if } \lim_{n,m} (a_n|b_m)_o \to \infty
\]

with \((a_n)_n, (b_n)_n \in X^\infty\).
This construction does not depend on the choice of the basepoint \(o\). The class of the sequence \((x_n)_n\) is denoted \(\lim_n x_n = \xi\).

Identifying stationary sequences with elements of \(X\) the Gromov product extends to \(\overline{X} = X \cup \partial X\) by the formula:

\[
(x|y)_o = \sup_{(x_n), (y_m)} \liminf_{n,m} (x_n|y_m)_o
\]
where the sup is taken over all sequences \((x_n)_n\) and \((y_m)_m\) that represent respectively \(x\) and \(y\) in \(\overline{X}\).

This extended product satisfies the following properties \(\blacksquare\):

1. \(\langle x|y\rangle_o = \infty\) if and only if \(x, y \in \partial X\) and \(x = y\);
2. \(\langle x|y\rangle_o \geq \min\{\langle x|z\rangle_o, (z|y\rangle_o\} - 2\delta\) for all \(x, y, z \in \overline{X}\);
3. for all \(\xi, \eta \in \partial X\) and \((x_n)_n, (y_m)_m\) with \(\lim_n x_n = \xi\) and \(\lim_m y_m = \eta\) one has:
   \[
   (\langle \xi|\eta\rangle_o - 2\delta) \leq \liminf_{n,m} \langle x_n|y_m\rangle_o \leq (\langle \xi|\eta\rangle_o.
   \]

Let \(\varepsilon > 0\) and define the kernel:

\[
\rho_{\varepsilon,o} : \overline{X} \times \overline{X} \to \mathbb{R}^+, \quad \rho_{\varepsilon}(x, y) = \begin{cases} e^{-\varepsilon(x|y)_o} & (x, y) \in \overline{X} \times \overline{X} \setminus \Delta_X \\ 0 & \text{otherwise} \end{cases}
\]

where \(\Delta_X = \{(x, x) \mid x \in X\}\) is the diagonal points that belong to \(X\).

Using the Gromov product properties observe that:

1. \(\rho_{\varepsilon,o}(x, y) = \rho_{\varepsilon,o}(y, x)\) for all \(x, y \in \overline{X}\);
2. \(\rho_{\varepsilon,o}(x, y) = 0\) if and only if \(x = y\);
3. for all \(x, y, z \in \overline{X}\):
   \[
   \rho_{\varepsilon,o}(x, y) \leq (1 + \varepsilon') \max\{\rho_{\varepsilon,o}(x, y), \rho_{\varepsilon,o}(y, z)\}
   \]
   with \(\varepsilon' = e^{2\delta\varepsilon} - 1\).

The \textit{chains-pseudo-distance} associated is defined as:

\[
\theta_{\varepsilon,o}(x, y) = \inf\left\{\sum_{i=1}^{n} \rho_{\varepsilon,o}(x_i, x_{i+1}) \mid x_1 = x, \ldots, x_{n+1} = y\right\}
\]

and for \(\varepsilon' < \sqrt{2} - 1\) satisfies:

\[
(1 - 2\varepsilon')\rho_{\varepsilon,o}(x, y) \leq \theta_{\varepsilon,o}(x, y) \leq \rho_{\varepsilon,o}(x, y)
\]

for all \((x, y) \in \overline{X}^2 \setminus \Delta_X\).

In particular for such choice of \(\varepsilon\), \(\theta_{\varepsilon,o}\) is a distance on \(\overline{X}\). In the rest \(\varepsilon'\) is assumed to be strictly lower than \(\sqrt{2} - 1\).

Since \(e^{2\delta\varepsilon} \leq \rho_{\varepsilon,o}(x, y)\) for all \(x \neq y\) in \(X\) and the induced topology on \(X\) by \(\theta_{\varepsilon,o}\) is discrete.

**Definition 2.2.** Let \((X, d)\) be a proper roughly geodesic hyperbolic space. A distance \(d'\) on boundary \(\partial X\) is called visual if there exist \(\varepsilon > 0, \lambda \geq 1\) and \(o \in X\) such that:

\[
\lambda^{-1}\rho_{\varepsilon,o}(\xi, \eta) \leq d'(\xi, \eta) \leq \lambda \rho_{\varepsilon,o}(\xi, \eta)
\]

for all \(\xi, \eta \in \partial X\).
Note that the restriction of $\theta_{\varepsilon,o}$ to $\partial X$ is a visual metric. Its restriction is denoted $d_{\varepsilon,o}$ or simply $d_{\varepsilon}$ in the rest.

A proper sequence $(x_n)_n$ in $X$ is a Cauchy sequence for $\theta_{\varepsilon,o}$ if and only if
\[ \lim_{n,m} (x_n|x_m)_o = +\infty \]
and another proper sequence converges to the same boundary point for $\theta_{\varepsilon,o}$ if and only if
\[ \lim_{n,m} (x_n|y_m)_o = +\infty \]

In other words the metric space $(\overline{X}, \theta_{\varepsilon,o})$ can be seen as the completion of $(X, \theta_{\varepsilon,o}|_X)$.

Endowed with this topology $\overline{X} = X \cup \partial X$ satisfies the following properties:

1. The topological space $\overline{X}$ is compact;
2. The embedding $X \hookrightarrow \overline{X}$ is a homeomorphism on his image, in particular $X$ is open in $\overline{X}$;
3. for all sequence $(\xi_n)_n$ in $\partial X$ one has $\xi_n \to \xi \in \partial X$ if and only if $(\xi_n|\xi)_o \to \infty$.

2.2. Busemann compactification and strongly hyperbolic metric spaces.

Definition 2.3. A proper roughly geodesic metric space $(X, d)$ is strongly hyperbolic if there exists $\varepsilon > 0$ such that
\[ \exp(-\varepsilon (x|y)_w) \leq \exp(-\varepsilon (x|z)_w) + \exp(-\varepsilon (z|y)_w) \]
for all $x, y, z, w \in X$.

Some remarkable consequences of the strong hyperbolicity are:

1. the Gromov product $(.|.)_w$ extends continuously to $\overline{X} = X \cup \partial X$ for all $w \in X$;
2. the kernel $\rho_{\varepsilon,o}$, introduced above, is an actual metric on $\partial X$ for $\varepsilon$ small enough.

These facts guarantee sharper conformal properties on the boundary $\partial X$.

Given a strongly hyperbolic space $(X, d)$, the Busemann function at $\xi \in \partial X$ is well defined as:
\[ \beta_\xi(x, y) = 2(\xi|y)_x - |y|_x = \lim_{z \to \xi} d(z, x) - d(z, y) \]
for all $x, y \in X$. In particular the Busemann boundary $\partial_\infty X$ and the Gromov boundary $\partial X$ coincide.

In the rest $\beta_\xi(o, x)$ is denoted $\beta_\xi(x)$ for all $\xi \in \partial X$ and $x \in X$. 

Another important feature of the strong hyperbolicity is the \textit{metric conformality} for the action of the group of isometries of $X$ on $\partial X$:
\[
d_{o,\varepsilon}(g\xi, g\eta) = \exp\left(\frac{\varepsilon}{2}\beta(\xi)\right) \exp\left(\frac{\varepsilon}{2}\beta_{\eta}(\xi)\right)d_{o,\varepsilon}(\xi, \eta)
\]
for all $\xi, \eta \in \partial X$ and $g \in \text{Isom}(X, d)$, where $d_{o,\varepsilon} = \rho_{\varepsilon, \alpha}$.

As a consequence if $\nu$ denote the Hausdorff measure of dimension $D_{\varepsilon}$ on $(\partial X, d_{o,\varepsilon})$ [30], the following measure conformal relation holds:
\[
\frac{dg_{\ast}\nu}{d\nu}(\xi) = \exp(\varepsilon D_{\varepsilon}\beta(\xi)) = \exp(\delta_{\varepsilon}\beta(\xi))
\]
for all $\xi \in \partial X$ and $g \in \text{Isom}(X, d)$.

\textbf{Remark 2.4.} The measure $\nu$ is independent of the parameter $\varepsilon$ chosen.

Examples of strongly hyperbolic space are given by Hadamard manifolds with pinched negative curvatures more generally CAT($-1$) metric spaces [30] and, as exposed in subsection 2.4, Green metric structures on hyperbolic groups.

\subsection*{2.3. Boundary retractions.}

As above $(X, d)$ is a proper roughly geodesic hyperbolic space. A \textit{boundary retraction} is defined as a continuous map $f : X \to \partial X$ such that $f|_{\partial X} = id_{\partial X}$. Such a retraction induces an isometric operator:
\[
E_f : (C^0(\partial X), \| \cdot \|_{\infty}) \to (C^0(X), \| \cdot \|_{\infty}), \quad \varphi \mapsto \varphi \circ f
\]
i.e. $\| \varphi \circ f \|_{\infty} = \| \varphi \|_{\infty}$ for all $\varphi \in C^0(\partial X)$, called \textit{boundary extension}.

\subsubsection*{2.3.1. Existence of boundary retractions.}

The \textit{shadow} of a ball centered at $x \in X$ with radius $R$ viewed at the basepoint $o \in X$ is defined as:
\[
O_o(x, R) = \{ \xi \in \partial X | \exists c \text{ s.t } c(0) = o, c(x) = \xi, c(\mathbb{R}_+) \cap B(x, R) \neq \emptyset \}
\]
where $c$ denote a rough geodesic in $X$.

A shadow can equivalently be described in terms of the Gromov product as:
\[
O_o(x, R) = \{ \xi \in \partial X | (\xi|x)_o \geq |x|_o - R \}
\]

Note that there exists $R_0 > 0$ which does not depend on $x \in X$ such that for all $R \geq R_0$, $O_o(x, R) \neq \emptyset$ [15].

An example of a boundary retraction can be obtained as follows:

\textbf{Lemma 2.5.} Let $R > 0$ be a large positive constant and $f : X \to \partial X$ such that $f(x) \in O_o(x, R)$ for all $x \in X$. Then $f$ is well defined and extends to a boundary retraction on $\overline{X}$ called \textit{linear boundary retraction}. 
Proof. Since $X$ is discrete it is enough to prove for every sequence $(x_n)_n$ in $X$ converging to a point $\xi \in \partial X$, $(f(x_n))_n$ converges to $\xi$.

The definition of $f$ implies $(f(x)|x)_o \geq |x|o - R$ for all $x \in X$. On the other hand $\theta_{\varepsilon,o}(y_n,\eta) \to 0$ if and only if $(y_n|\eta)_o \to +\infty$ for any $(y_n)_n$ in $\overline{X}$ and $\eta \in \partial X$.

Thus given $(x_n)_n$ with $x_n \to \xi$ since:

$$(f(x_n)|\xi)_o \geq \min\{(f(x_n)|x_n)_o, (x_n|\xi)_o\} - \delta$$

one has $\theta_{\varepsilon,o}(f(x_n), \xi) \to 0$, in other words $f$ can be extended to $\overline{X}$ by $f(\xi) = \xi$ for all $\xi \in \partial X$. $\square$

In some sense every boundary retraction is of this form:

**Lemma 2.6.** Let $f$ be a boundary retraction of $X$. Then there exists a proper function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that:

$$(f(x)|x)_o \geq \varphi(|x|o)$$

for all $x \in X$.

**Proof.** Let $R > 0$ be a positive constant and $f'$ a linear boundary retraction such that $(f'(x)|x)_o \geq |x|o - R$ for $x \in X$.

Observe that:

$$(f(x)|x)_o \geq \min\{(f'(x)|x)_o, (f(x)|f'(x))_o\} - \delta \geq \varphi(|x|o)$$

with

$$\varphi(t) = \inf_{|t| \leq |x| < |t| + 1} \min\{|t| - R, (f(x)|f'(x))_o\} - \delta$$

Therefore $\varphi$ is proper iff the map $x \mapsto (f(x)|f'(x))_o$ on $X$ is proper.

A similar argument than the one used in the proof of the next lemma, based on compactness of $\partial X$, the continuity of $f$ and $f'$ together with the fact that these two functions are identical on the boundary guarantees the properness of the last map. $\square$

2.3.2. Boundary retraction and sequence of measures with accumulation at infinity.

A sequence of probabilities on $\overline{X}$ has accumulation at infinity if for all compact $K \subset X$ one has $\mu_n(K) \to 0$. In other words any limit of a $(\mu_n)_n$ subsequence is supported in $\partial X$.

**Lemma 2.7.** Let $f$ be a boundary retractions and $(\mu_n)_n$ a sequence of probabilities on $\overline{X}$ with accumulation at infinity.

If $f_\star \mu_n$ converges weakly to $\mu \in \text{Prob}(\partial X)$, then for every other boundary retraction $f'$, $f'_\star \mu_n$ converges weakly to $\mu$. 

Proof. Let $f'$ be another boundary retraction. Since $E_f(\varphi)$ and $E_{f'}(\varphi)$ are continuous functions on the compact space $\overline{X}$ and thus uniformly continuous. Moreover $E_f(\varphi)|_{\partial X} = E_{f'}(\varphi)|_{\partial X} = \varphi$.

Therefore given $\varepsilon > 0$, there exists $\alpha > 0$ such that for all $\xi \in \partial X$ and $x \in B_{\theta,\alpha}(\xi, \alpha) \subset \overline{X}$, $|E_f(\varphi)(x) - \varphi(\xi)|, |E_{f'}(\varphi)(x) - \varphi(\xi)| \leq \frac{\varepsilon}{4}$.

Because $\partial X$ is compact one can find a finite number, $n$, of $\xi_i \in \partial X$, $i = 1, \ldots, n$ such that $\partial X \subset \cup_{i=1}^n B_{\theta,\alpha}(\xi_i, \alpha) = U$.

By construction of $U \subset \overline{X}$, $|E_{f'}(\varphi)(x) - E_{f}(\varphi)(x)| \leq \frac{1}{2}\varepsilon$ for all $x \in U$.

Let $K = U^c \subset X$ be the compact complement of $U$ in $\overline{X}$. Since $\mu_n$ has accumulation at infinity, there exists $n_0$, for all $n \geq n_0$, $\mu_n(K) \leq \frac{\varepsilon}{4|\varphi|_{\infty}}$.

Together these estimates lead to:

$$|\int_{\partial X} \varphi \cdot f_*\mu_n - \int_{\partial X} \varphi \cdot f'_{*}\mu_n| \leq \int_{K} E_f(\varphi) - E_{f'}(\varphi) \, d\mu_n| + \int_{U} E_f(\varphi) - E_{f'}(\varphi) \, d\mu_n|$$

$$\leq \mu_n(K)(\|E_{f'}(\varphi)\|_{\infty} + \|E_f(\varphi)\|_{\infty}) + \frac{1}{2}\varepsilon\mu_n(U)$$

$$\leq 2\mu_n(K)\|\varphi\|_{\infty} + \frac{1}{2}\varepsilon \leq \varepsilon$$

which concludes the proof. \(\square\)

In the rest a boundary retraction, $f$, is fixed and for all $x \in X$ we denote $\hat{x} = f(x) \in \partial X$. Moreover we extend this notation for $m \in \text{Prob}(\overline{X})$ (respectively, $\varphi \in C^0(\partial X)$) by $\hat{m} = f_*m \in \text{Prob}(\partial X)$ (respectively, $\hat{\varphi} = E_f(\varphi) \in C^0(\overline{X})$).

Note that for all $\varphi \in C^0(\partial X)$, $x \in \overline{X}$ and $m \in \text{Prob}(\overline{X})$ the following identities hold:

$$\hat{\varphi}(x) = \varphi(\hat{x}), \quad \hat{m}(\varphi) = m(\hat{\varphi})$$

2.4. Random walks and hyperbolic groups.

In the first part of this subsection $\Gamma$ is a general discrete group.

Given a probability $\mu \in \text{Prob}(\Gamma)$ on $\Gamma$ a (left) $\mu$-random walk on $\Gamma$ is a Markov chain with states space $\Gamma$ and transition probabilities $p_\mu(g, h) = \mu(g^{-1}h)$.

We call $\mu$-random walk starting at $g \in \Gamma$ the unique Markov chain (up to equivalence) with initial law $\delta_g$ and transition probabilities $p_\mu$.

2.4.1. Standard models of random walks.

Let $(\Gamma^{N^*}, \mathcal{B}, \mu^{N^*})$ be the probability space obtained as infinite product of $(\Gamma, \mathcal{P}(\Gamma), \mu)$, where $\mathcal{B}$ is the $\sigma$-algebra generated by cylinders. The coordinate projectors

$$H_k : \Gamma^{N^*} \to \Gamma, \quad H_k((\omega_i)_i) = \omega_k$$
form a sequence of $\mu$ distributed independent random variables and a model for random walk starting at $g_0$ is given by $X_0 = g_0$ and $X_n = H_1 \ldots H_n$ for $n \geq 1$. In other words:

$$
P(X_{n+1} = h | X_0 = h_0, \ldots, X_n = g) = \begin{cases} 0 & \text{if } h_0 \neq g_0 \\ p_\mu(g, h) & \text{if } h_0 = g_0 \end{cases}
$$

The canonical model for the random walk starting at $g \in \Gamma$ is defined as the image of $(\Gamma^{N^*}, \mu^{N^*})$ by:

$$
\mathcal{R}_g : \Gamma^{N^*} \to \Gamma^N, \quad \mathcal{R}_g((\omega_i)_i) = \begin{cases} g & \text{for } n = 0 \\ gH_1 \ldots H_n((\omega_i)_i) & \text{for } n \geq 1 \end{cases}
$$

where the random position at time $n$, $X_n$, is given by the $n$-th coordinate projection on $\Gamma$.

In the rest we denote $\Omega = \Gamma^N$ endowed with $\sigma$-algebra generated by cylinders and the probability:

$$
P = \mathcal{R}_{e*}\mu^{N^+} = \delta_e \otimes \bigotimes_{n=1}^{\infty} \mu^+\mu^n
$$

or $P^\mu$ to insist on the $\mu$ dependence.

**Remark 2.8.** If $(X_n)_n$ is a $\mu$-random walk starting at $e \in \Gamma$, then $(gX_n)_n$ gives a $\mu$-random walk starting at $g \in \Gamma$. Moreover the corresponding law on $\Omega$ is given by $P_g = g_*P = \mathcal{R}_{g*}\mu^{N^+} = \delta_g \otimes \bigotimes_{n=1}^{\infty} g_*\mu^n$ where $\Gamma$ acts on $\Omega$ by the left-diagonal translation.

2.4.2. **Filtrations and stopping times.**

A filtration on a measurable space $(\Omega, \mathcal{B})$ is an increasing sequence of sub-$\sigma$-algebra of $\mathcal{B}$, $(\mathcal{F}_n)_n$.

In our case the **canonical filtration** is defined by $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$ for $n \geq 0$ where $(X_n)_n$ is a $\mu$-random walk.

Given $(\Omega, \mathcal{B}, (\mathcal{F}_n)_n)$ a measurable map $T : \Omega \to \mathbb{N} \cup \{\infty\}$ is called **stopping time** with respect to the filtration $(\mathcal{F}_n)_n$ if $\{T = n\} \in \mathcal{F}_n$ for all $n$.

Together with this stopping time $T$ is associated the **stopping time $\sigma$-algebra**, $\mathcal{F}_T$, defined as:

$$
\mathcal{F}_T = \{B \in \mathcal{B} \mid B \cap \{T = n\} \in \mathcal{F}_n \text{ for all } n \geq 0\}.
$$

Assuming $T$ is almost surely finite the $\mu$-**random position at time** $T$ is the $\mathcal{F}_T$-measurable random variable given by:

$$
X_T : \Omega \to \Gamma, \quad X_T(\omega) = \begin{cases} X_n(\omega) & \text{if } T(\omega) = n \\ e & \text{if } T(\omega) = +\infty \end{cases}
$$
2.4.3. **Fundamental transformations associated to random walks.**

Given \( g \in \Gamma \) the map \( R_g \) defined above is an isomorphism of measured spaces with inverse \( R_g^{-1}((\omega_n)_n) = (X_{i-1}^{-1}X_i((\omega_n)_n))_{i \geq 1} \).

The shift, \( T \), on \((\mathbb{R}^*,\mu^*)\) given by \( T((\omega_i)_i) = (\omega_{i+1})_i \) is a measure preserving transformation that is mixing and thus ergodic. Moreover \( T \) is intertwined by \( R_g \) with the Bernoulli shift \( U_g \) on \((\Omega,P_g)\) given by \( U_g((\omega_n)_n) = (g\omega_1^{-1}\omega_{n+1})_n \). Since \( R_g \) is an \((T,U_g)\)-equivariant isomorphism of measured spaces it preserves spectral properties and thus \( U_g \) is also ergodic and mixing on \((\Omega,P_g)\).

Another transformation associated to a Markov chains is the *Markov shift*: 

\[
\Theta : (\Omega,P_g) \rightarrow (\Omega,P_g), \quad \Theta((\omega_n)_n) \rightarrow (\omega_{n+1})_n.
\]

More generally, given a stopping time \( T \) one can define on \( \{ T < +\infty \} \subset \Omega \) the Markov shift at time \( T \) by:

\[
\Theta_T : \{ T < +\infty \} \subset \Omega \rightarrow \Omega, \quad \Theta_T(\omega) = (\omega_{T(\omega)+n})_n
\]

Note that unlike the Bernoulli shift these Markov transformations are not measure preserving. Nevertheless the so called strong Markov property holds:

**Proposition (Strong Markov).** Let \( (\Omega,B,(\mathcal{F}_n)_n,P) \) be filtered space and \( \varphi \) a positive measurable function. Then the following relation between random variables holds:

\[
\mathbb{E}(1_{\{T < +\infty\}}\varphi \circ \Theta_T|\mathcal{F}_T) = 1_{\{T < +\infty\}}\mathbb{E}_{X_T}(\varphi)
\]

2.4.4. **Random walks and metric structures on hyperbolic groups.**

A discrete group \( \Gamma \) is called Gromov hyperbolic if it acts geometrically on a proper roughly geodesic \( \delta \)-hyperbolic space \((X,d)\).

Let \( \mathcal{D}(\Gamma) \) be the collection of left-invariant pseudo-metrics on \( \Gamma \) which are quasi-isometric to a locally finite word distance.

Since \( \Gamma \) acts geometrically on \((X,d)\), the Milnor lemma guarantees that for any locally finite word distance, \( d_\Gamma \), on \( \Gamma \) and \( w \in X \) the map:

\[
\varphi : (\Gamma,d_\Gamma) \rightarrow (O(w),d_X|_{O(w)}) \subset (X,d_X), \quad g \mapsto g.w
\]

is a \( \Gamma \)-equivariant quasi-isometry. In particular the distance \( d_X|_{O(w)} \) on \( O(w) \approx_{q.i} \Gamma \) is hyperbolic and belongs to \( \mathcal{D}(\Gamma) \). Therefore we will assume in the rest that \( X = \Gamma \) endowed with a distance \( d \in \mathcal{D}(\Gamma) \) upon which \( \Gamma \) acts by left-translations.

In order to make a distinction between \( \Gamma \) as a group acting on a space and \( \Gamma \) as metric space we denote the last one \((X,d)\) and fixe the basepoint \( o = e \in X \).
Let us introduce an important example of strong hyperbolic distances on \( \Gamma = X \) which arise from random walks called Green distance. Let \( \mu \in \text{Prob}(\Gamma) \) be a admissible symmetric probability measure on \( \Gamma \) and let us consider the stopping time:

\[
\tau_g : \Omega \to \mathbb{N} \cup \{\infty\}, \quad \tau_g((\omega)_n) = \inf\{n|\omega_n = g\}.
\]

The probability that a \( \mu \)-random walk starting at \( g \) ever hits \( g' \) is given by:

\[
F_\mu(g, g') = \mathbb{P}_g(\tau_{g'} < +\infty)
\]

and the associated Green distance is defined as \( d_\mu(g, g') = -\log F_\mu(g, g') \).
It is enough for \( \Gamma \) to be finitely generated non-amenable to guarantee that \( d_\mu \) is a left-invariant distance quasi-isometric to locally finite word distances [7].

Nevertheless the rough geodesic structure which is needed to deduce the hyperbolicity uses the Ancona criterion [2] which holds when \( \mu \) is finitely supported and \( \Gamma \) is hyperbolic:

**Theorem (Ancona [2]).** Given a non-elementary hyperbolic group \( \Gamma \), the Green distance induced by an admissible finitely supported symmetric probability on \( \Gamma \) is hyperbolic.

The strong hyperbolic property of such metrics was established in [30]:

**Theorem (Bogdan, Spakula [30]).** Given a non-elementary hyperbolic group \( \Gamma \), the Green distance induced by an admissible finitely supported symmetric probability on \( \Gamma \) is strongly hyperbolic.

The strong hyperbolicity of these type of distances together with the Patterson-Sullivan theory guarantee the conformal properties of stationary measures associated Theorem 1.5 [8]:

\[
\frac{d\mu^\nu}{d\nu}(\xi) = \exp(\delta_{\Gamma} \beta_{\xi}(g^{-1}o))
\]

for \( \xi \in \partial X \) and \( g \in \text{Isom}(X, d) \), where \( \delta_{\Gamma} \) is the critical exponent of \( \Gamma \).

**2.4.5. Spectral gaps and random walks.**
A symmetric probability measure on \( \Gamma \) is called non-elementary if the semigroup generated by its support in \( \Gamma \) is non-elementary.

Given an symmetric non-elementary probability measure, \( \mu \), on \( \Gamma \), the spectral radius, \( \rho(\mu) \), is defined as:

\[
\rho(\mu) = \limsup_n \mu^*(e)^{\frac{2}{n}}.
\]

It is related to the amenability of the subgroup generated its support by the following theorem:
Theorem (Kersten [26]). Let $\Gamma$ be a finitely generated group and $\mu$ a symmetric probability on $\Gamma$. Then $\mu$ has a spectral gap, i.e. $\rho(\mu) < 1$ if and only if $\mu$ is non-elementary.

Given a $\mu$-random walk starting at $e \in \Gamma$, $(X_n)_n$, for all $n, m \geq 0$ the following subadditive inequality holds:

$$|X_{n+m}| \leq |X_n| + |X_m \circ U^m|$$

Using the ergodicity of the Bernoulli shift, $U$, the Kingman subadditive [17] theorem guarantees the convergence almost surely and in $L^1(\Omega, \mathbb{P})$ of $(\frac{|X_n|}{n})_n$ to a constant $\ell$ called the drift of escape, with $\ell > 0$ whenever $\mu$ is non-elementary [35].

Let us prove the following useful estimate:

Lemma 2.9. Let $\mu \in \text{Prob}(\Gamma)$ be a probability measure on a group $\Gamma$. Then for all $n$ one has:

$$\mu^{*n}(g) \leq \max\{\mu^{*n-1}(e), \mu^{*n}(e)\}$$

for all $g \in \Gamma$.

Proof. Observe that:

$$\mu^{*2n}(x) = \sum_{g \in \Gamma} \mu^{*n}(g^{-1}g') \mu^{*n}(g')$$

$$\leq \sqrt{\sum_{g \in \Gamma} \mu^{*n}(g^{-1}g')^2} \sqrt{\sum_{g \in \Gamma} \mu^{*n}(g')^2}$$

$$\leq \sum_{g \in \Gamma} \mu^{*n}(g')^2 = \mu^{*2n}(e)$$

and

$$\mu^{*2n+1}(x) = \sum_{g \in \Gamma} \mu^{*2n}(g^{-1}g') \mu(g')$$

$$\leq \sum_{g \in \Gamma} \mu^{*2n}(e) \mu(g') = \mu^{2n}(e)$$

which imply the lemma.  

2.4.6. Random walks on Hyperbolic groups.

In addition to recalling basics about the asymptotic behavior of random walks over hyperbolic spaces we insist on ideas present in Theorem proof. Assume $\Gamma$ is a non-elementary group that acts properly by isometries on $(X, d)$ and $\mu \in \text{Prob}(\Gamma)$ an symmetric non-elementary probability on $\Gamma$. 

Proposition 2.10. For $\mathbb{P}$-almost every $\omega \in \Omega$ the sequence $(X_n(\omega)\cdot o)_n$ in $(X,d)$ converges to $Z(\omega) \in \partial X$.

Proof. It is enough to prove that for $\mathbb{P}$-almost every $\omega \in \Omega$, $(X_n(\omega)\cdot o)_n$ is a $\theta_{\varepsilon,o}$-Cauchy sequence.

Let us consider the random variables $Y_n = d(X_{n-1}\cdot o, X_n\cdot o) = |H_n|_o$ for $n \geq 1$. These are real independent variables with distribution $\sigma \in \text{Prob}(\mathbb{R}^+)$ given by cumulative function $f_\sigma(t) = \mathbb{P}(|X_1|_o \leq t)$ for $t \in \mathbb{R}$.

Since $\mu$ has a first moment observe that:

$$E(Y_n) = \int_{\mathbb{R}^+} t \, d\sigma(t) = \int_{\mathbb{R}^+} \sigma(t \geq a) \, d\lambda(a) = \int_{\mathbb{R}^+} \mathbb{P}(|H_1|_o \geq a) \, d\lambda(a) = \int_{\Omega} |H_1|_o \, \mathbb{P}(\omega) = \sum_{g \in \Gamma} \mu(g)|g\cdot o|_o$$

is finite and non-zero because $\mu$ is admissible.

The law of large numbers implies that for $\mathbb{P}$-almost all $\omega \in \Omega$, $S_n(\omega) = \frac{1}{n} \sum_{k=1}^{n} Y_k(\omega) \rightarrow E(|X_1|_o)$ and therefore $\frac{1}{n} Y_n \rightarrow 0$ $\mathbb{P}$-almost surely.

On the other hand, since $\mu$ is non-elementary, $\frac{1}{n}|X_n|_o$ converges $\mathbb{P}$-almost surely to $\ell > 0$ and thus $\lim \frac{1}{n}(X_n|X_{n+1})_o(\omega) = \ell$. It follows for all $m \leq n$ large enough:

$$\theta_{\varepsilon,o}(X_m(\omega), X_n(\omega)) \leq \sum_{k=m}^{\infty} \rho_{\varepsilon,o}(X_k(\omega), X_{k+1}(\omega)) \leq \sum_{k=m}^{\infty} e^{-\frac{1}{2} \ell k} \rightarrow 0, \quad \text{when } n, m \text{ go to infinity}$$

for $\mathbb{P}$-almost all $\omega \in \Omega$. \qed

Let us investigate the properties of the hitting random variable $Z$ and make some observations that will be useful in Theorem 1 proof.

Remark 2.11. For all $k \geq 0$ fixed, $\Theta^k$ preserves essentially the set upon which $(X_n)_n$ converges to $Z$ and as a consequence $X_n \circ \Theta^k(\omega) \rightarrow Z \circ \Theta^k(\omega)$.

On the other hand

$$(X_n \circ \Theta^k(\omega)|X_n(\omega))_o \geq \frac{1}{2}(|X_n|_o(\omega) - |H_{n+1} \ldots H_{n+k}|_o(\omega))$$

for all $\omega \in \Omega$ and

$$\frac{1}{n}|H_{n+1} \ldots H_{n+k}|_o(\omega) \leq \frac{1}{n} \sum_{i=1}^{k} |H_{n+i}|_o(\omega)$$
converges to 0 when \( n \) goes to infinity and \( k \) is fixed \( \mathbb{P} \)-almost surely by the law of large numbers. It follows that \((X_n \circ \Theta^k)_n\) and \((X_n)_n\) converge to the same limit:

\[
Z \circ \Theta^k(\omega) = Z(\omega)
\]

for \( \mathbb{P} \)-almost all \( \omega \in \Omega \). In other words the random variable \( Z \) is Markov shift invariant.

Moreover for all \( \omega \in \Omega \) one has \( X_n \circ \Theta^k(\omega) = X_{n+k}(\omega) = X_k(\omega).X'_n(\omega) \) where \( X'_n = X^{-1}_kX_{n+k} \) is a \( \mu \)-distributed random walk starting at \( \varepsilon \) independent of \( X_k \) for all \( k' \leq k \).

Therefore \( \mathbb{P} \)-almost surely:

\[
Z(\omega) = \lim_{n} X_n \circ \Theta^k(\omega) = X_k(\omega).\lim_{n} X'_n(\omega) = X_k(\omega).Z_k(\omega)
\]

where \( Z_k \) is independent of \( X_k \) for all \( k' \leq k \). If \( U \) denotes the Bernoulli shift at \( \varepsilon \) on \( \Omega \) for \( k \geq 0 \) one has:

\[
X_n \circ U^k = X^{-1}_kX_n \circ \Theta^k \to Z_k
\]

and since \( U \) is \( \mathbb{P} \)-measure preserving one deduce that the sequence of random variables: \((Z_k)_{k \geq 0}\) have the same law.

As a consequence if \( \nu \) is the law of \( Z \) on \( \partial X \) the above relation implies:

\[
\nu(A) = \mathbb{P}(Z \in A) = \mathbb{P}(X_kZ_k \in A)
\]

\[
= \int_{\Omega} 1_A(X_kZ_k)(\omega) \mathbb{P}(\omega)
\]

\[
= \sum_g \mu^{*k}(g) \int_{\Omega} 1_A(gZ_k(\omega)) \mathbb{P}(\omega)
\]

\[
= \sum_g \mu^{*k}(g)\nu(g^{-1}A)
\]

for all measurable set \( A \subset \partial X \). In other words \( \nu \) is \( \mu \)-stationary.

**Lemma 2.12.** Let \( \nu \) be a diffuse probability on \( \partial X \) and \((g_n)_n\) a proper sequence of elements of \( \Gamma \). Assume \( g_n\nu \) converges to \( \nu' \), then \( \nu' \) is a Dirac mass. Moreover if the limit probability satisfies \( \nu' = \delta_\xi \) then \( \lim_n g_n.o = \xi \).

**Proof.** Since \((g_n)_n\) is proper, after extraction of a subsequence, one may assume that there exist \( \xi_\pm \in \partial \Gamma \) such that \( g_n^\pm.o \to \xi_\pm \).

Let \( \eta \neq \xi_- \) be a boundary point distinct from \( \xi_- \). Then \((\eta|g^{-1}_n.o)_o \) is a bounded sequence and since \( |g^{-1}_n.o - (\eta|o)_g^{-1},o | \leq (\eta|g^{-1}_n.o)_o \), it follows that \((\eta|o)_{g^{-1}_n,o} = (g_n\eta|g_n.o)_o \to +\infty \). In other words \( g_n\eta \to (g_n.o) = \xi_+ \). Because \( \nu \) is diffuse for all \( \varphi \in C^0(\partial X) \) the dominated convergence theorem implies:

\[
\int_{\partial \Gamma} \varphi(\eta) d\nu \to \int_{\partial \Gamma} \varphi(\xi_+) d\nu = \varphi(\xi_+)
\]
Conversely if $\nu'$ is Dirac mass at $\xi$ then $\xi$ is the only possible cluster value of $(g_n)_n$ by the previous argument. \qed

**Proposition 2.13.** Let $Z$ be the random limit of a $\mu$-random walk $(X_n)_n$ on $\Gamma$. The law of $Z$, $\nu$, called hitting measure on $\partial X$ given by:

$$\nu = \int_{\partial X} \delta_{Z(\omega)} \mathbb{P}(\omega)$$

has no atoms and is the unique $\mu$-stationary probability on $\partial X$.

**Proof.** Assume $\nu$ is a $\mu$-stationary probability on $\partial X$ with an atom at $\xi \in \partial X$. Let $\varphi_n \in \ell^1(\Gamma)$ for $n \geq 0$ be the sequence of positive functions on $\Gamma$ given by $\varphi_n(g) = \mu^\ast n(g)^{-1} \nu(g^{-1} \xi)$. Since $\nu$ is $\mu$-harmonic one has:

$$\sum_{g \in \Gamma} \varphi_n(g) = \nu(\xi) > 0$$

for all $n \geq 0$.

On the other hand the support of $\mu$ generates an non-amenable subgroup of $\Gamma$. This implies that $\mu^\ast n(g) \leq \rho(\mu)^n$ with $\rho(\mu) < 1$ by Kersten theorem and therefore $(\varphi_n)_n$ converges pointwise to 0.

Since $\varphi_n(g) \leq u(g) = \nu(g^{-1} \xi)$ with $u \in \ell^1(\Gamma)$, the dominated convergence theorem implies that $\sum_g \varphi_n(g) \to 0$ but $\sum_g \varphi_n(g) = \nu(\xi) > 0$ for all $n \geq 0$. This is a contradiction and $\nu$ is necessarily diffuse on $\partial X$.

Since $\nu$ is diffuse Lemma 2.12 implies that $X_n(\cdot)_\ast \nu$ converges $\mathbb{P}$-almost surely to $\delta_{Z(\cdot)}$. The martingale convergence theorem implies:

$$\int_{\Omega} X_n(\omega)_\ast \nu \mathbb{P}(\omega) \to \int_{\Omega} \delta_{Z(\omega)} \mathbb{P}(\omega)$$

Indeed for $\varphi \in C^0(\partial X)$, $\varphi_n(\omega) = \int_{\partial X} \varphi(X_n(\omega) \xi) d\nu(\xi)$ with $\omega \in \Omega$ is a bounded martingale on $\Omega$ and thus converges in $L^1(\Omega, \mathbb{P})$.

One the other hand $\nu$ is $\mu$-stationary and thus:

$$\int_{\Omega} X_n(\omega)_\ast \nu \mathbb{P}(\omega) = \mu^\ast n \ast \nu = \nu$$

$\square$

3. **Probabilistic Margulis-Roblin equidistribution**

The next theorem is a probabilistic analogue of Roblin equidistribution Theorem 4.1.1 of [32] (see also [28]). In this section $\mu$ denote an arbitrary symmetric non-elementary probability measure on $\Gamma$ with 1st moment.
**Theorem 1.** Let \( \varphi_1, \varphi_2 \in C^0(\partial X) \) be two continuous functions on \( \partial X \), \( (X_n)_n \) a \( \mu \)-random walk starting at \( a \in X \) with \( \mu \) as above and \( \nu \) the unique \( \mu \)-stationary measure on \( \partial X \).

Then the following equidistribution holds:

\[
\mathbb{E}_\mu [\varphi_1(X_n, a) \varphi_2(X_n^{-1}, a)] = \frac{1}{n+1} \sum_{g \in \Gamma} \varphi_1(g, a) \varphi_2(g^{-1}, a) \mu^n(g) \to \int_{\partial X} \varphi_1 \, d\nu \cdot \int_{\partial X} \varphi_2 \, d\nu.
\]

Moreover the boundary retraction might be chosen differently for \( \varphi_1 \) and \( \varphi_2 \).

**Corollary 3.1.** Given \( \varphi \in C^0(\partial X \times \partial X) \) a continuous function on \( \partial X \times \partial X \), the following equidistribution holds:

\[
\mathbb{E}_\mu [\Phi(X_n, \xi, X_n^{-1}, \eta)] = \sum_{g \in \Gamma} \Phi(g, \xi, g^{-1}, \eta) \mu^n(g) \to \int_{\partial X \times \partial X} \Phi \, d\nu \otimes \nu.
\]

**Proof.** Observe that:

\[
C^0(\partial X) \otimes_{\text{alg}} C^0(\partial X) \subset C^0(\partial X) \widehat{\otimes}_b C^0(\partial X) = C^0(\partial X \times \partial X)
\]

where \( \otimes_{\text{alg}} \) and \( \widehat{\otimes}_b \) stand respectively for the algebraic tensor product and the injective one [33]. If \( J : C^0(\partial X \times \partial X) \to C^0(\partial X \times \partial X)^{**} \) denote the topological bidual injection one has:

\[
J[C^0(\partial X) \otimes_{\text{alg}} C^0(\partial X)]^w = C^0(\partial X \times \partial X)^{**}
\]

where \( \widehat{\otimes}^w \) stands for the weak closer on \( C^0(\partial X \times \partial X)^{**} \). Using the fact that probability measures on \( C^0(\partial X \times \partial X) \) are contractions one deduce the extension of Theorem 1 for arbitrary function on \( \partial X \times \partial X \) and choice of boundary retractions.

Take \( B : \Gamma = X \to \partial X \) given by \( B(g) = g.\xi_0 \) for some fixed \( \xi_0 \in \partial X \). Since \( \Gamma \) acts geometrically on \( X \), \( B \) extends to a boundary retraction on \( X \). The corollary follows from Theorem 1 applied to boundary retractions of this type. \( \square \)

**Proof of Theorem 1.** Let us denote \( I_n = \mathbb{E}_\mu [\varphi_1(X_n^{-1}, a) \varphi_2(X_n, a)] \) for \( n \geq 0 \) and \( I_+ = \lim \sup_n I_n \), which is bounded above by \( \|\varphi_1\|_x \|\varphi_2\|_x \).

It is enough to prove:

\[
I_+ \leq \int_{\partial X} \varphi_1 \, d\nu \cdot \int_{\partial X} \varphi_2 \, d\nu
\]

Indeed, exchanging \( \varphi_1 \) for \( -\varphi_1 \) leads to:

\[
\int_{\partial X} \varphi_1 \, d\nu \cdot \int_{\partial X} \varphi_2 \, d\nu \leq \lim \inf_n I_n
\]

and concludes the proof of Theorem 1.
Let us start by introducing the maps:

\[ F_{n,k} : \Omega \rightarrow \mathbb{R}_+, \quad \omega \mapsto (X_n(\omega).o|X \circ \Theta^k(\omega).o) \]

and

\[ F'_{n,k} : \Omega \rightarrow \mathbb{R}_+, \quad \omega \mapsto (X^{-1}_n \circ U^k(\omega).o|X^{-1}_n \circ \Theta^k(\omega).o) \]

for \( n, k \geq 0 \) where \( U \) and \( \Theta \) denote respectively the Bernoulli and the Markov shift on \( \Omega \).

Observe that:

\[ |F_{n,k} - \ell.n| = \frac{1}{2}(|X_.o_n - \ell.n| + (|X_{n+k}.o|_o - \ell.(n + k)) - (|H_{n+1} \cdots H_{n+k}.o|_o - \ell.k)); \]

\[ |F'_{n,k} - \ell.n| = \frac{1}{2}(|H_{n+1}^{-1} \cdots H_{k+1}^{-1}.o|_o - \ell.n) + (|X^{-1}_n.o_n - \ell.(n + k)) - (|H^{-1}_n \cdots H^{-1}_1.o|_o - \ell.k)| \]

for all \( n, k \geq 0 \). The subadditive ergodic theorem [17] guarantees that \( X_.o_n \rightarrow \ell \) pointwise and in \( L^1(\Omega, \mathbb{P}) \) where \( \ell > 0 \) denotes the drift of \( \mu \), that is non-zero since the support of \( \mu \) generates a non-amenable subgroup in \( \Gamma \). Moreover following Remark [2.11] the 1st moment assumption implies that \( \frac{F_{n,k}}{n}, \frac{F'_{n,k}}{n} \rightarrow \ell \) pointwise and in \( L^1(\Omega, \mathbb{P}) \), when \( k \) is fixed and \( n \) goes to infinity.

On the other hand since \( (\varphi_2(X^{-1}_n.o))_n \) defines a bounded sequence in \( L^\infty(\Omega, \mathbb{P}) = L^1(\Omega, \mathbb{P})^* \) one can assume that \( \varphi_2(X^{-1}_n.o) \) converges to some function \( \varphi_\infty \in L^\infty(\Omega, \mathbb{P}) \) for the weak-* topology \( \sigma(L^\infty, L^1) \).

We are going to prove the following inequality:

\[ I_+ \leq \int_\Omega \varphi_1(Z \circ U^k(\omega))\varphi_\infty(\omega)\mathbb{P}(\omega) \]

Let us fixe \( k \in \mathbb{N} \) and \( \varepsilon > 0 \). Using Egoroff theorem one can find \( B \subset \Omega \) with \( \mathbb{P}(B) \leq \varepsilon \) such that \((\frac{F_{n,k}}{n})_n\) and \((\frac{F'_{n,k}}{n})_n\) converge uniformly to \( \ell \) on \( A = \Omega \setminus B \). Because \( \hat{\varphi}_i, i = 1, 2, \) are uniformly continuous on the compact \( \bar{X} \) there exists \( \eta \leq \varepsilon \) such that for all \( x, y \in \bar{X} \) with \( \theta_{a,\varepsilon}(x, y) \leq \eta, |\hat{\varphi}_i(x) - \hat{\varphi}_i(y)| \leq \varepsilon \).

Take \( n_0 \) large enough such that \( e^{-\frac{1}{2}\ell n_0} \leq \eta \) and for all \( n \geq n_0, \)

\[ |\frac{F_{n,k}(\omega)}{n} - \ell| + |\frac{F'_{n,k}(\omega)}{n} - \ell| \leq \frac{1}{2} \ell \]

(1)
for all $\omega \in \Omega \setminus B$,
\[
|E_\mu[\varphi_1(Z \circ U^k)\hat{\varphi}_2(X_{n-1},o)] - \int_\Omega \varphi_1(Z \circ U^k(\omega))\varphi_{-\infty}(\omega)\mathbb{P}(\omega)| \leq \varepsilon
\]
and
\[
\int_\Omega |\varphi_1(Z(\omega)) - \hat{\varphi}_1(X_n(\omega),o)|\mathbb{P}(\omega) \leq \varepsilon.
\]
In particular Equation (1) implies:
\[
\theta_{o,\varepsilon}(X_n(\omega),o, X_n \circ \Theta^k(\omega),o), \theta_{o,\varepsilon}(X_n^{-1} \circ U^k(\omega),o, X_n^{-1} \circ \Theta^k(\omega),o) \leq \eta
\]
for $n \geq n_0$ and $\omega \in \Omega \setminus B$.

For $n \geq n_0$ one has:
\[
I_+ - \varepsilon \leq E_\mu[\hat{\varphi}_1(X_{n+k},o)\hat{\varphi}_2(X_{n+k},o)] = E_\mu[\hat{\varphi}_1(X_{n-1} \circ \Theta^k(\omega),o)\hat{\varphi}_2(X_n \circ \Theta^k(\omega),o)]
\]
Observe that:
\[
\int_\Omega \hat{\varphi}_1(X_{n-1} \circ \Theta^k(\omega),o)\hat{\varphi}_2(X_n \circ \Theta^k(\omega),o)\mathbb{P}(\omega) = \int_\Omega \hat{\varphi}_1(X_{n-1} \circ U^k(\omega),o)\hat{\varphi}_2(X_n(\omega),o)\mathbb{P}(\omega)
\]
\[
+ \int_{\Omega \setminus \Omega, B \cup B} (\hat{\varphi}_1(X_{n-1} \circ \Theta^k(\omega),o) - \hat{\varphi}_1(X_{n-1} \circ U^k(\omega),o))\hat{\varphi}_2(X_n \circ \Theta^k(\omega),o)\mathbb{P}(\omega)
\]
\[
+ \int_{\Omega \setminus \Omega, B \cup B} \hat{\varphi}_1(X_{n-1} \circ U^k(\omega),o)(\hat{\varphi}_2(X_n \circ \Theta^k(\omega),o) - \hat{\varphi}_2(X_n(\omega),o))\mathbb{P}(\omega)
\]
\[
\leq \int_\Omega \hat{\varphi}_1(X_{n-1} \circ U^k(\omega),o)\hat{\varphi}_2(X_n(\omega),o)\mathbb{P}(\omega) + 4\mathbb{P}(B)\|\varphi_1\|_\infty \|\varphi_2\|_\infty + 2\varepsilon \max_{i=1,2} \|\varphi_i\|_\infty
\]
In other words:
\[
I_+ + O(\varepsilon) \leq \int_\Omega \hat{\varphi}_1(X_n \circ U^k(\omega),o)\hat{\varphi}_2(X_{n-1}(\omega),o)\mathbb{P}(\omega)
\]
On the other hand, using the inequality $\|\varphi_{-\infty}\|_\infty \leq \|\varphi_2\|_\infty$ one obtain:
\[
|\int_\Omega \hat{\varphi}_1(X_n \circ U^k(\omega),o)\hat{\varphi}_2(X_{n-1}(\omega),o)\mathbb{P}(\omega) - \int_\Omega \varphi_1(Z \circ U^k(\omega))\varphi_{-\infty}(\omega)\mathbb{P}(\omega)|
\]
\[
\leq |\int_\Omega \hat{\varphi}_1(X_n \circ U^k(\omega),o)[\hat{\varphi}_2(X_{n-1}(\omega),o) - \varphi_{-\infty}(\omega)]\mathbb{P}(\omega)|
\]
\[
+ |\int_\Omega [\varphi_1(Z \circ U^k(\omega)) - \hat{\varphi}_1(X_n \circ U^k(\omega),o)]\varphi_{-\infty}(\omega)\mathbb{P}(\omega)|
\]
\[
\leq |\int_\Omega \hat{\varphi}_1(X_n \circ U^k(\omega),o)[\hat{\varphi}_2(X_{n-1}(\omega),o) - \varphi_{-\infty}(\omega)]\mathbb{P}(\omega)|
\]
\[
+ \|\varphi_2\|_\infty \int_\Omega |\varphi_1(Z(\omega)) - \hat{\varphi}_1(X_n(\omega),o)|\mathbb{P}(\omega) \leq \varepsilon(1 + \|\varphi_2\|_\infty)\]
and since \( \varepsilon \) can be chosen arbitrary small it follows that:

\[
I_+ \leq \int_\Omega \varphi_1(Z \circ U^k(\omega))\varphi_{-\infty}(\omega)\mathbb{P}(\omega)
\]

for all \( k \in \mathbb{N} \).

Eventually observe that the weak convergence of \((\varphi_2(X_n^{-1}.o))_n\) to \( \varphi_{-\infty} \) implies:

\[
\mathbb{E}^\mu(\varphi_2(X_n^{-1}.o)) \to \int_\Omega \varphi_{-\infty}(\omega)\mathbb{P}(\omega)
\]

and the symmetry of \( \mu \) gives:

\[
\mathbb{E}^\mu(\varphi_2(X_n^{-1}.o)) = \mathbb{E}^\mu(\varphi_2(X_n.o)) \to \int_{\partial X} \varphi_2 d\nu
\]

This imposes:

\[
\int_\Omega \varphi_{-\infty}(\omega)\mathbb{P}(\omega) = \int_{\partial X} \varphi_2 d\nu
\]

Since the Bernouilli shift \( U \) is mixing one deduce:

\[
I_+ \leq \int_\Omega \varphi_1(Z \circ U^k(\omega))\varphi_{-\infty}(\omega)\mathbb{P}(\omega) \xrightarrow{k \to +\infty} \int_{\partial X} \varphi_1 d\nu \int_\Omega \varphi_{-\infty}(\omega)\mathbb{P}(\omega)
\]

\[
= \int_{\partial X} \varphi_1 d\nu \int_{\partial X} \varphi_2 d\nu
\]

\[\square\]

4. A Probabilistic Approach of the Irreducibility

Let \( \nu \) be a Ahlfors-regular quasiconformal measure on \( \partial X \) for the action of \( \Gamma \) on \((X,d)\). The principal example in our framework are stationary measures on \( \partial X \) for finitely supported symmetric admissible probabilities on \( \Gamma \). The associated boundary representation of \( \Gamma \) is defined as:

\[
\pi_o : \Gamma \to \mathcal{U}[L^2[\partial X,\nu]], \quad \pi_o(g)\varphi(\xi) = \sqrt{r_o(g^{-1},\xi)}\varphi(g^{-1}\xi)
\]

where \( r_o \) denotes the Radon-Nikodym derivative cocycle given by the formula:

\[
r_{\nu}(g,\xi) = \frac{dg \ast \nu(\xi)}{d\nu(\xi)} = e^{\varepsilon D_{\partial X}(g^{-1}o)}
\]

with \( g \in \Gamma, \xi \in \partial X \).

The Harish-Chandra function on \( \Gamma \) is defined as the matrix coefficient given by:

\[
\Xi_o(g) = (\pi_o(g)1_{\partial \Gamma}|1_{\partial \Gamma}) = \|\sqrt{r_o(g^{-1},\cdot)}\|_{L^1(\partial X,\nu)} = \Xi_o(g^{-1})
\]
for $g \in \Gamma$.

In the rest one denote:

$$\tilde{\pi}_o(g) = \frac{\pi_o(g)}{\Xi_o(g)}$$

the renormalization of the representation $\pi_o$ by the Harish-Chandra function $\Xi_o$.

Let $P_{1,\partial X}$ be the one dimensional projector on $L^2[\partial X, \nu]$ given by:

$$P_{1,\partial X}(\varphi) = (\varphi|1_{\partial X})1_{\partial X}$$

for $\varphi \in L^2[\partial X, \nu]$.

The principal result of this section is:

**Theorem 2.** Let $\mu \in \text{Prob}(\Gamma)$ be a finitely supported, symmetric probability on $\Gamma$ and $\nu$ its associated stationary measure on $\partial X$.

Denote $P_n \in \mathcal{B}(L^2[\partial X, \nu])$ the operator defined as:

$$P_n = E^\mu[\tilde{\pi}_o(X_n)] = \sum_{g \in \Gamma} \frac{\pi_o(g)}{\Xi_o(g)} \mu^n(g).$$

Then $(P_n)_n$ defines an uniformly bounded sequence of operators which converges to the one dimensional projector $P_{1,\partial X}$ for the weak-* operator topology, in other words:

$$\sum_{g \in \Gamma} \frac{(\pi_o(g)\varphi|\psi)}{\Xi_o(g)} \mu^n(g) \xrightarrow{n \to +\infty} \int_{\partial X} \varphi d\nu \int_{\partial X} \psi d\nu$$

for all $\varphi, \psi \in L^2[\partial X, \nu]$.

Before we go to the proof of Theorem 2 let us give a direct consequence:

**Corollary 4.1.** Let $\mu \in \text{Prob}(\Gamma)$ be a finitely supported, symmetric probability on $\Gamma$ and $\nu$ its associated stationary measure on $\partial X$.

Then the associated boundary representation $(\Gamma, \pi_o, L^2[\partial X, \nu])$ is irreducible.

**Proof.** We start by proving that the vector $1_{\partial X}$ is cyclic for $(\Gamma, \pi_o, L^2[\partial X, \nu])$.

Indeed assume $\psi \in L^2[\partial X, \nu] \cap \text{span}\{\pi(g)1_{\partial X} | g \in \Gamma\}$, then Proposition 4.5 implies:

$$E^\mu[\tilde{w}(X_n)(\tilde{\pi}_o(X_n)1_{\partial X}|\psi)_{L^2}]$$

$$= \sum_{g \in \Gamma} \tilde{w}(g)(\tilde{\pi}_o(g)1_{\partial X}|\psi)_{L^2} \xrightarrow{n \to +\infty} \int_{\partial X} \psi(\xi)w(\xi) d\nu(\xi)$$

for all $w \in \text{Lip}(\partial X, d_{o,\cdot})$. Using the density of Lipschitz functions together with the fact that $E^\mu[\tilde{w}(X_n)(\tilde{\pi}_o(X_n)1_{\partial X}|\psi)_{L^2}] = 0$ for all $n$ it follows that $\psi = 0$.  

Let $V_0$ be a non-trivial subrepresentation of $(\Gamma, \pi, L^2[\partial X, \nu])$. Then there exists $\psi \in V_0$ such that $(1, \pi|\psi_0)_{L^2} \neq 0$. Otherwise using the invariance of this subrepresentation we would have that $V_0$ is orthogonal to $\text{span}\{\pi(g)1_{\partial X} | g \in \Gamma\} = L^2[\partial X, \nu]$.

Finally since $P_{1_{\partial X}}$ belongs to the von Neumann algebra of $\pi_0$ it follows that $P_{1_{\partial X}} \in \nu_{\partial X}$.

□

4.0.1. Proximal phenomenon in $L^2$ and boundary representations.

A fundamental estimate of the Harish-Chandra is given by the following lemma:

**Lemma 4.2** ([29] [20]). Let $\Xi_0$ be the Harish-Chandra function on $\Gamma$ associated to its geometric action on $(X, d)$. Then for all $g \in \Gamma$:

$$\Xi_0(g) = (1 + |g.o|_o) e^{-\frac{1}{2}D_\varepsilon |g.o|_o},$$

where $D_\varepsilon$ is the Hausdorff dimension of $\nu$.

As explained in the following lemma the sequence of absolutely continuous probabilities with respect to $\nu$ given by:

$$u_g d\nu = \frac{\sqrt{r_o(g^{-1})}}{\Xi_0(g)} d\nu = \frac{e^{\varepsilon D_\varepsilon (g.o|_o)}}{(1 + |g.o|_o)} d\nu$$

on $(\partial X, \nu)$ gives an approximation of the Dirac mass in the sense:

$$(\varphi|u_g)_{L^2} = \int \varphi(\xi) \frac{e^{\varepsilon D_\varepsilon (g.o|_o)}}{(1 + |g.o|_o)} d\nu(\xi) \to \varphi(\xi_0)$$

when $g$ goes to $\xi_0$. Moreover this convergence is controlled by the length of $g$:

**Lemma 4.3.** [20] Let $\Psi \in \text{Lip}(\partial X \times \partial X, d)$ be a Lipschitz function on $\partial X \times \partial X$ for the $\ell^1$-product distance: $d((\xi, \eta), (\xi', \eta')) = d_{o,e}(\xi, \xi') + d_{o,e}(\eta, \eta')$ for $\xi, \xi', \eta, \eta' \in \partial X$. Assume $\varepsilon > 0$ small enough such that the Hausdorff dimension, $D_\varepsilon$, of $(\partial X, d_{o,e})$ is strictly greater than 1. Then:

$$|(\Psi|u_g \otimes u_g^{-1})_{L^2} - \Psi(\tilde{g}, \tilde{g})| \leq \frac{2\lambda(\Psi)}{(1 + |g.o|_o)^{1/D_\varepsilon}}$$

for all $g \in \Gamma$, where $\lambda(\Psi)$ is the Lipschitz constant of $\Psi$ and $\tilde{g} = g^{-1}.o$.

**Proof.** Using the fact:

$$\beta(o, g.o; \xi) = 2(\xi|g.o)_o - |g.o|_o \leq 2(\xi|\tilde{g}.o)_o - |g.o|_o$$
for all \( g \in \Gamma, \xi \in \partial X \) together with Lemma 4.2 we deduce for any \( 0 < r < \text{Diam}(\partial X) \):

\[
|(|\Psi| u_g \otimes u_{g^{-1}})_{L^2} - \Psi(\hat{g}, \check{g})| = |(|\Psi - \Psi(\hat{g}, \check{g})| u_g \otimes u_{g^{-1}})_{L^2}|
\]

\[
\leq \lambda(\Psi) \int_{\partial X} [d_{o, \varepsilon}(\hat{g}, \xi) + d_{o, \varepsilon}(\check{g}, \eta)] u_g(\xi) u_{g^{-1}}(\eta) d\nu(\xi) d\nu(\eta)
\]

\[
\leq \lambda(\Psi) [2r + \int_{B_{\varepsilon}(\hat{g}; r)} \frac{d_{o, \varepsilon}(\hat{g}, \xi)^{1-D_\varepsilon}}{1 + |g.o|_o} d\nu(\xi) + \int_{B_{\varepsilon}(\check{g}; r)} \frac{d_{o, \varepsilon}(\check{g}, \eta)^{1-D_\varepsilon}}{1 + |g.o|_o} d\nu(\eta)]
\]

\[
\leq 2\lambda(\Psi) [r + \frac{r^{1-D_\varepsilon}}{1 + |g.o|_o}]
\]

where the second inequality is obtained by the decomposition \( \partial X = B_{\varepsilon}(\hat{g}, \check{g}); r) \cup B_{\varepsilon}(\hat{g}, \check{g}); r) \). The lemma follows by taking \( r = (1 + |g.o|_o)^{-1/D_\varepsilon} \). \( \square \)

**Remark 4.4.** Since \( \nu \) is Ahlfors-regular and the Lebesgue differentiation theorem guarantees the density of Lipschitz functions in \( L^2[\partial X, \nu] \).

**Proposition 4.5.** Given \( w, \varphi, \psi \in \text{Lip}(\partial X, d_{o, \varepsilon}) \), three Lipschitz functions on \( \partial X \), the following holds:

\[
\mathbb{E}^\mu [w(\tilde{X}_n,o)(\tilde{\pi}_o(X_n)\varphi|\psi) - w(\tilde{X}_n,o)\varphi(\tilde{X}_n^{-1},o)\psi(\tilde{X}_n,o)] \xrightarrow{n \to +\infty} 0
\]

This Proposition 4.5 together with Theorem 1 implies:

**Corollary 4.6.** Given any Lipschitz functions \( w, \varphi, \psi \in \text{Lip}(\partial X, d_{o, \varepsilon}) \) one has:

\[
\mathbb{E}^\mu [w(\tilde{X}_n,o)(\tilde{\pi}_o(X_n)\varphi|\psi)] \to \int_{\partial X} \varphi d\nu, \int_{\partial X} w.\psi d\nu
\]

when \( n \) goes to infinity.

**Proof Proposition 4.5.** Let \( w, \varphi, \psi \in \text{Lip}(\partial X, d_{o, \varepsilon}) \) be three Lipschitz functions.

Take \( 0 < s < 1 \) such that \( \rho_\mu \delta^s_\Gamma < 1 \), where \( \rho_\mu < 1 \) is the spectral radius of \( \mu \) and \( \delta_\Gamma \) the growth rate of \( \Gamma \). Observe that:

\[
\mathbb{P}(|X_n|o \leq sn) = \mu^{*n}(B_X(o, sn)) 
\]

\[
\leq \max\{\mu^{*n}(e), \mu^{*(n-1)}(e)\}|B_X(o, sn) \cap \Gamma.o|
\]

\[
\leq \rho_\mu^{n}\delta^{*n}_\Gamma = (\rho_\mu\delta^s_\Gamma)^n
\]
for all \( n \).
On the other hand:

\[
\begin{align*}
\sum_{g: |g|_o \geq sn} |w(\widehat{g}.o)(\pi_o(g)\varphi|\psi) - w(\widehat{g}.o)\varphi(\overline{g^{-1}}.o).\psi(\widehat{g}.o)|\mu^{*n}(g) \\
&= \sum_{g: |g|_o \geq sn} |w(\widehat{g}.o)(\pi_o(g)\varphi|\psi - \psi(\widehat{g}.o)1_{\partial X}) \\
&- w(\widehat{g}.o)(\varphi(\overline{g^{-1}}.o)1_{\partial X} - \varphi|\pi(\overline{g^{-1}}).\psi(\widehat{g}.o)|\mu^{*n}(g) \\
&\leq \|w\|_\infty \|\varphi\|_\infty \sum_{g: |g|_o \geq sn} |(\psi - \psi(\widehat{g}.o)1_{\partial X}|u_g)|\mu^{*n}(g) \\
&+ \|w\|_\infty \|\psi\|_\infty \sum_{g: |g|_o \geq sn} |(\varphi(\overline{g^{-1}}.o)1_{\partial X} - \varphi|u_{g^{-1}})|\mu^{*n}(g) \\
&\leq \|w\|_\infty \|\varphi\|_\infty \lambda(\psi) + \|\psi\|_\infty \lambda(\varphi) \\
&\quad \quad (1 + sn)^{1/D}.
\end{align*}
\]

The last inequality is a consequence of Lemma \[4.3\] It follows that:

\[
\begin{align*}
\mathbb{P}^\mu\left[w(\widehat{X}_n.o)(\pi_o(X_n)\varphi|\psi) - w(\widehat{X}_n.o)\varphi(\overline{X^{-1}}.o).\psi(\widehat{X}_n.o)\right] \\
= \sum_{g \in \Gamma} |w(\widehat{g}.o)(\pi_o(g)\varphi|\psi) - w(\widehat{g}.o)\varphi(\overline{g^{-1}}.o).\psi(\widehat{g}.o)|\mu^{*n}(g) \\
&\leq \sum_{g: |g|_o \geq sn} |w(\widehat{g}.o)(\pi_o(g)\varphi|\psi) - w(\widehat{g}.o)\varphi(\overline{g^{-1}}.o).\psi(\widehat{g}.o)|\mu^{*n}(g) \\
&+ \mathbb{P}(|X_n|_o \leq sn). \|w(\widehat{g}.o)(\pi_o(g)\varphi|\psi) - w(\widehat{g}.o)\varphi(\overline{g^{-1}}.o).\psi(\widehat{g}.o)\|_\infty(\Gamma) \\
&\leq \|w\|_\infty \|\varphi\|_\infty \lambda(\psi) + \|\psi\|_\infty \lambda(\varphi) \\
&\quad \quad (1 + sn)^{1/D} + (\rho \delta_\Gamma^2)^n \|w\|_\infty \|\varphi\|_\infty \|\psi\|_\infty,
\end{align*}
\]

which converges to 0 when \( n \) goes to infinity.

\[\square\]

The last ingredient of Theorem \[2\] proof and thus Corollary \[4.1\] is the uniform boundedness of the family of operators \( \{P_n \mid n \geq 0\} \) introduced in its statement. This is subject of the next subsection. Assuming this fact let us conclude the proof of Theorem \[2\]

**Proof of Theorem \[2\]**. Since \( (P_n)_n \) are uniformly bounded it is enough to prove Theorem \[2\] for \( \varphi, \psi \in \text{Lip}(\partial X, d_{o,\varepsilon}) \) which span a dense subspace in \( L^2[\partial X, \nu] \).
Observe that:
\[
|\langle \mathcal{P}_n \varphi, \psi \rangle| - \int \varphi \, d\nu \int \psi \, d\nu = |\mathbb{E}^{\mu}[\langle \tilde{\pi}_o(X_n) \varphi, \psi \rangle]| - \int_{\partial X} \varphi \, d\nu \int_{\partial X} \psi \, d\nu
\]
\[
\leq |\mathbb{E}^{\mu}[\langle \tilde{\pi}_o(X_n) \varphi, \psi \rangle]| - \mathbb{E}^{\mu}[\langle \tilde{\varphi}(X_n, o), \psi(X_n^{-1}, o) \rangle] + |\mathbb{E}^{\mu}[\langle \tilde{\varphi}(X_n, o), \psi(X_n^{-1}, o) \rangle] - \int_{\partial X} \varphi \, d\nu \int_{\partial X} \psi \, d\nu|
\]

On one hand Theorem 1 guarantees that:
\[
\mathbb{E}^{\mu}[\langle \tilde{\varphi}(X_n, o), \psi(X_n^{-1}, o) \rangle] \xrightarrow{n \to \infty} \int_{\partial X} \varphi \, d\nu \int_{\partial X} \psi \, d\nu.
\]

On the other hand Proposition 4.5 guarantees:
\[
\mathbb{E}^{\mu}[\langle \tilde{\pi}_o(X_n) \varphi, \psi \rangle] \xrightarrow{n \to \infty} \mathbb{E}^{\mu}[\langle \tilde{\varphi}(X_n, o), \psi(X_n^{-1}, o) \rangle]
\]
This concludes the proof of Theorem 2. \qed

4.0.2. Uniform boundedness of the average sequence \((\mathcal{P}_n)_n\). We start this section with several technical preliminary results and conclude the uniform boundedness in Proposition 4.10 and Corollary 4.11.

**Lemma 4.7.** Let \((\partial X, d_{o, \varepsilon}, \nu)\) be the Gromov boundary of \((X, d)\) together with \(\nu\) a \(D_\varepsilon\)-Ahlfors regular probability measure. Then
\[
\int_{\{\eta \mid (\eta, \xi) \leq k\}} d_{o, \varepsilon}^{-D_\varepsilon}(\xi, \eta) \, d\nu(\eta) = 1 + \varepsilon D_\varepsilon k
\]
for any \(k \geq 0\).

**Proof.** Using the \(D_\varepsilon\)-Ahlfors-regularity of \(\nu\) observe that:
\[
\int_{\{\eta \mid (\eta, \xi) \leq k\}} d_{o, \varepsilon}^{-D_\varepsilon}(\xi, \eta) \, d\nu(\eta)
\]
\[
= 1 + \varepsilon D_\varepsilon \int_0^k e^{\varepsilon D_\varepsilon t} \nu_0\{\{\xi, \cdot\} > t\} \, dt
\]
\[
= 1 + \varepsilon D_\varepsilon \int_0^k e^{\varepsilon D_\varepsilon t} \nu_0\{\{d_{o, \varepsilon}(\xi, \cdot) < e^{-\varepsilon t}\} \, dt
\]
\[
= 1 + \varepsilon D_\varepsilon \int_0^k e^{\varepsilon D_\varepsilon t} e^{-e\varepsilon D_\varepsilon t} \, dt = 1 + \varepsilon D_\varepsilon k.
\]
\qed

**Lemma 4.8.** Let \(C \geq \delta + 2\) be a positive constant. There exists \(C' > 0\), for any pair of distinct points, \(\eta_0 \neq \eta'_0\), in the boundary...
$\partial X$ of $X$ with $C \geq (\eta_0|\eta'_0)_o + \delta + 1$ the two neighborhoods of $\eta_0$ and $\eta'_0$ given respectively by:

$$U(\eta_0, C) = U = \{ x \in X | (x|\eta_0)_o \geq C \}$$

and

$$U(\eta'_0, C) = U' = \{ x \in X | (x|\eta'_0)_o \geq C \}$$

satisfy $U \cap U' = \emptyset$ and for any $\eta \in \partial X \cap U$, $\eta' \in \partial X \cap U'$ the following inequality holds:

$$(g.o|\xi)_o \leq \max\{(g.\eta|\xi)_o, (g.\eta'|\xi)_o\} + C'$$

for all $g \in \Gamma$ and $\xi \in \partial X$.

**Proof.** Assume there exists $x \in U \cap U'$. Then one has:

$$(\eta_0|\eta'_0)_o \geq \min\{(x|\eta_0)_o, (x|\eta'_0)_o\} - \delta \geq C - \delta \geq (\eta_0|\eta'_0)_o + 1$$

which is a contradiction, thus $U \cap U' = \emptyset$ and in particular $\overline{X} = U^c \cup U'^c$.

Let $g \in \Gamma$ and assume $g^{-1}.o \in U'^c$. One might also assume that $(g.o|\xi)_o \geq (g.\eta|g.o)_o$, otherwise the inequality:

$$(g.\eta|\xi)_o \geq \min\{(g.\eta|g.o)_o, (g.o|\xi)_o\} - \delta = (g.o|\xi)_o - \delta$$

guarantees the second part of the lemma with $C' = \delta$.

Since $d(o,g^{-1}.o) - (o|\eta)_{g^{-1}.o} \approx (g^{-1}.o|\eta)_o$ and $g^{-1}.o \in U'^c$ which implies that $(g^{-1}.o|\eta)_o \leq C$, one has:

$$(g.o|\xi)_o - C \leq |g.o|_o - C \leq d(o,g^{-1}.o) - (g^{-1}.o|\eta)_o$$

$$\approx (o|\eta)_{g^{-1}.o} = (g.o|g.\eta)_o \leq (g.\eta|\xi)_o + \delta$$

where the last inequality is a consequence of the hyperbolic inequality:

$$(g.\eta|\xi)_o \geq \min\{(g.o|g.\eta)_o, (g.o|\xi)_o\} - \delta$$

together with our assumptions. Eventually the lemma follows by taking $C + \delta \leq C'$ where the extra additive constant in the choice of $C'$ only depend on the geometry on $(X, d)$. $\square$

**Corollary 4.9.** Let $C \geq \delta + 2$ be a fixed constant. Following Lemma 4.8 notations, let $C'$ and $\eta \neq \eta'$ two distinct boundary points that belong respectively to $U$ and $U'$. 
There exists $\lambda > 0$ which only depends on $C$ such that:

\[
\mathbb{E}^{\mu}[\tilde{\pi}_o(X_n)]1_{\partial X}(\xi) = \sum_{g \in \Gamma} \frac{e^{\varepsilon D_c(g.o|\xi)_o}}{1 + |g.o|_o} \mu_n^*(g)
\]

\[
\leq \lambda \sum_{g \in \Gamma} [f(g\eta, \xi) + f(g\eta', \xi)] \mu_n^*(g)
\]

for all $\xi \in \partial X$ and $n \geq 0$ with

\[
f : \Gamma \times \partial X \times \partial X \to \mathbb{R}_+; \quad (g, \eta, \xi) \mapsto \frac{d^{-D_c}(\eta, \xi)}{1 + |g.o|_o}
\]

Proof. Using Lemma 4.8 one has:

\[
e^{\varepsilon D_c(g.o|\xi)_o} \leq e^{\varepsilon D_c[\max\{(g.\eta|\xi)_o, (g.\eta'|\xi)_o\} + C']}
\]

\[
\leq e^{\varepsilon D_c C'} \left[ e^{\varepsilon D_c(g.\eta|\xi)} + e^{\varepsilon D_c(g.\eta'|\xi)} \right]
\]

\[
= e^{\varepsilon D_c C'} \left[ d^{-D_c}(g.\eta, \xi) + d^{-D_c}(g.\eta', \xi) \right]
\]

for all $g \in \Gamma$. It follows that:

\[
e^{\varepsilon D_c(g.o|\xi)_o} \leq e^{\varepsilon D_c C'} \left[ \frac{d^{-D_c}(g.\eta, \xi)}{1 + |g.o|_o} + \frac{d^{-D_c}(g.\eta', \xi)}{1 + |g.o|_o} \right]
\]

Averaging with respect to $\mu_n^*$ proves the corollary with $\lambda = e^{2\delta_\Gamma C'}$. \qed

We can prove the principal ingredient of the uniform boundedness of the sequence of operators $(P_n)_n$:

**Proposition 4.10.** The sequence of functions $(\mathbb{E}^{\mu}[\tilde{\pi}_o(X_n)]1_{\partial X})_n$ is uniformly bounded in $L^\infty(\partial X, \nu)$.

**Proof.** Let $\lambda > \delta_\Gamma$ and $\frac{\ln E}{[\lambda - a]} \leq a$ where $\delta_\Gamma = \varepsilon D_\varepsilon$ denote the critical exponent of $\Gamma$ and $E = \mathbb{E}^{\mu}[e^{\lambda|X_1|_o}]$, that is finite since $\mu$ is finitely supported.

Using exponential Chebyshev inequality:

\[
e^{\lambda a(n+k)} \mathbb{P}(|X_n|_o \geq a(n+k)) \leq \mathbb{E}[e^{\lambda|X_1|_o}]
\]

\[
\leq \mathbb{E}[e^{\lambda|X_1|_o}]^n = E^n
\]

and therefore:

\[
e^{\delta a(n+k)} \mathbb{P}(|X_n|_o \geq a(n+k)) \leq e^{\delta a(n+k)} e^{-\lambda a(n+k)} E^n
\]

\[
= e^{[\delta - \lambda]ak + [\ln E + a\delta - a\lambda]n}
\]
It follows

$$\mathbb{E}[\tilde{\nu}_o(X_n)1_{|X_n| \geq an}]1_{\delta X}(\xi) \leq \sum_{k=0}^{+\infty} \sum_{g \in \Gamma, a(n+k) \leq \left|g\right|_o \leq a(n+k+1)} \mu^{sn}(g) e^{\varepsilon D_c(g,o)\xi} / (1 + \left|g\right|_o)$$

$$\leq \sum_{k=0}^{+\infty} \mathbb{P}(|X_n| \geq a(n + k)) e^{\varepsilon D_c a(n+k+1)}$$

$$\leq e^{[\ln E + a\varepsilon D_c - a\lambda]n} \sum_k e^{[\delta - \lambda]ak} = e^{[\ln E + a\varepsilon D_c - a\lambda]n}$$

On the other hand:

$$\mathbb{E}[\tilde{\nu}_o(X_n)1_{|X_n| \leq a'n}]1_{\delta X}(\xi) = \sum_{g \in \Gamma, |g|_o \leq a'n} \mu^{sn}(g) e^{\varepsilon D_c(g,o)\xi} / (1 + |g|_o)$$

$$\leq \mu^{sn}(e) \sum_{g \in \Gamma, |g|_o \leq a'n} e^{\varepsilon D_c(g,o)\xi} / (1 + |g|_o)$$

$$= \rho^n e^{\delta a'n} = e^{[\ln \rho \varepsilon D_c - a\lambda]n}$$

with $-\ln \rho / \varepsilon D_c \leq a'$. Let us denote:

$$f_k : \partial X \times \partial X \rightarrow \mathbb{R}^+; \quad (\eta, \xi) \mapsto \begin{cases} d^{\varepsilon D_c(\eta,\xi)} / [1 + \exp(\delta_k)] & \text{for } (\xi|\eta)_o \leq k \\ 1 & \text{otherwise} \end{cases}$$

for $k \geq 1$. Note that Lemma 4.7 implies that $\|f_k(\cdot,\xi)\|_1 \leq 1$ for all $\xi \in \partial X$ and $k \geq 1$.

Take $\eta, \eta' \in \partial X$ and $U, U' \subset \partial X$ as in Lemma 4.8 and observe that:

$$\mathbb{E}[\tilde{\nu}_o(X_n)1_{a'n \leq |X_n| \leq an}]1_{\delta X}(\xi) = \sum_{g \in \Gamma, a'n \leq |g|_o \leq an} \mu^{sn}(g) e^{\varepsilon D_c(g,o)\xi} / (1 + |g|_o)$$

$$\leq 1 / [1 + a'n] \sum_{g \in \Gamma, a'n \leq |g|_o \leq an} \mu^{sn}(g) f_n(g\eta,\xi)$$

$$+ 1 / [1 + a'n] \sum_{g \in \Gamma, a'n \leq |g|_o \leq an} \mu^{sn}(g) f_n(g\eta',\xi)$$
\[
\leq \frac{1}{(1 + a'n)\nu(U)} \sum_{g \in \Gamma} \mu^{**}(g) \int_{\partial X} f_n(\eta, \xi) d\eta
\]
\[
+ \frac{1}{(1 + a'n)\nu(U')} \sum_{g \in \Gamma} \mu^{**}(g) \int_{\partial X} f_n(\eta', \xi) d\eta'
\]
\[
\leq 2 \frac{\|f_n(\cdot, \xi)\|_1}{(1 + a'n) \min\{\nu(U), \nu(U')\}}
\]
\[
= 2 \frac{1 + an}{(1 + a'n) \min\{\nu(U), \nu(U')\}} = 2 \frac{1 + a}{a' \min\{\nu(U), \nu(U')\}}
\]

The proposition follows from the uniform bounds on each of these quantities.

\[\square\]

**Corollary 4.11.** Let \( \varphi, \psi \in L^2[\partial X, \nu] \) be two square integrable functions on \((\partial X, \nu)\). Then one has:
\[
|\langle \mathbb{E}^\mu[\tilde{\pi}_o(X_n)], \varphi|\psi \rangle| \leq \|\varphi\| \|\psi\|_2
\]
uniformly on \(n\). In particular the sequence of operator averages \( (\mathcal{P}_n) \) is uniformly bounded.

**Proof.** Take \( \varphi, \psi \in L^4(\partial X, \nu) \) and observe that:
\[
|\langle \mathbb{E}^\mu[\tilde{\pi}_o(X_n)], \varphi|\psi \rangle|^2 = \left| \int \varphi(\gamma^{-1} \xi) \psi(\xi) u_g(\xi) d\xi \mu^{**}(g) \right|^2
\]
\[
\leq \int |\varphi(\gamma^{-1} \xi)|^2 u_g(\xi) d\xi \mu^{**}(g) \int |\psi(\xi)|^2 u_g(\xi) d\xi \mu^{**}(g)
\]
\[
= \langle \mathbb{E}^\mu[\tilde{\pi}_o(X_n)]1_{\partial X}, |\varphi|^2 \rangle \langle |\varphi|^2 |\mathbb{E}^\mu[\tilde{\pi}_o(X_n)]1_{\partial X} \rangle^2
\]
\[
\leq \|\mathbb{E}^\mu[\tilde{\pi}_o(X_n)]1_{\partial X}\|_\mathcal{L}_2^\infty \|\varphi\|_2^2 \|\psi\|_2^2
\]
The corollary follows by density of \( L^4(\partial X, \nu) \) inside of \( L^2(\partial X, \nu) \).

\[\square\]

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