Good measures for non-simple dimension groups

Abstract Akin’s notion of good measure, introduced to classify measures on Cantor sets has been translated to dimension groups and corresponding traces by Bezuglyi and the author, but emphasizing the simple (minimal dynamical system) case. Here we deal with non-simple (non-minimal) dimension groups. In particular, goodness of tensor products of large classes of non-good traces (measures) is established. We also determine the pure faithful traces on the dimension groups associated to xerox type actions on AF C*-algebras; the criteria turn out to involve algebraic geometry and number theory.

We also deal with coproducts of dimension groups, wherein, despite expectations, goodness of direct sums is nontrivial. In addition, we verify a conjecture of [BeH] concerning good subsets of Choquet simplices, in the finite-dimensional case.

David Handelman

Introduction & definitions

Akin [Ak1, Ak2, ADMY, ... ] introduced and studied the notion of good measures, in connection with the classification of (probability) measures on Cantor sets up to homeomorphism. With the development in [Pu, HPS, GPS, etc], of classification and construction of minimal actions with respect to strong orbit and orbit equivalence via Vershik maps and ordered Grothendieck groups of AF C*-algebras, this and related properties were translated into the language of (traces on) dimension groups (a class of partially ordered abelian groups) in [BeH]. In particular, the characterizations therein of goodness of traces on simple dimension groups provided relatively easy constructions of good and non-good measures on minimal systems. For more details, see the discussion in the introduction to [BeH].

Recent work (e.g., [FO, P]) has extended Vershik action(s) to non-minimal systems, and correspondingly to non-simple dimension groups. Here we give computable criteria for goodness in the general (approximately divisible) case, and then use the criteria to give a surprising result that tensor products of (some) non-good traces are good; this applies to the ugly traces of [BeH]. We also completely determine the pure faithful traces on fixed point algebras under xerox actions of tori: these include Pascal’s triangle and variations corresponding to spatially and temporally homogeneous random walks with finite support on the lattice $\mathbb{Z}^d$.

From [H1, Theorem III.3], the pure faithful traces correspond to points $r = (r_i)$ in the strictly positive orthant of $\mathbb{R}^d$; those that are good are precisely the ones that satisfy two number-theoretic conditions, which in the case $d = 1$ reduce to (i) no other algebraic conjugate of $r = r_1$ is positive and (i) if the leading and terminal coefficients of the polynomial implementing the random walk are $a_0$ and $a_k$, then there exists $s$ such that $a_0^s/r$ and $a_k^s/r$ are both algebraic integers.

We also deal with a strict form of direct sum of dimension groups, determining when the corresponding sum of traces is good; there are some surprises here, as the direct sum can be good without either one being good (in fact, we find for each $m$, a collection of simple dimension groups with traces, $(G_i, \tau_i)$ such that for any strict direct sum of $m$ or fewer distinct summands, $\oplus_{i \in S} G_i$, the sum of the traces is not good, but for any direct sum of more than $m$ direct summands, the sum is good.

We then consider good sets of traces. The first problem is the definition; it should be consistent with the current definition in the simple case, and in the singleton case, and we discuss various possibilities; finally, we settle on one. We show that for the class of dimension groups considered above (arising from random walks on $\mathbb{Z}^d$), with any reasonable definition, the notion is surprisingly restrictive, and even order-unit goodness turns out to be sensitive to the Newton polyhedra of the polynomials (unlike the case for single traces).

\footnote{Supported in part by a Discovery grant from NSERC.}
There are two appendices. The first characterizes order unit good traces on simplicial dimension groups, and the resulting characterization suggests that there are no effective for goodness involving order unit goodness when there are discrete traces, in contrast to the approximately divisible situation discussed in the rest of this article. The second appendix verifies, in the finite-dimensional trace space case, a conjecture made in [BeH, section 7] concerning the structure of good subsets relative to a simplex.

**Definitions.** A partially ordered abelian group $G$ with positive cone $G^+$ is *unperforated* if whenever $n$ is a positive integer and $g \in G$, then $ng \in G^+$ entails $g \in G^+$. An order unit for $G$ is an element $u \in G^+$ such that for all $g \in G$, there exists a positive integer $K$ such that $-Ku \leq g \leq Ku$. A trace (formerly, state) is a nonzero positive group homomorphism $\tau : G \to \mathbb{R}$; if $\tau(u) = 1$ and $u$ is an order unit, we say $\tau$ is normalized (with respect to $u$). The trace $\tau$ is *faithful* if $\ker \tau \cap G^+ = \{0\}$ (this is much weaker than being one to one, and corresponds to faithful measure).

When $(G,u)$ is a partially ordered abelian group with order unit, we may form $S(G,u)$, the compact convex set of normalized traces, equipped with the weak (or point-open) topology. We denote by $Aff S(G,u)$ the Banach space of continuous convex-linear (affine) real-valued functions on $S(G,u)$. There is a natural representation $G \to Aff S(G,u)$, given by $g \mapsto \hat{g}$, where $\hat{g}(\tau) = \tau(g)$.

If $(G,u)$ is an unperforated order unit group, we say $G$ is *approximately divisible* if its range in $Aff S(G,u)$ is norm-dense; for dimension groups with order unit, this is equivalent to $\tau(G)$ being dense in $\mathbb{R}$ for all pure traces $\tau$, or equivalently, for all order units $g \in G$, there exist order units $a, b$ of $G$ such that $g = 2a + 3b$ (and there are many other equivalent formulations).

When $I$ is a subgroup (typically an order ideal) of a partially ordered abelian group $G$, we say $I$ has its own order unit $w$ or $w$ is a relative order unit of $I$ if $w \in I$ is an order unit of $I$ with respect to the relative ordering inherited from $G$. This is to emphasize the fact that $w$ is not an order unit for $G$, merely for $I$.

If $G$ is an unperforated ordered abelian group, we say $G$ is *nearly divisible* if for every order ideal $(I,w)$ which has its own order unit, $(I,w)$ is approximately divisible; an equivalent form that does without the order ideals is that for all $g \in G^+$, there exists $a,b \in G^+$ such that $g = 2a + 3b$ and $g \leq ka, kb$ for some positive integer $k$.

For example, if $G = H \otimes U$ where $H$ is a partially ordered unperforated abelian group and $U$ is a noncyclic subgroup of the rationals, $\mathbb{Q}$, then $G$ is nearly divisible, and it is approximately divisible if it has an order unit. We will see plenty of nearly divisible examples that are not of this type in later sections.

A trace on $G$ is *discrete* if its image $\tau(G)$ is a cyclic (that is, discrete) subgroup of $\mathbb{R}$. An alternative characterization of approximately divisible, for dimension groups, is that $(G,u)$ admit no discrete traces; for nearly divisible, the characterization is that no nonzero order ideal with order unit admits a discrete trace.

For general relevant results on partially ordered abelian groups, especially dimension groups, see [G].

An *interval* in a partially ordered group $G$, is a subset of the form $[0,b] := \{ g \in G \mid 0 \leq g \leq b \}$ for some $b \in G^+$.

Following [BeH], and based on Akin’s notion for measures on Cantor sets, a trace $\tau : G \to \mathbb{R}$ is good (as a trace of $G$) if for all $b \in G^+$, $\tau([0,b]) = [0,\tau(b)]$, that is, if $a' \in G$ and $0 \leq \tau(a') \leq \tau(b)$, there exists $a \in [0,b]$ such that $a - a' \in \ker \tau$. If $(G,u)$ is a partially ordered abelian group with order unit, we say $\tau$ is *order unit* good if in the definition of good, we restrict $b$ to be an order unit.

**1 Characterization of goodness**

Order unit goodness is relatively easy to characterize when $(G,u)$ is approximately divisible [BeH, Proposition 1.7]: $\tau$ is order unit good iff the image of $\ker \tau$ in $Aff S(G,u)$ is dense in $\tau^+ :=$
\{ h \in \text{Aff} S(G, u) \mid h(\tau) = 0 \} \text{ (the latter is closed and codimension one subspace of } \text{Aff} S(G, u)\). This makes examples and non-examples relatively easy to construct. There is a corresponding characterization for goodness, which we shall simplify a bit, and used to actually do something.

**PROPOSITION 1.1** Suppose \((G, u)\) is a dimension group with order unit. Let \(\tau\) be a faithful trace of \(G\). Then \(\tau\) is good iff for all nonzero order ideals with order unit \((I, w)\), both \(\tau(I) = \tau(G)\) and \(\tau|I\) is order unit good.

**Remark.** Necessity is shown in [BeH, Proposition 4.2]; although the statement hypothesizes that \(\tau\) be pure, this is not used in the proof; also shown there was that if \(\tau\) is good, then \(\tau|I\) is good (as a trace on the order ideal \(I\)), and this implies (in the case that \(I\) is approximately divisible) that \(\tau|I\) is order unit good, just from the definitions.

**Remark.** It is always possible to reduce to the case that \(\tau\) be faithful, by factoring out the maximal order ideal \(J\) contained in \(\ker \tau\) [BeH, Lemma 4.4]. In this case, the criteria apply to \(G/J\) (replacing \(G\)). This would make the statement somewhat more complicated.

**Proof.** Proof of necessity is given in [BeH; Proposition 4.2], requiring neither purity of \(\tau\) nor approximate divisibility.

Conversely, suppose \(a \in G, b \in G^+\) and \(0 < \tau(a) < \tau(b)\). Form the order ideal \(I\) generated by \(b\), that is, \(I = \{ c \in G \mid \exists N \in \mathbb{N} \text{ such that } -N b \leq c \leq N b \}\). Then \(I\) is an order ideal with its own order unit, \(b\). Since \(\tau(I) = \tau(G)\), there exists \(a_1 \in I\) such that \(\tau(a_1) = \tau(a)\). Now order unit goodness of \(\tau|I\) yields \(a' \in I\) such that \(\tau(a') = \tau(a_1) = \tau(a)\) and \(0 \leq a' \leq b\), verifying goodness of \(\tau\).

Let \(G\) be a dimension group, and let \(I\) and \(J\) be order ideals thereof. Then \(H := I + J\) (the set of sums of elements in \(I\) and \(J\)) and \(I \cap J\) are both order ideals. Most of the following are variations on [BeH, Lemma 1.3]. As in [BeH], and element \(v\) of \(G^+\) is \(\tau\)-good or \(\tau\)-order unit good if \(\tau([0, v]) = [0, \tau(v)]\).

**LEMMA 1.2** Suppose \(G\) is a dimension group, and \(I\) and \(J\) each have (relative) order units, \(w, y\) respectively. Then

(a) \(I + J\) is an order ideal of \(G\) with a (relative) order unit.

(b) Let \(\tau\) be a trace on \(G\) such that \(\ker \tau \cap G^+ = \{0\}\) and \(\tau(I) \cap \tau(J)\) is dense in \(\mathbb{R}\). If \(\tau|I\) and \(\tau|J\) are good (as traces on \(I\) and \(J\) respectively), then \(\tau\) is good.

(c) If \(I + J\) is approximately divisible, then every order unit \(b\) of \(I + J\) can be written in the form \(b = u + v\) where \(u, v\) are relative order units for \(I, J\) respectively.

(d) If \(v\) is \(\tau\)-order unit good (with respect to \(I\)) and \(w\) is \(\tau\)-order unit good (with respect to \(J\)), and \(\tau(I) \cap \tau(J)\) is dense in \(\mathbb{R}\), then \(v + w\) is \(\tau\)-order unit good with respect to \(I + J\).

(e) Suppose each of \(I, J\) and \(I + J\) are approximately divisible, and \(\tau\) is a trace on \(I + J\) such that each of \(\tau|I\) and \(\tau|J\) is order unit good, and \(\tau(I) \cap \tau(J)\) is dense in \(\mathbb{R}\). Then \(\tau\) is order unit good as a trace of \(I + J\).

**Remark.** Part (c) can fail if approximate divisibility is dropped; for example, take \(G = \mathbb{Z}^3\) with the usual simplicial ordering, let \(I\) be the order ideal generated by \((1, 1, 0)\) and let \(J\) be the order ideal generated by \((0, 1, 1)\); then \(I + J = G\) and the order unit \((1, 1, 1)\) cannot be realized as a sum of relative order units from \(I\) and \(J\) respectively.

**Proof.** (a) That \(I + J\) is an order ideal is ancient, e.g., [G]. If \(w\) and \(y\) are respective order units for \(I\) and \(J\), then \(z := w + y\) is an order unit for \(I + J\). To see this, let \(f \in (I + J)^+\); for dimension groups, \((I + J)^+ = I^+ + J^+\), hence we can find \(e \in I^+\) and \(g \in J^+\) such that \(f = e + g\). Since there exist positive integers \(k, k'\) such that \(e \leq kw\) and \(g \leq k'v\), we have \(f \leq k''z\) where \(k'' = \max \{k, k'\}\).
(b) Select \( b \in G^+ \) and \( a \in G \) such that \( \tau(a) < \tau(b) \). We may write \( b = i + j \) where \( i \in I^+ \) and \( j \in J^+ \) (see (G)). Then \( \tau(i), \tau(j) > 0 \). We may write \( \tau(a) = r + s \) where \( r \in \tau(I) \) and \( s \in \tau(J) \).

Assume \( \tau(a) \geq \tau(i) \). By density of \( \tau(I) \cap \tau(J) \), given \( 0 < \epsilon < \min \{ \tau(i), \tau(b) - \tau(a) \} \), there exists \( \delta \in \tau(I) \cap \tau(J) \) such that \( \tau(i) - \epsilon < r + \delta < \tau(i) \). Then \( s - \delta = \tau(a) - r - \delta \) satisfies \( \tau(a) - \tau(i) + \epsilon > s - \delta > \tau(a) - \tau(i) > 0 \)

Hence we can write \( \tau(a) = (r + \delta) + (s - \delta) \), where the parenthesized terms are respectively in the intervals \((0, \tau(i))\) and \((0, \tau(a) - \tau(i) + \epsilon)\). However, \( \epsilon < \tau(b) - \tau(a) \) entails \( \tau(a) - \tau(i) + \epsilon < \tau(b) - \tau(i) = \tau(j) \). Since \( \pm \delta \in \tau(I \cap J) \), we may thus find \( a_1 \in I \) and \( a_2 \in J \) such that \( 0 < \tau(a_1) < \tau(i) \) and \( 0 < \tau(a_2) < \tau(j) \). Since each of \( \tau/I \) and \( \tau/J \) is good, there exist \( c_1 \in [0, i] \) (the interval in \( I \)) and \( c_2 \in [0, j] \) such that \( \tau(c_1) < \tau(i) \) and \( \tau(c_2) < \tau(j) \). Hence we have \( c := c_1 + c_2 \in [0, b] \) and \( \tau(c) = \tau(c_1) + \tau(c_2) < \tau(i) + \tau(j) = \tau(b) \), verifying goodness in this case.

Reversing the roles of \( i \) and \( j \), the same conclusion results if \( \tau(a) \geq \tau(j) \), so we are reduced to the case that \( \tau(a) < \min \{ \tau(i), \tau(j) \} \). If \( \tau(a) = 0 \), there is nothing to do (except set \( c = 0 \)). Otherwise, choose \( 0 < \epsilon < \tau(a)/2 \) find real \( \delta \in \tau(I \cap J) \) such that \( \tau(a)/2 - \epsilon < \delta + r < \tau(a)/2 \), and consider \( \tau(a) = (r + \delta) + (s - \delta) \); then \( r + \delta \in (0, \tau(a)/2) \subset (0, \tau(i)) \), so \( s - \delta \in (\tau(a)/2, \tau(a)) \subset (0, \tau(j)) \). Now we can proceed as in the previous paragraph.

(c) Now let \( b \) be an order unit of \( I + J \). By approximate divisibility of \( I + J \), the range of \( I + J \) in \( \text{Aff} \mathcal{S}(I + J, b) \) is dense; hence given \( \epsilon > 0 \), we may find \( b_0 \in I + J \) such that \( (1/2 - \epsilon)1 < b_0 < 1/2 \) (where \( \hat{\tau} \) refers only to the representation on \( \mathcal{S}(I + J, b) \), that is, \( \hat{b} = 1 \)). Let \( \epsilon < 1/8 \), so that \( \hat{b}_0 \gg 0 \) and thus \( b_0 \) is an order unit of \( I + J \), and moreover, \( 2b_0 \leq b \), and \( b - b_0 \) is also an order unit for \( I + J \).

Now consider the set \( S := \{ c \in I^+ \mid c \leq b_0 \} \). This is directed, as if \( c, c' \in S \), then we have \( c, c' \leq b_0, c + c' \); interpolating, we obtain \( c'' \) such that \( c, c' \leq c'' \leq b_0, c + c' \); as \( c + c' \in I \), it follows that \( c'' \in I \), so \( c'' \in \mathcal{S} \). As there exists \( k \) such that \( w \leq kb_0 \), we can write \( w = \sum_{i=1}^{k} w_i \) where \( w_i \in I^+ \) and each \( w_i \leq b_0 \). Then \( w_i \in S \), so there exists \( u_0 \in I^+ \) such that \( w_i \leq u_0 \leq b_0 \) for all \( i \).

Since \( \sum w_i = w \) is an order unit for \( I \), \( ku_0 \) is an order unit for \( I \), and thus \( u_0 \) is too. Hence there exists an order unit \( u_0 \) of \( I \) such that \( u_0 \leq b_0 \).

Since \( b - b_0 \) is also an order unit for \( I + J \), applying the same process to \( J \) instead of \( I \) yields an order unit \( v_0 \) of \( J \) such that \( v_0 \leq b - b_0 \). Thus \( u_0 + v_0 \leq b_0 + (b - b_0) = b \). The element \( b - (u_0 + v_0) \) is in the positive cone of \( I + J \), so can be written \( b - (u_0 + v_0) = c + d \) where \( c \in I^+ \) and \( d \in J^+ \). This yields \( b = (u_0 + c) + (v_0 + d) \); setting \( u = u_0 + c \), we see that \( u \in I^+ \) and is larger than an order unit for \( I \), so is itself an order unit for \( I \); similarly \( v = v_0 + d \) is an order unit for \( J \).

(d) & (e) Select an order unit \( b \) for \( I + J \), and \( \alpha \in I + J \) such that \( 0 < \tau(\alpha) < \tau(b) \). By (c), we may write \( b = u + v \) where \( u \) and \( v \) are order units for \( I \) and \( J \) respectively. We can write \( a = r + s \) where \( r \in I \) and \( s \in J \), and set \( t = \tau(u) \) (as \( \tau(I) \) is order unit good, it does not vanish identically, hence \( t > 0 \)), so that \( \tau(v) = \tau(b) - t \), which is again positive. Now proceed as in the proof of (b).

The density requirement on \( \tau(I) \cap \tau(J) \) is essential.

**LEMMA 1.3** Suppose that \( u \) and \( v \) are elements of \( G^+ \), and let \( \tau \) be a trace such that each is \( \tau \)-order unit good on the order ideals they generate, \( I(u) \) and \( I(v) \) respectively.

(a) If \( u + v \) is \( \tau \)-order unit good on \( I(u) + I(v) = I(u + v) \) and \( \tau(I(u)) + \tau(I(v)) \) is dense in \( \mathbf{R} \), then \( \tau(I(u)) \cap \tau(I(v)) \neq \{0\} \);

(b) if additionally, both \( \tau(I(u)) \) and \( \tau(I(v)) \) are dense subgroups of \( \mathbf{R} \), then so is \( \tau(I(u)) \cap \tau(I(v)) \).

**Proof.** Suppose the intersection consists of just 0. We may find positive real numbers \( s \in \tau(I(u)) \) and \( t \in \tau(I(v)) \) such that \( s > \tau(u) \), \( t \) is \( \tau(v) \), and \( 0 < r := s - t < \tau(u + v) \) (since the value group
is dense). By order unit goodness, there exists \(a\) such that \(0 \leq a \leq u + v\) and \(\tau(a) = r\). Riesz decomposition entails \(a = a_1 + a_2\) where \(0 \leq a_1 \leq u\) and \(0 \leq a_2 \leq v\). Set \(s' = \tau(a_1) \geq 0\) and \(t' = \tau(a_2) \geq 0\). Then \(s - t = s' + t'\), so \(s - s' = t + t'\). The intersection consisting of 0 forces \(s = s'\) and \(t = -t'\); the latter forces \(t = t' = 0\), a contradiction.

Now suppose the intersection is nonzero and not dense. Then it is cyclic, so there exists \(x \in \mathbb{R}\), which we may assume positive, such that \(\tau(I(u)) \cap \tau(I(v)) = x\mathbb{Z}\). We may find \(0 < s, t < x\) with \(s \in \tau(I(u))\) and \(t \in \tau(I(v))\) such that \(0 < r := s - t\). Find \(a \leq u + v\) as above with \(r = \tau(a)\), similarly decompose \(a = a_1 + a_2\), and define \(s', t'\) as in the preceding paragraph. We deduce \(s - s' = t + t'\); hence there exists an integer \(m\) such that \(s - s' = mx = t + t'\); as \(t, t' \geq 0\), we have \(m \geq 0\), but as \(s < x\), we have \(m < 1\); hence \(m = 0\). This forces \(t = t' = 0\), again a contradiction.  

**COROLLARY 1.4** Let \(G\) be a nearly divisible dimension group with a faithful trace \(\tau\). Suppose that \(I\) and \(J\) are order ideals with their own order units such that each of \(\tau|I\), \(\tau|J\), and \(\tau(I + J)\) is order unit good. Then \(\tau(I) \cap \tau(J)\) is a dense subgroup of \(\mathbb{R}\).

**Proof.** Since \(\tau\) is faithful, \(\tau|I\) and \(\tau|J\) are nonzero, and since every trace on an order ideal with order unit is nondiscrete (as the order ideals are approximately divisible by definition), it follows that \(\tau(I)\) and \(\tau(J)\) are dense. Now Lemma 1.3(b) applies.

Let \((G, u)\) be a dimension group. Let \(J\) be a collection of nonzero order ideals each with their own order unit, such that every order ideal of \(G\) with order unit can be expressed as a sum of order ideals from \(J\) (such a sum can always be made finite, as the order ideal has an order unit); then we say \(J\) is a generating set of order ideals of \(G\).

The criteria in Proposition 1.2 for goodness can be reduced to that on a generating set of order ideals. This will make the computations of section 4 much simpler.

**LEMMA 1.5** Let \((G, u)\) be a nearly divisible dimension group, let \(J\) be a generating set of order ideals of \(G\), and let \(\tau\) be a faithful trace of \(G\). Sufficient for \(\tau\) to be a good trace of \(G\) is that it satisfy

(i) for all \(J \in J\), \(\tau(J) = \tau(G)\) and

(ii) for all \(J \in J\), \(\tau(I)\) is an order unit good trace of \(I\).

**Proof.** We can express a nonzero order ideal \(I\) with order unit as \(I = \sum J_\alpha\) for some \(J_\alpha \in J\). Thus \(\tau(I) = \sum \tau(J_\alpha) = \tau(G)\).

Since \(I\) has an order unit, the sum can be made finite; now we apply induction (on the number of summands) to 1.2(d); this verifies the second property in Proposition 1.1.

Verifying the various criteria for goodness and related properties is much simpler when the partially ordered abelian group is an ordered ring having 1 as an order unit.

**LEMMA 1.6** Let \((R, 1)\) be a (commutative) partially ordered commutative ring with 1 as order unit. If \(R\) is approximately divisible, then it is nearly divisible.

**Proof.** Approximate divisibility implies the existence of order units \(u\) and \(v\) such that \(1 = 2u + 3v\); for any \(r \in R^+ \setminus \{0\}\), we thus have \(r = 2(ru) + 3(rv)\). From \(1 \leq ku, kv\) for some positive integer \(k\), we deduce \(r \leq k(ru), k(rv)\), verifying the definition of nearly divisible.

The following is implicit in the proof of [BeH, Corollary 7.12].

**LEMMA 1.7** Let \((R, 1)\) be a partially ordered (commutative) unperforated ring with 1 as order unit, that is approximately divisible. Let \(\tau\) be a faithful pure trace. Then \(\tau\) is order unit good iff for all \(\sigma \in \partial_e S(R, 1) \setminus \{\tau\}\), \(\sigma(\ker \tau) \neq \{0\}\).

**Proof.** Since 1 is an order unit of the partially ordered ring, \(X := \partial_e S(R, 1)\) is compact and consists precisely of the normalized multiplicative traces of \(R\); moreover, \(\text{Aff} S(R, 1) = C(X, \mathbb{R})\).
with the affine representation re-interpreted as $\tilde{g}(\phi) = \phi(g)$ for $\phi \in X$ (note the use of $\sim$ rather than $\cong$, to distinguish them). By approximate divisibility, the image of $R$ is dense in $C(X, R)$. If $A$ is any ideal of $R$, then its closure in $C(X, R)$ is an ideal therein, hence of the form $\text{Ann}(Y) := \{f \in C(X, R) | f[Y] \equiv 0\}$ for a unique compact subset $Y$ of $X$.

Since $\tau$ is pure, it is multiplicative, and therefore $\ker \tau$ is an ideal of $R$ [not an order ideal, unless $\ker \tau = 0$, as $\ker \tau \cap R^+ = \{0\}$ is the definition of faithfulness]. The closure of the image of $\ker \tau$ in $C(X, R)$ can thus be written in the form $\text{Ann}(Y)$ for some compact subset $Y$.

If $\tau$ is order unit good, then $\text{Ann}(Y)$ is $\text{Ann}(\{\tau\})$ (corresponding to $\tau^+$ in $\text{Aff}S(R, 1)$), from which it follows that $Y = \{\tau\}$. Hence if $\sigma \in X \setminus \{\tau\}$, there exists continuous $f : X \to [0, 1]$ such that $f(\tau) = 0$ but $f(\sigma) = 1$; then $f \in \text{Ann}(\{\tau\})$, hence there exist $g_n \in R$ such that $g_n \in \ker \tau$ and $\tilde{g}_n \to f$ uniformly. Applying $\sigma$, there exists $n$ such that $\sigma(g_n) \neq 0$, so that $\sigma(\ker \tau) \neq \{0\}$.

Conversely, suppose for every $\sigma \in X \setminus \{\tau\}$, $\sigma(\ker \tau) \neq \{0\}$. Then $\sigma \not\in Y$; hence $Y = \{\tau\}$, so that the closure of the image of $\ker \tau$ is codimension one in $C(X, R)$, hence equal to $\tau^+$ in $\text{Aff}S(G, u)$. Thus $\tau$ is order unit good.

2 Tensor products

If $G$ and $H$ are partially ordered abelian groups, we may form the tensor product (as $\mathbb{Z}$-modules) $G \otimes_{\mathbb{Z}} H$ (usually, we delete the subscripted $\mathbb{Z}$); it is equipped with a cone which makes it into a partially ordered group, $\{\sum g_i \otimes h_i | g_i \in G^+ \text{ and } h_i \in H^+\}$ [GH2, Proposition 2.1]. If both are dimension groups, then so is $G \otimes H$, and if $u, v$ are respectively order units for $G, H$, then $u \otimes v$ is an order unit for $G \otimes H$. If $\sigma, \tau$ are respective (normalized) traces on $(G, u)$ and $(H, v)$, then $\sigma \otimes \tau$ (defined in the obvious way) is a (normalized) trace of $(G \otimes H, u \otimes v)$.

A special case occurs when we form the divisible hull of a dimension group, $G \otimes \mathbb{Q}$, the rational vector space that $G$ generates. Then $\tau$ extends to a trace $G \otimes \mathbb{Q}$ in the obvious way, denoted $\tau \otimes 1_{\mathbb{Q}}$. In general, $\tau$ being order unit good or good implies the corresponding property for $\tau \otimes 1_{\mathbb{Q}}$, but the converse fails practically generically. As a special case, we [BeH] defined a trace $\tau$ to be ugly if $\tau \otimes 1_{\mathbb{Q}}$ is good and $\ker \tau$ has discrete image in (the Banach space) $\text{Aff}S(G, u)$. Ugly traces exist in profusion.

In Akin’s original context of measures on Cantor sets, he showed that (what amounts to) the tensor product of good traces is good; in the context of simple dimension groups or more generally for approximately divisible dimension groups, the tensor product of order unit good traces was shown to be order unit good. Here, we show a somewhat surprising result for order unit goodness: if $(G, u)$ and $(H, v)$ are approximately divisible, and both $\sigma \otimes 1_{\mathbb{Q}}$ and $\tau \otimes 1_{\mathbb{Q}}$ are order unit good on their respective groups, then $\sigma \otimes \tau$ is order unit good (as a trace on $G \otimes H$). This means that the tensor product has a stronger property (in general) than its constituents. In particular, the tensor product of ugly traces is at least order unit good.

Using the criterion of Proposition 1.1, we then obtain a corresponding criterion for goodness of the tensor product ($G$ and $H$ are nearly divisible, $\sigma \otimes 1_{\mathbb{Q}}$ and $\tau \otimes 1_{\mathbb{Q}}$ are good, and a condition that guarantees the value groups on the order ideals is the same as the full value group).

**PROPOSITION 2.1** Let $(G, u)$ and $(H, v)$ be approximately divisible dimension groups with traces $\sigma$ and $\tau$ respectively. If each of $\sigma \otimes 1_{\mathbb{Q}}$ and $\tau \otimes 1_{\mathbb{Q}}$ on $G \otimes \mathbb{Q}$ and $H \otimes \mathbb{Q}$ respectively is order unit good, then the trace on $(G \otimes H, u \otimes v)$ given by $\sigma \otimes \tau$, is order unit good.

If we only require that $\sigma \otimes \tau \otimes 1_{\mathbb{Q}}$ (a trace on $G \otimes H \otimes \mathbb{Q}$) be order unit good (in place of each of $\sigma \otimes 1_{\mathbb{Q}}$ and $\tau \otimes 1_{\mathbb{Q}}$ being good), the conclusion may still be true. In any event, I know of no counter-examples.

We require a number of elementary results about tensor products. Here the tensors will be over one of the rings $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$; *torsion-free* (module) means torsion-free abelian group when the
underlying ring is $\mathbf{Z}$, otherwise is just means vector space over the relevant field.

**Lemma 2.2** Let $A$ and $B$ be torsion-free modules, and $A' \subset A$, $B' \subset B$ submodules such that $A/A'$ and $B/B'$ are torsion-free.

(a) The kernel of the map $A \otimes B \to A \otimes B/B'$ is $A \otimes B'$.

(b) The kernel of the map $A \otimes B \to A/A' \otimes B/B'$ is $A \otimes B' + A' \otimes B$.

**Proof.** (a) One inclusion is obvious. Because the quotient is torsion-free, $A \otimes B/B'$ is torsion-free. We have an induced map $(A \otimes B)/(A \otimes B') \to A \otimes B/B'$. If $z$ is in the kernel, find a nonzero integer $n$ such that $nz$ has a representative in $A \otimes B$ of least length (as $n$ varies over nonzero integers), say $nz = \sum a_i \otimes b_i + (A \otimes B')$. Then $\{a_i\}$ is rationally linearly independent, hence the image, $n\sigma\tau$, yields, $0 = \sum a_i \otimes (b_i + B')$. Since $B/B'$ is torsion-free, this easily implies all $b_i + B' = 0$ (tensor with $Q$ if necessary, so we are working over a field, then use a basis for $B'B_Q$, extended to $BQ$).

[This proof works for all fields.]

(b) First, $A \otimes B/(A \otimes B')$ is naturally isomorphic to $A \otimes B/B'$ by (a). Then another application of (a) with the order reversed yields a natural isomorphism $(A \otimes B/B')/(A' \otimes B/B') \cong A/A' \otimes B/B'$. Then the kernel of the first map is $A \otimes B'$, and of the second is $A' \otimes B/B'$, which pulls back to $A \otimes B' + A' \otimes B$.

**Proof.** (of Proposition 2.1) We will show that that the closure of the image of $\ker \sigma \otimes \tau$ in $\text{Aff}(S(G \otimes H, u \otimes v))$ is $(\sigma \otimes \tau)^*$; by [BeH, Proposition 1.7], $\sigma \otimes \tau$ is order unit good.

First, we identify $\text{Aff}(S(G, u) \otimes_R \text{Aff}(S(H, v))$ with a subspace of $\text{Aff}(S(G \otimes H, u \otimes v))$ in the obvious way. Standard results (e.g., pure traces are pure tensors) yields that it is a dense subspace.

We note that $(\ker \sigma) \otimes H + G \otimes (\ker \tau) \subset \ker \sigma \otimes \tau$. It easily follows that the closure of the image of $(\ker \sigma) \otimes H$ contains everything in $y \otimes \text{Aff}(S(H, v))$ (real tensors) where $y$ varies over the image of $\ker \sigma$ (in $(\sigma^*) \subset \text{Aff}(S(G, u))$. For $y$ fixed, $y \otimes \text{Aff}(S(H, v))$ is a real vector space, and this means that we can rewrite it as $yR \otimes \text{Aff}(S(H, v))$ (just approximate real multiples of $\tilde{v}$ by elements of $\tilde{H}$, and transfer through the tensor product). Taking finite sums, we see that the closure of the image of $\ker \sigma \otimes H$ includes the closure of $\text{Im}(\ker \sigma)Q \otimes \text{Aff}(S(H, v))$.

Now $\sigma \otimes 1_Q$ being order unit good implies $\ker \sigma \otimes Q$ has dense image in $\sigma^+$ (in $\text{Aff}(S(G, u))$).

If $e$ is an element of $G \otimes Q$, there exists a nonzero integer $m$ such that $me \in G$. If in addition, $\sigma \otimes 1_Q(e) = 0$, then $\sigma(me) = 0$; thus $\ker \sigma \otimes 1_Q \subset (\ker \sigma)Q$ (the reverse inclusion is trivial, but never needed).

Thus the closure of the image of $(\ker \sigma) \otimes H$ contains $\text{Im}(\ker \sigma)Q \otimes \text{Aff}(S(H, v))$, which in turn contains the closure of $\text{Im}(\ker \sigma)Q \otimes \text{Aff}(S(H, v))$, and thus includes $\sigma^+ \otimes \text{Aff}(S(H, v))$.

Similarly, the closure of the image of $G \otimes \ker \tau$ contains $\text{Aff}(S(G, u) \otimes \tau^+)$. Set $A = \text{Aff}(S(G, u), A' = \sigma^+, B = \text{Aff}(S(H, v))$, and $B' = \tau^+$; then each is a Banach space, and $A/A'$ and $B/B'$ are both one-dimensional, and the closure of the image of $\ker \sigma \otimes \tau$ contains $A' \otimes B + A \otimes B'$.

By (b) above, $A \otimes B/(A' \otimes B + A \otimes B')$ is one-dimensional. Let $W = A' \otimes B + A \otimes B'$ and $Z = \text{Aff}(S(G, u) \otimes \text{Aff}(S(H, v))$, so that $W$ is a codimension one subspace of $Z$. It is an easy exercise to show that when we complete $Z$ to $\text{Aff}(S(G \otimes H, u \otimes v)$, the closure, $\overline{W}$, is of at most codimension one. (This is a general Banach space result; if $\overline{W} \neq Z$, then $W = \overline{W} \cap Z$ as $W$ is codimension one in $Z$; choose $z \in Z \setminus W$; the functional sending $z \mapsto 1$ and $W \mapsto 0$ is continuous (essentially the closed graph theorem), hence extends to a bounded linear functional $p$ on $\overline{W}$; we may write arbitrary $y \in Z$ as $\lim_{n} y_n$; then $y_n = p(y_n)z + (y_n - p(y_n)z)$, and thus by continuity, $y = p(y)z + (y - p(y)z)$, and $y - p(y)z$ is in $\overline{W}$; hence $z + \overline{W} = Z$.

In particular, the closure of the image of $\ker \sigma \otimes \tau$ in $\text{Aff}(S(G \otimes H, u \otimes v)$ is codimension one. As it is contained in $(\sigma \otimes \tau)^*$, which is proper, it follows that the image of $\ker \sigma \otimes \tau$ is dense in $(\sigma \otimes \tau)^*$.

This explains a phenomenon exemplified in [BeH, Example 9]. Let $G$ be a critical dimension
group of rank $k + 1$ (that is, a free rank $k + 1$ abelian group densely embedded in $\mathbb{R}^d$, and equipped with the strict ordering therefrom [H4]). Then we say $G$ is basic (as a critical group) if it is order-

isomorphic to the subgroup of $\mathbb{R}^k$ spanned by $\{e_i; \sum \alpha_j e_j\}$ where $\{e_i\}$ is the standard basis and $\{1, \alpha_1, \ldots, \alpha_k\}$ is linearly independent over the rationals (this guarantees density of the subgroup).

Every critical group is topologically isomorphic to a group of the latter form.

For basic critical groups, every pure trace is ugly, as is immediate from the definitions. Hence if $G_i$ are basic critical groups (and there is more than one), their tensor product (a simple dimension group) $\otimes G_i$ has all of its pure traces good. In [BeH, Example 9], an example was given of a basic critical group of rank three, for which all pure traces on $G \otimes G$ are good. We also asked whether the pure traces on $G \otimes G \otimes G$ are good, and now we know that the answer is yes.

It is possible that among critical groups, basic ones are characterized by all pure traces being ugly. There are lots of critical groups for which all or some are bad, hence not ugly [BeH, section 2].

Now suppose that $(G, u)$ and $(H, v)$ are nearly divisible, and $\sigma, \tau$ are normalized traces on $G, H$ respectively such that $\sigma \otimes 1_\mathbb{Q}$ and $\tau \otimes 1_\mathbb{Q}$ are both good. We expect to obtain that $\sigma \otimes \tau$ is a good trace on $G \otimes H$.

**LEMMA 2.3** Let $(G, u)$ and $(H, v)$ be dimension groups with order unit.

(a) Then $G \otimes H$ is approximately divisible iff at least one of $G$ or $H$ is;

(b) $G \otimes H$ is nearly divisible iff at least one of $G$ or $H$ is.

**Proof.** (a) Suppose $G$ is approximately divisible. Every pure trace of $(G \otimes H, u \otimes v)$ is of the form $\sigma \otimes \tau [GH2, Lemma 4.1]$, where $\sigma, \tau$ are pure traces of $G, H$ respectively. Then $(\sigma \otimes \tau)(G \otimes H)$ is $\sigma(G) \cdot \tau(H)$ (the set of sums of terms of the form $\sigma(g) \cdot \tau(h)$); as $\sigma(G)$ is dense, obviously so is $\sigma(G) \cdot \tau(H)$, so that $G \otimes H$ has no discrete traces, and is thus approximately divisible. Obviously the same argument applies if $H$ is approximately divisible.

If neither $G$ nor $H$ is approximately divisible, then there exists a discrete trace $\sigma$ of $G$ and a discrete trace $\tau$ of $H$; as these are normalized (at $u, v$ respectively), $\sigma(G) = (1/n)\mathbb{Z}$ and $\tau(H) = (1/m)\mathbb{Z}$ for some positive integers $m$ and $n$; then $(\sigma \otimes \tau)(G \otimes H) = (1/mn)\mathbb{Z}$, which is discrete. Hence $G \otimes H$ admits a discrete trace, thus is not approximately divisible.

(b) Select $a = \sum g_i \otimes h_i \in (G \otimes H)^+$; from the definition of the ordering on the tensor product, we can assume each of $g_i$ and $h_i$ are positive in their respective groups. By definition, we can write $g_i = 2a_i + 3b_i$ where $0 \leq g_i \leq ka_i, kb_i$ for some positive integer $k$; since the sum is finite, we can take the same integer $k$ for all $i$. Set $c_1 = \sum a_i \otimes h_i$ and $c_2 = \sum b_i \otimes h_i$. Then $a = 2c_1 + 3c_2$; moreover, $\sum g_i \otimes h_i \leq k \sum a_i \otimes h_i$, that is, $a \leq kc_1$, and similarly $a \leq kc_2$.

If neither $G$ nor $H$ is nearly divisible, there exist an order ideal of $G$ with its own order unit, $(I, w)$ together with a discrete trace $(I, \phi)$, and an order ideal of $H$ with its own order unit, $(J, y)$ and a discrete trace on it, $\psi$. Then $\phi \otimes \psi$ is a discrete trace (as above) of $I \otimes J$; this being an order ideal of $G \otimes H$, the latter is not nearly divisible.

**LEMMA 2.4** Let $G$ and $H$ be nearly divisible, having faithful traces $\sigma$ and $\tau$ respectively such that $\sigma \otimes 1_\mathbb{Q}$ and $\tau \otimes 1_\mathbb{Q}$ are good as traces on $G \otimes \mathbb{Q}, H \otimes \mathbb{Q}$ respectively.

(a) Let $(I, w)$ be an order ideal of $G$ with its own order unit, and let $(J, y)$ be an order ideal of $H$ with its own order unit. Then $(\sigma \otimes \tau)(I \otimes J)$ is order unit good.

(b) Suppose for each order ideal $I$ of $G$, $\sigma(I) = \sigma(G)$, and similarly, for each order ideal $J$ of $H$, we have $\tau(J) = \tau(H)$. Then for every nonzero order ideal $L$ of $G \otimes H$, we have $(\sigma \otimes \tau)(L) = (\sigma \otimes \tau)(G \otimes H)$

(c) Suppose the hypotheses of (b) apply. Let $(L, v)$ be an arbitrary order ideal of $G \otimes H$ with its own order unit. Then $(\sigma \otimes \tau)|L$ is order unit good.
Proof. (a) Each of the restrictions of $\sigma \otimes 1_Q$ and $\tau \otimes 1_Q$ to $I \otimes Q$ and $J \otimes Q$ respectively is good, hence is order unit good, and thus $(\sigma \otimes \tau)|(I \otimes J)$ is an order unit good trace of $I \otimes J$.

(b) First, if $L = I \otimes J$ (where $I$ and $J$ are nonzero order ideals in $G$ and $H$ respectively), then $(\sigma \otimes \tau)(I \otimes J)$ is the subgroup of $R$ generated by all terms of the form $\sigma(a) \cdot \tau(b)$, where $a \in I$ and $b \in J$, and $(\sigma \otimes \tau)(G \otimes H)$ has the same form, except $a$ and $b$ are allowed to vary over $G$ and $H$ respectively. Since for all $a \in G$, there exists $a' \in I$ such that $\sigma(a') = \sigma(a)$, and similarly for $\tau$, the two groups are equal.

If $e \in L^+$, then by definition of the tensor product ordering, we can write $e = \sum g_i \otimes h_i$. For an element $x$ in the positive cone of a dimension group, let $I(x)$ be the order ideal it generates; then it is easy to check (since sums of order ideals are again order ideals in a dimension group) that $L = I(e) = \sum I(g_i) \otimes I(h_i)$; in particular, $L$ contains a tensor product of order ideals, so the previous paragraph applies.

(c) Every $e \in (G \otimes H)^+$ can be written in the form $e = \sum g_i \otimes h_i$ with $g_i \in G^+$ and $h_i \in H^+$. By (a), the restriction of $\sigma \otimes \tau$ to each of $I(g_i) \otimes I(h_i)$ is order unit good. Since $\sigma \otimes \tau(L) = (\sigma \otimes \tau)(G \otimes H)$, for any nonzero order ideal $L$ of $G \otimes H$, we may apply 1.2(e) (the intersection of the value groups is dense), so the restriction of $\sigma \otimes \tau$ to $L$ is order unit good.

•

PROPOSITION 2.5 Suppose that $(G, u, \sigma)$ and $(H, v, \tau)$ are nearly divisible dimension groups with faithful trace having the following properties:

(i) for all nonzero order ideals $I$ ($J$) of $G$ ($H$), $\sigma(I) = \sigma(G)$ ($\tau(J) = \tau(H)$);

(ii) each of $\sigma \otimes 1_Q$ and $\tau \otimes 1_Q$ is good on $G \otimes Q$, $H \otimes Q$ respectively.

Then $\sigma \otimes \tau$ is a good trace of $G \otimes H$.

•

3 Examples from xerox actions of tori on UHF algebras

We characterize the good faithful pure traces on the dimension groups arising from xerox product type actions of tori on UHF $C^*$-algebras. It turns out that there is a surprising number-theoretic component.

Form the Laurent polynomial ring in $d$ variables over the integers, $Z[x_1^{\pm 1}]$, and let $Z[x_1^{\pm 1}]^+$ denote the set of those with only nonnegative coefficients. As in [H1, H2], we adopt monomial notation, that is, for $w \in Z^d$, define $x^w = x_1^{w(1)} \cdot x_2^{w(2)} \cdots x_d^{w(d)}$. For any $f \in Z[x_1^{\pm 1}]$, we denote the coefficient of $x^w$ in $f$ by $(f, x^w)$ (inner product notation, which is consistent with the origins of the work), and we set $\text{Log } f := \{ w \in Z^d \mid (f, x^w) \neq 0 \}$. Let $P = \sum a_w x^w \in Z[x_1^{\pm 1}]^+$ (where $a_w \in Z^+$), and form the ring $R_P = Z[\{ x^w/P \}_{w \in \text{Log } P}]$, equipped with the partial ordering generated added and multiplicatively by $\{ x^w/P \mid w \in \text{Log } P \}$, this is a dimension group and an ordered ring with 1 as order unit, and many more properties. We may also form $Z[x_1^{\pm 1}, 1/P]$ (a subring of the field of fractions of the Laurent polynomial ring. It also has a partial ordering given by $\{ f/P^k \mid \exists N \text{ such that } P^N f \text{ has no negative coefficients} \}$). The restriction of this to $R_P$ yields the original ordering.

This arose from the following construction. Let $n = P(1, 1, 1, \ldots, 1)$, and form $A = \otimes M_n C$ (the UHF $C^*$-algebra). The Laurent polynomial $P$ is the character of an $n$-dimensional representation of the torus $T^d$, say given by $z \mapsto \text{diag } (z^w)$ (one for each $w$ that appears in $P$, with repetitions as indicated by the multiplicities, that is, the coefficients. This yields a map $\pi : T^d \to M_n C$ with nonzero entries along the diagonal. Form $\phi := \otimes \text{Ad } \pi : T^d \to \text{Aut } A$, and the corresponding fixed point subrings, $A^\phi(T^d)$, and $A \times \phi T^d$, the latter the $C^*$-crossed product. Then $(K_0(A^\phi(T^d)), [1])$ is naturally ordered ring isomorphic to $R_P$ and $K_0(A \times \phi T^d)$ similarly isomorphic to the ordered ring $Z[x_1^{\pm 1}, 1/P]$. This will play a role in what follows.

Renault [R] determined the positive cone and analyzed (inter alia) the structure of $R_P$ when
\[ P = 1 + x. \] That was in 1980; people are still obliviously reproving his and other results (concerning Pascal’s triangle Bratteli diagrams) 30+ years later!

We normally assume that \( P \) is projectively faithful, that is, \( \text{Log} \ P - \text{Log} \ P \) generates (as an abelian group) the standard copy of \( \mathbb{Z}^d \) in \( \mathbb{R}^d \) (we can reduce to this case anyway). This has the effect that whenever \( v \in \text{Log} \ P^k \cap \text{int} \text{cvx} \text{Log} \ P^k \) for some positive integer \( k \), \( x^v/P^k \) belongs to \( R_P \) and \( R_P \ [(x^v/P^k)^{-1}] = \mathbb{Z}[x_i^{\pm1}, 1/P] \), i.e., the larger ring is obtained by inverting \( x^v/P^k \).

We call an element of the form \( x^w/P \) with \( w \in \text{Log} \ P \) a formal monomial in \( R_P \). (It can happen that \( x^w/P \in R_P \) even if \( w \notin \text{Log} \ P \)—e.g., if \( w + \text{Log} P^k \subseteq \text{Log} P^{k+1} \) for some \( k \). These aren’t significant in what follows.)

In addition to the obvious facts about \( R_P \) (it is a commutative, finitely generated hence noetheriandomain), the following results are known [H1, H2]:

\( R_P = \{ g/P^k \mid g \in \mathbb{Z}[x] \}, \ \text{Log} \ g \subset \text{Log} P^k \}, R_P \) is a partially ordered ring with 1 as an order unit, and it is a dimension group [H1, section I];

all sums and finite intersections of order ideals are order ideals are order ideals (this is true for all dimension groups) [G];

products of order ideals are order ideals (this is not generally true for commutative partially ordered domains having 1 as an order unit and being dimension groups) [H1];

every order ideal is an order ideal (true in every partially ordered commutative ring in which 1 is an order unit) [H1, Proposition I.2];

if \( f \) is a formal monomial, then \( fR_P \) (the ideal generated by \( f \)) is an order ideal [H2; Proposition II.2A];

every order ideal is the finite sum of ideals, \( \sum f_i R_P \) where \( f_i \) are formal monomials, and all such sums are order ideals [H2, p19];

if \( f \) is a formal monomial and \( a \in R_P \), then \( fa \in R_P^+ \) implies \( a \in R_P^+ \) (follows from the definitions); the conclusion is also true if we replace \( \text{formal monomial} \) by \( \text{order unit} \), a result that is very special for \( R_P \) [H2; Proposition II.5];

the pure traces are exactly the multiplicative ones (true for any partially ordered ring with 1 as an order unit); the pure faithful traces are exactly those of the form \( \tau_r(g/P^k) = g(r)/P^k(r) \) where \( r = (r_i) \) is a strictly positive \( d \)-tuple in \( \mathbb{R}^d \), and these extend in the obvious way to positive homomorphisms \( \tau_r : \mathbb{Z}[x_i^{\pm1}; 1/P] \rightarrow \mathbb{R} \) (warning: although the ring \( \mathbb{Z}[x_i^{\pm1}; 1/P] \) is partially ordered, 1 is not an order unit for it) [H1, Theorem III.3];

the weighted moment map/Legendre transform corresponding to \( P \) implements a homeomorphism \( \partial \delta S(R_P, 1) = \text{cvx} \text{Log} \ P \) (the latter is the \( \text{Newton polytope} \) of \( P \)) sending the faithful pure traces onto the interior; unexpectedly, the set of \( \text{pure} \) traces admits a type of convex structure; in particular, the faces correspond to traces that factor through quotients in a particularly nice way [H2, Theorem IV.1];

In general, \( R_P \) is not a pure polynomial ring; only rarely does it have unique factorization [H2, Appendix A, Theorem A.8A].

Now let us consider the following property of a faithful pure trace \( \tau \equiv \tau_r \):

1. For every nonzero order ideal \( I \), \( \tau_r(I) = \tau_r(R_P) \).

By Proposition 1.1, this is one of the two necessary conditions for \( \tau_r \) to be a good trace.

Here \( r = (r_i) \in (\mathbb{R}^d)^{++} \) as described above. First we note that \( \{ fR_P \} \) (as \( f \) varies over all products of formal monomials) is a generating set of order ideals with order unit (they are given as ring ideals, but in fact are order ideals by the properties above, and every order ideal is a finite sum of these). Necessary and sufficient for (1) to hold is simply that it hold for all ideals of the form \( I_w = (x^w/P)R_P \) (where \( w \in \text{Log} \ P \), a finite set). To see this, note that \( \tau_r(I_w) = (r^w/P(r))\tau_r(R_P) \), hence \( \tau_r(I_w) = \tau_r(R_P) \) iff \( P(r)/r^w \in \tau_r(R_P) \); thus if this holds for all \( w \in \text{Log} \ P \), then each of
LEMMA 3.1 For \( r \in (\mathbb{R}^d)^{++} \), \( \tau_r \) satisfies (1) iff for all \( v \in \Log P \), \( P(r)/r^v \in Z[a^v/P]_{w \in \Log P} \).

This is a fairly drastic condition, even when \( d = 1 \) and \( P = 1 + x \) or \( 2 + 3x \).

For \( r \in (\mathbb{R}^d)^{++} \) and \( P \in Z[r_i]^+ \), let \( R_r = Z[r^v/P(r)]_{w \in \Log P} \); this is exactly \( \tau_r(R_P) \), and is a finitely generated unital subring of \( 

\text{LEMMA 3.2 Let } r = (r_i) \in (\mathbb{R}^d)^{++} \text{ and } P \in Z[r_i]^+ \text{ be projectively faithful. Then } r \text{ satisfies (1) iff } R_r = Z[r_i^{\pm 1}, P(r)^{-1}].

\text{Proof.} \text{We may construct } R_P \text{ by beginning with } Z[x_i^{\pm 1}] \text{ (the Laurent polynomial ring) instead of } Z[x_i]; \text{ this is in fact how it was originally constructed in } \text{[H1, H2]. By replacing } P \text{ by } x^v P^t \text{ for some } v \in \mathbb{Z}^d \text{ and positive integer } t \text{ (this has no effect on } R_P \text{, up to order isomorphism), we can arrange that } 0 \text{ is in the interior of } \text{cvx } \Log P \text{ and in } \Log P. \text{ Then } 1/P \in R_P \text{ and we may invert } 1/P, \text{ creating } R_P[P] = Z[x_i^{\pm 1}, P^{-1}] \text{ [H2]. Let } I = (1/P)R_P; \text{ this is an order ideal } ([H2, p19]), \text{ and } Z[x_i^{\pm 1}, P^{-1}] = \bigcup_{j \in \mathbb{Z}^*} P^j R_P.

\text{If } r \text{ satisfies (1) with respect to } P, \text{ then applying it to } I, \text{ we obtain } \tau_r(I) = \tau_r(1/P)\tau_r(R) = (1/P(r))\tau_r(R) = (1/P(r))R_r; \text{ by hypothesis, this is } R_r, \text{ so that } P(r) \in R_r. \text{ Thus } \tau_r(P^j R_P) = P^j(r)R_r \subseteq R_r. \text{ Taking the union, we obtain } \tau_r(Z[x_i^{\pm 1}, P^{-1}]) \subseteq R_r, \text{ and the reverse inclusion is trivial.}

Conversely, suppose } R_r = \tau_r(Z[x_i^{\pm 1}, P^{-1}]). \text{ Then } \tau_r(x_i^{\pm 1}) = r_i^{\pm 1} \text{ and } \tau_r(P^{\pm 1}) = P^{\pm 1}(r) \text{ belong to } R_r \text{ and are invertible therein. Hence if } f \text{ is any formal monomial, } \tau_r(f) \text{ is a product of terms of the form } r^w/P(r), \text{ hence is invertible in } R_r. \text{ Thus if } I \text{ is an order ideal, it contains a formal monomial, and } \tau_r(I) \text{ contains an invertible element in } R_r, \text{ and so } \tau_r(I) = R_r = \tau_r(R_P).

Thus } r \text{ satisfies (1).}

In other words, (1) holds iff the range of evaluation at } r \text{ on } R_P \text{ is the same as the range of the evaluation on the much larger ring } Z[x_i^{\pm 1}, 1/P].

Now we consider what (1) means in the special case of } d = 1.

Let } A \text{ be a unital subring of } 

\text{LEMMA 3.3 Let } P \text{ be a projectively faithful element of } Z[x]^+ \text{ with smallest and largest degree coefficients } a_0 \text{ and } a_k \text{ respectively. If } r \in R^{++} \text{ satisfies (1) with respect to } P, \text{ then there exist nonnegative integers } s \text{ and } t \text{ such that } a_0^s/r \text{ and } a_k^t/r \text{ are integral.}

\text{Proof.} \text{Write } P = a_0 + \sum_{0 < i < k} a_i x^i + a_k x^k \text{ where } a_i \text{ are nonnegative integers (some can be zero, but we still need } \gcd \{\{a_i \neq 0\} \cup \{k\}\} = 1. \text{ From } P(r) \in Z[r^j/P(r)]_{j \in \Log P}, \text{ we deduce an equation of the form } P(r)^{m+1} = p(r) \text{ where } p \in Z[x] \text{ and } \deg p \leq \deg P^m = km. \text{ The leading term of this expression is } a_k^{m+1} r^{(m+1)k}, \text{ and so } r \text{ satisfies a monic polynomial with coefficients from } A = Z[a_k^{-1}]. \text{ It follows that } a_k^{t/r} \text{ is integral for all sufficiently large } s.
Replacing $P$ by its reversal (also called reciprocal) $\hat{P}$ (defined by $\hat{P}(x) = P(x^{-1})x^k$), and redoing the process yields the other form, that $a_0^2/r$ is integral.

The following is true if we weaken the hypotheses on $P$ to be projectively faithful (instead of requiring all the intermediate coefficients to be strictly positive). The modifications to the proof will muddy an already-complicated but elementary argument; so we just outline it afterwards. We can replace $P$ by any power of itself, without changing anything, so the no gaps condition is just that the second largest and second smallest terms have nonzero coefficients.

**Proposition 3.4** Let $r \in \mathbb{R}^{++}$ and $P \in \mathbb{Z}[x]$ be $\sum_{i=0}^{k} a_i x^i$ where all $a_i \neq 0$. Let $a_0$ and $a_k$ be the coefficients of the least and greatest degree terms in $P$. Let $R_r = \mathbb{Z}[\{r^i/P\}_{i \in \log P}]$. Then the following are equivalent

(i) $r$ satisfies (1) with respect to $P$

(ii) there exist nonnegative integers $s$ and $t$ such that both $a_0^s r$ and $a_k^t/r$ are algebraic integers

(iii) $R_r = \mathbb{Z}[r^{\pm 1}, (P(r))^{\pm 1}]$

(iv) for all $j \in \log P$, $P(r)/r^j \in R_r$.

**Proof.** (ii) implies (iv). Without loss of generality, we may assume $P = a_0 + \sum_{0 < i < k} a_i x^i + a_k x^k$.

If $c$ is an algebraic integer, then $\mathbb{Z}[c]$ is free on the $\mathbb{Z}$-basis \{1, $c, c^2, \ldots, c^{k-1}\}$ where $e$ is the degree of $c$ (this is an alternative definition of integrality); in particular, for every positive integer $u$, we can write $e^u = \sum_{i=0}^{e-1} b_i c^i$, in other words, there exists a polynomial $p \in \mathbb{Z}[x]$ of degree at most $e - 1$ such that $c^u = p(c)$.

Apply this to $c = a_k^s r$; for each positive integer $u$, we can write $(a_k^s r)^u = p_u(a_k^s r) = q_u(r)$ where $\deg q_u \leq e - 1$. Multiplying this by $r^{-u(s-1)}$, we obtain $(a_k^s r)^u = r^{-u(s-1)} q_u$; setting $Q_u = r^{u(s-1)} q_u$, we have $(a_k^s r)^u = Q_u(r)$ where $Q_u \in \mathbb{Z}[x]$ and $\deg Q_u = u(s-1) + \deg q_u \leq u(s-1) + e - 1$. Hence (multiplying by an additional $r^j$), we have for every $j = 0, 1, 2, \ldots, Q_{u,j} \in \mathbb{Z}[x]$ such that $\deg Q_{u,j} = u(s-1) + j$ and $(a_k^s r)^{u+j} = Q_{u,j}(r)$. We will subsequently choose $u$ to be fairly large.

Now let $N$ be a (large) positive integer, and consider the $k$ leading coefficients of $P^N$, that is, the coefficients of the terms $x^{kN}, x^{kN-1}, \ldots, x^{kN-k+1}$. They are respectively divisible by $a_k^{N-1}, \ldots, a_k^{N-k+1}$ (as is trivially easy to see). Hence we may find integers $b_i$ (with $b_0 = 1$) such that

$$P^N - \sum_{i=0}^{k-1} (a_k x)^{N-i} x^{N(k-1)} b_i := G$$

is a polynomial of degree at most $Nk - k$. Assume (as we may) that $N - k = us$ for some integer $u$. Replace each $(a_k x^{N-i})$ by $Q_{u,k-i}$; this has no effect on the value at $r$. Setting $H = \sum_{i=0}^{k-1} b_i Q_{u,k-i} x^{(N-1)}$, we have $P^N(r) = (G + H)(r)$. Then

$$\deg(G + H) \leq \max \{\deg G, \deg H\}$$

$$\leq \max \left\{ \{Nk - k, \max \{\deg Q_{u,k-i} + Nk - N\}\} \right\}$$

$$\leq \max \{\{Nk - k, u(s-1) + e - 1 + Nk - N\}\} = \max \{Nk - k, Nk - N + e - 1 + N - k - u\}$$

$$\leq \max \{Nk - k, Nk - k - u + e - 1\}.$$ 

We can choose $u \geq e - 1$ at the outset, and so guarantee that $\deg(G + H) \leq Nk - k$. Thus $P(r) = (G + H)(r)/P^{N-1}(r)$. For every $0 \leq i \leq k$, $r^i/P(r) \in R_r$, and since $\deg(G + H) \leq Nk - k = \deg P^{N-1}$, we obtain $P(r) \in R_r$.

Now form the reversal of $P$, given by $\hat{P}(x) = P(x^{-1})x^k$; this reverses the roles of $a_k$ and $a_0$, and the same process (using $a_0^s/r$ being integral) yields after translating back, $P(r)/r^k \in R_r$. 12
From $P(r) \in R_r$, we obtain $r^i = (r^i/P(r)) \cdot P(r) \in R_r$ for $i \in \Log P$, and thus for all $i \geq 0$. Since $P(r)/r^k \in R_r$, we deduce $r^{-k} \in R_r$, hence $r^{-j} \in R_r$ for all $j \geq 0$; hence $P(r)/r^j \in R_r$.

Now (i) implies (ii) was done in the previous lemma, and the equivalence of (i), (iii), and (iv) follows from the general results preceding this. 

To prove the result when $P$ is projectively faithful, we can still write $P = a_0 + \sum_{1 \leq i \leq k-1} a_i x^i + a_k x^k$, only this time $\gcd \{i | a_i \neq 0\} = 1$ (equivalent after translation to projective faithfulness). Then it is elementary, and presumably well-known, that there exists $M$ such that for all $N \geq 0$, $(P^N, x^i) \neq 0$ if $M < i < kN - M$. Now in the construction above, make sure that when the multiplications by powers of $r$ take place, that the exponent lands in the interval where all the coefficients are guaranteed nonzero (we are of course free to take arbitrary large powers of $P$).

A strange consequence is that when the hypotheses on $P$ are satisfied, the set of $r$ such that $\tau_r$ satisfies (1) is closed under multiplication; this follows immediately from (ii), but not obviously from any of the other equivalent properties.

This does not appear to extend to more than one variable. For example, if $P = 2 + 3x + 5y$, and we restrict to $r = (m, n)$ with positive integer coordinates, it is tedious but routine to see that $\tau_r$ satisfies (1) with respect to $P$ iff for all primes $p$ and $q$,

$$p|m \implies p|(2+5n) \quad \text{and} \quad q|n \implies q|(2+3m).$$

For example, $(7,1), (3,11), (2^i, 2^j)$ (where both $i, j > 0$) satisfy these conditions, but $(14,2)$ does not. Of course, there may be another, more appropriate, notion of multiplication with respect to which the set is closed.

Another general property concerns approximate divisibility. Let $K = \cvx \Log P$; this is a compact convex polytope. Let $e \in K$ be an extreme point (we do not use the usual term, vertex, because this might be confused with lattice point); then $v \in \Log P$, and there is a pure trace associated with $v$, $\sigma^v$, given by $\sigma^v(g/P^k) = (g, x^{kv})/(P, x^k)$ (this can also be obtained as the limit along a path of $\tau_r$, via l’Hôpital’s rule, as in [H1, section III, especially just before III.3]).

Since every order ideal of $R_P$ is of the form $\sum f_i R_P$ (finite sum), if we assume that $R_P$ is approximately divisible, then $R_P$ is nearly divisible. Thus every order ideal has its own order unit and is approximately divisible. If $\tau$ is faithful, then $\tau(I \cap J) \neq 0$ (no finite intersections of order ideals can be zero since they are also ideals in a domain), and $I \cap J$ is itself approximately divisible, hence $\tau(I \cap J)$ is dense in $R$. Thus for any faithful trace that is order unit good for $R_P$, its restriction to any nonzero order ideal is also order unit good.

Thus we have the following.

**Lemma 3.5** The ordered ring $R_P$ is approximately divisible iff for all extreme points $v$ of $K = \cvx \Log P$, $(P, x^v) > 1$.

**Lemma 3.6** Let $P = \sum \lambda_w x^w \in \Z^{[x_i^{\pm 1}]+}$ with $(P, x^v) > 1$ for all extreme points of $K = \cvx \Log P$.

(a) Then $R_P$ is nearly divisible

(b) If $\tau$ is a faithful trace that is order unit good for $R_P$, then its restriction to any nonzero ideal is order unit good for that ideal.

If we replace $R_P$ by $S_P := R_P \otimes \Q = \Q[x^v/P]$, then it is divisible, which is of course stronger than nearly divisible, so that (a) holds automatically (without the hypothesis on the coefficients at extreme points), and (b) also holds by the same arguments.

**Proposition 3.7** Let $r = (r_i) \in (\R^d)^{++}$, and let $P \in \Z^{[x_i^{\pm 1}]+}$ be projectively faithful.

(a) the pure trace $\tau_r$ on $R_P$ is good iff
(i) \( \tau_r \) is order unit good for \( R_P \) and 
(ii) for all \( v \in \Log P, P(r)/r^v \in \Z[x^w/P(r)]_{w \in \Log P} \).

(b) the pure trace \( \tau_r \) on \( S_P \) is good iff 
(i) \( \tau_r \) is order unit good for \( R_P \).

**Remark.** Note the absence of (ii) from (b), and the appearance of \( R_P \) in (bi). It is known (along the same lines as in [BeH, Proposition 5.10]), that if \( \tau_r \) is order unit good (for either coefficient ring), then each \( r_i \) is algebraic. Since \( \Q[r_1, \ldots, r_d] \) is thus a field, (ii) is redundant in (b).

**Proof.** We show that if \( \tau_r \) is order unit good (which means that the closure of the image of \( \ker \tau_r \) in \( \Aff S(R, 1) \) is exactly \( \tau_r^+ = \{ h \in \Aff S(R, 1) \mid h(\tau_r) = 0 \} \)), then its restriction to any order ideal is also order unit good. It suffices to do this for \( I = f R_P \) where \( f \) is a formal monomial.

The map \( R_P \to f R_P \) given by \( r \mapsto fr \) is an order-isomorphism of \( R_P \) modules (this of course uses the the fact that \( fr \geq 0 \) in \( R_P \) entails \( r \geq 0 \)). Using \( f \) as an order unit for \( I \), the map on traces \( \tau \mapsto \tau/\tau(f) \) (restricted to those \( \tau \) such that \( \tau(f) \neq 0 \) sends \( \tau_r \to \tau_r/\tau_r(f) = \tau' \), and \( \ker \tau' = \ker \tau_r \cap f R_P = f \cdot \ker \tau_r \) (since \( f(r) \neq 0 \)). The map between \( R_P \) modules induces an affine homeomorphism between \( S(R_P, 1) \) and \( S(I, f) \), sending \( \tau_r \) to \( \tau' \), and it easily follows that \( \tau' \) is order unit good. But \( \tau' \) is just the normalization of \( \tau/I \), hence the latter is order unit good.

The rest follows from the preceding results.

In one variable, we can show that \( \tau_r \) is order unit good iff none of the algebraic conjugates of \( r \) (except itself) are positive real. In more than one variable, the situation is far more complicated, and there is no decisive theorem (yet).

**Example** Let \( d = 1 \) and \( P = 1 + x \); then we can rewrite \( R_P = \Z[1/P, x/P] = \Z[1 - X, X] \) where \( X = x/(1 + x) \), and the positive cone translates to \( \langle X, 1 - X \rangle \). This goes back to Renault [R]. The translation however, obscures some of the features, as we will see. First, \( R_P \) has two discrete pure traces, \( \tau_0 = \sigma^0 \) and \( \tau_\infty = \sigma^1 \) (0 and 1 are the extreme points of the convex set \( \Conv \Log P = [0, 1] \)), so is not approximately divisible. However, it is interesting to calculate the condition that \( \tau_r(I) = \tau_r(R_P) \) for all nonzero order ideals.

By 3.7 above, this amounts to \( 1 + r, 1 + 1/r \in \Z[1/(1 + r), r/(1 + r)] \); as \( r/(1 + r) = 1 - 1/(1 + r) \), the condition (1) is equivalent to \( 1 + r \pm 1 \in \Z[1/(1 + r)] \). Now for a real number \( s \), the condition \( s \in \Z[1/s] \) is equivalent to \( s \) be an algebraic integer (that is, satisfies a monic integer polynomial).

Hence we infer that if (1) holds for \( \tau_r \), then \( r \) has to be an algebraic unit (that is, not only is its minimal polynomial over the integers monic, but the constant term must be \( \pm 1 \) as well). Conversely, if \( r \) is an algebraic unit, then the desired membership property holds.

We conclude that \( \tau_r \) satisfies (1) iff \( r \) is an algebraic unit.

In particular, if \( r \) is an integer, then \( \tau_r \) satisfies (1) iff \( r = 1 \) (we are restricting ourselves to actual traces, hence excluding negative values for \( r \)).

The translation, \( X = x/(1 + x) \) converts \( r \) to \( r/(1 + r) \); then of course \( \tau(X) \) is a fractional linear transformation of an algebraic unit, but this characterization is not as pleasant as the pre-translation version.

Let \( V \subset \C^d \). For \( A \) a subring of \( \C \), define \( I_A(V) \) to be the ideal in the polynomial ring \( A[x_1, \ldots, x_d] \) consisting of polynomials that vanish at all points of \( V \). Given an ideal \( I \) of \( A[x_1, \ldots, x_d] \), define \( Z_A(I) \) to be the common zero set (in \( \C^d \)) of all elements of \( I \). The *variety generated by \( V \)* over \( A \) is simply \( Z_A I_A(V) \). If \( A = \Z \), we drop the subscript.

We say \( r = (r_i) \in (\R^d)^{++} \) is *really isolated* if \( ZI(\{r\}) \cap (\R^d)^{++} = \{r\} \). For example, if \( d = 1 \), then \( r \) is really isolated if \( r \) is algebraic and all algebraic conjugates of \( r \) other than \( r \) itself are not positive real. In general, \( r \) is really isolated means that the slice of the variety generated by \( r \) (or more simply, the Zariski closure of \( \{r\} \)) by the positive orthant contains only \( r \).
The argument in [BeH, 5.10] shows that if \( r \) is really isolated (or more generally, \( \{ r \} \) is an isolated point in \((\mathbb{R}^d)^+ \cap ZI(\{ r \})\), then all of its coordinates are algebraic (there is an assumption in [op cit] concerning interior points which is automatic here). We remind the reader that we have assumed that \( P \) is projectively faithful, which implies in particular, that its Newton polytope contains a \( d \)-ball.

The condition that \( r \) be really isolated appears in the examples in [BeH, Examples 5 & 10], for which the relevant dimension groups are remotely related to the ones appearing here.

**Proposition 4.8** Suppose \( R_P \) is approximately divisible, and \( \tau \) is a pure faithful trace. Then

(a) \( \tau \) is an order unit good trace of \( R_P \) if and only if \( \tau \) is an order unit good trace of \( R_P \).

(b) \( \tau_r \) is a good trace of \( R_P \) if and only if \( \tau_r \) is really isolated, and for all \( v \in \log P \), \( P(r)/v \) is of the form \( \mathbb{Z}\{r^w/P(r)\}_{w \in \log P} \).

(c) \( \tau_r \) is a good trace of \( R_P \otimes \mathbb{Q} \) if and only if \( \tau_r \) is really isolated.

**Proof.** Every pure faithful trace of \( R_P \) is of the form \( \tau_r \) for (unique) \( r \) in the positive orthant.

If \( r \) is not really isolated, then there exists \( r' \in (\mathbb{R}^d)^+ \) such that every polynomial that vanishes at \( r \) also vanishes at \( r' \). Suppose \( a := g/P^k \in R_P \); we may assume \( \log g \subseteq \log P^k \). Then \( \tau_r(a) = 0 \), then \( g(r') = 0 \), hence \( g(r') = 0 \), whence \( \tau_r(a) = 0 \); thus with \( \sigma = \tau_r \), we have \( \sigma \in \partial_s S(R, 1) \setminus \{ \tau_r \} \) such that \( \sigma \ker \tau_r \equiv 0 \). Hence \( \tau_r \) is not order unit good. The same of course applies with \( R_P \otimes \mathbb{Q} \) in place of \( R_P \).

Conversely, suppose that \( r \) is really isolated, but there exists \( \sigma \in \partial_s S(R, 1) \setminus \{ \tau_r \} \) such that \( \sigma \ker \tau_r = 0 \). Then \( \sigma \) cannot be faithful (as otherwise, \( \sigma = \tau_r \), for some \( \tau' \in (\mathbb{R}^d)^+ \), and \( \tau' \in ZI(\{ r \}) \)). Consider \( S = R_P \otimes \mathbb{Q} \), and let \( T_r, \Sigma \) be the extension to \( S \) of \( \tau_r \) and \( \Sigma \) (both extend, since the ranges are torsion-free abelian groups). Then \( T_r(S) = Q[r^w/P(r)] \), which is a field (since the coordinates are algebraic, so are all the \( r^w/P(r) \)). Then \( \ker \Sigma \) is a field, so \( \ker T_r \) is a maximal ideal. Also, \( \ker T_r \cap R_P = \ker \tau_r \) and \( \ker \Sigma \cap R_P = \sigma \). If \( \ker \tau_r \subseteq \ker \sigma \), then \( \ker T_r \subseteq \ker \Sigma \), but maximality of \( \ker T_r \) implies \( \ker T_r = \ker \Sigma \), and thus \( \ker \tau_r = \ker \sigma \). However, since \( \sigma \) is not faithful, ker \( \sigma \) contains a positive nonzero element of \( R_P \), whereas ker \( \tau_r \) does not, a contradiction.

Hence if \( r \) is really isolated, then \( \sigma \in \partial_s S(R, 1) \setminus \{ \tau_r \} \) implies \( \sigma(\ker \tau_r) \neq 0 \), and by Lemma 1.7 above, this implies \( \tau_r \) is order unit good. The same of course applies to \( T_r \) as a trace on \( S_P \). This yields (a), and contributes to (c).

Part (b) now follows from preceding results in this section.

Part (c) comes from \( Q[r^w/P(r)] \) being a field (which in turn arises because the coordinates of \( r \) are algebraic), so that condition (1) is automatic.

A particular consequence is that the set of good pure faithful traces of \( S_P = R_P \otimes \mathbb{Q} \) is the same for all choices (with \( d \) fixed) of faithfully projective \( P \in \mathbb{Z}[x_i]^+ \) (or \( P \in \mathbb{Q}[x_i]^+ \)), whereas for \( R_P \), there is dependence on \( P \).

When \( d = 1 \), the conditions for \( \tau_r \) to be good are precisely that no distinct algebraic conjugate of \( r \) be positive and the integrality condition, (ii), of Proposition 3.4.

**Example** Let \( d = 1 \) and \( P = 2 + 3x \). By Proposition 3.4, the positive real number \( r \) satisfies (1) iff there exists \( s \) such that both \( 2s/r \) and \( 3s/r \) are integral. Let \( K = Q(r) \), and \( Z_K \) the ring of integers in \( K \). The fractional ideal \( r Z_K \) factors as \( \prod P_i \cap \prod Q_j \) (where \( P_i \) and \( Q_j \) are prime ideals in \( Z_K \), and we allow repetitions; the products might also be over the empty set). The intersections \( P_i \cap Z \) and \( Q_j \cap Z \) determine primes in \( Z \), denoted respectively \( p_i \) and \( q_j \). Then (1) is equivalent to \( p_i = 2 \) and \( q_j = 3 \) for all \( i \) and \( j \). Hence \( \tau_r \) is good for \( R_P \) iff no non-identity algebraic conjugate is positive and the prime factorization of the fractional ideal \( r Z_K \) consists of primes sitting over 2 in the numerator and over 3 in the denominator.
In this section, we have restricted ourselves to pure faithful traces; this is a technical convenience. By the comment after Proposition 1.1, we can factor out the maximal order ideal contained in the kernel of a trace, and in the case that the dimension group is \( R_P \), these correspond to quotients corresponding to faces of the Newton polytope ([H1, section VII]). This amounts to a reduction to a lower dimensional lattice and vector space, that is, a polynomial in fewer variables.

There are related naturally occurring classes of dimension groups whose pure traces can be similarly analyzed. For example, for the matrix-valued random walks appearing in [H5], in non-degenerate cases, the pure faithful traces are similarly parameterized by the positive orthant (the non-faithful traces are generically terrible, but can be analyzed in reasonable cases). An amusing example appears in [P], where very specific local limit asymptotics were used to derive the one-degenerate cases, the pure faithful traces are similarly parameterized by the positive orthant (again via the large eigenvalue function, an algebraic function), which are indexed by the unit interval.

5 Direct sums and goodness

For (noncyclic) simple dimension groups, there is a notion of direct sum (corresponding to coproduct; see [BeH, Appendix 2] for a discussion). This actually extends to nearly divisible dimension groups. Let \( G \) and \( H \) be nearly divisible. Form the group direct sum \( G \oplus H \), and impose on it the ordering given by the positive cone

\[
\{(g, h) \mid g \in G^+ \setminus \{0\} \text{ and } h \in H^+ \setminus \{0\}\} \cup \{(0, 0)\}.
\]

When \( G \) and \( H \) are simple (and noncyclic) dimension groups, the resulting strict direct sum \( G \oplus_s H \) (s for strict) is also a simple dimension group. It is each to check that when both \( G \) and \( H \) are nearly divisible, then so is \( G \oplus_s H \) (trivial), and when both are additionally dimension groups, so is the strict direct sum. We suppress the subscript \( s \).

If \( K = G \oplus H \), and \( \sigma \) and \( \tau \) are traces on \( G \) and \( H \) respectively, we consider the possibility that \( \phi := \sigma \oplus \tau \) (defined by \((g, h) \mapsto \sigma(g) + \tau(h)) \) be good or order unit good. If we put the usual direct sum ordering on \( K \) and consider the question of characterizing when \( \phi \) is good, the answer is not exciting. However, if we put the strict direct sum ordering on \( K \), then order unit goodness is interesting. Iteration of this process yields some weird examples.

**Lemma 5.1** [A consequence of the method of proof of [BeH, Proposition 1.7]] Suppose \((K, \omega)\) is an approximately divisible dimension group with order unit, and \( \phi \) is an order unit good trace. Then whenever \( a \in G \), \( b \in G^{++} \) and \( 0 < \phi(a) < \phi(b) \), for all \( \epsilon > 0 \), there exists \( a' \in [0, b] \) such that \( \phi(a') = a \) and \( ||a' - \hat{b}\sigma(a)/\sigma(b)|| < \epsilon \).

**Proof.** Approximate divisibility implies density of \( G \) in \( \text{Aff} S(G, u) \). Set \( j = \sigma(b)\hat{b}/\sigma(a) \), so that \( j(\sigma) = \sigma(a) \) and \( \inf j = \sigma(a)\sigma(b)^{-1}\inf \hat{b} \). There exists \( g_n \in G \) such that \( \hat{g}_n \to j \) uniformly. If for infinitely many \( n \), \( g_n(\sigma) = \sigma(a) \), we are done (taking large enough \( n \)). Otherwise, select \( \sigma(a)(\sigma(2b)^{-1}\inf b > \epsilon > 0 \) and \( ||\hat{g}_n - j|| < \epsilon \), then \( \sigma(g_n) - \sigma(a) < \epsilon \) provided \( n \) is sufficiently large; if \( \sigma(g_n) > \sigma(a) \), set \( c_n = g_n - a \). There exists an order unit \( z_n \) such that \( 0 < \sigma(c_n)1 < z_n < 2\epsilon \). By order unit goodness, there exists \( v_n \ll z_n \) such that \( \sigma(c_n) = \sigma(v_n) \), and of course, \( ||v_n|| < ||z_n|| < 2\epsilon \). Then \( g_n - v_n \) has image within \( 3\epsilon \) of \( j \), and it is easy to check that \( g_n - v_n \) is strictly positive, hence is an order unit.
If instead, \( \sigma(a) > \sigma(c_n) \) for infinitely many \( n \), we obtain a corresponding \( c_n = g_n - a \) and \( v_n \ll z_n \), and this time, \( g_n + v_n \) has all the right properties. In both cases, by taking \( n \) sufficiently large, we make the error terms go to zero, hence obtain the \( a' \) as one of \( g_n \pm v_n \).

In the following, the function \( \psi \) will not be a group homomorphism (just a function, and usually a weird one, if it exists at all).

**Lemma 5.2** Suppose \( G \) and \( H \) are nearly divisible dimension groups, each with order unit, and respective trace \( \sigma \) and \( \tau \). Let \( K = G \oplus H \) with the strict ordering, and suppose that the trace on \( K \), \( \phi := \sigma \oplus \tau \) is order unit good. Then provided the following condition holds, \( \sigma \) is order unit good as a trace on \( G \):

there exists a function \( \psi : \tau^{-1}(\sigma(G) \cap \tau(H)) \to \sigma^{-1}(\sigma(G) \cap \tau(H)) \) that is pseudo-norm continuous with the additional property that \( \sigma \psi = \tau \).

**Remark.** As we will see below, without the weird extra condition, the result fails.

**Proof.** Select an order unit \( b \) in \( G \), and \( a \) in \( G \) such that \( 0 < \sigma(a) < \sigma(b) \). As \( H \) is approximately divisible, there is a sequence of order units in \( H \), \( (h_n) \), such that \( h_n \to 0 \) (with respect to the pseudo-norm topology on \( H \); equivalently, as functions on \( S(H, v) \), \( \hat{h}_n \) converges uniformly to zero). There also exists \( \delta \) in \( G \) such that \( \sigma(b - a)/4 < \hat{\delta} < \min \{ \sigma(b - a)/2, \inf_{b \in S(G, u)} \theta(b)/2 \} \) uniformly on \( S(G, u) \). Then \( B_n := (b - \delta, h_n) \) are order units of \( G \oplus H \), and \( \phi(a, 0) < \phi(B_n) = \sigma(b) - \sigma(\delta) + \tau(h_n) \).

Since \( \phi \) is order unit good and each \( B_n \) is an order unit, there exist \( (a_n, z_n) \) such that \( 0 \ll a_n \ll b - \delta \) and \( 0 \ll z_n \ll h_n \) with \( \phi(a_n, z_n) = \sigma(a) \), and by the previous lemma, \( \inf_{S(G, u)} \hat{a}_n \) is bounded below (as \( n \to \infty \)); in particular, \( \|z_n\|_H \to 0 \) and \( \sigma(a_n) + \tau(z_n) = \sigma(a) \). Thus \( z_n \in \tau^{-1}(\sigma(G) \cap \tau(H)) \), so we may consider the sequence \( \psi(z_n) \in \sigma^{-1}(\sigma(G) \cap \tau(H)) \). Since \( \psi \) is pseudo-norm continuous, \( \psi(z_n) \to 0 \) uniformly on \( S(G, u) \).

Consider \( a_n + \psi(z_n) \); its value at \( \sigma \) is \( \sigma(a_n) + \sigma(\psi(z_n)) = \sigma(a_n) + \tau(z_n) = \sigma(a) \). If we choose \( n \) sufficiently large that \( \|\psi(z_n)\| < \inf \delta \), then \( a_n + \psi(z_n) \ll b - \delta + \delta = b \). In addition, we can also choose \( n \) sufficiently large that \( \inf_{S(G, u)} \hat{a}_n \), by the uniform boundedness below of the \( a_n \) (there is no guarantee that \( \psi(z_n) \) is positive). Then \( a_n + \psi(z_n) \) is an order unit in the interval \([0, b]\) and we are done.

One advantage of not requiring normalization of \( \sigma \) and \( \tau \) is that we can replace them by any positive scalar multiples, in testing for order unit goodness of \( \lambda \sigma \oplus \mu \tau \); the first hypotheses are unchanged, but the second translates to density of \( (\lambda \sigma(G)) \cap (\mu \tau(G)) \) in \( R \). In the following, we cannot apply earlier results directly, since \( G \oplus 0 \) is not an order ideal of \( G \oplus H \) (strict ordering).

**Lemma 5.3** Suppose that \( \sigma \) is a trace on \( G \), \( \tau \) is a trace on \( H \), and \( \sigma \oplus \tau = \phi \) is order unit good for \( K = G \oplus H \) with the strict ordering, and moreover assume that each of \( G \) and \( H \) is approximately divisible. Then \( \sigma(G) \cap \tau(H) \) is dense in \( R \).

**Proof.** We use the characterization of order unit good traces on approximately divisible dimension groups, namely \( \ker \phi \) has dense image in \( \phi^* \) [BeH, Proposition 1.7].

Suppose the intersection is not dense; then exists real \( \delta \geq 0 \) such that \( \sigma(G) \cap \tau(H) = \delta Z \). We have that \( \ker \phi \) has dense range in \( \text{Aff} S(K, (u, v)) = \text{Aff} S(G, u) \times \text{Aff} S(H, v) \). But \( \ker \phi = \{(g, h) \in G \oplus H \mid \sigma(g) = -\tau(h)\} \).

If \( \delta = 0 \), then \( \ker \phi = \ker \sigma \oplus \ker \tau \) (since \( \sigma(g) = -\tau(h) \) implies \( \sigma(g) \in \tau(H) \cap \sigma(H) \), hence is zero). The image of \( \ker \phi \) is then contained in \( \sigma^* \times \tau^* \), which is closed and of codimension two in \( \text{Aff} S(K, (u, v)) \), and so \( \ker \phi \) cannot be dense in \( \phi^* \) (which as codimension one), hence \( \phi \) cannot be order unit good.

If \( \delta \neq 0 \), select \( g \) and \( h \) in \( G \) and \( H \) respectively such that \( \sigma(g) = \delta = \tau(h) \). Then it is easy to see that \( \ker \phi = (\ker \sigma \oplus \ker \tau) + (g, -h)Z \), and then its range is contained in \( (\sigma^* \times \tau^*) + (g, -h)Z \).
However, the latter is closed (easy to see), and so the image of \( \ker \phi \) is contained in a proper closed subspace (with a discrete direct summand) of \( \phi^\perp \), hence in this case as well, \( \phi \) is not order unit good.

Now we want to determine when \( \sigma \oplus \tau \) is good or order unit good. Let \( \pi_G : G \oplus H \to G \) and \( \pi_H : G \oplus H \to H \) be the obvious projection maps. Unlike the inclusions \( G, H \to G \oplus H \), these are order-preserving. First, consider \( \sigma \circ \pi_G : \ker \phi \to \sigma(G) \cap \tau(H) \subseteq R \). The kernel is exactly \( \ker \sigma \oplus \ker \tau \); we also note that \( \sigma \) extends to a map \( \Sigma : \phi^\perp \to R \) (sending \((j,l)\) to \(j(\sigma)\)), the kernel of which is \( \sigma^\perp \times \tau^\perp \). With the identification of \( \text{Aff} \, S(K, (u,v)) \) with \( \text{Aff} \, S(G, u) \times \text{Aff} \, S(H, v) \), we have the following diagram.

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \ker \sigma \oplus \ker \tau & \longrightarrow & \ker \phi & \sigma \circ \pi_G & \sigma(G) \cap \tau(H) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\ker \sigma \times \ker \tau & \longrightarrow & \ker \phi & \Sigma & \longrightarrow & \mathbb{R} & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \sigma^\perp \times \tau^\perp & \longrightarrow & \phi^\perp & \Sigma & \longrightarrow & \mathbb{R} & \longrightarrow & 0
\end{array}
\]

The long horizontal overlines indicate closure, as subgroups of the affine function vector spaces; of course, there is no requirement that any of the three overlined groups be real vector spaces (they are norm-complete subgroups). The two leftmost top vertical arrows are just induced by the affine representations; the right one is the inclusion, compatible with \( \Sigma \) restricted to the image of \( \ker \phi \). The two leftmost bottom vertical arrows are the obvious inclusions. The \( \Sigma \) in the middle row is an abuse of notation; it represents the restriction of \( \Sigma \) to \( \ker \phi \), the closed subgroup of \( \phi^\perp \), but the notation is already rather complex.

The middle row need not be exact at either end (for example, if \( \ker \phi \) has dense image in \( \phi^\perp \) but one or both of \( \ker \sigma \) or \( \ker \tau \) does not have dense image in \( \sigma^\perp \) or respectively \( \tau^\perp \), then it is not left exact; if \( \sigma(G) \cap \tau(H) \) is discrete, then the middle line is not right exact).

If \( \ker \phi \) has dense image in \( \phi^\perp \), then \( \sigma(G) \cap \tau(H) \) is a dense subgroup of \( R \): we simply note that density of the image of \( \ker \phi \) in \( \phi^\perp \), the latter being a closed and therefore a norm-complete subspace of \( \text{Aff} \, S(K, (u,v)) \), entails that for every bounded linear functional that is not zero on \( \phi^\perp \), its restriction to a dense subgroup must be not zero and have dense range in the reals.

It also leads to a straightforward proof that if \( \ker \sigma \) and \( \ker \tau \) have dense images in \( \sigma^\perp \) and \( \tau^\perp \) respectively, and if \( \sigma(G) \cap \tau(H) \) is a dense subgroup of \( R \), then \( \ker \phi \) has dense image in \( \phi^\perp \). We have that \( \sigma^\perp \times \tau^\perp = \ker \sigma \times \ker \tau \subseteq \ker \phi \subseteq \phi^\perp \). The left and right terms of these inclusions are vector spaces, and since \( \sigma^\perp \times \tau^\perp \) is a closed codimension two subspace of \( \text{Aff} \, S(K, (u,v)) \) and \( \phi^\perp \) is codimension one, it follows that \( \sigma^\perp \times \tau^\perp \) is a codimension one subspace of \( \phi^\perp \). (The proof does not stop here—we do not know that \( \ker \phi \) is a real vector space.)

The map \( \Sigma \) induces \( \ker \phi/(\ker \sigma \oplus \ker \tau) \) to a subgroup of the reals. However, this subgroup of the reals includes the dense subgroup \( \sigma(G) \cap \tau(H) \), and as \( \ker \phi \) is a norm-complete abelian group, the image must be complete, and thus must be onto. In addition, since \( \ker \Sigma = \sigma^\perp \times \tau^\perp = \ker \sigma \times \ker \tau \), it follows that \( \ker \Sigma \cap \ker \phi = \ker \sigma \times \ker \tau \). We thus have \( \ker \Sigma \subseteq \ker \phi \subseteq \phi^\perp \), but \( \Sigma \) induces equality \( \ker \phi/(\ker \Sigma \cap \ker \phi) = \phi^\perp/\ker \Sigma \). It follows immediately that \( \ker \phi = \phi^\perp \).
Now we can show that if the closure of the images of \( \ker \sigma \) and \( \ker \tau \) are real vector spaces, and if \( \ker \phi \) is order unit good, then \( \sigma \) and \( \tau \) are order unit good.

We wish to show \( \ker \sigma \times \ker \tau = \sigma^r \times \tau^r \) (from this it follows trivially that \( \ker \sigma = \sigma^r \) and \( \ker \tau = \tau^r \)). Since the left-hand is a vector space, and a complete normed abelian group (hence a closed vector subspace of \( \text{Aff}(S(K,(u,v))) \), if equality does not hold, there exists a bounded linear functional \( \alpha \) on \( \text{Aff}(S(K,(u,v))) \) that kills \( \ker \sigma \times \ker \tau \) but not \( \sigma^r \times \tau^r \); in particular, \( \alpha \) does not kill \( \phi^r \).

By composition with the affine representation, we “restrict” \( \alpha \) to a real-valued bounded group homomorphism \( \beta : G \oplus H \to \mathbb{R} \) (for a treatment of bounded group homomorphisms on dimension groups, see [G]; their behaviour is just what you’d expect). Since \( \alpha \) kills \( \ker \sigma \times \ker \tau \), it follows that \( \beta \) kills \( \ker \sigma \oplus \ker \tau \). We form the normed abelian group \( \ker \phi/(\ker \sigma \oplus \ker \tau) \), which via \( \sigma \), we know to be \( \sigma(G) \cap \tau(H) \subseteq \mathbb{R} \). Thus \( \beta \) induces a bounded real-valued group homomorphism on \( \ker \phi/(\ker \sigma \oplus \ker \tau) \), call it \( \overline{\beta} \). We thus have two bounded group homomorphisms on the quotient, \( \overline{\beta} \) and \( \overline{\sigma} \), but as the quotient is isomorphic (as a normed abelian group) to a subgroup of the reals, there must be a positive real number \( \lambda \) such that \( \overline{\beta} = \lambda \overline{\sigma} \).

This forces \( \beta = \lambda \cdot \sigma \circ \pi_G \) (as bounded group homomorphisms on \( \ker \phi \)). Since \( \ker \phi \) has dense image in its completion (!) which happens to be \( \phi^r \), we have that \( \alpha \mid \phi^r = \lambda \Sigma \). Thus \( \alpha \) kills \( \sigma^r \times \tau^r \), a contradiction.

To summarize, we have the following results.

**PROPOSITION 5.4** Suppose \((G,u,\sigma)\) and \((H,v,\tau)\) are approximately divisible dimension groups with order unit and distinguished trace, and form \( K = G \oplus H \), and the trace \( \phi = \sigma \oplus \tau : K \to \mathbb{R} \).

(a) If \( \phi \) is order unit good (with respect to either the usual or the strict ordering on \( K \)), then \( \sigma(G) \cap \tau(H) \) is a dense subgroup of the reals, and \( \sigma \otimes 1_Q \) and \( \tau \otimes 1_Q \) are order unit good as traces on \( G \otimes Q \) and \( H \otimes Q \) respectively.

(b) If the closure of the images of \( \ker \sigma \) and \( \ker \tau \) in \( \sigma^r \) and \( \tau^r \) respectively are real vector spaces, and if \( \phi \) is order unit good, then both \( \sigma \) and \( \tau \) are order unit good.

(c) If \( \sigma \) and \( \tau \) are order unit good and \( \sigma(G) \cap \tau(H) \) is dense in \( \mathbb{R} \), then \( \phi \) is order unit good.

Examples exist (given below) with \( G \) and \( H \) simple dimension groups to show that if \( \phi \) is order unit good, then neither \( \sigma \) nor \( \tau \) (or exactly one of them) need be order unit good.

This method suggest what we should do with multiple traces. Let \((G_i,u_i,\sigma_i), i = 1, 2, \ldots, n\), be approximately divisible dimension groups, each with order unit and (unnormalized) trace. Form \( K = \bigoplus G_i \) with the strict ordering, and \( \phi = \sigma_1 \oplus \sigma_2 \oplus \cdots \oplus \sigma_n : K \to \mathbb{R} \), and the map \( T : K \to \mathbb{R}^n \) defined by \( \phi((g_i)) = \sum \sigma_i(g_i) \) and \( T((g_i)) = (\sigma_1(g_1), \sigma_2(g_2), \ldots, \sigma_n(g_n)) \). Identify \( \text{Aff}(S(K,(u_i))) \) with the cartesian product \( \text{Aff}(S(G_1,(u_1))) \times \cdots \times \text{Aff}(S(G_n,(u_n))) \).

If \((g_i) \in \ker \phi\), then \( \sigma_n(g_n) = -\sum_{i=1}^{n-1} \sigma_i(g_i) \), and we can interchange \( n \) with any other integer less than \( n \). In particular, \( V := \sigma^{-1}_n(\sigma_n(G_n) \cap (\sum_{i=1}^n \sigma_i(G_i))) \) is independent of permutations and the range of \( T \) on \( \ker \phi \) is \( T(V) \).

Extend \( T \) to \( \mathcal{T} : \text{Aff}(S(K,(u_i))) \to \mathbb{R}^n \) (sending \((j_i)\) to \((j_i(\sigma_i))\)). Restricted to \( \phi^r \), the range of \( \mathcal{T} \) is exactly \((1,1,1,\ldots,1)^\perp \), i.e., the entries add to zero.

Now we can form the diagram analogous to the previous one.
(b) If the closure of the image of \( \ker \sigma \) functional homomorphism group homomorphism on the quotient, \( T \) closed codimension \( n \) of \( \sigma \) \( \ker \) closure are linear combinations of the coordinate function \( s \), which correspond to the of uniqueness arising from the relation that the sum of the coefficients is zero.

We quickly see that density of \( T(V) \) (a subgroup of \( \mathbb{R}^n \) contained in \( (1, \ldots, 1)^\perp \)) in \( (1, \ldots, 1)^\perp \) is necessary that \( \phi \) be order unit good, that is, it is necessary in order that \( \ker \phi \) have norm dense image in \( \phi^\perp \).

Suppose all the \( \sigma_i \) are order unit good and \( T(V) \) is dense in \( (1, \ldots, 1)^\perp \). Then \( \ker \sigma_1 \times \cdots \times \ker \sigma_n = \sigma_1^\perp \times \cdots \times \sigma_n^\perp \) is a closed subspace of \( \phi^\perp \), and the middle line yields that the codimension of \( \ker \phi \) in \( \text{Aff} S(K) \) is \( n - (n - 1) = 1 \), so being a closed subspace of the codimension one space \( \phi^\perp \), \( \ker \phi \) must equal it, and therefore \( \phi \) is order unit good.

Suppose \( \phi \) is order unit good (hence we have density of \( T(V) \) in \( (1, \ldots, 1)^\perp \)) and each of \( \ker \sigma_1 \) is a vector space. To show \( \sigma_i \) are all order unit good, sufficient is that \( \ker \sigma_i \) have dense image in \( \sigma_i^\perp \), and it is easy to show sufficient for this is that \( \ker \sigma_1 \times \cdots \times \ker \sigma_n \) equals \( \sigma_1^\perp \times \cdots \times \sigma_n^\perp \).

We note that the bounded real-valued group homomorphisms on \( T(V) \) and of course on its closure are linear combinations of the coordinate functions, which correspond to the \( \sigma_i \), with lack of uniqueness arising from the relation that the sum of the coefficients is zero.

By assumption, each \( \ker \sigma_i \) is a vector space (and of course closed in \( \text{Aff} S(G_i, u_i) \), whence the whole batch \( L := \ker \sigma_1 \times \cdots \times \ker \sigma_n \) is a closed subspace of \( M := \sigma_1^\perp \times \cdots \times \sigma_n^\perp \) (which is itself a closed codimension \( n \) subspace of \( \text{Aff} S(K) \)). If they are not equal, there exists a bounded linear functional \( \alpha \) on \( \text{Aff} S(K, (u_i)) \) that kills \( L \) but not \( M \). This induces a bounded real-valued group homomorphism \( \beta \) on \( \ker \phi \), which kills \( \ker \sigma_1 \oplus \cdots \oplus \ker \sigma_n \). Hence it induces a bounded real-valued group homomorphism on the quotient, \( T(V) \), \( B : T(V) \to \mathbb{R} \).

Each \( \sigma_i \) induces \( \Sigma_i \) on \( T(V) \), and these are the coordinate functions. Hence there exist real \( \lambda_i \) such that \( B = \sum \lambda_i \Sigma_i \). Thus \( \beta - \sum \lambda_i \sigma_i \) vanishes identically on \( \ker \phi \), and by density, \( \alpha = \sum \lambda_i \sigma_i \) (where \( \sigma_i \) is now interpreted as the map \( (j_i) \mapsto j_i(\sigma_i) \) on \( \text{Aff} S(K) \)). But this obviously kills \( \sigma_1^\perp \times \cdots \times \sigma_n^\perp \), a contradiction. Hence each \( \sigma_i \) is order unit good.

To summarize what happens with multiple traces:

**THEOREM 5.5** Let \((G_i, u_i, \sigma_i)\) be approximately divisible dimension groups with order unit \((u_i)\) and (unnormalized) trace \((\sigma_i)\). Form \( K = \oplus G_i \) (with the strict ordering), and the trace \( \phi = \oplus \sigma_i \) on \( K \). Set \( J = \sigma_n(G_n) \cap \left( \sum_{i \leq n-1} \sigma_i(G_i) \right) \), a subgroup of \( \mathbb{R} \).

(a) If \( \phi \) is order unit good, then \( T(\sigma_n^{-1}(J)) \) is dense in \((1,1,\ldots,1)^\perp \).
(b) If the closure of the image of \( \ker \sigma_i \) in \( \sigma_i^\perp \) is a real vector space for all \( i \), and if \( \phi \) is order unit good, then all \( \sigma_i \) are order unit good.
(c) If the image of \( \ker \sigma_i \) is dense in \( \sigma_i^\perp \) for all \( i \) (that is, each \( \sigma_i \) is order unit good), and if \( T(\sigma_n^{-1}(J)) \) is dense in \((1,1,\ldots,1)^\perp \), then \( \phi \) is order unit good.
The conditions for order unit goodness with \( n \) direct summands are slightly different, in that they involve the density of a subgroup of \( \mathbb{R}^{n-1} \) (identified with \((1,\ldots,1)^\perp\)), or simply that the closure of \( T(V) \) is a vector space of dimension \( n - 1 \) (in general, the closure need not be a vector space). To some extent, this explains some of the phenomena illustrated in the examples below, with direct sums of two not yielding an order unit good trace, but direct sums of three doing so. In fact, the argument in the example, \( G_n = \mathbb{Z} + (\sqrt{3} + n\sqrt{2})\mathbb{Z} \), essentially boils down to showing the closure of \( T(V) \) is a two-dimensional vector space. But actually calculating with \( T(V) \) seems awkward.

However, in some cases computation is moderately feasible. Suppose \( G_1 = \mathbb{Z} + \sqrt{6}\mathbb{Z}, G_2 = \mathbb{Z} + \sqrt{15}\mathbb{Z}, \) and \( G_3 = \mathbb{Z} + \sqrt{10}\mathbb{Z} \). Then \( T(V) \) is discrete, so \( \sigma_1 \oplus \sigma_2 \oplus \sigma_3 \) is not order unit good. However, if we add a fourth term, \( G_4 = \mathbb{Z} + (\sqrt{6} + \sqrt{15} + \sqrt{10})\mathbb{Z}, \) then \( \ker \phi \cap \sigma_4^{-1}(G_4 \cap (\sum G_i)) = \{(a + b\sqrt{6}, c + b\sqrt{15}, d + b\sqrt{10}, -(a + c + d) - b((\sqrt{6} + \sqrt{15} + \sqrt{10}))) \mid a, b, c, d \in \mathbb{Z}\} \). Let \( v_1 = (1, 0, 0, -1), v_2 = (0, 1, 0, -1), \) and \( v_3 = (0, 0, 1, -1) \); then \( \ker \phi \) is the \( \mathbb{Z} \)-span of \( \{v_1, v_2, v_3, \sqrt{6}v_1 + \sqrt{15}v_2 + \sqrt{10}v_3\} \).

The map from \( \ker \phi \) to \( \mathbb{R}^3 \) given by \( v_i \mapsto e_i \) (standard basis elements) has range the free abelian group on \( \{e_1, e_2, e_3, \sqrt{6}e_1 + \sqrt{15}e_2 + \sqrt{10}e_3\} \). Since \( \{1, \sqrt{6}, \sqrt{15}, \sqrt{10}\} \) is linearly independent over the rationals, this group is dense. It is trivial that \( \{v_i\} \) is a real basis for \( \phi^* \), so \( \phi \) is good. In this example, all the ker \( \sigma_i \) are trivial, so \( T(V) \) is all of ker \( \phi \).

On the other hand, if we omit any one or two of the \( G_i \), the resulting trace is not order unit good, since the resulting \( T(V) \) will be discrete.

We can similarly construct \((G_i, \sigma_i)\) (the \( G_i \) subgroups of the reals), \( i = 1, \ldots, n \) such that \( \oplus_{i=1}^n \sigma_i \) is order unit good, but for no proper subset \( J \) of \( \{1, 2, \ldots, n\} \) with \( |J| > 1 \) is \( \oplus_{i \in J} \sigma_i \) order unit good: Let \( \{p_i\}_{i=1}^n \) be distinct primes; set \( G_i = \mathbb{Z} + \sqrt{p_i}\mathbb{Z} \) for \( 1 \leq i \leq n - 1 \), and \( G_n = \mathbb{Z} + \left(\sum_{i=1}^{n-1} \sqrt{p_i}\right)\mathbb{Z} \). The resulting \( T(V) \) will be a critical group of rank \( n \), so all subgroups of lesser rank are discrete.

**Example 5.6** Simple dimension groups \((G, \sigma)\) and \((H, \tau)\) with traces such that \( \phi = \sigma + \tau \) is (order unit) good on the strict direct sum \( K = G \oplus H \), but \( \sigma \) is not good as a trace on \( G \) (and in one case, \( \tau \) is good, in another case, it is not).

**Proof.** For simple dimension groups (as \( G, H, \) and \( K \) are), order unit goodness is equivalent to goodness. Begin with three subgroups of the reals,

\[
\begin{align*}
G_1 &= \mathbb{Z} + \sqrt{3}\mathbb{Z} + \sqrt{5}\mathbb{Z} \\
G_2 &= \mathbb{Z} + \sqrt{2}\mathbb{Z} + \sqrt{5}\mathbb{Z} \\
G_3 &= \mathbb{Z} + (\sqrt{3} + \sqrt{2})\mathbb{Z}.
\end{align*}
\]

Let \( \tau_i \) denote the respective identifications of \( G_i \) with subgroups of the reals; these are traces on each of these three totally ordered dimension groups. Each \( \tau_i \) is the unique (up to scalar multiple) trace, so is good. Now form \( L = G_1 \oplus G_2 \) with the strict order; since both subgroups contain \( \mathbb{Z} + \sqrt{5}\mathbb{Z} \), which is dense, it follows from the criterion above that \( \tau_1 \oplus \tau_2 \) is a good trace thereon. Next, form \( K = L \oplus G_3 \), with the strict order; since the value group of \( \tau_1 \oplus \tau_2 \) (\( L \)) includes \( \mathbb{Z} + (\sqrt{3} + \sqrt{2})\mathbb{Z} \) and the latter is dense, we can apply the criterion again, and so deduce that \( \tau_1 \oplus \tau_2 \oplus \tau_3 \) is good, as a trace on \( K \).

However, we can obtain \( K \) by proceeding in a different order. Set \( G = G_1 \oplus G_3 \) with the strict order. Either by direct examination or by the necessity of the density condition, \( \tau_1 \oplus \tau_3 \) is not good (note in particular, that the intersection of the value groups is just \( \mathbb{Z} \)). Let \( H = G_2 \); then the obvious permutation order isomorphism which takes \( K \) to \( G \oplus H \) takes \( \tau_1 \oplus \tau_2 \oplus \tau_3 \) to \( \tau_1 \oplus \tau_3 \oplus \tau_2 \), hence the latter is good. But with \( \sigma = \tau_1 \oplus \tau_3 \) and \( \tau = \tau_2 \), we have that \( \sigma \) is not good whereas \( \sigma \oplus \tau \) (and \( \tau \)) is good.
To obtain an example wherein neither \(\sigma\) nor \(\tau\) is good, let \(G_4\) be another copy of \(G_3\), set 
\[ G = G_1 \oplus G_3 \quad \text{and} \quad H = G_2 \oplus G_4 \] 
(with the strict orderings of course); \(\tau = \tau_2 \oplus \tau_4\) is for the same reason as \(\sigma = \tau_1 \oplus \tau_3\), not good, but their direct sum is good.

6 Good sets of traces

As in \([BeH]\), a compact convex subset \(Y\) of \(S(G, u)\) is order unit good (with respect to \((G, u)\)) if given \(b \in G^+ \setminus \{0\}\) (\(b\) is an order unit of \(G\)) and \(a \in G\) such that 0 \(\leq aK \leq bK\), there exists \(a' \in G\) such that \(a'K = aK\) and \(0 \leq a' \leq b\). When \(Y\) is a face (it need not be; e.g., for any singleton subset of \(S(G, u)\), \(\{\tau\}\) is good iff the trace \(\tau\) is good as defined for individual traces), \(Y\) is order unit good iff ker \(Y := \cap_{\tau \in K}\ker \tau\) has dense range in 
\[ Y^\perp = \{h \in \operatorname{Aff} S(G, u) \mid hK = 0\}. \]
When \(G\) is simple, this was defined as good in \([BeH]\). When \(G = \operatorname{Aff} K\) (where \(K\) is a Choquet simplex), equipped with the strict ordering, goodness of subsets of \(K\) is an interesting geometric property. In Appendix B, we show that at least when \(K\) is finite-dimensional, the good subsets of \(K\) are as conjectured in \([BeH, \text{Conjecture, section 7}]\).

There is a problem in defining what a good subset \(Y\) should be in the non-simple case. It should be consistent with what has been defined in the simple case (where good = order unit good), and the singleton case (whence came the original definition of good); additionally, it would be desirable that if \(Y = S(G, u)\), then \(Y\) should be good whenever \(G\) is a dimension group such that \(\Inf G = \{0\}\).

We give a definition of good in more complicated situations, including for a set of traces; this extends some of the definitions in \([BeH]\). For any partially ordered abelian group \(H\) and \(h \in H^+\), recall the definition of the interval generated by \(h\), denoted \([0, h]\) (possibly with a subscript \(H\) if necessary to avoid ambiguity about the choice of group), to be \(\{j \in H \mid 0 \leq j \leq h\}\). Let \((G, u)\) be a dimension group (at this stage, we really only require that it be a partially ordered unperforated group) with order unit. Let \(L\) be a subgroup of \(G\); we say \(L\) is a good subgroup of \(G\) if the following hold:

(i) \(L\) is convex (that is, if \(a \leq c \leq b\) with \(a, b \in L\) and \(c \in G\), then \(c \in L\)), and \(G/L\) is unperforated

(ii) using the quotient map \(\pi : G \to G/L\), the latter equipped with the quotient ordering, for every \(b \in G^+\), \(\pi([0, b]) = [0, \pi(b)]\).

Convexity is required in order that the quotient positive cone be proper, that is, the only positive and negative elements are zero. Unperforation is often redundant; it may always be (there are no counter-examples; see the discussion concerning refinability in \([BeH]\)). The second property says that for all \(b \in G^+\), and for all \(a \in G\) such that \(0 \leq a + L \leq b + L\) (or equivalently, \((a + L) \cap G^+\) and \((b - a + L) \cap G^+\) are both nonempty), there exists \(a' \in G\) such that \(a - a' \in L\) and \(0 \leq a' \leq b\). This is a strong lifting property.

For example, if \(\tau\) is a trace, set \(L = \ker \tau\); this is convex, and is a good subgroup of \(G\) iff \(\tau\) is good (as a trace); in this case, \(G/L\) is naturally isomorphic to a subgroup of the reals, so unperforation is automatic.

For a subset of \(S(G, u)\), \(U\), define \(\ker U = \cap_{\sigma \in U}\ker \sigma\); for a subset of \(G\), \(W\), define \(Z(W) = \{\sigma \in S(G, u) \mid \sigma(w) = 0\}\). For good sets, we may as well assume that \(Y = Z(\ker Y)\) at the outset, in other words, \(\sigma \in Y\) iff \(\sigma(\ker \tau) = 0\), since in any reasonable definition for good or order unit good, the candidate set will satisfy \(Y = Z(\ker Y)\). As explained in \([BeH]\), these form the collection of closed sets with respect to a Zariski-like topology, and also extend the definition of facial topology (relative to \(G\)) on \(\partial_e S(G, u)\) to \(S(G, u)\). If \(Y \subseteq S(G, u)\), set \(\hat{Y} = Z(\ker Y) = \{\sigma \in S(G, u) \mid \sigma(\ker Y) = \{0\}\}\); this is a closure operation, corresponding to the facial topology and analogous to the Zariski topology from algebraic geometry. In many cases, we just assume \(Y = \hat{Y}\) already, since \(\ker Y = \ker \hat{Y}\).

We say \(\hat{Y}\) is a good subset of \(S(G, u)\) with respect to \((G, u)\) if \(Y = \hat{Y}\) and \(\ker Y\) is a good subgroup of \(G\). If \(Y = \{\tau\}\), and \(\tau\) is merely an order unit good trace, then \(\ker \tau\) has dense image
in \( \tau^+ \), and this implies \( Y = \tilde{Y} \).

If \( L \) is a subgroup of \( G \), then we may form \( Y \equiv Z(L) = \{ \sigma \in S(G, u) \mid \sigma(L) = \{0\} \} \). Then \( Y \) satisfies \( \tilde{Y} = Y \). However, it can happen that \( L \) is a good subgroup of \( G \), but \( Z(L) \) is not a good subset of \( S(G, u) \) with respect to \( G \).

For example, let \( (H, [\chi_H]) \) be the ordered Čech cohomology group of any noncyclic primitive subshift of finite type. It is known not to be a dimension group, but is unperforated and has numerous other properties [BoH1, BoH2]. There exists a dimension group \((G, u)\) such that \( H \cong G/\text{Inf}\ G \) with the quotient ordering. Set \( Y = S(G, u) \), so that \( \ker Y = \text{Inf}\ G \). Since the quotient \( H = G/\text{Inf}\ G \) is not a dimension group, it follows from results below that property (ii) fails. However, \( L = \{0\} \) is clearly a good subset of \( G \), and \( Z(L) = Y \), but \( \ker Y = \text{Inf}\ G \). So \( Y \) is not a good subset of \( S(G, u) \).

In the definition of a good subgroup, it may be that the relatively strong condition that \( G/L \) is unperforated can be replaced by the much weaker \( G/L \) is torsion-free, in the presence of (ii), the lifting property. This is the case in the situation described in [BeH, Proposition 7.6], dealing with simple dimension groups and \( L = \ker Y \). There are criteria for the quotient \( G/L \) to be unperforated [BeH, Lemma B1], but these are not always easy to verify.

The following is implicit in [BeH, Proposition 7.6].

**Lemma 6.1** Suppose \((G, u)\) is a dimension group and \( L \) is a convex subgroup of \( G \) satisfying (ii). Then \( G/L \) with the quotient ordering is an interpolation group, and its trace space is canonically affinely homeomorphic to \( L^+ \). The latter is a Choquet simplex.

**Proof.** If \( 0 \leq a + L \leq (b + L) + (c + L) \) in \( G/L \), first lift \( b \) and \( c \) separately to positive elements of \( G \); it doesn’t hurt to relabel them \( b \) and \( c \). Applying (ii) to \( 0 \leq a + L \leq (b + c) + L \), we can find \( a' \in [0, b + c] \) such that \( a - a' \in L \). Hence \( 0 \leq a' \leq b + c \); by interpolation in \( G \), we may find \( a_1 \in [0, b] \) and \( a_2 \in [0, c] \) such that \( a' = a_1 + a_2 \). Then \( a + L = a' + L = (a_1 + L) + (a_2 + L) \) and \( a_1 + L \) are positive elements of \( G/L \), and each is contained less \( b + L, c + L \) respectively. Thus \( G/L \) satisfies Riesz decomposition. The rest is standard.

If we attempt the simplest definition possible for goodness of a set, that is, \( Y \) is better (a facetious, but not inapt name) for \((G, u)\) if (i) \( Y = Z(\ker Y) \) and (ii) whenever \( a \in G, b \in G^+ \) and \( 0 \leq \tilde{a}|Y \leq \tilde{b}|Y \), there exists \( a' \in G^+ \) such that \( \tilde{a}|Y = \tilde{a}'|Y \) and \( a' \leq b \). This turns out to be much too restrictive (although it is an interesting property); for example, if \( Y = S(G, u) \), then \( Y \) is better implies \( G/\text{Inf}\ G \) (with the quotient ordering; this need not be a dimension group) is archimedean, which hardly ever occurs; and if \( G \) is simple, this is generally stronger than order unit good. If \( Y \) is a singleton, then strong goodness agrees with the original definition of good.

**Lemma 6.2** Let \((G, u)\) be a dimension group with order unit \( u \). If \( Y \subseteq S(G, u) \) is good, then \( G/\ker Y \) is a dimension group, with trace space canonically affinely homeomorphic to \( Y \).

**Proof.** As good implies order unit good, \( \ker Y \) has dense image in \( Y^+ \), and thus its closure is a vector space, so that by [BeH; Corollary B2], \( G/\ker Y \) is unperforated. Now suppose \( 0 \leq a + \ker Y \leq (b + \ker Y) + (b' + \ker Y) \), where the latter two terms are nonnegative. Hence we may assume \( b, b' \geq 0 \), and thus \( 0 \leq a + \ker Y \leq (b + b') + \ker Y \) implies there exists \( a' \in G^+ \) such that \( a' + \ker Y = a + \ker Y \) and \( a' \leq b + b' \). Riesz interpolation in \( G \) yields a decomposition \( a' = a_1 + a_2 \) where \( 0 \leq a_1, a_2 \) and \( a_1 \leq b \) and \( a_2 \leq b' \). Hence \( a + \ker Y = a' + \ker Y = (a_1 + \ker Y) + (a_2 + \ker Y) \), and \( a_1 + \ker Y \leq b + \ker Y \), and \( a_2 + \ker Y \leq b' + \ker Y \). Thus \( G/\ker Y \) satisfies interpolation.

Any trace \( \tau \) of \( G/\ker Y \), normalized at \( u + \ker Y \), induces a trace \( \tilde{\tau} \) of \((G, u)\) by composing with the quotient mapping. Conversely, if \( \sigma \) is a trace that kills \( \ker Y \), then from the definition, \( \sigma \in Y \). Hence the map \( S(G/\ker Y, u + \ker Y) \to S(G, u) \) is one to one and onto, and it is easy to
see that it is an affine homeomorphism.

LEMMA 6.3 If \( Y \) is a good subset of \( S(G,u) \), \((I,w)\) is an order ideal of \( G \) with its own order unit, and for all \( \sigma \in Y, \sigma|I \neq 0 \), then the map \( I/(I \cap \ker Y) \to G/\ker Y \) is an order isomorphism.

Proof. First we show \( I/(\ker Y \cap I) \) is unperforated, by showing the image of \( I \) is an order ideal in \( G/\ker Y \) (which is unperforated, by the preceding). Select \( 0 \leq a + \ker Y \leq b + \ker Y \), where \( b \in I \); we can write \( b = b_1 - b_2 \) where \( b_i \in I^+ \), and thus \( 0 \leq a + \ker Y \leq b_1 + \ker Y \), and now \( b_1 \in I^+ \). There thus exists \( a' \in [0,b_1] \) such that \( a - a_1 \in \ker Y \). As \( 0 \leq a' \leq b_1 \) and \( b_1 \in I \), it follows that \( a_1 \in I^+ \), so that \( a_1 + \ker Y \) is in the image of \( I \); the latter is thus a convex subgroup of \( G/\ker Y \). Directedness of the image is trivial, so \( I/(I \cap \ker Y) \) is an order ideal in \( G/\ker Y \).

Any order ideal in an unperforated partially ordered group is itself unperforated, so \( I/(\ker Y \cap I) \) is unperforated.

If \( \sigma \in Y \) and \( \sigma(w) = 0 \), then \( \sigma(I) = 0 \), contradicting the property of \( Y \); hence \( \hat{w}|Y \gg \delta \) for some \( \delta > 0 \). Since \( G/\ker Y \) is unperforated and its trace space is canonically identified with \( Y \), it follows that \( w + \ker Y \) is an order ideal for \( G/\ker Y \). Hence the order ideal generated by \( w + \ker Y \) is all of \( G/\ker Y \). Hence the image of \( I \) in \( G/\ker Y \) is onto.

So far, the map \( I/(I \cap \ker Y) \to G/\ker Y \) is one to one (by construction), order-preserving (by definition), and now we know that it is onto. To show it is an order-isomorphism, it suffices to show that the image of \( I^+ \) is all of the positive cone.

Select \( b \in G^+ \). Then \( b|Y \ll m \) for some integer \( m \), so there exists an integer \( N \) such that \( b \ll N \hat{w} \), and thus \( 0 \leq b + \ker Y \leq Nw + \ker Y \) (the latter by unperforation, again). By goodness, there exists \( a \in [0,Nw] \) such that \( a - b \in \ker Y \); \( 0 \leq a \leq Nw \) implies \( a \in I^+ \), and it maps to \( b + \ker Y \).

The latter property is the analogue of \( \tau(I) = \tau(G) \) for a single good trace \( \tau \) of \( G \). If we weakened the hypotheses, say to simply \( \ker Y \) does not contain \( I \), then the result is unclear. We have similar problems with the following characterization when some points of \( Y \) are not faithful.

LEMMA 6.4 Let \((I,w)\) be an order ideal of \( G \) with its own order unit, and suppose that every point of \( Y \) does not kill \( I \). Then the map \( \phi_I : Y \to S(I,w) \) given by \( \sigma \mapsto \sigma/\sigma(w)I \) is continuous. If \( Y \) is good with respect to \((G,u)\), then \( \phi_I(Y) \) is good with respect to \((I,w)\).

Proof. The restriction map on traces sends every point to a nonzero trace of \( I \), and thus the map is continuous (as \( Y \) is compact, \( \inf_{\sigma \in Y} \sigma(w) > 0 \)). Suppose \( \rho \) is a normalized trace on \((I,w)\) such that \( \rho(I \cap \ker \rho) \) is identically zero. Then \( \rho \) induces a trace on \( I/(I \cap \ker \rho) \), hence is a trace on \( G/\ker \rho \), and therefore \( \rho \) is the restriction of a trace from \( G \), necessarily killing \( Y \). If \( r \) is the lifted trace, we must have \( r \in Y \), and thus \( \rho \in \phi_I(Y) \). In particular, relative to \((I,w)\), \( \phi_I(Y) = Z(\ker \phi_I(Y)) \), and it follows immediately that \( \phi_I(Y) \) is good with respect to \((I,w)\).

The condition on \( Y \) in the next result, that every point be faithful, is rather strong, but makes things easier to deal with. The much weaker faithfulness condition \( (\ker Y \cap G^+ = \{0\}) \) is innocuous, as we can factor out the maximal order ideal contained in \( \ker Y \).

LEMMA 6.5 Let \((G,u)\) a dimension group, and \( Y \) a subset of \( S(G,u) \) for normalized traces \( \sigma \), \( \sigma|\ker Y \equiv 0 \) if \( \sigma \in Y \), and \( \ker Y \cap G^+ = \{0\} \).

(a) The trace space of the quotient abelian group \( G/\ker Y \) is canonically affinely homeomorphic to \( Y \).

(b) If \( G/\ker Y \) is unperforated and \( Y \) satisfies the additional condition that every element of \( Y \) is faithful, then \( G/\ker Y \) is simple.

Proof. Let \( \phi \) be a normalized trace of \((G/\ker Y,u + \ker Y)\), and let \( \pi : G \to G/\ker Y \) be the
exists an integer $N$.  

**Proof.**

Then $a$ that $G/L$ is an order unit. 

Let $v$ Then $a$ that $G/L$ is an order unit. 

**Lemma 6.6** Let $(G, u)$ be an approximately divisible dimension group, and let $L$ be a convex subgroup. If $G/L$ is unperforated, then order units lift.  

That is, given $a$ such that $a + L$ is an order unit of $G/L$, there exists an order unit $v$ of $G$ such that $a - v \in L$.  

**Proof.** The traces of $G/L$ are the traces of $G$ that kill $L$, $Z := Z(L) \subseteq S(G, u)$. As $a + L$ is an order unit, $a|L \gg \delta$ for some $\delta > 0$. As $G$ is approximately divisible, there exists $w \in G$ such that $\delta/3 < \hat{w} < \delta/2$. Then $(\hat{a} - \hat{w}) | Z \gg \delta/2$; as $G/L$ is unperforated, $a - w + L$ is in $(G/L)^\perp$. From the definition of quotient ordering, there exists $c \in G^+$ such that $c + L = a - w + L$. Set $v = c + w$. Then $v + L = a - w + w + L = a + L$; since $v \geq w$ and $w$ is an order unit, it follows that $v$ is an order unit.

If we drop approximate divisibility, we obtain that for all order units $a + L$ of $G/L$, there exists an integer $N$ such that for all $n \geq N$, there exist order units $v_n$ of $G$ such that $v_n - na \in L$.  

(Instead of using a small order unit $w$, we take $u$ or any other order unit we can find.)

The following gives a general result (without assuming $G/kery$ is unperforated, but instead, that $Y$ is a face) about lifting order units.

**Lemma 6.7** Suppose $Y = Z(kery)$ is a face of $S(G, u)$ such that the image of ker $Y$ is dense in $Y^\perp$. Let $a \in G$ satisfy $a + ker Y \geq 0$ and $\hat{a}|Y \gg \delta$ for some $\delta > 0$. Then there exists $a' \in G^{++}$ such that $a' + ker Y = a + ker Y$.  

**Proof.** From the quotient ordering, there exists $c \in G^+$ such that $c - a \in ker Y$. Let $F = \{\tau \in S(G, u) \mid \tau(c) = 0\}$; because $c \in G^+$, $F$ is a face, and is obviously closed. Since $\hat{c}|Y = \hat{a}|Y$, we must have $F \cap Y = \emptyset$. There thus exists $h \in Aff(S(G, u)^+)$ such that $h|Y \equiv 0$ and $h|F \equiv 1$.

As $h \in Y_\perp$, there exist $g_n \in ker Y$ such that $\hat{g_n} \to h$ uniformly. Hence $\hat{g_n} + c \to h + \hat{c}$ uniformly. The latter however is strictly positive (since $\hat{c} \geq 0$ and $\hat{c}^{-1}(0) = F$). Hence there exists $n$ such that $\hat{g_n} + c$ is strictly positive; as $G$ is unperforated, $a' := g_n + c$ is an order unit of $G$. Its image modulo ker $Y$ is $c + ker Y = a + ker Y$.

**Proposition 6.8** Suppose that $(G, u)$ is a nearly divisible dimension group, and $Y = Z(ker Y)$ is a subset of $S(G, u)$ such that for all $\sigma \in Y$, $ker \sigma \cap G^+ = \{0\}$. Suppose that
either $Y$ is a face or $G/\ker Y$ is unperforated. Then $Y$ is good (with respect to $(G,u)$) iff (a) $\phi_I(Y)$ is order unit good for all order ideals $I$ having their own order unit, and (b) for every nonzero order ideal $I$, $I + \ker Y = G + \ker Y$.

Remark. Condition (b) is just a restatement of the map $I/(I \cap \ker Y) \to G/\ker Y$ being onto. It does not require the stronger property, that it is an order isomorphism.

Proof. Sufficiency of the conditions. Suppose $b \in G^+$ and $a \in G$ and in addition, $0 \leq a + \ker Y \leq b + \ker Y$. Let $I \equiv I(b)$ be the order ideal generated by $b$ ($I(b) = \{g \in G \mid \exists N \in \mathbb{N} \text{ such that } -Nb \leq g \leq Nb\}$). By (b), there exists $a_1 \in I(b)$ such that $a_1 + \ker Y = a + \ker Y$. Since $I/(I \cap \ker Y)$ is simple, $0 \leq a_1 + \ker Y \leq b + \ker Y$ entails either $a_1 + \ker Y = 0 + \ker Y$ or $a_1 + \ker Y$ is an order unit. In the former case, set $a' = 0$.

Otherwise, if $Y$ is a face, there exists $a_2 \in I^{++}$ such that $a_2 + \ker Y = a_1 + \ker Y$. Similarly, either $b + \ker Y = a_1 + \ker Y$ (in which case, we take $a' = b$) or the difference $b + \ker Y - (a_2 + \ker Y)$ is an order unit in $I/(\ker Y \cap I)$.

If $G/\ker Y$ is unperforated, then $I/(I \cap \ker Y)$ is unperforated (follows from $I$ being an order ideal in $G$), and applying Lemma 6.5(b) to $\phi_I(Y)$, is simple with trace space canonically $\phi_I(Y)$. This means that the order-preserving one to one and onto map $I/(I \cap \ker Y) \to G/\ker Y$ induces an affine homeomorphism on their respective trace spaces; since the image in their affine function representations are the same, that of $I/(I \cap \ker Y)$ has dense range, and being simple (and $\phi_I(Y)$ being a simplex), the latter is a simple dimension group. A one to one order-preserving group homomorphism between simple dimension groups which induces an affine homeomorphism on the trace spaces is necessarily an order isomorphism.

Thus in either case, we have $0 \ll \hat{a}Y \ll \hat{b}Y$; now order unit goodness of $(I(b),b)$ yields $a' \in I^+$ such that $a' \leq b$.

Necessity of the conditions follows from the preceding results. 

Now we briefly examine examples in $R_p$. When $R$ is a partially ordered commutative unperforated ring with $1$ as an order unit, every closed face of $S(R,1)$ is uniquely determined by its extreme points and these form a compact subset of $X = \partial_e S(R,1)$ (and conversely, every closed subset of $X$ yields a closed face in this way). Thus, as a preliminary question, we can ask when the closed face obtained from the closed subset $Y$ of $X$ is good (for $R$) or order unit good. We say $Y$ generates an (order unit) good face when this occurs.

It is easy to see that $Y$ generates an order unit good face for $R$ iff for all pure traces $\sigma \notin Y$, $\sigma|\ker Y$ is not identically zero (we define $\ker Y = \cap_{\tau \in Y}\ker \tau$, as usual).

To see this, if $Y$ generates an order unit good face for $R$, then $\ker Y$ has dense image in $\text{Ann} Y := \{f \in C(X,\mathbb{R}) \mid f|Y \equiv 0\}$. There exists $f \in \text{Ann} Y$ such that $f(\sigma) \neq 0$, and there exist $a_n \in \ker Y$ such that $a_n \to f$ uniformly, so there exists $a \in \{a_n\}$ such that $0 \neq \hat{a}(\sigma) = \sigma(a)$, hence $\sigma|\ker Y$ is not identically zero.

Conversely, suppose $\sigma(\ker Y) \neq \{0\}$ for every $\sigma \in X \setminus Y$. It is trivial that $\ker Y$ is an ideal of $R$ (not generally an order ideal), so its closure in $C(X,\mathbb{R})$ is a closed ideal thereof, hence of the form $\text{Ann} Z$ for some closed $Z \subset X$. Obviously $Y \subset X$, but if $\sigma \in Z \setminus Y$, there exists $a \in \ker Y$ such that $\sigma(a) \neq 0$, so that $\hat{a} \notin \text{Ann} Z$, a contradiction. Hence $Z = Y$, so $\ker Y$ has dense image in $\text{Ann} Z$, and thus $Y$ is order unit good for $R$.

**LEMMA 6.9** Let $R$ be a partially ordered unperforated approximately divisible commutative ring, and let $Y$ be a compact subset of the set of faithful pure traces. Let $(I,v)$ be a nonzero order ideal with its own order unit.

(a) The set $Y$ maps by normalized restriction to a compact set of pure faithful traces on $(I,v), Y_I$. 

26
(b) If the closed face generated by $Y$ is order unit good for $R$, then the closed face of $S(I,w)$ generated by $Y_I$ is order unit good for $(I,v)$.

Good sets for $R_P$ (several variables) corresponding to faces (that is, closed subsets of the pure trace space) are highly dependent on the choice of coefficients. For example, as we will see below, if $V$ is the variety given by $f = (x - 3)^2 + (y - 3)^2 - 1$, the circle of radius one centred at $(3, 3)$ and $P = c_0 + c_1 x + c_2 y$, then $V$ (or its corresponding face in $S(R_P, 1)$ is order unit good, but not good, no matter what the choice of (positive) integers $c_0, c_1, c_2$. On the other hand, if $P_1 = P \cdot Q$ where $Q = c + x f + y g + x y h$ where $f$ is a polynomial in $x$ with no negative coefficients such that $(x - 3)^2 + 8$ divides some power of $c + x f$ (such exist!), $g$ is a polynomial in $y$ such that $(y - 3)^2 + 8$ divides some power of $c + y g$, and $h$ is a polynomial in $x y^{-1}$ such that $(1 + X^2)$ divides some power of $h(X)$, then $V$ is a good subset for $R_{P_1}$ (the conditions on the coefficients of monomials appearing in the faces of the Newton polytope described by the divisibility condition are necessary and sufficient for 6.8(b) to apply; however they are also extremely complicated).

Now we specialize to $R = R_P$ or $R_P \otimes \mathbb{Q}$, and to avoid severe complications, also assume that the compact $Y$ consists of pure faithful traces (that is, $Y$ is a compact subset of the positive orthant, $(\mathbb{R}^d)^{++}$, after identifying the pure faithful traces with points of the orthant). Then $\ker Y = \{ f/P^k \in R_P \mid f \in \mathbb{P} \}$. Recalling that for $f \in \mathbb{Z}[x_1, \ldots, x_d]$, $f/P^k \in R_P$ means there exists $l$ such that $\log f/P^l \subseteq \log P^{k+l}$, which means we may as well assume $\log f \subseteq \log P^k$.

Hence $Y$ is order unit good for $R$ iff whenever $\sigma$ is a pure trace not in $Y$, $\sigma|\ker Y \neq 0$. If we restrict $\sigma$ to the faithful pure traces, then we deduce a necessary condition: If $Y \subset (\mathbb{R}^d)^{++}$ is compact, then $Y$ is order unit good for $R_P$ implies

$$ZI(Y) \cap (\mathbb{R}^d)^{++} = Y.$$ 

That is, intersecting the Zariski closure of $Y$ with $(\mathbb{R}^d)^{++}$ gives no new points. In the singleton case, we have seen that this condition, real isolation, is sufficient. However, for general compact $Y$, it is no longer sufficient.

In fact, examples to illustrate this are ubiquitous (when $d > 1$). The very simplest one I could think of is the following. Let $P = 1 + xy + x$ (the coefficients, here all ones, are not terribly important); then $\log P$ is the triangle with vertices $\{(0, 0), (1, 1), (1, 0)\}$, and as rings $R_P \cong \mathbb{Z}[X, W]$ (the pure polynomial ring in two variables) via the transformation $X = x/P$ and $W = xy/P$. Let $f = (x - 3)^2 + (y - 3)^2 - 1$, so $Z(f) \cap \mathbb{R}^2$ is the circle of radius one centred at $(3, 3)$, and we set $Y$ to be this circle, sitting inside the positive quadrant of $\mathbb{R}^2$. In particular, $\log f = \{(0, 0), (1, 0), (2, 0), (0, 1), (0, 2)\}$. It is trivial that $ZI(Y) \cap (\mathbb{R}^2)^{++} = Y$. However, there exists $\sigma \in \partial_{c} S(R_P, 1) \backslash Y$ such that $\sigma|\ker Y = 0$.

Explicitly, $\sigma$ is the pure trace corresponding to the extreme point of $\text{cvxLog} P$ given by $(0, 0)$; $\sigma(g/P^k) = (g, x^{0,0})/(P, x^{0,0})$. Suppose $a = h/P^k \in R_P$; we may assume $\log h \subseteq \log P^k$. If $r \in \mathbb{P}$ implies $h(r)/P^k(r) = 0$, that is, $\tau(a) = 0$ for all $\tau \in \mathbb{P}$, then $h|\mathbb{P} \equiv 0$ (since $Y$ is in the positive quadrant, $P|\mathbb{P}$ vanishes nowhere). Hence there exists $e \in \mathbb{Q}[x, y]$ such that $h = e \cdot f$ (as $I_{\mathbb{Q}}(f) = f \mathbb{Q}[x, y]$); multiplying by a positive integer $N$, we may assume $Nh = e \cdot f$ where $e \in \mathbb{Z}[x, y]$.

We claim that this forces $h(0,0) = 0$, that is, its constant term must be zero, from which it would follow that $\sigma(a) = 0$, showing that $\ker Y \subset \ker \sigma$, as desired. If $h(0,0) \neq 0$, then as $\log h \subseteq \log P^k$, we would have to have $(0,0) \in \ker h$, and in particular, this point is an extreme point of $\text{cvxLog} h$. Since $(0,0)$ is also an extreme point of $\text{cvxLog} f$, it easily follows that $(0,0)$ is an extreme point of $\text{cvxLog} e$ (I am used to working with Laurent polynomials, hence this complicated argument, rather than the simple observation about evaluation). Now consider the coefficients of $e$ and of $f$ restricted to the line $x = 0$ (that is, throw away all the monomials with a power of $x$), $e_0$ and $f_0 = y^2 - 6y + 17$. The product is not zero, and cannot be a single monomial (since $f_0$ is
not), hence there must be, in addition to the constant term, a term of the form $y^j$ in the product. It is easy to check that this forces $(0,j) \in \text{Log } e \cdot f = \text{Log } h$. However, $\text{Log } P^k$ is contained in the lattice cone generated by $\{(0,0), (1,1), (1,0)\}$, which does not contain $(j,0)$. This contradicts $\text{Log } h \subseteq \text{Log } P^k$.

This example does not depend on the coefficients in $P$, that is, we could just as well have taken $P = 2 + 3xy + 5x$ (which guarantees that $R_P$ is approximately divisible), nor whether we take $R_P$ or $R_P \otimes Q$.

In contrast, if we take the same $f$, but $P = 2 + 3x + 5y$ (or with any other positive coefficients), then $f/P^2 \in R_P$ and for all non-faithful pure $\sigma$, $\sigma(f/P^2) > 0$, hence the same $Y$ is now order unit good for $R_P$. This is part of a more general criterion.

Let $h$ be a polynomial in $d$ variables, and let $S$ be a finite set of lattice points in $\mathbb{Z}^d$, and $K(S) = \text{cvx } S$. Suppose $F$ is a proper face of $K(S)$, and $\text{Log } h \subseteq kS$ (the set of sums of $k$ elements of $S$). We define the facial polynomial of $h$ relative to $F$ and $k$, $h_{F,k}$ by throwing away all the terms in $x^w$ of $h$ for which $w \notin kF$. In case $S = \text{Log } P$, we can form the element $h_{F,k}/(P_F)^k \in R_{P_F}$ (in fewer variables, the number being the dimension of $F$). This yields a positive homomorphism $R_P \to R_{P_F}$ as described in [H1].

Let $Y$ satisfy $ZI(Y) \cap (\mathbb{R}^d)^{++}$, and form the ideal $I(Y)$ of $\mathbb{Z}[x_1, \ldots, x_d]$. Let $P$ be a projectively faithful polynomial in $\mathbb{Z}[x_1, \ldots, x_d]$. We say that $Y$ can be fitted with respect to $P$ if there exists a polynomial $h \in I(Y)$ such that

(a) \( \log h \subseteq \text{Log } P^k \) for some $k$

(b) for every proper face $F$ of $\text{cvx } P$, $h_{F,k}$ has no negative coefficients.

This depends on $\log P$, but not very much on the coefficients $P$, as follows from [H2, Proposition II.5].

Condition (b) can be somewhat weakened, since we are permitted to multiply numerator and denominator of $h/P^k$ by powers of $P$, and apply eventual positivity criteria, e.g., [H1A]. The condition is equivalent to for all pure $\sigma$ that is not faithful, there exists $h \in I(Y)$ such that $\sigma(h/P_k) > 0$. For example, with $\log P = \{(0,0), (0,1), (1,0)\}$ and $Y$ the circle in $(\mathbb{R}^2)^{++}$ of radius 1 centred at $(3,3)$, $Y$ is fitted with respect to $P$. Just observe that $f$ has the three facial polynomials (corresponding to the three edges of $\text{cvx } \log P$ (the extreme points take care of themselves, so we need not worry about the zero-dimensional faces), $(x - 3)^2 + 17$, $(y - 3)^2 + 17$, $x^2 + y^2$. If we multiply the first two by a sufficiently high power, say $N$, of $1 + x$ (respectively $(1 + y)$), the outcome will have no negative coefficients. It follows that if $h = P^N f$, then $h$ will be positively fitted with respect to $P$, with $k = N + 2$.

Now the following is practically tautological.

**Proposition 6.10** Let $P$ be a faithfully projective element of $\mathbb{Z}[x_i]$, and $Y$ a compact subset of $((\mathbb{R})^d)^{++}$. Then $Y$ generates an order unit good face for $R_P$ (and simultaneously for $R_P \otimes Q$) iff

(i) \( ZI(Y) \cap (\mathbb{R}^d)^{++} = Y \) and

(ii) $Y$ can be fitted with respect to $P$.

Conditions on $Y$ to guarantee property (b) of Proposition 6.8 seem to be very difficult, involving divisibility of polynomials (and so are highly dependant on the actual coefficients). So goodness of subsets of $\partial_S(R_P,1)$ is still problematic.

**Appendix 1. Order unit good traces on $\mathbb{Z}^k$**

The criteria for goodness of traces on nearly divisible dimension groups depend on order unit goodness; and the usefulness of the former is a consequence of the relatively simple characterization of order unit good traces on approximately divisible dimension groups, namely density of the image of $\ker \tau$ in $\tau^+$ via the affine representation of $(G,u)$. 28
To obtain useful criteria for goodness on a larger class of dimension groups, it would be helpful to find an analogous characterization of order unit goodness in the presence of discrete traces. In this appendix, we consider the extreme dimension groups with discrete traces, namely the simplicial ones, $\mathbb{Z}^k$, with the usual ordering. It is already known that up to scalar multiple, the only good traces are given by left multiplication by a $0-1$ vector (that is, the entries consist only of zeros and ones) [H6, Lemma 6.2].

With the current definition of order unit good (really intended for approximately divisible groups), the order unit good traces on $\mathbb{Z}^k$ can be characterized, but the characterization makes it difficult to see how to obtain goodness criteria, as we did in the nearly divisible chase.

Let $v \in (\mathbb{R}^{k \times 1})^+ \setminus \{0\}$; then $v$ induces a trace on $\mathbb{Z}^k$, via left multiplication, $\phi_v : \mathbb{Z}^k \to \mathbb{R}$ sending $w \mapsto vw$ (we think of $\mathbb{Z}^k$ as a set of columns, so matrix multiplication makes sense). Obviously we can replace $v$ by any positive real multiple of itself without changing properties such as goodness or order unit goodness. In addition, we may apply any permutation to the entries, with the same lack of bad consequences. We may also discard any zeros (reducing the size of the vectors, that is, decreasing $k$).

Suppose $v$ has only integer entries; then we may order the nonzero entries, so that

$$v = (n(1), n(2), \ldots, n(r); 0, 0, \ldots, 0) \quad \text{where } n(1) \leq n(2) \leq \ldots .$$

We may also assume that $\gcd \{n(i)\} = 1$.

**Lemma A.1** With this choice of $v$, $\phi_v$ is order unit good iff $n(1) = 1$ and for all $r \geq j > 1$, $n(j) \leq 1 + \sum_{i < j} n(i)$.

**Proof.** Assume $v$ is in the form indicated, and $\phi_v$ is order unit good. Since $\gcd \{n(i)\} = 1$, there exists a vector $w$ such that $vw = 1$. Set $u = (1, 1, 1, \ldots, 1)$; we have that $u$ is an order unit, hence it is $\phi_v$-order unit good. Since $vu > 1$ (unless $v = (1, 0, 0, \ldots, 0)$ which is trivially good), there must exist $w_0 \in (\mathbb{Z}^k)^+$ such that $vw_0 = vu = 1 < vu$. Since the nonzero entries of $v$ are increasing, this forces the smallest one, $n(1)$, to be 1. Hence $n(1) = 1$.

Since $vu = \sum n(i) := N$, and there exists $w \in \mathbb{Z}^k$ such that $vw = 1$, there exists for each $s$ with $1 < s < N$ $w_s \in \{0, 1\}^k$ (as $0 \leq w_0 \leq u$) such that $vw_s = s$, by order unit goodness of $u$.

Now suppose that for some $j$, $n(j) > 1 + \sum_{i < j} n(i)$. Then $n(j) - 1$ cannot be realized as a sum of $n(i)$s (using at most one for each choice of $i$), since $n(j) - 1 > \sum_{i < j} n(i)$, and $n(j) \leq n(j')$ for all $j' > j$ (if there are any such $j'$). Hence no such $w_0$ can exist.

Thus, if $u$ is $\phi_v$-order unit good, then the constraint on growth must hold.

Conversely, suppose the inequalities hold. It is then an easy induction argument (on $r$, augmenting the vector by adjoining $n(k + 1)$) to show that $u$ is $\tau_v$-order unit good, by realizing every integer in the interval $(0, N)$. Finally, to show that every order unit is $\phi_v$-order unit good ($u$ was the smallest choice), it suffices to show that if we add a single one to a $\phi_v$-order unit good vector, the outcome is again $\phi_v$-order unit good. \hfill \blacktriangleleft

In particular, the choices for $v$, $(1, 2, 4, 8, 16)$ and $(1, 1, 1, 4)$ yield order unit good traces, but $(1, 3)$ and $(1, 1, 1, 5)$ do not. This rather complicated set of conditions, when applied to order ideals in dimension groups that have a simplicial quotient by an order ideal, makes order unit goodness likely unusable for the purposes we had in mind.

**Lemma A.2** If $\phi_v : \mathbb{Z}^k \to \mathbb{R}$ is an order unit good trace, then up to scalar multiple, $v \in (\mathbb{Z}^{k \times 1})^+$.

**Proof.** In $\mathbb{Z}^k$, all intervals of the form $[0, u]$ (where $u$ is an order unit) are finite sets. If there were an irrational ratio among the nonzero entries of $v$, we would obtain $\phi_v(\mathbb{Z}^k) \cap [0, N]$ is infinite, for any positive integer $N$. If order unit goodness held, this would be impossible. Hence all the ratios...
are rational, and it easily follows that after suitable scalar multiplication, we can convert \( v \) to an integer row.

**Proposition A.3** Let \( v \) be an element of \( \mathbf{R}^k \setminus \{0\} \). Then \( \phi_v \) is an order unit good trace, iff up to scalar multiple and after rearrangement so that \( v = (n(1), \ldots, n(r); 0, 0, \ldots) \) with \( n(i - 1) \leq n(i) \), we have \( n(i) \in \mathbb{N}, n(1) = 1 \), and for all \( 1 < j \leq r \),

\[
n(j) \leq 1 + \sum_{i<j} n(i).
\]

**Appendix 2. Good simplices**

In the finite-dimensional case, we verify a conjecture from [BeH, section 7] that good subsets of Choquet simplices are obtained as coproducts of faces with singleton subsets of disjoint faces.

Let \( K \) be a Choquet simplex. A nonempty subset \( J \) of \( K \) is said to be good (following [BeH]) if it satisfies the following (redundant set of) properties:

(i) \( J \) is a (compact) Choquet simplex

(ii) there exists a closed flat \( L \) such that \( J = L \cap K \)

(iii) if \( a \in \text{Aff}(J)^{++} \) and \( b \in \text{Aff}(K)^{++} \) are such that \( a \preccurlyeq b|J \), then there exists \( a' \in \text{Aff}(K)^{++} \) such that \( a'|J = a \) and \( a' \preccurlyeq b \).

We denote this relationship between \( J \) and \( K \), \( J \preccurlyeq K \) (there is an uppercase \( G \) inside the inclusion sign). If \( F \) is a closed face of \( K \), we denote it \( F \triangleleft K \). A question arising out of [BeH] is to characterize good subsets of Choquet simplices. For example, closed faces are good, and singleton sets are also good, and coproducts (within the category of simplices and good subsets) preserve these properties. A conjecture was made concerning the structure of good subsets; we verify this in the case that \( K \) is finite-dimensional.

Now (ii) is redundant, and only the compact convex part of (i) is necessary. This is based on the following simple construction.

If \( X \) is a subset of a real vector space, define the affine span of \( X \), denoted Aspan \( X \), as the set of finite sums \( \{ \sum r_i x_i \mid r_i \in \mathbf{R}, \sum r_i = 1, x_i \in X \} \).

If \( J \) is a singleton or a line segment, there is (almost) nothing to do. Define \( L_0 = \text{Aspan} J \). If there exists \( v \in (K \cap L_0) \setminus J \), we can write \( v = \sum \alpha_i v_i - \beta_j w_j \) where \( v_i, w_j \in J \), and \( \alpha_i, \beta_j > 0 \), and \( \sum \alpha_i - \sum \beta_j = 1 \). We can also arrange that cvx \( \{v_i\} \cap \text{cvx} \{w_j\} = \emptyset \). Hence for any positive \( \eta < 1 \), there exists \( a \in (\text{Aff} J)^{++} \) such that \( 1 - \eta < a|\text{cvx} \{w_j\} < 1 \) and \( a|\text{cvx} \{a_i\} < \eta \). Since \( a \) is continuous, it is bounded above, so (iii) applies with some constant \( b \in \text{Aff} K \).

Hence there exists \( a' \in (\text{Aff} K)^{++} \) such that \( a = a'|J \). Evaluating the equation at \( a' \), we obtain \( 0 < a'(w) = \sum \alpha_i a(v_i) - \sum \beta_j a(w_j) < \eta \sum \alpha_i - (1 - \eta) \sum \beta_j \). This entails \( \eta (\sum \alpha_i + \sum \beta_j) > \sum \beta_j \), now \( \sum \beta_j > 0 \), since otherwise \( v \in J \). Hence we can choose at the outset positive \( \eta < \sum \beta_j / (\sum \alpha_i + \sum \beta_j) \), which yields a contradiction.

Thus \( L_0 \cap K = J \). If \( x_n \in L_0 \) and \( x_n \rightarrow x \in K \), but \( x \notin J \), there exists a line segment joining \( x \) to an element of the relative interior of \( J \); it must pass through at least two points in \( J \), hence \( x \in L_0 \). In other words, with \( L \) equalling the closure of \( L_0 \), we have \( J = L_0 \cap K = L \cap K \).

To check that the compact convex set \( J \) must be a simplex if (iii) is satisfied, note that the quotient \( \text{Aff} K / J^{+} \) (with the strict ordering on \( \text{Aff} K \), \( J^{+} = \{ a \in \text{Aff} K \mid a|J = 0 \} \), and the quotient ordering) is order isomorphic to \( \text{Aff} J \) (with the strict ordering). But goodness implies ([BeH]) that it satisfies Riesz interpolation, which of course forces \( J \) to be a Choquet simplex.

Let \( K' \) and \( K'' \) be simplices (simplices mean Choquet simplices; but most of the time we will working in finite dimensions, so simplex means the usual simplex) sitting inside some common simplex \( K \) which in turn is contained in some topological vector space. Suppose that Aspan \( K' \cap
As \( \text{Span } K'' = \emptyset \); we write this as \( K' \land K'' = \emptyset \). Then the closure of \( \text{cvx}(K', K'') \) is itself a simplex, and we refer to this as the coproduct, written \( K' \lor K'' \) (this is more an internal coproduct, but we shall not distinguish internal from external). If \( K' \) and \( K'' \) are faces of \( K \), sufficient for \( K' \land K'' = \emptyset \) is that their intersection be empty (since \( K \) is a simplex); in this case, we say that \( K' \) and \( K'' \) are disjoint. If \( \{ K^i \} \) is a finite family of subsimplices, then disjointness of the set is defined inductively in the obvious way, so that \( \forall_i K^i \) makes sense and is a simplex.

We record elementary properties related to goodness.

**LEMMA B.1**  
(a) Suppose \( J \subset K \) and \( K \subset L \); then \( J \subset L \).
(b) If \( F \subset K \), then \( F \subset K \).
(c) If \( J \subset K \) and \( F \subset K \), then \( J \cap F \subset J \) and \( J \cap F \subset K \) whenever \( J \cap F \neq \emptyset \).
(d) If \( J_i \subset K_i \) for \( i = 1, 2 \) and \( K_1 \land K_2 = \emptyset \), then \( J_1 \lor J_2 \subset K_1 \lor K_2 \).

The crucial result is the following. Its proof rests heavily on finite-dimensionality, but is a minor modification of the previous argument.

**LEMMA B.2**  
Let \( K \) be a finite dimensional simplex, and suppose \( J \subset K \). Let \( J_i \) and \( J_2 \) be disjoint faces of \( J \). Set \( F_i \) \((i = 1, 2)\) to be the smallest face of \( K \) that contains \( J_i \). Then \( F_1 \) and \( F_2 \) are disjoint.

**Proof.**  
It suffices to show that \( F_1 \cap F_2 = \emptyset \). If not, the intersection is a face, hence contains a vertex (that is, extreme point) of \( K \), call it \( v \). We may suppose that \( v \not\in J_2 \) (since \( J_1 \land J_2 = \emptyset \)). Since \( J \) is itself a finite dimensional simplex and \( J_i \) are disjoint faces, for any \( \eta > 0 \) (which we will specify later), we may find \( \eta/\mu > 0 \) \((\text{since } \mu > 0, \mu \geq 0, \text{ and } \mu = 1 - \sum \mu_s)\). Evaluating at \( \lambda v \), we obtain \( \eta/\mu \). Evaluating at \( \lambda v \), we obtain \( \eta/\mu \).

Now working within \( F_i \), again since \( F_i \) is the smallest face containing \( J_i \), there must exist \( y \in J_i \) such that \( y = \mu v + \sum t \mu_t y_t \) \((\text{since } \mu > 0, \mu_s \geq 0, \text{ and } \mu = 1 - \sum \mu_s)\). Applying \( \lambda v \), we obtain \( \lambda a'(v) = \lambda a(v) - \sum \lambda s a'(v_s) \geq 1 - \eta - (1 - \lambda) \) (since \( a'(v_s) \leq b(v_s) = 1 \)). Thus \( a'(v) \geq 1 - \eta/\lambda \).

Thus the two inequalities force \( \eta/\mu + \eta/\lambda > 1 \). We reach a contradiction if we choose \( \eta < 1/(1/\mu + 1/\lambda) \).

One obstruction (among several) to extending this to infinite-dimensional simplices is the fact that the representing measures of relative interior points might vanish on the intersection of the faces. We would also have to restrict to closed faces in this case (since otherwise it is not clear that the smallest face exists), and this will present problems when we want to use it.

Let \( \{ F_i \} \) be a disjoint collection of faces (that is, for all \( i, F_i \land (\forall j \neq i) F_j = \emptyset \)) of the simplex \( K \), and for each \( i \), let \( v_i \) be a point in the relative interior of \( F_i \); we also assume that \( F_i \) are not themselves singletons. We may form \( J_0 := \text{cvx} \{ v_i \} \) and \( F_0 := \text{cvx} \{ F_i \} \); of course, this is the coproduct of \((\{ v_i \}, F_i)\), and \( J_0 \) is thus a good subset of \( F_0 \) (since each \( \{ v_i \} \subset F_i \)). As in [BeH], we call the \((v_i, F_i)\), together with \((F, F)\) (that is, the face \( F \subset F \) building blocks. It was conjectured (in the finite-dimensional case) that if \( J \subset K \), then there exists a face \( F \) of \( K \), together with a disjoint face \( F_0 \) obtained as the coproduct, such that \( J = F \lor J_0 \); in other words, that coproducts of the building blocks yield all good subsets; alternatively, that there is a face maximal of \( K \) sitting inside \( J \), and \( J \) is obtained by taking coproducts with respective singleton sets sitting inside pairwise disjoint faces. This now follows easily.

31
COROLLARY B.3 Suppose $K$ is a finite-dimensional simplex and $J \varsubsetneq K$. Then there exist a (possibly empty) face $F$ of $K$ together with a finite set of faces $F_i$ of dimension at least one such that \{\{F, F_1, \ldots \}\} is disjoint, together with $v_i$ in the relative interior of $F_i$ such that $J = \text{cvx} \{F, v_i\}$.

Proof. We proceed by induction on the dimension of $J$. Let $F$ be the convex hull of all the vertices of $K$ that lie in $J$; these are automatically vertices of $J$. If this exhausts the vertices of $J$, then $F = J$ and $F$ is a face (since $K$ is a finite-dimensional simplex), and there is nothing to do. Of course, $F$ can be empty.

Otherwise, there exists a vertex $v_1$ of $J$ that is not in $\partial_e K$; necessarily this belongs to a proper face (it cannot be in the interior, in fact by property (ii), but this can also be proved using only (i) and (iii)) of $K$, and let $F_1$ be the smallest face of $K$ containing $v_1$. Then $v_1$ is in the relative interior of $F_1$. Let $J^1$ be the complementary face to \{\{v_1\}\} in $J$ (that is, the convex hull of all the other vertices of $J$).

If $J^1$ is empty, then $J = J^1$ is already a singleton, and we are done.

If $J^1$ is not empty, then $J^1 \varsubsetneq J$, so $J^1 \varsubsetneq J$, and thus by transitivity, $J_1 \varsubsetneq K$. We can apply the previous lemma. Let $F^1$ be the smallest face of $K$ containing $J^1$; then $F^1 \cap F_1 = \emptyset$, and thus $J$ decomposes as the coproduct of $J^1$ and \{\{v_1\}\} (using faces $F^1$ and $F_1$), so by induction on the dimension of $J$, we are done.

The conjecture in the case that $K$ be infinite-dimensional is more complicated, and I have no idea how to proceed.

Acknowledgment
Discussions with my colleague Damien Roy concerning the material in section 3 were very helpful.

References

[Ak1] E Akin, Measures on Cantor space, Topology Proc, 24 (1999) 1–34.

[Ak2] E Akin, Good measures on Cantor space, Trans Amer Math Soc, 357 (2005) 2681–2722.

[ADMY] E Akin, R Dougherty, RD Mauldin, and A Yingst, Which Bernoulli measures are good measures?, Colloq Math, 110 (2008) 243–291.

[BeH] S Bezuglyi & D Handelman, Measures on Cantor set: the good, the ugly, the bad, Trans Amer Math Soc (to appear).

[BoH1] M Boyle & D Handelman, Ordered equivalence, flow equivalence and ordered cohomology, Israel J Math 95 (1996) 169–210.

[BoH2] M Boyle & D Handelman, unpublished drafts, 1993–2013.

[EHS] EG Effros, David Handelman, & Chao-Liang Shen, Dimension groups and their affine representations, Amer J Math 102 (1980) 385–407.

[FO] SB Frick & N Ormes, Dimension groups and invariant measures for polynomial odometers, Acta Appl Math (2013).

[GPS] T Giordano, IF Putnam, & CF Skau, Topological orbit equivalence and $C^*$-crossed products, J Reine Angew Math 469 (1995), 51–111.

[G] KR Goodearl, Partially ordered abelian groups with interpolation, Mathematical Surveys and Monographs, 20, American Mathematical Society, Providence RI, 1986.

[GH] KR Goodearl & David Handelman, Metric completions of partially ordered abelian groups, Indiana Univ J Math 29 (1980) 861–895.
[GH2] KR Goodearl & David Handelman, Tensor products of dimension groups and $K_0$ of unit-regular rings, Canad J Math 38 (1986), no. 3, 633–658.

[H1] David Handelman, Positive polynomials and product type actions of compact groups, Mem Amer Math Soc 54 (1985), 320, xi+79 pp.

[H1A] David Handelman, Deciding eventual positivity of polynomials, Ergodic theory and dynamical systems 6 (1985) 57–79.

[H2] David Handelman, Positive polynomials, convex integral polytopes, and a random walk problem, Lecture Notes in Mathematics, 1282, Springer–Verlag, Berlin, 1987, xii+136 pp.

[H3] D Handelman, Iterated multiplication of characters of compact connected Lie groups, J of Algebra 173 (1995) 67–96.

[H4] D Handelman, Free rank $n+1$ dense subgroups of $\mathbb{R}^n$ and their endomorphisms, J Funct Analysis 46 (1982), no. 1, 1–27.

[H5] D Handelman, Matrices of positive polynomials, Electronic J Linear Algebra 19 (2009) 2–89.

[H6] D Handelman, Realizing dimension groups, good measures, and Toeplitz factors, submitted. (Formerly known as Equal row and equal column sum realizations of dimension groups, on ArXiv.)

[HPS] RH Herman, IF Putnam, & CF Skau, Ordered Bratteli diagrams, dimension groups and topological dynamics, Internat J Math 3 (1992), no. 6, 827–864.

[P] K Petersen, An adic dynamical system related to the Delannoy numbers, Ergodic theory & dynamical systems 32 (2012) 809–823.

[Pu] IF Putnam, The $C^*$ algebras associated with minimal homeomorphisms of the Cantor set, Pacific J Math 136 (1989) 329–353.

[R] J Renault, A groupoid approach to $C^*$-algebras, Lecture Notes in Mathematics, 793, Springer–Verlag, Heidelberg, 1980.

Mathematics Dept, University of Ottawa, Ottawa K1N 6N5 ON, Canada; dehsg@uottawa.ca