CONVECTIVE COOLING AND FRAGMENTATION OF GRAVITATIONALLY UNSTABLE DISKS.

Roman R. Rafikov\textsuperscript{1,2}

Draft version September 23, 2018

ABSTRACT

Gravitationally unstable disks can fragment and form bound objects provided that their cooling time is short. In protoplanetary disks radiative cooling is likely to be too slow to permit formation of planets by fragmentation within several tens of AU from the star. Recently, convection has been suggested as a faster means of heat loss from the disk but here we demonstrate that it is only marginally more efficient than the radiative cooling. The crucial factor is the rate at which energy can be radiated from the disk photosphere, which is robustly limited from above in the convective case by the adiabatic temperature gradient (given a certain midplane temperature). Thus, although vigorous convection is definitely possible in disks, the inefficiency of radiative loss from the photosphere may create a bottleneck limiting the ability of the disk to form self-gravitating objects. Based on this argument we derive a set of analytical constraints which diagnose the susceptibility of an unstable disk to fragmentation and show that the formation of giant planets by fragmentation of protoplanetary disks is unlikely to occur at distances of tens of AU. At the same time these constraints do not preclude the possibility of fragmentation and star formation in accretion disks around supermassive black holes.

\textit{Subject headings:} planets and satellites: formation — solar system: formation — planetary systems: protoplanetary disks

1. INTRODUCTION.

It has been known for a long time (Safronov 1960; Toomre 1964) that massive differentially rotating disks are prone to gravitational instability (hereafter GI) which sets in when the destabilizing effect of the self-gravity in the disk exceeds the combined restoring action of the pressure and Coriolis forces. Instability condition is conveniently expressed in terms of the so-called Toomre $Q$-parameter as

$$Q = \frac{\Omega c_s}{\pi G \Sigma} \lesssim Q_0,$$  \hspace{1cm} (1)

where $\Omega$, $c_s$, and $\Sigma$ are the local angular frequency (disk is assumed to be in Keplerian rotation), sound speed, and surface density of the disk, and $Q_0 \approx 0.75 - 1.5$ (Kim, Ostriker, & Stone 2002; Boss 2002) is a threshold value of $Q$ at which GI sets in.

Nonlinear outcome of the GI depends on the thermodynamics of the disk, namely on its ability to cool rapidly (Gammie 2001; Johnson & Gammie 2003). Rapid cooling keeps pressure forces subdominant compared to the disk self-gravity in the non-linear stage of GI thus promoting the collapse of unstable parts of the disk and leading to the formation of bound objects. When the cooling is slow fragmentation is avoided and the disk settles to a quasi-steady regime of gravitoturbulence (Gammie 2001) characterized by dramatic perturbations of the disk surface density and fluid velocity (Durisen et al. 2006). Gammie (2001), Rice et al. (2003), and Rice, Lodato, & Armitage (2005) have demonstrated using simulations with fixed cooling time $t_{\text{cool}}$ that fragmentation requires

$$\Omega_{\text{cool}} \lesssim \zeta,$$  \hspace{1cm} (2)

where $\zeta \approx 3 - 12$ depending on the adiabatic index $\gamma$ of the disk material, although in some situations $\zeta$ can be much higher (Johnson & Gammie 2003).

Fragmentation resulting from GI is likely an important ingredient of the dynamics of accretion disks in the central regions of galaxies, around supermassive black holes (SMBHs; Paczynski 1978; Kolyikalov & Sunyaev 1980; Illarionov & Romanova 1988; Levin 2003; Goodman 2005; Goodman & Tan 2004; Thompson, Quataert, & Murray 2005), including the past disk around the black hole in the center of our Galaxy (Levin & Beloborodov 2003; Nayakshin 2005). On smaller scales, fragmentation of gravitationally unstable protoplanetary disks resulting in direct formation of gas giant planets has been put forward (Cameron 1978; Boss 1998) as an alternative to the conventional core instability model (Perri & Cameron 1974; Harris 1978; Mizuno 1980). Some simulations incorporating radiation transfer (Boss 2002; Mayer 2006) seem to demonstrate that planet formation by GI is possible. However, these results are in drastic conflict with the long radiative cooling times of optically thick protoplanetary disks (Rafikov 2005; hereafter R05) and the recent numerical results by another group (Cai et al. 2006; Boley et al. 2006).

To reconcile inefficient radiative cooling with fragmentation exhibited in their simulations Boss (2004) and Mayer et al. (2006) have come forward with a suggestion that gravitationally unstable protoplanetary disks can be efficiently cooled by convection so that fragmentation becomes possible. The goal of this paper is to critically revise this possibility by setting in a lower limit on the cooling timescale of the disk in which energy is transported away from the midplane by convective motions. We then use this result in \ref{eq:1} to constrain the properties of the disks in which GI and convective cooling lead to fragmentation.

2. CONDITIONS FOR CONVECTION.
The question of whether energy in the disk is transported by convection or radiation depends on the behavior of disk opacity. Bell & Lin (1994) have demonstrated that in some temperature intervals opacity \( \kappa(P, T) \) can be well represented as

\[
\kappa = \kappa_0 P^\alpha T^\beta,
\]

(\( P \) and \( T \) are the gas pressure and temperature) which we assume to hold throughout the paper.

In this section we focus on optically thick disks, for which \( \tau \equiv \int_{-\infty}^{\infty} \rho \kappa dz \gg 1 \). Following the approach of Lin & Papaloizou (1980) we demonstrate in Appendix A that some optically thick disks are guaranteed to be convective whenever

\[
\nabla_0 \geq \nabla_{ad},
\]

(4)

Here

\[
\nabla_0 \equiv \frac{1 + \alpha}{4 - \beta}, \quad \nabla_{ad} \equiv \frac{\gamma - 1}{\gamma}.
\]

Condition 4 directly follows from some assumptions about the spatial distribution of the energy dissipation rate within the disk (see Appendix A) and the Schwarzschild criterion for convection

\[
\nabla \geq \nabla_{ad}, \quad \nabla \equiv \frac{d \ln T}{d \ln P},
\]

(6)

where \( \nabla \) is a temperature gradient.

Using condition 4 one can infer that in the low-temperature regime disk is convective for \( T \leq 150 \) K, since then \( \kappa \) is dominated by ice grains (\( \alpha = 0, \beta \approx 2 \); Bell & Lin 1994) and \( \nabla_{ad} = 2/7 \) while \( \nabla_0 \approx 1/2 \) (\( \gamma = 7/5 \)) in this case. At higher temperatures, when ice grains sublime (but below the sublimation point of the silicate and metal grains) \( \alpha = 0, \beta \approx 1/2 \) and \( \nabla_0 \approx \nabla_{ad} \), so that the disk is marginally unstable for convection. At the same time, submillimeter observations of protoplanetary disks demonstrate that dust opacity at low temperatures is best described by \( \alpha = 0, \beta \approx 1 \) (Kitamura et al. 2002), most likely as a result of dust grain growth.

In this case \( \nabla_0 = 1/3 \) and disk is convective.

At higher temperatures, when all dust grains sublime, opacity is pressure-dependent (\( \alpha \neq 0 \)). In particular, \( \alpha \approx 2/3 \) and \( \beta \approx 7/3 \) in the regime of molecular opacity (\( T \approx 1.5 - 5 \times 10^3 \) K), so that \( \nabla_0 \approx 1 \) and disk is again convective. When using criterion 4 at even higher temperatures one has to bear in mind possible importance of the radiation pressure which we neglect in our analysis.

3. Upper limit on the convective cooling rate.

We first consider optically thick disks in which dense midplane regions cool in a two-stage fashion. Their heat first needs to be transported somehow to the disk photosphere (located at \( \tau \approx 1 \)) and then lost to space by radiation from the photosphere, irrespective of whether transport of energy inside the disk is done by radiation or convection.

In a convective disk the first leg of the cooling process can be rather fast but not instantaneous. Indeed, convection is driven by the buoyancy of the hot gas, but buoyancy cannot supply gas with the vertical acceleration higher than vertical gravitational acceleration \( g_z \).

Because of that, when the parcel of hot gas reaches a scale height above the midplane (which is roughly the location of the photosphere) its vertical speed would at most \( \sim c_s \) and it would take time at least \( \tau \sim \Omega^{-1} \) to get there. Thus, the heat transport inside the disk cannot occur in less than the dynamical timescale of the disk. If this were the end of story, criterion 2 would suggest that convective energy transport is marginally compatible with fragmentation.

However, the need to eventually lose the transported energy by radiation from the photosphere changes this conclusion, since the cooling time is in fact the sum of the interior heat transport timescale (which can be as short as \( \Omega^{-1} \)) and the timescale of radiative losses from the disk surface. For a given midplane temperature \( T_m \) the fastest cooling occurs for the highest possible photospheric temperature \( T_{ph} \), since the energy loss from the photosphere per unit area is \( \sigma T_{ph}^4 \), assuming blackbody emission. For a fixed \( T_m \) higher \( T_{ph} \) implies smaller temperature variation for a fixed change in pressure, or, in other words, lower value of \( \nabla \) [see equation (4)]. Thus, disk cools most efficiently when \( \nabla \) assumes its lowest possible value.

Whenever the radiative temperature gradient \( \nabla_r \) defined as

\[
\nabla_r \equiv \frac{3}{16} \frac{\kappa PF}{g_z \sigma T^4},
\]

(7)

(here \( \sigma \) is a Stephan-Boltzmann constant and \( F \) is the upward energy flux) satisfies the condition \( \nabla_r < \nabla_{ad} \), disk is convectively stable and \( \nabla = \nabla_r < \nabla_{ad} \). In this case energy is transported from the midplane of the disk to its photosphere radiatively, and equations 1 and 2 directly yield constraints on the properties of disks capable of forming bound objects by the GI (R05). As \( \nabla_r \) becomes larger than \( \nabla_{ad} \) convection sets in and the temperature gradient obeys \( \nabla_{ad} < \nabla < \nabla_r \). From this one can immediately see that the smallest possible value of \( \nabla \) (leading to highest \( T_{ph} \)) in a convective disk is \( \nabla_{ad} \). Situation when \( \nabla \approx \nabla_{ad} \) is possible only in the case of very efficient convection when the radiative flux is small compared to the convective flux, which typically requires high gas density. Whether this is the case depends on the situation at hand but using \( \nabla = \nabla_{ad} \) allows us to obtain a robust upper limit on the cooling rate of the disk.

Disk with \( \nabla = \nabla_{ad} \) is vertically isotropic and we assume this to hold all the way to the photosphere (this is verified in Appendix B). Then

\[
\frac{T_{ph}}{T_m} = \left( \frac{P_{ph}}{P_m} \right) \frac{\nabla_{ad}}{T_m},
\]

(8)

where \( P_{ph} \) and \( P_m \) are the gas pressure at the photosphere and at the midplane, respectively. For the purposes of this calculation it is enough to assume that vertical acceleration \( g_z \) is roughly constant in which case

\[
\frac{P_{ph}}{P_m} \approx \frac{\kappa(P_m; T_m)}{\kappa(P_{ph}; T_{ph})} \tau^{-1}.
\]

(9)

Then one finds from 4, 8, and 9 that

\[
\frac{T_{ph}}{T_m} \approx \tau^{-n/4}, \quad \eta \equiv \frac{4 \nabla_{ad}}{1 + \alpha + \beta \nabla_{ad}}.
\]

(10)

Thus, the fastest possible rate \( L \) of energy loss per unit surface area of an optically thick convective disk is \( L \approx \).
σT_4^3 τ^{-η}. Cassen (1993) used similar line of argument to calculate \( T_{ph} \) for the disk with constant opacity (which is, in fact, convectively stable). Equation (10) reduces to his result \( η = 4(γ - 1)/γ \) if we set \( α = β = 0 \).

In the optically thin case (\( τ ≤ 1 \)) disk loses energy to space by volumetric radiative cooling irrespective of whether it is convective or radiative. Then, according to the Kirchhoff’s law, \( L \approx σT_4^3 τ \). In general, for arbitrary \( τ \), one can interpolate between the optically thick and thin regimes so that the cooling rate per unit surface area is

\[
L \approx \frac{σT_4^3}{f_c(τ)} \quad f_c(τ) = \chi τ^η + \frac{φ}{τ} \tag{11}
\]

where \( χ, φ \approx 1 \) are constants. In Appendix E we present detailed calculation of \( χ \) and show that its value does not deviate strongly from unity.

Cooling rate is a ratio of the total thermal energy content per unit surface area of the disk \( E_{th} \sim Σc_s^2 \) where \( c_s \equiv (kT/m) \) is the isothermal sound speed corresponding to the midplane temperature \( T_m \), the same as used in (1); here \( k \) is a Boltzmann constant, \( m \) is the mean molecular weight to the cooling rate \( L \). Using (11) one finds the shortest possible cooling time in a convective disk:

\[
t_{cool} = \frac{Σc_s^2}{σT_m} f_c(τ) \tag{12}
\]

\[
\approx 2 \times 10^4 \text{ yrs} \quad 10^3 \text{g cm}^{-2} \quad \left( \frac{T_m}{100 \text{ K}} \right)^{-3} \frac{f_c(τ)}{10^3},
\]

where the numerical estimate is for the gas of Solar composition and one has mind that \( f_c(τ) \approx τ + 1 \) for the adopted fiducial values of \( Σ \) and \( T \).

When the disk is convectively stable energy is transported by radiation, however \( t_{cool} \) is still given by the expression (12) but with \( f_c(τ) \) replaced with \( f_r(τ) \approx τ + 1 \). In the optically thin case cooling is the same for both modes of heat transfer while in the optically thick regime one finds

\[
\frac{f_c(τ)}{f_r(τ)} \approx τ^{-η}, \quad ξ = \frac{(4 - β)(V_a - V_{ad})}{1 + α + βV_{ad}}. \tag{13}
\]

Thus, when the condition (14) is satisfied and the disk is convective, fastest possible convective cooling is more effective than the radiative cooling would be for the same disk parameters \( (Σ, T_m, Ω) \). However, this improvement is not very large for two reasons. First, in real disks only the condition \( V_{ad} < V_a < V_{ad} \) must be fulfilled, so that the real cooling rate is somewhere in between the radiative and the maximum possible convective regimes. Thus, \( t_{cool} \) is likely to be longer than equation (12) predicts. Second, even the shortest cooling time (12) is not so different from radiative \( t_{cool} \): for example, \( ξ = 3/11 \) for \( α = 0, β = 2 \), while for \( α = 0, β = 1 \) (as supported by observations of T Tauri disks) one finds \( ξ = 1/9 \). Thus, the difference between the radiative and convective \( t_{cool} \) is a very weak function of \( τ \): for \( τ = 10^4 \) the ratio of cooling times in two cases are \( f_c/f_r \approx 7 \) and \( 2 \) correspondingly.

4. Implications for disk fragmentation.

Combination of gravitational instability condition (14) with the cooling requirement (2) allows one to set a robust lower limit on \( Σ \) and \( T \) of the disk region which can fragment as a result of GI, which has been previously done in R05 for the case of radiative heat transfer. Repeating his arguments for the case of maximally efficient convective cooling [i.e. using eq. (12)] we find that self-gravitating objects can form only in disks satisfying the following properties:

\[
Σ ≥ Σ_{min} = Σ_{inf}(f_τ(τ))^{1/5}, \tag{14}
\]

\[
T_m ≥ T_{min} = T_{inf}(f_τ(τ))^{2/5}, \tag{15}
\]

where

\[
Σ_{inf} ≡ Ω^{7/5}(πGQ_0)^{-6/5} \left[ \frac{1}{ζσ} \left( \frac{k}{μ} \right) \right]^{1/5} \approx 6.6 \times 10^5 g \text{ cm}^{-2} a_{AU}^{-21/10} (Q_0^6 μ^4 ζ)^{-1/5} \left( \frac{M_*}{M_☉} \right)^{7/10} \tag{16}
\]

\[
T_{inf} ≡ Ω^{1/5}(ζπGQ_0 σ)^{-2/5} \left( \frac{k}{μ} \right)^{3/5} \approx 5800 K a_{AU}^{-6/5} μ^{-3/5} (Q_0ζ)^{-2/5} \left( \frac{M_*}{M_☉} \right)^{2/5}. \tag{17}
\]

Here \( a_{AU} \) is the semimajor axis \( a \) in units of AU, \( μ \equiv μ/m_H \) is the mean molecular weight relative to the atomic hydrogen mass \( m_H \), and \( M_* \) is the mass of the central star \( (M_☉ \) is the Solar mass). Needless to say, the condition (14) has to be satisfied simultaneously with (15) which additionally limits \( T \) from above.

One can clearly see that the only difference between the radiative case studied in R05 and convective case considered here is in the explicit form of the function \( f_c(τ) \), which changes to \( f_r(τ) \) in the radiative case. Bearing in mind that \( f_c(τ) \) is similar to \( f_r(τ) \) in that it reaches the minimum value \( f_r(τ) \approx 1 \) at \( τ \approx 1 \) we conclude that the lowest possible \( (infimum) \) values of \( Σ \) and \( T \) given by (10) and (17) are the same as in the radiative case. Thus, all the constraints set in R05 on the basis of \( Σ_{inf} \) and \( T_{inf} \) persist in the convective case as well. In particular, it follows from (10) and (17) that \( T_{ph} ≥ T_{inf} f_r(τ)^{3/20} \), so that the photospheric temperature of fragmenting disk \( T_{ph} \) is limited from below by at least \( T_{inf} \).

5. Application to protoplanetary disks and other environments.

For molecular gas of solar composition and \( ζ = 10, Q_0 = 1 \) one finds \( Σ_{inf} ≈ 2 \times 10^5 \text{ g cm}^{-2} \) and \( T_{inf} ≈ 1400 \text{ K} \) at 1 AU. These very extreme conditions \( Σ \) exceeds the corresponding value in the minimum mass Solar nebula (Hayashi 1981) by \( ≈ 10^2 \) are clearly incompatible with the observations of T Tauri disks which precludes the possibility of giant planet formation by GI within several AU from the parent star. At larger distances, however, the constraints set by \( Σ_{inf} \) and \( T_{inf} \) relax and become more acceptable from the observational point of view. At 10 AU, for example, \( Σ_{inf} ≈ 1.6 \times 10^3 \text{ g cm}^{-2} \) and \( T_{inf} ≈ 100 \text{ K} \) which is at (or above) the uppermost end of the observed distributions of \( Σ \) and \( T \).

However, in such environment \( f_c(τ) \approx τ \approx 10^3 \) so that equations (14)-(15) set much more severe constraints on the disk properties. Careful inspection of (14)-(15) demonstrates that planet formation by GI at 10 AU is only possible if \( T > 1500 \text{ K} \) so that dust has sublimated lowering the opacity — otherwise \( τ \)
would be so high that according to (13) $T$ cannot self-
consistently remain below the sublimation point. Thus,
planet formation by GI even at 10 AU from the cen-
tral star requires disk properties which are clearly inco-
sistent with the current observations of protoplanetary
disks (Kitamura et al. 2002). Only beyond 20 AU do we
find that $T_{\text{min}} < 1500$ K and $\Sigma_{\text{min}} < 2.4 \times 10^3$ g cm$^{-2}$
so that silicates condense. At 45 AU $T \lesssim 150$ K and ice
grains condense which increases opacity by a factor of
several. In general, we find that the disk properties lim-
ited by (12)-(15) become more or less compatible with
the observations of T Tauri disks only at $a \sim 100$ AU.
Only this far from the star can planets possibly form by
the GI, which contradicts the conclusion of Boss (2006).

Previous discussion focused on the local disk properties
necessary for planet formation by GI. It is however clear,
by analogy with R05, that the global parameters of the
fragmenting disk – its mass and luminosity – should also
be rather extreme. We find that the disk mass would
exceed several $M_\odot$ if planets were to form by GI at 20
AU, but it reduces to $\sim 0.1 M_\odot$ at 100 AU. The masses
of self-gravitating objects that would form as a result of
fragmentation are also going to be large, likely in the
brown dwarf regime (R05), although this can be reliably
clarified only by numerical simulations.

Fragmentation of gravitationally unstable disks around
SMBHs in galactic centers is also constrained by the condi-
tions (12)-(15) if these disks are convective. In this
case, unlike the situation with the protoplanetary disks,
no obvious conflict with the observations arises. Indeed,
let’s consider the possibility of star formation in a disk
at $a = 0.1$ ps from the SMBH in the Galactic Center.
For a black hole mass of $3 \times 10^6 M_\odot$ simultaneous frag-
mentation and GI at 0.1 pc require $\Sigma_{\text{inf}} \approx 6$ g cm$^{-2}$
and $T_{\text{inf}} \approx 4$ K. The real temperature is in fact limited
from below by the stellar irradiation at the level $\sim 40$ K
(Nayakshin 2005) and this automatically increases mini-
imum $\Sigma$ to $\sim 25$ g cm$^{-2}$ to satisfy the Toomre crite-
ron (1). This translates into $\tau \approx 8$ and cooling time
$\approx 0.25 \tau^{-1}$ which would lead to fragmentation. Char-
acteristic disk mass $\pi a^2 \Sigma \sim 4 \times 10^3 M_\odot$ is fully compatible
with the current constraints on the total stellar mass in
the two stellar disks around the central black hole in the
Galactic Center (Nayakshin et al. 2006).

6. DISCUSSION.

There is good agreement between the constraints on $\Sigma$
and $T$ presented in this paper and R05 and the results
of the three-dimensional grid-based simulations by Mejia
(2004), Cai et al. (2006) and Boley et al. (2006). Using
radiation transfer scheme based on the flux-limited diffu-
sion these authors find that planet formation by GI does
not occur at distances of $\sim 10$ AU from the star, in ac-
cord with our findings. This outcome is clearly caused
by the long radiative cooling timescale in dense and cold
disk models that are run by these authors. In particular,
Cai et al. (2006) find that lowering disk metallicity has
the effect of decreasing the cooling timescale, which is
obvious from equation (12) if one also notices that the
low-temperature opacity $\kappa$ is proportional to metallicity
and $f_c(\tau) \propto \kappa^\nu$ in the optically thick regime (typical at
distances of several tens of AU in massive protoplanetary
disks).

At the same time recent radiative hydrodynamical sim-
ulations by Boss (2004) and Mayer et al. (2006) demon-
strate efficient fragmentation of massive and cold proto-
planetary disks and thus disagree with both the analyt-
ical constraints presented in this paper and the simul-
ations of Mejia (2004), Cai et al. (2006) and Boley et al.
(2006). The arguments presented here do not support the
idea of Boss (2004) and Mayer et al. (2006) that energy
transport by convection can significantly speed up the
disk cooling and lead to its fragmentation. While there
are many factors that can cause this discrepancy (opac-
ties, treatment of shock dissipation, numerical scheme
employed, etc.) we believe that the major source of dis-
agreement is in the treatment of the external radiative
boundary condition by different groups.

To illustrate this point we note that Boss (2004), Mayer
et al. (2006), and Boley et al. (2006) do observe$^3$ efficient
convection in their simulations. However, rapid vertical
transport of energy from the midplane to the disk sur-
face is a necessary but not sufficient condition for rapid
cooling, since this energy must then be radiated away
from the photosphere. Thus, the photosphere presents a
bottleneck for the disk cooling such that the disk cannot
cool faster than its photospheric temperature permits.
For this reason it is very important that the external ra-
diative boundary condition is treated with extreme care$^4$
in simulations.

Some of the vertical motions seen in simulations and
attributed to convection may actually be due to the up-
ward propagating waves launched by the overdensities
cauised by the GI (Boley et al. 2006). The dissipation
of such waves may heat up the upper layers of the disk
which has an effect of increasing the photospheric tem-
perature. This, however, has nothing to do with the
cooling of transiently collapsing overdensities in the disk
as these waves do nothing to transport the thermal en-
ergy away from the dense gas near the midplane. Thus,
such wave motions cannot facilitate fragmentation of an
unstable disk.

Similarly, the external irradiation of the disk (by the
central star or the surrounding envelope) which acts to
increase the photospheric temperature does not acceler-
ate disk cooling. In this case, although the disk surface
loses more energy because of the higher $T_{\text{ph}}$, it also gains
energy from the absorbed external radiation which keeps
the net cooling rate unchanged. Cooling of overdensities
forming near the midplane as a result of GI depends only
on the temperature gradients that develop in the interior
and thus is not affected by external heating. Besides,
external illumination tends to stabilize temperature gra-
dient and to suppress convection.

Our final comment refers to the use of constant fac-
tor $\zeta \sim 1 - 10$ in the fragmentation condition (14).
Using simulations with nonlinear cooling rates devised to mim-
ic the effect of realistic opacities on disk cooling Johnson &
Gammie (2003) have demonstrated that $\zeta$ can be sig-
ificantly larger than $10$ just below the so-called opacity
gaps — regions in the $\rho-T$ space where opacity suddenly
changes. In particular, they find that $\zeta \sim 10^4$ at $T \sim 10^3$

$^3$ Boley et al. (2006) find convection to be disrupted during the active phase of the GI.
$^4$ In particular, the proper treatment of the photospheric transition between the optically thick and thin regions should be espe-
cially challenging for the SPH simulations such as used by Mayer et al. (2006).
K, near the point of silicate dust sublimation where opacity drops by a factor of \( \sim 10^3 \). Such an increase of \( \zeta \) significantly relaxes our constraints (16)-(17) which seems to allow fragmentation in disks of lower \( \Sigma \) and \( T \). However, the less conservative (but still robust) constraint (18)-(19) eliminates this concern because (similar to \( \zeta \)) \( \kappa \) and \( \tau \) are also very large on the verge of the opacity gap (compared to their values at the bottom of the gap) so that \( F \gg 1 \) in (18)-(19) and this offsets the decrease of \( \Sigma_{inj} \) and \( T_{inj} \) driven by large \( \zeta \). Besides, this issue only arises if \( T \sim 10^3 K \) which is atypical for a protoplanetary disk in T Tauri phase anyway.

The arguments presented in this paper and R05 are analytic in nature, however, they are based solely on the numerically verified criteria (1) and (2). Thus, keeping in mind previous discussion, we conclude that to within factors of order unity our constraints (16)-(19) on the properties of unstable disks capable to support the formation of self-gravitating objects should be valid in applications to realistic disks in which energy is transported by convection.

7. Conclusions.

We have investigated the possibility of rapid disk cooling by convection in the context of fragmentation of gravitationally unstable disks. We have shown that even the most extreme form of convective cooling (realized when \( \nabla = \nabla_{ad} \)) produces cooling rates which are only marginally higher than in the case of radiative energy transport. The major reason for the inefficient cooling of dense and cold convective disks is in the bottleneck caused by the need to eventually radiate from the photosphere the energy that has been transported there from the midplane, irrespective of how the latter has been done. Armed with this knowledge we demonstrate that inefficient cooling precludes direct formation of giant planets by GI in protoplanetary disks anywhere within several tens of AU from the parent star (analogous conclusion has been reached in R05). At the same time, disk cooling (convective or radiative) is fast enough to allow fragmentation of gravitationally unstable disks around black holes in the galactic centers, supporting the idea that the efficient star formation is possible in such disks.

I am grateful to Chris Matzner and Sergey Nayakshin for careful reading of the manuscript and making very useful suggestions, to Yuri Levin for helpful exchanges, and to Charles Gammie for bringing the work by Cassen (1993) to my attention. The financial support for this work is provided by the Canada Research Chairs program and a NSERC Discovery grant.

APPENDIX

DERIVATION OF CONDITION (11).

To derive the condition (11) in the case of opacity given by equation (3) we will follow the approach adopted by the Lin & Papaloizou (1980). Equation of hydrostatic equilibrium states that

\[
dP \over dz = -\rho g_z, \quad (A1)
\]

where vertical acceleration \( g_z = \Omega^2 z + g_z^{ph} \) is contributed both by the central object (\( \Omega^2 z \) term) and the disk self-gravity (\( g_z^{ph} \) term). The latter contribution satisfies Poisson equation

\[
dg_z^{ph} \over dz = 4\pi G\rho. \quad (A2)
\]

Radiative transport is described by

\[
(16\pi T^3) \frac{dT}{dz} = -F; \quad dF \over dz = \epsilon, \quad (A3)
\]

where \( F \) is the vertical radiative flux and \( \epsilon \) is the volumetric energy dissipation rate. Equations (A1) and (A2) result in \( \nabla_r \) given by equation (7), but they can also be combined in the following way:

\[
\nabla_0 \frac{16}{3} \frac{\sigma}{\kappa_0} \frac{d(P^{4-\beta})}{dz} = \frac{F}{g_z}. \quad (A4)
\]

We now note that if \( F/g_z \) monotonically decreases as \( z \) increases (and, accordingly, as \( P \) decreases), then, integrating equation (A4) down from the photosphere we arrive at the following inequality:

\[
\nabla_0 \frac{16}{3} \frac{\sigma}{\kappa_0} \left( P^{4-\beta} - P^{4-\beta}_{ph} \right) = \frac{F}{g_z} \int_{P_{ph}}^{P} \frac{d(P^{1+\alpha})}{dz} < \frac{F}{z} \left( P^{1+\alpha} - P^{1+\alpha}_{ph} \right), \quad (A5)
\]

since \( F/g_z \) is assumed to be largest at \( P \). This can be finally rewritten as

\[
\nabla_0 \left[ 1 - \left( \frac{P_{ph}}{P} \right)^{4-\beta} \right] < \nabla_r \left[ 1 - \left( \frac{P_{ph}}{P} \right)^{1+\alpha} \right], \quad (A6)
\]

which reduces to \( \nabla_0 < \nabla_r \) at high depth in the disk, where \( T \gg T_{ph} \) and \( P \gg P_{ph} \). Thus, if (1) the condition (11) is fulfilled and (2) \( F/g_z \) is a decreasing function of \( z \), then \( \nabla_r > \nabla_{ad} \) and convection has to operate in the bulk of the disk since, according to the Schwarzschild criterion (6), disk cannot be radiative.
We now find out under which circumstances \( F/g_z \) monotonically decreases as \( z \) grows. One has

\[
\frac{g_z^2}{\Omega^2} \frac{d}{dz} \left( \frac{F}{g_z^2} \right) = \int_0^z [\epsilon(z) - \epsilon(z')] dz' + \frac{4\pi G}{\Omega^2} \int_0^z [\epsilon(z)\rho(z') - \epsilon(z')\rho(z)] dz',
\]

(A7)

where equation (A2) has been used. The first term in the right-hand side is clearly negative when a reasonable and weak assumption of \( \epsilon \) decreasing with increasing \( z \) is made. Sign of the second term, which quantifies the effect of the disk self-gravity on stability, depends on how rapidly \( \epsilon \) decreases with \( z \). If, like in a conventional \( \alpha \)-disk (Shakura & Sunyaev 1973) with constant \( \alpha \) and sound speed, \( \epsilon(z) \propto \rho(z) \), then the second term is identically zero and \( d(F'/g_z)/dz < 0 \). When \( \epsilon(z) \) decays with \( z \) faster than \( \rho(z) \) the second term in (A7) is negative and \( d(F'/g_z)/dz \) is again negative. However, if \( \epsilon(z)/\rho(z) \) is an increasing function of \( z \) then the sign of the second term in (A7) is positive and one cannot tell for sure how \( F'/g_z \) behaves as a function of \( z \). In the latter case condition (4) may or may not be sufficient to determine whether disk is convective.

It may be possible that gravitationally unstable disks have \( \epsilon(z)/\rho(z) \) increasing with \( z \). This can be caused, for example, by dissipation of the upward-propagating waves driven by the instability, which would tend to deposit their energy high up in the disk stabilizing its structure. We can then conclude that in many situations (certainly in non-self-gravitating disks) condition (4) is sufficient for convection, while in some self-gravitating disks a more conservative criterion than (4) may be required.

**Cooling of an isentropic disk.**

Here we calculate the structure and the cooling time of a disk that has an isentropic interior (for which the condition \( \nabla = \nabla_{ad} \) is satisfied) with equation of state \( P = K \rho^\gamma \) smoothly matching the outer radiative layer near the photosphere. We will separately consider structure of non-self-gravitating disks with \( g_z = \Omega^2 z \) and self-gravitating disks with \( g_z = g_z^{sg} \) satisfying equation (A2).

**Non-self-gravitating disk.**

In the non-self-gravitating isentropic disk equation (A1) can be solved using \( g_z = \Omega^2 z \), resulting in the following disk structure:

\[
\frac{T(z)}{T_m} = \left[ \frac{P(z)}{P_m} \right]^{\nabla_{ad}} = \left[ \frac{\rho(z)}{\rho_m} \right]^{\gamma-1} = 1 - \frac{z^2}{H^2}, \quad H^2 = \frac{2}{\nabla_{ad}} \frac{\kappa_B T_m}{\mu \Omega^2},
\]

(B1)

where \( T_m, P_m, \rho_m \) are the midplane temperature, pressure, and density, while \( H \) is recognized as the height of the disk surface. Using (B1) and assuming disk to be optically thick (so that the isentropic part contains most of the mass) one can calculate the total surface density

\[
\Sigma = 2 \int_0^\infty \rho dz = I_1 \rho_m H, \quad I_1 \equiv 2 \int_0^1 (1 - x^2)^{1/(\gamma-1)} dx,
\]

(B2)

and the full optical depth

\[
\tau = 2 \int_0^\infty \rho \kappa dz = I_2 \rho_m \kappa_m H, \quad I_2 \equiv 2 \int_0^1 (1 - x^2)^{(\gamma+1)/2(\gamma-1)} dx,
\]

(B3)

where \( \kappa_m = \kappa(P_m,T_m) = \kappa_0 P_m^\beta T_m^\delta \) is the opacity at the midplane. The total thermal energy content per unit surface area is

\[
E_{th} = \frac{2}{\gamma - 1} \int_0^\infty P dz = I_3 P_m H, \quad I_3 \equiv \frac{2}{\gamma - 1} \int_0^1 (1 - x^2)^{1/\nabla_{ad}} dx,
\]

(B4)

while the midplane pressure \( P_m = (\nabla_{ad}/2) \Omega^2 H^2 \rho_m \) [following from eq. (B1) and the ideal gas law] can be expressed through the value of \( g_z \) at the disk surface \( g_z(H) = \Omega^2 H \) in the following way:

\[
P_m = I_4 \rho_m H g_z(H), \quad I_4 \equiv \frac{\nabla_{ad}}{2}.
\]

(B5)

**Self-gravitating disk.**

In a self-gravitating case one can solve the system (A1)-(A2) with the equation of state \( P = K \rho^\gamma \) to find the following implicit relation between \( \rho \) and \( z \):

\[
z = H I_0^{-1} \int_0^{\rho/\rho_m} x^{\gamma-2} (1 - x^\gamma)^{-1/2} dx,
\]

(B6)
occurs. We assume that this happens at the temperature $T_H$, in the outer radiative zone, so that

$$T = T_H \left[ 2^{(\beta-4)/4} + X \right]^{1/(4-\beta)}.$$  

(B13)

in the outer radiative zone, so that

$$\tau(z) = \int_0^z \kappa(p)\, dz = \frac{4}{3} \left[ \left( 2^{(\beta-4)/4} + X \right)^{4/(4-\beta)} - \frac{1}{2} \right],$$  

(B14)

[here we have used equation (A1)] and

$$\nabla_r = \frac{\partial \ln T}{\partial \ln P} = \frac{X}{2^{(\beta-4)/4} + X}.$$  

(B15)

Transition from radiative to convective energy transport occurs when $\nabla_r = \nabla_{ad}$, i.e. at

$$X_{tr} = \frac{2^{(\beta-4)/4} \nabla_{ad}}{(\nabla_0 - \nabla_{ad})},$$  

(B16)

while the photosphere ($\tau = 2/3$) lies at $X_{ph} = 1 - 2^{(\beta-4)/4}$, see equation (B14). From equations (B15), (B13), and (B16) one finds

$$T_{tr} = T_{ph} \frac{2^{(\beta-4)/4} \nabla_0}{(\nabla_0 - \nabla_{ad})}^{1/(4-\beta)},$$  

(B17)

$$P_{tr} = \left[ \frac{16 \times 2^{(\beta-4)/4} \nabla_0 \nabla_{ad} g_z(H)}{3(\nabla_0 - \nabla_{ad})} \kappa_0 T_{ph}^{\beta} \right]^{1/(1+\alpha)}.$$  

(B18)

Using these relations, equations (B19) and (B15), and the fact that $T_{tr}/T_m = (P_{tr}/P_m) \nabla_{ad}$ one obtains

$$\frac{T_{ph}}{T_m} = \lambda T^{-\eta/4}, \quad \lambda = \left[ \frac{16 \nabla_{ad} I_2}{3 I_4} \right]^{\frac{\nabla_{ad}}{\nabla_0 - \nabla_{ad}}} \left( 2^{(\beta-4)/4} \nabla_0 \nabla_{ad} - \nabla_0 \right)^{1/(1+\alpha + \beta \nabla_{ad})},$$  

(B19)
where $\eta$ was introduced in equation (10). Using equations (B3), (B4) and (B19) we finally derive cooling time in the optically thick case as

$$t_{\text{cool}} = \frac{E_{\text{th}}}{2 \sigma T_{\text{ph}}^4} = \frac{\chi \Sigma c_s^2}{\sigma T_m^4} \eta, \quad \chi = \frac{I_3}{2 I_1 \lambda^4}. \quad (B20)$$

In particular, in the case of a non-self-gravitating disk one finds for $\alpha = 0, \beta = 2$ ($\eta = 8/11$) using equations (B2)-(B5) that $\lambda = 1.37, \chi = 0.31$, while for $\alpha = 0, \beta = 1$ ($\eta = 8/9$) one gets $\lambda = 1.55, \chi = 0.19$. When the disk is self-gravitating one obtains using equations (B8)-(B11) that $\lambda = 1.24, \chi = 0.46$ for $\alpha = 0, \beta = 2$, while $\lambda = 1.37, \chi = 0.3$ for $\alpha = 0, \beta = 1$. In realistic unstable Keplerian disk the self-gravity of the disk and the central star contribute roughly equally to $g_z$, so that one should expect $\lambda = 1.24 - 1.37, \chi = 0.31 - 0.46$ for $\alpha = 0, \beta = 2$ and $\lambda = 1.37 - 1.55, \chi = 0.19 - 0.3$ for $\alpha = 0, \beta = 1$. Note that the small value of $\chi$ in some cases should not strongly affect our estimates of $\Sigma_{\text{min}}$ and $T_{\text{min}}$ as these quantities depend on $\chi$ rather weakly, see equations (14)-(15).

REFERENCES

Beckwith, S. V. W., Sargent, A. I., Chini, R. S., & Guesten, R. 1990, AJ, 99, 924
Bell, K. R. & Lin, D. N. C. 1994, ApJ, 427, 987
Boley, A. C., Mejia, A. C., Durisen, R. H., Cai, K., Pickett, M. K., & D’Alessio, P., 2006, astro-ph/0607112
Boss, A. P. 1998, ApJ, 503, 923
Boss, A. P. 2002, ApJ, 576, 462
Boss, A. P. 2004, ApJ, 610, 456
Boss, A. P. 2006, ApJL, 637, L137
Cai, K., Durisen, R. H., Michael, S., Boley, A. C., Mejia, A. C., Pickett, M. K., & D’Alessio, P. 2006, ApJL, 636, L149
Cameron, A. G. W. 1978, Moon and the Planets, 18, 5
Cassen, P. 1993, in “Lunar and Planetary Inst., Twenty-fourth Lunar and Planetary Science Conference”, Part 1: A-F, 261
Durisen, R. H., Boss, A. P., Mayer, L., Nelson, A. F., Quinn, T., & Rice, W. K. M. 2006, astro-ph/0603179
Gammie, C. F. 2001, ApJ, 553, 174
Goodman, J. 2003, MNRAS, 339, 937
Goodman, J. & Tan, J. C. 2004, ApJ, 608, 108
Harris, A. W. 1978, Lunar Planet. Sci., 9, 459
Hayashi, C. 1981, Progr. Theor. Phys. Suppl., 70, 35
Illarionov, A. F. & Romanova, M. M. 1988, Sov. Astr., 32, 148
Johnson, B. M. & Gammie, C. F. 2003, ApJ, 597, 131
Kim, W.-T., Ostriker, E. C., & Stone, J. M. 2003, ApJ, 595, 574
Kitamura, Y.,Momose, M., Yokogawa, S., Kawabe, R., Tamura, M. & Ida, S. 2002, ApJ, 581, 357
Kolykhalov, P. I. & Sunyaev, R. A. 1980, SovAL, 6, 357
Lin, D. N. C. & Papaloizou, J. 1980, MNRAS, 191, 37
Levin, Yu. 2003, [astro-ph/0307084]
Levin, Yu. & Beloborodov, A. M. 2003, ApJL, 590, L33
Mayer, L., Lufkin, G., Quinn, T., & Wadsley, J. 2006, astro-ph/0606361
Mayer, L., Quinn, T., Wadsley, J., & Stadel, J., 2002, Science, 298, 1756
Mizuno, H. 1980, Progr. Theor. Phys., 64, 544
Nayakshin, S. 2005, astro-ph/0512259
Nayakshin, S., Deluca, W., Guhathakurta, J., & Genzel, R. 2006, MNRAS, 366, 1410
Paczynski, B. 1978, AcA, 28, 91
Perri, F. & Cameron, A. G. W. 1974, Icarus, 22, 416
Pickett, B., Cassen, P., Durisen, R. H., & Link, R. 1998, ApJ, 504, 468
Rafikov, R. R. 2005, ApJL, 621, L69 (R05)
Rice, W. K. M., Armitage, P. J., Bate, M. R., & Bonnell, I. A. 2003, MNRAS, 339, 1025
Rice, W. K. M., Lodato, G., & Armitage, P. J. 2005, MNRAS, 364, L56
Safronov, V. S. 1960, Ann. d’Astrophysique, 23, 979
Shakura, N. I. & Sunyaev, R. A. 1973, A&A, 24, 337
Thompson, T. A., Quataert, E., & Murray, N. 2005, ApJ, 630, 167
Toomre, A. 1964, ApJ, 139, 1217