ENTIRE POSITIVE SOLUTIONS OF THE SINGULAR
EMDEN-FOWLER EQUATION WITH NONLINEAR
GRADIENT TERM

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Abstract
Let $p$ and $q$ be locally Hölder functions in $\mathbb{R}^N$, $p > 0$ and $q \geq 0$. We study the Emden-Fowler equation $-\Delta u + q(x)|\nabla u|^a = p(x)u^{-\gamma}$ in $\mathbb{R}^N$, where $a$ and $\gamma$ are positive numbers. Our main result establishes that the above equation has a unique positive solutions decaying to zero at infinity. Our proof is elementary and it combines the maximum principle for elliptic equations with a theorem of Crandall, Rabinowitz and Tartar.

Keywords: Emden-Fowler equation, singular elliptic equation, entire solution, maximum principle.

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1 Introduction and the main result

Singular semilinear elliptic problems have been intensively studied in the last decades. Such problems arise in the study of non-Newtonian fluids, boundary layer phenomena for viscous fluids, chemical heterogenous catalysts or in the theory of heat conduction in electrically conducting materials. For instance, problems of this type characterize some reaction-diffusion processes where the unknown $u \geq 0$ is viewed as the density of a reactant (see, e.g., [1]). In this framework a major place is played by the Emden-Fowler singular equation

$$-\Delta u = p(x)u^{-\gamma}, \quad x \in \Omega, \quad (1)$$

where $\Omega$ is an open set (bounded or unbounded) in $\mathbb{R}^N (N \geq 3)$, $\gamma > 0$, and $p : \Omega \rightarrow (0, \infty)$ is a continuous function. For a comprehensive study of the Emden-Fowler equation we refer to [6, 8, 9, 13, 14, 20, 21] and the references therein. If $\Omega$ is bounded, Lazer and McKenna proved in [16] that (1) has a unique positive solution if $p$ is a smooth positive function. The existence of entire positive solutions for $\gamma \in (0,1)$ and under certain additional hypotheses has been established in Edelson [8] and in Kusano-Swanson [14]. For instance, Edelson proved the existence of a solution provided that

$$\int_1^\infty r^{N-1+\lambda(N-2)} \max_{|x|=r} p(x)dr < \infty,$$

for some $\lambda \in (0,1)$. This result is generalized for any $\gamma > 0$ via the sub and super solutions method in Shaker [20] or by other methods in Dalmasso [6]. For further results related to singular elliptic equations we also refer to [3, 4, 5, 6, 11].
The purpose of this paper is to extend some of these results in the more general framework of singular elliptic equations with nonlinear gradient term. Problems of this type arise in stochastic control theory and have been first studied in Lasry and Lions [15]. The corresponding parabolic equation was considered in Quittner [19]. Elliptic problems with nonlinear gradient term have been also studied in various contexts (see, e.g., [2, 12, 17, 18]).

We study the problem

$$
\begin{cases}
-\Delta u + q(x)|\nabla u|^a = p(x)u^{-\gamma} & \text{in } \mathbb{R}^N \\
u > 0 & \text{in } \mathbb{R}^N \\
\lim_{|x| \to \infty} u(x) = 0,
\end{cases}
$$

where $N \geq 3$, $a > 0$ and $\gamma > 0$. We assume throughout this paper that $p, q \in C^0_{\text{loc}}(\mathbb{R}^N)$, $p > 0$ and $q \geq 0$ in $\mathbb{R}^N$. Set

$$
\Phi(r) = \max_{|x|=r} p(x).
$$

We impose no growth hypothesis on $q$ but we suppose that $p$ satisfies the following decay condition to zero at infinity:

$$
\int_0^\infty r\Phi(r)dr < \infty.
$$

In particular, potentials $p(x)$ which behave like $|x|^{-\alpha}$ as $|x| \to \infty$, with $\alpha > 2$, satisfy this assumption.

Our main result is the following:

**Theorem 1.1.** Under the above hypotheses, problem (2) has a unique classical solution.

### 2 Proof of Theorem 1.1

We first establish the existence of at least one solution of problem (2). For this purpose, for any integer $n \geq 1$, we consider the auxiliary boundary value problem

$$
\begin{cases}
-\Delta u + q(x)|\nabla u|^a = p(x)u^{-\gamma} & \text{in } B_n \\
u > 0 & \text{in } B_n \\
u = 0, & \text{on } \partial B_n,
\end{cases}
$$

where $B_n := \{x \in \mathbb{R}^N; |x| < n\}$. We observe that the function $u = \varepsilon \varphi_1$ is a subsolution of (3), provided that $\varepsilon > 0$ is sufficiently small, where $\varphi_1 > 0$ is the first eigenfunction of $(-\Delta)$ in $H^1_0(B_n)$. In order to find a supersolution of (3), we observe that any solution of

$$
\begin{cases}
-\Delta u = p(x)u^{-\gamma} & \text{in } B_n \\
u > 0 & \text{in } B_n \\
u = 0, & \text{on } \partial B_n
\end{cases}
$$

is a supersolution of (3). But problem (3) has a solution, by Theorem 1.1 in Crandall, Rabinowitz and Tartar [6]. Denote by $u_n$ this solution. By standard bootstrap arguments (see [10]), $u_n \in C^2(B_n) \cap C(\overline{B_n})$. Also, by the maximum principle, it follows that $u_n \leq u_{n+1}$ in $B_n$. Until now we know that there exists $u(x) := \lim_{n \to \infty} u_n(x) \leq +\infty$, for all $x \in \mathbb{R}^N$. 

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Next, we establish the existence of a positive smooth function $v$ such that $u_n \leq v$ in $\mathbb{R}^N$. Let $\Phi$ be defined by \( (3) \) and set

$$w(r) := K - \int_0^r \int_0^{\zeta} \sigma^{N-1} \Phi(\sigma)d\sigma d\zeta,$$

where

$$K := \int_0^\infty \int_0^{\zeta} \sigma^{N-1} \Phi(\sigma) d\sigma d\zeta \quad \text{for any } r > 0,$$

provided that the integral is convergent. Then $-\Delta w = \Phi(r)$ and $\lim_{r \to \infty} w(r) = 0$.

We prove in what follows that $K < +\infty$. An integration by parts yields

$$\int_0^r \int_0^{\zeta} \sigma^{N-1} \Phi(\sigma) d\sigma d\zeta = (2-N)^{-1} \int_0^r \frac{d}{d\zeta} \zeta^{2-N} \int_0^{\zeta} \sigma^{N-1} \Phi(\sigma) d\sigma d\zeta = (N-2)^{-1} \left( -r^{-(N-2)} \int_0^r \sigma^{N-1} \Phi(\sigma) d\sigma + \int_0^{\zeta} \zeta \Phi(\zeta) d\zeta \right).$$

Next, by L'Hospital's rule,

$$\lim_{r \to \infty} \left( -r^{-(N-2)} \int_0^r \sigma^{N-1} \Phi(\sigma) d\sigma + \int_0^{\zeta} \zeta \Phi(\zeta) d\zeta \right) = \lim_{r \to \infty} \frac{\int_0^{\zeta} \zeta \Phi(\zeta) d\zeta}{r^{N-2}} = \lim_{r \to \infty} \int_0^{\infty} \zeta \Phi(\zeta) d\zeta = \int_0^{\infty} \zeta \Phi(\zeta) d\zeta < \infty,$$

by our assumption \( (3) \). Thus we obtain that $K = (N-2)^{-1} \int_0^{\infty} \zeta \Phi(\zeta) d\zeta < \infty$. So, by the definition of $w$, $w(r) < (N-2)^{-1} \int_0^{\infty} \zeta \Phi(\zeta) d\zeta$, for any $r > 0$.

Set

$$v(r) := \left[ c(2+\gamma)w(r) \right]^{1/(2+\gamma)},$$

where

$$c := [K(2+\gamma)]^{1/(1+\gamma)}.$$

In particular, from $w(r) \to 0$ as $r \to \infty$, we deduce that $v(r) \to 0$ as $r \to \infty$. Since $w$ is a decreasing function, it follows that $v$ decreases, too. Hence

$$\int_0^{v(r)} t^{1+\gamma} dt \leq \int_0^{v(0)} t^{1+\gamma} dt = cw(0) = cK = \int_0^c t^{1+\gamma} dt.$$

It follows that $v(r) \leq c$ for all $r > 0$.

On the other hand,

$$\nabla w = \frac{1}{c} v^{1+\gamma} \nabla v \quad \text{and} \quad \Delta w = \frac{1}{c} v^{1+\gamma} \Delta v + \frac{1}{c} \left( v^{1+\gamma} \right)' |\nabla v|^2.$$
Hence
\[ \Delta v < cv^{-1-\gamma} \Delta w = -c v^{-1-\gamma} \Phi(r) \leq -v^{-\gamma} \Phi(r). \] (9)

By (6) and (9) we obtain that \( u_n \leq v \) in \( B_n \). Therefore
\[ u_1 \leq u_2 \leq \cdots \leq u_n \leq u_{n+1} \leq \cdots \leq v, \]
with \( v \) vanishing at infinity. Now, standard bootstrap arguments (see [10]) imply that \( u(x) := \lim_{n \to \infty} u_n(x) \) is well defined and smooth in \( \mathbb{R}^N \). Moreover, \( u \) is a classical solution of problem (2).

We justify in what follows the uniqueness of the solution to problem (2). Suppose that \( u \) and \( v \) are arbitrary solutions of (2). In order to establish the uniqueness, it is enough to show that \( u \leq v \) in \( \mathbb{R}^N \). Arguing by contradiction, it follows that \( \max_{x \in \mathbb{R}^N} (u(x) - v(x)) =: M > 0 \). Assume that \( u(x_0) - v(x_0) = M \). Then \( u(x_0) > v(x_0) > 0 \), \( \nabla u(x_0) = \nabla v(x_0) \) and \( \Delta (u - v)(x_0) \leq 0 \). But

\[ \Delta (u - v)(x_0) = q(x_0) \left| \nabla u(x_0) \right|^a - \left| \nabla v(x_0) \right|^a \right] + p(x_0) \left( v^{-\gamma}(x_0) - u^{-\gamma}(x_0) \right) = p(x_0) \left( v^{-\gamma}(x_0) - u^{-\gamma}(x_0) \right) > 0, \]

which is a contradiction. This implies that \( u \leq v \), and so \( u = v \) in \( \mathbb{R}^N \), \( \square \)

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