Continuous Nakayama Representations

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Abstract

We introduce continuous analogues of Nakayama algebras. In particular, we introduce the notion of (pre-)Kupisch functions, which play a role as Kupisch series of Nakayama algebras, and view a continuous Nakayama representation as a special type of representation of $\mathbb{R}$ or $S^1$. We investigate equivalences and connectedness of the categories of Nakayama representations. Specifically, we prove that orientation-preserving homeomorphisms on $\mathbb{R}$ and on $S^1$ induce equivalences between these categories. Connectedness is characterized by a special type of points called separation points determined by (pre-)Kupisch functions. We also construct an exact embedding from the category of finite-dimensional representations for any finite-dimensional Nakayama algebra, to a category of continuous Nakayama representations.

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1 Introduction

1.1 Background

A finite-dimensional algebra $A$ is Nakayama if it is both left and right serial. That is, left and right indecomposable projective $A$-modules have unique composition series. A basic Nakayama algebra is isomorphic to a quiver algebra $kQ/I$ where $Q$ is either an $A_n$ type,...
Fig. 1 Push-down of the interval $\mathbb{R}$-module $M_{[0,1.5]}$ to the $\mathbb{S}^1$-module $\overline{M}_{[0,1.5]}$

quiver with straight orientation, or an $\hat{\mathbb{A}}_n$ type quiver with cyclic orientation. Any basic Nakayama algebra can be determined by its Kupisch series $(l_1, l_2, \ldots, l_n)$, where $l_i$ encodes the length of the $i$-th indecomposable projective module. In representation theory, Nakayama algebras have finite representation type and are considered as one of the most well-known classes of algebras. Many homological properties of Nakayama algebras have been revealed. For instance, Gustafson [7] showed that the global dimension of a basic Nakayama algebra with $n$ non-isomorphic simple modules is bounded by $2n - 2$. In [14], Ringel characterized Gorenstein projective modules over Nakayama algebras. In [15] and [18], the authors studied the finitistic dimension of Nakayama algebras. In [17], Sen characterizes Nakayama algebras which are higher Auslander. In [13], the authors classify quasi-hereditary Nakayama algebras.

Persistence modules have been extensively studied in both representation theory and data science, specifically in the interest of persistence theory in topological data analysis. The pointwise finite-dimensional $\mathbb{R}$-representations, which appear as one-parameter persistence homology, are well understood classes of persistence modules. Any pointwise finite-dimensional $\mathbb{R}$-representation has a unique barcode decomposition as a direct sum of interval modules $MU_{[4, 5]}$. Recently, Hanson and Rock [9] considered pointwise finite-dimensional $\mathbb{S}^1$-representations and showed that any such representation can be uniquely decomposed as a direct sum of (possibly infinitely many) string modules $\overline{M}_U$ and finitely many Jordan cells.

1.2 Continuous Nakayama Representations

In this paper, we consider pointwise finite-dimensional (pwf) representations over $\mathbb{R}$ or over $\mathbb{S}^1$ subject to some relations given by a (pre-)Kupisch function. These $\mathbb{R}$- and $\mathbb{S}^1$-representations are continuous analogues of representations of Nakayama algebras, which we call continuous Nakayama representations.

Consider $\mathbb{R}$ as a category where the objects are points in $\mathbb{R}$ and there is a unique morphism $g_{xy} : x \to y$ if $x \leq y$. An $\mathbb{R}$-representation $M$ over a field $k$ is a covariant functor $M : \mathbb{R} \to k$-Vec. For any interval $U$, an interval module $M_U$ is an $\mathbb{R}$-representation such that

$$M_U(x) = \begin{cases} k & x \in U \\ 0 & x \notin U \end{cases}$$

and $M_U(x \xrightarrow{g_{xy}} y) = id_k$ for $x \leq y \in U$.

We similarly consider $\mathbb{S}^1$ as a category where the objects are elements of $\mathbb{S}^1$ and morphisms move counter-clockwise. A pwf $\mathbb{S}^1$-representation $M$ over a field $k$ is a covariant functor $M : \mathbb{S}^1 \to k$-Vec. A string module $\overline{M}_U$ is an $\mathbb{S}^1$-representation obtained from the “push-down” of the interval $\mathbb{R}$-module $M_U$ via the covering map $p : \mathbb{R} \to \mathbb{S}^1$.

Continuous Nakayama representations are pwf representations on an interval $I \subseteq \mathbb{R}$ or on $\mathbb{S}^1$ that are compatible with some relations given by (pre-)Kupisch functions (Definitions 3.1 and 3.9). Similar to the Kupisch series of Nakayama algebras, a (pre-)Kupisch
function \( \kappa \) determines an indecomposable projective representation \( M_{[t,t+\kappa(t)]} \) (or \( \overline{M}_{[t,t+\kappa(t)]} \)) at each point \( t \in \mathbb{R} \). Thus, each indecomposable representation of \( I \) (or \( \mathbb{S}^1 \)) is \( M_U \) (or \( \overline{M}_U \)) for some interval \( U \subset [t,t+\kappa(t)] \). We denote the full subcategory of pwf \( \mathbb{R} \)-representations compatible with a pre-Kupisch function \( \kappa \) on an interval \( I \) by \( (I, \kappa) \)-pwf and the full subcategory of pwf \( \mathbb{S}^1 \)-representations compatible with a Kupisch function \( \kappa \) by \( (\mathbb{S}^1, \kappa) \)-pwf.

The categorical structures of \((I, \kappa)\)-pwf and \((\mathbb{S}^1, \kappa)\)-pwf are similar to those of the representations of discrete Nakayama algebras. Additionally, Theorem 6.3 says that the module category of any basic connected Nakayama algebra can be regarded as an exact abelian subcategory of \( \mathbb{S}^1 \)-representations compatible with a Kupisch function \( \kappa \).

**1.3 Results**

We investigate the connectedness and equivalences of the categories \((I, \kappa)\)-pwf or \((\mathbb{S}^1, \kappa)\)-pwf among various (pre-)Kupisch functions and their relation to discrete Nakayama representations.

Recall that an orientation-preserving homeomorphism between intervals \( I \) and \( J \) is an increasing bijective map \( f : I \to J \). Given a pre-Kupisch function \( \kappa \) on an interval \( I \), an orientation preserving homeomorphism \( f : I \to J \) induces a push-forward pre-Kupisch function \( f_*\kappa \) on \( J \) (Definition 3.5). In fact, this gives rise to an equivalence of categories \((I, \kappa)\)-pwf \( \to (J, f_*\kappa)\)-pwf that sends interval modules \( M_U \) to \( M_{f(U)} \) (Theorem 3.7). Similarly, an orientation-preserving homeomorphism on \( \mathbb{S}^1 \) is a degree 1 circle homeomorphism \( f : \mathbb{S}^1 \to \mathbb{S}^1 \) ([10]). For a Kupisch function \( \kappa \), such an \( f \) induces a push-forward Kupisch function \( f_*\kappa \) (Definition 3.11). This also gives an equivalence \((\mathbb{S}^1, \kappa)\)-pwf \( \to (\mathbb{S}^1, f_*\kappa)\)-pwf (Theorem 3.15).

The converse of Theorems 3.7 and 3.15 may not hold in case the category \((I, \kappa)\)-pwf (or \((\mathbb{S}^1, \kappa)\)-pwf) is disconnected. This is because one can construct an equivalence by permuting the orthogonal components of the representation category (Example 5.12).

In order to describe the connectedness of the categories of continuous Nakayama representations, we introduce the notion of separation points, using an analytical property of (pre-)Kupisch functions (Definition 4.2). Denote by \( S(\kappa) \) the set of all separation points of \( \kappa \). We show the following result concerning the connectedness of categories \((I, \kappa)\)-pwf (or \((\mathbb{S}^1, \kappa)\)-pwf).

**Theorem 1.1** (1) Let \( \kappa \) be a pre-Kupisch function on an interval \( I \). Then \((I, \kappa)\)-pwf is a connected additive category if and only if \( S(\kappa) = \emptyset \).

(2) Let \( \kappa \) be a Kupisch function. Then \((\mathbb{S}^1, \kappa)\)-pwf is a connected additive category if and only if \( |S(\kappa) \cap [0, 1]| \leq 1 \).

When (pre-)Kupisch functions \( \kappa \) and \( \lambda \) have no separation points, any equivalence \((I, \kappa)\)-pwf \( \cong (J, \lambda)\)-pwf (or \((\mathbb{S}^1, \kappa)\)-pwf \( \cong (\mathbb{S}^1, \lambda)\)-pwf) is induced by a push-forward \( \lambda = f_*\kappa \) via some orientation preserving homeomorphism \( f \).

**Theorem 1.2** (1) Let \( F : (I, \kappa)\)-pwf \( \to (J, \lambda)\)-pwf be an equivalence of categories where \( \kappa \) and \( \lambda \) are pre-Kupisch functions on intervals \( I \) and \( J \) respectively, such that \( S(\kappa) = S(\lambda) = \emptyset \). Then \( F \) induces an orientation preserving homeomorphism \( f : I \to J \) such that \( \lambda = f_*\kappa \).

(2) Let \( F : (\mathbb{S}^1, \kappa)\)-pwf \( \to (\mathbb{S}^1, \lambda)\)-pwf be an equivalence of categories, where \( \kappa \) and \( \lambda \) are Kupisch functions such that \( S(\kappa) = S(\lambda) = \emptyset \). Then \( F \) induces an orientation preserving homeomorphism \( f : \mathbb{S}^1 \to \mathbb{S}^1 \) such that \( \lambda = f_*\kappa \).
To reveal the relation with representations of discrete Nakayama algebras, we associate to each basic Nakayama algebra $A$ having Kupisch series $(l_0, l_1, \cdots, l_{n-1})$ an associated Kupisch function $\kappa_A$ (Definition 6.1). Then we show the following result.

**Theorem 1.3** Let $A$ be a basic connected Nakayama algebra with a Kupisch series $(l_0, l_1, \cdots, l_{n-1})$ and $\kappa_A$ its associated Kupisch function. Then there is an exact embedding $F : A\text{-mod} \to (S^1, \kappa_A)\text{-pwf}$, which preserves projective objects. Moreover, if $A$ is a linear Nakayama algebra, then there is an exact embedding $L : A\text{-mod} \to (\mathbb{R}, \kappa_A)\text{-pwf}$, which preserves projective objects.

### 1.4 Further Investigations

It is also worth mentioning a connection of our results with dynamical systems. Pre-Kupisch (respectively Kupisch) functions naturally give rise to self-maps on $\mathbb{R}$ (respectively $S^1$) and hence defines a dynamical system on $\mathbb{R}$ (respectively $S^1$). The construction of push-forward of (pre-)Kupisch functions corresponds to the notion of topological conjugacy in dynamic systems. Therefore Theorem 1.2 asserts that the classification of categories $(S^1, \kappa)$-pwf and $(\mathbb{R}, \kappa)$-pwf up to categorical equivalence amounts to classifying their corresponding dynamical system up to positive topologically conjugacy.

### 2 $\mathbb{R}$- and $S^1$-representations

In this section we recall representations (persistence modules) over $\mathbb{R}$ and over $S^1$. We describe interval modules for $\mathbb{R}$, string modules for $S^1$, and the relationship between them via covering theory and orbit categories.

#### 2.1 $\mathbb{R}$-representations

Denote by $\mathbb{R}$ the category of real numbers, where objects are real numbers and there is a unique generating morphism $g_{xy} : x \to y$ if $x \leq y$ and $g_{xx} = id_x$. Composition is given by $g_{yz} \circ g_{xy} = g_{xz}$. It follows that

$$\text{Hom}_\mathbb{R}(x, y) = \begin{cases} \{g_{xy}\} & x \leq y \\ \emptyset & x > y \end{cases}. $$

An $\mathbb{R}$-representation over a field $k$ is a covariant functor $M : \mathbb{R} \to k\text{-Vec}$. A morphism of $\mathbb{R}$-representations is a natural transformation $f : M \to N$. That is, a morphism is a collection of $k$-linear maps $f(x)$ for each $x \in \mathbb{R}$ such that the following diagram commutes:

$$
\begin{array}{ccc}
M(x) & \xrightarrow{M(g_{xy})} & M(y) \\
\downarrow f(x) & & \downarrow f(y) \\
N(x) & \xrightarrow{N(g_{xy})} & N(y).
\end{array}
$$
Let \((a, b] \subseteq \mathbb{R}\) be an interval. The \(\mathbb{R}\)-representation \(M_{(a, b]}\) is defined as:

\[
M_{(a, b]}(t) = \begin{cases} 
 k & a < t \leq b \\
 0 & \text{otherwise}
\end{cases}
\]

In general, if \(U\) is any interval, we define the interval module \(M_U\) similarly.

**Example 2.1** There is a nonzero homomorphism \(f : M_{[1, 3]} \to M_{[0, 2]}\) with

\[
f(s) = \begin{cases} 
 0 = M_{[1, 3]}(s) \xrightarrow{0} M_{[0, 2]}(s) = k & 0 \leq s < 1 \\
 k = M_{[1, 3]}(s) \xrightarrow{\psi} M_{[0, 2]}(s) = k & 1 \leq s \leq 2 \\
 k = M_{[1, 3]}(s) \xrightarrow{0} M_{[0, 2]}(s) = 0 & 2 < s \leq 3
\end{cases}
\]

We see that \(\text{Im } f = M_{[1, 2]}\), \(\text{Ker } f = M_{(2, 3]}\) and \(\text{Coker } f = M_{[0, 1)}\).

We call a module \(M\) pointwise finite-dimensional (pwf) if \(\dim M(x) < \infty\), for each \(x \in \mathbb{R}\). Denote by \(\mathbb{R}\)-pwf the category of pwf \(\mathbb{R}\)-representations. Indecomposable objects in \(\mathbb{R}\)-pwf are exactly the interval modules.

Recall that an abelian category is **Krull–Remak–Schmidt** if every object decomposes uniquely up to isomorphism into a (possibly infinite) direct sum of indecomposables and each indecomposable has a local endomorphism ring.

**Theorem 2.2** \([4, 5]\) Any pointwise finite-dimensional \(\mathbb{R}\)-representation decomposes uniquely up to isomorphism into a direct sum of interval modules. In particular, \(\mathbb{R}\)-pwf is Krull–Remak–Schmidt.

For convenience, we introduce the following definition and lemma.

**Definition 2.3** Define the left intersection of intervals \(U\) and \(V\) as:

\[
U \cap_L V = \begin{cases} 
 U \cap V & \text{if } (\forall x \in V \setminus U, \forall y \in U, \ y < x) \ \text{and} \ (\forall x \in V, \forall y \in U \setminus V, \ y < x) \\
\emptyset & \text{otherwise}
\end{cases}
\]

**Lemma 2.4** Let \(M_U\) and \(M_V\) be interval modules in \(\mathbb{R}\)-pwf. Then

\[
\text{Hom}_{\mathbb{R}}(M_V, M_U) = \begin{cases} 
 k & U \cap_L V \neq \emptyset \\
 0 & U \cap_L V = \emptyset
\end{cases}
\]

### 2.2 \(S^1\)-representations

In this section we follow \([9]\) in our definition of \(S^1\).

Denote by \(S^1\) the category of points \(e^{2i\pi \theta}\) for \(\theta \in \mathbb{R}\). There is a unique generating morphism \(g_{xy} : x \to y\) for all pairs \(x, y \in S^1\) where \(g_{xx} = \text{id}_x\). Additionally, we have a unique map \(\omega_x : x \to x\) which captures maps going around \(S^1\) exactly once from \(x\) to \(x\). Consider points \(x = e^{2i\pi \theta}, y = e^{2i\pi \phi},\) and \(z = e^{2i\pi \psi},\) where \(\theta \leq \phi \leq \psi, \phi - \theta < 1,\) and \(\psi - \phi < 1\). Composition \(g_{yz}g_{xy}\) is defined as

\[
g_{yz}g_{xy} = \begin{cases} 
 g_{sz} & 0 \leq \psi - \theta < 1 \\
 g_{sz} \circ \omega_x = \omega_z \circ g_{sz} & 1 \leq \psi - \theta < 2
\end{cases}
\]
Define $\omega_0^x := id_x = g_{xx}$. In particular, for any $y \neq x$ in $S^1$, we have $\omega_x = g_{yx} \circ g_{xy}$. It follows that

$$\text{Hom}_{S^1}(x, y) = \left\{ g_{xy} \circ \omega_0^n = \omega_0^n \circ g_{xy} : n \in \mathbb{N} \right\}.$$ 

An $S^1$-representation over a field $k$ is a covariant functor $M : S^1 \to k\text{-Vec}$. An $S^1$-representation homomorphism is a natural transformation $f : M \to N$.

In fact, perspectives from covering theory (see [1, 3, 5, 6]) provide a convenient tool to understand the $S^1$-representations. Since the abelian group $\mathbb{Z}$ can act freely on $\mathbb{R}$ by $n \cdot x = x + n$. The category $S^1$ can be considered as the orbit category $\mathbb{R}/\mathbb{Z}$ via the Galois covering

$$p : \mathbb{R} \to S^1 \cong \mathbb{R}/\mathbb{Z}$$

$$t \mapsto t \mod 1.$$

Denote by $k\text{-Mod}$ and $S^1\text{-Mod}$ the category of all representations on $k$ and $S^1$, respectively. From covering theory, the Galois covering $p$ gives rise to two important functors between these two categories: (1) the pull-up functor $p^* : S^1\text{-Mod} \to k\text{-Mod}$ sending $M \mapsto M \circ p$ and (2) the push-down functor $p_* : k\text{-Mod} \to S^1\text{-Mod}$ which satisfies $p_*M(x) = \bigoplus_{i \in p^{-1}(x)} M(t)$. It is well-known that $(p_*, p^*)$ forms an adjoint pair and $p_*$ preserves finitely generated objects (see, for example, [2, Theorem 4.3]). However, we also remind the readers that $p_*$ does not preserve pointwise finite-dimensional modules. (For example, consider $p_* (\bigoplus_{j \in \mathbb{Z}} M_{[j, i])} \). In the following, we give an explicit description of indecomposable $S^1$-representations using the push-down functor.

**Remark 2.5** We remind the reader that classical covering theory is defined on $k$-linear categories. We may take the perspective of using the $k$-linearization of $\mathbb{R}$, denoted by $k\mathbb{R}$. Then the category of $k$-linear functors from $k\mathbb{R}$ to $k$-vector spaces is equivalent to $k\text{-Mod}$. We may use covering theory of $k\mathbb{R}$ to obtain the $k$-linearization of $S^1$, denoted by $kS^1$. The category of $k$-linear functors from $kS^1$ to $k$-vector spaces is equivalent to $S^1\text{-Mod}$.

Define an $S^1$ string module $\overline{M}_U$ as the push-down of an interval $\mathbb{R}$-representation $M_U$ for some bounded interval $U$. Explicitly, for each $x \in S^1$, denote by $p^{-1}_U(x) := p^{-1}(x) \cap U$. Then $\overline{M}_U(x) := \bigoplus_{b \in p^{-1}_U(x)} M_U(b)$.

Denote the elements of $p^{-1}_U(x)$ by $\{b_{i,x}\}_{i=1}^{p^{-1}_U(x)}$, such that $b_{i,x} < b_{i+1,x}$ for all $i$. For each generating morphism $g_{xy} : x \to y$, where $\overline{M}_U(x) \neq 0$, the linear maps $\overline{M}_U(g_{xy})$ are defined on the basis $\{b_{i,x}\}_{i=1}^{p^{-1}_U(x)}$.

$$\overline{M}_U(g_{xy})(b_{i,x}) = \begin{cases} b_{j,y} & \exists b_{j,y} \in p^{-1}_U(y) \text{ such that } 0 \leq b_{j,y} - b_{i,x} < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Due to $\overline{M}_U(\omega_x) = \overline{M}_U(g_{yx} g_{xy})$,

$$\overline{M}_U(\omega_x)(b_{i,x}) = \begin{cases} b_{i+1,x} & 1 \leq i < |p^{-1}_U(x)| \\ 0 & i = |p^{-1}_U(x)|. \end{cases}$$

If $f : M_U \to M_V$ is a morphism of $\mathbb{R}$-representations, where $U$ and $V$ are bounded intervals, then push-down functor induces a morphism $\overline{f} : \overline{M}_U \to \overline{M}_V$ as $\overline{f}(x) = \oplus f(b_{i,x}) : \oplus M_U(b_{i,x}) \to \oplus M_V(b'_{j,x})$, where $b_{i,x} \in p^{-1}_U(x)$ and $b'_{j,x} \in p^{-1}_V(x)$.
An $S^1$ band module (or Jordan cell) $B$ is an indecomposable $S^1$-representation such that $B(g_{x,y})$ is an isomorphism for all $x, y \in S^1$. For example, if $k$ is algebraically closed, then $B(\omega_x)$ can be written as a Jordan block.

**Theorem 2.6 [9]** Any pointwise finite-dimensional $S^1$-representation decomposes into a direct sum of string modules and finitely-many band modules. In particular, $S^1$-pwf is Krull–Remak–Schmidt.

As we are constructing a continuous analogue of Nakayama representations, we study $S^1$-representations compatible with Kupisch functions. Thus, we are only interested in string modules.

Let $U$ be an interval of $\mathbb{R}$. For any $i \in \mathbb{Z}$, denote by $U + i$ the interval $\{x + i \mid x \in U\}$. The $\mathbb{R}$-representations $M_{U+i}$ are called translated modules of $M_U$. Denote by $\widetilde{X} = X \circ p$ the pull-up of the $S^1$-representation $X$. One can check that $\widetilde{M}_U = \overline{M}_U \circ p = \bigoplus_{i \in \mathbb{Z}} M_{U+i}$. We have the following lemma which should be compared with Lemma 2.4.

**Lemma 2.7** Let $\overline{M}_U$ and $\overline{M}_V$ be $S^1$ string modules. Then

$$\text{Hom}_{S^1}(\overline{M}_U, \overline{M}_V) \cong \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathbb{R}}(M_U, M_{V+i}).$$

**Proof** Since the push-down is left adjoint to the pull-up, we have

$$\text{Hom}_{S^1}(\overline{M}_U, \overline{M}_V) \cong \text{Hom}_{\mathbb{R}}(M_U, \widetilde{M}_V)$$

$$= \text{Hom}_{\mathbb{R}}(M_U, \bigoplus_{i \in \mathbb{Z}} M_{V+i}) \cong \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathbb{R}}(M_U, M_{V+i}).$$

**Corollary 2.8** (to Lemma 2.7) Let $\overline{M}_U$ and $\overline{M}_V$ be $S^1$ string modules. Then

$$\dim_k \text{Hom}_{S^1}(\overline{M}_U, \overline{M}_V) = |\{i \mid U \cap (V + i) \neq \emptyset\}|.$$

**Proof** Since there are only finitely many $i$ such that $\text{Hom}_{\mathbb{R}}(M_U, M_{V+i}) \neq 0$ for bounded intervals $U$ and $V$, this is always a finite direct sum. Hence, the statement follows from Lemmas 2.4 and 2.7.

Recall that a module $M$ is called a brick if $\text{End}(M)$ is a division ring. In particular, bricks are indecomposable.

**Corollary 2.9** (to Corollary 2.8) An $S^1$ string module $\overline{M}_U$ is a brick if and only if $|x - y| < 1$ for $\forall x, y \in U$.

The push-down functor does not preserve pwf representations, so we instead use the concept of orbit categories and restrict the push-down functor to a subcategory of $\mathbb{R}$-pwf.

**Definition 2.10** Let $\mathbb{R}$-bpwf be the full subcategory of $\mathbb{R}$-pwf, where for each object $M$ in $\mathbb{R}$-bpwf, there exists $a \leq b \in \mathbb{R}$ such that $M(x) = 0$ if $x \leq a$ or $x \geq b$. (Here, the extra ‘b’ stands for “bounded.”)

Let $S^1$-spwf be the full subcategory of $S^1$-pwf whose objects do not have a band summand. (Here, the extra ‘s’ stands for “string.”)
It is a straightforward check that $R$-bpwf is a Serre subcategory of $R$-pwf and is thus abelian. Moreover, $R$-bpwf is Krull–Remak–Schmidt. Since we have the cyclic orientation on $S^1$, the category $S^1$-spwf is a Serre subcategory of $S^1$-pwf; in particular it is also abelian and Krull–Remak–Schmidt.

By Theorem 2.6, we see that every pwf $S^1$-representation that does not have a band summand is isomorphic to the push-down of some object $M$ in $R$-bpwf. By restricting the push-down functor to $R$-bpwf, we can view $S^1$-spwf as the orbit category of $R$-bpwf where the autoequivalence on $R$-bpwf is induced by the action of $\mathbb{Z}$.

3 (Pre-)Kupisch Functions and Compatible Representations

In this section we define a continuous analogue to Kupisch series called (pre-)Kupisch functions. Compatible modules form a subcategory of $R$-pwf (or $S^1$-pwf) that behaves like the category of finite-dimensional representations of a Nakayama algebra.

Throughout this paper, functors between $R$-pwf, $S^1$-pwf, and their subcategories are assumed to be $k$-linear and additive.

3.1 Pre-Kupisch functions and compatible $R$-representations

Definition 3.1 Let $I \subseteq \mathbb{R}$ be an interval.

(1) A pre-Kupisch function on $I$ is $\kappa : I \to \mathbb{R}^>0$ such that

- $K(t) := \kappa(t) + t$ is increasing on $I$ and
- For all $t \in I$, we have $[t, K(t)] = [t, \kappa(t) + t] \subseteq I$.

(2) We call an $R$-representation compatible with $\kappa$ if each of its indecomposable summands $M_U$ satisfies $U \subseteq [t, K(t)] \subseteq I$, for some $t \in I$. Let $(I, \kappa)$-pwf be the full subcategory of $R$-pwf consisting of precisely the modules compatible with $\kappa$.

Remark 3.2 We remark on some immediate facts about pre-Kupisch functions.

(1) For any pre-Kupisch function $\kappa$ on an interval $I$, the category $(I, \kappa)$-pwf is a Serre subcategory of $R$-pwf and is thus abelian and Krull–Remak–Schmidt.

(2) For a pre-Kupisch function $\kappa : \mathbb{R} \to \mathbb{R}^>0$, the restriction $\kappa|_I : I \to \mathbb{R}^>0$ on an interval $I$ is not always a pre-Kupisch function. In particular, there are no pre-Kupisch functions on intervals of the form $[a, b]$, $(a, b)$ or $(-\infty, b]$. This is because $[b, b + \kappa(b)] \not\subseteq I$.

(3) For any pre-Kupisch function $\kappa : \mathbb{R} \to \mathbb{R}^>0$, the restriction $\kappa|_{\mathbb{R}^\geq0} : \mathbb{R}^\geq0 \to \mathbb{R}^>0$ is a pre-Kupisch function on $\mathbb{R}^\geq0$.

(4) A pre-Kupisch function $\kappa$ determines the indecomposable projective modules in $(I, \kappa)$-pwf, which are the interval modules $M_{[t, K(t)]]}$ and $M_{(t, K(t))}$, for all $t \in I$.

We make use of the following fact.

Proposition 3.3 Let $I, J$ be intervals. An increasing bijection $f : I \to J$ is continuous and hence a homeomorphism.

Increasing bijections $f : I \to J$ are also called orientation-preserving homeomorphisms. Denote by $\text{Homeo}_+(I, J)$ the set of orientation-preserving homeomorphisms from $I$ to $J$ and $\text{Homeo}_+(I)$ the set of orientation-preserving homeomorphisms from $I$ to $I$. The following result shows that an orientation-preserving homeomorphism preserves subintervals and their left intersections.
Lemma 3.4 Let $I, J$ be intervals in $\mathbb{R}$, $f \in \text{Homeo}_{\text{+}}(I, J)$, and $U, V$ intervals in $I$. Then $f(U \cap_L V) = f(U) \cap_L f(V)$.

Proof First, we claim that $U \cap_L V = \emptyset$ if and only if $f(U) \cap_L f(V) = \emptyset$. Indeed, if $U \cap_L V \neq \emptyset$, then $\forall x \in V \setminus U$, $\forall y \in U, y < x$ and $\forall x \in V, \forall y \in U \setminus V, y < x$. Since $f$ is an increasing bijection, $\forall x \in f(V) \setminus f(U), \forall y \in f(U)$, it follows that $f^{-1}(x) \in V \setminus U$ and $f^{-1}(y) \in U$, hence $f^{-1}(y) < f^{-1}(x)$, which implies $y < x$. Similarly, $\forall x \in f(V), \forall y \in f(U) \setminus f(V)$, it follows that $y < x$. So $f(U) \cap_L f(V) \neq \emptyset$. Then the converse direction of the claim follows directly from the fact that $f^{-1} : J \rightarrow I$ is again an increasing homeomorphism.

To prove the lemma, notice that if $U \cap_L V = \emptyset$, then $f(U \cap_L V) = \emptyset = f(U) \cap_L f(V)$. If $U \cap_L V \neq \emptyset$, then $U \cap_L V = U \cap V$. So $f(U \cap_L V) = f(U \cap V) = f(U) \cap f(V) = f(U) \cap_L f(V)$. $\square$

Given a pre-Kupisch function $\kappa$ on an interval $I$ and an increasing bijection $I \rightarrow J$, we now define the push-forward $f_\ast \kappa$ and show that this is a pre-Kupisch function on $J$.

Definition 3.5 Let $\kappa$ be a pre-Kupisch function on interval $I$ and let $f : I \rightarrow J$ be an increasing bijection. We define the push-forward by $f_\ast \kappa(t) := f(\kappa \circ f^{-1}(t) + f^{-1}(t)) - t$.

Notice that by definition $f$ sends any interval $[t, t + \kappa(t)]$ to the interval $[f(t), f(t) + f_\ast \kappa(f(t))]$.

Lemma 3.6 If $\kappa$ is a pre-Kupisch function on an interval $I$ and $f \in \text{Homeo}_{\text{+}}(I, J)$, then $f_\ast \kappa$ is a pre-Kupisch function on $J$. Furthermore, $f$ defines a bijection from the set of pre-Kupisch functions on $I$ to the set of pre-Kupisch functions on $J$.

Theorem 3.7 Let $I, J$ be intervals of $\mathbb{R}$. If $\kappa$ is a pre-Kupisch function on $I$ and $f \in \text{Homeo}_{\text{+}}(I, J)$ then $(I, \kappa)$-pwf $\cong (J, f_\ast \kappa)$-pwf.

Proof An interval $U \subseteq [t, K(t)] \subseteq I$ if and only if $f(U) \subseteq [f(t), f(\kappa(t) + t)] = [f(t), f(t) + f_\ast \kappa(f(t))] \subseteq J$. Hence we construct a functor $F : (I, \kappa)$-pwf $\rightarrow (J, f_\ast \kappa)$-pwf which sends interval modules to interval modules: $F(M_U) = M_{f(U)}$. A nonzero morphism $\varphi : M_U \rightarrow M_V$ is determined by a scalar $c$ such that $\varphi(x) = c \cdot 1_{k} : M_U(x) \rightarrow M_V(x)$, for $\forall x \in V \cap_L U$. Thus, we define $F(\varphi) : M_{f(U)} \rightarrow M_{f(V)}$ as a morphism such that $F(\varphi)(x) = c \cdot 1_{k}$ for all $x \in f(V) \cap_L f(U)$. Notice that $F$ is a dense functor.

By Lemmas 2.4 and 3.4, we have an induced bijection $\text{Hom}(I, \kappa)$-pwf $(M_U, M_V) \cong \text{Hom}(J, f_\ast \kappa)$-pwf $(M_{f(U)}, M_{f(V)})$ $\varphi \mapsto F(\varphi)$ as $k$-vector spaces and so $F : \text{Hom}(M_U, M_V) \rightarrow \text{Hom}(M_{f(U)}, M_{f(V)}), \varphi \mapsto F(\varphi)$, is an isomorphism. Since all modules are direct sums of interval modules, extending additively, $F : (I, \kappa)$-pwf $\rightarrow (J, f_\ast \kappa)$-pwf is an equivalence.

Corollary 3.8 (to Theorem 3.7) Let $I$ be an interval with a pre-Kupisch function $\kappa$. Then

$$(I, \kappa)\text{-pwf} \cong \begin{cases} (\mathbb{R}_{\geq 0}, \lambda)\text{-pwf}, & I = [a, b) or [a, +\infty) \\ (\mathbb{R}, \mu)\text{-pwf}, & I \text{ is open} \end{cases}$$

for some pre-Kupisch function $\lambda$ on $\mathbb{R}_{\geq 0}$ or $\mu$ on $\mathbb{R}$.

We warn the reader that categories of the forms $(\mathbb{R}, \mu)$-pwf and $(\mathbb{R}_{\geq 0}, \lambda)$-pwf are possibly equivalent, when they are not connected (Example 5.14).
3.2 Kupisch functions and compatible \( S^1 \)-representations

**Definition 3.9** (1) A pre-Kupisch function \( \kappa : \mathbb{R} \to \mathbb{R}^{>0} \) on \( \mathbb{R} \) is called a **Kupisch function** if \( \kappa(t + 1) = \kappa(t) \) for all \( t \in \mathbb{R} \).

(2) We call a \( \text{pwf} \) \( S^1 \)-representation **compatible with a Kupisch function** \( \kappa \) if all of its indecomposable summands are strings and each of its indecomposable summands \( M_U \) satisfies \( U \subseteq [t, K(t)] = [t, t + \kappa(t)] \) for some \( t \in \mathbb{R} \). Let \( (S^1, \kappa) \)-pwf be the full subcategory of \( S^1 \)-pwf consisting of representations compatible with \( \kappa \).

**Remark 3.10** We remark on two immediate consequences of Definition 3.9.

(1) For any Kupisch function \( \kappa \), the category \( (S^1, \kappa) \)-pwf is a Serre subcategory of \( S^1 \)-pwf and is thus abelian and Krull–Remak–Schmidt. Furthermore, \( (S^1, \kappa) \)-pwf is a Serre subcategory of \( S^1 \)-spwf.

(2) For any periodic pre-Kupisch function \( \kappa \) with periodicity \( r > 0 \) there is a Kupisch function \( \lambda \) such that \((\mathbb{R}, \kappa)\)-pwf \( \cong \) \((\mathbb{R}, \lambda)\)-pwf. In fact, one can choose \( \lambda = f_\ast \kappa \) where \( f \in \text{Homeo}_+(\mathbb{R}) \) is given by \( f(t) = \frac{t}{r} \).

One should consider the Kupisch function \( \kappa \) as the lift of a function \( S^1 \to \mathbb{R}^{>0} \), which assigns the length of a projective string module at each point on the circle.

Given a Kupisch function \( \kappa \), we construct new Kupisch functions using orientation-preserving homeomorphisms \( f : S^1 \to S^1 \).

Recall that a homeomorphism \( f : S^1 \to S^1 \) is called **orientation-preserving** if its lift \( \tilde{f} \) through the universal covering \( p : \mathbb{R} \to S^1 \), \( p(t) = e^{2\pi it} \) is strictly increasing. An orientation-preserving homeomorphism has degree 1, namely \( \tilde{f}(t + 1) = \tilde{f}(t) = 1 \) for all \( t \) ((10)). Denote by \( \text{Homeo}_+(S^1) \) the set of all the orientation-preserving homeomorphisms on \( S^1 \). Notice that an orientation-preserving map need not be a simple rotation, for example consider \( f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \) given by \( f(x) = x + b \sin(2\pi x) \) for any \( 0 < b < \frac{1}{2\pi} \).

**Definition 3.11** Let \( f \in \text{Homeo}_+(S^1) \) and choose a lift \( \tilde{f} \) of \( f \). For any Kupisch function \( \kappa \) we define the push-forward by

\[
(f_\ast \kappa)(t) := (\tilde{f}_\ast \kappa)(t) = \tilde{f}(\kappa \circ \tilde{f}^{-1}(t) + \tilde{f}^{-1}(t)) - t.
\]

Straightforward computations show the right side is independent of the choice of \( \tilde{f} \) and it is again a Kupisch function. We have the following Lemma that follows from Lemma 3.6.

**Lemma 3.12** If \( \kappa \) is a Kupisch function and \( f \in \text{Homeo}_+(S^1) \) then \( f_\ast \kappa \) is a Kupisch function. Furthermore, \( f \) defines a bijection on the set of Kupisch functions.

**Construction 3.13** Let \( \kappa \) and \( \lambda \) be Kupisch functions. Consider the subcategory \( (\mathbb{R}, \kappa) \)-bpwf and \( (\mathbb{R}, \lambda) \)-bpwf of \( (\mathbb{R}, \kappa) \)-pwf and \( (\mathbb{R}, \lambda) \)-pwf, respectively. Recall from Remark 3.2(1) that these are abelian, Krull–Remak–Schmidt subcategories.

Recall that given an \( \mathbb{R} \)-module \( M \), the **translated module** \( M^{(i)} \) (under the \( \mathbb{Z} \) action) is an \( \mathbb{R} \)-representation such that \( M^{(i)}(x) = M(x - i) \). Hence for an interval module \( M_U \), its translated module \( M_U^{(i)} = M_{U+i} \). Since a Kupisch function \( \kappa \) is 1-periodic, the translation \( (-)^{(i)} : (\mathbb{R}, \kappa) \)-bpwf \( \to \) \( (\mathbb{R}, \kappa) \)-bpwf is an equivalence with inverse \( (-)^{(-i)} \). Let \( F : (\mathbb{R}, \kappa) \)-bpwf \( \to \) \( (\mathbb{R}, \lambda) \)-bpwf be an additive covariant functor which preserves indecomposable objects and satisfies \( F \circ (-)^{(1)} = (-)^{(1)} \circ F \).

We now define a functor \( \overline{F} : (S^1, \kappa) \)-pwf \( \to \) \( (S^1, \lambda) \)-pwf. Recall from Remark 3.10(1) that these are abelian, Krull–Remak–Schmidt subcategories. On indecomposables, define \( \overline{F}(M_U) := F(M_U) \). This is well-defined since \( F(M_U^{(1)}) = (FM_U)^{(1)} \).

\( \square \) Springer
Given a homomorphism \( f \in \text{Hom}_{(S^1, \kappa)}(\overline{M}_U, \overline{M}_V) \), we identify \( f \) as \((f_i : M_U \to M_{V+i})_{i \in \mathbb{Z}} \) due to Lemma 2.7. Then \((F(f_i)) : F(M_U) \to F(M_{V+i}) = (FM_V)_{i \in \mathbb{Z}} \) defines a homomorphism \( \overline{F}(f) \in \text{Hom}_{(S^1, \lambda)}(FM_U, FM_V) \).

To show \( \overline{F} \) is a well-defined functor, it remains to show that \( \overline{F} \) respects compositions. Let \( f = (f_i : \overline{M}_U \to \overline{M}_V) \) and \( g = (g_j : \overline{M}_V \to \overline{M}_W) \) be morphisms in \((S^1, \kappa)\)-pwf. Then the composition \( g \circ f = (h_k : \overline{M}_U \to \overline{M}_W) \) is given by \( h_k = \sum_{d \in \mathbb{Z}} g_{k-d}^{(d)} f_d : M_U \to M_{W+k} \), where \( g_{d}^{(d)} : M_V^d \to M_W^d \) are the translated morphisms.

Now \( \overline{F}(g \circ f) = (F(h_k))_{k \in \mathbb{Z}} \), where the \( k \)-th component \( F(h_k) = F(\sum_{d \in \mathbb{Z}} g_{k-d}^{(d)} f_d) = \sum_{d \in \mathbb{Z}} F(g_{k-d}^{(d)}) \circ F(f_d) = \sum_{d \in \mathbb{Z}} F(g_{k-d}) \circ F(f_d) \). On the other hand, since \( \overline{F}(f) = (F(f_i))_{i \in \mathbb{Z}} \) and \( \overline{F}(g) = (F(g_j))_{j \in \mathbb{Z}} \), the \( k \)-th component of \( \overline{F}(g) \circ \overline{F}(f) \) is given by \( \sum_{d \in \mathbb{Z}} F(g_{k-d}) \circ F(f_d) \). Hence \( \overline{F}(g \circ f) = \overline{F}(g) \circ \overline{F}(f) \).

**Lemma 3.14** Let \( \kappa \) and \( \lambda \) be Kupisch functions. Assume \( F : (\mathbb{R}, \kappa)\)-pwf \to (\mathbb{R}, \lambda)\)-pwf preserves indecomposable objects and satisfies \( F \circ (-)^{(1)} = (-)^{(1)} \circ F \). Let \( \overline{F} \) be as in Construction 3.13. Then the following diagram commutes:

\[
\begin{array}{ccc}
(\mathbb{R}, \kappa)\text{-pwf} & \xrightarrow{F} & (\mathbb{R}, \lambda)\text{-pwf} \\
p_* \downarrow & & \downarrow p_* \\
(\mathbb{S}^1, \kappa)\text{-pwf} & \xrightarrow{\overline{F}} & (\mathbb{S}^1, \lambda)\text{-pwf},
\end{array}
\]

where \( p_* \) denotes the push-down. Furthermore, if \( F \) is an equivalence then so is \( \overline{F} \).

**Proof** First, by Construction 3.13, for interval modules \( M_U \in (\mathbb{R}, \kappa)\)-pwf, we have \( \overline{F} p_* (M_U) = \overline{F}(M_U) = F(M_U) = p_* F(M_U) \). Second, let \( f : M_U \to M_V \) be a morphism of indecomposables in \((\mathbb{R}, \kappa)\)-pwf. Then \( p_* f = (f_i)_{i \in \mathbb{Z}} : \overline{M}_U \to \overline{M}_V \), where \( f_0 = f \) and \( f_i = 0 \) for \( i \neq 0 \). Hence by Construction 3.13, \( \overline{F}(p_* f) = (F(f_i)) \), where \( F(f_0) = F(f) \) and \( F(f_i) = 0 \) for \( i \neq 0 \), which coincides with \( p_* F(f) \). So the diagram commutes.

Now assume \( F \) is an equivalence. Then

\[
F : \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{(\mathbb{R}, \kappa)\text{-pwf}}(M_U, M_{V+i}) \to \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{(\mathbb{R}, \lambda)\text{-pwf}}(FM_U, FM_{V+i})
\]

is an isomorphism. So,

\[
\overline{F} : \text{Hom}_{(\mathbb{S}^1, \kappa)}(\overline{M}_U, \overline{M}_V) \to \text{Hom}_{(\mathbb{S}^1, \lambda)}(\overline{F}(M_U), \overline{F}(M_V))
\]

is also an isomorphism.

**Theorem 3.15** Let \( \kappa \) be a Kupisch function and \( f \in \text{Homeo}_+(S^1) \). Then \((S^1, \kappa)\text{-pwf} \cong (S^1, f_\kappa)\text{-pwf}\).

**Proof** Since \( \tilde{f} : \mathbb{R} \to \mathbb{R} \) is an orientation preserving homeomorphism, it induces an equivalence \( \tilde{F} : (\mathbb{R}, \kappa)\text{-pwf} \to (\mathbb{R}, f_\kappa)\text{-pwf} \), according to Theorem 3.7. It is clear \( \tilde{F} \) preserves indecomposables and \( \tilde{F} \circ (-)^{(1)} = (-)^{(1)} \circ \tilde{F} \) because \( \tilde{f} \) has degree 1. Hence, by Lemma 3.14, there is an induced equivalence \( F : (S^1, \kappa)\text{-pwf} \to (S^1, f_\kappa)\text{-pwf} \).

**Example 3.16** (1) For any constant \( c \), the shifted Kupisch function \( \kappa(t - c) \) is a pushforward \( f_\kappa(t) \) via the circle map \( f(t) = te^{2\pi ci} \). Therefore, \((S^1, \kappa(t - c))\text{-pwf} \cong (S^1, \kappa)\text{-pwf} \).
(2) If \( \kappa \) and \( \lambda \) are constant Kupisch functions with distinct values, then there is no orientation preserving homeomorphism \( f \) on \( \mathbb{R} \) (or \( S^1 \)) such that \( f_\ast \kappa = \lambda \). It follows from Theorem 5.5 that, in this case, \((S^1, \kappa)\)-pwf \( \not \cong \) \((S^1, \lambda)\)-pwf.

(3) If \( \kappa = \frac{1}{8} \sin(2\pi t) + \frac{1}{2}, \lambda = \frac{1}{2}, \) then \((S^1, \kappa)\)-pwf \( \not \cong \) \((S^1, \lambda)\)-pwf, since \((S^1, \kappa)\)-pwf contains non-period modules of infinite projective dimension whereas \((S^1, \lambda)\)-pwf does not.

4 Separation Points and Connectedness

In this section, we discuss the connectedness of \((\mathbb{R}, \kappa)\)-pwf and of \((S^1, \kappa)\)-pwf. We prove that the orthogonal components (Definition 4.6) of these categories are determined by the existence of certain discontinuities of the (pre-)Kupisch functions, which we call separation points (Definition 4.2).

4.1 Separation Points

Assume a real valued function \( f(x) \) is discontinuous at \( x_0 \). Recall that \( x_0 \) is called a discontinuity of first kind if both \( \lim_{x \to x_0^-} f(x) \) and \( \lim_{x \to x_0^+} f(x) \) exist but are not equal.

We need the following well-known result for monotone functions from analysis, which is usually referred to as the Darboux–Froda Theorem [16].

**Theorem 4.1** A real valued monotone function defined on an interval has at most countably many discontinuities and each discontinuity must be of the first kind.

Let \( \kappa(t) \) be a pre-Kupisch function. Then \( K(t) = \kappa(t) + t \) is an increasing function and \( K(t) > t \). So \( K(t) \) and hence \( \kappa(t) \) has at most countably many discontinuities, all of the first kind. In particular, \( \lim_{t \to a^-} \kappa(t) \) and \( \lim_{t \to a^+} \kappa(t) \) exist for every real number \( a \).

We show that the category \((\mathbb{R}, \kappa)\)-pwf decomposes into two orthogonal components when there is a sequence \( \{K^n(t)\}_{n=1}^\infty \) such that \( \lim_{n \to \infty} K^n(t) < \infty \). We characterize such limit points as separation points:

**Definition 4.2** Let \( \kappa \) be a pre-Kupisch function on \( I \). An interior point \( c \) in \( I \setminus \partial I \) is a separation point if

1. \( \lim_{t \to c^-} \kappa(t) = 0; \)
2. \( K(t) < c, \) for \( t \in I, t < c. \)

Notice \( \kappa \) may have discontinuities that are not separation points. For example,

\[
\kappa(t) = \begin{cases} 
1 & t < 0 \\
2 & t \geq 0 
\end{cases}
\]

is a pre-Kupisch function with one discontinuity at 0 but 0 is not a separation point.

The following proposition follows from Definition 4.2.

**Proposition 4.3** Let \( \kappa \) be a pre-Kupisch function on \( I \).

1. If \( \lim_{t \to c^-} \kappa(t) = 0, \) then \( K(t) \leq c \) for \( \forall t < c. \) Furthermore, if the equality holds for some \( t_0 < c, \) then \( K(t) = c \) for \( \forall t \in [t_0, c). \)
(2) If \( c \in (t, K(t)) \), for some \( t \in I \), then \( \lim_{t \to c^-} \kappa(t) \neq 0 \).

(3) For any \( t \in I \), there is no separation point in \( (t, K(t)] \).

Denote by \( S(\kappa) \) the set of separation points. Since \( \kappa(t) \) is discontinuous at any separation point, the cardinality of \( S(\kappa) \) is at most countable.

If \( \kappa(t) \) is a pre-Kupisch function on \( I \), then \( K(t) : I \to I \) is an increasing function. The value of \( K(t) \) indicates the socle \( S_K(t) \) (i.e. the simple sub-representation at \( K(t) \)) of the indecomposable projective module at \( t \). For any point \( t \) and \( n \geq 0 \), there is an inequality \( K^{n+1}(t) = \kappa(K^n(t)) + K^n(t) > K^n(t) \). Hence \( \{K^n(t)\}_{n=0}^{\infty} \) is a strictly increasing sequence.

**Lemma 4.4** Let \( \kappa \) be a pre-Kupisch function on \( I \). If \( \lim_{n \to \infty} K^n(a) = c \in I \), then \( c \) is the minimum separation point which is greater than \( a \).

**Proof** First we show that \( c \) is a separation point. Since \( \lim_{n \to \infty} \kappa(t) \) always exists, it can be computed by \( \lim_{t \to c^-} \kappa(K^n(a)) = \lim_{n \to \infty} K^{n+1}(a) - K^n(a) = 0 \). On the other hand, for any \( t < c \), there is some \( n > 0 \) such that \( t < K^n(a) < c \). Therefore, \( K(t) < K^{n+1}(a) < \lim_{n \to \infty} K^{n+1}(a) = c \) because \( K^n(a) \) strictly increase. So \( K(t) < c \) for \( t < c \) and hence \( c \) is a separation point. ss Next, according to Proposition 4.3 (3), there is no separation point in intervals \( (K^n(a), K^{n+1}(a)) \). So there is no separation point in the interval \( (a, c) = \bigcup_{n=0}^{\infty} (K^n(a), K^{n+1}(a)) \).

**Proposition 4.5** If \( S(\kappa) = \emptyset \), then for all \( t \in I \),

\[
\lim_{n \to \infty} K^n(t) = \begin{cases} 
\sup(I), & \text{sup}(I) < +\infty \\
+\infty, & \text{otherwise}.
\end{cases}
\]

### 4.2 Orthogonal decomposition of \((\mathbb{R}, \kappa)\)

Following [12], we have the following definition of connected additive category.

**Definition 4.6** Let \( \mathcal{A} \) be an additive category and \((\mathcal{A}_i)_{i \in I}\) a family of full additive subcategories. We have an orthogonal decomposition

\[
\mathcal{A} \cong \bigoplus_{i \in I} \mathcal{A}_i
\]

of \( \mathcal{A} \) if \( \mathcal{A} = \sum_i \mathcal{A}_i \) (each object can be written as a direct sum \( \bigoplus_{i \in I} X_i \) with \( X_i \in \mathcal{A}_i \)) and \( \text{Hom}(X_i, X_j) = 0 \) for \( X_i \in \mathcal{A}_i \) and \( X_j \in \mathcal{A}_j \) and \( i \neq j \). We call \( \mathcal{A}_i \)’s the orthogonal components of \( \mathcal{A} \). The additive category \( \mathcal{A} \) is **connected** if it admits no proper decomposition \( \mathcal{A} \cong \mathcal{A}_1 \oplus \mathcal{A}_2 \).

**Remark 4.7** A finite dimensional algebra \( \Lambda \) is connected (i.e. 0 and 1 are the only central idempotents) if and only if the module category \( \Lambda \)-mod is a connected additive category.

We remind the readers that there is another notion of connected category commonly used in category theory: any two objects can be connected with a finite zigzag of (nonzero) morphisms. Notice that every additive category is a “connected category” in this sense, since there are always nonzero morphisms \( X \leftarrow X \oplus Y \rightarrow Y \) for any objects \( X \) and \( Y \). To
emphasize the difference, we always use the terminology “connected additive category” for our definition of connectedness as in Definition 4.6. We compare these two notions in Lemma 4.8 below.

Lemma 4.8 Let $A = \bigoplus_{i \in I} A_i$ be an orthogonal decomposition. For any sequence of indecomposable objects $X_1, X_2, \ldots, X_n$ in $A$, if there are nonzero morphisms either $X_i \to X_{i+1}$ or $X_{i+1} \to X_i$ for all $0 < i < n$, then $X_1$ and $X_n$ are in the same orthogonal component.

Proof Since $X_1$ and $X_2$ are indecomposable, there must be $i, j$ such that $X_1 \in A_i$ and $X_2 \in A_j$. However, either $\text{Hom}(X_1, X_2) \neq 0$ or $\text{Hom}(X_2, X_1) \neq 0$ so we must have $i = j$. For the same reason, $X_2$ and $X_3$ and hence all $X_i$’s are in the same orthogonal component.

In this section, we are going to discuss the orthogonal decomposition of $(\mathbb{R}, \kappa)$-pwf (or $\text{(S}^1, \kappa)$-pwf) according to the (pre-)Kupisch function $\kappa$.

Lemma 4.9 Let $\kappa$ be a pre-Kupisch function on an interval $I$, then $(I, \kappa)$-pwf is a connected additive category if and only if $S(\kappa) = \emptyset$.

Proof First, assume $S(\kappa) \neq \emptyset$. If $c \in I$ is a separation point, let $C^−$ be the full subcategory of $(I, \kappa)$-pwf consisting of representations with indecomposable summands $M_U$ where $U \subseteq I \cap (-\infty, c)$, and $C^+$ be the full subcategory of $(I, \kappa)$-pwf consisting representations with indecomposable summands $M_U$ where $U \subseteq I \cap [c, +\infty)$. It follows that $\text{Hom}(C^−, C^+) = 0 = \text{Hom}(C^+, C^−)$. At the same time, for any interval module $M_U \in (I, \kappa)$-pwf, $U \subseteq [t, K(t)] \subseteq I$. According to Proposition 4.3 (3), the interval $[t, t + K(t)] \subseteq I \cap (-\infty, c)$ or $I \cap [c, +\infty)$. So $M_U$ is in $C^−$ or in $C^+$. Hence $(I, \kappa)$-pwf $\cong C^− \oplus C^+$ is a disconnected additive category.

Now, assume $S(\kappa) = \emptyset$. We first show that there is a finite sequence of nonzero morphisms between any two projective indecomposables $M_{[a, K(a)]}$ and $M_{[b, K(b)]}$. Without loss of generality, assume $a < b$. According to Proposition 4.5, $\lim_{n \to \infty} K^n(a) = \sup(I)$ or $+\infty$. It follows that $b \in [K^n(a), K^{n+1}(a)]$ for some $n$. Hence there is a sequence of nonzero homomorphisms

$$M_{[b, K(b)]} \to M_{[K^n(a), K^{n+1}(a)]} \to M_{[K^{n-1}(a), K^n(a)]} \to \cdots \to M_{[a, K(a)]}.$$ 

Second, notice that, for each $p < q \in I$, there are nonzero morphisms between indecomposables in $(I, \kappa)$-pwf:

$$M_{[p, q]} \to M_{[p, q]} \quad \text{and} \quad M_{[p, q]} \to M_{[p, q]}.$$ 

and the nonzero projective cover: $M_{[p, K(p)]} \to M_{[p, q]}$. It follows that there is a zigzag sequence of nonzero morphisms between any two indecomposables in $(I, \kappa)$-pwf. According to Lemma 4.8, any two indecomposables in $(I, \kappa)$-pwf are in the same orthogonal component. Therefore, $(I, \kappa)$-pwf is a connected additive category.

Let $\kappa$ be a pre-Kupisch function on $\mathbb{R}$. We now classify the orthogonal components of $(\mathbb{R}, \kappa)$-pwf.
Notation 4.10 For a separation point \( c \in S(\kappa) \), denote by \( c^* = \lim_{n \to \infty} K^n(c) \) when the limit exists.

The following proposition is proven by straightforward computations.

Proposition 4.11 Let \( \kappa \) be a pre-Kupisch function and \( c \in S(\kappa) \neq \emptyset \). Then \( \kappa|_{[c, c^*)} \) is a pre-Kupisch function on \([c, c^*)\). If \( q = \max S(\kappa) \) exists, then \( \kappa|_{[q, +\infty)} \) is a pre-Kupisch function on \([q, +\infty)\). If \( p = \min S(\kappa) \) exists, then \( \kappa|_{(-\infty, p)} \) is a pre-Kupisch function on \((-\infty, p)\).

Lemma 4.12 Whenever they exist, the subcategories \( C_c = ([c, c^*), \kappa|_{[c, c^*)})\)-pfw, \( C_{max} = ([q, +\infty), \kappa|_{[q, +\infty)})\)-pfw, and \( C_{min} = ((-\infty, p), \kappa|_{(-\infty, p)})\)-pfw are connected additive category.

Proof It follows from the definition that \( S(\kappa|_{[c, c^*)}) = S(\kappa) \cap (c, c^*) \). However, according to Lemma 4.4, \( S(\kappa) \cap (c, c^*) = \emptyset \). Hence, by Lemma 4.9, the subcategory \( C_c = ([c, c^*), \kappa|_{[c, c^*)})\)-pfw is a connected additive category.

Similarly, if \( q = \max S(\kappa) \) (respectively \( p = \min S(\kappa) \)) exists we have \( C_{max} = ([q, +\infty), \kappa|_{[q, +\infty)})\)-pfw (respectively \( C_{min} = ((-\infty, p), \kappa|_{(-\infty, p)})\)-pfw) is a connected additive category.

Therefore, if \( S(\kappa) \neq \emptyset \), whichever of \( C_c, C_{max}, \) and \( C_{min} \) exist are all of the orthogonal components of \((\mathbb{R}, \kappa)\)-pfw. Up to equivalence (Corollary 3.8), we have the following orthogonal decomposition of \((\mathbb{R}, \kappa)\)-pfw.

Theorem 4.13 Let \( \kappa \) be a pre-Kupisch function on \( \mathbb{R} \) and \( S(\kappa) \) the set of separation points. Then

\[
(\mathbb{R}, \kappa)-\text{pfw} \cong \bigoplus_{c \in S(\kappa)} \begin{cases} (\mathbb{R}^{\geq 0}, \kappa_c)-\text{pfw} & \inf(S(\kappa)) = -\infty, \\ (\mathbb{R}, \kappa_0)-\text{pfw} \oplus \bigoplus_{c \in S(\kappa)} (\mathbb{R}^{\geq 0}, \kappa_c)-\text{pfw} & \inf(S(\kappa)) > -\infty, \end{cases}
\]

where \( \kappa_0 \) and \( \kappa_c \)’s are pre-Kupisch functions on \( \mathbb{R} \) and \( \mathbb{R}^{\geq 0} \), respectively, which satisfy \( S(\kappa_0) = S(\kappa_c) = \emptyset \).

4.3 Orthogonal decomposition of \((S^1, \kappa)\)

Let \( \kappa \) be a Kupisch function and recall \( K(t) = \kappa(t) + t \). All the properties discussed above about separation points still hold for \( \kappa \) as a pre-Kupisch function on \( \mathbb{R} \). Furthermore, since \( \kappa \) is 1-periodic, it has the following property.

Lemma 4.14 If \( \kappa \) is a Kupisch function, then \( c \in S(\kappa) \) if and only if \( c + 1 \in S(\kappa) \).

Therefore \( S(\kappa) = \{c + n \mid c \in S(\kappa) \cap [0, 1), \; n \in \mathbb{Z}\} \).

If \( S(\kappa) = \emptyset \), by Theorem 4.9, \((\mathbb{R}, \kappa)-\text{pfw}\) is connected. Therefore, under the push-down, \((S^1, \kappa)-\text{pfw}\) is connected.

Lemma 4.15 Let \( \kappa \) be a Kupisch function, \( c \in S(\kappa) \), and \( I = [c, c + 1) \). Then \( \kappa|_{I} \) is a pre-Kupisch function on \( I \) and \((S^1, \kappa)-\text{pfw} \cong (I, \kappa|_{I})-\text{pfw}\).

Proof Let \( F : (I, \kappa|_{I})-\text{pfw} \to (S^1, \kappa)-\text{pfw} \) be the push-down functor sending \( F(M_U) = \overline{M_U} \). For intervals \( U \), \( V \subseteq [c, c + 1) \), \( U \cap (V + i) = \emptyset \) for \( i \neq 0 \). Hence \( \text{Hom}_{S^1}(\overline{M_U}, \overline{M_V}) \cong \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathbb{R}}(M_U, M_{V+i}) = \text{Hom}_{\mathbb{R}}(M_U, M_V) \). Therefore, \( F \) induces an equivalence \((I, \kappa|_{I})-\text{pfw} \cong (S^1, \kappa)-\text{pfw}\). 

\( \square \)
Recall that for any \( c \in I \subseteq \mathbb{R} \), \( c^* = \lim_{n \to \infty} K^n(c) \) when the limit exists (Notation 4.10).

**Theorem 4.16** Suppose \( c_0 \in S(\kappa) \neq \emptyset \). Then

\[
(\mathbb{S}^1, \kappa)\text{-}\text{pwf} \cong \bigoplus_{c \in S(\kappa) \cap [c_0, c_0+1]} \left( [c, c^*] \kappa \right)\text{-}\text{pwf} \cong \bigoplus_{c \in S(\kappa) \cap [c_0, c_0+1]} (\mathbb{R}^{\geq 0}, \kappa_c)\text{-}\text{pwf}
\]

for some pre-Kupisch functions \( \kappa_c \), satisfying \( S(\kappa_c) = \emptyset \).

Notice that \( S(\kappa) \cap [c_0, c_0+1) \) is in bijection with \( S(\kappa) \cap [0, 1) \), since \( \kappa \) is 1-periodic.

**Corollary 4.17** (to Theorem 4.16) \( (\mathbb{S}^1, \kappa)\text{-}\text{pwf} \) is a connected additive category if and only if \( |S(\kappa) \cap [0, 1)| \leq 1 \).

**Example 4.18** If \( \kappa(t_0) \geq 1 \) for some \( t_0 \), then \( (\mathbb{S}^1, \kappa)\text{-}\text{pwf} \) is a connected additive category

In Sect. 5 we show an example of a category \( (\mathbb{S}^1, \kappa)\text{-}\text{pwf} \) which is not a connected additive category (Example 5.12).

## 5 Orientation Preserving Homeomorphisms and Equivalences

In this section, we prove the converse of Theorems 3.7 and 3.15 for (pre-)Kupisch functions without separation points. First, we summarize some properties of equivalence functors, which we will use later.

**Lemma 5.1** Let \( F : \mathcal{C} \to \mathcal{D} \) be an equivalence between abelian categories. Then

1. \( F \) is exact;
2. \( F \) preserves isomorphism classes;
3. \( F \) preserves simple objects;
4. \( F \) preserves socle and top: (i.e. If \( S \) is a simple sub-object/quotient object of \( X \), then \( F(S) \) is a simple sub-object/quotient object of \( F(X) \).)
5. \( F \) preserves indecomposability.
6. \( F \) preserves bricks.
7. \( F \) preserves projective objects and preserves injective objects.
8. \( F \) preserves subfactors; (i.e, if \( Y \) is a subobject of a quotient object \( X \) in \( \mathcal{C} \) then \( F(Y) \) is a subobject of a quotient object of \( F(X) \) in \( \mathcal{D} \).)

Next, we show that if an equivalence functor sends an indecomposable interval module \( M_U \) to an indecomposable interval module \( N_V \), then these two intervals \( U \) and \( V \) have the same “open-close” type.

**Lemma 5.2** Let \( F : (I, \kappa)\text{-}\text{pwf} \to (J, \lambda)\text{-}\text{pwf} \) be an equivalence of categories where \( \kappa \) and \( \lambda \) are pre-Kupisch functions on intervals \( I \) and \( J \), respectively. Then for any \( [x, y] \subseteq [x, x + \kappa(x)] \subseteq I \), where \( x < y \), there is some \( [a, b] \subseteq [a, a + \lambda(a)] \subseteq J \), such that \( F(M_{[x,y]}) \cong N_{[a,b]} \), \( F(M_{[x,y]}) \cong N_{[a,b]} \), \( F(M_{[x,y]}) \cong N_{[a,b]} \), and \( F(M_{[x,y]}) \cong N_{[a,b]} \).

**Proof** Since equivalence functors preserve indecomposability, \( F(M_{[x,y]}) \) is an indecomposable representation in \( (J, \lambda)\text{-}\text{pwf} \). The equivalence functors also preserve the top and socle of a representation. Hence \( F(M_{[x,y]}) \cong N_{[a,b]} \) for some closed interval \( [a, b] \).

Because equivalent functors are exact, applying \( F \) to the exact sequence \( 0 \to M_{[x,y]} \to M_{[x,y]} \to M_{[x,y]} \to 0 \), we have \( F(M_{[x,y]}) \cong \ker(N_{[a,b]} \to N_{[a,a]}) \cong N_{[a,b]} \). The other statements follow in a similar manner.
A similar conclusion also holds for indecomposable string modules.

**Lemma 5.3** Let $F : (S^1, \kappa)$-pwf $\rightarrow (S^1, \lambda)$-pwf be an equivalence of categories where $\kappa$ and $\lambda$ are Kupisch functions. Then for any $[x, y] \subseteq [x, x+\kappa(x)]$, where $x < y$, there is some $[a, b] \subseteq [a, a+\lambda(a)]$, such that $F(\mathcal{M}_{[x,y]}) \cong \overline{M}_{[a,b]}$, $F(\mathcal{M}_{[x,y]}) \cong \overline{N}_{[a,b]}$, and $F(\overline{M}_{[x,y]}) \cong \overline{N}_{[a,b]}$.

**5.1 Proofs of the Theorems**

Now we proceed to prove the converse to Theorems 3.7 and 3.15 in the case that there are no separation points. Recall that for a pre-Kupisch function $\kappa$, $K(t) = \kappa(t) + t$.

**Theorem 5.4** Let $F : (I, \kappa)$-pwf $\rightarrow (J, \lambda)$-pwf be an equivalence of categories where $\kappa$ and $\lambda$ are pre-Kupisch functions on intervals $I$ and $J$, respectively, such that $S(\kappa) = S(\lambda) = \emptyset$. Then $F$ induces an orientation preserving homeomorphism $f \in \text{Homeo}_+(I, J)$ such that $\lambda = f_\ast \kappa$.

**Proof** We denote by $M_U$ and $N_V$ interval modules of $(I, \kappa)$-pwf and $(J, \lambda)$-pwf, respectively.

Since the equivalence functor $F$ preserves simple representations, for any simple representation $M_{[x,t]}$, $F(M_{[x,t]}) \cong N_{[a,a]}$ for some $a \in J$. Define $f(x) = a$ and we are going to show $f : I \rightarrow J$ is an orientation preserving homeomorphism.

The fact $f$ is a bijection follows immediately from Lemma 5.1(3) and the fact that $F$ is dense.

If $x < y$ in $I$, then since $S(\kappa) = \emptyset$, there exists a natural number $n$ such that $K^{n-1}(x) < y \leq K^n(x)$. So, there is a sequence of nonzero morphisms in $(I, \kappa)$-pwf:

$$M_{[y,y]} \rightarrow M_{[K^{n-1}(x), y]} \rightarrow M_{[K^{n-2}(x), K^{n-1}(x)]} \rightarrow \cdots \rightarrow M_{[x, K(x)]} \rightarrow M_{[x,x]}.$$ 

Since the equivalence functor $F$ also preserves indecomposable representations, for any interval module $M_U$, $F(M_U) = N_V$ for some interval $V \subseteq J$. Applying $F$ to the above sequence morphisms, we have a sequence of nonzero morphisms in $(J, \lambda)$-pwf:

$$N_{[f(y), f(y)]} \rightarrow N_{V_{n-1}} \rightarrow N_{V_{n-2}} \rightarrow \cdots \rightarrow N_{V_0} \rightarrow N_{[f(x), f(x)]}.$$ 

Hence, $[f(x), f(y)] \cap_L V_0 \neq \emptyset$, $V_0 \cap_L V_1 \neq \emptyset$, $\cdots$, $V_{n-1} \cap_L [f(y), f(y)] \neq \emptyset$, which implies $f(x) \leq f(y)$. So, $f$ is increasing and therefore $f \in \text{Homeo}_+(I, J)$.

Now we show $\lambda = f_\ast \kappa$. Notice $F$ preserves the top and socle of any interval module: $F(M_{[a,b]}) = N_{[f(a), f(b)]}$. Moreover, since $F$ also preserves projective representations. For each indecomposable projective $M_{[I, \kappa(t)+t]}$, $F(M_{[I, \kappa(t)+t]}) = N_{[f(t), f(t)+\lambda(f(t))]}$. So we have $f(t + \kappa(t)) = f(t) + \lambda(f(t))$, which means $\lambda = f_\ast \kappa$.

**Theorem 5.5** Let $F : (S^1, \kappa)$-pwf $\rightarrow (S^1, \lambda)$-pwf be an equivalence of categories where $\kappa$ and $\lambda$ are Kupisch functions such that $S(\kappa) = S(\lambda) = \emptyset$. Then $F$ induces an orientation preserving homeomorphism $f \in \text{Homeo}_+(S^1)$ such that $\lambda = f_\ast \kappa$.

Before we prove the theorem we provide a useful construction and some technical results.

We denote by $\overline{M}_U$ and $\overline{N}_V$ interval modules of $(S^1, \kappa)$-pwf and $(S^1, \lambda)$-pwf respectively.

**Construction 5.6** We define functions $\tilde{f} : \mathbb{R} \rightarrow \mathbb{R}$ and $f : S^1 \rightarrow S^1$ from the functor $F$. Since the equivalence functor $F$ preserves simple objects, there is a unique $h \in [0, 1)$ such that $F(\overline{M}_{[0,0]}) = \overline{N}_{[h,h]}$. For all $n \in \mathbb{Z}$, define $\tilde{f}(n) = h + n.$
For each $t \in (0, 1)$ there is a unique $b \in (h, h+1)$ such that $F(M_{t,t+1}) \cong N_{[b,b]}$. For all $t \in (0,1)$ and $n \in \mathbb{Z}$, define $\tilde{f}(t+n) = b + n$. Define $f : S^1 \to S^1$ as $f(e^{2\pi i t}) = e^{2\pi ib}$, for each $t \in [0,1)$.

Notice that $\tilde{f}$ is a lift of $f$.

Since $F$ is an equivalence, $f$ is a bijection. In the following, we show that $f \in \text{Homeo}_+(S^1)$ by showing $\tilde{f} : \mathbb{R} \to \mathbb{R}$ is an increasing bijection (we cannot apply the homotopy lifting theorem directly, since, a priori, we do not know $f$ is continuous).

**Lemma 5.7** Let $\tilde{f} : \mathbb{R} \to \mathbb{R}$ be the function defined in Construction 5.6. Then

1. $\tilde{f}(t+1) = \tilde{f}(t) + 1$.
2. $\tilde{f}$ is a bijection.
3. $\tilde{f}$ is increasing.

**Proof** (1) follows immediately from the definition. (2) Since $f$ is a bijection, restricting on each interval $[n, n+1)$, the function $\tilde{f}|_{(n,n+1)} : [n, n+1) \to [\tilde{f}(0)+n, \tilde{f}(0)+n+1)$ is a bijection. Hence $\tilde{f} : \mathbb{R} \to \mathbb{R}$ is a bijection. (3) First, we prove $\tilde{f}$ is increasing locally.

**Claim** For any $0 \leq s < t < 1$ and $t \leq K(s) = s + \kappa(s)$, it follows that $\tilde{f}(s) \leq \tilde{f}(t)$.

Indeed, consider the module $M_{[s,t]}$ in $(S^1, \kappa)$-pwf. According to Lemma 5.3, $F(M_{[s,t]} \cong \overline{N}_{[a,b]}$ for some interval $[a,b]$. Since $F$ preserves top and socle, $F(M_{[s,t]} \cong \overline{N}_{[a,a]}$ and $F(M_{[t,t]} \cong \overline{N}_{[b,b]}$. Hence $\tilde{f}(s) = a - i$ and $\tilde{f}(t) = b - j$ for some $i, j \in \mathbb{Z}$. Next, because $0 \leq t - s < 1$, $M_{[s,t]}$ is a brick. Hence, by Lemma 5.1(6), $\overline{N}_{[a,b]}$ is also a brick. This implies $0 \leq b - a < 1$ and so $i - j \leq \tilde{f}(t) - \tilde{f}(s) = i - j + 1$. But, $|\tilde{f}(t) - \tilde{f}(s)| < 1$, since both $\tilde{f}(s)$ and $\tilde{f}(t)$ are $\tilde{f}(0), \tilde{f}(0)+1$. Thus, $i - j = -1$ or $0$. If $i - j = 0$, it follows that $\tilde{f}(s) \leq \tilde{f}(t)$. We show $i - j$ cannot be $-1$ by finding a contradiction. If $i - j = -1$, it follows that $\tilde{f}(t) < \tilde{f}(s) \leq \tilde{f}(t) + 1$. Also from the definition of $\tilde{f}$, we have an inequality:

$$\tilde{f}(0) \leq \tilde{f}(t) < \tilde{f}(s) < \tilde{f}(0) + 1 \leq \tilde{f}(t) + 1.$$ 

Notice that, in this case, $\overline{N}_{[a,b]} \cong \overline{N}_{[\tilde{f}(s), \tilde{f}(t)+1]}$ and, from the above inequality, $\overline{N}_{[\tilde{f}(0)+1, \tilde{f}(0)+1]} \cong \overline{N}_{[\tilde{f}(0), \tilde{f}(0)+1]}$ is a subfactor of $\overline{N}_{[a,b]}$. Therefore $\overline{M}_{[0,0]}$ must be a subfactor of $\overline{M}_{[s,t]}$, which forces $s = 0$. But then $\tilde{f}(s) = \tilde{f}(0)$ contradicts the above inequality. Second, we prove that $\tilde{f}$ is increasing on $[0,1)$. Since $S(\kappa) = \emptyset$, for $0 \leq s < t < 1$, there is a natural number $m$ such that $K^m(s) < t \leq K^{m+1}(s)$. Apply the claim for pairs $K^{i-1}(s)$ and $K^i(s)$ for $i = 1, \ldots, m$ and for the pair $K^m(s)$ and $t$, we have $\tilde{f}(s) \leq \tilde{f}(K(s)) \leq \cdots \leq \tilde{f}(K^m(s)) \leq \tilde{f}(t)$. Therefore $\tilde{f}$ is increasing on $[0,1)$. Last, combining (1) with the fact that $\tilde{f}$ is increasing on $[0,1)$, it follows that $\tilde{f}$ is increasing on $\mathbb{R}$.

**Corollary 5.8** (to Lemma 5.7) The map $f$ defined in Construction 5.6 is an orientation preserving homeomorphism in $\text{Homeo}_+(S^1)$ with a lift $\tilde{f} \in \text{Homeo}_+(\mathbb{R})$.

**Lemma 5.9** If $M_{[s,t]}$ is a string module in $(S^1, \kappa)$-pwf, then $F(M_{[s,t]}) \cong \overline{N}_{[\tilde{f}(s), \tilde{f}(t)]}$.

**Proof** Suppose $F(M_{[s,t]}) \cong \overline{N}_{[a,b]}$. Then $a = \tilde{f}(s) + i$ and $b = \tilde{f}(t) + j$. It suffice to show $i = j$.

Suppose $d \leq t - s < d + 1$ for some integer $d$. This is equivalent to say $\dim_k \text{End}_{S^1}M_{[s,t]} = d + 1$. Hence $\dim_k \text{End}_{S^1}N_{[a,b]} = d + 1$ and therefore $b - a \in [d, d + 1]$. That is

$$\tilde{f}(t) - \tilde{f}(s) \in [d + i - j, d + 1 + i - j].$$
Example 5.12

For induces an equivalence functor \( \tilde{\text{interval exchanging map}} \) in Fig. 2 and see [19] for the formal definition.

Remark 5.11

An example of an interval exchanging map for intervals that are periodic in \( \mathbb{R} \) as in Remark 5.11. We push the interval exchange down to the circle using the periodicity. The orange interval is fixed. The red and blue intervals are swapped

Now, since \( s + d \leq t < s + d + 1 \) and \( \tilde{f} \) is increasing, we have

\[
\tilde{f}(s) + d = \tilde{f}(s + d) \leq \tilde{f}(t) < \tilde{f}(s + d + 1) = \tilde{f}(s) + d + 1.
\]

That is \( \tilde{f}(t) - \tilde{f}(s) \in [d, d + 1) \). This forces \( i - j = 0 \), which finishes the proof.

Proof of Theorem 5.5

It remains to show \( \lambda = f_\ast \kappa \). Notice the equivalence \( F \) preserves indecomposable projective representations. According to Lemma 5.9 \( F(M_{[f(x), f(x + \kappa(x))]})) \cong N_{[f(x), f(x + \kappa(x))]} \cong N_{[f(x), f(x) + \lambda(f(x))]}. \) So, \( \tilde{f}(x + \kappa(x)) = \tilde{f}(x) + \lambda(f(x)) \). That is,

\[
\lambda(t) = \tilde{f}(\tilde{f}^{-1}(t) + \kappa \circ \tilde{f}^{-1}(t)) - t.
\]

By definition, \( \lambda = f_\ast \kappa \).

Corollary 5.10 (to Theorem 5.5)

If \( F : (S^1, \kappa)\text{-pwf} \to (S^1, \lambda)\text{-pwf} \) is an equivalence, then \( F \) induces an equivalence functor \( \tilde{F} : (\mathbb{R}, \kappa)\text{-pwf} \to (\mathbb{R}, \lambda)\text{-pwf} \) such that \( F(M_U) = \tilde{F}(M_U) \) for each interval module \( M_U \in (S^1, \kappa)\text{-pwf} \).

Proof

This follows since \( \tilde{f} \) in Construction 5.6 is in \( \text{Homeo}_+(\mathbb{R}) \), by Lemmas 3.14 and 5.7.

We make some remarks about Theorem 5.5 in the cases when the Kupisch functions have at least one separation point. First, let us fix some notations: for any map \( T : [0, 1) \to [0, 1) \), we can extend it to \( \tilde{T} : \mathbb{R} \to \mathbb{R} \) by defining \( \tilde{T}(t) = T(t) + n \) for \( t \in [n, n + 1) \), \( n \in \mathbb{Z} \) and push it down to \( \tilde{T} : S^1 \cong \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \) by defining \( \tilde{T}([t]) = [\tilde{T}(t)] \), where \( [x] \) stands for the coset \( x + z \in \mathbb{R}/\mathbb{Z} \).

Remark 5.11

(1) When \( S(\kappa) \cap [0, 1) \) is a finite set, Theorem 5.5 can be modified using interval exchanging maps. That is, the equivalence \( F : (S^1, \kappa)\text{-pwf} \to (S^1, \lambda)\text{-pwf} \) induces a map \( f : S^1 \to S^1 \) such that for some interval exchanging map \( T : [0, 1) \to [0, 1) \), \( \tilde{T} \circ f \) is an orientation preserving homeomorphism and \( \lambda = [(\tilde{T} \circ f)_* \kappa] \circ \tilde{T} \). (See an example of interval exchanging map in Fig. 2 and see [19] for the formal definition.)

(2) In particular, since when \( |S(\kappa) \cap [0, 1)| \leq 2 \), \( \tilde{T} \) is just a circle rotation, and \( \tilde{T} = x + \theta \), for some rotation number \( \theta = T(0) \). Together with Example 3.16 (1), this implies \( \lambda(t) = (\tilde{T} \circ f)_* \kappa (t + \theta) \) is still a push-forward of \( \kappa \) by an orientation-preserving homeomorphism.

Example 5.12

For \( n \in \mathbb{N} \setminus \{0\} \), let \( \kappa_n(t) = \frac{k + 1}{2n} - \frac{t}{2} \) for \( \frac{k}{n} \leq t < \frac{k + 1}{n} \), \( k \in \mathbb{Z} \). We see that \( S(\kappa_n) = \frac{1}{2n} \mathbb{Z} \). The category \((S^1, \kappa_n)\text{-pwf} \) is disconnected for \( n > 1 \). See Fig. 3 for a picture of the orthogonal components in \((S^1, \kappa_n)\text{-pwf} \), for \( n \in \{1, 2, 3\} \).

By Theorem 4.16, \((S^1, \kappa_n)\text{-pwf} \) is equivalent to the direct sum of \( n \) copies of \((\mathbb{R}^\geq 0, \lambda)\text{-pwf} \).
There is an auto-equivalence $F : (S^1, \kappa_1)$-pwf $\to (S^1, \kappa_3)$-pwf defined in the following way on objects:

$$F \overline{M}_U = \begin{cases} 
\overline{M}_{U-\frac{1}{3}} & U \subseteq [0, \frac{1}{3}) \\
\overline{M}_U & U \subseteq [\frac{1}{3}, \frac{2}{3}) \\
\overline{M}_{U+\frac{1}{3}} & U \subseteq [\frac{2}{3}, 1).
\end{cases}$$

Consider the interval exchanging map

$$T(t) = \begin{cases} 
-\frac{1}{3} & t \in [0, \frac{1}{3}) \\
t & t \in [\frac{1}{3}, \frac{2}{3}) \\
+\frac{1}{3} & t \in [\frac{2}{3}, 1).
\end{cases}$$

Restricting the functor $F$ on the simple objects induces a circle map $\tilde{T}$, which is not an orientation preserving homeomorphism. It is easy to verify that $\tilde{T}^2 = \text{id}$ and $\kappa(t) = \kappa(\tilde{T}(t))$.

According to Theorem 4.13 and Theorem 4.16, the orthogonal components of categories $(R, \kappa)$-pwf or $(S^1, \lambda)$-pwf are of the following forms:

(i) $(R_{\geq 0}, \mu)$-pwf where $S(\mu) = \emptyset$;
(ii) $(R, \nu)$-pwf where $S(\nu) = \emptyset$; and
(iii) $(S^1, \lambda)$-pwf where $S(\lambda) = \emptyset$.

We verify that categories of the forms (i), (ii), (iii) above are pairwise non-equivalent:

**Proposition 5.13** Let $\mu$, $\nu$ be pre-Kupisch functions on $R_{\geq 0}$ and $R$, respectively, and $\lambda$ be a Kupisch function. Assume $S(\lambda) = S(\mu) = S(\nu) = \emptyset$. Then categories (i) $(R_{\geq 0}, \mu)$-pwf, (ii) $(R, \nu)$-pwf and (iii) $(S^1, \lambda)$-pwf are mutually non-equivalent.

**Proof** Since $R_{\geq 0}$ and $R$ are non-homeomorphic, categories of the forms (i) and (ii) are not equivalent, by Theorem 5.4.

Next, denote by $L(t) = \lambda(t) + t$. By Proposition 4.5, there is $N > 0$ such that $1 \in (L^N(0), L^{N+1}(0))$. Then there is a sequence of nonzero non-isomorphisms between indecomposable $S^1$-representations compatible with $\lambda$.

$$\overline{M}_{[0,0]} \cong \overline{M}_{[1,1]} \to \overline{M}_{[L^N(0),1]} \to \overline{M}_{[L^{N-1}(0),L^N(0)]} \to \cdots \to \overline{M}_{[L(0),L^2(0)]} \to \overline{M}_{[L(0),0]} \to \overline{M}_{[0,0]},$$

where $\overline{M}_{[0,0]}$ is simple. But there is no such a sequence for any simple $R$-representation. Therefore $(S^1, \lambda)$-pwf is not equivalent to any subcategory of $R$-pwf, hence not equivalent to categories (i) and (ii).

Proposition 5.13 may not be true if the (pre-)Kupisch functions have separation points, as shown in the following example.
**Example 5.14** For all $n \in \mathbb{Z}$ and $t \in [n, n+1)$, let $\nu(t) = \frac{n+1-t}{2}$ and let $\mu = \nu|_{[0, \infty)}$. Notice $\nu$ and $\mu$ are pre-Kupisch functions on $\mathbb{R}$ and $\mathbb{R}^{\geq 0}$, respectively. We see that $S(\nu) = \mathbb{Z}$ and $S(\mu) = \mathbb{N}$. The connected orthogonal components of both $(\mathbb{R}, \nu)$-pwf and $(\mathbb{R}^{\geq 0}, \mu)$-pwf are equivalent to $([0, 1), \nu|_{[0,1)})$-pwf $\cong (\mathbb{R}^{\geq 0}, \eta)$-pwf, for some pre-Kupisch function $\eta$.

To compute an $\eta$ explicitly, choose a homeomorphism $f : [0, 1) \to [0, \infty)$, say $f(t) = \frac{t}{1-t}$, and take the push-forward $\eta(t) = f_*(\nu|_{[0,1)}) = t + 1$. Then $(\mathbb{R}, \nu)$-pwf $\cong ((\mathbb{R}^{\geq 0}, \eta)$-pwf)$^\mathbb{Z} \cong ((\mathbb{R}^{\geq 0}, \eta)$-pwf)$^\mathbb{N} \cong (\mathbb{R}^{\geq 0}, \mu)$-pwf.

### 5.2 Relation to Dynamical Systems

One concludes that the classification of categories $(\mathbb{S}^1, \kappa)$-pwf and $(\mathbb{R}, \kappa)$-pwf up to categorical equivalence reduces to the classification of categories of types $(\mathbb{R}, \kappa)$-pwf, $(\mathbb{R}^{\geq 0}, \kappa)$-pwf, or $(\mathbb{S}^1, \kappa)$-pwf, where $S(\kappa) = \emptyset$. We now discuss how the classification is related to topological dynamical systems.

A topological dynamical system $(X, \sigma)$ contains a topological space $X$ and a self-map $\sigma : X \to X$. A topological conjugacy between systems $(X, \sigma)$ and $(Y, \tau)$ is a homeomorphism $f : X \to Y$ such that $f \circ \sigma = \tau \circ f$. When $X, Y$ are both intervals or $\mathbb{S}^1$, we say a topological conjugacy $f$ is positive if it is orientation preserving.

Notice that a (pre-)Kupisch function $\kappa$ on $\mathbb{R}$ or $\mathbb{R}^{\geq 0}$ gives rise to a dynamical system $(\mathbb{R}, K(t))$ or $(\mathbb{R}^{\geq 0}, K(t))$, where $K(t) = \kappa(t) + t$ is the self-map. And a Kupisch function $\kappa$ induces a circle map

$$\tilde{K} : \mathbb{S}^1 \to \mathbb{S}^1$$

$$\left(e^{2\pi i t}\right) \mapsto e^{2\pi(i\kappa(t)+t)},$$

hence a system $(\mathbb{S}^1, \tilde{K})$. One can check that a (pre-)Kupisch function $\lambda = f_\kappa \kappa$ is a push-forward by some orientation preserving homeomorphism $f$ if and only if $f$ is a positive topological conjugacy between the induced dynamical systems. Thus Theorem 5.4 and Theorem 5.5 says that classifying categories $(\mathbb{S}^1, \kappa)$-pwf and $(\mathbb{R}, \kappa)$-pwf is equivalent to classifying their corresponding dynamical systems up to positive topologically conjugacy. See [11] for more details about circle dynamics.

### 6 Continuous and Discrete Nakayama Representations

In this section, we discuss the relation between discrete and continuous representations. We show that the module category of any discrete Nakayama algebra can be embedded in $(\mathbb{S}^1, \kappa)$-pwf for some Kupisch function $\kappa$.

Recall that a basic connected Nakayama algebra $A$ is called **linear** if its Ext-quiver is $A_n$:

$$0 \twoheadrightarrow 1 \twoheadrightarrow 2 \twoheadrightarrow \cdots \twoheadrightarrow n - 1$$

and **cyclic** if its Ext-quiver is $C_n$:
Let $A$ be a basic connected Nakayama algebra of rank $n$ (i.e. $A$ has $n$ isomorphism classes of simple modules). The Kupisch series $(l_0, l_1, \cdots, l_{n-1})$ is a sequence with $l_i = l(P_i)$ the length of the indecomposable projective module $P_i$. By definition, $A$ is a (connected) linear Nakayama algebra if and only if $l_{n-1} = 1$. Since any $A_n$ representation $M$ can be viewed as a $C_n$ representation with $M(n-1) \to M(0)$ being zero, representations of any Nakayama algebra are representations on $C_n$.

**Definition 6.1** Let $A$ be a basic connected Nakayama algebra with a Kupisch series $(l_0, l_1, \cdots, l_{n-1})$. We define the associated Kupisch function $\kappa_A$ as follows. For any $k \in \mathbb{Z}$,

$$\kappa_A(t) := \left\{ \begin{array}{ll}
1 - t & 0 \leq t < \frac{1}{3} \\
\frac{4}{3} - t & \frac{1}{3} \leq t < 1.
\end{array} \right.$$ 

The relation between $A$-modules and $S^1$-representations compatible with $\kappa_A$ is described in Lemma 6.4 and Remark 6.5. Notice that the associated Kupisch function $\kappa_A$ has no separation points.

**Example 6.2** Let $A$ be a Nakayama algebra with Kupisch series $(3, 3, 2)$. Then the associated Kupisch function $\kappa_A$ on $[0, 1)$ is:

$$\kappa_A(t) = \begin{cases} 
1 - t & 0 \leq t < \frac{1}{3} \\
\frac{4}{3} - t & \frac{1}{3} \leq t < 1.
\end{cases}$$

The rest of this section is dedicated to proving the following theorem.

**Theorem 6.3** Let $A$ be a basic connected Nakayama algebra with a Kupisch series $(l_0, l_1, \cdots, l_{n-1})$ and $\kappa_A$ its associated Kupisch function. Then there is an exact embedding $F : A\operatorname{-mod} \to (S^1, \kappa_A)\operatorname{-pwf}$ that preserves projective objects. Moreover, if $A$ is a linear Nakayama algebra, then there is an exact embedding $L : A\operatorname{-mod} \to (\mathbb{R}, \kappa_A)\operatorname{-pwf}$, which preserves projective objects.

For the convenience of showing functoriality, we need to describe the image of this functor on objects first. Let $\mathcal{M} = \operatorname{add}[\overline{M}(\frac{a}{n}, \frac{b}{n}) \in (S^1, \kappa_A)\operatorname{-pwf} \mid a, b \in \mathbb{N}, a < n]$ be the full subcategory of $(S^1, \kappa_A)\operatorname{-pwf}$ consisting of direct sums of string modules of the form $\overline{M}(\frac{a}{n}, \frac{b}{n})$.

For $0 \leq i < n$, denote by $p_i = (\frac{i+1}{n} \mod \mathbb{Z}) \in \mathbb{R}/\mathbb{Z} \cong S^1$. For each string module $\overline{M}(\frac{a}{n}, \frac{b}{n}) \in \mathcal{M}$, we assign a representation $G\overline{M}(\frac{a}{n}, \frac{b}{n}) = (X_i, \varphi_i)$ of the quiver $C_n$ as the
following: $X_i = \overline{M}_{(\frac{i}{n}, \frac{i+1}{n})}(p_i)$ for $0 \leq i < n$ and $\varphi_i : X_i \to X_{i+1}$ is given by $\varphi_i = \overline{M}_{(\frac{i}{n}, \frac{i+1}{n})}(g_{pi}, p_{i+1})$, where $g_{pi}, p_{i+1}$ is a generating morphism on $\mathbb{S}^1$.

**Lemma 6.4** Let $\overline{M}_{(\frac{a}{n}, \frac{b}{n})} \in \mathcal{M}$, then $M = G\overline{M}_{(\frac{a}{n}, \frac{b}{n})}$ is an indecomposable $A$-module with top $M \cong S_a$ and length $l(M) = b - a$.

**Proof** According to the definition of string module $\overline{M}_{(\frac{a}{n}, \frac{b}{n})}$ as in Sect. 2.2, the representation $M = (X_i, \varphi_i)$ can be explicitly written in the following way.

Let $U = (\frac{a}{n}, \frac{b}{n})$, $x = p_{i+1}$, and $y = p_{i+2}$.

The vector space $X_k$ has a basis $p_U^{-1}(x) = \{b_{i,x}\}_{i=1}^{l}$ such that $b_{i,x} < b_{i+1,x}$ for all $i$. The linear transformation $\varphi_k = \overline{M}_U(g_{xy}) : X_k \to X_{k+1}$ is defined on the basis of $X_k$ as:

$$\varphi_k(b_{i,x}) = \begin{cases} b_{j,y} & \exists b_{j,y} \in p_U^{-1}(y) \text{ such that } 0 \leq b_{j,y} - b_{i,x} < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $M = (X_i, \varphi_i)$ is a string module on $C_n$ with top $M \cong S_a$ and $l(M) = b - a$. To see $M$ is an $A$-module, we just need to check $l(M) \leq l_a$ for the Kupisch series $(l_0, l_1, \cdots, l_a, \cdots, l_{n-1})$. But this follows immediately since $b - a \leq nK_A(a) - a \leq l_a$.

**Remark 6.5** We mention some important $A$-modules corresponding to the assignment $G$.

1. Each $G\overline{M}_{(\frac{i}{n}, \frac{i+1}{n})}$ is isomorphic to the simple module $S_i$.
2. Each $G\overline{M}_{(\frac{i}{n}, \kappa_A(\frac{i}{n}))}$ is isomorphic to the projective module $P_i$.

Furthermore, the assignment $G$ defines an additive functor: a morphism $f : \overline{M} \to \overline{N}$ in $\mathcal{M}$ gives rise to an $A$-module homomorphism $Gf : G\overline{M} = (X_i, \varphi_i) \to G\overline{N} = (Y_i, \psi_i)$ by taking $Gf_i : X_i \to Y_i$ to $f(p_i)$.

**Lemma 6.6** The additive functor $G : \mathcal{M} \to A\text{-mod}$ is an equivalence.

**Proof** We need to show that $G$ is fully-faithful and dense.

Let $f : \overline{M} \to \overline{N}$ be a morphism in $\mathcal{M}$ such that $Gf = 0$. By definition, $f(p_i) = 0$ for all $i$. For any $x, y \in (\frac{i}{n}, \frac{i+1}{n}]$ and for all $i$, $\overline{M}(g_{xy})$ and $\overline{N}(g_{xy})$ are identity maps. Since $f(p_i)\overline{M}(g_{xy}) = \overline{N}(g_{xy})f(x)$, it follows that $f(x) = f(p_i) = 0$ for all $x \in (\frac{i}{n}, \frac{i+1}{n}]$ and for all $i$. Hence, $f = 0$.

Secondly, for any homomorphism $g = (g_i) : G\overline{M} = (X_i, \varphi_i) \to G\overline{N} = (Y_i, \psi_i)$, we define a natural transformation $f : \overline{M} \to \overline{N}$ such that $f(x) = g_i$ for $x \in (\frac{i}{n}, \frac{i+1}{n}]$. It satisfies $Gf = g$ and, therefore, $G : \text{Hom}_\mathcal{M}(\overline{M}, \overline{N}) \to \text{Hom}_A(G\overline{M}, G\overline{N})$ is bijective.

Finally, since $A$ is a basic connected Nakayama algebra, any indecomposable $A$-module can be uniquely determined by its top and length up to isomorphism. Assume $X$ is an indecomposable $A$-module with top $X \cong S_a$ and $l(X) = l$. Then, according to Lemma 6.4, $G(\overline{M}_{(\frac{a}{n}, \frac{b}{n})}) \cong X$. Therefore, $G$ is dense.

**Corollary 6.7** Let $A$ be a basic connected Nakayama algebra with Kupisch series $(l_0, l_1, \cdots, l_{n-1})$, $\kappa_A$ be its associated Kupisch function, and $\overline{P} = \bigoplus_{i=0}^{n-1} \overline{M}_{(\frac{i}{n}, \kappa_A(\frac{i}{n}))}$ be in $\langle S^1, \kappa_A \rangle$-pfw. Then $A \cong \text{End}_{\mathcal{S}^1}(\overline{P})$.
**Proof** Since $G\overline{M}(\frac{k}{\n}, K_A(\frac{k}{\n})) = P_t$ are the indecomposable projective $A$-modules, it follows that $\overline{M} = \bigoplus_{t=0}^{n-1} P_t \cong \text{End}_A(\bigoplus_{t=0}^{n-1} P_t) \cong \text{End}_A(\bigoplus_{t=0}^{n-1} \overline{M}(\frac{k}{\n}, K_A(\frac{k}{\n})))$.

**Proof of Theorem 6.3** The embedding $\iota : \mathcal{M} \to (\mathcal{S}^1, \kappa_A)$-pwf is exact by construction. Therefore, there is an exact embedding $F = \iota \circ G^{-1} : A\text{-mod} \to (\mathcal{S}^1, \kappa_A)$-pwf. Each indecomposable projective $A$-module $P(a)$ has top $P(a) \cong S_0$ and $I(P(a)) = I_a$. Thus, $F(P(a)) \cong \overline{M}(\frac{a}{n}, \kappa A(\frac{k}{\n})) = \overline{M}(\frac{k}{\n}, K(\frac{k}{\n}))$, which is a projective object in $(\mathcal{S}^1, \kappa_A)$-pwf.

When $A$ is a linear Nakayama algebra, it follows that $I_{n-1} = 1$. Hence $K_A(t) \leq 1$ for all $0 \leq t < 1$. So for each string module $\overline{M}_U$ in $\mathcal{M}$, we have $U \subseteq (0, 1]$ and $\text{Hom}_A(\overline{M}_U, \overline{M}_V) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_R(M_U, M_{V+i}) = \text{Hom}_R(M_U, M_V)$. Therefore $\iota' : \mathcal{M} \to (\mathbb{R}, \kappa_A)$-pwf sending $\bigoplus \overline{M}_U$ to $\bigoplus M_U$ is a fully-faithful embedding and $L = \iota' \circ G^{-1} : A\text{-mod} \to (\mathbb{R}, \kappa_A)$-pwf is the desired exact embedding.

Continuous Nakayama representations have composition series in the sense of [8]. In particular, it is shown in [8] that indecomposable representations of $\mathbb{R}$ with straight orientation (and thus string representations of $\mathcal{S}^1$ with cyclic orientation) are uniserial. Thus, an indecomposable continuous Nakayama representation of $(\mathbb{R}, \kappa)$-pwf or of $(\mathcal{S}^1, \kappa)$-pwf is also uniserial, for some (pre-)Kupisch function $\kappa$. Although categories of both discrete and continuous Nakayama representations have uniserial indecomposable objects, the homological properties of continuous Nakayama representations are sometimes quite different. We conclude with some interesting examples, which demonstrates the difference of some homological properties between categories of discrete and continuous Nakayama representations.

**Example 6.8** Consider $\kappa(t) = \begin{cases} 0, & n \leq t \leq n + \frac{1}{2}, \; n \in \mathbb{Z} \\ 1 - (t - n), & n + \frac{1}{2} \leq t < n + 1, \; n \in \mathbb{Z}. \end{cases}$

The simple representation $\overline{M}_{[0,0]} = \overline{M}_{[1,1]}$ does not have an injective envelope. This is because, if there was an injective envelope $\overline{M}_{[1,1]} \to \overline{E}$, then $\overline{E}$ must be indecomposable and of the form $\overline{E} = \overline{M}_{[0,1]}$. But since any inclusion $\overline{M}_{[1,1]} \hookrightarrow \overline{M}_{\frac{1}{2}+\varepsilon,1}$ $(0 \leq \varepsilon < \frac{1}{2})$ cannot factor through $\overline{M}_{[1,1]} \hookrightarrow \overline{M}_{\frac{1}{2}+\varepsilon,1}$, there is no such injective envelope.

**Example 6.9** Consider the category $(\mathcal{S}^1, \kappa)$-pwf, where for $t \in (0, 1]$

$$K(t) = \kappa(t) + t = \begin{cases} \frac{1}{n} & n \leq t < \frac{1}{n}, \text{ for } n \geq 4 \\ \frac{1}{n} + 1 & \frac{1}{n+1} \leq t < \frac{1}{n}, \text{ for } n \geq 4 \end{cases}$$

Every string module $\overline{M}_U$ has finite projective dimension. But $\text{pd} \overline{M}_{\frac{1}{2^n+1}, \kappa}$ = $2k - 1$ for $k \in \mathbb{N}^{\geq 0}$. Hence $\text{pd} \left( \bigoplus_{k=1}^{\infty} M_{\frac{1}{2^n+1}, \frac{1}{2}} \right) = \infty$ and $\sup\{\text{pd} \overline{M}_U \mid \text{pd} \overline{M}_U < \infty\} = \infty$.

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