Teleportation fidelity and its deviation for an arbitrary two-qubit state when the classical communication is noisy

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In this work, we consider the following teleportation protocol: There is an arbitrary two-qubit resource state, shared between two spatially separated parties, Alice and Bob. Applying local unitary operators, they transform the resource state into the canonical form. To teleport an unknown qubit, Alice now measures her qubits in the Bell basis. Then, the measurement outcome is communicated by Alice to Bob via noisy classical channel(s). Finally, after receiving the classical message, Bob applies the necessary unitary operator to his qubit. Under this protocol, we find the exact formulae of teleportation fidelity and its deviation. We further find conditions for non-classical fidelity within this protocol. If the classical communication is noiseless in the above protocol then there are resource states which can lead to zero fidelity deviation. However, we show that such states may not lead to zero fidelity deviation when the classical communication is noisy in the same protocol. We also explore the opposite case, i.e., the states, which cannot lead to zero fidelity deviation in the above protocol when the classical communication is noiseless, may lead to zero fidelity deviation when the classical communication is noisy in the same protocol without compromising the non-classical fidelity. Moreover, we exhibit scenarios within the present protocol, where the fidelity deviation increases if the entanglement of the resource state is increased.

I. INTRODUCTION

Quantum teleportation [1–7] is a physical process via which it is possible to transfer the state of a quantum system from one location to another without knowing the state. To accomplish this process, one can use entanglement [8–10] as resource along with local operations and classical communication (LOCC) [11]. This process is important for many reasons. Two of the reasons are (i) state can be transferred from one location to another without transmitting the quantum system, (ii) any non-local task can be accomplished by a teleportation-based protocol [12, 13]. Here a nonlocal task means, the task which cannot be completed by LOCC only.

For perfect teleportation, it is required to have a maximally entangled state. However, if two resource states are there then it is important to find which resource state is the better one to teleport an unknown quantum state. For this purpose, one can use the standard figure of merit, the teleportation fidelity [14, 15]. We usually take the average value of the teleportation fidelity, known as the average fidelity. This average is taken over all input states, uniformly distributed in the Bloch sphere. We mention here that a different figure of merit is introduced in Ref. [16], known as the random teleportation robustness. But here we stick to teleportation fidelity because the fluctuations, associated with this figure of merit, are easier to realize.

Recently, fidelity deviation [17–22] in quantum teleportation has got considerable attention. It is important to study this quantity as it gives the idea of fluctuations associated with teleportation fidelity. In particular, due to unavoidable imperfections which occur in a practical situation, it is expected that fluctuations must be associated with teleportation fidelity. Thus, the duo, teleportation fidelity and its deviation contain more information to characterize quantum teleportation than the teleportation fidelity alone. However, fidelity deviation has been studied considering different resource states [18, 19, 21] and other constraints [20, 22]. In these articles, it has also been argued that the states which can lead to both non-classical fidelity and zero fidelity deviation, are the most desired states as resource in quantum teleportation. But it is yet to explore how noisy classical communication affects fidelity deviation in a teleportation protocol.

In a perfect teleportation process, while teleporting an unknown qubit, two classical bits are required to communicate. But in a practical situation one cannot deny the environmental interaction with the resource state (in other words, imperfect quantum channels [23, 24]) as well as the classical channels. There are a few papers
where noisy classical communication is consid-
ered in a teleportation scenario and corresponding fidelity has been calculated. Nevertheless, an easily calculable formula for fidelity considering an arbitrary two-qubit resource state is still unknown when the classical communication is noisy. In Ref. [25], it has been analyzed what is the sufficient amount of classical communication to achieve the non-classical teleportation fidelity while teleporting an unknown qubit. There the authors have con-
sidered noisy classical communication but the resource states are of particular types. Later, in Ref. [26], the authors have analyzed the amount of classical communi-
cation sufficient to beat the classical limit of teleportation fidelity while teleporting a qudit and the resource state is being a maximally entangled state in $C^d \otimes C^d$.

Here we consider the following teleportation protocol: There is an arbitrary two-qubit state as resource, shared between two spatially separated parties, Alice and Bob. Applying local unitary operators, the parties transform the state into the canonical form. In order to teleport an unknown qubit, Alice now measures her two qubits, the qubit to be teleported and a qubit of the resource state, in the Bell basis. Then, the measurement outcome is commu-
cicated by Alice to Bob via noisy classical channel(s). Finally, after receiving the classical message, Bob applies the necessary unitary operator to his qubit. So, basic-
ically after transforming an arbitrary two-qubit resource state into the canonical form by applying local unitary operators, the parties follow the standard teleportation protocol where the classical communication is noisy.

Under the above protocol, we find the exact formulae of teleportation fidelity and its deviation. We further find conditions for non-classical fidelity within this protocol. If the classical communication is noiseless then there are some resource states which can lead to zero fidelity deviation in the above teleportation protocol. However, we show that such resource states may not lead to zero fidelity deviation when the classical communication is noisy in the same protocol. We also explore the opposite case, i.e., the states, which cannot lead to zero fidelity deviation in the above protocol when the classical communication is noiseless, may lead to zero fidelity deviation when the classical communication is noisy in the same protocol without compromising the non-classical fidelity. Interestingly, we demonstrate scenarios within the present protocol, where the fidelity deviation increases if the entanglement of the resource state is increased.

We organize the paper in the following way: In Sec. II, a few preliminary concepts are provided. Thereafter, in Sec. III, we present the main results with corresponding discussions. Finally, in Sec. IV, the conclusion is drawn, mentioning some open problems.

II. PRELIMINARIES

Starting from an arbitrary two-qubit resource state $\rho$, if the parties (Alice and Bob, between which the resource state is shared), follow the standard teleportation proto-
ocol [1], then it may not always be possible to achieve the optimal teleportation fidelity [14]. But it is always possible to maximize the average fidelity, applying the local unitary operators. Thus, the maximum value of the average fidelity, $F_F$, is an optimized value over the local unitary operators and it is known as the maximal fidelity. This maximum value can be achieved through the local unitary (LU) strategies and the standard protocol, pro-
posed in Ref. [27]. In particular, to achieve the optimal value of the teleportation fidelity, the parties, first apply a LU strategy to transform the given resource state $\rho$ to the canonical form $\rho$ before starting the standard protocol [27]. So, the quantity of interest is the maximum value of $F_F$ (for a greater details, one can go through the paper [19], and the references therein).

Note that $F_F$ is simply the average fidelity corresponding to the state $\rho$. It is defined in a later portion.

Remember that in the above when we talk about the standard teleportation protocol, except the resource state, we do not consider any other imperfections. But in this work, we consider noisy classical communication. This makes the situation more complex and in fact, when the classical communication is noisy, it is not known how to find the maximum achievable value of the average fidelity. However, we stick to a teleportation protocol which includes LU strategies and the standard teleportation protocol with noisy classical communication as mentioned earlier. We now move forward for a greater math-
ematical details.

An arbitrary two-qubit state $\rho$ can be expressed in the following form, known as the Hilbert-Schmidt representation:

$$\rho = \frac{1}{4} \left( I_4 + R \cdot \sigma \otimes I_2 + I_2 \otimes S \cdot \sigma + \sum_{i,j=1}^{3} T_{ij} \sigma_i \otimes \sigma_j \right),$$  \hspace{1cm} (1)$$

where $I_4$ is an $n \times n$ identity matrix (in the whole manuscript), $R$, $S$ are the vectors of $\mathbb{R}^3$, $R(S) \cdot \sigma = \sum_{i=1}^{3} R_i (S_i) \sigma_i$, the elements $T_{ij} = \text{Tr}(\rho \sigma_i \otimes \sigma_j)$, $\forall i, j = 1, 2, 3$, build a $3 \times 3$ matrix $T$. This matrix is real and it is known as the correlation matrix. In the above expression $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices. We mention here that $I_2$ is sometimes considered as the zeroth Pauli matrix, $\sigma_0$. As mentioned earlier, there always exist local unitary operators $u, v$, such that the parties transform the state $\rho$ to $\rho$, i.e., $(u \otimes v)\rho (u \otimes v)^T = \rho$. Let $t_{11}, t_{22}, t_{33}$ be the eigenvalues of the correlation matrix $T$. Then the canonical form $\rho$ is given by:

$$\rho = \frac{1}{4} \left( I_4 + r \cdot \sigma \otimes I_2 + I_2 \otimes s \cdot \sigma + \sum_{i=1}^{3} \lambda_i |t_{ii}| \sigma_i \otimes \sigma_i \right),$$  \hspace{1cm} (2)$$
in the above $\lambda_i$ can be $\pm 1$. These values can be deter-
bined by the sign of the determinant of $T$, i.e., det$T$. If det$T \leq 0$, then we can take $\lambda_i = -1$ for all $|t_{ii}| \neq 0$, $i = 1, 2, 3$. Again, if det$T > 0$, then we can take $\lambda_1, \lambda_2 = -1$, and $\lambda_3 = +1$ for the choices where $i \neq j \neq k$,
\(i, j, k = 1, 2, 3\), while the eigenvalues of \(T\) obey the condition \(|t_{ii}| \geq |t_{jj}| \geq |t_{kk}|\) (also see [19] in this regard).

After the above transformation, the standard teleportation protocol begins. The very first step of which is a measurement in the Bell basis by Alice. To convey the measurement outcome, Alice sends two classical-bits (in other words, cbits) to Bob. Based on the classical message that Bob receives, he applies the appropriate unitary operator to his qubit. In the Table I, we provide the Bell states, the cbits which is to be communicated by Alice to Bob, and the unitary operators.

| The Bell states | cbits | Unitary operators |
|-----------------|------|------------------|
| \(|\phi_0\rangle = (1/\sqrt{2})(|00\rangle + |11\rangle)| | 00 | \(\varnothing\) |
| \(|\phi_1\rangle = (1/\sqrt{2})(|00\rangle - |11\rangle)| | 11 | \(\varsigma_1\) |
| \(|\phi_2\rangle = (1/\sqrt{2})(|01\rangle + |10\rangle)| | 01 | \(\varsigma_2\) |
| \(|\phi_3\rangle = (1/\sqrt{2})(|01\rangle - |10\rangle)| | 10 | \(\varsigma_3\) |

TABLE I. The Bell state \(|\phi_k\rangle\) corresponds to the measurement outcome \(k, \forall k = 0, \ldots, 3\). Corresponding to each outcome \(k\), cbits and the unitary rotation are also given.

As mentioned earlier, we use here noisy classical communication. So, it is expected that when Alice sends two-cbit through the noisy classical channel(s), the teleportation fidelity is reduced by some amount. In the result section, we show how much amount of fidelity gets reduced. However, in this work, we consider two different noise models.

Noise Model-I: Let us first consider a single noisy classical channel, through which Alice tries to send two-cbit of classical information. This channel is characterized by the following conditional probabilities:

\[
P(ab|\bar{a}b) = p_0, \quad P(\bar{a}b|ab) = p_1, \\
P(ab|\bar{b}a) = p_2, \quad P(\bar{b}a|ba) = p_3,
\]

where both \(a\) and \(b\) take the value either 0 or 1, \(\bar{a}\) and \(\bar{b}\) are the complements of \(a\) and \(b\) respectively. Alice’s input is one of the bit strings, drawn from the set \(\{00, 01, 10, 11\}\). The probabilities, \(p_i \geq 0\) for \(\forall i = 0, \ldots, 3\), \(\sum_{i=0}^{3} p_i = 1\). It is possible to quantify the amount of classical information, conveyed by Alice to Bob. The quantification is based on the mutual information between Alice’s input \(X\) and Bob’s output \(Y\). It is given as the following:

\[
I(X : Y) = 2 + \sum_{i=0}^{3} p_i \log_2(p_i).
\]

Notice that in the above definition if we use \(p_i = 1\) and \(p_j = 0\), when \(i\) takes any particular value from the set \(\{0, 1, 2, 3\}\) and \(j = \{0, 1, 2, 3\} - \{i\}\), then \(I = 2\), which means that the amount of classical information, communicated is two-cbit. On the other hand, if all probabilities \((p_i)\) are equal, then the amount of classical information communicated is zero. In fact, \(p_0\) is the probability, responsible for appropriate unitary rotation and \(p_1, p_2, p_3\) introduce the errors to Bob’s choice of unitary after receiving the information regarding the Alice’s measurement outcome.

In brief, here the teleportation protocol includes the following steps: (i) the parties apply the local unitary operators to transform the resource state \(\rho\) to the canonical form \(\varrho\), (ii) Alice performs the Bell measurement on her qubits, (iii) Alice sends two-cbit through the noisy classical channel, (iv) the classical message, sent by Alice, becomes noisy according to the above noise model, (v) finally, after receiving the noisy classical message, Bob applies the appropriate unitary operation. In the Table II, we summarize the measurement outcomes, corresponding cbits of Alice, the cbits, received by Bob and the unitary rotations.

Noise Model-II: We now consider that Alice sends the measurement outcome using two independent noisy classical channels, i.e., one cbits through each classical channel. These classical channels are labeled by \(C_l, l\) is either \(\eta\) or \(\eta’\). The conditional probabilities are given as:

\[
P(a|a) = l, \quad P(a|a) = 1 - l,
\]

where \(a\) takes the value 0 or 1. \(\bar{a}\) is the complement of \(a\), \(1/2 \leq l \leq 1\). Again, the amount of classical information which is communicated can be quantified using the concept of mutual information, given as the following:

\[
I_l = 1 - H(l),
\]

where \(H(l) = -l \log_2 l - (1 - l) \log_2 (1 - l)\), the binary Shannon entropy. In the teleportation protocol, the total amount of classical information which is communicated, is the sum of \(I_\eta\) and \(I_{\eta’}\) of the two noisy classical channels, i.e., \([2 - H(\eta) - H(\eta’)]\) cbits. If we compare the steps of the teleportation protocol with the steps, summarized in the above, then the difference is that we consider here a different noise model (Noise Model-II). Obviously, Bob gets the classical message accordingly and the unitary operators may act with different probabilities.

We now define two important quantities the average fidelity corresponding to the state \(\rho\), i.e., \(F_\varrho\) and its deviation \(\Delta F_\varrho\). The average fidelity can be defined as the following:

\[
F_\varrho = \langle \delta \rangle = \int f_\varrho \, d\psi,
\]

where \(f_\varrho\) is the fidelity between the unknown state which Alice wants to teleport (the input state) and the state which is prepared on Bob’s side after his unitary operation (the output state). Obviously, \(f_\varrho\) corresponds to the state \(\varrho\) in the canonical form. The integral is taken over a uniform distribution \(d\psi\) with respect to all input states, where \(f \, d\psi = 1\) (normalized Haar measure).

Previously, it has been argued that average fidelity alone may not characterize quantum teleportation completely [18]. The reason is that it does not provide any
idea of the fluctuations which is expected to be associated with the above figure of merit. Clearly, it is quite justified to consider fidelity deviation, the standard deviation of teleportation fidelity. This quantity can be regarded as a measure of the fluctuations, associated with fidelity. We consider here fidelity deviation \( \Delta_\rho \) corresponding to \( F_\rho \). It is defined as:

\[
\Delta_\rho = \sqrt{\langle \delta^2 \rangle - F_\rho^2}, \tag{8}
\]

where \( \langle \delta^2 \rangle = \int \delta^2 \, d\psi \). We want the above quantity to be minimum for a given resource state. In particular, if \( \Delta_\rho = 0 \) in the standard teleportation protocol, then the given resource state teleports all input states equally well. But for the present protocol, it is not known how to minimize the fidelity deviation corresponding to an arbitrary resource state. However, we stick to the pair of quantities \( \{ F_\rho, \Delta_\rho \} \) and examine the effect of noisy classical communication on these quantities.

In fact, a resource state is a desired resource state in quantum teleportation if such a resource state lead to non-classical teleportation fidelity and zero fidelity deviation. So, finding such resource states, is important.

| Measurement outcome | The Bell state | cbits, sent by Alice | cbits, received by Bob | Unitary operation |
|---------------------|---------------|---------------------|-----------------------|------------------|
| 0                   | \( |\psi_0\rangle \) | 00 with \( p_0 \)   | \( c_0 \) with \( p_0 \) |                  |
|                     |               | 10 with \( p_1 \)   | \( c_3 \) with \( p_1 \) |                  |
|                     |               | 01 with \( p_2 \)   | \( c_2 \) with \( p_2 \) |                  |
|                     |               | 11 with \( p_3 \)   | \( c_2 \) with \( p_3 \) |                  |
| 1                   | \( |\psi_1\rangle \) | 11 with \( p_0 \)   | \( c_1 \) with \( p_0 \) |                  |
|                     |               | 01 with \( p_1 \)   | \( c_1 \) with \( p_1 \) |                  |
|                     |               | 10 with \( p_2 \)   | \( c_1 \) with \( p_1 \) |                  |
|                     |               | 00 with \( p_3 \)   | \( c_1 \) with \( p_3 \) |                  |
| 2                   | \( |\psi_2\rangle \) | 01 with \( p_0 \)   | \( c_2 \) with \( p_0 \) |                  |
|                     |               | 11 with \( p_1 \)   | \( c_1 \) with \( p_1 \) |                  |
|                     |               | 00 with \( p_2 \)   | \( c_1 \) with \( p_2 \) |                  |
|                     |               | 10 with \( p_3 \)   | \( c_1 \) with \( p_3 \) |                  |
| 3                   | \( |\psi_3\rangle \) | 10 with \( p_0 \)   | \( c_1 \) with \( p_0 \) |                  |
|                     |               | 00 with \( p_1 \)   | \( c_1 \) with \( p_1 \) |                  |
|                     |               | 11 with \( p_2 \)   | \( c_1 \) with \( p_2 \) |                  |
|                     |               | 01 with \( p_3 \)   | \( c_1 \) with \( p_3 \) |                  |

TABLE II. In the above, we provide Alice’s measurement outcomes, corresponding Bell states and Alice’s cbits. Then, we provide the cbits which is received by Bob and the unitary operations, associated with certain probabilities.

### III. RESULTS

**Fidelity:** Let us consider the state which is to be teleported by Alice to Bob, is \( |\psi\rangle = \frac{1}{2}(I_2 + \mathbf{a} \cdot \mathbf{\sigma}) \) (Bloch sphere representation), \( \mathbf{a} \) is a unit vector in \( \mathbb{R}^3 \). After Alice’s measurement in the Bell basis \( \{ |\phi_k\rangle \}_k \), the state which is prepared on Bob’s side (before the application of the unitary operator), is given by:

\[
\rho_k = \frac{1}{s_k} \text{Tr}_A \left( (|\phi_k\rangle \langle \phi_k | \otimes I_2)(|\psi\rangle \langle \psi | \otimes \sigma)(|\phi_k\rangle \langle \phi_k | \otimes I_2) \right), \tag{9}
\]

where the resource state is \( \rho \), the trace is taken over Alice’s two-qubit system. ‘\( s_k \)’ is the probability of getting the outcome \( k \), it can be defined as: \( s_k = \text{Tr} \left( (|\phi_k\rangle \langle \phi_k | \otimes I_2)(|\psi\rangle \langle \psi | \otimes \sigma)(|\phi_k\rangle \langle \phi_k | \otimes I_2) \right) \). Using the Hilbert-Schmidt representations, \( \rho_k \) can be written as the following:

\[
\rho_k = \frac{1}{8s_k} \left( (1 + \mathbf{a}^\top \mathbf{T}_k \mathbf{r})I_2 + (\mathbf{s} + \mathbf{T}_k^\top \mathbf{r} \mathbf{a}) \cdot \mathbf{\sigma} \right), \tag{10}
\]

where the matrices \( \{ \mathbf{T}_k \} \) correspond to the Bell states \( |\phi_k\rangle \) \( \forall k = 0, \ldots, 3 \), they are given as: \( \mathbf{T}_0 = \text{diag}(1, 1, -1, -1) \) for \( |\phi_0\rangle \), \( \mathbf{T}_1 = \text{diag}(-1, 1, -1, 1) \) for \( |\phi_1\rangle \), \( \mathbf{T}_2 = \text{diag}(1, -1, -1, 1) \) for \( |\phi_2\rangle \), \( \mathbf{T}_3 = \text{diag}(-1, -1, 1, 1) \) for \( |\phi_3\rangle \), the matrix \( \mathbf{T} \) and the vectors \( \mathbf{r}, \mathbf{s} \) correspond to \( \rho \), \( \mathbf{a} \) is produced by taking the transpose of the unit vector \( \mathbf{a} \). Remember that originally, we have defined correlation matrix for any given state \( \rho \) but after the transformation to the canonical form, we use the state \( \rho \). So, henceforth \( \mathbf{T} \) belongs to \( \rho \), which is a \( 3 \times 3 \) diagonal matrix. After generation of \( \rho_k \), Bob applies the unitary operator on it based on the classical message that he receives from Alice. The details of the application of unitary operators are given in Table II (we follow the Noise Model-I first).
The states which are produced after the application of the unitary operators, are given by:

\[
\begin{align*}
\rho'_{0} &= p_0 \rho_{0}\rho_{0} + p_1 \rho_{1}\rho_{0} + p_2 \rho_{2}\rho_{2} + p_3 \rho_{3}\rho_{3}, \\
\rho'_{1} &= p_0 \rho_{1}\rho_{1} + p_1 \rho_{2}\rho_{1} + p_2 \rho_{3}\rho_{1} + p_3 \rho_{2}\rho_{2}, \\
\rho'_{2} &= p_0 \rho_{2}\rho_{2} + p_1 \rho_{1}\rho_{2} + p_2 \rho_{0}\rho_{2} + p_3 \rho_{3}\rho_{2}, \\
\rho'_{3} &= p_0 \rho_{3}\rho_{3} + p_1 \rho_{0}\rho_{3} + p_2 \rho_{1}\rho_{3} + p_3 \rho_{2}\rho_{2},
\end{align*}
\]

where each \(\rho'_{k}\) corresponds to the outcome \(k\). In this way, the average state which is produced on bob’s side, is given by-

\[
\begin{align*}
\rho_{\text{avg}} &= \sum_{k=0}^{3} s_k \rho'_{k} = \\
&= p_0 (s_0 \rho_{0}\rho_{0} + s_1 \rho_{1}\rho_{1} + s_2 \rho_{2}\rho_{2} + s_3 \rho_{3}\rho_{3}) + \\
&+ p_1 (s_0 \rho_{1}\rho_{1} + s_2 \rho_{2}\rho_{1} + s_3 \rho_{3}\rho_{1} + s_0 \rho_{0}\rho_{2}) + \\
&+ p_2 (s_0 \rho_{2}\rho_{2} + s_1 \rho_{1}\rho_{2} + s_2 \rho_{0}\rho_{2} + s_3 \rho_{3}\rho_{2}) + \\
&+ p_3 (s_0 \rho_{3}\rho_{3} + s_1 \rho_{0}\rho_{3} + s_2 \rho_{1}\rho_{3} + s_3 \rho_{2}\rho_{2}).
\end{align*}
\]

From the above equation, we now consider the following term: \(s_0 \rho_{0}\rho_{0} + s_1 \rho_{1}\rho_{1} + s_2 \rho_{2}\rho_{2} + s_3 \rho_{3}\rho_{3}\). This term can be written as:

\[
\begin{align*}
\frac{1}{8} & \left[ (1 + a^T T_0 r) I_2 + O'_0 (s + T^T T_0 a) \cdot \sigma + (1 + a^T T_1 r) I_2 + O'_1 (s + T^T T_1 a) \cdot \sigma \right] + \\
\frac{1}{8} & \left[ (1 + a^T T_2 r) I_2 + O'_2 (s + T^T T_2 a) \cdot \sigma + (1 + a^T T_3 r) I_2 + O'_3 (s + T^T T_3 a) \cdot \sigma \right].
\end{align*}
\]

In the above we use the following fact: \(\zeta_k(n, \sigma) \zeta_k = (O'_k(n, \sigma), \forall k = 0, \ldots, 3\). Each \(O_k\) is a rotation in \(\mathbb{R}^3\). Here \(O'_k = -T_k, \forall k = 0, \ldots, 3\) for a greater details one can go through theRefs. [19, 27]). Notice that \(T_i^T T_k = T_k^T T_i^T\) in the above expression, i.e., in Eq. (13). Moreover, we also use the relations \(\sum_{k=0}^{3} T_k = 0\) and \(T_k^2 = I_2, \forall k = 0, \ldots, 3\) in Eq. (13). We finally get \(s_0 \rho_{0}\rho_{0} + s_1 \rho_{1}\rho_{1} + s_2 \rho_{2}\rho_{2} + s_3 \rho_{3}\rho_{3} = \frac{1}{2} (I_2 - T^T a \cdot \sigma) = \frac{1}{2} (I_2 + T^T T_0 a \cdot \sigma),\) remembering the fact that \(T_3 = -T_2, \ldots\). Following the similar steps and using the relations \(T_0 T_3 = T_3 T_0 = -T_0 = T_2 T_1 = T_1 T_2 = T_2 T_0\), it is possible to show that \(s_0 \rho_{3}\rho_{3} + s_1 \rho_{1}\rho_{3} + s_2 \rho_{2}\rho_{3} + s_3 \rho_{0}\rho_{3} = \frac{1}{2} (I_2 + T^T T_0 a \cdot \sigma),\) using the relations \(T_0 T_2 = T_2 T_0 = -T_0 = T_1 T_3 = T_3 T_1 = T_1 T_3 = T_1 T_2 = T_2 T_1 = T_2 T_3,\) and \(s_0 \rho_{1}\rho_{1} + s_1 \rho_{0}\rho_{1} + s_2 \rho_{2}\rho_{1} + s_3 \rho_{3}\rho_{1} = \frac{1}{2} (T^T T_0 a \cdot \sigma),\) using the relations \(T_0 T_0 = T_0 T_0 = T_1 T_1 = T_2 T_2 = T_3 T_3,\) clearly, \(\rho_{\text{avg}}\) can be rewritten in the following way:

\[
\rho_{\text{avg}} = \frac{1}{2} \left[ (I_2 + T^T (p_0 T_0 + p_1 T_1 + p_2 T_2 + p_3 T_3) a \cdot \sigma) \right] = \frac{1}{2} \left[ (I_2 + T^T \rho_{\text{avg}} a \cdot \sigma) \right],
\]

where \(\sum_{i=0}^{3} p_i = 1\) and the matrix \(X = T^T (p_0 T_0 + p_1 T_1 + p_2 T_2 + p_3 T_3)\) is a \(3 \times 3\) real diagonal matrix because the correlation matrices are \(3 \times 3\) real diagonal matrices. Using \(\rho_{\text{avg}}\), we calculate \(f_\theta\) which is given by:

\[
f_\theta = \text{Tr} \left( \rho_{\text{avg}} |\psi\rangle \langle \psi| \right) = \frac{1}{2} \left( 1 + a^T X a \right),
\]

where \(|\psi\rangle \langle \psi| = \frac{1}{2} (I_2 + a \cdot \sigma)|\psi\rangle \langle \psi|\) is the unknown input state which Alice wants to teleport. We recall that we want to calculate the average fidelity \(F_\rho\), corresponding to the resource state \(\rho\) which is in the canonical form. For this purpose, we apply Schur’s lemma on \(\mathbb{R}^4\) for a detailed description of this lemma one can go through theRefs. [14, 18, 19] and the references therein). This lemma leads us to the expression of \(F_\rho\), given by:

\[
F_\rho = \int f_\theta \text{da} = \int \left( \frac{1 + a^T X a}{2} \right) \text{da} = \frac{1}{2} \left( 1 + \text{Tr}(X) \right),
\]

where we use \(\int a^T X a \text{ da} = \frac{1}{2} \text{Tr}(X)\) by Schur’s lemma. The above expression is the desired expression for teleportation fidelity for our protocol, starting from an arbitrary two-qubit state \(\rho\) when we use the Noise Model-I for noisy classical communication.

Using \(T = \text{diag}(\lambda_1 |t_{1i}|, \lambda_2 |t_{1j}|, \lambda_3 |t_{kk}|)\) and the forms of \(T_k\), we calculate \(\text{Tr}(X)\) which is given as:

\[
\text{Tr}(X) = (p_1 + p_3 - p_0 - p_2) \lambda_1 |t_{1i}| + (p_2 + p_3 - p_0 - p_1) \lambda_3 |t_{1j}|,
\]

\[
+ (p_1 + p_2 - p_0 - p_3) \lambda_2 |t_{kk}|,
\]

where \(i \neq j \neq k \in \{1, 2, 3\}\). Now, rearranging the above and using the relation \(\sum_{i=0}^{3} p_i = 1\), we get:

\[
\text{Tr}(X) = 2p_1 (\lambda_1 |t_{1i}| + \lambda_2 |t_{kk}|) + 2p_2 (\lambda_2 |t_{1j}| + \lambda_1 |t_{kk}|) + 2p_3 (\lambda_3 |t_{kj}| + \lambda_1 |t_{kk}|) - \sum_{i=1}^{3} \lambda_i |t_{ii}|.
\]

If \(\det T \leq 0\) then we take \(\lambda_1 = \lambda_2 = \lambda_3 = -1\) for all \(|t_{ii}| \neq 0, i = 1, 2, 3\) and \(T = \text{diag}(-|t_{1i}|, -|t_{2j}|, -|t_{3k}|)\). Hence, \(F_\rho\) becomes
where the term $\frac{1}{2} \left(1 + \frac{1}{3} \sum_{i=1}^{3} |t_{ii}| \right)$ is the optimal teleportation fidelity corresponding to a two-qubit resource state $\rho$ when the classical communication is noiseless. We

$$F_{\theta} = \frac{1}{2} \left(1 + \frac{1}{3} \sum_{i=1}^{3} |t_{ii}| \right) - \frac{1}{3} \left(p_1 (|t_{11}| + |t_{33}|) + p_2 (|t_{22}| + |t_{33}|) + p_3 (|t_{11}| + |t_{22}|) \right),$$  \hspace{1cm} (19)

Remember that the indices $i, j, k$ are determined, maintaining the ordering $|t_{ii}| \geq |t_{jj}| \geq |t_{kk}|$. Thus, the term $\frac{1}{2} \left(1 + \frac{1}{3} \sum_{i=1}^{3} |t_{ii}| \right)$ is due to the noise, associated with the classical communication. We say this term as $f_{\text{noise}}/3$ and using this term we find the condition for non-classical fidelity for our protocol when det $T < 0$. We take here the definition of non-classical fidelity as the fidelity which is greater than the maximum fidelity ($\frac{3}{4}$), achievable through the standard teleportation protocol [28, 29]. However, we consider only those states for which det $T$ is less than zero because it is known that in the standard teleportation protocol a resource state can lead to non-classical fidelity only when det $T < 0$, i.e., $\sum_{i=1}^{3} |t_{ii}| > 1$ (see Refs. [14, 28–30]). In the present case also, we use standard teleportation protocol but with noise in the classical communication. This is why the condition for non-classical fidelity gets modified here. The modified condition is given by-

$$\frac{1}{2} \left(1 + \frac{1}{3} \sum_{i=1}^{3} |t_{ii}| \right) - \frac{1}{3} f_{\text{noise}} > \frac{3}{4},$$  \hspace{1cm} (20)

Within our protocol, using Noise Model-I, if a resource state satisfies the above condition, than that state leads to non-classical fidelity. Notice that the quantity $f_{\text{noise}}$ is a nonzero positive number and therefore, not all entangled states satisfy the above condition.

We also find out expressions for teleportation fidelity using Noise Model-II in our protocol. For this purpose, we use the following transformation rules: $p_0 \to \eta \eta'$, $p_1 \to (1-\eta) \eta'$, $p_2 \to \eta (1-\eta')$, $p_3 \to (1-\eta)(1-\eta')$. Using these transformation rules, along with Eq. (16), we get the following expression for teleportation fidelity:

$$F_{\theta} = \frac{1}{2} (1 + \frac{\text{Tr}(X')}{3}),$$  \hspace{1cm} (22)

where $X' = T'_{11}[\eta \eta' T_{11} + (1-\eta)\eta' T_{11} + (1-\eta')(1-\eta') T_{12}]$ and $T_{1k}$ have their usual meaning $\forall k = 0, \ldots, 3$. Using $T = \text{diag}(\lambda_i |t_{ii}|, \lambda_j |t_{jj}|, \lambda_k |t_{kk}|)$ and the forms of $T_k$, we calculate $Tr(X')$ which is given as:

$$\text{Tr}(X') = (1-2\eta) \lambda_i |t_{ii}| + (1-2\eta') \lambda_j |t_{jj}| - (1-2\eta')(1-2\eta') \lambda_k |t_{kk}|,$$  \hspace{1cm} (23)

On the other hand, if det $T > 0$, then $\lambda_i = \lambda_j = -1$, $\lambda_k = 1$ for any choice of $i \neq j \neq k \in \{1,2,3\}$ satisfying the condition $|t_{ii}| \geq |t_{jj}| \geq |t_{kk}|$. Hence, $F_{\theta}$ becomes

$$F_{\theta} = \frac{1}{2} (1 + \frac{1}{3} \sum_{i=1}^{3} |t_{ii}|) - \frac{1}{3} |t_{ii}| (1-\eta) + \frac{1}{3} |t_{ii}| (1-\eta') + |t_{kk}| (\eta + \eta' - 2\eta\eta').$$  \hspace{1cm} (24)

Remember that the indices $i, j, k$ are determined in such a way that the ordering $|t_{ii}| \geq |t_{jj}| \geq |t_{kk}|$ is maintained. Thus, the term $\frac{1}{2} \left(1 + \frac{1}{3} \sum_{i=1}^{3} |t_{ii}| \right)$ is maximized.

Next, consider det $T > 0$, i.e., $\lambda_i = \lambda_j = -1$, $\lambda_k = 1$ for any choice of $i \neq j \neq k \in \{1,2,3\}$ satisfying the condition $|t_{ii}| \geq |t_{jj}| \geq |t_{kk}|$. Hence, $F_{\theta}$ becomes:

$$F_{\theta} = \frac{1}{2} (1 + \frac{\text{Tr}(X')}{3}),$$  \hspace{1cm} (25)

Within our protocol, using Noise Model-I, if a resource state satisfies the above condition, than that state leads to non-classical fidelity. Notice that the quantity $f_{\text{noise}}$ is a nonzero positive number and therefore, not all entangled states satisfy the above condition.
and using this term we find out the condition, given as the following:
\[
\frac{1}{2}(1 + \frac{1}{2} \sum_{i=1}^{3} |t_{ii}|) - \frac{1}{2} f'_{\text{noise}} > \frac{1}{2},
\]
\[
\Rightarrow \sum_{i=1}^{3} |t_{ii}| > (1 + 2f'_{\text{noise}}).
\] (26)

Remember that in Noise Model-II, \(\eta f'\) is greater than or equal to the other probabilities \((1 - \eta)\eta', \eta(1 - \eta')\), \((1 - \eta)(1 - \eta')\) as \(\frac{1}{2} \leq \eta, \eta' \leq 1\). We now need to calculate fidelity deviation for different noise models.

**Fidelity deviation:** The expression for fidelity deviation, \(\Delta_f\), corresponding to the fidelity \(F_f\), is given in Eq. (8). It has been pointed out in Ref. [19] that \(\Delta_f\) lies between 0 and \(\frac{1}{2}\), i.e., \(0 \leq \Delta_f \leq \frac{1}{2}\). In fact, \(\Delta_f = 0\) if and only if \(F_f = f_\phi\) for all input states \(|\psi\rangle\). The main motivation to study fidelity deviation is that the pair \(\{F_f, \Delta_f\}\) gives more information regarding quantum teleportation than just teleportation fidelity \([18]\). In order to calculate \(\Delta_f\) (we use Noise Model-I first), we first calculate \(\langle \delta^2 \rangle\). From Eq. (7) and Eq. (16), we get \(\langle \delta^2 \rangle = \int f_\phi^2 d\alpha\), where \(f_\phi = \frac{1}{2} (1 + a^T X a)\). So, \(\langle \delta^2 \rangle\) can be written as \(\frac{1}{4} (1 + 2a^T X a + (a^T X a)(a^T X a)) d\alpha\). Here, the expression for fidelity deviation under Noise Model-II, iff the following condition is satisfied:
\[
|t_{ii}|(2p_i - 1 + p_i) = |t_{jj}|(2p_j - 1 + p_j) = |t_{kk}|(2p_k - 1 + p_k).
\] (34)
Several important aspects

From Eq. (34), it is evident that even if $|t_{11}| = |t_{22}| = |t_{33}|$, we may not get zero fidelity deviation (considering Noise Model-I). This is because the condition for zero fidelity deviation is also dependent on $p_1$, $p_2$, and $p_3$ along with $|t_{11}|$, $|t_{22}|$, $|t_{33}|$. In the present teleportation protocol if there is no noise in the classical communication then a state can lead to zero fidelity deviation when $|t_{11}| = |t_{22}| = |t_{33}|$ [19]. Such states can lead to zero fidelity deviation in the same protocol with noisy classical communication when $p_1 = p_2 = p_3$. On the other hand, if $p_1 \neq p_2 \neq p_3$ then the states with $|t_{11}| = |t_{22}| = |t_{33}|$ must not lead to zero fidelity deviation. More importantly, the states which are not dispersion-free (i.e., cannot lead to zero fidelity deviation) in the present teleportation protocol with noiseless classical communication, can be dispersion-free (i.e., can lead to zero fidelity deviation) in the same protocol with noisy classical communication. Explicit example of which is given in a later portion.

For Noise Model-II, if $|t_{11}| = |t_{22}| = |t_{33}|$ for a given state, then from Eq. (35), we get $\eta = \eta' = 1$ to make fidelity deviation zero (we exclude the choice $\eta = \eta' = 1/2$ as it implies that the amount of classical information communicated by Alice to Bob, is zero). But $\eta = \eta' = 1$ means classical channels are noiseless. Therefore, in the range $1/2 < \eta, \eta' < 1$, we cannot make fidelity deviation zero for a state with $|t_{11}| = |t_{22}| = |t_{33}|$. Clearly, Noise Model-I is comparatively better than the Noise Model-II when a state with $|t_{11}| = |t_{22}| = |t_{33}|$, is given. We next consider some well-known two-qubit entangled states (det$\Gamma < 0$) and calculate the teleportation fidelity as well as fidelity deviation for those states.

**Pure entangled states:** We consider the state $|\Phi \rangle = a |00 \rangle + b |11 \rangle$, $a$ and $b$ are non-zero positive real numbers, such that $a^2 + b^2 = 1$. For this state $|t_{11}| = |t_{22}| = 2ab$ and $|t_{33}| = 1$. So, $F_\phi$ and $\Delta_\phi$, corresponding to the state $|\Phi \rangle$ under the Noise Model-I, are given by:

$$F_\phi = \frac{1}{2} (1 + ab)$$

$$\Delta_\phi = \frac{1}{\sqrt{30}} \left[ (p_0 + p_2 - p_1 - p_3)^2 a^2 b^2 + (p_0 + p_1 - p_2 - p_3)^2 a^2 b^2 + (p_0 + p_3 - p_1 - p_2)^2 + (p_0 + p_2 + p_1 + p_3) 2ab + (p_0 + p_1 - p_2 - p_3) 2ab + (p_0 + p_3 - p_1 - p_2) 2ab \right]^{1/2}. \quad (36)$$

In the above expressions if we put $a = b = 1/\sqrt{2}$, assuming $|\Phi \rangle$ as a Bell state (i.e., $|\Phi \rangle = |\phi_0 \rangle$) then we get $F_\phi = 1+2p_0$ and $\Delta_\phi = (2/3\sqrt{10}) [(p_1 - p_2)^2 + (p_1 - p_3)^2 + (p_2 - p_3)^2]^{1/2}$. So, for the Bell state, $\Delta_\phi = 0$ only when $p_1 = p_2 = p_3$. If we assume $p_1 = p_2 = p_3$, considering $a > b$ in the above expression of $\Delta_\phi$, then we get $\Delta_\phi = (1/3\sqrt{5})(1 - 4p_1)(1 - 2ab)$, which is decreasing function of the entanglement (we use concurrence [31] here as a measure of entanglement), contained by the state $|\Phi \rangle$, i.e., $2ab$. Nevertheless, if $p_0 \neq p_1 \neq p_2 \neq p_3$, then it is possible to demonstrate specific scenarios where the fidelity deviation increases with the increment of entanglement, contained in the given pure state as resource. One such scenario is given as the following: We consider $p_0 = 0.6, p_1 = 0.2, p_2 = 0.15$, and $p_3 = 0.05$ and plot $F_\phi, \Delta_\phi$ with respect to the concurrence of $|\Phi \rangle$, i.e., $2ab$. These plots are given in Fig. (1) and Fig. (2). From the figures, it is clear that the teleportation fidelity is non-classical when the concurrence of the state $|\Phi \rangle$ is greater than 0.64. Furthermore, in this range, the deviation is increasing with the increment of the concurrence of $|\Phi \rangle$.

![FIG. 1. Teleportation fidelity is plotted against the concurrence of a pure entangled state when a particular type of noisy classical channel ($p_0 = 0.6, p_1 = 0.2, p_2 = 0.15$, and $p_3 = 0.05$) is employed. In the plot, it is clearly shown that the teleportation fidelity belongs to the non-classical range.](image)

![FIG. 2. Fidelity deviation is plotted against the concurrence of a pure entangled state when a particular type of noisy classical channel ($p_0 = 0.6, p_1 = 0.2, p_2 = 0.15$, and $p_3 = 0.05$) is employed.](image)
entangled states are not dispersion-free. But these states might be dispersion-free when there is noise in the classical communication. We provide an explicit example where such thing is happening.

We assume \( a = \sqrt{0.9} \) and \( b = \sqrt{0.1} \), thus, the concurrence of the state \( |\Phi\rangle = 2ab = 0.6 \). To make this state dispersion-free, we search for a noisy classical channel which obeys the Noise Model-I. Using Eq. (34), we get \( p_1 = p_2 \) as \( |t_{11}| = |t_{22}| = 0.6 \). We also assume \( p_1 = p_2 = 0.15 \). This implies that \( p_3 = 0.017 \) to satisfy Eq. (34). Therefore, the state \( \sqrt{0.9} |00\rangle + \sqrt{0.1} |11\rangle \) is dispersion-free in our protocol if the available noisy classical channel is characterized by the probabilities \( p_1 = p_2 = 0.15, p_3 = 0.017, \) and \( p_0 = (1 - 0.317) = 0.683 \).

It is also important to check if this state can lead to non-classical fidelity in our protocol when the available noisy classical channel is characterized by the above probabilities. Putting the values \( |t_{11}| = |t_{22}| = 0.6, |t_{13}| = 1, p_1 = p_2 = 0.15, p_3 = 0.017 \) in Eq. (19), we get \( F_\rho = 0.7 \), which is greater than 2/3 and thus, it is non-classical fidelity. This example indicates the fact that even if a resource state is provided which is not dispersion-free in an optimal protocol, it might be dispersion-free if a suitable noisy classical channel is employed without compromising the non-classical fidelity.

To find the expressions of \( F_\rho \) and \( \Delta_\rho \) for \( |\Phi\rangle = a |00\rangle + b |11\rangle \), using Noise Model-II, we have to use the transformation rules: \( p_0 \rightarrow \eta p_0, p_1 \rightarrow (1 - \eta) p_1, p_2 \rightarrow \eta (1 - \eta)p_2 \), \( p_3 \rightarrow (1 - \eta)(1 - \eta)p_3 \) and have to modify Eq. (36). The modified expressions are given by:

\[
\begin{align*}
F_\rho &= \frac{1}{2}(1 + ab) - \frac{1}{4}[2ab(2 - \eta - \eta') + (\eta + \eta' - 2\eta\eta')] , \\
\Delta_\rho &= \frac{1}{4\sqrt{10}}[16a^2b^2(\eta - \eta')^2 + (2\eta' - 1)^2(2ab - 2\eta + 1)^2 + (2\eta - 1)^2(2ab - 2\eta' + 1)^2].
\end{align*}
\]

In the above expressions if we put \( a = b = 1/\sqrt{2} \), assuming \( |\Phi\rangle \) as a Bell state (i.e., \( |\Phi\rangle = |\phi_0\rangle \)) then we get \( F_\rho = \frac{1 + 2\eta}{2\sqrt{3}} \) and \( \Delta_\rho = (2/3\sqrt{3})(1 - 3\eta + 3\eta' + \eta'(-3 + 7\eta - 6\eta' + 4\eta^2)^{1/2} \). However, assuming \( |\Phi\rangle \) as a nonmaximally entangled state, we can observe the variation of fidelity deviation with the concurrence of \( |\Phi\rangle \). In particular, like Noise Model-I, here also we can demonstrate the increase of fidelity deviation with the increment of the concurrence, 2ab.

**Werner states:** We consider a family of Werner states [32], given by \( \rho_W = \epsilon |\phi_0\rangle \langle \phi_0| + 1/2(1 - \epsilon) I_4 \), where \( |\phi_0\rangle = (1/\sqrt{2})(|00\rangle + |11\rangle) \). The states \( \rho_W \) are entangled when \( 1/3 < \epsilon \leq 1 \) (we consider here only these values of \( \epsilon \)). For \( \rho_W \), it is known that \( |t_{11}| = |t_{22}| = |t_{13}| = \epsilon \). Using these values in Eq. (19), we get-

\[
F_\rho = \frac{1}{6}(3 - \epsilon + 4\epsilon p_0). \tag{38}
\]

We first use Noise Model-I. Again, using Eq. (29), we get the expression of \( \Delta_\epsilon \) for \( \rho_W \), which is given by-

\[
\Delta_\epsilon = \frac{4(\epsilon p_0 - 1)}{3\sqrt{10}}[(p_1 - p_2)^2 + (p_1 - p_3)^2 + (p_2 - p_3)^2]^{1/2}, \tag{39}
\]

where the term \( \frac{2\epsilon - 1}{\sqrt{10}} \) is the concurrence of \( \rho_W \). From the above expression, it is quite evident that for Noise Model-I, the fidelity deviation, corresponding to the state \( \rho_W \), is zero only when \( p_1 = p_2 = p_3 \). On the other hand, for the fixed values of \( p_1, p_2, p_3 \), where \( p_1 \neq p_2 \neq p_3 \), the fidelity deviation, corresponding to the state \( \rho_W \), is an increasing function of concurrence of the state. Using Noise Model-II also, one can get the expressions of teleportation fidelity and its deviation for \( \rho_W \). To serve this purpose, one has to use the transformation rules: \( p_0 \rightarrow \eta p_0, p_1 \rightarrow (1 - \eta)p_1, p_2 \rightarrow \eta(p_2 - 1) - \eta', p_3 \rightarrow (1 - \eta)(1 - \eta') \) in Eq. (38) and in Eq. (39). We mention here that when the classical communication is noiseless in the present protocol, pure states are not dispersion free but the Werner states \( (\rho_W) \) are. Therefore, one may expect that when the classical communication is noisy, Werner states may perform better compared to the pure states from fidelity deviation point of view. But we find here specific situations where the fidelity deviation increases with the increment of entanglement within the Werner states.

**Special cases:** From the expression, given in Eq. (19), it is expected that the maximum value of the teleportation fidelity is dependent on the given resource state as well as on the noisy classical channel. However, if the resource state and the noisy classical channel both are arbitrary, then it is difficult to provide exact conditions for which the fidelity is maximum. But we focus on two cases which are worth mentioning. (i) We assume that the resource state is given and it has the property: \( |t_{11}| = |t_{22}| = |t_{33}| \). In this case, if we consider the teleportation fidelity of Eq. (19), then the fidelity depends on the values of \( |t_{11}| \) and \( p_0 \). Notice that for the Noise Model-I, for a fixed value of \( p_0 \), it is possible to consider different noisy classical channels for which \( p_1 + p_2 + p_3 = 1 - p_0 \) (fixed) but the individual values of \( p_1, p_2, p_3 \) are different. We say this class of channels as \( \Lambda_{0\rho} \). Interestingly, the given resource state leads to same amount of teleportation fidelity in the present protocol for all the channels belonging to the class \( \Lambda_{0\rho} \) with a fixed \( p_0 \). In this sense, any state with the property \( |t_{11}| = |t_{22}| = |t_{33}| \), are not biased towards a particular channel of \( \Lambda_{0\rho} \) with a fixed \( p_0 \). (ii) We assume that the noisy classical channel is given (under Noise Model-I) and it has the property: \( p_1 = p_2 = p_3 \). In this case, if we consider the teleportation fidelity of Eq. (19), then the fidelity depends on the value of the sum \( (\sum_{i=1}^{3} |t_{ii}|) \) and on \( p_1 \). Now, for a fixed value of the sum \( (\sum_{i=1}^{3} |t_{ii}|) \), we can have different states. We say this class of states with a fixed value of the sum \( (\sum_{i=1}^{3} |t_{ii}|) \) as \( \Lambda_{\rho} \). Interestingly, the given channel leads to same amount of teleportation fidelity in the present protocol for all the states belonging to the class \( \Lambda_{\rho} \) with a fixed value of the sum \( (\sum_{i=1}^{3} |t_{ii}|) \). In this sense, any channel with the property \( p_1 = p_2 = p_3 \), are
not biased towards a particular state of $\Lambda_\rho$ with a fixed value of the sum $\sum_{i=1}^3 |t_{ii}|i$. We also mention that when a resource state with the property: $|t_{11}| = |t_{22}| = |t_{33}|$, is given, or a noisy classical channel with the property: $p_1 = p_2 = p_3$ (under Noise Model-I), is given, the expression of Eq. (19), provides the maximum achievable fidelity within the present protocol.

**Minimum classical communication cost:** From Eq. (21), we find the condition for non-classical fidelity under Noise Model-I. However, here our motivation is to obtain the minimum amount of classical communication which is required to communicate from Alice’s side to Bob, such that the condition for non-classical fidelity, given in Eq. (21), is satisfied. We say this minimum amount of classical communication as minimum classical communication cost (within the present protocol). The quantification of classical information is given in Eq. (4), $I = 2 + \sum_{i=0}^3 p_i \log_2(p_i)$, where $\sum_{i=0}^3 p_i = 1$. Now, we formulate the problem of minimum classical communication cost in the following way:

$$\begin{align*}
\text{minimize} & \quad 2 + \sum_{i=0}^3 p_i \log_2(p_i) \\
\text{subject to} & \quad \sum_{i=1}^3 |t_{ii}| > (1 + 2f_{\text{noise}}), \\
& \quad \sum_{i=0}^3 p_i = 1.
\end{align*}$$

(40)

To get the solution of the above problem, we first consider the constraint equations with equality. Then, we can apply Lagrange’s method of undetermined multipliers. It is difficult to get an exact solution of the problem. However, for a given state, it may possible to solve the above. For example, for a pure state $|\Phi\rangle$, we can put $|t_{11}| = |t_{22}| = 2ab$ and $|t_{33}| = 1$ in the above equation, where $|\Phi\rangle = a|00\rangle + b|11\rangle$, $a, b$ are nonzero positive numbers such that $a^2 + b^2 = 1$. Thus, we can have expressions of $p_i$ in terms of the entanglement of $|\Phi\rangle$ (i.e., $2ab \forall i = 0, \ldots, 3$). Putting these values of $p_i$, in Eq. (4), one can get the minimum classical communication cost. Using Noise Model-II also, similar type of formulation for minimum classical communication cost is possible.

**IV. CONCLUSION AND OPEN PROBLEMS**

In this work, we have explored the effect of noisy classical communication on teleportation fidelity and its deviation. Under the present teleportation protocol, we have derived the exact formulae of teleportation fidelity and its deviation. Furthermore, we have studied the conditions for non-classical fidelity within this protocol. If the classical communication is noiseless then there are some resource states which can lead to zero fidelity deviation in the present teleportation protocol. However, we have shown that such resource states may not lead to zero fidelity deviation when the classical communication is noisy in the same protocol. Clearly, if all types of imperfections are there in a teleportation protocol, then in general, it is a difficult problem to find the resource states which can lead to both non-classical fidelity and zero fidelity deviation. Nevertheless, we have explored an explicit case where the state, which cannot lead to zero fidelity deviation in an optimal protocol when the classical communication is noiseless, may lead to zero fidelity deviation when the classical communication is noisy in the similar protocol without compromising the non-classical fidelity. This is clearly depicting the fact that “noisy environment” may not always bad. Again, we demonstrate scenarios within the present protocol, where the fidelity deviation increases if the entanglement of the resource state is increased. So, higher amount of entanglement may not always certify the desired quantum teleportation.

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