Enumeration of monochromatic three term arithmetic progressions in two-colorings of any finite group

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Abstract

There are many extremely challenging problems about existence of monochromatic arithmetic progressions in colorings of groups. Many theorems hold only for abelian groups as results on non-abelian groups are often much more difficult to obtain. In this research project we do not only determine existence, but study the more general problem of counting them. We formulate the enumeration problem as a problem in real algebraic geometry and then use state of the art computational methods in semidefinite programming and representation theory to derive lower bounds for the number of monochromatic arithmetic progressions in any finite group.

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1 Introduction

In Ramsey theory colors or density are commonly used to force structures. There exist monochromatic arithmetic progressions of any length if we color the integers with a finite number of colors. Similarly there are arithmetic progressions of any length in any subset of the integers of positive density.

Getting finite, one can ask how large does \( n \) need to be for us to find a monochromatic progression of a desired length whenever \( \mathbb{Z}_n \) is colored by a fixed number of colors. A more difficult problem is to determine how many monochromatic progressions there are of desired length when the cyclic group \( \mathbb{Z}_n \) is colored by a fixed number of colors given \( n \).

Many results on monochromatic progressions are first obtained for cyclic groups and then extended to results for other groups. Many results can be extended to abelian groups using the same machinery as for the cyclic group, whereas results about non-abelian groups are usually very difficult and require a completely different set of tools.

In this paper we reformulate the problem of counting monochromatic arithmetic progressions to a problem in real algebraic geometry that can be attacked by state of the art optimization theory. The methods allows us to count monochromatic progressions in any finite group, including non-abelian groups.

In the next section we present our results. In later sections we present our methods and proofs.

2 Results

The main theorem of this paper holds for any finite group \( G \), including non-abelian groups for which very little is known about arithmetic progressions. The only information that is needed to get a lower bound for a specific group \( G \) is the number of elements of the different orders of \( G \). The lower bound is sharp for some groups, for example \( \mathbb{Z}_p \) with \( p \) prime, but is not optimal for most groups.

We have further included Table 1 where we have calculated the lower bound for some small symmetric groups.

**Theorem 2.1.** Let \( G \) be any finite group and let \( R(3, G, 2) \) denote the minimal number of monochromatic 3-term arithmetic progressions in any two-coloring of \( G \). Let \( G_k \) denote the set of elements of \( G \) of order \( k \), \( N = |G| \) and \( N_k = |G_k| \). Denote the Euler phi function \( \phi(k) = |\{ t \in \{1, \ldots, k\} : t \text{ and } k \text{ are coprime}\}| \). Let \( K = \{ k \in \{5, \ldots, n \} : \phi(k) \geq \frac{3k}{4}\} \). For any \( G \) there are \( \sum_{k=5}^{n} \frac{N \cdot N_k}{2} + \frac{N \cdot N_3}{24} \) arithmetic progressions of length 3. At least

\[
R(3, G, 2) \geq \sum_{k \in K} \frac{N \cdot N_k}{8} (1 - \frac{k - \phi(k)}{\phi(k)})
\]

of them are monochromatic in a 2-coloring of \( G \).
| Group G | Number of 3-APs | Lower bound for $R(3,G,2)$ |
|---------|----------------|-----------------------------|
| $S_5$   | 4540           | 90                          |
| $S_6$   | 205440         | 3240                        |
| $S_7$   | 11307660       | 306180                      |
| $S_8$   | 774278400      | 16208640                    |

Table 1: Calculation of $\sum_{k \in K} \frac{N_k \cdot N_{k}}{8} \left(1 - \frac{3^{k - \phi(k)}}{\phi(k)}\right)$ for small symmetric groups

3 Polynomial optimization

To obtain the main theorem we use methods from real algebraic geometry. In this article we only give the elementary definitions relating to polynomial optimization that are needed to prove the results. For a more extensive review of the topic we refer to [16].

Given polynomials $f(x), g_1(x), \ldots, g_m(x)$ we define a polynomial optimization problem to be a problem on the form

$$
\rho_\ast = \inf_x f(x)
\text{ subject to } g_1(x) \geq 0, \ldots, g_m(x) \geq 0,
$$

The problem can be reformulated as follows

$$
\rho_\ast = \sup_\lambda \lambda
\text{ subject to } f(x) - \lambda \geq 0, g_1(x) \geq 0, \ldots, g_m(x) \geq 0,
$$

where $f(x), g_1(x), \ldots, g_m(x)$ are the same polynomials as before and $\lambda \in \mathbb{R}$. We refer to the book by Lasserre [7] for an extensive discussion on the relationship between these problems. It is for example easy to see that $\rho_\ast = \rho_\ast$.

One of the most challenging problems in real algebraic geometry is to find the most useful relationships between nonnegative polynomials and sums of squares. These relationships are known as Positivstellensätze. Let

$$
\sigma_\ast = \sup_\lambda \lambda
\text{ subject to } f(x) - \lambda = \sigma_0 + \sum_{i=1}^m \sigma_i g_i,
$$

One can easily see that $\sigma_\ast \leq \rho_\ast$ holds, and under some technical conditions (Archimedean and compact) it was proven by Putinar that $\sigma_\ast = \rho_\ast$. The equality holds because of Putinar’s Positivstellensatz, which is discussed further in [16]. The optimization problem can be relaxed by bounding the degrees of all involved monomials to $d$, lets call the solution $\sigma^*_d$. It holds that $\sigma^*_d_1 \leq \sigma^*_d_2$ for $d_1 < d_2$, and it was proven by Lasserre [6] that if the Archimedean and compact conditions hold, then

$$
\lim_{d \to \infty} \sigma^*_d = \sigma^* = \rho^* = \rho_\ast.
$$
The positivity condition is rewritten as a sum of square condition because the latter is equivalent to a semidefinite condition: \( f(x) \) is a sum of squares of degree \( 2d \) if and only if \( f(x) = v_d^T Q v_d \) for some positive semidefinite matrix \( Q \), where \( v_d \) is the vector of all monomials up to degree \( d \). This makes it possible to use semidefinite programming to find \( \sigma_d^* \) as well as a sum of squares based certificate for the lower bound of our original polynomial, \( f(x) = \sigma_d^* + \sigma_0 + \sum_{i=1}^{m} \sigma_i g_i \geq \sigma_d^* \).

4 Exploiting symmetries in semidefinite programming

Let \( C \) and \( A_1, \ldots, A_m \) be real symmetric matrices and let \( b_1, \ldots, b_m \) be real numbers. To reduce the order of the matrices in the semidefinite programming problem

\[
\max \{ \mathrm{tr}(CX) \mid X \text{ positive semidefinite}, \mathrm{tr}(A_i X) = b_i \text{ for } i = 1, \ldots, m \}
\]

when it is invariant under all the actions of a group is the goal of this section.

As in [5] and [4], we use a \( * \)-representation to reduce the dimension of the problem. For the reader interested in \( * \)-algebras we recommend the book by Takesaki [17]. A collection of several new methods, including the one we use, to solve invariant semidefinite programs can be found in [1]. Other important recent contributions in this area include [3, 2, 18, 9, 10, 13].

**Definition 4.1.** A matrix \( * \)-algebra is a collection of matrices that is closed under addition, scalar multiplication, matrix multiplication and transposition.

Let \( G \) be a finite group that acts on a finite set \( Z \). Define a homomorphism \( h : G \to S_{|Z|} \), where \( S_{|Z|} \) is the group of all permutations of \( Z \). For every \( g \in G \) there is a permutation \( h_g = h(g) \) of the elements of \( Z \) with the properties \( h_{gg'} = h_g h_{g'} \) and \( h_{g^{-1}} = h_g^{-1} \). For every permutation \( h_g \), there is a corresponding permutation matrix \( M_g \in \{0,1\}^{|Z| \times |Z|} \), element-wise defined by

\[
(M_g)_{i,j} = \begin{cases} 
1 & \text{if } h_g(i) = j, \\
0 & \text{otherwise} 
\end{cases}
\]

for all \( i, j \in Z \). Let the span of these permutation matrices define the matrix \( * \)-algebra

\[
\mathcal{A} = \left\{ \sum_{g \in G} \lambda_g M_g \mid \lambda_g \in \mathbb{R} \right\}.
\]

The matrices \( X \) satisfying \( XM_g = M_g X \) for all \( g \in G \) are the matrices invariant under the action of \( G \). The \( * \)-algebra consisting of the collection of all such matrices,

\[
\mathcal{A}' = \{ X \in \mathbb{R}^{n \times n} \mid XM = MX \text{ for all } M \in \mathcal{A} \},
\]

is known as the commutant of \( \mathcal{A} \). We let \( d = \dim \mathcal{A}' \) denote the dimension of the commutant.
The commutant has a basis of \( \{0, 1\} \)-matrices, which we denote \( E_1, \ldots, E_d \), with the property that \( \sum_{i=1}^d E_i = J \), where \( J \) is the all-one matrix of size \( |Z| \times |Z| \).

We form a new basis for the commutant by normalizing the \( E_i \)s

\[
B_i = \frac{1}{\sqrt{\text{tr}(E_i^T E_i)}} E_i.
\]

The new basis has the property that \( \text{tr}(B_i^T B_j) = \delta_{i,j} \), where \( \delta_{i,j} \) is the Kronecker delta.

From the new basis we introduce multiplication parameters \( \lambda^k_{i,j} \) by

\[
B_i B_j = \sum_{k=1}^d \lambda^k_{i,j} B_k
\]

for \( i, j, k = 1, \ldots, d \).

We introduce a new set of matrices \( L_1, \ldots, L_d \) by

\[
(L_k)_{i,j} = \lambda^k_{i,j}
\]

for \( k, i, j = 1, \ldots, d \). The matrices \( L_1, \ldots, L_d \) are \( d \times d \) matrices that span the linear space

\[
\mathcal{L} = \{ \sum_{i=1}^d x_i L_i : x_1, \ldots, x_d \in \mathbb{R} \}.
\]

**Theorem 4.2** ([5]). The linear function \( \phi : A' \to \mathbb{R}^{d \times d} \) defined by \( \phi(B_i) = L_i \) for \( i = 1, \ldots, d \) is a bijection, which additionally satisfies \( \phi(XY) = \phi(X)\phi(Y) \) and \( \phi(X^T) = \phi(X)^T \) for all \( X, Y \in A' \).

**Corollary 4.3** ([5]). \( \sum_{i=1}^d x_i B_i \) is positive semidefinite if and only if \( \sum_{i=1}^d x_i L_i \) is positive semidefinite.

If it is possible to find a solution \( X \in A' \), then Corollary 4.3 can be used to reduce the size of the matrix in the linear matrix inequality. Let us show that this is possible:

**Lemma 4.4.** There is a solution \( X \in A' \) to a \( G \)-invariant semidefinite program

\[
\max \{ \text{tr}(CX) \mid X \text{ positive semidefinite, } \text{tr}(A_iX) = b_i \text{ for } i = 1, \ldots, m \}.
\]

**Proof.** Let \( C, A_1, \ldots, A_m \) be \( |Z| \times |Z| \) matrices commuting with \( M_g \) for all \( g \in G \). If \( X \) is an optimal solution to the optimization problem then the group average, \( X' = \frac{1}{|G|} \sum_{g \in G} M_g X M_g^T \), is also an optimal solution: It is feasible since

\[
\text{tr}(A_j X') = \text{tr}(A_j \left( \frac{1}{|G|} \sum_{g \in G} M_g X M_g^T \right)) = \text{tr}(\left( \frac{1}{|G|} \sum_{g \in G} M_g A_j X M_g^T \right)) = \text{tr}(\left( \frac{1}{|G|} \sum_{g \in G} A_j X \right)) = \text{tr}(A_j X),
\]

5
where we have used that the well-known fact that the trace is invariant under change of basis. By the same argument \( \text{tr}(CX') = \text{tr}(CX) \), which implies that \( X' \) is optimal. It is easy to see that \( X' \in A' \).

The following theorem follows, which gives a tremendous computational advantage when \( d \) is significantly smaller than \( |Z| \):

**Theorem 4.5** \((\text{5})\). The \( G \)-invariant semidefinite program

\[
\max \{ \text{tr}(CX) \mid X \succeq 0, \text{tr}(A_iX) = b_i \text{ for } i = 1, \ldots, m \}
\]

has a solution \( X = \sum_{i=1}^{d} x_i B_i \) that can be obtained by

\[
\max \{ \text{tr}(CX) \mid \sum_{i=1}^{d} x_i L_i \succeq 0, \text{tr}(A_i \sum_{j=1}^{d} B_j x_j) = b_i \text{ for } i = 1, \ldots, m \}.
\]

5 Problem formulated as a semidefinite program

Let \( \chi : G \to \{-1, 1\} \) be a 2-coloring of the finite group \( G \), and for simplicity let \( x_g = \chi(g) \) for all \( g \in G \). Furthermore, let \( x_G \) denote the vector of all variables \( x_g \). Let us introduce the polynomial

\[
p(x_a, x_b, x_c) = \frac{(x_a + 1)(x_b + 1)(x_c + 1) - (x_a - 1)(x_b - 1)(x_c - 1)}{8},
\]

where \( a, b, c \in G \), which has the property that

\[
p(x_a, x_b, x_c) = \begin{cases} 1 & \text{if } x_a = x_b = x_c \\ 0 & \text{otherwise.} \end{cases}
\]

The polynomial \( p \) is one when \( a, b, c \) are the same color and zero otherwise. It follows that

\[
R(3, G, 2) = \min_{x_G \in \{-1, 1\}^{|G|}} \sum_{\{a,b,c\} \text{ is an A.P. in } G} p(x_a, x_b, x_c).
\]

The integer problem is relaxed to a problem on the hypercube to obtain a lower bound for \( R(3, G, 2) \). Since any solution of the integer program is also a solution to the hypercube problem we have

\[
R(3, G, 2) \geq \min_{x_G \in [-1, 1]^{|G|}} \sum_{\{a,b,c\} \text{ is an A.P. in } G} p(x_a, x_b, x_c)
\]

\[
= \min_{x_G \in [-1, 1]^{|G|}} \sum_{\{a,b,c\} \text{ is an A.P. in } G} \frac{x_a x_b + x_a x_c + x_b x_c + 1}{4}
\]

\[
= \min_{x_G \in [-1, 1]^{|G|}} \frac{P_G}{4} + \sum_{\{a,b,c\} \text{ is an A.P. in } G} \frac{1}{4}
\]
We immediately get a lower bound for \( R(3, G, 2) \) by finding a lower bound for the homogeneous degree 2 polynomial \( p_G \). The coefficient of \( x_a x_b \) in \( p_G \) equals the number of times the pair \((a, b)\) occurs in a 3-arithmetic progression, which depends on the group \( G \). After the coefficients are found the state-of-the-art methods surveyed in Sections 3 and 4 are used to find a lower bound for \( \min_{G \in [-1,1]^G} p_G \).

Let us use the degree 3 relaxation of Putinar’s Positivstellensatz, and let \( \lambda^* \) denote the maximal lower bound using this relaxation. Denote the elements of \( G \) by \( g_1, \ldots, g_{|G|} \) and let \( v \) be the vector of all monomials of degree less or equal than one; \( v = [1, x_{g_1}, \ldots, x_{g_{|G|}}]^T \). We get

\[
\begin{align*}
\lambda^* = \max \lambda & \quad \text{subject to:} \\
& p_G - \lambda = v^T Q_0 v + \sum_{g \in G} v^T Q^+_g v(1 + x_g) + \sum_{g \in G} v^T Q^-_g v(1 - x_g), \\
x_G & \in [-1,1]^G \\
Q_0, Q^+_g, Q^-_g & \succeq 0 \text{ for all } g \in G.
\end{align*}
\]

For simplicity, let the \( Q \)-matrices entries be indexed by the set \( \{1, g_1, \ldots, g_{|G|}\} \). Arithmetic progressions are invariant under affine transformations: if \((a, b) \in G \rtimes \mathbb{Z}^*\), and \( \{i, j, k\} \) is an arithmetic progression, then \((a, b) \cdot \{i, j, k\} = \{a + bi, a + bj, a + bk\}\) is also an arithmetic progression (here + is the action of the group \( G \) and \( bi \) is the repeated action of the group \( bi = \sum_{j=1}^b i \)). It follows that the semidefinite program is invariant under affine transformations, and thus that we can find an invariant solution by Lemma 4.4. This implies that the degree 3 relaxation has a solution for which \( Q_0(g_i, g_j) = Q_0(a + bg_i, a + bg_j) \) and \( Q_0(1, g_i) = Q_0(g_i, 1) = Q_0(a + bg_i) \) for all \((a, b) \in G \rtimes \mathbb{Z}^*\) and \( g_i, g_j \in G \). Also \( Q^+_g(g_i, g_k) = Q^+_a(a + bg_i, a + bg_k) \) and \( Q^-_g(1, g_k) = Q^-_a(a + bg_k) \) and \( Q^+_g(1, 1) = Q^+_a(1, 1) \). By the same argument we get similar equalities for the indices of the matrices \( Q^-_g \). From the equalities it is easy to see that the matrices \( Q^+_g \) and \( Q^-_g \) are obtained by simultaneously permuting the rows and columns of \( Q_{g_i}^+ \) and \( Q_{g_i}^- \), respectively, and hence it is enough to require that \( Q^+_1 \) and \( Q^-_1 \) are positive semidefinite where \( g_1 \in G \) is the identity element.

In conclusion we see that the number of variables can be reduced significantly, and that only three \(|G| + 1 \times |G| + 1\)-matrices are required to be positive semidefinite instead of the \(2|G| + 1\) matrices required in the original formulation. The dimension of the commutant is small for some groups, and in those cases the size of these matrices can be reduced further using Theorem 4.5.

The certificates for the lower bounds we get from these methods are numerical, and to make them algebraic additional analysis of the numerical data, and possibly further restrictions, must be done. There is no general way to find algebraic certificates, and for many problem it might not even be possible. In this
paper it is a vital step to find as the numerical certificates only gives certificates for one group at the time whereas we need a certificate for all groups.

It is in general very difficult to go from numerical patterns to algebraic certificates. For a specific group all information required to find a lower bound can be found in an eigenvalue decomposition of the involved matrices, but there is no general way of finding the optimal algebraic lower bound when the different eigenvalues and eigenvectors have decimal expansions that cannot trivially be translated into algebraic numbers. If one is interested in a rational approximation to the lower bound one can use methods by Parrilo and Peyrl [11]. These methods gives a good certificate for a specific group but does not help when one want to find certificates for an infinite family of groups.

To find a certificate for all groups, one of the tricks we used was to restrict the SDP above by requiring that some further entries equal one another, in order to at least get a lower bound for $\lambda^*$. There are also many other ways to restrict the SDP further, including setting elements to zero. Restricting the SDP either decreases the optimal value or reduces the number of solutions to the original problem. In the ideal scenario one can keep restricting the SDP until there is only one solution, without any change in the optimal value.

6 Proof of Theorem 2.1

To make the computations in the proofs that follow readable we introduce additional notation:

$$\sigma(a; b_0, b_1, \ldots, b_{n-1}) = a + \sum_{i,j \in \mathbb{Z}_n} b_{j-i} x_i x_j.$$ 

By elementary calculations we have the following equalities, which are needed in the proofs:

$$I_1 = \sum_{i \in \mathbb{Z}_n} (1 - x_i^2) = \sigma(n; -1, 0, \ldots, 0),$$

$$I_2 = \left( \sum_{i \in \mathbb{Z}_n} x_i \right)^2 = \sigma(0; 1, 2, \ldots, 2),$$

$$I_3^j = \frac{1}{2} \sum_{i \in \mathbb{Z}_n} (x_i - x_{i+j})^2 = \sigma(0; 1, 0, \ldots, 0, -1, 0, \ldots, 0, -1, 0, \ldots, 0).$$

The first one is non-negative since we for the problem require that $-1 \leq x_i \leq 1$ and the other polynomials are non-negative since they are sums of squares.

**Proof of Theorem 2.1.** Let us count all arithmetic progressions $\{a, b \cdot a, b \cdot b \cdot a, \ldots\}$, with $(a, b) \in G \times G_k$. There is a cyclic subgroup of $G$ with elements $\{1, b, b^2, \ldots, b^{k-1}\}$. Let us write $k = p_1^{e_1} p_2^{e_2} \cdots p_s^{e_s}$ for $1 < p_1 < \cdots < p_s$ distinct primes and $e_1, \ldots, e_s$ positive integers. The elements $\{b^{p_i}, b^{2p_i}, \ldots, b^{(k-1)p_i}\}$ are of order less than $k$ for all $i$ whereas the elements $U = \{b^t : t \text{ and } k \text{ are coprime}\}$
are of order \( k \). Note that \( \phi(k) = |U| \). If \( k < 3 \) there are no arithmetic progressions, so in all the following calculations we will always assume that \( k \geq 3 \). If \( 3 \nmid k \), then all triples \( \{(a, b^i, b^{2i}) : b^i \in U\} \) will be distinct arithmetic progressions. Since there are \( \frac{Nk}{\phi(k)} \) different cyclic subgroups of \( G \) of this type there are \( \frac{Nk}{\phi(k)} \) arithmetic progressions of the form \( \{1, b^i, b^{2i}\} \), where \( b^i \) is an element of order \( k \). Since \( \{a, b \cdot a, b \cdot b \cdot a\} \) is an arithmetic progression if and only if \( \{1, b, b \cdot b\} \) is an arithmetic progression it follows that there are \( \frac{Nk}{\phi(k)} \) arithmetic progressions with \( (a, b) \in G \times G_k \) if \( 3 \nmid k \). If \( 3 \mid k \) we have to be careful so that we do not count arithmetic progressions of the form \( \{a, b^i \cdot a, b^{i+1} \cdot a\} \) three times. Since \( b^k \) will be of lower order than \( k \) if \( k > 3 \) it follows that this kind of arithmetic progressions only occur for \( (a, b) \in G \times G_k \) if \( k > 3 \), and \( \frac{Nk}{\phi(k)} \) if \( (a, b) \in G \times G_3 \).

The number of monochromatic arithmetic progressions is given by

\[
R(3, G, 2) = \sum_{\{a, b, c\} \text{ is an A.P. in } G} p(x_a, x_b, x_c) = \sum_{\{a, b, c\} \text{ is an A.P. in } G} \frac{x_a x_b + x_a x_c + x_b x_c + 1}{4}.
\]

Let us rewrite this as

\[
R(3, G, 2) = \frac{\sum_{k=1}^{n} \frac{Nk}{\phi(k)} + p_k}{4} + \frac{N \cdot N_3}{24} + \frac{p_3}{4}.
\]

where \( p_k \) is the sum of all polynomials \( x_s x_t x_s + x_s x_t x_s + x_t x_t x_s \) where \( t \) is of order \( k \). Let us also define the further reduced polynomial \( p_k^{(a, b)} \) for a fixed pair \( (a, b) \in G \times G_k \) by

\[
p_k^{(a, b)} = \sum_{0 \leq i < j \leq k-1} c_{b^i \cdot a, b^j \cdot a} x_{b^i \cdot a} x_{b^j \cdot a}
\]

where \( c_{b^i \cdot a, b^j \cdot a} \) denotes how many times the pair \( (b^i \cdot a, b^j \cdot a) \) is in an arithmetic progression \( \{s, t \cdot s, t \cdot t \cdot s\} \) with \( t \) of order \( k \). When \( a = 1 \) we will have a set \( \{b_i, \ldots, b_{N_k/\phi(k)}\} \) of elements of order \( k \) such that \( b_i^{tj} \neq b_i^{kj} \) for all \( j, k \in \{1, \ldots, N_k/\phi(k)\} \) and all \( c_1, c_2 \in \{1, \ldots, k-1\} \), and can write

\[
p_k = \frac{1}{k} \sum_{a \in G} \sum_{j=1}^{N_k/\phi(k)} p_k^{(a, b_{j+1})}
\]

It follows that if \( p_k^{(1, b)} \geq c \) for a \( b \) of order \( k \), then \( p_k \geq \frac{N_k}{\phi(k)} c \).

Let \( k \mod 2 = 1, k \mod 3 \neq 0 \): It is fairly easy to see that if \( b^i \) is of order \( k \), then so is \( b^{-i} \) and \( b^{2i} \). Also, if \( 2 \mid i \) then \( b^{i/2} \) is of order \( k \), if \( 2 \nmid i \) then \( b^{(i+1)/2} \) is of order \( k \). Hence any \( b^i \) is in both arithmetic progressions \( b^{-i}, 1, b^i \) and \( 1, b^i, b^{2i} \), and also in either \( 1, b^{i/2}, b^i \) or \( 1, b^{(i+1)/2}, b^i \). These are
all arithmetic progressions containing both 1 and \( b^i \). Note that the elements that are not of order \( k \) (apart from the identity element) are not in any of these arithmetic progressions, hence if \( k = p_1^{e_1} \cdots p_s^{e_s} \), then

\[
p_k^{(1,b)} = \sum_{i,j \in \mathbb{Z}_k} b_{j-i} x_i x_j = \sigma(0; b_0, b_1, \ldots, b_{k-1})
\]

where \( b_{tp_i} = 0 \) for \( t \in \{0, \ldots, k-1\} \) and \( i \in \{1, \ldots, s\} \), and \( b_k = 3 \) for all other \( t \). In particular we get

\[
p_k^{(1,b)} = \frac{3}{2} I_2 + \frac{3}{2} \sum_{t,i} I_{3t} b_{i} + \frac{3}{2} (1+k-\phi(k)-1) I_1 - \frac{3}{2} (k-\phi(k)) k \geq -\frac{3}{2} (k-\phi(k)) k,
\]

and it follows that

\[
p_k \geq -\frac{N \cdot N_k}{k} \frac{3}{2} (k-\phi(k)) k = -\frac{3 N \cdot N_k}{2} (k-\phi(k)) k,
\]

and furthermore that

\[\frac{N \cdot N_k}{2} + p_k \geq \frac{N \cdot N_k}{2} (1 - \frac{3}{2} \frac{k-\phi(k)}{\phi(k)})\]

In cases when \( 1 - \frac{3}{2} \frac{k-\phi(k)}{\phi(k)} < 0 \), that is when \( \phi(k) < \frac{4k}{3} \), we will instead use the trivial bound

\[\frac{N \cdot N_k}{2} + p_k \geq 0\]

**k mod 2 = 0:** \( b \in G_k \), and let us color all elements \( \{1, b^2, b^4, \ldots, b^{k-2}\} \) blue and \( \{b, b^3, b^5, \ldots, b^{k-1}\} \) red. In an arithmetic progression \( \{a, c \cdot a, c \cdot c \cdot a\} \) with \( (a, c) \in \{0, b, b^2, \ldots, b^{k-1}\} \times \{0, b, b^2, \ldots, b^{k-1}\} \cap G_k \) it holds that \( a \) and \( c \cdot a \) are of different colors. The coloring can be extended too all pairs \( (a, c) \in G \times G_k \), and thus there is a coloring without monochromatic arithmetic progressions. Hence we cannot hope to do better than the trivial bound using these methods:

\[\frac{N \cdot N_k}{2} + p_k \geq 0\]

**k mod 3 = 0:** Let us color the elements \( \{0, b, b^3, b^4, \ldots, b^{k-3}, b^{k-2}\} \) blue and \( \{b^2, b^5, b^8, \ldots, b^{k-1}\} \) red. As when \( k \) mod 2 = 0 we consider arithmetic progressions \( \{a, c \cdot a, c \cdot c \cdot a\} \) with \( (a, c) \in \{0, b, b^2, \ldots, b^{k-1}\} \times \{0, b, b^2, \ldots, b^{k-1}\} \cap G_k \). It is easy to see that either \( a \) and \( c \cdot a \) or \( a \) and \( c \cdot c \cdot a \) are of different colors. Again there is a coloring without monochromatic arithmetic progressions and so we cannot do better than the trivial bound:

\[\frac{N \cdot N_k}{2} + p_k \geq 0\]

for \( k > 3 \) and

\[\frac{N \cdot N_3}{6} + p_3 \geq 0\]
for \( k = 3 \).

Let \( K = \{ k \in \{ 5, \ldots, n \} : \varphi(k) \geq \frac{3k}{4} \} \). If 2 or 3 divide \( k \) then \( \varphi(k) < \frac{3k}{4} \), and hence none of those numbers are included in \( K \). Summing up all cases we get

\[
R(3, G, 2) \geq \sum_{k \in K} \frac{N \cdot N_k}{2} \left( 1 - \frac{3k - \varphi(k)}{\varphi(k)} \right).
\]

\[
\square
\]

7  Methods for longer arithmetic progressions

Let \( \chi : G \to \{ -1, 1 \} \) be a 2-coloring of the group \( G \), and let \( x_g = \chi(g) \) for all \( g \in G \). Let also \( x_G \) denote the vector of all variables \( x_g \). For \( a, b, c \in G \), let us introduce the polynomial

\[
p(a_1, \ldots, a_k) = \frac{(1 + x_{a_1}) \cdots (1 + x_{a_k}) + (1 - x_{a_1}) \cdots (1 - x_{a_k})}{2^k},
\]

which has the property that

\[
p(a_1, \ldots, a_k) = \begin{cases} 1 & \text{if } x_{a_1} = \cdots = x_{a_k} \\ 0 & \text{otherwise.} \end{cases}
\]

In other words, the polynomial \( p \) is one when \( \{a_1, \ldots, a_k\} \) is a monochromatic arithmetic progression and zero otherwise. It follows that

\[
R(k, G, 2) = \min_{x_G \in \{-1, 1\}^{|G|}} \sum_{\{a_1, \ldots, a_k\} \text{ is an A.P. in } G} p(a_1, \ldots, a_k).
\]

To find a lower bound for \( R(k, G, 2) \) we relax the integer quadratic optimization problem to an optimization problem on the hypercube. Since it is a relaxation, i.e. any solution of the integer program is also a solution to the hypercube problem, we have

\[
R(k, G, 2) \geq \min_{x_G \in [-1, 1]^{|G|}} \sum_{\{a_1, \ldots, a_k\} \text{ is an A.P. in } G} p(a_1, \ldots, a_k),
\]

an optimization problem that we can find lower bounds for using Putinar’s Positivstellensatz and the Lasserre Hierarchy.

8  Related problems

The methods developed in this article may also be applied in a wide variety of similar problems.

Let \( R(3, [n], 2) \) denote the minimal number of monochromatic arithmetic progressions of length 3 in a 2-coloring of \([n]\). Asymptotic bounds for \( R(3, [n], 2) \) have been found for large \( n \) using other methods [12]:

\[
\frac{1675}{32768} n^2 (1 + o(1)) \leq R(3, [n], 2) \leq \frac{117}{2192} n^2 (1 + o(1)).
\]
The lower bound can possibly be improved using the methods introduced in this article. The major challenge is the lack of symmetries in the problem.

Let \( R(3, \mathbb{Z}_n, 2) \) denote the minimal number of monochromatic arithmetic progressions of length 3 in a 2-coloring of the cyclic group \( \mathbb{Z}_n \). Optimal, or a constant from optimal, lower bounds for \( R(3, \mathbb{Z}_n, 2) \) have been found for all \( n \) [14]:

\[
n^2/8 - c_1 n + c_2 \leq R(3, \mathbb{Z}_n, 2) \leq n^2/8 - c_1 n + c_3,
\]

where the constants depends on the modular arithmetic and are tabulated in the following table.

| \( n \mod 24 \) | \( c_1 \) | \( c_2 \) | \( c_3 \) |
|-----------------|--------|--------|--------|
| 1, 5, 7, 11, 13, 17, 19, 23 | 1/2 | 3/8 | 3/8 |
| 8, 16 | 1 | 0 | 0 |
| 2, 10 | 1 | 3/2 | 3/2 |
| 4, 20 | 1 | 0 | 2 |
| 14, 22 | 1 | 3/2 | 3/2 |
| 3, 9, 15, 21 | 7/6 | 3/8 | 27/8 |
| 0 | 5/3 | 0 | 0 |
| 12 | 5/3 | 0 | 18 |
| 6, 18 | 5/3 | 1/2 | 27/2 |

A corollary is that we can find an optimal, or a constant from optimal, lower bound for the number of monochromatic arithmetic progressions for the dihedral group \( D_{2n} \) for any \( n \):

\[
R(3, D_{2n}, 2) = 2R(3, \mathbb{Z}_n, 2).
\]

In particular

\[
n^2/4 - 2c_1 n + 2c_2 \leq R(D_{2n}; 3) \leq n^2/4 - 2c_1 n + 2c_3
\]

where the constants can be found in the table above.

Let \( R(4, \mathbb{Z}_n, 2) \) denote the minimal number of monochromatic arithmetic progressions of length 4 in a 2-coloring of the cyclic group \( \mathbb{Z}_n \). Asymptotic bounds for \( R(4, \mathbb{Z}_n, 2) \) have been found for large \( n \) in [19], and the bounds have since then been improved in [8, Theorem 1.1, 1.2, 1.3] to:

\[
\frac{7}{192} p^2 (1 + o(1)) \leq R(4, \mathbb{Z}_p, 2) \leq \frac{17}{300} p^2 (1 + o(1))
\]

when \( p \) is prime, and for other \( n \)

\[
c_1 n^2 (1 + o(1)) \leq R(4, \mathbb{Z}_n, 2) \leq c_2 n^2 (1 + o(1))
\]

where the constants depends on the modular arithmetic on \( n \) in accordance with the following table.

| \( n \mod 4 \) | \( c_1 \) | \( c_2 \) |
|----------------|---------|---------|
| 1, 3 | 7/192 | 17/300 |
| 0 | 2/66 | 8543/1452000 |
| 2 | 7/192 | 8543/1452000 |
Furthermore [8, Theorem 1.5]

\[ \lim_{n \to \infty} R(4, \mathbb{Z}_n, 2) \leq \frac{1}{24}, \]

and it is conjectured that [8, Conjecture 1.1]:

\[ \inf_n \{ R(4, \mathbb{Z}_n, 2) \} = \frac{1}{24}. \]

It is possible to use the methods in this article to improve these results, and possibly give an affirmative answer to the conjecture. The challenges is that the solution seem to depend heavily on the modular arithmetic of \( n \), and so one needs to numerically find solutions for fairly high values of \( n \). This is difficult when one has to do a relaxation of degree at least 4 (and possibly even higher).

Let \( R(4, [n], 2) \) denote the minimal number of monochromatic arithmetic progressions of length 4 in a 2-coloring of \( [n] \). Let furthermore \( R(5, \mathbb{Z}_n, 2) \) and \( R(5, [n], 2) \) denote the minimal number of monochromatic arithmetic progressions of length 5 in \( \mathbb{Z}_n \) and \( [n] \) respectively. Upper bounds have been found for \( R(4, [n], 2), R(5, \mathbb{Z}_n, 2) \) and \( R(5, [n], 2) \). For \( n \) large enough [8, Equation (12)]:

\[ R(4, [n], 2) \leq \frac{1}{72} n^2 (1 + o(1)). \]

When \( n \) is odd and large enough we have [8, Theorem 1.4]:

\[ R(5, \mathbb{Z}_n, 2) \leq \frac{3629}{131424} n^2 (1 + o(1)). \]

When \( n \) is even and large enough we have [8, Theorem 1.4]:

\[ R(5, \mathbb{Z}_n, 2) \leq \frac{3647}{131424} n^2 (1 + o(1)). \]

Furthermore [8, Theorem 1.5]:

\[ \lim_{n \to \infty} R(5, \mathbb{Z}_n, 2) \leq \frac{1}{72}. \]

Finally when \( n \) is large enough we have [8, Equation (13)]:

\[ R(5, [n], 2) \leq \frac{1}{304} n^2 (1 + o(1)). \]

These are all upper bounds that have been obtained through good colorings. Lower bounds to all these problems can possibly be found using the methods developed in this article using a relaxation of high enough order.

Another type of problems that one can use the methods developed in this article to solve are enumeration problems in fixed density sets. As in this article it is of interest to count arithmetic progressions. Let \( W(k, S, \delta) \) denote the minimal number of arithmetic progressions of length \( k \) in any subset of \( S \) of
cardinality $|S|\delta$. One can for example let $S$ be a group or $[n]$. Note that if one could find strict lower bounds for $W(k,[n],\delta)$ for all $n$, $k$ and $\delta$ this would imply optimal quantitative bounds for Szemerédi’s theorem. Although it might be too ambitious to try to find strict lower bounds it might still be possible to find bounds good enough to generalize Szemerédi’s theorem. This kind of results can be obtained by methods very similar to the ones used in this article as discussed in [15].

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