An explicit upper bound of the argument of Dirichlet $L$-functions on the generalized Riemann hypothesis

Takahiro Wakasa *

Graduate School of Mathematics, Nagoya University, Chikusa-ku, Nagoya 464-8602, Japan;
e-mail: d11003j@math.nagoya-u.ac.jp

Abstract

We prove an explicit upper bound of the function $S(t, \chi)$, defined by the argument of Dirichlet $L$-functions attached to a primitive Dirichlet character $\chi \pmod{q > 1}$. An explicit upper bound of the function $S(t)$, defined by the argument of the Riemann zeta-function, have been obtained by A. Fujii [1]. Our result is obtained by applying the idea of Fujii’s result on $S(t)$. The constant part of the explicit upper bound of $S(t, \chi)$ in this paper does not depend on $\chi$.

1 Introduction

We consider the argument of Dirichlet $L$-functions. Let $L(s, \chi)$ be the Dirichlet $L$-function, where $s = \sigma + it$ is a complex variable, associated with a primitive Dirichlet character $\chi \pmod{q > 1}$. We denote the non-trivial zeros of $L(s, \chi)$ by $\rho(\chi) = \beta(\chi) + i\gamma(\chi)$, where $\beta(\chi)$ and $\gamma(\chi)$ are real numbers. Then, when $t \neq \gamma(\chi)$, we define

$$S(t, \chi) = \frac{1}{\pi} \arg L \left( \frac{1}{2} + it, \chi \right).$$

This is given by continuous variation along the straight line $s = \sigma + it$, as $\sigma$ varies from $+\infty$ to $\frac{1}{2}$, starting with the value zero. Also, when $t = \gamma(\chi)$, we define

$$S(t, \chi) = \frac{1}{2} \{ S(t + 0, \chi) + S(t - 0, \chi) \}.$$
In Selberg [2], it is known that

\[ S(t, \chi) = O(\log q(t + 1)) \]

and under the generalized Riemann hypothesis (GRH)

\[ S(t, \chi) = O\left( \frac{\log q(t + 1)}{\log \log q(t + 3)} \right). \]

The purpose of the present article is to prove the following result.

**Theorem 1.** Assuming GRH. Then,

\[ |S(t, \chi)| < 0.804 \cdot \frac{\log q(t + 1)}{\log \log q(t + 3)} + O\left( \frac{\log q(t + 3)}{\left(\log \log q(t + 3)\right)^2} \right). \]

The constant 0.804 obviously does not depend on \( \chi \). Also, the implied constant of the error term does not depend on \( q \). The details of the argument concerning error terms can be seen in the proof of this theorem. However, our result does not include the case of the function \( S(t) \) which is defined by the argument of the Riemann zeta-function since we assume \( q > 1 \). An explicit upper bound of the function \( S(t) \) is obtained by A. Fujii [1], where the value is 0.83.

The basic policy of the proof of this theorem is based on A. Fujii [1]. In the proof, \( S(t, \chi) \) is separated by three parts \( M_1, M_2 \) and \( M_3 \). Fujii’s idea of [1] is applied to all parts. But we need Lemma [1] which is an explicit formula for \( L'(s, \chi) \). This lemma is an analogue of Selberg’s result.

To prove our result, we introduce some notations and prove the aforementioned Lemma [1] in Section 2.

## 2 Some notations and a lemma

Here we introduce the following notations.

Let \( s = \sigma + it \). We suppose that \( \sigma \geq \frac{1}{2} \) and \( t \geq 2 \). Let \( x \) be a positive number satisfying \( 4 \leq x \leq t^2 \). Also, we put

\[ \sigma_1 = \frac{1}{2} + \frac{1}{\log x} \]

and

\[ \Lambda_x(n) = \begin{cases} 
\Lambda(n) & \text{for } 1 \leq n \leq x, \\
\Lambda(n) \frac{x^2}{\log x} & \text{for } x \leq n \leq x^2,
\end{cases} \]

with

\[ \Lambda(n) = \begin{cases} 
\log p & \text{if } n = p^k \text{ with a prime } p \text{ and an integer } k \geq 1, \\
0 & \text{otherwise.}
\end{cases} \]

Using these notations, we prove the following lemma.
Lemma 1. Assume the GRH. Let \( t \geq 2 \) and \( x > 0 \) such that \( 4 \leq x \leq t^2 \). Then for \( \sigma \geq \sigma_1 = \frac{1}{2} + \frac{1}{\log x} \) we have

\[
\frac{L'(s)}{L(s)}(\sigma, \cdot, \chi) = -\sum_{n < x^2} \frac{\Lambda_x(n)}{n^\sigma} \chi(n) - \frac{x^{\frac{1}{2}-\sigma}}{1 - \frac{1}{e} (1 + \frac{1}{e} \omega')} \Re \left( \sum_{n < x^2} \frac{\Lambda_x(n)}{n^{\sigma+it}} \chi(n) \right) + \frac{x^{\frac{1}{2}-\sigma}}{1 - \frac{1}{e} (1 + \frac{1}{e} \omega')} \cdot \frac{1}{2} \log q(t+1) + O(x^{\frac{1}{2}-\sigma}),
\]

where \( |\omega| \leq 1 \) and \( -1 \leq \omega' \leq 1 \).

This is an analogue of Lemma 2 of A. Fujii [1].

Lemma 2. Let \( a = 0 \) if \( \chi(-1) = 1 \), and \( a = 1 \) if \( \chi(-1) = -1 \). Then, for \( x > 1 \), \( s \neq -2q - a \) (\( q = 0, 1, 2, \cdots \)) and \( s \neq \rho(\chi) \), we have

\[
\frac{L'(s)}{L(s)}(\sigma, \cdot, \chi) = -\sum_{n < x^2} \frac{\Lambda_x(n)}{n^{\sigma}} \chi(n) + \frac{1}{\log x} \sum_{q=0}^{\infty} \int_{-\infty}^{a+\infty} \frac{x^{-2q-a-s} - x^{-2(2q+a+s)}}{(2q+a+s)^2} dz.
\]

Lemma 2 is similar to Lemma 15 of Selberg [2]. We write here only a sketch of the proof of Lemma 2.

If \( a = \max(1, \sigma) \), we have

\[
\sum_{n < x^2} \frac{\Lambda_x(n)}{n^{\sigma}} \chi(n) = \frac{1}{2 \pi i} \log x \int_{a-\infty i}^{a+\infty i} \frac{x^{z-s} - x^{2(z-s)}}{(z-s)^2} \cdot \frac{L'(s)}{L(s)}(z, \chi) dz.
\]

We consider residues which we encounter when we move the path of integration to the left. At the point \( z = s \), the residue is \( -\log x \frac{L'(s)}{L(s)}(s, \chi) \). At the zeros \( -2q - a \) (\( q = 0, 1, 2, \cdots \)), the residues are \( x^{-2q-a-s} - x^{-2(2q+a+s)} \) \( (2q+a+s)^2 \). At the zeros \( s = \rho \) of \( L(s, \chi) \), the residues are \( x^{\rho-s} - x^{2(\rho-s)} \) \( (s-\rho)^2 \). Thus, we obtain Lemma 2.

Proof of Lemma 7 Assume the GRH. In Lemma 2, since for \( \sigma \geq \sigma_1 = \frac{1}{2} + \frac{1}{\log x} \)

\[
\left| \frac{1}{\log x} \sum_{\rho} \frac{x^{\rho-s} - x^{2(\rho-s)}}{(s-\rho)^2} \right| = \frac{1}{\log x} \left| \sum_{\gamma} \frac{x^{\frac{1}{2}-\sigma}}{(\sigma-\frac{1}{2})^2 + (t-\gamma)^2} \right| \leq \frac{x^{\frac{1}{2}-\sigma}}{\log x} \sum_{\gamma} \frac{1 + x^{\frac{1}{2}-\sigma}}{(\sigma-\frac{1}{2})^2 + (t-\gamma)^2} \leq x^{\frac{1}{2}-\sigma} \left( 1 + x^{\frac{1}{2}-\sigma} \right) \sum_{\gamma} \frac{\sigma_1 - \frac{1}{2}}{(\sigma_1 - \frac{1}{2})^2 + (t-\gamma)^2}.
\]
we have

\[
\frac{1}{\log x} \sum_{\rho} \frac{x^{\rho-s} - x^{2(\rho-s)}}{(s - \rho)^2} = x^{\frac{1}{2} - \sigma} \left(1 + x^{\frac{1}{2} - \sigma}\right) \sum_{\gamma} \frac{\sigma_1 - \frac{1}{2}}{(\sigma_1 - \frac{1}{2})^2 + (t - \gamma)^2},
\]

where \(|\omega| \leq 1\). Hence by Lemma 2, we have for \(\sigma \geq \sigma_1\)

\[
\frac{L'}{L}(\sigma + it, \chi) = -\sum_{n < x^2} \frac{\Lambda_x(n)}{n^{\sigma_1 + it}} \chi(n) + O \left(\frac{x^{\frac{1}{2} - \sigma}}{\log x}\right)
\]

\[
+ x^{\frac{1}{2} - \sigma} \left(1 + x^{\frac{1}{2} - \sigma}\right) \sum_{\gamma} \frac{\sigma_1 - \frac{1}{2}}{(\sigma_1 - \frac{1}{2})^2 + (t - \gamma)^2}.
\]

(1)

In particular, by \(x^{\frac{1}{2} - \sigma} \leq x^{-\frac{\omega'}{2}} = \frac{1}{e}\) we get for \(\sigma \geq \sigma_1\)

\[
\mathcal{R} \frac{L'}{L}(\sigma_1 + it, \chi) = -\mathcal{R} \left(\sum_{n < x^2} \frac{\Lambda_x(n)}{n^{\sigma_1 + it}} \chi(n)\right) + O \left(\frac{1}{\log x}\right)
\]

\[
+ \frac{1}{e} \left(1 + \frac{1}{e}\right) \omega' \sum_{\gamma} \frac{\sigma_1 - \frac{1}{2}}{(\sigma_1 - \frac{1}{2})^2 + (t - \gamma)^2}.
\]

(2)

where \(-1 \leq \omega' \leq 1\).

Here, since by p. 46 of Selberg [2]

\[
\mathcal{R} \frac{L'}{L}(s, \chi) = \mathcal{R} \left(-\frac{1}{2} \log \frac{q}{\pi} - \frac{1}{2} \log \left(s + \frac{a}{2}\right)\right) + \sum_{\gamma} \frac{\sigma - \frac{1}{2}}{(\sigma - \frac{1}{2})^2 + (t - \gamma)^2} + O(1),
\]

we get for \(t \geq 2\)

\[
\mathcal{R} \frac{L'}{L}(\sigma_1 + it, \chi) = -\frac{1}{2} \log q(t + 1) + \sum_{\gamma} \frac{\sigma_1 - \frac{1}{2}}{(\sigma_1 - \frac{1}{2})^2 + (t - \gamma)^2} + O(1).
\]

(3)

By [2] and [3] we have

\[
\left(1 - \frac{1}{e}\right) \left(1 + \frac{1}{e}\right) \omega' \sum_{\gamma} \frac{\sigma_1 - \frac{1}{2}}{(\sigma_1 - \frac{1}{2})^2 + (t - \gamma)^2}
\]

\[
= -\mathcal{R} \left(\sum_{n < x^2} \frac{\Lambda_x(n)}{n^{\sigma_1 + it}} \chi(n)\right) + \frac{1}{2} \log q(t + 1) + O \left(\frac{1}{\log x}\right) + O(1).
\]

Inserting the above inequality to (1), we obtain Lemma 1.

\[\square\]

3 Proof of Theorem 1

The quantity \(S(t, \chi)\) is separated into the following three parts.

\[
S(t, \chi) = \frac{1}{\pi} \int_{\sigma_1}^{\infty} \mathcal{R} \frac{L'}{L}(\sigma + it, \chi) d\sigma
\]
say. Here, 

\[
\eta(M) = \frac{1}{\pi} \left\{ \Im \int_{\sigma_1}^{\infty} \frac{L'(\sigma + it, \chi)}{L(\sigma + it, \chi)} d\sigma + \Im \left\{ \left( \sigma_1 - \frac{1}{2} \right) \frac{L'(\sigma_1 + it, \chi)}{L(\sigma_1 + it, \chi)} \right\} \right. 
- \left. \Im \int_{\sigma_1}^{\sigma_1 + \frac{1}{2}} \left\{ \frac{L'(\sigma_1 + it, \chi)}{L(\sigma + it, \chi)} - \frac{L'(\sigma + it, \chi)}{L(\sigma + it, \chi)} \right\} d\sigma \right\} 
= \frac{1}{\pi} \Im (M_1 + M_2 + M_3), 
\]

First, we estimate \( M_1 \). By Lemma 1 we have

\[
M_1 = \int_{\sigma_1}^{\infty} \left\{ - \sum_{n < x^2} \frac{A_n(x)}{n^{\sigma+i\tau}} \chi(n) - \frac{1}{1 - \frac{1}{c}} \log x \sigma \sum_{n < x^2} \frac{A_n(x)}{n^{\sigma+i\tau}} \chi(n) \right\} d\sigma
\]

say.

\[
|\eta_1(t)| = \left| \int_{\sigma_1}^{\infty} \Re \omega \left( 1 + x^{\frac{1}{2} - \sigma} \right) x^{\frac{1}{2} - \sigma} d\sigma \right| \cdot \left| \Re \left( \sum_{n < x^2} \frac{A_n(x)}{n^{\sigma+i\tau}} \chi(n) \right) - \frac{1}{2} \log q(t + 1) \right|
+ O \left( \int_{\sigma_1}^{\infty} x^{\frac{1}{2} - \sigma} d\sigma \right)
\]

say. Here,

\[
|\eta_1(t)| \leq \left| \int_{\sigma_1}^{\infty} \Re \left( \sum_{n < x^2} \frac{A_n(x)}{n^{\sigma+i\tau}} \chi(n) \right) - \frac{1}{2} \log q(t + 1) \right|
\]

Next, applying Lemma 1 to \( M_2 \), we get

\[
|M_2| = \left| \frac{1}{\log x} \left\{ - \sum_{n < x^2} \frac{A_n(x)}{n^{\sigma+i\tau}} \chi(n) - \frac{1}{1 - \frac{1}{c}} \log x \sigma \sum_{n < x^2} \frac{A_n(x)}{n^{\sigma+i\tau}} \chi(n) \right\}
\right|
\]

say.
say. Hence we have

\[ \text{say.} \]

Next we estimate \( M_3 \). By Lemma 16 of Selberg [2] we get

\[ |\Im(M_3)| = \left| \int_{\frac{1}{2}}^{\sigma_1} \Im \left\{ \sum_{\rho} \frac{1}{\sigma_1 + it - \rho} - \sum_{\rho} \frac{1}{\sigma + it - \rho} + O(1) \right\} \, d\sigma \right| \]

\[ \leq \int_{\frac{1}{2}}^{\sigma_1} \sum_{\gamma} \frac{(t - \gamma) \left\{ (\sigma - \frac{1}{2})^2 - (\sigma_1 - \frac{1}{2})^2 \right\} \left\{ (\sigma - \frac{1}{2})^2 + (t - \gamma)^2 \right\}}{(\sigma - \frac{1}{2})^2 + (t - \gamma)^2} \, d\sigma + O \left( \frac{1}{\log x} \right) \]

\[ = N(\gamma) + O \left( \frac{1}{\log x} \right), \]

say. If \( t = \gamma \), we see \( N(\gamma) = 0 \) easily. If \( t \neq \gamma \), we have

\[ N(\gamma) \leq \sum_{\gamma} \frac{(\sigma_1 - \frac{1}{2})^2}{(\sigma - \frac{1}{2})^2 + (t - \gamma)^2} \int_{\frac{1}{2}}^{\sigma_1} \frac{|t - \gamma|}{(\sigma - \frac{1}{2})^2 + (t - \gamma)^2} \, d\sigma \]

\[ \leq \sum_{\gamma} \frac{(\sigma_1 - \frac{1}{2})^2}{(\sigma - \frac{1}{2})^2 + (t - \gamma)^2} \int_{\frac{1}{2}}^{\infty} \frac{|t - \gamma|}{(\sigma - \frac{1}{2})^2 + (t - \gamma)^2} \, d\sigma \]

\[ \leq \frac{\pi}{2 \log x} \sum_{\gamma} \frac{\sigma_1 - \frac{1}{2}}{(\sigma_1 - \frac{1}{2})^2 + (t - \gamma)^2} \]

since \( \sigma < \sigma_1 \) for \( M_3 \).

Here, by (2) and (3) we get

\[ \sum_{\gamma} \frac{\sigma_1 - \frac{1}{2}}{(\sigma_1 - \frac{1}{2})^2 + (t - \gamma)^2} = \frac{1}{1 - \frac{1}{e} \left( 1 + \frac{1}{e} \right) \omega'} \cdot \frac{1}{2} \log q(t + 1) \]

\[ + O \left( \left| \sum_{n<x^2} \frac{\Lambda_n(x)}{n^{\sigma_1 + it} \chi(n)} \right| \right) + O \left( \frac{1}{(\log x)^2} \right). \]

So,

\[ N(\gamma) = \frac{\pi}{4} \cdot \frac{1}{1 - \frac{1}{e} \left( 1 + \frac{1}{e} \right) \omega'} \cdot \frac{1}{\log x} \cdot \log q(t + 1) \]

\[ + O \left( \frac{1}{\log x} \left| \sum_{n<x^2} \frac{\Lambda_n(x)}{n^{\sigma_1 + it} \chi(n)} \right| \right) + O \left( \frac{1}{(\log x)^3} \right). \]

Hence we have

\[ |\Im(M_3)| \leq \frac{\pi}{4} \cdot \frac{1}{1 - \frac{1}{e} \left( 1 + \frac{1}{e} \right) \omega'} \cdot \frac{1}{\log x} \cdot \log q(t + 1) \]

6
\[ + O \left( \frac{1}{\log x} \left| \sum_{n<x^{1+\eta}} \Lambda_n(x) \chi(n) \right| \right) + O \left( \frac{1}{\log x} \right) \]

\[ = \eta_4(t) + O \left( \frac{1}{\log x} \left| \sum_{n<x^{1+\eta}} \Lambda_n(x) \chi(n) \right| \right) + O \left( \frac{1}{\log x} \right), \quad (7) \]

say.

Finally, we estimate the sums on right-hand sides of (4), (5), (6) and (7).

By definition of \( \Lambda_x(n) \) we have

\[ \left| \sum_{n<x} \frac{\Lambda_x(n)}{n^{\sigma_1+\eta} \log n} \chi(n) \right| \leq \sum_{n<x} \frac{\Lambda(n)}{n^{\sigma_1+\eta}} + \sum_{x \leq n \leq x^{1+\eta}} \frac{\Lambda(n) \log x^2}{n^{\sigma_1+\eta}} \cdot \frac{1}{\log x} \ll \frac{x}{\log x}. \]

Similarly,

\[ \left| \sum_{n<x^{1+\eta}} \frac{\Lambda_x(n)}{n^{\sigma_1+\eta} \log x} \chi(n) \right| \ll \frac{x}{(\log x)^2}. \]

So, we see

\[ |M_1| \leq \frac{\left( \frac{1}{e} + \frac{1}{2e} \right)}{1 - \frac{1}{e}(1 + \frac{1}{e})} \cdot \frac{1}{2} \cdot \log q(t+1) \frac{x}{\log x} + O \left( \frac{x}{(\log x)^2} \right), \]

\[ |M_2| \leq \frac{\left( \frac{1}{e} + \frac{1}{\epsilon} \right)}{1 - \frac{1}{e}(1 + \frac{1}{e})} \cdot \frac{1}{2} \cdot \log q(t+1) \frac{x}{\log x} + O \left( \frac{x}{(\log x)^2} \right), \]

and

\[ |M_3| \leq \eta_4(t) + O \left( \frac{x}{(\log x)^2} \right). \]

For \( \eta_1(t), \eta_2(t), \eta_3(t) \) and \( \eta_4(t) \), taking \( x = \log q(t+3) \sqrt{\log q(t+3)} \) we have

\[ |S(t, \chi)| < \frac{1}{\pi} \cdot \frac{1}{1 - \frac{1}{e}(1 + \frac{1}{e})} \left\{ \left( \frac{\left( \frac{1}{e} + \frac{1}{2e} \right)}{2} + \frac{\left( \frac{1}{e} + \frac{1}{\epsilon} \right)}{2} + \frac{\pi}{4} \right) \frac{\log q(t+1)}{\log x} \right\} \]

\[ + O \left( \frac{x}{(\log x)^2} \right) \]

\[ = 0.803986 \cdots \frac{\log q(t+1)}{\log \log q(t+3)} + O \left( \frac{\log q(t+3)}{(\log \log q(t+3))^2} \right). \]

Therefore we obtain the theorem. \( \square \)

**Acknowledgments**

I thank Prof. Kohji Matsumoto for his advice and patience during the preparation of this paper. I also thank Prof. Giuseppe Molteni, Prof. Yumiko Umegaki and Dr. Ryo Tanaka, who gave many important advice.
References

[1] A. Fujii, An explicit estimate in the theory of the distribution of the zeros of the Riemann zeta function, Comment. Math. Univ. Sancti Pauli, 53, (2004), 85-114.

[2] A. Selberg, Contributions to the theory of Dirichlet’s L-function, Avh. Norske Vir. Akad. Oslo I:1, (1946), No. 3, 1-62.

[3] A. Selberg, Collected Works, vol I, 1989, Springer.

[4] E. C. Titchmarsh, The theory of the Riemann zeta-function, Second Edition; Revised by D. R. Heath-Brown. Clarendon Press Oxford, 1986.