Importance of fluctuations of cross sections in muon-catalysed $t$-$t$ fusion reactions

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Abstract

We discuss the reaction rate of the muon-catalysed $t$-$t$ fusion. The reaction rate is determined as a function of the temperature using the model of “in flight” fusion. We especially take into account the effect of the fluctuation of the cross section due to the existence of the muon. The obtained reaction rate $5.0 \times 10^{-3}$ $\mu s^{-1}$ is a factor of $10^{-3}$ smaller than the experimental muonic cycling rate in the solid tritium target.

1 Introduction

In the liquid hydrogen isotopes mixture, muons assist the fusion through the formation of a muonic molecule, since the size of the muonic molecule is much smaller than that of the ordinary molecules and the fusing nuclei tend to stay closer. This whole process takes place at the thermal energies where the conventional measurements of the fusion cross section using a charged beam cannot be performed. This mechanism of the muon catalyzed fusion ($\mu$CF) might provide us an unique opportunity to investigate, in a rather direct way, the fusion cross section, i.e., the astrophysical $S$-factor, at extremely low energy. For this purpose we need to know quantitatively the effect that a muon shields the Coulomb potential between colliding nuclei [1]. And then this shielding effects of muons should be removed from the $S$-factor data, in order to asses the bare reaction rate correctly. From this point of view, the $t$-$t\mu$ fusion could provide us an elucidating example. It has been investigated experimentally in the gas, liquid [2] and solid [3] targets. Especially in the latter experiments the fusion neutron energy spectrum has been determined and its distribution suggests that the fusion is followed by a sequential decay of $^5\text{He}$:

$$t + t + \mu \rightarrow ^5\text{He}^* + n \rightarrow \alpha + n + n + Q(11.33\text{MeV}),$$

(1)

where $^5\text{He}^*$ is in the $3/2^-$ and $1/2^-$ resonant states. The $t$-$t$ fusion with muons has not received much attention as a candidate of an energy source in contrast to the $d$-$t$ and the $d$-$d$ $\mu$CF. This is partly because of the difficulty of tritium handling. Moreover it is because its reaction rate is expected to be much lower than the others due to the lack of the resonant muonic complex formation. In fact the cycling rate obtained experimentally: $3.3 \pm 0.7$ $\mu s^{-1}$ [3] ($15 \mu s^{-1}$ [4, 5]) of the $t$-$t\mu$ is smaller than that of the $d$-$t\mu$ of the order of 100. Another distinctive difference of the $d$-$t\mu$ and the $d$-$d$ $\mu$ reactions from the $t$-$t\mu$ reaction is that their cycling rates has target density and temperature effects, which are likely caused by 3-body collisions. Put another way, if the cycling rate of the $t$-$t\mu$ reaction
does not have the target density and temperature dependences, one can verify that the dependences originate from the formation of the resonant muonic complex.

On the other hand, the cross section of the reaction $^3\text{H}(^3\text{H},2p)^4\text{He}$ has been measured in the triton beam energy range $E_{\text{lab}} = 30-300$ (keV) \cite{6,7,8,9}. This energy range is much higher than thermal energies. The astrophysical $S$-factor of the reaction has been studied theoretically by means of DWBA \cite{10} and the generator coordinate method \cite{11}.

We determine the reaction rate of the $t$-$t\mu$CF by considering so-called “in flight” fusion \cite{12} and compare it with the experimental muon cycling rate. At thermal energies, where the $\mu$CF takes place, fluctuations of the cross section might play an important role. We investigate the influence of the fluctuations by using a semi-classical method, the constrained molecular dynamics (CoMD) approach \cite{13,14}. The molecular dynamics contains all possible correlations and fluctuations due to the initial conditions(events). In the CoMD, the constraints restrict the phase space configuration of the muon to fulfill the Heisenberg uncertainty principle. The results are given as an average and a variance over ensembles of the quantity of interest, which is determined in the simulation. We make use of the average and the variance of the enhancement factor of the cross section by the muon, that have been obtained in simulations of the $d$-$t\mu$CF \cite{1}, and convert them into the average and the variance of the effective potential shift. If so, one can use the same potential shift for the case of the $t$-$t\mu$CF, because of the isotope independence of the screening effect. We, thus, determine the reaction rate of the $t$-$t\mu$CF as a function of the temperature, taking into account the effect of the fluctuation of the cross section by the presence of the muon.

This paper is organized as follows. In Sec. 2 we derive the variance of the effective potential shift of the reactions between hydrogen isotopes by the muons. The reaction rate as a function of the temperature for the $t$-$t\mu$CF is determined in Sec. 3. We discuss possible origins of the discrepancies between the experimental muon cycling rate and the obtained reaction rate in Sec. 4. In Sec. 5 we summarize the paper.

2 Reaction cross sections in the presence of muons

The influence of the muonic degrees of freedom to the reaction cross section can be taken into account as an enhancement by the screening effect \cite{15,16}. We determine the enhancement factor at the incident center-of-mass(c.m.) energy $E$ as

$$f_\mu = \frac{\sigma(E)}{\sigma_0(E)},$$

where $\sigma(E)$, and $\sigma_0(E)$ are the cross sections in the presence, and in the absence, respectively, of the muon. The $\sigma(E)$ fluctuates depending on the dynamics of the 3(or N)-body system \cite{1}. This fluctuation of the cross section can be written in terms of the fluctuation of the enhancement factor: $\Delta f_\mu$ as

$$\Delta \sigma(E) = \sigma_0(E) \Delta f_\mu,$$

where we have assumed that the fluctuation of the bare cross section is negligible at low temperature compared to the screened one. Through the molecular dynamics simulation we obtain this fluctuation as a variance of the enhancement factor. At the same time we determine the average enhancement factor: $\bar{f}_\mu$. In our previous study we have simulated

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this enhancement of the cross section by the muon in the case of the \(d-t\) \(\mu\)CF reaction \[1\]. We utilize the results from this simulation for the case of \(t-t\) \(\mu\)CF reaction. In the left-top panel of Fig. 1, the ratio \(\Delta f_{\mu}/\bar{f}_{\mu}\) is shown as a function of the incident c.m. energy. In the high energy limit the ratio approaches zero, i.e., the \(f_{\mu}\) distribution becomes a \(\delta\)-function (\(\Delta f_{\mu} = 0\)) and the average \(f_{\mu}\) approaches 1; there is no effective enhancement. In the low energy limit the ratio \(\Delta f_{\mu}/\bar{f}_{\mu}\) is much larger than 1; this fact implies that the system exhibits a sensitive dependence of the dynamics on the initial conditions, i.e., the muonic motion becomes chaotic. The energy dependence of the ratio \(\Delta f_{\mu}/\bar{f}_{\mu}\) is approximated well by the function \(2.05 \times E_{\text{inc}}^{-0.52}\), where \(E_{\text{inc}}\) is in units of keV. This curve is shown by the dotted line in the left-top panel in Fig. 1.

We write down the enhancement factor in terms of a constant shift of the potential barrier. Here we have assumed that the enhancement is represented in terms of a constant shift of the potential barrier due to the presence of the muon. The average \(\bar{f}_{\mu}\) for the \(dt\mu\) is in good agreement with the exact adiabatic limit with the screening potential:

\[
U_{\mu}^{(AD)} = BE_t - BE_{5\text{He}} \sim 8.3\text{keV},
\]

where \(BE_t\) and \(BE_{5\text{He}}\) are the binding energies of muonic tritium and muonic \(5\text{He}\) ion, respectively \[1\]. We, therefore, assume that the average potential shift \(\bar{U}_{\mu}\) is equivalent to \(U_{\mu}^{(AD)}\). From the commonly used expression of the bare cross section:

\[
\sigma_0(E) = \frac{S(E)}{E} e^{-2\pi\eta(E)},
\]
where $S(E)$ and $\eta(E)$ are the astrophysical $S$-factor and Sommerfeld parameter. The cross section in the presence of the muon is expressed by

$$\sigma(E) = \frac{S(E)}{E + U_\mu} e^{-2\pi\eta(E+U_\mu)}, \quad (6)$$

in terms of the potential shift $U_\mu$. Taking the derivative of the potential shift:

$$\frac{\Delta \sigma}{\Delta U_\mu} = \frac{\sigma}{E + U_\mu} (\pi\eta(E + U_\mu) - 1). \quad (7)$$

Hereafter we substitute the potential shift $U_\mu$ in Eq. (7) by its average $\bar{U}_\mu$ and thus

$$\frac{\Delta \sigma}{\bar{f}_\mu} = \frac{\Delta U_\mu}{E + \bar{U}_\mu} (\pi\eta(E + \bar{U}_\mu) - 1). \quad (8)$$

One can deduce the average potential shift $\bar{U}_\mu$, and its fluctuation $\Delta U_\mu$ from the ratio $\Delta f_\mu/\bar{f}_\mu$.

$$\frac{\Delta U_\mu}{\bar{U}_\mu} = \frac{\Delta f_\mu}{f_\mu} \frac{E + \bar{U}_\mu}{(\pi\eta(E + \bar{U}_\mu) - 1)\bar{U}_\mu} \quad (9)$$

This potential shift is independent of the difference of isotopes, so that we make use of the same value in both cases of the $d$-$t$ and the $t$-$t$ $\mu$CF.

In the left-bottom panel of Fig. 1 the fluctuation of the potential shift divided by the average potential shift is shown as a function of the incident energy. It is striking that the slope of the ratio $\Delta U_\mu/\bar{U}_\mu$ changes at the ionization energy of the muonic tritium, which we indicate by the arrow in the figure. At this incident energy the total energy of the system is zero. The total system is unbound at the incident energies higher than this point, while the 3-body system might be bound at lower energies. By contrast, as it is shown in the right-bottom panel in Fig. 1 in the case of the bound electron screening the binding energy of the electron is much lower than the incident energy of our interest and the ratio $\Delta U_e/\bar{U}_e$ decrease monotonically as the incident energy decreases. Again the arrow indicates the ionization energy of the deuterium atom. The irregular muonic motion leads to smaller external classical turning point [1]. As a consequence the irregularity makes the enhancement factor larger opposed to the result of the electron screening [15, 16], where the irregular(chaotic) events give smaller enhancement factors. This contradiction is accounted for the fact that the system remains bound in the present case at low incident energies, while in the previous case even the lowest incident energy that was investigated is much higher than the binding energy of the electrons. Therefore the chaotic dynamics of the electrons causes to dissipate the kinetic energy between the target and the projectile and lowers the probability of fusion.

3 Reaction rate

In a liquid hydrogen tritium target at the temperature $T$, the velocity distribution, $\phi(v)$, of a pair of colliding particles is written as the Maxwellian distribution,

$$\Psi(E,T) dE = \phi(v,T) dv = \frac{2}{\sqrt{\pi} k_B T} e^{-\frac{v}{k_B T}} \frac{dE}{\sqrt{k_B T E}}, \quad (10)$$
where $E$ and $v$ are the relative energy and the velocity of the pair of colliding particles, in the present case two tritons, and $k_B$ is Boltzmann constant. Although the experiment in [3] has been performed using the solid target, in this paper we assume Eq. (10) as the velocity distribution for simplicity. We will reconsider the validity of this assumption afterwards. The reaction rate per pair of particles is given by [17]

$$< \sigma v > = \int \sigma(E)v\Psi(E,T)dE,$$

(11)

where $\sigma(E)$ is the reaction cross section in the presence of the muon. The reaction rate at the liquid hydrogen density $\rho_{LH} = 4.25 \times 10^{22} \text{cm}^{-2}$:

$$\lambda = \rho_{LH} < \sigma v >$$

(12)

is obtained as a function of the temperature.

As we have seen in Sec. 2, the cross section in the presence of the muon is expressed effectively with a potential shift $U_\mu$, hence we write down the reaction rate:

$$\lambda = \rho_{LH} \int \sigma_0(E + U_\mu)v\Psi(E,T)dE,$$

(13)

where $\sigma_0(E + U_\mu)$ fluctuates depending on the variance of $U_\mu$. This fluctuation of the screening potential can be incorporated through the following equation:

$$\lambda = \rho_{LH} \int \sigma_0(E + U_\mu)v\Psi(E,T)N(U_\mu) dEdU_\mu,$$

(14)

where $N(U_\mu)$ is the distribution of the screening potential. As the first guess, the distribution is likely to be a normal (Gaussian) distribution. In reality, however, we find that the distribution changes characteristically as a function of the incident energy. This change is observed at the point where the incident energy coincides with the ionization energy of the muon. In Fig. 2 the left panel shows a histogram of the screening potential at a relatively high incident energy $E = 60$ (keV). The abscissa is the screening potential normalized by the average value $U_\mu = 5.6$ (keV). The distribution of the histogram is approximated well

![Figure 2](image-url)

Figure 2: The histograms of the screening potential divided by the average at a relatively high incident energy (the left panel) and at an incident energy lower than the ionization energy (the right panel).
by the Gaussian distribution, as we expected. Moreover one sees that there are some components with negative screening potentials. This negative component of the screening potential means that muons can be kicked out to an unbound state in some cases and bring away the relative energy of the colliding nuclei. As the incident energy is reduced, the distribution changes. The right panel in Fig. 2 shows one of such a situation: a histogram of the screening potential at the incident energy \(E = 0.3\) (keV). In this case, because the muon is forced to remain bound the whole process in the entrance channel, the screening potential cannot be negative any more. In fact, we can approximate the distribution of the histogram with the distribution of Gaussian Orthogonal Ensemble (GOE) rather than a normal distribution. We, therefore, approximate this fluctuation of the screening potential by Gaussian distribution around the average potential shift \(\bar{U}_\mu\) and with variance \(\Delta U_\mu\):

\[
N(U_\mu) = \frac{1}{\sqrt{2\pi\Delta U_\mu}} \exp \left( -\frac{(U_\mu - \bar{U}_\mu)^2}{2\Delta U_\mu^2} \right). \tag{15}
\]

in the energy region \(E > BE_t\) and by GOE:

\[
N(U_\mu) = \frac{\pi}{2} \left( \frac{U_\mu}{\bar{U}_\mu} \right) \exp \left( -\frac{\pi}{4} \left( \frac{U_\mu}{\bar{U}_\mu} \right)^2 \right) \tag{16}
\]

in the energy region \(E < BE_t\). Using these distributions of the screening potential, we assess the Eq. (14). We substitute the bare cross section by Eq. (5) and we use the polynomial expression \[\text{S}(E) = 0.20 - 0.32E + 0.476E^2 \text{ (MeVb)}, \tag{17}\]
where the energy \(E\) in MeV, as the astrophysical \(S\)-factor in Eq. (5).

### 4 Results and discussions

The obtained reaction rate by Eq. (14) is shown in Fig. 3 by circles linked with a solid line as a function of the temperature. The circles linked with the dotted line are obtained by taking into account the enhancement of the cross section using the average potential shift \(\bar{U}_\mu\) alone, i.e., Eq. (13) substituted \(U_\mu\) by \(\bar{U}_\mu\). The triangles show the bare reaction rate. The circles deviate from Eq. (13) at the temperature lower than \(10^8\) K and do not show much temperature dependence from 0.1 K to \(10^5\) K. The deviation of the reaction rate by Eq. (14) from the one by Eq. (13) in the low temperature region indicates that the fluctuation of the enhancement factor has a strong influence on the reaction rate at the thermal energy. Including the fluctuation of the cross section, the reaction rate for the \(t-t\mu\)CF reaches at \(5.0 \times 10^{-3}\) \(\mu\text{s}^{-1}\). This is a factor of \(10^{-3}\) smaller than the experimental muonic cycling rate in the solid tritium target, \(3.3\pm0.7\ \mu\text{s}^{-1}\ [3],\) which is marked by the square in Fig. 3. We remark several possible explanations of this underestimation of the rate in the following. The first one is that the assumption of the velocity distribution of colliding nuclei to be the Maxwellian could be inappropriate. We should make use of the velocity distribution in the solid target. Otherwise the second possible explanation is that assuming that the procedure of the estimation of the reaction rate is correct, the result could mean that the actual bare \(S\)-factor must be greater than that given by Eq. (17) in the low energy region where the measurement has not been performed.
Figure 3: Reaction rate of the $tt\mu$ system at the liquid hydrogen density as a function of the target temperature. The circles linked with the dotted line are obtained taking into account the average enhancement of the cross section by the muon, alone. The circles linked with the solid line are obtained taking into account the fluctuation of the enhancement of the cross section by the muon. The triangles show the bare reaction rate. The square shows the experimental data of the muon cycling rate.

5 Conclusions and Future Perspectives

In this paper we have estimated the reaction rate in the muon-catalysed $t-t$ fusion as a function of the temperature in the framework of the "in flight" fusion. We have used the CoMD simulations to estimate the enhancement effect of the cross section due to the muon. We found that the estimated reaction rate does not show temperature dependence in the low temperature region in contrast to the experimental data of the $d-t$ and the $d-d\mu$CF. The obtained reaction rate in the low temperature: $5.0\times 10^{-3}\mu s^{-1}$ underestimates the experimental muonic cycling rate in the solid tritium target, $3.3\pm0.7\mu s^{-1}$, a factor of $10^{-3}$. This is either because our assumption that the velocity distribution is Maxwellian is not correct in the case with the solid target or because the $S$-factor, which we used to estimate the rate, in the low energy region is lower than the actual value.

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