STABLE PAIRS AND GOPAKUMAR-VAFA TYPE INVARIANTS ON HOLOMORPHIC SYMPLECTIC 4-FOLDS

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Abstract. As an analogy to Gopakumar-Vafa conjecture on Calabi-Yau 3-folds, Klemm-Pandharipande defined Gopakumar-Vafa type invariants of a Calabi-Yau 4-fold \( X \) using Gromov-Witten theory. When \( X \) is holomorphic symplectic, Gromov-Witten invariants vanish and one can consider the corresponding reduced theory. In a companion work, we propose a definition of Gopakumar-Vafa type invariants for such a reduced theory. In this paper, we give them a sheaf theoretic interpretation via moduli spaces of stable pairs.

Keywords: Gopakumar-Vafa type invariants, stable pairs, holomorphic symplectic 4-folds

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0. Introduction

0.1. Gopakumar-Vafa invariants. A smooth complex projective 4-fold \( X \) is holomorphic symplectic if it is equipped with a non-degenerate holomorphic 2-form \( \sigma \in H^0(X, \Omega^2_X) \). The ordinary Gromov-Witten invariants of \( X \) always vanish for non-zero curve classes. Instead a reduced Gromov-Witten theory is defined by Kiem-Li’s cosection localization \( [KL] \).

Given cohomology classes \( \gamma_i \in H^*(X, \mathbb{Z}) \), the (reduced) Gromov-Witten invariants of \( X \) in a non-zero curve class \( \beta \in H_2(X, \mathbb{Z}) \) are defined by

\[
GW_{g,\beta}(\gamma_1, \ldots, \gamma_l) = \int_{[\overline{M}_{g,l}(X,\beta)]^{vir}} \prod_{i=1}^l ev_i^*(\gamma_i),
\]

where

\( [\overline{M}_{g,l}(X,\beta)]^{vir} \in A_{2-g+l}(\overline{M}_{g,l}(X,\beta)) \)

is the (reduced) virtual class and \( ev_i : \overline{M}_{g,l}(X,\beta) \to X \) is the evaluation map at the \( i \)-th marking.

We refer to \( [DMS], [O21], [OSY] \) for some references on computations for (0.1). Gromov-Witten invariants are in general rational numbers because the moduli space \( \overline{M}_{g,l}(X,\beta) \) of stable maps is a Deligne-Mumford stack. It is an interesting question to find out integer-valued invariants which underlie them.

In \( [COT22] \), we studied this question and defined genus 0 Gopakumar-Vafa invariants

\[
n_{0,\beta}(\gamma_1, \ldots, \gamma_l) \in \mathbb{Q}
\]

for any non-zero curve class \( \beta \) and genus 1 and 2 Gopakumar-Vafa invariants

\[
n_{1,\beta}(\gamma) \in \mathbb{Q}, \quad \forall \gamma \in H^1(X, \mathbb{Z}); \quad n_{2,\beta} \in \mathbb{Q}
\]

for any primitive curve class \( \beta \) (i.e. it is not a multiple of a non-zero curve class in \( H_2(X, \mathbb{Z}) \)) from Gromov-Witten invariants \( (0.1) \) (see \( [11] \) for details). This may be compared with the previous works of Gopakumar and Vafa \( [GV] \) on Calabi-Yau 3-folds, Klemm and Pandharipande \( [KP] \) on Calabi-Yau 4-folds and Pandharipande and Zinger \( [PZ] \) on Calabi-Yau 5-folds.

In loc. cit., we conjectured the integrality of \( (0.2) \) and \( (0.3) \) and provided substantial evidence for it. The aim of this paper is to give a sheaf theoretic interpretation of these Gopakumar-Vafa invariants using moduli spaces of stable pairs, in analogy with the discussion of \( [CMT19], [CT19] \) on ordinary Calabi-Yau 4-folds.

0.2. GV/Pairs correspondence. Let \( F \) be a one dimensional coherent sheaf on \( X \) and \( s \in H^0(F) \) be a section. For an ample divisor \( \omega \) on \( X \), we denote the slope function by \( \mu(F) = \chi(F)/\omega \cdot [F] \). The pair \( (F, s) \) is called \( Z_t \)-stable \( (t \in \mathbb{R}) \) if

(i) for any subsheaf \( 0 \neq F' \subseteq F \), we have \( \mu(F') < t \),

(ii) for any subsheaf \( F'' \subseteq F \) such that \( s \) factors through \( F'' \), we have \( \mu(F/F'') > t \).
For a non-zero curve class $\beta \in H_2(X, \mathbb{Z})$ and $n \in \mathbb{Z}$, we denote by

$$P^t_n(X, \beta)$$

the moduli space of $\mathbb{Z}_t$-stable pairs $(F, s)$ with $((F), \chi(F)) = (\beta, n)$. It has a wall-chamber structure and for a general $t \in \mathbb{R}$ (i.e. outside a finite subset of rational numbers in $\mathbb{R}$), it is a projective scheme.

When $t < \frac{n}{\omega_\beta}$, $P^t_n(X, \beta)$ is empty. The first nontrivial chamber appears when $t = \frac{n}{\omega_\beta} + 0^+$, which we call Joyce-Song (JS) chamber (here $0^+$ denotes a sufficiently small positive number with respect to the fixed $\omega, \beta, n$). When $t \gg 1$, it recovers the moduli space of Pandharipande-Thomas (PT) stable pairs $\mathcal{P}^{\text{vir}}$ (Proposition 1.0).

For general $t \in \mathbb{R}$, by Theorem 1.7, we can define its DT$_4$ virtual class following [BJ, OT] (see also [CL14]). However, by a cosection argument the virtual class vanishes, see [KiP, Sav].

Using Kiem-Park’s cosection localization [KiP], we have a (reduced) virtual class

$$[P^t_n(X, \beta)]^{\text{vir}} \in A_{n+1}(P^t_n(X, \beta), \mathbb{Q}),$$

depending on the choice of orientation $\omega, \beta, n$ [CGJ, CL17]. More precisely, for each connected component of $P^t_n(X, \beta)$, there are two choices of orientation which affect the virtual class by a sign (component-wise). To define its counting invariants, let

$$\tau : H^m(X, \mathbb{Z}) \to H^{m-2}(P^t_n(X, \beta), \mathbb{Z}),$$

$$\tau(\gamma) := \pi_{*,*}(\pi_X^* \chi \cup \text{ch}_3(F)),$$

where $I = (O \to \mathbb{F})$ is the universal $\mathbb{Z}_t$-stable pair and $\pi_P, \pi_X$ are projections from $P^t_n(X, \beta) \times X$ onto its factors. For $\gamma_i \in H^m(X, \mathbb{Z})$, the $\mathbb{Z}_t$-stable pair invariants are defined by

$$P^t_{n,\beta}(\gamma_1, \ldots, \gamma_l) := \int_{[P^t_n(X, \beta)]^{\text{vir}}} \prod_{i=1}^l \tau(\gamma_i) \in \mathbb{Q}.$$

When $n = -1$, we also write

$$P^t_{-1,\beta} := \int_{[P^t_{-1}(X, \beta)]^{\text{vir}}} 1.$$

Here is the main conjecture of this paper, which gives a sheaf theoretic interpretation of all genus Gopakumar-Vafa invariants using $\mathbb{Z}_t$-stable pair invariants.

**Conjecture 0.1.** (Conjecture 1.10) Fix $n \in \mathbb{Z}$, $\beta \in H_2(X, \mathbb{Z})$ and let $t > \frac{n}{\omega_\beta}$ be generic. For certain choice of orientation, we have

1. If $n \geq 2$, then
   $$P^t_{n,\beta}(\gamma_1, \ldots, \gamma_l) = 0.$$

2. If $n = 1$, then
   $$P^t_{1,\beta}(\gamma_1, \ldots, \gamma_l) = n_0,\beta(\gamma_1, \ldots, \gamma_l).$$

3. If $n = 0$ and $\beta$ is primitive, then
   $$P^t_{0,\beta}(\gamma) = n_1,\beta(\gamma).$$

4. If $n = -1$ and $\beta$ is primitive, then
   $$P^t_{-1,\beta} = n_2,\beta.$$

We verify this conjecture by a computation in an ideal geometry where curves deform in families of expected dimensions and have expected generic properties (see 1.4). Besides this, we study several examples and prove our conjecture in those cases.

0.3. **Verification of conjectures I:** $K3 \times K3$. Let $X = S \times T$ be the product of two $K3$ surfaces. When the curve class $\beta \in H_2(S \times T, \mathbb{Z})$ is of non-trivial degree over both $S$ and $T$, then one can construct two linearly independent cosections for moduli spaces of stable maps, which imply that the (reduced) Gromov-Witten invariants of $X$ in this class vanish. Therefore we always restrict to consider curve classes of form

$$\beta \in H_2(S, \mathbb{Z}) \subseteq H_2(X, \mathbb{Z}).$$

**Theorem 0.2.** (Theorem 2.11) Let $X = S \times T$ be as above. Then Conjecture 0.1 holds for any primitive curve class $\beta \in H_2(S, \mathbb{Z}) \subseteq H_2(X, \mathbb{Z})$. 

**Remark 2.15** Let $X = S \times T$ be as above. Then Conjecture 0.1 holds for any primitive curve class $\beta \in H_2(S, \mathbb{Z}) \subseteq H_2(X, \mathbb{Z})$. 


In fact, by the global Torelli theorem (see e.g. [Ver, Huy]), primitive curve classes on K3 surfaces can be deformed to irreducible curve classes. By deformation invariance, we only need to deal with an irreducible curve class $\beta$, in which case we have an isomorphism (Proposition 2.3): 

$$P^t_0(X, \beta) \cong P^t_0(S, \beta) \times T,$$

and a forgetful map

$$P^t_0(S, \beta) \to M_n(S, \beta),$$

where $M_n(S, \beta)$ is the coarse moduli space of one dimensional stable sheaves $F$ on $S$ with $[F] = \beta$, $\chi(F) = n$. Both $P^t_0(S, \beta)$ and $M_n(S, \beta)$ are smooth schemes. We can then determine the DT$_4$ virtual class of $P^t_0(X, \beta)$ (Theorem 2.8) and its pushforward (under the forgetful map) by the Thom-Porteous formula (Proposition 2.9). This enables us to reduce the computation of $Z_t$-stable pair invariants to certain tautological integrals on $P^t_0$-plane class and let us identify $M_n$ where

$$\pi$$

where $\pi$ classes

$S$ schemes of points on

monodromy operators [M08], we relate such integrals to certain tautological integrals on Hilbert schemes of points on $S$ (see 2.6 for details), which we explicitly determine using [COT22] (see the proof of Theorem 2.14 2.15 for details).

0.4. Verification of conjectures II: $T^*\mathbb{P}^2$. Let $H \in H^2(T^*\mathbb{P}^2)$ be the pullback of the hyperplane class and let us identify $H_2(T^*\mathbb{P}^2, \mathbb{Z}) \cong \mathbb{Z}$ by taking the degree against $H$.

By explicitly describing the moduli spaces and virtual classes, we obtain:

Proposition 0.3. (Proposition 5.3) For certain choice of orientation, we have

$$P_{1,1}(H^2, H^2) = 1, \quad P_{1,2}(H^2, H^2) = -1, \quad P_{1,3}(H^2, H^2) = 0,$$

$$P_{0,1}(H^2) = P_{0,2}(H^2) = 0, \quad P_{0,3}(H^2) = 1, \quad P_{-1,1} = P_{-1,2} = P_{-1,3} = 0.$$ 

Moreover, $P^t_0(X, \beta)$ is independent of the choice of $t > n/d$ in the listed cases above.

In particular, for $X = T^*\mathbb{P}^2$, we have

- Conjecture (7.1) (2) holds when $d \leq 3$.
- Conjecture (7.1) (3), (4) hold.

0.5. Verification of conjectures III: exceptional curves on Hilb$^2(K3)$. Let $S$ be a K3 surface and Hilb$^2(S)$ be the Hilbert scheme of two points on $S$. Consider the Hilbert-Chow map

$$\pi : \text{Hilb}^2(S) \to \text{Sym}^2(S)$$

to the symmetric product of $S$. Let $D$ be the exceptional divisor fitting into Cartesian diagram:

$$\begin{array}{ccc}
D & \xrightarrow{\iota} & \text{Hilb}^2(S) \\
\pi \downarrow & & \downarrow \pi \\
S & \xrightarrow{\Delta} & \text{Sym}^2(S),
\end{array}$$

where $\Delta$ is the diagonal embedding and $\pi : D \to S$ is a $\mathbb{P}^1$-bundle. The following provides a verification of our (genus 0) conjecture for imprimitive curve classes.

Theorem 0.4. (Theorem 5.1) In the JS chamber, Conjecture (7.1) (1), (2) hold for multiple fiber classes $\beta = r[\mathbb{P}^1]$ ($r \geq 1$) of $\pi$ as above.

In fact, by the Jordan-Hölder filtration and a dimension counting, the JS pair invariants of $P^{JS}_n(X, r[\mathbb{P}^1])$ are zero unless $n = r$ and in which case we have

$$P^{JS}_n(X, n[\mathbb{P}^1]) \cong \text{Hilb}^n(S).$$

Then the proof makes use of the Chern class operator of tautological bundles by Lehn [Lehn].

0.6. Multiple fiber classes of elliptic fibrations. Let $p : S \to \mathbb{P}^1$ be an elliptic K3 surface and consider the elliptic fibration:

$$\bar{p} := p \times \text{id}_T : X := S \times T \to \mathbb{P}^1 \times T := Y,$$

where $T$ is a K3 surface. Denote $f$ to be a generic fiber of $\bar{p}$ and $p \in H_0(T)$ be the point class.

The following gives a closed formula of $Z_t$-stable pair invariants for multiple fiber classes.

Theorem 0.5. (Theorem 2.19) Let $t > 0$. Then for certain choice of orientation, we have

$$\sum_{r \geq 0} P^t_{0, r[f]}(\gamma) q^r = 24 \left( \int_{S \times p} \gamma \right) \cdot \sum_{m \geq 1} \sum_{n|m} n^2 q^m.$$
As for the proof, we note that there is an isomorphism 
\[ \hat{p}^* : \text{Hilb}^r(Y) \cong P_0^r(X, r[f]), \]
under which the (reduced) virtual classes
\[ (-1)^{n+1} [\text{Hilb}^r(Y)]^{vir} = [P_0^r(X, r[f])]^{vir} \in A_1(\text{Hilb}^r(Y)) \]
can be identified for certain choice of orientation on the right hand side. Then we are left to evaluate an integral on \([\text{Hilb}^r(Y)]^{vir}\) which can be done via the degeneration method and a Behrend function argument \([\text{B}, \text{OS}].\) We refer to Theorem 2.17 for a similar result for trivial elliptic fibration \(E \times E \times T \to E \times T\) and the proof therein for details.

The formula in Theorem 0.5 seems to support our speculation of a GV/Pairs correspondence in genus 1 for imprimitiv curve classes (see \([\text{COT}22]\) for details).

0.7. A conjectural virtual pushforward formula. Finally we remark that for a general holomorphic symplectic 4-fold \(X\) and an irreducible curve class \(\beta \in H_2(X, \mathbb{Z})\), we have a forgetful map as in (0.4):
\[ P_0^r(X, \beta) \to M_\alpha(X, \beta), \]
where \(M_\alpha(S, \beta)\) is the coarse moduli space of one dimensional stable sheaves \(F\) on \(X\) with \([F] = \beta, \chi(F) = n\). In Appendix §A we conjecture a virtual pushforward formula for this map (which we verify for the product of \(K3\) surfaces, see Proposition A.5). Together with Conjecture 0.1 (4), this formula implies a conjectural relation between genus 2 Gopakumar-Vafa invariants and certain descendent invariants on \(M_1(X, \beta)\) (Proposition A.3), which appears as \([\text{COT}22]\ Conj. 2.2 (iii)\).

0.8. Notation and convention. In this paper, all varieties and schemes are defined over \(\mathbb{C}\). For a morphism \(\pi : X \to Y\) of schemes, and for \(F, G \in D^b(\text{Coh}(X))\), we denote by \(R\pi_* R\text{Hom}(F, G)\) the functor \(R\pi_* R\text{Hom}_X(F, G)\).

A class \(\beta \in H_2(X, \mathbb{Z})\) is called effective if there exists a non-empty curve \(C \subset X\) with \([C] = \beta\). An effective class \(\beta\) is called irreducible if it is not the sum of two effective classes, and it is called primitive if it is not a positive integer multiple of an effective class.

A holomorphic-symplectic variety is a smooth projective variety with a non-degenerate holomorphic two form \(\sigma \in H^0(X, \Omega^2_X)\). A holomorphic-symplectic variety is irreducible hyperkähler if \(X\) is simply connected and \(H^0(X, \Omega^2_X)\) is generated by a symplectic form. A \(K3\) surface is an (irreducible) hyperkähler variety of dimension 2.

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1. Definitions and conjectures

1.1. Gopakumar-Vafa invariants. Let \(X\) be a holomorphic symplectic 4-fold and \(\overline{M}_{g, l}(X, \beta)\) be the moduli stack of genus \(g\), \(l\)-pointed stable maps to \(X\) with non-zero curve class \(\beta\). Its virtual class \([\overline{M}_{g, l}(X, \beta)]^{vir}\) vanishes due to a trivial factor in the obstruction sheaf. By Kiem-Li’s theory of cosection localization \([\text{KLL}],\) one can define a (reduced) virtual class\(^4\)
\[ [\overline{M}_{g, l}(X, \beta)]^{vir} \in A_{2-g+l}(\overline{M}_{g, l}(X, \beta)). \]

For integral classes
\[ \gamma_i \in H^{m_i}(X, \mathbb{Z}), \quad 1 \leq i \leq l, \]
the (primary) Gromov-Witten invariant is defined by
\[ GW_{g, \beta}(\gamma_1, \ldots, \gamma_l) = \int_{\overline{M}_{g, l}(X, \beta)]^{vir}} \prod_{i=1}^l ev_i^* (\gamma_i) \in \mathbb{Q}, \]

\(^4\)The virtual class mentioned in this paper is always assumed to be the reduced one.
where $ev_i : \overline{M}_{g,l}(X, \beta) \rightarrow X$ is the $i$-th evaluation map.

When $g = 0$, the virtual dimension of $\overline{M}_{0,l}(X, \beta)$ is $l + 2$, and (1.2) is zero unless
\[
(1.3) \quad \sum_{i=1}^l (m_i - 2) = 4.
\]

Similar to the case of Calabi-Yau 4-folds and 5-folds [KP, PZ], we make the following definition:

**Definition 1.1.** ([COT22 Def. 1.5]) For any $\gamma_1, \ldots, \gamma_l \in H^*(X, \mathbb{Z})$, we define the genus 0 Gopakumar-Vafa invariant $n_{0,\beta}(\gamma_1, \ldots, \gamma_l) \in \mathbb{Q}$ recursively by the multiple cover formula:
\[
GW_{0,\beta}(\gamma_1, \ldots, \gamma_l) = \sum_{k \geq 1, k|\beta} k^{n-3} n_{0,\beta/k}(\gamma_1, \ldots, \gamma_l).
\]

When $g = 1$, the virtual dimension of $\overline{M}_{1,l}(X, \beta)$ is $l + 1$, and (1.2) is zero unless
\[
(1.4) \quad \sum_{i=1}^l (m_i - 2) = 2.
\]

In this paper, we concentrate on the case when $l = 1$ and $m_1 = 4$. Because curves in imprimitive curve classes are very difficult to control, we restrict hereby to the case of a primitive curve class.

**Definition 1.2.** ([COT22 Def. 1.6]) Assume that $\beta \in H^2(X, \mathbb{Z})$ is primitive. For any $\gamma \in H^4(X, \mathbb{Z})$, we define the genus 1 Gopakumar-Vafa invariant $n_{1,\beta}(\gamma) \in \mathbb{Q}$ by
\[
GW_{1,\beta}(\gamma) = n_{1,\beta}(\gamma) - \frac{1}{24} GW_{0,\beta}(\gamma, c_2(X)),
\]
where $c_2(X)$ is the second Chern class of $T_X$.

When $g = 2$, the virtual dimension of $\overline{M}_{2,0}(X, \beta)$ is zero, so we can consider (1.2) without insertions:
\[
GW_{2,\beta} := \int_{\overline{M}_{2,0}(X, \beta)}^\text{vir} 1 \in \mathbb{Q}.
\]

**Definition 1.3.** ([COT22 Def. 1.7]) Assume that $\beta \in H^2(X, \mathbb{Z})$ is primitive. We define the genus 2 Gopakumar-Vafa invariant $n_{2,\beta} \in \mathbb{Q}$ by
\[
GW_{2,\beta} = n_{2,\beta} - \frac{1}{24} n_{1,\beta}(c_2(X)) + \frac{1}{2 \cdot 24^2} GW_{0,\beta}(c_2(X), c_2(X)) + \frac{1}{24} N_{\text{nodal,} \beta}.
\]

Here $n_{1,\beta}(\cdot)$ is given in Definition 1.5, and $N_{\text{nodal,} \beta} \in \mathbb{Q}$ is the virtual count of rational nodal curves [NO] as defined by
\[
(1.5) \quad N_{\text{nodal,} \beta} := \frac{1}{2} \left( \int_{\overline{M}_{0,2}(X, \beta)}(ev_1 \times ev_2)^*(\Delta_X) - \int_{\overline{M}_{0,1}(X, \beta)}^\text{vir} \frac{ev_1^*(c(X))}{1 - \psi_1} \right),
\]
where
- $\Delta_X \in H^8(X \times X)$ is the class of the diagonal, and
- $c(X) = 1 + c_2(X) + c_4(X)$ is the total Chern class of $T_X$.

1.2. $Z_t$-stable pair invariants. Let $\omega$ be an ample divisor on $X$ and $t \in \mathbb{R}$, we recall the following notion of $Z_t$-stable pairs.

**Definition 1.4.** ([CT19 Lem 1.7]) Let $F$ be a one dimensional coherent sheaf and $s : \mathcal{O}_X \rightarrow F$ be a section. For an ample divisor $\omega$, we denote the slope function by $\mu(F) = \chi(F)/(|\omega : [F]|)$.

We say $(F, s)$ is a $Z_t$-(semi)stable pair ($t \in \mathbb{R}$) if
- (i) for any subsheaf $0 \neq F' \subseteq F$, we have $\mu(F') < (\leq) t$,
- (ii) for any subsheaf $F' \subseteq F$ such that $s$ factors through $F'$, we have $\mu(F/F') > (\geq) t$.

There are two distinguished stability conditions appearing as special cases of $Z_t$-stability.

**Definition 1.5.** ([PT09, CT19 Def. 1.10])
- (i) A pair $(F, s)$ is a PT stable pair if $F$ is a pure one dimensional sheaf and $s$ is surjective in dimension one.
- (ii) A pair $(F, s)$ is a JS stable pair if $s$ is a non-zero morphism, $F$ is $\mu$-semistable and for any subsheaf $0 \neq F' \subseteq F$ such that $s$ factors through $F'$ we have $\mu(F') < \mu(F)$.

**Proposition 1.6.** ([CT19 Prop. 1.11]) For a pair $(F, s)$ with $[F] = \beta$ and $\chi(F) = n$, its
- (i) $Z_t$-stability with $t \rightarrow \infty$ is exactly PT stability,
- (ii) $Z_t$-stability with $t = \frac{1}{\omega(F)} + 0^+$ is exactly JS stability.
For $\beta \in H_2(X, \mathbb{Z})$ and $n \in \mathbb{Z}$, we denote by

$$P_n^t(X, \beta) \quad \text{(resp. } P_n^t(X, \beta))$$

the moduli stack of $Z_t$-stable (resp. $Z_t$-semistable) pairs $(F, s)$ with $[F] = \beta$ and $\chi(F) = n$.

By Proposition 1.6 there are two distinguished moduli spaces, PT moduli spaces and JS moduli spaces, by specializing $t \to \infty$ and $t = \frac{n}{\omega \cdot \beta} + \omega^+$ respectively:

$$P_n(X, \beta) := P_n^{t \to \infty}(X, \beta), \quad P_n^{JS}(X, \beta) := P_n^{t = \frac{n}{\omega \cdot \beta} + \omega^+}(X, \beta).$$

By a GIT construction, $P_n^t(X, \beta)$ is a quasi-projective scheme, and $P_n^t(X, \beta)$ admits a good moduli space

$$\mathcal{P}_n^t(X, \beta) \to \mathcal{P}_n^t(X, \beta),$$

where $\mathcal{P}_n(X, \beta)$ is a projective scheme which parametrizes $Z_t$-polystable objects. The following result shows that moduli stacks of $Z_t$-stable pairs are indeed open substacks of moduli stacks of objects in the derived categories of coherent sheaves.

**Theorem 1.7.** ([CT19, Thm. 0.1]) $P_n^t(X, \beta)$ admits an open immersion

$$P_n^t(X, \beta) \to \mathcal{M}_0, \quad (F, s) \mapsto (\mathcal{O}_X \to \mathcal{F})$$

to the moduli stack $\mathcal{M}_0$ of $E \in D^b \text{Coh}(X)$ with $\text{Ext}^2(0, E) = 0$ and $\det(E) \cong \mathcal{O}_X$.

Therefore for a general choice of $t$ (i.e., outside a finite subset of rational numbers in $\mathbb{R}$), $P_n^t(X, \beta)$ is a projective scheme which can be given a $(-2)$-shifted symplectic derived scheme structure [PTVV] and has a virtual class [BJ, OT] (see also [CL14]).

Parallel to GW theory, the virtual class of $P_n^t(X, \beta)$ vanishes [KIP, Sav]. One can define a reduced virtual class due to Kiem-Park [KIP, Def. 8.7, Lem. 9.4]:

$$[P_n^t(X, \beta)]^{\vir} \in A_{n+1}(P_n^t(X, \beta), \mathbb{Q}),$$

depending on the choice of orientation [CCJ, CL17]. To define its counting invariants, let

$$\tau : H^m(X, \mathbb{Z}) \to H^{m-2}(P_n^t(X, \beta), \mathbb{Z}),$$

$$\tau(\gamma) := \pi_{P*}(\pi_X^* \gamma \cup \text{ch}(\mathcal{F})),$$

where $I = (\mathcal{O} \to \mathcal{F})$ is the universal $Z_t$-stable pair and $\pi_P, \pi_X$ are projections from $P_n^t(X, \beta) \times X$ onto its factors.

**Definition 1.8.** Let $t \in \mathbb{R}$ be generic and $\gamma_i \in H^{m_i}(X, \mathbb{Z})$ ($1 \leq i \leq l$). The $Z_t$-stable pair invariants are

$$P_n^t(\gamma_1, \ldots, \gamma_l) := \int_{[P_n^t(X, \beta)]^{\vir}} \prod_{i=1}^l \tau(\gamma_i) \in \mathbb{Q}.$$  

When $n = -1$, we write

$$P_{-1, \beta} := \int_{[P_{-1}^t(X, \beta)]^{\vir}} 1.$$  

In PT and JS stabilities, we also write

$$P_n(\gamma_1, \ldots, \gamma_l) := P_n^{t \to \infty}(\gamma_1, \ldots, \gamma_l), \quad P_n^{JS}(\gamma_1, \ldots, \gamma_l) := P_n^{t = \frac{n}{\omega \cdot \beta} + \omega^+}(\gamma_1, \ldots, \gamma_l).$$

**Remark 1.9.** By Definition 1.2 and a dimension counting, $Z_t$-stable pair invariants are non-zero only if both of the following conditions hold:

$$t > \frac{n}{\omega \cdot \beta}, \quad \sum_{i=1}^l (m_i - 2) = 2n + 2.$$  

In [CMT19, CT19], similar invariants are used to give sheaf theoretic interpretations of Gopakumar-Vafa type invariants for ordinary Calabi-Yau 4-folds [KP]. Below, we give a parallel proposal for holomorphic symplectic 4-folds using Definition 1.8.
1.3. Conjecture. We state the main conjecture of this paper.

**Conjecture 1.10.** Let $X$ be a holomorphic symplectic 4-fold with an ample divisor $\omega$. Fix $n \in \mathbb{Z}$ and $\beta \in H_2(X,\mathbb{Z})$ and let $t > \frac{1}{2\omega}$ be generic. For certain choice of orientation, we have

1. If $n \geq 2$, then
   $$P^t_{n,\beta}(\gamma_1,\ldots,\gamma_t) = 0.$$

2. If $n = 1$, then
   $$P^t_{1,\beta}(\gamma_1,\ldots,\gamma_t) = n_{0,\beta}(\gamma_1,\ldots,\gamma_t) \in \mathbb{Z}.$$

3. If $n = 0$ and $\beta$ is primitive, then
   $$P^t_{0,\beta}(\gamma) = n_{1,\beta}(\gamma) \in \mathbb{Z}.$$

4. If $n = -1$ and $\beta$ is primitive, then
   $$P^t_{-1,\beta} = n_{2,\beta} \in \mathbb{Z}.$$

**Remark 1.11.** By the global Torelli theorem [Ver, Huy], primitive curve classes on irreducible hyperkähler varieties can be deformed to irreducible curve classes. Therefore $Z_t$-stable pair invariants are independent of the choice of $t > \frac{1}{2\omega}$ for such cases by [CT19] Prop. 1.12.

**Remark 1.12.** Our conjecture implies that there is no nontrivial wall-crossing for $Z_t$-stable pairs invariants when $t > \frac{1}{2\omega}$, contrary to the ordinary CY case [CT19, CT20a, CT20b].

**Remark 1.13.** Similarly to [CK19, Conj. 0.3], we may use counting invariants on Hilbert schemes $I_n(X,\beta)$ of curves to give a sheaf theoretic interpretation of Gopakumar-Vafa invariants in which case zero dimensional subschemes [CK18] (conjecturally) will not contribute, i.e. “$\text{DT} = \text{PT}$”. It is curious whether one can do a K-theoretic refinement as [CKM19].

1.4. Heuristic argument. In this section, we verify Conjecture 1.10 using heuristic argument in an ideal geometry (ref. [COT22 §1.4, §1.5]). To be specific, as the virtual dimension of $\overline{\mathcal{M}}_{g,0}(X,\beta)$ is $2 - g$, we assume that:

- Any genus $g$ curve moves in a smooth compact $(2 - g)$-dimensional family.

In particular, there are no curves of genus $g \geq 3$. Unfortunately, complicated phenomena still arise even in the ideal case, for example, one can have two (resp. one) dimensional families of reducible rational (resp. elliptic) curves, and any member of a rational curve family is expected to intersect nontrivially with some member in the same family (see [COT22 §1.4] for details).

However, things will be simplified if we make the following additional assumptions:

- $X$ is irreducible hyperkähler,
- the effective curve class $\beta \in H_2(X,\mathbb{Z})$ is primitive.

By the global Torelli for (irreducible) hyperkähler varieties [Ver, Huy], the pair $(X,\beta)$ is deformation equivalent (through a deformation with keeps $\beta$ of Hodge type) to a pair $(X',\beta')$, where $\beta' \in H_2(X,\mathbb{Z})$ is irreducible, so we may without loss of generality assume:

- the effective curve class $\beta \in H_2(X,\mathbb{Z})$ is irreducible.

Under these assumptions, our ideal geometry of curves simplifies to the following form:

1. The rational curves in $X$ of class $\beta$ move in a proper 2-dimensional smooth family of embedded irreducible rational curves. Except for a finite number of rational nodal curves, the rational curves are smooth, with normal bundle $\mathcal{O}_p \oplus \mathcal{O}_p \oplus \mathcal{O}_p(-2)$.

2. The arithmetic genus 1 curves in $X$ of class $\beta$ move in a proper 1-dimensional smooth family of embedded irreducible genus 1 curves. Except for a finite number of rational nodal curves, the genus one curves are smooth elliptic curves with normal bundle $L^{-1} \oplus \mathcal{O}$, where $L$ is a generic degree zero line bundle.

3. All genus two curves are smooth and rigid.

4. There are no curves of genus $g \geq 3$.

We need to compute $Z_t$-stable pair invariants in this ideal setting. The key heuristic we use is that only $Z_t$-stable pairs with connected support will ‘contribute’ to our invariants.

The observation is that for a $Z_t$-stable pair $I = (O_X \to F)$ such that $F$ is supported on a disconnected curve $C = C_1 \sqcup C_2$, we may write $F = F_1 \oplus F_2$ where $F_i$ is supported on $C_i$ ($i = 1, 2$). We set

$$I_1 = (O_X \to F_1), \quad I_2 = (O_X \to F_2).$$

Then the obstruction space satisfies

$$(1.8) \quad \text{Ext}^2(I, I) = \text{Ext}^2(I_1, I_1) \oplus \text{Ext}^2(I_2, I_2).$$
Indeed there is a distinguished triangle (see the argument of \cite[(2.13)]{CMT19}):

$$R\text{Hom}(I, F) \rightarrow R\text{Hom}(I, I)_0[1] \rightarrow R\text{Hom}(F, \mathcal{O}_X)[2],$$

and we have

$$R\text{Hom}(I, F) \cong R\text{Hom}(I, F_1) \oplus R\text{Hom}(I, F_2) \cong R\text{Hom}(I_1, F_1) \oplus R\text{Hom}(I_2, F_2),$$

where the second isomorphism follows since \(I\) is isomorphic to \(I_i\) near the support of \(F_i\). Combining with

$$R\text{Hom}(F, \mathcal{O}_X) = R\text{Hom}(F_1, \mathcal{O}_X) \oplus R\text{Hom}(F_2, \mathcal{O}_X),$$

we obtain

$$R\text{Hom}(I, I)_0[1] \cong R\text{Hom}(I_1, I_1)_0[1] \oplus R\text{Hom}(I_2, I_2)_0[1].$$

Hence \((1.8)\) holds.

Therefore the surjective isotropic cosections (see \cite[Lem. 9.4]{KIP}) of obstruction spaces in the RHS of \((1.8)\) give rise to a (mutually orthogonal) two dimensional isotropic cosection in the LHS. Hence \((1.8)\) holds.

By Definition 1.8 and above discussion, \(Z_t\)-stable pair invariants

$$P_{n, \beta}^t(\gamma_1, \ldots, \gamma_{l}) = \int_{\mathcal{P}_n^{\text{vir}}(\mathcal{X}, \beta)} \prod_{i=1}^{l} \tau(\gamma_i)$$

count \(Z_t\)-stable pairs whose support are connected and incident to cycles dual to \(\gamma_1, \ldots, \gamma_{l}\). Say such an incident \(Z_t\)-stable pair is supported on a \((2 - g)\)-dimensional family:

$$p : C_{\beta}^0 \rightarrow S_{\beta}^g$$

of genus \(g\) curves \((g = 0, 1, 2)\), where \(C_{\beta}^0\) is the total space of this family. Each cycle \(\gamma_i\) will cut down real dimension of \(S_{\beta}^g\) by \(\deg(\gamma_i) - 2\). As we have

$$\sum_{i=1}^{l} (\deg(\gamma_i) - 2) = 2n + 2,$$

so all insertions in total cut down real dimension of \(S_{\beta}^g\) by \(2n + 2\).

**The case** \(n \geq 1\). When \(n \geq 2\), the dimension cut down by insertions is bigger than the largest possible dimension of \(S_{\beta}^g\), so there can not be such incident stable pairs and

$$P_{n, \beta}^t(\gamma_1, \ldots, \gamma_{l}) = 0.$$

This confirms Conjecture \((1.8)\) (1).

When \(n = 1\), insertions cut down real dimension of \(S_{\beta}^g\) by 4, so any incident \(Z_t\)-stable pair \(I = (\mathcal{O}_X \rightarrow F)\) can only be supported on genus 0 family. As in \cite[§4.1]{CT19}, by Harder-Narasimhan and Jordan-Hölder filtration, we know

$$F \cong \mathcal{O}_C,$$

for some rational curve \(C\) in \(S_{\beta}^g\). Therefore incident \(Z_t\)-stable pairs (with \(\chi = 1\)) are in one to one correspondence with intersection points of \(C_{\beta}^0\) with cycles dual to \(\gamma_1, \ldots, \gamma_{l}\) and

$$P_{1, \beta}^t(\gamma_1, \ldots, \gamma_{l}) = \int_{S_{\beta}^g} \prod_{i=1}^{l} p_*(f^* \gamma_i),$$

where \(f : C_{\beta}^0 \rightarrow X\) is the evaluation map. Therefore Conjecture \((1.8)\) (2) is confirmed in this ideal case as both sides are (virtually) enumerating rational curves of class \(\beta\) incident to cycles dual to \(\gamma_1, \ldots, \gamma_{l}\).

**The case** \(n = 0\). Since \(Z_t\)-stable pairs \(I = (\mathcal{O}_X \rightarrow F)\) supported on genus 0 curves satisfy \(\chi(F) > 0\) and a 4-cycle \(\gamma \in H^4(X)\) misses genus 2 curves in general position, so when \([F] = \beta\) is irreducible, the pair must be scheme theoretically supported on an elliptic curve \(C\) and

$$I = (\mathcal{O}_X \rightarrow \mathcal{O}_C \xrightarrow{s} L),$$

where \(L\) is a line bundle on \(C\) with \(\chi(C, L) = 0\). By \(Z_t\)-stability, \(s\) is non-trivial, so \(s\) must be an isomorphism by the stability of line bundles. Therefore incident \(Z_t\)-stable pairs (with \(\chi = 0\)
are in one to one correspondence with intersection points of 4-cycle $\gamma$ with genus 1 curve family $C_{\beta}$ of class $\beta$ and

$$P_{n,\beta}(\gamma) = \int_{C_{\beta}} f^* \gamma,$$

where $f : C_{\beta} \to X$ is the evaluation map. Therefore Conjecture 1.8 (3) is confirmed in this ideal setting as both sides are (virtually) enumerating elliptic curves of class $\beta$ incident to $\gamma$.

The case $n = -1$. Any $Z_\ell$-stable pair $I = (O_X \to F)$ with irreducible curve class $[F] = \beta$ is scheme theoretically supported on a smooth rigid genus 2 curve $C$:

$$I = (O_X \to O_C \to L),$$

where $L$ is a line bundle on $C$ with $\chi(C, L) = -1$. As above, by $Z_\ell$-stability, $s$ must be an isomorphism. Hence $P^*_{g,1}(X, \beta)$ is identified with the set of all rigid genus 2 curves in class $\beta$ in the ideal geometry, whose count gives exactly genus 2 Gopakumar-Vafa invariant $I_{\beta}$. Therefore Conjecture 1.8 (4) is confirmed in the ideal setting.

1.5. Speculations for general curve classes. For a smooth projective Calabi-Yau 4-fold $X$ and $\gamma \in H^4(X, \mathbb{Z})$, we have genus 0, 1 Gopakumar-Vafa type invariants $n_{0,\beta}(\gamma), n_{1,\beta} \in \mathbb{Q}$ defined by Klemm and Pandharipande from Gromov-Witten theory [KP] and stable pair invariants $P_{n,\beta}(\gamma) \in \mathbb{Z}$ [CMT19] (here $P_{n,\beta}(\gamma)$ is a shorthand for $P_{n,\beta}(\gamma, \ldots, \gamma)$). They are related by the following conjectural formula [CMT19 §1.7]:

$$\sum_{n,\beta} \frac{P_{n,\beta}(\gamma)}{n!} q^n q^\beta = \prod_{\beta > 0} \left( \exp(y q^\beta)^{n_{0,\beta}(\gamma)} M(q^\beta)^{n_{1,\beta}} \right),$$

where $M(q) = \prod_{k \geq 1} (1 - q^k)^{-k}$ is the MacMahon function.

By taking logarithmic differentiation with respect to $y$, we obtain

$$y \frac{d}{dy} \log \left( \sum_{n,\beta} \frac{P_{n,\beta}(\gamma)}{n!} q^n q^\beta \right) = \sum_{\beta > 0} y \frac{d}{dy} n_{0,\beta}(\gamma) y q^\beta = \sum_{\beta > 0} n_{0,\beta}(\gamma) y q^\beta.$$

If we view it as an equality for corresponding reduced invariants on holomorphic symplectic 4-folds, it surprisingly recovers Conjecture 1.10 (i), (ii) (i.e. the genus zero part).

We do similar manipulations for genus one invariants. Note that $y^0 q^\beta$ parts of (1.9) are

$$\sum_{\beta} P_{0,\beta} q^\beta = \prod_{\beta > 0} M(q^\beta)^{n_{1,\beta}}.$$

This equality is written down by a computation in the “CY 4 ideal geometry” (ref. [CMT19 §2.5]), where rational curves contribute zero and each super-rigid elliptic curve (on an ideal CY 4) in class $\beta$ contributes by $M(q^\beta)$ (ref. [CMT19 Thm. 5.10]). Taking logarithmic differentiation with respect to $q$:

$$q \frac{d}{dq} \log (M(q)) = \sum_{d \geq 1} q^d \sum_{i \geq 1, i \mid d} i^2.$$

We then wonder whether in the holomorphic symplectic 4-folds setting, each ideal elliptic curve family in class $\beta$ contributes to $P_{0,\beta}(\gamma)$ by

$$\sum_{i \geq 1, i \mid d} i^2.$$

Summing over all elliptic curve families, this would imply

$$P_{0,\beta}(\gamma) = \sum_{d \geq 1, d \mid \beta} n_{1,\beta/d}(\gamma) \sum_{i \geq 1, i \mid d} i^2.$$

It is quite curious whether the above formula gives the correct PT/GV correspondence. For multiple fiber classes of elliptic fibrations, our computations show the formula seems correct (see Theorems 2.17, 2.19, Remark 2.20). As for $P_{-1,\beta}$ and genus 2 Gopakumar-Vafa invariants, we haven’t found analogous formula for general curve classes.
2. PRODUCT OF K3 SURFACES

In this section, we consider the product of two K3 surfaces:

\[ X = S \times T, \quad \text{with } \beta \in H_2(S, \mathbb{Z}) \subseteq H_2(X, \mathbb{Z}). \]

As observed in [COT22 §5], this contains all interesting curve classes on \( X \) because if \( \beta \in H_2(X, \mathbb{Z}) \) is of non-trivial degree over both \( S \) and \( T \), one can construct two linearly independent cosections, which imply that reduced Gromov-Witten invariants of \( X \) in this class vanish.

2.1. Gopakumar-Vafa invariants. Recall Gopakumar-Vafa invariants specified in Definitions 1.1, 1.2, 1.3. They are computed in [COT22 Prop. 5.1] as follows: write Gopakumar-Vafa invariants.

\[ \gamma, \gamma' \in H^4(X) \]

based on Künneth decomposition:

\[ H^4(X) \cong (H^0(S) \otimes H^4(T)) \oplus (H^2(S) \otimes H^2(T)) \oplus (H^4(S) \otimes H^0(T)). \]

Fix also a curve class

\[ \alpha = \theta_1 \otimes p + p \otimes \theta_2 \in H^6(X) \cong (H^2(S) \otimes H^4(T)) \oplus (H^4(S) \otimes H^2(T)). \]

**Proposition 2.1.** ([COT22 Prop. 5.1]) For \( \beta \in H_2(S, \mathbb{Z}) \subseteq H_2(X, \mathbb{Z}) \), we have

\[ n_{0, \beta}(\gamma, \gamma') = (D_1 \cdot \beta) \cdot (D_1' \cdot \beta) \cdot \int_T (D_2 \cdot D_2') \cdot N_0 \left( \frac{\beta^2}{2} \right), \]

\[ n_{0, \beta}(\alpha) = (\theta_1 \cdot \beta) N_0 \left( \frac{\beta^2}{2} \right). \]

If \( \beta \) is primitive, we have

\[ n_{1, \beta}(\gamma) = 24A_2 N_1 \left( \frac{\beta^2}{2} \right), \quad n_{2, \beta} = N_2 \left( \frac{\beta^2}{2} \right), \]

where

\[ (2.1) \quad \sum_{l \in \mathbb{Z}} N_0(l) q^l = \frac{1}{q} \prod_{n \geq 1} \left( 1 - q^n \right)^{24}, \]

\[ (2.2) \quad \sum_{l \in \mathbb{Z}} N_1(l) q^l = \frac{1}{q} \prod_{n \geq 1} \left( 1 - q^n \right)^{24} \left( \frac{d}{dq} G_2(q) \right), \]

\[ (2.3) \quad \sum_{l \in \mathbb{Z}} N_2(l) q^l = \left( \frac{1}{d} \prod_{n \geq 1} \frac{1}{(1 - q^n)^{24}} \right) \left( 24q \frac{d}{dq} G_2 - 24G_2 - 1 \right), \]

with Eisenstein series:

\[ G_2(q) = -\frac{1}{24} + \sum_{n \geq 1, d|n} d q^n. \]

2.2. Moduli spaces of \( Z_t \)-stable pairs. For a point \( t \in T \), let \( i_t : S \to S \times \{ t \} \hookrightarrow X \) be the inclusion. Consider the pushforward map

\[ (2.4) \quad i_* : P^t_n(S, \beta) \times T \to P^t_n(X, \beta), \]

\[ \left( \mathcal{O}_S \xrightarrow{i_t} F, t \right) \mapsto \left( \mathcal{O}_X \to i_t \mathcal{O}_S \xrightarrow{\nu_s \circ i_t} i_t F, \right), \]

where \( P^t_n(S, \beta) \) is the moduli space of \( Z_t \)-stable pairs \( (F, s) \) on \( S \) with \( [F] = \beta \) and \( \chi(F) = n \).

Without causing confusions, we use the same notation \( t \) for both the stability parameter and closed points in \( T \).

We restrict to the following setting.

**Setting 2.2.** We consider the case when the following conditions are satisfied:

1. The map \((2.4)\) is an isomorphism and \( P^t_n(S, \beta) \) is smooth of dimension \( \beta^2 + n + 1 \).
2. There is a well-defined forgetful map

\[ f : P^t_n(S, \beta) \to M_n(S, \beta), \quad (\mathcal{O}_S \to F) \mapsto F, \]

\[ \text{to the coarse moduli scheme } M_n(S, \beta) \] of one dimensional stable sheaves \( F \) on \( S \) with \( [F] = \beta \) and \( \chi(F) = n \).

**Proposition 2.3.** Setting 2.2 is satisfied when \( \beta \) is irreducible.
Proof. When $\beta$ is irreducible, $P_n^\beta(X, \beta)$ is independent of the choice of $t > \frac{2q}{m}$ [CT19 Prop. 1.12], so we can set $t \to \infty$ and work with PT stability. The isomorphism follows from similar argument as [CMT19 Prop. 3.11]. The key point is that for any such $Z_t$-stable pair $(F, s)$, $F$ is stable and therefore scheme theoretically supported on $S \times \{t\}$ for some $t \in T$ ([CMT18 Lem. 2.2]). The smoothness of $P_n^\beta(S, \beta)$ follows from [KY, PT10 Prop. C.2]. □

2.3. Virtual classes. We determine the virtual class of $P_n^\beta(X, \beta)$ in Setting 2.2. Firstly recall:

Definition 2.4. ([SW Ex. 16.52, pp. 410], [EG Lem. 5]) Let $E$ be a $SO(2n, \mathbb{C})$-bundle with a non-degenerate symmetric bilinear form $Q$ on a connected scheme $M$. Denote $E_+$ to be its positive real form. The half Euler class of $(E, Q)$ is 
$$c^{\frac{1}{2}}(E, Q) := \pm e(E_+) \in H^{2n}(M, \mathbb{Z}),$$
where the sign depends on the choice of orientation of $E_+$.

Definition 2.5. ([EG, KP Def. 8.7]) Let $E$ be a $SO(2n, \mathbb{C})$-bundle with a non-degenerate symmetric bilinear form $Q$ on a connected scheme $M$. An isotropic cession of $(E, Q)$ is a map 
$$\phi : E \to \mathcal{O}_M,$$
such that the composition 
$$\phi \circ \phi^\vee : \mathcal{O}_M \to E^\vee \cong E \to \mathcal{O}_M$$
is zero. If $\phi$ is furthermore surjective, we define the (reduced) half Euler class: 
$$c_{\text{red}}^{\frac{1}{2}}(E, Q) := c^{\frac{1}{2}}((\phi^\vee \mathcal{O}_M) \big/ (\phi^\vee \mathcal{O}_M), Q) \in H^{2n-2}(M, \mathbb{Z}),$$
as the half Euler class of the isotropic reduction. Here $Q$ denotes the induced non-degenerate symmetric bilinear form on $(\phi^\vee \mathcal{O}_M) \big/ (\phi^\vee \mathcal{O}_M)$.

We show reduced half Euler classes are independent of the choice of surjective cession.

Lemma 2.6. ([COT22 Lem. 5.5]) Let $E$ be a $SO(2n, \mathbb{C})$-bundle with a non-degenerate symmetric bilinear form $Q$ on a connected scheme $M$ and 
$$\phi : E \to \mathcal{O}_M$$
be a surjective isotropic cession. Then we can write the positive real form $E_+$ of $E$ as 
$$E_+ = \mathcal{E}_+ \oplus \mathbb{R}^2$$
such that 
$$c_{\text{red}}^{\frac{1}{2}}(E, Q) = \pm e(\mathcal{E}_+).$$
Moreover, it is independent of the choice of surjective cession.

In particular, when $E = \mathcal{O}^{\oplus 2} \oplus V$ such that $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus Q|_V$, we have 
$$c_{\text{red}}^{\frac{1}{2}}(E, Q) = \pm e^{\frac{1}{2}}(V, Q|_V).$$
Recall a $Sp(2r, \mathbb{C})$-bundle (or symplectic vector bundle) is a complex vector bundle of rank $2r$ with a non-degenerate anti-symmetric bilinear form. One class of quadratic vector bundles is given by tensor product of two symplectic vector bundles $V_1, V_2$. Their half Euler classes can be computed using Chern classes of $V_1, V_2$. For our purpose, we restrict to the following case.

Lemma 2.7. ([COT22 Lem. 5.6]) Let $(V_1, \omega_1)$, $(V_2, \omega_2)$ be a $Sp(2r, \mathbb{C})$ (resp. $Sp(2, \mathbb{C})$)-bundle) on a connected scheme $M$. Then 
$$(V_1 \otimes V_2, \omega_1 \otimes \omega_2)$$
defines a $SO(4r, \mathbb{C})$-bundle whose half Euler class satisfies 
$$e^{\frac{1}{2}}(V_1 \otimes V_2, \omega_1 \otimes \omega_2) = \pm (e(V_1) - c_{2r-2}(V_1) \cdot e(V_2)).$$

We determine the (reduced) virtual class of $P_n^\beta(X, \beta)$.

Theorem 2.8. In Setting 2.2 for certain choice of orientation, we have 
$$(2.5) \left[ P_n^\beta(X, \beta) \right]_{\text{vir}} = \left[ (P_n^\beta(S, \beta) \cap f^* e(T_{M_n(S, \beta)})) \times [T] - e(T) \left( (P_n^\beta(S, \beta) \cap f^* c_{2\beta}(T_{M_n(S, \beta)})) \right) \right],$$
where $f : P_n^\beta(S, \beta) \to M_n(S, \beta)$ is the map as in Setting 2.2.

2This means a real half dimensional subbundle such that $Q$ is real and positive definite on it. By homotopy equivalence $SO(m, \mathbb{C}) \sim SO(m, \mathbb{R})$, it exists and is unique up to isomorphisms.
Proof. The proof is similar as [CMT19] Prop. 4.7. Under the isomorphism \(2.3\): 
\[ P^n_\lambda(S, \beta) \times T \cong P^n_\lambda(X, \beta), \]
the universal stable pair \(I_X = (O \to F_X)\) of \(P^n_\lambda(X, \beta)\) satisfies 
\[ F_X = F_S \boxtimes O_{\Delta_T}, \]
where \(\mathbb{I}_S = (O \to F_S)\) is the universal stable pair of \(P^n_\lambda(S, \beta)\) and \(\Delta_T\) denotes the diagonal.

As in [CMT19] Eqn. (29), we have a distinguished triangle 
\[ (2.7) \quad R \text{Hom}_{\pi_p}(\mathbb{I}_X, F_X) \to R \text{Hom}_{\pi_p}(\mathbb{I}_X, \mathbb{I}_X)_0[1] \to R \text{Hom}_{\pi_p}(F_X, O)[2], \]
where \(\pi_p : P^n_\lambda(X, \beta) \times X \to P^n_\lambda(X, \beta)\) is the projection. 

From stable pair \(I_X = (O \to F_X)\) and Eqn. (2.6), we get a distinguished triangle 
\[ (2.8) \quad R \text{Hom}_{\pi_p}(F_S \boxtimes O_{\Delta_T}, F_S \boxtimes O_{\Delta_T}) \to R \text{Hom}_{\pi_p}(O, F_S \boxtimes O_{\Delta_T}) \to R \text{Hom}_{\pi_p}(\mathbb{I}_X, F_X). \]
By adjunction, we get an isomorphism 
\[ (2.9) \quad R \text{Hom}_{\pi_p}(F_S \boxtimes O_{\Delta_T}, F_S \boxtimes O_{\Delta_T}) \cong R \text{Hom}_{\pi_p}(F_S, F_S) \boxtimes (-1)^{t_1} T_T[-1], \]
where \(\pi_p : P^n_\lambda(S, \beta) \times S \to P^n_\lambda(S, \beta)\) is the projection.

Combining (2.3) and (2.9), we obtain 
\[ (2.10) \quad R \text{Hom}_{\pi_p}(\mathbb{I}_X, F_X) \cong R \text{Hom}_{\pi_p}(\mathbb{I}_S, F_S) \boxtimes (T_T \oplus O_{\Delta_T}[-1]). \]
Combining (2.7) and (2.10), we obtain 
\[ \mathcal{E}xt^1_{\pi_p}(\mathbb{I}_X, \mathbb{I}_X)_0 \cong \mathcal{E}xt^0_{\pi_p}(\mathbb{I}_S, F_S) \oplus T_T, \]
and an exact sequence 
\[ (2.11) \quad 0 \to \mathcal{E}xt^1_{\pi_p}(\mathbb{I}_S, F_S) \oplus \mathcal{E}xt^1_{\pi_p}(F_S, F_S) \boxtimes T_T \oplus \mathcal{E}xt^0_{\pi_p}(F_S, F_S) \boxtimes O_T \to \mathcal{E}xt^2_{\pi_p}(\mathbb{I}_X, \mathbb{I}_X)_0 \to \cdots. \]
We claim that the second arrow above is an isomorphism, which can be done by a dimension counting. In fact, let \(I = (O_S \to F) \in P^n_\lambda(S, \beta)\), the cohomology of the distinguished triangle 
\[ R \text{Hom}_{\lambda}(F, F) \to R \text{Hom}_{\lambda}(O_S, F) \to R \text{Hom}_{\lambda}(I, F) \]
implies that \(\text{Ext}^i_{\lambda}(I, F) = 0\) for \(i \geq 2\). In Setting \(2.2\) we know \(\text{ext}^0_{\lambda}(I, F) = \beta^2 + n + 1\), therefore 
\[ \text{ext}^1_{\lambda}(I, F) = 1. \]
As \(F\) is stable, we have 
\[ \text{ext}^2_{\lambda}(F, F) = \text{ext}^0_{\lambda}(F, F) = 1, \quad \text{ext}^1_{\lambda}(F, F) = \beta^2 + 2. \]
So the rank of the second term of (2.11) is \(2\beta^2 + 6\). One can easily check the rank of the third term in (2.11) is also \(2\beta^2 + 6\) by Riemann-Roch formula and first condition of Setting \(2.2\). To sum up, we get an isomorphism: 
\[ \mathcal{E}xt^1_{\pi_p}(\mathbb{I}_S, F_S) \oplus \mathcal{E}xt^1_{\pi_p}(F_S, F_S) \boxtimes T_T \oplus \mathcal{E}xt^0_{\pi_p}(F_S, F_S) \boxtimes O_T \cong \mathcal{E}xt^2_{\pi_p}(\mathbb{I}_X, \mathbb{I}_X)_0. \]
As in [CMT19] Prop. 4.7, one can show the decomposition in the LHS is also with respect to the Serre duality pairing on \(\mathcal{E}xt^2_{\pi_p}(\mathbb{I}_X, \mathbb{I}_X)_0\). Our the claim follows from Lemmata 2.6 and 2.7. \(\square\)

2.4. Thom-Porteous formula. As our insertion (2.7) depends only on the fundamental cycle of the universal sheaf, it is useful to know the pushforward of the virtual class \(2.5\) under the forgetful map. In this section, let \(\beta \in H^4(S, \mathbb{Z})\) be an irreducible curve class, then \(P^n_\lambda(X, \beta)\) is independent of the choice of \(t > \frac{n}{2\beta}\) [CMT19] Prop. 1.12, so we can set \(t \to \infty\) and work with PT stability. Consider the forgetful map 
\[ f : P_n(S, \beta) \to M_n(S, \beta), \quad (O_S \to F) \mapsto F. \]
Recall that \(P_n(S, \beta)\) is smooth of dimension \(\beta^2 + n + 1\) and \(M_n(S, \beta)\) is smooth of dimension \(\beta^2 + 2\). The image of \(f\) in \(M_n(S, \beta)\) is the locus 
\[ (2.12) \quad \{ F \in M_n(S, \beta) \mid h^0(F) \geq 1 \}, \]
where surjectivity follows since \(\beta\) is irreducible and \(F\) is pure, so any non-zero section \(s \in H^0(S, F)\) must have zero-dimensional cokernel. The expected dimension of sections is \(\chi(F) = n\), so the image is everything if \(n = 1\), a divisor if \(n = 0\) and a codimension 2 cycle if \(n = -1\). 

Let \(F_S\) be a (twisted) universal sheaf on \(M_n(S, \beta) \times S\). If \(n = 1\) (or more generally, there exists a \(K\)-theory class pairing with 1 with a sheaf parametrized by \(M_n(S, \beta)\)) the twisted sheaf can be taken to be an actual sheaf. For us here the difference will not matter, since we are only
interested in the Chern character of the universal sheaf, which can also be easily defined in the twisted case. We refer to [MOS] for a discussion.

Let \( \pi_M : M_n(S, \beta) \times S \to M_n(S, \beta) \) be the projection. We resolve the complex \( R\pi_M(\mathbb{F}_S) \) by a 2-term complex of vector bundles:

\[
R\pi_M(\mathbb{F}_S) \cong (E_0 \xrightarrow{\sigma} E_1).
\]

Then (2.12) is the degeneracy locus

\[
D_1(\sigma) = \{ x \in M_n(S, \beta) \mid \dim \ker(\sigma(x)) \geq 1 \}.
\]

By the Thom-Porteous formula [Ful] §14.4 (see [GT] Prop. 1) for a modern treatment and observe that \( P_n(S, \beta) \) is precisely what is called \( \tilde{D}_1(\sigma) \) there, we get the following:

**Proposition 2.9.**

\[
f_*[P_n(S, \beta)] = c_1(-R\pi_M(\mathbb{F}_S)) \cap [M_n(S, \beta)].
\]

We can calculate the right hand side above by Grothendieck-Riemann-Roch formula

\[
\text{ch}(-R\pi_M(\mathbb{F}_S)) = -\pi_M(\text{ch}(\mathbb{F}_S) \cdot \pi_S^* \text{td}(S)).
\]

We obtain the following:

\[
\begin{align*}
\text{ch}_{2}(P_4(S, \beta)) &= 1,  \\
\text{ch}_{2}(P_6(S, \beta)) &= -\pi_M(\text{ch}(\mathbb{F}_S)) - 2\pi_M(\text{ch}(\mathbb{F}_S)\pi_S^*p), \\
\text{ch}_{2}(P_{-1}(S, \beta)) &= \frac{1}{2} (c_1(-R\pi_M(\mathbb{F}_S))² + \pi_M(\text{ch}(\mathbb{F}_S)) + 2\pi_M(\text{ch}(\mathbb{F}_S)\pi_S^*p),
\end{align*}
\]

where we used Poincaré duality on the right to identify homology and cohomology and \( p \in H^4(S) \) denotes the point class. A small calculation shows that the right hand side is indeed independent of the choice of universal family \( \mathbb{F} \) (i.e. the formulae stay invariant under replacing \( \mathbb{F} \) by \( \mathbb{F} \otimes \pi_M^*\mathcal{L} \) for \( \mathcal{L} \in \text{Pic}(M_n(S, \beta)) \)). This will be useful later on.

### 2.5. Genus 0 in irreducible classes.

In this section, we prove Conjecture 1.10 (1), (2) for irreducible curve classes. We first recall a result of Fujiki [Fuj] and its generalization in [GHLJ] Cor. 23.17.

**Theorem 2.10.** ([Fuj], [GHLJ] Cor. 23.17) Let \( M \) be a hyperkähler variety of dimension \( 2n \). Assume \( \alpha \in H^{2d}\left(M, \mathbb{C}\right) \) is of type \((2j, 2j)\) on all small deformation of \( M \). Then there exists a constant \( C(\alpha) \in \mathbb{C} \) depending only on \( \alpha \) (called Fujiki constant of \( \alpha \)) such that

\[
\int_M \alpha \cdot \beta^{2n-2} = C(\alpha) \cdot q_M(\beta)^{n-j}, \quad \forall \beta \in H^2(M, \mathbb{C}),
\]

where \( q_M : H^2(M, \mathbb{C}) \to \mathbb{C} \) denotes the Beauville-Bogomolov-Fujiki form.

**Theorem 2.11.** Let \( X = S \times T \) and \( \beta \in H_2(S, \mathbb{Z}) \subseteq H_2(X, \mathbb{Z}) \) be an irreducible curve class. Then Conjecture 1.10 (1), (2) hold.

**Proof.** By Proposition 2.8, we have a forgetful map

\[
f = (f, \text{id}_T) : P_n(X, \beta) = P_n(S, \beta) \times T \to M_n(S, \beta) \times T.
\]

As our insertion 1.7 only involves fundamental cycle of the universal one dimensional sheaf, so it is the pullback \( f^* \) of a cohomology class from \( M_n(S, \beta) \times T \).

When \( n > 1 \), we have

\[
\dim_{\mathbb{C}} P_n(S, \beta) = \beta^2 + n + 1 > \beta^2 + 2 = \dim_{\mathbb{C}} M_n(S, \beta).
\]

By Theorem 2.8 and Proposition 2.9, it is easy to see

\[
P_{n, \beta}(\gamma_1, \ldots, \gamma_n) = 0, \quad n > 1.
\]

When \( n = 1 \), we take insertion \( \gamma, \gamma' \in H^4(X) \) for example (other cases follow from easier versions of the same argument). Based on Künneth decomposition:

\[
H^4(X) \cong (H^0(S) \otimes H^4(T)) \oplus (H^2(S) \otimes H^2(T)) \oplus (H^4(S) \otimes H^0(T)),
\]

we write

\[
\begin{align*}
\gamma &= A_1 \cdot 1 \otimes p + D_1 \otimes D_2 + A_2 \cdot p \otimes 1, \\
\gamma' &= A_1' \cdot 1 \otimes p + D_1' \otimes D_2' + A_2' \cdot p \otimes 1.
\end{align*}
\]

By Eqn. (2.10), the insertion becomes (see also [COT22] Proof of Thm. 5.8):

\[
(2.14) \quad \tau(\gamma) = (D_1 \cdot \beta) \otimes D_2 + A_2 f^* \pi_M(\pi_S^* p \cdot \text{ch}(\mathbb{F}_S)) \otimes 1,
\]
where $\pi_S, \pi_M$ are projections from $S \times M_n(S, \beta)$ to its factors. Hence

$$\tau(\gamma) \cdot \tau(\gamma') = (D_1 \cdot \beta) \cdot (D'_1 \cdot \beta) \otimes (D_2 \cdot D'_2) + A_2A'_2f^* (\pi_{\mathbb{P}^2} (\pi_S^* p \cdot \text{ch}(\mathcal{F}))^2 \otimes 1 + \text{others},$$

where “others” lie in $H^2(P_1(S, \beta)) \otimes H^2(T)$. By Theorem 2.8, we get

$$P_{1, \beta}(\gamma, \gamma') = (D_1 \cdot \beta) (D'_1 \cdot \beta) \int_T (D_2 \cdot D'_2) \int_{P_1(S, \beta)} f^* e(T_{M_1(S, \beta)})$$

$$- e(T)A_2A'_2 \int_{P_1(S, \beta)} f^* \left( c_{\beta^2} (T_{M_1(S, \beta)}) \cdot \pi_{\mathbb{P}^2} (\pi_S^* p \cdot \text{ch}(\mathcal{F}))^2 \right)$$

$$= (D_1 \cdot \beta) (D'_1 \cdot \beta) \int_T (D_2 \cdot D'_2) \int_{M_1(S, \beta)} e(T_{M_1(S, \beta)})$$

$$- e(T)A_2A'_2 \int_{M_1(S, \beta)} c_{\beta^2} (T_{M_1(S, \beta)}) \cdot \pi_{\mathbb{P}^2} (\pi_S^* p \cdot \text{ch}(\mathcal{F}))^2$$

$$= (D_1 \cdot \beta) (D'_1 \cdot \beta) \int_T (D_2 \cdot D'_2) e(M_1(S, \beta)),$$

where the second equality follows from Proposition 2.10 and the last equality is proved using Fujiki formula (Theorem 2.10) and the evaluation

$$q_M(\pi_{\mathbb{P}^2} (\pi_S^* p \cdot \text{ch}(\mathcal{F}))) = 0,$$

(which follows for example from [COT22, Proof of Thm. 5.8]). Conjecture 1.10 (2) then reduces to [COT22 Thm. 5.8].

2.6. Transport of integrals to the Hilbert schemes. To compute the stable pair theory on $P_n(S, \beta)$ for $n < 0$, we will need to handle more complicated descendent integrals on $M_n(S, \beta)$. As in [COT22, §4.4] which deals with the $n = 1$ case, we use here the general framework of monodromy operators of Markman [M08] (see also [O22] to reduce to the Hilbert schemes.

Consider the Mukai lattice, which is the lattice $\Lambda = H^*(S, \mathbb{Z})$ endowed with the Mukai pairing

$$(x, y) := - \int_S x^\vee y,$$

where, if we decompose an element $x \in \Lambda$ according to degree as $(r, D, n)$, we have written $x^\vee = (r, -D, n)$. Given a sheaf or complex $E$ on $S$ the Mukai vector of $E$ is defined by

$$v(E) = \sqrt{td_S} \cdot \text{ch}(E) \in \Lambda.$$

Let $M(v)$ be a proper smooth moduli space of stable sheaves on $S$ with Mukai vector $v \in \Lambda$ (where stability is with respect to some fixed polarization). We assume that there exists a universal family $\mathcal{F}$ on $M(v) \times S$. If it does not exist, everything below can be made to work by working with the Chern character $\text{ch}(\mathcal{F})$ of a quasi-universal family, see [M08] or [O22]. Let $\pi_M, \pi_S$ be the projections to $M(v)$ and $S$. One has the Mukai morphism

$$\theta_\mathcal{F} : \Lambda \to H^2(M(v)),$$

$$\theta_\mathcal{F}(x) = \left[ \pi_{M*} (\text{ch}(\mathcal{F}) \cdot \sqrt{td_S} \cdot x^\vee) \right]_{\text{deg}=2k},$$

where $[-]_{\text{deg}=k}$ stands for extracting the degree $k$ component and (as we will also do below) we have suppressed the pullback maps from the projection to $S$. Define the universal class

$$u_v = \exp \left( \frac{\theta_\mathcal{F}(v)}{(v, v)} \right) \text{ch}(\mathcal{F}) \sqrt{td_S},$$

which is independent of the choice of universal family $\mathcal{F}$. For $x \in \Lambda$, consider the normalized descendents:

$$B(x) := \pi_M(u_v \cdot x^\vee),$$

and let $B_k(x) = [B(x)]_{\text{deg}=2k}$ its degree $2k$ component.

Example 2.12. For $v = (1, 0, 1 - d)$, the moduli space becomes the punctual Hilbert scheme: $M(v) = S^{[d]}$. Then we have

$$u_v = \exp \left( \frac{-\delta}{2d - 2} \right) \text{ch}(\mathcal{I}_Z) \sqrt{td_S},$$

where we let $\delta = \pi_* \text{ch}_3(\mathcal{O}_Z)$ (so that $-2\delta$ is the class of the locus of non-reduced subschemes).

We define the standard descendents on the Hilbert scheme by

$$\mathfrak{S}_d(\alpha) = \pi_* (\pi_S^* (\alpha) \text{ch}_d(\mathcal{O}_Z)) \in H^*(S^{[d]}).$$
One obtains that
\[ B_1(p) = -\frac{\delta}{2d-2}, \]
\[ B_2(p) = \frac{1}{2} \frac{\delta^2}{(2d-2)^2} - \mathcal{G}_2(p). \]

For a divisor \( D \in H^2(S) \), one finds
\[ B_1(D) = \mathcal{G}_2(D), \]
\[ B_2(D) = \mathcal{G}_3(D) - \frac{\delta}{2d-2} \mathcal{G}_2(D). \]

Using the descendents \( B_k(x) \), one allows to move between any two moduli spaces of stable sheaves on \( S \) just by specifying a Mukai lattice isomorphism \( g : \Lambda \rightarrow \Lambda \). We give the details in the case of our interest, see [M08, O22] for the general case.

As before let \( \beta \in \operatorname{Pic}(S) \) be an irreducible effective class of square \( \beta \cdot \beta = 2d - 2 \), and let \( n \in \mathbb{Z} \). We want to connect the moduli spaces \( M_n(S, \beta) \) to \( S^{[d]} \).

Let \( \beta = e + (d - 1)f \) where \( e, f \in H^2(S, \mathbb{Z}) \) span a hyperbolic lattice: \( \mathbb{Z} e \oplus \mathbb{Z} f \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). We do not require \( e, f \) to be effective here. Define the isomorphism \( g : \Lambda \rightarrow \Lambda \) by
\[ 1 \mapsto (0, -e, n), \quad p \mapsto (0, f, 0), \quad e \mapsto (1, -n f, 0), \quad f \mapsto (0, 0, -1), \quad g|_{(1, p, c, f)^+} = \text{id}. \]

One sees that \( g \) is an isometry of the Mukai lattice and that \( g_* (0, \beta, n) = (1, 0, 1 - d) \).

Then one has:

**Theorem 2.13.** (Markman [M08, reformulation as in O22 Thm. 4]) For any \( k_i \geq 0 \) and \( \alpha_i \in H^*(S) \) and any polynomial \( P \),
\[ \int_{M_n(S, \beta)} P(B_{k_i}(\alpha_i), c_j(T_{M_n(S, \beta)})) = \int_{S^{[d]}} P(B_{k_i}(g \alpha_i), c_j(T_{S^{[d]}})). \]

**2.7. Genus 1 in irreducible classes.** Recall the genus 1 Gopakumar-Vafa invariants (Proposition 2.11). On the stable pair side, we have the following:

**Theorem 2.14.** Let \( \beta \in H_2(S, \mathbb{Z}) \subseteq H_2(X, \mathbb{Z}) \) be an irreducible curve class. Then for certain choice of orientation, we have
\[ P_{0, \beta}(\gamma) = c(T) N_1 \left( \frac{\beta^2}{2} \right) \int_{S \times \mathbb{P}} \gamma. \]

In particular, Conjecture 1.10 (3) holds in this case.

**Proof.** The strategy is as follows: First we write our stable pair invariants as integrals on the moduli spaces \( M_0(S, \beta) \), then express the integrand in terms of the classes \( B_k(x) \) and then use Markman’s Theorem 2.13 to reduce to an integral over the Hilbert scheme, which is known by the results of [CO12].

By Equ. (2.14) and Theorem 2.8 (choose the inverse orientation there), we have
\[ P_{0, \beta}(\gamma) = c(T) \int_{S \times \mathbb{P}} \gamma \cdot \int_{M_0(S, \beta)} f^* (c_\beta (T_{M_0(S, \beta)}) \cdot \pi_{M_0(S, \beta)}(\gamma \cdot \pi_{S_{\mathbb{P}}}(p) \cdot \operatorname{ch}(\mathbb{P} S))) \).

Using Proposition 2.9 we find
\[ P_{0, \beta}(\gamma) = c(T) \int_{S \times \mathbb{P}} \gamma \cdot \int_{M_0(S, \beta)} c_\beta (T_{M_0(S, \beta)}) \cdot c_1(-\mathbb{R}_{M_0(S, \beta)}(\gamma)) \cdot \pi_{M_0(S, \beta)}(\gamma \cdot \operatorname{ch}(\mathbb{P} S) \cdot \pi_{S_{\mathbb{P}}}(p)). \]

A calculation shows that we have
\[ B_1(p) = \pi_*(\operatorname{ch}(\mathbb{P} S) \cdot \pi_{S_{\mathbb{P}}}(p)). \]

Moreover, the expressions 2.13 are invariant under replacing \( \operatorname{ch}(\mathbb{P} S) \) by \( \operatorname{ch}(\mathbb{P} S) \cdot \exp(\ell) \) for any line bundle \( \ell \in H^2(M_0(S, \beta)) \). Hence we can use \( \operatorname{ch}(\mathbb{P} S) := \operatorname{ch}(\mathbb{P} S) \cdot \exp(\theta_{F_S}(v)/(v, v)) \) which shows that
\[ c_1(-\mathbb{R}_{M_0(S, \beta)}(\gamma)) = -\pi_{M_0(S, \beta)}(\gamma \cdot \pi_{S_{\mathbb{P}}}(p)) - 2\pi_{M_0(S, \beta)}(\gamma \cdot \operatorname{ch}(\mathbb{P} S) \cdot \pi_{S_{\mathbb{P}}}(p)) \]
\[ = -B_1 \left( \frac{\sqrt{\delta}}{2d-2} \right) - 2B_1(p) \]
\[ = -B_1(1 + p). \]
We obtain that:
\[
\int_{M_0(S,\beta)} c_{2d-2}(T_{M_0(S,\beta)}) \cdot c_1(-R\pi_{M_\ast}(\mathbb{P}^1)) \cdot \pi_{M_\ast} (\text{ch}_1(\mathbb{P}^1) \cdot \pi_2^\ast p)
\]
\[
= - \int_{M_0(S,\beta)} c_{2d-2}(T_{M_0(S,\beta)}) B_1(1+p)B_1(p)
\]
\[
= - \int_{S^{[d]}} c_{2d-2}(T_{S^{[d]}}) B_1(-e+f)B_1(f)
\]
\[
= - \sum_{S^{[d]}} c_{2d-2}(T_{S^{[d]}}) \Theta_2(-e+f)\Theta_2(f)
\]
\[
= - ((-e+f) \cdot C(c_{2d-2}(T_{S^{[d]}}))
\]
\[
=N_1(d-1),
\]
where we used the $k = 1$ case of [COT22, Thm. 4.2] in the last step. 

2.8. Genus 2 in irreducible classes. Let $\beta_4 \in H_2(S,\mathbb{Z}) \subseteq H_2(X,\mathbb{Z})$ be an irreducible curve class of square $\beta_4^2 = 2d - 2$. Below, we use similar method to compute stable pair invariants $P_{-1,\beta_4}$ on $X$ for all $d$.

**Theorem 2.15.** For certain choice of orientation, we have
\[
\sum_{d \in \mathbb{Z}} P_{-1,\beta_4} q^d = \left( \prod_{n=1} \left(1 - q^n\right)^{-24}\right) \left( 24q \frac{d}{dq}G_2(q) - 24G_2(q) -1 \right)
\]
\[
= 72q^2 + 1920q^3 + 28440q^4 + 305280q^5 + 2639760q^6 + 19450368q^7 + \cdots.
\]
In particular, Conjecture 1.10 (4) holds in this case.

**Proof.** As in the genus 1 case, by Theorem 2.8 and Proposition 2.9 we have:
\[
P_{-1,\beta} = -e(T) \int_{M_{-1}(S,\beta)} c_{2d-2}(T_{M_{-1}(S,\beta)}) \cdot c_2(-R\pi_{M_\ast}(\mathbb{P}^1)).
\]
With the same discussion as before one gets:
\[
c_2(-R\pi_{M_\ast}(\mathbb{P}^1)) = \frac{1}{2} B_1(1+p)^2 + B_2(1+p).
\]
Hence applying Markman’s Theorem 2.13 we conclude
\[
\int_{M_{-1}(S,\beta)} c_{2d-2}(T_{M_{-1}(S,\beta)}) \cdot c_2(-R\pi_{M_\ast}(\mathbb{P}^1))
\]
\[
= \int_{M_{-1}(S,\beta)} c_{2d-2}(T_{M_{-1}(S,\beta)}) \cdot \left( \frac{1}{2} B_1(1+p)^2 + B_2(1+p) \right)
\]
\[
= \int_{S^{[d]}} c_{2d-2}(T_{S^{[d]}}) \left( \frac{1}{2} B_1(-e+f-p)^2 + B_2(-e+f-p) \right)
\]
\[
= \int_{S^{[d]}} c_{2d-2}(T_{S^{[d]}}) \left[ \Theta_2(-e+f) + \frac{\delta}{2d-2} \right]^2
\]
\[
+ \int_{S^{[d]}} c_{2d-2}(T_{S^{[d]}}) \left( \Theta_3(-e+f) - \frac{\delta}{2d-2} \Theta_2(-e+f) - \frac{1}{2} \frac{\delta^2}{(2d-2)^2} + \Theta_2(p) \right)
\]
\[
= \frac{1}{2} \left( (-e+f)^2 + \frac{\delta \cdot \delta}{(2d-2)^2} \right) N_1(d-1) - \frac{1}{2} \frac{\delta \cdot \delta}{(2d-2)^2} N_1(d-1) + \int_{S^{[d]}} c_{2d-2}(T_{S^{[d]}}) \Theta_2(p)
\]
\[
= - N_1(d-1) + \int_{S^{[d]}} c_{2d-2}(T_{S^{[d]}}) \Theta_2(p).
\]
Thus we conclude that
\[
P_{-1,\beta} = e(T) \left( N_1(d-1) - \int_{S^{[d]}} c_{2d-2}(T_{S^{[d]}}) \Theta_2(p) \right).
\]
The desired formula now follows by the evaluation given in [COT22, Prop. 4.6]:
\[
\sum_{d \geq 0} q^d \int_{S^{[d]}} c_{2d-2}(T_{S^{[d]}}) \Theta_2(p) = \prod_{n=1}^{-24} \left( G_2(q) + \frac{1}{24} \right).
\]
Finally, comparing with Proposition 2.11 we are done. 

\[\square\]
Remark 2.16. By the global Torelli theorem, primitive curve classes on K3 surfaces can be deformed to irreducible curve classes. Combining Theorem 2.14, Theorem 2.17, Theorem 2.18 we know Conjecture 1.10 also holds for primitive curve classes $\beta \in H_2(S) \subseteq H_2(X)$.

2.9. Genus 1: multiple fiber classes of elliptic fibrations. Let $X = E \times E \times T$ be the product two copies of an elliptic curve $E$ and a K3 surface $T$. It gives the trivial elliptic fibration

\[ \pi : X \to Y : = E \times T. \]

For multiple fiber classes of $\pi$ (2.16), we have the following closed evaluation:

**Theorem 2.17.** Let $t > 0$ and $\gamma \in H^4(X)$. For certain choice of orientation, we have

\[ \sum_{\gamma_{\leq 0}} P^d_{\alpha_{\gamma_I}}(\gamma) q^n = 24 \left( \int_{E \times E \times T} \gamma \right) \cdot \sum_{m \geq 1} n^2 q^m. \]

**Proof.** By [CT19, Prop. 5.3], we know $P^d_0(X, n[\gamma])$ is independent of the choice of $t > 0$, so we may set $t \to \infty$ and work with PT stability. As in [CMT19, Lem. 3.5], there is an isomorphism

\[ \pi^* : \text{Hilb}^n(Y) \cong P_0(X, n[\gamma]), \quad I_Z \mapsto \pi^* I_Z. \]

For $I = \pi^* I_Z \in P_0(X, n[\gamma])$, by projection formula and

\[ R{\pi_*} \mathcal{O}_X \cong \mathcal{O}_Y \oplus K_Y[-1], \]

we obtain

\[ \mathcal{R} \text{Hom}_X(I, I) \cong \mathcal{R} \text{Hom}_Y(I_Z, I_Z) \oplus \mathcal{R} \text{Hom}_Y(I_Z, I_Z \oplus K_Y)[-1]. \]

By taking the traceless part, we get

\[ \text{Ext}_{X}^1(I, I_0) \cong \text{Ext}_{X}^1(I, I_0) \oplus \text{Ext}_{X}^1(I, I_0) \]

\[ \cong \text{Ext}_{Y}^1(I_Z, I_Z) \oplus \text{Ext}_{Y}^1(I_Z, I_Z), \]

where we use Serre duality in the second isomorphism.

Next we compare cosections on these obstruction spaces. By [KiP, Lem. 9.4], we have a surjective isotropic cosection

\[ \phi_X : \text{Ext}_{X}^1(I, I_0) \xrightarrow{\text{At}(I)} \text{Ext}_{X}^1(I, I \otimes T^* X) \xrightarrow{\text{tr}} H^3(X, T^* X) \xrightarrow{H \otimes 3} H^4(X, \wedge^3 T^* X) \xrightarrow{\int} \mathbb{C}, \]

where $\text{At}(I) \in \text{Ext}_{X}^1(I, I \otimes T^* X)$ denotes the Atiyah class of $I$, $H \in H^1(X, T^* X)$ is an ample divisor and $\sigma_X \in H^0(X, \wedge^3 T^* X)$ is a holomorphic symplectic form of $X$. The cosection $\phi_X$ is isotropic from the proof of [KiP, Cor. 9.5], and the surjectivity of $\phi_X$ also follows from loc. cit. together with (in the notation of [KiP, Lem. 9.4])

\[ \int_X \iota_{H \otimes X} \sigma_X \cup \beta \neq 0, \]

where $\beta$ is the Poincaré dual of the curve class of $I$, and the above non-vanishing follows since $H$ is ample and $\beta$ is effective.

By the compatibility of Atiyah classes with map $\pi : X \to Y$ (ref. [BuFl, Prop. 3.14]), we have a commutative diagram

\[ \text{Ext}_{X}^2(I, I_0) \xrightarrow{\text{At}(I)} \text{Ext}_{X}^3(I, I \otimes T^* X) \xrightarrow{\text{tr}} H^3(X, T^* X) \xrightarrow{\text{pr}} H^{1,1}(S) \otimes H^{0,2}(T) \]

\[ \text{Ext}_{Y}^2(I_Z, I_Z) \xrightarrow{\text{At}(I_Z)} \text{Ext}_{Y}^3(I_Z, I_Z \otimes T^* Y) \xrightarrow{\text{tr}} H^3(Y, T^* Y) \xrightarrow{\text{pr}} H^{1,1}(E) \otimes H^{0,2}(T), \]

where $i$ is the embedding in (2.18), $\text{tr}$ denotes the trace map and $\text{pr}$ is the projection with respect to Kunneth decomposition. We define a cosection

\[ \phi_Y : \text{Ext}_{Y}^2(I_Z, I_Z) \xrightarrow{\text{At}(I_Z)} \text{Ext}_{Y}^3(I_Z, I_Z \otimes T^* Y) \xrightarrow{\text{tr}} H^3(Y, T^* Y) \cong H^{1,1}(E) \otimes H^{0,2}(T) \xrightarrow{\epsilon} \mathbb{C}, \]

where $\epsilon(\alpha) = \int_X H \sigma_X \cdot \pi^* \alpha, \quad \alpha \in H^{1,1}(E) \otimes H^{0,2}(T)$. It is easy to see $\phi_Y$ is a positive multiple of the standard cosection of Hilb$^n(Y)$ (see e.g. [O18b, Eqn. (6)]), hence its reduced virtual class keeps the same.
By diagram (2.19), we have a commutative diagram:

\[
\begin{array}{ccc}
\text{Ext}_X^2(I, I)_0 & \xrightarrow{\phi_Y} & \mathbb{C} \\
\odownarrow{\phi_Y} & & \downarrow{\phi_Y} \\
\text{Ext}_X^2(I, I)_0 & \xrightarrow{Q_{\text{Cores}}} & \text{Ext}_X^2(I, I)_0 \cong \text{Ext}_Y^2(I_Z, I_Z)_0 \oplus \text{Ext}_Y^2(I_Z, I_Z)_0
\end{array}
\]

We claim that \(\ker(\phi_Y)\) is a maximal isotropic subspace of \(\ker(\phi_Y^X) / \image(\phi_Y^X)\). In fact, by taking dual, we have a commutative diagram

\[
\begin{array}{ccc}
\mathbb{C} & \xrightarrow{\phi_Y^*} & \text{Ext}_X^2(I, I)_0^\vee \xrightarrow{Q_{\text{Cores}}} \text{Ext}_X^2(I, I)_0^\vee \cong \text{Ext}_Y^2(I_Z, I_Z)_0^\vee \oplus \text{Ext}_Y^2(I_Z, I_Z)_0^\vee \\
\odownarrow{\phi_Y} & & \downarrow{\phi_Y} \\
\mathbb{C} & \xrightarrow{\phi_Y^*} & \text{Ext}_Y^2(I_Z, I_Z)_0^\vee.
\end{array}
\]

Since \(\phi_Y^*\) is surjective, so \(\phi_Y^*\) is injective, therefore

\[\image(\phi_Y^X) \cap \ker(\phi_Y) \subseteq \image(\phi_Y^X) \cap \text{Ext}_Y^2(I_Z, I_Z)_0 = 0,\]

and \(\ker(\phi_Y^X)\) defines a subspace of \(\ker(\phi_Y^X) / \image(\phi_Y^X)\). This is a maximal isotropic subspace as

\[i : \text{Ext}_Y^2(I_Z, I_Z)_0 \rightarrow \text{Ext}_X^2(I, I)_0\]

is so.

The above construction works in family and therefore we have

\[\left[ P_0(X, n[E]) \right]^\text{vir} = \left[ \text{Hilb}^n(Y) \right]^\text{vir} \in A_1(P_0(X, n[E])),\]

for certain choice of orientation in the LHS. Consider a commutative diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{\pi_X} & X \times P_0(X, n[E]) \xrightarrow{\pi_0} P_0(X, n[E]) \\
\mid & \mid \quad \pi = (\pi, (\pi^*)^{-1}) & \mid \mid (\pi^*)^{-1} = \pi' \mid \\
Y & \xleftarrow{\pi_Y} & Y \times \text{Hilb}^n(Y) \xrightarrow{\pi_M} \text{Hilb}^n(Y),
\end{array}
\]

and denote \(Z \hookrightarrow Y \times \text{Hilb}^n(Y)\) to be the universal 0-dimensional subscheme. Then

\[
P_{0, n[E]}(\gamma) = \int_{[P_0(X, n[E])]^\text{vir}} \pi_{P_0}^*(\pi_X^* \gamma \cdot \pi_0^* \text{ch}_3(\mathcal{O}_Z)) = \int_{[\text{Hilb}^n(Y)]^\text{vir}} \pi_{M_0}^*(\pi_X^* \gamma \cdot \pi_0^* \text{ch}_3(\mathcal{O}_Z)) = \int_{[\text{Hilb}^n(Y)]^\text{vir}} \pi_{M_0}^*(\text{ch}_3(\mathcal{O}_Z) \cdot \pi_0^* (\pi_X^* \gamma)) = \int_{[\text{Hilb}^n(Y)]^\text{vir}} \pi_{M_0}^*(\text{ch}_3(\mathcal{O}_Z) \cdot \pi_Y^* \pi_\gamma).
\]

The statement now follows from Proposition 2.13 below. \(\square\)

**Proposition 2.18.** Let \(\omega \in H^2(E, \mathbb{Z})\) be the class of point and \(D \in H^2(T, \mathbb{Q})\) any class. Then for any \(n \geq 1\) we have:

\[
\int_{[\text{Hilb}^n(E \times T)]^\text{vir}} \pi_{M_0}^*(\text{ch}_3(\mathcal{O}_Z) \pi_Y^* (\omega \otimes 1)) = (-1)^{n+1} e(T) \sum_{d|n} d^2,
\]

\[
\int_{[\text{Hilb}^n(E \times T)]^\text{vir}} \pi_{M_0}^*(\text{ch}_3(\mathcal{O}_Z) \pi_Y^* (1 \otimes D)) = 0.
\]

**Proof.** Write \(\text{Hilb} = \text{Hilb}^n(T \times E)\) and consider the diagram

\[
\begin{array}{ccc}
T \times E & \xrightarrow{\pi_{T \times E}} & \text{Hilb} \times T \times E \xrightarrow{\pi_M} \text{Hilb} \\
\delta \downarrow & & \downarrow p \\
\text{Hilb} \times T \times E & \xrightarrow{\pi_{M/E}} & \text{Hilb} / E,
\end{array}
\]

where the quotient by \(E\) is taken in the stacky sense. The universal subscheme \(Z \subset \text{Hilb} \times T \times E\) has a natural \(E\)-linearization and hence arises from the pullback of a subscheme \(Z / E \subset (\text{Hilb} \times T \times E) / E\). Moreover, as in \cite{O18b}, there exists a natural (0-dimensional) virtual class \([\text{Hilb} / E]^\text{vir}\) such that

\[\text{[Hilb]}^\text{vir} = p^* [\text{Hilb} / E]^\text{vir}.\]
Since the virtual class of \( \text{Hilb}/E \) arises from a symmetric obstruction theory (on an \( \text{étale} \) cover of \( \text{Hilb}/E \)), its degree can be computed by an Behrend weighted Euler characteristic \( [3] \):

\[
\int_{[\text{Hilb}/E]^{vir}} 1 = e(\text{Hilb}/E, \nu).
\]

We argue now as follows: Applying the pushpull formula and using \( p \circ \pi_M = \pi_{M/E} \circ \tilde{p} \) we have

\[
N_n := \int_{[\text{Hilb}]^{vir}} \pi_M*\left( ch_3(OZ)\pi_T^{*}\chi E(\omega \otimes 1) \right)
= \int_{[\text{Hilb}/E]^{vir}} \pi_{M/E*}\tilde{p}*(ch_3(OZ)\pi_T^{*}\chi E(\omega \otimes 1))
= \int_{[\text{Hilb}/E]^{vir}} \pi_{M/E*}\left( ch_3(OZ/E)\tilde{p}*(\pi_T^{*}\chi E(\omega \otimes 1)) \right).
\]

Then by checking on fibers we have \( \tilde{p}*(\pi_T^{*}\chi E(\omega \otimes 1)) = 1 \) as well as \( \pi_{M/E*}ch_3(OZ/E) = n \). This implies that

\[
N_n = n \int_{[\text{Hilb}/E]^{vir}} 1 = n \cdot e(\text{Hilb}/E, \nu) = 24(-1)^{n-1} \sum_{d|n} d^2,
\]

where for the last equality we have used \([51] \text{ Cor. 1}. \]

For the second integral we argue identically, but observe that we have

\[
ch_3(OZ)\pi_T^{*}\chi E(1 \otimes D) = \tilde{p}*(ch_3(OZ/E)\pi_T^{*}\chi E(D)),
\]

so when pushing forward by \( \tilde{p} \) the integral vanishes.

Similarly, we can consider a nontrivial elliptic fibration:

\[
\tilde{p} = (p, \text{id}_T) : X = S \times T \to \mathbb{P}^1 \times T,
\]

where \( p : S \to \mathbb{P}^1 \) is an elliptic \( K3 \) surface with a section \( i \). Let \( f \) be a generic fiber of \( \tilde{p} \).

**Theorem 2.19.** Let \( t > 0 \) and \( \gamma \in H^4(X) \). Then for certain choice of orientation, we have

\[
(2.20) \quad \sum_{r \geq 0} P_{0, r[f]}(\gamma) q^r = 24 \left( \int_{\mathcal{S}_x \gamma} \right) \cdot \sum_{m \geq 1} \sum_{n|m} n^2 q^m.
\]

**Proof.** The first proof is parallel to the proof of Theorem \( 2.17 \) For the second part, we need to evaluate

\[
(2.21) \quad \int_{[\text{Hilb}^{vir}(\mathbb{P}^1 \times T)]^{vir}} \pi_M*\left( ch_3(OZ) \pi_T^{*}\chi E(\omega \otimes 1) \right),
\]

where \( \omega \in H^2(\mathbb{P}^1) \) is the class of a point. We consider the degeneration of \( T \times \mathbb{P}^1 \) given by the product of \( T \) with the degeneration of \( \mathbb{P}^1 \) into a chain of three \( \mathbb{P}^1 \)’s. By specializing the insertion \( \omega \) to the middle factor, we are reduced to an integral of the relative Hilbert schemes \( \text{Hilb}^n(T \times \mathbb{P}^1/T_0 \cup T_\infty) \) with the same integrand. But this integral is also the outcome of applying the degeneration formula to the integrals considered in Proposition \( 2.18 \) (under the degeneration of \( E \) to a nodal \( \mathbb{P}^1 \)). Hence \( 2.21 \) is given by \( (-1)^{n+1}\gamma(T) \sum_{d|n} d^2 \) as well. For the analogue of the second integral in Proposition \( 2.18 \) the localization formula applied to the scaling action of \( C^* \) on \( \mathbb{P}^1 \) shows that it vanishes. \( \square \)

**Remark 2.20.** On the product of two \( K3 \) surfaces, genus 1 Gopakumar-Vafa invariants in imprimitive classes are defined in \([53] \text{ Def. A.1}. \) In particular, for multiple fiber classes \( \beta = r[f] \) above, by using \([53] \text{ Eqn. (5.7)}, \) we know \( n_{1, r[f]}(\gamma) = 0 \) if \( r > 1 \).

### 3. Hilbert schemes of two points on \( K3 \)

#### 3.1. Rational curves on exceptional locus

Let \( S \) be a \( K3 \) surface. Consider the Hilbert-Chow map

\[
\pi : \text{Hilb}^2(S) \to \text{Sym}^2(S)
\]

to the symmetric product of \( S \). Let \( D \) be the exceptional divisor fitting into Cartesian diagram:

\[
\begin{array}{ccc}
D & \xrightarrow{i} & \text{Hilb}^2(S) \\
\downarrow & & \downarrow \pi \\
S & \xrightarrow{\Delta} & \text{Sym}^2(S)
\end{array}
\]
where $\Delta$ is the diagonal embedding. Note that $\pi : D \to S$ is a $\mathbb{P}^1$-bundle and any fiber of it has normal bundle $O_{\mathbb{P}^1}(-2,0,0)$.

**Theorem 3.1.** When $t = \frac{m}{n \cdot 3} + 0^+$ (i.e. in JS chamber), Conjecture [CMT19] (1), (2) hold for multiple fiber classes $\beta = r[\mathbb{P}^1]$ $(r \geq 1)$ of $\pi$ as above.

**Proof.** As in [CMT19, Lem. 6.4], [CMT20, Lem. 3.1], by Jordan-Hölder filtration, the JS moduli space $P_{\alpha}^{IS}(X, d[\mathbb{P}^1])$ is nonempty only if
\[
d \mid n, \ n > 0.
\]
so we may assume $n = m \cdot d$ for $m \in \mathbb{Z}_{\geq 1}$. Consider the map
\[
(3.1) \quad f : P_{\alpha d}(X, d[\mathbb{P}^1]) \to \text{Sym}^d(S), \quad (F, s) \mapsto \pi_*[F].
\]
As the insertion [1.7] only involves fundamental cycle of the universal one dimensional sheaf $F$, we have
\[
[F] = \bar{f}^*[Z],
\]
where $[Z] \mapsto \text{Sym}^d(S) \times S$ is the class of incident subvariety and $\bar{f} = (f, id_S)$. Therefore
\[
\int_{P_{\alpha d}(X, d[\mathbb{P}^1])}^{vir} \prod_{i=1}^m \tau(\gamma_i) = \int_{P_{\alpha d}(X, d[\mathbb{P}^1])}^{vir} f^*\Phi,
\]
for some $\Phi \in H^{2(m+1)}(\text{Sym}^d(S))$. When $m > 1$, we have $md + 1 > 2d$, therefore $\Phi = 0$ and
\[
P_{\alpha d, d[\mathbb{P}^1]}^{IS}(\gamma_1, \ldots, \gamma_l) = 0, \quad \text{if } m > 1.
\]
For $m = 1$, by a Jordan-Hölder filtration argument as [CMT20, Lem. 3.1] again, we have
\[
\text{Hilb}^d(S) \cong P^{IS}(D, d[\mathbb{P}^1]) \cong P^{IS}(X, d[\mathbb{P}^1]),
\]
\[
I_Z \mapsto \pi_*I_Z \mapsto (O_X \to i_*\pi^*O_Z).
\]
For $I_X = (O_X \to i_*\pi^*O_Z)$, we write $I_D = (O_D \to \pi^*O_Z)$. As in [CMT19, Prop. 4.3], [CKM20, Prop. 4.2], we have a canonical isomorphism
\[
\text{Ext}^1_D(I_D, \pi^*O_Z) \cong \text{Ext}^1_X(I_X, I_X)_0,
\]
and an inclusion of maximal isotropic subspace
\[
(3.2) \quad \text{Ext}^2_D(I_D, \pi^*O_Z) \to \text{Ext}^2_X(I_X, I_X)_0.
\]
From distinguished triangle
\[
I_D \to O_D \to \pi^*O_Z,
\]
we obtain a distinguished triangle
\[
R\text{Hom}_D(\pi^*O_Z, \pi^*O_Z) \to R\text{Hom}_D(O_D, \pi^*O_Z) \to R\text{Hom}_D(I_D, \pi^*O_Z).
\]
By projection formula, we have
\[
R\text{Hom}_D(\pi^*O_Z, \pi^*O_Z) \cong R\text{Hom}_S(O_Z, O_Z), \quad R\text{Hom}_D(O_D, \pi^*O_Z) \cong R\text{Hom}_S(O_S, O_Z).
\]
Therefore we get an exact sequence
\[
(3.3) \quad 0 = H^1(S, O_Z) \cong H^1(D, \pi^*O_Z) \to \text{Ext}^1_D(I_D, \pi^*O_Z) \to \text{Ext}^2_S(O_Z, O_Z) \to 0.
\]
By Serre duality, we have
\[
(3.4) \quad \text{Ext}^2_S(O_Z, O_Z) \cong \text{Ext}^0_S(O_Z, O_Z) \cong H^0(S, O_Z)^\vee.
\]
Combining Equ. (3.2), (3.3), (3.4), we obtain a maximal isotropic subspace
\[
H^0(S, O_Z)^\vee \cong \text{Ext}^2_X(I_X, I_X)_0.
\]
Working in family, we see that the dual of tautological bundle $O_S^{d[\mathbb{P}^1]}$ on $\text{Hilb}^d(S)$ is a maximal isotropic subbundle of the obstruction bundle of $P_{\alpha d}^{IS}(X, d[\mathbb{P}^1])$. By Lemma 2.6 we obtain
\[
[P_{\alpha d}^{IS}(X, d[\mathbb{P}^1])]^{vir} = [\text{Hilb}^d(S)] \cap c_{d-1} \left( O_S^{d[\mathbb{P}^1]} \right),
\]
for certain choice of orientation. As for insertions, consider the following diagram

\[
\begin{array}{ccc}
S & \xrightarrow{\pi} & X \\
\downarrow{\pi_S} & & \downarrow{\pi_X} \\
S \times \text{Hilb}^d(S) & \xrightarrow{(\pi, \text{id})} & D \times \text{Hilb}^d(S) \xrightarrow{(\iota, \text{id})} X \times \text{Hilb}^d(S) \\
\downarrow{\pi_M} & & \downarrow{\pi_M} & \downarrow{\pi_M} \\
\text{Hilb}^d(S) & & \text{Hilb}^d(S) & \text{Hilb}^d(S),
\end{array}
\]

let \( Z \hookrightarrow \text{Hilb}^d(S) \times S \) denote the universal zero dimensional subscheme, then

\[
\tau(\gamma) = \pi_{M*} (\pi_X^* \gamma \cdot \ch_1(i_* \varphi^* \mathcal{O}_Z)) \\
= \pi_{M*} (\pi_X^* \gamma \cdot \iota_! \varphi^* [Z]) \\
= \pi_{M*} \tilde{i}_* (\tilde{i}^! \pi_X^* \gamma \cdot \varphi^* [Z]) \\
= \pi_{M*} \tilde{\pi}_* (\pi_D^* \gamma \cdot \varphi^* [Z]) \\
= \pi_{M*} (\pi_D^* \gamma \cdot [Z]) \\
= \pi_{M*} (\pi_D^* \gamma \cdot [Z]) \in H^2(\text{Hilb}^d(S)),
\]

which depends only on \([Z]\) and hence it is a pullback from \(\text{Sym}^d(S)\) by the Hilbert-Chow map

\[
\text{HC}: \text{Hilb}^d(S) \to \text{Sym}^d(S).
\]

To sum up, we have

\[
P^\text{IS}_{d,d[p]}(\gamma_1, \ldots, \gamma_l) = \int_{\text{Hilb}^d(S)} c_{d-1} \left( \mathcal{O}_S^{[d]} \cdot \prod_{i=1}^l \pi_{M*} (\pi_S^* \gamma_i \cdot [Z]) \right).
\]

When \( d = 1 \), this reduces to [CO122] Lem. 3.7. When \( d > 1 \), we claim the above integral is zero. In fact, by [Lehn] Thm. 4.6, we have the formula

\[
\sum_{m \geq 0} c_{d-1} \left( \mathcal{O}_S^{[m]} \right) z^m = \exp \left( \sum_{m \geq 1} \frac{(-1)^{m-1}}{m} q_m(1) z^m \right) \cdot 1
\]

where \( q_m(1) \) are linear maps (called Nakajima operators)

\[
q_m(1) \in \text{End}(\mathbb{H}), \quad \mathbb{H} = \bigoplus_{m \geq 0} H^*(\text{Hilb}^m(S), \mathbb{Q}),
\]

which is of bidegree \((m, 2m - 2)\). By looking at the bidegree \((d, 2d - 2)\)-part, we have

\[
c_{d-1} \left( \mathcal{O}_S^{[d]} \right) = q_d(1), \quad \text{where } 1 \in H^0(\text{Hilb}^0(S)).
\]

By the definition of \( q_d(1) \) in [Lehn] Def. 2.3, we have \( q_d(1)(1) = \pi_{M*}[\mathcal{Q}] \) where \( \mathcal{Q} \) is the cycle on \(\text{Hilb}^d(S) \times S\) supported on \((\xi, x)\) with \(\text{Supp}(\xi) = x\). Therefore we know \( c_{d-1} \left( \mathcal{O}_S^{[d]} \right) \) is supported on \(H^{-1}(\Delta)\), where

\[
\Delta = \{(x, \ldots, x) \in \text{Sym}^d(S) \} \subseteq \text{Sym}^d(S)
\]

is the small diagonal. Our insertion is a pullback from \(\text{Sym}^d(S)\) and gives \((d + 1)\)-dimensional constraint on \(\text{Sym}^d(S)\). If \( d > 1, d + 1 > 2 = \dim \Delta \), therefore the integral is zero.

3.2. **Small degree curve classes on** \(X = T^*\mathbb{P}^2\). When the \(K3\) surface \(S\) has a \((-2)\)-curve \(C \subset S\), the Hilbert scheme \(\text{Hilb}^2(S)\) contains \(\text{Sym}^2(C) \subset \text{Hilb}^2(S)\) as a Lagrangian subvariety. For curve classes coming from \(\text{Sym}^2(C) \cong \mathbb{P}^4\), our invariants can be studied on the local model \(X = T^*\mathbb{P}^2\).

We have an identification of curve classes:

\[
H_2(X, \mathbb{Z}) = H_2(\mathbb{P}^2, \mathbb{Z}) = \mathbb{Z}[\ell],
\]

where \(\ell \subset \mathbb{P}^2\) is a line. Let \(H \in H^2(T^*\mathbb{P}^2)\) be the pullback of hyperplane class and identify \(H_2(T^*\mathbb{P}^2, \mathbb{Z}) \cong \mathbb{Z}\) by its degree against \(H\). Gopakumar-Vafa invariants are given as follows:
Proposition 3.2. ([COT22 Cor. 6.2])

\[
n_{0,d}(H^2, H^2) = \begin{cases} 
1 & \text{if } d = 1, \\
-1 & \text{if } d = 2, \\
0 & \text{otherwise.} 
\end{cases}
\]

\[n_{1,1}(H^2) = 0, \quad n_{2,1} = 0.\]

In the stable pair side, we compute invariants for small degree curve classes.

Proposition 3.3. For certain choice of orientation, we have

\[P_{1,1}(H^2, H^2) = 1, \quad P_{1,2}(H^2, H^2) = -1, \quad P_{1,3}(H^2, H^2) = 0,\]

\[P_{0,1}(H^2) = P_{0,2}(H^2) = 0, \quad P_{0,3}(H^2) = 1, \quad P_{-1,1} = P_{-1,2} = P_{-1,3} = 0.\]

Moreover, \(P_{n}^3(X, d)\) is independent of the choice of \(t > n/d\) in the listed cases above.

In particular, for \(X = T^*\mathbb{P}^2\), we have

- Conjecture (2) holds when \(d \leq 3\).
- Conjecture (3), (4) hold.

Proof. As noted in [COT22 Proof of Lem. 6.3], we have a diagram

\[X = T^*\mathbb{P}^2 \overset{i}{\longrightarrow} O_{\mathbb{P}^2}(-1)^{\oplus 3} \quad \pi \quad T,\]

where \(i\) is a closed imbedding (coming from the Euler sequence) and \(\pi\) contracts \(\mathbb{P}^2\) to a point in an affine scheme \(T\). It is easy to see that any one dimensional closed subscheme \(C \subset X\) with \([C] = d (d = 1, 2)\) satisfies \(\chi(\mathcal{O}_C) \geq 1\). Therefore by [CT19 Prop. 1.12], we know for \(n = -1, 0, 1\) and \(d \leq 3\), the moduli space \(P_n^3(X, d)\) is independent of the choice of \(t > n/d\). So we may take \(t \to \infty\) and work with PT stability. Using similar analysis as [CKM20 Prop. 3.9], we know all stable pairs \((\mathcal{O}_X \to F)\) in the above cases are scheme theoretically supported on the zero section \(\mathbb{P}^2 \subset X\) and \(F\) are stable. Then obviously \(P_{-1}(X, d) = \emptyset\) if \(d \leq 3\) and corresponding invariants vanish.

When \(n = 1, d \leq 3\), the isomorphism

\[P_1(X, d) \cong M_1(X, d), \quad (\mathcal{O}_X \to F) \to F,\]

to the moduli space of one dimensional stable sheaves \(F\) with \([F] = d [\ell]\) and \(\chi(F) = 1\) will reduce the computation to the corresponding one on \(M_1(X, d)\) [COT22 Prop. 6.5].

When \(d = 1, 2\), we have \(P_0(X, d) = \emptyset\), so invariants are zero. For \(d = 3\), the support map

\[P_0(X, 3) \cong P_0(\mathbb{P}^2, 3) \overset{\bar{\pi}}{\cong} |O_{\mathbb{P}^2}(3)| \cong \mathbb{P}^9, \quad F \mapsto \text{supp}(F)\]

is an isomorphism. The universal one dimensional sheaf satisfies \(F = \mathcal{O}_C\) for the universal (1,3)-divisor \(C \hookrightarrow \mathbb{P}^9 \times \mathbb{P}^2\). Let \(\pi_M\) : \(P_0(X, 3) \times \mathbb{P}^2 \to P_0(X, 3)\) be the projection. Bott’s formula [OSS pp. 4] implies that

\[R\text{Hom}_{\pi_M}(\mathcal{O}, \mathcal{O}(-C) \boxtimes T^*\mathbb{P}^2) \cong \mathcal{O}_{\mathbb{P}^9}(-1)^{\oplus 8}, \quad R\text{Hom}_{\pi_M}(\mathcal{O}, \mathcal{O}(C) \boxtimes T^*\mathbb{P}^2) \cong \mathcal{O}_{\mathbb{P}^9}(-1)^{\oplus 8}, \quad R\text{Hom}_{\pi_M}(\mathcal{O}, \mathcal{O} \boxtimes T^*\mathbb{P}^2) \cong \mathcal{O}_{\mathbb{P}^9}[-1].\]

Therefore, we have

\[R\text{Hom}_{\pi_M}(\mathcal{O}_C, \mathcal{O}_C \boxtimes T^*\mathbb{P}^2)[1] \cong R\text{Hom}_{\pi_M}(\mathcal{O}(-C) \to \mathcal{O}, (\mathcal{O}(-C) \to \mathcal{O}) \boxtimes T^*\mathbb{P}^2)[1] \cong \mathcal{O}_{\mathbb{P}^9}(-1)^{\oplus 8} \oplus \mathcal{O}_{\mathbb{P}^9}(1)^{\oplus 8} \oplus \mathcal{O}_{\mathbb{P}^9} \oplus \mathcal{O}_{\mathbb{P}^9}.\]

By Grothendieck-Verdier duality, it is easy to see

\[\mathcal{O}_{\mathbb{P}^9}(-1)^{\oplus 8} \oplus \mathcal{O}_{\mathbb{P}^9}\]

is a maximal isotropic subbundle of \(R\text{Hom}_{\pi_M}(\mathcal{O}_C, \mathcal{O}_C \boxtimes T^*\mathbb{P}^2)[1]\).

Following the proof of Theorem 2.8 one can show the reduced virtual class of \(P_0(X, 3)\) can be calculated as the reduced half Euler class of the bundle \(R\text{Hom}_{\pi_M}(\mathcal{O}_C, \mathcal{O}_C \boxtimes T^*\mathbb{P}^2)[1]\). Therefore

\[[P_0(X, 3)]^{\text{vir}} = \pm \varepsilon (\mathcal{O}_{\mathbb{P}^9}(-1)^{\oplus 8}) \cap [\mathbb{P}^9] \in H_4(\mathbb{P}^9).\]
Let $h \in H^2(\mathbb{P}^9)$ denote the hyperplane class. It is straightforward to check
\[ \tau_0(H^2) = [h]. \]
By integration again the virtual class, we have the desired result. \hfill \square

**APPENDIX A. A CONJECTURAL VIRTUAL PUSHFORWARD FORMULA**

Let $\beta \in H_2(X, \mathbb{Z})$ be an irreducible curve class on a holomorphic symplectic 4-fold $X$. There is a well-defined forgetful map
\[ f : P_n(X, \beta) \to M_n(X, \beta), \quad (\mathcal{O}_X \to F) \mapsto [F], \]
to the coarse moduli scheme of one dimensional stable sheaves $F$ with $[F] = \beta$, $\chi(F) = n$.
Motivated by the Thom-Porteous formula (Proposition 2.9), we conjecture the following:

**Conjecture A.1.** In the above setting, there exists a choice of orientation such that
\[ f_*[P_n(X, \beta)]^\text{vir} = c_{1-n}(-R\pi_{M_*}(\mathcal{F})) \cap [M_n(X, \beta)]^\text{vir}, \]
where $\pi_M : M_n(X, \beta) \times X \to M_n(X, \beta)$ is the projection and $\mathcal{F}$ is a universal sheaf (if exists).

**Remark A.2.** This should be proved by adapting Park’s beautiful work on virtual pullback \cite{Park} to the cosection localized version.

We can rewrite the degree of $[P_{-1}(X, \beta)]^\text{vir}$ as a descendent integral on $[M_1(X, \beta)]^\text{vir}$. Let $\mathcal{F}_{\text{norm}}$ be the normalized universal sheaf on $M_1(X, \beta) \times X$, i.e. $\det(R\pi_{M_*}\mathcal{F}_{\text{norm}}) \cong \mathcal{O}_{M_1(X, \beta)}$.

**Proposition A.3.** Assume Conjecture A.1. For certain choice of orientation, we have
\[ P_{-1, \beta} = -\int_{[M_1(X, \beta)]^\text{vir}} \pi_{M_*}(\text{ch}_4(\mathcal{F}_{\text{norm}})) - \frac{1}{12} \int_{[M_1(X, \beta)]^\text{vir}} \pi_{M_*}(\text{ch}_4(\mathcal{F}_{\text{norm}}) \pi^* X(e_2(X))). \]
Using notations from \cite[§2.1]{COT22}, this is written as
\[ (A.1) \quad P_{-1, \beta} = -[\tau_3(1)_{\beta}^{DT_1} - \frac{1}{12} \tau_1(e_2(X))_{\beta}^{DT_1}]. \]

**Proof.** The derived dual gives an isomorphism
\[ M_{-1}(X, \beta) \cong M_1(X, \beta), \quad F \mapsto F^\vee. \]
Then the computation is finished by applying the Grothendieck-Riemann-Roch formula. \hfill \square

**Remark A.4.** Based on Conjecture A.1 (4), this reproduces genus 2 Gopakumar-Vafa invariants of $X$ and therefore providing a sheaf theoretic approach to them using descendent integrals on moduli spaces of one dimensional stable sheaves as \cite{CT20a}.

**Proposition A.5.** Conjecture A.1 holds on the product $X = S \times T$ of two $K3$ surfaces. In particular, Eqn. (A.1) holds in this case.

**Proof.** Say $\beta \in H_2(S, \mathbb{Z}) \subseteq H_2(X, \mathbb{Z})$, we have
\[ f = f_S \times \text{id}_T : P_n(X, \beta) \cong P_n(S, \beta) \times T \to M_n(X, \beta) \cong M_n(S, \beta) \times T, \]
for forgetful map $f_S : P_n(S, \beta) \to M_n(S, \beta)$.

By Theorem 2.9 and \cite[Thm. 5.7]{COT22}, for certain choice of orientation, we have
\[ (A.2) \quad [P_n(X, \beta)]^\text{vir} = ([P_n(S, \beta)] \cap f_S^*e(T_{M_n(S, \beta)})) \times [T] - e(T) ([P_n(S, \beta)] \cap f_S^*e_2(T_{M_n(S, \beta)})), \]
\[ [M_n(X, \beta)]^\text{vir} = ([M_n(S, \beta)] \cap e(T_{M_n(S, \beta)})) \times [T] - e(T) ([M_n(S, \beta)] \cap e_2(T_{M_n(S, \beta)})). \]
Also note that a universal sheaf $\mathcal{F}$ on $M_n(X, \beta) \times X$ (if exists) is of form
\[ \mathcal{F} = \mathcal{F}_S \boxtimes \mathcal{O}_{\Delta_T}, \]
where $\mathcal{F}_S$ is a universal sheaf on $M_n(S, \beta) \times S$ and $\Delta_T$ is the diagonal of $T \times T$. So
\[ (A.3) \quad R\pi_{M_*}\mathcal{F} = R\pi_{M_S*}\mathcal{F}_S, \]
where $\pi_{M_S} : M_n(S, \beta) \times S \to M_n(S, \beta)$ is the projection. Combining Eqs. (A.2), (A.3), we are reduced to Proposition 2.9. \hfill \square
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