Bose-Einstein condensation in tight-binding bands

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We present a theoretical study of condensation of bosons in tight binding bands corresponding to simple cubic, body centered cubic, and face centered cubic lattices. We have analyzed non-interacting bosons, weakly interacting bosons using Bogoliubov method, and strongly interacting bosons through a renormalized Hamiltonian approach valid for number of bosons per site less than or equal to unity. In all the cases studied, we find that bosons in a body centered cubic lattice has the highest Bose condensation temperature. The growth of condensate fraction of non-interacting bosons is found to be very close to that of free bosons. The interaction partially depletes the condensate at zero temperature and close to it, while enhancing it beyond this range below the Bose-Einstein condensation temperature. Strong interaction enhances the boson effective mass as the band-filling is increased and eventually localizes them to form a Bose-Mott-Hubbard insulator for integer filling.

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I. INTRODUCTION

In a many boson system, when the thermal de-Broglie wavelength of a particle becomes comparable to the inter-particle separation, a condensation in momentum space occurs at a finite temperature and a macroscopic number of particles occupy the lowest single particle energy level and enter into a phase locked state. This phenomenon predicted by Einstein\textsuperscript{1} by applying Bose statistics\textsuperscript{2} to a three dimensional homogeneous system of non-interacting atoms in the thermodynamic limit is well known as the Bose-Einstein condensation. Though it took seventy years to eventually observe Bose-Einstein condensation\textsuperscript{3–6} in metastable, inhomogeneous, finite, and three dimensional Bose atom vapors, progress ever since has been tremendous. Extensive investigations of various aspects of the condensate, a macroscopic quantum coherent state of atoms, is now a rapidly expanding field of research\textsuperscript{7–11}.

One of the interesting recent developments in this field is the observation\textsuperscript{12}, by Greiner and collaborators, of a transition between superfluid and Mott insulating phases of bosons in an optical lattice. Experimentalists have explored this superfluid-Mott transition and the excitation spectra in the superfluid\textsuperscript{12,13} and Mott insulating phases\textsuperscript{12}. In these experiments, the condensate is adiabatically transferred to a simple cubic optical lattice produced by counter propagating laser beams. By changing the characteristics of the laser beams, it has been possible to achieve great control over $t/U$, where $t$ is the inter-site hopping energy and $U$ the on-site boson-boson interaction energy. It has been shown that bosons in optical lattices can be adequately modeled employing a clean Bose-Hubbard model\textsuperscript{14}. That a band-width controlled transition from a superfluid state to a Bose-Mott-Hubbard (BMH) insulating state is possible at a critical value of $t/U$ for integer number of bosons per site was predicted in theoretical studies\textsuperscript{14–19} on Bose-Hubbard model. In the BMH insulator, the bosons are site localized and the single particle excitation spectrum acquires a gap. Recently, superfluid to Mott insulator transition was observed\textsuperscript{20} in finite one-dimensional optical lattices as well.

The experimental realization of bosons in optical lattices provides a microscopic laboratory for the exploration of collective behavior of quantum many particle systems in narrow energy bands with great control on $t$, $U$, and the number of boson per site ($n$). There is already theoretical studies\textsuperscript{21,22} on the possibility of creating different types of two dimensional lattices (triangular, square, hexagonal, for example). It has been proposed\textsuperscript{23} recently that trimerized optical Kagome lattice can be achieved experimentally and that a superfluid-Mott transition at fractional filling is possible for bosons in this lattice. It is reasonable to expect that three and two dimensional optical lattices of different symmetries will be created in near future. Many experimental groups have produced three dimensional optical lattices and experimental studies of Bose condensates in these lattices will surely receive increasing attention in the coming years. Motivated by such possibilities, we present a theoretical study of Bose-Einstein condensation in tight binding bands corresponding to simple cubic (sc), body centered cubic (bcc), and face centered cubic (fcc) lattices. We have analyzed non-interacting, weakly interacting, and strongly interacting bosons. The weakly interacting bosons were analyzed using a Bogoliubov type theory\textsuperscript{24}. For the strongly interacting bosons, we use a renormalized Hamiltonian valid for $n \leq 1$ obtained by projecting out on-site multiple occupancies. This analysis is presented in the next section and the conclusions are given in Sec. III.
II. BOSE CONDENSATION IN TIGHT BINDING BANDS

A. Non-interacting bosons

In this section we discuss the simplest of the three cases studied. The Hamiltonian of the non-interacting bosons in a tight-binding energy band is:

\[ H = \sum_{\mathbf{k}} [\epsilon(\mathbf{k}) - \mu] c_{\mathbf{k}}^\dagger c_{\mathbf{k}}, \]

where \( \epsilon(\mathbf{k}) \) is the band-structure corresponding to \( \text{sc}, \text{bcc}, \) and \( \text{fcc} \) lattices, \( \mu \) the chemical potential, and \( c_{\mathbf{k}}^\dagger \) is the boson creation operator. Confining to nearest neighbor Wannier functions overlaps, these band structures when lattice constant is set to unity are,

\[ \epsilon_{\text{sc}}(k_x, k_y, k_z) = -2t \sum_{\mu=x}^z \cos(k_{\mu}), \]

\[ \epsilon_{\text{bcc}}(k_x, k_y, k_z) = -8t \sum_{\mu=x}^z \cos \left( \frac{k_{\mu}}{2} \right), \]

and,

\[ \epsilon_{\text{fcc}}(k_x, k_y, k_z) = -2t \sum_{\mu=x, \mu \neq \nu \neq x}^z \sum_{\nu=x}^z \cos \left( \frac{k_{\mu}}{2} \right) \cos \left( \frac{k_{\nu}}{2} \right), \]

where \( t \) is the nearest neighbor boson hopping energy. The condensation temperature \( (T_B) \) for bosons in these bands can be calculated from the boson number equation:

\[ n = \frac{1}{N_x N_y N_z} \sum_{k_x} \sum_{k_y} \sum_{k_z} \frac{1}{2} \sum_{q} \delta(\epsilon(k_x, k_y, k_z)) - \mu) / k_B T - 1, \]

where \( N_x = N_y = N_z \) is the total number of lattice sites, \( k_B \) the Boltzmann constant, and \( T \) the temperature. At high temperature, the chemical potential is large and negative. As the temperature comes down, the chemical potential raises gradually to eventually hit the bottom of the band. Below this temperature there is macroscopic occupation of the band bottom and we have a Bose condensate. On further reduction of temperature, the chemical potential is pinned to the bottom of the band and bosons are progressively transferred from excited states in to the condensate. All the particles are in the condensate at absolute zero temperature. Fixing the chemical potential at the bottom of the band the solution of the number equation gives the Bose condensation temperature \( (T_B) \). For \( T > T_B \) a lower value of \( \mu \) satisfies the number equation, while for \( T < T_B \) the RHS of Eq. (4) is less than the number of bosons \( (n) \), the difference being the number of condensate particles \( (n_0) \). We determined \( T_B \) and \( n_0 \) for different lattices as a function of filling and temperature. Results of these calculations are shown in Figs. 1-5. We find that for bosons in the tight binding band corresponding to the \( \text{bcc} \) lattice has the highest Bose condensation temperature. Comparing the single boson Density Of States (DOS), we find that the band structure with smallest DOS near the bottom of the band has the highest \( T_B \). The physical reason behind it is that, as the temperature is lowered from above the Bose condensation temperature, the bosons are transferred from the high energy states to the low energy states following the Bose distribution function. To accommodate these bosons the chemical potential would touch the bottom of the boson band at a higher temperature for a system with smaller DOS near the bottom of the band compared to a system which has larger DOS there. Consequently the Bose condensation temperature for the former system would be higher than that for the latter. The growth of the condensate fraction for different lattices is shown in Figs. (1)-(3), and in Fig (4) we have compared condensate fraction growth for different lattices. Also shown by dotted lines is the condensate fraction for free bosons and they are found to be rather close. The variation of \( T_B \) with \( n \) is shown in Fig. (5) which shows an initial fast growth and a monotonic increase for higher values of \( n \).

B. Weakly interacting bosons

To study Bose condensation of weakly interacting Bosons in tight binding bands, we employ the following Hamiltonian:

\[ H = \sum_{\mathbf{k}} [\epsilon(\mathbf{k}) - \mu] c_{\mathbf{k}}^\dagger c_{\mathbf{k}} + \frac{U}{2N_s} \sum_{\mathbf{k}} \sum_{\mathbf{k}'} \sum \sum \sum_{\mathbf{q}} c_{\mathbf{k}+\mathbf{q}}^\dagger c_{\mathbf{k}'}^\dagger c_{-\mathbf{q}} c_{\mathbf{k}}. \]

Here \( \epsilon(\mathbf{k}) \) is the boson band structure, \( \mu \) the chemical potential, \( c_{\mathbf{k}}^\dagger \) the boson creation operator, \( U \) the boson-boson repulsive interaction energy taken to be a constant for simplicity, and \( N_s \) the number of lattice sites. In this section, we will deal with a range of \( U \) such that \( U < 2W \) where \( W \) is the half-band-width. Our aim is to get a boson number equation in terms of \( \epsilon(\mathbf{k}), U, \mu, \) and temperature. One can then calculate the condensate fraction and the transition temperature. To this end we follow Bogoliubov theory\(^\text{24}\). In this theory, one makes an assumption that the ground state of interacting bosons is a Bose condensate. Since the lowest single particle state (which is \( \mathbf{k} = 0 \) in our case of simple tight binding bands) has macroscopic occupation (say \( N_0 \)), we have \( < c_{\mathbf{0}}^\dagger c_{\mathbf{0}} > \approx c_{\mathbf{0}} c_{\mathbf{0}}^\dagger \). Then the the operators \( c_{\mathbf{0}}^\dagger \) and \( c_{\mathbf{0}} \) can be treated as complex numbers and one gets \( < c_{\mathbf{0}}^\dagger > = < c_{\mathbf{0}} > = \sqrt{N_0} \). This complex number substitution has been recently shown\(^\text{25}\) to be justified. Clearly we have a two-fluid system consisting of two subsystems of condensed and non-condensed...
bosons. There are interactions between particles within each subsystem and interaction between particles in the two subsystems. In the Bogoliubov approach, to obtain second order interaction terms one makes the substitution: \( c_{0}^{\dagger} \to \sqrt{N_{0}} + c_{0} \). In the ground state, the linear fluctuation terms terms must vanish and this fixes the chemical potential to \( \mu = U n_{0} + \epsilon_{0} \), where \( \epsilon_{0} \) is the minimum of the single particle energy spectrum (which is equal to \(-z \xi\) for a bi-partite lattice with co-ordination number \( z \)), and \( n_{0} = N_{0}/N_{s} \). Following Bogoliubov approach, after a mean-field factorization of the interaction term, we obtain for lattices with inversion symmetry the following mean-field Hamiltonian:

\[
H_{BMF} = -E_{0} + \sum_{k} \frac{[\xi(k) + U n_{0}]}{2} (c_{k}^{\dagger}c_{k} + c_{-k}^{\dagger}c_{-k}) + \frac{U n_{0}}{2} \sum_{k} \xi(k) c_{k}^{\dagger}c_{-k} + c_{-k}^{\dagger}c_{k}, \tag{6}
\]

where \( E_{0} \equiv -U n_{0} N_{0}/2 \) and \( \xi(k) \equiv \epsilon(k) - \epsilon_{0} \). As mentioned earlier our aim is to get an equation for number of particles. This can be obtained from the Green’s function\(^{26} \) \( G(k, \omega) \equiv \langle \langle c_{k}; c_{k}^{\dagger} \rangle \rangle_{\omega} \) using the relation:

\[
n_{k} = \text{lim}_{\eta \to 0} \int_{-\infty}^{\infty} [G(k, \omega + i\eta) - G(k, \omega - i\eta)] f(\omega) d\omega. \tag{7}
\]

where \( f(\omega) = 1/[\exp(\omega/k_{B}T) - 1] \). The Heisenberg equation of motion for \( G(k, \omega) \) is:

\[
\omega G(k, \omega) = [\epsilon_{k} c_{k}^{\dagger} + \langle \langle c_{k}; H; c_{k}^{\dagger} \rangle \rangle_{\omega}, \tag{8}
\]

where \( H \) is the Hamiltonian of the system. Using \( H_{BMF} \), we obtain:

\[
\omega G(k, \omega) = 1 + [\xi(k) + U n_{0}]G(k, \omega) + U n_{0} F(k, \omega), \tag{9}
\]

and the Green’s function to which \( G(k, \omega) \) is coupled \( F(k, \omega) \equiv \langle \langle c_{k}^{\dagger} ; c_{-k}^{\dagger} \rangle \rangle_{\omega} \) obeys the equation of motion:

\[
\omega F(k, \omega) = -[\xi(k) + U n_{0}]F(k, \omega) - U n_{0} G(k, \omega). \tag{10}
\]

Solving the above two equations one obtains:

\[
G(k, \omega) = \left( \frac{\omega + \xi(k) + U n_{0}}{2 E(k)} \right) \times \left( \frac{1}{\omega + E(k)} - \frac{1}{\omega - E(k)} \right), \tag{11}
\]

where the Bogoliubov quasiparticle energy \( E_{k} \) is:

\[
E(k) = \sqrt{\xi^{2}(k) + 2 U n_{0} \xi(k)}. \tag{12}
\]

Note that for the tight binding band dispersions used, \( E(k) \) is linear in \( k \) in the long wave-length limit. Now, the number of particles per site \( (n) \) is readily obtained using Eqs. (7) and (11) to be:

\[
n = n_{0} + \frac{1}{2 N_{s}} \sum_{k} \left( \frac{1 + \xi(k) + U n_{0}}{E_{k}} - \frac{1}{e^{\beta E(k)/k_{B}T} - 1} \right) + \frac{1}{2 N_{s}} \sum_{k} \left[ \frac{1 - \xi(k) + U n_{0}}{E_{k}} - \frac{1}{e^{-\beta E(k)/k_{B}T} - 1} \right]. \tag{13}
\]

It is useful to look at some limits of the above equation. When \( U = 0 \) and \( T = 0 \), we have \( n_{0} = n \) which means that all the particles are in the condensate at absolute zero in the non-interacting limit. When \( U = 0 \) and \( T \neq 0 \), on obtains:

\[
n_{0} = n - \frac{1}{N_{s}} \sum_{k} \frac{\xi(k)}{e^{\beta E(k)/k_{B}T} - 1}. \tag{14}
\]

The second term on the RHS of the above equation is the thermal depletion of the condensate. Further, when \( U \neq 0 \) and \( T = 0 \), we get:

\[
n_{0} = n - \frac{1}{N_{s}} \sum_{k} \frac{\xi(k) + U n_{0} - E(k)}{2 E(k)}. \tag{15}
\]

in which second term on the RHS is the interaction induced depletion of the condensate. The interaction has twin effects of leading to a modified excitation spectrum and to a partial depletion of the condensate. The gap-less and linear long-wavelength excitation spectrum is consistent with experimental measurements\(^{12} \) on interacting bosons in their bose-condensed state in optical lattices. In Fig. (6), we have shown the numerical solution of Eq. (13) for \( bcc \), \( fcc \) and \( sc \) lattices. The condensate fraction \( (n_{0}/n) \) is seen to be gradually suppressed with increasing \( U/W \). Note also that at \( U = 0 \), all the particles are in the condensate. Finally, we consider the case: \( U \neq 0 \) and \( T \neq 0 \). In Fig. (7), we have displayed the variation of the condensate fraction as a function of temperature for various values of \( U/W \). The interaction partially depletes the condensate at zero temperature and close to it, while enhancing it beyond this range below the Bose-Einstein condensation temperature. We have not plotted these curves all the way to \( T_{B} \) since the Bogoliubov approximation breaks down close to \( T_{B} \). The variation of \( T_{B} \) with \( n \) is same as in the case of non-interacting bosons as can be seen by setting \( n_{0} = 0 \) in Eq. (13). At very low temperature, thermal depletion is negligible and correlation induced depletion causes a reduction of \( n_{0} \) with increasing \( U \). At higher temperatures when thermal depletion is important, \( U \) plays another role. The energies of the excited states shift to larger values with increasing \( U \), consequently the population in the excited states decreases and an enhancement of \( n_{0} \) occurs with increasing \( U \).

The analysis presented in this section is reasonable provided the effect of interaction is perturbative. When the
interaction strength increases, there is a possibility for a correlation induced localization transition for interacting bosons in a narrow band. In the next section, we analyze this strongly interacting bosons case.

C. Strongly interacting bosons

We first write the Hamiltonian, Eq. (5), in real space. Then we have:

$$H = \sum_{ij} [-t - \mu \delta_{ij}] c_i^\dagger c_j + \frac{U}{2} \sum_i n_i (n_i - 1), \quad (16)$$

where $n_i = c_i^\dagger c_i$. For simplicity, let us confine to the case of $n \leq 1$. The effect of increasing interaction ($U$) is to make motion of the bosons in the lattice correlated so as to avoid multiple occupancy of the sites. In the dilute limit, the effect of $U$ is not serious since there are enough vacant sites. The effect of $U$ then is prominent when $n$ is close to unity or to an integer value in the general case. In the large $U$ limit, it becomes favorable for the bosons to localize on the sites to avoid the energy penalty of multiple site occupancy. Qualitatively then, one can see that increasing $U$ increases the effective mass (or decreases the band-width) of the bosons and eventually drives them, for integer filling, to a BMH insulator state. For large $U$, when the double or multiple occupancy is forbidden, one can calculate the band-width reduction factor $[\phi_B(n)]$ approximately following the spirit of Renormalized Hamiltonian Approach (RHA) to fermion Hubbard model based on Gutzwiller approximation. Within the RHA, the effect of projecting out double or multiple occupancies on a non-interacting boson wave function is taken into account by a a classical renormalization factor which is the ratio of the probabilities of the corresponding physical process in the projected and unprojected spaces. The probability of a hopping process in the projected space is given by $n(1-n)$ for $n \leq 1$. This simply implies that the site from which hopping takes place must be occupied and the target site must be empty in the projected space. In the unprojected space, the probability of hopping is just equal to the probability of the site from which hopping takes place being occupied. The hopping takes place for non-interacting bosons irrespective of the target site being empty or occupied by any number of bosons. This probability may be found out by calculating the number of ways a given number of non-interacting bosons is distributed in $N_S$ number of lattice sites and $(N_S - 1)$ number of lattice sites. The difference would give the number of configurations where a particular site is occupied. Following this route, the probability that a site is occupied is obtained as:

$$p(N; N_s) = 1 - \frac{(N + N_s - 2)! (N_s - 1)!}{(N_s - 2)! (N + N_s - 1)!}, \quad (17)$$

where $N$ and $N_s$ are the total number of bosons and lattice sites, respectively. In the thermodynamic limit:

$$p(n) = \frac{n}{1+n}. \quad (18)$$

So, the hopping probability is just $p(n)$. The above equation is valid for any $n$. Now, in the strongly correlated state (large $U$ limit), confining to the case of $n \leq 1$, the hopping probability is $n(1-n)$. Hence the $\phi_B(n)$ is obtained to be,

$$\phi_B(n) = 1 - n^2. \quad (19)$$

The above equation is valid only for $n \leq 1$ and in the large $U$ limit. It may be interesting for the reader to note that, in the Fermion case, $\phi_F(n) = 2(1-n)/(2-n)$ which has been used in the studies of superconductivity in strong coupling fermion Hubbard model of high temperature superconductors. Since in the large $U$ limit, double or higher site-occupancies are forbidden, one can write a renormalized Hamiltonian, valid for $n \leq 1$, for strongly correlated bosons as :

$$H_{sc} = \sum_k [\phi_B(n) c(k) - \mu] c_k^\dagger c_k. \quad (20)$$

The above $H_{sc}$ is clearly the Hamiltonian of non-interacting bosons in a narrow band which has undergone a strong correlation induced filling-dependent band-narrowing. One can see that as $n$ increases from 0 to 1, the boson effective mass increases to eventually diverge at $n = 1$ and a BMH insulator obtains. We do admit that there are limitations to this renormalized Hamiltonian approach. While it has a merit that detailed band structure information can be incorporated in the Bose condensation temperature calculation, it has the demerit that we have to restrict ourselves to large $U$ and $n \leq 1$. The above Hamiltonian is valid for $U/W > (U/W)_c$ where $(U/W)_c$ is the critical value for transition into the Mott insulating phase for $n = 1$. Fixing a precise lower limit on $(U/W)$ is not possible in the absence of either exact analytical or numerical solution of three dimensional Bose-Hubbard model. It may be mentioned that in this large $U$ limit and for $n \leq 1$, the Bose-Hubbard model is reduced to the lattice Tonks (hard-core boson) gas which is different from the classical gas of elastic hard spheres investigated by Tonks. Our results imply then that the effective mass of bosons in a lattice Tonks gas in a narrow energy band is strongly band-filling dependent. The variation of the Bose condensation temperatures with $n$ for bosons with correlation-induced renormalized energy bands corresponding to $sc$, $bcc$, and $fcc$ lattices are displayed in Fig. (8). In the region between each curve and the $n$-axis, the bosons are in their Bose-condensed state, and above each curve they are in their normal state (except at $n = 1$ and below $k_B T \approx U$). The variation of $T_B$ is a resultant of the combined effects of increasing density and increasing effective mass since $T_B$ is proportional to $n/m^*$. Beyond around twenty percent filling,
the increasing effective mass over-compensates the effect of increasing \( n \) and pulls down the growth of \( T_B \) eventually driving it to zero at \( n = 1 \) at which density one has a BMH insulator.

It is of some interest to make a comparison between Mott-Hubbard (MH) metal-insulator transitions observed in fermion systems in condensed matter physics. For a half-filled narrow band of fermions, one way to induce a MH insulator to metal transition is by reducing the ratio of Coulomb repulsion \( (U) \) to the band-width \( (2W) \). This was experimentally achieved\(^3\) in \( V_2O_3 \) by application of pressure. This then is a band-width controlled MH transition\(^3\). Another way to induce a MH insulator to metal transition is to start with a Mott insulator and reduce the band-filling which would then be filling-controlled MH transition. This was achieved\(^5\) in \( La_xSr_{1-x}TiO_3 \). For a simple Gutzwiller approximation based analysis of the properties of this material see Ref. 36. Now, the Mott transition observed\(^2\) in boson systems in optical lattices is the band-width controlled one. It would be interesting to look for filling-controlled Mott transition in boson systems in optical lattices which may be possible by starting with the BMH insulator and flipping a few atoms out of the trap.

III. CONCLUSIONS

In this paper we presented a theoretical study of condensation of bosons in tight binding bands corresponding to \( sc \), \( bcc \), and \( fcc \) lattices. We analyzed condensation temperature and condensate fraction of non-interacting bosons, weakly interacting bosons using Bogoliubov method, and strongly interacting bosons through a renormalized Hamiltonian approach (limited to \( n \leq 1 \)) capable of incorporating the detailed boson band structures. In all the cases studied, we find that bosons in a tight binding band corresponding to a \( bcc \) lattice has the highest Bose condensation temperature. The growth of condensate fraction of non-interacting bosons is found to be very close to that of free bosons. In the case of weakly interacting bosons, the interaction partially depletes the condensate at zero temperature and close to it, while enhancing it beyond this range below the Bose-Einstein condensation temperature. Strong interaction enhances the boson effective mass as the band-filling is increased and eventually localizes them to form a Bose-Mott-Hubbard insulator for \( n = 1 \). In the strongly interacting bosons case, we found that all bosons are in the condensate at absolute zero temperature. We also pointed out a possibility of a filling-controlled BMH transition for bosons in optical lattices.

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FIG. 1. The variation of condensate fraction with temperature for bosons in a sc lattice for $n = 0.8$ (top), $n = 0.6$ (middle), $n = 0.4$ (bottom). In this and other figures $W$ is the half-band-width.

FIG. 2. The same as in Fig. 1 for bosons in a bcc lattice for $n = 0.8$ (top), $n = 0.6$ (middle), $n = 0.4$ (bottom).

FIG. 3. The same as in Fig. 1 for bosons in a fcc lattice for $n = 0.8$ (top), $n = 0.6$ (middle), $n = 0.4$ (bottom).
FIG. 4. The variation of condensate fraction (for $n = 0.6$) for bcc (top), fcc (middle), and sc (bottom) lattices. The dots are plots of $1 - (T/T_c)^{3/2}$.

FIG. 5. The variation of Bose condensation temperature (of non-interacting or weakly interacting bosons) with $n$ for bcc (top), fcc (middle), and sc (bottom) lattices.

FIG. 6. The variation of condensate fraction (for $T = 0$ and $n = 0.4$) with $U/W$ for bcc (top), fcc (middle), and sc (bottom) lattices.

FIG. 7. Condensate fraction (for $n = 0.4$) vs temperature for various values of $U/W$ (shown on the curves) and for sc lattice.

FIG. 8. Bose condensation temperature (of strongly interacting bosons) vs $n$ for bcc (top), fcc (middle), and sc (bottom) lattices. The dots are for non-interacting bosons in a bcc lattice.