ADS/CFT String Duality and Conformal Gauge Theories.

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Abstract

Compactification of Type IIB superstring on an $AdS_5 \times S^5/\Gamma$ background leads to SU(N) gauge field theories with prescribed matter representations. In the 't Hooft limit of large N such theories are conformally finite. For finite N and broken supersymmetry ($\mathcal{N} = 0$) I derive the constraints to be two-loop conformal and examine the consequences for a wide choice of $\Gamma$ and its embedding $\Gamma \subset \mathbb{C}^3 (\supset S^5)$. 
Introduction. Recently the relationship of string theory to gauge theory received stimulus from the conjecture by Maldacena [1] (related earlier papers are [2–5]) stemming from string duality which makes in its strongest form the assertion that the information contained in superstring theory is encoded in a four-dimensional gauge field theory including its non-perturbative sector. This has been vigorously pursued by many authors, especially Witten [6–8]. A brief review is in [9].

This relationship appears ironic when one recalls that the earliest string theories, the dual resonance models for strong interactions, were abandoned in favor of an SU(3) gauge theory 25 years ago. String theory has generally been regarded as much more general than gauge field theory because of its far richer structure; however, that perception was based on perturbative arguments, and the new developments of Maldacena et al. are essentially non-perturbative.

The idea is to consider N coincident D3-branes with 4-dimensional world volume theories having superconformal symmetry. This is conjectured [1] to be dual (weak coupling related to strong coupling) to type IIB superstring theory in a spacetime with geometry AdS$_5 \times S^5$. The world volume theory is in this case an $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with gauge group SU(N). Originally it is U(N) but this is broken to SU(N).

The radii of the AdS$_5$ and $S^5$ are equal and both given by $R = \lambda^{1/4} l_s$ where $\lambda$ is the 't Hooft parameter [10] $\lambda = g_{YM}^2 N$ ($g_{YM}^2 = g_s$, the string coupling constant) and $l_s^2 = \alpha'$ the universal Regge slope. The string tension is $T = (2\pi \alpha')^{-1}$.

The $\mathcal{N} = 4$ SU(N) gauge theory has been known to be ultra-violet finite for many years [11]. This is true not only for $N \rightarrow \infty$, the conformal limit of Maldacena, but also for finite $N$.

Breaking supersymmetries. By factoring out a discrete group $\Gamma$ in $S^5/\Gamma$ it is possible to break some or all of the $\mathcal{N} = 4$ supersymmetries. The isometry of $S^5$ is SO(6) $\sim$ SU(4) which may be identified with the R-parity of the $\mathcal{N} = 4$ conformal gauge theory. The spinors are in the 4 and the scalars are in the 6 of this SU(4). I shall here consider only
abelian groups $\Gamma = Z_p$, although non-abelian $\Gamma$ are worth further study (see e.g. [12,13]). I am considering only $AdS_5 \times S^5/\Gamma$, although the second 5-dimensional orbifold can be more general e.g. the $T^{p,q}$ spaces considered in [14].

The number of unbroken symmetries has been studied in e.g. [15,16] with the result that if $\Gamma \subset SU(2)$ there remains $\mathcal{N} = 2$ supersymmetry; if that is not satisfied but $\Gamma \subset SU(3)$ there remains $\mathcal{N} = 1$ supersymmetry; finally if even that is not satisfied one is left with $\mathcal{N} = 0$ or no supersymmetry. This last case is of most interest here.

It has been demonstrated that the large $N$ limit of the resultant gauge theory coincides with that of the $\mathcal{N} = 4$ case. Such arguments have been made both using string theory [17] and directly at the field theory level [18]. In the latter case the proof involves a monodromy of the representation for the group $\Gamma$.

For finite $N$, however, there is no argument that the resultant gauge theory is conformal, especially for $\mathcal{N} = 0$ where there are no non-renormalization theorems.

Nevertheless, if there does exist a conformal gauge theory in four dimensions with $\mathcal{N} = 0$, it would be so tightly constrained as to be possibly unique and would be of interest especially if it could contain the standard $SU(3) \times SU(2) \times U(1)$ model with its peculiar representations for the quarks and leptons.

The representations which occur in the resultant $\mathcal{N} \leq 2$ gauge theories from the orbifold construction have been studied using quiver diagrams [15]. I will find that these diagrams, while convenient for the cases $\mathcal{N} \geq 1$ need augmentation for the case $\mathcal{N} = 0$.

To specify the potentially conformal gauge theory I need to state how the group $\Gamma$ is embedded in $C^3$. Let the three complex coordinates of $C^3$ be denoted by $\underline{X} = (X_1, X_2, X_3)$. The action of $Z_p$ is the specified by:

$$\underline{X} \rightarrow (\alpha^{a_1}X_1, \alpha^{a_2}X_2, \alpha^{a_3}X_3)$$

(1)

where $\alpha = exp(2\pi i/p)$ and the three integers $a_\mu = (a_1, a_2, a_3)$ specify the embedding.

In order to ensure an $\mathcal{N} = 0$ result, I must insist that $\Gamma$ is not contained in $SU(3)$ by the requirement that
\[ a_1 + a_2 + a_3 \neq 0 \pmod{p} \] (2)

At the same time, for the correct behavior of the spinors we need in addition

\[ a_1 + a_2 + a_3 = 0 \pmod{2} \] (3)

For any given \( p \), there is a finite \( \nu(p) \) number of choices satisfying Eq.(2) and Eq.(3). We shall indicate later how to enumerate these \( \nu(p) \).

**Matter representations.** Because the discrete group \( \mathbb{Z}_p \) leads to the identification of \( p \) points in \( \mathbb{C}^3 \) and the \( N \) coinciding D3-branes converge on all \( p \) copies, the gauge group becomes \( SU(N)^p \). The surviving states are invariant under the product of a gauge transformation and a \( \mathbb{Z}_p \) transformation defined as in Eq.(1) above.

For the scalars, it then follows that the scalars fall into the representations

\[ \sum_{\mu} (\bar{N}_i, N_{i \pm a_\mu}) \] (4)

For \( a_\mu \neq 0 \) these are bi-fundamentals and for \( a_\mu = 0 \) complex adjoints. If we focus on one \( SU(N) \) the only non-singlet representations (the same will be true for the fermions) are fundamentals, anti-fundamentals and adjoints. These representations also follow from the Douglas-Moore quiver diagram.

For the fermions we must consider the transformation of a 4-spinor by making four combinations \( A_\lambda (1 \leq \lambda \leq 4) \) of the \( a_\mu \)

\[ A_1 = (a_1 + a_2 + a_3)/2 \] (5)

\[ A_2 = (a_1 - a_2 - a_3)/2 \] (6)

\[ A_3 = (-a_1 + a_2 - a_3)/2 \] (7)

\[ A_4 = (-a_1 - a_2 + a_3)/2 \] (8)

Again the surviving states are invariant under a product of the \( \mathbb{Z}_p \) and gauge transformations. This leads to the fermion representation:
\[ \sum_{\lambda} (N_i, \bar{N}_{i+A}) \]  
which can, if required, be deduced from a (different) quiver diagram.

**Two-loop \( \beta \)-functions.** I may take the detailed formula for the gauge coupling \( \beta \)-function \( \beta_g \) from [14]. The two leading orders are:

\[
\beta_g = \beta_g^{(1)} + \beta_g^{(2)}
\]  
with

\[
\beta_g^{(1)} = -\frac{g^3}{(4\pi)^2} \left[ \frac{11}{3} C_2(G) - \frac{4}{3} \kappa S_2(F) - \frac{1}{6} S_2(S) \right]
\]  
and

\[
\beta_g^{(2)} = -\frac{g^5}{(4\pi)^4} \left[ \frac{34}{3} (C_2(G))^2 - \kappa \left( 4C_2(F) + \frac{20}{3} C_2(G) \right) S_2(F) - \left( 2C_2(S) + \frac{1}{3} C_2(G) \right) S_2(S) + \frac{2\kappa Y_4(F)}{g^2} \right]
\]  

Here \( C_2, S_2 \) are the quadratic Casimir, Dynkin index respectively for the representations indicated, \( \kappa \) is 1/2, 1 for Weyl, Dirac fermions respectively, products like \( C_2(R)S_2(R) \) imply a sum over irreducible representations and finally the Yukawa term is included naturally in the two-loop term (unlike in [19]) because here the Yukawa couplings are proportional to the gauge coupling. The crucial quantity \( Y_4(F) \) is defined in terms of the Yukawa matrix \( Y_{ij}^a \psi_i \zeta \psi_j \phi^a \) by

\[
Y_4(F) = Tr \left( C_2(F)Y^aY^a \right)
\]  

Looking first at \( \mathcal{N} = 4 \), the values are easily seen to \( C_2(G) = N, S_2(F) = 4N, S_2(S) = 6N \) while \( C_2(F)S_2(F) = 4N^2 \) and \( C_2(S)S_2(S) = 6N^2 \). Finally \( Y_4(F) = 24g^2N^2 \). It follows from Eq. (12) and Eq. (13) the \( \beta_g = 0 \) for \( \mathcal{N} = 4 \) at two loops, as is well known [11].

However, the situation for \( \mathcal{N} = 0 \) is much more complicated.

At one-loop level for \( \mathcal{N} = 0 \) the evaluation of \( \beta_g^{(1)} \) is the same term-by-term as for \( \mathcal{N} = 4 \). This is already in [20, 22] for the one-loop level and since the one-loop \( \beta \)-function is purely leading-order in \( N \) it conforms to the general arguments of [17,18].
At two-loop order I must examine the non-leading terms in $1/N$ in Eq.\((\text{13})\). The first, third and fifth terms are always the same for $N = 0$ as for $N = 4$, respectively $34N^2/3 - 40N^2/3 - 2N^2 = -4N^2$.

To evaluate the second, fourth and sixth terms I find it necessary to distinguish four cases which are designated $(\alpha, \beta, \gamma, \delta)$ as follows:

\[ a_1 = a_2; \quad a_3 = 0. \quad A_1 = -A_4 \neq 0; \quad A_2 = A_3 = 0. \quad \text{(Case } \alpha). \] (15)

\[ a_1 \neq a_2; \quad a_3 = 0. \quad A_1 = -A_4 \neq 0; \quad A_2 = -A_3 \neq 0. \quad \text{(Case } \beta). \] (16)

\[ \text{All } a_\mu \neq 0. \quad \text{One } A_\lambda = 0; \quad \text{three } A_\lambda \neq 0. \quad \text{(Case } \gamma). \] (17)

\[ \text{All } a_\mu \neq 0. \quad \text{All } A_\lambda = 0. \quad \text{(Case } \delta). \] (18)

These possibilities lead to fermion and scalar representations of $SU(N)$ which are different for the four cases. They exhaust the choices which leave $N = 0$ which requires that Eqs.\((\text{3})\),\((\text{4})\) are fulfilled. (Note that at least two $a_\mu$ must be non-zero).

The evaluation of the remaining terms in Eq.\((\text{13})\) can now be done case by case. In Case $\alpha$, where both fermions and scalar appear in both fundamentals and adjoints, we find that $C_2(F)S_2(F) = 4N^2(1 - 1/(2N^2))$, $C_2(S)S_2(S) = 6N^2(1 - 2/(3N^2))$ and $Y_4(F) = (24N^2 - 16)g^2$. Substituting in Eq.\((\text{13})\) leads, as generally expected to an non-vanishing $\beta_g$ for finite $N$ and a non-conformal gauge theory.

For the other cases, I find for the three group theory expressions $C_2(F)S_2(F)$, $C_2(S)S_2(S)$ and $Y_4(F)$ respectively the following:

\[ 4N^2(1 - 1/(N^2)), \quad 6N^2(1 - 2/(3N^2)) \quad \text{and} \quad (24N^2 - 24). \quad \text{(Case } \beta) \] (19)

\[ 4N^2(1 - 3/(4N^2)), \quad 6N^2(1 - 1/(N^2)) \quad \text{and} \quad (24N^2 - 18). \quad \text{(Case } \gamma) \] (20)

\[ 4N^2(1 - 1/(N^2)), \quad 6N^2(1 - 1/(N^2)) \quad \text{and} \quad (24N^2 - 24). \quad \text{(Case } \delta) \] (21)
Substituting in Eq. (13), I find that \( \beta_g^{(2)} \) is non-vanishing except in the Case \( \gamma \). For this surviving theory, the fermions are in both fundamentals and adjoints, while all scalars are in fundamentals. This is therefore the only combination of matter representations of further interest.

**Directions.** A subsequent question to be addressed is what happens at three-loop and even higher orders. Also one must consider running of the Yukawa and quartic Higgs self-couplings due to possible non-vanishing of their \( \beta \)-functions \( \beta_Y \) and \( \beta_H \). It is planned to publish a more complete analysis elsewhere; I conclude this proposal with comments and possible future directions.

Often low-energy supersymmetry is adopted in order to solve the hierarchy problem of the Planck or GUT scale to the weak scale. This hierarchy is theory-generated and one may instead be agnostic about physics at \( \gtrsim 1000\text{TeV} \) scale where there is no real information. For example, recent ideas about extra Kaluza-Klein dimensions at reduced scales *e.g.* [23–26] avoid the hierarchy altogether and hence remove the main motivation for low-energy supersymmetry.

The possible role of an \( \mathcal{N} = 0, d = 4 \) conformal gauge theory may be put in context by imagining the level of skepticism to infinite renormalization of QED in 1948 (and later of the standard model) if the example of [11] had been found four decades earlier.

The exciting possibility is that the standard model is part of such an \( \mathcal{N} = 0 \) conformal gauge theory. The mass scales \( \Lambda_{QCD} \) and \( M_W \) would arise from necessarily non-perturbative effects, and gravity would be accommodated through the holographic principle [27,3]. Using AdS/CFT duality could help identify the relevant conformal theory. If so, this could shed light on the outstanding questions (families, CP violation, etc.) posed by the standard model.
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