On Bound-State $\beta^-$–Decay Rate of the Free Neutron

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We calculate the bound-state $\beta^-$–decay rate of the free neutron. We show that hydrogen in the final state of the decay is produced with a probability of about 99% in the hyperfine state with zero orbital $\ell = 0$ and atomic angular momentum $F = 0$.

INTRODUCTION

The continuum-state $\beta^-$–decay of the free neutron $n \rightarrow p + e^- + \bar{\nu}_e$ is well measured experimentally \[^1\] and investigated theoretically \[^2, 3\]. Recently \[^4, 5\] Schott et al. have reported the experimental data on the bound-state $\beta^-$–decay of the free neutron $n \rightarrow H + \nu_e$. In this letter we apply the technique, which we used for the analysis of the weak decays of the H–like, bare heavy ions and mesic hydrogen \[^6-10\], to the calculation of the bound-state $\beta^-$–decay rate of the free neutron.

$V - A$ weak hadronic interactions

The weak interaction Hamilton density operator take in the form

$$\mathcal{H}_W(x) = \frac{G_F}{\sqrt{2}} V_{ud} [\bar{\psi}_p(x)\gamma_\mu (1 - g_A\gamma^5)\psi_n(x)]$$

$$\times [\bar{\psi}_e(x)\gamma^\mu (1 - \gamma^5)\psi_{\nu_e}(x)], \quad (1)$$

where $G_F = 1.166 \times 10^{-11}$ MeV$^{-2}$ is the Fermi weak constant, $V_{ud} = 0.97377$ is the CKM matrix element \[^1\], $g_A = 1.3$ is the axial–vector renormalisation constant and $\psi_p(x)$, $\psi_n(x)$, $\psi_e(x)$ and $\psi_{\nu_e}(x)$ are operators of interacting proton, neutron, electron and anti-neutrino, respectively. The $T$–matrix of weak interactions is equal to

$$T = - \int d^4x \mathcal{H}_W(x). \quad (2)$$

In the final state of the bound-state $\beta^-$–decay hydrogen can be produced only in the $ns$–states, where $n$ is a principal quantum number $n = 1, 2, \ldots \ [3, 10]$. The contribution of the excited $nl$-state with $l > 0$ is negligible small. Due to hyperfine interactions \[^10\] hydrogen can be in two hyperfine states $(ns)_F$ with $F = 0$ and $F = 1$.

The wave function of hydrogen $H$ in the $(ns)_F$–state we take in the form \[^12-14\]

$$|H^{(ns)}(\vec{q})\rangle = \frac{1}{(2\pi)^3} \sqrt{2E_H(\vec{q})}$$

$$\times \int \frac{d^3k_e}{\sqrt{2E_e(k_e)}} \frac{d^3k_p}{\sqrt{2E_p(k_p)}} \delta^{(3)}(\vec{q} - \vec{k}_e - \vec{k}_p)$$

$$\times \phi_{ns}(\frac{m_p\vec{k}_e - m_e\vec{k}_p}{m_p + m_e}) a_p^\dagger(\vec{k}_e, \sigma_e) a_p^\dagger(\vec{k}_p, \sigma_p)|0\rangle, \quad (3)$$

where $E_H(\vec{q}) = \sqrt{M_H^2 + \vec{q}^2}$ and $\vec{q}$ are the total energy and the momentum of hydrogen, $M_H = m_p + m_e + \epsilon_{ns}$ and $\epsilon_{ns}$ are the mass and the binding energy of hydrogen $H$ in the $(ns)_F$ hyperfine state; $\phi_{ns}(\vec{k})$ is the wave function of the $ns$–state in the momentum representation \[^13\] (see also \[^12-14\]). For the calculation of the bound state $\beta^-$–decay rate we can neglect the hyperfine splitting of the energy levels of the $ns$–states.

For the amplitude of the bound-state $\beta^-$–decay we obtain the following expression

$$M(n \rightarrow H^{(ns)} + \nu_e) = G_F V_{ud} \sqrt{2m_n2E_H2E_\nu}$$

$$\times \int \frac{d^3k}{(2\pi)^3} \phi_{ns}^\ast(\vec{k} - \frac{m_e}{m_p + m_e}\vec{q}) \left\{ [\varphi_p^\dagger \varphi_n] + [\varphi_e^\dagger \varphi_{\nu_e}] \right\}. \quad (4)$$

The integral over $\vec{k}$ of the wave function $\phi_{ns}^\ast(\vec{k})$ defines the wave function $\psi_{ns}(0)$ in the coordinate representation, equal to $\psi_{ns}(0) = \sqrt{\alpha^3 m_n^n/\hbar}$, where $m_n$ is the electron mass and $\alpha = 1/137.036$ is the fine–structure constant. This gives

$$M(n \rightarrow H^{(ns)} + \nu_e) = G_F V_{ud} \sqrt{2m_n2E_H2E_\nu}$$

$$\times \psi_{(ns)}^\ast(0) \left\{ [\varphi_p^\dagger \varphi_{\sigma_n}] [\varphi_{\sigma_n}^\dagger \chi_{\nu_e}] - g_A [\varphi_p^\dagger \varphi_{\nu_e}] [\varphi_e^\dagger \chi_{\nu_e}] \right\}. \quad (5)$$
The bound-state $\beta^-$--decay rate of the free neutron is

$$
\lambda_{\beta^-} = \frac{1}{2m_n} \int \frac{1}{2} \sum_{n=1}^{\infty} \sum |M(n \rightarrow H(ns) + \bar{\nu}_e)|^2 
\times (2\pi)^4 \delta^{(4)}(k_\nu + q - p) \frac{d^3q}{(2\pi)^32E_H} \frac{d^3k_\nu}{(2\pi)^32E_\nu}.
$$

(6)

Since the energy shifts of hyperfine interactions is rather small compared with the energy differences of hydrogen [10], we neglect the hyperfine splitting. Summing over all polarisations of the proton and the electron we take into account the contributions of the hyperfine states $(ns)_F$ of hydrogen with $F = 0$ and $F = 1$. Summing up over the principal quantum number and taking into account that the antineutrino is polarised parallel to its momentum we get

$$
\lambda_{\beta^-} = (1 + 3g_A^2) \zeta(3) G_F^2 |V_{ud}|^2 \alpha^3 m_e^2 \pi^2 
\times \sqrt{(m_p + m_e)^2 + E^2} m_n
$$

(7)

where $\zeta(3) = 1.202$ is the Riemann function, coming from the summation over the principal quantum number $n$, and $E_\nu$ is equal to

$$
E_\nu = Q_{\beta^-} = \frac{m_n^2 - (m_p + m_e)^2}{2m_n} = 0.782 \text{ MeV},
$$

(8)

where $Q_{\beta^-}$ is the $Q$--value of the continuum-state $\beta^-$--decay of the free neutron $[1]$. The theoretical value of the continuum-state $\beta^-$--decay rate of the free neutron is

$$
\lambda_{\beta^-} = (1 + 3g_A^2) G_F^2 |V_{ud}|^2 f(Q_{\beta^-}, Z = 1) =
1.131 \times 10^{-3} \text{s}^{-1},
$$

(9)

where the continuum-state $\beta^-$--decay rate of the free neutron is calculated for the experimental masses of the interacting particles [1] and the Fermi integral $f(Q_{\beta^-}, Z = 1)$ equal to

$$
f(Q_{\beta^-}, Z = 1) =
\int_{m_e}^{Q_{\beta^-} + m_e} \frac{2\pi\alpha E^2(Q_{\beta^-} + m_e - E)^2}{1 - e^{-2\pi\alpha E/\sqrt{Q_{\beta^-}^2 - m_e^2}}}\,dE
= 0.059 \text{ MeV}^5,
$$

(10)

where we have taken into account the contribution of the Fermi function [2]

$$
F(Z = 1, E) =
\frac{2\pi\alpha E}{\sqrt{E^2 - m_e^2}} \frac{1}{1 - e^{-2\pi\alpha E/\sqrt{E^2 - m_e^2}}}.
$$

(11)

The theoretical value of the lifetime $\tau_{\beta^-} = 1/\lambda_{\beta^-} = 884.1 \text{s}$ agrees well with the experimental data $\tau_{\beta^-}^{\exp} = 885.7(8) \text{s}$ [1].

For the ratio $R_{\beta^-} = \lambda_{\beta^-}/\lambda_{\beta^-}$ of the bound and continuum state $\beta^-$--decay rates of the free neutron we get the following expression

$$
R_{\beta^-} = \zeta(3) \frac{2\pi \alpha^3 m_e^2}{m_n} \sqrt{(m_p + m_e)^2 + E^2} f(Q_{\beta^-}, Z = 1) =
4.06 \times 10^{-6}.
$$

(12)

Our value for the ratio of the decay rates agrees well with the results obtained in [17] (see also [6, 5]): $R_{\beta^-} = 4.20 \times 10^{-6}$.

## Concluding discussion

Since our calculations are carried out for pure $V - A$ theory of weak interactions, our results should make corrections to the experimental analysis of the contribution of scalar and pseudoscalar weak interactions of hadrons [5, 6, 7]. We would like to emphasize that the continuum-state $\beta^-$--decay rate of the free neutron is sensitive to the value of the axial–vector constant $g_A$. The value $\tau_{\beta^-} = 884.2 \text{s}$ is obtained for $g_A = 1.3$. For the experimental value $g_A = 1.2695$ the lifetime is $\tau_{\beta^-} = 919.7 \text{s}$, agreeing with the experimental value with an accuracy better than 4%. However, the axial–vector constant $g_A$ is cancelled for the ratio $R_{\beta^-}/c$, therefore our prediction for the ratio of the bound- and continuum-state $\beta^-$--decay rates of the free neutron can be valid with an accuracy much better than 4%. Since the factor $(1 + 3g_A^2)$ cancels in the ratio $R_{\beta^-}/c$, this has no influence on the value $R_{\beta^-}/c = 4.06 \times 10^{-6}$. Apart from the radiative corrections to the continuum-state $\beta^-$--decay rate of the free neutron, which are of the same order of magnitude [18], the discrepancy of about 4% can be attributed to the contributions of the scalar and tensor versions of hadronic weak interactions [4] (see also [19]), but it is hardly worth to discuss these contributions in connection with the bound-state $\beta^-$--decay of the free neutron.

Using the amplitude Eq. (4) we can estimate the relative probabilities of the $n \rightarrow H + \bar{\nu}_e$ decays into the different hyperfine states of hydrogen. Let $(\lambda_{\beta^-})_F$ be the decay rate of the bound-state $\beta^-$--decay into the hyperfine state $(ns)_F$. The ratios of the decay rates are equal to

$$
R_{F=1} = \frac{(\lambda_{\beta^-})_{F=1}}{\lambda_{\beta^-}} = \frac{3 (1 - g_A)^2}{4 (1 + 3g_A^2)} = 0.01,
$$

where $g_A$ is the value of the axial–vector constant.
$\sigma_0 \quad \sigma_p \quad \sigma_e \quad \sigma_\nu \quad f$

$\frac{1}{2} \quad \frac{1}{2} \quad -\frac{1}{2} \quad \frac{1}{2} \quad 1 + g_A$

$\frac{1}{2} \quad \frac{1}{2} \quad \frac{1}{2} \quad \frac{1}{2} \quad 0$  

$\frac{1}{2} \quad -\frac{1}{2} \quad -\frac{1}{2} \quad \frac{1}{2} \quad 0$  

$\frac{1}{2} \quad -\frac{1}{2} \quad \frac{1}{2} \quad \frac{1}{2} \quad -2g_A$  

$\frac{1}{2} \quad \frac{1}{2} \quad -\frac{1}{2} \quad \frac{1}{2} \quad 0$  

$\frac{1}{2} \quad \frac{1}{2} \quad \frac{1}{2} \quad \frac{1}{2} \quad 0$  

$\frac{1}{2} \quad -\frac{1}{2} \quad -\frac{1}{2} \quad \frac{1}{2} \quad 1 - g_A$  

$\frac{1}{2} \quad -\frac{1}{2} \quad \frac{1}{2} \quad \frac{1}{2} \quad 0$

| TABLE I: The contributions of different spinorial states of the interacting particles to the amplitudes of the bound-state $\beta^-$-decay of the free neutron: $f$ is defined by $f = [\varphi_1^T \chi_\nu] [\varphi_\nu^T \varphi]\sigma - g_A [\varphi_1^T \sigma \chi_\nu] [\varphi_\nu^T \varphi_\nu]$. |

$R_{F=0} = \frac{(\lambda_{\beta^-})_{F=0}}{\lambda_{\beta^-}} = \frac{1}{4} \frac{(1 + 3g_A)^2}{1 + 3g_A^2} = 0.99,$  (13)

calculated for both the experimental value of the axial coupling constant $g_A = 1.2695$ and $g_A = 1.3$.

This means that in the final state of the $n \rightarrow H + \bar{\nu}_e$ decay hydrogen is produced in the hyperfine state with $F = 0$ with a probability $99\%$. The main part of this probability $83.36\%$ is caused by the transition to the ground hyperfine state $(1s)F=0$.

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