Noncommutative teleparallel gravity from a consistent deformation of gauge theory

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Abstract.
Starting from a standard noncommutative gauge theory and using the Seiberg-Witten map we propose a new version of a noncommutative gravity. We use consistent deformation theory starting from a free gauge action and gauging a killing symmetry of the background metric to construct a deformation of the gauge theory that we can relate with gravity. The result of this consistent deformation of the gauge theory is nonpolynomial in $A^\mu$. From here we can construct a version of noncommutative gravity that is simpler than previous attempts. Our proposal is consistent and is not plagued with the problems of other approaches like twist symmetries or gauging other groups.

1. Deformed gauge theory and Gravity
Consider the usual Yang-Mills action given by

$$S_{YM} = \int d^4x F_{\mu\nu}^A F_{\mu\nu}^A,$$

where the non abelian field is $A_\mu^A$ and its associated $F_{\mu\nu}^A$ is given by

$$F_{\mu\nu}^A = \partial_\mu A_\nu^A - \partial_\nu A_\mu^A + f_{BC}^A A_\mu^B A_\nu^C,$$

and the gauge symmetry

$$\delta A_\mu^A = D_\mu \omega^A,$$

where

$$D_\mu \omega^A = \partial_\mu \omega^A + A_\mu^B f_{BC}^A \omega^C.$$

Now suppose that we want to deform this action (using the consistent deformation approach given in [1, 2]) and construct the action that is invariant under the deformed transformations

$$\delta A_\mu^A = D_\mu \omega^A + \varepsilon^\nu \partial_\nu A_\mu^A + A_\mu^A \partial_\mu \varepsilon^\nu = D_\mu \omega^A + \mathcal{L}_\varepsilon A_\mu^A,$$

where

$$\varepsilon^\mu = \omega^A \varepsilon^\mu_A,$$
and $\xi^\mu_A$ a constant matrix. This deformation can be done and the result is (for details see [2, 3])

$$L = -\frac{1}{4}(1 + \xi^\rho_A A^A_\rho) \tilde{F}^A_{\mu\nu},$$

(7)

where

$$\tilde{F}^A_{\mu\nu} = E^\rho_{\mu} E^\rho_{\nu} F^A_{\rho\sigma},$$

(8)

and

$$E^\nu_{\mu} = \delta^\nu_{\mu} - \xi^\rho_A E^A_B A^B_{\mu},$$

(9)

Here $E^A_B$ are defined by

$$E^C_A (\delta^A_A + \xi^\mu_A A^A_\mu) = \delta^A_B,$$

(10)

that are nonpolynomial in the gauge fields $A^A_\mu$. At first sight this theory looks very complicated because among the nonlinearities of the YM theory we have infinite coupling terms encoded in the non polynomial (in the gauge fields) matrices $E^\nu_{\mu}$. Nevertheless this action have very nice properties. First of all, it can be written in a “geometric form”

$$L = -\frac{1}{4}\sqrt{-g} g^\mu\rho g^\nu\sigma F_{\mu\nu} A^A F^A_{\rho\sigma},$$

(11)

where

$$g_{\mu\nu} = \eta^{\alpha\beta} e^\alpha_{\mu} e^\beta_{\nu} = 1 + A^a + A A^T = \eta_{\mu\nu} + A_{\mu\nu} + A_{\mu\rho} A^\rho_{\nu} + A_{\nu\mu} + A_{\nu\rho} A^\rho_{\mu},$$

(12)

and

$$A^A_{\mu\nu} = \xi^\nu_A A^A_\mu.$$  

(13)

Surprisingly enough, the transformation properties of this “metric” are as a usual space-time metric, i.e., transform under the gauge transformations (5) as a symmetric two-tensor but now under the diffeomorphism generated by $\varepsilon^\mu = \omega^A \xi^\mu_A$

$$\delta g_{\mu\nu} = L_\varepsilon g_{\mu\nu}.$$  

(14)

The “vielbein” transforms as a space-time vector for the index $\mu$

$$\delta e^\nu_{\mu} = \varepsilon^\rho \partial^\rho e^\nu_{\mu} + e^\nu^\rho \partial^\rho \varepsilon^\rho.$$  

(15)

Notice that the metric that rise an lower spee-time indices is the flat Minkowski background $\eta_{\mu\nu}$ and

$$E^\nu_{\mu} e^\rho_{\nu} = \delta^\rho_{\mu}.$$  

(16)

So our question is: Can we relate this theory with Einstein Gravity? The answer is yes and we will show how we can do that.

Using the field redefinition that interchanges the role of the gauge field $A^\lambda_\mu$ with the vielbein (12) we can define a new “field strength” as

$$F^A_{\mu\nu} = \xi^\rho_A F^A_{\mu\nu} = \partial_{\mu} A^\rho_{\nu} - \partial_{\nu} A^\rho_{\mu} = \partial_{\mu} e^\rho_{\nu} - \partial_{\nu} e^\rho_{\mu},$$

(17)

where $F^A_{\mu\nu}$ is the usual Yang-Mills field strength given by (2). We are taking as our internal gauge group an abelian group and $\xi^A = \delta^A_A$ for simplicity. This imply that we are adding four copies of deformed Maxwell actions defined as the same deformed YM action with structure constants of the gauge group equal to zero.
By making the crucial observation that this redefined field strength (17) can be related to the Ricci rotation coefficients $\Omega_{\mu\nu\rho}$ of the standard construction of Weitzenböck gravity [4, 5], we can now build from the deformation of the abelian Yang-Mills theory the corresponding action of General Relativity. Up to an irrelevant numerical factor the torsion tensor is $T_{\mu\nu\rho} \sim \Omega_{\mu\nu\rho}$.

Now the explicit relation between the Ricci rotation coefficients and the field strength of the deformed gauge theory is

$$\Omega_{\sigma\kappa\rho} = \hat{F}_{\rho\sigma\kappa} = E_{\rho}^{\mu}E_{\sigma}^{\nu}\hat{F}_{\mu\nu\rho}. \quad (18)$$

In terms of $\Omega_{\mu\nu\rho}$ the deformed Yang-Mills Lagrangian (7) is

$$L = -\frac{1}{4} e \Omega_{\mu\nu}^{\rho} \Omega_{\rho\mu}^{\nu}, \quad (19)$$

where we have identified the factor $(1 + \text{Tr} A)$ as the determinant of the vielbein$^1$. Since there is no curvature tensor available in this construction the most general Lagrangian for this theory of gravity can be written as a combination of the so called Weitzenböck invariants

$$I_1 = \Omega_{\mu\nu\rho} \Omega^{\mu\nu\rho}, \quad I_2 = \Omega_{\mu\nu\rho} \Omega^{\rho\mu\nu}, \quad I_3 = \Omega_{\mu\rho}^{\rho} \Omega^{\mu} \sigma. \quad (20)$$

The Pellegrini-Plebański Lagrangian [6] is

$$L = e c^i I_i. \quad (21)$$

To fix the coefficients $c^i$ in order to have Einstein gravity it is instructive to write the linearized action. Denoting the symmetric part of $A_{\mu\nu}$ by $A_{\mu\nu}^S$ and the antisymmetric part as $A_{\mu\nu}^A$ and retaining terms up to second order in $A$, using the field redefinition $e \rightarrow 1 + A$ in the Pellegrini-Plebański Lagrangian (21), the result is [4], $c_1 = 1, c_2 = 2, c_3 = -4$. Of course we can construct the full nonlinear action by the use of our field redefinition to get

$$L = e(\Omega_{\mu\nu\rho} \Omega^{\mu\nu\rho} + 2\Omega_{\mu\nu\rho} \Omega^{\rho\mu\nu} - 4\Omega_{\mu\rho}^{\rho} \Omega^{\mu} \sigma). \quad (22)$$

This is the desired Hilbert-Einstein action. It is interesting to observe that all the invariance under diffeomorphisms is recovered only with this particular choice for the coefficients $c^i$. It is also worth noticing that this action is not only invariant under the local transformations (15) but also is actually invariant under the full local Poincaré transformations just because the Einstein-Hilbert action is invariant under this symmetry. With this form of the Einstein-Hilbert action in terms of the vielbein $e_{\mu}^{\nu}$ we can check that the combination of terms given by (22) is such that the action can be written in terms of the fields $g_{\mu\nu}$ and its derivatives as the standard action of General Relativity. In the following section we will construct the noncommutative counterpart of this action.

2. Noncommutative Gravity

As we have four copies of the deformed Maxwell action we can start this section by deforming in the noncommutative sense (using the Seiberg-Witten map) the action of our previous section for the case of an abelian internal group.

This new deformation is a noncommutative deformation with parameter $\vartheta$. The noncommutative deformation can be applied to the original gauge theory or to the nonpolynomial deformed one. We will try the noncommutative deformation of the original gauge theory because

$^1$ It is worth noticing that the Weiberg-Witten theorem [7] is not violated in our approach because we are not constructing the graviton directly from the field theory. To make contact with gravity we need to consider also other invariants (see below). We can mention also that the background of our noncommutative theory is not the Minkowski metric alone but we have also the background tensor $\theta^{\mu\nu}$. 
this deformation is now well understood and under control [8, 9]. We will develop the deformation of the
gauge theory using the Seiberg-Witten map. As is well known this map to first order in \( \vartheta \) is

\[
F^1_{\mu\nu} = -\frac{1}{2} \theta^{\rho\sigma} \left( \{ A_\sigma, \partial_\rho F_{\mu\nu} \} - \{ F_{\rho\sigma}, F_{\mu\nu} \} \right),
\]

where \( \{ \cdot, \cdot \} \) is the anticommutator and \( \theta^{\rho\sigma} \) is the noncommutative tensor defined by

\[
[x^\mu, x^\nu]_* = i \theta^{\mu\nu}.
\]

The effect in the noncommutative Maxwell theory

\[
L = -\frac{1}{4} \eta^{\mu\rho} \eta^{\nu\sigma} F_{\mu\nu} \star F_{\rho\sigma},
\]

of this field redefinition is to add a new effective vertex to the free Maxwell action. The result of this after the noncommutative deformation is (to first order in \( \theta^{\mu\nu} \))

\[
L = -\frac{1}{4} (1 + \xi^\rho A_\rho) \left( \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} + 2 \theta^{\mu\sigma} \hat{F}_{\mu\nu} \hat{T}^\sigma \right),
\]

where \( \hat{F}_{\mu\nu} = F^\rho_{\mu} F^\sigma_{\nu} F_{\rho\sigma} \) and \( \hat{T}^\sigma = -\frac{1}{4} \delta^\nu_{\sigma} \hat{F}^\alpha_{\mu} \hat{F}^\alpha_{\nu} + \hat{F}^\alpha_{\beta} \hat{F}^{\nu\beta} \). The deformed gauge symmetry is

\[
\delta_\lambda A_\mu = \partial_\mu \lambda + \lambda \xi^\nu \hat{F}_{\nu\mu},
\]

To show that the NC deformation is consistent we can calculate the gauge variation of the Lagrangian (26) under the deformed transformation (27). Under the basic assumption that the tensor \( \theta \) is invariant under the background metric killing vector that we want to gauge

\[
\mathcal{L}_\xi \theta^{\mu\nu} = 0,
\]

the result is

\[
\delta L = -\frac{1}{4} \partial_\rho \left( \xi^\rho (1 + \xi \cdot A) \left( \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} + 2 \theta^{\mu\sigma} \hat{F}_{\mu\nu} \hat{T}^\sigma \right) \right) = \partial_\rho (\xi^\rho L)
\]

So for example in the case of a translation \( \xi^\mu = a^\mu \) with \( a^\mu \) a constant, the condition (28) is automatically valid and the complete deformation of the action (26) and its gauge symmetry (27) is not obstructed.

2.1. NC Gravity from deformed NCYM Theory

The next step is to try the same trick but with the Deformed Yang-Mills theory. The
generalization of our idea, presented in the previous section, to the case of the YM theory
is straightforward. We start from the NCYM action in flat space-time

\[
L = -\frac{1}{4} \eta^{\mu\rho} \eta^{\nu\sigma} F^A_{\mu\nu} \star F_{\rho\sigma}^A
\]

where

\[
F^A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu]^A_{\star}
\]

\[2\] We will take the not very extended interpretation that \( \theta \) is a Lorentz tensor and that the \( \star \) product is covariant
under global Poincaré transformations. For details see [10]. We will absorb the deformation parameter \( \vartheta \) in the definition of \( \theta^{\mu\nu} \).
and the usual $\ast$ gauge symmetry. Applying the Seiberg-Witten map (up to first order in the deformation parameter)

$$F^C_{\mu\nu} \to F^C_{\mu\nu} + \frac{1}{2} \theta^{\alpha\beta} d^{ABC} \left( F_{\mu\alpha}^A F_{\nu\beta}^B - A^A_{\alpha} \partial_{\beta} F^B_{\mu\nu} + \frac{1}{2} f^{BDE} A^A_{\alpha} A^B_{\beta} F^D_{\mu\nu} \right), \quad (32)$$

we obtain

$$L = -\frac{1}{4} \left( \text{Tr} (F^C_{\mu\nu} F^C_{\mu\nu}) + \theta^{\alpha\beta} d^{ABC} F^A_{\mu\nu} \left( \frac{1}{4} F^{B}_{\beta\alpha} F^{C}_{\mu\nu} + F^{B}_{\mu\alpha} F^{C}_{\nu\beta} \right) \right), \quad (33)$$

where we are taking the trace over the group indices as usual and the coefficients $d^{ABC}$ are defined by

$$\{ T_A, T_B \} = d^{ABC} T_C. \quad (34)$$

Notice that this theory is defined on the global gauge algebra of the Poincaré group that comes from the algebra of the killing vectors of the background metric. This theory is invariant under the usual nonabelian gauge symmetry

$$\delta A^A_{\mu} = D_{\mu} \omega^A, \quad (35)$$

and the global symmetry defined by

$$\delta_B A^A_{\mu} = \xi^B_{\mu} F^A_{\mu\nu}, \quad (36)$$

for each $B$. Now we perform the same steps to deform this action (to first order is $\theta$) to all orders in $\kappa$. The new action is

$$L = -\frac{1}{4} \left( 1 + \xi^A_{\mu} A^A_{\mu} \right) \left( \text{Tr} (\hat{F}^C_{\mu\nu} \hat{F}^C_{\mu\nu}) + \theta^{\alpha\beta} d^{ABC} \hat{F}^A_{\mu\nu} \left( \frac{1}{4} \hat{F}^{B}_{\beta\alpha} \hat{F}^{C}_{\mu\nu} + \hat{F}^{B}_{\mu\alpha} \hat{F}^{C}_{\nu\beta} \right) \right), \quad (37)$$

where $\hat{F}^C_{\mu\nu} = F^C_{\mu\nu} + \kappa \xi^A_{\mu} A^A_{\nu}$. The deformed gauge symmetry is

$$\delta_{\omega} A^A_{\mu} = D_{\mu} \omega^A + \omega^{B}_{\xi^B_{\mu}} \hat{F}^A_{\mu\nu}, \quad (38)$$

or

$$\delta A^A_{\mu} = D_{\mu} \omega^A + \varepsilon^{\nu}_{\mu} A^A_{\mu} \partial_{\nu} A^A_{\mu} + A^A_{\mu} \partial_{\mu} \varepsilon^{A}_{\nu} = D_{\mu} \omega^A + \mathcal{L}_{\varepsilon} A^A_{\mu}, \quad (39)$$

where

$$\varepsilon^{\mu}_{A} = \omega^{A}_{\xi^A_{\mu}}. \quad (40)$$

The action is invariant under the deformed symmetry (38) provided

$$\omega^A \mathcal{L}_{\xi^A_{\mu}} \theta^{\mu\nu} = 0. \quad (41)$$

The condition (41) do not restrict in any sense the definition of noncommutativity as stated in (24) because this noncommutation of the coordinates is invariant under global translations. The content of the condition (41) is to gauge less symmetry than the deformation (27) is capable to gauge. This is a central point in the construction of our model for noncommutative gravity. The base gauge symmetry upon we are constructing gravity from the deformed NCYM action are only local translations. This versatility of the gauging procedure adopted here allow us to construct a model that in principle is free from the usual obstructions present in many models of noncommutative gravity\(^3\).

\(^3\) In the case of the gauge theory the generators of gauge symmetries are constructed from the global symmetries $\xi^A$ with local parameters $\omega^A$. For space-time symmetries the generators are local and given by $\varepsilon^{\mu}$. To relate them we need to a map from each generator $\xi^A_{\mu}$ to $\varepsilon^{\mu}$ given by (40).
As a consequence of the condition (41) the transformation rule for the vielbein (15) becomes
\[ \delta e^\nu_{\mu} = \varepsilon^\rho \partial_\rho e^\nu_{\mu} + e^\nu_{\rho} \partial_\mu \varepsilon^\rho. \] (42)

In that case the transformation rule associated with the vielbein comes only through its lower space-time index. In this sense we are turning off the local Lorentz rotations associated with the frames defined by the vielbeins. The frames are “rigid” and the associated theory of gravity that emerges from here is the teleparallel gravity [11]. This framework will be used here to build a noncommutative gravity theory. The reason why we need to impose the condition (41) is to preserve the gauge invariance of our noncommutative model.

As we will choose the translations as the only symmetries that will be gauged then we can use the same rules to construct noncommutative gravity as the ones that we have used to construct standard gravity from the commutative deformed action (7). Namely to identify the Ricci rotation coefficients \( \Omega \) with \( \tilde{F} \)
\[ \Omega_{\sigma\kappa}^\rho = \tilde{F}_{\sigma\kappa}^\rho = E_\sigma^\rho E_\kappa^\mu F_{\mu\nu}^\rho, \] (43)
and
\[ F_{\mu\nu}^\rho = \xi_\mu A_{\nu}^\rho - \partial_\nu A_\mu^\rho = \partial_\mu e_\nu^\rho - \partial_\nu e_\mu^\rho. \] (44)
The next move is to substitute this expressions in the noncommutative action (37) and build the analogs of the invariants \( I_i \) for the noncommutative case. In this way we will end with a version of a noncommutative Pellegrini-Plebanski Lagrangian that is the central result of our note
\[ L_{NCG} = L_{PP} + L_{NC}, \] (45)
where \( L_{PP} \) is the usual Pellegrini-Plebanski Lagrangian (with the coefficients adjusted in such a way to recover Einstein Gravity) and \( L_{NC} \) is the non commutative correction to gravity.

As we have essentially four copies of the Maxwell action the non commutative deformed YM action is (after the identification of the Ricci coefficients \( \Omega \) with \( \tilde{F} \))
\[ L_1 = e \left( \Omega_{\mu\nu}^\rho \Omega_{\rho\mu\nu} + \theta^\alpha\beta d_{\rho\sigma\delta} \Omega_{\rho\mu\nu}^\sigma \Omega_{\mu\nu}^\sigma + \Omega_{\mu\nu}^\rho \Omega_{\mu\nu}^\sigma \Omega_{\rho\nu\sigma} \right), \] (46)
for some suitable coefficients \( d_{\rho\sigma\delta} = 1 \) (when the indices are equal and zero otherwise, just the sum of four Maxwell terms as stated above) and the identification of \( \det e_{\nu}^\mu \) with the factor in front of the noncommutative Lagrangian (37).

This is a central result of this article. To identify the others invariants that correspond to the noncommutative version of the invariants \( I_i \) we read from (46) the replacement in terms of \( \Omega \) in the form
\[ \Omega_{\mu\nu}^\rho \rightarrow \Omega_{\mu\nu}^\rho + (\Omega_{NC})_{\mu\nu}^\rho, \] (47)
and we found
\[ (\Omega_{NC})_{\mu\nu}^\rho = \theta^\alpha\beta d_{\rho\sigma\delta} \left( 4 \Omega_{\gamma\delta}^\sigma \Omega_{\mu\nu}^\delta + \Omega_{\mu\sigma}^\beta \Omega_{\nu\delta}^\sigma \right). \] (48)
Using this dictionary we can write the corresponding noncommutative quadratic invariants
\[ L_2 = e \left( \Omega_{\mu\nu\rho} \Omega_{\rho\mu\nu} + \Omega_{\mu\nu\rho} \Omega_{\gamma\delta}^\sigma \Omega_{\mu\nu}^\delta + \Omega_{\mu\nu\rho} \Omega_{\gamma\delta}^\sigma \Omega_{\mu\nu}^\delta \right). \] (49)
Taking into account that the trace of \( \Omega_{NC} \) is
\[ (\Omega_{NC})_{\mu} = (\Omega_{NC})_{\mu}^\nu = \theta^\alpha\beta d_{\sigma\delta} \left( 4 \Omega_{\gamma\delta}^\sigma \Omega_{\mu\nu}^\delta + \Omega_{\mu\sigma}^\beta \Omega_{\nu\delta}^\sigma \right), \] (50)
the last noncommutative invariant is

\[ L_3 = e (\Omega_{\mu} \Omega^\mu + 2 \Omega^\mu (\Omega_{NC})_{\mu}). \] (51)

The searched noncommutative gravity Lagrangian is then

\[ L_{NC} = L + 2e (\Omega^{\rho \mu \nu} (\Omega_{NC})_{\rho \mu \nu} + \Omega^{\rho \mu \nu} (\Omega_{NC})_{\mu \nu \rho} + \Omega^{\mu \nu \rho} (\Omega_{NC})_{\rho \mu \nu} - 4 \Omega^\mu (\Omega_{NC})_{\mu}). \] (52)

As a final comment let us consider the field content and symmetries of this NC gravity Lagrangian. The first piece in (52) is the standard Einstein Hilbert Lagrangian and can be written in terms of the curvature associated to the Levi-Civita connection or if we want using the spin connection as usual. The second piece is much more tricky. The field content of this NC correction terms is just the vielbein \( e^\nu_{\mu} \) and its derivatives. Our present understanding of these terms does not allow us to write them as \( f(R) \) corrections to Einstein Gravity as we may expect if the theory were invariant under the full diffeomorphism symmetry. Here \( R \) is the curvature associated with the Levi-Civita connection. All that we can do is to write this piece in terms of the Torsion of the Weitzenböck connection. A possible implication of this fact is that this piece of the Lagrangian (52) breaks the complete diffeomorphism symmetry already present in the first piece (Einstein-Hilbert Lagrangian) to the local translation symmetry just as in teleparallel gravity. In turn this imply that the symmetry of the Lagrangian (52) is just the symmetry of the teleparallel gravity. Perhaps the reason behind this breakdown of the symmetry is the presence of the noncommutative parameter and its transformation properties under local Lorentz transformations. This issue and the geometrical and phenomenological implications of this noncommutative correction to the Einstein Gravity need further study that we leave for a future work.

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Another fact that need a further investigation comes from string theory. The noncommutative leading order correction of Einstein Gravity that comes from string theory start at second order terms in the noncommutative parameter \( \theta^{\mu \nu} \). We need to develop this second order correction terms in our formalism to see if we can write a Lagrangian with a complete diffeomorphism invariance.
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