ENERGY OF SURFACE STATES FOR 3D MAGNETIC SCHRÖDINGER OPERATORS

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Abstract. We establish a semi-classical formula for the sum of eigenvalues of a magnetic Schrödinger operator in a three-dimensional domain with compact smooth boundary and Neumann boundary conditions. The eigenvalues we consider have eigenfunctions localized near the boundary of the domain, hence they correspond to surface states. Using relevant coordinates that straighten out the boundary, the leading order term of the energy is described in terms of the eigenvalues of model operators in the half-axis and the half-plane.

1. Introduction and main result

1.1. Introduction. The computation of the number and the sum of eigenvalues of Schrödinger operators in various asymptotic regimes is a central question in mathematical physics. One motivation comes from the problem of stability of matter (see Lieb-Solovej-Yngvason \cite{27}). Another motivation is the calculation of the quantum current (see Fournais \cite{11}). The object of study in \cite{27} is mainly the Pauli operator with magnetic field and electric potential. The study of the finiteness of the number and the energy of negative eigenvalues of the Pauli operator has been the object of study of numerous papers, starting probably with the establishing of the Cwikel-Rozenblum-Lieb and Lieb-Thirring bounds, and followed up by many important papers such as \cite{27, 8, 34, 17}.

This paper aims at answering the same question as in \cite{27} but for the Schrödinger operator with magnetic field. The electric potential is removed but the operator is defined in a domain with boundary. This leads to a similar situation as in \cite{27}, but the geometry of the boundary will have a significant influence on the expression of the leading order terms (see Theorem 1.2 below). Details will be discussed at a later point of this introduction.

The similar analogy between the results of this paper and those of \cite{27} has been observed previously. In \cite{20}, while estimating the ground state energy of a Schrödinger operator in a domain with boundary, Helffer-Mohamed observed an analogy between the semi-classical analysis of Schrödinger operators with electric potentials and that of Schrödinger operators in domains with boundaries. Loosely speaking, this analogy can be summarized by saying that ‘boundaries’ play a similar role to ‘electric potentials’. More precisely, this analogy is established in \cite{20} for the question of computing the ground state energy for an operator in a domain with boundary. Guided by this analogy, several important applications to the analysis of the Ginzburg-Landau model of superconductivity are given. We refer the reader to the monograph \cite{12} and references therein.

It is natural to wonder whether the same type of analogy between ‘boundaries’ and ‘electric potentials’ still exists for the question of computing the energy, as done in \cite{27}. The paper of Fournais-Kachmar \cite{13} established this type of analogy between boundaries and electric potentials for two dimensional domains and Neumann boundary condition. The goal of this paper is to generalize the results of \cite{13} to the case of three dimensional smooth domains.

1.2. Earlier results. Let $d \in \{2, 3\}$ and $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain with compact and smooth boundary $\partial \mathcal{O}$. We will consider both the case of interior domains $\Omega = \mathcal{O}$ and exterior domains $\Omega = \mathbb{R}^d \setminus \overline{\mathcal{O}}$. 

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We consider a magnetic vector potential $\mathbf{A} \in C^\infty(\overline{\Omega}; \mathbb{R}^d)$ and introduce the magnetic field $\mathbf{B} = \text{curl} \mathbf{A}$ and the quantities
\begin{equation}
    b := \inf_{x \in \overline{\Omega}} |\mathbf{B}(x)|, \quad b' = \inf_{x \in \partial \Omega} |\mathbf{B}(x)|. \tag{1.1}
\end{equation}
We assume that $b > 0$. For $h > 0$, we introduce the Neumann Schrödinger operator $\mathcal{P}_h$ with magnetic field:
\begin{equation}
    \mathcal{P}_h = (-ih \nabla + \mathbf{A})^2 \quad \text{in} \quad L^2(\Omega), \tag{1.2}
\end{equation}
whose domain is,
\begin{equation}
    \mathcal{D}(\mathcal{P}_h) = \left\{ u \in L^2(\Omega) : (-ih \nabla + \mathbf{A})^2 u \in L^2(\Omega), \quad j = 1, 2, \quad \nu \cdot (-ih \nabla + \mathbf{A}) u = 0 \quad \text{on} \partial \Omega \right\}. \tag{1.3}
\end{equation}
Here, for $x \in \partial \Omega$, $\nu(x)$ denotes the unit interior normal vector to $\partial \Omega$ at $x$.

The operator (1.2) is the Friedrichs’ self-adjoint extension in $L^2(\Omega)$ associated with the semi-bounded closed quadratic form:
\begin{equation}
    Q_h(u) := \int_{\Omega} (-ih \nabla + \mathbf{A})u^2 dx, \quad D(Q_h) := \left\{ u \in L^2(\Omega) : (-ih \nabla + \mathbf{A})u \in L^2(\Omega) \right\}. \tag{1.4}
\end{equation}
If the domain $\Omega$ is bounded (interior case), it results from the compact embedding of $\mathcal{D}(Q_h)$ into $L^2(\Omega)$ that $\mathcal{P}_h$ has compact resolvent. Hence the spectrum is purely discrete consisting of a sequence of positive eigenvalues accumulating at infinity.

In the case of exterior domains, the operator $\mathcal{P}_h$ can have essential spectrum. It was established in [20, Theorem 3.1] that there exists a constant $C_d \geq 0$ such that for all $h \in (0, h_0]$, we have
\begin{equation}
    \int_{\Omega} |(-ih \nabla + \mathbf{A})u|^2 dx \geq h \int_{\Omega} |(\mathbf{B}(x) - C_d h^{1/4})|u(x)|^2, \quad \forall \ u \in C_0^\infty(\Omega). \tag{1.5}
\end{equation}
Notice that in the two dimensional case where $d = 2$, we have that $C_d = 0$. Using a magnetic version of Persson’s Lemma (see [3, 30]), we get that
\begin{equation}
    \inf \text{Spec}_{\text{ess}} \mathcal{P}_h \geq h(b - C_d h^{1/4}). \tag{1.6}
\end{equation}

If the magnetic field is constant and the domain $\Omega$ has a smooth boundary, it is established that:
\begin{equation}
    \inf \text{Spec} \mathcal{P}_h = h \Theta_0 b + o(h), \quad (h \to 0_+), \tag{1.7}
\end{equation}
where $\Theta_0 \in (0, 1)$ is the universal constant introduced in (1.6). In such a situation, we see that if $\Lambda \in [0, b)$, then the set
\begin{equation}
    \text{Spec} \mathcal{P}_h \cap [0, \Lambda h) \neq \emptyset.
\end{equation}
In general, we consider $\Lambda \in [0, b)$ and work under the assumption that $\text{Spec} \mathcal{P}_h \cap [0, \Lambda h) \neq \emptyset$ and denote the elements of this set as an increasing sequence of eigenvalues counting multiplicities,
\begin{equation}
    \text{Spec}(\mathcal{P}_h) \cap (-\infty, \Lambda h) = \{ e_1(h), e_2(h), \ldots \}.
\end{equation}

In [13], it is established the asymptotic behavior of the sum
\begin{equation}
    \sum_j (e_j(h) - \Lambda h)_- := \text{Tr} (\mathcal{P}_h - \Lambda h)_-
\end{equation}
in the semi-classical limit $h \to 0$. Here $(x)_- = \max(-x, 0)$ denotes the negative part of a number $x \in \mathbb{R}$, and, for a self-adjoint operator $\mathcal{H}$, the operator $\mathcal{H}_- = -\mathcal{H}_{(-\infty, 0)}(\mathcal{H})\mathcal{H}$ is defined via the spectral theorem.

The result of [13], valid in two dimensions ($d = 2$), is recalled in Theorem 1.1 below. In the statement of the theorem, notice that, if $\xi \in \mathbb{R}$, the number $\mu_1(\xi)$ is the lowest eigenvalue of the harmonic oscillator
\begin{equation}
    -\partial_t^2 + (t - \xi)^2, \quad \text{in} \quad L^2(\mathbb{R}_+),
\end{equation}
and $\Theta_0$ is the universal constant defined as follows
\begin{equation}
    \Theta_0 = \inf_{\xi \in \mathbb{R}} \mu_1(\xi). \tag{1.6}
\end{equation}
Theorem 1.1. (Fournais-Kachmar [13]; d = 2). As \( h \to 0_+ \), there holds,
\[
\lim_{h \to 0} h^{-1/2} \sum_j (e_j(h) - bh)_- = \frac{1}{2\pi} \int_{\partial \Omega \times \mathbb{R}} B(x)^{3/2} \left( -\frac{b}{B(x)} + \mu_1(\xi) \right)_- ds(x)d\xi,
\]
where \( ds(x) \) denotes the arc-length measure on the boundary.

1.3. Main results. We focus on the case where \( \Omega \subset \mathbb{R}^3 \) (\( d = 3 \)). The main result of this paper is a generalization of Theorem 1.1 valid when \( d = 3 \).

We shall need the following notation

- Given \( \eta \in \mathbb{R} \), \( \omega \) an open domain in \( \mathbb{R}^3 \) and a self-adjoint operator \( \mathcal{H} \) in \( L^2(\omega) \) such that the spectrum below \( \eta \) is discrete, we shall denote by
  \[
  \mathcal{N}(\eta; \mathcal{H}, \omega) := \text{Tr}(1_{(-\infty, \eta]}(\mathcal{H}))
  \]
  the number of eigenvalues less than \( \eta \), counting multiplicities, and by
  \[
  \mathcal{E}(\eta; \mathcal{H}, \omega) := \text{Tr}(\mathcal{H} - \eta)_-
  \]
  their corresponding sum below \( \eta \).
- If \( x \) is a point on the boundary of \( \Omega \), then \( \theta(x) \) denotes the angle in \([0, \pi/2]\) between the magnetic field \( B = \text{curl} \, A \) and the tangent plane to \( \partial \Omega \) at the point \( x \). More precisely,
  \[
  \partial \Omega \ni x \mapsto \theta(x) = \arcsin \left( \frac{|B(x) \cdot \nu(x)|}{|B(x)|} \right) \in [0, \pi/2].
  \]
- We let \( \mathbb{R}_+ = (0, \infty) \), \( \mathbb{R}_+^2 = \mathbb{R} \times (0, \infty) \) and \( \mathbb{R}_+^3 = \mathbb{R}^2 \times (0, \infty) \).
- For \( \xi \in \mathbb{R} \), we denote by \( \mu_1(\xi) \) the lowest eigenvalue of the harmonic oscillator
  \[
  -\partial_t^2 + (t - \xi)^2 \quad \text{in} \quad L^2(\mathbb{R}_+)
  \]
  with Neumann boundary conditions at \( t = 0 \).
- For \( \theta \in (0, \pi/2] \), we introduce the two-dimensional operator
  \[
  \mathcal{L}(\theta) := -\partial_t^2 - \partial_s^2 + (t \cos(\theta) - s \sin(\theta))^2 \quad \text{in} \quad L^2(\mathbb{R}_+^2).
  \]
  It is well known (see [22]) that the essential spectrum of \( \mathcal{L}(\theta) \) is the interval \([1, \infty)\), and we shall denote by \( \{\zeta_j(\theta)\}_j \) the countable set of eigenvalues of \( \mathcal{L}(\theta) \) in the interval \([\zeta_1(\theta), 1)\).
- We define the positive and negative parts of a real number \( x \) by \( (x)_\pm = \max(\pm x, 0) \).

The main theorem of this paper is:

Theorem 1.2. Suppose \( \Omega \subset \mathbb{R}^3 \) is either an interior or an exterior domain with compact smooth boundary \( \partial \Omega \). Given \( \Lambda \in [0, b) \), the following asymptotic formula holds,
\[
\sum_j (e_j(h) - \Lambda h)_- = \text{Tr}(\mathcal{P}_h - \Lambda h)_- = \int_{\partial \Omega} |B(x)|^2 E(\theta(x), \Lambda|B(x)|^{-1})d\sigma(x) + o(1), \quad \text{as} \quad h \to 0,
\]
where, for \( \lambda \in [0, 1) \), the function \( E(\theta, \lambda) \) is given by
\[
E(\theta, \lambda) = \begin{cases} 
\frac{1}{3\pi^2} \int_0^{\infty} (\mu_1(\xi) - \lambda)^{3/2} d\xi & \text{if} \; \theta = 0, \\
\frac{2\pi}{\sin(\theta)} \sum_j (\zeta_j(\theta) - \lambda)_- & \text{if} \; \theta \in (0, \pi/2], 
\end{cases}
\]
and \( d\sigma(x) \) denotes the surface measure on the boundary \( \partial \Omega \).

Remark 1.3. In the case \( \theta = \frac{\pi}{2} \), it is well known (see [21]) that the first eigenvalue \( \zeta_1(\frac{\pi}{2}) = 1 \) which implies that \( E(\frac{\pi}{2}, \lambda) = 0 \) for any \( \lambda \in [0, 1) \).

Remark 1.4. In the case \( \theta \in (0, \pi/2) \), we emphasize that the sum appearing in the formula of \( E(\theta, \lambda) \) above, is a finite sum. Indeed, in view of Lemma 6.8 below, we learn that the number of eigenvalues of \( \mathcal{L}(\theta) \), below a fixed \( \lambda \in [0, 1) \), is finite.
Remark 1.5. In Lemma 5.1 below, we show that the function
\[(\theta, \lambda) \mapsto E(\theta, \lambda),\]
is a continuous function as a function of two variables. Consequently, we obtain
\[
\lim_{\theta \to 0} \sin(\theta) \frac{1}{2\pi} \sum_j (\zeta_j(\theta) - \lambda) = \frac{1}{3\pi^2} \int_0^\infty (\mu_1(\xi) - \lambda)^{3/2} d\xi.
\]
Notice that this formula is connected to the formula for the number of eigenvalues given in \([29] \):
\[
\lim_{\theta \to 0} \left( \sin \theta \mathcal{N}(\lambda, \theta) \right) = \frac{1}{\pi} \int_0^\infty (\mu_1(\xi) - \lambda)^{3/2} d\xi,
\]
where \(\mathcal{N}(\lambda, \theta) = \text{Card} \{ \zeta_j(\theta) : \zeta_j(\theta) \leq \lambda \} \).

Using the technique to go from energies to densities (see \([9] \) for details), we can differentiate both sides of \((1.12)\) with respect to \(\Lambda h\) and get an asymptotic formula for the number of eigenvalues of \(\mathcal{P}_h\) below \(\Lambda h\). This is stated in the next corollary valid under the assumptions made in Theorem 1.2.

Corollary 1.6. Let \(\Lambda \in [0, b)\) and \(\sigma\) be the surface measure on \(\partial \Omega\). If,
\[
\sigma\left( \{ x \in \partial \Omega : \theta(x) \in (0, \pi/2), \ \Lambda|B(x)|^{-1} \in \text{Spec } L(\theta(x)) \} \right) = 0,
\]
then, the following asymptotic formula holds true as \(h \to 0_+\),
\[
\lim_{h \to 0} h \mathcal{N}(\Lambda h; \mathcal{P}_h, \Omega) = \int_{\partial \Omega} |B(x)| n(\theta(x), \Lambda|B(x)|^{-1}) d\sigma(x),
\]
where, for \(\lambda \in [0, 1)\), \(n(\theta, \lambda)\) is given by
\[
n(\theta, \lambda) = \begin{cases}
\frac{1}{2\pi^2} \int_0^\infty (\mu_1(\xi) - \lambda)^{1/2} d\xi & \text{if } \theta = 0, \\
\frac{\sin(\theta)}{2\pi} \mathcal{N}(\lambda, \theta) & \text{if } \theta \in (0, \pi/2].
\end{cases}
\]

The result of Corollary 1.6 is a generalization of the asymptotic formula given in \([16] \) for the number of edge states in two dimensional domains. However, it is worthy to notice that the result in two dimensions is valid without the geometric condition in Corollary 1.6.

The geometric condition in \((1.13)\) is satisfied when \(\Omega\) is the unit ball, the magnetic field \(B\) is constant of unit length and \(\Lambda\) is sufficiently close to the universal constant \(\Theta_0\). As we shall see, this is closely related to the behavior of the functions \((0, \pi/2] \ni \theta \mapsto \zeta_j(\theta)\).

In this concern, we recall the following two results.

Lemma 1.7. \([22] \) The functions \(\theta \mapsto \zeta_j(\theta)\) are increasing and continuous on \((0, \pi/2)\). Moreover,
\[
\zeta_1(0) = \Theta_0 \quad \text{and} \quad \forall \theta \in [0, \pi/2], \quad \zeta_1(\theta) < 1.
\]

The second Lemma is taken from \([31] \).

Lemma 1.8. Let \(N \geq 1\) be an integer and suppose that there exists \(\theta_* \in (0, \pi/2)\) such that the following assumptions are satisfied
\[
(1) \quad \zeta_N(\theta_*) < 1;
(2) \quad \text{The eigenvalues } \{\zeta_j(\theta_*)\}_{1 \leq n \leq N} \text{ are simple.}
\]

Define
\[
\theta_{\text{max}, N} := \sup \{ \theta \in (0, \pi/2], \ \zeta_N(\theta) < 1 \}.
\]
Then for all \(1 \leq j \leq N\), the functions \(\theta \mapsto \zeta_j(\theta)\) are strictly increasing on \((0, \theta_{\text{max}, N}).\)
It is pointed in [31] that, to each $N$, there is $\theta_*$ such that the two conditions of Lemmas 1.3 are satisfied. Thus, for every $N$, the conclusion of Lemma 1.3 is true. In particular, when $N = 2$ we get,

$$
\delta = \min \left( \frac{\zeta_2(0) - \zeta_1(0)}{2}, \frac{1 - \Theta_0}{2} \right) > 0.
$$

By continuity of the functions $\zeta_1(\theta)$ and $\zeta_2(\theta)$, there exists $\epsilon_0 \in (0, \theta_{\text{max},2})$ such that for all $\theta \in [0, \epsilon_0]$,

$$
\zeta_2(\theta) \geq \zeta_1(\theta) + \delta \geq \Theta_0 + \delta.
$$

Take $\Lambda \in (\Theta_0, \Theta_0 + \delta)$. That way we get that

$$
\forall \theta \in [0, \pi/2], \quad \zeta_2(\theta) \geq \Theta_0 + \delta > \Lambda,
$$

and $\zeta_1(\theta) = \Lambda$ has at most one solution in $[0, \theta_{\text{max},1}]$. Notice here that $\theta_{\text{max},1} = \pi/2$ is a consequence of Lemma 1.7.

Returning back the condition (1.13) and the above discussion, we see that when the magnetic field is constant of unit length and the domain $\Omega$ is the unit ball, the set

$$
\Sigma = \{ x \in \partial \Omega : \theta(x) \in (0, \pi/2), \quad \Lambda |\mathbf{B}(x)|^{-1} \in \text{Spec } \mathcal{L}(\theta(x)) \}
$$

consists of at most one circle (defined by the solution $\theta$ of $\zeta_1(\theta) = \Lambda$). That way the set $\Sigma$ has measure zero relative to the surface measure and the condition (1.13) is satisfied.

1.4. Perspectives. We list some natural questions for future research:

1. Inspection of the number $\mathcal{N}(\Lambda h; \mathcal{P}_h, \Omega)$ when the condition in (1.13) is violated.
2. Theorem 1.2 is established when the domains $\Omega$ has a smooth boundary. An interesting question is to study the case when the domain $\Omega$ has corners or wedges (see [31]). In two dimensions, this is done in [26].
3. The inspection of the effect of the boundary conditions might be interesting. Theorem 1.2 is established for the operator with Neumann boundary condition. A natural question is to consider the operator with Robin boundary condition

$$
\nu \cdot (-ih \nabla + \mathbf{A})u + \gamma u = 0 \quad \text{on } \partial \Omega,
$$

where $\gamma \in L^\infty(\partial \Omega; \mathbb{R})$ (see [24]).
4. The asymptotic formula in Theorem 1.2 holds for the energy of the eigenvalues below the energy level $\Lambda h$ with $\Lambda < b$. However, in two dimensions, such a restriction on $\Lambda$ does not appear ($\Lambda$ is allowed to be $b$). Removing the restriction on $\Lambda$ in three dimensions is an interesting question.

1.5. Organization of the paper. Compared to the situation in two dimensional domains, the analysis of the problem in three dimensional domains needs new ingredients. The reason is that the boundary in 3D is a surface and has richer geometry than that in 2D.

The paper is organized as follows. Section 2 contains some basic tools concerning the variational principle of the sum of negative eigenvalues. Section 3 is devoted to the spectral analysis of the model operator on a half-cylinder with Neumann boundary condition on one edge and Dirichlet boundary conditions on the other edges. Section 4 is devoted to the construction of the function $E(\theta, \lambda)$ as the limit of the energy of the operator in the half-cylinder. Continuity properties of this function are studied in Section 5. Explicit formulas of $E(\theta, \lambda)$ are established in Section 6. Section 7 contains the expression of the operator relative to local coordinates near the boundary of the domain $\Omega$. Section 8 concludes with the proof of Theorem 1.2 and Corollary 1.6.
2. Variational principle

In this section, we recall useful methods to establish upper and lower bounds on the energy of eigenvalues (see for example [36]).

**Lemma 2.1.** Let $\mathcal{H}$ be a semi-bounded self-adjoint operator on $L^2(\mathbb{R}^3)$ satisfying

$$\inf \text{Spec}_{\text{ess}}(\mathcal{H}) \geq 0.$$  \hspace{1cm} (2.1)

Let $\{\nu_j\}_{j=1}^{\infty}$ be the sequence of negative eigenvalues of $\mathcal{H}$ counting multiplicities. We have,

$$-\sum_{j=1}^{\infty} (\nu_j)^- = \inf \sum_{j=1}^{N} \langle \psi_j, \mathcal{H} \psi_j \rangle,$$  \hspace{1cm} (2.2)

where the infimum is taken over all $N \in \mathbb{N}$ and orthonormal families $\{\psi_1, \psi_2, \cdots, \psi_N\} \subset D(\mathcal{H})$.

The next lemma states another variational principle. It is used in several papers, e.g. [27].

**Lemma 2.2.** Let $\mathcal{H}$ be a self-adjoint semi-bounded operator satisfying the hypothesis (2.1). Suppose in addition that $(H)_{-}$ is trace class. For any orthogonal projection $\gamma$ with range belonging to the domain of $\mathcal{H}$ and such that $H \gamma$ is trace class, we have,

$$-\sum_{j=1}^{\infty} (\nu_j)^- \leq \text{Tr}(H \gamma).$$  \hspace{1cm} (2.3)

For later purposes, we include

**Corollary 2.3.** Let $\Omega$ be a subset of $\mathbb{R}^3$. Suppose that $P$ is a positive self-adjoint operator on $L^2(\Omega)$ such that its spectrum below 1 is discrete. Let $\lambda \in [0, 1)$ and $\zeta \in \mathbb{R}$ such that $-\lambda \leq \zeta < 1-\lambda$. We have

$$E(\lambda + \zeta; P, \Omega) \leq E(\lambda; P, \Omega) + \zeta N(\lambda + \zeta; P, \Omega),$$  \hspace{1cm} (2.4)

where $N(\cdot; P, \Omega)$ and $E(\cdot; P, \Omega)$ are introduced in (1.8) and (1.9) respectively.

**Proof.** Let $\{\lambda_k\}_{k=1}^{N}$ be the family of eigenvalues below $\lambda + \zeta$ for $P$ and $\{g_k\}_{k=1}^{N}$ are associated (normalized) eigenfunctions. Let us define the trial density matrix $\gamma : L^2(\Omega) \ni f \mapsto \gamma f \in L^2(\Omega)$,

$$\gamma f = \sum_{1 \leq k \leq N} \langle f, g_k \rangle g_k,$$

which satisfies $0 \leq \gamma \leq 1$ (in the sense of quadratic forms). By Lemma 2.2, it follows that

$$-E(\lambda; P, \Omega) := -\text{Tr}(P - \lambda) \leq \text{Tr}((P - \lambda) \gamma).$$  \hspace{1cm} (2.5)

On the other hand, we have

$$\text{Tr}((P - \lambda) \gamma) = \sum_{1 \leq k \leq N} (\lambda_k - \lambda) = \sum_{1 \leq k \leq N} (\lambda_k - \lambda - \zeta) + \zeta \sum_{1 \leq k \leq N} 1$$

$$= -E(\lambda + \zeta; P, \Omega) + \zeta N(\lambda + \zeta; P, \Omega).$$  \hspace{1cm} (2.6)

Inserting this into (2.5) yields (2.4). \qed

3. Model operator in the half-space

Our main goal in this section is to establish an upper bound on the number and the sum of eigenvalues below the infimum of the essential spectrum of a magnetic Schrödinger operator in a half-cylinder in terms of the area of the cylinder base.
3.1. **Reflection with respect to the boundary.** In order to state Lemma 3.1 below, we need to define the reflected magnetic Schrödinger operator in $L^2(\mathbb{R}^3)$ associated with the Neumann Schrödinger operator in $L^2(\mathbb{R}^3_+)$.  

Given $\theta \in [0, \pi/2]$, we consider the magnetic potential 

$$
\tilde{F}_\theta(r, s, t) := (0, 0, |t| \cos(\theta) - s \sin(\theta)), \quad (r, s, t) \in \mathbb{R}^3.
$$

Let 

$$
\tilde{P}_\theta = (-i \nabla + \tilde{F}_\theta)^2 \quad \text{in} \quad L^2(\mathbb{R}^3),
$$

be the self-adjoint operator defined by the closed quadratic form 

$$
\tilde{Q}_\theta(u) := \int_{\mathbb{R}^3} |(-i \nabla + \tilde{F}_\theta)u|^2 \, drdsdt, \quad \mathcal{D}(\tilde{Q}_\theta) := \{ u \in L^2(\mathbb{R}^3) : (-i \nabla + \tilde{F}_\theta)u \in L^2(\mathbb{R}^3) \}.
$$

We let $\beta = (0, \cos(\theta), \sin(\theta))$ denote the constant magnetic field generated by the vector potential: 

$$
F_\theta(r, s, t) = (0, 0, t \cos(\theta) - s \sin(\theta)), \quad (r, s, t) \in \mathbb{R}^3_+.
$$

Furthermore, let 

$$
P^N_\theta = (-i \nabla + \tilde{F}_\theta)^2 \quad \text{in} \quad L^2(\mathbb{R}^3_+),
$$

be the self-adjoint (Neumann) Schrödinger operator associated with the quadratic form 

$$
Q^N_\theta(u) := \int_{\mathbb{R}^3_+} |(-i \nabla + \tilde{F}_\theta)u|^2 \, drdsdt, \quad \mathcal{D}(Q^N_\theta) := \{ u \in L^2(\mathbb{R}^3_+) : (-i \nabla + \tilde{F}_\theta)u \in L^2(\mathbb{R}^3_+) \}.
$$

We establish in the next lemma estimates on the eigenvalue counting function and the energy of eigenvalues for a perturbation of $P^N_\theta$.

**Lemma 3.1.** Let $\mathcal{U}$ be a positive bounded potential in $L^2(\mathbb{R}^3)$ verifying $\mathcal{U}(\cdot, \cdot, -t) = \mathcal{U}(\cdot, \cdot, t)$. Assume that the spectrum of $P^N_\theta + \mathcal{U}$ below $\lambda$ is discrete. We have 

$$
\mathcal{N}(\lambda; P^N_\theta + \mathcal{U}, \mathbb{R}^3_+) \leq C_{CLR} \int_{\mathbb{R}^3} (\mathcal{U})^{3/2} \, drdsdt, \quad \mathcal{E}(\lambda; P^N_\theta + \mathcal{U}, \mathbb{R}^3_+) \leq C_{LT} \int_{\mathbb{R}^3} (\mathcal{U})^{5/2} \, drdsdt,
$$

where $C_{CLR}$ and $C_{LT}$ are two positive universal constants.

**Proof.** Let $n \in \mathbb{N}$. Let $\{u_j\}_{j=1}^n$ be an orthonormal family of eigenfunctions with corresponding eigenvalues $\{\lambda_j\}_{j=1}^n$ associated with the operator $P^N_\theta + \mathcal{U}$ in $L^2(\mathbb{R}^3_+)$. We define the extension to $\mathbb{R}^3$ of the function $u_j$ by:

$$
\tilde{u}_j(r, s, t) = \begin{cases} \sqrt{2}u_j(r, s, t) & t \geq 0 \\ u_j(r, s, -t) & t < 0. \end{cases}
$$

Since $\{u_j\}_{j=1}^n$ are normalized and pairwise orthogonal, we get for all $1 \leq j, k \leq n$,

$$
\langle \tilde{u}_j, \tilde{u}_k \rangle_{L^2(\mathbb{R}^3)} = \langle u_j, u_k \rangle_{L^2(\mathbb{R}^3_+)} = \delta_{j,k},
$$

where $\delta_{j,k}$ is the Kronecker symbol.

The bilinear form associated with $\tilde{Q}_\theta + \mathcal{U}$ is defined on the form domain by:

$$
\tilde{a}_{\theta, \mathcal{U}}(u, v) = \int_{\mathbb{R}^3} \left( (-i \nabla + \tilde{F}_\theta)u(-i \nabla + \tilde{F}_\theta)v + \mathcal{U}uv \right) \, drdsdt.
$$

Here the magnetic field $\tilde{F}_\theta$ is the same as in (3.1).

It is easy to see that the functions $\{\tilde{u}_j\}_{j}$ belong to the form domain $\mathcal{D}(\tilde{Q}_\theta)$, since, by construction, we have $\tilde{Q}_\theta(\tilde{u}_j) = Q^N_\theta(u_j)$. Since the potential $\mathcal{U}$ is symmetric in the $t$-variable, we obtain for all $1 \leq j, k \leq n$,

$$
\tilde{a}_{\theta, \mathcal{U}}(\tilde{u}_j, \tilde{u}_k) = \langle u_j, (P^N_\theta + \mathcal{U})u_k \rangle_{L^2(\mathbb{R}^3_+)}. \quad (3.10)
$$
Since the \( \{u_j\}_{j=1}^n \) are eigenfunctions of \( \mathcal{P}_\theta^N + \mathcal{U} \), we get using (3.9),

\[
\tilde{a}_{\theta,\mathcal{U}}(\tilde{u}_j, \tilde{u}_k) = \delta_{j,k}\mu_k, \quad \forall \ 1 \leq j, k \leq n. \tag{3.11}
\]

Let \( \tilde{\mu}_n \) be the \( n \)-th eigenvalue of \( \tilde{\mathcal{P}}_\theta^N + \mathcal{U} \) defined by the min-max principle. Owing to (3.9) and (3.11) we find,

\[
\tilde{\mu}_n = \inf_{v_1, \ldots, v_n \in \mathcal{D}(\bar{\mathcal{Q}}_\theta)} \max_{\|v\| = 1} \tilde{a}_{\theta,\mathcal{U}}(v, v) \leq \max_{v \in \text{span}[u_1, \ldots, u_n]} \tilde{a}_{\theta,\mathcal{U}}(v, v) = \mu_n,
\]

This yields

\[
\mathcal{N}(\lambda; \mathcal{P}_\theta^N + \mathcal{U}, \mathbb{R}_+^3) \leq \mathcal{N}(\lambda; \tilde{\mathcal{P}}_\theta + \mathcal{U}, \mathbb{R}^3), \tag{3.12}
\]

and

\[
\mathcal{E}(\lambda; \mathcal{P}_\theta^N + \mathcal{U}, \mathbb{R}_+^3) \leq \mathcal{E}(\lambda; \tilde{\mathcal{P}}_\theta + \mathcal{U}, \mathbb{R}^3). \tag{3.13}
\]

The lemma follows by applying CLR inequality (resp. Lieb-Thirring inequality) to the right-hand side of (3.12) (resp. (3.13)).

\[\square\]

### 3.2. Schrödinger operator in a half-cylinder.

Consider a positive real number \( L \), and define the domain

\[
\Omega^L = \left( -\frac{L}{2}, \frac{L}{2} \right)^2 \times \mathbb{R}_+ \tag{3.14}
\]

In this section, we will analyse the magnetic Schrödinger operator

\[
\mathcal{P}_\theta^L = (-i\nabla + \mathbf{F}_\theta)^2 \quad \text{in} \quad L^2(\Omega^L) \tag{3.15}
\]

with Neumann boundary conditions at \( t = 0 \), and Dirichlet boundary conditions at \( r \in \{-\frac{L}{2}, \frac{L}{2}\} \) and \( s \in \{-\frac{L}{2}, \frac{L}{2}\} \). Here, for \( \theta \in [0, \pi/2] \), \( \mathbf{F}_\theta \) is the magnetic potential introduced in (3.4).

More precisely, the operator \( \mathcal{P}_\theta^L \) is defined as the self-adjoint Friedrichs extension in \( L^2(\Omega^L) \) associated with the semi-bounded quadratic form

\[
Q_\theta^L(u) = \int_{\Omega^L} |(-i\nabla + \mathbf{F}_\theta)u|^2 drdsdt, \tag{3.16}
\]

defined for all functions \( u \) in the form domain of \( Q_\theta^L \),

\[
\mathcal{D}(Q_\theta^L) = \left\{ u \in L^2(\Omega^L) : (-i\nabla + \mathbf{F}_\theta)u \in L^2(\Omega^L), \quad u\left( -\frac{L}{2}, \cdot, \cdot \right) = u\left( \frac{L}{2}, \cdot, \cdot \right) = 0, \quad u\left( \cdot, -\frac{L}{2}, \cdot \right) = u\left( \cdot, \frac{L}{2}, \cdot \right) = 0 \right\}. \tag{3.17}
\]

The next lemma establishes super-additivity properties on the sum of eigenvalues for \( \mathcal{P}_\theta^L \).

**Lemma 3.2.** For all \( n \in \mathbb{N} \), \( \lambda \in [0, 1] \) and \( L > 0 \), we have,

\[
\frac{\mathcal{E}(\lambda; \mathcal{P}_{\theta}^{nL}, \Omega^{nL})}{n^2L^2} \geq \frac{\mathcal{E}(\lambda; \mathcal{P}_{\theta}^{L}, \Omega^{L})}{L^2}. \tag{3.18}
\]

**Proof.** Let \( j, k \in \mathbb{N} \) such that \( 0 \leq j, k \leq n - 1 \), we define the domain

\[
\Omega_{j,k}^L := \left( \frac{(-n + 2j)L}{2}, \frac{(-n + 2j + 2)L}{2} \right) \times \left( \frac{(-n + 2k)L}{2}, \frac{(-n + 2k + 2)L}{2} \right) \times \mathbb{R}_+.
\]

It is clear that \( \Omega^{nL} = \cup_{j,k=0}^{n-1} \Omega_{j,k}^L \). We next consider the (Friedrichs) self-adjoint operator \( \mathcal{P}_{\theta,j,k}^L \) defined by the closed quadratic form

\[
Q_{\theta,j,k}^L(u) = \int_{\Omega_{j,k}^L} |(-i\nabla + \mathbf{F}_\theta)u|^2 drdsdt, \tag{3.19}
\]
We define the partition of unity
\[
\mathcal{D}(\Omega_{\theta,j,k}) = \left\{ u \in L^2(\Omega^L) : (-iD + F_0)u \in L^2(\Omega^L_{\theta,j,k}) \right\},
\]
\[
u \left( \frac{(-n + 2j)L}{2}, \cdot \right) = \nu \left( \frac{(-n + 2j + 2)L}{2}, \cdot \right) = 0,
\]
\[
u \left( \cdot, \frac{(-n + 2k)L}{2} \right) = \nu \left( \cdot, \frac{(-n + 2k + 2)L}{2} \right) = 0. \quad (3.20)
\]
Taking boundary conditions into account, we observe that for all \( u = \sum_{j,k} u_{j,k} \in \oplus_{j,k} \mathcal{D}(\Omega_{\theta,j,k}) \),
\[
\mathcal{D}(\Omega_{\theta,j,k})(u) = \sum_{j,k} \mathcal{D}(\Omega_{\theta,j,k})(u_{i,j}).
\]
This implies, that \( \mathcal{D}(\Omega_{\theta,j,k}) \leq \oplus_{j,k} \mathcal{D}(\Omega_{\theta,j,k}) \) (in the sense of quadratic forms). From the min-max principle, it follows easily that
\[
\sum_{j,k} E(\lambda; \mathcal{D}(\Omega_{\theta,j,k}), \Omega^L_{\theta,j,k}) \leq E(\lambda; \mathcal{D}(\Omega_{\theta}), \Omega^L_{\theta}), \quad \forall 1 \leq j, k \leq n. \quad (3.21)
\]
Since the operator \( \mathcal{D}(\Omega_{\theta,j,k}) \) is unitarily equivalent to \( \mathcal{D}(\Omega_{\theta}) \) by magnetic translation invariance, (3.21) becomes,
\[
n^2 E(\lambda; \mathcal{D}(\Omega_{\theta}), \Omega^L_{\theta}) \leq E(\lambda; \mathcal{D}(\Omega_{\theta}), \Omega^L_{\theta}).
\]
This gives (3.18) upon dividing both sides by \( n^2L^2 \).

We show in the next lemma a rough bound on the number and the sum of eigenvalues of \( \mathcal{D}(\Omega_{\theta}) \) in terms of \( L^2 \).

**Lemma 3.3.** Let \( L > 0 \). There exists a constant \( C \) such that for all \( \lambda \in [0,1) \) and \( \theta \in [0,\pi/2] \), it holds true that
\[
\frac{N(\lambda; \mathcal{D}(\Omega_{\theta}^L), \Omega^L_{\theta})}{L^2} \leq \frac{C}{\sqrt{1-\lambda}}, \quad (3.22)
\]
and
\[
\frac{E(\lambda; \mathcal{D}(\Omega_{\theta}^L), \Omega^L_{\theta})}{L^2} \leq \frac{C}{\sqrt{1-\lambda}} \quad (3.23)
\]
where \( N(\lambda; \mathcal{D}(\Omega_{\theta}^L), \Omega^L_{\theta}) \) and \( E(\lambda; \mathcal{D}(\Omega_{\theta}^L), \Omega^L_{\theta}) \) are defined in (1.8) and (1.9) respectively.

**Proof.** Let \( (\psi_1(t), \psi_2(t)) \) be a partition of unity on \( \mathbb{R}^+ \) with \( \psi_1^2(t) + \psi_2^2(t) = 1 \) and:
\[
\left\{ \begin{array}{l}
\psi_1(t) = 1 \quad \text{if} \; 0 < t < 1, \\
\psi_1(t) = 0 \quad \text{if} \; t > 2.
\end{array} \right. \quad (3.24)
\]
Let \( T > 1 \) be a large number to be chosen later. We consider the following two sets
\[
\Omega^T_1 = \left( -\frac{L}{2}, \frac{L}{2} \right)^2 \times (0, 2T), \quad \Omega^T_2 = \left( -\frac{L}{2}, \frac{L}{2} \right)^2 \times (T, \infty).
\]
We define the partition of unity \( (\psi_1(T), \psi_2(T)) \) by
\[
\psi_1(T) = \psi_1\left( \frac{t}{T} \right), \quad \psi_2(T) = \psi_2\left( \frac{t}{T} \right)
\]
We have \( \psi_1^2(T) = \frac{T}{T^2} \psi_1^2(T) \). Thus we deduce that there exists a constant \( C_0 > 0 \) such that
\[
\sum_{k=1}^{2} |\psi_k^2(T)|^2 \leq \frac{C_0}{T^2} \quad (3.25)
\]
By the IMS formula and the fact that $\psi_1^2, T(t) + \psi_2^2, T(t) = 1$, we find for all $u \in D(Q_0^L)$

$$Q_0^L(u) = \sum_{k=1}^2 \left( Q_0^L(\psi_k, T) - \int_{\Omega_T} U_T(t)|\psi_k, T u|^2 \: drdsdt \right), \quad U_T(t) = \sum_{k=1}^2 |\psi_k, T(t)|^2. \tag{3.26}$$

Let us denote by $P_{\theta, 1}^L$ and $P_{\theta, 2}^L$ the self-adjoint operators associated with the following quadratic forms:

$$Q_{\theta, 1}^L(u) = \int_{\Omega_T^2} \left[ |\partial_t u|^2 + |\partial_x u|^2 + |(-i\partial_x + t \cos(\theta) - s \sin(\theta))u|^2 \right] \: drdsdt, \tag{3.27}$$

$$D(Q_{\theta, 1}^L) := \{ u \in L^2(\Omega_T^2) : (-i\nabla + F_\theta)u \in L^2(\Omega_T^2), \quad u(2T, \cdot, \cdot) = 0, \quad (-\frac{L}{2}, \cdot, \cdot) = u\left(\frac{L}{2}, \cdot, \cdot\right) = 0 \}, \tag{3.28}$$

and

$$Q_{\theta, 2}^L(u) = \int_{\Omega_T^2} \left[ |\partial_t u|^2 + |\partial_x u|^2 + |(-i\partial_x + t \cos(\theta) - s \sin(\theta))u|^2 \right] \: drdsdt, \tag{3.29}$$

$$D(Q_{\theta, 2}^L) := \{ u \in L^2(\Omega_T^2) : (-i\nabla + F_\theta)u \in L^2(\Omega_T^2), \quad u(T, \cdot, \cdot) = 0, \quad (-\frac{L}{2}, \cdot, \cdot) = u\left(\frac{L}{2}, \cdot, \cdot\right) = 0 \}. \tag{3.30}$$

respectively.

It is clear from (3.26) that,

$$Q_0^L(u) \geq Q_{\theta, 1}^L(\psi_1, T) + Q_{\theta, 2}^L(\psi_2, T) - \frac{C_0}{T^2} \int_{\Omega_T} |u|^2 \: drdsdt. \tag{3.31}$$

By the variational principle (cf. [6, Lemma 5.1]), we see that

$$\mathcal{N}(\lambda; P_{\theta}^L, \Omega^L) \leq \mathcal{N}(\lambda + C_0T^{-2}; P_{\theta, 1}^L \oplus P_{\theta, 2}^L, \Omega^L). \tag{3.32}$$

It follows that

$$\mathcal{N}(\lambda; P_{\theta}^L, \Omega^L) \leq \mathcal{N}(\lambda + C_0T^{-2}; P_{\theta, 1}^L, \Omega_{T}^2) + \mathcal{N}(\lambda + C_0T^{-2}; P_{\theta, 2}^L, \Omega_{T}^2). \tag{3.33}$$

The Dirichlet boundary conditions imposed at $r \in \{-\frac{L}{2}, \frac{L}{2}\}$, $s \in \{\frac{L}{2}, \frac{L}{2}\}$ and $t = T$ ensures that the estimate (3.33) remains true if we replace $\Omega_{T}^2$ by $\mathbb{R}^3$ in the definition of $Q_{\theta, 2}^L$.

Since the first eigenvalue of the Schrödinger operator with constant unit magnetic field in $L^2(\mathbb{R}^3)$ is equal to 1, we thus find

$$Q_{\theta, 2}^L(u) \geq \int_{\mathbb{R}^3} |u|^2 \: drdsdt. \tag{3.34}$$

Choose $T = 2\sqrt{\frac{C_0}{1 - \lambda}}$, it holds that $1 > \lambda + C_0T^{-2}$ and

$$Q_{\theta, 2}^L(u) > (\lambda + C_0T^{-2}) \int_{\mathbb{R}^3} |u|^2 \: drdsdt. \tag{3.35}$$

This clearly gives $\mathcal{N}(\lambda + C_0T^{-2}; P_{\theta}, \mathbb{R}^3) = 0$. Thus, it remains to estimate $\mathcal{N}(\lambda + C_0T^{-2}; P_{\theta, 1}^L, \Omega_{T}^2)$. To do this, we introduce a potential $V$ satisfying

$$\begin{cases} V \geq 0, \\ \text{supp } V \subset \mathbb{R}^3_+ \setminus \Omega_{T}^2. \end{cases} \tag{3.36}$$

Under these assumptions on $V$, we may write for all $u \in D(Q_{\theta, 1}^L)$,

$$\int_{\Omega_T^2} |(-i\nabla + F_\theta)u|^2 \: drdsdt = \int_{\Omega_T^2} |(-i\nabla + F_\theta)u|^2 \: drdsdt + \int_{\mathbb{R}^3} V(x)|u|^2 \: drdsdt. \tag{3.37}$$
Here, we have extended \( u \) by 0 to the whole of \( \mathbb{R}^3_+ \) in the last integral. Therefore, it follows from the min-max principle that:

\[
N(\lambda + C_0 T^{-2}; \mathcal{P}_0^L, \Omega_1^T) = N(\lambda + C_0 T^{-2}; \mathcal{P}_0^L + V, \Omega_1^T).
\]

Since any function \( u \) that belongs to the form domain of \( Q_{0,1}^L \) can be extended by 0 to the half space \( \mathbb{R}^3_+ \), we get using the bound in (3.25) and the min-max principle that

\[
N(\lambda + C_0 T^{-2}; \mathcal{P}_0^L + V, \Omega_1^T) \leq N(\lambda; \mathcal{P}_0^N + V_1, \mathbb{R}^3_+),
\]

where

\[
V_1 = V - \frac{C_0}{T^2}.
\]

We select the potential \( V \) in \( \mathbb{R}^3_+ \) as follows,

\[
V(r, s, t) := \left(1 + \frac{C_0}{T^2}\right) 1_{\mathbb{R}^3_+ \setminus \Omega_1^T}.
\]

It is easy to check that \( V \) satisfies the assumptions in (3.35). To \( V \), we associate the reflected potential in \( \mathbb{R}^3 \) defined by

\[
\tilde{V}(r, s, t) := \left(1 + \frac{C_0}{T^2}\right) 1_{\mathbb{R}^3 \setminus \tilde{\Omega}_1^T},
\]

with

\[
\tilde{\Omega}_1^T = \left\{(r, s, t) \in \left(-\frac{L}{2}, \frac{L}{2}\right)^2 \times \mathbb{R} : |t| < 2T\right\}.
\]

In view of Lemma 3.1, we have,

\[
N(\lambda; \mathcal{P}_0^N + V_1, \mathbb{R}^3_+) \leq C_{CLR} \int_{\mathbb{R}^3} (\lambda - \tilde{V}_1)^{3/2} drdsdt
\]

where

\[
\tilde{V}_1 = \tilde{V} - \frac{C_0}{T^2}.
\]

Next, we compute the integral

\[
\int_{\mathbb{R}^3} (\lambda - \tilde{V}_1)^{3/2} drdsdt = 2\lambda^{3/2} \int_{\tilde{\Omega}_1^T} drdsdt = 4\lambda^{3/2} L^2 T.
\]

Inserting this in (3.39), we obtain

\[
N(\lambda; \mathcal{P}_0^N + V_1, \mathbb{R}^3_+) \leq 4C_{CLR} \lambda^{3/2} L^2 T.
\]

Combining the estimates (3.33), (3.37), (3.38) and (3.40) gives (3.22) upon inserting the choice

\[
T = 2\sqrt{\frac{L}{\lambda}}.
\]

In a similar fashion, we can prove (3.23) by following the steps of the proof of (3.22), and using Lemma 3.1 (the energy case). \(\square\)

### 4. The large area limit

Consider \( \theta \in [0, \pi/2] \) and a large positive number \( L > 0 \). Recall the magnetic potential introduced in (3.4) and the magnetic Schrödinger operator \( \mathcal{P}_0^L \) given in (3.15). In accordance with the definition of \( \mathcal{E} \) in (1.9), we write, for \( \lambda \in [0, 1) \),

\[
\mathcal{E}(\lambda; \mathcal{P}_0^L, \Omega^L) = \sum_j (\zeta_j^L(\theta) - \lambda) _-,
\]

where \( \{\zeta_j^L(\theta)\}_j \) denotes the sequence of eigenvalues of \( \mathcal{P}_0^L \).

We are interested in the behaviour of \( \mathcal{E}(\lambda; \mathcal{P}_0^L, \Omega^L) \) as \( L \) approach \( \infty \). We will obtain a limiting function \( E(\theta, \lambda) \) (see Theorem 4.1 below) such that the leading order asymptotics

\[
\mathcal{E}(\lambda; \mathcal{P}_0^L, \Omega^L) \sim E(\theta, \lambda)L^2
\]
holds true as $L \to \infty$. This approach was adapted in [14, 15] to prove the existence of several limiting functions related to the Ginzburg-Landau functional. We aim to prove

**Theorem 4.1.** Let $\theta \in [0, \pi/2]$ and $\lambda \in [0, 1)$. There exists a constant $E(\theta, \lambda)$ such that

$$\lim_{L \to \infty} \inf \frac{\mathcal{E}(\lambda; P^L_0, \Omega^L)}{L^2} = \lim_{L \to \infty} \sup \frac{\mathcal{E}(\lambda; P^L_0, \Omega^L)}{L^2} = E(\theta, \lambda).$$

Moreover, for all $\lambda_0 \in [0, 1)$, there exist positive uniform constants $L_0$ and $C_0$ such that,

$$E(\theta, \lambda) - \frac{2C_0}{L^{2/3}} \leq \frac{\mathcal{E}(\lambda; P^L_0, \Omega^L)}{L^2} \leq E(\theta, \lambda),$$

for all $\theta \in [0, \pi/2]$, $\lambda \in [0, \lambda_0]$, $L \geq 2L_0$.

The proof of Theorem 4.1 relies on the following lemma, which is proved in [14, Lemma 2.2].

**Lemma 4.2.** Consider a decreasing function $d : (0, \infty) \to (-\infty, 0]$ such that the function $f : (0, \infty) \ni L \mapsto \frac{d(L)}{L} \in \mathbb{R}$ is bounded.

Suppose that there exist constants $C > 0$, $L_0 > 0$ such that the estimate

$$f(nL) \geq f((1+a)L) - C \left( a + \frac{1}{a^2L^2} \right),$$

holds true for all $a \in (0, 1)$, $n \in \mathbb{N}$, $L \geq L_0$. Then $f(L)$ has a limit $A$ as $L \to \infty$. Furthermore, for all $L \geq 2L_0$, the following estimate holds true,

$$f(L) \leq A + \frac{2C}{L^{2/3}}.$$

In order to use the result of Lemma 4.2, we establish the estimate in the Lemma 4.3 below.

**Lemma 4.3.** Let $\lambda_0 \in [0, 1)$ and $\theta \in [0, \pi/2]$. There exist universal constants $C_0 > 0$ and $L_0 \geq 1$ such that, for all $L \geq L_0$, $\lambda \in [0, \lambda_0]$, $n \in \mathbb{N}$ and $a \in (0, 1)$, we have,

$$\frac{\mathcal{E}(\lambda; P^L_0, \Omega^L)}{n^2L^2} \leq \frac{\mathcal{E}(\lambda; P^{(1+a)L}_0, \Omega^{(1+a)L})}{(1+a)^2L^2} + C_0 \left( \frac{1}{a^2L^2} + a \right).$$

Furthermore, the function

$$L \mapsto \mathcal{E}(\lambda; P^L_0, \Omega^L)$$

is monotone increasing.

**Proof.** Let $n \geq 2$ be a natural number. If $a \in (0, 1)$ and $j = (j_1, j_2) \in \mathbb{Z}^2$, let

$$K_{a,j} = I_{j_1} \times I_{j_2},$$

where

$$\forall p \in \mathbb{Z}, \quad I_p = \left( \frac{2p + 1 - n}{2}, \frac{2p + 1 - n}{2} \right) \cap \left( \frac{1 + a}{2}, \frac{1 + a}{2} \right).$$

Consider a partition of unity $(\chi_j)_j$ of $\mathbb{R}^2$ such that:

$$\sum_j |\chi_j|^2 = 1, \quad 0 \leq \chi_j \leq 1 \quad \text{in} \ \mathbb{R}^2, \quad \text{supp} \chi_j \subset K_{a,j}, \quad |\nabla \chi_j| \leq \frac{C}{a},$$

where $C$ is a universal constant. We define $\chi_{j,L}(r, s) = \chi_j \left( \frac{r}{L}, \frac{s}{L} \right)$. We thus obtain a new partition of unity $(\chi_{j,L})_{j \in J}$ such that $\text{supp} \chi_{j,L} \subset K_{a,j,L}$, with

$$K_{a,j,L} = \{(Lr, Ls) : (r, s) \in K_{a,j}\}.$$

Let $\mathcal{J} = \{j = (j_1, j_2) \in \mathbb{Z}^2 : 0 \leq j_1, j_2 \leq n-1\}$ and $K_{nL} = \left( -\frac{nL}{2}, \frac{nL}{2} \right)^2$. Then the family $\{K_{a,j,L}\}_{j \in \mathcal{J}}$ is a covering of $K_{nL}$, and is formed of exactly $n^2$ squares with side length $L$. 

We restrict the partition of unity \( \{ \chi_{j,L} \}_{j \in J} \) to the set \( K_{nL} = \left( -\frac{nL}{2}, \frac{nL}{2} \right)^2 \). Let \( \mathcal{Q}_{\theta}^{nL} \) be the quadratic form defined in (3.16) and \( \{ f_{k,n} \}_{k=1}^{N} \) be any orthonormal set in \( \mathcal{D}(Q_{\theta}^{nL}) \). By the IMS formula and the fact that \( \sum_{j} \chi_{j,L}^2 = 1 \), we have

\[
\sum_{k=1}^{N} \left( \mathcal{Q}_{\theta}^{nL}(f_{k,n}) - \lambda \| f_{k,n} \|^2_{L^2(\Omega \times L)} \right) = \sum_{k=1}^{N} \sum_{j \in J} \left\{ \left( \mathcal{Q}_{\theta}^{nL}(\chi_{j,L} f_{k,n}) - \lambda \| \chi_{j,L} f_{k,n} \|^2_{L^2(\Omega \times L)} \right) - \| \nabla \chi_{j,L} f_{k,n} \|^2_{L^2(\Omega \times L)} \right\}
\]

(4.6)

where \( \Omega^{nL} = K_{nL} \times \mathbb{R}_+ \) is as defined in (3.14). Using the bound on \( |\nabla \chi_{j}| \) in (4.5) we obtain

\[
\sum_{k=1}^{N} \left( \mathcal{Q}_{\theta}^{nL}(f_{k,n}) - \lambda \| f_{k,n} \|^2_{L^2(\Omega \times L)} \right) \geq \sum_{k=1}^{N} \sum_{j \in J} \left( \mathcal{Q}_{\theta}^{nL}(\chi_{j,L} f_{k,n}) - (\lambda + \frac{C}{a^2 L^2}) \| \chi_{j,L} f_{k,n} \|^2_{L^2(\Omega \times L)} \right).
\]

(4.7)

For \( j \in J \), we define the trial density matrix \( \gamma : L^2(K_{a,j,L}) \ni f \mapsto \gamma_j f \in L^2(K_{a,j,L}) \),

\[
\gamma_j f = \chi_{j,L} \sum_{k=1}^{N} \langle \chi_{j,L} f, f_{k,n} \rangle f_{k,n}.
\]

It is clear that \( \gamma_j \) is a finite rank operator satisfying \( \gamma_j = \gamma_j^2 \) and \( 0 \leq \gamma_j \leq 1 \) (in the sense of quadratic forms). Moreover, we note that \( \gamma_j \) is constructed so that we can write

\[
\text{Tr} \left[ \left( \mathcal{Q}_{\theta}^{nL} - \left( \lambda + \frac{C}{L^2 a^2} \right) \mathcal{Q}_{\theta}^{nL} \right) \right] = \sum_{k=1}^{N} \left( \mathcal{Q}_{\theta}^{nL}(\chi_{j,L} f_{k,n}) - (\lambda + \frac{C}{L^2 a^2}) \| \chi_{j,L} f_{k,n} \|^2_{L^2(\Omega \times L)} \right).
\]

(4.8)

Notice that each \( \chi_{j,L} \) is supported in a square with side length \( (1 + a)L \). Hence, using magnetic translation invariance and Lemma 2.2, it follows that

\[
-\mathcal{E} \left( \lambda + \frac{C}{L^2 a^2} ; \mathcal{Q}_{\theta}^{nL}, \Omega^{(1+a)L} \right) \leq \text{Tr} \left[ \left( \mathcal{Q}_{\theta}^{nL} - \left( \lambda + \frac{C}{L^2 a^2} \right) \right) \right], \quad \forall j \in J.
\]

Using (4.8), this reads

\[
-\mathcal{E} \left( \lambda + \frac{C}{L^2 a^2} ; \mathcal{Q}_{\theta}^{nL}, \Omega^{(1+a)L} \right) \leq \sum_{k=1}^{N} \left( \mathcal{Q}_{\theta}^{nL}(\chi_{j,L} f_{k,n}) - (\lambda + \frac{C}{L^2 a^2}) \| \chi_{j,L} f_{k,n} \|^2_{L^2(\Omega \times L)} \right).
\]

Substituting into (4.7), we obtain

\[
\sum_{k=1}^{N} \left( \mathcal{Q}_{\theta}^{nL}(f_{k,n}) - \lambda \| f_{k,n} \|^2_{L^2(\Omega \times L)} \right) \geq -n^2 \mathcal{E} \left( \lambda + \frac{C}{L^2 a^2} ; \mathcal{Q}_{\theta}^{nL}, \Omega^{(1+a)L} \right)
\]

(4.9)

for all orthonormal family \( \{ f_{k,n} \}_{k=1}^{N} \) and \( N \in \mathbb{N} \). Therefore we conclude, in view of Lemma 2.1, that

\[
\mathcal{E}(\lambda, \mathcal{Q}_{\theta}^{nL}, \Omega^{nL}) \leq n^2 \mathcal{E} \left( \lambda + \frac{C}{L^2 a^2} ; \mathcal{Q}_{\theta}^{nL}, \Omega^{(1+a)L} \right).
\]

(4.10)

Let \( L_0 \geq C/(a \sqrt{1 - L_0}) \) chosen so that \( \lambda + C a^{-2} L^{-2} < 1 \) for all \( L \geq L_0 \). Applying Lemma 2.3 with \( \varsigma = \frac{C}{a^2 L^2} \), we find,

\[
\mathcal{E} \left( \lambda + \frac{C}{a^2 L^2} ; \mathcal{Q}_{\theta}^{nL}, \Omega^{(1+a)L} \right) - \mathcal{E}(\lambda ; \mathcal{Q}_{\theta}^{nL}, \Omega^{(1+a)L}) \leq C \frac{a^2 L^2}{a^2 L^2} \mathcal{N} \left( \lambda + \frac{C}{L^2 a^2} ; \mathcal{Q}_{\theta}^{nL}, \Omega^{(1+a)L} \right).
\]

By (3.22), it follows that

\[
\mathcal{E}(\lambda ; \mathcal{Q}_{\theta}^{nL}, \Omega^{(1+a)L}) \leq \mathcal{E}(\lambda ; \mathcal{Q}_{\theta}^{nL}, \Omega^{(1+a)L}) + \frac{C}{a^2}.
\]
for all $L \geq L_0$ and $\lambda \in [0, \lambda_0]$. Inserting this into (4.10), we get,

$$E(\lambda; \mathcal{P}_\theta^{nL}, \Omega^{nL}) \leq n^2 E(\lambda; \mathcal{P}_\theta^{(1+a)L}, \Omega^{(1+a)L}) + Cn^2 / a^2. \quad (4.11)$$

Dividing both sides by $n^2L^2$, we find

$$\frac{E(\lambda; \mathcal{P}_\theta^{nL}, \Omega^{nL})}{n^2L^2} \leq \frac{E(\lambda; \mathcal{P}_\theta^{(1+a)L}, \Omega^{(1+a)L})}{L^2} + C \left( \frac{1}{a^2L^2} \right). \quad (4.12)$$

We infer from (3.23) the following upper bound,

$$\frac{E(\lambda; \mathcal{P}_\theta^{nL}, \Omega^{nL})}{n^2L^2} \leq \frac{E(\lambda; \mathcal{P}_\theta^{(1+a)L}, \Omega^{(1+a)L})}{(1+a)^2L^2} + C \left( \frac{1}{a^2L^2} \right), \quad (4.13)$$

for all $L \geq L_0$ and $\lambda \in [0, \lambda_0]$. This proves the first assertion of the lemma.

To obtain monotonicity of $E(\lambda; \mathcal{P}_\theta^{nL}, \Omega^{nL})$, we consider $L' \geq L > 0$. Since the extension by zero of a function in the form domain $\mathcal{P}_\theta^{nL}$ is contained in the form domain of $\mathcal{P}_\theta^{nL'}$ and the values of both forms coincide for such a function, we may write in the sense of quadratic forms

$$\mathcal{P}_\theta^{nL'} \leq \mathcal{P}_\theta^{nL}.$$ 

On account of Lemma 2.2, it follows that,

$$-\text{Tr}(\mathcal{P}_\theta^{nL'} - \lambda) \leq -\text{Tr}(\mathcal{P}_\theta^{nL} - \lambda).$$

This shows that $E(\lambda; \mathcal{P}_\theta^{nL}, \Omega^{nL})$ is monotone increasing with respect to $L$, thereby proving the statement of the lemma. \hfill \Box

**Proof of Theorem 4.1.** Let $f(L) = -\frac{E(\lambda; \mathcal{P}_\theta^{nL}, \Omega^{nL})}{L^2}$. Thanks to Lemma 4.3, we know that the functions $f(L)$ and $d(L) = -E(\lambda; \mathcal{P}_\theta^{nL}, \Omega^{nL})$ satisfy the assumptions in Lemma 4.2. Consequently, $f(L)$ has a limit as $L \to \infty$. Let us define

$$E(\theta, \lambda) := -\lim_{L \to \infty} f(L).$$

By Lemma 4.2, there exists $L_0 > 0$ such that

$$E(\theta, \lambda) \leq \frac{E(\lambda; \mathcal{P}_\theta^{nL}, \Omega^{nL})}{L^2} + \frac{2C_0}{L^{2/3}}, \quad (4.14)$$

for all $L \geq 2L_0$ and $\lambda \in [0, \lambda_0]$. It remains to establish the upper bound. According to Lemma 3.2, we know that the energy satisfies

$$\frac{E(\lambda; \mathcal{P}_\theta^{nL}, \Omega^{nL})}{n^2L^2} \geq \frac{E(\lambda; \mathcal{P}_\theta^{nL}, \Omega^{nL})}{L^2}.$$ 

Letting $n \to \infty$ gives us

$$E(\theta, \lambda) \geq \frac{E(\lambda; \mathcal{P}_\theta^{nL}, \Omega^{nL})}{L^2}. $$

This, together with (4.14), completes the proof of Theorem 4.1. \hfill \Box

5. **Properties of the function $E(\theta, \lambda)$**

In Theorem 4.1, we proved the existence of a limiting function $E(\theta, \lambda) \in [0, \infty)$ defined for $\theta \in [0, \pi/2]$ and $\lambda \in [0, 1)$. We aim in this section to study the properties of $E(\theta, \lambda)$ as a function of $\theta$ and $\lambda$. In particular, we establish continuity of $E(\theta, \lambda) > 0$ with respect to $\theta$ and $\lambda$.

**Lemma 5.1.** Let $\lambda_0 \in [0, 1)$ and $\delta > 0$. There exists $L^* > 0$ such that the following is true. Let $L \geq L^*$, there exists $\eta > 0$ such that, if

$$(\epsilon, \nu) \in (-\eta, \eta)^2 \quad \text{and} \quad (\theta, \lambda) \in [0, \pi/2] \times [0, \lambda_0],$$

then

$$E(\theta, \lambda) > 0.$$
then
\[
\left| \frac{\mathcal{E}(\lambda + \nu; \mathcal{P}_{\theta^+}, \Omega) - \mathcal{E}(\lambda; \mathcal{P}_{\theta}, \Omega^L)}{L^2} \right| \leq \frac{\delta}{2}.
\]

**Proof.** We introduce a partition of unity of \( \mathbb{R} \),
\[
\zeta_1^2 + \zeta_2^2 = 1, \quad \text{supp} \, \zeta_1 \subset [0, 1], \quad \text{supp} \, \zeta_2 \subset [1/2, \infty), \quad |\zeta_p| \leq C', \quad p = 1, 2. \tag{5.1}
\]
Let
\[
L \geq L^* := \max \left\{ \left( \frac{4C'}{1 - \lambda_0} \right)^{1/2}, \left( \frac{4C'C}{\delta \sqrt{(1 - \lambda_0)/2}} \right)^{1/2} \right\}, \tag{5.2}
\]
where \( C' \) and \( C \) are the universal constants appearing in (5.1) and (3.23) respectively. We put further,
\[
\zeta_{p,L}(r,s,t) = \zeta_p(t/L), \quad p = 1, 2, \quad (r,s,t) \in \mathbb{R}^3_+.
\]
Next, let \( \{ f_k \}_{k=1}^N \) be an orthonormal family of compactly supported functions in \( \mathcal{D}(\mathcal{Q}_{\theta^+L}) \). We have the following IMS decomposition formula
\[
\sum_{k=1}^N \left( \mathcal{Q}_{\theta^+L}(f_k) - (\lambda + \nu) \| f_k \|_{L^2(\Omega^L)}^2 \right)
= \sum_{k=1}^N \sum_{p=1}^2 \left( \mathcal{Q}_{\theta^+L}(\zeta_p f_k) - (\lambda + \nu) \| \zeta_p f_k \|_{L^2(\Omega^L)}^2 \right) - \| \nabla \zeta_p, L \| f_k \|_{L^2(\Omega^L)}^2 \right\} \tag{5.3}
\]
To estimate the last term we use the bound on \( \nabla \zeta_p \) in (5.1), and get, after inserting \( \zeta_1^2 + \zeta_2^2 = 1 \), that
\[
\sum_{k=1}^N \left( \mathcal{Q}_{\theta^+L}(f_k) - (\lambda + \nu) \| f_k \|_{L^2(\Omega^L)}^2 \right) \geq \sum_{k=1}^N \sum_{p=1}^2 \left( \mathcal{Q}_{\theta^+L}(\zeta_p f_k) - (\lambda + \nu + C'L^2) \| \zeta_p f_k \|_{L^2(\Omega^L)}^2 \right). \tag{5.4}
\]
Since the function \( \varphi = \zeta_{2,L} f_k \in \mathcal{C}_0^\infty(\Omega^L) \) is compactly supported in \( \Omega^L \), it can be extended by zero to all of \( \mathbb{R}^3 \). Hence, using the fact that the first eigenvalue of the Schrödinger operator in \( L^2(\mathbb{R}^3) \) is 1, and selecting \( |\nu| < \frac{\lambda_0}{2} \), the choice of \( L \) in (5.2) yields
\[
\mathcal{Q}_{\theta^+L}(\zeta_{1,L} f_k) \geq \int_{\Omega^L} |\zeta_{1,L} f_k|^2 drdsdt > (\lambda + \nu + C'L^2) \int_{\Omega^L} |\zeta_{1,L} f_k|^2 drdsdt. \tag{5.5}
\]
Consequently, we find that the term corresponding to \( p = 2 \) on the right hand side of (5.4) is strictly positive and can be neglected for a lower bound. What remains is to estimate the term corresponding to \( p = 1 \) in (5.4). Using Cauchy-Schwarz inequality (with \( \varphi \) arbitrary), we obtain,
\[
\mathcal{Q}_{\theta^+L}(\zeta_{1,L} f_k)
\geq (1 - \delta) \int_{\Omega^L} |(-i \nabla + \mathbf{F}_{\theta^+}) \zeta_{1,L} f_k|^2 drdsdt - \delta^{-1} \int_{\Omega^L} |(\mathbf{F}_{\theta^+} - \mathbf{F}_{\theta}) \zeta_{1,L} f_k|^2 drdsdt. \tag{5.6}
\]
where \( \mathbf{F}_{\theta} \) is the same as in (3.4). Using the bounds
\[
|\cos(\theta + \epsilon) - \cos(\theta)| \leq |\epsilon|, \quad |\sin(\theta + \epsilon) - \sin(\theta)| \leq |\epsilon|,
\]
we get
\[
|\mathbf{F}_{\theta^+}(r,s,t) - \mathbf{F}_{\theta}(r,s,t)| \leq |\epsilon| (|s| + |t|), \quad \forall \, (r,s,t) \in \mathbb{R}^3_+.
\]
Taking the support of \( \zeta_{1,L} \) into consideration, we infer from (5.6) the following bound,
\[
\mathcal{Q}_{\theta^+L}(\zeta_{1,L} f_k) \geq (1 - \delta) \int_{\Omega^L} |(-i \nabla + \mathbf{F}_{\theta}) \zeta_{1,L} f_k|^2 drdsdt - \delta^{-1} \epsilon^2 L^2 \int_{\Omega^L} |\zeta_{1,L} f_k|^2 drdsdt.
\]
Inserting this into (5.3), we get, using the bound on $|\zeta_{1,L}^L|$, that

$$
\sum_{k=1}^N \left( Q_{\theta+\epsilon}^L(f_k) - (\lambda + \nu) \|f_k\|^2_{L^2(\Omega^L)} \right) 
\geq \sum_{k=1}^N \left( (1 - \theta) Q_{\theta}^L(\zeta_{1,L}f_k) - (\lambda + \nu + \epsilon^2 \theta^{-1}(L)^2 + C' L^{-2}) \|\zeta_{1,L}f_k\|^2_{L^2(\Omega^L)} \right). 
$$

(5.7)

We choose $\theta = |\epsilon|$ and define the trial density matrix $L^2(\mathbb{R}^3) \ni f \mapsto \gamma f \in L^2(\mathbb{R}^3)$,

$$
\gamma f = \sum_{k=1}^N \langle f, \zeta_{1,L}f_k \rangle \zeta_{1,L}f_k.
$$

It is clear that $0 \leq \gamma \leq 1$ in the sense of quadratic forms and that $\mathcal{P}_{\theta}^L \gamma$ is trace class (actually this is a finite-rank operator). By Lemma 2.2 we see that

$$
-\mathcal{E} \left( \frac{\lambda + \nu + |\epsilon|L^2 + C'L^{-2}}{1 - |\epsilon|} ; \mathcal{P}_{\theta}^L, \Omega^L \right) \leq \text{Tr} \left[ \left( \mathcal{P}_{\theta}^L - \frac{\lambda + \nu + |\epsilon|L^2 + C'L^{-2}}{1 - |\epsilon|} \right) \gamma \right]
:= \sum_{k=1}^N \left( Q_{\theta}^L(\zeta_{1,L}f_k) - \left( \frac{\lambda + \nu + |\epsilon|L^2 + C'L^{-2}}{1 - |\epsilon|} \right) \|\zeta_{1,L}f_k\|^2_{L^2(\Omega^L)} \right). 
$$

(5.8)

Inserting this into (5.7), we obtain

$$
\sum_{k=1}^N \left( Q_{\theta+\epsilon}^L(f_k) - (\lambda + \nu) \|f_k\|^2_{L^2(\Omega^L)} \right) \geq -(1 - |\epsilon|) \mathcal{E} \left( \frac{\lambda + \nu + |\epsilon|L^2 + C'L^{-2}}{1 - |\epsilon|} ; \mathcal{P}_{\theta}^L, \Omega^L \right). 
$$

(5.9)

Consequently, it follows from Lemma 2.1 that,

$$
\mathcal{E} \left( \frac{\lambda + \nu + |\epsilon|L^2 + C'L^{-2}}{1 - |\epsilon|} ; \mathcal{P}_{\theta}^L, \Omega^L \right) \leq (1 - |\epsilon|) \mathcal{E} \left( \frac{\lambda + \nu + |\epsilon|L^2 + C'L^{-2}}{1 - |\epsilon|} ; \mathcal{P}_{\theta}^L, \Omega^L \right). 
$$

(5.10)

Fix $|\epsilon| < \frac{1-\lambda_0}{4(1+L^2)}$. Applying Lemma 2.3 with $\xi = \frac{|\epsilon|(L^2 + \lambda) + \nu + C'L^{-2}}{1 - |\epsilon|}$, we get,

$$
\mathcal{E} \left( \frac{\lambda + \nu + |\epsilon|L^2 + C'L^{-2}}{1 - |\epsilon|} ; \mathcal{P}_{\theta}^L, \Omega^L \right) \leq \mathcal{E}(\lambda; \mathcal{P}_{\theta}^L, \Omega^L) 
+ \frac{|\epsilon|(L^2 + \lambda) + \nu + C'L^{-2}}{1 - |\epsilon|} N \left( \frac{\lambda + \nu + |\epsilon|L^2 + C'L^{-2}}{1 - |\epsilon|} ; \mathcal{P}_{\theta}^L, \Omega^L \right). 
$$

(5.11)

Plugging (5.11) into (5.10), we obtain from (3.23) that

$$
\mathcal{E}(\lambda + \nu; \mathcal{P}_{\theta+\epsilon}^L, \Omega^L) \leq (1 - |\epsilon|) \mathcal{E}(\lambda; \mathcal{P}_{\theta}^L, \Omega^L) + \frac{C}{\sqrt{(1 - \lambda_0)/2}}(|\epsilon|(L^2 + \lambda) + \nu + C'L^{-2})L^2, 
$$

(5.12)

where $C$ is the constant from (3.22). Interchanging the roles of $\theta$ and $\theta + \epsilon$ we arrive at

$$
\left| \mathcal{E}(\lambda + \nu; \mathcal{P}_{\theta+\epsilon}^L, \Omega^L) - \mathcal{E}(\lambda; \mathcal{P}_{\theta}^L, \Omega^L) \right|
\leq |\epsilon| \mathcal{E}(\lambda; \mathcal{P}_{\theta}^L, \Omega^L) + \frac{C}{\sqrt{(1 - \lambda_0)/2}}(|\epsilon|(L^2 + \lambda) + |\nu| + C'L^{-2})L^2. 
$$

(5.13)
Dividing both sides by $L^2$, we get,
\[
\left| \frac{\mathcal{E}(\lambda + \nu; \mathcal{P}_{\theta=e}^L, \Omega^L)}{L^2} - \frac{\mathcal{E}(\lambda; \mathcal{P}_{\theta}^L, \Omega^L)}{L^2} \right| \leq \left| \epsilon \right| \frac{\mathcal{E}(\lambda; \mathcal{P}_{\theta}^L, \Omega^L)}{L^2} + \frac{C(|\epsilon|((L)^2 + \lambda) + |\nu| + C'L^{-2})}{\sqrt{(1 - \lambda_0)/2}}.
\]
Using the estimate in (3.23), we further obtain
\[
\left| \frac{\mathcal{E}(\lambda + \nu; \mathcal{P}_{\theta=e}^L, \Omega^L)}{L^2} - \frac{\mathcal{E}(\lambda; \mathcal{P}_{\theta}^L, \Omega^L)}{L^2} \right| \leq \eta \left| \frac{\mathcal{E}(\lambda; \mathcal{P}_{\theta}^L, \Omega^L)}{L^2} + \frac{C(\eta(L^2 + \lambda + 1) + C'L^{-2})}{\sqrt{(1 - \lambda_0)/2}}. \right.
\]
Selecting $\eta < \min\left( \frac{1-\lambda_0}{4(1+L^2)}, \frac{\delta\sqrt{(1-\lambda_0)/2}}{4C(\lambda_0+C'L^2)} \right)$ and using the condition (5.2) on $L$, we conclude that,
\[
\left| \frac{\mathcal{E}(\lambda + \nu; \mathcal{P}_{\theta=e}^L, \Omega^L)}{L^2} - \frac{\mathcal{E}(\lambda; \mathcal{P}_{\theta}^L, \Omega^L)}{L^2} \right| \leq \frac{\delta}{2},
\]
thereby proving the assertion of the lemma.

We have the following corollary of Lemma 5.1.

**Corollary 5.2.** Given $\lambda_0 \in [0, 1)$, the function
\[
[0, \pi/2] \times [0, \lambda_0] \ni (\theta, \lambda) \mapsto E(\theta; \lambda)
\]
is continuous.

**Proof.** In view of Theorem 4.1, there exist constants $C_0$ and $L_0$ such that for all $L \geq 2L_0$ and $(\nu, \epsilon)$ satisfying $\lambda + \nu \in [0, \lambda_0]$ and $\theta + \epsilon \in [0, \pi/2]$, one has
\[
|E(\theta + \epsilon, \lambda + \nu) - E(\theta, \lambda)| \leq \frac{|\mathcal{E}(\lambda + \nu; \mathcal{P}_{\theta+\epsilon}^L, \Omega^L) - \mathcal{E}(\lambda; \mathcal{P}_{\theta}^L, \Omega^L)|}{L^2} + \frac{2C_0}{L^{2/3}}.
\]
Let $\delta > 0$ and select $L^*$ as in Lemma 5.1. Let $L \geq \max\{2L_0, L^*, (4C_0/\delta)^{3/2}\}$. We assign to $L$ a constant $\eta > 0$ as described in Lemma 5.1. Consequently, if
\[
(\nu, \epsilon) \in (-\eta, \eta)^2, \quad \lambda + \nu \in [0, \lambda_0], \quad \text{and} \quad \theta + \epsilon \in [0, \pi/2],
\]
then
\[
|E(\theta + \epsilon, \lambda + \nu) - E(\theta, \lambda)| \leq \delta/2 + \delta/2 = \delta.
\]
This finishes the proof.

We conclude this section by the following lemma.

**Lemma 5.3.** Let $\theta \in [0, \pi/2]$. The function
\[
[0, 1) \ni \lambda \mapsto E(\theta, \lambda)
\]
is locally Lipschitz.

**Proof.** Fix $\lambda_0 \in [0, 1)$, and let $\lambda_1, \lambda_2 \in [0, \lambda_0]$ be such that $\lambda_1 < \lambda_2$. Let $L > 0$ and $\mathcal{P}_{\theta}^L$ be as defined in (3.15). We infer from Lemma 2.3 that
\[
\mathcal{E}(\lambda_2; \mathcal{P}_{\theta}^L, \Omega^L) - \mathcal{E}(\lambda_1; \mathcal{P}_{\theta}^L, \Omega^L) \leq (\lambda_2 - \lambda_1)N(\lambda_2; \mathcal{P}_{\theta}^L, \Omega^L).
\]
In view of (3.22), there exists a constant $C_0$ independent of $\theta$ such that
\[
N(\lambda_2; \mathcal{P}_{\theta}^L, \Omega^L) \leq C_0L^2.
\]
This implies
\[
\mathcal{E}(\lambda_2; \mathcal{P}_{\theta}^L, \Omega^L) - \mathcal{E}(\lambda_1; \mathcal{P}_{\theta}^L, \Omega^L) \leq C_0L^2(\lambda_2 - \lambda_1).
\]
Dividing both sides by $L^2$, we get, after taking $L \to \infty$,
\[
E(\lambda_2; \lambda) - E(\theta, \lambda_1) \leq C_0(\lambda_2 - \lambda_1).
\]
Interchanging the roles of $\lambda_1$ and $\lambda_2$, we get further,

$$|E(\theta, \lambda_2) - E(\theta, \lambda_1)| \leq C_0|\lambda_2 - \lambda_1|,$$

which gives the assertion of the lemma. □

6. Explicit formula of $E(\theta, \lambda)$

Recall the constant $E(\theta, \lambda)$ defined in (4.1). The aim of this section is to provide an explicit formula for $E(\theta, \lambda)$ using the projectors on the eigenfunctions of the Neumann Schrödinger operator given in (3.5). We shall consider the cases $\theta = 0$ and $\theta \in (0, \pi/2]$ separately. Indeed, the construction of eigenprojectors in the case $\theta = 0$ is similar in spirit to the two-dimensional case (cf. [13, Section 4]), whereas in the case $\theta \in (0, \pi/2]$, the projectors are constructed using the spectral decomposition of the two-dimensional model operator $\mathcal{L}(\theta)$.

6.1. $E(\theta, \lambda)$ in the case $\theta = 0$. We start by recalling the family of one-dimensional harmonic oscillators $H(\xi)$, $\xi \in \mathbb{R}$, defined by :

$$H(\xi) = -\partial_t^2 + (t - \xi)^2 \quad \text{in} \quad L^2(\mathbb{R}_+).$$

on their common Neumann domain:

$$\{ v \in H^2(\mathbb{R}_+), \ t^2v \in L^2(\mathbb{R}_+), \ v'(0) = 0 \}.$$

We denote by $(u_j(\cdot;\xi))_{j=1}^\infty$ the orthonormal family of real-valued eigenfunctions of the operator $H(\xi)$, i.e.,

$$H(\xi)u_j(t;\xi) = \mu_j(\xi)u_j(t;\xi), \quad u'_j(0;\xi) = 0, \quad \int_{\mathbb{R}_+} u_j(t;\xi)^2 dt = 1. \quad (6.2)$$

The lowest eigenvalue $\mu_1(\xi)$ is studied in [2, 7]. We collect in the following proposition some of the properties of $\mu_1(\xi)$ as a function of $\xi$ :

**Proposition 6.1.** The function $\mathbb{R} \ni \xi \mapsto \mu_1(\xi)$ is continuous and satisfies

1. $\mu_1(\xi) > 0$, for all $\xi \in \mathbb{R}$.
2. At $-\infty$ we have the limit

$$\lim_{\xi \to -\infty} \mu_1(\xi) = +\infty. \quad (6.3)$$

3. At the origin the value is

$$\mu_1(0) = 1. \quad (6.4)$$

4. At $+\infty$ we have

$$\lim_{\xi \to +\infty} \mu_1(\xi) = 1. \quad (6.5)$$

5. $\mu_1$ has a minimum $\Theta_0 \in (0, 1)$ at a unique $\xi_0 \in (0, 1)$,

$$\Theta_0 := \inf_{\xi \in \mathbb{R}} \mu_1(\xi) = \mu_1(\xi_0) < 1. \quad (6.6)$$

Moreover, this minimum is non-degenerate and $\mu_1(\xi)$ is strictly decreasing on $(-\infty, \xi_0]$ from $+\infty$ to $\Theta_0$ and strictly increasing on $[\xi_0, \infty)$ from $\Theta_0$ to 1.

The next Lemma is taken from [16, Lemma 2.1].

**Lemma 6.2.** The second eigenvalue $\mu_2(\xi)$ satisfies,

$$\inf_{\xi \in \mathbb{R}} \mu_2(\xi) > 1.$$

Thanks to Proposition 6.1, one can easily prove the following:

**Lemma 6.3.** Let $\mu_1(\xi)$ be defined as in (6.2). We have

$$\int_{\mathbb{R}^2} (\mu_1(\xi) + \tau^2 - \lambda) \chi d\xi d\tau = \frac{4}{3} \int_0^\infty (\mu_1(\xi) - \lambda)^{3/2} d\xi,$$

and the integrals are finite for all $\lambda \in [0, 1)$. 

For later reference, we include Agmon-type estimates on the eigenfunction $u_1(t; \xi)$ (cf. [25, Theorem 2.6.2]).

**Lemma 6.4.** Let $\lambda \in [0, 1)$. For all $\epsilon \in (0, 1)$, there exists a constant $C_\epsilon$ such that, for all $\xi \in \mathbb{R}_+$ satisfying $\mu_1(\xi) \leq \lambda$, we have
\[
\left\| e^{\frac{it\epsilon^2}{2}} u_1(t, \xi) \right\|^2_{H^1 \left( \{ t \in \mathbb{R}_+: (t-\epsilon)^2 \geq C_\epsilon \} \right)} \leq C_\epsilon .
\] (6.7)

Next, we consider the Schrödinger operator (3.5) in the particular case $\theta = 0$, i.e.,
\[
\mathcal{P}_0^N = -\partial_t^2 - \partial_r^2 + (-i\partial_r + t)^2 \quad \text{in} \quad L^2(\mathbb{R}_+^3).
\] (6.8)

Let $(\xi, \tau) \in \mathbb{R}^2$ and denote by $\mathcal{F}_{r\rightarrow\xi}$ (resp. $\mathcal{F}_{s\rightarrow\tau}$) the partial Fourier transform with respect to the variable $r$ (resp. $s$). We define the bounded function $\mathbb{R}_+^3 \ni (r, s, t) \mapsto v_j(r, s, t; \xi, \tau)$ by:
\[
v_j(r, s, t; \xi, \tau) = \frac{1}{2\pi} e^{-ir\xi} e^{i\tau s} u_j(t; \xi).
\] (6.9)

Then, we introduce the projectors $\Pi_j(\xi, \tau)$ on the functions $v_j$:
\[
L^2(\mathbb{R}_+^3) \ni \varphi \mapsto (\Pi_j(\xi, \tau)\varphi)(r, s, t) = v_j(r_1, s_1, t_1; \xi, \tau) \int_{\mathbb{R}_+^3} \overline{v_j(r_2, s_2, t_2; \xi, \tau)} \varphi(r_2, s_2, t_2) dr_2 ds_2 dt_2
\] (6.10)
so that we can write, in terms of quadratic forms,
\[
\langle \varphi, \Pi_j(\xi, \tau)\varphi \rangle_{L^2(\mathbb{R}_+^3)} = \left| \langle \varphi, v_j(\cdot; \xi, \tau) \rangle_{L^2(\mathbb{R}_+^3)} \right|^2 = 2\pi \left| \mathcal{F}_{r\rightarrow-\xi} \left( \mathcal{F}_{s\rightarrow-t} \varphi(\cdot, \cdot, t) \right)(-\tau) \right| (-\xi, u_j(t; \xi))_{L^2(\mathbb{R}_+^3)}^2
\] (6.11)
We state in the next lemma useful properties of the family $\{ \Pi_j(\xi, \tau) \}_{(j, \xi, \tau) \in \mathbb{N} \times \mathbb{R}^2}$.

**Lemma 6.5.** For all $\varphi \in D(\mathcal{P}_0^N)$, we have
\[
\langle \mathcal{P}_0^N \Pi_j(\xi, \tau)\varphi, \varphi \rangle_{L^2(\mathbb{R}_+^3)} = (\mu_j(\xi) + \tau^2) \langle \Pi_j(\xi, \tau)\varphi, \varphi \rangle_{L^2(\mathbb{R}_+^3)}.
\] (6.12)

Moreover, for all $\varphi \in L^2(\mathbb{R}_+^3)$, one has
\[
\sum_j \int_{\mathbb{R}^2} \langle \varphi, \Pi_j(\xi, \tau)\varphi \rangle_{L^2(\mathbb{R}_+^3)} d\xi d\tau = 2\pi \| \varphi \|^2_{L^2(\mathbb{R}_+^3)}.
\] (6.13)

**Proof.** Let $(\tau, \xi) \in \mathbb{R}^2$. By the definition of $v_j$ in (6.9), we find,
\[
\mathcal{P}_0^N v_j(r, s, t; \xi, \tau) = (\mu_j(\xi) + \tau^2) v_j(r, s, t; \xi, \tau).
\]

The definition of the projectors in (6.10) immediately gives us (6.12).

Using the fact that $u_j(\cdot; \xi)$ is an orthonormal basis of $L^2(\mathbb{R}_+)$ for all $\xi \in \mathbb{R}$, we find, using the representation in (6.11),
\[
\sum_j \langle \varphi, \Pi_j(\xi, \tau)\varphi \rangle_{L^2(\mathbb{R}_+^3)} = 2\pi \int_{\mathbb{R}_+} \left| \mathcal{F}_{r\rightarrow-\xi} \left( \mathcal{F}_{s\rightarrow-t} \varphi(\cdot, \cdot, t) \right)(-\tau) \right|^2 dt.
\]

Integrating with respect to $\xi$ and $\tau$, and applying Plancherel’s identity twice, we obtain
\[
\int_{\mathbb{R}^2} \sum_j \langle \varphi, \Pi_j(\xi, \tau)\varphi \rangle_{L^2(\mathbb{R}_+^3)} d\xi d\tau = 2\pi \| \varphi \|^2_{L^2(\mathbb{R}_+^3)}.
\]

The proof of the lemma is thus completed. $\Box$

We will prove
Theorem 6.6. Given \( \lambda \in (0, 1) \), the following formula holds true:

\[
E(0, \lambda) = \frac{1}{3\pi} \int_0^{\infty} (\mu_1(\xi) - \lambda)^{3/2} d\xi, \tag{6.14}
\]

where \( \mu_1(\xi) \) is defined in (6.2).

Proof. We start by obtaining an upper bound on \( E(0, \lambda) \). Let \( L > 0 \). Pick an arbitrary positive integer \( N \) and let \( \{f_1, \cdots, f_N\} \) be any \( L^2 \) orthonormal set in \( D(P_0^L) \). In view of (6.12) and (6.13), we have the following splitting (recall the domain \( \Omega^L \) from (3.14)),

\[
\sum_{j=1}^N \langle f_j, (P_0^L - \lambda)f_j \rangle_{L^2(\Omega^L)} = \frac{1}{2\pi} \sum_{j=1}^N \sum_{p=1}^\infty \int_{\mathbb{R}^2} (\mu_p(\xi) + \tau^2 - \lambda) \langle f_j, \Pi_p(\xi, \tau)f_j \rangle_{L^2(\mathbb{R}_+^3)} d\xi d\tau,
\]

where we have extended \( f_j \) by \( 0 \) to \( \mathbb{R}_+^3 \setminus \Omega^L \). Since \( \lambda \in [0, 1) \), Lemma 6.2 gives that \( \mu_p(\xi) + \tau^2 > \lambda \) for \( p \geq 2 \). Hence, we obtain

\[
\sum_{j=1}^N \langle f_j, (P_0^L - \lambda)f_j \rangle_{L^2(\Omega^L)} \geq - \frac{1}{2\pi} \int_{\mathbb{R}^2} (\mu_1(\xi) + \tau^2 - \lambda) - \sum_{j=1}^N \langle f_j, \Pi_1(\xi, \tau)f_j \rangle_{L^2(\mathbb{R}_+^3)}. \tag{6.15}
\]

Since \( \{f_j\}_{j=1}^N \) is an orthonormal family in \( L^2(\Omega^L) \), we deduce that

\[
\sum_{j=1}^N \langle f_j, \Pi_1(\xi, \tau)f_j \rangle_{L^2(\mathbb{R}_+^3)} = \sum_{j=1}^N |\langle v_1, f_j \rangle|^2 \leq \|v_1\|^2 = \frac{1}{2\pi} L^2. \tag{6.16}
\]

The last equality comes from the fact that the function \( u_1(\cdot; \xi) \) is normalized in \( L^2(\mathbb{R}_+) \) for all \( \xi \). Substituting (6.16) into (6.15) yields

\[
\sum_{j=1}^N \langle f_j, (P_0^L - \lambda)f_j \rangle_{L^2(\Omega^L)} \geq - \frac{L^2}{4\pi^2} \int_{\mathbb{R}^2} (\mu_1(\xi) + \tau^2 - \lambda) d\xi d\tau.
\]

Then, on account of Definition 1.9 and Lemma 2.1, we have

\[
\frac{\mathcal{E}(\lambda; P_0^L, \Omega^L)}{L^2} \leq \frac{1}{4\pi^2} \int_{\mathbb{R}^2} (\mu_1(\xi) + \tau^2 - \lambda) d\xi d\tau.
\]

Letting \( L \to \infty \) and using Lemma 6.3, we arrive at

\[
E(0, \lambda) \leq \frac{1}{3\pi} \int_0^{\infty} (\mu_1(\xi) - \lambda)^{3/2} d\xi. \tag{6.17}
\]

We give the proof of the lower bound on \( E(0, \lambda) \). Let \( M(\xi, \tau) \) be a function with \( 0 \leq M \leq 1 \) and consider the trial density matrix

\[
\gamma = \int_{\mathbb{R}^2} M(\xi, \tau) \Pi_1(\xi, \tau) d\xi d\tau.
\]

We will prove that \( 0 \leq \gamma \leq 2\pi \) in the sense of quadratic forms. Consider \( g \in L^2(\Omega^L) \). Using that \( 0 \leq M \leq 1 \), we have

\[
0 \leq \langle g, \gamma g \rangle_{L^2(\mathbb{R}_+^3)} \leq \int_{\mathbb{R}^2} |\langle g, v_1 \rangle|^2 d\xi d\tau \leq \sum_j \int_{\mathbb{R}^2} |\langle g, v_j \rangle|^2 d\xi d\tau = 2\pi \|g\|^2.
\]

The last step follows from Plancherel’s identity and the fact that \( u_j(\cdot, \xi) \) is an orthonormal basis of \( L^2(\mathbb{R}_+) \). Recall the quadratic form \( Q_0^L \) from (3.16). It is easy to check that

\[
Q_0^L(v_1) = \frac{L^2}{2\pi} (\mu_1(\xi) + \tau^2). \tag{6.18}
\]
We choose $M$ to be the characteristic function of the set $\{(\xi, \tau) \in \mathbb{R}^2 : \lambda - \mu_1(\xi) - \tau^2 \geq 0\}$. We compute, using (6.18),

$$\text{Tr}[(P_{\Omega}^L - \lambda)\gamma] = \int_{\mathbb{R}^2} M(\xi, \tau) (Q_{\tau}^L (v_1) - \lambda \|v_1\|^2_{L^2(\Omega^L)}) d\xi d\tau = -\frac{L^2}{2\pi} \int_{\mathbb{R}^2} (\mu_1(\xi) + \tau^2 - \lambda) - d\xi d\tau.$$ 

In view of Lemma 2.2, we get

$$-\text{Tr}(P_{\Omega}^L - \lambda) - \frac{1}{2\pi} \text{Tr}((P_{\Omega}^L - \lambda)\gamma) = -\frac{L^2}{4\pi^2} \int_{\mathbb{R}^2} (\mu_1(\xi) + \tau^2 - \lambda) - d\xi d\tau.$$ 

This gives us

$$\mathcal{E}(\lambda; P_{\Omega}^L, \Omega^L) \geq \frac{1}{4\pi^2} \int_{\mathbb{R}^2} (\mu_1(\xi) + \tau^2 - \lambda) - d\xi d\tau = \frac{1}{3\pi^2} \int_0^{\infty} (\mu_1(\xi) - \lambda)^{3/2} d\xi.$$ 

Letting $L \to \infty$ yields the desired upper bound.

\[ \square \]

6.2. $E(\theta, \lambda)$ in the case $\theta \in (0, \pi/2]$. The purpose of this subsection is to provide an explicit formula for $E(\theta, \lambda)$ in the case $\theta \in (0, \pi/2]$ and $\lambda \in [0, 1)$. However, we have not been able to compute it directly as in the case $\theta = 0$. Our approach is to find an alternative limiting function $F(\theta, \lambda)$ (see (6.35) below), which can be constructed and computed explicitly using the eigenprojectors on the eigenfunctions of the two-dimensional model operator from (1.11):

$$\mathcal{L}(\theta) = -\partial_\xi^2 - \partial_\tau^2 + (t \cos(\theta) - s \sin(\theta))^2 \quad \text{in} \quad L^2(\mathbb{R}^2_+).$$

(6.19)

Let us recall some fundamental spectral properties of $\mathcal{L}(\theta)$ when $\theta \in (0, \pi/2]$ (see [12] for details and references). We denote by $\zeta_1(\theta)$ the infimum of the spectrum of $\mathcal{L}(\theta)$:

$$\zeta_1(\theta) := \inf \text{Spec}(\mathcal{L}(\theta)).$$

(6.20)

The function $(0, \pi/2) \ni \theta \mapsto \zeta_1(\theta)$ is monotone increasing and the essential spectrum is the interval $[1, \infty)$. We denote the non-decreasing sequence of eigenvalues of $\mathcal{L}(\theta)$ in $(-\infty, 1)$ counting multiplicities by $(\zeta_j(\theta))_{j \in \mathbb{N}}$. The associated orthonormal sequence of eigenfunctions is denoted by $(u_{\theta,j})_{j \in \mathbb{N}}$ and satisfies,

$$\mathcal{L}(\theta) u_{\theta,j} = \zeta_j(\theta) u_{\theta,j}, \quad \langle u_{\theta,j}, u_{\theta,k} \rangle_{L^2(\mathbb{R}^2_+)} = \delta_{j,k}. \quad (6.21)$$

Using the technique of ‘Agmon estimates’, it is proved in [5, Theorem 1.1] that the eigenfunctions of $\mathcal{L}(\theta)$ decay exponentially at infinity. For later use, we record this as

\[ \text{Lemma 6.7.} \quad \text{Let } \theta \in (0, \pi/2]. \quad \text{Given } \lambda \in (0, 1) \text{ and } \alpha \in (0, \sqrt{1 - \lambda}), \text{ there exists a positive constant } C_{\alpha, \lambda} \text{ such that, for any eigen-pair } (\zeta_j(\theta), u_{\theta,j}) \text{ of } \mathcal{L}(\theta) \text{ with } \zeta_j(\theta) < \lambda, \text{ we have} \]

$$Q_\theta(e^{\alpha \sqrt{\zeta_j - \lambda}} u_{\theta,j}) \leq C_{\alpha, \lambda} \|u_{\theta,j}\|^2_{L^2(\mathbb{R}^2_+)},$$

where $Q_\theta$ is the quadratic form associated with $\mathcal{L}(\theta)$.

\[ \text{Lemma 6.8.} \quad \text{Let } \theta \in (0, \pi/2]. \quad \text{There exists a constant } C \text{ such that} \]

$$\mathcal{N}(1; \mathcal{L}(\theta)) \leq \frac{C}{\sin(\theta)}.$$ 

Next, we define the function $\mathbb{R}_+^3 \ni (r, s, t) \mapsto v_{\theta,j}(r, s, t; \xi)$ by

$$v_{\theta,j}(r, s, t; \xi) = \frac{1}{\sqrt{2\pi}} e^{i\xi t} u_{\theta,j}\left(s - \frac{\xi}{\sin(\theta)}, t\right). \quad (6.22)$$
where \( \{ u_{\theta,j} \}_j \) are the eigenfunctions from (6.21). We define the projectors \( \pi_{\theta,j} \) by
\[
(\pi_{\theta,j}(\xi)) \varphi(s_1, t_1) = u_{\theta,j} \left( s_1 - \frac{\xi}{\sin(\theta)}, t_1 \right) \int_{\mathbb{R}^2_+} u_{\theta,j} \left( s_2 - \frac{\xi}{\sin(\theta)}, t_2 \right) \varphi(s_2, t_2) ds_2 dt_2.
\]
(6.23)

We then introduce a family of operators \( \Pi_{\theta,j} \) defined by
\[
L^2(\mathbb{R}^3_+) \ni f \mapsto \Pi_{\theta,j} f(r, s, t_1)
= \int_{\mathbb{R}} v_{\theta,j}(r, s, t_1; \xi) \left( \int_{\mathbb{R}^3_+} v_{\theta,j}(r_2, s_2, t_2; \xi) f(r_2, s_2, t_2) dr_2 ds_2 dt_2 \right) d\xi.
\]
(6.24)

In terms of quadratic forms, we have,
\[
L^2(\mathbb{R}^3_+) \ni f \mapsto (\Pi_{\theta,j} f, f)_{L^2(\mathbb{R}^3_+)} = \int_{\mathbb{R}} \left\{ v_{\theta,j}(\cdot, \xi) f, f \right\}_{L^2(\mathbb{R}^3_+)}^2 d\xi
= \int_{\mathbb{R}} \left( F_{\xi \to r}(\pi_{j,\theta}(\xi)(F_{r \to \xi} f(\cdot, s, t)(\xi))(r), f(r, s, t))^2_{L^2(\mathbb{R}^3_+)} d\xi.
\]
(6.25)

Since the Fourier transform is a unitary transform and \( \pi_{\theta,j}(\xi) \) is a projection, it is easily to be seen that the operator \( \Pi_{\theta,j} \) is a projection too. The following Lemma illustrates relevant properties of the family of projectors \( \{ \Pi_{\theta,j} \}_j \).

**Lemma 6.9.** For all \( f \in \mathcal{D}(\mathcal{P}_{\theta}^N) \), we have
\[
(\mathcal{P}_{\theta}^N \Pi_{\theta,j} f, f)_{L^2(\mathbb{R}^3_+)} = \zeta_j(\theta)(\Pi_{\theta,j} f, f)_{L^2(\mathbb{R}^3_+)},
\]
(6.26)
and for all \( f \in L^2(\mathbb{R}^3_+) \), one has
\[
\left( \sum_j \Pi_{\theta,j} f, f \right)_{L^2(\mathbb{R}^3_+)} \leq \|f\|_{L^2(\mathbb{R}^3_+)}^2.
\]
(6.27)

Moreover, for any smooth cut-off function \( \chi \in C^\infty_0(\mathbb{R}^2) \), it holds true that
\[
\text{Tr}(\chi \Pi_{\theta,j} \chi) = \frac{\sin(\theta)}{2\pi} \int_{\mathbb{R}^2} \chi^2(r, s) dr ds.
\]
(6.28)

**Proof.** Applying the operator \( \mathcal{P}_{\theta}^N \) to the function \( v_{\theta,j} \), we find
\[
\mathcal{P}_{\theta}^N v_{\theta,j}(r, s, t; \xi) = \zeta_j(\theta) v_{\theta,j}(r, s, t; \xi).
\]

The assertion (6.26) then follows from the definition of \( \Pi_{\theta,j} \) in (6.24).

To prove (6.27), we rewrite (6.25) as
\[
\left( \sum_j \Pi_{\theta,j} f, f \right)_{L^2(\mathbb{R}^3_+)} = \int_{\mathbb{R}} \left( \pi_{\theta,j}(\xi)(F_{r \to \xi} f(\cdot, s, t)(\xi)), F_{r \to \xi} f(\cdot, s, t)(\xi) \right)_{L^2(\mathbb{R}^3_+)} d\xi.
\]
(6.29)

It can be easily shown that \( \sum_j \pi_{\theta,j} \) is a projection. Hence, by Plancherel’s identity, we see that
\[
\left( \sum_j \Pi_{\theta,j} f, f \right)_{L^2(\mathbb{R}^3_+)} \leq \int_{\mathbb{R}^3} \int_{\mathbb{R}} |F_{r \to \xi} f(\cdot, s, t)|^2 d\xi ds dt = \int_{\mathbb{R}^3} |f(r, s, t)|^2 dr ds dt.
\]
(6.28)

We come to the proof of (6.28). For this, we notice that
\[
\text{Tr}(\chi \Pi_{\theta,j} \chi) = \frac{1}{2\pi} \int_{\mathbb{R}^3} \chi^2(r, s) \left( \int_{\mathbb{R}} \left| e^{i\xi}\tau u_{\theta,j}(s - \frac{\xi}{\sin(\theta)}, t) \right|^2 d\xi \right) dr ds dt.
\]

Performing the change of variable \( \nu = s - \frac{\xi}{\sin(\theta)} \) and using that the functions \( \{ u_{\theta,j} \}_j \) are normalized, we get
\[
\text{Tr}(\chi \Pi_{\theta,j} \chi) = \frac{1}{2\pi} \sin(\theta) \int_{\mathbb{R}^2} \chi^2(r, s) dr ds \int_{\mathbb{R}^3} |u_{\theta,j}(\nu, t)|^2 d\nu dt = \frac{1}{2\pi} \sin(\theta) \int_{\mathbb{R}^2} \chi^2(r, s) dr ds.
\]
(6.30)
Thereby completing the proof of the lemma.

Let $a > 0$. In order to define $F(\theta, \lambda)$ below, we need to introduce the cut-off function $\chi_a \in C_0^\infty(\mathbb{R}^2)$, which satisfies

\[ 0 \leq \chi_a \leq 1, \text{ in } \mathbb{R}^2, \supp \chi_a \in \left(-\frac{1 + a}{2}, \frac{1 + a}{2}\right), \chi_a = 1 \text{ in } \left(-\frac{1}{2}, \frac{1}{2}\right), |\nabla \chi_a| \leq Ca^{-1}. \tag{6.31} \]

Let $L > 0$. We set

\[ \chi_{a,L}(r, s) = \chi_a \left(\frac{r}{L}, \frac{s}{L}\right), \ (r, s) \in \mathbb{R}^2, \tag{6.32} \]

and,

\[ \mu_a = \int_{\mathbb{R}^2} \chi_a^2(r, s) dr ds. \tag{6.33} \]

Notice that $\mu_a$ satisfies

\[ \mu_a = 1 + \mathcal{O}(a), \quad (a \to 0_+). \tag{6.34} \]

We define $F(\theta, \lambda; a)$ to be

\[ F(\theta, \lambda; a) := \limsup_{L \to \infty} \frac{\text{Tr}(\chi_{a,L}(\mathcal{P}_\theta^N - \lambda)\chi_{a,L})}{L^2}, \tag{6.35} \]

where $\mathcal{P}_\theta^N$ is the self-adjoint operator given in (3.5). We now formulate the main theorem of this section.

**Theorem 6.10.** Let $\theta \in (0, \pi/2]$, $\lambda \in [0, 1)$ and $E(\theta, \lambda)$ be as introduced in (4.1). We have the following explicit formula of $E(\theta, \lambda)$

\[ E(\theta, \lambda) = \frac{1}{2\pi} \sin(\theta) \sum_j (\zeta_j(\theta) - \lambda)_-, \tag{6.36} \]

where the $\{\zeta_j(\theta)\}_j$ are the eigenvalues from (6.21).

The proof of Theorem 6.10 is splitted into two lemmas.

**Lemma 6.11.** Let $a \in (0, 1)$, $\lambda \in [0, 1)$ and $\theta \in (0, \pi/2]$. The following limit exists,

\[ \lim_{L \to \infty} \frac{\text{Tr}(\chi_{a,L}(\mathcal{P}_\theta^N - \lambda)\chi_{a,L})}{L^2}, \]

and its value is $F(\theta, \lambda; a)$ introduced in (6.35). Furthermore, there holds,

\[ F(\theta, \lambda; a) = \frac{1}{2\pi} \mu_a \sin(\theta) \sum_j (\zeta_j(\theta) - \lambda)_-, \tag{6.37} \]

where $\mu_a$ is the constant defined in (6.33).

**Proof.** Let $\mathcal{P}_\theta^N$ be the self-adjoint operator given in (3.5), and let $\{g_1, \ldots, g_N\}$ be any orthonormal set in $\mathcal{D}(\mathcal{P}_\theta^N)$. It follows from Lemma 6.9 that

\[ \sum_{k=1}^N \langle \chi_{a,L}g_k, (\mathcal{P}_\theta^N - \lambda)\chi_{a,L}g_k \rangle_{L^2(\mathbb{R}^3_+)} \geq - \sum_j (\zeta_j(\theta) - \lambda) - \sum_{k=1}^N \langle \chi_{a,L}g_k, \Pi_{\theta,j}\chi_{a,L}g_k \rangle_{L^2(\mathbb{R}^3_+)} \tag{6.38} \]

Since $\{g_k\}_{k=1}^N$ is an orthonormal family in $L^2(\mathbb{R}^3_+)$ and performing a similar calculation to that in (6.30), we deduce that

\[ \sum_{k=1}^N \langle \chi_{a,L}g_k, \Pi_{\theta,j}\chi_{a,L}g_k \rangle_{L^2(\mathbb{R}^3_+)} \leq \int_{\mathbb{R}} \sum_{k=1}^N \left| \langle g_k, v_{j,\theta}\chi_{a,L} \rangle_{L^2(\mathbb{R}^3_+)} \right|^2 d\xi \leq \frac{1}{2\pi} \mu_a L^2 \sin(\theta). \]
Implementing this in \((6.38)\), we obtain
\[
\sum_{k=1}^{N} \langle \chi_{a,L} g_k, (\mathcal{P}^N_{\theta} - \lambda)\chi_{a,L} g_k \rangle_{L^2(\mathbb{R}^3_+)} \geq -\frac{1}{2\pi} L^2 \mu_a \sin(\theta) \sum_j (\zeta_j(\theta) - \lambda)_- . \tag{6.39}
\]
By the variational principle in Lemma 2.1, we find
\[
\text{Tr}(\chi_{a,L}(\mathcal{P}^N_{\theta} - \lambda)\chi_{a,L})_- \leq \frac{1}{2\pi} L^2 \mu_a \sin(\theta) \sum_j (\zeta_j(\theta) - \lambda)_- . \tag{6.40}
\]
Dividing by \(L^2\) on both sides, we get after passing to the limit \(L \to \infty\),
\[
\limsup_{L \to \infty} \frac{\text{Tr}(\chi_{a,L}(\mathcal{P}^N_{\theta} - \lambda)\chi_{a,L})_-}{L^2} \leq \frac{1}{2\pi} \mu_a \sin(\theta) \sum_j (\zeta_j(\theta) - \lambda)_- . \tag{6.41}
\]
To prove a lower bound, we consider the density matrix
\[
\gamma = \sum_{\{j: \zeta_j(\theta) \leq \lambda\}} \Pi_{\theta,j} . \tag{6.42}
\]
It is easy to see that \(\gamma \geq 0\), and in view of \((6.27)\), it follows that
\[
\langle f, \gamma f \rangle_{L^2(\mathbb{R}^3_+)} \leq \|f\|_{L^2(\mathbb{R}^3_+)}^2 . \tag{6.43}
\]
Next, by Cauchy-Schwarz inequality, we have,
\[
\text{Tr}(\chi_{a,L}(\mathcal{P}^N_{\theta} - \lambda)\chi_{a,L}\Pi_{\theta,j}) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (|(-i\nabla + F_{\theta})\chi_{a,L} v_{\theta,j}|^2 - \lambda |\chi_{a,L} v_{\theta,j}|^2)drdsdt\xi \\
\leq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left\{ \chi_{a,L}^2(r,s) \left| (-i\nabla + F_{\theta}) v_{\theta,j} \right|^2 - \lambda |v_{\theta,j}|^2 \right\}drdsdt\xi . \tag{6.44}
\]
Performing the change of variable \(v = s - \frac{\xi}{\sin(\theta)}\) in \((6.44)\) and using that the function \(u_{j,\theta}\) is normalized, we arrive at
\[
\text{Tr}(\chi_{a,L}(\mathcal{P}^N_{\theta} - \lambda)\chi_{a,L}\gamma) \leq \frac{\sin(\theta)}{2\pi} \sum_{\{j: \zeta_j(\theta) \leq \lambda\}} \left\{ |\zeta_j(\theta) - \lambda| \mu_a L^2 + \int_{\mathbb{R}^2} |\nabla \chi_{a,L}(r,s)|^2 drds \right\} , \tag{6.45}
\]
where \(\mu_a\) is the constant from \((6.33)\). Dividing both sides of the aforementioned inequality by \(L^2\), we see that,
\[
\frac{\text{Tr}(\chi_{a,L}(\mathcal{P}^N_{\theta} - \lambda)\chi_{a,L}\gamma)}{L^2} \leq \frac{1}{2\pi} \sin(\theta) \sum_{\{j: \zeta_j(\theta) \leq \lambda\}} \left\{ \mu_a (\zeta_j(\theta) - \lambda) + L^{-2} \int_{\mathbb{R}^2} |\nabla \chi_a|^2 drds \right\} . \tag{6.46}
\]
Here we point out that the number \(N(\lambda; L(\theta), \mathbb{R}^3_+)\) is controlled by \(C/\sin(\theta)\) according to Lemma 6.8. From Lemma 2.2, it follows that
\[
- \frac{\text{Tr}(\chi_{a,L}(\mathcal{P}^N_{\theta} - \lambda)\chi_{a,L})_-}{L^2} \leq - \frac{1}{2\pi} \mu_a \sin(\theta) \sum_j (\zeta_j(\theta) - \lambda)_- + C(2\pi)^{-1} L^{-2} \int_{\mathbb{R}^2} |\nabla \chi_a|^2 drds .
\]
Taking \(\liminf\) \(L \to \infty\), we deduce that
\[
\liminf_{L \to \infty} \frac{\text{Tr}(\chi_{a,L}(\mathcal{P}^N_{\theta} - \lambda)\chi_{a,L})_-}{L^2} \geq \frac{1}{2\pi} \mu_a \sin(\theta) \sum_j (\zeta_j(\theta) - \lambda)_- .
\]
This together with \((6.41)\) and the definition of \(F(\theta, \lambda; a)\) in \((6.35)\), finish the proof of the Lemma. \(\square\)
Our next goal is to establish a connection between the function $F(\theta, \lambda)$ obtained in Lemma 6.11 and $E(\theta, \lambda)$ from (4.1).

**Theorem 6.12.** Let $a \in (0, 1)$, $\theta \in (0, \pi/2)$ and $\lambda \in [0, 1)$. There holds,
\[
\mu_a E(\theta, \lambda) \leq F(\theta, \lambda; a) \leq (1 + a)^2 E(\theta, \lambda),
\]
where $\mu_a$ is the constant from (6.32).

**Proof.** Let $L \gg \ell \gg 1$. We consider the domain
\[
\Omega_{j,k,\ell} = (j\ell, (j + 1)\ell) \times (k\ell, (k + 1)\ell) \times \mathbb{R}_+, \quad (j, k) \in \mathbb{Z}^2.
\]
We will denote by $N_B$ the number of boxes of the form $\Omega_{j,k,\ell}$ intersecting $\text{supp} \chi_{a,L}:
\[
N_B = |\{ (j, k) \in \mathbb{Z}^2 : \text{supp} \chi_{a,L} \cap \Omega_{j,k,\ell} \neq \emptyset \}|. \tag{6.47}
\]
Recall the magnetic potential $F_\theta$ defined in (3.4). Consider the self-adjoint operator $P^\ell_{\theta,j,k}$ defined by the closed quadratic form
\[
\mathcal{Q}^\ell_{\theta,j,k}(u) = \int_{\Omega_{j,k,\ell}} |(-i\nabla + F_\theta)u|^2 drdsdt,
\]
with domain,
\[
\mathcal{D}(\mathcal{Q}^\ell_{\theta,j,k}) = \left\{ u \in L^2(\Omega_{j,k,\ell}) : (-i\nabla + F_\theta)u \in L^2(\Omega_{j,k,\ell}), \right. \nonumber
\]
\[
\left. u(j\ell, \cdot, \cdot) = u((j + 1)\ell, \cdot, \cdot) = 0, u(\cdot, k\ell, \cdot) = u(\cdot, (k + 1)\ell, \cdot) = 0 \right\}. \tag{6.48}
\]
Since any function that belongs to the form domain $\bigoplus_{j,k} \mathcal{D}(\mathcal{Q}^\ell_{\theta,j,k})$ lies in the form domain $\mathcal{D}(\mathcal{Q}^N_{\theta})$ and the values of both quadratic forms coincide for such a function, we have the operator inequality
\[
\chi_{a,L}(P^\ell_{\theta,j,k} - \lambda)\chi_{a,L} \leq \bigoplus_{j,k} \chi_{a,L}(P^\ell_{\theta,j,k} - \lambda)\chi_{a,L}. \tag{6.49}
\]
It follows that
\[
-\text{Tr}(\chi_{a,L}(P^\ell_{\theta,j,k} - \lambda)\chi_{a,L}) - \leq -\sum_{j,k} \text{Tr}(\chi_{a,L}(P^\ell_{\theta,j,k} - \lambda)\chi_{a,L}) - . \tag{6.50}
\]
Let $S_{jk} \in \mathbb{N}$ be the number of eigenvalues of $P^\ell_{\theta,j,k}$ that are below $\lambda$, denoted by $\{\lambda_m\}_{m=1}^{S_{jk}}$, and let $\{f_m\}_{m=1}^{S_{jk}} \in \mathcal{D}(\mathcal{Q}^\ell_{\theta,j,k})$ be associated (normalized) eigenfunctions. We consider the density matrix
\[
\gamma_{j,k} f = \sum_{m=1}^{S_{jk}} \langle f, f_m \rangle f_m, \quad f \in L^2(\Omega_{jk}).
\]
We compute
\[
\text{Tr}(\chi_{a,L}(P^\ell_{\theta,j,k} - \lambda)\chi_{a,L} \gamma_{j,k}) = \sum_{m=1}^{S_{jk}} \mathcal{Q}^\ell_{\theta,j,k}(\chi_{a,L}f_m) - \lambda \|\chi_{a,L}f_m\|^2
\]
\[
= \sum_{m=1}^{S_{jk}} \left\{ (\lambda_m - \lambda)\|\chi_{a,L}f_m\|^2 + \|\nabla \chi_{a,L} |f_m|^2 \right\}, \tag{6.51}
\]
where the last step follows since $\{f_m\}_{m=1}^{S_{jk}}$ are eigenfunctions.

Let us denote by $x^*_{j,k,\ell}$ the point belonging to the interval $(j\ell, (j + 1)\ell) \times (k\ell, (k + 1)\ell)$ where the function $\chi^2_{a,L}$ attains its minimum:
\[
\chi^2_{a,L}(x^*_{j,k,\ell}) = \min_{(r,s) \in (j\ell, (j + 1)\ell) \times (k\ell, (k + 1)\ell)} \chi^2_{a,L}(r, s).
It follows that
\[
\text{Tr} \left( \chi_{a,L}(\mathcal{P}_{\theta,j,k}^\ell - \lambda) \chi_{a,L} \gamma_{j,k} \right) \leq \chi_{a,L}^2(x^f_{j,k,\ell}) \sum_{m=1}^{S_{jk}} (\lambda_m - \lambda) + \sum_{m=1}^{S_{jk}} \| \nabla \chi_{a,L} f_m \|^2, \tag{6.52}
\]
where have used that the term \( \sum_{m=1}^{S_{jk}} (\lambda_m - \lambda) \) is negative. Inserting this into (6.52) and using the bound \( | \nabla \chi_{a,L} | \leq C(aL)^{-1} \), we find
\[
\text{Tr} \left( \chi_{a,L}(\mathcal{P}_{\theta,j,k}^\ell - \lambda) \chi_{a,L} \gamma_{j,k} \right) \leq \chi_{a,L}^2(x^f_{j,k,\ell}) \sum_{m=1}^{S_{jk}} (\lambda_m - \lambda) + C S_{jk} (aL)^{-2}.
\]
By (3.22), we have \( S_{jk} \leq C \ell^2 \). Using (4.2), we obtain
\[
\text{Tr} \left( \chi_{a,L}(\mathcal{P}_{\theta,j,k}^\ell - \lambda) \chi_{a,L} \gamma_{j,k} \right) \leq \chi_{a,L}^2(x^f_{j,k,\ell})(-E(\theta, \lambda) + C \ell^{-2/3} \ell^2 + C \ell^2 (aL)^{-2}. \tag{6.53}
\]
By (6.50) and Lemma 2.2, it follows that
\[
- \text{Tr} \left( \chi_{a,L}(\mathcal{P}_{\theta,j,k}^N - \lambda) \chi_{a,L} \right) \leq \left\{ \sum_{j,k} \chi_{a,L}^2(x^f_{j,k,\ell}) \ell^2 \right\} (-E(\theta, \lambda) + C \ell^{-2/3} + CN_B \ell^2 (aL)^{-2}. \tag{6.54}
\]
The sum \( \sum_{j,k} \chi_{a,L}^2(x^f_{j,k,\ell}) \ell^2 \) is a (lower) Riemannian sum. Thus, we have
\[
\left| \sum_{j,k} \chi_{a,L}^2(x^f_{j,k,\ell}) \ell^2 - \int_{\mathbb{R}^2} \chi_{a,L}^2(r,s) dr ds \right| = \mu_a L^2 \leq C \ell L.
\]
Substituting this into (6.54), we obtain
\[
- \text{Tr} \left( \chi_{a,L}(\mathcal{P}_{\theta,j,k}^N - \lambda) \chi_{a,L} \right) \leq E(\theta, \lambda) L^2 \left( - \mu_a + C \ell L^{-1} \right) + C \ell^{-2/3}(\mu_a L^2 + C \ell L) + CN_B \ell^2 (aL)^{-2.}
\]
Dividing both sides by \( L^2 \), we get
\[
- \frac{\text{Tr} \left( \chi_{a,L}(\mathcal{P}_{\theta,j,k}^N - \lambda) \chi_{a,L} \right)}{L^2} \leq E(\theta, \lambda) \left( - \mu_a + \frac{C \ell L^{-1}}{L^2} \right) + C \ell^{-2/3}(\mu_a + C L^{-1}) + CN_B \ell^{-2} a^{-2} L^{-4}.
\]
We make the following choice of \( \ell \),
\[
\ell = L^n, \quad \eta < 1.
\]
Since \( N_B \sim ((1 + a))^2 L^2 \ell^{-2} \) as \( L \to \infty \), we get, after taking \( \liminf \), the following lower bound,
\[
\liminf_{L \to \infty} \frac{\text{Tr} \left( \chi_{a,L}(\mathcal{P}_{\theta,j,k}^N - \lambda) \chi_{a,L} \right)}{L^2} \geq E(\theta, \lambda) \mu_a. \tag{6.55}
\]
It remains to prove the upper bound. Using Lemma 2.2 and that the trace is cyclic, we see that
\[
- \text{Tr} \left( \chi_{a,L}(\mathcal{P}_{\theta,j,k}^N - \lambda) \chi_{a,L} \right) = \inf_{0 \leq \gamma \leq 1} \text{Tr} \left[ (\mathcal{P}_{\theta,j,k}^N - \lambda) \gamma \chi_{a,L} \right].
\]
Since the function \( \chi_{a,L} \) is supported in \( \left( \frac{-1}{2}, \frac{1}{2} \right)^2 \), it follows that
\[
- \text{Tr} \left( \chi_{a,L}(\mathcal{P}_{\theta,j,k}^N - \lambda) \chi_{a,L} \right) \geq \inf_{0 \leq \gamma \leq 1} \text{Tr} \left[ (\mathcal{P}_{\theta,j,k}^{(1+a)L} - \lambda) \gamma \right] \geq -((1 + a) L)^2 E(\theta, \lambda). \tag{6.56}
\]
By taking \( \limsup \), this yields that,
\[
\limsup_{L \to \infty} \frac{\text{Tr} \left( \chi_{a,L}(\mathcal{P}_{\theta,j,k}^N - \lambda) \chi_{a,L} \right)}{L^2} \leq E(\theta, \lambda) (1 + a)^2. \tag{6.57}
\]
Combining (6.55) and (6.57), we obtain
\[
\mu_a E(\theta, \lambda) \leq \liminf_{L \to \infty} \frac{\text{Tr}(\chi_{a,L}(\mathcal{P}_0^N - \lambda)\chi_{a,L})}{L^2} \\
\leq \limsup_{L \to \infty} \frac{\text{Tr}(\chi_{a,L}(\mathcal{P}_0^N - \lambda)\chi_{a,L})}{L^2} \leq (1 + a)^2 E(\theta, \lambda). \tag{6.58}
\]

Recalling the definition of \( F(\theta, \lambda) \) in (6.35) and Lemma 6.11 finishes the proof of Lemma 6.12. \( \square \)

**Proof of Theorem 6.10.** The proof follows easily by combining the results of Lemmas 6.11, 6.12, and by sending the parameter \( a \) to \( 0^+ \). \( \square \)

### 6.3. Dilatation

Let us define the unitary operator
\[
U_{h,b} : L^2(\mathbb{R}^3_+) \ni u \mapsto U_{h,b} u(z) = h^{-3/4}b^{3/4}u(h^{-1/2}b^{1/2}z) \in L^2(\mathbb{R}^3_+), \tag{6.59}
\]

Let \( h, b > 0 \) and \( \theta \in [0, \pi/2] \). We introduce the self-adjoint operator
\[
\mathcal{P}^N_{\theta,h,b} = (-ih\nabla + bF_\theta)^2, \quad \text{in} \quad L^2(\mathbb{R}^3_+), \tag{6.60}
\]

with Neumann boundary conditions at \( t = 0 \). With \( \mathcal{P}^N_0 \) being the operator from (3.5), it is easy to check that
\[
\mathcal{P}^N_{\theta,h,b} = h^2U_{h,b}^{-1}\mathcal{P}^N_0 U_{h,b}. \tag{6.61}
\]

For \( j \in \mathbb{N} \) and \((\xi, \tau) \in \mathbb{R}^2\), we introduce the family of projectors
\[
\Pi_j(\xi, \tau; h, b) = U_{h,b}\Pi_j(\xi, \tau)U_{h,b}^{-1} \tag{6.62}
\]

and, for \( \theta \in (0, \pi/2] \),
\[
\Pi_{\theta,j}(h, b) = U_{h,b}\Pi_{\theta,j}U_{h,b}^{-1} \tag{6.63}
\]

where, \( \Pi_j(\xi, \tau) \) and \( \Pi_{\theta,j} \) are introduced in (6.10) and (6.25) respectively. We deduce the following two generalizations of Lemma 6.5 and Lemma 6.9.

**Lemma 6.13.** For all \( \varphi \in D(\mathcal{P}^N_{0,h,b}) \), we have
\[
\langle \mathcal{P}^N_{0,h,b}(\Pi_j(\xi, \tau; h, b)\varphi), \varphi \rangle_{L^2(\mathbb{R}^3_+)} = h^2\mu_j(\xi) + \tau^2\langle \Pi_j(\xi, \tau; h, b)\varphi, \varphi \rangle_{L^2(\mathbb{R}^3_+)}, \tag{6.64}
\]

and for all \( f \in L^2(\mathbb{R}^3_+) \),
\[
\sum_j \int_{\mathbb{R}^2} \langle \varphi, \Pi_j(\xi, \tau; h, b)\varphi \rangle_{L^2(\mathbb{R}^3_+)} d\xi d\tau = 2\pi \| \varphi \|^2_{L^2(\mathbb{R}^3_+)}. \tag{6.65}
\]

Moreover, for any smooth cut-off function \( \chi \in C^\infty_0(\mathbb{R}^2) \), it holds true that
\[
\text{Tr}(\chi \Pi_j(\xi, \tau; h, b)\chi) = bh^{-1}(2\pi)^{-1}\int_{\mathbb{R}^2} \chi^2(r,s)drds. \tag{6.66}
\]

**Lemma 6.14.** Let \( f \in D(\mathcal{P}^N_{\theta,h,b}) \), we have
\[
\langle \mathcal{P}^N_{\theta,h,b}\Pi_{\theta,j}(h, b)f, f \rangle_{L^2(\mathbb{R}^3_+)} = h^2\zeta_j(\theta)\langle \Pi_{\theta,j}(h, b)f, f \rangle_{L^2(\mathbb{R}^3_+)}, \tag{6.67}
\]

and for all \( f \in L^2(\mathbb{R}^3_+) \),
\[
\langle \sum_j \Pi_{\theta,j}(h, b)f, f \rangle_{L^2(\mathbb{R}^3_+)} \leq \| f \|^2_{L^2(\mathbb{R}^3_+)}. \tag{6.68}
\]

Moreover, for any smooth cut-off function \( \chi \in C^\infty_0(\mathbb{R}^2) \), it holds true that
\[
\text{Tr}(\chi \Pi_{\theta,j}(h, b)\chi) = bh^{-1}(2\pi)^{-1}\sin(\theta)\int_{\mathbb{R}^2} \chi^2(r,s)drds. \tag{6.69}
\]
7. Boundary coordinates

7.1. Local coordinates. We denote the standard coordinates on \( \mathbb{R}^3 \) by \( x = (x_1, x_2, x_3) \). The standard Euclidean metric is given by

\[
g_0 = dx_1^2 + dx_2^2 + dx_3^2. \tag{7.1}
\]

We introduce a system of coordinates valid near a point of the boundary. These coordinates are used in \cite{23} and then in \cite{32} in order to estimate the ground state energy of a magnetic Schrödinger operator with large magnetic field (or with small semi-classical parameter). Consider a point \( x_0 \in \partial \Omega \). Let \( \mathcal{V}_{x_0} \) be a neighbourhood of \( x_0 \) such that there exist local boundary coordinates \((r, s)\) in \( W = \mathcal{V}_{x_0} \cap \partial \Omega \), i.e., there exist an open subset \( U \) of \( \mathbb{R}^2 \) and a diffeomorphism \( \phi_{x_0} : W \to U \), \( \phi_{x_0}(x) = (r, s) \), such that \( \phi_{x_0}(x_0) = 0 \) and \( D\phi_{x_0}(x_0) = Id_2 \) where \( Id_2 \) is the \( 2 \times 2 \) identity matrix. Then for \( t_0 > 0 \) small enough, we define the coordinate transformation \( \Phi_{x_0}^{-1} \) as

\[
\quad U \times (0, t_0) \ni (r, s, t) \mapsto x := \Phi_{x_0}^{-1}(r, s, t) = \phi_{x_0}^{-1}(r, s) + tv,
\]

where \( v \) is the interior normal unit vector at the point \( \phi_{x_0}^{-1}(r, s) \in \partial \Omega \). This defines a diffeomorphism of \( U \times (0, t_0) \) onto \( \mathcal{V}_{x_0} \) and its inverse \( \Phi_{x_0} \) defines local coordinates on \( \mathcal{V}_{x_0} \), \( \mathcal{V}_{x_0} \ni x \mapsto \Phi_{x_0}(x) = (r(x), s(x), t(x)) \) such that

\[
t(x) = \text{dist} \left( x, \partial \Omega \right).
\]

It is easily to be seen that the \( \Phi_{x_0} \) is constructed so that

\[
\quad \Phi_{x_0}(x_0) = 0, \quad D\Phi_{x_0}(x_0) = Id_3, \tag{7.3}
\]

where \( Id_3 \) denotes the \( 3 \times 3 \) identity matrix. For convenience, we shall henceforth write \((y_1, y_2, y_3)\) instead of \((r, s, t)\). Let us consider the matrix

\[
g := g_{x_0} = (g_{pq})_{p,q=1}^3, \tag{7.4}
\]

with

\[
g_{pq} = \begin{pmatrix} \partial x / \partial y_p & \partial x / \partial y_q \end{pmatrix}, \quad (X, Y) = \sum_{1 \leq p, q \leq 3} g_{pq} \tilde{X}_p \tilde{Y}_q, \tag{7.5}
\]

where \( X = \sum_p \tilde{X}_p \partial \) and \( Y = \sum_q \tilde{Y}_q \partial \).

The Euclidean metric (7.1) transforms to

\[
g_0 = \sum_{1 \leq p, q \leq 3} g_{pq} dy_p \otimes dy_q = dy_3 \otimes dy_3 + \sum_{1 \leq p, q \leq 2} \left[ G_{pq}(y_1, y_2) - 2y_3 K_{pq}(y_1, y_2) + y_3^2 L_{pq}(y_1, y_2) \right] dy_p \otimes dy_q,
\]

where

\[
G = \sum_{1 \leq p, q \leq 2} G_{pq} dy_p \otimes dy_q = \sum_{1 \leq p, q \leq 2} \begin{pmatrix} \partial x / \partial y_p & \partial x / \partial y_q \end{pmatrix} dy_p \otimes dy_q,
\]

\[
K = \sum_{1 \leq p, q \leq 2} K_{pq} dy_p \otimes dy_q = \sum_{1 \leq p, q \leq 2} \begin{pmatrix} \partial \nu / \partial y_p & \partial \nu / \partial y_q \end{pmatrix} dy_p \otimes dy_q,
\]

\[
L = \sum_{1 \leq p, q \leq 2} L_{pq} dy_p \otimes dy_q = \sum_{1 \leq p, q \leq 2} \begin{pmatrix} \partial \nu / \partial y_p & \partial \nu / \partial y_q \end{pmatrix} dy_p \otimes dy_q
\]

are the first, second and third fundamental forms on \( \partial \Omega \).

Note that if \( x \in \mathcal{V}_{x_0} \cap \partial \Omega \), i.e., \( t(x) = 0 \), \( g_0 \) reduces to

\[
g_0 = dy_3 \otimes dy_3 + G. \tag{7.6}
\]
We denote by $g^{-1} := (g_{p q})_{p,q=1}^3$ the matrix inverse of $g$. Let $y_0$ be such that $\Phi_{x_0}^{-1}(y_0) \in \mathcal{V}_{x_0} \cap \partial \Omega$. By virtue of (7.3), we may assume, by taking $\mathcal{V}_{x_0}$ small enough, that

$$\frac{1}{2} \text{Id}_3 \leq g^{-1}(y_0) \leq 2 \text{Id}_3.$$  

(7.7)

Using this and the Taylor expansion of the matrix $g^{-1} = (g_{p q})_{p=1}^3$ about $y_0$, we find

$$1 - 2c|y - y_0|g^{-1}(y_0) \leq g^{-1}(y) \leq (1 + 2c|y - y_0|)g^{-1}(y_0).$$  

(7.8)

Let $|g| = \det(g)$. The Lebesgue measure transforms into $dx = |g|^{1/2}dy$. The Taylor expansion of $|g|^{1/2}$ in $\mathcal{V}_{x_0}$, together with (7.7), gives us:

$$1 - 2c|y - y_0||g|^{1/2}(y_0) \leq |g|^{1/2}(y) \leq (1 + 2c|y - y_0||g|^{1/2}(y_0).$$  

(7.9)

Here the constant $c$ appearing in (7.8) and (7.9) can be chosen uniformly (i.e., independently of $x_0$) by compactness and regularity of $\partial \Omega$. The magnetic potential $\tilde{A} = (\tilde{A}_1, \tilde{A}_2, \tilde{A}_3)$ is transformed to a magnetic potential in the new coordinates $\tilde{A} = (\tilde{A}_1, \tilde{A}_2, \tilde{A}_3)$ given by

$$\tilde{A}_p(y) = \sum_{k=1}^3 A_k(\Phi_{x_0}^{-1}(y)) \frac{\partial x_k}{\partial y_p}, \quad p = 1, 2, 3.$$  

(7.10)

The approximation of the magnetic potential in the new coordinates is done by replacing $\tilde{A}$ by its linear part at $y_0$, which we denote $\tilde{A}^{\mathrm{lin}} = (\tilde{A}_1^{\mathrm{lin}}, \tilde{A}_2^{\mathrm{lin}}, \tilde{A}_3^{\mathrm{lin}})$, so that

$$|\tilde{A}_p(y) - \tilde{A}_p^{\mathrm{lin}}(y)| \leq C|y - y_0|^2,$$  

(7.11)

for all $p = 1, 2, 3$, where

$$\tilde{A}^{\mathrm{lin}}(y) = \tilde{A}(y_0) + \sum_{p=1}^3 (y_p - y_0) \frac{\partial \tilde{A}}{\partial y_p}(y_0).$$  

(7.12)

The following identity (cf. [21, formula (7.23)]) gives the strength of the magnetic field in terms of the new coordinates,

$$|b(\Phi_{x_0}^{-1}(y_0))|^2 = \tilde{B}(y_0)^2 = |g(y_0)|^{-1} \left[ \sum_{p,q=1}^3 g_{p q}(y_0) \alpha_p \alpha_q \right],$$  

(7.13)

where $\alpha = \text{curl}(\tilde{A}^{\mathrm{lin}}) = (\alpha_1, \alpha_2, \alpha_3)$ is given by

$$\alpha_1 = \frac{\partial \tilde{A}_3}{\partial y_2}(y_0) - \frac{\partial \tilde{A}_2}{\partial y_3}(y_0), \quad \alpha_2 = \frac{\partial \tilde{A}_1}{\partial y_3}(y_0) - \frac{\partial \tilde{A}_3}{\partial y_1}(y_0), \quad \alpha_3 = \frac{\partial \tilde{A}_2}{\partial y_1}(y_0) - \frac{\partial \tilde{A}_1}{\partial y_2}(y_0).$$  

(7.14)

Let $u \in L^2(\mathcal{V}_{x_0})$, we define the map

$$y \mapsto \tilde{u}(y) := u(\Phi_{x_0}^{-1}(y)).$$  

(7.15)

The next Lemma expresses, in terms of the new coordinates, the quadratic form and the $L^2$-norm of a function $u$ supported in a neighborhood of $x_0$.

**Lemma 7.1.** Let $u \in \mathcal{D}(\mathcal{Q}_h)$ such that supp $u \subset \mathcal{V}_{x_0}$. We have

$$\mathcal{Q}_h(u) = \int_{\mathcal{V}_{x_0}} |(-i h \nabla + \mathbf{A}) u|^2 dx = \int_{\mathbb{R}^4} g_{p q} \left[ (-i h \nabla_{y_p} + \tilde{A}_p) \tilde{u} \right] \left[ (-i h \nabla_{y_q} + \tilde{A}_q) \tilde{u} \right] |g|^{1/2} dy,$$  

(7.16)

and

$$\|u\|_{L^2(\Omega)}^2 = \int_{\Omega} |u(x)|^2 dx = \int_{\mathbb{R}^4} |g|^{1/2}|\tilde{u}(y)|^2 dy.$$  

(7.17)
7.2. Approximation of the quadratic form. The starting point is to simplify the expression of the quadratic form given in (7.16) in terms of the new coordinates. To achieve this, we proceed as follows. Let \( \ell, T > 0 \) (\( T \) and \( \ell \) depend on \( h \) and tend to 0 as \( h \to 0 \)). Consider the sets
\[
Q_{0,\ell,T} = (-\ell/2, \ell/2)^2 \times (0, T), \quad Q_{0,\ell} = (-\ell/2, \ell/2)^2 \times \{0\},
\]
(7.18)
such that \( \Phi^{-1}_{x_0}(Q_{0,\ell,T}) \subset V_{x_0} \).

Next, consider an arbitrary point \( y_0 \in Q_{0,\ell} \). Then \( \Phi^{-1}_{x_0}(y_0) \) lies on the boundary and the metric \( g_0 \) has the form (7.6). Consequently, the matrix \( g(y_0) \) can be orthogonally diagonalized (being symmetric in this case), and such a diagonalization amounts to a rotation of the coordinate system. After performing such a diagonalization, we may assume that the matrix \( g(y_0) \) is a diagonal matrix given as follows
\[
g(y_0) = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]
(7.19)

By virtue of (7.7), it is easy to see that \( \lambda_1, \lambda_2 > 0 \). Moreover, we have \( |g(y_0)|^{1/2} = \sqrt{\lambda_1 \lambda_2} \).

Denote
\[
\tilde{Q}_{0,\ell,T} := \left( -\frac{\lambda_1^{1/2} \ell}{2}, \frac{\lambda_1^{1/2} \ell}{2} \right) \times \left( -\frac{\lambda_2^{1/2} \ell}{2}, \frac{\lambda_2^{1/2} \ell}{2} \right) \times (0, T).
\]

Let \( z = (z_1, z_2, z_3) = (\lambda_1^{1/2} y_1, \lambda_2^{1/2} y_2, y_3) \) and consider a function \( u \in L^2(V_{x_0}) \) such that \( \tilde{u} \), defined in (7.15), satisfies
\[
\text{supp } \tilde{u} \subseteq Q_{0,\ell,T}.
\]
(7.20)

We then define
\[
\tilde{u}(z) := \tilde{u}(\lambda_1^{1/2} z_1, \lambda_2^{1/2} z_2, z_3)
\]
(7.21)

We will approximate \( Q_h(u) \) via the quadratic form in the half-space corresponding to a constant magnetic field.

Lemma 7.2. Let \( F_\theta \) be the magnetic potential given in (3.4) and let \( y_0 \in Q_{0,\ell} \). There exists a constant \( C > 0 \) (independent of \( y_0 \)) and a function \( \phi_0 := \phi_{y_0} \in C^\infty(Q_{0,\ell,T}) \) such that, for all \( \varepsilon \in (0, 1] \) satisfying \( \varepsilon \geq (\ell + T) \) and for all \( u \) satisfying (7.20) one has
\[
\left| Q_h(u) - \int_{\tilde{Q}_{0,\ell,T}} |(-i h \nabla_z + b_0 F_{\theta_0})e^{i\phi_0/h} \tilde{u}|^2 \, dz \right| \\
\leq C \varepsilon \int_{\tilde{Q}_{0,\ell,T}} |(-i h \nabla_z + b_0 F_{\theta_0})e^{i\phi_0/h} \tilde{u}|^2 \, dz + C (\ell^2 + T^2)^2 \varepsilon^{-1} \int_{\tilde{Q}_{0,\ell,T}} |\tilde{u}|^2 \, dz,
\]
(7.22)

and,
\[
(1 - C(\ell + T)) \int_{\tilde{Q}_{0,\ell,T}} |\tilde{u}|^2 \, dz \leq |u|_{L^2(V_{x_0})}^2 \leq (1 + C(\ell + T)) \int_{\tilde{Q}_{0,\ell,T}} |\tilde{u}|^2 \, dz.
\]
(7.23)

Here \( b_0 = |\tilde{B}(y_0)| \), \( \theta_0 = \tilde{\theta}(y_0) \), and to a function \( v(x) \) we associate the functions \( \tilde{v}(y) \) and \( v(z) \) by means of (7.15) and (7.21) respectively.

The proof of Lemma 7.23 is given in the appendix.

8. Proof of Theorem 1.2

8.1. Lower bound. In this section, we shall prove the lower bound in Theorem 1.2.
8.1.1. Splitting into bulk and surface terms. Let

\[ h^{1/2} \ll \varsigma \ll 1 \]  

be a positive number to be chosen later (see (8.47) below) as a positive power of \( h \). We consider smooth real-valued functions \( \psi_1 \) and \( \psi_2 \) satisfying

\[ \psi_1^2(x) + \psi_2^2(x) = 1 \quad \text{in} \quad \Omega, \]  

where

\[ \psi_1(x) := \begin{cases} 
1 & \text{if} \quad \text{dist}(x, \partial \Omega) < \varsigma/2 \\
0 & \text{if} \quad \text{dist}(x, \partial \Omega) > \varsigma, 
\end{cases} \]  

and such that there exists a constant \( C_1 > 0 \) so that

\[ \sum_{k=1}^{2} |\nabla \psi_k|^2 \leq C_1 \varsigma^{-2}. \]  

Let \( \{ f_j \}_{j=1}^{N} \) be any \( L^2 \) orthonormal set in \( \mathcal{D}(\mathcal{P}_h) \) and \( Q_h \) be the quadratic form introduced in (1.3). To prove a lower bound for \( \sum_{j}(e_j(h) - \Lambda h) - \), we use the variational principle in Lemma 2.1. Namely, we seek a uniform lower bound of

\[ \sum_{j=1}^{N} (Q_h(f_j) - \Lambda h). \]

The following Lemma shows that the bulk contribution is negligible compared to the expected leading order term.

**Lemma 8.1.** Let \( \Lambda \in [0, b) \) with \( b \) from (1.1). The following lower bound holds true

\[ \sum_{j=1}^{N} (Q_h(f_j) - \Lambda h) \geq \sum_{j=1}^{N} \left( Q_h(\psi_1 f_j) - (\Lambda h + C_1 h^2 \varsigma^{-2}) \|\psi_1 f_j\|^2 \right), \tag{8.5} \]

where \( \{ f_j \}_{j=1}^{N} \) is an \( L^2 \) orthonormal set in \( \mathcal{D}(\mathcal{P}_h) \) and \( \psi_1 \) is the function from (8.3).

**Proof.** By the IMS formula, we find

\[ Q_h(f_j) = \sum_{k=1}^{2} \left( Q_h(\psi_k f_j) - h^2 \|\nabla \psi_k f_j\|^2 \right). \]

Using the fact that \( \psi_1^2 + \psi_2^2 = 1 \) and the bound on \( |\nabla \psi_k| \) in (8.4), it follows that

\[ \sum_{j=1}^{N} (Q_h(f_j) - \Lambda h) \geq \sum_{k=1}^{2} \sum_{j=1}^{N} \left( Q_h(\psi_k f_j) - (\Lambda h + C_1 h^2 \varsigma^{-2}) \|\psi_k f_j\|^2 \right). \tag{8.6} \]

Let us now examine the term corresponding to \( k = 2 \) in the right hand side of (8.6). Using the inequality (1.4) for \( u := \psi_2 f_j \), we see that

\[ \int_{\Omega} \left| (-ih \nabla + A)\psi_2 f_j \right|^2 dx \geq h(b - C h^{1/4}) \int_{\Omega} |\psi_2 f_j|^2 dx. \]

We write

\[ h(b - C h^{1/4}) \int_{\Omega} |\psi_2 f_j|^2 dx = h \Lambda \int_{\Omega} |\psi_2 f_j|^2 dx + h(b - \Lambda - C h^{1/4}) \int_{\Omega} |\psi_2 f_j|^2 dx. \]

This yields, in view of (8.1),

\[ Q_h(\psi_2 f_j) \geq (\Lambda h + C_1 h^2 \varsigma^{-2}) \int_{\Omega} |\psi_2 f_j|^2 dx. \tag{8.7} \]

This gives that the bulk term in (8.6) is positive, and the lemma follows. \( \square \)
8.1.2. Partition of unity of the boundary. Recall the cut-off function \( \psi_1 \) from (8.3), which is supported near a neighborhood of the boundary \( \partial \Omega \). Let

\[
\mathcal{O}_1 := \text{supp } \psi_1 = \{ x \in \Omega : \text{dist}(x, \partial \Omega) \leq \varsigma \}, \tag{8.8}
\]

where \( \varsigma \) is, as introduced in (8.1).

Given a point \( x \) of the boundary, we let \( \Phi_x^{-1} \) be the coordinate transformation valid near a small neighbourhood of \( x \) (these coordinates are introduced in Section 7). Since the boundary is smooth, there exists \( \delta_x > 0 \) such that

\[
\Phi_x^{-1} : \tilde{\Omega}_{\delta_x} \to \mathcal{O}_x,
\]

where,

\[
\tilde{\Omega}_{\delta_x} := (-\delta_x, \delta_x)^2 \times (0, \delta_x), \quad \mathcal{O}_x = \Phi_x^{-1}(\tilde{\Omega}_{\delta_x}).
\]

Next, we consider the subset \( \Omega_{\delta_x} \) of \( \tilde{\Omega}_{\delta_x} \) to be

\[
\Omega_{\delta_x} := \left( -\frac{\delta_x}{2}, \frac{\delta_x}{2} \right)^2 \times (0, \delta_x),
\]

and a covering of \( \mathcal{O}_1 \) by open sets \( \{ \mathcal{O}_x \}_{x \in \partial \Omega} \). Using the compactness of the boundary, it follows that there exists an integer \( K \) and an index set \( J = \{ 1, \ldots, K \} \), such that the sets \( \{ \mathcal{O}_{x_l} \}_{l \in J} \) form a finite covering of \( \mathcal{O}_1 \). For ease of notation, we write \( \delta_l \) (respectively \( \mathcal{O}_l, \Phi_l \)) instead of \( \delta_{x_l} \) (respectively \( \mathcal{O}_{x_l}, \Phi_{x_l} \)). We emphasize here that the \( \delta_l \)'s are fixed and independent of \( h \). Thus, by choosing \( \varsigma = \varsigma(h) \) sufficiently small (see (8.47) below), we may assume that

\[
\varsigma \ll \delta_0 := \min_{l \in J} \delta_l. \tag{8.9}
\]

Next, we choose \( \{ \chi_l \}_{l \in J} \) to be non-negative, smooth, compactly supported functions such that

\[
\sum_{l \in J} \chi_l^2(x) \equiv 1 \quad \text{in } \mathcal{O}_1, \quad \text{supp } \chi_l \subset \mathcal{O}_l, \tag{8.10}
\]

and such that there exists a constant \( C_2 > 0 \) (independent of \( h \)) so that

\[
\sum_{l \in J} | \nabla \chi_l(x) |^2 \leq C_2, \tag{8.11}
\]

for all \( x \in \Omega \). Next, consider the lattice \( \{ F^m_\varsigma \}_{m \in \mathbb{Z}^2} \) of \( \mathbb{R}^2 \) generated by the square:

\[
F_\varsigma := \left( -\frac{\varsigma}{2}, \frac{\varsigma}{2} \right)^2.
\]

If \( m \in \mathbb{Z}^2 \), denote by \( (r_m, s_m) = m \varsigma \in \mathbb{R}^2 \) the center of the square \( F^m_\varsigma \) so that we can write

\[
F^m_\varsigma = \left( -\frac{\varsigma}{2} + r_m, \frac{\varsigma}{2} + r_m \right) \times \left( -\frac{\varsigma}{2} + s_m, \frac{\varsigma}{2} + s_m \right).
\]

We let \( \mathcal{I}_l = \{ m \in \mathbb{Z}^2 : F^m_\varsigma \cap \left( -\frac{\delta_l}{2}, \frac{\delta_l}{2} \right)^2 \neq \emptyset \} \). If \( m \in \mathcal{I}_l \) and \( \eta > 0 \), we will write

\[
F^m_\eta.l = \left( -\frac{\eta}{2} + r_m, \frac{\eta}{2} + r_m \right) \times \left( -\frac{\eta}{2} + s_m, \frac{\eta}{2} + s_m \right), \quad Q^m_\eta.l := F^m_\eta.l \times (0, \varsigma). \tag{8.12}
\]

Let \( a \ll 1 \) to be chosen later as a positive power of \( h \) (see (8.47) below). We introduce a new partition of unity of the square \( \left( -\frac{\delta_l}{2}, \frac{\delta_l}{2} \right)^2 \) by smooth functions \( \{ \tilde{\varphi}_{m,l} \}_{m \in \mathcal{I}_l} \) with the following properties

\[
\sum_{m \in \mathcal{I}_l} \tilde{\varphi}_{m,l}^2 \equiv 1 \quad \text{in } \left( -\frac{\delta_l}{2}, \frac{\delta_l}{2} \right)^2, \quad \text{supp } \tilde{\varphi}_{m,l} \subset F^{m,l}_{(1+a)\varsigma}, \quad \tilde{\varphi}_{m,l} = 1 \quad \text{in } F^{m,l}_{(1-a)\varsigma}, \tag{8.13}
\]

and such that there exists a constant \( C_3 > 0 \) so that

\[
\sum_{m \in \mathcal{I}_l} | \nabla \tilde{\varphi}_{m,l} |^2 \leq C_3 (a \varsigma)^{-2}. \tag{8.14}
\]
We set 
\[ \varphi_{m,l}(x) = \tilde{\varphi}_{m,l}(\Phi_l(x)). \]
Let \( y_{m,l} \) be an arbitrary point of \( Q_{(1 + a) \varsigma}^{m,l} \). As we did in Section 7, we may assume, after performing a diagonalization, that \( g_l(y_{m,l}) \) (\( g_l \) is the short notation of \( g_{x_l} \)) is a diagonal matrix given by
\[

\begin{pmatrix}
\lambda_{m,l,1} & 0 \\
0 & \lambda_{m,l,2} \\
0 & 0
\end{pmatrix}.
\]
(8.15)

For \( y = (y_1, y_2, y_3) \in \mathbb{R}^3 \), we denote \( y^\perp = (y_1, y_2) \in \mathbb{R}^2 \). Applying (7.9) with \( y_0 := y_{m,l} = (y_{m,l}^\perp, 0) \in F_{(1 + a) \varsigma}^{m,l} \times \{0\} \), we immediately see that
\[
||g_l||^{1/2}(y) - \lambda_{m,l,1}^{1/2}\lambda_{m,l,2}^{1/2} \leq c_\varsigma \lambda_{m,l,1}^{1/2}\lambda_{m,l,2}^{1/2}.
\]
(8.16)

We also note that we can approximate the function \( \tilde{\chi}^2_l \) within the domain \( Q_{(1 + a) \varsigma}^{m,l} \) by \( \tilde{\chi}^2_l(y_{m,l}) \). Indeed, by Taylor expansion, we obtain that for some positive constant \( c_\varsigma > 0 \)
\[
||\tilde{\chi}^2_l(y) - \tilde{\chi}^2_l(y_{m,l})|| \leq c_\varsigma. 
\]
(8.17)

Put \( z = (z_1, z_2, z_3) = (\lambda_{m,l,1}y_1, \lambda_{m,l,2}y_2, y_3) \) and denote by
\[
\tilde{Q}_{(1 + a) \varsigma}^{m,l} := \left( -\frac{S_{m,l,1}}{2}, \frac{S_{m,l,1}}{2} \right) \times \left( -\frac{S_{m,l,2}}{2}, \frac{S_{m,l,2}}{2} \right) \times (0, \varsigma), \quad \varsigma_{m,l,k} = \frac{\lambda_{m,l,k}^2(1 + a) \varsigma}{2}, \quad k = 1, 2.
\]

In the following lemma, we apply localization formulas to restrict the analysis into small boxes where we can approximate the quadratic form using Lemma 7.2.

**Lemma 8.2.** Let \( \Lambda \in [0, b) \), \( b \) the constant in (1.1), \( F_\theta \) the magnetic potential given in (3.4) and \( y_{m,l} \in F_{(1 + a) \varsigma}^{m,l} \times \{0\} \). There exists a function \( \phi_{m,l} := \phi_{y_{m,l}} \in C^\infty(\tilde{Q}_{(1 + a) \varsigma}^{m,l}) \) and a constant \( \tilde{C} > 0 \) such that for all \( \varepsilon \in (0, 1] \) satisfying \( \varepsilon \gg \varsigma \), one has
\[
\sum_{j=1}^N (Q_h(\psi_1 f_j) - \Lambda h) 
\geq (1 - \tilde{C}\varepsilon) \sum_{j=1}^N \sum_{l \in J} \sum_{m \in I_l} \left\{ \int_{\tilde{Q}_{(1 + a) \varsigma}^{m,l}} \left| -i h \nabla_z z + b_{m,l} F_{\theta_{m,l}} \right| e^{i\phi_{m,l}/h} \tilde{\varphi}_{m,l} \tilde{\psi}_1 \tilde{\chi} f_j \right|^2 dz \right.
\]
\[
- \Lambda_1(h, a, \varsigma, \varepsilon) \int_{\tilde{Q}_{(1 + a) \varsigma}^{m,l}} |\tilde{\varphi}_{m,l} \tilde{\psi}_1 \tilde{\chi} f_j|^2 dz \right\}, 
\]
(8.18)

where
\[
\Lambda_1(h, a, \varsigma, \varepsilon) = \frac{(\Lambda h + \tilde{C} h^2 (a \varsigma)^{-2})(1 + \tilde{C} \varsigma) + \tilde{C} \varsigma^4 \varepsilon^{-1}}{1 - \tilde{C}\varepsilon}, 
\]
(8.19)

\( b_{m,l} = |\tilde{B}(y_{m,l})| \), \( \theta_{m,l} = \tilde{\theta}(y_{m,l}) \) and to a function \( v(x) \), we associate the function \( \tilde{v}(z) \) by means of (7.21).

8.1.3. The leading order term. For \( h, b > 0 \) and \( \theta \in [0, \pi/2] \), we recall the operator \( \mathcal{P}_{\theta, h, b}^N \) from (6.60). Let us rewrite (8.18) as
\[
\sum_{j=1}^N (Q_h(f_j) - \Lambda h) \geq I_1 + I_2.
\]
(8.20)
where

\[ I_1 = (1 - \tilde{C}\varepsilon) \times \sum_{j=1}^{N} \sum_{l \in J} \sum_{m,l \in I_{\theta,\varepsilon} \cap [0,\pi/2]} \langle e^{i\phi_{m,l}/h}\tilde{\varphi}_{m,l} \tilde{\chi}_{l}\tilde{\psi}_{1} \tilde{f}_{j}, (p_{m,l,h,b_{m,l}} - \Lambda_{1}(h,a,\zeta,\varepsilon)) e^{i\phi_{m,l}/h}\tilde{\varphi}_{m,l} \tilde{\chi}_{l} \tilde{\psi}_{1} \tilde{f}_{j} \rangle, \] (8.21)

and

\[ I_2 = (1 - \tilde{C}\varepsilon) \times \sum_{j=1}^{N} \sum_{l \in J} \sum_{m,l \in I_{\theta,\varepsilon} \cap [0,\pi/2]} \langle e^{i\phi_{m,l}/h}\tilde{\varphi}_{m,l} \tilde{\chi}_{l} \tilde{\psi}_{1} \tilde{f}_{j}, (p_{m,l,h,b_{m,l}} - \Lambda_{1}(h,a,\zeta,\varepsilon)) e^{i\phi_{m,l}/h}\tilde{\varphi}_{m,l} \tilde{\chi}_{l} \tilde{\psi}_{1} \tilde{f}_{j} \rangle. \] (8.22)

Below in (8.47), the parameters \(a,\zeta\) and \(\varepsilon\) are chosen so that, when \(h\) is sufficiently small, one has

\[ h^{-1}\Lambda_{1}(h,a,\zeta,\varepsilon) < b, \] (8.23)

where \(b\) is defined in (1.1). We first start by estimating \(I_1\). Using Lemma 6.14, we see that

\[ I_1 \geq -h(1 - \tilde{C}\varepsilon) \sum_{l \in J} \sum_{m,l \in I_{\theta,\varepsilon} \cap [0,\pi/2]} b_{m,l} \sum_{k} (\zeta_{k}(\theta_{m,l}) - h^{-1}b_{m,l}^{1/2}\Lambda_{1}(h,a,\zeta,\varepsilon)) \times \]

\[ \sum_{j=1}^{N} \langle e^{i\phi_{m,l}/h}\tilde{\psi}_{1} \tilde{\varphi}_{m,l} \tilde{\chi}_{l} \tilde{\psi}_{1} \tilde{f}_{j}, \Pi_{\theta_{m,l},k}(h,b_{m,l}) e^{i\phi_{m,l}/h}\tilde{\psi}_{1} \tilde{\varphi}_{m,l} \tilde{\chi}_{l} \tilde{\psi}_{1} \tilde{f}_{j} \rangle_{L^{2}(\tilde{Q}^{m,l}_{1})}. \] (8.24)

Here, for \(\theta \in (0,\pi/2)\) and \(b > 0\), \(\{\zeta_{k}(\theta)\}_{k}\) are the eigenvalues from (6.21) and \(\Pi_{\theta,k}(h,b)\) is the projector defined in (6.63). Using (8.16) and that \(dz = \lambda_{m,l}^{1/2} \lambda_{m,l}^{1/2} dy\), we obtain that for some constant \(C_{4} > 0\)

\[ \sum_{j=1}^{N} \langle e^{i\phi_{m,l}/h}\tilde{\psi}_{1} \tilde{\varphi}_{m,l} \tilde{\chi}_{l} \tilde{\psi}_{1} \tilde{f}_{j}, \Pi_{\theta_{m,l},k}(h,b_{m,l}) e^{i\phi_{m,l}/h}\tilde{\psi}_{1} \tilde{\varphi}_{m,l} \tilde{\chi}_{l} \tilde{\psi}_{1} \tilde{f}_{j} \rangle_{L^{2}(\tilde{Q}^{m,l}_{1})} \]

\[ \leq (1 + C_{4}) \sum_{j=1}^{N} \langle f_{j}, H(m,l,k,h,b_{m,l},\theta_{m,l}) f_{j} \rangle_{L^{2}(\Omega)}. \] (8.25)

Here \(H(m,l,k,h,b_{m,l},\theta_{m,l})\) is a positive operator, which is given by,

\[ H(m,l,k,h,b_{m,l},\theta_{m,l}) := \psi_{1} \chi_{l} \tilde{\varphi}_{m,l} U_{\Phi_{1}} V_{z\rightarrow y} e^{i\phi_{m,l}/h} \Pi_{\theta_{m,l},k}(h,b_{m,l}) e^{i\phi_{m,l}/h} V_{z\rightarrow y}^{-1} U_{\Phi_{1}}^{-1} \psi_{1} \chi_{l} \tilde{\varphi}_{m,l}, \]

where, for a function \(v\), \(V_{z\rightarrow y}\) is defined by

\[ (V_{z\rightarrow y}v)(y) = v(\lambda_{m,l}^{1/2}y_{1}, \lambda_{m,l}^{1/2}y_{2}, y_{3}), \] (8.26)

and, for a function \(u\), the transformation \(U_{\Phi_{1}}\) is given by

\[ (U_{\Phi_{1}}u)(x) = u(\Phi_{1}(x)). \] (8.27)

Since \(\{f_{j}\}_{j=1}^{N}\) is an orthonormal family in \(L^{2}(\Omega)\), we deduce that

\[ \sum_{j=1}^{N} \langle f_{j}, H(m,l,k,h,b_{m,l},\theta_{m,l}) f_{j} \rangle_{L^{2}(\Omega)} \leq \text{Tr}(H(m,l,k,h,b_{m,l},\theta_{m,l})). \] (8.28)
Combining (8.24), (8.25) and (8.28), and using that \( \varepsilon \gg \varsigma \) (see (8.47) below), we obtain that for some constant \( C_5 > 0 \)

\[
I_1 \geq - (1 - C_5 \varepsilon) h \sum_{l \in J} \sum_{m \in I} \sum_{k} b_{m,l} \left( \zeta_k(\theta_{m,l}) - h^{-1} b_{m,l}^{-1} A_1(h, a, \varsigma, \varepsilon) \right) \times
\]

\[
\text{Tr}(H(m, l, h, b_{m,l}, \theta_{m,l})).
\]

(8.29)

It is straightforward to show that

\[
\text{Tr}(H(m, l, k, h, b_{m,l}, \theta_{m,l})) = b_{m,l}^{3/2} h^{-3/2} \int_{\mathbb{R}} \int_{\mathbb{R}_+^3} |g_r(y)|^{1/2} \psi^2(y) \tilde{\psi}_m^2(y) \tilde{\psi}_m^2(y_{1}, y_{2}, 0) |V_{z} - y(v_{\theta_{m,l}, k(h^{-1/2} b_{m,l}^{1/2})} \xi)|^2 dy d\xi,
\]

(8.30)

where, for \( \theta \in [0, \pi/2] \), \( v_{\theta,k} \) is the function defined in (6.22). Using (8.16) and (8.17), and that \( \psi_1(x) \leq 1 \) for all \( x \in \Omega \), it follows that

\[
\text{Tr}(H(m, l, k, h, b_{m,l}, \theta_{m,l})) \leq (2\pi)^{-1} (1 + c_5) (\chi^2(y_{m,l}) + c_5 \varsigma) \lambda_{m,l,1}^{1/2} \lambda_{m,l,2}^{1/2} h^{-3/2} b_{m,l}^{3/2} \times
\]

\[
\int_{\mathbb{R}} \int_{\mathbb{R}_+^3} \tilde{\varphi}_m^2(y_{1}, y_{2}, 0) |u_{\theta_{m,k}, h^{-1/2} b_{m,l}^{1/2} \lambda_{m,l,2}^2 y_2} - \frac{\xi}{\sin(\theta_{m,l})}|^2 dy d\xi,
\]

(8.31)

where, for \( \theta \in [0, \pi/2] \), the functions \{\( u_{\theta,k} \)\} are introduced in (6.21). Performing similar calculations to that in (6.30), we deduce that

\[
\text{Tr}(H(m, l, k, h, b_{m,l}, \theta_{m,l})) \leq (2\pi)^{-1} h^{-1} b_{m,l}^{1/2} \lambda_{m,l,1}^{1/2} \lambda_{m,l,2}^{1/2} \chi^2(y_{m,l}) + c_5 \varsigma)(1 + c_5) \sin(\theta_{m,l}) \int_{\mathbb{R}} |\tilde{\varphi}_m^2| dy_1 dy_2
\]

\[
\leq (2\pi)^{-1} h^{-1} b_{m,l}^{1/2} \lambda_{m,l,1}^{1/2} \lambda_{m,l,2}^{1/2} \chi^2(y_{m,l}) + c_5 \varsigma)(1 + c_5) \sin(\theta_{m,l})(1 + a)^2 \varsigma^2,
\]

(8.32)

where in the last step, we used that the function \( \tilde{\varphi}_m \) is less than one and supported in the square \( F_{m,l}^{(1+a)\varsigma} \). Recalling (6.36) and substituting (8.32) into (8.29), we obtain that for some positive constant \( C_6 > 0 \)

\[
I_1 \geq -(1 - C_6 \varepsilon)(1 + a)^2 \sum_{l \in J} \sum_{m \in I} \sum_{\theta_{m,l} \in (0, \pi/2]} \sum_{k = 0} b_{m,l} \left( \mu_k(\xi) + \tau^2 - h^{-1} b_{m,l}^{-1} A_1(h, a, \varsigma, \varepsilon) \right) \times
\]

\[
E(\theta_{m,l}, h^{-1} b_{m,l}^{-1} A_1(h, a, \varsigma, \varepsilon)) b_{m,l}^{1/2} \lambda_{m,l,1}^{1/2} \lambda_{m,l,2}^{1/2} \chi^2(y_{m,l}) + c_5 \varsigma)^2.
\]

(8.33)

We now proceed in a similar manner to get a lower bound on \( I_2 \). By virtue of Lemma 6.13, it follows that

\[
I_2 \geq -h(2\pi)^{-1} (1 - \tilde{C} \varepsilon) \sum_{l \in J} \sum_{m \in I} \sum_{\theta_{m,l} = 0} b_{m,l} \int_{\mathbb{R}^2} \sum_{k} \left( \mu_k(\xi) + \tau^2 - h^{-1} b_{m,l}^{-1} A_1(h, a, \varsigma, \varepsilon) \right) \times
\]

\[
\sum_{j=1}^{N} \langle e^{i\phi_{m,l}/h} \tilde{\psi}_1 \tilde{\psi}_m \tilde{\psi}_m \tilde{J}_j \rangle \Pi_k(h, b_{m,l}; \xi, \tau)e^{i\phi_{m,l}/h} \tilde{\psi}_1 \tilde{\psi}_m \tilde{J}_j \rangle \int_{L^2(Q_{m,l}^{(1+a)\varsigma})} d\xi d\tau,
\]

(8.34)
where, for \((\xi, \tau) \in \mathbb{R}^2\) and \(b > 0\), \(\{\mu_k(\xi)\}_k\) are the eigenvalues from (6.2) and \(\Pi_1(\xi, \tau; h, b)\) is the projector defined in (6.62). Using Lemma 6.2, it follows that

\[
I_2 \geq - h(2\pi)^{-1}(1 - \tilde{C}\varepsilon) \sum_{l \in J} \sum_{m \in \mathcal{I}_l} b_{m,l} \int_{\mathbb{R}^2} \left( \mu_1(\xi) + \tau^2 - h^{-1}b_{m,l}^2 \Lambda_1(h, a, \xi, \tau) \right) \text{d}x \text{d}y, \quad (8.35)
\]

By an equality similar to that in (8.29), we find

\[
I_2 \geq - h(1 - C_5\varepsilon)(2\pi)^{-1} \sum_{l \in J} \sum_{m \in \mathcal{I}_l} b_{m,l} \times \int_{\mathbb{R}^2} \sum_k \left( \mu_k(\xi) + \tau^2 - h^{-1}b_{m,l}^2 \Lambda_1(h, a, \xi, \tau) \right) \text{Tr}(H'(m, l, h, b_{m,l}; \xi, \tau)) \text{d}x \text{d}y. \quad (8.36)
\]

Here \(H'(m, l, h, b_{m,l}; \xi, \tau)\) is a positive operator, which is given by,

\[
H'(m, l, h, b_{m,l}; \xi, \tau) := \psi_1 \chi_l \bar{\varphi}_{m,l} U_\Phi, V_{z \rightarrow y} e^{i\phi_{m,l}/h} \Pi_1 (\xi, \tau; h, b_{m,l}) e^{i\phi_{m,l}/h} V_{z \rightarrow y}^{-1} U_\Phi^{-1} \psi_1 \chi_l \bar{\varphi}_{m,l},
\]

where \(U_\Phi\) and \(V_{z \rightarrow y}\) are the same as defined in (8.27) and (8.26) respectively. It is easy to see that

\[
\text{Tr}(H'(m, l, h, b_{m,l}; \xi, \tau)) = \int_{\mathbb{R}^3} |g(y)|^{1/2} \bar{\varphi}_{m,l}(y) \chi_l(y) \tilde{\chi}_l^2(y) \tilde{\varphi}_{m,l}(y_1, y_2, 0) V_{z \rightarrow y} (v_1 (h^{-1/2} b_{m,l}^{1/2}; \xi, \tau)) |^2 \text{d}y, \quad (8.37)
\]

where the function \(v_1\) is defined in (6.9). Using (8.16) and (8.17), and that \(\psi_1(x) \leq 1\) for all \(x \in \Omega\), it follows that

\[
\text{Tr}(H'(m, l, h, b_{m,l}; \xi, \tau)) \leq (2\pi)^{-1} (1 + \varepsilon \kappa) (\tilde{\chi}_l^2(y_{m,l}) + c_5 \varepsilon) \chi_l^{1/2} m_{l,1}^{1/2} \chi_l m_{l,2} h^{-3/2} b_{m,l}^{3/2} \int_{\mathbb{R}^3} \tilde{\varphi}_{m,l}(y_1, y_2, 0) |u_1 (h^{-1/2} b_{m,l}^{1/2}; \xi, \tau)|^2 \text{d}y, \quad (8.38)
\]

Using that the function \(u_1(\cdot; \xi)\) (from (6.2)) is normalized in \(L^2(\mathbb{R}^3)\), we get

\[
\text{Tr}(H'(m, l, h, b_{m,l}; \xi, \tau)) \leq (2\pi)^{-1} h^{-1} b_{m,l} \chi_l m_{l,1}^{1/2} \chi_l m_{l,2} (\tilde{\chi}_l^2(y_{m,l}) + c_5 \varepsilon) (1 + \varepsilon (1 + a)^2 \varepsilon^2. \quad (8.39)
\]

Inserting (8.39) in (8.36) yields

\[
I_2 \geq - h(1 - C_6\varepsilon)(4\pi^2)^{-1} \sum_{l \in J} \sum_{m \in \mathcal{I}_l} b_{m,l}^2 \chi_l m_{l,1}^{1/2} \chi_l m_{l,2} (\tilde{\chi}_l^2(y_{m,l}) + C_\varepsilon (1 + a)^2 \varepsilon^2 \times \int_{\mathbb{R}^2} \left( \mu_1(\xi) + \tau^2 - h^{-1}b_{m,l} \Lambda_1(h, a, \xi, \tau) \right) \text{d}x \text{d}y. \quad (8.40)
\]

Using (6.14), it follows that

\[
I_2 \geq -(1 - C_6\varepsilon)(1 + a)^2 \sum_{l \in J} \sum_{m \in \mathcal{I}_l} E(0, \Lambda_1(h, a, \xi, \tau)) b_{m,l}^2 \chi_l m_{l,1}^{1/2} \chi_l m_{l,2} (\tilde{\chi}_l^2(y_{m,l}) + C_\varepsilon) \varepsilon^2. \quad (8.41)
\]
Therefore, combining (8.33) and (8.41), and using (8.20), we obtain
\[
\sum_{j=1}^{N} (Q_h(f_j) - \Lambda h) \geq - (1 - C_6 \varepsilon)(1 + a)^2 \times 
\sum_{l \in J} \sum_{m \in I_l} b_{m,l}^{1/2} \lambda_{m,l,1}^{1/2} \lambda_{m,l,2}^{1/2} \chi_l^2(y_{m,l}) + c_5 \varepsilon E(\theta_{m,l}, h^{-1} b_{m,l}^{-1} \Lambda_1(h, a, \varsigma, \varepsilon)) \varepsilon^2. 
\tag{8.42}
\]

Using the fact that for all \( \lambda_0 \in (0, 1) \), the function \( (0, \lambda_0] \times [0, \pi/2] \to E(\theta, \lambda) \) is bounded by Lemma 5.1, we see that the term
\[
C_\varsigma (1 + a)^2 \sum_{l \in J} \sum_{m \in I_l} b_{m,l}^2 \varepsilon E(\theta_{m,l}, h^{-1} b_{m,l}^{-1} \Lambda_1(h, a, \varsigma, \varepsilon)) \varepsilon^2
\]
is bounded by \( C_\varsigma \sum_{l \in J} \sum_{m \in I_l} \lambda_{m,l,1}^{1/2} \lambda_{m,l,2}^{1/2} \varepsilon^2 \sim C_\varsigma |\partial \Omega|. \) This leads to
\[
\sum_{j=1}^{N} (Q_h(f_j) - \Lambda h) \geq - (1 - C_6 \varepsilon)(1 + a)^2 \times 
\sum_{l \in J} \sum_{m \in I_l} \lambda_{m,l,1}^{1/2} \lambda_{m,l,2}^{1/2} \chi_l^2(y_{m,l}) b_{m,l}^2 E(\theta_{m,l}, h^{-1} b_{m,l}^{-1} \Lambda_1(h, a, \varsigma, \varepsilon)) \varepsilon^2 + O(\varsigma). 
\tag{8.43}
\]

By (7.19), we have \( \lambda_{m,l,1}^{1/2} \lambda_{m,l,2}^{1/2} = |g_l(y_{m,l})|^{1/2}. \) For \( y = (y_1, 0) \in F_{\varsigma}^{m,l} \times \{0\}, \) we define the function
\[
G(y) := |g_l(y)|^{1/2} \chi_l(y)\|B(y)\|^2 E(\theta(y), h^{-1} |B(y)|^{-1} \Lambda_1(h, a, \varsigma, \varepsilon)). 
\tag{8.44}
\]

We pick \( y_{m,l} \in F_{\varsigma}^{m,l} \times \{0\} \) so that
\[
\min_{y \in F_{\varsigma}^{m,l} \times \{0\}} G(y) = G(y_{m,l}).
\]

Then the right-hand side of (8.43) is a lower Riemann sum. Hence, we find
\[
\sum_{m \in I_l} |g_l(y_{m,l})|^{1/2} \chi_l^2(y_{m,l}) b_{m,l}^2 E(\theta_{m,l}, h^{-1} b_{m,l}^{-1} \Lambda_1(h, a, \varsigma, \varepsilon)) \varepsilon^2 = \sum_{m \in I_l} G(y_{m,l}) \varepsilon^2 \leq 
\int_{(-\delta, \delta)^2} G(y_1, y_2, 0) dy_1 dy_2 = \int_{\partial \Omega} \chi_l^2(x)\|B(x)\|^2 E(\theta(x), h^{-1} |B(x)|^{-1} \Lambda_1(h, a, \varsigma, \varepsilon)) d\sigma(x). 
\tag{8.45}
\]

Plugging this into (8.43), and using that \( \sum_{l \in J} \chi_l^2(x) = 1, \) we obtain
\[
\sum_{j=1}^{N} (Q_h(f_j) - \Lambda h) \geq - (1 - C_6 \varepsilon)(1 + a)^2 \int_{\partial \Omega} \chi_l^2(x)\|B(x)\|^2 E(\theta(x), h^{-1} |B(x)|^{-1} \Lambda_1(h, a, \varsigma, \varepsilon)) d\sigma(x) + O(\varsigma). 
\tag{8.46}
\]

We make the following choice of \( \varepsilon, a \) and \( \varsigma, \)
\[
\varepsilon = h^{1/4}, \quad a = h^{1/16} \quad \varsigma = h^{3/8}. 
\tag{8.47}
\]

This choice yields that for some constant \( C_7 > 0, \) one has
\[
h^{-1} \Lambda_1(h, a, \varsigma, \varepsilon) \sim \frac{A + C_7 h^{1/8}}{1 - C h^{1/4}} \quad \text{as} \quad h \to 0.
\]

The function \( [0, 1] \times [0, \pi/2] \to E(\theta, \lambda) \) is locally Lipschitz according to Lemma 5.3. This gives
\[
\left| E(\theta(x), h^{-1} \Lambda_1(h, a, \varsigma, \varepsilon) |B(x)|^{-1}) - E(\theta(x), \Lambda |B(x)|^{-1}) \right| \leq C_8 h^{1/8} b^{-1}, 
\tag{8.48}
\]
for some constant $C_8 > 0$. The constant $b$ is introduced in (1.1). It follows that for some constant $C_9 > 0$, we have
\[
\sum_{j=1}^{N} \left( Q_h(f_j) - Ah \right) \geq -(1 + C_9 h^{1/8}) \int_{\Omega} |B(x)|^2 E(\theta(x), \Lambda|B(x)|^{-1}) \, dx + O(h^{1/8}),
\]
uniformly with respect to $N$ and the orthonormal family $\{f_j\}$. As a consequence, we may use Lemma 2.1 to obtain the desired lower bound.

### 8.2. Upper bound

Let $\zeta > 0$ be as in (8.1) and $F_{\zeta}^{m,l}$ be the set defined in (8.12) with $l \in J$ and $m \in I_l$ being the indices corresponding to the partitions $\{\chi_l\}_{l \in J}$ and $\{\tilde{\varphi}_{m,l}\}_{m \in I_l}$ introduced in (8.10) and (8.13) respectively. Let $\{y_{m,l}\}$ be a finite family of points in $F_{\zeta}^{m,l} \times \{0\}$ to be specified later at the end of this section. To each point $y_{m,l}$ we associate $b_{m,l} = |\tilde{B}(y_{m,l})|$ and define the operator $\left( \Omega \right)$ as in (8.12) and define the operator $\left( \Omega \right)$ respectively. Let $y \in Q_{(1+a)\zeta}^{m,l}$ (see the definition of the set in (8.12)) and $\lambda_{m,l,1}, \lambda_{m,l,2}$ be the diagonal components of the matrix $g_1(y_{m,l})$ from (8.15). We put $z = (\lambda_{m,l,1}^1 y_1, \lambda_{m,l,2}^2 y_2, y_3)$. Let $(\xi, \tau) \in \mathbb{R}^2$. Recall the tilde-notation from (7.15) and define the functions
\[
\tilde{f}_{j,l,m}(y; \xi, \tau; h) := h^{-3/4} f_{j,l,m}^{1/4} \left( h^{-1/2} b_{m,l}: \xi \right) \left( \tilde{\varphi}_{m,l} \tilde{v}_{1} \right)(y) \quad \text{if} \quad \theta_{m,l} \in (0, \pi/2]
\]
\[
\tilde{g}_{l,m}(y; \xi, \tau; h) := (2\pi)^{-1/2} h^{-3/4} b_{m,l}^{1/4} \left( h^{-1/2} b_{m,l}^{-1/2} \xi, \tau \right) \left( \tilde{\varphi}_{m,l} \tilde{v}_{1} \right)(y) \quad \text{if} \quad \theta_{m,l} = 0.
\]
where $v_{j,\rho}(:, \xi), v_{1}(::, \xi, \tau)$ and $\tilde{v}_{1}$ are the functions found in (6.22), (6.9) and (8.3) respectively.

Recall the coordinate transformation $\Phi_l$ valid near a neighborhood of the point $x_l$ (see Subsection 7), and let $x = \Phi_l^{-1}(y)$. We define $f_{j,l,m}(x; \xi, \tau; h) := \tilde{f}_{j,l,m}(y; \xi, \tau; h)$ and $g_{l,m}(x, \xi, \tau; h) := \tilde{g}_{l,m}(y, \xi, \tau; h)$. Let $\Lambda \in (0, b)$ and $L > 0$ be a sufficiently large number to be selected later so that $\zeta h^{-1/2} \gg L$. We put
\[
M_{j,l,m,\xi} = 1_{\left\{ (j,l,m,\xi) \in N \times I_l \times \mathbb{R} : \xi \left( \theta_{m,l} \right) - b_{m,l}^{-1} \Lambda \leq 0, \quad \frac{\xi}{\Lambda} \left( \theta_{m,l} \right) \leq L \right\}},
\]
and
\[
M'_{m,l,\xi,\tau} = 1_{\left\{ (j,m,l,\xi,\tau) \in N \times I_l \times \mathbb{R}^2 : \quad \mu_1(\xi) + \tau^2 - b_{m,l}^{-1} \Lambda \leq 0 \right\}}.
\]
Here we note that the condition $\mu_1(\xi) + \tau^2 - b_{m,l}^{-1} \Lambda \leq 0$ implies, in view of Proposition 6.1 and the fact that $b_{m,l}^{-1} \Lambda \leq b^{-1} \Lambda < 1$, that there exists a constant $K > 0$ (independent of $m, l$) such that
\[
(\xi, \tau) \in I_{\xi,\tau} := (0, K) \times (-1, 1).
\]

Define, for $f \in L^2(\Omega), 
\[
(\gamma_1 f)(x) = \sum_{l \in J} \sum_{m \in I_l} \sum_{\theta_{m,l} \in (0, \pi/2]} \int_{\mathbb{R}} M_{j,l,m,\xi} \langle f, f_{j,l,m}(\cdot, \xi, \tau; h) \rangle f_{j,l,m}(x, \xi, \tau; h) \, d\xi,
\]
and
\[
(\gamma_2 f)(x) = \sum_{l \in J} \sum_{m \in I_l} \int_{\mathbb{R}^2} M'_{m,l,\xi,\tau} \langle f, g_{l,m}(\cdot, \xi, \tau; h) \rangle g_{l,m}(x, \xi, \tau; h) \, d\xi d\tau.
\]
We have the following lemma.

**Lemma 8.3.** Let $f \in L^2(\Omega)$ and define the operator $\gamma$ by
\[
\gamma f = \gamma_1 f + \gamma_2 f.
\]
There exists a constant $C_{10} > 0$ such that the quadratic form associated to $\gamma$ satisfies
\[
0 \leq \langle \gamma f, f \rangle_{L^2(\Omega)} \leq (1 + C_{10}) \|f\|_{L^2(\Omega)}^2.
\]
Proof. Consider \( f \in L^2(\Omega) \). It is easy to see that \( \langle \gamma f, f \rangle \geq 0 \). Next, using that \( M_{j,m,\ell,\xi} \leq 1 \), we see that
\[
\langle f, \gamma_1 f \rangle_{L^2(\Omega)} \leq \sum_{l \in J} \sum_{m \in I_l} \sum_{\theta_m, \ell \in (0, \pi/2]} \int_{\mathbb{R}^3} |\bar{\chi}_l \bar{\psi}_m \bar{\varphi}_m \bar{f}||^2 d\xi. \tag{8.52}
\]
By (7.17) and (8.16), it follows that there exists a constant \( C_{11} > 0 \) such that
\[
\left| \langle f, f_{j,l,m}(x,\xi,h) \rangle_{L^2(\Omega)} \right|^2 \leq (1 + C_{11}s)\lambda_{m,l,1}\lambda_{m,l,2} \int_{\mathbb{R}^3} \left| \bar{\chi}_l \bar{\psi}_1 \bar{\varphi}_m \bar{f} \right|^2 \tag{8.53}
\]
and for a function \( u, \bar{u} \) is associated to \( u \) using (7.21) and (7.15).

Substituting (8.53) into (8.52), we find
\[
\langle f, \gamma_1 f \rangle_{L^2(\Omega)} \leq (1 + C_{11}s) \sum_{l \in J} \sum_{m \in I_l} \sum_{\theta_m, \ell \in (0, \pi/2]} \int_{\mathbb{R}^3} |\bar{\chi}_l \bar{\psi}_m \bar{\varphi}_m \bar{f}||^2 d\xi. \tag{8.54}
\]
In a similar fashion, one can show that
\[
\langle f, \gamma_2 f \rangle_{L^2(\Omega)} \leq (1 + C_{11}s) \sum_{l \in J} \sum_{m \in I_l} \sum_{\theta_m, \ell \in (0, \pi/2]} \int_{\mathbb{R}^3} |\bar{\chi}_l \bar{\psi}_m \bar{\varphi}_m \bar{f}||^2 d\xi d\tau. \tag{8.55}
\]
Next, we recall the definition of \( v_{j,\theta}(y; \xi) \) from (6.22) (resp. \( v_j \) from (6.9)) and use the fact that \( \{v_{j,\theta_m,\ell}\}_j \) (resp. \( u_j(\cdot,\xi) \) for all \( \xi \in \mathbb{R} \)) is an orthonormal set of eigenfunctions, we thus find
\[
\langle f, \gamma_1 f \rangle_{L^2(\Omega)} \leq (1 + C_{11}s) \sum_{l \in J} \sum_{m \in I_l} \int_{\mathbb{R}^3} |\bar{\chi}_l \bar{\psi}_m \bar{f}(z)|^2 dz. \tag{8.56}
\]
Implementing this into (8.55), and using (8.10) and (8.13), yields the claim of the lemma. \( \square \)

By the variational principle in Lemma 2.2, an upper bound of the sum of eigenvalues of \( \mathcal{P}_h \) below \( \Delta h \) follows if we can prove an upper bound on
\[
(1 + C_{12}s)^{-1} \text{Tr}\left[(\mathcal{P}_h - \Delta h) \gamma_1 \right] = (1 + C_{12}s)^{-1} \left( \text{Tr}\left[(\mathcal{P}_h - \Delta h) \gamma_1 \right] + \text{Tr}\left[(\mathcal{P}_h - \Delta h) \gamma_2 \right] \right)
\]
Recall the quadratic form \( \mathcal{Q}_h \) defined in (1.3). We start by estimating
\[
\text{Tr}\left[(\mathcal{P}_h - \Delta h) \gamma_1 \right] := \sum_{l \in J} \sum_{m \in I_l} \sum_{\theta_m, \ell \in (0, \pi/2]} \int_{\mathbb{R}} M_{j,m,\ell,\xi} \xi f_{j,l,m}(x,\xi,h) - \Delta h \|f_{j,l,m}(x,\xi,h)\|^2 d\xi. \tag{8.56}
\]
Recall the transformation $V_z \rightarrow y$ introduced in (8.26). Using (7.17), it follows from (8.16) and (8.17) that there exists a constant $C_{13} > 0$ such that

$$\int_{\Omega} |f_{j,l,m}(x, \xi; h)|^2 \, dx = \int_{\mathbb{R}^3_+} |g_l|^{1/2}(y)|\hat{f}_{j,l,m}(y, \xi; h)|^2 \, dy$$

$$\geq (\chi^2(y_{m,l}) - C_{13})\lambda^1_{m,l}^{1/2} \lambda^0_{m,l,2} b^3 m_l^{-1} h^{-3/2} \int_{\mathbb{R}^3} |V_z \rightarrow y(v_{j,l,m}(h^{-1/2} b^1 m_l^{1/2} z; \xi))\bar{\varphi}_{m,l}\psi_1|^2 \, dy. \quad (8.57)$$

Let us write the last integral as

$$\int_{Q_{(1+a)\lambda}^{m,l}} |V_z \rightarrow y(v_{j,l,m}(h^{-1/2} b^1 m_l^{1/2} z; \xi))(\bar{\varphi}_{m,l}\psi_1)(y)|^2 \, dy = \int_{Q_{(1+a)\lambda}^{m,l}} |V_z \rightarrow y(v_{j,l,m}(h^{-1/2} b^1 m_l^{1/2} z; \xi))\bar{\varphi}_{m,l}\psi_1|^2 \, dy$$

$$+ \int_{Q_{(1+a)\lambda}^{m,l}} \left[ -1 + \varphi_{m,l}^2 \bar{\psi}_1^2 \right] |V_z \rightarrow y(v_{j,l,m}(h^{-1/2} b^1 m_l^{1/2} z; \xi))\bar{\varphi}_{m,l}\psi_1|^2 \, dy. \quad (8.58)$$

As we shall work on the support of $M_{j,m,l}$ in view of (8.56), we may restrict ourselves to the indices $(j, m, l, \xi)$ satisfying $\zeta_j(\theta_{m,l}) \leq b^1_{m,l} - b^{-1}$ and $|\xi|/\sin(\theta_{m,l})| \leq L$. Recalling that $L$ is chosen so that $L \ll h^{-1/2} \zeta$ and using Lemma 6.7, it follows that for all $\alpha \in \sqrt{1 - \Lambda b^{-1}}$, there exists a constant $C_{14} > 0$ such that

$$\int_{Q_{(1+a)\lambda}^{m,l}} |V_z \rightarrow y(v_{j,l,m}(h^{-1/2} b^1 m_l^{1/2} z; \xi))(\bar{\varphi}_{m,l}\psi_1)(y)|^2 \, dy$$

$$\geq (1 - e^{-\alpha(C_{14} b^{-1/2} - L)}) \int_{Q_{(1+a)\lambda}^{m,l}} |V_z \rightarrow y(v_{j,l,m}(h^{-1/2} b^1 m_l^{1/2} z; \xi))\bar{\varphi}_{m,l}\psi_1|^2 \, dy, \quad (8.59)$$

where we have used (8.8) and (8.13).

Implementing (8.59) in (8.57), we obtain

$$\int_{\Omega} |f_{j,l,m}(x, \xi; h)|^2 \, dx \geq (1 - e^{-\alpha(C_{14} b^{-1/2} - L)})(\chi^2(y_{m,l}) - C_{13})\lambda^1_{m,l}^{1/2} \lambda^0_{m,l,2} b^3 m_l^{-1} h^{-3/2} \times$$

$$\int_{Q_{(1+a)\lambda}^{m,l}} |V_z \rightarrow y(v_{j,l,m}(h^{-1/2} b^1 m_l^{1/2} z; \xi))\bar{\varphi}_{m,l}\psi_1|^2 \, dy. \quad (8.60)$$

Let us estimate $Q_h(f_{j,l,m})$. Applying Lemma 7.2 with $u = f_{j,l,m}$, we find, for all $\varepsilon \gg \zeta$,

$$Q_h(f_{j,l,m})$$

$$\leq (1 + C \varepsilon) \int_{Q_{(1+a)\lambda}^{m,l}} \left[ \left| -i h \nabla z + b_{m,l} F_{\theta_{m,l}} \right| e^{i \varphi_{m,l} / h} \hat{f}_{j,l,m} \right|^2 \, dz + C \varepsilon^{-1} \int_{Q_{(1+a)\lambda}^{m,l}} |\hat{f}_{j,l,m}|^2 \, dz$$

$$\leq (1 + C \varepsilon) h^{-3/2} b_{m,l}^{3/2} \int_{Q_{(1+a)\lambda}^{m,l}} (\varphi_{m,l}^{1/2} \hat{\chi}_l)^2 \left[ \left| -i h \nabla z + b_{m,l} F_{\theta_{m,l}} v_{j,l,m}(h^{-1/2} b^1 m_l^{1/2} z; \xi) \right| \right]^2 \, dz$$

$$+ h^{-3/2} b_{m,l}^{3/2} \int_{Q_{(1+a)\lambda}^{m,l}} \left[ \left| \nabla (\hat{\chi}_l \varphi_{m,l}) \right|^2 + C \varepsilon \left| \varphi_{m,l}^{1/2} \hat{\psi}_1^{1/2} \right|^2 \right] |v_{j,l,m}(h^{-1/2} b^1 m_l^{1/2} z; \xi)|^2 \, dz, \quad (8.61)$$

where $C$ is the constant from Lemma 7.2.
By (8.4), (8.11), (8.14), and approximating $\tilde{\chi}^2$ using (8.17), it follows that for some constant $C_{15} > 0$,

$$Q_h(f_{j,l,m})$$

$$\leq \hbar^{-3/2}m_{l,m}^2(\tilde{\chi}^2(y_{m,l}) + C_{15}\varepsilon)\int_{Q_{m,l}^{(1+a)}} |(-i\hbar \nabla_z + b_{m,l}F_{\theta,m,l})(h^{-1/2}b_{m,l}^2z; \xi)|^2 dz$$

$$+ C_{15}h^{-3/2}m_{l,m}^2(\varepsilon^4 - 1 + h^2(a\varsigma)^2)\int_{Q_{m,l}^{(1+a)}} |v_{j,\theta,m,l}(h^{-1/2}b_{m,l}^2z; \xi)|^2 dz.$$

$$\leq \lambda_{m,l,1}^{1/2}(h^{-1/2}b_{m,l}^2(\tilde{\chi}^2(y_{m,l}) + C_{15}\varepsilon))(\lambda_{m,l,1}\lambda_{m,l,2}^{1/2} + C_{15}h^{-3/2}m_{l,m}^2(\varepsilon^4 - 1 + h^2(a\varsigma)^2)\times$$

$$\int_{Q_{m,l}^{(1+a)}} |V_{z=\theta,j,\theta,m,l}(h^{-1/2}b_{m,l}^2z; \xi)|^2 dy. \quad (8.62)$$

Combining (8.60) and (8.61), we obtain in view of (8.56) that,

$$\text{Tr}_{(\mathcal{P} - \Lambda h)\gamma_1} \leq \sum_{l \in \mathcal{L}} \sum_{m \in \mathcal{N}} \sum_{j \in \mathcal{J}} \int_{\mathbb{R}} M_{j,m,l,\xi} \left(Q_h(f_{j,l,m}) - \Lambda h \|f_{j,l,m}\|^2 \right) d\xi$$

$$\leq \sum_{l \in \mathcal{L}} \sum_{m \in \mathcal{N}} \sum_{j \in \mathcal{J}} \int_{\mathbb{R}} M_{j,m,l,\xi} \left(h^{-1/2}b_{m,l}^2(\tilde{\chi}^2(y_{m,l}) + C_{15}\varepsilon)\lambda_{m,l,1}\lambda_{m,l,2}^{1/2} + C_{15}(h^{-1/2}b_{m,l}^2z; \xi)|^2 dy\right) d\xi. \quad (8.63)$$

Performing the translation $\nu = h^{-1/2}b_{m,l}^2z - \frac{\xi}{\sin \theta_{m,l}}$ and let $\nu_\pm = \pm h^{-1/2}b_{m,l}^2(1 + a)\varsigma \pm L$, it follows that

$$\text{Tr}_{(\mathcal{P} - \Lambda h)\gamma_1} \leq -(1 + a)^2 \sum_{l \in \mathcal{L}} \sum_{m \in \mathcal{N}} \sum_{j \in \mathcal{J}} \tilde{\chi}^2(y_{m,l})\lambda_{m,l,1}\lambda_{m,l,2}^{1/2} b_{m,l}^2 \times$$

$$\left\{ \frac{1}{(2\pi)^{-1}} \sin(\theta_{m,l})(\zeta_j(\theta_{m,l}) - \Lambda b_{m,l}^{-1}) \right\} \int_{\nu_-}^{\nu_+} |u_{j,\theta,m,l}(\nu, t)| dt + I_{\text{rest}}^{(1)}.$$

where

$$I_{\text{rest}}^{(1)} = (1 + a)^2 \sum_{l \in \mathcal{L}} \sum_{m \in \mathcal{N}} \sum_{j \in \mathcal{J}} \tilde{\chi}^2(y_{m,l})\lambda_{m,l,1}\lambda_{m,l,2}^{1/2} b_{m,l}^2 \times$$

$$\left\{ C_{15}(\varepsilon \zeta_j(\theta_{m,l}) + h^{-1}b_{m,l}^{-1}(\varepsilon^4 - 1 + h^2(a\varsigma)^2) + \Lambda b_{m,l}^{-1}(e^{-\alpha(C_{14}d_{L}^{-1/2} - L)}(\tilde{\chi}^2)(y_{m,l}) - C_{13}\varsigma)) + C_{13}\varsigma \right\}.$$

(8.64)

By virtue of Lemma 6.10, (8.64) reads

$$\text{Tr}_{(\mathcal{P} - \Lambda h)\gamma_1} \leq -(1 + a)^2 \sum_{l \in \mathcal{L}} \sum_{m \in \mathcal{N}} \sum_{j \in \mathcal{J}} \tilde{\chi}^2(y_{m,l})\lambda_{m,l,1}\lambda_{m,l,2}^{1/2} b_{m,l}^2 \times$$

$$E(\theta_{m,l}, \Lambda b_{m,l}^{-1}) \int_{\nu_-}^{\nu_+} |u_{j,\theta,m,l}(\nu, t)| dt + I_{\text{rest}}^{(1)}.$$

(8.65)
It remains to estimate
\[
\text{Tr}[(\mathcal{P}_h - \Lambda h) \gamma_2] := \sum_{l \in J} \sum_{m \in I_x} M'_{m,l,\xi,\tau} \int_{\mathbb{R}} \left( \mathcal{Q}(g_{l,m}(x,\xi,\tau;h)) - \Lambda h \|g_{l,m}(x,\xi,\tau;h)\|^2 \right) d\xi.
\]
\[
(8.66)
\]
We start by estimating \(\|g_{l,m}(x,\xi,\tau;h)\|^2\). It follows from (7.17) (8.16) and (8.17) that there exists a constant \(C_{13} > 0\) such that
\[
\int_{\Omega} \|g_{l,m}(x,\xi,\tau;h)\|^2 dx \\
\geq (2\pi)^{-1}(\tilde{\chi}_l^2(y_{m,l}) - C_{13}\varsigma)\lambda_{m,l,1}^{1/2} b_{m,l}^{3/2} h^{-3/2} \int_{\mathbb{R}^3} |V_{z \rightarrow y}(v_1(h^{-1/2} b_{m,l}^{1/2};\xi)) \tilde{\varphi}_{m,l,\psi_1}(y)|^2 dy \\
= (2\pi)^{-2}(\chi_l^2(y_{m,l}) - C_{13}\varsigma)\lambda_{m,l,1}^{1/2} b_{m,l}^{3/2} h^{-3/2} \int_{\mathbb{R}^3} |u_1(h^{-1/2} b_{m,l}^{1/2};\xi) \tilde{\varphi}_{m,l,\psi_1}(y)|^2 dy,
\]
where the function \(u_1(\cdot,\xi)\) is introduced in (6.2). Let us write the last integral as
\[
\int_{Q_{m,l}^{(1+a)\varsigma}} |u_1(h^{-1/2} b_{m,l}^{1/2} y_3;\xi) \tilde{\varphi}_{m,l,\psi_1}|^2 dy = \int_{Q_{m,l}^{(1+a)\varsigma}} |u_1(h^{-1/2} b_{m,l}^{1/2} y_3;\xi)|^2 dy \\
+ \int_{Q_{m,l}^{(1+a)\varsigma}} [-1 + \tilde{\varphi}_{m,l,\psi_1}^2] |u_1(h^{-1/2} b_{m,l}^{1/2} y_3;\xi)|^2 dy.
\]
Due to the support of \(\tilde{\psi}_1\), we note that the integral on the right hand side is restricted to the set where \(y_3 \geq \varsigma/2\). Recalling (8.50) and selecting \(\varsigma\) as in (8.47), one has for \(h\) sufficiently small,
\[
(b_{m,l}^{1/2} h^{-1/2} \varsigma - \xi)^2 \geq (b_{m,l}^{1/2} h^{-1/2} \varsigma - \xi)^2 \gg \frac{1}{16} h \ll 1.
\]
Using this and Lemma 6.4, we obtain for some constant \(C_{16} > 0\)
\[
\int_{Q_{m,l}^{(1+a)\varsigma}} |u_1(h^{-1/2} b_{m,l}^{1/2} y_3;\xi)(\tilde{\varphi}_{m,l,\psi_1}(y)|^2 dy \geq (1 - e^{-C_{16} h^{-1}}) \int_{Q_{m,l}^{(1+a)\varsigma}} |u_1(h^{-1/2} b_{m,l}^{1/2} y_3;\xi)|^2 dy.
\]
where we have used (8.8) and (8.13).
Implementing (8.70) in (8.67), we obtain
\[
\int_{\Omega} \|g_{l,m}(x,\xi,\tau;h)\|^2 dx \geq (2\pi)^{-2}(1 - e^{-C_{16} h^{-1}}) (\tilde{\chi}_l^2(y_{m,l}) - C_{13}\varsigma)\lambda_{m,l,1}^{1/2} b_{m,l}^{3/2} h^{-3/2} \times \\
\int_{Q_{m,l}^{(1+a)\varsigma}} |u_1(h^{-1/2} b_{m,l}^{1/2} y_3;\xi)|^2 dy.
\]
Using the same arguments that have led to (8.62), one can show that
\[
\mathcal{Q}_h(g_{l,m}) \leq h^{-3/2} b_{m,l}^{3/2} (2\pi)^{-1} (\tilde{\chi}_l^2(y_{m,l}) + C_{15}\varsigma) \int_{Q_{m,l}^{(1+a)\varsigma}} \left| (-i\hbar \nabla_z + b_{m,l} \mathbf{F}_{\theta_{m,l}}) v_1(h^{-1/2} b_{m,l}^{1/2} z;\xi) \right|^2 dz \\
+ C_{15} h^{-3/2} b_{m,l}^{3/2} (2\pi)^{-1} (\varsigma^4 e^{-1} + h^2 (a\varsigma)^{-2}) \int_{Q_{m,l}^{(1+a)\varsigma}} |v_1(h^{-1/2} b_{m,l}^{1/2} z;\xi)|^2 dz \\
\leq \sum_{l \in J} \sum_{m \in I_x} \lambda_{m,l,1}^{1/2} \lambda_{m,l,2}^{1/2} b_{m,l}^{2} (1 + a)^2 \varsigma^2 (2\pi)^{-2} \times \\
\left\{ (\tilde{\chi}_l^2(y_{m,l}) + C_{15}\varsigma)(\mu_1(\xi) + \varsigma^2) + C_{15} h^{-1} b_{m,l}^{1}(\varsigma^4 e^{-1} + h^2 (a\varsigma)^{-2}) \right\}.
\]
(8.72)
Integrating in $\xi$ and $\tau$ and taking into account (8.50), it follows that (recall (8.66))

$$
\text{Tr} \left[ (\mathcal{P}_h - \Lambda h) \gamma_2 \right] \leq \sum_{l \in J} \sum_{m \in I_l} M'_{m,l,\xi,\tau} \lambda^{1/2} m_{l,1} \lambda^{1/2} m_{l,2} \lambda^{2} m_{l} (2\pi)^{-2} \xi^2 (1 + a)^2 \times
$$

$$
\int_{\mathbb{R}^2} \left( (\tilde{\chi}^2_l(y_{m,l}) + C_1 \xi)(\mu_1(\xi) + \tau^2) + C_1 h^{-1} b_{m,l}^{-1} (s^4 e^{-1} + h^2 a^{-2} \xi^2) - \Lambda b_{m,l}^{-1} (1 - e^{-C_2 h^{-1}}) (\tilde{\chi}^2_l(y_{m,l}) - C_{13} s) \right) d\xi d\tau
$$

$$
= - (1 + a)^2 \sum_{l \in J} \sum_{m \in I_l} \tilde{\chi}^2_l(y_{m,l}) \lambda^{1/2} m_{l,1} \lambda^{1/2} m_{l,2} \lambda^{2} m_{l} s^2 (2\pi)^{-2} \int_{\mathbb{R}^2} (\mu_1(\xi) + \tau^2 - \Lambda b_{m,l}^{-1}) \gamma_2 (1 + a)^2 \times
$$

$$
+ I^{(2)}_{\text{rest}}, \quad (8.73)
$$

where

$$
I^{(2)}_{\text{rest}} = \sum_{l \in J} \sum_{m \in I_l} \lambda^{1/2} m_{l,1} \lambda^{1/2} m_{l,2} h^2 m_{l} (2\pi)^{-2} \xi^2 (1 + a)^2 \int_{\mathbb{R}^2} M'_{m,l,\xi,\tau} \left( C_1 \xi (\mu_1(\xi) + \tau^2) + C_1 h^{-1} b_{m,l}^{-1} (s^4 e^{-1} + h^2 a^{-2} \xi^2) - \Lambda b_{m,l}^{-1} (1 - e^{-C_2 h^{-1}}) (\tilde{\chi}^2_l(y_{m,l}) - C_{13} s) + C_{13} s \right) d\xi d\tau. \quad (8.74)
$$

Taking into account the support of $M'_{m,l,\xi,\tau}$, we deduce the following bound on $|I^{(2)}_{\text{rest}}|$,

$$
|I^{(2)}_{\text{rest}}| \leq \sum_{l \in J} \sum_{m \in I_l} \lambda^{1/2} m_{l,1} \lambda^{1/2} m_{l,2} h^2 m_{l} (2\pi)^{-2} \xi^2 (1 + a)^2 4K \left( C_1 \xi \Lambda b_{m,l}^{-1}
$$

$$
+ C_1 h^{-1} b_{m,l}^{-1} (s^4 e^{-1} + h^2 a^{-2} \xi^2) - \Lambda b_{m,l}^{-1} (1 - e^{-C_2 h^{-1}}) (\tilde{\chi}^2_l(y_{m,l}) - C_{13} s) + C_{13} s \right). \quad (8.75)
$$

In view of Lemma 6.6, the estimate (8.73) reads

$$
\text{Tr} \left[ (\mathcal{P}_h - \Lambda h) \gamma_2 \right] \leq -(1 + a)^2 \sum_{l \in J} \sum_{m \in I_l} \tilde{\chi}^2_l(y_{m,l}) \lambda^{1/2} m_{l,1} \lambda^{1/2} m_{l,2} h^2 m_{l} s^2 (2\pi)^{-2} E(0, \Lambda b_{m,l}^{-1}) + I^{(2)}_{\text{rest}}. \quad (8.76)
$$

Combining (8.65) and (8.76), and recalling (6.36), we obtain

$$
\text{Tr} \left[ (\mathcal{P}_h - \Lambda h) \frac{\gamma_1}{1 + C_{10} s} \right]
$$

$$
\leq -(1 + a)^2 \sum_{l \in J} \sum_{m \in I_l} \tilde{\chi}^2_l(y_{m,l}) \lambda^{1/2} m_{l,1} \lambda^{1/2} m_{l,2} h^2 m_{l} s^2 (2\pi)^{-2} E(\theta_{m,l}, \Lambda b_{m,l}^{-1}) \int_{\nu_+}^{\nu_+} |\mu_{j,m_l}(\nu, t)| dt - I^{(1)}_{\text{rest}}
$$

$$
+ (1 + a)^2 \sum_{l \in J} \sum_{m \in I_l} \tilde{\chi}^2_l(y_{m,l}) \lambda^{1/2} m_{l,1} \lambda^{1/2} m_{l,2} h^2 m_{l} s^2 E(0, \Lambda b_{m,l}^{-1}) - I^{(2)}_{\text{rest}}. \quad (8.77)
$$
By Lemma 2.2, it is easy to see that
\[- \sum_l (e_j(h) - \Lambda h)_- \leq -(1 + C_{10}\varsigma)^{-1} (1 + a)^2 \sum_{l \in J} \sum_{m \in \mathcal{I}_l} \tilde{\chi}_l^2(y_{m,l}) \lambda_{m,l,1}^{1/2} \lambda_{m,l,2}^{1/2} b_{m,l,2}^2 \times \]
\[E(\theta_{m,l}, \Lambda b_{m,l}^{-1}) \int_{\nu_-} \int_{\nu_+} |u_{j,\theta_{m,l}}(\nu, t)|dvdt - I_{\text{rest}}^{(1)} + (1 + a)^2 \sum_{l \in J} \tilde{\chi}_l^2(y_{m,l}) \lambda_{m,l,1}^{1/2} \lambda_{m,l,2}^{1/2} b_{m,l,2}^2 E(0, \Lambda b_{m,l}^{-1}) - I_{\text{rest}}^{(2)}. \]  \( (8.78) \)

Using the upper bound estimate in Lemma 6.8, together with the fact that |B| is bounded on \( \partial\Omega \) and that \( \sum_{l \in J} \sum_{m \in \mathcal{I}_l} \varsigma \lambda_{m,l,1}^{1/2} \lambda_{m,l,2}^{1/2} \sim |\partial\Omega| \), we find
\[|I_{\text{rest}}^{(1)}| + |I_{\text{rest}}^{(2)}| = O(\varepsilon + h^{-1/4} |\varepsilon - 1| + h(\varsigma) - 2 + e^{-a(C_{14}h^{-1/2} - L)} + e^{-C_{15}h^{-1}}). \]  \( (8.79) \)

Recall the choice of \( \varepsilon, \varsigma, a \) in (8.47) and choose in addition \( L = h^{-1/16} \), we thus obtain
\[|I_{\text{rest}}^{(1)}| + |I_{\text{rest}}^{(2)}| = O(h^{1/8}), \]
and
\[\lim_{h \to 0} \int_{\nu_-} \int_{\nu_+} |u_{j,\theta_{m,l}}(\nu, t)|dvdt = \int_{-\infty}^{\infty} |u_{j,\theta_{m,l}}(\nu, t)|dvdt = 1. \]

We thus get, when taking \( \limsup_{h \to 0} \) on both sides of (8.78) the following estimate
\[\limsup_{h \to 0} \left\{ - \sum_j (e_j(h) - \Lambda h)_- \right\} \leq - \sum_{l \in J} \sum_{m \in \mathcal{I}_l} \tilde{\chi}_l^2(y_{m,l}) \lambda_{m,l,1}^{1/2} \lambda_{m,l,2}^{1/2} b_{m,l,2}^2 E(\theta_{m,l}, \Lambda b_{m,l}^{-1}). \]  \( (8.80) \)

By (8.15), we have \( \lambda_{m,l,1}^{1/2} \lambda_{m,l,2}^{1/2} = |g_l(y_{m,l})|^{1/2} \). For \( y = (y_1, y_2, 0) \in F_{c,m,l} \times \{0\} \), we define the function
\[v(y) := |g_l(y)|^{1/2} \tilde{\chi}_l(y) |\tilde{B}(y)|^2 E(\tilde{\theta}(y), |\tilde{B}(y)|^{-1} \Lambda). \]  \( (8.81) \)

We choose the points \( y_{m,l} \in F_{c,m,l} \times \{0\} \) so that
\[\max_{y \in F_{c,m,l} \times \{0\}} v(y) = v(y_{m,l}). \]

Then the right-hand side of (8.78) is an upper Riemann sum. We thus get
\[\sum_{m \in \mathcal{I}_l} |g_l(y_{m,l})|^{1/2} \tilde{\chi}_l^2(y_{m,l}) b_{m,l}^2 E(\theta_{m,l}, b_{m,l}^{-1} \Lambda) \varsigma^2 \geq \sum_{m \in \mathcal{I}_l} v(y_{m,l}) \varsigma^2 \geq \int_{\delta_1 \delta_2} v(y_1, y_2, 0)dy_1dy_2 \geq \int_{\partial\Omega} \tilde{\chi}_l^2(x) |\tilde{B}(x)|^2 E(\theta(x), |\tilde{B}(x)|^{-1} \Lambda) d\sigma(x). \]  \( (8.82) \)

Inserting this and (8.82) into (8.78), and using (8.10), we obtain
\[\limsup_{h \to 0} \left\{ - \sum_j (e_j(h) - \Lambda h)_- \right\} \leq - \int_{\partial\Omega} |B(x)|^2 E(\theta(x), \Lambda |B(x)|^{-1}) d\sigma(x), \]
which is the desired upper bound.
8.3. Proof of Corollary 1.6. Let us start by computing the left- and right- derivatives of the function \((0, 1) \ni \lambda \to E(\theta, \lambda)\). Using the formula of \(E(\theta, \lambda)\) given in Theorem 1.2, we find

\[
\frac{\partial E}{\partial \lambda_+}(\theta, \lambda) = \begin{cases} 
\frac{1}{2\pi} \int_0^\infty \frac{(\mu_1(\xi) - \lambda)^{1/2}}{\sin(\theta)} d\xi & \text{if } \theta = 0, \\
\frac{1}{2\pi} \int_0^\infty \frac{(\mu_1(\xi) - \lambda)^{1/2}}{\sin(\theta)} \text{card}\{j : \zeta_j(\theta) \leq \lambda\} & \text{if } \theta \in (0, \pi/2),
\end{cases}
\]  

(8.83)

and

\[
\frac{\partial E}{\partial \lambda_-}(\theta, \lambda) = \begin{cases} 
\frac{1}{2\pi} \int_0^\infty \frac{(\mu_1(\xi) - \lambda)^{1/2}}{\sin(\theta)} d\xi & \text{if } \theta = 0, \\
\frac{1}{2\pi} \int_0^\infty \frac{(\mu_1(\xi) - \lambda)^{1/2}}{\sin(\theta)} \text{card}\{j : \zeta_j(\theta) < \lambda\} & \text{if } \theta \in (0, \pi/2),
\end{cases}
\]  

(8.84)

Notice that the condition in (1.13) ensures the equality of \(\frac{\partial E}{\partial \lambda_+}(\theta, \lambda)\) and \(\frac{\partial E}{\partial \lambda_-}(\theta, \lambda)\) when \(\lambda \in \{\Lambda|B(x)|^{-1} : x \in \partial \Omega\}\).

Let \(\varepsilon > 0\). Using Corollary 2.3, we obtain

\[
\text{Tr}(\mathcal{P}_h - (\Lambda + \varepsilon)h)_- - \text{Tr}(\mathcal{P}_h - \Lambda h)_- \geq \varepsilon h N(\Lambda h; \mathcal{P}_h, \Omega).
\]  

(8.85)

On the other hand, by the formula in (1.12), we have

\[
\text{Tr}(\mathcal{P}_h - (\Lambda + \varepsilon)h)_- - \text{Tr}(\mathcal{P}_h - \Lambda h)_- = \int_{\partial \Omega} |B(x)|^2 \left( E(\theta(x), (\Lambda + \varepsilon)|B(x)|^{-1}) - E(\theta(x), \Lambda|B(x)|^{-1}) \right) d\sigma(x) + o(1), \text{ as } h \to 0.
\]

Implementing this into (8.85), then taking \(\lim \sup_{h \to 0^+} \), we see that

\[
\lim \sup_{h \to 0^+} h N(\Lambda h; \mathcal{P}_h, \Omega) \leq \int_{\partial \Omega} |B(x)| \frac{E(\theta(x), (\Lambda + \varepsilon)|B(x)|^{-1}) - E(\theta(x), \Lambda|B(x)|^{-1})}{\varepsilon |B(x)|^{-1}} d\sigma(x).
\]

We recall here that \(|B(x)| > 0\) for all \(x \in \partial \Omega\). Taking the limit \(\varepsilon \to 0^+\), we deduce using (8.83), and Lebesgue’s dominated convergence Theorem, that

\[
\lim \sup_{h \to 0^+} h N(\Lambda h; \mathcal{P}_h, \Omega) \leq \int_{\partial \Omega} |B(x)| \frac{\partial E}{\partial \lambda_+}(\theta(x), \Lambda|B(x)|^{-1}) d\sigma(x).
\]  

(8.86)

Replacing \(\varepsilon\) by \(-\varepsilon\) in (8.85) and following the same arguments that led to (8.86), we find

\[
\lim \inf_{h \to 0^+} h N(\Lambda h; \mathcal{P}_h, \Omega) \geq \int_{\partial \Omega} |B(x)| \frac{\partial E}{\partial \lambda_-}(\theta(x), \Lambda|B(x)|^{-1}) d\sigma(x).
\]  

(8.87)

It follows by the assumption (1.13) that

\[
\int_{\partial \Omega} |B(x)| \frac{\partial E}{\partial \lambda_+}(\theta(x), \Lambda|B(x)|^{-1}) d\sigma(x) = \int_{\partial \Omega} |B(x)| \frac{\partial E}{\partial \lambda_-}(\theta(x), \Lambda|B(x)|^{-1}) d\sigma(x).
\]  

(8.88)

Combining (8.86) and (8.87) we obtain

\[
\lim_{h \to 0^+} h N(\Lambda h; \mathcal{P}_h, \Omega) = \int_{\partial \Omega} |B(x)| \frac{\partial E}{\partial \lambda_+}(\theta(x), \Lambda|B(x)|^{-1}) d\sigma(x),
\]  

(8.89)

which finishes the proof.

ACKNOWLEDGEMENTS

This paper is a major part of the author’s Ph.D. dissertation. The author wishes to thank her advisors S. Fournas and A. Kachmar. Financial support through the Lebanese University and CNRS as well as through the grant of S. Fournas from Lundbeck foundation.
Appendix A. Proof of Lemma 7.2

Using (7.16), (7.8) and (7.9), we obtain that for some constant $c_1 > 0$

\[
(1 - c_1(\ell + T)) \left\{ \int_{Q_{0,t,T}} \sum_{p,q=1}^{3} g^{pq}(y_0) \left[ (-i\hbar \nabla y_p + \tilde{A}_p^\text{lin}) \tilde{u} \right] \left[ (-i\hbar \nabla y_q + \tilde{A}_q^\text{lin}) \tilde{u} \right] |g(y_0)|^{1/2} dy \right\} 
\leq Q_h(u) \leq (1 + c_1(\ell + T)) \left\{ \int_{Q_{0,t,T}} \sum_{p,q=1}^{3} g^{pq}(y_0) \left[ (-i\hbar \nabla y_p + \tilde{A}_p^\text{lin}) \tilde{u} \right] \left[ (-i\hbar \nabla y_q + \tilde{A}_q^\text{lin}) \tilde{u} \right] |g(y_0)|^{1/2} dy \right\}.
\]

Similarly, using (7.9) and (7.17), we have for some constant $c_2 > 0$

\[
(1 - c_2(\ell + T)) \int_{Q_{0,t,T}} |g(y_0)|^{1/2} |\tilde{u}|^2 dy \leq ||u||_{L^2(V_{\nu_0})}^2 \leq (1 + c_2(\ell + T)) \int_{Q_{0,t,T}} |g(y_0)|^{1/2} |\tilde{u}|^2 dy.
\]

By the Cauchy-Schwarz inequality, we get using (7.11) that there exists a constant $c_3 > 0$ such that

\[
(1 - \varepsilon) \int_{\mathbb{R}^3} \sum_{p,q=1}^{3} g^{pq}(y_0) \left[ (-i\hbar \nabla y_p + \tilde{A}_p^\text{lin}) \tilde{u} \right] \left[ (-i\hbar \nabla y_q + \tilde{A}_q^\text{lin}) \tilde{u} \right] |g(y_0)|^{1/2} dy
\]

\[-c_3(\ell^2 + T^2)^2 \varepsilon^{-1} \int_{\mathbb{R}^3} |\tilde{u}|^2 |g(y_0)|^{1/2} dy
\]

\[
\leq \int_{\mathbb{R}^3} \sum_{p,q=1}^{3} g^{pq}(y_0) \left[ (-i\hbar \nabla y_p + \tilde{A}_p^\text{lin}) \tilde{u} \right] \left[ (-i\hbar \nabla y_q + \tilde{A}_q^\text{lin}) \tilde{u} \right] |g(y_0)|^{1/2} dy
\]

\[
\leq (1 + \varepsilon) \int_{\mathbb{R}^3} \sum_{p,q=1}^{3} g^{pq}(y_0) \left[ (-i\hbar \nabla y_p + \tilde{A}_p^\text{lin}) \tilde{u} \right] \left[ (-i\hbar \nabla y_q + \tilde{A}_q^\text{lin}) \tilde{u} \right] |g(y_0)|^{1/2} dy
\]

\[+ c_3(\ell^2 + T^2)^2 \varepsilon^{-1} \int_{\mathbb{R}^3} |\tilde{u}|^2 |g(y_0)|^{1/2} dy. \quad (A.3)
\]

for any $\varepsilon > 0$. Next, we perform the change of variables $z = (z_1, z_2, z_3) = \left( \lambda_1^{1/2} y_1, \lambda_2^{1/2} y_2, y_3 \right)$. We thus infer using (7.19) the following quadratic form in the $(z_1, z_2, z_3)$ variables

\[
\int_{\mathbb{R}^3} \sum_{p,q=1}^{3} g^{pq}(y_0) \left[ (-i\hbar \nabla y_p + \tilde{A}_p^\text{lin}) \tilde{u} \right] \left[ (-i\hbar \nabla y_q + \tilde{A}_q^\text{lin}) \tilde{u} \right] |g(y_0)|^{1/2} dy
\]

\[= \sum_{p=1}^{3} \int_{Q_{0,t,T}} |(-i\hbar \nabla z_p + \mathbf{F}_p) \tilde{u}|^2 dz, \quad (A.4)
\]

where $\mathbf{F} = (\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3)$ is the magnetic potential given by

\[
\mathbf{F}_1(z) = \lambda_1^{-1/2} \tilde{A}_1^\text{lin}(z), \quad \mathbf{F}_2(z) = \lambda_2^{-1/2} \tilde{A}_2^\text{lin}(z), \quad \mathbf{F}_3(z) = \tilde{A}_3^\text{lin}(z).
\]

Also, we have

\[
\int_{Q_{0,t,T}} |\tilde{u}|^2 |g(y_0)|^{1/2} dy = \int_{Q_{0,t,T}} |\tilde{u}|^2 dz \quad (A.5)
\]

Substituting this into (A.3) yields

\[
(1 - c_2(\ell + T)) \int_{Q_{0,t,T}} |\tilde{u}|^2 dz \leq ||u||_{L^2(V_{\nu_0})}^2 \leq (1 + c_2(\ell + T)) \int_{Q_{0,t,T}} |\tilde{u}|^2 dz. \quad (A.6)
\]
Let $\beta = (\beta_1, \beta_2, \beta_3) = \text{curl}_x(F(z))$ and note that the coefficients of $\beta$ and $\alpha$ (see (7.14)) are related by
\[
\beta_1 = \lambda_2^{-1/2}\alpha_1, \quad \beta_2 = \lambda_1^{-1/2}\alpha_2, \quad \beta_3 = (\lambda_1\lambda_2)^{-1/2}\alpha_3.
\]
The relation (7.13) gives that
\[
|\beta| = |\text{curl}_x(F(z))| = (\beta_1^2 + \beta_2^2 + \beta_3^2)^{1/2} = |\widetilde{\mathbf{B}}|(y_0)
\]
We thus perform a gauge transformation so that there exists a function $\phi_0 \in C^\infty(\widetilde{Q}_{0,t,T})$ such that
\[
F(z) = b_0F_{\theta_0}(z) + \nabla\phi_0, \quad b_0 = |\widetilde{B}(y_0)|,
\]
where, for $\theta \in [0, \pi/2]$, $F_\theta$ is the magnetic field from (3.4) and
\[
\theta_0 := \tilde{\theta}(y_0) = \arcsin\left(\frac{z_3}{|\beta|}\right).
\]
We emphasize here that (A.9) is compatible with the definition of $\theta(x)$ given in (1.10), i.e., $\tilde{\theta}(y_0) = \theta(F_z^{-1}(y_0))$. Combining (A.3), (A.4) and (A.8), we obtain, using (A.5),
\[
(1 - \varepsilon)\int_{\tilde{Q}_{0,t,T}} |(-ih\nabla_z + b_0F_{\theta_0})e^{i\phi_0/h} \tilde{u}|^2 dz - c_3(\epsilon^2 + T^2)^2\int_{\tilde{Q}_{0,t,T}} |\tilde{u}|^2 dz
\leq \int_{\mathbb{R}^3} \sum_{p,q=1}^3 g^{pq}(y_0)\left[(-ih\nabla_{y_p} + \hat{A}_p)\tilde{u}\right] \left[(-ih\nabla_{y_q} + \hat{A}_q)\tilde{u}\right] |g(y_0)|^{1/2} dy
\leq (1 + \varepsilon)\int_{\tilde{Q}_{0,t,T}} |(-ih\nabla_z + b_0F_{\theta_0})e^{i\phi_0/h} \tilde{u}|^2 dz + c_3(\epsilon^2 + T^2)^2\varepsilon^{-1}\int_{\tilde{Q}_{0,t,T}} |\tilde{u}|^2 dz. \quad (A.10)
\]
for any $\varepsilon > 0$. Choose $\varepsilon \geq \ell + T$. Inserting (A.10) into (A.1), we obtain that for some constant $c_4 > 0$
\[
(1 - c_4\varepsilon)\int_{\tilde{Q}_{0,t,T}} |(-ih\nabla_z + b_0F_{\theta_0})e^{i\phi_0/h} \tilde{u}|^2 dz - c_4(\epsilon^2 + T^2)^2\varepsilon^{-1}\int_{\tilde{Q}_{0,t,T}} |\tilde{u}|^2 dz
\leq \mathcal{Q}_h(u) \leq (1 + c_4\varepsilon)\int_{\tilde{Q}_{0,t,T}} |(-ih\nabla_z + b_0F_{\theta_0})e^{i\phi_0/h} \tilde{u}|^2 dz + c_4(\epsilon^2 + T^2)^2\varepsilon^{-1}\int_{\tilde{Q}_{0,t,T}} |\tilde{u}|^2 dz. \quad (A.11)
\]
Recall (A.6) and choose $C = \max\{c_2, c_4\}$, thereby establishing (7.22) and (7.23).

**APPENDIX B. PROOF OF LEMMA 8.2**

According to Lemma 8.1, the lemma follows if we can prove a lower bound on the right-hand side of (8.5). We start by estimating $\mathcal{Q}_h(\psi_1f_j)$. Using the IMS decomposition formula, it follows that
\[
\mathcal{Q}_h(\psi_1f_j) = \sum_{l \in J} \left(\mathcal{Q}_h(\chi_l\psi_1f_j) - \hbar^2 \|\nabla\chi_l|\psi_1f_j\|^2_{L^2(\Omega)}\right).
\]
Using (8.11), and implementing (8.10), we get
\[
\mathcal{Q}_h(\psi_1f_j) - (\mathcal{A}h + C_1h^2\varsigma^{-2}) \|\psi_1f_j\|^2_{L^2(\Omega)}
\geq \sum_{l \in J} \left(\mathcal{Q}_h(\chi_l\psi_1f_j) - (\mathcal{A}h + (C_1 + C_2)h^2\varsigma^{-2}) \|\psi_1\chi_lf_j\|^2_{L^2(\Omega)}\right), \quad (B.2)
\]
where we used that $\varsigma^{-2} \gg 1$ (see (8.47) below).
Applying the IMS formula once again, we then find, using that $a \ll 1$,
\[
\mathcal{Q}_h(\psi_1 \chi f_j) = \sum_{m \in \mathcal{I}_j} \left\{ \mathcal{Q}_h(\varphi_{m,l} \psi_1 \chi f_j) - h^2 \| (\nabla \varphi_{m,l} \psi_1 \chi f_j) \|_{L^2(\Omega)}^2 \right\} \\
\geq \sum_{m \in \mathcal{I}_j} \left\{ \mathcal{Q}_h(\varphi_{m,l} \psi_1 \chi f_j) - (C_1 + C_2 + C_3^2) h^2 (a \varepsilon)^2 \| \varphi_{m,l} \psi_1 \chi f_j \|_{L^2(\Omega)}^2 \right\}.
\]
(B.3)

The last inequality follows from (8.14) and $C_3^2 := C_3 \sup_{l \in \mathcal{J}} \| D \Phi_l \|^2$. Inserting this into (B.2), it follows that
\[
\mathcal{Q}_h(\psi_1 f_j) - (\Lambda h + C_1 h^2 \varepsilon^2) \| \psi_1 f_j \|_{L^2(\Omega)}^2 \\
\geq \sum_{l \in \mathcal{J}} \sum_{m \in \mathcal{I}_l} \left\{ \mathcal{Q}_h(\varphi_{m,l} \psi_1 \chi f_j) - (\Lambda h + (C_1 + C_2 + C_3^2) h^2 (a \varepsilon)^2) \| \varphi_{m,l} \psi_1 \chi f_j \|_{L^2(\Omega)}^2 \right\}.
\]
(B.4)

Applying Lemma 7.2 with $y_0$ replaced by $y_{m,l}$, $u = \varphi_{m,l} \psi_1 \chi f_j$, $\ell = (1 + a) \varsigma$, $T = \varsigma$, we then deduce that there exists a function $\phi_{m,l} := \phi_{y_{m,l}} \in C^\infty(\overline{Q}_{(1+a)\varsigma})$ such that, for all $\varepsilon \in (0, 1]$ satisfying $\varepsilon \gg \varsigma$, one has, using $a \ll 1$,
\[
\mathcal{Q}_h(\psi_1 f_j) - (\Lambda h + C_1 h^2 \varepsilon^2) \| \psi_1 f_j \|_{L^2(\Omega)}^2 \\
\geq (1 - C \varepsilon) \sum_{l \in \mathcal{J}} \sum_{m \in \mathcal{I}_l} \int_{\overline{Q}_{(1+a)\varsigma}} |(-ih \nabla_z + b_{m,l} F_{\theta_{m,l}}) e^{ih \varsigma \phi_{m,l}} \varphi_{m,l} \psi_1 \chi f_j |^2 dz \\
- (\Lambda h + (C_1 + C_2 + C_3^2) h^2 (a \varepsilon)^2) (1 + 3C \varsigma) + 25C \varsigma^4 \varepsilon^{-1}) \sum_{l \in \mathcal{J}} \sum_{m \in \mathcal{I}_l} \int_{\overline{Q}_{(1+a)\varsigma}} | \varphi_{m,l} \psi_1 \chi f_j |^2 dz,
\]
(B.5)

where $C$ is the constant from Lemma 7.2. Put $\tilde{C} = \max \{ C_1 + C_2 + C_3^2, 25C \}$. Inserting (B.5) into (8.5) yields the desired estimate of the lemma.

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