Boundedness theorem for Fano log-threefolds

Alexandr Borisov
Department of Mathematics
Pennsylvania State University
e-mail: borisov@math.psu.edu

(Revised English version)
Jan 6, 1994

Abstract
The main purpose of this article is to prove that the family of all Fano threefolds with log-terminal singularities with bounded index is bounded.

1 Introduction
First of all, let me recall necessary definitions and list some known results and conjectures in direction of bondedness of Fano manifolds. All varieties in this paper are over field of complex numbers.

Definition 1.1 Normal variety $X$ is called the variety with log-terminal singularities if $mK_X$ is a Cartier divisor for some integer $m$ and there exists a resolution $\pi : Y \rightarrow X$ of singularities of $X$ such that exceptional divisors $F_i$ of $\pi$ have simple normal crossings and in formula $K_Y = \pi^* K_X + \sum (a_i F_i)$ all $a_i > -1$

Definition 1.2 Index (or Gorenstein index) of variety $X$ is a minimal natural number $m$, s.t. $mK_X$ is a Cartier divisor. Of course, index is defined for $Q$-Gorenstein varieties only.
Definition 1.3 Three-dimensional algebraic variety $X$ is called Fano log-threefold if the following conditions hold.

1) $X$ has log-terminal singularities,
2) $X$ is $Q$-factorial,
3) Picard number $\rho(X) = 1$,
4) $-K_X$ is ample.

Remark 1.1 This is only my terminology inspired by the term "$Q$-Fano threefolds".

The following statement will be proven in this paper.

Main Theorem.

For an arbitrary natural $n$ all Fano log-threefolds of index $n$ lie in finite number of families.

Remark 1.2 Unfortunately, no effective bound on any invariant of $X$ will be given because of Noetherian induction in the section 4.

Here are some results in the direction of boundedness of Fano manifolds.

1) Boundedness theorem for smooth Fano manifolds of an arbitrary direction is proven by Kollár, Miyaoka and Mori in [11]. Before this result there were several proofs with extra condition $\rho(X) = 1$. Three-dimensional smooth case was also treated before by a long work of many authors beginning with Fano itself. See [7] for discussion.

2) Two-dimensional Fano varieties are traditionally called Del Pezzo surfaces. Smooth (=terminal) case is fairly easy and the answer is the following.

$P^1 \times P^1, P^2$ with 0 to 8 blown-up points in general position. (The generality of position may be stated precisely.)

I should notice here that there are many difficult problems concerning Del Pezzo surfaces if basic field is NOT algebraically closed.

Log-terminal case (with arbitrary Picard number) was studied by Alexeev and Nikulin (see [15]). One of the main results in this direction is boundedness under the condition of bounded multiplicity of singularities. Let me mention that by using methods of this paper one can obtain a new simple proof of some intermediate result, namely boundedness under the condition of bounded index.
3) The model case of toric varieties of arbitrary dimension is treated in [3], see also [8].

4) Boundedness of Fano threefolds with $\rho = 1$ and terminal singularities is proven by Kawamata in [8].

5) Boundedness of Fano threefolds with terminal singularities with no extra conditions is announced by Mori.

All these results justify the following conjecture.

Conjecture 1.1 The family of all Fano varieties of given dimension with discrepancies of singularities greater (or greater or equal) $-1 + \epsilon$, where $\epsilon$ is an arbitrary given positive real number, is bounded.

Remark 1.3 This conjecture is so natural that probably many people suspected it but I didn’t see it published. Batyrev proposed the weaker variant of this conjecture, where the condition on discrepancies is replaced by the condition of boundedness of index. (\[3\]) Very recently Alexeev told me that he also stated the above conjecture as a part of a general phenomenon noticed by Shokurov that some geometric invariants (in this case minimal discrepancies of Fano varieties) can accumulate only from above (below). See [3] for a discussion.

I am expressing my thanks to V. Iskovskih who encouraged me to work in this direction. I am glad to thank V. Shokurov and V. Alexeev who invited me to the geometry seminar at Johns Hopkins University and whose remarks simplified and even corrected this paper. I also want to thank my brother Lev for helpful discussions.

2 Preliminary remarks and first lemmas

In [10] Kollár proved that all three-dimensional normal varieties $X$ with an ample Cartier divisor $D$ lie in finite number of families if two higher coefficients of Hilbert polynomial $P(m) = \chi(mD)$ are bounded. In our case of three-dimensional Fano varieties of index $n$ it works as follows. Let $D$ be equal to $-nK_X$. Then it is a Cartier divisor and it follows from general theory of Riemann-Roch that
\[ \chi(O_X(-mnK_X)) = \frac{1}{12}(-K_X)^3nn(n+1)(2nm+1) + \alpha m + \beta, \]
where \(\alpha\) and \(\beta\) are some constants depending on \(X\).

Therefore in order to prove the Main Theorem we only need to prove that \((-K_X)^3\) is bounded. The following lemma shows that in our case it is also equivalent to the condition that \(h^0(-2nK_X)\) is bounded.

**Lemma 2.1** For arbitrary Fano log-threefold \(X\) of index \(n\) (actually, only conditions (1) and (3) are used) the following inequality holds.

\[ h^0(-2nK_X) \geq (-K_X)^3 \left( \frac{5}{3}n^3 + \frac{1}{2}n^2 \right) - 1 \]

**Proof** By the Kawamata-Vieweg vanishing theorem \(h^i(-mnK_X) = 0, i > 0, m \geq 0\). Therefore \(h^0(-mnK_X) = \frac{1}{12}(-K_X)^3nm(n+1)(2nm+1) + \alpha m + \beta\) for \(m \geq 0\). Let us consider "the second derivative at 1".

\[ h^0(-2nK_X) - 2h^0(-nK_X) + h^0(O_X) = (-K_X)^3 \left( \frac{5}{3}n^3 + \frac{1}{2}n^2 \right). \]

Now the statement of lemma follows from the fact that \(h^0(O_X) = 1\) and \(h^0(-nK_X) \geq 0\).

**Lemma 2.2** Suppose \(v \in V\) - is a closed point of \(k\)-dimensional variety with multiplicity of local ring \(r\), \(D\) is a semiample \(Q\)-Cartier divisor on \(V\). Suppose further that the general point \(x\) of \(V\) can be connected to \(v\) by some curve \(\gamma_x\), such that \(\gamma_x \cdot D \leq d\)

Then \(D^k \leq r \cdot d^k\).

**Proof** For sufficiently large \(m\) such that \(mD\) is a Cartier divisor one have that \(h^0(O_V(mD)) = m^kB^k + O(m^{k-1})\). Therefore if \(D^k > r \cdot d^k\) then for \(m \gg 0\) one can find a non-zero global section \(s \in H^0(O_V(mD))\) such that its image by trivialization map of \(O_V(mD)\) in \(v\) lies in \((md + 1)\)th power of maximal ideal of point \(v\). Then every curve \(\gamma_x\) lies in \(\text{Supp}(s)\), that is impossible.

**Remark 2.1** The above lemma is very general. In applications \(V\) will be our Fano log-threefold \(X\) and \(D\) will be \((-K_X)\).

### 3 Covering family and first division into cases

**Remark 3.1** (about notations). We will often consider birational varieties. Doing this we will usually identify curves on different varieties if they coincide in their general points. Namely, let \(X \leftarrow X' \) and \(L \subset X, L' \subset X'\)
be curves. Then $L$ and $L'$ are identified if there are Zariski open subsets $U \subset X$ and $U' \subset X'$, such that the above rational map is defined on them and $U \cong U'$, $L \cap U \cong L' \cap U' \neq \emptyset$ via it. The identified curves will be usually denoted by the same symbol. The same convention will be used for two-dimensional subvarieties. If it is necessary to point out that, say, simple divisor $S$ is considered on variety $X$ it will be denoted by $S_X$. Another convention is that $\{l\}$ will denote the family of curves with general element $l$ and $\{H\}$ will denote the linear system of Weil divisors with general element $H$. It will be clear in every particular case why these conventions agree with each other.

Now we start to prove our Main Theorem. Suppose $X$ is a Fano log-threelfold, $\pi_Y : Y \to X$ is its $Q$-factorial terminal modification, $\pi_{Y_1} : Y_1 \to Y$ is a resolution of isolated singularities of $Y$. By the Miyaoka-Mori theorem (\cite{13}, see also \cite{9}) there exists a covering family of rational curves $\{l\}$, such that $l \cdot (-K_X) \leq 6$. The family $\{l\}$ is free on $Y_1$ that is small full deformation of $l$ covers small neighborhood of $l$. (See \cite{14}.) We can and will denote by $\{l_{Y_1}\}$ full family, that is (some Zariski open subset of) a component of the scheme of morphisms from $P^1$ to $Y_1$. Consider the RC-fibration $\varphi : Y_1 \to Z$, associated with $\{l\}$. (See \cite{11}, \cite{12}.) The following cases are possible.

(0) $\dim Z = 0$. In \cite{12} such $X$ are called primitive. It implies that two general points of $Y_1$ can be joined by chain of no more than 3 curves from $\{l\}$. It follows now from one of the ”gluing lemmas” (\cite{12}) that we can glue them together and obtain new family $\{l'\}$. Then we can apply to it lemma 2.2 and obtain that

$$(-K_X)^3 \leq (3 \cdot l \cdot (-K_X))^3 \leq (3 \cdot 6)^3$$

(1) $\dim Z = 1$. In this case after some additional blowing-up $\tilde{Y} \to Y_1$ we obtain a morphism $\varphi_{\tilde{Y}} : \tilde{Y} \to Z$. Here $Z \cong P^1$, because $X$ is rationally connected (see \cite{12}).

(2) $\dim Z = 2$. In this case general $l \in \{l\}$ is smooth and does not intersect with another general $l$ on $Y_1$. And it is exactly the general fiber of the RC-fibration.

We will proceed by the following way. First of all we will treat the case (1). Doing this we will require $l \cdot (-K_X)$ to be bounded not by 6 but only by an arbitrary constant depending on $n$. After that we will reduce the case (2) to the case (1) but for some new family $\{l'\}$ where $l' \cdot (-K_X)$ will be bounded.
4 The treatment of case (1)

Let $S$ be a general fiber of our RC-fibration. As we already mentioned, the image of RC-fibration is rational. This implies that $S$ are linear equivalent on $Y$ and therefore on $X$. Notice that it can happen that $\{l\}$ does not connect two general points of $S$ immediately. But it will always be true if we glue two examples of $\{l\}$. (See [12].) Therefore we will assume, that $\{l\}$ is a connecting family on $S$. Evidently, $l^2 \geq 1$ on a smooth surface $\tilde{S} = S_Y$. The condition that $X$ is Q-factorial with Picard number 1 implies that $S_X = \alpha H, \alpha > 0$, where $H = (-2nK_X)$. We will assume up to the end of this section that $l \cdot H \leq \rho$, where $\rho$ is some constant depending on $n$.

Proposition 4.1 If $h^0(H) > 2(\rho + 1)^2$ then $\alpha \leq \frac{1}{2}$.

Proof Let $S_1$ and $S_2$ be two general surfaces from $\{S_X\}$. Let $l_1, l_2, ... l_{\rho+1} \subset S_1$ and $l_{\rho+2}, l_{\rho+3}, ... l_{2\rho+2} \subset S_2$ be general curves from $\{l\}$. We have that $H \cdot l \leq \rho$, therefore

$\dim H^0(O_X(H)) - \dim H^0(J_{l_i} \cdot O_X(H)) \leq \dim H^0(O_{l_i}(H)) \leq \rho + 1.$

(Here $J_{l_i}$ is an ideal sheaf of the curve $l_i \subset X$.)

This implies that

$\text{codim}\left( \bigcap_{i=1}^{2\rho+2} H^0(J_{l_i} \cdot O_X(H)) \right) \leq \sum_{i=1}^{2\rho+2} (\rho + 1) = 2(\rho + 1)^2.$

If $h^0(H) > 2(\rho + 1)^2$ then there exists a divisor $H^* \in |H|$, such that all $l_i \subset H^*$. Suppose $\pi_1$ is a composition of birational morphism $\tilde{S}_1 \to S_1$ and embedding $\tilde{S}_1 \to X$. If $H^*$ does not contain $S_1$ then $(\pi_1)\ast(H^*)$ does not contain $\tilde{S}_1$ but at the same time contains preimages of $l_i$. On $\tilde{S}_1$ we have $(\pi_1^*H^*) \cdot l \geq \sum l_i \cdot l \geq \rho + 1$. It contradicts to the fact that $(\pi_1^*H^*) \cdot l = H \cdot l \leq \rho$. Therefore $H^*$ contains $S_1$ and, by the same arguments, $S_2$. This implies that $\alpha \leq \frac{1}{2}$.

We will always assume below that $\alpha \leq \frac{1}{2}$.

Proposition 4.2 For arbitrary $S \in \{S\}$ on $X$, arbitrary positive integer $k$

$h^i(X, J_S \cdot O_X(kH)) = 0$ for every $i > O$. 

6
Proof $S$ is a simple divisor, therefore $J_S = O_X(-S)$, where $O_X(-S)$ is a
divisorial sheaf sheaf, associated with Weil divisor $(-S)$. After that, $O_X(kH)$
is an invertible sheaf, therefore $J_S \cdot O_X(kH) = O_X(kH - S)$. Now one can
apply Kawamata-Vieweg vanishing theorem (see the reformulation of it in
$\llbracket \rrbracket$) because $kH - S - K_X = (k - \alpha + \frac{1}{2n})H$ is ample for $k \geq 1$.

Proposition 4.3 For all $k > 0$, $l > 0$ $h^i(S, O_S(kH)) = O$.

Proof It follows from exact sequence

$$0 \rightarrow J_S \cdot O_X(kH) \rightarrow O_X(kH) \rightarrow O_S(kH) \rightarrow 0,$$

vanishing theorem and proposition 4.2.

Proposition 4.4 All surfaces $S_X$ for given $n$ and $\rho = c(n)$ lie in finite
number of families.

Proof By a result of Kollár ($\llbracket 10 \rrbracket$) and proposition 4.3 it is enough to prove
the boundedness of coefficients of Hilbert polynomial $P(k) = \chi(O_S(kH)) = h^0(O_S(kH))$, $k \geq 1$. For this purpose we will prove that there exists some constant $c_1(n, \rho)$ such that for all $k \geq 1$ it is true that $h^0(O_S(kH)) \leq k^2 \cdot c_1(n, \rho)$.

It implies the boundedness of coefficients by the following arguments. Suppose $P(k) = a_2k^2 + a_1k + a_0$. Evidently, $0 \leq a_2 \leq c_1$. Therefore

$$|a_1| = |P(2) - P(1) - 3a_2| \leq 4c_1.$$  

After that, $|a_0| = |P(1) - a_2 - a_1| \leq 5c_1$.

In order to prove that $h^0(O_S(kH)) \leq k^2 \cdot c_1(n, \rho)$ consider the following
construction. By applying several times gluing lemma to a free family $\{l\}$
on $\tilde{S}$ ($\llbracket 12 \rrbracket$) we obtain families $\{l_k\}$ such that $l_k = k \cdot l$ as divisors on $\tilde{S}$
and therefore on $S$. (Here ”=“ means algebraic equivalence.) Notice that
the natural map $\mu_k : H^0(S, O_S(kH)) \rightarrow H^0(l_{k\rho+1}, O_{l_{k\rho+1}}(kH))$ is injective.

Otherwise there should have been some $D \in |kH|$ containing $l_{k\rho+1}$ but not
containing $S$. As in the proof of proposition 4.1 we obtain a contradiction
by intersecting with general $l$. The fact that $\mu_k$ is injective implies that

$$h^0(O_S(kH)) \leq h^0(l_{k\rho+1}, O_{l_{k\rho+1}}(kH)) \leq l_{k\rho+1}(kH) + 1 = (k\rho + 1)k\rho + 1,$$

that is what we need.

Proposition 4.5 In the condition of the above proposition there is a con-
stant $c_2(n, \rho)$, such that on every general $S_X$ EVERY two points can be joined
by some irreducible curve $\gamma$, such that $\gamma \cdot (-K_X) \leq c_2$.

Proof It is a straightforward consequence of boundedness of $S_X$ with $H|_{S_X}$.
Indeed, it is true for a general element of every one of families in proposition
4.2 and Noetherian induction on base completes the proof.
Remark 4.1  Of course, two GENERAL points of $S_X$ are already connected
by $l$, but the above proposition gives much more.

Now we can complete the treatment of the case (1). By the definition of Fano
log-threefold $\rho(X) = 1$ therefore two general $S_X$ intersect with each other.
Moreover, they intersect along some curve $C$ because $X$ is Q-factorial. We
know that $\{S_X\}$ is a linear system, therefore all of them contain $C$. It may
happen that $C$ lies in $\text{Sing}(X)$, but the multiplicity of $X$ in a general point
$x_0 \in C$ is bounded by $2n$, because the index of $X$ is bounded by $n$. (By
canonical cover trick it is a factor of $CDV$ singularity that is analytically
isomorphic to $(DV - \text{point}) \times (disk)$.) Therefore we can apply lemma 2.2 to
$X, (-K_X), x_0$ to obtain a bound on $(-K_X)^3$.

5  Two lemmas

In this section we will prove some adjunction lemma and a lemma about
accurate resolution that will be used in next section to treat the case (2).
However, these lemmas themself are interesting enough to deserve a separate
section.

Lemma 5.1 (adjunction) Suppose $X$ is a three-dimensional variety and $S$
is simple Weil divisor on it, such that $(K_X + S)$ is $Q$-Cartier. Suppose $\{L\}$ is
a covering family of curves on $S$, $\hat{S}$ is a minimal resolution of normalization
of $S$. Then $K_{\hat{S}} \cdot L \leq (K_X + S) \cdot L$.

Proof Denote by $\pi$ the natural morphism $\hat{S} \rightarrow X$. Then by the proposition
3.2.2 of $K_{\hat{S}} = \pi^*(K_X + S) - D$, where $D$ is an effective divisor. The rest
is trivial.

Remark 5.1 The above lemma is due to Shokurov. In the first variant of
this paper I formulated and proved it only under condition that singularities
of $X$ were isolated which is enough for applications.

Lemma 5.2 (accurate resolution)

Suppose $X$ is a $Q$-factorial three-dimensional variety, $E \subset X$ is a simple
Weil divisor, $\{L\}$ is a covering family of curves on $E$. Suppose further that
there exists a covering family $\{l\}$ on $X$, such that $l \cdot E \geq 1$ and a linear
system $|H|$ on $X$, such that the following inequalities hold true. ($c_i$ are some nonnegative constants.)

1) $H \cdot l \leq c_1$
2) $H \cdot L \leq c_2$
3) $K_X \cdot L \leq c_3$
4) $-E \cdot L \leq c_4$

Then $h^0(H) > 1 + (c_1 + 1)(c_2 + c_1c_4 + 1)$ implies that there exists a resolution $Y \to X$, such that $\{L\}$ have no base points on $E_Y$ and $K_Y \cdot L \leq c_3 + 2(c_2 + c_1c_4)$.

**Remark 5.2** The proof of this lemma will be pretty long. It will take the rest of the section.

**Remark 5.3** In some sense this lemma is a very weak substitute for the following conjecture for which I have a lot of evidence.

**Accurate Resolution Conjecture** For an arbitrary Q-Gorenstein threefold $X$ there exists a resolution of singularities $\pi : Y \to X$, such that for EVERY Q-Cartier divisor $H$ on $X$ containing a curve $L_X$ not lying in $\text{Sing}(X)$ the following inequality holds true.

$$(K_Y + D_Y) \cdot L_Y \leq (K_X + D_X) \cdot L_X$$

First of all we will introduce some convenient notations. Let $\{D\}$ be a linear system of Weil divisors. We will denote by $H^0(\{D\})$ the corresponding vector subspace in $H^0(O_X(D))$, where $O_X(D)$ is a divisorial sheaf, associated with $D$. Reversely, for a linear subspace $V \subset H^0(\{D\})$ let $|V|$ be the corresponding linear system. Divisor that corresponds to $s \in H^0(O_X(D))$ will be denoted by $(s)$. Section that determines divisor $D$ will be called “equation” of $D$. Of course, it is defined up to multiplicative constant. By definition $h^0(\{D\}) = \dim H^0(\{D\}) = \dim \{D\} + 1$.

For the purpose of convenience we introduce the concept of $L$-base of linear system in the following way. Suppose $\{D\}$ is a linear system of Weil divisors, $\{L\}$ is a family of curves parameterized by base $S$. For every nonempty Zariski open subset $U \subset S$ let $V(U, \{D\})$ be a linear subspace in $H^0(\{D\})$, spanned by $s$, such that $(s)$ contains $L_u$ for some $u \in U$. 

9
Evidently, \( V(U' \cap U'', \{D\}) \subset V(U', \{D\}) \cap V(U'', \{D\}) \) and \( H^0(\{D\}) \) is finite-dimensional. Therefore there exists the minimal \( V(U^*, \{D\}) \), such that \( V(U^*, \{D\}) \subset V(U, \{D\}) \) for every \( U \subset S \). Then \( |V(U^*, \{D\})| \) will be called \( L \)-base of \( \{D\} \) and denoted by \( \{D\}^L \).

**Proposition 5.1** \( h^0(\{D\}^L) \geq h^0(\{D\}) - 1 \cdot D - 1 \)

**Proof** Suppose \( \{D\}^L = |V(U^*, \{D\})| \), \( u \in U^* \). We can also assume that \( L_u \) is not contained in \( \text{Sing}(X) \). Choose on \( L_u \) points \( x_1, x_2, \ldots, x_d \), \( L \cdot D < d \leq L \cdot D + 1 \) lying in nonsingular part of \( X \). The condition of vanishing in \( x_1, x_2, \ldots, x_d \) determines a subspace in \( H^0(\{D\}) \) of codimension no greater than \( d \) and \( d \leq L \cdot D + 1 \). Now we just notice that for every \( s \) from this subspace \( (s) \) contains \( L_u \), because otherwise we would have a contradiction by intersecting it with \( L_u \).

Define a new linear system \( \{H_s\} \) by the following procedure. Denote \( |H| \) by \( \{H_0\} \) and for every nonnegative integer \( i \) let \( \{H_{i+1}\} \) be a movable part of \( \{H_i\}^L \). Evidently, \( \{H_i\} \) will eventually stabilize. This stabilized \( \{H_i\} \) will be our \( \{H_s\} \). It is evident that \( \{H_s\} \) is movable and \( \{H_s\} = \{H_s\}^L \). (Here we set as definition that trivial linear systems \( \emptyset \) and \( |O_X| \) are movable.)

**Proposition 5.2** If \( h^0(H) > 1 + (c_1 + 1)(c_2 + c_1c_4 + 1) \) then \( \{H_s\} \) is not trivial

**Proof** First of all, let \( \{H\}^L = a_iE + D_i + \{H_{i+1}\} \), where \( a_i \geq 0 \), \( D_i \) does not contain \( E \). Notice that if \( a_i = 0 \) then \( \{H_{i+1}\}^L = \{H_{i+1}\} \) and the procedure stabilizes. From the other hand, \( \sum a_i \leq c_1 \) because \( E \cdot L \geq 1 \) and \( H \cdot l \leq c_1 \). Therefore \( \{H_s\} = \{H_{[c_1]+1}\} \). It is easy to see that for all \( i \) \( H_i \cdot L \leq H \cdot L + c_1(-E \cdot L) \leq c_2 + c_1c_4 \). Therefore by proposition 5.1 we have that \( h^0(\{H_s\}) \geq h^0(H) - (c_1 + 1)(c_2 + c_1c_4 + 1) > 1 \). This implies \( \{H_s\} \) is not trivial.

We also have from the above proof that \( H_s \cdot L \leq c_2 + c_1c_4 \). Apply to \( K_X + 2\{H_s\} \) Alexeev Minimal Model Program (\[\Box\]). Namely, let \( \pi : Y_1 \to X \) be a terminal modification of \( K_X + 2\{H_s\} \) in sense of Alexeev.

**Proposition 5.3** Under the above notations the following is true.

1. \( Y_1 \) is \( Q \)-factorial and have at worst terminal singularities.
(2) \( \{ \pi' H_\ast \} \) is free. Here \( \{ \pi' H_\ast \} \) is a inverse image of linear system \( \{ H \} \) in sense of Alexeev, that is general element of \( \{ \pi' H_\ast \} \) is \( \pi' H_\ast \) for general \( H_\ast \in \{ H \} \).

(3) \( K_{Y_1} \cdot L \leq c_3 + 2(c_2 + c_1c_4) \)

**Proof** Parts (1) and (2) are proved the same way as lemma 1.22 in [1]. Part (3) is a corollary of the following chain of inequalities.

\[
K_{Y_1} \cdot L \leq (K_{Y_1} + 2(\pi' H_\ast)) \cdot L \leq (K_X + 2H_\ast) \cdot L \leq c_3 + 2(c_2 + c_1c_4)
\]

Here the middle inequality is due to the following argument. By definition of terminal modification \( K_{Y_1} + 2(\pi' H_\ast) \) is \( \pi - \text{nef} \) and therefore in adjunction formula \( K_{Y_1} + 2(\pi' H_\ast) = \pi^*(K_X + 2H_\ast) + \sum a_i D_i \), where \( D_i \) are exceptional divisors, all \( a_i \leq 0 \).

For the rest of the section we will use the following notations. Suppose \( D_i, i = 1, ..., k \) are exceptional divisors of morphism \( \pi \). For an arbitrary Weil divisor \( F \) on \( X \) we will say that discrepancy of \( F \) is a \( k \)-tuple \( \{ \text{discr}_{D_i}(F) \} \) of discrepancies of \( F \) in \( D_i \), that is numbers \( \text{discr}_{D_i}(F) \) from the formula \( \pi^* F = \pi'(F) + \sum \text{discr}_{D_i}(F) D_i \). In these notations we have the following lemma.

**Lemma 5.3** Suppose \( F = (s), s \in H^0(O_X(F)) \). Suppose \( s = \sum \alpha_j s_j \), where \( (s_j) = F_j \). Then for all \( D_i \) \( \text{discr}_{D_i}(F) \geq \min_j \text{discr}_{D_i}(F_j) \) and for a general \( \{ \alpha_j \} \) for given \( \{ s_j \} \) this inequality becomes an equality.

**Proof** Suppose \( rF \) is a Cartier divisor. In a neighborhood of generic point \( \pi(D_i) \) the sheaf \( O_X(rF) \) can be trivialized. With respect to this trivialization the local equation \( f \) of divisor \( rF \) is, by Newton binomial formula, a linear combination of local equations \( f(\gamma) \) of divisors \( \sum \gamma_j F_j \), where \( \sum \gamma_j = r, \gamma_j \in \mathbb{Z}_{\geq 0} \). By the definition, \( \text{discr}_{D_i}(F)^{\frac{1}{r}} = \frac{1}{r} \text{discr}_{D_i}(rF) \) and \( \text{discr}_{D_i}(rF) \) is just an image of \( f \) by a valuation on function field \( \mathbb{C}(X) \) of variety \( X \) corresponding to \( D_i \). Therefore for arbitrary \( \{ \alpha_j \} \) \( \text{discr}_{D_i}(rF) \geq \frac{1}{r} \min_j \text{discr}_{D_i}(\sum \gamma_j F_j) \geq \min_j \text{discr}_{D_i}(F_j) \) and for general \( \{ \alpha_j \} \) it becomes an equality.

Suppose now that \( P_1 \subset \{ H_\ast \} \) is a set of all divisors \( H_\ast \) containing some \( L \in \{ L \} \). Suppose a general element of \( P_1 \) has discrepancy \( \{ d_i \} \). Denote the set of all divisors from \( P_1 \) with such discrepancy by \( P \).

**Proposition 5.4** "Equations" of \( H_\ast, H_\ast \in P \), span \( H^0(\{ H_\ast \}) \).
Proof For a general $L \in \{L\}$ divisors $H_\ast \in P$, containing $L$ constitute a nonempty Zariski open subset in linear system of divisors from $\{H\}$ containing $L$. Therefore their ”equations” span the corresponding subspace in $H^0(\{H_\ast\})$. By definition $\{H_\ast\} = \{H_\ast\}^L$, so we are done.

**Proposition 5.5** $\{L\}$ have no base points on $E_{Y_1}$.

**Proof** Proposition 5.4 and lemma 5.3 applied together imply that discrepancy of general element of linear system $\{H_\ast\}$ equals $\{d_i\}$. Therefore for every $H_\ast \in P \pi'_H \ast \in \{\pi'_H\}$. Moreover, the linear equivalence between divisors $\pi'_H \ast$ is given by the same functions from $C(Y_1) = C(X)$ as between corresponding divisors $H_\ast$. Therefore the proposition 5.4 implies that ”equations” of $\pi'_H \ast$, where $H_\ast \in P$, span $H^0(\{\pi'_H\})$.

Suppose all $L$ on $Y_1$ pass through some point $y$. Then all $\pi'_H \ast$, where $H_\ast \in P$, contain $y$. But it is in contradiction with proposition 5.3, (2), so proposition 5.5 is proven.

To complete the proof of the whole Accurate Resolution Lemma it is enough to choose an arbitrary resolution of singularities $Y \rightarrow Y_1$. Then $Y \rightarrow X$ will satisfy all the requirements of accurate resolution.

### 6 Treatment of case (2)

Now we are in situation and notations of case (2). (See section 3.)

**Proposition 6.1** On $Y_1$ there exists a divisor $E$ which is exceptional with respect to morphism $\pi_{Y_1}^Y : Y_1 \rightarrow X$ such that $E \cdot l \geq 1$.

**Proof** Suppose $C$ is some general enough curve on the image $Z$ of RC-fibration $\varphi$. Suppose $D \subset X$ is an image by $\pi_{Y_1}^Y$ of the surface $[\varphi^{-1}(C)]$. (Here parenthesis means Zariski closure.) The general $l_{Y_1}$ does not intersect $\varphi^{-1}(C)$ and, therefore, $[\varphi^{-1}(C)]$. ($Y_1$ is smooth therefore $\{l\}$ is free, see [14].) So, if $l_{Y_1}$ does not intersect with exceptional divisors of $\pi_{Y_1}^Y$ then $l_X \cdot D = 0$, that is impossible because $X$ is Q-factorial and $\rho(X) = 1$. Q.E.D.

Notice that if $E \cdot l \geq 1$ then general $l_{Y_1}$ intersects with $E$ in general points because $\{l_{Y_1}\}$ is free. Two cases are possible.
(A) There exists such $E \subset Y_1$ that is exceptional with respect to the morphism $\pi^{Y_1}_Y$.

(B) Family $\{l_Y\}$ is free. Then there exists $E \subset Y$ that is exceptional with respect to $\pi^Y_X$.

The proof is generally the same in both cases but some technical details are different. We begin with the case (A). By the relative version of the usual Minimal Model Program morphism $\pi^{Y_1}_Y$ can be decomposed into extremal contractions and flips, relative over $Y$. Suppose $\pi^{Y_3}_Y$ is the first that contract some divisor $E_Y$, for which $l \cdot E_Y \geq 1$. Suppose $\tilde{E}_Y$ is a minimal resolution of $E_Y$.

**Proposition 6.2 (Case (A))** There exists a covering family $\{L\}$ of rational curves on $E_Y$, such that the following conditions hold true.

1. $L \cdot K_{E_Y} < 0$
2. $-L \cdot E_Y < 3$
3. $L$ does not admit a nontrivial 2-point deformation on $\tilde{E}_Y$, that is a deformation with two fixed points, whose image is not in $L$.

**Proof** Suppose $\pi^{Y_3}_Y(E_Y)$ is a curve. Then we can choose $\{L\}$ to be the fibers of $\pi^{Y_3}_Y|_{E_Y}$. Then (1) is true by the definition of extremal contraction. Suppose $\tilde{E}_Y$ is a normalization of $E_Y$. Then $\{L\}$ does not have base points on $\tilde{E}_Y$ and therefore $L$ does not pass through its singularities. This easily implies (3). The condition (2) follows from the fact that (by lemma 5.3)

$$ (K_{Y_3} + E_Y) \cdot L \geq K_{E_Y} \cdot L = -2 > -3. $$

Suppose now that $\pi^{Y_3}_Y(E_Y)$ is a point. Consider a minimal model $F$ of $\tilde{E}_Y$. The surface $\tilde{E}_Y$ is birationally ruled or rational therefore we have two possibilities for $F$:

1. $F \cong P^2$
2. $F$ is ruled, there is a morphism $\theta : F \to C$

We let $\{L\}$ be the family of planes on $P^2$ in the first case and the family of fibers of $\theta$ in the second one. It evidently satisfies the condition (3). The condition (1) holds for arbitrary curve on $E_Y$. The condition (2) again follows from the fact that

$$ (K_{Y_3} + E_Y) \cdot L \geq K_{E_Y} \cdot L \geq -3. $$
The proposition is proven.

Now we can apply the Accurate Resolution Lemma (lemma 5.2.) Here $X$ means $Y_3$, $H$ means $(\pi_X^Y)^*(-2nK_X)$ and constants will be as follows.

c_1 = 12n, c_2 = 0, c_3 = 0, c_4 = 3.

We see that if $h^0(-2nK_X)$ is big enough there exists a resolution $Y_4 \rightarrow Y_3$ such that $K_{Y_4} \cdot L \leq 2(3 \cdot 12n) = 72n$ and $\{L\}$ have no base points on $E_{Y_4}$.

**Proposition 6.3** $L$ does not admit a nontrivial 2-point deformation on $Y_4$.

**Proof** If such deformation existed it would be a deformation on $E_{Y_4}$ by rigidity lemma. (About this lemma see [6], section 1. I must only notice that it is not stated there correctly, one should add a condition of flatness of morphism $f$. It was noticed by several people, my attention was brought to it by Iskovskikh.) The system $\{L\}$ has no base points on $E_{Y_4}$ therefore $L$ does not pass through the singularities of normalization $\hat{E}_{Y_4}$ of the surface $E_{Y_4}$.

Resolution of singularities $\hat{E}_{Y_4}$ is naturally mapped to $\hat{E}_{Y_3}$ therefore 2-point deformation of $L$ on $E_{Y_4}$ gives deformation on $E_{Y_4}$ and then on $E_{Y_4}$, and then on $\hat{E}_{Y_3}$. The last is impossible by the choice of $L$, Q.E.D.

Now we can apply to $\{L\}$ and $\{l\}$ the gluing lemma on $Y_4$ (see [11]) to obtain a new covering family of rational curves $\{l'\}$. But now the image of RC-fibration corresponding to $\{l'\}$ has dimension 1 or 0. And $l' \cdot (-K_X) \leq \left(1 + \dim Y_4 + L \cdot K_{Y_4}\right)(l \cdot (-K_Y)) \leq 6(4 + 72n)$. So we managed to reduce the case (2A) to cases (1) and (0), as it was promised at the end of section 3.

Now we consider the case (B). Similarly to the case A, we have the following statement.

**Proposition 6.4 (Case (B))** There exists a covering family $\{L\}$ of rational curves on $E_Y$, such that the following conditions hold true.

1. $L \cdot K_Y < 0$
2. $-L \cdot E_Y < 3$
3. $L$ does not admit a 2-point nontrivial deformation on $\hat{E}_{Y_3}$.
4. $\pi_X^Y(L)$ is a point

**Proof** If $\pi_X^Y(E_Y)$ is a curve let $\{L\}$ be the family of fibers of $\pi_X^Y|_{E_Y}$. If $\pi_X^Y(E_Y)$ is a point then let it come from the minimal model of $\hat{E}_Y$ as in the proof of
proposition 6.2. As in the case (A), \(K_{E_Y} \cdot L\) is \(-2\) or \(-3\). Conditions (3) and (4) are evidently satisfied, we only need to prove (1) and (2). In order to do it consider the adjunction formula for \(\pi_X\), multiplied by \(L\):

\[
K_Y \cdot L = \sum_{E_i \neq E_Y} a_i E_i L + aE_Y \cdot L
\]

(*)

Here \(a_i\) and \(a\) are discrepancies, they are of form \((-\frac{m}{n})\), \(m \in \{0, 1, ..., n-1\}\), where \(n\) is an index of \(X\). (Discrepancies are nonpositive because \(Y\) is a terminal modification of \(X\).) We have the following chain of inequalities.

\[-3 \leq K_{E_Y} \cdot L \leq (1 + a)E_Y \cdot L + \sum_{E_i \neq E_Y} a_i E_i L \leq (1 + a)E_Y \cdot L\]

Here the middle inequality follows from lemma 5.1 and formula (*), and the right from nonpositivity of \(a_i\). Therefore 1 + \(\frac{1}{n}\) implies that either \(-E_Y \cdot L \leq 0\) or \(-E_Y \cdot L \leq 3n\). Therefore \(-E_Y \cdot L \leq 3n\). Now the condition (1) follows from the following chain of inequalities.

\[K_Y \cdot L = \sum_{E_i \neq E_Y} a_i E_i L + aE_Y \cdot L \leq aE_Y L \leq 3n\]

Here the right inequality holds because of the following argument. We know that \(-1 < a \leq 0\) therefore \(E_Y L \geq 0\) implies \(aE_Y L \leq 0\) and \(E_Y L < 0\) implies \(aE_Y L \leq -E_Y L\). Q.E.D.

Again, as in case (A), we apply the Accurate Resolution Lemma (lemma 5.2). The only difference is that now we have \(Y\) instead of \(Y_3\) and constants are as follows.

\(c_1 = 12n, c_2 = 0, c_3 = 3n, c4 = 3n\).

Again if \(h^0(-2nK_X)\) is big enough there exists an accurate resolution \(Y_4\). We have again that \(L\) does not admit nontrivial 2-point deformation on \(Y_4\). (Arguments from the proof of proposition 6.3 work without any problems because of condition (4) of proposition 6.4.) So we can apply gluing lemma from [11]. The bound on \(l' \cdot (-K_X)\) will be the following.

\[l' \cdot (-K_X) \leq (4 + L \cdot K_{Y_4}) (l' \cdot (-K_X)) \leq (4 + 3n + 2(12n \cdot 3n)) \cdot 12n = 12n(4 + 3n + 72n^2)\]

So we completed the treatment of case (2B). Our Main Theorem is finally proven.
References

[1] V. Alexeev, General elephants of Q-Fano 3-folds. Preprint (1993), to appear in Compositio Math.

[2] V. Alexeev, Boundedness and $K^2$ for log surfaces. Preprint (1993).

[3] V. V. Batyrev, Ogranichennost’ stepeni mnogomernyh toricheskikh mnogoobrazii Fano. Vestn. MGU Ser. 1, matem., mech. (1982), no 1, 22-27. (in Russian).

[4] V. V. Batyrev, The cone of effective divisors of threefolds. Contemporary Math. 131 (1989), Part 3, 337-352.

[5] A. A. Borisov, L. A. Borisov, Osobye toricheskie mnogoobraziiia Fano. Matem. Sbornik (1992), no 2, 134-141. (in Russian).

[6] H. Clemens, J. Kollár, S. Mori, Higher dimensional complex geometry. Astérisque 166 (1988).

[7] V. A. Iskovskikh, Lektsii po trekhmernym algebraicheskim mnogoobraziiam. Mnogoobraziiia Fano. Izd. Mosk. universitet (1988). (book, in Russian).

[8] Y. Kawamata, Boundedness of Q-Fano threefolds. Contemporary Math. 131 (1989), Part 3, 439-445.

[9] Y. Kawamata, On the length of an extremal rational curve. Invent. Math. 105 (1991), N3, 609-611.

[10] J. Kollár, Toward Moduli of singular varieties. Compositio Math. 56 (1985), 369-392.

[11] J. Kollár, Y. Miyaoka, S. Mori, Rational connectedness and boundedness of Fano manifolds. J. Differential geom. 36 (1992), no 3, 765-779.

[12] J. Kollár, Y. Miyaoka, S. Mori, Rationally connected varieties. J. Algebraic geom. 1 (1992), 429-448.

[13] Y. Miyaoka, S. Mori, A numerical criterion for uniruledness. Ann. of Math. 124 (1986), 65-69
[14] A. M. Nadel, The boundedness of degree of Fano manifold with Picard number one. Preprint

[15] V. V. Nikulin, Del Pezzo surfaces with log terminal singularities III. Math. USSR Izv. 35 (1990).

[16] V. V. Shokurov, 3-fold log flips. Math. USSR Izv. 56 (1992), 105-203.