Mixed Tate motives and the unit equation

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Abstract. This is the second installment in a sequence of articles devoted to “explicit Chabauty-Kim theory” for the thrice punctured line. Its ultimate goal is to construct an algorithmic solution to the unit equation whose halting will be conditional on Goncharov’s conjecture about exhaustion of mixed Tate motives by motivic iterated integrals (refined somewhat with respect to ramification), and on Kim’s conjecture about the determination of integral points via $p$-adic iterated integrals. In this installment we explain what this means while developing basic tools for the construction of the algorithm. We also work out an elaborate example, which goes beyond the cases that were understood before, and allows us to verify Kim’s conjecture in a range of new cases.

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1. Introduction

1.1. Let $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, let $S$ be a finite set of primes, and let $S = \text{Spec} \mathbb{Z}[S^{-1}]$. The equation cutting out the set of $S$-valued points

$$X(S) = \{(x, y) \in \mathbb{Z}[S^{-1}]^* \times \mathbb{Z}[S^{-1}]^* \mid x + y = 1\}$$

inside $(\mathbb{G}_m \times \mathbb{G}_m)(S)$ is known as the $S$-unit equation. Although $X(S)$ has been known to be finite for a long time, the problem of constructing an algorithmic solution remains open.

Let $p \in S$ be a closed point. The category of mixed Tate motives over $S$ with $\mathbb{Q}$-coefficients possesses a canonical fiber functor $\text{[DG]}$; the associated group is of the form

$$G(S) = \mathbb{G}_m \rtimes U(S)$$

with $U(S)$ free prounipotent on the set of generators $S \cup N$ where

$$N = \{\nu_{-3}, \nu_{-5}, \nu_{-7}, \ldots\}.$$  

Similarly, the category of mixed Tate filtered $\phi$ modules over $\mathbb{Q}_p$ studied by Chatzistamatiou–Ünver $\text{[CU]}$ has an associated group of the form

$$G(F \phi) = \mathbb{G}_m \rtimes U(F \phi),$$

with $U(F \phi)$ canonically free prounipotent on generators $v_i$ indexed by the integers $i \leq -1$. Our study of explicit Chabauty-Kim theory for the thrice punctured line, begun in $\text{[DCWI]}$, revolves around a certain diagram, which places the localization map

$$X(S) \rightarrow X(\mathbb{Z}_p)$$

in a commuting square together with filtered $\phi$ realization at the level of certain nonabelian cohomology varieties for the two groups $G(S)$, $G(F \phi)$.

$$\begin{array}{ccc}
X(S) & \xrightarrow{k} & X(\mathbb{Z}_p) \\
\downarrow & & \downarrow \\
H^1(G(S), U) & \xrightarrow{F \phi} & H^1(G(F \phi), U^{F \phi}) \\
\alpha & & \downarrow \\
& & U^{F \phi}
\end{array}$$

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We refer to this diagram as “Kim’s cutter”. Based on the “higher Chabauty’s method” pioneered by Minhyong Kim [Kim1, Kim2], Kim’s cutter serves as a powerful tool for carving out the finite set of $S$-integral points $X(S)$ inside the analytic space $X(\mathbb{Z}_p)$ of $p$-adic points.

1.2. Under Kim’s conjecture [BDCKW], if

$$U(X) \twoheadrightarrow U$$

is a sufficiently large quotient of the unipotent fundamental group of Deligne-Goncharov [DG], and if $U^F\phi$ is its realization in filtered $\phi$ modules, then

$$(K) \quad \alpha^{-1}\left(F\phi\left(H^1(G(S), U)\right)\right) = X(S).$$

In this sequence of articles, we focus on a certain sequence of quotients

$$U^n = \mathbb{Q}(1) \ltimes \prod_{i=1}^{n} \mathbb{Q}(i)$$

constructed by Deligne [Del1, Section 16]. Our goal is to construct an algorithm for computing $p$-adic approximations of the sets

$$\alpha^{-1}\left(F\phi\left(H^1(G(S), U^n)\right)\right)$$

for $S$, $n$ and $p$ arbitrary, and hence, conjecturally, for solving the $S$-unit equation. In Explicit Chabauty-Kim theory [DCW1] we focused on the case $n = 2$. Here we lay the foundations that will be needed for the general case with $S$, $n$, and $p$ arbitrary. We then investigate two new special cases, given by $S = \emptyset$, $n$, $p$ arbitrary, and by $S = \{2\}$, $n = 4$, $p$ arbitrary.

1.3. We refer the reader to Explicit Chabauty-Kim theory for a more thorough discussion of background. As explained in the introduction, and documented in the appendix, the prospects of a direct approach are unclear. The difficulty lies in the relative structurelessness of the nonabelian cohomology variety

$$H^1(G(S), U^n);$$

although abstractly isomorphic to affine space, it lacks a natural set of coordinates. Even making direct effective use of its known dimension is hard, since the latter depends on Borel’s computations of higher $K$-groups [Bor1, Bor2], which are real-analytic in nature.

1.4. Our approach is to factor the filtered $\phi$ realization map $F\phi$ through a certain auxiliary vector space. Let $\overline{S}$ denote a set of primes containing $S$ but not $p$, let $\overline{S} = \text{Spec}\mathbb{Z}[\overline{S}^{-1}]$ and let

$$A(\overline{S}) = \mathcal{O}(U(\overline{S}))$$

denote the graded Hopf algebra of functions on $U(\overline{S})$. Similarly, let

$$A(F\phi) = \mathcal{O}(U(F\phi))$$

denote the graded hopf algebra of functions on $U(F\phi)$. 
Theorem. The filtered $\varphi$ realization map $F\phi$ factors through $A(\overline{S})_1 \times \prod_1^n A(\overline{S})_i$ as in the following diagram.

\[
\begin{array}{cccc}
H^1(G(S),U^n) & \xrightarrow{F\phi} & H^1(G(F\phi),U^{n,F\phi}) & \xrightarrow{F\phi^A} \\
\downarrow{\lambda} & & \downarrow{\lambda} & \downarrow{ev_{u-1}} \\
A(\overline{S})_1 \times \prod_1^n A(\overline{S})_i & \xrightarrow{F\phi^A} & A(F\phi)_1 \times \prod_1^n A(F\phi)_i & \\
\end{array}
\]

Here $F\phi^A$ denotes the map induced by filtered $\phi$ realization on the level of graded Hopf algebras; see Section 5 below for the definitions of the maps $\lambda$ and $ev_{u-1}$, as well as a discussion of the maps which form Kim’s cutter.

1.5. The problem of constructing bases for the vector spaces $A(\overline{S})_i$ is well known. Candidate elements are provided by Goncharov’s theory of motivic iterated integrals (see, for instance, [Gon2]). Goncharov conjectured that in the two extreme cases $\overline{S} = \text{Spec } \mathbb{Z}$ and $\overline{S} = \text{Spec } \mathbb{Q}$, each $A(\overline{S})_i$ is spanned by iterated integrals on the plane, suitably punctured (c.f. his ICM lectures [Gon1]). The case $\overline{S} = \text{Spec } \mathbb{Z}$ is now a theorem due to Francis Brown [Bro2]. It is reasonable to hope that $A(\overline{S})$ is exhausted by iterated integrals in certain intermediate cases as well. One case has been known since before Brown’s theorem: the case $\overline{S} = \text{Spec } \mathbb{Z} \setminus \{2\}$ follows from Deligne’s work [Del2].

1.6. After a brief review of free prounipotent groups, our work begins by exploring the possibility that a certain condition on $\overline{S}$, which makes use of the Archimedean absolute value, suffices to ensure exhaustion of $A(\overline{S})$ by iterated integrals. We offer a modicum of evidence for this possibility coming from level $n = 2$ in Proposition 3.4. The Archimedean condition is that $\overline{S}$ be of the form $\mathbb{Z} \setminus \{\text{primes} \leq m\}$.

Proposition. Suppose $n = 1$ or 2, and suppose $\overline{S}$ obeys the Archimedean condition. Then $A(\overline{S})_n$ is spanned by iterated integrals on $X$.

1.7. In a section devoted to $p$-adic periods, we discuss the category of mixed Tate filtered $\phi$ modules over $\mathbb{Q}_p$. One of the main goals is the construction of a certain special element $u \in U(F\phi)$. We also introduce the notion of a filtered $\phi$ iterated integral, a filtered $\phi$ analog of Goncharov’s motivic iterated integrals. These are compatible with motivic iterated integrals via filtered $\phi$ realization on the one hand, and give rise to Coleman functions by evaluation on the special element $u$ on the other.

1.8. Next, we turn to the construction of the diagrams (G) and (L). In order to construct the map $\lambda$, we make two preliminary constructions. The first is a proposition in nonabelian cohomology (segments 5.2–5.2.3): the second is Deligne’s representation $\rho^D$, a map $U^n \rightarrow U^{(n+1)\times(n+1)}$ to the group of endomorphisms of $\bigoplus_1^{n+1} \mathbb{Q}(i)$ which are unipotent with respect to the weight filtration (Segment 5.3). According to the proposition, we have

\[H^1(G(S),U^n) = \text{Hom}_{\mathbb{G}_m}(U(S),U^n)\,.
\]

We define $\lambda$ (Segment 5.4.1) by sending a $\mathbb{G}_m$-equivariant homomorphism $\rho : G(S) \rightarrow U^n$.
to its composite with a certain set of coordinates on $U^n$ which arise as appropriate matrix entries in Deligne’s representation. As the name suggests, the map $e_{v^{-1}}$ is given by evaluation at $u^{-1} \in U(F\phi)(\mathbb{Q}_p)$. The map $\kappa$ of Kim’s cutter is given roughly by sending an integral point $b \in X(S)$ to the path torsor $bP_0$ \cite{Kim2}. The map $\alpha$ is basically the “higher $p$-adic Albanese map” considered by Furusho \cite{Furusho1, Furusho2} \cite{Furusho6}. Of course, both $\alpha$ and $\kappa$ need to be projected from the full unipotent fundamental group onto our quotient of choice $U^n$; we give an explicit formula for the projection, and an ensuing formula for $\alpha$ based on Furusho’s computations in Segment \cite{Furusho6}.

1.9. Armed with these constructions, we may give a sketch of our future algorithm \cite{DCW1}. At each level $n$, it has two main steps. In step one, it searches among the motivic iterated integrals on the affine line, suitably punctured, for bases of $A(S)_i$, $i \leq n$; in step two, it produces equations for the image of $\lambda$ in terms of the corresponding coordinates. An important point is that step one will not depend on making effective use of Borel’s computations of higher $K$-groups. In this sense, our construction here is less precise than the one made in \cite{DCW1} where we made effective use of the vanishing of $K_2(\mathbb{Q}) \otimes \mathbb{Q}$ to construct a preferred basis of $A(S)_2$.

If our program comes to fruition, then we will have at our disposal an algorithm for computing the set $X(S)$. Our algorithm will be usable in practice, but with one caveat — it will not be guaranteed to halt. The main theorem of explicit Chabauty-Kim theory for the thrice punctured line, as we now see it, will have two parts. (1) If the algorithm halts for the input $S$, then the output is equal to the set $X(S)$ of solutions to the $S$-unit equation. (2) If Kim’s conjecture holds for $S$, and exhaustion by iterated integrals holds for an open subscheme $\mathfrak{S} \subset S$,

then the algorithm halts for the input $S$.

1.10. The remainder of the article is devoted to our two examples. The case $S = \mathrm{Spec} \mathbb{Z}$, although new, does not depend on the constructions outlined above. Our result does depend, however, on the conjectured nonvanishing

$$\zeta^p(n) \neq 0$$

for $n$ odd $\geq 3$. An immediate corollary of Proposition \cite{DCW1} is the following

**Theorem.** Suppose $S = \mathrm{Spec} \mathbb{Z}$. Assume $\zeta^p(n) \neq 0$ for $n$ odd $\geq 3$. Then the ideal of Coleman functions defining the locus

$$\alpha^{-1}\left( F\phi\left( H^1(G(S), U^\infty) \right) \right) \subset X(\mathbb{Z}_p)$$

is generated by the functions

$$\log^p z \quad \log^p (1 - z) \quad \text{and} \quad \text{Li}^p_n(z) \text{ for } n \text{ odd.}$$

This result should be compared with Section 4 of *A nonabelian conjecture* \cite{BDCKW}, where Kim’s conjecture

$$\emptyset = \alpha^{-1}\left( F\phi\left( H^1(G(S), U^2) \right) \right)$$

for $S = \mathrm{Spec} \mathbb{Z}$ is shown to hold already in depth $n = 2$ for many primes $p$. 

1.11. The case $S = \text{Spec } \mathbb{Z} \setminus \{2\}$, $n = 4$ ($p$ arbitrary) is substantially more interesting. Segments 7.1–7.2.4 are devoted to constructing a basis of $A(S)_i$ for each $i \leq 4$. A fundamental computational tool for working with the vector spaces $A(S)_i$ is the family of complexes

$$A(m) = 0 \to A_m \to \bigoplus_{i+j=m, i,j \geq 1} A_i \otimes A_j \to \cdots \to A_1 \otimes A_0 \to 0$$

as well as the formula

$$H^i(A(m)) = \text{Ext}^i(\mathbb{Q}(0), \mathbb{Q}(m))$$

(see [BGSV] Segment 3.16), which holds in any mixed Tate category. If we restrict attention to classical zeta values and classical polylogarithms, then Deligne’s representation may be used to give a very simple formula for the first boundary map, which is just the reduced coproduct (that is, the coproduct minus its components in $A_0 \otimes A_m$ and $A_m \otimes A_0$); compare our formula 7.1.2 with Goncharov’s Theorem 1.2 of Galois symmetries [Gon2]. A second tool at our disposal is the filtered $\phi$-realization morphism

$$F\phi : A(S)_i \to A(F\phi)_i.$$

1.12. Recall that $UU(X)^{dR} = UU(X)^w$ is canonically of the form $\mathbb{Q}\langle\langle x, y \rangle\rangle$. Here $x$ corresponds to monodromy around 0 while $y$ corresponds to monodromy around 1. For $w$ a word in $x, y$, and $b$ either an integral point or a tangent vector at a puncture, we define $\text{Li}_w(b)$ to be the motivic iterated integral of $w$ from 0 to $b$; see Segment 7.1.3 for a precise definition. We set

$$\log b = \text{Li}_x b, \quad \text{Li}_i b = -\text{Li}_{x+y} b,$$

and

$$\zeta(i) = \text{Li}_i(1).$$

The main result here is as follows (7.2.4).

**Proposition.** The set $\{(\log 2)^4, (\log 2)^3, \zeta(3), \text{Li}_4(1/2)\}$ forms a basis of $A(\text{Spec } \mathbb{Z} \setminus \{2\})_4$.

A large part of what makes the case $n = 4$ interesting the role played by $\text{Li}_4$ of an $S$-integral point.

1.13. If we fix arbitrary generators of $U(S)$, these endow both $H^1(G(S), U^n)$ and $A(S)_i$ with coordinates; computing the image of

$$\lambda : H^1(G(S), U^n) \to A(S)_1 \times \prod_{i=1}^4 A(S)_i$$

in terms of these is a purely formal matter (Segments 7.2.5–7.3.1). More interesting is the problem of rewriting the resulting equations in terms of our concrete polylogarithmic basis (7.3.3). In doing so, we make use of the identity

$$\text{Li}_3(1/2) = \frac{7}{8} \zeta(3) + \frac{1}{6} (\log 2)^3,$$

in which $\frac{7}{8}$ is a rational number $p$-adically close to $7/8$ for several small $p$; see Segment 7.2.3.4.

The final result consists of a pair of explicit Coleman functions on $X(\mathbb{Z}_p)$ (7.4).
Theorem. Assume $\zeta_p(3) \neq 0$\footnote{This is known for $p$ regular, and conjectured in general. See Example 2.19(b) of Furusho [Fur1].} Suppose $S = \text{Spec} \mathbb{Z} \setminus \{2\}$ and $n = 4$. Then the ideal of Coleman functions defining the locus

$$\alpha^{-1} \left( F_\phi \left( H^1(G(S), U^n) \right) \right) \subset X(\mathbb{Z}_p)$$

is generated by the functions

$$F_2(z) = \text{Li}_2^p(z) + \frac{1}{2}(\log^p z \log(1 - z))$$

and

$$F_4(z) = \text{Li}_4^p(z) + \frac{8}{7} CP^p(\log^p z) \text{Li}_3^p(z) + \left( \frac{4}{21} C^p + \frac{1}{24} \right) (\log^p z)^3 \log^p (1 - z),$$

where

$$C^p = \frac{(\log^p 2)^3}{24 \zeta^p(3)} + \frac{\text{Li}_4^p(1/2)}{(\log^p 2) \zeta^p(3)}.$$ 

The first was already visible in depth 2, the second is new. As a corollary, we obtain three identities.

Corollary. Suppose $b = 2, 1/2$, or $-1$. Then $F_4(b) = 0$.

1.14. The case $S = \text{Spec} \mathbb{Z} \setminus \{2\}$ provides a new testing ground for Kim’s conjecture. Our computations, documented in Section 6, provide overwhelming evidence that the conjectured equality (1.2(K)) holds for all primes $p$ in the range $3 \leq p \leq 29$, after which the computation time becomes too long. The problem of turning our overwhelming numerical evidence into a proof nevertheless requires more work. This will be subsumed under a general study of approximation of roots of $p$-adic functions defined by iterated integrals in the near future.

1.15. Actually, following arguments given to us by Hidekazu Furusho, we can prove that $\frac{7}{8} = \frac{7}{8}$. We relegate this to an appendix in order to emphasize the fact that in working out our future algorithm in the present special case, we don’t need to rely on this equality.

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2. Free prounipotent groups

2.1. We make constant use of free prounipotent groups throughout this work. We begin with a brief review of their structure. Let $k$ be a field, and $N$ a set. For any $k$-algebra $R$ we let $R(\langle N \rangle)$ denote the ring of noncommutative formal power series in $N$, endowed with the comultiplication induced by $\Delta(n) = 1 \otimes n + n \otimes 1$ for $n \in N$. An element $F \in R(\langle N \rangle)$ is grouplike if

$$\Delta(F) = F \otimes F$$

and Lie algebra like if

$$\Delta(F) = 1 \otimes F + F \otimes 1.$$
2.2. Let $I$ denote the augmentation ideal of $k\langle\langle N\rangle\rangle$, and let $A$ denote the topological dual 
\[ k\langle\langle N\rangle\rangle^\vee = \lim_{\to} (k\langle\langle N\rangle\rangle/I^n)^\vee. \]

Let $U$ denote the group 
\[ R \mapsto \{ \text{group-like elements in } R\langle\langle N\rangle\rangle \}. \]

Let $\mathfrak{n}$ denote the space of Lie algebra like elements in $k\langle\langle N\rangle\rangle$. Then $U$ is a proalgebraic $k$-group with coordinate ring $\mathcal{O}(U) = A$, Lie algebra 
\[ \text{Lie } U = \mathfrak{n}, \]
and completed universal enveloping algebra 
\[ \mathcal{U}U = k\langle\langle N\rangle\rangle. \]

We call $U$ the free prounipotent group on $N$. It has the implied universal mapping property with respect to (pro)unipotent groups over $k$. In particular, a representation of $U$ on a finite dimensional $k$-vector space $E$ is the same as a nilpotent representation of $\mathfrak{n}$, which is the same as a map of sets 
\[ r : N \to \mathfrak{gl} E \]
whose image is contained in a nilpotent subalgebra.

2.3. If $P$ is a $U$-torsor (necessarily trivial), we let $\mathcal{U}P$ denote its universal enveloping module: 
\[ \mathcal{U}P := \mathcal{U}U \times_U P, \]
a free $\mathcal{U}U$-module of rank one.

3. Exhaustion of mixed Tate motives by iterated integrals

3.1. Review of motivic iterated integrals. We begin by recalling the construction of motivic iterated integrals given by Goncharov \cite{Gon2}.

3.1.1. As in the introduction, we let $\mathfrak{S}$ denote a finite set of prime numbers and we let $\mathfrak{S} = \text{Spec } \mathbb{Z} \setminus \mathfrak{S}$. Let $\text{Mot}(\mathfrak{S})$ denote the category of mixed Tate motives over $\mathfrak{S}$ with $\mathbb{Q}$-coefficients. $\text{Mot}(\mathfrak{S})$ possesses a canonical fiber functor 
\[ \omega : \text{Mot}(\mathfrak{S}) \to \text{Vect}(\mathbb{Q}) \]
given by 
\[ \omega(E) = \bigoplus_i \text{Hom} \left( \mathbb{Q}(-i), \text{gr}^{W}_{2i} E \right). \]

Claim. The canonical fiber functor $\omega$ is canonically isomorphic to the de Rham fiber functor $dR$.

Proof. If $E$ is a mixed Tate motive over $\mathfrak{S}$, then $E^{dR}$ is a $\mathbb{Q}$-vector space equipped with Hodge and weight filtrations. From the fact that the associated mixed Hodge structure is mixed Tate, it follows that the map 
\[ \bigoplus_i F^i E^{dR} \cap W_{2i} E^{dR} \to E^{dR} \]
is bijective. Subsequently, the composite 
\[ F^i E^{dR} \cap W_{2i} E^{dR} \subset W_{2i} E^{dR} \to \text{gr}^{W}_{2i} E^{dR} = \text{Hom}_{\text{Vect}(\mathbb{Q})} \left( \mathbb{Q}(-i)^{dR}, \text{gr}^{W}_{2i} E^{dR} \right) \]
\[ = \text{Hom}_{\text{Mot}(\mathfrak{S})} \left( \mathbb{Q}(-i), \text{gr}^{W}_{2i} E \right) \]
is bijective, which completes the construction. \qed
**Definition.** We define the *mixed Tate fundamental group of* $\mathfrak{S}$ *to be the group*

$$G(\mathfrak{S}) = \mathbb{G}_m \ltimes U(\mathfrak{S})$$

*associated by Tannaka duality to the canonical fiber functor* $\omega$.

3.1.2. Let $S'$ denote a finite subset of $\mathbb{Q}$, let $w$ denote a word of length $n$ in the elements of $S'$ and let $a, b$ be either $\mathbb{Q}$-rational points of $\mathbb{A}^1 \setminus S'$ or tangent vectors to $\mathbb{A}^1$ at the points of $S'$. We wish to construct an element

$$\int_a^b w|_{U(S)}$$

of the $n^{th}$ graded piece $A(S)$ of $A(\text{Spec} \mathbb{Q})_n$. This element will always be contained $A(S)$ for some $S \subset \text{Spec} \mathbb{Z}$ open; this will hold, for instance, if $\mathfrak{S}$ is contained in the locus of good reduction for $\mathbb{A}^1 \setminus S'$, $a$, and $b$. We write

$$\int_a^b w|_{U(S)}$$

for the iterated integral above, considered as an element of $A(S)$.

3.1.3. We let $U_a = U_a(\mathbb{A}^1 \setminus S')$ denote the unipotent fundamental group of $\mathbb{A}^1 \setminus S'$ at $a$, we let $bP_a = bP_a(\mathbb{A}^1 \setminus S')$ denote the unipotent path torsor, we let $UU_a$ denote the completed universal enveloping algebra, and we let $bP_a$ denote the completed universal enveloping module, a $UU_a$-module object of $\text{Mot}(\mathfrak{S})$.

3.1.4. The de Rham realizations $U^{\text{dR}}_a$, $bP^{\text{dR}}_a$ were studied extensively for instance by Deligne [Del1] (start with segment 15.52). Similar conclusions were reached in a somewhat different way by Goncharov (see in particular Proposition 3.2 of [Gon2]). The path torsor $bP^{\text{dR}}_a$ is canonically trivialized by a certain $\mathbb{Q}$-point $b1_a \in bP^{\text{dR}}_a(\mathbb{Q})$. In terms of Deligne’s Tannakian interpretation of $bP^{\text{dR}}_a$, this special path may be constructed as follows: each unipotent flat bundle $(E, \nabla)$ on $\mathbb{A}^1_{\mathbb{Q}} \setminus S'$ extends uniquely to a flat bundle $(\tilde{E}, \tilde{\nabla})$ on $\mathbb{P}^1_{\mathbb{Q}}$ with log poles along $S' \cup \{\infty\}$; the maps

$$E(a) \leftarrow E(\mathbb{P}^1) \rightarrow E(b)$$

are bijective, and $b1_a(E)$ is the composite

$$E(a) \rightarrow E(b).$$

Turning to the fundamental group $U^{\text{dR}}_a$ itself, it is canonically free prounipotent on $S'$. Hence

$$UU^{\text{dR}}_a = \mathbb{Q}\langle\langle S'\rangle\rangle.$$

The weight filtration is given by

$$W_{-2d}UU^{\text{dR}}_a = \mathbb{Q}\langle\langle S'\rangle\rangle_{\geq d},$$

the subspace spanned by words of length not less than $d$. The Hodge filtration is given by

$$F^{-i}UU^{\text{dR}}_a = \mathbb{Q}\langle\langle S'\rangle\rangle_{\leq i}.$$

Finally, the trivialization

$$bP^{\text{dR}}_a = bP^{\text{dR}}_a$$

is compatible with Hodge and weight filtrations. In particular, we have

$$\left(F^0bP_a(\mathbb{Q})\right) \cap bP_a(\mathbb{Q}) = \{b1_a\}.$$
3.1.5. According to one point of view, the iterated integral
\[ \int_a^b w \]
begins its life as a function on the de Rham path torsor \( bP^\text{dR}_a = bP^\text{dR}_a (\mathbb{A}^1 \setminus S') \). Recall that the space of functions \( O(bP^\text{dR}_a) \) is equal to the topological dual \( (U_b P^\text{dR}_a)^\vee = \lim_{\rightarrow} (\mathbb{Q} \langle S' \rangle / I^n) \) of the universal enveloping algebra. We define the iterated integral above to be the function corresponding to the linear functional dual to \( w \).

3.1.6. Let \( o(1_a) : U(\text{Spec} \mathbb{Q}) \to bP^\text{dR}_a \) denote the orbit map. Then we define
\[ \int_a^b w|_{U(\text{Spec} \mathbb{Q})} := o(1_a)^2 \left( \int_a^b w \right). \]
In a language more native to the land of mixed Tate categories, we have
\[ \int_a^b w|_{U(\text{Spec} \mathbb{Q})} = (U_b P^\text{w} a, b1_a, f_w), \]
the framed object of \( \text{Mot}(\mathcal{S}) \) given by
\[ \mathbb{Q} \xrightarrow{\mathcal{L}_1} \text{gr} W_0 U_b P^\text{w} a \xrightarrow{\text{gr} W_{-2n}} U_b P^\text{w} a \xrightarrow{f_w} \mathbb{Q}, \]
where \( f_w \) denotes the linear functional dual to \( w \).

3.2. Definition. Given \( \mathcal{S} \subset \text{Spec} \mathbb{Z} \) and \( n \in \mathbb{N} \), we may ask if \( A(\mathcal{S})_n \) is spanned by elements of the form \( \int_a^b w|_{U(\mathcal{S})} \). If so, we say that exhaustion by iterated integrals holds for \( \mathcal{S} \) at level \( n \).

3.3. In his ICM lectures \cite{Gon}, Goncharov states the conjectures that exhaustion holds for \( \mathcal{S} = \text{Spec} \mathbb{Z} \) and for \( \mathcal{S} = \text{Spec} \mathbb{Q} \). As mentioned in the introduction, the former is now a theorem due to Francis Brown \cite{Bro2}, while the case \( \mathcal{S} = \text{Spec} \mathbb{Z} \setminus \{2\} \) follows from Deligne’s work \cite{Del2}. Moreover, in the latter case we have
\[ S' = \{1, 0, -1\}. \]

3.4. Naive hope. For \( S' \) arbitrary, exhaustion is not likely to hold, but we believe that the following condition on \( \mathcal{S} \) may be sufficient.

Archimedean condition. \( \mathcal{S} \) is of the form \( \mathcal{S} = \text{Spec} \mathbb{Z} \setminus \{\text{primes} \leq m\} \) for some \( m \in \mathbb{N} \).

Moreover, we believe it may be sufficient to consider only iterated integrals on
\[ \mathbb{A}^1 \setminus \{0, 1, \text{primes} \leq m\}. \]

3.5. In this direction, we have the following proposition, which follows easily from our work \cite{DCW1}.
Proposition. Suppose $n = 1$ or $2$, and suppose $S$ obeys the Archimedean condition. Then exhaustion holds for $S$ at level $n$. Moreover, we may set

$$S' = \{0, 1\}$$

and consider only the logarithmic and dilogarithmic integrals

$$\log b = \int_0^b 0 \quad \text{and} \quad \text{Li}_2 b = \int_0^b 01$$

for $b \in X(S)$, as well as products of pairs of logarithms.

Proof. We know that $A(S)_1$ is spanned by $(\log q')'$s, for $q' \in S$. This establishes the first assertion for $n = 1$, with no assumption on $S$. However, the requirement

$$S' = \{0, 1\}$$

of the Moreover clause places a harsh condition on the integral points which may intervene. Special for $S$ obeying the Archimedean condition is that $q' - 1$ is a product of powers of primes $\in S$; hence $q'$ is an $S$-valued point of

$$X = A^1 \setminus \{0, 1\}.$$

Tate’s computation of $K_2(\mathbb{Q})$ tells us that

$$A(S)_1 \otimes A(S)_1 = \mathbb{Z}[S^{-1}]_\mathbb{Q} \otimes \mathbb{Z}[S^{-1}]_\mathbb{Q}$$

is spanned by elements of the forms

$$u \otimes v + v \otimes u$$

and

$$t \otimes (1 - t).$$

Here, again, we use the Archimedean condition on $S$. Now we have an isomorphism

$$A_2 \cong A_1 \otimes A_1$$

which sends

$$uv \mapsto u \otimes v + v \otimes u$$

and

$$- \text{Li}_2(t) \mapsto t \otimes (1 - t).$$

So our generators of $A_1 \otimes A_1$ lift to (products of) motivic iterated integrals on the punctured line. $\square$

4. $p$-adic periods

4.1. A mixed Tate filtered $\phi$-module, in its natural context, is a mixed Tate object of the category of weakly admissible filtered $\phi$ modules. Chatzistamatiou-Ünver [CU] explain how this definition can be greatly simplified. If we restrict attention to mixed Tate filtered $\phi$-modules over $\mathbb{Q}_p$, we can give an even simpler definition, which, moreover, expresses the symmetry between the Hodge and Frobenius structures in this case.

Definition. A mixed Tate filtered $\phi$-module over $\mathbb{Q}_p$ is a vector space $E$ equipped with an increasing filtration $W$ indexed by $2\mathbb{Z}$ called the weight filtration, plus two decreasing filtrations denoted $F$ and $F'$ both indexed by $\mathbb{Z}$. $F$ is called the Hodge filtration, $F'$ is the filtration induced by Frobenius. These are required to be opposite to the weight filtration, meaning that the composite maps

$$F^i E \hookrightarrow E \twoheadrightarrow E/W_{2(i-1)}E$$

are bijective for all $i$. We set

$$E_i := F^i E \cap W_{2i}E$$

and

$$E'_i := F'^i E \cap W_{2i}E.$$
4.2. Mixed Tate filtered $\phi$ modules form a mixed Tate category, which we denote by $F\phi(Q_p)$. Morphisms are simply morphisms of vector spaces respecting all three filtrations. The unit object $Q_p(0)$ is given by $Q_p$ in degree 0 for each filtration. The tensor product is defined by taking tensor product filtrations in the usual way. The Tate object $Q_p(1)$ is $Q_p$ in degree $-1$ for each filtration. As with any mixed Tate category, there is a canonical fiber functor
\[ \omega_{\text{can}}: F\phi(Q_p) \to \text{Vect}(Q_p) \]
given by
\[ \omega_{\text{can}}(E) = \bigoplus_i \text{Hom}(Q_p(-i), \text{gr}^{W_i} E). \]
The associated fundamental group comes equipped with a semidirect product decomposition
\[ G(F\phi) = \mathbb{G}_m \ltimes U(F\phi) \]
with $U(F\phi)$ pronipotent.

4.3. There is a second naturally occurring fiber functor $\omega_{\text{underlying}}$ underlyng on $F\phi(Q_p)$ which sends a mixed Tate filtered $\phi$ module to the underlying vector space. There are also two natural Tannakian paths
\[ \omega_{\text{can}} \rightarrow \omega_{\text{underlying}} \]
connecting the two natural fiber functors. These are given by taking direct sums of the natural bijections
\[ E_i \xrightarrow{p} \text{Hom}(Q_p(-i), \text{gr}^{W_i} E) \xleftarrow{p'} E'_i. \]
Let $u = p \circ (p')^{-1}$. More explicitly, for each object $E$, $u(E)$ is the unique element of $\text{GL}(E)$, unipotent with respect to the weight filtration, such that
\[ u(F') = F. \]
Since $u(Q_p(1)) = 1$, $u$ belongs to the unipotent radical $U(F\phi)$.

4.4. The completed universal enveloping algebra $U(U(F\phi))$ inherits a grading from the $\mathbb{G}_m$-action. Let $v_i$ denote the component of $\exp u$ in graded degree $i$, $i \in \mathbb{Z}_{\leq -1}$. Then it is straightforward to check that $U(F\phi)$ is free pronipotent on the $v_i$’s \cite{22}.

4.5. Let $S'$ denote a finite set of disjoint sections of $\mathbb{A}^1$ over $\text{Spec } Z_p$. By a $\mathbb{Z}_p$-integral base point of $\mathbb{A}^1 \setminus S'$, we mean either a $\mathbb{Z}_p$-point, or a nowhere vanishing tangent vector to a point of $S'$. Consider $\mathbb{Z}_p$-integral base points $a, b$. Assume $S', a, b$ are disjoint. The theory of the de Rham fundamental group of Deligne \cite{Del1} and Wojtkowiak \cite{Woj} has a $p$-adic variant due to Olsson \cite{Ols2, Ols1} which takes values in the category of weakly admissible filtered $\phi$ modules. In the case at hand, the fundamental group $U_{a}^{F\phi} = U_{a}^{F\phi}(\mathbb{A}^1 \setminus S')$ is a unipotent group object in $F\phi(Q_p)$ and the path torsor $bP_{a}^{F\phi}$ is a torsor object in $F\phi(Q_p)$.

4.6. Suppose briefly that the situation of segment \cite{145} comes from a global situation, given by a finite subset $S' \subset \mathbb{Q}$ with good reduction over $S \subset \text{Spec } Z$, and by $S$-integral base points $a, b$ of $\mathbb{A}^1 \setminus S'$. Then it follows from Olsson’s theory \cite{Ols2} that the filtered $\phi$ path torsor $bP_{a}^{F\phi}$ is equal to the filtered $\phi$ realization of the unipotent path torsor $bP_{a}$ in $\text{Mot}(S)$.

4.7. After forgetting the (filtration induced by) frobenius, $U_{a}^{F\phi}$ and $bP_{a}^{F\phi}$ are the usual de Rham fundamental group and path torsor. In particular, the comments of segment \cite{3.1.4} apply with $Q_p$ in place of $\mathbb{Q}$. 
4.8. As in the motivic setting, we define the iterated integral
\[ \int_a^b w \]
to be the function
\[ b P_a^F \phi \rightarrow \mathbb{A}^1_{\mathbb{Q}_p} \]
induced by the linear functional
\[ Q_p(\langle S' \rangle) \rightarrow \mathbb{Q}_p \]
dual to \( w \), and the associated filtered \( \phi \) iterated integral
\[ \int_a^b w|_{U(F\phi)} \]
to be the composite of the iterated integral above with the orbit map
\[ o(b1_a) : U(F\phi) \rightarrow b P_a^F \phi. \]

4.9. Motivic and filtered \( \phi \) iterated integrals are compatible in the following sense. If our local situation comes from a global situation as in paragraph 4.6, then the diagram
\[
\begin{array}{ccc}
U(S)_{\mathbb{Q}_p} & \xrightarrow{o(b1_a)} & U(F\phi) \\
\downarrow & & \downarrow \scriptstyle{o(b1_a)} \\
(b P_a^{dR}(\mathbb{A}^1_Q \setminus S'))_{\mathbb{Q}_p} & \xrightarrow{\int_a^b w_{|U(F\phi)}} & b P_a^{F\phi}(\mathbb{A}^1_{\mathbb{Q}_p} \setminus S')
\end{array}
\]
commutes. In particular, the similarity between the notation of paragraph 4.8 and paragraphs 3.1.5–3.1.6 poses no danger.

4.10. Under the orbit map
\[ o(b1_a) : U(F\phi) \rightarrow b P_a^{F\phi}, \]
the inverse special element \( u^{-1} \in U(F\phi)(\mathbb{Q}_p) \) maps to the unique Frobenius invariant path. Indeed, we have
\[
(F^0 U(U(F\phi)) \cap U(F\phi)) = u^{-1}(F^0 U(U(F\phi)) \cap U(F\phi)) = \{ u^{-1}(b1_a) \}.
\]
It follows from Besser’s interpretation of Coleman integration [Bes] that the function
\[ b \mapsto \int_a^b w|_{U(F\phi)}(u^{-1}) \]
is a Coleman function on \( (\mathbb{A}^1 \setminus S')(\mathbb{Z}_p) \).
5. Blueprints for an algorithm

5.1. Let $S$ denote a finite set of primes, $\overline{S} \supset S$ a possibly larger set of primes. Let $S = \text{Spec} \mathbb{Z} \setminus S$, $\overline{S} = \text{Spec} \mathbb{Z} \setminus \overline{S}$. Let $G(S) = \mathbb{G}_m \ltimes U(S)$ denote the fundamental group of the category of mixed Tate motives over $S$ at the canonical fiber functor, let $U^n$ denote the group object

$$U^n := \mathbb{Q}(1) \ltimes \prod_{1}^{n} \mathbb{Q}(i),$$

let $A(S) = \mathcal{O}(U(S))$. We fix a prime $p \in S$. Let $G(F\phi) = \mathbb{G}_m \ltimes U(F\phi)$ denote the fundamental group of the category of mixed Tate filtered $\phi$ modules over $\mathbb{Q}_p$ at the canonical fiber functor, and let $A(F\phi) := \mathcal{O}(U(F\phi))$. Let $U^{(n+1) \times (n+1)}$ denote the group of endomorphisms of

$$V^n := \bigoplus_{1}^{n+1} \mathbb{Q}(i)$$

which are unipotent with respect to the weight filtration. We then have the two diagrams discussed in the introduction; glued together, they look like so.

![Diagram](attachment:image.png)

Our first goal for this section is to discuss each of the arrows appearing in this diagram. In particular, a certain proposition in nonabelian group cohomology will yield the construction of $\lambda$, and with it the proof of Theorem 1.4, as an immediate corollary.

5.2. A proposition in nonabelian group cohomology. Let $U, U'$ be pronipotent groups over a field $k$, which are equipped with a $\mathbb{G}_m$-action such that the associated Lie algebras $\mathfrak{n}, \mathfrak{n}'$ are graded in purely negative degrees. Let $G = \mathbb{G}_m \ltimes U, G' = \mathbb{G}_m \ltimes U'$. We consider the action of $G$ on $U'$ through the projection

$$\chi : G \to \mathbb{G}_m.$$

**Proposition.** Each equivalence class of cocycles $[c]$ in $H^1(G, U')$ contains a unique representative $c_0$ such that

$$c_0(\mathbb{G}_m) = \{1\};$$

its restriction to $U$ is a $\mathbb{G}_m$-equivariant homomorphism

$$c_0|U : U \to U'.$$

The map

$$[c] \mapsto c_0|_{\mathbb{G}_m}$$

defines a bijection

$$H^1(G, U') = \text{Hom}_{\mathbb{G}_m}(U, U').$$
5.2.1. For the proof, it is convenient to start with an intermediate construction — a bijection
\[ H^1(G, U') = \text{Hom}^\chi(G, G')/U', \]
which we denote by \( c \mapsto \rho_c \). By \( \text{Hom}^\chi \), we mean homomorphisms such that the triangle
\[
\begin{array}{c}
G \\
\downarrow
\end{array}
\rightarrow
\begin{array}{c}
G' \\
\downarrow
\end{array}
\rightarrow
\begin{array}{c}
\mathbb{G}_m \\
\downarrow
\end{array}
\]
commutes; \( U' \) acts by conjugation. Recall that \( Z^1(G, U') \) is the set of morphisms of sheaves \( c : G \rightarrow U' \)
satisfying the cocycle condition
\[ c(g_1g_2) = c(g_1)(g_1c(g_2)) \]
We define \( \rho_c \) by
\[ \rho_c(g) := c(g)\chi(g). \]
It is straightforward to check that this defines a \( U' \)-equivariant bijection
\[ Z^1(G, U') = \text{Hom}^\chi(G, G'), \]
and hence a bijection as stated.

5.2.2. We claim that the set of splittings of the projection \( G' \rightarrow \mathbb{G}_m \) forms a torsor under \( U' \). The latter acts by conjugation. To check transitivity, we apply Segment 5.2.1 with \( U = 1 \) trivial, to obtain
\[ \{\text{splittings}\}/U' = \text{Hom}^\chi(\mathbb{G}_m, G')/U' = H^1(\mathbb{G}_m, U') = \ast, \]
the trivial pointed set. Having checked transitivity, we may check freeness against the given splitting \( \mathbb{G}_m \subset G' \). Suppose \( u' \) is a point of its stabilizer (with values in a \( k \)-algebra \( k' \)). Then for all \( t \in \mathbb{G}_m(k''), \) a \( k' \)-algebra), we have
\[ u'tu'^{-1} = t, \]
so
\[ u' = tu't^{-1}, \]
so \( u' \in U'^{\mathbb{G}_m}(k'') \), which, according to our condition on the grading, is trivial.

5.2.3. It follows from Segment 5.2.2 that every orbit for the action of \( U' \) on \( \text{Hom}^\chi(G, G') \) contains a unique element \( \rho \) for which the reverse triangle
\[
\begin{array}{c}
G \\
\downarrow
\end{array}
\rightarrow
\begin{array}{c}
G' \\
\downarrow
\end{array}
\rightarrow
\begin{array}{c}
\mathbb{G}_m \\
\downarrow
\end{array}
\]
commutes. Using Segment 5.2.1 again, we have
\[ H^1(G, U') = \text{Hom}^\chi(G, G')/U' \]
\[ = \{\text{homomorphisms making both triangles commute}\} \]
\[ = \text{Hom}^{\mathbb{G}_m}(U, U'), \]
which concludes the proof of Proposition 5.2.
5.3. Deligne’s representation. The subrepresentation of the adjoint representation
\[ U^{n+1} \to \text{GL}(V^n) \]
given by the Lie ideal
\[ V^n \subset n^{n+1} = \text{Lie} U^{n+1} \]
in which the factor \( \mathbb{Q}(1) \) is eliminated from the product, factors through
\[ U^{n+1} \to U^n. \]
We refer to the resulting map
\[ \rho^D : U^n \to U^{(n+1)\times(n+1)} \]
as Deligne’s representation. In terms of the natural generators \( x, y \) of \( n^{n+1} \), a basis of \( V^n \) is given by
\[ \{x, (ad x)y, \ldots, (ad x)^n y\}. \]
For \( n \geq 2 \), \( \rho^D \) is faithful. The diagonal maps \( \alpha \) and \( ev_u^{-1} \) as we construct them, will actually land in the image \( \rho^D(U^{n,F\phi}) \).

5.4. Proof of Theorem 1.4. All that remains to be done is to make two simple constructions.

5.4.1. Construction of \( \lambda, \lambda^{F\phi} \). In constructing the maps \( \lambda, \lambda^{F\phi} \), as well as the two maps \( ev_u^{-1} \), we make constant use of the isomorphism of Proposition 5.2 as well as of Deligne’s representation. Applied with \( U = U(S) \) and \( U' = U^{n,\omega} \) the canonical realization of \( U^n \), the proposition reads
\[ H^1(G(S), U^n) = \text{Hom}^{G_m}(U(S), U^{n,\omega}). \]
Let \( \pi_{i,j} \) denote the projection
\[ U^n \to \text{Hom}(\mathbb{Q}(-j), \mathbb{Q}(-i)) = \mathbb{Q}(j-i). \]
For \( \rho \in \text{Hom}^{G_m}(U(S), U^{n,\omega}) \) we define
\[ \lambda(\rho) = ((\pi_{-2,-3}) \circ \rho^D \circ \rho, (\pi_{-1,-2}) \circ \rho^D \circ \rho, (\pi_{-1,-3}) \circ \rho^D \circ \rho, \ldots, (\pi_{-1,-n-1}) \circ \rho^D \circ \rho). \]
The filtered \( \phi \) variant, \( \lambda^{F\phi} \) is defined similarly.

5.4.2. Remark. To clarify the definition, we note that the composite
\[ U^n \xrightarrow{\rho^D} U^{(n+1)\times(n+1), F\phi} \]
is an isomorphism; moreover, each \( \pi_{i,j} \circ \rho^D \) is homogeneous of degree \( j-i \). So \( \lambda \), in other words, simply maps \( \rho \) to its components under a set of homogeneous coordinates on \( U^n \).

5.4.3. Construction of \( ev_u^{-1} \). As the symbols suggest, both maps \( ev_u^{-1} \) are given by evaluation on the inverse special element \( u^{-1} \in U(F\phi)(\mathbb{Q}_p) \) of segment 1.3. Applied to the product \( A(F\phi) \times \prod^I A(F\phi) \), this takes us to affine \( n+1 \)-space; we may then reverse the coordinates of Remark 5.4.2 to obtain a map to \( U^{n,F\phi} \) as shown.\(^2\)

Since commutativity of the lower portion of the diagram is clear, this completes the proof of Theorem 1.4.

\(^2\)Deligne’s representation allows us to work with the endomorphism algebra \( \text{End} V^n \) where otherwise we would have to work in the enveloping algebra \( U^{n,F\phi} \). This will simplify some computations below. With the somewhat awkward formulation here, we pay a small price.
5.5. Review of $\kappa$. We turn our attention to the upper portion of the diagram. Let $U(X) = U_0(X)$ denote the unipotent fundamental group of $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ at the usual tangent vector at 0. Recall the depth filtration $D$ of Deligne-Goncharov [DG], Segment 6.1: we consider the map

$$U(X) \to \mathbb{G}_a$$

induced by the natural inclusion $X \subset \mathbb{G}_m$, we let $D^1$ denote its kernel, and $D^n$ the descending central series of $D^1$. Deligne uses “Deligne’s representation” $\rho^D$ to establish an isomorphism, necessarily canonical,

$$U(X)/D^2 = U^\infty := \mathbb{Q}(1) \times \prod_1^\infty \mathbb{Q}(i).$$

Given $z \in X(S)$, we have the path torsor object $\mathcal{P}_0$, constructed, for instance, in Deligne-Goncharov [DG]. If we push forward to $U^n$ and apply the canonical fiber functor, we obtain a $G(S)$-equivariant $U^n,\omega$-torsor; its isomorphism class gives us a point

$$\kappa(z) \in H^1(G(S), U^n).$$

The filtered $\phi$ variant $\kappa^\phi$ is defined similarly. The commutativity of the upper square is discussed at length in Dan-Cohen–Wewers [DCW1] for $n = 2$; the same discussion applies here, without change.

5.6. Discussion of $\alpha$. We define $\alpha$ to be the composite

$$X(\mathbb{Z}_p) \xrightarrow{\alpha^{\text{full}}} U(X)^{\phi} \xrightarrow{\text{projection}} U^n,\phi$$

where $\alpha^{\text{full}}$ is the unipotent Albanese map of Furusho [Fur1, Fur2]. Our use of $u^{-1}$ (in place of $u$) ensures that the upper triangle commutes.

5.6.1. We wish to give an explicit formula for the projection, and subsequently, an explicit formula for $\alpha$. Recall that $U(X)^{\phi}$ is canonically free on two generators $x, y$ representing monodromy around 0 and 1 respectively. As such, it embeds as the functor of grouplike elements in the noncommutative formal power series ring space

$$\mathbb{A}^1(\langle x, y \rangle) : R \mapsto R(\langle x, y \rangle).$$

We consider the latter as a Hopf algebra for the comultiplication induced by

$$x \mapsto 1 \otimes x + x \otimes 1 \quad y \mapsto 1 \otimes y + y \otimes 1.$$  

Let $n \subset \mathbb{Q}(\langle x, y \rangle)$ denote the Lie algebra of Lie elements, and let $V$ denote the Lie ideal with basis

$$x, [xy], [x^2y], [x^3y], \ldots.$$  

We use the notational convention $[abc] = [a, [b, c]],$ etc.

Claim. The representation

$$U \to \text{GL}(V)$$

induced by the adjoint representation is given by

$$\sum_w L_\omega^w w \mapsto \begin{pmatrix} \cdots & 1 & L_x & 1 & \cdots & \cdots & -L_{xy} & 1 & \cdots & \cdots & 1 & \cdots & \cdots \end{pmatrix}.$$
Proof. This is merely a verification, using the grouplike property of $\sum L_w w$ and beginning with the associated map $n \to gl(V)$ of Lie algebras. The latter is given by

\[
\begin{pmatrix}
\vdots & \vdots & \vdots \\
0 & l_x & -l_{xxxy} \\
0 & l_x & -l_{xy} \\
0 & -l_y & 0
\end{pmatrix}
\]

As a corollary, we obtain the following

Proposition.

\[
\alpha(b) = \begin{pmatrix}
1 & \log^p(b) & \frac{(\log^p(b))^2}{2} & \frac{(\log^p(b))^3}{3!} & \text{Li}^p_1(b) \\
1 & \log^p(b) & \frac{(\log^p(b))^2}{2} & \text{Li}^p_2(b) \\
1 & \log^p(b) & \text{Li}^p_3(b) \\
1 & -\log(1-b) & 1
\end{pmatrix}
\]

5.7. In terms of the construction summarized in the diagram of Segment 5.1, the algorithm we hope to construct will proceed according to the following rubric.

Algorithm (sketch). Initiate a search for integral points. This gives us a gradually increasing list $S' \subset X(S)$ of points. Simultaneously with this search, we compute a $p$-adic approximation of $X(Z_p)_n$; we gradually increase $n$. This, in turn, gives us a gradually decreasing set of possible locations for integral points. The algorithm halts when (and if) all possible locations are accounted for by integral points that have already been found.

The computation of $X(Z_p)_n$ for a given $n$ and a given level of precision takes place in three steps.

5.7.1. We search for a basis of $A(S_i)$ for $i \leq n$ among the motivic iterated integrals which are unramified over $\overline{S}$. Such a search can be made algorithmic by using, for instance, $p$-adic approximations of $p$-adic periods.

5.7.2. We compute the scheme-theoretic image of $\lambda$ in terms of the basis constructed in the previous step. We can then use the resulting equations to find relations between the components of $F \phi^H$, and to translate those into Coleman functions on $X(Z_p)$, without difficulty.

5.7.3. We compute $p$-adic approximations of the roots of those functions. An algorithm constructed by Besser–de Jeu for computing polylogarithms provides a method which can be generalized to arbitrary iterated integrals.
6. **The case** \( S = \text{Spec } \mathbb{Z} \)

6.1. **Proposition.** Assume \( \zeta^p(n) \neq 0 \) for \( n \) odd \( \geq 3 \). Then the image of

\[
H^1\left(G(\mathcal{S}), U^m\right)_{\mathbb{Q}_p} \xrightarrow{\phi^H} H^1\left(G(\Phi), U^{m,F\phi}\right) \xrightarrow{ev_{u-1}} U^{n,F\phi} = \mathbb{Q}_p(1) \times \prod_i \mathbb{Q}_p(i)
\]

consists precisely of the coordinate-plane

\[
\prod_{i \in [3,n] \text{ odd}} \mathbb{Q}_p(i).
\]

**Proof.** For the category of mixed Tate motives over \( \text{Spec } \mathbb{Z} \), we have

\[
H^1(\mathbb{Q}(1)) = H^0(\mathbb{Q}(1)) = 0,
\]

so the long exact sequence associated to the extension

\[
1 \to \prod_i \mathbb{Q}(i) \to U^m \to \mathbb{Q}(1) \to 1
\]

gives us an isomorphism

\[
H^1\left(\prod_i \mathbb{Q}(i)\right) = H^1(U^n).
\]

We also have \( H^1(\mathbb{Q}(i)) = 0 \) for \( i \) even. On the other hand, for \( i \) odd \( \geq 3 \), the map

\[
H^1(\mathbb{Q}(i)) \otimes \mathbb{Q}_p \to \mathbb{Q}_p(i)
\]

has been amply studied: the cohomology group is one-dimensional, generated by a special zeta motive \( \zeta(i) \); its image in \( \mathbb{Q}_p(i) \) is Furusho’s \( p \)-adic special zeta value \( \zeta_p(i) \). \( \square \)

6.2. We may now pull back along \( \alpha \) using Proposition 5.6.1 to obtain Theorem 1.10. The condition \( \zeta^p(n) \neq 0 \) for \( n \) odd \( \geq 3 \) is known for \( p \) regular and conjectured by Iwasawa theorists for \( p \) arbitrary. See Examples 2.19(b) of Furusho [Fur1].

7. **The case** \( S = \{2\} \) **at level** \( n = 4 \)

7.1. **Preliminaries.** Let \( U_0(X) \) denote the unipotent fundamental group of \( X = \mathbb{P}^1 \setminus \{0, 1, \infty\} \) at the usual tangent vector at 0. We sometimes omit either the 0 or the \( X \) from the notation. Let \( 1P_0 \) denote the torsor of paths to the tangent vector \(-1\) at 1. Recall again the depth filtration \( D \) of Deligne-Goncharov [DG], Segment 6.1: we consider the map

\[
U(X) \to G_a
\]

induced by the natural inclusion \( X \subset G_m \), we let \( D^1 \) denote its kernel, and \( D^n \) the descending central series of \( D^1 \); Deligne shows that

\[
U(X)/D^2 = \prod_i \mathbb{Q}(i) \rtimes \mathbb{Q}(1).
\]
7.1.1. **The motivic iterated integrals** \( \kappa_{i,j} \). Let \( \text{Mot}(S) \) denote the category of mixed Tate motives over \( S = \text{Spec} \mathbb{Z}[S^{-1}] \) with \( \mathbb{Q} \)-coefficients, let \( U(S) \) denote the unipotent radical of the fundamental group at the de Rham fiber functor, and let \( A = A(S) \) denote the graded Hopf algebra of functions on \( U(S) \). Given \( b \in X(S) \), we construct functions \( \kappa_{i,j}(b) \in A_{j-i} \). Let \( V \subset \text{Lie} \prod \mathbb{Q}(i) \times \mathbb{Q}(1) \) be the Lie ideal obtained by eliminating the factor \( \mathbb{Q}(1) \) in the product, and let

\[
V_b := b P_1 U(S) \times V
\]
denote the twist of \( V \) by the path torsor \( b P_1 \), which is a representation of the fundamental group \( U_b(X) \) at \( b \). The twist by \( b P_1 \) does not affect the associated graded for the weight filtration, so we have

\[
\text{gr}_b^{-2i} V_b = \mathbb{Q}(i) .
\]

We set

\[
\kappa_{i,j}(b) = (V_b, v_j, f_i)
\]

with \( v_j : \mathbb{Q}(-j) \xrightarrow{\sim} \text{gr}_b^W V_b \) and \( f_i : \text{gr}_b^W V_b \xrightarrow{\sim} \mathbb{Q}(-i) \). We are here referring to the notion of a framed object of a mixed Tate category; see Appendix A of Galois symmetries [Gon2].

7.1.2. **Coprodut formula.** One advantage of passing to Deligne’s representation is that we obtain the following simple formula for the reduced coproduct

\[
d : A_n \to A_1 \otimes A_{n-1} + A_2 \otimes A_{n-2} + \cdots + A_{n-1} \otimes A_1
\]

for the \( \kappa_{i,j} \)'s.

**Lemma.** We have for \( b \in X(S) \) and \( i < j \leq -1 \),

\[
d(\kappa_{i,j}(b)) = \sum_{i<l<k} \kappa_{i,l}(b) \cdot \kappa_{l,j}(b) .
\]

**Proof.** This follows from the usual formula for the coproduct of a framed object. \( \square \)

7.1.3. **Motivic polylogarithms.** Certain special iterated integrals on \( \mathbb{A}^1 \setminus \{0,1\} \) belong to the old tradition of classical polylogarithms, lifted, by the construction of segment 3.1.6, to the motivic setting. For \( b \) an \( S \)-integral base point and \( w \) a word in \( \{0,1\} \) of length \( n \), let us write

\[
\text{Li}_w(b) = \int_0^b w ,
\]

an element of \( A(S)_n \). The traditional notation is then given by

\[
\text{Li}_x(b) =: \log(b) \quad \text{Li}_y(b) =: \log(1-b) \quad -\text{Li}_{x+y}(b) =: \text{Li}_{i+1}(b) .
\]

We add a superscript \( p \) or \( \infty \) to denote the associated \( p \)-adic or complex polylogarithm.

7.1.4. **Proposition.** Arranging the \( \kappa_{i,j} \) as a matrix, we have

\[
\begin{pmatrix}
\kappa_{-2,-2}(b) & \kappa_{-2,-1}(b) & \kappa_{-1,-1}(b)
\end{pmatrix}
= 
\begin{pmatrix}
\kappa_{-2,-2}(b) & \cdots & \kappa_{-1,-1}(b) \\
\kappa_{-2,-1}(b) & \kappa_{-1,-1}(b) & \kappa_{1,0}(b) \\
\kappa_{-1,-1}(b) & \kappa_{1,0}(b) & \kappa_{2,0}(b)
\end{pmatrix}
= 
\begin{pmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots \\
1 & (\log(b))^2/2 & (\log(b))^3/3! & \kappa_{2,0}(b) & \kappa_{3,0}(b) & \cdots \\
1 & \log(b) & (\log(b))^2/2 & \kappa_{1,0}(b) & \kappa_{2,0}(b) & \kappa_{3,0}(b) & \cdots \\
1 & \log(b) & \log(b) & \kappa_{1,0}(b) & \kappa_{2,0}(b) & \kappa_{3,0}(b) & \kappa_{4,0}(b) & \cdots \\
1 & \log(b) & \log(b) & \log(b) & \kappa_{1,0}(b) & \kappa_{2,0}(b) & \kappa_{3,0}(b) & \kappa_{4,0}(b) & \cdots \\
1 & \log(b) & \log(b) & \log(b) & \log(b) & \kappa_{1,0}(b) & \kappa_{2,0}(b) & \kappa_{3,0}(b) & \kappa_{4,0}(b) & \kappa_{5,0}(b) & \cdots \\
1 & \log(b) & \log(b) & \log(b) & \log(b) & \log(b) & \kappa_{1,0}(b) & \kappa_{2,0}(b) & \kappa_{3,0}(b) & \kappa_{4,0}(b) & \kappa_{5,0}(b) & \kappa_{6,0}(b) & \cdots \\
1 & \log(b) & \log(b) & \log(b) & \log(b) & \log(b) & \log(b) & \kappa_{1,0}(b) & \kappa_{2,0}(b) & \kappa_{3,0}(b) & \kappa_{4,0}(b) & \kappa_{5,0}(b) & \kappa_{6,0}(b) & \kappa_{7,0}(b) & \cdots \\
1 & \log(b) & \log(b) & \log(b) & \log(b) & \log(b) & \log(b) & \log(b) & \kappa_{1,0}(b) & \kappa_{2,0}(b) & \kappa_{3,0}(b) & \kappa_{4,0}(b) & \kappa_{5,0}(b) & \kappa_{6,0}(b) & \kappa_{7,0}(b) & \kappa_{8,0}(b) & \cdots \\
\end{pmatrix}
\]
Proof. Let $c_{b,1_0}$ denote the cocycle $U(S) \to U_0(X)^{dR}$ associated to the de Rham realization $\_bP_0(X)^{dR}$ of the path torsor. Let $\rho^{dR}$ denote the de Rham realization of Deligne’s representation $U_0(X)^{dR} \to GL(V^{dR}) = GL\left( \bigoplus_{i=1}^{\infty} \mathbb{Q}(i)^{dR} \right)$.

Then on the one hand, we have $c_{b,1_0} = \sum Li_w(b) w$ as grouplike elements of $A(S)(\langle x, y \rangle)$, and on the other hand, $\rho^{dR} \circ c_{b,1_0} \in GL_{A(S)}(A(S) \otimes V^{dR}) = GL_{A(S)}\left( \bigoplus_{i=1}^{\infty} A(S)(i) \right)$ is the the matrix of $\kappa_{i,j}(b)$’s. So the proposition follows from the calculation of Segment 5.6.1. □

7.1.5. Variant. The above discussion goes thru with a tangent vector in place of the integral point $b$. The only change, aside from the notational one:

$\zeta(w) := Li_w(1)$

is the known vanishing: $\zeta(x) = \zeta(y) = 0$, and $\zeta(n) = 0$ for $n$ even. The result, written in the traditional notation, is as follows.

**Proposition.** We have

$$\begin{pmatrix} \ldots \ldots \ldots \\ \kappa_{-2,-2}(1) \kappa_{-2,-1}(1) \kappa_{-1,-1}(1) \end{pmatrix} = \begin{pmatrix} \ldots \ldots \ldots \\ 1 \zeta(5) \\ 1 \zeta(3) \\ 1 \end{pmatrix}.$$

7.1.6. The functions $f_w$. We let $f_w$ denote the linear functional $A^1(\langle v_{-1}, v_{-2}, \ldots \rangle) \to A^1$ sending a power series to the coefficient of $w$. Recall that $U(F \phi)$ is realized as the space of grouplike elements; we continue to denote by $f_w$ the restriction of $f_w$ to the grouplike elements, which then belongs to $A(F \phi)_n$ with $n$ equal to the degree of $w$ (we are here referring to the graded degree, equal to the sum of the subscripts, or, which is the same, to half the weight). In this notation, a basis of $A(F \phi)_n$ is given by

$$\{f_w \mid w \text{ of degree } n\}.$$

To simplify notation, we write $w$ as a word in $\mathbb{N}$, with “.” denoting multiplication in the power-series ring; so for instance,

$$f_{1.2} = f_{v_{-1}v_{-1}}.$$
7.1.7. The \( p \)-adic period of a motivic polylogarithm. We denote \( F\phi \)-realization

\[
A \to A(F\phi)
\]

by \( F\phi \), placing the symbol \( F\phi \) sometimes on the left, as in \( F\phi(L_i w b) \), and sometimes as an exponent, as in \( L_i F\phi(b) \) or \( (L_i w b)^{F\phi} \). We have

\[
(L_i w b)^{F\phi}(u^{-1}) = L_i^p b,
\]

the \( p \)-adic polylogarithmic value of Furusho \( \text{[Fur1]} \).

7.2. Determination of bases; computation of \( F\phi \). We now begin to work our way up the latter, developing more tools as they become necessary. In Segment \( [7,2] \) \( \text{[n=1,2,3,4]} \) we find a basis of \( A(S)_n \) \( \text{(S = Spec Z \setminus \{2\})} \). For \( n = 1, 2, 3 \) we also compute the matrix associated to \( F\phi \):

\[
A(S)_n \to A(F\phi)_n.
\]

Let us abbreviate \( A(S) =: A \).

7.2.1. A basis of \( A_1 \) is given by \( \{\log 2\} \). A basis of \( A(F\phi)_1 \) is given by \( \{f_1\} \). The \( 1 \times 1 \) matrix associated to \( F\phi_1 \) is given by

\[
(\log 2)^{F\phi}(v_{-1}) = \log^p(2).
\]

7.2.2. Our discussion of the level \( n = 2 \) begins with a preliminary item.

7.2.2.1. Shuffle product. Recall that under the bijection \( A(F\phi) = UU(F\phi)^\vee \), multiplication corresponds to “shuffle product”. Here \( U \) denotes the completed universal enveloping algebra, and the superscript \( \vee \) denotes the topological dual. The terminology is confusing, since in this case, with generators in multiple degrees, there are no shuffles involved. Let us record the formula. If \( w \) is a word, a subword \( w' \) is the same as a subsequence. Its complement \( w'' \) is the complementary subsequence, that is, those letters omitted from \( w \) when passing to \( w' \). Then for \( w \) a word and \( f, g \) linear functionals

\[
A^1_{\langle \langle v_{-1}, v_{-2}, \ldots \rangle \rangle} \to A^1
\]

we have

\[
(fg)(w) = \sum_{\text{subwords } w' \text{ with complement } w''} f(w')g(w'').
\]

One checks that if \( u \in A^1_{\langle \langle v_{-1}, v_{-2}, \ldots \rangle \rangle} \) is grouplike, then

\[
(fg)(u) = f(u)g(u)
\]

as expected.

7.2.2.2. A basis of \( A_2 \) is given by \( (\log 2)^2 \). A basis of \( A(F\phi)_2 \) is given by \( \{f_1, f_2\} \). Using the “shuffle product” of Segment \( [7,2,2,1] \) we find that

\[
(\log F\phi)^2(v_{-1}) = 2((\log F\phi)(v_{-1}))^2 = 2(\log^p 2)^2,
\]

while

\[
(\log F\phi)^2(v_{-2}) = 0,
\]

from which we conclude that the \( 1 \times 2 \) matrix associated to \( (F\phi)_2 \) is given by

\[
\begin{pmatrix}
2(\log^p 2)^2 \\
0
\end{pmatrix}.
\]
7.2.2.3. Identity for $\text{Li}_2(1/2)$. If $b \in X(\mathbb{Z}[S^{-1}])$, then $\text{Li}_2 b = r(\log 2)^2$ with $r \in \mathbb{Q}$. We compute $r$ in two ways. Our first method is to evaluate (the $F\phi$ realization) at $v_{-1}^2$. Since

$$v_{-1}(V_{3,b}) = \begin{pmatrix} 0 & \log p b & 0 \\ 0 & 0 & -\log p (1 - b) \\ 0 & 0 & 0 \end{pmatrix}$$

we find that $(\text{Li}_2 b)(v_{-1}^2) = (v_{-1}(V_{3,b}))_1 = -(\log p b) \log p (1 - b)$, so

$$r = \frac{-(\log p b) \log p (1 - b)}{2(\log p)^2}.$$

Our second method is to apply the boundary map $A_2 \to A_1 \otimes A_1$. We apply the comultiplication map to the $(3, 2, 1)$-framed object $V_{3,b}$ with associated $k$-framed objects $\kappa_{i,j}$ with $k = j - i = 0, 1, 2$ given by

$$\begin{pmatrix} 1 & \log b & \text{Li}_2 b \\ 1 & -\log (1 - b) \\ 1 & \end{pmatrix}$$

and eliminate the edge-terms in $A_0 \otimes A_2 + A_2 \otimes A_0$ to obtain

$$d(\text{Li}_2 b) = -(\log b) \otimes \log (1 - b).$$

On the other hand, we have

$$d((\log 2)^2) = 2(\log 2) \otimes (\log 2),$$

so we again obtain

$$-(\log b) \otimes (\log (1 - b)) = r \cdot 2(\log 2) \otimes (\log 2),$$

as expected. For instance, for $b = 1/2$, we get a well-known identity.

**Lemma.**

$$\text{Li}_2(1/2) = -\frac{1}{2}(\log 2)^2.$$ 

7.2.3. We begin our discussion of the level $n = 3$.

7.2.3.1. Let us record in detail how we go about computing the value $\text{Li}_3^{F\phi}(b)(w)$, for $w \in \{v_{-1}^3, v_{-1}v_{-2}, v_{-2}v_{-1}, v_{-3}\}$ a word of degree 3. The lower-right $4 \times 4$ block

$$M = \begin{pmatrix} 1 & \log b & (\log b)^2/2 & \text{Li}_3 b \\ 1 & \log b & \text{Li}_2 b \\ 1 & -\log (1 - b) \\ 1 & \end{pmatrix}$$

of the matrix appearing in Proposition [7.1.4] reminds us that

$$\text{Li}_3 b = \kappa_{-4,-1}(b)$$

may be represented by the $(-4, -1)^{st}$ matrix entry of the representation $(V_4)_b$, where $V_4$ denotes the quotient

$$\bigoplus_{1}^{4} \mathbb{Q}(i)$$
of $V$. The action $v_i((V_4)_b)$ of $v_i$ is given by the $i$th superdiagonal of $\log(M^{F\phi}(u^{-1}))$. We find that

$$v_{-1}(V_4,b) = \begin{pmatrix} 0 & \log^p b \\ 0 & \log^p b \\ 0 & -\log^p(1-b) \\ 0 & 0 \end{pmatrix}$$

$$v_{-2}(V_4,b) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}(\log^p b) \log^p(1-b) + \text{Li}_2^p b \\ 0 & 0 & 0 \end{pmatrix}$$

$$v_{-3}(V_4,b) = \begin{pmatrix} 0 & 0 & -\frac{1}{12}(\log^p b)^2 \log^p(1-b) - \frac{1}{2}(\log^p b) \text{Li}_2^p b + \text{Li}_3^p b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\text{Li}_3^F(b)(w)$ is given by the $(1,4)$th matrix entry of $w(V_4,b)$. So multiplying the above matrices and reading off the north-eastern corner, we find

$$\text{Li}_3^F(b)(v_{-1}^3) = -(\log^p b)^2 \log^p(1-b)$$

$$\text{Li}_3^F(b)(v_{-1}v_{-2}) = \frac{1}{2}(\log^p b)^2 \log^p(1-b) + (\log^p b) \text{Li}_2^p b$$

$$\text{Li}_3^F(b)(v_{-2}v_{-1}) = 0$$

$$\text{Li}_3^F(b)(v_{-3}) = -\frac{1}{12}(\log^p b)^2 \log^p(1-b) - \frac{1}{2}(\log^p b) \text{Li}_2^p b + \text{Li}_3^p b .$$

7.2.3.2. Proposition. A basis of $A_3$ is given by $\{(\log 2)^3, \zeta(3)\}$.

Proof. We use the complex $A(3)$ [111] to conclude that $\dim A_3 = 2$. It is an old story that $\zeta(3)$ generates the subspace

$$\text{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(3)) \subset A_3 .$$

On the other hand, denoting the coproduct by $\nu$, we have

$$\nu((\log 2)^3) = (\nu(\log 2))^3$$

$$= (1 \otimes (\log 2) + (\log 2) \otimes 1)^3$$

$$= 1 \otimes (\log 2)^3 + 3(\log 2) \otimes (\log 2)^2 + 3(\log 2)^2 \otimes (\log 2) + (\log 2)^3 \otimes 1 ,$$

so

$$d((\log 2)^3) = 3(\log 2) \otimes (\log 2)^2 + 3(\log 2)^2 \otimes (\log 2)$$

$$\neq 0 .$$

Since

$$\ker d = H^1(A(3)) = \text{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(3)) ,$$

the proposition follows. □
A basis of $A(F\phi)_3$ is given by $\{f_{1,3}, f_{1,2}, f_{2,1}, f_3\}$. Applying the above computations, we find that the matrix of $(F\phi)_3$ is given by

$$
\begin{pmatrix}
  (\log^p 2)^3 & 0 \\
  0 & 0 \\
  0 & 0 \\
  0 & \zeta(p^3)
\end{pmatrix}.
$$

7.2.3.4. **Identity for $\text{Li}_3(1/2)$**. Let $b \in X(S)$ be an $S$-integral point ($S = \{2\}$). The computations of segments [7.2.2.3] [7.2.3.1] are also useful for expanding $\text{Li}_3(b)$ as a linear combination

$$
\text{Li}_3(b) = s\zeta(3) + t(\log 2)^3
$$

of our basis elements. Evaluating at $v_3^{-1}$, we find that

$$
t = -\frac{(\log^p b)^2 \log^p (1 - b)}{6(\log^p 2)^3}.
$$

Evaluating at $v_{-3}$, we find that

$$
s = -\frac{1}{12}(\log^p b)^2 \log^p (1 - b) - \frac{1}{2}(\log^p b) \text{Li}_2^p b + \text{Li}_3^p b
$$

$$
= \frac{1}{6}(\log^p b)^2 \log^p (1 - b) + \text{Li}_2^p b
$$

$$
\zeta(p^3).
$$

For instance, for $b = 1/2$, we have

$$
\text{Li}_3(1/2) = -\frac{1}{6}(\log^p 2)^3 + \frac{\text{Li}_2^p(1/2)}{\zeta(p^3)} \zeta(3) + \frac{1}{6}(\log 2)^3.
$$

In particular, we’ve shown that

$$
-\frac{1}{6}(\log^p 2)^3 + \frac{\text{Li}_2^p(1/2)}{\zeta(p^3)}
$$

a priori $\in \mathbb{Q}_p$, is actually $\in \mathbb{Q}$. Computations based on Lip service [BdJ] show that it is $p$-adically close to $7/8$ for several small primes $p$. We will denote this number by $\widetilde{7}/8$. In this notation, we have

$$
\text{Li}_3(1/2) = \widetilde{(7/8)}\zeta(3) + \frac{1}{6}(\log 2)^3
$$

on the motivic level.

7.2.3.5. **Remark.** It is arguably harder to prove that the coefficient of $\zeta(3)$ is nonzero using complex periods. The reason is that the complex periods of motivic polylogarithms in the present sense (as opposed to Brown’s sense) are a priori only well-defined up to periods of lower weight polylogarithms. Nevertheless, this problem has been resolved by Francis Brown [Bro1].

7.2.4. **Proposition.** The set $\mathcal{B} = \{(\log 2)^4, (\log 2)\zeta(3), \text{Li}_4(1/2)\}$ forms a basis of $A_4$.

**Proof.** A glance at the complex $A(4)$ reveals that the dimension is 3. By the computations above, a basis for

$$
A_1 \otimes A_3 + A_2 \otimes A_2 + A_3 \otimes A_1
$$

is given by

$$
\{(\log 2) \otimes \zeta(3), (\log 2) \otimes (\log 2)^3, (\log 2)^2 \otimes (\log 2)^2, (\log 2)^3 \otimes (\log 2), \zeta(3) \otimes (\log 2)\}.
$$

Using Lemma [7.1.2] together with Proposition [7.1.4] to compute reduced coproducts

$$
d : A_4 \to A_1 \otimes A_3 + A_2 \otimes A_2 + A_3 \otimes A_1
$$
we find that

(M) \[ d(B) = \begin{pmatrix} 0 & 1 & -\frac{7}{8} \\ 4 & 0 & -1/6 \\ 6 & 0 & -1/4 \\ 4 & 0 & -1/6 \\ 0 & 1 & 0 \end{pmatrix}. \]

We can verify numerically that \( \frac{7}{8} \neq 0 \), so the proposition follows.

7.2.5. **Summary.** We’ve found that a basis of \( A(S)_{1} \times \prod_{1}^{4} A(S)_{i} \) is given by

(C) \[ \{ (\log 2)', \log 2, (\log 2)^2, (\log 2)^3, \zeta(3), (\log 2)^4, (\log 2)\zeta(3), \text{Li}_4(1/2) \} , \]

where \( (\log 2)' \) denotes \( \log 2 \) regarded as an element of the left-hand copy of \( A_1 \).

7.3. **Computation of image of \( \lambda \).** Our next task is to compute the scheme-theoretic image of the map

\[ \lambda : H^1(G(S), U^n) \to A(S)_{1} \times \prod_{1}^{4} A(S)_{i} \]

in terms of these coordinates.

7.3.1. We begin by considering arbitrary generators \( \nu_i, i \text{ odd } \leq -1 \), of \( U(S) \). These arbitrary generators give us a second, abstract basis of the target space. We write \( \phi_{1,2} \) for the element of \( A(S) \) dual to \( (\nu_{-1})^2 \), and similarly for any word in the \( \nu_i \); in this notation, the abstract basis is given by

(A) \[ \{ \phi_1', \phi_1, \phi_{1,2}, \phi_{1,3}, \phi_3, \phi_{1,4}, \phi_{1,3}, \phi_{3,1} \} . \]

We wish to compute the image of \( \lambda \) in terms of this basis.

7.3.2. We use the notation

\[ \lambda = \lambda_1' \times \lambda_1 \times \lambda_2 \times \lambda_3 \times \lambda_4 \]

for the components of \( \lambda \). We write \( r^D = \text{Lie } \rho^D \). An arbitrary element \( \rho \) of

\[ \text{Hom}^{G_m}(U(S), U^4) \]

may be specified by setting

\[ r^D r(v_{-1}) = \begin{pmatrix} 0 & a & 0 \\ 0 & a & 0 \\ 0 & -b & 0 \end{pmatrix}, \quad r^D r(v_{-3}) = \begin{pmatrix} 0 & 0 \\ 0 & -d \\ 0 & 0 \end{pmatrix}, \]

with \( a, b, d, \in \mathbb{Q} \). We wish to solve

\[ \lambda_1'(\rho) = x_1' \phi_1' \]
for \( x'_1 \in \mathbb{Q} \). In the following diagram, \( \lambda'_1(\rho) \) appears as the composite from upper left to the far right.

\[
\begin{array}{ccc}
U(S) & \xrightarrow{\rho} & U^4 \\
\downarrow & & \downarrow \rho^D & \downarrow \\
\mathfrak{n}(S) & \xrightarrow{R} & \mathfrak{n}^4 \\
\downarrow & & \downarrow R^D & \downarrow \\
\pi_{-2,-3}^5 \mathbb{Q}(i) & \xrightarrow{\pi_{-2,-3}} & A^1 \\
\end{array}
\]

We need to evaluate it on the element \( v_{-1} \in \mathfrak{n}(S) \) on the lower left, not a priori in its domain of definition. To this end we consider the induced maps of enveloping algebras, as in the central row. We find that the central composite \( \pi_{-2,-3} \circ R^D \circ R \) is the unique linear functional on \( \mathfrak{n}(S) \) restricting to \( \lambda'_1(\rho) \) on \( U(S) \), and so we have

\[
x'_1 = (\pi_{-2,-3} \circ R^D \circ R)(v_{-1}) = (\pi_{-2,-3} \circ r^D \circ r)(v_{-1}) = a.
\]

Similarly, by computing appropriate words in the two matrices above and extracting appropriate matrix entries, we find that

\[
\lambda(\rho) = a\phi'_1 - b\phi_1 - ab\phi_{1,2} - a^2b\phi_{1,3} - d\phi_3 - a^3b\phi_{1,4} - ad\phi_{1,3}
\]

If we denote an arbitrary element of \( A_1 \times \prod_1^4 A_i \) by

\[
x'_1\phi'_1 + x_1\phi_1 + x_{1,2}\phi_{1,2} + \cdots + x_{3,1}\phi_{3,1}
\]

then we’ve found that equations for the image of \( \lambda \) are as follows.

1. \( x_{1,2} = x_1x'_1 \)
2. \( x_{1,3} = x_1x'_1^2 \)
3. \( x_{1,4} = x_1x'_1^3 \)
4. \( x_{1,3} = x'_1x_3 \)
5. \( x_{3,1} = 0 \)

7.3.3. Our next task is to rewrite equations \( 7.3.1 \) (1-5) in terms of the concrete basis \( L_{2,5}^1(C) \), allowing ourselves to impose certain conditions on our arbitrary generators \( \nu_i \) of \( U(S) \). Since \( \dim A_1 = 1 \), \( \log 2 \) is a scalar multiple of \( \phi_1 \), so after possibly replacing \( \nu_{-1} \) by a scalar multiple, we may assume

1. \( (\log 2)' = \phi'_1 \)
2. \( \log 2 = \phi_1. \)

The formula \( (\phi_1)^n = n!\phi_{1,n} \) then implies

3. \( (\log 2)^2 = 2\phi_{1,2} \)
4. \( (\log 2)^3 = 6\phi_{1,3}. \)

\[\text{We have learned that the computation which follows is quite similar to the algorithm constructed by Francis Brown in On the decomposition of motivic multiple zeta values [Bro3].}\]
Similarly, $\phi_3$ and $\zeta(3)$ belong to the one-dimensional subspace $\text{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(3))$ of $A_3$, so after possibly replacing $\nu_{-3}$ by a scalar multiple, we may assume
\begin{equation}
(\zeta(3) = \phi_3).
\end{equation}

Subsequently, we have
\begin{equation}
(\log 2)^4 = 24\phi_1.
\end{equation}
\begin{equation}
(\log 2)\zeta(3) = \phi_1 + \phi_3.
\end{equation}
We use our computation (7.2.4) of the injection
d : $A_4 \to A_1 \otimes A_3 + A_2 \otimes A_2 + A_3 \otimes A_1$
to expand $\text{Li}_4(1/2)$ in terms of the abstract basis:
d($\text{Li}_4(1/2)$) = $-7/8\phi_1 \otimes \phi_3 - \phi_1 \otimes \phi_1 - \phi_1 \otimes \phi_3 - \phi_1 \otimes \phi_1$.
\begin{equation}
\text{Li}_4(1/2) = -7/8\phi_1 - \phi_1.
\end{equation}

7.3.4. If we denote an arbitrary linear combination of our concrete basis elements (7.2.5) by
\begin{equation}
y = y_1'(\log 2) + y_i \log 2 + y_1(\log 2)^2 + y_1^3(\log 2) + y_1\zeta(3) + y_1^4(\log 2)^4 + y_1^3(\log 2)\zeta(3) + y_1\text{Li}_4(1/2),
\end{equation}
then in terms of these coordinates, equations (7.3.1) become
\begin{equation}
2y_1 = y_1',
\end{equation}
\begin{equation}
6y_1^3 = y_1^2y_1',
\end{equation}
\begin{equation}
24y_1^4 - y_1 = y_1^3y_1',
\end{equation}
\begin{equation}
-7/8y_1 = y_1^3y_1,
\end{equation}
\begin{equation}
y_1\zeta = 0.
\end{equation}

7.4. **Proof of Theorem 1.13.** The map
\begin{equation}
A(S)_1 \times \prod_{1}^{4} A(S)_i \to A(F\phi)_1 \times \prod_{1}^{4} A(F\phi)_i \to \mathbb{A}^5
\end{equation}
sends $y$ to
\begin{equation}
\begin{pmatrix}
(\log^p 2)y_1' \\
(\log^p 2)y_1 \\
(\log^p 2)^2y_1^2 \\
(\log^p 2)^3y_1^3 + \zeta^p(3)y_1^3 \\
(\log^p 2)^4y_1^4 + (\log^p 2)\zeta(3)y_1^3 + \text{Li}_4(1/2)y_1
\end{pmatrix}
\end{equation}
which, according to equations 7.3.4(1-5), equals
\[
\begin{pmatrix}
(\log^2 y_1^l) y_l^l \\
(\log^2 y_1^l) y_1^l \\
\frac{1}{2} (\log^2 y_1^l) y_1^l y_l^l \\
(\log^2 y_1^l)^3 y_l^l y_l^l + \zeta_p(3) y_l^l \\
\frac{24}{6} (\log^2 y_1^l)^3 y_l^l y_l^l - \frac{8}{7} \left(\frac{\log^2 y_1^l}{24} + \text{Li}_k^p(1/2)\right) y_l^l y_l^l \\
\end{pmatrix}.
\]

Denoting the standard coordinates of \(\mathbb{A}^5\) by \(X_1, Y_1, Y_2, Y_3, Y_4\), we obtain the two equations
\[
Y_2 = \frac{1}{2} X_1 Y_1 \\
Y_4 = \frac{X_1^3 Y_1}{24} - \frac{8}{7} \left(\frac{(\log^2 y_1^l)^3 + \text{Li}_4^p(1/2)\zeta_p(3)}{(\log^2 y_1^l)\zeta_p(3)}\right) \left( X_1 Y_3 - \frac{X_1^3 Y_1}{6} \right).
\]

We pull back along \(X(Z_p) \xrightarrow{\alpha} U^{4, F \phi} \hookrightarrow U^{5 \times 5, F \phi} \xrightarrow{p^n} \mathbb{A}^5\) using Proposition 5.6.1 to obtain the two functions given in the theorem.

8. Experimental verification of Kim’s conjecture for \(S = \{2\}\)

8.1. It is an easy exercise to show that the set \(X(S)\) of \(S\)-integral points of \(X = \mathbb{P}^1 \setminus \{0, 1, \infty\}\) consists of the three points \(1/2, 2, -1\). Kim’s conjecture predicts that for \(n\) sufficiently large we have
\[
X(S) = \alpha^{-1}\left(F \phi(H^1(G(S), U^n))\right) \subset X(Z_p).
\]

We are able to verify this experimentally for \(n = 4\) and for small primes \(p\).

8.2. We use the computer algebra system \texttt{sage}. The code we have written consists of two files, \texttt{localanalytic.sage} and \texttt{Lip.sage} [DCW2]. The file \texttt{localanalytic.sage} contains the definition of a sage class \texttt{pAdicLocalAnalyticFunction}. An object in this class represents a locally analytic function on an open subset of \(\mathbb{A}^1(Z_p) = \mathbb{Z}_p\). In plainer words, this means that the function is defined on a subset of all residue disks, and on each such residue disk it is given by a formal power series over \(\mathbb{Q}_p\). The main feature of these objects is a routine that determines the (finite) set of zeroes of a given function. The file \texttt{Lip.sage} contains an implementation of the algorithm described in [BRJ] which creates polylogarithmic functions \(\text{Li}_k^p\) for \(k = 1, \ldots, n\).

8.3. Here is a sample session where we verify Kim’s conjecture for \(S = \{2\}, n = 4\) and \(p = 11\). The commands
\[
\begin{align*}
\text{runfile localanalytic.sage} \\
\text{runfile Lip.sage}
\end{align*}
\]
load the code. With
\[
p=11 \\
\text{LIP} = \text{make Li}(p, \text{’all’,} 4, 30)
\]
we construct the functions \(\text{Li}_k^p\) for \(k = 1, \ldots, 4\), which are stored in the list \texttt{LIP}. The last parameter is the relative precision used for computations in \(\mathbb{Q}_p\). As a side effect, we also have commands available to compute \(\log^p(x)\) for any \(p\)-adic integer \(x\) and \(\zeta^p(k)\) for \(k = 2, \ldots, 4\). For example:

\[
\begin{align*}
\text{sage:} & \quad \logp(2^{-3}) - 3*\logp(2) \\
0(11^{-30}) & \\
\text{sage:} & \quad \text{zetap}(3)
\end{align*}
\]
\[ 6 \cdot 11^3 + 6 \cdot 11^4 + 9 \cdot 11^5 + 5 \cdot 11^6 + 11^7 + 11^8 + 10 \cdot 11^{10} + 2 \cdot 11^6 + 11^7 + 11^8 + 5 \cdot 11^9 + 11^{10} + 2 \cdot 11^{11} + 2 \cdot 11^{20} + 2 \cdot 11^{21} + 11^{22} + 2 \cdot 11^{23} + 7 \cdot 11^{24} + 9 \cdot 11^{25} + 5 \cdot 11^{26} + 0(11^{27}) \]

Now we check that the rational constant \( \tilde{7}/8 \) from Segment 5.2.3.4 is \( p \)-adically close to \( 7/8 \):

\[
\text{sage: } \text{zetap}(4)
\]
\[0(11^{-33})\]

We create the two functions \( F_1 \) and \( F_2 \) from Segment 5.4.

\[
\text{sage: } R.<x1,y1,y2,y3,y4> = \mathbb{Q}_p(p,30)[x,y]
\]
\[
\text{sage: } c2 = c1^(-1)*(\logp(2)^3/(24*\text{zetap}(3)) + \text{Lip}(4,1/2)/(\logp(2)*\text{zetap}(3)))
\]
\[
\text{sage: } F1 = \text{subs}_\text{pol}(y2-1/2*x1*y1, \text{LIP})
\]
\[
\text{sage: } F2 = \text{subs}_\text{pol}(y4-x1^3*y1/24+c2*(x1*y3-x1^3*y1/6), \text{LIP})
\]

We compute the set of zeroes of both functions:

\[
\text{sage: } Z1 = F1.\text{zeroes}()
\]
\[
\text{sage: } \text{len}(Z1)
\]
\[9\]
\[
\text{sage: } Z2 = F2.\text{zeroes}()
\]
\[
\text{sage: } \text{len}(Z2)
\]
\[17\]

Finally, we check that the set of common zeroes is exactly the set of \( S \)-integral points:

\[
\text{sage: } Z = \text{padic}_\text{common}_\text{zeroes}([F1,F2],20)
\]
\[
\text{sage: } [z.\text{rational}_\text{reconstruction()} \text{ for } z \text{ in } Z]
\]
\[[2, 1/2, -1] \]

8.4. We have run the previous commands for all primes \( p = 3, 5, \ldots, 29 \) and always got the same results:

(a) The constant \( \tilde{7}/8 \) is \( p \)-adically as close to \( 7/8 \) as can be expected from the chosen precision.

(b) The two functions \( F_1, F_2 \) have exactly the set \( \{2, 1/2, -1\} \) of common zeroes.

Even though (a) gives very convincing evidence that \( \tilde{7}/8 = 7/8 \), it does not and cannot provide a proof. In contrast, (b) can in principle be rigourously proved with our methods. In order to do this, we would have to make sure that the results of our calculation hold up to the stated precision (for instance, that \( c_1 \equiv 7/8 \pmod{11^{24}} \) in line 8-9). The point is that we can determine the exact number of zeroes of a \( p \)-adic analytic function from a sufficiently close approximation. Since we know by Theorem 1.8 that the \( S \)-integral points are zeroes of \( F_1 \) and \( F_2 \), we can hope to prove that there are no more.
Appendix

1.1. In this section we give a proof of the relation

\[ \text{Li}_p^3(1/2) = \frac{7}{8} \zeta_p(3) + \frac{1}{6} \log^p(2)^3. \]

As an immediate corollary, we obtain:

\[ \tilde{7}/8 = 7/8. \]

In other words, the motivic identity follows from the corresponding identity of \( p \)-adic periods. See segment 7.2.3.4. The proof, which was suggested to us by H. Furusho, is based on the proof of its complex cousin:

\[ \text{Li}_\infty^3(1/2) = \frac{7}{8} \zeta_\infty(3) + \frac{1}{6} \log^\infty(2)^3 - \frac{1}{2} \log^\infty(2)^2 \zeta_\infty(2), \]

which is proved, for instance, in segment 6.12 of Lewin [Lew].

Proving identities as above over the \( p \)-adics is actually easier than over the complex numbers, because the polylogarithmic functions are, unlike in the complex case, single valued. There is, however, a subtle point having to do with the value of \( \text{Li}_p^n(z) \) at \( z = 1 \) (which is, by definition, \( \zeta_p(n) \)). Explaining this argument in detail is essentially the main point of this section. Our main reference is [Fur1].

1.2. Let \( \mathbb{C}_p \) denote the completion of an algebraic closure of \( \mathbb{Q}_p \). For the remainder of this appendix, we shed the superscript \( p \) above our \( p \)-adic polylogarithms. On the open unit disk, the polylogarithmic functions can be easily defined as convergent power series. In fact, one sets

\[ \text{Li}_n(z) := \sum_{k \geq 1} \frac{z^k}{k^n}, \]

for \( n \geq 1 \) and \( z \in \mathbb{C}_p, |z| < 1 \). It is easy to see that the series converges uniformly on every closed disk \( |z| \leq r \) with radius \( r < 1 \). Therefore, the functions \( \text{Li}_n(z) \) are analytic on the open unit disk. They satisfy the differential equation

\[ \frac{d}{dz} \text{Li}_n(z) = \begin{cases} \frac{1}{z} \text{Li}_{n-1}(z), & n \geq 2, \\ \frac{1}{1-z}, & n = 1. \end{cases} \]

(*)

1.3. Using Coleman’s theory of \( p \)-adic integration, one can extend \( \text{Li}_n \) to functions on \( Y := \mathbb{P}^1(\mathbb{C}_p) - \{0, 1, \infty\} \), see [Fur1]. To do this we choose an arbitrary element \( a \in \mathbb{C}_p \) and let \( \log^a : \mathbb{C}_p^\times \to \mathbb{C}_p \) denote the branch of the \( p \)-adic logarithm defined by \( \log^a(p) := a \). Then there exist unique functions

\[ \text{Li}_n^a : Y \to \mathbb{C}_p \]

with the following properties.

(a) \( \text{Li}_n^a \) is a Coleman function in the sense of [Bes] and [Fur1]. (Note that the definition of Coleman functions depends on the choice of the branch \( \log^a \) of the \( p \)-adic logarithm.)

(b) \( \text{Li}_n^a \) satisfies the differential equation

\[ \frac{d}{dz} \text{Li}_n^a(z) = \begin{cases} \frac{1}{z} \text{Li}_{n-1}^a(z), & n \geq 2, \\ \frac{1}{1-z}, & n = 1. \end{cases} \]

(c) On the open unit disk \( |z| < 1 \), \( \text{Li}^a(z) \) agrees with \( \text{Li}_n(z) = \sum_k z^k/k^n \).

\[ ^4\text{Francis Brown as pointed out to us that the same result could also be deduced directly from the complex identity via the construction of [Bro1].} \]
For instance,

\[ \text{Li}_a^n(z) = -\log^a(1 - z). \]

1.4. **Remark.** Let

\[ V := \{ z \in Y \mid |z|, |1 - z|, |z^{-1}| = 1 \} \subset Y \]

be the complement of the open unit disks around 0, 1, \( \infty \). The restriction of the functions \( \log^a(z) \) and \( \text{Li}_a^n(z) \) to \( V \) are independent of the chosen branch of the logarithm. For a point \( z \in V \) we may therefore write \( \log(z) \) and \( \text{Li}_a^n(z) \).

1.5. In order to define \( p \)-adic zeta values, Furusho has shown that the functions \( \text{Li}_a^n(z) \) have a well-defined value for \( z = 1 \) if \( n \geq 2 \). We need to recall what this means. Let \( x \in \{ 0, 1, \infty \} \) and \( z_x \) be a local parameter at \( x \) (e.g. \( z_0 := z, z_1 := 1 - z, z_\infty = z^{-1} \)). If \( f \) is a Coleman function on \( Y \) then there exists a radius \( 0 < r < 1 \) such that \( f \) is defined on the open admissible subset

\[ U_x(r) := \{ z_x \in \mathbb{C}_p \mid r \leq |z_x| < 1 \} \subset Y. \]

Furthermore, \( f \) admits an expansion of the form

\[ f(z_x) = f_0(z_x) + f_1(z_x) \log^a(z_x) + \ldots + f_m(z_x) \log^a(z_x)^m, \]

where \( f_0, \ldots, f_m \) are analytic functions on \( U_x(r) \). We say that \( f \) is *defined* at \( z = x \) if the functions \( f_0, \ldots, f_m \) extend to analytic functions on the whole open disk

\[ U_x = \{ z_x \in \mathbb{C}_p \mid |z_x| < 1 \} \]

and, moreover, \( f_i(x) = 0 \) for \( i = 1, \ldots, m \). If this is the case, then \( f(x) := f_0(x) \) is called the *value* of \( f \) at \( z = x \).

It is clear that the sum \( f = g + h \) of two Coleman functions is defined at \( x \) if both \( g \) and \( h \) are defined at \( x \). Moreover, if this is the case then \( f(x) = g(x) + h(x) \). Furthermore, the identity principle for Coleman functions implies that a Coleman function \( f \) vanishes identically if and only if \( df = 0 \) and \( f(x) = 0 \), for some \( x \in \{ 0, 1, \infty \} \). This argument will play a crucial role in the proof of Proposition 1.7 below.

1.6. We recall Theorem 2.13 of Furusho [Fur1].

**Theorem (Furusho).** For \( n \geq 2 \), \( \text{Li}_a^n \) is defined at \( z = 1 \), and the value

\[ \zeta(n) := \text{Li}_a^n(1) \in \mathbb{Q}_p \]

is independent of the choice of \( a \). Moreover, \( \zeta(n) = 0 \) if \( n \) is even.

By Remark 1.3 and the independence result above, our final result does not depend on the choice of the branch of the logarithm. From now on, we will therefore ignore the choice of \( a \) and omit it from the notation.

1.7. The following proposition states several functional equations for \( \text{Li}_2 \) and \( \text{Li}_3 \) whose complex counterparts correspond to formulas (1.11), (1.12), (6.10) and (6.4) of Lewin [Lew].
Proposition. The following relations between Coleman functions hold.

(1) \[ \text{Li}_2(z) + \text{Li}_2(1 - z) + \log(z) \log(1 - z) = 0, \]

(2) \[ \text{Li}_2(z) + \text{Li}_2\left(\frac{z}{z - 1}\right) + \frac{1}{2} \log(1 - z)^2 = 0, \]

(3) \[ \text{Li}_3(z) + \text{Li}_3(1 - z) + \text{Li}_3\left(\frac{z}{z - 1}\right) - \frac{1}{6} \log(1 - z)^3 + \frac{1}{2} \log(z) \log(1 - z)^2 = \zeta(3), \]

(4) \[ \text{Li}_3(z^2) - 4 \text{Li}_3(z) - 4 \text{Li}_3(-z) = 0. \]

Proof. To prove (1) we set \( f(z) := \text{Li}_2(z) + \text{Li}_2(1 - z) + \log(z) \log(1 - z). \)

This is a sum of three Coleman function on \( Y \). Each of these functions is defined at \( z = 0 \) and takes the value 0. It follows that \( f(z) \) is itself a Coleman function, defined at \( z = 0 \), and that \( f(0) = 0 \).

We compute its derivative with respect to \( z \), using (1.2*):

\[
\frac{d}{dz} f(z) = -\frac{\log(1 - z)}{z} + \frac{\log(z)}{1 - z} + \frac{\log(1 - z)}{z} - \frac{\log(z)}{1 - z} = 0.
\]

Since \( f(z) \) is locally analytic, this means that \( f(z) \) is constant on every residue disk. But then the identity principle for Coleman functions implies that \( f(z) \) is constant everywhere. Since \( f(0) = 0 \), \( f \) vanishes identically. This proves (1). The proof of (2) is very similar and left to the reader.

To prove (3) we set \( g(z) := \text{Li}_3(z) + \text{Li}_3(1 - z) + \text{Li}_3\left(\frac{z}{z - 1}\right) - \frac{1}{6} \log(1 - z)^3 + \frac{1}{2} \log(z) \log(1 - z)^2. \)

All terms in this sum are Coleman functions which are defined at \( z = 0 \). Moreover, all terms vanish at \( z = 0 \) except for \( \text{Li}_3(1 - z) \) which takes the value \( \zeta(3) \). Therefore, \( g \) is a Coleman function defined at \( z = 0 \) such that \( g(0) = \zeta(3) \). We compute the derivative of \( g \):

\[
\frac{d}{dz} g(z) = \frac{1}{z} \text{Li}_2(z) + \frac{1}{z - 1} \text{Li}_2(1 - z) - \frac{1}{z(z - 1)} \text{Li}_2\left(\frac{z}{z - 1}\right) - \frac{1}{2z(z - 1)} \log(1 - z)^2 + \frac{1}{z - 1} \log(z) \log(1 - z)
\]

From (1) and (2) we obtain expressions for \( \text{Li}_2(1 - z) \) and \( \text{Li}_2(z/(z - 1)) \) in terms of \( \text{Li}_2(z) \), \( \log(z) \) and \( \log(1 - z) \). Plugging these expressions into (3), we obtain, after a short computation,

(6) \[ \frac{d}{dz} g(z) = 0. \]

As before, this shows that \( g \) is constant, and then the identity \( g(0) = \zeta(3) \) proves (3).

The proof of (4) is again very similar. In addition to the arguments used already, one has to use that pullbacks of Coleman functions via rational functions are again Coleman functions. Thus, \( \text{Li}_3(z^2) \) is a Coleman function on \( \mathbb{P}^1(\mathbb{C}_p) \) \( \{0, \pm 1, \infty\} \) which vanishes for \( z = 0 \).

1.8. Corollary. We have

\[ \text{Li}_3(1/2) = 7/8 \zeta(3) + \frac{1}{6} \log(2)^3. \]

Proof. Evaluation of (1.7.3) at \( z = 1/2 \) yields the identity \n
(\star) \[ 2 \text{Li}_3(1/2) + \text{Li}_3(-1) - \frac{1}{3} \log(2) = \zeta(3). \]

Evaluation of (4) at \( z = 1 \) yields \n
(\star\star) \[ \text{Li}_3(-1) = -\frac{3}{4}. \]
Combining (1) and (2) gives the desired result.

\[
\square
\]

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