THE PHASE DIAGRAM OF
THE COMPLEX BRANCHING BROWNIAN MOTION
ENERGY MODEL

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ABSTRACT. We complete the analysis of the phase diagram of the complex branching Brownian motion energy model by studying Phases I, III and boundaries between all three phases (I-III) of this model. For the properly rescaled partition function, in Phase III and on the boundaries I/III and II/III, we prove a central limit theorem with a random variance. In Phase I and on the boundary I/II, we prove an a.s. and $L^1$ martingale convergence. All results are shown for any given correlation between the real and imaginary parts of the random energy.

1. INTRODUCTION

Random energy models (REM) suggested by Derrida [13], [14] turned out to be a useful and instructive “playground” in the studies of strongly correlated random systems on large/high-dimensional state spaces, see, e.g., the recent reviews [29], [23], [11]. In this context, branching Brownian motion (BBM) viewed as a random energy model plays a special rôle. It turns out that BBM has correlations which are exactly at the borderline between the regime of weak correlations (REM universality class $^1$) and the one of strong correlations $^2$. Apart from that, BBM is a particularly transparent representative for a whole class of models with similar (so-called logarithmic) correlation strength: Gaussian free field [51], [9], [10]; Gaussian multiplicative chaos/cascades [50], [7]; characteristic polynomials of random matrices and number-theoretic models [17], [3], [4], cover times [8], etc.

In this paper, we focus on the complex-valued BBM energy model and show that this model lies exactly at the borderline of the complex REM universality class. This means that the phase diagram of the model is the same as in the complex REM, cf. Derrida [15] and [21]. However, the fluctuations of the partition function of this model are already influenced by the strong correlations and differ from those of the REM in all phases of the model, as we show in this work (and in [18]).

The motivation to consider the complex-valued setup is two-fold:

(1) Critical phenomena. Lee and Yang [27] observed that phase transitions (= analyticity breaking of the log-partition function) occur at critical points due to the

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$^1$ = the same phase diagram as for the field of independent random energies.

$^2$ = different phase diagram comparing to the REM one, due to the strictly larger leading order of the minimal energy than the one for the independent field of random energies.
accumulation of complex zeros of the partition function (viewed as a function of the temperature) around the critical points on the real line, as the size of the system tends to infinity (= thermodynamic limit).

(2) Quantum physics and interference phenomena. The formalism of quantum physics is based on the sums (and integrals) of complex exponentials which naturally leads to cancellations between the magnitudes of the summands in the partition function. This is a manifestation of the interference phenomenon, see, e.g., [16].

1.1. Branching Brownian motion. Before stating our results, let us briefly recall the construction of a BBM. Consider a canonical continuous branching process: a continuous time Galton-Watson (GW) process [6]. It starts with a single particle located at the origin at time zero. After an exponential time of parameter one, this particle splits into \(k \in \mathbb{Z}_+\) particles according to some probability distribution \((p_k)_{k \geq 0}\) on \(\mathbb{Z}_+\). Then, each of the newborn particles splits independently at independent exponential (parameter 1) times again according to the same \((p_k)_{k \geq 0}\), and so on. We assume that \(\sum_{k=1}^{\infty} p_k = 1\). In addition, we assume that \(\sum_{k=1}^{\infty} kp_k = 2\) (i.e., the expected number of children per particle equals two). Finally, we assume that \(K := \sum_{k=1}^{\infty} k(k-1)p_k < \infty\) (finite second moment). At time \(t = 0\), the GW process starts with just one particle.

For given \(t \geq 0\), we label the particles of the process as \(i_1(t), \ldots, i_{n(t)}(t)\), where \(n(t)\) is the total number of particles at time \(t\). Note that under the above assumptions, we have \(\mathbb{E}[n(t)] = e^t\). For \(s \leq t\), we denote by \(i_k(s, t)\) the unique ancestor of particle \(i_k(t)\) at time \(s\). In general, there will be several indices \(k, l\) such that \(i_k(s, t) = i_l(s, t)\). For \(s, r \leq t\), define the time of the most recent common ancestor of particles \(i_k(r, t)\) and \(i_l(s, t)\) as

\[
d(i_k(r, t), i_l(s, t)) := \sup\{u \leq s \land r : i_k(u, t) = i_l(u, t)\}.
\]

For \(t \geq 0\), the collection of all ancestors naturally induces the random tree

\[
\mathbb{T}_t := \{i_k(s, t) : 0 \leq s \leq t, 1 \leq k \leq n(t)\}
\]
called the GW tree up to time \(t\). We denote by \(\mathcal{F}_{\mathbb{T}_t}\) the \(\sigma\)-algebra generated by the GW process up to time \(t\).

In addition to the genealogical structure, the particles get a position in \(\mathbb{R}\). Specifically, the first particle starts at the origin at time zero and performs Brownian motion until the first time when the GW process branches. After branching, each new-born particle independently performs Brownian motion (started at the branching location) until their respective next branching times, and so on. We denote the positions of the \(n(t)\) particles at time \(t \geq 0\) by \(x_1(t), \ldots, x_{n(t)}(t)\).

We define BBM as a family of Gaussian processes,

\[
x_t := \{x_1(s, t), \ldots, x_{n(t)}(s, t) : s \leq t\}
\]

indexed by time horizon \(t \geq 0\). Note that conditionally on the underlying GW tree these Gaussian processes have the following covariance

\[
\mathbb{E}[x_k(s, t)x_l(r, t) \mid \mathcal{F}_{\mathbb{T}_t}] = d(i_k(s, t), i_l(r, t)), \quad s, r \in [0, t], \quad k, l \leq n(t).
\]

In what follows, to lighten the notation, we will simply write \(x_i(s) := x_i(s, t), i \leq n(t),\) \(s \leq t\) hoping that this will not cause confusion about the parameter \(t \geq 0\).

\(\text{This implies that } p_0 = 0, \text{ so none of the particles ever dies.}\)
1.2. A model of complex-valued random energies. In this section, we introduce the complex BBM random energy model.

Let $\rho \in [-1, 1]$. For any $t \in \mathbb{R}_+$, let $X(t) := (x_k(t))_{k \leq n(t)}$ and $Y(t) := (y_k(t))_{k \leq n(t)}$ be two BBMs with the same underlying GW tree such that, for $k \leq n(t)$,

$\text{Cov}(x_k(t), y_k(t)) = |\rho| t.$

(1.5)

Note that

$Y(t) \overset{D}{=} \rho X(t) + \sqrt{1 - \rho^2} Z(t),$

(1.6)

where $\overset{D}{=} \text{denotes equality in distribution and } Z(t) := (z_i(t))_{i \leq n(t)}$ is a branching Brownian motion independent from $X(t)$ and with the same underlying GW process. Representation (1.6) allows us to handle arbitrary correlations by decomposing the process $Y$ into a part independent from $X$ and a fully correlated one.

We define the partition function for the complex BBM energy model with correlation $\rho$ at inverse temperature $\beta := \sigma + i\tau \in \mathbb{C}$ by

$X_{\beta, \rho}(t) := \sum_{k=1}^{n(t)} e^{\sigma x_k(t) + i\tau y_k(t)}.$

(1.7)

1.3. Notation. By $\mathcal{L}[\cdot], \mathcal{L}[: \cdot :], \text{and } \implies \text{or } \text{wlim, we denote the law, conditional law, and weak convergence respectively.}$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{phase_diagram.png}
\caption{Phase diagram of the REM and the BBM energy model. The grey curves are the level lines of the limiting log-partition function, cf. (1.9). This paper deals with phases $B_1$ and $B_3$ and the boundaries. For a treatment of phase $B_2$, see \cite{18}.}
\end{figure}

1.4. Main results. Let us specify the three domains depicted on Figure 1 analytically:

$B_1 := \mathbb{C} \setminus \overline{B_2 \cup B_3}, \quad B_2 := \{\sigma + i\tau \in \mathbb{C}: 2\sigma^2 > 1, |\sigma| + |\tau| > \sqrt{2}\},$

$B_3 := \{\sigma + i\tau \in \mathbb{C}: 2\sigma^2 < 1, \sigma^2 + \tau^2 > 1\}.$

(1.8)
Remark. Some of our results will be stated under the *binary branching* assumption (i.e., \( p_k = 0 \) for all \( k > 2 \)). Existence of all moments for the number of children of a given particle would also suffice for all our proofs and will not require essential changes.

Our first result states that the complex BBM energy model indeed has the phase diagram depicted on Figure 1.

**Theorem 1.1** (Phase diagram). For any \( \rho \in [-1,1] \), and any \( \beta \in \mathbb{C} \), the complex BBM energy model with binary branching has the same log-partition function and the phase diagram (cf., Figure 1) as the complex REM, i.e.,

\[
\lim_{t \to \infty} \frac{1}{t} \log \mathcal{X}_{\beta, \rho}(t) = \begin{cases} 
1 + \frac{1}{2}(\sigma^2 - \tau^2), & \beta \in \overline{B}_1, \\
\sqrt{2}|\sigma|, & \beta \in \overline{B}_2, \\
\frac{1}{2} + \sigma^2, & \beta \in \overline{B}_3 
\end{cases} \tag{1.9}
\]

in probability.

See Section 5 for a proof.

**Remark.** It is conjectured that the convergence in (1.9) also holds in \( L^1 \), see [21, Theorem 2.15] for a related result for the REM.

1.5. A class of martingales. In the centre of our analysis are the following martingales

\[
\mathcal{M}_{\sigma, \tau}(t) = e^{-t \left(1 + \frac{2 \sigma^2}{\sigma^2 + \tau^2}\right)} \mathcal{X}_{\beta, \rho}(t) = \sum_{k=1}^{n(t)} e^{-t \left(1 + \frac{2 \sigma^2}{\sigma^2 + \tau^2}\right)} e^{\sigma x_k(t) + i \tau y_k(t)}. \tag{1.10}
\]

Note that, for \( \beta = \sigma \in [0, \frac{1}{\sqrt{2}}) \), \( \mathcal{M}_{\sigma, 0}(t) \) coincides with the McKean martingale introduced in [12], where it was proven that these martingales converge almost surely and in \( L^1 \) to a non-degenerate limit.

The next theorem states that for \( \beta \in B_1 \) the martingales \( \mathcal{M}_{\sigma, \tau}(t) \) are in \( L^p \) for some \( p > 1 \). For \( |\beta| < 1 \), this was already proven in [18, Proposition A.1].

**Theorem 1.2** (\( L^p \) martingale convergence in \( B_1 \)). For \( \beta = \sigma + i \tau \) with \( \beta \in B_1, |\beta| \geq 1 \), and any \( \rho \in [-1,1] \), \( \mathcal{M}_{\sigma, \tau}(t) \) is a martingale with expectation 1 and it is in \( L^p \) for \( p \leq \frac{\sqrt{2}}{\sigma} \). Hence, the limit

\[
\lim_{t \to \infty} \mathcal{M}_{\sigma, \tau}(t) =: \mathcal{M}_{\sigma, \tau} \tag{1.11}
\]

exists a.s., in \( L^1 \), and is non-degenerate.

See Section 2 for a proof.

**Remark.** For \( \beta \in B_1, |\beta| < 1 \), and any \( \rho \in [-1,1] \), it has been proven in [18, Proposition A.1] that \( \mathcal{M}_{\sigma, \tau}(t) \) is \( L^2 \)-bounded.

On the boundary \( B_{1,2} \) between phases \( B_1 \) and \( B_2 \), i.e., on the set

\[
B_{1,2} := \overline{B}_1 \cap \overline{B}_2 = \{ \sigma + i \tau \in \mathbb{C} : |\sigma| > 1/\sqrt{2}, |\sigma| + |\tau| = \sqrt{2} \} \tag{1.12}
\]

a similar result still holds.

**Theorem 1.3** (\( L^p \) martingale convergence on \( B_{1,2} \)). For \( \beta \in B_{1,2} \) and any \( \rho \in [-1,1] \), we have that \( \mathcal{M}_{\beta}(t) \) is a \( L^p \)-bounded martingale, for any \( p < \frac{\sqrt{2}}{\sigma} \) with expectation 1. Hence, the limit

\[
\lim_{t \to \infty} \mathcal{M}_{\sigma, \tau}(t) =: \mathcal{M}_{\sigma, \tau} \tag{1.13}
\]

exists a.s. in \( L^1 \), and is non-degenerate.
Theorem 1.4 (CLT with random variance for $|\sigma| < 1/\sqrt{2}$). Let $\beta = \sigma + i\tau$ with $|\sigma| < 1/\sqrt{2}$ and $\rho \in [-1, 1]$. For $\beta \in B_1$, 

$$\lim_{r \to \infty} \lim_{t \to \infty} \mathcal{L} \left[ \frac{M_{\sigma,\tau}(t + r) - M_{\sigma,\tau}(r)}{e^{r(1-\sigma^2-\tau^2)}} \bigg| M_{2\sigma,0} \right] = \mathcal{N} \left( 0, C_1 M_{2\sigma,0} \right),$$

where $C_1 > 0$ is some constant.

Remark. A result resembling Theorem 1.4(i) was obtained by Iksanov and Kabluchko in [19] for $\beta \in \mathbb{R}$.

Remark. The appearance of the random variance in Theorem 1.4 (and in the following ones) is in sharp contrast with the REM [21] and generalized REM [22], where CLTs with deterministic variance hold for $\beta$ in the strip $|\sigma| < 1/\sqrt{2}$.

Theorem 1.5 (CLT with random variance in $B_3$). For $\beta \in B_3$, $\rho \in [-1, 1]$ and binary branching,

$$\mathcal{L} \left[ \frac{X_{\beta,\rho}(t)}{\sqrt{t(1/2+\sigma^2)}} \bigg| M_{2\sigma,0} \right] \xrightarrow{t \to \infty} \mathcal{N} \left( 0, C_2 M_{2\sigma,0} \right),$$

where $C_2 > 0$ is some constant.

Remark. A similar result also holds on the boundary between phases $B_1$ and $B_3$, i.e., on the set 

$$B_{1,3} := \overline{B}_1 \cap \overline{B}_3 = \{ \sigma + i\tau \in \mathbb{C}: \sigma^2 + \tau^2 = 1, |\sigma| < 1/\sqrt{2} \}.\quad (1.18)$$

Theorem 1.6 (CLT with random variance on $B_{1,3}$). For $\beta \in B_{1,3}$, $\rho \in [-1, 1]$, and binary branching,

$$\mathcal{L} \left[ \frac{X_{\beta,\rho}(t)}{\sqrt{t(1/2+\sigma^2)}} \bigg| M_{2\sigma,0} \right] \xrightarrow{t \to \infty} \mathcal{N} \left( 0, C_3 M_{2\sigma,0} \right),$$

where $C_3 > 0$ is some constant.
Recall that the behaviour of the partition function at $\beta = \sqrt{2}$ is determined by the martingale $M_{1,0}(t)$, which is related to another martingale – the so-called derivative martingale $Z(t)$:

$$Z(t) := \sum_{i=1}^{n(t)} (\sqrt{2}t - x_k(t)) e^{-\sqrt{2}(\sqrt{2}t - x_k(t))},$$  \hspace{1cm} (1.20)

Lalley and Sellke proved in [26] that $Z(t)$ converges a.s. as $t \to \infty$ to a non-trivial limit $Z$ which is a positive and a.s. finite random variable. At the boundary, $B_{2,3} := B_2 \cap B_3 = \{ \sigma + i\tau \in \mathbb{C} : |\sigma| = 1/\sqrt{2}, |\tau| \geq 1/\sqrt{2} \}$, \hspace{1cm} (1.21)

after appropriate rescaling, we have the following CLT with random variance.

**Theorem 1.7** (CLT with random variance for $|\sigma| = 1/\sqrt{2}$). Let $\beta = \sigma + i\tau$ with $|\sigma| = 1/\sqrt{2}$ and $\rho \in [-1, 1]$ and assume binary branching. Then:

(i) For $\tau > 1/\sqrt{2}$,

$$\text{wlim}_{r \uparrow \infty} \text{wlim}_{t \uparrow \infty} \mathcal{L} \left[ \frac{1}{\sqrt{t}} \cdot \frac{X_{\beta,\rho}(t + r)}{e^{(t+r)(1/2+\sigma^2)}} \bigg| \mathcal{F}_r \right] = \mathcal{N} \left( 0, C_2 \sqrt{\frac{2}{\pi} \mathcal{Z}} \right).$$ \hspace{1cm} (1.23)

(ii) For $\tau = 1/\sqrt{2}$,

$$\text{wlim}_{r \uparrow \infty} \text{wlim}_{t \uparrow \infty} \mathcal{L} \left[ \frac{1}{\sqrt{t}} \cdot \frac{X_{\beta,\rho}(t + r)}{e^{(t+r)(1/2+\sigma^2)}} \bigg| \mathcal{F}_r \right] = \mathcal{N} \left( 0, C_3 \sqrt{\frac{2}{\pi} \mathcal{Z}} \right).$$ \hspace{1cm} (1.24)

See Section 4.3 for a proof.

### 1.6. Organization of the rest of the paper

The remainder of the paper is organized as follows. In Section 2, we prove Theorems 1.2 and 1.4 concerning the behaviour of the partition function in Phase $B_1$. In Section 3, we treat Phase $B_3$. We start with a second moment computation which is then in the next subsection generalised to a constrained higher moment computation. Finally, in Section 3.3, we prove Theorem 1.5. The boundaries $B_{1,3}, B_{2,3}$ (Theorems 1.6 and 1.7) are proved in Section 4. Section 5 contains the proof of Theorem 1.1.

### 2. Proof of results for phase $B_1$

We start by proving the martingale convergence of $\mathcal{M}_{\sigma,\tau}(t)$.

**Proof of Theorem 1.2** One readily checks that $\mathcal{M}_{\sigma,\tau}(t)$ is a martingale with expectation $1$. Next, we compute the $\sqrt{2}/\sigma$-moment of $\mathcal{M}_\beta(t)$. To do this, first consider

$$E \left[ \left| \sum_{k=1}^{n(t)} e^{\sigma x_k(t) + i\tau y_k(t)} \right|^{\sqrt{2}/\sigma} \right] = E \left[ \left| \sum_{k=1}^{n(t)} e^{(\sigma + i\tau) x_k(t) + i\tau \sqrt{1-\rho^2} z_k(t)} \right|^{\sqrt{2}/\sigma} \right],$$ \hspace{1cm} (2.1)

where we used Representation (1.6). The right-hand side of (2.1) is equal to

...
Next, we bound (2.7) from above, using Jensen’s inequality for the conditional expectation since

\[ k \]

By the branching property,

We set

\[ q_{k,j} := d(x_k(t), x_j(t)). \] (2.4)

By the branching property,

\[ x_k(t) - x_j(t) \equiv x_k^{(1)}(t - q_{k,j}) - x_j^{(2)}(t - q_{k,j}), \] (2.5)

where \( k' \) and \( j' \) are two BBM particles at time \( t - q_{k,j} \) from two independent copies \( X^{(1)}(\cdot) \) and \( X^{(2)}(\cdot) \) of a BBM and let \( n^{(1)}(s) \) and \( n^{(2)}(s) \) denote the number of particles of \( X^{(1)} \), resp. \( X^{(2)} \), at time \( s \). Using (2.5), we rewrite (2.3) as

\[ \mathbb{E} \left[ \left( e^{(1 - \rho^2)^2 (t - q_{k,j})} \sum_{k=1}^{n^{(1)}} x_k^{(1)} (t - q_{k,j}) x^{(2)}_j(t - q_{k,j}) \right)^{1/2\sigma} \right]. \] (2.6)

Using again Jensen’s inequality for the conditional expectation \((\sum(\ldots))^{\sqrt{2\sigma}} \leq (\sum(\ldots))^{\sqrt{2\sigma}} \)

since \( \sqrt{2\sigma} > 1 \), we bound (2.6) from above by

\[ \mathbb{E} \left[ \sum_{k=1}^{n^{(1)}} x_k^{(1)} (t - q_{k,j}) x^{(2)}_j(t - q_{k,j}) \right]^{1/2\sigma}. \] (2.7)

Next, we bound (2.7) from above, using Jensen’s inequality for the conditional expectation \((\sum(\ldots))^{\sqrt{2\sigma}} < 1 \), by

\[ \mathbb{E} \left[ \sum_{k=1}^{n^{(1)}} e^{(\sigma + i\rho \tau)x_k^{(1)}(t - q_{k,j}) + (\sigma - i\rho \tau)x^{(2)}_j(t - q_{k,j})} \mathbb{E} \left[ \sum_{k' \leq n^{(1)}(t - q_{k,j}) \atop j' \leq n^{(2)}(t - q_{k,j})} e^{x_k^{(1)}(t - q_{k,j})x_j^{(2)}(t - q_{k,j})} \right]^{1/2\sigma} \right]. \] (2.8)
where $\mathcal{F}_{q,k,t}$ is the $\sigma$-algebra generated by the BBM $X$ up to time $q_{k,t}$, in particular we condition on $q_{k,t}$. Calculating the inner expectations in (2.8), gives

$$
E\left[ \sum_{k'=n(1)(q_{k,t})}^{n(2)(q_{k,t})} e^{(\sigma+i\tau) x_y^{(1)}(t-q_{k,t}) + (\sigma-i\tau) x_y^{(2)}(t-q_{k,t})} \bigg| \mathcal{F}_{q,k,t} \right] = K e^{2(t-q_{k,t})} \int_{-\infty}^{\infty} dy e^{(\sigma+i\tau) y + (\sigma-i\tau) y'} e^{-\frac{2+t-q_{k,t}^2}{2(t-q_{k,t})}} \frac{1}{2\pi(t-q_{k,t})} \] (2.9)

by completing the square. Hence, (2.8) is equal to

$$
K \mathbb{E} \left[ \left( e^{\frac{\sigma^2 - \tau^2 + 2 + (t-q_{k,t})}{\sqrt{2} \sigma}} \right)^{n(q_{k,t})} \right] = K \int_0^t dq e^{\frac{\sigma^2 - \tau^2 + 2 + (t-q)}{\sqrt{2} \sigma}} \int_{-\infty}^{\infty} dx \ e^{\sqrt{2}x - \frac{x^2}{2\sigma}} \frac{1}{\sqrt{2\pi \sigma}} = K \int_0^t dq e^{\frac{\sigma^2 - \tau^2 + 2 + (t-q)}{\sqrt{2} \sigma}} e^{2q}, \] (2.10)

by computing the Gaussian integral. Using (2.10) and noticing that the normalization factor in (1.10) is equal to $e^{-\frac{2+t-q_{k,t}^2}{\sqrt{2} \sigma}}$, we bound the $\frac{\sigma^2}{\sigma}$-moment of $\mathcal{M}_{\sigma,\tau}(t)$ by

$$
K \int_0^t dq e^{\frac{\sigma^2 - \tau^2 + 2 + (t-q)}{\sqrt{2} \sigma}} e^{2q} = K \int_0^t dq e^{\frac{\sigma^2 - \tau^2 + 2 + (t-q)}{\sqrt{2} \sigma}}. \] (2.11)

For $|\tau| + |\sigma| < \sqrt{2}$, the right-hand side of (2.11) is uniformly bounded by some constant $C$. Since $\mathcal{M}_{\sigma,\tau}(t)$ is bounded in $L^p$ for some $p > 1$, the a.s. limit exists and the convergence also holds in $L^1$. Moreover, $\mathbb{E}[\mathcal{M}_{\sigma,\tau}(t)] = 1$ and hence the limit is non-degenerate.

Next, we turn to proving the central limit theorem when $\sigma < 1/\sqrt{2}$.

**Proof of Theorem 1.4** We start with the proof of (1.16). Let

$$a_k(r) := e^{-r(1 + \frac{\sigma^2}{\sigma} - \tau^2)} e^{\sigma x_k(r) + i\tau y_k(r)}. \] (2.12)

Then, we can rewrite $\mathcal{M}_{\sigma,\tau}(t)$ as

$$
\mathcal{M}_{\sigma,\tau}(t + r) = \sum_{k=1}^{n(r)} a_k(r) \mathcal{M}^{(k)}_{\sigma,\tau}(t), \] (2.13)

where $\mathcal{M}^{(k)}_{\sigma,\tau}(t)$ are i.i.d. copies of $\mathcal{M}_{\sigma,\tau}(t)$. Hence, conditional on $\mathcal{F}_r$, $\mathcal{M}_{\sigma,\tau}(t)$ can be written as a sum of independent random variables. To prove a CLT, we want to use the two-dimensional Lindeberg-Feller condition (conditional on $\mathcal{F}_r$). First, we take the limit $t \uparrow \infty$. For $\sigma < 1/\sqrt{2}$ and $\beta \in B_1$, we have $\sigma^2 + \tau^2 < 1$. Then, by [18], Proposition A.1, $\mathcal{M}^{(k)}_{\sigma,\tau}(t)$ is $L^2$-bounded and

$$
\lim_{t \uparrow \infty} \mathbb{E} \left[ \left| \mathcal{M}^{(k)}_{\sigma,\tau}(t) \right|^2 \right] = C_1. \] (2.14)

Hence, the a.s. limit $\mathcal{M}_{\sigma,\tau}$ exists in $L^2$ and as $t \uparrow \infty$ the right-hand side of (2.13) converges a.s. to

$$
\mathcal{M}_{\sigma,\tau} = \sum_{k=1}^{n(r)} a_k(r) \mathcal{M}^{(k)}_{\sigma,\tau}, \] (2.15)
where $\mathcal{M}^{(k)}_{\sigma,\tau}$ are i.i.d. copies of $\mathcal{M}_{\sigma,\tau}$. To compute the variance of (2.15), consider

$$\sum_{k=1}^{n(r)} \mathbb{E} \left[ \left| a_k(r) \mathcal{M}^{(k)}_{\sigma,\tau} \right|^2 \bigg| \mathcal{F}_r \right].$$

(2.16) is equal to

$$\sum_{k=1}^{n(r)} |a_k(r)|^2 \mathbb{E} \left[ \left| \mathcal{M}^{(k)}_{\sigma,\tau} \right|^2 \right] = C_1 \sum_{k=1}^{n(r)} |a_k(r)|^2,$$

by (2.14). Now,

$$C_1 \sum_{k=1}^{n(r)} |a_k(r)|^2 = C_1 \sum_{k=1}^{n(r)} e^{2\sigma x_k(r)-2\rho(1+\frac{\sigma^2}{2})} = C_1 \mathcal{M}_{2\sigma,0}(r)e^{-r(1-(\sigma^2+\tau^2))}.$$  

(2.18) together with the extra rescaling in (1.16),

$$C_1 e^{(1-\sigma^2-\tau^2)r} \sum_{k=1}^{n(r)} |a_k(r)|^2 = C_1 \mathcal{M}_{2\sigma,0}(r),$$

which converges a.s. as $r \uparrow \infty$ to $C_1 \mathcal{M}_{2\sigma,0}$.

It remains to check the Lindeberg-Feller condition. We set

$$b_k(r) := a_k(r)e^{-(1-\sigma^2-\tau^2)r}.$$  

(2.20)

Let $\epsilon > 0$ and consider

$$\frac{1}{C_1 \mathcal{M}_{2\sigma,0}(r)} \sum_{i=1}^{n(r)} \mathbb{E} \left[ \left| b_k(r) \left( \mathcal{M}^{(k)}_{\sigma,\tau} - 1 \right) \right|^2 \right] \times \mathbb{1} \{ \left| b_k(r) \left( \mathcal{M}^{(k)}_{\sigma,\tau} - 1 \right) \right| > \epsilon \sqrt{C_1 \mathcal{M}_{2\sigma,0}(r)} \bigg| \mathcal{F}_r \right].$$

(2.21)

We rewrite (2.21) as

$$\frac{1}{C_1 \mathcal{M}_{2\sigma,0}(r)} \sum_{i=1}^{n(r)} b_k(r)b_k(r) \mathbb{E} \left[ \left| \left( \mathcal{M}^{(k)}_{\sigma,\tau} - 1 \right) \right|^2 \right] \times \mathbb{1} \{ \left| \left( \mathcal{M}^{(k)}_{\sigma,\tau} - 1 \right) \right|^2 > \epsilon^2 \left| b_k(r) \right|^2 C_1 \mathcal{M}_{2\sigma,0}(r) \bigg| \mathcal{F}_r \right].$$

(2.22)

We consider for a fixed $k$

$$\mathbb{E} \left[ \left| \left( \mathcal{M}^{(k)}_{\sigma,\tau} - 1 \right) \right|^2 \mathbb{1} \{ \left| \left( \mathcal{M}^{(k)}_{\sigma,\tau} - 1 \right) \right|^2 > \epsilon^2 \left| b_k(r) \right|^2 C_1 \mathcal{M}_{2\sigma,0}(r) \} \bigg| \mathcal{F}_r \right].$$

(2.23)

Using again that by [18, Proposition A.1]

$$\mathbb{E} \left[ \left| \left( \mathcal{M}^{(k)}_{\sigma,\tau} - 1 \right) \right|^2 \right] = C_1 < \infty,$$

we have that (2.23) converges to zero as $r \uparrow \infty$ if

$$|b_k(r)|^{-2} C_1 \mathcal{M}_{2\sigma,0}(r) \xrightarrow{r \uparrow \infty} \infty.$$  

(2.25)

Observe that $\mathcal{M}_{2\sigma,0}(r)$ is a $L^2$-bounded martingale with mean one, if $\sigma < 1/\sqrt{2}$. Hence, it converges a.s. and in $L^1$. Consider

$$|b_k(r)|^{-2} = e^{-2\sigma x_k(r)+2(\frac{1}{2}+\sigma^2)r},$$

(2.26)
since $x_k(r) < \sqrt{2}r$ a.s. (by Lalley-Selke argument in [26]). On this event, we have
\[
|b_k(r)|^{-2} \geq e^{(-2\sqrt{2}\sigma+1+2\sigma^2)r} = e^{(1-\sqrt{2}\sigma)^2r},
\]
which converges to infinity as $r \uparrow \infty$. Hence, (2.25) holds a.s. \hfill \square

3. Proof of CLT for Phase $B_3$

In this section, we deal with phase $B_3$ and prove Theorem 1.5.

3.1. Second moment computations. We start by controlling the second moment of $N_{\sigma,\tau}(t)$ defined in (1.19) in phase $B_3$ and its appropriately scaled version
\[
\hat{N}_{\sigma,\tau}(t) := t^{-1/2}N_{\sigma,\tau}(t)
\]
on the boundary $B_{1,3}$.

**Lemma 3.1.** It holds:

(i) For $\beta \in B_3$ or $\beta \in B_{2,3} \setminus \{(1+i)/\sqrt{2}\}$ and any $\rho \in [-1, 1],$
\[
\lim_{t \uparrow \infty} \mathbb{E} \left[ |N_{\sigma,\tau}(t)|^2 \right] = C_2 < \infty,
\]
for some positive constant $0 < C_2 < \infty$.

(ii) For $\beta \in B_{1,3}$ or $\beta = \frac{1}{\sqrt{2}}(1+i)$ and any $\rho \in [-1, 1],$
\[
\lim_{t \uparrow \infty} \mathbb{E} \left[ |\hat{N}_{\sigma,\tau}(t)|^2 \right] = C_3 < \infty,
\]
for some positive constant $0 < C_3 < \infty$.

**Proof.** (i) We have
\[
\mathbb{E} \left[ |N_{\sigma,\tau}(t)|^2 \right] = e^{-2t(1/2+\sigma^2)} \mathbb{E} \left[ \sum_{k,l=1}^{n(t)} e^{\sigma(x_k(t)+x_l(t))+i\tau(y_k(t)-y_l(t))} \right].
\]
Using Representation (1.6), we rewrite the right-hand side of (3.4) as
\[
e^{-2t(1/2+\sigma^2)} \mathbb{E} \left[ \sum_{k,l=1}^{n(t)} e^{\lambda x_k(t)+\lambda x_l(t)+i\tau \sqrt{1-\rho^2}(z_k(t)-z_l(t))} \right],
\]
where $\lambda = \sigma + i\rho\tau$ and $(z_k(t))_{k \leq n(t)}$ are the particles of a BBM on $\mathbb{T}_t$ that is independent from $X(t)$. By conditioning on $\mathcal{F}^t$, we have that (3.5) is equal to
\[
e^{-2t(1/2+\sigma^2)} \mathbb{E} \left[ e^{-(1-\rho^2)\tau^2(t-d(x_k(t),x_l(t)))} \sum_{k,l=1}^{n(t)} e^{\lambda x_k(t)+\lambda x_l(t)} \right].
\]
The expectation in (3.6) is equal to
\[
K \int_0^t dq \ e^{2t-q-(1-\rho^2)\tau^2(t-q)} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi q}} \int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi(t-q)}} \times \int_{-\infty}^{\infty} \frac{dy'}{\sqrt{2\pi(t-q)}} \ e^{2\sigma x+\sigma(y+y'+i\rho(y-y'))} e^{-\frac{q^2+q'2}{2(t-q)}} e^{-x^2/2q}.
\]

\[4\] $C_2$ depends on $\sigma$ and $\tau$ but not on $\rho$. We do not make this dependence explicit in our notation.
Lemma 3.2. The following two Lemmata ensure that we can introduce the desired constraint. These consist of computing constrained moments.

\begin{equation}
K \int_0^t dq \ e^{2i - q - (1 - \rho^2)q^2 + (\sigma^2 - \rho^2)q^2 + (t - q)} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi q}} e^{2\pi x^2/2q}
= K \int_0^t dq \ e^{2i - q - \tau q^2 + \sigma^2 q^2 + (t - q)} \ e^{2\sigma^2 q^2}.
\end{equation}

(3.8)

Plugging (3.8) back into (3.6), we get that (3.6) is equal to

\begin{equation}
e^{-2i(1/2 + \sigma^2)} K \int_0^t dq \ e^{2i - q - \tau q + \sigma^2 q^2} = K \int_0^t dq \ e^{(t - q)(1 - \tau - \sigma^2)} = \frac{K}{1 - \tau - \sigma^2} \left(e^{((1 - \tau - \sigma^2)} - 1\right).
\end{equation}

(3.9)

As \( t \uparrow \infty \), the term in (3.9) converges to \( \frac{K}{\tau + \sigma^2 - 1} \), which we call \( C_2 \) from now on.

(ii) Proceeding as in Part (i), we get that

\begin{equation}
\mathbb{E} \left[ |\tilde{N}_{\sigma, \tau}(t)|^2 \right] = t^{-1} e^{-2i(1/2 + \sigma^2)} \mathbb{E} \left[ e^{-(1 - \rho^2)\tau^2(t - d(x_k(t), x_l(t)))} \sum_{k,l=1}^{n(t)} e^{\lambda x_k(t) + \lambda x_k(t)} \right].
\end{equation}

(3.10)

Plugging (3.8) into (3.10), we get that (3.10) is equal to

\begin{equation}
K t^{-1} \int_0^t dq \ e^{(t - q)(1 - \tau - \sigma^2)} = K t^{-1} \int_0^t dq = K,
\end{equation}

(3.11)

since \( \sigma^2 + \tau^2 = 1 \) in \( B_{1,3} \).

\[ \square \]

3.2. Constrained moment computation in \( B_3 \). In this section, we continue our preparations for the proof of Theorem 1.5. These consist of computing constrained moments. The following two Lemmata ensure that we can introduce the desired constraint.

Lemma 3.2. Let \( \beta \in B_3 \). Then for all \( \epsilon > 0 \) and \( \delta > 0 \), uniformly for all \( t \) large enough, there exists \( A_0 \) such that for all \( A > A_0 \)

\begin{equation}
\mathbb{P} \left\{ \left| \sum_{k=1}^{n(t)} e^{\sigma x_k(t) + i \tau y_k(t) - (1/2 + \sigma^2)t} \mathbb{1}_{\{x_k(t) > 2\sigma t + A\sqrt{t}\}} \right| > \delta \right\} < \epsilon.
\end{equation}

(3.12)

Proof. Using a second moment Chebyshev inequality, we bound the probability in (3.12) from above by

\begin{equation}
e^{-2i(1/2 + \sigma^2)} \mathbb{E} \left[ \sum_{k,l=1}^{n(t)} e^{\sigma x_k(t) + i \tau y_k(t) - (1/2 + \sigma^2)q} \mathbb{1}_{\{x_k(t) > 2\sigma t + A\sqrt{t}\}} \right].
\end{equation}

(3.13)

Continuing as in the proof of Lemma 3.1, we rewrite (3.13) as

\begin{equation}
e^{-2i(1/2 + \sigma^2)} \mathbb{E} \left[ e^{-(1 - \rho^2)\tau^2(t - d(x_k(t), x_l(t)))} \sum_{k,l=1}^{n(t)} e^{\lambda x_k(t) + \lambda x_k(t)} \mathbb{1}_{\{x_k(t) > 2\sigma t + A\sqrt{t}\}} \right].
\end{equation}

(3.14)

We rewrite the expectation in (3.14)
Observe that by the computations in Lemma 3.1 for $r$ sufficiently large

\[ K \int_{0}^{t-r} dq \ e^{2t-q-(1-\rho^2)r^2(t-q)} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi q}} \int_{2\sigma t + A\sqrt{t-x}}^{\infty} \frac{dy}{\sqrt{2\pi (t-q)}} \times \int_{2\sigma t - x}^{\infty} \frac{dy'}{\sqrt{2\pi (t-q)}} e^{2\sigma x + \sigma(y+y') + i\tau y} e^{-\frac{y^2}{2(t-q)}} e^{-x^2/2q}. \quad (3.15) \]

Hence, it suffices to take consider the integration domain $q > t - r$. Now, $P(y > r) < e^{-r/2}$. To have $x + y > 2\sigma t + A\sqrt{t}$ on that event, $x > 2\sigma t + A\sqrt{t} - r$ must hold. By inserting this into the second moment, we have that the bound is smaller than $\epsilon/2$ for $A$ sufficiently large.

**Lemma 3.3.** Let $\beta \in B_3$, $\rho \in [-1, 1]$ and $\gamma > \frac{1}{2}$. Let $A > 0$. Then, for all $\epsilon > 0$ and $d > 0$, there exists $r_0 > 0$ such that, for all $r > r_0$, uniformly for all $t$ sufficiently large,

\[ P \left\{ \left| \sum_{k=1}^{n(t)} e^{\sigma x_k(t) + i\tau y_k(t)} - (\frac{1}{2} + \sigma^2) t \right| \times 1 \{ x_k(t) < 2\sigma t + A\sqrt{t}, \exists s \in [r, t]: x_k(s) > 2\sigma s + s^\gamma \} > \delta \right\} < \epsilon. \quad (3.17) \]

**Proof.** We use again a second moment bound. Similarly to the proof of Lemma 3.2, we bound the probability in (3.17) from above by

\[ e^{-2t(1/2 + \sigma^2)} E \left[ e^{-(1-\rho^2)r^2(t-d(x_k(t), x_l(t)))} \sum_{k,l=1}^{n(t)} e^{\lambda x_l(t) + \lambda x_k(t)} \times 1 \{ x_k(t), x_l(t) < 2\sigma t + A\sqrt{t}, \exists s \in [r, t]: x_k(s) > 2\sigma s + s^\gamma, \exists s' \in [r, t]: x_l(s') > 2\sigma s' + (s')^\gamma \} \right]. \quad (3.18) \]

By only keeping track of the path event for one of the particles, we get that (3.18) is bounded from above by

\[ e^{-2t(1/2 + \sigma^2)} E \left[ e^{-(1-\rho^2)r^2(t-d(x_k(t), x_l(t)))} \sum_{k,l=1}^{n(t)} e^{\lambda x_l(t) + \lambda x_k(t)} \times 1 \{ x_k(t), x_l(t) < 2\sigma t + A\sqrt{t}, \exists s \in [r, t]: x_k(s) > 2\sigma s + s^\gamma \} \right]. \quad (3.19) \]

We rewrite (3.19) as

\[ K \int_{0}^{t} dq \ e^{2t-q} e^{2t-q-(1-\rho^2)r^2(t-q)} E \left[ e^{\lambda x_1(t) + \lambda (x_1(q) + x_2(t-q))} \times 1 \{ x_1(t), x_1(q) + x_2(t-q) < 2\sigma t + A\sqrt{t}, \exists s \in [r, t]: x_1(s) > 2\sigma s + s^\gamma \} \right], \quad (3.20) \]
where $x_1(\cdot)$ is a standard Brownian motion and $x_2(t-q)$ is an independent $\mathcal{N}(0, t-q)$ distributed random variable. Calculation of the expectation in (3.20) with respect to $x_2(t-q)$ yields

$$K \int_0^t dq \, e^{2(t-q)\frac{\sigma^2}{2}} e^{-t-q-(1-\rho^2)^2(t-q)} \mathbb{E} \left[ e^{\lambda x_1(t)+\lambda x_1(q)} \right] .$$

As in the proof of Lemma 3.2, we can first choose $r_1$ large enough such that the above integral from 0 to $t-r_1$ is bounded by $\epsilon/3$. Moreover, $x_1(t) = x_1(q) + \tilde{x}(t-q)$, where $\tilde{x}(t-q)$ is normal distributed with mean zero and variance $t-q$ that is independent from $x_1(s)$ for $s \leq q$. Then, for all $R > R_2$,

$$\mathbb{P}\{|\tilde{x}(t-q)| > R\} < \frac{\epsilon}{3} .$$

Observe that the intersection of the event $\{\tilde{x}(t-q) > R\}$ and the event in the indicator in (3.21) is contained in the event

$$\{x_1(t) < 2\sigma t + A\sqrt{t}, x_1(s) > 2\sigma s + s^\gamma, \tilde{x}(t-q) > R\} \subset \left\{ x_1(s) - \frac{s}{q} x_1(q) < s^\gamma - \frac{(A\sqrt{t} - R)s}{q} \right\} .$$

Using that $x_1(s) - \frac{s}{q} x_1(q) = \xi_k(s)$ is a Brownian bridge that is independent from $x_1(q)$ and also from $\tilde{x}(t-q)$, we bound (3.21) from above by

$$K \int_0^t dq \, e^{2(t-q)\frac{\sigma^2}{2}} e^{2(t-q-(1-\rho^2)^2)} \mathbb{E} \left[ e^{\lambda x_1(t)+\lambda x_1(q)} \right] \times \mathbb{P}\left\{ x_1(s) > s^\gamma - \frac{(A\sqrt{t} - R)s}{q} \right\} .$$

By the same computations as in (3.7) and (3.8), we can bound (3.24) from above by

$$C_2 \mathbb{P}\{ x_1(s) > s^\gamma - \frac{(A\sqrt{t} - R)s}{t-R} \} .$$

It is a well known fact for Brownian bridges (see, e.g., [12] Lemma 2.3) for a precise statement) that by choosing $r$ sufficiently large (3.25) can be made as small as we want. This finishes the proof of Lemma 3.3.

Define

$$N^{c,A}_{\sigma,\tau}(t) := \sum_{k=1}^{n(t)} e^{-(t/2+\sigma^2)} e^{\sigma x_k(t)+i\tau y_k(t)}$$

as

$$\times \mathbb{1}\{x_k(t) < 2\sigma t + A\sqrt{t}, \forall s \in [r,t]: x_k(s) \leq 2\sigma s + s^\gamma \} .$$

The following lemma provides the asymptotics for all moments of (3.26) in the $t \to \infty$ limit.

**Lemma 3.4** (Moment asymptotics). Consider a branching Brownian motion with binary splitting. For $\beta \in B_3$, for any $A > 0$

$$\lim_{t \to \infty} \mathbb{E}\left[ |N^{c,A}_{\sigma,\tau}(t)|^2 \right] = C_{2,A} .$$
with \( \lim_{A \to \infty} C_{2,A} = C_2 \) and, for \( k \in \mathbb{N} \), we have

\[
\lim_{r \to \infty} \lim_{t \to \infty} \mathbb{E} \left[ N^{\sigma,A}_{r,t} (t) \right]^{2k} | \mathcal{F}_r \right] = k! (C_{2,A} \mathbb{M}_{2\sigma,0})^k \quad \text{a.s. and in } L^1. \tag{3.28}
\]

Moreover, for \( k' < k \),

\[
\lim_{r \to \infty} \lim_{t \to \infty} \mathbb{E} \left[ N^{\sigma,A}_{r,t} (t) \right]^{k} N^{\sigma,A}_{r,t} (t)^{k'} | \mathcal{F}_r \right] = 0 \quad \text{a.s. and in } L^1. \tag{3.29}
\]

**Proof.** We proceed by induction over \( k \). For \( k = 1 \), we observe that the claim follows directly from Lemma [3.1] together with the second moment computation done in the proof of Lemma 3.2 and Lemma 3.3.

To bound the \( 2k \)-moment, we rewrite (3.28) as

\[
\mathbb{E} \left[ \sum_{l_1,\ldots,l_{2k} \leq n(t)} 2^k \prod_{j=1}^{2k} e^{-t (1/2+\sigma^2)} e^{\sigma x_{l_j}(t)} + \gamma y_{l_j}(t) \right]
\]

For \( l_1, \ldots, l_{2k} \leq n(t) \), we can find a matching using the following algorithm:

1. Choose the two labels \( j, j' \) such that \( d(x_{l_j}, x_{l_{j'}}) \) is maximal. Call them \( l_1 \) and \( l_{\sigma(1)} \) from know on.
2. Delete them.
3. Pick \( l_1 \) in the remaining set and match it with the remaining \( l_{j'} \) such that \( d(x_{l_j}, x_{l_{j'}}) \) is maximal. Iterate.

The pairs obtained in this way we call \((l_1, l_{\sigma(1)}), \ldots, (l_k, l_{\sigma(k)})\). We rewrite (3.30) as

\[
\mathbb{E} \left[ \sum_{l_2,\ldots,l_k \leq n(t)} k \sum_{j=1}^{k} e^{-t (1/2+\sigma^2)} e^{\sigma (x_{l_j}(t) + x_{l_{\sigma(j)}}(t))} + \gamma (y_{l_j}(t) + y_{l_{\sigma(j)}}(t)) \right]
\]

Using (1.6), we can rewrite for \( j \in \{1, \sigma(1)\} \)

\[
y_{l_j}(t) = \rho y_{l_1}(t) + \sqrt{1 - \rho^2} z_{l_j}(t), \tag{3.32}
\]

where \((z_k(t))_{k \leq n(t)}\) are particles of a BBM on the same Galton-Watson tree as \((x_k(t))_{k \leq n(t)}\) but independent from it. Observe that using the requirement that \( d(x_{l_1}, x_{l_{\sigma(1)}}) \) is chosen maximal, we have

\[
i \gamma (y_{l_1}(t) - y_{l_{\sigma(1)}}(t)) = i \sqrt{1 - \rho^2} \left( z_1(t - d(x_{l_1}(t), x_{l_{\sigma(1)}}(t))) - z_2(t - d(x_{l_1}(t), x_{l_{\sigma(1)}}(t))) \right)
\]

where \( z_1, z_2 \) are two independent \( \mathcal{N}(0, (t - d(x_{l_1}(t), x_{l_{\sigma(1)}}(t)))) \)-distributed random variables. Plugging (3.33) into (3.31) and computing the expectation with respect to \( z_1, z_2 \), we
We can rewrite (3.36) as
\[ E \left[ \sum_{l_2, \ldots, l_k \leq n(t)} \prod_{j=2}^{k} e^{-t(1+2\sigma^2)} \exp \left( \sigma (x_{l_j} (t) + x_{l_{(j)}} (t)) + i \tau (y_{l_j} (t) + y_{l_{(j)}} (t)) \right) \right. \]
\[ \times 1 \{ x_{l_{(j)}} (t), x_{l_j} (t) < 2\sigma t + A \sqrt{t}, \forall s \in [r, t]: x_{l_{(j)}} (s), x_{l_j} (s) \leq 2\sigma s + s^\gamma \} \]
\[ \times e^{-t(1+2\sigma^2)-\tau^2(t-\delta(x_{l_1}, x_{l_{(1)}}))} e^{(\sigma+i\tau)p x_{l_1} (t)+ (\sigma-i\tau)p x_{l_{(1)}}} \]
\[ \left. \times 1 \{ x_{l_{(1)}} (t), x_{l_1} (t) < 2\sigma t + A \sqrt{t}, \forall s \in [r, t]: x_{l_{(1)}} (s), x_{l_1} (s) \leq 2\sigma s + s^\gamma \} \right] \].

We decompose
\[ x_{l_{(1)}} (t) = x_{l_1} d(x_{l_1}, x_{l_{(1)}}) + x^{(1)} (t - d(x_{l_1}, x_{l_{(1)}})); \]
\[ x_{l_1} (t) = x_{l_1} d(x_{l_1}, x_{l_{(1)}}) + x^{(2)} (t - d(x_{l_1}, x_{l_{(1)}})); \] (3.35)

where \( x^{(1)}, x^{(2)} \) are two independent \( \mathcal{N}(0, t - d(x_{l_1}, x_{l_{(1)}})) \)-distributed random variables. By Step one of our matching procedure, we can plug (3.34) into (3.35) and compute the expectation with respect to \( x^{(1)} \) and \( x^{(2)} \), we obtain that (3.34) is bounded from above by
\[ E \left[ \sum_{l_2, \ldots, l_k \leq n(t)} \prod_{j=2}^{k} e^{-t(1+2\sigma^2)} e^{\sigma (x_{l_j} (t) + x_{l_{(j)}} (t)) + i \tau (y_{l_j} (t) + y_{l_{(j)}} (t))} \right. \]
\[ \times 1 \{ x_{l_{(j)}} (t), x_{l_j} (t) < 2\sigma t + A \sqrt{t}, \forall s \in [r, t]: x_{l_{(j)}} (s), x_{l_j} (s) \leq 2\sigma s + s^\gamma \} \]
\[ \times e^{-t(1+2\sigma^2)-\tau^2(t-\delta(x_{l_1}, x_{l_{(1)}})) + \sigma^2(t-d(x_{l_1}, x_{l_{(1)}}))} e^{2\sigma x_{l_1} d(x_{l_1}, x_{l_{(1)}})} \]
\[ \left. \times 1 \{ \forall s \in [r, t]: d(x_{l_1}, x_{l_{(1)}}); x_{l_{(1)}} (s), x_{l_1} (s) \leq 2\sigma s + s^\gamma \} \right]. \] (3.36)

We now introduce the event
\[ A_r = \left\{ \exists s \in [r, d(x_{l_1}, x_{l_{(1)}})], \exists j \in \{2, \ldots, k, \sigma(2), \ldots, \sigma(k)\}: d(x_{l_1}, x_{l_j}) = s \right\} . \] (3.37)

We can rewrite (3.36) as
\[ E\left[ \ldots \times 1_{A_r} \right] + E\left[ \ldots \times 1_{A_r^c} \right] =: J_{A_r} + J_{A_r^c}. \] (3.38)

We will prove that the first summand is of a smaller order than the second one. We need the following lemma.

**Lemma 3.5.** Let \( x, y \) be \( \mathcal{N}(0, q) \) distributed random variables. Then, for any \( m_1, m_2 \geq 1 \) and constant \( C > 0 \),
\[ E \left[ e^{(m_1+2)\sigma x+i\tau m_2 x} 1 \{ x < 2\sigma q + C q^\gamma \} \right] \]
\[ = o \left( e^{2\sigma q} E \left[ e^{-2m_1 \sigma x+i\tau m_2 x} 1 \{ x < 2\sigma q + C q^\gamma \} \right] \right) \]
\[ \times E \left[ e^{(m_1+1)\sigma x+i\tau (m_2+1) x} 1 \{ x < 2\sigma q + C q^\gamma \} \right] \]
\[ = o \left( e^{2\sigma q} E \left[ e^{-m_1 \sigma x+i\tau m_2 x} 1 \{ x < 2\sigma q + C q^\gamma \} \right] \right) \]
\[ \times E \left[ e^{(m_1+1)\sigma x+i\tau (m_2+1) x} 1 \{ x < 2\sigma q + C q^\gamma \} \right]. \] (3.39)

and similarly
\[ E \left[ e^{(m_1+1)\sigma x+i\tau (m_2+1) x} 1 \{ x < 2\sigma q + C q^\gamma \} \right] \]
\[ = o \left( e^{2\sigma q} E \left[ e^{-m_1 \sigma x+i\tau m_2 x} 1 \{ x < 2\sigma q + C q^\gamma \} \right] \right) \]
\[ \times E \left[ e^{(m_1+1)\sigma x+i\tau (m_2+1) x} 1 \{ x < 2\sigma q + C q^\gamma \} \right]. \] (3.40)

\(^5\)A corresponding lower bound also holds due to the second moment computation in Lemma 3.4.
Proof. The l.h.s. in (3.39) is equal to
\[
\int_{-\infty}^{2\sigma q+Cq^\gamma} \frac{dx}{\sqrt{2\pi q}} e^{(m_1+2)\sigma x+i\tau m_2 x} e^{-\frac{x^2}{2q}}. \tag{3.41}
\]
Making a change of variable \(y = (m_1 + 2)\sigma q + i\tau m_2 q + x\), we obtain that (3.41) equals to
\[
e^{((m_1+2)\sigma+2\tau m_2 q)q/2} \int_{-\infty}^{(m_1+2)\sigma q+i\tau m_2 q+Cq^\gamma} \frac{dy}{\sqrt{2\pi q}} e^{-\frac{y^2}{2q}}. \tag{3.42}
\]
For \(m_1 \geq 1\), by the Gaussian tail asymptotics, (3.42) is bounded from above by
\[
e^{2m_1\sigma^2 q+2\sigma q+m_2^2 \tau^2 q} e^{m_1 \sigma Cq^\gamma}. \tag{3.43}
\]
The expectation on the right hand side of (3.39) is equal to
\[
\int_{-\infty}^{2\sigma q+Cq^\gamma} dx \ e^{m_1 \sigma x+i\tau m_2 x} e^{-\frac{x^2}{2q}} \int_{-\infty}^{2\sigma q+Cq^\gamma} \frac{dy}{\sqrt{2\pi q}} e^{-\frac{y^2}{2q}}. \tag{3.44}
\]
If \(m_1 > 2\), (3.44) is asymptotically equal to
\[
\frac{1}{\sqrt{2\pi (m_1 - 2)q}} e^{2m_1\sigma^2 q - 2\sigma q + m_2^2 \tau^2 q} e^{2\sigma^2 q} e^{(m_1-2)\sigma Cq^\gamma}. \tag{3.45}
\]
Comparing (3.45) with (3.43) yields the claim of Lemma (3.39). For \(m_1 = 1\) or \(m_2 = 1\), we bound the integral in (3.44) by \(e^{(m_1+\sigma m_2)q/2+2\sigma^2 q/2}\).

The proof of (3.40) follows along the same lines. \(\Box\)

We continue the proof of Lemma 3.4. Consider \(J_{\mathcal{A},r}\). Consider the skeleton generated by the leaves \(l_1, l_{\sigma(1)}, \ldots, l_k, l_{\sigma(k)}\) of the Galton-Watson tree. By path(\(\cdot\)) we denote the unique path from a leave \(\cdot\) to the root. To each edge in the Galton-Watson tree, we associate the following number
\[
m(e) := \sum_{j \in \{1, \sigma(1), \ldots, k, \sigma(k)\}} 1_{e \in \text{path}(l_j)}. \tag{3.46}
\]

For \(k, j \in [n(t)]\), define (cf. Fig. 2)
\[
\text{length}(x_k(t), x_j(t)) := d(x_1(t), x_k(t)) - d(x_1(t), x_j(t)), \quad t \in \mathbb{R}_+. \tag{3.47}
\]
Looking at the path of \( x_{l_1}(t) \), the quantity \( m(\cdot) \) for \( e \subset \text{path}(l_1) \) before time \( d(x_{l_1}, x_{l_2}) \) where \( l_{j*} \) satisfies the following conditions

(i) \( m \) is constant between \( l_{j*} \) and \( l_{j*-1} \) and the piece has length \( > 2r \).
(ii) \( \sum_{i=1}^{j*-1} \text{length}(x_{i-1}, x_i) < (\text{length}(x_{j*-1}, x_{j*}))^\gamma \), where length is defined in the Fig. 2.

Such a \( l_{j*} \) exists for all \( t > t_0(r) \) because there are at most \( 2k - 2 \) points, where \( m \) it is allowed to change. We call the value of \( m \) on the path where \( l_{j*} \) and \( l_{j*-1} \) between \( m^* \). \( m^* \) corresponds to an time interval \( [R, R + \ell] \), where

\[
\ell = \ell(j^*, t) := \text{length} (x_{l_{j*}-1}(t), x_{l_{j*}}(t)).
\]

Then, up to time \( R \) the minimal particle is a.s. \( > -\sqrt{2R} \). Hence,

\[
x_{l_{j*}}(R + \ell) - x_{l_{j*}}(R) < x_{l_{j*}}(R + \ell) + \sqrt{2R}.
\]

(3.49)

Since we compute an expectation conditional on \( x_{l_{j*}}(R + \ell) < 2\sigma(R + \ell) + (R + \ell)^\gamma \), we obtain on this event

\[
x_{l_{j*}}(R + \ell) - x_{l_{j*}}(R) < 2\sigma(R + \ell) + (R + \ell)^\gamma + \sqrt{2R}.
\]

(3.50)

Due to our choice of \( j^* \), we have \( 2\sigma R + \sqrt{2R} < C'(\ell) \gamma \) for some positive constant \( C' \).

By taking the expectation with respect to \( x_{l_{j*}}(R + \ell) - x_{l_{j*}}(R) \) only, we can write extract from \( J_A \) the factor

\[
\mathbb{E} \left[ e^{(m^* + i\tau m') x_{l_{j*}}(R + \ell) - x_{l_{j*}}(R)} \chi \{ x_{l_{j*}}(R + \ell) - x_{l_{j*}}(R) < 2\sigma \ell + (C' + 1)(\ell)^\gamma \} \right].
\]

(3.51)

By Lemma 3.5, \( 3.5(3.51) \) is

\[
o(\ell e^{\sigma R}) \mathbb{E} \left[ e^{((m^* - 2\sigma + i\tau m') x_{l_{j*}}(R + \ell) - x_{l_{j*}}(R))} \chi \{ x_{l_{j*}}(R + \ell) - x_{l_{j*}}(R) < 2\sigma \ell + (C' + 1)(\ell)^\gamma \} \right] \times \mathbb{E} \left[ e^{2\sigma(x_{l_{j*}}(R + \ell) - x_{l_{j*}}(R))} \chi \{ x_{l_{j*}}(R + \ell) - x_{l_{j*}}(R) < 2\sigma \ell + (C' + 1)(\ell)^\gamma \} \right].
\]

(3.52)

for \( l \) large (which by Assumption (i) on \( l \) corresponds to \( r \) large). Note that the quantity, inside the brackets in \( 3.52 \), corresponds to the same expectation but where in the underlying tree \( l_1 \) branched off before time \( R \).

Iteratively, that leads to

\[
J_A, \quad \ell_r, t \to \infty \quad o(J_{A^*}).
\]

(3.53)

Since \( k \) was chosen arbitrary, we know that the main contribution to the \( 2k \)-th moment comes from the term where \( l_1, \ldots, l_k \) have split before time \( r \) for \( r \) large enough. We condition on \( F_r \) and compute:

\[
\mathbb{E} \left[ \sum_{l_1, l_2, \ldots, l_k \leq n(t)} \prod_{j=2}^{k} e^{-t(1 + 2\sigma^2)} e^{\sigma(x_{l_1}(t) + x_{l_{i}(t)}) + i\tau(y_{l_1}(t) + y_{l_{j}(t)}(t))} \right.
\]

\[
\times \chi \{ x_{l_{i}(t)}(t), x_{l_{j}}(t) < 2\sigma t + A\sqrt{t}, \forall s \in [r, t] : x_{l_{i}(s)}, x_{l_{j}} \leq 2\sigma s + s^\gamma \}
\]

\[
\times \chi \{ \sup_{j, j' \leq k} d(l_{j}, l_{j'}) < r \} \left. \right| F_r
\]

\[
\mathbb{E} \left[ \sum_{l_1, l_2, \ldots, l_k \leq n(t)} \prod_{j=2}^{k} b_{l_{j}}(r) \mathbb{E} \left[ \left( (N_{\sigma, r}^{\gamma, A}(t - r))^{(j)} \right)^2 \left. \right| F_r \right] \right],
\]

(3.54)
where $b_{lj}(r)$ is defined in (2.20) and $(N_{\sigma,\tau}^{c,A}(t - r))^{(j)}$ are i.i.d. copies of $(N_{\sigma,\tau}^{c,A}(t - r))^{(j)}$. By our second moment computations (Case $k = 1$), as mentioned at the beginning of this proof,
\begin{equation}
\lim_{t \to \infty} \mathbb{E} \left[ \left( (N_{\sigma,\tau}^{c,A}(t - r))^{(j)} \right)^2 \right] = C_{2,A}.
\end{equation}
Moreover, by invariance under permutation (in the labelling procedure),
\begin{equation}
\sum_{l_1,l_2,...,l_k \leq n(t)} \prod_{j=2}^{k} b_{lj}(r) \tilde{b}_{lj}(j) = k! \left( \sum_{k=1}^{n(r)} e^{2\sigma x_k(r) - (1 + 2\sigma^2)r} \right)^k.
\end{equation}
Observe that $\sum_{k=1}^{n(r)} e^{2\sigma x_k(r) - (1 + 2\sigma^2)r} = \mathcal{M}_{2\sigma,0}$ which converges almost surely to $\mathcal{M}_{2\sigma,0}$. This proves (3.28).

The case $k' < k$ follows similarly. Take an optimal matching of the first $k'$ particles. The other particles will not be matched. Take one $l_1$ that has not been matched. Along its path, we can again find the first macroscopic piece on which $m(\cdot)$ is constant. Applying Lemma 3.5, we get that the contribution is the largest if $\lim_{t \to \infty} \mathbb{E} \left[ \left( (N_{\sigma,\tau}^{c,A}(t - r))^{(j)} \right)^2 \right] = C_{2,A}$.

3.3. Proof of Theorem 1.5

Proof of Theorem 1.5 By Lemma 3.4 conditionally on $\mathcal{F}_r$, the moments of $N_{\sigma,\tau}^{c,A}(t)$ converge to the moments of a $\mathcal{N}(0, C_{2,A} \mathcal{M}_{2\sigma,0})$ a.s. as $t \to \infty$ and then $r \to \infty$. Since the normal distribution is uniquely characterised by its moments, this implies convergence in distribution. Moreover, by Lemma 3.3 and Lemma 3.3
\begin{equation}
\lim_{t \to \infty} \mathbb{E} \left[ N_{\sigma,\tau}^{c,A}(t) - N_{\sigma,\tau}^{c,A}(t) \right] = \delta_0,
\end{equation}
and $\lim_{A \to \infty} C_{2,A} = C_2$. The claim of Theorem 1.5 follows.

4. The boundaries

In this section, we provide the proofs of Theorems 1.3, 1.6 and Proposition 1.7 describing the limiting fluctuations of the partition function on all boundaries between the phases, i.e., on the 1D manifolds $B_{1,2} = \overline{B_1} \cap \overline{B_2}$, $B_{1,3} = \overline{B_1} \cap \overline{B_3}$, and $B_{2,3} = \overline{B_2} \cap \overline{B_3}$.

4.1. The boundary between phases $B_1$ and $B_3$.

Proof of Theorem 1.6 The proof of Theorem 1.6 works as in phase $B_3$. Observe first that
\begin{equation}
\mathbb{E} \left[ \frac{X_{\beta,\delta}(t)}{\sqrt{t e^{(1/2 + \sigma^2)}}} \right] = \frac{1}{\sqrt{t}}.
\end{equation}
Moreover, let
\begin{equation}
\hat{N}_{\sigma,\tau}(t) := t^{-1/2} N_{\sigma,\tau}(t) \quad \text{and} \quad \hat{N}_{\sigma,\tau}^{c,A}(t) := t^{-1/2} N_{\sigma,\tau}^{c,A}(t).
\end{equation}
By Lemma 3.1 (ii),
\begin{equation}
\lim_{t \to \infty} \mathbb{E} \left[ |\hat{N}_{\sigma,\tau}(t)|^2 \right] = C_3.
\end{equation}
Now, we need the following.

**Lemma 4.1.** For $\beta \in B_3$,

$$\lim_{t \to \infty} \mathbb{E} \left[ |\hat{N}_{\sigma,\tau}^{c,A}(t)|^2 \right] = C_{3,A}, \quad (4.4)$$

with $\lim_{t \to \infty} C_{3,A} = C_3$ and, for $k \in \mathbb{N}$, we have

$$\lim_{t \to \infty} \mathbb{E} \left[ |\hat{N}_{\sigma,\tau}^{c,A}(t)|^{2k} | \mathcal{F}_t \right] = k! (C_{3,M_{2,\sigma,0}})^k \quad \text{a.s. and in } L^1. \quad (4.5)$$

Moreover, for $k' < k$,

$$\lim_{t \to \infty} \mathbb{E} \left[ \hat{N}_{\sigma,\tau}^{c,A}(t)^{k} \hat{N}_{\sigma,\tau}^{c,A}(t)^{k'} | \mathcal{F}_t \right] = 0 \quad \text{a.s. and in } L^1. \quad (4.6)$$

**Proof.** The proof of Lemma 4.1 is a rerun of the proof of Lemma 4.4.

The claim of Theorem 1.6 follows with the very same arguments as the proof of Theorem 1.4.

4.2. **Real critical point** $\beta = \sqrt{2}$. For $|\sigma| = 1/\sqrt{2}$, the following scaling of the martingale $\mathcal{M}_{1,0}(t)$ plays an important role

$$\mathcal{M}_{1,0}^{SH}(t) := \sqrt{t} \sum_{i=1}^{n(t)} e^{-\sqrt{2}(t-x_i(t))}. \quad (4.7)$$

$\mathcal{M}_{1,0}^{SH}(t)$ is called critical additive martingale and the rescaling appearing in the r.h.s. of (4.7) is referred to as Seneta-Heyde scaling. The limiting behaviour of $\mathcal{M}_{1,0}^{SH}$ in the setting of branching random walks has been first analysed in [1]. An alternative proof is given in [25]. As $t \to \infty$, (4.7) converges in probability to a limiting random variable $\mathcal{M}_{1,0}^{SH}$.

**Lemma 4.2.** Denote $\mathcal{M}_{1,0}^{SH}(t) := \sqrt{t} \sum_{i=1}^{n(t)} e^{-\sqrt{2}(t-x_i(t))}$ and $\mathcal{M}_{1,0}^{SH} := \left( \frac{2}{\pi} \right)^{1/2} \mathcal{Z}$, where $\mathcal{Z}$ is the limit of the derivative martingale, cf. (1.20). Then, for $\beta = \sqrt{2}$, the following convergence holds in probability

$$\mathcal{M}_{1,0}^{SH}(t) \xrightarrow{p \ t \to \infty} \mathcal{M}_{1,0}^{SH}. \quad (4.8)$$

**Proof.** The proof is just an adaptation of the result for the branching random walk (see [25, Section 6.5]).

4.3. **The boundary between phases** $B_2$ and $B_3$; and the triple point $\beta = (1 + i)/\sqrt{2}$. In this section, we prove the convergence of the moments of the rescaled partition function on the boundary between phases $B_2$ and $B_3$ to the moments of a Gaussian random variable with random variance in probability which is the content of Theorem 1.7.

**Proof of Theorem 1.7 (i)** The proof of Theorem 1.7 (i) is a modification of the proof of Theorem 1.4(ii) in the following way.

**Lemma 4.3.** For $\beta$ with $\sigma = \frac{1}{\sqrt{2}}, \rho \in [-1,1]$ and binary branching

$$\lim_{t \to \infty} \mathbb{E} \left[ |N_{o,\tau}^{c,A}(t)|^2 \right] = C_{2,A}, \quad (4.9)$$

and, for $k \in \mathbb{N}$, we have

$$\lim_{t \to \infty} \mathbb{E} \left[ |N_{o,\tau}^{c,A}(t)|^{2k} | \mathcal{F}_t \right] = k! (C_{2,A,M_{2,\sigma,0}})^k \quad \text{in probability,} \quad (4.10)$$
where $\mathbb{M}^{\text{SH}}_{1,0}$ is the martingale defined in (4.8). Moreover, for $k' < k$,

$$
\lim_{r \uparrow \infty} \lim_{t \to \infty} \mathbb{E} \left[ N^{\varepsilon, A}_{\sigma, \tau}(t)^k N^{\varepsilon, A}_{\sigma, \tau}(t)^{k'} \bigg| \mathcal{F}_r \right] = 0 \quad \text{in probability.} \tag{4.11}
$$

**Proof.** The proof is a rerun of the proof of Lemma 3.4 with the only difference that the martingale $\mathbb{M}^{\text{SH}}_{1,0}$ only converges in probability towards $\mathbb{M}^{\text{SH}}_{1,0}$ as $t \uparrow \infty$ and that there is an additional factor $r^{1/4}$ needed. □

Since (4.10) and (4.11) only hold in probability, using the same method as in the proof of Theorem 1.4, we get the corresponding weak convergence result.

(ii) For the triple point, the argument is similar to (i) but with the moments as given in Lemma 4.1 with $\mathbb{M}^{\sigma,0}_2$ replaced by $\mathbb{M}^{\text{SH}}_{1,0}$. □

### 4.4. The boundary between phases $B_1$ and $B_2$

In this section, we prove Theorem 1.3.

**Proof of Theorem 1.3.** For $\beta \in \bar{B}_1 \cap \bar{B}_2 \setminus \{\beta = \sqrt{2}, \beta = \frac{1}{\sqrt{2}}(1 + i)\}$, consider in the same way as in the proof of Theorem 1.2 the $\frac{\sigma^2}{\gamma}$-moment for some $\gamma > \sigma$ and $\sqrt{2}\gamma > 1$. Then, a rerun of the computation starting from (2.1) up to (2.11) bounds the $\frac{\sigma^2}{\gamma}$-moment from above by

$$
\int_0^t dq \, e^{(\sigma^2 - \gamma^2)(\sigma^2 - \tau^2) \frac{\sigma^2}{\gamma^2}} \frac{\sigma^2}{\gamma^2} e^{-\frac{\sigma^2}{\gamma^2} q + \frac{\gamma^2}{\gamma^2} q} = \int_0^t dq \, e^{-\frac{\sigma^2}{\gamma^2} q} = \int_0^t dq \, e^{-\frac{\gamma^2}{\gamma^2} q},
$$

since $|\tau| + |\sigma| = \sqrt{2}$. The r.h.s. of (4.12) is uniformly bounded by a constant. Hence, $\mathcal{M}_{\sigma, \tau}(t)$ is in $L^p$ for some $p > 1$. Hence, it converges a.s. and in $L^1$. The limit is non-degenerate because $\mathbb{E} [\mathcal{M}_{\sigma, \tau}(t)] = 1$ and Theorem 1.3 follows. □

## 5. Proof of Theorem 1.1

In this section, as a consequence of the fluctuation results of the previous sections, we derive the phase diagram shown on Fig. 1.

**Proof of Theorem 1.1.** Convergence in probability for $\beta \in B_1$ and $B_3$ in (1.9) follows from Theorems 1.2 and 1.4 (ii) by [21, Lemma 3.9 (1)]. Convergence for the glassy phase $\beta \in \bar{B}_2$ was shown in [18]. For the boundaries between all three phases, the formula (1.9) follows from the continuity of the limiting log-partition function. □

### References

[1] E. Aïdékon and Zh. Shi. The Seneta-Heyde scaling for the branching random walk. *Ann. Probab.*, 42(3):959–993, 2014.

[2] G. Alsmeyer and M. Meiners. Fixed points of the smoothing transform: two-sided solutions. *Probab. Theory Related Fields*, 155(1-2):165–199, 2013.

[3] L.-P. Arguin, D. Belius, and P. Bourgade. Maximum of the characteristic polynomial of random unitary matrices. *Preprint*, 2015. Available at [http://arxiv.org/abs/1511.07399](http://arxiv.org/abs/1511.07399).

[4] L.-P. Arguin, D. Belius, and A.J. Harper. Maxima of a randomized Riemann Zeta function, and branching random walks. *Preprint*, 2015. Available at [http://arxiv.org/abs/1506.00629](http://arxiv.org/abs/1506.00629).
[5] L.-P. Arguin, A. Bovier, and N. Kistler. The extremal process of branching Brownian motion. *Probab. Theory Relat. Fields*, 157:535–574, 2013.

[6] K.B. Athreya and P.E. Ney. *Branching processes*. Springer-Verlag, New York-Heidelberg, 1972.

[7] J. Barral, X. Jin, and B. Mandelbrot. Convergence of complex multiplicative cascades. *Ann. Appl. Probab.*, 20(4):1219–1252, 2010.

[8] D. Belius and N. Kistler. The subleading order of two dimensional cover times. *Probab. Theory Relat. Fields*, 167(1):461–552, 2017.

[9] M. Biskup and O. Louidor. Extreme local extrema of two-dimensional discrete Gaussian free field. *Preprint*, 2013. Available at [https://arxiv.org/abs/1306.2602](https://arxiv.org/abs/1306.2602).

[10] M. Biskup and O. Louidor. Full extremal process, cluster law and freezing for two-dimensional discrete Gaussian Free Field. *Preprint*, 2016. Available at [https://arxiv.org/abs/1606.00510](https://arxiv.org/abs/1606.00510).

[11] A. Bovier. *Gaussian Processes on Trees: From Spin-Glasses to Branching Brownian Motion*. Cambridge University Press, 2016.

[12] A. Bovier and L. Hartung. The extremal process of two-speed branching Brownian motion. *Electron. J. Probab.*, 19(18):1–28, 2014.

[13] B. Derrida. Random-Energy Model: Limit of a Family of Disordered Models. *Phys. Rev. Lett.*, 45:79–82, 1980.

[14] B. Derrida. A generalization of the Random Energy Model which includes correlations between energies. *J. Physique Lett.*, 46(9):401–407, 1985.

[15] B. Derrida. The zeroes of the partition function of the random energy model. *Physica A: Stat. Mech. Appl.*, 177:31–37, 1991.

[16] A. Dobrinevski, P. Le Doussal, and K.J. Wiese. Interference in disordered systems: A particle in a complex random landscape. *Phys. Rev. E*, 83(6):061116, 2011.

[17] Y.V. Fyodorov, G.A. Hiary, and J.P. Keating. Freezing transition, characteristic polynomials of random matrices, and the Riemann zeta function. *Phys. Rev. Lett.*, 108(17):170601, 2012.

[18] L. Hartung and A. Klimovsky. The glassy phase of the complex branching Brownian motion energy model. *Electron. Commun. Probab.*, 20, 2015.

[19] A. Iksanov and Z. Kabluchko. A central limit theorem and a law of the iterated logarithm for the Biggins martingale of the supercritical branching random walk. *Preprint*, 2015. Available at [https://arxiv.org/abs/1507.08458](https://arxiv.org/abs/1507.08458).

[20] A. Iksanov and M. Meiners. Fixed points of multivariate smoothing transforms with scalar weights. *ALEA Lat. Am. J. Probab. Math. Stat.*, 12(1):69–114, 2015.

[21] Z. Kabluchko and A. Klimovsky. Complex random energy model: zeros and fluctuations. *Probab. Theory Relat. Fields*, 158(1-2):159–196, 2014.

[22] Z. Kabluchko and A. Klimovsky. Generalized random energy model at complex temperatures. *Preprint*, 2014. Available at [http://arxiv.org/abs/1402.2142](http://arxiv.org/abs/1402.2142).

[23] N. Kistler. Derrida’s random energy models. From spin glasses to the extremes of correlated random fields. In V. Gayrard and N. Kistler, editors, *Correlated Random Systems: Five Different Methods*. Springer, 2015.

[24] K. Kolesko and M. Meiners. Convergence of complex martingales in the branching random walk: The boundary. *Preprint*, 2016. Available at [https://arxiv.org/abs/1611.05220](https://arxiv.org/abs/1611.05220).

[25] A. E. Kyprianou and T. Madaule. The Seneta-Heyde scaling for homogeneous fragmentations. *Preprint*, 2015. Available at [https://arxiv.org/abs/1507.01559](https://arxiv.org/abs/1507.01559).

[26] S. P. Lalley and T. Sellke. A conditional limit theorem for the frontier of a branching Brownian motion. *Ann. Probab.*, 15(3):1052–1061, 1987.

[27] T. D. Lee and C. N. Yang. Statistical Theory of Equations of State and Phase Transitions. II. Lattice Gas and Ising Model. *Phys. Rev.*, 87:410–419, 1952.

[28] M. Meiners and S. Mentemeier. Solutions to complex smoothing equations. *Probab. Theory Relat. Fields*, pages 1–70, 2016.

[29] D. Panchenko. *The Sherrington-Kirkpatrick model*. Springer, 2013.

[30] R. Rhodes and V. Vargas. Gaussian multiplicative chaos and applications: a review. *Probability Surveys*, 11, 2014.

[31] O. Zeitouni. Branching random walks and Gaussian free fields. Lecture notes, 2013.
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