On the state space and dynamics selection in linear stochastic models: a spectral factorization approach

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Abstract—Matrix spectral factorization is traditionally described as finding spectral factors having a fixed analytic pole configuration. The classification of spectral factors then involves studying the solutions of a certain algebraic Riccati equation which parametrizes their zero structure. The pole structure of the spectral factors can be also parametrized in terms of solutions of another Riccati equation. We study the relation between the solution sets of these two Riccati equations and describe the construction of general spectral factors which involve both zero- and pole-flipping on an arbitrary reference spectral factor.

I. INTRODUCTION

An important and widely used class of models in control engineering and signal processing describes an $m$-dimensional observed random signal $\{y(t)\}$ as output of a linear system driven by white noise:

\[
\begin{align*}
  x(t+1) &= Ax(t) + Bw(t) \\
  y(t)   &= Cx(t) + Dw(t)
\end{align*}
\]

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{m \times n}$, $D \in \mathbb{R}^{m \times m}$, $w$ is a normalized white noise. The $n$-dimensional signal $x$ is the state vector. The basic steps for the constructions of models of the form \textnormal{(1)} from observations of $\{y(t)\}$ lead to the following three problems which in various forms permeate all linear systems and control theory:

1) Estimate the spectral density $\Phi_y(z)$ of $y$, see \textnormal{[5], [8], [9], [15]-[17]} and references therein.

2) Compute a stochastically minimal\textsuperscript{1} spectral factor of $\Phi_y(z)$, i.e. a matrix transfer function $W(z)$ such that

\[
\Phi_y(z) = W(z)W^\dagger(z^{-1}),
\]

see \textnormal{[2], [3]} and references therein.

3) Fix a minimal realization $W(z) = C(zI - A)^{-1}B + D$ to provide a parametrization of the model \textnormal{(1)}.

The literature on these topics being enormous we have chosen to quote only a few recent papers in which one can find a more extensive bibliography. The study of models \textnormal{(1)} of the signal $y$ without a priori constraints of causality or analyticity is exposed in the recent book \textnormal{[11]}. The objective of this paper is to continue the analysis and study in more depth the relations among different models \textnormal{(1)} which are in a sense equivalent as they serve to represent the same process but may have different system-theoretic structure and properties.

\textsuperscript{1}Stochastic minimality means that we are only interested in models of minimal complexity so that we only consider spectral factors $W(z)$ of minimal McMillan degree.

Indeed, representations \textnormal{(1)} have several degrees of freedom. The most obvious (and least interesting) one is the choice of basis in the input and in the state space. In particular, the matrices $A, B, C, D$ in step 3. are determined up to a transformation of the form $T^{-1}A, T^{-1}B, CT, DU$ where $T$ is an arbitrary invertible matrix and $U$ is an arbitrary orthogonal matrix. Once these degrees of freedom are factored out, we are left with two more interesting objects:

A. The state space as a coordinate free representative of a model \textnormal{(1)}

B. The (dynamical) causality structure (related in particular the choice of direction of the time arrow) of equivalent models.

One of the key result of stochastic realization theory (see \textnormal{[11]}) is that these two choices correspond, respectively, to the \textit{choice of zeros and poles} of the spectral factor $W(z)$ in \textnormal{(2)}. Each pole configuration of the spectral factor corresponds to a certain causality structure so that, once this configuration is fixed, one is left with the choice of the zero structure of the spectral factor, which just means choosing a (minimal) state space of the realization.

Matrix spectral factorization is traditionally described as finding spectral factors having a fixed analytic pole configuration so that all corresponding models are causal, and classifying different models corresponds to parametrizing all possible zero structures of $W$. However, a zero structure fixes, independent of causality, a possible minimal state space\textsuperscript{2} for $y$. Hence, once a minimal state space (i.e. the zero structure of $W$) is fixed, there is a whole family of possible causality structures which can be parametrized by the allowed pole locations of a spectral factor $W$.

If some minimal \textit{reference} spectral factor is fixed, minimal spectral factorization can be seen as a \textit{zero- or pole- flipping} transformation performed on the reference factor. In this paper we analyse the interplay between this two operations in relation to the solutions sets of two families of algebraic Riccati equations. We derive closed-form formulas that allow to compute the model corresponding to a given causality structure and state space. This may be viewed as the completion of an endeavour first undertaken in \textnormal{[12]} in continuous time but not pushed to the final consequences. Here we shall address the discrete-time situation and give a complete solution.

Although our main motivation is stochastic modelling, our contribution can also be viewed as related to algebraic Riccati equations and to spectral factorization. Both have important applications in several areas of control, signal processing and

\textsuperscript{2} We stress that the choice of the state space must not be confused with the choice of basis in $\mathbb{R}^n$.}
II. BACKGROUND ON SPECTRAL FACTORIZATION AND ALGEBRAIC RICCATI EQUATIONS

Let $\Phi(z)$ be a $m \times m$ rational spectral density matrix of a regular stationary process, where regularity is meant in the sense explained in [11] Sec. 6.8 and let

$$W(z) := C(zI - A)^{-1}B + D$$

be a minimal realization of a minimal square spectral factor of $\Phi(z)$ so that $\Phi(z) = W(z)W(z)^*$, where $W(z)^* := W(z^{-1})^\top$ is the conjugate transpose. By regularity the matrix $D$ is non-singular, [6]; it will be assumed to be symmetric and positive definite: this rules out the uninteresting degree of freedom corresponding to multiplying a spectral factor on the right side by a constant orthogonal matrix.

By regularity the numerator matrix $\Gamma := A - BD^{-1}C$ is non-singular (see Theorem 6.8.2 in [11]). In this paper we shall moreover assume that both $A$ and $\Gamma$ are unimodular. Note in particular that we do not assume analyticity of $W(z)$ outside of the unit disk. For the relevant definitions and facts about spectral factorization in this context we shall refer to Chap 16 of the book [11].

**Definition 2.1:** Let $W_i(z)$; $i = 1, 2$ be minimal spectral factors of the same rational spectral density. We shall say that $W_1(z)$ and $W_2(z)$ have the same pole structure if they admit a state space realization with the same state transition matrix. Likewise, we say that $W_1(z)$ and $W_2(z)$ have the same zero structure if they admit a state space realization with the same numerator matrix.

In classical spectral factorization one assumes that the state matrix $A$ has all eigenvalues inside the unit circle and one aims at classifying all different minimal spectral factors having a fixed (analytic) pole structure, in terms of their zero structure, equivalently, in terms of invariant subspaces for the transpose of the numerator matrix $\Gamma$. It is well-known that this involves the study of an algebraic Riccati equation. In the present context we have the following result, which has appeared in several places in the literature.

**Proposition 2.1:** Let $W_0(z) := C(zI - A)^{-1}B + D$ be a minimal realization of a square reference spectral factor.

1) There is a one-to-one correspondence between symmetric solutions of the homogeneous algebraic Riccati equation

$$P = \Gamma P \Gamma^\top - \Gamma P C^\top (D D^\top + C P C^\top)^{-1} C P \Gamma^\top$$

(4)

and minimal spectral factors of $\Phi(z)$ having the same pole structure of $W_0(z)$. This correspondence is defined by the map assigning to each solution $P$ the spectral factor

$$W_P(z) := C(zI - A)^{-1}B_P + D_P$$

(5)

where

$$B_P := (BD^\top + APC^\top)(DD^\top + CPC^\top)^{-1/2};$$

$$D_P := (DD^\top + CPC^\top)^{1/2}.$$  

2) There is a one-to-one correspondence between symmetric solutions of (4) and $\Gamma^\top$-invariant subspaces which is defined by the map assigning to each solution $P$ the $\Gamma^\top$-invariant subspace $\ker(P)$.

For a proof we shall just refer the reader to Corollary 16.5.7 and Lemma 16.5.8 in [11] where the equation differs by an inessential change of sign. A similar Riccati equation although in a different context is studied in [12].

In particular, let $P_+$ be the unique non singular solution of (4), then the corresponding $\Gamma^\top-$invariant subspace $\ker P_+$ is trivial and the zeros of $W_0(z)$ are all flipped to reciprocal positions. This follows from standard Riccati theory. We shall denote the corresponding spectral factor by $W_+(z)$.

Zero-flipping can also be visualized as right-multiplication of $W_0(z)$ by a suitable square all-pass function so as to preserve minimality. The entailed factorization of $W_P(z)$ is in turn uniquely identified by the existence of a $\Gamma^\top-$invariant subspace [4].

On the other hand, we have the following fact which describes the pole-flipping relation among spectral factors keeping a fixed zero structure. The result can be traced back to Theorem 16.4.2 of [11].

**Proposition 2.2:** Let $W_0(z) := C(zI - A)^{-1}B + D$ be a minimal realization of a square reference spectral factor.

1) There is a one-to-one correspondence between symmetric solutions of the algebraic Riccati equation

$$Q = A^\top QA - A^\top QB (I + B^\top QB)^{-1} B^\top QA,$$  

(7)

and minimal normalized spectral factors having the same zero structure of $W_0(z)$. This correspondence is defined by the map assigning to each solution $Q$ the spectral factor

$$W_Q(z) := C_Q(zI - A_Q)^{-1}B_Q + D_Q,$$  

(8)

where

$$\Delta_Q := I + B^\top QB,$$

$$C_Q := C - D \Delta_Q^{-1} B^\top QA,$$

$$A_Q := A - B \Delta_Q^{-1} B^\top QA,$$

$$B_Q := B \Delta_Q^{-1/2} U,$$

$$D_Q := D \Delta_Q^{-1/2} U,$$

and $U$ is the orthogonal matrix

$$U := (D \Delta_Q^{-1/2})^\top ((D \Delta_Q^{-1/2})(D \Delta_Q^{-1/2})^\top)^{-1/2}$$

which is selected in such a way that $D_Q$ is symmetric and positive definite.

2) There is a one to one correspondence between symmetric solutions of (7) and $A$-invariant subspaces which is defined by the map assigning to each solution $Q$ the $A$-invariant subspace $\ker(Q)$.

Anym solution $P$ can actually be seen as the difference say $X - X_0$ of two arbitrary solutions of an equivalent Riccati equation parametrizing the minimal spectral factors which is defined directly in terms of a realization of $\Phi$ and does not involve a reference spectral factor, see [11] Sect. 16.5]. Here $X_0$ is kept fixed as a reference solution and $\Gamma$ describes the zero structure of the reference spectral factor $W_0$.  

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Proof. That the zero structures of $W_Q(z)$ and of $W_0(z)$ coincide is the content of Theorem 16.4.5 in [11]. The rest is readily checked.

III. COMBINING POLE AND ZERO FLIPPING

We want to understand the combination of zero- and pole-flipping leading to an arbitrary minimal square spectral factor $W$. To this end let's consider the spectral factor $W_Q(z)$ defined in [8] as a reference spectral factor and describe the zero-flipping process on $W_Q(z)$. By direct computation we easily find that the numerator matrix of $W_Q(z)$ is the same of the numerator matrix of $W_0(z)$, i.e. the matrix $\Gamma$. Hence the Riccati equation (4) corresponding to $W_Q(z)$ takes the form

$$P_Q = \Gamma P_Q \Gamma^\top - \Gamma P_Q C_Q \,(D_Q D_Q^\top + C_Q P_Q C_Q^\top)^{-1} C_Q P_Q \Gamma^\top$$

(10)

where $C_Q$ is as defined in [9] and $D_Q := D_{\Delta_{Q}}^{-1/2, U}$. Notice that, since equations (4) and (10) involve the same matrix $\Gamma$ and each symmetric solution of either equation is uniquely attached to a $\Gamma^\top$-invariant subspace [13], the map assigning to each solution $P$ of (4) the solution $P_Q$ of (10) such that $\ker(P) = \ker(P_Q)$ is a one to one correspondence between the set $\mathcal{P}$ and the set $\mathcal{P}_Q$ of symmetric solutions of (4) and (10).

Our main contribution is to analyze the relations between $\mathcal{P}$ and $\mathcal{P}_Q$ and to provide an explicit formula to compute the solution $P_Q$ from a give pair $P, Q$. In this way once parametrized the solutions of (4) and (7), we do not need to solve (10) and we have a closed-form formula for the spectral factor with assigned pole and zero structure, or equivalently for the model with assigned state-space and causality structure.

We may of course consider a dual path to transform $W_0$ into $W$ by taking instead the zero-flipped $W_{\bar{P}}$ as a reference and flipping poles by considering a solution $Q_P$ of a Riccati equation similar to (7) so as to make the following diagram commutative:

$$
\begin{array}{c}
W_0 \xrightarrow{P} W_P \\
Q \downarrow \quad \downarrow Q_P \\
W_Q \xrightarrow{P_Q} W
\end{array}
$$

(11)

The resulting spectral factor should have been denoted $W_{P_{\bar{Q}}}$ but the simplified notation $W$ here should not be cause of confusion.

It is well-known that both (4) and (10) have a unique non-singular solution which we denote by $P_+$ and $P_{Q+}$, respectively. The relation between these two solutions is the content of the following lemma.

**Lemma 3.1:** The nonsingular solutions $P_+$ and $P_{Q+}$ are related by the formula

$$P_{Q+}^{-1} = Q + P_+^{-1}.$$  \hspace{1cm} (12)

**Proof.** It is immediate to check that $P_+^{-1}$ is the (unique) solution of the discrete-time Lyapunov equation

$$P_+^{-1} - \Gamma^\top P_+^{-1} \Gamma + C^\top D^{-\top} D^{-1} C = 0.$$  \hspace{1cm} (13)

Similarly, $P_{Q+}^{-1}$ is the (unique) solution of the discrete-time Lyapunov equation

$$P_{Q+}^{-1} - \Gamma^\top P_{Q+}^{-1} \Gamma + C_Q^\top D_Q^{-\top} D_Q^{-1} C_Q = 0.$$  \hspace{1cm} (14)

Therefore, the difference $\Delta := P_{Q+}^{-1} - P_+^{-1}$ is the (unique) solution of the discrete-time Lyapunov equation

$$\Delta - \Gamma^\top \Delta \Gamma + C_Q^\top D_Q^{-\top} D_Q^{-1} C_Q - C^\top D^{-\top} D^{-1} C = 0.$$  \hspace{1cm} (15)

We now compute

$$R_Q := C_Q^\top D_Q^{-\top} D_Q^{-1} C_Q - C^\top D^{-\top} D^{-1} C = C^\top D^{-\top} B Q B D^{-1} C - C^\top D^{-\top} B^\top Q A D^{-1} C + A^\top Q B D^{-1} C - A^\top Q B D^{-1} C.$$  \hspace{1cm} (16)

This equation, together with (15) gives

$$\Delta - \Gamma^\top \Delta \Gamma = Q - \Gamma^\top Q \Gamma,$$  \hspace{1cm} (17)

and, by uniqueness, $\Delta := P_{Q+}^{-1} - P_+^{-1} = Q$, so that (12) follows.

Since $\ker(P_+ = \{0\}$, all zeros of the corresponding spectral factor, denoted by the symbol $W_+(z)$, are those of $W_0$ flipped to their reciprocals. The same happens for $W_Q$ whatever solution $Q$ of (7) is chosen. In particular, denoting the nonsingular solution of (7) by $Q_+$, all poles of the corresponding spectral factor, say $W_0(z)$, will be the reciprocals of those of $W_0(z)$. The commutative diagram (11) takes on the form

$$
\begin{array}{c}
W_0 \xrightarrow{P_+} W_+ \\
Q_+ \downarrow \quad \downarrow [Q_{P_+}]_+ \\
W_0 \xrightarrow{[P_{Q_+}]_+} W_+
\end{array}
$$

where $[Q_{P_+}]_+$ and $[P_{Q_+}]_+$ are the invertible solutions of the Riccati equations which are respectively flipping the poles of $W_+(z)$ and the zeros of $W_0(z)$. Hence, both poles and zeros of $W_+$ are the reciprocals of those of $W_0(z)$. This corresponds to “total” flipping of singularities.

We would now like to derive an explicit formula generalizing (12) to a generic solution $P_Q$ to a function of $P$ and $Q$. To this end we shall use the following lemma which is a particular case of [11] Theorem 2.2]. An analogous result is Statement 1. (iii) of Theorem 3.1 in [12] although referring to the specific case of all-pass functions.

**Lemma 3.2:** Any solution $P$ of the Riccati equation (4) corresponding to a $\Gamma^\top$-invariant subspace $\mathcal{S}$ can be expressed by the formula

$$P = [(I - \Pi_\mathcal{S}) P_+^{-1} (I - \Pi_\mathcal{S})]_{+}^\top$$

(17)
where \(^\dagger\) denotes Moore-Penrose pseudoinverse and \(\Pi_S\) is the orthogonal projector onto the subspace \(S = \ker P\).

We are now ready to present our main result.

**Theorem 3.1:** Let \(P\) be an arbitrary solution of \(4\). Then the unique solution \(P_Q\) of \(10\) such that \(\ker(P) = \ker(P_Q)\) can be expressed by the formula

\[
P_Q = [PP^\dagger QQP^\dagger P^\dagger]^\dagger
\]

which generalizes \(12\).

**Proof.** Since \((I - \Pi_S)\) projects onto the range space of \(P\), a basic property of the Moore-Penrose pseudoinverse \(10\) (p. 421) implies that \((I - \Pi_S) = PP^\dagger\) so that \(17\) can be rewritten

\[
P = [PP^\dagger P^\dagger P^\dagger]^\dagger
\]

and hence

\[
P^\dagger = PP^\dagger P^\dagger P^\dagger.
\]

Now, since \(P\) and \(P_Q\) have the same kernel they also have the same image so that the orthogonal projectors on this image may be written in two ways as:

\[
I - \Pi_S = PP^\dagger = P_Q P_Q^\dagger.
\]

Thus, the analog of formula \(17\) for \(P_Q\) yields

\[
P_Q = \left[(I - \Pi_S)P_Q^\dagger (I - \Pi_S)\right]^\dagger = \left[PP^\dagger P^\dagger P^\dagger\right]^\dagger
\]

where \(P_Q^\dagger\) is the only non-singular solution of \(10\) (such a solution corresponds to the \(\Gamma^\top\)-invariant subspace \(\{0\}\)).

Hence, after inserting \(12\), we get

\[
P_Q = [PP^\dagger (Q + P^\dagger P^\dagger)]^\dagger
\]

and, finally, by using \(19\) we obtain the following explicit expression for \(P_Q\) depending only on \(P\) and \(Q\):

\[
P_Q = [PP^\dagger QQP^\dagger P^\dagger + P^\dagger]^\dagger.
\]

\[
\square
\]

Finally, let us consider two arbitrary \(\Gamma^\top\) and \(A\)-invariant subspaces \(X\) and \(Y\) which is to say two arbitrary zero and pole flipping transformations of the singularities of \(W_0(z)\) or, equivalently, an arbitrary state space and causality configuration for the model \(\Pi\). Suppose we want to compute the corresponding minimal spectral factor \(W(z)\), or equivalently the corresponding model \(\Pi\). Let \(P\) and \(Q\) be the solutions of the Riccati equations \(4\) and \(7\) corresponding to the invariant subspaces \(X\) and \(Y\) and consider the left lower path in the commutative diagram \(11\) so that the zero flipping is done after a pole flipping defined by \(Q\). The relevant Riccati solution \(P_Q\) is given in formula \(18\) so that the desired realization of \(W(z)\) can be explicitly written in closed form as

\[
W_Q(z) := C_Q(zI - A_Q)^{-1}B_{P_Q} + D_{P_Q}
\]

where

\[
B_{P_Q} := (B_QD_Q^\top + A_QP_QC_Q^\top)(D_QD_Q^\top + C_QP_QC_Q^\top)^{-1/2}
\]

\[
D_{P_Q} := (D_QD_Q^\top + C_QP_QC_Q^\top)^{1/2},
\]

\(P_Q\) is given by \(18\) and \(A_Q, B_Q, C_Q, D_Q\) are given by \(6\).

Naturally, an analogous procedure would work by following the upper right path; i.e. computing first \(P\) and then performing the appropriate pole flipping defined by \(Q\).

**Conclusion**

We have discussed the classification of general (not necessarily analytic) square spectral factors in terms of the solutions of two algebraic Riccati equations. We have also described the construction of general spectral factors which involve both zero- and pole-flipping on an arbitrary reference spectral factor.

**References**

[1] D. Alpago and A. Ferrante, Families of Solutions of Algebraic Riccati Equations. Submitted to *Automatica*. Preprint available at [arXiv:1801.09557](https://arxiv.org/abs/1801.09557) 2018.

[2] G. Baggio, and A. Ferrante. On Minimal Spectral Factors with Zeros and Poles lying on Prescribed Regions. *IEEE Trans. Automatic Control*, Vol. AC-61(8):2251–2255, DOI: 10.1109/TAC.2015.2484330, 2016.

[3] G. Baggio, and A. Ferrante. On the Factorization of Rational Discrete-Time Spectral Densities. *IEEE Trans. Automatic Control*, Vol. AC-61(4):969–981, DOI: 10.1109/TAC.2015.2446851, 2016.

[4] H. Bart, I. Gohberg and R. Kaashoek, *Minimal Factorization of Matrices and Operator Functions*, Operator Theory 1, Birkhäuser Verlag, Basel 1984.

[5] A. Ferrante, C. Masiero, and M. Pavon. Time and Spectral Domain Relative Entropy: A New Approach to Multivariate Spectral Estimation. *IEEE Trans. Automatic Control*, Vol. AC-57(10):2561–2575, 2012.

[6] A. Ferrante, G. Picci, and S. Pinzoni. Silverman Algorithm and the Structure of Discrete-Time Stochastic Systems. *Linear Algebra and its Applications*, Vol. 351–352:219–242, 2002.

[7] A. Ferrante and G. Picci, Representation and Factorization of Discrete-Time Rational All-Pass Functions, *IEEE Transactions on Automatic Control*, vol. 62, No. 7, July 2017.

[8] T. Georgiou. Spectral analysis based on the state covariance: the maximum entropy spectrum and linear fractional parameterization. *IEEE Trans. Aut. Control*, 47:1811–1823, 2002.

[9] T. Georgiou. Relative entropy and the multivariable dimensionless moment problem. *IEEE Trans. Inform. Theory*, 52:1052–1066, 2006.

[10] R. A. Horn and C.R. Johnson *Matrix Analysis* Cambridge U.P. 1985.

[11] A. Lindquist and G. Picci, *Linear Stochastic Systems: A Geometric Approach to Modeling, Estimation and Identification*, Springer series in Contemporary Mathematics, Heidelberg, Germany: Springer Verlag, 2015.

[12] G. Picci and S. Pinzoni: Acausal Models and Balanced realizations of stationary processes *Linear Algebra and its Applications*, vol. 205–206, pp. 957–1003, 1994.

[13] J. C. Willems, Least squares stationary optimal control and the algebraic Riccati equations, *IEEE Transactions on Automatic Control*, vol. 16: 621–634, 1971.

[14] Harald K. Wimmer, A Parametrization of Solutions of the Discrete-time Algebraic Riccati Equation Based on Pairs of Opposite Unmixed Solutions *SIAM Journal on Control and Optimization* Vol. 44 n. 6, 2006.

[15] M. Zorzi. A new family of high-resolution multivariate spectral estimators. *IEEE Trans. Autom. Control*, 59(4):892–904, Apr. 2014.

[16] M. Zorzi. Multivariate Spectral Estimation based on the concept of Optimal Prediction. *IEEE Trans. Autom. Control*, 60:1647–1652, Jun. 2015.

[17] M. Zorzi. Rational approximations of spectral densities based on the alpha divergence. *Mathematics of Control, Signals, and Systems*, 26(2):259–278, 2014.