An invariant analytic orthonormalization procedure with applications

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Abstract

We apply the orthonormalization procedure previously introduced by two of us and adopted in connection with coherent states to Gabor frames and other examples. For instance, for Gabor frames we show how to construct \( g(x) \in \mathcal{L}^2(\mathbb{R}) \) in such a way the functions \( g_n(x) = e^{ian_1x}g(x+an_2), n \in \mathbb{Z}^2 \) and \( a \) some positive real number, are mutually orthogonal. We discuss in some details the role of the lattice naturally associated to the procedure in this analysis.
I Introduction and mathematical results

In the mathematical and physical literature many examples of complete sets of vectors in a given Hilbert space \( \mathcal{H} \) are constructed starting from a single normalized element \( f_0 \in \mathcal{H} \), acting on this vector several times with a given set of unitary operators. This is exactly what happens, for instance, for coherent states and for wavelets, as well as for Shannon systems and Gabor frames. All these examples can be considered as particular cases of a general procedure in which a certain set of vectors is constructed acting on a fixed element of \( \mathcal{H} \), \( f_0 \), with a certain set of unitary operators, \( A_1, \ldots, A_N \): \( f_{k_1,\ldots,k_N} := A_{k_1}^{1} \cdots A_{k_N}^{N} f_0 \), \( k_j \in \mathbb{Z} \) for all \( j = 1, 2, \ldots, n \). These vectors may or may not be orthogonal: we consider here the problem of orthonormalizing this set, i.e. the problem of producing a new set of vectors which share with the original one most of its features (we will be more precise in the following) and, moreover, are also orthonormal. In [1] we have constructed a general strategy for achieving this aim and we have applied it to coherent states, producing a new set of square-integrable functions which have many of the properties of coherent state and, moreover, are mutually orthogonal. These results will be reviewed in Section II, where we also describe the method.

In Section III we apply the strategy to Shannon frame, while Section IV is dedicated to a deep analysis of Gabor frames. In particular we will discuss the role of what we call the lattice spacing in the orthonormalization procedure.

Section V is devoted to some examples, which we will discuss in detail, and to our conclusions.

II Stating the problem and first results

Let \( \mathcal{H} \) be a Hilbert space, \( f_0 \in \mathcal{H} \) a fixed element of the space and \( A_1, \ldots, A_N \), \( N \) given unitary operators: \( A_j^{-1} = A_j^{1} \), \( j = 1, 2, \ldots, N \). Let \( \mathcal{H}_N \) be the closure of the linear span of the set
\[
\mathcal{N}_N = \{ f_{k_1,\ldots,k_N} := A_{k_1}^{1} \cdots A_{k_N}^{N} f_0, \ k_1, \ldots k_N \in \mathbb{Z} \} \tag{2.1}
\]
which we assume to consist of linearly independent vectors in order not to be trivial. Of course the vectors \( f_{k_1,\ldots,k_N} \) are complete in \( \mathcal{H}_N \) by definition and \( \mathcal{H}_N \) could coincide or not with all of \( \mathcal{H} \). In general there is no reason why the vectors in \( \mathcal{N}_N \) should be mutually orthogonal. On the contrary, without a rather clever choice of both \( f_0 \) and \( A_1, \ldots, A_N \), it is very unlikely to obtain such an o.n. set. Our aim is to discuss some general technique
which produces another normalized vector $\varphi \in \mathcal{H}_N$ such that the set

$$\mathcal{M}_N = \{ \varphi_{k_1, \ldots, k_N} := A_{k_1}^{k_1} \cdots A_{k_N}^{k_N} \varphi, \ k_1, \ldots, k_N \in \mathbb{Z} \}$$

(2.2)
is made of orthogonal vectors. Moreover, we would like this set to share as much of the original features of $\mathcal{N}_N$ as possible. For instance, if the set $\mathcal{N}_N$ is a set of coherent states, we have shown in [1] that the new vectors $\varphi_{k_1, \ldots, k_N}$ are, among the other features, eigenstates of a (sort of) annihilation operator, give rise to a resolution of the identity and they saturate the Heisenberg uncertainty relation: so they appear, in some way, as a set of generalized orthonormal coherent states.

The easiest situation, $N = 1$, goes like this: in this case we have $\mathcal{N}_1 = \{ f_k := A^k f_0, \ k \in \mathbb{Z} \}$ with $< f_k, f_l > \neq \delta_{k,l}$ (otherwise we have already solved the problem!). Since $\mathcal{N}_1$ is complete in $\mathcal{H}_1$, any element in $\mathcal{H}_1$ can be written in terms of the vectors of $\mathcal{N}_1$. Let $\varphi_0 \in \mathcal{H}_1$ be the following linear combination $\varphi_0 = \sum_{k \in \mathbb{Z}} c_k f_k$, and let us define more vectors of $\mathcal{H}_1$ as $\varphi_n = A^n \varphi_0 = \sum_{k \in \mathbb{Z}} c_k f_{k+n} = X f_n$, where we have introduced the operator $X = \sum_{k \in \mathbb{Z}} c_k A^k$. The coefficients $c_k$ should be fixed by the following orthogonalization requirement: $< \varphi_n, \varphi_0 > = \delta_{n,0}$, for $n \in \mathbb{Z}$. As we have discussed in [1], these expansions are, up to this stage, purely formal. However, in many concrete relevant situations they converge and, if this is so, we find that the coefficients are

$$c_l = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-ipl} dp}{\sqrt{\alpha(p)}}$$

(2.3)

where the function $\alpha(p)$ is defined as $\alpha(p) = \sum_{l \in \mathbb{Z}} a_l e^{ipl}$, with $a_j = < A^j f_0, f_0 >$. A sufficient condition for $\alpha(p)$ to exist is that $\{ a_j \} \in l^1(\mathbb{Z})$, which, in turns, is related to the nature of both $A$ and $f_0$. We refer to [1] and to the next Section for more results and examples concerning $N = 1$.

Let us now take $N = 2$. We quickly review here our previous results on coherent states.

Let $\hat{q}$ and $\hat{p}$ be the position and momentum operators on the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$, $[\hat{q}, \hat{p}] = i \mathbb{I}$, and let us now introduce the following unitary operators:

$$U(n) = e^{ia(n_1 \hat{q} - n_2 \hat{p})}, \quad D(n) = e^{iz_\mathbb{N} l_1 - z_\mathbb{N} b}, \quad T_1 := e^{i\hat{a} \hat{q}}, \quad T_2 := e^{-i\hat{a} \hat{p}}.$$  
(2.4)

Here $a$ is a real constant satisfying $a^2 = 2\pi L$ for some $L \in \mathbb{N}$, while $z_\mathbb{N}$ and $b$ are related to $n = (n_1, n_2)$ and $\hat{q}$, $\hat{p}$ via the following equalities:

$$z_\mathbb{N} = \frac{a}{\sqrt{2}} (n_2 + in_1), \quad b = \frac{1}{\sqrt{2}} (\hat{q} + i\hat{p}).$$

(2.5)
With these definitions it is clear that

\[ U(n) = D(n) = (-1)^{L_{n_1}n_2}T_1^{n_1}T_2^{n_2} = (-1)^{L_{n_1}n_2}T_2^{n_2}T_1^{n_1}, \]

where we have also used the commutation rule \([T_1, T_2] = 0\).

Let \(\varphi_0\) be the vacuum of \(b, b\varphi_0 = 0\), and let us define the following coherent states:

\[ \varphi^{(L)}_{\mathbf{n}} := T_1^{n_1}T_2^{n_2}\varphi_0 = T_2^{n_2}T_1^{n_1}\varphi_0 = (-1)^{L_{n_1}n_2}U(n)\varphi_0 = (-1)^{L_{n_1}n_2}D(n)\varphi_0. \]

1. It is very well known that the set of these vectors, \(\mathcal{C}^{(L)} = \{\varphi^{(L)}_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}^2\}\), satisfies a number of relevant properties. However, it is also well known that they are not mutually orthogonal. Indeed we have \(I^{(L)}_{\mathbf{n}} := \langle \varphi^{(L)}_{\mathbf{n}}, \varphi^{(L)}_{\mathbf{0}} \rangle = (-1)^{L_{n_1}n_2}e^{-\frac{L}{2}(n_1^2+n_2^2)}\), which is only nearly zero if \(L\) is large enough and \((n_1, n_2) \neq (0, 0)\).

Our aim is to construct a family of vectors \(\mathcal{E}^{(L)}\) which shares with \(\mathcal{C}^{(L)}\) most of the above features and which, moreover, is made of orthonormal vectors. We have shown in \([1]\) that this is possible, in suitable Hilbert spaces, if \(L > 1\), while complications arise for \(L = 1\). Incidentally, we have shown that the set \(\mathcal{C}^{(L)}\) is complete in \(\mathcal{H}\) if and only if \(L = 1\).

Let us now define, for each \(L \geq 1\), the following set:

\[ h_L := \text{linear span}\left\{ \varphi^{(L)}_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}^2 \right\}. \]

We know that \(h_1 = \mathcal{H}\), while, for \(L > 1\), \(h_L \subseteq \mathcal{H}\). It is further clear that \(h_L\) is an Hilbert space for each \(L\), since it is a closed subspace of \(\mathcal{H}\).

Now we define

\[ \Psi^{(L)}_{\mathbf{n}} := \sum_{k \in \mathbb{Z}^2} c_k^{(L)} \varphi^{(L)}_{\mathbf{k}+\mathbf{n}} \]

Of course this means that \(\Psi^{(L)}_{\mathbf{n}} := \sum_{k \in \mathbb{Z}^2} c_k^{(L)} \varphi^{(L)}_{\mathbf{k}}\) and, because of the commutativity of \(T_1\) and \(T_2\), we also have

\[ \Psi^{(L)}_{\mathbf{n}} = T_1^{n_1}T_2^{n_2}\Psi^{(L)}_{\mathbf{0}}. \]

Therefore the new set constructed in this way, \(\mathcal{E}^{(L)} := \{\Psi^{(L)}_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}^2\}\), is invariant under the action of \(T_1\) and \(T_2\), exactly as the set \(\mathcal{C}^{(L)}\), independently of the choice of the coefficients of the expansion \(c_k^{(L)}\). These coefficients, however, must not be chosen freely:

\(^1\text{(a) \(\mathcal{C}^{(L)}\) is invariant under the action of \(T_j^{n_j}\), for all \(n_j, j = 1, 2\); (b) each \(\varphi^{(L)}_{\mathbf{n}}\) is an eigenstate of \(b\): \(b\varphi^{(L)}_{\mathbf{n}} = z_n\varphi^{(L)}_{\mathbf{n}}\); (c) they satisfy the resolution of the identity on a certain Hilbert space \(h_L\), see \([2, 3]\), \(\sum_{\mathbf{n} \in \mathbb{Z}^2} |\varphi^{(L)}_{\mathbf{n}}\rangle\langle \varphi^{(L)}_{\mathbf{n}}| = \mathbb{1}_L\), where \(\mathbb{1}_L\) is the identity on \(h_L\); (d) They saturate the Heisenberg uncertainty principle: if we call \((\Delta X)^2 = \langle X^2 \rangle - \langle X \rangle^2\) for \(X = \hat{q}, \hat{p}\), then \(\Delta \hat{q}\Delta \hat{p} = \frac{1}{2}\).}
they are fixed requiring that the vectors in the set $\mathcal{E}^{(L)}$ are orthonormal: $\langle \Psi_{\underline{L}}^{(L)}, \Psi_{\underline{L}}^{(L)} \rangle = \delta_{\underline{L},\underline{L}}$. This will fix (not uniquely!) the value of the $c_{\underline{L}}^{(L)}$’s, with a procedure which extends formula (2.3). These coefficients, if they exist, can be found as follows:

$$c_{\underline{L}}^{(L)} = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \frac{e^{-iP \cdot k}}{\sqrt{F_L(P)}} dP.$$  

(2.11)

where $F_L(P) = \sum_{m \in \mathbb{Z}} (-1)^{Lm_1m_2} e^{-\frac{\pi}{2} L(m_1^2+m_2^2)} e^{iP \cdot m}$.

As already mentioned, in [1] we have seen that $L = 1$ and $L > 1$ are really different situations: if $L > 1$ the above formulas are well defined and produce a set $\mathcal{E}^{(L)}$ which has the same properties of the vectors in $\mathcal{C}^{(L)}$ and, moreover, is made of orthonormal vectors. On the contrary, if $L = 1$, the procedure does not work: the series are not convergent and $F_L(P)$ has a zero in $[0, 2\pi \times [0, 2\pi]$. This is not related to our method but it is rather a well known feature of coherent states: indeed it is a standard result in functional analysis that in $\mathcal{H}$ no orthonormal set of coherent states can be constructed, [2]!

III Frames of translated

We begin this section by recalling few known results concerning frames of translated. After that, we will use our method to construct an o.n. set of translated.

Assume that $\phi \in L^2(\mathbb{R})$ and consider the set of the form $\mathcal{F} = \{\phi(\cdot - k)\}_{k \in \mathbb{Z}} = \{\hat{T}_k \phi\}$ where $\hat{T}_k$ is related to the translation operator in (2.4) as follows: $\hat{T}_k := T_k^a$, with $a = 1$. Christensen, Deng, and Heil in [4] by using the Beurling densities proved that $\mathcal{F}$ cannot be complete in all of $L^2(\mathbb{R})$ and that it can, at most, be a frame for a proper subspace of $L^2(\mathbb{R})$. This is in line with what we have discussed in the previous section: indeed the closure of the set $\mathcal{N}_1$ was not required to be all of $\mathcal{H}$. What our strategy can produce here is, at most, an o.n. set spanning the same Hilbert space which is spanned by the set $\mathcal{F}$.

Let us consider the function

$$\Phi : \mathbb{R} \to \mathbb{R}, \quad \Phi(p) = \sum_{k \in \mathbb{Z}} |\hat{\phi}(p+k)|^2$$  

(3.1)

where $\hat{\phi}$ is the Fourier transform of $\phi$, $\hat{\phi}(p) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \phi(x) e^{-ipx} dx$. The function $\Phi$ is 1-periodic and belongs to $L^1([0, 1])$. This function can be used to check whether the set $\mathcal{F}$ is or not a frame of translates. Indeed, see for instance [5] and references therein, the following proposition holds true:

**Proposition 1** Let $\phi \in \mathcal{L}^2(\mathbb{R})$ then for $A, B > 0$ the set $\mathcal{F}$ is an $(A, B)$-frame if and only if

$$A \leq \Phi(p) \leq B$$
a.e. in \( p \in [0, 1] \setminus \mathcal{C} \), where \( \mathcal{C} = \{ p \in [0, 1] : \Phi(p) = 0 \} \).

**Remark:** we want to stress that we are here adopting a notation which is slightly different when compared with [5]: in particular we still call frame a certain set even if it only spans a subset of the original Hilbert space, at least when this aspect is clear and does not lead to confusion. In other words, whenever the situation is clear from the context, we will say that a certain set of translated is a frame even if it does not spans all of \( \mathcal{H} \).

Other results on frames of translates are discussed in many details in [5], to which we refer for more details.

Let us now apply our procedure for \( N = 1 \) to a set of translated. For that we suppose that an \((A,B)\)-frame \( N_1 = \{ \varphi_j \in \mathcal{H} \}_{j \in \mathbb{Z}} \) is generated by a single fixed vector \( \varphi \) and a single unitary operator \( T \equiv T_2 \):

\[
\varphi_j = T^j \varphi, \ j \in \mathbb{Z}.
\]

So there exist \( A, B > 0 \) such that

\[
A \| f \|^2 \leq \sum_{j \in \mathbb{Z}} |\langle T^j \varphi, f \rangle|^2 \leq B \| f \|^2, \ \forall f \in \mathcal{H},
\]

where \( \mathcal{H} \) is the subspace of \( \mathcal{H} \) spanned by the vectors of \( N \). The construction of the o.n. set goes as in the first part of the previous section. Therefore we can define the coefficients \( a_j \) as usual, \( a_j = \langle T^j \varphi, \varphi \rangle \), and from these we find the function \( \alpha(p) \) and finally the coefficients \( c_k \)'s which produce the vector \( \varphi_0 = \sum_{k \in \mathbb{Z}} c_k f_k \) as in (2.3).

An example of this construction was already essentially discussed in [1]. We consider this same example here from a slightly different perspective.

**Example:** Let \( g(x) = \chi_{[0, a]}(x) \) be the characteristic function in the interval \( [0, a] \), with \( 1 < a < 2 \). Of course we have

\[
N_1 = \{ g_n(x) := T^n g(x) = \chi_{[n, n+a]}(x), \ n \in \mathbb{Z} \}.
\]

The overlap coefficients \( a_j \) can be written as \( a_j = a \delta_{j,0} + (a - 1) (\delta_{j,-1} + \delta_{j,1}) \), so that \( \alpha(p) = a + 2(a - 1) \cos(p) \). This is a nonnegative, real and 2\( \pi \)-periodic function and is never zero in \( [0, 2\pi] \) since it has a minimum in \( p = \pi \) and \( \alpha(\pi) = 2 - a > 0 \). If we fix, as an example, \( a = \frac{3}{2} \), we can compute analytically \( \sum_{l \in \mathbb{Z}} |c_l|^2 = \frac{1}{2\pi} \int_0^{2\pi} \frac{dp}{\alpha(p)} = \frac{2}{\sqrt{3}} \). We refer to [1] for further considerations on this example. Here we just want to comment that the function \( \Phi(p) = \sum_{k \in \mathbb{Z}} |\hat{g}(p + k)|^2 \) in (3.1) can be written as \( \Phi(p) = a + 2(a - 1) \cos(2\pi p) \),
and this satisfies the inequality $2 - a < \Phi(p) < 3a - 2$. Therefore, using Proposition 1 above, we conclude that the set $\{g_n(x), n \in \mathbb{Z}\}$ is a $(2 - a, 3a - 2)$-frame (which, of course, spans a proper subspace of $L^2(\mathbb{R})$). So we have a frame which, by means of our strategy, produces an o.n. set which spans the same Hilbert subspace of $L^2(\mathbb{R})$ and which is stable under translations.

We would also like to mention that a different orthonormalization procedure is well known to people working on wavelets since it may be useful in the very first steps of a multi-resolution analysis to construct an orthonormal basis of a certain subspace of $L^2(\mathbb{R})$, $V_0$, starting from a given Riesz basis of $V_0$. This technique, which is reviewed in [6], is mainly based on the following fact: in $\varphi(x)$, together with its integer translated $\varphi(x + n)$, $n \in \mathbb{Z}$ is a Riesz basis in $V_0 \subset L^2(\mathbb{R})$, then the inverse Fourier transform $\tilde{\varphi}(x)$ of the function $\frac{1}{\sqrt{2\pi}} (\sum_{l \in \mathbb{Z}} |\hat{\varphi}(p + 2\pi l)|^2)^{-1/2} \hat{\varphi}(p)$ is such that the set $\{\tilde{\varphi}(x + n), n \in \mathbb{Z}\}$ is an orthonormal basis of the same set $V_0$.

It is clear that this assumption exactly coincides with our original hypothesis: any Riesz basis is a set of vectors $\{T_k\varphi\}_{k \in \mathbb{Z}}$ which are linearly independent and a system of generator in the Hilbert space they generate, by definition.

It is also worth mentioning that the two techniques both rely on Fourier analysis and, from this point of view, they look similar. Nevertheless they are different and inequivalent since they produce, in general, different results starting from the same seed function $g(x)$ and, moreover, since the difficulties of the computations are not necessarily comparable. For instance, if we consider the previous example, our orthonormalization procedure produces, at the end, the following function: $\sum_n \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-imp} dp}{a + 2(a-1) \cos(p)} \right) \chi[n,n+a](x)$, and the integrals can be easily computed (Remember that we need to compute only very few of them, because general arguments on Fourier series show that the coefficients $c_m$ go to zero very fast with $n$, see below). A bit more difficult, in our opinion, is to compute the function $\tilde{\varphi}(x)$ using formula above. However both these approaches in general can only be applied within an approximation scheme. Finding an exact solution is, but for few examples, quite difficult if not impossible. From this point of view our strategy seems convenient with respect to the other one since it is very natural to construct a perturbative approach. Indeed we only need to check if the function $\alpha(p)$ has some zero in $[0, 2\pi]$. If this is not so, in fact, we know that the coefficients $c_m$ go to zero faster than any inverse power in $|m|$ and, therefore, the series defining the new o.n. set converges very fast. This means that only the first few $c_m$'s are required to get a reasonable approximation of the solution we are looking for, and this can be easily done numerically.

**Remark:** It is maybe not surprising to notice that both in our and in the above
approaches the following integrals play an important role

\[ a_k = \int_{-\infty}^{+\infty} \phi(t)\overline{\phi(t+k)} dt. \]

This is because \( a_k \) is a measure of the non orthogonality between the original functions: the smaller is the difference of \( a_k \) from \( \delta_{k,0} \), the closer is the set \( \{ \phi(x+k), k \in \mathbb{Z} \} \) to an orthonormal set.

**IV  Gabor frames**

**IV.1  What we get from \((k,q)\)-representation**

We begin this section by recalling few results on the generalized \((k,q)\)-representation which we have introduced in [1], and which extends analogous results discussed, for instance, in [7].

Let \( T(a) = e^{i\hat{a}a} \) and \( \tau(b) = e^{i\hat{q}b} \), with \( ab = 2\pi L \), for some natural \( L \). It is clear that for all possible \( L \in \mathbb{N} \) the two operators still commute: \([T(a), \tau(b)] = 0\). For each given \( A > 0 \) let us define the set of (generalized) functions

\[ \Phi^{(A,a)}_{k,q}(x) = \sqrt{\frac{A}{2\pi}} \sum_{l \in \mathbb{Z}} e^{iklA} \delta(x - q - la), \quad (4.1) \]

where \((k, q) \in \square^{(A)} := [0, \frac{2\pi}{A}] \times [0, a[\). If \( \xi_x \) is the generalized eigenvector of the position operator \( \hat{q} \), \( \hat{q} \xi_x = x \xi_x \), we write \( \Phi^{(A,a)}_{k,q}(x) \) as \( \Phi^{(A,a)}_{k,q}(x) = \langle \xi_x, \Phi^{(A,a)}_{k,q} \rangle \).

**Proposition 2** With the above definitions the following statements hold true:

1. \[ T(a)\Phi^{(A,a)}_{k,q}(x) = e^{ikA}\Phi^{(A,a)}_{k,q}(x), \quad \tau(b)\Phi^{(A,a)}_{k,q}(x) = e^{iqb}\Phi^{(A,a)}_{k,q}(x), \quad (4.2) \]

2. \[ \int \int_{\square^{(A)}} \overline{\Phi^{(A,a)}_{k,q}(x)} \Phi^{(A,a)}_{k,q}(x') dk dq = \delta(x - x'), \quad (4.3) \]

3. \[ \int \int_{\square^{(A)}} |\Phi^{(A,a)}_{k,q} \rangle \langle \Phi^{(A,a)}_{k,q} | = \mathbb{I}, \quad (4.4) \]

where the usual Dirac bra-ket notation has been adopted;

4. \[ \int_{\mathbb{R}} \overline{\Phi^{(A,a)}_{k,q}(x)} \Phi^{(A,a)}_{k',q'}(x) dx = \delta(k - k') \delta(q - q'). \quad (4.5) \]
The proof of these statements does not differ significantly from the standard one, and
will be omitted here. We just want to remark that, for general \( a \) and \( a' \), we find that
\[ T(a)\Phi^{(A,a')}_{k,q} \neq e^{ikA}\Phi^{(A,a')}_{k,q}(x). \]
In other words, in general \( \Phi^{(A,a')}_{k,q}(x) \) is not an eigenstate of
\( T(a) \) if \( a \neq a' \).

Also, it should be noticed that the value of the parameter \( b \) entering in the definition
of \( \tau(b) \), is fixed by requiring that \( T \) and \( \tau \) commute but play no role in the definition of
the lattice cell \( \Box^{(A)} \), which on the other way is defined by an extra positive parameter, \( A \),
which needs not to be related to \( b \) itself. However, quite often in applications \( A \) coincides
with \( a \) and with \( b \).

We now use this generalized \((k,q)\)-representation in connection with Gabor frames and
in the attempt of getting an o.n. set of functions in \( L^2(\mathbb{R}) \). The starting point is not very
different from that in Section II: let \( \hat{q} \) and \( \hat{p} \) be the position and momentum operators on
the Hilbert space \( \mathcal{H} = L^2(\mathbb{R}) \), \([\hat{q},\hat{p}]=i\mathbb{I} \), and let
\[ U(\bar{n}) = e^{i\alpha(n_1\hat{q}-n_2\hat{p})}, \quad D(\bar{n}) = e^{z_\omega b^\dagger - \tau(b)}, \]
\[ T_1 := e^{i\alpha (n_1)} \quad \text{and} \quad T_2 := e^{-i\alpha (n_2)} \quad \text{as in (2.1)} \]
As before, \( a \) is a real constant such that \( a^2 = 2\pi L \)
for some \( L \in \mathbb{N} \), while \( z_\omega \) and \( b \) are related to \( \bar{n} = (n_1, n_2) \) and \( \hat{q}, \hat{p} \) as in (2.5). As in (2.6)
we have \( U(\bar{n}) = D(\bar{n}) = (-1)^{L_{n_1} n_2} T_1^{n_1} T_2^{n_2} = (-1)^{L_{n_1} n_2} T_2^{n_2} T_1^{n_1} \),
where we have again used the commutativity \( [T_1, T_2] = 0 \).

Let now \( g(x) \) be a fixed function in \( \mathcal{H} \), and let us define the following functions:
\[ g^{(L)}_\bar{n}(x) := T_1^{n_1} T_2^{n_2} g(x) = T_2^{n_2} T_1^{n_1} g(x) = (-1)^{L_{n_1} n_2} U(\bar{n}) g(x) = (-1)^{L_{n_1} n_2} D(\bar{n}) g(x). \]

(4.6)

We call \( \mathcal{C}^{(L)} = \{g^{(L)}_\bar{n}(x), \bar{n} \in \mathbb{Z}^2 \} \) the set of these vectors, which is the standard set of
coherent states if \( g(x) \) is just the vacuum of the operator \( b \). These functions, which quite
often in the literature are written as \( e^{i\omega x} g(x-n_2x_0) \) \( g(x) \) for some positive \( x_0 \) and \( \omega_0 \), are not
mutually orthogonal for a generic \( g(x) \). For instance, they are not orthogonal for coherent
states. Here we will assume always that the functions in \( \mathcal{C}^{(L)} \) are linearly independent.

Our aim is to construct a family of vectors \( \mathcal{E}^{(L)} \) which shares with \( \mathcal{C}^{(L)} \) most of its
features and which, moreover, is made of orthonormal vectors: we are specializing our
original settings of Section II to \( N = 2 \). Of course, we are interested here in considering
a starting function \( g(x) \) such that \( < g^{(L)}_\bar{n}, g \neq \delta_{\bar{n}}. \)
Also, we would like to know if for some choice of \( g(x) \in L^2(\mathbb{R}) \) and \( L \geq 1 \) it is possible that \( \mathcal{C}^{(L)} \) is a complete set in \( L^2(\mathbb{R}) \).
In other words, if we define \( \mathcal{H}_{g,L} := \text{linear span} \{g^{(L)}_\bar{n}(x), \bar{n} \in \mathbb{Z}^2 \} \), we wonder whether a
clever choice of \( g \) and \( L \) produces \( \mathcal{H}_{g,L} = L^2(\mathbb{R}) \). These questions can be answered rather
easily by making use of the generalized \((k,q)\)-representation.

We first remark that, introducing the generalized common eigenstates of \( T_1 \) and \( T_2 \)
as in the beginning of this section, we have
\[ T_1 \Phi^{(A,a)}_{k,q} = e^{i\alpha_q} \Phi^{(A,a)}_{k,q} \quad \text{and} \quad T_2 \Phi^{(A,a)}_{k,q} = e^{i\alpha_k} \Phi^{(A,a)}_{k,q}. \]
Therefore \( < g^{(L)}_n, \Phi_{k,q}^{(A,a)} > = e^{-in_1 a} e^{-ikn_2 A} < g, \Phi_{k,q}^{(A,a)} > \) which, in turns, using the resolution of the identity in Eq. (4.4), implies that

\[
I^{(L)}_{g^{(L)}_n} = < g^{(L)}_n, g > = \int_{0}^{2\pi/A} dk e^{-ikn_2 A} \int_{0}^{a} dq e^{-iqn_1 a} |g(k,q)|^2 \tag{4.7}
\]

where we have introduced the \((k,q)\) representation of the function \(g(x)\) as \(g(k,q) = < \Phi_{k,q}^{(A,a)} , g >\). We will now see that, as in \([1]\), the role of \(L\) is crucial here.

If \(L = 1\) the set \(S^{(1)} = \{e^{-ikn_2 A} e^{-iqn_1 a}, (n_1,n_2) \in \mathbb{Z}\}\) is complete in \(L^2(\square^{(A)})\), so that the functions in \(C^{(1)}\) can be mutually orthogonal if and only if \(g(k,q)\) is a phase so that \(|g(k,q)|\) is constant almost everywhere (a.e.) in \(\square^{(A)}\). This is the reason why the coherent states are not mutually orthogonal: the \((k,q)\) representation of the vacuum of the annihilation operator \(b\) is not simply a phase!

If \(L > 1\) the conclusion changes. Let us consider, as an example, \(L = 2\). In this case the set \(S^{(2)}\) is no longer complete in \(L^2(\square^{(A)})\) so that equation (4.7) does not automatically implies that \(|g(k,q)|\) must be constant. On the contrary, splitting the double integral on \(\square^{(A)}\) in two contributions over \((k,q) \in [0,2\pi/A] \times [0,a/2]\): \(\square^{(A)}_{1/2}\) and \((k,q) \in [0,2\pi/A] \times [a/2,a]\), we can write \(I^{(2)}_{g^{(2)}_n}\) as

\[
I^{(2)}_{g^{(2)}_n} = \int_{0}^{2\pi/A} dk e^{-ikn_2 A} \int_{0}^{a/2} dq e^{-iqn_1 a} \left( |g(k,q)|^2 + |g(k,q + a/2)|^2 \right),
\]

which is equal to \(\delta_{n,0}\) if and only if \(|g(k,q)|^2 + |g(k,q + a/2)|^2\) is constant almost everywhere for \((k,q) \in \square^{(A)}_{1/2}\) since \(S^{(2)}\) is complete in \(L^2(\square^{(A)}_{1/2})\). This result is easily generalized:

**Proposition 3** The set \(C^{(L)}\) generated by a given square-integrable function \(g(x)\) is made of orthonormal functions if and only if in the \((k,q)\)-representation \(g\) satisfies the following equality almost everywhere

\[
|g(k,q)|^2 + \left| g\left(k, q + \frac{a}{L}\right) \right|^2 + \left| g\left(k, q + \frac{2a}{L}\right) \right|^2 + \cdots + \left| g\left(k, q + \frac{(L-1)a}{L}\right) \right|^2 = \frac{LA}{2\pi a} \tag{4.8}
\]

Of course, once such a \(g(k,q)\) has been found, then the related \(g(x)\) can be simply obtained as

\[
g(x) = \int \int_{\square^{(A)}} dk dq \Phi_{k,q}^{(A,a)}(x) g(k,q) \tag{4.9}
\]

Any \(g(x)\) which cannot be written in this form cannot produce an o.n. set \(C^{(L)}\). As a consequence, this is exactly the kind of functions we will consider in the rest of this section, since we are interested here in producing an orthonormal set starting from a different set \(C^{(L)}\) which is not orthogonal from the very beginning.
The generalized \((k, q)\) representation can also be used to discuss the problem of the completeness of \(C(L)\) in \(L^2(\mathbb{R})\). Before doing this, however, it may be worth observing that by definition \(C(L)\) is complete in \(H_{g,L}\) but, since in general \(H_{g,L}\) is only a subset of \(L^2(\mathbb{R})\), we have no information about the completeness of \(C(L)\) on this larger Hilbert space. In other words, if we find conditions for \(C(L)\) to be complete in \(L^2(\mathbb{R})\) we also find conditions for \(H_{g,L}\) to coincide with \(L^2(\mathbb{R})\).

To answer these questions we first recall that the set \(C(L)\) is complete in \(L^2(\mathbb{R})\) if, given a square integrable function \(h(x)\) which is orthogonal to \(g^{(L)}_n(x)\) for all \(n \in \mathbb{Z}^2\), then \(h(x) = 0\) a.e. in \(\mathbb{R}\). Using the properties of the functions \(\Phi^{(A,a)}_{k,q}\) we can write

\[
<g^{(L)}_n, h> = \int_0^{2\pi/A} dk e^{-ikn_2A} \int_0^a dq e^{-iqn_1a} g(k, q) h(k, q),
\]

where, as usual, \(g(k, q) =< \Phi^{(A,a)}_{k,q}, g >\) and \(h(k, q) =< \Phi^{(A,a)}_{k,q}, h >\). Again, to find conditions for these scalar products to be zero \(\forall n \in \mathbb{Z}^2\), it is convenient to consider separately the two cases: \(L = 1\) and \(L > 1\).

If \(L = 1\), using as before the completeness of the set \(\mathcal{S}^{(1)}\) in \(L^2(\Box^{(A)})\), we deduce that \(< g^{(L)}_n, h > = 0 \forall n \in \mathbb{Z}^2\) if and only if \(\overline{g(k, q)} h(k, q) = 0\) a.e. in \(\Box^{(A)}\). Therefore, if \(g(k, q)\) is zero at most on a set of zero measure, this implies that \(h(k, q) = 0\) a.e. in \(\Box^{(A)}\) and, as a consequence, that \(h(x) = 0\) a.e. in \(\mathbb{R}\).

**Remark:** the vacuum of \(b\), which generates the set of coherent states, satisfies this condition, \(\mathbb{R}\), and therefore the related set \(C^{(1)}\) is complete in \(L^2(\mathbb{R})\). Examples of functions which **do not** satisfy this condition can be easily constructed. Consider, for instance, a vector \(\hat{g}\) which in the \((k, q)\)-representation is equal to 1 for \((k, q) \in \Box^{(A)}_{1/2}\) and zero otherwise. In the \(x\)-representation this function looks like \(\hat{g}(x) = \sqrt{\frac{2\pi}{\pi}} \chi_{[0,a/2]}(x)\), where \(\chi_I(x)\) is, as usual, the characteristic function of the set \(I\). For what we have just shown this function does not generate a complete set in \(L^2(\mathbb{R})\) since non-zero functions \(h(k, q)\) for which \(\overline{g(k, q)} h(k, q) = 0\) a.e. in \(\Box^{(A)}\) can be easily found.

Let us now consider what happens for \(L > 1\) and, to be concrete, let us fix \(L = 2\). Splitting the integral as we have done before, we deduce that \(< g^{(L)}_n, h > = 0\) for all \(n \in \mathbb{Z}^2\) if and only if is zero the following combination \(\overline{g(k, q)} h(k, q) + g\left(k, q + \frac{a}{2}\right) h\left(k, q + \frac{a}{2}\right)\).

In other words, \(< g^{(L)}_n, h > = 0\) for all \(n \in \mathbb{Z}^2\) if and only if

\[
\overline{g(k, q)} h(k, q) + g\left(k, q + \frac{a}{2}\right) h\left(k, q + \frac{a}{2}\right) = 0 \quad \text{a.e. in} \quad \Box^{(A)}_{1/2} \quad (4.10)
\]

and, as a consequence, \(C(L)\) is complete in \(L^2(\mathbb{R})\) for some given \(g(k, q)\) if and only if the only solution of equation (4.10) is \(h(k, q) = 0\) a.e. in \(\Box^{(A)}_{1/2}\). Of course, any function
\( g(k, q) \) which is zero on a set \( \mathcal{D} \subset \mathbb{D}^{(A)}_{1/2} \) of non zero measure cannot produce a complete set, because equation (4.10) would have a non trivial solution: it is enough to choose a function \( h(k, q) \) which is zero only outside \( \mathcal{D} \)! Hence, in order to obtain a complete set, it is surely necessary to assume that \( g(k, q) \) is zero at most on a set of zero measure. However, even under this assumption it is easy to check that other non trivial solutions of this equation do exist, and therefore \( \mathcal{C}(L) \) cannot be complete in \( L^2(\mathbb{R}) \). Let indeed \( s_o(x) \) be a fixed function in \( L^2(\mathbb{R}) \) and let \( \mu(k, q) := e^{iqa/2} \sum_{n \in \mathbb{Z}} e^{inkA} s_o(q + na/2) \). \( \mu(k, q) \) belongs to \( L^2(\mathbb{D}^{(A)}_{1/2}) \) since \( \int_{\mathbb{D}^{(A)}_{1/2}} |\mu(k, q)|^2 dk dq = \frac{2\pi}{A} \int_{\mathbb{R}} |s_o(x)|^2 dx \), which is finite because \( s_o(x) \in L^2(\mathbb{R}) \). Using now the boundary conditions \( g(k, q + a) = e^{ikA} g(k, q) \) and \( \mu(k, q + a/2) = -e^{ikA} \mu(k, q) \), it is easy to check that \( h(k, q) := g(k, q + a/2) \mu(k, q) \) satisfies equation (4.10) and is different from zero a.e. if \( s_o(x) \) is chosen, e.g., positive, since in this case \( g(k, q) \neq 0 \) a.e. in \( \mathbb{D}^{(A)}_{1/2} \).

Let us now summarize the above results: the set \( \mathcal{C}(L) \) consists of o.n. functions if and only if \( g(k, q) \) satisfies equation (4.8). Moreover, if \( g(k, q) \neq 0 \) a.e. in \( \mathbb{D}^{(A)} \), then \( \mathcal{C}(1) \) is complete in \( L^2(\mathbb{R}) \). Finally, if \( L > 1 \), there is no choice of \( g(k, q) \) for which the set \( \mathcal{C}(L) \) is complete in \( L^2(\mathbb{R}) \). Therefore, for any given \( g \) and for all \( L \geq 2 \) we have \( \mathcal{H}_{g,L} \subset L^2(\mathbb{R}) \).

This result reflects, in a certain sense, what happens for frames of translates which are surely not complete in all of \( L^2(\mathbb{R}) \), as we have discussed in the previous section.

**IV.2 Back to the orthonormalization problem**

We are now ready to apply our orthonormalization procedure to the set \( g_\underline{\alpha}(x) \) in [4.6], with the only requirement that \( \mathcal{C}(L) \) must consists of functions which are not orthogonal from the very beginning and linearly independent. As in [1] and in Section II we define

\[
\Psi^{(L)}(x) := \sum_{k \in \mathbb{Z}^2} c_k^{(L)} g_{k+n}^{(L)}(x),
\]  

(4.11)

which implies that \( \Psi^{(L)}(x) := \sum_{k \in \mathbb{Z}^2} c_k^{(L)} g_{k+n}^{(L)}(x) \) and, because of the commutativity of \( T_1 \) and \( T_2 \), that

\[
\Psi^{(L)}(x) = T_1^{n_1} T_2^{n_2} \Psi^{(L)}(x).
\]

(4.12)

Therefore the new set constructed in this way, \( \mathcal{E}(L) := \{ \Psi^{(L)}(x), \underline{n} \in \mathbb{Z}^2 \} \), is invariant under the action of \( T_1 \) and \( T_2 \), exactly as the set \( \mathcal{C}(L) \), independently of the choice of the coefficients of the expansion \( c_k^{(L)} \). These coefficients \( c_k^{(L)} \) are, as usual, fixed (non uniquely) by requiring that the vectors in the set \( \mathcal{E}(L) \) are orthonormal, \( \langle \Psi^{(L)}_{\underline{n}}, \Psi^{(L)}_{\underline{m}} \rangle = \delta_{\underline{n},\underline{m}} \), or, equivalently, that \( \langle \Psi^{(L)}_{\underline{n}}, \Psi^{(L)}_{\underline{m}} \rangle = \delta_{\underline{n},\underline{m}} \). Indeed, if they exist, the coefficients can be found
as before:
\[ c_k^{(L)} = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \frac{e^{-iP_k}}{\sqrt{F_L(P)}} dP, \]  \hspace{1cm} (4.13)
where \( F_L(P) = \sum_{m \in \mathbb{Z}^2} I_m^{(L)} e^{iP \cdot m} \). As in [1], the behavior of the coefficients in (4.13) is directly related to the nature of the convergence and to the zeros of \( F_L(P) \), which, in turns, are related to the coefficients of the overlap between \( g_m^{(L)}(x) \) and \( g(x) \), \( I_m^{(L)} = \langle g_m^{(L)}, g \rangle \). In particular the following results immediately follows from basic facts in the Fourier series analysis and from the results in [1]:

1. if \( I_m^{(L)} \in L^1(\mathbb{Z}^2) \) then \( F_L(P) \) is continuous, non negative, \((2\pi, 2\pi)\)-periodic and bounded;

2. if \( I_m^{(L)} \in s(\mathbb{Z}^2) \) then \( F_L(P) \) is a \( C^\infty \) non negative function;

3. if \( I_m^{(L)} \) are such that \( \sum_{m \in \mathbb{Z}^2} |I_m^{(L)}| < 1 \) then \( F_L(P) \neq 0 \) for all \( P \in [0, 2\pi] \times [0, 2\pi] \) and, as a consequence, \( \{c_k^{(L)}\} \in s(\mathbb{Z}^2) \);

4. if \( I_m^{(L)} \in s(\mathbb{Z}^2) \) then the sequence \( \Psi_n^{(L)} := \sum_{k:||k|| \leq N} c_k^{(L)} \Psi_k^{(L)} \) converges in the \( \|\cdot\|_2 \)-norm so that its limit, \( \Psi_n^{(L)} \), belongs to \( H_{g,L} \).

If we introduce the operator \( X_L = \sum_{m \in \mathbb{Z}^2} c_m^{(L)} T_1^{m_1} T_2^{m_2} \) then \( \Psi_n^{(L)} = X_L g_n^{(L)} \). It is also possible to extend the following proposition, originally given in [1]:

**Proposition 4** Suppose that \( \{c_m^{(L)}\} \in L^1(\mathbb{Z}^2) \) and that \( F_L(P) \neq 0 \) for all \( P \in [0, 2\pi] \times [0, 2\pi] \). Then \( E^{(L)} \) is complete in \( H_{g,L} \) if and only if \( X_L \) admits a bounded inverse.

The proof is very close to that given in [1] and will not be repeated here.

It is rather interesting to observe that the same sum rule which was proved, under the assumptions of this Proposition, in [1], that is \( \sum_{k \in \mathbb{Z}^2} \alpha_k^{(L)} c_k^{(L)} = 1 \), also holds true in the present settings. Here the \( \alpha_k^{(L)} \)’s are the coefficients of the expansion of the non-orthogonal functions \( g_m^{(L)} \) in terms of the orthogonal ones, \( g_m^{(L)}(x) = \sum_{m \in \mathbb{Z}^2} \alpha_m^{(L)} \Psi_m^{(L)}(x) \). Explicitly we find \( \alpha_m^{(L)} = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} e^{-iP \cdot m} \sqrt{F_L(P)} \). Again, we don’t give here the proof of this result which is based on the Poisson summation formula since it is very close to that given in [1]. We just want to stress that for each starting function \( g(x) \) we get a different sum rule.

The vectors of the set \( E^{(L)} \) have the following properties:

\[ \sum_{n \in \mathbb{Z}^2} |\Psi_n^{(L)}| = \sum_{m \in \mathbb{Z}^2} |g_n^{(L)}| = 1_{g,L}, \hspace{1cm} (4.14) \]
where 1_{g,L} is the identity operator in \( \mathcal{H}_{g,L} \), which, for what we have proven at the beginning of this section, can be the identity in \( \mathcal{L}^2(\mathbb{R}) \) only if \( L = 1 \) and \( g \) is taken conveniently.

Also, if by chance \( g^{(L)}_n(x) \) is an eigenstate of some operator \( \hat{k}_L \) (as it happens for coherent states), then \( \Psi^{(L)}_n(x) \) is an eigenstate of the operator \( \hat{K}_L = X_L \hat{k}_L X_L^{-1} \), with the same eigenvalue.

Here as in [1] the role of \( L \) is very important and in particular we are not sure a priori that \( X_L^{-1} \) exists for all values of \( L \). For instance, if \( L = 1 \) it is possible to adapt the same redutio ad absurdum argument given in [1] to conclude that our orthogonalization procedure cannot work. More in detail, suppose that we have constructed the o.n. set \( \mathcal{E}^{(1)} \) which is a basis of \( \mathcal{L}^2(\mathbb{R}) \). Then we conclude, as in [1], that the original set \( \mathcal{C}^{(1)} \) is an o.n. basis of \( \mathcal{L}^2(\mathbb{R}) \) as well, which is false.

Before considering some examples a final remark is in order: for Gabor frames it is widely discussed in the literature, see [6] for instance and references therein, that 3 different regions appear in their analysis and they give rise to different mathematics. More in details: the set \( \{ g_n(x) = e^{ian_1x}g(x + an_2), n_1, n_2 \in \mathbb{Z} \} \) can be an orthonormal basis only if \( a^2 = 2\pi \) (but with poor localization properties). It can be a frame for all of \( \mathcal{H} \) if \( a^2 < 2\pi \) (and it may have good localization properties). Finally, it cannot be a frame for all of \( \mathcal{H} \) if \( a^2 > 2\pi \). Since our strategy works for \( a^2 = 2\pi L, L = 1, 2, 3, \ldots \), this automatically excludes the case \( a^2 < 2\pi \). This constraint, in our approach, is due to the important requirement that \([T_1, T_2] = 0\) which we hope to weaken in a close future. However, as we have widely discussed here and in [1], already for \( a^2 \geq 2\pi \) we find a very rich mathematical setting, which produces many interesting results.

V Examples and conclusions

The first example of this construction produces the classical coherent states. We don’t discuss this example here since few results have already been given in Section II and a complete analysis can be found in [1]. In this paper we concentrate our attention on other examples whose computations can be carried out almost completely analytically. Other examples can be discussed in a totally analogous way.

**Example 1.**

Let us consider the following normalized compactly supported function,

\[
g(x) = \begin{cases} 
\frac{1}{\sqrt{3a/2}}, & |x| \leq \frac{3a}{4} \\
0, & \text{otherwise}.
\end{cases}
\]
It is clear that the different \( g_n(x) = e^{i\pi_1 x} g(x + an_2) \) are not all automatically orthogonal, even if \( < g_n, g_0 > \) is surely zero if \( |n_2| \geq 2 \) because, in this case, the supports of the two functions have empty intersection. It is easy to check that, choosing \( L = 4 \) so that \( a^2 = 8\pi \) to simplify the computations, \( F_L(P) = 1 + \frac{2}{3} \cos(P_2) =: F_L(P_2) \) which is independent of \( P_1 \). As a consequence, the coefficients \( c_{k}^{(L)} \) in (4.13) can be written as \( c_{k}^{(L)} = \delta_{k,0} \hat{c}_{k_2} \), where

\[
\hat{c}_{k_2} = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{e^{-iP_2k_2}}{\sqrt{F_L(P_2)}} dP_2.
\]

These coefficients can be easily computed: \( \hat{c}_0 = 1.11308, \hat{c}_{\pm1} = -0.216769, \hat{c}_{\pm2} = 0.0625106 \) and so on. Needless to say, because of the analytic properties of \( F_L(P_2) \) which is never zero in \([9, 2\pi]\[, they tend to zero faster than any inverse power of \( k_2 \) so that all the series which appear along our computations are surely convergent (in the strongest topology).

From formula (4.11) we find that \( \Psi(x) = \sum_k \hat{c}_k g(x + ka) \). We plot in the following figure \( \Psi_N(x) = \sum_{k=-N}^{N} \hat{c}_k g(x + ka) \) for different values of \( N \).

![Figure 1: \( \Psi(x) \) for \( N = 0, N = 1, N = 2 \) (first row) and \( N = 3, N = 4 \) (second row)](image)

It is just a technicality to check that the overlap between \( \Psi(x) \) and \( T_1^{n_1} T_2^{n_2} \Psi(x) \) can be written in terms of the coefficients \( \hat{c}_k \) as follows:

\[
< \Psi_n^{(L)}, \Psi > = \sum_{l \in \mathbb{Z}} [\hat{c}_l (\hat{c}_{l+n_2} + (\hat{c}_{l+n_2+1} + \hat{c}_{l+n_2-1})/3)] \delta_{n_1,0} \tag{5.1}
\]

and we find that, replacing \( \sum_{l \in \mathbb{Z}} \) with \( \sum_{l=-N}^{N} \) in (5.1), for \( N = 3 \) we have \( \|\Psi\|^2 = 1.00001 \) while the modulus of the overlap between two different wave-functions does not exceed
0.0605858. Therefore we conclude that this is already a very good approximation, which can however be improved if we take \( N = 4 \): in this case we find \( \| \Psi \|^2 = 1 \) and the modulus of the overlap between two different wave-functions is always less than 0.00208293.

The computation of \( X_L \) directly follows from its definition: \( X_L = \sum_{n_2 \in \mathbb{Z}} \hat{c}_{n_2}^{(L)} T_{n_2}^2 \). Its inverse, \( X_L^{-1} \), can be computed in complete analogy: since \( \hat{g}^{(L)}_n(x) = \sum_{m \in \mathbb{Z}^2} \hat{c}_m^{(L)} \Psi_m^{(L)}(x) = X_L^{-1} \Psi_m^{(L)}(x) \), we get \( X_L^{-1} = \sum_{n_2 \in \mathbb{Z}} \hat{c}_{n_2}^{(L)} T_{n_2}^{-2} \), where \( \hat{c}_{n_2}^{(L)} = \frac{1}{2\pi} \int_0^{2\pi} dp \, e^{-ipn_2} \sqrt{F_L(P)} \). We find, for instance, \( \hat{c}_{0}^{(L=4)} = 0.96857 \), \( \hat{c}_{+1}^{(L=4)} = 0.175095 \), \( \hat{c}_{-1}^{(L=4)} = -0.016400 \), and so on.

**Example 2.**

Let us consider now the following function:

\[
g(x) = \begin{cases} N_b \exp \left( \frac{-1}{x^2+b^2} \right), & |x| \leq b \\ 0, & \text{otherwise,} \end{cases}
\]

where \( b = \frac{3a}{4} \), \( a^2 = 2\pi L \), has been introduced only to simplify the notation. \( N_b \) is a normalization constant which depends obviously on \( b \) and, as a consequence, on \( L \). Once again we consider a function with compact support just to simplify the computations, since with our choice \( \langle g \mid g \rangle = 0 \) is zero if \( |n_2| \geq 2 \). Hence, the function \( F_L(P) \) can be written as \( F_L(P) = f_0(P_1) + 2f_1(P_1) \cos(P_2) \), where \( f_0(P_1) = \sum_{n_1 \in \mathbb{Z}} I_{n_1,0}^{(L)} \exp(iP_1n_1) \) and \( f_1(P_1) = \sum_{n_1 \in \mathbb{Z}} I_{n_1,1}^{(L)} \exp(iP_1n_1) \). The simplest results are obtained when both these functions can be reasonably well replaced by their constant main contributions, that is those contributions coming from \( n_1 = 1 \). Indeed, if this can be done, then we get \( F_L(P) \approx 1 + 2I_{0,1}^{(L)} \cos(P_2) \). \( F_L(P) \) which does not depend on \( P_1 \), so that, once again, \( c_k^{(L)} = \delta_{k_1,0} \hat{c}_{k_2}^{(L)} \), where \( \hat{c}_{k_2}^{(L)} = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-iP_2k_2}}{\sqrt{F_L(P_2)}} dP_2 \) and \( \Psi^{(L)}(x) = \sum_k \hat{c}_{k}^{(L)} g^{(L)}(x + ka) \). Of course, replacing \( f_0(P_1) \) and \( f_1(P_1) \) respectively with their first contributions \( I_{0,0}^{(L)} = 1 \) and \( I_{0,1}^{(L)} \) is possible only if we have some control on what we are neglecting. However, if \( L = 1 \), it is easy to check that \( I_{0,1}^{(L)} \) is smaller than that part of \( f_2(P_2) \) we are neglecting, \( \delta f_2(P_2) \), so that replacing \( F_L(P) \) with \( F_L(P_2) \) is a dangerous operation! This will be reflected on the fact that, as we will see, the orthonormalization procedure does not work in this case: we get a new family of functions which are not all mutually orthogonal as we would like. On the other hand, if we consider \( L = 2 \), then \( |\delta f_2(P_2)| < |I_{0,1}^{(L)}| \) but they are of the same order of magnitude: we have some chance that the procedure works but we cannot be sure at this stage. For \( L \geq 3 \), finally, \( |\delta f_2(P_2)| \ll |I_{0,1}^{(L)}| \) so that we expect to get a good orthonormal set for \( H_{a,L} \).

These are exactly the conclusions that we obtain at the end of the computations: if \( L = 1 \) the function \( \Psi^{(1)}(x) = \sum_k \hat{c}_{k}^{(1)} g^{(1)}(x + ka) \) is such that \( \| \Psi^{(1)} \|^2 \simeq 0.96 \), while, for instance, \( \langle \Psi^{(1)} , \Psi^{(1)}_{(0,1)} \rangle \simeq 0.1 \). We see that the related functions do not produce an
orthogonal set.

Let us see what happens if $L = 2$. In this case we find $\|\Psi^{(2)}\|^2 \simeq 0.99996$ and $\langle \Psi^{(2)}, \Psi^{(2)}_{(0,1)} \rangle \simeq 0.0001$, which are quite promising. However, we also find that $\langle \Psi^{(2)}, \Psi^{(2)}_{(1,0)} \rangle \simeq 0.037$ which shows that for $L = 2$ the approximated $\Psi^{(2)}_n$'s constructed as shown above are orthogonal up to corrections which are small but not too much. However, as expected, this out of orthogonality parameter gets smaller and smaller when $L$ increases: already for $L = 3$ we find $\langle \Psi^{(3)}, \Psi^{(3)}_{(1,0)} \rangle \simeq -0.012$, and all the other scalar products are much smaller. The norm of $\Psi^{(3)}$ is essentially 1. In the following figure we plots the different approximations of $\Psi^{(L)}(x)$ for $L = 3$.

![Figure 2: $\Psi(x)$ for $N = 0, N = 1, N = 2$](image)

**Example 3.**

Let us consider now the following function:

$$g(x) = \begin{cases} \frac{1}{\sqrt{a}} \cos \left(\frac{\pi}{2\pi} x\right), & |x| \leq a \\ 0, & \text{otherwise.} \end{cases}$$

Also in this example we consider a function with compact support since it allows to perform almost all the computations analytically. As before $\langle g_\omega, g \rangle$ is zero if $|n_2| \geq 2$, and this makes the computation of $F_L(P)$ simple. Indeed, if we introduce the function $f(P_1) = \sum_{m_1 \in \mathbb{Z}} e^{i P_1 m_1 \pi / 4\pi L^2 m_1^2}$ then we get $F_L(P) = 1 + 2 f(P_1) \cos(P_2)$. Of course, due to this formula, $c^{(L)}_{\omega}$ is not the product of $\delta_{n_1,0}$ and a coefficient which only depends on $n_2$. Nevertheless, this is exactly what happens if we consider the first approximation of $f(P_1) \simeq \frac{1}{\pi}$ in analogy with what we have done in the previous example. Again, this replacement is justified especially for values of $L$ larger than 2 and we find $c^{(L)}_{\omega} = \delta_{k_1,0} \hat{c}_{k_2}$, where $\hat{c}_{k_2} = \frac{1}{2\pi} \int_0^{2\pi} e^{-iP_2 k_2} dP_2$, $F_L(P_2) \simeq 1 + \frac{2}{\pi} \cos(P_2)$, and $\Psi(x) = \sum_k \hat{c}_k g(x + ka)$.

The coefficients can again be easily computed: $\hat{c}_0 = 1.0997$, $\hat{c}_{\pm 1} = -0.20105$, $\hat{c}_{\pm 2} = 0.0545131$ and so on. Of course, because of the analytic properties of $F_L(P_2)$, they tend to
zero faster than any inverse power of \( k \) so that even in this example all the series which appear along our computations are surely convergent.

We plot in the following figure \( \Psi_N(x) = \sum_{k=-N}^{N} c_k g(x + ka) \) for different values of \( N \) and for \( L = 1 \). Not many differences appear in the shapes of the functions for larger values of \( L \). The only major difference is a bigger support.

![Figure 3: \( \Psi(x) \) for \( N = 0, N = 1, N = 2 \) (first row) and \( N = 3, N = 4 \) (second row)](image)

For what concerns the check of the orthonormality, we first notice that formula (5.1) does not hold here and should be replaced by the following one, which can be easily analytically derived:

\[
\langle \Psi^{(L)}_{n_1,0}, \Psi \rangle = \delta_{n_1,0} \sum_{l \in \mathbb{Z}} \hat{c}_l \hat{c}_{l-n_2} + \frac{1}{\pi(1 - 4n_1^2L^2)} \sum_{l \in \mathbb{Z}} \hat{c}_l (\hat{c}_{l-n_2+1} + \hat{c}_{l-n_2-1}) \quad (5.2)
\]

As we see, this formula explicitly depends on \( L \), as it should. Also, replacing the sum on \( l \in \mathbb{Z} \) with a finite sum on \( l = -N, \ldots, N \), we can compute the norm of \( \Psi(x) \), which already for \( N = 4 \) is equal to 1 but for an error smaller than \( O(10^{-6}) \). An interesting result concerns the overlap between two different functions \( \Psi^{(L)}_{n_1,0} \) and \( \Psi(x) \). If \( n_1 = 0 \) we find that \( \left| \langle \Psi^{(L)}_{0,n_2}, \Psi \rangle \right| \leq 10^{-3} \) already for \( N = 4 \), while if \( N = 5 \) it is even smaller than \( 5 \times 10^{-4} \), \( \forall n_2 \in \mathbb{Z} \), independently of \( L \). However, \( L \) plays a role in the computation of \( \left| \langle \Psi^{(L)}_{n_1,n_2}, \Psi \rangle \right| \) if \( n_1 \neq 0 \). Indeed in this case we can easily check that, for instance, \( \left| \langle \Psi^{(L=1)}_{1,1}, \Psi \rangle \right| = .155 \). This result shows that, as it was already discussed, our procedure cannot produce an on basis when \( L = 1 \). A different conclusion is obtained if \( L > 1 \). Indeed, already for \( L = 2 \), we find that the maximum overlap is given by \( \left| \langle \Psi^{(L=2)}_{1,1}, \Psi \rangle \right| = \).
.031, already for $N = 4$. This is a reasonable approximation which gets better and better as $L$ increases: the same quantity equals 0.013 if $L = 3$, 0.007 if $L = 4$ and so on. To improve the approximation we should consider a better approximation for $f(P)$. However, such an improvement necessarily produce a more complicated expression for $c_{n}^{(L)}$, and will not be considered here.

More examples are easily constructed starting from functions with compact support in $[-a, a]$. For these functions the analytic expressions of $F_{L}(P)$ can be found with no particular difficulties, and reasonable approximations can also be deduced in most of the cases. The situations becomes technically more difficult when the starting function $g(x)$ has no compact support, as in [11]. In this case all the computations are usually more delicate even if, in principle, they still produce an o.n. basis in the Hilbert space $H_{g,L}$. More than constructing other examples we are interested in extending the orthonormalization procedure to a slightly different situation, i.e. to the case in which the original set of functions $\{T_{n_{1}}^{m_{1}}T_{n_{2}}^{m_{2}}g(x), n_{1}, n_{2} \in \mathbb{Z}\}$ is constructed using two unitary operators $T_{1}$ and $T_{2}$ which do not commute. This analysis, in fact, would produce interesting outputs related to Gabor systems for all of $L^{2}(\mathbb{R})$ and to systems of wavelets.

Acknowledgements

This work was partially supported by M.U.R.S.T. and partially by the M.S.R.T. of Iran. MRA wishes to thank the people at the Dipartimento di Metodi e Modelli Matematici for their hospitality.

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