Monopole Percolation and The Universality Class of the Chiral Transition in Four Flavor Noncompact Lattice QED

by

A. Kocic
Theory Division
CERN
CH-1211 Geneva 23, Switzerland

J. B. Kogut
Physics Department
1110 West Green Street
University of Illinois
Urbana, IL 61801-3080

K. C. Wang
Australian National University
Box 4, G.P.O.
Canberra, A.C.T. 2600 Australia

March 20, 2018
Abstract

We simulate four flavor noncompact lattice QED using the Hybrid Monte Carlo algorithm on $10^4$ and $16^4$ lattices. Measurements of the monopole susceptibility and the percolation order parameter indicate a transition at $\beta = 1/e^2 = .205(5)$ with critical behavior in the universality class of four dimensional percolation. We present accurate chiral condensate measurements and monitor finite size effects carefully. The chiral condensate data supports the existence of a power-law transition at $\beta = .205$ in the same universality class as the chiral transition in the two flavor model. The resulting equation of state predicts the mass ratio $m_\pi^2/m_\sigma^2$ in good agreement with spectrum calculations while the hypothesis of a logarithmically improved mean field theory fails qualitatively.
1 Introduction

Computer simulations of noncompact lattice QED have proven to be very intriguing and surprising. For two and four flavors of light dynamical fermions a continuous chiral symmetry breaking transition was found some time ago\[^1\]. Recent large scale simulations of the $N_f = 2$ flavor model have given results consistent with a non-mean field transition which might signal the existence of a non-trivial fixed point\[^2\]. It was also found that the chiral transition is essentially coincident with a monopole condensation transition whose critical properties are in the universality class of four dimensional percolation\[^3\]. Since conventional physical pictures of chiral symmetry breaking\[^4\] indicate that monopole condensation implies chiral symmetry breaking, these results suggest that theories of fundamental charges and monopoles provide a natural scenario for nontrivial ultraviolet behavior. The purpose of the present article is to investigate whether these results extend to the $N_f = 4$ theory where fermion screening is even more severe. We shall provide evidence for the remarkably simple and intriguing result that the $N_f = 2$ and 4 theories lie in the same universality class. Although fermion screening will shift the theory’s critical point to stronger coupling than seen in the $N_f = 2$ theory, the critical indices governing both monopole condensation and chiral symmetry breaking will be essentially identical. These results underscore the fact that renormalization group studies of noncompact lattice QED which are based on perturbative ideas borrowed from weakly coupled continuum QED are not justified here. Theories with fundamental monopoles are intrinsically nonperturbative because of the Dirac quantization condition. The numerical evidence that we shall present supporting the claim that the $N_f = 2$ and 4 theories share the same critical behavior is inexplicable perturbatively.

We begin by summarizing the main points of the later sections of this paper. The reader is referred to previous simulations and analytic papers for additional background. In Sec. 2 of this paper we consider the evidence for a monopole condensation transition in the $N_f = 4$ theory. Finite size scaling analyses of our $10^4$ and $16^4$ data produce estimates for the critical indices of this transition which place it in the universality of four dimensional percolation. In Sec. 3 we consider our $10^4$ and $16^4$ measurements of the chiral condensate. The $10^4$ measurements are particularly extensive and involved couplings $\beta = 1/e^2 = .22, .215, .21, .205, .20, .195, .19, .185$ and bare fermion masses of $m = .005, .01, .02, .03, .04, .05, .06$ and .07. Comparing
these results with past measurements and our $16^4$ measurements at $m = .03$, we conclude that finite size effects are sufficiently small only for $m \geq .03$ on the $10^4$ lattice to permit quantitative analysis. The $10^4$ data at $m \geq .03$ are consistent with a power-law chiral transition between $\beta = .210$ and $.205$ with a critical index $\delta = 2.3(1)$. The chiral condensate data falls on a universal Equation of State with the critical point $\beta_c = .205$ and the critical indexes $\delta = 2.31$ and $\beta_{mag} = .764$ which deviate significantly from mean field values. In Sec. 4 we consider the spectroscopy data of Ref. [5] and concentrate on the ratio $R(\beta,m) = m_\pi^2/m_\sigma^2$ since it can be predicted from the model’s Equation of State with no free parameters[6]. We show that the Equation of State of Sec. 3 accounts for the data well while the Equation of State of Ref. [5] which is based on a logarithmically trivial model fails qualitatively. Finally, in Sec. 5 we observe that the Equation of State found in Sec. 2 for the four flavor model is essentially identical to that of the two flavor model studied for different parameters, on different lattice sizes by different algorithms on different computers. This result indicates that the two models are in the same universality class. In Sec. 6 additional work is proposed to challenge the simple physical picture that the results of Sec. 5 suggest.

2 Monopole Condensation in the Four Flavor Model

We have discussed monopole percolation in quenched noncompact lattice QED in Ref. [2] and in the two flavor model in Ref. [3]. The quenched model was analyzed with considerable accuracy because photon configurations could be generated using FFT methods. Thousands of statistically independent configurations could be generated and analyzed on lattices ranging up to $24^4$. Finite size scaling analyses then produced critical exponents of the monopole percolation transition at coupling $\beta = .244$ in excellent agreement with four dimensional percolation. The same measurements on two flavor configurations on $8^4 - 18^4$ lattices produced the same set of critical indices, although the statistical uncertainties were considerably larger due to the relative slowness of the hybrid molecular dynamics algorithm. Fermion feedback shifted the monopole condensation transition to stronger coupling, $\beta = .225(5)$. In the four flavor model the hybrid molecular dynam-
ics algorithm was replaced by the hybrid monte carlo scheme which is free of systematic errors. It proved easy to keep the time step of the algorithm reasonably large while maintaining a high (50%-80%) acceptance rate so that the algorithm explored phase space relatively efficiently. The time step $dt$ was tuned proportional to the bare fermion mass $m$: for $m = .005$, $dt = .005$; for $m = .01$, $dt = .01$; for $m \geq .02$, $dt = .02$. Measurements were taken after unit time intervals, and between 250–500 time intervals were generated at each $m$ and $\beta$. In fact, at crucial couplings such as $\beta = .205$ we doubled our statistics. In that case we required that the product of the acceptance rate times the number of trial trajectories (a trajectory is defined to be a unit time interval, or $(dt)^{-1}$ sweeps) be at least 500. Statistical errors in the observables were then estimated by standard binning methods and those statistical errors are recorded in our tables below of raw data.

As in past studies we measured the monopole percolation order parameter $M = n_{\text{max}}/n_{\text{tot}}$, its susceptibility $\chi$ and the monopole concentration $\langle \ell \rangle / N \ell [7]$. Recall that $n_{\text{max}}$ is the number of links with nonvanishing monopole current in the largest connected monopole cluster on the lattice and $n_{\text{tot}}$ is the total number of such links. The susceptibility $\chi$ is essentially the variance in the order parameter $M$. The monopole concentration $\langle \ell \rangle / N \ell$ is the average amount of monopole current on the lattice per link. These observables and bond percolation in general have been discussed more fully in Ref. [7] and [3], and the reader should consult those discussions for additional theoretical background. The monopole susceptibility, order parameter and concentration measurements on our $10^4$ lattices are recorded in Tables 1–3. In Table 4 we list our $16^4$ data which we collected at only one fermion mass value $m = .03$ due to our limited computer resources. Consider the monopole susceptibility first. In Fig. 1 we plot the $10^4$ and $16^4$ data vs. $\beta = 1/e^2$ at $m = .03$. Clearly the peak is considerably higher and sharper on the $16^4$ lattice. We can estimate the ratio of the susceptibility to the correlation length critical indices for monopole condensation, $\gamma_{\text{mon}}/\nu_{\text{mon}}$, from finite size scaling which states that the maximum of $\chi$ on a $L^4$ lattice should grow as,

$$\chi_{\text{max}} \sim L^{\gamma_{\text{mon}}/\nu_{\text{mon}}}$$  \hspace{1cm} (1)

From the tables on Fig. 1 we calculate,

$$\gamma_{\text{mon}}/\nu_{\text{mon}} = 2.29(9)$$  \hspace{1cm} (2)
which should be compared to the value 2.25(1) found in the quenched model\[3\]. The peak in $\chi$ occurs at $\beta = .210$ in the four flavor model as compared to $\beta = .244$ in the quenched model. So fermion, screening has shifted the monopole percolation transition to stronger coupling but apparently has not affected it otherwise. The scaling behavior of the order parameter at the critical point also yields another ratio of critical indices,

$$M(\beta = 0.210, L) \sim L^{-\beta_{\text{mon}}/\nu_{\text{mon}}} \tag{3}$$

where $\beta_{\text{mon}}$ is the “magnetic” exponent which governs the $\beta$ dependence of the order parameter in the condensed phase in the thermodynamic limit,

$$M(\beta) = C(\beta_c - \beta)^{\beta_{\text{mon}}}, \quad \beta < \beta_c \approx 0.210 \tag{4}$$

Since $M = .239(9)$ on the $10^4$ lattice at $\beta = .210$ and $M = .1452(41)$ on the $16^4$ lattice at $\beta = .210$, Eq. (3) implies that

$$\beta_{\text{mon}}/\nu_{\text{mon}} = 0.9(1) \tag{5}$$

This result should be compared to the quenched model’s value of $0.88(2)$ and the supposedly exact value for four dimensional percolation of $0.875$. Again, fermion screening appears to have left the critical behavior of the monopole percolation transition unchanged.

Finally, in Fig. 2 we plot the order parameter $M$ and the monopole concentration $\langle \ell \rangle / N_\ell$ on the $16^4$ lattice. Note that the monopole concentration varies smoothly between a value of $0.12$ at $\beta = 0.22$ and $0.17$ at $\beta = 0.19$ while the order parameter turns on abruptly at $\beta_c = 0.205(5)$. Following the discussion of bond percolation in Ref. [7], recall that the critical concentration of occupied bonds that induces the percolation transition in four dimensions is $0.16(1)$. Since $\langle \ell \rangle / N_\ell$ varies smoothly as a function of $\beta$, one could parametrize monopole condensation as a function of $\langle \ell \rangle / N_\ell$ and change the language of the transition to resemble that of bond percolation in greater detail. Since Eq. (2) and (5) are compatible with the critical indices of the traditional percolation transition, the parallel between the two transitions seems very reasonable.

In the quenched and two flavor model much more extensive measurements of the monopole transition were made, so the scaling laws Eq. (1) and (3) were checked in greater detail\[2\]. It would be worthwhile to do the same for the four flavor model as well, given more computer resources.
A final comment on the bare fermion mass dependence of the monopole data. We note from Table 1 that as $m$ increases from .005 to .07, the peak in $\chi$ shifts from .205 to .215. This is the expected trend—as $m$ increases fermion screening is suppressed and the monopole transition moves toward its quenched coupling of .244. Note, however, that as $m$ varies the height of the peak in $\chi$ does not change significantly.

3 Chiral Condensate Measurement and the Equation of State

Now consider the chiral transition. We accumulated extensive data on the average plaquette and the chiral condensate $\langle \bar{\Psi} \Psi \rangle$ recorded in Tables 5 and 6 respectively. The relatively small error bars were deduced from standard binning procedures. As we have seen in $N_f = 0$ and 2 studies, accuracy is essential in searching for the critical behavior in noncompact lattice QED $\langle \bar{\Psi} \Psi \rangle$ measurements. It is also necessary to have data over a wide range in $m$ and $\beta$, to meaningfully distinguish between different theoretical models. Since $10^4$ is not a particularly large lattice, we must study finite size effects carefully. Recall that the finite size effects in the quenched $\langle \bar{\Psi} \Psi \rangle$ data were very small and bare fermion masses in the range .001–.005 could be used to search for critical behavior$^8$. For $N_f = 2$ the finite size effects were larger and $\langle \bar{\Psi} \Psi \rangle$ data at $m = .005$ were slightly suppressed on $10^4$ lattices as compared to $16^4$ $^2$. Data at $m \geq .01$ did not display statistically significant finite size effects. The finite size effects for the $N_f = 4$ model are even larger. This is illustrated in Table 7 where we compare a sample of our $10^4$ data with the $12^4$ data of Ref. [5]. The $m = .04$ data show no size dependence while the $m = .02$ and .01 $\langle \bar{\Psi} \Psi \rangle$ measurements are suppressed on the smaller lattice. Since the lattice has only been scaled by 20% in linear dimensions between $10^4$ and $12^4$, these finite size effects in $\langle \bar{\Psi} \Psi \rangle$ are very dangerous. Therefore, we must discard our $10^4$ data at $m = .005$, .01 and .02 when searching for the bulk critical behavior in $\langle \bar{\Psi} \Psi \rangle$. This finding convinced us to accumulate data out to $m = .07$. We did not proceed further because once $m$ becomes a fair fraction of unity, non-universal lattice artifacts will plague the values of $\langle \bar{\Psi} \Psi \rangle$. The reader can check that the $m = .03 \langle \bar{\Psi} \Psi \rangle$ measurements of Table 6 on a $10^4$ lattice and the $m = .03 \langle \bar{\Psi} \Psi \rangle$ measurements of Table 4 are in
good agreement.

In order to argue that there is a chiral transition hiding in the data of Table 6, we must make a hypothesis concerning the character of the critical point. The hypothesis we have been pursuing in recent work is that the critical behavior is controlled by a power-law divergent correlation length \( \xi \sim |\beta - \beta_c|^{-\nu} \) in the chiral \( m \to 0 \) limit. This is the simplest nontrivial hypothesis we can make and since it occurs many times in statistical mechanics, effective data analysis strategies are known and well-understood. In particular, there should be a scaling region around the critical point where \( \langle \Psi \bar{\Psi} \rangle \) satisfies a universal equation of state,

\[
\frac{m}{\langle \Psi \bar{\Psi} \rangle^\delta} = f \left( \frac{\Delta \beta}{\langle \Psi \bar{\Psi} \rangle^{1/\beta_{mag}}} \right)
\]  

(6)

where \( \Delta \beta = \beta - \beta_c \) and the parameters \( \delta \) and \( \beta_{mag} \) are familiar critical indices. In particular, at \( \beta = \beta_c \), \( \delta \) controls the response of the chiral condensate to the symmetry breaking field \( m \),

\[
\langle \bar{\Psi} \Psi \rangle \sim m^{1/\delta} \quad (\beta = \beta_c)
\]  

(7)

The index \( \beta_{mag} \) controls the shape of \( \langle \bar{\Psi} \Psi \rangle \) as a function of coupling \( \beta = 1/e^2 \) within the broken-symmetry phase,

\[
\langle \bar{\Psi} \Psi \rangle = D(\beta - \beta_c)^{\beta_{mag}} \quad (\beta < \beta_c)
\]  

(8)

in the chiral limit \( m \to 0 \). In order to find \( \beta_c \) and measure \( \delta \) we consider Eq. (7) and plot \( \ln \langle \bar{\Psi} \Psi \rangle \) vs. \( \ln(m) \) from Table 6. The fixed-\( \beta \) plots are shown in Fig. 3. The coupling \( \beta_c = .205 \) fits a power-law Eq. (7) excellently with a value of \( \delta = 2.31 \). Since the data at other \( \beta \) values do not deviate far from straight lines, we see again the need for accurate \( \langle \bar{\Psi} \Psi \rangle \) data to proceed. Another slightly different and better way to find \( \delta \) and \( \beta_c \) is to plot \( -1/\ln(m) \) vs. \( -1/\ln(\langle \bar{\Psi} \Psi \rangle) \) at each coupling. At the critical coupling such a curve should be linear with a slope of \( \delta \) and it should pass through the origin. The data of Table 6 for \( m = .03-.07 \) are plotted in this fashion in Fig. 4 and only \( \beta_c = .21 \) or .205 emerge as candidates for the critical point. The straight lines in the figure give \( \delta = 2.2(1) \) in good agreement, not surprisingly, with Fig. 3. The reader should note that the data for other \( \beta \) values do not lie on straight lines which pass through the origin in Fig. 4.
Two comments are in order at this point. First, if the analysis of Figs. 3–4 is correct, then monopole condensation and chiral symmetry breaking would be coincident transitions. This is very satisfying and would fit into the physical picture of Ref. [4] nicely. The estimate of $\beta_c = .205 - .210$ agrees with our earlier, cruder work and also with the interesting and different approach of Ref. [9]. It disagrees with the chiral critical point of Ref. [5] and we shall discuss this point in detail below. The second comment consists of the observation that $\delta = 2.31$ was exactly the critical exponent found by similar procedures in a much larger scale simulation of the two flavor model[2]. This result, which may be a crucial key into the physics of noncompact lattice QED, will be discussed in Sec. 5 below.

In order to investigate the Equation of State Eq. (6) we need an estimate of $\beta_{mag}$ in addition to $\beta_c$ and $\delta$. Recall that $\beta_{mag}$ is related to $\gamma$, the “magnetic” susceptibility index and $\delta$ through the relation $\beta_{mag} = \gamma/(\delta - 1)$. We first investigate the hypothesis $\gamma = 1$ so that $\beta_{mag} = .764$. The hypothesis $\gamma = 1$ will prove to be self-consistent. It is a reasonable choice because it is true in the quenched model, as seen in simulations[8] and analytic Schwinger-Dyson studies[10], and it is also true in mean field theory. Regardless of the motivation for $\beta_{mag} = .764$, Fig. 5 of Eq. (6) follows. The data appear to lie around a straight line which is approximately,

$$f(x) = -5.3125x + 1.15$$  \hspace{1cm} (9)

As discussed in Ref. [6] a linear universal function $f$ is only possible if $\gamma$ is precisely unity, so the self-consistency of this discussion is certainly pleasing.

If Fig. 5 and Eq. (9) are really true, then the chiral transition is coincident with monopole condensation and its critical indices deviate significantly from mean field theory (where $\delta = 3$, $\beta_{mag} = 1/2$). However, since Fig. 5 can be viewed as a “fit” depending on several parameters ($\beta_c$, $\delta$ and $\beta_{mag}$) it’s significance and uniqueness are open to argument. Other hypotheses for the character of the phase transition, complete with their parameters, might describe the data just as well or even better. It is important to confront the Equation of State fit to other independent challenges.
4 Spectroscopy and the Equation of State

There is much more content in the Equation of State and the universal function of $f$ than the chiral condensate. In Ref. [6] we develop the scaling theory of mass ratios and apply it to chiral transitions. In particular, the ratio of the squares of the pion and sigma masses,

$$R(\beta, m) = \frac{m_\pi^2}{m_\sigma^2}$$  \hspace{1cm} (10)

is particularly illuminating because it is a dimensionless renormalization group invariant quantity and because it is highly constrained by chiral symmetry. In fact, as discussed in Ref. [6] it is completely determined within the scaling region by the universal function $f^{[6]}$,

$$R = \frac{m_\pi^2}{m_\sigma^2} = \left( \delta - \frac{x}{\beta_{mag}} \frac{f'(x)}{f(x)} \right)^{-1}$$  \hspace{1cm} (11)

The general shapes of the curves of $R(\beta, m)$ at fixed couplings are informative and easily understood. At the critical point $x = 0$, so $R$ reduces to

$$R(\beta_c, m) = \frac{1}{\delta}$$  \hspace{1cm} (12)

So, the curve should be flat at $\beta_c$ and give another estimate of $\delta$. For $\beta < \beta_c$ in the broken symmetry phase, the curves should be concave downward and intersect the origin because the pion is massless in the $m \rightarrow 0$ limit. For $\beta < \beta_c$ in the symmetric phase, the curves should be concave upward and intersect $R = 1$ at $m = 0$ because the pion and the sigma should become partners of a chiral multiplet.

In Ref. [2] we showed that $R$, as determined analytically from that theory’s Equation of State, fit the two flavor model’s spectroscopy data very well and had the general features expected from chiral symmetry consideration. To do the same exercise for the four flavor model we borrow the $m_\pi$ and $m_\sigma$ spectroscopy data from Ref. [5]. Tables of pion and sigma masses can be found there for $\beta = .22, .21, .20, .19, .18, .17,$ and $m = .16, .09, .04, .02$ and .01. We accept this data and plot the ratio $R(\beta, m)$ vs. $M$ at fixed $\beta$ values in Fig. 6 with the error bars as given in Ref. [5]. We note that $R$ is flat for $\beta = .20 - .21$ with a value near .4 implying $\delta \approx 2.5$. In Fig. 7 we plot Eq. (11)
with the universal function Eq. (9) choosing the same couplings $\beta = .22 - .17$ as the spectroscopy data of Fig 6. The general agreement between the plots is very satisfying, although the theoretical plot lies slightly higher than the data.

We close with a remark about the meson spectrum calculation and the ratio $R$. The computer calculation of $m_\pi^2$ and $m_\sigma^2$ are done in the “valence quark” picture which ignores possible mixing with pure multi-photon states. That mixing is proportional to the overlap of the lowest lying two-fermion (positronium) and photonium (glueball) states. If the lowest lying photonium state is relatively heavy, as expected, the mixing would be small. The success of our calculation Fig. 7 certainly suggests this, but an explicit verification would be best.

5 The Failure of Logarithmically Improved Mean Field Theory

As discussed in Sec. 3, the chiral condensate data may be fit by very different hypotheses. In particular, in Ref. [5] a logarithmically improved $O(2)$ sigma model is used for fitting purposes and it fits the data very well. The Equation of State reads,

$$m = 2\sigma V'_{eff}(\sigma^2) = \tau \frac{\sigma}{\ell \ln|\sigma^{-1}|} + \theta \frac{\sigma^3}{\ell \ln|\sigma^{-1}|}$$

where $\sigma = \langle \bar{\psi} \psi \rangle$, $\tau = \tau_1 \theta(1 - \beta/\beta_c)$ and $\theta^{-1} = \theta_o + \theta_1(1 - \beta/\beta_c)$. Choosing specific values for the five parameters $\beta_c$, $p$, $\tau_1$, $\theta_c$ and $\theta_1$, a very good fit to the data is found. The resulting chiral transition occurs at $\beta_c = .186(1)$ with mean field critical indices built in. Two comments are in order. First, since the fit involves five parameters, its significance is certainly debatable. However, our new $\langle \bar{\psi} \psi \rangle$ data for $\beta = .22 - .185$ and $m = .03 - .07$ satisfies Eq. (13) very well as shown in Fig. 8. This is certainly expected since our range of $\beta$ and $m$ values lies within those considered in the original fit of Ref. [5]. Second, $\beta_c = .186(1)$ lies deep within the monopole condensate phase found here. We would expect that all the physics is pushed to the cutoff at such a strong coupling and that the size of bound states, etc. would be on the order of the lattice spacing for $\beta \approx .186$. 
Let us subject Eq. (13) to the same test that Eq. (6), (9) and (11) have just passed. In particular, from Sec. 8 of Ref. [5] we read off a formula for 

\[ R = \frac{m_e^2}{m^2} = \left( 1 + 2\sigma^2 V_{\text{eff}}''(\sigma^2)/V_{\text{eff}}'(\sigma^2) \right)^{-1} \]  

(14a)

and

\[ V_{\text{eff}}''(\sigma^2) = \frac{p\tau}{4\sigma^2 \ln^{p+1} |\sigma^{-1}|} + \frac{\theta}{2\ln |\sigma^{-1}|} + \frac{\theta}{4\ln^2 |\sigma^{-1}|} \]  

(14b)

The important point about Eq. (14a) is that it involves no additional parameters beyond those used to obtain a chiral Equation of State fit. Either Eq. (14a) fits the spectroscopy data or the hypothesis of logarithmic triviality is wrong. Eq. (14a) simplifies at the critical point \( \beta = \beta_c \) to,

\[ R(\beta_c, m) = (3 + \ln^{-1} |\sigma^{-1}|)^{-1} \]  

(15)

The “three” in Eq. (15) occurs because \( \delta = 3 \) in mean field theory. So, in this model as opposed to a nontrivial fixed point theory, \( R \) is not quite constant at the critical point and it approaches its chiral limit of \( \frac{1}{3} \) from below. In Fig. 9 we plot Eq. (14) and see that it differs qualitatively from the spectroscopy data. For example, the spectroscopy data for \( R \) falls as \( m \) decreases at \( \beta = .190 \) while the logarithmically-improved \( O(2) \) sigma model predicts that it should rise. The data changes from concave downward to concave upward for \( \beta \) between .20 and .21 indicating that the critical point is in the interval while the logarithmically-improved \( O(2) \) model curves change their character for \( \beta \) between .18 and .19 because \( \beta_c = .186 \) is incorporated in the chiral Equation of State Eq. (13).

6 The Universality Class of Noncompact Lattice QED

We noted in Sec. 3 above that the critical indices of our power-law fits to the \( N_f = 4 \) chiral condensate data were essentially the same as those obtained previously for the \( N_f = 2 \) model. Thus, the two models should be in the same universality class and should have identical universal functions \( f \) (up to a choice of scale) and Equation of State. We, therefore, plot in Fig. 9 the
Equation of State (6) for both the $N_f = 2$ data of Ref. [2] and the $N_f = 4$ data discussed here. A linear fit and universality is quite compelling. Note that the $N_f = 2$ data was obtained at different couplings, different bare fermion masses, by a different algorithm, on a lattice of different size, on a different computer.

This intriguing result begs for further justification. Larger scale more accurate mass spectrum and chiral condensate measurements are imperative. Will these results persist on larger lattices or will triviality eventually be found? Can we implicate the monopoles more directly in the physics of the chiral phase transition? How can it be that the critical indices are essentially independent of the number of flavors? How can fermion screening shift the critical couplings as a function of flavor but not effect the dynamics of the critical point? This is just a sampling of the array of questions the results of Ref. [2] and this paper produce.

We note with interest that the methods of Ref. [9] have predicted critical couplings in excellent agreement with ours.
Acknowledgement

The simulation done here used the CRAY facilities of NERSC and PSC as well as the now defunct ST-100 of Argonne National Laboratory. J. B. Kogut is supported in part by NSF-PHY97-00148.

References

1. E. Dagatto, A. Kocic and J. B. Kogut, Phys. Rev. Lett. 60, 772 (1988); 61, 2416 (1988).

2. S. J. Hands, R. L. Renken, A. Kocic, J. B. Kogut, D. K. Sinclair and K.C. Wang, Phys. Lett. B261, 294 (1991) ILL-(TH)-92-#16, Aug., 1992.

3. S. J. Hands, A. Kocic, and J.B. Kogut, ILL-(TH)-92-7 (to appear in Phys. Lett. B)

4. S. J. Hands, A. Kocic, and J. B. Kogut, Nuc. Phys. B357, 467 (1991).

5. M. Gockeler, R. Horsley, P. Rakow, G. Schierholz and R. Sommer, Nuc. Phys. B371, 713 (1992)

6. A. Kocic, J. B. Kogut and M.-P. Lombardo, ILL-(TH)-92-18, Aug., 1992.

7. S. J. Hands and R. Wensley, Phys. Rev. Lett. 63, 2169 (1989).

8. A. Kocic, J. B. Kogut, M.-P. Lombardo and K. C. Wang, ILL-(TH)-92-12, CERN-TH. 6542/92, June, 1992.

9. A. Azcoiti, G. Di Carlo and A. F. Grillo, DFTUZ.91/34 and references contained therein.

10. S.J. Hands, A. Kocic, E. Dagatto and J.B. Kogut, Nuc. Phys. B347, 217 (1990).
Table 1: Monopole Susceptibility $\chi$ on a $10^4$ Lattice.

| $\beta/m$ | .005 | .01  | .02  | .03  | .04  | .05  | .06  | .07  |
|-----------|------|------|------|------|------|------|------|------|
| .22       | 32.1(7) | 32.9(6) | —    | 36.5(7) | 37.3(1.2) | 41.2(9) | 43.3(9) | 45.7(8) |
| .215      | 39.0(8) | 41.5(1.4) | —    | 47.5(2.6) | 49.9(1.5) | 48.5(1.3) | 49.6(2.3) | 50.8(1.6) |
| .21       | 48.2(1.6) | 49.7(1.4) | —    | 48.3(2.5) | 45.8(2.2) | 48.7(2.0) | 45.3(1.9) | 44.6(2.3) |
| .205      | 52.9(3.0) | 49.1(1.8) | —    | 42.1(3.3) | 34.6(1.9) | 35.9(2.1) | 27.2(1.9) | 30.3(1.9) |
| .20       | 41.4(2.6) | 34.9(2.6) | —    | 28.1(3.2) | 17.8(1.5) | 16.4(1.6) | 11.6(7) | 10.3(7) |
| .195      | 17.1(1.6) | 16.0(1.5) | —    | 9.2(6) | 7.6(4) | 6.6(3) | 5.9(2) | 5.3(1) |
| .19       | 8.4(6) | 7.0(4) | —    | 4.9(3) | 4.3(3) | 3.5(1) | 3.4(1) | 3.2(1) |
| .185      | —    | —    | —    | 3.2(1) | 2.8(1) | 2.6(1) | 2.3(1) | 2.2(1) | 2.0(1) |

Table 2: Monopole Percolation Order Parameter on a $10^4$ Lattice.

| $\beta/m$ | .005 | .01  | .02  | .03  | .04  | .05  | .06  | .07  |
|-----------|------|------|------|------|------|------|------|------|
| .22       | .070(2) | .077(2) | —    | .083(2) | .104(9) | .110(5) | .117(5) | .130(5) |
| .215      | .104(3) | .104(3) | —    | .131(4) | .147(6) | .176(5) | .186(8) | .208(6) |
| .21       | .154(4) | .159(6) | —    | .223(9) | .259(11) | .283(8) | .298(8) | .336(7) |
| .205      | .225(8) | .270(8) | —    | .347(9) | .396(7) | .417(7) | .457(8) | .480(4) |
| .20       | .373(9) | .407(8) | —    | .478(7) | .533(5) | .565(6) | .589(4) | .605(4) |
| .195      | .523(7) | .548(9) | —    | .619(5) | .642(4) | .666(6) | .681(3) | .695(3) |
| .19       | .639(4) | .659(4) | —    | .708(3) | .723(3) | .744(2) | .752(1) | .760(2) |
| .185      | —    | —    | —    | .759(2) | .776(2) | .788(2) | .798(2) | .807(2) | .816(1) |
### Table 3: Monopole Concentration on a $10^4$ Lattice.

| $\beta$ | $m$ | .005 | .01 | .02 | .03 | .04 | .05 | .06 | .07 |
|---------|-----|------|-----|-----|-----|-----|-----|-----|-----|
| .22     | .1163(2) | .1171(2) | —   | .1194(4) | .1207(3) | .1223(3) | .1237(2) | .1249(2) |
| .215    | .1224(2) | .1229(3) | —   | .1260(3) | .1275(3) | .1289(3) | .1301(3) | .1315(2) |
| .21     | .1287(2) | .1292(3) | —   | .1331(4) | .1349(4) | .1361(2) | .1371(3) | .1388(3) |
| .205    | .1351(3) | .1360(2) | —   | .1401(3) | .1422(3) | .1437(3) | .1452(3) | .1469(2) |
| .20     | .1422(3) | .1432(4) | —   | .1472(3) | .1501(2) | .1518(3) | .1530(3) | .1548(3) |
| .19     | .1489(4) | .1513(3) | —   | .1560(3) | .1582(3) | .1602(4) | .1617(3) | .1636(3) |
| .195    | .1584(2) | .1599(4) | —   | .1653(3) | .1670(4) | .1695(3) | .1708(2) | .1723(3) |
| .19     | —     | —     | —   | .1723(3) | .1748(3) | .1770(3) | .1784(3) | .1802(2) | .1820(3) |

### Table 4: $16^4$ data at $m = .03$ for the plaquette $S_o$, the chiral condensate $\langle \bar{\psi} \psi \rangle$, the monopole susceptibility $\chi$, the monopole order parameter $M$ and the monopole concentration $\langle l \rangle / N_l$.

| $\beta$ | $S_o$ | $\langle \bar{\psi} \psi \rangle$ | $\chi$ | $M$ | $\langle l \rangle / N_l$ |
|---------|-------|-------------------------------|------|----|---------------------|
| .22     | .9751(5) | .1625(5) | 62.7(8) | .0287(7) | .1199(1) |
| .215    | .9950(3) | .1746(5) | 97.4(1.2) | .0550(15) | .1257(1) |
| .21     | 1.0181(4) | .1899(6) | 146.0(4.6) | .1452(41) | .1326(1) |
| .205    | 1.0442(4) | .2076(6) | 61.6(4.2) | .3533(45) | .1405(2) |
| .20     | 1.0710(5) | .2259(5) | 17.2(5) | .5196(25) | .1486(2) |
| .195    | 1.0992(5) | .2450(6) | 8.3(2) | .6294(21) | .1568(2) |
| .19     | 1.1288(6) | .2658(8) | 4.7(1) | .7112(12) | .1655(2) |

### Table 5: Plaquette averages on a $10^4$ Lattice

| $\beta$ | $m$ | .005 | .01 | .02 | .03 | .04 | .05 | .06 | .07 |
|---------|-----|------|-----|-----|-----|-----|-----|-----|-----|
| .22     | .9638(6) | .9661(5) | .9679(4) | .9735(10) | .9779(10) | .9829(8) | .9869(6) | .9896(8) |
| .215    | .9842(6) | .9851(7) | .9904(6) | .9952(8) | 1.0003(7) | 1.0046(8) | 1.0087(8) | 1.0136(6) |
| .21     | 1.0060(5) | 1.0073(6) | 1.0135(6) | 1.0197(9) | 1.0247(10) | 1.0287(5) | 1.0320(6) | 1.0374(7) |
| .205    | 1.0263(5) | 1.0301(7) | 1.0361(60) | 1.0430(8) | 1.0485(7) | 1.0536(7) | 1.0585(6) | 1.0636(6) |
| .20     | 1.0505(7) | 1.0541(9) | 1.0621(8) | 1.0671(8) | 1.0766(7) | 1.0810(8) | 1.0858(7) | 1.0913(6) |
| .195    | 1.0761(9) | 1.0805(9) | 1.0896(9) | 1.0969(9) | 1.1036(10) | 1.1098(11) | 1.1152(7) | 1.1216(10) |
| .19     | 1.1051(6) | 1.1113(12) | 1.1118(7) | 1.1282(9) | 1.1349(8) | 1.1424(9) | 1.1469(6) | 1.1515(10) |
| .185    | —     | —     | —   | 1.1526(8) | 1.1612(7) | 1.1681(10) | 1.1729(9) | 1.1790(7) | 1.1848(8) |
Table 6: Chiral condensate on a $10^4$ Lattice

| $\beta$ | 0.005 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 |
|---------|-------|------|------|------|------|------|------|------|
| 0.22    | 0.0325(4) | 0.0658(6) | 0.1144(7) | 0.1588(16) | 0.1921(15) | 0.2173(10) | 0.2387(9) | 0.2572(8) |
| 0.215   | 0.0370(5) | 0.0729(7) | 0.1292(8) | 0.1737(10) | 0.2054(11) | 0.2303(12) | 0.2519(9) | 0.2695(8) |
| 0.21    | 0.0411(12) | 0.0883(8) | 0.1466(10) | 0.1901(9) | 0.2213(9) | 0.2534(11) | 0.2640(9) | 0.2823(8) |
| 0.205   | 0.0580(13) | 0.1050(10) | 0.1623(12) | 0.2058(13) | 0.2325(13) | 0.2575(9) | 0.2781(8) | 0.2969(8) |
| 0.20    | 0.0754(17) | 0.1244(18) | 0.1875(13) | 0.2214(13) | 0.2523(9) | 0.2754(8) | 0.2938(6) | 0.3104(8) |
| 0.195   | 0.0986(21) | 0.1533(21) | 0.2065(14) | 0.2420(9) | 0.2696(12) | 0.2905(13) | 0.3105(9) | 0.3271(8) |
| 0.19    | 0.1335(23) | 0.1820(21) | 0.2282(16) | 0.2650(12) | 0.2893(14) | 0.3111(15) | 0.3257(8) | 0.3400(1) |
| 0.185   | —     | —     | 0.2579(10) | 0.2887(10) | 0.3095(12) | 0.3261(9) | 0.3410(9) | 0.3554(8) |

Table 7: Finite size effects in $\langle \bar{\psi}\psi \rangle$ on $10^4$ and $12^4$ Lattices

| $m$ \ $\beta$ | $10^4$ Lattice | $12^4$ Lattice |
|---------------|---------------|---------------|
| 0.04          | 0.2893(14) | 0.2523(9) | 0.2213(9) | 0.1921(15) | 0.2892(6) | 0.2514(5) | 0.2197(4) | 0.1917(4) |
| 0.02          | 0.2282(16) | 0.1875(13) | 0.1466(10) | 0.1144(7) | 0.2340(7) | 0.1891(6) | 0.1550(6) | 0.1213(4) |
| 0.01          | 0.1820(21) | 0.1244(18) | 0.0883(8) | 0.0658(6) | —          | 0.1322(10) | 0.0917(6) | —          |
Figure Captions

Figure 1. Monopole Susceptibility on $10^4$ (circles) and $16^4$ (triangles) lattices at $m = .03$

Figure 2. Monopole Order Parameter $M$ (circles) and Concentration (triangles) on a $16^4$ Lattice at $m = .03$

Figure 3. $\ln\langle\bar{\psi}\psi\rangle$ vs. $\ln(m)$ at Various Couplings ($\beta = .185$ (dark circle), .190 (dark square), .195 (inverted dark triangle), .200 (dark triangle), .205 (cross), .210 (triangle), .215 (inverted triangle), .220 (square)) on a $10^4$ Lattice

Figure 4. $-1/\ln(m)$ vs. $-1/\ln\langle\bar{\psi}\psi\rangle$ on a $10^4$ Lattice

Figure 5. Chiral Equation of State Eq. (6) for $10^4 \langle\bar{\psi}\psi\rangle$ data $m = .03 - .07$, $\beta = .22-.185$

Figure 6. $R(\beta, m)$ vs. $m$ for Various $\beta$ for the $12^4$ Spectroscopy Data of Ref. [5]

Figure 7. $R(\beta, m)$ vs. $m$ for Various $\beta$ of Fig. 6 From the Fixed Point Equation of State (EOS) of Eq. (6) and (9)

Figure 8. Chiral Equation of State Fit of the $\beta = .22-.185, m = .03-.07 10^4$ Data Using the Logarithmically-improved O(2) Model of Ref. [5]

Figure 9. $R(\beta, m)$ vs. $m$ for Various $\beta$ of Fig. 6 From the Logarithmically-improved O(2) Sigma Model of Ref. [5]

Figure 10. Fixed Point Equation of State for $N_f = 2$ Data of Ref. [2] and $N_f = 4$ Data of Table 6