CONSTRUCTIBLE SHEAVES ON SIMPLICIAL COMPLEXES
AND KOSZUL DUALITY

MAXIM VYBORNOV

Abstract. We obtain a linear algebra data presentation of the category \( \mathcal{S}_c(X, \delta) \)
of constructible with respect to perverse triangulation sheaves on a finite simplicial complex \( X \). We also establish Koszul duality between \( \mathcal{S}_c(X, \delta) \) and the category \( \mathcal{M}_c(X, \delta) \) of perverse sheaves constructible with respect to the triangulation.

Introduction

Let \( X \) be a finite simplicial complex. There is a well known linear algebra data description of (constructible with respect to the triangulation) sheaves of vector spaces on \( X \). A sheaf corresponds to a gadget (called cellular sheaf) which assigns vector spaces to simplices and linear maps to pairs of incident simplices.

Let \( \delta \) be some perversity, that is a function \( \delta : \mathbb{Z}_{\geq 0} \to \mathbb{Z} \) satisfying some additional properties. R. MacPherson [M93] obtained a linear algebra data description of the category of (constructible with respect to the triangulation) \( \delta \)-perverse sheaves of vector spaces on \( X \), generalizing the above description. MacPherson’s linear algebra gadgets are called cellular perverse sheaves. The category of cellular perverse sheaves is denoted by \( \mathcal{P}(X, \delta) \).

Given \( X \) and a perversity \( \delta \) we can construct certain subsets of \( X \) called perverse simplices. The first version of such subsets was introduced in [GM80]. One could say that the role of perverse simplices in the intersection homology theory is the same as the role of usual simplices in the usual homology theory. In this paper we introduce the category of constructible with respect to \( (-\delta) \)-perverse simplices sheaves of vector spaces on \( X \), which we call \( \delta \)-sheaves. If \( \delta(k) = -k, \ k \geq 0 \), is the bottom perversity, \( (-\delta) \)-perverse simplices are just usual simplices, so we obtain usual constructible sheaves. We believe that \( \delta \)-sheaves should be as natural for the study of intersection homology as classical constructible sheaves are for the study of classical homology. Our main result (Theorem B) gives a description of the category of \( \delta \)-sheaves in terms of linear algebra data. Our linear algebra gadgets assign vector spaces to perverse simplices and linear maps to pairs of “incident” perverse simplices. The category of such gadgets is denoted by \( \mathcal{R}(X, \delta) \).
It is useful to identify a category of perverse sheaves or a category of linear algebra data with the category of modules over some underlying algebra. The algebra underlying the category $\mathcal{P}(X, \delta)$ (resp. $\mathcal{R}(X, \delta)$) is denoted by $A(X, \delta)$ (resp. $B(X, \delta)$). In many important cases underlying algebras turn out to be Koszul. Koszulity was established for algebras underlying perverse sheaves (middle perversity) on certain algebraic varieties in [BGSo] and [PS]. The Koszulity of $A(X, \delta)$ and $B(X, \delta)$ was conjectured by the author and established by A. Polishchuk in [P]. Moreover, it was shown in [V] that the categories $\mathcal{P}(X, \delta)$ and $\mathcal{R}(X, \delta)$ are (in some sense) Koszul dual to each other. This implies the Koszul duality between the category of perverse sheaves and the category of $\delta$-sheaves. Therefore, we could say that the two categories carry the same amount of information about the topology of $X$.

We believe that the category of $\delta$-sheaves should be considered for more complex stratified objects and that it may prove to be as important for the study of singular spaces as the corresponding category of perverse sheaves. The principal technical advantage of considering $\delta$-sheaves vs. perverse sheaves is that we do not have to deal with the derived category explicitly. The price we have to pay for this is more complicated topology. It may be said that replacing perverse sheaves by $\delta$-sheaves we move the intrinsic complexity of the theory from homological algebra to topology.

The paper is organized as follows. In part 1 we recall some notions from [GM80], [M93], [V] and [BBD]. In part 2 we state the theorems. In part 3 we prove our main theorem.

Throughout the paper $\mathbb{F}$ stands for a (commutative) field, char $\mathbb{F} = 0$. All vector spaces are considered to be over $\mathbb{F}$.

1. Preliminaries

1.1. Perversities and Perverse skeleta.

Let $X$ be a connected finite simplicial complex. Let $\hat{X}$ be the first barisentric subdivision of $X$. If two simplices $\Delta$ and $\Delta'$ of $X$ are incident we write $\Delta \leftrightarrow \Delta'$. We assume that

1. $\dim X = n$
2. $X$ is a regular cell complex
3. For every set of 0-dimensional simplices $\{v_0, v_1, \ldots, v_r\}$ there is either exactly zero or exactly one $r$-dimensional simplex $\Delta = \{v_0, v_1, \ldots, v_r\}$ with $\{v_0, v_1, \ldots, v_r\}$ as its vertices
4. all simplices are nondegenerate

**Definition 1.1.1 ([M93]).** A perversity $\delta : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ is a function from the non-negative integers $\mathbb{Z}_{\geq 0}$ to the integers such that $\delta(0) = 0$ and $\delta$ takes every interval $\{0, 1, \ldots, k\} \subset \mathbb{Z}_{\geq 0}$ bijectively to an interval $\{a, a + 1, \ldots, a + k\} \subset \mathbb{Z}$ for some $a \in \mathbb{Z}_{\leq 0}$. In other words, a perversity is such a function $\delta$ that $\delta(0) = 0$ and for
\( k \in \mathbb{Z}_{\geq 0} \),
\[
\delta(k) = \begin{cases} 
\max_{i \in [0,k-1]} \delta(i) + 1 \\
\min_{i \in [0,k-1]} \delta(i) - 1
\end{cases}
\]

**Definition 1.1.2 ([M93], cf. [GM80]).** Given a perversity \( \delta \), we define \( \delta(\Delta) = \delta(\dim \Delta) \), where \( \Delta \) is a simplex of \( X \). Given a simplex \( \hat{\Delta} = \{c_1, c_2, \ldots, c_s\} \) of \( \hat{X} \) where \( \{c_1, c_2, \ldots, c_s\} \) are baricenters of \( \Delta_1, \Delta_2, \ldots, \Delta_s \), \( \dim \Delta_1 < \dim \Delta_2 < \cdots < \dim \Delta_s \) we denote by \( \max \Delta \) such vertex \( c_i \) that \( \delta(\Delta_i) = \max\{\delta(\Delta_1), \delta(\Delta_2), \ldots, \delta(\Delta_s)\} \).

Given a simplex \( \Delta \) with the baricenter \( c \) we define the corresponding perverse simplex
\[
\delta\Delta = \bigcup_{\max \Delta = c} \hat{\Delta}
\]

We define the \( k \)-th perverse skeleton \( X^\delta_k \), \( \min[0,n] \delta \leq k \leq \max[0,n] \delta \), as follows
\[
X^\delta_k = \bigsqcup_{\delta(\Delta) \leq k} \delta\Delta \subset X
\]

Thus, we have a filtration
\[
X^\delta_i \subset X^\delta_{i+1} \subset \cdots \subset X^\delta_{i+n-1} \subset X^\delta_{i+n} = X
\]
where \( i = \min[0,n] \delta \). It is easy to see that perverse simplices are connected components of \( X^\delta_k - X^\delta_{k-1} \), \( \min[0,n] \delta \leq k \leq \max[0,n] \delta \). The decomposition of \( X \) into a disjoint union of perverse simplices is called \( \delta \)-perverse triangulation of \( X \).

From now on we will fix a perversity \( \delta \).

**1.2. Quiver algebras and categories of their modules.**

In this section we recall some definition from [V].

**Definition 1.2.1.** Let \( Q(X, \delta) \) be a quiver (i.e. finite simple oriented tree) whose vertices are indexed by simplices of \( X \) and there is an arrow from \( \Delta \) to \( \Delta' \) if and only if \( \delta(\Delta) = \delta(\Delta') + 1 \) and \( \Delta \leftrightarrow \Delta' \). There is a standard construction of a quiver algebra \( \mathbb{F} Q(X, \delta) \) associated to \( Q(X, \delta) \).

**Definition 1.2.2.** Let \( A(X, \delta) \) be the quotient of \( \mathbb{F} Q(X, \delta) \) by the chain complex relations: if \( \Delta' \) is any simplex such that \( \delta(\Delta') = k + 1 \) and \( \Delta'' \) is any simplex such that \( \delta(\Delta'') = k - 1 \), then
\[
\sum_{\Delta : \delta(\Delta) = k, \Delta' \leftrightarrow \Delta \leftrightarrow \Delta''} a(\Delta, \Delta'') \cdot a(\Delta', \Delta) = 0
\]
where \( a(\Delta, \Delta') \) are generators of \( A(X, \delta) \).
**Definition 1.2.3.** Let $B(X, \delta)$ be the quotient of $FQ(X, \delta)$ by the equivalence relations: Suppose that $\Delta', \Delta'', \Delta_1$ and $\Delta_2$ are cells of $X$ such that:

1. $\delta(\Delta') = k + 1$, $\delta(\Delta'') = k - 1$ and $\delta(\Delta_1) = \delta(\Delta_2) = k$
2. $\Delta' \leftrightarrow \Delta_1 \leftrightarrow \Delta''$ and $\Delta' \leftrightarrow \Delta_2 \leftrightarrow \Delta''$

then

$$b(\Delta_1, \Delta'') \cdot b(\Delta', \Delta_1) = b(\Delta_2, \Delta'') \cdot b(\Delta', \Delta_2)$$

where $b(\Delta_\alpha, \Delta_\beta)$ are generators of $B(X, \delta)$.

The category of left finite dimensional modules over $A(X, \delta)$ will be denoted by $\mathcal{P}(X, \delta)$ and the category of left finite dimensional modules over $B(X, \delta)$ will be denoted by $\mathcal{R}(X, \delta)$. The category $\mathcal{P}(X, \delta)$ was introduced by R. MacPherson in [M93] for an arbitrary finite regular cell complex. It is called the category of **cellular perverse sheaves**.

**1.3. Constructible sheaves.**

**Definition 1.3.1.** A sheaf $\mathcal{A}$ of $F$-vector spaces is called **constructible** with respect to $\delta$-perverse triangulation if for all simplices $\Delta$, $i_\Delta^* \mathcal{A}$ is a constant sheaf on $\delta\Delta$ associated to a finite dimensional vector space over $F$, where $i_\Delta : \delta\Delta \hookrightarrow X$ is the inclusion of the corresponding perverse simplex $\delta\Delta$.

We will denote the category of all constructible with respect to $(-\delta)$-perverse triangulation sheaves (\delta-sheaves) by $\mathcal{G}h_c(X, \delta)$.

**Definition 1.3.2([GM80]).** A ”classical” perversity $p : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$ is a non-decreasing function from the set of non-negative integers $\mathbb{Z}_{\geq 0}$ to itself such that $p(0) = 0$ and $p(k) - p(k - 1)$ is either 0 or 1.

It is easy to see that there is a one-to-one correspondence between the set of all perversities and the set of all ”classical” perversities. Let $p$ be a classical perversity corresponding to our fixed perversity $\delta$. We will denote by $\mathcal{M}_c(X, \delta)$ the category of (homologically) constructible with respect to the triangulation $p$-perverse sheaves introduced in [BBD].

**2. Theorems**

**Theorem A.** The following categories

$$\mathcal{R}(X) \simeq \mathcal{G}h_c(X)$$

are (canonically) equivalent.

This result is rather well known. A version of it was presented in [M93], another version is offered as exercise VIII.1 in [KS]. A detailed discussion will appear in [GMMV]. The primary goal of this paper is to generalize this result to the case of an arbitrary perversity.
**Theorem B.** The following categories

\[ \mathcal{R}(X, \delta) \simeq \mathcal{S}h_c(X, \delta) \]

are (canonically) equivalent

Canonically equivalent means that the equivalence functor does not depend on anything but the triangulation of \( X \) and perversity.

**Proof.** The proof will appear in part 3. \( \Box \)

The following result was obtained by R. MacPherson ([M93], [M95]). It may be considered as another generalization of the Theorem A.

**Theorem C.** The following categories

\[ \mathcal{P}(X, \delta) \simeq \mathcal{M}_c(X, \delta) \]

are equivalent.

**Proof.** It follows from [P] that \( \mathcal{M}_c(X, \delta) \) is equivalent to the category of modules over the quadratic dual to \( B(X, \delta)^{\text{opp}} \). Such a quadratic dual algebra is exactly \( A(X, \delta) \) \( \Box \)

**Theorem D (cf. [P], [V]).** (a) The category \( \mathcal{S}h_c(X, \delta) \) is Koszul;

(b) The category \( \mathcal{M}_c(X, \delta) \) is Koszul;

(c) There exists a functor

\[ K : D^b(\mathcal{M}_c(X, \delta)) \to D^b(\mathcal{S}h_c(X, -\delta)) \]

which is an equivalence of triangulated categories.

**Proof.** The proof of the fact that \( B(X, \delta) \) is Koszul (up to changing the right multiplication to the left one) is given in section 3 of [P]. Then (a) is implied by Theorem B. Since the quadratic dual \( A(X, \delta) \) is equivalent to \( B(X, \delta) \) it follows that \( A(X, \delta) \) is Koszul as the quadratic dual of a Koszul algebra. Therefore (b) is implied by Theorem C. Quadratic duality of \( A(X, \delta) \) and \( B(X, -\delta) \) and the fact that \( B(X, \delta) = B(X, -\delta)^{\text{opp}} \) implies that \( E(A(X, \delta)) = B(X, \delta) \) by Theorem 2.10.1 of [BGSo], where \( E(A(X, \delta)) \) is the Koszul dual of \( A(X, \delta) \). Using the construction of [BGSo] it is not hard to see (cf. [V]) that the Koszul duality functor restricts to an equivalence functor

\[ K : D^b(\mathcal{P}(X, \delta)) \to D^b(\mathcal{R}(X, -\delta)) \]

Thus, (c) follows from Theorem B and Theorem C. \( \Box \)

**Remark.** Note that the Koszul duality functor transforms simple objects in \( \mathcal{M}_c(X, \delta) \) (intersection homology sheaves on closed simplices) to indecomposable projective objects in \( \mathcal{S}h_c(X, \delta) \) (constant sheaves on certain subsets of \( X \)).
3. Proof of the Theorem B

3.1. Linear algebra data.

We will redefine the category $\mathcal{R}(X, \delta)$ in terms of linear algebra data. We leave it to the reader to check that the two definitions coincide.

Definition 3.1.1. An object $S$ of $\mathcal{R}(X, \delta)$ is the following data:

1. (Stalks) For every simplex $\Delta$ in $X$, a finite dimensional vector space $S(\Delta)$ called the stalk of $S$ at $\Delta$.
2. (Restriction maps) For every pair of simplices $\Delta$ and $\Delta'$ in $X$ such that $\delta(\Delta) = \delta(\Delta') + 1$, and $\Delta \leftrightarrow \Delta'$, a linear map $s(\Delta, \Delta') : S(\Delta) \to S(\Delta')$ called the restriction map.

subject to the equivalence axiom (TEA):

Suppose that $\Delta', \Delta'', \Delta_1$ and $\Delta_2$ are simplices of $X$ such that:

1. $\delta(\Delta') = k + 1$, $\delta(\Delta'') = k - 1$ and $\delta(\Delta_1) = \delta(\Delta_2) = k$
2. $\Delta' \leftrightarrow \Delta_1 \leftrightarrow \Delta''$ and $\Delta' \leftrightarrow \Delta_2 \leftrightarrow \Delta''$

then

$$s(\Delta_1, \Delta'') \circ s(\Delta', \Delta_1) = s(\Delta_2, \Delta'') \circ s(\Delta', \Delta_2)$$

The morphisms in this category are stalkwise linear maps commuting with the restriction maps.

We define another category $\mathcal{S}(X, \delta)$ of linear algebra data.

Definition 3.1.2. The category $\mathcal{S}(X, \delta)$ is a full subcategory of the abelian category $\mathcal{R}(\hat{X})$. An object $T$ of $\mathcal{R}(\hat{X})$ belongs to $\mathcal{S}(X, \delta)$ if for any $\hat{\Delta}, \hat{\Delta}' \subseteq -\delta \Delta$ we have

1. $T(\hat{\Delta}) = T(\hat{\Delta}')$
2. if $\hat{\Delta}$ is a codim-1 face of $\hat{\Delta}'$ then $t(\hat{\Delta}, \hat{\Delta}') = \text{Id}_{T(\hat{\Delta})}$.

3.2. $\mathcal{R}(X, \delta) = \mathcal{S}(X, \delta)$.

Definition 3.2.1. We will consider a partial order on the set of simplices of $X$ which is uniquely defined by the following: $\Delta < \Delta'$ if

1. $\Delta$ and $\Delta'$ are incident
2. $\delta(\Delta) = \delta(\Delta') + 1$

we say that $\Delta \leq \Delta'$ if $\Delta < \Delta'$ or $\Delta = \Delta'$. It is easy to see that if $\Delta \leftrightarrow \Delta'$ and $\delta(\Delta) > \delta(\Delta')$ then $\Delta < \Delta'$.

Definition 3.2.2. Let $\Delta_I \leq \Delta_T$ be two simplices of $X$. Let $S$ be an object of $\mathcal{R}(X, \delta)$. We define a linear map $f(\Delta_I, \Delta_T) : S(\Delta_I) \to S(\Delta_T)$ in the following way

1. if $\Delta_I = \Delta_T$ then $f(\Delta_I, \Delta_T) = \text{Id}_{S(\Delta_I)}$
2. if $\Delta_I < \Delta_T$ then by definition there is a sequence

$$(3.2.1) \quad \Delta_I = \Delta_0 < \Delta_1 < \Delta_2 < \cdots < \Delta_r = \Delta_T$$
\[ \delta(\Delta_i) = \delta(\Delta_{i+1}) + 1, \quad 0 \leq i \leq r - 1. \]

We set
\[ f(\Delta_I, \Delta_T) = s(\Delta_{r-1}, \Delta_r) \circ \cdots \circ s(\Delta_0, \Delta_1) \]

It is easy to see that \( f(\Delta_I, \Delta_T) \) does not depend on the choice of sequence (3.2.1); and moreover, that all \(<\) in (3.2.1) could be changed to \(\leq\). In particular, if \(\Delta_1 \leq \Delta_2 \leq \Delta_3\) then
\[ f(\Delta_1, \Delta_3) = f(\Delta_2, \Delta_3) \circ f(\Delta_1, \Delta_2) \quad (3.2.2) \]

**Lemma 3.2.3.** Let \(\hat{\Delta}, \hat{\Delta}'\) of \(\hat{X}\) be such that \(\hat{\Delta}\) is a codim 1 face of \(\hat{\Delta}'\). Let \(\hat{\Delta} \subseteq -\delta \hat{\Delta}\), \(\hat{\Delta}' \subseteq -\delta \hat{\Delta}'\). Then \(\Delta \leftrightarrow \Delta'\) and \(\Delta \leq \Delta'\).

**Proof.** Let \(\hat{\Delta} = \{v_1, \ldots, v_k\}, \hat{\Delta}' = \{v_1, \ldots, v_k, e\}\). Case I: \(e = \max(\hat{\Delta}')\). Then \(e\) is a baricenter of \(\hat{\Delta}'\). By definitions \(-\delta(\hat{\Delta}') > -\delta(\hat{\Delta}) \implies \delta(\hat{\Delta}') < \delta(\hat{\Delta})\). Let \(v_i = \max(\hat{\Delta}), v_i\) is the baricenter of \(\hat{\Delta}\). Then \(1\)-simplex \(\{v_i, e\}\) is a simplex of \(\hat{X}\), thus \(\Delta \leftrightarrow \Delta'\). Case II: \(e \neq \max(\hat{\Delta}')\). Then \(v_i = \max(\hat{\Delta}) = \max(\hat{\Delta}')\), \(v_i\) is the baricenter of both \(\Delta\) and \(\hat{\Delta}'\), thus \(\Delta = \Delta'\). \(\square\)

**Theorem 3.2.4.** The category \(\mathcal{R}(X, \delta)\) is isomorphic to the category \(\mathcal{S}(X, \delta)\)

\[ \mathcal{R}(X, \delta) = \mathcal{S}(X, \delta) \]

**Proof.** (a) The functor \(\Phi : \mathcal{R}(X, \delta) \to \mathcal{S}(X, \delta)\). If \(S\) is an object of \(\mathcal{R}(X, \delta)\) then \(T = \Phi(S)\) is constructed as follows

\[ T(\hat{\Delta}) = S(\Delta), \quad \text{for } \hat{\Delta} \subseteq -\delta \hat{\Delta} \]

If \(\hat{\Delta}'\) is a codim 1 face of \(\hat{\Delta}''\) then \(\Delta' \leq \Delta''\) by Lemma 3.2.3 and we set

\[ t(\hat{\Delta}', \hat{\Delta}'') = f(\Delta', \Delta''), \quad \text{for } \hat{\Delta}' \subseteq -\delta \hat{\Delta}' \text{ and } \hat{\Delta}'' \subseteq -\delta \hat{\Delta}'' \]

Now let \(\hat{\Delta}' \subseteq -\delta \hat{\Delta}', \hat{\Delta}_1 \subseteq -\delta \hat{\Delta}_1, \hat{\Delta}_2 \subseteq -\delta \hat{\Delta}_2, \hat{\Delta}'' \subseteq -\delta \hat{\Delta}''\) be such a quadruple for which we have to check TEA. Then by Lemma 3.2.3 we have

\[ \Delta' \leq \Delta_1 \leq \Delta'' \]
\[ \Delta' \leq \Delta_2 \leq \Delta'' \]

By definitions we have

\[ t(\hat{\Delta}_1, \hat{\Delta}'') \circ t(\hat{\Delta}', \hat{\Delta}_1) \overset{\text{def}}{=} f(\Delta_1, \Delta'') \circ f(\Delta', \Delta_1) \overset{(3.2.2)}{=} f(\Delta', \Delta'') \]

\[ \overset{(3.2.2)}{=} f(\Delta_2, \Delta'') \circ f(\Delta', \Delta_2) \overset{\text{def}}{=} t(\hat{\Delta}_2, \hat{\Delta}'') \circ t(\hat{\Delta}', \hat{\Delta}_2) \]

\( \square \)
Φ on morphisms is defined in the obvious way.

(b) The functor $S(X, \delta) \to R(X, \delta)$. If $T$ is an object of $S(X, \delta)$ then $S = \Psi(T)$ is constructed as follows

$S(\Delta) = T(\hat{\Delta}), \quad \text{for } \hat{\Delta} \subseteq ^{-\delta}\Delta$

$S(\Delta)$ is well defined due to constructibility of $T$. Let $\Delta'$ and $\Delta''$ be two incident simplices of $X$ such that $\delta(\Delta') = \delta(\Delta'') + 1$. We have to construct the map $s(\Delta', \Delta'')$. Let $c'$ be a baricenter of $\Delta'$ and $c''$ be a baricenter of $\Delta''$. We set

$s(\Delta', \Delta'') = t(c', \{c', c''\})$

where $\{c', c''\}$ is a simplex of $\hat{X}$. Let $\Delta' < \Delta_m < \Delta''$, $\delta(\Delta') = \delta(\Delta_m) + 1 = \delta(\Delta'') + 2$. Let $c'$, $c_m$ and $c''$ be baricenters of $\Delta'$, $\Delta_m$ and $\Delta''$ respectively. Let us assign special names to the following four simplices of $\hat{X}$: $l = \{c', c''\}$, $l_m' = \{c', c_m\}$, $l_m'' = \{c'', c_m\}$ and $tr = \{c', c_m, c''\}$. We have

$s(\Delta', \Delta'') = s(l', l_m) = t(l', l_m)\circ t(c', l_m')$

The equivalence axiom follows.

$Ψ$ on morphisms is defined in the obvious way.

(c) It follows from our explicit construction that $Ψ \circ Φ = Id$. Using constructibility of $T$, definitions and (3.2.2) it is easy to see that $Φ \circ Ψ = Id$. □

3.3. $S(X, \delta) \simeq Sh_c(X, \delta)$.

**Definition 3.3.1.** Let $S$ be an object of $R(\hat{X})$. Let $i_A : A \hookrightarrow X$ be a closed union of simplices of $\hat{X}$. We define a functor $i_A^* : R(\hat{X}) \to R(A)$ as follows: if $T = i_A^* S$ then

1. $T(\hat{\Delta}) = S(\hat{\Delta})$
2. $t(\hat{\Delta}', \hat{\Delta}'') = s(\hat{\Delta}', \hat{\Delta}'')$

**Lemma 3.3.2.** The following functorial diagram commutes i.e. two possible compositions of functors are isomorphic

$$
\begin{array}{ccc}
\mathcal{R}(\hat{X}) & \xleftarrow{\sim} & Sh_c(\hat{X}) \\
\downarrow i_A^* & & \downarrow i_A^* \\
\mathcal{R}(A) & \xleftarrow{\sim} & Sh_c(A)
\end{array}
$$

where $i_A^* : Sh_c(\hat{X}) \to Sh_c(A)$ is a standard sheaf theory functor.

**Proof.** Proof is left to the reader □
Theorem 3.3.3. The category \( \mathcal{S}(X, \delta) \) is (canonically) equivalent to the category \( \mathcal{S}_c(X, \delta) \)

\[ \mathcal{S}(X, \delta) \simeq \mathcal{S}_c(X, \delta) \]

Proof. By definitions and functoriality of \( i^* \) the category \( \mathcal{S}_c(X, \delta) \) is a full subcategory of \( \mathcal{S}_c(\hat{X}) \). Lemma 3.3.2 implies that the equivalence functor \( R(\hat{X}) \simeq \mathcal{S}_c(\hat{X}) \) restricts to an equivalence functor \( \mathcal{S}(X, \delta) \simeq \mathcal{S}_c(X, \delta) \). □

Theorem 3.2.4 and Theorem 3.3.3 imply Theorem B.

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Department of Mathematics, Yale University, 10 Hillhouse Ave, New Haven, CT 06520

E-mail address: mv@math.yale.edu