Boundary conformal field theory at the extraordinary transition: The layer susceptibility to $O(\varepsilon)$

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Abstract

We present an analytic calculation of the layer (parallel) susceptibility at the extraordinary transition in a semi-infinite system with a flat boundary. Using the method of integral transforms put forward by McAvity and Osborn [Nucl. Phys. B 455 (1995) 522] in the boundary CFT we derive the coordinate-space representation of the free mean-field propagator at the transition point. The simple algebraic structure of this function provides a practical possibility of higher-order calculations. Thus we calculate the explicit expression for the layer susceptibility at the extraordinary transition in the one-loop approximation. Our result is correct up to order $O(\varepsilon)$ of the $\varepsilon = 4 - d$ expansion and holds for arbitrary width of the layer and its position in the half-space. We discuss the general structure of our result and consider the limiting cases related to the boundary operator expansion and (bulk) operator product expansion. We compare our findings with previously known results and less complicated formulas in the case of the ordinary transition. We believe that analytic results for layer susceptibilities could be a good starting point for efficient calculations of two-point correlation functions. This would be of great importance in view of recent breakthrough in the bulk and boundary conformal field theory in general dimensions.

Keywords: boundary conformal field theory, extraordinary transition, ordinary transition, boundary operator expansion, operator product expansion, free propagator, layer susceptibility, epsilon expansion

1 Introduction

The conformal invariance has appeared in theoretical physics more than 100 years ago — see [1, 2]. However, its importance for critical phenomena associated with the second-order phase transitions has been realized due to the seminal 1970 letter by Polyakov [3].1 About 1 The scale invariance inherent in systems at critical points, along with the invariance under translations and rotations, usually entails the existence of an invariance under a larger symmetry group, that of conformal transformations [4, 5, 6]. The interrelations between the scale and conformal invariance are thoroughly discussed in the recent review paper [7].
the same time the conformal bootstrap equations came into play [3, 8, 9], along with the short-distance operator product expansions by Wilson [10] in the high-energy physics and Kadanoff [11, 12] in the theory of critical phenomena. The relevant relations have been the self-consistent equations for full correlation functions of the Schwinger-Dyson type. The term bootstrap implies an intrinsic self-consistency [13]. These equations allowed conformally invariant solutions for correlation functions, and the Operator Product Expansion (OPE) has been recognized as an efficient tool to treat them. An early review has been given by Mack [14]; a very recent review on Conformal Field Theory (CFT) by the same author can be found in [15].

In what followed, there has been a lot of formal development, see [16, 17, 18, 19, 20, 21] and the reviews [22, 23], but the practical output like explicit expressions or numerical values of critical exponents appeared to be relatively modest [24, 25, 20] and [22, Sec. 15]. In this respect, the greatest achievement of the classical conformal bootstrap programme have been the calculations of critical exponents up to $O(N^3)$ in the large-$N$ expansion [26, 27, 28], see also the book by Vasiliev [29] and the recent review [30]. Moreover, the traditional bootstrap approach and OPE have been used in a comprehensive study of the operator algebra and multi-point correlation functions in a series of papers by Lang and Rühl including [31].

A breakthrough in the development of the CFT occurred with the appearance of the seminal paper by Belavin, Polyakov and Zamolodchikov [32](see also [33 and [34]). In two space dimension, where the group of conformal transformations is infinite dimensional, the combination of the conformal bootstrap and the OPE was able to yield the exact solutions of the so-called minimal models. The (infinite) family of these models included the well-known exactly solvable two-dimensional Ising, three-state Potts, and Ashkin-Teller models as special cases. It is hard to overestimate the impact of this work. According to Google Scholar, by now the paper [32] has more than 5700 citations, it has strongly influenced the Les Houches proceedings [35], it was in the basis of the compilation [36] reprinted there along with a series of related papers, its methods and results have been discussed in numerous review articles and books, in particular [4, 37, 5, 6, 38]. The two-dimensional CFT influenced several areas of mathematics and benefited from their development [38, 39, 40].

The success of the CFT in two dimensions stimulated its application in different branches of the Quantum Field Theory [41], (super)strings [42, 43, 44, 45, 46], statistical mechanics. In the latter, essential progress has been achieved in bulk [47, 48] and boundary critical phenomena [49, 50, 51, 52], finite-size scaling [53], critical Casimir effect [54, 55, 56] polymer physics [57, 58], etc. (see [5, 6]). In view of the substantial progress of CFT, Henkel [59] proposed the hypothesis of generalized local scale invariance intended to describe the dynamic and strongly anisotropic static systems at criticality. This conjecture and its implications in the case of uniaxial Lifshitz points have been critically analysed in [60].

A new explosion of interest in conformal bootstrap approach has occurred in about the last ten years, see the reviews [61, 62]. It started with the work [63] where new ideas and numerical methods have been applied to old equations. The research activities moved to space dimensions higher than two, especially $d = 3$. The scaling dimensions of basic operators and hence the critical exponents have been extracted from the fundamental consistency conditions and conformal symmetry considerations without any microscopic input [64, 65, 66, 67, 68]. The highlight of the ”new” bootstrap approach are the most precise calculations of critical exponents of the three-dimensional Ising model [69], namely $(\eta, \nu) = (0.0362978(20), 0.629971(4))$. 


As acknowledged by the authors of this last reference, their results for the $O(N)$ model with $N = 2$ and 3 are still less accurate as compared to the best Monte Carlo estimates [70, 71, 72].

Again, this development stimulated numerous research activities in diverse scientific fields. Merely search "bootstrapping" in Google Scholar and you will see hundreds of interesting contributions. A big number of references can be found in the review papers [61, 62]. Among of them, of special interest for us are recent investigations of systems with boundaries [62, Sec. V.B.6].

The early fundamental work by Cardy "Conformal Invariance and Surface Critical Behavior" [49] appeared in 1984. The general formulation has been given in general $d$ dimensions and some exact results have been derived in $d = 2$. Approximately at the same time, the implications of the conformal invariance in semi-infinite systems have been noticed in several explicit calculations (see the references in [4, p. 82]). A new essential portion of fundamental knowledge has appeared in a series of papers by McAvity and Osborn [77, 50, 51]. The last of these references provided a basis for building up in [78] the modern bootstrap program for boundary CFT in $d$ dimensions.

Liendo, Rastelli, and van Rees [78] formulated the self-consistent equations by comparing two different short-distance expansions for the same two-point correlation function. These are the Operator Product Expansion and the Boundary-Operator Expansion (BOE) worked out in [51] in a semi-infinite $d$ dimensional Euclidean space bounded by a flat $d - 1$ dimensional (codimension-one) surface. The equations could be checked analytically in two cases. It was possible to reproduce the free Neumann and Dirichlet propagators in $d$ dimensions. Also, the non-trivial $O(\varepsilon)$ contributions to the two-point functions in $d = 4 - \varepsilon$ dimensions could be derived for the same boundary conditions (corresponding to the special and ordinary transitions) in accord with previous calculations [79, 50, 51]. In [78, Appendix B] the free propagator of the extraordinary transition in $d = 4$ has been reproduced (see [80, (4.96)] and below). The expression [78, (B.37)] for the one-point function $\langle \phi \rangle$ could be compared with the order-parameter profile of [81]. Moreover, using the linear programming methods the bootstrap constraints have been studied in $d = 3$ for the special and extraordinary transitions. This have led to numerical estimates some of which have been compared with the results of two-loop calculations of [82] (see also [83, 84]).

Further progress in extracting the conformal data from the bootstrap equations in three-dimensional semi-infinite systems has been achieved in the subsequent papers [85, 86]. Here the analysis of bootstrap equations differs from that of [78]. The method of determinants is used relying on a truncation of the operator spectrum. In the case of the extraordinary transition, the bootstrap yielded certain information on the bulk operator spectrum [85]. For the ordinary and special transitions precise estimates have been obtained for relevant scaling dimensions, which compared well both with field-theoretic calculations [82] and a series of Monte Carlo results: see [86, Table 1].

Other types of "defects" such as line (codimension-two) defects as well as certain general features of the boundary CFT have been studied in [87, 88, 89] and [90, 91, 92]; see also the recent proceedings [93].

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2For an exhaustive comparison with other data, especially that of the six-loop epsilon expansion, see also [73].

3Exhaustive reviews on surface critical behavior in semi-infinite systems can be found in [74, 75, 76].

4To the best of our knowledge, the $O(\varepsilon)$ correction to this function still remains unknown.
Very recently essential progress has been achieved in analytic solution of boundary-CFT bootstrap equations [94]. First, the known anomalous scaling dimensions of the leading boundary operators corresponding to Neumann and Dirichlet boundary conditions were reproduced up to order $O(\varepsilon^2)$ [94, p. 12]. This is (unfortunately) the highest order of the $\varepsilon$-expansion known up to now for critical exponents of the special and ordinary transitions. Moreover, just "using the analytic structure of the two point function and symmetries of the BCFT\textsuperscript{\textsuperscript{\textsuperscript{T}}} the $O(\varepsilon^2)$ contribution to the scaling function of the two-point correlator has been found and written down in [94, Sec. 4.4] (without any comparison or further discussion). This function was not previously known in the statistical physics literature. Regrettably, the two-point correlation function of the extraordinary transition is by now not known even to order $O(\varepsilon)$. In the rest of this paper we shall concentrate on this last type of transition.

The general information and relevant original literature on the extraordinary transition can be found in the standard references [74, 75, 76]. Its special feature is the presence of the non-vanishing order-parameter profile $\langle \phi(z) \rangle$ both above and below the critical temperature $T_c$ for $z \geq 0$. By $z$ we denote the coordinate normal to the hyperplane constraining the half-space and located at $z = 0$. All surface critical exponents (or corresponding scaling dimensions) can be expressed in terms of the bulk exponents. The recognition of this feature goes back to Bray and Moore [95] (see also [96] where the conformal-invariance arguments are involved). The most important feature from the conformal bootstrap perspective is that the most relevant terms of the boundary-operator expansion of a general bulk operator contain the identity operator and the $zz$-component of the stress-energy tensor, $T_{zz}(0)$, with the scaling dimension $d$ [55], see also [76, Sec. 3.3].

We note also that there are several interesting examples of physical systems undergoing phase transitions which belong to the universality class of the extraordinary transition. The most prominent example is the experimentally observed phenomenon of critical adsorption, see [97, 98, 99]. It occurs close to the demixing point in liquid binary mixtures confined by a hard wall or interface, and in constrained polymer solutions [58].

On the other hand, we should also acknowledge that the theoretical description of the extraordinary transitions is far from completeness. This is to large extent due to the mathematical difficulties one immediately encounters. So, presenting the new analytic results in this non-trivial case we hope to stimulate some progress in this direction.

In Sec. 2 we recall the known $pz$-representation of the free mean-field propagator at the extraordinary transition. Using one of the methods of the boundary conformal field theory we obtain the coordinate representation of this function at criticality and discover its simple and elegant structure. In Sec. 3 we present an explicit calculation of the layer susceptibility $\chi(z, z')$ for arbitrary distances $z$ and $z'$ from the surface in the one-loop approximation. An explicit expression is obtained correct up to $O(\varepsilon)$ order of the $\varepsilon = 4 - d$ expansion. The singularities it contains have their interpretation in terms of short-distance operator product expansions. The results are compared with the previously known data and with analogous expressions of the more common case of the ordinary transition. In Sec. 4 we discuss the relevance of present results for eventual future calculations of coordinate representations of two-point correlation functions. This is an interesting issue given the great recent progress achieved in the conformal field theory both in the bulk and semi-infinite systems. Some technical detail can be found in the Appendix.
2 Mean-field correlation function at the extraordinary transition

At the critical temperature, the free mean-field propagator in the "mixed" $p_z$ representation is given by [52, (2.5)]

$$G_0(p; z < z') = \frac{1}{2p} \left[ W(-p_z) - W(p_z) \right] W(p_z')$$  

(1)

where the function $W$ is

$$W(x) = e^{-x} \left( 1 + \frac{3}{x} + \frac{3}{x^2} \right).$$  

(2)

As before, $z \geq 0$ is the coordinate normal to the $(d-1)$-dimensional boundary that constrains the semi-infinite space at $z = 0$. By $p$ we denote the absolute value of the momentum $\mathbf{p}$ conjugated to the $(d-1)$-dimensional position vector $\mathbf{r}$ parallel to the surface. The exhaustive information on the underlying $\phi^4_d$ model and other background information can be found in [75, 81, 52].

Though the above expression is valid only in four spatial dimensions $d = 4$, our goal now will be to find its counterpart in the coordinate representation by using the dimensional continuation to $d \leq 4$ with $d$ only slightly differing from 4. This may be achieved by the direct Fourier transformation of $G(p; z, z')$ in $d-1$ directions parallel to the boundary. However, we find this way a bit annoying and not very instructive. Instead of this, we shall use another integral transformation which employs the conformal invariance of the underlying free theory. It was introduced to the subject in [51] and [100]. For convenience, we reproduce the needed information in a few words below.

In the semi-infinite critical system, the conformal invariance implies that the two-point correlation function of fields can be written in the scaling form

$$G(x, x') = \frac{1}{(4zz')^{\Delta_\phi}} g(\xi).$$  

(3)

Here $\Delta_\phi$ is the usual full scaling dimension of the field$^5\Delta_\phi = \frac{1}{2}(d-2+\eta)$ and the variable $\xi$ is one of the possible invariants of the (restricted) conformal transformations preserving the boundary of the system [49]:

$$\xi = \frac{|x - x'|^2}{4zz'}. $$  

(4)

Integration of the correlation function $G(x, x')$ over directions parallel to the surface in a strip between the parallel hyperplanes at $z$ and $z'$ $(z \leq z')$ defines the layer (or parallel) susceptibility

$$\chi(z, z') = \int d^{d-1}r G(r; z, z').$$  

(5)

By $r$ we denote the distance between the points $x$ and $x'$ in the "parallel" directions: the vector $x' - x$ has the components $(r, z' - z)$. Using this convention in the definition of the

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$^5\eta$ is the familiar (bulk) correlation function exponent, and $\eta/2$ is often called the anomalous dimension of the field. The $\varepsilon$-expansion of $\eta$ starts with $O(\varepsilon^2)$. 

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variable $\xi$ in (4) and reporting the scaling form (3) into (5) we obtain, after performing the angular integration,

$$\chi(z, z') = \frac{1}{(4zz')^{d-1}} S_{d-1} \int_0^\infty dr r^{d-2} g \left( \frac{r^2 + |z - z'|^2}{4zz'} \right).$$

(6)

Here $S_{d-1}$ is the area of a unit sphere in $d - 1$ dimensions. For general $d$, the well-known formula reads

$$S_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma \left( \frac{d}{2} \right)}.$$  

(7)

A slight modification of the integration variable in (6) leads to the representation

$$\chi(z, z') = \frac{1}{(4zz')^{d-1}} \frac{S_{d-1}}{2} \int_0^\infty du u^{\frac{d-3}{2}} g(u + \rho)$$

(8)

which defines the general scaling form for the layer susceptibility\(^6\) (cf. (64) below)

$$\chi(z, z') = \frac{1}{4zz'} \hat{g}(\rho).$$

(9)

In the last two expressions $\rho$ denotes the combination

$$\rho = \frac{|z - z'|^2}{4zz'}$$

(10)

explicitly appearing in (6).

An important feature is that the equations (8) and (9) define an integral transform

$$\hat{g}(\rho) = \frac{\pi^{\frac{d-1}{2}}}{\Gamma \left( \frac{d-1}{2} \right)} \int_0^\infty du u^{\frac{d-3}{2}} g(u + \rho)$$

(11)

which can be inverted via

$$g(\xi) = \frac{\pi^{\frac{d-1}{2}}}{\Gamma \left( \frac{d-1}{2} \right)} \int_0^\infty d\rho \rho^{\frac{d-3}{2}} \hat{g}(\rho + \xi).$$

(12)

This inversion formula \([51, 100]\) allows to calculate the scaling function $g(\xi)$ of the correlator $G(x, x')$ starting from the scaling function $\hat{g}(\rho)$ of the layer susceptibility $\chi(z, z')$ which is normally easier accessible in the explicit (perturbative) calculations.

### 2.1 A warm-up: Ordinary transition

Let us demonstrate the above procedure with a simplest example of reproducing the well-known Dirichlet propagator $G_D(x, x')$. The massless Dirichlet propagator in the $pz$ representation $G_D(p; z < z')$ is recovered from $G(p; z < z')$ in (1) when we take there into account\(^6\)This is equivalent to the scaling representation of the same function discussed in \([101]\).
only the pure exponential terms in the functions \( W(x) \). The limit \( p \to 0 \) of \( G_D(p; z < z') \) leads us to the layer susceptibility \( \chi(z < z') = z \). In more general, symmetric, form

\[
\chi_D(z, z') = \min(z, z').
\] (13)

This result should be represented in the scaling form (9). We have

\[
\chi_D(z, z') = \sqrt{4zz'} \frac{1}{2} \left( \frac{\min(z, z')}{\max(z, z')} \right)^{1/2} \equiv \sqrt{4zz' \frac{1}{2} \zeta^{1/2}}.
\] (14)

Here we introduced a new variable \( \zeta \),

\[
\zeta = \frac{\min(z, z')}{\max(z, z')}.
\] (15)

It will play an important role in the subsequent one-loop calculations. But for the present purpose we need to identify the factor \( \frac{1}{2} \zeta^{1/2} \) as the scaling function depending on the scaling variable \( \rho \). In terms of \( \zeta \), the definition (10) of \( \rho \) reads

\[
\rho = \frac{1}{4} (\zeta + \zeta^{-1} - 2),
\] (16)

and thereof follows

\[
\zeta = (\sqrt{\rho + 1} - \sqrt{\rho})^2.
\] (17)

Hence we obtain for the required scaling function

\[
\hat{g}_D(\rho) = \frac{1}{2} \left( \sqrt{\rho + 1} - \sqrt{\rho} \right).
\] (18)

Thus, in order to calculate \( g_D(\xi) \) through (12) we should evaluate the Euler integral in

\[
g_v(\xi) = -\frac{\pi^{-\frac{d-1}{2}}}{\Gamma(-\frac{d-1}{2})^2} \int_0^\infty \rho^{-\frac{d-1}{2}-1} \sqrt{\rho + \xi} \, dp
\] (19)

and subtract the same function \( g_v \) of the argument \( \xi + 1 \). The result is

\[
g_D(\xi) = C_d \left[ \xi^{1-\frac{d}{2}} - (\xi + 1)^{1-\frac{d}{2}} \right].
\] (20)

Multiplying this scaling function through the overall factor \((4zz')^{-\Delta_{\phi}}\) according to (3)7 and using the explicit expression (4) for the scaling variable \( \xi \) we obtain the Dirichlet propagator in its usual form given in terms of the Euclidean spatial coordinates in \( d \) dimensions:

\[
G_D^{(d)}(r; z, z') = C_d \left( R_{-}^{2-d} - R_{+}^{2-d} \right) = G_v^{(d)}(r; |z - z'|) - G_v^{(d)}(r; z + z').
\] (21)

The constant \( C_d \) arising from the expression (19) is given by

\[
C_d = \frac{1}{4} \pi^{-d/2} \Gamma\left(\frac{d-2}{2}\right) = S_d^{-1} d.\]

\(\text{footnote}{7}\)Remember that in the present case of the free theory \( \eta = 0 \).
This is the usual "geometric factor" appearing in the coordinate representation of the free propagators. Finally, we indicated in (21) that the Dirichlet propagator consists of the difference of two usual bulk propagators $G^{(d)}_v$ depending respectively on two distances $R_-$ and $R_+$ in $d$-dimensional space,

$$R_\mp = \sqrt{r^2 + (z \mp z')^2}, \quad r^2 = \sum_{\alpha=1}^{d-1} r_{\alpha}^2.$$  \hspace{1cm} (23)

### 2.2 The "extraordinary" case

Once we have carefully done the above exercise, there remains not very much work to do in order to derive a similar coordinate representation of the free propagator at the extraordinary transition. This will correspond to the Fourier transform over the parallel directions of the "$p_z$" propagator given in (1). Again, the $p \to 0$ limit of this expression yields the corresponding layer susceptibility at the extraordinary transition in the mean-field approximation. It is given by

$$\chi_0(z, z') = \frac{1}{5} \left[ \min(z, z') \right]^3 \frac{1}{\max(z, z')}^2 = \frac{4}{10} \frac{1}{\xi^2}.$$  \hspace{1cm} (24)

Hence, the scaling function to be integrated in (12) is

$$\hat{g}(\rho + \xi) = \frac{1}{10} \left( \sqrt{\rho + \xi + 1} - \sqrt{\rho + \xi} \right)^5.$$  \hspace{1cm} (25)

The integral is not as complicated as it might appear at first glance. Simple algebraic properties

$$\left( \sqrt{\rho + \xi + 1} - \sqrt{\rho + \xi} \right) \left( \sqrt{\rho + \xi + 1} + \sqrt{\rho + \xi} \right) = 1$$  \hspace{1cm} (26)

and

$$\left( \sqrt{\rho + \xi + 1} - \sqrt{\rho + \xi} \right)^2 + \left( \sqrt{\rho + \xi + 1} + \sqrt{\rho + \xi} \right)^2 = 2 \left( 1 + 2 \rho + 2 \xi \right)$$  \hspace{1cm} (27)

allow us to reduce $\hat{g}(\rho + \xi)$ to a linear combination of square roots thus leading us to several simple integrals similar to that in (19). We write the fifth power $a^5$ in (25) as $a^2 a^2 a^2$, express the first $a^2$ through (27) and eliminate the resulting "mixed" terms using (26). Then we perform the same trick with the remaining factor $a^2$. The result is

$$\left( \sqrt{\rho + \xi + 1} - \sqrt{\rho + \xi} \right)^5 = \left[ -5 - 20(\rho + \xi) - 16(\rho + \xi)^2 \right] \sqrt{\rho + \xi}$$  \hspace{1cm} (28)

$$+ \left[ 5 - 20(\rho + \xi + 1) + 16(\rho + \xi + 1)^2 \right] \sqrt{\rho + \xi + 1}.$$

We see that taking only the first terms $-5$ and $+5$ in square brackets again reproduces the "Dirichlet" contribution (20) in the present scaling function. Performing explicitly the remaining integrations related to the first line in (28) we obtain

$$g^{(1)}(\xi) = C_d \xi^{1-\frac{d}{2}} - \frac{6}{\pi} C_{d-2} \xi^{2-\frac{d}{2}} + \frac{12}{\pi^2} C_{d-4} \xi^{3-\frac{d}{2}}.$$  \hspace{1cm} (29)
Obviously, the second line of (28) produces, up to the signs, the contributions of the same form but with replacements $\xi \rightarrow (\xi + 1)$. Adding up all terms we obtain for the whole scaling function $g(\xi)$

$$g(\xi) = C_d \left[ \xi^{1-\frac{d}{2}} - (\xi + 1)^{1-\frac{d}{2}} \right] - \frac{6}{\pi} C_{d-2} \left[ \xi^{2-\frac{d}{2}} + (\xi + 1)^{2-\frac{d}{2}} \right] + \frac{12}{\pi^2} C_{d-4} \left[ \xi^{3-\frac{d}{2}} - (\xi + 1)^{3-\frac{d}{2}} \right].$$

(30)

Note that the increase of powers of $\xi$ by one in the three subsequent terms in the last equation can be formally traced back to the corresponding decrease of the parameter $d$ in these powers by two. This was readily anticipated in the coefficients $C_d$, $C_{d-2}$, and $C_{d-4}$ which all are given by the same formula (22). Thus we may write the function $g(\xi)$ in the form

$$g(\xi) = g_{D}^{(d)}(\xi) - \frac{6}{\pi} g_{N}^{(d-2)}(\xi) + \frac{12}{\pi^2} g_{D}^{(d-4)}(\xi).$$

(31)

Here $g_{D}^{(d)}(\xi)$ is the scaling function of the $d$-dimensional Dirichlet propagator given in (20), while $g_{N}^{(d-2)}$ and $g_{D}^{(d-4)}$ are similar functions of the same argument, which would appear in the Neumann propagator in $d - 2$ and the Dirichlet propagator in $d - 4$ dimensions.

Multiplying the last function by $(4\pi z)_{-d}^\phi$ like before, we can express the final result as

$$G_{0}^{(d)}(r; z, z') = G_{D}^{(d)}(r; z, z') - \frac{3}{2\pi^2} G_{N}^{(d-2)}(r; z, z') + \frac{3}{(2\pi)^2} \frac{1}{(zz')^2} G_{D}^{(d-4)}(r; z, z').$$

(32)

As noted above, the first term is given by the familiar Dirichlet propagator in $d$ dimensional space (21). This function depends on two distances $R_-$ and $R_+$ in $d$ dimensions (23) whose powers as well as the overall geometric factor $C_d$ (22) are parametrized by $d$. The remaining "Neumann" and "Dirichlet" functions in the last two terms have analogous functional forms. They are parametrized by $d - 2$ and $d - 4$ in the same manner but still depend on the same distances $R_-$ and $R_+$ (23) defined in $d$ dimensional space, and hence are not the true correlation functions in $d' = d - 2$ or $d - 4$ dimensions. For all of them we may write generally

$$G_{D/N}^{(d')} (r; z, z') = G_{D}^{(d)} (r; |z - z'|) + G_{N}^{(d')} (r; z + z') = C_{d'} (R_{-}^{2-d'} - R_{+}^{2-d'}).$$

(33)

Let us note that different terms of functions defined by the last equation may be generated by differentiation. For example,

$$\frac{\partial G_{V}^{(d-2)} (r, |z|)}{\partial z^2} = -\pi G_{V}^{(d)} (r, |z|).$$

(34)

Once again we stress that the function (32) is not the true propagator of the free theory in $d < 4$ dimensions, since it has been obtained by a $d$-dimensional integral transformation of the four-dimensional function in the spirit of the dimensional regularization. However, it must reproduce the correct mean-field two-point function in $d = 4$.

To see this we consider the limit $\varepsilon \rightarrow 0$ in the above result. This yields

$$G_{0}^{(d=4)} (x, x') = \frac{1}{4\pi^2} \left[ \frac{1}{\xi} - \frac{1}{\xi + 1} + 12 + 6(1 + 2\xi) \ln \frac{\xi}{\xi + 1} \right].$$

(35)
The first term inside the square brackets in is related to the bulk propagator in four dimensions, the first two terms correspond to the Dirichlet propagator, and the last two contributions give the correction specific for the present case of the extraordinary transition. The logarithmic term arises in the limit \( \varepsilon \to 0 \) due to the simple poles in \( \varepsilon \) in the coefficients \( C_{d-2} \) and \( C_{d-4} \) in (30). Similar logarithms appeared also in [51] in the investigation of a correlation function in the non-linear \( O(N) \) model using the large-\( N \) expansion, and their origin was discussed using the OPE arguments. It seems that the mechanisms of the occurrence of logarithmic terms in the both cases as well as their interpretations from the point of view of short-distance expansions could be similar.

The result (35) is compatible with the analogous expression in [80, (4.96)](while the same propagator given in [102, (12)] is not correct). This can be seen, for example, by rewriting short-distance expansions could be similar.

Another observation suggested by the above calculation is that the two well known functions: \( W \) from (2) and \( K_{5/2} \) may be related through

\[
W(x) = \sqrt{\frac{2x}{\pi}} K_{\frac{5}{2}}(x). \tag{40}
\]

Obviously, the coordinate representation (32) is considerably simpler than (1) or (36) and can be much more advantageous for eventual higher-order calculations. In the next section we employ it in a one-loop calculation of the layer susceptibility.
An interesting question is also, whether it could be possible to obtain a similarly structured real-space representation for the ”massive” free propagator away from criticality. For \( T > T_c \) its \( p_z \) representation is given by \(^8\) \[ G_0(p; z < z'; m_0) = \frac{1}{2\kappa} \frac{p^2}{\kappa^2 - 4m_0^2} [W(-z, p, m_0) - W(z, p, m_0)] W(z', p, m_0) \] (41) with \( \kappa = \sqrt{p^2 + m_0^2} \) and \[ W(z, p, m_0) = e^{-\kappa z} \left( 1 + 3 \frac{\kappa m_0}{p^2} \coth (m_0 z) + 3 \frac{m_0^2}{p^2} \coth^2 (m_0 z) \right). \] (42) It is easy to see that in the limit of zero mass \( m_0 \) the last expressions reduce to (1) and (2).

3 Layer susceptibility

In this section we discuss an explicit calculation of the layer susceptibility \( \chi(z, z') \) at the extraordinary transition to the one-loop order.

3.1 Perturbation theory

The susceptibility of a layer confined between the planes \( z = z \) and \( z = z' \) (suppose \( z \leq z' \)) is defined as the integral of the (connected) pair correlation function \( G(r; z, z') \) with respect to the parallel coordinates \( r \), see \(^[105, 52, 51] \) and \(^[110, \text{Sec. III.B}] \). Obviously, it coincides with the \( p = 0 \) limit of the correlation function in its \( p_z \) representation:

\[ \chi(z, z') = \int d^{d-1}r G(r; z, z') = G(p = 0; z, z'). \] (43)

Thus, the first-order Feynman-diagram expansion for the layer susceptibility is given in the graphic form by \(^9\)

\[ \chi(z, z') = \begin{array}{c}
\text{(a)} \\
\text{(b)} \\
\text{(c)} \\
\text{(d)}
\end{array} + \begin{array}{c}
\text{(e)} \\
\text{(f)}
\end{array}. \] (44)

The first contribution \( (a) \) is the mean-field layer susceptibility

\[ \chi_a(z, z') = \chi_0(z, z') = G_0(p = 0; z, z') = \frac{1}{5} \left[ \frac{\min(z, z')}{|z - z'|} \right]^3 = \frac{1}{10} \frac{(z + z' - |z - z'|)^3}{(z + z' + |z - z'|)^2} = \frac{1}{5} \frac{z^3}{z^2} \text{ if } z < z'. \] (45)

The ”one-loop” terms are

\[ \chi_b(z, z') = -u_0 \int_0^\infty dy G_0(p = 0; z, y) G_0(p = 0; y, z') m_0(y) m_1(y), \] (46)

\(^8\)Formulas in Refs. \([107, 81]\) contain misprints.

\(^9\)For more detail on the loop expansion see \([52]\); for convenience, we shall follow here the notation of this reference very closely.
\[
\chi_c(z, z') = -\frac{u_0}{2} \int_0^\infty dy G_0(p = 0; z, y) G_0(p = 0; y, z') G_0(r = 0; y, y), \tag{47}
\]

and

\[
\chi_d(z, z') = \frac{u_0^2}{2} \int_0^\infty dy \int_0^\infty dy' G_0(p = 0; z, y)m_0(y) \left[ \int dr G_0^2(r; y, y') \right] m_0(y') G_0(p = 0; y', z'). \tag{48}
\]

A great simplifying feature here is that the propagators at the external lines are simply the corresponding mean-field layer susceptibilities (45). The vertices are associated with the bare coupling constant \(u_0\) of the underlying scalar \(\phi^4\) theory.

In the above formulae \(m_0(z)\) and \(m_1(z)\) are the zero- and one-loop contributions of the magnetization profile at the critical point:

\[
m(z) = m_0(z) + m_1(z) = \sqrt{\frac{12}{u_0}} \frac{1}{z} - \frac{\sqrt{3u_0}}{(1+\varepsilon)(4-\varepsilon)} z G_0(r=0; z, z). \tag{49}
\]

In the dimensional regularization which implies the prescription \(G_{\text{bulk}}(R = 0) = 0\) (see Eqs. (21) – (22)) the self-energy ”tadpole” \(G_0(r=0; z, z)\) is given by

\[
G_0(r=0; z, z) = \int \frac{d^{d-1}p}{(2\pi)^{d-1}} G_0(p; z, z) = \mathcal{P} z^{-2+\varepsilon}, \tag{50}
\]

with the constant

\[
\mathcal{P} = \mathcal{P}_D \frac{(4-\varepsilon)(6-\varepsilon)}{2+\varepsilon} \frac{1}{\varepsilon}. \tag{51}
\]

Here, the overall numerical factor

\[
\mathcal{P}_D = -C_d 2^{-2+\varepsilon} \quad \text{with} \quad C_d = \frac{S_d^{-1}}{d-2} \tag{52}
\]

just coincides with the analogous proportionality constant \(\mathcal{P}_D\) from the tadpole graph including the Dirichlet propagator (21):

\[
G_D(r=0; z, z) = \int \frac{d^{d-1}p}{(2\pi)^{d-1}} G_D(p; z, z) = \mathcal{P}_D z^{-2+\varepsilon}. \tag{53}
\]

An important difference between the one-loop tadpole graphs in the theory of the extraordinary and ordinary transitions is that \(\mathcal{P}\) contains a pole in \(\varepsilon\), while \(\mathcal{P}_D\) is of order \(O(\varepsilon^0)\). The factor \(\varepsilon^{-1}\) in the present case appears due to the presence of a non-zero order parameter profile at \(T = T_c\). This prevents one from calculating the individual first-order Feynman graphs at \(d = 4\) as it is usually possible in the less complicated theories. Here, one should do all the loop integrations in the dimensionally regularized form and perform the cancellations of such \(O(1/\varepsilon)\) terms between the different graphs in the whole contribution of \(O(u_0)\).

The one-loop graphs (b) and (c) contain actually just the same Feynman integrals owing to the definitions of \(m_0(z)\) and \(m_1(z)\) in (49). They are easy to compute, and their sum is explicitly given in [52, (4.6)]:

\[
\chi_b(z, z') + \chi_c(z, z') = \chi_0(z, z') \frac{u_0}{10} \mathcal{P} \left[ \frac{12}{(1+\varepsilon)(4-\varepsilon)} - 1 \right] z^\varepsilon \left( \frac{\varepsilon}{5+\varepsilon} + \frac{1-\varepsilon}{\varepsilon} + \frac{1}{5-\varepsilon} \right). \tag{54}
\]
Here again,
\[ \zeta = \frac{\min(z, z')}{\max(z, z')} = \frac{z + z' - |z - z'|}{z + z' + |z - z'|} \leq 1 \]  
(55)
and with the convention \( z \leq z' \), we have \( \zeta = z/z' \leq 1 \).

The calculation of the contribution of the graph (d) is much more involved, and it was performed in [52] only in the asymptotic regime \( \zeta \ll 1 \). Nevertheless, using our real-space propagator function (32) we were able to find the explicit expression for this contribution. The most complicated issue was the calculation of the inner loop of the graph (d) as it stands in the square brackets in (48). The resulting expression is given by a rather involved function written down in (A.3). The final result for the contribution of the graph (d) is given by
\[ \chi_d(z, z') = \chi_0(z, z') \cdot \frac{24}{5} u_0 (-\mathcal{P}_D) z^{\epsilon} \left[ \frac{\zeta^\epsilon K}{5 + \varepsilon} + \frac{L - \zeta^\epsilon K}{\varepsilon} + \frac{L}{5 - \varepsilon} + H_d(\zeta) \right] \]  
(56)
where
\[ H_d(\zeta) = f_0(\zeta) + \frac{25 f_1(\zeta)}{5 + \varepsilon} \left[ (1 - \zeta)^{3+\varepsilon} + (1 + \zeta)^{3+\varepsilon} \right] + \frac{25 f_2(\zeta)}{5 + \varepsilon} \left[ (1 - \zeta)^{3+\varepsilon} - (1 + \zeta)^{3+\varepsilon} \right] \]  
(57)
is the new function which was absent in the calculations of [52]. It is responsible for the dependence of \( \chi_d(z, z') \) (as well as of the entire layer susceptibility) on arbitrary values of \( \zeta \) in the whole interval \((0, 1)\). The somewhat bulky explicit expressions for the functions \( f_0(\zeta), f_1(\zeta) \) and \( f_2(\zeta) \) are written down in the Appendix along with some more calculational detail.

The function \( H_d(\zeta) \) is regular at \( \zeta \to 0 \) and behaves as \( \zeta^4 \) which was predicted in [52], namely
\[ H_d(\zeta) = \frac{1}{20736} (2 - \varepsilon)(6 - \varepsilon)(11 - \varepsilon) \zeta^4 + O(\zeta^6). \]  
(58)
It is also regular as \( \varepsilon \to 0 \) but has apparent singularities at \( \zeta = \pm 1 \). While \( \zeta = -1 \) is outside the definition range of \( \zeta \), the singularity at \( \zeta = +1 \) arises on the mutual approach of coordinates \( z \) and \( z' \). Thus, it is related to the short-distance behavior inside the bulk of the semi-infinite system which is described by the OPE. We shall return to this issue below.

Returning to (56) we acknowledge that the first three terms in square brackets have been present in [52, (4.10)]. They contain the leading contribution to \( \chi_d(z, z') \) in the limit \( \zeta \to 0 \) and are responsible for the corresponding singularity. The limiting behavior \( \zeta \to 0 \) is related to the short-distance behavior associated with the BOE, see [52]. The epsilon expansion
\[ \frac{\zeta^\epsilon K}{5 + \varepsilon} + \frac{L - \zeta^\epsilon K}{\varepsilon} + \frac{L}{5 - \varepsilon} = \frac{25}{288} \frac{5}{16} \ln \zeta + O(\varepsilon) \]  
(59)
shows up the logarithmic behavior. The explicit expressions for the constants \( K \) and \( L \) are
\[ K = \frac{(2 - \varepsilon)(72 - \varepsilon^2)}{6\varepsilon(2 + \varepsilon)(4 + \varepsilon)(6 + \varepsilon)}, \quad L = \frac{(2 - \varepsilon)(4 - \varepsilon)(6 - \varepsilon)}{48\varepsilon(2 + \varepsilon)}. \]  
(60)
The constants \( K \) and \( L \) have to be multiplied by the factor \( 2 \cdot 2^2 \Gamma(2 - \varepsilon) = 2 + O(\varepsilon) \) in order to match their counterparts from [52] (see Appendix). In this way \( L \) conforms with \( \mathcal{L} \) from [52, (4.11)]. The constant \( \mathcal{K} \) has been calculated in [52] only to order \( O(\varepsilon) \), and (60) exhibits its explicit form.
3.2 One-loop result and $\varepsilon$ expansion

For the complete expression for the layer susceptibility $\chi(z, z')$ which is the sum of contributions (45), (54) and (56) we obtain

$$\chi(z, z') = \frac{1}{5} z^3 + 6^{\frac{S_d-1}{d-2}} \frac{2^\varepsilon z^3}{z^2} \left[ \frac{K+S}{5+\varepsilon} \zeta^\varepsilon + \frac{(L+S) - \zeta^\varepsilon (K+S)}{\varepsilon} + \frac{L+S}{5-\varepsilon} + H_d(\zeta) \right]$$

(61)

where the constant $S$ stems from (54) and is given by (cf. [52, (4.8)])

$$S = -\frac{(4-\varepsilon)(6-\varepsilon)}{48\varepsilon(2+\varepsilon)} \left[ \frac{12}{(1+\varepsilon)(4-\varepsilon)} - 1 \right].$$

(62)

We recall that the function $H_d(\zeta)$ is defined in (58) while $f_0(\zeta), f_1(\zeta)$ and $f_2(\zeta)$ are listed in (A.5)-(A.7). Again, the first three terms in the square brackets of (61) agree with the results of [52, (4.6), (4.10)].

In order to perform the $\varepsilon$ expansion of $\chi(z, z')$ in (61) we use the standard coupling constant renormalization of the massless theory and the corresponding value of the one-loop fixed point. This implies the following chain of transformations:

$$u_0 \to \bar{u}_0 = u_0 \mu^{-\varepsilon} \to u = \bar{u}_0 K_d \to u^* = \frac{2}{3}\varepsilon.$$

(63)

Here $\mu$ is an arbitrary momentum scale and $K_d = (2\pi)^{-d} S_d$. The area of a unit $d$-dimensional sphere $S_d$ can be found in (7).

Now, performing the $\varepsilon$ expansion up to $O(\varepsilon)$, factoring out the overall constant amplitude and exponentiating the $\ln \zeta$ term from (59) we finally obtain the layer susceptibility $\chi(z, z')$ in the scaling form:

$$\chi(z, z') = \frac{1}{5} \left( 1 + \frac{5\varepsilon}{36} \right) \sqrt{zz'} \zeta^{\frac{2\varepsilon}{3}} \left[ 1 + \varepsilon g(\zeta) \right] + O(\varepsilon^2).$$

(64)

The function $g(\zeta)$ is given by

$$g(\zeta) = \frac{\zeta^{-6}}{1260} \left[ g_0(\zeta) - g_1(\zeta)(1+\zeta)^3 \ln(1+\zeta) - g_1(-\zeta)(1-\zeta)^3 \ln(1-\zeta) \right]$$

(65)

with

$$g_0(\zeta) = 2700\zeta^2 - 1030\zeta^4 + 3343\zeta^6,$$

$$g_1(\zeta) = 30(8 + 25\zeta - 36\zeta^2 + 25\zeta^3 + 8\zeta^4);$$

it is easy to see that $g(\zeta)$ is an even function of its argument. Stemming directly from $H_d(\zeta)$ in (58), the function $g(\zeta)$ starts with $O(\zeta^4)$ as $\zeta \to 0$,

$$g(\zeta) = \frac{11}{1080} s^4 + O(s^6)$$

(66)

as it should.
3.3 Scaling

The explicit expression (64) agrees with the generally expected scaling form

\[
\chi(z, z') = (zz')^{\frac{\eta + 1}{2}} \zeta^{\frac{\eta - 1}{2}} X(\zeta)
\]

which can be compared with [101, (8)], [110, Sec. III.B] and (9). The power of \(zz'\) in front of (67) comes directly in integrating the conformal invariant form of the two-point correlation function (3) over parallel directions according to (5). The power of \(\zeta\) is needed to correctly reproduce the singular behavior of \(\chi(z, z')\) as \(\zeta \to 0\) which is related to the critical exponent of parallel correlations at \(T_c, \eta_\parallel\). The remaining function \(X(\zeta)\) is regular in the limit \(\zeta \to 0\).

We recall finally that in (9) which is a direct consequence of the conformal invariance the argument \(\rho\) of the scaling function \(\hat{g}(\rho)\) is expressed in terms of the currently used variable \(\zeta\) via

\[
\rho = \frac{1}{4}(\zeta + \frac{1}{\zeta} - 2), \quad \text{and inversely,} \quad \zeta = \frac{\sqrt{\rho + 1} - \sqrt{\rho}}{\sqrt{\rho + 1} + \sqrt{\rho}}.
\]

Hence, \(\rho \to \infty\) as \(\zeta \to 0\) and \(\rho \to 0\) in the limit \(\zeta \to 1\).

Now we recall that the surface critical exponents of the extraordinary transition all are expressed in terms of the bulk exponents and the space dimension \(d\) [95, Sec. 5.3], [75, Sec. III.C.15]. The exact value of the correlation exponent \(\eta_\parallel\) is \(\eta_\parallel = d + 2\) or \(\eta_\parallel = 6 - \varepsilon\) if we parametrize \(d\) with \(\varepsilon = 4 - d\). Hence, for the power exponent \((\eta_\parallel - 1)/2\) we obtain \((5 - \varepsilon)/2\), precisely as in (64). This implies also that we can cast the layer susceptibility \(\chi(z, z')\) of (64) in the form

\[
\chi(z, z') = 5 \left(1 + \frac{5\varepsilon}{36}\right) (zz')^{\frac{\eta + 1}{2}} \zeta^{\frac{\eta - 1}{2}} \left[1 + \varepsilon g(\zeta)\right] + O(\varepsilon^2).
\]

where the exact correlation exponents \(\eta\) and \(\eta_\parallel\) appear. No further higher powers of \(\varepsilon\) should contribute to the factor \(\zeta^{(d+1)/2}\).

3.4 Singular behavior at \(\zeta \to 1\)

As promised before, now we turn to the discussion of the layer susceptibility in the alternative limit \(\zeta \to 1\). The associated singular behavior of \(\chi(z, z')\) is related to the term \(\sim \ln(1 - \zeta)\) in (65). When \(\zeta \to 1\) the coordinate \(z\) approaches \(z'\) and the thickness of the layer becomes small. Thus, the non-analyticity arises owing to a bulk short-distance singularity and associates, via OPE, with the singular behavior of the energy density. The limit \(\zeta \to 1\) was inaccessible in [52] where the calculations have been done only in the asymptotic regime \(\zeta \ll 1\).

In order to perform the explicit exponentiation of the \(\ln(1 - \zeta)\) term we express our scaling function in (64) in a form symmetric with respect to the interchange of coordinates \(z\) and \(z'\). For this purpose we write the variable \(\zeta\) as

\[
\zeta = \frac{z + z' - |z - z'|}{z + z' + |z - z'|} \equiv \frac{1 - x}{1 + x}
\]

where we define the variable \(x\) as

\[
x = \frac{|z - z'|}{z + z'}.
\]
Expanding $\chi(z, z')$ in powers of small $x$ and exponentiating the logarithmic term we obtain

$$\chi(z \leftrightarrow z') = \frac{z + z'}{10} \left[ a_0 - 5x + a_2x^2 + a_3x^{3-2}\varepsilon + O(x^4, \varepsilon^2) \right]$$

(72)

with the "amplitudes"

$$a_0 = 1 + \frac{\varepsilon}{7} \left( \frac{1297}{45} - 40 \ln 2 \right) + O(\varepsilon^2),$$

(73)

$$a_2 = 12 \left[ 1 + \frac{\varepsilon}{21} \left( \frac{3593}{60} - 86 \ln 2 \right) \right] + O(\varepsilon^2),$$

(74)

$$a_3 = -20 \left[ 1 - \frac{\varepsilon}{36} (29 + 24 \ln 2) \right] + O(\varepsilon^2).$$

(75)

The singular contribution $\sim |z - z'|^{3-2\varepsilon/3}$ can be understood from an OPE argument. When the distance $|x - x'|$ between the points $x$ and $x'$ goes to zero the OPE of the product $\phi(x)\phi(x')$ has a contribution proportional to the energy density operator $\epsilon(\bar{x}) = -\frac{1}{2} \phi^2(\bar{x}),$

$$C_{\phi\phi}(|x - x'|) \phi^2(\bar{x}).$$

(76)

The expansion point $\bar{x}$ is usually defined to be the midpoint $\bar{x} = (x + x')/2$. In the semi-infinite geometry of the type considered here the one-point function of the energy density has a non-vanishing profile at the transition point (for a very careful discussion of this and similar profiles see [111, 112, 109])

$$\langle \phi^2(\bar{x}) \rangle = \frac{A_{\phi^2}}{(2\varepsilon)^{2\Delta_{\phi^2}}}. $$

(77)

Owing to the translational invariance in directions parallel to the boundary only the dependence on the normal distance to the boundary appears. The scaling dimension $\Delta_{\phi^2}$ is related to the usual bulk critical exponents of the specific heat and correlation length via

$$\Delta_{\phi^2} = \frac{1 - \alpha}{\nu} = d - \frac{1}{\nu}. $$

(78)

In the special case of the one-component scalar field considered throughout the paper $\Delta_{\phi^2} = 2 - 2\varepsilon/3 + O(\varepsilon^2)$. It follows from the equations (76)–(77) that the coordinate dependence of the short-distance coefficient $C_{\phi\phi}(|x - x'|)\phi^2(\bar{x})$ is proportional to $|x - x'|^{-2\Delta_{\phi^2} + \Delta_{\phi^2}}$. This implies in turn that the scaling function $g(\xi)$ of the two-point function (3) has to have a contribution $\sim \xi^{-\Delta_{\phi^2} + \Delta_{\phi^2}/2}$ as $\xi \to 0$. Integrated over $r$ in $d - 1$ directions parallel to the surface along the lines of Sec. 2 this power leads to the singular contribution

$$\chi_{\text{sing}}(z \leftrightarrow z') \sim |z - z'|^{d - 2\Delta_{\phi^2} + \Delta_{\phi^2}} = |z - z'|^{1 - \eta + \Delta_{\phi^2}}$$

(79)

in the layer susceptibility. With $\Delta_{\phi^2} = 2 - 2\varepsilon/3 + O(\varepsilon^2)$, the exponent of the last power is just $3 - 2\varepsilon/3 + O(\varepsilon^2)$ in full agreement with (72).
3.5 A cool-down: Ordinary transition

At the ordinary transition the layer susceptibility $\chi(z, z')$ is given, within the same one-loop order, only by two graphs (a) and (c) from (44). As before, these are is easy to calculate, and the result is (again, with $z < z'$)

$$\chi^{ORD}(z, z') = z - \frac{u_0}{2} P_D z' z^e \left( \frac{\zeta^e}{1+\varepsilon} + \frac{1-\zeta^e}{\varepsilon} + \frac{1}{1-\varepsilon} \right).$$

(80)

Using the explicit expressions for $P_D$ from (52) and for the fixed point (63) we obtain the $\varepsilon$ expansion

$$\chi^{ORD}(z, z') = z \left[ 1 + \frac{\varepsilon}{6} (2 - \ln \zeta) \right] + O(\varepsilon^2) = \left( 1 + \frac{\varepsilon}{3} \right) \sqrt{zz'} \zeta^{1-\varepsilon/3} + O(\varepsilon^2).$$

(81)

We have put the last expression in the same scaling form as in (64). The power of $\zeta$ here agrees again with the expected one from (67). Indeed, in the present case $\eta_\parallel = 2 - \varepsilon/3 + O(\varepsilon^2)$ which follows, for example, from [75, (3.155a)] at $n = 1$. Hence $(\eta_\parallel - 1)/2 = (1-\varepsilon/3)/2 + O(\varepsilon^2)$ thus reproducing the exponent of $\zeta$ from (81). A similar calculation could be easily done for the special transition. In this case $\eta_\parallel^{sp} = -\varepsilon/3 + O(\varepsilon^2)$ [75, (3.156a)].

In (81), there is no correction of order $O(\varepsilon)$ to the pure power behavior in $\zeta$ as it was in the case of the extraordinary transition, see (64). The origin of that correction in (64) was the specific contribution of the graph (d) which appears due to the presence of a non-vanishing order-parameter profile at the extraordinary transition. Accordingly, the singular contribution to the layer susceptibility (64) as $\zeta \to 1$ could be interpreted as a consequence of the mutual approach of two $\phi^3$ vertices in the graph (d). In the case of the ordinary transition, an analogous correction to the pure power law behavior of (81) is expected to show up in order $O(\varepsilon^2)$ due to the two-loop sunset graph.

4 Discussion and outlook

In 1995, McAvity and Osborn applied the integral transformations [51, (4.18)-(4.19)] in their prominent investigation of ”Conformal field theories near a boundary in general dimensions” in the context of a large-$N$ expansion. They acknowledged an analogy to the Radon transform. In doing so they refered to the book [113]. Some other mathematical references on the subject can be mentioned [114, 115, 116]. The statement was that by integrating (3) over planes parallel to the boundary the transform function (11) is obtained, which can be subsequently inverted to reproduce (3). Apart of the application of this procedure in [51, Sec. 4] and its subsequent discussion in [100] we did not find any other source where it was applied.

In the present paper we have presented a successful application of the Radon transform as it was formulated in [51, Sec. 4]. This allowed to obtain the dimensionally continued version of the mean-field free propagator whose simple and symmetric form enabled us to perform further explicit calculations of the parallel susceptibility $\chi(z, z')$.

Actually, an inverse transform of $\chi(z, z')$ from the section 3.2 would produce the two-point correlation function at the extraordinary transition to $O(\varepsilon)$ which has never been calculated before. As a simple exercise, the inversion formula could be applied to $\chi^{ORD}(z, z')$ from the section 3.5 to reproduce the known $O(\varepsilon)$ expression at the ordinary transition [79, 50, 51, 94].
Very recently in this less involved case the two point function has been analytically obtained to \(O(\varepsilon^2)\) from the boundary conformal bootstrap [94]. There is no other result available for this function to the second order of the epsilon expansion. For sure, it would be very useful to derive it by other means and to perform the appropriate comparison. We believe that the application of the presented approach should be capable to reach such a result which would be useful both for the statistical mechanics and conformal field theory. On the other side, an explicit calculation of the \(O(\varepsilon)\) two-point function at the extraordinary transition would be a challenge for the boundary conformal bootstrap.

Finally, we stress once again that the parallel susceptibility (5) (cf. (43)) is a physical object which is much easier to handle (at least perturbatively) as the full two-point correlation function. At the same time its investigation gives access both to surface and bulk short-distance singularities controlled by the BOE and OPE. Hence a reasonable question arises, whether this function could be directly accessible in conformal bootstrap approach and eventually Monte Carlo simulations.

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Appendix

In this Appendix we give some technical detail of our calculation of the ”hard” graph (d) and list the explicit expressions for the functions \(f_0(\zeta), f_1(\zeta)\) and \(f_2(\zeta)\) appearing in the function \(H_d(\zeta)\) in (58).

For the explicit calculation of the contribution \(\chi_d(z, z')\) in (48) we need an exact expression for the inner loop of the graph (d),

\[
\mathcal{B}(z, y) = \int d^{d-1}r \ G_0^2(r; z, y) .
\]  

(A.1)

Substituting here the function \(G_0(r; z, y)\) from (32) and performing the integration over parallel coordinates in a straightforward fashion we obtain

\[
\mathcal{B}(z, y) = C_d \ 2^\varepsilon \ b(z, y)
\]  

(A.2)
that the function $g_L$ and $b$ form (A.3) for the function one. This was done in [52, (4.9)]. Using this single-integral representation and the explicit to the interchange of its arguments the double integration in (48) c an be reduced to a single to retrieve their counterparts from [52].

An important thing is that in view of the full symmetry of the function $b(z, y)$ with respect to the interchange of its arguments the double integration in (48) can be reduced to a single one. This was done in [52, (4.9)]. Using this single-integral representation and the explicit form (A.3) for the function $b$ we obtain the result (56)-(57) for the contribution (d). Note that the function $g(Z)$ in [52] is related to $b(y/z)\equiv b(1, y/z)$ through the normalization

$$g\left(\frac{y}{z}\right) = 2^{1+\varepsilon}\Gamma(2-\varepsilon)z^{1-\varepsilon}b(z, y|z<y) \equiv 2^{1+\varepsilon}\Gamma(2-\varepsilon)b\left(\frac{y}{z}\right) \quad \text{for} \quad \frac{y}{z} > 1.$$  

This means that all the integrals including the function $b$ and, in particular, the constants $K$ and $L$ (see (56) and below) have to be multiplied by the factor $2\cdot 2^{1-\varepsilon}\Gamma(2-\varepsilon) = 2 + O(\varepsilon)$ in order to retrieve their counterparts in [52].

The functions $f_i(\zeta)$ (with $\zeta = z/y'$, $\zeta \leq 1$) from (58) are

$$f_0(\zeta) = \frac{5(2-\varepsilon)(7-\varepsilon)}{48\varepsilon(1-\varepsilon^2)} - \frac{25(5-\varepsilon}\zeta^{-2}}{12(1-\varepsilon^2)(2 + \varepsilon)(3 + \varepsilon)} + \frac{75(3-\varepsilon)\zeta^{-4}}{2\varepsilon(1-\varepsilon^2)(3 + \varepsilon)(4 + \varepsilon)(5 + \varepsilon)}$$

$$+ \frac{300\zeta^{-6}}{\varepsilon(1+\varepsilon)(2 + \varepsilon)(3 + \varepsilon)(5 + \varepsilon)(6 + \varepsilon)(7 + \varepsilon)}, \quad \text{(A.5)}$$

$$f_1(\zeta) = -\frac{24(1-\varepsilon)(4+\varepsilon)(\zeta^{-2} + \zeta^{-6}) - (432 + 282\varepsilon + 67\varepsilon^2 + 2\varepsilon^3 + \varepsilon^4)\zeta^{-4}}{4\varepsilon(1-\varepsilon^2)(2 + \varepsilon)(3 + \varepsilon)(4 + \varepsilon)(6 + \varepsilon)(7 + \varepsilon)}$$

$$f_2(\zeta) = \frac{(50 + \varepsilon + 5\varepsilon^2)(\zeta^{-3} + \zeta^{-5})}{2\varepsilon(1-\varepsilon^2)(2 + \varepsilon)(4 + \varepsilon)(6 + \varepsilon)(7 + \varepsilon)}. \quad \text{(A.7)}$$

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