Projective normality of abelian surfaces of type 
\((1, 2d)\).

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Abstract: We show that an abelian surface embedded in \(\mathbb{P}^N\) by a very ample line bundle \(\mathcal{L}\) of type \((1, 2d)\) is projectively normal if and only if \(d \geq 4\). This completes the study of the projective normality of abelian surfaces embedded by complete linear systems.

Mathematics Subject Classifications (2000): Primary, 14K05; secondary, 14N05, 14E20.

Key Words: Abelian surfaces, quadrics, double coverings.

1 Introduction

Let \(A\) be an abelian surface. Let \(\mathcal{L}\) be an ample line bundle on \(A\) of type \((n_1, n_2)\). It induces a rational map \(\phi_\mathcal{L} : A \rightarrow \mathbb{P}^{n_1 n_2 - 1}\). We want to study when this map is a projectively normal embedding. The known results are the following:

1. If \(n_1 \geq 3\) then \(\phi_\mathcal{L}\) is a projectively normal embedding (see [9], [1]).

2. If \(n_1 = 2\) then \(\phi_\mathcal{L}\) is a projectively normal embedding if and only if no point of \(K(\mathcal{L})\) is a base point of \(\mathcal{L}'\), where \(\mathcal{L} = \mathcal{L}'^2\) (see [11], [1]).

3. If \(n_1 = 1\) then \(\mathcal{L}\) is a primitive bundle of type \((1, n_2)\). In this case:
   
   (a) If \(n_2 = 7, 9, 11\) or \(n_2 \geq 13\) then \(\phi_\mathcal{L}\) is a projectively normal embedding if and only if \(\mathcal{L}\) is very ample (see [10]).
   
   (b) If \(n_2 \geq 7\) and \(A\) is generic (in particular \(NS(A) \simeq \mathbb{Z}\)) then \(\phi_\mathcal{L}\) is a projectively normal embedding (see [7]).
   
   (c) If \(n_2 > 8\) and \(A\) is not isogenous to product of elliptic curves then \(\phi_\mathcal{L}\) is a projectively normal embedding (see [8]).

*Supported by EAGER.
Note that if $n_1n_2 < 7$ the embedding can never be projectively normal, because the dimension of $\text{Sym}^2 H^0(L)$ is less than that of $H^0(L^2)$. Thus the open cases are when the line bundle $L$ is of type $(1,8)$, $(1,10)$ or $(1,12)$. In this paper we study these cases. The main result is the following:

**Theorem 1.1** Let $A$ be an abelian surface and let $L$ be a line bundle on $A$ of type $(1,n)$. The induced map $\phi_L : A \to \mathbb{P}^{n-1}$ is a projectively normal embedding if and only if $L$ is very ample and $n \geq 7$.

It is well known that a necessary condition for $\phi_L$ to be a projectively normal embedding is the very ampleness of $L$. Furthermore, by Proposition 2.3, [8] it is sufficient to analyze the 2-normality of the map $\phi_L$, that is, study the quadrics containing the image of the abelian surface.

We will work with a very ample line bundle $L$ of type $(1,2d)$. Taking a nontrivial 2-torsion point $x \in K(L)$, we construct an involution on $A$ that extends to $\mathbb{P}^{2d-1}$. From this, there are two disjoint spaces of fixed points in $\mathbb{P}^{2d-1}$ of complementary dimension. Moreover there is an induced involution in the space of quadrics of $\mathbb{P}^{2d-1}$. This space decomposes into two subspaces of invariant quadrics. We call one of them the space of base quadrics: these are quadrics containing the two spaces of fixed points. The other one is the space of harmonic quadrics: they have the property that the spaces of fixed points are polar spaces with respect to them. We will see how this decomposition is related to the decomposition into Heisenberg modules.

On the other hand, we will consider the scroll $R$ obtained by joining with lines the points of $\phi_L(A)$ related by the involution. We will see that $R$ and $\phi_L(A)$ are contained in the same number of independent base quadrics. We will modify the arguments of R. Lazarsfeld in [10] to bound this number. This will allow us to solve the case $(1,10)$. Moreover, we will obtain that the abelian surface embedded by a very ample line bundle of type $(1,6)$ is never contained in quadrics.

Finally, we will work with the space of harmonic quadrics containing the abelian surface to complete the cases of the polarizations of type $(1,8)$ and $(1,12)$.

**Acknowledgement** I thank K. Hulek and H.-Ch. von Bothmer for their useful comments and suggestions. I am also grateful to the Institut für Mathematik of Hannover for its hospitality and to the project EAGER for financial support.
2 Preliminaries.

First, let us recall some general facts about involutions and double covers of smooth varieties:

**Definition 2.1** Let $X, Y$ be two smooth varieties. A map $\pi : X \rightarrow Y$ is called a double cover of $Y$ if it is a finite map of degree 2.

**Theorem 2.2** Let $\pi : X \rightarrow Y$ be a double cover. Then:

1. $\pi_*\mathcal{O}_X \simeq \mathcal{O}_Y \oplus \mathcal{M}$ where $\mathcal{M}$ is a line bundle on $Y$.
2. $\mathcal{M}^2 \simeq \mathcal{O}_Y(-B)$ where $B$ is the branch divisor of $\pi$.

**Proof:** See [12].

**Definition 2.3** We say that a line bundle $\mathcal{L}$ on $X$ is invariant by the double cover $\pi : X \rightarrow Y$ when $\mathcal{L} \simeq \pi^*\mathcal{L}$, where $\mathcal{L}$ is a line bundle on $Y$. Equivalently, $\mathcal{L}$ is invariant by $\pi$ if and only if $\mathcal{L} \simeq \mu^*(\mathcal{L})$, where $\mu$ is the involution of $X$ induced by $\pi$.

From now on we will work with a double cover $\pi : X \rightarrow Y$ and an invariant line bundle $\mathcal{L}$ on $X$. The map $\mu : X \rightarrow X$ will be the involution induced by $\pi$.

Let us consider the rank 2 vector bundle $E = \pi^*\mathcal{L}$ and let $S = \mathbb{P}(E)$ be the corresponding $\mathbb{P}^1$ bundle on $Y$. Moreover, take $E' = \pi^*(\mathcal{L}^2)$. From the definition of invariant line bundle, the projection formula and Theorem 2.2 it follows:

$E = \pi_*\mathcal{L} = \pi_*\pi^*\mathcal{L}_1 = \mathcal{L}_1 \oplus \mathcal{L}_2$

$E' = \pi_*\mathcal{L}^2 = \pi_*\pi^*(\mathcal{L}_1^2) = \mathcal{L}_1^2 \oplus (\mathcal{L}_1 \otimes \mathcal{L}_2)$

where $\mathcal{L}_2 = \mathcal{L}_1 \otimes \mathcal{M}$.

From this, we have canonical isomorphisms of vector spaces:

$H^0(X, \mathcal{L}) \simeq H^0(S, \mathcal{O}_S(1)) \simeq H^0(Y, \mathcal{E}) \simeq H^0(Y, \mathcal{L}_1) \oplus H^0(Y, \mathcal{L}_2)$

$H^0(X, \mathcal{L}^2) \simeq H^0(Y, \mathcal{E}') \simeq H^0(Y, \mathcal{L}_1^2) \oplus H^0(Y, \mathcal{L}_1 \otimes \mathcal{L}_2)$

$H^0(S, \mathcal{O}_S(2)) \simeq H^0(Y, \mathcal{S}^2\mathcal{E}) \simeq H^0(Y, \mathcal{L}_1^2) \oplus H^0(Y, \mathcal{L}_1 \otimes \mathcal{L}_2) \oplus H^0(Y, \mathcal{L}_2^2)$.

Note that $H^0(Y, \mathcal{L}_1)$ and $H^0(Y, \mathcal{L}_2)$ correspond to the $\pm 1$-eigenspaces of $H^0(X, \mathcal{L})$ with respect to the involution $\mu$.

In order to study the quadrics containing the projective images of $X$ and $S$, we consider the maps:

$\alpha : \text{Sym}^2(H^0(X, \mathcal{L})) \rightarrow H^0(X, \mathcal{L}^2)$

$\beta : \text{Sym}^2(H^0(S, \mathcal{O}_S(1))) \rightarrow H^0(S, \mathcal{O}_S(2))$. 

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Using the previous isomorphisms, we see that they decompose into the following way:

\[ \alpha \simeq \alpha_h \oplus \alpha_b \]
\[ \beta \simeq \beta_h \oplus \beta_b \]

with,

\[ \alpha_h : \text{Sym}^2(H^0(Y, L_1^2)) \oplus \text{Sym}^2(H^0(Y, \mathcal{L}_2^2)) \to H^0(Y, \mathcal{L}_1^2) \]
\[ \alpha_b : H^0(Y, L_1) \otimes H^0(Y, \mathcal{L}_2) \to H^0(Y, \mathcal{L}_1 \otimes \mathcal{L}_2) \]

and

\[ \beta_h : \text{Sym}^2(H^0(Y, \mathcal{L}_1^2)) \oplus \text{Sym}^2(H^0(Y, \mathcal{L}_2^2)) \to H^0(Y, \mathcal{L}_1^2) \oplus H^0(Y, \mathcal{L}_2^2) \]
\[ \beta_b : H^0(Y, L_1) \otimes H^0(Y, \mathcal{L}_2) \to H^0(Y, \mathcal{L}_1 \otimes \mathcal{L}_2) \]

Let us remark that \( \alpha_h \) and \( \beta_h \) correspond to the restriction of \( \alpha \) and \( \beta \) to the \( 1 \)-eigenspaces with respect to the involutions induced by \( \mu \) on \( \text{Sym}^2(H^0(Y, \mathcal{L}_1^2)) \) and \( \text{Sym}^2(H^0(S, \mathcal{O}_S(1))) \). On the other hand, \( \alpha_b \) and \( \beta_b \) correspond to the restriction to the \(-1\)-eigenspaces. To understand the map \( \alpha_h \), note that, by Theorem 2.2, there is a natural inclusion of linear systems

\[ |\mathcal{L}_2^2| \subset |\mathcal{L}_1^2| \]

given by the addition of the branch divisor \( B \).

There is a nice geometrical interpretation of these maps. Suppose that the line bundle \( \mathcal{L} \) is very ample. Then the linear system \( |\mathcal{O}_S(1)| \) separates the points related by the involution on each ruling of \( S \). Therefore, \( |\mathcal{O}_S(1)| \) is base point free linear system. Furthermore, the \( \mathbb{P}^1 \)-bundle \( S \) has two natural sections given by the epimorphisms \( \mathcal{L}_1 \oplus \mathcal{L}_2 \to \mathcal{L}_1 \) and \( \mathcal{L}_1 \oplus \mathcal{L}_2 \to \mathcal{L}_2 \). The linear system \( |\mathcal{O}_S(1)| \) restricted to them correspond to the linear systems \( |\mathcal{L}_1| \) and \( |\mathcal{L}_2| \) on \( Y \), so they are base point free too.

We have an embedding \( X \to \mathbb{P}^N \) where \( \mathbb{P}^N = \mathbb{P}(H^0(X, \mathcal{L})^*) \). We will identify \( X \) with its image. On the other hand, we have a map \( S \to \mathbb{P}(H^0(X, \mathcal{L})^*) \). This is not necessarily birational, but anyway, the image is a scroll. We will denote it by \( R \). The involution \( \mu \) on \( X \) extends to \( \mathbb{P}^N \). There are two subspaces of fixed points: \( \mathbb{P}(H^0(Y, \mathcal{L}_1^*) \) and \( \mathbb{P}(H^0(Y, \mathcal{L}_2^*) \).

Moreover, we have an induced involution in the space of quadrics of \( \mathbb{P}^N \). This space decomposes into two subspaces of invariant quadrics corresponding to the \( \pm1 \)-eigenspaces:

\[ H^0(\mathcal{O}_{\mathbb{P}^N}(2)) \simeq H^0(\mathcal{O}_{\mathbb{P}^N}(2))_h \oplus H^0(\mathcal{O}_{\mathbb{P}^N}(2))_b \]

where

\[ H^0(\mathcal{O}_{\mathbb{P}^N}(2))_h \simeq \text{Sym}^2(H^0(Y, \mathcal{L}_1)) \oplus \text{Sym}^2(H^0(Y, \mathcal{L}_2)) \]
\[ H^0(\mathcal{O}_{\mathbb{P}^N}(2))_b \simeq H^0(Y, \mathcal{L}_1) \otimes H^0(Y, \mathcal{L}_2). \]
**Definition 2.4** With the previous notation, we call $H^0(\mathcal{O}_{\mathbb{P}^N}(2))_h$ the space of harmonic quadrics of $\mathbb{P}^N$ and we call $H^0(\mathcal{O}_{\mathbb{P}^N}(2))_b$ the space of base quadrics of $\mathbb{P}^N$.

**Remark 2.5** The base quadrics correspond to quadrics containing the spaces of fixed points of $\mathbb{P}^N$; on the other hand, these spaces are polar spaces respect to the harmonic quadrics.

Now, the spaces $\ker(\alpha_b)$ and $\ker(\beta_b)$ corresponds to the spaces of base quadrics containing $X$ and $R$ respectively and the spaces $\ker(\alpha_h)$ and $\ker(\beta_h)$ corresponds to the spaces of harmonic quadrics containing $X$ and $R$ respectively.

Note, that $\ker(\alpha_b) \simeq \ker(\beta_b)$. This corresponds to the following geometrical fact: a quadric containing $X$ and a space of fixed points meets each generator of $R$ in three points, so it contains the scroll. Thus, the spaces of base quadrics containing $X$ and $R$ are the same.

We finish this section by proving an useful result to bound the number of independent quadrics containing a projective variety.

**Lemma 2.6** Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerated $d$-dimensional variety of degree $m$. Let $k$ be the number of independent quadrics containing $X$. Then:

1. If $m > 2(N - d)$ then $k \leq \binom{N-d+2}{2} - 2(N - d) - 1$
2. If $m \leq 2(N - d)$ then $k \leq \binom{N-d+2}{2} - m$.

**Proof:** Let $\mathbb{P}^{N-d}$ be a generic $(N - d)$-dimensional space in $\mathbb{P}^N$. It cuts $X$ in a set $M$ of $m$ points in general position. They are contained in at least the same number of independent quadrics than $X$. Moreover, it is well known that $2r + 1$ points in $\mathbb{P}^r$ in general position impose independent conditions to the quadrics. From this if $m > 2(N - d)$ we have $k \leq h^0(\mathcal{O}_{\mathbb{P}^{N-d}}(2)) - 2(N - d) - 1$ and if $m \leq 2(N - d)$ then $k \leq h^0(\mathcal{O}_{\mathbb{P}^{N-d}}(2)) - m$.

### 3 The Heisenberg decomposition of the space of quadrics.

Let $V$ be a $n$-dimensional vector space. Let us take coordinates $\{x_0, \ldots, x_{n-1}\}$ in $V$. Let us consider the Schrödinger representation of the Heisenberg group $H_n$. We take generators $\sigma, \tau \in GL(V)$, defined by:

\[
\sigma(x_j) = x_{j-1}, \\
\tau(x_j) = \varepsilon^j x_j, \text{ with } \varepsilon = e^{2\pi i/n}.
\]
The Heisenberg group acts on $\text{Sym}^2 V$. This corresponds to the space of quadrics of $\mathbf{P}(V)$. We will denote it by $W$. The dimension of $W$ is $n(n+1)/2$. We will describe explicitly the Heisenberg decomposition of $W$. We distinguish two cases depending on the parity of $n$:

1. Suppose that $n$ is odd.

In this case $W$ decomposes into $\frac{n+1}{2}$ $n$-dimensional irreducible $\mathbf{H}_n$-modules:

$$W \simeq \bigoplus W_l$$

where

$$W_l = \langle x_i x_j / i = j + l; i, j \in \mathbb{Z}_n \rangle$$

and $l = 0, \ldots, \frac{n+1}{2} - 1$. Because all of them are isomorphic as $\mathbf{H}_n$-modules, we will write $W_l \simeq V'$.

2. Suppose that $n$ is even. Let us take $n = 2d$.

By using the Schrödinger representation, we find the following decomposition of $W$ in four $\mathbf{H}_n$-modules, $W \simeq W^+_0 \oplus W^-_0 \oplus W^+_1 \oplus W^-_1$, with,

$$W^+_0 = \langle x_i x_j + x_{i+d} x_{j+d} / i + j \equiv 0; i, j \in \mathbb{Z}_{2d} \rangle$$
$$W^-_0 = \langle x_i x_j - x_{i+d} x_{j+d} / i + j \equiv 0; i, j \in \mathbb{Z}_{2d} \rangle$$
$$W^+_1 = \langle x_i x_j + x_{i+d} x_{j+d} / i + j \equiv 2; i, j \in \mathbb{Z}_{2d} \rangle$$
$$W^-_1 = \langle x_i x_j - x_{i+d} x_{j+d} / i + j \equiv 2; i, j \in \mathbb{Z}_{2d} \rangle$$

They are not irreducible, but each one of them decomposes into isomorphic $d$-dimensional irreducible $\mathbf{H}_n$-modules:

$$W^+_l \simeq \bigoplus W^\pm_{l,m}$$

where

$$W^\pm_{l,m} = \langle x_i x_j \pm x_{i+d} x_{j+d} / i + j \equiv l; i = j + m; i, j \in \mathbb{Z}_{2d} \rangle$$

and $l = 0, 1; m = 0, \ldots, d; l \equiv m$. If we fix $l$ and the sign $\pm$, the spaces $W^\pm_{l,m}$ are isomorphic as $\mathbf{H}_n$-modules. We will write $W^\pm_{l,m} \simeq V^\pm_l$.

We summarize these results in the following theorem.

**Theorem 3.1** Let $V$ be a vector space of dimension $n$. Let $\mathbf{H}_n$ be the $n$-dimensional Heisenberg group acting on $V$. The space $W = \text{Sym}^2 V$ has the following decomposition into $\mathbf{H}_n$-modules:

1. If $n$ is odd then

$$W \simeq \frac{n+1}{2} V'$$

where $V'$ is a $n$-dimensional irreducible $\mathbf{H}_n$-module.
2. If \( n = 2d \) is even then:

(a) If \( d \) is odd,
\[
W \simeq \frac{d+1}{2} V_0^+ \oplus \frac{d+1}{2} V_0^- \oplus \frac{d+1}{2} V_1^+ \oplus \frac{d-1}{2} V_1^-.
\]

(b) If \( d \) is even,
\[
W \simeq \frac{d+2}{2} V_0^+ \oplus \frac{d}{2} V_0^- \oplus \frac{d}{2} V_1^+ \oplus \frac{d}{2} V_1^-.
\]

The spaces \( V_\pm^k \) are \( d \)-dimensional \( H_n \)-modules and no two of them are isomorphic.

Remark 3.2 Let us suppose that \( n = 2d \). In this case \( \sigma^d \) and \( \tau^d \) are elements of order two in \( H_n \) and they define involutions \( \mu_\sigma, \mu_\tau \) in \( P(V) \). Moreover, \( \sigma^d \tau^d \) is not necessarily of order two, but \( (\sigma^d \tau^d)^2 = (-1)^d \text{Id}_V \) so it defines an involution \( \mu_{\sigma \tau} \) in \( P(V) \).

The space of quadrics \( W \) decomposes with respect to these involutions into two subspaces of invariant quadrics. With the previous notation we have:

1. With respect to the involution \( \mu_\sigma \), \( W = \langle W_0^+, W_1^+ \rangle \oplus \langle W_0^-, W_1^- \rangle \) of dimensions \( d(d+1) \) and \( d^2 \) respectively.

2. With respect to the involution \( \mu_\tau \), \( W = \langle W_0^+, W_0^- \rangle \oplus \langle W_1^+, W_1^- \rangle \) of dimensions \( d(d+1) \) and \( d^2 \) respectively.

3. With respect to the involution \( \mu_{\sigma \tau} \), \( W = \langle W_0^+, W_1^- \rangle \oplus \langle W_0^-, W_1^+ \rangle \) of dimensions \( d^2 \) and \( d(d+1) \) respectively if \( d \) is odd and \( d(d+1) \) and \( d^2 \) respectively if \( d \) is even.

Moreover, we know that the dimension of the spaces of harmonics quadrics and base quadrics is \( d(d+1) \) and \( d^2 \) respectively. If we compare these dimensions with the dimensions of the invariant spaces in each case we see the following: when \( d \) is odd the space of base quadrics must contain \( W_1^- \); when \( d \) is even the space of harmonic quadrics must contain \( W_0^+ \).

4 Quadrics containing abelian varieties of type \((1, \ldots, 1, 2d)\).

Let \( A \) be an abelian variety of dimension \( g \) and let \( L \) be an ample line bundle of type \((1, \ldots, 1, 2d)\). Let \( K(L) \) be the kernel of the isogeny \( \lambda_L : A \to \hat{A} \simeq \text{Pic}^0(A) \)
determined by the line bundle $L$. It is well known that $K(L) \simeq \mathbb{Z}_{2d} \times \mathbb{Z}_{2d}$. The natural extension of $K(L)$ by $\langle \varepsilon \rangle \subset \mathbb{C}^*$, with $\varepsilon = e^{2\pi i/(2d)}$, is the Heisenberg group $H_{2d}$ of dimension $2d$.

Suppose that $|L|$ is very ample. It induces an embedding $\phi_L : A \rightarrow \mathbb{P}^{2d-1} = \mathbb{P}(V)$, with $V = H^0(A, L)^*$. We will identify $A$ with its image. Let $I$ be the space of quadrics containing $A$. This is an $H_{2d}$-submodule of the space of quadrics of $\mathbb{P}^{2d-1}$. By Theorem 3.1, the space $I$ decomposes into the following $H_{2d}$-modules:

$$I = I^+_0 \oplus I^-_0 \oplus I^+_1 \oplus I^-_1$$

where $I^\pm_j \subset W^\pm_j$.

On the other hand, let $\{x_1, x_2, x_3\}$ be the three nontrivial 2-torsion points in $K(L)$. Let $x = x_i$ be one of them. We can consider the double cover $\pi : A \rightarrow \tilde{A} \simeq A/x$. $L$ is invariant by $\pi$. We are in the situation described in section 2. Now, $\tilde{A}$ is an abelian variety; $L = \pi^* L_1$, where $L_1$ is a line bundle in $\tilde{A}$ of type $(1, \ldots, 1, d)$; $L_2 = L_1 \otimes M$, where $M \in Pic^0(\tilde{A})$ verifies $M^2 \simeq O_{\tilde{A}}$. Joining by lines the points of $A$ related by the involution we obtain a scroll $R \subset \mathbb{P}^{2d-1}$. $R$ is the image of the $\mathbb{P}^1$ bundle $S = \mathbb{P}(L_1 \oplus L_2)$ by the tautological linear system $|O_S(1)|$ (see [3] for a detailed construction). Moreover, $I$ decomposes into a space $I_b$ of base quadrics and a space $I_h$ of harmonic quadrics:

$$I_b = ker(\alpha_b : H^0(L_1) \oplus H^0(L_2) \rightarrow H^0(L_1^2))$$

$$I_h = ker(\alpha_h : Sym^2(H^0(L_1)) \oplus Sym^2(H^0(L_2)) \rightarrow H^0(L_1 \otimes L_2)).$$

Note that this decomposition depends on the choice of the 2-torsion point $x \in K(L)$. In fact, the three involutions defined on $\mathbb{P}(V)$ by $x_1, x_2, x_3$ correspond to the three involutions described in Remark 3.2. In this way, we know how this decomposition is related to the Heisenberg decomposition:

**Proposition 4.1** The spaces $I_h$ and $I_b$ decompose into the sum of two $H_{2d}$-modules. In particular:

1. If $d$ is odd, $I_h = I^+_1 \oplus I^-_1$, $I_b = I^-_1 \oplus I^+_1$ or $I_h = I^+_1 \oplus I^-_0$ where each possibility occurs for exactly one of the 2-torsion points $x_1$, $x_2$ and $x_3$.

2. If $d$ is even, $I_h = I^+_0 \oplus I^-_1$, $I_b = I^+_0 \oplus I^-_0$ or $I_h = I^+_0 \oplus I^-_1$ where each possibility occurs for exactly one of the 2-torsion points $x_1$, $x_2$ and $x_3$. ■
5 Quadrics containing abelian surfaces of type 

\((1, 2d)\).

Let \(A\) be an abelian surface and let \(L\) be a very ample line bundle of type \((1, n)\). We want to study the projective normality of the embedding given by the line bundle \(L\). By Proposition 2.3, [8] it is sufficient to analyze the normality with respect to quadrics.

Note that, if \(n < 5\), \(L\) is never very ample. When \(n = 5, 6, L\) cannot be 2-normal because the dimension of \(\text{Sym}^2(H^0(L))\) is less than the dimension of \(H^0(L^2)\). However, one can ask if the map

\[
\text{Sym}^2(H^0(L)) \rightarrow H^0(L^2)
\]

has the expected behavior, that is, if it is injective. This is equivalent to study whether there are quadrics containing the abelian surface.

The case \(n = 5\) is immediate. \(A\) is embedded as a surface of degree 10 in \(\mathbb{P}^4\). By the Heisenberg decomposition of the space of quadrics, if the ideal of quadrics containing \(A\) is not empty, it has at least 5 independent quadrics, and this is not possible.

When \(n > 5\), we use the idea of R. Lazarsfeld in [10]. He applies Lemma 2.6 to the abelian surface and the fact that the space of quadrics decomposes into \(d\)-dimensional \(H_n\) irreducible modules, with \(d = n/m.c.d.(n, 2)\). This provides directly the projective normality when \(n = 7, 9, 11\) and \(n \geq 13\).

However, when \(n = 6, 8, 10, 12\) this argument does not imply projective normality, but it reduces the possibilities for the number of independent quadrics containing \(A\). If we call this number \(k\), we obtain:

| Polarization | \(k\) |
|--------------|------|
| (1, 6)       | 0, 3 |
| (1, 8)       | 4, 8 |
| (1, 10)      | 15, 20 |
| (1, 12)      | 30, 36 |

We will analyze these cases in detail. We work with very ample line bundles \(L\) of type \((1, n)\) with \(n = 2d\). We will use the construction developed in the previous sections.

Let us consider the involution given by a nontrivial 2-torsion point in \(K(L)\). The line bundles \(L_1\) and \(L_2\) on the abelian surface \(\tilde{A}\) are of type \((1, d)\). We have seen that they are base-point-free so they define regular maps \(\phi_1 : \tilde{A} \rightarrow \mathbb{P}^{d-1}\) and \(\phi_2 : \tilde{A} \rightarrow \mathbb{P}^{d-1}_2\). In particular, when \(d \geq 4\) they are birational embeddings or \(2:1\) maps onto elliptic scrolls (see [2]; [4]; [13]). When \(d = 3\), they are \(6:1\)
coverings of $\mathbf{P}^2$. Moreover, $\mathbf{P}_1^{d-1}$ and $\mathbf{P}_2^{d-1}$ are disjoint subspaces of $\mathbf{P}^{n-1}$. The scroll $R$ meets each one of these subspaces exactly in the image of the maps $\phi_1$ and $\phi_2$.

We will study the number of independent base quadrics and the number of independent harmonic quadrics containing the abelian surface. We will denote these numbers by $k_b$ and $k_h$ respectively. We will need the following result:

**Lemma 5.1** Let $C_2$ (resp $C_1$) be a generic curve in $|L_2|$ (resp. $|L_1|)$. If $d \geq 4$, then the restriction of $\phi_1$ (resp. $\phi_2$) to $C_2$ (resp. $C_1$) is a birational map.

**Proof:** If the map $\phi_1$ is birational the result is immediate, because $L_1, L_2$ differ by translation.

Suppose that $\phi_1$ (and then $\phi_2$) is not birational. Because $L_1$ and $L_2$ are base point free and $d \geq 4$, it is known that $\phi_1, \phi_2$ are $2:1$ maps onto elliptic scrolls (see [4]). In particular there are involutions $\sigma_1, \sigma_2$ of $A$ such that $\sigma_i^* L_i = L_i$ and $\phi_i = \phi_i \circ \sigma_i$.

Suppose that the restriction of $\phi_1$ to any divisor $C_2 \in |L_2|$ is not a birational map. This means that $\sigma_1 = \sigma_2$ and in particular, $\sigma_1^* L_2 = L_2$. But $L_2 = L_1 \otimes \mathcal{M}$, so it holds that $\sigma_1^* \mathcal{M} = \mathcal{M}$. The abelian surface $A$ is given locally by an equation $z_j^2 - 1 = 0$ in the line bundle $\mathcal{M} = \tilde{A}$ (see [12]). Since $\mathcal{M}$ is invariant by $\sigma_1$, this involution can be lifted to an involution $\sigma$ in $A$. From this:

$$\sigma^* L = \sigma^* \pi^* L_1 = \pi^* \sigma_1^* L_1 = \pi^* L_1 = L.$$ 

We see that $L$ is invariant by $\sigma$. Moreover, because $\phi_i = \phi_i \circ \sigma_i$ for $i = 1, 2$, the map $\phi$ defined by $L$ verifies $\phi = \phi \circ \sigma$. But this contradicts its very ampleness. $\blacksquare$

**Remark 5.2** When $d = 3$, the maps $\phi_1, \phi_2$ are $6:1$ coverings over $\mathbf{P}^2$. Since $L_2 \not\cong L_1$, the generic divisor of $|L_2|$ (resp. $|L_1|$) is mapped by $\phi_1$ (resp. $\phi_2$) to a nondegenerate curve of $\mathbf{P}^2$. $\blacksquare$

### 5.1 Abelian surfaces of type $(1,6)$ and $(1,10)$.

First, we will use the fact that there are the same number of independent base quadrics containing the abelian surface and the scroll $R$. Then we will use a particular version of Lemma 2.6 to bound this number.

Let us consider generic hyperplanes $H_1 \in \mathbf{P}_1^{d-1}$ and $H_2 \in \mathbf{P}_2^{d-1}$. Let $C_i = \phi_i^* H_i \in |L_i|$. Note that $C_1, C_2 = 2d$. Consider the space $H = \langle H_1, H_2 \rangle$. It is a $\mathbf{P}^{n-3}$ and it is invariant by the involution. The base quadrics of $\mathbf{P}^{n-1}$ intersected with $H$ are base quadrics of $H$. 10
Let $F$ be the intersection of $H$ with the scroll $R$. $F$ is contained in at least $k_b$ independent base quadrics. Because the lines of $R$ are lines joining the image of the points of $A$ by the maps $\phi_i$, we have:

$$\bigcup_{P \in C_1 \cap C_2} \langle \phi_1(P), \phi_2(P) \rangle \subset F.$$ 

The spaces $H_1$ and $H_2$ are generic, so when $d > 3$ we can apply Lemma 5.1. We see that $\phi_i(C_1 \cap C_2)$ are $2d$ points in general position. The base quadrics of rank 2 in $H$ are formed by two hyperplanes containing $H_1$ and $H_2$ respectively. It follows that $F$ is not contained in base quadrics of rank 2.

The space of base quadrics of $H$ has projective dimension $(d - 1)^2 - 1$ and the subvariety of base quadrics of rank 2 is of dimension $2(d - 2)$. Thus,

$$k_b \leq (d - 1)^2 - 1 - 2(d - 2) = d(d - 4) + 4,$$ when $d > 3$.

When $d = 3$ we know that $\phi_i(C_1 \cap C_2)$ are points spanning a line (see Remark 5.2). It follows that $F$ is contained in at most a finite number of base quadrics of rank 2. From this

$$k_b \leq 2$$ when $d = 3$.

On the other hand, $k_b = \dim(\ker(\alpha_b))$. Since $h^0(\mathcal{L}_1) = h^0(\mathcal{L}_2) = d$ and $h^0(\mathcal{L}_1 \otimes \mathcal{L}_2) = 4d$, it follows that

$$k_b \geq d^2 - 4d = d(d - 4),$$ when $d > 3$.

Finally, by Proposition 4.1 we know that $k_b$ must be multiple of $d$. Combining the previous bounds with this fact, we obtain the following possibilities for $k_b$:

| Polarization | $k_b$ |
|--------------|-------|
| $(1, 6)$     | 0     |
| $(1, 8)$     | 0, 4  |
| $(1, 10)$    | 5     |
| $(1, 12)$    | 12    |

We can repeat this argument with the three nontrivial torsion points of $K(\mathcal{L})$. Let $I_0^+ \oplus I_0^- \oplus I_1^+ \oplus I_1^-$ be the decomposition of the ideal of quadrics containing $A$ into $\mathbb{H}_{2d}$-modules. Suppose that $d$ is odd. By Proposition 4.1 we know:

$$k_b = \dim(I_1^-) + \dim(I_0^-) = \dim(I_1^+) + \dim(I_0^+) = \dim(I_1^-) + \dim(I_0^+) = \dim(I_1^-) + \dim(I_0^+).$$

Using this fact and the previous bounds for $k$ and $k_b$, one can compute directly the number of independent quadrics containing the abelian surface $A$ of type $(1, 6)$ and $(1, 10)$:
Theorem 5.3 Let $A$ be an abelian surface embedded in $\mathbb{P}^{n-1}$ by a very ample line bundle $\mathcal{L}$ of type $(1,n)$. Then,

1. If $n = 6$, $A$ is not contained in quadrics.
2. If $n = 10$, $A$ is contained in $15$ independent quadrics, so it is projectively normal.

5.2 Abelian surfaces of type $(1,8)$ and $(1,12)$.

Now we will compute the number of independent harmonic quadrics containing the abelian surface $A$. This is equivalent to study the kernel of the map:

$$\alpha_h : \text{Sym}^2(H^0(\mathcal{L}_1)) \oplus \text{Sym}^2(H^0(\mathcal{L}_2)) \rightarrow H^0(\mathcal{L}_1^2).$$

We also consider the maps:

$$\alpha_1 : \text{Sym}^2(H^0(\mathcal{L}_1)) \rightarrow \text{Sym}^2(H^0(\mathcal{L}_1)) \oplus \text{Sym}^2(H^0(\mathcal{L}_2)) \rightarrow H^0(\mathcal{L}_1^2),$$

$$\alpha_2 : \text{Sym}^2(H^0(\mathcal{L}_2)) \rightarrow \text{Sym}^2(H^0(\mathcal{L}_1)) \oplus \text{Sym}^2(H^0(\mathcal{L}_2)) \rightarrow H^0(\mathcal{L}_1^2).$$

We call the images of these maps $\text{Im}_1, \text{Im}_2$ respectively.

First we study the polarization of type $(1,8)$. In this case, the abelian surface $A$ is embedded in $\mathbb{P}^7$, $h^0(\mathcal{L}^2) = 32$ and $h^0(O_{\mathbb{P}^7}(2)) = 36$. The possibilities for $k$ are $k = 4$ or $k = 8$. The line bundle is projectively normal when $k = 4$. The images of $\bar{A}$ by the maps defined by the line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ are singular octics in $\mathbb{P}^3$ or elliptic quartic scrolls (see [2]). They are not contained in quadrics, so the maps $\alpha_1$ and $\alpha_2$ are injective. One immediately sees that $\dim(\text{Im}_1) = \dim(\text{Im}_2) = 10$ and $h^0(\mathcal{L}_1^2) = 16$. The image of $\alpha_h$ has at least dimension 10.

We want to show that $\alpha_h$ is a surjection. Suppose that this is not true. Then $\dim(\ker(\alpha_h)) > 4$. Moreover, since $\ker(\alpha_h)$ is an $\mathbb{H}_8$-module, its dimension must be exactly 8 and the image of $\alpha_h$ is a 12-dimensional space. This means that the subspaces $\text{Im}_1$ and $\text{Im}_2$ intersect in an 8-dimensional space.

Let us fix a generic divisor $C_2$ in $|\mathcal{L}_2|$. By Lemma 5.1, $\phi_1(C_2)$ is a non-degenerate curve of degree 8 in $\mathbb{P}^3$ so it lies at most in a quadric. On the other hand, $\phi_2(C_2)$ lies in a plane, so it is contained in at least 4 independent quadrics of rank 2 in $\mathbb{P}^3$. Thus, there is a 4-dimensional subspace $U$ of $\text{Im}_2$ corresponding to quadrics containing $\phi_2(C_2)$. If $\dim(\text{Im}_1 \cap \text{Im}_2) = 8$, then $\dim(U \cap \text{Im}_1) \geq 2$. Thus, there at least two independent quadrics containing $\phi_1(C_2)$, but this is not possible.

We have checked that the number of harmonic quadrics containing the abelian surface is exactly 4. We can repeat this argument for the three 2-torsion points in $K(\mathcal{L})$. Applying Proposition 4.1, we see that:

$$4 = k_h = \dim(I_0^+) + \dim(I_1^-) = \dim(I_0^+) + \dim(I_1^-) = \dim(I_0^+) + \dim(I_1^-).$$
It follows that $A$ must be contained in exactly 4 independent quadrics and we obtain the following result:

**Theorem 5.4** An abelian surface embedded in $\mathbb{P}^7$ by a very ample line bundle of type $(1, 8)$ is projectively normal.

Let us study the polarization of type $(1, 12)$. Now, the abelian surface $A$ is embedded in $\mathbb{P}^{11}$, $h^0(\mathcal{L}^2) = 48$ and $h^0(\mathcal{O}_{\mathbb{P}^{11}}(2)) = 78$. The possibilities for $k$ are $k = 30$ or $k = 36$. The line bundle is projectively normal when $k = 30$.

The images of $\bar{A}$ by the maps defined by the line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ are abelian surfaces in $\mathbb{P}^5$ or elliptic normal scrolls of degree 6 (see [4]).

In the first case, we have seen that they are not contained in quadrics (Theorem 5.3). Thus, $\dim(\text{Im} \mathcal{L}_1) = \dim(\text{Im} \mathcal{L}_2) = 21$ and $h^0(\mathcal{L}_1^2) = 24$. It follows that the kernel of $\alpha_h$ has at most dimension 21. But $\ker(\alpha_h)$ is an $\mathbb{H}_{12}$-module, so its dimension is multiple of 6. We deduce that there are exactly 18 independent harmonic quadrics containing the abelian surface $A$.

In the second case, it is known that the normal elliptic scrolls in $\mathbb{P}^5$ are contained in 3 independent quadrics (see [5], [6]). From this, $\dim(\text{Im} \mathcal{L}_1) = \dim(\text{Im} \mathcal{L}_2) = 18$ and $h^0(\mathcal{L}_1^2) = 24$. The kernel of $\alpha_h$ must be of dimension 18 or 24. Suppose that it is 24. Then $\text{Im} \mathcal{L}_1 = \text{Im} \mathcal{L}_2$ in $H^0(\mathcal{L}_1^2)$. Let us take two generic curves $C, C' \in |\mathcal{L}_2|$. The images $\phi_2(C), \phi_2(C')$ are contained in a degenerate quadric. Because $\text{Im} \mathcal{L}_1 = \text{Im} \mathcal{L}_2$, the curves $\phi_1(C), \phi_1(C')$ must be contained in a quadric that does not contain $\phi_1(A)$. But $\phi_1(A)$ is an elliptic surface of degree 6 and by lemma 5.1, $\phi_1(C), \phi_1(C')$ are non-degenerate curves of degree 12, so this is not possible. We conclude that the number of independent harmonic quadrics containing the abelian surface $A$ is again 18.

Finally, using the number of base quadrics computed in the previous section we see that $A$ is contained exactly in 30 independent quadrics and we obtain the following result.

**Theorem 5.5** An abelian surface embedded in $\mathbb{P}^{11}$ by a very ample line bundle of type $(1, 12)$ is projectively normal.

**References**

[1] Birkenhake, Ch.; Lange, H. *Complex abelian varieties*. Springer-Verlag (1992).

[2] Birkenhake, Ch.; Lange, H; van Straten, D.; *Abelian surfaces of type $(1, 4)$*. Math. Ann. 285, 625-646 (1989).
[3] Ciliberto, C.; Hulek, K. A bound on the irregularity of abelian scrolls in projective space. Bauer, Ingrid (ed.) et al., Complex geometry. Collection of papers dedicated to Hans Grauert on the occasion of his 70th birthday. Berlin: Springer. 85-92 (2002).

[4] Hulek, K.; Lange, H. Examples of abelian surfaces in $\mathbb{P}^4$. J. Reine Angew. Math. 363, 200-216 (1985).

[5] Homma, Y. Projective normality and the defining equations of ample invertible sheaves on elliptic ruled surfaces with $e \geq 0$. Nat. Sci. Rep. Ochanomizu Univ. 31, 61-73 (1980).

[6] Homma, Y. Projective normality and the defining equations of an elliptic ruled surface with negative invariant. Nat. Sci. Rep. Ochanomizu Univ. 33, 17-26 (1982).

[7] Iyer, J.N. Projective normality of abelian surfaces given by primitive bundles. Manuscr. Math. 98, No.2, 139-153 (1999).

[8] Iyer, J.N. Projective normality of abelian varieties. Trans. Am. Math. Soc. 355, No.8, 3209-3216 (2003).

[9] Koizumi, S. Theta relations and projective normality of Abelian varieties. Am. J. Math. 98, 865-889 (1976).

[10] Lazarsfeld, R. Projectivite normale des surface abeliennes. Redige par O. Debarre. Prepublication No. 14, Europroj- C.I.M.P.A.,Nice (1990).

[11] Ohbuchi, A. A note on the projective normality of special line bundles on abelian varieties. Tsukuba J. Math. 12, No.2, 341-352 (1988).

[12] Persson, U.; Double covers and surfaces of general type. Algebraic geometry proceedings, Tromsø, Norway 1977. Springer Lecture notes in mathematics, 687 (1978).

[13] Ramanan, S. Ample divisors on abelian surfaces. Proc. Lond. Math. Soc. 51, 231-245 (1985).