Determination of the Metric from the Connection

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Abstract

As is well known, a metric on a manifold determines a unique symmetric connection for which the metric is parallel: the Levi-Civita connection. In this paper we investigate the inverse problem: to what extent is the metric of a Riemannian manifold determined by its Levi-Civita connection? It is shown that for a generic Levi-Civita connection of a metric $h$ there exists a set of positive semi-definite tensor fields $h_a$ such that the parallel metrics are the positive-definite linear combinations of the $h_a$. Moreover, the set of all parallel metrics may be constructed by a solely algebraic procedure.
1 Introduction

One of the fundamental principles of differential geometry is that a Riemannian manifold \((M, h)\) uniquely determines a connection: the Levi-Civita connection of the metric. This paper examines how much information on the metric \(h\) is retained by the associated Levi-Civita connection. Specifically, we inquire to what extent the metric of a Riemannian manifold is determined by the Levi-Civita connection and seek a method for constructing the general form of a parallel metric on the manifold.

We begin, in the following section, by introducing a genericity condition on the set of connections, defined in terms of the Riemann curvature tensor. This shall enable us to investigate the structure of the space of local parallel metrics; that is, metrics defined on an open set of the manifold, which are parallel with respect to the given Levi-Civita connection \(\nabla\). This, in turn, shall lead to a decomposition of the tangent space of the manifold into a direct sum of orthogonal subbundles. The original metric \(h\), restricted to the subbundles of the decomposition, defines a set of positive semi-definite tensor fields \(h_a\). It is proved that a metric is parallel with respect to \(\nabla\) if and only if it is a positive-definite linear combination of the \(h_a\). Furthermore, the complete set of parallel metrics may be obtained by a purely algebraic procedure; integration of differential equations is not required. Lastly, we provide an example that illustrates the method described herein.

In [1], it is determined when an arbitrary analytic symmetric connection is a Levi-Civita connection for some metric. The problem of finding the metric from the Ricci curvature has been studied by DeTurk [2], [3], [4] and [5], and from the curvature in general relativity by Hall [6] and Hall and McIntosh [7].
2 Parallel metrics for generic Levi-Civita connections

In what follows, $M$ shall denote a connected manifold of dimension $n$. This is not essential but it shall lead to concepts that are more familiar. Moreover, the scope of the paper is not thereby limited since the results below may be applied to the separate components of a general manifold.

It will also be convenient to represent a metric in terms of contravariant indices:

$$<, > = g = \sum_{i,j=1}^{n} g^{ij} X_i \otimes X_j$$

where $(X_1, ..., X_n)$ is a local frame.

This shall be the convention throughout.

Define a connection $\nabla$ on $M$ to be generic at a point $m \in M$ if there exist tangent vectors $\xi_1, \xi_2 \in T_mM$ such that the linear transformation $R(\xi_1, \xi_2) : T_mM \rightarrow T_mM$ has $n$ distinct (complex) eigenvalues, where $R$ denotes the Riemann curvature tensor of $\nabla$. A connection is generic if it is generic at every point $m \in M$. Henceforth $\nabla$ shall denote a generic symmetric connection on $M$ with parallel metric $h$.

Consider a connected open set $U$ of $M$ for which there exist vector fields $\xi_1, \xi_2$ on $U$ such that

$$R(\xi_1(u), \xi_2(u)) : T_uM \rightarrow T_uM$$

has $n$ distinct eigenvalues $\lambda_i = \lambda_i(u)$, for each $u \in U$, and an associated frame of (complex) eigenvector fields $(Z_1, ..., Z_n)$ defined on $U$:

$$R(\xi_1, \xi_2)(Z_i) = \lambda_i Z_i$$

Let $\mathcal{U}$ denote the set of all such connected open sets of $M$. Since $\nabla$ is generic, $\mathcal{U}$ is an open covering of $M$. If $V \in \mathcal{U}$ then any connected open subset $W \subseteq V$ is also in $\mathcal{U}$. 
Lemma 1

(i) The frame of eigenvector fields \((Z_1, \ldots, Z_n)\) of \(R(\xi_1, \xi_2)\), after a possible reordering, has the form

\[X_1 + iX_2, X_1 - iX_2, \ldots, X_{2m - 1} + iX_{2m}, X_{2m - 1} - iX_{2m}, (X_n)\]

where the \(X_i\) are vector fields on \(U\) and \(X_n\), the eigenvector field corresponding to the zero eigenvalue, is included if \(n\) is odd. Furthermore,

(ii) \(X_1, \ldots, X_{2m}, (X_n)\) is an orthogonal frame on \(U\) for any metric \(g\) on \(U\), parallel with respect to \(\nabla\). That is, \(g\) is expressible in the form \(g = \sum_{i=1}^{n} g_i X_i \otimes X_i\) for some functions \(g_i: U \to \mathbb{R}^+\).

Proof:

(i) Since \(\nabla\) is the Levi-Civita connection of the metric \(h\), there exists an orthonormal basis of \(T_u M\) at each \(u \in U\), with respect to which \(R(\xi_1, \xi_2)\) is represented as a skew-symmetric matrix. Hence the eigenvalues \(\lambda_i = \lambda_i(u)\) are purely imaginary with associated eigenvector fields \(Z_{2k - 1} = X_{2k - 1} + iX_{2k}\) for \(\lambda_{2k - 1}\) and \(Z_{2k} = X_{2k - 1} - iX_{2k}\) for \(\lambda_{2k} = -\lambda_{2k - 1}\), \(1 \leq k \leq m\), except for \(\lambda_n = 0\) when \(n\) is odd, with associated eigenvector field \(Z_n = X_n\).

(ii) Let \(g = \sum_{i,j=1}^{n} g^{ij} Z_i \otimes Z_j\) be a metric parallel on \(U\); thus \(R(\xi_1, \xi_2)(g) = 0\). The explicit form of \(g\) gives

\[
R(\xi_1, \xi_2)(g) = [\nabla_{\xi_1}, \nabla_{\xi_2}](g) - \nabla_{[\xi_1, \xi_2]}(g) \\
= \sum_{i,j=1}^{n} g^{ij} (R(\xi_1, \xi_2)((Z_i) \otimes Z_j + Z_i \otimes R(\xi_1, \xi_2)(Z_j))) \\
= \sum_{i,j=1}^{n} g^{ij} (\lambda_i Z_i \otimes Z_j + Z_i \otimes \lambda_j Z_j) \\
= \sum_{i,j=1}^{n} g^{ij} (\lambda_i + \lambda_j) Z_i \otimes Z_j
\]

Therefore \(g^{ij}(\lambda_i + \lambda_j) = 0\) for all \(1 \leq i, j \leq n\). The eigenvalues \(\lambda_i\) are distinct, by hypothesis, and \(\lambda_{2k} = -\lambda_{2k - 1}\), \(1 \leq k \leq m\), except for \(\lambda_n = 0\) when \(n\) is
odd. Hence \( g^{ij} = 0 \), unless \((i, j) = (2k - 1, 2k)\) or \((i, j) = (2k, 2k - 1)\) for some \( k \in \{1, \ldots, m\} \), or \( i = j = n \) when \( n \) is odd. It follows that \( g^{ij} \) is block diagonal with \( 2 \times 2 \) blocks down the main diagonal and with a single \( 1 \times 1 \) block for odd \( n \). In the \((X_1, \ldots, X_n)\) frame the \( k^{th} \) \( 2 \times 2 \) block has the form

\[
\sum_{i,j=2k-1}^{2k} g^{ij} Z_i \otimes Z_j = g^{2k-1,2k} Z_{2k-1} \otimes Z_{2k} + g^{2k,2k-1} Z_{2k} \otimes Z_{2k-1}
\]

\[
= g^{2k-1,2k} (X_{2k-1} + iX_{2k}) \otimes (X_{2k-1} - iX_{2k}) +
\]

\[
= g^{2k,2k-1} (X_{2k-1} - iX_{2k}) \otimes (X_{2k-1} + iX_{2k})
\]

\[
= g_{2k-1} X_{2k-1} \otimes X_{2k-1} + g_{2k} X_{2k} \otimes X_{2k}
\]

where \( g_{2k-1} = g_{2k} := g^{2k-1,2k} + g^{2k,2k-1} \). Defining \( g_n := g^{nn} \) for odd \( n \),

\[
g = \sum_{i=1}^{n} g_i X_i \otimes X_i
\]

q.e.d.

Let \( \theta = \theta_j^i \) denote the connection form of \( \nabla \) in the \((X_1, \ldots, X_n)\) frame:

\[
\nabla X_j = \sum_{i=1}^{n} X_i \otimes \theta_j^i
\]

The following lemma indicates the amount of variation allowed among parallel metrics.

**Lemma 2** Let \(<,>^1 = \sum_{i=1}^{n} g_i X_i \otimes X_i \) and \(<,>^2 = \sum_{i=1}^{n} k_i X_i \otimes X_i \) be two arbitrary metrics on \( U \), parallel with respect to \( \nabla \). Then there exist constants \( c_i \in \mathbb{R}^+ \) such that \( g_i = c_i k_i \), for all \( 1 \leq i \leq n \).

**Proof:**

Since \(<,>^1 \) is parallel,

\[
0 = \nabla (\sum_{i=1}^{n} g_i X_i \otimes X_i)
\]

\[
= \sum_{i,j=1}^{n} X_i \otimes X_j \otimes (\delta_{ij} dg_i + g_j \theta_j^i + g_i \theta_j^i)
\]

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When $i = j$ this gives, $\theta_i^i = -\frac{1}{2}d \log g_i$. Similarly, $\theta_i^i = -\frac{1}{2}d \log k_i$. Therefore
\[d \log g_i = d \log k_i\]
and since $U$ is connected, $g_i = c_i k_i$ for some $c_i \in \mathbb{R}^+$. q.e.d.

Next, we shall seek to make this observation stronger. In light of Lemma 1, the metric $h$ can be written locally as
\[h|_U = \sum_{i=1}^{n} \rho_i X_i \otimes X_i\]
for some functions $\rho_i : U \to \mathbb{R}^+$. Define the orthonormal basis of local vector fields $(Y_1, ..., Y_n)$ by $Y_i := \sqrt{\rho_i} X_i$, for $1 \leq i \leq n$. Then $h|_U = Y_1 \otimes Y_1 + \cdots + Y_n \otimes Y_n$. Let $\omega = \omega_j^i$ denote the connection form of $\nabla$ with respect to the frame $(Y_1, ..., Y_n)$:
\[\nabla Y_j = \sum_{i=1}^{n} Y_i \otimes \omega_j^i\]

**Lemma 3** Consider $g = \sum_{i=1}^{n} c_i Y_i \otimes Y_i$, where $c_i \in \mathbb{R}$. $\nabla g = 0$ if and only if $(c_i - c_j) \omega_j^i(u) = 0$ for all $u \in U$ and $1 \leq i, j \leq n$.

**Proof:**

Taking the covariant derivative of $g$ gives,
\[\nabla g = \nabla \left( \sum_{i=1}^{n} c_i Y_i \otimes Y_i \right) = \sum_{i,j=1}^{n} Y_i \otimes Y_j \otimes (c_j \omega_j^i + c_i \omega_j^i)\]

Therefore $\nabla g = 0$ if and only if $c_j \omega_j^i + c_i \omega_i^j = 0$ for all $1 \leq i, j \leq n$. Since $\omega_j^i = -\omega_i^j$, this holds if and only if $(c_i - c_j) \omega_j^i = 0$ for all $1 \leq i, j \leq n$. q.e.d.
Define $r(U)$ to be the equivalence relation on $\{1, \ldots, n\}$ generated by the relations \{$(i, j) \mid \omega_i^j(u) \neq 0$ for some $u \in U$\}. Thus $ir(U)j$ for $i \neq j$ if and only if there exists a sequence $i = i_1, \ldots, i_{k-1} = j$ in $\{1, \ldots, n\}$ and $u_{i_1}, \ldots, u_{i_{k-1}} \in U$ such that $\omega_{i_{l+1}}^{i_l}(u_{i_l}) \neq 0$ for all $1 \leq l \leq k-1$. $r(U)$ partitions $\{1, \ldots, n\}$ into $\beta(U)$ disjoint subsets $P_1(U), \ldots, P_{\beta(U)}(U)$.

Define tensor fields $h_i(U)$ on $U$ by

$$h_i(U) := \sum_{j \in P_i(U)} Y_j \otimes Y_j,$$

for $1 \leq i \leq \beta(U)$. Note that

$$h|U = \sum_{i=1}^{\beta(U)} h_i(U)$$

**Lemma 4** If

$$g = \sum_{i=1}^{\beta(U)} a_i h_i(U)$$

for constants $a_i \in \mathbb{R}$ then $g$ is parallel with respect to $\nabla$.

**Proof:**

Suppose that $g = \sum_{i=1}^{n} c_i Y_i \otimes Y_i = \sum_{k=1}^{\beta(U)} a_k h_k(U)$ for constants $c_i, a_k \in \mathbb{R}$. If $ir(U)j$ then $i$ and $j$ belong to the same equivalence class $P_k(U)$, say. Hence $c_i = c_j = a_k$. On the other hand, if $i$ and $j$ are not $r(U)$-related then $\omega_i^j(u) = 0$ for all $u \in U$. In either case, $(c_i - c_j) \omega_i^j(u) = 0$ for all $u \in U$. Therefore by Lemma 3, $g$ is parallel with respect to $\nabla$.

q.e.d.

**Lemma 5** If $g$ is a metric on $U$ parallel with respect to $\nabla$ then

$$g = \sum_{i=1}^{\beta(U)} a_i h_i(U)$$

for some constants $a_i \in \mathbb{R}^+$.  

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Proof:
Let $g$ be a metric on $U$ parallel with respect to $\nabla$. By Lemma 1, $g$ may be written $g = \sum_{i=1}^{n} g_i X_i \otimes X_i$ for some functions $g_i : U \to \mathbb{R}^+$. Then Lemma 2 and the fact that $h|_U = \sum_{i=1}^{n} \rho_i X_i \otimes X_i$ imply that $g$ is of the form

$$g = \sum_{i=1}^{n} c_i \rho_i X_i \otimes X_i = \sum_{i=1}^{n} c_i Y_i \otimes Y_i$$

for some constants $c_i \in \mathbb{R}^+$. By Lemma 3, $(c_i - c_j)\omega^i_j(u) = 0$ for all $u \in U$ and $1 \leq i, j \leq n$. Hence $c_i = c_j$ whenever $ir(U)_j$. This means that $g$ may be expressed as $g = \sum_{i=1}^{\beta(U)} a_i h_i(U)$ for constants $a_i \in \mathbb{R}^+$.

q.e.d.

Up to this point we have explored the structure of the set of parallel metrics on a single open set. We shall now investigate how these structures relate to each other on intersecting sets. The definition of $h_i(U)$ described above depends upon (1) the choice of vector fields $\xi_1$ and $\xi_2$ on $U$ enforcing the genericity condition, (2) the ordering of the associated orthonormal frame $(Y_1, ..., Y_n)$, and (3) the ordering of the blocks, $P_1(U), ..., P_{\beta(U)}(U)$, of the associated partition. For each $U$ in $\hat{U}$ we shall require that such choices have been made and define the tensor fields $h_i(U), 1 \leq i \leq \beta(U)$, accordingly.

**Lemma 6** Let $W \subseteq U$ be sets in $\hat{U}$. Then there exist constants $c_{iq} \in \mathbb{R}$ such that

$$h_i(U)|_W = \sum_{q=1}^{\beta(W)} c_{iq} h_q(W)$$

for $1 \leq i \leq \beta(U)$.

**Proof:**
By Lemma 5,

$$\sum_{j=1}^{\beta(U)} h_j(U)|_W = \sum_{q=1}^{\beta(W)} a_q h_q(W)$$
for some $a_q \in \mathbb{R}^+$. By Lemmas 4 and 5,

$$h_i(U)|_W + \sum_{j=1}^{\beta(U)} h_j(U)|_W = \sum_{q=1}^{\beta(W)} b_{iq} h_q(W)$$

for some $b_{iq} \in \mathbb{R}^+$. Subtracting gives

$$h_i(U)|_W = \sum_{q=1}^{\beta(W)} c_{iq} h_q(W)$$

where $c_{iq} := b_{iq} - a_q \in \mathbb{R}$.

q.e.d.

Let $\tilde{Q}_i(U)$ be the subbundle of $TU$ spanned by the vector fields $\{Y_j : j \in P_i(U)\}$, for each $i \in 1, ..., \beta(U)$. We then have a decomposition of the tangent space $TU = \tilde{Q}_1(U) \oplus \cdots \oplus \tilde{Q}_\beta(U)$. Denote the dual subbundles by $\tilde{Q}^*_i(U)$. Then $T^*_U = \tilde{Q}^*_1(U) \oplus \cdots \oplus \tilde{Q}^*_\beta(U)$. Observe that $h_i(U)$ is a section of $\tilde{Q}_i(U) \otimes \tilde{Q}_i(U)$, defining a positive semi-definite bilinear form $h_i(U) : T^*_u U \times T^*_u U \rightarrow \mathbb{R}$ for each $u \in U$. Set

$$h_i(U)^\perp := \bigcup_{u \in U} \{\sigma \in T^*_u U : h_i(U)(\sigma, \tau) = 0, \quad \forall \tau \in T^*_u U\}$$

Then

$$h_i(U)^\perp = \bigoplus_{j \neq i} \tilde{Q}^*_j(U)$$

**Lemma 7** Let $W \subseteq U$ be sets in $\tilde{U}$. Then there exists a partition

$$\Gamma(1), ..., \Gamma(\beta(U))$$

of the set of integers $\{1, ..., \beta(W)\}$ such that

$$\tilde{Q}_i(U)|_W = \bigoplus_{q \in \Gamma(i)} \tilde{Q}_q(W)$$
and
\[ h_i(U)|_W = \sum_{q \in \Gamma(i)} h_q(W) \]
for \(1 \leq i \leq \beta(U)\).

**Proof:**

By Lemma 6, there exist constants \(c_{iq} \in \mathbb{R}\) such that
\[ h_i(U)|_W = \sum_{q=1}^{\beta(W)} c_{iq}h_q(W) \]
for \(1 \leq i \leq \beta(U)\). Define subsets \(\gamma(i)\) and \(\Gamma(i)\) of \(\{1, \ldots, \beta(W)\}\) by
\[ \gamma(i) := \{q : c_{iq} \neq 0\} \]
and
\[ \Gamma(i) := \{1, \ldots, \beta(W)\} - \bigcup_{j \neq i} \gamma(j) \]
\[ = \{q : c_{jq} = 0, \ \forall j \neq i\} \]
for \(1 \leq i \leq \beta(U)\). Now,
\[ \bar{Q}_i^*(U)|_W = \bigcap_{j \neq i} (h_j(U)|_W)^\perp \]
\[ = \bigcap_{j \neq i} (\sum_{q=1}^{\beta(W)} c_{jq}h_q(W))^\perp \]
\[ = \bigcap_{j \neq i} (\sum_{q \in \gamma(j)} h_q(W))^\perp \]
\[ = \bigcap_{j \neq i} \bigoplus_{q \in \gamma(j)} \bar{Q}_q^*(W) \]
\[ = \bigoplus_{q \in \Gamma(i)} \bar{Q}_q^*(W) \]

Hence,
\[ \bigoplus_{q=1}^{\beta(W)} \bar{Q}_q^*(W) = T^*W = \bigoplus_{i=1}^{\beta(U)} \bar{Q}_i^*(U)|_W = \bigoplus_{i=1}^{\beta(U)} \bigoplus_{q \in \Gamma(i)} \bar{Q}_q^*(W) \]

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It follows that $\Gamma(1), ..., \Gamma(\beta(U))$ is a partition of \{1, ..., $\beta(W)$\} and
\[
\bar{Q}_i(U)|_W = \bigoplus_{q \in \Gamma(i)} \bar{Q}_q(W)
\]
for $1 \leq i \leq \beta(U)$.

Furthermore,
\[
h_i(U)|_W = \sum_{q=1}^{\beta(W)} c_{iq} h_q(W) = \sum_{j=1}^{\beta(U)} \sum_{q \in \Gamma(j)} c_{iq} h_q(W) = \sum_{q \in \Gamma(i)} c_{iq} h_q(W)
\]
since $c_{iq} = 0$ for $q \in \Gamma(j)$ when $j \neq i$. Consequently,
\[
\sum_{q=1}^{\beta(W)} h_q(W) = h|_W = \sum_{i=1}^{\beta(U)} h_i(U)|_W = \sum_{i=1}^{\beta(U)} \sum_{q \in \Gamma(i)} c_{iq} h_q(W)
\]
Therefore $c_{iq} = 1$ for all $q \in \Gamma(i)$ and so
\[
h_i(U)|_W = \sum_{q \in \Gamma(i)} h_q(W)
\]
q.e.d.

We now turn to the problem of how to piece together these local parallel metrics into a global one. This is done by means of an appropriately defined equivalence relation. Let $\mathcal{U}$ be any subset of $\tilde{\mathcal{U}}$ that covers $M$ and has the property that the intersection of pairs of sets in $\mathcal{U}$ is connected. For instance, we can choose $\mathcal{U}$ to be a good refinement of $\tilde{\mathcal{U}}$. Let $I := \{(i, U) : U \in \mathcal{U} \text{ and } 1 \leq i \leq \beta(U)\}$ and define $\sim$ to be the equivalence relation on $I$ generated by the relations
\[
(i, U) \sim (j, V) \quad \text{if} \quad U \cap V \neq \emptyset \quad \text{and} \quad \bar{Q}_i(U) \cap \bar{Q}_j(V) \neq \emptyset
\]
where "0" means the zero distribution on $U \cap V$. These relations are required in order to join the $\overline{Q}_i$ distributions together in a smooth way. The equivalence partitions $I$ into $A$ blocks denoted $I_1, \ldots, I_A$.

**Lemma 8** For each $a \in \{1, \ldots, A\}$ and $U \in \mathcal{U}$ the set $\{i : (i, U) \in I_a\}$ is non-empty.

**Proof:**
Let $C(U)$ denote the subset of $\mathcal{U}$ consisting of all sets $U'$ for which there exists a sequence of sets $U' = U_1, U_2, \ldots, U_k = U$ in $\mathcal{U}$ such that $U_l \cap U_{l+1} \neq \emptyset$ for all $1 \leq l \leq k - 1$. We claim that $C(U) = \mathcal{U}$. First observe that if $V \in \mathcal{U}$ and $V \cap U' \neq \emptyset$ for some $U' \in C(U)$ then $V \in C(U)$ also. Let $S$ denote the union of the sets in $C(U)$. Suppose that $S$ is not equal to $M$. Since $M$ is connected the boundary $\partial S$, of $S$ is non-empty. $\mathcal{U}$ covers $M$, so there exists an open set $V \in \mathcal{U}$ such that $V \cap \partial S \neq \emptyset$. This means that $V \cap S \neq \emptyset$ and $V \cap S^c \neq \emptyset$, where $S^c$ denotes the compliment of $S$ in $M$. Hence $V \cap U' \neq \emptyset$ for some $U' \in C(U)$ and $V \notin C(U)$, which is a contradiction. Therefore $S = M$. Let $W \in \mathcal{U}$. Then $W \cap U' \neq \emptyset$ for some $U' \in C(U)$ and so $W \in C(U)$. This demonstrates the claim.

Let $(i_1, U_1)$ be any representative of $I_a$. We have shown that there exists a sequence of sets $U_2, \ldots, U_k = U$ in $\mathcal{U}$ such that $U_l \cap U_{l+1} \neq \emptyset$ for all $1 \leq l \leq k - 1$. For each $l \in \{1, \ldots, k-1\}$ and $p \in \{1, \ldots, \beta(U_l)\}$ there exists at least one $q \in \{1, \ldots, \beta(U_{l+1})\}$ such that $\overline{Q}_p(U_l) \cap \overline{Q}_q(U_{l+1}) \neq 0$. Therefore we can find a sequence $i_2, \ldots, i_k$ such that $(i_1, U_1) \sim (i_2, U_2) \sim \cdots \sim (i_k, U_k) = (i_k, U)$. Hence $i_k \in \{i : (i, U) \in I_a\}$.

**q.e.d.**

By Lemma 8, we may define a non-trivial distribution $Q_a(U)$ on $U$ by

$$Q_a(U) := \bigoplus_{\{i : (i, U) \in I_a\}} \overline{Q}_i(U) = \bigoplus_{\{i : (i, U) \in I_a\}} \text{span}\{Y_j : j \in \mathbb{P}_i(U)\}$$
for all $a \in \{1, ..., A\}$ and $U \in \mathcal{U}$. Define the distributions $Q_a$ on $M$ by specifying their restriction on each $U \in \mathcal{U}$ to be

$$Q_a|_U := Q_a(U)$$

Lemma 9  The $Q_a$ are well-defined.

Proof:
Let $V$ be another set in $\mathcal{U}$ having non-zero intersection with $U$. We must show that on the intersection, $Q_a(U)|_{U\cap V} = Q_a(V)|_{U\cap V}$. By Lemma 7, the set $\{1, ..., \beta(U \cap V)\}$ partitions into $\Gamma_U(1), ..., \Gamma_U(\beta(U))$ in such a way that

$$\bar{Q}_i(U)|_{U\cap V} = \bigoplus_{q \in \Gamma_U(i)} \bar{Q}_q(U \cap V)$$

$\{1, ..., \beta(U \cap V)\}$ also partitions into $\Gamma_V(1), ..., \Gamma_V(\beta(V))$ in such a way that

$$\bar{Q}_j(V)|_{U\cap V} = \bigoplus_{q \in \Gamma_V(j)} \bar{Q}_q(U \cap V)$$

Observe that if $\Gamma_U(i) \cap \Gamma_V(j) \neq \emptyset$ then $\bar{Q}_i(U)|_{U\cap V} \cap \bar{Q}_j(V)|_{U\cap V} \neq 0$ and so $(i, U) \sim (j, V)$. Hence,

$$Q_a(U)|_{U\cap V} := \bigoplus_{\{i: (i, U) \in I_a\}} \bar{Q}_i(U)|_{U\cap V}$$

$$= \bigoplus_{\{i: (i, U) \in I_a\}} \sum_{q \in \Gamma_U(i)} \sum_{\{j: \Gamma_U(i) \cap \Gamma_V(j) \neq \emptyset\}} \bar{Q}_q(U \cap V)$$

$$\subseteq \sum_{\{i: (i, U) \in I_a\}} \sum_{\{j: (i, U) \sim (j, V)\}} \sum_{q \in \Gamma_V(j)} \bar{Q}_q(U \cap V)$$

$$= \bigoplus_{\{j: (j, V) \in I_a\}} \sum_{q \in \Gamma_V(j)} \bar{Q}_q(U \cap V)$$

$$= \bigoplus_{\{j: (j, V) \in I_a\}} \bar{Q}_j(V)|_{U\cap V}$$

$$:= Q_a(V)|_{U\cap V}$$
Similarly, \( Q_a(V)|_{U \cap V} \subseteq Q_a(U)|_{U \cap V} \) and so the two distributions are equal on \( U \cap V \).

q.e.d.

The tangent space has the direct sum decomposition into subbundles:

\[
TM = Q_1 \oplus \cdots \oplus Q_A
\]

Denote the dual subbundles by \( Q^*_a \) and the fibre of \( Q^*_a \) over \( m \in M \) by \( Q^*_a(m) \). Then the cotangent space at \( m \) has the decomposition:

\[
T^*_mM = Q^*_1(m) \oplus \cdots \oplus Q^*_A(m)
\]

\( h \) is a section of \( TM \otimes TM \) and therefore determines a bilinear map \( h(m) : T^*_mM \times T^*_mM \to \mathbb{R} \) for each \( m \in M \). Define positive semi-definite metrics \( h_a \) on \( M \) for \( 1 \leq a \leq A \) by

\[
h_a(m)|_{Q^*_b(m) \times Q^*_c(m)} := \delta_{ab}\delta_{ac}h(m)|_{Q^*_b(m) \times Q^*_c(m)}
\]

for each \( m \in M \). On \( U \in \mathcal{U} \),

\[
h_a|_U = \sum_{\{k:(k,U)\in I_a\}} h_k(U)
\]

Therefore \( h = \sum_{a=1}^A h_a \).

We may now describe the parallel metrics of \( \nabla \) on \( M \).

**Theorem 10** \( g \) is a parallel metric on \( M \) if and only if it can be written as \( g = \sum_{a=1}^A c_a h_a \) for some \( c_a \in \mathbb{R}^+ \).

**Proof:**

\( \iff \) By Lemma 4 and the fact that \( h_a|_U = \sum_{\{k:(k,U)\in I_a\}} h_k(U) \), \( \nabla h_a = 0 \) for all \( 1 \leq a \leq A \). Therefore any \( g = \sum_{a=1}^A c_a h_a \), where \( c_a \in \mathbb{R}^+ \), is a parallel
metric on $M$.

$\implies$ Suppose that $g$ is a parallel metric on $M$. Let $U$ and $V$ be two sets in $\mathcal{U}$ with non-empty intersection. By Lemma 7, the set $\{1, ..., \beta(U \cap V)\}$ partitions into $\Gamma_{U}(1), ..., \Gamma_{U}(\beta(U))$ in such a way that

\[
\tilde{Q}_{i}(U)|_{U \cap V} = \bigoplus_{q \in \Gamma_{U}(i)} \tilde{Q}_{q}(U \cap V)
\]

and

\[
h_{i}(U)|_{U \cap V} = \sum_{q \in \Gamma_{U}(i)} h_{q}(U \cap V)
\]

$\{1, ..., \beta(U \cap V)\}$ also partitions into $\Gamma_{V}(1), ..., \Gamma_{V}(\beta(V))$ in such a way that

\[
\tilde{Q}_{j}(V)|_{U \cap V} = \bigoplus_{q \in \Gamma_{V}(j)} \tilde{Q}_{q}(U \cap V)
\]

and

\[
h_{j}(V)|_{U \cap V} = \sum_{q \in \Gamma_{V}(j)} h_{q}(U \cap V)
\]

Suppose that $\tilde{Q}_{i}(U) \cap \tilde{Q}_{j}(V) \neq 0$ for some $i$ and $j$. Then there exists an element $p \in \Gamma_{U}(i) \cap \Gamma_{V}(j)$.

For any $W \in \mathcal{U}$, Lemma 5 defines positive constants $c_{k}(W)$ by

\[
g|_{W} = \sum_{k=1}^{\beta(W)} c_{k}(W)h_{k}(W)
\]

Therefore on $U \cap V$, $g$ may be expressed as

\[
g|_{U \cap V} = \sum_{k=1}^{\beta(U)} c_{k}(U)h_{k}(U)|_{U \cap V} = \sum_{k=1}^{\beta(U)} c_{k}(U) \sum_{q \in \Gamma_{U}(k)} h_{q}(U \cap V)
\]

Similarly,

\[
g|_{U \cap V} = \sum_{l=1}^{\beta(V)} c_{l}(V)h_{l}(V)|_{U \cap V} = \sum_{l=1}^{\beta(V)} c_{l}(V) \sum_{q \in \Gamma_{V}(l)} h_{q}(U \cap V)
\]
The coefficient of \( h_p(U \cap V) \) is \( c_i(U) = c_j(V) \). Therefore

\[
\bar{Q}_i(U) \cap \bar{Q}_j(V) \neq 0 \implies c_i(U) = c_j(V)
\]

From the definition of the equivalence \( \sim \) it follows that

\[
(i, U_1) \sim (j, U_2) \implies c_i(U_1) = c_j(U_2)
\]

This allows us to define the positive constants \( c_a := c_k(U) \), where \((k, U)\) is any representative of \( I_a \), for \( 1 \leq a \leq A \).

Thus,

\[
g_{|U} = \sum_{k=1}^{\beta(U)} c_k(U) h_k(U)
\]

\[
= \sum_{a=1}^{A} \sum_{\{k:(k,U) \in I_a\}} c_k(U) h_k(U)
\]

\[
= \sum_{a=1}^{A} c_a \sum_{\{k:(k,U) \in I_a\}} h_k(U)
\]

\[
= \sum_{a=1}^{A} c_a h_a|_U
\]

It follows that \( g = \sum_{a=1}^{A} c_a h_a \).

\[\text{q.e.d.}\]

**Corollary 11** The manifold of parallel metrics on \( M \) has dimension \( A \).

Given the Riemannian manifold \((M, h)\) with generic Levi-Civita connection \( \nabla \), the determination of the \( h_a \) is an algebraic construction. Therefore Theorem 10 enables one to obtain all parallel metrics of \( \nabla \) by purely algebraic means; integration of differential equations is unnecessary.

Let us summarize the steps in the procedure.
1. Find an open cover \( \mathcal{U} \subseteq \hat{U} \) of \( M \) having the property that the intersection of pairs of sets in \( \mathcal{U} \) is connected.
For each \( U \in \mathcal{U} \), follow steps 2-5:

2. Find a frame of eigenvector fields \((Z_1, ..., Z_n)\) on \( U \) for \( R(\xi_1, \xi_2) \).

3. Obtain the associated frame of orthogonal vector fields \((X_1, ..., X_n)\).

4. Define the orthonormal frame \((Y_1, ..., Y_n)\).

5. Obtain the decomposition \( TU = \overline{Q}_1(U) \oplus \cdots \oplus \overline{Q}_\beta(U)(U) \).

6. Construct the subbundles \( Q_a \).

7. Define the tensor fields \( h_a \) and apply the theorem.

**Example** Consider the Riemannian manifold \((M, h)\) where \( M := \mathbb{R}^4 \) and \( h := dx^2 + e^{2x} dy^2 + du^2 + e^{2u} dv^2 \), which in contravariant form is \( \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial x} + e^{-2x} \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial y} + \frac{\partial}{\partial u} \otimes \frac{\partial}{\partial u} + e^{-2u} \frac{\partial}{\partial v} \otimes \frac{\partial}{\partial v} \). \((M, h)\) is a Cartesian product of two isomorphic irreducible Riemannian manifolds and therefore it is expected that the general parallel metric would be a positive-definite linear combination of the pullbacks onto \( M \) of the component metrics, \( \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial x} + e^{-2x} \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial y} \) and \( \frac{\partial}{\partial u} \otimes \frac{\partial}{\partial u} + e^{-2u} \frac{\partial}{\partial v} \otimes \frac{\partial}{\partial v} \).

The Christoffel symbols for the Levi-Civita connection \( \nabla \) of \( h \) are

\[
\begin{align*}
\Gamma^x_{yy} &= -e^{2x} \\
\Gamma^y_{xy} &= \Gamma^y_{yx} = 1 \\
\Gamma^u_{vv} &= -e^{2u} \\
\Gamma^v_{uv} &= \Gamma^v_{vu} = 1
\end{align*}
\]

and all others zero. Define a good cover \( \mathcal{U} := \{U, U'\} \) of \( M \) by

\[
\begin{align*}
U &:= \{(x, y, u, v) \in \mathbb{R}^4 : x > u \} \\
U' &:= \{(x, y, u, v) \in \mathbb{R}^4 : x < u + \log 2 \}
\end{align*}
\]

On \( U \) let \( \xi_1 := \frac{\partial}{\partial x} + \frac{\partial}{\partial u} \) and \( \xi_2 := \frac{\partial}{\partial y} + \frac{\partial}{\partial v} \). The Riemann curvature with
respect to the frame \( (\partial / \partial x, \partial / \partial y, \partial / \partial u, \partial / \partial v) \) on \( U \) is

\[
R(\xi_1, \xi_2) = \begin{pmatrix}
0 & -e^{2x} & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -e^{2u} \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

The eigenvalues are \( \lambda = -ie^x, ie^x, -ie^u \) and \( ie^u \), which are distinct on \( U \). Corresponding eigenvector fields on \( U \) are

\[
Z_1 = e^x \frac{\partial}{\partial x} + i \frac{\partial}{\partial y}, \quad Z_2 = e^x \frac{\partial}{\partial x} - i \frac{\partial}{\partial y}, \quad Z_3 = e^u \frac{\partial}{\partial u} + i \frac{\partial}{\partial v}, \quad Z_4 = e^u \frac{\partial}{\partial u} - i \frac{\partial}{\partial v}
\]

which defines

\[
X_1 := e^x \frac{\partial}{\partial x}|_U \\
X_2 := \frac{\partial}{\partial y}|_U \\
X_3 := e^u \frac{\partial}{\partial u}|_U \\
X_4 := \frac{\partial}{\partial v}|_U
\]

The orthonormal frame \( (Y_1, \ldots, Y_4) \) is then given by

\[
Y_1 := \frac{\partial}{\partial x}|_U \\
Y_2 := e^{-x} \frac{\partial}{\partial y}|_U \\
Y_3 := \frac{\partial}{\partial u}|_U \\
Y_4 := e^{-u} \frac{\partial}{\partial v}|_U
\]

The curvature form \( \omega = \omega^i_j \) with respect to \( (Y_1, \ldots, Y_4) \) satisfies \( \omega^2_1(Y_2) = \omega^3_4(Y_4) = 1 \). Therefore \( 2r(U)1 \) and \( 4r(U)3 \). Furthermore, \( \omega^i_j = 0 \) for \( i \in \{1, 2\} \) and \( j \in \{3, 4\} \). Thus there are exactly two equivalence classes for the equivalence relation \( r(U) \):

\[
P_1(U) = \{1, 2\} \quad \text{and} \quad P_2(U) = \{3, 4\}
\]
This gives,

\[ Q_1(U) = \text{span}\{Y_1, Y_2\} = \text{span}\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}|_U \]
\[ Q_2(U) = \text{span}\{Y_3, Y_4\} = \text{span}\{\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\}|_U \]
\[ h_1(U) = Y_1 \otimes Y_1 + Y_2 \otimes Y_2 = \left(\frac{\partial}{\partial x} \otimes \frac{\partial}{\partial x} + e^{-2x} \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial y}\right)|_U \]
\[ h_2(U) = Y_3 \otimes Y_3 + Y_4 \otimes Y_4 = \left(\frac{\partial}{\partial u} \otimes \frac{\partial}{\partial u} + e^{-2u} \frac{\partial}{\partial v} \otimes \frac{\partial}{\partial v}\right)|_U \]

On \( U' \) let \( \xi_1' := \frac{\partial}{\partial x} + 2 \frac{\partial}{\partial u} \) and \( \xi_2' := \frac{\partial}{\partial y} + \frac{\partial}{\partial v} \). The Riemann curvature with respect to the frame \( \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right) \) on \( U' \) is

\[
R(\xi_1', \xi_2') = \begin{pmatrix}
0 & -e^{2u} & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -2e^{2u} \\
0 & 0 & 2 & 0
\end{pmatrix}
\]

The eigenvalues are \( \lambda' = -ie^x, ie^x, -2ie^u \) and \( 2ie^u \), which are distinct on \( U' \).

Corresponding eigenvector fields on \( U' \) are

\[ Z_1' = e^x \frac{\partial}{\partial x} + i \frac{\partial}{\partial y}, \quad Z_2' = e^x \frac{\partial}{\partial x} - i \frac{\partial}{\partial y}, \quad Z_3' = e^u \frac{\partial}{\partial u} + i \frac{\partial}{\partial v}, \quad Z_4' = e^u \frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \]

which defines

\[ X_1' := e^x \frac{\partial}{\partial x}|_{U'} \]
\[ X_2' := \frac{\partial}{\partial y}|_{U'} \]
\[ X_3' := e^u \frac{\partial}{\partial u}|_{U'} \]
\[ X_4' := \frac{\partial}{\partial v}|_{U'} \]

The orthonormal frame \( (Y_1', ..., Y_4') \) is then given by

\[ Y_1' := \frac{\partial}{\partial x}|_{U'} \]
\[ Y_2' := e^{-x} \frac{\partial}{\partial y}|_{U'} \]
\[ Y_3' := \frac{\partial}{\partial u}|_{U'} \]
\[ Y_4' := e^{-u} \frac{\partial}{\partial v}|_{U'} \]

Continuing the analysis as above gives

\[ P_1(U') = \{1, 2\} \quad \text{and} \quad P_2(U') = \{3, 4\} \]
and
\[ \tilde{Q}_1(U') = \text{span}\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \}|_{U'} \]
\[ \tilde{Q}_2(U') = \text{span}\{ \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \}|_{U'} \]
\[ h_1(U') = \left( \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial x} + e^{-2x} \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial y} \right)|_{U'} \]
\[ h_2(U') = \left( \frac{\partial}{\partial u} \otimes \frac{\partial}{\partial u} + e^{-2u} \frac{\partial}{\partial v} \otimes \frac{\partial}{\partial v} \right)|_{U'} \]

Next we consider the equivalence relation \( \sim \). Restricted to \( U \cap U' \),
\[ \tilde{Q}_1(U)|_{U \cap U'} = \tilde{Q}_1(U')|_{U \cap U'} \]
\[ \tilde{Q}_2(U)|_{U \cap U'} = \tilde{Q}_2(U')|_{U \cap U'} \]
\[ Q_1(U) \cap Q_2(U') = 0 \]
\[ Q_2(U) \cap Q_1(U') = 0 \]

Therefore (1, \( U \)) \( \sim \) (1, \( U' \)), (2, \( U \)) \( \sim \) (2, \( U' \)) and there are no other non-trivial relations. This gives
\[ I_1 = \{ (1, U), (1, U') \} \]
\[ I_2 = \{ (2, U), (2, U') \} \]
\[ Q_1 = \text{span}\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \} \]
\[ Q_2 = \text{span}\{ \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \} \]
\[ h_1 = \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial x} + e^{-2x} \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial y} \]
\[ h_2 = \frac{\partial}{\partial u} \otimes \frac{\partial}{\partial u} + e^{-2u} \frac{\partial}{\partial v} \otimes \frac{\partial}{\partial v} \]

By Theorem 10, \( g \) is a parallel metric with respect to \( \nabla \) if and only if
\[ g = c_1 h_1 + c_2 h_2 \] for some constants \( c_1, c_2 \in \mathbb{R}^+ \); the anticipated result.
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