New Approach for Vorticity Estimates of Solutions of the Navier-Stokes Equations

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Abstract

We develop a new approach for regularity estimates, especially vorticity estimates, of solutions of the three-dimensional Navier-Stokes equations with periodic initial data, by exploiting carefully formulated linearized vorticity equations. An appealing feature of the linearized vorticity equations is the inheritance of the divergence-free property of solutions, so that it can intrinsically be employed to construct and estimate solutions of the Navier-Stokes equations. New regularity estimates of strong solutions of the three-dimensional Navier-Stokes equations are obtained by deriving new explicit a priori estimates for the heat kernel (i.e., the fundamental solution) of the corresponding heterogeneous drift-diffusion operator. These new a priori estimates are derived by using various functional integral representations of the heat kernel in terms of the associated diffusion processes and their conditional laws, including a Bismut-type formula for the gradient of the heat kernel. Then the a priori estimates of solutions of the linearized vorticity equations are established by employing a Feynman-Kac-type formula. The existence of strong solutions and their regularity estimates up to a time proportional to the reciprocal of the square of the maximum initial vorticity are established. All the estimates established in this paper contain known constants that can be explicitly computed.

Key words: Gradient estimates, vorticity estimates, approach, iteration scheme, a priori estimates, strong solutions, Navier-Stokes equations, vorticity equations, heat kernel, fundamental solution, stochastic diffusion process, heterogeneous drift-diffusion operator, conditional laws, Bismut-type formula, Feynman-Kac-type formula.

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1 Introduction

We are concerned with the quantitative regularity estimates of solutions of the Navier-Stokes equations in $\mathbb{R}^3$. In this paper, we consider the Cauchy problem for the Navier-Stokes equations for $x \in \mathbb{R}^3$ and $t \geq 0$:

$$\begin{align*}
\begin{cases}
\partial_t u + u \cdot \nabla u + \nabla P &= \nu \Delta u, \\
\nabla \cdot u &= 0,
\end{cases}
\end{align*}$$

subject to the periodic initial condition:

$$u|_{t=0} = u_0,$$

satisfying that $u_0(x+Lk) = u_0(x)$ for all $k \in \mathbb{Z}^3$ and $\nabla \cdot u_0 = 0$, where $\nu > 0$ is the kinematic constant, and $L > 0$ is the period of the initial data. Although periodic flows are special in nature, the periodic solutions can be considered as ideal solutions for turbulent motions away from their physical boundaries, or as models for homogeneous turbulent flows.

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The Cauchy problem (1.1)–(1.2) for the Navier-Stokes equations (1.1) with periodic initial data (1.2) seeks for a velocity vector field $u(x,t)$ and a scalar pressure $P(x,t)$ that are periodic functions with period $L$ satisfying the initial condition that $u(x,0) = u_0(x)$, where $u_0(x)$ is the periodic initial velocity in (1.2), which is divergence-free, i.e., $\nabla \cdot u_0 = 0$.

The mathematical study of global solutions of the Navier-Stokes equations (1.1) was initiated in the work of Leray [23, 24] and Hopf [16, 17], in which global weak solutions were constructed and investigated. Since then, many properties and features of the solutions of the Navier-Stokes equations (1.1) have been understood; see [14, 15, 21, 36, 39, 40] and the references cited therein. The Navier-Stokes equations (1.1) have been studied by using various methods; the mathematical analysis of (1.1) has been mainly based on several functional analysis methods and on the results on certain functional spaces such as the Sobolev spaces and the Besov spaces. Great progress has been made in the past decades, the existences of local and global solutions of the Navier-Stokes equations (1.1) have been studied (cf. [10, 13, 20, 26, 34, 35], besides the references cited therein and above). The partial regularity of the weak solutions of the Navier-Stokes equations (1.1) has also received intensive study; see [5, 25, 30, 31, 32, 33] and the references cited therein. However, the quantitative regularity of solutions, the global existence of strong solutions, and the uniqueness of weak solutions of the three-dimensional (3-D) Navier-Stokes equations remain to be the major open problems in Mathematics.

1.1 Main theorem

Assume that the periodic initial data function $u_0$ is smooth and divergence-free. Then any strong solution $u(x,t)$ of the Cauchy problem (1.1)–(1.2) must have a constant mean velocity, so that it is assumed without loss of generality that

$$\int_{[0,L]^d} u(x,t) \, dx = 0, \quad (1.3)$$

due to the Galilean invariance of system (1.1). Denote the vorticity:

$$\omega(x,t) = \nabla \times u(x,t) \quad (1.4)$$

which is the curl of the velocity. Then $\omega_0 = \nabla \times u_0$ is the initial vorticity.

In this paper, we develop a new approach, i.e., an iteration scheme, to construct and estimate strong solutions of the Cauchy problem (1.1)–(1.2) more sharply than the existing results, by developing an array of useful mathematical tools in Analysis. The main results for the quantitative estimates of strong solutions can be stated in the following theorem.

**Theorem 1.1 (Main Theorem).** There are two universal constants $C_1 > 0$ and $C_2 > 0$ such that there exists a unique strong solution $u(x,t)$ of the Cauchy problem (1.1)–(1.2) for all $t \in [0,T_0]$ with

$$T_0 = C_1 v^2 L^{-4} \| \omega_0 \|_{\infty}^{-2}, \quad (1.5)$$

so that the following estimates for $u(x,t)$ hold: For all $0 < t \leq T_0$ and $x \in \mathbb{R}^3$,

$$|u(x,t)| \leq C_2 L \| \omega_0 \|_{\infty}, \quad |\nabla u(x,t)| \leq \frac{C_2 L}{\sqrt{vt}} \| \omega_0 \|_{\infty}, \quad (1.6)$$

$$|\omega(x,t)| \leq C_2 \| \omega_0 \|_{\infty}, \quad |\nabla \omega(x,t)| \leq \frac{C_2}{\sqrt{vt}} \| \omega_0 \|_{\infty} \quad (1.7)$$

where $\| \omega_0 \|_{\infty}$ is the $L^\infty$-norm of the initial vorticity $\omega_0$.

The two positive constants $C_1$ and $C_2$ in Theorem 1.1 are computable, which can be worked out by tracing all the universal constants in the proof.

The Cauchy problem (1.1)–(1.2) of the Navier-Stokes equations with periodic initial data $u_0(x)$ has been studied traditionally in the Fourier space, which is particularly the case in turbulence literature; see, for example, [2, 7, 11]. Our approach is necessary to departure from the well-known methods. The quantitative regularity estimates, Theorem 1.1, will be proved by developing a new approach via an array of mathematical tools from the theory of partial differential equations, stochastic analysis, and the Hodge theory on the torus.
1.2 New approach – An iteration scheme for the construction and estimates of strong solutions

We now describe briefly the new approach – an iteration scheme – developed in this paper to prove Theorem 1.1; see §6–§7 for details.

First of all, by using the dimensionless scaling,

\[ U(x,t) = \frac{L}{2v} u(Lx, \frac{L^2}{2v}t) \]

has period 1 and solves the Navier-Stokes equations (1.1) with \( v = \frac{1}{2} \). Thus, without loss of generality, we will assume that the viscosity constant \( v = \frac{1}{2} \) and \( L = 1 \) in what follows. Furthermore, from now on, by a (time-dependent) periodic tensor field \( f(x,t) \) on \( \mathbb{R}^3 \), or equivalently by saying that a tensor field \( f(x,t) \) is periodic, we mean that \( f(x,t) \) is a tensor field on \( \mathbb{R}^3 \) depending on the time parameter \( t \) and satisfies that \( f(x + k, t) = f(x,t) \) for all \( x \in \mathbb{R}^3 \), \( t \geq 0 \), and \( k \in \mathbb{Z}^3 \).

Our new iteration scheme for the construction of strong solutions of the Cauchy problem (1.1)–(1.2) is based on the vorticity equation for \( \omega = \nabla \wedge u \):

\[ \partial_t \omega + (u \cdot \nabla) \omega - A(u)\omega - \frac{1}{2} \Delta \omega = 0, \quad (1.8) \]

where \( A(u) \) is the total derivative of \( u \), a tensor field with components \( A(u)^i_j = \partial_i u^j \). Although there are several formulations of the vorticity equations, one of our main observations is that this version of formulation serves our aims particularly well.

Suppose that \( b(x,t) \) is a periodic, smooth, and divergence-free vector field such that \( b(x,0) = u_0(x) \). Then we define a vector field \( w(x,t) \), which should be a candidate of the vorticity (while \( w \) is not in general the vorticity of \( b \)), by solving the Cauchy problem of the following linear parabolic equations:

\[
\begin{cases}
\partial_t w + (b \cdot \nabla) w - A(b)w - \frac{1}{2} \Delta w = 0, \\
w(\cdot, 0) = \omega_0,
\end{cases}
\]

where \( A(b) = (A(b)^i_j) = (\partial_i b^j) \) is the total derivative of \( b \). The unique solution \( w(x,t) \) has two properties that are important to our approach:

(i) It can be shown that \( \nabla \cdot w = 0 \) (divergence-free again); this is the key property that makes the linear parabolic equations (1.9) appealing and workable to our task.

(ii) \( \int_{[0,1]^3} w(x,t) \, dx = 0 \) for all \( t > 0 \), which is satisfied when \( t = 0 \).

With these, we define the candidate \( v(x,t) \) for the velocity by solving the Poisson equation:

\[
\begin{cases}
\Delta v = -\nabla \wedge w, \\
\int_{[0,1]^3} v(x,t) \, dx = 0 \quad \text{for any } t > 0.
\end{cases}
\]

Then \( w = \nabla \wedge v \), according to the Hodge theory on the torus.

In this way, we construct a mapping \( V \) that sends \( b(x,t) \) to \( v(x,t) \). That is, the iteration for obtaining a strong solution is defined as

\[ u^{(n)} = V(u^{(n-1)}) \quad \text{for } n = 1, 2, \ldots, \quad (1.10) \]

with the initial iteration defined by \( u^{(0)}(x,t) = u_0(x) \) for all \( x \) and \( t \geq 0 \).

Let us point out that the computational schemes for simulations of turbulent flows based on different formulations of the vorticity equations have been very fruitful in the past, cf. [6, 27] for an overview. Our analysis below shows that the iteration scheme developed in this paper does converge to the strong solution with inherent gradient estimates of the iteration solutions uniformly. This shows that the iteration scheme should also be useful for developing numerical algorithms to compute turbulent solutions.
1.3 Assumptions and notations

In order to describe the technical aspects in our study, we introduce a few notations and assumptions that will be used throughout the paper. By a function or a tensor field we mean a function defined on $\mathbb{R}^d$ or a tensor field on the torus $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$; in the latter case, it is identified with a periodic field on $\mathbb{R}^d$ with period 1 in each coordinate variable.

Suppose that $f(x,t)$ for $x \in \mathbb{R}^d$ is a tensor field depending on a time parameter $t \geq 0$. Then we assume that $f$ is Borel measurable on $\mathbb{R}^d \times [0, \infty)$. For $I \subset [0, \infty)$, the $L^\infty$-norm of $f$ over $\mathbb{R}^d \times I$ is defined to be

$$ \|f\|_{L^\infty(I)} = \sup_{(x,t) \in \mathbb{R}^d \times I} |f(x,t)|, $$

and for the case when $I = [0, \infty)$, the previous norm is simply denoted by $\|f\|_\infty$.

If $0 \leq \tau < T$, the parabolic $L^\infty$-norm of $f$ over an interval $[\tau, T]$ will play an important role, which is defined by

$$ \|f\|_{\tau \rightarrow T} = \sup_{(x,t) \in \mathbb{R}^d \times [\tau, T]} |\sqrt{T-\tau} f(x,t)| = \|\sqrt{T-\tau} f\|_{L^\infty([\tau, T])}. \quad (1.11) $$

In the case when $T = \infty$, the interval $[\tau, T]$ is replaced by $[\tau, \infty)$. From the definition, it is clear that

$$ \|f\|_{\tau \rightarrow T} \leq \sqrt{T-\tau} \|f\|_{\tau \rightarrow \infty} \leq \sqrt{T-\tau} \|f\|_\infty. \quad (1.12) $$

Throughout the paper, the probability density function (PDF) of a normal random variable with mean zero and variance $t > 0$ is denoted by

$$ G_t(x) = \frac{1}{(2\pi t)^{d/2}} \exp(-|x|^2/2t) \quad \text{for} \ x \in \mathbb{R}^d. \quad (1.13) $$

Notice that $G_{t-\tau}(y-x)$ is the fundamental solution of the heat operator $\partial_t - \frac{1}{2} \Delta$ in the Euclidean space $\mathbb{R}^d$.

As a convention, the Laplacian $\Delta$ and the gradient $\nabla$ (in particular, the divergence operation $\nabla \cdot$ and the curl $\nabla \wedge$), when operating on time-dependent tensor fields on $\mathbb{R}^d$, apply to space variable $x$ only.

If $b(x,t)$ is a time-dependent vector field on $\mathbb{R}^d$ for $t \geq 0$, the heterogeneous drift-diffusion differential operator of second order:

$$ L_{b(x,t)} = \frac{1}{2} \Delta + b(x,t) \cdot \nabla \quad (1.14) $$

will play an important role in our study. When no confusion arises, $L_{b(x,t)}$ is denoted simply by $L_b$. Among the technical assumptions on $b(x,t)$, the most essential one is the assumption that $b(x,t)$ is solenoidal (i.e., divergence-free). That is, for every $t$, the divergence $\nabla \cdot b(\cdot, t)$ vanishes identically in the distributional sense. Under this assumption, the formal adjoint operator:

$$ L_b^* = L_{-b}, $$

which is again an elliptic operator of the same type. The second assumption is technical for the construction of probabilistic structures. For simplicity, we assume that $b(x,t)$ is Borel measurable and bounded over any finite interval, i.e., $\|b\|_{L^\infty([0, T])} < \infty$ for every $T > 0$. The probability density function of the $L_b$-diffusion (i.e., the fundamental solution or heat kernel to the parabolic operator $L_{b(x,t)}^* + \partial_t$; see §2) is denoted by

$$ p_b(\tau, x, t, y) \quad \text{for} \ t > \tau \geq 0 \ \text{and} \ x, y \in \mathbb{R}^d. $$

Throughout the paper, universal constants (the constants depending only on the dimension $d$, or some parameters $\beta, \gamma$, etc. introduced in proofs) are denoted by $C_1, C_2,$ etc. which may be different at each occurrence.
1.4 A priori estimates of solutions of the parabolic equations

In order to prove Theorem 1.1, the main effort is to derive precise a priori estimates of solutions of the Cauchy problem (1.9) for the parabolic equations. To achieve this, there are two tasks to be carried out.

1. The main task is to derive explicit a priori estimates for the fundamental solution (or called the heat kernel) of the parabolic operator $\partial_t - L_b$ and its gradient in terms of the bound of $b$ and the parabolic norm of $\nabla b$, when $b(x,t)$ is a bounded, divergence-free, and smooth vector field on $\mathbb{R}^d$.

Although the regularity theory for linear parabolic equations has been well established (cf. [12, 22, 38]), our a priori estimates contain the universal constants depending only on the dimension $d$ and a parameter $\beta > 1$ fixed in our estimates. In addition to its explicit form, for a divergence-free vector field $b(x,t)$, the gradient estimate for the heat kernel associated with the parabolic operator $\partial_t - L_b$, depends only on the bound of $b$ and its first-order derivative $\nabla b$.

**Theorem 1.2.** Let $b(x,t)$ be a smooth, divergence-free, and bounded time-dependent vector field on $\mathbb{R}^d$. Then, for every $\beta > 1$, there are constants $C_1$ and $C_2$ depending only on $\beta$ and dimension $d$ such that

\[ p_b(\tau, x, t, y) \leq C_1 e^{C_2(t-\tau)} |y - x|^{1/2} G_{\beta(t-\tau)}(y - x), \]

where $\tau, x, t, y \in \mathbb{R}^d$, and $G_{\beta(t-\tau)}$ is the heat kernel associated with the parabolic operator $\partial_t - L_b$.

These estimates are quite delicate to derive: They are obtained by introducing substantial tools from stochastic analysis, mainly various functional integral representations for the fundamental solutions (cf. [29, 28]) and a new kind of Bismut’s formulas (cf. [3, 4]), together with careful and explicit computations. Estimates (1.15)–(1.16) will be proved in §4 and §5, respectively.

2. The second task in our study is to prove the explicit a priori estimates to the iteration $u^{(n)} = V(u^{(n-1)})$ in (1.10), or equivalently, to derive the a priori estimates of the solutions of the Cauchy problem (1.9) for the linear parabolic equations that define the nonlinear mapping $V$, namely the solution of the Cauchy problem for the parabolic equations:

\[
\begin{cases}
(\partial_t - L_{-b})w = A(b)w, \\
w(\cdot, 0) = \omega_0,
\end{cases}
\]

where, as before, $L_{-b}$ denotes the time-dependent elliptic operator $\frac{1}{2} \Delta - b \cdot \nabla$.

The crucial observation is that the term on the right-hand side, $A(b)w$, is a linear zero-order term, which differs fundamentally from the linearized Navier-Stokes equations. This crucial difference allows us to apply the Feynman-Kac-type formula, obtained in this context in [28], to derive the necessary explicit a priori estimates.

The a priori estimates and technical tools are worked out for a general dimension $d$ and a general vector field $b(x,t)$ that is divergence-free on $\mathbb{R}^d$. Therefore, they have independent interests and are likely useful for both treating the Navier-Stokes equations with other boundary conditions and dealing with other linear/nonlinear PDEs.

1.5 Organisation of the paper

In §2, several probabilistic structures associated with a time-dependent vector field $b(x,t)$ are reviewed, and then a functional integration representation formula for the heat kernel of $\frac{1}{2} \Delta - b \cdot \nabla$ and a Bismut-type formula for the gradient of the heat kernel are established, which are the tools for deriving the a priori estimates in Theorem 1.2 we need. In §3, several technical potential estimates are established, which will be used in the proof of Theorem 1.2 that will be carried out in §4–§5. In §6, the linearized vorticity equations are carefully analyzed, and the main regularity results for the strong solutions of the Cauchy problem (1.1)–(1.2) with periodic initial data (1.2) for the Navier-Stokes equations (1.1) in $\mathbb{R}^3$ will be proved in §7.
2 Probabilistic Tools for the New Approach

In this section, we first introduce several probabilistic structures associated with a time-dependent vector field \( b(x,t) \in \mathbb{R}^d \) that is divergence-free (in the distributional sense), bounded, and Borel measurable. Then we establish a functional integration representation formula and a Bismut-type formula for the heat kernel of the corresponding heterogeneous drift-diffusion operator.

2.1 Fundamental solutions and diffusions

Let \( \Gamma_b(x,t,\xi,\tau) \), for \( 0 \leq \tau < t \) and \( \xi,x \in \mathbb{R}^d \), denote the fundamental solution of the forward parabolic operator \( L_b - \partial_t \), and let \( \Gamma_b^*(x,t,\xi,\tau) \), for \( 0 \leq t < \tau \) and \( x,\xi \in \mathbb{R}^d \), denote the fundamental solution of the backward parabolic equation \( L_b^* + \partial_t \); see [12] for their definitions and basic constructions. Since \( b(x,t) \) is bounded, \( \Gamma_b(x,t,\xi,\tau) \) and \( \Gamma_b^*(x,t,\xi,\tau) \) exist and unique, and

\[
\Gamma_b(x,t,\xi,\tau) = \Gamma_b^*(\xi,t,x,t) \quad \text{for any } t > \tau \geq 0 \text{ and } x,\xi \in \mathbb{R}^d. \tag{2.1}
\]

Moreover, \( \Gamma_b(x,t,\xi,\tau) \) is positive and continuous in \( \tau < t \) and \( x,\xi \in \mathbb{R}^d \), and \( \Gamma_b^*(x,t,\xi,\tau) \) is positive and continuous in \( t < \tau \) and \( x,\xi \in \mathbb{R}^d \).

Let \( p_b(\tau,\xi,t,y) \), for \( 0 \leq \tau < t \) and \( \xi,y \in \mathbb{R}^d \), denote the transition probability density function of the \( L_b \)-diffusion (cf. [18, 37, 38]). Since \( \nabla \cdot b = 0 \),

\[
p_b(\tau,\xi,t,y) = \Gamma_b^*(\xi,\tau,y,t) = \Gamma_b(y,t,\tau,\xi) \quad \text{for } t > \tau \geq 0 \text{ and any } \xi,y \in \mathbb{R}^d. \tag{2.2}
\]

For \( T > 0 \), \( b^T(x,t) \) denotes a bounded divergence-free vector field such that \( b^T(x,t) \) coincides with \( b(x,T-t) \) for all \( 0 \leq t < T \) and \( x \in \mathbb{R}^d \). Then

\[
p_{b^T}(T-t,y,T-\tau,\xi) = \Gamma_b(y,t,\tau,\xi), \tag{2.3}
\]

\[
p_b(\tau,\xi,t,y) = p_{-b^T}(T-t,y,T-\tau,\xi), \tag{2.4}
\]

for all \( 0 \leq \tau < t < T \) and \( \xi,y \in \mathbb{R}^d \).

Let \( \Omega = C([0,\infty),\mathbb{R}^d) \) be the space of continuous paths \( \varphi : [0,\infty) \to \mathbb{R}^d \), equipped with the natural filtration denoted by \( \mathcal{F}_t^0 \) for \( t \geq 0 \). For \( \tau \geq 0 \) and \( \xi \in \mathbb{R}^d \), there is a unique probability measure \( \mathbb{P}^{\tau,\xi}_b \) on \( \Omega \) of all continuous paths such that

\[
\mathbb{P}^{\tau,\xi}_b [\varphi \in \Omega : \varphi(t) = \xi \text{ for all } 0 \leq t \leq \tau] = 1,
\]

and the marginal distribution for any finite partition \( \tau = t_0 < t_1 < \cdots < t_k \):

\[
\mathbb{P}^{\tau,\xi}_b [\varphi(t_1) \in dx_1, \cdots, \varphi(t_k) \in dx_k]
\]

is given by

\[
p_b(\tau,\xi,t_1,x_1)p_b(t_1,x_1,t_2,x_2) \cdots p_b(t_{k-1},x_{k-1},t_k,x_k)dx_1 \cdots dx_k.
\]

The collection \( \mathbb{P}^{\tau,\xi}_b \), for \( \tau \geq 0 \) and \( \xi \in \mathbb{R}^d \), is called the diffusion with infinitesimal generator \( L_b \). The construction of \( L_b \)-diffusion, or equivalently of \( p_b(s,x,t,y) \), can be based on the Cauchy problem for the stochastic differential equation (SDE):

\[
\begin{align*}
\frac{dX}{dt} &= b(X,t)dt + dB, \\
X_\tau &= \xi,
\end{align*} \tag{2.5}
\]

where \( B = (B^1, \cdots, B^d) \) is a Brownian motion on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) (cf. [18, 37] for the details). If \( b(x,t) \) is jointly continuous and global Lipschitz continuous in \( x \) (uniformly in \( t \)), then the Cauchy problem (2.5) has a unique strong solution, whose distribution gives rise to the probability measure \( \mathbb{P}^{\tau,\xi}_b \).
2.2 Diffusion bridges and a Feynman-Kac-type formula

Let $\tau \geq 0$, $T > \tau$, and $\xi, \eta \in \mathbb{R}^d$ be fixed. The conditional law $\mathbb{P}_{\text{b}}^{\xi, \tau \to \eta, T}$, also called the pinned measure or $L_b$-diffusion bridge measure, is formally defined to be $\mathbb{P}_{\text{b}}^{\xi, \tau \to \eta, T} \{ \cdot \mid w(T) = \eta \}$, which is again Markovian. According to (14.1) in [8], $\mathbb{P}_{\text{b}}^{\xi, \tau \to \eta, T}$ is the unique probability measure on $\Omega$ with time non-homogeneous transition probability density function:

$$q_b(s,x,t,y) = \frac{p_b(s,x,t,y)p_b(t,y,T,\eta)}{p_b(s,x,T,\eta)} \quad \text{for } \tau < s < t \text{ and } x,y \in \mathbb{R}^d. \quad (2.6)$$

It can be shown (cf. [29]) that

$$\frac{d\mathbb{P}_{\text{b}}^{\xi, \tau \to \eta, T}}{d\mathbb{P}_{\text{b}}^{\xi, \tau \to \eta, T}}_{\mathcal{F}_T} = \frac{p_b(t,\varphi(t),T,\eta)}{p_b(\tau,\xi,T,\eta)} \quad \text{for } t \in [\tau, T), \quad (2.7)$$

where $\varphi(t)$ denotes the general sample point and the canonical process on $\Omega$.

**Theorem 2.1.** The pinned measures satisfy the following duality relation:

$$\mathbb{P}_{\text{b}}^{\eta, 0 \to \xi, T} = \mathbb{P}_{\text{b}}^{\xi, 0 \to \eta, T} \circ \tau_T,$$

where $\tau_T$ is the time reversal operator at $T$, that is, $\tau_T : \Omega \to \Omega$ which sends $w$ to $\tau_T w(t) = w(T-t)$ for $t \in [0, T]$.

Therefore, if $\{X_t\}$ is an $L_b$-diffusion and $\{Y_t\}$ is an $L_{-b'}$-diffusion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then

$$\mathbb{P} [F(X_t) \mid X_0 = \xi, X_T = \zeta] = \mathbb{P} [F(Y_T) \mid Y_0 = \zeta, Y_T = \eta]$$

for any bounded or positive Borel measurable function $F$ on $\Omega$.

The following is a Feynman-Kac-type formula that was established in [28] in this version:

**Theorem 2.2.** Suppose that $f(x,t)$ is a strong solution to the parabolic equations:

$$\left( \partial_t - L_{-b(x,t)} \right) f^i(x,t) = A^i_j(x,t) f^j(x,t) \quad \text{in } \mathbb{R}^d \times (0, \infty), \quad (2.8)$$

subject to the initial condition: $f(x,0) = f_0(x)$, and $A^i_j(x,t)$ are joint continuous in $(x,t)$, Lipschitz continuous in $x$ (uniformly in $t$ in any finite interval, where $i, j = 1, \ldots, d$). Then

$$f(x,t) = \int_{\mathbb{R}^d} f_0(\xi) p_{\text{b}}(0, \xi, t, x) \mathbb{P}_{\text{b}}^{\xi, 0 \to \eta, t} \left[ Q(0,t) \right] d\xi, \quad (2.9)$$

where, for every $t > 0$, $Q(s,t,\varphi) = \{Q^i_j(s,t,\varphi)\}$ (but the sample point $\varphi \in \Omega$ will be suppressed if no confusion arises) is the unique solution of the Cauchy problem for the differential equations:

$$\begin{cases}
\frac{\partial}{\partial s} Q^i_j(s,t,\varphi) = -Q^i_k(s,t,\varphi) A^k_j(\varphi(s),s), \\
Q^i_j(t,t,\varphi) = \delta^i_j, & \text{for } s \leq t,
\end{cases}$$

for $i, j = 1, \ldots, d$, and $\varphi \in \Omega$.

2.3 Functional integral representation for fundamental solutions

Let $B$ be a standard Brownian motion of dimension $d$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For $\tau \geq 0$ and $\xi \in \mathbb{R}^d$,

$$X^\tau_\xi = \begin{cases}
B_t - B_\tau + \xi & \text{for } t \geq \tau, \\
\xi & \text{for } t \leq \tau.
\end{cases}$$

Then $X^\tau_\xi$ is a Brownian motion stated at $\xi$ at the initial time $\tau$. According to Theorem 6.4.2 in [37], the $L_b$-diffusion may also be constructed by using the Cameron-Martin density:

$$U_{\text{b}}^{\xi, \tau}(t) = \exp \left[ \int_{t \land \tau}^t b(X^\tau_\xi, s) \cdot dB_s - \frac{1}{2} \int_{t \land \tau}^t |b|^2(X^\tau_\xi, s) \, ds \right]. \quad (2.10)$$
Theorem 2.3. The following representation holds for all \( t > \tau \):

\[
p_b(\tau, \xi, t, y) = G_{t-\tau}(y - \xi) + \int_\tau^t \mathbb{P} \left[ U_b^{T, \xi}(s) G_{t-s} (y - X_s^{T, \xi}) b(X_s^{T, \xi}, s) \cdot \frac{y - X_s^{T, \xi}}{t-s} \right] ds.
\] (2.11)

Proof. The theorem was established in [29] for the time homogeneous case, and their method can be developed to be applied to our case. For the completeness, we outline the proof here. According to the Cameron-Martin formula,

\[
\int_{\mathbb{R}^d} p_b(\tau, \xi, t, x) f(x) \, dx = \mathbb{P} \left[ U_b^{T, \xi}(t) f(X_t^{T, \xi}) \right] = \int_{\mathbb{R}^d} \mathbb{P} \left[ U_b^{T, \xi}(t) f(x) \right] |X_t^{T, \xi} = x| \mathbb{P} \left[ X_t^{T, \xi} \in dx \right] = \int_{\mathbb{R}^d} \mathbb{P} \left[ U_b^{T, \xi}(t) |X_t^{T, \xi} = x\right] G_{t-\tau}(x - \xi) f(x) \, dx.
\]

Choose \( f(z) = \delta_z(\,dz\, \) to obtain

\[
\frac{p_b(\tau, \xi, t, y)}{G_{t-\tau}(y - \xi)} = \mathbb{P} \left[ U_b^{T, \xi}(t) |X_t^{T, \xi} = y\right].
\] (2.12)

Now we notice that both

\[
R(s) = \frac{G_{t-s}(y - X_s^{T, \xi})}{G_{t-\tau}(y - \xi)}
\]

(see (2.7)) and \( U_b^{T, \xi}(s) \) are exponential martingales so that

\[
dR(s) = R(s) \nabla \ln G_{t-s}(y - X_s^{T, \xi}) \cdot dB(s), \quad R(\tau) = 1,
\]

\[
dU_b^{T, \xi}(s) = U_b^{T, \xi}(s) b(X_s^{T, \xi}, s) \cdot dB(s), \quad U_b^{T, \xi}(\tau) = 1.
\]

Therefore, integrating by parts, together with (2.7), yields

\[
\frac{p_b(\tau, \xi, t, y)}{G_{t-\tau}(y - \xi)} = \lim_{s \uparrow t} \mathbb{P} \left[ R(s) U_b^{T, \xi}(s) \right] = 1 + \mathbb{P} \left[ (R, U_b^{T, \xi}) \right] = 1 + \frac{1}{G_{t-\tau}(y - \xi)} \mathbb{E} \left[ \int_\tau^t U_b^{T, \xi}(s) b(X_s^{T, \xi}, s) \cdot \nabla G_{t-s}(y - X_s^{T, \xi}) \, ds \right],
\]

which leads to the representation formula in (2.11). \( \square \)

2.4 A Bismut-type formula

From now on, we make a further assumption that \( b(x, t) \) is a time-dependent, bounded, and divergence-free \( C^1 \)-vector field.

Using the idea of Bismut (cf. Chapter 14, §14.1 in Bismut-Lebeau [4], and also cf. Elworthy-Li [9]), we first establish a Bismut-type formula for the gradient of the fundamental solution:

\[
\partial_{\tau} \ln p_b(\tau, \xi, t, x).
\]

Let \( \tau \geq 0 \) and \( \xi \in \mathbb{R}^d \) be fixed. Bismut’s idea is based on the following observation: Since \( b \) is divergence-free, as a function of \( t > \tau \) and \( x \in \mathbb{R}^d \), \( p_b(\tau, \xi, t, x) \) solves the forward parabolic equation \( (L_b - \partial_t)p_b = 0 \), so that

\[
(L_b - \partial_t) \ln p_b = -\frac{1}{2} |\nabla \ln p_b|^2 \quad \text{on} \ \mathbb{R}^d \times (\tau, \infty).
\] (2.13)
Let $T > 0$. Consider $f(x,t) = \ln p_b(\tau, \xi, \tau + T - t, x)$ for $0 \leq t < T$. Then

$$\left(L_{-b} + \partial T\right)f = -\frac{1}{2} |\nabla f|^2,$$

(2.14)

where both sides are evaluated at $(\tau, \xi, \tau + T - t, x)$. Now we consider the following Cauchy problem of the stochastic differential equations:

$$\begin{cases}
\ud Y_t^x = db(Y_t^x, T + \tau - t) \ud r,

Y_0^x = x,
\end{cases}$$

(2.15)

which determines a diffusion process with generator $L_{-b} + \partial T$, where $B$ is a standard $d$-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, whose filtration is denoted by $(\mathcal{F}_t^0)_{t \geq 0}$. Then (2.14) implies that

$$R_t^\xi = p_b(\tau, \xi, T + \tau - t, Y_t^x) = e^{f(Y_t^x, t)} - f(0, 0)$$

for $t \in [0, T)$

(2.16)

is a positive martingale and

$$\begin{cases}
\ud R_t^\xi = R_t^\xi \nabla f(Y_t^x, t) \cdot \ud B_t,

R_0^\xi = 1.
\end{cases}$$

(2.17)

Finally, we set $Z_j^k(x,t) = \partial_t Y_t^{x,k}$ for $j, k = 1, \cdots, d$.

**Theorem 2.4** (Bismut-Type Formula for the Forward Variable). Suppose that $b(x,t)$ is bounded and $C^1$ with bounded derivative over any finite interval so that $\nabla \cdot b = 0$. Then

$$\partial_t \ln p_b(\tau, \xi, T + \tau, x) = \mathbb{Q} \left[ \int_0^T \frac{\rho(t)}{\rho(T)} Z_j^k(x,t) \ud B_t^k \right],$$

(2.18)

where $\mathbb{Q}$ is the probability measure on $(\Omega, \mathcal{F}_T^0)$ such that

$$\left. \frac{\ud \mathbb{Q}}{\ud \mathbb{P}} \right|_{\mathcal{F}_T^0} = R_T^\xi$$

for $t \in [0, T)$,

and $\rho(t)$ is any continuous and piecewise differentiable function with $\rho(0) = 0$ and $\rho(t) > 0$ for $t > 0$.

**Proof.** Since $t \to R_t^\xi$ is a martingale so that

$$p_b(\tau, \xi, T + \tau, x) = \mathbb{P} \left[ p_b(\tau, \xi, T + \tau - t, Y_t^x) \right],$$

where $\mathbb{P}$ (similarly for $\mathbb{Q}$) also means taking expectation with respect to $\mathbb{P}$ (resp. $\mathbb{Q}$). By using the previous fact,

$$\partial_t \ln p_b(\tau, \xi, T + \tau, x) = \left[ \frac{\partial_t p_b(\tau, \xi, T + \tau, x)}{p_b(\tau, \xi, T + \tau, x)} \right] p_b(\tau, \xi, T + \tau, x)$$

$$= \mathbb{P} \left[ \partial_t p_b(\tau, \xi, T + \tau - t, Y_t^x) \right]$$

$$= \mathbb{P} \left[ \frac{\partial_b p_b(\tau, \xi, T + \tau - t, Y_t^x)}{p_b(\tau, \xi, T + \tau, x)} \partial_\tau \ln p_b(\tau, \xi, T + \tau - t, Y_t^x) \right]$$

$$= \mathbb{P} \left[ R_t^\xi Z_j^k(x,t) \partial_\tau \ln p_b(\tau, \xi, T + \tau - t, Y_t^x) \right]$$

for all $0 < t < T$,

where $Z_j^k(x,t) = \partial_t Y_t^{x,k}$ for $j, k = 1, \cdots, d$. Let

$$M_t^j = \int_0^t \rho(s) Z_j^k(s) \ud B_t^k$$

for $0 < t < T$. 

\[ 
\]
Observe that

$$\langle R, M^j \rangle_t = \int_0^t \rho'(s)Z^j(s)R_s \partial_\ell \ln p_b(\tau, \xi, T + \tau - s, Y_s) \, ds.$$ 

Thus we have

$$\partial_\ell \ln p_b(\tau, \xi, T + \tau, x) = \frac{1}{\rho(T)} \mathbb{E} \left[ \int_0^T \rho'(s)Z^j(s)R_s \partial_\ell \ln p_b(\tau, \xi, T + \tau - s, Y_s) \, ds \right]$$

$$= \frac{1}{\rho(T)} \mathbb{E} \left[ \langle R^k, M^j \rangle_T \right]$$

$$= \frac{1}{\rho(T)} \mathbb{E} \left[ R^k_T \int_0^T \rho'(t)Z^j(t) \, dB^k_t \right].$$

This completes the proof. □

3 Basic Estimates for the Heat Kernel

In this section, we establish several estimates for the heat kernel. In particular, Lemma 3.3 contains the technical estimates needed for the proof of Theorem 4.1 below.

Recall that

$$G_t(x) = (2\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{2t}}$$

for $x \in \mathbb{R}^d$ and $t > 0$. Clearly,

$$\nabla_x \ln G_{t-s}(y-x) = \frac{y-x}{t-s} \quad \text{for } t > s \geq 0,$$

which leads to the following equality:

$$|\sqrt{t-s} \nabla_x G_{t-s}(y-x)| = \frac{|y-x|}{\sqrt{t-s}} G_{t-s}(y-x). \quad (3.1)$$

Now we notice that, for $\beta > 0$,

$$G_{t-s}(y-x) = (2\pi t)^{-\frac{d}{2}} e^{-\frac{1}{2} \frac{y-x^2}{t}} G_{\beta(t-s)}(y-x), \quad (3.2)$$

so that, for $\beta > 1$,

$$\frac{|y-x|}{\sqrt{t-s}} G_{t-s}(y-x) = (2\pi t)^{-\frac{d}{2}} e^{-\frac{1}{2} \frac{y-x^2}{t}} G_{\beta(t-s)}(y-x)$$

$$= (2\pi t)^{-\frac{d}{2}} \frac{|y-x|}{\sqrt{t-s} \left( 1 + \frac{1}{\beta} \frac{y-x^2}{t} \right) + \cdots} G_{\beta(t-s)}(y-x)$$

$$\leq \frac{(2\pi t)^{-\frac{d}{2}}}{\sqrt{2(\beta-1)}} G_{\beta(t-s)}(y-x).$$

Therefore, using (3.2) and the same argument, we have the following lemma which will be used in our computations later.

**Lemma 3.1.** For the heat kernel in $\mathbb{R}^d$, the following estimates hold:

(i) For $\beta > 1$,

$$\frac{|y-x|}{\sqrt{t-s}} G_{t-s}(y-x) \leq \frac{(2\pi t)^{-\frac{d}{2}}}{\sqrt{2(\beta-1)}} G_{\beta(t-s)}(y-x) \quad \text{for all } t > s \geq 0 \text{ and } x, y \in \mathbb{R}^d. \quad (3.4)$$

(ii) For $\alpha \geq 0$ and $\beta > 1$,

$$\left( \frac{|y-x|}{\sqrt{t-s}} \right)^\alpha G_{t-s}(y-x) \leq \frac{(2\pi t)^{-\frac{d}{2}}}{\sqrt{2(\beta-1)}} \left( \frac{2\beta}{\beta-1} \right)^k G_{\beta(t-s)}(y-x) \quad \text{for all } t > s \geq 0 \text{ and } x, y \in \mathbb{R}^d, \quad (3.5)$$

where $k = \left[ \frac{\alpha}{2} \right] + 1$ if $\frac{\alpha}{2}$ is not an integer, and $k = \frac{\alpha}{2}$ otherwise.
Lemma 3.2. Let \( B \) be an \( \mathbb{R}^d \)-Brownian motion on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Consider
\[
I_{\alpha, \beta}(\tau, x, s,t,y) = \mathbb{P} \left[ \left( \frac{y - X_t}{t-s} \right)^\alpha G_{t-s}(y - X_t) \right] \text{ for } t > s > \tau \text{ and } x, y \in \mathbb{R}^d,
\]
where \( \alpha \geq 0, \beta > 0 \) and \( X_t = B_t - B_\tau + x \). Then
\[
I_{\alpha, \beta}(\tau, x, s,t,y) = C_1 t_1^{-\frac{d(\beta-1)}{2} + \alpha \beta} G_{t_1 + t_2}(y - x) \int_{\mathbb{R}^d} \left| \sqrt{\frac{t_1 t_2}{t_1 + t_2} z + \frac{t_1(x-y)}{t_1 + t_2}} \right|^\alpha G_1(z) \, dz,
\]
where
\[
t_1 = \beta^{-1}(t-s), \quad t_2 = s - \tau,
\]
and
\[
C_1 = (2\pi)^{-\frac{d(\beta-1)}{2}} \beta^{-\frac{(\beta+\alpha)\beta}{2}}
\]
depend only on \( \beta > 0 \) and \( d \).

Proof. The proof follows from an elementary computation. First notice that
\[
G_{t_1}(z - y) G_{t_2}(z - x) = G_{\frac{t_1 + t_2}{t_1 t_2}}(z - \frac{t_1 x + t_2 y}{t_1 + t_2}) G_{t_1 + t_2}(y - x) \text{ for } t_1 > 0 \text{ and } t_2 > 0.
\]
Now, for \( s > \tau \), the law of \( X_s \) is normal with PDF \( G_{s-\tau}(z - x) \), and
\[
(G_{t-s}(y - z))^\beta = \beta^{-\frac{(\beta+\alpha)\beta}{2}} (2\pi)^{-\frac{d(\beta-1)}{2}} t_1^{-\frac{d(\beta-1)}{2} + \alpha \beta} G_{t_1}(z - y) G_{t_2}(z - x)
\]
where \( t_1 = \beta^{-1}(t-s) \), so that
\[
I_{\alpha, \beta}(\tau, x, s,t,y) = \beta^{-\frac{(\beta+\alpha)\beta}{2}} (2\pi)^{-\frac{d(\beta-1)}{2}} t_1^{-\frac{d(\beta-1)}{2} + \alpha \beta} \int_{\mathbb{R}^d} |z-y|^\alpha G_{t_1}(z - y) G_{t_2}(z - x) \, dz.
\]
Then
\[
I_{\alpha, \beta}(\tau, x, s,t,y) = \frac{G_{t_1 + t_2}(y - x)}{(2\pi)^{\frac{d(\beta-1)}{2}} \beta^{\frac{(\beta+\alpha)\beta}{2}} t_1^{\frac{d(\beta-1)}{2} + \alpha \beta}} \int_{\mathbb{R}^d} |z-y|^\alpha G_{\frac{t_1 + t_2}{t_1 t_2}}(z - \frac{t_1 x + t_2 y}{t_1 + t_2}) \, dz
\]
\[
= \frac{G_{t_1 + t_2}(y - x)}{(2\pi)^{\frac{d(\beta-1)}{2}} \beta^{\frac{(\beta+\alpha)\beta}{2}} t_1^{\frac{d(\beta-1)}{2} + \alpha \beta}} \int_{\mathbb{R}^d} \left| \sqrt{\frac{t_1 t_2}{t_1 + t_2} z + \frac{t_1(x-y)}{t_1 + t_2}} \right|^\alpha G_1(z) \, dz,
\]
where the second step follows by changing the variable in the integral, which leads to (3.7).

Lemma 3.3. Let \( \alpha \geq 0 \) and \( \beta \geq 1 \).

(i) For \( t > s > \tau \geq 0 \),
\[
\sqrt{I_{\alpha, \beta}(\tau, x, s,t,y)} \leq C_3 (t - \tau)^{\frac{d(\beta-1)}{2} + \frac{\alpha}{2}} (t-s)^{-\frac{d(\beta-1)}{2}} \left[ \kappa_1 \left( \frac{s - \tau}{t - \tau} \right)^{\frac{\alpha}{2}} + \frac{|x-y|}{\sqrt{t-\tau}} \right] G_{\beta(t-\tau)}(y - x),
\]
where
\[
\kappa_1 = \sqrt{\int_{\mathbb{R}^d} |z|^\alpha G_1(z) \, dz}, \quad C_3 = \beta^{\frac{2d+\alpha}{2} + d(\beta-1)} \frac{\alpha}{2} (2\pi)^{\frac{d(\beta-1)}{2} + \frac{\alpha}{2}} C_1^{\frac{1}{\beta}},
\]
and \( C_1 \) given in (3.9) depends only on \( \alpha \geq 0, \beta \geq 1, \) and \( d \).
(ii) If $\alpha \geq 0$ and $\beta \geq 1$ such that $\frac{d(\beta - 1)}{2p} + \frac{\alpha}{2} < 1$, then

$$\int_t^\infty \sqrt[p]{I_{\alpha,\beta}(\tau, x, s, t, y)} ds \leq C_4(t - \tau)^{1 - \frac{\alpha}{2}} G_{\beta(t_1 + t_2)}(y - x) \left(1 + \frac{|x - y|^\alpha}{(t - \tau)^\frac{\alpha}{2}}\right) \quad \text{for } t > \tau \geq 0,$$

where $C_4 = \max(\kappa_3, \kappa_0) C_3$ depends only on $\beta, \alpha$, and $d$ (note that $\kappa_3, \kappa_0$ and $C_3$ can be traced in the proof below).

**Proof.** By (3.7) and the triangle inequality, we obtain

$$\sqrt[p]{I_{\alpha,\beta}(\tau, x, s, t, y)} = C_1 t_1^{-\frac{d(\beta - 1)}{2p} + \alpha} \left(\int_{\mathbb{R}^d} \left| \frac{t_1 t_2}{t_1 + t_2} y + t_1(x - y) + t_1 \frac{\alpha}{2} G_1(z) dz \right|^\frac{p}{p - 1} G_{\beta(t_1 + t_2)}(y - x) \right)^{\frac{p}{p - 1}} \leq C_1 t_1^{-\frac{d(\beta - 1)}{2p} + \alpha} \left(\kappa_1 \left(\frac{t_1 t_2}{t_1 + t_2}\right)^\frac{\alpha}{2} + \left(t_1 \frac{|x - y|^\alpha}{t_1 + t_2}\right)\right) \frac{\sqrt[p]{G_{\beta(t_1 + t_2)}(y - x)}}{G_{\beta(t_1 + t_2)}(y - x)}.$$  

Using the identity

$$\sqrt[p]{G_{\beta(t_1 + t_2)}(y - x)} = \beta^\frac{d}{2}(2\pi)^{-\frac{d(\beta - 1)}{2p}} \left(\int_{\mathbb{R}^d} \left(1 + \frac{|x - y|^\alpha}{(t_1 + t_2)^\frac{\alpha}{2}}\right) \right)^{\frac{p}{p - 1}} G_{\beta(t_1 + t_2)}(y - x),$$

and substituting this into the previous inequality, we therefore obtain

$$\sqrt[p]{I_{\alpha,\beta}(\tau, x, s, t, y)} \leq C_2 (t_1 + t_2)^{-\frac{d(\beta - 1)}{2p}} \left(\beta^\frac{d}{2}(2\pi)^{-\frac{d(\beta - 1)}{2p}} G_{\beta(t_1 + t_2)}(y - x)\right) \quad \text{for } t > \tau \geq 0,$$

where

$$C_2 = \beta^\frac{d}{2}(2\pi)^{-\frac{d(\beta - 1)}{2p}} C_1^{\frac{p}{p - 1}}.$$  

Since

$$t_1 = \beta^{-1}(t - s), \quad t_2 = s - \tau,$$

$$t - \tau \geq t_1 + t_2 = \beta^{-1}(t - s) + s - \tau \geq \beta^{-1}(t - \tau),$$

and

$$G_{\beta(t_1 + t_2)}(y - x) \leq \beta^\frac{d}{2} G_{\beta(t - \tau)}(y - x),$$

substituting these relations into the previous inequality, we thus deduce

$$\sqrt[p]{I_{\alpha,\beta}(\tau, x, s, t, y)} \leq C_3 (t - \tau)^{-\frac{d(\beta - 1)}{2p} - \frac{\alpha}{2}} \left(\beta^\frac{d}{2}(2\pi)^{-\frac{d(\beta - 1)}{2p}} G_{\beta(t - \tau)}(y - x)\right) \quad \text{for } t > \tau \geq 0,$$

where

$$C_3 = C_2 \beta^\frac{2p - \alpha}{2}(2\pi)^{-\frac{d(\beta - 1)}{2p}} C_1^{\frac{p}{p - 1}}.$$  

which implies (i).

To prove (ii), we observe that the integral of the first term in the bracket in (3.15) can be written as

$$\kappa_1 \int_\tau^t \frac{\beta^\frac{d}{2}(2\pi)^{-\frac{d(\beta - 1)}{2p}}}{(t - s)^{\frac{d(\beta - 1)}{2p} + \frac{\alpha}{2}}} ds = \kappa_1 \int_\tau^t \frac{\beta^\frac{d}{2}(2\pi)^{-\frac{d(\beta - 1)}{2p}}}{(t - \tau - (t - s))^{\frac{d(\beta - 1)}{2p} + \frac{\alpha}{2}}} ds = \kappa_0 (t - \tau)^{-\frac{d(\beta - 1)}{2p}} \quad \text{for } t > \tau \geq 0,$$

where

$$\kappa_0 = \kappa_1 \int_0^1 (1 - s)^{\frac{d}{2}(2\pi)^{-\frac{d(\beta - 1)}{2p}} - \frac{\alpha}{2}} ds.$$
which is finite when
\[
\frac{d\beta - d}{2\beta} + \frac{\alpha}{2} < 1, \quad \frac{\alpha}{2} > -1.
\]
Similarly, the second integral can be written as
\[
\int_\tau^t (t - s)^{-\beta} ds = \kappa_3(t - \tau)^{1 - \frac{d(\beta - 1)}{2\beta}},
\]
where \(\kappa_3 = \int_0^1 s^{-\beta} ds\) is finite when \(\frac{d(\beta - 1)}{2\beta} < 1\). Then
\[
\int_\tau^t \sqrt{I_{1,\beta}(\tau, x, s, t, y)} ds \leq C_3(t - \tau)^{1 - \frac{\beta}{2}} G_{\beta(t - \tau)}(y - x) \left( \kappa_0 + \kappa_3 |x - y| \alpha \right)
\]
and (3.13) follows immediately.

\[ \square \]

4 Explicit Estimates for the Fundamental Solution

We retain the basic assumption on the vector field \(b(x, t)\) that is bounded and Borel measurable. Notice that the divergence-free assumption on \(b\) in the following theorem, Theorem 4.1, is not needed.

**Theorem 4.1.** For every \(\beta > 1\), there are constants \(C_1\) and \(C_2\) depending only on \(\beta\) and the dimension \(d\) such that
\[
p_b(\tau, x, t, y) \leq C_1 e^{C_2(t - \tau)\|b\|_Y^2} G_{\beta(t - \tau)}(y - x) \quad \text{for } t > \tau \geq 0 \text{ and } x, y \in \mathbb{R}^d.
\]

**Proof.** According to Theorem 2.3,
\[
p_b(\tau, x, t, y) = G_{t - \tau}(y - x) + \mathbb{E} \left[ \int_\tau^t U_s b(X_s, s) \frac{y - X_s}{t - s} G_{t - s}(y - X_s) ds \right],
\]
where \(X_s = B_t - B_\tau + x\), \(B\) is the standard Brownian motion of dimension \(d\), and \(U\) is the Cameron-Martin density of \(L_b\)-diffusion with respect to the Brownian motion. Consider the second term on the right-hand side:
\[
J = \mathbb{E} \left[ \int_\tau^t U_s b(X_s, s) \cdot \frac{y - X_s}{t - s} G_{t - s}(y - X_s) ds \right].
\]
By the Hölder inequality, we have
\[
|J| \leq \|b\|_\infty \int_\tau^t \sqrt{\mathbb{P} [U_s]} \sqrt{I_{1,\beta}(\tau, x, s, t, y)} ds,
\]
where \(\frac{1}{\gamma} + \frac{1}{\beta} = 1\) and
\[
U_s = \exp \left[ \int_{\tau/s}^s b(X_r, r) \cdot dB_r - \frac{1}{2} \int_{\tau/s}^s |b|^2(X_r, r) dr \right].
\]
Since \(U\) is an exponential martingale, then
\[
\mathbb{P} [U_s] \leq e^{\frac{1}{\beta} \gamma (\gamma - 1) \|b\|_Y^2 (s - \tau)},
\]
so that
\[
|J| \leq \|b\|_\infty e^{\frac{1}{\beta} \gamma (\gamma - 1) \|b\|_Y^2 (t - \tau)} \int_\tau^t \sqrt{I_{1,\beta}(\tau, x, s, t, y)} ds,
\]
where
\[
I_{1,\beta}(\tau, x, s, t, y) = \mathbb{E} \left[ \left( \frac{y - X_s}{t - s} h(s, X_s, t, y) \right)^\beta \right] \quad \text{(4.2)}
\]
and \( X_s = B_s - B_{\tau} + x \). Thus, by Lemma 3.3,
\[
|J| \leq C_3 \|b\|_{2,\infty} e^{\frac{1}{2} (\gamma - 1) \|b\|_{2,\infty}^2 (t-\tau)} (\sqrt{1-\tau} + |y-x|) \gamma_{\beta(t_1+\tau)}(y-x),
\]
where \( C_3 > 1 \) depends only on \( d \) and \( \beta \), and
\[
t_1 = \beta^{-1}(t-s), \quad t_2 = s - \tau.
\]
The estimate follows from the following inequality: For every \( \gamma > 1 \), there are constants \( C_5 \) and \( C_6 \) depending only on \( \gamma, \beta, \) and \( d \) such that
\[
\sqrt{\tau} \|b\|_{2,\infty} e^{\frac{1}{2} (\gamma - 1) \|b\|_{2,\infty}^2 \tau} (1 + \frac{|\gamma|}{\sqrt{\tau}}) \gamma_{\beta\tau}(z) \leq C_5 e^{C_6 \|b\|_{2,\infty}^2} \gamma_{\beta\tau}(z).
\]
\[\]
As a consequence, we may deduce the following estimate that will be used in the proof of the gradient estimate for \( p_{b}(s,x,t,y) \).

**Lemma 4.2.** Let \( b \) be a bounded time-dependent vector field with bound \( \|b\|_{\infty} \), let \( x, \xi \in \mathbb{R}^d \) and \( t > 0 \) be fixed, and let \( \gamma \geq 1 \). Consider the following integral:
\[
J(\varepsilon) = \sqrt{\int_{\mathbb{R}^d} (p_{b}(\tau, \xi, \tau + \varepsilon, z))^{\gamma} p_{-b}(0, x, t - \varepsilon, z) \, dz}.
\]
Then, for every \( \beta > \gamma \), there are \( C_1 \) and \( C_2 \) depending only on \( \beta, \gamma \), and the dimension \( d \) such that
\[
J(\varepsilon) \leq C_1 e^{C_2 \|b\|_{2,\infty}^2 \left( \frac{d}{\varepsilon} \right)^{d/2 - 1}} \gamma_{\beta\tau}(\xi - x) \quad \text{for any } 0 < \varepsilon < t \text{ and } x, \xi \in \mathbb{R}^d.
\]

**Proof.** Without loss of generality, we may assume that \( \tau = 0 \). Let \( \lambda_1 \geq 1, \lambda_2 \geq 1 \), and
\[
t_1 = \gamma^{-1} \lambda_1 \varepsilon, \quad t_2 = \lambda_2 (t - \varepsilon).
\]
Then
\[
\frac{|z - \xi|^2}{t_1} + \frac{|z - x|^2}{t_2} = \frac{|z - \left( \frac{t_1}{t_1 + t_2} \xi + \frac{t_2}{t_1 + t_2} x \right)|^2}{t_1 + t_2} + \frac{|\xi - x|^2}{t_1 + t_2},
\]
so that the product
\[
(G_{\lambda_1 \varepsilon}(z - \xi))^\gamma G_{\lambda_2 (t - \varepsilon)}(z - x)
\]
equals
\[
\left( \frac{t_1 + t_2}{t_1} \right)^{d/2 - 1} \gamma_{\beta\tau} \left( G_{\gamma(t_1 + t_2)}(\xi - x) \right)^\gamma G_{\beta \tau}(z - \xi) - \frac{t_2 \xi}{t_1 + t_2} - \frac{t_1 x}{t_1 + t_2}.
\]
Let \( \beta > 1 \) and \( \gamma \geq 1 \) be fixed. Then, by (4.1),
\[
p_{b}(\tau, \xi, \tau + \varepsilon, z) \leq C_1 e^{C_2 \|b\|_{2,\infty}^2} \gamma_{\beta \tau}(z - \xi)
\]
and
\[
p_{-b}(0, x, t - \varepsilon, z) \leq C_1 e^{C_2 (t - \varepsilon)} \|b\|_{2,\infty}^2 \gamma_{\beta \tau}(z - x),
\]
so that
\[
E \equiv (p_{b}(\tau, \xi, \tau + \varepsilon, z))^{\gamma} p_{-b}(0, x, t - \varepsilon, z)
\]
\[
\leq C_1^{\gamma + 1} e^{C_2 (\gamma + t - \varepsilon)} \|b\|_{2,\infty}^2 \left( \gamma_{\beta \tau}(z - \xi) \right)^\gamma G_{\beta \tau}(z - x)
\]
\[
= C_1^{\gamma + 1} e^{C_2 (\gamma + t - \varepsilon)} \|b\|_{2,\infty}^2 \left( \frac{t_1 + t_2}{t_1} \right)^{d/2 - 1} \gamma_{\beta \tau}(z - x).
\]
\[
\times \left(G_{\gamma(t_1+t_2)}(\xi-x)\right)^\gamma G_{\frac{\alpha_2}{\alpha_1}t_2} \left(\frac{t_1 \xi}{t_1 + t_2} - \frac{t_1 x}{t_1 + t_2}\right),
\]

where
\[
t_1 = \gamma^{-1} \beta \epsilon, \quad t_2 = \beta (t - \epsilon).
\]

Thus, we have
\[
J(\epsilon) \leq C_{\gamma}^{1+\frac{1}{\gamma}} e^{\frac{C_{\gamma}}{2}} \|b\|^2 \left(\frac{t_1 + t_2}{t_1}\right)^{\frac{\gamma}{2}} G_{\gamma(t_1+t_2)}(\xi-x).
\]

Since
\[
\gamma^{-1} \beta t \leq t_1 + t_2 = \gamma^{-1} \beta \epsilon + \beta (t - \epsilon) \leq \beta t,
\]
then
\[
J(\epsilon) \leq \gamma^{\frac{\gamma}{2}} C_{\gamma}^{1+\frac{1}{\gamma}} e^{\frac{C_{\gamma}}{2}} \|b\|^2 \left(\frac{1}{\epsilon}\right)^{\frac{\gamma}{2}} G_{\gamma\beta}(\xi-x),
\]

which yields the required estimate. \qed

5 An Explicit Gradient Estimate for the Fundamental Solution

In this section, we assume that the vector field \( b(x, t) \) is smooth, divergence-free, and bounded, and its derivative is also bounded. Under these assumptions, the \( L_b \)-diffusion may be constructed by solving Itô’s stochastic differential equations. The goal of this section is to establish an explicit gradient estimate for \( p_b(\tau, \xi, t + \tau, x) \) with respect to \( x \). Recall that, for the heat kernel on \( \mathbb{R}^d \),

\[
|\nabla_x G_t(x - \xi)| \leq \frac{\beta^{\frac{d+1}{2}}}{\sqrt{2(\beta - 1)}} \frac{1}{\sqrt{t}} G_{\beta t}(x - \xi) \quad \text{for any } t > 0,
\]

where \( \beta > 1 \) is any constant. We aim to achieve a similar bound for \( p_b(\tau, \xi, t + \tau, x) \). To this end, we first prove the following estimate.

**Lemma 5.1.** Let \( T > \tau \geq 0 \) be fixed, and let \( Y \) be the strong solution of the Cauchy problem of the stochastic differential equations:

\[
\begin{cases}
\frac{dY^i}{dt} = dB^i - b^i(Y, T + \tau - t)dt, \\
Y^i(0) = x^i,
\end{cases}
\]

for \( i = 1, \ldots, d \), \( (5.2) \)

where \( B = (B^1_t, B^2_t, \ldots, B^d_t) \) is the standard Brownian motion of dimension \( d \) on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \). Then \( Z^i = \partial_{Y^i} Y, i, j = 1, \ldots, n \), form the solution of the Cauchy problem of the differential equations:

\[
\begin{cases}
\frac{dZ^i}{dt} = -Z_j^i \frac{\partial b^j}{\partial Y^i}(Y, T + \tau - t)dt, \\
Z^i(0) = \delta^i_j,
\end{cases}
\]

and satisfy

\[
|Z(t)| \leq |Z(0)| e^{2(\sqrt{T - \tau} - \sqrt{T - t})\|\nabla b\|_{\tau + \tau - t}} \quad \text{for } t \in [0, T].
\]

**Proof.** From \((5.3)\), we deduce

\[
\begin{cases}
\frac{d|Z|^2}{dt} = -2Z^i Z_j^i \frac{\partial b^i}{\partial Y^j}(Y, T + \tau - t)dt, \\
Z^i(0) = \delta^i_j.
\end{cases}
\]

By definition, we have

\[
|\sqrt{s - \tau} \nabla b(x, s)| \leq \|\nabla b\|_{\tau + \tau + T} \quad \text{for } s \in [\tau, \tau + T],
\]

\[
|\frac{dV}{dt}| \leq \frac{1}{\sqrt{2(\beta - 1)}} V \quad \text{for } t > 0,
\]

where
\[
V(t) = \frac{1}{\sqrt{t}} G_{\beta t}(\xi - x).
\]

Hence, we have

\[
|\nabla_x \left(\frac{1}{\sqrt{t}} G_{\beta t}(x - \xi)\right)| \leq \frac{\beta^{\frac{d+1}{2}}}{\sqrt{2(\beta - 1)}} \frac{1}{\sqrt{t}} G_{\beta t}(x - \xi) \quad \text{for any } t > 0,
\]

which completes the proof.
which yields that
\[
|\nabla b(x, T + \tau - s)| \leq \frac{1}{\sqrt{T - s}} \|\nabla b\|_{t \to t + T} \quad \text{for } s \in [0, T).
\]

Therefore, we have
\[
|Z(t)|^2 = d^2 - 2 \int_0^t Z_i'(s)Z_j'(s) \frac{\partial b_i}{\partial y^k}(Y, T + \tau - s) \, ds
\]
\[
\leq d^2 + 2 \|\nabla b\|_{t \to t + T} \int_0^t \frac{|Z(s)|^2}{\sqrt{T - s}} \, ds \quad \text{for } 0 \leq t \leq T.
\]
Let
\[
f(t) = \int_0^t \frac{|Z(s)|^2}{\sqrt{T - s}} \, ds.
\]
Then the previous integral inequality may be written as
\[
|Z(t)|^2 = f'(t)\sqrt{T - t} \leq d^2 + 2 \|\nabla b\|_{t \to t + T} f(t) \quad \text{for } 0 \leq t \leq T,
\]
so that
\[
f'(t) \leq \frac{\frac{d^2}{\sqrt{T - t}} + 2 \|\nabla b\|_{t \to t + T} f(t)}{\sqrt{T - t}} \quad \text{for all } 0 \leq t < T.
\]
Define
\[
q(t) := \exp \left( -2 \|\nabla b\|_{t \to t + T} \int_0^t \frac{1}{\sqrt{T - s}} \, ds \right) = \exp \left( -4 \|\nabla b\|_{t \to t + T} (\sqrt{T} - \sqrt{T - t}) \right).
\]
Then \(q(t)\) satisfies
\[
\begin{align*}
q'(t) &= \frac{2|\nabla b|_{t \to t + T}}{\sqrt{T - t}} q(t), \\
q(0) &= 1.
\end{align*}
\]
This implies that
\[
(fq)'(t) \leq \frac{d^2}{\sqrt{T - t}} q(t).
\]
After integrating from 0 to \(t \leq T\), we have
\[
f(t) \leq d^2 e^{-4\|\nabla b\|_{t \to t + T}\sqrt{T - t}} \int_0^t \frac{e^{4\|\nabla b\|_{t \to t + T}\sqrt{T - s}}}{\sqrt{T - s}} \, ds = d^2 e^{4\|\nabla b\|_{t \to t + T}(\sqrt{T} - \sqrt{T - t}) - 1}.\]
Then
\[
|Z(t)|^2 \leq d^2 + 2 \|\nabla b\|_{t \to t + T} f(t) = d^2 e^{4\|\nabla b\|_{t \to t + T}(\sqrt{T} - \sqrt{T - t})},
\]
that is,
\[
|Z(t)| \leq |Z(0)| e^{2\|\nabla b\|_{t \to t + T}(\sqrt{T} - \sqrt{T - t})} \quad \text{for all } t \in [0, T],
\]
where \(|Z(0)| = d\) is the dimension, or the norm of the identity matrix. \(\square\)

**Theorem 5.2 (Explicit Gradient Estimate).** For any \(\beta > 1\), there are constants \(C_1\) and \(C_2\) depending only on \(\beta\) and the dimension \(d\) such that
\[
|\nabla p_b|(\tau, \xi, t + \tau, x) \leq \frac{C_1}{\sqrt{t}} e^{C_2\|b\|_{t \to t + \tau}^{\beta + \frac{1}{2}} \sqrt{\tau\|b\|_{t \to t + \tau}}} G_{\beta}(\xi - x)
\]
for all \(t > 0\), \(\tau \geq 0\), and \(x, \xi \in \mathbb{R}^d\).
Proof. According to the forward Bismut-type formula (see Theorem 2.4),
\[
\partial_t \ln p_b(\tau, \xi, T + \tau, x) = \mathbb{Q} \left[ \int_0^T \frac{\rho'(t)}{\rho(T)} Z_j^b(t) dB^j(t) \right],
\]
where \( \mathbb{Q} \) is the conditional law that \( Y_\tau = \xi \) and \( Y_{T+\tau} = x \), and \( \rho(t) \) can be any continuous and piecewise differentiable function with \( \rho(0) = 0 \) and \( \rho(t) > 0 \) for \( t > 0 \). Let \( \varepsilon > 0 \) be small, and let \( \rho \) be the function such that \( \rho(t) = t \) for \( t \in [0, T - \varepsilon] \) and \( \rho(t) = T - \varepsilon \) for \( t > T - \varepsilon \). Then
\[
\partial_t \ln p_b(\tau, \xi, T + \tau, x) = \frac{1}{T - \varepsilon} \mathbb{E} \left[ \int_0^T Z_j^b(t) dB^j(t) \right] = \frac{1}{T - \varepsilon} \mathbb{E} \left[ \int_0^{T - \varepsilon} Z_j^b(t) dB^j(t) \right],
\]
where
\[
R_{T - \varepsilon} = \frac{p_b(\tau, \xi, T + \tau, x)}{p_b(\tau, \xi, T + \tau, x)}
\]
according to (2.7), and \( Y \) is the solution of the Cauchy problem (5.2) for the stochastic differential equations. Therefore, we have
\[
|\nabla \ln p_b| (\tau, \xi, T + \tau, x) \leq \frac{C_q}{T - \varepsilon} \sqrt{\mathbb{E} \left[ \left( \frac{p_b(\tau, \xi, \tau + \varepsilon, Y_{T - \varepsilon})}{p_b(\tau, \xi, T + \tau, x)} \right)^p \right]} \leq \frac{1}{T - \varepsilon} \mathbb{E} \left[ \int_0^{T - \varepsilon} |Z(t)|^2 dt \right],
\]
where \( C_q \) is the constant in the Burkholder-Davis-Gundy inequality (cf. [18]) that is applied to handle the Itô integral with \( \frac{1}{p} + \frac{1}{q} = 1 \). Thanks to (5.4),
\[
|Z(t)| \leq |Z(0)| e^{2\|\nabla b\|_{T - \varepsilon}^2} \leq |Z(0)| e^{2\|\nabla b\|_{T - \varepsilon}^2} T - \varepsilon \quad \text{for all } t \in [0, T],
\]
where \( |Z(0)| = d \) is the dimension, or the norm of the identity matrix. It follows that
\[
\sqrt{\mathbb{E} \left[ \left( \int_0^{T - \varepsilon} |Z(t)|^2 dt \right)^{\frac{q}{2}} \right]} \leq |Z(0)| \sqrt{\mathbb{E} \left[ \left( \int_0^{T - \varepsilon} e^{A t} dt \right)^{\frac{q}{2}} \right]} \leq |Z(0)| \sqrt{\frac{e^{A(T - \varepsilon)} - 1}{A(T - \varepsilon)}},
\]
where \( A := \frac{2\|\nabla b\|_{T - \varepsilon}^2}{\sqrt{T - \varepsilon}} \).

Plugging into the previous inequality, we obtain
\[
|\nabla \ln p_b| (\tau, \xi, T + \tau, x) \leq \frac{C_q |Z(0)|}{T - \varepsilon} \sqrt{\frac{e^{A(T - \varepsilon)} - 1}{A(T - \varepsilon)}} \sqrt{\mathbb{E} \left[ \left( \frac{p_b(\tau, \xi, \tau + \varepsilon, Y_{T - \varepsilon})}{p_b(\tau, \xi, T + \tau, x)} \right)^p \right]}.
\]
The estimate is true for any \( p > 1 \), so is for \( 1 \leq q < \infty \). It follows that
\[
|\nabla p_b| (\tau, \xi, T + \tau, x) \leq \sqrt{2C_q |Z(0)|} \sqrt{T - \varepsilon} \frac{e^{A(T - \varepsilon)} - 1}{A(T - \varepsilon)} \mathbb{E} \left[ \frac{1}{\sqrt{T}} e^{\frac{1}{4} \int_0^{T(T - \varepsilon)} (p_b(\tau, \xi, \tau + \varepsilon, z))^p p_{-b}(0, x, T - \varepsilon, z) dz} \right]
\]
for any \( p > 1 \). By choosing \( \varepsilon = \frac{T}{2} \) and using the fact \( \frac{e^x - 1}{x} \leq e^x \) for \( x > 0 \), the previous inequality yields that
\[
|\nabla p_b| (\tau, \xi, T + \tau, x) \leq \sqrt{2C_q |Z(0)|} \frac{1}{\sqrt{T}} e^{\frac{T(T - \varepsilon)}{2} \mathbb{E}^{\frac{1}{4} \int_0^{T(T - \varepsilon)} (p_b(\tau, \xi, \tau + \varepsilon, z))^p p_{-b}(0, x, T - \varepsilon, z) dz} \frac{1}{\sqrt{T}}}
\]
for any \( p > 1 \). Finally, according to Lemma 4.2, for any \( \beta > p \),
\[
J(\varepsilon) = \frac{1}{\sqrt{T}} e^{K_2 T |\nabla b|^2} (\frac{T}{\varepsilon})^{\frac{1}{1 - \frac{1}{2}}} G_{\beta T} (\xi - x).
\]
Therefore, we conclude that
\[
|\nabla p_b| (\tau, \xi, T + \tau, x) \leq \frac{K_4}{\sqrt{T}} e^{K_2 T |\nabla b|^2} (\frac{T}{\varepsilon})^{\frac{1}{1 - \frac{1}{2}}} G_{\beta T} (\xi - x),
\]
which yields the estimate. \( \square \)
6 Linearized Vorticity Equations

In this section, we exploit the explicit estimates in §3–§5 to the study of the linearized vorticity equations for the the Navier-Stokes equations (1.1) with the viscosity constant \( \nu = \frac{1}{2} \) (without loss of generality).

We make the following identification: Any tensor field \( F \) defined on \([0,1]^3\) satisfying the periodic condition that \( F(x+k) = F(x) \), for \( i = 1, 2, 3 \), \( x \in [0,1]^3 \), and \( k \in \mathbb{Z}^3 \), will be identified with its periodic extension (with period 1) on \( \mathbb{R}^3 \).

6.1 Linear parabolic equations arising from the vorticity equations

Our construction of strong solutions of the Cauchy problem (1.1)–(1.2) of the Navier-Stokes equations (1.1) with \( \nu = \frac{1}{2} \) will be based on the vorticity equation for \( \omega = \nabla \wedge u \):

\[
\partial_t \omega + (u \cdot \nabla) \omega - A(u) \omega - \frac{1}{2} \Delta \omega = 0,
\]

where \( A(u) \) is the total derivative of \( u \), which is a tensor with its components \( A(u)^{ij} = \partial_i u^j \). Although there are several formulations of the vorticity equations, we will explain that this version serves our aims well. The crucial observation is based on an elementary identity:

\[
\nabla \wedge ((u \cdot \nabla)u) = (u \cdot \nabla) \omega - A(u) \omega \tag{6.2}
\]

so that the vorticity equations are equivalent to

\[
\nabla \wedge (\partial_t u + (u \cdot \nabla)u - \frac{1}{2} \Delta u) = 0,
\]

which is not surprising though. One can even argue that this is from where the vorticity equations come. However, this formulation allows us to define an iteration for the construction of strong solutions of the Navier-Stokes equations.

More precisely, we start with a periodic, smooth, and divergence-free vector field \( b(x,t) \) such that \( b(x,0) = \mathbf{u}_0(x) \), and we then want to construct a nonlinear mapping that sends \( b(x,t) \) to \( v(x,t) \), which will be denoted by \( V \), so that \( v = V(b) \). This is achieved by the two steps:

Step 1. For given \( b(x,t) \), we define a vector field \( w(x,t) \) by solving the following Cauchy problem of the linear parabolic equations:

\[
\begin{cases}
\partial_t w + (b \cdot \nabla)w - A(b)w - \frac{1}{2} \Delta w = 0, \\
w(\cdot,0) = \omega_0,
\end{cases}
\]

where \( A(b) = (\partial_{ij} b^i)_{1 \leq i,j \leq 3} \) is the total derivative of \( b \); this \( w \) should be a candidate of the vorticity, while it is not the vorticity of \( b \). There are two properties of the unique solution \( w(x,t) \) which are important to our approach.

First, \( w \) is again divergence-free. In fact, the divergence of \( w(x,t) \), denoted by \( f(x,t) \), \( i.e. \), \( f = \nabla \cdot w \), satisfies the following parabolic equation:

\[
\begin{cases}
\partial_t f + (b \cdot \nabla)f - \frac{1}{2} \Delta f = 0, \\
f(\cdot,0) = 0.
\end{cases}
\]

This equation may be obtained by differentiating (6.4) and using the following elementary vector identity:

\[
\nabla \cdot ((b \cdot \nabla)w) = b \cdot \nabla (\nabla \cdot w) + \nabla \cdot (A(b)w),
\]

which holds for any vector field \( w \) and any \( b \) such that \( \nabla \cdot b = 0 \). By the uniqueness of linear parabolic equation, we conclude that \( f = 0 \), so that \( w(x,t) \) is divergence-free.
Second, we show that the mean: \( m(t) = \int_{[0,1]^3} w(x,t) \, dx = 0 \). In fact, since both \( b \) and \( w \) are divergence-free and periodic, then

\[
\int_{[0,1]^3} (b \cdot \nabla) w \, dx = \int_{[0,1]^3} A(b) w \, dx = \frac{1}{2} \int_{[0,1]^3} \Delta w \, dx = 0,
\]

so that \( m(t) = m(0) = 0 \).

**Step 2.** We define the candidate for the velocity \( \nu(x,t) \) by solving the Poisson equation:

\[
\Delta \nu = -\nabla \wedge w = \nu \quad (6.7)
\]

at any instance such that \( \int_{[0,1]^3} \nu(x,\cdot) \, dx = 0 \), which has a unique solution satisfying the periodic condition. Applying \( \nabla \cdot \) to both sides of the Poisson equation, we obtain that \( \Delta (\nabla \cdot \nu) = 0 \), so that \( \nabla \cdot \nu \) must be constant for every \( t \). Since \( \int_{[0,1]^3} \nabla \cdot \nu \, dx = 0 \), then \( \nabla \cdot \nu = 0 \). Therefore, \( \nabla \cdot \nu = 0 \) and \( \nabla \cdot w = 0 \). The Poisson equation for \( \nu \) implies that

\[
\begin{align*}
\nabla \wedge (\nabla \cdot \nu - w) &= -\Delta \nu - \nabla \wedge w = 0, \\
\nabla \cdot (\nabla \wedge v - w) &= -\nabla \cdot w = 0.
\end{align*}
\]

Then, according to the Hodge theory, \( \nabla \wedge v - w \) vanishes identically, so it follows that \( w = \nabla \wedge v \). In particular, \( \nu(x,0) = u_0(x) \).

Therefore, in this way, we are able to construct an iteration via the nonlinear mapping \( V \) so that \( \nu = V(b) \).

The advantage for using this iteration via the nonlinear mapping \( b \to \nu = V(b) \) can now be put forward in the following: Observe that the parabolic equation (6.4) can be rewritten as

\[
\begin{cases}
(\partial_t - L_{-b}) \, w = A(b) w, \\
w(\cdot,0) = \omega_0,
\end{cases}
\]

where, as before, \( L_{-b} \) denotes the time-dependent elliptic operator \( \frac{1}{2} \Delta - b \cdot \nabla \). The crucial difference from the linearised Navier-Stokes equations lies in the fact that the term on the right-hand side, \( A(b) w \), is a linear zero-order term. This crucial difference allows us to apply the Feynman-Kac-type formula to obtain the necessary \textit{a priori} estimates, which will be derived in the remainder of this section.

### 6.2 Brownian motion on \( \mathbb{T}^3 \)

Let \( h(\tau,x,t,y) \) denote the heat kernel on \( \mathbb{T}^3 \), which is the transition probability density function of the Brownian motion on \( \mathbb{T}^3 \), a diffusion on \( \mathbb{T}^3 \) with its infinitesimal generator \( \frac{1}{2} \Delta \). Then

\[
h(\tau,x,t,y) = \sum_{k \in \mathbb{Z}^3} G_{t-\tau}(y-x+k) \quad \text{for } x,y \in [0,1]^3,
\]

where the series on the right-hand side and its derivative series indeed converge uniformly for \((x,y) \in [0,1]^3\).

Since \( \mathbb{T}^3 \) is a compact manifold without boundary, there is a Green function \( C(x,y) \) associated with the Laplacian \( \Delta \), denoted by \( C(x,y) \) (cf. [11, page 108]), which possesses the following properties:

(i) \( C(x,y) \) is smooth out off the diagonal, periodic in \( x,y \in [0,1]^3 \), and

\[
|\nabla_x^k C(x,y)| \leq \frac{C_1}{|x-y|^k} \quad \text{for } x,y \in [0,1]^3,
\]

where \( k = 0,1,2, C_1 \) is a constant, and \( \int_{[0,1]^3} C(x,y) \, dy = 0 \). In particular,

\[
\int_{[0,1]^3} |\nabla_x^k C(x,y)| \, dy \leq C_2 \quad \text{for all } x \in \mathbb{R}^3 \text{ and } k = 0,1,2,
\]

where \( C_2 \) is a universal constant.
(ii) For every periodic $C^2$ function $\psi$, the Green formula holds:

$$\psi(x) = \int_{[0,1]^3} \psi(y) \, dy + \int_{[0,1]^3} C(x,y) \Delta \psi(y) \, dy.$$ 

It follows that, for any periodic function $f$ with mean zero: $\int_{[0,1]^3} f(y) \, dy = 0$, the Poisson equation:

$$\Delta \psi = f$$

has a unique periodic solution with mean zero, given by the Green formula:

$$\psi(x) = \int_{[0,1]^3} C(x,y) f(y) \, dy \quad \text{for } x \in \mathbb{R}^3.$$ 

Let $b(\cdot, t)$ be a vector field on $[0,1]^3$ depending on $t \geq 0$, which is identified with a vector field $b(x,t) = (b^1(x,t), b^2(x,t), b^3(x,t))$ on $\mathbb{R}^3$ with period 1 along each coordinate. Let $L_b = \frac{1}{2} \Delta + b(\cdot, t) \cdot \nabla$ be an elliptic operator on $[0,1]^3$. Let $h_b(\tau, x, t, y)$ denote the transition probability density function of the $L_b$-diffusion on $[0,1]^3$. Then

$$h_b(\tau, x, t, y) = \sum_{k \in \mathbb{Z}^3} p_b(\tau, x, t, y + k) \quad \text{for } x, y \in [0,1]^3,$$

where $p_b(\tau, x, t, y)$ is the transition probability density function on $\mathbb{R}^3$ with infinitesimal generator $L_b = \frac{1}{2} \Delta + b(\cdot, t) \cdot \nabla$ on $\mathbb{R}^3$.

### 6.3 A priori estimates for the linearized vorticity equations

In this subsection, we establish several a priori estimates for the first step of the iteration for solving the vorticity equation, defined in the following: Let $u_0$ be a given smooth, periodic initial velocity vector field with divergence-free, and $\omega_b = \nabla \times u_0$, and let $b(x, t)$ be a smooth time-dependent divergence-free periodic vector field on $\mathbb{R}^3$ with $b(x, 0) = u_0(x)$. Hence, $u_0(x + k) = u_0(x)$ and $b(x + k, t) = b(x, t)$ for $k \in \mathbb{Z}^3$ and $x \in \mathbb{R}^3$. Define the periodic vorticity field $w(x, t)$ to be the unique solution of the Cauchy problem (6.4) for the linear vorticity equations.

The vector field $v(x, t)$ is the unique solution with mean zero to the Poisson equation (6.7) on $\mathbb{T}^3$, which may be given by the Green formula:

$$v^j(x, t) = \int_{[0,1]^3} \varepsilon^{ijk} C(x, y) \partial_j w^k(y, t) \, dy \quad \text{for } t \geq 0 \text{ and } x \in \mathbb{T}^3.$$ 

We have shown that both $v$ and $w$ are divergence-free, and $w = \nabla \times v$.

Moreover, the solution $w(x, t)$ of the Cauchy problem (6.4) satisfies the following implicit equation:

$$w(x,t) = \int_{[0,1]^3} h_{-b}(\tau, \xi, t, x) w(\xi, \tau) \, d\xi + \int_{\tau}^{t} \int_{[0,1]^3} h_{-b}(s, \xi, t, x) A(b)(\xi, s) w(\xi, s) \, d\xi \, ds \quad (6.9)$$

for $t > \tau \geq 0$ and $x \in [0,1]^3$. On the other hand, by the forward Feynman-Kac’s formula (2.9), we have the following nonlinear representation:

$$w^k(x, t) = \int_{[0,1]^3} \omega^j_k(\xi) h_b(0, \xi, t, x) \mathbb{P}_x^\xi \left[ Q^j(0, t) \mid X_t = x \right] \, d\xi \quad \text{for } k = 1, 2, 3, \quad (6.10)$$

where $X$ is the canonical process on the path space $\Omega = C([0, \infty), \mathbb{T}^3)$, $\mathbb{P}_x^\xi = \mathbb{P}^{0, \xi}$ is the diffusion family on $\Omega$ with infinitesimal generator $L_b$, and $Q(s) = Q(s, t)$ is the solution to

$$\begin{cases}
\frac{d}{ds} Q^j(s) = -Q^j(s) A(b)_k^j(X_s, s) & \text{for } s \leq t, \\
Q^j(t) = \delta^j_i.
\end{cases} \quad (6.11)$$

We now derive several a priori estimates.
Lemma 6.1. Let $\eta = (\eta(s))_{s \geq 0}$ be any continuous curve in $\mathbb{R}^3$, and let $G(s) = (G_j(s))_{s \leq j}$ be the unique solution of the Cauchy problem for the ordinary differential equations:

$$
\begin{align*}
\frac{d}{ds} G_j^i(s) &= -G_k^j(s)A(b)^j_i(\eta(s),s) \quad \text{for } 0 \leq s \leq t, \\
G_j^i(t) &= \delta_j^i,
\end{align*}
$$

(6.12)

for $i, j = 1, 2, 3$. Then

$$
\sum_{i,j} G_j^i(0)G_j^i(0) \leq 9e^{4\sqrt{7}\|v\|_{0 \rightarrow t}}.
$$

(6.13)

Proof. For simplicity of notation, we set

$$
f(s) := \sum_{i,j} G_j^i(s)G_j^i(s),
$$

which is the squared Hilbert-Schmidt’s norm of $(G_j^i(s))$. Then

$$
\frac{df(s)}{ds} = 2G_j^i(s)\frac{d}{ds} G_j^i(s)
$$

$$
= -2G_k^j(s)G_j^i(s)A(b)^j_i(\eta(s),s)
$$

$$
\geq -2|\nabla b(\eta(s),s)|f(s)
$$

$$
\geq -\frac{2}{\sqrt{s}} ||\nabla b||_{0 \rightarrow t} f(s) \quad \text{for } s \in [0,t].
$$

After integration, we obtain

$$
\ln f(t) - \ln f(0) \geq -2||\nabla b||_{0 \rightarrow t} \int_0^t \frac{1}{\sqrt{s}} ds = -4\sqrt{t} ||\nabla b||_{0 \rightarrow t}.
$$

Then

$$
G(0) \leq 9e^{4\sqrt{7}||v||_{0 \rightarrow t}} \quad \text{for all } t \geq 0.
$$

\hfill \square

Lemma 6.2. For every $\beta > 1$, there are universal constants $C_1$ and $C_2$ depending only on $\beta$ such that

$$
|w(x,t)| \leq C_1 e^{C_2 \beta ||b||_{0 \rightarrow t}^2 + 2\sqrt{7}||v||_{0 \rightarrow t}} G_{\beta t}(\omega_h)(x) \quad \text{for all } t > 0 \text{ and } x \in \mathbb{T}^3.
$$

(6.14)

Proof. The estimate in Lemma 6.1 allows us to control the conditional expectation in (6.10) and to obtain

$$
|w(x,t)| \leq 3e^{2\sqrt{7}||v||_{0 \rightarrow t}} \int_{[0,1]^3} |\omega_h(\xi)| h_h(0,\xi,t,x) d\xi
$$

$$
= 3e^{2\sqrt{7}||v||_{0 \rightarrow t}} \int_{[0,1]^3} |\omega_h(\xi)| \sum_{k \in \mathbb{Z}^3} p_h(0,\xi,t,x+k) d\xi.
$$

Using the uniform estimate (cf. Theorem 4.1):

$$
p_h(0,\xi,t,x) \leq C_1 e^{C_2 \beta ||b||_{0 \rightarrow t}^2} G_{\beta t}(x-\xi),
$$

we obtain

$$
|w(x,t)| \leq 3e^{2\sqrt{7}||v||_{0 \rightarrow t}} \int_{[0,1]^3} |\omega_h(\xi)| \sum_{k \in \mathbb{Z}^3} G_{\beta t}(x+k-\xi) d\xi
$$

$$
= 3C_1 e^{C_2 \beta ||b||_{0 \rightarrow t}^2 + 2\sqrt{7}||v||_{0 \rightarrow t}} \int_{[0,1]^3} |\omega_h(\xi)| \sum_{k \in \mathbb{Z}^3} G_{\beta t}(x+k-\xi) d\xi
$$

$$
= 3C_1 e^{C_2 \beta ||b||_{0 \rightarrow t}^2 + 2\sqrt{7}||v||_{0 \rightarrow t}} G_{\beta t}(\omega_h)(x).
$$

\hfill \square

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With Lemmas 6.1–6.2, we now establish the following three theorems.

**Theorem 6.3.** For every \( \beta > 1 \), there are two positive constants \( C_1 \) and \( C_2 \) depending only on \( \beta \) such that
\[
|\nabla w(x,t)| \leq \frac{C_1}{\sqrt{t}} e^{C_2||b||^2_\infty + 3\sqrt{t}||\nabla b||_{0-\infty}} G_{\beta t}(|\omega_0|)(x) \quad \text{for all } t > 0 \text{ and } x \in \mathbb{T}^3.
\]

**Proof.** Recall that \( w \) satisfies the following equality:
\[
w(x,t) = \int_{[0,1]^3} h_{-b}(0, \xi, t, x) \omega_0(\xi) \, d\xi + \int_0^t \int_{[0,1]^3} h_{-b}(s, \xi, t, x) A(b)(\xi, s) w(\xi, s) \, d\xi \, ds.
\]
for \( t > 0 \) and \( x \in [0,1]^3 \). Differentiating both sides in \( x \) to obtain
\[
\nabla w(x,t) = \int_{[0,1]^3} \nabla_s h_{-b}(0, \xi, t, x) \omega_0(\xi) \, d\xi + \int_0^t \int_{[0,1]^3} \nabla_s h_{-b}(s, \xi, t, x) A(b)(\xi, s) w(\xi, s) \, d\xi \, ds
\]
\[
= \int_{[0,1]^3} \sum_{k \in \mathbb{Z}^3} \nabla_s p_{-b}(0, \xi, t, x + k) \omega_0(\xi) \, d\xi
\]
\[
+ \int_0^t \int_{[0,1]^3} \sum_{k \in \mathbb{Z}^3} \nabla_s p_{-b}(s, \xi, t, x + k) A(b)(\xi, s) w(\xi, s) \, d\xi \, ds.
\]
According to Theorem 5.2, for every \( \beta > 1 \), there are two universal constants \( C_2 \) and \( C_3 \) such that
\[
|\nabla_s p_b|(s, \xi, t, x) \leq C_3 \frac{e^{C_2(t-s)||b||^2_\infty + 2\sqrt{t-s}||\nabla b||_{0-\infty}} \, G_{\beta(t-s)}(x-\xi)}{\sqrt{t-s}} \quad \text{for any } t > s \geq 0. \tag{6.15}
\]
Thanks to this estimate, using the triangle inequality, we obtain
\[
|\nabla w|(x,t) \leq \int_{[0,1]^3} |\omega_0(\xi)| \sum_{k \in \mathbb{Z}^3} |\nabla_s p_{-b}|(0, \xi, t, x + k) \, d\xi
\]
\[
+ \int_0^t \int_{[0,1]^3} |A(b)(\xi, s)| w(\xi, s) \sum_{k \in \mathbb{Z}^3} |\nabla_s p_{-b}|(s, \xi, t, x + k) \, d\xi \, ds
\]
\[
\leq C_3 \frac{e^{\frac{C_2}{\sqrt{t}}||b||^2_{0-\infty} + C_2t||b||^2}}{\sqrt{t}} \int_{[0,1]^3} |\omega_0(\xi)| \sum_{k \in \mathbb{Z}^3} G_{\beta t}(\xi - x - k) \, d\xi
\]
\[
+ C_3 \int_0^t \frac{e^{\frac{C_2}{\sqrt{t-s}}||b||^2_{0-\infty} + C_2(t-s)||b||^2}}{\sqrt{t-s}} \int_{[0,1]^3} |A(b)(\xi, s)| w(\xi, s) \sum_{k \in \mathbb{Z}^3} G_{\beta(t-s)}(\xi - x - k) \, d\xi \, ds
\]
\[
= C_3 \frac{e^{\frac{C_2}{\sqrt{t}}||b||^2_{0-\infty} + C_2t||b||^2}}{\sqrt{t}} \int_{\mathbb{R}^3} |\omega_0(\xi)| G_{\beta t}(\xi - x) \, d\xi
\]
\[
+ C_3 \int_0^t \frac{e^{\frac{1}{\sqrt{t-s}}||b||^2_{0-\infty} + C_2(t-s)||b||^2}}{\sqrt{t-s}} \int_{\mathbb{R}^3} |A(b)(\xi, s)| w(\xi, s) G_{\beta(t-s)}(\xi - x) \, d\xi \, ds.
\]
By definition of \( S = S(b) \), we have
\[
|A(b)(\xi, s)w(\xi, s)| \leq \frac{1}{\sqrt{t}} ||\nabla b||_{0-\infty} |w(\xi, s)|.
\]
Together with estimate (6.14), we then deduce
\[
|A(b)(\xi, s)w(\xi, s)| \leq \frac{3C_3}{\sqrt{t}} ||\nabla b||_{0-\infty} e^{C_2t||b||^2_\infty + 2\sqrt{t}||\nabla b||_{0-\infty}} G_{\beta t}(|\omega_0|)(\xi) \quad \text{for all } t > s > 0.
\]

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It follows that
\[
I_2(x,t) = \int_0^t \frac{e^{\frac{1}{2}\sqrt{t-s}}|\nabla b|_{0\to t} + C_2|b|^2_{0\to t}}{\sqrt{t-s}} \left( \int_{\mathbb{R}^d} |A(b)(\xi,s)||w(\xi,s)|G_{\beta t}(\xi-x)\,d\xi \right)\,ds
\]
\[
\leq 3C_3 |\nabla b|_{0\to t} G_{\beta t}(|\omega_0|)(x) \int_0^t \frac{e^{\frac{1}{2}\sqrt{t-s}}|\nabla b|_{0\to t} + 2\sqrt{t}|\nabla b|_{0\to t} + C_2|b|^2_{0\to t}}{\sqrt{t-s}}\,ds
\]
\[
\leq 3C_3 |\nabla b|_{0\to t} G_{\beta t}(|\omega_0|)(x) e^{C_2|b|^2_{0\to t} + \frac{1}{2}\sqrt{t}|\nabla b|_{0\to t}}
\]
\[
\leq \frac{C_5}{\sqrt{t}} e^{C_2|b|^2_{0\to t} + 3\sqrt{t}|\nabla b|_{0\to t}} G_{\beta t}(|\omega_0|)(x).
\]
Similarly, we have
\[
I_1(x,t) = \int_{\mathbb{R}^d} |\nabla h_{b}(0,\xi,0,\omega_0(\xi))|\,d\xi \leq \frac{C_1}{\sqrt{t}} e^{\sqrt{|\nabla b|_{0\to t}} + C_2|b|^2_{0\to t}} G_{\beta t}(|\omega_0|)(x).
\]
The claimed estimate follows immediately. \qed

**Theorem 6.4.** Let \( v \) be the unique solution with mean zero of the Poisson equation:
\[
\Delta v = -\nabla \wedge w \quad \text{in} \quad \mathbb{R}^3,
\]
which is also periodic. Then there are constants \( C_1 \) and \( C_2 \), depending only on \( \beta \), such that
\[
|v(x,t)| \leq C_1 e^{C_2|b|^2_{0\to t} + 2\sqrt{t}|\nabla b|_{0\to t}} \|\omega_0\|_{\infty},
\]
\[
|\nabla v(x,t)| \leq \frac{C_1}{\sqrt{t}} e^{C_2|b|^2_{0\to t} + 3\sqrt{t}|\nabla b|_{0\to t}} \|\omega_0\|_{\infty},
\]
for all \( t \geq 0 \) and \( x \in [0,1]^3 \).

**Proof.** By Green’s formula,
\[
v(x,t) = -\int_{[0,1]^3} C(x,y)\nabla \wedge w(y,t)\,dy,
\]
\[
\nabla_x v(x,t) = \int_{[0,1]^3} \nabla_x C(x,y)\nabla \wedge w(y,t)\,dy,
\]
where \( C(x,y) \) is the Green function of \( T^3 \). Hence, we conclude that there exist universal constants \( C_4 \) and \( C_5 \) such that
\[
|v(x,t)| \leq C_4 e^{2\sqrt{t}|\nabla b|_{0\to t} + C_2|b|^2_{0\to t}} \|\omega_0\|_{\infty},
\]
\[
|\nabla v(x,t)| \leq \frac{C_5}{\sqrt{t}} e^{3\sqrt{t}|\nabla b|_{0\to t} + C_2|b|^2_{0\to t}} \|\omega_0\|_{\infty},
\]
as \( G_{\beta t}(|\omega_0|)(x) \leq \|\omega_0\|_{\infty} \). \qed

**Theorem 6.5.** There are two universal constants \( C_1, C_2 > 0 \) such that, if
\[
\|b\|_{L^\infty([0,T] \times \mathbb{R}^3)} \leq C_1 \|\omega_0\|_{\infty}, \quad \|\nabla b\|_{0\to T} \leq C_2 \|\omega_0\|_{\infty},
\]
then, for
\[
T = \frac{C_1}{\|\omega_0\|_{\infty}^2},
\]
the following estimates hold:
\[
\|v\|_{L^\infty([0,T] \times \mathbb{R}^3)} \leq C_2 \|\omega_0\|_{\infty}, \quad \|w\|_{L^\infty([0,T] \times \mathbb{R}^3)} \leq C_2 \|\omega_0\|_{\infty},
\]
\[
\|\nabla v\|_{0\to T} \leq C_2 \|\omega_0\|_{\infty}, \quad \|\nabla w\|_{0\to T} \leq C_2 \|\omega_0\|_{\infty}.
\]
Proof. Let us choose \( T > 0 \) such that
\[
C_0 \lambda e^{C_1 T \mu^2 \lambda^2 + 3 \sqrt{T} \mu C_0 \lambda} = \mu C_0 \lambda.
\]
Then we solve \( T \) to obtain
\[
T \mu^2 C_0^3 \lambda^2 + \frac{3}{C_2} \sqrt{T} \mu C_0 \lambda - \frac{1}{C_2} \ln \mu = 0,
\]
so that
\[
\sqrt{T} = \frac{2 \ln \mu}{C_0 C_2 \mu (\sqrt{9 + 4 \ln\mu + 3}) \lambda},
\]
where \( C_0 = C_4 \vee C_5 \vee \ldots \) and \( \lambda = \max G_{BT}(\omega_0)(x) \). Choose \( \mu = e \). Then
\[
\sqrt{T} = \frac{2}{C_0 C_2 e (\sqrt{13} + 3) \lambda}
\]
so that \( T = \frac{C}{T} \).

7 Navier-Stokes Equations

Thanks to the explicit \emph{a priori} estimates established in §3–§6, we are now in a position to study the strong solutions of the Cauchy problem:
\[
\begin{aligned}
\partial_t u + (u \cdot \nabla) u - \frac{1}{2} \Delta u &= -\nabla p, \\
\nabla \cdot u &= 0,
\end{aligned}
\]
(7.1)
with initial data:
\[
u(x, 0) = u_0(x),
\]
(7.2)
such that the periodic condition that \( u_0(x + e_i) = u_0(x) \) for \( x \in [0, 1]^3 \), where \( i = 1, 2, 3 \), \( (e_i) \) is the standard basis of \( \mathbb{R}^3 \).

**Theorem 7.1.** There are universal constants \( C_1 > 0 \) and \( C_2 > 0 \) such that, for any periodic initial data \( u_0 \) with mean zero and \( \omega_0 = \nabla \wedge u_0 \), there exists a unique strong solution \( u(x, t) \) of the Cauchy problem (7.1)–(7.2) for the Navier-Stokes equations (7.1) with periodic initial data (7.2) for \( t \leq T \) so that \( T = \frac{C}{\|\omega_0\|_{L^2}^2} \), and
\[
\|u\|_{L^\infty([0, T] \times \mathbb{R}^3)} \leq C_2 \|\omega_0\|_{L^\infty}, \quad \|\omega\|_{L^\infty([0, T] \times \mathbb{R}^3)} \leq C_2 \|\omega_0\|_{L^\infty},
\]
\[
\|\nabla u\|_{L^\infty([0, T] \times \mathbb{R}^3)} \leq C_2 \|\omega_0\|_{L^\infty}, \quad \|\nabla \omega\|_{L^\infty([0, T] \times \mathbb{R}^3)} \leq C_2 \|\omega_0\|_{L^\infty},
\]
where \( \|V\|_{0 \rightarrow T} = \sup_{(x, t) \in \mathbb{R}^3 \times [0, T]} \sqrt{\mathbf{V}(x, t)} \).

**Proof.** To construct the strong solution of the Cauchy problem (7.1)–(7.2), we construct the following iterations:

Set \( u^{(0)}(x, t) = u_0(x) \) for all \( x \) and \( t \geq 0 \), and define \( u^{(n)} = V(u^{(n-1)}) \) inductively for \( n \geq 1 \). Then \( \nabla \cdot u^{(n)} = 0 \) and \( \nabla \cdot w^{(n)} = 0 \), and
\[
\omega^{(n)} = \nabla \wedge u^{(n)} = w^{(n)} \quad \text{for all} \ n = 1, 2, \ldots.
\]

Hence, for each \( n \geq 1 \), \( \omega^{(n)} \) solves the linear parabolic equations on the torus
\[
\begin{aligned}
\partial_t \omega^{(n)} + (u^{(n-1)} \cdot \nabla) \omega^{(n)} - A(u^{(n-1)}) \omega^{(n)} - \frac{1}{2} \Delta \omega^{(n)} &= 0, \\
\omega^{(n)}(\cdot, 0) &= \omega_0,
\end{aligned}
\]
where \( \omega^{(n)} = \nabla \wedge u^{(n)} \) and \( \nabla \cdot u^{(n)} = 0 \).

Let \( T = \frac{C_1}{\|\omega_0\|_\infty} \) in Theorem 6.5.

Then

\[
|u^{(n)}(x,t)| \leq C_2 \|\omega_0\|_\infty, \\
|\nabla u^{(n)}(x,t)| \leq \frac{C_2}{\sqrt{T}} \|\omega_0\|_\infty, \\
|\omega^{(n)}(x,t)| \leq C_2 \|\omega_0\|_\infty, \\
|\nabla \omega^{(n)}(x,t)| \leq \frac{C_2}{\sqrt{T}} \|\omega_0\|_\infty,
\]

where \( C_1 \) and \( C_2 \) are universal positive constants. By the standard parabolic regularity theory (cf. [22]),

\[
\sup_{\mathbb{R}^3 \times [\delta, T]} \left| \partial_t^{k} \nabla^l u^{(n)} \right| \leq C_{k,l} \quad \text{for all } n,
\]

where the constant, \( C_{k,l} \), depends on \( \|\omega_0\|_\infty \) and \( \delta > 0 \) only, for every \( \delta > 0 \) and \( k, l \in \mathbb{N} \). These a priori estimates allow us to conclude that, if necessary for a convergent subsequence, \( u^{(n)} \to u \) and \( \omega^{(n)} \to \omega \) (in a space with a norm including high derivatives) so that

\[
\partial_t \omega + (u \cdot \nabla) \omega - A(u) \omega - \frac{1}{2} \Delta \omega = 0,
\]

where \( \nabla \cdot u = 0 \) and \( \omega = \nabla \wedge u \). Then

\[
\nabla \wedge \left( \partial_t u + (u \cdot \nabla) u - \frac{1}{2} \Delta u \right) = \partial_t \omega + (u \cdot \nabla) \omega - A(u) \omega - \frac{1}{2} \Delta \omega = 0,
\]

which implies that there is a scalar function \( P \) such that

\[
\partial_t u + (u \cdot \nabla) u - \frac{1}{2} \Delta u = -\nabla P.
\]

Together with \( \nabla \cdot u = 0 \), we conclude that \( u \) is the unique strong solution of the Cauchy problem (7.1)–(7.2) for the Navier-Stokes equations (7.1). This completes the proof. \( \square \)

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