G-TORSORS AND UNIVERSAL TORSORS
OVER NONSPLIT DEL PEZZO SURFACES

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Abstract. Let $S$ be a smooth del Pezzo surface that is defined over a field $K$ and splits over a Galois extension $L$. Let $G$ be either the split reductive group given by the root system of $S_L$ in Pic$S_L$, or a form of it containing the Néron–Severi torus. Let $\mathcal{G}$ be the $G$-torsor over $S_L$ obtained by extension of structure group from a universal torsor $\mathcal{T}$ over $S_L$. We prove that $\mathcal{G}$ does not descend to $S$ unless $\mathcal{T}$ does. This is in contrast to a result of Friedman and Morgan that such $\mathcal{G}$ always descend to singular del Pezzo surfaces over $\mathbb{C}$ from their desingularizations.

1. Introduction

One of the most famous theorems of 19th century algebraic geometry states that cubic surfaces over $\mathbb{C}$ contain precisely 27 lines. The symmetry group of their configuration is the Weyl group $W$ of the root system $E_6$.

Cubic surfaces over fields $K$ that are not algebraically closed exhibit a substantially wider range of phenomena. Given such a cubic surface $S$ over $K$, there is a finite Galois extension $L/K$ over which $S$ splits (i.e., over which all 27 lines are defined). Its Galois group $\Gamma$ acts on the configuration of lines via $W$. This action plays a major role in the geometry of $S$.

An important tool in the study of such cubic surfaces $S$ are the universal torsors introduced and studied by Colliot-Thélène and Sansuc [2]. These are certain torsors over $S$ under the Néron–Severi torus. Such universal torsors do not always exist in the nonsplit case, and they do not descend to singular cubic surfaces from their desingularizations. In particular, this does not provide degenerations of universal torsors to singular cubic surfaces.

Friedman and Morgan suggested to replace the Néron–Severi torus here by a reductive algebraic group $G$ of type $E_6$. They construct natural $G$-torsors over singular cubic surfaces over $\mathbb{C}$ [6]. These $G$-torsors are compatible with degenerations of split cubic surfaces [5].

In this paper, we deal with the existence of natural $G$-torsors over non-split smooth cubic surfaces. Here, the natural candidates for $G$ are a certain split group of type $E_6$, and forms of it that contain the Néron–Severi torus, because both reflect the combinatorial information coming from $S$. Motivated by the result of Friedman and Morgan, there has been a hope that such $G$-torsors could exist also in situations where universal torsors do not exist. However, the main result of the present paper is that such $G$-torsors

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do not exist unless universal torsors exist, and even if they exist, they are not more canonical than universal torsors.

More generally, let \( S \) be a smooth del Pezzo surface over a field \( K \). By [17, Theorem 1.6], there is a finite Galois extension \( L/K \) such that the base change \( S_L \) is split. Then the Galois group \( \Gamma = \text{Gal}(L/K) \) acts on \( \Lambda_S := \text{Pic} S_L \).

Manin [11, §25] discovered that the Picard group \( \Lambda_S \) of the split del Pezzo surface \( S_L \) together with the intersection form \( \langle -,- \rangle \) provide us with a reduced root datum \( \Phi S \subset \Lambda_S, \Phi^\vee S \subset \Lambda^\vee_S \) given as follows: The set of roots is

\[
\Phi_S := \{ \alpha \in \Lambda_S | (\alpha, \alpha) = -2, (\alpha, -K_{S_L}) = 0 \}.
\]

With \( \alpha^\vee \in \Lambda^\vee_S \) given by \( (\alpha^\vee, \lambda) := (-\alpha, \lambda) \) for \( \lambda \in \Lambda_S \), the set of coroots is

\[
\Phi^\vee_S := \{ \alpha^\vee \in \Lambda^\vee_S | \alpha \in \Phi_S \}.
\]

Its Weyl group \( W_S \subset \text{Aut} \Lambda_S \) is the symmetry group of the configuration of \((1, -1)\)-curves on \( S_L \).

By a \( \Gamma \)-twisted root datum, we mean a root datum \( \Phi \subset \Lambda, \Phi^\vee \subset \Lambda^\vee \) together with a left action \( \Gamma \times \Lambda \to \Lambda \) such that \( \gamma(\Phi) \subset \Phi \) and \( \gamma^*(\Phi^\vee) \subset \Phi^\vee \) for all \( \gamma \in \Gamma \); for example, each reductive algebraic group over \( K \) that splits over \( L \) yields a \( \Gamma \)-twisted root datum \( \Phi S \subset \Lambda_S, \Phi^\vee S \subset \Lambda^\vee_S \) (cf. [4, Exposée XXII, Définition 1.9 and Proposition 1.10]).

Let

\[
(\Phi_S \subset \Lambda_S, \Phi^\vee_S \subset \Lambda^\vee_S, \Gamma \times \Lambda_S \to \Lambda_S)
\]

be the \( \Gamma \)-twisted root datum given by \( (\Phi_S \subset \Lambda_S, \Phi^\vee_S \subset \Lambda^\vee_S) \) endowed with the action \( \Gamma \times \Lambda_S \to \Lambda_S \) coming from the fact that \( S \) is defined over \( K \).

The Néron–Severi torus for \( S_L \) is the split \( L \)-torus \( T \) with character lattice \( \text{Hom}(T, \mathbb{G}_{m,L}) = \Lambda_S \) as abelian groups. A universal torsor for \( S_L \) is a \( T \)-torsor \( T \) over \( S_L \) whose type \( \text{Hom}(T, \mathbb{G}_{m,L}) \to \text{Pic} S_L \) (in the sense of [2], see Section 3) is the identity on \( \Lambda_S \). Such a \( T \) exists and is unique up to isomorphism.

The Néron–Severi torus for \( S \) is the \( K \)-torus \( T \) that splits over \( L \) with character lattice \( \text{Hom}(T, \mathbb{G}_{m,K}) = \Lambda_S \) as \( \Gamma \)-modules. A universal torsor for \( S \) is a \( T \)-torsor \( T \) over \( S \) whose base change \( T_L \) is a universal torsor for \( S_L \). Such a \( T \) exists if \( S(K) \neq \emptyset \) [2, Remarque 2.2.9], but otherwise may or may not exist [2, Exemples 2.2.11 and 2.2.12]. For example, the blow-up of a Severi–Brauer surface \( B \) of index 3 in an effective 0-cycle of degree 6 is a cubic surface \( S \) without a universal torsor, since the relevant elementary obstructions [2, Définition 2.2.1 and Proposition 2.2.8(iii)] for \( B \) and for \( S \) coincide. If universal torsors exist, they may or may not be unique up to isomorphism [2 (2.0.2)].

In this situation, our main result on the existence and classification of such \( G \)-torsors on del Pezzo surfaces \( S \) is:

**Theorem.** Let \( G \) be a reductive group over \( K \) with maximal torus \( \iota : T \to G \) such that

(i) \( T \) is split over \( K \) with character lattice \( \text{Hom}(T, \mathbb{G}_{m,K}) = \Lambda_S \), and

the root datum given by \( G \supset T \) is \( (\Phi_S \subset \Lambda_S, \Phi^\vee_S \subset \Lambda^\vee_S) \), or
(ii) \( T \) is the Néron–Severi torus for \( S \), and the \( \Gamma \)-twisted root datum given by \( G \supset T \) is \( (\Phi_S \subset \Lambda_S, \Phi^\vee_S \subset \Lambda^\vee_S, \Gamma \times \Lambda_S \to \Lambda_S) \).

Let \( T \) be a universal \( T_L \)-torsor over \( S_L \), and let \( G := (\iota_L)_* T \) be the \( G_L \)-torsor over \( S_L \) obtained by extension of structure group. Then the groupoid of \( G \)-torsors \( G^\circ \) over \( S \) such that \( G^\circ_L \cong G \) is equivalent to the groupoid of universal torsors over \( S \).

In particular, \( G \) descends to a \( G \)-torsor over \( S \) if and only if universal torsors for \( S \) exist; if this is the case, then the \( K \)-forms of \( G \) are in bijection to universal torsors. Part (ii) of this theorem is contained in Theorem 4.2 below, and part (i) is contained in its Corollary 4.3.

The proof of Theorem 4.2 is based on descent from the split situation. It turns out that the relevant \( T \)-torsors and \( G \)-torsors have the same descent data because they have the same group of global automorphisms, essentially by Proposition 3.1, which might be of independent interest since it applies to torsors under general reductive groups \( G \) over smooth projective schemes. Corollary 4.3 then follows using that the split group \( G \) over \( K \) contains a Néron–Severi torus, a result due independently to Gille [7] and Raghunathan [12].

For another connection between the del Pezzo surface \( S \) and this reductive group \( G \) via universal torsors, see [13, 15] over algebraically closed fields, and [14, Theorem 4.4] over nonclosed fields. Regarding the Hasse principle for the existence of rational points and weak approximation, torsors under connected linear algebraic groups \( G \) do not provide finer obstructions than the Brauer–Manin obstruction [8, Théorème 2].

The structure of our paper is as follows. In Section 2, we collect a few basic facts about \( G \)-torsors, with a view towards their descent. We circumvent the difficulty of two Galois actions (on \( S_L \) and \( G_L \)) as follows: Instead of working with \( G_L \)-torsors over \( S_L \), we view them as \( G \)-torsors over \( S_L \) regarded as a \( K \)-scheme. In Section 3, we compare automorphisms of \( T \)-torsors and \( G \)-torsors in the split case. Finally, in Section 4, we apply our descent setup to prove the main results.

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2. Torsors

Let \( X \) be a smooth projective scheme over \( K \). We assume that \( X \) is connected (but not necessarily geometrically connected over \( K \)). Let \( G \) be a linear algebraic group over \( K \).
Definition 2.1. A $G$-torsor over $X$ is a faithfully flat morphism $G \to X$ of finite type together with a right action $G \times_K G \to G$ over $X$ such that the map $G \times_K G \to G \times_Y G$ given by $(u, g) \mapsto (u, ug)$ is an isomorphism.

Definition 2.2. Let $\phi : G \to H$ be a homomorphism of linear algebraic groups over $K$. The extension of structure group of $G$ along $\phi$ is the $H$-torsor
\[ \phi_* G := G \times^H H = (G \times_K H)/G \to X, \]
where $G$ acts on $G$ from the right, and on $H$ via $\phi$ from the left.

Let $f : Y \to X$ be a morphism of connected smooth projective $K$-schemes.

Definition 2.3. The pullback of $G$ along $f$ is the $G$-torsor
\[ f^* G := G \times_X Y \to Y. \]

Remark 2.4. Generalizing Definition 2.2 if $\phi$ is a homomorphism of group schemes over $X$ from $G_X := G \times_K X$ to $H_X := H \times_K X$, then
\[ \phi_* G := (G \times_X H_X)/G_X \]
is still an $H$-torsor over $X$. Its pullback along $f$ satisfies
\[ f^* (\phi_* G) \cong (f^* \phi)_* (f^* G), \]
where $f^* \phi : G_Y \to H_Y$ is the pullback of $\phi$.

Definition 2.5. A morphism $f : Y \to X$ between connected smooth projective schemes over $K$ is a Galois covering if $f$ is finite, surjective and étale such that the natural map
\[ \Sigma \times Y \to Y \times_X Y, \quad (\sigma, y) \mapsto (\sigma y, y) \]
is an isomorphism for the discrete group $\Sigma := \text{Aut}_X Y$ \cite{1} §6.2B.

Here, each $\sigma \in \Sigma$ is a $K$-morphism $\sigma : Y \to Y$. Hence any $G$-torsor $G$ over $Y$ can be pulled back along $\sigma$, resulting in another $G$-torsor $\sigma^* G$ over $Y$.

Suppose that $G = f^* G^\circ$ for some $G$-torsor $G^\circ$ over $X$. For each $\sigma \in \Sigma$, let $c_\sigma : \sigma^* G \to G$ be the canonical isomorphism $\sigma^* G = \sigma^* f^* G^\circ = f^* \sigma^* G^\circ = G$ of $G$-torsors over $Y$ coming from the fact that $f \circ \sigma = f : Y \to X$. By construction, they satisfy the cocycle condition
\[ c_{\sigma \rho} = c_\sigma \circ \sigma^* (c_\rho) \]
as isomorphisms $(\rho \sigma)^* G = \sigma^* \rho^* G \to \sigma^* G \to G$ of $G$-torsors over $Y$ for every $\rho, \sigma \in \Sigma$. Conversely, we have the following descent result.

Lemma 2.6. Given a $G$-torsor $G$ over $Y$, and isomorphisms $c_\sigma : \sigma^* G \to G$ for $\sigma \in \Sigma$ satisfying (3) for all $\rho, \sigma \in \Sigma$, there is a unique $G$-torsor $G^\circ$ over $X$ such that $f^* G^\circ$ with the induced cocycle is isomorphic to $G$ with $(c_\sigma)_{\sigma \in \Sigma}$.

Proof. The relatively affine scheme $G$ over $Y$ is the relative spectrum of a quasi-coherent $O_Y$-algebra $A$, which descends to a unique quasi-coherent $O_X$-algebra $A^\circ$ according to \cite{16} Lemma 0DIV. We define $G^\circ$ as the relative spectrum of $A^\circ$ over $X$. 

The group action $G \times_K G \to G$ corresponds to a morphism $A \to A \otimes_K K[G]$ of quasi-coherent $O_Y$-algebras, where $K[G] := \Gamma(G, O_G)$ is the coordinate ring of $G$. Since $\sigma$ is a $K$-morphism, we have a canonical isomorphism $\sigma^*(A \otimes_K K[G]) = (\sigma^*A) \otimes_K K[G]$, which allows us to use [16, Lemma 0D1V] again to descend the given group action to a unique group action $G \circ \times_K G \to G \circ$. This turns $G \circ$ into a $G$-torsor over $X$, as required. □

3. AUTOMORPHISMS OF $T$-TORSORS AND $G$-TORSORS

Let $X$ be a smooth projective scheme over a field $K$. Let $L_1, \ldots, L_n$ be line bundles over $X$ such that $\text{Hom}(L_i, L_j) = 0$ for $i \neq j$, and hence in particular $L_i \not\cong L_j$ for $i \neq j$. Then every automorphism of the vector bundle $E := L_1 \oplus \cdots \oplus L_n$ over $X$ is of the form $c_1 \oplus \cdots \oplus c_n$ for automorphisms $c_i$ of $L_i$. This is the special case $G = \text{GL}_n$ of the following proposition.

**Proposition 3.1.** Let $G$ be a reductive group over $K$ with split maximal torus $i : T \to G$ and resulting Weyl group $W_G$. Let $T$ be a $T$-torsor over $X$ satisfying the following two conditions:

(i) For every root $\alpha : T \to \mathbb{G}_m$ of $G$, the line bundle $L_\alpha$ given by the $\mathbb{G}_m$-torsor $\alpha^*T$ over $X$ has no nonzero global section.

(ii) For every $1 \neq w \in W_G$, the $T$-torsor $w^*T$ obtained by extension of structure group along $w : T \to T$ is not isomorphic to $T$.

Let $G := i^*T$ denote the $G$-torsor over $X$ obtained from $T$ by extension of structure group. Viewing the total space $T$ as contained in $G$, every automorphism of the $G$-torsor $G \to X$ restricts to an automorphism of the $T$-torsor $T \to X$.

**Proof.** Since $T$ is split over $K$, the assumptions (i) and (ii) still hold after base change to the algebraic closure $\overline{K}$ of $K$. This allows us to assume $K = \overline{K}$ without loss of generality. Working on each connected component of $X$ individually, we may also assume that $X$ is connected.

We consider the natural map

$$T = \text{Aut } T \to \text{Aut } G.$$ 

Here the group scheme of global automorphisms $\text{Aut } G$ over $K$ is by definition the Weil restriction $\text{Res}_{X/K}(\text{Aut}_X G)$ of the group scheme of local automorphisms $\text{Aut}_X G := G \times^G \text{Ad } G$

over $X$, where $G$ acts via the homomorphism

$$\text{Ad} : G \to \text{Aut } G, \quad g \mapsto \text{conj}_g := g \cdot \cdot g^{-1},$$

and $\text{Aut } T$ is defined similarly. These Weil restrictions exist as linear algebraic group schemes over $K$ according to [9, §1.4].

Let $K[\epsilon]$ be the ring of dual numbers, with $\epsilon^2 = 0$. Since our map

$$T = \text{Aut } T \to \text{Aut } G$$

is injective on $K$-points and on $K[\epsilon]$-points, it is a closed immersion by [4, Exposé VI\textsc{B}, Corollaire 1.4.2]. We have to prove that it is an isomorphism.
Let $g$ denote the Lie algebra of $G$. Let $G_{K[e]}$ be the base change of $G$ from $K$ to $K[e]$. Its Weil restriction $G[e] := \text{Res}_{K[e]/K}(G_{K[e]})$ is a smooth linear algebraic group and appears in the natural short exact sequence
\[ 0 \to g \to G[e] \to G \to 1 \]
by [3, II, §4, Théorème 3.5]. This induces an exact sequence
\[ 0 \to G \times G g \to G \times G G[e] \to \text{Aut}_X G \to 1 \]
of associated group schemes over $X$. Since the Lie algebra Lie($\text{Aut}_G$) of $\text{Aut}_G$ consists of its $K[e]$-valued points (i.e., sections from $X$ into $G \times G G[e]$) that reduce to the identity modulo $e$, we deduce
\[ \text{Lie}(\text{Aut}_G) = H^0(X, G \times G g). \]
Since $G = \iota_* T$, we thus obtain
\[ \text{Lie}(\text{Aut}_G) = H^0(X, T \times T g). \]

Let
\[ g = g_0 \oplus \bigoplus_{\alpha} g_\alpha \]
be the root space decomposition of $g$ into its eigenspaces under $T$. For any root $\alpha$, the line bundle
\[ T \times T g_\alpha \cong L_\alpha \]
has no nonzero global sections by assumption (i). Therefore,
\[ H^0(X, T \times T g) = H^0(X, T \times T g_0) = H^0(X, \mathcal{O}_X \otimes g_0) = g_0 = t, \]
the Lie algebra of $T$. Since $\dim \text{Aut}_G \geq \dim T = \dim t$, it follows that $\text{Aut}_G$ is smooth with tangent spaces $t$. This shows that our closed immersion $T \hookrightarrow \text{Aut}_G$ is an open embedding.

Being also connected, $T$ is the connected component of the identity in $\text{Aut}_G$. In particular, $T$ is normal in $\text{Aut}_G$. We view elements of $\text{Aut}_G$ as global sections of
\[ \text{Aut}_X G = G \times G^{\text{Ad}} G = T \times T^{\text{Ad}} G \to X. \]
Then the elements of $T \subset \text{Aut}_G$ become the constant global sections of
\[ X \times T = T \times T^{\text{Ad}} T \subset T \times T^{\text{Ad}} G. \]

Let $T_x$ be the fiber of $T$ over a point $x \in X$. Since each global section $s \in \text{Aut}_G$ normalizes the constant sections $T$, its value $s(x) \in T_x \times T^{\text{Ad}} G$ normalizes $T = T_x \times T^{\text{Ad}} T$. Hence $s(x) \in T_x \times T^{\text{Ad}} N$ for the normalizer $N$ of $T$ in $G$, and therefore, $s$ is a global section of
\[ T \times T^{\text{Ad}} N \subset T \times T^{\text{Ad}} G. \]

Let $p : N \to N/T = W_G$ be the canonical projection. Since $T$ acts trivially on $W_G$, the projection $p$ induces a map
\[ T \times T^{\text{Ad}} N \to X \times W_G \]
of group schemes over $X$. Here the image of $s$ is a global section of $X \times W_G$, which is automatically a constant $w \in W_G$ since $W_G$ is discrete. Therefore, $s$ is now a global section of the open subscheme

$$T \times^{T,Ad} p^{-1}(w) \subset T \times^{T,Ad} N.$$ 

Hence $s : T \times^T G \to T \times^T G$ restricts to an isomorphism from the closed subscheme $T \times^T T \subset T \times^T G$ to the closed subscheme $T \times^T p^{-1}(w) \subset T \times^T G$, where now $T$ acts on the $p^{-1}(w) \subset G$ once more by multiplication from the left. This restriction becomes an isomorphism of $T$-torsors when we let $T$ act on $p^{-1}(w)$ by multiplication from the right.

Choose an element $n \in N(K)$ with $p(n) = w$, and define $\phi : T \to p^{-1}(w)$ by $\phi(t) = nt$. Then $\phi$ intertwines the two $T$-actions by multiplication from the right. Moreover, the relation $\phi(n^{-1}tnt') = t\phi(t')$ for $t, t' \in T$ shows that $\phi$ also intertwines the two $T$-actions where $t \in T$ acts on $T$ as multiplication by $\text{conj}_{n^{-1}}(t)$ from the left, and on $p^{-1}(w)$ as multiplication by $t$ from the right. Hence $\phi$ induces an isomorphism of $T$-torsors

$$(w^{-1})_*T = (\text{conj}_{n^{-1}})_*T = T \times^T,\text{conj}_{n^{-1}} T \to T \times^T p^{-1}(w).$$

As the latter is isomorphic to $T$ via $s$, we conclude that $(w^{-1})_*T$ is isomorphic to $T$. Using assumption (ii), we deduce that $w = 1$.

Hence every automorphism of $G$ is a global section of $T \times^{T,Ad} T = X \times T$, and thus an element of $T$.

Now we assume that $S$ is a split del Pezzo surface over $K$. Let $T$ be a split torus over $K$ with character lattice $\Lambda_T$, and let $T$ be a $T$-torsor over $S$. Recall from [2] that the type of $T$ is the homomorphism $\tau : \Lambda_T \to \Lambda_S$ that sends a character $\chi : T \to \mathbb{G}_m$ to the class $[\chi_*T]$ of the line bundle given by the $\mathbb{G}_m$-torsor $\chi_*T$ over $S$. For a homomorphism $\phi : T \to T'$ of split $K$-tori, it is easy to check that the extension of structure group $\phi_*T$ has type $\tau \circ \phi^* : \Lambda_T' \to \Lambda_T \to \Lambda_S$.

Let $G$ be a split reductive group over $K$ with maximal torus $\iota : T \to G$ and resulting reduced root datum $(\Phi_G \subset \Lambda_T, \Phi_G^\vee \subset \Lambda_T^\vee)$. We assume that the type $\tau : \Lambda_T \to \Lambda_S$ of $T$ is an isomorphism from this root datum to $(\Phi_S \subset \Lambda_S, \Phi_S^\vee \subset \Lambda_S^\vee)$, i.e., $\tau$ is bijective, $\tau(\Phi_G) = \Phi_S$, and $\tau(\Phi_G^\vee) = \Phi_S^\vee$. This determines $G$ up to isomorphism [3] Exposé XXV, Corollaire 1.2]. The assumption that $\tau$ is bijective also means that $T$ is a universal torsor.

**Remark 3.2.** This group $G$ can be described more explicitly as follows (see also [6], §2). The anticanonical class $-K_S \in \Lambda_S$ vanishes on all coroots and hence defines a character

$$\chi := \tau^{-1}(-K_S) : G \to \mathbb{G}_m.$$ 

If the degree of $S$ is at most 6, then the subgroup $\{\phi \in \Lambda_S^\vee \mid \phi(-K_S) = 0\}$ is generated by the coroots $\alpha^\vee \in \Lambda_S^\vee$, so we obtain a short exact sequence

$$1 \to [G, G] \to G \xrightarrow{\chi} \mathbb{G}_m \to 1,$$

in which $\ker(\chi)$ is simply connected. Therefore, the commutator subgroup $[G, G]$ is the semisimple and simply connected algebraic group of type $E_8$, $E_7$, $E_6$, $D_5$, $A_4$, $A_2 + A_1$ if the degree of $S$ is 1, 2, 3, 4, 5, 6, respectively.
In degree 7, we have \( G \cong \mathbb{G}_m \times \text{GL}_2 \). In degree 8, we have \( G \cong \text{GL}_2 \) if \( S \cong \mathbb{P}^1 \times \mathbb{P}^1 \), and \( G \cong \mathbb{G}_m^2 \) if \( S \) is the blow-up of \( \mathbb{P}^2 \) in a point. Finally, in degree 9, we have \( G \cong \mathbb{G}_m \).

Let \( \mathcal{G} := \iota_* \mathcal{T} \) denote the \( G \)-torsor over \( S \) obtained from \( \mathcal{T} \) by extension of structure group. We can view the total space \( \mathcal{T} \) as contained in \( \mathcal{G} \).

**Corollary 3.3.** Let \( \mathcal{T}_1 \) be another \( T \)-torsor over \( S \) of the same type \( \tau \) as \( \mathcal{T} \), and \( \mathcal{G}_1 = \iota_* \mathcal{T}_1 \). Every isomorphism \( \mathcal{G}_1 \to \mathcal{G} \) of \( G \)-torsors over \( S \) restricts to an isomorphism \( \mathcal{T}_1 \to \mathcal{T} \) of \( T \)-torsors over \( S \).

**Proof.** We can choose an isomorphism \( \mathcal{T} \cong \mathcal{T}_1 \) since both are torsors of the same type \( \tau \) under the split torus \( T \). Hence it suffices to verify that \( \mathcal{T} \) satisfies the assumptions of Proposition 3.1.

Indeed, for any root \( \alpha \in \Phi_G \), the line bundle \( L_\alpha \) over \( S \) has an isomorphism class \( \tau(\alpha) \) since \( \mathcal{T} \) has type \( \tau \). In particular, the anticanonical degree of \( L_\alpha \) is \((-K_S, \tau(\alpha)) = 0 \), and \( L_\alpha \) is not \( \mathcal{O}_S \). Hence \( H^0(S, L_\alpha) = 0 \) for each root \( \alpha \).

Further, as \( \mathcal{T} \) has type \( \tau \), the type of \( w_* \mathcal{T} \) is \( \tau \circ w^* : \Lambda_T \to \Lambda_T \to \Lambda_S \).

If \( w \neq 1 \), then \( \tau \neq \tau \circ w^* \), and hence \( \mathcal{T} \neq w_* \mathcal{T} \). \( \square \)

4. PROOF OF THE MAIN RESULTS

We use the notation of Section 1. In particular, \( S \) and \( S_L \) are both connected smooth projective schemes over \( K \) (even though \( S_L \) is not geometrically connected over \( K \) unless \( L = K \)), and the projection \( \pi : S_L \to S \) is a \( K \)-morphism. More precisely, \( \pi \) is a Galois covering with group

\[ \Sigma = \{ \text{id}_S \times \gamma^* \mid \gamma \in \Gamma \} \cong \Gamma^{\text{op}} \]

in the sense of Definition 2.5.

Let \( \mathcal{G} \) be a \( G \)-torsor over the \( K \)-scheme \( S_L \), as in Definition 2.1. Then the same total space \( \mathcal{G} \) can also be viewed as a \( G_L \)-torsor over the \( L \)-scheme \( S_L \).

Indeed, the required action \( G \times_L G_L \to \mathcal{G} \) of \( G_L \) comes from the given action \( G \times_K G \to \mathcal{G} \) of \( G \), since the two fiber products are canonically isomorphic and the axioms are clearly equivalent. Conversely, every \( G_L \)-torsor over the \( L \)-scheme \( S_L \) can be viewed as a \( G \)-torsor over the \( K \)-scheme \( S_L \).

Let \( T \) be a \( K \)-torus that splits over \( L \) with character lattice \( \Lambda_T \), and let \( \mathcal{T} \) be a \( T \)-torsor over \( S_L \). Then we can in particular view \( \mathcal{T} \) as a \( T_L \)-torsor over \( S_L \), which has a type \( \tau : \Lambda_T \to \Lambda_S \) in the sense of Section 3.

**Lemma 4.1.** For \( \gamma \in \Gamma \), put \( \sigma = \text{id}_S \times \gamma^* \in \Sigma \). Then the \( T \)-torsor \( \sigma^* \mathcal{T} \) over \( S_L \) has type

\[ \sigma^* \circ \tau \circ (\gamma^*_L)^{-1} : \Lambda_T \to \Lambda_S, \]

where \( \gamma^*_L : \Lambda_T \to \Lambda_T \) is induced by \( \gamma \).

**Proof.** Every element of \( \Lambda_T \) has the form \( \gamma^*_L \chi \) for a unique \( \chi \in \Lambda_T \). The type of \( \sigma^* \mathcal{T} \) sends \( \gamma^*_L \chi \) to the class of the line bundle

\[ (\gamma^*_L \chi)_* (\sigma^* \mathcal{T}) \cong \sigma^* (\chi_* \mathcal{T}) \]

over \( S_L \), where the isomorphism is a special case of (2). This class is \((\sigma^* \circ \tau)(\chi)\), as required. \( \square \)
Theorem 4.2. Let \( \iota : T \hookrightarrow G \) be a maximal torus in a reductive group over \( K \) such that \( T \) splits over \( L \), with resulting \( \Gamma \)-twisted root datum
\[
(\Phi_G \subset \Lambda_T, \Phi_S^\vee \subset \Lambda_S^\vee, \Gamma \times \Lambda_T \to \Lambda_T).
\]
Let \( T \) be a \( T \)-torsor over \( S_L \) whose type \( \tau : \Lambda_T \to \Lambda_S \) is an isomorphism to the \( \Gamma \)-twisted root datum
\[
(\Phi_S \subset \Lambda_S, \Phi_S^\vee \subset \Lambda_S^\vee, \Gamma \times \Lambda_S \to \Lambda_S)
\]
given by \( S \). We consider the \( G \)-torsor \( \mathcal{G} := \iota_* T \) over \( S_L \) obtained by extension of structure group.

Then the functor \( \iota_* \) from the groupoid of \( T \)-torsors \( T^0 \) over \( S \) such that \( T^0_L \cong T \to \) the groupoid of \( G \)-torsors \( G^0 \) over \( S \) such that \( G^0_L \cong \mathcal{G} \) is an equivalence of categories.

Proof. Let \( T^0 \) be a \( T \)-torsor over \( S \) such that \( T^0_L \cong T \). Let \( G^0 := \iota_* T^0 \).

Then the base change of \( G^0 \) to \( L \) is
\[
G^0_L = (\iota_*(T^0))(L) \cong \iota_*(T^0_L) \cong \iota_*(T^0_L).
\]
Clearly any isomorphism \( \phi \) between such \( T \)-torsors over \( S \) induces an isomorphism \( \iota_* \phi \) between \( G \)-torsors over \( S \). Hence \( \iota_* \) is indeed a functor.

Let \( T^* \) be another \( T \)-torsor over \( S \) such that \( T^*_L \cong T \), and \( G^0 := \iota_* T^* \). Let \( \psi : G^* \to G^0 \) be an isomorphism of \( G \)-torsors over \( S \). Then Corollary 4.3 applies to the base change
\[
\psi_L : \iota_*(T^*_L) = G^*_L \to G^*_L = \iota_*(T^*_L)
\]
and shows that \( \psi_L \) restricts to an isomorphism \( T^*_L \to T^*_L \). In particular, \( \psi_L(T^*_L) = T^*_L \) as closed subschemes of \( G^*_L \), and hence \( \psi(T^*) = T^0 \) as closed subschemes of \( G^0 \). The restriction \( T^* \to T^0 \) of \( \psi \) becomes an isomorphism of \( T \)-torsors over \( S_L \) after base change, and therefore is an isomorphism of \( T \)-torsors over \( S \). This proves that the functor \( \iota_* \) is fully faithful.

Now assume that the \( G \)-torsor \( \mathcal{G} = \iota_* T \) descends to a \( G \)-torsor \( G^0 \) over \( S \). As explained in Section 2, we have an isomorphism
\[
c_{\sigma} : \sigma^* \mathcal{G} \to \mathcal{G}
\]
of \( G \)-torsors over \( S_L \) for each \( \sigma \in \Sigma \), satisfying the cocycle conditions [5]. Here, we have \( \sigma^* \mathcal{G} = \iota_* \sigma^* T \). Since \( \tau \) is \( \Gamma \)-equivariant, \( \sigma^* T \) has the same type \( \tau \) as \( T \) due to Lemma 4.1. Therefore, Corollary 4.3 shows that \( c_{\sigma} \) restricts to an isomorphism
\[
c_{\sigma} : \sigma^* T \to T.
\]
The \( c_{\sigma} \) satisfy the cocycle condition because the \( c_{\sigma} \) do. Therefore, \( T \) descends to a \( T \)-torsor \( T^0 \) over \( S \). By construction, \( \iota_* T^0 \cong G^0 \). This proves that the functor \( \iota_* \) is essentially surjective.

Corollary 4.3. Let \( \iota : T \hookrightarrow G \) be a split maximal torus in a split reductive group over \( K \), with resulting root datum \( (\Phi_G \subset \Lambda_T, \Phi_S^\vee \subset \Lambda_S^\vee) \). Let \( T \) be a \( T \)-torsor over \( S_L \) whose type \( \tau : \Lambda_T \to \Lambda_S \) is a \( \mathbb{Z} \)-linear isomorphism from this root datum to \( (\Phi_S \subset \Lambda_S, \Phi_S^\vee \subset \Lambda_S^\vee) \). We consider the \( G \)-torsor \( \mathcal{G} := \iota_* T \) over \( S_L \) obtained by extension of structure group.

Then the groupoid of \( G \)-torsors \( G^0 \) over \( S \) such that \( G^0_L \cong \mathcal{G} \) is equivalent to the groupoid of universal torsors over \( S \).
Proof. Let $N \subset G$ denote the normalizer of $T$ in $G$. The exact sequence
\[ 1 \to T \to N \to W_G \to 1 \]
defines the Weyl group $W_G$ of $G$. For $g \in G(L)$, we denote by
\[ \text{conj}_g := g \cdot \cdot g^{-1} : G_L \to G_L \]
the conjugation. For $n \in N(L)$, $\text{conj}_n$ restricts to an automorphism of $T_L$, which depends only on the image of $n$ in $W_G$. This defines a left action of $W_G$ on $T_L$, and consequently a right action of $W_G$ on $\Lambda_T$, where $w \in W_G$ acts via the pullback $w^* : \Lambda_T \to \Lambda_T$ of characters. Sending $w \in W_G$ to $(w^{-1})^* : \Lambda_T \to \Lambda_T$, we identify $W_G$ with the Weyl group $W(\Phi_G) \subset \text{Aut} \Lambda_T$ of our root datum of $G$.

Since $\tau$ is an isomorphism of root data, it induces an isomorphism
\[ \tau_W : W_G \to W_S, \]
where $W_S := W(\Phi_S) \subset \text{Aut} \Lambda_S$ is the Weyl group of our root datum of $S$.

The Galois action of $\Gamma$ on $\Lambda_S$ factors through the Weyl group action of $W_S$ on $\Lambda_S$ via a natural map
\[ \rho_S : \Gamma \to W_S. \quad (4) \]

Applying [7, Théorème 5.1(b)] or [12, Theorem 1.1] to the semisimple commutator subgroup $G' := [G,G] \subset G$, there is a $g \in G'(L) \subset G(L)$ such that, for each $\gamma \in \Gamma$, the element $g^{-1}\gamma(g) \in G(L)$ is in $N(L)$, and $\tau_W$ maps its image in $W_G$ to $\rho_S(\gamma)$. The former implies that there is a unique maximal torus $T' : T' \hookrightarrow G$ over $K$ such that $T'_L = gT_Lg^{-1}$, and the latter says that the square on the right in the following diagram commutes:
\[ \begin{array}{ccc}
\Lambda_T' & \xrightarrow{\text{conj}_g^\ast} & \Lambda_T \\
\gamma_T' \downarrow & & \downarrow \gamma_T \\
\Lambda_T' & \xrightarrow{\text{conj}_g^\ast} & \Lambda_T \\
\gamma_T' \downarrow & & \downarrow \gamma_T \\
\end{array} \quad \tau \quad \begin{array}{ccc}
\Lambda_T & \xrightarrow{\tau} & \Lambda_S \\
\downarrow \gamma_T & & \downarrow \rho_S(\gamma) \\
\Lambda_T & \xrightarrow{\tau} & \Lambda_S \\
\end{array} \]

In order to determine the Galois action $\gamma_T^{T'}$ coming from the fact that $T'$ is defined over $K$, we note that the commutative diagram
\[ \begin{array}{ccc}
G_L & \xrightarrow{\gamma(g-)} & G_L \\
\downarrow \gamma^* & & \downarrow \gamma^* \\
G_L & \xrightarrow{g \cdot \cdot g^{-1}} & G_L \\
\end{array} \quad \begin{array}{ccc}
G_L & \xrightarrow{\gamma(g-)} & G_L \\
\downarrow \gamma^* & & \downarrow \gamma^* \\
G_L & \xrightarrow{g \cdot \cdot g^{-1}} & G_L \\
\end{array} \]

restricts to a commutative diagram
\[ \begin{array}{ccc}
T_L & \xrightarrow{\text{conj}_{\gamma(g)}} & T'_L \\
\downarrow \gamma & & \downarrow \gamma \\
T_L & \xrightarrow{\text{conj}_g} & T'_L, \\
\end{array} \]
which in turn induces the commutative diagram of character lattices

\[
\begin{array}{ccc}
\Lambda'_T & \xrightarrow{\text{conj}_{g}^*} & \Lambda_T \\
\gamma'_T & \cong & \gamma_T \\
\Lambda_T & \xleftarrow{\text{conj}_{g}^*} & \Lambda'_T,
\end{array}
\]

Since \( T \) is split over \( K \), the map \( \gamma'_T \) here is the identity, and hence

\[
\gamma'_T = \text{conj}_{g}^*(g^{-1}).
\]

This shows that the square on the left in (5) commutes as well. Therefore, the composition

\[
(6) \Lambda'_T \xrightarrow{\text{conj}_{g}^*} \Lambda_T \xrightarrow{\tau} \Lambda_S
\]

is Galois-equivariant. In particular, \( T' \) is a Néron–Severi torus.

Now Theorem 4.2 applies to \( \iota' : T' \hookrightarrow G \), the \( T' \)-torsor \( T' := (\text{conj}_{g})_T \) and the resulting \( G \)-torsor \( G' := (\iota')_T' \) over \( S_L \). Indeed, the type of \( T' \) is by construction the composition (6). This composition is Galois-equivariant, as we have just seen. It is also an isomorphism of root data since \( \text{conj}_{g}^* \) preserves the root datum of \( G \) and \( \tau \) respects the relevant root data by assumption. So the hypotheses of Theorem 4.2 are satisfied.

Therefore, the functor \( \iota'_L \) is an equivalence from the groupoid of \( T' \)-torsors \( T\circ \) over \( S \) such that \( T\circ \cong T' \) to the groupoid of \( G \)-torsors \( G\circ \) over \( S \) such that \( G\circ \cong G' \). The former groupoid is equivalent to the groupoid of universal \( T' \)-torsors over \( S \) because \( T' \) is a Néron–Severi torus and the type of \( T' \) is a Galois-equivariant isomorphism. The latter groupoid is the groupoid in the claim because \( G' \cong G \). Indeed, \( G' = (\text{conj}_{g})_L G \) because \( (\iota')_L \circ \text{conj}_{g} = \text{conj}_{g} \circ (\iota_L \circ \text{conj}_{g}) \), and \( \text{conj}_{g} \) is an inner automorphism of \( G_L \).

\[\square\]

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