FROM PARTICLE TO KINETIC AND HYDRODYNAMIC DESCRIPTIONS OF FLOCKING

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Abstract. We discuss the Cucker-Smale’s (C-S) particle model for flocking, deriving precise conditions for flocking to occur when pairwise interactions are sufficiently strong long range. We then derive a Vlasov-type kinetic model for the C-S particle model and prove it exhibits time-asymptotic flocking behavior for arbitrary compactly supported initial data. Finally, we introduce a hydrodynamic description of flocking based on the C-S Vlasov-type kinetic model and prove flocking behavior without closure of higher moments.

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1. Introduction

Collective self-driven motion of self-propelled particles such as flocking of birds and mobile agents, schooling of fishes, swarming of bacteria, appears in many context, e.g., biological organism [2, 7, 8, 9, 17, 18, 21, 24, 25], mobile network [1, 4, 10, 11] appears in many contexts of biological system, mobile and human network [5, 6]. The flocking dynamics
of self-propelled particles is important to understand the nature of the aforementioned self-propelled particles. The terminology "flocking" represents the phenomenon in which self-propelled individuals using only limited environmental information and simple rules, organize into an ordered motion (see [23], and it was a subject of biologists [2, 18]. The study of flocking mechanism based on mathematical models was first started from the work of Vicsek et al [25], and was further motivated by the hydrodynamic approach [23].

Our starting point is a particle description, proposed recently by Cucker-Smale [5, 6], as a new simple dynamical system to explain the emergency of flocking mechanism with birds, with language evolution in primitive societies etc. The Cucker-Smale’s system is different from previous flocking models, e.g., [25], in the sense that the collisionless momentum transfer between particles, \(\{(x_i(t), v_i(t))\}_{i=1}^{N}\), is done through a long-range bi-particle interaction potential, \(r(x, y) = r(|x - y|)\) depending on the distance \(|x - y|\),

\[
\frac{d}{dt}v_i(t) = \frac{\lambda}{N} \sum_{1 \leq j \leq N} r(x_i(t), x_j(t))(v_j(t) - v_i(t)).
\]

The Cucker-Smale’s flocking system (in short C-S system) is reviewed in Section 2. Here we revisit the formation of flocking in C-S dynamics in terms of the fluctuations relative to the center of mass \(x_c(t) := 1/N \sum x_i(t)\). The dynamics of fluctuations makes transparent the flocking dynamics. Our main result, summarized in theorem 2.1 improves [5] for slowly decaying interaction potential, \(r(|x - y|) \sim |x - y|^{-2\beta}, 2\beta \leq 1\). It is shown that flocking emerges in the sense that the following two main features occur: (i) the diameter \(\max |x_i(t) - x_j(t)|\) remains uniformly bounded thus defining the “flock”, and (ii) the “flock” is traveling with a bulk mean velocity which is asymptotically particle-independent, \(v_i(t) \approx v_c := 1/N \sum v_i(0)\).

When the number of particles is sufficiently large, it is not economical to keep track of the motion of each particle through the Cucker-Smale’s system. Instead, one is forced to study the mean field limit of C-S system and we introduce a kinetic description for flocking, in analogy with the Vlasov equation in plasma and astrophysics. In Section 3 we present a Vlasov type mean field model, which is derived from the C-S system using the BBGKY hierarchy in statistical mechanics. The formal derivation, carried in Section 3, follows by taking the limit of an \(N\)-particle interacting system consisting of self-propelled particles governed by C-S flocking dynamics. To this end, let \(f = f(x, v, t)\) denote the one-particle distribution function of such particles positioned at \((x, t) \in \mathbb{R}^d \times \mathbb{R}_+\) with a velocity \(v \in \mathbb{R}^d\); the dynamics of the distribution function \(f\) is determined by

\[
\partial_t f + v \cdot \nabla_x f + \lambda \nabla_v \cdot Q(f, f) = 0,
\]

where \(\lambda\) is a positive constant, and \(Q(f, f)\) is the interaction term

\[
Q(f, f)(x, v, t) := \int_{\mathbb{R}^{2d}} r(x, y)(v_s - v)f(x, v, t)f(y, v_s, t)dv_sdy,
\]

dictated by a prescribed interparticle interaction kernel, \(r = r(x, y)\). We refer to Degond and Motsch [7, 8, 9] for recent kinetic description of Vicsek type model of flocking. The dynamics of particle trajectories of the proposed kinetic description of flocking is analyzed in Section 3.2 in Section 3.3 we prove the global existence of smooth solutions to the kinetic model with arbitrary smooth compactly supported initial data. In Section 4 we show that the kinetic model reveals the time-asymptotic flocking behavior when the bounded interparticle interaction rate has a sufficiently strong long range. Our results are summarized in the
main theorem proving the decay of energy fluctuations, $\Lambda[f](t)$, around the mean bulk velocity, $u_c$,

$$\Lambda[f](t) := \int_{\mathbb{R}^2} |v - u_c|^2 f(x,v,t) dv dx, \quad u_c(t) = \frac{\int_{\mathbb{R}^2} vf(x,v,t) dv dx}{\int_{\mathbb{R}^2} f(x,v,t) dv dx} = u_c(0).$$

Flocking is proved for the restricted range, $2\beta < 1/2$, realized by the asymptotic decay estimate, $\Lambda[f](t) \rightarrow 0$ as $t \rightarrow \infty$.

In Section 5 we turn our attention to the hydrodynamic description of flocking, furnished by moments of the kinetic distribution function. We study the dynamics of the resulting system of balanced laws related to the moments of Vlasov model. Despite the lack of closure, we present a fundamental estimate which enables to conclude the flocking mechanism at the macroscopic hydrodynamic scales. Theorem 5.1 states that the energy-related functional, $\Gamma(t)$

$$\Gamma(t) := \int_{\mathbb{R}^2} \left( \frac{1}{2} |u(x) - u(y)|^2 + e(x) + e(y) \right) \rho(x) \rho(y) dy dx.$$

decays provided the interparticle interaction, $\varphi(s) = \inf r(x(s), y(s))$ decays slowly enough so that its primitive, $\Phi(t)$, diverges. This in turn in related to the increase of entropy

$$\frac{d}{dt} \int_{\mathbb{R}^2} f \log(f) dx dv \geq 0,$$

as particles with increasingly highly correlated velocities flock towards particle-independent bulk velocity.

2. A PARTICLE DESCRIPTION OF FLOCKING

2.1. The Cucker-Smale model. In this section, we briefly review the Cucker-Smale’s flocking system in [5, 6, 22], which manifests the time-asymptotic flocking behavior of many particle systems. We reinterpret the C-S system in terms of fluctuations relative to the center of mass coordinates, which enables us to simplify and sharpen the derivation of sufficient conditions for flocking to occur.

Consider an $N$-particle interacting system consisting of identical particles with mass $m$ to be assumed to be unity. Let $[x_i(t), v_i(t)] \in \mathbb{R}^2$ be the phase space position of an $i$-particle. The Cucker-Smale dynamical system [5, 6] takes the form

$$(2.1) \quad \frac{d}{dt} x_i(t) = v_i(t), \quad \frac{d}{dt} v_i(t) = \frac{\lambda}{N} \sum_{1 \leq j \leq N} r(x_i(t), x_j(t)) (v_j(t) - v_i(t)).$$

Here $\lambda$ is a positive constant, and $r(x, y)$ is a symmetric, bi-particle interaction kernel,

$$(2.2) \quad r(x, y) = r(y, x) \leq A.$$

To discuss the time asymptotic flocking behavior, we will restrict our attention to inter-particle interactions which are decreasing functions of the distance$^1$.

$$(2.3a) \quad r(x, y) = r(|x - y|), \quad r(\cdot) \mathrm{isdecreasing}.$$

A prototype example is the interaction kernel with a polynomial decay of order $2\beta$, [3],

$$(2.3b) \quad r(|x - y|) \geq \frac{A}{(1 + |x - y|^2)^{\beta}}, \quad \beta \geq 0.$$

$^1$To emphasize this point, we therefore continue to refer to general symmetric kernels, $r(x, y) = r(y, x)$, whenever translation invariance is not necessary.
We note in passing that only a lower-bound of the interaction kernel matters.

For notational simplicity we often omit $t$-dependence from the particle identification, abbreviating $x_i \equiv x_i(t)$ and $v_i \equiv v_i(t)$.

Let $m_j(t), \ j = 0, 1, 2$ denote the moments.

$$m_0 := \sum_{i=1}^{N} 1 = N, \quad m_1(t) := \sum_{i=1}^{N} v_i(t), \quad m_2(t) := \sum_{i=1}^{N} |v_i(t)|^2.$$ 

Regarding the dynamics of these moments, we have the following estimate.

**Proposition 2.1.** Let $(x_i(t), v_i(t))$ be the solution to the C-S system (2.1), (2.2). Then the following estimates hold.

(2.4a) \[ \frac{d}{dt} m_1(t) = 0. \]

(2.4b) \[ \frac{d}{dt} m_2(t) = -\frac{\lambda}{N} \sum_{1 \leq i, j \leq N} r(x_i, x_j) |v_j - v_i|^2. \]

(2.4c) \[ m_2(t) \geq m_2(0)e^{-2\lambda At} + \frac{|m_1(0)|^2}{m_0} \left( 1 - e^{-2\lambda At} \right). \]

**Remark 2.1.** Proposition 2.1 tells us that although the kinetic energy $m_2(t)$ is monotonically decreasing, (2.4b), it has the following nonzero lower bound if initial momentum $m_1(0) \neq 0$,

(2.5) \[ m_2(t) \geq \frac{|m_1(0)|^2}{m_0}. \]

Flocking occurs when equality takes place in the Cauchy-Schwarz inequality (2.5).

**Proof.** Conservation of momentum in (2.4a) follows from the symmetry $r(x_i, x_j) = r(x_j, x_i)$, for

$$\frac{d}{dt} \left( \sum_{i=1}^{N} v_i \right) = \frac{\lambda}{N} \sum_{1 \leq i, j \leq N} r(x_i, x_j)(v_j - v_i) = 0.$$

Moreover, symmetry also implies

$$\sum_{1 \leq i, j \leq N} r(x_i, x_j)v_i \cdot (v_i - v_j) = -\sum_{1 \leq i, j \leq N} r(x_i, x_j)v_j \cdot (v_i - v_j)$$

and hence the energy dissipation (2.4b) follows

$$\frac{d}{dt} \left( \sum_{i=1}^{N} |v_i|^2 \right) = -\frac{2\lambda}{N} \sum_{1 \leq i, j \leq N} r(x_i, x_j)v_i \cdot (v_i - v_j)$$

$$= \frac{2\lambda}{N} \sum_{1 \leq i, j \leq N} r(x_i, x_j)v_j \cdot (v_i - v_j) = -\frac{\lambda}{N} \sum_{1 \leq i, j \leq N} r(x_i, x_j)|v_i - v_j|^2.$$
Finally, to prove (2.4c), we use the energy dissipation in (2.4b), the fact that \( r(x_i, x_j) \leq A \) and the conservation of momentum in (2.4a) to find
\[
\frac{d}{dt} m_2(t) = -\frac{\lambda}{N} \sum_{1 \leq i,j \leq N} r(x_i, x_j)|v_i - v_j|^2 \geq -\frac{\lambda}{N} A \sum_{1 \leq i,j \leq N} |v_i - v_j|^2 \\
= -2\lambda A \left( m_2(t) - \frac{|m_1(t)|^2}{N} \right) = -2\lambda A \left( m_2(t) - \frac{|m_1(0)|^2}{m_0} \right).
\]
Gronwall’s lemma yields (2.4c). \( \square \)

2.2. Asymptotic behavior of fluctuations — flocking. We now turn to study the asymptotic time behavior of solutions to C-S system (2.1), (2.3a). To this end, we introduce a center of mass system \((x_c(t), v_c(t))\),
\[
x_c(t) := \frac{1}{N} \sum_{i=1}^{N} x_i(t), \quad v_c(t) := \frac{1}{N} \sum_{i=1}^{N} v_i(t).
\]
Then, thanks to conservation of momentum, the velocity \(v_c\) is constant in \(t\), and the trajectory of center of mass \(x_c\) is a straight line:
\[
v_c(t) = v_c(0), \quad x_c(t) = x_c(0) + tv_c(0).
\]
Observe that the fluctuations around their center of mass,
\[
x_i(t) \mapsto x_i(t) - x_c(t), \quad v_i(t) \mapsto v_i(t) - v_c(t),
\]
satisfy the same C-S system (2.1), (2.3a); it is here that we take into account the fact that the interparticle kernel depends on the distance, \(r(x, y) = r(|x - y|)\). We shall show that under appropriate conditions, flocking occurs in the sense that these fluctuations decay in time. Thus, the time-asymptotic dynamics of C-S solutions emerges as a linear movement with a fixed velocity dictated by the coordinates of center of mass.

To proceed, we introduce the two auxiliary functions which measure the fluctuations of the fluctuations around their center of mass,
\[
X(t) := \sum_{1 \leq i \leq N} |x_i(t) - x_c(t)|^2, \quad V(t) := \sum_{1 \leq i,j \leq N} |v_i(t) - v_c(0)|^2,
\]
subject to initial conditions \((X_0, V_0) = (X(0), V(0))\). The flocking behavior will depend in an essential way on the behavior of the minimal value of the interparticle interaction at time \(t\),
\[
\varphi(t) := \min_{1 \leq i,j \leq N} r(x_i(t), x_j(t)).
\]
We begin with the fluctuations of velocities.

Lemma 2.1. [Fluctuations of velocities]. Let \((x_i(t), v_i(t))\) be the solution of the system (2.1), (2.3a). Then we have
\[
V(t) \leq V_0 e^{-2\lambda \Phi(t)}, \quad \Phi(t) := \int_0^t \varphi(\tau)d\tau.
\]

Proof. We invoke (2.4b) with \(v_i(t) - v_c(t)\) replacing \(v_i(t)\) to find that
\[
\frac{d}{dt} \sum_{i=1}^{N} |v_i - v_c|^2 = -\frac{\lambda}{N} \sum_{1 \leq i,j \leq N} r(x_i, x_j)|v_j - v_i|^2.
\]
Since $\sum_i (v_i - v_c) = 0$, we have $\sum_{1 \leq i, j \leq N} |v_i - v_j|^2 = 2NV(t)$, and the result follows from Gronwall’s integration of
\[
\frac{d}{dt} V(t) = -\frac{\lambda}{N} \sum_{1 \leq i, j \leq N} r(x_i, x_j)|v_j - v_i|^2 \leq -2\lambda \varphi(t)V(t).
\]

**Remark 2.2.** Lemma 2.1 implies the sufficient condition for flocking is that the interparticle interaction potential decays sufficiently slow, so that its primitive, $\Phi(t)$, diverges, i.e.,
\[
\lim_{t \to \infty} \Phi(t) = \infty \quad \text{then} \quad \lim_{t \to \infty} |v_i(t) - v_c| = 0, \quad i = 1, \ldots, N.
\]

The answer whether $\varphi(t)$ decays sufficiently slow to enforce flocking depends on the variance of positions $x_i(t)$.

**Lemma 2.2.** [Fluctuations of positions]. Let $(x_i(t), v_i(t))$ be the solution of the system (2.1), (2.3a). Then we have
\[
X(t) \leq 2X_0 + V_0 t^2, \quad t \geq 0.
\]

**Proof.** We use Cauchy-Schwartz’s inequality to see
\[
\frac{d}{dt} \sum_{i=1}^{N} |x_i - x_c|^2 = \sum_{i=1}^{N} (x_i - x_c) \cdot \left( \frac{dx_i}{dt} - \frac{dx_c}{dt} \right)
= \sum_{i=1}^{N} (x_i - x_c) \cdot (v_i - v_c) \leq \sqrt{\sum_{i=1}^{N} |x_i - x_c|^2} \sqrt{\sum_{i=1}^{N} |v_i - v_c|^2}.
\]

Using Lemma 2.1 we obtain
\[
\frac{d}{dt} X(t) \leq \sqrt{V(t)} \sqrt{X(t)} \leq \sqrt{V_0} e^{-\lambda \varphi(t)} \sqrt{X(t)},
\]
and the solution of this differential inequality yields
\[
X(t) \leq 2X_0 + \frac{V_0}{2} \left[ \int_{0}^{t} e^{-\lambda \varphi(\tau)} d\tau \right]^2 \leq 2X_0 + V_0 t^2.
\]

As a corollary of Lemma 2.2 we now obtain the desired lower bound for $\varphi(t)$.

**Corollary 2.1.** Let $(x_i(t), v_i(t))$ be the solutions to (2.1), (2.3a). Then $\varphi(t)$ satisfies
\[
\varphi(t) \geq r(\sqrt{2X(t)}) \geq r(\sqrt{4X_0 + V_0 t^2}).
\]

**Proof.** We use Lemma 2.2 to see that for each $i, j \in \{1, \ldots, N\}$,
\[
|x_i - x_j|^2 \leq 2(|x_i - x_c|^2 + |x_j - x_c|^2) \leq 2X(t) \leq 4X_0 + V_0 t^2.
\]
Since $r(\cdot)$ is decreasing, we have $\varphi(t) = \min_{1 \leq i, j \leq N} r(x_i(t), x_j(t)) \geq r(\sqrt{4X_0 + V_0 t^2})$. □

The asymptotic flocking now depends on the specific decay of the interparticle interaction $r(\cdot)$. As an example, consider the C-S system with the $2\beta$ interaction (2.1), (2.3b), where
\[
\varphi(t) \geq A(1 + 4X_0 + V_0 t^2)^{-\beta} \geq A\kappa_1 (1 + t)^{-2\beta}, \quad \kappa_1 := \left( \max\{1 + 4X_0, V_0\} \right)^{-\beta}.
\]
We conclude that the divergence of \( \Phi(t) = \int_0^t \varphi(\tau)d\tau \) and hence, by (2.8) that flocking occurs, for \( 2\beta < 1 \). This recovers the Cucker-Smale result [5, 6]. Below we improve the Cucker-Smale result proving unconditional flocking result for \( \beta = 1/2 \).

**Theorem 2.1.** Let \((x_i(t), v_i(t))\) be the solutions to (2.1), (2.3b) with \( V_0 > 0 \). Then the following holds.

(i) There exist a positive constant, \( C_2 \) (depending only on \( \kappa_1, A \) and \( \beta \) as specified in (2.12) below), such that

\[
|x_i(t) - x_c| \lesssim |x_i(0) - x_c| + C_2.
\]

(ii) There exists constants, \( \kappa_i > 0, i = 1, 2 \), such that

\[
|v_i(t) - v_c| \lesssim \sqrt{V_0} \times \begin{cases} 
 e^{-\lambda A\kappa_2 t}, & \beta \in [0, \frac{1}{2}), \\
 (1 + t)^{-\lambda A\kappa_1}, & \beta = \frac{1}{2}, \\
 & \kappa_1 = \max\{1 + 4X_0, V_0\}^{-\beta}.
\end{cases}
\]

**Remark 2.3.** Theorem 2.1 shows the two main features of flocking occur with the \( 2\beta \)-interaction potential, \( 2\beta \leq 1 \), namely, the diameter \( \max |x_i(t) - x_j(t)| \) remains uniformly bounded thus defining the traveling “flock” with velocity which is asymptotically particle-independent, \( v_i(t) \approx v_c \).

**Proof.** We begin with the case \( 0 \leq \beta < \frac{1}{2} \). To get the optimal exponential convergence rate, we employ a bootstrapping argument in three steps.

**Step 1.** We first obtain a weak integrable decay rate for \( |v_i - v_c| \). Using (2.10) we find

\[
- \int_0^t \varphi(\tau)d\tau \leq -A\kappa_1 \int_0^t (1 + \tau)^{-2\beta}d\tau \lesssim C_1 \left( 1 - (1 + t)^{1-2\beta} \right), \quad 0 \leq \beta < \frac{1}{2}.
\]

The above estimate together with Lemma 2.1 yield \( V(t) \lesssim V_0 e^{-2C_1(1+t)^{1-2\beta}} \).

**Step 2.** Next, we improve lemma 2.2 observing that for \( \beta < 1/2 \), the position \( X(t) \) remains uniformly bounded in time. Indeed, we have,

\[
x_i(t) = x_i(0) + \int_0^t v_i(\tau)d\tau, \quad x_c(t) = x_c(0) + \int_0^t v_c d\tau.
\]

Time integrability of \( |v_i(t) - v_c(t)| \) then yields (2.11a),

\[
|x_i(t) - x_c(t)| \leq |x_i(0) - x_c(0)| + \int_0^t |v_i(\tau) - v_c|d\tau
\]

\[
\lesssim |x_i(0) - x_c(0)| + \int_0^\infty e^{-C_1(1+t)^{1-2\beta}}dt \leq |x_i(0) - x_c(0)| + C_2,
\]

where

\[
C_2 \lesssim \int_0^\infty e^{-C_1(1+t)^{1-2\beta}}dt < \infty, \quad 0 < \beta < \frac{1}{2}.
\]

**Step 3.** The uniform bound of \( X(t) \) implies an improved estimate for the interparticle interaction \( \varphi(t) \): corollary 2.1 implies

\[
\varphi(t) \geq r(\sqrt{2X_0}) \geq A(1 + 4X_0 + 8C_2^2)^{-\beta} = A\kappa_2,
\]

which in turn, using lemma 2.1 yields the optimal exponential convergence rate (2.11b)

\[
|v_i(t) - v_c(t)|^2 < V(t) \leq V_0 e^{-2\lambda \Phi(t)} \lesssim e^{-2\lambda A\kappa_2 t}.
\]
It remains to deal with the case $\beta = \frac{1}{2}$. Here, we have

$$-2\lambda \Phi(t) \leq -2\lambda A \left( \max\{1+4X_0, V_0\} \right)^{-\frac{1}{2}} \int_0^t (1+\tau)^{-1}d\tau = -2\lambda A \kappa_1 \ln(1+t),$$

which in turn implies \((2.1b)\), $V(t) \leq V_0 e^{-2\lambda \Phi(t)} \leq V_0 (1+t)^{-2\lambda A \kappa_1}$.

Remarks.

1. Consider the borderline case $\beta = \frac{1}{2}$ with initial configuration satisfying

$$\lambda \kappa_1 > 1, \quad \text{i.e.}, \quad \sqrt{\max\{1+4X_0, V_0\}} < \lambda,$$

then the same bootstrapping argument used for $\beta \in [0, \frac{1}{2})$ gives the exponential convergence:

$$|v_i(t) - v_c| \leq \sqrt{V_0 e^{-A \kappa_2 t}}.$$

2. Flocking occurs even if $\beta > \frac{1}{2}$, but only for special initial configurations. Sufficient flocking conditions for such initial profiles is presented in [5].

3. From particle to kinetic description of flocking

3.1. Derivation of a mean-field model. We assume that the number of particles involved in the C-S model \((2.1), (2.2)\) is large enough that it becomes meaningful to observe the $N$-particle distribution function,

\[(3.1a) \quad f^N = f^N(x, v_1, \ldots, x_N, v_N, t), \quad (x_i, v_i) \in \mathbb{R}^d \times \mathbb{R}^d.\]

Since particles are indistinguishable, the probability density $f^N = f^N(\cdot)$ is symmetric in its phase-space arguments,

\[(3.1b) \quad f^N(\cdot, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot, t) = f^N(\cdot, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot, t),\]

so we can ‘probe’ $f^N$ by any of its $N$ pairs of phase-variables. Let $f^N(\cdot, \cdot, t)$ denote the marginal distribution

$$f^N(x_1, v_1, t) := \int_{\mathbb{R}^{2d(N-1)}} f^N(x_1, v_1, x_-, v_-, t) d x_- d v_-, \quad (x_-, v_-) := (x_2, v_2, \ldots, x_N, v_N).$$

The formal derivation of a kinetic description for the C-S particle system \((2.1), (2.2)\) is carried out below using the BBGKY hierarchy, e.g., [3, 19, 20], based on the Liouville equation, [15]

\[(3.2) \quad \partial_t f^N + \sum_{i=1}^N v_i \cdot \nabla x_i f^N + \frac{\lambda}{N} \sum_{i=1}^N \nabla v_i \cdot \left( \sum_{j=1}^N r(x_i, x_j)(v_j - v_i) f^N \right) = 0.\]

To this end, one study the marginal distribution $f^N(x_1, v_1, t)$ by integration of \((3.2)\) with respect to $d x_- d v_- = dv_2 dx_2 \cdots dv_N dx_N$ (to simplify the notations, we now suppress the time-dependence whenever it is clear by the context, denoting $f^N(x_1, v_1, \ldots, x_N, v_N, t) = f^N(x_1, v_1, \ldots, x_N, v_N)$). Since $f^N(\cdot, \cdot)$ is rapidly decaying at infinity, the transport term in \((3.2)\) amounts to

\[(3.3) \quad \int_{\mathbb{R}^{2d(N-1)}} \sum_{i=1}^N v_i \cdot \nabla x_i f^N d x_- d v_- = v_1 \cdot \nabla x_1 f^N(x_1, v_1).\]
The corresponding integration of the forcing term in (3.2), yields
\[
\frac{\lambda}{N} \sum_{i=1}^{N} \int_{\mathbb{R}^{2d(N-1)}} \sum_{j=1}^{N} \nabla_{v_{i}} \cdot \left( r(x_{i}, x_{j})(v_{j} - v_{i}) f^{N} \right) dx_{-} dv_{-}
\]
\[
= \frac{\lambda}{N} \int_{\mathbb{R}^{2d(N-1)}} \sum_{2 \leq j \leq N} \nabla_{v_{1}} \cdot \left( r(x_{1}, x_{j})(v_{j} - v_{1}) f^{N} \right) dx_{-} dv_{-}.
\]

But the symmetry of \( f^{N} \), (3.1b), implies that the integrals being summed above are the same for \( j = 2, 3 \ldots, N \). Consequently, it will suffice to consider \( j = 2 \):
\[
\frac{\lambda}{N} \sum_{i=1}^{N} \int_{\mathbb{R}^{2d(N-1)}} \sum_{j=1}^{N} \nabla_{v_{i}} \cdot \left( r(x_{i}, x_{2})(v_{2} - v_{i}) f^{N} \right) dx_{-} dv_{-}
\]
\[
= \frac{\lambda}{N} (N - 1) \int_{\mathbb{R}^{2d(N-1)}} r(x_{1}, x_{2}) \nabla_{v_{1}} \cdot \left( (v_{2} - v_{1}) f^{N} \right) dx_{2} dv_{2} \cdots dx_{N} dv_{N}
\]
\[
= \left( \lambda - \frac{\lambda}{N} \right) \nabla_{v_{1}} \cdot \left( \int_{\mathbb{R}^{2d}} r(x_{1}, x_{2})(v_{2} - v_{1}) g^{N} dx_{2} dv_{2} \right).
\]

Here \( g^{N} \) is the two-particle marginal function
\[
g^{N}(x_{1}, v_{1}, x_{2}, v_{2}, t) := \int_{\mathbb{R}^{2d(N-2)}} f^{N} dx_{3} dv_{3} \cdots dx_{N} dv_{N}.
\]

Thus, in view of (3.3) and (3.4), marginal integration of (3.2) over \((x_{-}, v_{-})\) implies that the one-particle density function, \( f^{N}(x_{1}, v_{1}, t) \), satisfies
\[
\partial_{t} f^{N} + v_{1} \cdot \nabla_{x_{1}} f^{N} + \left( \lambda - \frac{\lambda}{N} \right) \nabla_{v_{1}} \cdot \left( \int_{\mathbb{R}^{2d}} r(x_{1}, x_{2})(v_{2} - v_{1}) g^{N} dx_{2} dv_{2} \right) = 0.
\]

We now take the mean-field limit \( N \rightarrow \infty \): we end up with the one- and two-particle limiting densities, \( f := \lim_{N \rightarrow \infty} f^{N}(x_{1}, v_{1}) \) and \( g := \lim_{N \rightarrow \infty} g^{N}(x_{1}, v_{1}, x_{2}, v_{2}) \), which satisfy
\[
\partial_{t} f + v_{1} \cdot \nabla_{x_{1}} f + \lambda \nabla_{v_{1}} \cdot \left( \int_{\mathbb{R}^{2d}} r(x_{1}, x_{2})(v_{2} - v_{1}) g dx_{2} dv_{2} = 0 \right).
\]

To close the above equation we make the “molecular chaos” assumption about the independence of the two-point particle distribution,
\[
g(x_{1}, v_{1}, x_{2}, v_{2}, t) = f(x_{1}, v_{1}, t) f(x_{2}, v_{2}, t);
\]
Relabel, \((x_{1}, v_{1}) \mapsto (x, v)\) and \((x_{2}, v_{2}) \mapsto (y, v_{*})\). We conclude that the one-particle distribution function \( f(x, v, t) \) satisfies the Vlasov-type mean-field model,
\[
\partial_{t} f + v \cdot \nabla_{x} f + \lambda \nabla_{v} \cdot Q(f, f) = 0,
\]
\[
Q(f, f)(x, v, t) := \int_{\mathbb{R}^{2d}} r(x, y)(v_{*} - v)f(x, v, t)f(y, v_{*}, t) dv_{*} dy.
\]

Here, \( Q(f, f) \) is the quadratic interaction which can be expressed in the equivalent form
\[
Q(f, f)(x, v, t) = fL[f], \quad L[f](x, v, t) := \int_{\mathbb{R}^{2d}} r(x, y)(v_{*} - v)f(y, v_{*}, t) dv_{*} dy.
\]
3.2. A priori estimates. We begin our study with a series of a priori estimates on the solution of the mean-field model (3.6), and the growth rate of the $x$ and $v$-support of $f$. We first set

$$
\psi_0(\xi) := \xi, \quad \psi_i(\xi) := \xi_i \quad i = 1, \ldots, d, \quad \text{and} \quad \psi_{d+1}(\xi) := |\xi|^2.
$$

Let $f$ be a classical solution to (3.6) with a rapid decay in phase space $\mathbb{R}^{2d}$. A straightforward integration of (3.6) yields

\begin{equation}
\frac{d}{dt} \int_{\mathbb{R}^{2d}} \psi_i(x,v) f(x,v) dv dx = \int_{\mathbb{R}^{2d}} \nabla_v \psi_i(x,v) \cdot Q(f,f) dv dx,
\end{equation}

\begin{equation}
\frac{d}{dt} \int_{\mathbb{R}^{2d}} \psi(x,v) f(x,v) dv dx = \int_{\mathbb{R}^{2d}} \nabla_x (\psi(x,v) \cdot f(x,v)) dv dx.
\end{equation}

Using (3.7) we obtain

**Proposition 3.1.** Let $f$ be a classical solutions decaying fast enough at infinity in phase space. Then following macroscopic quantities associated with $f$, satisfy

\begin{equation}
\frac{d}{dt} \int_{\mathbb{R}^{2d}} v f(x,v) dv dx = 0;
\end{equation}

\begin{equation}
\frac{d}{dt} \int_{\mathbb{R}^{2d}} |v|^2 f(x,v) dv dx = - \int_{\mathbb{R}^{2d}} r(x,y) |v - v_s|^2 f(x,v) f(y,v_s) dv y dv dx;
\end{equation}

\begin{equation}
\frac{d}{dt} \int_{\mathbb{R}^{2d}} f^p(x,v) dv dx = -d(p-1) \int_{\mathbb{R}^{4d}} r(x,y) f(y,v_s) f^p(x,v) dv_y dv dx dv dy dx.
\end{equation}

**Proof.** Equality (3.8a) follows from (3.7a) with $\psi_i(v) = v_i$,

$$
\int_{\mathbb{R}^{2d}} \nabla_v \psi_i(v) \cdot Q(f,f) dv dx = \int_{\mathbb{R}^{4d}} r(x,y)(v_{s_i} - v_i) f(y,v_s) f(x,v) dv_y dv dx dv = 0.
$$

The last integral vanishes due to antisymmetry of the integrand, realized by the interchange of variables $(x,v) \leftrightarrow (y,v_s)$. The statement of (3.8b) follows from (3.7b) with $\psi_{d+1}(v) = |v|^2$, and observing that

$$
\int_{\mathbb{R}^{2d}} \nabla_v \psi_{d+1}(v) \cdot Q(f,f) dv dx = 2 \int_{\mathbb{R}^{2d}} v \cdot Q(f,f) dv dx
$$

$$
= 2 \int_{\mathbb{R}^{4d}} r(x,y) v \cdot (v_s - v) f(y,v_s) f(x,v) dv_y dv dx dv
$$

$$
= -2 \int_{\mathbb{R}^{4d}} r(x,y) v_s \cdot (v_s - v) f(y,v_s) f(x,v) dv_y dv dx dv
$$

$$
= - \int_{\mathbb{R}^{4d}} r(x,y) |v - v_s|^2 f(x,v) f(y,v_s) dv_y dv dx dv dx.
$$

Finally, we note the two identities, $f^{p-1} v \cdot \nabla_v f = \frac{1}{p} v \cdot \nabla_v f^p$, and $f^{p-1} \nabla_v \cdot Q(f,f) \equiv \nabla_v \cdot \left( \frac{L[f]}{p} \right) f^p + \left( 1 - \frac{1}{p} \right) \left( \nabla_v \cdot L[f] \right) f^p$, $Q(f,f) = fL[f]$.

Integration of (3.6) against $f^{p-1}$ then yields

$$
\frac{d}{dt} \int_{\mathbb{R}^{2d}} f^p dv dx = -p \int_{\mathbb{R}^{2d}} f^{p-1} \left( v \cdot \nabla_x f + \nabla_v \cdot Q(f,f) \right) dv dx
$$

$$
= -(p-1) \int_{\mathbb{R}^{2d}} \left( \nabla_v \cdot L[f] \right) f^p dv dx.
\[ -d(p - 1) \int_{\mathbb{R}^{4d}} r(x, y)f(y, v_s)f^p(x, v)dvdydx. \]

Let \( f \) be a classical kinetic solution of (3.6). The statement of (3.8c) shows that its \( L^1(dx dv) \)-norm, the total macroscopic mass is conserved in time (while according to (3.8c), higher \( L^p(dx dv) \)-norms of \( f \) decay in time),

\[
\begin{align*}
(3.9a) \quad & M_0(t) := \int_{\mathbb{R}^{2d}} f(x, v, t)dxdv \equiv M_0. \\
(3.9b) \quad & M_1(t) := \int_{\mathbb{R}^{2d}} vf(x, v, t)dxdv \equiv M_1.
\end{align*}
\]

Similarly, (3.8a) tells us that the total macroscopic momentum is conserved in time,

\[
M_2(t) := \int_{\mathbb{R}^{2d}} |v|^2f(x, v, t)dxdv \leq M_2(0).
\]

Finally, (3.8b) tells us that the total amount of macroscopic energy is non-increasing in time,

\[
M_0 := M_0(0), \quad M_1 := M_1(0) \text{ and } M_2(0) \text{ denote, respectively, the initial amounts of mass, momentum and energy at } t = 0. \text{ Next, we turn to the following a priori bound on the kinetic velocity.}
\]

**Lemma 3.1.** Let \( [x(t), v(t)] \) be the particle trajectory issued from \( (x, v) \in \text{supp}_{x,v}f_0 \) at time 0. Then the \( i \)-component of velocity trajectory, \( v_i(t) = v_i(t; 0, x, v), \ i = 1, \cdots, d, \) satisfies

\[
v_i(t) \in \left( v_i(0)e^{-\lambda AM_0t} - \frac{J_0}{M_0}(1 - e^{-\lambda AM_0t}), \ v_i(0)e^{-\lambda AM_0\Phi(t)} + \lambda AJ_0 \int_0^t e^{-\lambda AM_0(\Phi(t) - \Phi(s))}ds \right).
\]

Here, \( \Phi(t) := \int_0^t \varphi(s)ds, \ M_0 = \|f_0\|_{L^1_{x,v}} \) is the initial total mass and \( J_0 := \sqrt{\lambda M_0 M_2} \).

**Proof.** For given \( (x, v) \in \text{supp}_{x,v}f_0 \), we set \( x(s) \equiv x(s; 0, x, v) \) and \( v(s) \equiv v(s; 0, x, v) \). Note that for each \( i = 1, \cdots, d, \) we have

\[
(3.10) \quad L_i[f(x(t), v(t), t)] = \int_{\mathbb{R}^{2d}} r(x, y)(v_{si}(t) - v_i)f(y, v_s)dvdy
\]

\[
= v_i(t) + \int_{\mathbb{R}^{2d}} r(x, y)v_{si}f(y, v_s)dvdy - \left( \int_{\mathbb{R}^{2d}} r(x, y)f(y, v_s)dvdy \right).
\]

Lower and upper bounds for the kinetic velocities are obtained in terms of the estimates,

\[
\varphi(t)M_0 \leq \int_{\mathbb{R}^{2d}} r(x, y)f(y, v_s)dvdy \leq AM_0,
\]

\[
\left| \int_{\mathbb{R}^2} r(x, y)vf(y, v_s)dvdy \right| \leq A \sqrt{\|f_0\|_{L^1_{x,v}}} \sqrt{\|v\|^2f_0\|_{L^1_{x,v}}} = AJ_0.
\]

It then follows from (3.10) that

\[
-\lambda AM_0v_i(s) - \lambda AJ_0 \leq \frac{d}{dt}v_i(t) = \lambda L_i[f(x(t), v(t), t)] \leq -\lambda \varphi(t)M_0v_i(t) + \lambda AJ_0,
\]

and the desired result follows by Gronwall's integration.
Remark 3.1. Let $\Omega(t)$ denote the $v$-projection of $\text{supp} f(\cdot, t)$,
\begin{equation}
\Omega(t) := \{v \in \mathbb{R}^d : \exists (x, v) \in \mathbb{R}^{2d} \text{ such that } f(x, v, t) \neq 0\}.
\end{equation}

Lemma 3.7 shows that if $f_0(x, \cdot)$ is compactly supported, then $\text{supp} f(x, \cdot, t)$ remains finite, with a weak growth estimate for the velocity trajectory,
\begin{equation}
|v_i(t)| \leq \max \left\{ \eta_0 + \frac{J_0}{M_0}, \eta_0 + \lambda J_0 t \right\} \leq \eta_0 + \frac{J_0}{M_0} + \lambda J_0 t,
\end{equation}

3.3. Global existence of classical solutions. In this section we develop a global existence theory for classical solutions of the Vlasov-type flocking equation,
\begin{align}
\partial_t f + v \cdot \nabla_x f + \lambda \nabla_v \cdot \left( f L[f] \right) &= 0, \quad x, v \in \mathbb{R}^d, t > 0, \\
L[f](x, v, t) &= \int_{\mathbb{R}^{2d}} r(x, y)(v_* - v)f(y, v_*, t)dv_*dy, \quad r(x, y) = \frac{A}{(1 + |x - y|^2)\gamma}, \\
\text{subject to initial datum} \quad f(x, v, 0) &= f_0(x, v).
\end{align}

We begin by noting that the kinetic solution $f$ remains uniformly bounded. To this end, rewrite the mean-field model (3.13) in a 'non-conservative' form,
\begin{equation}
\partial_t f + v \cdot \nabla_x f + \lambda L[f] \cdot \nabla_v f = -\lambda f \nabla_v \cdot L[f], \quad x, v \in \mathbb{R}^d, t > 0.
\end{equation}

Consider the particle trajectories, $[x(t), v(t)] \equiv [x(t; t_0, x_0, v_0), v(t; t_0, x_0, v_0)]$, passing through $(x_0, v_0) \in \mathbb{R}^d \times \mathbb{R}^d$ at time $t_0 \in \mathbb{R}_+$,
\begin{equation}
\frac{dx}{dt}(t) = v(t), \quad \frac{dv}{dt}(t) = \lambda L[f(x(t), v(t), t)].
\end{equation}

Noting that $-\nabla_v \cdot L[f] = d \int_{\mathbb{R}^{2d}} r(x, y)f(y, v_*, t)dv_*dy$, we find
\[\|\nabla_v \cdot L[f]\|_{L_2^{\infty}} \leq dA\|f\|_{L_2^{1,\infty}} = dA M_0,\]

which implies that the following inequality holds along the particle trajectories,
\[\frac{df}{dt}(x(t), v(t), t) \leq \lambda dA M_0 f(x(t), v(t), t).\]

It follows that as long as initial data $f_0$ has a finite mass, there will be no finite time blow-up for $f(\cdot, t)$,
\begin{equation}
\|f(t)\|_{L_2^{\infty}} \leq e^{\lambda dA M_0 t}\|f_0\|_{L_2^{\infty}}.
\end{equation}

Next, we turn to study the smoothness of $f(\cdot, t)$. Since the local existence theory will be followed from the standard fixed point argument, e.g., [3], we only obtain a priori $C^1$-norm bound of $f$ to conclude a global existence of classical solutions.

Theorem 3.1. Consider the flocking kinetic model (3.13). Suppose that the initial datum $f_0 \in (C^1 \cap W^{1,\infty})(\mathbb{R}^{2d})$ satisfies

1. Initial datum is compactly supported in the phase space, $\text{supp}_{(x,v)} f_0(\cdot)$ is bounded, and in particular, $\Omega(0) \subset B_{r_0}(0)$.

2. Initial datum is $C^1$-regular and bounded:
\[\sum_{0 \leq |\alpha| + |\beta| \leq 1} \|\nabla_x^\alpha \nabla_v^\beta f_0\|_{L_2^{\infty}} < \infty.\]
Then, for any $T \in (0, \infty)$, there exists a unique classical solution $f \in C^1([0, T) \times \mathbb{R}^d)$. 

**Proof.** We express the non-conservative kinetic model (3.14) in terms of the nonlinear transport operator $\partial_t + v \cdot \nabla_x + \lambda L[f] \cdot \nabla_v$, 

\begin{equation}
Tf = -\lambda f \nabla_v \cdot L[f], \quad T := \partial_t + v \cdot \nabla_x + \lambda L[f] \cdot \nabla_v.
\end{equation}

We claim that there exist (possibly different) positive constants, $C = C(d, \lambda, M_0, J_0) > 0$, such that 

\begin{align}
|T(f)| &\leq C|f|, 
|T(\partial_x f)| \leq C \left( |f| + (\eta(t) + 1)|\nabla_v f| + |\partial_x f| \right), 
\eta(t) := \max_{v \in \Omega(t)} |v|,
\end{align}

(3.19c)To verify these inequalities, observe that by (3.18) 

$\frac{d}{dt} \int_{\mathbb{R}^d} (1 + |x - y|^2)^{\beta+1} \frac{f(y, v_s, t)dv_sdy}{2\beta dA(x_i - y_i)} (v_s - v)f(y, v_s, t)dv_sdy,$

and (3.19a) follows with $C := \lambda d A M_0 \geq \lambda \|\nabla_v \cdot L[f](t)\|_{L^\infty_{v'}}.$

Next, differentiating (3.18) we obtain 

$T(\partial_x f) = -\lambda(\partial_x L[f]) \cdot \nabla_v f - \lambda(\partial_x \nabla_v \cdot L[f])f - \lambda(\nabla_v \cdot L[f])\partial_x f.$

Straightforward calculation yields 

$\partial_x L[f] = -\int_{\mathbb{R}^d} \frac{2\beta Ad(x_i - y_i)}{(1 + |x - y|^2)^{\beta+1}} (v_s - v)f(y, v_s, t)dv_sdy,$

and since the variation of the relevant kinetic velocities at time $t$ do not exceed $|v - v_s| \leq 2\eta(t)$, we find $\|\partial_x L[f](t)\| \leq 4\beta A n(t) M_0$; similarly, 

$\partial_x \nabla_v \cdot L[f] = \int_{\mathbb{R}^d} \frac{2\beta Ad(x_i - y_i)}{(1 + |x - y|^2)^{\beta+1}} f(y, v_s, t)dv_sdy \quad \Rightarrow \quad \|\partial_x \nabla_v \cdot L[f](t)\|_{L^\infty_{v'}} \leq 2\beta d A M_0.$

We conclude that (3.19b) holds with, say, $C = \lambda d A M_0 (1 + 2\beta + 4\eta(t))$.

Finally, we differentiate (3.15) with respect to $v_i$, noting that $\partial_{v_i} \nabla_v \cdot L[f] = 0$ 

$T(\partial_{v_i} f) = -\lambda \partial_{v_i} f - \lambda(\partial_{v_i} L[f]) \cdot \nabla_v f - \lambda(\nabla_v \cdot L[f])\partial_{v_i} f.$

Straightforward calculation then yields, 

$\partial_{v_i} L[f] = -\int_{\mathbb{R}^d} r(x, y) f(y, v_s, t)dv_sdy \quad \Rightarrow \quad \|\partial_{v_i} L[f](t)\|_{L^\infty_{v'}} \leq M_0,$

and (3.19c) follows with $C = \lambda + \lambda(d + 1) A M_0$.

Now, let $F(t)$ measure the $W^{1,\infty}$-norm of $f(\cdot, t)$ 

$F(t) := \sum_{0 \leq |\alpha| + |\beta| \leq 1} \|\nabla_x^\alpha \nabla_v^\beta f(t)\|_{L^\infty_{v'}}.$

The inequalities (3.19) imply 

$\frac{d}{dt} F(t) \leq (\eta(t) + 1) F(t).$

Lemma 3.1 (see remark 3.1), tells us that $\eta(t) \lesssim \eta_0 + t$, and we end up with the energy bound 

$F(t) \leq F(0)e^{C(t + t^2)}, \quad C = C(\eta_0, M_0, J_0, \beta, d, A).$
Equipped with this a priori $W^{1,\infty}$ estimate, standard continuation principle yields a global extension of local classical solutions.

**Remarks.**

1. The above a priori estimate need not be optimal. Since we used a rough estimate \[3.12\] for the size of $\Omega(t)$,
\[
\max_{v \in \Omega(t)} |v| \lesssim \eta_0 + t,
\]
we end with the quadratic exponential growth, $e^{C(t+t^2)}$. An optimal bound, however, could be $e^{Ct}$. Of course, one cannot expect a uniform bound for $C^1$-norm, because the one-particle distribution function may grow exponentially along the particle trajectory (see \[3.16\]).

2. The global existence of classical solution can be improved for more general kernels.

3. For related works on kinetic granular type dissipative systems, we refer to \[12, 13, 14\].

### 4. Time-asymptotic behavior of kinetic flocking

In this section, we present the time-asymptotic flocking behavior of the kinetic model for flocking \[3.13\]. As in the case with particle description discussed in Section 2, we will show that the velocity of particles will contracted to the mean bulk velocity $u_c$, which corresponds to the velocity at the center of mass:

\[
u_c(t) := \frac{1}{M_0} \int_{\mathbb{R}^{2d}} v f(x, v,t) dv, \quad u_c(t) \equiv u_c(0).
\]

We recall that the energy decay in \[3.8\]

\[
\frac{d}{dt} M_2(t) \leq 0.
\]

We note that unlike granular flows, for example, e.g. \[3\], the energy decay \[4.1\] does *not* drive the energy to zero: if the initial momentum $M_1 \neq 0$, then the kinetic energy $M_2(t)$ has a nonzero lower bound, in analogy with the discrete case, consult remark 2.1. Indeed, since the total mass and momentum are conserved, $M_i(t) \equiv M_i, i = 0, 1$, \[3.8\] implies

\[
\frac{d}{dt} M_2(t) \geq -A \int_{\mathbb{R}^{2d}} |v - v_s|^2 f(x, v)f(y, v_s) dv dv_y dydx = -2A M_0 M_2(t) + 2A |M_1|^2,
\]

and Gronwall’s lemma yields the following kinetic analog of \[2.46, 2.5\] \[4.2\]

\[
M_2(t) \geq M_2(0)e^{-2AM_0t} + \frac{|M_1|^2}{M_0} (1 - e^{-2AM_0t}) \geq \frac{|M_1|^2}{M_0}.
\]

Thus, energy decay by itself does not assert flocking. As in the particle description, the emergence of the time-asymptotic flocking behavior depends on the sufficiently slow decay rate of the interparticle interaction $\varphi(s) = r(x(s), y(s))$. To this end, we let $\Sigma$ denote the $x$-projection of $\supp f(\cdot, t)$

\[
\Sigma(t) := \{ x \in \mathbb{R}^d : \exists (x, v) \in \mathbb{R}^{2d} \text{ such that } f(x, v, t) \neq 0 \},
\]

and denote its initial size, $\zeta_0 := \max_{x \in \Sigma_0} |x|$. 
Lemma 4.1. Let $f$ be a global classical solution to \((3.13)\). Then, there exists a $\kappa_3 > 0$ such that $\varphi(t) = \inf_{(x,y) \in \Sigma(t)} \tau(x,y)$ satisfies
\begin{equation}
\varphi(t) \geq A\kappa_3^{-\beta}(1 + t^2 + t^4)^{-\beta}.
\end{equation}

The constant $\kappa_3$ is given by
\begin{equation}
\kappa_3 := \max \left\{ 1 + 12 \zeta_0^2, 12 \left( \eta_0 + \frac{J_0}{M_0} \right)^2, 3\lambda^2 J_0^2 \right\}.
\end{equation}

Proof. Let $(x,v) \in \text{supp}_{x,v} f$. It follows from remark 3.1 that
\begin{equation}
|v_i(t)| \leq \eta_0 + J_0 M_0 + \lambda J_0 t,
\end{equation}
and hence
\begin{equation}
|x_i(t;0,x,v)| \leq |x_i| + \int_0^t |v_i(s;0,x,v)| ds \leq \zeta_0 + \left( \eta_0 + \frac{J_0}{M_0} \right) t + \frac{\lambda J_0 t^2}{2}.
\end{equation}
This gives an estimate on the size of $x$-support $\Sigma(t)$ of $f$. For $x,y \in \Sigma(t)$,
\begin{align*}
\zeta(t) &\leq 1 + |x - y|^2 \\
 &\leq 1 + 2(|x|^2 + |y|^2) \\
 &\leq 1 + 4 \left[ \zeta_0 + \left( \eta_0 + \frac{J_0}{M_0} \right) t + \frac{\lambda J_0 t^2}{2} \right]^2 \\
 &\leq 1 + 12 \zeta_0^2 + 12 \left( \eta_0 + \frac{J_0}{M_0} \right)^2 t^2 + 3\lambda^2 J_0^2 t^4 \\
 &\leq \kappa_3 (1 + t^2 + t^4),
\end{align*}
and (4.4) follows, $\varphi(t) \geq r(\zeta(t))$. \qed

Let $v - u_c$ denote the fluctuation (or peculiar) kinetic velocity. We will quantify the emergence of the time-asymptotic flocking behavior in term of the corresponding energy fluctuation
\begin{equation}
\Lambda[f(t)] := \int_{\mathbb{R}^d} |v - u_c|^2 f(x,v,t) dv dx.
\end{equation}

The time-evolution estimate of $\Lambda[f(t)]$, will depends on the decay rate of $\varphi(t)$.

Let $f$ be a classical solution of \((3.13)\) with compact support in $x$ and $v$. Direct calculation implies
\begin{equation}
\frac{d}{dt} \Lambda[f(t)] = \int_{\mathbb{R}^{2d}} |v - u_c|^2 \partial_t f(x,v) dv dx \\
= -\int_{\mathbb{R}^{2d}} |v - u_c|^2 v \cdot \nabla_x f dv dx - \lambda \int_{\mathbb{R}^{2d}} |v - u_c|^2 \nabla_v \cdot Q(f,f) dv dx =: I_1 + I_2.
\end{equation}

The first term on the right vanishes by the divergence theorem
\begin{align*}
I_1 &= -\int_{\mathbb{R}^{2d}} |v - u_c|^2 v \cdot \nabla_x f dv dx = -\int_{\mathbb{R}^{2d}} \text{div}_x \left( |v - u_c|^2 v f \right) dv dx = 0.
\end{align*}
The second term is simplified as follows.

\[
\mathcal{I}_2 = 2\lambda \int_{\mathbb{R}^d} (v - u_c) \cdot Q(f, f) dv dx
\]

\[
= -2\lambda \int r(x, y)(v - u_c) \cdot (v - v_\ast)f(y, v_\ast)f(x, v) dv_\ast dv dy dx
\]

\[
= -2\lambda \int r(x, y)v \cdot (v - v_\ast)f(y, v_\ast)f(x, v) dv_\ast dv dy dx
\]

\[
= -\lambda \int r(x, y)|v - v_\ast|^2f(y, v_\ast)f(x, v) dv_\ast dv dy dx.
\]

We summarize the last three equalities in the following lemma.

Lemma 4.2. Let \( f \) be a classical solution of (3.13) subject to compactly supported initial conditions \( f_0 \). Then, the decay of the energy functional \( \Lambda[f(t)] \) in (4.4) is governed by

\[
\frac{d}{dt} \Lambda[f(t)] = -\lambda \int r(x, y)|v - v_\ast|^2f(y, v_\ast)f(x, v) dv_\ast dv dy dx.
\]

Equipped with lemma 4.2 we can state the main result of this section.

Theorem 4.1. Let \( f \) be the classical kinetic solution constructed in Theorem 3.1. Then, the decay of its energy fluctuations around the mean bulk velocity \( u_c \), is given by

\[
\Lambda[f(t)] \lesssim \Lambda[f_0] \times \begin{cases} C_3 e^{-\kappa_4 t^{1-\delta}} , & 0 \leq \beta < \frac{1}{4}, \\ (1 + t)^{-\kappa_5} , & \beta = \frac{1}{4}. \end{cases}
\]

The constants involved are \( \kappa_4 = 1/(3\kappa_3)^\beta (1 - 4\beta) > 0 \) and \( \kappa_5 = 2\lambda A/\sqrt{3\kappa_3} > 0 \).

Proof. Lemma 4.2 implies

\[
\frac{d}{dt} \Lambda[f(t)] = -\lambda \int_{\mathbb{R}^d} r(x, y)|v - v_\ast|^2f(y, v_\ast)f(x, v) dv_\ast dv dy dx
\]

\[
\leq -\lambda \varphi(t) \int_{\mathbb{R}^d} |v - v_\ast|^2f(y, v_\ast)f(x, v) dv_\ast dv dy dx = -2\lambda \varphi(t) \mathcal{M}_0 \Lambda[f(t)].
\]

As before, the identity

\[
|v - v_\ast|^2 = |v - u_c|^2 + |v_\ast - u_c|^2 + 2(v - u_c) \cdot (v_\ast - u_c)
\]

induces the corresponding decomposition of the integrand on the right. Noting that

\[
\int_{\mathbb{R}^d} (v - u_c) f(x, v, t) dv dx = 0.
\]

We conclude

\[
\frac{d}{dt} \Lambda[f(t)] \leq -2\lambda \varphi(t) \mathcal{M}_0 \Lambda[f(t)]
\]

and Gronwall’s integration yields

\[
\Lambda[f(t)] \leq \Lambda[f_0] e^{-2\Phi(t)}, \quad \Phi(t) = \int_0^t \varphi(s) ds.
\]

We distinguish between two cases.

Case 1 \([0 \leq \beta < \frac{1}{4}]\). According to Lemma 4.1

\[
\varphi(t) \geq A\kappa_3^{-\beta}(1 + t^2 + t^4)^{-\beta} \geq A(3\kappa_4)^{-\beta} t^{-4\beta}, \quad t \geq 1.
\]
We compute for \( t \geq 1 \),

\[
-\Phi(t) \lesssim -\int_1^t \varphi(\tau)d\tau \leq -\frac{A}{(3\kappa_1)^\beta(1-4\beta)}(t^{1-4\beta} - 1)
\]

which yields the first part of (4.8).

**Case 2** \([\beta = \frac{1}{4}]\). For \( t \geq 1 \) we have

\[
\exp\left(-2\lambda A\Phi(t)\right) \leq \exp\left(-\frac{2\lambda \ln t}{(3\kappa_1)^{1/4}}\right) \lesssim (1 + t)^{-\frac{2\lambda}{\sqrt{\kappa_1}}},
\]

and the second part of (4.8) follows. \( \square \)

5. From kinetic to hydrodynamic description of flocking

In this section we discuss the hydrodynamic description for flocking, which is formally obtained by taking moments of the kinetic model (5.13).

\[(5.1a)\quad \partial_t f + v \cdot \nabla_x f + \lambda \nabla_v \cdot Q(f, f) = 0, \quad x, v \in \mathbb{R}^d, \quad t > 0,
\]

\[(5.1b)\quad Q(f, f)(x, v, t) = \int_{\mathbb{R}^{2d}} r(x, y)(v_s - v) f(x, v, t)f(y, v_s, t)dv dv dy.
\]

We first set hydrodynamic variables: the mass \( \rho := \int_{\mathbb{R}^d} f dv \), the momentum, \( \rho u := \int_{\mathbb{R}^d} vf dv \), and the energy, \( \rho E := \int_{\mathbb{R}^d} |v|^2 f dv \), which is the sum of kinetic and internal energies (corresponding to the first two terms in the decomposition of kinetic velocities \( |v|^2 = |u|^2 + |v - u|^2 + 2(v - u) \cdot u \))

\[(5.2)\quad \rho E = \rho e + \frac{1}{2}\rho|u|^2, \quad \rho e := \frac{1}{2} \int |v - u|^2 f(x, v, t)dv.
\]

For notational simplicity, we suppress time-dependence, denoting \( \rho(x) \equiv \rho(x, t) \), \( u(x) \equiv u(x, t) \) and \( E(x) = E(x, t) \) when the context is clear.

We compute the \( v \)-moments of (5.1): multiply (5.1) against \( 1, v \) and \( |v|^2/2 \) and integrate over the velocity space \( \mathbb{R}^d \). We end up with the system of equations,

\[(5.3a)\quad \partial_t \rho + \nabla_x \cdot (\rho u) = 0,
\]

\[(5.3b)\quad \partial_t (\rho u) + \nabla_x \cdot \left((\rho u \otimes u + P)\right) = \mathcal{S}^{(1)},
\]

\[(5.3c)\quad \partial_t (\rho E) + \nabla_x \cdot \left(\rho Eu + Pu + q\right) = \mathcal{S}^{(2)}.
\]

Here, \( \mathcal{S}^{(j)}, j = 1, 2 \), are the nonlocal source terms given by

\[(5.3d)\quad \mathcal{S}^{(1)}(x, t) := -\lambda \int_{\mathbb{R}^d} r(x, y)(u(x) - u(y))\rho(x)\rho(y)dy,
\]

\[(5.3e)\quad \mathcal{S}^{(2)}(x, t) := -\lambda \int_{\mathbb{R}^d} r(x, y)\left[E(x) + E(y) - u(x) \cdot u(y)\right]\rho(x)\rho(y)dy,
\]

and \( P = (p_{ij}), q = (q_i) \) denote, respectively, the stress tensor and heat flux vector,

\[(5.3f)\quad p_{ij} := \int_{\mathbb{R}^d} (v_i - u_i)(v_j - u_j)dv, \quad q_i := \int_{\mathbb{R}^d} (v_i - u_i)|v - u|^2 dv.
\]
Remark 5.1. The total mass of the source term $S^{(1)}$ vanishes: exchange of variables $x \leftrightarrow y$ yields
\begin{equation}
(5.4a) \quad \int_{\mathbb{R}^d} S^{(1)}(x, t) dx = -\lambda \int_{\mathbb{R}^d} r(x, y)(u(x) - u(y))\rho(x)\rho(y) dy dx = 0.
\end{equation}

The source term $S^{(2)}$ is non-positive: using (5.3b) we find
\begin{equation}
(5.4b) \quad S^{(2)}(t) = -\lambda \int_{\mathbb{R}^d} r(x, y) \left[ \frac{1}{2} |u(x) - u(y)|^2 + e(x) + e(y) \right] \rho(x)\rho(y) dy \leq 0.
\end{equation}

We conclude that the total mass and momentum, $\int \rho(x, t) dx$ and $\int \rho(x, t) u(x, t) dx$, are conserved in time. The total energy, however,
\[ E(t) = \int_{\mathbb{R}^d} \rho(x, t) E(x, t) dx = \frac{1}{2} \int_{\mathbb{R}^d} |v|^2 f(x, v, t) dv dx, \]
is dissipating, which is responsible for the formation of time-asymptotic flocking behavior.

We turn to quantify this decay. We first write the total energy as the sum of total kinetic and potential energies, corresponding to (5.2),
\[ E(t) = E_K(t) + E_P(t), \quad E_K(t) := \frac{1}{2} \int_{\mathbb{R}^d} \rho(x, t)|u(x, t)|^2 dx, \quad E_P(t) := \frac{1}{2} \int_{\mathbb{R}^d} |v-u(x, t)|^2 f dv dx. \]

Lemma 5.1. The time evolution of the total, kinetic and internal energies is governed by
\begin{align}
(5.5a) & \quad \frac{d}{dt} E(t) = -\lambda \int_{\mathbb{R}^d} r(x, y) \left[ \frac{1}{2} |u(x) - u(y)|^2 + e(x) + e(y) \right] \rho(x)\rho(y) dy dx;
(5.5b) & \quad \frac{d}{dt} E_P(t) = -\lambda \int_{\mathbb{R}^d} r(x, y) \left( e(x) + e(y) - u(x) \cdot u(y) \right) \rho(x)\rho(y) dy dx - 2 \int_{\mathbb{R}^d} (\nabla_x \cdot u) \rho dx;
(5.5c) & \quad \frac{d}{dt} E_K(t) = -\frac{\lambda}{2} \int_{\mathbb{R}^d} r(x, y) \left( |u(x)|^2 + |u(y)|^2 \right) \rho(x)\rho(y) dy dx + 2 \int_{\mathbb{R}^d} (\nabla_x \cdot u) \rho dx.
\end{align}

Proof. The equality (5.5a) follows from integration of (5.3d) and invoking (5.4b). For the decay rate of the total internal energy $E_P(t)$ in (5.5b), we calculate
\[ \frac{d}{dt} E_P(t) = \int_{\mathbb{R}^d} \partial_t \left( \frac{|v-u|^2}{2} \right) f dv dx + \int_{\mathbb{R}^d} \left| \frac{v-u}{2} \right| \partial_t f dv dx =: K_1 + K_2. \]

We estimate $K_1$ separately. The first term, $K_1$ vanishes, for
\[ K_1 = \int_{\mathbb{R}^d} (u-v) \cdot \partial_t u f dv dx = \int_{\mathbb{R}^d} u \cdot \partial_t u f dv dx - \int_{\mathbb{R}^d} \partial_t u \cdot (vf) dv dx \]
\[ = \int_{\mathbb{R}^d} u \cdot \partial_t u pf dx - \int_{\mathbb{R}^d} \partial_t u \cdot (pu) dx = 0. \]

For the second term, $K_2$, we use the kinetic model to find
\[ K_2 = \frac{1}{2} \int_{\mathbb{R}^d} |v-u|^2 \partial_t f dv dx = -\frac{1}{2} \int_{\mathbb{R}^d} |v-u|^2 \left( v \cdot \nabla_x f + \lambda \nabla_v \cdot Q(f, f) \right) dv dx \]
\[ = -\frac{1}{2} \int_{\mathbb{R}^d} |v-u|^2 v \cdot \nabla_x f dv dx + \lambda \int_{\mathbb{R}^d} (v-u) \cdot Q(f, f) dv dx =: K_{21} + \lambda K_{22}. \]

The term $K_{21}$ amounts to
\[ K_{21} = -\frac{1}{2} \int_{\mathbb{R}^d} |v-u|^2 v \cdot \nabla_x f dv dx = \frac{1}{2} \int_{\mathbb{R}^d} \left( \nabla_x |v-u|^2 \right) \cdot (vf) dv dx \]
\[ = -\int_{\mathbb{R}^d} (v - u) \cdot ((\nabla_x \cdot u)vf) \,dvdx = -2\int_{\mathbb{R}^d} (\nabla_x \cdot u)(\rho e) \,dx. \]

A lengthy calculation shows that the remaining term, \( K_{22} \), equals

\[ K_{22} = \int_{\mathbb{R}^d} x \cdot (v - u(x)) \cdot (v_\alpha - v) f(x, v_\alpha) dv_\alpha dvdydx \]

Finally (5.5c) follows by subtracting (5.5b) from (5.5a).

Next, we present a fundamental estimate for the flocking behavior to the system (5.3). We set

\[ (5.6) \quad \Gamma(t) := \int_{\mathbb{R}^d} \left( \frac{1}{2} |u(x) - u(y)|^2 + e(x) + e(y) \right) \rho(x) \rho(y) \,dydx. \]

The functional \( \Gamma(t) \) can be expressed in terms of the moments \( M_i \) in (3.9) (corresponding to the splitting of its integrand \( \frac{1}{2} |u(x) - u(y)|^2 + e(x) + e(y) \equiv \rho(x) E(x) + \rho(y) E(y) - u(x) \cdot u(y) \)),

\[ \Gamma(t) \equiv 2E(t) M_0 - \| \mathcal{M}_1 \|^2, \quad E(t) = \mathcal{M}_1(t) = 2E(t). \]

Since \( \mathcal{M}_i, i = 0, 1 \) are constants, this reveals that \( \Gamma(t) \) is essentially the total energy. We arrive at the main theorem of this section.

**Theorem 5.1.** Assume \((\rho, u, e)\) is a smooth solution of the system (5.3), \((\rho, u, e) \in C^1([0, T] \times \mathbb{R}^d)\). Then we have

\[ \Gamma(t) \leq \Gamma(0) e^{-2\mathcal{M}_0 \lambda(t)}, \quad \Phi(t) = \int_0^t \varphi(s) \,ds, \quad \varphi(s) := \inf_{(x_0, y_0)} r(x(s), y(s)) \,ds. \]

Here, \( \mathcal{M}_0 \) is the initial total mass, \( \mathcal{M}_0 = \| \rho_0 \|_{L^1} \) and the infimum is taken over all particle trajectories, \((x_0, y_0) \mapsto (x(s), y(s))\).

**Proof.** We use (5.5a) and the relation \( \Gamma(t) = 2E(t) M_0 - \| \mathcal{M}_1 \|^2 \) to find

\[ \frac{d}{dt} \Gamma(t) = -\lambda M_0 \int_{\mathbb{R}^2} \varphi(t) \int_{\mathbb{R}^d} \left( |u(x) - u(y)|^2 + 2u(x) + 2u(y) \right) \rho(x) \rho(y) \,dydx \]

\[ \leq -\lambda M_0 \varphi(t) \int_{\mathbb{R}^2} \left( |u(x) - u(y)|^2 + 2u(x) + 2u(y) \right) \rho(x) \rho(y) \,dydx \]

\[ = -2\lambda M_0 \varphi(t) \Gamma(t). \]

Gronwall’s inequality then yields the desired result. \( \square \)
We conclude that whenever the interparticle interaction, \( \varphi(s) = \inf r(x(s), y(s)) \) decays slowly enough so that its primitive, \( \Phi(t) \), diverges, then flocking occurs, \( |u(x, t) - u(y, t)| \to 0 \), in agreement with the flocking behavior of the C-S particle model, consult remark 2.2. It is remarkable that the emergence of flocking is deduced here independently of the constitutive relation for \( P \). In this context we observe that energy dissipation, driven by the negative source term \( S^{(2)} \) in (5.3c) vanishes as \( t \to \infty \). Indeed, theorem 5.1 tells us that by (5.4b),

\[ |S^{(2)}| \leq \Lambda \Gamma(t) \to 0. \]

6. Epilogue: flocking dissipation and entropy

We have seen that the self-propelled flocking dynamics is driven by energy dissipation. The dissipation mechanism reveals itself through energy decay in the particle description (2.7), in the kinetic description (3.8b) and equivalently, in the hydrodynamic description (5.7). Observe that,

\[ \Gamma(t) = M_2 M_0 - |M_1|^2 \geq 0. \]

The right of (6.8) is the usual Cauchy-Schwartz inequality

\[ \left| \int_{\mathbb{R}^2} v f(x, v) dv dx \right|^2 \leq \int_{\mathbb{R}^2} |v|^2 f(x, v) dv dx \times \int_{\mathbb{R}^2} f(x, v) dv dx. \]

Thus, by theorem 5.1, \( \Phi(t) \to \infty \) implies time asymptotic flocking by letting \( \Gamma(t) \to 0 \) which in turn, enforces an approximate Cauchy-Schwartz equality. It then follows that \( v \) approaches the bulk velocity, \( v \to \bar{u} \) as \( t \to \infty \). We refer to this mechanism as flocking dissipation. It is intimately related to the entropy increase in the kinetic model (3.6). To this end we compute

\[ \frac{d}{dt} \int_{\mathbb{R}^2} f \log(f) dx dv = -\lambda \int_{\mathbb{R}^2} r(x, y) \nabla_v \log(f) \cdot (v - \bar{u}) f f^* dv dv dx dy dx \]

\[ = -\lambda \int_{\mathbb{R}^2} r(x, y) \nabla_v f \cdot (v - \bar{u}) f f^* dv dv dx dy dx \]

\[ = \lambda \int_{\mathbb{R}^2} r(x, y) f f^* dv dv dx dy dx = \lambda \int_{\mathbb{R}^4} r(x, y) \rho(x) \rho(y) dy dx. \]

This is a reversed \( H \)-theorem. Entropy increases due to the “improbable” statistical behavior of particles with increasingly highly correlated velocities, as they flock towards particle-independent bulk velocity.

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