A Nearly Optimal Deterministic Online Algorithm for Non-Metric Facility Location

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Abstract

In the online non-metric variant of the facility location problem, there is a given graph consisting of set $F$ of facilities (each with a certain opening cost), set $C$ of potential clients, and weighted connections between them. The online part of the input is a sequence of clients from $C$, and in response to any requested client, an online algorithm may open an additional subset of facilities and must connect the given client to an open facility.

We give the first online, polynomial-time deterministic algorithm for this problem, with competitive ratio of $O(\log |F| \cdot (\log |C| + \log \log |F|))$. The result is optimal up to loglog factors. Previously, the only known solution for this problem with a sub-linear competitive ratio was randomized [Alon et al., TALG 2006]. Our approach is based on solving a different fractional relaxation than that of Alon et al., where we combine dual fitting and multiplicative weight updates approaches. By maintaining certain monotonicity properties of the created fractional solution, we are able to handle the dependencies between facilities and connections in a rounding routine.

Our result, combined with the algorithm by Naor et al. [FOCS 2011] implies the first deterministic algorithm for the online node-weighted Steiner tree problem. The resulting competitive ratio is $O(\log k \cdot \log^2 \ell)$ on graphs of $\ell$ nodes and $k$ terminals.

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1 Introduction
The facility location (FL) problem [ABM16] is one of the best known examples of network design problems, extensively studied both in operations research and in computer science. Its simple definition, NP-hardness and rich combinatorial structure have led to developments of tools and solutions in key areas of approximation algorithms, combinatorial optimization and linear programming.

An instance of the FL problem consists of a set $F$ of facilities, each with a certain opening cost, and a set $C$ of clients. $F$ and $C$ can be seen as two sides of a bipartite graph. The undirected edges between them have lengths that can either satisfy the triangle inequality (metric FL) or be arbitrary (non-metric FL). The goal is to open a subset of facilities and connect each client to an open facility. The total cost (the sum of opening and connection costs) is subject to minimization. In the metric scenario, by taking a metric closure, one can assume that each facility is reachable by each client, but it is not the case for the non-metric variant.

Instances and objectives. In this paper, we focus on an online variant of the non-metric FL problem. We first formalize the offline variant in a way that makes a connection to the online variant more apparent.

A facility-client graph $G = (F, C, E, \text{cost})$ is a bipartite graph, whose one side is the set $F$ of facilities and another side is the set of clients $C$. Set $E \subseteq F \times C$ contains available facility-client
connections (edges). We use function cost to denote both costs of opening facilities and connection costs (edge lengths). All costs are non-negative.

An instance of the non-metric FL is a pair \((G, A)\), where \(G = (F, C, E, \text{cost})\) is a facility-client graph and \(A \subseteq C\) is a subset of active clients. A feasible solution to such instance is a set of open (purchased) facilities \(F' \subseteq F\) and a subset of purchased edges \(E' \subseteq E\), such that any active client \(c \in A\) is connected by a purchased edge to an open facility. The cost of such solution is equal to \(\sum_{f \in F'} \text{cost}(f) + \sum_{e \in E'} \text{cost}(e)\).

**Online scenario.** In an online variant of the FL problem, the facility-client graph \(G\) is known in advance, but neither elements of \(A\) nor its cardinality are known up-front by an online algorithm \(\text{Alg}\). The clients from \(A\) appear one by one. Upon seeing a new active client, \(\text{Alg}\) may purchase additional facilities and edges, with the requirement that facilities and edges purchased so far must constitute a feasible solution to all presented active clients. The total cost of \(\text{Alg}\) is denoted by \(\text{Alg}(G, A)\). (We sometimes use \(\text{Alg}(G, A)\) to denote also the solution computed by \(\text{Alg}\).) Purchase decisions are final and cannot be revoked later. The goal is to minimize the competitive ratio, defined as \(\sup_{(G,A)} \{ \text{Alg}(G, A)/\text{Opt}(G, A) \} \), where \(\text{Opt}\) is the optimal (offline) algorithm.

### 1.1 Related work

Most of the prior work has been devoted to the offline scenario. While the metric variant of the FL problem admits O(1)-approximation algorithm [Li13], the best competitive ratio for the non-metric one is \(O(\log |A|)\) [Hoc82], and it cannot be asymptotically improved unless \(\text{NP} \subseteq \text{DTIME}[n^{O(\log \log n)}]\) [Fei98]. For a more comprehensive treatments of the offline scenario, including multitude of variants, we refer the reader to the entry in the encyclopedia of algorithms [ABM16] or the survey by Shmoys [Shmoo].

For the online metric FL, the problem was resolved over ten years ago by Meyerson [Mey01] and Fotakis [Foto08]: the lower and upper bounds on the competitive ratio are \(\Theta(\log |A|/ \log \log |A|)\), both for deterministic and randomized algorithms. Simpler deterministic algorithms attaining slightly worse competitive ratio of \(O(\log |A|)\) were given by Anagnostopoulos et al. [ABUH04] and Fotakis [Foto07]. Note that for the metric variant, the set \(C\) of potential clients can be arbitrarily large.

Less is known about algorithms for the online non-metric FL. To the best of our knowledge, the only result concerning this variant is a randomized \(O(\log |F| \cdot \log |A|/ \log \log |F|)\)-competitive algorithm by Alon et al. [AAA+06]. We describe some of the involved ideas in the next subsection.

### 1.2 Our results and techniques

The main result of this paper is the first deterministic online algorithm for the non-metric FL problem, achieving a non-trivial (sub-linear) competitive ratio. In particular, we show the following theorem.

**Theorem 1.** There exists a deterministic polynomial-time \(O(\log |F| \cdot (\log |C| + \log \log |F|))\)-competitive algorithm \(\text{Det}\) for the online non-metric facility location problem on set \(F\) of facilities and set \(C\) of clients.

Our algorithm attains a nearly optimal competitive ratio, as no deterministic algorithm can have ratio smaller than \(\Omega(\log |F| \cdot \log |C|/ (\log \log |F| + \log \log |C|))\). This follows by the lower bound for the online set cover problem [AAA+06, AAA03] and holds even for uniform facility
costs. If we restrict our attention to the polynomial-time deterministic solution, then a higher lower bound of $\Omega(\log |F| \cdot \log |C|)$ holds (assuming BPP $\neq$ NP) [Koro04].

**Randomized vs deterministic solutions.** For the reader unfamiliar with the landscape of the competitive analysis, a non-obvious phenomenon is worth mentioning. Unlike in the offline regime, the existence of the randomized solution only sometimes implies the existence of the deterministic solution with a similar ratio. Most notably, for many famous online problems, e.g., the $k$-server problem [Lee18], metrical task systems [BCLL19], or paging problems [ACER19], the gaps between competitive ratios achievable by randomized and deterministic solutions are exponential. For many other cases, e.g., the metric facility location problem [Fot11] or some edge-weighted Steiner problems [Umb15], deriving deterministic solutions with asymptotically same ratio required different techniques and substantially larger effort.

**Previous randomized solution.** Before we describe our approach, we sketch the current state-of-the-art randomized algorithm for the non-metric FL problem by Alon et al. [AAA+06]. We call a facility $f$ to which a client $c$ could be connected a covering facility for $c$. Their solution involves solving a natural fractional relaxation of the problem: there is a fractional opening variable $y_f$ for each facility $f$ and a connection variable $x_{c,f}$ for a client $c$ and a facility $f$ covering $c$. Once a client $c$ arrives, for each covering facility $f$ independently, their algorithm increases either $y_f$ or $x_{c,f}$, whichever is smaller, using multiplicative update method (see, e.g., [AHK12]). The client $c$ is considered fractionally served once the sum of terms $\min\{x_{c,f}, y_f\}$ over all covering facilities is at least 1. The resulting competitive ratio is $O(\log |F|)$.

The computed fractional solution can be then randomly rounded using techniques borrowed from approximation algorithms. The authors of [AAA+06] select a random threshold $\theta_f$ common for an opening variable $y_f$ and connection variables involving facility $f$. Once any variable exceeds its threshold, it is rounded up to 1 and the corresponding object (facility or connection) is purchased. Choosing $\theta_f$ to have expectation $\Theta(1/\log |A|)$ guarantees that the resulting integral solution is feasible with high probability, and the rounding part incurs a factor of $O(\log |A|)$ in the competitive ratio.

**Why deterministic rounding is challenging.** The description of the randomized algorithm by Alon et al. [AAA+06] given above may seem deceptively simple, but it hides an important and subtle property, implicitly exploited by the authors. Namely, the threshold $\theta_f$ is common for facility $f$ and all connections to it. This ensures the necessary dependency: once $\min\{x_{c,f}, y_f\} \geq \theta_f$, the rounding purchases both facility $f$ and connection from $c$ to $f$. (Note that the left hand side of this inequality is the amount that their fractional solution controls.)

It is unclear how to extend this property to deterministic rounding. One of the straightforward approaches would be to deterministically round facilities to ensure necessary coverage of each client. However, neglecting the connection costs in the rounding process easily leads to a situation, where the facilities are rounded “correctly”, but the cost of connecting a client to the closest open facility in the integral solution is incomparably larger than the corresponding fractional cost. A different approach would be to apply the text-book reduction of non-metric FL to the set cover problem [Hoc82] which yields exponentially many sets. While the rounding part is then efficient, the competitive ratio of the fractional solution becomes polynomial in $|F|$.

We note that all known deterministic schemes that round fractional solutions generated by
the multiplicative updates operate in rather limited scenarios, where clients have to be covered or packed and all important interactions between clients are handled at the time of constructing the fractional solution. This is the case for the deterministic rounding for the set cover problem [AAA03, BN09a] and for the throughput-competitive virtual circuit routing problem [AAP93, BN09b]. These methods are based on derandomizing the method of pessimistic estimators [Rag88] in online manner, by transforming a pessimistic estimator into a potential function [You95] that can be controlled by the deterministic rounding process. The paper by Buchbinder and Naor [BN09b], which itself describes some of these deterministic rounding methods, lists the online network design problems (including the non-metric FL problem) as unresolved challenges (see the discussion in Section 1.1 of [BN09b]).

**Our techniques.** In our solution, we create a new linear relaxation of the problem. We first round the graph distances to powers of 2. For any client, we cluster facilities that have same distance to this client. (Note that such clusters are client-dependent.) To solve the fractional variant, we run two schemes in parallel: we increase connection variables $x_{c,t}$ corresponding to clusters at distance $t$ and increase facility variables $y_f$ for all facilities in “reachable” clusters (where the corresponding connection variables are 1). The increases of these variables use two different frameworks: dual fitting for linear increases of connection variables and primal-dual scheme involving multiplicative updates for facility variables. Ensuring appropriate balance between these two different type of updates and is one of the technical difficulties that we tackle in this paper.

We stop increasing variables once there exists a collection of clusters that are both “fractionally open” (sum of variables $y_f$ within these clusters is $\Omega(1)$) and “reachable” by the considered client. To argue about the existence of such collection, we use both LP inequalities and structural properties of our fractional algorithm.

Finally, we construct a deterministic rounding routine. We focus on facilities only, neglecting whether particular clients are active or not and how far they are from a given facility. However, we strengthen rounding properties, ensuring, for (some) collections of clusters, that if the sum of opening variables in these collections is $\Omega(1)$, then the integral solution contains an open facility in one of these clusters. This ensures that in the integral solution, there will be a facility whose distance from the considered client is asymptotically not larger than the cost invested in the fractional solution for connecting this client. This gives us the desired dependency between facilities and connections.

**Application to online node-weighted Steiner tree.** Our result has an immediate application for the online node-weighted Steiner tree (NWST) problem, where the graph consists of $\ell$ nodes and an online algorithm is given $k$ terminals to be connected. Namely, the randomized solution for the online NWST problem by Naor et al. [NPS11] is in fact a deterministic polynomial-time “wrapper” around randomized routine solving the non-metric FL problem. To solve an instance of the NWST problem, their algorithm constructs a sub-instance of non-metric FL with $O(\ell)$ facilities, $O(\ell)$ potential clients and $O(k)$ active clients. Such instance can be solved by the algorithm of Alon et al. [AAA+06] with the competitive ratio of $O(\log k \cdot \log \ell)$. The wrapper adds another $O(\log k)$ factor in the ratio, resulting in $O(\log^2 k \cdot \log \ell)$-competitive algorithm.

Our deterministic algorithm, when applied to this setting would be $O(\log^2 \ell)$-competitive on the constructed non-metric FL sub-instance. Therefore, by replacing the randomized algorithm by Alon et al. [AAA+06] with our deterministic one, we immediately obtain the first online deterministic solution for online NWST.
Corollary 2. There exists a polynomial-time deterministic online algorithm for the node-weighted Steiner tree problem, which is $O(\log k \cdot \log^2 \ell)$-competitive on graphs with $\ell$ nodes and $k$ terminals.

We note that the currently best solution for the node-weighted Steiner tree is randomized and achieves the ratio of $O(\log \ell)$ [HLP17, HLP14] and the best known lower bound for deterministic algorithms is $\Omega(\log \ell \cdot \log k / (\log \log \ell + \log \log k))$ [NPS11, AAA03].

Note about up-front knowledge of facility-client graph. Unlike for the randomized variant, obtaining sub-linear guarantees for a deterministic solution requires knowing a priori the set of potential client-facility connections. To see this, consider a graph of $|F|$ facilities with unit opening costs and the set of $|C| = |F|$ clients. The graph edges are constructed dynamically as clients are activated and all revealed possible connections are of cost 0. The first active client can be connected to all facilities. Each subsequent client can be connected to all facilities but the ones already open by an algorithm. This way an algorithm needs to eventually open all facilities, for a total cost of $|F|$. On the other hand, the offline optimal algorithm can open the last facility opened by an algorithm and connects all clients to this facility at the total cost of 1. Thus, under the unknown-graph assumption, the competitive ratio of any deterministic algorithm would be at least $|F|$.

1.3 Preliminaries and paper organization

For any facility-client graph $G$, we define its aspect ratio $\Delta_G$ as the ratio of the largest to smallest positive cost in $G$. These costs include both facilities and connection costs.¹ Note that the aspect ratio is a property of $G$ and is independent of the set of active clients $A$.

Let $T_G$ contain all powers of two between the largest and the smallest positive distance ( inclusively) and also number 0. In particular, $T_G$ contains all distances in $G$ and $|T_G| \leq 2 + \log \Delta_G$. Whenever $G$ is clear from the context, we drop $G$ subscript.

We may assume that $F$ contains at least two facilities and $C$ contains at least two clients, as otherwise the problem becomes trivial. For a facility $f \in F$, let $\text{set}(f)$ be the set of clients that may be connected to $f$. For any client $c \in C$ and distance $t \in T$, cluster $F_{c,t}$ contains all facilities that are incident to $c$ using edges of cost $t$. Note that for a fixed $c$, clusters $F_{c,t}$ are disjoint (no client has two connections of different costs to the same facility).

Powers of two assumption. In the whole paper, we assume that all facilities and connection costs are either equal to 0 or are powers of 2 and are at least 1. This can be easily achieved by initial scaling of positive costs and distances, so that they are at least 1 and rounding positive ones up to the nearest power of two. This transformation changes the competitive ratio at most by a factor of 2.

Sections overview. Our core approach is to solve a carefully crafted fractional relaxation of the problem (Section 2), and then round it in a deterministic fashion (Section 3). This way, we obtain a deterministic online algorithm $\text{Int}$ that on any input $(G = (F, C, E, \text{cost}), A)$ computes a feasible solution of cost

$$\text{Int}(G, A) \leq O(\log |F| \cdot (\log |C| + \log \log \Delta_G)) \cdot \text{Opt}(G, A) + 2 \cdot \max_{f \in F} \text{cost}(f).$$

¹In the standard definition of the aspect ratio, only distances are taken into account.
Moreover Int runs in time \(\text{poly}(|G|, |A|, \max_{e \in E} \text{cost}(e), \max_{f \in F} \text{cost}(f))\). In Section 4, we apply doubling and edge pruning techniques, to get rid of dependencies on costs in the running time and on \(\Delta_G\) in the competitive ratio, achieving guarantees of Theorem 1.

## 2 Fractional solution

We fix an instance \((G = (F, C, E, \text{cost}), A)\) of the online non-metric facility problem. For each facility \(f\), we introduce an opening variable \(y_f \geq 0\) (fractional opening of \(f\)) and for each client \(c\) and each distance \(t \in T\) a connection variable \(x_{c,t} \geq 0\). Intuitively, \(x_{c,t}\) denotes how much, fractionally, client \(c\) invests into connections to facilities from cluster \(F_{c,t}\). For any set \(F'\) of facilities we use \(y(F')\) as a shorthand for \(\sum_{f \in F'} y_f\).

### Primal program.

After \(k\) clients from \(A\) arrive (we denote their set by \(A_k\)), we consider the following linear program \(P_k\).

\[
\begin{align*}
\text{minimize} & \quad \sum_{f \in F} \text{cost}(f) \cdot y_f + \sum_{c \in A_k} \sum_{t \in T} t \cdot x_{c,t} \\
\text{subject to} & \quad x_{c,t} \geq z_{c,t} & \text{for all } c \in A_k, t \in T, \\
& \quad y(F_{c,t}) \geq z_{c,t} & \text{for all } c \in A_k, t \in T, \\
& \quad \sum_{t \in T} z_{c,t} \geq 1 & \text{for all } c \in A_k
\end{align*}
\]

and non-negativity of all variables.

### Serving constraints.

The LP constraints combined together are equivalent to the set of the following (non-linear) requirements

\[
\sum_{t \in T} \min\{x_{c,t}, y(F_{c,t})\} \geq 1 \quad \text{for all } c \in A_k. 
\tag{1}
\]

We call (1) for client \(c\) the \textit{serving constraint} for client \(c\). In our description, we omit variables \(z_{c,t}\) and the original constraints, ensuring only that the serving constraints hold and implicitly setting \(z_{c,t} = \min\{x_{c,t}, y(F_{c,t})\}\).

The LP above is indeed a valid relaxation of the FL problem. To see this, take any feasible integral solution. For any facility \(f\) opened in the integral solution, set variable \(y_f\) to 1. For each client \(c\) connected to facility \(f\), set variable \(x_{c,T}\) to 1, where \(T = \text{cost}(f,c)\). This guarantees that \(\min\{x_{c,T}, y(F_{c,T})\} = 1\), and thus the serving constraint (1) is satisfied for each client \(c\).

### Dual program.

The program \(D_k\) dual to \(P_k\) is

\[
\begin{align*}
\text{maximize} & \quad \sum_{c \in A_k} \gamma_c \\
\text{subject to} & \quad \gamma_c \leq \alpha_{c,t} + \beta_{c,t} & \text{for all } c \in A_k, t \in T, \\
& \quad \alpha_{c,t} \leq t & \text{for all } c \in A_k, t \in T, \\
& \quad \sum_{c \in \text{set}(f) \cap A_k} \beta_{c,\text{cost}(f,c)} \leq \text{cost}(f) & \text{for all } f \in F,
\end{align*}
\]

and non-negativity of all variables.
2.1 Overview

Our algorithm FRAC creates a solution to $P_k$, ensuring that the serving constraint (1) holds for all clients $c \in A_k$. As outlined in the introduction, the computed solution guarantees some additional properties that are useful for the rounding part later.

Whenever a client $c$ arrives, FRAC increases connection variables $x_{c,t}$ one by one starting from the smallest $t$, at the pace proportional to $1/t$. We ensure that $x_{c,t} \in [0,1]$, i.e., once any of these variables reaches 1, FRAC stops increasing them. A distance $t$, for which $x_{c,t} = 1$, is called saturated.

In parallel to manipulating variables $x_{c,t}$, FRAC increases all variables $y_f$ for facilities reachable from client $c$ using saturated distances. The variables $y_f$ are increased using the multiplicative update rule (scaled appropriately to take costs of facilities into account).

2.2 Algorithm FRAC

At the very beginning, before any client arrives, FRAC sets all variables $y_f$ to 0 for all positive-cost facilities and to 1 for zero-cost ones. There are no other variables as the set $A_0$ of active clients is empty. Note that the dual program already contains the last type of constraints, but the sums on their left-hand side range over empty sets of $\beta$ variables, and hence these constraints are trivially satisfied.

Whenever a new client $c$ arrives in step $k$, FRAC updates the primal (dual) programs from $P_{k-1}$ ($D_{k-1}$) to $P_k$ ($D_k$), and then computes a feasible solution to $P_k$ (on the basis of the already created solution to $P_{k-1}$) and a nearly-feasible solution to $D_k$.

New variables in primal and dual programs: FRAC sets $x_{c,t} \leftarrow 0$ for all $t \in T \setminus \{0\}$ and sets $x_{c,0} \leftarrow 1$. In the dual solution, it sets $\gamma_c \leftarrow 0$, $\alpha_{c,t} \leftarrow 0$ and $\beta_{c,t} \leftarrow 0$ for all $t \in T$.

Update primal program: A new serving constraint $\sum_{t \in T} \min\{x_{c,t}, y(F_c(t))\} \geq 1$ appears in the primal program (and is violated unless $y(F_{c,0}) \geq 1$). As we never decrease primal variables, the already existing serving constraints (1) are satisfied and will never be violated.

Update dual program: New constraints appear in the dual program and new variables $\beta_{c,t}$ appear on the left-hand side of the already existing inequalities. Since the new variables are initialized to 0, the validity of all dual constraints is unaffected.

Update primal and dual solutions: Let $T_c^1 = \{t \in T : x_{c,t} \geq 1\}$ be the set of saturated distances, i.e., initially FRAC sets $T_c^1 \leftarrow \{0\}$. While the serving constraint for $c$ is violated, FRAC executes the update operation consisting of the following steps:

1. Set $\gamma_c \leftarrow \gamma_c + 1$.
2. For each $t \in T$, independently, adjust one dual variable: if $t \in T_c^1$, then set $\beta_{c,t} \leftarrow \beta_{c,t} + 1$ and otherwise set $\alpha_{c,t} \leftarrow \alpha_{c,t} + 1$.
3. If $T_c^1 \subseteq T$, choose active distance $t^* \leftarrow \min(T \setminus T_c^1)$ to be the smallest non-saturated distance, and then set $x_{c,t^*} \leftarrow x_{c,t^*} + 1/t^*$. (Note that $0 \notin T_c^1$, and thus $t^* > 0$.)
4. For any facility $f \in \bigcup_{t \in T_c^1} F_{c,t}$, independently, perform augmentation of $y_f$, setting
   \[ y_f \leftarrow \left(1 + \frac{1}{\text{cost}(f)}\right) \cdot y_f + \frac{1}{|F| \cdot \text{cost}(f)}. \]
5. Update the set of saturated distances, setting $T_c^1 \leftarrow \{t \in T : x_{c,t} \geq 1\}$. 

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We note that if variable \( y_f \) is augmented in Step 4, then \( \text{cost}(f) > 0 \) (i.e., Step 4 is well defined). This follows as the distance \( \tau = \text{cost}(c, f) \) is saturated then, and hence \( x_{c, \tau} = 1 \). If \( \text{cost}(f) \) was equal to 0, then \( y_f \) would be initialized to 1, and therefore \( y(F_{c, \tau}) \geq 1 \), in which case the serving constraint for \( c \) would be satisfied.

\section*{Sidenote about T.}
Note that Frac increases also connection variables \( x_{c, t} \) where \( F_{c, t} \) is empty, i.e., invests into distances to non-existing facilities. This could be avoided, but the resulting algorithm and analysis would be slightly more complicated and it would not lead to asymptotic improvement of the performance.

\subsection*{2.3 Structural properties}

We focus on a single client \( c \) processed by Frac. We start with a property of connection variables \( x_{c, t} \). The distances from \( T \) that are neither saturated nor active, we call inactive. The following claim follows by an immediate induction on update operations performed by Frac.

\textbf{Lemma 3.} At all times when a client \( c \) is considered, \( x_{c, t} \in [0, 1] \) for any \( t \in T \). In particular, \( x_{c, t} = 1 \) for any saturated distance \( t \in T^1 \). Furthermore,

1. either all distances are saturated,
2. or there exists an active distance \( t^* > 0 \), such that (i) all smaller distances are saturated, and (ii) all larger distances are inactive and the corresponding \( x_{c, t} \) variables are equal to zero.

Augmentation is performed on variables \( y_f \) corresponding to facilities whose distance from \( c \) is saturated.

\textbf{Lemma 4.} On any input \((G = (F, C, E, \text{cost}), A)\), Frac returns a feasible solution and runs in time \( \text{poly}(|G|, |A|, \max_{e \in E} \text{cost}(e), \max_{f \in F} \text{cost}(f)) \).

\textit{Proof.} Fix any client \( c \in A \). By the definition of Frac, it takes \( t \) update operations to increase value \( x_{c, t} \) from 0 to 1. Hence after \( \sum_{t \in T} t < 2 \cdot \max_{e \in E} \text{cost}(e) \) update operations, all connection variables are equal to 1. Latest at that point, all variables \( y_f \) for \( f \in \bigcup_{t \in T} F_{c, t} \) are augmented in each update operation. Each variable \( y_f \) can be augmented at most \(|F| \cdot \text{cost}(f)\) times till it reaches or exceeds 1. That is, after at most \( 2 \cdot \max_{e \in E} \text{cost}(e) + |F| \cdot \max_{f \in F} \text{cost}(f) \) update operations, the serving constraint is satisfied, i.e., the generated solution is feasible. \hfill \Box

The following lemma shows the crucial property of Frac. Namely for any client \( c \), there exist a “good” distance \( \tau \), such that the collection of clusters \( F_{c, t} \) at distance \( t \leq \tau \) is together fractionally half-open and that Frac invested \( \Omega(\tau) \) into connecting client \( c \). For any client \( c \) and distance \( t \in T \), we define a set \( S_{c, t} \) to be a collection of clusters alluded to in the introduction.

\[ S_{c, t} = \bigcup_{t' \in T : t' \leq t} F_{c, t'}. \]

\textbf{Lemma 5.} Once Frac finishes serving client \( c \), there exists a distance \( \tau \in T \), such that \( y(S_{c, \tau}) \geq 1/2 \) and \( \sum_{t \in T} t \cdot x_{c, t} \geq \tau/2 \).

\textit{Proof.} We consider the state of variables once Frac finishes serving client \( c \). Let \( t^* > 0 \) be the largest distance from \( T \) for which \( x_{c, t^*} > 0 \). As the serving constraint for client \( c \) is satisfied, we have

\[ 1 \leq \sum_{t \in T} \min \{ x_{c, t}, y(F_{c, t}) \} = \min \{ x_{c, t^*}, y(F_{c, t^*}) \} + \sum_{t \in T : t < t^*} \min \{ x_{c, t}, y(F_{c, t}) \}. \] (2)
We pick $\tau$ depending on the value of the last term of (2).

If $\min\{x_{c,t^*}, y(F_{c,t^*})\} \geq 1/2$, we set $\tau = t^*$. Then, $y(S_{c,\tau}) \geq y(F_{c,\tau}) \geq \min\{x_{c,\tau}, y(F_{c,\tau})\} \geq 1/2$, and the first condition of the lemma. Furthermore, $\sum_{t \in T} t \cdot x_{c,t} \geq \tau \cdot x_{c,\tau} \geq \tau/2$.

Otherwise, $\min\{x_{c,t^*}, y(F_{c,t^*})\} < 1/2$, and then, by (2), $\sum_{t \in T : t < t^*} \min\{x_{c,t}, y(F_{c,t})\} \geq 1/2$. In such case, we choose $\tau$ as the largest distance from $T$ smaller than $t^*$. Then

$$y(S_{c,\tau}) = \sum_{t \in T : t \leq \tau} y(F_{c,t}) \geq \sum_{t \in T : t \leq \tau} \min\{x_{c,t}, y(F_{c,t})\} \geq 1/2,$$

i.e., the first condition of the lemma holds. By Lemma 3, either $t^*$ is active at the end of processing $c$ or all distances become saturated and $t^*$ is the largest distance from $T$. In either case, $x_{c,t} = 1$ for any distance $t < t^*$, and thus in particular $x_{c,\tau} = 1$. Hence, the second part of the lemma holds as $\sum_{t \in T} t \cdot x_{c,t} \geq \tau \cdot x_{c,\tau} = \tau$. \hfill $\square$

2.4 Dual solution is almost feasible

Using primal-dual analysis, we may show that the generated dual solution violates each constraint at most by a factor of $O(\log |F|)$.

**Lemma 6.** For any facility $f$, FRAC augments $y_f$ at most $O(\log |F|) \cdot \text{cost}(f)$ times.

**Proof.** First, we observe that variable $y_f$ can be augmented only if prior to augmentation it is smaller than 1. To show that, observe that the augmentation of $y_f$ occurs only when FRAC processes an active client $c \in \text{set}(f)$. Let $\tau = \text{cost}(f,c)$, i.e., $f \in F_{c,\tau}$. As FRAC augments $y_f$, $\tau$ must be saturated, i.e., $x_{c,\tau} = 1$. On the other hand, the serving constraint (1) is not satisfied when $y_f$ is augmented, and thus $\min\{x_{c,\tau}, y(F_{c,\tau})\} < 1$ which implies that $y_f$ must be strictly smaller than 1.

In particular, if $\text{cost}(f) = 0$, then $y_f$ is set to 1 immediately at the beginning, and hence no augmentation of $y_f$ is ever performed, and the lemma follows trivially. As all non-zero costs are at least 1, below we assume $\text{cost}(f) \geq 1$.

During the first $\text{cost}(f)$ augmentations, the value of $y_f$ increases from 0 to at least $1/|F|$ (due to additive increases). Next, during the subsequent $[\log_{1+1/\text{cost}(f)} |F|]$ augmentations, the value of $y_f$ reaches at least 1 (due to multiplicative increases), and thus it will not be augmented any more. In total, the number of augmentations is upper-bounded by $\text{cost}(f) + [\log_{1+1/\text{cost}(f)} |F|] = O(\log |F|) \cdot \text{cost}(f)$. In the last inequality, we used $\text{cost}(f) \geq 1$. \hfill $\square$

**Lemma 7.** FRAC violates each dual constraint at most by a factor of $O(\log |F|)$.

**Proof.** We show the claim for all types of constraints in the dual program.

1. Each dual constraint $\gamma_c \leq \alpha_{c,t} + \beta_{c,t}$ always holds with equality as together with $\gamma_c$, for each $t \in T$, FRAC increments either $\alpha_{c,t}$ or $\beta_{c,t}$.

2. Consider a constraint $\alpha_{c,t} \leq t$. Initially $\alpha_{c,t} = 0$ when client $c$ appears, and it is incremented in an update operation only if $t$ is not saturated. Distances are processed from the smallest to the largest, and it takes exactly $t'$ update operations for a distance $t' \in T$ to become saturated. Therefore, $\alpha_{c,t}$ can be incremented at most $\sum_{t' \leq t} t'$ times. If $t = 0$, then $\alpha_{c,t} = 0$ trivially. Otherwise, we use the fact that $T \setminus \{0\}$ contains only powers of 2, and hence $\alpha_{c,t} \leq \sum_{t' \leq t} t' < 2 \cdot t$. 10
3. Finally, fix any facility $f^* \in F$ and consider the constraint \( \sum_{c \in \text{set}(f^*) \cap A_k} \beta_{c, \text{cost}(f^*, c)} \leq \text{cost}(f^*) \).

We want to show that this constraint is violated at most by \( O(\log |F|) \), i.e., that
\[
\sum_{c \in \text{set}(f^*) \cap A_k} \beta_{c, \text{cost}(f^*, c)} \leq O(\log |F|) \cdot \text{cost}(f^*). \tag{3}
\]

The left-hand side of (3) is initially 0 and it is incremented only when Frac processes some active client $c^* \in \text{set}(f^*)$. In a single update operation, Frac may increment multiple $\beta$ variables, but only one of them, namely $\beta_{c^*, \text{cost}(f^*, c^*)}$, contributes to the growth of the left-hand side of (3). If variable $\beta_{c^*, \text{cost}(f^*, c^*)}$ is incremented, it means that the distance $\tau = \text{cost}(f^*, c^*)$ is already saturated, i.e., $\tau \in T_{c^*}^*$. Thus, in the same update operation, Frac augments all variables $y_f$ for $f \in \bigcup_{t \in T_{c^*}^*} F_{c^*, t}$. This set of facilities includes cluster $F_{c^*, t}$ and thus also facility $f^*$. By Lemma 6, augmentation of $f^*$ may happen at most $O(\log |F|) \cdot \text{cost}(f^*)$ times, which implies our claim. \(\square\)

### 2.5 Competitive ratio of Frac

Finally, we show that in each update operation the growth of the primal cost is at most \(O(1)\) times the growth of the dual cost. This will imply the competitive ratio of Frac.

**Lemma 8.** For any step $k$, the value of the solution to $P_k$ computed by Frac is at most 3 times the value of its solution to $D_k$.

**Proof.** As the values of both solutions are initially zero, it suffices to analyze the growth of the primal and dual objectives for a single update operation. The value of the dual solution grows by 1 as $\gamma_c$ is incremented only for the requested client $c$. Thus, it is sufficient to show that the primal solution increases at most by 3.

By $y_f, x_{c,t}$ and $T_{c}^1$, we understand the values of these variables before an update operation. Let $F_1 = \bigcup_{t \in T_{c}^1} F_{c,t}$. As the serving constraint for client $c$ is not satisfied at that point, \[1 > \sum_{t \in T} \min \{x_{c,t}, y(F_{c,t})\} \geq \sum_{t \in T_{c}^1} \min \{x_{c,t}, y(F_{c,t})\} \geq \sum_{t \in T_{c}^1} y(F_{c,t}) = y(F_1). \tag{4}\]

In the last inequality we used that (by Lemma 3), $T_{c}^1 = \{t \in T : x_{c,t} = 1\}$. The last equality follows as sets $F_{c,t}$ are disjoint for different $t$.

Within a single update operation, let $\Delta x_{c,t}$ and $\Delta y_f$ be the increases of variables $x_{c,t}$ and $y_f$, respectively. By Lemma 3, Frac increases one connection variable $x_{c,t^*}$ for an active distance $t^*$ (and no connection variable if there is no active distance) and performs augmentations of $y_f$ for all $f \in F_1$. The increase of the primal value is then
\[
\Delta P = \sum_{t \in T} t \cdot \Delta x_{c,t} + \sum_{f \in F_1} \text{cost}(f) \cdot \Delta y_f
\[
\leq 1 + \sum_{f \in F_1} \text{cost}(f) \cdot \left( \frac{y_f}{\text{cost}(f)} + \frac{1}{|F| \cdot \text{cost}(f)} \right)
\[
= 1 + y(F_1) + \frac{|F_1|}{|F|} < 3,
\]

where the last inequality follows by (4). \(\square\)
Lemma 9. For any input $G = (F, C, E, \text{cost}), A)$, it holds that $\text{FRAC}(G, A) \leq O(\log |F|) \cdot \text{Opt}(G, A)$.

Proof. Let $k$ be the total number of active clients in $A$, and let $\text{val}(P_k)$ and $\text{val}(D_k)$ be the values of the final primal and dual solutions generated by $\text{FRAC}$. Then,

$$\text{FRAC}(G, A) = \text{val}(P_k) \leq 3 \cdot \text{val}(D_k) \quad \text{(by Lemma 8)}$$

$$\leq O(\log |F|) \cdot \text{Opt}(G, A) \quad \text{(by Lemma 7 and weak duality).}$$

3 Deterministic rounding

Now we define our deterministic algorithm $\text{INT}$, which rounds the fractional solution computed by $\text{FRAC}$. For a client $c \in A$, $\text{INT}$ observes the actions of $\text{FRAC}$ while processing $c$ and on this basis makes its own decisions. First, $\text{INT}$ processes augmentations of variables $y_f$ performed by $\text{FRAC}$, and purchases some facilities. Once $\text{FRAC}$ finishes handling client $c$, $\text{INT}$ connects $c$ to the closest open facility. (We show below that such facility exists.)

3.1 Purchasing facilities

Purchasing facilities by $\text{INT}$ is based solely on graph $G$ and on updates of $y_f$ variables produced by $\text{FRAC}$. In particular, it neglects whether a given client is active or not. We use integral variables $\hat{y}_f \in \{0,1\}$ to denote whether $\text{INT}$ opened facility $f$. Furthermore, for any set $F'$ we use $\hat{y}(F')$ as a shorthand for $\sum_{f \in F'} \hat{y}_f$. On the basis of the facility-client graph $G$, we define the set $C \times T$ of elements.

The following lemma is an adaptation of the deterministic rounding routine for the set cover problem by Alon et al. [AAA03]. In the proof, we cover artificially constructed elements, each being a pair $(c, t) \in C \times T$ by sets corresponding to facilities from $F$. For completeness, the proof is given in Appendix A.

Lemma 10. Fix any input $G = (F, C, E, \text{cost}), A)$. Initially, $\hat{y}_f = y_f = 0$ for any $f \in F$. There exists a deterministic polynomial-time online algorithm $\text{INT}FAC$ that transforms increments of fractional variables $y_f$ to increments of integral variables $\hat{y}_f \in \{0,1\}$, so that

- Condition $y(S_{c,t}) \geq 1/2$ implies $\hat{y}(S_{c,t}) \geq 1$ for any client $c \in C$ (active or inactive) and any $t \in T$,
- $\sum_{f \in F} \text{cost}(f) \cdot \hat{y}_f \leq O(\log |C \times T|) \cdot \sum_{f \in F} \text{cost}(f) \cdot y_f + 2 \cdot \max_{f \in F} \text{cost}(f)$.

3.2 Connecting clients

Recall that once $\text{INT}$ purchases facilities using deterministic routine $\text{INT}FAC$ (cf. Lemma 10), it connects client $c$ to the closest open facility. Now we show that such facility indeed exists and we bound the competitive ratio of $\text{INT}$.

Lemma 11. On any input $(G, A)$, the solution generated by $\text{INT}$ is feasible and the total cost of connecting clients by $\text{INT}$ is at most $2 \cdot \text{FRAC}(G, A)$.

Proof. Fix any client $c \in A$. By Lemma 5, there exists distance $\tau \in T$ such that $y(S_{c,\tau}) \geq 1/2$ and $\sum_{t \in T} t \cdot x_{c,t} \geq \tau/2$. By Lemma 10, once $\text{INT}$ purchases facilities, it holds that $\hat{y}(S_{c,\tau}) \geq 1$. It means that at least one facility is opened in set $S_{c,\tau}$, i.e., at the distance at most $\tau$ from $c$. 
Therefore, \(\text{INT}\) is feasible and by connecting \(c\) to the closest open facility, it ensures that the connection cost is at most \(\tau \leq 2 \cdot \sum_{t \in T} t \cdot x_{c,t}\). The proof is concluded by observing that \(\sum_{t \in T} t \cdot x_{c,t}\) is the connection cost of \(\text{FRAC}\) that can be attributed solely to connection of client \(c\).

\[\square\]

**Lemma 12.** For any input \((G = (F, C, E, \text{cost}), A)\), it holds that \(\text{INT}(G, A) \leq q \cdot \log |F| \cdot (\log |C| + \log \log \Delta_G) \cdot \text{OPT}(G, A) + 2 \cdot \max_{f \in F} \text{cost}(f)\), where \(q\) is a universal constant not depending on \(G\) or \(A\). Furthermore, \(\text{INT}\) runs in time \(\text{poly}(|G|, |A|, \max_{e \in E} \text{cost}(e), \max_{f \in F} \text{cost}(f))\).

**Proof.** Let \(\rho = \max_{f \in F} \text{cost}(f)\). Then,

\[
\text{INT}(G, A) \leq \sum_{f \in F} \text{cost}(f) \cdot \hat{y}_f + 2 \cdot \text{FRAC}(G, A) \leq O(\log |C \times T|) \cdot \text{FRAC}(G, A) + 2 \cdot \rho
\]

(by Lemma 11)

\[
\leq O\left(\left(\log |C| + \log |T|\right) \cdot \log |F|\right) \cdot \text{OPT}(G, A) + 2 \cdot \rho
\]

(by Lemma 10)

The bound on the cost of \(\text{INT}\) is concluded by using \(|T| \leq 2 + \log \Delta_G\).

By Lemma 4, \(\text{FRAC}\) running time is \(\text{poly}(|G|, |A|, \max_{e \in E} \text{cost}(e), \max_{f \in F} \text{cost}(f))\). On top of that, \(\text{INT}\) adds its own computations (in particular the rounding scheme of \(\text{INTFac}\)), whose runtime is polynomial in \(|G|\) and \(|A|\). This implies the second part of the lemma (the running time of \(\text{INT}\)).\[\square\]

## 4 Handling large aspect ratios

The guarantee of Lemma 12 has two deficiencies: (i) the bound on the competitive ratio of \(\text{INT}\) depends on the aspect ratio of \(G\) and on the cost of the most expensive facility, (ii) the running time of \(\text{INT}\) depends on the maximal cost in the graph \(G\) (which can be exponentially large in the input description). We show how to use cost doubling and edge pruning to handle these issues.

**Theorem 1 (restated).** There exists a deterministic polynomial-time \(O(\log |F| \cdot (\log |C| + \log \log |F|))\)-competitive algorithm \(\text{DET}\) for the online non-metric facility location problem on set \(F\) of facilities and set \(C\) of clients.

**Proof.** Fix facility-client graph \(G = (F, C, E, \text{cost})\) for the non-metric facility location problem. Recall that we assumed that all non-zero costs and distances in \(G\) are powers of 2 and are at least 1. Let \(R = \log |F| \cdot (\log |C| + \log \log (|F| \cdot |C|))\).

We now construct a deterministic algorithm \(\text{DET}\) which is \(O(R)\)-competitive on an input \((G, A)\). Let \(q\) be the constant from Lemma 12. \(\text{DET}\) operates in phases, numbered from 0. In phase \(j\), it executes the following operations.

1. \(\text{DET \ pre-purchases}\) all facilities and edges of \(G\) whose cost is smaller than \(2^j / (|F| \cdot |C|)\).

2. \(\text{DET}\) creates an auxiliary facility-client graph \(\tilde{G}_j\) applying the following modifications to \(G\).

   - First, \(\text{DET}\) creates graph \(G_j\) containing only edges and facilities from \(G\) whose individual cost is at most \(2^j\). It also removes connections to facilities that have been removed in this process.
   - Second, the costs of all facilities and edges that have been pre-purchased by \(\text{DET}\) are set to zero in \(G_j\). In result, \(G_j\) is a sub-graph of \(G\) with adjusted distances and costs of facilities, has the same set of clients, its set of facilities is a subset of \(F\), and \(\Delta_{G_j} \leq |F| \cdot |C|\).
3. **Det** simulates algorithm **Int** on input \((\tilde{G}_j, A)\). That is, for a client \(c \in A\), **Det** verifies whether the overall cost of **Int** (including serving \(c\)) remains at most \(h_j \cdot (q \cdot R + 2) \cdot 2^j\). In such case, **Det** outputs the choices of **Int** for client \(c\) as its own. We emphasize that **Int** is run also on clients that have been already served in the previous phases; in effect, **Det** may purchase the same facilities or connections multiple times.

4. Eventually, either the sequence \(A\) of active clients ends and the total cost of **Int** on \((G_j, A)\) is at most \(h_j \cdot (q \cdot R + 2) \cdot 2^j\) (in which case **Det** terminates as well) or the purchases made by **Int**, while handling a client \(c \in A\), caused its cost to exceed \(h_j \cdot (q \cdot R + 2) \cdot 2^j\). (This includes the special case where \(c\) is disconnected from all facilities in \(\tilde{G}_j\), because all edges that connected \(c\) to facilities in \(G\) were more expensive than \(2^j\) or connected \(c\) to facilities with opening cost greater than \(2^j\).) In the case of exceeded cost, **Det** disregards the decisions of **Int** for client \(c\), terminates **Int**, and starts phase \(j + 1\), processing also all clients that were already served in phase \(j\).

We now analyze the performance of **Det**. Let \(k = \lceil \log(\text{Opt}(G, A)) \rceil \geq 0\). We show that **Det** terminates latest in phase \(k\). Assume that **Det** has not finished within phases 0, 1, \ldots, \(k - 1\). In phase \(k\), **Det** creates auxiliary graphs \(G_k\) and \(\tilde{G}_k\), and runs **Int** on graph \(\tilde{G}_k\). Graph \(G_k\) contains all edges of \(G\) of cost at most \(2^k\); their cost in \(G_k\) is the same or reset to zero. As \(\text{Opt}(G, A) \leq 2^k\), \(\text{Opt}(G_k, A)\) purchases only edges that are in \(G_k\), and thus \(\text{Opt}(G, A)\) is also a feasible solution to instance \((G_k, A)\). Thus, \(\text{Opt}(G_k, A) \leq \text{Opt}(G, A) \leq 2^k\). As \(\tilde{G}_k\) is the scaled-down copy of \(G_k\), \(\text{Opt}(\tilde{G}_k, A) = h_k \cdot \text{Opt}(G_k, A) \leq h_k \cdot 2^k\).

Let \(F_k\) be the set of facilities of graph \(\tilde{G}_k\) and \(\tilde{c}ost_k(f)\) is the cost of opening facility \(f\) in graph \(\tilde{G}_k\). Clearly, \(|F_k| \leq |F|\) and \(\tilde{c}ost_k(f) \leq h_k \cdot \text{cost}(f)\) for any \(f \in F\). By our construction, \(\Delta_{\tilde{G}_k} = \Delta_{\tilde{G}} \leq |F| \cdot |C|\). Hence, Lemma 12 implies that

\[
\text{Int}(\tilde{G}_k, A) \leq q \cdot \log |F_k| \cdot (\log |C| + \log \log \Delta_{\tilde{G}_k}) \cdot \text{Opt}(\tilde{G}_k, A) + 2 \cdot \max_{f \in F_k} \tilde{c}ost_k(f)
\leq h_k \cdot q \cdot \log |F| \cdot (\log |C| + \log \log(|F| \cdot |C|)) \cdot 2^k + 2 \cdot h_k \cdot 2^k
= h_k \cdot (q \cdot R + 2) \cdot 2^k.
\]

Therefore, **Int** is not terminated prematurely within phase \(k\) because of high cost and it finishes the entire sequence \(A\). This implies feasibility of **Int**: it serves all clients latest in phase \(k\).

To bound the total cost of **Det**, recall that at the beginning of phase \(j\), **Det** purchases at most \(|F| \cdot |C|\) edges and at most \(|F|\) facilities, each of cost at most \(2^j / (|F| \cdot |C|)\). The associated overall cost is at most \(2 \cdot 2^j\). The cost of the subsequent execution of algorithm **Int** on \(G_j\) is, by our termination rule, at most \(h_j \cdot (q \cdot R + 2) \cdot 2^j\), and thus the cost incurred by repeating **Int**’s actions on \(G\) is at most \((q \cdot R + 2) \cdot 2^j\). The overall cost is then \(\text{Det}(G, A) \leq \sum_{j=0}^{k} (q \cdot R + 4) \cdot 2^j = O(R) \cdot 2^k = O(R) \cdot \text{Opt}(G, A) = O(|F| \cdot (\log |F| \cdot (\log |C| + \log \log |F|)) \cdot \text{Opt}(G, A)\).

For the running time of **Det**, we note that in phase \(j\), **Int** is run on a graph \(\tilde{G}_j\) whose smallest cost is 1, and hence the largest cost is at most \(\Delta_{\tilde{G}_j} = \Delta_{\tilde{G}} \leq |F| \cdot |C|\). Thus, by Lemma 12, the running time of **Int** in a single phase is polynomial in \(|G|\) and \(|A|\), and the number of phases is logarithmic in the maximum cost occurring in \(G\), and thus also polynomial in \(|G|\). □
5 Final remarks

The presented deterministic solution to the non-metric facility location problem is one of the successful examples where the performance of the deterministic solution (nearly) matches the performance of the randomized one. By clustering facilities, we encoded dependencies between facilities and clients, which allowed us later to apply rounding to facilities only, neglecting the actual active clients. It would be however interesting and useful to have an online deterministic rounding routine able to handle such dependencies internally (e.g., by creating a pessimistic estimator that can be computed and handled in online manner), as it is the case for the set cover problem or throughput-competitive virtual circuit routing [BN09b].

That said, we believe that our distance clustering techniques can be extended to other network design problems for which only randomized algorithms existed so far, e.g., online multicast problems on trees [AAA+06], online group Steiner problem on trees [AAA+06] or variants of the facility location problem that are used as building blocks for solutions to other node-weighted Steiner problems [HLP14, HLP17]. Finally, another open problem is whether these techniques could be also applied more directly for the node-weighted Steiner tree, resulting in a better deterministic competitive ratio.

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A Algorithm INTFAC for rounding facilities (Proof of Lemma 10)

We start with a technical claim and later we define our rounding procedure. As we mentioned earlier, this part is an adaptation of the deterministic rounding procedure for the set cover problem by Alon et al. [AAA03].

**Lemma 13.** Fix any \( q \in [0, 1/2] \) and any \( r \geq 0 \). Let \( X \) be a binary variable being 0 with probability \( p > 0 \). Then, \( E[\exp(q \cdot X)] \leq \exp(-(3/2) \cdot q \cdot \ln p) \).

**Proof.** Using the definition of \( X \), we have

\[
E[\exp(q \cdot X)] = p \cdot e^0 + (1 - p) \cdot e^q = \exp(\ln p) + (1 - \exp(\ln p)) \cdot e^q \leq 1 + \ln p - e^q \cdot \ln p = 1 - \ln p \cdot (e^q - 1) \leq 1 - (3/2) \cdot q \cdot \ln p \leq \exp(-(3/2) \cdot q \cdot \ln p).
\]

In the first inequality, we used that \( e^x \cdot 1 + (1 - e^x) \cdot z \leq (1 + x) \cdot 1 + (-x) \cdot z \) for any \( x \leq 0 \) and \( z \geq 1 \) and in the second one, we used that \( e^x - 1 \leq 3x/2 \) for any \( x \in [0, 1/2] \). \( \square \)

**A.1 Algorithm description**

Let \( \ell = |C| \cdot |T| \). We consider the potential function \( \Phi = \Phi_1 + \Phi_2 \), where

\[
\Phi_1 = \sum_{(c,t) : y(S_{ct}) = 0} \ell^4 y(S_{ct}),
\]

\[
\Phi_2 = \ell \cdot \exp \left( \sum_{f \in F} \frac{\text{cost}(f)}{2\rho} \cdot \hat{y}_f - \sum_{f \in F} \frac{b \cdot \text{cost}(f)}{2\rho} \cdot y_f \right),
\]

where \( \rho = \max_{f \in F} \text{cost}(F) \) and \( b = 6 \cdot \ln \ell = O(\log |C \times T|) \).

Assume that \( \text{FAC} \) augmented variable \( y_f \). Then our algorithm INTFAC chooses whether to set \( \hat{y}_f \) to 1 or not (purchase \( f \) or not), so that the potential \( \Phi \) does not increase. (We again emphasize that this choice neglects the current set of active clients.)
A.2 Correctness and performance

Lemma 14. Assume \( y_f \) is increased by \( \delta \). If \( \hat{y}_{f^*} = 1 \), then \( \Phi \) does not increase. Otherwise, there is a choice to either set \( \hat{y}_{f^*} \) to 1 or not, such that \( \Phi \) does not increase.

Proof. By \( y_f \) and \( \hat{y}_{f^*} \), we mean the values of these variables before an update operation of \( \text{FRAC} \).

First, we assume \( \hat{y}_{f^*} = 1 \). Augmenting variable \( y_{f^*} \) affects values of \( y(S_{c,t}) \) for \( f^* \in S_{c,t} \): all such \( y(S_{c,t}) \) increase by \( \delta \). However, for any element \((c,t)\), such that \( f^* \in S_{c,t} \), it holds that \( \hat{y}(S_{c,t}) \geq \hat{y}_{f^*} = 1 \), i.e., element \((c,t)\) is not counted in the sum occurring in \( \Phi_1 \). Therefore, augmenting variable \( y_{f^*} \) does not affect \( \Phi_1 \). Furthermore, augmenting \( y_f \) and keeping \( \hat{y}_{f^*} \) unchanged can only decrease \( \Phi_2 \). Thus, \( \Phi = \Phi_1 + \Phi_2 \) does not increase when \( \hat{y}_{f^*} = 1 \).

Second, we consider the case \( \hat{y}_{f^*} = 0 \). To show that either setting \( \hat{y}_{f^*} \) to 1 or leaving it at 0, does not increase the potential, we use probabilistic method and show that if we pick such action randomly (setting \( \hat{y}_{f^*} = 1 \) with probability \( 1 - \ell^{-4} - \delta \)), then, on expectation, neither \( \Phi_1 \) nor \( \Phi_2 \) increases.

- As observed above, only elements \((c,t)\) for which \( S_{c,t} \) contain \( f^* \) are affected by augmentation of \( y_{f^*} \) and possible change of \( \hat{y}_{f^*} \). Let \( Q = \{ (c,t) : f^* \in S_{c,t} \} \) be the set of elements contributing to \( \Phi_1 \) that could be affected by augmentation of \( y_{f^*} \) or a possible change of \( \hat{y}_{f^*} \).

Fix any element \((c,t) \in Q \). Its initial contribution towards \( \Phi_1 \) is \( \ell^4 y(S_{c,t}) \) and when \( y_{f^*} \) is augmented, its contribution increases to \( \ell^4 y(S_{c,t}) + \delta \). However, with probability \( 1 - \ell^{-4} - \delta \), variable \( \hat{y}_{f^*} \) is set to 1, thus \( y(S_{c,t}) \) grows from 0 to 1, and in effect element \((c,t)\) stops contributing to \( \Phi_1 \). Hence, the expected final contribution of element \((c,t)\) towards \( \Phi_1 \) is thus \( \ell^4 y(S_{c,t}) + \ell^{-4} + 0 \cdot (1 - \ell^{-4} - \delta) = \ell^4 y(S_{c,t}) \), i.e., is equal to its initial contribution. Therefore, in expectation, the value of \( \Phi_1 \) is unchanged.

- It remains to bound the expected value of \( \Phi_2 \). Recall that we assumed that \( y_{f^*} = 0 \). Let \( \bar{Y} \) be the random variable equal to the value of \( \hat{y}_{f^*} \) after random choice (i.e., \( \bar{Y} = 1 \) with probability \( 1 - \ell^{-4} - \delta \)) and \( \Phi_2' \) denote the value of \( \Phi_2 \) after augmenting \( y_{f^*} \) and after the random choice. Then,

\[
\Phi_2' = \ell \cdot \exp \left( \sum_{f \in F} \frac{\text{cost}(f)}{2\rho} \cdot \hat{y}_f + \frac{\text{cost}(f^*)}{2\rho} \cdot \bar{Y} - \sum_{f \in F} \frac{b \cdot \text{cost}(f)}{2\rho} \cdot y_f - \frac{b \cdot \text{cost}(f)}{2\rho} \cdot \delta \right)
\]

\[
= \Phi_2 \cdot \exp \left( \frac{\text{cost}(f^*)}{2\rho} \cdot \bar{Y} \right) \cdot \exp \left( - \frac{b \cdot \text{cost}(f^*)}{2\rho} \cdot \delta \right)
\]

To estimate \( \mathbb{E}[\Phi_2'] \), we upper-bound the expected value of expression \( \exp(\bar{Y} \cdot \text{cost}(f^*)/(2\rho)) \), using Lemma 13 with \( q = \text{cost}(f^*)/(2\rho) \leq 1/2 \) and \( p = \ell^{-4} - \delta \), obtaining that

\[
\mathbb{E} \left[ \exp \left( \frac{\text{cost}(f^*)}{2\rho} \cdot \bar{Y} \right) \right] \leq \exp \left( - \frac{(3/2) \cdot \text{cost}(f^*) \cdot \ln p}{2\rho} \right) = \exp \left( \frac{6 \cdot \ln \ell \cdot \text{cost}(f^*) \cdot \delta}{2\rho} \right)
\]

Therefore, \( \mathbb{E}[\Phi_2'] \leq \Phi_2 \) and the lemma follows.
Lemma 10 (restated). Fix any input \((G = (F, C, E, \text{cost}), A)\). Initially, \(y_f = \hat{y}_f = 0\) for any \(f \in F\). There exists a deterministic polynomial-time online algorithm \textsc{IntFac} that transforms increments of fractional variables \(y_f\) to increments of integral variables \(\hat{y}_f\in\{0,1\}\), so that

- Condition \(y(S_{c,t}) \geq 1/2\) implies \(\hat{y}(S_{c,t}) \geq 1\) for any client \(c \in C\) (active or inactive) and any \(t \in T\),
- \(\sum_{f \in F} \text{cost}(f) \cdot \hat{y}_f \leq O(\log |C \times T|) \cdot \sum_{f \in F} \text{cost}(f) \cdot y_f + 2 \cdot \max_{f \in F} \text{cost}(f)\).

Proof of Lemma 10. Initially, all variables \(y_f\) and \(\hat{y}_f\) are zero, and thus \(\Phi = \sum_{(c,t) \in C \times T} \ell^0 + \ell \cdot \exp(0) = 2 \cdot \ell\). By Lemma 14, the potential never increases. Since \(\Phi_2\) is non-negative, any summand of \(\Phi_1\) is always at most \(2 \cdot \ell \leq \ell^2\). Therefore, \(4 \cdot y(S_{c,t}) \geq 2\) always implies \(\hat{y}(S_{c,t}) > 0\), i.e., the first part of the lemma follows.

To show the second part, we again use that \(\Phi = \Phi_1 + \Phi_2 \leq 2 \cdot \ell\) at any time. As \(\Phi_1\) is non-negative, \(\Phi_2 \leq 2 \cdot \ell\). Substituting the definition of \(\Phi_2\), dividing by \(\ell\), and taking natural logarithm of both sides yields

\[
\frac{1}{2\rho} \cdot \sum_{f \in F} (\hat{y}_f \cdot \text{cost}(f) - b \cdot y_f \cdot \text{cost}(f)) \leq \ln(2) < 1.
\]

Therefore, \(\sum_{f \in F} \hat{y}_f \cdot \text{cost}(f) \leq 2\rho + b \cdot \sum_{f \in F} y_f \cdot \text{cost}(f)\). \(\square\)