A GEOMETRIC MODEL OF TUBE CATEGORIES

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Abstract. We give a geometric model for a tube category in terms of homotopy classes of oriented arcs in an annulus with marked points on its boundary. In particular, we interpret the dimensions of extension groups of degree 1 between indecomposable objects in terms of negative crossing numbers between corresponding arcs, giving a geometric interpretation of the description of an extension group in the cluster category of a tube as a symmetrized version of the extension group in the tube. We show that a similar result holds for finite dimensional representations of the linearly oriented quiver of type $A_\infty$.

1. Introduction

Let $k$ be an algebraically closed field. A tube category $\mathcal{T}_n$ of rank $n$ over $k$ can be defined as the category of finite dimensional nilpotent representations of an oriented $n$-cycle over $k$. We introduce a geometric model of such a category using an annulus with $n$ marked points on the outer boundary and none on the inner boundary. The indecomposable objects of $\mathcal{T}_n$ are in bijection with certain oriented arcs in the annulus, i.e. those which are oriented anticlockwise and which are not homotopic to a boundary arc between successive marked points; we refer to them as admissible arcs. The AR-quiver of the tube can be realised via the geometry of the arcs. We then show that:

Theorem 1.1. Let $\alpha$ and $\beta$ be admissible arcs in an annulus with $n$ marked points on its outer boundary, and let $A$ and $B$ be the corresponding indecomposable objects in a tube $\mathcal{T}_n$ of rank $n$. Then the dimension of $\text{Ext}^{1}_{\mathcal{T}_n}(A, B)$ coincides with the minimal number of negative crossings between $\alpha$ and $\beta$.

Cluster categories were introduced in [BMRRT] (and in [CCS] for type $A$) in order to model the cluster algebras [FZ] of Fomin-Zelevinsky. They have since been used widely in representation theory (see e.g. [K2] for a survey). Since a tube is skeletally small it follows from Keller’s theorem [K1, §9.9] that the cluster category $\mathcal{C}_{\mathcal{T}_n}$ associated to a tube (which we call a cluster tube) is triangulated (see also [BKL] §5 and [BMV] §2). We show that the minimal number of crossings between $\alpha$ and $\beta$ is equal to the sum of the minimal number of positive crossings and the minimal number of negative crossings. This total number coincides with $\text{Ext}^{1}_{\mathcal{C}_{\mathcal{T}_n}}(A, B)$. We thus obtain a geometric interpretation of the fact that $\text{Ext}^{1}_{\mathcal{C}_{\mathcal{T}_n}}(A, B) \cong \text{Ext}^{1}_{\mathcal{T}_n}(A, B) \oplus D \text{Ext}^{1}_{\mathcal{T}_n}(B, A)$, (which can be proved as in [BMRRT] Prop. 1.7) for indecomposable objects $A$ and $B$ in $\mathcal{T}_n$, where $D = \text{Hom}_k(\cdot, k)$. We also show that similar results hold for the category of finite dimensional representations of the linear orientation of the quiver $A_\infty$, a hereditary category whose AR-quiver is of type $ZA_\infty$. 

1
We note that geometric models of cluster categories and of the generalised cluster categories of Amiot [A] use unoriented arcs in marked Riemann surfaces: see [BZ, CCS]. Such models were used for cluster algebras in [FZ, FST]. Our perspective here is that by using oriented arcs we can model the classical categories \( T_n \) arising in representation theory. The 2-Calabi-Yau \( \text{Ext}^1 \)-symmetry of the cluster category then arises by forgetting the orientation on the arcs. We note that oriented arcs have also been used in [BM] to obtain a geometric model for the root category \( \mathbb{H} \) of type \( A_1 \) (the quotient of the bounded derived category by the square of the shift).

We remark that [G] and [BZ, 3.18] give a construction of the AR-quiver of a cluster tube (in the latter case, as a subcategory of a generalised cluster category) using unoriented arcs in an annulus. Also, it follows from [BZ, 5.1, 5.3] that if the intersection number of two admissible arcs in the annulus is non-zero then the corresponding \( \text{Ext}^1 \)-group in the cluster tube is non-zero (with equivalence in the case where the arcs are equal). Thus, in the case of the cluster tube, we obtain a strengthening of this last result.

In Section 2 we recall tube categories and indicate how the AR-quiver of the tube can be constructed geometrically using admissible arcs in the annulus. In Section 3 we compute intersection numbers between admissible arcs in the annulus by lifting them to the universal covering space, and prove the main result. In Section 4 we give an interpretation of our model using doubled arcs which, in particular, gives a connection with a model for the rigid part of the cluster tube given in [BMV, §3]. In Section 5 we consider the case of the category of finite dimensional representations of the linear orientation of \( A_\infty \).

Since completing this work, we learnt of interesting independent work in the diploma thesis [W] which, in particular, gives a bijection between string modules over a quiver of type \( \tilde{A}_n \) (i.e. a quiver whose underlying graph is a cycle, but is not an oriented cycle itself) and certain oriented arcs in the annulus (regarded as a cylinder). A geometric interpretation of the AR-quiver is given and it is shown that the negative intersection number of two arcs coincides with the dimension of the extension group of the corresponding modules. Since tubes can be obtained as subcategories of the module categories of such quivers, this gives an alternative proof of Theorem 1.1. The difference here is that we work directly with the tube, via finite dimensional nilpotent representations of a cyclic quiver, allowing a simpler proof for this subcategory. We also include a topological description of the relationship between intersection numbers in the annulus and its universal cover (see Section 3). Our approach links up interpretations of intersection numbers with and without orientation, thus strengthening results of [BZ] in this case, and also gives an explanation for the doubled arc model for exceptional objects in the tube appearing in [BMV].

2. The AR-quiver of a tube

2.1. Categories of tube shape. We refer to [ARS] for background on the theory of finite dimensional algebras and their representations. A stable translation quiver \( \Gamma \) without multiple arrows is said to be of \emph{tube type} if it is of the form \( \Gamma = \mathbb{Z} A_\infty/n \) for some \( n > 0 \), cf. [Ri, §3]. In that case we say that the tube \( \Gamma \) has rank \( n \) and we call the additive hull of the mesh category of \( \Gamma \) a \emph{tube of rank} \( n \), denoting it \( T_n \). If \( \mathcal{C}_n \) is a cyclically oriented quiver with \( n \) vertices then the category of finite-dimensional nilpotent representations of \( k\mathcal{C}_n \) is equivalent to \( T_n \) (this follows from [Ri, 3.6(6)]; see also [RvdB, III.1.1]).

The category \( T_n \) is Hom-finite, abelian, hereditary and uniserial [Ri, §3]. We have the duality \( \text{Ext}^1_{T_n}(X, Y) \cong D \text{Hom}_{T_n}(Y, \tau X) \) in \( T_n \), where \( \tau \) denotes the AR-translate. We can form its cluster category as in [BMRRT, §1], obtaining the
cluster tube $C_{T_n}$ of rank $n$ as the orbit category $D^b(T_n)/\tau^{-1}[1]$. As noted above, Keller’s theorem [K1] §9.9 applies, so the cluster tube is triangulated. This category has also been studied in [BKL1] [BKL2] [BMV]. Its AR-quiver is the same as the AR-quiver of $T_n$ (arguing as in [BMRRT] Prop. 1.3). To abbreviate we write $\Gamma_{T_n} := \Gamma_{AR}(T_n)$ for the AR-quiver of $T_n$.

2.2. Arcs in the annulus and its universal cover. Let $\mathbb{A}(n) = \mathbb{A}(n,0)$ be an annulus with $n$ marked points on the outer boundary. Our goal is to use oriented arcs in $\mathbb{A}(n)$ as a geometric model for the tube and the cluster tube as introduced in Subsection 2.1.

Remark 2.1. In [FST] Example 6.9, the authors consider annuli of the form $\mathbb{A}(r,s)$ with $r, s \geq 1$. These have $r$ marked points on the outer boundary, $s$ marked points on the inner boundary and unoriented arcs between these marked points. More generally, the authors study triangulated surfaces as a geometric model for cluster algebras. In particular, the annuli $\mathbb{A}(r,s)$ give rise to cluster algebras associated to quivers of type $A_{r+s-1}$ with $r$ arrows in one direction and $s$ arrows in the other direction. Thus it makes sense to use the annulus $\mathbb{A}(n)$, with $n$ marked points on one boundary component only, to model $T_n$, whose objects can be considered as (nilpotent) representations of a cyclic quiver with $n$ arrows all in the same direction.

We consider smooth oriented arcs in surfaces with boundary. Recall that two such arcs $\alpha, \beta$ which start at the same point and finish at the same point are said to be homotopy equivalent if there is a homotopy between them fixing the starting point and the finishing point. We write $\alpha \sim \beta$ to denote this.

Let us now consider oriented arcs between marked points of the boundary of $\mathbb{A}(n)$. We start by labelling the marked points of $\mathbb{A}(n)$ with $0, 1, \ldots, n-1$, in an anti-clockwise direction. We will always take arcs up to homotopy fixing the starting and ending points. For simplicity, we will assume that the marked points are equally spaced around the outer boundary of $\mathbb{A}(n)$. An oriented arc from a marked point $i$ to a marked point $l$, for $i, l \in \{0, 1, \ldots, n-1\}$, is homotopically equivalent to the boundary arc which starts at $i$, goes around the boundary $k$ times (for some $k \geq 0$) and then ends in $l$; we shall denote this arc by $i_k l$.

It will be most convenient to describe oriented arcs and their intersections using the universal cover of $\mathbb{A}(n)$. To be precise, observe that we may identify the annulus $\mathbb{A}(n)$ with a (bounded) cylinder $Cyl(n)$ of height 1 with $n$ marked points on the lower boundary: we regard the cylinder as a rectangle in $\mathbb{R}^2$ with the vertical sides identified, cf. Figure 1. As in the annulus, we label the marked points on the lower boundary of $Cyl(n)$ from left to right and assume that arcs are oriented anti-clockwise (when viewed from the top of the cylinder), i.e. left to right in the rectangle in $\mathbb{R}^2$.

In a second step, we move from the cylinder $Cyl(n)$ to its universal covering space $U = (U, \pi_n)$, with $U = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1\}$, an infinite strip in the plane. We adopt the orientation on $U$ inherited from its embedding in the plane. The covering map $\pi_n: U \to Cyl(n)$ is induced from wrapping $U$ around $Cyl(n)$, i.e. the effect of $\pi_n$ on $(x, y) \in U$ is to take the first coordinate modulo $n$:

$$\pi_n: U \to Cyl(n), \ (x, y) \mapsto (x \mod n, y),$$

so $(\lambda n, y)$ is mapped to $(0, y)$ for any $\lambda \in \mathbb{Z}$. We choose an orientation on $Cyl(n)$ such that $\pi_n$ is orientation-preserving. Via the identification of $Cyl(n)$ with $\mathbb{A}(n)$, this induces an orientation on $\mathbb{A}(n)$. We will also write $\pi_n$ to indicate the covering map from $U$ to $\mathbb{A}(n)$ directly. We take the integers $(r, 0) \in U$, $r \in \mathbb{Z}$, as marked points on the lower boundary of $U$. 


We define $\sigma: U \to U$ to be the translation which adds $n$ to the $x$-coordinate of a point in $U$, with inverse $\sigma^{-1}$ subtracting $n$ from the $x$-coordinate of a point, so that $\pi_n(\sigma^r(u)) = \pi_n(u)$ for all $u \in U$ and for all $r \in \mathbb{Z}$. We write $G := \langle \sigma \rangle$ for the group generated by $\sigma$ which thus acts naturally on $U$.

Observe that for any $s \in \mathbb{R}$, the image of $U_s = \{(x,y) \in U \mid s \leq x < s + n\}$ under $\pi_n$ is the cylinder $\text{Cyl}(n)$.

It is clear that the oriented arc in the cylinder $\text{Cyl}(n)$ corresponding to $i_kl$ starts at a marked point $i$ and then wraps around the cylinder $k$ times before ending at a marked point $l$. We now introduce an alternative notation for arcs in $U$ and $\mathbb{A}(n)$.

Definition 2.2. Let $i, j$ be integers with $j > i + 1$. An admissible arc in $U$ is an oriented arc in $U$ starting at the marked point $i = (i, 0)$ and ending in $j = (j, 0)$; we denote it by $[i, j]$. An oriented arc in $\mathbb{A}(n)$ is said to be admissible if it is of the form $\pi_n([i, j])$ where $[i, j]$ is admissible in $U$.

Definition 2.3. Let $[i, j]$ be an admissible arc in $U$. Then we define the combinatorial length of $[i, j]$ to be the integer $j - i$.

Remark 2.4. 1) If $\alpha$ is an admissible arc in $\mathbb{A}(n)$ with starting point $i$, $0 \leq i < n$, then it has a unique lift $\tilde{\alpha}$ in $U$ which starts at $i = (i, 0)$. We call this lift the canonical lift of $\alpha$. Observe that all other liftings of $\alpha$ can be obtained by iteratedly applying $\sigma$ or $\sigma^{-1}$ to $\tilde{\alpha}$. So we get $\pi_n^{-1}(\alpha) = G\tilde{\alpha}$.

2) Observe that the winding number of the arc $\pi_n([i, j])$ around the inner boundary of $\mathbb{A}(n)$ is given by $k = \lfloor \frac{j - i}{n} \rfloor$.

2.3. A quiver of arcs. We can associate a stable translation quiver to $\mathbb{A}(n)$, called $\Gamma(\mathbb{A}(n)) = \Gamma_{\text{arc}}(\mathbb{A}(n))$. The vertices of $\Gamma(\mathbb{A}(n))$ are the admissible arcs in $\mathbb{A}(n)$. The arrows of $\Gamma(\mathbb{A}(n))$ are as follows. Let $\alpha$ and $\beta$ be admissible arcs in $\mathbb{A}(n)$. Then there is an arrow from $\alpha$ to $\beta$ if and only if there are admissible arcs $[a, b]$ and $[c, d]$ in $U$ with $\pi_n([a, b]) = \alpha$, $\pi_n([c, d]) = \beta$ and either $c = a + 1$ and $d = b$, or $c = a$ and $d = b + 1$. 

![Figure 1. $\alpha = \pi_n([2,24])$ in the annulus $\mathbb{A}(8)$ and in the cylinder $\text{Cyl}(8)$](image)

![Figure 2. The universal cover $U$ with a lift of the arc $\alpha = \pi_n([2,24])$](image)
The translation $\tau$ on $\Gamma(\mathbb{A}(n))$ is induced by the map $i \mapsto i - 1 \pmod{n}$ for $i$ a marked point of $\mathbb{A}(n)$. This clearly defines a structure of stable translation quiver on the set of admissible arcs in $\mathbb{A}(n)$. The case $n = 5$ is shown in Figure 3 as an example.

2.4. Isomorphism of translation quivers. Here we show that the quivers $\Gamma(\mathbb{A}(n))$ and $\Gamma(\mathcal{T}_n)$ are isomorphic and thus get the geometric model for tubes. This isomorphism also appears in the Master’s thesis [G] of B. Gehrig and in [BZ, §3.4] (in these two cases, using unoriented arcs), and in [W, §4.18] (using oriented arcs).

To describe the AR-quiver $\Gamma(\mathcal{T}_n)$, recall that $\mathcal{T}_n$ is uniserial. We denote the simple objects by $S_0, S_1, \ldots, S_{n-1}$. Then, if $M$ is an indecomposable object in $\mathcal{T}_n$, it has a unique composition series of the form $S_{i_1} \rightarrow S_{i_2} \rightarrow \cdots \rightarrow S_{i_k}$, for some $k \geq 1$, where $i_1, i_2, \ldots, i_k \in \{0, 1, \ldots, n-1\}$ and $i_{j+1} = (i_j + 1) \pmod{n}$, where $7$ denotes the reduction of $a$ modulo $n$. If $a, b \in \mathbb{Z}$ and $a + 1 < b$, we denote the unique indecomposable object of $\mathcal{T}_n$ with composition series $S_{i_1} \rightarrow S_{i_2} \rightarrow \cdots \rightarrow S_{i_k}$ by $M[a, b]$. Note that every indecomposable object arises uniquely in this way up to a shift of both entries by $n$, i.e. $M[a, b] \cong M[a + n, b + n]$; thus we obtain a unique parametrization of the indecomposable objects of $\mathcal{T}_n$ (up to isomorphism) by restricting $a$ to be in the set $\{0, 1, 2, \ldots, n-1\}$.

The AR-quiver of $\mathcal{T}_n$ is then easily described. It has vertices given by the indecomposable objects $M[a, b]$ of $\mathcal{T}_n$, for $0 \leq a \leq n-1$ and $a + 1 < b$ (note that $M[a, a + 2] \cong S_{a+1}$, for $a = 0, 1, \ldots, n-1$). For any such $a, b$, the inclusion $M[a, b] \rightarrow M[a, b + 1]$ and the surjection $M[a, b] \rightarrow M[a + 1, b]$ (if $a + 2 < b$) correspond to arrows in the AR-quiver (using the above isomorphism if necessary); all arrows in the AR-quiver arise in this way. The AR-translate $\tau$ takes $M[a, b]$ to $M[a + 1, b - 1]$.

It is then straightforward to check that we have the following.

**Lemma 2.5.** The quivers $\Gamma(\mathbb{A}(n))$ and $\Gamma(\mathcal{T}_n)$ are isomorphic as stable translation quivers. On vertices, the isomorphism is given by the map:

$$
\varphi : \{\text{admissible arcs in } \mathbb{A}(n)\} \rightarrow \text{ind}(\mathcal{T}_n)
$$

$$
\pi_n([a, b]) \mapsto M[a, b]
$$

where $a \in \{0, 1, \ldots, n-1\}$, $b \in \mathbb{Z}$ and $a + 1 < b$. 

\[\vspace{1cm}\]

**Figure 3.** The translation quiver $\Gamma(\mathbb{A}(5))$. 

\[\vspace{1cm}\]
3. Intersection Numbers as Dimensions of Extension Groups

In this section we show that (negative) intersection numbers between admissible arcs in $A(n)$ can be interpreted as dimensions of extension groups between the corresponding objects in $T_n$ via the isomorphism in Lemma 2.5. To do this we need to compute these intersection numbers; we do it by showing that they can be computed in the universal cover $U$ of $A(n)$. We were unable to find the exact statement we needed in the literature so include here a full proof.

We have the following (see e.g. [C, 5.4]).

**Theorem 3.1.** (Monodromy Theorem) Let $\pi : E \to X$ be a covering space of a surface $X$ and let $\gamma, \delta$ be paths in $E$ with common starting point. Then $\gamma \sim \delta$ if and only if $\pi(\gamma) \sim \pi(\delta)$. It follows that, in this situation, $\gamma$ and $\delta$ have a common finishing point.

A crossing between two oriented arcs $\alpha, \beta$ in an oriented surface $X$ is called **positive** if the orientation arising from the pair of tangents to the arcs at the crossing point is compatible with the orientation of $X$; it is **negative** otherwise. Figure 4 illustrates the two kinds of crossing for arcs in the plane (or $U$). We remark that the sign of a crossing in $A(n)$ is the same for either the orientation we gave $A(n)$ in Section 2.2 (i.e. compatible with that on $\text{Cyl}(n)$) or the orientation inherited from its embedding into the plane.

Two arcs $\alpha$ and $\beta$ in a surface $X$ are said to be in **general position** provided they intersect each other only transversely and have no multiple points of intersection. We denote this by writing $\alpha \bowtie \beta$. The **geometric intersection number** of two arcs $\alpha$ and $\beta$, denoted $I_X(\alpha, \beta)$, is defined by:

$$I_X(\alpha, \beta) = \min_{\alpha' \sim \alpha, \beta' \sim \beta} |\alpha' \cap \beta'|.$$  

This is finite (see e.g. [GP, §3]). We similarly define $I^+_{X}(\alpha, \beta)$ as the minimum number of positive crossings and $I^-_{X}(\alpha, \beta)$ in terms of the number of negative crossings (in the case where the arcs are oriented).

Recall that a **lift** of an arc $\alpha$ in $A(n)$ (to $U$) is an arc $\tilde{\alpha}$ in $U$ such that $\pi_n(\tilde{\alpha}) = \alpha$.

**Lemma 3.2.** Let $\tilde{\alpha}, \tilde{\beta}$ be lifts of oriented arcs $\alpha, \beta$ in $A(n)$ respectively. Then

$$I_{A(n)}(\alpha, \beta) = \min_{\gamma \sim \tilde{\alpha}, \delta \sim \tilde{\beta}} |\gamma \cap G\delta|.$$  

Furthermore, $I^+_{A(n)}(\alpha, \beta)$ (respectively, $I^-_{A(n)}(\alpha, \beta)$) can be computed in the same way except that on the right hand side, only positive (respectively, negative) crossings in the intersection are counted.
Proof. We note that the right hand side of the above is always finite as $\gamma, \delta$ are compact. We first prove the following claim:

Claim: Let $\gamma$ and $\delta$ be lifts in $U$ of $\alpha'$ and $\beta'$, respectively. Then the following are equivalent.

(a) $\gamma \sim \tilde{\alpha}$, $\delta \sim \tilde{\beta}$, and $G\gamma \pitchfork G\delta$.
(b) $\alpha^\prime \sim \alpha$, $\beta^\prime \sim \beta$, and $\alpha^\prime \pitchfork \beta^\prime$.

Proof of claim: (a) implies (b). Given $\gamma, \delta$ as in (a), the fact that $\alpha' \sim \alpha$ and $\beta' \sim \beta$ follows from Theorem 3.1. If $\alpha'$ and $\beta'$ intersected non-transversely, two lifts $\sigma^a(\gamma)$ and $\sigma^b(\delta)$ of $\alpha'$ and $\beta'$ would also intersect non-transversely, a contradiction. And if $\alpha'$ and $\beta'$ intersected in a multiple point then, taking the pre-image of an admissible neighbourhood of that point, we'd have that $G\gamma$ and $G\delta$ intersected in a multiple point, a contradiction. Hence (b) holds.

Conversely, suppose that $\alpha'$ and $\beta'$ are as in (b). Then $\gamma \sim \tilde{\alpha}$ and $\delta \sim \tilde{\beta}$ follow from Theorem 3.1. If $G\gamma$ and $G\delta$ intersected non-transversely, the image of a non-transverse intersection would be a non-transverse intersection of $\alpha'$ and $\beta'$, a contradiction. Similarly, if $G\gamma$ and $G\delta$ intersected in a multiple point, the image would be a multiple intersection of $\alpha'$ and $\beta'$, a contradiction. Hence (a) holds.

Now in order to prove the lemma we must show that, in the circumstances of the claim, we have

$$|\gamma \cap G\delta| = |\alpha' \cap \beta'|.$$ 

Let $f$ be $\pi_n$ restricted to $\gamma \cap G\delta$. It is clear that the image of $f$ is contained in $\alpha' \cap \beta'$. If $x \in \alpha' \cap \beta'$ there is $u \in U$ such that $\pi_n(u) = x$. Then we have:

$$u \in G\gamma \cap G\delta$$

since $\pi_n\gamma = \alpha'$ and $\pi_n\delta = \beta'$. So $u \in \sigma^a(\gamma) \cap \sigma^b(\delta)$ for some $a, b \in \mathbb{Z}$. Hence $\sigma^{-a}(u) \in \gamma \cap G\delta$ and $\pi_n(\sigma^{-a}(u)) = \pi_n(u) = x$ so $f$ is surjective. If $u \neq u' \in \gamma \cap G\delta$ and $f(u) = f(u')$ then $u, u' \in \gamma$, $u \in \sigma^a(\delta)$ and $u' \in \sigma^b(\delta)$ for some $a, b \in \mathbb{Z}$. But then $\pi_n(u)$ is a multiple point in the intersection of $\alpha'$ and $\beta'$, a contradiction. Hence $f$ is injective and thus a bijection. We have the seen that the first part of the lemma holds. The other cases can be dealt with in a similar way, since $\pi_n$ preserves orientation.

To compute the intersection number of two admissible arcs $\alpha$ and $\beta$ in $A(n)$, we will later exhibit explicit arcs in $U$ homotopic to lifts of $\alpha$ and $\beta$ giving an upper bound for $I_{A(n)}(\alpha, \beta)$ using Lemma 3.2. The following result gives a lower bound which we will use to get equality. This lower bound is easily computable as it just involves intersection numbers in $U$.

Corollary 3.3. Let $\tilde{\alpha}, \tilde{\beta}$ be lifts of oriented arcs $\alpha, \beta$ in $A(n)$ respectively. Then

$$I_{A(n)}(\alpha, \beta) \geq \sum_{m \in \mathbb{Z}} I_U(\tilde{\alpha}, \sigma^m(\tilde{\beta})).$$

Similar results hold for $I^+_A(n)(\alpha, \beta)$ and $I^-_A(n)(\alpha, \beta)$.

Proof. Note that the sum on the right hand side is finite since $\tilde{\alpha}$ and $\tilde{\beta}$ are compact, so only finitely many of the intersection numbers are non-zero. Using Lemma 3.2
we have:

\[
I_{\delta(n)}(\alpha, \beta) = \min_{\gamma_{\sim \delta} \in G_{\sim \delta}} |\gamma \cap G_{\delta}|
\]

\[
\geq \sum_{m \in \mathbb{Z}} \min_{\gamma_{\sim \delta} \in G_{\sim \delta}} |\gamma \cap \sigma^m(\delta)|
\]

\[
\geq \sum_{m \in \mathbb{Z}} \min_{\gamma_{\sim \delta} \in G_{\sim \delta}} |\gamma \cap \delta|
\]

\[
= \sum_{m \in \mathbb{Z}} I_{\delta}(\tilde{\alpha}, \sigma^m(\beta)),
\]

as required.

We can now put the above results together to obtain a computation of intersection numbers of admissible arcs in the annulus.

**Proposition 3.4.** Let \([a, b]\) and \([c, d]\) be admissible arcs in \(\mathbb{U}\) and suppose that \(d - c \geq b - a\), i.e. the combinatorial length of \([c, d]\) is greater than or equal to the combinatorial length of \([a, b]\). Then:

(a) \[I_{\delta(n)}^+(\pi_n([a, b]), \pi_n([c, d])) = |\{m \in \mathbb{Z} : a < \sigma^m(c) < b\}|;\]

(b) \[I_{\delta(n)}^-(\pi_n([a, b]), \pi_n([c, d])) = |\{m \in \mathbb{Z} : a < \sigma^m(d) < b\}|;\]

(c) \[I_{\delta(n)}(\pi_n([a, b]), \pi_n([c, d])) = I_{\delta(n)}^+(\pi_n([a, b]), \pi_n([c, d])) + I_{\delta(n)}^-(\pi_n([a, b]), \pi_n([c, d])) .\]

**Proof.** We note that \([c, d]\) is homotopy equivalent to the oriented arc \(\delta\) consisting of two line segments, the first, \(L_1\), joining \((c, 0)\) to \((\frac{c+d}{2}, h)\) and the second, \(L_2\), joining \((\frac{c+d}{2}, h)\) to \((d, 0)\), where \(0 < h < 1\) is fixed. Note that in order to ensure the arc is smooth, we round it off at the point \((\frac{c+d}{2}, h)\); it is clear that we can do this within a ball of radius \(\varepsilon > 0\) for arbitrarily small \(\varepsilon\) so it will not affect the rest of the proof.

By translating \([a, b]\) if necessary, we may assume that \(c \leq a < c + n\). Let \(M'_1\) be the line segment joining \((a, 0)\) to the point \((\frac{1}{2}(c + d + n), h)\), i.e. to the mid-point between the top of \(\delta\) and the top of \(\sigma(\delta)\).

Let \(\delta'\) be the \(\sigma\)-translate of \(\delta\) whose end-point is \((y, 0)\) with \(y < b\) maximal. Let \(M'_2\) be the line segment joining \((b, 0)\) with \((y + (c - d + n)/2, h)\), i.e. with the mid-point between the top of \(\delta'\) and the top of \(\sigma(\delta')\). Let \(v\) be the point where \(M'_1\) and \(M'_2\) meet; then let \(M_1\) be the line joining \((a, 0)\) to \(v\) and let \(M_2\) be the line joining \(v\) to \((b, 0)\). Then \(\gamma\), the oriented arc which starts at \((a, 0)\), travels along \(M_1\) to \(v\) and then along \(M_2\) to \((b, 0)\) is homotopy equivalent to \([a, b]\) (rounded off at \(v\) as before). See Figure 5 for an example.

It is clear that \(\gamma\) crosses \(G\delta\) transversely, and hence so does \(G\gamma\). Note that \(M_1\) crosses only \(\sigma\)-translates of \(L_2\) and \(\sigma\)-translates of \(M_2\). Similarly, \(M_2\) crosses only \(\sigma\)-translates of \(L_1\) and \(\sigma\)-translates of \(M_1\). The same is true of \(\sigma\)-translates of \(M_1\) and \(M_2\). It follows that there are no multiple crossings in the intersection of \(G\gamma\) and \(G\delta\).

Let \(\alpha = \pi_n([a, b])\) and \(\beta = \pi_n([c, d])\). By Lemma 3.2 we have that \(I_{\delta(n)}^+(\alpha, \beta)\) is less than or equal to the number of positive crossings between \(\gamma\) and \(G\delta\). Using the
Figure 5. Computing the intersection numbers of $\pi_4([0,26])$ and $\pi_4([3,17])$. We have $I_{\Lambda^{(4)}}^+(\pi_4([0,26]), \pi_4([3,17])) = 3$, $I_{\Lambda^{(4)}}^+(\pi_4([0,26]), \pi_4([3,17])) = 4$ and $I_{\Lambda^{(4)}}^+(\pi_4([0,26]), \pi_4([3,17])) = 7$.

Explicit description of $\gamma$ and $\delta$ above, the latter number equals $\sum_{m \in \mathbb{Z}} I_U^+(\gamma, \sigma^m \delta)$, which equals $|\{m \in \mathbb{Z} : a < \sigma^m(c) < b\}|$. By Corollary 3.3, we have $I_{\Lambda^{(n)}}^+(\alpha, \beta) \geq \sum_{m \in \mathbb{Z}} I_U^+(\tilde{\alpha}, \sigma^m \tilde{\beta})$.

Since $\gamma \sim [a,b]$ and $\delta \sim [c,d]$, it follows that $I_{\Lambda^{(n)}}^+(\alpha, \beta) \geq \sum_{m \in \mathbb{Z}} I_U^+(\gamma, \sigma^m \delta)$.

Putting this together with the above, we have equality and that part (a) of the proposition holds.

Parts (b) and (c) of the proposition are proved in a similar way, using the same representatives $\gamma$ and $\delta$ as above. □

For an example of Proposition 3.4 in the case $n = 4$, see Figure 5.

We note that it follows from the above proof that the inequality in Corollary 3.3 is an equality, i.e. that intersection numbers between admissible arcs in $\mathcal{A}(n)$ can be computed in $\mathcal{U}$.

Corollary 3.5. Let $\tilde{\alpha}, \tilde{\beta}$ be lifts of admissible arcs $\alpha, \beta$ in $\mathcal{A}(n)$ respectively. Then $I_{\Lambda^{(n)}}^+(\alpha, \beta) = \sum_{m \in \mathbb{Z}} I_{U}^+(\tilde{\alpha}, \sigma^m(\tilde{\beta}))$.

Similar results hold for $I_{\Lambda^{(n)}}^-(\alpha, \beta)$ and $I_{\Lambda^{(n)}}^-(\alpha, \beta)$.

Remark 3.6. Let $c$ be a single oriented cycle in $\mathcal{A}(n)$ from 0 to itself, i.e. a generator of the fundamental group at 0. We note that it follows from Proposition 3.4 that, for $r, s > 0$, $I_{\Lambda^{(n)}}^+(c^r, c^s) = I_{\Lambda^{(n)}}^-(c^r, c^s) = \min(r, s) - 1$ and $I_{\Lambda^{(n)}}^+(c^r, c^s) = 2 \min(r, s) - 2$.

Recall the bijection between admissible arcs in $\mathcal{A}(n)$ and indecomposable objects in $\mathcal{T}_n$ from Lemma 2.5 taking $\pi_n([a,b])$ to $M[a,b]$ for an admissible arc $\pi_n([a,b])$. The object $M[a,b]$ has a unique composition series (ordered from the socle upwards), $S_{a+1}, S_{a+2}, \ldots, S_b$. We now have all we need in order to show that the dimensions of extension groups (of degree 1) in the tube can be interpreted as intersection numbers (of negative intersections) of the corresponding admissible arcs in the annulus, our main result.

Theorem 3.7. Let $\pi_n([a,b])$ and $\pi_n([c,d])$ be admissible arcs in $\mathcal{A}(n)$. Then
\[ \dim \text{Ext}^1_{\mathcal{T}_\tau}(M[a, b], M[c, d]) = I_{\mathcal{A}_{(n)}}(\pi_n([a, b]), \pi_n([c, d])); \]
\[ \dim \text{Ext}^1_{\mathcal{C}_\tau}(M[c, d], M[a, b]) = I^+_n(\pi_n([a, b]), \pi_n([c, d])); \]
\[ \dim \text{Ext}^1_{\mathcal{C}_\tau}(M[a, b], M[c, d]) = I_{\mathcal{A}_{(n)}}(\pi_n([a, b]), \pi_n([c, d])); \]

Proof. We first prove (a). We have
\[ \text{Ext}^1_{\mathcal{T}_\tau}(M[a, b], M[c, d]) \cong D \text{Hom}_{\mathcal{T}_\tau}(M[c, d], \tau M[a, b]) \]
Assume first that the combinatorial length of \([c, d]\) is greater than or equal to that of \([a, b]\). Then a homomorphism in \(\text{Hom}_{\mathcal{T}_\tau}(M[c, d], M[a - 1, b - 1])\) must map the top, \(S_{\mathcal{A}_{(n)}}\), of \([c, d]\) onto a composition factor \(S_{\mathcal{A}_{(n)}}\) of \([a - 1, b - 1]\) and is uniquely determined up to a scalar by this choice. Therefore, the dimension of the last space above is the number of times \(\overline{d - 1}\) appears in the sequence \([\pi, \pi + 1, \ldots, \pi - 2]\), i.e. the number of times \(\pi\) appears in the sequence \(n + 1, n + 2, \ldots, n - 1\). This is equal to \(I_{\mathcal{A}_{(n)}}(\pi_n([a, b]), \pi_n([c, d]))\) by Proposition 3.3.

If the combinatorial length of \([c, d]\) is less than or equal to that of \([a, b]\), a homomorphism in \(\text{Hom}_{\mathcal{T}_\tau}(M[c, d], M[a - 1, b - 1])\) must map a composition factor \(S_{\mathcal{A}_{(n)}}\) of \([c, d]\) onto the socle \(S_{\mathcal{A}_{(n)}}\) of \([a - 1, b - 1]\), and is uniquely determined up to a scalar by this choice. Hence the dimension of this Hom-space is the number of times \(\pi\) appears in the sequence \([c + 1, c + 2, \ldots, d - 1]\), i.e. \(I^+_n(\pi_n([c, d]), \pi_n([a, b]))\) by Proposition 3.3. This is the same as \(I_{\mathcal{A}_{(n)}}(\pi_n([a, b]), \pi_n([c, d]))\) as required. Part (a) is proved.

The proof of part (b) is similar. Part (c) follows from the fact that:
\[ \text{Ext}^1_{\mathcal{C}_\tau}(M[a, b], M[c, d]) \cong \text{Ext}^1_{\mathcal{T}_\tau}(M[a, b], M[c, d]) \oplus D \text{Ext}^1_{\mathcal{T}_\tau}(M[c, d], M[a, b]), \]
(proved as in [BMV, Prop. 1.7]) where \(D = \text{Hom}_{\mathcal{A}_{(n)}}(-, k)\) is the linear duality, and Proposition 3.4 (c).

Remark 3.8. In the light of Theorem 3.7, Proposition 3.4 (c) can be regarded as a geometric version of equation \((\mathbb{Z})\) for the cluster tube, i.e. passing to the cluster category (of the tube) corresponds to passing from oriented to unoriented arcs in the annulus.

Using the results [LYZ 5.1,5.3], (in the case of an annulus \(X\) with a single marked point on the inner boundary and \(n\) marked points on the outer boundary), we get that, in the situation of Theorem 3.7, \(I_{\mathcal{A}_{(n)}}(\pi_n([a, b]), \pi_n([c, d])) \neq 0\) implies that \(\text{Ext}^1_{\mathcal{C}_\tau}(M[a, b], M[c, d]) \neq 0\) and \(\text{Ext}^1_{\mathcal{C}_\tau}(M[c, d], M[a, b]) \neq 0\) (with equivalence in the case where \(\pi_n([a, b]) = \pi_n([c, d])\)). We use the fact (see [BMV 5.2]) that the cluster tube is a thick subcategory of the cluster category associated to \(X\). Thus Theorem 3.7 (c) can be regarded as a strengthening of this result in the case of a tube. See also [W 4.23] for an alternative proof of parts (a) and (b).

4. Doubled arc model

In [BMV 3.3], the additive subcategory of a tube of rank \(n\) generated by the rigid indecomposable objects is modelled by pairs of unoriented arcs in a regular \(2n\)-sided polygon. We now indicate how this model is related to the model presented here.

For an integer \(n \geq 2\), consider the annulus \(\mathcal{A}_{(n)}\) as being embedded in the complex plane with its centre at the origin with marked points (on the outer boundary) at the \(n\)th roots of unity. Then the map \(\psi: z \mapsto \overline{z}\) (followed by a scaling to map the new inner boundary onto the same circle it was to start with) induces a map
from $A_{2n}$ to $A(n)$, mapping the marked points of $A_{2n}$ onto those of $A(n)$. We see that $\psi$ maps $i, i + n$ onto $i$ for each $i$. The preimage under $\psi$ of $\pi_n([a, b])$ is $\pi_{2n}([a, b]) \cup \pi_{2n}([a + n, b + n])$, up to homotopy equivalence. This pair of oriented arcs corresponds to a pair of oriented arcs in a regular 2n-gon as in [BMV] §3 (if the orientation is dropped), except that here the diameters are represented as a pair.

**Proposition 4.1.** Let $[a, b], [c, d]$ be admissible arcs in $U$. Then

\[(a)\quad I_{A_{2n}}^+(\psi^{-1}(\pi_n([a, b]), \psi^{-1}(\pi_n([c, d]))) = 2I_{A(n)}^+(\pi_n([a, b]), \pi_n([c, d])).\]

\[(b)\quad I_{A_{2n}}^-(\psi^{-1}(\pi_n([a, b]), \psi^{-1}(\pi_n([c, d]))) = 2I_{A(n)}^-(\pi_n([a, b]), \pi_n([c, d])).\]

\[(c)\quad I_{A_{2n}}(\psi^{-1}(\pi_n([a, b]), \psi^{-1}(\pi_n([c, d]))) = 2I_{A(n)}(\pi_n([a, b]), \pi_n([c, d])).\]

**Proof.** We note that for oriented arcs $\gamma, \gamma’, \delta$ in $U$ which do not self-intersect, it is easy to see that

$I_{U}^{+}(\gamma \cup \gamma’, \delta) = I_{U}^{+}(\gamma, \delta) + I_{U}^{+}(\gamma’, \delta)$.

It follows from Corollary 3.5 that $I_{A(n)}^+$ has the same additivity property. Let $[a, b], [c, d]$ be admissible arcs in $U$. Then:

$I_{A_{2n}}^+(\psi^{-1}(\pi_n([a, b]), \psi^{-1}(\pi_n([c, d]))) = I_{A_{2n}}^+(\pi_{2n}([a, b]), \pi_{2n}([c, d]))$

$+ I_{A_{2n}}^-(\pi_{2n}([a, b]), \pi_{2n}([c + n, d + n]))$

$+ I_{A_{2n}}^+(\pi_{2n}([a + n, b + n]), \pi_{2n}([c, d]))$

$+ I_{A_{2n}}^+(\pi_{2n}([a + n, b + n]), \pi_{2n}([c + n, d + n])).$

Hence

$I_{A_{2n}}^+(\psi^{-1}(\pi_n([a, b]), \psi^{-1}(\pi_n([c, d]))) = |\{m \in \mathbb{Z} : a < \sigma^{2m}(c) < b\}|

+ |\{m \in \mathbb{Z} : a < \sigma^{2m+1}(c) < b\}|

+ |\{m \in \mathbb{Z} : \sigma(a) < \sigma^{2m}(c) < \sigma(b)\}|

+ |\{m \in \mathbb{Z} : \sigma(a) < \sigma^{2m+1}(c) < \sigma(b)\}|,

so

$I_{A_{2n}}^+(\psi^{-1}(\pi_n([a, b]), \psi^{-1}(\pi_n([c, d]))) = 2I_{A(n)}^+(\pi_n([a, b]), \pi_n([c, d])), \]

as required for (a). Part (b) is proved similarly, and part (c) follows. \hfill \square

**Corollary 4.2.** Let $[a, b], [c, d]$ be admissible arcs in $U$. Then

$2\dim \text{Ext}^1_{C_{T_n}}(M[a, b], M[c, d]) = I_{A_{2n}}(\psi^{-1}(\pi_n([a, b]), \psi^{-1}(\pi_n([c, d]))).$

**Proof.** This follows from Proposition (4.1)(c) and Theorem 3.7. \hfill \square

We thus recover [BMV] 3.2 in the case where $\pi_n([a, b])$ and $\pi_n([c, d])$ are non self-intersecting, i.e. the corresponding indecomposable objects in $T_n$ are rigid. Note that the doubled diameters that appear here should have been used in [BMV] also in order for [BMV] 3.2 to hold.
5. AR-quivers of type $\mathbb{Z}A_\infty$

By [RvdR III.1.1], the category $\mathcal{T}_\infty$ of finite dimensional representations of the quiver $A_\infty$:

$$
\cdots \longrightarrow -3 \longrightarrow -2 \longrightarrow -1 \longrightarrow 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 3 \cdots
$$

is a connected Hom-finite noetherian hereditary abelian category with almost split sequences whose AR-quiver is of type $\mathbb{Z}A_\infty$. Hence one can form the cluster category $\mathcal{C}_{\mathcal{T}_\infty}$ as the quotient $D^b(\mathcal{T}_\infty)/(\tau^{-1}[1])$; see [KR1 2.1]. Since $\mathcal{T}_\infty$ is skeletally small, it follows from [K1 9.9] that it is triangulated and thus also 2-Calabi-Yau. As pointed out to us by P. Jorgensen, it follows from [KR 2 Thm. 2.1] that this category coincides with the category $D$ considered by Holm and Jorgensen in [HJ] (referred to there as a cluster category of type $A_\infty$).

The indecomposable objects in $\mathcal{T}_\infty$ (and thus, also, in $\mathcal{C}_{\mathcal{T}_\infty}$) are, up to isomorphism, of the form $X[a, b]$ where $[a, b]$ is an admissible arc in $U$. The representation $X[a, b]$ has a unique composition series (ordered from the socle upwards) given by $S_{a+1}, S_{a+2}, \ldots, S_b - 1$. There is an irreducible map in $\mathcal{T}_\infty$ from $X[a, b]$ to $X[c, d]$, for admissible arcs $[a, b]$ and $[c, d]$ in $U$, if and only if $c = a$ and $d = b + 1$ or $c = a + 1$ and $d = b$ (compare with Subsection 2.3). The AR-translate takes $X[a, b]$ to $X[a - 1, b - 1]$. The irreducible maps and AR-translate have the same description in $\mathcal{T}_\infty$ and in $\mathcal{C}_{\mathcal{T}_\infty}$; see [BMRRT 1.3].

Theorem 5.1. Let $[a, b]$ and $[c, d]$ be admissible arcs in $U$. Then

(a) $\dim \operatorname{Ext}^1_{\mathcal{T}_\infty}(X[a, b], X[c, d]) = I_U([a, b], [c, d])$;

(b) $\dim \operatorname{Ext}^1_{\mathcal{T}_\infty}(X[c, d], X[a, b]) = I_U^+[([a, b], [c, d])$;

(c) $\dim \operatorname{Ext}^1_{\mathcal{C}_{\mathcal{T}_\infty}}(X[a, b], X[c, d]) = I_U([a, b], [c, d])$.

Proof. This result is proved in the same way as Theorem 3.7 (except that the reduction modulo $n$ is omitted). Note that the same proof of formula (1) holds in this context. □

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