THE EXISTENCE AND NONEXISTENCE RESULTS OF GROUND STATE NODAL SOLUTIONS FOR A KIRCHHOFF TYPE PROBLEM

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Abstract. In this paper, we investigate the existence and nonexistence of ground state nodal solutions to a class of Kirchhoff type problems

\[-(a + b \int_{\Omega} |\nabla u|^2 \, dx) \Delta u = \lambda u + |u|^2 u, \quad u \in H^1_0(\Omega),\]

where \(a, b > 0\), \(\lambda < a \lambda_1\), \(\lambda_1\) is the principal eigenvalue of \((-\Delta, H^1_0(\Omega))\). With the help of the Nehari manifold, we obtain that there is \(\Lambda > 0\) such that the Kirchhoff type problem possesses at least one ground state nodal solution \(u_b\) for all \(0 < b < \Lambda\) and \(\lambda < a \lambda_1\) and prove that its energy is strictly larger than twice that of ground state solutions. Moreover, we give a convergence property of \(u_b\) as \(b \downarrow 0\). Besides, we firstly establish the nonexistence result of nodal solutions for all \(b \geq \Lambda\). This paper can be regarded as the extension and complementary work of W. Shuai (2015)[21], X.H. Tang and B.T. Cheng (2016)[22].

1. Introduction and main results. In this paper, we are concerned with the existence and nonexistence of ground state nodal solutions of the following Kirchhoff type problem:

\[
\begin{aligned}
-\left(a + b \int_{\Omega} |\nabla u|^2 \, dx \right) \Delta u &= \lambda u + |u|^2 u, \quad x \in \Omega, \\
u &= 0, \quad x \in \partial \Omega,
\end{aligned}
\]

where \(\Omega\) is a smooth bounded domain in \(\mathbb{R}^N\), \(N = 1, 2, 3\), \(a, b\) are positive constants, \(\lambda < a \lambda_1\), \(\lambda_1\) is the principal eigenvalue of \((-\Delta, H^1_0(\Omega))\).

Problem (1) is called nonlocal because of the presence of the term

\[-(a + b \int_{\Omega} |\nabla u|^2 \, dx),\]
which implies that the equation in (1) is no longer a point-wise identity. This phenomenon provokes some mathematical difficulties, which make the study of such a class of problems particularly interesting. Problem (1) is related to the stationary analogue of the equation

\[
\begin{cases}
  u_{tt} - \left( a + b \int_{\Omega} |\nabla u|^2 \, dx \right) \Delta u = h(x, u), & x \in \Omega, \\
  u = 0, & x \in \partial \Omega.
\end{cases}
\]

proposed by Kirchhoff [12] in 1883 as a generalization of the well-known D’Alembert wave equation for free vibrations of elastic strings. We have to point out that nonlocal problems also appear in other fields, such as biological systems, please refer to [1, 5]. However, problem (2) received great attention only after Lions [15] proposed an abstract functional analysis framework for the problem.

When \( b = 0 \) in problem (1), it reduces to the classic semilinear elliptic problem. Bartsch, Weth and Willem [4] have obtained a ground state nodal solution. After that many authors are devoted to the investigations for a variety of elliptic equations on bounded domain or whole space. Remarkably, (1) is a nonlocal problem which causes that the energy functional has totally different properties from the case \( b = 0 \), which makes the study of problem (1) particularly interesting.

Kirchhoff type problems have been paid much attention to various authors, especially on the existence of positive solutions, multiple solutions, ground state solutions, semiclassical states and the concentration behavior of positive solutions, see for example, [10, 11, 13, 14, 16, 20, 23, 26] and the references therein. However, regarding the existence of nodal solutions for the following Kirchhoff type problem

\[
\begin{cases}
  - \left( a + b \int_{\Omega} |\nabla u|^2 \, dx \right) \Delta u = f(u), & x \in \Omega, \\
  u = 0, & x \in \partial \Omega,
\end{cases}
\]

to the best of our knowledge, there are a few results in the context, such as [8, 9, 17, 18, 19, 21, 22, 25, 28]. For the case that \( f \) satisfies asymptotically 3-linear growth condition, Zhang and Perera [28], Mao and Luan [18] studied the existence of one nodal solution via invariant sets of descent flow. For the case that the nonlinearity \( f \) satisfies super-3-linear growth condition, by constraint variational methods and the quantitative deformation lemma, Figueiredo and Nascimento [9] studied the existence of ground state nodal solutions for problem (3), where \( f \) satisfies the (AR)-condition: \( 0 < \theta \int_{0}^{s} f(t) \, dt \leq f(s)s \) for some \( \theta \in (4, 6) \), \( \forall |s| > 0 \). After that, Shuai [21] studied the existence and asymptotic behavior of ground state nodal solutions for problem (3), where \( f \in C^1(\mathbb{R}, \mathbb{R}) \) satisfies the following conditions:

1. \((f_1)\) \( f(s) = o(|s|) \) as \( s \to 0 \);
2. \((f_2)\) \( \lim_{s \to \pm \infty} \frac{f(s)}{|s|^p} = 0 \) for some constant \( 4 < p < 2^* \), where \( 2^* = +\infty \) for \( N = 1, 2 \) and \( 2^* = 6 \) for \( N = 3 \);
3. \((f_3)\) \( \lim_{s \to \pm \infty} \frac{F(s)}{|s|^{2^*}} = +\infty \), where \( F(s) = \int_{0}^{s} f(t) \, dt \);
4. \((f_4)\) \( \frac{f(s)}{|s|^{2^*}} \) is an increasing function of \( s \in \mathbb{R} \setminus \{0\} \).

Recently, Tang and Cheng [22] improved and generalized some results of Shuai [21] by replacing the monotonicity condition \((f_4)\) with the weaker condition
th there exists a $\theta_0 \in (0, 1)$ such that for all $t > 0$ and $\tau \in \mathbb{R} \setminus \{0\}$,

$$\left[ \frac{f(\tau)}{\tau^3} - \frac{f(t\tau)}{(t\tau)^3} \right] \sign(1 - t) + \frac{\alpha \theta_0 \lambda_1 |1 - t^2|}{(t\tau)^2} \geq 0.$$ 

We must point out that Shuai [21], Tang and Cheng [22] studied the existence of ground state nodal solutions to problem (3) when $f$ satisfies super-3-linear growth condition at infinity and superlinear growth at zero. So, a natural question is whether these conditions can be relaxed to obtain the same results. Motivated by the previously mentioned works, in the present paper, we shall consider the case $f$ satisfies super-3-linear growth condition at infinity and linear growth at zero, in other words, we will investigate the existence and nonexistence of ground state nodal solutions to problem (1) and give a more refined analysis about the existence and nonexistence of ground state nodal solutions to problem (1).

When dealing with problem (1), we delicately analyze the behaviors of the term $b \left( \int_\Omega |\nabla u|^2 dx \right) \Delta u$ and the term $|u|^2 u$, and find that both $b \left( \int_\Omega |\nabla u|^2 dx \right)^2$ and $\int_\Omega u^4 dx$ are 4-order. This observation indicates that problem (1) does not always have nodal solutions or even solutions for all $b > 0$. Hence, we divide the issue into two circumstances and give the existence and nonexistence results of ground state nodal solutions to problem (1). On the other hand, this observation also indicates that the methods used in above papers cannot be used here directly. Indeed, we give a more refined analysis about the existence and nonexistence of ground state nodal solutions to problem (1).

Next, we give some notations. Throughout this paper, let $H^1_0(\Omega)$ be the usual Sobolev space equipped with the norm $\|u\| = \left( \int_\Omega |\nabla u|^2 dx \right)^{\frac{1}{2}}$, $|\cdot|_s$ be the usual Lebesgue space $L^s(\Omega)$ norm. Let $S^2$ be the principal eigenvalue of the following nonlinear eigenvalue problem

$$\begin{align*}
\left\{ 
& -||u||^2 \Delta u = \mu |u|^2 u, \quad x \in \Omega, \\
& u = 0, \quad x \in \partial\Omega.
\right.
\end{align*}$$

It is well known that $S$ is obtained by an associated eigenfunction $e_1$ which is strictly positive in $\Omega$ by [28]. In particular, $S$ is defined as

$$S = \inf_{u \in H^1_0(\Omega) \setminus \{0\}} \frac{\|u\|^2}{\|u\|_{4}^4}, \quad |u| \leq S^{-\frac{1}{2}} \|u\|. \quad (4)$$

Define an energy functional $J_b$ on the space $H^1_0(\Omega)$ by

$$J_b(u) = \frac{a}{2} \|u\|^2 + \frac{b}{4} \|u\|^4 - \frac{\lambda}{2} \int_\Omega u^2 dx - \frac{1}{4} \int_\Omega u^4 dx, \quad \forall u \in H^1_0(\Omega).$$

Then $J_b$ is well defined on $H^1_0(\Omega)$ and is of $C^1$, and for each $u, \ v \in H^1_0(\Omega)$, we have

$$\langle J'_b(u), v \rangle = \left( a + b \int_\Omega |\nabla u|^2 dx \right) \int_\Omega \nabla u \cdot \nabla v - \lambda \int_\Omega uv dx - \int_\Omega |u|^2 uv dx.$$

where $\langle \cdot, \cdot \rangle$ denotes the usual duality. It is standard to verify that the weak solutions of problem (1) correspond to the critical points of the functional $J_b$. Furthermore, if $u \in H^1_0(\Omega)$ is a solution of problem (1) and $u^\pm \neq 0$, then $u$ is a nodal solution of problem (1), where

$$u^+(x) := \max\{u(x), 0\} \quad \text{and} \quad u^-(x) := \min\{u(x), 0\}.$$
Here a solution is called a ground state (or least energy) nodal one if it possesses the least energy among all nodal solutions. By a simple calculation, we can obtain that

\[ J_b(u) = J_b(u^+) + J_b(u^-) + \frac{b}{2} \| u^+ \|^2 \| u^- \|^2, \quad (5) \]

\[ \langle J'_b(u), u^+ \rangle = \langle J'_b(u), u^- \rangle + b \| u^+ \|^2 \| u^- \|^2, \quad (6) \]

\[ \langle J'_b(u), u^- \rangle = \langle J'_b(u^-), u^- \rangle + b \| u^+ \|^2 \| u^- \|^2, \quad (7) \]

When \( b = 0 \), problem (1) does not depend on the nonlocal term \( \left( \int_{\Omega} |\nabla u|^2 dx \right) \bigtriangleup u \) any more, i.e., it becomes

\[
\begin{cases}
-a\Delta u = |u|^2u, & x \in \Omega, \\
u = 0, & x \in \partial\Omega,
\end{cases}
\]

which corresponds to the energy functional \( J_0 : H^1_0(\Omega) \to \mathbb{R} \) by

\[ J_0(u) = \frac{a}{2} \| u \|^2 - \frac{\lambda}{2} \int_{\Omega} u^2 dx - \frac{1}{4} \int_{\Omega} u^4 dx, \quad \forall u \in H^1_0(\Omega). \]

Similarly, \( J_0 \) is well defined on \( H^1_0(\Omega) \) and is of \( C^1 \), and

\[ \langle J'_0(u), v \rangle = a \int_{\Omega} \nabla u \cdot \nabla v - \lambda \int_{\Omega} uv dx - \int_{\Omega} |u|^2uv dx, \quad \forall u, v \in H^1_0(\Omega). \]

From (5), (6), (7), it is easy to see that there are some essential differences in studying the nodal solutions for problem (1) between \( b > 0 \) and \( b = 0 \) because the so-called nonlocal term \( b \left( \int_{\Omega} |\nabla u|^2 dx \right) \bigtriangleup u \). Therefore, the methods of seeking nodal solutions for problems as (8) seem to be not applicable to problem (1). Inspired by the above mentioned works, we will consider the following minimization problems:

\[ m_0 := \inf \{ J_0(u) : u \in M_0 \}, \quad m_b := \inf \{ J_b(u) : u \in M_b \}, \]

where

\[ M_b = \{ u \in H^1_0(\Omega) : u^\perp \neq 0, \langle J'_b(u), u^\perp \rangle = \langle J'_b(u), u^- \rangle = 0 \}, \]

\[ M_0 = \{ u \in H^1_0(\Omega) : u^\perp \neq 0, \langle J'_0(u), u^\perp \rangle = \langle J'_0(u), u^- \rangle = 0 \}, \]

whose minimizers are corresponding to the nodal solutions for problems (1) and (8), respectively.

Another aim of the paper is to show the energy of any nodal solutions of problem (1) is strictly larger than twice that of the ground state solutions of problem (1), and establish the convergence of the ground state nodal solution as \( b \searrow 0 \). As usual, we seek the ground state solutions of problems (1) and (8) as minimizers of corresponding energy functionals \( J_b \) and \( J_0 \) on the following Nehari manifolds:

\[ N_b = \{ u \in H^1_0(\Omega) \setminus \{0\} : \langle J'_b(u), u \rangle = 0 \}, \]

\[ N_0 = \{ u \in H^1_0(\Omega) \setminus \{0\} : \langle J'_0(u), u \rangle = 0 \}, \]

respectively. Similarly, let

\[ c_0 := \inf \{ J_0(u) : u \in N_0 \}, \quad c_b := \inf \{ J_b(u) : u \in N_b \}. \]

Our main results can be stated as follows.
Theorem 1.1. There exists $\Lambda > 0$ such that for each $\lambda < a\lambda_1$,

(i) if $0 < b < \Lambda$, problem (1) has at least one ground state nodal solution which has precisely two nodal domains, and $m_b > 2\varepsilon$;

(ii) if $b \geq \Lambda$, problem (1) does not admit any nodal solution, and $\Lambda < \frac{1}{2\pi^2}$.

Theorem 1.2. For each $\lambda < a\lambda_1$, for any sequence $\{b_n\}$ with $b_n \not< 0$ as $n \to \infty$, there exists a subsequence, still denoted by $\{b_n\}$, such that $u_{b_n}$ convergent to $u_0$ strongly in $H^1_0(\Omega)$, where $u_0$ is a ground state nodal solution of problem (8) which has precisely two nodal domains.

Remark 1. Our results make good explanation for the existence and nonexistence of ground state nodal solutions to problem (1) if $N = 1, 2, 3$. However, if $N = 4$, problem (1) involves the critical nonlinearity $|u|^2u$ because $2^* = 4$. As far as we know, Daisuke Naimen [20] proved that if $0 < \lambda < a\lambda_1$, problem (1) has a solution if and only if $0 < b < \frac{1}{2\pi^2}$. On the other hand, according to the proof of Theorem 1.1 (ii), we can obtain that if $N = 4$, problem (1) does not admit any nodal solution for all $\lambda < a\lambda_1$ and $b \geq \frac{1}{2\pi^2}$. So, we propose an open question whether problem (1) has a nodal solution if $0 < b < \frac{1}{2\pi^2}$, $N = 4$.

Remark 2. Comparing with [21] and [22], we investigate the existence of ground state nodal solutions to problem (1) and give a convergence property of ground state nodal solutions as $b \not< 0$ when the nonlinearity satisfies 3-linear growth condition at infinity and linear growth at zero. However, Shuai [21], Tang and Cheng [22] considered the case the nonlinearity satisfies super-3-linear growth condition at infinity and superlinear growth at zero. Since both $\left(\int_\Omega |\nabla u|^2dx\right)^2$ and $\int_\Omega u^4dx$ are 4-order, we introduce some new ideas to prove that $M_b \not= \emptyset$. Moreover, we firstly give the nonexistence result of nodal solutions to problem (1). Consequently, our results can be regarded as the extension and supplementary work of [21] and [22].

We organize this paper as follows. In Section 2 we present some notations and prove some useful preliminary lemmas which pave the way for getting one ground state nodal solution. Then Section 3 is devoted to proving Theorem 1.1 and Theorem 1.2, and obtaining the existence and nonexistence results of nodal solutions.

2. Some preliminary lemmas. In this section, we give some preliminary lemmas which are crucial for proving our results. Firstly, we will check that $M_b \not= \emptyset$ in $H^1_0(\Omega)$ if there exists $u$ with some conditions.

Lemma 2.1. If $\lambda < a\lambda_1$, $u \in H^1_0(\Omega)$ satisfies $u^\pm \not= 0$ and

\[
\begin{cases}
 b||u^+|^4 + b||u^+||^2||u^-||^2 < \int_\Omega |u^+|^4dx, \\
 b||u^-|^4 + b||u^+||^2||u^-||^2 < \int_\Omega |u^-|^4dx,
\end{cases}
\]

(9)

there is a unique pair $(s_u, t_u)$ of positive numbers such that $s_u u^+ + t_u u^- \in M_b$ and $J_b(s_u u^+ + t_u u^-) = \max_{s,t \geq 0} J_b(su^+ + tu^-)$.

Proof. Let $\lambda < a\lambda_1$, $u \in H^1_0(\Omega)$ with $u^\pm \not= 0$ and (9), then $su^+ + tu^- \in M_b$ if and only if

\[
\begin{cases}
 as^2||u^+|^2 + bs^4||u^+|^4 + bs^2t^2||u^+||^2||u^-||^2 = \lambda s^2 \int_\Omega |u^+|^2dx + s^4 \int_\Omega |u^+|^4dx, \\
 at^2||u^-|^2 + bt^4||u^-|^4 + bs^2t^2||u^+||^2||u^-||^2 = \lambda t^2 \int_\Omega |u^-|^2dx + t^4 \int_\Omega |u^-|^4dx.
\end{cases}
\]
Hence, we only need to show that there is only one positive solution \((S, T)\) to the following system

\[
\begin{align*}
& S \left( \int_{\Omega} |u^+|^4 \, dx - b|u^+|^4 \right) - bT|u^+|^2|u^-|^2 = a\|u^+\|^2 - \lambda \int_{\Omega} |u^+|^2 \, dx, \\
& T \left( \int_{\Omega} |u^-|^4 \, dx - b|u^-|^4 \right) - bS|u^+|^2|u^-|^2 = a\|u^-\|^2 - \lambda \int_{\Omega} |u^-|^2 \, dx.
\end{align*}
\] (10)

It is easy to see from (9) that

\[
\begin{align*}
& b|u^+|^2|u^-|^2 < \int_{\Omega} |u^+|^4 \, dx - b|u^+|^4, \\
& b|u^+|^2|u^-|^2 < \int_{\Omega} |u^-|^4 \, dx - b|u^-|^4.
\end{align*}
\] (11)

Consequently,

\[
D = \left| \int_{\Omega} |u^+|^4 \, dx - b|u^+|^4 - b|u^+|^2|u^-|^2 \int_{\Omega} |u^-|^4 \, dx - b|u^-|^4 \right| > 0.
\]

Together with \( \lambda < a\lambda_1 \), we have \( a\|u^\pm\|^2 > \lambda \int_{\Omega} |u^\pm|^2 \, dx \) and

\[
D_S = \left| a\|u^+\|^2 - \lambda \int_{\Omega} |u^+|^2 \, dx - b|u^+|^2|u^-|^2 \int_{\Omega} |u^-|^4 \, dx - b|u^-|^4 \right| > 0,
\]

\[
D_T = \left| \int_{\Omega} |u^+|^4 \, dx - b|u^+|^4 - a\|u^+\|^2 - \lambda \int_{\Omega} |u^+|^2 \, dx \int_{\Omega} |u^-|^2 \, dx + b|u^+|^2|u^-|^2 \int_{\Omega} |u^-|^4 \, dx - b|u^-|^4 \right| > 0.
\]

Let \( S = \frac{D_S}{D_T} \) and \( T = \frac{D_T}{D_S} \), then \((S, T) \in (0, +\infty) \times (0, +\infty)\) is the unique solution to system (10). Choosing \( s_u = \sqrt{S} \) and \( t_u = \sqrt{T} \), we can obtain that \((s_u, t_u)\) is the unique pair of positive numbers such that \( s_u u^+ + t_u u^- \in M_0 \).

Furthermore, since

\[
J_b(su^+ + tu^-) = \frac{as^2}{2} |u^+|^2 + \frac{bs^4}{4} |u^+|^4 - \frac{\lambda s^2}{2} \int_{\Omega} |u^+|^2 \, dx - \frac{s^4}{4} \int_{\Omega} |u^+|^4 \, dx
\]

\[
+ \frac{at^2}{2} |u^-|^2 + \frac{bt^4}{4} |u^-|^4 - \frac{\lambda t^2}{2} \int_{\Omega} |u^-|^2 \, dx - \frac{t^4}{4} \int_{\Omega} |u^-|^4 \, dx
\]

\[
+ \frac{bs^4}{2} |u^+|^2 |u^-|^2,
\]

it is not difficult to verify that

\[
\frac{\partial^2 J_b(su^+ + tu^-)}{\partial s^2} = \left( a\|u^\pm\|^2 - \lambda \int_{\Omega} |u^\pm|^2 \, dx \right) + 3s^2 \left( b\|u^\pm\|^4 - \int_{\Omega} |u^\pm|^4 \, dx \right)
\]

\[
+ bt^2\|u^\pm\|^2 |u^\pm|^2,
\]

\[
\frac{\partial^2 J_b(su^+ + tu^-)}{\partial t^2} = \left( a\|u^\pm\|^2 - \lambda \int_{\Omega} |u^\pm|^2 \, dx \right) + 3t^2 \left( b\|u^\pm\|^4 - \int_{\Omega} |u^\pm|^4 \, dx \right)
\]

\[
+ bs^2\|u^\pm\|^2 |u^\pm|^2.
\]
From the fact that \((x_u^2, t_u^2)\) is the solution of system (10), we have
\[
\frac{\partial^2 J_b(su^+ + tu^-)}{\partial s^2} \bigg|_{(s_u, t_u)} = -2s_u^2 \left( \int_{\Omega} |u^+|^4 dx - b||u^+||^4 \right) < 0, 
\]
(12)
\[
\frac{\partial^2 J_b(su^+ + tu^-)}{\partial t^2} \bigg|_{(s_u, t_u)} = -2t_u^2 \left( \int_{\Omega} |u^-|^4 dx - b||u^-||^4 \right) < 0, 
\]
(13)
\[
\frac{\partial^2 J_b(su^+ + tu^-)}{\partial s \partial t} \bigg|_{(s_u, t_u)} = 2bs_ut_u||u^+||^2||u^-||^2 > 0. 
\]
(14)

We consider the Hessian matrix of \(J_b(su^+ + tu^-)\), i.e.
\[
H(s_u, t_u) = \left( \begin{array}{cc}
\frac{\partial^2 J_b(su^+ + tu^-)}{\partial s^2} & \frac{\partial^2 J_b(su^+ + tu^-)}{\partial s \partial t} \\
\frac{\partial^2 J_b(su^+ + tu^-)}{\partial t \partial s} & \frac{\partial^2 J_b(su^+ + tu^-)}{\partial t^2}
\end{array} \right)_{(s_u, t_u)}.
\]

Combining with (11), one can obtain that
\[
\det H(s_u, t_u) = 4s_u^2t_u^2 \left( \int_{\Omega} |u^+|^4 dx - b||u^+||^4 \right) \left( \int_{\Omega} |u^-|^4 dx - b||u^-||^4 \right)
- 4b^2s_u^2t_u^2||u^+||^4||u^-||^4
= 4s_u^2t_u^2 \left( \int_{\Omega} |u^+|^4 dx - b||u^+||^4 \right) \left( \int_{\Omega} |u^-|^4 dx - b||u^-||^4 \right)
- (b||u^+||^2||u^-||^2)^2 \geq 0.
\]
(15)

Therefore, it follows from (12), (13), (14), (15) that
\[
J_b(s_uu^+ + t_uu^-) = \max_{s,t \geq 0} J_b(su^+ + tu^-),
\]
and we complete the proof.

**Lemma 2.2.** Assume that \(\lambda < a\lambda_1\) and \(u \in M_b\), then (9) holds.

**Proof.** Let \(u \in M_b\), we have from the definition of \(M_b\) that \(u^\pm \neq 0\) and
\[
\begin{aligned}
& a||u^+||^2 + b||u^+||^4 + b||u^+||^2||u^-||^2 = \lambda \int_{\Omega} |u^+|^2 dx + \int_{\Omega} |u^+|^4 dx, \\
& a||u^-||^2 + b||u^-||^4 + b||u^+||^2||u^-||^2 = \lambda \int_{\Omega} |u^-|^2 dx + \int_{\Omega} |u^-|^4 dx.
\end{aligned}
\]
(16)

Since \(\lambda < a\lambda_1\) and \(\lambda_1\) is the principal eigenvalue of \((-\triangle, H^1_0(\Omega))\), we can obtain that
\[
a||u^+||^2 > \lambda \int_{\Omega} |u^+|^2 dx, \quad a||u^-||^2 > \lambda \int_{\Omega} |u^-|^2 dx,
\]
which implies from (16) that
\[
\begin{aligned}
& b||u^+||^4 + b||u^+||^2||u^-||^2 < \int_{\Omega} |u^+|^4 dx, \\
& b||u^-||^4 + b||u^+||^2||u^-||^2 < \int_{\Omega} |u^-|^4 dx.
\end{aligned}
\]
Then we have completed the proof.
Lemma 2.3. Assume that $\lambda < a\lambda_1$, $u \in H^1_0(\Omega)$ with $u^\pm \neq 0$ and $\langle J'_\beta(u), u^\pm \rangle \leq 0$, then there is a unique pair $(s_u, t_u) \in (0, 1] \times (0, 1]$ such that

$$s_u u^+ + t_u u^- \in M_\beta.$$  

Proof. If $u \in H^1_0(\Omega)$ satisfies $u^\pm \neq 0$ and $\langle J'_\beta(u), u^\pm \rangle \leq 0$, we have

$$
\begin{align*}
\begin{cases}
    a\|u^+\|^2 + b\|u^+\|^4 + b\|u^+\|^2|u^-|^2 \leq \lambda \int_\Omega |u^+|^2 dx + \int_\Omega |u^+|^4 dx, \\
    a\|u^-\|^2 + b\|u^-\|^4 + b\|u^+\|^2|u^-|^2 \leq \lambda \int_\Omega |u^-|^2 dx + \int_\Omega |u^-|^4 dx,
\end{cases}
\end{align*}
$$

(17)

Then by Lemma 2.1, there is a unique pair $(s_u, t_u)$ of positive numbers such that

$$s_u u^+ + t_u u^- \in M_\beta.$$  

It means that $(s^2_u, t^2_u)$ is the solution of system (10). Similar to the argument of Lemma 2.1, we have from (17) that

$$D_{s^2_u} = \left( a\|u^+\|^2 - \lambda \int_\Omega |u^+|^2 dx \right) \left( \int_\Omega |u^-|^4 dx - b\|u^-\|^4 \right) + b\|u^+\|^2|u^-|^2 \left( a\|u^-\|^2 - \lambda \int_\Omega |u^-|^2 dx \right) \leq \left( \int_\Omega |u^+|^4 dx - b\|u^+\|^4 - b\|u^+\|^2|u^-|^2 \right) \left( \int_\Omega |u^-|^4 dx - b\|u^-\|^4 \right) + b\|u^+\|^2|u^-|^2 \left( \int_\Omega |u^-|^4 dx - b\|u^-\|^4 - b\|u^+\|^2|u^-|^2 \right),$$

$$= \left( \int_\Omega |u^+|^4 dx - b\|u^+\|^4 \right) \left( \int_\Omega |u^-|^4 dx - b\|u^-\|^4 \right) - b^2\|u^+\|^4|u^-|^4 = D.$$  

Therefore, $s^2_u = \frac{D_{s^2_u}}{D} \leq 1$. Similarly, $t^2_u = \frac{D_{t^2_u}}{D} \leq 1$. Then there is a unique pair $(s_u, t_u) \in (0, 1] \times (0, 1]$ such that $s_u u^+ + t_u u^- \in M_\beta$. \hfill \square

Lemma 2.4. If $\lambda < a\lambda_1$, for any $u \in H^1_0(\Omega)$ with $b\|u\|^4 < \int_\Omega |u|^4 dx$, there exists a unique $\tilde{s}_u > 0$ such that $\tilde{s}_u u \in N_\beta$. Moreover, $J_\beta(\tilde{s}_u u) > J_\beta(s u)$ for all $s \geq 0$ and $s \neq \tilde{s}_u$.

Proof. If $\lambda < a\lambda_1$, for any $u \in H^1_0(\Omega)$ with $b\|u\|^4 < \int_\Omega |u|^4 dx$, one can get that $su \in N_\beta$ if and only if

$$as^2\|u\|^2 + bs^4\|u\|^4 = \lambda s^2 \int_\Omega u^2 dx + s^4 \int_\Omega u^4 dx,$$
it is easy to see that there exists a unique \( s_u = \left( \frac{a\|u\|^2 - \lambda \int_{\Omega} u^2 dx}{J_b(u)^2} \right)^{\frac{1}{2}} \) such that \( s_u u \in N_b \). Furthermore, since

\[
\frac{\partial J_b(su)}{\partial s} = s \left( a\|u\|^2 - \lambda \int_{\Omega} u^2 dx \right) - s^2 \left( \int_{\Omega} u^4 dx - b\|u\|^4 \right),
\]

we have \( J_b(s_u u) > J_b(su) \) for all \( s \geq 0 \) and \( s \neq s_u \).

Define

\[
\Lambda = \sup\{b > 0 \mid M_b \neq \emptyset\},
\]

then the following results hold.

**Lemma 2.5.** Assume that \( \lambda < a\lambda_1 \), we have

(i) if \( 0 < b < \frac{1}{\sqrt{2}} \), \( c_b > 0 \) is attained by some \( v_b \in N_b \) and \( v_b \) is a positive critical point of \( J_b \);

(ii) if \( 0 < b < \Lambda \), \( m_b > 0 \) is attained by some \( u_b \in M_b \) and \( u_b \) is a nodal critical point of \( J_b \).

**Proof.** (i) Firstly, to show \( N_b \neq \emptyset \) for all \( 0 < b < \frac{1}{\sqrt{2}} \), we only need to show that for all \( 0 < b < \frac{1}{\sqrt{2}} \), there exists at least one \( u \in H^1_0(\Omega) \) with \( b\|u\|^4 < \int_{\Omega} u^4 dx \) by Lemma 2.4. In fact, we can pick up the extremal function \( e_1 \) for \( S \) by (4), then for each \( 0 < b < \frac{1}{\sqrt{2}} \),

\[
b\|e_1\|^4 < \frac{1}{S^2}\|e_1\|^4 = \int_{\Omega} e_1^4 dx.
\]

It follows that \( e_1 \) is the one that meets the requirements, which means that \( N_b \neq \emptyset \).

For each \( u \in N_b \), it follows from \( \lambda < a\lambda_1 \), \( 0 < b < \frac{1}{\sqrt{2}} \) and Sobolev inequality (4) that

\[
a\|u\|^2 + b\|u\|^4 = \lambda \int_{\Omega} u^2 dx + \int_{\Omega} u^4 dx \leq \frac{\lambda}{\lambda_1} \|u\|^2 + \frac{1}{S^2}\|u\|^4.
\]

Then

\[
\|u\| \geq \left( \frac{a - \frac{\lambda}{\lambda_1}}{\frac{1}{S^2} - b} \right)^{\frac{1}{2}} > 0,
\]

and

\[
J_b(u) = \frac{1}{4} \left( a\|u\|^2 - \lambda \int_{\Omega} u^2 dx \right) \geq \frac{1}{4} \left( a - \frac{\lambda}{\lambda_1} \right) \|u\|^2.
\]

Therefore,

\[
c_b = \inf_{u \in N_b} J_b(u) \geq \frac{1}{4} \left( a - \frac{\lambda}{\lambda_1} \right)^2 \frac{S^2}{1 - bS^2} > 0,
\]

and \( J_b \) is coercive and bounded below on \( N_b \) for all \( \lambda < a\lambda_1 \) and \( 0 < b < \frac{1}{\sqrt{2}} \).

Let \( \{v_n\} \subset N_b \) be a minimizing sequence for \( J_b \). Obviously, \( J_b(v_n) = J_b(\|v_n\|) \) and \( \|v_n\| \in N_b \) and therefore we can assume from the beginning that \( v_n(x) \geq 0 \) a.e. in \( \Omega \) and for all \( n \). It follows from the fact \( J_b \) is coercive on \( N_b \) that the sequence \( \{v_n\} \) is bounded in \( H^1_0(\Omega) \), so that, up to subsequences, \( v_n \rightharpoonup v_b \) in \( H^1_0(\Omega) \) and \( v_b(x) \geq 0 \). We now prove that \( v_n \to v_b \) strongly in \( H^1_0(\Omega) \). Supposing the contrary, then \( \|v_b\| < \liminf_{n \to +\infty} \|v_n\| \), we get

\[
a\|v_b\|^2 + b\|v_b\|^4 < \lambda \int_{\Omega} v_b^2 dx + \int_{\Omega} v_b^4 dx.
\]
which means that \( v_b(x) \neq 0 \) in \( \Omega \) and \( b||v_b||^4 < \int_{\Omega} v_b^4 dx \) by \( \lambda < a\lambda_1 \). By Lemma 2.4, there exists a unique \( \bar{s}_v > 0 \) such that \( \bar{s}_v v_b \in N_b \). Moreover, \( J_b(\bar{s}_v v_n) \leq J_b(v_n) \) for all \( v_n \in N_b \). Therefore, we obtain

\[
c_b \leq J_b(\bar{s}_v v_b)
\]

\[
= \frac{a}{2} \|\bar{s}_v v_b\|^2 + \frac{b}{4} \|\bar{s}_v v_b\|^4 - \lambda \frac{2}{2} \int_{\Omega} |\bar{s}_v v_b|^2 dx - \frac{4}{4} \int_{\Omega} |\bar{s}_v v_b|^4 dx
\]

\[
< \liminf_{n \to \infty} \left[ \frac{a}{2} \|\bar{s}_v v_n\|^2 + \frac{b}{4} \|\bar{s}_v v_n\|^4 - \lambda \frac{2}{2} \int_{\Omega} |\bar{s}_v v_n|^2 dx - \frac{4}{4} \int_{\Omega} |\bar{s}_v v_n|^4 dx \right]
\]

\[
= \liminf_{n \to \infty} J_b(\bar{s}_v v_n)
\]

\[
\leq \liminf_{n \to \infty} J_b(v_n) = c_b,
\]

which leads to a contradiction. Thus \( v_n \to v_b \) strongly in \( H^1_0(\Omega) \), \( v_b \in N_b \) and \( J_b(v_b) = c_b \). Similar to the argument in Brown and Zhang [7], we can conclude \( v_b \) is a critical point of \( J_b \). And by the strong maximum principle, \( u \) is strictly positive.

(ii) For each \( u \in H^1_0(\Omega) \) with \( u^\pm \neq 0 \), (9) holds for small enough \( b > 0 \), which means from Lemma 2.1 that \( M_b \neq \emptyset \). Thus

\[
\{ b > 0 \mid M_b \neq \emptyset \} \neq \emptyset \text{ and } \Lambda > 0.
\]

We first verify that \( M_b \neq \emptyset \) for all \( 0 < b < \Lambda \). Obviously, it follows from the definition of \( \Lambda \) that \( M_{\Lambda - \zeta} \neq \emptyset \) for sufficiently small \( \zeta > 0 \). Therefore, by the fact that \( \zeta \) is arbitrary, it suffices to show that \( M_b \neq \emptyset \) for all \( 0 < b < \Lambda - \zeta \). If there exists \( u \in M_{\Lambda - \zeta} \), we deduce from Lemma 2.2 that

\[
\left\{ \begin{aligned}
(\Lambda - \zeta) ||u^+||^4 + (\Lambda - \zeta)||u^-||^2 ||u^-||^2 &< \int_{\Omega} |u^+|^4 dx, \\
(\Lambda - \zeta) ||u^-||^4 + (\Lambda - \zeta)||u^+||^2 ||u^-||^2 &< \int_{\Omega} |u^-|^4 dx.
\end{aligned} \right.
\]

Thus, for all \( 0 < b < \Lambda - \zeta \), we have that

\[
\left\{ \begin{aligned}
b||u^+||^4 + b||u^+||^2 ||u^-||^2 &< \int_{\Omega} |u^+|^4 dx, \\
b||u^-||^4 + b||u^+||^2 ||u^-||^2 &< \int_{\Omega} |u^-|^4 dx.
\end{aligned} \right.
\]

It follows from Lemma 2.1 that there is a pair \( (s_b, t_b) \) of positive numbers such that \( s_b u^+ + t_b u^- \in M_b \) which means that \( M_b \neq \emptyset \). Therefore, \( M_b \neq \emptyset \) for all \( 0 < b < \Lambda \).

Secondly, if \( u \in M_b \), we have \( \langle J'_b(u), u^\pm \rangle = 0 \), that is

\[
a||u^\pm||^2 + b||u^\pm||^4 + b||u^+||^2 ||u^-||^2 = \lambda \int_{\Omega} |u^\pm|^2 dx + \int_{\Omega} |u^\pm|^4 dx.
\]

Then

\[
a||u^\pm||^2 + b||u^\pm||^4 = \lambda \int_{\Omega} |u^\pm|^2 dx + \int_{\Omega} |u^\pm|^4 dx,
\]

similarly, we can obtain that

\[
||u^\pm||^2 \geq \frac{a - \frac{\lambda}{\lambda_1}}{S^2 - b} > 0.
\]

(18)

it follows from \( u \in M_b \subset N_b \) that \( J_b(u) \geq \frac{1}{4} \left( a - \frac{\lambda}{\lambda_1} \right) ||u||^2 \), which means that \( m_b > 0 \).
Thirdly, assume that \( \{u_n\} \subset \mathcal{M}_b \) is a minimizing sequence for \( J_b \), namely such that \( J_b(u_n) \to m_b \). We have already observed that \( J_b \) is coercive on \( \mathcal{N}_b \), this implies that the sequence \( \{u_n\} \) is bounded in \( H^1_0(\Omega) \), going if necessary to a subsequence, still denoted by \( \{u_n\} \), we can assume that there exists an \( u_b \in H^1_0(\Omega) \) such that, for \( n \) sufficiently large,

\[
\begin{aligned}
&u_n^\pm \to u_b^\pm \text{ weakly in } H^1_0(\Omega), \\
u_n(x) \to u_b(x) \text{ almost everywhere on } \Omega,
&u_n^\pm \to u_b^\pm \text{ strongly in } L^s(\Omega) \text{ for } 1 \leq s < 6.
\end{aligned}
\]

Next since \( \{u_n\} \subset \mathcal{M}_b \subset \mathcal{N}_b \), we have \( (J'_b(u_n), u_n^\pm) = 0 \), that is

\[
\begin{aligned}
&\left\{ \begin{array}{l}
a||u_n^+||^2 + b||u_n^+||^4 + b||u_n^-||^2||u_n^-||^2 = \lambda \int_{\Omega} |u_n^+|^2 dx + \int_{\Omega} |u_n^+|^4 dx, \\
a||u_n^-||^2 + b||u_n^-||^4 + b||u_n^+||^2||u_n^+||^2 = \lambda \int_{\Omega} |u_n^-|^2 dx + \int_{\Omega} |u_n^-|^4 dx.
\end{array} \right\
\end{aligned}
\]

Then

\[
\left( a - \frac{\lambda}{\lambda_1} \right) \|u_n^+\|^2 + b\|u_n^+\|^4 + b\|u_n^-\|^2\|u_n^-\|^2 \leq \int_{\Omega} |u_n^+|^4 dx.
\]

Passing to the limit, we obtain from (18) and \( \lambda < a\lambda_1 \) that

\[
0 < \liminf_{n \to \infty} \left[ \left( a - \frac{\lambda}{\lambda_1} \right) \|u_n^+\|^2 + b\|u_n^+\|^4 + b\|u_n^-\|^2\|u_n^-\|^2 \right] \leq \int_{\Omega} |u_n^+|^4 dx,
\]

which implies that \( u_b^\pm \neq 0 \) and

\[
\begin{aligned}
&\left\{ \begin{array}{l}
a||u_b^+||^2 + b||u_b^+||^4 + b||u_b^-||^2||u_b^-||^2 \leq \lambda \int_{\Omega} |u_b^+|^2 dx + \int_{\Omega} |u_b^+|^4 dx, \\
a||u_b^-||^2 + b||u_b^-||^4 + b||u_b^+||^2||u_b^+||^2 \leq \lambda \int_{\Omega} |u_b^-|^2 dx + \int_{\Omega} |u_b^-|^4 dx.
\end{array} \right\
\end{aligned}
\]

Then by Lemma 2.3, there is a unique pair \((s_u, t_u) \in (0, 1) \times (0, 1)\) such that \( s_u u_b^+ + t_u u_b^- \in \mathcal{M}_b \). And thus

\[
J_b(s_u u_b^+ + t_u u_b^-) \geq m_b.
\]

It follows from \( \lambda < a\lambda_1 \) that

\[
\begin{aligned}
&J_b(s_u u_b^+ + t_u u_b^-) \\
= &J_b(s_u u_b^+ + t_u u_b^-) - \frac{1}{4} J_b(s_u u_b^+ + t_u u_b^-), s_u u_b^+ + t_u u_b^-) \\
= &\frac{1}{4} \left( a||s_u u_b^+ + t_u u_b^-||^2 - \lambda \int_{\Omega} |s_u u_b^+ + t_u u_b^-|^2 dx \right) \\
= &\frac{1}{4} \left[ s_u^2 \left( a||u_b^+||^2 - \lambda \int_{\Omega} |u_b^+|^2 dx \right) + t_u^2 \left( a||u_b^-||^2 - \lambda \int_{\Omega} |u_b^-|^2 dx \right) \right] \\
\leq &\frac{1}{4} \left( a||u_b^+||^2 - \lambda \int_{\Omega} |u_b^+|^2 dx \right) + \left( a||u_b^-||^2 - \lambda \int_{\Omega} |u_b^-|^2 dx \right) \\
= &\frac{1}{4} \left( a||u_b||^2 - \lambda \int_{\Omega} |u_b|^2 dx \right) \\
\leq &\liminf_{n \to \infty} \left[ J_b(u_n) - \frac{1}{4} J_b(u_n) \right] \\
= &m_b,
\end{aligned}
\]
Let minimizer. \[ b \text{ problem (1)}. \] Then Lemma 2.2 yields 2.4 that there exist \( s \).

It follows from Lemma 2.1 that

\[ (s,t) \in D. \]

It follows from Lemma 2.1 that

\[ (s,t) \in D. \]

Lastly, if \( J_b'(u_b) \neq 0 \), there exist \( \delta > 0 \) and \( \alpha > 0 \) such that

\[ u \in H_0^1(\Omega), \quad \| J_b'(u_b) \| \geq \alpha, \quad \| u - u_b \| \leq 3\delta. \]

Let \( D = (1 - \sigma, 1 + \sigma) \times (1 - \sigma, 1 + \sigma), \) \( 0 < \sigma < 1 \) and \( \psi(s, t) = su^+ + tu^-, (s, t) \in D. \)

It follows from Lemma 2.1 that

\[ m := \max_{\partial D} J_b(\psi) < m_b. \]

Let \( \varepsilon = \min \{ mb - m, \frac{\alpha \delta}{8} \} \) and \( S_\delta = \{ u \in H_0^1(\Omega) \mid \| u - u_b \| \leq \delta \} \), there exists a deformation \( \eta \in C([0, 1] \times H_0^1(\Omega), H_0^1(\Omega)) \) such that

\[ \begin{align*}
(a) \quad \eta(1, u) = u & \quad \text{if } u \notin J_b^{-1}([mb - 2\varepsilon, mb + 2\varepsilon]) \cap S_\delta; \\
(b) \quad \eta(1, J_b^0 \cap S_\delta) & \subset J_b^0; \\
(c) \quad J_b(\eta(1, u)) & \leq J_b(u), \forall u \in H_0^1(\Omega). 
\end{align*} \]

By Lemma 2.1 and (b), we obtain that

\[ \max_{(s,t)\in D} J_b(\eta(1, \psi(s,t))) < m_b. \]

We prove that \( \eta(1, \psi(D)) \cap M_b \neq \emptyset \), contradicting to the definition of \( m_b \). Let us define \( \gamma(s, t) = \eta(1, \psi(s, t)) \) and

\[ \begin{align*}
\varphi_0(s, t) &= \left( \langle J_b'(su^+ + tu^-), su^+ \rangle, \langle J_b'(su^+ + tu^-), tu^- \rangle \right), \\
\varphi_1(s, t) &= \left( \langle J_b'(\gamma(s, t)), \gamma^+(s, t) \rangle, \langle J_b'(\gamma(s, t)), \gamma^-(s, t) \rangle \right). 
\end{align*} \]

Lemma 2.1 and the degree theory now yields \( \deg(\varphi_0, D, 0) = 1 \). It follows from (19) and (a) that \( \psi = \gamma \) on \( \partial D \). Consequently, we obtain \( \deg(\varphi_1, D, 0) = \deg(\varphi_0, D, 0) = 1 \). Therefore, \( \varphi_1(s_0, t_0) = 0 \) for some \( (s_0, t_0) \in D \), so that \( \eta(1, \psi(s_0, t_0)) = \gamma(s_0, t_0) \in M_b \), which is a contradiction. Hence, \( u_b \) is a nodal critical point of \( J_b \), and we complete the proof. \( \square \)

3. Proof of the main results. In this section, we will prove the main results. To begin with, we will show that the ground state nodal solution \( u_b \) of problem (1) has precisely two nodal domains, and its energy is strictly larger than twice that of the ground state energy.

Proof of Theorem 1.1. (i) In view of Lemma 2.5, there exists a \( u_b \in M_b \) such that \( m_b = J_b(u_b) \) and \( J_b'(u_b) = 0 \). In other words, \( u_b \) is a ground state nodal solution to problem (1). Then Lemma 2.2 yields \( b \| u_b^+ \|^4 < \int_{\Omega} |u_b^+|^4 dx \), it follows from Lemma 2.4 that there exist \( s_1, t_1 > 0 \) such that \( s_1 u_b^+, t_1 u_b^- \in N_b \). Then we can deduce from Lemma 2.1 that

\[ \begin{align*}
m_b &= J_b(u_b) \geq J_b(s_1 u_b^+ + t_1 u_b^-) \\
&= J_b(s_1 u_b^+) + J_\lambda(t_1 u_b^-) + \frac{bs_1^2 t_1^2}{2} \| u_b^+ \|^2 \| u_b^- \|^2 \\
&> J_b(s_1 u_b^+) + J_\lambda(t_1 u_b^-) \geq 2c_0. 
\end{align*} \]

Now, we show that \( u_b \) has exactly two nodal domains. We assume by contradiction that \( u_b = u_1 + u_2 + u_3 \) with

\[ u_i \neq 0, \quad u_1 \geq 0, \quad u_2 \leq 0, \quad \text{supp}(u_i) \cap \text{supp}(u_j) = \emptyset, \quad i \neq j \quad (i, j = 1, 2, 3). \]
Moreover, using the fact that \( J'_b(u_b) = 0 \), we get
\[
\begin{align*}
\{ (J'_b(u_1 + u_2), u_1) &= (J'_b(u_b), u_1) - b\|u_1\|^2\|u_3\|^2 < 0, \\
(\bar{J}'_b(u_1 + u_2), u_2) &= (\bar{J}'_b(u_b), u_2) - b\|u_2\|^2\|u_3\|^2 < 0.
\end{align*}
\]
Consequently, by Lemma 2.3, there exist \((\bar{s}, \bar{t}) \in (0, 1) \times (0, 1)\) such that \(\bar{s}u_1 + \bar{t}u_2 \in M_b\), \(\bar{s}u_1 + \bar{t}u_2 \geq m_b\).
Noting that \(\lambda < a\Lambda\), \((\bar{J}'_b(u_b), u_b) = 0\) and \((\bar{J}'_b(\bar{s}u_1 + \bar{t}u_2), \bar{s}u_1 + \bar{t}u_2) = 0\), we have
\[
m_b = J_b(u_b) - \frac{1}{4}(J'_b(u_b), u_b) = \frac{1}{4}\left[a\|u_b\|^2 - \lambda \int_\Omega u_b^2 dx \right] + \frac{1}{4}\left[a\|u_3\|^2 - \lambda \int_\Omega u_3^2 dx \right] + \frac{1}{4}\left[a\|u_1\|^2 - \lambda \int_\Omega u_1^2 dx \right] + \frac{1}{4}\left[a\|u_2\|^2 - \lambda \int_\Omega u_2^2 dx \right]
\geq \frac{1}{4}\left[a\|u_1\|^2 - \lambda \int_\Omega u_1^2 dx \right] + \frac{1}{4}\left[a\|u_2\|^2 - \lambda \int_\Omega u_2^2 dx \right] = J_b(\bar{s}u_1 + \bar{t}u_2) - \frac{1}{4}(\bar{J}'_b(\bar{s}u_1 + \bar{t}u_2), \bar{s}u_1 + \bar{t}u_2)
\geq m_b,
\]
which leads to a contradiction, and thus the minimizer \(u_b\) has precisely two nodal domains.

(ii) Let \(\lambda < a\Lambda\) and \(b \geq \Lambda\), we claim that problem (1) does not admit any nodal solution. Clearly, by the definition of \(\Lambda\), \(M_b = \emptyset\) for all \(b > \Lambda\). Now we will prove \(M_b = \emptyset\) if \(b = \Lambda\). Suppose the thesis is false, then there exists \(u \in M_\Lambda\) and, (1.1) is the positive solution of system (10) when \(b = \Lambda\). According to the proof of Lemma 2.1,
\[
D = \left| \int_\Omega |u^+|^4 dx - \Lambda \|u^+\|^4 \right| - \Lambda \|u^+\|^2 \|u^-\|^2 \int_\Omega |u^-|^4 dx - \Lambda \|u^-\|^4 \right| > 0.
\]
Since \(D\) is continuous at \(\Lambda\) for the fixed \(u\), we can deduce that there exists \(\kappa > 0\) such that for all \(b \in (\Lambda - \kappa, \Lambda + \kappa)\),
\[
D_b = \left| \int_\Omega |u^+|^4 dx - b \|u^+\|^4 \right| - b \|u^+\|^2 \|u^-\|^2 \int_\Omega |u^-|^4 dx - b \|u^-\|^4 \right| > 0.
\]
Then by the proof of Lemma 2.1, one finds that system (10) possesses one positive solution, denoted by \((S_b, T_b)\), and thus \(\sqrt{S_b}u^+ + \sqrt{T_b}u^- \in M_b\), which contradicts the definition of \(\Lambda\). Therefore, problem (1) does not admit any nodal solution for all \(\lambda < a\Lambda\) and \(b > \Lambda\).

Furthermore, we will show that \(M_b = \emptyset\) for all \(b \geq \frac{1}{2S_b}\), which means that \(\Lambda \leq \frac{1}{2S_b}\). If \(b \geq \frac{1}{2S_b}\), there exists \(u \in M_b\), we have from Lemma 2.2 that (9) holds. Then it follows from Sobolev inequality (4) and \(b \geq \frac{1}{2S_b}\) that
\[
\int_\Omega |u^+|^4 dx \leq \frac{1}{2S_b} \|u^+\|^4 \leq 2b \|u^+\|^4.
\]
Combining with (20) and the first inequality of (9), one can get that
\[ b\|u^+\|^2 + b\|u^+\|^2\|u^-\|^2 < 2b\|u^+\|^4, \]
which means that
\[ b\|u^+\|^2\|u^-\|^2 < 2b\|u^+\|^4 - b\|u^+\|^4 = b\|u^+\|^4. \]
Then we can obtain from \( \|u^+\| \neq 0 \) that
\[ \|u^-\|^2 < \|u^+\|^2. \tag{21} \]
On the other hand, by Sobolev inequality (4) and \( b \geq \frac{1}{2S^2} \), we can easily get that
\[ \int |u^-|^4dx \leq \frac{1}{S^2}\|u^-\|^4 \leq 2b\|u^-\|^4. \tag{22} \]
Similarly, we deduce from (22) and the second inequality of (9) that
\[ b\|u^+\|^4 + b\|u^+\|^2\|u^-\|^2 < 2b\|u^-\|^4, \]
then \( \|u^+\|^2 < \|u^-\|^2 \), which contradicts (21). Hence, \( M_b = \emptyset \) for all \( b \geq \frac{1}{2S^2} \), and we have proved that \( \Lambda \leq \frac{1}{2S^2} \).

Now, we are in a situation to prove Theorem 1.2. In the following, we regard \( b > 0 \) as a parameter in problem (1). We shall analyze the convergence property of \( ub \) as \( b \searrow 0 \).

**Proof of Theorem 1.2.** For any \( b \searrow 0 \), let \( u_b \in M_b \) be the ground state nodal solution to problem (1), which changes sign only once.

Choose a nonzero function \( w_0 \in C^\infty_0(\Omega) \) such that \( w_0^\pm \neq 0 \), there exists a \( 0 < \beta < \Lambda \) such that
\[
\begin{cases}
\beta\|w_0^+\|^4 + \beta\|w_0^+\|^2\|w_0^-\|^2 < \frac{1}{2} \int_\Omega |w_0^+|^4dx, \\
\beta\|w_0^-\|^4 + \beta\|w_0^+\|^2\|w_0^-\|^2 < \frac{1}{2} \int_\Omega |w_0^-|^4dx.
\end{cases}
\tag{23}
\]
Thus, for any \( b \in [0, \beta] \), it follows from \( \lambda < a\lambda_1 \) and (23) that
\[
J_b(sw_0^+ + tw_0^-) = \frac{s^2}{2} \left( a\|w_0^+\|^2 - \lambda \int_\Omega |w_0^+|^2dx \right) + \frac{bs^4}{4}\|w_0^+\|^4
- \frac{s^4}{4} \int_\Omega |w_0^+|^4dx + \frac{t^2}{2} \left( a\|w_0^-\|^2 - \lambda \int_\Omega |w_0^-|^2dx \right)
+ \frac{bt^4}{4}\|w_0^-\|^4 - \frac{t^4}{4} \int_\Omega |w_0^-|^4dx + \frac{bs^2t^2}{2}\|w_0^+\|^2\|w_0^-\|^2
\leq \frac{s^2}{2} \left( a\|w_0^+\|^2 - \lambda \int_\Omega |w_0^+|^2dx \right) + \frac{bs^4}{4}\|w_0^+\|^4
- \frac{s^4}{4} \int_\Omega |w_0^+|^4dx + \frac{t^2}{2} \left( a\|w_0^-\|^2 - \lambda \int_\Omega |w_0^-|^2dx \right)
+ \frac{bt^4}{4}\|w_0^-\|^4 - \frac{t^4}{4} \int_\Omega |w_0^-|^4dx
\leq \frac{s^2}{2} \left( a\|w_0^+\|^2 - \lambda \int_\Omega |w_0^+|^2dx \right) - \frac{\beta s^4}{2}\|w_0^+\|^2\|w_0^-\|^2
+ \frac{t^4}{2} \left( a\|w_0^-\|^2 - \lambda \int_\Omega |w_0^-|^2dx \right) - \frac{\beta t^4}{2}\|w_0^+\|^2\|w_0^-\|^2.
\]
It is not difficult to see that there exists $\theta > 0$ such that for all $s, t > 0$,

$$m_b = J_b(u_b) \leq \max_{s, t \geq 0} J_b(sw_0^+ + tw_0^-) < \theta,$$

where $\theta = \max_{s, t \geq 0} h(s, t)$ and

$$h(s, t) = \frac{s^2}{2} \left( a\|w_0^+\|^2 - \lambda \int_{\Omega} |w_0^+|^2 dx \right) - \frac{bs^4}{2}\|w_0^+\|^2\|w_0^-\|^2$$

$$+ \frac{t^2}{2} \left( a\|w_0^-\|^2 - \lambda \int_{\Omega} |w_0^-|^2 dx \right) - \frac{bt^4}{2}\|w_0^+\|^2\|w_0^-\|^2.$$

For any sequence $\{b_n\}$ with $b_n \searrow 0$ as $n \to \infty$, one can obtain from Theorem 1.1 that for large $n$, there exists $u_{b_n} \in M_{b_n}$ is a nodal critical point of $J_{b_n}$, then

$$\theta + 1 \geq J_{b_n}(u_{b_n}) - \frac{1}{4}\langle J'_{b_n}(u_{b_n}), u_{b_n} \rangle \geq \frac{1}{4} \left( a - \frac{\lambda}{\lambda_1} \right) \|u_{b_n}\|^2.$$

This shows that $\{u_{b_n}\}$ is bounded in $H^1_0(\Omega)$, then there exists a subsequence of $\{b_n\}$, still denoted by $\{b_n\}$, such that $u_{b_n} \rightharpoonup u_0$ weakly in $H^1_0(\Omega)$. By the compactness of the embedding $H^1_0(\Omega) \hookrightarrow L^s(\Omega)$ for $1 \leq s < 6$, using a standard argument, we can prove that $u_{b_n}^\pm \rightharpoonup u_0^\pm$ strongly in $H^1_0(\Omega)$, and $u_0^\pm \neq 0$. Furthermore, we deduce that for all $u \in H^1_0(\Omega)$,

$$0 = \lim_{n \to \infty} \langle J'_{b_n}(u_{b_n}), u \rangle$$

$$= \lim_{n \to \infty} \left[ (a + b_n\|u_{b_n}\|^2) \int_{\Omega} \nabla u_{b_n} \cdot \nabla u dx - \lambda \int_{\Omega} u_{b_n} udx - \int_{\Omega} |u_{b_n}|^2 u_{b_n} udx \right]$$

$$= a \int_{\Omega} \nabla u_0 \cdot \nabla u dx - \lambda \int_{\Omega} u_0 udx - \int_{\Omega} |u_0|^2 u_0 udx$$

$$= \langle J'_0(u_0), u \rangle,$$

which implies that

$$J'_0(u_0) = 0, \quad u_0 \in M_0, \quad J_0(u_0) \geq m_0. \quad (24)$$

In the proof of Theorem 1.1, $b = 0$ is allowed. Then there exists a $v_0 \in M_0$ such that

$$J_0(u_0) = m_0 = \inf_{u \in M_0} J_0(u),$$

and $v_0$ is a nodal solution to problem (1) which changes sign only once. Similarly, we can pick up $\epsilon \in (0, \Lambda)$ which is independent on $b_n$ such that

$$\left\{ \begin{array}{l}
\epsilon\|v_0^+\|^4 + \epsilon\|v_0^+\|^2\|v_0^-\|^2 < \int_{\Omega} |v_0^+|^4 dx,
\epsilon\|v_0^-\|^4 + \epsilon\|v_0^+\|^2\|v_0^-\|^2 < \int_{\Omega} |v_0^-|^4 dx.
\end{array} \right.$$
According to Lemma 2.1, there is a unique pair \((s_0, t_0)\) of positive numbers such that \(s_0v_0^+ + t_0v_0^- \in \mathcal{M}_n\). Let \(b_n \in [0, \epsilon]\), we can know that
\[
\langle J_n'(s_0v_0^+ + t_0v_0^-), s_0v_0^+ \rangle = a\|s_0v_0^+\|^2 + b_n\|s_0v_0^+\|^4 + b_n\|s_0v_0^-\|^2\|t_0v_0^-\|^2
\]
\[
- \lambda \int_\Omega |s_0v_0^+|^2 dx - \int_\Omega |s_0v_0^-|^4 dx
\]
\[
\leq a\|s_0v_0^+\|^2 + \epsilon\|s_0v_0^+\|^4 + \epsilon\|s_0v_0^-\|^2\|t_0v_0^-\|^2
\]
\[
- \lambda \int_\Omega |s_0v_0^+|^2 dx - \int_\Omega |s_0v_0^-|^4 dx
\]
\[
= \langle J_n'(s_0v_0^+ + t_0v_0^-), s_0v_0^+ \rangle = 0.
\]
In the same way, we can obtain that
\[
\langle J_n'(s_0v_0^+ + t_0v_0^-), t_0v_0^- \rangle \leq \langle J_n'(s_0v_0^+ + t_0v_0^-), t_0v_0^- \rangle = 0.
\]
It follows from Lemma 2.3 that for all \(b_n \in [0, \epsilon]\), there is a unique pair \((s_n, t_n)\) in \((0, s_0] \times (0, t_0]\) such that \(s_n v_0^+ + t_n v_0^- \in \mathcal{M}_{b_n}\). Then for any sequence \(\{b_n\}\) with \(b_n \to 0\) as \(n \to \infty\), we have as \(n \to \infty\),
\[
b_n s_n^4 \|v_0^+\|^4 \to 0, \quad b_n^2 t_n^2 \|v_0^-\|^2 \|v_0^-\|^2 \to 0, \quad b_n t_n^4 \|v_0^-\|^4 \to 0,
\]
together with \(\langle J_n'(s_n v_0^+ + t_n v_0^-), s_n v_0^+ \rangle = \langle J_n'(s_n v_0^+ + t_n v_0^-), t_n v_0^- \rangle = 0\), we can get
\[
\begin{align*}
\frac{a\|v_0^+\|^2 + o(1)}{a\|v_0^-\|^2 + o(1)} &= \lambda \int _\Omega \frac{|v_0^+|^2 dx + s_n^4 \int _\Omega |v_0^+|^4 dx}{|v_0^-|^2 dx + t_n^2 \int _\Omega |v_0^-|^4 dx},
\end{align*}
\]
and by \(\langle J_n'(v_0), v_0^\pm \rangle = 0\), we have
\[
\begin{align*}
\frac{a\|v_0^+\|^2}{a\|v_0^-\|^2} &= \lambda \int _\Omega \frac{|v_0^+|^2 dx + \int _\Omega |v_0^+|^4 dx}{|v_0^-|^2 dx + \int _\Omega |v_0^-|^4 dx},
\end{align*}
\]
Combining with (25) and (26), one has \(s_n \to 1, t_n \to 1\) as \(n \to \infty\). Now, we only need to show \(J_0(u_0) = J_0(v_0)\), then by (24), \(u_0\) is a ground state nodal solution of problem (8) which changes sign only once. In fact,
\[
J_0(v_0) \leq J_0(u_0) = \lim _{n \to \infty} J_{b_n}(u_{b_n})
\]
\[
\leq \lim _{n \to \infty} J_{b_n}(s_n v_0^+ + t_n v_0^-) = J_0(v_0^+ + v_0^-) = J_0(v_0).
\]
This completes the proof of Theorem 1.2. \(\square\)

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