ON EQUITABLE COLORING OF BOOK GRAPH FAMILIES

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Abstract. A proper vertex coloring of a graph is equitable if the sizes of color
classes differ by atmost one. The notion of equitable coloring was introduced
by Meyer in 1973. A proper $h$-colorable graph $K$ is said to be equitably
$h$-colorable if the vertex sets of $K$ can be partitioned into $h$ independent color
classes $V_1, V_2, \ldots, V_h$ such that the condition $|V_i| - |V_j| \leq 1$ holds for all
different pairs of $i$ and $j$ and the least integer $h$ is known as equitable chromatic
number of $K$. In this paper, we find the equitable coloring of book graph,
middle, line and central graphs of book graph.

1. INTRODUCTION

The idea of equitable coloring was discovered by Meyer [4] in 1973. Hajmal
and Szemeredi [3] proved that graph $K$ with degree $\Delta$ is equitable $h$-colorable, if
$h \geq \Delta + 1$. Later Equitable Coloring Conjecture for bipartite graphs was proved.
Equitable vertex coloring of corona graphs is NP-hard.

The graphs considered here are simple. Vertex coloring is a particular case of
Graph coloring. The collection of vertices receiving same color is known as color
class. A proper $h-$colorable graph $K$ is said to be equitably $h-$colorable if the
vertex sets of $K$ can be partitioned into $h$ independent color classes $V_1, V_2, \ldots, V_h$
such that the condition $|V_i| - |V_j| \leq 1$ holds for all different pairs of $i$ and $j$ [1].
And the least integer $h$ is known as equitable chromatic number of $K$ [1]. Here
we found equitable coloring of book graph, middle, line and central graphs of book graph.

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2. Preliminaries

Line graph \([2]\) of \(K_L(K)\) is attained by considering the edges of \(K\) as the vertices of \(L(K)\). The adjacency of any two vertices of \(L(K)\) is a consequence of the corresponding adjacency of edges in \(K\).

Middle graph \([5]\) of \(K, M(K)\) is attained by adding new vertex to all the edges of \(K\). The adjacency of any two new vertices of \(M(K)\) is a consequence of the corresponding adjacency of edges in \(K\) or adjacency of a vertex and an edge incident with it.

Central graph \([6]\) of \(K, C(K)\) is attained by the insertion of new vertex to all the edges of \(K\) and connecting any two new vertices of \(K\) which were previously non-adjacent.

The q-book graph is defined as the graph Cartesian product \(S_{(q+1)} \times P_2\), where \(S_q\) is a star graph and \(P_2\) is the path graph.

3. Results

3.1. On Equitable Coloring of Middle Graph of Book Graph.

- Order of \(M(B_q)\) is \(5q + 3\)
- Number of incidents of \(M(B_q)\) is \(q^2 + 9q + 2\)
- Maximum degree of \(M(B_q)\) is \(2(q+1)\)
- Minimum degree of \(M(B_q)\) is \(2\)

Algorithm A

**Input:** The value \(q\) of \(B_q\), for \(q \geq 3\)

**Outcome:** Equitably colored \(V[M(B_q)]\)

**Procedure:**

\[
\text{start} \\
\{ \\
V_a = \{g, h, z\}; \\
C(g) = C(h) = 1; \\
C(z) = q + 2; \\
f\text{or } s = 1 \text{ to } q \\
\{ \\
V_b = \{g_s, h_s\}; \\
C(g_s) = s; \\
C(h_s) = s; \\
\} \\
f\text{or } s = 1 \text{ to } q \\
\{ \\
V_c = \{k_s, l_s\}; \\
C(k_s) = s + 1; \\
C(l_s) = s + 1; \\
\}
\]
Theorem 3.1. For any book graph $M(B_q)$ the equitable chromatic number,

$$\chi_{M(B_q)} = q + 2, \forall q \geq 3$$

Proof. For $q \geq 3$, $V(B_q) = \{g, h, g_s, h_s : 1 \leq s \leq q\}$.

$V[M(B_q)] = \{g, h, z \cup \{g_s : 1 \leq s \leq q\} \cup \{h_s : 1 \leq s \leq q\} \cup \{k_s : 1 \leq s \leq q\} \cup \{l_s : 1 \leq s \leq q\} \cup \{m_s : 1 \leq s \leq q\} \}$, where $z, k_s, l_s$ and $m_s$ are the subdivision of the edges $gh, gg_s, hh_s$ and $g_h$ respectively.

Let us consider $V[M(B_q)]$ and the color set $C = \{c_1, c_2, ..., c_{q+2}\}$. Assign the equitable coloring by Algorithm A. Therefore,

$$\chi_{M(B_q)} \leq q + 2.$$ 

And since, there exists a maximal induced complete subgraph of order $q + 2$ by the vertices $z, g, k_s$ and therefore $\chi_{M(B_q)} \geq q + 2$.

$c_1, c_2, ..., c_{q+2}$ are independent sets of $M(B_q)$. And $|c_i| - |c_j| \leq 1$, for every different pair of $i$ and $j$. Hence,

$$\chi_{M(B_q)} = q + 2.$$ 

3.2. On Equitable Coloring of Central Graph of Book Graph. Features of Central Graph of Book Graph

- Order of $C(B_q)$ is $5q + 3$
- Number of incidents of $C(B_q)$ is $2(q^2 + 3q + 1)$
- Maximum degree of $C(B_q)$ is $2q + 1$
- Minimum degree of $C(B_q)$ is $2$

Algorithm B

Input: The value ‘$q$’ of $B_q$, for $q \geq 3$
Outcome: Equitably colored $V[C(B_q)]$
Procedure:

start

for $s = 1$ to $q$

\{
$V_d = \{m_s\}$;
if $s$ is odd
$C(m_s) = q + 1$;
else
$C(m_s) = q + 2$;
\}
\}
$V = V_a \cup V_b \cup V_c \cup V_d$
end
\{
    for \( s = 1 \) to \( q \)
    \( V_a = \{ g, h, m_s \} \);
    \{
      if \( s = 1 \) to \( 3 \)
      \( C(g) = C(h) = C(m_s) = 1; \)
      else
      \( C(m_s) = s - 1; \)
    }  
    for \( s = 1 \) to \( q \)
    \{  
      \( V_b = \{ g_s, h_s \};  \)
      \( C(g_s) = s + 1; \)
      \( C(h_s) = s + 1; \)
    }  
    if \( q \) is odd
    \{
      \( V_c = \{ k_s, l_s, z \}; \)
      for \( s = 1 \) to \( q \)
      \{  
        if \( s = 1 \) to \( q - 1 \)
        \{
          \( C(k_s) = s + 2; \)
          \( C(l_s) = s + 2; \)
        }  
        else
        \( C(z) = C(k_s) = C(l_s) = 2; \)
      }  
    }  
    else
    \{  
      for \( s = 1 \) to \( q \)
      \( V_c = \{ k_s, l_s, z \}; \)
      \( C(z) = 2; \)
      \( C(k_s) = C(l_s) = q - s + 2; \)
    }  
  }  
\}
\text{end}

**Theorem 3.2.** For any book graph \( C(B_q) \) the equitable chromatic number,

\[ \chi_{eq}[C(B_q)] = q + 1, \forall q \geq 3 \]
Proof. For $q \geq 3$,

$$V(B_q) = \{g, h, g_s, h_s : 1 \leq s \leq q\}.$$ 

$$V[C(B_q)] = \{g, h, z\} \cup \{g_s : 1 \leq s \leq q\} \cup \{h_s : 1 \leq s \leq q\} \cup \{k_s : 1 \leq s \leq q\} \cup \{l_s : 1 \leq s \leq q\} \cup \{m_s : 1 \leq s \leq q\},$$

where $z$, $k_s$, $l_s$, and $m_s$ are the subdivision of the edges $gh$, $gg_s$, $hh_s$, and $gs.hs$ respectively.

Let us consider $V[C(B_q)]$ and the color set $C = \{c_1, c_2, ..., c_{q+1}\}$. Assign the equitable coloring by Algorithm B. Therefore,

$$\chi = |C(B_q)| \leq q + 1$$

And $\chi[C(B_q)] = q + 1$. That is, $\chi[C(B_q)] \geq \chi[C(B_q)] = q + 1$. Therefore,

$$\chi = |C(B_q)| \geq q + 1.$$ 

c_1, c_2, ..., c_{q+1} are independent sets of $C(B_q)$. And $|c_i| - |c_j| \leq 1$, for every different pair of $i$ and $j$. Thus,

$$\chi = |C(B_q)| = q + 1.$$

\[
\]

3.3. On Equitable Coloring of Line Graph of Book Graph.

- Order of $L(B_q)$ is $3q + 1$
- Number of incidents of $L(B_q)$ is $q(q + 3)$
- Maximum degree of $L(B_q)$ is $2q$
- Minimum degree of $L(B_q)$ is $2$

Algorithm C
Input: The value 'q' of $B_q$, for $q \geq 3$
Outcome: Equitably coloring $V[L(B_q)]$

Procedure:
begin

```{begin}

\begin{algorithm}
\begin{algorithmic}
\EndProcedure
\end{algorithm}
\end{algorithm}
```
Theorem 3.3. For any book graph $L(B_q)$ the equitable chromatic number, 

$$\chi_{eq}[L(B_q)] = q + 1, \forall q \geq 3$$

Proof. For $q \geq 3$, 

$$V(B_q) = \{g, h, g_s, h_s : 1 \leq s \leq q\}.$$ 

The edge set of $B_q$ is \{z, k_s, l_s, m_s : 1 \leq s \leq q\} where z be the edge corresponding to the vertices gh, each $k_s$ be the edge corresponding to the vertex $gg_s$, each edge $l_s$ be the edge corresponding to the vertex $hh_s$, each edge $m_s$ be the edge corresponding to the vertex $g_s h_s$. By the definition of line graph, the edge set of line graph is converted into vertices of $L(B_q)$.

$$V[L(B_q)] = \{z\} \cup \{k_s : 1 \leq s \leq q\} \cup \{l_s : 1 \leq s \leq q\} \cup \{m_s : 1 \leq s \leq q\}.$$ 

Let us consider the $V[L(B_q)]$ and the color set $C = \{c_1, c_2, ..., c_{q+1}\}$. Assign the equitable coloring by Algorithm C. Therefore, 

$$\chi_{eq}[L(B_q)] \leq q + 1$$

And since, there exists a maximal induced complete subgraph of order $q+1$ by the vertices $z, k_s$ and therefore 

$$\chi_{eq}[L(B_q)] \geq q + 1.$$ 

c_1, c_2, ..., c_{q+1} are independent sets of $L(B_q)$. And $|c_i| - |c_j| \leq 1$, for every different pair of i and j. Thus, 

$$\chi_{eq}[L(B_q)] = q + 1.$$ 

3.4. On Equitable Coloring of Book Graph. Features of Book Graph.

- Order of $B_q$ is $2(q + 1)$
- Number of incidents of $B_q$ is $3q + 1$
- Maximum degree of $B_q$ is $q + 1$
- Minimum degree of $B_q$ is $2$

Algorithm D

Input: The value 'q' of $B_q$, for $q \geq 3$

Outcome: Equitably colored $V(B_q)$

Procedure:

```
start
{
for s = 1 to q
{
Va = \{g_s, h_s\};
C(h) = 1;
}
```
\( C(g_s) = 1; \)
\[
\begin{align*}
&\text{for } s = 1 \text{ to } q \\
&\{ \\
&V_b = \{g, h_s\}; \\
&C(g) = 2; \\
&C(h_s) = 2; \\
&\}
\end{align*}
\]
\( V = V_a \cup V_b \)
\]~

**Theorem 3.4.** For any book graph \( B_q \) the equitable chromatic number,
\[ \chi_e(B_q) = 2, \forall q \geq 3. \]

**Proof.** For \( n \geq 3 \),
\[ V(B_q) = \{g, h\} \cup \{g_s : 1 \leq s \leq q\} \cup \{h_s : 1 \leq s \leq q\}. \]

Let us consider the \( V(B_q) \) and the color set \( C = \{c_1, c_2\} \). Assign the equitable coloring by Algorithm D. Therefore,
\[ \chi_e(B_q) \leq 2. \]

And since, there exists a maximal induced complete subgraph of order 2 in \( B_q \) (say path \( P_2 \)). Therefore,
\[ \chi_e(B_q) \geq 2 \]

\( c_1, c_2 \) are independent sets of \( B_q \). And \( ||c_i| - |c_j|| \leq 1 \), for every different pair of \( i \) and \( j \). Hence,
\[ \chi_e(B_q) = 2. \]

\[ \square \]

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