A new surface tension VOF evaluation by using variational representation and Galerkin interpolation projection

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Abstract. In this work we propose a new algorithm for studying two-phase interface advection problems dominated by surface tension. We use a Volume Of Fluid (VOF) algorithm for studying the evolution of the two-phase interface on a Cartesian grid and a finite element numerical scheme for the velocity-pressure state. The velocity field that drives the evolution of this interface is obtained by solving the weak form of the Navier-Stokes equation where the surface tension force is not defined in a singular way. With standard numerical approaches that solve the strong form of the Navier-Stokes equations the surface force is determined by taking the divergence of the surface tension tensor. The computation of the divergence term results in a force which is non-convergent when the grid is refined since the tensor is computed in a discontinuous cell-by-cell way. In the past this approach was proposed with artificial different smoothing schemes in order to compute such a singular force. In this work we use the variational formulation of the Navier-Stokes equation and avoid differentiation. The tensor which is a function of the unit normal is evaluated by a Galerkin projection over regular Sobolev spaces. This allows the piece-wise continuous representation of the surface tensor and the unit normal based on the VOF reconstruction. Tests on convergence for two and three-dimension in the static and dynamical cases are reported to show the correct representation in the desired spaces. This method is also natural for coupling non uniform grid computation of the fluid with Cartesian grid of the VOF algorithm.

Keywords: Navier-Stokes equations; finite element methods; VOF; two-phase flows; capillary force.

1. Introduction

Two-phase flows are of great interest in many engineering applications ranging from environmental sciences to the oil and nuclear industries. In recent years this interest has inspired a great number of algorithms for the solution of problems involving moving interfaces and surface tension forces, see [1, 2, 3, 4, 5] and references therein. All these methods provide good solutions to the problem of interface advection but an accurate representation of surface forces remain a challenging problem with fixed grid representation. As a matter of fact, a surprising feature of the simulation of a single droplet in equilibrium in another phase without any volume force, is the appearance of the so-called spurious currents resulting from an unbalanced representation of capillary forces due to surface tension and the related pressure jump across the interface. This leads to the problem of an accurate velocity field description at the interface which otherwise may lead to numerical instabilities and artificial interface breakups.
In this paper the Navier-Stokes equations are discretized by using a variational formulation in order to cope with the discontinuities of the fluid properties. The variational formulation of the Navier-Stokes equations allows the treatment of the capillary force in its original form. The derivative of the surface tension tensor, which is inherently singular in its standard classic representation, is avoided. The surface tension tensor is computed by extending the unit normal and projecting the indicator phase into a more regular functional spaces by a Galerkin projection with the purpose of reproducing this force more accurately on fixed grids. Spurious currents are greatly reduced and drop dynamics can be numerically simulated with great accuracy. Furthermore the proposed approach does not need an explicit computation of the curvature which is a very challenging task. The basics of statics and dynamics of the surface tension force is reproduced with well known tests in two and three-dimensional cases.

In this paper we consider two-phase flows over a two-dimensional open bounded domain $\Omega$. Let the reference phase 1 be contained in $\Omega_f$ and $\chi(x,t)$ be the indicator function for the reference phase defined in such a way that is one for all $x \in \Omega_f$ and zero on $\Omega - \Omega_f$. Over the interval of time $[0,T]$ the indicator function $\chi$ behaves like a passive scalar and satisfies the following advection equation

$$ \frac{\partial \chi}{\partial t} + u \cdot \nabla \chi = 0 \quad \text{in } \Omega \times [0,T],$$

where the velocity field $u$ obeys the incompressibility constraint

$$ \nabla \cdot u = 0 \quad \text{in } \Omega \times [0,T]$$

and the Navier-Stokes equations over the whole domain $\Omega$ are given by the momentum conservation equation [6]

$$ \rho \left( \frac{\partial u}{\partial t} + u \cdot \nabla u \right) = -\nabla p + \nabla \cdot \left[ \mu (\nabla u + \nabla u^T) \right] + F + F_s \quad \text{in } \Omega \times [0,T].$$

$F$ and $F_s$ are the external body (e.g., gravity) and surface tension forces, respectively. The force $F_s$ per unit of volume is defined as $F_s(x) = f_s \delta_s$ where $\delta_s$ is one on the surface and zero otherwise. The surface force $f_s$ can be written clearly on the surface points $x_s$ as

$$ f_s(x_s) = -\sigma \kappa n \quad \text{on } \Gamma_f$$

where $n$ is the unit external normal, $\kappa$ the interface mean curvature and $\sigma$ the surface tension coefficient. The coefficient $\sigma$ is assumed to be constant. The normal $n$, the mean curvature $\kappa$ and the interface line $\Gamma_f$ are all geometrical quantities.

The geometrical quantities can be computed in a standard geometrical way but it should be more convenient a direct computation from the indicator function $\chi$. It is clear that the function $\chi$ is a discontinuous function and its derivative is a distribution. In fact we can relate the normal direction vector $d$ (non unitary vector) with the gradient of $\chi$ through a distribution function. By using the generalized definition of derivatives we have

$$ \int_\Omega \nabla \chi \cdot w \, dx = - \int_\Omega \chi \nabla \cdot w \, dx = - \int_{\Gamma_f} \nabla \cdot w \, dx = - \int_{\Gamma_f} n \cdot w \, ds =$$

$$ = - \int_\Omega \delta_s n \cdot w \, dx \quad \forall w \in C^\infty_0(\Omega).$$

Formally this implied the extension of the normal direction vector $d$ from the surface $\Gamma_f$ ($d$) to the whole domain $\Omega$ ($\tilde{d}$) in the following form

$$ \tilde{d} = n \delta_s = -\nabla \chi \quad \text{on } \Omega. $$
We remark that the extended normal direction $\hat{d}$ is not a unitary vector and takes values equal to zero on all points except those on the surface $\Gamma_f$. This normal direction extended function is a distribution and it can be formally written only by introducing the surface Dirac distribution $\delta_s$ defined in (5). In the rest of the paper we use the symbol $\hat{\cdot}$ to denote the extension of functions from interface/phase to the rest of the domain.

The evaluation of the curvature $\kappa$ through the geometric normal is standard. For each point on the surface $\Gamma_f$ we have

$$k = \nabla_s \cdot \mathbf{n} \quad \text{on } \Gamma_f,$$

(7)

where $\nabla_s = (I - \mathbf{nn})\n$. Therefore we can write

$$F_s(x_s) = -\sigma \kappa \mathbf{n} = -\sigma \mathbf{n} \nabla_s \cdot \mathbf{n} \quad x_s \in \Gamma_f.$$  

(8)

It is easy to note that the surface divergence on the tensor $T_s = \sigma (I - \mathbf{nn})$ gives the same result

$$\nabla_s \cdot T_s = -\sigma (\nabla_s \cdot \mathbf{n}) \mathbf{n} - \sigma (\mathbf{n} \cdot \nabla_s) \mathbf{n} = -\sigma \kappa = F_s(x_s) \quad \text{on } \Gamma_f.$$  

(9)

The evaluation of the curvature $\kappa$ through the $\chi$ indicator is rather difficult and could be understood only in a distributional sense. In fact, since $\nabla_n \cdot T_s = 0$, we can write

$$\int_{\Gamma_f} (\nabla_s \cdot T_s) : \mathbf{w} \, dx = \int_{\Gamma_f} T_s : \nabla_s \mathbf{w} \, dx = \int_{\Gamma_f} T_s : \nabla \mathbf{w} \, dx = \int_{\Omega} T_s \delta_s : \nabla \mathbf{w} \, dx$$

(10)

for all $\mathbf{w} \in C^\infty_0(\Omega)$ with $\nabla_n \cdot \mathbf{w} = 0$ on the surface $\Gamma_f$. Therefore we can write formally the extended force over the whole domain $\hat{\mathbf{F}}$ as

$$\hat{\mathbf{F}} = T_s \delta_s = \sigma (I - \mathbf{nn}) \delta_s \quad \text{on } \Omega,$$

(11)

where we have again introduced the surface distribution $\delta_s$. The extended force $\hat{\mathbf{F}}$ can be computed only by (10) where the derivative on discontinuous functions are considered in their generalized form.

The two fluids, the reference phase 1 and the secondary phase 2, have both constant densities, $\rho_1$ and $\rho_2$, and dynamical viscosities, $\mu_1$ and $\mu_2$, respectively. The density $\rho$ and viscosity $\mu$ in (3) are functions of the indicator function $\chi$

$$\rho = \chi \rho_1 + (1 - \chi) \rho_2,$$

(12)

$$\mu = \chi \mu_1 + (1 - \chi) \mu_2 \quad \text{on } \Omega.$$  

(13)

Since the derivatives of the discontinuous function $\chi$ are distributions, a treatment of the problem in its weak formulation, which will be introduced in the next section, is more appropriate.

2. Variational formulation of the two-phase Navier-Stokes equations

Let $\Omega \subset \mathbb{R}^2$ be an open set with boundary $\Gamma$. We denote with $H^d(\Omega)$ the Sobolev space of all functions having square integrable derivatives up to order $d$ on $\Omega$. We also denote with $\| \cdot \|_d$ and $(\cdot, \cdot)_d$ the standard Sobolev norm and inner product, respectively. When $d = 0$, we write $L^2(\Omega)$ instead of $H^0(\Omega)$ and drop the index from the inner product designation. For vector-valued functions and spaces, we use boldface notation. For example, $\mathbf{H}^d(\Omega)$ denotes the space of $\mathbb{R}^2$-valued functions such that each of the two components belongs to $H^d(\Omega)$. We recall that $H^1(\Omega) \subset L^2(\Omega) = H^0(\Omega)$ [7].
In order to define a weak form of the Navier-Stokes equations, we multiply by the test functions \( w \) and integrate over the domain in order to write [3, 4, 7]

\[
\int_{\Omega} \rho \left( \frac{\partial u}{\partial t} + u \cdot \nabla u \right) \cdot w \, dx = -\int_{\Omega} \mu (\nabla u + \nabla u^T) : \nabla w \, dx + \int_{\Omega} p \nabla \cdot w \, dx + \int_{\Omega} F \cdot w \, dx + \int_{\Omega} \hat{T} : \nabla w \, dx
\]

(14)

\[
\int_{\Omega} \nabla \cdot u \psi = 0 \, dx \quad \text{in } \Omega \times [0, T]
\]

(15)

for all test functions \((w, \psi) \in H^1(\Omega) \times L^2(\Omega)\). The symbol ":" is the double product operator, namely \( A : B = \sum_i \sum_j A_{ij} B_{ji} \). The system (14-15) includes vanishing velocity or vanishing pressure boundary conditions along the domain boundary \( \Gamma \). It is important to remark that in (14-15) no curvature values are involved and only the extension of \( \hat{T} \) with no derivatives is required.

The variational form of the two-phase equation can be written as [4, 8]

\[
\frac{\partial}{\partial t} \int_{\Omega} \chi \varphi \, dx = -\int_{\Omega} \varphi \nabla \cdot (u \chi) \, dx \quad \text{in } \Omega \times [0, T],
\]

(16)

for all test functions \( \varphi \in L^2(\Omega) \). The problem is completed once the extended tensor \( \hat{T} \) is known.

From the extended normal direction \( \hat{d} \), defined by

\[
\int_{\Omega} \hat{d} \cdot w \, dx = -\int_{\Omega} \nabla \chi \cdot w \, dx = \int_{\Omega} \chi \nabla \cdot w \, dx \quad \forall w \in C^\infty_0(\Omega),
\]

(17)

we should compute the normal unit vector and the extended tensor \( \hat{T} \). Formally we can write \( \hat{n} = \hat{d} / |\hat{d}| \) and \( \hat{T} = \sigma (\hat{I} - \hat{n} \hat{n}) \) but all these terms are meaningful only in distributional spaces.

It clear that it is difficult to work with distributions and it seems reasonable to regularize the problem in more regular spaces by projecting the function \( \chi \) from \( L^2(\Omega) \) into \( H^1(\Omega) \) where all the derivatives are well defined. Therefore it is convenient to compute the normal and surface tension by projecting \( \chi \) from \( L^2(\Omega) \) into \( \chi^* = P \chi \in H^1(\Omega) \). For example we can use a simple Galerkin projection as

\[
\delta \nabla^2 \chi^* + \chi^* = \chi
\]

(18)

where the derivatives of \( \chi^* \) are well defined for \( \delta > 0 \). From \( \chi^* \) we can compute the extended normal direction \( \hat{d} = -\nabla \chi^* \), an extension of the normal \( \hat{n} = \hat{d} / |\hat{d}| \) and the extended tensor \( \hat{T} = \hat{I} - \hat{n} \hat{n} \).

3. Finite element and VOF spatial discretization

Let \( \Omega_h \) be an open bounded domain with boundary \( \Gamma_h \). The reference phase is contained in the region \( \Omega_{fh} \subset \Omega_h \) whose boundary \( \Gamma_{fh} \) consists of a chain of connected segments. Let \( X^h \subset H^1(\Omega_h) \) and \( S^h \subset L^2(\Omega_h) \) be two families of finite dimensional subspaces parameterized by \( h \) that tends to zero. Also we denote by \( S^h_0 \) the family of finite dimensional subspaces that contains piecewise constant functions. We make the usual approximation assumptions on \( X^h \) and \( S^h \) including the LBB condition (see, e.g., [7]), and use standard cubic Taylor-Hood elements. Under these conditions, the velocity field \( u_h \in X^h \) and the pressure field \( p_h \in S^h \).
satisfy the following discrete Navier-Stokes equations

\[
\frac{1}{\Delta t} \int_{\Omega_h} \rho \mathbf{u}_h \mathbf{v}_h \, d\mathbf{x} + \int_{\Omega_h} \left( \rho \mathbf{u}_h \cdot \nabla \mathbf{u}_h \right) \cdot \mathbf{v}_h \, d\mathbf{x} + \int_{\Omega_h} \mu (\nabla \mathbf{u}_h + \nabla \mathbf{u}_h^T) : \nabla \mathbf{v}_h \, d\mathbf{x} + \int_{\Omega_h} p_h \nabla \cdot \mathbf{v}_h \, d\mathbf{x} = \frac{1}{\Delta t} \int_{\Omega_h} \rho \mathbf{u}^{(n-1)}_h \mathbf{v}_h \, d\mathbf{x} + \sigma \int_{\Omega_h} \left( \mathbf{I}_h - \hat{\mathbf{n}}_h \hat{\mathbf{n}}_h^T \right) : \nabla \mathbf{v}_h \, d\mathbf{x} + b(q_h, \mathbf{u}_h) = 0, \tag{19}
\]

for all test functions \((\mathbf{v}_h, q_h) \in H^1(\Omega_h) \times S^h(\Omega_h)\). The term with the time derivative has been discretized with a simple backward Euler scheme, then the velocity \(\mathbf{u}_h^{(n-1)}\) is defined at the previous time step \(n - 1\), while all the other quantities are at time step \(n\). Furthermore the Navier-Stokes equations are discretized by using SUPG and GLS discretization schemes. The GLS is not necessary here since Taylor-Hood finite elements are considered but it may be necessary to reduce pressure errors from the surface tension term which is not discretized in agreement with the LBB condition.

For the phase equation we use a VOF piecewise linear approximation. Let \(S^h_0\) be the discrete color function defined by

\[
C_h = \frac{1}{\Omega_h} \int_{\Omega_h} \chi_h(x,t) \, d\mathbf{x}, \tag{21}
\]

where \(\Omega_h\) is the volume of the finite element. Clearly, the color function \(C_h\) is one for all elements located in the interior of \(\Omega_h\), zero for all external elements and between zero and one in the elements cut by the interface. The density \(\rho_h \in S^h\) and the viscosity \(\mu_h \in S^h\) become

\[
\rho_h = \rho_1 C_h + \rho_2 (1 - C_h), \quad \mu_h = \mu_1 C_h + \mu_2 (1 - C_h) \tag{22}
\]

The color function \(C_h\) solves [4]

\[
\frac{\partial}{\partial t} \int_{\Omega} C_h \varphi_h \, d\mathbf{x} = - \int_{\Omega} \varphi_h \nabla \cdot \left( \mathbf{u} C_h \right) \, d\mathbf{x} \quad \text{in} \ \Omega_h \times [0, T], \tag{23}
\]

for all test functions \(\varphi_h \in S^h_0\). In order to regularize the computation of the normal and surface tension we project \(C_h\) from \(S^h_0\) into \(X^*_h\) by using the simple Galerkin projection. We set

\[
\alpha \nabla^2 C^*_h + C^*_h = C_h \tag{24}
\]

for a small positive \(\delta\). We can compute the extended normal direction \(\hat{\mathbf{d}}_h^* = -\nabla C^*_h \in S^h\) and the unit normal as \(\hat{\mathbf{n}}^* = \hat{\mathbf{d}}^*/|\hat{\mathbf{d}}^*| \in S^h\). Once the extended normal is known in \(S^h\) also the extended tensor \(\hat{T} = \left( \mathbf{I}_h - \hat{\mathbf{n}}_h \hat{\mathbf{n}}_h^T \right)\) is known in \(S^h\).

4. Numerical tests

4.1. Steady solutions and stability tests

The numerical discretization of the surface tension term generates parasitic or spurious currents. These spurious currents are generated in the vicinity of fluid interfaces whose magnitude can be estimated by the simulation of the static case, i.e. a static spherical droplet/bubble immersed in another fluid in zero gravity conditions. The analytical solution of the static case is characterized by a zero velocity field and a pressure jump between the two phases \(\Delta p = \sigma \kappa\), where \(\kappa\) is the mean curvature. However, due to the singularity of the force, the forcing term results in small pressure-velocity fluctuations near the interface. Depending on the numerical representation, on
viscous damping and on the value of the surface tension coefficient \( \sigma \), these fluctuations give rise to flows that, in many cases, do not decrease with mesh refinement. In this work we introduce a numerical scheme that does not need the explicit curvature computation and represents the

**Figure 1.** Two-dimensional steady case (A). The projected color function \( C_h^* \) (left) and the pressure distribution \( p_h \) (center and right) for VOF grid resolution 160 \( \times \) 160.

**Figure 2.** Three-dimensional steady case (A). Plane reconstruction of the interface (top left), the pressure \( p_h \) (top right) and the velocity field \( u_h \) on a plane (bottom) for a 40 \( \times \) 40 \( \times \) 40 VOF grid resolution.
We present two different tests: case A, where we consider known the exact value of the curvature and case B where the surface tension term is computed directly from the numerical representation of the interface. In order to show the robustness of the approach we show computation in two and three-dimensional spaces.

First we consider Case A with assigned curvature. If the curvature \( \kappa = \kappa_0 \) is constant we can extend this value outside the interface \( \tilde{\kappa} = \kappa_0 \) and, for the surface tension term in (14), we have

\[
\sigma \int_{\Omega_h} \left( \tilde{I}_h - \tilde{\nu}_h \tilde{n}_h \right) : \nabla v_h \, dx = \sigma \int_{\Omega_h} \left( C_h \tilde{\kappa} \right) \tilde{I} : \nabla v_h \, dx.
\] (25)

For the two-dimensional steady case A we consider a unitary box with a two-dimensional droplet radius of 0.15. In Figure 1 the two-dimensional steady case A with assigned constant curvature is shown. On the left one can see the projected color function \( C^*_h \), while the pressure distribution \( p_h \) is shown in the center and on the right for VOF grid resolution 160 × 160. In this case we set \( \mu_1/\mu_2 = \rho_1/\rho_2 = 100 \) with \( \mu_2 = 1.e^-4 \) and \( \rho_2 = 1 \). If the curvature is constant and assigned as
Figure 5. Three-dimensional steady case (B). Surface plane with PLIC-VOF reconstruction (left) and the projected color function \( C_h^* \) for \( 40 \times 40 \times 40 \) VOF grid resolution.

Figure 6. Three-dimensional steady case (B). Pressure \( p_h \) (left) and velocity field \( u_h \) (right) for \( 40 \times 40 \times 40 \) VOF grid resolution.

in (25) the velocity field is zero \( (\approx 10^{-18}) \) and no spurious currents are present.

For the three-dimensional steady case A we consider again a unitary box with a droplet radius of 0.15. In Figure 2 the three-dimensional steady case A with assigned constant curvature is shown. On the left top one can see the reconstruction of the interface of the color function \( C_h^* \) and on the right top the pressure distribution \( p_h \) for VOF grid resolution \( 40 \times 40 \times 40 \). In this case we set \( \mu_1/\mu_2 = \rho_1/\rho_2 = 100 \) with \( \mu_2 = 1.e - 3 \) and \( \rho_2 = 1 \). On the bottom of Figure 2 it is possible to see the velocity field over the central plane. If the curvature is constant, and assigned as in (25), the velocity field is in the range of \( 10^{-23}-10^{-10} \) and no spurious currents are present. In fact, since the pressure term has the form in (25), \( p_h = \sigma C_h \kappa \) when the interpolation space for \( p_h \) and \( C_h \) is the same and respects the LBB condition.

This test aims to check the correct distribution of the interface forces. However, if the curvature \( \kappa_h \) is not available analytically the numerical noise alters the interface force balance and spurious currents are generated. The computation of the surface tension force can be seen in a great variety of implementations, see for example [9]. Almost all VOF methods for different numerical techniques, which are based on the same discrete color function \( C_h \), do not usually
approximate the interface local curvature in an accurate way.

Now we consider Case B where the curvature is not known and the surface tension term is computed numerically as in (14). We propose an approach where the color function field \( C_h \in S_h \) is projected into \( C^*_h \in X_h \subset H^1(\Omega) \) where its gradient is well defined. In order to project the function we use a Galerkin type interpolation, for example as in (24) with \( \alpha = 5 \times 10^{-4} \). We remark that the curvature is not computed explicitly but only the unit normal is needed both in two and the three-dimensional cases.

Figures 3-4 show the results for the pressure and velocity for the two-dimensional steady case with numerical surface tension term. The geometry is the same as in Case A. On the left and on the right of Figure 3 the pressure for VOF resolution 320 \( \times \) 320 and square norm \( \| u_h - u \|^2_{L^2} \) of the velocity error as a function of time for VOF grid resolution 160 \( \times \) 160 are shown, respectively. The error is evaluated in \( L^2 \) norm in agreement with the functional spaces considered. It is important to remark that the spurious currents do not vanish or increase but show an oscillating behavior as a function of time. The numerical solution is a non linear solution that takes into account the numerical reconstruction of the interface and the pressure field. From Figure 3 on the right one can see that the corrections on the interface position move the pressure field and this adjusts back the interface position with oscillating behavior. If one improves the resolution the behavior remains the same as can be seen in Figure 4. On Figure 4 velocity \( \| u_h - u \|^2 \) and pressure \( \| p_h - p \|^2 \) errors are shown for different VOF grid resolutions 40 \( \times \) 40 (A), 80 \( \times \) 80 (B) 160 \( \times \) 160 (C) and 320 \( \times \) 320 (D) as a function of time. The error decreases with higher resolutions but does not vanish in time. The steady solution error oscillates depending from many factors such as temporal step and location of the interface.

For the three-dimensional case B the results are shown in Figures 5-7. The data and the geometry are the same as in the previous case A. In Figure 5 one can see surface plane with PLIC-VOF reconstruction and the projected color function \( C^*_h \) for 40 \( \times \) 40 \( \times \) 40 VOF grid resolution. In Figure 6 the pressure \( p_h \) and the velocity field \( u_h \) for 40 \( \times \) 40 \( \times \) 40 VOF grid resolution are shown on the left and right, respectively. Finally the velocity \( \| u_h - u \|^2_{L^2} \) error for different VOF grid resolutions 20 \( \times \) 20 \( \times \) 20 (A) and 40 \( \times \) 40 \( \times \) 40 (B) as a function of time is reported in Figure 7. As in the previous case the \( L^2 \)-norm error decreases with the resolution but oscillates in time as the reconstruction algorithm moves back and forward the plane inside the cell.
Figure 8. Interface evolution for $t = 0, 3.5, 6$ for VOF resolution $80 \times 80 \times 80$ and $\rho_1/\rho_2 = 100$, $\mu_1/\mu_2 = 100$, $\mu_1 = 0.002$.

Figure 9. Radius oscillation amplitude as a function of time for different density ratio $\rho_1/\rho_2 = 10$ (A), 100(B) for VOF resolution $64 \times 64$ and table of the time period $T$ for different resolutions and time step $dt = 0.025$ (C) and $dt = 0.01$ (D) in comparison with the analytic value for $\mu_2 = \mu_1 = 0.002$.

4.2. Two-dimensional droplet oscillations

In the previous section we have considered a simulation based on the Laplace’s law and test the behavior of our surface tension algorithm with respect to the generation of parasitic currents. The results for two and three-dimensional pressure-velocity field can be considered satisfactory. However, even if the spurious currents remain bounded for small time steps in the equilibrium solution for different mesh resolution and density ratio, we need to study the system during its evolution in time. In this section we consider the two-dimensional case. We consider a two-dimensional geometry, as shown in figure 8, with cylindrical coordinates $(r, \theta)$. The interface
Figure 10. Pressure time evolution for \( t = 0, 2, 3.5, 6 \) for VOF resolution \( 80 \times 80 \times 80 \) and \( \rho_1/\rho_2 = 100, \mu_1/\mu_2 = 100 \) \( \mu_1 = 0.002 \).

line \( r = r(\theta) \), in its initial configuration, is given by

\[
r = R (1 - \epsilon/4 + \epsilon P_2(\cos \theta)),
\]

where \( R \) is the droplet radius at equilibrium, \( \epsilon \) the relative deformation factor and \( P_2(x) \) the Legendre Polynomial of order 2 \([10]\). This test is focused on small oscillations released by the second spherical harmonic. In particular, the oscillations are characterized by an angular frequency \( \omega_2 \) given by the expression \([11]\)

\[
\omega_2^2 = \frac{6 \sigma}{(\rho_1 + \rho_2) R_0^3}.
\]  

(26)

Our simulations are performed over a two-dimensional unit square domain. Homogeneous Neumann boundary conditions are enforced over the rest of the boundary. The droplet with density \( \rho_1 \) and viscosity \( \mu_1 \) is surrounded by a medium with density \( \rho_2 \) and viscosity \( \mu_2 \). Different density ratios are obtained by fixing the reference phase density \( \rho_1 = 1 \) and changing the density \( \rho_2 \). The viscosity for reference phase is set to \( 2 \times 10^{-3} \). In this test we assume \( R = 0.195, \epsilon = 0.02, \sigma = 0.25, \rho_1/\rho_2 = 10-100, \mu_1/\mu_2 = 1 \) and \( \alpha = 0.0002 \). We also consider different time steps with \( dt = 0.025-0.01 \). In Figure 4.2, on the left, we plot the oscillating radial position of the interface point at \( \theta = 0 \) as a function of time for VOF resolution \( 64 \times 64 \) and different density ratios. One can see the results obtained for \( \rho_1/\rho_2 = 10 \) (case A) and for \( \rho_1/\rho_2 = 100 \) (case B).
On the right of Figure 4.2 we have the table of the time period $T = \frac{2\pi}{\omega_2}$ with $\rho_1/\rho_2 = 100$ for different resolutions and time step $dt = 0.025$ (C) and $dt = 0.01$ (D) in comparison with the analytic value. By changing the viscosity the oscillation are damping quickly. We consider the same geometry and we set $\rho_1/\rho_2 = 100$, $\mu_1/\mu_2 = 100$, $\mu_1 = 0.002$. The reconstructed interface evolution for $t = 0, 3.5, 6$ for VOF resolution $80 \times 80 \times 80$ is shown in Figure 8. In Figure 10 the time evolution of the pressure field for $t = 0, 2., 3.5, 6$ for VOF resolution $80 \times 80 \times 80$ is reported.

4.3. Three-dimensional droplet oscillations

![Figure 11. Interface reconstruction at $t = 0$.](image)

![Figure 12. Interface reconstruction at $t = 0.4$.](image)

![Figure 13. Color function on the central plane at $t = 1$.](image)

In this section we consider the two-dimensional case. We consider, in axisymmetric coordinates $(r, \theta)$, the initial ellipsoidal shape of figure 8, but now $r = R(1 + \epsilon P_2(\cos\theta))$, where $R$ is the radius of the spherical droplet at equilibrium, $\epsilon$ the relative deformation and $P_2(x)$ the Legendre Polynomial of order 2 [10]. In [11], an analytical solution for the small amplitude
oscillations of an incompressible inviscid droplet in the vacuum is obtained. The author wrote the fundamental mode of oscillation associated to the Legendre polynomial of order 2 a with the corresponding angular frequency

\[ \omega_2^2 = \frac{8\sigma}{\rho R^3}. \]  

(27)

Our simulations are performed over a three-dimensional cubic unit domain. Homogeneous Neumann boundary conditions are enforced over the rest of the boundary. As in the two-dimensional case the droplet with density \( \rho_1 \) and viscosity \( \mu_1 \) is surrounded by a media with density \( \rho_2 \) and viscosity \( \mu_2 \). The density and viscosity for reference phase are set to 1 and \( 2 \times 10^{-3} \), respectively. In this test we assume \( R = 0.195, \epsilon = 0.01, \sigma = 0.25, \rho_1/\rho_2 = 10\text{-}100, \mu_1/\mu_2 = 1 \) and \( \alpha = 0.0002 \).

| mesh       | analytic \( T \)  | \( \Delta t = 0.025 \) \( T \) |
|------------|-------------------|------------------|
| 32 \times 32 \times 32 | 3.182             | 3.361            |
| 64 \times 64 \times 64 | 3.182             | 3.255            |

Table 1. Oscillation period \( T \) for different mesh resolutions.

In Table 1 we show the oscillation period \( T \) for different mesh resolutions in comparison with the analytical one. By changing the viscosity the oscillation are damping quickly. We consider the same geometry and we set \( \rho_1/\rho_2 = 100, \mu_1/\mu_2 = 100 \) and \( \mu_1 = 0.002 \).

The results, obtained with a 32 \times 32 \times 32 mesh resolution and a time step \( dt = 0.025 \) are shown in Figures 11-13. For each Figure we report from the left to right the reconstructed interface, the color function and the pressure field on the central plane.

5. Conclusions

A new numerical approach for computing the surface tension term for two-phase incompressible flows in two and three-dimensional geometries has been proposed. This algorithm computes the surface tension tensor directly from the normal vector and a Galerkin projection of the indicator function. In the framework of a variational formulation of the Navier-Stokes equations it is possible to evaluate the surface tension force without the explicit evaluation of the local mean curvature. Classical tests, reproducing the solution of the static case and the dynamic of the droplet oscillations, are simulated in two and three-dimensional geometries. This method can be classified as continuous surface stress (CSS) method. For these methods the direct computation of the curvature is not necessary. Our proposed method has several advantages over the other CSS methods: the numerical implementation of the two-dimensional and three-dimensional cases is basically the same and no derivative of the tensor stress is required. On the contrary continuous surface force (CSF) methods, such as methods of smoothing the discontinuity of surface tension forces and convolution with a suitable kernel, need the curvature computation. This is a very hard task when the second derivative of the color function should be evaluated since the accuracy and convergence of the numerical evaluation of a distribution function remain an open question. The main weakness of our approach is the cost of computation: the extended field should be computed over all the domain and the projection in \( H^1 \) need the solution of a linear system. At the same time this projection can be used easily in non-Cartesian grids with moving meshes [5, 12].
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