SOLUTION OF A WAVE EQUATION BY A MIXED FINITE ELEMENT — FICTITIOUS DOMAIN METHOD

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Abstract — The main goal of this article is to investigate the capability of fictitious domain methods to simulate the scattering of linear waves by an obstacle whose shape does not fit the mesh. The space-time discretization relies on a combination of a mixed finite element method à la Raviart-Thomas with a fairly standard finite difference scheme for the time discretization. The numerical results described in the article point to a good performance of the numerical method investigated here.

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1. Introduction

In this article we discuss the numerical solution of a linear wave equation using a combined fictitious domain — a mixed finite element methodology. We consider a linear wave equation with a constant coefficient in a domain which is a rectangle containing a circular obstacle (Fig.1). The Dirichlet boundary conditions are imposed on the obstacle boundary, while the absorbing boundary conditions are prescribed on the external part of the boundary. We use a mixed variational formulation to construct a discrete problem. This discrete scheme is “explicit” in time and is of the lowest-order finite element approximation in space. To avoid the difficulties concerning the implementation of a discrete scheme in a curvilinear domain, we construct a finite element approximation of the mixed problem in a rectangular domain containing the original domain (see, e.g., [1]). The interface conditions are treated by Lagrange multipliers. The conjugate gradient algorithm (discussed in [2,3]) is used for solving the resulting system of linear algebraic equations (of the “saddle-point” type).

The paper is organized as follows: In Section 2 we formulate the wave problem, then provide its variational formulation and discretization, assuming that there is no obstacle. We introduce the obstacle and a fictitious domain method in Section 3. Finally, numerical results are presented in Section 4.
2. Formulation of the problem. Discretization

Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ with boundary $\Gamma_{ext}$. We consider then the following wave problem:

$$\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \Delta \phi = 0 \quad \text{in} \quad \Omega \times (0, T), \quad (1)$$

with the absorbing boundary condition

$$\frac{1}{c} \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial n} = 0 \quad \text{on} \quad \Gamma_{ext} \times (0, T), \quad (2)$$

and the initial conditions

$$\phi(0) = \phi_0, \quad \phi_t(0) = \phi_1. \quad (3)$$

Here $n$ is the outward unit normal vector on $\Gamma_{ext}$. Now let us consider an equivalent approach based on a mixed formulation. We introduce the new variables:

$$p = \nabla \phi, \quad v = \frac{\partial \phi}{\partial t}. \quad (4)$$

Clearly, $p$ and $v$ verify the following equations:

$$\frac{\partial p}{\partial t} = \nabla v \quad \text{and} \quad \frac{\partial v}{\partial t} = c^2 \nabla \cdot p \quad \text{in} \quad \Omega \times (0, T),$$

$$v + c p \cdot n = 0 \quad \text{on} \quad \Gamma_{ext} \times (0, T), \quad (5)$$

$$p(0) = \nabla \phi_0, \quad v(0) = \phi_1. \quad (6)$$

Multiplying the first equation in $(5)$ by $q \in H(\Omega, div) = \{ q \in (L^2(\Omega))^2 : \nabla \cdot q \in L^2(\Omega) \}$, the second equation by $w \in L^2(\Omega)$, and then applying the divergence theorem, we get the
variational formulation of problem (5)–(7):
\[
\int_\Omega \frac{\partial p}{\partial t} \cdot q dx + \int_\Omega v \nabla \cdot q dx + c \int_{\Gamma_{\text{ext}}} (p \cdot n)(q \cdot n) d\Gamma = 0, \quad \forall q \in H(\Omega, \text{div}),
\]
(8)
\[
\int_\Omega \frac{\partial v}{\partial t} w dx - c^2 \int_\Omega \nabla \cdot p w dx = 0, \quad \forall w \in L^2(\Omega),
\]
(9)
\[
p(0) = \nabla \phi_0, \quad v(0) = \phi_1.
\]
(10)
We do not specify the spaces where the functions \( p(x, t) \) and \( v(x, t) \) belong, because the main goal of the paper is to construct a discrete problem which approximates (5)–(7), retains the main properties of the original problem and is easy to implement.

One of the important properties of problem (5)–(7) is the energy dissipation property. Namely, let us denote by
\[
E(t) = \frac{1}{2} \int_\Omega |p(x, t)|^2 dx + \frac{1}{2c^2} \int_\Omega v^2(x, t) dx
\]
the energy of the system at time \( t > 0 \). Then it is easy to verify that \( dE/dt \) satisfy the inequality
\[
\frac{dE(t)}{dt} = -c \int_{\Gamma_{\text{ext}}} (p \cdot n)^2 d\Gamma \leq 0,
\]
(11)
therefore we have a monotonic decrease (dissipation) of the energy.

In order to get a fully discrete problem which should be easy to implement, we first construct a time “explicit” semi-discrete problem.

Let \( \Delta t > 0 \) be a time step and let \( p^n \approx p(n \Delta t), \ v^{n+1/2} \approx v((n+1/2)\Delta t) \) for \( n = 0, 1, 2, \ldots \) approximate the corresponding functions \( p(t) \) and \( v(t) \). The time discretization of problem (8)–(10) reads as follows:

For \( n \geq 1 \), find \((p^n, v^{n+1/2}) \in H(\Omega, \text{div}) \times L^2(\Omega)\) such that
\[
\int_\Omega \frac{p^n - p^{n-1}}{\Delta t} \cdot q dx + \int_\Omega v^{n-1/2} \nabla \cdot q dx + c \int_{\Gamma_{\text{ext}}} \left( \frac{p^n + p^{n-1}}{2} \cdot n \right)(q \cdot n) d\Gamma = 0,
\]
(12)
\[
\forall q \in H(\Omega, \text{div}),
\]
\[
\int_\Omega \frac{v^{n+1/2} - v^{n-1/2}}{\Delta t} w dx - c^2 \int_\Omega \nabla \cdot p^n w dx = 0, \quad \forall w \in L^2(\Omega),
\]
(13)
\[
p^0 = \nabla \phi_0, \quad \frac{v^{1/2} - v^0}{\Delta t/2} = c^2 \nabla \cdot p^0.
\]
(14)
Now, we construct a fully discrete problem.

For convenience we set \( \Omega \) to be a rectangular domain with boundaries parallel to the coordinate axes, namely \( \Omega = (0, x_L) \times (0, y_L) \). We define \( R_x : 0 = x_0 < x_1 < \ldots < x_{N_x} = x_L \) and \( R_y : 0 = y_0 < y_1 < \ldots < y_{N_y} = y_L \) be partitions of \([0, x_L]\) and \([0, y_L]\), respectively, and denote by \( \Delta_{i,j} \) a mesh cell - rectangle \([x_i, x_{i+1}] \times [y_j, y_{j+1}]\). We use \( T = \{ \Delta_{i,j} \} \) to denote the corresponding partition of \( \Omega \).
Let $P_h$ be a finite dimensional subspace of $H(\Omega, \text{div})$ constructed via the lowest-order Raviart-Thomas finite element method, namely,

$$P_h = \{ p_h \in H(\Omega, \text{div}) : (p_{1h}, p_{2h}) \in P_{1,0} \times P_{0,1} \text{ for each } \Delta_{i,j} \in \mathcal{T} \},$$

where $P_{k,l}$ is the space of polynomials of degree $k$ in $x$ and $l$ in $y$. By $V_h \subset L_2(\Omega)$ we denote the space of piecewise constant functions, which are constant on every $\Delta_{i,j}$.

Further, let $\Delta_{i,j} \in \mathcal{T}$, then we denote by

$$a^-_{i,j} = \left( x_i, \frac{1}{2}(y_j + y_{j+1}) \right), \quad a^+_{i,j} = \left( x_{i+1}, \frac{1}{2}(y_j + y_{j+1}) \right),$$

$$b^-_{i,j} = \left( \frac{1}{2}(x_i + x_{i+1}), y_j \right), \quad b^+_{i,j} = \left( \frac{1}{2}(x_i + x_{i+1}), y_{j+1} \right)$$

the mid–points of the corresponding edges (Fig. 2). The values of the fluxes $p_h \cdot n$ at the mid–points of the finite elements $\Delta_{i,j} \in \mathcal{T}$ are taken as degrees of freedom for $p_h \in P_h$. Thus, for rectangular elements they are $p_{1h}(a^-_{i,j}), p_{2h}(b^-_{i,j})$ etc. For $v_h \in V_h$ the values at the center of the cells $\Delta_{i,j} \in \mathcal{T}$ are taken as the degrees of freedom.

Further we denote by $P_h$ and $V_h$ the vectors whose components are the degrees of freedom for the corresponding functions $p_h \in P_h$ and $v_h \in V_h$.

We use the following quadrature formula to approximate $\int_\Omega p_h \cdot q_h \, dx$:

$$S_\Delta(p_h \cdot q_h) = \frac{1}{4} \text{meas } \Delta_{i,j} \left( p_{1h}(a^-_{i,j}) q_{1h}(a^-_{i,j}) + p_{1h}(a^+_{i,j}) q_{1h}(a^+_{i,j}) \right.$$

$$\left. + p_{2h}(b^-_{i,j}) q_{2h}(b^-_{i,j}) + p_{2h}(b^+_{i,j}) q_{2h}(b^+_{i,j}) \right);$$

the above integral is approximated then by

$$S_\Omega(p_h \cdot q_h) = \sum_{\Delta_{i,j} \in \mathcal{T}} S_\Delta(p_h \cdot q_h).$$
The space discretized formulation of problem (12)–(14) involves the pair \((p^n_h, v^{n+1/2}_h) \in P_h \times V_h\) satisfying

\[
S_\Omega \left( \frac{p^n_h - p^{n-1}_h}{\Delta t} \cdot \mathbf{q}_h \right) + \int_\Omega v^{n-1/2}_h \nabla \cdot \mathbf{q}_h \, dx + c \int_{\Gamma_{\text{ext}}} \left( \frac{p^n_h + p^{n-1}_h}{2} \cdot \mathbf{n} \right) (\mathbf{q}_h \cdot \mathbf{n}) \, d\Gamma = 0,
\]

\((\forall \mathbf{q}_h \in P_h, \forall \mathbf{v}_h \in V_h)\) \quad (15)

\[
\int_\Omega \frac{v^{n+1/2}_h - v^{n-1/2}_h}{\Delta t} w_h \, dx - c^2 \int_\Omega \nabla \cdot p^n_h w_h \, dx = 0,
\]

\((\forall \mathbf{w}_h \in V_h)\) \quad (16)

with the initial conditions

\[
p^0_h = (\nabla \phi_0)_h, \quad \frac{v^{1/2}_h - v^0_h}{\Delta t/2} = c^2 \nabla \cdot p^0_h,
\]

\((17)\)

where \((\nabla \phi_0)_h\) is an appropriate \(P_h\)-approximation of \((\nabla \phi_0)\).

Obviously, there exists a unique solution \((p^n_h, v^{n+1/2}_h)\) of problem (15), (16). Moreover, \((p^n_h, v^{n+1/2}_h)\) can be found recurrently from a system of linear algebraic equations, namely

\[
M_p P_h = G_1, \quad M_v V_h = G_2
\]

with \(M_p\) and \(M_v\) diagonal matrices defined by

\[
(M_p P_h, \mathbf{Q}) = S_\Omega \left( \frac{p^n_h}{\Delta t} \cdot \mathbf{q}_h \right) + c \int_{\Gamma_{\text{ext}}} \left( \frac{p^n_h}{2} \cdot \mathbf{n} \right) (\mathbf{q}_h \cdot \mathbf{n}) \, d\Gamma \quad (\forall \mathbf{p}_h \in P_h, \forall \mathbf{q}_h \in P_h),
\]

\[
(M_v V_h, \mathbf{W}) = \frac{1}{c^2} \int_\Omega \frac{v^{n+1/2}_h}{\Delta t} w_h \, dx \quad (\forall \mathbf{v}_h \in V_h, \forall \mathbf{w}_h \in V_h).
\]

Proposition 1. Let \(h_{\text{min}}\) denote the minimal step size. There exists a positive number \(\chi\) such that the condition

\[
\Delta t \leq \frac{\chi}{c} h_{\text{min}} \quad (18)
\]

ensures the positive definiteness of the quadratic form

\[
\varepsilon^{n+1} = \frac{1}{2} S_\Omega ((p^n_h)^2) + \frac{1}{2c^2} \int_\Omega (v^{n+1/2}_h)^2 \, dx - \frac{\Delta t}{2} \int_\Omega \nabla \cdot p^n_h v^{n+1/2}_h \, dx,
\]

\((19)\)

which we call the discrete energy. System (15)–(17) satisfies the discrete energy identity

\[
\varepsilon^{n+1} - \varepsilon^n = -c \Delta t \int_{\Gamma_{\text{ext}}} \left( \frac{p^n_h + p^{n-1}_h}{2} \cdot \mathbf{n} \right)^2 \, d\Gamma.
\]

\((20)\)

Thus,

\[
\varepsilon^{n+1} \leq \varepsilon^n, \forall n,
\]

and the discrete problem (15)–(17) is stable.
Proof. Let us choose \( q = \Delta t \frac{p_h^n + p_h^{n-1}}{2} \) in (15) and \( w_h = \Delta t \frac{v_h^{n+1/2} - v_h^{n-1/2}}{2\varepsilon} \) in (16). By adding these equalities we obtain identity (20).

Now, by direct calculations it is easy to get the inequality

\[
\int_{\Delta_{i,j}} |\nabla \cdot p_h|^2 \, dx \leq \frac{c_1}{h_{\min}} S_{\Delta_{i,j}}(|p_h|^2)
\]

for any \( p_h \in P_h \) and any \( \Delta_{i,j} \), with a constant \( c_1 \) which does not depend on the mesh size. From here we derive the estimate

\[
\left| \int_\Omega \nabla \cdot p_h^n v_h^{n+1/2} \, dx \right| \leq \frac{c_1}{h_{\min}} \left( S_{\Delta_{i,j}}(|p_h|^2) \right)^{1/2} \left( \int_\Omega |v_h^{n+1/2}|^2 \, dx \right)^{1/2},
\]

thus, under condition (18) with \( \chi = 2/c_1 \) we get the positive definiteness of \( \varepsilon^n \) for all \( n \). \( \square \)

3. A fictitious domain method with a boundary-supported Lagrange multiplier

Now we consider a domain \( \omega \subset \Omega \) with a piecewise smooth boundary \( \gamma \). Let equation (1) and the initial conditions (3) be valid in \( \Omega \setminus \bar{\omega} \) and problem (1)–(3) be completed by the following Dirichlet boundary condition on \( \gamma \):

\[
\phi = 0 \quad \text{on } \gamma \times (0, T).
\]

(21)

As before, we introduce the new functions \( p \) and \( v \) and get an initial boundary-value problem similar to (5)–(7) with the additional boundary condition

\[
v = 0 \quad \text{on } \gamma \times (0, T).
\]

(22)

Now, let the spaces \( P_h \) and \( V_h \) be defined as above and let \( \Lambda_\eta \) be a finite dimensional subspace of \( L_2(\gamma) \), which is defined below.

First, let us define a family \( T_\gamma \) of cells \( \Delta \in T \), which we call ”boundary elements”. In \( T_\gamma \) the following cells \( \Delta \) are included:

- \( \Delta \) such that \( \gamma \cap \hat{\Delta} \neq \emptyset \);
- if \( \gamma \cap \hat{\Delta} = \emptyset \), but \( \gamma \cap \partial \Delta \neq \emptyset \), then we include \( \Delta \) in \( T_\gamma \) only if \( \Delta \in \omega \) (Fig. 3).

Now, let \( \gamma = \{ \delta \gamma_i \} \) be a partitioning of \( \gamma \) such that every interval \( \delta \gamma_i \) contains at least one interval \( \gamma \cap \Delta \) of the intersection of \( \gamma \) with a cell \( \Delta \) from \( T_\gamma \). Below we denote by \( \Lambda_\eta \) a space of piecewise constant functions which are constant on every \( \delta \gamma_i \).

We also denote by \( \Pi_\gamma w_h \) a ”trace” of \( w_h \in V_h \) on \( \gamma \) which is defined as a function piecewise constant on \( \gamma \), which is equal to \( w_h \) on each \( \gamma \cap \Delta \) for \( \Delta \in T_\gamma \).
A fully discrete problem reads as follows: find a triple \((p_h^n, v_h^{n+1/2}, \lambda_\eta^n) \in P_h \times V_h \times \Lambda_\eta\) satisfying

\[
S_\Omega \left( \frac{p_h^n - p_h^{n-1}}{\Delta t} \cdot q_h \right) + \int_\Omega v_h^{n-1/2} \nabla \cdot q_h \, dx + c \int_{\Gamma_{ext}} \left( \frac{p_h^n + p_h^{n-1}}{2} \cdot n \right) (q_h \cdot n) \, d\Gamma = 0, \quad \forall q_h \in P_h,
\]

\[
\int_\Omega \frac{v_h^{n+1/2} - v_h^{n-1/2}}{\Delta t} w_h \, dx - c^2 \int_\Omega \nabla \cdot p_h^n w_h \, dx + c^2 \int_\gamma \lambda_\eta^n \Pi_\gamma w_h \, dx = 0, \quad \forall w_h \in V_h,
\]

\[
\int_\gamma \mu_\eta \Pi_\gamma v_h^{n+1/2} \, d\gamma = 0, \quad \forall \mu_\eta \in \Lambda_\eta,
\]

\[
p_0^h = (\nabla \phi_0)_h, \quad \frac{v_h^{1/2} - v_h^0}{\Delta t/2} = c^2 \nabla \cdot p_0^h.
\]

Denote by \(\tilde{p}_h, \tilde{v}_h\) and \(\tilde{\Lambda}_h\) the vectors whose components are the degrees of freedom for the corresponding functions \(p_h \in P_h, v_h \in V_h\) and \(\lambda_\eta \in \Lambda_\eta\).

Then for a fixed time \(t_n = n \Delta t\) the first equation (23) is a system of linear algebraic equations with a mass matrix \(M_v\) defined above, while for \(\tilde{V}_h\) and \(\tilde{\Lambda}_h\) we get a system of linear algebraic equations:

\[
\begin{align*}
M_v \tilde{V}_h + B^T \tilde{\Lambda}_h &= F \\
B \tilde{V}_h &= 0
\end{align*}
\]

with \(M_v\) defined above and the rectangular matrix \(B\) defined by

\[
(B \mathbf{W}, \Lambda_h) = \int_\gamma \lambda_\eta \Pi_\gamma w_h \, dx \quad \forall \lambda_\eta \in \Lambda_\eta, w_h \in V_h.
\]
Proposition 2. System (27) has a unique solution $(\tilde{V}_h, \tilde{\Lambda}_h)$.

Proof. Since $M_v$ is a regular matrix, it is enough to prove that $\text{Ker} B^T = \{0\}$. Suppose $B^T \tilde{\Lambda}_h = 0$, i.e.,

$$\int_{\gamma} \lambda_{\gamma} \Pi_{\gamma} w_h dx = 0 \quad \forall w_h \in V_h.$$

Let us take $w_h \in V_h$ such that $\Pi_{\gamma} w_h = 1$ for one $\Delta_{i,j} \in T_{\gamma}$, while it equals zero in other cells $\Delta$. Then $\lambda_{\gamma} = 0$ for a $\delta_{\gamma_k}$ which contains $\gamma \cap \Delta_{i,j}$. Because each $\delta_{\gamma_k}$ contains at least one interval $\gamma \cap \Delta$ with $\Delta \in T_{\gamma}$, then $\tilde{\Lambda}_h \equiv 0$. \hfill $\square$

Proposition 3. Let assumption (18) be fulfilled. Then system (23)–(25) satisfies the energy identity (20), where the discrete energy $\varepsilon_{n+1}$ is defined by (11). Thus, problem (23)–(25) is stable for $\Delta t \leq \frac{\chi}{c} h_{\text{min}}$.

Proof. We proceed as in the proof of Proposition 1. Namely, let us choose $q = \Delta t \frac{p_{n+1}^{h} + p_{n-1}^{h}}{2}$ in (23), $w_h = \Delta t \frac{v_{n+1/2}^{h} + v_{n-1/2}^{h}}{2 v^2}$ in (24) and $\mu_{\gamma} = \lambda_{h}^{n}$ in equation

$$\int_{\gamma} \mu_{\gamma} (\Pi_{\gamma} v_{n+1/2}^{h} + \Pi_{\gamma} v_{n-1/2}^{h}) d\gamma = 0.$$

Adding these equalities, we get identity (20). Now, the rest of the proof is the same as in Proposition 1. \hfill $\square$

In order to solve the system of linear equations (27) at each time step, we use the conjugate gradient algorithm in the form given in [3].

4. Numerical experiments

For our numerical experiments we have taken $\Omega = (0,1) \times (0,1)$. The initial condition for the function $\phi$ (Fig. 4) has been set to be

$$\phi_0(x, y) = \begin{cases} \cos^2 2\pi r, & \text{if } r \leq 1/4, \\ 0, & \text{otherwise,} \end{cases}$$

and $\phi_1 = 0$,

where $r = \sqrt{(x - 1/2)^2 + (y - 1/2)^2}$. We have considered the following cases:

1. Wave problem in the domain $\Omega$ without an obstacle (Fig. 5–7). For the \{h, $\Delta t$\} we took $h_x = h_y = 1/100$ and $\Delta t = 1/500$.

2. Wave problem in the domain $\Omega$ with $\omega$ as a quarter of a disc in the right lower corner.

We have taken $h_x = h_y = 1/100$ and $\Delta t = 1/500$. The radius of the disk has been set to $R(\omega) = 2/5$. We divided the boundary of $\omega$ into 25 intervals, i.e. $\delta_{\gamma} = 1/25$. The fictitious domain method described in Section 4 has been used. Figures 8–12 show the wave propagation from the initial state (Fig. 4) and its reflection from the obstacle. We observe very good agreement of the energy dissipation curves for different mesh and time step sizes in Fig. 13.
Solution of a wave equation.

Figure 4. Initial function $\phi$

Figure 5. Domain without an obstacle, time=0.5
Figure 6. Domain without an obstacle, time=0.7

Figure 7. Domain without an obstacle, time=1
Figure 8. Domain with a circular obstacle, time=0.2

Figure 9. Domain with a circular obstacle, time=0.4
Figure 10. Domain with a circular obstacle, time=0.6

Figure 11. Domain with a circular obstacle, time=0.8
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Figure 12. Domain with a circular obstacle, time=1

Figure 13. Energy dissipation for $\Delta t = 1/250, h = 1/50$ (dashed line) and $\Delta t = 1/500, h = 1/100$ (solid line)
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