BFV extensions and their obstructions in mechanical systems with Lie-2 symmetry

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We study mechanical models with first class constraints, not restricting to only regular and irreducible ones. We show that while the BFV charge always exists, the BFV extension of the Hamiltonian may be obstructed. Somewhat astonishingly this happens already for a free particle in $\mathbb{R}^3$ when constraining its angular momentum. We determine a complex whose cohomology governs the extendability and provide examples with interesting geometrical extensions.

INTRODUCTION

One of the main assumptions in textbooks about constrained systems—see, e.g., [1]—is that the constraint functions $G_a$ are regular on the unconstrained symplectic manifold. Important existence theorems about BRST-BV [2, 3] or BFV [4, 5] extensions have been proven in this setting only. But not a single physically realistic system satisfies this condition. (In a system where it is satisfied, one hardly needs a gauge theoretic description, albeit the non-linearity of the quotient space, if smooth, may, admittedly, also pose problems.) In realistic situations, group actions often have fixed points and there the corresponding constraints are not regular. In Yang-Mills gauge theories, for example, the constraints become non-regular at reducible connections.

On the other hand, without irreducibility and, in particular, regularity assumptions, often the BV and BFV extensions become hard mathematical problems. Let us illustrate this statement in the finite dimensional setting: Suppose you are given a set $v_a$ of vector fields on a manifold $M$ satisfying

$$[v_a, v_b] = C^c_{ab} v_c$$

for some functions $C^c_{ab}$. Then the corresponding ”BRST differential” $Q$ is readily written down as

$$Q = \xi^a v_a - \frac{1}{2} C^a_{bc} \xi^b \partial_c$$

where $\xi^a$ are the ”ghosts”, odd coordinates on an extended graded manifold. At first sight, a short calculation, using (1) together with the Jacobi identity for the Lie bracket of vector fields, seems to establish $Q^2 = 0$. However, if the vector fields $v_a$ are not linearly independent at each $x \in M$, there exist functions $t^a_f$ such that

$$t^a_f v_a = 0. \quad (3)$$

This happens already for the standard three vector fields $L_a$ generating rotations in $\mathbb{R}^3$ where $x^a L_a = 0$. Then the structure functions $C^a_{bc}$ entering (1) are not unique:

$$C^a_{bc} \sim C^a_{bc} + t^a_f B^f_{bc}. \quad (4)$$

The question if, within this equivalence class, there exists a choice of the structure functions $C^a_{bc}$ such that $Q$ as defined in (2) squares to zero is, by the work of Vaintrob [6], equivalent to the question if the singular foliation generated by the vector fields $v_a$ comes from a Lie algebroid covering it. [22] And this question is, as of this day, an unsolved mathematical problem.

In the present letter, we want to consider a Hamiltonian constrained system defined on a cotangent bundle $T^*M$, with first class constraints $G_a$ linear in the momenta and a Hamiltonian $H$ quadratic in them. In fact, linear constraints are always of the form

$$G_a(x, p) \equiv v^i_a(x) p_i. \quad (5)$$

So we can identify them with the vector fields $v_a$ on $M$ above, since the first class property of the constraints $G_a$ is tantamount to precisely (1). We do not restrict to merely regular constraints—otherwise one dealt with regular foliations (on $M$ as well as $T^*M$) only—but we permit leaves of different dimensions in principle. We consider reducibilities of one level between the constraints: [4] captures all dependencies between the vector fields and there are no dependencies between the functions $t^a_f$.

In more mathematical terms, this means that one has a sequence of vector bundles

$$0 \to F \xrightarrow{\iota} E \xrightarrow{\rho} TM, \quad (6)$$

such that $0 \to \Gamma(F) \to \Gamma(E) \to \Gamma(0)$ is exact. Here $F$ denotes the subset of vector fields obtained as the image of $\rho$, a generating set of which corresponds to the vector fields $v_a = \rho(e_a)$. Likewise, the map $t$ gives rise to the functions $t^a_f$ after the choice of local bases in $e_a$ and $b_f$ in $E$ and $F$, respectively.

And now the situation is different: Under these assumptions, it has been proven recently [11] that there does exist a homological vector field $Q$, an extension of (2), which describes a Lie 2-algebroid covering the singular foliation generated by the vector fields $v_a$. And its Hamiltonian lift to the cotangent bundle of the graded space underlying $Q$ yields precisely the BFV charge $Q_{BFV}$ of the constrained system, including the ghost-for-ghosts required from the reducibility of the constraints. The condition $Q^2 \equiv \frac{1}{2}[Q, Q] = 0$ gives rise—for free due to the property of a Hamiltonian lift—to the BFV master equation

$$(Q_{BFV}, Q_{BFV})_{BFV} = 0. \quad (7)$$
What we want to address here is the question of the existence of a BFV extension $H_{BFV}$ of the Hamiltonian

$$\text{Ham} = \frac{1}{2} g^{ij}(x)p_i p_j,$$

(8)

corresponding to a metric $g$ on $M$. We assume the usual compatibility of the constraints with the dynamics,

$$\{G_a, \text{Ham}\} = \omega^b_a G_b$$

(9)

for some functions $\omega^b_a$ on $T^*M$. This equips the base manifold $M$ with the structure of a singular Riemannian foliation (see, e.g., [12]). An extension such that one has

$$(Q_{BFV}, \mathcal{H}_{BFV})_{BFV} = 0$$

(10)

is the required second corner stone in the BFV formalism.

Since the constraints are not assumed to be regular, we can again not rely on any textbook proof for the existence. But this time, it will even not hold true in general. This will be illustrated by a well-known mechanical system: a free particle in $\mathbb{R}^3$, whose angular momenta we use as constraints. The Hamiltonian does not have a BFV extension in this case.

The central technical tool will be another sequence of vector bundles, but this one turns out to not always be exact, even not on the level of sections. As the complex \([\xi, \eta, 0]\) governs the non-uniqueness of the structure functions \([4]\), this second complex determines, in addition to the obstructions of an extension, also the non-uniqueness of the coefficient functions $\omega^b_a$. The presentation will be kept short, more details will be provided elsewhere.

**BFV Phase Space and Charge**

To construct the BFV phase space $\mathcal{M}_{BFV}$, one extends the classical variables $(x^i, p_i) \in T^*M$ by a ghost $\xi^a$ (odd and of degree one) for each of the constraints and a ghost-for-ghosts $\eta^i$ (even and of degree two) for each of the dependencies $t^a_j G_a \equiv 0$ (which here follow directly from \([4]\) or \([9]\)). Both of the latter two ghost families are accompanied by their momenta of opposite degrees. Together they give rise to the BFV symplectic form

$$\omega_{BFV} = dx^i \wedge dp_i + d\xi^a \wedge d\pi_a + d\eta^i \wedge dP_I.$$

(11)

It is of degree zero and thus also the graded Poisson bracket $(\cdot, \cdot)_{BFV}$ it induces.

The usual procedure to construct a BFV charge is to let it start as $Q_{BFV} := \xi^a G_a + \eta^i t^a_j \pi_a - \frac{1}{2} \xi^a \xi^b C^c_{ab} \pi_c + \ldots$ and to add higher order terms in $\xi$ and $\eta$ preserving the total degree so that (7) is satisfied. If we pose the additional condition, which is admissible here, that it should be linear in the momenta, the most general expression obtained in this way is of the form

$$Q_{BFV} = (\xi^a \rho^a) p_i + (\eta^i \eta^j - \frac{1}{2} \xi^a \xi^b C^c_{ab}) \pi_a + \left(\frac{1}{2} \xi^a \xi^b \xi^c \eta_{abc} - \Gamma^I_{Ja} \eta^i \xi^a\right) P_I,$$

(12)

where we use the notation $\rho^a_i \equiv v^a_i$. Viewing the vector fields $v_a$ as functions on $T^*M$, we obtain $G_a$; likewise, viewing the vector field $Q$ on the graded manifold

$$\mathcal{M} := E[1] \oplus F[2],$$

(13)

as a function on its cotangent bundle $T^*\mathcal{M} \cong \mathcal{M}_{BFV}$, yields (12). The numbers in the brackets in (13) denote the degrees carried by the local fiber-linear coordinates $\xi^a$ and $\eta^i$ on $E$ and $F$, respectively. In fact, every homological vector field $Q$ of degree 1 on $\mathcal{M}$ defines a Lie 2-algebroid structure (see, e.g., [13]) on the vector bundle $E \oplus F$—much as (2) defines a Lie algebraic on $E$ \([6]\).

The coefficient functions entering (12) all have some algebraic and/or geometrical meaning: One way of viewing them is that the terms linear in the ghosts $(\xi, \eta)$ give rise to a complex \([6]\), those quadratic in them are 2-brackets, which are not tensorial but satisfy a Leibniz rule, and the cubic one is a 3-bracket, which, if non-zero, reflects the fact that the 2-bracket between sections of $E$ then does not satisfy the Jacobi identity. For some introduction to Lie 2-algebras see \([8]\), for the more general $L_\infty$-algebras see \([9, 10]\). Alternatively, we may view, e.g., the coefficients $\Gamma^I_{Ja}$ as a local expression for an $E$-covariant derivative on $F$. The tensor $h \in \Gamma(F \otimes \Lambda^2 E^*)$ then satisfies $E Dh = 0$, where $E D$ denotes the corresponding exterior covariant $E$-derivative. There is a vast literature on the geometry of Lie $\infty$-algebras, see, e.g., \([14, 10]\); for more details about the geometry of Lie 2-algebroids see \([13, 17]\).

The fact that such a Lie 2-algebroid structure—and thus a BFV charge satisfying (7)—exists for every singular foliation generated by vector fields $v_a$ on $M$ has been proven in \([11]\).

The classical constrained system on $T^*M$ considered here is already uniquely determined by the underlying singular foliation. This is in contrast to the coefficient functions entering (12): they are constrained by $Q^2 = 0$ but, at each step of an extension, they are not unique. This should be also reflected on the BFV level: Indeed, e.g. a change \([4]\) of the almost Lie bracket on $E$ can be obtained by lifting the degree preserving diffeomorphism

$$\eta^I \mapsto \eta^I + \frac{1}{2} B_{abc} \xi^a \xi^b \xi^c$$

(14)

from $\mathcal{M}$ to $T^*\mathcal{M}$. Such a change of the structure functions $C^a_{bc}$ does not come for free: also other quantities entering $Q_{BFV}$ are then changed correspondingly. E.g., $h$ receives an additive contribution by the $E$-covariant exterior derivative of $B \in \Gamma(F \otimes \Lambda^2 E^*)$, $h \mapsto h + E DB \ldots$, where the dots denote terms quadratic in $B$ (see [13]).

**Geometrical Interpretation of $H_{BFV}$**

We first observe that the BFV bracket decreases the polynomial degree $\text{pol}$ of momenta $(p_i, \pi_a, P_I)$. Since, in addition, both $Q_{BFV}$ and Ham are homogeneous with
with 2-brackets and one 3-bracket. A (non-degenerate) 
H-structure equips it in the lowest order with a metric \( g \) 
on the base and in the next order—the terms linear in the momenta \( p_i \) or \( \Omega \)—with a connection defined on all of \( \mathcal{Q} \) (note that we can always equip \( TM \) with the canonical Levi-Civita connection of \( g \)).

In the case of an almost HQ-structure, where we have both \( Q \) and \( H \), compatibility tensors—between connections and brackets, for example—follow by looking at the components of \( \mathcal{L}_Q H \). To obtain covariant formulas on the nose, it is useful to re-express \( \mathcal{L}_Q H \) in terms of the variables \( \{ \xi_i \} \). Then, e.g., the new coefficient of the term quadratic in \( \xi_i \), \( \xi_i \xi_j \), has the geometric interpretation of an \( E \)-torsion \( E^T \in \Gamma(E \otimes \Lambda^2 E^*) \) [18]. The bracket of two covariant vector fields \( \mathcal{L}_Q H \) yields the curvatures of \( \nabla \), both on \( E \) and \( F \) (as well as a contribution proportional to \( D^\gamma \)). To lowest orders we find:

\[
\mathcal{L}_Q H = \frac{1}{2} \xi^a (E \nabla_{\nu} g)_{ij} \partial^\nu \partial^\gamma (E \nabla_{\nu} g)_{ij} - \frac{1}{2} \xi^a \xi^c \left( S_{\nu \rho}^{\alpha \beta} - \xi^b \xi^c \right) \partial^\alpha (E \nabla_{\nu} g)_{ij} \partial^\beta (E \nabla_{\nu} g)_{ij} + b^i (E \nabla_{\nu} g)_{ij} - \xi^b \xi^c \partial^\nu \partial^\gamma (E \nabla_{\nu} g)_{ij} + \ldots .
\]  

Here \( \partial^\nu \) is the vector field corresponding to \( \mathcal{L}_Q H \) and \( E \nabla \) denotes the \( E \)-covariant derivative acting on \( TM \) according to \( E \nabla_{\nu} s^i = \left[ \rho(s), v \right] + \rho(\nabla_{\nu} s^i), s \in \Gamma(E), v \in \Gamma(TM) \). \( \Sigma_{\nu \rho}^{ab} \) has been decomposed according to its degree, \( \Sigma_{\nu \rho}^{ab} = \frac{1}{2} \Sigma_{\nu \rho}^{e \xi \xi} \xi^e + \Sigma_{\nu \rho}^{a b} \eta^a \), a semicolon denotes a covariant derivative, and we introduced the abbreviation

\[
S_{\nu \rho}^{ab} = E T_{\nu \rho}^{ab} - \rho^b \nabla_{\nu} g_{ij} - \rho^a \nabla_{\rho} g_{ij}.
\]  

The tensor \( S \) measures the compatibility of the 2-bracket and the connection \( \nabla \) on \( E \), see [18]. As for Jacobi identities of the brackets in a higher Lie algebroid, also here we do not expect [18] to ensure \( S \) to vanish on the nose, but to instead do so up to some appropriate boundary term only; what this means precisely, is subject of the next section.

Let us mention that the current method is efficient in obtaining Bianchi type of identities for quantities such as \( S, Q^2 \) = 0 implies that the application of another \( \mathcal{L}_Q \) to \( S \) vanishes identically. From this one can read off, for example, that \( S \) satisfies

\[
E D S + \nabla (\xi, h) = 0 .
\]  

Here \( E D \) denotes the \( E \)-exterior covariant derivative associated naturally to \( E \nabla \) on \( TM \) it acts as specified above, on \( E \) according to \( E \nabla_{\nu} s^i = \left( [s, s] \right)_{\nu} + \nabla_{\nu} (s, s), s, s \in \Gamma(E) \), and on elements of \( \Lambda^2 E^* \) by a straightforward generalization of the Cartan formula for the de Rahm differential (13, 19). The identity (22) specialized to a Lie algebroid, where \( h = 0 \) was of essential importance in the construction of the BV-extension [20] and the above derivation constitutes a significant simplification of the one provided there.
**COMPLEX GOVERNING THE EXTENSION**

In the construction of \([12]\), the sequence of vector bundles \([3]\) and its exactness on the level of sections plays a crucial role. There is a similar sequence which governs the extension problem \([3, 19, 10]\), but despite being constructed from the previous one, this one is in general no longer exact on the level of sections.

Let us denote the complex \([8]\) by \(E^\bullet\) (so, in particular, \(E^0 = TM, E^{-1} = E, \) and \(E^{-2} = F\)) and tensor it with itself shifted to the right, \(F^\bullet := E^\bullet \otimes E^\bullet[-1]\), where, by definition, \(E^[-1] = E^{-1}\). At degree zero, e.g., one has \(F^0 = TM \otimes E \oplus E \otimes TM\), where in the first term both factors carry degree 0, while in the second one, elements in \(E\) enter with degree -1 and those in \(TM\) with degree +1, again adding up to 0.

By a standard construction (see, e.g., \([21]\)), \(F^\bullet\) is again a complex. This, however, is not yet the sequence \(G^\bullet\) we are interested in:

\[
G^\bullet := 0 \rightarrow S^2F \xrightarrow{\delta} F \otimes E \xrightarrow{\delta} F \otimes TM \oplus \Lambda^2E \xrightarrow{\delta} E \otimes TM \rightarrow S^2TM,
\]

where the degrees are such that \(G^0 = S^2TM, G^{-1} = E \otimes TM, \) etc. Now one observes that \(G^\bullet \rightarrow F^\bullet\) [1]. For example, typical elements \(\Phi \in F^0\) and \(\varphi \in G^{-1}\) are of the form \(\Phi = \Phi^a \partial_\alpha \otimes e_a + \Phi^a \otimes \partial_\alpha\) and \(\varphi = \varphi^a e_a \otimes \partial_\alpha\), respectively, and then \(G^{-1}\) is embedded into \(F^0\) by the diagonal map, \(\Phi^a \rightarrow \varphi^a, \Phi^a \rightarrow \varphi^a\). The codifferential \(\delta\) is easily identified with

\[
\delta := \rho^a b_i \frac{\partial}{\partial a_i} + t^a_i \pi_a \frac{\partial}{\partial y^a},
\]

when replacing the vector fields \([17]\), which provide a basis for \(G^\bullet\), by the corresponding frame.

In general, for the cohomology of a tensor product of two complexes there is a Künneth formula. It says that even if the cohomology of each of the complexes is trivial, which is the case here, there can still be a non-zero contribution called "torsion". Below we will provide an example that, in general, \(G^\bullet\) is not exact, even not on the level of sections. It is remarkable that \(\delta\) also generates the coboundary operator of the complex \([8]\). In the general construction of \(B(F)V\) with regular constraints \([1]\), it is called the Koszul-Tate differential. But while in the present context of non-regular constraints, it has no cohomology upon restriction to fiber-linear functions on \(T^*M\), it can change upon restriction to quadratic ones.

On the other hand, if \(G^\bullet\) is exact, then the existence of the BFV extension \(H_{BFV}\) is guaranteed. This follows from a standard consideration: \((Q_{BFV}, \cdot) = \delta + X_{rest}\). At each step when adding terms from the right to the left in \([24]\), one finds an expression that is already \(\delta\)-closed. Now, \(\delta\) having no cohomology, one can always add a \(\delta\)-exact contribution from the next level so as to cancel it. In general, the resulting expression for \(H_{BFV}\) will contain all possible terms compatible with degrees, as is also the case for \([12]\); together they then define an HQ-structure on \([13]\).

The absence of obstructions also leads to remarkable geometrical formulas. To illustrate this, assume that

\[
H^1(\Gamma(G^\bullet), \delta) = 0.
\]

Then for every \(\varphi \in \Gamma(G^{-1})\) such that \(\delta \varphi = 0\), there is some \(\psi \equiv \Psi^i b_i \otimes \partial_\alpha + \frac{1}{2} M^{ab} e_a \wedge e_b \in \Gamma(G^{-2})\) such that \(\varphi = \delta \psi\). Concretely,

\[
\rho^a \varphi^a + \rho^a \varphi^a = 0 \Rightarrow \varphi^a = t^a_i \psi^i + \rho^a_i M^{ab},
\]

where \(M^{ab}(x) = -M^{ba}(x)\). The exactness at a given degree is preserved if one tensors a complex with a fixed \(C^\infty(M)\)-module; this permits adding spectator indices to the quantities in \([26]\). Using a connection \(\nabla\) such that \(E \nabla g = 0\), the tensor \(S \in \Gamma(E \otimes \Lambda^2E^* \otimes T^*M)\) defined in \([21]\) is always \(\delta\)-closed and, assuming \([25]\), can therefore be decomposed into some \(\gamma \in \Gamma(F \otimes T^*M \otimes \Lambda^2E^*)\) and \(\Sigma \in \Gamma(\Lambda^2E \otimes \Lambda^2E^*)\) as follows:

\[
S_{abc} = t^a_i g_{bc} + \rho^a_i g_{ij} \Sigma^{da}.
\]

There are, in general, no explicit expressions for these tensors. In fact, they are even not uniquely determined: We can change \(\gamma\) and \(\Sigma\) simultaneously by

\[
\Sigma_{ab} \rightarrow \Sigma_{ab} + t^a_i V_{cd} - t^a_i V_{cd}b_i, \quad \gamma_{abc} \rightarrow \gamma_{abc} + \gamma_{abc}c_i V_{ab},
\]

for any \(V \in \Gamma(F \otimes E \otimes \Lambda^2E^*)\) without changing \(S\).

As another application, we return to the initial compatibility condition \([19]\). It can be shown \([12]\) that \([19]\) holds true in arbitrary local patches iff there exists a connection \(\nabla\) on \(E\) such that its induced \(E\)-connection annihilates the metric, \(E \nabla g = 0\). If \(\Gamma^a_{\alpha\beta}\) denote the connection coefficients of \(\nabla\), we can choose \(\omega^a_{\alpha\beta} := \Gamma^a_{\alpha\beta}(x) g^{ij} p_j\). The difference between two connections is a tensor field and then \([19]\) implies that this tensor is \(\delta\)-closed. This provides the following equivalence for the choice of the connection coefficients (\(M^a_{bc} = -M^a_{cb}\))

\[
\omega^a_{\alpha\beta} \sim \omega^a_{\alpha\beta} + t^a_i \Psi_{\alpha\beta} + \rho^a_i M^{bc}
\]

and if \([24]\) holds true, these are all the ambiguities.

Finally, the identity \(\{ t^a_i \Gamma_{Ga}, H \} = 0\) yields

\[
t^a_i \Gamma_{Ga} \rightarrow t^a_i \Gamma_{Ga} + \rho^a_i M^{bc}.
\]

Thus, \(t^a_i \Gamma_{Ga}\) satisfies the condition of \([24]\) and, if \(H^1 = 0\), there exist tensors \(\Psi\) and \(M\) such that

\[
t^a_i \Gamma_{Ga} = t^a_i \Psi_{Ga} + \rho^a_i M^{ab}.
\]
EXAMPLES

The prototype of a singular constraint $T^*\mathbb{R} \cong \mathbb{R}^2$ is $G = xp$. It is singular at the origin, the constraint surface has the shape of a cross. The corresponding BFV charge is simply $Q_{BFV} = \xi xp$. There is no compatible (non-degenerate) Hamiltonian in this case. If we admit a Hamiltonian $\text{Ham} = \frac{1}{2}h(x)p^2$, dropping the condition that $h = 1/g$, then a BFV extension exists as long as $h = O(x^2)$: $H_{BFV} = \frac{1}{2}h(x)p^2 + x^2 \left( \frac{\partial}{\partial x} \right)^2 \xi xp$.

More generally, if $t = 0$, the anchor map $\rho: E \to TM$ needs to be injective on a dense open subset of $M$ (this follows from injectivity of $\rho$: $\Gamma(E) \to \Gamma(TM)$) and, if one insists on a non-degenerate metric $g$ satisfying (9), even everywhere. Then the foliation on $M$ and the constraints (5) on $T^* M$ are regular.

Consider even $E = TM$, $\rho = \text{id}$. While the BFV charge is very simple in this case, $Q_{BFV} = \xi^i p^i$, there still remains some geometry in the BFV extension of the Hamiltonian: To satisfy the compatibility (9), we need a connection $\nabla$ on $TM$ such that

$$g_{ij,k} + T_{ij,k} + T_{jik} = 0,$$

(32)

where $T_{ijk} = g_{il}T^l_{jk}$; here $T^l_{jk}$ denote the components of the torsion tensor. The Levi-Civita connection satisfies this condition evidently; the ambiguity (29) translates into the freedom of a torsion of the connection, but maintains (22) instead of metricity. One then finds

$$H_{BFV} = \frac{1}{2}g^{ij}_{\pi_1} \nabla_i \nabla_j + \frac{1}{2} R^k_{i j k l} \xi^i \xi^j \pi_k \pi_l + \frac{1}{2} T^i_{j k} : \xi^i \xi^j \pi_k \pi_l$$

(33)

where $R^i_{j k l}$ denote the components of the Riemann tensor. Choosing the Levi-Civita connection in $p_i^\nabla = p_i + \Gamma^i_{kl} \xi^k j_{\pi j}$, (32) is satisfied identically, and the last term in (33) disappears.

Let us now consider the phase space $T^* \mathbb{R}^3$ with constrained angular momentum, $G_a = \varepsilon_{abc} x^b p^c$. Then one has

$$Q_{BFV} = \varepsilon_{abc} x^a b^c + \frac{1}{2} \varepsilon_{abc} \xi^a \xi^b \pi_c + \eta \varepsilon^{a} \pi_a .$$

(34)

It is easy to verify that it squares to zero. The last term is often suppressed, but it is essential to take the dependancy of the constraints $x^a G_a = 0$ into account: Ghosts-for-ghosts—here just one, $\eta$—are needed in general to ensure the correct $Q_{BFV}$-cohomology of observables already in the regular case (see, e.g., [1]).

In this example, $M = \mathbb{R}^3$, $E = M \times \mathbb{R}^3$, $F = M \times \mathbb{R}$, and the maps $\rho$ and $t$ in (34) can be identified with the sections $\rho = \varepsilon_{abc} x^b e^c \otimes \frac{\partial}{\partial x^a}$ and $t = x^a b^* \otimes e_a$, where $(e^a)^3_{a=1}$ and $b^*$ denote the basis in $E^*$ and $F^*$, respectively. The kernel of $\rho$ is one-dimensional outside the origin, while it is all of $E_0 = \mathbb{R}^3$ at the origin $0 \in M$. The map $t$ spans the radial vectors in ker $\rho$ for all $x \neq 0$, but, for continuity reasons, vanishes at 0. Thus, the complex (34) has no cohomology outside the origin, but $H^1_{t=0}(\mathcal{C}, \delta) = \mathbb{R}^3$. And still, again for continuity reasons (every radial vector field has to vanish at the origin), it is exact on the level of sections: $H^1(\Gamma(\mathcal{C}^*), \delta) = 0$. Correspondingly, the extension problem for the BFV charge (34) is not obstructed.

Let us equip the bundles $E$ and $F$ with their canonical flat connections. Then one has

$$\nabla t = dx^a \otimes b^* \otimes e_a \simeq \frac{\partial}{\partial x^a} \otimes b^* \otimes e_a ,$$

(35)

using the standard metric of $M = \mathbb{R}^3$ in the identification, which is an element in $\Gamma(E \otimes TM) \otimes_{C^\infty(M)} \Gamma(F^*)$. It is easy to see that $\delta(\nabla t) = 0$ which is equivalent to (30): applying $\rho$ to $e_a$ and symmetrizing over the two ensuing entries in $TM$ gives zero due to the contraction with the $\varepsilon$-tensor. But $\nabla t$ cannot be $\delta$ exact, i.e. it cannot be of the form (31), since both $t$ and $\rho$ vanish at the origin.

This shows, on the one hand, that here

$$H^1(\Gamma(\mathcal{C}^*), \delta) \neq 0 ,$$

(36)

and, on the other hand, that the BFV extension of

$$\text{Ham} = \frac{1}{2} p_a p_a$$

(37)

is obstructed: As we learn from the last line in (20), we need $\nabla t$ to be exact for the BFV-extension of (37).

It is remarkable that such a simple system—angular momenta as constraints together with the rotation-invariant Hamiltonian (37), so (9) being satisfied with $\omega_a = 0$—does not have a proper BFV formulation when taking the reducibility of the constraints into account.

As a final example, we keep the constraint structure (12) with one-level reducibilities, but truncate the expansion (19) for the BFV extension of the Hamiltonian:

$$H_{BFV} = \frac{1}{2} g^{ij}_{\pi_1} \nabla_i \nabla_j - \frac{1}{2} \varepsilon_{abc} \xi^a \xi^b \pi_a \pi_b ,$$

(38)

where $p^i_1$ is given by (18) and $\pi_a = \varepsilon_a - \lambda^i_{abc} \varepsilon_d \varepsilon^e \nabla_d \varepsilon^e \nabla_b \pi^c$, is one-dimensional outside the origin, while it is all of $E_0 = \mathbb{R}^3$ at the origin $0 \in M$. The map $t$ spans the radial vectors in ker $\rho$ for all $x \neq 0$, but, for continuity reasons, vanishes at 0. Thus, the complex (34) has no cohomology outside the origin, but $H^1_{t=0}(\mathcal{C}, \delta) = \mathbb{R}^3$. And still, again for continuity reasons (every radial vector field has to vanish at the origin), it is exact on the level of sections: $H^1(\Gamma(\mathcal{C}^*), \delta) = 0$. Correspondingly, the extension problem for the BFV charge (34) is not obstructed.

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$$\nabla t = dx^a \otimes b^* \otimes e_a \simeq \frac{\partial}{\partial x^a} \otimes b^* \otimes e_a ,$$

(35)

using the standard metric of $M = \mathbb{R}^3$ in the identification, which is an element in $\Gamma(E \otimes TM) \otimes_{C^\infty(M)} \Gamma(F^*)$. It is easy to see that $\delta(\nabla t) = 0$ which is equivalent to (30): applying $\rho$ to $e_a$ and symmetrizing over the two ensuing entries in $TM$ gives zero due to the contraction with the $\varepsilon$-tensor. But $\nabla t$ cannot be $\delta$ exact, i.e. it cannot be of the form (31), since both $t$ and $\rho$ vanish at the origin.

This shows, on the one hand, that here

$$H^1(\Gamma(\mathcal{C}^*), \delta) \neq 0 ,$$

(36)

and, on the other hand, that the BFV extension of

$$\text{Ham} = \frac{1}{2} p_a p_a$$

(37)

is obstructed: As we learn from the last line in (20), we need $\nabla t$ to be exact for the BFV-extension of (37).

As a final example, we keep the constraint structure (12) with one-level reducibilities, but truncate the expansion (19) for the BFV extension of the Hamiltonian:

$$H_{BFV} = \frac{1}{2} g^{ij}_{\pi_1} \nabla_i \nabla_j - \frac{1}{2} \varepsilon_{abc} \xi^a \xi^b \pi_a \pi_b ,$$

(38)

where $p^i_1$ is given by (18) and $\pi_a = \varepsilon_a - \lambda^i_{abc} \varepsilon_d \varepsilon^e \nabla_d \varepsilon^e \nabla_b \pi^c$, is one-dimensional outside the origin, while it is all of $E_0 = \mathbb{R}^3$ at the origin $0 \in M$. The map $t$ spans the radial vectors in ker $\rho$ for all $x \neq 0$, but, for continuity reasons, vanishes at 0. Thus, the complex (34) has no cohomology outside the origin, but $H^1_{t=0}(\mathcal{C}, \delta) = \mathbb{R}^3$. And

This implies in particular that $t$ needs to have constant rank: but also vice versa: whenever $t: E \to H$ has constant rank, we can choose a connection $\nabla$ on $E$ such that the section $t$ becomes covariantly constant—using (29) so as to keep $E \nabla g = 0$ unaltered. The last condition is not overly restrictive, since always $\delta S = 0$.

A priori, there are many more conditions to be satisfied for (10) to hold, but with the following trick, one may show that they can be all reduced to only one: For this we first remark that within (39) we can always change the two quantities $\gamma$ and $\Sigma$—which are then to enter the extension (36)—according to (28). Since $t$ has a constant rank, we can choose some $C \subset E$ such that $E = C \oplus t(F)$. 
One may now see that the transformations (28) can be used to always assure \( \Sigma \in \Gamma(\Lambda^2 C \otimes \Lambda^2 E^\ast) \). Then the only remaining condition is:

\[
ED_\lambda \Sigma = 0. \tag{40}
\]

Here \( ED_\lambda \) is defined as \( ED \) but replacing the 2-bracket on \( E \) by \([s,s']_\lambda = [s,s'] + t(\lambda(s,s'))\). We finally remark that from (22) one can conclude an equation similar to (40): One first observes that the operators \( ED \) and (21) commute, \([ED,\delta] = 0\). Using (22) one then finds that \( \delta(ED(\gamma + \Sigma)) = -(t,\nabla h)_F \), where the \( h\)-contribution only adds to \( ED\gamma \). If \( H^2(\Gamma(G^\ast),\delta) = 0 \) holds, moreover, one finds that \( ED\Sigma \) is part of a coboundary for some \( \tilde{V} \in \Gamma(\Lambda^1 E^* \otimes E \otimes F) \). (10) then translates into the condition that \( \tilde{V} \) needs to be a contraction of \( \lambda \) with \( \Sigma \).

**SUMMARY AND OUTLOOK**

In this letter we considered mechanical models of constrained systems where, as is usually the case in physics, the constraints are not necessarily everywhere regular and where they are permitted to have reducibilities of the first level. We showed that while the BFV charge \( Q_{BFV} \) always exists, this is not the case for the BFV extension of the Hamiltonian. We identified the complex (28) governing the extension problem, providing sufficient conditions for the exitence of \( \mathcal{H}_{BFV} \) in this way.

We showed that the Hamiltonian (37) does not have a BFV extension if we choose angular momenta as constraints. There now is the challenge of how to adapt the BFV formalism so as to cover such systems as well.

It would be interesting to better understand the geometro-algebraic nature of HQ-structures, encoded in the BFV formalism of such systems—also in the context of higher level reducibilities, bringing Lie \( \infty \)-algebrds into the game. We intend to come back to this elsewhere.

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[25] In the case of the angular momenta \( L_a \), there is a preferred choice of structure functions, namely the constant ones. They can be identified with the structure constants of the Lie algebra so(3). Indeed this example of rotations comes from a simple Lie algebroid, namely the action Lie algebroid so(3) \( \times \mathbb{R}^3 \).
[26] The formulas in this last example are an improvement of a result of [24] obtained in the context of Cartan-Lie algebroids, where the \( \Sigma \)-term in (35) is absent. During the preparation of the present work, we were informed by N. Ikeda that he found similar formulas in the context of Courant algebroids, see [24].