A GENERALIZATION OF THE SPACE OF COMPLETE QUADRICS

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To any homogeneous polynomial $h$ we naturally associate a variety $\Omega_h$ which maps birationally onto the graph $\Gamma_h$ of the gradient map $\nabla h$ and which agrees with the space of complete quadrics when $h$ is the determinant of the generic symmetric matrix. We give a sufficient criterion for $\Omega_h$ being smooth which applies for example when $h$ is an elementary symmetric polynomial. In this case $\Omega_h$ is a smooth toric variety associated to a certain generalized permutohedron. We also give examples when $\Omega_h$ is not smooth.

1. Introduction and results

Let $h \in \mathbb{R}[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree $d$. We will always assume that there is no invertible linear change of coordinates $T$ such that $h(Tx) \in \mathbb{R}[x_1, \ldots, x_{k-1}]$. The gradient map of $h$ is the rational map

$$\nabla h : \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}, \ x \mapsto [\nabla h(x)] = [\frac{\partial}{\partial x_1} h(x) : \cdots : \frac{\partial}{\partial x_n} h(x)].$$

It is a regular map on the open subset $U \subset \mathbb{P}^{n-1}$ of all points where $h$ does not vanish. Its graph $\Gamma_h$ is the Zariski closure of all pairs $(x, \nabla h(x))$ in $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ with $x \in U$. In this note we will study resolutions of singularities of $\Gamma_h$ for certain $h$ and thereby address Question 43 in [Stu20]:

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Question 43. Can we define a generalization of the space of complete quadrics where the role of the symmetric determinant is played by an arbitrary hyperbolic polynomial \( h \)? Such a manifold could be a canonical resolution of the graph of the gradient map of \( h \).

One motivation for this question is the study of so-called *hyperbolic exponential families* in [MSUZ16] where the gradient map plays a prominent role. While the construction we present in this article is motivated by the theory of hyperbolic polynomials, hyperbolicity plays a subordinate role in the rest of the paper. Instead, other properties such as \( M \)-convexity will matter.

In the case when \( h = \det(X) \) is the determinant of the \( n \times n \) generic symmetric matrix \( X \), such a resolution of singularities is given by the space of complete quadrics. For any integer \( 0 < i < n \) and any symmetric matrix \( A \in S^n \) we denote by \( \wedge^i A \in S^\binom{n}{i} \) the representing matrix of the linear map \( \wedge^i \mathbb{R}^n \to \wedge^i \mathbb{R}^n \) induced by \( A \). Note that \( \wedge^i A \) is nonzero if \( \det(A) \neq 0 \). Now the space of complete quadrics \( \Omega_{\det X} \) is the Zariski closure of all tuples \( ([A], [\wedge^2 A], \ldots, [\wedge^{n-1} A]) \) in \( \mathbb{P}(S^n) \times \mathbb{P}(S^\binom{n}{2}) \times \cdots \times \mathbb{P}(S^\binom{n}{n-2}) \times \mathbb{P}(S^n) \) with \( A \) invertible. The projection of \( \Omega_{\det X} \) onto the first and the last coordinate is a birational map onto \( \Gamma_{\det(X)} \). Moreover, it was shown for example in [Lak87] that \( \Omega_{\det X} \) is smooth.

In this note we will define a variety \( \Omega_h \) for an arbitrary homogeneous polynomial \( h \in \mathbb{R}[x_1, \ldots, x_n] \) together with a regular and birational map to \( \Gamma_h \) which agrees with the space of complete quadrics when \( h = \det(X) \) is the determinant of the generic symmetric matrix. Before we give the definition of \( \Omega_h \), we recall the definition of a hyperbolic polynomial.

**Definition 1.1.** A homogeneous polynomial \( h \in \mathbb{R}[x_1, \ldots, x_n] \) is hyperbolic with respect to \( e \in \mathbb{R}^n \) if the univariate polynomial \( h(te - v) \in \mathbb{R}[t] \) has only real zeros for all \( v \in \mathbb{R}^n \). The *hyperbolicity cone* of \( h \) at \( e \) is

\[
\Lambda_e(h) = \{ v \in \mathbb{R}^n : h(te - v) \text{ has only nonnegative roots} \}.
\]

The prototype of a hyperbolic polynomial is the determinant of the generic symmetric matrix \( \det(X) \). Indeed, since a real symmetric matrix has only real eigenvalues, the polynomial \( \det(X) \) is hyperbolic with respect to the identity matrix \( I \). The hyperbolicity cone of \( \det(X) \) at \( I \) is the cone of positive semidefinite matrices.

The entries of \( \wedge^{k+1} X \) cut out the variety of symmetric matrices with rank at most \( k \). For a real symmetric matrix \( A \) the algebraic and geometric multiplicity of an eigenvalue agree. Thus the rank of \( A \) equals to the degree of the univariate polynomial \( \det(tI + A) \).

In fact the same holds true when we replace \( I \) by any positive definite matrix. This shows that we can express the degeneracy locus of the rational map...
\[ \mathbb{P}(\mathbb{S}^n) \rightarrow \mathbb{P}(\mathbb{S}^{(n+1)/2}), [A] \mapsto [\wedge^{k+1}A] \] in terms of the hyperbolic rank function of \( \det(X) \):

**Definition 1.2.** Let \( h \in \mathbb{R}[x_1, \ldots, x_n] \) be hyperbolic with respect to \( e \in \mathbb{R}^n \). The hyperbolic rank function of \( h \) is defined as

\[
\text{rank}_{h,e} : \mathbb{R}^n \rightarrow \mathbb{N}, v \mapsto \deg(h(e+tv)).
\]

It was shown in [Brä11, Lemma 4.4] that \( \text{rank}_{h,e} = \text{rank}_{h,a} \) for any \( a \in \text{int}(\Lambda_e(h)) \). We let \( d = \deg(h) \), \( 0 \leq k < d \) and \( v \in \mathbb{R}^n \). Then we have \( \text{rank}_{h,e}(v) \leq d-1 \) if and only if all \( k \)th order partial derivatives \( \frac{\partial^k h}{\partial x_{i_1} \cdots \partial x_{i_k}} \) of \( h \) vanish in \( v \).

Let's denote by \( D^1_1, \ldots, D^k_{m_k} \) a basis of the span of all \( k \)th order partial derivatives of \( h \). We consider the rational map

\[
\Delta h : \mathbb{P}^{n-1} \rightarrow \mathbb{P}^{m_1-1} \times \cdots \times \mathbb{P}^{m_d-1-1},
\]

\[
[x] \mapsto ([D^1_1(x) : \cdots : D^1_{m_1}(x)], \ldots, [D^{d-1}_1(x) : \cdots : D^{d-1}_{m_d-1}(x)]).
\]

We define the variety \( \Omega_h \) to be the normalisation of the image of this rational map. The projection on the first and the last coordinate gives a birational morphism \( \omega_h : \Omega_h \rightarrow \Gamma_h \). Moreover, when \( h = \det(X) \) is the determinant of the generic symmetric matrix, then \( \Omega_{\det(X)} \) is isomorphic to the space of complete quadrics as defined above and thus \( \Omega_{\det(X)} \) is smooth in that case.

Another important example for hyperbolic polynomials are the elementary symmetric polynomials.

**Theorem 1.1.** Let \( \sigma_{d,n} \) be the elementary symmetric polynomial of degree \( d \) in \( n \) variables. Then \( \Omega_{\sigma_{d,n}} \) is a smooth toric variety.

It is well-known that \( \sigma_{d,n} \) is hyperbolic with respect to every point in the positive orthant. Such polynomials are called stable. The theory of stable polynomials connects nicely to discrete convex analysis [Mur03]. We denote by \( \delta_k \in \mathbb{Z}^n \) the \( k \)th unit vector.

**Definition 1.3.** A nonempty set of integer points \( B \subset \mathbb{Z}^n \) is called \( M \)-convex if for all \( x, y \in B \) and every index \( i \) with \( x_i > y_i \), there exists an index \( j \) with \( x_j < y_j \) such that \( x - \delta_i + \delta_j \in B \) and \( y + \delta_i - \delta_j \in B \).

**Theorem 1.2** (Theorem 3.2 in [Brä07]). Let \( h \in \mathbb{R}[x_1, \ldots, x_n] \) be a homogeneous stable polynomial. Then the support of \( h \) is \( M \)-convex.

In Section 5 we will give a sufficient criterion for \( \Omega_h \) being smooth when the support of \( h \) is \( M \)-convex. We will apply this criterion for proving Theorem 1.1.

However, there are also stable (and thus hyperbolic) polynomials \( h \) for which \( \Omega_h \) is not smooth.
Example 1.4. Consider the polynomial
\[ h = w(2x + 4y + 7z)(4x + 2y + 7z) 
+ x^3 + 11x^2y + 11xy^2 + y^3 + 15x^2z + 46xyz + 15y^2z + 37x^2z + 37yz^2 + 21z^3. \]
One can check that \( h \) is stable. Further, using the computer algebra system Macaulay2 [M2], one checks that \( \Omega_h \) is not smooth.

2. A simple polymatroid

In this section we prepare the proof of Theorem 1.1. Recall that a polymatroid on the ground set \([n] = \{1, \ldots, n\}\) is a function \( r : 2^{[n]} \to \mathbb{Z}_{\geq 0} \) such that for all \( S, T \subset [n] \) we have:

1. \( r(S) \leq r(T) \) if \( S \subset T \),
2. \( r(S \cup T) + r(S \cap T) \leq r(S) + r(T) \), and
3. \( r(\emptyset) = 0 \).

If further \( r(\{i\}) \leq 1 \) for all \( i \in [n] \), then \( r \) is called a matroid. The second property is usually called submodularity. We call the number \( d = r([n]) \) the rank of \( r \). See [Wel76, Chapter 18] for a general reference on the theory of polymatroids.

Example 2.1. Let \( h \in \mathbb{R}[x_1, \ldots, x_n] \) be hyperbolic with respect to \( e \in \mathbb{R}^n \). The function that sends \( S \subset [n] \) to \( \text{rank}_{h,e}(\sum_{i \in S} \delta_i) \) is a polymatroid [Brä11, Proposition 3.2].

For all \( 0 \leq k \leq d \) the \( k \)th truncation \( r_k \) is the polymatroid defined by
\[ r_k(S) = \min(d - k, r(S)) \]
for all \( S \subset [n] \). It follows directly from the definition that the sum of polymatroids is again a polymatroid. We define the following polymatroid
\[ \mathcal{T} = r_0 + \ldots + r_d. \]

To every polymatroid \( r \) one associates the independence polytope
\[ P(r) = \{ x \in (\mathbb{R}_{\geq 0})^n : \sum_{i \in S} x_i \leq r(S) \text{ for all } S \subset [n] \}. \]

The goal of this section is to show that for every polymatroid \( r \) on \([n]\) the polytope \( P(\mathcal{T}) \) is simple. A characterization of polymatroids, whose independence polytope is simple, was given in [GK96, Theorem 2]. The following lemmata will enable us to apply this criterion.
**Definition 2.2.** Let \( r \) be a polymatroid on \([n]\). We say that a subset \( S \subseteq [n] \) is \( r \)-inseparable if for every two disjoint and nonempty subsets \( S_1, S_2 \subseteq [n] \) with \( S = S_1 \cup S_2 \) we have \( r(S) < r(S_1) + r(S_2) \).

**Remark 2.1.** If \(|S| \leq 1\), then \( S \) is \( r \)-inseparable for every polymatroid \( r \).

**Lemma 2.2.** Let \( r, r' \) be polymatroids on \([n]\). If \( S \subseteq [n] \) is \( r \)-inseparable, then \( S \) is \((r + r')\)-inseparable.

**Proof.** Assume that \( S \) is not \((r + r')\)-inseparable. Let \( 0 \neq S_1, S_2 \subseteq [n] \) such that \( S \) is the disjoint union of \( S_1 \) and \( S_2 \). If \( r(S) + r'(S) \geq r(S_1) + r'(S_1) + r(S_2) + r'(S_2) \), then by submodularity of \( r' \) we get \( r(S) \geq r(S_1) + r(S_2) \) which shows that \( S \) is not \( r \)-inseparable.

**Remark 2.3.** Let \(|S| \geq 2\) and let \( x \in [n] \) be a loop of \( r \), i.e. \( r(\{x\}) = 0 \). If \( x \in S \), then \( S \) is not \( r \)-inseparable: \( r(S) = r(S \setminus \{x\}) + r(\{x\}) \).

**Lemma 2.4.** Let \( S \subseteq [n] \) with \(|S| \geq 2\) and \( r \) a polymatroid on \([n]\). Then \( S \) is \( \overline{r} \)-inseparable if and only if \( S \) does not contain a loop of \( r \).

**Proof.** We first observe that \( x \in [n] \) is a loop of \( r \) if and only if \( x \) is a loop of all truncations \( r_k \) and thus of \( \overline{r} \). Now the “only if” direction follows from Remark 2.3. For the “if” direction assume that \( S \) does not contain any loop of \( r \). By Lemma 2.2 it suffices to show that \( S \) is \( r_{d-1} \)-inseparable. This is clear since

\[
r_{d-1}(S) = 1 < 2 = r_{d-1}(S_1) + r_{d-1}(S_2)
\]

for all nonempty subsets \( S_1, S_2 \subseteq S \).

**Lemma 2.5.** Let \( r \) be a polymatroid on \([n]\) of rank \( d \). Let \( S, T \subseteq [n] \) such that

1. \( S \cap T \neq 0, S \not\subseteq T, T \not\subseteq S \),
2. \( \overline{r}(S \cap T) < \overline{r}(S), \overline{r}(S \cap T) < \overline{r}(T) \), and
3. the sets \( S, T, S \cup T \) are \( \overline{r} \)-inseparable.

Then \( \overline{r}(S \cap T) + \overline{r}(S \cup T) < \overline{r}(S) + \overline{r}(T) \).

**Proof.** We proceed by induction on \( d \). We first show that for \( d \leq 1 \) there are no subsets \( S, T \subseteq [n] \) satisfying (1), (2), (3). If \( d = 0 \), then \( r \) and \( \overline{r} \) are both the zero function. Thus there are no subsets \( S, T \subseteq [n] \) satisfying (2). If \( d = 1 \), we still have \( r = \overline{r} \). Condition (1) implies that \(|S| \geq 2 \). Thus (3) and Lemma 2.4 imply that \( S \) contains no loop of \( r \). Therefore, we have \( r(S) = r(S \cap T) = 1 \) contradicting (2).
Now let \( d > 1 \) and assume that the claim is true for the polymatroid \( r_1 \) of rank \( d - 1 \). We assume for the sake of a contradiction that \( S, T \subseteq [n] \) satisfy (1), (2), (3) but \( \mathcal{P}(S \cap T) + \mathcal{P}(S \cup T) = \mathcal{P}(S) + \mathcal{P}(T) \). Again (1) implies that \( |S| \geq 2 \). So by (3) and Lemma \([2.4]\) the set \( S \cup T \) contains no loop of \( r \). Since \( d > 1 \), this implies that \( S \cup T \) contains no loop of \( r_1 \) as well. Thus again by Lemma \([2.4]\) the sets \( S, T, S \cup T \) are \( \mathcal{P}_T \)-inseparable. By submodularity and because \( \mathcal{P} = r + \mathcal{P}_T \) we have

\[
r(S) + r(T) = r(S \cap T) + r(S \cup T) \quad \text{and} \quad \mathcal{P}_T(S) + \mathcal{P}_T(T) = \mathcal{P}_T(S \cap T) + \mathcal{P}_T(S \cup T).
\]

So by induction hypothesis we have without loss of generality that \( \mathcal{P}_T(S \cap T) = \mathcal{P}_T(S) \), which implies \( r_1(S \cap T) = r_1(S) \), and \( r(S \cap T) < r(S) \). Thus we must have \( r(S) = d \) and the equation

\[
d + r(T) = r(S) + r(T) = r(S \cap T) + r(S \cup T) = r(S \cap T) + d
\]

implies that \( r(T) = r(S \cap T) \). This in turn shows that \( \mathcal{P}(T) = \mathcal{P}(S \cap T) \) contradicting (2).

**Lemma 2.6.** Let \( r \) be a polymatroid on \([n]\) of rank \( d \). Let \( k \geq 2 \) and \( S_1, \ldots, S_k \subseteq [n] \) nonempty and pairwise disjoint. Let \( S \subseteq [n] \) \( \mathcal{P}_T \)-inseparable with \( \cup_{i=1}^k S_i \subseteq S \) and \( \mathcal{P}(\cup_{i=1}^k S_i) = \mathcal{P}(S) \). Then \( \mathcal{P}(\cup_{i=1}^k S_i) < \sum_{i=1}^k \mathcal{P}(S_i) \).

**Proof.** We first observe that since \( |S| \geq 2 \) and \( S \) is \( \mathcal{P}_T \)-inseparable, Lemma \([2.4]\) implies that \( S \) contains no loop of \( r \). Thus each \( S_i \) also contains no loop of \( r \).

We proceed again by induction on \( d \). If \( d = 0 \), then there every element is a loop contradicting the assumptions. If \( d = 1 \), then we have

\[
\mathcal{P}(\cup_{i=1}^k S_i) = 1 < 2 \leq k = \sum_{i=1}^k \mathcal{P}(S_i).
\]

Now let \( d > 1 \). Then because \( S \) contains no loop of \( r \), it also contains no loop of \( r_1 \) which shows that \( S \) is \( \mathcal{P}_T \)-inseparable. Further \( \cup_{i=1}^k S_i \subseteq S \) and \( \mathcal{P}(\cup_{i=1}^k S_i) = \mathcal{P}(S) \) imply that \( \mathcal{P}_T(\cup_{i=1}^k S_i) = \mathcal{P}_T(S) \). By induction hypothesis we have \( \mathcal{P}_T(\cup_{i=1}^k S_i) < \sum_{i=1}^k \mathcal{P}_T(S_i) \) which implies the claim because \( r = r_0 \) is submodular.

**Theorem 2.7.** Let \( r \) be a polymatroid on \([n]\). Then the polytope \( P(\mathcal{P}) \) is simple.

**Proof.** A characterization of simple independence polytopes of polymatroids was given in \([\text{GK96}], \text{Theorem 2}\). It says that the polytope \( P(\mathcal{P}) \) is simple if and only if the conclusion of the two preceding Lemmas \([2.5]\) and \([2.6]\) holds.
We will be interested in the base polytope of a polymatroid rather than in its independent polytope. If \( r : 2^{[n]} \to \mathbb{R} \) is a submodular function, then its base polytope \( B(r) \) is defined as

\[
B(r) = \{ x \in (\mathbb{R}_{\geq 0})^n : \sum_{i \in S} x_i \leq r(S) \text{ for all } S \subseteq [n] \text{ and } \sum_{i=1}^n x_i = r([n]) \}. 
\]

**Corollary 2.8.** Let \( r \) be a polymatroid on \([n]\). Then the polytope \( B(\mathcal{T}) \) is simple.

*Proof.* Clearly, the base polytope is a face of the independence polytope. Thus the claim follows from Theorem 2.7. \( \square \)

**Remark 2.9.** Taking the sum of submodular functions is compatible with taking the Minkowski sum of their base polytopes [Mur03, Theorem 4.23(1)]. Thus if \( r \) is a polymatroid on \([n]\) of rank \( d \), then we have that \( B(\mathcal{T}) = B(r_0) + \ldots + B(r_d) \).

We end this section with describing the polytope \( B(\mathcal{T}) \) explicitly when \( r \) is the rank function of a matroid. We start with the following easy lemma.

**Lemma 2.10.** Let \( r = r_{\mathcal{M}} \) be the rank function of a matroid \( \mathcal{M} \) of rank \( d \) on \([n]\). There is a basis \( B \) of \( \mathcal{M} \) such that for all \( i \in [n] \) we have

\[
a_i := r([i]) - r([i-1]) = \begin{cases} 1 & \text{if } i \in B \\ 0 & \text{otherwise.}\end{cases}
\]

*Proof.* Since \( \mathcal{M} \) is a matroid of rank \( d \), we have \( a_i \in \{0,1\} \) and \( B = \{ i \in [n] : a_i = 1 \} \) has cardinality \( d \). Let \( k_1 < \cdots < k_d \) the elements of \( B \). We show by induction on \( m \) that \( I_m := \{ k_1, \ldots, k_m \} \) is independent. Assume that \( I_{m-1} \) is independent. Since \( r([k_m]) = m \), there is an independent subset \( I \) of \([k_m]\) of cardinality \( m \). Thus there is an element \( e \in I \setminus I_{m-1} \) such that \( I_{m-1} \cup \{ e \} \) is independent. Since \( r([k_m - 1]) = m - 1 \), we must have \( e = k_m \). \( \square \)

**Proposition 2.11.** Let \( r = r_{\mathcal{M}} \) be the rank function of a matroid \( \mathcal{M} \) of rank \( d \) on \([n]\). The vertices of the polytope \( B(\mathcal{T}) \) are exactly those points \( v \in \mathbb{R}^n \) whose support is a basis of \( \mathcal{M} \) and whose nonzero entries comprise the numbers \( 1, \ldots, d \).

*Proof.* Let \( v \) be a vertex of \( B(\mathcal{T}) \). Then \( v \) is also a vertex of \( P(\mathcal{T}) \). Then by [Wel73, §18.4, Theorem 1] there exists an integer \( 0 \leq k \leq n \) and a bijection \( \pi : [n] \to [n] \) such that \( v_{\pi(j)} = \pi(\pi(1), \ldots, \pi(j)) - \pi(\{\pi(1), \ldots, \pi(j-1)\}) \) if \( j \in [k] \) and \( v_{\pi(j)} = 0 \) otherwise. Since \( v \) lies in \( B(\mathcal{T}) \), we can assume without loss of generality that \( k = n \). Further, after relabeling, we can assume that \( \pi \) is the identity map. Now let \( B = \{ k_1, \ldots, k_d \} \) with \( k_1 < \cdots < k_d \) be the basis of \( \mathcal{M} \)
as in the preceding lemma. Then we have \( v_j = d - m + 1 \) if \( j = k_m \) and zero if \( j \notin B \). This shows that \( v \) is of the desired form.

Conversely, take \( v \in \mathbb{R}^n \) whose support \( \{i_1, \ldots, i_d\} \) is a basis of \( M \) such that \( v_{ij} = d - j + 1 \) if \( j = k_m \) and zero if \( j \neq k_m \). This shows that \( v \) is of the desired form.

Example 2.3. For instance when \( M = U(2, 4) \) is the uniform matroid on 4 elements of rank 2, then \( B(r_1) \) is the standard 3-simplex in \( \mathbb{R}^4 \) and \( B(r_0) \) is the octahedron whose vertices are the permutations of \( (1, 1, 0, 0) \) (and thus is not simple). The Minkowski sum \( B(\tau) = B(r_0) + B(r_1) \) is simple by Corollary 2.8. It is the truncated tetrahedron whose vertices are the permutations of \( (2, 1, 0, 0) \).

Figure 1: Lattice polytopes from Example 2.3 (left to right): \( B(r_0) \), \( B(r_1) \) and \( B(\tau) \). The figures were created using polymake [PM].

3. Polynomials with \( M \)-convex support

Let \( h \in \mathbb{R}[x_1, \ldots, x_n] \) be a homogeneous polynomial of degree \( d \) and assume that its support \( \text{supp}(h) \subset \mathbb{Z}^n \) is \( M \)-convex (see Definition 1.3). Recall that the Newton polytope \( \text{Newt}(h) \) of \( h \) is defined as the convex hull of \( \text{supp}(h) \) in \( \mathbb{R}^n \). The statements in the following theorem are standard in the literature of polymatroids. Proofs can be found for example in [Mur03, §4.4].

**Theorem 3.1.** Consider the function \( \rho_h : 2^{[n]} \rightarrow \mathbb{Z}_{\geq 0} \) defined by

\[
\rho_h(S) = \max \left\{ \sum_{i \in S} \alpha_i : \alpha \in \text{supp}(h) \right\}
\]

for all \( S \subset [n] \). Then \( \rho_h \) is a polymatroid of rank \( d \), \( \text{Newt}(h) = B(\rho_h) \) and \( \text{supp}(h) = B(\rho_h) \cap \mathbb{Z}^n \).

**Remark 3.2.** If \( h \) is stable, then all coefficients of \( h \) have the same sign, see e.g. [Brä11, Lemma 4.3]. This implies that for every \( e \in (\mathbb{R}_{>0})^n \) we have that

\[
\rho_h(S) = \text{rank}_{h,e}(\sum_{i \in S} \delta_i)
\]
as there can be no cancellation of terms.

An intriguing class of polynomials with $M$-convex support are Lorentzian polynomials.

**Definition 3.1.** Let $h \in \mathbb{R}[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree $d$ whose support is $M$-convex and all of whose coefficients are nonnegative. Then $h$ is **Lorentzian** if for every $i_1, \ldots, i_{d-2} \in [n]$ the Hessian of the derivative
\[
\frac{\partial^{d-2}}{\partial x_{i_1} \cdots \partial x_{i_{d-2}}} h
\]
has at most one positive eigenvalue.

**Remark 3.3.** Let us clarify the relations between the different classes of polynomials that appeared so far. Let $h \in \mathbb{R}[x_1, \ldots, x_n]$ be a homogeneous polynomial with nonnegative coefficients. Then we have the following implications:

- $h$ is stable $\iff$ $h$ is hyperbolic w.r.t. every $a \in (\mathbb{R}_{>0})^n$
- $\Rightarrow$ $h$ is Lorentzian
- $\Rightarrow$ $\text{supp}(h)$ is $M$-convex

There are Lorentzian polynomials that are not stable [BH20, Example 2.3]. Furthermore, not every homogeneous polynomial with $M$-convex support and nonnegative coefficients is Lorentzian.

**Theorem 3.4** (Theorem 3.10 in [BH20]). A subset $B \subset (\mathbb{Z}_{\geq 0})^n$ is $M$-convex if and only if there is a Lorentzian polynomial $h \in \mathbb{R}[x_1, \ldots, x_n]$ with $B = \text{supp}(h)$.

**Lemma 3.5** (Corollary 2.11 in [BH20]). Let $h \in \mathbb{R}[x_1, \ldots, x_n]$ be a Lorentzian polynomial and $e \in (\mathbb{R}_{\geq 0})^n$. The derivative
\[
D_e h = \sum_{i=1}^n e_i \frac{\partial h}{\partial x_i}
\]
is Lorentzian as well. In particular, the support of $D_e h$ is $M$-convex.

**Lemma 3.6.** If $h \in \mathbb{R}[x_1, \ldots, x_n]$ is Lorentzian of degree $d$ and $e \in (\mathbb{R}_{>0})^n$, then we have for all $0 \leq k \leq d$ that $(\rho_h)_k = \rho_{D_e h}$.

**Proof.** It suffices to prove the claim in the case $k = 1$ because the general case follows from an iterative application of this case.

Let $S \subset [n]$. Since $D_e h$ has degree $d-1$, we have $\rho_{D_e h}(S) \leq d - 1$. If $\rho_h(S) = d$, then there is an $\alpha \in \text{supp}(h)$ such that $\sum_{i \in S} \alpha_i = d$. For any $j \in [n]$ with $\alpha_j > 0$
we have $\alpha' = \alpha - \delta_j \in \text{supp}(D_e h)$ and thus $\rho_{D_e h}(S) \geq d - 1$. Now let $\rho_h(S) < d$ and $\alpha \in \text{supp}(h)$ such that $\sum_{i \in S} \alpha_i = \rho_h(S)$. Since the degree of $h$ is $d$, there must be an index $j \in [n] \setminus S$ such that $\alpha_j > 0$. We have $\alpha' = \alpha - \delta_j \in \text{supp}(D_e h)$ and thus $\rho_{D_e h}(S) \geq \rho_h(S)$. If $\beta \in \text{supp}(D_e h)$ satisfies $\rho_{D_e h}(S) = \sum_{i \in S} \beta_i$, then there is a $j \in [n]$ such that $\beta + \delta_j \in \text{supp}(h)$ so $\rho_{D_e h}(S) \leq \rho_h(S)$.

The following lemma connects the polymatroid $\overline{\rho_h}$ with the variety $\Omega_h$.

**Proposition 3.7.** Let $h \in \mathbb{R}[x_1, \ldots, x_n]$ be homogeneous of degree $d$ with $\text{supp}(h)$ being $M$-convex. Consider the polymatroid $r = \rho_h$. For each $0 \leq k \leq d$ the set $B(r_k) \cap \mathbb{Z}^n$ agrees with the set $B_k$ of all $\alpha \in \mathbb{Z}^n$ such that the monomial $\prod_{i=1}^n x_i^{\alpha_i}$ is in the support of a $k$th order partial derivative of $h$.

**Proof.** Both $r_k$ and $B_k$ only depend on the support of $h$. Thus we can assume without loss of generality that $h$ is Lorentzian by Theorem 3.4. Then for any $e \in (\mathbb{R}_{>0})^n$ we have that $B_k$ is the support of $D^k h$ because $h$ has nonnegative coefficients. Thus $B_k$ is $M$-convex by Lemma 3.5 and the result follows from Theorem 3.1 and the preceding lemma. \hfill \square

4. Preliminaries from algebraic and toric geometry

In this section we revisit some notions and results from algebraic geometry that will be used in the final section. Let $A$ be a nonzero $(m+1) \times (n+1)$ matrix. Recall that a rational map $\pi : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ of the form $[x] \mapsto [Ax]$ is called a linear projection and that the centre of $\pi$ is the linear subspace $E$ of all $[x] \in \mathbb{P}^n$ such that $Ax = 0$. Clearly, the rational map $\pi$ is regular on $\mathbb{P}^n \setminus E$. Thus if $X \subset \mathbb{P}^n$ is a projective variety with $X \cap E = \emptyset$, the restriction $f = \pi\vert_X : X \to \pi(X)$ is a morphism. This map $f$ is finite, so in particular it has only finite fibers. See [Sha77, §I.5.3] for the definition and proofs. We will use the following standard facts on finite morphisms which follow for example from [Har95, Lemma 14.8].

**Lemma 4.1.** Let $f_i : X_i \to Y_i$, $i = 1, 2$, be finite morphisms of projective varieties. The product map $f_1 \times f_2$ is also finite. If $Y_1 = X_2$, then $f_2 \circ f_1$ is finite. If $Z \subset X_1$ is closed, then $f_1\vert_Z : Z \to f_1(Z)$ is finite.

Given a projective variety $X$, a normalisation of $X$ is a normal variety $X^\nu$ with a finite birational morphism $\nu : X^\nu \to X$. The normalisation is unique up to isomorphism. In particular, if $Y \to X$ is a finite birational morphism from a smooth variety $Y$, then $Y$ is the normalisation of $X$ because every smooth variety is normal. See [Sha77, §II.5] for definitions and proofs.

We will especially consider toric varieties. The book [CLS11] gives a comprehensive introduction to toric varieties and we will adopt their notation. For
example, given a lattice polytope $P \subset \mathbb{R}^n$, we denote by $X_P$ the associated toric variety \cite{CLS11, §2.3}. Similarly, for any finite set $A \subset \mathbb{Z}^n$ of lattice points we denote by $X_A \subset \mathbb{P}^{|A|-1}$ the image of the monomial map whose exponents are given by the elements of $A$. Note that in general $X_{P\cap\mathbb{Z}^n}$ is not necessarily isomorphic to $X_P$ but when $P$ is a smooth polytope this is the case by \cite[Proposition 2.4.4]{CLS11}: A lattice polytope $P \subset \mathbb{R}^n$ is called smooth if its associated toric variety $X_P$ is smooth \cite{CLS11, §2.4}. We further have:

**Corollary 4.2.** Let $r$ be a polymatroid on $[n]$. Then the polytope $B(r)$ is a smooth lattice polytope.

**Proof.** By the Corollary to \cite[§18.4, Theorem 1]{Wel76} the independence polytope $P(r)$ is a lattice polytope. Since $B(r)$ is a face of $P(r)$, it is a lattice polytope as well. Corollary 2.3 states that $B(r)$ is simple and the Submodularity Theorem\footnote{The Submodularity Theorem was proved by Edmonds \cite{Edm70} although not stated in the language of generalized permutahedra. An alternative, combinatorial proof is given in \cite[Theorem 1.2]{CL20}.} \cite{Edm70} states that $B(r)$ is a so-called generalized permutahedron. Now the claim follows from \cite[Corollary 3.10]{PRW08} which says that a simple lattice polytope, which is a generalized permutahedron, is smooth.

**Remark 4.3.** Let $h \in \mathbb{R}[x_1, \ldots, x_n]$ be homogeneous of degree $d$ with $\text{supp}(h)$ being $M$-convex and let $B_k$ the set of all $\alpha \in \mathbb{Z}^n$ such that the monomial $\prod_{i=1}^n x_i^{\alpha_i}$ is in the support of a $k$th order partial derivative of $h$. Then it follows from Proposition 3.7 and Corollary 4.2 that the Minkowski sum $B_1 + \ldots + B_{d-1}$ is the set of lattice points in a smooth polytope. In general, if we drop the assumption of $M$-convexity, this is no longer true. Consider for example $h = a \cdot x_1 x_2^2 + b \cdot x_3^3$ with nonzero $a, b$. Then $B_1 + B_2$ is the set of lattice points in a simple polytope that is not smooth.

5. **A sufficient criterion for smoothness**

Let $h \in \mathbb{R}[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree $d$ whose support is $M$-convex with nonnegative coefficients and $r = \rho_h$. Recall that we denote by $D_1^k, \ldots, D_{mk}^k$ a basis of the span of all $k$th order partial derivatives of $h$. For all $1 \leq k < d$ consider the rational map

$$\Delta^k h : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{mk-1}, [x] \mapsto [D_1^k(x) : \ldots : D_{mk}^k(x)].$$

By Proposition 3.7 we can decompose the map $\Delta^k h$ as $\pi_k \circ f_k$ where $f_k$ is the monomial map associated to the polytope $B(r_k)$ (whose image is $X_{B(r_k)\cap\mathbb{Z}^n}$) and $\pi_k$ the linear projection given by summing the monomials in each $D_i^k$. 


Example 5.1. Let $h = x_1^2x_2 + x_1x_2^3 + x_1x_2x_3 + x_2^3x_3$. Then $\Delta^1 h(x)$ equals:

$$[2x_1x_2 + x_2^2 + 2x_1x_3 + x_2x_3 : x_1^2 + 2x_1x_2 + x_1x_3 + 2x_2x_3 : x_1^2 + x_1x_2 + x_2^2].$$

We further have $f_1 : \mathbb{P}^2 \rightarrow \mathbb{P}^4, [x_1 : x_2 : x_3] \mapsto [x_1^2 : x_1x_2 : x_1x_3 : x_2^3 : x_2x_3].$ We label the coordinates on $\mathbb{P}^4$ by $z_{ij}$ where $i$ and $j$ keep track of the exponent of $x_1$ and $x_2$ respectively. The image $X$ of $f_1$ is cut out by $z_{10}z_{02} - z_{11}z_{10}z_{10} - z_{20}z_0$ and $z_{11}^2 - z_{20}z_{02}$. Furthermore, the projection $\pi_1$ sends $[z_{20} : z_{11} : z_{10} : z_{02} : z_{01}]$ to $[2z_{11} + z_{02} + 2z_{10} + z_{01} : z_{20} + 2z_{11} + z_{10} + 2z_{01} : z_{20} + z_{11} + z_{02}].$

The centre of $\pi_1$ is spanned by $[0 : -1 : 0 : 1 : 1]$ and $[1 : -1 : 1 : 0 : 0]$ and is disjoint from $X$. Thus $\pi_1$ restricts to a finite morphism $X \rightarrow \mathbb{P}^2$.

Proposition 5.1. If the centre of the linear projection $\pi_k$ is disjoint from $X_{B(r_k) \cap \mathbb{Z}^n}$ for each $1 \leq k < d$, then $\Omega_h$ is smooth. More precisely, it is isomorphic to the smooth toric variety $X_{B(\pi)}$.

Proof. Let $P = B(\pi)$. By definition $\Omega_h$ is the normalisation of the image $Y \subseteq \prod_{i=1}^{d-1} \mathbb{P}^{m_i - 1}$ of the birational map $\Delta h(x) = (\Delta^1 h(x), \ldots, \Delta^{d-1} h(x)).$ Consider the rational map $f : \mathbb{P}^{n-1} \dashrightarrow \prod_{i=1}^{d-1} \mathbb{P}^{B(r_i) \cap \mathbb{Z}^n} \dashrightarrow \prod_{i=1}^{d-1} \mathbb{P}^{m_i - 1}$ given by $f(x) = (f_1(x), \ldots, f_{d-1}(x))$ and let $\pi = \prod_{i=1}^{d-1} \pi_i.$ By construction $\Delta h$ factors as $\pi \circ f$. If we compose $f$ with the Segre embedding of $\prod_{i=1}^{d-1} \mathbb{P}^{B(r_i) \cap \mathbb{Z}^n}$, we obtain the monomial map associated to the Minkowski sum $\sum_{i=1}^{d-1} B(r_i)$ which is $P$ by Remark 2.9. Thus the image of $f$ can be identified with $X_{P \cap \mathbb{Z}^n}$ and $Y$ is the image of $X_{P \cap \mathbb{Z}^n}$ under the rational map $\pi : \prod_{i=1}^{d-1} \mathbb{P}^{B(r_i) \cap \mathbb{Z}^n} \dashrightarrow \prod_{i=1}^{d-1} \mathbb{P}^{m_i - 1}$. Because $P$ is smooth by Corollary 4.2, the variety $X_{P \cap \mathbb{Z}^n}$ is the smooth toric variety $X_P$. Letting $E_i$ be the centre of $\pi_i$, this rational map $\pi$ is regular on $U = \prod_{i=1}^{d-1} (\mathbb{P}^{B(r_i) \cap \mathbb{Z}^n} \setminus E_i)$. Since the projection of $X_P$ on the $i$th factor $\mathbb{P}^{B(r_i) \cap \mathbb{Z}^n}$ is $X_{B(r_i) \cap \mathbb{Z}^n}$, which is disjoint from $E_i$ by assumption, it follows that $X_P \subset U$. Thus restricting $\pi$ gives a surjective morphism $p = \pi|_{X_P} : X_P \rightarrow Y$ which is finite since each $\pi_i|_{X_{B(r_i) \cap \mathbb{Z}^n}}$ is finite and by Lemma 4.1 since $\Delta h = p \circ f$ is birational and $f$ is birational, it follows that $p$ is also birational. Thus $X_P$ is the normalisation of $Y$. \hfill $\Box$

Example 5.2. Consider the polynomial

$$h = x^3 + 11x^2y + 11xy^2 + y^3 + 15x^2z + 46xyz + 15y^2z + 37xz^2 + 37yz^2 + 21z^3 + w(29x^2 + 90xy + 29y^2 + 150xz + 150yz + 137z^2).$$

One can check that $h$ is stable. Note that $h$ has the same support as the polynomial in Example 4.4 but different coefficients. Using Macaulay2, one checks that the conditions of Proposition 5.1 are fulfilled and thus $\Omega_h$ is smooth. It is the toric variety associated to the triangular frustum whose vertices are obtained by permuting the last three entries of $(0, 3, 0, 0)$ and $(2, 1, 0, 0)$. Here the coordinates correspond to the variables in alphabetic order.
Now let $h = \sigma_{d,n}$ be the elementary symmetric polynomial of degree $d$.

**Lemma 5.2.** The centre of the linear projection $\pi_k$ is disjoint from $X_{B(r_k)\cap \mathbb{Z}^n}$ for each $1 \leq k < d$.

**Proof.** The lattice points of $B(r_k)$ are exactly the points $v \in \mathbb{R}^n$ with $d - k$ entries equal to 1 and all other entries 0. Thus $B(r_k)$ is the hypersimplex $\Delta_{d-k}$. We denote $X = X_{\Delta_{d-k} \cap \mathbb{Z}^n} \subset \mathbb{P}^{(n-k)-1}$ and we label the coordinates on $\mathbb{P}^{(n-k)-1}$ by $z_S$ for $S \subset [n]$ of size $d - k$. Every $k$th order derivative of $\sigma_{d,n}$ is an elementary symmetric polynomial of degree $d - k$ in the variables indexed by some subset $T \subset [n]$ of size $n - k$. Thus the centre $E$ of $\pi_k$ is the common zero set of all linear forms $L_T = \sum_{S \subset T, |S| = d-k} z_S$ for subsets $T \subset [n]$ of size $n - k$. The statement of [MSUZ16, Lemma 6.4] is that $X$ is disjoint from the common zero set $E'$ of all linear forms $H_i = \sum_{S \subset [n] \setminus \{i\}, |S| = d-k} z_S$ for $i \in [n]$. We have for all $i \in [n]$

$$\begin{pmatrix} n + k - 1 - d \\ n - d \end{pmatrix} \cdot H_i = \sum_{T \subset [n] \setminus \{i\}, |T| = n - k} L_T.$$

This implies $E \subset E'$ and thus $X$ is also disjoint from $E$. \hfill \square

**Proof of Theorem 1.1** This follows from Lemma 5.2 and Proposition 5.1 \hfill \square

**Remark 5.3.** By Proposition 2.11 we have that $\Omega_{\sigma_{d,n}}$ is the smooth toric variety $X_P$ where $P$ is the convex hull of all permutations of $(1, \ldots, d - 1, 0, \ldots, 0) \in \mathbb{R}^n$.

**Remark 5.4.** Fix some $M$-convex set $S \subset (\mathbb{Z}_{\geq 0})^n$ and let $V$ be the vector space of polynomials $h$ with $\text{supp}(h) \subseteq S$. Note that there is an integer $d$ such that any $h \in V$ is homogeneous of degree $d$. There are matrices $A_h^k$ whose entries depend linearly on the coefficients of $h$ such that, when $\text{supp}(h) = S$, the centre of $\pi_k$ is the set of all $[x]$ with $A_h^k x = 0$. Consider the incidence correspondence

$$\Sigma_k = \{ ([h], [x]) \in \mathbb{P}(V) \times X_{B(r_k) \cap \mathbb{Z}^n} : A_h^k x = 0 \}.$$

This is a projective variety. Thus the projection of $\Sigma_k$ onto the first factor is a closed subvariety $Y_k$ of $\mathbb{P}(V)$. By construction the criterion from Proposition 5.1 applies to a polynomial $h \in V$ with $\text{supp}(h) = S$ if and only if $[h]$ is not contained in any of the $Y_k$. Therefore, depending on $S$, either Proposition 5.1 applies to no polynomial with support $S$, or to a generic such polynomial. We say that $S$ is *torically smoothable* if the latter is the case. The support of the elementary symmetric polynomial $\sigma_{d,n}$ is torically smoothable by Lemma 5.2. Based on experiments we conjecture that $S$ is torically smoothable at least when $S$ contains the support of $\sigma_{d,n}$. This condition is empty when $d > n$. 

Remark 5.5. If $h$ has nonnegative coefficients, then we can assume the same for each $D^{k_i}_t$. Then the linear projection $\pi_k$ is at least regular on the nonnegative part of $X_{B(r_k)}$ as there can be no cancellation of terms. Thus we have at least a regular map on the nonnegative part of $X_{B(\mathcal{P})}$ that maps birationally onto the graph $\Gamma_{h,+}$ of $\nabla h$ restricted to the nonnegative orthant. In general, even when $h$ is stable, we cannot expect $\pi_k$ to be regular on all of $X_{B(r_k)}$. Take for instance the stable polynomial from Example 1.4. In this case $B(r_2)$ is the triangular frustum whose vertices are obtained by permuting the last three entries of $(0, 2, 0, 0)$ and $(1, 1, 0, 0)$. Using Macaulay2 [M2] one checks that the centre of $\pi_2$ intersects the toric variety $X_{B(r_2)}$ in a real point of the torus orbit corresponding to the face with vertices $(1, 1, 0, 0)$, $(1, 0, 1, 0)$, and $(1, 0, 0, 1)$.

Remark 5.6. Let $h \in \mathbb{R}[x_1, \ldots, x_n]$ be hyperbolic with respect to $e \in \mathbb{R}^n$. In the spirit of the preceding remark one can speculate whether hyperbolicity of $h$ guarantees smoothness of $\Omega_h$ at least at some distinguished subset. To make this more precise let $U \subset \mathbb{P}^{n-1}$ be the set of all $[p]$ such that $p$ is in the interior of $\Lambda_e(h)$. Since $h$ does not vanish on $U$, the gradient map $\nabla h$ is regular on $U$. We can thus consider the subset $C = \omega_h^{-1}(\nabla h(U))$ of $\Omega_h$. We think it is reasonable to ask whether the Euclidean closure of $C$ contains only smooth points of $\Omega_h$.

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