ON SMOOTH DIVISORS OF A PROJECTIVE HYPERSURFACE.

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Dedicated to Christian Peskine.

INTRODUCTION.

This paper deals with the existence of smooth divisors of a projective hypersurface $\Sigma \subset \mathbb{P}^n$ (projective space over an algebraically closed field of characteristic zero). According to a celebrated conjecture of Hartshorne, at least when $n \geq 7$, any such a variety should be a complete intersection. Since the existence of smooth, non complete intersection, subcanonical $X \subset \mathbb{P}^n$ of codimension two is equivalent, via the correspondence of Serre, to the existence of indecomposable rank two vector bundles on $\mathbb{P}^n$ and since no indecomposable vector bundle of $\mathbb{P}^n$, $n \geq 5$, is presently known, it is widely believed that any smooth, subcanonical subvariety of $\mathbb{P}^n$, $n \geq 5$, of codimension two is a complete intersection. Furthermore recall that, by a theorem of Barth, the subcanonical condition is automatically satisfied if $n \geq 6$. This in turn implies that a smooth (subcanonical if $n = 5$) divisor of a projective hypersurface $\Sigma \subset \mathbb{P}^n$, $n \geq 5$, is a complete intersection too.

In this paper we show that, roughly speaking, for any $\Sigma \subset \mathbb{P}^n$ there can be at most finitely many exceptions to the last statement. Indeed our main result is:

**Theorem 0.1.** Let $\Sigma \subset \mathbb{P}^n$, $n \geq 5$ be an integral hypersurface of degree $s$. Let $X \subset \Sigma$ be a smooth variety with $\dim(X) = n - 2$. If $n = 5$, assume $X$ subcanonical. If $X$ is not a complete intersection in $\mathbb{P}^n$, then:

$$d(X) \leq \frac{s(s-1)(s-1)^2 - n + 1}{n-1} + 1.$$

In other words a smooth codimension two subvariety of $\mathbb{P}^n$, $n \geq 5$ (if $n = 5$, we assume $X$ subcanonical) which is not a complete intersection cannot lie on a hypersurface of too low degree (too low with respect to its own degree) and, on a fixed hypersurface, Hartshorne’s conjecture in codimension two is “asymptotically” true.

The starting point is Severi-Lefschetz theorem which states that if $n \geq 4$ and if $X$ is a Cartier divisor on $\Sigma$, then $X$ is the complete intersection of $\Sigma$ with another hypersurface. For instance if $\Sigma$ is either smooth or singular in a finite set of points and if $n \geq 5$, the picture is very clear:

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(1) there exists smooth $X \subset \Sigma$ with $\text{dim}(X) = n - 2$ and with degree arbitrarily large;

(2) any smooth $X \subset \Sigma$ with $\text{dim}(X) = n - 2$ is a complete intersection of $\Sigma$ with another hypersurface

(3) no smooth $X \subset \Sigma$ with $\text{dim}(X) = n - 2$ can meet the singular locus of $\Sigma$.

Using Theorem 0.1 we get (the first statement comes again from an easy application of the Theorem of Severi-Lefschetz-Grothendieck):

**Theorem 0.2.** Let $\Sigma \subset \mathbb{P}^n$, $n \geq 5$, be an integral hypersurface of degree $s$ with $\text{dim}\text{Sing}(\Sigma) \geq 1$.

1. If $n \geq 6$ and $\text{dim}\text{Sing}(\Sigma) \leq n - 5$ then $\Sigma$ does not contain any smooth variety of dimension $n - 2$.

2. Suppose $\text{dim}\text{Sing}(\Sigma) \geq n - 4$. If $X \subset \Sigma$ is smooth, subcanonical, with $\text{dim}(X) = n - 2$ then $d(X) \leq s^{(s-1)^2-n+1}_{n-1} + 1$.

We point out a consequence of this result.

**Corollary 0.3.** Let $\Sigma \subset \mathbb{P}^n$, $n \geq 5$, be an integral hypersurface s.t. $\text{dim}\text{Sing}(\Sigma) \geq 1$.

1. If $n \geq 6$ and $\text{dim}\text{Sing}(\Sigma) \leq n - 5$ then $\Sigma$ does not contain any smooth variety of dimension $n - 2$.

2. Suppose $\text{dim}\text{Sing}(\Sigma) \geq n - 4$. Then there are only finitely many components of $\text{Hilb}(\Sigma)$ containing smooth, subcanonical varieties of dimension $n - 2$.

Last but not least, at the end of the paper we show how this circle of ideas allows to improve the main results of [3] about subcanonical varieties of $\mathbb{P}^5$ and $\mathbb{P}^6$:

**Theorem 0.4.** Let $X \subset \mathbb{P}^5$ be a smooth threefold with $\omega_X \simeq \mathcal{O}_X(e)$. If $h^0(\mathcal{I}_X(5)) \neq 0$, then $X$ is a complete intersection.

**Theorem 0.5.** Let $X \subset \mathbb{P}^6$ be a smooth fourfold. If $h^0(\mathcal{I}_X(6)) \neq 0$, then $X$ is a complete intersection.

Theorem 0.1 follows, thanks to a crucial remark essentially proved in [4] (see Lemma 1.6), from a bound of $e$ (where $\omega_X \simeq \mathcal{O}_X(e)$), see Theorem 2.4 which can be viewed as a strong (since the degree is not involved) generalization of the "Speciality theorem" of Gruson-Peskine [6]. The proof of this bound is quite simple if $X \cap \text{Sing}(\Sigma)$ has the right dimension. This is done in the first section where a weaker version of Theorem 2.4 and hence of Theorem 0.1 is proved (if $n = 5$ we assume $\text{Pic}(X) \simeq \mathbb{Z}.H$). In the second section we show how a refinement of the proof yields our final result. Finally let’s observe that our approach doesn’t apply to the case $n = 4$. 
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1. Reduction and the speciality theorem, weak version.

Notations 1.1. Given a projective scheme $Y \subset \mathbb{P}^n$ we denote by $d(Y)$ the degree of $Y$.

Notations 1.2. In this section, $X \subset \mathbb{P}^n$, $n \geq 5$, will denote a smooth, non degenerate, codimension two subvariety which is not a complete intersection. We will always assume $X$ subcanonical: $\omega_X \cong \mathcal{O}_X(e)$; notice that this condition is fulfilled if $\text{Pic}(X) \cong \mathbb{Z}.H$; finally, thanks to a theorem of Barth, this last condition is automatically fulfilled if $n \geq 6$.

By Serre’s construction we may associate to $X$ a rank two vector bundle:

$$0 \to \mathcal{O} \to E \to \mathcal{I}_X(e + n + 1) \to 0$$

The Chern classes of $E$ are: $c_1(E) = e + n + 1, c_2(E) = d(X) =: d$.

Let $\Sigma$ be an hypersurface of degree $s$ containing $X$. Then $\Sigma$ gives a section of $\mathcal{I}_X(s)$ which lifts to a section $\sigma_\Sigma \in H^0(E(-e - n - 1 + s))$ (notice that $\sigma_\Sigma$ is uniquely defined if $e + n + 1 - s < 0$). Assume that $Z$, the zero-locus of $\sigma_\Sigma$, has codimension two. Notice that since $X$ is not a complete intersection, this certainly holds if $s = \min\{t \mid h^0\mathcal{I}_X(t) \neq 0\}$. Anyway, if $Z$ has codimension two, then $d(Z) = c_2(E(-e - n - 1 + s)) = d - s(e + n + 1 - s)$ and $\omega_Z \cong \mathcal{O}_Z(-e - 2n - 2 + 2s)$.

Remark 1.3. By [10], if $X \subset \Sigma \subset \mathbb{P}^n$, $n \geq 3$, with $\omega_X \cong \mathcal{O}_X(e)$ and $d(\Sigma) \leq n - 2$ then $X$ is complete intersection, hence in the remainder of this paper we will assume $s \geq n - 1$.

Remark 1.4. Notice that $E(-e - n - 1) |_X \cong \mathcal{N}_X^\vee$. It is well known that the scheme $X \cap Z$ is the base locus of the jacobian system of $\Sigma$ on $X$: $X \cap Z = X \cap \text{Jac}(\Sigma)$. So, the fundamental cycle (5 1.5) of $Z$ in $\text{A}_*(X)$ is $c_2(\mathcal{N}_X^\vee(s))$ as soon as $X$ and $Z$ intersect in the expected codimension.

The main goal of this section is to prove:

**Theorem 1.5** (Speciality theorem, weak version). Let $X \subset \mathbb{P}^n$, $n \geq 5$ be a smooth codimension two subvariety. If $n = 5$ assume $\text{Pic}(X) \cong \mathbb{Z}.H$. Let $\Sigma$ be an hypersurface of degree $s$ containing $X$. If $X$ is not a complete intersection, then:

$$e \leq \frac{(s - 1)((s - 1)^2 - n + 1)}{n - 1} - n + 1$$

where $\omega_X \cong \mathcal{O}_X(e)$.

Let’s see how this is related with a bound of the degree. First recall the following:
Lemma 1.6. Let \( X \subset \mathbb{P}^n \), \( n \geq 4 \), be a smooth codimension two subvariety which is not a complete intersection. Let \( \Sigma \) be an hypersurface of minimal degree containing \( X \). Set \( s := d(\Sigma) \).

1. \( n - 4 \leq \dim(X \cap \text{Sing}(\Sigma)) \leq n - 3 \).
2. If \( \omega_X \cong \mathcal{O}_X(e) \), then \( d(X) \leq s(n - 1 + e) + 1 \).
3. If \( \dim(X \cap \text{Sing}(\Sigma)) = n - 3 \) and if \( \text{Pic}(X) \cong \mathbb{Z}.H \), then \( d(X) \leq (s - 2)(n - 1 + e) + 1 \).

Proof. The first item is \([3] \text{ Lemma 2.1} \); 2) is \([3] \text{ Lemma 2.2 (i)} \) and the last item is \([3] \text{ Lemma 2.2 (ii)} \) with \( l = 2 \) (thanks to Severi and Zak theorems \( h^1(\mathcal{I}_X(1)) = 0 \), \([11]) \).

Theorem 1.7. Let \( \Sigma \subset \mathbb{P}^n \), \( n \geq 5 \), be an integral hypersurface of degree \( s \). Let \( X \subset \Sigma \) be a smooth subvariety with \( \dim(X) = n - 2 \). If \( n = 5 \) assume \( \text{Pic}(X) \cong \mathbb{Z}.H \). If \( X \) is not a complete intersection, then \( d(X) < \frac{s(s-1)(s-1)^2-n+1}{n-1} + 1 \).

In order to prove Theorem 1.7 we need some preliminary results.

Lemma 1.8. Let \( \Sigma \) denote an hypersurface of degree \( s \) containing \( X \). With assumptions \((\text{codim}(\sigma\Sigma)_0 = 2) \) and notations as in \([1,2] \) assume \( \dim(X \cap Z) = n - 4 \). Then \( Y := X \cap Z \) is a subcanonical, l.c.i. scheme with \( \omega_Y \cong \mathcal{O}_Y(2s - n - 1) \). Moreover \( Y \) is the base locus of the jacobian system of \( \Sigma \) in \( X \).

Proof. We are assuming that \( Y \) is a proper intersection between \( X \) and \( Z \) hence

\[
0 \to \mathcal{O} \to E |{}_{X^{-e-n-1+s}} \to \mathcal{I}_{Y,X}(-e-n-1+2s) \to 0
\]

so \( N^s_{Y,X} \cong E |{}_{X^{-s}} \) and the first statement follows by adjunction. For the last statement, use \([1,4] \).

Notations 1.9. Keep the assumptions of Lemma 1.8 and denote by \( \Sigma_1 \) and \( \Sigma_2 \) two general partials of \( \Sigma \). Since \( \dim(X \cap Z) = n - 4 \), \( C := X \cap \Sigma_1 \cap \Sigma_2 \) is a subcanonical, l.c.i. scheme containing \( Y \) such that \( N_{C,X} \cong \mathcal{O}_X(s - 1) \oplus \mathcal{O}_X(s - 1) \). We have \( \omega_C \cong \mathcal{O}_C(e + 2s - 2) \). The scheme \( C \) is a complete intersection in \( X \) which links \( Y \) to another subscheme.

Lemma 1.10. With notations as in Lemma 1.8 denote by \( R \) the residual to \( Y \) with respect to \( C \). Then \( C = Y \cup R \) is a geometric linkage and \( \Delta := R \cap Y \) is a Cartier divisor of \( Y \) such that: \( \mathcal{I}_{\Delta,Y} \cong \mathcal{O}_Y(-e-n+1) \).

Furthermore: \( d(\Delta) \leq (s - 1)d(X)((s - 1)^2 - d(Z)) \) and:

\[
d(Z)(e + n + 1) \leq (s - 1)((s - 1)^2 - d(Z)).
\]
Proof. Denote by $Y_{red}$ the support of $Y$ and set $Y_{red} = Y_1 \cup \cdots \cup Y_r$ where $Y_i$, $1 \leq i \leq r$, are the irreducible components of $Y_{red}$. Furthermore, denote by $P_i$ the general point of $Y_i$. Since $Y$ is l.c.i. in $X$ and since $I_{Y,X}(s-1)$ is globally generated by the partials of $\Sigma$, we can find two general elements in $\text{Jac}(\Sigma)$ generating the fibers of $N_{Y,X}^s(s-1)$ at each $P_i$, $1 \leq i \leq r$. This implies that $R \cup Y$ is a geometric linkage.

Now consider the local Noether sequence (exact sequence of liaison):

$$0 \rightarrow I_C \rightarrow I_R \rightarrow \omega_Y \otimes \omega_C^{-1} \rightarrow 0.$$ 

we get

$$\omega_Y \otimes \omega_C^{-1} \cong \frac{I_R}{I_C} \cong \frac{I_R + I_Y}{I_C + I_Y} \cong \frac{I_{\Delta}}{I_Y} \cong I_{\Delta,Y}$$

(the second isomorphism follow by geometric linkage, since $I_R \cap I_Y = I_C$) hence $\omega_Y \otimes \omega_C^{-1} \cong \mathcal{O}_Y(-e - n + 1) \cong I_{\Delta,Y}$ and we are done.

For the last statement, the scheme $\Delta \subset R$ is the base locus of the jacobian system of $\Sigma$ in $R$, hence $\Delta \subset \overline{\Sigma} \cap R$ with $\overline{\Sigma}$ a general element of $\text{Jac}(\Sigma)$ and $d(\Delta) \leq d(R) \cdot (s - 1)$. We conclude since $d(R) \cdot (s - 1) = (d(C) - d(Z)) \cdot (s - 1) = ((s - 1)^2 d(X) - d(Z)d(X)) \cdot (s - 1)$. The last inequality follows from $d(\Delta) = d(Y) \cdot (e + n + 1) = d(X) \cdot d(Z) \cdot (e + n + 1)$.

Now we can conclude the proof of Theorem 1.5 (and hence of Theorem 1.7).

Proof of Theorem 1.6. It is enough to prove the theorem for $s$ minimal. Let $\Sigma$ be an hypersurface of minimal degree containing $X$, we set $s := d(\Sigma)$ and $d := d(X)$. According to Lemma 1.6 we distinguish two cases.

1) $\dim(X \cap \text{Sing}(\Sigma)) = n - 3$. In this case, by Lemma 1.6 we have $d \leq (s - 2)(n - 1 + e) + 1$. On the other hand $d(Z) = d - s(e + n + 1 - s)$ (see 1.2). It follows that $d(Z) \leq (s - 1)^2 - 2(n - 1 + e)$. Since $d(Z) \geq n - 1$ by 1.10, we get: 

$$\frac{(s - 1)^2 - n + 1}{d(Z)} - 1 - n + 1 \geq e.$$ 

One checks (using $s \geq n - 1$) that this implies the bound of Theorem 1.6.

2) $\dim(X \cap \text{Sing}(\Sigma)) = n - 4$. By the last inequality of Lemma 1.10 $e \leq (s - 1)[\frac{(s - 1)^2 - n + 1}{d(Z)} - 1] - n + 1$. Since $d(Z) \geq n - 1$ by 1.10, we get the result. \qed

2. THE SPECIALITY THEOREM.

In this section we will refine the proof of Theorem 1.5 for $n = 5$ in order to prove Theorem 0.4 of the introduction. For this we have to assume only that $X$ is subcanonical, which, of course, is weaker than assuming $\text{Pic}(X) \cong \mathbb{Z}.H$. The assumption $\text{Pic}(X) \cong \mathbb{Z}.H$ is used just to apply the last statement of Lemma 1.6 in order to settle the case $\dim(X \cap \text{Sing}(\Sigma)) = n - 3$. Here instead we will argue like in the proof of the case $\dim(X \cap \text{Sing}(\Sigma)) = n - 4$, but working modulo the divisorial part (in $X$) of $X \cap \text{Sing}(\Sigma)$; this will introduce some technical complications, but conceptually, the proof runs as before. Since the proof works for every $n \geq 5$ we will state it in this generality giving thus an alternative proof of Theorem 1.5.
Notations 2.1. In this section, with assumptions and notations as in 1.2, we will assume furthermore that \( \dim(X \cap Z) = n - 3 \) and will denote by \( L \) the dimension \( n - 3 \) part of \( X \cap Z \subset X \); moreover we set \( \mathcal{L} = \mathcal{O}_X(L) \).

Set \( Y' := \text{res}_L(X \cap Z) \), we have \( \mathcal{I}_{Y',X} := (\mathcal{I}_{X \cap Z,X} : \mathcal{I}_{L,X}) \). Since we have:

\[
0 \to \mathcal{O} \to E|_X (-e - n - 1 + s) \otimes \mathcal{L}^* \to \mathcal{I}_{Y',X}(-e - n - 1 + 2s) \otimes (\mathcal{L}^*)^2 \to 0
\]

it follows that \( \mathcal{N}^*_{\mathcal{L}'} \simeq \mathcal{E}|_X (-s) \otimes \mathcal{L} \) and \( Y' \) is a l.c.i. scheme with \( \omega_{Y'} \simeq \mathcal{O}_Y(2s - n - 1) \otimes (\mathcal{L}^*)^2 \).

Denote by \( \Sigma_1 \) and \( \Sigma_2 \) two general partials of \( \Sigma \). Since \( X \cap Z = X \cap \text{Sing}(\Sigma) \), \( \Sigma_1 \) and \( \Sigma_2 \) both contain \( L \). Let \( C' := \text{res}_L(X \cap \Sigma_1 \cap \Sigma_2) \). Since \( \mathcal{N}_{C',X} \simeq (\mathcal{O}_{C'}(s - 1) \oplus \mathcal{O}_{C'}(s - 1)) \otimes \mathcal{L}^* \). We have \( \omega_{C'} \simeq \mathcal{O}_{C'}(e + 2s - 2) \otimes (\mathcal{L}^*)^2 \).

Lemma 2.2. Denote by \( R' \) the residual to \( Y' \) with respect to \( C' \). Then \( C' = Y' \cup R' \) is a geometric linkage and \( \Delta' := R' \cap Y' \) is a Cartier divisor of \( Y' \) such that: \( \mathcal{I}_{\Delta',Y'} \simeq \mathcal{O}_Y(-e - n + 1) \).

Proof. We argue as in the proof of Lemma 1.10 denote by \( Y'_{\text{red}} \) the support of \( Y' \), set \( Y'_{\text{red}} = Y'_1 \cup \cdots \cup Y'_{r} \), where \( Y'_i \), \( 1 \leq i \leq r \), are the irreducible components of \( Y'_{\text{red}} \), and denote by \( P_i \) the general point of \( Y'_i \). Choose the partials \( \Sigma_1 \) and \( \Sigma_2 \) in such a way that they generate the ideal sheaf of \( X \cap Z \) at each \( P_i \), \( 1 \leq i \leq r \). In order to check that \( R' \cup Y' \) is a geometric linkage we only need to consider the components contained in \( L \). Consider a point \( P_i \in L \). Since \( L \subset X \cap Z \subset \Sigma_1 \cap \Sigma_2 \), the local equations of \( X \cap Z \) in \( (\mathcal{I}_{Y',X}(s - 1))_{P_i} \) have the form \( (lf, lg) \) where \( l \) is the equation of \( L \), \( lf \) is the equation of \( \Sigma_1 \) and \( lg \) the equation of \( \Sigma_2 \). Since \( Y' := \text{res}_L(X \cap Z) \) and \( C' := \text{res}_L(X \cap \Sigma_1 \cap \Sigma_2) \) then the ideals of both \( Y' \) and \( C' \) at \( P_i \) are equal to \( (f, g) \subset (\mathcal{I}_{Y',X}(s - 1))_{P_i} \). This implies that \( R' \cup Y' \) is a geometric linkage and the remainder of the proof is similar as above.

\[ \square \]

Lemma 2.3. Let \( \Sigma \subset \mathbb{P}^n \), \( n \geq 5 \), be an hypersurface of degree \( s \) containing \( X \), a smooth variety with \( \dim(X) = n - 2 \) and \( \omega_X \simeq \mathcal{O}_X(e) \). Assume \( \sigma_{\Sigma} \) vanishes in codimension two and \( \dim(X \cap \text{Sing}(\Sigma)) = n - 3 \) (see 1.2). Then \( e < s - n \) or \( d(Z) \cdot (e + n + 1) \leq (s - 1)[(s - 1)^2 - d(Z)] \).

Proof. We keep back the notations of 2.1 Notice that the fundamental cycle of \( Y' \) in \( \mathbb{A}_{n-4}(X) \) is

\[
c_2(E|_X (-e - n - 1 + s) \otimes \mathcal{L}^*) = d(Z)H^2 + (e + n + 1 - 2s)H \cap L + L^2 \quad (+)
\]

(\( H \) represents the hyperplane class and \( \cap \) denotes the cap product in \( \mathbb{A}_*(X) \). By abuse of notations, for any \( A \in \mathbb{A}_*(X) \subset \mathbb{A}_*(X) \) we denote by \( d(A) \in \mathbb{Z} \) the degree of \( A \): \( d(A) := d(A \cap H^i) \), \( A \cap H^i \in \mathbb{A}_i(\mathbb{P}^n) \simeq \mathbb{Z} \).

For any closed subscheme \( \Gamma \subset X \) we still denote by \( \Gamma \in \mathbb{A}_*(X) \) the fundamental cycle of \( \Gamma \) (1.5).
We claim that:

$$d(\Delta') \leq (s-1)d(X)((s-1)^2-d(Z)) - [(s-1)(e+n-1) + (s-1)^2 - d(Z)]d(H^2 \cap L) +$$

$$+ (e+n-1)d(H \cap L^2) \quad (*)$$

Assume the claim for a while and let’s show how to conclude the proof. Combining \(2.2\) with (*) we get

$$d(\Delta') = d(Y')(e+n-1) \leq$$

$$\leq (s-1)d(X)((s-1)^2 - d(Z)) - [(s-1)(e+n-1) + (s-1)^2 - d(Z)]d(H^2 \cap L) +$$

$$+ (e+n-1)d(H \cap L^2)$$

and by (+) above

$$d(\Delta') = (e+n-1)d(H \cap (d(Z)H^2 + (e+n+1-2s)H \cap L + L^2)) \leq$$

$$\leq (s-1)d(X)((s-1)^2 - d(Z)) - [(s-1)(e+n-1) + (s-1)^2 - d(Z)]d(H^2 \cap L) +$$

$$+ (e+n-1)d(H \cap L^2).$$

If \(e < s - n\) we are done, so we can assume \(e + n \geq s\). We have

$$d(X)d(Z)(e+n-1) \leq (s-1)d(X)((s-1)^2 - d(Z)) +$$

$$+[e+n-1](s-e-n) - (s-1)^2 + d(Z)]d(L)$$

To conclude it is enough to check that \((e+n-1)(e-e-n) - (s-1)^2 + d(Z) \leq 0\).

Since \(d(Z) = d-s(e+n+1-s)\) (see \([1.2]\) and since \(d \leq s(e+n+1) + 1\) by Lemma \([1.6]\) this follows from: \(s(n-1+e) + 1 \leq s(e+n+1-s) + (s-1)^2 + (e+n-s)(e+n-1)\).

A short computation shows that this is equivalent to \(0 \leq (e+n-s)(e+n-1)\), which holds thanks to our assumption \(e+n \geq s\).

**Proof of the claim:**

Denote by \(|M|\) the moving part of the Jacobian of \(\Sigma\) in \(X\) and by \(M\) the corresponding line bundle. The scheme \(\Delta'\) is the base locus of \(|M|_{R'}\) hence \(\Delta' \subset \tilde{M} \cap R'\) where \(\tilde{M}\) is a general element of \(|M|\). We have

$$d(\Delta') \leq d(\tilde{M} \cap R') = d(c_1(\mathcal{M}_{R'})).$$

In order to prove the statement we need to calculate the cycle \(c_1(\mathcal{M}_{R'}) \in \mathbb{A}_{n-5}(X)\). First of all we calculate the fundamental cycle of \(R'\) in \(\mathbb{A}_{n-4}(X)\):

$$R' \sim C' - Y' \sim ((s-1)H - L)^2 - (d(Z)H^2 + (e+n+1-2s)H \cap L + L^2) =$$

$$= ((s-1)^2 - d(Z))H^2 - (e+n-1)H \cap L.$$

Finally, the cycle \(c_1(\mathcal{M}_{R'}) \in \mathbb{A}_{n-5}(X)\) is:

$$c_1(\mathcal{M}_{R'}) \sim ((s-1)H - L) \cap R' \sim$$

$$\sim (s-1)((s-1)^2 - d(Z))H^3 - (s-1)(e+n-1) + (s-1)^2 - d(Z))H^2 \cap L + (e+n-1)H \cap L^2.$$
Now we can state the improved version of Theorem 1.5.

**Theorem 2.4** (Speciality theorem). Let $X \subset \mathbb{P}^n$, $n \geq 5$, be a smooth variety with $\dim(X) = n - 2$ and $\omega_X \cong \mathcal{O}_X(e)$. Let $\Sigma \subset \mathbb{P}^n$ denote an hypersurface of degree $s$ containing $X$. If $X$ is not a complete intersection, then:

$$e \leq \frac{(s - 1)[(s - 1)^2 - n + 1]}{n - 1} - n + 1.$$  

**Proof.** It is sufficient to prove the theorem for $s$ minimal. We distinguish two cases (see Lemma 1.6).

If $\dim(X \cap \text{Sing}(\Sigma)) = n - 4$, then we argue exactly as in the proof of Theorem 1.5.

If $\dim(X \cap \text{Sing}(\Sigma)) = n - 3$, then by Lemma 2.3 we have $e < s - n$ or $d(Z) \cdot (e + n + 1) \leq (s - 1)[(s - 1)^2 - d(Z)]$. In the first case we conclude using $s \geq n - 1$ (Remark 1.6) and, in the second case, we conclude using the fact that $d(Z) \geq n - 1$ by 10.

**Proof of Theorem 0.1**. As explained in the Section 1, it follows from Theorem 2.4 and Lemma 1.6.

3. **Proofs of 0.2 and of 0.3**

**Proof of Theorem 0.2**. If $X$ is not a complete intersection, this follows from Theorem 1.1. Assume $X$ is a complete intersection. Let $F$ and $G$ $(d(F) = f, d(G) = g)$ be two generators of the ideal of $X$. Then the equation of $\Sigma$ has the form $PF + QG$. But since $\Sigma$ is irreducible and since $X \cap \text{Sing}(\Sigma) \neq \emptyset$, then both $P$ and $Q$ have degree $> 0$. This implies $s - 1 \geq f$ and $s - 1 \geq g$ hence $d = fg \leq (s - 1)^2 < s\frac{(s - 1)[(s - 1)^2 - n + 1]}{n - 1} + 1$.

**Proof of Corollary 0.3**. The argument goes as in the proof of 2 Lemma 4.3: by 8 the coefficients of the Hilbert polynomial of $X$ can be bounded in terms of the degree $d$ hence in terms of $s$, by 0.2 and there are finitely many components of $\text{Hilb}(\Sigma)$ containing smooth varieties of dimension $n - 2$.

4. **Proof of 0.4 and 0.5**

**Notations 4.1.** By 34, we may assume that $X$ lies on an irreducible hypersurface $\Sigma$ of degree $n$, $5 \leq n \leq 6$ and that $h^0(\mathcal{I}_X(n - 1)) = 0$. The assumption of 1.2 is satisfied and by Lemma 1.10 and Lemma 2.3 we get: $e < s - n$ or $d(Z) \cdot (e + n + 1) \leq (s - 1)[(s - 1)^2 - d(Z)]$. The first case cannot occur in our situation, we may assume $e \geq 3$ if $n = 5$ by 11 (resp. $e \geq 8$ if $n = 6$ by 17 Cor. 6.2). So we may assume $d(Z) \cdot (e + n + 1) \leq (s - 1)[(s - 1)^2 - d(Z)] (+).$ Now if $e \geq E$, from (+) we get: $d(Z) \leq \frac{(s - 1)^2}{E + n + s}$.
Proof of Theorem 0.4. Applying (+) with $n = s = 5$ and $E = 3$ we get $d(Z) \leq 4$, hence $d(Z) = 4$ (10). Arguing as in 3 Lemma 2.6, every irreducible component of $Z_{red}$ appears with multiplicity, so $Z$ is either a multiplicity four structure on a linear space or a double structure on a quadric. In both cases it is a complete intersection: in the first case this follows from 9 and in the second one, from the fact that $Z$ is given by the Ferrand construction since $\text{endim}(Z_{red}) \leq 4$. □

Proof of Theorem 0.5. Applying (+) with $n = s = 6$ and $E = 8$, we get $d(Z) \leq 6$. If $d(Z) = 6$, (*) implies $e \leq 8$. So $e = 8$ and $6 = d(Z) = d - 6e - 6$. It follows that $d = 60$ and we conclude with 10 Theorem 1.1. So $d(Z) \leq 5$, hence (10), $d(Z) = 5$. Now (*) yields $e \leq 13$. Moreover $5 = d(Z) = d - 6e - 6$ yields $d = 6e + 11$. If $e \leq 10$, again, we conclude with Theorem 1.1 of 9. We are left with the following possibilities: $(d, e) = (77, 11), (83, 12), (89, 13)$. We conclude with 14 (list on page 216). □

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