Bi-Hamiltonian Structure of Super KP Hierarchy

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ABSTRACT

We obtain the bi-Hamiltonian structure of the super KP hierarchy based on the even super KP operator $\Lambda = \theta^2 + \sum_{i=-2}^{\infty} U_i \theta^{-i-1}$, as a supersymmetric extension of the ordinary KP bi-Hamiltonian structure. It is expected to give rise to a universal super $W$-algebra incorporating all known extended superconformal $W_N$ algebras by reduction. We also construct the super BKP hierarchy by imposing a set of anti-self-dual constraints on the super KP hierarchy.
1 Introduction

The integrable nonlinear (partial) differential KdV [1] and KP [2] equations are known to possess bi-Hamiltonian structures. This was first conjectured by Adler [3] and then proved by Gelfand and Dickey [4] for the KdV hierarchy, via the famous Gelfand-Dickey bracket. For the KP hierarchy, which is a set of multi-time evolution equations in Lax form

$$\frac{\partial L}{\partial t_m} = [(L^m)_+, L], \quad m = 1, 2, 3, \ldots$$

(1.1)

based on the pseudo-differential operator

$$L = D + \sum_{r=-1}^{\infty} u_r D^{-r-1}, \quad D \equiv \frac{\partial}{\partial x}$$

(1.2)

and contains all the $N$-th generalized KdV hierarchies by reduction

$$(L^m)_- = 0$$

(1.3)

where $(L^m)_+, (L^m)_-$ denote the purely differential and pseudo-differential parts of $L^m$ respectively, the first Hamiltonian structure was initially constructed by Watanabe [5] and the extension to the bi-Hamiltonian structure was later worked out by Dickey [6].

There are two types of supersymmetric extensions of the KP hierarchy based on the odd pseudo-super-differential operator [7]

$$\bar{\Lambda} = \theta + \sum_{i=-1}^{\infty} \bar{U}_i \theta^{-i-1}, \quad \theta \equiv \frac{\partial}{\partial \zeta} + \zeta \frac{\partial}{\partial x}$$

(1.4)

and the even pseudo-super-differential operator [8]

$$\Lambda = \theta^2 + \sum_{i=-2}^{\infty} U_i \theta^{-i-1}$$

(1.5)

respectively. (Note one can not obtain the odd (even) operator from the even (odd) one.) In the former case, the discussion of its first Hamiltonian structure [9] appeared to be pathological: Because of an inversion of the even-odd parity of the nontrivial conserved supercharges (the integration of the super Hamiltonian functions $SRes\bar{\Lambda}^{2n+1}$), neither these supercharges nor the corresponding super Hamiltonian formalism can be reduced
to the ones of the ordinary KP hierarchy. To be of the correct parity, one needs to start with the letter super hierarchy. It precisely leads to the accomplishment of an explicit construction of the first super KP Hamiltonian structure [10], which is indeed a supersymmetric extension of the first ordinary KP Hamiltonian structure [5]. The reduction of this super KP hierarchy gives rise to all the generalized $N$-th (even) super KdV hierarchies which have been shown to be bi-Hamiltonian [8]. Thus it remains to explore the bi-Hamiltonian structure of the super KP hierarchy.

Of physical interests, the KP hierarchy via its Hamiltonian structures is known to incorporate all known extended conformal $W$-algebras. In particular, the second Hamiltonian structure of the $N$-th KdV hierarchy was found [11] to be identical to the $W_N$ algebra [12]; and then the first Hamiltonian structure of the KP hierarchy itself was shown [13] to generate the $W_{1+\infty}$ and $W_{\infty}$ algebras [14] that are the general linear deformations of $W_N$ in the usual large $N$-limit [15]. Further, the second KP Hamiltonian structure was suggested to be isomorphic to the unique nonlinear deformation of $W_\infty$ – the $\hat{W}_\infty$ algebra [16], which is expected to constitute a universal $W$-algebra containing all $W_N$ algebras through the natural reduction of KP to KdV’s. Upon supersymmetrization, the newly established first super KP Hamiltonian structure [10] naturally gives rise to a linear super $W_{1+\infty}$ algebra whose bosonic sector is identified with $W_{1+\infty} \oplus W_{1+\infty}$, and a subalgebra of it turns out to be isomorphic to the known $N = 2$ super $W_\infty$ algebra [17]. Similar to the ordinary case [16], one intends to obtain a universal super $W$-algebra incorporating all known extended superconformal $W_N$ algebras by reduction, and it is very likely to be the general nonlinear deformation of super $W_\infty$ which retains the characteristic nonlinearity of super $W_N$. While there is evidence showing that super $W_N$ are related to the super KdV hierarchies [18], this universal super $W$-algebra is strongly expected to be the yet to be determined second Hamiltonian structure of the super KP hierarchy.

The main issue of this paper is to obtain a one-parametric family of super Hamiltonian forms based on the even operator (1.5), and identify the two limiting cases of it as the first [10] and second Hamiltonian structures of the corresponding super KP hierarchy. We will present a comprehensive proof of this super bi-Hamiltonian structure in a way similar to the ordinary KP case [6]. In addition, we will construct a super
BKP hierarchy by imposing a number of super anti-self-dual constraints on super KP, generalizing the ordinary BKP hierarchy [19]. It may provide another class of super \( W \)-algebras.

Since both KP (KdV) hierarchy and its Hamiltonian structures – \( W \)-algebras have played essential roles in 2d quantum gravity and noncritical string theories [20,21,22], the determination of the super KP bi-Hamiltomian structure will naturally provide a promising framework of studying 2d quantum supergravity and noncritical superstrings.

2 Super KP Hierarchy in Hamiltonian Formalism

We will formulate the super KP hierarchy on \((1 \mid 1)\) superspace. The basic variables are \( x \) and \( \zeta \) with parity even and odd respectively. The supercovariant derivative \( \theta = \partial/\partial \zeta + \zeta \partial/\partial x \) satisfies \( \theta^2 = \partial(\equiv \partial/\partial x) \). The basic ingredient of super KP hierarchy is a pseudo-super-differential operator with even parity given by eq.(1.5), where the superfield coefficients \( U_i \) are functions of \( x, \zeta \) and various (even) time variables \( t_m (m = 1, 2, 3, \ldots) \). The parity of a function \( F \) will be indicated by \( p(F) \) which is equal to zero for \( F \) being even and one for \( F \) being odd. Accordingly, \( p(U_i) = i + 1 \). Recall that an arbitrary pseudo-super-differential operator \( P \) has the formal expression

\[
P = \sum_{i=-\infty}^{N} V_i \theta^i \tag{2.1}
\]

and

\[
P_+ = \sum_{i=0}^{N} V_i \theta^i, \quad P_- = \sum_{i=-\infty}^{-1} V_i \theta^i. \tag{2.2}
\]

The multiplication of two such operators \( P \) and \( Q \) is determined by the associativity and the basic relation \( \theta U V = (\theta U)V + (-1)^{p(U)}U \theta V \) for \( \theta \) acting on arbitrary superfield functions \( U \) and \( V \); in particular,

\[
\theta^i U = \sum_{l=0}^{\infty} (-1)^{p(U)(i-l)} \begin{bmatrix} i \\ l \end{bmatrix} U^{[l]} \theta^{i-l} \tag{2.3}
\]
where $U_i^{[l]} \equiv (\theta^l U)$ that is different from $\theta^l U$. The super-binomial coefficients $\binom{i}{k}$ turn out to be for $i \geq 0$

$$
\binom{i}{k} = \begin{cases} 
0 & \text{for } k < 0 \text{ or } k > i \text{ or } (i, k) = (0, 1) \text{ mod } 2 \\
\binom{[i/2]}{[k/2]} & \text{for } 0 \leq k \leq i \text{ and } (i, k) \neq (0, 1) \text{ mod } 2
\end{cases}; \quad (2.4)
$$

and are expressed for $i < 0$ by the identity

$$
\binom{i}{k} = (-1)^{[k/2]} \binom{-i + k - 1}{k}. \quad (2.5)
$$

The super KP hierarchy in the Lax form is a system of infinitely many evolution equations for the functions $U_i$

$$
\frac{\partial \Lambda}{\partial t_m} = [(\Lambda^m)_+, \Lambda]. \quad (2.6)
$$

It is easy to check that the different time evolutions in eq.(2.6) are consistent, for these flows actually commute with each other:

$$
\frac{\partial^2 \Lambda}{\partial t_m \partial t_n} = \frac{\partial^2 \Lambda}{\partial t_n \partial t_m}. \quad (2.7)
$$

This super KP hierarchy has four times as many degrees of freedom as its bosonic counterpart has. By letting $U_{2r+1} = v_{2r} = 0$ ($U_i(x, \zeta) \equiv v_i(x) + \zeta u_i(x)$), it reduces, though not manifestly, indeed to the ordinary KP hierarchy [2]. Particularly, it owns infinitely many independent conserved supercharges $\int \int \frac{1}{n} SRes\Lambda^n dxd\zeta \equiv \int \frac{1}{n} SRes\Lambda^n dX$:

$$
\frac{\partial}{\partial t_m} \int \frac{1}{n} SRes\Lambda^n dX = 0 \quad n = 1, 2, 3, \ldots, \quad (2.8)
$$

which are truly the supersymmetric extension of the conserved charges in the KP hierarchy. Here the super-residue of a super operator $P$, $SResP$, means the coefficient of $\theta^{-1}$ term in $P$. The proof of eq.(2.8) is indebted to the powerful theorem on the super commutator of two pseudo-super-differential operators $P$ and $Q$ [7]:

$$
\int SRes[P, Q]dX = 0, \quad (2.9)
$$

where $[P, Q]$ is defined to be $PQ - (-1)^{p(P)p(Q)}QP$. 

5
In searching for the super KP Hamiltonian structures, one tries to put eq.(2.6) into the Hamiltonian form

\[
\frac{\partial \Lambda}{\partial t_m} = K \frac{\delta \Pi_m}{\delta U} = \sum_{i,j=-2}^{\infty} \frac{(-1)^j K_{ij}}{\delta U_j} \frac{\delta \Pi_m}{\delta U_i} \delta^{-i-1}, \tag{2.10}
\]

where the factor \((-1)^j\) is introduced to maintain the correct parity and \(\delta/\delta U_i\) stands for the variational derivative for superfield functions

\[
\frac{\delta F}{\delta U_i} = \sum_{k=0}^{i} (-1)^{(i+1)k+k(k+1)/2} \left( \frac{\partial F}{\partial U_i}[k] \right) \right|^{(i+1)} \tag{2.11}
\]

The infinite dimensional supermatrix \(K_{ij}\) in eq.(2.10) is said to be a Hamiltonian structure if the super Poisson brackets associated with it

\[
\{U_i(X), U_j(Y)\} = K_{ij}(X) \delta(X - Y) \tag{2.12}
\]

form an algebra, that is equivalent to proving the brackets between arbitrary superfield functions \(F(U_i)\) and \(G(U_j)\)

\[
\{ \int F(U_i(X))dX, \int G(U_j(Y))dY \} = \sum_{i,j=-2}^{\infty} \int (-1)^{(p(F)+1)(i+1)} \frac{\delta F(X)}{\delta U_i(Z)} K_{ij}(Z) \frac{\delta G(Y)}{\delta U_j(Z)} dZ \tag{2.13}
\]

satisfy the super Jacobi identities and are super-antisymmetric. Here \(X \equiv (x, \zeta_x)\) and \(\delta(X - Y) \equiv \delta(x - y)\delta(\zeta_x - \zeta_y)\). Correspondingly, \(\Pi_m\) in eq.(2.10) are regarded as super Hamiltonian functions of this Hamiltonian structure. Now with respect to eq.(2.12), one is able to rewrite the Hamiltonian form (2.10) into algebraic brackets

\[
\frac{\partial U_i}{\partial t_m} = [(\Lambda^m)_{+}, \Lambda]_i = \{U_i, \int \Pi_m(Y)dY\} \tag{2.14}
\]

which will appear to be a convenient form in identifying the Hamiltonian structures later.

To proceed, let us expand \(\Lambda^m\) as

\[
\Lambda^m = \sum_{j=-\infty}^{2m} \theta^j \lambda_j(m). \tag{2.15}
\]
It follows that [10]
\[\lambda_j(m) = (-1)^j \frac{1}{(m+1)} \frac{\delta SRes\Lambda^{m+1}}{\delta U_j} \] \quad j \geq 0. \quad (2.16)

Then use the above expression to calculate \([[(\Lambda^m)_+, \Lambda]].\) This leads directly to the first Hamiltonian form of eq.(2.6)

\[
[(\Lambda^m)_+, \Lambda]_{i} = \frac{1}{(m+1)} \sum_{j=-2}^{\infty} (-1)^j K_{ij}^{(1)} \frac{\delta SRes\Lambda^{m+1}}{\delta U_j}
\]

\[
= \{U_i, \frac{1}{m+1} \int SRes\Lambda^{m+1}(Y)dY\}_1
\] \quad (2.17)

where \(K_{ij}^{(1)}\) (the index 1 denotes the first one, similar for 2 below) has been proved to be the first Hamiltonian structure in ref.[10] and the conserved supercharge densities \((1/m+1)SRes\Lambda^{m+1}\) are identified with the super Hamiltonian functions \(\Pi_m^{(1)}\) as expected.

To obtain the second Hamiltonian form of eq.(2.6), let us express \([[(\Lambda^m)_+, \Lambda]]\) into a bilinear form of \(\Lambda\) (similar to the ordinary KP case)

\[
[(\Lambda^m)_+, \Lambda] = (\Lambda\Lambda^{m-1})_+\Lambda - \Lambda(\Lambda^{m-1}\Lambda)_+.
\] \quad (2.18)

Then we substitute eqs.(2.15)-(2.16) for \(\Lambda^{m-1}\) into the right hand side of eq.(2.18), and it turns out to be

\[
((\Lambda\Lambda^{m-1})_+\Lambda - \Lambda(\Lambda^{m-1}\Lambda)_+)_{i} = \frac{1}{m} \sum_{j=-2}^{\infty} (-1)^j K_{ij}^{(2)} \frac{\delta SRes\Lambda^{m}}{\delta U_j}
\]

\[
= \{U_i, \frac{1}{m} \int SRes\Lambda^{m}(Y)dY\}_2
\] \quad (2.19)

where \(K_{ij}^{(2)}\) is the yet to be determined second super KP Hamiltonian structure with \((1/m)SRes\Lambda^{m}\) begin the associated super Hamiltonian functions \(\Pi_m^{(2)}\). We leave the proof of this second Hamiltonian structure to the next section.

### 3 Bi-Hamiltonian Structure

We observe from eq.(2.19) that to obtain the second Hamiltonian structure of eq.(2.6) it is crucial to analyze the mapping from the set of all pseudo-super-differential operators

\[
R \equiv \{M = \sum_i \theta^i m_i\}
\] \quad (3.1)
to a special set of operators $S \equiv \{ \sum_{i=-2}^{\infty} V_i \theta^{-i-1} \}$ (note $\Lambda \in S$):

$$K(M) = (\Lambda M)_+ \Lambda - \Lambda (M \Lambda)_+ = \Lambda (M \Lambda)_- - (\Lambda M)_- \Lambda. \quad (3.2)$$

By a constant shift of $\Lambda$ to $\hat{\Lambda} \equiv \Lambda - c$, the mapping

$$\hat{K}(M) = (\hat{\Lambda} M)_+ \hat{\Lambda} - \hat{\Lambda} (M \hat{\Lambda})_+ = \sum_{i=-2}^{\infty} K_i(M) \theta^{-i-1} \quad (3.3)$$

will actually give rise to a one-parametric family of super KP Hamiltonian forms; by taking $c = 0$, eq.(3.3) obviously goes back to eq.(3.2) and in the $c \to \infty$ limit, eq.(3.3) is reduced to the first Hamiltonian form with $M = \Lambda^n$:

$$\lim_{c \to \infty} (-\frac{1}{c} \hat{K}(M)) = M_+ \Lambda + (\Lambda M)_+ - (M \Lambda)_- \Lambda M_+ = [M_+, \Lambda] + [\Lambda, M] = [\Lambda^n_+, \Lambda]. \quad (3.4)$$

Thus in general, one may consider the bi-Hamiltonian mapping (3.3).

To begin with, let us assign a super-differentiation $\theta_\alpha$ to each pseudo-super-differential operator $\alpha = \sum_{i=-2}^{\infty} \alpha_i \theta^{-i-1} \in S$. It acts on superfield function $F$ as

$$\theta_\alpha F = \sum_{i=0}^{\infty} \sum_{j=-2}^{\infty} (-1)^{p(\alpha)} \alpha_j^{[i]} \frac{\partial F}{\partial U_j^{[i]}}, \quad (3.5)$$

and the action can directly extend onto an arbitrary pseudo-super-differential operator $P = \sum_i V_i \theta^i$ as

$$\theta_\alpha P = \sum_i (\theta_\alpha V_i) \theta^i. \quad (3.6)$$

Nontrivially, this super-differentiation supercommutes with $\theta$:

$$[\theta_\alpha, \theta] = 0 \quad (3.7)$$

and therefore can be well-defined on the super-functionals $\int F(U_i(X))dX$ under appropriate boundary conditions as

$$\theta_\alpha \int F dX = \int \theta_\alpha F dX. \quad (3.8)$$

Now we are going to prove the following proposition:

$$[\theta_{\hat{K}(M)}, \theta_{\hat{K}(N)}] = \theta_{\hat{K}(M(\Lambda N)_- - (M \Lambda)_+ N + \theta_{\hat{K}(M)} N - (-1)^{p(M)} p(N) (M \leftrightarrow N))}. \quad (3.9)$$
Proof: First we expand the right hand side of eq.(3.9) on superfield $F$:

$$[\theta_{\hat{K}(M)},\theta_{\hat{K}(N)}]F = \sum_{i,i'=0} \sum_{j,j'=0} (-1)^{p(N)i+p(M)i'} K_{j,j'}^{[\eta]}(M) \frac{\partial (K_{j,j'}^{[\eta]}(N))}{\partial U_{[\eta]}^{j,j'}} - (-1)^{p(M)p(N)}(M \leftrightarrow N)$$

$$= \sum_{i,i'=0} \sum_{j,j'=0} (-1)^{p(N)i+p(M)i'} K_{j,j'}^{[\eta]}(M) \frac{\partial (K_{j,j'}^{[\eta]}(N))}{\partial U_{[\eta]}^{j,j'}} - (-1)^{p(M)p(N)}(M \leftrightarrow N)$$

$$= \sum_{i} (-1)^{p(M)+p(N)}(\theta_{\hat{K}(M)}(K_{j}(N)))^{[i]} \frac{\partial F}{\partial U_{[j]}^{i}} - (-1)^{p(M)p(N)}(M \leftrightarrow N)$$

$$= \theta_{\hat{K}(M)}(K(N)) - (-1)^{p(M)p(N)}\theta_{\hat{K}(N)}(\hat{K}(M))^F$$

(3.10)

where we have used eq.(3.7) and have taken into account the $(-1)^{p(M)p(N)}(M \leftrightarrow N)$ part. Next one only needs to evaluate

$$A \equiv \theta_{\hat{K}(M)}(\hat{K}(N)) - (-1)^{p(M)p(N)}\theta_{\hat{K}(N)}(\hat{K}(M))$$

$$= \theta_{\hat{K}(M)}(\hat{K}(N)) + \hat{\Lambda}(N\hat{\Lambda})+ - (-1)^{p(M)p(N)}(M \leftrightarrow N).$$

Noticing that $\theta_{\alpha}\Lambda = \alpha$ implied by eq.(3.6) and using eq.(3.7) again, we have

$$A = (\hat{K}(M)N)_+\hat{\Lambda} + (\hat{\Lambda}\theta_{\hat{K}(M)}(N))_+\hat{\Lambda} + (-1)^{p(M)p(N)}(\hat{\Lambda}N)_+\hat{K}(M) - \hat{K}(M)(N\hat{\Lambda})_+$$

$$-\hat{\Lambda}((\theta_{\hat{K}(M)}N)_+\hat{\Lambda} - (-1)^{p(M)p(N)}\Lambda(N\hat{K}(M))_+ - (-1)^{p(M)p(N)}(M \leftrightarrow N)$$

$$= ((\hat{\Lambda}M)_+\hat{\Lambda} - \hat{\Lambda}(M\hat{\Lambda})_+)N)_+\hat{\Lambda} + (-1)^{p(M)p(N)}(\hat{\Lambda}N)_+((\hat{\Lambda}M)_+\hat{\Lambda} - \hat{\Lambda}(M\hat{\Lambda})_)_+$$

$$-((-\hat{\Lambda}M)_+\hat{\Lambda} - \hat{\Lambda}(M\hat{\Lambda})_+)N)_+ - (-1)^{p(M)p(N)}\hat{\Lambda}(N((\hat{\Lambda}M)_+\hat{\Lambda} - \hat{\Lambda}(M\hat{\Lambda})_+)_+$$

$$+\hat{K}(\theta_{\hat{K}(M)}N) - (-1)^{p(M)p(N)}(M \leftrightarrow N).$$

It follows by taking the $(-1)^{p(M)p(N)}(M \leftrightarrow N)$ terms into account more frequently that

$$A = ((\hat{\Lambda}M)_+\hat{\Lambda}N - \hat{\Lambda}(M\hat{\Lambda})_+)N)\hat{\Lambda} + (-1)^{p(M)p(N)}(\hat{\Lambda}N)_+(\hat{\Lambda}M)_+\hat{\Lambda}$$

$$+\hat{\Lambda}(M\hat{\Lambda})_+(N\hat{\Lambda})_+ - (-1)^{p(M)p(N)}\hat{\Lambda}(N(\hat{\Lambda}M)_+\hat{\Lambda} - N\hat{\Lambda}(M\hat{\Lambda})_+)_+$$

$$+\hat{K}(\theta_{\hat{K}(M)}N) - (-1)^{p(M)p(N)}(M \leftrightarrow N)$$

$$= ((\hat{\Lambda}M)_+\hat{\Lambda}N - \hat{\Lambda}(M\hat{\Lambda})_+)N - (\hat{\Lambda}M)_+(\hat{\Lambda}N)_+\hat{\Lambda} + \hat{\Lambda}((\hat{\Lambda}M)_+(N\hat{\Lambda})_+$$

$$+M(\hat{\Lambda}N)_+\hat{\Lambda} - M(\hat{\Lambda}N)_+)\hat{\Lambda} + \hat{K}(\theta_{\hat{K}(M)}N) - (-1)^{p(M)p(N)}(M \leftrightarrow N)$$

$$= (\hat{\Lambda}M(\hat{\Lambda}N)_+ - \hat{\Lambda}(M\hat{\Lambda})_+)\hat{\Lambda} - \hat{\Lambda}((\hat{\Lambda}M)_+N\hat{\Lambda} - M(\hat{\Lambda}N)_+)\hat{\Lambda} + \hat{K}(\theta_{\hat{K}(M)}N) - (-1)^{p(M)p(N)}(M \leftrightarrow N)$$

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\[ + \hat{K}(\theta_{K(M)}N) - (-1)^{p(M)p(N)}(M \leftrightarrow N) \]

\[ = \hat{K}(M(\hat{\Lambda}N)_- - (M\hat{\Lambda})_+ N) + \hat{K}(\theta_{K(M)}N) - (-1)^{p(M)p(N)}(M \leftrightarrow N) \]

\[ = \hat{K}(M(\hat{\Lambda}N)_- - (M\hat{\Lambda})_+ N + \theta_{K(M)}N - (-1)^{p(M)p(N)}(M \leftrightarrow N)) \quad (3.11) \]

as desired. (QED)

This proposition allows us to define a closed 2-form $\Omega$ on the set of super-differentiations \{\(\theta_{K(M)}\)\}:

\[
\Omega(\theta_{K(M)}, \theta_{K(N)}) \equiv \int S\text{Res}(\hat{K}(M)N)dx, \quad (3.12)
\]

which is necessarily super-antisymmetric:

\[
\Omega(\theta_{K(N)}, \theta_{K(M)}) = -(-1)^{p(M)p(N)}\Omega(\theta_{K(M)}, \theta_{K(N)}). \quad (3.13)
\]

The closeness of eq.(3.12) constitutes another proposition.

Proof: With a suitable parity factor in front of each term, we have

\[
d\Omega(\theta_{K(M)}, \theta_{K(N)}, \theta_{K(L)}) = (-1)^{p(M)p(N)}(\theta_{K(M)}\Omega(\theta_{K(N)}, \theta_{K(L)}) - \Omega([\theta_{K(M)}, \theta_{K(N)}], \theta_{K(L)})) + \text{c.p.} \quad (3.14)
\]

where c.p. is understood as the overall cyclic permutations among \(M, N, L\). The first term of eq.(3.14) is well-defined due to eq.(3.8) and is evaluated as follows:

\[
\theta_{K(M)}\Omega(\theta_{K(N)}, \theta_{K(L)}) = \int S\text{Res}(\hat{K}(M)(\hat{\Lambda}N)_+ + (M\hat{\Lambda})_+ L)dX
\]

\[
= \int S\text{Res}\{(\hat{K}(M)(\hat{\Lambda}N)_+ + (M\hat{\Lambda})_+ L)\hat{K}(M) - \hat{K}(M)(\hat{\Lambda}N)_+ - \hat{K}(M)(N\hat{\Lambda})_+ \hat{K}(M)\}N + \hat{K}(M)(N\hat{\Lambda})_+ L + (\hat{\Lambda}N)\hat{K}(N)\theta_{K(M)}L\}dX
\]

\[
= \int S\text{Res}\{\hat{K}(M)(\hat{\Lambda}N)_+ - (M\hat{\Lambda})_+ L - (\hat{\Lambda}N)\hat{K}(N)\theta_{K(M)}L\}dX. \quad (3.15)
\]

The evaluation of the second term of eq.(3.14) needs the proposition (3.9) and it turns out to be

\[
-\Omega([\theta_{K(M)}, \theta_{K(N)}], \theta_{K(L)}) = \int S\text{Res}\{(-1)^{p(M)p(N)}\hat{K}(L)(\hat{\Lambda}N)_+ - (M\hat{\Lambda})_+ N \hat{K}(L)\}dX + \theta_{K(M)}N - (-1)^{p(M)p(N)}(M \leftrightarrow N))dX. \quad (3.16)
\]
Now substituting eqs.(3.15)-(3.16) into (3.14), we find

\[
\begin{align*}
    d\Omega(\hat{K}(M), \hat{K}(N), \hat{K}(L)) &= \int S\text{Res}\{((e^{-1})^{p(M)p(L)}\hat{K}(M)(N\hat{L})_+ - (N\hat{A})_+ L - (e^{-1})^{p(N)p(L)}(N \leftrightarrow L))
    + (M \leftrightarrow L, N \leftrightarrow M, L \leftrightarrow N)) \\
    & \quad + (e^{-1})^{p(M)(p(N)+p(L))}\hat{K}(N)\theta_{\hat{K}(M)} L - (e^{-1})^{p(N)(p(L)+p(M))}\hat{K}(L)\theta_{\hat{K}(N)} M\}dX + c.p.
    \\
    &= 2\int S\text{Res}\{(e^{-1})^{p(M)p(L)}((\hat{A}M)_{+}\hat{A} - \hat{A}(M\hat{A})_{+})(N\hat{A}L)_- - (N\hat{A})_+ L \\
    & \quad - (e^{-1})^{p(N)p(L)}(N \leftrightarrow L))\}dX + c.p.. \tag{3.17}
\end{align*}
\]

Further, by using eq.(2.9) and the following lemma:

\[
\begin{align*}
    (e^{-1})^{p(P)p(R)}\int S\text{Res}(P-QR)_{+}dX + c.p. &= (e^{-1})^{p(P)p(R)}\int S\text{Res}(PQR)dX, \tag{3.17}
\end{align*}
\]

we conclude that

\[
\begin{align*}
    d\Omega(\hat{K}(M), \hat{K}(N), \hat{K}(L)) &= 2(e^{-1})^{p(M)p(L)}\int S\text{Res}((\hat{A}M)_{+}\hat{A} - \hat{A}(M\hat{A})_{+})(N\hat{A}L)_- - (N\hat{A})_+ L \\
    & \quad - (e^{-1})^{p(N)p(L)}(N \leftrightarrow L))dX + c.p.
    \\
    &= 2(e^{-1})^{p(M)p(L)}\int S\text{Res}(\hat{A}M\hat{A}N\hat{L} - M\hat{A}N\hat{L}\hat{A} - (e^{-1})^{p(N)p(L)}(N \leftrightarrow L))dX \\
    &= 0. \tag{3.18}
\end{align*}
\]

(QED)

Before we define the super Poisson brackets associated with this closed 2-form, let us introduce the variational derivative with respect to the super KP operator \(\Lambda\) as an generalization of eq.(2.11):

\[
\frac{\delta F}{\delta \Lambda} = \sum_{i=-2}^{\infty} (-1)^{p(F)+i+1}\theta_{U_i} \frac{\delta F}{\delta U_i}. \tag{3.19}
\]

It follows that

\[
\theta_{\alpha} \int FdX = \int S\text{Res}(\alpha\frac{\delta F}{\delta \Lambda})dX. \tag{3.20}
\]

Comparing eq.(3.20) with (3.12), one finds eq.(3.12) with \(N\) chosen to be \(\delta F/\delta \Lambda\) may be expressed in terms of the super-differentiation \(\theta_{\hat{K}(M)}\) on the functional \(\int FdX\):

\[
\Omega(\hat{K}(M), \theta_{\hat{K}(\frac{\delta F}{\delta \Lambda})}) = \int S\text{Res}(\hat{K}(M)\frac{\delta F}{\delta \Lambda})dX = \theta_{\hat{K}(M)} \int FdX. \tag{3.21}
\]
Note eqs. (3.20)-(3.21) are well-defined due to eq. (3.8) so that we can denote $\theta_{K(\delta F/\delta \Lambda)}$ as $\theta_{\int FdX}$. Now taking $M$ to be $\delta G/\delta \Lambda$, we are led to the following definition of super Poisson brackets between the functionals $\int FdX$ and $\int GdX$:

$$\{ \int F(X)dX, \int G(Y)dY \} = \Omega(\theta_{\int FdX}, \theta_{\int GdY}) = \theta_{\int FdX} \int GdY.$$  \hspace{1cm} (3.22)

Implied by eqs. (3.13), eq. (3.22) is super-antisymmetric as required. Furthermore, with eq. (3.21), one can easily show the next proposition:

$$\theta_{\{ \int FdX, \int GdY \}} = [\theta_{\int FdX}, \theta_{\int GdY}].$$  \hspace{1cm} (3.23)

Proof: It is equivalent to show

$$I \equiv \Omega(\theta_{K(L)}, -[\theta_{\int FdX}, \theta_{\int GdY}] + \theta_{\{ \int FdX, \int GdY \}}) = 0$$  \hspace{1cm} (3.24)

for arbitrary $L$. Indeed, by adding some proper null terms,

$$I = \theta_{K(L)} \{ \int FdX, \int GdY \} - \Omega(\theta_{K(L)}, [\theta_{\int FdX}, \theta_{\int GdY}])$$

$$= -(-1)^{(p(F)+1)(p(G)+1)} \theta_{K(L)} \theta_{\int GdY} \int FdX - \theta_{K(L)} \theta_{\int FdX} \int GdY$$

$$+ (-1)^{(p(L)(p(F)+1)} \theta_{\int FdX} \theta_{K(L)} \int GdY + \theta_{K(L)} \theta_{\int FdX} \int GdY$$

$$+ (-1)^{(p(L)(p(F)+p(G)+2)} \Omega([\theta_{\int FdX}, \theta_{\int GdY}], \theta_{K(L)})$$

$$= -(-1)^{(p(G)+1)(p(F)+p(L)+1)} \theta_{K(L)} \theta_{\int GdY} \int FdX$$

$$- (-1)^{(p(F)+1)(p(G)+1)} \theta_{K(L)} \theta_{\int GdY} \int FdX$$

$$- \theta_{K(L)} \theta_{\int FdX} \int GdY + (-1)^{(p(L)(p(F)+1)} \theta_{\int FdX} \theta_{K(L)} \int GdY$$

$$+ [\theta_{K(L)}, \theta_{\int FdX}] \int GdY + (-1)^{(p(L)(p(F)+p(G)+2)} \Omega([\theta_{\int FdX}, \theta_{\int GdY}], \theta_{K(L)})$$

$$= -\theta_{K(L)} \Omega(\theta_{\int FdX}, \theta_{\int GdY}) - \Omega([\theta_{K(L)}, \theta_{\int FdX}], \theta_{\int GdY})$$

$$- (-1)^{(p(L)(p(F)+p(G)+2)} \theta_{\int FdX} \Omega(\theta_{K(L)}, \theta_{\int GdY}) - \Omega([\theta_{\int FdX}, \theta_{\int GdY}], \theta_{K(L)})$$

$$- (-1)^{(p(G)+1)(p(F)+p(L)+1)} \theta_{\int GdY} \Omega(\theta_{K(L)}, \theta_{\int FdX}) - \Omega([\theta_{\int GdY}, \theta_{\int FdX}], \theta_{K(L)})$$

$$= (-1)^{(p(L)(p(G)+1)+d\Omega(\theta_{K(L)}, \theta_{\int FdX}, \theta_{\int GdY})}$$  \hspace{1cm} (3.25)

which vanishes because of eq. (3.18). (QED)
As a consequence, the super Poisson brackets (3.22) satisfy the super Jacobi identities. Theorem:

\[ J \equiv (-1)^{(p(F)+1)(p(H)+1)} \{ \int F dX, \int G dY, \int H dZ \} + c.p. = 0. \]  

(3.26)

Proof: On one hand,

\[ J = -(p(F)+1)(p(H)+1) \int F dX, \int G dY, \int H dZ \} + c.p. 
= -(p(F)+1)(p(H)+1) \int F dX, \int G dY, \int H dZ \} + c.p. 
= -(p(F)+1)(p(H)+1) \int F dX, \int G dY, \int H dZ \} + c.p., \]

on the other hand, by using eq.(3.25),

\[ J = (p(F)+1)(p(H)+1) \int H dZ + c.p. \]
\[ = (p(F)+1)(p(H)+1) \int H dZ + c.p. \]

Thus,

\[ 2J = -d\Omega(\int F dX, \int G dY, \int H dZ) = 0. \]  

(3.27)

(QED)

So we have proved the super Poisson algebra (3.22) based on the general Hamiltonian form (3.3) for arbitrary pseudo-super-differential operator \( M \). It remains to specify \( M \) so that one may use this superalgebra to indeed obtain the bi-Hamiltonian structure of the super KP hierarchy. Let us first set \( M = \Lambda^{m-1} = \sum_j \theta^j \lambda_j (m-1) \) and the parameter \( c = 0 \). Note only the \( j \geq -2 \) part of \( M \) is involved in eq.(3.3); we denote it as \( \Lambda^{m-1}_\perp \).

From eq.(2.16), we have

\[ \Lambda^{m-1}_\perp = \sum_{j=-2}^{\infty} \theta^j \lambda_j (m-1) \frac{\delta S \text{Res} \Lambda^m}{\delta U_j} 
= \frac{1}{m} \frac{\delta S \text{Res} \Lambda^m}{\delta \Lambda} \]  

(3.28)

Hence, we have correspondingly chosen the function \( F \) in eq.(3.22) to be the super Hamiltonian function \( \Pi^{(2)}_m = (1/m) S \text{Res} \Lambda^m \). Then,

\[ \{ \frac{1}{m} \int S \text{Res} \Lambda^m dX, \int G dY \} \]
\[
\theta(1/m) \int SRes\Lambda^m dX \int GdY = \int SRes(K(\Lambda^{m-1})\frac{\delta G}{\delta \Lambda})dX
\]
\[
= \int SRes([\Lambda^{m}],\Lambda)\frac{\delta G}{\delta \Lambda}dX
\]
which is equivalent to
\[
\left\{ \frac{1}{m} \int SRes\Lambda^m dX, U_i \right\}_2 = -\{U_i, \int \Pi_m(1) dX \}_2 = [(\Lambda^m)_+, \Lambda].
\]
(3.30)

Except for an insignificant negative sign (which can be absorbed into a redefinition of eq.(3.22)), eq.(3.30) precisely coincides with the Hamiltonian form (2.19) and thus gives rise to the second Hamiltonian structure of the super KP hierarchy (2.6).

Now we set \( M = \Lambda^m \) and let \( c \to \infty \). This amounts to choose \( F = (1/(m+1))SRes\Lambda^{m+1} \) and then
\[
\left\{ \frac{1}{m+1} \int SRes\Lambda^{m+1} dX, \int GdY \right\}_1 = \lim_{c \to \infty} \frac{1}{c} \int SRes(\hat{K}(\Lambda^{m})\frac{\delta G}{\delta \Lambda})dX = \int SRes([\Lambda, (\Lambda^m)_+]\frac{\delta G}{\delta \Lambda})dX
\]
(3.31)
or
\[
- \left\{ \frac{1}{m+1} \int SRes\Lambda^{m+1} dX, U_i \right\}_1 = \{U_i, \int \Pi_m(1) dX \}_1 = [(\Lambda^m)_+, \Lambda].
\]
(3.32)

It is identical to eq.(2.17). By taking \( U_{-2} = U_{-1} = 0 \) in \( \Lambda \) in eq.(3.32), which are actually first-class constraints, (3.32) is trivially reduced to the version of the first super KP Hamiltonian structure obtained in ref.[10]. Therefore we have achieved the super KP bi-Hamiltonian structure with eq.(3.22) as its associated superalgebra.

This bi-Hamiltonian structure (3.30) and (3.32) naturally leads to a set of Lenard recursion relations connecting the conserved supercharges \( \int \Pi_m dX \equiv \int \Pi_m^{(2)} dX = \int \Pi_m^{(1)} dX \):
\[
\{U_i, \int \Pi_{m+1} dX \}_1 = -\{U_i, \int \Pi_m dX \}_2.
\]
(3.33)

Furthermore, this set of conserved supercharges is in involution with respect to the bi-Hamiltonian structure (3.30) and (3.32). For example,
\[
\left\{ \int \Pi_n(X)dX, \int \Pi_m(Y)dY \right\}_2 = \frac{1}{mn} \int SRes(K(\Lambda^{m-1})\frac{\delta SRes\Lambda^m}{\delta \Lambda})dX
\]
\[
= \int SRes([\Lambda^{m-1}, \Lambda]\Lambda^{n-1})dX = 0.
\]
(3.34)
The supercommutivity among $\int \Pi_n dX$’s under eq.(3.32) holds similarly or due to the recursion relation (3.33). This proves the formal complete integrability of the super KP hierarchy (2.6).

4 Super BKP Hierarchy

Finally in this section, we construct the super BKP hierarchy – a supersymmetric extension of the ordinary BKP hierarchy [19]. It is a system of nonlinear super-differential evolution equations obtained from the super KP hierarchy (2.6)

$$\frac{\partial \Lambda}{\partial t_m} = [(\Lambda^m)_+, \Lambda]$$

with $m$ being odd positive integers and with the following anti-self-dual constraints imposing on $\Lambda$:

$$\Lambda = -\theta^{-1} \Lambda^* \theta.$$  

Here the “dual” operation $*$ is defined to be

$$\theta^* = -\theta, \quad F(U_i)^* = F(U_i)$$

for any function $F(U_i)$ and

$$(PQ)^* = (-1)^{p(P)p(Q)} Q^* P^*$$

for any two pseudo-super-differential operators $P$ and $Q$. It follows that for $n \geq 0$,

$$\theta^{n*} = (-1)^{(n+1)/2} \theta^n, \quad \theta^{-n*} = (-1)^{[n/2]} \theta^{-n}$$

and for any $P$,

$$SRes P^* = SRes P.$$  

Now we see the constraints (4.2) is equivalent to letting $U_i$ satisfy

$$U_i = \sum_{k=0}^{i}(-1)^{[k/2]+[(i-k)/2]} \left[ \begin{array}{c} i + 1 \\ k + 1 \end{array} \right] U_k^{[i-k]}.$$  

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For the first a few of them, we have explicitly

\begin{align*}
U_{-2} &= 0, \quad U_{-1} = 0, \quad U_0 \equiv U_0, \quad U_1 \equiv U_1, \\
U_2 &= \frac{1}{2}(U_1^{[1]} - U_0^{[2]}), \quad U_3 = -U_1^{[2]}, \\
U_4 &\equiv U_4, \quad U_5 \equiv U_5, \quad \ldots, \\
\end{align*}

(4.8)

and in general, half of the degrees of freedom of \( U_i \) are eliminated by eq.(4.7). Exactly, we have the following proposition: Eq.(4.2) holds if and only if

\[ SRes(\Lambda^{2n+1}\theta^{-1}) = 0 \] (4.9)

and

\[ SRes\Lambda^{2n} = \frac{1}{2}SRes(\Lambda^{2n}\theta^{-1})^{[1]}, \] (4.10)

where \( n = 0, 1, 2, \ldots \) and \( \Lambda^0 \equiv U_{-2}\theta^{-1} \).

Proof: For necessity, by using eq.(4.6) we have

\[ SRes\Lambda^m = SRes\Lambda^{m*} = SRes(-\theta\Lambda\theta^{-1})^m = (-1)^{m+1}SRes\Lambda^m + (-1)^mSRes(\Lambda^m\theta^{-1})^{[1]}, \]

which leads to eq.(4.10) with even \( m \); with \( m \) being odd, we proceed slightly differently: from eq(4.2)

\[ SRes(\Lambda^m\theta^{-1}) = -SRes(\theta^{-1}\Lambda^{m*}) \]

and from eq.(4.6)

\[ SRes(\Lambda^m\theta^{-1}) = SRes(\Lambda^m\theta^{-1})^* = SRes(\theta^{-1}\Lambda^{m*}), \]

hence eq.(4.9) is true.

For sufficiency, we notice from the leading equations of (4.9) and (4.10) that \( U_{-2} = U_{-1} = 0 \), so we can always write

\[ B \equiv \Lambda + \theta^{-1}\Lambda^*\theta = b_i\theta^{-i} + \text{lower order terms} \] (4.11)
where $i$ is certain positive integer. Because $B^* = \theta^{-1}B\theta$, we find

$$(-1)^{[i/2]} b_i \theta^{-i} = (-1)^{i} b_i \theta^{-i} + \text{lower order terms},$$

therefore $b_i$ vanish for $i = 4n + 1,4n + 2 (n = 0,1,2,\ldots)$ as identities. Moreover, for $i = 4n$,

$$SRes(\Lambda^{2n+1}\theta^{-1}) = SRes( (B - \theta^{-1}\Lambda^*\theta)^{2n+1}\theta^{-1})$$

$$= (-1)^{2n+1} SRes( (\theta^{-1}\Lambda^*\theta)^{2n+1}\theta^{-1}) + (2n + 1)b_{4n}$$

$$= -SRes(\Lambda^{2n+1}\theta^{-1}) + (2n + 1)b_{4n},$$

which yields $b_{4n} = 0$ due to eq.(4.9). Similarly for $i = 4n + 3$,

$$SRes\Lambda^{2n+2} = SRes(B - \theta^{-1}\Lambda^*\theta)^{2n+2} = SRes(\theta^{-1}\Lambda^*^{2n+2}\theta) + (2n + 2)b_{4n+3}$$

$$= SRes(\Lambda^{2n+2}\theta^{-1})^{[1]} - SRes\Lambda^{2n+2} + (2n + 2)b_{4n+3},$$

which leads to $b_{4n+3} = 0$ from eq.(4.10). Overall, eq.(4.11) is equal to zero as desired. (QED)

Now we consider the consistency of the super BKP hierarchy, that is obtained via our final proposition: Eq.(4.1) is compatible with eq.(4.2).

Proof: We need to show

$$\frac{\partial(\Lambda + \theta^{-1}\Lambda^*\theta)}{\partial t_m} = 0, \quad m = 1,3,5,\ldots$$

(4.12)

which is equivalent to

$$\theta^{-1}[(\Lambda^m)_+,\Lambda]^*\theta = -[(\Lambda^m)_+,\Lambda]$$

(4.13)

with $\Lambda + \theta^{-1}\Lambda^*\theta = 0$. Indeed,

$$[(\Lambda^m)_+,\Lambda]^* = \Lambda^*(\Lambda^m)^+_+ - (\Lambda^m)^+_+\Lambda^*$$

$$= \theta^\Lambda\theta^{-1}(\theta^m\theta^{-1})_+ - (\theta^m\theta^{-1})_+\theta\Lambda\theta^{-1}$$

$$= \theta^\Lambda\theta^{-1}(\theta^m)_+\theta^{-1} - SRes(\Lambda^m\theta^{-1})^{[1]}\theta^{-1}$$

$$- (\theta^m)_+\theta^{-1} - SRes(\Lambda^m\theta^{-1})^{[1]}\theta^{-1})\theta\Lambda\theta^{-1}$$

$$= -\theta[(\Lambda^m)_+,\Lambda]\theta^{-1}$$
where we have used eq.(4.9). (QED)

In conclusion we note through the natural reduction (4.2) the super KP bi-Hamiltonian structure obtained in earlier sections is expected to give rise to its super BKP counterpart. We conjecture the constraints (4.2) are of first class so that this Hamiltonian reduction can be accomplished without Dirac’s prescription.

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