SIMULTANEOUS LARGE VALUES AND DEPENDENCE OF
DIRICHLET L-FUNCTIONS IN THE CRITICAL STRIP

SHÔTA INOUE AND JUNXIAN LI

Abstract. We consider the joint value distribution of Dirichlet $L$-functions in the critical strip $\frac{1}{2} < \sigma < 1$. We show that the values of distinct Dirichlet $L$-functions are dependent in the sense that they do not behave like independently distributed random variables and they prevent each other from obtaining large values. Nevertheless, we show that distinct Dirichlet $L$-functions can achieve large values simultaneously infinitely often.

1. Introduction and statements of results

Many problems in number theory revolve around determining the size of $L$-functions. Let $\frac{1}{2} < \sigma < 1$. Montgomery [13] conjectured for the Riemann zeta-function $\zeta(s)$ that

$$\max_{t \in [T, 2T]} \log |\zeta(\sigma + it)| \asymp \frac{(\log T)^{1-\sigma}}{\log \log T}.$$  \hspace{1cm} (1.1)

Even under the Riemann Hypothesis we only know that $\log \zeta(\sigma + it) \ll (\log t)^{2-2\sigma} / \log \log t$, which is still far from the conjecture (1.1). However, we do know that the lower bound in (1.1) holds unconditionally from the work of Montgomery [13] using Diophantine approximation, which improved the previous work of Titchmarsh [16, Theorem 8.12]. Later Aistleitner [1] gave a different proof of the lower bound in (1.1) using the resonance method. We would expect that (1.1) holds for other $L$-functions as well, but both methods break down in establishing the lower bound in (1.1) when the coefficients of the $L$-function are not positive. For more general $L$-functions, the best known $\Omega$-result is due to Aistleitner-Pańkowski [2] where they showed that

$$\max_{t \in [T, 2T]} \log |L(\sigma + it)| \gg \frac{(\log T)^{1-\sigma}}{\log \log T},$$ \hspace{1cm} (1.2)

as long as $L(s)$ satisfies some suitable assumptions. On the other hand, one could obtain the lower bound as in (1.1) for the Dedekind zeta-function $\zeta_K(s)$ of a number field $K$ due to the positivity of the coefficients in the Dirichlet series (see e.g. [3, 14]). The Dedekind zeta-function of a number field of degree greater than two can be factorized, for instance, when $K = \mathbb{Q}(\eta_q)$ with $\eta_q$ the primitive $q$-th root of unity we know that $\zeta_K(s)$ can be factored as $\prod_{\chi \pmod{q}} L(s, \chi)$, where $\chi$ runs over all Dirichlet characters modulo $q$. The fact that Dedekind zeta-functions can obtain large values shows that the product of certain $L$-functions can be large, and thus it is natural to ask if all of these $L$-functions can achieve large values simultaneously along vertical lines. The existence of simultaneous large values of general $L$-functions is an open problem mentioned in [12]. We give an affirmative answer to this question in the case of Dirichlet $L$-functions.

2010 Mathematics Subject Classification. Primary 11M06; Secondary 60B12.

Key words and phrases. simultaneous large values, dependence of Dirichlet $L$-functions.
Theorem 1. Let $\frac{1}{2} < \sigma < 1$, $\theta = (\theta_1, \ldots, \theta_r) \in \mathbb{R}^r$, and let $\chi = (\chi_1, \ldots, \chi_r)$ be an $r$-tuple of distinct primitive Dirichlet characters. There exists some constant $c = c(\sigma, r) > 0$ such that for any $T \geq T_0(\sigma, \chi, \theta)$ sufficiently large we have

$$\max_{t \in [T; 2T]} \min_{1 \leq j \leq r} \Re e^{-i\theta_j} \log L(\sigma + it, \chi_j) \geq c \frac{(\log T)^{1-\sigma}}{\log \log T}.$$ 

Remark 1. The lower bound agrees with the lower bound in (1.2) (except the constant $c$) obtained by Aistleitner-Pańkowski for a single general $L$-function whose coefficients are not necessarily positive.

Remark 2. For comparison, we mention the recent work of Mahatab-Pańkowski-Vatwani [12] where they used Diophantine approximation to prove existence of joint large values of a class of $L$-functions in a small neighborhood: they showed that there exist $t_1, \ldots, t_r \in [T, 2T]$ such that

$$\Re e^{-i\theta_j} \log L_j(\sigma + it) \gg \frac{(\log T)^{1-\sigma}}{\log \log T}$$

with $|t_i - t_j| \leq 2(\log T)^{(1+\sigma)/2}(\log \log T)^{1/2}$ when each of the $L$-functions $L_j$ satisfies suitable assumptions.

To prove Theorem 1, we consider the joint value distribution of Dirichlet $L$-functions in the critical strip $\frac{1}{2} < \sigma < 1$. Let us recall the value distribution of the Riemann zeta-function. The study of the value distribution of $\zeta(s)$ in the critical strip dates back to the work of Bohr-Jessen [4], where a continuous limiting distribution of values of $\zeta(\sigma + it)$ in the case $\sigma > \frac{1}{2}$ was established. Precisely, for any fixed $\sigma > \frac{1}{2}$, there exists a probability measure $P_\sigma$ on $(\mathbb{R}, B(\mathbb{R}))$ such that for any fixed $V \in \mathbb{R}$

$$\Psi(T, V) := \frac{1}{T} \meas \{ t \in [T, 2T] : \log |\zeta(\sigma + it)| > V \} \sim P_\sigma((V, +\infty)) \text{ as } T \to +\infty.$$ (1.3)

However, the precise behavior of $P_\sigma$ had not been determined until some 60 years later when Hattori-Matsumoto [7] first showed that for $\frac{1}{2} < \sigma < 1$

$$P_\sigma((V, +\infty)) = \exp \left( -A(\sigma) V^{1-\sigma} (\log V)^{1-\sigma} (1 + o(1)) \right), \ V \to +\infty$$ (1.4)

where $A(\sigma)$ is defined by

$$A(\sigma) = \left( \frac{\sigma^{2\sigma}}{(1-\sigma)^{2\sigma-1} G(\sigma)^\sigma} \right)^{1/\sigma}, \ G(\sigma) = \int_0^\infty \frac{\log I_0(u)}{u^{1+\frac{1}{\sigma}}} du.$$ (1.5)

Here $I_0(z) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(z \cos \theta) d\theta = \sum_{n=0}^{\infty} (z/2)^{2n}/n!^2$ is the modified 0-th Bessel function. Later Lamzouri [9] gave an effective bound for the $o(1)$ in (1.4) uniformly for $V \ll (\log T)^{1-\sigma}$ by comparing $\zeta(s)$ with its random Euler product model.

We would like to generalize (1.3) and (1.4) for the joint value distribution of Dirichlet $L$-functions. To do this, we introduce the following notation. Let $\frac{1}{2} < \sigma < 1$. For an $r$-tuple of Dirichlet characters $\chi = (\chi_1, \ldots, \chi_r)$, $V = (V_1, \ldots, V_r) \in \mathbb{R}^r$, and $\theta \in \mathbb{R}^r$ we denote

$$\Psi(T, V, \chi, \theta) := \frac{1}{T} \meas \{ t \in [T, 2T] : \Re e^{-i\theta_j} \log L(\sigma + it, \chi_j) > V_j \text{ for } j = 1, \ldots, r \}.$$ 

From [15, Theorem 12.1], we know that there exists a probability measure $P_\sigma^\chi$ on $(\mathbb{R}^r, B(\mathbb{R}^r))$ such that

$$\Psi(T, V, \chi, \theta) \sim P_\sigma^\chi((V_1, \infty) \times \cdots \times (V_r, \infty)), \ T \to \infty.$$
The existence of the limiting distribution can be used to study the joint universality theorem (see e.g. [10]). We are interested in determining the behavior of $P^X_\theta$ when $V_j \to \infty$. To state the results, we define for $\alpha = (\alpha_1, \ldots, \alpha_r) \in (\mathbb{R}_{>0})^r$

$$\xi(\sigma, \chi, \theta; \alpha) := \frac{1}{\phi(d)} \sum_{u \in (\mathbb{Z}/d\mathbb{Z})^r} \left| \sum_{j=1}^r \alpha_j e^{-i\theta_j} \chi_j(u) \right|^2$$

(1.6)

with $\phi$ the Euler totient function and $d$ the least common multiple of the moduli of $\chi_1, \ldots, \chi_r$.

Throughout this paper, we denote the vector with all components being one by $1 \in \mathbb{R}^r$ for some suitable positive integer $r$ which should be clear from the content and so we omit it in this notation. We say that two Dirichlet characters $\chi_1$ and $\chi_2$ are equivalent denoted by $\chi_1 \sim \chi_2$ if they are induced by the same primitive character.

We first state a result concerning the joint value distribution for two Dirichlet-$L$-functions.

**Theorem 2.** Let $\frac{1}{2} < \sigma < 1$ and $\theta \in \mathbb{R}^2$. Let $\chi = (\chi_1, \chi_2)$ with $\chi_1 \not\sim \chi_2$. Then, there exists a positive constant $a_1 = a_1(\sigma)$ such that for any large $T, V$ satisfying $V \leq a_1 \frac{\log T}{\log \log T}$, we have

$$\Psi(T, V, \chi, \theta) = \exp \left( -\tilde{\xi}(\sigma, \chi) A(\sigma) V \frac{1}{\sigma} (\log V)^{\frac{1}{1-\sigma}} \left( 1 + O_{\sigma, \chi, \theta} \left( \frac{\log \log V}{\log V} \right)^{1-\sigma} \right) \right)$$

where $V = (V, V)$ and

$$\tilde{\xi}(\sigma, \chi, \theta) = 2^{\frac{1}{1-\sigma}} \left( \xi(\sigma, \chi, \theta; 1) \right)^{\frac{\sigma}{1-\sigma}},$$

and $\xi(\sigma, \chi, \theta; 1) = \text{defined in (1.6)}$ and $A(\sigma)$ is defined in (1.5).

The joint value distribution of $L$-functions on the critical line $\sigma = \frac{1}{2}$ was considered by Bombieri-Hejhal [5], who showed that normalized values of $L$-functions satisfying certain assumptions behave like independent Gaussian distributed random variables. One may wonder if the independence still holds when we move into the critical strip away from the critical line. From work of Voronin [17] and Lee-Nakamura-Pańkowski [11], we know that different $L$-functions under suitable assumptions satisfy joint universality properties, which indicate some independence between these $L$-functions. However, we show that the “independence” breaks down when considering their value distributions and distinct Dirichlet $L$-functions “repel” each other in the sense that they prevent each other obtaining large values simultaneously.

**Theorem 3.** Let $\frac{1}{2} < \sigma < 1$, $\theta \in \mathbb{R}^2$ and $\chi = (\chi_1, \chi_2)$ with $\chi_1 \not\sim \chi_2$. Then $\tilde{\xi}(\sigma, \chi, \theta) > 2$.

Note that from work of Lamzouri, we can infer that (1.3) still holds if one replace the Riemann zeta-function by any fixed Dirichlet $L$-function. Thus if the values of distinct Dirichlet $L$-functions behave like independent random variables, we would have $\tilde{\xi}(\sigma, \chi, \theta) = 2$ contrary to Theorem 3.

The joint distribution for a generic set of $r$-tuples of Dirichlet $L$-functions is more complicated. We need to further introduce the arithmetic factors $\Xi_j(\sigma, \chi, \theta; \alpha)$ for $1 \leq j \leq r$, $\alpha = (\alpha_1, \ldots, \alpha_r) \in (\mathbb{R}_{>0})^r$

$$\Xi_j(\sigma, \chi, \theta; \alpha) := \frac{1}{\phi(d)} \sum_{u \in (\mathbb{Z}/d\mathbb{Z})^r} \left| \sum_{\ell=1}^r \alpha_\ell e^{-i\theta_\ell} \chi_\ell(u) \right|^2 \sum_{k=1}^r \alpha_k e^{-i\theta_k} \chi_k(u),$$

(1.7)
where $d$ is the least common multiple of the moduli of $\chi_1, \ldots, \chi_r$ and $\sum_{u \in (\mathbb{Z}/d\mathbb{Z})^*}^*$ means the sum is over the irreducible residue class of $d$ such that $\sum_{\ell=1}^r \alpha_\ell e^{-i\theta_\ell \chi_\ell (u)} \neq 0$. Note that we have the relation
\[ \xi(\sigma, \chi, \theta; \alpha) = \sum_{j=1}^r \alpha_j \Xi_j(\sigma, \chi, \theta; \alpha). \] (1.8)

With this notation, we can state the result for the joint distribution of $r$ Dirichlet $L$-functions.

**Theorem 4.** Let $\frac{1}{2} < \sigma < 1$, $\boldsymbol{\theta} \in \mathbb{R}^r$, and let $\chi = (\chi_1, \ldots, \chi_r)$ be an $r$-tuple of pairwise inequivalent Dirichlet characters. Suppose $\alpha = (\alpha_1, \ldots, \alpha_r) \in (\mathbb{R}_{>0})^r$ satisfies $\Xi_j(\sigma, \chi, \theta; \alpha) > 0$ for all $1 \leq j \leq r$. Then there exists a positive constant $a_1 = a_1(\sigma, \alpha)$ such that for any sufficiently large $T$ and for $V = (\Xi_1(\sigma, \chi, \theta; \alpha)V, \ldots, \Xi_r(\sigma, \chi, \theta; \alpha)V)$ with $V$ sufficiently large satisfying $V \leq a_1(\log T)^{1-\sigma}$, we have
\[ \Psi(T, V, \chi, \theta) = \exp \left( -\xi(\sigma, \chi, \theta; \alpha)A(\sigma)V^{1-\sigma} (\log V)^{\frac{\sigma}{1-\sigma}} \left( 1 + O_{\sigma, \chi, \theta, \alpha} \left( \left( \frac{\log \log V}{\log V} \right)^{1-\sigma} \right) \right) \right). \]

Remark 3. Here the arithmetic factors $\Xi_j(\sigma, \chi, \theta; \alpha)$ appear naturally in the partial derivatives of the cumulant-generating function and we need their positivity when applying the saddle point method. To apply Theorem 4 unconditionally, we need to find $\alpha$ such that $\Xi_j(\sigma, \chi, \theta; \alpha) > 0$ for all $1 \leq j \leq r$, but the choice of $\alpha$ depends highly on $\chi, \theta,$ and $\sigma$. When $r = 1, 2$ we can simply take $\alpha = 1$. However, the choice $\alpha = 1$ does not work for all choices of $\chi$ in the full range $1/2 < \sigma < 1$. For example, we can find 8 characters modulo 13 such that $\Xi_j(\sigma, 1) < 0$ for some $j$ when $\sigma$ is close to 1. This is the reason that we introduce the parameter $\alpha$ and our choice of $\alpha$ is in Theorem 5 below.

Remark 4. Here we assume that $\chi_i \neq \chi_j$ for all $i \neq j$ since we allow $\theta_j$ to be arbitrary. We could relax the conditions on $\chi_j$ if we put more restrictions on $\theta$. For instance, if $\chi_1, \ldots, \chi_r$ is the set of all Dirichlet characters modulo $d$ and $\theta = (\theta, \ldots, \theta)$, then one can show that $\Xi_j(\sigma, \chi, 1) > 0$ for all $j$ and the conclusion in Theorem 4 still holds. (See the remark after Lemma 5.1.)

Next we give our choice of $\alpha$ which can be used to verify the conditions in Theorem 4 as well as dependence of value distributions of distinct Dirichlet $L$-functions.

**Theorem 5.** Let $\frac{1}{2} < \sigma < 1$, $r \geq 2$, $\boldsymbol{\theta} \in \mathbb{R}^r$, and let $\chi$ be an $r$-tuple of pairwise inequivalent Dirichlet characters. For $\alpha = (\alpha, 1, \ldots, 1)$ with $\alpha \geq \alpha_0(\sigma, r)$ sufficiently large depending only on $\sigma$ and $r$, we have $\Xi_j(\sigma, \chi, \theta; \alpha) > 0$ for all $j$ and
\[ \xi(\sigma, \chi, \theta; \alpha) > \sum_{j=1}^r \Xi_j(\sigma, \chi, \theta; \alpha)^{\frac{1}{1-\sigma}}. \]

Remark 5. If the values of Dirichlet $L$-functions behave like independently distributed random variables, then we would have $\xi(\sigma, \chi, \theta; \alpha) = \sum_{j=1}^r \Xi_j(\sigma, \chi, \theta; \alpha)^{\frac{1}{1-\sigma}}$ contrary to Theorem 5.

To prove Theorem 4, we consider the joint value distribution for the corresponding Dirichlet polynomials. Define $P_\chi(\sigma + it, X) = \sum_{p \leq X} \chi(p) p^{\sigma+it}$ and $\Psi(T, V, X) = \Psi(T, V, X; \chi, \theta)$ by
\[ \Psi(T, V, X) := \frac{1}{T} \meas \left\{ t \in [T, 2T] : \Re e^{-i\theta_j} P_{\chi_j}(\sigma + it, X) > V_j \text{ for all } j = 1, \ldots, r \right\}. \]
Lemma 2.1. Let a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) such that for any large numbers \(T, V, X = (\log T)^L\) with \(V \leq a_2^{(\log T)^{1-\sigma}}\), we have

\[
\Psi(T, V, X) = \exp \left( -\xi(\sigma, \chi; \alpha)A(\sigma)V^{-\sigma} \left( \log V \right)^{-\sigma} \left( 1 + O_{\sigma, \chi, \theta, \alpha} \left( \frac{\log \log V}{\log V} \right)^{1-\sigma} \right) \right)
\]

for \(V = (\Xi_1(\sigma, \chi, \theta; \alpha)V, \ldots, \Xi_r(\sigma, \chi, \theta; \alpha)V)\).

To conclude the introduction, we give some heuristics on the dependence of the joint value distribution in the critical strip \(\frac{1}{2} < \sigma < 1\) by comparing it with the situation on the critical line \(\sigma = \frac{1}{2}\). Roughly speaking, the value distribution of \(\log L(\sigma + it, \chi)\) can be compared with a truncated Dirichlet series over primes with random coefficients: \(\sum_{p \leq X} \frac{\chi(p)\Lambda(p)}{p^{1/2}}\) where \(\{\Lambda(p)\}_p\) is a sequence of independent random variables uniformly distributed on the unit circle and \(X\) is certain parameter that will be chosen depending on the distribution of zeros close to the line \(\text{Re}(s) = \sigma\). On the critical line \(\sigma = \frac{1}{2}\), we need a long Dirichlet polynomial (e.g., with length \(X = t^{1/(\log t)^{O(1)}})\) to determine the distribution of \(\log L \left( \frac{1}{2} + it, \chi \right)\), since there are many zeros close to the critical line and we need \(\frac{1}{2} \sum_{p \leq X} \frac{|\Lambda(p)|^2}{p} \sim \frac{1}{2} \log \log t\) to capture the variance. In this case, the orthogonality of Dirichlet characters is enough to conclude that the joint value distributions of different Dirichlet \(L\)-functions behave like independent random variables asymptotically. One of the key points of the independency comes from the following asymptotic formula:

\[
\mathbb{E} \left[ \left( \sum_{j=1}^r z_j \text{Re} \sum_{p \leq X} \frac{\chi_j(p)\Lambda(p)}{p^{1/2}} \right)^2 \right] \sim \sum_{j=1}^r z_j^2 \mathbb{E} \left[ \left( \text{Re} \sum_{p \leq X} \frac{\chi_j(p)\Lambda(p)}{p^{1/2}} \right)^2 \right]
\]

which holds for any \(z_1, \ldots, z_r \in \mathbb{C}\) as \(X \to \infty\). When \(\sigma > \frac{1}{2}\), we only need a short Dirichlet polynomial (e.g., with length \(\log t^{O(1)}\)) to determine the distribution of \(\log L(\sigma + it, \chi)\), since there are fewer zeros close to the line \(\text{Re} = \sigma\) and the variance \(\frac{1}{2} \sum_{p \leq X} \frac{|\Lambda(p)|^2}{p^{1/2}}\) converges. In contrast to the case of the critical line, for example, we can find some \(z_1, \ldots, z_r \in \mathbb{C}\) such that

\[
\mathbb{E} \left[ \left( \sum_{j=1}^r z_j \text{Re} \sum_{p \leq X} \frac{\chi_j(p)\Lambda(p)}{p^{1/2}} \right)^2 \right] \not\sim \sum_{j=1}^r z_j^2 \mathbb{E} \left[ \left( \text{Re} \sum_{p \leq X} \frac{\chi_j(p)\Lambda(p)}{p^{1/2}} \right)^2 \right]
\]

as \(X \to \infty\) when \(\sigma > \frac{1}{2}\), which then prevent distinct Dirichlet \(L\)-functions from being independent. The choice for \(z_1, \ldots, z_r\) is related to the choice of \(\alpha\) in Theorem 5.

2. Preliminaries

From now on, we use \(\{\Lambda(p)\}_p \in \mathcal{P}\) to denote a sequence of independent random variables on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) uniformly distributed on the unit circle in \(\mathbb{C}\).

Lemma 2.1. Let \(T \geq 5\). Let \(\{a(p)\}\) be a complex sequence over prime numbers. For any \(X \geq 3, k \in \mathbb{Z}_{\geq 1}\) such that \(X^k \leq T\), we have

\[
\int_T^{2T} \left| \sum_{p \leq X} \frac{a(p)}{p^{it}} \right|^{2k} dt \ll Tk! \left( \sum_{p \leq X} |a(p)|^2 \right)^k,
\]

(2.1)
and for any $X \geq 3$, $k \in \mathbb{Z}_{\geq 1}$

$$
\mathbb{E} \left[ \left( \sum_{p \leq X} a(p) \chi(p) \right)^{2k} \right] \leq k! \left( \sum_{p \leq X} |a(p)|^2 \right)^k. 
$$

(2.2)

Here, the above sums run over prime numbers, and $\mathbb{E}[]$ is the expectation.

Proof. This follows from [8, Lemma 3.6] and [8, Lemma 4.3]. □

Lemma 2.2. Let \( \{b_1(m)\}, \ldots, \{b_n(m)\} \) be complex sequences. For any \( q_1, \ldots, q_s \) distinct prime numbers and any \( k_{1,1}, \ldots, k_{1,n}, \ldots, k_{s,1}, \ldots, k_{s,n} \in \mathbb{Z}_{\geq 1} \), we have

\[
\frac{1}{T} \int_{T}^{2T} \prod_{\ell=1}^{s} \left( \text{Re} b_1(q_{\ell}^{k_{1,\ell}}) q_{\ell}^{-it k_{1,\ell}} \right) \cdots \left( \text{Re} b_n(q_{\ell}^{k_{n,\ell}}) q_{\ell}^{-it k_{n,\ell}} \right) dt = \mathbb{E} \left[ \sum_{\ell=1}^{s} \left( \text{Re} b_1(q_{\ell}^{k_{1,\ell}}) \chi(q_{\ell}^{k_{1,\ell}}) \right) \cdots \left( \text{Re} b_n(q_{\ell}^{k_{n,\ell}}) \chi(q_{\ell}^{k_{n,\ell}}) \right) + O \left( \frac{1}{T} \prod_{\ell=1}^{s} \prod_{j=1}^{n} q_{\ell}^{j} |b_j(q_{\ell}^{j})| \right) \right].
\]

Proof. This is [8, Lemma 4.2]. □

Lemma 2.3. Let \( \chi \) be a Dirichlet character modulo \( q \). Let \( \sigma > \frac{1}{2} \). There exist positive constants \( \delta_{\chi} \), \( A = A(\sigma, \chi) \) such that for any \( k \in \mathbb{Z}_{\geq 1} \), \( 3 \leq X \leq T^{1/k} \),

\[
\frac{1}{T} \int_{T}^{2T} \left| \log L(\sigma + it, \chi) - \sum_{2 \leq n \leq X} \frac{\Lambda(n) \chi^*(n)}{n^{\sigma + it} \log n} - \sum_{p \mid q} \log \left( 1 - \frac{\chi^*(p)}{p^{\sigma + it}} \right) \right|^{2k} dt
\]

\[
\leq A^k k^4 X^{\delta_{\chi}(1-2\sigma)} + A^k k^k \left( \frac{X^{1-2\sigma}}{\log X} \right)^k.
\]

Here, \( \Lambda(n) \) is the von Mangoldt function, and \( \chi^* \) is the primitive character that induces \( \chi \).

Proof. We consider the case when \( \chi \) is primitive first. In this case we know that \( L(s, \chi) \) belongs to the Selberg class, satisfies a strong zero density estimate (see [6, Lemma 1]), and \( \sum_{p \mid q} \log \left( 1 - \frac{\chi^*(p)}{p^{\sigma}} \right) = 0 \). Hence, we derive from [8, Proposition 3.3] that

\[
\frac{1}{T} \int_{T}^{2T} \left| \log L(\sigma + it, \chi) - \sum_{2 \leq n \leq X} \frac{\Lambda(n) \chi(n)}{n^{\sigma + it} \log n} \right|^{2k} dt
\]

\[
\leq A^k k^4 T^{\delta_{\chi}(1-2\sigma)} + A^k k^k \left( \sum_{X < p \leq T^{1/k}} \frac{|\chi(p)|^2}{p^{2\sigma}} \right)^k
\]

for some constants \( \delta_{\chi} > 0 \), \( A = A(\sigma, \chi) > 0 \). Using the prime number theorem and partial summation, we see that

\[
\sum_{X < p \leq Y} \frac{|\chi(p)|^2}{p^{2\sigma}} \leq \sum_{p > X} \frac{1}{p^{2\sigma}} \ll_{\sigma} \frac{X^{1-2\sigma}}{\log X},
\]

which completes the proof in the case of primitive characters.
When \( \chi \) is imprimitive, we use the formula
\[
\log L(s, \chi) = \log L(s, \chi^*) + \sum_{p \mid q} \log \left( 1 - \frac{\chi^*(p)}{p^s} \right),
\]
which together with the primitive case completes the proof. \( \square \)

3. Approximate formulae for moment generating functions

In this section, we give an approximate formula for moment generating functions of an \( r \)-tuple of Dirichlet polynomials supported on primes, in analogue to [8, Section 4.1]. Throughout this section, we suppose that \( a(p) = (a_1(p), \ldots, a_r(p)) \) is an \( r \)-tuple of bounded sequences supported on prime numbers. We define \( \| \cdot \| \) stands for the maximum norm, that is \( \| z \| = \max_{1 \leq j \leq r} |z_j| \) if \( z = (z_1, \ldots, z_r) \in \mathbb{C}^r \). We also write \( \| a_j \|_\infty := \sup_p |a_j(p)| \) and \( \| a \| = \| \| a_1 \|_\infty, \ldots, \| a_r \|_\infty \| \). For every \( z = (z_1, \ldots, z_r) \in \mathbb{C}^r \), \( \sigma, t \in \mathbb{R} \), and any prime number \( p \), we define
\[
K_a(p, z) := \sum_{j=1}^r z_j a_j(p) \sum_{k=1}^r z_k a_k(p),
\]
and
\[
P_j(\sigma + it, X) := \sum_{p \leq X} \frac{a_j(p)}{p^{\sigma+it}}.
\]
To compute the moment generating function of \( (P_j(\sigma + it, X))_{j=1}^r \), we work on a subset \( \mathcal{A} \) where the Dirichlet polynomials do not obtain large values. Precisely, let \( \mathcal{A} = \mathcal{A}(T, X, a) \) be
\[
\mathcal{A} = \bigcap_{j=1}^r \left\{ t \in [T, 2T] : |P_j(\sigma + it, X)| \leq \left( \frac{\log T}{\log \log T} \right)^{1-\sigma} \right\}.
\]
(3.1)

We first show that the measure of \( \mathcal{A} \) is close to \( T \).

**Lemma 3.1.** Let \( L \geq 1 \). Let \( T, X \) be large numbers with \( X \leq (\log T)^L \). Then, there exists a positive number \( b_0 = b_0(\sigma, L, \| a \|, r) \) such that
\[
\frac{1}{T} \text{meas}([T, 2T] \setminus \mathcal{A}) \leq \exp \left( -b_0 \frac{\log T}{\log \log T} \right).
\]

**Proof.** Let \( k \) be a positive integer. If \( X \leq k \log 2k \), then we see that
\[
\frac{1}{T} \int_0^T \left| \sum_{p \leq X} \frac{a_j(p)}{p^{\sigma+it}} \right|^{2k} dt \ll \left( \sum_{p \leq k \log 2k} \frac{a_j(p)}{p^\sigma} \right)^{2k} \ll \left( \frac{C_1 k^{1-\sigma}}{(\log 2k)^\sigma} \right)^{2k}
\]
for some constant \( C_1 > 0 \) depending on \( \sigma \) and \( \| a_j \|_\infty \). Suppose that the inequality \( k \log 2k < X \) holds. We then write
\[
\int_T^{2T} |P_j(\sigma + it, X)|^{2k} dt \leq 4^k \left( \int_T^{2T} \left| \sum_{p \leq k \log 2k} \frac{a_j(p)}{p^{\sigma+it}} \right|^{2k} dt + \int_T^{2T} \left| \sum_{k \log 2k < p \leq X} \frac{a_j(p)}{p^{\sigma+it}} \right|^{2k} dt \right).
\]
By estimate (2.1) and the prime number theorem, it holds that, for \(1 \leq k \leq \frac{\log T}{L \log \log T}\),
\[
\frac{1}{T} \int_{T}^{2T} \left| \sum_{k \log 2k < p \leq X} \frac{a_j(p)}{p^\sigma+it} \right|^{2k} \, dt \ll k! \left( \sum_{k \log 2k < p \leq X} \frac{|a_j(p)|^2}{p^{2\sigma}} \right)^k \ll \left( \frac{C_2 k^{1-\sigma}}{\log (2k)^\sigma} \right)^{2k},
\]
where \(C_2\) is a positive constant which may depend on \(\sigma\) and \(\|a_j\|_\infty\). Furthermore, by the prime number theorem it follows that
\[
\frac{1}{T} \int_{T}^{2T} \left| \sum_{p \leq \log 2k} \frac{a_j(p)}{p^\sigma+it} \right|^{2k} \, dt \ll \left( \sum_{p \leq \log 2k} \frac{|a_j(p)|}{p^\sigma} \right)^{2k} \ll \left( \frac{C_3 k^{1-\sigma}}{(\log 2k)^\sigma} \right)^{2k}
\]
for some positive constant \(C_3\) which may depend on \(\sigma\) and \(\|a_j\|_\infty\). Hence, we have
\[
\frac{1}{T} \int_{T}^{2T} |P_j(\sigma + it, X)|^{2k} \, dt \ll \left( \frac{B_j k^{1-\sigma} \log \log T}{(\log 2k)^\sigma (\log T)^{1-\sigma}} \right)^{2k}
\]
for any \(1 \leq k \leq \frac{\log T}{L \log \log T}\). Here \(B_j = B(a_j, \sigma) := \max_{1 \leq i \leq 3} C_i\). Therefore, there exist positive constants \(B_j = B_j(\sigma, \|a_j\|_\infty)\) such that
\[
\frac{1}{T} \text{meas} \left\{ t \in [T, 2T] : |P_j(\sigma + it, X)| > \left( \frac{(\log T)^{1-\sigma}}{\log \log T} \right) \right\} \leq \left( \frac{B_j k^{1-\sigma} \log \log T}{(\log 2k)^\sigma (\log T)^{1-\sigma}} \right)^{2k}
\]
holds for any \(2 \leq k \leq \frac{\log T}{L \log \log T}\). Hence, we have
\[
\frac{1}{T} \text{meas}([T, 2T] \setminus A) \leq \frac{1}{T} \sum_{j=1}^{r} \text{meas} \left\{ t \in [T, 2T] : |P_j(\sigma + it, X)| > \left( \frac{(\log T)^{1-\sigma}}{\log \log T} \right) \right\}
\]
\[
\leq \left( \frac{B_j k^{1-\sigma} \log \log T}{(\log k)^\sigma (\log T)^{1-\sigma}} \right)^{2k},
\]
where \(B = r \cdot \max_{1 \leq j \leq r} B_j\). By choosing \(k = \lfloor c \log T/\log \log T \rfloor\) for a suitably small constant \(c = c(\sigma, L, \|a\|, r)\), we complete the proof. \(\square\)

Next we compute the moment generating function of \((P_j(\sigma + it, X))_{j=1}^{r}\) on \(A\).

**Proposition 2.** Let \(\frac{1}{2} < \sigma < 1\), \(L \geq 1\) be fixed. There exists a positive constant \(b_1 = b_1(\sigma, L, \|a\|, r)\) such that for large \(T\), \(X = (\log T)^L\), and \(z = (z_1, \ldots, z_r) \in \mathbb{C}^r\) with \(\|z\| \leq b_1(\log T)^\sigma\), we have
\[
\frac{1}{T} \int_{A} \exp \left( \sum_{j=1}^{r} z_j \text{Re} P_j(\sigma + it, X) \right) \, dt
\]
\[
= \prod_{p \leq X} I_0 \left( \frac{\sqrt{K_a(p, z)} / p^{2\sigma}}{b_1 \log \log T} \right) + O \left( \exp \left( -b_1 \frac{\log T}{\log \log T} \right) \right).
\]
Here \(I_0(z)\) is the 0-th modified Bessel function.

To prove Proposition 2, we require two lemmas.

**Lemma 3.2.** For every \(1 \leq j \leq r\), define
\[
P_j(\sigma, X, X) = \sum_{p \leq X} \frac{a_j(p) X(p)}{p^\sigma}. \tag{3.3}
\]
Then there exists a positive constant $C = C(\sigma, \|a_j\|_\infty)$ such that for any $X \geq 3$, $k \in \mathbb{Z}_{\geq 1}$

$$
\mathbb{E}\left[ |P_j(\sigma, X)|^k \right] \leq \left( \frac{C k^{1-\sigma}}{(\log 2 k)^\sigma} \right)^k.
$$

**Proof.** Using the Cauchy-Schwarz inequality, we have

$$
\mathbb{E}\left[ |P_j(\sigma, X)|^k \right] \leq \left( \mathbb{E}\left[ |P_j(\sigma, X)|^{2k} \right] \right)^{1/2}.
$$

Using (2.2) instead of (2.1) in the proof of (3.2), we can show that

$$
\mathbb{E}\left[ |P_j(\sigma, X)|^{2k} \right] \leq \left( \frac{C k^{1-\sigma}}{(\log 2 k)^\sigma} \right)^{2k}
$$

for any $X \geq 3$, $k \in \mathbb{Z}_{\geq 1}$ which completes the proof. \(\square\)

**Lemma 3.3.** For any $z = (z_1, \ldots, z_r) \in \mathbb{C}^r$, we have

$$
\mathbb{E}\left[ \exp \left( \sum_{j=1}^r z_j \text{Re} \left( \frac{a_j(p)}{p^{\sigma}} \mathcal{X}(p) \right) \right) \right] = I_0(\sqrt{K_a(p, z)/p^{2\sigma}}).
$$

**Proof.** By the Taylor expansion of $\exp(\cdot)$ and the identity $\text{Re} w = \frac{w + \overline{w}}{2}$, we can write

$$
\mathbb{E}\left[ \exp \left( \sum_{j=1}^r z_j \text{Re} \left( \frac{a_j(p)}{p^{\sigma}} \mathcal{X}(p) \right) \right) \right] = \sum_{n=0}^\infty \frac{1}{2^n p^{n \sigma n!}} \mathbb{E}\left[ \left( \sum_{j=1}^r z_j (a_j(p) \mathcal{X}(p) + \overline{a_j(p) \mathcal{X}(p)}) \right)^n \right].
$$

Using the binomial expansion, we find that

$$
\left( \sum_{j=1}^r z_j (a_j(p) \mathcal{X}(p) + \overline{a_j(p) \mathcal{X}(p)}) \right)^n = \sum_{\ell=0}^n \binom{n}{\ell} \left( \sum_{j=1}^r z_j a_j(p) \right)^\ell \left( \sum_{k=1}^r z_k a_k(p) \right)^{n-\ell} \mathcal{X}(p)^{2\ell-n}.
$$

Since $\mathcal{X}(p)$ is uniformly distributed on the unit circle in $\mathbb{C}$, we have

$$
\mathbb{E}[\mathcal{X}(p)^a] = \begin{cases} 1 & \text{if } a = 0, \\ 0 & \text{otherwise} \end{cases}
$$

for any $a \in \mathbb{Z}$. Hence, we obtain

$$
\mathbb{E}\left[ \exp \left( \sum_{j=1}^r z_j \text{Re} \left( \frac{a_j(p)}{p^{\sigma}} \mathcal{X}(p) \right) \right) \right] = \sum_{n=0}^\infty \frac{1}{2^n p^{2n \sigma} (2n)!} \binom{2n}{n} K_a(p, z)^n
$$

$$
= \sum_{n=0}^\infty \frac{1}{(n!)^2} \left( \sqrt{K_a(p, z)/p^{2\sigma}} \right)^{2n} = I_0(\sqrt{K_a(p, z)/p^{2\sigma}}).
$$

\(\square\)
Proof of Proposition 2. Let $L \geq 1$ be fixed and $T$, $X$, and $Y$ be large numbers such that $X \leq (\log T)^L$ and $Y = \frac{\log T}{4L \log \log T}$. Let $z = (z_1, \ldots, z_r) \in \mathbb{C}^r$ with $\|z\| \leq b_1(\log T)^\sigma$, where $b_1 = b_1(\sigma, L, \|a\|, r)$ is a suitably small constant to be chosen later. From (3.1), we have

$$\frac{1}{T} \int_A \exp \left( \sum_{j=1}^r z_j \text{Re} P_j(\sigma + it, X) \right) dt$$

$$= \frac{1}{T} \sum_{0 \leq k \leq Y} \frac{1}{k!} \int_A \left( \sum_{j=1}^r z_j \text{Re} P_j(\sigma + it, X) \right)^k dt + O\left( \sum_{k>Y} \frac{1}{k!} \left( r\|z\| \frac{(\log T)^{1-\sigma}}{\log \log T} \right)^k \right).$$

From the choice of $Y$ and the bound for $\|z\|$, we see that this $O$-term is bounded by $\ll \exp \left( -\frac{\log T}{\log \log T} \right)$. Using the Cauchy-Schwarz inequality, we find that

$$\frac{1}{T} \int_A \left( \sum_{j=1}^r z_j \text{Re} P_j(\sigma + it, X) \right)^k dt$$

$$= \frac{1}{T} \int_T^{2T} \left( \sum_{j=1}^r z_j \text{Re} P_j(\sigma + it, X) \right)^k dt$$

$$+ O\left( \frac{1}{T} (\text{meas}([T, 2T] \setminus A))^{1/2} \left( \int_T^{2T} \left| \sum_{j=1}^r z_j \text{Re} P_j(\sigma + it, X) \right|^{2k} dt \right)^{1/2} \right).$$

By Lemma 3.1, estimate (3.2), and the bound for $\|z\|$, this $O$-term is

$$\ll \exp \left( -\frac{b_0}{2} \frac{\log T}{\log \log T} \right) \left( C_1 b_1(\log T)^\sigma \frac{\log T^{1-\sigma}}{(\log 2k)^\sigma} \right)^k$$

$$\ll \exp \left( -\frac{b_0}{2} \frac{\log T}{\log \log T} \right) \left( 2C_1 b_1 \frac{\log T}{\log \log T} \right)^k$$

for $0 \leq k \leq Y$, where $C_1 = C_1(\sigma, \|a\|) > 0$ is a positive constant. It follows that

$$\frac{1}{T} \int_A \exp \left( \sum_{j=1}^r z_j \text{Re} P_j(\sigma + it, X) \right) dt$$

$$= \frac{1}{T} \sum_{0 \leq k \leq Y} \frac{1}{k!} \int_T^{2T} \left( \sum_{j=1}^r z_j \text{Re} P_j(\sigma + it, X) \right)^k dt$$

$$+ O\left( \exp \left( -\frac{b_0}{2} \frac{\log T}{\log \log T} \right) \sum_{0 \leq k \leq Y} \frac{1}{k!} \left( 2C_1 b_1 \frac{\log T}{\log \log T} \right)^k \right).$$

This $O$-term is

$$\ll \exp \left( -\frac{b_0}{2} \frac{\log T}{\log \log T} \right) \sum_{k=0}^{\infty} \frac{1}{k!} \left( 2C_1 b_1 \frac{\log T}{\log \log T} \right)^k = \exp \left( -\left( \frac{b_0}{2} - 2C_1 b_1 \right) \frac{\log T}{\log \log T} \right).$$

Hence the $O$-term on the right hand side of (3.4) is $\ll \exp \left( -\frac{b_0}{4} \frac{\log T}{\log \log T} \right)$ when $b_1$ is sufficiently small.
Next, we write

$$
\int_T^{2T} \left( \sum_{j=1}^r z_j \Re P_j(\sigma + it, X) \right)^k dt
\quad = \quad \sum_{1 \leq j_1, \ldots, j_k \leq r} z_{j_1} \cdots z_{j_k} \sum_{p_1, \ldots, p_k \leq X} \frac{1}{(p_1 \cdots p_k)^\sigma} \int_T^{2T} \left( \Re a_{j_1}(p_1)p_1^{-it} \right) \cdots \left( \Re a_{j_k}(p_k)p_k^{-it} \right) dt.
$$

From this equation and Lemma 2.2, we have

$$
\frac{1}{T} \int_T^{2T} \left( \sum_{j=1}^r z_j \Re P_j(\sigma + it, X) \right)^k dt
\quad = \quad E \left[ \left( \sum_{j=1}^r z_j \Re P_j(\sigma, \mathcal{X}, X) \right)^k \right]
\quad + \quad O \left( \frac{1}{T} \sum_{1 \leq j_1, \ldots, j_k \leq r} |z_{j_1} \cdots z_{j_k}| \sum_{p_1, \ldots, p_k \leq X} |a_{j_1}(p_1) \cdots a_{j_k}(p_k)|(p_1 \cdots p_k)^{1-\sigma} \right),
$$

where $P_j(\sigma, \mathcal{X}, X)$ is defined by (3.3). Additionally, we see that this $O$-term is

$$
\ll \frac{1}{T} \left( \sum_{j=1}^r |z_j| \sum_{p \leq X} |a_j(p)| p^{1-\sigma} \right)^k \leq \frac{(C\|z\|X^2)^k}{T} \leq \frac{C^{2k}}{T^{1/3}} \leq \exp \left( -\frac{\log T}{\log \log T} \right)
$$

for $0 \leq k \leq Y$ when $T$ is sufficiently large. Therefore, we have

$$
\frac{1}{T} \int_A \exp \left( \sum_{j=1}^r z_j \Re P_j(\sigma + it, X) \right) dt
\quad = \quad \sum_{0 \leq k \leq Y} \frac{1}{k!} E \left[ \left( \sum_{j=1}^r z_j \Re P_j(\sigma, \mathcal{X}, X) \right)^k \right] + O \left( \exp \left( -\frac{b_0}{4} \log \log \log T \right) \right)
\quad = \quad E \left[ \exp \left( \sum_{j=1}^r z_j \Re P_j(\sigma, \mathcal{X}, X) \right) \right] - \sum_{k > Y} \frac{1}{k!} E \left[ \left( \sum_{j=1}^r z_j \Re P_j(\sigma, \mathcal{X}, X) \right)^k \right]
\quad + \quad O \left( \exp \left( -\frac{b_0}{4} \log \log \log T \right) \right).
$$

Using Lemma 3.2 and the bound for $\|z\|$, we obtain

$$
E \left[ \left( \sum_{j=1}^r z_j \Re P_j(\sigma, \mathcal{X}, X) \right)^k \right] \leq b_1^k (\log T)^{\sigma k} \left( \frac{Ck^{1-\sigma}}{(\log 2k)^{\sigma}} \right)^k
$$
for some constant \( C = C(a, \sigma) > 0 \). Hence it holds that
\[
\left| \sum_{k > Y} \frac{1}{k!} \mathbb{E} \left[ \left( \sum_{j=1}^{r} z_j \text{Re} P_j(\sigma, X) \right)^k \right] \right| \leq \sum_{k > Y} \frac{(Ch_1(\log T)^\sigma)^k}{(k \log k)^{\sigma k}} \leq \sum_{k > Y} e^{-k} \leq \exp \left( -\frac{\log T}{4L \log \log T} \right).
\]

We find from these that for any \( \|z\| \leq b_1(\log T)^\sigma \) with \( b = b_1(\sigma, L, \|a\|, r) > 0 \) sufficiently small
\[
\frac{1}{T} \int_{A} \exp \left( \sum_{j=1}^{r} z_j \text{Re} P_j(\sigma + it, X) \right) dt
= \mathbb{E} \left[ \exp \left( \sum_{j=1}^{r} z_j \text{Re} P_j(\sigma, X) \right) \right] + O \left( \exp \left( -b \frac{\log T}{\log \log T} \right) \right)
= \prod_{p \leq X} \mathbb{E} \left[ \exp \left( \sum_{j=1}^{r} \text{Re} \frac{a_j(p)}{p^\sigma} \chi_j(X) \right) \right] + O \left( \exp \left( -b \frac{\log T}{\log \log T} \right) \right)
\]
by using the independence of \( \chi_j(\cdot) \). By this equation and Lemma 3.3, we complete the proof of Proposition 2. \( \square \)

4. Distribution of Dirichlet polynomials in the strip \( \frac{1}{2} < \sigma < 1 \)

Let \( \chi = (\chi_1, \ldots, \chi_r) \) be an \( r \)-tuple of Dirichlet characters, and \( \theta \in \mathbb{R}^r \). For \( x = (x_1, \ldots, x_r) \in \mathbb{R}^r \) define
\[
K_{\chi, \theta}(n, x) := \left| \sum_{j=1}^{r} x_j e^{-i\theta_j} \chi_j(n) \right|^2,
F_{\chi, \theta, \sigma}(x) := \frac{1}{\phi(d)} \sum_{u \in \mathbb{Z} / d \mathbb{Z}^r} K_{\chi, \theta}(u, x)^{\frac{1}{\sigma}}. \tag{4.1}
\]

We evaluate the main term in Proposition 2 for \( K_{\chi, \theta}(p, x) \) in this section.

Proposition 3. Under the notation above, for any \( X \geq 30, 3 \leq x_j \leq X^{\frac{2\sigma}{r}} \), we have
\[
\prod_{p \leq X} I_0 \left( \sqrt{K_{\chi, \theta}(p, x)/p^{2\sigma}} \right) = \exp \left( \frac{G(\sigma)}{\log \|x\|} \left( F_{\chi, \theta, \sigma}(x) + O_{\sigma, \chi} \left( \frac{\|x\|^\frac{3}{2} \log \log \|x\|}{\log \|x\|} \right) \right) \right),
\]
where \( G(\sigma) \) is defined in (1.5).

Lemma 4.1. Let \( d, u \in \mathbb{Z} \setminus \{0\} \) with \( (d, u) = 1 \). Let \( \frac{1}{2} < \sigma < 1 \) be fixed. Denote
\[
B(x, X; u, d) := \sum_{p \leq X, p \equiv u \pmod{d}} \log I_0 \left( \frac{x}{p^\sigma} \right).
\]
For \( X \geq 3 \) and \( 0 \leq x \leq X^{\frac{2\sigma}{r}} \), we have
\[
B(x, X; u, d) = \frac{G(\sigma)x^{\frac{1}{\sigma}}}{\phi(d) \log (x + 2)} \left( 1 + O_{d, \sigma} \left( \frac{1}{\log (x + 2)} \right) \right).
\]
Proof. Using the Taylor expansion of $I_0$ we find that for $0 \leq x \leq 2$

$$B(x, X; u, d) \ll \sum_{p \leq X} \frac{x^2}{p^{2\sigma}} \ll x^2 \ll \frac{x^\frac{1}{2}}{(\log(x+2))^2},$$

which proves the lemma when $0 \leq x \leq 2$. From now on, we assume that $x \geq 2$. Let $y_0, y_1$ be some parameters to be chosen later. We split the range of $p \leq X$ into three ranges

$$B(x, X; u, d) = \left( \sum_{p \leq y_0} + \sum_{y_0 < p \leq X} + \sum_{y_0 < p \leq y_1} \right) \log I_0 \left( \frac{x}{p^\sigma} \right) =: S_1 + S_2 + S_3.$$

From the prime number theorem in arithmetic progressions we have

$$A(y; u, d) := \sum_{p \leq y} \frac{\phi(y)}{\phi(d)} + O_d \left( y \exp \left( -c\sqrt{\log y} \right) \right) \quad (4.2)$$

where $\text{li}(y) := \int_2^y \frac{du}{\log u}$. By partial summation, we find that

$$S_3 = - \int_{y_0}^{y_1} A(\xi; u, d) \left( \frac{d}{d\xi} \log I_0 \left( \frac{x}{\xi^\sigma} \right) \right) d\xi + A(y_1; u, d) \log I_0 \left( \frac{x}{y_1^\sigma} \right) \quad (4.3)$$

By equation (4.2), the integral on the right hand side is equal to

$$- \frac{1}{\phi(d)} \int_{y_0}^{y_1} \text{li}(\xi) \left( \frac{d}{d\xi} \log I_0 \left( \frac{x}{\xi^\sigma} \right) \right) d\xi + O_d \left( \int_{y_0}^{y_1} \xi e^{-c\sqrt{\log \xi}} \left( \frac{d}{d\xi} \log I_0 \left( \frac{x}{\xi^\sigma} \right) \right) d\xi \right).$$

Note that we used the monotonicity of $I_0$ in the above deformation. We also have

$$- \int_{y_0}^{y_1} \text{li}(\xi) \left( \frac{d}{d\xi} \log I_0 \left( \frac{x}{\xi^\sigma} \right) \right) d\xi$$

$$= - \text{li}(y_1) \log I_0 \left( \frac{x}{y_1^\sigma} \right) + \text{li}(y_0) \log I_0 \left( \frac{x}{y_0^\sigma} \right) + \int_{y_0}^{y_1} \frac{1}{\log \xi} \log \frac{x}{\xi^\sigma} d\xi,$$

and

$$\int_{y_0}^{y_1} \xi e^{-c\sqrt{\log \xi}} \left( \frac{d}{d\xi} \log I_0 \left( \frac{x}{\xi^\sigma} \right) \right) d\xi$$

$$\ll y_1 e^{-c\sqrt{\log y_1}} \log I_0 \left( \frac{x}{y_1^\sigma} \right) + y_0 e^{-c\sqrt{\log y_0}} \log I_0 \left( \frac{x}{y_0^\sigma} \right) + \int_{y_0}^{y_1} e^{-c\sqrt{\log \xi}} \log I_0 \left( \frac{x}{\xi^\sigma} \right) d\xi$$

$$\ll x^2 y_1^{-1-2\sigma} e^{-c\sqrt{\log y_1}} + x y_0^{-1-\sigma} e^{-c\sqrt{\log y_0}}.$$

Substituting the above estimates to (4.3) and using (4.2), we obtain

$$S_3 = \frac{1}{\phi(d)} \int_{y_0}^{y_1} \frac{\log I_0 \left( \frac{x}{\xi^\sigma} \right)}{\log \xi} d\xi + O_d \left( x^2 y_1^{-1-2\sigma} e^{-c\sqrt{\log y_1}} + x y_0^{-1-\sigma} e^{-c\sqrt{\log y_0}} \right).$$
By making the change of variables $u = \frac{x}{\xi}$, we have
\[
\int_{y_0}^{y_1} \frac{\log I_0\left(\frac{x}{\xi}\right)}{\log \xi} d\xi = x^{\frac{1}{\sigma}} \int_{x/y_0^\sigma}^{x/y_1^\sigma} \frac{\log I_0(u)}{u^{1+\frac{1}{\sigma}}} \log (x/u) du.
\]
We have that \(\frac{1}{\log(x/u)} = \frac{1}{\log x} + O\left(\frac{\|u\|}{\log x}\right)\) for \(x^{-1/2} \leq u \leq x^{1/2}\). Therefore, by choosing \(y_0 = x^{\frac{1}{2\sigma}}, y_1 = x^{\frac{3}{2\sigma}}\) we find that the main term of \((4.4)\) is equal to
\[
\frac{x^{\frac{1}{\sigma}}}{\log x} \int_{x/y_0^\sigma}^{x/y_1^\sigma} \frac{\log I_0(u)}{u^{1+\frac{1}{\sigma}}} du + O\left(\frac{x^{\frac{1}{\sigma}}}{\log x}\right) \int_{x/y_1^\sigma}^{x/y_0^\sigma} \frac{\log I_0(u) \|u\|}{u^{1+\frac{1}{\sigma}}} du).
\]
Moreover, we find that the main term of \((4.4)\) is equal to
\[
\frac{x^{\frac{1}{\sigma}}}{\log x} \int_{0}^{\infty} \frac{\log I_0(u) \|u\|}{u^{1+\frac{1}{\sigma}}} du = O_{\sigma}\left(\frac{x^{\frac{1}{\sigma}}}{(\log x)^{\frac{1}{2}}} \int_{0}^{\infty} \frac{\log I_0(u) \|u\|}{u^{1+\frac{1}{\sigma}}} du\right).
\]
and that the \(O\)-term of \((4.4)\) is
\[
\ll \frac{x^{\frac{1}{\sigma}}}{(\log x)^{\frac{1}{2}}} \int_{0}^{\infty} \frac{\log I_0(u) \|u\|}{u^{1+\frac{1}{\sigma}}} du \ll_{\sigma} \frac{x^{\frac{1}{\sigma}}}{(\log x)^{\frac{1}{2}}}.
\]
Hence, we have
\[
S_3 = \frac{G(\sigma)x^{\frac{1}{\sigma}}}{\phi(d) \log x}\left(1 + O_{d,\sigma}\left(\frac{1}{\log x}\right)\right).
\]
We also find that
\[
S_1 \leq \sum_{p \leq y_0} \frac{x}{p^{\sigma}} \ll x^{\frac{1+\sigma}{\sigma}} \ll_{\sigma} \frac{x^{\frac{1}{\sigma}}}{(\log x)^{\frac{1}{2}}}
\]
by the inequality \(I_0(x/p^{\sigma}) \leq \exp(x/p^{\sigma})\), and that
\[
S_2 \ll \sum_{p > y_1} \frac{x^2}{p^{2\sigma}} \ll_{\sigma} \frac{x^2}{y_1^{2\sigma-1}} \ll_{\sigma} \frac{x^{\frac{1}{\sigma}}}{(\log x)^{\frac{1}{2}}}
\]
by using the Taylor expansion of \(I_0\). Thus, we obtain Lemma 4.1. \(\Box\)

**Proof of Proposition 3.** Let \(x = (x_1, \ldots, x_r) \in (\mathbb{R}_{\geq 3})^r\), and \(\theta \in \mathbb{R}^r\). When \(\chi_j\) is a Dirichlet character modulo \(q_j\), we take \(d = \text{lcm}(q_1, \ldots, q_r)\) and split the sum over \(p\) into residue classes modulo \(d\) to obtain
\[
\sum_{p \leq X} \log I_0\left(\sqrt{K_{\chi,\theta}(p, x)/p^{2\sigma}}\right)
= \sum_{u \in (\mathbb{Z}/d\mathbb{Z})^r} \sum_{\substack{p \leq X \atop p \equiv u \mod d}} \log I_0\left(\sqrt{K_{\chi,\theta}(u, x)/p^{2\sigma}}\right) + \sum_{\substack{p \leq X \atop p|d}} \log I_0\left(\sqrt{K_{\chi,\theta}(p, x)/p^{2\sigma}}\right).
\]
Since \(|\log I_0(x)| \leq x\) and \(|K_{\chi,\theta}(p, x)| \leq \|x\|^2\) it holds that
\[
\sum_{p|d} \log I_0\left(\sqrt{K_{\chi,\theta}(u, x)/p^{2\sigma}}\right) \leq \sum_{p|d} \frac{1}{p^{\sigma}} \|x\| \ll_{\chi} \|x\|.
\]
By Lemma 4.1, we have
\[
\sum_{p \leq X} \log I_0 \left( \sqrt{K_{\chi, \theta}(u, x)/p^{2\sigma}} \right) = \frac{G(\sigma)K_{\chi, \theta}(u, x)^{\sigma}}{\phi(d) \log K_{\chi, \theta}(u, x)^{1/2}} \left( 1 + O_{\sigma, \chi} \left( \frac{1}{\log(K_{\chi, \theta}(u, x)^{1/2} + 2)} \right) \right). \tag{4.5}
\]
If \( K_{\chi, \theta}(u, x) \geq \frac{\|x\|^2}{(\log \|x\|)^2} \), then (4.5) becomes
\[
\frac{G(\sigma)}{\phi(d) \log \|x\|} \left( K_{\chi, \theta}(u, x)^{\sigma} + O_{\sigma, \chi} \left( \frac{\|x\|^\sigma \log \log \|x\|}{\log \|x\|} \right) \right)
\]
by using the trivial bound \( K_{\chi, \theta}(u, x) \leq \|x\|^2 \). If \( K_{\chi, \theta}(u, x) \leq \frac{\|x\|^2}{(\log \|x\|)^2} \), then we have (4.5) is \( \ll_{\sigma, \chi} \frac{\|x\|^\sigma}{(\log \|x\|)^2} \). Hence, we obtain
\[
\sum_{p \leq X} \log I_0 \left( \sqrt{K_{\chi, \theta}(p, x)/p^{2\sigma}} \right) = \frac{G(\sigma)}{\log \|x\|} \left( F_{\chi, \theta, \sigma}(x) + O_{\sigma, \chi} \left( \frac{\|x\|^\sigma \log \log \|x\|}{\log \|x\|} \right) \right),
\]
which completes the proof of Proposition 3. \( \square \)

Now, we are ready to prove Proposition 1.

**Proof of Proposition 1.** Let \( \alpha \in (\mathbb{R}^+)^r \) such that \( \Xi_1(\sigma, \chi, \theta; \alpha), \ldots, \Xi_r(\sigma, \chi, \theta; \alpha) > 0 \). Let \( L \geq 2 \). Let \( T \) be large, and \( X = (\log T)^L \). Let \( V \) be large with \( V \leq \frac{a_2(\log T)^{1-\sigma}}{\log \log V} \), where \( a_2 = a_2(\sigma, L, \alpha) \) is a suitably positive constant to be chosen later. Here, we let the positive parameters \( x_1, \ldots, x_r \) to be the solutions of the equations
\[
V = \frac{G(\sigma)}{\sigma \log (x_1/\alpha_1)}^{\frac{1}{\sigma - 1}} = \cdots = \frac{G(\sigma)}{\sigma \log (x_r/\alpha_r)}^{\frac{1}{\sigma - 1}}.
\]
When \( V \) is sufficiently large, we can verify that \( x_i/\alpha_i = x_j/\alpha_j \) for all \( i, j = 1, \ldots, r \), and that
\[
\frac{x_j}{\alpha_j} = A(\sigma) \left( V \log V \right)^\frac{1}{1-\sigma} \left( 1 + O_{\sigma, \chi, \alpha} \left( \frac{\log \log V}{\log V} \right) \right). \tag{4.6}
\]
To use Propositions 2, 3, we choose \( a_2 = a_2(\sigma, L, \alpha) \) such that \( \|x\| \leq \frac{b_1}{2}(\log T)^\sigma \), where \( b_1 = b_1(\sigma, L, r) \) is the same constant as in Proposition 2 with \( a_j(p) = \chi_j(p)e^{-i\theta_j} \).

For any Lebesgue measurable set \( S \subset [T, 2T] \) and \( \mathbf{v} = (v_1, \ldots, v_r) \in \mathbb{R}^r \), denote
\[
\Psi_S(T, \mathbf{v}, X) := \frac{1}{T} \text{meas} \left\{ t \in S : \Re \sum_{p \leq X} \frac{\chi_j(p)e^{-i\theta_j}}{p^{\sigma+it}} > \Xi_j(\sigma, \chi, \theta; \alpha)v_j \text{ for all } j = 1, \ldots, r \right\}. \tag{4.7}
\]
Let \( y_j = x_j \Xi_j(\sigma, \chi; \alpha) \). Then we find that
\[
y_1 \cdots y_r \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{y_1 v_1 + \cdots + y_r v_r} \tilde{\Psi}_A(T, (v_1, \ldots, v_r), X) dv_1 \cdots dv_r
= \frac{1}{T} \int_\mathcal{A} \exp \left( \sum_{j=1}^{r} x_j \text{Re} \left( \sum_{p \leq X} \frac{\chi_j(p) e^{-ip\theta}}{p^{\sigma+it}} \right) \right) dt,
\]
where \( \mathcal{A} = \mathcal{A}(T, X, \chi, \theta) \) is the set defined by (3.1) with \( a_j(p) = \chi_j(p) e^{-ip\theta} \). By this equation and Propositions 2, 3, we have
\[
\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{y_1 v_1 + \cdots + y_r v_r} \tilde{\Psi}_A(T, (v_1, \ldots, v_r), X) dv_1 \cdots dv_r
= \exp \left( \frac{G(\sigma)}{\log ||x||} \left( F_{X, \theta, \sigma}(x_1, \ldots, (1 + \varepsilon)x_k, \ldots, x_r) + O_{\sigma, X} \left( \left\| x \right\| \frac{\varepsilon \log \log \|x\|}{\log \|x\|} \right) \right) \right).
\]

Next, we divide the range of the integral of (4.8) as follows:
\[
\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} = \int_{V(1-\delta)}^{V(1+\delta)} \cdots \int_{V(1-\delta)}^{V(1+\delta)} + \sum_{k=1}^{r} \left( \int \cdots \int_{D_k^+} + \int \cdots \int_{D_k^-} \right),
\]
where
\[
\int \cdots \int_{D_k^+} = \int_{V(1-\delta)}^{V(1+\delta)} \cdots \int_{V(1-\delta)}^{V(1+\delta)} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty},
\]
\[
\int \cdots \int_{D_k^-} = \int_{V(1-\delta)}^{V(1+\delta)} \cdots \int_{V(1-\delta)}^{V(1+\delta)} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty},
\]
for some small \( \delta \) to be chosen later. By equation (4.8), we find that for any \( \varepsilon > 0 \)
\[
\int \cdots \int_{D_k^{\pm}} e^{y_1 v_1 + \cdots + y_r v_r} \tilde{\Psi}_A(T, (v_1, \ldots, v_r), X) dv_1 \cdots dv_r
\leq e^{\varepsilon y_k V(1+\delta)} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{y_1 v_1 + \cdots + (1+\varepsilon)y_k v_k + \cdots + y_r v_r} \tilde{\Psi}_A(T, (v_1, \ldots, v_r), X) dv_1 \cdots dv_r
= e^{\varepsilon y_k V(1+\delta)} \exp \left( \frac{G(\sigma)}{\log ||x||} \left( F_{X, \theta, \sigma}(x_1, \ldots, (1 + \varepsilon)x_k, \ldots, x_r) + O_{\sigma, X} \left( \left\| x \right\| \frac{\varepsilon \log \log \|x\|}{\log \|x\|} \right) \right) \right).
\]

It follows from the mean value theorem that there exist \( c_\pm \in (0, 1) \) such that
\[
F_{X, \theta, \sigma}(x_1, \ldots, (1 + \varepsilon)x_k, \ldots, x_r)
= F_{X, \theta, \sigma}(x) \pm \frac{\partial F_{X, \theta, \sigma}}{\partial x_k} (x_1, \ldots, (1 \pm c_\pm \varepsilon)x_k, \ldots, x_r) \varepsilon x_k.
\]
From the definition of $F_{\chi, \theta, \sigma}$, we find that
\[
\frac{\partial F_{\chi, \theta, \sigma}}{\partial x_k}(x_1, \ldots, (1 \pm c_{\pm \varepsilon})x_k, \ldots, x_r)
= \frac{1}{\phi(d)} \sum_{u \in (Z/dZ) \times} \frac{1}{\sigma} K_{\chi, \theta}(u, x_1, \ldots, (1 \pm c_{\pm \varepsilon})x_k, \ldots, x_r) \frac{1}{2\sigma - 1} \frac{\partial K_{\chi, \theta}}{\partial x_k}(u, x_1, \ldots, (1 \pm c_{\pm \varepsilon})x_k, \ldots, x_r)
= \frac{1}{\sigma \phi(d)} \sum_{u \in (Z/dZ) \times} \left( \sum_{j=1}^{r} \left| \frac{x_j}{\alpha_j^2} \right| \sum_{j=1}^{r} \alpha_j e^{-i\theta_j} \chi_j(u) \pm \alpha_k c_{\pm \varepsilon} e^{-i\theta_k} \chi_k(u) \right)^2 \frac{1}{2\sigma - 1}
\times \left( \frac{x_k}{\alpha_k} \left( \text{Re} e^{-i\theta_k} \chi_k(u) \right) \sum_{\ell=1}^{r} \alpha_{\ell} e^{-i\theta_{\ell}} \chi_{\ell}(u) \pm \alpha_k c_{\pm \varepsilon} \right)
= \frac{(x_k/\alpha_k)^{\frac{1}{2\sigma} - 1}}{\sigma \phi(d)} \sum_{u \in (Z/dZ) \times} \left( \sum_{j=1}^{r} \alpha_j e^{-i\theta_j} \chi_j(u) \right)^{\frac{1}{2\sigma} - 2} \times (1 + O_{\sigma, \chi, \alpha}(\varepsilon))^{\frac{1}{2\sigma} - 2}
\times \text{Re} e^{-i\theta_k} \chi_k(u) \sum_{\ell=1}^{r} \alpha_{\ell} e^{-i\theta_{\ell}} \chi_{\ell}(u) (1 + O_{\sigma, \chi, \alpha}(\varepsilon)) + O_{\sigma, \chi, \alpha} \left( (\varepsilon x_k)^{\frac{1}{2\sigma} - 1} \right)
= \sigma^{-1} \Xi_k(\sigma, \chi, \theta; \alpha)(x_k/\alpha_k)^{\frac{1}{2\sigma} - 1} + O_{\sigma, \chi, \alpha} \left( (\varepsilon \|x\|)^{\frac{1}{2\sigma} - 1} \right)
\] where $\Xi(\sigma, \chi, \theta; \alpha)$ is defined in (1.7). Applying this in (4.11) and using the definition of $x$, we find that (4.10) is
\[
\leq \exp \left( \frac{G(\sigma)}{\log \|x\|} \left( F_{\chi, \theta, \sigma}(x) - \frac{\varepsilon \delta}{2\sigma} \Xi_k(\sigma, \chi, \theta; \alpha) x_k^{\frac{1}{\sigma}} \alpha_k^{\frac{1}{\sigma}} \right) \right)
\] by taking $\delta = K \varepsilon^{\frac{1}{\sigma} - 1}$ with $\varepsilon = (\log \log \|x\|/\log \|x\|)^{\sigma}$ and $K$ sufficiently large depending on $\sigma, \chi, \alpha$. From the assumption that $\Xi_k(\sigma, \chi, \theta; \alpha) > 0$ we can choose $K$ large enough depending only on $\sigma, \chi, \theta$, and $\alpha$ so that
\[
\int_{V(1+\delta)}^{V(1-\delta)} \cdots \int_{V(1+\delta)}^{V(1-\delta)} e^{y_1 v_1 + \cdots + y_r v_r} \tilde{\Psi}_A(T, (v_1, \ldots, v_r), X) dv_1 \cdots dv_r \quad (4.12)
= \exp \left( \frac{G(\sigma)}{\log \|x\|} \left( F_{\chi, \theta, \sigma}(x) + O_{\sigma, \chi, \alpha} \left( \|x\|^{\frac{\sigma}{2}} \log \log \|x\| \right) \right) \right).
\] by equation (4.8) and (4.9).

Note that
\[
\int_{V(1+\delta)}^{V(1-\delta)} \cdots \int_{V(1+\delta)}^{V(1-\delta)} e^{y_1 v_1 + \cdots + y_r v_r} dv_1 \cdots dv_r = \exp((y_1 + \cdots + y_r)V(1 + O(\delta))),
\]
which, together with (4.12), gives
\[
\tilde{\Psi}_A(T, (V(1 + \delta), \ldots, V(1 + \delta)), X)
\leq \exp \left( \frac{G(\sigma)}{\log \|x\|} \left( F_{\chi, \theta, \sigma}(x) + O_{\sigma, \chi, \alpha} \left( \|x\|^{\frac{\sigma}{2}} \log \log \|x\| \right) \right) \right) - (y_1 + \cdots + y_r)V(1 + O(\delta))
\leq \tilde{\Psi}_A(T, (V(1 - \delta), \ldots, V(1 - \delta)), X).
\]
It follows by (1.6), (1.7), and \( \frac{x_1}{a_1} = \frac{x_2}{a_2} = \cdots = \frac{x_r}{a_r} \) that

\[
y_1 + \cdots + y_r V = \left( \alpha_1 \Xi_1(\sigma, \chi; \theta; \alpha) \frac{x_1}{a_1} + \cdots + \alpha_r \Xi_r(\sigma, \chi; \theta; \alpha) \frac{x_r}{a_r} \right) V = \xi(\sigma, \chi; \theta; \alpha) \frac{x_1}{a_1} V.
\]

Using (4.1) and the equation

\[
\frac{G(\sigma) F_{\chi, \theta, \alpha}(x)}{\log ||x||} = \left( \frac{1}{\phi(d)} \sum_{u \in \langle \mathbb{Z}/d\mathbb{Z} \rangle^r} \left| \sum_{\ell=1}^r \alpha_\ell e^{-i \ell u} \chi(\ell) \right|^\frac{1}{\sigma} \right) \times \frac{G(\sigma)}{\log x_1} \left( \frac{x_1}{a_1} \right)^{1/\sigma} \left( 1 + O_{\sigma, \chi, \alpha} \left( \frac{1}{\log x_1} \right) \right)
\]

we also have

\[
\sigma \xi(\sigma, \chi; \theta; \alpha) \frac{x_1}{a_1} V \left( 1 + O_{\sigma, \chi, \alpha} \left( \frac{1}{\log x_1} \right) \right).
\]

Hence, the middle in (4.13) is equal to

\[
\exp \left( -(1 - \sigma) \xi(\sigma, \chi; \theta; \alpha) \frac{x_1}{a_1} V \left( 1 + O_{\sigma, \chi, \alpha} \left( \frac{1}{\log x_1} \right) \right) \right),
\]

which combined with the inequalities (4.13) yields that

\[
\tilde{\Psi}_A(T, (V, \ldots, V), X)
\]

\[
= \exp \left( -(1 - \sigma) \xi(\sigma, \chi; \theta; \alpha) \frac{x_1}{a_1} V \left( 1 + O_{\sigma, \chi, \alpha} \left( \frac{1}{\log ||x||} \right) \right) \right).
\]

Using (4.6), we obtain

\[
\tilde{\Psi}_A(T, (V, \ldots, V), X)
\]

\[
= \exp \left( -(\xi(\sigma, \chi; \theta; \alpha) A(\sigma) V \frac{1}{\sigma} (\log V)^{1-\sigma} \left( 1 + O_{\sigma, \chi, \alpha} \left( \frac{\log \log V}{\log V} \right)^{1-\sigma} \right) \right)
\]

By this equation and Lemma 3.1, we obtain that when \( a_2 \) is suitably small,

\[
\Psi(T, V, \chi; \theta; \alpha) = \tilde{\Psi}_A(T, (V, \ldots, V), X) + O \left( \frac{1}{T} \text{ meas}([T, 2T] \setminus A) \right)
\]

\[
= \exp \left( -(\xi(\sigma, \chi; \theta; \alpha) A(\sigma) V \frac{1}{\sigma} (\log V)^{1-\sigma} \left( 1 + O_{\sigma, \chi, \alpha} \left( \frac{\log \log V}{\log V} \right)^{1-\sigma} \right) \right)
\]

for \( V = (\Xi_1(\sigma, \chi; \theta; \alpha) V, \ldots, \Xi_r(\sigma, \chi; \theta; \alpha) V) \). This completes the proof of Proposition 1. \( \square \)

5. Value distribution of Dirichlet L-functions in the strip \( \frac{1}{2} < \sigma < 1 \)

**Proof of Theorem 4.** Let \( \frac{1}{2} < \sigma < 1 \), \( \theta \in \mathbb{R}^r \), and let \( \chi = (\chi_1, \ldots, \chi_r) \) be an \( r \)-tuple of Dirichlet characters modulo \( q_j \). Assume that there exists an \( \alpha = (\mathbb{R}_{>0})^r \) such that \( \Xi_1(\sigma, \chi; \theta; \alpha), \ldots, \Xi_r(\sigma, \chi; \theta; \alpha) > 0 \). Let \( T \) be a large parameter, and let \( X = (\log T)^L \) with \( L = \frac{10}{2^{a_2}} \). Moreover, let \( V = (\Xi_1(\sigma, \chi; \theta; \alpha) V, \ldots, \Xi_r(\sigma, \chi; \theta; \alpha) V) \) with \( V \) a large with \( V \leq a_1 (\log T)^{1-\sigma} \). Here, we choose \( a_1(\sigma, \alpha) = a_2(\sigma, L, \alpha) > 0 \) where \( a_2 \) is the same constant.
as in Proposition 1. Using Lemma 2.3, we see that
\[
\max_{1 \leq j \leq r} \frac{1}{T} \int_{T}^{2T} \left| \log L(\sigma + it, \chi_j) - \sum_{2 \leq n \leq X} \frac{\Lambda(n) \chi_j(n)}{n^{\sigma + it} \log n} - \sum_{p \mid q_j} \log \left( 1 - \frac{\chi_j(p)}{p^{\sigma + it}} \right) \right|^2 dt \tag{5.1}
\]
\[
\leq A_k k^{4kT(1-2\sigma)} \delta_\chi + A_k k^k \left( \frac{X^{1-2\sigma}}{\log X} \right)^k \leq A_k k^{4kT(1-2\sigma)} \delta_\chi + \left( \frac{k}{(\log T)^9} \right)^k
\]
for \(1 \leq k \leq \frac{1}{2} \frac{\log T}{\log \log T}\) with \(\delta_\chi\) and \(A = A(\sigma, \chi)\) some positive constants. We see that
\[
\left| \sum_{2 \leq n \leq X} \frac{\Lambda(n) \chi_j(n)}{n^{\sigma + it} \log n} - \sum_{2 \leq n \leq X} \frac{\Lambda(n) \chi_j^*(n)}{n^{\sigma + it} \log n} \right| \leq \sum_{p \mid q_j, \ell \geq 1} \frac{1}{p^{\sigma - 1}}
\]
and that
\[
\left| \sum_{2 \leq n \leq X} \frac{\Lambda(n) \chi_j(n)}{n^{\sigma + it} \log n} - \sum_{p \leq X} \frac{\chi_j(p)}{p^{\sigma + it}} \right| \leq \sum_{p \leq X} \frac{|\chi_j(p)|}{\ell p^{\sigma}}.
\]
Applying inequality (5.1) with \(k = \lfloor c \frac{\log T}{\log \log T} \rfloor, c = \min\{1, \delta_\chi\}(2\sigma - 1)/20\) and the above two inequalities, we can find that there exists a set \(C \subset [T, 2T]\) such that \(\frac{1}{T} \text{meas}([T, 2T] \setminus C) \leq T^{-\delta}\)
with \(d = d(\sigma, \chi)\) some positive constant, and for all \(t \in C\) and \(j = 1, \ldots, r\),
\[
\left| \log L(\sigma + it, \chi_j) - \sum_{p \leq X} \frac{\chi_j(p)}{p^{\sigma + it}} \right| \leq 1 + C_j, \tag{5.2}
\]
where \(C_j\) is the positive constants given by \(C_j = \sum_{p \mid q_j} \frac{1}{p^{\sigma - 1}} + \sum_{p \leq X} \frac{|\chi_j(p)|}{\ell p^{\sigma}}\). In particular, we have
\[
\frac{1}{T} \text{meas}([T, 2T] \setminus C) \leq \exp \left( -2\xi(\sigma, \chi, \theta; \alpha)A(\sigma)V^{\frac{1}{1-\sigma}} (\log V)^{\frac{\alpha}{1-\sigma}} \right) \tag{5.3}
\]
when \(T\) is sufficiently large. By the inequality (5.2) we also find that
\[
\tilde{\Psi}_C(T, (V + (1 + C_1)/\Xi_1(\sigma, \chi, \theta; \alpha), \ldots, V + (1 + C_r)/\Xi_r(\sigma, \chi, \theta; \alpha)), X)
\]
\[
\leq \frac{1}{T} \text{meas} \left\{ t \in C : \Re e^{-i\theta_j} \log L(\sigma + it, \chi_j) > \Xi_j(\sigma, \chi, \theta; \alpha)V_j \right\} \text{ for } j = 1, \ldots, r
\]
\[
\leq \tilde{\Psi}_C(T, (V - (1 + C_1)/\Xi_1(\sigma, \chi, \theta; \alpha), \ldots, V - (1 + C_r)/\Xi_r(\sigma, \chi, \theta; \alpha)), X)
\]
where \(\tilde{\Psi}_C(T, V, X)\) is defined by (4.7). By these inequalities and Proposition 1, we have
\[
\Psi(T, V, \chi, \theta) = \exp \left( -\xi(\sigma, \chi, \theta; \alpha)A(\sigma)V^{\frac{1}{1-\sigma}} (\log V)^{\frac{\alpha}{1-\sigma}} \left( 1 + O_{\sigma, \chi, \theta, \alpha} \left( \left( \frac{\log \log V}{\log V} \right)^{1-\sigma} \right) \right) \right)
\]
\[
+ O \left( \frac{1}{T} \text{meas}([T, 2T] \setminus C) \right)
\]
which together with the inequality (5.3) yields Theorem 4. \qed

Next we prove Theorem 2. We fist prove the positivity of \(\Xi_1(\sigma, \chi, 1)\) when \(r = 2\).

**Lemma 5.1.** Let \(\chi = (\chi_1, \chi_2)\) with \(\chi_1 \napprox \chi_2\) and let \(\theta = (\theta_1, \theta_2) \in \mathbb{R}^2\). Then we have
\[
\Xi_1(\sigma, \chi, \theta; 1) = \Xi_2(\sigma, \chi, \theta; 1) = \xi(\sigma, \chi, \theta; 1)/2 > 0.
\]
Proof. First, we prove \( \xi(\sigma, \chi; \theta; 1) > 0 \). Since \( \chi_1 \) and \( \chi_2 \) are inequivalent, we see that
\[
\xi(\frac{1}{2}, \chi, \theta; 1) = \frac{1}{\phi(d)} \sum_{u \in (\mathbb{Z}/d\mathbb{Z})^*} \left( \sum_{1 \leq j, k \leq 2} e^{-i(\theta_j - \theta_k)} \chi_j \chi_k(u) = 2. \right.
\]
Therefore, there exists some \( u_0 \in (\mathbb{Z}/d\mathbb{Z})^* \) such that \( \sum_{j=1}^2 e^{-i\theta_j} \chi_j(u_0) \neq 0 \), which yields the positivity of \( \xi(\sigma, \chi, \theta; 1) \).

Next, we show that \( \Xi_1(\sigma, \chi, \theta; 1) = \Xi_2(\sigma, \chi, \theta; 1) = \xi(\sigma, \chi, \theta; 1)/2 \) when \( r = 2 \). The second equation follows from the first one since we have
\[
\xi(\sigma, \chi, \theta; 1) = \sum_{j=1}^r \Xi_j(\sigma, \chi, \theta; 1)
\]
by (1.8). If \( \chi_1(u) \neq 0, \chi_2(u) \neq 0 \), we see that
\[
\Re \left( e^{-i\theta_1} \chi_1(u) (e^{-i\theta_1} \chi_1(u) + e^{-i\theta_2} \chi_2(u)) \right) = \Re \left( 1 + \chi_1(u) \chi_2(u) e^{-i(\theta_2 - \theta_1)} \right) = \Re \left( 1 + e^{-i(\theta_1 - \theta_2)} \chi_1(u) \chi_2(u) \right) = \Re \left( e^{-i\theta_2} \chi_2(u) (e^{-i\theta_1} \chi_1(u) + e^{-i\theta_2} \chi_2(u)) \right).
\]
Thus
\[
\Xi_1(\sigma, \chi, \theta; 1) = \frac{1}{\phi(d)} \sum_{u \in (\mathbb{Z}/d\mathbb{Z})^*} \left| \sum_{\ell=1}^2 e^{-i\theta_\ell} \chi_\ell(u) \right|^\frac{1}{r-2} \Re \left( \sum_{k=1}^2 e^{-i\theta_k} \chi_k(u) \right) = \frac{1}{\phi(d)} \sum_{u \in (\mathbb{Z}/d\mathbb{Z})^*} \left| \sum_{\ell=1}^2 e^{-i\theta_\ell} \chi_\ell(u) \right|^\frac{1}{r-2} \Re \left( e^{-i\theta_1} \chi_1(u) \sum_{k=1}^2 e^{-i\theta_k} \chi_k(u) \right) = \Xi_2(\sigma, \chi, \theta; 1).
\]

Remark 6. Note that the proof of Lemma 5.1 can be generalized when \( \{\overline{\chi} \chi k\}_{k=1, \ldots, \phi(q)} \) runs over all characters modulo \( d \). We also note that with the choice \( \theta = (\theta, \ldots, \theta) \) and \( \alpha = (1, \ldots, 1) \)
\[
\xi(\sigma, \chi, \theta; 1) = \phi(q) \left( \frac{1}{\phi(q)} \right)^{\frac{1}{r-1}}.
\]
by orthogonality of Dirichlet characters. By the choice of \( \chi \), we see as in the proof of Lemma 5.1 that
\[
\Xi_j(\sigma, \chi, \theta; 1) = \xi(\sigma, \chi, \theta; 1)/\phi(q) > 0.
\]
Proof of Theorem 2. Combing Theorem 4 and Lemma 5.1, we derive Theorem 2. □

Next we give the proof of Theorem 3 which shows that distinct Dirichlet characters “repels” each other.

Proof of Theorem 3. From Theorem 2, it suffices to show that \( 0 < \xi(\sigma, \chi, \theta; 1) < 2 \). Since we have
\[
\xi(\sigma, (\chi_1, \chi_2), (\theta_1, \theta_2); 1) = \xi(\sigma, (1, \overline{\chi_1} \chi_2), (0, \theta_2 - \theta_1); 1),
\]
it is enough show that for any non-principal character \( \chi \) (mod \( q \)) and \( \theta \in \mathbb{R} \)
\[
\frac{1}{\phi(q)} \sum_{a \in (\mathbb{Z}/q\mathbb{Z})^*} |1 + e^{-i\theta} \chi(a)|^{1/\sigma} < 2.
\] (5.4)
The positivity follows from Lemma 5.1. Next we consider the upper bound. We have \(|1 + e^{-i\theta} \chi(a)|^\frac{2}{2} = 2\Re(1 + Re e^{-i\theta} \chi(a))^\frac{2}{2}\), and \((1 + x)^{1/2} \leq 1 + \frac{1}{2\sigma} x\) for \(-1 \leq x \leq 1\). Using these, we have

\[
\frac{1}{\phi(q)} \sum_{a \in (\mathbb{Z}/q\mathbb{Z})^*} |1 + e^{-i\theta} \chi(a)|^{1/\sigma} \leq \frac{2^{1/2\sigma}}{\phi(q)} \sum_{a \in (\mathbb{Z}/q\mathbb{Z})^*} \left(1 + \frac{1}{2\sigma} \Re e^{-i\theta} \chi(a)\right) = 2^{1/2\sigma} < 2,
\]

which is inequality (5.4). Thus, we obtain Theorem 3.

As mentioned in Remark 3, the choice \(\alpha = 1\) cannot guarantee that \(\Xi_j(\sigma, \chi, \theta; 1) > 0\) for all \(\chi, \theta\) and all \(\frac{1}{2} < \sigma < 1\). Nevertheless we can find a particular choice for \(\alpha\) such that \(\Xi_j(\sigma, \chi, \theta; \alpha) > 0\). To state the following results, we introduce the quantity \(B(\chi, \theta)\) defined by

\[
B(\chi, \theta) = \sum_{\ell = 2}^{r} \sum_{\chi \sim \chi_{1}\chi_{\ell}} \cos(2\theta_1 - \theta_j - \theta_\ell).
\]

**Proposition 4.** Let \(\frac{1}{2} < \sigma < 1\), \(\theta \in \mathbb{R}^r\), and let \(\chi = (\chi_1, \ldots, \chi_r)\) be an \(r\)-tuple of Dirichlet characters. Suppose \(\chi_i \neq \chi_j\) for \(i \neq j\). For \(\alpha = (\alpha, 1, \ldots, 1)\) with \(\alpha \geq \alpha_0(\sigma, r)\) sufficiently large, we have

\[
\Xi_j(\sigma, \chi, \theta; \alpha) = \begin{cases} 
\alpha^{\frac{1}{\sigma} - 1} \left(1 + \frac{r - 1 - (2\sigma - 1)B(\chi, \theta)}{4\sigma^2} \left(\frac{1}{\sigma} - 2\right) + O_{\sigma, r}(\alpha^{-3})\right) & \text{if } j = 1, \\
\alpha^{\frac{1}{\sigma} - 2} \left(1 - (2\sigma - 1)\cos(2\theta_1 - \theta_j - \theta_\ell)\right) + O_{\sigma, r}(\alpha^{-1}) & \text{if } \chi_j \sim \chi_{1}\chi_{\ell} \text{ for some } 2 \leq \ell \leq r, \\
\alpha^{\frac{1}{\sigma} - 2} \left(\frac{1}{2\sigma^2} + O_{\sigma, r}(\alpha^{-1})\right) & \text{otherwise.}
\end{cases}
\]

In particular, we have \(\Xi_j(\sigma, \chi, \theta; \alpha) > 0\) for all \(j\) when \(\alpha = (\alpha, 1, \ldots, 1)\) for \(\alpha \geq \alpha_1(\sigma, r)\) sufficiently large depending only on \(\sigma\) and \(r\).

**Proof.** Let \(\alpha_1 = \alpha\) and \(\alpha_2 = \cdots = \alpha_r = 1\). It follows from the Taylor expansion that

\[
\left|\sum_{\ell = 1}^{r} \alpha_\ell e^{-i\theta_\ell} \chi_\ell(u)\right|^{\frac{2}{\sigma^2}} = \alpha^{\frac{1}{\sigma} - 2} \left|1 + \frac{1}{\alpha} \sum_{\ell = 2}^{r} e^{i(\theta_1 - \theta_\ell)} \chi_1\chi_\ell(u)\right|^{\frac{2}{\sigma^2}}
\]

\[
= \alpha^{\frac{1}{\sigma} - 2} \left\{1 + \frac{1}{\alpha} \left(\frac{1}{\sigma} - 2\right) \Re \sum_{\ell = 2}^{r} e^{i(\theta_1 - \theta_\ell)} \chi_1\chi_\ell(u) - \frac{1}{2\alpha^2} \left(\frac{1}{\sigma} - 2\right) \Re \left(\sum_{\ell = 2}^{r} e^{i(\theta_1 - \theta_\ell)} \chi_1\chi_\ell(u)\right)^2 \right\} + O_{\sigma, r}(\alpha^{\frac{1}{\sigma} - 5}).
\]
By this formula and the fact that \( \sum_{\ell=1}^{r} \alpha_\ell e^{-i\theta_\ell} \chi_\ell(u) \) is not zero for all \( u \in (\mathbb{Z}/d\mathbb{Z})^\times \) when \( \alpha \) is large enough (e.g. \( \alpha > r - 1 \)), we find that

\[
\Xi_j(\sigma, \chi, \theta; \alpha) = \frac{\alpha^{\frac{1}{\sigma} - 2}}{\phi(d)} \sum_{u \in (\mathbb{Z}/d\mathbb{Z})^\times} \left\{ 1 + \frac{1}{\alpha} \left( \frac{1}{\sigma} - 2 \right) \Re \sum_{\ell=2}^{r} e^{i(\theta_1 - \theta_\ell)} \overline{\chi_1} \chi_\ell(u) \right. \\
- \frac{1}{2\alpha^2} \left( \frac{1}{\sigma} - 2 \right) \Re \left( \sum_{\ell=2}^{r} e^{i(\theta_1 - \theta_\ell)} \overline{\chi_1} \chi_\ell(u) \right)^2 \\
+ \frac{1}{2\alpha^2} \left( \frac{1}{\sigma} - 2 \right)^2 \left( \Re \sum_{\ell=2}^{r} e^{i(\theta_1 - \theta_\ell)} \overline{\chi_1} \chi_\ell(u) \right)^2 \left\} \Re e^{-i\theta_j} \chi_j(u) \sum_{k=1}^{r} \alpha_k e^{-i\theta_k} \chi_k(u) \\
+ O_{\sigma, r} \left( \alpha^{\frac{1}{\sigma} - 4} \right),
\]

where

\[
A_{j,1}(u) := \Re e^{-i\theta_j} \chi_j(u) \sum_{k=1}^{r} \alpha_k e^{-i\theta_k} \chi_k(u), \\
A_{j,2}(u) := \sum_{\ell=2}^{r} \sum_{k=1}^{r} \alpha_k \Re \left( e^{i(\theta_1 - \theta_\ell)} \overline{\chi_1} \chi_\ell(u) \right) \Re \left( e^{i(\theta_j - \theta_k)} \chi_j \chi_k(u) \right), \\
A_{j,3}(u) := \Re \left( e^{-i(\theta_1 - \theta_j)} \chi_1 \chi_j(u) \right) \Re \left( \sum_{\ell=2}^{r} e^{i(\theta_1 - \theta_\ell)} \overline{\chi_1} \chi_\ell(u) \right)^2, \\
A_{j,4}(u) := \Re \left( e^{-i(\theta_1 - \theta_j)} \chi_1 \chi_j(u) \right) \left( \Re \sum_{\ell=2}^{r} e^{i(\theta_1 - \theta_\ell)} \overline{\chi_1} \chi_\ell(u) \right)^2.
\]

Since \( \chi_1, \ldots, \chi_r \) are pairwise inequivalent, there exists one \( \chi_i \) such that \( \chi_i \overline{\chi_i} \) is principal, and there exists at most one \( \chi_i \) such that \( \chi_i^2 \overline{\chi_i} \) is principal. By this observation and the orthogonality of characters, we have

\[
\sum_{u \in (\mathbb{Z}/d\mathbb{Z})^\times} A_{j,1}(u) = \sum_{u \in (\mathbb{Z}/d\mathbb{Z})^\times} \Re e^{-i\theta_j} \chi_j(u) \sum_{k=1}^{r} \alpha_k e^{-i\theta_k} \chi_k(u) \\
= \Re \sum_{k=1}^{r} e^{i(\theta_j - \theta_k)} \alpha_k \sum_{u \in (\mathbb{Z}/d\mathbb{Z})^\times} \overline{\chi_j \chi_k}(u) = \alpha_j \phi(d) = \begin{cases} 
\alpha \phi(d) & \text{if } j = 1, \\
\phi(d) & \text{if } j \neq 1,
\end{cases}
\]
and
\[ \sum_{u \in (\mathbb{Z}/d\mathbb{Z})^r} A_{j,2}(u) \]
\[ = \frac{1}{2} \mathbb{E}_{\ell=2} \sum_{k=1}^{r} \alpha_k \sum_{\ell=2}^{r} \Re \left( e^{i(\theta_1 - \theta_j - \theta_\ell)} \chi_{j} \chi_{\ell} \chi_k(u) + e^{i(\theta_1 - \theta_j - \theta_\ell)} \chi_{j} \chi_{\ell} \chi_k(u) \right) \]
\[ = \begin{cases} \frac{1}{2} \left( B(\chi, \theta) + r - 1 \right) & \text{if } j = 1, \\
\frac{\alpha \phi(d)}{2} \left( 1 + \cos(2\theta_1 - \theta_j - \theta_\ell) \right) + O_r(\phi(d)) & \text{if } \chi_j \sim \chi_{1} \chi_{\ell} \text{ for some } 2 \leq \ell \leq r, \\
\frac{\alpha \phi(d)}{2} + O_r(\phi(d)) & \text{otherwise.} \end{cases} \]

Moreover, we also find when \( j = 1 \) that
\[ \sum_{u \in (\mathbb{Z}/d\mathbb{Z})^r} A_{j,3}(u) = \Re \sum_{j=2}^{r} \sum_{\ell=2}^{r} e^{i(2\theta_1 - \theta_j - \theta_\ell)} \sum_{u \in (\mathbb{Z}/d\mathbb{Z})^r} \chi_{j} \chi_{\ell} \chi(u) = \phi(d) B(\chi, \theta), \]
\[ \sum_{u \in (\mathbb{Z}/d\mathbb{Z})^r} A_{j,4}(u) = \frac{1}{2} \Re \sum_{j=2}^{r} \sum_{\ell=2}^{r} \sum_{u \in (\mathbb{Z}/d\mathbb{Z})^r} \left( e^{i(2\theta_1 - \theta_j - \theta_\ell)} \chi_{j} \chi_{\ell} \chi(u) + e^{i(\theta_j - \theta_\ell)} \chi_{j} \chi_{\ell} \chi(u) \right) \]
\[ = \frac{1}{2} \left( B(\chi, \theta) + r - 1 \right) \phi(d). \]

From these formulas and the trivial bounds \( \sum_{u \in (\mathbb{Z}/d\mathbb{Z})^r} A_{j,i}(u) \ll_r \phi(d), i = 3, 4 \) for \( j \neq 1 \), we obtain Proposition 4. \( \square \)

**Proof of Theorem 5.** Put \( \alpha_1 = \alpha \) and \( \alpha_2 = \cdots = \alpha_r = 1 \) with \( \alpha \) sufficiently large. Then Proposition 4 shows that \( \Xi_j(\sigma, \chi, \theta; \alpha) > 0 \) for all \( j \). By (1.8) and Proposition 4, we have
\[ \xi(\sigma, \chi, \theta; \alpha) = \sum_{j=1}^{r} \alpha_j \Xi_j(\sigma, \chi, \theta; \alpha) = \alpha^{\frac{1}{\alpha}} + \frac{1}{4 \sigma^2} (r - 1 - (2\sigma - 1) B(\chi, \theta)) \alpha^{\frac{1}{\alpha} - 2} + O_{r, \sigma} (\alpha^{\frac{1}{\alpha} - 3}). \]

On the other hand, we find by Proposition 4 that
\[ \sum_{j=1}^{r} \Xi_j(\sigma, \chi; \alpha) \]
\[ = \alpha^{\frac{1}{\alpha}} + \frac{1 - 2 \sigma}{4 \sigma^2 (1 - \sigma)} (r - 1 - (2\sigma - 1) B(\chi, \theta)) \alpha^{\frac{1}{\alpha} - 2} + O_{r, \sigma} \left( \alpha^{\frac{1}{\alpha - \sigma} \left( \frac{1}{\alpha} - 2 \right)} + \alpha^{\frac{1}{\alpha} - 3} \right). \]

Note that we have the trivial bound \( B(\chi, \theta) \leq r - 1 \) since there is at most one \( \ell \in \{2, \ldots, r\} \) such that \( \chi_j \sim \chi_{1} \chi_{\ell} \) for each \( j \). Hence, it holds that \( r - 1 - (2\sigma - 1) B(\chi, \theta) > 0 \) for \( \frac{1}{2} < \sigma < 1 \). Thus, we obtain
\[ \xi(\sigma, \chi; \alpha) - \sum_{j=1}^{r} \Xi_j(\sigma, \chi; \alpha) \]
\[ = \frac{1}{4 \sigma (1 - \sigma)} \left( r - 1 - (2\sigma - 1) B(\chi, \theta) \right) \alpha^{\frac{1}{\alpha} - 2} + O_{\sigma, r} \left( \alpha^{\frac{1}{\alpha - \sigma} \left( \frac{1}{\alpha} - 2 \right)} + \alpha^{\frac{1}{\alpha} - 3} \right) > 0 \]
for any \( \alpha \geq \alpha_0(r, \sigma) \) with \( \alpha_0(r, \sigma) \) sufficiently large depending only on \( r \) and \( \sigma \). This completes the proof. \( \square \)
Proof of Theorem 1. From Theorem 4 and Proposition 4 we see that there exists \( c = c(\sigma, r) \) such that for \( V = \frac{(\log T)^{1-\sigma}}{\log \log T} \) with \( T \) sufficiently large depending on \( \sigma, \chi, \theta \) we have that
\[
\Psi(T, V, \chi, \theta) > 0
\]
for \( V = (\Xi_1(\sigma, \chi; \theta; \alpha)V, \ldots, \Xi_r(\sigma, \chi; \theta; \alpha)V) \) where \( \alpha = (\alpha_1(\sigma, r), 1, \ldots, 1) \) and \( \alpha_1 \) is the same constant as in Proposition 4. This gives the lower bound in Theorem 1. □

Acknowledgments. The authors would like to thank Winston Heap for helpful comments on improving the exposition of the paper and Masahiro Mine for providing valuable remarks on a dependence property of \( L \)-functions. We would also like to thank Lukasz Pańkowski for pointing out a mistake in an earlier version of the paper. The first author was supported by Grant-in-Aid for JSPS Research Fellow (Grant Number: 21J00425). The second author is supported by Germany’s Excellence Strategy grant EXC-2047/1 - 390685813 as well as DFG grant BL 915/5-1.

References
[1] C. Aistleitner, Lower bounds for the maximum of the Riemann zeta function along vertical lines, *Math. Ann.* 365 (2016) 473–496.
[2] C. Aistleitner and L. Pańkowski, Large values of \( L \)-functions from the Selberg class. *J. Math. Anal. Appl.* 446 (2017), no. 1, 345–364.
[3] U. Balakrishnan, Extreme values of the Dedekind zeta function, *Acta Arith.* 46 (1986) 199–209.
[4] H. Bohr and B. Jessen, Über die Werteverteilung der Riemannschen Zetafunktion, Erste Mitteilung, *Acta Math.* 54 (1930), 1–35; Zweite Mitteilung, ibid. 58 (1932), 1–55.
[5] E. Bombieri and D. A. Hejhal, On the distribution of zeros of linear combinations of Euler products, *Duke Math. J.* 80 no.3 (1995), 821–862.
[6] A. Fujii, On the zeros of Dirichlet \( L \)-functions. I, *Trans. Amer. Math. Soc.* 196 (1974), 225–235.
[7] T. Hattori and K. Matsumoto, A limit theorem for Bohr-Jessen’s probability measures of the Riemann zeta-function, *J. Reine Angew. Math.* 507 (1999), 219–232.
[8] S. Inoue and J. Li, Joint value distribution of \( L \)-functions on the critical line, preprint, arXiv:2102.12724.
[9] Y. Lamzouri, On the distribution of extreme values of zeta and \( L \)-functions in the strip \( \frac{1}{2} < \sigma < 1 \), *Int. Math. Res. Not. IMRN* 2011, no.23, 5449–5503.
[10] A. Laurinčikas and K. Matsumoto, The joint universality of twisted automorphic \( L \)-functions, *J. Math. Soc. Jpn.* 56 (2004), 923–939.
[11] Y. Lee, T. Nakamura, and L. Pańkowski, Selberg’s orthonormality conjecture and joint universality of \( L \)-functions, *Math. Z.* 286 (2017), 1–18.
[12] K. Mahatab, L. Pańkowski, and A. Vatwani, Joint Extreme values of \( L \)-functions, *Math. Z.* 302 (2022), no. 2, 1177–1190.
[13] H. L. Montgomery, Extreme values of the Riemann zeta function, *Comment. Math. Helvetici* 52 (1977), 511–518.
[14] A. Sankaranarayanan and J. Sengupta, Omega theorems for a class of \( L \)-functions, *Funct. et Approx.* 36 (2006), 119–131.
[15] J. Steuding, Value-distribution of \( L \)-functions. Berlin: Springer, 2007.
[16] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, Second Edition, Edited and with a preface by D. R. Heath-Brown, The Clarendon Press, Oxford University Press, New York, 1986.
[17] S. M. Voronin, Analytic Properties of Dirichlet Generating Functions of Arithmetic Objects, Ph.D. Thesis, Moscow, Steklov Math. Institute (1977) (Russian).

(S. Inoue) DEPARTMENT OF MATHEMATICS, TOKYO INSTITUTE OF TECHNOLOGY, 2-12-1 OOKAYAMA, MEGURO-KU, TOKYO 152-8551, JAPAN
Email address: inoue.s.bd@m.titech.ac.jp

(J. Li) MATHEMATISCHES INSTITUT, ENDENICHER ALLEE 60, 53115 BONN
Email address: jli135@math.uni-bonn.de