Periodic orbit theory allows calculations of long time properties of chaotic systems from traces, dynamical zeta functions and spectral determinants of deterministic evolution operators, which are in turn evaluated in terms of periodic orbits. For the case of stochastic dynamics a direct numerical evaluation of the trace of an evolution operator is possible as a multidimensional integral. Techniques for evaluating such path integrals are discussed. Using as an example the logistic map \( f(x) = \lambda x (1-x) \) with moderate to strong additive Gaussian noise, rapid convergence is demonstrated for all values of \( \lambda \) with strong noise as well as at fixed \( \lambda = 5 \) for all noise levels.

The result of these investigations is a weak noise perturbation theory, representing the trace of the evolution operator and derived quantities as a power series expansion in \( \sigma^2 \), the noise level. The coefficients are combinations of higher derivatives of the map evaluated at the periodic orbits of the deterministic unperturbed system. Numerically, the coefficients themselves converge at a similar rate to the classical periodic orbit theory, but the power series in \( \sigma \) is useful only for weak noise, say \( \sigma < 0.03 \), suggesting the following question, the subject of this paper:

**To what extent does periodic orbit theory survive strong noise, and how fast does it converge?**

Strong noise differs qualitatively from weak noise in a number of respects: The stochastic dynamics is equally close to many slightly different deterministic dynamical systems, so the concept of a unique perturbation theory becomes less defined, in addition to the lack of convergence of such a theory. Also, Gaussian noise has no preferred status; for weakly stochastic systems, all types of noise distributions with a given variance \( \sigma^2 \) are identical to order \( \sigma^2 \).

The approach taken here is that the relevant quantity, the trace of an evolution operator, is evaluated numerically, using very little detailed information about the dynamics, in particular without reference to periodic orbits. The method is general enough to include any type of dynamics (hyperbolic, intermittent, attracting) and uncorrelated noise, subject to smoothness of both dynamics and the noise distribution, with the latter decaying exponentially at large distances. Here, as in Refs. [11], the noise is additive, but this is not a necessary condition.

From the trace, it is straightforward to construct the spectral determinant, and hence highly convergent expansions for escape rates and dynamical averages, in the spirit of cumulant expansions, as in standard periodic orbit theory. This has some similarities to Ref. [12], where various approximations to the quantum trace are compared.

Section II outlines the formalism required for the calculation, in particular casting the trace as a multidimensional integral. Section III discusses numerical approaches for evaluating this integral. Finally, the results and their ramifications are discussed in Section IV.

### II. FORMALISM

The goal is to determine the long time properties of stochastic dynamical systems, here one dimensional maps with additive noise:

\[
x_{n+1} = f(x_n) + \sigma \xi_n ,
\]  

(1)
where \( f(x) \) is a known function, for example the logistic map 
\[
f(x) = \lambda x(1 - x) ,
\]
(2)
\( \sigma \) is a measure of the strength of the noise, and \( \xi_n \) are independent identically distributed random variables with unit variance,
\[
\langle \xi_m \xi_n \rangle = \delta_{mn} ,
\]
(3)
such as a normalized Gaussian distribution. The methods used here are equally applicable to \( \sigma \) and \( \xi \) that depend on \( x \), and non-Gaussian noise distributions.

Instead of the Langevin form (1) it is more convenient to consider the discrete Fokker-Planck equation for a probability distribution \( \rho(x) \) transported by the dynamics and diffusing due to the noise:
\[
\rho_{n+1}(x) \equiv \mathcal{L}[\rho_n](x) = \int \delta_\sigma(x - f(x')) \rho_n(x') dx'
\]
(4)
where \( \delta_\sigma(y) \) is the noise kernel, for example
\[
\delta_\sigma(y) = \frac{e^{-y^2/(2\sigma^2)}}{\sigma\sqrt{2\pi}},
\]
(5)
reducing to a Dirac \( \delta \) in the deterministic \( \sigma = 0 \) limit.

Long time properties of the dynamics are obtained from the leading eigenvalue(s) of the linear evolution operator \( \mathcal{L} \), which are (the inverses of) solutions of the characteristic equation
\[
\text{det}(1 - z\mathcal{L}) = 0 .
\]
(6)
For example, the probability of a point initially in an open system remaining there after \( n \) iterations is typically proportional to \( e^{-\gamma n} \) where the escape rate \( \gamma \) is related to the leading zero \( z_0 \) by
\[
\gamma = -\ln z_0 .
\]
(7)
Dynamical averages and diffusion coefficients can be obtained from the leading zero of appropriately weighted evolution operators \( \mathcal{L} \).

The spectral determinant (3) of an infinite dimensional operator may be defined by its cumulant expansion in powers of \( z \), using the matrix relation \( \ln \text{det} = \text{tr} \ln \) and Taylor expanding the logarithm:
\[
\text{det}(1 - z\mathcal{L}) = \exp \left( - \sum_{n=1}^\infty \frac{z^n}{n} \text{tr}\mathcal{L}^n \right) \\
= 1 - z \text{tr}\mathcal{L} + \frac{z^2}{2} \left[ (\text{tr}\mathcal{L})^2 - \text{tr}\mathcal{L}^2 \right] - \ldots \\
= \sum_{n=0}^\infty C_n z^n
\]
(8)
\( C_n \) may be obtained from all the traces \( \text{tr}\mathcal{L}^m \) with \( m \leq n \), and an approximation for \( z_0 \) is obtained by numerical root finding on the \( n \)'th degree polynomial given by the truncation of the determinant. The above expression for \( C_n \) quickly gets complicated; it is easier to expand the derivative
\[
- z \frac{d}{dz} \text{det}(1 - z\mathcal{L}) = \text{det}(1 - z\mathcal{L}) \sum_{n=1}^\infty z^n \text{tr}\mathcal{L}^n ,
\]
(9)
which leads to the recursive equation
\[
C_n = \frac{1}{n} \left( \text{tr}\mathcal{L}^n - \sum_{m=1}^{n-1} C_m \text{tr}\mathcal{L}^{n-m} \right) .
\]
(10)
The trace is straightforward to write down as an \( n \)-dimensional integral, a discrete periodic chain reminiscent of a path integral, obtained in Ref. [13],
\[
\text{tr}\mathcal{L}^n = \int \prod_{j=0}^{n-1} dx_j \prod_{j=0}^{n-1} \delta_\sigma(x_{j+1} - f(x_j))
\]
(11)
where the index \( j \) is cyclic, so \( x_n = x_0 \). In the noiseless \( (\sigma = 0) \) limit, the integrand is a product of Dirac \( \delta \)-functions, and the trace is given by a sum over the fixed points of \( f^n \), that is, the \( n \)-cycles of \( f \). In Refs. [10,11] the weak noise limit is obtained by a saddlepoint expansion of the integral around these cycles. Here, the integral is performed numerically, up to \( n = 5 \), as described in the following section.

III. NUMERICAL METHODS

The required quantities, \( \text{tr}\mathcal{L}^n \), are \( n \)-dimensional integrals, which in the case of weak noise \( (\sigma \ll 1) \) have a large number of sharp peaks surrounding the periodic points of the deterministic map. Obtaining an accurate numerical estimate of the integral for any \( n > 2 \) seems prohibitively difficult, since Monte Carlo approaches take too long to converge, and direct integration schemes require a small step size, but cover a large configuration space. See Ref. [14] for more discussion.

In the case of Gaussian (or similar) noise and smooth dynamics \( f(x) \) the integrand is smooth and decays exponentially fast at the boundaries. This in turn implies that the simplest possible integration algorithm, summing the integrand at a cubic array of coordinate values, converges faster than any power of the step size, and is typically exponential once the step size is smaller than \( \sigma \).

This remarkable convergence rate for smooth, exponentially decaying integrands follows from the observation that by multiplying the terms near the boundary by appropriate factors, it is possible to obtain algorithms of higher and higher order in the step size [14]. Exponentially decaying integrands are impervious to any such coefficients, and so converge faster than any power of the step size.
Note also, that having chosen, say \( x_0 \) and \( x_1 \), and the argument of the exponential, \(- (x_1 - f(x_0))^2/(2 \sigma^2)\) happens to be too small, it is not necessary to consider the other \( x_j \). This provides a very substantial saving in time for \( n > 2 \).

Finally, the integral is symmetric under a cyclic interchange of the \( x_j \); this implies an additional saving of a factor \( n \). The logic required here is not trivial since the contribution differs depending on whether some of the \( x_j \) are identical. For example, for \( n = 4 \), choose two values \( x_{\text{min}} \) and \( x_{\text{max}} \) beyond which there is no possible contribution. Then sum \( x_{\text{min}} \leq x_0 \leq x_{\text{max}} \), defining \( x_0 \) to be the largest of the \( x_j \), and the one occurring first, if more than one are maximum. Sum \( x_{\text{min}} \leq x_1 \leq x_0 \), checking that the argument of the exponential is not too small. Sum \( x_{\text{min}} \leq x_2 \leq x_0 \), again checking the argument of the exponential. Then sum \( x_{\text{min}} \leq x_3 < x_0 \), and multiply each contribution by 4. If \( x_2 = x_0 \) the \( x_1 \) could form a 2-cycle repeated twice, so when \( x_3 = x_1 \) count the term twice instead of four times, and stop the sum over \( x_3 \) to avoid double counting. Finally, the repeated fixed point \( x_0 = x_1 = x_2 = x_3 \) has been excluded, so sum this explicitly and count it once. The case \( n = 5 \) is simpler as there is only a repeated fixed point, but there are more possibilities for which of the \( x_j \) are maximum.

Even with the above short cuts, large \( n \), small \( \sigma \), and stringent precision requirements can lead to sums of \( 10^9 \) terms. This means it is advisable to group them in size (using the argument of the exponential) as they are summed, then combine the groups from smallest to largest to minimize roundoff error.

Given the above algorithm the step size \( h \) is decreased until two successive estimates agree to within a specified precision (for example 10 digits). Since the amount of time increases as \( h^{-n} \) the optimal sequence is probably \( h_j = h_0 e^{-j/n} \). Note that large initial values of \( h \) can lead to a zero result as the entire contribution region may be missed.

With the above algorithm, calculation of the trace up to \( n = 5 \) with \( \sigma \geq 0.01 \), and \( n = 6 \) for somewhat higher values of \( \sigma \), is feasible for the case of Gaussian noise and smooth one dimensional dynamics.

IV. RESULTS

The logistic map \( f(x) = \lambda x (1-x) \) for various values of \( \lambda \) exhibits most of the behaviors observed in one dimensional maps. For all \( \lambda \geq 1 \) any initial \( x \) outside the range \([0, 1]\) ends up at \(-\infty\), while the behavior of points within this range depend of \( \lambda \) as follows: For \( 0 \leq \lambda \leq 1 \), the point \( x = 0 \) is a stable fixed point, marginally so at \( \lambda = 1 \), and then unstable for \( \lambda > 1 \). For \( 1 \leq \lambda \leq 3 \), the fixed point \( x = 1 - 1/\lambda \) is stable, and then bifurcates to a stable cycle of period 2. This cycle in turn becomes unstable, bifurcating to a 4-cycle, then an 8-cycle, and so on, to \( \lambda \approx 3.57 \) at which point a chaotic attractor forms. The period doubling cascade in the presence of weak noise may be described by the renormalization approach of Ref. \([13]\). At larger values of \( \lambda \) more stable cycles are created, including a 3-cycle which is stable at \( \lambda = 3.84 \), leading to a pattern of alternating stable “windows” surrounded by non-attracting unstable cycles and chaotic attractors containing many unstable cycles. At \( \lambda = 4 \) the attractor fills the interval \([0, 1]\), and in this case, the Ulam map, the dynamics is exactly solvable. For \( \lambda > 4 \) almost all initial conditions leave the interval, but infinitely many unstable cycles remain, forming a fractal repeller, with a well defined escape rate.

Imposing additive noise to the logistic map leads to escape for all \( \lambda > 0 \), although this may be very unlikely if \( \sigma \) is small. At \( \lambda = 2 \), for example, every point (except the endpoints) is attracted to the stable fixed point at \( x = 1/2 \), and the noise must move the trajectory out of the interval to escape. In cases like this, the stochastic behavior is analogous to quantum tunneling, and is exponentially suppressed for small \( \sigma \). At bifurcation points, including \( \lambda = 1 \), the stability of the relevant cycles is marginal, leading to intermittency. Marginal cycles are difficult to treat using cycle expansions, and it is one of the goals of this work to understand how this poor convergence is modified by the presence of noise.

The results of numerical evaluation of \( \text{tr} \mathcal{E}^n \) up to \( n = 5 \) are shown in Tab. \([11]\). The spectral determinant is evaluated using \([11]\), and \( C_5 \), the coefficient of \( z^5 \) is noted. Since for the parameters shown the first zero of the determinant is close to \( 1, - \log_{10} |C_5|\) gives roughly the number of significant digits of \( z \), and hence the escape rate, evaluated to \( n = 4 \). It also gives the approximate range of \( z \) over which the \( n = 4 \) approximation is valid.

It is seen that, for the trivial case \( \lambda = 0 \), corresponding to pure noise, and for strong noise \( \sigma = 1 \), the calculation is limited by the double precision arithmetic: evaluation of the trace beyond \( n = 4 \) is superfluous at this level of precision. Almost as precise is the case \( \lambda = 5 \) which has a repeller with complete binary symbolic dynamics in the absence of noise, and hence is an ideal candidate for cycle expansion methods. Nine significant digits are obtained at \( n = 4 \), corresponding to just 8 cycles. The presence of noise makes methods based on enumerating these cycles more difficult \([10,11]\), but convergence is rapid at any noise level.

The other cases, where escape is induced by the presence of noise, do rather poorly for small noise. The significance of \( \lambda = 1, 2, 3, 3.57, 3.84, 4 \) are discussed above; the other values in Tab. \([11]\) are \( \lambda = 3.5 \) which contains a stable 4-cycle, and \( \lambda = 3.72 \) which is not near any large stable window, and numerically exhibits a chaotic attractor, although mathematical proof is difficult. The nature of the underlying attractor seems to have little effect on the rate of convergence, except that the intermittent case (\( \lambda = 3 \) and particularly \( \lambda = 1 \)) is divergent.
at $\sigma = 0.01$ to this level of approximation; the escape rate probably converges at impossibly large $n$, either for the current numerical approach, or for standard cycle expansion techniques. In the other cases, particularly towards larger $\lambda$ the expansion appears to be converging, albeit slowly.

In conclusion: What are the optimal methods for determining the long time properties of stochastic systems? The strong noise case is best treated by numerical evaluation of the trace, described here, requiring little knowledge of the underlying dynamics. The elements of periodic orbit theory, traces and determinants indeed survive strong noise, and converge rapidly, without reference to periodic orbits. The intermittent case with weak noise remains an open problem; the results here show that weak noise does not substantially regularize cycle expansions of intermittent systems, at least with respect to the rate of convergence.

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| $\lambda$ | Type               | $0.01$ | $0.03$ | $0.1$ | $0.3$ | $1$ |
|----------|--------------------|-------|-------|-------|-------|----|
| 0        | Pure noise         | 12.7  | 12.7  | 12.4  | 12.7  | 12.6|
| 1        | Intermittent       | -2.3  | -0.8  | 1.2   | 3.8   | 8.5 |
| 2        | Stable 1-cycle     | 2.5   | 2.2   | 2.1   | 5.9   | 11.8|
| 3        | Bifurcation        | -0.3  | 0.7   | 2.8   | 7.4   | 13.2|
| 3.5      | Stable 4-cycle     | 0.3   | 1.4   | 3.4   | 7.8   | 13.2|
| 3.57     | $\infty$-cycle     | 0.4   | 1.1   | 3.6   | 7.8   | 13.3|
| 3.72     | Chaos              | 1.5   | 1.4   | 4.1   | 8.0   | 13.4|
| 3.84     | Stable 3-cycle     | 1.6   | 2.4   | 4.6   | 8.1   | 13.4|
| 4        | Ulam map           | 2.2   | 2.9   | 4.9   | 8.2   | 13.8|
| 5        | Repeller           | 9.2   | 9.1   | 8.4   | 9.1   | 13.3|

| Type               | $\sigma$ |
|--------------------|----------|
| Pure noise         | 12.7     |
| Intermittent       | -2.3     |
| Stable 1-cycle     | 2.5      |
| Bifurcation        | -0.3     |
| Stable 4-cycle     | 0.3      |
| $\infty$-cycle     | 0.4      |
| Chaos              | 1.5      |
| Stable 3-cycle     | 1.6      |
| Ulam map           | 2.2      |
| Repeller           | 9.2      |

TABLE I. Convergence of the spectral determinant, as measured by $-\log_{10} |C_5|$, where $C_5$ is the coefficient of $z^5$ in the cumulant expansion (8) for various types of dynamics of the logistic map (2). Larger numbers imply faster convergence, giving roughly the number of converged digits in the escape rate calculated to $n = 4$.

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