On problem of determining the stress state of elasto-plastic compressible area weakened by ellipsoidal cavity

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Abstract. The work is one of the stages of solving the problem of determining the stress state for a compressible mass weakened by cavities. In the problem statement, there is no pressure inside the cavity, and mutually perpendicular forces are applied at infinity. The problem is solved by the small parameter method, in a spherical coordinate system, in dimensionless length units. The previously found values are taken as the known stress components in the zero approximation in the elastic area and as the perturbed stress components in the plastic area. Using the conjugation of stresses at the boundary of elasto-plastic zone, the stress components for the elastic area are determined in the first approximation. The obtained results can be used in the field of mining, construction mechanics and other related fields.

1. Introduction

Determination of the stress and strain state of the mass around cavities and recesses is important in the practice of mining, construction mechanics and other related fields. The problems of determining the stress state near excavations are related to the study of rock pressure, which theoretical research can be presented in three directions. The first direction includes the researches based on practical data and special hypotheses using the simplified methods of structural mechanics. The second direction includes the researches based on the methods of mechanics of deformable solids. The third direction is related to the concept of rock pressure as a process of loss of mass stability.

2. Problem statement and the solution of the problem

This paper considers a weakened by an ellipsoidal cavity under mutually perpendicular forces at infinity mass of friable material with the properties of internal friction and cohesion. The case of the limit state of friable material has the form of [1] \( f(\sigma'_{ij}) = k_0 + a\sigma \), where \( \sigma'_{ij} \) are components of the stress deviator, \( k_0 \) is a cohesion coefficient, \( a = \tan\alpha \) is a coefficient of internal friction, \( \alpha \) is an angle of internal friction. Development of the theory considering the behavior of a material in the form [1] is associated with the names of S.A. Khristianovich, A.Yu. Ishlinsky [2], V.V. Sokolovsky [1], E.I. Shemyakin, V.G. Berezantsev, R. Shild, D.D. Ivlev [3-5], L.V. Ershov [6, 7].

Previously, the stress state of an ideal-plastic compressible area weakened by a spherical cavity was determined [8], the stress states of a mass weakened by cavities were determined for three cases [9-11]
satisfying the plasticity condition [4], the stress state of an ideal-plastic compressible area weakened by an ellipsoidal cavity was determined [12]. By means of the conjugation of stresses at the boundary of the elasto-plastic zone, the problem of determining the stress components in the first approximation for the elastic area was set. The previously found values [8], [12] are taken as stress components in the zero approximation and perturbed stress components in the plastic area. The problem is solved by the small parameter method, in a spherical coordinate system, in dimensionless units of length. All values having the length dimension are referred to the radius of spherical cavity $\rho_0$.

3. Results and discussion
The conditions of conjugation of solutions for the elastic and plastic zones are used to determine the perturbed stress components in the elastic area. It is assumed that the boundary of the plastic zone $F_s$ never crosses the cavity.

The values of components in square brackets are equal to the gap while crossing the surface $F_s$.

If the equation $F_s$ is represented as:

$$\rho = \rho_0 + \delta \beta_1(\theta, \phi),$$

where $\beta_0$ is a previously determined boundary of the elasto-plastic zone [8] in the zero approximation, then decomposing the stresses into the Taylor series by a small parameter $\delta$ and bringing them to the surface $\rho = \rho_0$, we obtain the conjugation conditions for perturbed components:

$$\left[\sigma'_{\rho}\right] = 0, \left[\tau'_{\rho \theta}\right] = 0, \left[\tau'_{\rho \phi}\right] = 0, \left[\sigma'_{\theta} + \frac{\partial \sigma_{0}^{\theta}}{\partial \rho} \beta_{1}\right] = 0.$$

Then, using the first three expressions (3) and the previously found values [12] of perturbed stress components in the plastic area, we obtain for the perturbed components on the inner surface of elastic zone:

$$\sigma'_{\rho} = H \left[ (c_1 + c_2)(1 - 3 \cos^2 \theta) + (c_1 - c_2) \sin^2 \theta \cos 2\phi \right],$$

$$\tau'_{\rho \theta} = F \frac{\partial P^2_{2}(\cos \theta)}{\partial \theta} \left[ 3(c_1 + c_2) + (c_1 - c_2) \cos 2\phi \right],$$

$$\tau'_{\rho \phi} = -2 F(c_1 - c_2) \sin 2\phi \sin \theta,$$

Where $H$ and $F$ are constants: $H = -A \frac{(C_1 \beta_1 \beta_0^\gamma_1 + C_2 \gamma_2 \beta_0^\gamma_2)}{\beta_0^3}$, $F = C_1 \beta_0^\gamma_1 + C_2 \beta_0^\gamma_2$, and $c_1, c_2, C_1, C_2, A, \gamma_1, \gamma_2$ are previously determined constants [8].

According to A.I. Lurie, the vector of external forces $P_{\rho}|_{\rho = \beta_0}$ [13] given on the surface $\rho = \beta_0$ can be decomposed into a series of surface spherical vectors:
\[ \tilde{P}_\rho|_{\rho=\beta_0} = \sum_{n=0}^{\infty} \tilde{Y}_n(\theta, \varphi), \]  

(5)

where \( \tilde{Y}_n(\theta, \varphi) \) if \( \rho = \beta_0 \), is presented as:

\[ \tilde{Y}_n(\theta, \varphi) = Y_{n\rho} \rho e^\rho + Y_{n\varphi} \varphi e^\varphi + \sigma_{\rho\theta} \rho \theta e^\theta + \tau_{\rho\varphi} \rho \varphi e^\varphi. \]  

(6)

The function \( \tilde{Y}_n(\theta, \varphi) \) can be represented as the sum of values if \( n=0 \) and \( n=2 \):

\[ \tilde{Y}_n(\theta, \varphi) = \tilde{Y}_0(\theta, \varphi) + \tilde{Y}_2(\theta, \varphi). \]  

(7)

The harmonic vector in the elastic zone (\( \rho > \beta_0 \)) is represented as [13]:

\[ \tilde{\Pi}_\rho = \sum_{n=0}^{\infty} \tilde{\Pi}_\rho^{-n-1} = \sum_{n=0}^{\infty} \left( \frac{\beta_0}{\rho} \right)^{n+1} \tilde{Y}_n(\theta, \varphi). \]  

(8)

Then, on any spherical surface that is concentric with the cavity the vector \( \tilde{P}_\rho \) is found by the formula:

\[ \tilde{P}_\rho = \frac{1}{\rho} \tilde{\Pi}_\rho - \frac{\beta_0^2 - \rho^2}{2 \rho} \sum_{n=0}^{m(n+3) \text{grad div} \Pi n^{-1}} \frac{m(n+3)}{2 \rho^2 + (3m-2)n + 3(m-1)} \]

(9)

where \( m \) is the Poisson number.

Then, solving together (4), (6), (7) and (9), we obtain:

\[\sigma'_{\rho} = \frac{\beta_0}{\rho^2} (c_1 + c_2)(1 - 3 \cos^2 \theta) \left[ H + \frac{m}{m-1} \left( 1 - \frac{\beta_0}{\rho^2} \right)(18F-H) \right] + \frac{\beta_0^3}{\rho^4} (c_1 - c_2) \cos 2\varphi \sin^2 \theta \left[ H + \frac{10m}{13m-7} \left( 1 - \frac{\beta_0}{\rho^2} \right)(18F+H) \right],\]

\[\tau'_{\rho\theta} = 3 \frac{\beta_0}{\rho^2} (c_1 + c_2) \sin 2\theta \left[ 3F - \frac{m}{2(m-1)} \left( 1 - \frac{\beta_0}{\rho^2} \right)(18F-H) \right] + \frac{\beta_0^3}{\rho^4} (c_1 - c_2) \cos 2\varphi \sin 2\theta \left[ F - \frac{5m}{2(13m-7)} \left( 1 - \frac{\beta_0}{\rho^2} \right)(18F+H) \right],\]

\[\tau'_{\rho\varphi} = -3 \frac{\beta_0^3}{\rho^4} (c_1 - c_2) \sin 2\varphi \sin \theta \left[ 6F - \frac{5m}{13m-7} \left( 1 - \frac{\beta_0}{\rho^2} \right)(18F + H) \right].\]

(10)

First, determining the remaining three components of the stress tensor \( \sigma'_{\theta}, \sigma'_{\varphi}, \tau'_{\theta\varphi} \), we find the components of the displacement vector \( \tilde{U}(U_{\rho}, U_{\theta}, U_{\varphi}) \):
\[2G \frac{U}{U} = -\sum_{n=0}^{\infty} \frac{1}{n+1} \frac{1}{\rho} \hat{\rho} \times \text{rot} \hat{\Pi}^{-n-1} + \frac{1}{(n+1)(n+2)} \hat{\rho} \cdot \text{div} \hat{\Pi}^{-n-1} \]
\[-\frac{(m-4)(n+1) + 2(m-1)}{(n+1)(mn^2 + (3m-2)n + 3(m-1))} \rho \cdot \text{div} \hat{\Pi}^{-n-1} \]
\[-\frac{2n-m+3}{(n+1)(n+2)(mn^2 + (3m-2)n + 3(m-1))} \rho^2 \text{grad} \text{div} \hat{\Pi}^{-n-1} \]
\[+ \frac{\rho^2 - \beta_0^2}{2} m \cdot \text{grad} \hat{\Pi}^{-n-1} \]
\[= \left(1 + \frac{1}{m^2 - 1} \right) \frac{m}{\rho} \frac{\partial U}{\partial \rho} + \frac{\partial U}{\partial \theta} + \frac{\partial U}{\partial \phi} \]

(11)

where \( \hat{\rho} \) is a radius-vector in the spherical coordinate system, \( G \) is a shear module.

Then, using the obtained components of the displacement vector \( \hat{U}(U_\rho, U_\theta, U_\phi) \) and the Hooke law written in inverse form, we determine the missing components of the stress tensor:

\[\sigma'_\theta = \frac{2G}{m-2} \text{div} \hat{U} + \frac{2G}{\rho} \left( \frac{\partial U}{\partial \theta} + U_\rho \right)\]
\[\sigma'_\phi = \frac{2G}{m-2} \text{div} \hat{U} + \frac{2G}{\rho} \left( \frac{1}{\sin \theta} \frac{\partial U}{\partial \phi} + U_\rho + U_\theta \cot \theta \right)\]
\[\tau'_\phi = \frac{G}{\rho} \left( \frac{\partial U}{\partial \phi} + U_\phi \cot \theta + \frac{1}{\sin \theta} \frac{\partial U}{\partial \phi} \right)\]

(12)

Solving (11) and (12) together, we obtain three remaining components of the stress tensor:

\[\sigma'_\theta = \frac{\beta_0 (c_1 + c_2)}{3\beta_0 \rho^2 (m-1)(m-2)} \left( m \beta_0 \left( H(6-4m + 3 \cos^2 \theta(3m-4)) + 18F(4m^2 - 9m \cos^2 \theta) \right) \right) \]
\[+ 9mH(m-1+\cos^2 \theta(1-2m))+54F(3-3m+\cos^2 \theta(m^2 + 3m - 1)) + \frac{\beta_0^3 (c_1 - c_2) \cos 2\phi}{3\rho^4 (m-2)(13m-7)} \]
\[\times (-3m \frac{\beta_0^2}{\rho^2} (H+18F)(m-2+\sin^2 \theta(11-4m))+(H+18F)(m-2)(7-4m) \]
\[+ 5H \sin^2 \theta(-3m^2+13m-7)+18F \sin^2 \theta(-2m^2+19m-14))\]

(13)
In the work of T.D. Semykina [14], the elasto-plastic stress state of triaxially strained area weakened by a spherical cavity is determined at the absence of stresses in the plastic zone. The stress state of area with an ellipsoidal cavity and triaxial compression at infinity can be represented in the linearized formulation as the superposition of two stress states – the stress state of area caused by disturbances on the boundary of the ellipsoidal cavity under the absence of forces at infinity, and the stress state of area weakened by a spherical cavity free from forces, and triaxial compression at infinity. Then, summing up the results obtained by T. D. Semykinain [14] and (13), it is possible to determine stresses in the first approximation in the elastic zone for a triaxially compressed area weakened by an ellipsoidal cavity:

$$
\sigma_{\varphi} = \frac{\beta_0 (c_1 + c_2)}{\rho^2 (m-1)(m-2)} (m \frac{\beta_0}{\rho} (H - 18F)(-m + 6 \cos^2 \theta(m - 1)) + 54F (-m^2 + 3m - 5
+ \cos^2 \theta(2m^2 - 3m + 7)) + 9mH (1 - \cos^2 \theta(m + 1)) + \frac{\beta_0^3 (c_1 - c_2) \cos 2\varphi}{\rho^4 (m - 2)(13m - 7)}
\times (3m \frac{\beta_0^2}{\rho^2} (H + 18F)(m - 2 + 3 \sin^2 \theta(m - 3)) + (H + 18F)(m - 2)(4m - 7))
+ H \sin^2 \theta(-19m^2 + 80m - 49)m \sin^2 \theta(m - 1)(3m - 14),

\epsilon'_{\varphi} = \frac{2 \beta_0^3 (c_1 - c_2) \sin 2 \varphi \cos \theta}{3 \rho^4 (13m - 7)} (18F + H) \left[ 4m - 7 + 3m \frac{\beta_0}{\rho^2} \right].
$$

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$$
\sigma^e_\rho = \frac{p_1' + p_2' + p_3'}{3} \rho^3 \beta_0^3 \sigma_0 + \frac{2}{3} \left[ 1 - 5 \frac{\beta_0^3}{\rho^3} + 4 \frac{\beta_0^5}{\rho^5} \right]
\times \left[ p_1' P_2 (\cos \theta_1) + p_2' P_2 (\cos \theta_2) + p_3' P_2 (\cos \theta_3) \right] + \sigma^e_\rho,
$$

$$
\sigma^e_\theta = \frac{p_1'}{3} \left[ \frac{2}{3} \cos^2 \theta \cdot \cos^2 \varphi \cdot \frac{\beta_0^3}{6 \rho^3} \cos 2\chi + \frac{2}{3} \frac{\beta_0^5}{\rho^5} \left[ P_2 (\cos \theta_1) - \cos^2 \theta_1 \cos 2\chi \right] \right]
+ \frac{p_2'}{3} \left[ \frac{2}{3} \cos^2 \theta \cdot \sin^2 \varphi \cdot \frac{\beta_0^3}{6 \rho^3} \cos 2\chi - \frac{2}{3} \frac{\beta_0^5}{\rho^5} \left[ P_2 (\cos \theta_2) - \cos^2 \theta_2 \cos 2\nu \right] \right]
+ \frac{p_3'}{3} \left[ \frac{2}{3} \rho^3 \frac{\beta_0^3}{\rho^3} - \frac{2}{3} \frac{\beta_0^5}{\rho^5} \left[ P_2 (\cos \theta_3) + \frac{2}{3} \frac{\beta_0^5}{\rho^5} \cos^2 \theta \right] \right] + \sigma^e_\theta.
$$
\[ \sigma_{\varphi}^e = p_1' \left( \frac{2}{3} \sin^2 \varphi + \frac{\beta^3}{6 \rho^3} + \frac{2 \beta^5}{3 \rho^5} \left[ P_2(\cos \vartheta_1) - \cos^2 \vartheta_1 \cos 2 \chi \right] \right) \]

\[ + p_2' \left[ \cos^2 \varphi + \frac{\beta^3}{6 \rho^3} - \frac{2 \beta^5}{3 \rho^5} \left[ P_2(\cos \vartheta_2) - \cos^2 \vartheta_2 \cos 2 \nu \right] \right] \]

\[ + p_3' \left[ \frac{1 \beta^3}{6 \rho^3} - \frac{2 \beta^5}{3 \rho^5} \left[ P_2(\cos \vartheta) + \cos^2 \vartheta \right] \right] + \sigma' \varphi, \]

\[ \tau_{\rho \varphi}^e = \left[ \frac{5 \beta^3}{3 \rho^3} - \frac{8 \beta^5}{3 \rho^5} \right] \left[ p_1' \cos \varphi + p_2' \sin \varphi \right] \sin \theta \cdot \cos \theta + \tau' \rho \varphi, \]

\[ \tau_{\rho \varphi}^e = \left[ \frac{5 \beta^3}{3 \rho^3} - \frac{8 \beta^5}{3 \rho^5} \right] \left[ - p_1' \sin \varphi + p_2' \cos \theta \right] \sin \theta \cdot \cos \varphi + \tau' \rho \varphi, \]

\[ \tau_{\theta \varphi}^e = p_1' \sin \chi \cdot \cos \chi \left[ \sin^2 \vartheta_1 + \frac{4 \beta^5}{3 \rho^5} \cos^2 \vartheta_1 \right] \]

\[ + p_2' \left[ \frac{1}{2} \sin 2 \varphi \cdot \cos \theta - \frac{1}{3} \sin 2 \nu \left[ \frac{\beta^3}{\rho^3} - \frac{2 \beta^5}{\rho^5} P_2(\cos \vartheta_2) \right] \right] + \tau' \theta \varphi, \]

where \( \sigma' \rho, \sigma' \theta, \sigma' \varphi, \tau' \rho \theta, \tau' \rho \varphi, \tau' \theta \varphi \) are determined from (10), (13); \( p_1', p_2', p_3' \) are determined from \( p_1 = p_0 + \delta P_1, p_2 = p_0 + \delta p_2, p_3 = p_0 + \delta p_3 \), where \( p_1, p_2, p_3 \) are mutually perpendicular forces at infinity, \( P_2(\cos \theta) \) is the Legendre polynomial of the second order;

\[ \cos \theta_1 = \sin \theta \cdot \cos \varphi, \cos \chi = \frac{\cos \theta \cdot \cos \varphi}{\sin \theta_1}, \sin \chi = \frac{\sin \varphi}{\sin \theta_1}, \]

\[ \cos \theta_2 = \sin \theta \cdot \sin \varphi, \cos \nu = \frac{\cos \theta \cdot \cos \varphi}{\sin \theta_2}, \sin \nu = \frac{\cos \varphi}{\sin \theta_2}. \]

4. Conclusion

In this work, the components of the stress tensor for the case of elastic zone are determined in the first approximation: three normal components \( \sigma_{\rho}^e, \sigma_{\theta}^e, \sigma_{\varphi}^e \) and three tangent components \( \tau_{\rho \theta}^e, \tau_{\rho \varphi}^e, \tau_{\theta \varphi}^e \) for the mass of friable material with the properties of internal friction and cohesion, and weakened by an ellipsoidal cavity under mutually perpendicular forces at infinity. The results can be used in calculating the stress and strain states of areas near cavities and recesses caused by rock pressure.

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