Level of noises and long time behavior of the solution for space-time fractional SPDE in bounded domains

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Abstract

In this paper we study the long time behavior of the solution to a certain class of space-time fractional stochastic equations with respect to the level \( \lambda \) of a noise and show how the choice of the order \( \beta \in (0, 1) \) of the fractional time derivative affects the growth and decay behavior of their solution. We consider both the cases of white noise and colored noise. Our results extend the main results in Foondun [16] to fractional Laplacian as well as higher dimensional cases.

Keywords: Space-time fractional SPDE, space-time white noise, space colored noise, moment bounds in bounded domains.
1 Introduction and Statement of the Main results

In 1855, Adolf Fick [11] developed his now-famous principle governing the transport of mass through diffusive media. In his second law, Fick showed that $u(t, x)$ obeys the classical diffusion equation $\partial_t u = Du_{xx}$ in the spatial dimension, which estimates how the concentration $u(t, x)$ of a diffusible substance varies with space and time, where $D$ is the diffusion coefficient. The space-time fractional diffusion equation which is obtained when integer-order derivative operators in space and time are replaced by fractional counterparts (in Caputo or Riemann-Liouville sense) has been recently treated by several authors (see, for example, Mijena and Nane [20], Saichev and Zaslavsky [28], and Mainardi, Luchko, and Pagnini[19]).

The typical form of the space-time fractional equation is
\[ \partial_t^\beta u = (-\Delta)^{\alpha/2}u, \]
where $\partial_t^\beta$ is the Caputo fractional derivative with $\beta \in (0, 1)$, $\alpha \in (0, 2)$ and $\Delta = \sum_{i=1}^d \partial_{x_i}^2$, is the Laplacian. These equations can be used to model anomalous diffusion processes or diffusion processes in non-homogeneous media with random fractal structures (see, for instance, Meerschaert et al. [22], Meerschaert, Nane, and Vellaisamy [24], Baeumer, Lueks, and Meerschaert [4], Chen, Kim and Kim [7] and the references therein). Since these space-time fractional equations depend on the fractional parameters $\beta$ and $\alpha$, in [9] the importance of the continuity of their solutions with respect to these parameters is discussed. For example, if the partial derivative in time $\partial_t$ in the classical heat equation $\partial_t u = -\Delta u$ is substituted with fractional derivatives $\partial_t^\beta$ for $0 < \beta < 1$, the processes explains the sticking and trapping behavior of particle, while if the Laplacian $\Delta$ is replaced with fractional power $(-\Delta)^{\alpha/2}$ for $0 < \alpha < 2$, it describes long particle jumps (see [1]). In [20], Mijena and Nane have recently introduced time fractional SPDEs, which can be utilized to represent phenomena with random effects and thermal memory.

Consider the following space-time fractional equation with Dirichlet boundary conditions (see (2.5) below for a representation of the solution).

$$
\begin{cases}
\partial_t^\beta u_t(x) = (-\Delta)^{\alpha/2}u(x), \ x \in B, \ t > 0, \\
u_t(x) = 0, \ x \in B^C, \ t > 0,
\end{cases}
$$

(1.1)

where $\alpha \in (0, 2)$ and $\beta \in (0, 1)$. The fractional time derivative is the Caputo derivative which first appeared in [3] and is defined by

$$
\partial_t^\beta u_t(x) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\partial u_r(x)}{\partial r} (t-r)^{\beta-1} dr.
$$

(1.2)

If $u_0(x)$ denotes the initial condition to equation (1.1), then the solution can be written as

$$
u_t(x) = \int_B G_B^{(\beta)}(t, x, y)u_0(y) dy,
$$

where $G_B^{(\beta)}(t, x, y)$ is the space-time fractional heat kernel defined in (2.7).

Now consider

$$
\partial_t^\beta u_t(x) = (-\Delta)^{\alpha/2}u(x) + f(t, x),
$$

(1.3)

with the same initial condition $u_0(x)$ and $f(t, x)$ is some nice function. To get the correct version of (1.3) we will make use of [29, 30, 31]. The fractional Duhamel principle implies that the mild solution to (1.3) for $t > 0$ is given by

$$
u_t(x) = \int_B G_B^{(\beta)}(t, x, y)u_0(y) dy + \int_0^t \int_B G_B^{(\beta)}(t-s, x, y)\partial_s^{1-\beta} f(s, y) dy ds.
$$

(1.4)

Using the fractional order integral $I_\gamma^t$ defined by

$$
I_\gamma^t f(t) := \frac{1}{\Gamma(\gamma)} \int_0^t (t-\tau)^{\gamma-1} f(\tau) d\tau,
$$

(2.6)
and the property
\[ \partial_t^\beta I_t^\beta g(t) = g(t), \]
for every \( \beta \in (0, 1) \), and \( g \in L_\infty(\mathbb{R}_+) \) or \( g \in C(\mathbb{R}_+) \), then by the Duhamel’s principle, the mild solution to (1.3) where the force is \( f(t, x) = I_t^{1-\beta}g(t, x) \), will be given by
\[
\begin{align*}
    u_t(x) &= \int_B G_B^{(\beta)}(t, x, y)u_0(y) \, dy + \int_0^t \int_B G_B^{(\beta)}(t-s, x, y)\partial_s^{1-\beta}(I_s^{1-\beta}g(s, y)) \, dy \, ds \\
    &= \int_B G_B^{(\beta)}(t, x, y)u_0(y) \, dy + \int_0^t \int_B G_B^{(\beta)}(t-s, x, y)g(s, y) \, dy \, ds. \quad (1.5)
\end{align*}
\]

Recently, Mijena and Nane [20], Foondun, Mijena, and Nane [14], and Foondun [16] considered the following time fractional stochastic heat equation on the interval \((0, L)\) with Dirichlet boundary condition:
\[
\begin{cases}
    \partial_t^\beta u_t(x) = \frac{1}{2} \partial_{xx} u_t(x) + I_t^{1-\beta}[\lambda \sigma(u_t(x))\hat{W}(t, x)] & \text{for } 0 < x < L \text{ and } t > 0 \\
    u_t(0) = u_t(L) = 0 & \text{for } t > 0,
\end{cases} \quad (1.6)
\]
where the initial condition \( u_0 : [0, L] \to \mathbb{R}_+ \) is non-random and non-negative bounded function which is strictly positive on a set of positive measures in \([0, L]\). \( \hat{W} \) denotes a space-time white noise and \( \sigma : \mathbb{R} \to \mathbb{R} \) is a globally Lipschitz function satisfying \( L_p|x| \leq \|\sigma(x)\| \leq L_\sigma|x| \) where \( L_p \) and \( L_\sigma \) are positive constants. \( \lambda \) is a positive parameter known as the level of the noise and will play a significant part in this paper.

Using Walsh [32], we define the mild solution to (1.6) as the random field \( u = \{u_t(x)\}_{t>0,x \in B} \) satisfying
\[
    u_t(x) = (P_B^\beta u_0)(x) + \lambda \int_0^L \int_0^t p_B^{\beta}(t-s, x, y)\sigma(u_s(y))W(ds, dy), \quad (1.7)
\]
where \( p_B^{\beta}(t, x, y) \) denotes the probability density function of the time-changed killed Brownian motion upon exiting the domain \([0, L]\) associated with the fractional time operator, and
\[
(P_B^\beta u_0)(x) := \int_0^L p_B^\beta(t, x, y)u_0(y) \, dy.
\]

In Foondun and Nualart [15] and Foondun, Guerngar, and Nane [13], the authors looked at the behavior of the solution to equation (1.6) for small and large \( \lambda \) when \( \beta = 1 \). They showed that if \( \lambda \) is large enough, the second moment of the solution \( u_t \) grows exponentially fast; while if \( \lambda \) is small, the second moment of the solution \( u_t \) eventually decays exponentially. Nualart [27] and Xie [33] have used precise heat kernel estimates to sharpen the results in [15]. However, in [16] Foondun has shown that a more complicated situation will occur instead of such phase transition if fractional time derivative replaces the usual time derivative. That is, for any fixed \( \beta \in (0, 1) \), the long time behavior of the supremum of the second moment of the solution to equation (1.6) behaves differently by considering the cases when \( \beta \in (0, \frac{1}{2}) \) and \( \beta \in (\frac{1}{2}, 1) \) separately, and the reasons for considering these cases are also explained. For more details about the interpretation of the results, one can refer [16]. These findings are interesting from an application standpoint since fractional time derivatives are commonly used in the modeling of various systems with memory. Therefore, it is very important to realize that the use of such derivatives can result in considerable change in the qualitative properties of the solution. The main aim of this work is to investigate the long time behavior of the solution to (1.8) with respect to the level of the noise \( \lambda \).

Our work extend the main results in Foondun [16] to fractional Laplacian as well as higher dimensional cases.
Consider the following stochastic heat equation on a regular bounded domain $B$ in $\mathbb{R}^d$, $d \geq 1$ with Dirichlet boundary condition:

\[
\begin{cases}
    \partial_t^2 u_t(x) = -(-\Delta)^{\frac{\alpha}{2}} u_t(x) + t^{1-\beta}[\lambda \sigma(u_t(x)) \dot{W}(t, x)] & \text{for } x \in B \text{ and } t > 0 \\
    u_t(x) = 0 & \text{for } x \notin B \text{ and } t > 0,
\end{cases}
\]

(1.8)

and the initial condition $u_0 : B \to \mathbb{R}_+$ is a non-random measurable and bounded function that has support with positive measure inside B. The operator $-(-\Delta)^{\frac{\alpha}{2}}$, where $0 < \alpha \leq 2$, is the $L^2$-generator of a symmetric $\alpha$-stable process $X_t^B$ killed when exiting $B$. $\dot{W}$ denotes a space-time white noise and $\sigma : \mathbb{R} \to \mathbb{R}$ is a globally Lipschitz function satisfying $l_\sigma |x| \leq |\sigma(x)| \leq L_\sigma |x|$ where $l_\sigma$ and $L_\sigma$ are positive constants. The positive parameter $\lambda$ is called the level of the noise.

$\dot{W}(t, x)$ is a space-time white noise with $x \in B$, which is assumed to be adapted with respect to a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, where $\mathcal{F}$ is complete and the filtration $\{\mathcal{F}_t, t \geq 0\}$ is right continuous. $\dot{W}(t, x)$ is a generalized process with covariance given by

\[
\mathbb{E}[\dot{W}(t, x)\dot{W}(s, y)] = \delta(t - s)\delta(x - y).
\]

That is, $W(f)$ is a random field indexed by function $f \in L^2((0, \infty) \times B)$ and for all $f, g \in L^2((0, \infty) \times B)$, we have

\[
\mathbb{E}[W(f)W(g)] = \int_0^\infty \int_B f(t, x)g(t, x) \, dx \, dt.
\]

Hence $W(f)$ can be represented

\[
W(f) = \int_0^\infty \int_B f(t, x)W(dx \, dt).
\]

Note that $W(f)$ is $\mathcal{F}_t$-measurable whenever $f$ is supported on $[0, t] \times B$.

The Walsh-Dalang Integrals [8, 32] that is used in equation (1.10) is defined as follows. We use the Brownian Filtration $\{\mathcal{F}_t\}$ and the Walsh-Dalang integrals defined as follows:

- $(t, x) \to \Phi_t(x)$ is an elementary random field when $\exists 0 < a < b$ and an $\mathcal{F}_a$-measurable $X \in L^2(\Omega)$ and $\phi \in L^2(B)$ such that

\[
\Phi_t(x) = X1_{(a, b)}(t)\phi(x) \quad t > 0, x \in B.
\]

- If $h = h_t(x)$ is non-random and $\Phi$ is elementary, then

\[
\int h \Phi dW := X \int_{(a, b) \times B} h_t(x)\phi(x)W(dx \, dt).
\]

- The stochastic integral is Wiener’s, and it is well defined iff $h_t(x)\phi(x) \in L^2([a, b] \times B)$.

- We have Walsh isometry,

\[
\mathbb{E}\left(\left\|\int h \Phi dW \right\|^2\right) = \int_0^\infty ds \int_B dy \mathbb{E}[|h_s(y)|^2\mathbb{E}(|\Phi_s(y)|^2)].
\]

(1.9)

We can make sense of equation (1.8) using Walsh theory [32] again by using the integral equation below.

\[
u_t(x) = (g_B^{(\beta)}u_0)_t(x) + \lambda \int_0^t \int_B G_B^{(\beta)}(t - s, x, y)\sigma(u_s(y))W(ds, dy),
\]

(1.10)
where \( G^{(\beta)}_B(t, x, y) \) denotes the heat kernel of the space-time fractional diffusion equation with Dirichlet boundary conditions in (1.1), and

\[
(G^{(\beta)}_B u_0)(x) := \int_B G^{(\beta)}_B(t, x, y)u_0(y) \, dy.
\]

If \( d < (2 \wedge \beta^{-1})\alpha \), the proof for the existence of a unique random-field solution of (1.8) satisfying

\[
\sup_{x \in B} \mathbb{E}|u_t(x)|^2 \leq c_1 e^{c_2 t \lambda^{\frac{2\alpha}{\alpha-d}}}
\]

for all \( t > 0 \) can be found in Foondun et al. [14].

Next we state our main results.

**Theorem 1.1.** Suppose that \( d < (2 \wedge \beta^{-1})\alpha \). Let \( u_t \) denote the unique solution to (1.10). Then the second moment of \( u_t \) cannot decay exponentially fast, regardless of what \( \lambda \) is. Indeed, if we further suppose that \( \beta \in (0, \frac{1}{2}] \), then as \( t \) gets large, \( \sup_{x \in B} \mathbb{E}|u_t(x)|^2 \) grows exponentially fast for any \( \lambda \).

**Theorem 1.2.** In Theorem 1.1 if \( \beta \in (\frac{1}{2}, 1) \), then there exist a strictly positive real number \( \lambda_u \) such that for all \( \lambda > \lambda_u \), \( \sup_{x \in B} \mathbb{E}|u_t(x)|^2 \) grows exponentially fast as time gets large.

Since the Mittag-Leffler function \( E_\beta(-t^\beta) \) (Gorenflo et al. [18]) behaves as a stretched exponential for \( t \to 0 \);

\[
E_\beta(-t^\beta) \approx 1 - \frac{t^\beta}{\Gamma(\beta + 1)} \approx e^{-t^{\beta/\Gamma(\beta+1)}}, \quad \text{for } 0 < t \leq 1,
\]

and as a polynomial decay for \( t \to \infty \),

\[
E_\beta(-t^\beta) \approx \frac{\sin(\beta\pi)}{\pi} t^{\beta}, \quad \text{for } t \geq 1,
\]

the polynomial decay behaviour of \( E_\beta(-t^\beta) \) illustrates the need for the sharp condition of \( \beta \in (0, \frac{1}{2}] \) in Theorem 1.1. So the representation \( G^{(\beta)}_B \) in equation (2.7) which is defined in terms of \( E_\beta(.) \) is crucial to our results.

**Theorem 1.3.** In Theorem 1.1, suppose that \( \beta \in (\frac{1}{2}, 1) \). Suppose also that either \( d < \alpha/2\beta \) or \( \{\varphi_n\}_{n \geq 1} \) are uniformly bounded by a constant \( C(B) \), then there exist a strictly positive real number \( \lambda_l \) such that for \( \lambda < \lambda_l \) the quantity \( \sup_{t>0} \sup_{x \in B} \mathbb{E}|u_t(x)|^2 \) is finite.

**Remark 1.4.** The assumption of a uniform bound \( C(B) \) on the eigenfunctions \( \{\varphi_n\}_{n \geq 1} \) look artificial, however there is no known bound for these eigenfunctions except the cases given in Lemma 2.1 and the uniform bounds as in Example 2.9 for the Brownian motion in higher dimensional rectangular boxes.

From the three results above in line with the interpretation given in Foondun [16], we note that whenever \( \beta \in (0, \frac{1}{2}] \), the killed time changed \( \alpha \)-stable process \( X^{\beta}_E \) defined in Section 2 reaches the boundary of \( B \) slower which allows the non-linear term to grow exponentially for any \( \lambda \) when \( t \) becomes large. On the other hand, for \( \beta \in (\frac{1}{2}, 1) \), this process does not have time to generate such growth unless \( \lambda \) is large enough, since it proceeds quickly enough to the boundary.

Fractional parameters play an important role in many problems involving space-time fractional equations. However, in modeling problems, these fractional parameters are unknown a priori. As a result, continuity of the solutions with respect to these parameters is essential for modeling purposes. The following continuity theorem of the unique solution \( u^{\beta}_t(x) \) of equation (1.5) with respect to the parameter \( \beta \) is Theorem 4.3(b) of Dang et al. [9].
Theorem 1.5 (Dang, Nane and Nguyen [9]). Let \( u^{(\gamma)}_t \) and \( u^{(\beta)}_t \) denote the solution to the following equation for parameters \( \gamma, \beta \in (0, 1) \) with \( \gamma \to \beta \). The initial condition \( u_0 \) is the same for both equations.

$$
\begin{aligned}
&\frac{\partial^{\beta}}{\partial t^{\beta}} u_t(x) = \Delta u_t(x), \quad x \in B, \quad t > 0, \\
&u_t(x) = 0, \quad x \in \partial D, \quad t > 0, \\
u(0, x) = f(x), \quad x \in B.
\end{aligned}
$$

(1.11)

Then, we have

$$
\lim_{\gamma \to \beta} ||u^{(\gamma)}_t(x) - u^{(\beta)}_t(x)||_H^2 = 0.
$$

where \( H \) is the Hilbert space of all functions with the bounded norm induced by

$$
||f||_H = \sqrt{\sum_{k=1}^{\infty} \mu_k^2 |\langle f, \varphi_k \rangle|^2}.
$$

Remark 1.6. Meerschaert et al. [24] established the existence of a unique solution for equation (1.11).

The following continuity theorem of the unique solution \( u^{(\beta)}_t(x) \) of equation (1.6) with respect to the parameter \( \beta \) is Theorem 1.3 of Foondun [16].

Theorem 1.7 (Foondun [16]). Let \( u^{(\beta)}_t \) and \( u_t \) denote the solution to equation (1.6) and the solution to equation (1.10) for \( \beta = 1 \) respectively. The initial condition \( u_0 \) is the same for both equations. Then, for any \( p \geq 2 \), we have

$$
\lim_{\beta \to 1} \sup_{x \in [0,L]} E|u_t(x) - u^{(\beta)}_t(x)|^p = 0.
$$

In the next theorem, we show the continuity of the solution \( u_t(x) \) of (1.10) with respect to the fractional parameter \( \beta \).

Theorem 1.8. Assume that \( \{\varphi_n\}_{n \geq 1} \) are uniformly bounded by a constant \( C(B) \). For \( d < \frac{1}{2} \min\left(\frac{\beta}{p}, \frac{1}{1} \right) \alpha \), let \( u^{(\gamma)}_t \) and \( u^{(\beta)}_t \) denote solutions to equation (1.10) for parameters \( \beta, \gamma \in (\frac{1}{2}, 1) \) respectively with \( \gamma \to \beta \). The initial condition \( u_0 \) is the same for both equations. Then, for any \( p \geq 2 \), we have

$$
\lim_{\gamma \to \beta} \sup_{x \in B} E|u^{(\gamma)}_t(x) - u^{(\beta)}_t(x)|^p = 0.
$$

Here we observe that Theorem 1.8 extends Theorem 1.3 in Foondun [16] to fractional Laplacian case, and Theorem 4.3(b) of Dang et al. [9] to stochastic case.

The other class of equations that we consider in this paper is equation with space colored noise stated as

$$
\begin{aligned}
&\partial_\gamma^\beta u_t(x) = -(-\Delta)^{\frac{\sigma}{2}} u_t(x) + I_t^{1-\beta}[\lambda \sigma(u_t(x))\hat{F}(t, x)] \quad \text{for } x \in B \quad \text{and } t > 0, \\
u_t(x) = 0 \quad \text{for } x \notin B \quad \text{and } t > 0,
\end{aligned}
$$

(1.12)

and the initial condition \( u_0 : B \to \mathbb{R}_+ \) is a non-random measurable and bounded function that has support with positive measure inside B. The operator \(-(-\Delta)^{\frac{\sigma}{2}}\), where \( 0 < \alpha \leq 2 \), is the \( L^2 \)-generator of a symmetric \( \alpha \)-stable process \( X_t^B \) killed when exiting \( B \). The function \( \sigma : \mathbb{R} \to \mathbb{R} \) is a globally Lipschitz function satisfying \( \sigma(x) \leq |\sigma(x)| \leq L_\sigma |x| \) where \( L_\sigma \) and \( L_\sigma \) are positive constants. The positive parameter \( \lambda \) is called the level of the noise. The noise \( \hat{F}(t, x) \) is white in time and colored in space satisfying

$$
\text{Cov}(\hat{F}(t, x), \hat{F}(s, y)) = \delta_0(t-s)f(x, y),
$$
where \(0 < f(x, y) \leq g(x - y)\) and \(g\) is a locally integrable function on \(\mathbb{R}^d\) with possible singularity at 0 satisfying
\[
\int_{\mathbb{R}^d} \frac{\hat{g}(\xi)}{1 + |\xi|^{\alpha}} d\xi < \infty,
\]
(1.13)
where \(\hat{g}\) denotes the Fourier transform of \(g\).

We will need the following non degeneracy condition on the spatial correlation of the noise.

**Assumption 1.9.** Assume there exists some positive number \(K_f\) such that
\[
\inf_{x, y \in B} f(x, y) \geq K_f.
\]

This assumption is very mild as it is shown by the following examples.

**Example 1.10.** For the following list of examples Assumption 1.13 is satisfied.

- **Riesz Kernel:**
  \[
  f(x, y) = \frac{1}{|x - y|^{\gamma}} \text{ with } \gamma < d \wedge \alpha.
  \]

- **The Exponential-type kernel:**
  \[
  f(x, y) = \exp[-(x \cdot y)].
  \]

- **The Ornstein-Uhlenbeck-type kernels:**
  \[
  f(x, y) = \exp[-|x - y|^{\delta}] \text{ with } \delta \in (0, 2].
  \]

- **Poisson Kernels:**
  \[
  f(x, y) = \sum_{j=1}^{d} \left( \frac{1}{1 + (x_j - y_j)^2} \right).
  \]

- **Cauchy Kernels:**
  \[
  f(x, y) = \sum_{j=1}^{d} \left( \frac{1}{1 + (x_j - y_j)^2} \right).
  \]

Following Walsh [32], \(u_t\) is a mild solution to (1.12) if
\[
u_t(x) = (G_B^{(\beta)} u_0)_t(x) + \lambda \int_0^t \int_B G_B^{(\beta)}(t - s, x, y) \sigma(u_s(y)) F(ds, dy),
\]
(1.14)
where \(G_B^{(\beta)}(t, x, y)\) denotes the heat kernel of the space-time fractional diffusion equation with Dirichlet boundary conditions in (1.12), and
\[
(G_B^{(\beta)} u_0)_t(x) := \int_B G_B^{(\beta)}(t, x, y) u_0(y) dy.
\]

**Theorem 1.11.** Suppose that the Dalang condition (1.13) holds. Let \(u_t\) denote the unique solution to (1.10). Then no matter what \(\lambda\) is, the second moment of \(u_t\) cannot decay exponentially fast. In fact, if we further assume that \(\beta \in (0, \frac{1}{2}]\) and Assumption 1.9 holds, then as \(t\) gets large, \(\sup_{x \in B} \mathbb{E} |u_t(x)|^2\) grows exponentially fast for any \(\lambda\).

**Theorem 1.12.** In Theorem 1.11, if \(\beta \in (\frac{1}{2}, 1)\) and Assumption 1.9 hold, then there exist a strictly positive real number \(\lambda_u\) such that for all \(\lambda > \lambda_u\), \(\sup_{x \in B} \mathbb{E} |u_t(x)|^2\) grows exponentially fast as time gets large.

**Theorem 1.13.** In Theorem 1.11, suppose that \(\beta \in (\frac{1}{2}, 1)\), \(d < \frac{\alpha}{2\beta}\), and \(\int_{B \times B} f(x, y) dx \, dy < \infty\). If \(\{\varphi_n\}_{n \geq 1}\) are uniformly bounded by a constant \(C(B)\), then there exist a strictly positive real number \(\lambda_l\) such that for \(\lambda < \lambda_l\) the quantity \(\sup_{t > 0} \sup_{x \in B} \mathbb{E} |u_t(x)|^2\) is finite.
Corollary 1.14. In Theorem 1.13, if $d = 1$ and $\alpha = 2$ then the conclusion of theorem follows.

Corollary 1.15. In Theorem 1.13, if $d = 1$ and $f$ is Riesz Kernel function, then the conclusion of theorem follows.

Theorem 1.16. Assume that $\{\varphi_n\}_{n \geq 1}$ are uniformly bounded by a constant $C(B)$, and
\[ \int_{B \times B} f(x, y) \, dx \, dy < \infty. \]
For $d < \frac{4}{\min\left(\frac{1}{2}, \frac{3}{2}\right)}$, let $u^{(\beta)}(t)$ and $u^{(\gamma)}(t)$ denote solutions to (1.10) for parameters $\beta, \gamma \in \left(\frac{1}{2}, 1\right)$ with $\gamma \rightarrow \beta$. The initial condition $u_0$ is the same for both equations. Then, for any $p \geq 2$, we have
\[ \limsup_{\gamma \rightarrow \beta} E|u^{(\gamma)}(t) - u^{(\beta)}(t)|^p = 0. \]

We now briefly give an outline of the paper. In this paper we employ similar methods as in Foondun [16] with crucial changes to prove our main results. This method was also used in [12] and [15] among others. Section 2 contains estimates and some preliminary results needed for the proof of main results. Section 3 is devoted to the proof of Theorem 1.1, Theorem 1.2, and Theorem 1.3 while the proofs of Theorem 1.8 is given in Section 4. Further, in Section 5 we give the proof of Theorem 1.11, Theorem 1.12, Theorem 1.13, and Theorem 1.16.

2 Preliminaries

In this section we give some preliminary results, which are needed for the proofs of the main theorems. Let $X_t$ denote a symmetric stable process of index $\alpha \in (0, 2)$ in $\mathbb{R}^d$ and $B$ be a regular bounded open subset of $\mathbb{R}^d$. Let $X_t^B$ denote the symmetric stable process killed upon exiting $B$. The following Hilbert-Schmidt expansion (see Davies [10]) holds for the probability density function $p_B$ of $X_t^B$,
\[ p_B(t, x, y) := \sum_{n=1}^{\infty} e^{-\mu_n t} \varphi_n(x) \varphi_n(y), \]
(2.1)
for all $x, y \in B$, $t > 0$, where $\{\varphi_n\}_{n \geq 0}$ is an orthonormal basis of $L^2(B)$, and $0 < \mu_1 < \mu_2 \leq \mu_3 \leq \cdots$ is a sequence of positive real numbers such that, for every $n \geq 1$, $P_t^B \varphi_n = e^{-\mu_n t} \varphi_n$.

From Theorem 2.3 of Blumenthal and Getoor [5] and the proof of Theorem 5.1 of Chen, Meerschaert, and Nane [6], we infer the following lemma:

Lemma 2.1. [5, 6] For any bounded open subset $B$ of $\mathbb{R}^d$, the system of eigenfunctions $\{\varphi_n\}_{n \geq 1}$ and the corresponding eigenvalues $\{\mu_n\}_{n \geq 1}$ satisfy the following:

a) $C_1 n^{\frac{d}{2}} \ll \mu_n \ll C_2 n^{\frac{d}{2}}$ for every $n \in \mathbb{N},$

b) $|\varphi_n(x)| \leq C \mu_n^{\frac{d}{2}}$ for every $x \in B,$

where $C, C_1,$ and $C_2$ are positive real numbers.

It is also known that
\[ u_t(x) := E^x[u_0(X^B_t)] \]
\[ = \int_{B} p_B(t, x, y) u_0(y) \, dy, \]
solves the heat equation $\partial_t u_t(x) = -(-\Delta)^{\alpha/2} u_t(x)$ defined on $B$ with Dirichlet boundary condition and initial condition $u_0$.

Let $D = \{D_r, r \geq 0\}$ denote a $\beta$-stable subordinator and $E_t$ to be the inverse of a stable subordinator of index $\beta \in (0, 1)$. The process $X^B_{E_t}$ is just a time-changed of the killed $\alpha$-symmetric stable process $X^B_t$. 

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and since $\beta \in (0, 1)$, $X_{E_t}^B$ moves more slowly than $X_t^B$. It is known that the density of the time changed process $X_{E_t}^B$ is given by the $G^{(\beta)}(t, x)$. By conditioning, we have
\[ G^{(\beta)}(t, x) = \int_0^\infty p_B(s, x)f_t(s)ds, \quad (2.2) \]
where
\[ f_t(s) = t\beta^{-1}s^{-1-1/\beta}g_{\beta}(ts^{-1/\beta}), \quad (2.3) \]
where $g_{\beta}(.)$ (Cf. Meerschaert and Straka [23]) is the density function of $D_1$ and is infinitely differentiable on the entire real line, with $g_{\beta}(u) = 0$ for $u \leq 0$; see Meerschaert and Scheffler [25] for more information about the inverse stable subordinator $E_t$.

The function $v_t(x) := \mathbb{E}^x[u_0(X_{E_t}^B)]$ solves the space-time fractional equation
\[
\begin{cases}
\partial_t^\beta v_t(x) = -(-\Delta)^{\alpha/2}v_t(x) \quad \text{for } x \in B \text{ and } t > 0 \\
v_t(x) = 0 \quad \text{for } x \notin B,
\end{cases}
\quad (2.4)
\]
with initial condition $u_0$ (Chen, Meerschaert, and Nane [6]). Thus, we get the following representation of $v_t(x)$
\[ v_t(x) := \mathbb{E}^x[u_0(X_{E_t}^B)] = \int_B \int_0^\infty \sum_{n=1}^\infty e^{-\mu_n s}\varphi_n(x)\varphi_n(y)f_t(s)u_0(y)dsdy \]
\[ = \int_B \sum_{n=1}^\infty E_\beta(-\mu_n t^\beta)\varphi_n(x)\varphi_n(y)u_0(y)dy \]
\[ = \int_B G_B^{(\beta)}(t, x, y)u_0(y)dy, \quad (2.5) \]
this follows since the Laplace transform of $f_t(s)$ is $\hat{f}_t(\lambda) = E_\beta(-\lambda t^\beta)$ where $E_\beta(x) = \sum_{k=0}^\infty \frac{x^k}{\Gamma(1+k\beta)}$ is the Mittag-Leffler function and have the following property,
\[ \frac{1}{1 + \Gamma(1-\beta)x} \leq E_\beta(-x) \leq \frac{1}{1 + \Gamma(1+\beta)^{-1}x} \quad \text{for } x > 0, \quad (2.6) \]
where $\Gamma(\cdot)$ is the gamma function. Thus, we get the following expansion,
\[ G_B^{(\beta)}(t, x, y) := \sum_{n=1}^\infty E_\beta(-\mu_n t^\beta)\varphi_n(x)\varphi_n(y). \quad (2.7) \]

Using (2.3) and a change of variable from (2.7) we obtain
\[ G_B^{(\beta)}(t, x, y) = \int_0^\infty \sum_{n=1}^\infty e^{-\mu_n(t/u)^\beta}\varphi_n(x)\varphi_n(y)g_{\beta}(u)du \]
\[ = \int_0^\infty p_B\left(\frac{t}{u}\right)\beta^\beta, \quad x, y)g_{\beta}(u) \ du, \quad (2.8) \]

The following Lemma from Mijena and Nane [20] about the $L^2$-norm of the heat kernel which is a crucial ingredient for the existence and uniqueness of solution of equation (1.8) is also used in proving Theorem 1.8.
Lemma 2.2. (Lemma 1 in [20]) Suppose that \( d < 2\alpha \), then
\[
\int_{\mathbb{R}^d} |G^\beta_t(x)|^2\,dx = C^* t^{\frac{\alpha d}{\alpha}} ,
\]
where the constant \( C^* \) is given by
\[
C^* = \frac{\nu^{-\frac{d}{2}} 2\pi^\frac{d}{2}}{\alpha \Gamma\left(\frac{d}{2}\right)} \frac{1}{(2\pi)^d} \int_0^\infty z^{-\frac{d}{\alpha}} (\mathbb{E}_\beta(-z))^2 \,dz.
\]

The following three results are helpful in proving the finiteness result in Theorem 1.3, and Theorem 1.13.

Lemma 2.3. For \( p \geq 2 \) and \( \beta \in \left(\frac{1}{p}, 1\right) \), we have
\[
\int_0^t E_\beta(-\mu_n(t-s)^\beta)^p \,ds \lesssim \mu_n^{-\frac{\beta}{p}},
\]
for every \( t > 0 \).

Proof. Using the Mittag-Leffler function bounds given in (2.6), we get
\[
\int_0^t E_\beta(-\mu_n(t-s)^\beta)^p \,ds \leq C \int_0^t \left( 1 + \Gamma(1+\beta)^{-1} \mu_n(t-s)^\beta \right)^{-p} \,ds \leq C \int_0^t \left( 1 \wedge (\mu_n(t-s)^\beta)^{-p} \right) \,ds,
\]
where
\[
1 \wedge (\mu_n(t-s)^\beta)^{-p} = \begin{cases} 1 & \text{if } (\mu_n(t-s)^\beta)^{-p} > 1, \\ (\mu_n(t-s)^\beta)^{-p} & \text{if } (\mu_n(t-s)^\beta)^{-p} \leq 1. \end{cases}
\]
Now
\[
(\mu_n(t-s)^\beta)^{-p} \leq 1 \implies \mu_n(t-s)^\beta \geq 1 \implies (t-s)^\beta \geq \frac{1}{\mu_n} \implies t-s \geq \left(\frac{1}{\mu_n}\right)^{\frac{1}{\beta}} \implies t - \mu_n^{-\frac{1}{\beta}} \geq s.
\]
So, from (2.13) using (2.14) we obtain

\[
\int_0^t E_\beta(-\mu_n(t-s)^\beta)^p \, ds \leq C \int_0^t \left(1 \wedge \left(\mu_n(t-s)^\beta\right)^{-p}\right) \, ds \tag{2.15}
\]

\[
= C \left(\int_0^{t-\frac{1}{\mu_n}} \left(\mu_n(t-s)^\beta\right)^{-p} \, ds + \int_{t-\frac{1}{\mu_n}}^t \, ds\right)
\]

\[
= C \left(\frac{1}{\mu_n^{p-1}} \int_0^{t-\mu_n^{-1}} (t-s)^{-p\beta} \, ds + \mu_n^{\frac{1}{\beta}}\right)
\]

\[
= C \left(\frac{1}{\mu_n^{p-1}(1-p\beta)} \left(t^{-p\beta+1} - \left(\frac{1}{\mu_n}\right)^{-p+\frac{1}{\beta}}\right) + \mu_n^{\frac{1}{\beta}}\right)
\]

\[
\leq C \left(\frac{\mu_n^{p-1}}{\mu_n^{p-1}(p\beta - 1)} + \mu_n^{\frac{1}{\beta}}\right)
\]

\[
\leq C \left(\frac{1}{p\beta - 1} + 1\right) \mu_n^{\frac{1}{\beta}}
\]

\[
\lesssim \mu_n^{-\frac{1}{\beta}}, \tag{2.16}
\]

since \(\frac{t^{-p\beta+1}}{1-p\beta} < 0\).

\[\square\]

**Lemma 2.4.** For \(p \geq 2\), \(\beta \in \left(\frac{1}{p}, 1\right)\) and \(d < \frac{\alpha}{\beta}\), we have

\[
\int_0^t \sum_{n=1}^\infty E_\beta(-\mu_n(t-s)^\beta)^p \, ds < \infty, \tag{2.17}
\]

for every \(t > 0\).

**Proof.** Using Lemma 2.1(a) and Lemma 2.3, we obtain the following:

\[
\int_0^t \sum_{n=1}^\infty E_\beta(-\mu_n(t-s)^\beta)^p \, ds = \sum_{n=1}^\infty \int_0^t E_\beta(-\mu_n(t-s)^\beta)^p \, ds
\]

\[
\lesssim \sum_{n=1}^\infty \frac{-\frac{1}{\beta}}{\mu_n}
\]

\[
\lesssim \sum_{n=1}^\infty n^{-\frac{\alpha}{p\beta}} < \infty, \tag{2.18}
\]

since \(\frac{\alpha}{p\beta} > 1\).

\[\square\]

**Lemma 2.5.** For \(p \geq 2\), \(\beta \in \left(\frac{1}{p}, 1\right)\), if \(d < \frac{\alpha}{2\beta}\) then we have

\[
\int_0^t \sum_{n=1}^\infty E_\beta(-\mu_n(t-s)^\beta)^p \varphi_n^2(x) \, ds < \infty, \tag{2.19}
\]

for every \(t > 0\).
Proof. We prove the result using by Lemma 2.1(b) and Lemma 2.3 as follows

\[
\int_0^t \sum_{n=1}^{\infty} E_\beta(-\mu_n(t-s)\beta) p \varphi_n^2(x) \, ds = \sum_{n=1}^{\infty} \int_0^t E_\beta(-\mu_n(t-s)\beta) p \varphi_n^2(x) \, ds \\
\leq C \sum_{n=1}^{\infty} \frac{\mu_n^{-\frac{1}{\beta}}}{\mu_n} \\
\leq \sum_{n=1}^{\infty} n^{\frac{d}{\alpha} - \frac{1}{\beta}} \\
\leq \sum_{n=1}^{\infty} n^{1-\frac{\alpha}{\beta d}} < \infty, \quad (2.20)
\]

since \( \frac{\alpha}{\beta d} > 2 \).

The following continuity Lemma is crucial to justify the conclusion of Theorem 1.8.

Lemma 2.6. Assume that \( \{ \varphi_n \}_{n \geq 1} \) are uniformly bounded by a constant \( C(B) \), depending on the geometrical characteristics of the domain \( B \), i.e.,

\[
C(B) = \sup_{n \geq 1, x \in B} \varphi_n(x).
\]

Fix \( t > 0 \), then for \( d < \alpha \) we have

\[
\lim_{\gamma \to \beta} \sup_{x \in B} \left| (G_B^{(\gamma)} u_0)_t(x) - (G_B^{(\beta)} u_0)_t(x) \right| = 0,
\]

where

\[
(G_B^{(\gamma)} u_0)_t(x) := \int_B G_B^{(\gamma)}(t, x, y) u_0(y) \, dy.
\]

Proof. Since the initial datum \( u_0 \) and \( G_B^{(\gamma)}(t, x, y) \) are given to be bounded from above, it suffices to show that

\[
\lim_{\gamma \to \beta} \sup_{x \in B} \left| G_B^{(\gamma)}(t, x, y) - G_B^{(\beta)}(t, x, y) \right| = 0.
\]

Now using the Laplace transform of the Mittag-Leffler function, for any \( \theta > 0 \) we have

\[
\lim_{\gamma \to \beta} \int_0^\infty e^{-\theta t} E_\gamma(-\mu_n t^\gamma) \, dt = \lim_{\gamma \to \beta} \frac{\theta^{\gamma-1}}{\theta^{\gamma} + \mu_n} = \int_0^\infty e^{-\theta t} E_\beta(-\mu_n t^\beta) \, dt.
\]

This implies that

\[
\lim_{\gamma \to \beta} E_\gamma(-\mu_n t^\gamma) = E_\beta(-\mu_n t^\beta). \quad (2.21)
\]

Now using the expansions for the heat kernel, we have

\[
\left| G_B^{(\beta)}(t, x, y) - G_B^{(\gamma)}(t, x, y) \right| \leq \sum_{n=1}^{\infty} \left| E_\beta(-\mu_n t^\beta) - E_\gamma(-\mu_n t^\gamma) \right| \left( \varphi_n(x) | \varphi_n(y) | \right). \quad (2.22)
\]

Taking limit as \( \gamma \to \beta \) on both sides of (2.22) we obtain the required result, since each terms in the summation can be bounded by a quantity independent of \( \gamma \) and \( \beta \). This is due to the uniform boundedness of the eigenfunctions \( \{ \varphi_n \}_{n \geq 1} \) and the bounds on the Mittag-Leffler function together with Lemma 2.1(a),
and summable as shown below.
For any \( t > 0 \), we obtain
\[
\sum_{n=1}^{\infty} \left| E_\beta(-\mu_n t^\beta) - E_\gamma(-\mu_n t^\gamma) \right| \leq |C(B)|^2 \sum_{n=1}^{\infty} \left| E_\beta(-\mu_n t^\beta) - E_\gamma(-\mu_n t^\gamma) \right|
\]
\[
\leq |C(B)|^2 \sum_{n=1}^{\infty} \left[ \frac{1}{1+\Gamma(1+\beta)^{-1}} \mu_n t^\beta + \frac{1}{1+\Gamma(1+\gamma)^{-1}} \mu_n t^\gamma \right]
\]
\[
\leq |C(B)|^2 \sum_{n=1}^{\infty} \frac{1}{\mu_n} \left( \frac{1}{t^\beta} + \frac{1}{t^\gamma} \right)
\]
\[
\leq 2|C(B)|^2 \max(1,1/t) \sum_{n=1}^{\infty} n^{-2} < \infty,
\]
where \( C \) is a constant positive real number.

The following lemma is useful in the proof of Lemma 2.8(b) and Proposition 6.1.

**Lemma 2.7.** If \( \beta \in (\beta_0, \beta_1) \) for some \( \beta_0, \beta_1 \in (0,1) \) and \( \frac{4}{\beta} < \lambda < \frac{1}{2\beta} \), then using the inequality \( e^{-z} \leq C \mu z^{-\mu} \), we bound the following integral as follows:
\[
\int_0^\infty e^{-2s/p} \frac{1}{s^{2\lambda \beta}} ds \leq p^{1-2\lambda \beta_0} \left[ \frac{C_{\mu_0}^2 \theta^{-2\mu_0}}{1-2\lambda - 2\mu_0} + \frac{C_{\mu_1}^2 \theta^{-2\mu_1}}{2\lambda \beta_0 + 2\mu_1 - 1} \right],
\]
where \( 0 < \mu_0 < \min \left( \frac{1}{2} - \lambda, \frac{1-2\beta_0 \lambda}{2} \right) \) and \( \mu_1 > \frac{1-2\beta_0 \lambda}{2} \).

**Proof.** Consider the integral
\[
\int_0^\infty e^{-2s/p} \frac{1}{s^{2\lambda \beta}} ds = p^{1-2\lambda \beta} \int_0^\infty e^{-2r/p} \frac{1}{r^{2\lambda \beta}} dr
\]
\[
= p^{1-2\lambda \beta} \left[ \int_0^1 e^{-2r/p} \frac{1}{r^{2\lambda \beta}} dr + \int_1^\infty e^{-2r/p} \frac{1}{r^{2\lambda \beta}} dr \right]
\]
\[
\leq p^{1-2\lambda \beta} \left[ C_{\mu_0}^2 \theta^{-2\mu_0} \int_0^1 \frac{1}{r^{2\lambda \beta + 2\mu_0}} dr + C_{\mu_1}^2 \theta^{-2\mu_1} \int_1^\infty \frac{1}{r^{2\lambda \beta + 2\mu_1}} dr \right]
\]
\[
\leq p^{1-2\lambda \beta_0} \left[ \frac{C_{\mu_0}^2 \theta^{-2\mu_0}}{1-2\lambda - 2\mu_0} + \frac{C_{\mu_1}^2 \theta^{-2\mu_1}}{2\lambda \beta_0 + 2\mu_1 - 1} \right],
\]
where \( \beta \in (\beta_0, \beta_1) \) for some \( \beta_0, \beta_1 \in (0,1) \), \( \mu_0 < \min \left( \frac{1}{2} - \lambda, \frac{1-2\beta_0 \lambda}{2} \right) \) and \( \mu_1 > \frac{1-2\beta_0 \lambda}{2} \), which implies that the proper integral \( \int_0^1 \frac{1}{r^{2\lambda \beta + 2\mu_0}} dr \) and \( \int_1^\infty \frac{1}{r^{2\lambda \beta + 2\mu_1}} dr \) are convergent.

**Lemma 2.8** (Nane and Tuan, 2022 [26]). Assume that \( \{ \varphi_n \}_{n \geq 1} \) are uniformly bounded by a constant \( C(B) \), depending on the geometrical characteristics of the domain \( B \), i.e.,
\[
C(B) = \sup_{n \geq 1, x \in B} \varphi_n(x).
\]

a) Let \( \beta \in (\frac{1}{2}, 1) \), and \( \frac{4}{\beta} < \lambda < \frac{1}{2\beta} \). Then there exists a constant \( C = C(\lambda) \) independent of \( \beta \) such that
\[
|G_B^{(\beta)}(t, x, y)| \leq Ct^{-\beta \lambda}.
\]
b) Let $\beta \in (\beta_0, \beta_1)$ for some $\beta_0, \beta_1 \in (0,1)$. Let also $0 < \mu_0 < \min \left( \frac{1}{2} - \lambda, \frac{1-2\beta\lambda}{2} \right)$ and $\mu_1 > \frac{1-2\beta\lambda}{2}$.

Then for $d < \frac{\alpha}{23}$, there exists positive constant $C$ which depends on $\mu_0, \mu_1, p$ and independent of $\beta$ such that

$$
\left[ \int_0^\infty \int_B e^{-2\beta s/p} |G_B(\beta)(s, x, y)|^2 dy ds \right]^{p/2} \lesssim C p^{p(1-2\beta\lambda)/2} (\theta^{-2\mu_0} + \theta^{-2\mu_1})^{p/2}. \tag{2.27}
$$

Proof. (a) Using the fact that $E_\beta(-\mu_0 t^2) \leq \frac{C}{(1+\mu_n^2 s)^{\beta}}$ for any $\frac{d}{\alpha} < \lambda < \frac{1}{23}$ and $C$ is independent of $\beta$, we find that

$$
|G_B(\beta)(t, x, y)| = \sum_{n=1}^{\infty} E_\beta(-\mu_n s^\beta) \varphi_n(x) \varphi_n(y) \lesssim C \sum_{n=1}^{\infty} |E_\beta(-\mu_n s^\beta)|
$$

$$
\lesssim C \sum_{n=1}^{\infty} \frac{1}{\mu_n^{\beta}} \lesssim C t^{-\beta\lambda}, \tag{2.28}
$$

where we used the eigenfunction expansion of $G_B(\beta)(t, x, y)$ and that the series $\sum_{n=1}^{\infty} \frac{1}{\mu_n^{\beta}}$ is convergent since $\lambda > \frac{d}{\alpha}$. (b) Using part (a) of this lemma together with Lemma 2.7, we obtain

$$
\int_0^\infty \int_B e^{-2\beta s/p} |G_B(\beta)(s, x, y)|^2 dy ds \lesssim \int_0^\infty e^{-2\beta s/p} \frac{1}{s^{2\beta\lambda}} ds
$$

$$
\lesssim p^{1-2\beta\lambda_0} \left[ \frac{\varphi_{\mu_0}^2 \theta^{-2\mu_0}}{1 - 2\lambda - 2\mu_0} + \frac{\varphi_{\mu_1}^2 \theta^{-2\mu_1}}{2\lambda\beta_0 + 2\mu_1 - 1} \right], \tag{2.29}
$$

where $\beta \in (\beta_0, \beta_1)$ for some $\beta_0, \beta_1 \in (0,1)$, $\mu_0 < \min \left( \frac{1}{2} - \lambda, \frac{1-2\beta\lambda}{2} \right) < \min \left( \frac{1}{2} - \lambda, \frac{1-2\beta\lambda}{2} \right)$ and $\mu_1 > \frac{1-2\beta\lambda}{2}$, which implies that the proper integral $\int_0^1 \frac{1}{r^{2\lambda+2\mu_0}} dr$ and $\int_1^\infty \frac{1}{r^{2\lambda+2\mu_1}} dr$ are convergent. It follows from (2.29) that for any $p \geq 0$

$$
\left[ \int_0^\infty \int_B e^{-2\beta s/p} |G_B(\beta)(s, x, y)|^2 ds dy \right]^{p/2} \lesssim p^{p(1-2\beta\lambda)/2} \left[ \frac{\varphi_{\mu_0}^2 \theta^{-2\mu_0}}{1 - 2\lambda - 2\mu_0} + \frac{\varphi_{\mu_1}^2 \theta^{-2\mu_1}}{2\lambda\beta_0 + 2\mu_1 - 1} \right]^{p/2},
$$

which allows us to deduce (2.27). That is, each terms in the summation can be bounded by a quantity independent of $\beta$.

Next we give an example where the sufficient condition of Lemma 2.6 is satisfied.

**Example 2.9.** Let $X_t$ denote a Brownian motion in $\mathbb{R}^2$ and $X_t^B$ denote the Brownian motion killed upon exiting the rectangular domain $B := [0, L_1] \times [0, L_2]$. The eigenvalues of the Dirichlet Laplacian are $\mu_{m,n} = \left( \frac{m\pi}{L_1} \right)^2 + \left( \frac{n\pi}{L_2} \right)^2$ and $\varphi_{m,n}(x, y) := \varphi_m(x) \varphi_n(y) = \frac{2}{\sqrt{L_1 L_2}} \sin \left( \frac{m\pi x}{L_1} \right) \sin \left( \frac{n\pi y}{L_2} \right)$ are the corresponding eigenfunction so that $|\varphi_{m,n}(x, y)| \leq \frac{2}{\sqrt{L_1 L_2}}$ for all $(x, y) \in B$. A similar result is valid for the Brownian motion in higher dimensional rectangular boxes.

### 3 Proof of Theorem 1.1, Theorem 1.2, and Theorem 1.3

In this section we give the proof of first three main theorems of the paper. We denote

$$
\Lambda(\theta) := \int_0^\infty e^{-\theta t} E_\beta(-\mu_1 t^\beta)^2 dt, \tag{3.1}
$$

for $\theta > 0$. Here it is crucial to note that we can use the bounds in (2.6) to observe that $\Lambda(\theta)$ tends to infinity as $\theta$ approaches to zero if and only if $2\beta \leq 1$. 

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**Proof of Theorem 1.1.** Let us first establish the second statement of the theorem. From the mild formulation given in (1.10) and using Stochastic Fubini theorem we set

\[
\langle u_t, \varphi_1 \rangle = \int_B u_t(x) \varphi_1(x) \, dx \\
= E\beta(-\mu_1 t^\beta)\langle u_0, \varphi_1 \rangle \\
+ \lambda \int_B \int_0^t E\beta(-\mu_1 (t-s)^\beta)\varphi_1(y) \sigma(u_s(y)) W(ds, dy). \tag{3.2}
\]

Taking the second moment of (3.2) and using the Ito-Walsh isometry, we get

\[
E\langle u_t, \varphi_1 \rangle^2 = E\beta(-\lambda_1 t^\beta)^2\langle u_0, \varphi_1 \rangle^2 \\
+ \lambda^2 \int_B \int_0^t E\beta(-\lambda_1 (t-s)^\beta)^2 \varphi_1^2(y) E[\sigma(u_s(y))]^2 ds dy. \tag{3.3}
\]

Now using (3.3) and the assumption on \( \sigma \) gives

\[
\int_0^\infty e^{-\theta t} E\langle u_t, \varphi_1 \rangle^2 \, dt = \Lambda(\theta) \langle u_0, \varphi_1 \rangle^2 + \lambda^2 \Lambda(\theta) \int_0^\infty e^{-\theta t} E[\sigma(u_s(y))]^2 \varphi_1^2(y) \, dy \, dt \\
\geq \Lambda(\theta) \langle u_0, \varphi_1 \rangle^2 + \lambda^2 \Lambda(\theta) \int_0^\infty e^{-\theta t} E\langle u_t, \varphi_1 \rangle^2 \, dt. \tag{3.4}
\]

If \( \beta \in (0, \frac{1}{2}] \), since \( \Lambda(\theta) \) tends to infinity as \( \theta \) goes to zero, we can choose \( \theta \) small enough and from (3.4) we obtain

\[
\int_0^\infty e^{-\theta t} E\langle u_t, \varphi_1 \rangle^2 \, dt \geq \Lambda(\theta) \langle u_0, \varphi_1 \rangle^2 + 2 \int_0^\infty e^{-\theta t} E\langle u_t, \varphi_1 \rangle^2 \, dt. \tag{3.5}
\]

This implies that that for sufficiently small \( \theta \), we have

\[
\int_0^\infty e^{-\theta t} E\langle u_t, \varphi_1 \rangle^2 \, dt = \infty,
\]

which implies that for sufficiently large \( t \), \( E\langle u_t, \varphi_1 \rangle^2 \) grows exponentially. Now using Cauchy-Schwarz inequality we get the following,

\[
E\langle u_t, \varphi_1 \rangle^2 \leq E \left( \left( \int_B |u_t(x)|^2 \, dx \right) \left( \int_B |\varphi_1(x)|^2 \, dx \right) \right) \\
\leq \sup_{x \in B} E|u_t(x)|^2.
\]

We can thus conclude that \( \sup_{x \in B} E|u_t(x)|^2 \) too grows exponentially fast for all values of \( \lambda \).

Note that the first part of the conclusion of the theorem merely follows from the fact that the second term of (3.5) is positive and that the first term cannot have exponential decay.

**Proof of Theorem 1.2.** Following the lines for the proof of Theorem 1.1, we get the inequality (3.4);

\[
\int_0^\infty e^{-\theta t} E\langle u_t, \varphi_1 \rangle^2 \, dt \geq \Lambda(\theta) \langle u_0, \varphi_1 \rangle^2 + \lambda^2 \Lambda(\theta) \int_0^\infty e^{-\theta t} E\langle u_t, \varphi_1 \rangle^2 \, dt. \tag{3.6}
\]

Since \( \beta \in (\frac{1}{2}, 1) \), the function \( \Lambda(\theta) \) is bounded. Thus, for any fixed \( \theta > 0 \) we can find sufficiently large \( \lambda_u \) so that for all \( \lambda \geq \lambda_u \), the above inequality (3.6) yields

\[
\int_0^\infty e^{-\theta t} E\langle u_t, \varphi_1 \rangle^2 \, dt \geq \Lambda(\theta) \langle u_0, \varphi_1 \rangle^2 + \frac{3}{2} \int_0^\infty e^{-\theta t} E\langle u_t, \varphi_1 \rangle^2 \, dt. \tag{3.7}
\]

The result of the theorem is obtained by following the arguments of the remaining part of the proof of Theorem 1.1. \( \square \)
Proof of Theorem 1.3. We establish the required result using Ito-Walsh isometry and the global Lipschitz assumption on $\sigma$. The second moment of the mild formulation given by (1.10) becomes:

$$\mathbb{E}|u_t(x)|^2 = |(G_{\beta}^{(\beta)}u)_t(x)|^2 + \lambda^2 \int_0^t \int_B G_{\beta}^{(\beta)}(t-s, x, y)^2 \mathbb{E}|u_s(y)|^2 ds dy$$

$$\leq |(G_{\beta}^{(\beta)}u)_t(x)|^2 + \lambda^2 \mathcal{L}_t \int_0^t G_{\beta}^{(\beta)}(t-s, x, y)^2 \mathbb{E}|u_s(y)|^2 ds dy$$

$$:= J_1 + J_2. \quad (3.8)$$

We observe $J_1$ is bounded by the square of the same constant as initial condition, since $\int_B G_{\beta}^{(\beta)}(t, x) dx \leq \int_{\mathbb{R}^d} G_{\beta}^{(\beta)}(t, x) dx = 1$ and the initial datum is assumed to be bounded above by a constant. It remains to bound $J_2$. Since $\{\varphi_n\}_{n \geq 1}$ is an orthonormal sequence, for each fixed $t > 0$ we obtain

$$J_2 := \lambda^2 \mathcal{L}_t \int_0^t \int_B G_{\beta}^{(\beta)}(t-s, x, y)^2 \mathbb{E}|u_s(y)|^2 ds dy$$

$$\leq a_t \lambda^2 \mathcal{L}_t \int_0^t \sum_{n=1}^{\infty} E_{\beta}(-\mu_n(t-s)^\beta) \varphi_n^2(x) ds < \infty, \quad (3.9)$$

by Lemma 2.5 in the case of $d < \alpha/2\beta$, where $a_t = \sup_{0 < s < t} \sup_{x \in B} \mathbb{E}|u_s(x)|^2$. In the case the eigenfunctions $\{\varphi_n\}_{n \geq 1}$ are uniformly bounded by a constant $C(B)$, it follows from the above that

$$J_2 := a_t \lambda^2 \mathcal{L}_t \int_0^t \sum_{n=1}^{\infty} E_{\beta}(-\mu_n(t-s)^\beta) \varphi_n^2(x) ds$$

$$\leq a_t \lambda^2 \mathcal{L}_t (C(B))^2 \int_0^t \sum_{n=1}^{\infty} E_{\beta}(-\mu_n(t-s)^\beta) ds \leq a_t \lambda^2 \mathcal{L}_t (C(B))^2 C(ML), \quad (3.10)$$

where $a_t = \sup_{0 < s < t} \sup_{x \in B} \mathbb{E}|u_t(x)|^2$ and $C(ML) = \int_0^t \sum_{n=1}^{\infty} E_{\beta}(-\mu_n(t-s)^\beta) ds$. By Lemma 2.4 $C(ML) < \infty$ is a finite constant independent of $t$. That is, since $\beta \in (\frac{1}{2}, 1)$, we can choose $\lambda_t$ sufficiently small that for all $\lambda \leq \lambda_t$, the above estimates shows

$$\sup_{0 < s < t} \sup_{x \in B} \mathbb{E}|u_t(x)|^2 \leq 1 + \frac{1}{2} \sup_{0 < s < t} \sup_{x \in B} \mathbb{E}|u_t(x)|^2.$$

This shows that $\sup_{0 < t < \infty} \sup_{x \in B} \mathbb{E}|u_t(x)|^2$ is finite and hence the conclusion of the theorem follows. \Box

4 Proof of Theorem 1.8

In this section we give the proof of Theorem 1.8 and the following proposition is used in its proof. We can use the bounds of $G_{\beta}^{(\beta)}(t, x, y)$ to show the following proposition holds:

**Proposition 4.1.** For $d < \frac{\alpha}{\beta}$, let $u_t^{(\beta)}$ be a solution to equation (1.10) for parameters $\beta \in \left(\frac{1}{2}, 1\right)$. Then for some $\theta$, the supremum on the $p^{th}$ moment of the solution $u_s^{(\beta)}(x)$ is given by

$$\sup_{t > 0, x \in B} e^{-\theta t} \mathbb{E}|u_t^{(\beta)}(x)|^p, \quad (4.1)$$

is bounded above by a constant independent of $\beta$. 

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Proof. We use the mild formulation in equation (1.10) and follow similar computations to those used in the above proof. Now consider

\[ u_t^{(\beta)}(x) = (G_B^{(\beta)} u_0)(x) + \lambda \int_0^t \int_B G_B^{(\beta)}(t-s, x, y)\sigma(u_s^{(\beta)}(y))W(ds, dy), \tag{4.2} \]

Now applying the inequality \((a+b)^p \leq 2^p(a^p + b^p)\) for any \(a, b \geq 0\) and using Burkholder-Davis-Gundy inequality, the \(p\)th moment of (4.2) becomes

\[ \mathbb{E}|u_t^{(\beta)}(x)|^p \leq 2^p|G_B^{(\beta)} u_0|(x)|^p + (2\lambda)^p \left[ \int_B \int_0^t |G_B^{(\beta)}(t-s, x, y)|^2 \left[ \mathbb{E}|\sigma(u_s^{(\beta)}(y))|^p \right]^{2/p} ds dy \right]^{p/2}, \tag{4.3} \]

\[ e^{-\theta t}\mathbb{E}|u_t^{(\beta)}(x)|^p \leq e^{-\theta t}|G_B^{(\beta)} u_0|(x)|^p + e^{-\theta t} \left[ \int_B \int_0^t |G_B^{(\beta)}(t-s, x, y)|^2 \left[ \mathbb{E}|\sigma(u_s^{(\beta)}(y))|^p \right]^{2/p} ds dy \right]^{p/2} \]

\[ \quad := J_1 + J_2. \tag{4.4} \]

Using Lemma 2.2 and since \(\sigma\) is globally Lipschitz inequality, the second term becomes

\[ J_2 := e^{-\theta t} \left[ \int_B \int_0^t |G_B^{(\beta)}(t-s, x, y)|^2 \left[ \mathbb{E}|\sigma(u_s^{(\beta)}(y))|^p \right]^{2/p} ds dy \right]^{p/2} \]

\[ \leq L^p \left[ \int_B \int_0^t G_B^{(\beta)}(t-s, x, y)^2 e^{\frac{\alpha\theta(t-s)}{p}} \left[ \mathbb{E}|u_s^{(\beta)}(y)|^p \right]^{2/p} ds dy \right]^{p/2} \]

\[ \leq L^p \left[ \int_B \int_0^t G_B^{(\beta)}(t-s, x, y)^2 e^{-\theta(t-s)} ds dy \right]^{p/2} \]

\[ \lesssim L^p b_t(\theta) \Gamma \left( 1 - \frac{\beta d}{\alpha} \right) \theta^{d+1} \frac{\beta d}{\alpha} \quad (4.5) \]

where \(b_t(\theta) = \sup_{0 < s < t} \sup_{y \in B} e^{-\theta s} \mathbb{E}|u_s^{(\beta)}(y)|^p\). Now we fix \(\theta > 0\) sufficiently large so that

\[ J_2 \lesssim \frac{1}{2} b_t(\theta). \tag{4.6} \]

Thus, we have

\[ e^{-\theta t}\mathbb{E}|u_t^{(\beta)}(x)|^p \lesssim 1 + \frac{1}{2} b_t(\theta). \tag{4.7} \]

Taking the supremum of the left side of (4.7) the result follows.

We are now ready to prove Theorem 1.8.

Proof of Theorem 1.8. From the mild formulation of the solutions, we have

\[ u_t^{(\gamma)}(x) - u_t^{(\beta)}(x) = (G_B^{(\gamma)} u_t)(x) - (G_B^{(\beta)} u_t)(x) + \lambda \int_B \int_0^t G_B^{(\gamma)}(t-s, x, y)\sigma(u_s^{(\gamma)}(y))W(ds, dy) \]

\[ - \lambda \int_B \int_0^t G_B^{(\beta)}(t-s, x, y)\sigma(u_s^{(\beta)}(y))W(ds, dy) \]

\[ = (G_B^{(\gamma)} u_t)(x) - (G_B^{(\beta)} u_t)(x) \]

\[ + \lambda \int_B \int_0^t [G_B^{(\gamma)}(t-s, x, y) - G_B^{(\beta)}(t-s, x, y)]\sigma(u_s^{(\gamma)}(y))W(ds, dy) \]

\[ + \lambda \int_B \int_0^t G_B^{(\beta)}(t-s, x, y)[\sigma(u_s^{(\gamma)}(y)) - \sigma(u_s^{(\beta)}(y))]W(ds, dy). \tag{4.8} \]
Now applying the inequality \((a + b + c)^p \leq 3^p(a^p + b^p + c^p)\) for any \(a, b, c \geq 0\) and using Burkholder-Davis-Gundy inequality, the \(p\)th moment of \((4.8)\) becomes

\[
\mathbb{E}|u_t^{(\gamma)}(x) - u_t^{(\beta)}(x)|^p \leq 3^p([G_B^{(\gamma)}(x)] - [G_B^{(\beta)}(x)])^p + (3\lambda)^p \left( \int_0^t \int_B |G_B^{(\gamma)}(t - s, x, y) - G_B^{(\beta)}(t - s, x, y)|^2 \mathbb{E} |\mathbb{E}[\sigma(u_s^{(\gamma)}(y))]|^{2/p} dy ds \right)^{p/2} + (3\lambda)^p \left( \int_0^t \int_B G_B^{(\beta)}(t - s, x, y)^2 \mathbb{E} |\mathbb{E}[\sigma(u_s^{(\beta)}(y))] - \sigma(u_s^{(\beta)}(y))|^{2/p} dy ds \right)^{p/2}.
\]

For a fixed \(\theta > 0\) the inequality in \((4.9)\) becomes

\[
e^{-\theta t} \mathbb{E}|u_t^{(\gamma)}(x) - u_t^{(\beta)}(x)|^p \leq e^{-\theta t} \left( [G_B^{(\gamma)}(x)] - [G_B^{(\beta)}(x)] \right)^p + e^{-\theta t} \int_0^t \int_B |G_B^{(\gamma)}(t - s, x, y) - G_B^{(\beta)}(t - s, x, y)|^2 \mathbb{E} |\mathbb{E}[\sigma(u_s^{(\gamma)}(y))]|^{2/p} dy ds \right)^{p/2} + e^{-\theta t} \left( \int_0^t \int_B G_B^{(\beta)}(t - s, x, y)^2 \mathbb{E} |\mathbb{E}[\sigma(u_s^{(\beta)}(y))] - \sigma(u_s^{(\beta)}(y))|^{2/p} dy ds \right)^{p/2}.
\]

Let us first work on the third term. Using Lemma 2.2 and since \(\sigma\) is globally Lipschitz, we have

\[
J_3 := e^{-\theta t} \left( \int_0^t \int_B G_B^{(\beta)}(t - s, x, y)^2 \mathbb{E} |\mathbb{E}[\sigma(u_s^{(\beta)}(y))] - \sigma(u_s^{(\beta)}(y))|^{2/p} dy ds \right)^{p/2} \\
\leq L_p^\alpha b_t(\theta) \left( \int_0^t \int_B G_B^{(\beta)}(t - s, x, y)^2 e^{-2\theta s/p} dy ds \right)^{p/2} \\
\leq L_p^\alpha b_t(\theta) \left[ \int_0^t \int_{\mathbb{R}^n} G_B^{(\beta)}(s, x, y)^2 e^{-2\theta s/p} ds \right]^{p/2} \\
\leq L_p^\alpha b_t(\theta) \left[ \int_0^\infty \int_{\mathbb{R}^n} s^{-\beta d/2} e^{-2\theta s/p} ds \right]^{p/2} \\
\leq L_p^\alpha b_t(\theta) \left[ \int_0^\infty \int_{\mathbb{R}^n} s^{-\beta d/2} e^{-2\theta s/p} ds \right]^{p/2} \\
\leq L_p^\alpha b_t(\theta) \left[ \Gamma \left( 1 - \frac{\beta d}{\alpha} \right) \theta^{-\beta \alpha - 1} \right]^{p/2}.
\]

where \(b_t(\theta) = \sup_{0 < s < t} \sup_{x \in B} e^{-\theta s} \mathbb{E}|u_s^{(\gamma)}(z) - u_s^{(\beta)}(z)|^p\). Now we fix \(\theta > 0\) sufficiently large so that

\[
J_3 \leq \frac{1}{b_t(\theta)}.
\]

Next we work on \(J_2\). Since \((a + b)^2 \leq 4(a^2 + b^2)\) for \(a, b \geq 0\), using Proposition 4.1, Lemma 2.8(b) and
the assumption on $\sigma$, we obtain

$$J_2 := e^{-\theta t} \left[ \int_0^t \int_B \left| G_B^{(\gamma)}(t-s,x,y) - G_B^{(\beta)}(t-s,x,y) \right|^2 \mathbb{E}|\sigma(u_s^{(\beta)}(y))|^{2/p} dy \right]^{p/2}$$

$$\lesssim \sup_{t>0, x \in B} e^{-\theta t} \mathbb{E}|u_s^{(\beta)}(x)|^p \left[ \int_0^\infty \int_B e^{-2\theta s/p} |G_B^{(\gamma)}(s,x,y) - G_B^{(\beta)}(s,x,y)|^2 dy ds \right]^{p/2}$$

$$\lesssim \left[ \int_0^\infty \int_B e^{-2\theta s/p} |G_B^{(\gamma)}(s,x,y) - G_B^{(\beta)}(s,x,y)|^2 dy ds \right]^{p/2}$$

$$\lesssim \left[ \int_0^\infty \int_B e^{-2\theta s/p} |G_B^{(\gamma)}(s,x,y)|^2 dy ds + \int_0^\infty \int_B e^{-2\theta s/p} |G_B^{(\beta)}(s,x,y)|^2 dy ds \right]^{p/2}$$

$$\lesssim \left[ \int_0^\infty \int_B e^{-2\theta s/p} |G_B^{(\gamma)}(s,x,y)|^2 dy ds \right]^{p/2}$$

$$\lesssim \left[ \int_0^\infty \int_B e^{-2\theta s/p} |G_B^{(\gamma)}(s,x,y) - G_B^{(\beta)}(s,x,y)|^2 dy ds \right]^{p/2}$$

$$\lesssim \left[ \int_0^\infty \int_B e^{-2\theta s/p} |G_B^{(\gamma)}(s,x,y)|^2 dy ds + \int_0^\infty \int_B e^{-2\theta s/p} |G_B^{(\beta)}(s,x,y)|^2 dy ds \right]^{p/2}$$

$$\lesssim \left[ \int_0^\infty \int_B e^{-2\theta s/p} |G_B^{(\gamma)}(s,x,y) - G_B^{(\beta)}(s,x,y)|^2 dy ds \right]^{p/2}$$

(4.12)

where $\beta \in (\beta_0, \beta_1)$ and $\gamma \in (\gamma_0, \gamma_1)$ for some $\beta_0, \beta_1, \gamma_0, \gamma_1 \in (0, 1)$, $\alpha < \lambda < \frac{1}{2\beta}$, $\frac{\alpha}{\alpha} < \lambda' < \frac{1}{2\beta}$,

$\mu_0 < \min \left( \frac{1}{2} - \lambda, \frac{1-2\beta\lambda}{\lambda'} \right)$, $\mu_0' < \min \left( \frac{1}{2} - \lambda', \frac{1-2\beta\lambda'}{\lambda} \right)$, $\mu_1 > \frac{1-2\beta\lambda}{\lambda'}$, and $\mu_1' > \frac{1-2\beta\lambda}{\lambda}$, where the bound on right hand side of (4.12) doesn’t depend on $\beta$ and $\gamma$.

We combine the above estimates and obtain the following:

$$\sup_{x \in B} e^{-\theta s} \mathbb{E}|u_s^{(\gamma)}(x) - u_s^{(\beta)}(x)|^p \lesssim e^{-\theta t} |(G_B^{(\gamma)} u_t)(x) - (G_B^{(\beta)} u_t)(x)|^p + \frac{1}{2} b_1(t)$$

$$+ \left[ \int_0^\infty \int_B e^{-2\theta s/p} |G_B^{(\gamma)}(s,x,y) - G_B^{(\beta)}(s,x,y)|^2 dy ds \right]^{p/2}$$

Since Lemma 2.6, and Lemma 2.8(b) allows us to use the dominated convergence theorem, taking $\gamma \to \beta$ the result of the theorem follows. \hfill \Box

5. Proof of Theorem 1.11, Theorem 1.12, and Theorem 1.13

Proof of Theorem 1.11. First, we establish the second statement of the theorem. From the mild formulation given in (1.10) and using Stochastic Fubini theorem, we set

$$\langle u_t, \varphi_1 \rangle = E_\beta(-\mu_1 t^\beta)(u_0, \varphi_1) + \lambda \int_B \int_0^t E_\beta(-\mu_1(t-s)^\beta) \varphi_1(y) \sigma(u_s(y)) F(ds, dy).$$

(5.1)

Taking the second moment of (5.1), we get

$$E(u_t, \varphi_1)^2 = E_\beta(-\lambda_1 t^\beta)^2(u_0, \varphi_1)^2$$

$$+ \lambda^2 \int_{B \times B} \int_0^t E_\beta(-\lambda_1(t-s)^\beta)^2 \varphi_1(y) \varphi_1(z) E[\sigma(u_s(y)) \sigma(u_s(z))] f(y, z) ds dy dz.$$
Now using (5.2) and the assumption on $\sigma$ gives
\[
\int_{0}^{\infty} e^{-\theta t} \mathbb{E} \langle u_t, \varphi_1 \rangle^2 \, dt = \Lambda(\theta) \langle u_0, \varphi_1 \rangle^2 + \lambda^2 \Lambda(\theta) \int_{0}^{\infty} e^{-\theta t} \int_{\mathbb{R} \times \mathbb{R}} \varphi_1(y) \varphi_1(z) \times \\
\mathbb{E} |\sigma(u_t(y))\sigma(u_t(z))| f(y, z) \, dy \, dz \, dt.
\]
\[
\geq \Lambda(\theta) \langle u_0, \varphi_1 \rangle^2 + \lambda^2 \beta^2 K_f \Lambda(\theta) \int_{0}^{\infty} e^{-\theta t} \int_{\mathbb{R} \times \mathbb{R}} \varphi_1(y) \varphi_1(z) \times \\
\mathbb{E} |u_t(y)u_t(z)| \, dy \, dz \, dt
\]
\[= \Lambda(\theta) \langle u_0, \varphi_1 \rangle^2 + \lambda^2 \beta^2 K_f \Lambda(\theta) \int_{0}^{\infty} e^{-\theta t} \mathbb{E} \langle u_t, \varphi_1 \rangle^2 \, dt. \tag{5.3}
\]

If $\beta \in (0, \frac{1}{2}]$, since $\Lambda(\theta)$ tends to infinity as $\theta$ goes to zero, we can choose $\theta$ small enough and from (5.3) we obtain
\[
\int_{0}^{\infty} e^{-\theta t} \mathbb{E} \langle u_t, \varphi_1 \rangle^2 \, dt \gtrsim \Lambda(\theta) \langle u_0, \varphi_1 \rangle^2 + \frac{1}{2} \int_{0}^{\infty} e^{-\theta t} \mathbb{E} \langle u_t, \varphi_1 \rangle^2 \, dt. \tag{5.4}
\]

This implies that for sufficiently small $\theta$, we have
\[
\int_{0}^{\infty} e^{-\theta t} \mathbb{E} \langle u_t, \varphi_1 \rangle^2 \, dt = \infty,
\]
which also implies that for sufficiently large $t$, $\mathbb{E} \langle u_t, \varphi_1 \rangle^2$ grows exponentially. Now using Cauchy-Schwarz inequality we get the following,
\[
\mathbb{E} \langle u_t, \varphi_1 \rangle^2 = \mathbb{E} \left| \int_{B} u_t(x) \varphi_1(x) \, dx \right|^2 \\
\leq \mathbb{E} \left( \left( \int_{B} |u_t(x)|^2 \, dx \right) \left( \int_{B} \varphi_1^2(x) \, dx \right) \right) \\
\leq \sup_{x \in B} \mathbb{E} |u_t(x)|^2.
\]
We can thus conclude that $\sup_{x \in B} \mathbb{E} |u_t(x)|^2$ too grows exponentially fast for all values of $\lambda$.

Note that the first part of the conclusion of theorem merely follows from the fact that the second term of (5.3) is positive and that the first term cannot have exponential decay. \qed

**Proof of Theorem 1.12.** Using the same notations as in the proof of Theorem 1.11, we observe that the inequality (5.3) holds;
\[
\int_{0}^{\infty} e^{-\theta t} \mathbb{E} \langle u_t, \varphi_1 \rangle^2 \, dt \gtrsim \Lambda(\theta) \langle u_0, \varphi_1 \rangle^2 + \lambda^2 \beta^2 K_f \Lambda(\theta) \int_{0}^{\infty} e^{-\theta t} \mathbb{E} \langle u_t, \varphi_1 \rangle^2 \, dt. \tag{5.5}
\]

Since $2\beta > 1$, the function $\Lambda(\theta)$ is bounded. Thus, for any fixed $\theta > 0$ there exists sufficiently large $\lambda_u$ so that for all $\lambda \geq \lambda_u$, from the inequality (5.5) we obtain
\[
\int_{0}^{\infty} e^{-\theta t} \mathbb{E} \langle u_t, \varphi_1 \rangle^2 \, dt \gtrsim \Lambda(\theta) \langle u_0, \varphi_1 \rangle^2 + 2 \int_{0}^{\infty} e^{-\theta t} \mathbb{E} \langle u_t, \varphi_1 \rangle^2 \, dt. \tag{5.6}
\]
Again, following similar lines of arguments as in the proof of Theorem 1.11, we get result of the current theorem. \qed

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Proof of Theorem 1.13. Using \((a + b)^2 \leq 4(a^2 + b^2)\) for \(a, b \geq 0\), and applying Burkholder’s inequality, the second moment of the mild formulation given by (1.14) becomes:

\[
\mathbb{E}|u_t(x)|^2 \leq 4|G_B^{(\beta)} u_t(x)|^2 + 4\lambda^2 \int_{B \times B} \int_0^t G_B^{(\beta)}(t - s, x, y)G_B^{(\beta)}(t - s, x, z) \times \\
\mathbb{E}[\sigma(u_s(y))\sigma(u_s(z))| f(y, z) ds dy dz \\
\leq 4|G_B^{(\beta)} u_t(x)|^2 + 4\lambda^2 L_\sigma^2 \int_{B \times B} \int_0^t G_B^{(\beta)}(t - s, x, y)G_B^{(\beta)}(t - s, x, z) \times \\
\mathbb{E}[u_s(y)u_s(z)| f(y, z) ds dy dz \\
:= J_1 + J_2.
\]

(5.7)

We observe \(J_1\) is bounded by the square of the same constant as initial condition, since \(\int_B G_B^{(\beta)}(t, x) dx \leq \int_{B^2} G^{(\beta)}(t, x) dx = 1\) and the initial datum is assumed to be bounded above by a constant. It remains to bound \(J_2\). Now if \(\{\varphi_n\}_{n \geq 1}\) are uniformly bounded by a constant \(C(B)\), for each fixed \(t > 0\), by Cauchy-Schwarz inequality, Lemma 2.1 and Lemma 2.3 we obtain

\[
J_2 := 4\lambda^2 L_\sigma^2 \int_{B \times B} \int_0^t G_B^{(\beta)}(t - s, x, y)G_B^{(\beta)}(t - s, x, z) \mathbb{E}[u_s(y)u_s(z)| f(y, z) ds dy \\
\leq 4\lambda^2 L_\sigma^2 \int_{B \times B} \int_0^t G_B^{(\beta)}(t - s, x, y)G_B^{(\beta)}(t - s, x, z) (\mathbb{E}[u_s(y)^2])^{\frac{1}{2}} (\mathbb{E}[u_s(z)^2])^{\frac{1}{2}} f(y, z) ds dy \\
\leq 4\lambda^2 L_\sigma^2 \int_0^t \sup_{x \in B} \mathbb{E}[|u_s(x)|^2] \int_{B \times B} G_B^{(\beta)}(t - s, x, y)G_B^{(\beta)}(t - s, x, z) f(y, z) ds dy \\
\leq 4\lambda^2 L_\sigma^2 a_t \int_0^t \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} E_B(-\mu_n(t - s)^2)E_B(-\mu_k(t - s)^2) \varphi_n(x)\varphi_k(x) \times \\
\int_{B \times B} \varphi_n(y)\varphi_k(z) f(y, z) ds dy \\
\leq 4\lambda^2 L_\sigma^2 a_t \int_0^t \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left( \int_0^t E_B(-\mu_n(t - s)^2) ds \right)^{\frac{1}{2}} \left( \int_0^t E_B(-\mu_k(t - s)^2) ds \right)^{\frac{1}{2}} \varphi_n(x)\varphi_k(x) \times \\
\int_{B} \varphi_n(y) \left[ \int_{B} \varphi_k(z) f(y, z) dz \right] dy \\
\leq C_3 C(B)^4 \lambda^2 L_\sigma^2 a_t \int_0^t \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left( \frac{1}{\mu_n^{\frac{1}{2}}} \right) \left( \frac{1}{\mu_k^{\frac{1}{2}}} \right) \int_{B} \left[ \int_{B} f(y, z) dz \right] dy \\
\leq C_3 \lambda^2 L_\sigma^2 a_t \int_0^t \left( \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}} \right) \left( \sum_{k=1}^{\infty} \frac{1}{k^{\frac{1}{2}}} \right) \int_{B} f(y, z) dz dy \\
\leq C \lambda^2 L_\sigma^2 a_t,
\]

(5.8)

where \(C = C_3 \left( \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}} \right) \left( \sum_{k=1}^{\infty} \frac{1}{k^{\frac{1}{2}}} \right) \int_{B \times B} f(y, z) dz dy < \infty\), and

\(a_t = \sup_{0 < s < t} \sup_{x \in B} \mathbb{E}[|u_s(x)|^2]\). Since \(\beta \in \left( \frac{1}{3}, 1 \right)\), we can choose \(\lambda_t\) sufficiently small that for all \(\lambda < \lambda_t\), the above estimates shows

\[
\sup_{0 < s < t} \sup_{x \in B} \mathbb{E}[|u_t(x)|^2] \lesssim 1 + \frac{1}{2} \sup_{0 < s < t} \sup_{x \in B} \mathbb{E}[|u_t(x)|^2].
\]

This shows that \(\sup_{0 < t < \infty} \sup_{x \in B} \mathbb{E}[|u_t(x)|^2]\) is finite and hence the conclusion of the theorem follows. \(\square\)
Proof of Corollary 1.14. Since \( \frac{1}{\beta} > 1, \) if \( \alpha = 2 \) and \( d = 1 \), (for the case of the eigenvalues \( \mu_n = \left( \frac{\alpha \pi}{L} \right)^2 \) and eigenvectors \( \varphi_n(x) = \left( \frac{2}{L} \right)^{\frac{1}{2}} \sin \left( \frac{\alpha \pi x}{L} \right) \) of Dirichlet Laplacian corresponding to a Brownian Motion killed upon exiting the domain \([0, L]\) so that \( |\varphi_n(x)| \leq \left( \frac{2}{L} \right)^{\frac{1}{2}} \) for each \( n \)) from (3.9) using Lemma 2.1 we obtain the result of the corollary. \( \square \)

Proof of Corollary 1.15. In this case, we note that \( C_4 \int_B \int f(y, z) \, dz \, dy < \infty, \) since
\[
\int_{B \times B} f(y, z) \, dz \, dy = 2 \frac{L^{2-\gamma}}{(1-\gamma)(2-\gamma)} < \infty. \tag{5.9}
\]

6 Proof of Theorem 1.16

In this section we give the proof of Theorem 1.16 and the following result is useful to guarantee the use of dominated convergence theorem in the proof of Theorem 1.16.

Proposition 6.1. Assume that \( \{\varphi_n\}_{n \geq 1} \) are uniformly bounded by a constant \( C(B) \), depending on the geometrical characteristics of the domain \( B \), i.e.,
\[
C(B) = \sup_{n \geq 1, x \in B} \varphi_n(x).
\]
Suppose that \( \int_{B \times B} f(x, y) \, dx \, dy < \infty. \) For \( \beta \in (\frac{1}{2}, 1) \) and \( \frac{d}{\alpha} < \lambda < \frac{1}{2 \gamma} \), let \( 0 < \mu_0 < \min \left( \frac{1}{2} - \lambda, \frac{1-2\beta\lambda}{2} \right) \) and \( \mu_1 > \frac{1-2\beta\lambda}{2} \) where \( \beta \in (\beta_0, \beta_1) \) for some \( \beta_0, \beta_1 \in (0, 1) \). Also for \( \gamma \in (\frac{1}{2}, 1) \) and \( \frac{d}{\alpha} < \lambda' < \frac{1}{2 \gamma} \), let \( 0 < \mu'_0 < \min \left( \frac{1}{2} - \lambda', \frac{1-2\beta'\lambda'}{2} \right) \) and \( \mu'_1 > \frac{1-2\beta'\lambda'}{2} \) where \( \gamma \in (\gamma'_0, \gamma'_1) \) for some \( \gamma'_0, \gamma'_1 \in (0, 1) \). Then for \( d < \frac{1}{2} \min \{ \frac{1}{\beta}, \frac{1}{\gamma} \} \), there exist positive constants \( C \) and \( C_1 \) such that
\[
\left[ \int_0^t \int_{B \times B} e^{-2(t-s)/p} |G_B^{(\gamma)}(t-s, x, y) - G_B^{(\beta)}(t-s, x, y)| |G_B^{(\gamma)}(t-s, x, z) - G_B^{(\beta)}(t-s, x, z)| \times f(y, z) \, dy \, dz \, ds \right]^{\frac{1}{2}} \leq C \left[ p^{1-2\lambda'_0} \left( \theta^{-2\mu'_0} + \gamma^{-2\mu'_1} \right) + p^{1-2\beta\lambda} \left( \theta^{-2\mu_0} + \gamma^{-2\mu_1} \right) \right]^{p/2}. \tag{6.1}
\]
Proof. Since \( \{\varphi_n\}_{n \geq 1} \) are uniformly bounded by a constant \( C(B) \) and \( \int_{B \times B} f(x, y) \, dx \, dy < \infty, \) for each \( t > 0, \) using Lemma 2.7, Lemma 2.8(a) and the fact that \( (a+b)^2 \leq 2(a^2 + b^2) \) for \( a, b \geq 0, \) we bound the integral term as follows:
\[
\int_0^t \int_{B \times B} e^{-2(t-s)/p} |G_B^{(\gamma)}(t-s, x, y) - G_B^{(\beta)}(t-s, x, y)| |G_B^{(\gamma)}(t-s, x, z) - G_B^{(\beta)}(t-s, x, z)| \times f(y, z) \, dy \, dz \, ds \leq C \left[ p^{1-2\lambda'_0} \left( \theta^{-2\mu'_0} + \gamma^{-2\mu'_1} \right) + p^{1-2\beta\lambda} \left( \theta^{-2\mu_0} + \gamma^{-2\mu_1} \right) \right]^{p/2}. \tag{6.2}
\]
where \( \mu_0 \) and \( \mu'_0 \) are as given in the hypothesis of the proposition. It follows from (6.2) that for any \( p \geq 0 \) we obtain

\[
\left[ \int_0^t \int_{B \times B} e^{-2q(t-s)/p} |G_B^{(\gamma)}(t-s, x, y) - G_B^{(\beta)}(t-s, x, y)||G_B^{(\gamma)}(t-s, x, z) - G_B^{(\beta)}(t-s, x, z)| \times \right.
\]

\[
f(y, z) \, dy \, dz \, ds \bigg]\bigg]^{p/2}
\]

which allows us to deduce (6.1). That is, each terms in the summation can be bounded by a quantity independent of \( \gamma \) and \( \beta \).

\[ \square \]

**Proof of Theorem 1.16.** From the mild formulation of the solutions, we have

\[
u_t^{(\gamma)}(x) - u_t^{(\beta)}(x) = (G_B^{(\gamma)} u_t)(x) - (G_B^{(\beta)} u_t)(x) + \lambda \int_0^t \int_B \left( G_B^{(\gamma)}(t-s, x, y) \sigma(u_s^{(\gamma)}(y)) \right) F(ds \, dy)
\]

\[
- \lambda \int_B \int_0^t G_B^{(\beta)}(t-s, x, y) \sigma(u_s^{(\beta)}(y)) F(ds \, dy) = (G_B^{(\gamma)} u_t)(x) - (G_B^{(\beta)} u_t)(x)
\]

\[
+ \lambda \int_B \int_0^t [G_B^{(\gamma)}(t-s, x, y) - G_B^{(\beta)}(t-s, x, y)] \sigma(u_s^{(\gamma)}(y)) F(ds \, dy)
\]

\[
+ \lambda \int_B \int_0^t [G_B^{(\gamma)}(t-s, x, y) - G_B^{(\beta)}(t-s, x, y)] \sigma(u_s^{(\gamma)}(y)) - \sigma(u_s^{(\beta)}(y)) F(ds \, dy).
\]

Now applying the inequality \((a + b + c)^p \leq 3^p(a^p + b^p + c^p)\) for any \( a, b, c \geq 0 \) and using Burkholder-Davis-Gundy inequality, the \( p \text{th} \) moment of (6.3) becomes

\[
E|u_t^{(\gamma)}(x) - u_t^{(\beta)}(x)|^p \leq 3^p|G_B^{(\gamma)} u_t(x) - (G_B^{(\beta)} u_t(x)|^p
\]

\[
+(3\lambda)^p E \left| \int_B \int_0^t \left( G_B^{(\gamma)}(t-s, x, y) - G_B^{(\beta)}(t-s, x, y) \right) \sigma(u_s^{(\gamma)}(y)) F(ds \, dy) \right|^p
\]

\[
+(3\lambda)^p E \left| \int_B \int_0^t \left( G_B^{(\gamma)}(t-s, x, y) - G_B^{(\beta)}(t-s, x, y) \right) \sigma(u_s^{(\gamma)}(y)) - \sigma(u_s^{(\beta)}(y)) F(ds \, dy) \right|^p.
\]

\[
E|u_t^{(\gamma)}(x) - u_t^{(\beta)}(x)|^p \leq 3^p|G_B^{(\gamma)} u_t(x) - (G_B^{(\beta)} u_t(x)|^p
\]

\[
+(3\lambda)^p \left[ \int_0^t \int_{B \times B} |G_B^{(\gamma)}(t-s, x, y) - G_B^{(\beta)}(t-s, x, y)||G_B^{(\gamma)}(t-s, x, z) - G_B^{(\beta)}(t-s, x, z)| \times
\]

\[
||\sigma(u_s^{(\gamma)}(y)) \sigma(u_s^{(\gamma)}(z))||^{p/2} f(y, z) \, ds \, dy \, dz \bigg]^{p/2}
\]

\[
+(3\lambda)^p \left[ \int_{B \times B} \int_0^t G_B^{(\beta)}(t-s, x, y)G_B^{(\beta)}(t-s, x, z) \times
\]

\[
||\sigma(u_s^{(\gamma)}(y)) - \sigma(u_s^{(\beta)}(y))||^{p/2} f(y, z) \, ds \, dy \, dz \bigg]^{p/2}.
\]

(6.5)
For a fixed $\theta > 0$ the inequality in (6.4) becomes

\[
e^{-\theta t} \mathbb{E}|u_s^{(\gamma)}(x) - u_s^{(\beta)}(x)|^p \lesssim e^{-\theta t}|(G_B^{(\gamma)} u_s)(x) - (G_B^{(\beta)} u_s)(x)|^p \\
+ e^{-\theta t} \left[ \int_0^t \int_{B \times B} |G_B^{(\gamma)}(t-s, x, y) - G_B^{(\beta)}(t-s, x, y)||G_B^{(\gamma)}(t-s, x, z) - G_B^{(\beta)}(t-s, x, z)|| \times \right. \\
\left[ \mathbb{E}[\sigma(u_s^{(\gamma)}(y))\sigma(u_s^{(\gamma)}(z))]^{p/2}]^2/f(y, z) \, ds \, dy \, dz \right]^{p/2} \\
+ e^{-\theta t} \left[ \int_{B \times B} \int_0^t G_B^{(\beta)}(t-s, x, y)G_B^{(\beta)}(t-s, x, z) \times \right. \\
\left[ \mathbb{E}[\sigma(u_s^{(\gamma)}(y)) - \sigma(u_s^{(\beta)}(y)))(\sigma(u_s^{(\gamma)}(z)) - \sigma(u_s^{(\beta)}(z)))]^{p/2}]^2/f(y, z) \, ds \, dy \, dz \right]^{p/2}.
\]

\[\quad := J_1 + J_2 + J_3. \quad (6.7)\]

Let us first work on the third term. Since $\sigma$ is globally Lipschitz, using Lemma 2.4, and Lemma 2.8 (a), we have

\[J_3 := e^{-\theta t} \left[ \int_{B \times B} \int_0^t G_B^{(\beta)}(t-s, x, y)G_B^{(\beta)}(t-s, x, z) \times \right. \\
\left[ \mathbb{E}[\sigma(u_s^{(\gamma)}(y)) - \sigma(u_s^{(\beta)}(y)))(\sigma(u_s^{(\gamma)}(z)) - \sigma(u_s^{(\beta)}(z)))]^{p/2}]^2/f(y, z) \, ds \, dy \, dz \right]^{p/2} \\
\lesssim \left[ \int_{B \times B} \int_0^t G_B^{(\beta)}(t-s, x, y)G_B^{(\beta)}(t-s, x, z) \times \right. \\
e^{-\theta \frac{2^d}{p}} [\mathbb{E}[(u_s^{(\gamma)}(y) - u_s^{(\beta)}(y))(u_s^{(\gamma)}(z) - u_s^{(\beta)}(z))]^{p/2}]^2/f(y, z) \, ds \, dy \, dz \right]^{p/2} \\
\lesssim \left[ \int_{B \times B} \int_0^t G_B^{(\beta)}(t-s, x, y)G_B^{(\beta)}(t-s, x, z) \times \right. \\
e^{-\theta \frac{2^d}{p}} [\mathbb{E}[(u_s^{(\gamma)}(y) - u_s^{(\beta)}(y))(u_s^{(\gamma)}(z) - u_s^{(\beta)}(z))]^{p/2}]^2/f(y, z) \, ds \, dy \, dz \right]^{p/2} \\
\lesssim \left[ \int_{B \times B} \int_0^t G_B^{(\beta)}(t-s, x, y)G_B^{(\beta)}(t-s, x, z) \times \right. \\
e^{-\theta \frac{2^d}{p}} [\mathbb{E}[(u_s^{(\gamma)}(y) - u_s^{(\beta)}(y))(u_s^{(\gamma)}(z) - u_s^{(\beta)}(z))]^{p/2}]^2/f(y, z) \, ds \, dy \, dz \right]^{p/2} \\
\lesssim \left[ \int_{B \times B} \int_0^t G_B^{(\beta)}(t-s, x, y)G_B^{(\beta)}(t-s, x, z) \times \right. \\
e^{-\theta \frac{2^d}{p}} [\mathbb{E}[(u_s^{(\gamma)}(y) - u_s^{(\beta)}(y))(u_s^{(\gamma)}(z) - u_s^{(\beta)}(z))]^{p/2}]^2/f(y, z) \, ds \, dy \, dz \right]^{p/2} \\
\lesssim L_\sigma b_t(\theta) \left[ \int_{B \times B} \int_0^t G_B^{(\beta)}(t-s, x, y)G_B^{(\beta)}(t-s, x, z) e^{-\theta \frac{2^d(t-s)}{p}} \, f(y, z) \, ds \, dy \, dz \right]^{p/2} \\
\lesssim L_\sigma b_t(\theta) \left[ \int_0^t (t-s)^{-2\lambda} e^{-\theta \frac{2^d(t-s)}{p}} \, f(y, z) \, dy \, dz \right]^{p/2} \\
\lesssim L_\sigma b_t(\theta) \left[ \int_0^t (t-s)^{-2\lambda} e^{-\theta \frac{2^d(s)}{p}} \, ds \, f(y, z) \, dy \, dz \right]^{p/2} \\
\lesssim L_\sigma b_t(\theta) \left[ \int_0^\infty e^{-2\theta r} \, ds \, f(y, z) \, dy \, dz \right]^{p/2} \\
\lesssim L_\sigma b_t(\theta) \left[ \int_0^\infty \frac{e^{-2\theta r}}{r^{2\lambda}} \, dr \, f(y, z) \, dy \, dz \right]^{p/2} \\
\lesssim L_\sigma b_t(\theta) \left[ \int_0^\infty \frac{e^{-2\theta r}}{r^{2\lambda}} \, dr \, f(y, z) \, dy \, dz \right]^{p/2} \\
\lesssim L_\sigma b_t(\theta) \left[ \int_0^\infty e^{-2\theta r} \, f(y, z) \, dy \, dz \right]^{p/2}, \quad (6.8)\]

where $b_t(\theta) = \sup_{0 < s < t} \sup_{z \in B} e^{-\theta s} \mathbb{E}|u_s^{(\gamma)}(z) - u_s^{(\beta)}(z)|^p$. For $t > 0$, we fix $\theta > 0$ sufficiently large so that

\[J_3 \lesssim \frac{1}{2} b_t(\theta). \quad (6.9)\]
Next we work on $J_2$. Using Proposition 6.2 below and the assumption on $\sigma$, we obtain

\[
J_2 := e^{-\theta t} \left[ \int_0^t \int_{B \times B} \left| G_B^{(\gamma)}(t - s, x, y) - G_B^{(\beta)}(t - s, x, y) \right| \left| G_B^{(\gamma)}(t - s, x, z) - G_B^{(\beta)}(t - s, x, z) \right| \times \right.
\[
\left[ \mathbb{E} |\sigma(u_s^{(\gamma)}(y))\sigma(u_s^{(\gamma)}(z))|^{p/2} \right]^{2/p} f(y, z) \, dy \, dz \right]^{1/2}.
\]

We combine the above estimates and obtain the following.

\[
\sup_{t > 0, x \in B} e^{-\theta s} \mathbb{E} |u_s^{(\gamma)}(x) - u_s^{(\beta)}(x)|^p \lesssim e^{-\theta t} |(G_B^{(\gamma)} u)_t(x) - (G_B^{(\beta)} u)_t(x)|^p + \frac{1}{2} b_t(\theta) +
\]

\[
\sup_{t > 0, x \in B} e^{-\theta t} \mathbb{E} |u_s^{(\gamma)}(x)|^p \left[ \int_0^t \int_{B \times B} e^{-2\theta(t-s)/p} |G_B^{(\gamma)}(t - s, x, y) - G_B^{(\beta)}(t - s, x, y)| \times \right.
\]

\[
\left. |G_B^{(\gamma)}(t - s, x, z) - G_B^{(\beta)}(t - s, x, z)| f(y, z) \, dy \, dz \right]^{1/2}.
\]

Since Lemma 2.6 and Proposition 6.1 allow us to use dominated convergence theorem, taking $\gamma \to \beta$ the result of the theorem follows. \hfill \square

The following proposition is used in the proof of Theorem 1.16. We use the bounds of $G_B^{(\beta)}(t, x, y)$ to show the proposition holds:

**Proposition 6.2.** Assume that $\{\varphi_n\}_{n \geq 1}$ are uniformly bounded by a constant $C(B)$. Suppose that $d < \frac{\beta}{2\gamma}$, and $\int_{B \times B} f(x, y) \, dx \, dy < \infty$. For some $\theta$, the supremum on the $p$th moment of the solution $u_s^{(\beta)}(x)$ to equation (1.12) given by

\[
\sup_{0 \leq s \leq t, x \in B} e^{-\theta s} \mathbb{E} |u_s^{(\beta)}(x)|^p
\]

is bounded above by a constant independent of $\beta$.

**Proof.** We use the mild formulation in equation (1.14) and follow similar computations to those used in the above proof. Now consider

\[
u_t^{(\beta)}(x) = (G_B^{(\beta)} u_0)_t(x) + \lambda \int_B \int_0^t G_B^{(\beta)}(t - s, x, y) \sigma(u_s^{(\beta)}(y)) \sigma^2(\nu_s^{(\beta)}(y)) f(\, ds \, dy,
\]

(6.12)

Now applying the inequality $(a + b)^p \leq 2^p(a^p + b^p)$ for any $a, b \geq 0$ and using Burkholder-Davis-Gundy inequality, the $p$th moment of (6.12) becomes

\[
\mathbb{E} |\nu_t^{(\beta)}(x)|^p \leq 2^p |(G_B^{(\beta)} u_0)_t(x)|^p + (2\lambda)^p \left[ \int_{B \times B} \int_0^t |G_B^{(\beta)}(t - s, x, y) G_B^{(\beta)}(t - s, x, z)| \times \right.
\]

\[
\left. \left[ \mathbb{E} |\sigma(u_s^{(\beta)}(y))\sigma(u_s^{(\beta)}(z))|^{p/2} \right]^{2/p} f(y, z) \, ds \, dy \, dz \right]^{p/2},
\]

(6.13)

\[
e^{-\theta t} |\nu_t^{(\beta)}(x)|^p \lesssim e^{-\theta t} |(G_B^{(\beta)} u_0)_t(x)|^p + e^{-\theta t} \left[ \int_{B \times B} \int_0^t |G_B^{(\beta)}(t - s, x, y) G_B^{(\beta)}(t - s, x, z)| \times \right.
\]

\[
\left. \left[ \mathbb{E} |\sigma(u_s^{(\beta)}(y))\sigma(u_s^{(\beta)}(z))|^{p/2} \right]^{2/p} f(y, z) \, ds \, dy \, dz \right]^{p/2},
\]

(6.14)

\[
:= J_1 + J_2.
\]
Since $\sigma$ is globally Lipschitz, using Lemma 2.4, and Lemma 2.8(a), the second term becomes

$$J_2 := e^{-\theta t} \left[ \int_{B \times B} \int_0^t |G_B^{(\beta)}(t-s, x, y)G_B^{(\beta)}(t-s, x, z)| \left( \mathbb{E} \left| \sigma\left( u_s^{(\beta)}(y) \right) \right| \right)^{p/2} f(y, z) \, ds \, dy \, dz \right]^{2/p},$$

$$\lesssim L_\sigma^p \left[ \int_{B \times B} \int_0^t G_B^{(\beta)}(t-s, x, y)G_B^{(\beta)}(t-s, x, z) \times e^{-\frac{2\theta t}{p}} \left( \mathbb{E} \left| u_s^{(\beta)}(y) \right| \right)^{p/2} \left( \mathbb{E} \left| u_s^{(\beta)}(z) \right| \right)^{p/2} f(y, z) \, ds \, dy \, dz \right]^{p/2},$$

$$\lesssim L_\sigma^p b_t(\theta) \left[ \int_{B \times B} \int_0^t G_B^{(\beta)}(t-s, x, y)G_B^{(\beta)}(t-s, x, z) e^{-\frac{2\theta(t-s)}{p}} f(y, z) \, ds \, dy \, dz \right]^{p/2},$$

$$\lesssim L_\sigma^p b_t(\theta) \left[ \int_{B \times B} \int_0^t (t-s)^{-2\beta \lambda} e^{-\frac{2\theta(t-s)}{p}} ds \, f(y, z) dy \, dz \right]^{p/2},$$

$$\lesssim L_\sigma^p b_t(\theta) \left[ \int_0^\infty s^{-2\beta \lambda} e^{-\frac{2\theta s}{p}} ds \, f(y, z) dy \, dz \right]^{p/2},$$

$$\lesssim L_\sigma^p b_t(\theta) \left[ \int_0^\infty e^{-\frac{2\theta r}{p}} r^{-2\beta \lambda} ds \, f(y, z) dy \, dz \right]^{p/2},$$

$$\lesssim L_\sigma^p \frac{2^{(1-2\beta \lambda)}}{2} b_t(\theta) (2\theta)^{\frac{p(2\beta \lambda - 1)}{2}} \Gamma(1 - 2\beta \lambda)^{\frac{p}{2}} \left[ \int_{B \times B} f(y, z) dy \, dz \right]^{p/2}, \quad (6.15)$$

where $b_t(\theta) = \sup_{0<s<t} \sup_{x \in B} e^{-\theta s} \mathbb{E} |u_s^{(\beta)}(x)|^p$. For $t > 0$, we fix $\theta > 0$ sufficiently large so that

$$J_2 \lesssim \frac{1}{2} b_t(\theta). \quad (6.16)$$

Thus, we have

$$e^{-\theta t} \mathbb{E} |u_t^{(\beta)}(x)|^p \lesssim 1 + \frac{1}{2} b_t(\theta). \quad (6.17)$$

Taking the supremum of the left side of (6.17) the result follows.
References

[1] V.V. Anh, N.N. Leonenko, and M.D. Ruiz-Medina (2016) *Space-time fractional stochastic equations on regular bounded open domains*. Fractional Calculus and Applied Analysis, 19(5), pp. 1161-1199.

[2] R.F. Bass *Probabilistic Techniques in Analysis* ISBN 0-387-94387-0, Springer-Verlag, 1995.

[3] M. Caputo, *Linear models of dissipation whose Q is almost frequency independent, Part II*. Geophys. J. R. Astr. Soc. 13 (1967), 529–539.

[4] B. Baeumer, T. Luks, and M.M. Meerschaert, *Space-time fractional Dirichlet problems*, Mathematische Nachrichten, Vol. 291 (2018), pp. 2516–2535.

[5] R.M. Blumenthal, R.K. Getoor, *Asymptotic distribution of the eigenvalues for a class of Markov operators*, Pacific J. Math. 9 (1959) 399–408.

[6] Z-Q. Chen, M. M. Meerschaert and E. Nane. *Space-time fractional diffusion on bounded domains* J. Math. Anal. Appl., 393:479–488, 2012.

[7] Z.Q. Chen, K.H. Kim, and P. Kim, *Fractional time stochastic partial differential equations*. Stoch. Process. Appl. 125(4), 1470–1499 (2015)

[8] R.C. Dalang. *Extending the martingale measure stochastic integral with applications to spatially homogeneous SPDEs*. Electron. J. Probab. 4(6), 29 (1999)

[9] D.T. Dang, E. Nane, D.M. Nguyen, N.H. Tuan. *Continuity of solutions of a class of fractional equations*. Potential Anal., 49 (2018), pp. 423-47.

[10] E.B. Davies. *Heat Kernels and Spectral Theory*. Cambridge Univ. Press, Cambridge, 1989.

[11] A. Fick. *On liquid diffusion*. Phil. Mag. 10 (1855) (63) : 30–39. doi:10.1080/14786445508641925.

[12] M. Foondun and D. Khoshnevisan, *Intermittence and nonlinear parabolic stochastic partial differential equations*, Electron. J. Probab. 14 (2009), no. 21.

[13] M. Foondun, N. Guerngar, and E. Nane. *Some properties of non-linear fractional stochastic heat equations on bounded domains*. Chaos, Solitons and Fractals 102 (2017), 86-93

[14] M. Foondun, J.B. Mijena, and E. Nane. *Non-linear noise excitation for some space-time fractional stochastic equations in bounded domains*. Fract. Calc. Appl. Anal. 19(6), 1527–1553 (2016)

[15] M. Foondun and E. Nualart, *On the behaviour of stochastic heat equations on bounded domains*, ALEA Lat. Am. J. Probab. Math. Stat. 12 (2015), no. 2, 551–571

[16] M. Foondun, *Remarks on a fractional-time stochastic equation*, Proc. Amer. Math. Soc. 149 (2021), 2235-2247

[17] M. Foondun, W. Liu, and M. Omaba, *Moment bounds for a class of fractional stochastic heat equations*, The Annals of Probability 45 (2017), 2131–2153. MR-3693959

[18] R. Gorenflo, F. Mainardi, M. Raberto,and E. Scalas, *Fractional Diffusion in Finance: Basic Theory*. Modelli Dinamici in Economia e Finanza, Italy, September 28-30,2000

[19] F. Mainardi, Y. Luchko, and G. Pagnini, *The fundamental solution of the space-time fractional diffusion equation*. Fractional Calculus and Applied Analysis, vol. 4, no. 2, pp. 153–192, 2001.
[20] J.B. Mijena and E. Nane. *Space-time fractional stochastic partial differential equations*, Stochastic Process. Appl. 125 (2015), no. 9, 3301–3326.

[21] M.M. Meerschaert, E. Nane, Y. Xiao, *Fractal dimensions for continuous time random walk limits*, Statist. Probab. Lett. 83 (2013) 1083–1093.

[22] M.M. Meerschaert, D.A. Benson, H.-P. Scheffler, B. Baeumer, *Stochastic solution of space-time fractional diffusion equations*. Phys. Rev. E 65, (2002), 1103–1106.

[23] M.M. Meerschaert, P. Straka, *Inverse stable subordinators*, Math. Model. Nat. Phenom. 8 (2) (2013) 1–16.

[24] Meerschaert, M.M., Nane, E., Vellaisamy, P., *Fractional Cauchy problems on bounded domains*. Ann. Probab. 37, 979–1007 (2009)

[25] M.M. Meerschaert and H.P. Scheffler. *Limit theorems for continuous time random walks with infinite mean waiting times*. J. Applied Probab. 41 (2004), No. 3, 623–638.

[26] , E. Nane and N.H. Tuan, *New results on continuity problems of fractional order for a family of fractional stochastic heat equations* Preprint, 2022.

[27] Eulalia Nualart, *Moment bounds for some fractional stochastic heat equations on the ball*, Electron. Commun. Probab. 23 (2018), Paper No. 41, 12, DOI 10.1214/18-ECP147. MR3841402

[28] A. Saichev and G. Zaslavsky, *Fractional kinetic equations: solutions and applications*. Chaos 7 (1997), 753-764.

[29] S. Umarov, E. Saydamatov, *A fractional analog of the Duhamel principle*, Fract. Calc. Appl. Anal., 9 (1) (2006), 57–70.

[30] S.R. Umarov, E.M. Saidamatov, *Generalization of the Duhamel principle for fractional-order differential equations* (In Russian). Dokl. Akad. Nauk 412, No 4 (2007), 463–465; Transl. in Dokl. Math. 75, No 1 (2007), 94–96.

[31] S. Umarov, *On fractional Duhamel’s principle and its applications*. J. Differential Equations 252, No 10 (2012), 5217–5234.

[32] J. B. Walsh; *An introduction to stochastic partial differential equations*, École d’été de probabilités de Saint-Flour, XIV—1984, 265–439, Lecture Notes in Math. 1180, Springer, Berlin, 1986.

[33] B. Xie. *Some effects of the noise intensity upon non-linear stochastic heat equations on $[0, 1]$, Stochastic Process. Appl. 126 (2016), no. 4, 1184–1205, DOI 10.1016/j.spa.2015.10.014. MR346119