TWO GEOMETRIC INTERPRETATIONS OF HARDY SUMS

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Abstract. The problem of finding the number of lattice points in a triangle has a classical solution if the lattice is \( \mathbb{Z}^2 \) and the vertices of the triangle have integer valued coordinates. We consider what happens when we replace the lattice by \( \mathbb{Z}^2 \) instead and give an explicit formula for the number of lattice points inside a triangle in terms of Hardy sums. Moreover, we give a second geometric interpretation of the Hardy sums as signed intersection numbers with a certain oriented net of geodesics. Using this geometric realization, we prove a generalized reciprocity law for Hardy sums by an elementary argument.

1. Introduction

Problems of counting points with integer coordinates in geometric objects are challenging and well-studied problems in mathematics and give an interesting connection between number theory and geometry (famous examples include counting integer points on the 3-sphere, the Gauss circle problem or Hardy and Littlewood’s work on counting points in a right-angled triangle [5]). These problems are typically very hard. However, when one considers two-dimensional polygons whose vertices have integer coordinates, the problem has an elegant and simple solution.

Theorem 1.1 (Pick [8]). Let \( P \) be a polygon whose vertices lie in \( \mathbb{Z}^2 \). We have

\[
\text{area}(P) = \#(P^\circ \cap \mathbb{Z}^2) + \frac{1}{2}\#(\partial P \cap \mathbb{Z}^2) + 1,
\]

where \( P^\circ \) denotes the interior and \( \partial P \) denotes the boundary of \( P \).

In this note, we limit ourselves to the case of triangles. For a pair of positive integers \( d, c \), let \( T(d, c) \) be the triangle in \( \mathbb{R}^2 \) with vertices \((0, 0), (d, 0), \) and \((0, c)\). The set \( T(d, c) \) is the set of points \((x, y) \in \mathbb{R}^2 \) such that \( 0 \leq \frac{x}{d} + \frac{y}{c} < 1 \) (we exclude the acute angles and the hypotenuse). For simplicity, we shall assume that \( (d, c) = 1 \).

By Theorem 1.1, we only need to identify the points on the boundary of \( T(d, c) \) if we want to find \( \#(T(d, c) \cap \mathbb{Z}^2) \), namely

\[
\#(T(d, c) \cap \mathbb{Z}^2) = \text{area}(T(d, c)) + \frac{1}{2}\#(\partial T(d, c) \cap \mathbb{Z}^2) - 1.
\]

As we assumed that \( d, c \) are coprime there are no lattice points \((x, y) \in \mathbb{Z}^2 \) that satisfy \( \frac{x}{d} + \frac{y}{c} = 1 \) with both \( x, y \) non-zero. Hence, the points on the boundary are given by \((0, 0), (1, 0), (0, 1), \ldots \), i.e. \( \#(\partial T(d, c) \cap \mathbb{Z}^2) = c + d + 1 \).
We then find by equation (1.1):

\[(1.2) \quad \#(T(d, c) \cap \mathbb{Z}^2) = \frac{1}{2}(c+1)(d+1) - 1.\]

The problem of finding the lattice points in \(T(d, c)\) becomes more difficult if we consider the lattice \((2\mathbb{Z})^2\) instead of \(\mathbb{Z}^2\). Obviously, the problem can be rescaled and solved by Pick’s Theorem if both \(d, c\) are even. So again, let us assume that \(d, c\) are positive coprime integers. In this case, at least one of the vertices of \(T(d, c)\) is no longer a lattice point and we cannot use Theorem 1.1.

Here, we shall prove that \(\#(T(d, c) \cap (2\mathbb{Z})^2)\) can be expressed in terms of the Hardy sums

\[(1.3) \quad S_4(d, c) = \sum_{k=1}^{c-1}((-1)^{\frac{k|d|}{c}})\text{ for } (d, c) = 1, c > 0, d \text{ odd.}\]

The Hardy sums \(S_4(d, c)\) are integer-valued analogs of the classical Dedekind sums

\[(1.4) \quad s(d, c) = \sum_{k=1}^{c-1} \left(\left(\frac{k}{c}\right)\left(\frac{kd}{c}\right)\right) \text{ for } (d, c) = 1, c > 0,\]

where \((x) = x - [x] - \frac{1}{2}\) for \(x \not\in \mathbb{Z}\) and \((x) = 0\) for \(x \in \mathbb{Z}\). Equivalently, the Dedekind sums \(s(d, c)\) are uniquely determined by the following properties:

1. \(s(0, 1) = 0,\)
2. \(s(d + c, c) = s(d, c),\)
3. \(s(d, c) + s(c, d) = \frac{1}{12} \left(\frac{c^2 + d^2 + 1}{cd}\right) - \frac{1}{4}\) (the reciprocity formula).

One can, in fact, express the Hardy sums \(S_4(d, c)\) as a non-trivial linear combination of Dedekind sums, namely

\[(1.5) \quad S_4(d, c) = 8s(d, 2c) - 4s(d, c) \quad \text{for } (d, c) = 1, d \text{ odd.}\]

Equation (1.5) can be proved in an elementary way [9] and is well-understood in the context of Eisenstein cohomology [10, Ch. 2]. The expression of \(\#(T(d, c) \cap (2\mathbb{Z})^2)\) for \( (d, c) = 1 \) in terms of \(S_4(d, c)\) is given in the next theorem.

**Theorem 1.2.** Let \(d, c\) be positive coprime integers.

1. If \(d, c\) are both odd, then
   \[
   \#(T(d, c) \cap (2\mathbb{Z})^2) = \frac{1}{8}(cd + 2d + 2c + S_4(d, c) + S_4(c, d)).
   \]

2. If \(d\) is even, then
   \[
   \#(T(d, c) \cap (2\mathbb{Z})^2) = \frac{1}{4}\left(c + d + \frac{cd}{2} - \frac{1}{2cd} + S_4(c, d/2)\right).
   \]

The proof of Theorem 1.2 relies on a theorem of Mordell [7], which we will present in the next section.
The properties of Dedekind sums can be explained from a cohomological viewpoint as well. Consider the integer-valued Dedekind symbol

\begin{equation}
\Phi \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{cases} b, & \text{for } c = 0, \\
\frac{a+d}{c} - 12 s(d, c), & \text{for } c \neq 0. \end{cases}
\end{equation}

Asai [1] constructed the Dedekind symbol as a splitting of the cohomology class in \( H^2(\text{SL}_2(\mathbb{R}), \mathbb{R}) \) associated with the central extension of the universal covering group. The space \( H^2(\text{SL}_2(\mathbb{R}), \mathbb{R}) \) is a one-dimensional \( \mathbb{R} \)-vector space and generated by the cocycle

\begin{equation}
w(A, B) = -\text{sign}(c_A c_B c_{AB})
\end{equation}

for \( A = \begin{pmatrix} * & * \\ c_A & * \end{pmatrix}, B = \begin{pmatrix} * & * \\ c_B & * \end{pmatrix} \in \text{SL}_2(\mathbb{Z}), \) and \( AB = \begin{pmatrix} * & * \\ c_{AB} & * \end{pmatrix} \). Following the terminology of Kirby and Melvin [6], we call \( w(A, B) \) the Rademacher cocycle. Asai’s ”splitting” refers to the fact that there exists a unique function \( \Phi : \text{SL}_2(\mathbb{Z}) \rightarrow \mathbb{R} \) such that

\[ w(A, B) = \Phi(AB) - \Phi(A) - \Phi(B), \]

which is exactly the Dedekind symbol \( \Phi \) in (1.6). The uniqueness of \( \Phi \) follows easily from \( H^1(\text{SL}_2(\mathbb{Z}), \mathbb{R}) = \{0\} \). The reciprocity law for Dedekind sums corresponds to the case where \( B = S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \).

The Hardy sums \( S_d(c) \), defined in (1.3), and

\begin{equation}
S(d, c) = \sum_{k=1}^{\frac{|c|-1}{2}} (-1)^{\frac{|c|-1}{2}+k+1}, \quad (d, c) = 1, \ c + d \text{ odd},
\end{equation}

however, need to be viewed in terms of the cohomology of certain subgroups of the modular group, namely

\[ \Gamma_0^0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : b \equiv 0 \pmod{2} \right\} \text{ resp.} \]

\[ \Gamma_{\theta} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : a \equiv d \pmod{2}, \ b \equiv c \pmod{2} \right\}. \]

Clearly, we can view the Hardy sums as functions on the subgroups \( \Gamma_{\theta} \) and \( \Gamma_0^0(2) \) directly by defining \( S(A) = S(d, c) \) for \( A = \begin{pmatrix} * & * \\ c & * \end{pmatrix} \in \Gamma_{\theta} \) with \( c \neq 0 \). For the subgroups \( \Gamma_{\theta} \) and \( \Gamma_0^0(2) \), the first cohomology group is non-trivial.

Hardy sums appear as the correction factors in the transformation law of the logarithms of \( \theta \)-functions, so can be studied from an automorphic point of view. In section 3, however, we will show that these Hardy sums can be defined as a signed intersection number, without any reference to automorphic forms; see Theorem 3.7. We use this geometric interpretation to prove by an elementary argument that, say, \( S(A) \) on \( \Gamma_{\theta} \), satisfies

\[ S(AB) - S(A) - S(B) = w(A, B) \]
for $A, B \in \Gamma_0$, where $w(A, B)$ is the Rademacher cocycle [1.7]. Equivalently, this identifies the Hardy sums as restrictions of the Dedekind symbol $\Phi$ to the subgroups $\Gamma$ resp. $\Gamma_0$ up to addition by a homomorphism.

**Theorem 1.3.** The functions $\Phi - S : \Gamma_0 \to \mathbb{Z}$ and $\Phi - S : \Gamma_0 \to \mathbb{Z}$ are homomorphisms.

We would like to remark that in an upcoming paper of Burrin and von Essen [4], Burrin’s generalized Dedekind sums [2, 3] are similarly realized as winding numbers of closed geodesics on the modular surface around the cusp $i\pi$.

2. **Proof of Theorem 1.2**

To give a closed expression of $\#(T(d, c) \cap (2\mathbb{Z})^2)$, we consider the problem as a three-dimensional one. Namely, we reformulate

\[
\#(T(d, c) \cap (2\mathbb{Z})^2) = \#\left\{(x, y) \in \mathbb{Z}^2 : 0 \leq \frac{2x}{d} + \frac{2y}{c} < 1\right\}
\]  
\[(2.1)\]

For positive integers $u, v, w$, let $D(u, v, w)$ be the tetrahedron in $\mathbb{R}^3$ with vertices $(u, 0, 0)$, $(0, v, 0)$, and $(0, 0, w)$, i.e.

\[D(u, v, w) = \left\{(x, y, z) \in \mathbb{R}^3 : 0 < \frac{x}{u} + \frac{y}{v} + \frac{z}{w} < 1\right\}.\]

Note that we excluded the vertices in our definition of $D(u, v, w)$.

Counting $\#(T(d, c) \cap (2\mathbb{Z})^2)$ is equivalent to counting the lattice points in $\mathbb{Z}^3$ of the tetrahedron $D(d, c, 2)$ with vertices $(d, 0, 0)$, $(0, c, 0)$, and $(0, 0, 2)$ intersected with the plane $\{(x, y, z) \in \mathbb{R}^3 : z = 1\}$ by (2.1). Geometrically, this corresponds to projecting $T(d, c) \cap (2\mathbb{Z})^2$ onto the plane $\{(x, y, z) \in \mathbb{R}^3 : z = 1\}$ with base point $(0, 0, 2)$.

Mordell expressed $\#(D(u, v, w) \cap \mathbb{Z}^3)$ in a pretty closed form in terms of Dedekind sums, which we will use for our solution to the problem.

**Theorem 2.1** (Mordell). Let $u, v, w$ be pairwise coprime positive integers. We have

\[
\#(D(u, v, w) \cap \mathbb{Z}^3) = \frac{uvw}{6} + \frac{1}{4}(uw + uw + vw) + \frac{1}{4}(u + v + w) + \frac{1}{12}\left(\frac{uw}{w} + \frac{uw}{v} + \frac{vw}{u}\right)
\]

\[\quad + \frac{1}{12uvw} - 2 - (s(uw, w) + s(uw, v) + s(vw, u)).\]

**Remark 2.2.** One should note that the right hand side in Theorem 2.1 is computationally easier than the left hand side. Indeed, if one were to calculate the left hand side without exploiting symmetries, the computational complexity would be $O(uvw)$. The right hand side, however, is much faster to compute, as Dedekind sums can be computed through the Euclidean algorithm (see the reciprocity formula in the introduction).
2.1. Proof of the First Part of Theorem 1.2. Let \( d, c \) be odd and coprime integers. By Theorem 2.1 we have

\[
\#(D(d, c, 2) \cap \mathbb{Z}^3) = \frac{cd}{3} + \frac{1}{4}(2c + 2d + cd) + \frac{1}{4}(c + d + 2) + \frac{1}{12} \left( \frac{2c}{d} + \frac{2d}{c} + \frac{cd}{2} \right) \\
+ \frac{1}{24cd} - 2 - (s(2d, c) + s(2c, d) + s(cd, 2)).
\]

(2.2)

Since \( cd \) is odd, we have \( s(cd, 2) = s(1, 2) = 0 \).

We split the number of lattice points in \( D(d, c, 2) \) into those contained in the plane \( \{ (x, y, z) \in \mathbb{R}^3 : z = 0 \} \) and those contained in the plane \( \{ (x, y, z) \in \mathbb{R}^3 : z = 1 \} \). The latter is, as we already said, equal to \( \#(T(d, c) \cap (2\mathbb{Z})^2) \). The former is equal to \( \# \left\{ (x, y) \in \mathbb{Z}^2 : 0 < \frac{x}{d} + \frac{y}{c} < 1 \right\} = \frac{1}{2}(c + 1)(d + 1) - 2 \)

by (1.2) with the exclusion of the point \((0, 0)\).

To write \( \#(T(d, c) \cap (2\mathbb{Z})^2) \) in terms of Hardy sums, we use the following easy lemma.

Lemma 2.3. Let \( d, c \) be odd and coprime integers. We have

\[ s(2c, d) + s(2d, c) = \frac{4c^2 + 4d^2 + 1}{24cd} - \frac{1}{8}(S_4(d, c) + S_4(c, d)) - \frac{3}{8}. \]

Proof. We use formula (1.3) for the Hardy sum \( S_4(d, c) \). A calculation yields

\[
S_4(d, c) + S_4(c, d) = 8s(d, 2c) - 4s(d, c) + 8s(c, 2d) - 4s(c, d) \\
= 8(s(d, 2c) + s(c, 2d)) - 4(s(d, c) + s(c, d)) \\
= 8 \left( -s(2c, d) - s(2d, c) + \frac{1}{12} \frac{4c^2 + d^2 + 1}{2cd} + \frac{1}{12} \frac{c^2 + 4d^2 + 1}{2cd} - \frac{1}{2} \right) \\
- 4 \left( \frac{1}{12} \frac{c^2 + d^2 + 1}{cd} - \frac{1}{4} \right) \\
= \frac{4c^2 + 4d^2 + 1}{3cd} - 3 - 8(s(2c, d) + s(2d, c)),
\]

where we used the reciprocity formula twice. \( \square \)

Putting (2.2) and Lemma 2.3 together yields the first part of Theorem 1.2.

2.2. Proof of the Second Part of Theorem 1.2. Let \( d, c \) be coprime integers and \( d \) an even integer. As in section 2.1, we would like to apply Mordell’s Theorem 2.1 to find \( \#(T(d, c) \cap (2\mathbb{Z})^2) \). However, Mordell’s Theorem 2.1 is no longer true if the integers \( u, v, w \) are not pairwise coprime. We hence need to adjust the statement.
Proposition 2.4. Let \(d, c\) be coprime integers and suppose that \(d\) is even and \(c\) is odd. The number of lattice points in the tetrahedron \(D(d, c, 2)\) is given by
\[
\#(D(d, c, 2) \cap \mathbb{Z}^3) = \frac{cd}{3} + \frac{1}{4}(2c + 2d + cd) + \frac{1}{4}(c + d + 2) + \frac{1}{12}\left(\frac{2c}{d} + \frac{2d}{c} + \frac{cd}{2}\right) + \frac{1}{24cd} - \frac{5}{2} - (2s(d, c) + s(2c, d)).
\]

Proof. The proof is along the lines of Mordell’s proof [7]. We only indicate where adjustments need to be made.

One starts with the formula
\[
\#(D(d, c, 2) \cap \mathbb{Z}^3) = \frac{1}{2} \sum'_{x, y, z} \left(\left\lfloor \frac{x}{d} + \frac{y}{c} + \frac{z}{2} \right\rfloor - 1\right) \left(\left\lfloor \frac{x}{d} + \frac{y}{c} + \frac{z}{2} \right\rfloor - 2\right),
\]
where the sum goes over \(0 \leq x < d, 0 \leq y < c, \) and \(0 \leq z < 2\) and \(\sum'\) means that we omit \(x = y = z = 0\).

There exists exactly one lattice point \((x, y, z)\) with \(\frac{x}{d} + \frac{y}{c} + \frac{z}{2} = 1\), namely \(x = \frac{d}{2}\) and \(z = 1\) (this would not be the case if \(c, d, 2\) were pairwise coprime). There exist no lattice points with \((x, y, z)\) with \(\frac{x}{d} + \frac{y}{c} + \frac{z}{2} = 2\).

Now rewrite
\[
\#(D(d, c, 2) \cap \mathbb{Z}^3) = \frac{1}{2} \sum'_{x, y, z} (E - ((E)) - 3/2)(E - ((E)) - 5/2) - \frac{3}{8},
\]
where \(E = \frac{x}{d} + \frac{y}{c} + \frac{z}{2}\) and the term \(-\frac{3}{8}\) is the correction factor for the unique case where \(E = 1\).

We split up the sum \(\sum'_{x, y, z}(E - ((E)) - 3/2)(E - ((E)) - 5/2) = A + B + C\) with
\[
A = \frac{1}{2} \sum'_{x, y, z} (E - 3/2)(E - 5/2), \quad B = -\sum'_{x, y, z} (E - 2) ((E)), \quad C = \frac{1}{2} \sum'((E))^2.
\]

It is straightforward to calculate \(A\), which is equal to \(\frac{3cd}{12} + \frac{e}{c} + \frac{d}{2c} + \frac{3c}{2} + \frac{3d}{2} - \frac{11}{4}\). For \(B\), we obtain
\[
B = \frac{1}{2} \sum' x d \left(\frac{x}{d} + \frac{y}{c} + \frac{z}{2}\right) + \frac{1}{2} \sum' y c \left(\frac{x}{d} + \frac{y}{c} + \frac{z}{2}\right) + \frac{1}{2} \sum z d \left(\frac{x}{d} + \frac{y}{c} + \frac{z}{2}\right)
\]
\[
= \sum x d \left(\frac{2x}{d} + \frac{2y}{c}\right) + \sum y c \left(\frac{2x}{d} + \frac{2y}{c}\right) + \frac{1}{2} \sum y c \left(\frac{2y}{c}\right)
\]
\[
= d^{-1} \sum x d \left(\frac{2cx}{d}\right) + 2 \sum y c \left(\frac{dy}{c}\right)
\]
\[
= s(2c, d) + 2s(d, c),
\]
where in the second last line we used that \(2y\) runs through a complete set of residues modulo \(c\) – thus also \(\sum_{y=1}^{c-1} \left(\frac{2y}{c}\right) = 0\), and that \(2x\) runs through a complete set of residues of \(d/2\) twice.
To calculate $C$, we may write
\[ C = \frac{1}{2} \sum_{x,y,z} \left( \left( \frac{x}{d} + \frac{y}{c} + \frac{z}{2} \right)^2 \right) - \frac{1}{4}, \]
as $2cx + 2dy + cdz$ runs through a complete set of residues modulo $2cd$ with the exception that $2cx + 2dy + cdz \equiv 0 \pmod{2cd}$ is attained twice. It is then easy to calculate $C = \frac{cd}{12} + \frac{1}{2cd}$. □

Similarly as in Lemma 2.3, the linear combination $s(2c,d) + 2s(d,c)$ in Proposition 2.4 can be rewritten in terms of Hardy sums.

**Lemma 2.5.** Let $d \geq 1$ be an even integer and $c \geq 1$ such that $(d,c) = 1$. We have
\[ 2s(d,c) + s(2c,d) = \frac{d^2 + c^2 + 1}{6cd} - \frac{1}{2} - \frac{1}{4}S_4(c,d/2). \]

**Proof.** We use formula (1.5) for the Hardy sums $S_4(c,d)$. Applying the reciprocity formula for Dedekind sums, we calculate
\[ 2s(d,c) + s(2c,d) = 2 \left( \frac{1}{12} \frac{d^2 + c^2 + 1}{cd} - \frac{1}{4} s(c,d) \right) + s(c,d/2) = \frac{d^2 + c^2 + 1}{6cd} - \frac{1}{2} - \frac{1}{4}S_4(c,d/2), \]
where we used that $s(2c,2d') = s(c,d')$ for all $d' \in \mathbb{Z}$. □

Proposition 2.4 and Lemma 2.5 prove the second part of Theorem 1.2.

### 3. Hardy Sums as Intersection Numbers and the Rademacher Cocycle

As mentioned in the introduction, the Dedekind sums $s(a,c)$ in (1.8) are uniquely characterized by the following three properties for $(a,c) = 1$:

1. $s(0,1) = 0$,
2. $s(a+c,c) = s(a,c)$,
3. $s(a,c) - s(-c,a) = \frac{1}{12} \frac{a^2 + c^2 + 1}{ac} - \frac{1}{4}$ for $a,c \neq 0$.

Since for each coprime pair of integers $a,c$ there exists a a matrix $A = \begin{pmatrix} a & * \\ c & * \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$, we may write $s(A)$ for $s(a,c)$, i.e. view the Dedekind sums as a function $s : \text{SL}_2(\mathbb{Z}) \to \mathbb{Q}$. By definition, the Dedekind sum $s(A)$ is invariant under right-multiplication by $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$; moreover, we set $s(1,0) = s(I) = 0$. The three defining properties above then correspond to the matrix operations under the generators $T$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ of $\text{SL}_2(\mathbb{Z})$, namely for $A = \begin{pmatrix} a & * \\ c & * \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ with $a,c \neq 0$:

1. $s(S) = s(I) = 0$,
2. $s(T^n A) = s(A)$ for all $n \in \mathbb{Z}$,
3. $s(A) - s(SA) = \frac{1}{12} \frac{a^2 + c^2 + 1}{ac} - \frac{1}{4}$. 

Each matrix in $$\text{SL}_2(\mathbf{Z})$$ can be written as a word in $$T$$, $$S$$ and therefore we get that the three properties uniquely define the function $$s : \text{SL}_2(\mathbf{Z}) \to \mathbf{Q}$$.

Similarly, we may view the Hardy sums $$S(a,c)$$ and $$S_i(a,c)$$ as functions on their respective groups $$\Gamma_\theta = \langle T^2, S \rangle$$ and $$\Gamma^0(2) = \langle T^2, V \rangle$$ with $$V = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$. To avoid confusion with the matrix $$S$$, we denote these functions in gothic print, i.e. $$\mathfrak{S} : \Gamma_\theta \to \mathbf{Z}$$ and $$\mathfrak{S}_4 : \Gamma^0(2) \to \mathbf{Z}$$. Put $$\mathfrak{S}(A) = S(a,c)$$ for $$A = \begin{pmatrix} a & * \\ c & * \end{pmatrix} \in \Gamma_\theta$$. The function $$\mathfrak{S} : \Gamma_\theta \to \mathbf{Z}$$ is uniquely determined by the properties:

1. $$\mathfrak{S}(S) = \mathfrak{S}(T) = \mathfrak{S}(I) = 0$$,
2. $$\mathfrak{S}(T^{2n}A) = \mathfrak{S}(A)$$ for all $$n \in \mathbf{Z}$$,
3. $$\mathfrak{S}(A) - \mathfrak{S}(SA) = \operatorname{sign}(ac)$$ for $$A = \begin{pmatrix} a & * \\ c & * \end{pmatrix} \in \Gamma_\theta$$;

and the function $$\mathfrak{S}_4 : \Gamma^0(2) \to \mathbf{Z}$$ (similarly defined) is uniquely determined by the properties:

1. $$\mathfrak{S}_4(V) = \mathfrak{S}_4(T) = \mathfrak{S}_4(J) = 0$$,
2. $$\mathfrak{S}_4(AT^{2n}) = \mathfrak{S}_4(A)$$ for all $$n \in \mathbf{Z}$$,
3. $$\mathfrak{S}_4(A) + \mathfrak{S}_4(AV) = -\operatorname{sign}(c(a + c))$$ for $$A = \begin{pmatrix} a & * \\ c & * \end{pmatrix} \in \Gamma^0(2)$$.

The uniqueness of this characterization will be seen in the proof of Theorem 3.7. Our goal in this section is to find a geometric characterization of the functions $$\mathfrak{S} : \Gamma_\theta \to \mathbf{Z}$$ and $$\mathfrak{S}_4 : \Gamma^0(2) \to \mathbf{Z}$$, which satisfies these three corresponding properties and prove a generalized reciprocity law for $$\mathfrak{S}$$ and $$\mathfrak{S}_4$$.

### 3.1. Net of Oriented Geodesics

Consider the set of rationals $$\frac{a}{c}$$ with $$(a,c) = 1$$, $$c > 1$$, and $$a,c$$ both odd and the net of geodesics connecting two such rationals $$\frac{a}{c}$$ and $$\frac{b}{d}$$ if $$ad - bc = \pm 2$$. Note that these geodesics are the image of the geodesic connecting $$-1$$ and $$1$$ under $$\Gamma_\theta$$.

Now orient the geodesic connecting $$\frac{a}{c}$$ and $$\frac{b}{d}$$ such that the geodesic goes from $$\frac{a}{c}$$ to $$\frac{b}{d}$$ if $$c < d$$ and from $$\frac{b}{d}$$ to $$\frac{a}{c}$$ if $$d < c$$. In the case $$c = d$$, which can only occur if the geodesic connects two integers, we give it two opposing orientations. We will denote the resulting net of oriented geodesics by $$\mathcal{N}_\theta$$.

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1To make our notations simpler, we consider $$S(a,c)$$ instead of the usual $$S(d,c)$$. The change is insubstantial, as $$S(d,c) = -\mathfrak{S}(A^{-1}) = \mathfrak{S}(A)$$ for $$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\theta$$; see Proposition 3.13.
Lemma 3.1. Let $a, b, c, d$ be odd integers with $c, d \geq 1$ and $(a, c) = 1$, $(b, d) = 1$ such that $ad - bc = \pm 2$. We have $c = d$ if and only if $c = d = 1$. We have $a = b$ if and only if $a, b = \pm 1$. If $a \neq b$, the condition $c < d$ is satisfied if and only if $|a| < |b|$.

Proof. If $a = b$, then $ad - bc = a(d - c) = \pm 2$, and since $a$ is odd, we have $a = \pm 1$. Analogously, it is showed that $c = d$ implies $c = d = 1$.

Let us thus suppose that $c, d > 1$. Then the integers $a, b$ need to have the same sign to satisfy the equation $ad - bc = \pm 2$. By multiplying the equation $ad - bc = \pm 2$ on both sides with $-1$ if necessary, we may assume w.l.o.g. that $a, b > 0$.

Suppose that $ad - bc = -2$ and that $a \neq b$. If $c < d$, then $ad = bc - 2$ implies that $ad < bc < bd$, hence $a < b$. If $d < c$, then $2 = bc - ad$ implies that $d(b - a) < 2$ and moreover that $b - a < 0$, since $b - a$ is an even integer and we excluded $a = b$. The case $ad - bc = 2$ is similar. □

Lemma 3.1 has the following consequence for the oriented net of geodesics $N_\theta$.

Lemma 3.2. The oriented net $N_\theta$ is invariant under the maps $t_1(z) = -\overline{z}$ and $t_2(z) = z + 2$. Moreover, under $t_3(z) = \frac{z}{|z|^2}$ any oriented geodesic $g \in N_\theta$ whose endpoints are not both of the form $\frac{1}{n}$ for $n \in \mathbb{Z} - \{0, \pm 1\}$ or both integers not equal to $\pm 1$ is mapped to an oriented geodesic in $N_\theta$.

Proof. Suppose we have a geodesic from $\frac{a}{c}$ to $\frac{b}{d}$ in $N_\theta$ with $c, d > 1$ odd integers, $d > c > 1$, $(a, c) = 1$ and $(b, d) = 1$ with $ad - bc = \pm 2$. We will only show the case where $a, b > 1$.

Under the orientation-reversing map $t_1$ the geodesic is mapped to the geodesic from $-\frac{a}{c}$ to $-\frac{b}{d}$, which still lies in $N_\theta$.

Under the orientation-preserving $t_2$ the geodesic is mapped to the one from $\frac{a + 2c}{c}$ to $\frac{b + 2d}{d}$, which also still lies in $N_\theta$.

Finally, the orientation-reversing map $t_3$ sends the geodesic to the geodesic from $\frac{a}{c}$ to $\frac{b}{d}$, which as a non-oriented geodesic still lies in $N_\theta$. By Lemma 3.1 the orientation is preserved in $N_\theta$ under $t_3$, as $a < b$. □
Remark 3.3. If \( g \in \mathcal{N}_\theta \) in Lemma 3.2 connects \( \frac{1}{n} \) and \( \frac{1}{n+2} \) for some \( n \in \mathbb{Z} \), then \( t_3(g) \) is a geodesic connecting \( n \) and \( n + 2 \), i.e. two integers. But geodesics connecting two integers in \( \mathcal{N}_\theta \) are oriented with two opposing orientations – by Lemma 3.1 those are, in fact, the only geodesics in \( \mathcal{N}_\theta \) which have opposing orientations. Hence, the oriented geodesic \( t_3(g) \) does not have two opposing orientations and does not lie in \( \mathcal{N}_\theta \).

Remark 3.4. More generally, as \( t_2(z) = T^2.z \) and \( t_1(t_3(z)) = S.z \) and \( \Gamma_\theta = \langle T^2, S \rangle \), the proof of Lemma 3.2 shows that the map arising from each matrix \( A \in \Gamma_\theta \) preserves geodesics in \( \mathcal{N}_\theta \) with the exception of the doubly-oriented geodesics connecting two integers and their preimages under \( A \).

3.2. Intersection Numbers. Let \( g, h \) be any two oriented geodesics in \( \mathcal{H} \). Define \( \varphi_g(h) \) as follows:

\[
\varphi_g(h) = \begin{cases} 
0, & \text{if } g \cap h = \emptyset, \\
+1, & \text{if } g \text{ intersects } h \text{ on the right}, \\
-1, & \text{if } g \text{ intersects } h \text{ on the left}.
\end{cases}
\]

By ” \( g \) intersects \( h \) from the right” we mean that if we follow the path of \( h \), then the orientation of \( g \) at the intersection point is directed to the right. If the geodesic \( g \) is doubly-oriented (as we allowed in section 3.1), then we set \( \varphi_g(h) = 1 - 1 = 0 \). We will define the Hardy sums as a signed intersection number with the net \( \mathcal{N}_\theta \), which we defined in section 3.1. For a geodesic \( h \), define the signed intersection number

\[(3.1) \quad I_\theta(h) = \sum_{g \in \mathcal{N}_\theta} \varphi_g(h),\]

whenever the sum exists.

We will denote an oriented geodesic \( h \) from two points \( z_1, z_2 \in \mathcal{H} \cup \mathbb{Q} \cup \{\infty\} \) by \( h(z_1, z_2) \). To simplify notation, we will write \( h_x = h(i \infty, x) \). Let \( \mathbb{Q}_\theta = \Gamma_\theta.i \infty \) be the set of rationals \( \frac{a}{c} \) with \( (a, c) = 1 \) and such that \( a + c \) is odd. For each \( x \in \mathbb{Q}_\theta \), let \( I_\theta(x) = I_\theta(h_x) \) be the signed intersection number of \( h_x \) with the net \( \mathcal{N}_\theta \). We first need to show that the function \( I_\theta(x) \) is well-defined.

Lemma 3.5. For \( x \in \mathbb{Q}_\theta \), the geodesic \( h_x \) only intersects finitely many geodesics in \( \mathcal{N}_\theta \).

Proof. Since the sign of intersection does not matter to prove the claim, we will ignore the orientation on the geodesics \( \mathcal{N}_\theta \) for the remainder of this proof and refer to \( \mathcal{N}_\theta \) as a set of non-oriented geodesics.

Suppose we have a geodesic in \( \mathcal{N}_\theta \) connecting \( 0 < \frac{a}{c} < \frac{b}{d} \), which intersects \( h_x \). The maximal radius of any semi-circular geodesic in \( \mathcal{N}_\theta \) is 1, since the radius is given by

\[
\frac{1}{2} \left( \frac{b}{d} - \frac{a}{c} \right) = \frac{1}{2} \frac{bc - ad}{cd} = \frac{1}{cd} \leq 1.
\]

Consider the triangle \( \Delta \) with endpoints \( -1, 1 \) and \( \infty \). Since \( \Delta \) is the boundary of a fundamental domain for \( \Gamma_\theta \), the orbit of \( \Delta \) under \( \Gamma_\theta \) gives a triangulation of the upper
half plane $H$. Denote the collection of geodesic boundaries of the triangulation in $H$ by $\mathcal{N}$. In fact, the set $\mathcal{N}_\theta$ is contained in $\mathcal{N}$. We will prove the stronger claim that the geodesic $h_x$ will intersect only finitely many geodesics of $\mathcal{N}$ in $H$.

Any geodesic in $\mathcal{N}$ which does intersect $h_x$ is one whose endpoints are such that one is to the left of $x$ and one is to the right of $x$.

Write $x = \frac{\alpha}{\gamma}$ with $(\alpha, \gamma) = 1$ and $\alpha + \gamma$ odd. Let $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_\theta$ be the unique matrix (up to sign) such that $\Delta_x = A.\Delta$ is a triangle, whose interior is intersected by $h_x$ and has $x$ as a vertex. Explicitly, the matrix $A = \pm \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_\theta$ is determined by $\gamma > 0$ and $-\gamma < \delta < \gamma$.

**Figure 2:** Picture of the case $x = \frac{3}{4}$. The red geodesics are those in $\mathcal{N}_\theta$, which are intersected by $h_{3/4}$.

Let $r_x$ be the greatest radius of a geodesic bounding $\Delta_x$. Since $\Gamma_\theta.\Delta$ is a triangulation, the radius of a geodesic in $\mathcal{N}$ whose endpoints lie to the left and right of $x$ is bounded from below by $r_x$: Any geodesic in $\mathcal{N}$, which is such that $x$ lies in between its endpoints, with smaller radius than $r_x$ would have to intersect the triangle $\Delta_x$, which cannot happen. Thus, for any geodesic from $\frac{\alpha}{\gamma}$ to $\frac{1}{\gamma}$ in $\mathcal{N}$ intersecting $h_x$ there is a lower bound on its radius $\frac{1}{cd}$ (and a fortiori in $\mathcal{N}_\theta$). More precisely, we have

$$1 \leq cd \leq \frac{1}{r_x}.$$

Hence, there are only finitely many possible geodesics in $\mathcal{N}$ intersecting $h_x$. \qed

**Remark 3.6.** For instance, the geodesic $h_0$ from $i\infty$ to 0 intersects only one geodesic in $\mathcal{N}_\theta$, the one connecting $-1$ and 1. In the notation of the proof of Lemma 3.5, the triangle $\Delta_0$ is given by $S.\Delta$.

Now we prove that the function $I_\theta(x)$ actually satisfies the three properties defining the Hardy sums $S(a, c)$.

**Theorem 3.7.** The signed intersection number $I_\theta : \mathbb{Q}_ \theta \to \mathbb{Z}$ satisfies the following properties for $x \in \mathbb{Q}_ \theta$:

1. $I_\theta(-x) = -I_\theta(x)$,
(2) \( I_\theta(x + 2) = I_\theta(x) \),

(3) if \( x \neq 0 \), then \( I_\theta(x) + I_\theta(\frac{1}{x}) = \text{sign}(x) \).

Moreover, the function \( I_\theta \) is uniquely determined by the properties (1)-(3).

As an immediate corollary, we get that the Hardy sums \( \mathcal{G} : \Gamma_\theta \to \mathbb{Z} \) can be realized as a signed intersection number.

**Corollary 3.8.** For \( A \in \Gamma_\theta \), we have

\[
\mathcal{G}(A) = I_\theta(A.i\infty).
\]

**Proof of Theorem 3.7.** We first show that \( I_\theta(x) \) in (3.1) satisfies these properties. For the first property, note that reflection along the imaginary axis sends the geodesic \( h_x \) to \( h_{-x} \).

In particular, reflection does not change the orientation of \( h_x \). However, reflection does change the orientation of the geodesics \( g \in \mathcal{N}_\theta \) which intersect \( h_x \). Since \( \mathcal{N}_\theta \) is invariant under reflections along the imaginary axis by Lemma 3.2, we have \(-I_\theta(x) = I_\theta(-x)\).

For the second property, we use that \( \mathcal{N}_\theta \) is invariant under translations \( z \mapsto z + 2 \) again, by Lemma 3.2— and that \( h_x \mapsto h_{x+2} \) under translation. Hence \( I_\theta(x + 2) = I_\theta(x) \).

For the third property, we may assume w.l.o.g. that \( x \in (-1, 1) \) by the second property. Let \( n_0 \) be an odd integer, such that \( n_0 < \frac{1}{x} < n_0 + 2 \). Let \( g_0 \) be the geodesic connecting \( \frac{1}{n_0} \) and \( \frac{1}{n_0 + 2} \).

In \( \mathcal{N}_\theta \) the geodesic \( g_0 \) is oriented from \( \frac{1}{n_0} \) to \( \frac{1}{n_0 + 2} \) if \( n_0 \) is positive (i.e. \( \text{sign}(x) > 0 \)) and from \( \frac{-1}{|n_0| - 2} \) to \( \frac{1}{n_0} \) if \( n_0 \) is negative (i.e. \( \text{sign}(x) < 0 \)). The geodesic \( g_1 \) from \( n_0 \) to \( n_0 + 2 \) has two opposing orientations in \( \mathcal{N}_\theta \).

We write

\[
I_\theta(h_x) = \varphi_{g_0}(h_x) + \sum_{g \in \mathcal{N}_\theta - \{g_0\}} \varphi_g(h_x) = \text{sign}(x) + \sum_{g \in \mathcal{N}_\theta - \{g_0\}} \varphi_g(h_x).
\]

We have \( t_3(h_x) = h(0, 1/x) \) and by Lemma 3.2 we see that

\[
\sum_{g \in \mathcal{N}_\theta - \{g_0\}} \varphi_g(h_x) = -\sum_{g \in \mathcal{N}_\theta - \{g_1\}} \varphi_g(h(0, 1/x)) = -I_\theta(h(0, 1/x)),
\]

since \( t_3 \) preserves the orientations on \( \mathcal{N}_\theta \), except for geodesics connecting points \( \frac{1}{n_0} \) and \( \frac{1}{n_0 + 2} \) like \( g_0 \); see Remark 3.3. It should also be noted that the doubly-oriented geodesic connecting \(-1 \) and \( 1 \), which is intersected by both \( h_x \) and \( h(0, 1/x) \) gets mapped to itself under \( t_3 \).

Consider the triangle of oriented geodesics from \( h_0 \), \( h(0, 1/x) \) and \( h(1/x, i\infty) \). It is clear that \( I_\theta(h(1/x, i\infty)) = -I_\theta(h_1/x) \) and that \( I_\theta(h_0) + I_\theta(h(0, 1/x)) + I_\theta(h(1/x, i\infty)) = 0 \). But by Remark 3.6 we have \( I_\theta(h_0) = 0 \), hence \( I_\theta(h(0, 1/x)) = I_\theta(h_1/x) \), which proves the third property.

Let us now show uniqueness: Let \( I_1(x) \) and \( I_2(x) \) be two functions satisfying the three properties. From the first property, it follows that \( I_1(0) = I_2(0) = 0 \). By the second property the difference \( I_3(x) = I_1(x) - I_2(x) \) is zero at every integer \( x \in 2\mathbb{Z} \).
Suppose that \( I_3(a/c) \) is zero for all \( \frac{a}{c} \in \mathbb{Q}_0 \), \((a, c) = 1\) with \( 1 \leq c < c' \). By what we just showed, this is true for \( c' = 2 \). We now show that the function \( I_3 \) also vanishes for \( \frac{a'}{c'} \) with \((a', c') = 1\) and \( a' + c' \) being odd, from which the claim will follow by induction. By the second property, we may assume w.l.o.g. that \(-c' < a' < c'\). Using the third property, we see that \( I_3(x) = -I_3 \left( \frac{1}{x} \right) \). But

\[
I_3 \left( \frac{a'}{c'} \right) = -I_3 \left( \frac{c'}{a'} \right) = 0
\]

by our assumption (since \( |a'| < c' \)). \( \square \)

**Remark 3.9.** There is an analog for \( \mathcal{S}_4 : \Gamma^0(2) \to \mathbb{Q} \) which is obtained as an intersection number by shifting the net \( \mathcal{N}_\theta \) with \( z \mapsto z + 1 \) and reversing the orientation of the geodesics. This follows from the fact \( \mathcal{S}(x + 1) = -\mathcal{S}_4(x) \).

### 3.3. The Rademacher Cocycle

We continue our investigation into the geometric nature of the function \( \mathcal{S} \). We will now show that it is in fact a splitting of the Rademacher cocycle \([1,7]\) restricted to the subgroup \( \Gamma_\theta \). On the full modular group \( \text{SL}_2(\mathbb{Z}) \), the Rademacher cocycle splits to the Dedekind symbol \([1,6]\), i.e. for \( A, B \in \text{SL}_2(\mathbb{Z}) \) with

\[
A = \begin{pmatrix} * & * \\ c_A & * \end{pmatrix}, \quad B = \begin{pmatrix} * & * \\ c_B & * \end{pmatrix}, \quad \text{and} \quad AB = \begin{pmatrix} * & * \\ c_{AB} & * \end{pmatrix},
\]

we have

\[
(3.3) \quad \Phi(AB) - \Phi(A) - \Phi(B) = -\text{sign}(c_A c_B c_{AB}).
\]

**Remark 3.10.** The sign function in \((3.3)\) is defined such that \( \text{sign}(0) = 0 \).

**Theorem 3.11.** Let \( A, B \in \Gamma_\theta \) with \( A = \begin{pmatrix} * & * \\ c_A & * \end{pmatrix}, \quad B = \begin{pmatrix} * & * \\ c_B & * \end{pmatrix}, \quad \text{and} \quad AB = \begin{pmatrix} * & * \\ c_{AB} & * \end{pmatrix} \). We have

\[
\mathcal{S}(AB) - \mathcal{S}(A) - \mathcal{S}(B) = -\text{sign}(c_A c_B c_{AB}).
\]

From property (3) of Theorem 3.7 (the reciprocity formula), it follows that

\[
(3.4) \quad \mathcal{S}(SA) - \mathcal{S}(A) = \text{sign}(ac)
\]

for \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\theta \), which, since \( \mathcal{S}(S) = 0 \), proves Theorem 3.11 in the case \( B = S \).

We now proceed to prove Theorem 3.11 through a series of proposition and lemmas.

**Lemma 3.12.** Let \( A, B \in \Gamma_\theta \) and write \( A = \begin{pmatrix} a_A & b_A \\ c_A & d_A \end{pmatrix}, \quad B = \begin{pmatrix} a_B & b_B \\ c_B & d_B \end{pmatrix}, \quad \text{and} \quad AB = \begin{pmatrix} * & * \\ c_{AB} & * \end{pmatrix} \).

We have

\[
\mathcal{S}(AB) - \mathcal{S}(A) - \mathcal{S}(B) = -\text{sign}(c_A c_B c_{AB})
\]

if and only if

\[
\mathcal{S}(A'B') - \mathcal{S}(A') - \mathcal{S}(B') = -\text{sign}(c_A c_B c_{A'B'}),
\]

where \( A' = AS = \begin{pmatrix} a_{A'} & b_{A'} \\ c_{A'} & d_{A'} \end{pmatrix}, \quad B' = SB = \begin{pmatrix} a_{B'} & b_{B'} \\ c_{B'} & d_{B'} \end{pmatrix} \) and \( A'B' = \begin{pmatrix} * & * \\ c_{A'B'} & * \end{pmatrix} \).
Proof. Due to the symmetry of the claim, it suffices to prove that if $\mathcal{G}(A'B') - \mathcal{G}(A') - \mathcal{G}(B') = -\text{sign}(c_Ac_Bc_A'c_B')$ holds, then $\mathcal{G}(AB) - \mathcal{G}(A) - \mathcal{G}(B) = -\text{sign}(c_Ac_Bc_Ac_B)$ holds as well.

Up to replacing $A$ with $-A$ and $B$ with $-B$ we may assume w.l.o.g. that $c_A, c_B > 0$. For the matrices $A'$ and $B'$ we have $A'B' = -AB$, hence $c_{A'B'} = -c_{AB}$, $c_{A'} = d_A$ and $c_{B'} = a_B$. By the reciprocity formula (3.4), we have

$$\mathcal{G}(B') = \mathcal{G}(B) - \text{sign}(a_Bc_B)$$

and

$$\mathcal{G}(A') = \mathcal{G}(A) - \text{sign}(d_Ac_A).$$

It hence follows that

$$\mathcal{G}(-AB) - \mathcal{G}(A') - \mathcal{G}(B') = \mathcal{G}(AB) - \mathcal{G}(A) - \mathcal{G}(B) + \text{sign}(c_Ad_A) + \text{sign}(a_Bc_B).$$

If $\text{sign}(c_Ac_B) < 0$, then

$$\text{sign}(c_Ad_A) + \text{sign}(a_Bc_B) = \text{sign}(c_A') + \text{sign}(c_{B'}) = 0,$$

thus

$$\mathcal{G}(AB) - \mathcal{G}(A) - \mathcal{G}(B) = \mathcal{G}(A'B') - \mathcal{G}(A') - \mathcal{G}(B') = \text{sign}(c_{A'B'}) = -\text{sign}(c_{AB}).$$

If $\text{sign}(c_Ac_B) > 0$, then

$$\text{sign}(c_Ad_A) + \text{sign}(a_Bc_B) = \text{sign}(c_A') + \text{sign}(c_{B'}) = 2 \text{sign}(c_A') = 2 \text{sign}(c_{B'}).$$

However, as $-c_{A'B'} = c_{AB} = c_Ac_A + c_{B'B}$, we have $\text{sign}(c_{A'B'}) = -\text{sign}(c_A') = -\text{sign}(c_{B'}).$

Thus,

$$\mathcal{G}(AB) - \mathcal{G}(A) - \mathcal{G}(B) = -\text{sign}(c_{A'B'}) - 2 \text{sign}(c_A') = \text{sign}(c_{A'B'}) = -\text{sign}(c_{AB}).$$

$\square$

With Lemma 3.12, we may directly prove Theorem 3.11 for the special case $B = A^{-1}$.

**Proposition 3.13.** For all $A \in \Gamma_\theta$, we have $\mathcal{G}(A^{-1}) = -\mathcal{G}(A)$.

Proof. Since $\mathcal{G}(A)$ is invariant under multiplication on the right by $T^2$ and under multiplication by $-I$, we may assume w.l.o.g. that $A = T^{2n_1}ST^{2n_2}S \ldots T^{2n_r}S$ for integers $n_1, \ldots, n_r$ and $r \geq 1$. The proof is by induction on $r$.

Suppose $r = 1$. Then $\mathcal{G}(T^{2n_1}S) = \mathcal{G}(2n_1) = 0$ and $\mathcal{G}(S^{-1}T^{-2n_1}) = \mathcal{G}(0) = 0$ by Theorem 3.11, so the claim is true.

Suppose that $r > 1$ and the claim holds for $r - 1$. We have

$$\mathcal{G}(A^{-1}) = \mathcal{G}(ST^{-2n_r} \ldots T^{-2n_2}ST^{-2n_1}) = \mathcal{G}(ST^{-2n_r} \ldots T^{-2n_2}S).$$

By the induction hypothesis we have

$$\mathcal{G}((T^{2n_2}S \ldots T^{2n_r}S)^{-1}) = -\mathcal{G}(T^{2n_2}S \ldots T^{2n_r}S),$$

where
which by Lemma 3.12 implies that
\[ \mathcal{S}(T^{2n_2}S \cdots T^{2n_r}S)^{-1}S) = \mathcal{S}(ST^{-2n_r} \cdots T^{-2n_2}S) = -\mathcal{S}(ST^{2n_3}S \cdots T^{2n_r}S). \]

Since \( \mathcal{S}(ST^{2n_3}S \cdots T^{2n_r}S)^{-1} = \mathcal{S}(T^{2n_1}ST^{2n_2}S \cdots T^{2n_r}S) \) by Theorem 3.11 the claim follows. \( \square \)

**Lemma 3.14.** Let \( A = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_\theta \) and write \( A = \pm T^{2n_0}S \cdots T^{2n_r}ST^{2n_{r+1}} \) for integers \( n_0, \ldots, n_{r+1} \) and \( n_1, n_2, \ldots, n_r \neq 0 \). We have \( \frac{a}{c} \in (-1, 1) \) if and only if \( n_0 = 0 \).

**Proof.** By induction on \( r \). For \( r = 0 \), we have \( \frac{a}{c} = T^{2n_0}S. i \infty = 2n_0 \), for which the claim is obviously true. Suppose \( r > 0 \) and that the claim holds for \( 1, \ldots, r-1 \). Let
\[ A = \pm T^{2n_0}ST^{2n_1}S \cdots T^{2n_r}ST^{2n_{r+1}} \] with \( n_1, \ldots, n_r \neq 0 \);
since \( S^2 = -I \), there is always such a representation. Let \( \frac{a'}{c'} = T^{2n_1}S \cdots T^{2n_r}S. i \infty \), which by the induction hypothesis lies outside \((-1, 1)\). Then
\[ \frac{a}{c} = T^{2n_0}ST^{2n_1} \cdots T^{2n_r}S. i \infty = T^{2n_0}S. \frac{a'}{c'} = -\frac{c'}{a'} + 2n_0 \]
and, as \(-\frac{c'}{a'} \in (-1, 1)\), it is easy to see that \( \frac{a}{c} \in (-1, 1) \) if and only if \( n_0 = 0 \). \( \square \)

The next lemma proves Theorem 3.11 under certain extra conditions, to which the general case will be reduced.

**Lemma 3.15.** Let \( A, B \in \Gamma_\theta \) and write \( A = \left( \begin{smallmatrix} a_A & b_A \\ c_A & d_A \end{smallmatrix} \right) \), \( B = \left( \begin{smallmatrix} a_B & b_B \\ c_B & d_B \end{smallmatrix} \right) \), and \( AB = \left( \begin{smallmatrix} a_{AB} & b_{AB} \\ c_{AB} & d_{AB} \end{smallmatrix} \right) \). Suppose that \( -\frac{d_A}{c_A} \in (-1, 1) \), \( \alpha = \frac{a_A}{c_A} \in (-1, 1) \), and \( \beta = \frac{a_B}{c_B} \notin (-1, 1) \).

Then
\[ \mathcal{S}(AB) - \mathcal{S}(A) - \mathcal{S}(B) = -\text{sign}(c_A c_B c_{AB}). \]

**Proof.** Upon replacing \( A \) by \(-A\) and \( B \) by \(-B\), we may assume w.l.o.g. that \( c_A, c_B > 0 \).
Under these conditions \( c_{AB} = c_A a_B + d_A c_B > 0 \) if and only if \( c_A \beta + d_A > 0 \), i.e. \( \beta > -\frac{d_A}{c_A} \).
Note that \( \frac{a_{AB}}{c_{AB}} = A. \beta = \frac{a_A}{c_A} - \frac{1}{c_A(c_A \beta + d_A)} \). Since \( |c_A \beta + d_A| \geq c_A |\beta| - |d_A| > |c_A| - |d_A| \geq 1 \)
by assumption, it follows that
\[ \left| \frac{a_{AB}}{c_{AB}} \right| = \left| \frac{a_A}{c_A} - \frac{1}{c_A(c_A \beta + d_A)} \right| < \left| \frac{a_A}{c_A} \right| + \frac{1}{c_A} \leq 1, \]
since \( |a_A| + 1 \leq |c_A| \).
Consider the geodesic \( h = h(a_{AB}/c_{AB}, \alpha) \). The geodesic \( h \) is the image of \( h(\beta, i \infty) \) under application of \( A \). For the oriented triangle consisting of the edges \( h(i \infty, a_{AB}/c_{AB}), h \) and \( h(\alpha, i \infty) \), we have
\[ I_\theta(h(i \infty, a_{AB}/c_{AB})) + I_\theta(h) + I_\theta(h(\alpha, i \infty)) = 0. \]

Since \( I_\theta(h(i \infty, a_{AB}/c_{AB})) = \mathcal{S}(AB) \) and \( I_\theta(h(\alpha, i \infty)) = -\mathcal{S}(A) \), it suffices to prove that \( I_\theta(h) = -\mathcal{S}(B) + \text{sign}(c_{AB}) \).
By Remark 3.4, the sign of each signed intersection of \( h(\beta, ix) \) with geodesics in \( \mathcal{N}_\theta \) does not change under the application of \( A \) with the exception of the single doubly-oriented geodesics that \( h(\beta, ix) \) intersects (as \( h = A.h(\beta, ix) \) has both endpoints in \((-1,1)\), the geodesic \( h(\beta, ix) \) does not intersect any preimage of a geodesic connecting two integers under \( A \)). For the change of the signed intersection due to \( h(\beta, ix) \) intersecting a doubly-oriented geodesic, we need to introduce a correction factor to \( I_\theta(h(\beta, ix)) \) to find \( I_\theta(h) \).

Thus consider the geodesic \( \tilde{g} \) from \( n_0 \) to \( n_0 + 2 \) with \( n_0 \) an odd integer such that \( n_0 < \beta < n_0 + 2 \), which is the only doubly-oriented geodesic in \( \mathcal{N}_\theta \) that \( h(\beta, ix) \) intersects. Under the matrix \( A \), the geodesic \( \tilde{g} \) is mapped to the geodesic in \( A.\tilde{g} \) connecting \( \frac{a_A n_0 + b_A}{c_A n_0 + d_A} \) and \( \frac{a_A (n_0 + 2) + b_A}{c_A (n_0 + 2) + d_A} \). To find the orientation of the geodesic \( A.\tilde{g} \) in \( \mathcal{N}_\theta \), we need to find out whether \( |c_A n_0 + d_A| < |c_A n_0 + d_A + 2c_A| \) or the contrary holds. Since \( c_A > 0 \), this only depends on whether \( c_A n_0 + d_A \) is negative or not.

Indeed, the linear function \( R \to R \), \( x \mapsto c_A x + d_A \) switches from negative to positive values at \( x = -\frac{d_A}{c_A} \). But this can’t happen in the interval \((n_0, n_0 + 2)\), as \( -\frac{d_A}{c_A} \in (-1,1) \) and \( \beta \in (n_0, n_0 + 2) \) does not lie in \((-1,1)\) by assumption.

Thus, the orientation of \( A.\tilde{g} \) in \( \mathcal{N}_\theta \) only depends on the sign of \( c_A \beta + d_A = c_{AB} \). Namely, as \( A.(n_0 + 2) - A.n_0 = \frac{1}{(c_A(n_0 + 2) + d_A)(c_A n_0 + d_A)} > 0 \), the orientation of \( A.\tilde{g} \) is positive, if \( c_{AB} \) is negative, and vice versa. Note further that \( A.\infty - A.x = \frac{1}{c_A(x + d_A)} \) for any \( x \in R \). Since \( c_A n_0 + d_A \) and \( c_A(n_0 + 2) + d_A \) have the same sign, the numbers \( \alpha - A.n_0 \) and \( \alpha - A.(n_0 + 2) \) have the same sign as well. In other words, \( \alpha \) lies outside the interval bounded by \( A.n_0 \) and \( A.(n_0 + 2) \) or the geodesic \( h \) from \( \frac{a_A b_A}{c_{AB}} \) to \( \alpha \) is oriented outwards of the geodesic \( A.\tilde{g} \).

Now if the orientation is positive, then the geodesic \( h \) intersects \( A.\tilde{g} \) with \( A.\tilde{g} \) pointing to the left and if the orientation is negative, then \( h \) intersects \( A.\tilde{g} \) with \( A.\tilde{g} \) pointing to the right, i.e.

\[
I_\theta(h) = \text{sign}(c_{AB}) + I_\theta(h(\beta, ix)) = \text{sign}(c_{AB}) - \mathcal{G}(B).
\]

All cases that we need to consider to prove Theorem 3.11 are covered in the lemmas and propositions above. We are only left with reducing the general case to them.

**Proof of Theorem 3.11.** Suppose \( A \neq B^{-1} \) otherwise the claim follows by Proposition 3.13.

Write

\[
A = \pm T^{2n_0} ST^{2m_1} S \ldots T^{2n_t} ST^{2m_{t+1}} \text{ and } B = \pm T^{2m_{0}} ST^{2m_1} S \ldots T^{2m_t} ST^{2m_{t+1}}
\]

for \( r, t \geq 0 \) and \( n_1, \ldots, n_r, m_1, \ldots, m_t \neq 0 \). We may assume that \( -\frac{d_A}{c_A} \in (-1,1) \) and \( \frac{d_A}{c_A} \in (-1,1) \) by Lemma 3.13 (i.e. that \( n_0 = n_{r+1} = 0 \)) upon replacing \( A \) with \( T^{-2n_0} AT^{-2m_{r+1}} \) and \( B \) with \( T^{2m_{r+1}} B \). Again, we may assume w.l.o.g. that \( c_A, c_B > 0 \).
If $|B'.\mathcal{I}| > 1$ (i.e. $m_0 \neq 0$ in (3.3) by Lemma 3.14), the claim follows from Lemma 3.15. Suppose thus that $|B'.\mathcal{I}| < 1$, i.e. that $B = \pm ST^{2m_1} S \cdots T^{2m_t} ST^{2m_{t+1}}$. We may now replace $A$ by $A' = AS$ and $B$ by $B' = SB$ as in Lemma 3.12. Again, by replacing $A' = \pm ST^{2m_1} S \cdots ST^{2r}$ by $A'T^{-2r}$ and $B'$ by $T^{2n_r} B'$, we may assume that $-\frac{d_{A'}}{c_{A'}} \in (-1, 1)$ and that $c_{A'}, c_{B'} > 0$. If $|B'.\mathcal{I}| > 1$, i.e. if $m_1 \neq -n_r$, we may use Lemma 3.15. Otherwise, repeat the argument until we reach $|B'.\mathcal{I}| > 1$.

This process will end, for if $r \neq t$, one of the matrices will at some point be equal to the identity, for which the claim is trivial. If $r = t$, we must have $-m_i \neq n_{t+1-i}$ for an $i = 0, \ldots, t + 1$, since we excluded the case $A = B^{-1}$. □

**Corollary 3.16.** Let $\Phi : \Gamma_\theta \rightarrow \mathbb{Z}$ be the classical Dedekind symbol restricted to the subgroup $\Gamma_\theta$. For the map $\mathcal{G} : \Gamma_\theta \rightarrow \mathbb{Z}$ we have

$$\mathcal{G} = \Phi - \chi_\theta,$$

where $\chi_\theta$ is the unique homomorphism $\Gamma_\theta \rightarrow \mathbb{Z}$ given by $\chi_\theta(T^2) = 2$ and $\chi_\theta(S) = 0$.

**Proof.** We have $\Phi(AB) - \Phi(A) - \Phi(B) = -\text{sign}(c_A c_B c_{AB})$ by (3.3). Hence, $\chi_\theta = \Phi - \mathcal{G}$ is a group homomorphism by Theorem 3.11. The values of $\chi_\theta$ at the generators of $\Gamma_\theta$ are obtained by $\Phi(T^2) = 2$, $\Phi(S) = 0$ and $\mathcal{G}(T^2) = 0$, $\mathcal{G}(S) = 0$. □

**Remark 3.17.** The same arguments work for the function $\mathcal{G}_4$ as well. In particular, the function $\mathcal{G}_4$ satisfies the generalized reciprocity law in Theorem 3.11

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