Complexity of Bondage and Reinforcement*

Fu-Tao Hu, Jun-Ming Xu†
Department of Mathematics
University of Science and Technology of China
Hefei, 230026, China

Abstract

Let $G = (V, E)$ be a graph. A subset $D \subseteq V$ is a dominating set if every vertex not in $D$ is adjacent to a vertex in $D$. A dominating set $D$ is called a total dominating set if every vertex in $D$ is adjacent to a vertex in $D$. The domination (resp. total domination) number of $G$ is the smallest cardinality of a dominating (resp. total dominating) set of $G$. The bondage (resp. total bondage) number of a nonempty graph $G$ is the smallest number of edges whose removal from $G$ results in a graph with larger domination (resp. total domination) number of $G$. The reinforcement number of $G$ is the smallest number of edges whose addition to $G$ results in a graph with smaller domination number. This paper shows that the decision problems for bondage, total bondage and reinforcement are all NP-hard.

Key words: Complexity; NP-completeness; NP-hardness; Domination; Bondage; Total bondage; Reinforcement

AMS Subject Classification (2000): 05C69

*The work was supported by NNSF of China (No.10671191).
† Correspondence to: J.-M. Xu; e-mail: xujm@ustc.edu.cn
1 Introduction

In this paper, we follow Xu [17] for graph-theoretical terminology and notation. A graph $G = (V, E)$ always means a finite, undirected and simple graph, where $V = V(G)$ is the vertex-set and $E = E(G)$ is the edge-set of $G$.

A subset $D \subseteq V$ is a dominating set of $G$ if every vertex not in $D$ is adjacent to a vertex in $D$. The domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of $G$. A dominating set $D$ is called a $\gamma$-set of $G$ if $|D| = \gamma(G)$. The bondage number of $G$, denoted by $b(G)$, is the minimum number of edges whose removal from $G$ results in a graph with larger domination number of $G$. The reinforcement number of $G$, denoted by $r(G)$, is the smallest number of edges whose addition to $G$ results in a graph with smaller domination number of $G$. Domination is a classical concept in graph theory. The bondage number and the reinforcement number were introduced by Fink et at. [3] and Kok, Mynhardt [12], respectively, in 1990. The reinforcement number for digraphs has been studies by Huang, Wang and Xu [11]. Domination as well as related topics is now well studied in graph theory. The literature on these subjects have been surveyed and detailed in the two excellent domination books by Haynes, Hedetniemi, and Slater [7, 8].

Theory of domination has been applied in many research fields. For different applications, many variations of dominations were proposed in the research literature by adding some restricted conditions to dominating sets, for example, the total domination and the restrained domination.

A dominating set $D$ is called a total dominating set if every vertex in $D$ is adjacent to another vertex in $D$. The total domination number, denoted by $\gamma_t(G)$, of $G$ is the minimum cardinality of a total dominating set of $G$. Use the symbol $D_t$ to denote a total dominating set. A total dominating set $D_t$ is called a $\gamma_t$-set of $G$ if $|D_t| = \gamma_t(G)$. The total bondage number of $G$, denoted by $b_t(G)$, is the minimum number of edges whose removal from $G$ results in a graph with larger total domination number of $G$. The total domination was introduced by Cockayne et al. [1]. Total domination in graphs
has been extensively studied in the literature. A survey of selected recent results on
total domination in Henning [9]. The total bondage number of a graph was first studied
by Kulli and Patwari [13] and further studied by Sridharan, Elias, Subramanian [15],
Huang and Xu [10].

Analogously, a dominating set \( D \) is called a *restrained dominating set* if every vertex
not in \( D \) is adjacent to another vertex not in \( D \). The *restrained domination number*,
denoted by \( \gamma_r(G) \), of \( G \) is the minimum cardinality of a total dominating set of \( G \). The
*restrained bondage number* of \( G \), denoted by \( b_r(G) \), is the minimum number of edges
whose removal from \( G \) results in a graph with larger restrained domination number of
\( G \). The restrained domination was introduced by Telle and Proskurowski [16], and the
restrained bondage number was defined by Hattingh and Plummer [6].

Why is a graph-theoretical parameter is proposed at once is to determine the
exact value of this parameter for all graphs. However, the problem determining dom-
ination for general graphs has been proved to be NP-complete (see GT2 in Appendix
in Garey and Johnson [4]); the problems determining total domination and restrained
domination for general graphs have been also proved to be NP-complete by Laskar et
al. [14], and by Domke et al. [2], respectively.

As regards the bondage problem, Hattingh et al. [6] showed that the restrained
bondage problem is NP-complete even for bipartite graphs. For the general bondage
problem, from the algorithmic point of view, Hartnell et al. [5] designed a linear time
algorithm to compute the bondage number of a tree. However, the complexity of this
problem is still unknown for other classes of graphs.

In this paper, we will show that the decision problems for bondage, total bondage
and reinforcement are all NP-hard. Their proofs are Section 3, Section 4 and Section
5 in this paper, respectively.
2 3-satisfiability problem

Following Garey and Johnson’s techniques for proving NP-hardness [1], we prove our results by describing a polynomial transformation from the known NP-complete problem: 3-satisfiability problem. To state the 3-satisfiability problem, we, in this section, recall some terms we will use in describing it.

Let \( U \) be a set of Boolean variables. A truth assignment for \( U \) is a mapping \( t : U \to \{T, F\} \). If \( t(u) = T \), then \( u \) is said to be “true” under \( t \); if \( t(u) = F \), then \( u \) is said to be “false” under \( t \). If \( u \) is a variable in \( U \), then \( u \) and \( \bar{u} \) are literals over \( U \). The literal \( u \) is true under \( t \) if and only if the variable \( u \) is true under \( t \); the literal \( \bar{u} \) is true if and only if the variable \( u \) is false.

A clause over \( U \) is a set of literals over \( U \). It represents the disjunction of these literals and is satisfied by a truth assignment if and only if at least one of its members is true under that assignment. A collection \( \mathcal{C} \) of clauses over \( U \) is satisfiable if and only if there exists some truth assignment for \( U \) that simultaneously satisfies all the clauses in \( \mathcal{C} \). Such a truth assignment is called a satisfying truth assignment for \( \mathcal{C} \).

The 3-satisfiability problem is specified as follows.

3-satisfiability problem:

**Instance:** A collection \( \mathcal{C} = \{C_1, C_2, \ldots, C_m\} \) of clauses over a finite set \( U \) of variables such that \( |C_j| = 3 \) for \( j = 1, 2, \ldots, m \).

**Question:** Is there a truth assignment for \( U \) that satisfies all the clauses in \( \mathcal{C} \)?

**Theorem 2.1** (Theorem 3.1 in [1]) The 3-satisfiability problem is NP-complete.

3 NP-hardness of bondage

In this section, we will show that the problem determining the bondage numbers of general graphs is NP-hard. We first state the problem as the following decision problem.
Bondage problem:

Instance: A nonempty graph $G$ and a positive integer $k$.

Question: Is $b(G) \leq k$?

**Theorem 3.1** The bondage problem is NP-hard.

**Proof.** We show the NP-hardness of the bondage problem by transforming the 3-satisfiability problem to it in polynomial time.

Let $U = \{u_1, u_2, \ldots, u_n\}$ and $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ be an arbitrary instance of the 3-satisfiability problem. We will construct a graph $G$ and a positive integer $k$ such that $\mathcal{C}$ is satisfiable if and only if $b(G) \leq k$. Such a graph $G$ can be constructed as follows.

For each $i = 1, 2, \ldots, n$, corresponding to the variable $u_i \in U$, associate a triangle $T_i$ with vertex-set $\{u_i, \bar{u}_i, v_i\}$. For each $j = 1, 2, \ldots, m$, corresponding to the clause $C_j = \{x_j, y_j, z_j\} \in \mathcal{C}$, associate a single vertex $c_j$ and add edge-set $E_j = \{c_jx_j, c_jy_j, c_jz_j\}$. Finally, add a path $P = s_1s_2s_3$, join $s_1$ and $s_3$ to each vertex $c_j$ with $1 \leq j \leq m$ and set $k = 1$.

Figure 1 shows an example of the graph obtained when $U = \{u_1, u_2, u_3, u_4\}$ and $\mathcal{C} = \{C_1, C_2, C_3\}$, where $C_1 = \{u_1, u_2, \bar{u}_3\}, C_2 = \{\bar{u}_1, u_2, u_4\}, C_3 = \{\bar{u}_2, u_3, u_4\}$.

To prove that this is indeed a transformation, we must show that $b(G) = 1$ if and only if there is a truth assignment for $U$ that satisfies all the clauses in $\mathcal{C}$. This aim can be obtained by proving the following four claims.

**Claim 3.1** $\gamma(G) \geq n + 1$. Moreover, if $\gamma(G) = n + 1$, then for any $\gamma$-set $D$ in $G$, $D \cap V(P) = \{s_2\}$ and $|D \cap V(T_i)| = 1$ for each $i = 1, 2, \ldots, n$, while $c_j \notin D$ for each $j = 1, 2, \ldots, m$.

**Proof.** Let $D$ be a $\gamma$-set of $G$. By the construction of $G$, the vertex $s_2$ can be dominated only by vertices in $P$, which implies $|D \cap V(P)| \geq 1$; for each $i = 1, 2, \ldots, n$, the vertex $v_i$ can be dominated only by vertices in $T_i$, which implies $|D \cap V(T_i)| \geq 1$. It follows that $\gamma(G) = |D| \geq n + 1$. 
Claim 3.2 \( \gamma(G) = n + 1 \) if and only if \( \mathcal{C} \) is satisfiable.

**Proof.** Suppose that \( \gamma(G) = n + 1 \) and let \( D \) be a \( \gamma \)-set of \( G \). By Claim 3.1, for each \( i = 1, 2, \ldots, n \), \( |D \cap V(T_i)| = 1 \), it follows that \( D \cap V(T_i) = \{ u_i \} \) or \( D \cap V(T_i) = \{ \bar{u}_i \} \) or \( D \cap V(T_i) = \{ v_i \} \). Define a mapping \( t : U \to \{ T, F \} \) by

\[
t(u_i) = \begin{cases} 
T & \text{if } u_i \in D \text{ or } v_i \in D, \\
F & \text{if } \bar{u}_i \in D,
\end{cases} \quad i = 1, 2, \ldots, n. \tag{3.1}
\]

We will show that \( t \) is a satisfying truth assignment for \( \mathcal{C} \). It is sufficient to show that every clause in \( \mathcal{C} \) is satisfied by \( t \). To this end, we arbitrarily choose a clause \( C_j \in \mathcal{C} \) with \( 1 \leq j \leq m \). Since the corresponding vertex \( c_j \) in \( G \) is adjacent to neither \( s_2 \) nor \( v_i \) for any \( i \) with \( 1 \leq i \leq n \), there exists some \( i \) with \( 1 \leq i \leq n \)
such that $c_j$ is dominated by $u_i \in D$ or $\bar{u}_i \in D$. Suppose that $c_j$ is dominated by $u_i \in D$. Since $u_i$ is adjacent to $c_j$ in $G$, the literal $u_i$ is in the clause $C_j$ by the construction of $G$. Since $u_i \in D$, it follows that $t(u_i) = T$ by (3.1), which implies that the clause $C_j$ is satisfied by $t$. Suppose that $c_j$ is dominated by $\bar{u}_i \in D$. Since $\bar{u}_i$ is adjacent to $c_j$ in $G$, the literal $\bar{u}_i$ is in the clause $C_j$. Since $\bar{u}_i \in D$, it follows that $t(u_i) = F$ by (3.1). Thus, $t$ assigns $u_i$ the truth value $T$, that is, $t$ satisfies the clause $C_j$. By the arbitrariness of $j$ with $1 \leq j \leq m$, we show that $t$ satisfies all the clauses in $\mathcal{C}$, that is, $\mathcal{C}$ is satisfiable.

Conversely, suppose that $\mathcal{C}$ is satisfiable, and let $t : U \rightarrow \{T, F\}$ be a satisfying truth assignment for $\mathcal{C}$. Construct a subset $D' \subseteq V(G)$ as follows. If $t(u_i) = T$, then put the vertex $u_i$ in $D'$; if $t(u_i) = F$, then put the vertex $\bar{u}_i$ in $D'$. Clearly, $|D'| = n$. Since $t$ is a satisfying truth assignment for $\mathcal{C}$, for each $j = 1, 2, \ldots, m$, at least one of literals in $C_j$ is true under the assignment $t$. It follows that the corresponding vertex $c_j$ in $G$ is adjacent to at least one vertex in $D'$ since $c_j$ is adjacent to each literal in $C_j$ by the construction of $G$. Thus $D' \cup \{s_2\}$ is a dominating set of $G$, and so $\gamma(G) \leq |D' \cup \{s_2\}| = n + 1$. By Claim 3.1, $\gamma(G) \geq n + 1$, and so $\gamma(G) = n + 1$.

**Claim 3.3** $\gamma(G - e) \leq n + 2$ for any $e \in E(G)$.

**Proof.** Let $E_1 = \{s_2s_3, s_1c_j, u_i\bar{u}_i, u_iv_i : i = 1, 2, \ldots, n; j = 1, 2, \ldots, m\}$ (induced by heavy edges in Figure I) and let $E_2 = E(G) \setminus E_1$. Assume $e \in E_2$. Let $D' = \{u_1, u_2, \ldots, u_n, s_1, s_2\}$. Clearly, $D'$ is a dominating set of $G - e$ since every vertex not in $D'$ is incident with some vertex in $D'$ via an edge in $E_1$. Hence, $\gamma(G - e) \leq |D'| = n + 2$. Now assume $e \in E_1$. Let $D'' = \{u_1, u_2, \ldots, u_n, s_2, s_3\}$. If $e$ is either $s_2s_3$ or incident with the vertex $s_1$, then $D''$ is a dominating set of $G - e$, clearly. If $e$ is either $u_i\bar{u}_i$ or $u_iv_i$ for some $i$ ($1 \leq i \leq n$), then we use the vertex either $v_i$ or $\bar{u}_i$ instead of $u_i$ in $D''$ to obtain $D'''$; and hence $D'''$ is a dominating set of $G - e$. These facts imply that $\gamma(G - e) \leq n + 2$. 


Claim 3.4 \( \gamma(G) = n + 1 \) if and only if \( b(G) = 1 \).

**Proof.** Assume \( \gamma(G) = n + 1 \) and consider the edge \( e = s_1s_2 \). Suppose \( \gamma(G) = \gamma(G - e) \). Let \( D' \) be a \( \gamma \)-set in \( G - e \). It is clear that \( D' \) is also a \( \gamma \)-set of \( G \). By Claim 3.1 we have \( c_j \notin D' \) for each \( j = 1, 2, \ldots, m \) and \( D' \cap V(P) = \{ s_2 \} \). But then \( s_1 \) is not dominated by \( D' \), a contradiction. Hence, \( \gamma(G) < \gamma(G - e) \), and so \( b(G) = 1 \).

Now, assume \( b(G) = 1 \). By Claim 3.1, we have that \( \gamma(G) \geq n + 1 \). Let \( e' \) be an edge such that \( \gamma(G) < \gamma(G - e') \). By Claim 3.3, we have that \( \gamma(G - e') \leq n + 2 \). Thus, \( n + 1 \leq \gamma(G) < \gamma(G - e') \leq n + 2 \), which yields \( \gamma(G) = n + 1 \).

By Claim 3.2 and Claim 3.4, we prove that \( b(G) = 1 \) if and only if there is a truth assignment for \( U \) that satisfies all the clauses in \( \mathcal{C} \). Since the construction of the bondage instance is straightforward from a 3-satisfiability instance, the size of the bondage instance is bounded above by a polynomial function of the size of 3-satisfiability instance. It follows that this is a polynomial transformation.

The theorem follows.

### 4 NP-hardness of total bondage

In this section, we will show that the problem determining the total bondage numbers of general graphs is NP-hard. We first state it as the following decision problem.

**Total bondage problem:**

**Instance:** A nonempty graph \( G \) and a positive integer \( k \).

**Question:** Is \( b_t(G) \leq k \)?

**Theorem 4.1** The total bondage problem is NP-hard.

**Proof.** We show the NP-hardness of the total bondage problem by reducing the 3-satisfiability problem to it in polynomial time.
Let $U = \{u_1, u_2, \ldots, u_n\}$ and $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ be an arbitrary instance of the 3-satisfiability problem. We will construct a graph $G$ and an integer $k$ such that $\mathcal{C}$ is satisfiable if and only if $b_t(G) \leq k$. Such a graph $G$ can be constructed as follows.

For each $i = 1, 2, \ldots, n$, corresponding to the variable $u_i \in U$, associate a graph $H_i$ with vertex-set $V(H_i) = \{u_i, \bar{u}_i, v_i, v'_i\}$ and edge-set $E(H_i) = \{v_iu_i, u_i\bar{u}_i, \bar{u}_iv_i, v_iv'_i\}$. For each $j = 1, 2, \ldots, m$, corresponding to the clause $C_j = \{x_j, y_j, z_j\} \in \mathcal{C}$, associate a single vertex $c_j$ and add edge-set $E_j = \{c_jx_j, c_jy_j, c_jz_j\}$, $1 \leq j \leq m$. Finally, add a graph $H$ with vertex-set $V(H) = \{s_1, s_2, s_3, s_4, s_5\}$ and edge-set $E(H) = \{s_1s_2, s_1s_4, s_2s_3, s_2s_4, s_4s_5\}$, join $s_1$ and $s_3$ to each vertex $c_j$, $1 \leq j \leq m$ and set $k = 1$.

Figure 2 shows an example of the graph obtained when $U = \{u_1, u_2, u_3, u_4\}$ and $\mathcal{C} = \{C_1, C_2, C_3\}$, where $C_1 = \{u_1, u_2, \bar{u}_3\}$, $C_2 = \{\bar{u}_1, u_2, u_4\}$ and $C_3 = \{\bar{u}_2, u_3, u_4\}$.

![Figure 2](image)

**Figure 2:** An instance of the total bondage problem resulting from an instance of the 3-satisfiability problem, in which $U = \{u_1, u_2, u_3, u_4\}$ and $\mathcal{C} = \{C_1, C_2, C_3\}$, where $C_1 = \{u_1, u_2, \bar{u}_3\}$, $C_2 = \{\bar{u}_1, u_2, u_4\}$ and $C_3 = \{\bar{u}_2, u_3, u_4\}$. Here $k = 1$ and $\gamma_t = 10$, where the set of bold points is a $\gamma_t$-set.

It is easy to see that the construction can be accomplished in polynomial time. All that remains to be shown is that $\mathcal{C}$ is satisfiable if and only if $b_t(G) = 1$. This aim can be obtained by proving the following four claims.
Claim 4.1 \( \gamma_t(G) \geq 2n + 2 \). For any \( \gamma_t \)-set \( D_t \) of \( G \), \( s_4 \in D_t \) and \( v_i \in D_t \) for each \( i = 1, 2, \ldots, n \). Moreover, if \( \gamma_t(G) = 2n + 2 \), then \( D_t \cap V(H) = \{s_2, s_4\} \) and \( |D_t \cap V(H_i)| = 2 \) for each \( i = 1, 2, \ldots, n \), while \( c_j \notin D_t \) for each \( j = 1, 2, \ldots, m \).

**Proof.** Let \( D_t \) be a \( \gamma_t \)-set of \( G \). By the construction of \( G \), it is clear that \( v_i \) is certainly in \( D_t \) to dominate \( v'_i \), and \( v_i \) can be dominated only by another vertex in \( H_i \). It follows that \( v_i \in D_t \) and \( |D_t \cap V(H_i)| \geq 2 \) for each \( i = 1, 2, \ldots, n \). It is also clear that \( s_4 \) is certainly in \( D_t \) to dominate \( s_5 \), and \( s_4 \) can be dominated only by another vertex in \( H \). This fact implies that \( s_4 \in D_t \) and \( |D_t \cap V(H)| \geq 2 \). Thus, \( \gamma_t(G) = |D_t| \geq 2n + 2 \).

Suppose that \( \gamma_t(G) = 2n + 2 \). Then \( |D_t \cap V(H_i)| = 2 \) for each \( i = 1, 2, \ldots, n \), and \( |D_t \cap V(H)| = 2 \). Consequently, \( c_j \notin D_t \) for each \( j = 1, 2, \ldots, m \). As a result, \( s_3 \) can be dominated only by the vertex \( s_2 \) in \( S \), that is, \( s_2 \in D_t \). Noting \( s_4 \in D_t \) and \( |D_t \cap V(H)| = 2 \), we have \( D_t \cap V(H) = \{s_2, s_4\} \).

Claim 4.2 \( \gamma_t(G) = 2n + 2 \) if and only if \( \mathcal{C} \) is satisfiable.

**Proof.** Suppose that \( \gamma_t(G) = 2n + 2 \) and let \( D_t \) be a \( \gamma_t \)-set of \( G \). By Claim 4.1, \( D_t \cap V(H) = \{s_2, s_4\} \) and for each \( i = 1, 2, \ldots, n \), \( |D_t \cap V(H_i)| = 2 \), it follows that \( D_t \cap V(H_i) = \{u_i, v_i\} \) or \( \{\bar{u}_i, v_i\} \) or \( \{v_i, v'_i\} \). Define a mapping \( t : U \to \{T, F\} \) by

\[
 t(u_i) = \begin{cases} 
 T & \text{if } u_i \in D_t \text{ or } v'_i \in D_t, \\
 F & \text{if } \bar{u}_i \in D_t, 
\end{cases} \quad i = 1, 2, \ldots, n. \tag{4.1}
\]

We will show that \( t \) is a satisfying truth assignment for \( \mathcal{C} \). It is sufficient to show that \( t \) satisfies every clause in \( \mathcal{C} \). To this end, we arbitrarily choose a clause \( C_j \in \mathcal{C} \). Since the corresponding vertex \( c_j \) is not adjacent to any member of \( \{s_2, s_4\} \cup \{v_i, v'_i : 1 \leq i \leq n\} \), there exists some \( i \) with \( 1 \leq i \leq n \) such that \( c_j \) is dominated by \( u_i \in D_t \) or \( \bar{u}_i \in D_t \).

Suppose that \( c_j \) is dominated by \( u_i \in D_t \). Then \( u_i \) is adjacent to \( c_j \) in \( G \), that is, the literal \( u_i \) is in the clause \( C_j \) by the construction of \( G \). Since \( u_i \in D_t \), we have \( t(u_i) = T \) by (4.1), which implies that \( t \) satisfies the clause \( C_j \).
Suppose that $c_j$ is dominated by $u_i \in D_t$. Then $u_i$ is adjacent to $c_j$ in $G$, that is, the literal $u_i$ is in the clause $C_j$. Since $u_i \in D_t$, we have $t(u_i) = F$ by (1.1), which implies that $u_i$ is assigned the truth value $T$ by $t$, so the clause $C_j$ is satisfied by $t$.

The arbitrariness of $j$ with $1 \leq j \leq m$ shows that all the clauses in $\mathcal{C}$ is satisfied, that is, $\mathcal{C}$ is satisfiable.

Conversely, suppose that $\mathcal{C}$ is satisfiable, and let $t : U \rightarrow \{T, F\}$ be a satisfying truth assignment for $\mathcal{C}$. Construct a subset $D' \subseteq V(G)$ as follows. If $t(u_i) = T$, then put the vertex $u_i$ in $D'$; if $t(u_i) = F$, then put the vertex $\bar{u}_i$ in $D'$. Clearly, $|D'| = n$. Since $t$ is a satisfying truth assignment for $\mathcal{C}$, for each $j = 1, 2, \ldots, m$, at least one of literals in $C_j$ is true under the assignment $t$. It follows that the corresponding vertex $c_j$ in $G$ is adjacent to at least one vertex in $D'$ since $c_j$ is adjacent to each literal in $C_j$ by the construction of $G$. Let $D'_t = D' \cup \{s_2, s_4, v_1, \ldots, v_n\}$. Clearly, $D'_t$ is a dominating set of $G$ and $|D'_t| = 2n + 2$. Since $s_2$ and $s_4$ are dominated by each other, $u_i$ and $\bar{u}_i$ are dominated by $v_i \in D'_t$ for each $i = 1, 2, \ldots, n$. $D'_t$ is also a dominating set of $G$. Hence, $\gamma_t(G) \leq |D'_t| = 2n + 2$. By Claim 4.1, $\gamma(G) \geq 2n + 2$. Therefore, $\gamma_t(G) = 2n + 2$.  

**Claim 4.3** For any $e \in E(G)$, $\gamma_t(G - e) \leq 2n + 3$.

**Proof.** We first assume $e = s_2s_3$ or $e = v_i\bar{u}_i$ for some $i$ with $1 \leq i \leq n$, and let $D'_t = (\cup_{i=1}^{n}\{u_i, v_i\}) \cup \{c_1, s_1, s_4\}$. It is easy to see that $D'_t$ is a dominating set of $G - e$. Secondly, assume $e = s_1c_j$ for some $j$ with $1 \leq j \leq m$, and let $D'_t = (\cup_{i=1}^{n}\{u_i, v_i\}) \cup \{s_2, s_3, s_4\}$. Then $D'_t$ is a dominating set of $G - e$. Otherwise, let $D'_t = (\cup_{i=1}^{n}\{v_i, \bar{u}_i\}) \cup \{s_1, s_2, s_4\}$. Then $D'_t$ is a dominating set of $G - e$. Hence, $\gamma_t(G - e) \leq |D'_t| = 2n + 3$. 

**Claim 4.4** $\gamma_t(G) = 2n + 2$ if and only if $b_t(G) = 1$.

**Proof.** Assume $\gamma_t(G) = 2n + 2$ and take $e = s_2s_4$. Suppose that $\gamma_t(G - e) = \gamma_t(G)$. Let $D'_t$ be a $\gamma_t$-set of $G - e$. As $D'_t$ is also a $\gamma_t$-set of $G$, by Claim 4.1 we
have $c_j \notin D'_t$ for every $j$ and $D'_t \cap V(H) = \{s_2, s_4\}$, which contradicts the fact that $s_2$ and $s_4$ could not be dominated by each other in $G-e$. This contradiction shows that $\gamma_t(G-e) > \gamma_t(G)$, whence $b_t(G) = 1$.

Now, assume $b_t(G) = 1$. By Claim 4.1, we have that $\gamma_t(G) \geq 2n + 2$. Let $e'$ be an edge such that $\gamma_t(G-e') > \gamma_t(G)$. By Claim 4.3, we have that $\gamma_t(G-e) \leq 2n + 3$. Thus, $2n + 2 \leq \gamma_t(G) < \gamma_t(G-e') \leq 2n + 3$, which yields $\gamma_t(G) = 2n + 2$.

It follows from Claim 4.2 and Claim 4.4 that $b_t(G) = 1$ if and only if $\mathcal{C}$ is satisfiable.

The theorem follows.

5 NP-hardness of reinforcement

In this section, we will show that the problem determining the reinforcements of general graphs is NP-hard. We first state it as the following decision problem.

Reinforcement problem:

Instance: A graph $G$ and a positive integer $k$.

Question: Is $r(G) \leq k$?

Theorem 5.1 The reinforcement problem is NP-hard.

Proof. The reinforcement problem is clearly in NP. In the following, we show the NP-hardness of the reinforcement problem by reducing the 3-satisfiability problem to it in polynomial time.

Let $U = \{u_1, u_2, \ldots, u_n\}$ and $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ be an arbitrary instance of the 3-satisfiability problem. We will construct a graph $G$ and an integer $k$ such that $\mathcal{C}$ is satisfiable if and only if $r(G) \leq k$. Such a graph $G$ can be constructed as follows.

For each $i = 1, 2, \ldots, n$, corresponding to the variable $u_i \in U$, associate a triangle $T_i$ with vertex-set $\{u_i, \bar{u}_i, v_i\}$. For each $j = 1, 2, \ldots, m$, corresponding to the clause $C_j = \{x_j, y_j, z_j\}$, associate a single vertex $c_j$ and add edges $(c_j, x_j), (c_j, y_j)$ and $(c_j, z_j)$, $1 \leq j \leq m$. Finally, add a vertex $s$ and join $s$ to every vertex $c_j$ and set $k = 1$. 


Figure 3: An instance of the reinforcement problem resulting from an instance of the 3-satisfiability problem, in which $U = \{u_1, u_2, u_3, u_4\}$ and $C = \{\{u_1, u_2, \bar{u}_3\}, \{\bar{u}_1, u_2, u_4\}, \{\bar{u}_2, u_3, u_4\}\}$. Here $k = 1$ and $\gamma = 5$, where the set of bold points is a $\gamma$-set.

Figure 3 shows an example of the graph obtained when $U = \{u_1, u_2, u_3, u_4\}$ and $C = \{C_1, C_2, C_3\}$, where $C_1 = \{u_1, u_2, \bar{u}_3\}$, $C_2 = \{\bar{u}_1, u_2, u_4\}$, $C_3 = \{\bar{u}_2, u_3, u_4\}$.

It is easy to see that the construction can be accomplished in polynomial time. All that remains to be shown is that $C$ is satisfiable if and only if $r(G) = 1$. To this aim, we first prove the following two claims.

Claim 5.1 $\gamma(G) = n + 1$.

**Proof.** Use the symbol $N[s]$ to denote the closed-neighborhood of $s$ in $G$, that is, $N[s] = \{u \in V(G) : us \in E\} \cup \{s\}$. On the one hand, let $D$ be a $\gamma$-set of $G$, then $\gamma(G) = |D| \geq n + 1$ since $|D \cap V(T_i)| \geq 1$ and $|D \cap N[s]| \geq 1$. On the other hand, $D' = \{s, u_1, u_2, \ldots, u_n\}$ is a dominating set of $G$, which implies that $\gamma(G) \leq |D'| = n + 1$. It follows that $\gamma(G) = n + 1$. 

Claim 5.2 If there exists an edge $e \in E(G)$ such that $\gamma(G + e) = n$, and let $D_e$ be a $\gamma$-set of $G + e$, then $|D_e \cap V(T_i)| = 1$ for each $i = 1, 2, \ldots, n$, while $c_j \notin D_e$ for each $j = 1, 2, \ldots, m$.

**Proof.** Suppose to the contrary that $|D_e \cap V(T_{i_0})| = 0$ for some $i_0$ with $1 \leq i_0 \leq n$. Then one end-vertex of the edge $e$ should be $v_{i_0}$ since $D_e$ dominates it via the
edge \( e \) in \( G + e \), and for every \( i \neq i_0 \), \( |D_e \cap V(T_i)| \geq 1 \) since \( D_e \) dominates \( v_i \).

By the hypotheses, two literals \( u_{i_0} \) and \( \bar{u}_{i_0} \) do not simultaneously appear in the same clause in \( \mathcal{C} \), they are not incident with the same vertex \( c_j \) in \( G \) for some \( j \). Since \( u_{i_0} \) and \( \bar{u}_{i_0} \) should be dominated by \( D_e \), there exist two distinct vertices \( c_j, c_l \in D_e \) such that \( c_j \) dominates \( u_{i_0} \) and \( c_l \) dominates \( \bar{u}_{i_0} \). Thus, \( |D_e| \geq n + 1 \), a contradiction. Hence, \( |D_e \cap V(T_i)| = 1 \) for each \( i = 1, 2, \ldots, n \), and \( c_j \notin D_e \) for every \( j \) since \( |D_e| = n \).

We now show that \( \mathcal{C} \) is satisfiable if and only if \( r(G) = 1 \).

Suppose that \( \mathcal{C} \) is satisfiable, and let \( t : U \to \{ T, F \} \) be a satisfying truth assignment for \( \mathcal{C} \). We construct a subset \( D' \subseteq V(G) \) as follows. If \( t(u_i) = T \) then put the vertex \( u_i \) in \( D' \); if \( t(u_i) = F \) then put the vertex \( \bar{u}_i \) in \( D' \). Then \( |D'| = n \). Since \( t \) is a satisfying truth assignment for \( \mathcal{C} \), for each \( j = 1, 2, \ldots, m \), at least one of literals in \( C_j \) is true under the assignment \( t \). It follows that the corresponding vertex \( c_j \) in \( G \) is adjacent to at least one vertex in \( D' \) since \( c_j \) is adjacent to each literal in \( C_j \) by the construction of \( G \). Without loss of generality let \( t(u_1) = T \), then \( D' \) is a dominating set of \( G + su_1 \), and hence \( \gamma(G + su_1) \leq |D'| = n \). By Claim 5.1, we have \( \gamma(G) = n + 1 \). It follows that \( \gamma(G + su_1) \leq n < n + 1 = \gamma(G) \), which implies \( r(G) = 1 \).

Conversely, assume \( r(G) = 1 \). Then there exists an edge \( e \) in \( G \) such that \( \gamma(G + e) = n \). Let \( D_e \) be a \( \gamma \)-set of \( G + e \). By Claim 5.2, \( |D_e \cap V(T_i)| = 1 \) for each \( i = 1, 2, \ldots, n \), and \( c_j \notin D_e \) for each \( j = 1, 2, \ldots, m \). Define \( t : U \to \{ T, F \} \) by

\[
t(u_i) = \begin{cases} 
T & \text{if } u_i \in D_e \text{ or } v_i \in D_e, \\
F & \text{if } \bar{u}_i \in D_e , 
\end{cases} \quad i = 1, 2, \ldots, n. \tag{5.1}
\]

We will show that \( t \) is a satisfying truth assignment for \( \mathcal{C} \). It is sufficient to show that every clause in \( \mathcal{C} \) is satisfied by \( t \).

Consider arbitrary clause \( C_j \in \mathcal{C} \) with \( 1 \leq j \leq m \). By Claim 5.2, the corresponding vertex \( c_j \) in \( G \) is dominated by \( u_i \) or \( \bar{u}_i \) in \( D_e \) for some \( i \). Suppose that \( c_j \) is dominated by \( u_i \in D_e \). Then \( u_i \) is adjacent to \( c_j \) in \( G \), that is, the literal \( u_i \) is in the clause \( C_j \) by the construction of \( G \). Since \( u_i \in D_e \), we have \( t(u_i) = T \) by (5.1), which implies that
$C_j$ is satisfied by $t$. Suppose that $c_j$ is dominated by $\bar{u}_i \in D_e$. Then $\bar{u}_i$ is adjacent to $c_j$ in $G$, that is, the literal $\bar{u}_i$ is in the clause $C_j$. Since $\bar{u}_i \in D_e$, we have $t(u_i) = F$ by (5.1), which implies that $\bar{u}_i$ is assigned the truth value $T$ by $t$, so the clause $C_j$ is satisfied. The arbitrariness of $j$ with $1 \leq j \leq m$ shows that all the clauses in $\mathcal{C}$ is satisfied by $t$, that is, $\mathcal{C}$ is satisfiable.

The theorem follows.

References

[1] E. J. Cockayne, R.M. Dawes, S.T. Hedetniemi, Total domination in graphs. Networks, 10 (1980) 211-219.

[2] G. S. Domke, J. H. Hattingh, S. T. Hedetniemi, R. C. Laskar, L. R. Markus, Restrained domination in graphs. Discrete Mathematics, 203 (1999) 61-69.

[3] J. F. Fink, M. S. Jacobson, L. F. Kinch, J. Roberts, The bondage number of a graph. Discrete Mathematics, 86 (1990) 47-57.

[4] M. R. Garey, D. S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, Freeman, San Francisco, 1979.

[5] B. L. Hartnell, L. K. Jorgensen, P. D. Vestergaard and C. Whitehead, Edge stability of the $k$-domination number of trees. Bulletin of the ICA, 22 (1998), 31-40.

[6] J. H. Hattingh, A. R. Plummer, Restrained bondage in graphs. Discrete Mathematics, 308 (2008), 5446-5453.

[7] T. W. Haynes, S. T. Hedetniemi, P. J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, New York, 1997.

[8] T. W. Haynes, S. T. Hedetniemi, P. J. Slater, Domination in Graphs: Advanced Topics, Marcel Dekker, New York, 1997.

[9] M. A. Henning, A survey of selected recent results on total domination in graphs. Discrete Mathematics, 309(1) (2009), 32-63.
[10] J. Huang and J.-M. Xu, The total domination and bondage numbers of extended de
bruijn and Kautz digraphs. *Computer and Mathematics with Applications*, 53(8)
(2007), 1206-1213.

[11] J. Huang, J.-W. Wang and J.-M. Xu, Reinforcement numbers of digraphs. *Discrete
Applied Mathematics*, 157(8) (2009), 1938-1946.

[12] J. Kok and C. M. Mynhardt, Reinforcement in graphs, *Congr. Numer.* 79 (1990)
225-231.

[13] V. R. Kulli, D.K. Patwari, The total bondage number of a graph, in: V. R. Kulli
(Ed.), *Advances in Graph Theory*, Vishwa, Gulbarga, (1991) 227-235.

[14] R. C. Laskar, J. Pfaff, S. M. Hedetniemi, S. R. Hedetniemi, On the algorithmic
complexity of total domination. *SIAM J. Algebraic Discrete Methods*, 5 (1984),
420-425.

[15] N. Sridharan, M. D. Elias, V. S. A. Subramanian, Total bondage number of a
graph. *AKCE Int. J. Graphs Combin.* 4 (2)(2007), 203-209.

[16] J. A. Telle, and A. Proskurowski, Algorithms for vertex partitioning problems on
partial $k$-trees. *SIAM J. Discrete Mathematics*, 10 (1997), 529-550.

[17] J.-M. Xu, *Theory and Application of Graphs*. Kluwer Academic Publishers, Dor-
drecht/Boston/London, 2003.