Stochastic averaging for the non-autonomous mixed stochastic differential equations with locally Lipschitz coefficients

Ruifang Wang\textsuperscript{a}, Yong Xu\textsuperscript{a,b,*}

\textsuperscript{a}Department of Applied Mathematics, Northwestern Polytechnical University, Xi’an, 710072, China
\textsuperscript{b}MIIT Key Laboratory of Dynamics and Control of Complex Systems, Northwestern Polytechnical University, Xi’an, 710072, China

Abstract

This work concerned with a non-autonomous slow-fast system, which is generalized by stochastic differential equations (SDEs) with locally Lipschitz coefficients, subjected to standard Brownian motion (Bm) and fractional Brownian motion (fBm) with Hurst parameter \(1/2 < H < 1\). As for fBm and locally Lipschitz coefficients, the pathwise approach and the Itô stochastic calculus are combined and the technique of stopping time is used very frequently. Then, the averaging principle for this system is obtained by the technique of time discretization and truncation.

Keywords. Averaging principles, fractional Brownian motion, non-autonomous system, generalised Riemann-Stieltjes integral, Itô stochastic integral

Mathematics subject classification. 60G22, 60H10, 34C29, 37B55

1. Introduction

In this paper, we study the following SDEs driven by fBm and standard Bm:

\[
\begin{align*}
\frac{du_t^\epsilon}{\epsilon} &= b_1 (t, u_t^\epsilon, v_t^\epsilon) dt + f_1 (t, u_t^\epsilon) dW_t^1 + g_1 (t, u_t^\epsilon) dB_t^H, & u_0^\epsilon &= x \in \mathbb{R}^n, \\
\frac{dv_t^\epsilon}{\epsilon} &= b_2 (t, u_t^\epsilon, v_t^\epsilon) dt + f_2 (t, u_t^\epsilon, v_t^\epsilon) dW_t^2, & v_0^\epsilon &= y \in \mathbb{R}^m,
\end{align*}
\]  

where \(\epsilon \in (0, 1]\) is a small positive parameter which represents the ratio of the natural time scale between the slow variable \(u_t^\epsilon \in \mathbb{R}^n\) and fast variable \(v_t^\epsilon \in \mathbb{R}^m\). Moreover, \(B_t^H = \{ B_t^H, t \in [0, T] \} \) (\(H \in (1/2, 1)\)) is \(d_1\)-dimensional fractional Brownian motion, \(W_t^1 = \{ W_t^1, t \in [0, T] \}\) and \(W_t^2 = \{ W_t^2, t \in [0, T] \}\) are \(d_2\) and \(d_3\) dimensional standard Brownian motions, respectively. Assume that the processes \(W_t^1, W_t^2\) and \(B_t^H\) are mutually independent, and initial variable \(x, y\) are fixed and independent of \((W_t^1, W_t^2, B_t^H)\).

Recall that fBm with Hurst index \(H \in (0, 1)\) is a zero mean Gaussian process \(\{ B_t^H, t \geq 0 \}\) with covariance function

\[
R_H(s, t) = \frac{1}{2} \left( s^{2H} + t^{2H} - |t - s|^{2H} \right).
\]

*Corresponding author

Email addresses: wrfjj@yahoo.com (Ruifang Wang), hsux3@nwpu.edu.cn (Yong Xu)
Notice that if $H = 1/2$, the process $B^H$ is a standard Bm, but if $H \neq 1/2$, it does not have independent increments. Moreover, from (1.2) we deduce that, $E |B_t - B_s|^2 = |t - s|^{2H}$. As a consequence, the process $B^H$ has $\alpha$-Hölder continuous paths for all $\alpha \in (0, H)$. It was introduced by Kolmogorov [1] in 1940, and later was named as fractional Brownian motions by Mandelbrot and Van Ness [2] in 1968.

Under some reasonable assumptions, the purpose of this paper is to show the averaging principle for the system (1.1):

$$\limsup_{\epsilon \to 0, t \in [0, T]} E \|u^\epsilon_t - \bar{u}_t\|^2_\alpha = 0, \quad (1.3)$$

where $\bar{u}_t$ is the solution of the corresponding averaged equation (see equation (2.4) below).

The averaging principle is a effective method to analysis stochastic dynamical systems with different time-scales. The rigorous result for the first related result about stochastic case was studied by Khasminskii [3] in 1968. Since then, the averaging principle has been investigated by many scholars. Now, let us recall a short literature about the averaging principle. For the autonomous case: Givon [4], Freidlin and Wentzell [5], Duan [6], Xu and his co-workers [7–9] studied the averaging principle of SDEs. In addition, Cerrai [10, 11], Wang and Roberts [12], Pei and Xu [13–15] and other scholars also investigated the averaging principle of stochastic partial differential equations (SPDEs) in recent years. For the non-autonomous case: Cerrai [16], Liu and his co-workers [17] studied the averaging principle for non-autonomous slow-fast systems driven by Brownian motion. Xu [18] also studied the averaging principle for non-autonomous slow-fast systems driven by Gaussian noises and Poisson random measures. However, both studies of Cerrai [16], Liu [17] and Xu [18] are driven by Gaussian noises or Poisson random measures, which can not describe the disturbances with long-range dependence.

We concerned with the averaging principle for a non-autonomous slow-fast system, which is driven by standard Bm and fBm with Hurst parameter $1/2 < H < 1$. The self-similar and long-range dependence properties of $B^H$ (if $H > 1/2$) make this process a useful driving noise in models arising in physics, telecommunication networks, finance and other fields. Since $B^H$ is not a semimartingale if $H \neq 1/2$ [19], the classical Itô theory to construct a stochastic calculus with respect to the fBm is no longer available. At present, some new techniques for the definition of stochastic integrals with respect to fBm have been studied by several authors [20–22]. To overcome this problem in our work, the integral $\int_0^t f_1 (r, u^\epsilon_r) dW^1_r$, $\int_0^t f_2 (r, u^\epsilon_r, u^\epsilon_r) dW^2_r$ should be interpreted as an Itô stochastic integral and the integral $\int_0^t g_1 (r, u^\epsilon_r) dB^H_r$ as a generalised Riemann-Stieltjes integral in the sense of Zähle[23].

Comparing with the work of Guerra and Nualart [24], in which the existence and uniqueness theorem for solutions of multidimensional, time dependent, SDEs driven by fBm with Hurst parameter $H > 1/2$ and standard Bm have been proved, the conditional expectation is extended to general expectation on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ (where $\{\mathcal{F}_t\}_{t \geq 0}$ is the $\sigma$-field generated by the random variables $W^1_t, W^2_t, B^H_t$ and the $\mathbb{P}$-null sets) and the coefficients are assumed satisfy local Lipschitz conditions in our work. In order to estimates the norm of an integral with respect to $B^H$ (it will produce some higher order terms) and to deal with the local Lispchitz continuity, some coefficients are assumed to be bounded and the technique of stopping time is used very frequently. Further, the existence and uniqueness of solutions is
proved at first and the averaging principle for this system is obtained by using the generalized Khasminskii method.

This paper is organized as follows. In Section 2, we introduce some notations and assumptions that will be used in the analysis of equation (1.1) and presents the main results. Section 3 is devoted to proving the existence and uniqueness of solutions. In Section 4, the averaged equation is defined by studying the equation associated to the fast equation and the detailed proof of the strong convergence result is presented by using the technique of time discretization and truncation. Note that, $C > 0$ with or without subscripts represents a general constant, the value of which may vary for different cases in this paper.

2. Preliminaries, assumptions and main result

Now, we recall some definitions and results that will be used throughout the paper. Let $| \cdot |$ be the Euclidean norm, $\langle \cdot , \cdot \rangle$ be the Euclidean inner product and $\| \cdot \|$ be the matrix norm.

Let $1/2 < H < 1, 1 - H < \alpha < 1/2$ and $d \in \mathbb{N}^+$, denote by $W_0^{\alpha, \infty}$ the space of measurable functions $f : [0, T] \to \mathbb{R}^d$ such that

$$
\| f \|_{\alpha, \infty} := \sup_{t \in [0, T]} \| f (t) \|_\alpha < \infty,
$$

where

$$
\| f (t) \|_\alpha := |f(t)| + \int_0^t \frac{|f(t) - f(s)|}{(t-s)^{\alpha+1}} ds.
$$

For $0 < \eta \leq 1$, let $C^n$ be space of $\eta$-Hölder continuous functions $f : [0, T] \to \mathbb{R}^d$, equipped with the norm

$$
\| f \|_{\eta} := \sup_{t \in [0, T]} |f(t)| + \sup_{0 \leq s < t \leq T} \frac{|f(t) - f(s)|}{(t-s)^\eta} < \infty.
$$

Denote by $W_T^{1-\alpha, \infty}$ ($0 < \alpha < 1/2$) the space of measurable functions $g : [0, T] \to \mathbb{R}^d$ such that

$$
\| g \|_{1-\alpha, \infty, T} := \sup_{0 < s < t \leq T} \left( \frac{|g(t) - g(s)|}{(t-s)^{1-\alpha}} + \int_s^t \frac{|g(r) - g(s)|}{(r-s)^{2-\alpha}} dr \right) < \infty.
$$

Moreover, denote by $W_0^{\alpha, 1}$ the space of measurable functions $f : [0, T] \to \mathbb{R}^d$ such that

$$
\| f \|_{\alpha, 1} := \int_0^T \frac{|f(s)|}{s^\alpha} ds + \int_0^T \int_s^T \frac{|f(s) - f(r)|}{(s-r)^{\alpha+1}} dr ds < \infty.
$$

Then, if $f \in W_0^{\alpha, 1}$ and $g \in W_T^{1-\alpha, \infty}$, for any $t \in [0, T]$, we know that $\int_0^t f dg$ exists, and have

$$
\left| \int_0^t f dg \right| \leq \Lambda_0^{0,t}(g) \| f \|_{\alpha, 1},
$$

where

$$
\Lambda_0^{0,T}(g) := \frac{1}{\Gamma(1-\alpha)} \sup_{0 < s < t < T} \left| (D_{t-}^{1-\alpha} g_{t-}) (s) \right| \leq \frac{1}{\Gamma(1-\alpha) \Gamma(\alpha)} \| g \|_{1-\alpha, \infty, T} < \infty,
$$

and $(D_{t-}^{1-\alpha} g_{t-}) (s)$ is the Weyl derivatives [25] of $g$.  


Remark 2.1. In particular, the fractional Brownian motion \( B^H \) with \( H > 1/2 \) have their trajectories in \( W_T^{1-\alpha, \infty} (1 - H < \alpha < 1/2) \). As a consequence, if \( u = \{u_t, t \in [0, T]\} \) is a stochastic process whose trajectories belong to the space \( W_0^{\alpha, 1} \), the generalised Riemann-Stieltjes integrals \( \int_0^T u_s dB^H_s \) exists, and we have the following estimate

\[
\left\| \int_0^T u_s dB^H_s \right\| \leq \Lambda^{0,T}_\alpha (B^H) \|u\|_{\alpha,1},
\]

where \( \Lambda^{0,T}_\alpha (B^H) := \frac{1}{\Gamma(1-\alpha)} \sup_{0<s<t<T} \left| (D_{t-s}^\alpha B^H_s) (s) \right| \) has moments of all orders (see Lemma 7.5 in Nualart and Răşcanu [26]).

The following lemma is the so-called Garsia-Rodemich-Rumsey inequality (see Theorem 1.4 in [27]):

Lemma 2.2. For any \( p \geq 1 \) and \( \theta > p^{-1} \), there exists some constant \( C_{\theta,p} > 0 \) such that for any continuous function \( f \in C [0, T] \), have

\[
|f(t) - f(s)|^p \leq C_{\theta,p} |t - s|^{\theta p - 1} \int_0^T \int_0^T \frac{|f(x) - f(y)|^p}{|x - y|^{\theta p + 1}} dx dy.
\]

In this paper, the following maps

\[
\begin{align*}
b_1 &: [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \times \Omega \to \mathbb{R}^n; \\
f_1 &: [0, \infty) \times \mathbb{R}^n \times \Omega \to \mathbb{R}^{n \times d_2}; \\
g_1 &: [0, \infty) \times \mathbb{R}^n \times \Omega \to \mathbb{R}^{n \times d_1}; \\
b_2 &: [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \times \Omega \to \mathbb{R}^m; \\
f_2 &: [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \times \Omega \to \mathbb{R}^{m \times d_3}
\end{align*}
\]

are continuous. Then, we give the following assumptions, which are supposed to hold for \( \mathbb{P} \)-almost all \( \omega \in \Omega \):

(A1) (a) For any \( R \in \mathbb{R} \), \( y \in \mathbb{R}^m \) and \( x_i \in \mathbb{R}^n \) with \( |x_i| \leq R \), there exist some constants \( \theta_1 \geq 0 \), such that

\[
|b_1(t, x_1, y) - b_1(t, x_2, y)| + \|f_1(t, x_1) - f_1(t, x_2)\| \leq C_{R,T} (1 + |y|^{\theta_1}) |x_1 - x_2|.
\]

(b) For any \( x \in \mathbb{R}^n \) and \( y_i \in \mathbb{R}^m \), there exist some constants \( \theta_2, \theta_3 \geq 0 \) and \( 0 < \kappa \leq 1 \), such that

\[
\begin{align*}
|b_1(t, x, y_1) - b_1(t, x, y_2)| &\leq C_T |y_1 - y_2| (1 + |x|^{\theta_2} + |y_1|^{\theta_3} + |y_2|^{\theta_3}); \\
|b_1(t, x_1, y_1) - b_1(s, x_1, y_1)| + \|f_1(t, x_1) - f_1(s, x_1)\| &\leq C_T |t - s|^\kappa (1 + |x_1|^{\theta_2} + |y_1|^{\theta_3}); \\
|b_1(t, x_1, y_1)| + \|f_1(t, x_1)\| &\leq C_T (1 + |x_1| + |y_1|).
\end{align*}
\]
(A2) (a) The mapping $g_1$ is continuously differentiable in $x \in \mathbb{R}^n$. For any $R \in \mathbb{R}$ and $x_i \in \mathbb{R}^n$ with $|x_i| \leq R$, there exist some constants $0 < \gamma \leq 1$, such that

$$\|g_1(t, x_1) - g_1(t, x_2)\| \leq C_{R,T} |x_1 - x_2|;$$

$$\|\nabla_x g_1(t, x_1) - \nabla_x g_1(t, x_2)\| \leq C_{R,T} |x_1 - x_2|^\gamma,$$

where $\nabla_x$ is the standard gradient with respect to the variable $x$.

(b) For any $x \in \mathbb{R}^n$ and $s, t \in [0, T]$, there exist some constants $0 < \beta \leq 1$, such that

$$\|g_1(t, x)\| \leq C_T (1 + |x|);$$

$$\|\nabla_x g_1(t, x) - \nabla_x g_1(s, x)\| + \|g_1(t, x) - g_1(s, x)\| \leq C_T |t - s|^\beta.$$

(A3) (a) For any fixed $t \in [0, \infty)$, the mapping $b_2(\cdot, \cdot, \cdot)$ is locally Lipschitz continuous and $f_2(t, \cdot, \cdot)$ is Lipschitz continuous.

(b) For any $x_i \in \mathbb{R}^n$, $y_i \in \mathbb{R}^m$ and $t \in [0, T]$, there exist some constants $\alpha_1, \alpha_2 > 0$ and $\epsilon \in (0, 1]$, such that

$$|b_2(t, x_1, y_1) - b_2(t, x_2, y_1)| \leq C_T |x_1 - x_2| \left(1 + |x_1|^{\alpha_1} + |x_2|^{\alpha_1} + |y_1|^{\alpha_2}\right);$$

$$|b_2(t, x_1, y_1) - b_2(s, x_1, y_1)| + |f_2(t, x_1, y_1) - f_2(s, x_1, y_1)| \leq C_T |t - s| \left(1 + |x_1|^{\alpha_1} + |y_1|^{\alpha_2}\right);$$

$$|b_2(t, x_1, y_1)| + \|f_2(t, x_1, y_1)\| \leq C_T (1 + |x_1| + |y_1|).$$

(A4) Assume that for any $x \in \mathbb{R}$, $y \in \mathbb{R}$, $b_1(t, x, y)$ and $f_1(t, x)$ are bounded.

(A5) (Strict monotonicity condition:) For any $t \in [0, +\infty)$, $x \in \mathbb{R}^n$, $y_1, y_2 \in \mathbb{R}^m$, there exist constants $\beta_1 > 0$, such that

$$2 \langle y_1 - y_2, b_2(t, x, y_1) - b_2(t, x, y_2) \rangle + \|f_2(t, x, y_1) - f_2(t, x, y_2)\|^2 \leq -\beta_1 |y_1 - y_2|^2.$$

(Strict coercivity condition:) For some fixed $p \geq 2$, any $t \in [0, +\infty)$, $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, there exist constants $C_{p,T}$, $\beta_p > 0$, such that

$$2 \langle y, b_2(t, x, y) \rangle + \|f_2(t, x, y)\|^2 \leq -\beta_p |y|^2 + C_{p,T}(1 + |x|^2).$$

**Remark 2.3.** The assumptions (A1)-(A4) ensures the existence and uniqueness of the solution of system (1.1). Strict monotonicity condition guarantee the exponential ergodicity (see Lemma 4.3 in Section 4) holds and strict coercivity condition is used to ensures the existence of invariant measures for the frozen equation (see Lemma 4.1 in Section 4).

Under the above assumptions, the main result of this paper as follows:

**Theorem 2.4.** Assume that the conditions (A1)-(A4) hold. Then, for any $\alpha \in (1 - H, 1/2 \wedge \beta \wedge \gamma/2)$, there exists a unique strong solution $(u^x_t, v^x_t)$ of equation (1.1).
**Theorem 2.5.** Assume that the conditions (A1)-(A5) hold. Then, for any \( \alpha \in (1 - H, 1/2 \land \beta \land \gamma/2) \), we have
\[
\lim_{\epsilon \to 0} \sup_{t \in [0, T]} \mathbb{E} \|u_t^\epsilon - \bar{u}_t\|^2 = 0,
\]
where \( \bar{u}_t \) is the solution of the corresponding averaged equation:
\[
d\bar{u}_t = \bar{b}_1(t, \bar{u}_t) \, dt + f_1(t, \bar{u}_t) \, dW_t^1 + g_1(t, \bar{u}_t) \, dB_t^H, \quad \bar{u}_0 = x \in \mathbb{R}^n,
\]
with \( \bar{b}_1(s, x) = \int_{\mathbb{R}^m} b_1(s, x, z) \mu^{s,x}(dz) \) and \( \mu^{s,x} \) is the unique invariant measure for the equation associated to the fast equation by fixed \( s > 0 \) and frozen slow component \( x \in \mathbb{R}^n \):
\[
dv_t = b_2(s, x, v_t) \, dt + f_2(s, x, v_t) \, dW_t^2, \quad v_0 = y \in \mathbb{R}^m.
\]

3. Existence, uniqueness of the solutions

In this section, we study the unique solutions for a class of mixed stochastic differential equation (1.1) driven by fBm and Bm (Theorem 2.4). Firstly, we construct an auxiliary equation. Secondly, we give some estimates for this auxiliary equation. Finally, the existence and uniqueness of solutions for original equation (1.1) is proved by defining the stopping time.

3.1. Some a-priori estimates of \((u_t^{\epsilon,n}, v_t^{\epsilon,m})\)

For any \( n \in \mathbb{N} \), we define the following stopping time
\[
\tau_n := \inf \left\{ t \geq 0 : A_\alpha^{0,t} (B^H) \geq n \right\},
\]
and study the following equation:
\[
\left\{
\begin{array}{l}
   du_t^\epsilon = b_{1,n}(t, u_t^\epsilon, v_t^\epsilon) \, dt + f_{1,n}(t, u_t^\epsilon) \, dW_t^1 + g_{1,n}(t, u_t^\epsilon) \, dB_t^{H,n}, \quad u_0^\epsilon = x \in \mathbb{R}^n; \\
   dv_t^\epsilon = \frac{1}{\epsilon} b_{2,n}(t, u_t^\epsilon, v_t^\epsilon) \, dt + \frac{1}{\sqrt{\epsilon}} f_2(t, u_t^\epsilon, v_t^\epsilon) \, dW_t^2,
\end{array}
\right.
\]
where \( B_t^{H,n} = B_t^{H,\tau_n} \) and
\[
b_{i,n}(t, x, y) = \begin{cases} 
   b_i(t, x, y), & |x| \leq n \text{ and } |y| \leq n, \\
   b_i(t, x, y/n/|y|), & |x| \leq n \text{ and } |y| > n, \\
   b_i(t, x/n/|x|, y), & |x| > n \text{ and } |y| \leq n, \\
   b_i(t, x/n/|x|, y/n/|y|), & |x| > n \text{ and } |y| > n,
\end{cases}
\]
and
\[
f_{i,n}(t, x) = \begin{cases} 
   f_i(t, x), & |x| \leq n, \\
   f_i(t, x/n/|x|), & |x| > n.
\end{cases}
\]

It is easy to know that the mapping \( b_{i,n}(t, \cdot, \cdot), f_{1,n}(t, \cdot) \) and \( g_{1,n}(t, \cdot) \) are Lipschitz continuous and satisfy all conditions in (A1)-(A3). Moreover, for any \( m > n \), we also have
\[
|x| \leq n \text{ and } |y| \leq n \Rightarrow \begin{cases} 
   b_{i,m}(t, x, y) = b_{i,n}(t, x, y) = b_i(t, x, y), \\
   f_{1,n}(t, x) = f_{1,m}(t, x) = f_1(t, x), \\
   g_{1,n}(t, x) = g_{1,m}(t, x) = g_1(t, x).
\end{cases}
\]
and
\[ \Lambda^0_{\alpha,t} (B^H) \leq n \Rightarrow B_t^{H,n} = B_t^{H,m} = B_t^H. \] 

(3.4)

Then, using the same argument as [24], it is easy to get that for any fixed \( n \in \mathbb{N} \), there exists a unique strong solution \((u_t^{\epsilon,n}, v_t^{\epsilon,n})\) to equation (3.2).

For any fixed \( \epsilon \in (0, 1] \), we study the solution \( u_t^{\epsilon,n} \) and \( v_t^{\epsilon,n} \) of equation (3.2) are bounded.

**Lemma 3.1.** Under the assumptions (A1)-(A4), for any \( \alpha \in (1 - H, 1/2 \land \beta \land \gamma/2) \) and \( p \geq 1 \), there exists some positive constant, such that

\[ E\|u^{\epsilon,n}\|_{p,\infty}^p \leq C_{\alpha,p,x,T}, \] 

(3.5)

and

\[ E\|v^{\epsilon,n}\|_{p,\infty}^p \leq C_{\alpha,p,\epsilon,x,y,T}. \] 

(3.6)

**Proof:** First, we estimate \( E\|u^{\epsilon,n}\|_{p,\infty}^p \). For brevity, we denote

\[ \Psi_t^{\epsilon} (\lambda, u^{\epsilon,n}) = \sup_{r \in [0,t]} e^{-\lambda r} |u_r^{\epsilon,n}|, \] 

(3.7)

and

\[ \Phi_t^{\epsilon} (\lambda, u^{\epsilon,n}) = \sup_{r \in [0,t]} e^{-\lambda r} \int_0^r \frac{|u_r^{\epsilon,n} - u_s^{\epsilon,n}|}{(r-s)^{\alpha+1}} ds. \] 

(3.8)

In order to estimate \( \|u^{\epsilon,n}\|_{p,\infty} \), we first estimate \( \Psi_t^{\epsilon} (\lambda, u^{\epsilon,n}) \). Thanks to the assumption (A4) and (2.1), it yields

\[ \Psi_t^{\epsilon} (\lambda, u^{\epsilon,n}) \leq C_{x,T} (1 + \Lambda^0_{\alpha,t} (B^H \sup_{r \in [0,t]} \int_0^r e^{-\lambda (r-s)} (s^{-\alpha} \Psi_s^{\epsilon} (\lambda, u^{\epsilon,n}) + \Phi_s^{\epsilon} (\lambda, u^{\epsilon,n})) ds \] 

\[ + \sup_{r \in [0,t]} \left| \int_0^r f_{1,n} (s, u_s^{\epsilon,n}) dW_s^1 \right| \] 

\[ \leq C_{x,T} (1 + \Lambda^0_{\alpha,t} (B^H \sup_{r \in [0,t]} \int_0^r e^{-\lambda (r-s)} (s^{-\alpha} \Psi_s^{\epsilon} (\lambda, u^{\epsilon,n}) + \Phi_s^{\epsilon} (\lambda, u^{\epsilon,n})) ds \] 

\[ + \sup_{r \in [0,t]} \left| \int_0^r f_{1,n} (s, u_s^{\epsilon,n}) dW_s^1 \right| \] 

\[ \leq C_{x,T} (1 + \Lambda^0_{\alpha,t} (B^H \sup_{r \in [0,t]} \int_0^r e^{-\lambda (r-s)} (s^{-\alpha} \Psi_s^{\epsilon} (\lambda, u^{\epsilon,n}) + \Phi_s^{\epsilon} (\lambda, u^{\epsilon,n})) ds \] 

\[ + \sup_{r \in [0,t]} \left| \int_0^r f_{1,n} (s, u_s^{\epsilon,n}) dW_s^1 \right| \) \],

(3.9)

where the last equation used the following estimate [26, page 66]

\[ \int_0^t e^{-\lambda (t-r)} r^{-\alpha} dr \leq C \lambda^{-1}. \]
Then, we estimate $\Phi^\epsilon_t (\lambda, u^\epsilon,n)$. According to the equation (4.17) in [26], we can get

$$
\Phi^\epsilon_t (\lambda, u^\epsilon,n) = \sup_{r \in [0,t]} e^{-\lambda r} \left| \int_0^r \left[ \int_s^r b_{1,n} (\sigma, u^\epsilon_{\sigma}, u^\epsilon_{\sigma}) \, d\sigma \right] \, ds + \int_0^r \left[ \int_s^r f_{1,n} (\sigma, u^\epsilon_{\sigma}) \, dW^1_\sigma \right] \, ds \right|
$$

$$
+ \int_0^r \left[ \int_s^r g_{1,n} (\sigma, u^\epsilon_{\sigma}) \, dB^{H,n}_\sigma \right] \, ds
$$

$$
\leq C_{\alpha,x,T} \left( 1 + \sup_{r \in [0,t]} \int_0^r \left[ \int_s^r f_{1,n} (\sigma, u^\epsilon_{\sigma}) \, dW^1_\sigma \right] \, ds \right)
$$

$$
+ A^{0,t}_{\alpha} (B^{H,n}) \sup_{r \in [0,t]} e^{-\lambda r}
$$

$$
\times \left( \int_0^r g_{1,n} (s, u^\epsilon_s) \, ds \right) + \int_0^r \left[ g_{1,n} (s, u^\epsilon_s) - g_{1,n} (\sigma, u^\epsilon_{\sigma}) \right] \, d\sigma ds
$$

$$
\leq C_{\alpha,x,T} (1 + A^{0,t}_{\alpha} (B^{H,n})) \left( 1 + \sup_{r \in [0,t]} \int_0^r (r - s)^{-\alpha - 1} \left| \int_s^r f_{1,n} (\sigma, u^\epsilon_{\sigma}) \, dW^1_\sigma \right| \, ds \right)
$$

$$
+ \lambda^{\alpha - 1} \Psi^\epsilon_t (\lambda, u^\epsilon,n) + \lambda^{\alpha - 1} \Phi^\epsilon_t (\lambda, u^\epsilon,n) \right).
$$

(3.10)

where the last equation used the estimate [26, page 66]

$$
\int_0^r e^{-\lambda (r - s)} (r - s)^{-\alpha} \, ds = \Gamma (1 - \alpha) \lambda^{\alpha - 1}.
$$

Let $C := C_{x,T} \cap C_{\alpha,x,T}$ and $\lambda = (4C (1 + A^{0,t}_{\alpha} (B^{H,n})))^{1/\alpha}$. Combine (3.9) and (3.10), making simple transformations as [28], it is easy to get that

$$
\Psi^\epsilon_t (\lambda, u^\epsilon,n) + \Phi^\epsilon_t (\lambda, u^\epsilon,n) \leq C_{\alpha,x,T} (1 + A^{0,t}_{\alpha} (B^{H,n})) \frac{1}{1 - \alpha} (1 + \Upsilon^\epsilon_t (\lambda, u^\epsilon,n)),
$$

where

$$
\Upsilon^\epsilon_t (\lambda, u^\epsilon,n) := \sup_{r \in [0,t]} \left( \left| \int_0^r f_{1,n} (s, u^\epsilon_s) \, dW^1_s \right| + \int_0^r (r - s)^{-\alpha - 1} \left| \int_s^r f_{1,n} (\sigma, u^\epsilon_{\sigma}) \, dW^1_\sigma \right| \, ds \right).
$$

Hence

$$
\|u^\epsilon,n\|_{\alpha,\infty} \leq e^{\lambda T} \left( \Psi^\epsilon_t (\lambda, u^\epsilon,n) + \Phi^\epsilon_t (\lambda, u^\epsilon,n) \right)
$$

$$
\leq C_{\alpha,x,T} e^{A^{0,t}_{\alpha} (B^{H,n})} \frac{1}{1 - \alpha} (1 + A^{0,t}_{\alpha} (B^{H,n})) \frac{1}{1 - \alpha} (1 + \Upsilon^\epsilon_t (\lambda, u^\epsilon,n)).
$$

Take expectations on both sides of the above equation, we can get

$$
\mathbb{E} \|u^\epsilon,n\|_{\alpha,\infty}^p \leq C_{\alpha,x,T} \left[ \mathbb{E} \left[ e^{3p C_{\alpha,x,T} (A^{0,t}_{\alpha} (B^{H,n}))^{1/\alpha}} \right] (1 + \mathbb{E} [A^{0,t}_{\alpha} (B^{H,n})]^{3p}) \frac{1}{(1 - \eta) \frac{1}{\alpha}} \right] \frac{1}{\beta}.
$$

(3.11)

We need to prove that $\mathbb{E} [\Upsilon^\epsilon_t (\lambda, u^\epsilon,n)]^p$ is bounded for any $p \geq 1$. Applying the Garsia-Rodemich-Rumsey inequality (2.2) with $p = 2/\eta$ and $\theta = (1 - \eta)/2$ (where $\eta \in (0, 1/2 - \alpha)$), it deduces that

$$
\left| \int_s^r f_{1,n} (r, u^\epsilon_{r,s}) \, dW^1_r \right| \leq C \eta \, (t - s)^{1/2 - \eta} \zeta,
$$

(3.12)
where
\[
\zeta = \left( \int_0^T \int_0^T |\sigma - r|^{-1/\eta} \left| \int_r^\sigma f_{1,n}(z, u^\epsilon_{r,n}) dW^1_z \right|^{2/\eta} d\sigma dr \right)^{\eta/2}.
\]

Then, take expectations for \( \zeta \), we can get
\[
E \zeta^p = \int_0^T \int_0^T |\sigma - r|^{-p/2} E \left| \int_r^\sigma f_{1,n}(z, u^\epsilon_{r,n}) dW^1_z \right|^p d\sigma dr
\]
\[
= \int_0^T \int_0^T |\sigma - r|^{-p/2} E \left( \int_r^\sigma \left| f_{1,n}(z, u^\epsilon_{r,n}) \right|^2 dz \right)^{p/2} d\sigma dr
\]
\[
\leq C_T.
\]

Fixed \( \eta \in (0, 1/2 - \alpha) \), it follows that
\[
E \left[ T^t_x (\lambda, u^\epsilon_{r,n}) \right]^p \leq C_p \int_0^T E \left| f_{1,n}(r, u^\epsilon_{r,n}) \right|^p dr + C_p E \zeta^p \left( \sup_{t \in [0,T]} \left| \int_0^t (t - s)^{-\eta - 1/2} ds \right|^p \right)
\]
\[
\leq C_{\alpha,p,T}. \tag{3.13}
\]

Thanks to (3.11), (3.13) and \( \Lambda^0_T \left( B^{H,n} \right) \) has moments of all orders [26], it yields (3.5).

Next, we estimate \( E \| v^\epsilon_{r,n} \|_2^2 \)
\[
E \| v^\epsilon_{r,n} \|_{p,\infty}^2 = E \left( \sup_{t \in [0,T]} \left| y + \frac{1}{\epsilon} \int_0^t b_{2,n}(r, u^\epsilon_{r,n}, v^\epsilon_{r,n}) dr + \frac{1}{\sqrt{\epsilon}} \int_0^t f_2(r, v^\epsilon_{r,n}) dW_r^2 \right|^p \right)
\]
\[
\leq C_{p,y,T} \left( 1 + E \left[ \sup_{t \in [0,T]} \left| \int_0^t b_{2,n}(r, u^\epsilon_{r,n}, v^\epsilon_{r,n}) dr \right|^p \right] + T^c (b_{2,n}) 
\]
\[
+ E \left[ \sup_{t \in [0,T]} \left| \int_0^t f_2(r, v^\epsilon_{r,n}) dW_r^2 \right|^p \right] + T^c (f_2) \right)
\]
\[
\leq C_{p,y,T} \left( 1 + \int_0^T E \left| u^\epsilon_{r,n} \right|^p dr + \int_0^T E \left| v^\epsilon_{r,n} \right|^p dr + T^c (b_{2,n}) + T^c (f_2) \right). \tag{3.14}
\]

where
\[
T^c (b_{2,n}) := E \left( \sup_{t \in [0,T]} \left| \int_0^t (t - s)^{-\alpha - 1} \left| \int_s^t b_{2,n}(r, u^\epsilon_{r,n}, v^\epsilon_{r,n}) dr \right| ds \right|^p \right),
\]

and
\[
T^c (f_2) := E \left( \sup_{t \in [0,T]} \left| \int_0^t (t - s)^{-\alpha - 1} \left| \int_s^t f_2(r, v^\epsilon_{r,n}) dW_r^2 \right| ds \right|^p \right).
\]

For \( T^c (b_{2,n}) \), applying the Garsia-Rodemich-Rumsey inequality (2.2) with \( p = 2/\varrho \) and \( \theta = (1 - \varrho)/2 \) (where \( \varrho \in (0, 1/2 - \alpha) \)) again, we also can get
\[
\left| \int_s^t b_{2,n}(r, u^\epsilon_{r,n}, v^\epsilon_{r,n}) dr \right| \leq C_{\varrho} |t - s|^{-1/2 - \varrho} \zeta,
\]
where
\[ \varsigma = \left( \int_0^T \int_0^T |\sigma - r|^{-1/2} \left| \int_r^\sigma b_{2,n}(z, u_{z,n}^\epsilon, v_{z,n}^\epsilon) dz \right|^{2/\theta} \frac{d\sigma dr}{\theta} \right)^{\theta/2}. \]

Then, take expectations for \( \varsigma \) and using Fubinis theorem, we can get
\[
\mathbb{E}\varsigma^p = \int_0^T \int_0^T |\sigma - r|^{-p/2} \mathbb{E} \left| \int_r^\sigma b_{2,n}(z, u_{z,n}^\epsilon, v_{z,n}^\epsilon) dz \right|^p d\sigma dr
\]
\[
\leq \int_0^T \int_0^T |\sigma - r|^{-1+p/2} \int_r^\sigma \mathbb{E} |b_{2,n}(z, u_{z,n}^\epsilon, v_{z,n}^\epsilon)|^p dz d\sigma dr
\]
\[
= \int_0^T \int_0^T \int_r^\sigma |\sigma - r|^{-1+p/2} d\sigma dr \mathbb{E} |b_{2,n}(z, u_{z,n}^\epsilon, v_{z,n}^\epsilon)|^p dz
\]
\[
\leq C_{p,T} \left( 1 + \int_0^T \mathbb{E} |u_{r,n}^\epsilon|^p dr + \int_0^T \mathbb{E} |v_{r,n}^\epsilon|^p dr \right). \] 

Hence
\[
\Gamma_T^\epsilon (b_{2,n}) \leq C \mathbb{E}\varsigma^p \left( \sup_{t \in [0,T]} \int_0^t (t - s)^{-\alpha-1/2} ds \right)^p
\]
\[
\leq C_{\alpha,p,T} \left( 1 + \int_0^T \mathbb{E} |u_{r,n}^\epsilon|^p dr + \int_0^T \mathbb{E} |v_{r,n}^\epsilon|^p dr \right). \tag{3.15} \]

Moreover, use the same argument as (3.13) and (3.15), we also can obtain
\[
\Lambda_T^\epsilon (f_2) \leq C_{\alpha,p,T} \left( 1 + \int_0^T \mathbb{E} |u_{r,n}^\epsilon|^p dr \right). \tag{3.16} \]

Substituting (3.15) and (3.16) into (3.14), thanks to (3.5), we have
\[
\mathbb{E}\|v_{\epsilon,n}^\alpha\|_{\alpha,\infty}^p = \mathbb{E} \left( \sup_{t \in [0,T]} \|v_{t,n}^\epsilon\|_\alpha \right)^p \leq C_{\alpha,p,\epsilon,x,y,T} \left( 1 + \int_0^T \mathbb{E} \left( \sup_{\sigma \leq r} \|v_{\sigma,n}^\epsilon\|_\alpha \right)^p dr \right). \tag{3.17} \]

Then, by Gronwall inequality, we have (3.6). The proof is complete. \( \square \)

3.2. The existence and uniqueness of solutions

Now, we study the existence and uniqueness of solutions for original equation (1.1):

**Proof of Theorem 2.4:** In order to prove the existence of the solution for (1.1), for any \( n \in \mathbb{N} \), we define the following stopping time
\[
\tau_n^1 := \inf \{ t \geq 0 : \| u_t^\epsilon_n \|_\alpha + \| v_t^\epsilon_n \|_\alpha \geq n \} \wedge \tau_n, \tag{3.18} \]
and we let
\[
\tau := \sup_{n \in \mathbb{N}} \tau_n^1. \tag{3.19} \]
It is easy to know that the sequence of stopping times \( \{\tau_n^1\} \) is non-decreasing and \( P(\tau = +\infty) = 1 \). Indeed,

\[
P(\tau < +\infty) = \lim_{T \to +\infty} P(\tau \leq T),
\]

and for each \( T > 0 \), thanks to Lemma 3.1 and Lemma 7.5 in [26], we can get

\[
P(\tau \leq T) = \lim_{n \to +\infty} P(\tau_n^1 \leq T)
\]

\[
= \lim_{n \to +\infty} \mathbb{P}(\sup_{t \in [0,T]} \|u_{n}^\epsilon\|_\alpha + \sup_{t \in [0,T]} \|v_{n}^\epsilon\|_\alpha + \sup_{t \in [0,T]} A_{n}^{0,t}(B^H) \geq n)
\]

\[
= \lim_{n \to +\infty} \frac{1}{n^2} \mathbb{E}\left(\sup_{t \in [0,T]} \|u_{n}^\epsilon\|_\alpha^2 + \sup_{t \in [0,T]} \|v_{n}^\epsilon\|_\alpha^2 + \frac{1}{\Gamma^2(1 - \alpha)} \sup_{0 < s < t} |(D_{t}^{1-\alpha} B_{s}^{H})| s|^{2}\right)
\]

\[
= 0.
\]

Hence, \( P(\tau < +\infty) = 0 \), that is, \( P(\tau = +\infty) = 1 \). Further, for any \( t \in [0, T] \) and \( \omega \in \{\tau = +\infty\} \), there exists \( m \in \mathbb{N} \) such that \( t \leq \tau_n^1(\omega) \). Then, we define

\[
u_{t}^\epsilon(\omega) := u_{t}^{\epsilon,m}(\omega) \quad \text{and} \quad v_{t}^\epsilon(\omega) := v_{t}^{\epsilon,m}(\omega).
\]

(3.20)

This is a good definition, as for any \( t \leq \tau_{n}^1 \wedge \tau_{m}^1 \), we have

\[
u_{t}^\epsilon n = u_{t}^{\epsilon,m} \quad \text{and} \quad v_{t}^\epsilon n = v_{t}^{\epsilon,m}, \quad \mathbb{P} - a.s.
\]

(3.21)

Actually, for any \( n \geq m \) and \( t \leq \tau_{n}^1 \wedge \tau_{m}^1 \), thanks to (3.3) and (3.4), we have

\[
u_{t}^\epsilon n - u_{t}^{\epsilon,m} + v_{t}^\epsilon n - v_{t}^{\epsilon,m}
\]

\[
= \int_{0}^{t} [b_{1,n}(r, u_{r}^{\epsilon,n}, v_{r}^\epsilon n) - b_{1,m}(r, u_{r}^{\epsilon,m}, v_{r}^{\epsilon,m})]dr + \int_{0}^{t} [f_{1,n}(r, u_{r}^{\epsilon,m}) - f_{1,m}(r, u_{r}^{\epsilon,m})]dW_{1}^{r} + \int_{0}^{t} g_{1,n}(r, u_{r}^{\epsilon,n})dB_{r}^{H,n} - \int_{0}^{t} g_{1,m}(r, u_{r}^{\epsilon,m})dB_{r}^{H,m} + \frac{1}{\epsilon} \int_{0}^{t} [b_{2,n}(r, u_{r}^{\epsilon,n}, v_{r}^\epsilon n) - b_{2,m}(r, u_{r}^{\epsilon,m}, v_{r}^{\epsilon,m})]dr + \frac{1}{\sqrt{\epsilon}} \int_{0}^{t} [f_{2}(r, u_{r}^{\epsilon,n}, v_{r}^\epsilon n) - f_{2}(r, u_{r}^{\epsilon,m}, v_{r}^{\epsilon,m})]dW_{2}^{r}.
\]

According to the paper [26] and [24], we know that the trajectories of \( u_{t}^{\epsilon,n} \) and \( u_{t}^{\epsilon,m} \) are \( \eta \)-Hölder continuous for all \( \eta < 1/2 \). Now, let \( \eta \in (\alpha/\gamma, 1/2) \) and consider the set \( \Omega_N \subset \Omega \) with \( N \in \mathbb{N} \), such that

\[
\Omega_N := \{\omega \in \{\tau = +\infty\} : \|u_{t}^{\epsilon,n}\|_{\eta} \leq N \text{ and } \|u_{t}^{\epsilon,m}\|_{\eta} \leq N\}.
\]
It is clear that $\Omega_N \not\supset \{\tau = +\infty\}$. Then, by proceeding as Proposition 3.4, Proposition 3.6 and Proposition 3.9 in [24], we can get
\[
\mathbb{E} \left[ \| u^{\epsilon,n}_{t} - v^{\epsilon,m}_{t} \|_{\alpha}^2 1_{\Omega_N} \right] + \mathbb{E} \left[ \| v^{\epsilon,n}_{t} - u^{\epsilon,m}_{t} \|_{\alpha}^2 1_{\Omega_N} \right] 
\leq C_{\alpha,T} \mathbb{E} \left( \int_{0}^{t} (t - r)^{-\alpha} \| b_{1,n}(r, u^{\epsilon,n}_{r}, v^{\epsilon,n}_{r}) - b_{1,n}(r, u^{\epsilon,m}_{r}, v^{\epsilon,m}_{r}) \|_{\alpha} 1_{\Omega_N} dr \right)^2 
+ C_{\alpha,T} \int_{0}^{t} (t - r)^{-\frac{1}{2} - \alpha} \mathbb{E} \left[ \| f_{1,n}(r, u^{\epsilon,n}_{r}) - f_{1,n}(r, u^{\epsilon,m}_{r}) \|_{\alpha}^2 1_{\Omega_N} \right] dr 
+ C_{\alpha,T} \mathbb{E} \left[ A^0_{\alpha} \left( B^{H,n} \right) \int_{0}^{t} (t - r)^{-2\alpha} + r^{-\alpha} \right] \left( 1 + \Delta u^{\epsilon,n}_{r} + \Delta u^{\epsilon,m}_{r} \right) \| u^{\epsilon,n}_{r} - u^{\epsilon,m}_{r} \|_{\alpha} 1_{\Omega_N} dr \right)^2 
+ C_{\alpha,T} \int_{0}^{t} (t - r)^{-\alpha} \mathbb{E} \left[ \| f_{2}(r, u^{\epsilon,n}_{r}, v^{\epsilon,n}_{r}) - f_{2}(r, u^{\epsilon,m}_{r}, v^{\epsilon,m}_{r}) \|_{\alpha}^2 1_{\Omega_N} \right] dr 
+ C_{\alpha,T} \int_{0}^{t} (t - r)^{-\frac{1}{2} - \alpha} \mathbb{E} \left[ \| f_{2}(r, u^{\epsilon,n}_{r}, v^{\epsilon,n}_{r}) - f_{2}(r, u^{\epsilon,m}_{r}, v^{\epsilon,m}_{r}) \|_{\alpha}^2 1_{\Omega_N} \right] dr 
\leq C_{\alpha,T} \int_{0}^{t} (t - r)^{-2\alpha} + (t - r)^{-\frac{1}{2} - \alpha} \left( \mathbb{E} \left[ \| u^{\epsilon,n}_{r} - u^{\epsilon,m}_{r} \|_{\alpha}^2 1_{\Omega_N} \right] + \mathbb{E} \left[ \| v^{\epsilon,n}_{r} - v^{\epsilon,m}_{r} \|_{\alpha}^2 1_{\Omega_N} \right] \right) dr 
+ C_{\alpha,n,T} \int_{0}^{t} (t - r)^{-2\alpha} + r^{-\alpha} \mathbb{E} \left[ \| u^{\epsilon,n}_{r} - u^{\epsilon,m}_{r} \|_{\alpha}^2 1_{\Omega_N} \right] dr 
+ C_{\alpha,T} \int_{0}^{t} (t - r)^{-\frac{1}{2} - \alpha} r^{-\frac{1}{2} - \alpha} \left( \mathbb{E} \left[ \| u^{\epsilon,n}_{r} - u^{\epsilon,m}_{r} \|_{\alpha}^2 1_{\Omega_N} \right] + \mathbb{E} \left[ \| v^{\epsilon,n}_{r} - v^{\epsilon,m}_{r} \|_{\alpha}^2 1_{\Omega_N} \right] \right) dr,
\]
where the last estimate is because $t \leq \tau^1_n \land \tau^1_m$ and $\Omega_N \not\supset \{\tau = +\infty\}$, we know that $A^0_{\alpha} \left( B^{H,n} \right) \leq n$ and
\[
1 + \Delta u^{\epsilon,n}_{r} + \Delta u^{\epsilon,m}_{r} = 1 + \int_{0}^{r} \frac{|u^{\epsilon,n}_{r} - u^{\epsilon,m}_{s}|}{(r - s)^{\alpha + 1}} ds + \int_{0}^{r} \frac{|u^{\epsilon,m}_{r} - u^{\epsilon,m}_{s}|}{(r - s)^{\alpha + 1}} ds 
\leq 1 + \left( \| u^{\epsilon,n}_{r} \|_{\gamma} + \| u^{\epsilon,m}_{r} \|_{\gamma} \right) \int_{0}^{r} (r - s)^{\gamma - \alpha - 1} ds \leq C_{N}.
\]
Then, for any $n \in \mathbb{N}$, by the Gronwall-type lemma (Lemma 7.6 in [26]), we deduce that
\[
\mathbb{E} \left[ \| u^{\epsilon,n}_{t} - u^{\epsilon,m}_{t} \|_{\alpha}^2 1_{\Omega_N} \right] + \mathbb{E} \left[ \| u^{\epsilon,n}_{t} - u^{\epsilon,m}_{t} \|_{\alpha}^2 1_{\Omega_N} \right] = 0, \quad t \leq \tau_n \land \tau_m.
\] (3.22)
Then, let $N \to +\infty$, as $\Omega_N \not\supset \{\tau = +\infty\}$, we can get (3.21).

Recalling that if $\omega \in \{\tau = +\infty\}$, and $t \leq \tau_m$, we denote $u^\epsilon_t$ is equal to $u^{\epsilon,m}_t$ and $v^\epsilon_t$ is equal to $v^{\epsilon,m}_t$, thanks to (3.3) and (3.4), it follows that
\[
\begin{aligned}
\left\{ \begin{array}{l}
\frac{d}{dt} u^{\epsilon}_{t} = x + \int_{0}^{t} b_{1}(r, u^{\epsilon}_{r}, v^{\epsilon}_{r}) dr + \int_{0}^{t} f_{1}(r, u^{\epsilon}_{r}) dW^{1}_{r} + \int_{0}^{t} g_{1}(r, u^{\epsilon}_{r}) dB^{H}_{r}, \\
\frac{d}{dt} v^{\epsilon}_{t} = y + \int_{0}^{t} b_{2}(r, u^{\epsilon}_{r}, v^{\epsilon}_{r}) dr + \int_{0}^{t} f_{2}(r, u^{\epsilon}_{r}, v^{\epsilon}_{r}) dW^{2}_{r},
\end{array} \right.
\end{aligned}
\] (3.23)
P-a.s., that is, $(u^\epsilon_t, v^\epsilon_t)$ is a solution of equation (1.1).

Finally, denote another solution of system (1.1) is $(u^{\epsilon,*}_t, v^{\epsilon,*}_t)$, use the same argument as (3.21), we can also get that
\[
u^{\epsilon}_t = v^{\epsilon,*}_t, \quad \mathbb{P} - a.s.
\] (3.24)
Thus, we prove the solution of system (1.1) is unique. The proof is complete. \hfill \Box
4. Proof of the main result

In this section, we prove the main Theorem 2.5, i.e. the slow process $u_t^\epsilon$ strongly converges to the averaged process $\bar{u}_t$ in the mean square sense, as $\epsilon \to 0$. Firstly, we need to define the averaged equation and give some properties of the averaged coefficient. Secondly, we construct an auxiliary process $\hat{v}_t^\epsilon$ by the technique of time discretization and give some estimates about it on the basis of some a-priori estimates for the solution $(u_t^\epsilon, v_t^\epsilon)$ of original equation (1.1) are given. Finally, we construct the stopping time and obtain appropriate control of $u_t^\epsilon - \bar{u}_t$ before and after the stopping time respectively.

4.1. The averaged equation

To define the averaged equation, we first consider the equation (2.5) associated to the fast equation. Under the assumptions (A1)-(A5), it is easy to prove that the equation (2.5) has a unique strong solution $v_{s,x,y}^t$, which is a time homogeneous Markov process. Moreover, use the same argument as [17], there exists some constant $\beta_1^* > 0$ such that the following estimates hold and we will not give a detailed proof here:

$$E |v_{s,x,y}^t|^p \leq C_{p,T} (1 + |x|^p) + e^{-\beta_1^* t} |y|^p,$$

and

$$E |v_{s,x,y_1}^t - v_{s,x,y_2}^t|^2 \leq e^{-\beta_1 t} |y_1 - y_2|^2,$$

and

$$E |v_{s_1,x_1,y}^t - v_{s_2,x_2,y}^t|^2 \leq C_T \left( |s_1 - s_2|^{2t} + |x_1 - x_2|^2 \right) \left( 1 + |x_1|^{2\alpha_1} + |x_2|^{2\alpha_1 \vee 2\alpha_2} + |y|^{2\alpha_2} \right).$$

Let $\{P_{s,x}^t\}_{t \geq 0}$ be the transition semigroup of $\{v_{s,x,y}^t\}_{t \geq 0}$, that is

$$P_{s,x}^t \varphi (y) := E \varphi (v_{s,x,y}^t), \quad s > 0, \ y \in \mathbb{R}^m,$$

where $\varphi : \mathbb{R}^m \to \mathbb{R}$ is a bounded measurable function.

Then, we can establish the following crucial lemma:

**Lemma 4.1.** Assume that the conditions (A1)-(A5) hold. Then, for any fixed $s > 0$ and $x \in \mathbb{R}^n$, there exists a unique invariant measure $\mu^{s,x}$ for the equation (2.5), and

$$\int_{\mathbb{R}^m} |z|^p \mu^{s,x} (dz) \leq C_{p,T} (1 + |x|^p).$$

Moreover, for any $t > 0$ and $y \in \mathbb{R}^m$, we obtain

$$\left| E b_1 (s, x, v_{s,x,y}^t) - \int_{\mathbb{R}^m} b_1 (s, x, z) \mu^{s,x} (dz) \right| \leq C_T e^{-\frac{\beta_1 t}{4}} \left( 1 + |x|^{2(\theta_2 \vee \theta_3 \vee 1)} + |y|^{2(\theta_3 \vee 1)} \right).$$

**Proof:** The detailed proof will be given in the Appendix.

Further, the averaged equation can be defined as (2.4) by the unique invariant measure $\mu^{s,x}$. Moreover, we can give some properties of the averaged coefficient $b_1$, where the detailed proof of Lemma 4.2 will be given in the Appendix.
Lemma 4.2. Assume that the conditions (A1)-(A5) hold. Then, for any $t \geq 0$ and $x \in \mathbb{R}^n$, we have
\[|\bar{b}_1(t,x)| \leq C(1 + |x|).\] (4.7)
Moreover, for any $s_1, s_2 \in [0,T]$, $R \in \mathbb{R}$ and $x_i \in \mathbb{R}^n$ with $|x_i| \leq R$, we have
\[|\bar{b}_1(s_1, x_1) - \bar{b}_1(s_2, x_2)| \leq C_R T (|s_1 - s_2| + |s_1 - s_2|^\alpha + |x_1 - x_2|).\] (4.8)

Lemma 4.3. Assume that the conditions (A1)-(A5) hold. Then, for any $s, \Lambda > 0, x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, we have
\[E \left| \frac{1}{A} \int_0^A b_1(s,x,v^{s,x,y}_r) \, dr - \bar{b}_1(s,x) \right|^2 \leq \frac{C}{A} (1 + |x|^{4(\theta_2 \vee \theta_3 \vee 1)} + |y|^{4(\theta_3 \vee 1)}).\] (4.9)

Proof: Using the Markov property of $\{v^{s,x,y}_t\}_{t \geq 0}$, we have
\[
E \left| \frac{1}{A} \int_0^A b_1(s,x,v^{s,x,y}_r) \, dr - \bar{b}_1(s,x) \right|^2 = \frac{2}{A^2} \int_0^A \int_r^A E \left[ (b_1(s,x,v^{s,x,y}_r) - \bar{b}_1(s,x)) (b_1(s,x,v^{s,x,y}_\sigma) - \bar{b}_1(s,x)) \right] d\sigma dr
\]
\[
= \frac{2}{A^2} \int_0^A \int_r^A E \left[ (b_1(s,x,v^{s,x,y}_r) - \bar{b}_1(s,x)) P^{s,x}_{\sigma-r} (b_1(s,x,v^{s,x,y}_\sigma) - \bar{b}_1(s,x)) \right] d\sigma dr
\]
\[
\leq \frac{2}{A^2} \int_0^A \int_r^A \left[ E \left| b_1(s,x,v^{s,x,y}_r) - \bar{b}_1(s,x) \right|^2 \right] E \left[ P^{s,x}_{\sigma-r} (b_1(s,x,v^{s,x,y}_\sigma) - \bar{b}_1(s,x))^2 \right] d\sigma dr.
\]
By assumption (A1) and thanks to (4.1), (4.7), we can get
\[
E \left| b_1(s,x,v^{s,x,y}_r) - \bar{b}_1(s,x) \right| \leq 2E \left| b_1(s,x,v^{s,x,y}_r) \right|^2 + 2E \left| \bar{b}_1(s,x) \right|^2
\]
\[
\leq C \left( 1 + |x|^2 + E \left| v^{s,x,y}_r \right|^2 \right)
\]
\[
\leq C \left( 1 + |x|^2 + e^{-\beta_2^2 t} |y|^2 \right).
\]

Then, according to the equation (4.6) and (4.1), we obtain
\[
E \left| P^{s,x}_{\sigma-r} (b_1(s,x,v^{s,x,y}_r) \, dr - \bar{b}_1(s,x)) \right|^2 = E \left| \mathbb{E} b_1(s,x,v^{s,x,y}_{\sigma-r}) - \int_{\mathbb{R}^m} b_1(s,x,z) \mu^{s,x}(dz) \right|^2
\]
\[
\leq C T e^{-\frac{\beta_1^2}{2}(\sigma-r)} E \left| 1 + |x|^{2(\theta_2 \vee \theta_3 \vee 1)} + E \left| v^{s,x,y}_r \right|^{2(\theta_3 \vee 1)} \right|^2
\]
\[
\leq C T e^{-\frac{\beta_1^2}{2}(\sigma-r)} (1 + |x|^{4(\theta_2 \vee \theta_3 \vee 1)} + |y|^{4(\theta_3 \vee 1)}).
\]
Hence
\[
E \left| \frac{1}{A} \int_0^A b_1(s,x,v^{s,x,y}_r) \, dr - \bar{b}_1(s,x) \right|^2 \leq \frac{C}{A} \left( 1 + |x|^{4(\theta_2 \vee \theta_3 \vee 1)} + |y|^{4(\theta_3 \vee 1)} \right) \int_0^A \int_r^A e^{-\frac{\beta_1^2}{2}(\sigma-r)} d\sigma dr
\]
\[
\leq \frac{C}{A} \left( 1 + |x|^{4(\theta_2 \vee \theta_3 \vee 1)} + |y|^{4(\theta_3 \vee 1)} \right).
\]
The proof is complete. \[\square\]

Thanks to the assumptions and Lemma 4.2, by proceeding as Theorem 2.4 and Lemma 3.1, it is easy to get that there exists a unique solution $\bar{u}_t$ to equation (2.4) and we have
\[E \| \bar{u} \|^p_{a,\infty} \leq C_{a,p,x,T}, \quad p \geq 1.\] (4.10)
4.2. Some a-priori estimates

To prove the Theorem 2.5, some priori estimates for the solution \((u_t^\epsilon, v_t^\epsilon)\) of original equation (1.1) need to be given at first.

**Lemma 4.4.** Assume that the conditions (A1)-(A5) hold. Then, for any \(\alpha \in (1 - H, 1/2 \wedge \beta \wedge \gamma/2)\), \(p \geq 1\) and \(t \in [0, T]\), we have

\[
\mathbb{E}\|u_t^\epsilon\|_\alpha^p \leq C_{\alpha,p,x,T} \quad \text{and} \quad \mathbb{E}|v_t^\epsilon|^2 \leq C_{\alpha,x,y,T}.
\]  

Moreover, for any \(h \in (0,1]\), it yields

\[
\mathbb{E}\left|u_{t+h}^\epsilon - u_t^\epsilon\right|^2 \leq C_{\alpha,x,y,T}h.
\]

**Proof:** According to the assumptions and use the same argument as Lemma 3.1 and [17, Lemma 3.1], it is easy to get that equation (4.11) hold. Moreover, using the Itô isometry for Brownian motion term and by proceeding as Lemma 4.2 in [28], we also can establish the equation (4.12). Here, we omit the detailed proof. \(\square\)

Then, inspired by Khasminskii’s idea in [3], for any \(\epsilon > 0\), we divide the interval \([0, T]\) into subintervals of size \(\delta_\epsilon > 0\), where \(\delta_\epsilon\) is a fixed number depending on \(\epsilon\). Now, we construct a process \(\hat{v}_t^\epsilon\) with initial value \(\hat{v}_0^\epsilon = v_0^\epsilon = y\), and for \(t \in [k\delta_\epsilon, \min\{(k+1)\delta_\epsilon, T\}]\), we have

\[
\hat{v}_t^\epsilon = \hat{v}_{k\delta_\epsilon}^\epsilon + \frac{1}{\epsilon} \int_{k\delta_\epsilon}^{t} b_2(k\delta_\epsilon, u_{k\delta_\epsilon}^\epsilon, \hat{v}_r^\epsilon) \, dr + \frac{1}{\sqrt{\epsilon}} \int_{k\delta_\epsilon}^{t} f_2(k\delta_\epsilon, u_{k\delta_\epsilon}^\epsilon, \hat{v}_r^\epsilon) \, dW_r^2,
\]

i.e.,

\[
\hat{v}_t^\epsilon = y + \frac{1}{\epsilon} \int_{0}^{t} b_2(r(\delta_\epsilon), u_{r(\delta_\epsilon)}^\epsilon, \hat{v}_r^\epsilon) \, dr + \frac{1}{\sqrt{\epsilon}} \int_{0}^{t} f_2(r(\delta_\epsilon), u_{r(\delta_\epsilon)}^\epsilon, \hat{v}_r^\epsilon) \, dW_r^2,
\]

where \(r(\delta_\epsilon) = \lfloor r/\delta_\epsilon \rfloor \delta_\epsilon\) is the nearest breakpoint preceding \(r\). By the construction of \(\hat{v}_t^\epsilon\), we have an estimate analogous to Lemma 4.4 hold, i.e., for any \(t \in [0, T]\), we have

\[
\mathbb{E}\left|\hat{v}_t^\epsilon\right|^2 \leq C_{\alpha,x,y,T}.
\]

Moreover, thanks to the assumptions and Lemma 4.4, by proceeding as Lemma 3.4 in [17], it is easy to get that

\[
\mathbb{E}\left|v_t^\epsilon - \hat{v}_t^\epsilon\right|^2 \leq C_{\alpha,x,y,T} \delta_\epsilon^{2\wedge 1}.
\]

4.3. The proof of the main result

Now, we construct the following stopping time \(\tau_R^\epsilon\) for each \(R \in \mathbb{R}\):

\[
\tau_R^\epsilon := \inf\{t \geq 0 : \|u_t^\epsilon\|_\alpha + \|\bar{u}_t\|_\alpha + A_\alpha^{0,t} (B^H) \geq R\}.
\]

Moreover, due to the trajectories of \(u_t^\epsilon\) and \(\bar{u}_t\) are \(\eta\)-Hölder continuous for all \(\eta < 1/2\). As the proof in Theorem 2.4, let \(\eta \in (\alpha/\gamma, 1/2)\) and consider the following set \(\Omega_N \subset \Omega\) with \(N \in \mathbb{N}\), such that

\[
\Omega_N := \{\omega \in \Omega : \|u^\epsilon\|_\eta \leq N \text{ and } \|\bar{u}\|_\eta \leq N\}.
\]

It is clear that \(\Omega_N \not\supset \Omega\).

First, we estimate the error of \(u_t^\epsilon - \bar{u}_t\) before a stopping time:
Lemma 4.5. Assume that the conditions (A1)-(A5) hold. Then, for any $\alpha \in (1 - H, 1/2 \wedge \beta \wedge \gamma/2)$, we have

$$\sup_{t \in [0,T]} \mathbb{E} \left[ \|u_t^\epsilon - \tilde{u}_t\|^2 \mathbf{1}_{\Omega_N \cap \{ T \leq \tau_h \}} \right] \leq C_{\alpha,x,y,R,N,T} \left( \delta_r^{2\alpha \wedge 2\epsilon \wedge (1/2 - \alpha)} + \epsilon / \delta_r \right). \quad (4.17)$$

Proof: Note that

$$u_t^\epsilon - \tilde{u}_t = \int_0^t (b_1 (r, u_r^\epsilon, \nu_r^\epsilon) - \tilde{b}_1 (r, \tilde{u}_r)) \, dr + \int_0^t (f_1 (r, u_r^\epsilon) - f_1 (r, \tilde{u}_r)) \, dW_r^1$$
$$+ \int_0^t (g_1 (r, u_r^\epsilon) - g_1 (r, \tilde{u}_r)) \, dB_r^H.$$

It is easy to know that

$$\mathbb{E} \left[ \|u_t^\epsilon - \tilde{u}_t\|^2 \mathbf{1}_{\Omega_N \cap \{ T \leq \tau_h \}} \right] \leq 3 \mathbb{E} \left[ \left\| \int_0^t (b_1 (r, u_r^\epsilon, \nu_r^\epsilon) - \tilde{b}_1 (r, \tilde{u}_r)) \, dr \right\|^2 \mathbf{1}_{\Omega_N \cap \{ T \leq \tau_h \}} \right]$$
$$+ 3 \mathbb{E} \left[ \left\| \int_0^t (f_1 (r, u_r^\epsilon) - f_1 (r, \tilde{u}_r)) \, dW_r^1 \right\|^2 \mathbf{1}_{\Omega_N \cap \{ T \leq \tau_h \}} \right]$$
$$+ 3 \mathbb{E} \left[ \left\| \int_0^t (g_1 (r, u_r^\epsilon) - g_1 (r, \tilde{u}_r)) \, dB_r^H \right\|^2 \mathbf{1}_{\Omega_N \cap \{ T \leq \tau_h \}} \right]$$
$$:= 3\mathcal{I}^1_t + 3\mathcal{I}^2_t + 3\mathcal{I}^3_t. \quad (4.18)$$

For $\mathcal{I}^2_t$, by Proposition 3.9 in [24], we obtain

$$\mathcal{I}^2_t \leq C_{R,T} \int_0^t (t - r)^{-\frac{1}{2} - \alpha} \mathbb{E} \left[ \|u_r^\epsilon - \tilde{u}_r\|^2 \mathbf{1}_{\Omega_N \cap \{ T \leq \tau_h \}} \right] \, dr. \quad (4.19)$$

Moreover, by Proposition 3.6 in [24], we also can get

$$\mathcal{I}^3_t \leq C_{\alpha,R,T} \int_0^t ((t - r)^{-2\alpha} + r^{-\alpha}) \mathbb{E} \left[ (1 + (\Delta u_r^\epsilon)^2 + (\Delta \tilde{u}_r)^2) \|u_r^\epsilon - \tilde{u}_r\|^2 \mathbf{1}_{\Omega_N \cap \{ T \leq \tau_h \}} \right] \, dr,$$

where

$$1 + (\Delta u_r^\epsilon)^2 + (\Delta \tilde{u}_r)^2 = 1 + \left( \int_0^r \frac{|u_r^\epsilon - u_s^\epsilon|^{\gamma}}{(r-s)^{\alpha+1}} \, ds \right)^2 + \left( \int_0^r \frac{|\tilde{u}_r - \tilde{u}_s|^{\gamma}}{(r-s)^{\alpha+1}} \, ds \right)^2.$$

If $\omega \in \Omega_N$ and $\eta \in (\alpha/\gamma, 1/2)$, we have

$$1 + (\Delta u_r^\epsilon)^2 + (\Delta \tilde{u}_r)^2 \leq 1 + (\|u_r^\epsilon\|^{2\gamma}_\eta + \|\tilde{u}_r\|^{2\gamma}_\eta) \left( \int_0^r (r-s)^{\eta-\alpha-1} \, ds \right)^2 \leq C_{\alpha,N,T}.$$

Hence

$$\mathcal{I}^3_t \leq C_{\alpha,R,N,T} \int_0^t ((t - r)^{-2\alpha} + r^{-\alpha}) \mathbb{E} \left[ \|u_r^\epsilon - \tilde{u}_r\|^2 \mathbf{1}_{\Omega_N \cap \{ T \leq \tau_h \}} \right] \, dr. \quad (4.20)$$
For $I^1_t$, we have
\[
I^1_t \leq 4E\left[\left\| \int_0^t b_1(r, u^c_r, v^c_r) - b_1\left(x(\delta_r), u^c_{\delta_r}, \dot{v}^c_r\right) dr \right\|_\alpha^2 \mathbf{1}_{\Omega_N \cap \{T \leq \tau_k\}}\right]
+ 4E\left[\left\| \int_0^t b_1\left(x(\delta_r), u^c_{\delta_r}, \dot{v}^c_r\right) - b_1\left(x(\delta), u^c_{\delta}, \dot{v}^c\right) dr \right\|_\alpha^2 \mathbf{1}_{\Omega_N \cap \{T \leq \tau_k\}}\right]
+ 4E\left[\left\| \int_0^t b_1\left(x(\delta), u^c_{\delta}, \dot{v}^c\right) - b_1\left(r, u^c_r\right) dr \right\|_\alpha^2 \mathbf{1}_{\Omega_N \cap \{T \leq \tau_k\}}\right]
+ 4E\left[\left\| \int_0^t b_1\left(r, u^c_r\right) - b_1\left(r, \bar{u}^c_r\right) dr \right\|_\alpha^2 \mathbf{1}_{\Omega_N \cap \{T \leq \tau_k\}}\right]
:= 4\mathcal{J}^1_t + 4\mathcal{J}^2_t + 4\mathcal{J}^3_t + 4\mathcal{J}^4_t. \tag{4.21}
\]

Thanks to Lemma 4.4 and equation (4.16), we can get
\[
E\left[\left\| \int_0^t b_1(r, u^c_r, v^c_r) - b_1\left(x(\delta_r), u^c_{\delta_r}, \dot{v}^c_r\right) dr \right\|_\alpha^2 \mathbf{1}_{\Omega_N \cap \{T \leq \tau_k\}}\right]
\leq E\left[\left\| \int_0^t b_1(r, u^c_r, v^c_r) - b_1\left(x(\delta), u^c_{\delta}, v^c_r\right) dr \right\|_\alpha^2 \mathbf{1}_{\Omega_N \cap \{T \leq \tau_k\}}\right]
+ E\left[\left\| \int_0^t b_1\left(x(\delta), u^c_{\delta}, v^c_r\right) - b_1\left(x(\delta), u^c_{\delta}, v^c_r\right) dr \right\|_\alpha^2 \mathbf{1}_{\Omega_N \cap \{T \leq \tau_k\}}\right]
+ E\left[\left\| \int_0^t b_1\left(x(\delta), u^c_{\delta}, v^c_r\right) - b_1\left(r, u^c_r\right) dr \right\|_\alpha^2 \mathbf{1}_{\Omega_N \cap \{T \leq \tau_k\}}\right]
\leq C_T \int_0^t |r - r(\delta)|^{2\alpha} E\left[\left\| (1 + |u^c_r|^{2q_2} + |v^c_r|^{2q_3}) \mathbf{1}_{\Omega_N \cap \{T \leq \tau_k\}}\right\|_\alpha dr
+ C_{R,T} \int_0^t E\left[\left\| u^c_r - u^c_{\delta_r} \right\|_\alpha^2 \mathbf{1}_{\Omega_N \cap \{T \leq \tau_k\}}\right] dr \int_s^t E\left[\left\| (1 + |v^c_r|^{2q_2}) \mathbf{1}_{\Omega_N \cap \{T \leq \tau_k\}}\right\|_\alpha dr
+ C_{R,T} \int_0^t E\left[\left\| v^c_r - \dot{v}^c_r \right\|_\alpha^2 \mathbf{1}_{\Omega_N \cap \{T \leq \tau_k\}}\right] dr \int_s^t E\left[\left\| (1 + |u^c_r|^{2q_2} + |v^c_r|^{2q_3} + |\dot{v}^c_r|^{2q_2}) \mathbf{1}_{\Omega_N \cap \{T \leq \tau_k\}}\right\|_\alpha dr
\leq C_{\alpha,x,y,T} \left(\delta^{2\alpha} + \delta + \delta^{2\alpha+1}\right) (t - s). \tag{4.22}
\]

Hence
\[
\mathcal{J}^1_t \leq 2E\left[\left\| \int_0^t b_1(r, u^c_r, v^c_r) - b_1\left(x(\delta_r), u^c_{\delta_r}, \dot{v}^c_r\right) dr \right\|_\alpha^2 \mathbf{1}_{\Omega_N \cap \{T \leq \tau_k\}}\right]
+ 2E\left[\left\| \int_0^{t_r} (t-r)^{-\alpha-1} \left\| \int_r^t b_1\left(s, u^c_s, v^c_s\right) - b_1\left(s(\delta_r), u^c_{\delta_r}, \dot{v}^c_s\right) ds \right\|_\alpha^2 \mathbf{1}_{\Omega_N \cap \{T \leq \tau_k\}}\right\|_\alpha dr
\leq C_{\alpha,x,y,T} \left(\delta^{2\alpha} + \delta + \delta^{2\alpha+1}\right) + C \int_0^t (t-r)^{-\frac{1}{2}} \frac{dr}{2}
\times \int_0^{t_r} (t-r)^{-\alpha} \left\| \int_r^t b_1\left(s, u^c_s, v^c_s\right) - b_1\left(s(\delta_r), u^c_{\delta_r}, \dot{v}^c_s\right) ds \right\|_\alpha^2 \mathbf{1}_{\Omega_N \cap \{T \leq \tau_k\}}\right\|_\alpha dr
\leq C_{\alpha,x,y,T} \left(\delta^{2\alpha} + \delta + \delta^{2\alpha+1}\right) \leq C_{\alpha,x,y,T} \delta^{2\alpha+2}. \tag{4.23}
\]
Then, due to (4.16) and by proceeding as (4.23), it is easy to get that
\[ J_t^3 + J_t^4 \leq C_{\alpha,R,x,y,T} \delta^2 \alpha^{2n_1} \alpha^{2} + C_{\alpha,R,T} \int_0^t (t - r)^{-2\alpha} \mathbb{E} \left[ \| u_r^\epsilon - \bar{u}_r^\epsilon \|_\alpha^2 1_{B_N \cap \{T \leq r\}} \right] dr. \] (4.24)

In order to prove the approximation result of the equation (4.17), we must estimate \( J_t^2 \).

**Lemma 4.6.** Assume that the conditions (A1)-(A5) hold. Then, for any \( t \in [0,T] \), we have
\[ \mathbb{E} \left\| \int_0^t b_1 \left( r \left( \delta_r \right), u_{r(\delta_r)}^\epsilon, \dot{v}_r^\epsilon \right) - \bar{b}_1 \left( r \left( \delta_r \right), u_{r(\delta_r)}^\epsilon \right) \right\|_\alpha^2 \leq C_{\alpha,x,y,T} \left( \frac{\epsilon}{\delta_r} + \delta_r^{\frac{\epsilon}{2}} \right). \] (4.25)

**Proof:** By elementary inequality, we have
\[
\mathbb{E} \left\| \int_0^t b_1 \left( r \left( \delta_r \right), u_{r(\delta_r)}^\epsilon, \dot{v}_r^\epsilon \right) - \bar{b}_1 \left( r \left( \delta_r \right), u_{r(\delta_r)}^\epsilon \right) \right\|_\alpha^2 \leq 3 \mathbb{E} \left| \int_0^t \left( r \left( \delta_r \right), u_{r(\delta_r)}^\epsilon, \dot{v}_r^\epsilon \right) - \bar{b}_1 \left( r \left( \delta_r \right), u_{r(\delta_r)}^\epsilon \right) \right| \leq C_T \mathbb{E} \left[ \int \int_0^t \left( r \left( \delta_r \right), u_{r(\delta_r)}^\epsilon, \dot{v}_r^\epsilon \right) - \bar{b}_1 \left( r \left( \delta_r \right), u_{r(\delta_r)}^\epsilon \right) \right] \leq C_T \mathbb{E} \left[ \int \int_0^t \left( r \left( \delta_r \right), u_{r(\delta_r)}^\epsilon, \dot{v}_r^\epsilon \right) - \bar{b}_1 \left( r \left( \delta_r \right), u_{r(\delta_r)}^\epsilon \right) \right].
\]

According to Lemma 4.3 and Lemma 4.4, we can get
\[
K_t^3 \leq 3 \max_{0 \leq k \leq \frac{|T/\delta_t| - 1}{\delta_t}} \mathbb{E} \left[ \int_0^t \left( r \left( \delta_r \right), u_{r(\delta_r)}^\epsilon, \dot{v}_r^\epsilon \right) - \bar{b}_1 \left( r \left( \delta_r \right), u_{r(\delta_r)}^\epsilon \right) \right] \leq C_T \mathbb{E} \left[ \int \int_0^t \left( r \left( \delta_r \right), u_{r(\delta_r)}^\epsilon, \dot{v}_r^\epsilon \right) - \bar{b}_1 \left( r \left( \delta_r \right), u_{r(\delta_r)}^\epsilon \right) \right].
\] (4.26)

The last equation is thanks to the distribution of \( u_{r(\delta_r)}^\epsilon, \dot{v}_{r(\delta_r)}^\epsilon \) coincides with the distribution of \( u_{r(\delta_r)}^\epsilon, \dot{v}_{r(\delta_r)}^\epsilon \) in the interval \( r \in [0, \delta_t] \) [17], where \( Y_{r, u_{r(\delta_r)}^\epsilon, v_{r(\delta_r)}^\epsilon}^\epsilon \) is the solution of the fast equation (2.5) by fixed \( k \delta_\epsilon > 0 \), frozen slow component \( u_{k \delta_\epsilon}^\epsilon \), and with initial datum \( u_{k \delta_\epsilon}^\epsilon \), and noise \( W^2 \) independent of both of them.

Moreover, for \( K_t^2 \), thanks to (4.7), (4.11) and (4.15), we obtain
\[
K_t^2 \leq C \delta_\epsilon \int_0^t \mathbb{E} \left[ \int_0^t \left( r \left( \delta_r \right), u_{r(\delta_r)}^\epsilon, \dot{v}_r^\epsilon \right) - \bar{b}_1 \left( r \left( \delta_r \right), u_{r(\delta_r)}^\epsilon \right) \right] \leq C \left( \frac{\epsilon}{\delta_t} + \delta_t^{\frac{\epsilon}{2}} \right).
\] (4.27)
\[ \leq C_T \delta_t \int_{[t/\delta_t] \delta_t}^t (1 + \mathbb{E} |u_r^{e(\delta_t)}|^2 + \mathbb{E} |\dot{v}_r^{e(\delta_t)}|^2) dr \]
\[ \leq C_{\alpha,x,y,T} \delta_t^2. \] (4.28)

For \( \mathcal{K}_t^3 \), using the Hölder inequality, we have

\[ \mathcal{K}_t^3 \leq 3 \int_0^t (t - r)^{-\alpha - \frac{1}{2}} dr \int_0^t (t - r)^{-\alpha - \frac{1}{2}} \mathbb{E} \left| \int_r^t b_1(s(\delta_t), u_s^{e(\delta_t)}, \dot{v}_s^{e(\delta_t)}) - \tilde{b}_1(s(\delta_t), u_s^{e(\delta_t)}) ds \right|^2 dr \]
\[ \leq C_{\alpha,T} \int_0^t (t - r)^{-\alpha - \frac{1}{2}} \mathbb{E} \left| \int_r^t b_1(s(\delta_t), u_s^{e(\delta_t)}, \dot{v}_s^{e(\delta_t)}) - \tilde{b}_1(s(\delta_t), u_s^{e(\delta_t)}) ds \right|^2 dr \]
\[ + C_{\alpha,T} \int_0^t (t - r)^{-\alpha - \frac{1}{2}} \mathbb{E} \left| \int_r^t b_1(s(\delta_t), u_s^{e(\delta_t)}, \dot{v}_s^{e(\delta_t)}) - \tilde{b}_1(s(\delta_t), u_s^{e(\delta_t)}) ds \right|^2 dr \]
\[ := \mathcal{K}_t^{31} + \mathcal{K}_t^{32}. \]

Using the same argument as the proof of (4.27) and the fact \( |\lambda_1| - |\lambda_2| \leq \lambda_1 - \lambda_2 + 1 \). Then, thanks to the Lemma 4.3, we can get

\[ \mathcal{K}_t^{31} \leq C_{\alpha,T} \int_0^t (t - r)^{-\alpha - \frac{1}{2}} \mathbb{E} \left| \int_r^t b_1(s(\delta_t), u_s^{e(\delta_t)}, \dot{v}_s^{e(\delta_t)}) - \tilde{b}_1(s(\delta_t), u_s^{e(\delta_t)}) ds \right|^2 dr \]
\[ + C_{\alpha,T} \int_0^t (t - r)^{-\alpha - \frac{1}{2}} \mathbb{E} \left| \int_r^t b_1(s(\delta_t), u_s^{e(\delta_t)}, \dot{v}_s^{e(\delta_t)}) - \tilde{b}_1(s(\delta_t), u_s^{e(\delta_t)}) ds \right|^2 dr \]
\[ + C_{\alpha,T} \int_0^t (t - r)^{-\alpha - \frac{1}{2}} \mathbb{E} \left| \int_r^t b_1(s(\delta_t), u_s^{e(\delta_t)}, \dot{v}_s^{e(\delta_t)}) - \tilde{b}_1(s(\delta_t), u_s^{e(\delta_t)}) ds \right|^2 dr \]
\[ \leq C_{\alpha,T} \int_0^t (t - r)^{-\alpha - \frac{1}{2}} \left| \int_r^t b_1(s(\delta_t), u_s^{e(\delta_t)}, \dot{v}_s^{e(\delta_t)}) - \tilde{b}_1(s(\delta_t), u_s^{e(\delta_t)}) ds \right|^2 dr \]
\[ + C_{\alpha,T} \int_0^t (t - r)^{-\alpha - \frac{1}{2}} \left| \int_r^t b_1(s(\delta_t), u_s^{e(\delta_t)}, \dot{v}_s^{e(\delta_t)}) - \tilde{b}_1(s(\delta_t), u_s^{e(\delta_t)}) ds \right|^2 dr \]
\[ \leq C_{\alpha,x,y,T} \max_{|\delta_t| + 1 \leq k \leq |\delta_t| - 1} \mathbb{E} \left| \int_{k\delta_t}^{(k+1)\delta_t} b_1(k\delta_t, u_{k\delta_t}^{e(\delta_t)}, \dot{v}_{k\delta_t}^{e(\delta_t)}) - \tilde{b}_1(k\delta_t, u_{k\delta_t}^{e(\delta_t)}) ds \right|^2 dr \]
\[
\leq \frac{C_{\alpha,T}}{\delta^2} \int_0^t \frac{1}{(t-r)^{1/2}} \frac{d}{dr} \max_{0 \leq k \leq \lfloor \frac{t}{\delta} \rfloor - 1} \mathbb{E} \left| \int_{k \delta}^{(k+1) \delta} b_1 (k \delta_e, u^\epsilon_{k \delta_e}, \hat{v}^\epsilon_{k \delta_e}) - \bar{b}_1 (k \delta_e, u^\epsilon_{k \delta_e}) \, ds \right|^2 \\
+ C_{\alpha,x,y,T} \delta^\frac{1}{2} \epsilon \\
\leq C_{\alpha,x,y,T} \left( \delta^\frac{1}{2} \epsilon + \epsilon / \delta_e \right).
\]

For $K^3_{\alpha,T}$, using the assumption (A2) and thanks to (4.11) and (4.15), it yields
\[
K^3_{\alpha,T} \leq C_{\alpha,T} \int_0^t (t-r)^{-\alpha - \frac{1}{2}} \int_r^t \mathbb{E} \left| b_1 (s \delta_e, u^\epsilon_{s \delta_e}, \hat{v}^\epsilon_{s \delta_e}) - \bar{b}_1 (s \delta_e, u^\epsilon_{s \delta_e}) \right|^2 \, ds \, dr \\
\leq C_{\alpha,x,y,T} \int_0^t (t-r)^{-\alpha} \, dr \leq C_{\alpha,x,y,T} \delta^\frac{1}{2} \epsilon.
\]

Hence
\[
K^3 \leq C_{\alpha,x,y,T} \left( \delta^\frac{1}{2} \epsilon + \epsilon / \delta_e + \delta^\frac{1}{2} \epsilon \right)
\]

Substituting (4.27)-(4.29) into (4.26), it follows that
\[
\mathbb{E} \left\| \int_0^t b_1 (r \delta_e, u^\epsilon_{r \delta_e}, \hat{v}^\epsilon_{r \delta_e}) - \bar{b}_1 (r \delta_e, u^\epsilon_{r \delta_e}) \, dr \right\|_\alpha^2 \leq C_{\alpha,x,y,T} \left( \frac{\epsilon}{\delta_e} + \delta^2 + \delta^\frac{1}{2} \epsilon + \delta^\frac{3}{2} \epsilon \right).
\]

The proof is complete. \(\square\)

So, thanks to (4.23), (4.24) and Lemma 4.6, it follows that
\[
\mathcal{T}_t^1 \leq C_{\alpha,R,T} \int_0^t (t-r)^{-2\alpha} \, dr \\
+ C_{\alpha,x,y,R,T} \left( \delta^2 \epsilon^{2\alpha \wedge (1/2-\alpha)} + \epsilon / \delta_e \right).
\]

Then, substituting (4.19), (4.20) and (4.30) into (4.18), it yields
\[
\mathbb{E} \left[ \left\| u^\epsilon - \bar{u} \right\|_\alpha^2 1_{\Omega_N \cap \{ T \leq \tau^\epsilon \}} \right] \leq C_{\alpha,R,N,T} \int_0^t (t-r)^{-\alpha} \frac{1}{r^{1-\alpha}} \, dr \\
+ C_{\alpha,R,N,T} \left( \delta^2 \epsilon^{2\alpha \wedge (1/2-\alpha)} + \epsilon / \delta_e \right).
\]

According to the Gronwall-type lemma (Lemma 7.6 in [26]), we deduce (4.17). The proof is complete. \(\square\)

Finally, the proof of our main result can be finished.

**Proof of Theorem 2.5:** Selecting $\delta_e = \epsilon \ln^{\epsilon^{-\epsilon}}$ (0 < $\epsilon$ < 1), we can get
\[
\lim_{\epsilon \to 0} \sup \mathbb{E} \left[ \left\| u^\epsilon - \bar{u} \right\|_\alpha^2 1_{\Omega_N \cap \{ T \leq \tau^\epsilon \}} \right] = 0.
\]

Then, let $N \to +\infty$, as $\Omega_N \nearrow \Omega$, it yields
\[
\lim_{\epsilon \to 0} \sup \mathbb{E} \left[ \left\| u^\epsilon - \bar{u} \right\|_\alpha^2 1_{\{ T \leq \tau^\epsilon \}} \right] = 0. 
\]

(4.31)
Using the Hölder inequality and Chebushev’s inequality, thanks to (4.10), (4.11) and Lemma 7.5 in [26], it yields

\[
\sup_{t \in [0,T]} \mathbb{E} \left[ \left\| u'_t - \bar{u}_t \right\|_{\alpha}^2 1_{\{T > \tau_{\alpha}^t\}} \right] \leq \sup_{t \in [0,T]} \left[ \mathbb{E} \left[ \left\| u'_t - \bar{u}_t \right\|_{\alpha}^4 \right] \frac{1}{2} \mathbb{P} \left( T > \tau_{\alpha}^t \right) \right]^{\frac{1}{2}} \\
\leq \sup_{t \in [0,T]} R^{-1} \left[ \mathbb{E} \left[ \left\| u'_t \right\|_{\alpha}^4 + \mathbb{E} \left[ \left\| \bar{u}_t \right\|_{\alpha}^4 \right] \frac{1}{2} \left[ \mathbb{E} \left( \sup_{t \in [0,T]} \left\| u'_t \right\|_{\alpha}^2 + \sup_{t \in [0,T]} \left\| \bar{u}_t \right\|_{\alpha}^2 + \sup_{t \in [0,T]} \left( A_{0,t} (B^R)^2 \right) \right) \right] \right]^{\frac{1}{2}} \\
\leq C_{\alpha,x,T} R^{-1}. \quad (4.32)
\]

Note that

\[
\sup_{t \in [0,T]} \mathbb{E} \left[ \left\| u'_t - \bar{u}_t \right\|_{\alpha}^2 \right] = \sup_{t \in [0,T]} \mathbb{E} \left[ \left\| u'_t - \bar{u}_t \right\|_{\alpha}^2 1_{\{T \leq \tau_{\alpha}^t\}} \right] + \sup_{t \in [0,T]} \mathbb{E} \left[ \left\| u'_t - \bar{u}_t \right\|_{\alpha}^2 1_{\{T > \tau_{\alpha}^t\}} \right]. \quad (4.33)
\]

Thanks to (4.31) and (4.32), let \( \epsilon \to 0 \) firstly and \( R \to \infty \) secondly, we can get the desired estimate (2.3). This completes the proof of Theorem 2.5. \( \square \)

Acknowledgments

This work was partly supported by the National Natural Science Foundation of China under Grant No. 11772255, the Fundamental Research Funds for the Central Universities, the Research Funds for Interdisciplinary Subject of Northwestern Polytechnical University, the Shaanxi Project for Distinguished Young Scholars, the Shaanxi Provincial Key R&D Program 2020KW-013 and 2019TD-010.

Appendix

In this section, we give the detailed proofs of Lemma 4.1 and Lemma 4.2:

**Proof of Lemma 4.1:** According to the estimate (4.1) and the classical Bogoliubov-Krylov argument, it is possible to get that the existence of an invariant measure \( \mu^{s,x} \). Then, thanks to (4.1), for all \( t > 0 \), we have

\[
\int_{\mathbb{R}^m} |z|^p \mu^{s,x} (dz) = \int_{\mathbb{R}^m} \mathbb{E} |v_{s,x,z}^t|^p \mu^{s,x} (dz) \leq C_{p,T} (1 + |x|^p) + \int_{\mathbb{R}^m} e^{-\beta \epsilon t} |y|^p \mu^{s,x} (dz).
\]

Therefore, if we take \( t > 0 \) such that \( e^{-\beta \epsilon t} \leq 1/2 \), we can get (4.5). Moreover, for any Lipschitz function \( \varphi : \mathbb{R}^m \to \mathbb{R} \), thanks to (4.2) and (4.5), we obtain

\[
\left| P_t^{s,x} \varphi (y) - \int_{\mathbb{R}^m} \varphi (z) \mu^{s,x} (dz) \right| \leq \int_{\mathbb{R}^m} \left| P_t^{s,x} \varphi (y) - P_t^{s,x} \varphi (z) \right| \mu^{s,x} (dz) \leq L_{\varphi} \int_{\mathbb{R}^m} \mathbb{E} |Y_t^{s,x,y} - Y_t^{s,x,z}| \mu^{s,x} (dz) \\
\leq L_{\varphi} \int_{\mathbb{R}^m} e^{-\frac{\beta \epsilon t}{2}} |y - z| \mu^{s,x} (dz).
\]
where $L \varphi = \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{|x-y|}$. So, $\mu^{s,x}$ is the unique invariant measure and is strong mixing.

Due to (4.1), (4.2) and (4.5), we can get

\[
\leq C L \varphi e^{-\frac{\beta}{2} t} (1 + |x| + |y|),
\]

The proof is complete. \qed

**Proof of Lemma 4.2**: For any $x_1, x_2 \in \mathbb{R}^n$ with $|x_i| \leq R$, thanks to the assumptions and equation (4.1), (4.3) and (4.6), we can get

\[
|\tilde{b}_1(s_1, x_1) - \tilde{b}_1(s_2, x_2)| \leq \left| \int_{\mathbb{R}^m} b_1(s_1, x_1, z) \mu^{s_1,x_1} (dz) - \mathbb{E} b_1(s_1, x_1, v_{t}^{s_1,x_1,0}) \right|
\]

\[
+ \left| \mathbb{E} b_1(s_1, x_1, v_{t}^{s_1,x_1,0}) - \mathbb{E} b_1(s_2, x_1, v_{t}^{s_1,x_1,0}) \right|
\]

\[
+ \left| \mathbb{E} b_1(s_2, x_1, v_{t}^{s_1,x_1,0}) - \mathbb{E} b_1(s_2, x_2, v_{t}^{s_1,x_1,0}) \right|
\]

\[
+ \left| \mathbb{E} b_1(s_2, x_2, v_{t}^{s_1,x_1,0}) - \int_{\mathbb{R}^m} b_1(s_2, x_2, z) \mu^{s, x_2} (dz) \right|
\]

\[
\leq C T e^{-\frac{\beta}{2} t} (1 + |x_1|^{2(\theta_2 \vee \theta_1) + 1} + |x_2|^{2(\theta_2 \vee \theta_1) + 1})
\]

\[
+ C T |s_1 - s_2|^\kappa (1 + |x_1|^{\theta_2} + \mathbb{E} |v_{t}^{s_1,x_1,0}|^{\theta_1})
\]

\[
+ C_{R,T} |x_1 - x_2| (1 + \mathbb{E} |v_{t}^{s_1,x_1,0}|^{\theta_1})
\]

\[
+ C T \mathbb{E} \left[ |v_{t}^{s_1,x_1,0} - v_{t}^{s_2,x_2,0}| (1 + |x_2|^{\theta_2} + |v_{t}^{s_1,x_1,0}|^{\theta_1} + |v_{t}^{s_2,x_2,0}|^{\theta_1}) \right]
\]

\[
\leq C T e^{-\frac{\beta}{2} t} (1 + |x_1|^{2(\theta_2 \vee \theta_1) + 1} + |x_2|^{2(\theta_2 \vee \theta_1) + 1})
\]

\[
+ C T |s_1 - s_2|^\kappa (1 + |x_1|^{\theta_2} + |x_1|^{\theta_1}) + C_{R,T} |x_1 - x_2| (1 + |x_1|^{\theta_1})
\]

\[
+ C T (|s_1 - s_2| + |x_1 + x_2|) (1 + |x_1|^{2(\alpha_1 \vee \theta_3)} + |x_2|^{2(\alpha_1 \vee \theta_2 \vee \theta_3)})
\]

\[
\leq C_{R,T} e^{-\frac{\beta}{2} t} + C_{R,T} (|s_1 - s_2|^\kappa + |s_1 - s_2|^{\ell} + |x_1 - x_2|).
\]
Let $t \to +\infty$, we obtain (4.8). Moreover, thanks to (4.5), we also have

$$
|b_1(t, x)| \leq \int_{\mathbb{R}^m} |b_1(t, x, z)| \mu^{t,x}(dz) \leq C_T \int_{\mathbb{R}^m} (1 + |x| + |z|) \mu^{t,x}(dz) \leq C_T (1 + |x|).
$$

The proof is complete. \qed

References

[1] A. N. Kolmogorov, Wienershe spiralen und einige andere interessante kurven in hilbertschen raum, C. R. (Dokl.) Acad. Sci. URSS (NS) 26 (1940) 115–118.

[2] B. B. Mandelbrot, J. W. Van Ness, Fractional brownian motions, fractional noises and applications, SIAM Review 10 (4) (1968) 422–437.

[3] R. Khasminskii, On the averaging principle for stochastic differential Itô equations, Kybernetika. 4 (1968) 260–279.

[4] D. Givon, Strong convergence rate for two-time-scale jump-diffusion stochastic differential systems, Multiscale. Model. Sim. 6 (2007) 577–594.

[5] M. Freidlin, A. Wentzell, Random Perturbations of Dynamical Systems, Springer Science and Business Media, Berlin Heidelberg, 2012.

[6] J. Q. Duan, W. Wang, Effective Dynamics of Stochastic Partial Differential Equations, Elsevier, 2014.

[7] Y. Xu, J. Q. Duan, W. Xu, An averaging principle for stochastic dynamical systems with Lévy noise, Physica D. 240 (2011) 1395–1401.

[8] Y. Xu, B. Pei, Y. G. Li, Approximation properties for solutions to non-Lipschitz stochastic differential equations with Lévy noise, Math. Method Appl. Sci. 38 (2015) 2120–2131.

[9] Y. Xu, B. Pei, J. L. Wu, Stochastic averaging principle for differential equations with non-Lipschitz coefficients driven by fractional Brownian motion, Stoch. Dynam. 17 (2017) 1750013.

[10] S. Cerrai, A khasminskii type averaging principle for stochastic reaction-diffusion equations, Ann. Appl. Probab. 19 (2009) 899–948.

[11] S. Cerrai, Averaging principle for systems of reaction-diffusion equations with polynomial nonlinearities perturbed by multiplicative noise, SIAM. J. Math. Anal. 43 (2011) 2482–2518.

[12] W. Wang, A. Roberts, Average and deviation for slow-fast stochastic partial differential equations, J. Differ. Equations. 253 (2012) 1265–1286.

[13] B. Pei, Y. Xu, J. L. Wu, Two-time-scales hyperbolic-parabolic equations driven by Poisson random measures: Existence, uniqueness and averaging principles, J. Math. Anal. Appl. 447 (2017) 243–268.
[14] B. Pei, Y. Xu, G. Yin, Stochastic averaging for a class of two-time-scale systems of stochastic partial differential equations, Nonlinear Anal. 160 (2017) 159–176.

[15] B. Pei, Y. Xu, G. Yin, Averaging principles for SPDEs driven by fractional Brownian motions with random delays modulated by two-time-scale Markov switching processes, Stoch. Dynam. 18 (2018) 1850023.

[16] S. Cerrai, A. Lunardi, Averaging principle for non-autonomous slow-fast systems of stochastic reaction-diffusion equations: The almost periodic case, SIAM. J. Math. Anal. 49 (2017) 2843–2884.

[17] W. Liu, M. Röckner, X. Sun, Y. Xie, Averaging principle for slow-fast stochastic differential equations with time dependent locally lipschitz coefficients, J. Differ. Equations 268 (6) (2020) 2910–2948.

[18] Y. Xu, R. F. Wang, Averaging principles for non-autonomous two-time-scale stochastic reaction-diffusion equations with jump, Complexity (2020) In Press.

[19] L. C. G. Rogers, Arbitrage with fractional brownian motion, Mathematical Finance 7 (1997) 95–105.

[20] D. Nualart, The Malliavin Calculus and Related Topics, Second Edition, Springer-Verlag, Berlin, 2006.

[21] F. Biagini, Y. Z. Hu, B. Øksendal, T. S. Zhang, Stochastic Calculus for Fractional Brownian Motion and Applications, Springer Science & Business Media, 2008.

[22] Y. Mishura, Stochastic Calculus for Fractional Brownian Motion and Related Processes, Springer, Berlin, 2008.

[23] M. Zahle, Integration with respect to fractal functions and stochastic calculus, I. Prob. Theory Relat. Fields 111 (3) (1998) 333–374.

[24] J. Guerra, D. Nualart, Stochastic differential equations driven by fractional Brownian motion and standard Brownian motion, Stoch. Anal. Appl. 26 (5) (2008) 1053–1075.

[25] S. Samko, A. Kilbas, O. Marichev, Fractional integrals and derivatives, theory and applications, Gordon and Breach Science Publishers, Yvendon, 1993.

[26] D. Nualart, A. Răşcanu, Differential equations driven by fractional Brownian motion, Collectanea Mathematica 53 (1) (2002) 55–81.

[27] A. Garsia, E. Rodemich, Monotonicity of certain functionals under rearrangement, Annales Institut Fourier 24 (2) (1974) 67–116.

[28] B. Pei, Y. Inahama, Y. Xu, Averaging principles for mixed fast-slow systems driven by fractional Brownian motion, arXiv preprint arXiv:2001.06945.