An \textbf{FPT}-algorithm for recognizing $k$-apices of minor-closed graph classes$^1$

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\textbf{Abstract}

Let $\mathcal{G}$ be a graph class. We say that a graph $G$ is a $k$-\textit{apex} of $\mathcal{G}$ if $G$ contains a set $S$ of at most $k$ vertices such that $G \setminus S$ belongs to $\mathcal{G}$. We prove that if $\mathcal{G}$ is minor-closed, then there is an algorithm that either returns a set $S$ certifying that $G$ is a $k$-apex of $\mathcal{G}$ or reports that such a set does not exist, in $2^{\text{poly}(k)} \cdot n^3$ time. Here \text{poly} is a polynomial function whose degree depends on the maximum size of a minor-obstruction of $\mathcal{G}$, i.e., the minor-minimal set of graphs not belonging to $\mathcal{G}$. In the special case where $\mathcal{G}$ excludes some apex graph as a minor, we give an alternative algorithm running in $2^{\text{poly}(k)} \cdot n^2$ time.

\textbf{Keywords}: Graph modification problems, irrelevant vertex technique, graph minors, parameterized algorithms, flat wall theorem.

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1 Introduction

Graph modification problems are fundamental in algorithmic graph theory. Typically, such a problem is determined by a graph class $G$ and some prespecified set $M$ of local modifications, such as vertex/edge removal or edge addition/contraction or combinations of them, and the question is, given a graph $G$ and an integer $k$, whether it is possible to transform $G$ to a graph in $G$ by applying $k$ modification operations from $M$. A plethora of graph problems can be formulated for different instantiations of $G$ and $M$. Applications span diverse topics such as computational biology, computer vision, machine learning, networking, and sociology [25]. As reported by Roded Sharan in [50], already in 1979, Garey and Johnson mentioned 18 different types of modification problems [26, Section A1.2]. For more on graph modification problems, see [9,25], as well as the running survey in [13]. In this paper we focus our attention on the vertex deletion operation. We say that a graph $G$ is a $k$-apex of a graph class $G$ if there is a set $S \subseteq V(G)$ of size at most $k$ such that the removal of $S$ from $G$ results in a graph in $G$. In other words, we consider the following meta-problem.

**Vertex Deletion to $G$**

**Input:** A graph $G$ and a non-negative integer $k$.

**Objective:** Find, if exists, a set $S \subseteq V(G)$, certifying that $G$ is $k$-apex of $G$.

To illustrate the expressive power of Vertex Deletion to $G$, if $G$ is the class of edgeless (resp. acyclic, planar, bipartite, (proper) interval, chordal) graphs, we obtain the Vertex Cover (resp. Feedback Vertex Set, Vertex Planarization, Odd Cycle Transversal, (proper) Interval Vertex Deletion, Chordal Vertex Deletion) problem.

By the classical result of Lewis and Yannakakis [41], Vertex Deletion to $G$ is NP-hard for every non-trivial graph class $G$. To circumvent its intractability, we study it from the parameterized complexity point of view and we parameterize it by the number $k$ of vertex deletions. In this setting, the most desirable behavior is the existence of an algorithm running in time $f(k) \cdot n^{O(1)}$, where $f$ is a function depending only on $k$. Such an algorithm is called fixed-parameter tractable, or FPT-algorithm for short, and a parameterized problem admitting an FPT-algorithm is said to belong to the parameterized complexity class FPT. Also, the function $f$ is called parametric dependence of the corresponding FPT-algorithm, and the challenge is to design FPT-algorithms with small parametric dependencies [14,18,21,44].

Unfortunately, we cannot hope for the existence of FPT-algorithms for every graph class $G$. Indeed, the problem is W-hard\(^1\) for some classes $G$ that are closed under induced subgraphs [42] or, even worse, NP-hard, for $k = 0$, for every class $G$ whose recognition problem is NP-hard, such as some classes closed under subgraphs or induced subgraphs (for instance 3-colorable graphs), edge contractions [11], or induced minors [19].

On the positive side, a very relevant subset of classes of graphs does allow for FPT-algorithms. These are classes $G$ that are closed under minors\(^2\), or minor-closed. To see this, we define $G_k$ as

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\(^1\)Implying that an FPT-algorithm would result in an unexpected complexity collapse; see [18].

\(^2\)A graph $H$ is a minor of a graph $G$ if it can be obtained from a subgraph of $G$ by contracting edges, see Subsection 2.2 for the definitions.
the class of the $k$-apices of $\mathcal{G}$, i.e., the \textit{yes}-instances of \textsc{Vertex Deletion to $\mathcal{G}$}, and observe that if $\mathcal{G}$ is minor-closed then the same holds for $\mathcal{G}_k$, for every $k$. This, in turn, implies that for every $k$, $\mathcal{G}_k$ can be characterized by a set $\mathcal{F}_k$ of minor-minimal graphs not in $\mathcal{G}_k$; we call these graphs the \textit{obstructions} of $\mathcal{G}_k$ and we know that they are finite because of the Robertson and Seymour theorem [48]. In other words, we know that the obstruction set of $\mathcal{G}_k$ is bounded by some function of $k$. Then one can decide whether a graph $G$ belongs to $\mathcal{G}_k$ by checking whether $G$ excludes all members of the obstruction set of $\mathcal{G}_k$, and this can be checked by using the FPT-algorithm in [47] (see also [20]).

As the Robertson and Seymour theorem [48] does not construct $\mathcal{F}_k$, the aforementioned argument is not constructive, i.e., it is not able to construct the claimed FPT-algorithm. An important step towards the constructibility of such an FPT-algorithm was done by Adler et al. [2], who proved that the parametric dependence of the above FPT-algorithm is indeed a constructible function.

The task of specifying (or even optimizing) this parametric dependence for different instantiations of $\mathcal{G}$ occupied a considerable part of research in parameterized algorithms. The most general result in this direction says that, for every $t$, there is some $c$ such that if the graphs in $\mathcal{G}$ have treewidth at most $t$, then $\textsc{Vertex Deletion to $\mathcal{G}$}$ admits an FPT-algorithm that runs in $c^k \cdot n^{O(1)}$ time [23,36]. Reducing the constant $c$ in this running time has attracted research on particular problems such as $\textsc{Vertex Cover}$ [12] (with $c = 1.2738$), $\textsc{Feedback Vertex Set}$ [38] (with $c = 3.619$), $\textsc{Apex-Pseudoforest}$ [10] (with $c = 3$), $\textsc{Pathwidth 1 Vertex Deletion}$ (with $c = 4.65$) [15] (see also [31] for further related results). The first step towards a parameterized algorithm for $\textsc{Vertex Deletion to $\mathcal{G}$}$ for cases where $\mathcal{G}$ has unbounded treewidth was done in [43] and later in [32] for the $\textsc{Planarization}$ problem, and the best parameterized dependence for this problem is $2^{O(k \log k)} \cdot n$, achieved by Jansen et al. [30]. These results were later extended by Kociumaka and Marcin Pilipczuk [39], who proved that if $\mathcal{G}_g$ is the class of graphs of Euler genus at most $g$, then $\textsc{Vertex Deletion to $\mathcal{G}_g$}$ admits a $2^{O(g^2 \log k)} \cdot n^{O(1)}$ step\textsuperscript{3} algorithm.

**Our results.** In this paper we give an explicit FPT-algorithm for $\textsc{Vertex Deletion to $\mathcal{G}$}$ for every minor-closed graph class $\mathcal{G}$. In particular, our main results are the following:

**Theorem 1.** If $\mathcal{G}$ is a minor-closed graph class, then $\textsc{Vertex Deletion to $\mathcal{G}$}$ admits an algorithm of time $2^{\text{poly}(k)} \cdot n^3$, for some polynomial $\text{poly}$ whose degree depends on $\mathcal{G}$.

We say that a graph $H$ is an \textit{apex} graph if it is a 1-apex of the class of planar graphs.

**Theorem 2.** If $\mathcal{G}$ is a minor-closed graph class excluding some apex graph, then $\textsc{Vertex Deletion to $\mathcal{G}$}$ admits an algorithm of time $2^{\text{poly}(k)} \cdot n^2$, for some polynomial $\text{poly}$ whose degree depends on $\mathcal{G}$.

In Subsection 8.2 we explain how the algorithms of Theorem 1 and Theorem 2 can be modified in order to apply to a series of variants of $\textsc{Vertex Deletion to $\mathcal{G}$}$.

\textsuperscript{3}Given a tuple $t = (x_1, \ldots, x_t) \in \mathbb{N}^t$ and two functions $\chi, \psi : \mathbb{N} \to \mathbb{N}$. We write $\chi(n) = \mathcal{O}(\psi(n))$ in order to denote that there exists a computable function $\phi : \mathbb{N}^t \to \mathbb{N}$ such that $\chi(n) = \mathcal{O}(\phi(t) \cdot \psi(n))$. 

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Our techniques. We provide here just a very succinct enumeration of the techniques that we use in order to achieve Theorem 1 and Theorem 2; a more detailed description with the corresponding definitions is provided, along with the algorithms, in the next sections.

Our starting point to prove Theorem 1 is to use the standard iterative compression technique of Reed et al. [46]. This allows us to assume that we have at hand a slightly too large set $S \subseteq V(G)$ such that $G \setminus S \in \mathcal{G}$. We then run the algorithm of Lemma 11 (whose proof uses [1,3,33,45]) that either reports that we have a no-instance, or concludes that the treewidth of $G$ is polynomially bounded by $k$, or finds a large wall $W_0$ in $G$. In the second case, we use the main algorithmic result of Baste et al. [5] (Proposition 4) to solve the problem parameterized by treewidth, achieving the claimed running time. In the latter case, we apply Proposition 14 (whose proof uses [8,34,35]) to find in $W_0$ a large flat wall $W$ together with an apex set $A$. Proposition 14 is an improved version of the original “Flat Wall Theorem” of Robertson and Seymour [47].

We find in $W$ a packing of an appropriate number of pairwise disjoint large enough subwalls. Two possible scenarios may occur. If the “interior” of each of these subwalls has enough neighbors in the set apex $S \cup A$, we apply a combinatorial result (Lemma 18) that guarantees that every possible solution should intersect $S \cup A$, and we can branch on it. On the other hand, if there exists a subwall whose interior $W$ has few neighbors in $S \cup A$, we argue that we can define from it a flat wall in which we can apply the irrelevant vertex technique of Robertson and Seymour [47] (Lemma 16). We stress that this flat subwall is not precisely a subwall of $W$ but a tiny “tilt” of a subwall of $W$, a new concept that is necessary for our proofs. The application of the irrelevant vertex technique requires a lot of technical care. For this, we use and enhance some of the ingredients introduced by Baste et al. [5].

In order to achieve the improved running time claimed in Theorem 2, we do not use iterative compression. Instead, we directly invoke Lemma 11. If the treewidth is small, we proceed as above. If a large wall is found, we apply Proposition 14 and we now distinguish two cases. If a large flat wall whose flaps have bounded treewidth is found, we find an irrelevant vertex using Lemma 16. Otherwise, inspired by an idea of Marx and Schlotter [43], we exploit the fact that $G$ excludes an apex graph, and we use flow techniques to either find a vertex that should belong to the solution, or to conclude that we are dealing with a no-instance.

Organization of the paper. In Section 2 we given the basic definitions and some preliminary results. Section 3 we introduce flat walls along with all the concepts and results around the Flat Wall Theorem. In Section 4 we present several algorithmic and combinatorial results that will be used during for the main algorithm, when finding an irrelevant vertex or when applying the branching step. In Section 5 we present the main algorithms claimed in Theorem 1 and Theorem 2. Among the results presented in Section 4, the proofs of the one supporting the irrelevant vertex technique is postponed in Section 6, while the proofs of those supporting the branching argument are postponed in Section 7. In Section 8 we explain how to modify our algorithms so to deal with a series of variants of the Vertex Deletion to $\mathcal{G}$ problem. We conclude in Section 9 with some directions for further research.
2 Definitions and preliminary results

Our first step is to restate the problem in a more convenient way. We next give some basic definitions and preliminary results.

2.1 Restating the problem

Let $\mathcal{F}$ be a finite non-empty collection of non-empty graphs. We use $\mathcal{F} \leq_m G$ to denote that some graph in $\mathcal{F}$ is a minor of $G$.

Let $\mathcal{G}$ be a minor-closed graph class and $\mathcal{F}$ be the set of its minor-obstructions. Clearly, VERTEX DELETION TO $\mathcal{G}$ is the same problem as asking, given a graph $G$ and some $k \in \mathbb{N}$, for a vertex set $S$ of at most $k$ vertices such that $\mathcal{F} \not\leq_m G \setminus S$. Following the terminology of [4–7, 23, 24, 36, 37], we call this problem $\mathcal{F}$-M-DELETION. In order to prove Theorem 1, we apply the iterative compression technique (introduced in [46]; see also [14]) and we give a $2^{\text{poly}(k)} \cdot n^2$ time algorithm for the following problem.

| $\mathcal{F}$-M-DELETION-COMPRESSION |
|------------------------------------|
| **Input:** | A graph $G$, a $k \in \mathbb{N}$, and a set $S$ of size $k + 1$ such that $\mathcal{F} \not\leq_m G \setminus S$. |
| **Objective:** | Find, if exists, a set $S' \subseteq V(G)$ of size at most $k$ such that $\mathcal{F} \not\leq_m G \setminus S'$. |

Some conventions. In what follows we always denote by $\mathcal{F}$ the obstruction set of the minor-closed class $\mathcal{G}$ of the instantiation of VERTEX DELETION TO $\mathcal{G}$ that we consider. Also, given a graph $G$, we define its apex number to be the smallest integer $a$ for which $G$ is an $a$-apex of the class of planar graphs. We define three constants depending on $\mathcal{F}$ that will be used throughout the paper whenever we consider such a collection $\mathcal{F}$. We define $a_\mathcal{F}$ as the minimum apex number of a graph in $\mathcal{F}$, we set $s_\mathcal{F} = \max\{|V(H)| \mid H \in \mathcal{F}\}$, and we set $\ell_\mathcal{F} = \max\{|E(H)| + |V(H)| \mid H \in \mathcal{F}\}$. We also agree that $n$ is the size of the input graph $G$. We can always assume that $G$ has $O(s_\mathcal{F} \cdot k \cdot n)$ edges, otherwise we can directly conclude that $(G, k)$ is a no-instance (for this, use the fact that $\mathcal{F}$-minor free graphs are sparse [40, 52]).

2.2 Preliminaries

Sets and integers. We denote by $\mathbb{N}$ the set of non-negative integers. Given two integers $p$ and $q$, the set $[p, q]$ refers to the set of every integer $r$ such that $p \leq r \leq q$. For an integer $p \geq 1$, we set $[p] = [1, p]$ and $\mathbb{N}_{\geq p} = \mathbb{N} \setminus [0, p − 1]$. For a set $S$, we denote by $2^S$ the set of all subsets of $S$. If $\mathcal{S}$ is a collection of objects where the operation $\cup$ is defined, then we denote $\bigcup \mathcal{S} = \bigcup_{X \in \mathcal{S}} X$.

Basic concepts on graphs. All graphs considered in this paper are undirected, finite, and without loops or multiple edges. We use standard graph-theoretic notation and we refer the reader to [17] for any undefined terminology. Let $G$ be a graph. We say that a pair $(L, R) \in 2^{V(G)} \times 2^{V(G)}$ is a separation of $G$ if $L \cup R = V(G)$ and there is no edge in $G$ between $L \setminus R$ and $R \setminus L$. Given a vertex $v \in V(G)$, we denote by $N_G(v)$ the set of vertices of $G$ that are adjacent to $v$ in $G$. For $S \subseteq V(G)$, we use the shortcut $G \setminus S$ to denote $G[V(G) \setminus S]$. Given a vertex $v \in V(G)$ with
neighbors \( u \) and \( w \), we define the *dissolution* of \( v \) to be the operation of deleting \( v \) and, if \( u \) and \( w \) are not adjacent, adding the edge \( \{u, w\} \). Given two graphs \( H, G \), we say that \( H \) is a *dissolution* of \( G \) if \( H \) can be obtained from \( G \) after dissolving vertices of \( G \). Given an edge \( e = \{u, v\} \in E(G) \), we define the *subdivision* of \( e \) to be the operation of deleting \( e \), adding a new vertex \( w \) and making it adjacent with \( u \) and \( v \). Given two graphs \( H, G \), we say that \( H \) is a *subdivision* of \( G \) if \( H \) can be obtained from \( G \) after subdividing edges of \( G \).

**Treewidth.** A *tree decomposition* of a graph \( G \) is a pair \((T, \chi)\) where \( T \) is a tree and \( \chi : V(T) \to 2^{V(G)} \) such that

1. \( \bigcup_{t \in V(T)} \chi(t) = V(G) \),
2. for every edge \( e \) of \( G \) there is a \( t \in V(T) \) such that \( \chi(t) \) contains both endpoints of \( e \), and
3. for every \( v \in V(G) \), the subgraph of \( T \) induced by \( \{t \in V(T) \mid v \in \chi(t)\} \) is connected.

The *width* of \((T, \chi)\) is defined as \( w(T, \chi) := \max \{ |\chi(t)| - 1 \mid t \in V(T) \} \). The *treewidth* of \( G \) is defined as

\[
\text{tw}(G) := \min \{ w(T, \chi) \mid (T, \chi) \text{ is a tree decomposition of } G \}.
\]

The following is the main result of [8]. We will use it to compute a tree decomposition of a graph of bounded treewidth.

**Proposition 3.** There is an algorithm that, given an graph \( G \) on \( n \) vertices and an integer \( k \), it outputs either a report that \( \text{tw}(G) > k \), or a tree decomposition of \( G \) of width at most \( 5k + 4 \). Moreover, this algorithm runs in \( 2^{O(k)} \cdot n \) steps.

**Contractions and minors.** The *contraction* of an edge \( e = \{u, v\} \) of a simple graph \( G \) results in a simple graph \( G' \) obtained from \( G \setminus \{u, v\} \) by adding a new vertex \( uv \) adjacent to all the vertices in the set \( N_G(u) \cup N_G(v) \setminus \{u, v\} \). A graph \( G' \) is a *minor* of a graph \( G \), denoted by \( G' \leq_m G \), if \( G' \) can be obtained from \( G \) by a sequence of vertex removals, edge removals, and edge contractions. If only edge contractions are allowed, we say that \( G' \) is a *contraction* of \( G \). If \( H \) is a minor of \( G \) then for every vertex \( v \in V(H) \) there is a set of vertices in \( G \) that are the endpoints of the edges of \( G \) contracted towards creating \( v \). We call this set *model* of \( v \) in \( G \). Given a finite collection of graphs \( \mathcal{F} \) and a graph \( G \), we use notation \( \mathcal{F} \leq_m G \) to denote that some graph in \( \mathcal{F} \) is a minor of \( G \).

We present here the main result of [5]. We will use this in order to solve \( \mathcal{F} \)-M-DELETION on instances of bounded treewidth.

**Proposition 4** (Baste et al. [5]). Let \( \mathcal{F} \) be a finite collection of graphs and let \( s_\mathcal{F} = \max \{|V(G)| \mid G \in \mathcal{F}\} \). There exists an algorithm that given a triple \((G, \text{tw}, k)\) where \( G \) is a graph on \( n \) vertices and of treewidth at most \( \text{tw} \), and \( k \) is a non-negative integer, it outputs, if it exists, a vertex set \( S \) of \( G \) of size at most \( k \) such that \( \mathcal{F} \not\leq_m G \setminus S \). Moreover, this algorithm runs in \( 2^{O_{\mathcal{F}}(\text{tw} \log \text{tw})} \cdot n \) steps.
Walls. An elementary $r$-wall, for some odd $r \geq 3$, is the graph obtained from a $(2r \times r)$-grid with vertices $(x, y) \in [2r] \times [r]$, after the removal of the “vertical” edges $\{(x, y), (x, y+1)\}$ for odd $x+y$, and then the removal of all vertices of degree one. Notice that, as $r \geq 3$, an elementary $r$-wall is a planar graph that has a unique (up to topological isomorphism) embedding in the plane $\mathbb{R}^2$ such that all its finite faces are incident to exactly six edges. The perimeter of an elementary $r$-wall is the cycle bounding its infinite face, while the cycles bounding its finite faces are called bricks. A brick that does not have common vertices with the perimeter is called internal brick. Also, the vertices in the perimeter of an elementary $r$-wall that have degree two are called pegs, while the vertices $(1,1), (2,r), (2r-1,1), (2r, r)$ are called corners (notice that the corners are also pegs).

Given an elementary $r$-wall $\bar{W}$, some $i \in \{1, 3, \ldots, 2r-1\}$, and $i' = (i + 1)/2$, the $i'$-th vertical path of $\bar{W}$ is the one whose vertices, in order of appearance, are $(i, 1), (i, 2), (i + 1, 2), (i + 1, 3), (i, 3), (i, 4), (i + 1, 4), (i + 1, 5), (i, 5), \ldots, (i, r-2), (i, r-1), (i + 1, r-1), (i + 1, r)$. Also, given some $j \in [2, r-1]$ the $j$-th horizontal path of $\bar{W}$ is the one whose vertices, in order of appearance, are $(1, j), (2, j), \ldots, (2r, j)$.

Figure 1: An 11-wall and its five layers.

An $r$-wall is any graph $W$ obtained from an elementary $r$-wall $\bar{W}$ after subdividing edges. A subgraph $W$ of a graph $G$ is called a wall of $G$ if $W$ is an $r$-wall for some odd $r \geq 3$ and we refer to $r$ as the height of the wall $W$. We insist that, for every $r$-wall, the number $r$ is always odd: for this, whenever an $r$-wall appears with $r$ even, we agree to round it up to the next odd $r+1$.

We call the vertices of an $r$-wall $W$ that are also vertices of $\bar{W}$ branch vertices. A cycle of $W$ is a brick (resp. the perimeter) of $W$ if its branch vertices are the vertices of a brick (resp. the perimeter) of $\bar{W}$. We denote by $\mathcal{C}(W)$ the set of all cycles of $W$. We use $D(W)$ in order to denote the perimeter of the wall $W$. A brick of $W$ is internal if it is disjoint from $D(W)$.

A vertical (resp. horizontal) path of $W$ is one whose branch vertices are the vertices of a vertical (resp. horizontal) path of $\bar{W}$. Notice that the perimeter and the bricks of an $r$-wall $W$ are uniquely defined regardless of the choice of the elementary $r$-wall $\bar{W}$. A subwall of $W$ is any subgraph $\bar{W}$ of $W$ that is an $r'$-wall, with $r' \leq r$, and such the vertical (resp. horizontal) paths of $\bar{W}$ are subpaths of the vertical (resp. horizontal) paths of $W$. A subwall of $W$ is internal if it is disjoint from the perimeter of $W$. 
The layers of an $r$-wall $W$ are recursively defined as follows. The first layer of $W$ is its perimeter. For $i = 2, \ldots, (r - 1)/2$, the $i$-th layer of $W$ is the $(i - 1)$-th layer of the subwall $W'$ obtained from $W$ after removing from $W$ its perimeter and removing recursively all occurring vertices of degree one. We refer to the $(r - 1)/2$-th layer as the inner layer of $W$. The central vertices of an $r$-wall are its two branch vertices that do not belong to any of its layers. See Figure 1 for an illustration of the notions defined above.

Given an $r$-wall $W$ and an odd $q \in \mathbb{N}_{\geq 3}$ where $r \geq q$, we define the central $q$-subwall of $W$, denoted by $W^{(q)}$, to be the $q$-wall obtained from $W$ after removing from $W$ its first $(r - q)/2$ layers and all occurring vertices of degree one.

The following result is derived from [1]. We will use it in order to find a wall in a bounded treewidth graph, given a tree decomposition of it.

**Proposition 5.** There is an algorithm that, given a graph $G$ on $m$ edges, a graph $H$ on $h$ edges without isolated vertices, and a tree decomposition of $G$ of width at most $k$, it outputs, if it exists, a minor of $G$ isomorphic to $H$. Moreover, this algorithm runs in $2^{O(k \log k)} \cdot h^{O(k)} \cdot 2^{O(h)} \cdot m$ steps.

## 3 Flat walls

In this section we deal with flat walls. More precisely, in Subsection 3.1 we define flat walls and some notions related to them, using the terminology of [35]. There, we also stress that a subwall of a flat wall is not necessarily a flat wall itself. To resolve this, in Subsection 3.2, we prove that given an internal subwall $W'$ of a flat wall $W$, there exists a small “tilt” of $W'$ that is indeed a flat wall. Finally, in Subsection 3.3, using the terminology of [5], we define homogeneous subwalls of flat walls and present some results related to this notion that are proved in [5].

### 3.1 Definitions

**Renditions.** Let $\Delta$ be a closed disk, i.e., a set homeomorphic to the set $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$. Given a subset $X$ of $\Delta$, we denote its closure by $\overline{X}$ and its boundary by $\text{bor}(X)$. A $\Delta$-painting is a pair $\Gamma = (U, N)$ where $N$ is a finite set of points of $\Delta$, $N \subseteq U \subseteq \Delta$, $U \setminus N$ has finitely many arcwise-connected components, called cells, such that, for every cell $c$, $\overline{c}$ is a closed disk, $\text{bor}(c) \cap \Delta \subseteq N$, and $|\text{bor}(c) \cap N| \leq 3$. We use the notation $U(\Gamma) := U$, $N(\Gamma) := N$ and denote the set of cells of $\Gamma$ by $C(\Gamma)$.

Notice that, given a $\Delta$-painting $\Gamma$, the pair $(N(\Gamma), \{c \cap N \mid c \in C(\Gamma)\})$ is a hypergraph whose hyperedges have cardinality at most three, and $\Gamma$ can be seen as a plane embedding of this hypergraph in $\Delta$.

Let $G$ be a graph, and let $\Omega$ be a cyclic permutation of a subset of $V(G)$ that we denote by $V(\Omega)$. By an $\Omega$-rendition of $G$ we mean a triple $(\Gamma, \sigma, \pi)$, where (a) $\Gamma$ is a $\Delta$-painting for some closed disk $\Delta$, (b) $\pi : N(\Gamma) \to V(G)$ is an injection, and (c) $\sigma$ assigns to each cell $c \in C(\Gamma)$ a subgraph $\sigma(c)$ of $G$, such that

1. $G = \bigcup_{c \in C(\Gamma)} \sigma(c),$
2. for distinct $c, c' \in C(\Gamma)$, $\sigma(c)$ and $\sigma(c')$ are edge-disjoint,
Figure 2: A graph $G$ together with a tight $\Omega$-rendition of $G$.

(3) for every cell $c \in C(\Gamma)$, $\pi(c \cap N) \subseteq V(\sigma(c))$,

(4) for every cell $c \in C(\Gamma)$, $V(\sigma(c)) \cap \bigcup_{c' \in C(\Gamma) \setminus \{c\}} V(\sigma(c')) \subseteq \pi(c \cap N)$, and

(5) $\pi(N(\Gamma) \cap \text{bor}(\Delta)) = V(\Omega)$, such that the points in $N(\Gamma) \cap \text{bor}(\Delta)$ appear in $\text{bor}(\Delta)$ in the same ordering as their images, via $\pi$, in $\Omega$.

See Figure 2 for an example of a graph $G$ together with an $\Omega$-rendition of $G$.

We say that an $\Omega$-rendition $(\Gamma, \sigma, \pi)$ of $G$ is tight if the following conditions are satisfied:

(i) For every $c \in C(\Gamma)$, the graph $\sigma(c) \setminus \pi(c \cap N)$, when non-null, is connected and the neighborhood of its vertex set in $G$ is $\pi(c \cap N)$.

(ii) For every $c \in C(\Gamma)$ there are $|c \cap N|$ vertex-disjoint paths in $G$ from $\pi(c \cap N)$ to the set $V(\Omega)$.

(iii) If there are two points $x, y$ of $N$ such that $e = \{\pi(x), \pi(y)\} \in E(G)$, then there is a $c \in C(\Gamma)$ such that $\sigma(c)$ is the two-vertex connected graph $(e, \{e\})$.

For an example of a tight rendition, see Figure 2. It is easy to see that given an $\Omega$-rendition of a graph $G$ where $V(\Omega)$ contains at least three vertices that are in a cycle of $G$, a tight $\Omega$-rendition of $G$ can be constructed in $O(n + m)$ steps. The first property is imposed by breaking every cell that violates it into as many cells as the connected components of $\sigma(c) \setminus \pi(c \cap N)$. The second property can be achieved as follows: we first construct an auxiliary planar graph $G'$ by substituting in $G$ each $\sigma(c)$ by a clique on $\pi(c \cap N)$ and by adding a new vertex $v_{\text{new}}$ adjacent to all the vertices in $V(\Omega)$; then the new rendition can be easily constructed using the triconnected component of $G'$ that contains $v_{\text{new}}$ (for this, one may use the classic algorithm of Hopcroft and Tarjan [28] that runs in $O(n + m)$ time). The enforcement of the third property is straightforward and can be done in $O(n)$ time.
Flat walls. Let $G$ be a graph and let $W$ be an $r$-wall of $G$. We say that a pair $(P, C) \subseteq D(W) \times D(W)$ is a choice of pegs and corners for $W$ if $W$ is the subdivision of an elementary $r$-wall $\bar{W}$ where $P$ and $C$ are the pegs and the corners of $\bar{W}$, respectively (clearly, $C \subseteq P$). To get more intuition, notice that a wall $W$ can occur in several ways from the elementary wall $\hat{W}$, depending on the way the vertices in the perimeter of $\hat{W}$ are subdivided. Each of them gives a different selection $(P, C)$ of pegs and corners of $W$.

We say that $W$ is a flat $r$-wall of $G$ if there is a separation $(X, Y)$ of $G$ and a choice $(P, C)$ of pegs and corners for $W$ such that:

- $V(W) \subseteq Y$,
- $P \subseteq X \cap Y \subseteq V(D(W))$, and
- if $\Omega$ is the cyclic ordering of the vertices $X \cap Y$ as they appear in $D(W)$, then there exists an $\Omega$-rendition $(\Gamma, \sigma, \pi)$ of $G[Y]$.

Given a graph $G$, we say that the pair $(A, W)$ is an $(a, r)$-apex wall pair of $G$ if $A$ is a subset of $a$ vertices from $G$ and $W$ is a flat $r$-wall of $G \setminus A$.

Figure 3: A graph $G$ consisting of an “orange” 7-wall $W$ and some black edges between its vertices.

In Figure 3, the “orange” 7-wall $W$ is a flat wall of $G$. The squared vertices are the pegs of $W$ while the fat squared vertices are its corners. Notice that $W$ contains only one internal 5-subwall $W' = W^{(5)}$ and many 3-subwalls, among them the wall $W''' = W^{(3)}$ (depicted in blue). Of course, the graph $G$ in Figure 3 contains also other walls as subgraphs such as the wall $W''$ consisting of the purple, green, and blue edges. Notice that $W'$ and $W'''$ are not a flat wall, while $W''$ is.

Compass and flaps. Given a flat wall $W$ of a graph $G$ as above, we call $G[Y]$ the compass of $W$ in $G$, denoted by $\text{compass}(W)$. We call $(X, Y)$ the separation certifying the flat wall $W$ and $X \cap Y$ is called the frontier of $W$. The ground set of $W$ is $\text{ground}(W) := \pi(N(\Gamma))$. We clarify that
ground(W) may contain vertices of the compass of W that are not necessarily vertices of W (this is not the case in the example of Figure 3 where the ground vertices of W is a subset of the vertices of W).

We also call the graphs in \( \text{flaps}(W) := \{ \sigma(c) \mid c \in C(\Gamma) \} \) flaps of the wall W. For each flap \( F \in \text{flaps}(W) \) we define its base as the set \( \partial F := V(F) \cap \text{ground}(W) \). A flap \( F \in \text{flaps}(W) \) is trivial if \( |\partial F| = 2 \) and it consists of one edge between the two vertices in \( \partial F \). As an example, the wall \( W'' \) in Figure 3, formed by all the fat edges (purple, green, and blue), is a flat wall. The pegs are the diamond vertices, the corners are the fat diamond vertices, and the rendition has two types of flaps: those whose base has three vertices, that are inside the light-blue disks, and those that are trivial flaps and are the purple fat edges that are outside of the light-blue disks (see also Figure 4 for the rendition of \( W'' \)). Notice that none of the internal subwalls of W is a flat wall.

![Figure 4: The rendition of the compass of the flat 5-wall W'' of Figure 3. Trivial flaps are depicted in purple.](image)

We also refer to the triple \( (\Gamma, \sigma, \pi) \) as a rendition of the compass of W in G (for some choice of pegs and corners). The linear-time transformation of a rendition to a tight one allows us to assume (but also to demand) that every rendition of the compass of a flat wall that we consider in this paper is tight.

**Tilts.** Given a wall \( W' \), we define its inpegs \( L_{W'} \) as the vertices of its perimeter that are incident to edges of W that are not in its perimeter. The interior of \( W' \) is the graph

\[
W'[\{V(W') \setminus V(P)\} \cup L_{W'}] \setminus E(P).
\]

We say that a wall \( W'' \) is a tilt of a wall \( W' \) if \( W'' \) and W have identical interiors. For instance, in Figure 3 the wall \( W'' \) is a tilt of \( W' = W^{(5)} \).

At this point, we ascertain that an internal subwall of a wall W is not necessarily a flat wall (see the 7-wall \( W' \) of the graph G in Figure 3 where, for instance, the walls \( W'' = W^{(3)} \) and \( W' = W^{(5)} \) are subwalls of W but they are not flat). On the positive side, as we prove later in Lemma 7 in Subsection 3.2, every internal subwall of a wall W has a tilt that is a flat wall.
Partially disk-embedded graphs. We say that a graph \( G \) is partially disk-embedded in some closed disk \( \Delta \), if there is some subgraph \( K \) of \( G \) that is embedded in \( \Delta \) in a way that \((V(G) \cap \Delta, V(G) \setminus \text{int}(\Delta))\) is a separation of \( G \), where \text{int} is used to denote the interior of a subset of the plane. From now on, we use the term partially \( \Delta \)-embedded graph \( G \) to denote that a graph \( G \) is partially disk-embedded in some closed disk \( \Delta \). We call the graph \( K = G \cap \Delta \) compass of the \( \Delta \)-embedded graph \( G \) and we assume that \( G \) is accompanied by an embedding of its compass in \( \Delta \), that is the set \( G \cap \Delta \). We say that \( G \) is a \( \Delta \)-embedded graph if it is partially \( \Delta \)-embedded graph and \( G \subseteq \Delta \) (the whole \( G \) is embedded in \( \Delta \)).

Levelings. Let \( W \) be a flat wall of a graph \( G \). Following [5], we define the leveling of \( W \) in \( G \), denoted by \( \tilde{W} \), as the bipartite graph where one part is the ground set of \( W \), the other part is the set of flaps of \( W \), and, given a pair \((v, F) \in \text{ground}(W) \times \text{flaps}(W)\), the set \( \{v, F\} \) is an edge of \( \tilde{W} \) if and only if \( v \in \partial F \). Again, keep in mind that \( \tilde{W} \) may contain (many) vertices that are not in \( W \). Notice that the incidence graph of the plane hypergraph \((N(\Gamma), \{c \cap N \mid c \in C(\Gamma)\})\) is isomorphic to \( \tilde{W} \) via an isomorphism that extends \( \pi \) and, moreover, bijectively corresponds cells to flaps. This permits us to treat \( \tilde{W} \) as a \( \Delta \)-embedded graph where \( \text{bor}(\Delta) \cap \tilde{W} \) is the frontier of \( W \). We call the vertices of \( \text{ground}(W) \) (resp. \( \text{flaps}(W) \)) ground-vertices (resp. flap-vertices) of \( \tilde{W} \). See Figure 5 for an example of a leveling. Recall that each edge \( e \) of \( \text{compass}(W) \) belongs to exactly one flap of \( W \).

![Leveling of a flat wall](image)

Figure 5: The leveling of the flat 5-wall \( W'' \) of Figure 3. The green vertices are the flap-vertices and the non-green vertices are the ground-vertices.

If both endpoints of \( e \) are in the boundary of this flap, then this flap should be a trivial one and we say that \( e \) is a short edge of \( \text{compass}(W) \). We define the graph \( W^\bullet \) as the graph obtained from \( W \) if we subdivide once every short edge in \( W \). The next observation is a consequence of the following three facts: flap-vertices of \( \tilde{W} \) have degree at most three, all the vertices of a wall have degree at most three, and every separation \((A, B)\) of order at most three of a wall is trivial.

**Observation 6.** If \( W \) is a flat wall of a graph \( G \), then the leveling \( \tilde{W} \) of \( W \) in \( G \) contains a subgraph \( W^R \) that is isomorphic to some subdivision of \( W^\bullet \) via an isomorphism that maps each ground vertex to itself.
We call the graph $W^R$ as in Observation 6 representation of the flat wall $W$ in the $\Delta$-embedded graph $\tilde{W}$, and therefore we can see it as a $\Delta$-embedded subgraph of $\tilde{W}$. Notice that the above observation permits to bijectively map each cycle of $W$ to a cycle of $W^R$ that is also a cycle of $W$. That way, each cycle $C$ of $W$ corresponds to a cycle $C'$ of $W^R$ denoted by $C^R$ and we call $C^R$ the representation of $C$ in $\tilde{W}$. From now on, we reserve the superscript $-^{R\prime}$-notation to denote the correspondence between $W$ (resp. $C$) and $W^R$ (resp. $C^R$).

We define the function $\text{flaps} : C(W) \to 2^{\text{flaps}(W)}$ so that, for each cycle $C$ of $W$, $\text{flaps}(C)$ contains each flap $F$ of $W$ that, when seen as a flap-vertex of the $\Delta$-embedded graph $\tilde{W}$, belongs to the closed disk bounded by $C^R$.

3.2 Creating flat walls from internal subwalls of flat walls

The following result is very similar to [35, Lemma 6.1]. Its proof is strongly based on the notion of levelings.

**Lemma 7.** Let $a, r, r' \in \mathbb{N}$, where $r > r' \geq 3$. Also, let $G$ be a graph, let $(A, W)$ be an $(a, r)$-apex wall pair of $G$, and let $W'$ be an internal $r'$-subwall of $W$. Then $W'$ has a tilt $W''$ such that $(A, W'')$ is an $(a, r')$-apex wall pair of $G$. Moreover,

1. the compass of $W''$ in $G \setminus A$ is a subgraph of the compass of $W$ in $G \setminus A$ and

2. if $P'$ is the perimeter of $W'$, then the vertex set of the compass of $W''$ in $G \setminus A$ is a subset of $\bigcup \text{flaps}(P')$.

Moreover, given $G$, $(A, W)$, and $W'$, the $(a, r')$-apex wall pair $(A, W'')$ can be constructed in $O(n)$ time.

**Proof.** Let $(\Gamma, \sigma, \pi)$ be a rendition of the compass of $W$ in $G \setminus A$. Let $\tilde{W}$ be the leveling of $W$ in $G \setminus A$ and let $W^R$ be the representation of $W$ in $\tilde{W}$. Let $W'^R$ be the subwall of $W^R$ corresponding to $W'$ and let $P'^R$ be the perimeter of $W'^R$. We call the vertices of $\tilde{W}$ that are inside the closed disk bounded by $P'^R$ cropped vertices of $\tilde{W}$.

![Figure 6: A boundary flap $F$ in the rendition of $W$ (depicted in gray in the leftmost figure). The green cells in the rightmost figure are the cells of the rendition that certified the flatness of $W''$.](image)

Let now $F$ be a flap-vertex of $V(P'^R)$ where $\partial F = \{x, y, z\}$ and $z$ is not a ground vertex of $W'^R$. We call such flaps boundary flaps and take into account that, when seen as vertices of $\tilde{W}$, they are cropped. Given a boundary flap $F$ where $\partial F = \{x, y, z\}$ and $z$ is not a ground vertex of $W'^R$, we define $P_F$ as a shortest path between $x$ and $y$ in $F$. We also define $w$ as a vertex of this path for
which there exists a path in $F$ from $w$ to $z$ avoiding all other vertices of $P_F$. We call $w$ the peg of $P_F$. We now consider the wall $W''$ of $G \setminus A$ that is created by substituting in $W'$, for each boundary flap $F$, the path $P' \cap F$ by the path $P_F$, where $P'$ is the perimeter of $W'$. Clearly $W''$ is a tilt of $W'$. We claim that $W''$ is a flat wall of $G \setminus A$. To see this, we define a new set of pegs and corners and we construct a rendition of the compass of $W''$ as follows. We first construct a painting $\Gamma'$ by removing from $\Gamma$ all the cells corresponding either to boundary flaps or to non-cropped flap vertices of $\bar{\bar{\hat{W}}}$, as well as all the points that correspond to non-cropped ground vertices of $\bar{\bar{\hat{W}}}$ (in Figure 6 this painting is depicted in green – notice that the cell corresponding to the cropped flap-vertex is not included as $F$ is a boundary flap). We also define $\pi'$ by projecting $\pi$ to the points of $\Gamma'$ and $\sigma'$ by projecting $\sigma$ to the cells of $\Gamma'$. We also declare as pegs of $W''$ all the ground vertices of $W''$ that are incident to edges of $W$ that do not belong to $W''$. The corners of $W''$ are the pegs of $W''$ that are corners of $W''$. Notice that $(\Gamma', \sigma', \pi')$ is not yet a rendition of $W''$ and that this choice of pegs and corners may be incomplete because the shortest paths $P_F$, corresponding to the boundary flaps in the perimeter of $W''$, are not yet taken into account. To complete the construction, for every boundary flap $F$, we enhance $\Gamma'$, with at most $|E(P_F)|$ additional cells each corresponding to a flap consisting of a maximal induced subpath of $P_F$ and we add the peg $w$ of $P_F$ in the set of pegs. Notice that we insist that the obtained rendition is tight, and this is the reason why we partition $P_F$ into maximal induced subpaths, instead of creating one flap for each of its edges. In the special case where the flap vertex $F$ is a corner of $W''$ we additionally declare the peg $w$ as a corner of $W''$. It is now easy to see that, under this choice of pegs and corners, $(\Gamma', \sigma', \pi')$ is a rendition of $W''$. This proves that $W''$ is indeed a flat wall. The fact that the compass of $W''$ is a subgraph of the compass of $W$ follows directly from the fact that all the flaps of $W''$ are either flaps of $W$ or subgraphs of flaps of $W$. The second statement follows by the construction of $W''$. \hfill \Box

Notice that the construction of $W''$ given in the proof of Lemma 7 is not unique, as the choice of the shortest paths $P_F$ is not unique.

From now on we refer to a pair $(A, W'')$ as in Lemma 7 as an $(a, r')$-apex wall pair generated by the internal $r'$-subwall $W'$ of $W$, and we keep in mind that the compasses of all such flat walls $W''$ may differ only on their perimeter.

The proof of Lemma 7 also implies the following.

**Observation 8.** Let $W$ be a flat wall of a graph $G$, and $W^R$ be the representation of $W$ in the leveling $\hat{W}$ of $W$ in $G$. Then for every internal subwall $\bar{\bar{\hat{W}}}$ of $W^R$ there exist an internal subwall $\hat{W}'$ of $\hat{W}$ and a tilt $W''$ of $W'$ such that

- $\hat{W}$ is the representation of $W'$ in the leveling $\hat{W}$ of $W$,
- $W''$ is a flat wall, and
- the vertex set of the compass of $W''$ in $G$ is a subset of $\bigcup \text{flaps}(P')$, where $P'$ is the perimeter of $W'$.

Moreover, given $G$, $W$, and $\hat{W}$, the flat wall $W''$ can be constructed in $O(n)$ steps.
3.3 Homogeneous walls

We first present some definitions on boundaried graphs and folios that will be used to define the notion of homogeneous walls. Following this, we present some results concerning homogeneous walls that are key ingredients in our proofs.

Boundaried graphs. Let \( t \in \mathbb{N} \). A \( t \)-boundaried graph is a triple \( G = (G, B, \rho) \) where \( G \) is a graph, \( B \subseteq V(G) \), \( |B| = t \), and \( \rho : B \rightarrow [t] \) is a bijection. We say that \( G_1 = (G_1, B_1, \rho_1) \) and \( G_2 = (G_2, B_2, \rho_2) \) are isomorphic if there is an isomorphism from \( G_1 \) to \( G_2 \) that extends the bijection \( \rho_2^{-1} \circ \rho_1 \). The triple \((G, B, \rho)\) is a boundaried graph if it is a \( t \)-boundaried graph for some \( t \in \mathbb{N} \). As in [47], we define the detail of a boundaried graph \( G = (G, B, \rho) \) as \( \text{detail}(G) := \max\{|E(G)|, |V(G) \setminus B|\} \). We denote by \( \mathcal{B}(t) \) the set of all (pairwise non-isomorphic) \( t \)-boundaried graphs. We also set \( \mathcal{B} := \bigcup_{t \in \mathbb{N}} \mathcal{B}(t) \).

Folios. We say that a \( t \)-boundaried graph \( G_1 = (G_1, B_1, \rho_1) \) is a minor of a \( t \)-boundaried graph \( G_2 = (G_2, B_2, \rho_2) \), denoted by \( G_1 \preceq_m G_2 \), if there is a sequence of removals of non-boundary vertices, edge removals, and edge contractions in \( G_2 \), disallowing contractions of edges with both endpoints in \( B_2 \), that transforms \( G_2 \) to a boundaried graph that is isomorphic to \( G_1 \) (during edge contractions, boundary vertices prevail). Note that this extends the usual definition of minors in graphs without boundary.

We say that \((M, T)\) is a tm-pair if \( M \) is a graph, \( T \subseteq V(M) \), and all vertices in \( V(M) \setminus T \) have degree two. We denote by \( \text{diss}(M, T) \) the graph obtained from \( M \) by dissolving all vertices in \( V(M) \setminus T \). A tm-pair of a graph \( G \) is a tm-pair \((M, T)\) where if \( M \) is a subgraph of \( G \). We call the vertices in \( T \) branch vertices of \((M, T)\).

If \( M = (M, B, \rho) \in \mathcal{B} \) and \( T \subseteq V(M) \) with \( B \subseteq T \), we call \((M, T)\) a btm-pair and we define \( \text{diss}(M, T) = (\text{diss}(M, T), B, \rho) \). Note that we do not permit dissolution of boundary vertices, as we consider all of them to be branch vertices. If \( G = (G, B, \rho) \) is a boundaried graph and \((M, T)\) is a tm-pair of \( G \) where \( B \subseteq T \), then we say that \((M, T)\), where \( M = (M, B, \rho) \), is a btm-pair of \( G = (G, B, \rho) \). Let \( G_i = (G_i, B_i, \rho_i) \), \( i \in [2] \). We say that \( G_1 \) is a topological minor of \( G_2 \), denoted by \( G_1 \preceq_{tm} G_2 \), if \( G_2 \) has a btm-pair \((M, T)\) such that \( \text{diss}(M, T) \) is isomorphic to \( G_1 \). We define the \( \ell \)-folio of \( G = (G, B, \rho) \in \mathcal{B} \) as

\[
\text{\ell-folio}(G) = \{G' \in \mathcal{B} \mid G' \preceq_{tm} G \text{ and } G' \text{ has detail at most } \ell \}.
\]

Homogeneous walls. Let \( G \) be a graph and \( W \) be a flat wall of \( G \). Let also \((\Gamma, \sigma, \pi)\) be a rendition of the compass of \( W \) in \( G \). Recall that \( \Gamma = (U, N) \) is a \( \Delta \)-painting for some closed disk \( \Delta \). Given a flap \( F \), we denote by \( \Omega(F) \) the counter-clockwise ordering of the vertices of \( \partial F \) as they appear in the corresponding cell of \( C(\Gamma) \). Notice that as \( |\partial F| \leq 3 \), this cyclic ordering is significant only when \( |\partial F| = 3 \), in the sense that \((x_1, x_2, x_3)\) remains invariant under shifting, i.e., \((v_1, v_2, v_3) \equiv (v_2, v_3, v_1)\) but not under inversion, i.e., \((v_1, v_2, v_3) \neq (v_3, v_2, v_1)\).

Let \( G \) be a graph and let \((A, W)\) be an \((a, r)\)-apex wall pair of \( G \). For each cell \( F \in \text{flaps}(W) \) with \( t_F = |\partial F| \), we fix \( \rho_F : \partial F \rightarrow [a + 1, a + t_F] \) such that \( \rho_F^{-1}(a + 1), \ldots, \rho_F^{-1}(a + t_F) \equiv \Omega(c) \). We also fix a bijection \( \rho_A : A \rightarrow [a] \). For each flap \( F \in \text{flaps}(W) \), we define the boundaried graph
\( \mathcal{F}^A := (G[A \cup F], A \cup \partial F, \rho_A \cup \rho_F) \) and we denote by \( F^A \) the underlying graph of \( \mathcal{F}^A \). We call \( \mathcal{F}^A \) an augmented flap of \( (A, W) \). Notice that \( G[V(\text{compass}(W)) \cup A] = \bigcup_{F \in \text{flaps}(W)} F^A \).

Given a \( \ell \in \mathbb{N} \), we say that two flaps \( F_1, F_2 \in \text{flaps}(W) \) are \((A, \ell)\)-equivalent, denoted by \( F_1 \sim_{A,\ell} F_2 \), if
\[
\ell\text{-folio}(\mathcal{F}^A_1) = \ell\text{-folio}(\mathcal{F}^A_2).
\]

For each \( F \in \text{flaps}(W) \), we define the \((a, \ell)\)-color of \( F \), denoted by \((a, \ell)\text{-color}(F)\), as the equivalence class of \( \sim_{A,\ell} \) to which \( \mathcal{F}^A \) belongs.

Let \( \tilde{W} \) be the leveling of \( W \) in \( G \setminus A \) and let \( W^R \) be the representation of \( W \) in \( \tilde{W} \). Recall that \( \tilde{W} \) is a \( \Delta \)-embedded graph. For each cycle \( C \) of \( W \), we define the \((a, \ell)\)-palette of \( C \), denoted by \((a, \ell)\text{-palette}(C)\), as the set of all the \((a, \ell)\)-colors of the flaps that appear as vertices of \( \tilde{W} \) in the closed disk bounded by \( C^R \) in \( \Delta \) (recall that by \( C^R \) we denote the representation of \( C \) in \( \tilde{W} \)).

Let \( a, \ell, r, q \in \mathbb{N} \), where \( r > q \geq 3 \) and let \((A, W)\) be an \((a, r)\)-apex wall pair of a graph \( G \). We say that \((A, W)\) is an \((\ell, q)\)-homogeneous \((a, r)\)-apex wall pair of \( G \) if every internal brick \( B \) of \( W \) that is not a brick of \( W^{(q)} \) has the same \((a, \ell)\)-palette (seen as a cycle of \( W \)). If we drop the demand that “\( B \) is not a brick of \( W^{(q)} \)” then we simply say that \((A, W)\) is an \( \ell\)-homogeneous \((a, r)\)-apex wall pair of \( G \).

The following observation is a consequence of the fact that, given a wall \( W \) and an internal subwall \( W' \) of \( W \), every internal brick of a tilt \( W'' \) of \( W' \) is also an internal brick of \( W \).

**Observation 9.** Let \( a, r, r' \in \mathbb{N} \), where \( r > r' \geq 3 \). Also, let \( G \) be a graph, let \((A, W)\) be an \((a, r)\)-apex wall pair of \( G \), and let \( W' \) be an internal \( r'\)-subwall of \( W \). If \((A, W)\) is \((\ell, q)\)-homogeneous for some \( \ell, q \in \mathbb{N} \) where \( r > q \geq 3 \), then every \((a, r')\)-apex wall pair \((A, W'')\) generated by \( W' \) is \((\ell, q)\)-homogeneous.

**The price of homogeneity.** Our purpose is to apply the irrelevant vertex technique and, for this, we need homogeneous walls. Finding a homogeneous flat wall inside a flat wall has a price, as it requires a polynomially larger wall. For this we restate [5, Lemma 4.3] as follows.

**Proposition 10.** There is a function \( f_1 : \mathbb{N}^3 \to \mathbb{N} \) such that if \( \ell, r, a \in \mathbb{N} \), where \( r \geq 3 \), \( G \) is a graph, and \((A, W)\) is an \((a, f_1(\ell, r, a))\)-apex wall pair of \( G \), then \( W \) has an \( r\)-subwall \( W' \) such that every \((a, r)\)-apex wall pair of \( G \) that is generated by \( W' \) is \( \ell\)-homogeneous. Moreover, it holds that \( f_1(\ell, r, a) = O(r^{c_{a,\ell}}) \), for some constant \( c_{a,\ell} \) depending on \( a \) and \( \ell \).

We point out that the constant \( c_{a,\ell} \) of **Proposition 10** is equal to the number of different folios that can be rooted through the augmented flaps of an apex wall pair \((A, W)\). In general, it follows that \( c_{a,\ell} = 2^{O((a+\ell) \log(a+\ell))} \). However, by using the notion of representatives instead of folios as in [5], we can obtain a smaller bound of \( c_{a,\ell} = 2^{O((a+\ell) \log(a+\ell))} \). We call \( c_{a,R,\ell,\mathcal{F}} \) \((a, \ell)\)-palette-variety of \( \mathcal{F} \). For simplicity, we write \( c_{\mathcal{F}} \) instead of \( c_{a,R,\ell,\mathcal{F}} \).

### 4 Auxiliary algorithmic and combinatorial results

In this section we provide some algorithmic and combinatorial results that will support the main algorithms of this paper. In **Subsection 4.1**, we present two algorithmic results that deal with the
problem of finding a (flat) wall in a graph. Then, in Subsection 4.2 we provide an algorithm that finds an irrelevant vertex inside a “large enough” flat wall. In Subsection 4.3, we present some combinatorial results that allow our algorithms to branch. Finally, in Subsection 4.4, we define canonical partitions of walls, a notion that will be useful for the application of the combinatorial results in our algorithms.

4.1 Two algorithmic results

In this subsection we provide some algorithmic results that, in general, deal with the problem of finding a (flat) wall in a graph and that will support our algorithms. We first prove Lemma 11 that intuitively states that there is an algorithm that, given a graph $G$ and two non-negative integers $r$ and $k$, outputs either that $(G,k)$ is a no-instance of $F$-M-Deletion, or a report that the treewidth of $G$ is polynomially bounded by $r$ and $k$, or an $r$-wall of $G$. Following this, we conclude this subsection by presenting a result derived from [35] and [34] implying the existence of an algorithm that, given a graph $G$ and a wall of $G$, outputs either a clique or a flat wall of $G$. Recall that $s_F = \max\{|V(H)| \mid H \in \mathcal{F}\}$.

**Lemma 11.** There exist a function $f_2 : \mathbb{N} \to \mathbb{N}$ and an algorithm as follows:

**Find-Wall**($G,r,k$)

**Input:** A graph $G$, an odd $r \in \mathbb{N}_{\geq 3}$, and an $k \in \mathbb{N}$.

**Output:** One of the following:

- Either a report that $G$ has treewidth at most $f_2(s_F) \cdot r + k$, or
- an $r$-wall $W$ of $G$, or
- a report that $(G,k)$ is a no-instance of $F$-M-Deletion.

Moreover, this algorithm runs in $2^{O(s_F(r^2 + (k + r) \log(k + r)))} \cdot n$ steps.

We will need some additional results in order to prove Lemma 11, whose claimed algorithm is a recursive one. Namely, given an instance of this algorithm, we compute a smaller size instance and recurse. This is achieved by using the following result that is derived from [45]. For a excellent analysis of the results of [45], see [3].

**Proposition 12.** There exists an algorithm with the following specifications:

**Input:** A graph $G$ and a non-negative integer $k$ such that $|V(G)| \geq 12k^3$.

**Output:** A graph $G^*$ such that $|V(G^*)| \leq (1 - \frac{1}{12k^2})|V(G)|$ and:

- Either $G^*$ is a subgraph of $G$ such that $\text{tw}(G) = \text{tw}(G^*)$, or
- $G^*$ is obtained from $G$ after identifying the vertices of a matching of $G$.

Moreover, this algorithm runs in $2^{O(k)} \cdot n$ steps.

The following theorem of Kawarabayashi and Kobayashi [33] provides a linear relation between the treewidth and the height of a largest wall in a minor free graph.
Case 1. In both cases, we recursively call the algorithm on that one may choose that does not contain Proposition 13. There is a function \( f_3 : \mathbb{N} \to \mathbb{N} \) such that, for every \( h, r \in \mathbb{N} \) and every graph \( G \) that does not contain \( K_h \) as a minor, if \( \text{tw}(G) \geq f_3(h) \cdot r \), then \( G \) contains an \( r \)-wall. In particular, one may choose \( f_3(h) = 2^{O(h^2 \log h)} \).

We now have all the ingredients to prove Lemma 11.

Proof of Lemma 11. We set \( c := f_3(h) \cdot r + k \). We now describe a recursive algorithm as follows.

In the base case, when \( |V(G)| \leq 12c^3 \), we can easily return one of the three possible outputs. If \( |V(G)| \geq 12c^3 \), then we call \( G \) the algorithm of Proposition 12 which outputs a graph \( G^* \) such that \( |V(G^*)| \leq (1 - \frac{1}{16c^3})|V(G)| \) and:

- Either \( G^* \) is a subgraph of \( G \) such that \( \text{tw}(G) = \text{tw}(G^*) \), or
- \( G^* \) is obtained from \( G \) after identifying the vertices of a matching \( M \) of \( G \).

In both cases, we recursively call the algorithm on \( G^* \) and we proceed as follows.

Case 1: \( G^* \) is a subgraph of \( G \) such that \( \text{tw}(G) = \text{tw}(G^*) \).

(a) If the recursive call on \( G^* \) reports that \( \text{tw}(G^*) \leq c \), then it also holds that \( \text{tw}(G) \leq c \).

(b) If the recursive call on \( G^* \) outputs an \( r \)-wall \( W \) of \( G^* \), then this is also a wall of \( G \).

(c) If \( (G^*, k) \) is a no-instance, then \( (G, k) \) is also a no-instance.

Case 2: \( G^* \) is obtained from \( G \) after identifying the vertices of a matching of \( G \).

(a) If the recursive call on \( G^* \) reports that \( \text{tw}(G^*) \leq c \), then we do the following: We first notice that the fact that \( \text{tw}(G^*) \leq c \) implies that \( \text{tw}(G) \leq 2c \), since we can obtain a tree decomposition of \( G \) from a tree decomposition \((T^*, \chi^*)\) of \( G^* \), by replacing, in every \( t \in T^* \), every occurrence of a vertex of \( G^* \) that is a result of an edge contraction by its endpoints on \( G \). Thus, we can call the algorithm of Proposition 4 to in order to find, if exists, a set \( S \) such that \( |S| \leq k \) and \( F \nsubseteq_m G \setminus S \).

- If the set \( S \) does not exist, then we algorithm reports that \( (G, k) \) is a no-instance.
- If the set \( S \) exists, then we apply the algorithm of Proposition 3 in \( G \setminus S \) (which runs in \( 2^{O(c)} \cdot n \) steps) and we get a tree decomposition of \( G \setminus S \) of width at most \( 10c + 4 \). Using this decomposition, we call the algorithm of Proposition 5 for \( G \setminus S \) in order to check whether it contains an \( r \)-wall \( W \) as a minor. This algorithm runs in \( 2^{O_h((r+k)^{\log(r+k)})} \cdot r^{O_h(r+k)} \cdot 2^{O(r^2) \cdot n} = 2^{O_h(r^2+(r+k)\log(r+k))} \cdot n \) steps, since \( |E(G \setminus S)| = O(n) \) and \( |E(W)| = O(r^2) \). If this algorithm outputs an \( r \)-wall \( W \) of \( G \setminus S \), then we output \( W \). Otherwise, following Proposition 13, we report that \( \text{tw}(G) \leq f_3(h) \cdot r + k = c \).

(b) If the recursive call on \( G^* \) outputs an \( r \)-wall \( W^* \) of \( G^* \), then by uncontracting the edges of \( M \) in \( W^* \) we can return an \( r \)-wall of \( G \).

(c) If \( (G^*, k) \) is a no-instance, then we also know that \( (G, k) \) is a no-instance.
It is easy to see that the running time of the above algorithm is

\[ T(n, k, r) \leq T((1 - \frac{1}{12r^2}) \cdot n, k, r) + 2^{\Theta_h(r^2 + (r+k) \log(r+k))} \cdot n, \]

which implies that \( T(n, k, r) = 2^{\Theta_h(r^2 + (r+k) \log(r+k))} \cdot n. \)

The next result follows from [35, Theorem 1.9] and the proof of [34, Theorem 5.2].

**Proposition 14.** There are functions \( f_4, f_5 : \mathbb{N} \to \mathbb{N} \) and an algorithm as follows:

**Clique-Or-Flat-Wall** \((G, r, t, W)\)

**Input:** A graph \( G \) on \( n \) vertices and \( m \) edges, an odd integer \( r \geq 3 \), a \( t \in \mathbb{N} \geq 1 \), and an \( R \)-wall \( W \) in \( G \), where \( R = f_4(t) \cdot r \).

**Output:** Either a minor of \( G \) isomorphic to \( K_t \), or

1. a set \( A \subseteq V(G) \) of size at most \( 12288t^24 \),
2. a flat \( r \)-wall \( \tilde{W} \) of \( G \setminus A \) such that \( V(\tilde{W}) \cap A = \emptyset \), and
3. a separation \((X, Y)\) of \( G \setminus A \) that certifies that \( \tilde{W} \) is a flat wall and an \( \Omega \)-rendition of \( G[Y] \) with flaps of treewidth at most \( f_5(t) \cdot r \), where \( \Omega \) is a cyclic ordering of \( X \cap Y \) determined by the order on the perimeter of \( \tilde{W} \).

Moreover, this algorithm runs in \( 2^{\Theta(r)}(m + n) \) time.

Proposition 14, without the bound on the treewidth of the flaps, has been proven in [35, Theorem 1.9]. However, in [35, Theorem 1.9] \( \tilde{W} \) is additionally a tilt of some \( r \)-subwall of \( W \) with a different function \( f_4' \) for the relation between \( R \) and \( r \) and has running time \( O(t^{24}(n + m)) \). This variant is stated as Proposition 21 in Subsection 5.2.

In Proposition 14 the bound on the treewidth of the flaps is obtained if we plug [35, Theorem 1.9] in the proof of [34, Theorem 5.2], taking into account the linear dependence between \( R \) and \( r \). The parametric dependence \( 2^{\Theta(r)} \) of the algorithm follows because of the use of the linear FPT-approximation algorithm for treewidth in [8] so as to compute the tree decompositions of the flaps. Another stronger version of Proposition 14, where no \( R \)-wall \( W \) is given in the input, appears in [24, Lemma 3.2], running in \( 2^{\Omega(t^{38})} \cdot n \log^2 n \) time. We do not need this stronger version here as, for our problem, the \( R \)-wall \( W \) will be found by the algorithm of Lemma 11.

### 4.2 Finding an irrelevant vertex

The irrelevant vertex technique was introduced in [47] for providing an FPT-algorithm for the **Disjoint Paths** problem. Moreover, this technique has appeared to be quite versatile and is now a standard tool of parameterized algorithm design (see e.g., [14,51]). The applicability of this technique for **F-M-Deletion** is materialized in this section by the algorithm of Lemma 16.

For the proof of Lemma 16 we need the next combinatorial result, Lemma 15, whose proof is presented in Section 6. Lemma 15 intuitively states that given an \( \ell \)-homogeneous apex wall pair \((A, W)\), where \( W \) is a big enough flat wall, the compass of every apex wall pair \((A, \tilde{W})\)
generated by \(W(q)\) can be “safely” removed from the input graph \(G\) in the way that \((G, k)\) and \((G \setminus V(\text{compass}(\hat{W})), k)\) are equivalent instances of \(\mathcal{F}\)-\textit{M-Deletion}.

\textbf{Lemma 15.} There is a function \(f_6 : \mathbb{N}^4 \to \mathbb{N}\) such that if \(a, \ell, q, k \in \mathbb{N}, q \geq 3, G\) is a graph, and \((A, W)\) is an \(\ell\)-homogeneous \((a, f_6(a, \ell, q, k))\)-apex wall pair of \(G\), then for every \((a, q)\)-apex wall pair \((A, \hat{W})\) generated by \(W(q)\), it holds that \((G, k)\) and \((G \setminus V(\text{compass}(\hat{W})), k)\) are equivalent instances of \(\mathcal{F}\)-\textit{M-Deletion}. Moreover, \(f_6(a, \ell, q, k) = O(a, \ell, q, k)\).

The following algorithm outputs a flat wall \(\hat{W}\) such that \((G, k)\) and \((G \setminus V(\text{compass}(\hat{W})), k)\) are equivalent instances of \(\mathcal{F}\)-\textit{M-Deletion}. In fact, in the rest of the paper, we use a slightly weaker version of \textbf{Lemma 16}, refered as \textbf{Corollary 17}, that outputs an irrelevant vertex. Here, we prove this more general result for future use. Recall that \(\ell_{\mathcal{F}} = \max\{|E(H)| + |V(H)| : H \in \mathcal{F}\}\).

\textbf{Lemma 16.} There exist a function \(f_7 : \mathbb{N}^4 \to \mathbb{N}\) and an algorithm with the following specifications:

\textit{Find-Irrelevant-Wall} \((G, q, k, b, A, W)\)

\textbf{Input:} A graph \(G\) on \(n\) vertices, two integers \(q, k \in \mathbb{N}\), an \((a, f_7(a, \ell, b, k))\)-apex wall pair \((A, W)\) of \(G\) whose all flaps have treewidth at most \(q\).

\textbf{Output:} A flat \(b\)-wall \(\hat{W}\) of \(G \setminus A\) such that \((G, k)\) and \((G \setminus V(\text{compass}(\hat{W})), k)\) are equivalent instances of \(\mathcal{F}\)-\textit{M-Deletion}.

Moreover, \(f_7(a, \ell, b, k) = O(a, \ell, q, k)\) for some constant \(c_a, \ell, q, k\) depending on \(a\) and \(\ell, q, k\). This algorithm runs in \(2^{O(a, \ell, q, k) \log (q + b)(k + b)} \cdot n\) time.

\textbf{Proof.} We set \(f_7(a, \ell, b, k) := f_1(\ell_{\mathcal{F}}, r, a)\), where \(r = f_6(a, \ell, b, k)\). The algorithm considers each one of the \((f_1(a, \ell, b, k))\) internal \(r\)-subwalls \(W'\) of \(W\) and constructs an \((a, r)\)-wall pair \((A, W'')\) generated by \(W'\). This can be done in \(O(n)\) time because of \textbf{Lemma 7}.

From \textbf{Proposition 10} there is a choice of \(W''\) such that \((A, W'')\) is \(\ell_{\mathcal{F}}\)-homogeneous. To check whether \((A, W'')\) is \(\ell_{\mathcal{F}}\)-homogeneous we do the following. Let \(B\) be the set of all flaps of \(W''\) that, when seen as flap vertices of \(\hat{W}\), appear in the closed disk bounded by the representation of \(B\) in \(\hat{W}\), where \(B\) is an internal brick of \(W''\) that is not a brick of \(W''(b)\), i.e., the central \(b\)-subwall of \(W''\).

For every flap \(F \in B\), we consider the boundaried graph \(F^A\). Using the fact that \(\text{tw}(F^A) \leq q + a\), we apply the algorithm of \textbf{Proposition 3} which outputs a tree decomposition of \(F^A\) of width at most \(5(q + a) + 4\). Then by applying the algorithm of \textbf{Proposition 5}, we compute \(\ell_{\mathcal{F}}\)-\text{folio}(\(F^A\)) in \(2^{O(a, \ell, q, k) \log (q + k)}\) time. Then, it is easy to check in linear time whether \((A, W'')\) is \(\ell_{\mathcal{F}}\)-homogeneous.

After we find \(W''\), we again use \textbf{Lemma 7} in order to construct a flat \(b\)-wall \(\hat{W}\) of \(G \setminus A\) generated by \(W''(b)\). The algorithm outputs \(\hat{W}\), and this is correct because of \textbf{Lemma 15}. 

Notice that \textbf{Lemma 16} implies \textbf{Corollary 17} if we set \(b = 3\) and output a central vertex of the obtained 3-wall.

\textbf{Corollary 17.} There exist a function \(f_7 : \mathbb{N}^3 \to \mathbb{N}\) and an algorithm as follows:

\textit{Find-Irrelevant-Vertex} \((G, q, k, A, W)\)

\textbf{Input:} A graph \(G\), two integers \(k, q \in \mathbb{N}\), and an \((a, f_7(a, \ell, k))\)-apex wall pair \((A, W)\) of \(G\) whose all flaps have treewidth at most \(q\).

\textbf{Output:} A vertex \(v \in V(G)\) such that \((G, k)\) and \((G \setminus v, k)\) are equivalent instances of \(\mathcal{F}\)-\textit{M-Deletion}. Moreover, \(f_7(a, \ell, k) = O(a, \ell, q, k)\) for some constant \(c_a, \ell, q, k\) depending on \(a\) and \(\ell, q, k\). This algorithm runs in \(2^{O(a, \ell, q, k) \log (q + k)} \cdot n\) time.
4.3 Branching in graphs with an apex grid

We now give a combinatorial result that will justify a branching step of our algorithm, i.e., its recursive application on a set of $O_s(k)$ vertices.

**Grids.** Let $k,r \in \mathbb{N}$. The $(k \times r)$-grid is the Cartesian product of two paths on $k$ and $r$ vertices, respectively. A vertex of a $(k \times r)$-grid is called *internal* if it has degree four, otherwise it is called *external*. We use the term $k$-grid for the $(k \times k)$-grid. Given that $k,r \geq 2$, we also define the *perimeter* of a $(k \times r)$-grid to be the unique cycle of the grid of length at least three that does not contain internal vertices.

![Figure 7: A 9-grid and its central 5-subgrid.](image)

Given a $(k \times r)$-grid $H$ with vertices $(x,y) \in [2r] \times [r]$, and some $i \in [k]$, the *$i$-th vertical path* of $H$ is the one whose vertices, in order of appearance, are $(i,1), (i,2), \ldots, (i,r)$. Also, given some $j \in [r]$ the *$j$-th horizontal path* of $H$ is the one whose vertices, in order of appearance, are $(1,j), (2,j), \ldots, (k,j)$.

Let $r \in \mathbb{N}_{\geq 2}$ and $H$ be an $r$-grid. Given an $i \in \lceil \frac{r}{2} \rceil$, we define the *$i$-th layer* of $H$ recursively as follows. The first layer of $H$ is its perimeter, while, if $i \geq 2$, the *$i$-th layer* of $H$ is the $(i-1)$-th layer of the grid created if we remove from $H$ its perimeter. Given two odd integers $r,q \in \mathbb{N}_{\geq 3}$ such that $q \leq r$ and an $r$-grid $H$, we define the *$q$-central subgrid* of $H$ to be the graph obtained from $H$ if we remove from $H$ its $\frac{r-q}{2}$ first layers. See Figure 7 for an illustration of the notions defined above.

Given a graph $G$ and a set $A \subseteq V(G)$, we say that a graph $H$ is an *$A$-fixed minor* of $G$ if $H$ can be obtained from a subgraph $G'$ of $G$ where $A \subseteq V(G')$, after contracting edges without endpoints in $A$. A graph $H$ is an *$A$-apex $r$-grid* if it can be obtained by an $r$-grid $\Gamma$ after adding a set $A$ of new vertices and some edges between the vertices of $A$ and $V(\Gamma)$. We call $\Gamma$ underlying grid of $H$.

Next we identify a combinatorial structure that guarantees the existence of a set of $q = O_s(k)$ vertices that intersects every solution $S$ of $\mathcal{F}$-$\text{M-Deletion}$ on input $(G,k)$. This will permit branching on $q$ simpler instances of the form $(G',k-1)$. Recall that $a_\mathcal{F}$ is the minimum apex number of a graph in $\mathcal{F}$.

**Lemma 18.** There exist three functions $f_8, f_9, f_{10} : \mathbb{N}^2 \to \mathbb{N}$, such that if $\mathcal{F}$ is a finite set of graphs, $G$ is a graph, $k \in \mathbb{N}$, and $A \subseteq V(G)$, $|A| = a_\mathcal{F}$, such that $G$ contains as an $A$-fixed
minor an $A$-apex $f_8(s_F,k)$-grid $H$ where each vertex $v \in A$ has at least $f_9(s_F,k)$ neighbors in the central $(f_8(s_F,k) - f_{10}(s_F,k))$-grid of $H \setminus A$, then for every solution $S$ of $F$-$M$-Deletion for the instance $(G,k)$, it holds that $S \cap A \neq \emptyset$. Moreover, $f_8(s_F,k) = O_{s_F}(k^{3/2})$, $f_9(s_F,k) = O_{s_F}(k^3)$, and $f_{10}(s_F,k) = O_{s_F}(k)$. 

We postpone the proof of Lemma 18 to Subsection 7.4. We conjecture that Lemma 18 is tight, in the sense that it cannot be proved for some $f_9(s_F,k) = O_{s_F}(k^{3-\epsilon})$.

Notice that in the special case where $a_F = 1$, then Lemma 18 can be improved by using the main combinatorial result of [16]. In particular [16, Lemma 3.1] easily implies that, in this case, $f_8(s_F,k) = O_{s_F}(k)$, $f_9(s_F,k) = O_{s_F}(k^2)$, and $f_{10}(s_F,k) = O_{s_F}(\sqrt{k})$. In Subsection 7.5, we improve these bounds as follows.

**Lemma 19.** There exist three functions $f_{11}, f_{12} : \mathbb{N}^2 \to \mathbb{N}$, and $f_{13} : \mathbb{N} \to \mathbb{N}$ such that if $F$ is a finite family of graphs containing an apex graph, $(G,k)$ is an instance of $\mathcal{F}$-$M$-Deletion, and $a \in V(G)$ such that $G$ contains as an $(\{a\}$-fixed minor an $\{a\}$-apex $f_{11}(s_F,k)$-grid $H$ where $a$ has at least $f_{12}(s_F,k)$ neighbors in the central $(f_{11}(s_F,k) - f_{13}(s_F,k))$-grid of $H \setminus a$, then $(G,k)$ and $(G \setminus a, k-1)$ are equivalent instances of $\mathcal{F}$-$M$-Deletion. Moreover, $f_{11}(s_F,k) = O_{s_F}(\sqrt{k})$, $f_{12}(s_F,k) = O_{s_F}(k)$, and $f_{13}(s_F) = O_{s_F}(1)$.

We will use the improved bounds of Lemma 19 for the proof of Theorem 2.

### 4.4 Canonical partitions

We conclude this section with an additional definition that will be useful for the application of Lemma 18 in our main algorithm.

Let odd integer $r \geq 3$. Let $W$ be an $r$-wall and let $P_1, \ldots, P_r$ (resp. $L_1, \ldots, L_r$) be its vertical (resp. horizontal) paths. For every even (resp. odd) $i \in [2, r-1]$ and every $j \in [2, r-1]$, we define $A^{(i,j)}$ to be the subpath of $P_i$ that starts from a vertex of $P_i \cap L_j$ and finishes at a neighbor of a vertex in $L_{j+1}$ (resp. $L_{j-1}$), such that $P_i \cap L_j \subseteq A^{(i,j)}$ and $A^{(i,j)}$ does not intersect $L_{j+1}$ (resp. $L_{j-1}$). Similarly, for every $i, j \in [2, r-1]$, we define $B^{(i,j)}$ to be the subpath of $L_j$ that starts from a vertex of $P_i \cap L_j$ and finishes at a neighbor of a vertex in $P_{i-1}$, such that $P_i \cap L_j \subseteq A^{(i,j)}$ and $A^{(i,j)}$ does not intersect $P_{i-1}$.

![Figure 8: A 5-wall and its canonical partition $Q$. The orange bag is the external bag $Q_{\text{ext}}$.](image)

For every $i, j \in [2, r-1]$, we denote by $Q^{(i,j)}$ the graph $A^{(i,j)} \cup B^{(i,j)}$ and $Q_{\text{ext}}$ to be the graph $W \setminus \bigcup_{i,j \in [2, r-1]} Q_{i,j}$. Now consider the collection $\mathcal{Q} = \{Q_{\text{ext}}\} \cup \{Q_{i,j} \mid i, j \in [2, r-1]\}$ and observe that the graphs in $\mathcal{Q}$ are connected subgraphs of $W$ and their vertex sets form a partition of $V(W)$.

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We call $\mathcal{Q}$ the canonical partition of $W$. Also, we call every $Q_{i,j}$, $i, j \in [2, r - 1]$ an internal bag of $\mathcal{Q}$, while we refer to $Q_{\text{ext}}$ as the external bag of $\mathcal{Q}$. See Figure 8 for an illustration of the notions defined above.

Let $(A, W)$ be an $(a, r)$-apex wall pair of a graph $G$ and let $\tilde{W}$ be the $\Delta$-embedded graph that is the leveling $\tilde{W}$ of $W$ in $G \setminus A$. Let also $W^R$ be the representation of $W$ in $\tilde{W}$. Consider a canonical partition $\mathcal{Q}$ of $W^R$. We enhance the graphs of $\mathcal{Q}$ so to include in them all the vertices of $\tilde{W}$ by applying the following procedure. We set $\mathcal{Q} := \mathcal{Q}$ and, as long as there is a vertex $x \in V(\tilde{W}) \setminus V(\bigcup \mathcal{Q})$ that is adjacent to a vertex of a graph $Q \in \mathcal{Q}$, update $\mathcal{Q} := \mathcal{Q} \setminus \{Q\} \cup \{\tilde{Q}\}$, where $\tilde{Q} = \tilde{W}[\{x\} \cup V(Q)]$. We just defined a partition of the vertices of $\tilde{W}$ into subsets inducing connected graphs. We call such a partition canonical partition of $\tilde{W}$. Notice that a canonical partition of $\tilde{W}$ is not unique (since the sets in $\mathcal{Q}$ can be “expanded” arbitrarily when introducing vertex $x$).

5 The algorithms

In this section we present the main algorithms of the paper. In Subsection 5.1 we present the general algorithm solving $\mathcal{F}$-M-Deletion-Compression while in Subsection 5.2 we present an improved algorithm solving $\mathcal{F}$-M-Deletion-Compression in the case where $a_\mathcal{F} = 1$. The existence of these two algorithms prove Theorem 1 and Theorem 2, respectively.

5.1 The general algorithm

Lemma 20. Let $\mathcal{F}$ be a finite collection of graphs. There is an algorithm solving $\mathcal{F}$-M-Deletion-Compression in $2^{O_s(k^{2(c_\mathcal{F} + 2)} \log k)} \cdot n^2$ time.

Proof. For simplicity, in this proof, we use $c$ instead of $c_\mathcal{F}$, $s$ instead of $s_\mathcal{F}$, $\ell$ instead of $\ell_\mathcal{F}$, $a$ instead of $a_\mathcal{F}$, and remember that $\ell = O_s(1)$ and $a = O_s(1)$. We set

$$b = f_7(k, a, \ell) = O(k^c), \quad x = f_9(s, k), \quad l = (12288s^{24} + k + 1) \cdot x,$$

$$m = f_8(s, k), \quad p = f_{10}(s, k), \quad h = \max\{m - p, \lceil \sqrt{l + 1} \cdot b \rceil\},$$

$$r = h + p + 2, \quad R = f_4(s) \cdot r, \quad \text{and notice that } R = O_s(k^{c+2}).$$

Recall that, in the definition of $R$, the constant $c$ is the palette-variety of $\mathcal{F}$ (defined in Subsection 3.3).

We present the algorithm Solve-Compression, whose input is a quadruple $(G, k', k, S)$ where $G$ is a graph, $k'$ and $k$ are non-negative integers where $k' \leq k$, and $S$ is a subset of $V(G)$ such that $|S| = k$ and $\mathcal{F} \not \subseteq_m G \setminus S$. The algorithm returns, if it exists, a solution for $\mathcal{F}$-M-Deletion on $(G, k')$. Certainly, we may assume that $k' < k$, otherwise $S$ is already a solution and we are done. The steps of the algorithm are the following:

Step 1. Run the algorithm Find-Wall of Lemma 11 with input $(G \setminus S, R, 0)$. This outputs, in $2^{O_s(k^{2(c+2)})} \cdot n$ time, either a report that $\text{tw}(G \setminus S) \leq f_2(s) \cdot R$, or an $R$-wall $W_0$ of $G \setminus S$. Notice
that \text{Find-Wall}(G \setminus S, R, s) never outputs the third case, since \((G \setminus S, 0)\) is a \text{yes}-instance of \text{F-M-Deletion}. In the first possible output, we know that \(tw(G) \leq f_2(s) \cdot R + k = O_s(k^{c+2})\), and we call the algorithm of Proposition 4 with input \((G, f_2(s) \cdot R + k, k')\) and return a correct answer in \(2^{O_s(k^{c+2} \log k)} \cdot n\) steps. In the second possible output, the algorithm moves to the second step.

\textbf{Step 2.} Call \text{Clique-Or-Flat-Wall} of Proposition 14 on \((G \setminus S, r, s, W_0)\). Since \(\mathcal{F} \not\leq_m G \setminus S\) and \(\mathcal{F} \leq_m K_s\), the algorithm outputs, in time \(2^{O_s(r)} \cdot (m + n) = 2^{O_s(k^{c+2})} \cdot n\), the following:

\begin{itemize}
  \item A set \(A \subseteq V(G) \setminus S\) of size at most \(12288s^{24}\),
  \item a separation \(r\)-wall \(W = G \setminus (S \cup A)\) such that \(V(W) \cap A = \emptyset\), and
  \item a \((X, Y)\) of \(G \setminus A\) that certifies that \(W\) is a flat wall, and an \(\Omega\)-rendition of \(G[Y]\) with flaps of treewidth at most \(q = f_5(s) \cdot r\), where \(\Omega\) is a cyclic ordering of \(X \cap Y\) determined by the order on the outer cycle of \(W\).
\end{itemize}

Let \(\hat{W}\) be the leveling of \(W\) and let \(W^R\) be the representation of \(W\) in the \(\Delta\)-embedded graph \(\hat{W}\) \((\hat{W}\) and \(W^R\) can straightforwardly be constructed in \(O(n)\) steps using the \(\Omega\)-rendition of \(G[Y]\)). Let also \(\hat{W} = (W^R)^{(r-p)}\), i.e., \(\hat{W}\) is the central \((r-p)\)-subwall of \(W^R\).

Consider a family \(\hat{W} = \{\hat{W}_1, \ldots, \hat{W}_{l+1}\}\) of \(l+1\) internal \(b\)-subwalls of \(\hat{W}\), such that if \(D_i\) is the closed disk in \(\Delta\) bounded by the perimeter, denoted by \(\hat{P}_i\), of \(\hat{W}_i\), then \(D_i \cap D_j = \emptyset\), for \(i \neq j\). We are allowed to do this since \(r - p - 2 \geq \lceil \sqrt{l+1} \cdot b \rceil\). By \(\text{flaps}(D_i)\) we denote all the flaps corresponding to flap-vertices of \(\hat{W}\) that are inside \(D_i\) in the embedding of \(\hat{W}\) in \(\Delta\). For every \(i \in [l+1]\), we compute, in \(O(n)\) time, the set

\[A_i = \{v \in S \cup A \mid v\text{ is adjacent in } G\text{ to a vertex of } \bigcup \text{flaps}(D_i)\}\]

and we proceed to the last step.

\textbf{Step 3.} The algorithm examines two cases depending on the sizes of the \(A_i\)'s. In the first case, the \textbf{branching case}, the outcome is a set of vertices, the set \(S \cup A\), that should intersect every possible solution. In the second case, the \textbf{irrelevant vertex case}, the outcome is an irrelevant vertex.

[Branching case]. For every \(i \in \{l+1\}\), it holds that \(|A_i| > a\). In this case the algorithm recursively calls \text{Solve-Compression} with input \((G \setminus \{x, k' \cdot \Delta - 1, |S \setminus x|, S \setminus x\})\), for every \(x \in S \cup A\), and if one of these new instances is a \text{yes}-instance, certified by a set \(\tilde{S}\), then return \(\tilde{S} \cup \{x\}\), otherwise return that \((G, k')\) is a no-instance.

We now prove that the above branching step of the algorithm is correct. Let \(\tilde{W}\) be the leveling of \(W\) in \(G \setminus (S \cup A)\). We define \(\tilde{G}\) as the graph obtained from \(G \setminus (S \cup A)\) if we remove all the vertices of the compass of \(W\) and take the union of the resulting graph with \(\tilde{W}\). Notice that \(\tilde{G}\) is partially \(\Delta\)-embedded in the sense that the part of \(G\) that is embedded in \(\Delta\) is \(\tilde{W}\). Notice that \(\tilde{G}\) is not necessarily a contraction of \(G \setminus (S \cup A)\), and this is because the trivial flaps appear in \(\tilde{W}\) as induced paths of length two instead of edges. Therefore, if \(\tilde{G}\) is the graph obtained from \(\tilde{G}\) after dissolving each flap-vertex corresponding to a trivial flap, then
• \( \bar{G} \) is a contraction of \( G \setminus (S \cup A) \),

• \( \bar{G} \) is a partially \( \Delta \)-embedded graph whose compass is a dissolution of \( \bar{W} \), and

• \( \bar{G} \) contains an \( r \)-wall \( \bar{W} \) that is a dissolution of \( W^R \) and \( \bar{W} \) is embedded in \( \Delta \) so that its perimeter is a dissolution of the perimeter of \( W^R \).

Consider a canonical partition \( \bar{Q} \) of \( \bar{W} \). Let \( \bar{Q} \) be the collection of connected subgraphs of \( \bar{G} \) that are obtained if we apply to the graphs in \( \bar{Q} \) the same dissolutions that we used to transform \( G \) to \( \bar{G} \) (we just take care that the edge contracted during each dissolution has both endpoints in some bag). Moreover, we enhance \( \bar{Q} \) by adding in its external bag all the vertices of \( \bar{G} \) that are not points of \( \Delta \). Notice that the vertex sets of the graphs in this new \( \bar{Q} \) define a partition of \( V(\bar{G}) \). Let now \( \bar{G}^+ \) be the graph obtained if we apply in \( G \) the same contractions that transform \( G \setminus (S \cup A) \) to its contraction \( \bar{G} \). Let \( a^* = |S \cup A| \leq 12288^{24} + k + 1 \). We now construct a minor of \( \bar{G}^+ \) by contracting all edges of each member of \( \bar{Q} \) to a single vertex and removing the vertex to which the external bag was contracted. We denote the resulting graph by \( \bar{G} \) and we observe that \( \bar{G} \) contains as a spanning subgraph an \( S \cup A \)-apex \((r-2)\)-grid \( \Gamma \). Recall that \( \Gamma \) is a \( S \cup A \)-fixed minor of \( G \). Let \( \bar{\Gamma} \) be the underlying grid of \( \Gamma \) and let \( \bar{\Gamma}_1, \ldots, \bar{\Gamma}_{l+1} \) be the “packing” of the \( h \)-central grid \( \bar{\Gamma}' \) of \( \bar{\Gamma} \), corresponding to the walls in \( \bar{W} \), where each \( \bar{\Gamma}_i \) is a \( b \)-grid. We can assume the existence of this packing because \( h \geq \lceil \sqrt{l+1} \cdot b \rceil \). The initial assumption that \( |A_i| > a \), for \( i \in [l+1] \), implies that \( \forall i \in [l+1], \) there are more than \( a \) apices of \( \Gamma \) that are adjacent to vertices of \( \bar{\Gamma}_i \).

For every \( v \in S \cup A \), let \( N_v \) be the set of neighbors of \( v \) in \( \bar{\Gamma} \) and let \( \bar{N} = \bigcup_{v \in S \cup A} N_v \). Let \( A^* \) be the set of vertices of \( S \cup A \) with \( |N_v| \geq x \). We claim that \( |A^*| \geq a \). Suppose to the contrary that \( |A^*| < a \). This implies that the vertices in \( (S \cup A) \setminus A^* \) are adjacent to at most \( x \cdot |(S \cup A) \setminus A^*| \leq l \) vertices in \( \bar{N} \). This, in turn implies that there is an \( i \in [l+1] \) such that there are no vertices in \( (S \cup A) \setminus A^* \) adjacent to vertices of \( \bar{\Gamma}_i \). Thus, for this \( i \), there are at most \( a \) apex vertices of \( \bar{G} \) that are adjacent to vertices of \( \bar{\Gamma}_i \), a contradiction to the conclusion of the previous paragraph. We arbitrarily remove vertices from \( A^* \) so that \( |A^*| = a \).

Consider now the \( A^* \)-apex \((r-2)\)-grid \( H = \Gamma \setminus ((S \cup A) \setminus A^*) \), and as each vertex in \( A^* \) has at least \( x \) neighbors in \( V(\bar{\Gamma}') \) and \( r-2 \geq m \), Lemma 18 can be applied for \( k', A^*, H \). This implies that \( (G, k') \) is a yes-instance of \( \mathcal{F} \)-M-Deletion if and only if there is some \( v \in A^* \) such that \( (G \setminus v, k' - 1) \) is a yes-instance of \( \mathcal{F} \)-M-Deletion. This completes the correctness of the branching step.

[Irrelevant vertex case]. There is an \( i \in [l+1] \) such that \( |A_i| \leq a \). Since \( \bar{W}_i \) is an internal \( b \)-subwall of \( W^R \), there is a subgraph of \( \text{compass}(W) \) that is a flat \( b \)-wall \( W_i'' \) of \( G \setminus (S \cup A) \) such that the set of vertices of the compass of \( W_i'' \) is a subset of \( \text{flaps}(D_i) \) (as we argued in Subsection 4.2, \( W_i'' \) is a tilt of the subwall of \( W \) represented by \( \bar{W}_i \) and can be found in \( O(n) \) time). This implies that if \( A_i'' \) is the set of vertices from \( S \cup A \) that are adjacent with vertices of the compass of \( W_i'' \) in \( G \setminus (S \cup A) \), then \( A_i'' \subseteq A_i \). Thus \( (A_i'', \bar{W}_i'') \) is an \( (|A_i''|, b) \)-apex wall pair in \( G \).

We now apply Find-Irrelevant-Vertex of Corollary 17 for \( (G, k, q, A_i'', W_i'') \) and obtain a vertex \( v \) such that \( (G, k) \) and \( (G \setminus v, k) \) are equivalent instances of \( \mathcal{F} \)-M-Deletion. According to Corollary 17, this vertex can be detected in \( 2^{O_{a,b}(k \log k)} \cdot n \) time and the algorithm correctly
recursively calls Solve-Compression with input \( (G \setminus v, k', k, S) \). This completes the irrelevant vertex case.

Recall that \(|S \cup A| \leq k + 1 + 12288s^{24} = O_s(k)\). Therefore, if \( T(n, k', k) \) is the running time of the above algorithm, then

\[
T(n, k', k) \leq 2^{O_s(k^{2(c+2)})} n + \max\{T(n - 1, k', k), O_s(k) \cdot T(n, k - 1, k)\}
\]

which, given that \( k' \leq k \), implies that \( T(n, k', k) = 2^{O_s(k^{2(c+2)})} n^2 \).

Notice now that the output of Solve-Compression on \( (G, k, k + 1, S) \) gives a solution for \( \mathcal{F} \)-M-Deletion-Compression on this instance. □

### 5.2 The apex-minor free case

In this subsection we prove that, in the case where \( a_\mathcal{F} = 1 \), there is an algorithm that solves \( \mathcal{F} \)-M-Deletion in time \( 2^{O_{s_\mathcal{F}}(k^{2(c+1)})} \cdot n^2 \), where \( c = c_{a, \ell_\mathcal{F}} \) and \( a = 12288s_{\mathcal{F}}^{24} \). The existence of such an algorithm implies Theorem 2.

Let \( q, r \in \mathbb{N}_{\geq 3} \) where \( r \geq q \). Also, let \( G \) be graph and let \( W \) be an \( r \)-wall in \( G \). The drop, denoted by \( D_{W'} \), of an \( q \)-subwall \( W' \) of \( W \) is defined as follows. Contract in \( G \) the perimeter of \( W \) to a single vertex \( v \), \( D_{W'} \) is the unique biconnected component of the resulting graph that contains the interior of \( W' \). We call the vertex \( v \) the pole of the drop \( D_{W'} \).

Our algorithm avoids iterative compression in the fashion that this is done by Marx and Schlotter in [43] for the Planarization problem. The algorithm has three main steps. We first set \( a = 12288s_{\mathcal{F}}^{24}, b' = f_7(a, k, \ell_\mathcal{F}) = O_{s_\mathcal{F}}(k^{c_{a, \ell_\mathcal{F}}}) \), and we define

\[
\begin{align*}
b &= f_4(s_{\mathcal{F}}) \cdot 2b' + 2 & l &= f_9(s_{\mathcal{F}}, k) \cdot k & p &= f_{10}(s_{\mathcal{F}}, k) \\
h &= \max\{f_8(s_{\mathcal{F}}, k) - p, b \cdot \sqrt{1 + 1}\} & r &= h + p + 2 & R &= f_{14}(s_{\mathcal{F}}) \cdot r + k = O_{s_\mathcal{F}}(k^{c+1}).
\end{align*}
\]

For the function \( f_4' \), see Proposition 21 after the description of this algorithm.

**Step 1.** Run the algorithm Find-Wall of Lemma 11 with input \((G, R, k)\) and, in \( 2^{O_{s_\mathcal{F}}(k^{2(c+1)})} \cdot n\) time, either report a no-answer, or conclude that \( tw(G) \leq f_2(s_{\mathcal{F}}) \cdot R + k \) and solve \( \mathcal{F} \)-M-Deletion in \( O(2^{O_{s_\mathcal{F}}(k^{c+1})} \log k \cdot n) \) time using the algorithm of Proposition 4, or obtain an \( R \)-wall \( W^* \) of \( G \). In the third case, consider all the \((R, b) = 2^{O_{s_\mathcal{F}}(k \log k)} b\)-subwalls of \( W \) and for each one of them, say \( W' \), construct its drop \( D_{W'} \), consider in \( D_{W'} \) the central \((b - 2)\)-subwall \( \tilde{W} \) of \( W' \), and run the algorithm Clique-Or-Flat-Wall of Proposition 14 with input \( D_{W'}, 2b', s_{\mathcal{F}}, \) and \( \tilde{W} \). This takes time \( 2^{O_{s_\mathcal{F}}(k^c)} \cdot n \). If for some of these drops the result is an \((|A'|, 2b')\)-apex wall pair \((W'', A')\) where \(|A'| \leq a\) and its flaps have treewidth at most \( q = f_5(2b') \cdot r \), then apply Step 2, otherwise apply Step 3.

**Step 2.** Consider the leveling \( \tilde{W} \) of \( W'' \) and, in the representation \( W^R \) of \( W'' \) in \( \tilde{W} \), pick a \( b'\)-wall \( \tilde{W} \) whose flap vertices do not correspond to a flap containing the pole of \( D_{W'} \). Then use \( \tilde{W} \) in order to find an \((|A'|, b')\)-apex wall pair \((W'''', A')\) of \( D_{W'} \) whose compass does not contain the pole of \( D_{W'} \). Notice that \((A', W''')\) is also an \((|A'|, b')\)-apex wall pair of \( G \), therefore the algorithm can apply...
Find-Irrelevant-Vertex of Corollary 17 for \((G, k, q, A', W')\) and obtain, in \(2^{O_{s,F}(k \log k)} \cdot n\) time, an (irrelevant) vertex \(v\) such that \((G, k)\) and \((G \setminus v, k)\) are equivalent instances of \(F\)-M-Deletion. Then the algorithm runs recursively on the equivalent instance \((G \setminus v, k)\).

Notice that Step 2 can be seen as the irrelevant vertex case of our algorithm.

Step 3. Consider all the \(\binom{R}{1}\) \(r\)-subwalls of \(W^\bullet\), and for each one \(W'\) of them, consider its central \(h\)-subwall \(\tilde{W}\) and compute the canonical partition \(Q\) of \(\tilde{W}\). Then for each internal bag \(Q\) of \(Q\) add a new vertex \(v_Q\) and make it adjacent with all vertices in \(Q\), then add a new vertex \(x\) and make it adjacent with all \(x_Q\)'s, and in the resulting graph, for every vertex \(y\) of \(G\) that is not in the union of the internal bags of \(Q\), check, in time \(O(k \cdot m) = O_{s,F}(k \cdot n)\) (using standard flow techniques), if there are \(f_6(s,F,k)\) internally vertex-disjoint paths from \(x\) to \(y\). If this is indeed the case for some \(y\), then \(y\) should belong to every solution of \(F\)-M-Deletion for the instance \((G, k)\) and the algorithm runs recursively on the equivalent instance \((G \setminus y, k - 1)\).

If no such \(y\) exists, then report that \((G, k)\) is a no-instance of \(F\)-M-Deletion.

Note that Step 3 can be seen as a trivial branching case where the only choice is vertex \(y\).

Notice that the third step of the algorithm, when applied takes time \(2^{O_{s,F}(k \log k)} \cdot n^2\). However, it cannot be applied more than \(k\) times during the course of the algorithm. As the first step runs in time \(2^{O_{s,F}(k^2(c+1)) \log k \cdot n}\), and the second step runs in time \(2^{O_{s,F}(k \log k)} \cdot n\), they may be applied at most \(n\) times, and the claimed time complexity follows.

We now proceed to the proof of correctness of the algorithm. The following result is proved in [35, Theorem 1.9]. We restate it here using the concept of tilt introduced in Subsection 3.1. Notice that the statement is slightly different than the one of Proposition 14. Here we need that the obtained flat wall is a tilt of a subwall of original wall since we exploit this in our algorithm. The price paid for this is that we do not have anymore the bound on the treewidth of the flaps; fortunately, this is not required in the proof of correctness.

**Proposition 21.** There is a function \(f_4' : \mathbb{N} \to \mathbb{N}\) such that if \(G\) is a graph, \(r \geq 3\) is an odd integer, \(t \in \mathbb{N}_{\geq 1}\), and \(W\) is an \(R\)-wall in \(G\), where \(R = f_4'(t) \cdot r\), then either \(K_r\) is a minor of \(G\) or there exist

1. a set \(A \subseteq V(G)\) of size at most \(12288t^{24}\),
2. a flat \(r\)-wall \(W_0\) of \(G \setminus A\) that is a tilt of a subwall of \(W\) such that \(V(W_0) \cap A = \emptyset\), and
3. a separation \((X,Y)\) of \(G\) such that \(W_0\) is an flat wall, and an \(\Omega\)-rendition of \(G[Y]\), where \(\Omega\) is a cyclic ordering of \(X \cap Y\) determined by the order on the outer cycle of \(W_0\).

**Proof of correctness of the algorithm.** Recall that \(s = s_F, a = 12288s^{24}, b' = f_7(a, k, \ell) = O_s(k^{\ell a \cdot F}),\) and

\[
\begin{align*}
    b &= f_4(s)(2b') + 2, \\
    l &= f_9(s, k) \cdot (k + a), \\
    p &= f_{10}(s, k), \\
    h &= \max\{ f_8(s, k) - p, b \cdot \sqrt{1 + 1}\}, \\
    r &= h + p + 2, \\
    R &= f_4'(s) \cdot r + k = O_s(k^{c+1}).
\end{align*}
\]
Let \((G, k)\) is a yes-instance and let \(S\) be a solution, i.e., \(|S| \leq k\) and \(F \not\leq_m G \setminus S\), and let \(W^\bullet\) be an \(R\)-wall of \(G\). Then some, say \(W\), of the \((f'_1(s) \cdot r)\)-subwalls of \(W^\bullet\) will not contain vertices of \(S\), hence \(W\) will be an \((f'_1(s) \cdot r)\)-subwall of \(G = G \setminus S\). Therefore, following Proposition 21, there is a set \(A \subseteq V(G)\), where \(|A| \leq a\) and an \(r\)-subwall \(W'\) of \(W\) that has a tilt \(W''\) such that \((A, W'')\) is an \((|A|, r)\)-apex wall pair in \(G'\).

Consider a flat tilt \(\tilde{W}\) of the central \(h\)-subwall of \(W''\) and the canonical partition \(Q\) of its leveling. Then for each internal bag \(Q\) of \(Q\) add a new vertex \(v_Q\) and make it adjacent to all vertices in \(Q\), then add a new vertex \(x_{all}\) and make it adjacent to all \(x_Q\)'s. In the resulting graph, if there are \(f_9(s, k)\) internally vertex-disjoint paths from \(x_{all}\) to a vertex \(v \in S \cup A\) then this is checked in Step 3 and the algorithm correctly (due to Lemma 19) runs recursively on the equivalent instance \((G \setminus v, k - 1)\). If this is not the case, then for each vertex \(v \in S \cup A\) there are less than \(f_9(s, k)\) internal bags of \(Q\) that contain flap-vertices whose corresponding flaps contain vertices adjacent to \(v\). This means that the internal bags of \(Q\) that contain flap-vertices whose corresponding flaps contain vertices adjacent to some vertex of \(S \cup A\) are at most \(l\).

Notice now that, as \(\tilde{W}\) is an \(h\)-wall, some, say \(\Gamma\), of its \(l + 1\) \(b\)-subwalls will have a tilt \(\tilde{W}\) that will be flat in \(G \setminus (S \cup A)\) and its compass will have no neighbors in \(S \cup A\). Then we consider the graph \(\tilde{G}\) obtained if in \(G\) we contact all vertices of the perimeter of \(\tilde{W}\) and the remove the components that do not contain the interior of \(\tilde{W}\). We know that \(\tilde{G}\) will be an \(F\)-free graph and by applying the algorithm of Proposition 14 on \(\tilde{G}\) we must find a flat \(2b'\)-subwall of \(G\) that has bounded treewidth flaps. Therefore, this wall should be detected in Step 2.

### 6 Existence of an irrelevant wall inside a flat wall

The correctness of the algorithm of Lemma 16 in Subsection 4.2 follows from the fact that in the compass of a “large enough” flat wall there exists an flat wall whose compass is irrelevant. This fact is asserted by Lemma 15 and this section is devoted to its proof. We first give some additional definitions and present a result that we derive from [5].

Let \(a, \ell, r \in \mathbb{N}\), where \(r \geq 3\) is an odd number. Let also \((A, W)\) be an \((a, r)\)-apex wall pair of a graph \(G\). We say that \((A, W)\) is an \(\ell\)-irrelevant \((a, r)\)-apex wall pair of \(G\) if for every graph \(H\) where \(\text{detail}(H) \leq \ell\), \(H\) is a minor of \(G\) if an only if \(G\) is a minor of \(G \setminus \text{compass}(W)\).

We also derive the following result from [5].

**Proposition 22.** There is a function \(f_{14} : \mathbb{N}^3 \rightarrow \mathbb{N}\) such that, for every \(a, \ell, q \in \mathbb{N}\) and every graph \(G\), if \((A, W)\) is an \((\ell, q)\)-homogeneous \((a, f_{14}(a, \ell, q))\)-apex wall pair of \(G\), then every \((a, q)\)-apex wall pair of \(G\) generated by \(W^{(q)}\) is \(\ell\)-irrelevant.

In fact, Proposition 22 is stated in [5, Theorem 5.2] for boundaryed graphs and finds an \(\ell\)-irrelevant vertex. Proposition 22 is derived by the same proof if we consider graphs with empty boundary.

#### 6.1 Proof of Lemma 15

**Proof of Lemma 15.** Let \(b = f_{14}(a, \ell, q)\), \(b' = \lceil b/2 \rceil\), \(r = (k + 1) \cdot b + q\), and \(f_6(a, \ell, q, k) = r + 2\). Also, let \(\tilde{W}\) be leveling of \(W\) in \(G \setminus A\) and let \(W'^R\) be the representation of \(W\) in \(W\). Let \(\tilde{W}\) be the
internal $r$-subwall of $W^R$.

For every $i \in [r]$, we denote by $P_i$ (resp. $L_i$) the $i$-th vertical (resp. horizontal) path of $\tilde{W}$. We also define, for every $i \in [k+1]$ the graph

$$Q_i := \bigcup_{j \in [b'-1]} P_{j+(i-1)b'} \cup \bigcup_{j \in [b']} P_{j+(k+1-i)b'} \cup \bigcup_{j \in [b'-1]} L_{j+(i-1)b'} \cup \bigcup_{j \in [b']} L_{j+(k+1-i)b'}.$$ 

For every $i \in [k+1]$, we define $\bar{W}_i$ to be the graph obtained from $Q_i$ after repeatedly removing from $Q_i$ all vertices of degree one (see Figure 9 for an example). Since $b = 2b' - 1$, for every $i \in [k+1]$, $\bar{W}_i$ is a $b$-subwall of $\tilde{W}$. Therefore, every $\bar{W}_i, i \in [k+1]$ is an internal $b$-subwall of $W^R$. For every $i \in [k+1]$, let $\bar{C}_i$ be the perimeter of $\bar{W}_i$ and let $D_i$ be the closed disk bounded by $\bar{C}_i$. Also, let $P(q)$ be the inner layer of $W(q)$ and let $\bar{P}(q)$ be the representation of $P(q)$ in $\tilde{W}$.

Figure 9: An illustration of the construction of the graphs $\bar{W}_i$.

Recall that $\tilde{W}$ is a $\Delta$-embedded graph. We define the function $fv : C(\tilde{W}) \to V(\tilde{W})$ so that, for each cycle $\bar{C}$ of $\tilde{W}$, $fv(\bar{C})$ contains all flap-vertices of $\tilde{W}$ that belong to the closed disk bounded by $\bar{C}$. Notice that if $C$ is a cycle of $W$ and $C^R$ is its representation in $\tilde{W}$, then $\flaps(C)$ is the set of all flaps of $W$ that, when seen as flap-vertices in $\tilde{W}$, are in $fv(C^R)$. Now observe that, for every $i \in [k]$, the fact that $r = (k + 1) \cdot b + q$ implies that $D_i$ contains the closed disk bounded by $\bar{P}(q)$. Therefore, for every $i \in [k+1]$, it holds that

$$fv(\bar{P}(q)) \subseteq fv(\bar{C}_i). \quad (1)$$

Let $(A, \bar{W}(q))$ be an $(a, q)$-apex wall pair generated by $W(q)$. Notice that, due to Lemma 7,

$$V(\compass(\bar{W}(q))) \subseteq \bigcup \flaps(P(q)). \quad (2)$$

Let $\hat{G} = G \setminus \compass(\bar{W}(q))$. We now aim to prove that if $(\hat{G}, k)$ is a yes-instance of $\mathcal{F}$-M-DELETION, then $(G, k)$ is also a yes-instance of $\mathcal{F}$-M-DELETION, since the reverse implication holds trivially.
Let $S$ be a subset of $V(\hat{G})$ of size at most $k$ such that $\mathcal{F} \subseteq_m \hat{G} \setminus S$. Suppose, towards a contradiction, that $\mathcal{F} \leq_m G \setminus S$ and let $H \in \mathcal{F}$ be a minor of $G \setminus S$. For every $i \in [k + 1]$, since $\tilde{W}_i$ is an internal $b$-wall of $W^R$, it follows, by Observation 8, that there exists an $(a, b)$-apex wall pair $(A, W_j^r)$ of $G$. Also, due to Observation 9 we can derive that $(A, W_j^r)$ is $(\ell, q)$-homogeneous in $G$. Notice that since $|S| \leq k$, there exists a $j \in [k + 1]$ such that $S$ does not intersect the vertex set of the flaps corresponding to flap-vertices of $\tilde{W}_j$. This implies that for every brick $B$ of $W_j^r$ that is internal and is not a brick of $W_j^{r(q)}$ (i.e., the central $q$-subwall of $W_j^r$), the set flaps($B$) remains intact after the removal of the vertex set $S$ from $G$. Thus, $(A, W_j^r)$ is an $(\ell, q)$-homogeneous $(a, b)$-apex wall pair of $G \setminus S$. We are now in position to apply Proposition 22 on $G \setminus S$ and $(A, W_j^r)$, which implies that every $(a, q)$-apex wall pair $(A, W''_j)$ of $G \setminus S$ generated by $W_j^{r(q)}$ is $\ell$-irrelevant. Therefore, $H$ is also a minor of $(G \setminus V(\text{compass}(W''_j))) \setminus S$. Also, it is easy to observe that if $C_j$ is the inner layer of $W_j^r$ then

$$\bigcup \text{flaps}(C_j) \subseteq V(\text{compass}(W''_j)).$$

As a consequence of (1), (2), and (3), it holds that $V(\text{compass}(\hat{W}^{r(q)})) \subseteq V(\text{compass}(W''_j))$ and therefore $H \leq_m \hat{G} \setminus S$, which is a contradiction. \hfill \qed

7 Combinatorial results for branching

In this section we prove a series of combinatorial results on the existence of branching sets in graphs containing a sufficiently big apex grid. In particular, in Subsection 7.1 we prove a combinatorial result that will be useful for the proof of Lemma 24, presented in Subsection 7.2. The latter, together with a result proved in Subsection 7.3 that allows us to find a branching structure in a given graph, imply Lemma 18, which is proved in Subsection 7.4. Also, in Subsection 7.5, we prove a version of Lemma 18 for the case where $|A| = 1$, with improved bounds (Lemma 19).

7.1 Supporting combinatorial result

In the rest of this section, we always assume that the vertex set of a $(k \times r)$-grid is $[k] \times [r]$ and each vertex $(i, j) \in [k] \times [r]$ is embedded at the point $(i, j)$ in a coordinate system whose horizontal axis refers to the first coordinate, whose vertical axis refers to the second coordinate, and each edge of the grid is represented by a straight line segment. Given a $(2m + 1 \times n)$-grid $H$, we refer to the $(m + 1)$-th horizontal path of $H$ as the middle horizontal path of $H$, which we denote by $P_H$. Let $(i, j, j') \in [-m, m] \times [n]^2$ where $j \neq j'$. We denote by $P_{i,j \rightarrow j'}$ the subpath of the $(m + 1 + i)$-th horizontal path of $H$ starting from the vertex $(m + 1 + i, j)$ and finishing in $(m + 1 + i, j')$. Let $(i, i', j) \in [-m, m]^2 \times [n]$ where $i \neq i'$. We denote by $P_{i \rightarrow i', j}$ the subpath of the $j$-th vertical path of $H$ starting from the vertex $(m + 1 + i, j)$ and finishing in $(m + 1 + i', j)$. See Figure 10 for an illustration of the above definitions.

Given a path $P$ and three integers $r, h, d \geq 1$, we say that a collection $\mathcal{C}$ of subsets of $V(P)$ is $(r, h, d)$-scattered in $P$, if $\mathcal{C} = \{C_1, \ldots, C_h\}$, where for every $i \in [h]$ $C_h$ is a subset of $V(P)$ of cardinality $r$, such that $\forall i, j \in [h], C_i \cap C_j = \emptyset$ and $\forall u, v \in \bigcup_{i \in [h]} C_i$, $\text{dist}_P(u, v) > d$. 

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Lemma 23. There exists a function \( f_{15} : \mathbb{N} \to \mathbb{N} \) such that for every \( r, a, d \in \mathbb{N} \), where \( d \geq 2r^2 \), if \( H \) is an \((m \times n)\)-grid, where \( m \geq f_{15}(r) \) and \( n \geq r^2 \cdot a \cdot (d + 1) \), and \( \mathcal{C} = \{C_1, \ldots, C_a\} \) is a collection of subsets of vertices of \( P_H \) that is \((r^2, a, d)\)-scattered in \( P_H \), then \( H \) contains as a minor an \( r \)-grid \( R \) such that the model of each vertex of \( R \) intersects every \( C_i, i \in [a] \). Moreover, \( f_{15}(r) = \mathcal{O}(r^2) \).

Proof. We set \( f_{15}(r) = 2(r^2 + r + 1) + 1 \) and \( d = 2r^2 \). Let \( H \) be an \((m \times n)\)-grid, where \( m \geq f_{15}(r) \) and \( n \geq r^2 \cdot a \cdot (d + 1) \), and \( \mathcal{C} = \{C_1, \ldots, C_a\} \) be a collection of subsets of vertices of \( P_H \) that is \((r^2, a, 2r^2)\)-scattered in \( P_H \). Notice that the existence of the collection \( \mathcal{C} \) follows from the fact that \( n \geq r^2 \cdot a \cdot (d + 1) \).

We define a function \( p : \mathcal{U} \mathcal{C} \to [n] \) that maps every vertex \( v \in \mathcal{U} \mathcal{C} \) to an integer \( i \in [n] \) if \( v \) belongs to the intersection of the \( i \)-th vertical path of \( H \) with \( P_H \). Intuitively, \( p(v) \) indicates the position of the vertex \( v \) on the middle horizontal path of \( H \). Observe that since \( \mathcal{C} \) is \((r^2, a, 2r^2)\)-scattered, it follows that for every \( u, v \in \mathcal{U} \mathcal{C} \), where \( u \neq v \), it holds that \( |p(u) − p(v)| > 2r^2 \).

We define the relation \( <_p \) on the vertices of \( \mathcal{U} \mathcal{C} \) such that for every \( u, v \in \mathcal{U} \mathcal{C} \), \( u <_p v \iff p(u) < p(v) \). For every \( i \in [a] \), we fix an ordering of the elements of \( C_i \) with respect to \( <_p \), i.e. \( C_i = \{v_{i1}^1, \ldots, v_{ip}^i\} \) where for every \( j, j' \in [r^2] \), \( j < j' \iff v_{ij}^j <_p v_{ij'}^j \). Intuitively, we can see the set \( C_i \) as the vertices in \( \mathcal{U} \mathcal{C} \) colored with color \( i \) and \( v_{ij}^j \) as the \( j \)-th vertex of color \( i \) that we encounter while traversing \( P_H \) from left to right.

We now aim to construct the vertices of the desired \( r \)-grid. To do this, we define a collection of pairwise vertex-disjoint trees that are subgraphs of \( H \) and every tree contains a vertex of every \( C_i \). The edges of each tree will be contracted to a single vertex that will be a vertex of the desired \( r \)-grid.

Towards this direction, we first consider a partition \( X_{1}, \ldots, X_{r^2} \) of \( \mathcal{U} \mathcal{C} \) such that for every \( j \in [r^2], X_j := \{v_{1j}^j, \ldots, v_{r^2j}^j\} \). Intuitively, each set \( X_j \) contains the \( j \)-th vertex (with respect to the ordering defined by \( <_p \)) of each color (i.e., of each \( C_i, i \in [a] \)). Observe that \( \forall (i,j) \in [h] \times [r^2], C_i \cap X_j = \{v_{ij}^j\} \).

For every \( j \in [r^2] \), let \( x_{j}^\text{left} \) (resp. \( x_{j}^\text{right} \)) be the vertex in \( X_j \) such that \( \forall x \in X_j, \text{ if } x \neq x_{j}^\text{left} \) (resp. \( x \neq x_{j}^\text{right} \)) then \( x_{j}^\text{left} <_p x \) (resp. \( x <_p x_{j}^\text{right} \)). For every \( j \in [r^2] \), we set \( T_j \) to be the graph

\[
\bigcup_{x \in V(X_j)} \{P_{j,p(x_{j}^\text{left}) \to p(x_{j}^\text{right})} \cup P_{j \to r^2+1,p(x_{j}^\text{left})} \cup P_{r^2+1 \to 0,p(x_{j}^\text{right})} \cup \bigcup_{x \in V(X_j)} P_{0 \to j,p(x)} \}.
\]
Also, we set \( s_j = p(x_j^{\text{left}}) \) and \( t_j = p(x_j^{\text{right}}) \). See Figure 11 for an illustration of the above definitions. Observe that \( T_j \) is a tree whose leaves are the vertices in \( (X_j \setminus \{b_j\}) \cup \{s_j, t_j\} \).

![Figure 11: Visualization of the graph \( T_j \) (depicted in orange) for \( h = 5 \).](image)

We stress that we can consider the graphs \( T_j \) since \( m \geq f_{15}(r) \). Observe that every \( T_j \) is a tree and for \( j \neq j' \), \( T_j \) and \( T_{j'} \) are not necessarily vertex-disjoint. To get a collection of pairwise vertex-disjoint trees, we have to resolve possible intersections.

Notice that if \( j < j' \), then \( T_j \) intersects \( T_{j'} \) only in the vertices \( (p(v_i^{j'}), j), i \in [a] \) where \( v_i^{j'} <_p x_j^{\text{right}} \) (see Figure 12). For every \( j \in [r^2 - 1] \) we set

\[
I_j = \{ h \in [n] \mid (i, j') \in [a] \times [j + 1, r^2] : h = p(v_i^{j'}) \land v_i^{j'} <_p x_j^{\text{right}} \}.
\]

Intuitively, these are the positions (in \( P_H \)) of the vertices of every \( T_{j'} \), \( j' > j \) that are on the left of \( x_j^{\text{right}} \) (see Figure 12).

![Figure 12: Visualization of the graph \( T_j \) (depicted in orange) and \( T_{j'} \) (depicted in purple) for \( h = 5 \). Here, \( I_j = \{p(v_2^{j'}), p(v_3^{j'}), p(v_5^{j'})\} \).](image)

For every \( h \in I_j \), we set \( h^{\text{left}} := h - (r^2 - j) \), \( h^{\text{right}} := h + r^2 - j \), and \( U_h \) to be the graph in Figure 13. More precisely,

\[
U_h = P_{-(r^2 - j) \to j, h^{\text{left}}} \cup P_{-(r^2 - j), h^{\text{left}} \to h^{\text{right}}} \cup P_{-(r^2 - j) \to j, h^{\text{right}}}. 
\]
Figure 13: Visualization of the graph $U_h$.

Also, we set $T^*_j$ to be the graph

$$
\left( T_j \setminus \bigcup_{h \in I_j} P_{j,h}^{h_{\text{left}} \to h_{\text{right}}} \right) \cup \bigcup_{h \in I_j} U_h.
$$

Observe that since $C$ is $(r^2, a, 2r^2)$-scattered, $T^*_1, \ldots, T^*_r$ are pairwise vertex-disjoint trees (see Figure 14) each containing a vertex of every $C_i, i \in [a]$.

Figure 14: The trees $T^*_j$ (depicted in orange) and $T^*_j$ (depicted in purple).

Towards the construction of the desired $r$-grid, we already mentioned that each $T^*_j, j \in [r^2]$ will be contracted to a single vertex. Our aim now is to “connect” these vertices in order to form an $r$-grid.

We set $l^t = \lfloor m/2 \rfloor + r^2 + 1$ and $l^t = \lfloor m/2 \rfloor - (r^2 + 1)$. To get more intuition, observe that for every $j \in [r^2]$, the set of vertices of degree one of $T^*_j$ is $X_j \cup \{(s_j, l^t), (t_j, l^t)\}$.

Now, for every odd $i \in [r - 1]$, we define $L^*_i$ to be the graph

$$
\bigcup_{j \in [r]} \left( P_{l^t \to l^t+j, s_i \cdot (r-j+1)} \cup P_{l^t+j, s_i \cdot (r-j+1) \to s_i \cdot (r+j)} \cup P_{l^t \to l^t+j, s_i \cdot (r+j)} \right) \cup P_{s_i \cdot (i-1) \cdot r+1 \to s_i \cdot r+r \cdot l^t}.
$$

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Also, for every even \(i \in [r - 1]\), we define \(L^\downarrow_i\) to be the graph
\[
\bigcup_{j \in [r]} \left( P_{i \to i-j, t_i, (r-j+1)} \cup P_{i \to i-j, t_i, (r-j+1) \to t_i, (r+j)} \cup P_{i \to i-j, t_i, (r-j+1) \to t_i, (r+j)} \right) \cup P_{(i-1) \to i, t_{i-1}, t_i, (r+1)}.
\]
Then, we consider the graph \(R^*\)
\[
\bigcup_{j \in [r^2]} T^*_j \cup \bigcup_{\text{odd } i \in [r-1]} L^\downarrow_i \cup \bigcup_{\text{even } i \in [r-1]} L^\uparrow_i.
\]
See Figure 15 for an illustration of the above graphs.

\[\text{Figure 15: Visualization of the graph } R^* \text{. Notice that the trees } T^*_4, T^*_5, \text{ and } T^*_6 \text{ intersect both } L^\uparrow_1 \text{ and } L^\downarrow_2.\]

Notice that the graph obtained from \(R^*\) if for every \(i \in [r^2]\) we contract all edges of \(T^*_i\) is a subdivision of an \(r\)-grid.

7.2 Finding a complete apex grid

An \(A\)-apex \(r\)-grid is complete if it contains every edge between the vertices of \(A\) and the vertices of the grid.
Lemma 24. There exist three functions \( f_{16}, f_{17} : \mathbb{N}^2 \to \mathbb{N}, \) \( f_{18} : \mathbb{N} \to \mathbb{N} \) such that if \( r, a \in \mathbb{N}, \) \( H \) is an \( f_{16}(r, a) \)-grid, and \( S = \{ S_1, \ldots, S_a \} \) is a collection of a subsets of vertices in the central \((f_{16}(r, a) - f_{18}(r))\)-subgrid of \( H \) such that \( \forall i \in [a], |S_i| \geq f_{17}(r, a), \) then \( H \) contains as a minor an \( r \)-grid \( R \) such that the model of each vertex of \( R \) in \( H \) intersects every \( S_i, i \in [a]. \) Moreover, \( f_{16}(r, a) = \mathcal{O}(r^3 \cdot 2^a), \) \( f_{17}(r, a) = \mathcal{O}(r^6 \cdot 2^a), \) and \( f_{18}(r) = \mathcal{O}(r^2). \)

Proof. Let \( f_{15} \) be the function of Lemma 23. Also, let

\[
\ell := \max\{2r^2, f_{15}(r)\} = f_{15}(r) \quad \text{(recall that} \quad f_{15}(r) > 2r^2),
\]

\[
b := \ell \cdot a + 3,
\]

\[
s := 2^{a-1} \cdot r^2 \cdot b^2,
\]

\[
z := \lceil \sqrt{s} \rceil,
\]

\[
f_{16}(r, a) := b \cdot z + \ell,
\]

\[
f_{17}(r, a) := s, \quad \text{and}
\]

\[
f_{18}(r) := \ell.
\]

For the sake of simplicity, we let \( m = f_{16}(r, a) - \ell = b \cdot z. \) Let \( R' \) be the \( \ell \times n \)-grid, where \( n = r^2 \cdot a \cdot (\ell + 1). \) We begin by arguing that the following holds:

**Claim 1:** \( R' \subseteq_m H \) and there is a collection \( \mathcal{V} = \{ V_1, \ldots, V_a \} \) of subsets of the vertices of the middle horizontal path of \( R' \), where

- for every \( i \in [a] \), the model of each vertex \( v \in V_i \) in \( H \) intersects \( S_i, \)
- for every \( i \in [a] \), \( |V_i| \geq 2^{a-1} \cdot r^2, \) and
- \( \forall u, v \in \bigcup \mathcal{V}, \text{dist}_{R'}(u, v) \geq 2r^2. \)

**Proof of Claim 1:** Let \( \tilde{H} \) be the central \( m \)-subgrid of \( H \). Also, let \( \mathcal{P} = \{ P_1, \ldots, P_m \} \) be the set of the vertical paths of \( \tilde{H} \), where \( P_i \) is the \( i \)-th vertical path of \( \tilde{H} \). For every \( j \in [b] \), let \( \mathcal{P}_j = \bigcup_{i \in [a]} P_{j+b(i-1)}. \) For every \( i \in [a], \) let \( x_i := \arg \max_{j \in [b]} \{|P_j \cap S_i|\} \). Intuitively, we partition \( \mathcal{P} \) into \( z \) sets of \( b \) consecutive vertical paths and let \( x_i \) be the index \( j \) maximizing the size of the intersection of the \( j \)-th paths of all said path sets with \( S_i. \) Observe that \( |P_{x_i} \cap S_i| \geq s/b. \)

Now, let \( \mathcal{L} = \{ L_1, \ldots, L_m \} \) be the set of the horizontal paths of \( \tilde{H} \), where \( L_i \) is the \( i \)-th horizontal path of \( \tilde{H} \). For every \( j \in [b] \), let \( \mathcal{L}_j = \bigcup_{i \in [a]} L_{j+b(i-1)}. \) For every \( i \in [a], \) let \( y_i := \arg \max_{j \in [b]} \{|L_j \cap P_{x_i} \cap S_i|\} \). Intuitively, we partition \( \mathcal{L} \) into \( z \) sets of \( b \) consecutive horizontal paths and let \( y_i \) be the index \( j \) maximizing the size of the intersection of the \( j \)-th paths of all said path sets with \( P_{x_i} \cap S_i. \) Observe that \( |L_{y_i} \cap P_{x_i} \cap S_i| \geq s/b^2. \) For every \( i \in [a], \) let \( Q_i := \mathcal{P}_{x_i} \cap L_{y_i}. \) See Figure 16 for an illustration of the above. For further intuition, observe that for every \( i \in [a], \) \( Q_i \) is a set of \( z^2 \) vertices in \( \tilde{H} \) and \( \forall u, v \in V(Q_i), \) where \( u \neq v, \) it holds that \( \text{dist}_{\tilde{H}}(u, v) \geq b. \)

Notice that since \( b = \ell \cdot a + 3, \) there exists a \( t \in [2, b - \ell] \) such that \( \forall i \in [a], x_i \notin [t, t + \ell - 1]. \) Let \( \mathcal{P} := \bigcup_{i \in [t, t+\ell-1]} P_i. \) Also, there exists a \( t' \in [2, b - \ell] \) such that \( \forall i \in [a], y_i \notin [t', t' + \ell - 1]. \) Let \( \mathcal{L} := \bigcup_{i \in [t', t'+\ell-1]} L_i. \) Intuitively, \( \mathcal{P} \) (resp. \( \mathcal{L} \)) is the union of \( z \) sets of \( \ell \) consecutive vertical (resp. horizontal) paths of \( \tilde{H} \) whose indices “avoid” \( x_i \) (resp. \( y_i \)) for every \( i \in [a]. \) In Figure 16, we
assume that $\widehat{P}$ (resp. $\widehat{L}$) is the set of the vertical (resp. horizontal) paths of $H$ that do not contain yellow or green subpaths.

We denote by $\widehat{P}$ (resp. $\widehat{L}$) the set of the vertical (resp. horizontal) paths of $H$ that intersect $P \setminus \widehat{P}$ (resp. $L \setminus \widehat{L}$). Let $H'$ be the graph obtained from $H$ after contracting every edge of the horizontal (resp. vertical) paths of $H$ whose endpoints are vertices of $\widehat{P}$ (resp. $\widehat{L}$). In Figure 16, the graph $H'$ is obtained if we contract every edge between the paths of $H$ that contain $j$-th yellow and the $j$-th green path, $j \in [5]$. Therefore, $H'$ is a contraction of $H$ and the fact that $m = b \cdot z$ implies that $H'$ is an $((\ell + 1) \cdot z)$-grid.

We call a vertex of $H'$ heavy if its model in $H$ intersects $\widehat{P} \cap \widehat{L}$. Notice that the model of each heavy vertex of $H'$ contains exactly one vertex of each $Q_i$, $i \in [a]$ and the distance in $H'$ between every two heavy vertices of $H'$ is at least $\ell$ in $H'$. For every $i \in [a]$, we set $V_i$ to be the set of vertices of $H'$ whose model in $H$ intersects $S_i$ and observe that for every $i \in [a]$, since $|Q_i \cap S_i| \geq 2^{a-1} \cdot r^2$, it follows that $|V_i| \geq 2^{a-1} \cdot r^2$. Let $\mathcal{V} := \{V_1, \ldots, V_a\}$ and observe that for every $u, v \in \bigcup \mathcal{V}$, where $u \neq v$, it holds that $\dist_{H'}(u, v) \geq \ell$ and, since $\ell \geq 2r^2$, it follows that $\dist_{H'}(u, v) \geq 2r^2$.

To conclude the proof of Claim 1, observe that since $H'$ is an $((\ell + 1) \cdot z)$-grid, it follows that $R' \leq_m H'$ (see Figure 17) and $\mathcal{V}$ is a collection of subsets of vertices of the middle horizontal path of $R'$ satisfying the claimed conditions. Claim 1 follows.
Following Claim 1, let $\mathcal{V}$ be a collection of vertex sets satisfying the properties above. It is easy to see that the sets in $\mathcal{V}$ are not necessarily disjoint. For each vertex $v \in \bigcup \mathcal{V}$, we define the trace of $v$ in $\mathcal{S}$ to be the set $I_v := \{i \in [a] \mid \text{the model of } v \text{ in } H \text{ intersects } S_i\}$. We say that a set $U \subseteq \bigcup \mathcal{V}$ is full with respect to $\mathcal{S}$ if $\bigcup_{v \in U} I_v = [a]$. We now argue that the following holds.

Claim 2: There is a collection $\mathcal{C} = \{C_1, \ldots, C_q\}$ of $q$ pairwise disjoint subsets of $\bigcup \mathcal{V}$, each of cardinality $r^2$, such that if we pick a vertex from every set $C_j$ then the resulting set is full with respect to $\mathcal{S}$.

Proof of Claim 2: Notice that each $V_i$ can be partitioned into a collection $\mathcal{V}_i$ of $2^{a-1}$ subsets such that every two vertices are in the same subset if and only if they have the same trace. For every $i \in [a]$, we set $C_i := \arg \max_{S \in \mathcal{V}_i} \{|S|\}$. Since $|V_i| \geq 2^{a-1} \cdot r^2$, it follows that $|C_i| \geq r^2$. Moreover, we can assume that every $C_i$ contains exactly $r^2$ vertices (by removing extra vertices). Notice that there may exist $i, j \in [q]$ such that $C_i = C_j$. Therefore, the collection $\mathcal{C} = \{C_1, \ldots, C_q\}$, $q \leq a$ is the desired one. Claim 2 follows.

To conclude the proof, consider the graph $R'$ from Claim 1 and the collection $\mathcal{C}$ from Claim 2. Following Claims 1 and 2, $\mathcal{C}$ is a collection of subsets of vertices of the middle horizontal path of $R'$ that is also $(r^2, q, 2r^2)$-scattered in the middle horizontal path of $R'$. The lemma follows by applying Lemma 23.

7.3 Finding a branching structure

The following easy observation (see e.g., [49]) intuitively states that every planar graph $H$ is a minor of a big enough grid, where the relationship between the size of the grid and $|V(H)|$ is linear.

Proposition 25 (Robertson and Seymour [49]). There is a function $f_{19} : \mathbb{N} \to \mathbb{N}$ such that every planar graph on $n$ vertices is a minor of the $f_{19}(n)$-grid. Moreover, $f_{19}(n) = \mathcal{O}(n)$.

Lemma 26. There is a function $f_{20} : \mathbb{N}^3 \to \mathbb{N}$ such that if $\mathcal{F}$ is a finite family of graphs, $k \in \mathbb{N}$, and $G$ is a graph that contains a complete $A$-apex $f_{20}(a_{\mathcal{F}}, s_{\mathcal{F}}, k)$-grid as an $A$-fixed minor for some $A \subseteq V(G)$ where $|A| = a_{\mathcal{F}}$, then for every solution $S$ of $\mathcal{F}$-M-DELETION for the instance $(G, k)$, it holds that $S \cap A \neq \emptyset$. Moreover $f_{20}(a_{\mathcal{F}}, s_{\mathcal{F}}, k) = \mathcal{O}_{s_{\mathcal{F}}} (\sqrt{k})$.

Proof. For simplicity, we set $s = s_{\mathcal{F}}$ and $a = a_{\mathcal{F}}$. Let $G$ be a graph, $m = f_{19}(s - a)$, where $f_{19}$ is the function of Proposition 25, and $r = \lceil \sqrt{(k + a^2 + 1) \cdot m} \rceil$. We set $f_{20}(a, s, k) = r$. Let also $H$ be the complete $A$-apex $r$-grid that is an $A$-fixed minor of $G$, for some $A \subseteq V(G)$, where $|A| = a$.

Observe that since $r = \lceil \sqrt{(k + a^2 + 1) \cdot m} \rceil$, $H \setminus A$ can be partitioned into $k + 1 + a^2$ $m$-grids $H_i$, such that $V(H_i) \cap V(H_j) = \emptyset$, for $i \neq j$. Let

$$\mathcal{H} = \{H[V(H_i) \cup A] \mid i \in [k + a^2 + 1]\}.$$ 

Notice that every $H \in \mathcal{H}$ is an $A$-apex $m$-grid. Our aim is to prove that if $S$ is a subset of $V(G)$ of size at most $k$ such that $\mathcal{F} \not\subseteq_m G \setminus S$, then $S \cap A \neq \emptyset$. Suppose towards a contradiction that $S \cap A = \emptyset$. Since $|S| \leq k$ and $|\mathcal{H}| = k + a^2 + 1$, there is a collection $\mathcal{H}' \subseteq \mathcal{H}$ of size $a^2 + 1$ such that for every $H' \in \mathcal{H}'$ and every $v \in V(H')$, $S$ does not intersect the model of $v$ in $G$. This implies that $\bigcup \mathcal{H}' \not\subseteq_m G \setminus S$. Let $L$ be a graph in $\mathcal{F}$ such that $a(L) = a$. We arrive to a contradiction by proving
that \( L \leq m \bigcup H' \). To see why \( L \leq m \bigcup H' \), fix a \( H' \in H' \) and observe that, since \( m = f_{19}(s-a) \), Proposition 25 implies that every planar graph on \( s-a \) vertices is a minor of \( H' \setminus A \) and every graph on \( a \) vertices is a minor of \( \bigcup(H' \setminus \{H'\}) \). The latter is a consequence of the fact that \( |H' \setminus \{H'\}| = a^2 \) and every vertex in \( A \) is adjacent to every vertex of the underlying grid of each \( H'' \in H' \). \( \square \)

### 7.4 The proof of Lemma 18

**Proof of Lemma 18.** Let \( G \) be a graph and \( F \) be a finite set of graphs. For simplicity, we set \( a = a_F \) and \( s = s_F \). We set \( r = f_{20}(a, s, k) \),

\[
\begin{align*}
    f_8(a, s, k) &= f_{16}(r, a), \\
    f_9(a, s, k) &= f_{17}(r, a), \\
    f_{10}(a, s, k) &= f_{18}(r).
\end{align*}
\]

Let \( A \) be a subset of \( V(G) \) of size \( a \) and let \( H \) be an \( A \)-apex \( f_{8}(a, s, k) \)-grid such that

- \( H \) is an \( A \)-fixed minor of \( G \) and
- each \( v \in A \) has at least \( f_{9}(a, s, k) \) neighbors in the central \( (f_{8}(a, s, k) - f_{10}(a, s, k)) \)-grid of \( H \setminus A \).

For every \( v \in A \), let \( S_v \) be the set of neighbors of \( v \) in the central \( (f_{8}(a, s, k) - f_{10}(s, k)) \)-grid of \( H \setminus A \). Keep in mind that \( |S_v| \geq f_{9}(a, s, k) \). By applying Lemma 24 in \( H \setminus A \) and \( \{S_v \mid v \in A\} \), we get that \( H \setminus A \) contains a minor an \( r \)-grid \( R \) such that the model of each vertex of \( R \) in \( H \) intersects every \( S_i, i \in [a] \). Therefore, let \( R^A \) be the graph obtained from \( R \) by adding in \( V(R) \) the vertex set \( A \) and the edges \( \{\{v, u\} \mid v \in A, u \in R, \text{ and } v \text{ is adjacent to a vertex in the model of } u \} \). Notice that, since the model of each vertex of \( R \) in \( H \) intersects every \( S_i, i \in [a] \), \( R^A \) is a complete \( A \)-apex \( r \)-grid. Therefore, since \( r = f_{20}(a, s, k) \), by Lemma 26 it follows that then for every solution \( S \) of \( F \)-M-DELETION for the instance \( (G, k) \), it holds that \( S \cap A \neq \emptyset \). \( \square \)

### 7.5 Improved bounds for the apex case

We conclude this section with an improvement of the bounds of Lemma 18 for the case where we have only one apex. Given an \( r \)-grid \( H \), we define the graph \( H^{+v}_k \) to be the graph obtained if we take \( k \) disjoint copies of \( H \) and a new vertex \( v \) and make \( v \) adjacent to every vertex of every copy of \( H \).

**Proof of Lemma 19.** Let \( f_{19} \) be the function of Proposition 25. For simplicity, we set \( s = s_F \). Also,
we set

\[ h = f_{19}(s), \]
\[ f_{13}(s) = \max\{2h^2, f_{15}(h)\} = f_{15}(h), \quad \text{(recall that } f_{15}(h) > 2h^2), \]
\[ b = 2 \cdot f_{15}(h) + 1, \]
\[ z = \left\lceil \sqrt{h^2 \cdot (k + 1)} \right\rceil, \]
\[ m = b \cdot z, \]
\[ f_{11}(s, k) = m + f_{13}(s), \text{ and} \]
\[ f_{12}(s, k) = b^2 \cdot h^2 \cdot (k + 1). \]

Let \((G, k)\) be an instance of \(\mathcal{F}\)-M-DELETION, \(a\) be a vertex in \(V(G)\), and \(H\) be an \(\{a\}\)-apex \(f_{11}(s, k)\)-grid that is an \(\{a\}\)-fixed minor of \(G\) such that \(a\) has at least \(f_{12}(s_{\mathcal{F}}, k)\) neighbors in the central \(m\)-grid \(\bar{H}\) of \(H \setminus a\). Also, let \(R\) be an \(h\)-grid.

We first prove the following:

\textbf{Claim:} \(G\) contains \(R_{k+1}^{a}\) as an \(\{a\}\)-fixed minor.

\textbf{Proof of Claim:} Let \(N_a\) be the set of neighbors of \(a\) in \(\bar{H}\) and keep in mind that \(|N_a| \geq f_{12}(s, k)\).

As in Lemma 24, we can consider a collection \(\mathcal{P}\) (resp. \(\mathcal{L}\)) of horizontal (resp. vertical) paths of \(\bar{H}\) such that if \(Q = V(\mathcal{U}\mathcal{P}) \cap (\mathcal{U}\mathcal{L}))\), then

\begin{itemize}
  \item \(Q\) is a subset of \(V(\bar{H})\) such that for every \(u, v \in Q\), \(\text{dist}_{\bar{H}}(u, v) \geq b\) and
  \item \(|Q \cap N_a| \geq h^2 \cdot (k + 1)\).
\end{itemize}

We set \(V = Q \cap N_a\) and observe that since \(m = b \cdot z = b \cdot \left\lceil \sqrt{h^2 \cdot k} \right\rceil\), it follows that there is a collection \(\mathcal{H} = \{H_1, \ldots, H_{k+1}\}\) of \(k + 1\) subgraphs of \(G\), such that

\begin{itemize}
  \item for every \(i \in [k + 1]\), each \(H_i\) is a subdivision of an \((m \times n)\)-grid, where \(m = f_{15}(h)\) and \(n = h^2 \cdot b\),
  \item for every \(j \neq i\), \(V(H_i) \cap V(H_j) = \emptyset\), and
  \item for every \(i \in [k + 1]\), if \(V_i\) is the set of all vertices of \(H_i\) whose model intersects \(V\), then \(V_i\) is a subset of the vertex set of the middle horizontal path of \(H_i\) of size \(h^2\).
\end{itemize}

Also, notice that \(\{V_i\}\) is \((h^2, 1, 2h^2)\)-scattered in the middle horizontal path of \(H_i, i \in [k + 1]\). Therefore, for every \(i \in [k + 1]\), we can apply Lemma 23 on \(H_i\) and \(\{V_i\}\) and derive that \(H_i\) contains \(R\) as a minor such that the model of each vertex of \(R\) in \(H_i\) intersects \(V\). Therefore, we conclude that \(G\) contains \(R_{k+1}^{a}\) as an \(\{a\}\)-fixed minor. The claim follows.

Following the Claim above, \(G\) contains \(R_{k+1}^{a}\) as an \(\{a\}\)-fixed minor. Therefore, as in Lemma 26, we observe that for every solution \(S\) of \(\mathcal{F}\)-M-DELETION for the instance \((G, k)\), it holds that \(a \in S\).

To see this, suppose towards a contradiction that there exists a solution \(S\) of \(\mathcal{F}\)-M-DELETION for the instance \((G, k)\) such that \(a \notin S\), and consider a graph \(H \in \mathcal{F}\). The fact that \(G\) contains \(R_{k+1}^{a}\) as an \(\{a\}\)-fixed minor implies that \(G \setminus S\) contains an \(\{a\}\)-apex \(h\)-grid as an \(\{a\}\)-fixed minor and therefore, since \(h = f_{19}(s)\), it follows that \(H \leq_m G \setminus S\), a contradiction. \(\square\)
8 Algorithms for variants of the Vertex Deletion to $\mathcal{G}$

We now present how our approach can be modified so to deal with several variants of the Vertex Deletion to $\mathcal{G}$ problem.

8.1 The general framework

Notice that both algorithms in Subsection 5.1 and Subsection 5.2 are based on one of the following three scenarios for Vertex Deletion to $\mathcal{G}$ with input $(G, k)$.

[Bounded treewidth case] A tree decomposition of $G$ of width $k^{O(1)}$, or

[Branching case] a set $B$ where $|B| = O(k)$, such that $(G, k)$ is a yes-instance if and only if, for some $x \in B$, $(G \setminus x, k - 1)$ is a yes-instance, or

[Irrelevant vertex case] a vertex $x$ such that $(G, k)$ is a yes-instance if and only if $(G \setminus x, k)$ is a yes-instance,

For each of the variants of Vertex Deletion to $\mathcal{G}$ that we consider, the algorithm recursively runs on an equivalent instance with one vertex less (irrelevant vertex case), or branches on $O(k)$ equivalent instances where both $k$ and the size of the graph are one less (branching case). The eventual outcome is to reduce the problem to the bounded treewidth case, producing a tree decomposition of $G$ of width $k^{O(1)}$ (bounded treewidth case). In each variant of the problem, the bounded treewidth case can be treated by a suitable modification of the dynamic programming algorithm of [4], taking into account the main combinatorial result in [5]. For each variant that we treat, the algorithm of Subsection 5.1 assumes that we have at hand a solution of Vertex Deletion to $\mathcal{G}$ of size $k$, which can be found by the algorithm in Theorem 1.

We next present the problem variants and explain how to adapt the branching case and the irrelevant vertex case for each of them.

8.2 Variants of Vertex Deletion to $\mathcal{G}$

A common part of the inputs of all problems below is the pair $(G, k)$, where $G$ is a graph and $k$ is a non-negative integer, i.e., an input of Vertex Deletion to $\mathcal{G}$.

Annotated. In the annotated version of Vertex Deletion to $\mathcal{G}$, the input is a triple $(G, k, R)$, where $R \subseteq V(G)$ and the problem asks for a solution $S$ where $S \subseteq R$.

[Branching case]: we branch on $(G \setminus x, k - 1, R \setminus x)$ for all the annotated vertices of $B$, i.e., for every $x \in B \cap R$. If there is no such a vertex we report that $(G, k, R)$ is a no-instance.

[Irrelevant vertex case]: we recurse on $(G \setminus x, k, R \setminus x)$, as every irrelevant vertex for the original problem is also an irrelevant vertex for its annotation variant.
Modulo. In the modulo version of Vertex Deletion to \( G \), the input is a quadruple \((G, k, q, p)\) where \( q, p \) are integers, \( p \) is a prime, and \( q < p \). The question is whether there is a solution \( S \) of size at most \( k \) where \(|S| \equiv q \pmod{p}\).

[Branching case]: we branch on \((G \setminus x, k - 1, q - 1 \pmod{p}, p)\) for every \( x \in B \).

[Irrelevant vertex case]: it is the same as every irrelevant vertex for the original problem is also an irrelevant vertex for this variant.

Weighted. In the weighted version of Vertex Deletion to \( G \), the input is a triple \((G, k, w)\) where \( w : V(G) \to \mathbb{R} \) is a weight function assigning positive real weights to the vertices of \( G \). The problem asks for a solution \( S \) where \( \sum_{v \in S} w(v) \leq k \).

[Branching case]: we branch on \((G \setminus x, k - w(x), w \setminus \{(x, w(x))\})\), for every \( x \in B \).

[Irrelevant vertex case]: it is the same as every irrelevant vertex for the original problem is also an irrelevant vertex for this variant.

For the above problem, if \( \epsilon = \min \{w(x) \mid x \in V(G)\} \), then the parametric dependence of the derived algorithm is \( 2^{\text{poly}(k/\epsilon)} \), as the size of the solution \( S \) is at most \( k/\epsilon \).

Counting. In the counting version of Vertex Deletion to \( G \) with input \((G, k)\), the output is the number \( \#_G(G, k) \) of all solutions of Vertex Deletion to \( G \) of size (at most) \( k \). We treat the case where we count solutions of size exactly \( k \) as the “\( \leq k \)”-case can be easily reduced to this.

[Branching case]: return \( \sum_{x \in B} \#_G(G \setminus x, k - 1) \).

[Irrelevant vertex case]: return \( \#_G(G \setminus x, k - 1) + \#(G \setminus x, k) \).

The above creates \( T(n, k) \) subproblems on bounded treewidth graphs, where

\[
T(n, k) = \max \{O(k) \cdot T(n - 1, k - 1), T(n - 1, k - 1) + T(n - 1, k)\}.
\]

This makes a total of \( 2^{O(k)} \cdot n \) problems, each solvable in \( 2^{kO(1)} \cdot n \) time by the counting version of the dynamic programming algorithm of [4], taking into account the analysis of [5].

Colored. In the colored version of Vertex Deletion to \( G \), the input is a triple \((G, k, \chi)\) where \( \chi : V(G) \to [k] \) is a function assigning colors from \([k]\) to the vertices of \( G \). The problem asks for a solution \( S \) to Vertex Deletion to \( G \) that carries all \( k \) colors, i.e., for each \( i \in [k] \), \(|S \cap \chi^{-1}(i)| = 1\). (Notice that the requested solution must have size exactly \( k \).) To deal with this problem, we deal with its annotated version where we permit \( \chi : V(G) \to \{0, 1, \ldots, k\} \), i.e., the vertices in \( R := \bigcup_{i \in [k]} \chi^{-1}(i) \) are annotated, while the vertices in \( \chi^{-1}(0) \) cannot participate in a solution (we call these vertices black vertices).
and Theorem 2, running in time $O(n^4)$ for some universal constant $c$ (i.e., not depending on the class $G$). Clearly, this is not the case of the algorithms of Theorem 1 and Theorem 2, running in time $2^{O_{s,k}(k^2c+1)}n^3$ and $2^{O_{s,k}(k^2c+1)}n^2$, respectively, where $c$ is the the palette-variety of the minor-obstruction set $F$ of $G$ which, from the corresponding proofs, is estimated to be $c = 2^{O(s^2 \log s)}$ and $c = 2^{O(s^4 \log s)}$, respectively (recall that $s$ is the maximum size of a minor-obstruction of $G$). We tend to believe that this dependence is unavoidable if we want to use the irrelevant vertex technique, as it reflects the price of homogeneity, mentioned in Subsection 3.3. Having homogeneous walls is critical for the application of this technique (see Lemma 16) when $G$ is more general than surface embeddable graphs (in the bounded genus case, all subwalls are already homogeneous). Is there a way to prove that this behavior is unavoidable subject to some complexity assumption? An interesting result of this flavor concerning the existence of polynomial

4The definition is as minors, except that only edges incident to degree-two vertices are contracted.
kernels for \textsc{Vertex Deletion to} $\mathcal{G}$ was given by Giannopoulou et al. \cite{27} who proved that, even for minor-closed families $\mathcal{G}$ that exclude a planar graph, the dependence on $\mathcal{G}$ on the degree of the polynomial kernel, which exists because of \cite{23}, is unavoidable subject to reasonable complexity assumptions.

\textbf{Other modification operations.} Another direction is to consider graph modification to a minor-closed graph class for different modification operations. Our approach becomes just simpler in the case where the modification operation is edge removal or edge contraction. In these two cases, we immediately get rid of the branching part of our algorithms, and only the irrelevant vertex part needs to be applied. Another challenge is to combine all aforementioned modifications. This is more complicated (and tedious) but not more complex. What is really more complex is to additionally consider edge additions. We leave it as an open research challenge (a first step was done for the case of planar graphs \cite{22}).

\textbf{Lower bounds.} Concerning lower bounds for \textsc{Vertex Deletion to} $\mathcal{G}$ under the Exponential Time Hypothesis \cite{29}, we are not aware of any lower bound stronger than $2^{o(k)} \cdot n^{O(1)}$ for any minor-closed class $\mathcal{G}$. This lower bound already applies when $\mathcal{F} = \{K_2\}$, i.e., for the \textsc{Vertex Cover} problem \cite{7,29}.

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\textbf{References}

\begin{enumerate}
\item Isolde Adler, Frederic Dorn, Fedor V. Fomin, Ignasi Sau, and Dimitrios M. Thilikos. Faster parameterized algorithms for minor containment. \textit{Theoretical Computer Science}, 412(50):7018–7028, 2011. URL: \url{https://doi.org/10.1016/j.tcs.2011.09.015}.
\item Isolde Adler, Martin Grohe, and Stephan Kreutzer. Computing excluded minors. In \textit{Proc. of the 19th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)}, pages 641–650, 2008. URL: \url{http://dl.acm.org/citation.cfm?id=1347082.1347153}.
\item Ernst Althaus and Sarah Ziegler. Optimal Tree Decompositions Revisited: A Simpler Linear-Time FPT Algorithm. \textit{CoRR}, abs/1912.09144, 2019. \url{arXiv:1912.09144}.
\item Julien Baste, Ignasi Sau, and Dimitrios M. Thilikos. Optimal algorithms for hitting (topological) minors on graphs of bounded treewidth. In \textit{Proc. of the 12th International Symposium on Parameterized and Exact Computation (IPEC)}, volume 89 of LIPIcs, pages 4:1–4:12, 2017. URL: \url{https://doi.org/10.4230/LIPIcs.IPEC.2017.4}.
\item Julien Baste, Ignasi Sau, and Dimitrios M. Thilikos. A complexity dichotomy for hitting connected minors on bounded treewidth graphs: the chair and the banner draw the boundary.
\end{enumerate}
In Proc. of the 31st Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), pages 951–970, 2020. URL: https://doi.org/10.1137/130947374.

[6] Julien Baste, Ignasi Sau, and Dimitrios M. Thilikos. Hitting minors on bounded treewidth graphs. II. single-exponential algorithms. Theoretical Computer Science, 814:135–152, 2020. doi:10.1016/j.tcs.2020.01.026.

[7] Julien Baste, Ignasi Sau, and Dimitrios M. Thilikos. Hitting minors on bounded treewidth graphs. III. lower bounds. Journal of Computer and System Sciences, 109:56–77, 2020. doi:10.1016/j.jcss.2019.11.002.

[8] Hans L. Bodlaender, Pål Grønås Drange, Markus S. Dregi, Fedor V. Fomin, Daniel Lokshtanov, and Michal Pilipczuk. A $\mathcal{O}(n)$ 5-Approximation Algorithm for Treewidth. SIAM Journal on Computing, 45(2):317–378, 2016. URL: https://doi.org/10.1137/130947374.

[9] Hans L. Bodlaender, Pinar Heggernes, and Daniel Lokshtanov. Graph modification problems (dagstuhl seminar 14071). Dagstuhl Reports, 4(2):38–59, 2014. URL: https://doi.org/10.4230/DagRep.4.2.38.

[10] Hans L. Bodlaender, Hirotaka Ono, and Yota Otachi. A faster parameterized algorithm for pseudoforest deletion. Discrete Applied Mathematics, 236:42–56, 2018. URL: https://doi.org/10.1016/j.dam.2017.10.018.

[11] Andries E. Brouwer and Henk Jan Veldman. Contractibility and NP-completeness. Journal of Graph Theory, 11(1):71–79, 1987. doi:10.1002/jgt.3190110111.

[12] Jianer Chen, Iyad A. Kanj, and Ge Xia. Improved upper bounds for vertex cover. Theoretical Computer Science, 411:3736–3756, September 2010. URL: http://dx.doi.org/10.1016/j.tcs.2010.06.026, doi:10.1016/j.tcs.2010.06.026.

[13] Christophe Crespelle, Pål Grønås Drange, Fedor V. Fomin, and Petr A. Golovach. A survey of parameterized algorithms and the complexity of edge modification. CoRR, abs/2001.06867, 2013. URL: https://arxiv.org/abs/2001.06867.

[14] Marek Cygan, Fedor V. Fomin, Lukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, Michal Pilipczuk, and Saket Saurabh. Parameterized Algorithms. Springer, 2015. doi:10.1007/978-3-319-21275-3.

[15] Marek Cygan, Marcin Pilipczuk, Michal Pilipczuk, and Jakub Onufry Wojtaszczyk. An improved FPT algorithm and a quadratic kernel for pathwidth one vertex deletion. Algorithmica, 64(1):170–188, 2012. doi:10.1007/s00453-011-9578-2.

[16] Erik D. Demaine, Fedor V. Fomin, MohammadTaghi Hajiaghayi, and Dimitrios M. Thilikos. Bidimensional parameters and local treewidth. SIAM Journal on Discrete Mathematics, 18(3):501–511, 2004. URL: https://doi.org/10.1137/S0895480103433410.

[17] Reinhard Diestel. Graph Theory, volume 173. Springer-Verlag, 4th edition, 2010.
[18] Rodney G. Downey and Michael R. Fellows. *Fundamentals of Parameterized Complexity*. Texts in Computer Science. 2013. doi:10.1007/978-1-4471-5559-1.

[19] Michael R. Fellows, Jan Kratochvíl, Matthias Middendorf, and Frank Pfeiffer. The complexity of induced minors and related problems. *Algorithmica*, 13(3):266–282, 1995. doi:10.1007/BF01190507.

[20] Michael R. Fellows and Michael A. Langston. Nonconstructive tools for proving polynomial-time decidability. *Journal of the ACM*, 35(3):727–739, 1988. doi:10.1145/44483.44491.

[21] Jörg Flum and Martin Grohe. *Parameterized Complexity Theory*. Texts in Theoretical Computer Science. An EATCS Series. Springer, 2006. doi:10.1007/3-540-29953-X.

[22] Fedor V. Fomin, Petr A. Golovach, and Dimitrios M. Thilikos. Modification to planarity is fixed parameter tractable. In Proc. of the 36th International Symposium on Theoretical Aspects of Computer Science (STACS), volume 126 of LIPIcs, pages 28:1–28:17, 2019. doi:10.4230/LIPIcs.STACS.2019.28.

[23] Fedor V. Fomin, Daniel Lokshtanov, Neeldhara Misra, and Saket Saurabh. Planar $F$-Deletion: Approximation, Kernelization and Optimal FPT Algorithms. In Proc. of the 53rd Annual IEEE Symposium on Foundations of Computer Science (FOCS), pages 470–479, 2012. doi:10.1109/FOCS.2012.62.

[24] Fedor V. Fomin, Daniel Lokshtanov, Fahad Panolan, Saket Saurabh, and Meirav Zehavi. Hitting Topological Minors is FPT. *CoRR*, abs/1904.02944, 2019. To appear in Proc. of the 52nd Annual ACM Symposium on Theory of Computing (STOC 2020). URL: http://arxiv.org/abs/1904.02944, arXiv:1904.02944.

[25] Fedor V. Fomin, Saket Saurabh, and Neeldhara Misra. Graph modification problems: A modern perspective. In Proc. of the 9th International Frontiers in Algorithmics Workshop (FAW), volume 9130 of LNCS, pages 3–6, 2015. URL: https://doi.org/10.1007/978-3-319-19647-3_1, doi:10.1007/978-3-319-19647-3\_1.

[26] Michael R. Garey and David S. Johnson. *Computers and intractability. A guide to the theory of NP-completeness*. W. H. Freeman and Co., San Francisco, 1979.

[27] Archontia C. Giannopoulou, Bart M. P. Jansen, Daniel Lokshtanov, and Saket Saurabh. Uniform kernelization complexity of hitting forbidden minors. *ACM Transactions on Algorithms*, 13(3):35:1–35:35, 2017. doi:10.1145/3029051.

[28] John E. Hopcroft and Robert Endre Tarjan. Dividing a graph into triconnected components. *SIAM J. Comput.*, 2(3):135–158, 1973. URL: https://doi.org/10.1137/0202012, doi:10.1137/0202012.

[29] Russell Impagliazzo, Ramamohan Paturi, and Francis Zane. Which problems have strongly exponential complexity? *Journal of Computer and System Sciences*, 63(4):512–530, 2001. URL: https://doi.org/10.1006/jcss.2001.1774.
[30] Bart M. P. Jansen, Daniel Lokshtanov, and Saket Saurabh. A near-optimal planarization algorithm. In Proc. of the 25th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), pages 1802–1811, 2014. URL: https://doi.org/10.1137/1.9781611973402.130.

[31] Gwenaël Joret, Christophe Paul, Ignasi Sau, Saket Saurabh, and Stéphan Thomassé. Hitting and harvesting pumpkins. SIAM Journal on Discrete Mathematics, 28(3):1363–1390, 2014. URL: https://doi.org/10.1137/1012083736.

[32] Ken-ichi Kawarabayashi. Planarity allowing few error vertices in linear time. In Proc. of the 50th Annual IEEE Symposium on Foundations of Computer Science (FOCS), pages 639–648, 2009. doi:10.1109/FOCS.2009.45.

[33] Ken-ichi Kawarabayashi and Yusuke Kobayashi. Linear min-max relation between the treewidth of $H$-minor-free graphs and its largest grid. In Proc. of the 29th International Symposium on Theoretical Aspects of Computer Science (STACS), volume 14 of LIPIcs, pages 278–289, 2012. URL: https://doi.org/10.4230/LIPIcs.STACS.2012.278.

[34] Ken-ichi Kawarabayashi, Yusuke Kobayashi, and Bruce Reed. The disjoint paths problem in quadratic time. Journal of Combinatorial Theory, Series B, 2011. doi:10.1016/j.jctb.2011.07.004.

[35] Ken-ichi Kawarabayashi, Robin Thomas, and Paul Wollan. A new proof of the flat wall theorem. Journal of Combinatorial Theory, Series B, 129:204–238, 2018. URL: https://doi.org/10.1016/j.jctb.2017.09.006.

[36] Eun Jung Kim, Alexander Langer, Christophe Paul, Felix Reidl, Peter Rossmanith, Ignasi Sau, and Somnath Sikdar. Linear kernels and single-exponential algorithms via protrusion decompositions. ACM Transactions on Algorithms, 12(2):21:1–21:41, 2016. doi:10.1145/2797140.

[37] Eun Jung Kim, Maria J. Serna, and Dimitrios M. Thilikos. Data-compression for parametrized counting problems on sparse graphs. In Proc. of the 29th International Symposium on Algorithms and Computation (ISAAC), volume 123 of LIPIcs, pages 20:1–20:13, 2018. doi: 10.4230/LIPIcs.ISAAC.2018.20.

[38] Tomasz Kociumaka and Marcin Pilipczuk. Faster deterministic feedback vertex set. Information Processing Letters, 114(10):556–560, 2014. doi:10.1016/j.ipl.2014.05.001.

[39] Tomasz Kociumaka and Marcin Pilipczuk. Deleting Vertices to Graphs of Bounded Genus. Algorithmica, 81(9):3655–3691, 2019. doi:10.1007/s00453-019-00592-7.

[40] Alexandr V. Kostochka. Lower bound of the Hadwiger number of graphs by their average degree. Combinatorica, 4:307–316, 1984. URL: http://dx.doi.org/10.1007/BF02579141, doi:10.1007/BF02579141.

[41] John M. Lewis and Mihalis Yannakakis. The node-deletion problem for hereditary properties is NP-complete. Journal of Computer and System Sciences, 20(2):219–230, 1980. URL: https://doi.org/10.1016/0022-0000(80)90060-4.
[42] Daniel Lokshtanov. Wheel-Free Deletion Is W[2]-Hard. In Proc. of the 3rd International Workshop on Parameterized and Exact Computation (IWPEC), volume 5018 of LNCS, pages 141–147, 2008. doi:10.1007/978-3-540-79723-4\_14.

[43] Dániel Marx and Ildikó Schlotter. Obtaining a planar graph by vertex deletion. Algorithmica, 62(3-4):807–822, 2012. URL: https://doi.org/10.1007/s00453-010-9484-z.

[44] Rolf Niedermeier. Invitation to fixed parameter algorithms, volume 31. Oxford University Press, 2006. doi:10.1093/ACPROF:DSO/9780198566076.001.0001.

[45] Ljubomir Perkovic and Bruce Reed. An Improved Algorithm for Finding Tree Decompositions of Small Width. International Journal of Foundations of Computer Science, 11:365–371, 01 2000. URL: https://doi.org/10.1142/S0129054100000247.

[46] Bruce Reed, Kaleigh Smith, and Adrian Vetta. Finding odd cycle transversals. Operations Research Letters, 32(4):299–301, 2004. URL: https://doi.org/10.1016/j.orl.2003.10.009.

[47] Neil Robertson and Paul D. Seymour. Graph Minors. XIII. The Disjoint Paths Problem. Journal of Combinatorial Theory, Series B, 63(1):65–110, 1995. doi:10.1006/jctb.1995.1006.

[48] Neil Robertson and Paul D. Seymour. Graph Minors. XX. Wagner’s conjecture. Journal of Combinatorial Theory, Series B, 92(2):325–357, 2004. doi:10.1016/j.jctb.2004.08.001.

[49] Neil Robertson, Paul D. Seymour, and Robin Thomas. Quickly excluding a planar graph. Journal of Combinatorial Theory, Series B, 62(2):323–348, 1994. URL: https://doi.org/10.1006/jctb.1994.1073.

[50] Roded Sharan. Graph Modification Problems and their Applications to Genomic Research. PhD thesis, Sackler Faculty of Exact Sciences, School of Computer Science, 2002.

[51] Dimitrios M. Thilikos. Graph minors and parameterized algorithm design. In The Multivariate Algorithmic Revolution and Beyond - Essays Dedicated to Michael R. Fellows on the Occasion of His 60th Birthday, pages 228–256, 2012.

[52] Andrew Thomason. The extremal function for complete minors. Journal of Combinatorial Theory, Series B, 81(2):318–338, 2001. URL: https://doi.org/10.1006/jctb.2000.2013.