Numerical evaluation of spherical GJMS determinants

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A numerical expression in the form of an integral is given for the determinant of the scalar GJMS operator on an odd–dimensional sphere. Manipulation yields a curious sum formula for the logdet in terms of the logdets of the ordinary conformal Laplacian for other dimensions.
1. Introduction

This short note has the restricted goal of presenting an easy-to-calculate formula for the determinant of the GJMS conformally invariant operator, $P_{2k}$, on an odd-dimensional sphere. Expressions for this are actually already known but it is always helpful to have alternative representations if only for variety and checking. Knowing the values on one geometry allows one to find them on conformally related spaces, using Branson’s $Q$ curvature if necessary.

In [1] I gave a direct spectral evaluation of $\log \det P_{2k}$ which yielded an integral over a Plancherel measure and agreed with other derivations done using dimensional regularisation in the context of the AdS/CFT correspondence by Diaz, [2], and Diaz and Dorn, [3], and via the Selberg $\zeta$–function by Guillarmou, [4]. The method here is an application of that in [5] which involved the ordinary conformally invariant Laplacian, $P_2$. A drawback is that $k$ is restricted to be an integer smaller than $d/2$ unlike some other formulae which allow the continuation of $k$.

2. The general method

I need to repeat quickly some definitions and basic facts. Branson’s construction of $P_{2k}$, [6], in the special case of the (round) unit $d$–sphere is a simple product,

$$P_{2k} = \prod_{j=0}^{k-1} (B^2 - \alpha_j^2), \quad \alpha_j = j + 1/2$$

$$= \frac{\Gamma(B + 1/2 + k)}{\Gamma(B + 1/2 - k)} = B^{|2k+1|} - 1,$$

expressed as a central factorial.² Here $B \equiv \sqrt{\frac{P_2 + 1}{4}}$ with $P_2 = -\Delta_2 + ((d - 1)^2 - 1)/4$ the Yamabe–Penrose operator, sometimes denoted by $Y_d$, on the sphere.

In [5], for a different purpose, I calculated the determinant of the operator $B^2 - \alpha^2$, where $\alpha$ was a general parameter. Because, as shown in [1], there is no multiplicative anomaly for odd dimensions³, the result can be used to find $\log \det P_{2k}$ very easily as the sum,

$$\log \det P_{2k} = \sum_{j=0}^{k-1} \log \det (B^2 - \alpha_j^2).$$

² It is amusing to note that the product formula of Juhl, [7] lemma 6.1, is the central version of Elphinstone’s theorem of 1858, [8].

³ There is one for the hemisphere.
It will turn out very shortly that this sum can be done.

3. The calculation

The method involves a Bessel function representation for the $\zeta$–function of the operator $B^2 − \alpha^2$ on the odd $d$–sphere, cf. [9], [10]. A contour technique then provides the expression, (cf equation (11) in [5]),

$$\log \det (B^2 − \alpha^2_j) = -\frac{1}{2^{d-2}} \int_0^\infty dx \text{ Re} \left( \frac{\cosh \frac{\tau}{2} \cosh \alpha_j \tau}{\tau \sinh^{d/2} \tau} \right) \sinh \frac{\alpha_j x}{2} \cosh \frac{d+1}{2} \frac{x}{2} \right),$$

(3)

On the first line $\tau = x + iy$, $(0 \leq y \leq 2\pi)$. The second line results from the choice $y = \pi$ and the fact that $\alpha_j$ is a half–integer.

The geometric sum over $j$ in (2) gives,

$$\log \det P_{2k} = \frac{(-1)^{(d-1)/2+k}}{2^{d-1}} \int_0^\infty dx \frac{\pi}{x^2 + \pi^2} \sinh \frac{x}{2} \sinh \frac{kx}{2} \sinh \frac{x}{2} \cosh \frac{d+1}{2} \frac{x}{2},$$

(4)

which is my main computational result. Because of the factor $(-1)^k$ it works sensibly only for $k$ an integer. Furthermore, the integral diverges if $k > d/2$. $^4$

4. A curious Chebyshev rearrangement

For $k$ an integer, there is the expression,

$$\frac{\sinh kx}{\sinh \frac{x}{2}} = U_{2k-1} \left( \cosh \frac{x}{2} \right),$$

in terms of Chebyshev polynomials and (4) reads,

$$\log \det P_{2k}(d) = \frac{(-1)^{(d-1)/2+k}}{2^{d-1}} \int_0^\infty dx \frac{\pi}{x^2 + \pi^2} \sinh^2 \frac{x}{2} \frac{U_{2k-1} \left( \cosh \frac{x}{2} \right)}{\cosh^{d+1} \frac{x}{2}}.$$ 

(5)

I have exhibited the dimension $d$ because expanding the Chebyshev polynomial will produce a sum of ordinary log det $s$ of $P_2(d')$ for dimension $d'$ in the range $d$ to $d - 2k + 2$.

$^4$ This is a consequence of the appearance of negative eigenvalues of $B - \alpha_j$ but I will not try to remedy this here.
Making this explicit, one has, first,

\[ U_{2k-1}(x) = x (u_0 + u_1 x^2 + \ldots + u_{k-1} x^{2k-2}) , \]

where the coefficients, \( u_j(k) \), are known, both in general, e.g. [11], [12], and by recursion for a given \( k \). (There are tables of them.) Then,

\[ \det P_{2k}(d) = \det P_{2v_0}^v(d) \det P_{2v_1}^v(d-2) \ldots \det P_{2v_{k-1}}^v(d-2k+2) , \]

where the powers \( v_j(k) \) are simply related to the \( u_j(k) \) by,

\[ v_j(k) = (-1)^{k-1} \frac{(-1)^j}{2j+1} u_j(k) . \]

The coefficients \( u_j \) alternate in sign so the \( v_j \) all have the same sign, and, moreover, are integers. It’s best to give a few examples. Trivially, \( v_0(1) = 1 \) and, non–trivially, one finds the determinant ‘product rules’,

\begin{align*}
 P_4(d) & \sim P_2^2(d) P_2(d-2) \\
 P_6(d) & \sim P_2^3(d) P_2^4(d-2) P_2(d-4) \\
 P_8(d) & \sim P_2^4(d) P_2^{10}(d-2) P_2^6(d-4) P_2(d-6) \\
 P_{10}(d) & \sim P_2^5(d) P_2^{20}(d-2) P_2^{21}(d-4) P_2^8(d-6) P_2(d-8) ,
\end{align*}

where \( \sim \) stands for equality of determinants (but not of operators!) and \( d \) is such that the final factor is never \( P_2(1) \).

Equation (7) is my second computational formula, although it is not very efficient. It can, however, be used to express the determinants in terms Riemann \( \zeta \)–functions since such expressions are known for the \( P_2 \)s. For example, for the Paneitz–Fradkin–Tseytlin–Riegert\(^5\) operator on the 5– and 7–spheres,

\begin{align*}
 \log \det P_4(5) &= \frac{1}{32} \left( 7 \log 2 - 13 \frac{\zeta(3)}{\pi^2} + 18 \frac{\zeta(5)}{\pi^4} \right) \approx 0.104642 \\
 \log \det P_4(7) &= -\frac{1}{256} \left( 3 \log 2 + 59 \frac{\zeta(3)}{30 \pi^2} - 55 \frac{\zeta(5)}{2 \pi^4} + 63 \frac{\zeta(7)}{4 \pi^6} \right) \approx -0.008297 ,
\end{align*}

which, of course, check numerically against the quadrature, (4). No doubt these expressions can also be deduced from the other representations of the determinants.

\(^5\) This designation is often reserved just for the four dimensional case.
5. Graphs

To give an indication of numbers, in Fig.1 and Fig.2 I plot $P_{2k}(35)$ for the allowed $k$, i.e. $1 \leq k \leq 17$. (There is nothing special about the dimension 35.) The values oscillate about zero with increasing amplitude, maximum to minimum being about $10^{10}$. 

**Fig.1. GJMS logdet, d=35**

**Fig.2. GJMS logdet, d=35**
Fig. 3 shows the logdets at the limiting $k = (d - 1)/2$ for $d = 3$ to $d = 21$ and Fig. 4 the logdet of the Paneitz operator ($k = 2$) against dimension. The values oscillate about zero with decreasing amplitude so that the determinant tends to unity with increasing $d$, as is clear from (4). This behaviour is typical.
6. Conclusion

The results of this note are the quadrature (4) (equivalently (5)) and the product formula, (6), for the determinant of the scalar GJMS operator on an odd sphere. The computation for even spheres is somewhat harder and involves a multiplicative anomaly, calculated in [1].

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