CAPABILITY OF NILPOTENT LIE ALGEBRAS WITH SMALL DERIVED SUBALGEBRA

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Abstract. In this paper, we classify all capable nilpotent Lie algebras with derived subalgebra of dimension at most 1.

1. Introduction

It’s about seventy years past when P. Hall uttered his famous problem. "For which group $G$ there exists a group $H$ with $G \cong H/Z(H)$?" He also noticed that finding such groups are important in classifying $p$-groups. Following Hall and Senior [9] a group $G$ with the above property is called capable. One of the famous results on this concept is due to Baer [1] where he classified all capable groups among the direct sums of cyclic groups and hence determined all capable finitely generated abelian groups. Capable groups in the class of extra special $p$-groups were characterized in [4]. They studied the notion of capability and introduced a central subgroup denoted by $Z^*(G)$ to be

$$\bigcap\{\phi(Z(E)) \mid (E, \phi) \text{ is a central extension}\},$$

and showed a group $G$ is capable if and only if $Z^*(G) = 1$.

Another notion having relation to capability is the exterior square of groups which was introduced in [5]. Using this concept G. Ellis [6], introduced the subgroup $Z^\wedge(G)$ to be the set of all elements $g$ of $G$ for which $g \wedge h = 1$ for all $h \in G$. He could prove $Z^\wedge(G) = Z^*(G)$ which is an interesting result.

Recently several properties of finite $p$-groups has found analogues results for nilpotent Lie algebras. For instance one can see [12] which introduced the notion $Z^*(L)$ for a Lie algebra $L$ similar to what Beyl et. al. [3] introduced for groups. They also could prove that a necessary and sufficient condition for a Lie algebra $L$ to be capable is $Z^*(L) = 0$.

In this paper we intend to prove some results for Lie algebras. First of all, we show that the two notions $Z^\wedge(L)$ and $Z^*(L)$ for a nilpotent finite dimensional Lie algebra $L$ are the same then we give a necessary and sufficient condition for an abelian finite dimensional Lie algebra to be capable. We also classify capable Heisenberg Lie algebras and finally using a result obtained by the first author in a joint paper [11] we give a necessary and sufficient conditions for nilpotent finite dimensional Lie algebras with $\dim(L^2) = 1$, to be capable.

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2. Preliminaries

In this section we state some lemmas to use in main results. Throughout this paper all Lie algebras are finite dimensional, $A(n)$ and $H(m)$ denote the abelian Lie algebra of dimension $n$ and the Heisenberg Lie algebra of dimension $2m + 1$, respectively.

**Lemma 2.1.** (See [12, Theorem 4.4]). Let $L$ be a Lie algebra and $N$ be an ideal of $L$. Then $N \subseteq Z^*(L)$ if and only if the natural map $M(L) \rightarrow M(L/N)$ is monomorphism.

**Lemma 2.2.** (See [7, Proposition 13] and [12, Proposition 4.1 (iii)]). Let $L$ be a Lie algebra and $N$ be a central ideal of $L$. Then the following sequences are exact.

(i) $L \wedge N \rightarrow L \wedge L \rightarrow L/N \wedge L/N \rightarrow 0$.

(ii) $M(L/N) \rightarrow N \cap L^2 \rightarrow 0$.

**Corollary 2.3.** $N \subseteq Z^\wedge(L)$ if and only if the natural map $L \wedge L \rightarrow L/N \wedge L/N$ is monomorphism.

**Lemma 2.4.** (See [8]). Let $L$ be a Lie algebra then the following sequence is a central extension.

$0 \rightarrow M(L) \rightarrow L \wedge L \rightarrow L^2 \rightarrow 0$.

**Corollary 2.5.** Let $A$ be a finite dimensional abelian Lie algebra. Then $M(A) \cong A \wedge A$.

The following lemma describes the Schur multipliers of abelian and Heisenberg algebras.

**Lemma 2.6.** (See [3, Lemma 3], [2, Example 3] and [10, Theorem 24]).

(i) $\dim M(A(n)) = \frac{1}{2}n(n-1)$

(ii) $\dim M(H(1)) = 2$.

(iii) $\dim M(H(m)) = 2m^2 - m - 1$ for all $m \geq 2$.

**Theorem 2.7.** (See [3, Theorem 1]). Let $L_1$ and $L_2$ be finite dimensional Lie algebras. Then

$\dim (M(L_1 \oplus L_2)) = \dim (M(L_1)) + \dim (M(L_2)) + \dim (L_1/L_1^2 \otimes L_2/L_2^2)$.

**Proposition 2.8.** Let $L_1$ and $L_2$ be two Lie algebras. Then

(i) $(L_1 \oplus L_2)^\wedge(L_1 \oplus L_2) \cong (L_1^\wedge L_1) \oplus (L_2^\wedge L_2) \oplus (L_1/L_1^2 \otimes L_2/L_2^2)$.

(ii) $Z^\wedge(L_1 \oplus L_2) \subseteq Z^\wedge(L_1) \oplus Z^\wedge(L_2)$

**Proof.** (i). It is a consequence of [7, Proposition 8].

(ii) It is obtained directly by using (i). \qed

3. Main Results

In this section, we classify all nilpotent Lie algebras with derived subalgebra of dimension at most 1. These Lie algebras are abelian Lie algebras and nilpotent non-abelian Lie algebras with derived subalgebra of dimension 1. It was proved in [12] that a Lie algebra $L$ is capable if and only if $Z^*(L) = 0$. The equality $Z^*(G) = Z^\wedge(G)$ holds for any group $G$. Here, we prove similar result for Lie algebras and deduce that a Lie algebra $L$ is capable if and only if $Z^\wedge(L) = 0$. Also, we state some lemmas for Heisenberg Lie algebras to use them in main results.
Lemma 3.1. For any Lie algebra $L$, $Z^*(L) = Z^\wedge(L)$

Proof. Considering Lemmas 2.1 and 2.2(i), we can see that $M(L) \longrightarrow M(L/Z^\wedge(L))$ is a monomorphism, so $Z^\wedge(L) \subseteq Z^*(L)$. On the other hand, by Lemmas 2.1 and 2.2(ii), we have $\dim M(L/Z^*(L)) = \dim M(L) + \dim (L^2 \cap Z^*(L))$. But Lemma 2.4 shows that

$$\dim (L \wedge L) = \dim M(L) + \dim L^2$$

and

$$\dim \left( L/Z^*(L) \wedge L/Z^*(L) \right) = \dim M(L/Z^*(L)) + \dim (L/Z^*(L))^2.$$

Using the isomorphism $(L/Z^*(L))^2 \cong L^2/(L^*(L) \cap L^2)$, we have

$$\dim (L \wedge L) = \dim \left( L/Z^*(L) \wedge L/Z^*(L) \right),$$

hence $Z^*(L) \subseteq Z^\wedge(L)$ due to Corollary 2.3. $\square$

Lemma 3.2. Let $H(m)$ be the Heisenberg Lie algebra. Then

(i) $H(1) \wedge H(1) \cong A(3)$.

(ii) $H(m) \wedge H(m) \cong A(2m^2 - m)$ for all $m \geq 2$.

Proof. Since $\dim H(m)^2 = 1$, Lemma 2.4 follows that $H(m) \wedge H(m)$ is abelian. Invoking Lemmas 2.4 and 2.6, we should have $\dim (H(1) \wedge H(1)) = 3$ and $\dim (H(m) \wedge H(m)) = 2m^2 - m$ for all $m \geq 2$. $\square$

Among the Lie algebras the simplest ones are abelian Lie algebras. Here we classify all abelian Lie algebras of finite dimension which are capable.

Theorem 3.3. $A(n)$ is capable if and only if $n \geq 2$.

Proof. Since $M(A(1)) = 0$, Lemma 2.1 implies $A(1)$ is not capable. Now, let $n \geq 2$ and $I$ be a $k$-dimensional ideal of $A(n)$. Then, we have

$$\dim M(A(n)/I) = \frac{1}{2}(n-k)(n-k-1)$$

and $\dim M(A(n)) = \frac{1}{2}n(n-1)$,

so Lemma 2.1 implies $I \subseteq Z^\wedge(A(n))$ if and only if $k = 0$. Hence, we should have $Z^\wedge(A(n)) = 0$ and the result holds. $\square$

Ignoring abelian Lie algebras, Heisenberg Lie algebras are probably the simplest Lie algebras to work with. The following theorem classifies all capable Heisenberg Lie algebras.

Theorem 3.4. $H(m)$ is capable if and only if $m = 1$.

Proof. First suppose that $m = 1$. Owning to Lemma 3.3, we have $\dim (H(1) \wedge H(1)) = 3$. On the other hand, for any nonzero ideal of $H(1)$ such as $I$, Corollary 2.3 and Lemma 2.6(i) follows that

$$\dim \left( (H(1)/I) \wedge (H(1)/I) \right) = 1.$$

Hence $Z^\wedge(H(1))$ contains no nonzero ideal and must be trivial.

Now, assume that $m \geq 2$ similar to the case $m = 1$, by using Lemma 3.3, $\dim (H(m) \wedge H(m)) = 2m^2 - m$ and

$$\dim \left( H(m)/H(m)^2 \wedge H(m)/H(m)^2 \right) = \dim M(H(m)/H(m)^2)$$

$$= 2m(2m-1)/2 = 2m^2 - m$$

which follows $H(m)^2 \subseteq Z^\wedge(H(m))$. $\square$
The direct sum of an abelian Lie algebra and a Heisenberg Lie algebra has the derived subalgebra of dimension 1 and its interesting to know which of them are capable. The following theorem gives a necessary and sufficient condition for capability of such Lie algebras.

**Theorem 3.5.** Let $L \cong H(m) \oplus A(k)$ then $L$ is capable if and only if $m = 1$.

**Proof.** We consider three cases as follows

(i) $m = k = 1$;
(ii) $m = 1$ and $k \geq 2$;
(iii) $m \geq 2$.

In case (i), $L \cong H(1) \oplus A(1)$ and Proposition 2.8 and Theorems 3.3, 3.4 follow that $Z^\wedge(L) \subseteq Z^\wedge(H(1)) \oplus Z^\wedge(A(1)) = A(1)$.

But $\dim M(L) = \dim M(H(1)) + \dim M(A(1)) + \dim (H(1)/H(1)^2 \otimes A(1)) = 2 + 0 + 2 = 4$.

On the other hand, $\dim M(L/A(1)) = \dim M(H(1)) = 2$, and so $A(1) \not\subseteq Z^\wedge(L)$ which implies that $Z^\wedge(L) = 0$.

In case (ii), Proposition 2.8 and Theorems 3.3, 3.4 deduce that $Z^\wedge(L) \subseteq Z^\wedge(H(1)) \oplus Z^\wedge(A(k)) = 0$,

as required.

Finally in case (iii), we claim that $L \wedge L \cong L/H(m)^2 \wedge L/H(m)^2$,

and hence $L$ is not capable.

Since $L/H(m)^2 \cong H(m)/H(m)^2 \oplus A(k)$, we have

$L/H(m)^2 \wedge L/H(m)^2 \cong (H(m)/H(m)^2 \wedge H(m)/H(m)^2) \oplus (A(k) \wedge A(k)) \oplus (H(m)/H(m)^2 \otimes A(k))$.

Thus

$\dim (L/H(m)^2 \wedge L/H(m)^2) = 2m(2m - 1)/2 + n(n - 1)/2 + 2mn$.

On the other hand,

$\dim L \wedge L = \dim M(L) + \dim H(m)^2 + n(n - 1)/2 + 2mn$.

Hence $\dim (L/H(m)^2 \wedge L/H(m)^2) = \dim L \wedge L$, and the result holds. \hfill \square

Now the following theorem by the first author in his joint paper [11] shows nothing remains to prove.

**Theorem 3.6.** Let $L$ be an $n$-dimensional nilpotent Lie algebra and $\dim L^2 = 1$ then $L \cong H(m) \oplus A(n - 2m - 1)$.
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