EXISTENCE OF SOLUTIONS FOR FRACTIONAL $p$–KIRCHHOFF TYPE EQUATIONS WITH A GENERALIZED CHOQUARD NONLINEARITIES

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Abstract: In this article, we establish the existence of solutions to the fractional $p$–Kirchhoff type equations with a generalized Choquard nonlinearities without assuming the Ambrosetti-Rabinowitz condition.

Keywords: fractional $p$-Kirchhoff type equations; Choquard equation; without the (AR) condition.

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1. INTRODUCTION AND STATEMENT OF MAIN RESULT

In this work, we consider the following fractional $p$–Kirchhoff equation

\begin{equation}
M(\|u\|_W^p)(-\Delta)_p u + V(x)|u|^{p-2}u = \lambda (I_\mu * F(u)) f(u), \quad \text{in } \mathbb{R}^N
\end{equation}

where $1 < p < N$, $M : \mathbb{R}^+_0 \to \mathbb{R}^+$ is a Kirchhoff function,

\begin{equation}
\|u\|_W = [\|u\|_{\mathcal{L}^p}^p + \int_{\mathbb{R}^N} V(x)|u|^p dx]^{1/p}
\end{equation}

with $[u]_{\mathcal{L}^p} = \left( \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} \,dx \,dy \right)^{1/p}$,

the potential function $V : \mathbb{R}^N \to \mathbb{R}^+$ is continuous, $f \in C(\mathbb{R}, \mathbb{R})$ and $F \in C(\mathbb{R}, \mathbb{R})$ with $F(u) = \int_0^u f(t) \,dt$, here $I_\mu(x) = |x|^{-\mu}$ is the Riesz potential of order $\mu \in (0, ps)$, and $(-\Delta)_p^s$ is the fractional $p$–Laplacian operator which, up to a normalization constant, is defined as

\begin{align}
(-\Delta)_p^s \varphi(x) = 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B(\varepsilon)} \frac{\varphi(x) - \varphi(y)}{|x-y|^{N+ps}} \,dy, \quad x \in \mathbb{R}^N,
\end{align}

along functions $\varphi \in C_0^\infty(\mathbb{R}^N)$, where $B_\varepsilon(x)$ denotes the ball of $\mathbb{R}^N$ centered at $x \in \mathbb{R}^N$ and radius $\varepsilon > 0$.

On the one hand, this paper is motivated by some works that has been focused on the study of Kirchhoff type problems. Fiscella and Valdinoci [15] first proposed a stationary fractional Kirchhoff variational model as follows

\begin{equation}
\begin{cases}
M \left( \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \,dx \,dy \right)(-\Delta)_s u(x) = \lambda f(x, u) + |u|^{2^*-2}u & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\end{equation}

where $\Omega \subset \mathbb{R}^N$ is an open bounded set, $2^* = \frac{2N}{N-2s}$, $N > 2s$ with $s \in (0, 1)$, $M$ and $f$ are two continuous functions under some suitable assumptions. In [15], the authors first provided a detail discussion about the physical meaning underlying the fractional Kirchhoff problems and their applications. They supposed that $M : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing and continuous function, and there exists $m_0 > 0$ such that $M(t) \geq m_0 = M(0)$ for all $t \in \mathbb{R}^+$. Based on the truncated skill and the mountain pass theorem, they obtained the existence of a non-negative solution to problem (1.3) for any $\lambda > \lambda^* > 0$, where $\lambda^*$ is an appropriate threshold.
Awtorui et al. [5] established the existence and the asymptotic behavior of non-negative solutions to problem (1.3) under different assumptions on $M$, the Kirchhoff function $M$ can be zero at zero, that is, the problem is degenerate case.

Moreover, there is a lot of literature concerning the existence and multiplicity of solutions for the fractional $p$–Laplacian Kirchhoff type problems. Xiang et al. in [35] investigated the existence of solutions for Kirchhoff type problems involving the fractional $p$–Laplacian by variational methods, where the nonlinearity is subcritical and the Kirchhoff function is non-degenerate. Combining the mountain pass theorem with Ekeland variational principle, Xiang et al. in [36] established the existence of two solutions for a degenerate fractional $p$–Laplacian Kirchhoff equation in $\mathbb{R}^N$ with concave-convex nonlinearity. By the same methods as in [36], Pucci et al. in [28] obtained the existence of two solutions for a nonhomogenous Schrödinger-Kirchhoff type equation involving the fractional $p$–Laplacian in $\mathbb{R}^N$ on a nondegenerate situation. Furthermore, nonexistence and multiplicity of solutions for a nonhomogeneous fractional $p$–Kirchhoff type problem involving critical exponent in $\mathbb{R}^N$ were studied in [37]. The existence of infinitely many solutions was proved in [29, 34] by using Krasnoselskii’s genus theory under degenerate frameworks. Recently, Song and Shi considered the existence of infinitely many solutions for degenerate $p$–fractional Kirchhoff equations with critical Sobolev-Hardy nonlinearities in [32, 33].

On the other hand, there are some results about the Choquard equation, consider the following Choquard or nonlinear Schrödinger-Newton equation

\begin{equation}
-\Delta u + V(x)u = (I_{\mu} * u^2)u + \lambda f(x, u) \quad \text{in } \mathbb{R}^N,
\end{equation}

which was elaborated by Pekar [27] in the framework of quantum mechanics. The first investigation for the existence and symmetry of solutions to (1.4) went back to the works of Lieb [18]. Equations of type (1.4) have been extensively studied, see e.g. [3, 23, 24] and references therein. Moroz and van Schaftingen in [24] considered the existence of ground-states for a generalized Choquard equation. The existence, multiplicity and concentration of solutions for a generalised quasilinear Choquard equation were studied by Alves and Yang in [1, 2]. We refer to [26] for a good survey of the Choquard equation.

In the setting of the fractional Choquard equations, the fractional Choquard equations,

\begin{equation}
(-\Delta)^s u + V(x)u = (I_{\mu} * F(u))f(u) \quad \text{in } \mathbb{R}^N,
\end{equation}

Wu [38] investigated existence and stability of solutions to (1.5) with $f(u) = u$ and $\mu \in (N - 2s, N)$. Subsequently, D’Avenia and Squassina in [10] studied the existence, regularity and asymptotic behavior of solutions to (1.5) with $f(u) = u^q$ and $V(x) \equiv \text{const}$. In particular, they claimed the nonexistence of solutions as $q \in \left(\frac{2N}{N - 2s}, \frac{2N}{s}\right)$. If $V(x) = 1$ and $f$ satisfies Berestycki-Lions type assumptions, the existence of ground state solutions for a fractional Choquard equation has been established in [31]. Very recently, Ambrosio studied the concentration phenomena of solutions for a fractional Choquard equation with magnetic field in [4].

Recently, Belchior et al. in [6] applied the mountain pass theorem without PS condition and a characterization of the infimum more suitable to the Nehari manifold naturally attached to the problem to study the existence of ground state, regularity and polynomial decay for the following fractional Choquard equation

\begin{equation}
(-\Delta)^s u + A|u|^{p-2} u = (I_{\mu} * F(u))f(u) \quad \text{in } \mathbb{R}^N,
\end{equation}
where $A$ is a positive constant, $f$ is a $C^1$ positive function on $(0, \infty)$, $\lim_{t \to 0} \frac{|f(t)|}{t^{p-1}} = 0$ for some $p < q < \frac{(2N-\mu)p}{2N-2p}$, and

$$f'(t)^2 - (p-1)f(t) > 0 \quad \text{for all } t > 0. \quad (1.7)$$

An example of a function $f$ satisfying these hypotheses is given by $f(t) = |t|^{q_1-1}t^* + |t|^{q_2-1}t^*$, where $p < q_1 < q_2 < \frac{(N-\mu)p}{N-p}$ and $t^* = \max\{|t|/2, 0\}$. From (1.7), $f$ satisfies the Ambrosetti-Rabinowitz condition (AR) for short:

$$pF(t) < tf(t) \quad \text{for all } t > 0, \quad (1.8)$$

and the function $\frac{f(t)}{t^{p-1}}$ is increasing. It is well known that the (AR)–condition is quite natural and important not only to ensure that an Euler-Lagrangian functional has the mountain pass geometry structure, but also to ensure that the Palais-Smale sequence of the functional is bounded. However, there are many functions which are superlinear at infinity, but do not satisfy the (AR)–condition, for example, the function $f(t) = |t|^{p-2}t \log(1 + |t|)$. Thus, many researchers have tried to drop the (AR)–condition for elliptic equations involving the $p$–Laplacian, see [13], [16], [17], [20] and references therein.

In particular, Lee et al. in [16] considered the existence of nontrivial weak solutions for the quasilinear Choquard equation with the nonlinearity $f$ does not satisfy the (AR)–condition.

Motivated by the above results, in the present paper, we are interested in the existence of solutions for the fractional $p$–Kirchhoff type equation (1.1) with a generalized Choquard nonlinearities without assuming the Ambrosetti-Rabinowitz condition. We first give the following assumptions on the potential function $V$ and the Kirchhoff function $M$.

$(V)$ $V : \mathbb{R}^N \to \mathbb{R}^+$ is a continuous function and there exists $V_0 > 0$ such that $\inf_{\mathbb{R}^N} V \geq V_0$.

$(M_1)$ $M : \mathbb{R}_0^+ \to \mathbb{R}^+$ is a continuous function and there exists $m_0 > 0$ such that $\inf_{t \geq 0} M(t) = m_0$.

$(M_2)$ There exists $\theta \in [1, \frac{2N-\mu}{N-\mu})$ such that

$$M(t)t \leq \theta M(t), \quad \forall t \geq 0,$$

where $\mathcal{M}(t) = \int_0^t M(\tau)d\tau$.

A typical example is $M(t) = m_0 + bt^{p-1}$, where $b \geq 0$, $t \geq 0$.

Moreover, we impose the following assumption on the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ that

$(F_1)$ $F \in C^1(\mathbb{R}, \mathbb{R})$.

$(F_2)$ There exist a constant $c_0 > 0$ and $p < q_1 \leq q_2 < \frac{(N-\mu)p}{N-p}$ such that for all $t \in \mathbb{R}$,

$$|f(t)| \leq c_0(|t|^{q_1-1} + |t|^{q_2-1}).$$

$(F_3)$ $\lim_{|x| \to \infty} F(|x|) = \infty$ uniformly for $x \in \mathbb{R}^N$.

$(F_4)$ There exist $c_1 \geq 0$, $r_0 \geq 0$ and $\kappa > \frac{N}{ps}$ such that

$$|F(t)|^\kappa \leq c_1 |t|^\kappa \mathcal{F}(t)$$

for all $t \in \mathbb{R}$ and $|t| \geq r_0$, where $\mathcal{F}(t) = \frac{1}{p} f(t) t - \frac{1}{2} F(t)$ $\geq 0$.

The main result is as follows.

**Theorem 1.1.** Let $0 < \mu < ps < N$, and $(V)$, $(M_1)$ – $(M_2)$ and $(F_1)$ – $(F_4)$ hold. Then problem (1.1) has a nontrivial weak solution for any $\lambda > 0$. 
The paper is organized as follows. In Section 2, we give some definitions and preliminaries. Section 3 is devoted to prove Theorem 1.1; we obtain the existence of solution to problem (1.1) by the mountain pass theorem.

2. Preliminaries

We introduce some useful notations. The fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ is defined by

$$W^{s,p}(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N) : [u]_{s,p} < \infty \right\},$$

where $[u]_{s,p}$ denotes the Gagliardo norm defined by

$$[u]_{s,p} = \left( \int \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, dx \, dy \right)^{1/p},$$

and $W^{s,p}(\mathbb{R}^N)$ is equipped with the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} = \left( \|u\|^p_p + [u]_{s,p}^p \right)^{1/p},$$

where and hereafter we denote by $\| \cdot \|_q$ the norm of Lebesgue space $L^q(\mathbb{R}^N)$. As it is well-known, $W^{s,p}(\mathbb{R}^N) = (W^{s,p}(\mathbb{R}^N),\|u\|_{W^{s,p}(\mathbb{R}^N)})$ is a uniformly convex Banach space. Let $L^p(\mathbb{R}^N)$, $V$ denote the Lebesgue space of real valued functions, with $V(x)|u|^p \in L^1(\mathbb{R}^N)$, equipped with norm

$$\|u\|_{p,V} = \left( \int_{\mathbb{R}^N} V(x)|u|^p \, dx \right)^{1/p} \quad \text{for all } u \in L^p(\mathbb{R}^N), V.$$

Let $W^{s,p}_V(\mathbb{R}^N)$ denote the completion of $C_0^\infty(\mathbb{R}^N)$, with respect to the norm

$$\|u\|_V = \left( [u]_{s,p}^p + \|u\|^p_{p,V} \right)^{1/p}.$$

The embedding $W^{s,p}_V(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ is continuous for any $q \in [p, \frac{Np}{N-p}]$ by [13] Theorem 6.7, namely there exists a positive constant $C_q$ such that

$$\|u\|_V \leq C_q\|u\|_V \quad \text{for all } u \in W^{s,p}_V(\mathbb{R}^N).$$

Next, we recall the Hardy-Littlewood-Sobolev inequality.

Theorem 2.1. [13] Theorem 4.3] Assume that $1 < r, t < \infty$, $0 < \mu < N$ and

$$\frac{1}{r} + \frac{1}{t} + \frac{\mu}{N} = 2.$$

Then there exists $C(N,\mu, r, t) > 0$ such that

$$\int \int_{\mathbb{R}^{2N}} \frac{|g(x)| \cdot |h(y)|}{|x-y|^\mu} \, dx \, dy \leq C(N,\mu, r, t)\|g\|_r\|h\|_t$$

for all $g \in L^r(\mathbb{R}^N)$ and $h \in L^t(\mathbb{R}^N)$.

In particular, $F(t) = |t|^q$ for some $q_1 > 0$, by the Hardy-Littlewood-Sobolev inequality, the integral

$$\int \int_{\mathbb{R}^{2N}} \frac{F(u(x))F(u(y))}{|x-y|^\mu} \, dx \, dy$$

is well defined if $F \in L^t(\mathbb{R}^N)$ for some $t > 1$ satisfying

$$\frac{2}{t} + \frac{\mu}{N} = 2, \quad \text{that is } t = \frac{2N}{2N-\mu}.$$
Hence, by the fractional Sobolev embedding theorem, if $u \in W^{s,p}_V(\mathbb{R}^N)$, we must require that $\ell q_1 \in [p, \frac{Np}{N- ps}]$. Thus, for the subcritical case, we must assume

$$\tilde{p}_{\mu,s} = \frac{(N - \mu/2)p}{N} < q_1 \leq q_2 < \frac{(N - \mu/2)p}{N - ps} = p_{\mu,s}^*.$$ 

Hence, $\tilde{p}_{\mu,s}$ is called the lower critical exponent and $p_{\mu,s}^*$ is said to be the upper critical exponent in the sense of the Hardy-Littlewood-Sobolev inequality.

Equation (1.1) has a variational structure and its associated energy functional $\mathcal{J}_\lambda : W^{s,p}_V(\mathbb{R}^N) \to \mathbb{R}$ is defined by

$$\mathcal{J}_\lambda(u) = \Phi(u) - \lambda \Psi(u).$$

with

$$\Phi(u) := \frac{1}{p} \mathcal{M}(\|u\|^p_W), \quad \text{and} \quad \Psi(u) := \frac{1}{2} \int_{\mathbb{R}^{2N}} \frac{F(u(x)) F(u(y))}{|x-y|^{\mu}} \, dx \, dy.$$ 

Under the assumption $(F_2)$, $\mathcal{J}_\lambda$ is of class $C^1(W^{s,p}_V(\mathbb{R}^N), \mathbb{R})$. We say that $u \in W^{s,p}_V(\mathbb{R}^N)$ is a weak solution of problem (1.1), if

$$\mathcal{M}(\|u\|^p_W) \bigl(\{u, \varphi\}_{s,p} + \int_{\mathbb{R}^N} V|u|^{p-2}u \varphi \, dx \bigr) = \lambda \int_{\mathbb{R}^N} (I_\mu * F(u)) \varphi \, dx,$$

for all $\varphi \in W^{s,p}_V(\mathbb{R}^N)$, where

$$\langle u, \varphi \rangle_{s,p} = \int_{\mathbb{R}^{2N}} \frac{[u(x) - u(y)]^{p-2}(u(x) - u(y)) \cdot [\varphi(x) - \varphi(y)]}{|x-y|^{N+ps}} \, dx \, dy.$$

Clearly, the critical points of $\mathcal{J}_\lambda$ are exactly the weak solutions of problem (1.1).

**Lemma 2.2.** [28] Lemma 2], Let $(V)$ and $(M_1)$ hold. Then $\Phi$ is of class $C^1(W^{s,p}_V(\mathbb{R}^N), \mathbb{R})$ and

$$\langle \Phi'(u), \varphi \rangle = \mathcal{M}(\|u\|^p_W) \left[ \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y)) \cdot (\varphi(x) - \varphi(y))}{|x-y|^{N+ps}} \, dx \, dy \right.$$

$$\left. + \int_{\mathbb{R}^N} V(x)|u(x)|^{p-2}u(x) \varphi(x) \, dx \bigr],$$

for all $u, \varphi \in W^{s,p}_V(\mathbb{R}^N)$. Moreover, $\Phi$ is weakly lower semi-continuous in $W^{s,p}_V(\mathbb{R}^N)$.

The next result is stated in [1].

**Lemma 2.3.** Assume $(F_2)$ holds, then there exists $K > 0$ such that

$$|I_\mu * F(v)| \leq K \quad \text{for} \ v \in W^{s,p}_V(\mathbb{R}^N).$$

**Lemma 2.4.** Let $(V)$ and $(F_1) - (F_2)$ hold. Then $\Psi$ and $\Psi'$ are weakly strongly continuous on $W^{s,p}_V(\mathbb{R}^N)$.

**Proof.** Let $\{u_n\}$ be a sequence in $W^{s,p}_V(\mathbb{R}^N)$ such that $u_n \to u$ in $W^{s,p}_V(\mathbb{R}^N)$ as $n \to \infty$. Then $\{u_n\}$ is bounded in $W^{s,p}_V(\mathbb{R}^N)$, and then there exists a subsequence denoted by itself, such that

$$u_n \to u \quad \text{in} \ L^\theta(\mathbb{R}^N) \cap L^\theta(\mathbb{R}^N), \quad \text{and} \quad u_n \to u \quad \text{a.e. in} \ \mathbb{R}^N \ \text{as} \ n \to \infty,$$

and by [7] Theorem IV-9] there exists $\ell \in L^\theta(\mathbb{R}^N) \cap L^\theta(\mathbb{R}^N)$ such that

$$|u_n(x)| \leq \ell(x) \quad \text{a.e. in} \ \mathbb{R}^N.$$
First, we show that $\Psi$ is weakly strongly continuous on $W^{1,p}_V(\mathbb{R}^N)$. Since $F \in C^1(\mathbb{R}, \mathbb{R})$, we see that $F(u_n) \to F(u)$ as $n \to \infty$ for almost all $x \in \mathbb{R}^N$, and so $(I_\mu * F(u_n))F(u_n) \to (I_\mu * F(u))F(u)$ as $n \to \infty$ for almost all $x \in \mathbb{R}^N$. From Lemma 2.3 and (F2), we have
\[
|\langle I_\mu * F(u_n)F(u_n) \rangle| \leq Kc_0 \left( \frac{|u_n(x)|^{q_1}}{q_1} + \frac{|u_n(x)|^{q_2}}{q_2} \right) \in L^1(\mathbb{R}^N).
\]
By Lebesgue dominated convergence theorem, we get
\[
\int_{\mathbb{R}^N} |\langle I_\mu * F(u_n)F(u_n) \rangle| \, dx \to \int_{\mathbb{R}^N} |\langle I_\mu * F(u)F(u) \rangle| \, dx \quad \text{as} \quad n \to \infty,
\]
which implies that $\Psi(u_n) \to \Psi(u)$ as $n \to \infty$. Thus $\Psi$ is weakly strongly continuous on $W^{1,p}_V(\mathbb{R}^N)$.

We next note that $\Psi'$ is weakly strongly continuous on $W^{1,p}_V(\mathbb{R}^N).$ Since $u_n(x) \to u(x)$ as $n \to \infty$ for almost all $x \in \mathbb{R}^N$, $F(u_n) \to F(u)$ for almost all $x \in \mathbb{R}^N$ as $n \to \infty$. Then
\[
\langle I_\mu * F(u_n)F(u_n) \rangle \to \langle I_\mu * F(u)F(u) \rangle \quad \text{a.e. in} \quad \mathbb{R}^N, \quad \text{as} \quad n \to \infty.
\]
By (F2) and Hölder inequality, we have that for any $\varphi \in W^{1,p}_V(\mathbb{R}^N)$,
\[
\int_{\mathbb{R}^N} |\langle I_\mu * F(u_n)F(u_n) \rangle| \varphi(x) \, dx \\
\leq c_0 K \int_{\mathbb{R}^N} \left( |u_n|^{q_1-1} + |u_n|^{q_2-1} \right) \varphi(x) \, dx \\
\leq c_0 K \left( |u_n|^{q_1} \| \varphi \|_{q_1} + |u_n|^{q_2} \| \varphi \|_{q_2} \right) \\
\leq c_0 K \left( C_{q_1} \| \varphi \|_{q_1}^{q_1-1} + C_{q_2} \| \varphi \|_{q_2}^{q_2-1} \right) \| \varphi \|_W.
\]
Then by Lebesgue dominated convergence theorem, we obtain
\[
|\langle \Psi'(u_n) - \Psi'(u) \rangle| \left( W^{1,p}_V(\mathbb{R}^N) \right)' \\
= \sup_{\|\varphi\|_{W^{1,p}_V(\mathbb{R}^N)} = 1} |\langle \Psi'(u_n) - \Psi'(u) \rangle, \varphi | \\
= \sup_{\|\varphi\|_{W^{1,p}_V(\mathbb{R}^N)} = 1} \int_{\mathbb{R}^N} |\langle I_\mu * F(u_n)F(u) \rangle| \varphi(x) - \langle I_\mu * F(u)F(u) \rangle \varphi(x) | \, dx \\
\to 0 \quad \text{as} \quad n \to \infty.
\]
Therefore, we get that $\Psi'(u_n) \to \Psi'(u)$ in $\left( W^{1,p}_V(\mathbb{R}^N) \right)'$ as $n \to \infty$. This completes the proof.

\hfill \Box

3. Proof of the main result

In this section, we will prove our main result. First, we introduce the following definition.

**Definition 3.1.** For $c \in \mathbb{R}$, we say that $J_\lambda$ satisfies the $(C)_c$ condition if for any sequence $\{u_n\} \subset W^{1,p}_V(\mathbb{R}^N)$ with
\[
J_\lambda(u_n) \to c, \quad \|J_\lambda'(u_n)\|(1 + \|u_n\|_W) \to 0,
\]
there is a subsequence $\{u_n\}$ such that $\{u_n\}$ converges strongly in $W^{1,p}_V(\mathbb{R}^N)$.

We will use the following mountain pass theorem to prove our result.
Lemma 3.2 (Theorem 1 in [9]). Let $E$ be a real Banach space, $I \in C^1(E, \mathbb{R})$ satisfies the $(C)_c$ condition for any $c \in \mathbb{R}$, and

(i) There are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho} \geq \alpha$.
(ii) There is an $e \in E \setminus B_\rho$ such that $I(e) \leq 0$.

Then,

$$c = \inf_{y \in \mathbb{R}} \max_{0 \leq t \leq 1} I(y(t)) \geq \alpha$$

is a critical value of $I$, where

$$\Gamma = \{y \in C([0, 1], E): y(0) = 0, y(1) = e\}.$$

We first show that the energy functional $J_1$ satisfies the geometric structure.

Lemma 3.3. Assume that $(V), (M_1) - (M_2)$ and $(F_1) - (F_3)$ hold. Then

(i) There exists $\alpha, \rho > 0$ such that $J_1(u) \geq \alpha$ for all $u \in W^{p,2}_{0}$. Then, $\lambda$ is bounded for any $\lambda > 0$.

(ii) $J_1(u)$ is unbounded from below on $W_{0}^{p,2}$.\]

Proof. (i) From Lemma 3.2 and $(M_1) - (M_2), (F_2)$, we have

$$J_1(u) = \frac{1}{p} \mathcal{M}(||u||_{W}^p) - \frac{\lambda}{2} \int_{\mathbb{R}^N} \frac{F(u(x))F(u(y))}{|x-y|^\mu} \, dx \, dy$$

$$\geq \frac{1}{p} \mathcal{M}(||u||_{W}^p) ||u||_{W}^p - \frac{\lambda \alpha}{2} \int_{\mathbb{R}^N} \frac{|u(x)|^{q_1} + |u(y)|^{q_2}}{q_2} \, dx$$

$$\geq \left[ \frac{\lambda}{p} \frac{\alpha}{q_2} \left( C_{q_1}^0 ||u||_W^{q_1-p} + C_{q_2}^0 ||u||_W^{q_2-p} \right) \right] ||u||_W^p.$$

Since $q_2 > q_1 > p$, the claim follows if we choose $\rho$ small enough.

(ii) From $(M_2)$, we have

$$M(t) \leq M(1)^{t} \quad \text{for all } t \geq 1.$$ 

By the assumption $(F_3)$, we can take that $t_0$ such that $F(t_0) \neq 0$, we find

$$\int_{\mathbb{R}^N} (I_{\mu} \ast F(t_0 \chi_{B_1})) F(t_0 \chi_{B_1}) \, dx = F(t_0) \int_{B_1} \int_{B_1} I_{\mu}(x-y) \, dx \, dy > 0,$$

where $B_1$ denotes the open ball centered at the origin with radius $r$ and $\chi_{B_1}$ denotes the standard indicator function of set $B_1$. By the density theorem, there will be $v_0 \in W_{0}^{p,2}$ with

$$\int_{\mathbb{R}^N} (I_{\mu} \ast F(v_0)) F(v_0) \, dx > 0.$$ 

Define the function $v_t(x) = (v_0(x))^t$, then

$$J_1(v_t) = \frac{1}{p} \mathcal{M}(||v_t||_{W}^p) - \frac{\lambda}{2} \int_{\mathbb{R}^N} \frac{F(v_t(x))F(v_t(y))}{|x-y|^\mu} \, dx \, dy$$

$$\leq \frac{1}{p} \mathcal{M}(1) ||v_0||_{W}^p - \frac{\lambda}{2} \int_{\mathbb{R}^N} \frac{F(v_0(x))F(v_0(y))}{|x-y|^\mu} \, dx \, dy$$

$$= \frac{1}{p} \mathcal{M}(1) \left[ t^{N-\mu} ||v_0||_{W}^p + t^N \int_{\mathbb{R}^N} V(tx)v_0 \, dx \right] - t^{2N-\mu} \frac{1}{2} \int_{\mathbb{R}^N} \frac{F(v_0(x))F(v_0(y))}{|x-y|^\mu} \, dx \, dy,$$

for sufficiently large $t$. Therefore, we have that $J_1(v_t) \to -\infty$ as $t \to \infty$ since $1 \leq \theta < \frac{2N-\mu}{N}$ gives that $2N-\mu > N\theta > (N-p)s\theta$. Hence we obtain that the functional $J_1$ is unbounded from below.

Lemma 3.4. Assume that $(V), (M_1) - (M_2)$ and $(F_1) - (F_4)$ hold. Then $(C)_c$ sequence of $J_1$ is bounded for any $\lambda > 0$.\]
Proof. Suppose that \( \{u_n\} \subset W^{k,p}_V(\mathbb{R}^N) \) is a \( (C)_c \)-sequence for \( \mathcal{F}(u) \), that is,

\[
\mathcal{F}(u_n) \to c, \quad \|\mathcal{F}'(u_n)\|_W(1 + \|u_n\|_W) \to 0,
\]

which shows that

\[
(3.2) \quad c = \mathcal{F}(u_n) + o(1), \quad \langle \mathcal{F}'(u_n), u_n \rangle = o(1)
\]

where \( o(1) \to 0 \) as \( n \to \infty \). We now prove that \( \{u_n\} \) is bounded in \( W^{k,p}_V(\mathbb{R}^N) \). We argue by contradiction. Suppose that the sequence \( \{u_n\} \) is unbounded in \( W^{k,p}_V(\mathbb{R}^N) \), then we may assume that

\[
(3.3) \quad \|u_n\|_W \to \infty, \quad \text{as} \quad n \to \infty.
\]

Let \( \omega_n(x) = \frac{\omega}{\|\omega\|_W} \), then \( \omega_n \in W^{k,p}_V(\mathbb{R}^N) \) with \( \|\omega_n\|_W = 1 \). Hence, up to a subsequence, still denoted by itself, there exists a function \( \omega \in W^{k,p}_V(\mathbb{R}^N) \) such that

\[
(3.4) \quad \omega_n(x) \to \omega(x) \quad \text{a.e. in} \quad \mathbb{R}^N, \quad \text{and} \quad \omega_n(x) \to \omega(x) \quad \text{a.e. in} \quad L^p(\mathbb{R}^N)
\]

as \( n \to \infty \), for \( p \leq r < \frac{Np}{N-p} \).

Let \( \Omega_1 = \{x \in \mathbb{R}^N : \omega(x) \neq 0\} \), then

\[
\lim_{n \to \infty} \omega_n(x) = \lim_{n \to \infty} \frac{u_n(x)}{\|u_n\|_W} = \omega(x) \neq 0 \quad \text{in} \quad \Omega_1,
\]

and (3.3) implies that

\[
(3.5) \quad \|u_n\| \to \infty \quad \text{a.e. in} \quad \Omega_1.
\]

So from the assumption \((F_3)\) and Lemma 2.3, we have

\[
(3.6) \quad \lim_{n \to \infty} \left( \frac{\mathcal{I}_\mu * F(u_n(x)))F(u_n(x))}{|u_n(x)|^p} \right) = \infty, \quad \text{for} \quad x \in \Omega_1.
\]

Moreover, by \((F_3)\), there exists \( t_0 > 0 \) such that

\[
\frac{F(t)}{|t|^p} > 1,
\]

for all \( |t| > t_0 \). Since \( F \) is continuous, then there exists \( C > 0 \) such that \( |F(t)| \leq C \) for all \( t \in [-t_0, t_0] \). Thus, we see that there is a constant \( C_0 \) such that for any \( t \in \mathbb{R} \), we have \( F(t) \geq C_0 \), which show that there is a constant \( C \) such that

\[
\frac{(\mathcal{I}_\mu * F(u_n))F(u_n) - C}{\|u_n\|_W^p} \geq 0.
\]

This means that

\[
(3.7) \quad \frac{(\mathcal{I}_\mu * F(u_n))F(u_n(x))}{|u_n(x)|^p} \|\omega_n(x)\|^p - \frac{C}{\|u_n\|_W^p} \geq 0.
\]

By (3.2) we have that

\[
(3.8) \quad c = \mathcal{F}(u_n) + o(1) = \frac{1}{p} \mathcal{M}(\|u_n\|_W^p) - \frac{1}{2} \int_{\mathbb{R}^N} (\mathcal{I}_\mu * F(u_n))F(u_n)dx + o(1).
\]

Using this and \((M_1) - (M_2)\), we find

\[
\frac{1}{p} \int_{\mathbb{R}^N} (\mathcal{I}_\mu * F(u_n))F(u_n)dx \geq \frac{1}{p\lambda} \mathcal{M}(\|u_n\|_W^p) - \frac{c}{\lambda} + \frac{\|u_n\|_W^p}{p\lambda} \to \infty, \quad \text{as} \quad n \to \infty.
\]
We claim that \( \text{meas}(\Omega_1) = 0 \). Indeed, if \( \text{meas}(\Omega_1) \neq 0 \), from (3.1), (3.6), (3.7), (3.8) and Fatou's lemma, we have

\[
\begin{align*}
&\liminf_{n \to \infty} \int_{\Omega_1} \frac{(I_\mu \ast F(u_n(x)))F(u_n(x))}{|u_n(x)|^{p^*}} |\omega_n(x)|^{p^*} \, dx - \int_{\Omega_1} \limsup_{n \to \infty} \frac{C}{||u_n||^{p^*}_W} \, dx \\
&\leq \int_{\Omega_1} \liminf_{n \to \infty} \left( \frac{(I_\mu \ast F(u_n(x)))F(u_n(x))}{|u_n(x)|^{p^*}} |\omega_n(x)|^{p^*} - \frac{C}{||u_n||^{p^*}_W} \right) \, dx \\
&\leq \liminf_{n \to \infty} \int_{\Omega_1} \left( \frac{(I_\mu \ast F(u_n(x)))F(u_n(x))}{|u_n(x)|^{p^*}} |\omega_n(x)|^{p^*} - \frac{C}{||u_n||^{p^*}_W} \right) \, dx \\
&= \liminf_{n \to \infty} \int_{\Omega_1} \left( \frac{(I_\mu \ast F(u_n(x)))F(u_n(x))}{|u_n(x)|^{p^*}} - \frac{C}{||u_n||^{p^*}_W} \right) \, dx \\
&\leq \liminf_{n \to \infty} \int_{\Omega_1} \frac{M(1)(I_\mu \ast F(u_n(x)))F(u_n(x))}{M(||u_n||^{p^*}_W)} \, dx - \liminf_{n \to \infty} \int_{\Omega_1} \frac{C}{||u_n||^{p^*}_W} \, dx \\
&\leq \liminf_{n \to \infty} \int_{\Omega_1} \frac{M(1)(I_\mu \ast F(u_n(x)))F(u_n(x))}{M(||u_n||^{p^*}_W)} \, dx - \liminf_{n \to \infty} \int_{\Omega_1} \frac{C}{||u_n||^{p^*}_W} \, dx \\
&= \frac{\mathcal{M}(1)}{p} \liminf_{n \to \infty} \int_{\mathbb{R}^N} \frac{(I_\mu \ast F(u_n(x)))F(u_n(x))}{pM(||u_n||^{p^*}_W)} \, dx \\
&= \frac{\mathcal{M}(1)}{p} \liminf_{n \to \infty} \int_{\mathbb{R}^N} (I_\mu \ast F(u_n(x)))F(u_n(x)) \, dx + c - o(1) \tag{3.10}
\end{align*}
\]

So by (3.9) and (3.10), we get

\[
\lim_{n \to \infty} = \frac{2\mathcal{M}(1)}{pA}.
\]

This is a contradiction. This shows that \( \text{meas}(\Omega_1) = 0 \). Hence \( \omega(x) = 0 \) for almost all \( x \in \mathbb{R}^N \). The convergence in (3.4) means that

\[
\omega_n(x) \to 0 \quad \text{a.e. in } \mathbb{R}^N, \quad \text{and} \quad \omega_n(x) \to 0 \quad \text{a.e. in } L^r(\mathbb{R}^N) \quad \text{as } n \to \infty, \tag{3.11}
\]

for \( p \leq r < \frac{Np}{N-\alpha p^*} \).

Using (3.2) and (M2), we get

\[
\begin{align*}
c + 1 &\geq \mathcal{J}_g(u_n) - \frac{1}{p^\theta} \langle \mathcal{J}_g'(u_n), u_n \rangle \\
&= \frac{1}{p} \mathcal{M}(||u_n||^{p^*}_W) - \frac{1}{p^\theta} M(||u_n||^{p^*}_W) ||u_n||^{p^*}_W \\
&\quad + \lambda \int_{\mathbb{R}^N} (I_\mu \ast F(u_n)) \left( \frac{1}{p^\theta} f(u_n)u_n - \frac{1}{2} F(u_n) \right) \, dx \\
&\quad \geq \lambda \int_{\mathbb{R}^N} (I_\mu \ast F(u_n)) \left( \frac{1}{p^\theta} f(u_n)u_n - \frac{1}{2} F(u_n) \right) \, dx \\
&= \lambda \int_{\mathbb{R}^N} (I_\mu \ast F(u_n)) \mathcal{F}(u_n) \, dx, \tag{3.12}
\end{align*}
\]

for \( n \) large enough.
Let us define \( \Omega_n(a, b) := \{ x \in \mathbb{R}^N : a \leq |u_n(x)| \leq b \} \) for \( a, b \geq 0 \). From \((M_1)\) and \((M_2)\), we have that

\[
(3.13) \quad \mathcal{M}(|u_n|_p^p) \geq \frac{1}{\theta} \mathcal{M}(|u_n|_W^p)|u_n|_W^p \geq \frac{m_0}{\theta} |u_n|_W^p.
\]

This together with \((3.3)\) and \((3.8)\) yields that

\[
0 < \frac{2}{p+1} \leq \limsup_{n \to \infty} \frac{\int_{\Omega_n} (I_{\mu} \ast F(u_n))F(u_n) \, dx}{\mathcal{M}(|u_n|_W^p)} = \limsup_{n \to \infty} \int_{\Omega_n} \frac{(I_{\mu} \ast F(u_n))F(u_n)}{\mathcal{M}(|u_n|_W^p)} \, dx = \limsup_{n \to \infty} \left( \int_{\Omega_n \cap (0, r_n)} + \int_{\Omega_n \cap (r_n, \infty)} \right) \frac{(I_{\mu} \ast F(u_n))F(u_n)}{\mathcal{M}(|u_n|_W^p)} \, dx.
\]

(3.14)

On the one hand, by Lemma \((2.8)\), \((3.15)\), \((P_2)\) and \((3.11)\), we obtain

\[
\int_{\Omega_n \cap (0, r_n)} \frac{(I_{\mu} \ast F(u_n))F(u_n)}{\mathcal{M}(|u_n|_W^p)} \, dx \leq \frac{K\theta}{m_0} \int_{\Omega_n \cap (0, r_n)} \frac{|F(u_n)|}{|u_n|_W^p} \, dx \leq \frac{c_0 K\theta}{m_0} \int_{\Omega_n \cap (0, r_n)} \left( \frac{|u_n|_{q_1}^q}{q_1 |u_n|_W} + \frac{|u_n|_{q_2}^p}{q_2 |u_n|_W} \right) \, dx = \frac{c_0 K\theta}{m_0} \int_{\Omega_n \cap (0, r_n)} \left( \frac{|u_n|_{q_1}^{q-\rho}}{q_1} + \frac{|u_n|_{q_2}^{q-\rho}}{q_2} \right) \, dx \rightarrow 0, \quad \text{as } n \to \infty.
\]

(3.15)

On the other hand, using Hölder inequality, \((3.11)\), \((3.12)\) and \((F_4)\), we find

\[
\int_{\Omega_n \cap (r_n, \infty)} \frac{|I_{\mu} \ast F(u_n)|F(u_n)}{\mathcal{M}(|u_n|_W^p)} \, dx \leq \frac{\theta}{m_0} \int_{\Omega_n \cap (r_n, \infty)} \frac{|I_{\mu} \ast F(u_n)|F(u_n)}{|u_n|_W^p} \, dx \leq \frac{\theta}{m_0} \int_{\Omega_n \cap (r_n, \infty)} \frac{|I_{\mu} \ast F(u_n)|F(u_n)}{|u_n|_W^p} |\omega_n(x)|^p \, dx \leq \frac{\theta}{m_0} \left( \int_{\Omega_n \cap (r_n, \infty)} \frac{|I_{\mu} \ast F(u_n)|F(u_n)}{|u_n|_W^p} \right)^\frac{1}{q} \left( \int_{\Omega_n \cap (r_n, \infty)} |\omega_n(x)|^\frac{pq}{p-q} \, dx \right)^\frac{p-q}{p}
\]

\[
\leq \frac{\theta}{m_0} c_1^\frac{1}{q} K^{\frac{pq}{p-q}} \left( \int_{\Omega_n \cap (r_n, \infty)} |I_{\mu} \ast F(u_n)|F(u_n) \, dx \right)^\frac{1}{q} \left( \int_{\Omega_n \cap (r_n, \infty)} |\omega_n(x)|^\frac{pq}{p-q} \, dx \right)^\frac{p-q}{p} \leq \frac{\theta}{m_0} c_1^\frac{1}{q} K^{\frac{pq}{p-q}} \left( \int_{\Omega_n \cap (r_n, \infty)} |I_{\mu} \ast F(u_n)|F(u_n) \, dx \right)^\frac{1}{q} \leq \frac{\theta}{m_0} c_1^\frac{1}{q} K^{\frac{pq}{p-q}} \left( \int_{\Omega_n \cap (r_n, \infty)} |\omega_n(x)|^\frac{pq}{p-q} \, dx \right)^\frac{p-q}{p} \rightarrow 0, \quad \text{as } n \to \infty.
\]

(3.16)

Here we used the fact that \( \frac{sp}{p-q} \in (p, \frac{Np}{N-p-q}) \) if \( k > \frac{N}{p} \). Thus, we get a contradiction from \((3.14)\), \((3.16)\). The proof is complete. \( \square \)
Lemma 3.5. Assume that $(V), (M_1) - (M_2)$ and $(F_1) - (F_4)$ hold. Then the functional $\mathcal{J}_\lambda$ satisfies $(C)_\lambda$-condition for any $\lambda > 0$.

Proof. Suppose that $\{u_n\} \subset W^{s,p}_V(\mathbb{R}^N)$ is a $(C)_\lambda$-sequence for $\mathcal{J}_\lambda(u)$, from Lemma 3.4 we have that $\{u_n\}$ is bounded in $W^{s,p}_V(\mathbb{R}^N)$, then if necessary to a subsequence, we have

$$u_n \rightarrow u \text{ in } W^{s,p}_V(\mathbb{R}^N), \quad u_n \rightarrow u \text{ a.e. in } \mathbb{R}^N,$$

(3.17)\]

$$u_n \rightarrow u \text{ in } L^q(\mathbb{R}^N) \cap L^{p_2}(\mathbb{R}^N),$$

$$|u_n| \leq \ell(x) \text{ a.e. in } \mathbb{R}^N, \text{ for some } \ell(x) \in L^q(\mathbb{R}^N) \cap L^{p_2}(\mathbb{R}^N).$$

For simplicity, let $\varphi \in W^{s,p}_V(\mathbb{R}^N)$ be fixed and denote by $B_\varphi$ the linear functional on $W^{s,p}_V(\mathbb{R}^N)$ defined by

$$B_\varphi(v) = \int_{\mathbb{R}^N} \frac{|\varphi(x) - \varphi(y)|^{p-2}(\varphi(x) - \varphi(y))}{|x-y|^{N+sp}}(v(x) - v(y)) \, dx \, dy.$$ 

for all $v \in W^{s,p}_V(\mathbb{R}^N)$. By H"older inequality, we have

$$|B_\varphi(v)| \leq |\varphi|_{p,p}^1 |v|_{s,p} \leq ||\varphi||_{W^{s,p}}^p ||v||_W,$$

for all $v \in W^{s,p}_V(\mathbb{R}^N)$. Hence, (3.18) gives that

$$\lim_{n \to \infty} \left( M(||u_n||_{W^{s,p}_V}) - M(||u||_{W^{s,p}_V}) \right) B_\varphi(u_n - u) = 0,$$

since $\{M(||u_n||_{W^{s,p}_V}) - M(||u||_{W^{s,p}_V})\}$ is bounded in $\mathbb{R}$.

Since $\mathcal{J}_\lambda'(u_n) \rightarrow 0$ in $(W^{s,p}_V(\mathbb{R}^N)^\prime$ and $u_n \rightarrow u$ in $W^{s,p}_V(\mathbb{R}^N)$, we have

$$\langle \mathcal{J}_\lambda'(u_n) - \mathcal{J}_\lambda'(u), u_n - u \rangle \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

That is,

\begin{align*}
\langle \mathcal{J}_\lambda'(u_n) - \mathcal{J}_\lambda'(u), u_n - u \rangle & = M(||u_n||_{W^{s,p}_V}) B_{u_n}(u_n - u) + \int_{\mathbb{R}^N} V(x)|u_n|^{p-2}u_n(u_n - u) \, dx \\
& \quad - M(||u||_{W^{s,p}_V}) B_u(u_n - u) + \int_{\mathbb{R}^N} V(x)|u|^{p-2}u(u_n - u) \, dx \\
& \quad - \lambda \int_{\mathbb{R}^N} \left[ (I_\mu F(u)) f(u) - (I_\mu F(u)) f(u) \right] (u_n - u) \, dx \\
& = M(||u_n||_{W^{s,p}_V}) B_{u_n}(u_n - u) - B_u(u_n - u) \\
& \quad + \left( M(||u_n||_{W^{s,p}_V}) - M(||u||_{W^{s,p}_V}) \right) B_n(u_n - u) \\
& \quad + M(||u_n||_{W^{s,p}_V}) \int_{\mathbb{R}^N} V(x)|u_n|^{p-2}u_n - |u|^{p-2}u(u_n - u) \, dx \\
& \quad + [M(||u_n||_{W^{s,p}_V}) - M(||u||_{W^{s,p}_V})] \int_{\mathbb{R}^N} V(x)|u|^{p-2}u(u_n - u) \, dx \\
& \quad - \lambda \int_{\mathbb{R}^N} \left[ (I_\mu F(u)) f(u) - (I_\mu F(u)) f(u) \right] (u_n - u) \, dx.
\end{align*}

(3.19)

From Lemma 3.4 we have

$$\int_{\mathbb{R}^N} \left[ (I_\mu F(u)) f(u) - (I_\mu F(u)) f(u) \right] (u_n - u) \, dx \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$ 

(3.20)
Moreover, using H"older inequality and (3.14), we have
\begin{equation}
(3.21) \quad [M(||u_n||_W^p) - M(||u||_W^p)] \int_{\mathbb{R}^N} V(x)|u|^{p-2}u(u_n - u)\,dx \to 0, \quad \text{as } n \to \infty.
\end{equation}
From (3.18)-(3.21) and (M1), we obtain
\[\lim_{n \to \infty} M(||u_n||_W^p)\left[B_{u_n}(u_n - u) - B_{u}(u_n - u)\right] + \int_{\mathbb{R}^N} V(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)\,dx = 0.\]
Since \(M(||u_n||_W^p)[B_{u_n}(u_n - u) - B_{u}(u_n - u)] \geq 0\) and \(V(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u) \geq 0\) for all \(n\) by convexity, (M1) and (V1), we have
\[\lim_{n \to \infty} \left[B_{u_n}(u_n - u) - B_{u}(u_n - u)\right] = 0,
\]
\begin{equation}
(3.22) \quad \lim_{n \to \infty} \int_{\mathbb{R}^N} V(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)\,dx = 0.
\end{equation}

Let us now recall the well-known Simon inequalities. There exist positive numbers \(c_p\) and \(C_p\), depending only on \(p\), such that
\begin{equation}
(3.23) \quad |\xi - \eta|^p \leq \begin{cases} 
C_p((|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi - \eta) & \text{for } p \geq 2, \\
C_p((|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi - \eta))^{p/2}(|\xi|^p + |\eta|^{2-p})^{(2-p)/2} & \text{for } 1 < p < 2,
\end{cases}
\end{equation}
for all \(\xi, \eta \in \mathbb{R}^N\). According to the Simon inequality, we divide the discussion into two cases.

Case \(p \geq 2\): From (3.22) and (3.23), as \(n \to \infty\), we have
\[
[u_n - u]_{2,p}^p = \int_{\mathbb{R}^N} \left[|u_n(x) - u(x) - u_n(y) + u(y)|^p / |x - y|^{N+ps}\right] \,dxdy \\
\leq c_p \int_{\mathbb{R}^N} \left[|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y)) - |u(x) - u(y)|^{p-2}(u_n - u)\right] \,dxdy \\
\times \left(u_n(x) - u(x) - u_n(y) + u(y)\right) \,dxdy \\
= c_p \left[B_{u_n}(u_n - u) - B_{u}(u_n - u)\right] = o(1),
\]
and
\[||u_n - u||_{p,V}^p \leq c_p \int_{\mathbb{R}^N} V(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)\,dx = o(1).\]

Consequently, \(||u_n - u||_{W} \to 0\) as \(n \to \infty\).

Case \(1 < p < 2\): taking \(\xi = u_n(x) - u_n(y)\) and \(\eta = u(x) - u(y)\) in (3.23), as \(n \to \infty\), we have
\[
[u_n - u]_{1,p}^p \leq C_p \left[B_{u_n}(u_n - u) - B_{u}(u_n - u)\right]^{p/2}((u_n)_{1,p}^p + [u]_{1,p}^{p(2-p)/2} \\
\leq C_p \left[B_{u_n}(u_n - u) - B_{u}(u_n - u)\right]^{p/2}((u_n)_{1,p}^{p(2-p)/2} + [u]_{1,p}^{p(2-p)/2}) \\
\leq C \left[B_{u_n}(u_n - u) - B_{u}(u_n - u)\right]^{p/2} = o(1).
\]

Here we used the fact that \([u_n]_{1,p}\) and \([u]_{1,p}\) are bounded, and the elementary inequality
\[(a + b)^{2-p/2} \leq a^{2-p/2} + b^{2-p/2} \quad \text{for all } a, b \geq 0 \text{ and } 1 < p < 2.
\]

Moreover, by H"older inequality and (3.22), as \(n \to \infty\),
\[
||u_n - u||_{p,V}^p \leq C_p \int_{\mathbb{R}^N} V(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u) \,dx
\]
Thus $\|u_n - u\|_W \to 0$ as $n \to \infty$. The proof is complete. \hfill \Box

Now we are ready to prove our main result.

**Proof of Theorem 1.1** By Lemmas 3.3-3.5 and using Lemma 3.2, we obtain that there exists a critical point of functional $J_\lambda$, so problem (1.1) has a nontrivial weak solution for any $\lambda > 0$.

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**References**

[1] C. O. Alves, M. Yang, Existence of semiclassical ground state solutions for a generalized Choquard equation, *J. Differential Equations*, 257 (2014) 4133-4164.
[2] C. O. Alves, M. Yang, Multiplicity and concentration of solutions for a quasilinear Choquard equation, *Journal of Mathematical Physics*, 55 (2014), 061502.
[3] C. Alves, G. Figueiredo, M. Yang, Existence of solutions for a nonlinear Choquard equation with potential vanishing at infinity, *Advances in Nonlinear Analysis*, 5 (2016), 331-346.
[4] V. Ambrosio, Concentration phenomena for a fractional Choquard equation with magnetic field, Preprint, arXiv:1807.07442.
[5] G. Autuori, A. Fiscella, P. Pucci, Stationary Kirchhoff problems involving a fractional elliptic operator and a critical nonlinearity, *Nonlinear Analysis*, 125 (2015), 699-714.
[6] P. Belchior, H. Bueno, O. Miyagaki, G. Pereira, Remarks about a fractional Choquard equation: Ground state, regularity and polynomial decay, *Nonlinear Analysis*, 164 (2017) 38-53.
[7] H. Brezis, Analyse fonctionelle. Théorie et applications, *Masson*, Paris (1983).
[8] M. Caponi, P. Pucci, Existence theorems for entire solutions of stationary Kirchhoff fractional p-Laplacian equations, *Ann. Mat. Pura Appl.*, 195 (2016), 2099-2129.
[9] D. Costa, O. Miyagaki, Nontrivial solutions for perturbations of the p–Laplacian on unbounded domains, *J. Math. Anal. Appl.* 193 (1995) 737-755.
[10] P. D’Avenia, M. Squassina, On fractional Choquard equations, *Math. Models Methods Appl. Sci.*, 25 (2015), 1447-1476.
[11] W. Chen, S. Deng, The Nehari manifold for a non-local elliptic operator involving concave-convex nonlinearities, *Z. Angew. Math. Phys.*, 66 (2015), 1387-1400.
[12] W. Chen, S. Deng, The Nehari manifold for a fractional p-Laplacian system involving concave-convex nonlinearities, *Nonlinear Analysis Series B: Real World Applications*, 27 (2016), 80-92.
[13] E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhiker guide to the fractional Sobolev spaces, *Bull. Sci. Math.* 136 (2012), 521-573.
[14] F. Fang, S. Liu, Nontrivial solutions of superlinear p–Laplacian equations, *J. Math. Anal. Appl.* 351 (2009), 138-140.
[15] A. Fiscella, E. Valdinoci, A critical Kirchhoff type problem involving a nonlocal operator, *Nonlinear Analysis*, 94 (2014), 156-170.
[16] J. Lee, J. Kim, J. Bae, K. Park, Existence of nontrivial weak solutions for a quasilinear Choquard equation, *Journal of Inequalities and Applications*, (2018) 2018:42.
[17] G. Li, C. Yang, The existence of a nontrivial solution to a nonlinear elliptic boundary value problem of p-Laplacian type without the Ambrosetti-Rabinowitz condition, *Nonlinear Analysis*, 72 (2010) 4602-4613.
[18] E. Lieb, Existence and uniqueness of the minimizing solution of Choquard’s nonlinear equation, *Stud. App. Math.*, **57** (1977), 93-105.

[19] E. Lieb, M. Loss, *Analysis*, Graduate Studies in Mathematics, vol. 14, Amer. Math. Soc., Providence, Rhode Island, 2001.

[20] S. H. Miyagaki, M. Souto, Superlinear problems without Ambrosetti and Rabinowitz growth condition, *J. Differ. Equ.* **245**(12)(2008), 3628-3638.

[21] G. Molica Bisci, V. Rădulescu, Ground state solutions of scalar field fractional Schrödinger equations, *Calculus of Variations and Partial Differential Equations*, **54** (2015), 2985-3008.

[22] G. Molica Bisci, V. Rădulescu, R. Servadei, *Variational Methods for Nonlocal Fractional Equations*, Encyclopedia of Mathematics and its Applications, **162**, Cambridge University Press, Cambridge, 2016.

[23] V. Moroz, J. Van Schaftingen, Ground states of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics, *J. Funct. Anal.*, **265** (2013), 153-184.

[24] V. Moroz, J. Van Schaftingen, Groundstates of nonlinear Choquard equations: Hardy-Littlewood-Sobolev critical exponent, *Comm. Contemp. Math.*, **17**(2015), 1550005 12pp.

[25] V. Moroz, J. Van Schaftingen, Existence of groundstates for a class of nonlinear Choquard equations, *Trans. Am. Math. Soc.* **367** (2015), 6557-6579.

[26] V. Moroz, J. Van Schaftingen, A guide to the Choquard equation, *J. Fixed Point Theory Appl.*, **19**(1)(2017), 773-813.

[27] S. Pekar, *Untersuchung uber die Elektronentheorie der Kristalle*, Akademie Verlag, 1954.

[28] P. Pucci, M. Xiang, B. Zhang, Multiple solutions for nonhomogeneous Schrödinger-Kirchhoff type equations involving the fractional $\ p-$Laplacian in $\mathbb{R}^N$, *Calc. Var. Partial Differential Equations*, **54** (2015), 2785-2806.

[29] P. Pucci, M. Xiang, B. Zhang, Existence and multiplicity of entire solutions for fractional $\ p-$Kirchhoff equations, *Adv. Nonlinear Anal.*, **5**(2016), 27-55.

[30] P. Pucci, M. Xiang, B. Zhang, Existence results for Schrödinger-Choquard-Kirchhoff equations involving the fractional $\ p-$Laplacian, *Adv. Calc. Var.*, doi: 10.1515/acv-2016-0049.

[31] Z. Shen, F. Gao, M. Yang, Ground states for nonlinear fractional Choquard equations with general nonlinearities, *Math. Methods Appl. Sci.*, **39**, no.14(2016), 4082-4098.

[32] Y. Song, S. Shi, Existence of infinitely many solutions for degenerate $\ p-$fractional Kirchhoff equations with critical Sobolev-Hardy nonlinearities, *J. Math. Anal. Appl.*, **411** (2014), 530-542.

[33] Y. Song, S. Shi, On a degenerate $\ p-$fractional Kirchhoff equations with critical Sobolev-Hardy nonlinearities, *Mediterr. J. Math.*, **15**(2018), 17.

[34] M. Xiang, G. Molica Bisci, G. Tian, B. Zhang, Infinitely many solutions for the stationary Kirchhoff problems involving the fractional $\ p-$Laplacian, *Nonlinear Anal.*, **29** (2016), 357-374.

[35] M. Xiang, B. Zhang, M. Ferrara, Existence of solutions for Kirchhoff type problem involving the non-local fractional $\ p-$Laplacian, *J. Math. Anal. Appl.*, **424**(2015), 1021-1041.

[36] M. Xiang, B. Zhang, M. Ferrara, Multiplicity results for the nonhomogeneous fractional $\ p-$Kirchhoff equations with concave-convex nonlinearities, *Proc. R. Soc. A.*, **471**(2015), 14.

[37] M. Xiang, B. Zhang, X. Zhang, A nonhomogeneous fractional $\ p-$Kirchhoff type problem involving critical exponent in $\mathbb{R}^N$, *Adv. Nonlinear Stud.*, (2016).

[38] D. Wu, Existence and stability of standing waves for nonlinear fractional Schrödinger equations with Hartree type nonlinearity, *J. Math. Anal. Appl.*, **411** (2014), 530-542.

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