Photon-photon scattering: a tutorial

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Abstract

Long-established results for the low-energy photon-photon scattering, $\gamma\gamma \to \gamma\gamma$, have recently been questioned. We analyze that claim and demonstrate that it is inconsistent with experience. We demonstrate that the mistake originates from an erroneous manipulation of divergent integrals and discuss the connection with another recent claim about the Higgs decay into two photons. We show a simple way of correctly computing the low-energy $\gamma\gamma$ scattering.

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1 Introduction

After Dirac proposed the theory of negative energy solutions of his equation \[1\], it was realized that photons can interact with other photons by polarizing the vacuum. Photon-photon scattering was qualitatively considered in this context by Halpern \[2\], and its cross section, for the case of photon energies low compared to the electron mass, was determined by Euler and Kockel in 1935 \[3\]-\[4\]. If the energy of each of the colliding photons is $\omega$ in the frame in which their total momentum vanishes, the low-energy differential cross section is

$$d\sigma/d\Omega = \frac{139\alpha^4}{(180\pi)^2} \left(\frac{\omega^6}{m^8}\right) (3 + \cos^2\theta)^2,$$

where $\alpha \simeq 1/137$ is the fine structure constant and $m$ is the electron mass.

High energy scattering was considered soon afterward \[5\]-\[6\]. A thorough analysis of the scattering at all energies, including partial cross sections for various polarization states, was carried out in \[7\]-\[8\], using the then new diagrammatic technique of Feynman. Since then, the photon-photon scattering cross section has been confirmed with other methods, and even higher-order QED corrections have been computed \[9\]. Results obtained up to 1971 are reviewed in \[10\] and more recent developments are summarized in \[11\].

Very recently, the classic result for the low-energy cross section \[11\] has been questioned \[12\]-\[13\]. In those papers, the cross section is found to be many orders of magnitude larger, since it is not suppressed by powers of $(\omega/m)$, but is proportional to $1/\omega^2$,

$$d\sigma_{PK}/d\Omega = \frac{\alpha^4}{(12\pi)^2} \left(3 + 2\cos^2\theta + \cos^4\theta\right).$$

As we will demonstrate in this paper, this claim is incorrect. It has already been pointed out \[14\] that it contradicts existing laboratory bounds on the the photon-photon cross section, obtained by colliding laser beams. We show in addition that a cross section increasing with the inverse squared energy of the colliding photons limits the mean free path of visible light due to collisions with the cosmic
microwave background radiation (CMBR) to less than the distance between Earth and Jupiter. Thus
the fact that we can sharply see much more distant astronomical objects proves that the low-energy
photon-photon scattering must be significantly suppressed, as predicted by eq. (1).

The matrix element for the photon-photon scattering is absent at the tree level since photons are
neutral. It arises only at the loop level. The sum of all contributing loop diagrams must be finite
since there is no parameter in the QED Lagrangian whose renormalization could absorb a divergence.
Refs. [12, 13] found an incorrect result because of the assumption that if the sum of those diagrams is
finite, they can be calculated without any regularization. In fact, even though the sum of the diagrams
is finite, each of them separately is divergent. Calculating the sum is somewhat delicate and is easiest
done with regularized loop integrals (see, however, an alternative calculation in [15] and another point
of view on avoiding regularization in [16]).

Interestingly, a similar error [17, 18] has recently cast doubt over the rate of the Higgs boson decay
into two photons. That process, too, is loop induced, and the sum of contributing loops is finite.
But individual loop integrals are divergent and must be regularized, as has already been thoroughly
discussed in this context [19, 20, 21, 22, 23].

2 Mean free path of photons in a microwave background

The CMBR is a gas of photons with the spectrum of a black body at a temperature of $1/\beta = 2.725$
K. Here we want to compute how far a visible-light photon with energy $E_\gamma \simeq 2.5$ eV can travel in
such a gas before scattering, from the point of view of an observer in whose frame the CMBR is
isotropic. (We will call it the LAB frame. For the purposes of this discussion an Earth-based observer
is a good approximation.) Consider one mode of the CMBR radiation, characterized by its energy
$E$ and inclination angle $\theta$ with respect to the direction from which the visible photon is incident.
The relative velocity of the two photons (as seen in the LAB frame) is $\vec{v}_1 + \vec{v}_2 = (1 + \cos \theta, -\sin \theta)$,
$|\vec{v}_1 + \vec{v}_2| = 2 \cos \frac{\theta}{2}$ (we use the units $c = \hbar = k_B = 1$). In the frame where the total momentum of
the photons vanishes, each has the energy $\omega$ given by

$$\omega = \frac{1}{2} \sqrt{2EE_\gamma (1 + \cos \theta)} = \cos \frac{\theta}{2} \sqrt{EE_\gamma}.$$  \hspace{1cm} (3)

That energy determines the scattering cross section. Collisions with photons in this particular mode
will occur at the rate

$$d\Gamma_{E\theta} = |\vec{v}_1 + \vec{v}_2| \sigma d\rho (E)$$ \hspace{1cm} (4)

where

$$d\rho (E) = \frac{E^2}{2\pi^2} \frac{dE d\cos \theta}{\exp (\beta E) - 1}$$ \hspace{1cm} (5)

is the density of CMBR photons with energy $E$, and $\sigma$ is the scattering cross section. Integrating over
the energies and directions of the CMBR photons we find the mean free path. Between collisions, the
visible-light photon will travel on average the distance

$$\lambda = \pi^2 \left[ \int_0^\infty dE \int_{-1}^{1} d\cos \theta \cos \frac{\theta}{2} \frac{E^2 \sigma}{\exp (\beta E) - 1} \right]^{-1}. \hspace{1cm} (6)$$

We now consider the two formulas for the low-energy cross section. If we use the classical result (1),
we find the total cross section

$$\sigma (\gamma\gamma \rightarrow \gamma\gamma) = \frac{973\alpha^4 \omega^6}{10125\pi m^8}, \hspace{1cm} (7)$$
and the mean free path

\[
\lambda = \pi^2 \left[ \frac{973\alpha^4 E_3^2}{10125\pi m^8} \int_{-1}^{1} d\cos \theta \cos \frac{\theta}{2} \int_{0}^{\infty} dE \frac{E^5}{\exp(\beta E) - 1} \right]^{-1}
\]

\[
= \pi^2 \left[ \frac{973\alpha^4 E_3^2}{10125\pi m^8} \cdot \frac{4}{9} \cdot \frac{8\pi^6}{63^{3/2}} \right]^{-1} = \frac{820125m^8}{4448\pi^3 \alpha^4 E_3^2}.
\]

(8)

Using \( m = 0.511 \text{ MeV} \) we find \( \lambda \simeq 7 \cdot 10^{68} \) meters, a distance that would take light about \( 10^{43} \) times more time to travel than the age of the Universe. In other words, the CMBR is a rather transparent medium at visible frequencies.

However, if we take instead the cross section suggested in [12, 13], we find from eq. (2)

\[
\sigma_{FK} = \frac{29\alpha^4}{540\pi \omega^2},
\]

(9)

which gives a much shorter mean free path,

\[
\lambda_{FK} = \pi^2 \left[ \frac{29\alpha^4}{540\pi E_\gamma} \int_{-1}^{1} d\cos \theta \cos \frac{\theta}{2} \int_{0}^{\infty} dE \frac{E}{\exp(\beta E) - 1} \right]^{-1}
\]

\[
= \pi^2 \left[ \frac{29\alpha^4}{540\pi E_\gamma} \cdot \frac{\pi^2}{63^{3/2}} \right]^{-1} = \frac{810\pi\beta^2 E_\gamma}{29\alpha^4},
\]

(10)

or \( \lambda_{FK} = 3 \cdot 10^{11} \) meters, equivalent to about 15 light minutes. For comparison, the orbital radius of Jupiter is about \( 8 \cdot 10^{11} \) meters. If the mean free path of the visible light were so much shorter than even the radius of Jupiter’s orbit, no stars would be visible on the night sky. Clearly, the result eq. (2) is at odds with experience.

The situation with eq. (2) is actually even worse. Since the cross section is not suppressed by the mass of the electron, there would be additional positive contributions from other charged fermions that would differ only by the coupling constant, and would further decrease the mean free path. This lack of suppression by the inverse mass of the loop particle contradicts the Appelquist-Carazzone decoupling theorem [24].

3 Determination of the photon-photon scattering

In this section we present a derivation of the photon-photon scattering matrix element in two regularization schemes: dimensional and Pauli-Villars. We consider the box diagram shown in Fig. 1. External photons carry momenta \( k_1, \ldots, k_4 \) which we will consider as incoming, \( k_1 + k_2 + k_3 + k_4 = 0 \).

There are six ways in which the four momenta can be arranged around the oriented electron loop. However, diagrams that differ only by the direction of the electron line give identical results so it is enough to compute three of them, corresponding to three cyclic permutations of \( k_1, k_2, \) and \( k_3 \). (If there was an odd number of photons coupling to the electron loop, the diagrams differing by the direction of the electron would cancel one other, resulting in a vanishing amplitude. This is the theorem due to Furry [25].)

At low energies of external photons it is especially easy to compute the diagrams in Fig. 1. We simply Taylor-expand the electron propagators in the external momenta, so that each propagator’s denominator becomes simply \( (q - m)^{-1} = (q + m)/(q^2 - m^2) \) where \( q \) is the loop momentum. Such expansion does not lead to any spurious divergences, and commutes with the integration over \( q \). We now explain how this integration is performed in two regularization schemes.
3.1 Dimensional regularization

Now that $q$ is present in the denominators only through $q^2$, also in the numerator we can replace all scalar products of $q$ with other vectors by powers of $q^2$ times products not involving $q$,

$$q^\mu_1 \ldots q^\mu_{2n} \rightarrow \frac{\Gamma(D)}{2^n \Gamma(D+n)} (q^2)^n S(g^\mu_1 \mu_2 \ldots g^\mu_{2n-1} \mu_{2n}).$$

Here $D$ is the space-time dimension and $S(g^\mu_1 \mu_2 \ldots g^\mu_{2n-1} \mu_{2n})$ is a sum of products of $n$ metric tensors $g$, totally symmetric in all indices $\mu_i$; it has $(2n-1)!$ terms. Terms odd in $q$ vanish upon integration.

The powers of $q^2$ resulting from (11) can be canceled against the denominators and the loop integration can be completed using

$$\int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2 - m^2 + i0)^a} = \frac{(-1)^a i}{(4\pi)^{D/2} m^{D-2n}} \Gamma(a-\frac{D}{2}) \Gamma(a).$$

Each of the three diagrams contains terms with the exponent $a = 2$, leading to a divergence $\Gamma(2 - \frac{D}{2}) \sim 1/(D - 4)$. The divergences cancel when we add all three contributions. But individual diagrams containing singularities $1/(D - 4)$ have also $D$-dependent factors, arising from the averaging in eq. (11). The resulting finite contributions do not cancel among themselves.

How do these remaining terms depend on $m$? We remember that they arise from the $a = 2$ sector, therefore they scale like $m^0$ (the overall dimension of the $\gamma\gamma \rightarrow \gamma\gamma$ amplitude). There are other terms that scale with this power, arising from convergent integrals like $m^2 \int d^4 q / (q^2 - m^2)^3$. The essential point is that the sum of all $m^0$ terms, including the remnants of singularities, adds up to zero. The total result for the amplitude turns out to be suppressed by four powers of $1/m$.

3.2 Pauli-Villars regularization

Another way of carrying out this calculation is to stay in four dimensions but add another amplitude, with the electron replaced by a very heavy particle of mass $M$, and with an opposite sign than the electron loop. The calculation proceeds very similarly to the case of dimensional regularization, with two changes. In averaging over the loop momentum directions (11) we replace $\frac{\Gamma(D+n)}{\Gamma(D)}$ by its value at $D = 4$, $(2n + 2)!/6$. The formula for the loop integration (12) is also replaced by its $D = 4$ value, except in the divergent case $a = 2$. In the dimensional regularization, this divergent integral gives $m^{D-4} \Gamma(2 - \frac{D}{2}) \rightarrow 2^{D-4} / \Gamma(D) - \ln m^2$. In the Pauli-Villars approach one finds a convergent combination

$$\int \frac{d^4 q}{(2\pi)^4} \left[ \frac{1}{(q^2 - m^2 + i0)^2} - \frac{1}{(q^2 - M^2 + i0)^2} \right] = -\frac{16\pi^2}{i} \ln \frac{M^2}{m^2}. $$

Figure 1: Virtual electron loop inducing the four-photon coupling.
When the three diagrams are added, this logarithm cancels, but now there are no finite remnants of the singularities. Instead, the $m^0$ terms from the convergent diagrams are canceled by the $M^0$ terms from the Pauli-Villars subtraction. Since they are independent of the electron mass, they are the same in the amplitudes with the electron and with the very heavy particle, and cancel in the difference.

In both regularization schemes, the only remaining result is suppressed by the electron mass.

### 3.3 Potential error from neglecting regularization

We have just seen that the regularization is crucial in computing the photon-photon scattering amplitude, even though the final result does not contain divergences. We now want to inspect more closely the part of the amplitude that does not contain external photon momenta, and thus scales like the zeroth power of the electron mass,

$$M_{m^0} \sim \int \frac{d^Dq}{(q^2 - m^2 + i0)} \left[ m^4 S_1^{\mu\nu\rho\sigma} + 2m^2 (2S_2^{\mu\nu\rho\sigma} - q^2 S_1^{\mu\nu\rho\sigma}) + 24q^\mu q^\nu q^\rho q^\sigma + (q^2)^2 S_1^{\mu\nu\rho\sigma} - 4q^2 S_2^{\mu\nu\rho\sigma} \right] \epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4,$$

with

$$S_1^{\mu\nu\rho\sigma} = g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\rho\nu},$$

$$S_2^{\mu\nu\rho\sigma} = g^{\mu\nu} q^\rho q^\sigma + \text{five other terms},$$

where the terms not shown in $S_2$ have the other five distributions of indices so that both $S_1$ and $S_2$ are totally symmetric in $\mu, \nu, \rho, \sigma$. The second line in (14) contains four powers of the loop momentum $q$ and thus represents divergent integrals. Without regularization, these divergent integrals simply do not have a meaning. If we apply the averaging procedure (11) to these terms, we find

$$\langle S_2^{\mu\nu\rho\sigma} \rangle = \frac{2q^2}{D} S_1^{\mu\nu\rho\sigma},$$

$$\langle q^\mu q^\nu q^\rho q^\sigma \rangle = \frac{(q^2)^2}{D(D + 2)} S_1^{\mu\nu\rho\sigma},$$

so that if $D = 4$, the second line of (14) vanishes, as does the term $\sim m^2$ in its first line. Thus, if the regularization is neglected, one is left with only the first term $m^4 S_1$ which, after the $q$ integration, gives a result independent of the electron mass, scaling like $m^0$.

$$iM_{m^0} = -\frac{4}{3} \alpha^2 S_1^{\mu\nu\rho\sigma} \epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4 \epsilon_5,$$

where $\epsilon_i$ are the polarization vectors of the four photons. This dependence of the amplitude on the polarization vectors (and not on the photon momenta) means that the induced coupling of the photons involves only their vector potential (the induced effective operator is proportional to $(A^2)^2$), and not its derivatives. It is not possible to construct such a coupling in a gauge invariant way.

This violation of gauge invariance may also generate photon’s mass. For example, if two of the external photon lines in Fig. 1 are contracted, the resulting two-loop diagram generates an operator $\sim A^2$, thus giving the photon a mass.
In order to see how the cross section in eq. (2) follows from the amplitude (17), we define two transverse polarization vectors for each photon, $\vec{\epsilon}_1^i, \vec{\epsilon}_2^i$, with $\vec{\epsilon}_1^i$ perpendicular to the scattering plane and $\vec{\epsilon}_2^i$ lying in that plane. We do not include here the longitudinal photon polarizations, present if the photon becomes massive, even though they may dominate the cross section; however, our goal here is merely to explain how the result (2) is related to the gauge-invariance violating amplitude (17). For the eq. (17) to give a non-zero result, each polarization must be represented an even number of times (otherwise there will always be a factor 0 in every term). There are eight possible such polarization assignments, giving the following values of the three terms in (17),

$$
\mathcal{M}_{1111} \sim 1 + 1 + 1,
\mathcal{M}_{2222} \sim 1 + \cos^2 \theta + \cos^2 \theta,
\mathcal{M}_{1122} \sim 1 + 0 + 0,
\mathcal{M}_{1212} \sim 0 + \cos \theta + 0,
\mathcal{M}_{1221} \sim 0 + 0 + \cos \theta,
$$

and the last three amplitudes enter with a weight factor of 2, due to the symmetry $1 \leftrightarrow 2$. The various amplitudes differ by the polarization of some photons, so they do not interfere. The sum of their squares gives $3^2 + (1 + 2 \cos^2 \theta)^2 + 2 + 4 \cos^2 \theta = 4 \left(3 + 2 \cos^2 \theta + \cos^4 \theta\right)$, the angular structure of the (incorrect) result quoted in (2). In fact, a sum over the polarizations of the final state photons and an average over the polarizations of the initial state photons, leads to the cross section given in eq. (2).

What went wrong in the above procedure? The formulas (15) cannot be applied to the divergent integrals in the second line of (14) in $D = 4$, without regularization. If we stay in $D$ dimensions, the terms we found to be zero in the $D \to 4$ limit given finite contributions that cancel against the first term of the integrand, $\sim m^4 S_1$. In this correct treatment the resulting amplitude is suppressed by four powers of $1/m$.

The recent incorrect claim about the decay $H \to \gamma \gamma$ [17, 18] originated with a similar, but somewhat simpler integral. An example of a contribution to that process is shown in Fig. (2). There are only three propagators, and the divergent integrals are present in the combination [18]

$$
I_{\mu\nu} (D) = \int d^D q \frac{q^2 g_{\mu\nu} - 4 q_{\mu} q_{\nu}}{\left(q^2 - m^2 + i0\right)^3}.
$$

Without dimensional regularization, if we take $D = 4$, it seems that this integral vanishes after averaging over $q$ with help of (11). As we have seen with the example of $\gamma \gamma$ scattering, and as has already been discussed in the literature [21, 19, 10], such manipulations with unregulated, divergent integrals are unjustified. In the case of the Higgs decay, they lead to the incorrect conclusion that $I_{\mu\nu} (D \to 4)$ vanishes. In fact, in the limit of a very heavy Higgs boson, the correct finite result of $I_{\mu\nu}$ gives the most important contribution.

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Figure 2: An example of a $W$ boson loop mediating the Higgs boson decay into two photons.
3.4 Results for polarized photons

The correct result of the loop integration in the $\gamma \gamma \rightarrow \gamma \gamma$ amplitude contains scalar products among photon momenta $k_i$, in addition to the polarization vectors $\epsilon^\lambda_i$. The effective photon-photon coupling induced in this way is described by operators involving the electromagnetic field tensor and is gauge invariant. We know calculate the scattering cross sections for various polarization situations. Instead of the linear polarizations we have just considered, the scattering amplitudes will be presented in terms of circular polarization states. Thus we introduce

\[ \vec{\epsilon}^{\pm} = \frac{1}{\sqrt{2}} (\vec{\epsilon} \pm i \vec{\epsilon}^2), \]

describing right- and left-handed polarization states, respectively. There are four possible initial polarization states, but it is sufficient to consider just two of them, $++$ and $+\mp$. We get three independent scattering amplitudes, $M_{++\mp}$, $M_{++\pm}$, and $M_{++\mp}$. We describe kinematics in terms of Mandelstam variables $s = (k_1 + k_2)^2 = 4E^2$ and $t = (k_1 + k_3)^2 = -2E^2 (1 - \cos \theta)$ and find

\[ iM_{++\mp} = \frac{2 \alpha^2 (s^2 + st + t^2)}{15m^4} = \frac{8 \alpha^2 \omega^4 (3 + \cos^2 \theta)}{15m^4}, \]
\[ iM_{++\pm} = \frac{2 \alpha^2 st (s + t)}{315m^6} = -\frac{16 \alpha^2 \sin^2 \theta \omega^6}{315m^6}, \]
\[ iM_{++\mp} = \frac{-11 \alpha^2 s^2}{45m^4} = -\frac{176 \alpha^2 \omega^4}{45m^4}. \]  

(18)

We note that the amplitude $M_{++\mp}$ vanishes at the leading order in $E/m$ expansion at which the other amplitudes are finite. In order to compute it, we have to evaluate two more terms in the Taylor expansion.

All other amplitudes can be obtained from eq. (18) using space and/or time reversal and the crossing symmetry. For example, $iM_{++\pm} = -\frac{11 \alpha^2 t^2}{45m^4}$ and $iM_{++\mp} = -\frac{11 \alpha^2 (s+t)^2}{45m^4}$.

3.5 Total cross section

Once the polarized amplitudes have been evaluated, the unpolarized cross section can be easily found. We quote here only the leading low-energy result (thus we neglect $M_{++\mp}$ and seven amplitudes related to it) for the cross section averaged over initial and summed over final polarizations,

\[ \frac{d\sigma (\gamma \gamma \rightarrow \gamma \gamma)}{d\Omega} = \frac{1}{256\pi^2 \omega^2} \frac{|M_{++\pm}|^2 + |M_{++\mp}|^2 + |M_{++\pm}|^2 + |M_{++\mp}|^2}{2} \]
\[ = \frac{139 \alpha^4 \omega^6 (3 + \cos^2 \theta)^2}{(180\pi)^2 m^8}, \]

(19)
(20)

in agreement with the classic result (1). When integrated over both $\theta$ and $\phi$ from 0 to $\pi$ (we integrate only over one hemisphere since the two final-state photons are identical), this gives the total photon-photon scattering cross section,

\[ \sigma (\gamma \gamma \rightarrow \gamma \gamma) = \frac{973 \alpha^4 \omega^6}{10125 \pi m^8}, \]

(21)
in agreement with [26]. Other texts seem to have misprints in these results [27, 28].
4 Conclusions

We have demonstrated that the recently claimed result for the $\gamma \gamma \rightarrow \gamma \gamma$ scattering cross-section must be wrong. The photon-photon coupling is induced by virtual loops with charged particles and is suppressed at low photon energy by the inverse power of the electron mass. The result lacks this suppression and yields a very large cross-section, therefore a short mean free path of visible photons even in the rare cosmic microwave radiation background. Such short path would obscure all astronomical objects as close as Jupiter.

We have showed that the error resulted from manipulating unregulated divergent integrals. A similar error misled the authors of [17, 18] in the context of the Higgs decay to two photons. Both processes arise only at the loop level and their amplitudes must be finite, since there are no parameters in the Lagrangian that could absorb a divergence. However, both processes are usually computed from a sum of several diagrams, among which some are divergent. For this reason, a regularization of individual contributions is necessary.

Interestingly, the mistakes in these recent studies of $\gamma \gamma \rightarrow \gamma \gamma$ and $H \rightarrow \gamma \gamma$ led to confusions about various types of decoupling. The correct result for the former process does respect Appelquist-Carazzone decoupling theorem in the limit of low photon energies or large electron mass, whereas the incorrect result of [12, 13] does not. On the other hand, the correct result for the Higgs decay does not vanish, as one could naively expect, in the limit of large Higgs mass (or, equivalently, low W boson mass; this type of decoupling affects for example quarks but not the longitudinal W components), while part of the reason why [17, 18] believed their result was that it did vanish in that limit.

We have also showed how the $\gamma \gamma \rightarrow \gamma \gamma$ amplitude can be calculated in the low-energy regime, and how an expansion in powers of the photon energy to the electron mass ratio can be organized. This tutorial illustrates useful techniques of loop calculations: averaging over loop momentum direction, loop momentum integration, and various regularizations. We hope it will be helpful for other similar loop calculations.

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References

[1] P. A. M. Dirac, Proc. Roy. Soc. Lond. A126, 360 (1930).
[2] O. Halpern, Phys. Rev. 44, 855 (1933).
[3] H. Euler and B. Kockel, Naturwissenschaften 23, 246 (1935).
[4] H. Euler, Ann. Physik 26, 398 (1936).
[5] A. Akhieser, L. Landau, and I. Pomeranchnook, Nature 138, 206 (1936).
[6] A. Achieser, Phys. Zeit. Sowjetunion 11, 263 (1937).
[7] R. Karplus and M. Neuman, Phys. Rev. 80, 380 (1950).
[8] R. Karplus and M. Neuman, Phys. Rev. 83, 776 (1951).
[9] W. Dittrich and H. Gies, Springer Tracts Mod. Phys. 166, 1 (2000).
[10] V. Costantini, B. De Tolls, and G. Pistoni, Nuovo Cim. A2, 733 (1971).
[11] L. C. Martin, C. Schubert, and V. M. Villanueva Sandoval, Nucl. Phys. B668, 335 (2003), hep-th/0301022.
[12] N. Kanda, Light-Light Scattering, arXiv:1106.0592, 2011.
[13] T. Fujita and N. Kanda, A Proposal to Measure Photon-Photon Scattering, arXiv:1106.0465, 2011.
[14] D. Bernard, Comment on: A Proposal to Measure Photon-Photon Scattering, arXiv:1106.0610, 2011.
[15] J. Schwinger, Particles, sources and fields, vol. II (Addison-Wesley, Redwood City, CA, 1973).
[16] R. Jackiw, Int. J. Mod. Phys. B14, 2011 (2000), hep-th/9903044.
[17] R. Gastmans, S. L. Wu, and T. T. Wu, Higgs Decay $H \rightarrow \gamma\gamma$ through a $W$ Loop: Difficulty with Dimensional Regularization, arXiv:1108.5322, 2011.
[18] R. Gastmans, S. L. Wu, and T. T. Wu, Higgs Decay into Two Photons, Revisited, arXiv:1108.5872, 2011.
[19] M. Shifman, A. Vainshtein, M. B. Voloshin, and V. Zakharov, Higgs Decay into Two Photons through the W-boson Loop: No Decoupling in the $m_W \rightarrow 0$ Limit, arXiv:1109.1785, 2011.
[20] D. Huang, Y. Tang, and Y.-L. Wu, Note on Higgs Decay into Two Photons $H \rightarrow \gamma\gamma$, arXiv:1109.4846, 2011.
[21] W. J. Marciano, C. Zhang, and S. Willenbrock, Higgs Decay to Two Photons, arXiv:1109.5304, 2011.
[22] F. Jegerlehner, Comment on $H \rightarrow \gamma\gamma$ and the role of the decoupling theorem and the equivalence theorem, arXiv:1110.0869, 2011.
[23] H.-S. Shao, Y.-J. Zhang, and K.-T. Chao, Higgs Decay into Two Photons and Reduction Schemes in Cutoff Regularization, arXiv:1110.6925, 2011.
[24] T. Appelquist and J. Carazzone, Phys. Rev. D11, 2856 (1975).
[25] W. H. Furry, Phys. Rev. 51, 125 (1937).
[26] V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, Relativistic Quantum Theory (Pergamon Press, Oxford, 1982).
[27] C. Itzykson and J.-B. Zuber, Quantum Field Theory (McGraw-Hill, Singapore, 1988).
[28] A. I. Akhiezer and V. B. Berestetskii, Quantum Electrodynamics (Interscience Publ., New York, 1965).