Truncation of lattice fractional quantum Hall Hamiltonians derived from conformal field theory

Dillip K. Nandy,1 N. S. Srivatsa,2 and Anne E. B. Nielsen2

1Department of Physics and Astronomy, Aarhus University, Ny Munkegade 120, DK-8000 Aarhus C
2Max-Planck-Institut für Physik komplexer Systeme, D-01187 Dresden, Germany

Conformal field theory has recently been applied to derive few-body Hamiltonians whose ground states are lattice versions of fractional quantum Hall states. The exact lattice models involve interactions over long distances, which is difficult to realize in experiments. It seems, however, that such long-range interactions should not be necessary, as the correlations decay exponentially in the bulk. This poses the question, whether the Hamiltonians can be truncated to contain only local interactions without changing the physics of the ground state. Previous studies have in a couple of cases with particularly much symmetry obtained such local Hamiltonians by keeping only a few local terms and numerically optimizing the coefficients. Here, we investigate a different strategy to construct truncated Hamiltonians, which does not rely on optimization, and which can be applied independent of the choice of lattice. We test the approach on two models with bosonic Laughlin-like ground states with filling factor 1/2 and 1/4, respectively. We first investigate how the coupling strengths in the exact Hamiltonian depend on distance, and then we study the truncated models. For the model at 1/2 filling, we find that the truncated model has an approximate twofold ground state degeneracy on the torus, and the overlap per site between these states and the states constructed from conformal field theory is higher than 0.98 for all the studied lattices. For the model at 1/4 filling, the overlap is again high, but some of the states with high overlap are found higher up in the spectrum. We expect that this is due to finite size effects.

I. INTRODUCTION

The fractional quantum Hall (FQH) effect is the precursor to a wide variety of systems that exhibit topological order and showed the limitations of Landau’s theory of symmetry breaking. Among several attempts to comprehend the phenomenon, Laughlin’s intuitive ansatz\textsuperscript{1} for the wavefunction successfully captured the physics to a great extent. In addition to studying the FQH effect in continuum systems, it is also interesting to realize FQH physics on lattices.\textsuperscript{2,3} In lattices, the physical magnetic field can be replaced by an artificial magnetic field, and a larger gap in the spectrum can be achieved. Another strong motivation for studying lattices is the interest in realizing FQH physics in ultracold atoms in optical lattices, which would pave the way for very detailed experimental investigations of the effect. In addition, lattice systems are known to host qualitatively new phases of matter.\textsuperscript{4–15}

One route to obtain FQH physics in lattices is to engineer flat bands that bear nonzero Chern numbers. The bands mimic Landau levels, and FQH like states can be achieved by adding interactions.\textsuperscript{2,16} Another route is to construct lattice versions of FQH trial wavefunctions\textsuperscript{17} and derive parent Hamiltonians for these states.\textsuperscript{18–20} The Hamiltonians obtained this way can be different from the fractional Chern insulator Hamiltonians.

It has turned out that a number of analytical FQH trial wavefunctions can be expressed in terms of conformal field theory (CFT) correlation functions, and a closely related construction can be used for lattice states. Using null fields in the CFT provides a route to construct parent Hamiltonians for the lattice states.\textsuperscript{21} The CFT approach to derive exact lattice Hamiltonians begins by finding a set of operators $\Lambda_i$ which annihilate the lattice FQH state $|\Psi_{\text{FQH}}\rangle$, i.e.

$$\Lambda_i |\Psi_{\text{FQH}}\rangle = 0. \quad (1)$$

The Hamiltonian, whose ground state is $|\Psi_{\text{FQH}}\rangle$, is then given by the following positive semi-definite operator

$$H^{\text{Exact}} = \sum_i \Lambda_i^\dagger \Lambda_i. \quad (2)$$

Typically, this Hamiltonian consists of few-body interactions and is nonlocal.

There are more reasons why it would be desirable to have a local Hamiltonian rather than a nonlocal Hamiltonian. First, long-range interactions are difficult to achieve in experiments. Second, it would be natural for the terms of a local Hamiltonian to only depend on the local structure of the lattice and not on all sites in the whole lattice, which also naturally leads to a well-defined thermodynamic limit. This simplifies the description and the interpretation. Third, the local Hamiltonians are easier to work with numerically, e.g. when studying the systems with exact diagonalization or tensor networks.\textsuperscript{22}

The FQH states have correlations that decay exponentially with distance in the bulk. This suggests that it should, in fact, not be necessary to have long-range interactions in the Hamiltonians. In previous work\textsuperscript{19,20} local Hamiltonians for a bosonic Laughlin state and a bosonic Moore-Read state with SU(2) symmetry were achieved using the following procedure. First, all terms except a selection of local terms are removed from the Hamiltonian. The coefficients of the remaining terms are then adjusted so that all terms that only differ by a lattice translation have the same strengths. Finally, the relative strengths of terms that do not only differ by a lattice
translation are determined by a numerical optimization of the overlap between the CFT states and the ground states of the truncated models. The latter step involves exact diagonalization and is done for small system sizes. This way of truncating the Hamiltonian is easier for models that have SU(2) symmetry, since the symmetry reduces the number of possible terms in the Hamiltonian and hence also the number of coefficients that need to be optimized. In models that do not possess such symmetries, however, many more terms in the Hamiltonian are possible, and this makes it difficult to find a suitable choice of parameters, which gives a high overlap per site independent of the system size. In this article, we investigate a different procedure to truncate Hamiltonians derived from CFT. The procedure is more general and can be applied to Hamiltonians with or without symmetries on all lattice geometries. The idea is to truncate the operator $\Lambda_i$ in the first place and then proceed to construct the Hamiltonian as in Eq. (2). There are more advantages of this approach. The $\Lambda_i$ operator is simpler to truncate than the Hamiltonian, since it contains fewer terms, and since the coefficients of the terms depend only on the relative positions of the involved sites and not on the rest of the lattice. This means that the local terms of the resulting Hamiltonians are independent of the size of the lattice used to compute them. Another advantage is that it is clear how to obtain Hamiltonians with either periodic or open boundary conditions as desired. Finally, there is no optimization involved, and once the lattice has been chosen, the only input to the procedure is the truncation radius for the $\Lambda_i$ operator.

We test the proposed truncation procedure numerically for two types of bosonic Laughlin states, one with SU(2) symmetry and one without. We do this by computing the overlap between the analytical states and the ground states of the truncated Hamiltonians. We do computations on square and triangular lattices with periodic boundary conditions, and in all considered cases we find the overlap per site to be higher than 0.98 for the lowest energy state. The considered models should have, respectively, a two and four fold degeneracy on the torus. For the former case, we indeed find that the two lowest energy states are separated by a gap to the first excited state. For the latter case, we find that some of the states with high overlap are higher up in the spectrum. We expect that this is due to finite size effects, which may be significant on the modest lattice size studied.

The paper is structured as follows. In Sec. II we introduce the exact models that we use for testing the truncation approach. We also show that truncating the Hamiltonian directly does not give an unequivocal result. In Sec. III we derive the local models. In Sec. IV we report our numerical results for the overlap between the analytical states and eigenstates of the truncated models. We also investigate the ground state degeneracies. Section V concludes the paper, and the appendix provides expressions for the exact wavefunctions on the torus.

II. EXACT MODEL

A. Exact Wavefunction

Let us first introduce the exact form of the considered wavefunctions on a lattice on the plane (i.e. with open boundary conditions) before constructing the exact Hamiltonians. The lattice Laughlin state with one particle per $q$ flux lines, where $q$ is a positive integer, can be expressed as

$$|\Psi^P(n_1,n_2,\ldots,n_N)\rangle = \Psi(n_1,n_2,\ldots,n_N)|n_1,n_2,\ldots,n_N\rangle$$

with

$$\Psi(n_1,n_2,\ldots,n_N) \propto \delta_n \chi_n \prod_{i<j} (z_i - z_j)^{q(n_i - n_j)}.$$

The prefactor $\delta_n$ fixes the number of particles in the state to $N/q$ and is defined by

$$\delta_n = \begin{cases} 1 & \text{for } \sum_i n_i = N/q, \\ 0 & \text{otherwise}. \end{cases}$$

The other prefactor $\chi_n = (-1)^{\sum_i (j-1)n_j}$ is a phase factor, and the complex number $z_i$ denotes the position of the $i^{th}$ site in the two-dimensional plane. $n_i \in \{0,1\}$ denotes the particle occupation number at the $i^{th}$ site. The amplitude $\Psi(n_1,n_2,\ldots,n_N)$ can also be expressed as a CFT correlator.

B. Exact Hamiltonian

In the CFT construction of the Hamiltonian, null fields are used to derive a family of operators that annihilate the considered state. These operators are of increasing complexity, and for the lattice Laughlin states the three simplest are

$$\Upsilon = \sum_{j=1}^{N} d_j,$$

$$\Omega = \sum_{j=1}^{N} (qn_j - 1),$$

$$\Lambda_i = \sum_{j \neq i} w_{ij}[d_j - d_i(qn_j - 1)].$$

Here, $d_j$ is the hardcore bosonic (fermionic) particle annihilation operator at site $j$ for $q$ even (odd), and $w_{ij} = \frac{1}{z_i - z_j}$. If we construct the Hamiltonian from $\Upsilon$ or $\Omega$, we do not obtain a unique ground state. We therefore construct the Hamiltonian from $\Lambda_i$ using Eq. (2).
After multiplying out all the factors, we have

\[
H^{\text{Exact}} = \sum_{i \neq j} C_1(i, j) d_i d_j + \sum_{i \neq j} C_2(i, j) n_i n_j + \sum_{i \neq j \neq k} C_3(i, j, k) d_i d_j n_k + \sum_{i \neq j \neq k} C_4(i, j, k) n_i n_j n_k + \sum_i C_5(i) n_i, \tag{8}
\]

where the coefficients are given by

\[
C_1(i, j) = 2 w_{ij}^* w_{ij} + \sum_{k(\neq i, \neq j)} (w_{ki}^* w_{kj} + w_{ji}^* w_{jk} + w_{ik}^* w_{ij}),
\]

\[
C_2(i, j) = (q^2 - 2q) w_{ij}^* w_{ij} - q \sum_{k(\neq i, \neq j)} (w_{ij}^* w_{ik} + w_{ij}^* w_{jk}),
\]

\[
C_3(i, j, k) = -q (w_{ij}^* w_{jk} + w_{ik}^* w_{ij}),
\]

\[
C_4(i, j, k) = q^2 w_{ik}^* w_{ij},
\]

\[
C_5(i) = \sum_{j(\neq i)} (w_{ij}^* w_{ji} + w_{ij}^* w_{ij}) + \sum_{j, k(\neq i)} w_{ij}^* w_{ik}.
\]

Note that the Hamiltonian conserves the number of particles, and we shall assume throughout that the number of particles in the system is fixed to \(N/q\). The state \(|\psi\rangle\) is the ground state of this Hamiltonian for any choice of lattice in the plane (i.e. for any choice of \(z_j\)).

C. Behavior of the coefficients of the Hamiltonian

There are five different types of operators in the Hamiltonian: a one-body term, two two-body terms, and two three-body terms with different coupling coefficients. The one-body term is local, since each term acts only on a single site. The coefficients \(C_3\) and \(C_4\) of the three-body terms do not involve any summation over indices. They hence depend only on the relative positions of the involved lattice sites, and their behavior is the same everywhere on the lattice. The \(C_3\) coefficient has two terms both of which have power law decay, while the \(C_4\) coefficient has one term which has a power law decay with distance. It is also seen that the decaying behavior of the three-body coefficients is independent of the system size.

The coefficients of the two-body terms do, however, have a more complicated behavior. \(C_1\) and \(C_2\) both contain two terms. The first decays as the square of the distance between the sites, and the second is a sum. Due to these sums, the coefficients depend on the positions of all the lattice sites. We demonstrate the behavior explicitly for the case of a square lattice in Fig. 1. In this figure, we plot only the term containing the sum, and for \(C_2\) we plot the sum of the contributions from \(n_i n_j\) and \(n_i n_k\), since these two operators are the same. We consider two cases. In the first case, we fix \(i\) to be an edge lattice point and then let \(j\) run along the edge. In the second, we fix \(i\) to be a lattice point in the middle of the lattice and then let \(j\) run towards the edge. The two cases are illustrated in Fig. 1(a).

From the results in Fig. 1(b) and 1(c), we make the fol-
The exact model is defined for open boundary conditions. This means that all the $\Lambda_i^{(L)}$ operators in the bulk have the same form, but the $\Lambda_i^{(L)}$ operators on the edge are different, because there are fewer neighbors. It is, however, straightforward to modify the truncated model to have periodic boundary conditions. We just need to use the bulk form of $\Lambda_i^{(L)}$ for all lattice sites and let the lattice wrap around the torus. In the following, we only consider the truncated model with periodic boundary conditions. We do this to reduce the finite size effects on the relatively small lattices we can investigate with exact diagonalization. For the numerical results presented below, we consider square and triangular lattices, and we take the local region to consist of the nearest neighbor sites (i.e. $s = 1$).

IV. OVERLAP OF THE EIGENSTATES

In order to quantify how well our constructed local models reproduce the exact states, we would like to compare eigenstates of $H^{\text{Local}}$ computed from exact diagonalization to the exact states. For this we need the exact states on the torus $|\Psi_{\text{Exact}}^{T,L}\rangle$ given in Appendix A. As a result of the topological order, there are $q$ states in total. For the $j^{\text{th}}$ energy eigenstate $|E_j\rangle$ of $H^{\text{Local}}$, we therefore define the overlap with the exact state to be

$$\Delta = \sum_{l=0}^{q-1} |\langle E_j | \tilde{\Psi}_{\text{Exact}}^{T,l} \rangle|^2,$$

where $|\tilde{\Psi}_{\text{Exact}}^{T,l}\rangle$ are the states in (11) after Gram-Schmidt orthonormalization. Due to the exponential growth of the Hilbert space dimension with system size, we expect the overlap to show an exponential decay with system size. This motivates us to also consider the overlap per site $\Delta^{1/N}$, which is expected to be roughly independent of system size for large enough systems.

Table III and IV show the overlaps for the lowest energy state $|E_1\rangle$ for the square and the triangular lattice, respectively. We consider the cases of half ($q = 2$) and quarter ($q = 4$) filling. All the computed overlaps per site are larger than 0.98, which shows that the truncated Hamiltonian indeed produces almost the same lowest energy state as the exact Hamiltonian. The overlaps are slightly better for $q = 2$ than for $q = 4$. For the square lattice, the overlap per site is largest for the $4 \times 4$ lattice, which has $L_x = L_y$.

So far we have seen that the lowest energy state has a high overlap with the exact state, which is topological. This suggests that the system remains in the same topological phase after truncating the Hamiltonian. We could now insert fluxes or create anyons, move them around cycles of the torus, and annihilate them again, and in this way obtain another $q-1$ states, which have the same local properties as the lowest energy state. Since the Hamiltonian consists of local terms, the $q$ states would all have

![Diagram](image-url)
shows the exact state on the torus and the lowest energy eigenstate of $H^{\text{local}}$ for the square lattice with $L_x \times L_y$ unit cells.

| $L_x \times L_y$ | $q = 2$ | $q = 4$ |
|-----------------|---------|---------|
| $\Delta$ | $\Delta^{1/N}$ | $\Delta$ | $\Delta^{1/N}$ |
| 3 × 4 | 0.8679 | 0.9883 | 0.8317 | 0.9848 |
| 4 × 4 | 0.9692 | 0.9980 | 0.9431 | 0.9963 |
| 4 × 5 | 0.9239 | 0.9961 | 0.9122 | 0.9954 |
| 4 × 6 | 0.9226 | 0.9966 | 0.7657 | 0.9889 |
| 5 × 6 | 0.9164 | 0.9971 | | |

TABLE II. Overlap $\Delta$ and overlap per site $\Delta^{1/N}$ between the exact state on the torus and the lowest energy eigenstate of $H^{\text{local}}$ for the triangular lattice with $L_x \times L_y$ unit cells.

| $L_x \times L_y$ | $q = 2$ | $q = 4$ |
|-----------------|---------|---------|
| $\Delta$ | $\Delta^{1/N}$ | $\Delta$ | $\Delta^{1/N}$ |
| 3 × 4 | 0.8400 | 0.9856 | 0.9317 | 0.9941 |
| 4 × 4 | 0.9507 | 0.9968 | 0.8710 | 0.9913 |
| 4 × 5 | 0.9098 | 0.9953 | 0.7512 | 0.9857 |
| 4 × 6 | 0.8913 | 0.9952 | 0.6827 | 0.9842 |
| 5 × 6 | 0.8210 | 0.9934 | | |

TABLE III. Overlap per site $\Delta^{1/N}$ between the exact state on the torus and selected eigenstates $|E_j\rangle$ of $H^{\text{local}}$ on the square and triangular lattices with 4 × 4 unit cells. We only consider eigenstates, which have the same momentum quantum numbers as the exact states, and we list the overlap per site for the $q$ lowest of these.

| | | $q = 2$ | $q = 4$ |
|---|---|---|---|
| | Square | Triangular | Square | Triangular |
| 0.9980 (|$E_1\rangle$) | 0.9968 (|$E_1\rangle$) | 0.9963 (|$E_1\rangle$) | 0.9913 (|$E_1\rangle$) |
| 0.9938 (|$E_2\rangle$) | 0.9935 (|$E_2\rangle$) | 0.9944 (|$E_4\rangle$) | 0.9865 (|$E_2\rangle$) |
| | | 0.9857 (|$E_{10}\rangle$) | 0.9745 (|$E_6\rangle$) | 0.9705 (|$E_{10}\rangle$) | 0.9624 (|$E_{12}\rangle$) |

FIG. 3. (Color online) The first 40 energy eigenvalues computed for $q = 2$ (top row) and $q = 4$ (bottom row) on a square (left column) and triangular (right column) lattice with 4 × 4 unit cells. Only the states that have the same momentum quantum numbers as the analytical states can have a nonzero overlap, and the red points mark the $q$ lowest of these states. For the $q = 4$ case, the fourth state with nonzero overlap is outside the displayed range.

the same energy in the thermodynamic limit. For the small system sizes that we are considering, however, the global properties of the states can have a significant influence on the energy, and the states may not be exactly degenerate. In fact, the study in [13] suggests that for systems with up to 30 sites there may not be a clear ground state degeneracy.

Let us first investigate the model with $q = 2$ on a lattice with 4 × 4 unit cells. The top row in Fig. 3 shows the low-energy part of the spectrum of the truncated Hamiltonian. For both the square and the triangular lattice, the two lowest energy states are separated by a gap from the rest of the spectrum. The overlap per site of these two states with the analytical states is higher than 0.993 in all cases as detailed in Tab. III. The ground state degeneracy is hence already visible on this small lattice, although not perfect.

We next consider the model with $q = 4$. For $q = 4$, the 5 × 6 lattice is not valid, since the number of lattice sites must be divisible by 4. To avoid that the lattice gets too elongated, we therefore consider the lattice with 4 × 4 unit cells. The spectrum is given in Fig. 3 and the overlaps are given in Tab. III. This time the states with high overlap are not the lowest energy states. The exact states all have the same momenta, and if we select the eigenstates of the local Hamiltonian, which have the same momenta as the exact states, then the states with high overlap are the lowest energy states. There is not a
clear gap between the $4^{th}$ and the $5^{th}$ state in this case. As argued above, however, the $q$-fold degeneracy must be present in the thermodynamic limit, unless the system is not in the expected topological phase.

V. CONCLUSION

We have investigated a general procedure to truncate lattice Hamiltonians derived from CFT null fields. Each of the null fields leads to an operator $\Lambda_i$ that annihilates the state, and the approach is to truncate this operator to a local form instead of truncating the Hamiltonian itself.

Truncating the $\Lambda_i$ operator has a number of advantages compared to truncating the Hamiltonian directly. First, the terms in the operator depend only on the relative coordinates of the involved sites, and not on the rest of the lattice. The local terms are hence unaltered, when we approach the thermodynamic limit or in other ways modify the rest of the lattice. On the contrary, some of the terms in the Hamiltonian depend on the whole lattice, and the local terms resulting from truncating the Hamiltonian directly will depend on the choice and size of the whole lattice. Second, it is clear how to construct models with open or periodic boundary conditions, unless one manually changes the local terms and optimizes them as desired, while truncating the Hamiltonian directly only gives models with open boundary conditions, unless one manually changes the local terms and optimizes the coupling strengths numerically. The fact that numerical optimization is not needed means that we can construct the local Hamiltonian even if the local regions are not small compared to the lattice sizes that can be investigated with exact diagonalization.

We have applied the truncation procedure to two Hamiltonians with $q = 2$ and $q = 4$ Laughlin type ground states on square and triangular lattices on the torus. In all cases, we find that the lowest energy eigenstate of the truncated model has a high overlap per site with the ground state of the original model. This suggests that the truncated model is in the same topological phase as the exact model. For the $q = 2$ model, we find an approximate two-fold ground state degeneracy, while for the $q = 4$ model, some of the states with high overlap are higher up in the spectrum. We expect that this is due to finite size effects.

An SU(2) invariant local Hamiltonian was constructed for the $q = 2$ state on the square lattice and the kagome lattice in [18] by truncating the Hamiltonian directly and numerically optimizing the coefficients. The same procedure does, however, fail to produce a simple Hamiltonian on the triangular lattice. In contrast, the approach of truncating the $\Lambda_i$ operator gives good overlap values on both the square and the triangular lattice without optimization.

Constructing parent Hamiltonians using tools from CFT provides an alternative route to obtain FQH physics in lattice systems, compared to fractional Chern insulators. One advantage is that the Hamiltonians by construction have states with a particular topology as ground states. Local Hamiltonians are easier to work with both experimentally and theoretically, and it is therefore important to find suitable truncation schemes. In the future, it would be interesting to test how large the local regions need to be to give good overlaps for other FQH models derived from CFT.

ACKNOWLEDGMENTS

This work has in part been supported by the Villum Foundation. DKN would like to thank the Max Planck Institute for the Physics of Complex Systems for hospitality during visits to the institute.

Appendix A: Exact state on the torus

The wavefunction $|\Psi^{T,l}\rangle$ can be expressed in terms of a CFT correlation function, and by evaluating this correlation function on the torus, it is possible to also obtain the states on the torus $|\Psi^{T,l}\rangle$. They have the following form

$$|\Psi^{T,l}\rangle = \sum_{n_1, n_2, \ldots, n_N} \Psi^{T,l}_{\text{Exact}}(n_1, n_2, \ldots, n_N) |n_1, n_2, \ldots, n_N\rangle \quad (A1)$$

with

$$\Psi^{T,l}_{\text{Exact}}(n_1, n_2, \ldots, n_N) \propto \delta_n \chi_n \Theta \left[ \frac{l}{q} + a_A \right] b_A \left( \sum_i N_i \right) q^n \prod_{i,j} \left( \frac{1}{2} - \frac{1}{2} \xi_i - \frac{1}{2} \xi_j \right)^{N_i - N_j} \cdot (A2)$$

Here, $\delta_n$ fixes the number of particles in the state to $N/q$, and $\chi_n$ is the same phase factor as for the states in the plane. There are $q$ states labeled by $l$, where $l \in \{0, 1, \ldots, q - 1\}$. The parameters $a_A$ and $b_A$ in general depend on the number of lattice sites and the value of $q$, but when $N$ and $q$ are both even, as for all the cases studied here, $a_A = b_A = 0$. For the square lattice $\tau = i L_y/L_x$, and for the triangular lattice $\tau = e^{\pi i/3} L_y/L_x$. Here, $L_x$ is the number of unit cells along the real axis, and $L_y$ is the number of unit cells in the $e^{i\pi/3}$ direction (the $e^{i\pi/3}$ direction) for the square (triangular) lattice. The rescaled coordinates are given by $\xi_i = z_i/L_x$, where we assume that the lattice constant is set to unity. The Riemann theta function is defined as

$$\Theta \left[ \frac{a}{b} \right] (\xi, \tau) = \sum_{n \in \mathbb{Z}} e^{i \pi \pi (n+a)^2 + 2i \pi (n+a)(\xi+b)} \cdot (A3)$$

The states in (A1) are not necessarily orthogonal, and we use Gram-Schmidt orthonormalization to numerically
obtain an orthonormal set of states spanning the same space.

* On leave from Department of Physics and Astronomy, Aarhus University, Ny Munkegade 120, DK-8000 Aarhus C

1 R. B. Laughlin, Phys. Rev. Lett. 50, 1395 (1983).
2 L. Hormozi, G. Möller, and S. H. Simon, Phys. Rev. Lett. 108, 256809 (2012).
3 M. Hafezi, A. S. Sørensen, E. Demler, and M. D. Lukin, Phys. Rev. A 76, 023613 (2007).
4 N. Goldman, A. Kubasiak, A. Bermudez, P. Gaspard, M. Lewenstein, and M. A. Martin-Delgado, Phys. Rev. Lett. 103, 035301 (2009).
5 R. N. Palmer and D. Jaksch, Phys. Rev. Lett. 96, 180407 (2006).
6 E. J. Bergholtz and Z. Liu, Int. J. Mod. Phys. B 27, 1330017 (2013).
7 S. Yang, Z. C. Gu, K. Sun, and S. D. Sarma, Phys. Rev. B 86, 241112(R) (2012).
8 T. Neupert, L. Santos, C. Chamon, C. Mudry, Phys. Rev. Lett. 106, 236804 (2011).
9 D. N. Sheng, Z. C. Gu, K. Sun, L. Sheng, Nat. Comm. 2, 389 (2011).
10 Y. F. Wang, Z. C. Gu, C. D. Gong, D. N. Sheng, Phys. Rev. Lett. 107, 146803 (2011).
11 V. Kalmeyer and R. B. Laughlin, Phys. Rev. Lett. 59, 2095 (1987).
12 D. F. Schroeter, E. Kapit, R. Thomale, and M. Greiter, Phys. Rev. Lett. 99, 097202 (2007).
13 R. Thomale, E. Kapit, D. F. Schroeter, and M. Greiter, Phys. Rev. B 80, 104406 (2009).
14 E. Kapit and E. Mueller, Phys. Rev. Lett. 105, 215303 (2010)
15 A. E. B. Nielsen, J. I. Cirac, and G. Sierra, Phys. Rev. Lett. 108, 257206 (2012).
16 G. Moore and N. Read, Nuclear Physics B360, 362 (1991).
17 J.-Y. Chen, L. Vanderstraeten, S. Capponi, and D. Poilblanc, Phys. Rev. B 98, 184409 (2018).
18 A. E. B. Nielsen, G. Sierra, and J. I. Cirac, Nat. Commun. 4, 2864 (2013).
19 I. Glasser, J. I. Cirac, G. Sierra, and A. E. B. Nielsen, New J. Phys. 17, 082001 (2015).
20 H.-H. Tu, A. E. B. Nielsen, J. I. Cirac, and G. Sierra, New J. Phys. 16, 033025 (2014).
21 A. Deshpande and A. E. B. Nielsen, J. Stat. Mech. 2016, 013102 (2016).