ON THE BEILINSON-HODGE CONJECTURE FOR $H^2$ AND RATIONAL VARIETIES

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ABSTRACT. The Beilinson-Hodge conjecture asserts the surjectivity of the cycle map

$$\bar{c}_n : H^n_M(X, \mathbb{Q}(n)) \to \text{Hom}_{\text{MHS}}(\mathbb{Q}(-n), H^n(X, \mathbb{Q}))$$

for all integers $n \geq 1$ and every smooth complex algebraic variety $X$. For $n = 2$, we prove the conjecture if $X$ is rational.

CONTENTS

Introduction \hfill 1
Acknowledgements \hfill 2
1. Cycle class to absolute Hodge cohomology \hfill 2
1.1. Higher Chow groups and absolute Hodge cohomology \hfill 2
1.2. Cycle class map \hfill 3
1.3. Beilinson-Hodge conjecture \hfill 3
2. Beilinson-Hodge conjectures for the generic point \hfill 4
2.1. Coniveau spectral sequences \hfill 4
2.2. $E_1$ complexes of the coniveau spectral sequence \hfill 8
2.3. An exact sequence for projective varieties with vanishing $H^1$ \hfill 12
2.4. Main theorem \hfill 12
2.5. An exact sequence for projective varieties \hfill 13
References \hfill 14

INTRODUCTION

The Beilinson-Hodge conjecture ($\text{BH}(X, n)$) asserts the surjectivity of the cycle map

$$\bar{c}_{n,n} : H^n_M(X, \mathbb{Q}(n)) \to \text{Hom}_{\text{MHS}}(\mathbb{Q}(-n), H^n(X, \mathbb{Q}))$$

for all integers $n \geq 1$ and every smooth complex algebraic variety $X$. Informally speaking, the conjecture means that every complex holomorphic $n$-form with logarithmic poles along the boundary divisor of every compactification of $X$ and rational cohomology class comes from a meromorphic form of the shape

$$\frac{1}{(2\pi i)^n} \sum_j m_j \cdot \frac{df_{j1}}{f_{j1}} \wedge \cdots \wedge \frac{df_{jn}}{f_{jn}}$$

with $f_{jk} \in \mathbb{C}(X)^*$ and $m_j \in \mathbb{Q}$.

This work has been supported by the SFB/TR 45 “Periods, moduli spaces and arithmetic of algebraic varieties”.

1
It is well-known that the conjecture holds for $n = 1$ (see, for example, [EV88, Proposition 2.12]). For $n \geq 2$, Asakura and Saito provided evidence for the conjecture by studying the Noether-Lefschetz locus of Beilinson-Hodge cycles (see [AS06], [AS07], [AS08]). By work of Arapura and Kumar, BH($X, n$) is known to hold for every $n$ provided that $X$ is a semi-abelian variety or a product of curves [AK09].

In our paper we consider only the case $n = 2$, and make two observations. First, for a smooth and projective variety $X$ the Beilinson-Hodge conjecture BH($\eta, 2$) for the generic point $\eta$ of $X$ is equivalent to the injectivity of the cycle map

$$H^3_M(X, \mathbb{Q}(2)) \rightarrow H^1_M(X, \mathbb{Q}(1)) \cdot H^2_M(X, \mathbb{Q}(1))$$

to absolute Hodge cohomology (Proposition 2.5.1). The left hand side is called the group of indecomposable cycles and has been studied by Müller-Stach [MS97]. In general, indecomposable cycles exist, but the image via the cycle class map is a countable group. By BH($\eta, 2$) we mean the surjectivity of the cycle class map

$$H^2_M(C(X), \mathbb{Q}(2)) \rightarrow \lim_{U \subset X} \text{Hom}_{\mathcal{MHS}}(\mathbb{Q}(-2), H^2(U, \mathbb{Q})),$$

where $U$ runs over all open subsets of $X$, and $C(X)$ denotes the function field.

The second observation is that if $X$ satisfies $H^1(X, \mathbb{C}) = 0$ then BH($U, 2$) for all open sets $U \subset X$ is equivalent to BH($\eta, 2$) (Proposition 2.5.3). The statement makes perfect sense when $H^1(X, \mathbb{C}) \neq 0$, but we can prove it only in the case $H^1(X, \mathbb{C}) = 0$.

Combining the two observations we obtain the main theorem of the paper.

**Theorem** (cf. Theorem 2.4.1). Let $X$ be smooth and connected. Let $\bar{X}$ be a smooth compactification of $X$. We denote by $\text{CH}(\bar{X}) \otimes \mathbb{Q}$ the Chow group of zero cycles on $\bar{X}$. If $\text{deg} : \text{CH}(\bar{X}) \otimes \mathbb{Q} \rightarrow \mathbb{Q}$ is an isomorphism then BH($X, 2$) holds.

For the proof we use a theorem of Bloch and Srinivas [BS83] which states that the indecomposable cycles vanish whenever the assumptions of our theorem are satisfied.

**Acknowledgements.** It is a pleasure to thank Hélène Esnault for her strong encouragement. I thank Donu Arapura, Florian Ivorra and Manish Kumar for useful discussions.

1. **Cycle class to absolute Hodge cohomology**

1.1. **Higher Chow groups and absolute Hodge cohomology.** Let $X$ be a smooth algebraic variety over the complex numbers.

Absolute Hodge cohomology was introduced by Beilinson ([Beî86], cf. [Jan88, §2]). Beilinson constructs for every complex algebraic variety $X$ an object $\underline{RT}(X, \mathbb{Q})$ in the derived category of mixed Hodge structures $D^b(MHS)$ such that

$$H^i(\underline{RT}(X, \mathbb{Q})) = H^i(X, \mathbb{Q}) \quad \text{for all } i.$$ 

Absolute Hodge cohomology $H^*_A(X, \mathbb{Q}(\bullet))$ is defined as follows:

$$H^*_{A}(X, \mathbb{Q}(p)) = \text{Hom}_{D^b(MHS)}(\mathbb{Q}, \underline{RT}(X, \mathbb{Q})(p)[q]),$$

for all $p, q$. 
The natural spectral sequence
\[ E_2^{ij} = \text{Ext}^i_{MHS}(\mathbb{Q}, H^j(X, \mathbb{Q})(p)) \Rightarrow H^{i+j}_M(X, \mathbb{Q}(p)), \]
and vanishing of \( \text{Ext}^i \) for \( i > 1 \), induces short exact sequences
\[ (1.1.1) \quad 0 \to \text{Ext}^1(\mathbb{Q}, H^{q-1}(X, \mathbb{Q})(p)) \to H^q_M(X, \mathbb{Q}(p)) \to \text{Hom}(\mathbb{Q}, H^q(X, \mathbb{Q})(p)) \to 0. \]
Note that Hom and Ext are taken in the category of mixed Hodge structures.

If \( X \) is smooth and proper then we have a comparison isomorphism with Deligne cohomology
\[ H^q_M(X, \mathbb{Q}(p)) \cong H^q_D(X, \mathbb{Q}(p)), \]
provided that \( q \leq 2p \) [Jan88, §2.7].

1.2. Cycle class map. Let \( DM_{gm, \mathbb{Q}} \) be Voevodsky’s triangulated category of motivic complexes with rational coefficients over \( \mathbb{C} \) ([Voe00], [MVW06]). Denoting by \( Sm/\mathbb{C} \) the category of smooth complex algebraic varieties, there is a functor
\[ Sm/\mathbb{C} \to DM_{gm, \mathbb{Q}}, \quad X \mapsto M_{gm}(X). \]
Motivic cohomology is defined by
\[ H^q_M(X, \mathbb{Q}(p)) = \text{Hom}_{DM_{gm}}(M_{gm}(X), \mathbb{Q}(p)[q]), \]
for \( X \) smooth and \( p \geq 0, q \in \mathbb{Z} \). There is a comparison isomorphism [Voe02] with Bloch’s higher Chow groups
\[ H^q_M(X, \mathbb{Q}(p)) \cong \text{CH}^p(X, 2p - q) \otimes \mathbb{Q}. \]
By Levine [Lev05] and Huber ([Hub00], [Hub04]) we have realizations
\[ (1.2.1) \quad r_M : DM_{gm} \to D^b(MHS) \]
at disposal, such that \( Rr_M(X, \mathbb{Q}) \) is the dual of \( r_M(M_{gm}(X)) \) and \( r_M(\mathbb{Q}(1)) = \mathbb{Q}(1) \). The realizations are triangulated \( \otimes \)-functors and therefore induce cycle maps
\[ (1.2.2) \quad c_{p, 2p-q} : H^q_M(X, \mathbb{Q}(p)) \to H^q_H(X, \mathbb{Q}(p)) \]
which are compatible with the localization sequence. An explicit construction of a cycle map by using currents is presented in [KL07]. Composition of \( c_{p, 2p-q} \) with the projection from \( (1.1.1) \) yields
\[ (1.2.3) \quad \tilde{c}_{p, 2p-q} : H^q_M(X, \mathbb{Q}(p)) \to \text{Hom}_{MHS}(\mathbb{Q}, H^q(X, \mathbb{Q})(p)). \]

1.3. Beilinson-Hodge conjecture. Let \( X \) be a smooth algebraic variety over \( \mathbb{C} \), and \( n \geq 0 \) an integer.

**Conjecture 1.3.1** (Beilinson-Hodge conjecture). \( BH(X,n) \): The cycle map \( \tilde{c}_{n,n} \) \[ (1.2.3) \] is surjective.

**Remark 1.3.2.** If \( X \) is smooth then
\[ c_{1,1} : H^1_M(X, \mathbb{Q}(1)) \to H^1_H(X, \mathbb{Q}(1)) \]
is an isomorphism [EVSS Proposition 2.12]. In particular, \( BH(X,1) \) holds.
2. Beilinson-Hodge conjectures for the generic point

2.1. Coniveau spectral sequences. The main technical tool of our paper is the coniveau spectral sequence for motivic and absolute Hodge cohomology. The existence and construction of the coniveau spectral is well-known and follows from the yoga of exact couples as in [BO74]. Because we couldn’t provide a reference for the case of absolute Hodge cohomology we will explain the construction in this section.

2.1.1. In the following, ? will stand for $M$ or $H$. For $p \geq 0$ we denote by $X(\cdot, p)$ the set of codimension $p$ points of $X$. For a point $x \in X$ (not necessarily closed) we define

$$H^q_? (x, \mathbb{Q}(p)) := \lim_{\longrightarrow} H^q_? (U, \mathbb{Q}(p)),$$

where $U$ runs over all open neighborhoods of $x$. For all $n \geq 0$, the coniveau spectral sequence reads:

$$E_{1}^{p,q} = \bigoplus_{x \in X(\cdot, p), p \leq n} H^{q-p}_? (x, \mathbb{Q}(n-p)) \Rightarrow H^{p+q}_? (X, \mathbb{Q}(n)).$$

The terms $E_{1}^{p,q}$ with $p > n$ are zero.

2.1.2. In order to construct the coniveau spectral sequence we use the category of finite correspondences $\text{Cor}_C$ [MVW06, Lecture 1]. There is an obvious functor

$$\text{Sm}/\mathbb{C} \rightarrow \text{Cor}_C, \quad X \mapsto [X].$$

We denote by $\mathcal{H}^b(\text{Cor}_C)$ the homotopy category of bounded complexes [Voe00, §2.1]. By construction [Voe00, Definition 2.1.1] there is a triangulated functor

$$\mathcal{H}^b(\text{Cor}_C) \rightarrow D\text{M}_{gm,\mathbb{Q}}.$$

Definition 2.1.3. Let $X$ be smooth and $Y \subset X$ a closed set. We define $c_Y X$ to be the complex

$$[X \setminus Y] \xrightarrow{j} [X] \quad \text{deg}=0$$

in $\mathcal{H}^b(\text{Cor}_C)$. The map $j$ is the open immersion.

Let $Y \subset X$ be a closed subset. For an open subset $U$ of $X$ and a closed subset $Y'$ of $U$ such that $Y \cap U \subset Y'$ we get a morphism of complexes

$$c_{Y'} U \rightarrow c_Y X.$$

Lemma 2.1.4. Let $X$ be smooth. Let $Y_1, Y_2$ be closed subsets of $X$ with $Y_2 \subset Y_1$.

1. The morphisms

$$c_{Y_1 \setminus Y_2} (X \setminus Y_2) \rightarrow c_{Y_1} X \rightarrow c_{Y_2} X \xrightarrow{+1} c_{Y_1 \setminus Y_2} (X \setminus Y_2)[1]$$
induced by \((2.1.3)\) and
\[
\begin{array}{c}
c_{Y_2}X \\ [X]
\end{array}
\xrightarrow{\text{id}}
\begin{array}{c}
\left[ X \setminus Y_2 \right] \\ \left[ X \setminus Y_2 \right]
\end{array}
\xrightarrow{\text{incl}}
\begin{array}{c}
\left[ X \setminus Y_1 \right]
\end{array}
\]
form a distinguished triangle in \(\mathcal{H}^b(\text{Cor}_C)\).

(2) If \(Y'_2 \subset Y'_1\) are closed subsets of \(X\) such that \(Y_i \subset Y'_i\), for \(i = 1, 2\), then the morphisms from \((2.1.3)\) induce a morphism of distinguished triangles
\[
\begin{array}{c}
c_{Y'_1 \setminus Y'_2}(X \setminus Y'_2) \\ c_{Y'_1 \setminus Y'_2}(X \setminus Y'_2)
\end{array}
\xrightarrow{\text{id}}
\begin{array}{c}
c_{Y'_1 \setminus Y'_2}(X \setminus Y'_2) \\ c_{Y'_1 \setminus Y'_2}(X \setminus Y'_2)
\end{array}
\xrightarrow{\text{incl}}
\begin{array}{c}
c_{Y'_1 \setminus Y'_2}(X \setminus Y'_2) \\ c_{Y'_1 \setminus Y'_2}(X \setminus Y'_2)
\end{array}
\]

Proof. For (1). equivalent to 0. There is an obvious isomorphism in \(\mathcal{H}^b(\text{Cor}_C)\) :
\[
\begin{array}{c}
\text{cone}(c_{Y_1 \setminus Y_2}(X \setminus Y_2)[-1]) = c_{Y_1 \setminus Y_2}(X \setminus Y_2), \\
[X]
\end{array}
\]
\[
\begin{array}{c}
\left[ X \setminus Y_2 \right] \\ \left[ X \setminus Y_2 \right]
\end{array}
\xrightarrow{\text{pr}_2}
\begin{array}{c}
\left[ X \setminus Y_2 \right] \\ \left[ X \setminus Y_2 \right]
\end{array}
\xrightarrow{\text{incl}}
\begin{array}{c}
\left[ X \setminus Y_2 \right] \\ \left[ X \setminus Y_2 \right]
\end{array}
\]
rendering commutative the diagram
\[
\begin{array}{c}
c_{Y_2}[X_2][-1] \\ c_{Y_2}[X_2][-1]
\end{array}
\xrightarrow{\text{cone}(c_{Y_1 \setminus Y_2}(X \setminus Y_2)[-1]) = c_{Y_1 \setminus Y_2}(X \setminus Y_2),}
\]
\[
\begin{array}{c}
\left[ X \setminus Y_2 \right] \\ \left[ X \setminus Y_2 \right]
\end{array}
\xrightarrow{\text{incl}, \text{incl}}
\begin{array}{c}
\left[ X \setminus Y_2 \right] \\ \left[ X \setminus Y_2 \right]
\end{array}
\xrightarrow{\text{incl}, \text{incl}}
\begin{array}{c}
\left[ X \setminus Y_2 \right] \\ \left[ X \setminus Y_2 \right]
\end{array}
\]

For (2). Straight-forward. \(\square\)

Definition 2.1.5. Let \(X\) be smooth and \(Y \subset X\) a closed subset. For all \(n \geq 0\) and \(q \in \mathbb{Z}\) we define
\[
H^q_{Y,\mathcal{M}}(X, \mathbb{Q}(n)) := \text{Hom}_{DM_{gm}}(c_Y X, \mathbb{Q}(n)[q])
\]
\[
H^q_{Y,\mathcal{H}}(X, \mathbb{Q}(n)) := \text{Hom}_{D^b(MHS)}(r_Y c_Y X, \mathbb{Q}(n)[q]).
\]
We implicitly used the functor \((2.1.2)\).

From \((2.1.3)\) we obtain a map
\[
H^q_{Y,\mathcal{M}}(X, \mathbb{Q}(n)) \to H^q_{Y,\mathcal{H}}(X, \mathbb{Q}(n))
\]
if \( U \) is an open subset of \( X \) and \( Y \cap U \subset Y' \).

2.1.6. For \( p \geq 0 \) we denote by \( Z^p = Z^p(X) \) the set closed subsets of \( X \) of codimension \( \geq p \), ordered by inclusion. Let \( Z^p/Z^{p+1} \) denote the ordered set of pairs \((Z, Z') \in Z^p \times Z^{p+1}\) such that \( Z \supset Z' \), with the ordering

\[
(Z, Z') \geq (Z_1, Z'_1) \quad \text{if} \quad Z \supset Z_1 \text{ and } Z' \supset Z'_1.
\]

We can form for all \( n \geq 0 \) and \( p \in \mathbb{Z} \):

\[
H^\ast_{Z^p,\gamma}(X, \mathbb{Q}(n)) := \begin{cases} 
\lim_{Z \in Z^p} H^\ast_{Z,\gamma}(X, \mathbb{Q}(n)) & \text{if } p \geq 0, \\
H^\ast_{\gamma}(X, \mathbb{Q}(n)) & \text{if } p \leq 0.
\end{cases}
\]

\[
H^\ast_{Z^p/Z^{p+1},\gamma}(X, \mathbb{Q}(n)) := \begin{cases} 
\lim_{(Z, Z') \in Z^p/Z^{p+1}} H^\ast_{Z \backslash Z',\gamma}(X \backslash Z', \mathbb{Q}(n)) & \text{if } p \geq 0, \\
0 & \text{if } p < 0.
\end{cases}
\]

In view of Lemma 2.1.4 we obtain for every \((Z, Z') \in Z^p/Z^{p+1}\) a long exact sequence

\[
H^\ast_{Z^p,\gamma}(X, \mathbb{Q}(n)) \to H^\ast_{Z,\gamma}(X, \mathbb{Q}(n)) \to H^\ast_{Z \backslash Z',\gamma}(X \backslash Z', \mathbb{Q}(n)) \xrightarrow{\partial_1} 0,
\]

and we can take the limit to get a long exact sequence

\[
H^\ast_{Z^p,\gamma}(X, \mathbb{Q}(n)) \to H^\ast_{Z,\gamma}(X, \mathbb{Q}(n)) \to H^\ast_{Z^p/Z^{p+1},\gamma}(X, \mathbb{Q}(n)) \xrightarrow{\partial_1} 0.
\]

This also holds for \( p < 0 \) for trivial reasons. We form an exact couple as follows

\[
D := \bigoplus_{p \in \mathbb{Z}} H^\ast_{Z^p,\gamma}(X, \mathbb{Q}(n)),
\]

\[
E := \bigoplus_{p \geq 0} H^\ast_{Z^p/Z^{p+1},\gamma}(X, \mathbb{Q}(n)),
\]

and the exact triangle induced by 2.1.6:

\[
\begin{tikzcd}
D \ar{dr} \ar{rr} & & D \\
& E. &
\end{tikzcd}
\]

Setting

\[ E_1^{p, q} := H^{p+q}_{Z^p/Z^{p+1},\gamma}(X, \mathbb{Q}(n)), \]

the exact couple yields a spectral sequence

\[
E_1^{p, q} \Rightarrow H^p_\gamma(X, \mathbb{Q}(n)),
\]

for all \( n \geq 0 \), such that

\[
E_\infty^{p, q} = \frac{N^p H^{p+q}_\gamma(X, \mathbb{Q}(n))}{N^{p+1} H^{p+q}_\gamma(X, \mathbb{Q}(n))},
\]

with

\[ N^p H^p_\gamma(X, \mathbb{Q}(n)) = \text{image}(H^p_\gamma(X, \mathbb{Q}(n)) \to H^p_\gamma(X, \mathbb{Q}(n))). \]

Lemma 2.1.7. Let \( X \) be smooth and \( n \geq 0 \).

1. If \( p \leq n \) then

\[
H^{q-p}_{Z^p/Z^{p+1},\gamma}(X, \mathbb{Q}(n)) \cong \bigoplus_{x \in X^{(p)}} H^{q-p}(x, \mathbb{Q}(n-p)), \quad \text{for all } q \in \mathbb{Z}.
\]
(2) If \( p > n \) then
\[
H^q_{Z^p/Z^p+1}(X, \mathbb{Q}(n)) = 0, \quad \text{for all } q \in \mathbb{Z}.
\]

Proof. The set
\[
S = \{(Z, Z') \in Z^p/Z^p+1 \mid Z \setminus Z' \text{ is smooth of pure codimension } = p\}
\]
is a cofinal subset of \( Z^p/Z^p+1 \); thus
\[
H^{q+p}_{Z^p/Z^p+1}(X, \mathbb{Q}(n)) = \lim_{(Z, Z') \in S} H^*_Z \mathcal{Z}(X \setminus Z', \mathbb{Q}(n))
\]
From the Gysin triangle \cite[Proposition 3.5.4]{Voe00} we obtain for all \((Z, Z') \in S\) a natural isomorphism
\[
c_Z(X \setminus Z') \cong M_{gm}(Z \setminus Z')(p)[2p],
\]
in \( DM_{gm} \).

For (1). By using cancellation we obtain
\[
H^{q+p}_{Z^p/Z^p+1}(X, \mathbb{Q}(n)) = H^{q-p}_Z(Z \setminus Z', \mathbb{Q}(n) - p)
\]
for all \((Z, Z') \in S\), and the restriction maps
\[
H^{q-p}_Z(Z \setminus Z', \mathbb{Q}(n) - p) \to \bigoplus_{x \in X(p)} H^{q-p}_x(x, \mathbb{Q}(n) - p)
\]
induce the desired isomorphism.

For (2). Suppose \( p > n \). We claim that
\[
(2.1.8) \quad H^*_Z(X, \mathbb{Q}(n)) = 0
\]
for all \( Z \in Z^p(X) \). In view of the long exact sequence \eqref{2.1.4} this will prove the claim.

By definition the vanishing of \( H^*_Z(X, \mathbb{Q}(n)) \) follows if the restriction map
\[
H^*_Z(X, \mathbb{Q}(n)) \to H^*_Z(X \setminus Z, \mathbb{Q}(n))
\]
is an isomorphism. Set \( U := X \setminus Z \). For \( ? = M \) we can use the comparison isomorphism with higher Chow groups. It is sufficient to prove that the restriction induces an isomorphism of complexes
\[
(2.1.9) \quad Z^n(U, \bullet) \xrightarrow{\cong} Z^n(U, \bullet),
\]
where \( Z^n(\cdot, \bullet) \) denotes Bloch’s cycle complex. Since \( X \setminus U \) has codimension \( > n \), the map \( (2.1.9) \) is injective. For the surjectivity, let \( A \in Z^n(U, m) \) be the class of an irreducible subvariety of \( U \times \Delta^m \). By definition \( A \) has codimension \( n \) and meets all faces \( U \times \Delta^i \) properly. Let \( \bar{A} \) be the closure of \( A \) in \( X \times \Delta^m \). Since
\[
\bar{A} \cap (X \times \Delta^i) \subset (A \cap (U \times \Delta^i)) \cup ((X \setminus U) \times \Delta^i),
\]
and \((X \setminus U) \times \Delta^i \) has codimension \( > n \) in \( X \times \Delta^i \), we conclude that \( \bar{A} \in Z^n(X, m) \).

For \( ? = H \). In view of \eqref{1.1.1}, we need to prove that the restriction induces isomorphisms
\[
(2.1.10) \quad \text{Hom}_{\text{MHS}}(\mathbb{Q}(-n), H^q(X, \mathbb{Q})) \xrightarrow{\cong} \text{Hom}_{\text{MHS}}(\mathbb{Q}(-n), H^q(U, \mathbb{Q})),
\]
(2.1.11) \quad \text{Ext}^1_{\text{MHS}}(\mathbb{Q}(-n), H^q(X, \mathbb{Q})) \xrightarrow{\cong} \text{Ext}^1_{\text{MHS}}(\mathbb{Q}(-n), H^q(U, \mathbb{Q})),
\]
for all $q$. In order to prove (2.1.10) and (2.1.11) we use the exact sequence

\begin{equation}
0 \to \frac{H^q_{X \setminus U}(X, \mathbb{Q})}{\text{im}(H^{q-1}(U, \mathbb{Q}))} \to H^q(X, \mathbb{Q}) \to H^q(U, \mathbb{Q}) \to \ker \left( \frac{H^{q+1}_{X \setminus U}(X, \mathbb{Q})}{H^{q+1}(X, \mathbb{Q})} \right) \to 0.
\end{equation}

Note that $\frac{H^q_{X \setminus U}(X, \mathbb{Q})}{\text{im}(H^{q-1}(U, \mathbb{Q}))}$ and $\ker(H^{q+1}_{X \setminus U}(X, \mathbb{Q}) \to H^{q+1}(X, \mathbb{Q}))$ are Hodge structures of weight $\geq 2p$. If $E$ is any mixed Hodge structure of weight $\geq 2p$ then

$$\text{Hom}(Q(-n), E) = 0, \quad \text{Ext}^1(Q(-n), E) = 0,$$

because $p > n$. Therefore (2.1.12) implies the statement.

\begin{proposition}
Let $X$ be smooth and $? = M$ or $? = \mathcal{H}$. Let $n \geq 0$ be an integer.

1. There is a spectral sequence

$$E^{p,q}_{1,?} = \bigoplus_{x \in X^{(p)}, p \leq n} H^{q-p}_?(x, \mathbb{Q}(n-p)) \Rightarrow H^{p+q}(X, \mathbb{Q}(n))$$

such that

$$E^{p,q}_{\infty,?} = \frac{N^pH^{p+q}_?(X, \mathbb{Q}(n))}{N^{p+1}H^{p+q}_?(X, \mathbb{Q}(n))},$$

with

$$N^pH^?_?(X, \mathbb{Q}(n)) = \bigcup_{U \subset X, \text{codim}(X \setminus U) \geq p} \ker(H^?_?(X, \mathbb{Q}(n)) \to H^?_?(U, \mathbb{Q}(n))),$$

where $U$ runs over all open subsets with $\text{codim}(X \setminus U) \geq p$.

2. The cycle map induces a morphism of spectral sequences

$$[E^{p,q}_{1,\mathcal{M}} \Rightarrow H^{p+q}_\mathcal{M}(X, \mathbb{Q}(n))] \to [E^{p,q}_{1,\mathcal{H}} \Rightarrow H^{p+q}_\mathcal{H}(X, \mathbb{Q}(n))].$$

\end{proposition}

\begin{proof}
For (1). The statement follows from the spectral sequence (2.1.7) and Lemma 2.1.7.

For (2). The realization $r_\mathcal{H}$ (1.2.1) induces a morphism of the exact couples (2.1.6).
\end{proof}

2.2. $E_1$ complexes of the coniveau spectral sequence. Let $X$ be smooth and connected, we denote by $\eta$ the generic point of $X$. The cycle map induces a morphism between the $E_1^{\bullet,2}$ complexes of the coniveau spectral sequence (Proposition 2.1.8) for $n = 2$:

\begin{equation}
\begin{array}{c}
E^{\bullet,2}_{1,\mathcal{M}}: H^2_\mathcal{M}(\eta, \mathbb{Q}(2)) \to \bigoplus_{x \in X^{(1)}} H^1_\mathcal{M}(x, \mathbb{Q}(1)) \to \bigoplus_{x \in X^{(2)}} \mathbb{Q} \\
E^{\bullet,2}_{1,\mathcal{H}}: H^2_\mathcal{H}(\eta, \mathbb{Q}(2)) \to \bigoplus_{x \in X^{(1)}} H^1_\mathcal{H}(x, \mathbb{Q}(1)) \to \bigoplus_{x \in X^{(2)}} \mathbb{Q} \\
\text{Hom}(\mathbb{Q}, H^2(\eta, \mathbb{Q})(2)) \to \bigoplus_{x \in X^{(1)}} \text{Hom}(\mathbb{Q}, H^1(x, \mathbb{Q})(1)) \to \bigoplus_{x \in X^{(2)}} \mathbb{Q}
\end{array}
\end{equation}
We call the complex in the first line $G_M(X, 2)$, the complex in the second line $G_H(X, 2)$, and finally the complex in the third line is called $G_{HS}(X, 2)$. The complex $G_{HS}(X, 2)$ is induced by $G_H(X, 2)$ via $\mathbb{L}$. 

For $G_M(X, 2)$ the group $H_M^2(\eta, \mathbb{Q}(2))$ is the component in degree $= 0$, and the grading is defined similarly for $G_H(X, 2)$ and $G_{HS}(X, 2)$. Via Gersten-Quillen resolution we have

\[(2.2.2) \quad H^1(G_M(X, 2)) = H^1(X, \mathbb{K}_2) \otimes \mathbb{Q},\]

where $\mathbb{K}_2$ is Quillen’s K-theory Zariski sheaf associated to the presheaf $U \mapsto K_2(\mathcal{O}_X(U))$.

**Proposition 2.2.1.** Let $X$ be smooth and connected.

1. There is a natural isomorphism $H^1(G_M(X, 2)) \xrightarrow{\cong} H_M^3(X, \mathbb{Q}(2))$.
2. There is a natural injective map $H^1(G_H(X, 2)) \rightarrow H^3_H(X, \mathbb{Q}(2))$. We call the image $H^3_{H, \text{alg}}(X, \mathbb{Q}(2))$.
3. There is a natural isomorphism $H^1(G_{HS}(X, 2)) \rightarrow H^3_{H, \text{alg}}(X, \mathbb{Q}(2)) / (H^1_H(\mathbb{C}, \mathbb{Q}(1)) \cdot H^2_H(X, \mathbb{Q}(1)))$.
4. The above maps form a commutative diagram

\[
\begin{array}{ccc}
H^1(G_M(X, 2)) & \xrightarrow{\cong} & H_M^3(X, \mathbb{Q}(2)) \\
\downarrow & & \downarrow c_{2, 1} \\
H^1(G_H(X, 2)) & \xrightarrow{\cong} & H^3_{H, \text{alg}}(X, \mathbb{Q}(2)) \\
\downarrow & & \\
H^1(G_{HS}(X, 2)) & \xrightarrow{\cong} & H^3_{H, \text{alg}}(X, \mathbb{Q}(2)) / H^1_H(\mathbb{C}, \mathbb{Q}(1)) \cdot H^2_H(X, \mathbb{Q}(1)),
\end{array}
\]

and

\[c_{2, 1} : H^3_M(X, \mathbb{Q}(2)) \rightarrow H^3_{H, \text{alg}}(X, \mathbb{Q}(2))\]

is surjective.

**Proof.** Statement (i) is proved in [MS97].

**Proof of (i) and (ii).** We use the coniveau spectral sequence (Proposition 2.1.8)

\[(2.2.3) \quad E^{p, q}_1 = \bigoplus_{x \in X(p)} H^p_q(x, \mathbb{Q}(n - p)) \Rightarrow H^{p+q}_M(X, \mathbb{Q}(n)),\]

where $? = M$ or $H$, and $0 \leq p \leq n$. We have

\[G_M(X, 2) = E^{0, 2}_1\]

for $n = 2$. We get $E^{1, 2}_1 = E^{1, 2}_\infty$ and $E^{2, 1}_\infty = E^{2, 0}_\infty$ for obvious reasons. Therefore we obtain an exact sequence

\[(2.2.4) \quad 0 \rightarrow E^{1, 2}_\infty \rightarrow H^3_M(X, \mathbb{Q}(2)) \rightarrow E^{0, 3}_\infty \rightarrow 0,\]

with

\[E^{1, 2}_\infty = \ker(H^3_M(X, \mathbb{Q}(2)) \rightarrow H^3_M(\eta, \mathbb{Q}(2))).\]

For $? = M$ we have $H^3_M(\eta, \mathbb{Q}(2)) = 0$; for $? = H$ we define

\[H^3_{H, \text{alg}}(X, \mathbb{Q}(2)) := \ker(H^3_H(X, \mathbb{Q}(2)) \rightarrow H^3_H(\eta, \mathbb{Q}(2))) = N^1H^3_H(X, \mathbb{Q}(2)).\]
For (iii). If $X$ is smooth then it is not difficult to see that
\[ \text{Pic}(X) \otimes \mathbb{Q} = H^2_M(X, \mathbb{Q}(1)) \cong H^2_R(X, \mathbb{Q}(1)). \]
It follows that via the isomorphism $H^1(G_H(X, 2)) \cong H^3_{R, \text{alg}}(X, \mathbb{Q}(2))$ the subgroup $H^1_M(\mathbb{C}, \mathbb{Q}(1)) \cdot H^3_R(X, \mathbb{Q}(1))$ of $H^3_{R, \text{alg}}(X, \mathbb{Q}(2))$ corresponds to the image of $\bigoplus_{x \in X^{(1)}} \mathbb{C}^* \otimes \mathbb{Q}$ in $H^1(G_H(X, 2))$. For every point $x \in X^{(1)}$ we have an exact sequence
\[ 0 \to \mathbb{C}^* \otimes \mathbb{Q} \to H^1_M(x, \mathbb{Q}(1)) \to \text{Hom}_{MHS}(\mathbb{Q}, H^1(x, \mathbb{Q}(1))) \to 0, \]
and therefore
\[ \ker(H^1(G_H(X, 2)) \to H^1(G_{HS}(X, 2))) = \text{im}(\bigoplus_{x \in X^{(1)}} \mathbb{C}^* \otimes \mathbb{Q} \to H^1(G_H(X, 2))). \]
This implies the claim.
Statement (iv) is obvious. \hfill \square

Remark 2.2.2. For a smooth projective variety $X$ we know that
\[ H^1(G_{HS}(X, 2)) \cong H^3_{R, \text{alg}}(X, \mathbb{Q}(2))/ (H^1_M(\mathbb{C}, \mathbb{Q}(1)) \cdot H^3_R(X, \mathbb{Q}(1))) \]
is a countable group \cite{MS97}. This is a consequence of the fact that deformations $a'$ of a class $a \in H^3_M(X, \mathbb{Q}(2))$ have the same image via $c_{2,1}$ modulo the group $H^1_M(\mathbb{C}, \mathbb{Q}(1)) \cdot H^3_R(X, \mathbb{Q}(1))$. There exist examples of K3-surfaces $X$ such that $H^1(G_{HS}(X, 2)) \neq 0$ \cite{MS97}.

Definition 2.2.3. Let $X$ be smooth, connected and projective. We denote by
\[ \text{image}(H^1_M(\mathbb{C}, \mathbb{Q}(1)) \cdot H^3_M(X, \mathbb{Q}(1))) =: H^3_M(X, \mathbb{Q}(2))_{\text{dec}} \subset H^3_M(X, \mathbb{Q}(2)) \]
the subgroup of decomposable cycles. In the same way we define $H^3_R(X, \mathbb{Q}(2))_{\text{dec}}$.

Note that
\begin{align}
(2.2.5) & \quad \mathbb{C}^* \otimes \mathbb{Q} = H^1_M(\mathbb{C}, \mathbb{Q}(1)) \cong H^1_R(\mathbb{C}, \mathbb{Q}(1)), \\
(2.2.6) & \quad \text{Pic}(X) \otimes \mathbb{Q} = H^3_M(X, \mathbb{Q}(1)) \cong H^3_R(X, \mathbb{Q}(1)).
\end{align}

Lemma 2.2.4. If $X$ is smooth, projective, and $H^1(X) = 0$, then the maps
\[ H^1_M(\mathbb{C}, \mathbb{Q}(1)) \otimes \mathbb{Q} H^3_M(X, \mathbb{Q}(1)) \to H^3_M(X, \mathbb{Q}(2)) \]
\[ H^1_R(\mathbb{C}, \mathbb{Q}(1)) \otimes \mathbb{Q} H^3_R(X, \mathbb{Q}(1)) \to H^3_R(X, \mathbb{Q}(2)) \]
are injective. In particular,
\[ H^3_M(X, \mathbb{Q}(2))_{\text{dec}} \to H^3_R(X, \mathbb{Q}(2))_{\text{dec}} \]
is an isomorphism.

Proof. By using the cycle map it is sufficient to prove the statement for absolute Hodge cohomology. The assumption $H^1(X) = 0$ implies
\[ H^3_R(X, \mathbb{Q}(1))) \cong \text{Hom}(\mathbb{Q}(-1), H^2(X, \mathbb{Q})). \]
The pure Hodge structure $H^2(X, \mathbb{Q})$ is polarizable and therefore
\[ \text{Hom}(\mathbb{Q}(-1), H^2(X, \mathbb{Q})) \otimes \mathbb{Q}(-1) \subset H^2(X, \mathbb{Q}) \]
is a direct summand which we call $H_{g,1}^{1,1}$. We get
\[ (H_{g,1}^{1,1} \otimes \mathbb{C})/(2\pi i)^2 \cdot H_{g,1}^{1,1} = \text{Ext}^1(\mathbb{Q}(-2), H_{g,1}^{1,1}) \]
\[ \subset \text{Ext}^1(\mathbb{Q}(-2), H^2(X, \mathbb{Q})) \subset H^3_R(X, \mathbb{Q}(2)) \]
and clearly $H^1_\varphi(C, \mathbb{Q}(1)) \otimes \mathbb{Q} \to H^2_\varphi(X, \mathbb{Q}(1))$ is mapping isomorphically onto $(H^1_{\mathrm{g}, \mathbb{Q}} \otimes \mathbb{C})/((2\pi i)^2 \cdot H^1_{\mathrm{g}, \mathbb{Q}}).$ Because of (2.2.5) and (2.2.6) we conclude that

$$H^3_\mathrm{MF}(X, \mathbb{Q}(2)) \to H^3_\varphi(X, \mathbb{Q}(2))$$

is an isomorphism. □

**Proposition 2.2.5.** Let $X$ be smooth and connected. Restriction to the generic point yields the following equalities:

$$H^2_\varphi(X, \mathbb{Q}(2)) \to H^0(G_M(X, 2))$$

$$H^2_\varphi(X, \mathbb{Q}(2)) \to H^0(G_H(X, 2)).$$

**Proof.** We use the coniveau spectral sequence (Proposition 2.1.8) for $n = 2$. We have

$$E^0_{1, 2} = H^0(G_1(X, 2))$$

for $? = M$ and $? = H$. Note that $E^0_{1, 2} = 0$ for $q \neq 2$. Thus $E^0_{1, 2} = E^0_{\infty, 2}$ and $E^0_{2, 0} = 0$. Moreover, $E^1_{1, 1} = 0$, because $H^0(U, \mathbb{Q}(1)) = 0$ for every $U$. It follows that $E^1_{\infty, 2} = 0$ and

$$H^2_\varphi(X, \mathbb{Q}(2)) = E^0_{\infty, 2} = E^0_{1, 2}.$$

□

**Lemma 2.2.6.** Let $X$ be smooth and connected. We denote by $\eta$ the generic point of $X$. If

$$H^2_\varphi(\eta, \mathbb{Q}(2)) \to H^2_\varphi(\eta, \mathbb{Q}(2))$$

is surjective then

$$H^2_\varphi(X, \mathbb{Q}(2)) \to H^2_\varphi(X, \mathbb{Q}(2))$$

is surjective.

**Proof.** We use Proposition 2.2.5 and need to prove that

$$H^0(G_M(X, 2)) \to H^0(G_H(X, 2))$$

is surjective. Since $H^1_\varphi(x, \mathbb{Q}(1)) = H^1_\varphi(x, \mathbb{Q}(1))$ for every point $x \in X$ of codimension $= 1$ this follows immediately from diagram (2.2.1). □

**Proposition 2.2.7.** If $X$ is smooth, connected and projective then

$$H^0(G_{\mathrm{HS}}(X, 2)) = 0.$$

**Proof.** Let $\eta \in X$ be the generic point. For

$$a \in H^0(G_{\mathrm{HS}}(X, 2)) \subset \text{Hom}(\mathbb{Q}(-2), H^2(\eta, \mathbb{Q}))$$

we can find an effective divisor $D$ such that $a$ is induced by a cohomology class $a' \in \text{Hom}(\mathbb{Q}(-2), H^2(X \setminus D, \mathbb{Q})).$ Let $S$ be a closed subset of $X$ of codimension $\geq 2$, such that $D \setminus S$ is smooth. Denoting $X' := X \setminus S$, we claim that $a' |_{X' \setminus D}$ maps to zero in $H^1(D \setminus S, \mathbb{Q})(-1)$ via the boundary map of the localization sequence for singular cohomology. Indeed, the map

$$\text{Hom}(\mathbb{Q}(-1), H^1(D \setminus S, \mathbb{Q})) \to \bigoplus_{x \in X'(1)} \text{Hom}(\mathbb{Q}(-1), H^1(x, \mathbb{Q}))$$

is injective and therefore the claim follows from $a \in H^0(G_{\mathrm{HS}}(X, 2)).$
Now \( a' \mid_{X' \setminus D} \) defines an extension of Hodge structures
\[
0 \to \text{image}(\mathbb{Q}(-1)^{\gamma_0(D \setminus S)}) \to E \to \mathbb{Q}(-2) \to 0,
\]
with \( E \subset H^2(X', \mathbb{Q}) \). We note that \( H^2(X', \mathbb{Q}) = H^2(X, \mathbb{Q}) \) is a pure Hodge structure of weight = 2, and therefore the same holds for \( E \). Thus the extension is trivial and \( a' \mid_{X' \setminus D} \) lifts to \( \text{Hom}(\mathbb{Q}(-2), H^2(X', \mathbb{Q})) = 0 \). This proves that \( a' \mid_{X' \setminus D} = 0 \) and implies \( a = 0 \).

### 2.3. An exact sequence for projective varieties with vanishing \( H^1 \).

**Lemma 2.3.1.** Let \( X \) be smooth, projective and connected. Suppose that \( H^1(X) = 0 \). We denote by \( \eta \) the generic point of \( X \). There is an exact sequence

\[
H^2_M(\eta, \mathbb{Q}(2)) \to H^3_H(\eta, \mathbb{Q}(2)) \to H^3_M(X, \mathbb{Q}(2)) \to H^3_H(X, \mathbb{Q}(2)).
\]

**Proof.** Via Proposition 2.2.1 we identify \( H^3_M(X, \mathbb{Q}(2)) \cong H^1(G_M(X, 2)) \) and need to show that there is an exact sequence

\[
H^2_M(\eta, \mathbb{Q}(2)) \to H^2_H(\eta, \mathbb{Q}(2)) \to H^1(G_M(X, 2)) \to H^1(G_H(X, 2)).
\]

We work with diagram (2.2.1). The map

\[
H^2_H(\eta, \mathbb{Q}(2)) \to H^1(G_M(X, 2))
\]

is defined by using \( E_{1,2}^{1,2} = E_{1,H}^{1,2} \). The assumptions on \( X \) imply that \( H^2_H(X, \mathbb{Q}(2)) = 0 \) and therefore \( H^0(G_H(X, 2)) = 0 \) by Proposition 2.2.5. The rest of the proof involves only diagram chasing. \( \square \)

### 2.4. Main theorem.

**Theorem 2.4.1.** Let \( X \) be smooth and connected. Let \( \tilde{X} \) be a smooth compactification of \( X \). We denote by \( \text{CH}_0(\tilde{X}) \otimes \mathbb{Q} \) the Chow group of zero cycles on \( \tilde{X} \). If \( \text{deg} : \text{CH}_0(\tilde{X}) \otimes \mathbb{Q} \to \mathbb{Q} \) is an isomorphism then \( BH(X, 2) \) holds.

**Proof.** In view of Lemma 2.2.6 it is sufficient to show that \( \tilde{C}_{2,2} : H^2_M(\eta, \mathbb{Q}(2)) \to H^3_H(\eta, \mathbb{Q}(2)) \) is surjective, where \( \eta \) is the generic point on \( X \).

Now, choose a smooth projective model \( Y \) of \( \eta \). Since \( \tilde{X} \) and \( Y \) are birational we conclude that \( \text{CH}_0(Y) \otimes \mathbb{Q} \cong \mathbb{Q} \). It follows that \( H^1(Y) = 0 \).

We claim that

\[
H^3_M(Y, \mathbb{Q}(2))_{\text{dec}} = H^3_M(Y, \mathbb{Q}(2)).
\]

This implies the theorem by using Lemma 2.3.1 because

\[
H^3_M(Y, \mathbb{Q}(2))_{\text{dec}} \subset H^3_H(Y, \mathbb{Q}(2))_{\text{dec}}
\]

by Lemma 2.2.4.

In view of Proposition 2.2.1 and 2.2.2 it is sufficient to prove that the cokernel of

\[
\mathbb{C}^* \otimes \mathbb{Z} \text{Pic}(X) \to H^1(X, \mathcal{K}_2)
\]

is torsion. This is proved in \( \text{[BSS83, Theorem 3(i)\]} \). \( \square \)
2.5. An exact sequence for projective varieties.

**Proposition 2.5.1.** Let $X$ be smooth, projective and connected. We denote by $\eta$ the generic point of $X$. There is an exact sequence

$$
\begin{align*}
H_M^3(\eta, \mathbb{Q}(2)) &\to \text{Hom}(\mathbb{Q}(-2), H^2(\eta, \mathbb{Q})) \\
&\to H^3_M(X, \mathbb{Q}(2))/H^3_M(X, \mathbb{Q}(2))_{\text{dec}} \\
&\to H^3_H(X, \mathbb{Q}(2))/H^3_H(X, \mathbb{Q}(2))_{\text{dec}}.
\end{align*}
$$

**Proof.** Via Proposition 2.2.1 we identify

$$H^1(G_M(X, 2)) \cong H^3_M(X, \mathbb{Q}(2)),$$

$$H^1(G_{HS}(X, 2)) \subset H^3_H(X, \mathbb{Q}(2))/H^3_H(X, \mathbb{Q}(2))_{\text{dec}}.$$

We work with diagram (2.2.1). For the map

$$\text{Hom}(\mathbb{Q}(-2), H^2(\eta, \mathbb{Q})) \to H^3_M(X, \mathbb{Q}(2))/H^3_M(X, \mathbb{Q}(2))_{\text{dec}}$$

we observe that

$$0 \to \bigoplus_{x \in X^{(1)}} \mathbb{C}^* \otimes \mathbb{Q} \to \bigoplus_{x \in X^{(1)}} H^3_M(x, \mathbb{Q}(1)) \to \bigoplus_{x \in X^{(1)}} \text{Hom}(\mathbb{Q}, H^1(x, \mathbb{Q})(1)) \to 0$$

is exact and

$$H^3_M(X, \mathbb{Q}(2))_{\text{dec}} = \text{im}(\bigoplus_{x \in X^{(1)}} \mathbb{C}^* \otimes \mathbb{Q} \to H^3_M(X, \mathbb{Q}(2))).$$

The assumptions on $X$ imply $H^0(G_{HS}(X, 2)) = 0$ by Proposition 2.2.7. The rest of the proof involves only diagram chasing. \[\square\]

**Remark 2.5.2.** Proposition 2.5.1 has also been proved by de Jeu and Lewis in [dJL11, Corollary 4.14], and more generally with integral coefficients in [dJL11, Corollary 6.5].

**Proposition 2.5.3.** Let $X$ be smooth, projective and connected. Suppose that $H^1(X, \mathbb{Q}) = 0$. The following statements are equivalent.

1. $BH(U, 2)$ holds for all open subsets $U$ of $X$.
2. $BH(\eta, 2)$ holds for the generic point $\eta$ of $X$.

**Proof.** Only (2) $\Rightarrow$ (1) is interesting. Proposition 2.5.1 implies that

$$H^3_M(X, \mathbb{Q}(2))/H^3_M(X, \mathbb{Q}(2))_{\text{dec}} \to H^3_H(X, \mathbb{Q}(2))/H^3_H(X, \mathbb{Q}(2))_{\text{dec}}$$

is injective. From Lemma 2.2.4 we see that

$$H^3_M(X, \mathbb{Q}(2))_{\text{dec}} \to H^3_H(X, \mathbb{Q}(2))_{\text{dec}}$$

is an isomorphism. Thus

$$H^3_M(X, \mathbb{Q}(2)) \to H^3_H(X, \mathbb{Q}(2))$$

is injective. It follows from Lemma 2.3.1 that

$$H^2_M(\eta, \mathbb{Q}(2)) \to H^2_H(\eta, \mathbb{Q}(2))$$

is surjective. Lemma 2.2.4 applied to $U$ implies the claim. \[\square\]

**Question 2.5.4.** Suppose $X$ is smooth, connected, projective, but $H^1(X) \neq 0$. We denote by $\eta$ the generic point of $X$. What is the relation between $BH(U, 2)$, for all $U$ open in $X$, and the surjectivity of

$$H^2_M(\eta, \mathbb{Q}(2)) \to \text{Hom}(\mathbb{Q}(-2), H^2(\eta, \mathbb{Q}))?$$
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