Homological Mirror Symmetry for Toric del Pezzo Surfaces

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Abstract

We prove the homological mirror conjecture for toric del Pezzo surfaces. In this case, the mirror object is a regular function on an algebraic torus \((\mathbb{C}^*)^2\). We show that the derived Fukaya category of this mirror coincides with the derived category of coherent sheaves on the original manifold.

1 Introduction

Mirror symmetry started as a mysterious relationship between complex geometry of a Calabi-Yau 3-fold and symplectic geometry of another Calabi-Yau 3-fold called the mirror manifold. In 1994, Kontsevich \cite{kontsevich} proposed a program to understand various mirror phenomena as a consequence of the following homological mirror conjecture: Calabi-Yau manifolds always come in pairs in such a way that the derived category of coherent sheaves on one manifold is equivalent as a triangulated category to the derived Fukaya category of the other.

Although mirror symmetry was first discovered for Calabi-Yau manifolds, there are also variants of these phenomena for other classes of manifolds. One such example is the Givental’s theorem \cite{givental} giving integral representations of \(J\)-functions of toric Fano manifolds. See also \cite{givental}, \cite{givental2}, \cite{givental3} and \cite{givental4}.

We can also formulate the homological mirror conjecture for toric Fano manifolds. Let \(X\) be a toric Fano manifold of dimension \(n\). Then the mirror partner is a regular function \(W\) on an algebraic torus \((\mathbb{C}^*)^n\) of the same dimension equipped with a symplectic structure (along with an additional data called a grading). Here, the function \(W\) is a Newton polynomial for the convex hull of the generators of 1-dimensional cone of the fan of \(X\). Coefficients of this polynomial do not matter as long as they are chosen general enough. By Kouchnirenko \cite{kouchnirenko}, \(W\) has exactly \(\dim H^*(X, \mathbb{C})\) critical points. Take a regular value \(t\) of \(W\) and a distinguished basis of vanishing
cycles in $W^{-1}(t)$. We also have to choose a grading and a spin structure on each of these vanishing cycles. The directed Fukaya category $\mathcal{F}_{\text{Fuk}} \rightarrow W$ of $W$ (along with the choice of gradings and spin structures) is an $A_{\infty}$-category whose objects are vanishing cycles and whose morphisms are Floer complexes. Roughly speaking, the Floer complex between two vanishing cycles are the vector space spanned by intersection points between them, and the compositions of morphisms are given by “counting polygons.” By Seidel [15], the derived category $D^b \mathcal{F}_{\text{Fuk}} \rightarrow W$ of $\mathcal{F}_{\text{Fuk}} \rightarrow W$ is independent of the choice of a distinguished basis of vanishing cycles.

The following is the main result of this paper:

**Theorem 1.1.** The derived category of coherent sheaves on a toric del Pezzo surface $X$ is equivalent as a triangulated category to the derived category of the directed Fukaya category of the mirror $W$ of $X$;

$$D^b \text{coh}(X) \cong D^b \mathcal{F}_{\text{Fuk}} \rightarrow W.$$  

Our proof is based on an explicit computation. The above Theorem 1.1 extends the work of Seidel [16] and Auroux, Katzarkov and Orlov [1], where the cases of $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$, and $\mathbb{P}^2$ blown-up at one point is treated. See also the paper by Hori, Iqbal and Vafa [8], where mirror symmetry for Fano manifolds are discussed from a physics point of view.

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## 2 Derived category of coherent sheaves

We describe the structure of the bounded derived category $D^b \text{coh}(Y)$ of coherent sheaves on $Y$ in this section, where $Y$ is the projective plane blown-up at three points $p_1$, $p_2$ and $p_3$ in general position. Let $\phi : Y \rightarrow \mathbb{P}^2$ be this blow-up and $E_1$, $E_2$ and $E_3$ be the exceptional divisors corresponding to $p_1$, $p_2$ and $p_3$ respectively. It is a toric surface and the generators of one-dimensional cones of its fan is drawn in Figure 1.
Definition 2.1.  
1. An object \( E \) in a triangulated category is exceptional if
\[
\text{Ext}^i(E, E) = \begin{cases} 
\mathbb{C} & \text{if } i = 0, \\
0 & \text{otherwise.} 
\end{cases}
\]

2. An ordered set of objects \((E_i)_{i=1}^N\) in a triangulated category is an exceptional collection if each \( E_i \) is exceptional and \( \text{Ext}^k(E_i, E_j) = 0 \) for any \( i > j \) and for any \( k \).

By combining theorems of Beilinson [3] and Orlov [12], we have the following generators of \( D^b \text{coh}(Y) \):

Theorem 2.2. Let
\[
\mathcal{C} = (E_1, E_2, E_3, E_4, E_5, E_6)
\]
where
\[
E_1 = \mathcal{O}_{E_1}(-1)[-1], \quad E_2 = \mathcal{O}_{E_2}(-1)[-1], \quad E_3 = \mathcal{O}_{E_3}(-1)[-1], \\
E_4 = \phi^* \mathcal{O}_{P^2}(-1), \quad E_5 = \phi^* \Omega_{P^2}(1), \quad E_6 = \mathcal{O}_Y.
\]
(1)

Then \( \mathcal{C} \) is an exceptional collection generating \( D^b \text{coh}(Y) \).

Here, \( \mathcal{O}_{E_i}(-1) \) is the sheaf supported on \( E_i \) and is isomorphic to the tautological sheaf \( \mathcal{O}_{P^1}(-1) \) on \( E_i \cong P^1 \), \([\bullet]\) is the shift operator in the derived category, \( \phi^* \) is the derived pull-back, \( \mathcal{O}_{P^2}(-1) \) is the tautological sheaf on \( P^2 \), and \( \Omega_{P^2}(1) \) is the cotangent sheaf of \( P^2 \) tensored with the hyperplane sheaf \( \mathcal{O}_{P^2}(1) \).

Proposition 2.3. All the non-zero Ext-groups within the exceptional collec-
tion \mathcal{C} are

\begin{align*}
\text{Hom}(\mathcal{E}_i, \mathcal{E}_4) &= \mathbb{C}p_i, \\
\text{Hom}(\mathcal{E}_i, \mathcal{E}_5) &= \text{Ker}(V^\vee \to (\mathbb{C}p_i)^\vee), \\
\text{Hom}(\mathcal{E}_i, \mathcal{E}_6) &= \mathbb{C}, \\
\text{Hom}(\mathcal{E}_4, \mathcal{E}_5) &= \Lambda^2 V^\vee, \\
\text{Hom}(\mathcal{E}_4, \mathcal{E}_6) &= V^\vee, \\
\text{Hom}(\mathcal{E}_5, \mathcal{E}_6) &= V,
\end{align*}

where \( i = 1, 2, 3 \), \( V \cong \mathbb{C}^3 \) is the three-dimensional vector space such that \( \mathbb{P}^2 = \mathbb{P}(V) \), the check denotes the dual vector space, \( \mathbb{C}p_i \subset V \) is the one-dimensional subspace corresponding to \( p_i \in \mathbb{P}(V) \), and the map \( V^\vee \to (\mathbb{C}p_i)^\vee \) is the dual of the inclusion \( \mathbb{C}p_i \hookrightarrow V \). Compositions of morphisms are given by

\begin{align*}
\text{Hom}(\mathcal{E}_i, \mathcal{E}_4) \times \text{Hom}(\mathcal{E}_4, \mathcal{E}_5) &\to \text{Hom}(\mathcal{E}_i, \mathcal{E}_5) \\
\psi \quad (v, \omega) &\mapsto -\iota_v \omega,
\end{align*}

\begin{align*}
\text{Hom}(\mathcal{E}_i, \mathcal{E}_4) \times \text{Hom}(\mathcal{E}_4, \mathcal{E}_6) &\to \text{Hom}(\mathcal{E}_i, \mathcal{E}_6) \\
\psi \quad (v, \omega) &\mapsto \omega(v),
\end{align*}

\begin{align*}
\text{Hom}(\mathcal{E}_i, \mathcal{E}_5) \times \text{Hom}(\mathcal{E}_5, \mathcal{E}_6) &\to \text{Hom}(\mathcal{E}_i, \mathcal{E}_6) \\
\psi \quad (\omega, v) &\mapsto \omega(v),
\end{align*}

\begin{align*}
\text{Hom}(\mathcal{E}_4, \mathcal{E}_5) \times \text{Hom}(\mathcal{E}_5, \mathcal{E}_6) &\to \text{Hom}(\mathcal{E}_4, \mathcal{E}_6) \\
\psi \quad (\omega, v) &\mapsto \iota_v \omega,
\end{align*}

where \( \iota_v : \wedge^2 V^\vee \to V^\vee \) for \( v \in V \) is the interior product.

Proof. All the computations reduce to those on \( \mathbb{P}^2 \) in the following way: Let \( \mathcal{E} \) and \( \mathcal{F} \) be vector bundles on \( \mathbb{P}^2 \). Then

\begin{align*}
\mathcal{R}\text{Hom}(\phi^* \mathcal{E}, \phi^* \mathcal{F}) &= \mathcal{R}\text{Hom}(\mathcal{E}, \mathcal{R}\phi_* \phi^* \mathcal{F}) \\
&= \mathcal{R}\text{Hom}(\mathcal{E}, \mathcal{F} \otimes \mathcal{R}\phi_* \mathcal{O}_V) \\
&= \mathcal{R}\text{Hom}(\mathcal{E}, \mathcal{F}),
\end{align*}

\begin{align*}
\mathcal{R}\text{Hom}(\phi^* \mathcal{E}, \mathcal{O}_{E_i}(-1)) &= \mathcal{R}\text{Hom}(\mathcal{E}, \mathcal{R}\phi_* \mathcal{O}_{E_i}(-1)) \\
&= 0,
\end{align*}
and
\[ \mathbb{R}\text{Hom}(\mathcal{O}_{E_i}(-1)[-1], \phi^*\mathcal{E}) = \mathbb{R}\text{Hom}(\{\mathcal{O}_Y \to \mathcal{O}_Y(E_i)\}, \phi^*\mathcal{E}) \]
\[ = \mathbb{R}\Gamma(\{\mathcal{O}_Y(-E_i) \to \mathcal{O}_Y\} \otimes \phi^*\mathcal{E}) \]
\[ = \mathbb{R}\Gamma(\{\mathcal{I}_{p_i} \to \mathcal{O}_{P^2}\} \otimes \mathcal{E}) \]
\[ = \mathbb{R}\Gamma(\mathcal{O}_{p_i} \otimes \mathcal{E}) \]
\[ = \mathcal{E}|_{p_i} \]

where \( \mathcal{I}_{p_i} \) is the ideal sheaf of \( p_i \) and the last line is the fiber at \( p_i \). Here, we have used the exact sequence
\[ 0 \to \mathcal{O}_Y \to \mathcal{O}_Y(E_i) \to \mathcal{O}_{E_i}(-1) \to 0. \]

We can further use the exact sequence
\[ 0 \to \Omega_{P^2}(1) \to V^\vee \otimes \mathcal{O}_{P^2} \to \mathcal{O}_{P^2}(1) \to 0 \]
to reduce the computations involving \( \Omega_{P^2}(1) \) to those involving \( \mathcal{O}_{P^2}(i) \), \( i \in \mathbb{Z} \):
\[ \mathbb{R}\text{Hom}(\mathcal{O}_{E_i}(-1)[-1], \phi^*\Omega_{P^2}(1)) = \mathbb{R}\Gamma(\mathcal{O}_{p_i} \otimes \{V^\vee \otimes \mathcal{O}_{P^2} \to \mathcal{O}_{P^2}(1)\}) \]
\[ = \{V^\vee \to (\mathbb{C}_{p_i})^\vee\}, \]

\[ \mathbb{R}\text{Hom}(\phi^*\mathcal{O}_{P^2}(-1), \phi^*\Omega_{P^2}(1)) = \mathbb{R}\text{Hom}(\mathcal{O}_{P^2}(-1), \{V^\vee \otimes \mathcal{O}_{P^2} \to \mathcal{O}_{P^2}(1)\}) \]
\[ = \mathbb{R}\Gamma(\mathcal{O}_{P^2}(1) \otimes \{V^\vee \otimes \mathcal{O}_{P^2} \to \mathcal{O}_{P^2}(1)\}) \]
\[ = \mathbb{R}\Gamma(\{V^\vee \otimes \mathcal{O}_{P^2}(1) \to \mathcal{O}_{P^2}(2)\}) \]
\[ = \{V^\vee \otimes V^\vee \to \text{Sym}^2 V^\vee\} \]
\[ = \mathcal{A}^2 V^\vee, \]

\[ \mathbb{R}\text{Hom}(\phi^*\Omega_{P^2}(1), \mathcal{O}_Y) = \mathbb{R}\text{Hom}(\{V^\vee \otimes \mathcal{O}_{P^2} \to \mathcal{O}_{P^2}(1)\}, \mathcal{O}_{P^2}) \]
\[ = \mathbb{R}\Gamma(\{\mathcal{O}_{P^2}(-1) \to V \otimes \mathcal{O}_{P^2}\}) \]
\[ = V. \]

Compositions of morphisms can be easily read off from the above computations. \( \square \)

## 3 Fukaya category

First we recall the definition of an \( A_\infty \)-category. For a \( \mathbb{Z} \)-graded vector space \( V \) and \( i \in \mathbb{Z} \), \( V[i] \) denotes the shift of \( V \) by \( i \); \( V[i]^j = V^{i+j} \).
**Definition 3.1.** An $A_{\infty}$-category $\mathcal{A}$ consists of

1. the set of objects $\text{Ob}(\mathcal{A})$,

2. for any $c_1, c_2 \in \text{Ob}(\mathcal{A})$, a $\mathbb{Z}$-graded $\mathbb{C}$-vector space $\text{hom}_{\mathcal{A}}(c_1, c_2)$ called the set of morphisms,

3. for any positive integer $k$ and for any set of objects $\{c_i\}_{i=0}^k$, the composition

$$m_k : \text{hom}_{\mathcal{A}}(c_0, c_1)[1] \otimes \cdots \otimes \text{hom}_{\mathcal{A}}(c_{k-1}, c_k)[1] \longrightarrow \text{hom}_{\mathcal{A}}(c_0, c_k)[1]$$

which is a linear map of degree 1

such that for any positive integer $k$, any set of objects $\{c_i\}_{i=0}^k$ and any set of morphisms $\{a_i\}_{i=1}^k$, $a_i \in \text{hom}_{\mathcal{A}}(c_{i-1}, c_i)$, the following $A_{\infty}$-relations holds:

$$\sum_{i=0}^{k-1} \sum_{j=i+1}^k (-1)^{\deg a_1 + \cdots + \deg a_i} m_{i-j+k+1}(a_1 \otimes \cdots \otimes a_i \otimes m_{j-i}(a_{i+1} \otimes \cdots \otimes a_j) \otimes a_{j+1} \otimes \cdots \otimes a_k) = 0.$$ 

Here degrees are counted after shifts, i.e., if $a \in V^i$, $\deg a = i - 1$ in $V[1]$. Since $m_1^2 = 0$, we define

$$\text{Hom}_{\mathcal{A}}(c_1, c_2) = H^0(\text{hom}_{\mathcal{A}}(c_1, c_2), m_1).$$

The Fukaya category of Lagrangian submanifolds in a symplectic manifold is defined in [6]. We use the following adaptation for exact Morse fibrations by Seidel [15]. Let $W : Z \to \mathbb{C}$ be a regular function on an affine algebraic manifold $Z$ of complex dimension $n$ with a Kähler structure. Assume the following conditions:

- The Kähler metric is complete.

- The symplectic form $\omega$ of $Z$ is exact, i.e., there exists a one form $\theta$ on $Z$ such that $\omega = d\theta$.

- At any critical points of $W$, the Hessian of $W$ is non-degenerate.

- All the critical values are distinct.

Such $W$ gives rise to an exact Morse fibration in the terminology of [15]. Using the Kähler structure, we can define the lift $\tilde{c}_p : [0, 1] \to Z$ of a path $c : [0, 1] \to \mathbb{C}$ starting from a point $p \in Z$ such that $W(p) = c(0)$ by using the
horizontal distribution defined as the orthogonal complement of the tangent space along the fiber of $W$.

Assume that the origin is a regular value of $W$ and fix an order $(p_i)_{i=1}^N$ on the set of critical points of $W$. A distinguished set $(c_i)_{i=1}^N$ of vanishing paths is a set of smooth paths $c_i : [0, 1] \to \mathbb{C}$ satisfying

1. $c_i(0) = 0$, $c_i(1) = W(p_i)$,
2. $c_i$ has no self-intersection,
3. images of $c_i$ and $c_j$ intersects only at the origin,
4. $c'_i(0) \neq 0$, and
5. $\text{arg } c'_{i+1}(0) < \text{arg } c'_i(0)$, $i = 1, \ldots, N - 1$, for some choice of the branch of $\text{arg}(\cdot)$.

Given a distinguished set $(c_i)_{i=1}^N$ of vanishing paths, the corresponding vanishing cycles $(C_i)_{i=1}^N$ are defined by

$$C_i = \{ p \in W^{-1}(0) \mid \lim_{t \to 1} \tilde{c}_p(t) = p_i \}.$$ 

They are Lagrangian submanifolds of $W^{-1}(0)$.

The directed Fukaya category $\mathcal{Fuk}^{-\to} W$ for $W$ is roughly an $A_\infty$-category whose objects are vanishing cycles and whose morphisms are Lagrangian intersection Floer complexes. To define a $\mathbb{Z}$-grading on the Floer complex, we need the concept of grading on Lagrangian submanifolds introduced by Kontsevich [10], which we now recall. See also Seidel [14].

Let $(M, \omega)$ be a symplectic manifold of dimension $2n$. An almost complex structure on $M$ is a section $J \in \Gamma(M, \text{End}(TM))$ such that $J^2 = -\text{id}$. $J$ is called $\omega$-compatible if $g(V_1, V_2) = \omega(V_1, JV_2)$ defines a Hermitian metric on the tangent bundle. Fix an $\omega$-compatible almost complex structure $J$ of $M$ and let $S$ be the principal $U(1)$-bundle associated to the complex line bundle $(\Lambda^n(T^*M, J))^\otimes 2$. The fiber of $S$ at $p \in M$ is $(\Lambda^n(T^*_pM, J))^\otimes 2 / \mathbb{R} > 0$. We assume that the first Chern class of this complex line bundle vanishes, so that $S$ has a section. A grading of $M$ is a choice of a section $\Theta : M \to S$. Fix a grading $\Theta$ on $M$. Let $\text{Lag}_M \to M$ be the Lagrangian Grassmannian bundle on $M$, whose fiber at $p \in M$ is the Grassmannian of Lagrangian subspaces in the symplectic vector space $T_pM$. Define $\det^2_\Theta : \text{Lag}_M \to U(1)$ as follows: For a Lagrangian subspace $L \subset T_pM$, pick any basis $(e_i)_{i=1}^n$ of $L$ and take the square of their exterior product; $(e_1 \wedge e_2 \wedge \cdots \wedge e_n)^\otimes 2 \in (\Lambda^n(T^*_pM, J))^\otimes 2$. $\det^2_\Theta(L)$ is the image of this element by $\Theta(p)$. A Lagrangian submanifold $L \subset M$ gives a canonical section $s_L : L \ni p \mapsto T_pL \in \text{Lag}_M|_p$. Denote the
composition of $s_L$ and $\det_2^\Theta$ by $\phi_L$. A grading of a Lagrangian submanifold is a lift $\tilde{\phi}_L : L \to \mathbb{R}$ of $\phi_L$ to the universal cover $\mathbb{R}$ of $U(1) \cong \mathbb{R}/\mathbb{Z}$.

Now we define the Maslov index. A smooth path $\Lambda : [0, 1] \to \text{Lag}(V, \omega)$ in the Lagrangian Grassmannian of a fixed symplectic vector space $(V, \omega)$ of dimension $2n$ is called crossingless if $\Lambda(0) \cap \Lambda(t) = \Lambda(0) \cap \Lambda(1)$ for all $t \in (0, 1)$. For a crossingless path $\Lambda_t$, its differential $\Lambda'(0)$ at $t = 0$ gives an element of the tangent space $T_{\Lambda(0)}\text{Lag}(V, \omega) \subset T_{\Lambda(0)}\text{Gr}(n, V) \cong \text{Hom}(\Lambda(0), V/\Lambda(0))$ where $\text{Gr}(n, V)$ is the Grassmannian of $n$-dimensional subspaces in $V$. The composition of $\Lambda'(0)$ and $\omega$ defines a quadratic form $\Lambda(0) \ni v \mapsto \omega(v, \Lambda'(0)v)$ on $\Lambda(0)$, which descends to a quadratic form on $\Lambda(0)/\Lambda(0) \cap \Lambda(1)$ since it vanishes on $\Lambda(0) \cap \Lambda(1)$. The resulting form on $\Lambda(0)/\Lambda(0) \cap \Lambda(1)$ is called the crossing form [13]. For an intersection $p \in L_1 \cup L_2$ of two graded Lagrangian submanifolds $(L_1, \tilde{\phi}_L_1)$ and $(L_2, \tilde{\phi}_L_2)$, its Maslov index is defined as follows: Choose a crossingless path $\Lambda : [0, 1] \to \text{Lag}_M|_p$ from $T_p L_0$ to $T_p L_1$ such that the corresponding crossing form at $t = 0$ is negative definite. There is a unique lift $\tilde{\alpha} : [0, 1] \to \mathbb{R}$ of the composition $\alpha$ of $\Lambda : [0, 1] \to \text{Lag}_M|_p$ and $\det_2^\Theta(p) : \text{Lag}_M|_p \to U(1)$ such that $\tilde{\alpha}(0) = \tilde{\phi}_L_0(p)$, and the Maslov index $I(p)$ of the intersection $p$ of two graded Lagrangian submanifolds is defined by

$$I(p) = \tilde{\phi}_L_1(p) - \tilde{\alpha}(1).$$

Now we come back to our exact Morse fibration $W$. A relative Maslov map is a section $\Theta$ of the second tensor power $(\Omega^{n-1}Z/\mathbb{C})^\otimes 2$ of the top exterior product of the relative cotangent bundle away from the critical points. Since 0 is a regular value, the restriction of $\Theta$ to $W^{-1}(0)$ gives a grading of $W^{-1}(0)$. Assume that all the vanishing cycles can be graded, and fix a grading on each vanishing cycle. We also need to choose a spin structure on each vanishing cycle in order to orient the moduli spaces of pseudoholomorphic maps [6]. Since each vanishing cycle is homeomorphic to a sphere, the choice of a spin structure is unique except for $\dim_C W^{-1}(0) = 1$, where one has as many as $H^1(S^1, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ choices. We take the non-trivial spin structure in such a case. Let $C_i^\natural$ denote the vanishing cycle $C_i$ endowed with the above grading and spin structure. We assume that vanishing cycles intersects each other transversally. This condition can always be met by moving vanishing cycles within their Hamiltonian isotopy classes if necessary.

**Definition 3.2.** Given a function $W$ on an affine Kähler manifold $Z$ together with a relative Maslov map $\Theta$ and a choice of a distinguished basis of vanishing cycles $(C_i^\natural)_{i=1}^N$ with gradings and spin structures, its directed Fukaya category $\mathfrak{Fut}^{-\infty}W$ is an $A_\infty$-category such that
• the set of objects is the distinguished basis of vanishing cycles;

\[ \Omega \! \mathfrak{f}(\mathfrak{u}^W) = (C^0_1, \ldots, C^0_N), \]

• the set of morphisms between \( C^0_i \) and \( C^0_j \) is the \( \mathbb{Z} \)-graded vector space

\[
\text{hom}_{\mathfrak{u}^W}(C^0_i, C^0_j) = \begin{cases} 
0 & i > j, \\
\mathbb{C} \cdot \text{id}_{C^0_i} & i = j, \\
\bigoplus_{p \in C^0_i \cap C^0_j} \text{span}_\mathbb{C}\{p\} & i < j
\end{cases}
\]

where \( \deg p = I(p) \) (the Maslov index), and

• for a positive integer \( k \), a set of objects \( (C^0_{i_0}, \ldots, C^0_{i_k}) \) and morphisms \( p_l \in C_{i_l-1} \cap C_{i_l} \) for \( l = 1, \ldots, k \), the composition \( m_k \) is given by

\[
m_k(p_1, \ldots, p_k) = \sum_{p_0 \in C_{i_0} \cap C_{i_k}} \#\mathcal{M}_{k+1}(C_{i_0}, \ldots, C_{i_k}; p_0, \ldots, p_k)p_0.
\]

Here, \( \mathcal{M}_{k+1}(C_{i_0}, \ldots, C_{i_k}; p_0, \ldots, p_k) \) is the stable compactification of \( \mathcal{M}_{k+1}(C_{i_0}, \ldots, C_{i_k}; p_0, \ldots, p_k) \) defined as follows: A disk with \( k + 1 \) marked points on the boundary is a pair \( (D^2, (z_0, \ldots, z_k)) \) of a closed unit disk \( D^2 = \{ z \in \mathbb{C} \mid |z| \leq 1 \} \) and an ordered set \( (z_0, \ldots, z_k) \) of \( k + 1 \) points on the boundary respecting the cyclic order. Let \( \partial_l D^2 \subset \partial D^2 \) be the interval between \( z_l \) and \( z_{l+1} \), where we set \( z_{k+1} = z_0 \). Fix an almost complex structure \( J \) on \( W^{-1}(0) \). A smooth map \( \varphi : D^2 \rightarrow M \) is called pseudoholomorphic if

\[
d\varphi \circ J_{D^2} = J \circ d\varphi
\]

where \( J_{D^2} \) is the canonical complex structure on \( D^2 \). \( \mathcal{M}_{k+1}(C_{i_0}, \ldots, C_{i_k}; p_0, \ldots, p_k) \) is the moduli space of pairs \( ((D^2, (z_0, \ldots, z_k)), \varphi) \) such that

1. \( (D^2, (z_0, \ldots, z_k)) \) is a disk with \( k + 1 \) marked points on the boundary,
2. \( \varphi : D^2 \rightarrow W^{-1}(0) \) is a pseudoholomorphic map,
3. \( \varphi(\partial_l D^2) \subset C_{i_l} \) for \( l = 0, \ldots, k \), and
4. \( \varphi(z_l) = p_l \) for \( l = 0, \ldots, k \).

Although \( \overline{\mathcal{M}}_{k+1}(C_{i_0}, \ldots, C_{i_k}; p_0, \ldots, p_k) \) is not an honest manifold in general due to subtleties in the definition of the moduli space of pseudoholomorphic maps, it has a Kuranishi structure with corners and it makes sense to "count" numbers \( \#\overline{\mathcal{M}}_{k+1}(C_{i_0}, \ldots, C_{i_k}; p_0, \ldots, p_k) \) of points. See [6] for details. These numbers are counted with signs determined by the orientation
of the moduli space, which in turn is determined by the spin structures on vanishing cycles. \( \#\mathcal{M}_{k+1}(C_{i_0}, \ldots, C_{i_k}; p_0, \ldots, p_k) \) is zero if \( \dim \mathcal{M}_{k+1}(C_{i_0}, \ldots, C_{i_k}; p_0, \ldots, p_k) \neq 0. \)

These moduli spaces must be compact so that it makes sense to count the numbers of points. To prove compactness of the moduli space, one needs the boundedness of the energy. Usually, this is achieved by introducing the Novikov ring as the coefficient ring of the Floer complex in order to control energies of pseudoholomorphic maps. This is not necessary in our case since vanishing cycles are exact Lagrangian submanifolds, i.e., \( [\theta] = 0 \in H^1(C_1, \mathbb{R}) \) for the primitive \( \theta \) of \( \omega; \omega = d\theta \). To see this, let \( C \) be the vanishing cycle corresponding to a vanishing path \( c : [0, 1] \to \mathbb{C} \). Then \( C \) bounds the disk \( D = \bigcup_{p \in C} \widetilde{c}_p([0, 1]); \partial D = C \). Since \( d\theta|_D = \omega|_D = 0, [\theta] = 0 \in H^1(\partial D, \mathbb{R}) \).

Therefore, there exists a function \( K_i \) on each vanishing cycle such that \( \theta|_C = dK_i \). Then for any \( \varphi \in \mathcal{M}_{k+1}(C_{i_0}, \ldots, C_{i_k}; p_0, \ldots, p_k) \), the energy of \( \varphi \) is

\[
E(\varphi) = \int_{D^2} \varphi^* \omega = \int_{\partial D^2} \varphi^* \theta = \sum_{l=0}^{k} \int_{\partial_l D^2} \varphi^* \theta = \sum_{l=0}^{k} (K_i(p_{l+1}) - K_i(p_l))
\]

and does not depend on the homotopy class of \( \varphi \). This also implies the vanishing of \( m_0 \). Non-compactness of \( W^{-1}(0) \) does not cause any problem either. Since \( W^{-1}(0) \) is Stein, the maximum principle prevents pseudoholomorphic disks from running away to infinity, assuring the compactness of the moduli space.

Seidel proved that although the directed Fukaya category \( \mathfrak{su}^\to W \) depends on the choice of the distinguished basis of vanishing cycles, different choices are related by mutations, hence its derived category is invariant.

Now we explain the mirror construction of toric Fano manifolds after Givental [7]. Given a fan of an \( n \)-dimensional toric Fano manifold, let \( \{v_i\}_{i=1}^r, v_i = (v_{i_1}, \ldots, v_{in}) \in \mathbb{Z}^n \), be the set of generators of its one-dimensional cones. Then the mirror object for this toric Fano manifold is the regular function

\[
W(x_1, \ldots, x_n) = \sum_{i=1}^{r} q_i x_1^{v_{i1}} \cdots x_n^{v_{in}}
\]

on the algebraic torus \( \text{Spec} \mathbb{C}[x_i^{\pm 1}]_{i=1}^n \). Here, \( q_i \)'s are parameters corresponding to the deformation of symplectic structures on the toric Fano manifold.

Therefore, the mirror of our \( Y \) is the regular function

\[
W(x, y) = q_1 x + q_2 y + \frac{q_3}{xy} + \frac{q_4}{x} + \frac{q_5}{y} + q_6 xy
\]
on the algebraic torus \((\mathbb{C}^\times)^2 = \text{Spec}\mathbb{C}[x, x^{-1}, y, y^{-1}]\) (See Figure 1). The Fukaya category does not depend on a general choice of \(q_i\)'s. We have used \((q_1, q_2, q_3, q_4, q_5, q_6) = (1, 1, 1, 0.215, 0.25, 0.3)\) to draw the figures appearing below. Equip \((\mathbb{C}^\times)^2\) with the symplectic form \(\frac{dx}{|x|} \wedge \text{d}(\text{arg } x) + \frac{dy}{|y|} \wedge \text{d}(\text{arg } y)\) and the relative Maslov map

\[
\Theta = (\text{Res } \frac{dx \wedge dy}{x y W(x, y)})^{\otimes 2} = \left( \frac{dx}{\partial_y (x y W(x, y))} \right)^{\otimes 2}.
\]

\(W^{-1}(0)\) is an affine elliptic curve, which can be compactified by adding six points. We depict the critical values of \(W\) and our choice of a distinguished set of vanishing paths in Figure 2. Figure 3 shows the corresponding vanishing cycles.

Figure 2: Distinguished basis of vanishing paths

The opposite sides of the square in Figure 3 are identified to form a two-dimensional torus. The open circles denote the points which are missing due to the non-compactness of \((\mathbb{C}^\times)^2\). We have drawn Figure 3 in the following way: First regard \(W^{-1}(t)\) as a branched double cover of \(\mathbb{C}^\times\) by the projection \(\pi_t : W^{-1}(t) \ni (x, y) \mapsto y \in \mathbb{C}^\times\). For any \(i = 1, \ldots, 6\), the branch points of \(\pi_{c_i(t)}\) moves in \(\mathbb{C}^\times\) as we vary \(t\), until two of them finally collides at \(t = 1\). The vanishing cycle \(C_i\) is the circle in \(W^{-1}(0)\) over this trajectory of collision. This determines the vanishing cycle \(C_i\) up to isotopy. These vanishing cycles can be straightened within their Hamiltonian isotopy classes. Here, being straight refers to the flat metric on the torus \(W^{-1}(0)\), where \(\bullet\) denotes the
Figure 3: Vanishing cycles in the fiber at the origin
completion. Note that this flat metric on \( W^{-1}(0) \) has nothing to do with the Kähler metric on \( W^{-1}(0) \) induced from that of \( (\mathbb{C}^*)^2 \). This determines \( C_i \) up to translation. This translational ambiguity can be fixed by imposing exactness, which requires the knowledge of the primitive \( \theta \) of the symplectic form \( \omega \) on \( W^{-1}(0) \). We do not try to carry this out since translations of straight \( C_i \)'s do not alter the combinatorial structure of intersections, which is all we need in our computation of the Fukaya category below. The grading on \( W^{-1}(0) \) given by \( \Theta \) is the grading coming from the restriction of the second tensor power of the holomorphic 1-form on \( W^{-1}(0) \). Then one can choose a grading \( \tilde{\phi} \) on each vanishing cycle \( C_i \) so that all the Maslov indices \( C_i \cap C_j \) is zero for \( i < j \). To give a spin structure on \( C_i \) is the same as to give a two-fold covering of \( C_i \). We take the non-trivial cover such that two branches interchange on the black dots in Figure 3. Different choices for spin structures lead to different categories, and it turns out that the above choice gives a category derived equivalent to the category of coherent sheaves on the del Pezzo surface \( Y \). Let \( C_i^\circ \) denote the vanishing cycle \( C_i \) with the above grading and spin structure. Since the Maslov index of all the intersection points are zero, the sign for the counting of a triangle is \(-1\) if the boundary of the triangle hits odd numbers of these dots, and \(+1\) otherwise (see Seidel [17]). Note that although the choice of spin structures is important, the choice of the positions of the black dots is irrelevant. The change of the signs caused by a change of the positions of the black dots can be absorbed by a redefinition of the signs of the basis for the Floer cohomologies. \( m_k \) is non-zero only for \( k = 2 \) (polygons with more than three edges in Figure 3 do not contribute since \( \text{hom}_{\text{Fuk}}(C_i^\circ, C_j^\circ) = 0 \) for \( i > j \)).

**Theorem 3.3.** There exists an isomorphism

\[
\phi_{ij} : \text{Hom}_{D^b\text{coh}(Y)}(E_i, E_j) \to \text{Hom}_{\text{Fuk}(W)}(C_i^\circ, C_j^\circ)
\]

as a \( \mathbb{C} \)-vector space for \( i, j = 1, \ldots, 6 \) such that the diagrams

\[
\text{Hom}_{D^b\text{coh}(Y)}(E_i, E_j) \times \text{Hom}_{D^b\text{coh}(Y)}(E_j, E_k) \longrightarrow \text{Hom}_{D^b\text{coh}(Y)}(E_i, E_k)
\]

\[
\begin{array}{c}
\phi_{ij} \times \phi_{jk} \\
\phi_{ik}
\end{array}
\]

(4)

commute.

**Proof.** We omit the lower suffix of \( \text{Hom}(\bullet, \bullet) \) since there is no danger of confusion. We construct the above \( \phi_{ij} \)'s explicitly. Since \( m_1 = 0 \), \( \text{Hom}(C_i^\circ, C_j^\circ) \) is isomorphic to \( \text{hom}(C_i^\circ, C_j^\circ) \) and spanned by intersection points of \( C_i \) and \( C_j \), to which we assign the basis of \( \text{Hom}(E_i, E_j) \) as in Figure 3. Notations
for the basis of \( \text{Hom}(\mathcal{E}_i, \mathcal{E}_j) \) is given in Appendix. This defines the linear isomorphisms \( \phi_{ij} \)'s. The commutativity of (4) is verified by counting triangles. Let us illustrate this with a few examples. Take \( x_1 \in \text{Hom}(\mathcal{E}_1, \mathcal{E}_4) \) and \( x_1^\vee \in \text{Hom}(\mathcal{E}_4, \mathcal{E}_6) \), whose composition is \( e_1 \in \text{Hom}(\mathcal{E}_1, \mathcal{E}_6) \). On the Fukaya side, we count the number of triangles whose edges are contained in \( C_1 \cup C_4 \cup C_6 \) and two of whose vertices are \( x_1 \) and \( x_1^\vee \). Such a triangle exists uniquely, whose remaining vertex is \( e_1 \) (Figure 4).

![Figure 4: Counting triangle](image)

Next we try to compose \( x_3 \in \text{Hom}(C_3^b, C_4^b) \) and \( x_1^\vee \in \text{Hom}(C_4^b, C_6^b) \) in the Fukaya category. The only possibility would be the triangle with \( e_3 \) as the remaining vertex. However, this is not allowed because of the missing point (Figure 5).
This shows that the composition of $x_3$ and $x_1^\vee$ in the Fukaya category is zero, which matches the table in Appendix calculated in $D^b\text{coh}(Y)$. Finally, as an example of counting with signs, we compute

$$\text{Hom}(C_1^\flat, C_4^\flat) \times \text{Hom}(C_4^\flat, C_5^\flat) \ni (x_1, x_1^\vee \wedge x_2^\vee) \mapsto -x_2^\vee \in \text{Hom}(C_1^\flat, C_5^\flat).$$

Indeed, the triangle whose vertices are $x_1, x_1^\vee \wedge x_2^\vee$, and $x_2^\vee$ contains one black dot on its edge (Figure 6).
We can do similar analyses for the rest of the triangles. The result perfectly agrees with the computations in $D^b \text{coh}(Y)$.

Although the derived category of an $A_\infty$-category is usually defined using twisted complexes, we adopt the following definition in this paper since our $\mathfrak{fr} W$ satisfies $\text{hom}^k_{\mathfrak{fr} W}(C_i, C_j) = 0$ for $k \neq 0$ and $m_k = 0$ for $k \neq 2$.

**Definition 3.4.** Let $A = \bigoplus_{i,j} \text{Hom}_{D^b \mathfrak{fr} W}(C_i, C_j)$ be the total morphism algebra. The derived Fukaya category $D^b \mathfrak{fr} W$ is the bounded derived category $D^b(\text{mod-}A)$ of the category of right finite-dimensional modules over the algebra $A$.

Now we can state our main theorem:

**Theorem 3.5.** There exists an equivalence of triangulated categories

$$D^b \text{coh}(Y) \cong D^b \mathfrak{fr} W.$$ 

**Proof.** From Theorem 3.3, we have

$$A \cong \bigoplus_{i,j} \text{Hom}_{D^b \text{coh}(Y)}(\mathcal{E}_i, \mathcal{E}_j).$$

Theorem 3.5 follows immediately from the theorem of Bondal [4] that $D^b \text{coh}(Y) \cong D^b(\text{mod-}(\bigoplus_{i,j} \text{Hom}(\mathcal{E}_i, \mathcal{E}_j))).$

We can perform similar analyses for all the other toric del Pezzo surfaces to obtain Theorem 1.1.

4 **Appendix**

Here we give the table of compositions of morphisms in the derived category of coherent sheaves on the projective plane $\mathbb{P}(V)$ blown-up at $p_1 = [1 : 0 : 0]$, $p_2 = [0 : 1 : 0]$, $p_3 = [0 : 0 : 1]$. We use the following notations: $\mathcal{E}_i$’s are defined in [1], $\{x_i\}_{i=1}^3$ is the basis of $V$, $\{x_i^\vee\}_{i=1}^3$ is the dual basis of $V^\vee$, and $e_i$ is the generator of $\text{Hom}(\mathcal{E}_i, \mathcal{E}_6) = \mathbb{C}$ for $i = 1, 2, 3$.

$\text{Hom}(\mathcal{E}_1, \mathcal{E}_4) \times \text{Hom}(\mathcal{E}_4, \mathcal{E}_5) \to \text{Hom}(\mathcal{E}_1, \mathcal{E}_5)$

$$
\begin{array}{c|cc|c}
  & x_1^\vee \wedge x_2^\vee & x_2^\vee \wedge x_3^\vee & x_3^\vee \wedge x_1^\vee \\
\hline
x_1 & -x_2^\vee & 0 & x_3^\vee \\
x_2 & x_3^\vee & 0 & -x_1^\vee \\
x_3 & -x_1^\vee & x_2^\vee & 0 \\
\end{array}
$$

$\text{Hom}(\mathcal{E}_1, \mathcal{E}_4) \times \text{Hom}(\mathcal{E}_4, \mathcal{E}_6) \to \text{Hom}(\mathcal{E}_1, \mathcal{E}_6)$

$$
\begin{array}{c|cc|c}
  & x_1^\vee & x_2^\vee & x_3^\vee \\
\hline
x_1 & e_1 & 0 & 0 \\
x_2 & 0 & e_1 & 0 \\
x_3 & 0 & 0 & e_1 \\
\end{array}
$$
\[
\begin{align*}
\text{Hom}(\mathcal{E}_1, \mathcal{E}_5) \times \text{Hom}(\mathcal{E}_5, \mathcal{E}_6) & \rightarrow \text{Hom}(\mathcal{E}_1, \mathcal{E}_6) \\
| & x_1 & x_2 & x_3 \\
x_2^\vee & 0 & e_1 & 0 \\
x_3 & 0 & 0 & e_1
\end{align*}
\]

\[
\begin{align*}
\text{Hom}(\mathcal{E}_2, \mathcal{E}_4) \times \text{Hom}(\mathcal{E}_4, \mathcal{E}_5) & \rightarrow \text{Hom}(\mathcal{E}_2, \mathcal{E}_5) \\
| & x_1^\vee \wedge x_2^\vee & x_2^\vee \wedge x_3^\vee & x_3^\vee \wedge x_1^\vee \\
x_2 & x_1^\wedge & x_2^\wedge & x_3^\wedge & 0
\end{align*}
\]

\[
\begin{align*}
\text{Hom}(\mathcal{E}_2, \mathcal{E}_4) \times \text{Hom}(\mathcal{E}_4, \mathcal{E}_6) & \rightarrow \text{Hom}(\mathcal{E}_2, \mathcal{E}_6) \\
| & x_1^\vee & x_2^\vee & x_3^\vee \\
x_2 & 0 & e_2 & 0
\end{align*}
\]

\[
\begin{align*}
\text{Hom}(\mathcal{E}_2, \mathcal{E}_5) \times \text{Hom}(\mathcal{E}_5, \mathcal{E}_6) & \rightarrow \text{Hom}(\mathcal{E}_2, \mathcal{E}_6) \\
| & x_1^\vee & x_2 & x_3 \\
x_1^\vee & e_2 & 0 & 0 \\
x_3 & 0 & 0 & e_2
\end{align*}
\]

\[
\begin{align*}
\text{Hom}(\mathcal{E}_3, \mathcal{E}_4) \times \text{Hom}(\mathcal{E}_4, \mathcal{E}_5) & \rightarrow \text{Hom}(\mathcal{E}_3, \mathcal{E}_5) \\
| & x_1^\vee \wedge x_2^\vee & x_2^\vee \wedge x_3^\vee & x_3^\vee \wedge x_1^\vee \\
x_3 & 0 & x_2^\wedge & -x_1^\wedge
\end{align*}
\]

\[
\begin{align*}
\text{Hom}(\mathcal{E}_3, \mathcal{E}_4) \times \text{Hom}(\mathcal{E}_4, \mathcal{E}_6) & \rightarrow \text{Hom}(\mathcal{E}_3, \mathcal{E}_6) \\
| & x_1^\vee & x_2^\vee & x_3^\vee \\
x_3 & 0 & 0 & e_3
\end{align*}
\]

\[
\begin{align*}
\text{Hom}(\mathcal{E}_3, \mathcal{E}_5) \times \text{Hom}(\mathcal{E}_5, \mathcal{E}_6) & \rightarrow \text{Hom}(\mathcal{E}_3, \mathcal{E}_6) \\
| & x_1^\vee & x_2 & x_3 \\
x_1^\vee & e_2 & 0 & 0 \\
x_2^\vee & 0 & e_3 & 0
\end{align*}
\]

\[
\begin{align*}
\text{Hom}(\mathcal{E}_4, \mathcal{E}_5) \times \text{Hom}(\mathcal{E}_5, \mathcal{E}_6) & \rightarrow \text{Hom}(\mathcal{E}_4, \mathcal{E}_6) \\
| & x_1^\vee \wedge x_2^\vee & x_2^\vee & -x_1^\vee & 0 \\
x_3^\vee & x_1^\wedge & x_3^\wedge & -x_2^\wedge & 0 \\
x_3^\vee & -x_3^\wedge & 0 & x_1^\wedge & 0
\end{align*}
\]

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