The solution to a multichannel Bethe potential and its application to pion-nucleus reactions.

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Abstract

We solve a radial Schrödinger equation for the case of a multichannel square well plus an exponential potential in one of the channels. The solution is obtained by summing exactly the infinite terms of the perturbative series for the evolution operator of the system. Analytical expressions for the scattering matrix and the scattering length are given, a Deser-like formula which connects approximate ground energies and scattering length is obtained for a particular case. Possible physical applications of these results are discussed briefly.
1 Introduction

The interactions of pions with nuclei or with other pions have received considerable attention since the sixties as possible tools for the study of various aspects of the nuclear structure or in order to check the validity of diverse effective theories.

In any process involving bound pions, it is crucial to know the value of the pion atomic wave function in the nucleus. The pions are generally trapped in atomic orbits with large radial quantum numbers, from which they are very unlikely to be absorbed by the nucleus. However, it is possible they cascade downwards, first making Auger transitions between higher levels and subsequently by X-ray transitions. It has been known since long that for nuclei with $Z \leq 10$ some of the pions succeed in reaching the 1s orbit before being absorbed by the nucleus. The absorption rate is roughly proportional to the probability of finding the meson inside the nucleus. This probability for the 1s state pions can be altered significantly both by the pion-nucleus strong interaction and by the finite charge radius of the nucleus. Early numerical studies and experimental data in pionic 2p-1s transitions (for example [1, 2, 3]) made clear that standard perturbation theory was not adequate for the calculation of nuclear pionic wave functions, energy level shifts and widths, even for relatively light nuclei.

Two experimental effects relevant to the study of the interactions of pions with nuclei have recently appeared in nuclear physics: the existence of deeply bound narrow pionic states in large $Z$ nuclei and the nuclear halo.

The deeply bound pionic states in Pb, recently discovered at GSI ([4]) and previously predicted in ([5, 6]), have stimulated a renewed interest in the nature of the s-wave part of the pion-nucleus optical potential. For such states to exist with sufficiently long lifetime, there must be a subtle cancellation between the $\pi^-$ coulomb and strong interactions in the bulk nucleus. The effective s-wave $\pi$-nucleon repulsion must roughly compensate the attractive Coulomb force in the interior of the nucleus. In this way the 1s pionic bound states in heavy systems may provide strong quantitative constrains on the s-wave $\pi$ interactions with nucleons.

The spectra of deeply-bound pionic atoms have been calculated by finding the numerical solution of a Klein-Gordon equation with a standard pion-nucleus optical potential ([3]). The widths of the deeply bound states were found to be narrow even for the 1s state of very large $Z$. As an alternative to Auger transitions, “pion transfer” reactions were proposed by the same authors as a practical way of producing deeply bound pionic states. An incoming sufficiently energetic neutron incident on a target nucleus changes into an outgoing proton emitting a $\pi^-$, which then gets captured by the target nucleus and forms a deeply-bound pionic state. The process $n \to p\pi^-$ does not take place in free space. Hence, at least one of the three particles must be necessarily off-shell, i.e. the $\pi^-$ with highest probability. The $(n, p)$ differential cross section can be written, under the Born approximation and supposing plane wave functions for the incoming and outgoing neutron and proton, as

$$\frac{d\sigma}{d\Omega} \propto |\Psi(q)|^2;$$
where $\Psi(q)$ is the Fourier transform of the pionic wave function. It is then important to have reliable methods for the computation of this wave function.

The neutron halo is a relatively recent discovery in nuclear physics, which affects neutron rich nuclei. This halo effect, which can maintain a significant neutron density even at distances of several nuclear radius, can be relevant in diverse neutron-capture reactions by neutron-rich nuclei. The neutron halo, being essentially repulsive for $\pi^-$ and existing over a large distance, could favor the existence of exceptionally narrow pion bound states for relatively low $Z$ nuclei or at least modify substantially the properties of such pionic atoms. Although numerical calculations of Schrödinger or Klein Gordon equations with arbitrary complex potentials can be performed easily, it is also convenient to have exact analytical solutions, even for simplified models, in order to study systematically such effects. These solutions can in fact help us to understand some general properties and serve as a benchmark for perturbative approximations and numerical calculations. The characteristics of the pion transfer reactions also make highly desirable to count on a multichannel framework as an alternative to optical potential methods.

In this work we present the technical details of the solution of a Schrödinger equation with a multichannel well potential plus a Bethe potential in one of the channels. In future publications we intend to use such a potential, complemented with a Coulombic potential in an external region, to describe the general properties of pionium and pionic bound states in exotic atoms as a function of $A$ and $Z$: pionic wave function, level shifts and widths. The exponential Bethe potential can be used at will to parametrize locally the quadratic electromagnetic potential inside an uniformly charged nucleus of arbitrary size or the decaying potential originated by an extended neutron density. The shortcomings of using a non-relativistic formalism are compensated by the advantages of having an exact analytical multichannel computation.

Bethe was the first to use an exponential potential to make quantitative calculations and explain various general features of cross sections for processes involving nuclei and nucleons. He used a function of the form $V(r) \sim e^{-r}$ to describe the complex absorptive part of the short range nuclear potential. This choice was motivated essentially by the possibility of solving exactly the corresponding Schrödinger equation. The same potential has also been employed to estimate nuclear transmission functions. The model comprising two square-well potentials coupled by a square-well interaction has been used by many authors in simple specific models of nuclear, atomic and molecular physics or as a test of various reaction theories. By comparing the results obtained from reaction theory approximate calculations with the exact analytic solutions one gets information about the rates of convergence of the various formulations.

The work presented here combines multichannel square-wells with the exponential Bethe potential. The method which has been used, i.e., summing over the infinite terms of a perturbative series for the evolution operator of the system, has at least the following advantages. First of all it is of intrinsic interest; it is, to our knowledge, the first time that such a method has been used for the solution of a second order Schrödinger equation. Moreover it is easily generalizable to other multichannel systems and allows the straight-
forward introduction of constant non-local derivative potentials. Finally the evolution operator formalism is specially suited, by its own nature, for systems where internal and external potentials coexist, as well as the use in conjunction with R-matrix methods.

2 Solving the Schrödinger equation. One channel case.

Consider the radial s-wave Schrödinger equation for a central potential of exponential profile:

$$\frac{d^2 y}{dr^2} + (q^2 + \rho e^{-\lambda r})y(r) = 0.$$  \hspace{1cm} (1)

The constants $q$, $\rho$ and $k$ are defined by the expressions $q^2 \equiv k^2 - 2mV_N \equiv 2m(E - V_N)$ and $\rho = 2m\rho_0$. The two quantities $m$ and $E$ are respectively the mass and the energy of the system; $V_N$ is a constant term which is introduced for convenience and $y(r) \equiv rR(r)$, where $R(r)$ is the radial part of the wave function of the system.

One can reduce Eq. (1) to a first order system of two equations, by defining a two dimensional vector $\Psi(r)$ (formed by the function $y(r)$ and its derivative) as follows:

$$\Psi(r) = \begin{pmatrix} dy(r)/dr \\ y(r) \end{pmatrix}.$$ \hspace{1cm} (2)

With this definition, Eq. (1) is equivalent to the system:

$$\frac{d\Psi(r)}{dr} = \left( H_0 - \rho e^{-\lambda r} V \right) \Psi(r),$$ \hspace{1cm} (3)

where:

$$H_0 = \begin{pmatrix} 0 & -q^2 \\ 1 & 0 \end{pmatrix}; \hspace{0.5cm} V = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \hspace{1cm} (4)
$$

We perform a linear transformation $\Psi = u \Psi_D$ in order to diagonalize $H_0$. We arrive at the equation:

$$i \frac{d\Psi_D}{dr} = H(r) \Psi_D(r) = (H_D - \rho_D e^{-\lambda r} W) \Psi_D,$$ \hspace{1cm} (5)

where $\rho_D = \rho/2q$ and

$$W = 2iq \ u^{-1} V u = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}; \hspace{0.5cm} H_D = i \ u^{-1} H_0 u = \begin{pmatrix} -q \\ 0 \\ 0 \\ q \end{pmatrix}; \hspace{1cm} (6)
$$

$$u = \begin{pmatrix} iq \\ -iq \end{pmatrix}.$$ \hspace{1cm} (7)

Note that in the process of passing from the second order equation to the first order one, the Hamiltonian has become “squared rooted”: the eigenvalues of $H_0$ are the roots of the “operator” $q^2$ and the full Hamiltonian fulfills the relation:

$$\left( H_D - \rho_D e^{-\lambda r} W \right)^2 = q^2 + \rho e^{-\lambda r}. \hspace{1cm} (8)$$
We have reduced our original problem to the resolution of a “time-dependent” Schrödinger equation with a non hermitic Hamiltonian. The evolution operator $U$ of the differential system (5) is given by the path-ordered exponential:

$$U(r, r_0) = P \exp \left[ -i \int_{r_0}^{r} ds H(s) \right].$$

(9)

At least formally, it is always possible to give a perturbative expansion for the operator $U$:

$$U(r, r_0) = U^{(0)}(r, r_0) + \sum_{n=1}^{\infty} U^{(n)}(r, r_0); \quad U^{(0)}(r, r_0) = \exp \left[ -i H_D(r - r_0) \right].$$

(10)

The $n$-term $U^{(n)}$ in the expansion is given by the integral

$$U^{(n)}(r, r_0) = (-i)^n \int_{\Gamma} d\tau_n \cdots d\tau_1 U^0(r, \tau_n)W(\tau_n) \cdots U^0(\tau_2, \tau_1)U^0(\tau_1, r_0).$$

(11)

In the previous formula we used the notation: $W(\tau) = -\rho_D \exp(-\lambda \tau)W$. The domain of integration is defined by:

$$\Gamma \equiv r > \tau_n > \ldots > \tau_1 > r_0.$$

One can show (following the same arguments as in [15]) that the computation of $U$ in the limit when $r \to \infty$ suffices in order to obtain the general evolution operator. The evolution operator for a finite interval $(r, r_0)$ can be expressed as the product of two operators defined respectively at the points $r$ and $r_0$, which are essentially instances of the asymptotic $U$.

We get the following expression for the elements of $U^{(n)}$ in a basis of eigenvectors of $H_D$ through elementary manipulations of Eq.(11):

$$U^{(n)}_{ba} = (-i)^n \exp(-i(E_b r - E_a r_0)) \times \sum_{k_1, \ldots, k(n-1)} \int_{\Gamma} d^n \tau \exp(i(\tau_n \omega_{k_1} + \ldots + \tau_1 \omega_{k(n-1)a}))W_{k_1}(\tau_n) \cdots W_{k(n-1)a}(\tau_1)$$

(12)

where $\omega_{k_1k_2} = E_{k_1} - E_{k_2}$; $E_k$ denotes one of the two eigenvalues of $H_D$. The matrix $W$ is such that $W^2 = 0$. It has a dyadic form ($W_{ij} = A_i B_j$) and hence the product of two contiguous factors in the integral (12) is

$$W_{ij}W_{jl} = W_{il}W_{jj}.$$

By applying repeatedly the previous relation, Eq.(12) becomes ($B_k = W_{kk}$):

$$U^{(n)}_{ba} = (-i)^n \exp(-i(E_b r - E_a r_0)) W_{ba} \times \sum B_{k_1} \cdots B_{k(n-1)} A_{k_1 \cdots k(n-1)};$$

$$(13)$$

$$A_{k_1 \cdots k(n-1)} = \int_{\Gamma} d^n \tau \exp(\tau_n (i \omega_{k_1} - \lambda) + \ldots + \tau_1 (i \omega_{k(n-1)a} - \lambda))(-\rho_D)^n.$$
The multiple sum runs independently over all the eigenvalues of the Hamiltonian. The limit \( r \to \infty \) is understood in the domain of integration. One can use then the following identity

\[
I_n(\omega_1, \ldots, \omega_n) \equiv \int_{r_0}^\infty \cdots \int_{r_0}^{x_2} d^n x \exp(\sum_i \omega_i x_i) = \frac{(-1)^n \exp(r_0 \sum_i \omega_i)}{\omega_n(\omega_n + \omega_{n-1}) \cdots (\omega_n + \omega_{n-1} + \cdots \omega_1)}
\]

in order to arrive at the expression:

\[
A_{k1 \ldots k(n-1)} = \frac{\rho_D^n \epsilon^{n_0}(iw_{ba} - n\lambda)}{i w_{ba} - n\lambda} \left( \frac{1}{(i w_{bk1} - \lambda) \cdots (i w_{bk(n-1)} - (n-1)\lambda)} \right). \tag{15}
\]

Inserting the expression (15) in Eq.(13), we obtain for the n-order matrix elements of U:

\[
U_{ba}^{(n)} = e^{-iE_b(r-r_0)} W_{ba}(-i\rho_D)^n e^{-nr_0\lambda} \times \frac{1}{iw_{ba} - n\lambda} \\
\times \sum B_{k1} \cdots B_{k(n-1)} \frac{1}{(i w_{bk1} - \lambda) \cdots (i w_{bk(n-1)} - (n-1)\lambda)}
= e^{-iE_b(r-r_0)} W_{ba} \frac{(-i\rho_D)^n e^{-n\lambda_0}}{i w_{ba} - n\lambda} \prod_{m=1}^{n-1} \sum_{k=1,2} \frac{B_k}{iw_{bk} - m\lambda}
= e^{-iE_b(r-r_0)} W_{ba} \frac{(-i\rho_D)^n e^{-n\lambda_0}}{i w_{ba} - n\lambda} \prod_{m=1}^{n-1} \frac{i w_{12}}{(-i w_{b1} + m\lambda)(-i w_{b2} + m\lambda)}
= e^{-iE_b(r-r_0)} W_{ba} \left( \frac{\rho_D/\lambda e^{-\lambda_0}}{(-i\rho_D/\lambda)} \right)^n \frac{i (w_{12}/\lambda)^{n-1}}{(-i w_{b1}/\lambda + n)(-i w_{b2}/\lambda + n)(n-1)(-i w_{b1}/\lambda + n-1)(-i w_{b2}/\lambda + n-1)} \tag{16}.
\]

In the third line, we have used the fact that \( B_1 + B_2 = 0 \) and that the expression \( B_1 w_{b2} + B_2 w_{b1} = w_{12} \) holds for any value of \( b \). The expressions \( (a)_{(b)} = a(a+1) \cdots (a+n-1) \) are the Pochhammer symbols. The matrix elements of U are given by a convergent series which can be summed readily into generalized hypergeometric functions. We have for example for \( U_{11}, U_{12} \):

\[
U_{11} = e^{-iE_1(r-r_0)} \sum_{n=0}^{\infty} \left( \frac{\rho_D w_{12}}{\lambda^2} e^{-\lambda_0} \right)^n \frac{1}{n!} \frac{1}{(-i w_{12}/\lambda)^n} \tag{17}.
\]

\[
U_{12} = e^{-iE_1(r-r_0)} \sum_{n=0}^{\infty} \left( \frac{\rho_D w_{12}}{\lambda^2} e^{-\lambda_0} \right)^n \frac{1}{-i w_{12}/\lambda + n} \frac{1}{(n-1)!} \frac{1}{(-i w_{12}/\lambda)^n} \tag{18}.
\]
The expressions for the other two elements of the matrix $U$ are similar. The matrix $U$ in the limit $r \to \infty$ can be written as:

$$U(r \to \infty, r_0) = \exp(-i H D r) U_{\text{red}}(r_0),$$

$$U_{\text{red}}(r_0) = \exp(i H D_0) U_{s}(r_0),$$

where the matrix $U_s$ is:

$$U_s = \begin{pmatrix}
0 F_1(\alpha, z) & 0 F_1(\alpha, z) - 0 F_1(1 + \alpha, z) \\
0 F_1(-\alpha, z) & 0 F_1(-\alpha, z) - 0 F_1(-1 - \alpha, z)
\end{pmatrix}.$$ (21)

The variables appearing in Eq. (22) are: $\alpha = -iw_{12}/\lambda$, $z = \rho D w_{12}/\lambda^2 \exp(-\lambda r_0)$. More explicitly: $\alpha = 2iq/\lambda$, $z = -\rho/\lambda^2 \exp(-\lambda r_0)$. The trace of the Hamiltonian in Eq. (5) is zero. This property alone guarantees that, although $U$ is non unitary, its determinant $\det U_s = 1$. Hence a relation for the absolute values of the hypergeometric functions appearing in Eq. (21) can be obtained immediately as a by-product.

For finite $r$ the evolution operator for Eq. (5) is given by:

$$U(r, r_0) = U_{\text{red}}(r)^{-1} U_{\text{red}}(r_0).$$ (22)

The evolution operator for Eq. (3) is obtained by applying an inverse transformation with the matrix $u$ defined by Eq. (7):

$$U(r, r_0) = u U_{\text{red}}(r)^{-1} U_{\text{red}}(r_0) u^{-1}. $$ (23)

The general solution of Eq. (1) is given, in terms of initial conditions at $r_0$, by

$$y(r) = U_{21}(r, r_0)y'(r_0) + U_{22}(r, r_0)y(r_0). $$ (24)

Thus, a regular particular solution, that is, one which vanishes at origin, is given by

$$y(r) = C U_{21}(r, 0),$$

where $C$ is an arbitrary constant.

We have supposed throughout the computations that the parameter $\lambda$ is bigger than zero. However the solution is still valid for any value of $\lambda$ by analytic continuation. One can check that the function (24) is effectively a solution of our equation and coincides with the solution given by Bethe (10).

2.0.1 The scattering matrix and scattering length.

The solution to the radial Schrödinger equation may be written as

$$y(r) = y^+(r) + S(q)y^-(r),$$ (25)
where \( y^-(r) \) and \( y^+(r) \) denote appropriate ingoing and outgoing solutions to the Schrödinger equation with suitable asymptotic behavior at infinity. All the relevant information about the system as bound states and scattering parameters can be obtained from the function \( S(q) \). One can give an explicit formula for \( S(q) \) for the case in which the exponential potential extends from zero to infinity. In this case the asymptotic states can be read directly from Eq.(19) where the matrix \( H_D \) is diagonal (Eq.(6)). We can write, taking into account that \( \Psi = u \Psi_D \),

\[
y(r) = \Psi_{D,1}(r) + \Psi_{D,2}(r) .
\]

In order to have a function \( y(r) \) regular at the origin, the functions \( \Psi_{D,1}(r) \) and \( \Psi_{D,2}(r) \) have to be chosen as

\[
\Psi_{D}(r) \equiv \begin{pmatrix} \Psi_{D,1}(r) \\ \Psi_{D,2}(r) \end{pmatrix} = U(r,0)u^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} .
\]  

(26)

It follows from Eq.(19) that the asymptotic behavior of these functions is

\[
\Psi_{D,2}(r \to \infty) \sim A e^{-iqr}; \quad \Psi_{D,1} \sim B e^{iqr}.
\]  

(27)

The functions \( \Psi_{D1,2} \) are called the “Jost” functions of the system. The asymptotic behavior of \( y(r) \) is:

\[
y(r) = \frac{1}{2iq} \left[ e^{iqr} (U_{red,11}(0) - U_{red,12}(0)) + e^{-iqr} (U_{red,21}(0) - U_{red,22}(0)) \right] .
\]  

(28)

Given Eq.(25) and the expressions for the elements of \( U_{red} \) in Eqs.(20,21), we obtain the following expression for the scattering function:

\[
S(q) = - {}_0F_1(1 - 2iq/\lambda, -\rho/\lambda^2) / {}_0F_1(1 + 2iq/\lambda, -\rho/\lambda^2) .
\]  

(29)

The bound states correspond to poles of \( S(q) \) and the eigenvalue equation becomes

\[
{}_0F_1(1 + 2iq/\lambda, -\rho/\lambda^2) = 0 .
\]  

(30)

The scattering length \( A \) is defined by the limit

\[
\frac{1}{A} = - \lim_{q \to 0} \frac{1}{q} \tan \left( \arg S(q)/2 \right) .
\]

We can compute explicitly \( A \) in this case and from Eqs.(29) we deduce that:

\[
\frac{1}{A} = i \lim_{q \to 0} \frac{1}{q} {}_0F_1(1 + 2iq/\lambda, -\rho/\lambda^2) / {}_0F_1(1 + 2iq/\lambda, -\rho/\lambda^2) .
\]  

(31)

The hypergeometric function \( {}_0F_1(a,z) \) is analytic both in \( z \) and in \( a \). We can evaluate readily the limit in terms of its derivative with respect to the parameter \( a \):

\[
\frac{1}{A} = - 2 \frac{\partial {}_0F_1(1, -\rho/\lambda^2)}{\lambda} \frac{1}{\lambda} .
\]  

(32)
The value of the derivative $\partial_a F_1(a, z)$ for $a = 1$ can be computed deriving term by term the series of the hypergeometric function and inserting that value; the result is:

$$\partial_a F_1(1, z) = -\sum_{n=1}^{\infty} \frac{z^n}{n!n!} \sum_{m=1}^{n} \frac{1}{m}. \quad (33)$$

This series can be summed in terms of Bessel functions:

$$\partial_a F_1(1, z) = J_0(2i\sqrt{z})(\log i\sqrt{z} + C) - \frac{\pi}{2} N_0(2i\sqrt{z}); \quad (34)$$

$J_0(z)$ and $N_0(z)$ are the Bessel functions of the first and second kind respectively. In Eq. (34) $C$ is the Catalan number (see [16] for definitions).

It follows from the eigenvalue equation (30) and from the formula (32) that we can obtain a Deser-like formula [17] which connects the approximate first eigenvalue shift in $\pi$-mesic atoms to $\pi - N$ scattering data. If $q_0$ is a sufficiently small root of Eq. (30) then one finds approximately:

$$F_1(1, z) + \frac{2iq_0}{\lambda} \partial_a F_1(1, z) \simeq 0 \quad (35)$$

or

$$q_0 \simeq \frac{i\lambda}{2} \frac{\partial_a F_1(1, z)}{\partial_a F_1(1, z)} = -iA. \quad (36)$$

Compare this expression with the Deser formula which, under rather general assumptions for the form of the nuclear interaction, can be written as

$$\frac{\Delta q}{q} \propto A_0 \quad (37)$$

where, $q$ and $\Delta q$ are respectively the energy of the s-state Coulomb bound states and the shift produced by the short range nuclear interaction, $A_0$ is the scattering length defined in the absence of the long-range (Coulomb) field. This formula can be obtained treating the nuclear interaction in first Born approximation. Sophisticated extensions of the Deser formula and of Eq. (36) can be obtained by exploiting the analytical properties of the scattering function and the solutions of the Schroedinger equation; further development will appear in a independent publication.

3 The multichannel case.

We proceed now to the generalization of Eq. (1) to a multichannel case. Although for simplicity and economy of notation we will only work out the two channel case, the general case is obviously analogous. We consider the following matrix equation for the two component vector $y(r)$:

$$\frac{d^2 y}{dr^2} = \left( H_0^{(2)} - pe^{-\lambda r} V_0^{(2)} \right) y(r); \quad y(r) \equiv \begin{pmatrix} y_1(r) \\ y_2(r) \end{pmatrix}, \quad (38)$$
where $H_0^{(2)}$ and $V_0^{(2)}$ are the matrices:

$$H_0^{(2)} = 2m(V_N - E); \quad V_0^{(2)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (39)$$

In Eq.(39) $m$ and $E$ are this time the diagonal matrices: $m = \text{Diag}(m_1, m_2)$, $E = \text{Diag}(E_1, E_2)$; $V_N$ is an arbitrary constant matrix.

Eq.(38) can be used to describe a two-channel system where only one of the channels is “charged” in the sense that it feels the exponential potential. In realistic applications the free parameters of such a potential, $\rho$ and $\lambda$, can be adjusted to approximate, at least qualitatively, the desired potential. For more quantitative descriptions, the full range of the potential can be divided into a number of subintervals and the potential itself approximated in each of them by the exponential function. In this way an arbitrary degree of accuracy can be obtained. The evolution operator formalism, due to its factorizing property, is especially well suited for this numerical approach based on the division into intervals.

As we did with the one channel case, the second order system given by Eq.(38) will be transformed into a four dimensional, first order system, whose evolution operator will be computed. It is convenient to perform first a diagonalization of $H_0^{(2)}$ by applying a transformation $u$:

$$H_D^{(2)} = u^{-1}H_0^{(2)}u = \text{Diag}(H_1, H_2); \quad V_1^{(2)} = u^{-1}V_0^{(2)}u; \quad y(r) = u y_I(r). \quad (40)$$

Defining a vector $\Psi(r)$ containing $y_I(r)$ and its derivative, one arrives at the four-dimensional, first order system:

$$\frac{d\Psi}{dr} = \left(H_0 - \rho e^{-\lambda r} V\right)\Psi; \quad \Psi(r) \equiv \left(\frac{dy_I(r)/dr}{y_I(r)}\right); \quad (41)$$

$$H_0 = \begin{pmatrix} 0 & H_D^{(2)} \\ V_1^{(2)} & 0 \end{pmatrix}; \quad V = \begin{pmatrix} 0 & V_1^{(2)} \\ 0 & 0 \end{pmatrix}. \quad (42)$$

One proceeds then to a new diagonalization of the $4 \times 4$ matrix $H_0$.

$$H_D = w^{-1}H_0w; \quad W = w^{-1}Vw; \quad \Psi = w\Psi_D. \quad (43)$$

We are finally left with the system

$$\partial_r \Psi_D = \left(H_D - \rho e^{-\lambda r} W\right)\Psi_D. \quad (44)$$

The matrix $H_D$ is

$$H_D = \text{Diag}[k_1, k_2, -k_1, -k_2], \quad k_{1,2} = \sqrt{H_{1,2}}. \quad (45)$$

The explicit form of the matrix $w$ is

$$w \equiv \begin{pmatrix} iq & -iq \\ i2 & 12 \end{pmatrix}, \quad (46)$$
where \( q \) is the following matrix:

\[
q \equiv \begin{pmatrix}
-ik_1 & 0 \\
0 & -ik_2
\end{pmatrix}.
\]

The \( 4 \times 4 \) matrix \( W \) is the tensor product of two orthogonal vectors \( W = \vec{a} \otimes \vec{b} \) and \( \vec{a} \cdot \vec{b} = 0 \). The elements of each of these vectors are essentially combinations of the elements of the matrix \( u \):

\[
\vec{a} = \frac{1}{\sqrt{2 \det u}} \begin{pmatrix}
-u_{21}k_2 & u_{22}k_1 & u_{21}k_1 & -u_{22}k_2
\end{pmatrix},
\]

\[
\vec{b} = \frac{1}{\sqrt{2 \det u}} (u_{12}, u_{11}, u_{12}, u_{11}).
\]

Some essential properties of \( W \), that will be used later, are: \( W^2 = 0, Tr(W) = 0 \). The diagonal elements of \( W \) are given explicitly by

\[
(W_{kk}) = 1/2 \ (C_3/k_2, C_1/k_1, -C_3/k_2, -C_1/k_1),
\]

with

\[
C_1 = u_{11}u_{22}/\det u; \quad C_3 = -u_{21}u_{12}/\det u; \quad C_1 + C_3 = 1.
\]

The computation of the evolution operator for the two channel version of Eq. (14) follows the same steps as the one channel case which we have seen in the previous section. In this case, because of the conventions we have used, the factor \((-i)^n\) appearing in front of the integrals (11) disappears. We obtain an equation similar to the third line of Eq. (16), but this time containing a sum over four eigenvalues:

\[
U_{ba}^{(n)} = e^{E_b(r-r_0)}W_{ba} \rho^n e^{-n\lambda x_0} \prod_{m=1}^{n} \sum_{k=1}^{4} \frac{W_{kk}}{-\omega_{ba} - n\lambda} \prod_{m=1}^{n-1} \frac{(-C_1)}{(-C_3)} \prod_{m=1}^{n-1} \left( \frac{(-C_1)}{(-C_3)} \frac{(-C_1)}{(-C_3)} \right) \prod_{m=1}^{n-1} \left( \frac{(-C_1)}{(-C_3)} \frac{(-C_1)}{(-C_3)} \right) = e^{E_b(r-r_0)}(\lambda W_{ba}) \rho^n e^{-n\lambda x_0} \prod_{m=1}^{n-1} \frac{(\omega_{ba}/\lambda + n)}{(\omega_{ba}/\lambda + n)} \prod_{m=1}^{n-1} \frac{(\omega_{ba}/\lambda + n)}{(\omega_{ba}/\lambda + n)} \prod_{m=1}^{n-1} \frac{(\omega_{ba}/\lambda + n)}{(\omega_{ba}/\lambda + n)} \prod_{m=1}^{n-1} \frac{(\omega_{ba}/\lambda + n)}{(\omega_{ba}/\lambda + n)}
\]

In the previous expressions we have used the following symbols: \( E_b = H_{D,bb} \). The factors \( s_b \) are respectively \( s_{1,3} = C_3(k_2^2 - k_1^2)/\lambda^2 \) and \( s_{2,4} = C_1(k_1^2 - k_2^2)/\lambda^2 \). \( x_{1,2} \) are the roots of
the second order polynomial in \( m \) appearing in the third line; the roots \( x_{1,2} \) depend on \( b \) but not on \( a \).

To get an explicit expression for the diagonal elements of \( U \), we take into account the definitions of \( x_1, x_2 \) and the quantities \( W_{bb} \); in this way we obtain

\[
U_{bb}^{(n)} = e^{E_b(r-r_0)} \frac{(-\rho/\lambda^2)^n e^{-n\lambda r_0}}{n!} \frac{(-x_1(n)(-x_2(n))}{(\omega_{ba}/\lambda)(n)(w_{b\beta}/\lambda)(n)(\omega_{b\gamma}/\lambda)(n)}. \tag{50}
\]

The symbols \( \alpha, \beta, \gamma \) represent here the three non-repeated indices not equal to \( b \).

For the non-diagonal elements, we use the fact that for any \( b \) (for any row of the matrix) the following relation is true:

\[
\frac{\omega_{ba}\omega_{b\beta}\omega_{b\gamma}}{x_1 x_2} = \frac{\lambda^2}{W_{bb}}.
\]

Hence, we have:

\[
U_{ba}^{(n)} = e^{E_b(r-r_0)} \frac{(-\rho/\lambda^2)^n e^{-n\lambda r_0}}{w_{ba}/\lambda + n} \frac{1}{(n-1)!} \frac{(-x_1(n)(-x_2(n))}{(\omega_{ba}/\lambda)(n)(w_{b\beta}/\lambda)(n)(\omega_{b\gamma}/\lambda)(n)}. \tag{51}
\]

Once we have obtained Eqs.(51,50), it is straightforward to perform the summation of the perturbative series in terms of generalized hypergeometric functions as was done for the one channel case. The evolution operator for the system given by Eq.(44) is defined in terms of the auxiliary matrices \( U_{red}, U_s \) as:

\[
U(r \to \infty, r_0) = \exp(H_D r) U_{red}(r_0),
\]

with the matrix \( U_{red} \) given by:

\[
U_{red}(r_0) = \exp(-H_D r_0) U_s(r_0).
\]

The expressions for the elements of the matrix \( U_s \) in terms of hypergeometric functions are easily deduced by inspection; for any row \( "b" \) the diagonal and non-diagonal elements are:

\[
U_{s,bb} = _2F_3(-x_{b1},-x_{b2}; \omega_{ba}/\lambda, \omega_{b\beta}/\lambda, \omega_{b\gamma}/\lambda; z), \tag{53}
\]

\[
U_{s,bj}(j \neq b) = W_{bj}/W_{bb} [U_{s,b1} - _2F_3(-x_{b1},-x_{b2}; 1 + \omega_{bj}/\lambda, \omega_{b\beta}, \omega_{b\gamma}; z)], \tag{54}
\]

where \( \omega_{ij} = (H_D)_{ii} - (H_D)_{jj} \) and \( \alpha, \beta \) and \( \gamma \) represent non-repeated indices as before. Moreover \( x_{b1}, x_{b2} \) are the roots of the equation:

\[
x^2 + \frac{2H_{D,bb}}{\lambda} x + s_b = 0, \tag{55}
\]

where the \( s_b \) are the quantities defined previously.
The evolution operator for a finite interval is the product of two matrices $U_{\text{red}}$ defined at the first and last points of the interval:

$$U(r, r_0) = U_{\text{red}}(r)^{-1} U_{\text{red}}(r_0).$$  

(56)

In the “physical” basis, where $H_0^{(2)}$ is diagonal, the general solution with arbitrary initial conditions at $r = r_0$ is given by:

$$\left( \frac{dy_I(r)}{dr} \right)_{y_I(r)} = w U_{\text{red}}^{-1}(r) U_{\text{red}}(r_0) w^{-1} \left( \frac{dy_I(r_0)}{dy_I(r_0)} \right).$$  

(57)

By using the expression for $w$, a general solution with regular behavior at origin is given by:

$$y_I(r) = \left( U_{[11]} - U_{[12]} + U_{[21]} - U_{[22]} \right) w^{-1} R;$$  

(58)

where $R$ is an arbitrary constant two dimensional vector. We have denoted with $U_{[ij]}$ ($1 \leq i, j \leq 2$) the $2 \times 2$ block element of the matrix $U$.

The asymptotic behavior for $y_I(r)$ is obtained from Eq.(52):

$$y_I(r \to \infty) = \left( \exp(iqr) \epsilon + \exp(-iqr) \sigma \right) (iq)^{-1} R;$$  

(59)

with the matrices:

$$\epsilon = \left( U_{s,[11]} - U_{s,[12]} \right); \quad \sigma = \left( U_{s,[21]} - U_{s,[22]} \right).$$  

(60)

According to Eq.(25), the scattering matrix is given in terms of the matrices $\epsilon, \sigma$ as:

$$S_I(q) = \sigma \epsilon^{-1}.$$  

(61)

The amplitude matrix $K_I$ is obtained by expanding the exponentials appearing in Eq.(59) in trigonometric functions. Hence Eq.(54) is equivalent to the following:

$$y_I(r \to \infty) = \left( \cos(qr)(\epsilon + \sigma) + i \sin(qr)(\epsilon - \sigma) \right) (iq)^{-1} R.$$  

(62)

Then, according to the usual definition,

$$K_I \equiv (\epsilon - \sigma)(\epsilon + \sigma)^{-1}.$$  

(63)

It is easy to check that the standard relation between the scattering matrix $S_I$ and the amplitude matrix $K_I$ holds:

$$S_I = (1 + iK_I)(1 - iK_I)^{-1}.$$  

(64)

The expressions for $S$ and $K$ in any other basis are obtained by a linear transformation. In the original basis where Eq.(58) was written, the matrices $S, K$ are given by

$$S = u^{(4)} S_I u^{(4),-1}, \quad K = u^{(4)} K_I u^{(4),-1},$$

the matrix $u^{(4)}$ is the 4x4 version of $u$, in a $2 \times 2$ block form is given by:

$$u^{(4)} = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}.$$  

(65)
4 Conclusions

In this work we have solved exactly an s-wave radial Schrödinger equation which in addition to a multichannel square-well potential contains a potential of exponential profile in one of the channels. The solution has been obtained by summing the infinite perturbative series for the evolution operator of the system. The scattering matrix has been computed explicitly for the special case where the square-well and the exponential potential extend to infinity. The scattering length for the one channel case has been obtained in an explicit analytical form. Due to the factorizing property of the evolution operator, the results are easily applicable to situations in which short and long range potentials coexist.

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