The $Z$ boson in the Framed Standard Model

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Abstract

The framed standard model (FSM), constructed initially for explaining the existence of three fermion generations and the hierarchical mass and mixing patterns of quarks and leptons [1][2], suggests also a “hidden sector” of particles [3] including some dark matter candidates. It predicts in addition a new vector boson $G$, with mass of order TeV, which mixes with the $\gamma$ and $Z$ of the standard model yielding deviations from the standard mixing scheme, all calculable in terms of a single unknown parameter $m_G$. Given that standard mixing has been tested already to great accuracy by experiment, this could lead to contradictions, but it is shown here that for the three crucial and testable cases so far studied (i) $m_Z - m_W$, (ii) $\Gamma(Z \rightarrow \ell^+\ell^-)$, (iii) $\Gamma(Z \rightarrow \text{hadrons})$, the deviations are all within the present stringent experimental bounds provided $m_G > 1$ TeV, but should soon be detectable if experimental accuracy improves. This comes about because of some subtle cancellations, which might have a deeper reason that is not yet understood. By virtue of mixing, $G$ can be produced at the LHC and appear as a $\ell^+\ell^-$ anomaly. If found, it will be of interest not only for its own sake but serve also as a window on to the “hidden sector” into which it will mostly decay, with dark matter candidates as most likely products.

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1 Introduction

The framed standard model (FSM) is constructed from the standard model (SM) by adding to the usual gauge boson and matter fermion fields the frame vectors in internal gauge space as dynamical variables (called framons), thereby seemingly assigning a geometrical meaning to the Higgs field, suggesting an origin for the three fermion generations, giving an explanation to the special patterns of fermion masses and mixing as seen in experiment, and providing a solution to the strong CP problem, as well as posing and answering some other questions of interest, all in terms of a small number of adjustable parameters [1], [2], [3].

Though having thus apparently explained, as it was originally intended to do, some outstanding peculiarities that the SM takes for granted as inputs from experiment, the FSM now faces the question whether the new framon degrees of freedom that it introduces might not lead to deviations from the SM beyond what is allowed by experiment. Some of the resulting deviations are addressed in [3], but the most acute and pressing is the following, which needs special attention. The FSM predicts a new vector boson, say $G$, with mass of order TeV, which mixes with the photon and the $Z$ in the standard sector. The urgent question is whether such mixing would spoil the present, near-perfect, agreement with experiment of the standard model mixing scheme.

In brief (for details see [3]), this situation comes about as follows. The framon as conceived in the FSM is a scalar field in two parts:

- the flavour framon:
  \[ \phi = (\phi_r), \quad r = 1, 2, \]

  and

- the colour framon:
  \[ \Phi = \left( \phi_\tilde{a} \right), \quad a = 1, 2, 3; \quad \tilde{a} = \tilde{1}, \tilde{2}, \tilde{3}. \]

Here we have left out some spacetime independent factors which do not enter into the calculations in this paper.

The flavour framon $\phi$ (written as a vector here) transforms as a doublet in local $su(2)$, while the colour framon $\Phi$ (written as a matrix here) transforms as a triplet in local $su(3)$ but as an anti-triplet in global $\tilde{su}(3)$.

The colour framon, being coloured and colour being confining, cannot propagate as a particle in free space, but can combine with a colour anti-triplet to form a colour singlet particle which propagates. In particular, it can combine with an antiframon
to give freely propagating bosons. Here, we are interested only in the $p$-wave bound states: $\Phi^\dagger D_\mu \Phi$, which we call $G$s. There are 8 of them, conveniently labelled by the Gell-Mann matrices $\lambda_1, \ldots, \lambda_8$. They are the colour analogues of the $W$-bosons labelled by the Pauli matrices $\tau_1, \tau_2, \tau_3$ in the electroweak sector. The analogy is not so clear in the conventional picture where the electroweak symmetry $su(2)$ is considered as spontaneously broken. But in the confinement picture of 't Hooft, which he showed in an illuminating paper [4] to be a “mathematically equivalent” interpretation, the $W$ appear also as $\phi^\dagger D_\mu \phi$ bound states, but here via $su(2)$ flavour confinement. Now in the electroweak theory, $W_3^\mu$ mixes with the $u(1)$ gauge field $A_\mu$ to form $\gamma$ and $Z$. So, in parallel, one would not be surprised that in the FSM when the symmetry is extended, the mixing is extended to include also $G^8$.

Explicitly how this mixing goes will be worked out in the next section. We start here with some general observations. The mixing of $A_\mu$ and $W_3^\mu$ in the electroweak theory involves three parameters which we may take as $g_1$, $g_2$ and $\zeta_W$, the last being the vacuum expectation value of the standard Higgs field, which is essentially the same as the flavour framon $\phi$ above. When extended to include $G^8$ in the FSM, the mixing involves two more parameters: $g_3$ and $\zeta_S$, the former being the colour coupling as measured in several processes [5], the latter the vacuum value of the colour framon $\Phi$. The fit to data performed in [2] suggests that $\zeta_S$ has a value of order TeV, leaving the new extended mixing scheme with rather little freedom.

As we shall see, the parameters $\zeta_W$ and $\zeta_S$ appear in the extended mixing formulae in the combination $\zeta_W^2/\zeta_S^2$ which is of order $10^{-2}$. If this should lead to deviations from the standard mixing formulae by amounts of that order, then it would be a disaster, for the standard mixing scheme has already been checked by experiment to an accuracy of several orders higher. This explains why tests of the new mixing scheme are of particular acuteness and urgency. Fortunately, as will be shown, in the predictions of the new mixing scheme of the mass and decay widths of the $Z$, the quantities so far studied, there are subtle cancellations which mean that, explicitly in the mass of $Z$, and effectively in its decay widths, the deviations of order $\zeta_W^2/\zeta_S^2$ of the FSM from standard mixing vanish. Indeed, it is shown that so long as $\zeta_S \geq 2$ TeV, the deviations of the new mixing from the standard one are all within the present experimental errors. In other words, the FSM has survived so far the tests against experiment to which it has been subjected.

Turning the argument around and taking the positive view that these deviations of FSM from SM are new physics to be searched for, we note that they depend on only one parameter of uncertain value, $\zeta_S$, and are thus all correlated. Besides, they are also correlated to the $G$ mass (which is proportional to $\zeta_S$ to a very good approximation), while $G$ itself can appear in LHC experiments as a lepton-antilepton
enhancement in the multi-TeV range. The whole complex would thus appear to be a fruitful region for future experiments to explore.

2 Mixing in the $\gamma - Z - G$ Complex

The mass (squared) matrices of the vector bosons can be extracted from the kinetic energy terms of the framons in the Lagrangian \[3\], namely:

\[
[(D_\mu \phi)^\dagger D_\mu \phi], \quad D_\mu = \partial_\mu + \frac{1}{2}ig_1 A_\mu - \frac{1}{2}ig_2 B_\mu, \\
\text{Tr}[(D_\mu \Phi)^\dagger D_\mu \Phi], \quad D_\mu = \partial_\mu - ig_1 \Gamma A_\mu - \frac{1}{2}ig_3 C_\mu,
\]

(3)

where the flavour framon $\phi$ has charge $-\frac{1}{2}$, $A_\mu, B_\mu, C_\mu$ are respectively the gauge potentials of the gauge symmetries $u(1), su(2), su(3)$, and $\Gamma$ is the charge matrix of the colour framon$^2$

\[
\Gamma = \begin{pmatrix}
-\frac{1}{3} & 0 & 0 \\
0 & -\frac{1}{3} & 0 \\
0 & 0 & +\frac{2}{3}
\end{pmatrix},
\]

(4)

in which the charges have been chosen explicitly to keep the photon massless.

First consider the flavour framons. Adopting the confinement picture of 't Hooft (which is more immediately extendable later to the colour framons) we can use the gauge freedom to fix the gauge by rotating, with an $su(2)$ transformation $\Omega(x)$, the doublet scalar field $\phi$ to point, at every spacetime point $x$, in the first direction and to be real,

\[
\phi = \Omega \begin{pmatrix}
\rho \\
0
\end{pmatrix} = \Omega \phi_{GF},
\]

(5)

with $\rho$ real. We can thus write:

\[
\rho = \zeta_W + h_W,
\]

(6)

where, $\zeta_W$ is the vacuum expectation value of $\rho$, and $h_W$, as its fluctuation about the vacuum value, is the Higgs boson field \[4\], \[6\].

Using the gauge-fixed field $\phi_{GF}$ and introducing the gauge invariant quantity:

\[
\frac{1}{2} \tilde{B}_\mu = \frac{1}{g_2} \Omega^\dagger (\partial_\mu - \frac{1}{2}ig_2 B_\mu) \Omega,
\]

(7)

so that

\[
\Omega^\dagger D_\mu \Omega = \frac{1}{2}ig_1 A_\mu - \frac{1}{2}ig_2 \tilde{B}_\mu,
\]

(8)

$^2$It is understood that, in our convention where in the matrix $\Phi$, rows are labelled by local colour $a$ but columns by global colour $\tilde{a}$, the matrix $\Gamma$ is to operate on $\Phi$ from the right.
we can rewrite the kinetic energy term to leading order (since $\phi_{GF}$ is constant to that order),

$$[D_\mu \phi]^\dagger [D_\mu \phi] = \phi_{GF}^\dagger [+ig_1 \frac{1}{2} A_\mu - ig_2 \tilde{B}_\mu]^\dagger [+ig_1 \frac{1}{2} A_\mu - ig_2 \tilde{B}_\mu] \phi_{GF}.$$ \hfill (9)

This gives for the mass term, $\tilde{B}_\mu$ being hermitian:

$$(\zeta_W, 0) \frac{1}{4} \left[ g_1^2 A_\mu^2 + g_2^2 \tilde{B}_\mu^2 - 2g_1g_2 A_\mu \tilde{B}_\mu \right] \begin{pmatrix} \zeta_W \\ 0 \end{pmatrix},$$ \hfill (10)

that is, $\zeta^2_W$ times the 11 element of the quantity inside the square bracket. We thus obtain the following mass squared matrix

$$\frac{\zeta^2_W}{4} \begin{pmatrix} g_2^2 & 0 & 0 & 0 \\ 0 & g_2^2 & 0 & 0 \\ 0 & 0 & g_1^2 & -g_1g_2 \\ 0 & 0 & -g_1g_2 & g_1^2 \end{pmatrix},$$ \hfill (11)

with only the $\tilde{B}^3$ component mixing with the $A_\mu$, giving the massless photon and the SM $Z$ boson:

$$\gamma_\mu = \frac{1}{\sqrt{g_1^2 + g_2^2}} \left( g_2 A_\mu + g_1 \tilde{B}_\mu^3 \right)$$

$$Z_\mu = \frac{1}{\sqrt{g_1^2 + g_2^2}} \left( -g_1 A_\mu + g_2 \tilde{B}_\mu^3 \right),$$ \hfill (12)

with

$$\frac{1}{e^2} = \frac{1}{g_1^2} + \frac{1}{g_2^2}.$$ \hfill (13)

Next we consider the kinetic energy term of the colour framons \(^2\). Again we can use an $su(3)$ matrix $\Omega(x)$ to fix the gauge of the framons $\Phi$,

$$\Phi = \Omega \Phi_{GF},$$ \hfill (14)

though not completely, there not being sufficient degrees of freedom in $su(3)$ to do so, but still enough to make its vacuum value take on a diagonal form as follows:

$$\Phi_{GF} \rightarrow \frac{\zeta_S}{\sqrt{3}} \begin{pmatrix} \sqrt{1 - R} & 0 & 0 \\ 0 & \sqrt{1 - R} & 0 \\ 0 & 0 & \sqrt{1 + 2R} \end{pmatrix} = \Phi_{\text{vac}},$$ \hfill (15)
and it is this last form which gives us the mass matrix of the vector bosons. Here $R$ is a scale-dependent quantity made out of coupling constants of the terms in the FSM framgon potential, and its actual value at any scale has been obtained by fitting FSM to fermion masses and mixing data [2]. For $\zeta_S \geq 2$ TeV which interests us here, one can effectively put $R = 0$.

Proceeding as in the flavour case, we define

$$\frac{1}{2} \tilde{C}_\mu = \frac{i}{g_3} \bar{\Omega} (\partial \mu - \frac{1}{2} i g_3 C_\mu) \Omega.$$  \hspace{1cm} (16)

so that the kinetic energy term can be rewritten to leading order as

$$\text{Tr} \left[ \Phi_{GF} \bar{\Omega} D^\dagger \mu \Omega \Omega^\dagger D^\mu \Omega \Phi_{GF} \right]$$

$$= \text{Tr} \left[ \Phi_{GF}^\dagger \left( -i g_1 \Gamma A_\mu - \frac{1}{2} i g_3 \tilde{C}_\mu \right)^\dagger \left( -i g_1 A_\mu \Gamma - \frac{1}{2} i g_3 \tilde{C}_\mu \right) \Phi_{GF} \right],$$  \hspace{1cm} (17)

We now expand $\tilde{C}_\mu$ in terms of the usual Gell-Mann matrices:

$$\tilde{C}_\mu = \sum_K \tilde{C}_\mu^K \lambda_K$$  \hspace{1cm} (18)

and rewrite the kinetic energy term as

$$\text{K.E.} = \text{Tr} \left[ \Phi_{vac}^\dagger \Phi_{vac} \left( g_1^2 A_\mu A^\mu \Gamma \Gamma^\dagger + \frac{1}{2} g_1 g_3 A_\mu \Gamma \left( \sum \tilde{C}_\mu^K \lambda_K \right) \right) + \frac{1}{2} g_1 g_3 A_\mu \left( \sum \tilde{C}_\mu^K \lambda_K \right) \Gamma \Gamma^\dagger + \frac{1}{4} g_3^2 \left( \sum \tilde{C}_\mu^K \lambda_K \right) \left( \sum \tilde{C}_\mu^K \lambda_K \right) \right].$$  \hspace{1cm} (19)

Now because we have chosen

$$\Gamma = -\frac{1}{\sqrt{3}} \lambda_8$$  \hspace{1cm} (20)

the resulting mass matrix is block-diagonal, with only the 0-8 block non-diagonal:

$$\frac{2}{3} (1 + R) \zeta_S^2 \left( -\frac{1}{\sqrt{3}} g_1 g_3 \begin{pmatrix} \frac{1}{3} g_1^2 & -\frac{1}{\sqrt{3}} g_1 g_3 \\ -\frac{1}{\sqrt{3}} g_1 g_3 & \frac{1}{g_3^2} \end{pmatrix} \right).$$  \hspace{1cm} (21)

This has a zero mode, as expected.

Finally we consider the sum of the two kinetic energy terms, and concentrate on the $(3 \times 3)$ $A - \tilde{B}^3 - \tilde{C}^8$ mixing matrix as (where the order of the rows and columns
are labelled according to the symmetries $u(1), su(2), su(3))$

$$M = \begin{pmatrix}
(\ell + \frac{1}{3}k)g_1^2 & -\ell g_1 g_2 & -k \frac{1}{2\sqrt{3}} g_1 g_3 \\
-\ell g_1 g_2 & \ell g_2^2 & 0 \\
-k \frac{1}{2\sqrt{3}} g_1 g_3 & 0 & \frac{k}{4} g_3^2 \\
\end{pmatrix}$$

(22)

where

$$\ell = \frac{1}{4}\zeta_W^2, \quad k = \frac{2}{3}(1 + R)\zeta_S^2$$

(23)

Crucially this has again a zero mode, which is indispensable for a viable theory containing the photon.

Apart from the zero mode, the other two eigenvalues ($\lambda_{\pm}$) are given by the roots of the following quadratic equation

$$\lambda^2 - \lambda \left(\ell g_2^2 + \frac{k}{4} g_3^2 + (\ell - \frac{k}{4})g_1^2\right) + k\ell \left(\frac{1}{4} g_1^2 g_3^2 + \frac{1}{3} g_2^2 g_3^2 + \frac{1}{3} g_1^2 g_2^2\right) = 0.$$  

(24)

These are then the mass squared of respectively the $Z$ boson and another boson we call $G$ of higher mass not present in the SM spectrum. The discriminant of (24) is positive for all values of the coupling constants, as it can be re-written as:

$$\left(\ell g_2^2 + (\ell - \frac{k}{4})g_1^2 - \frac{k}{4} g_3^2\right)^2 + \frac{4}{3} k\ell g_1^4.$$  

(25)

In this form, one sees immediately that both roots are positive for all values of the coupling constants.

The normalized zero eigenvector (which corresponds to the photon) is given by

$$v_1 = \begin{pmatrix}
\frac{e}{g_1} \\
\frac{e}{g_2} \\
\frac{2}{\sqrt{3}} \frac{e}{g_3} \\
\end{pmatrix}$$

(26)

where

$$\frac{1}{e^2} = \frac{1}{g_1^2} + \frac{1}{g_2^2} + \frac{1}{\frac{3}{4} g_3^2}.$$  

(27)

That this is indeed the correct normalization can be checked by writing the $su(2) \times u(1)$ neutral current in terms of the mass eigenstates and identifying the electromagnetic current as the piece that is coupled to the photon, the coefficient of which will give the electric charge $e$.  

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Comparison with equation (13) immediately tells us that the coupling \( g_1 \) in FSM differs from that of the SM. The exact relation is given in equation (36) below.

Then, after some more algebra, the other two eigenvectors can be found and one obtains the mixing matrix:

\[
\begin{pmatrix}
\gamma_{\mu} \\
Z_{\mu} \\
G_{\mu}
\end{pmatrix} = \begin{pmatrix}
g_1 \frac{e}{g_2} & \frac{e}{g_2} & \frac{2}{\sqrt{3}} \frac{e}{g_3} \\
-X_- & Y_- & -W_- \\
X_+ & -Y_+ & W_+
\end{pmatrix} \begin{pmatrix}
A_{\mu} \\
\tilde{B}_{\mu}^3 \\
\tilde{C}_{\mu}^8
\end{pmatrix},
\tag{28}
\]

where

\[
X_\pm = (kA - 3N_2^2 \lambda_\pm) \frac{g_1}{N_2 N_\pm} - \frac{k g_1^2 g_2 \sqrt{A}}{N_3 N_\pm}
\]

\[
Y_\pm = (kA - 3N_2^2 \lambda_\pm) \frac{g_2}{N_2 N_\pm} + \frac{k g_2^2 \sqrt{A}}{N_3 N_\pm}
\]

\[
W_\pm = \frac{\sqrt{3}}{2} \left( k g_1^2 \sqrt{A} \right) \left( \frac{g_1^2 + g_2^2}{g_1 g_2} \right) \frac{g_3}{N_3 N_\pm}
\tag{29}
\]

And the normalization factors are

\[
N_2^2 = g_1^2 + g_2^2
\]

\[
N_3^2 = \frac{g_1^2 + g_2^2}{g_1^2 g_2^2} A
\]

\[
N_\pm^2 = (kA - 3N_2^2 \lambda_\pm)^2 + k^2 g_1^4 A
\]

with

\[
A = g_1^2 g_2^2 + \frac{3}{4} g_1^2 g_3^2 + \frac{3}{4} g_2^2 g_3^2.
\tag{30}
\]

These mixing formulae of the FSM differ considerably in form from those of the SM. On the other hand, the mixing scheme of the SM has been tested already to great accuracy by experiment over a wide range of phenomena. Hence, for the FSM to remain viable, it has to be hoped that, despite the difference in form, the deviations in predicted values of the measured quantities would still remain within the present experimental errors over the whole range of data on which the SM mixing has been tested. However, to check if this is true will be a long and arduous process requiring a similar degree of sophistication to that applied in checking the standard model, which is not yet available in the FSM where the necessary tools have not
been developed. Nevertheless, we believe that it is imperative to do such checks to
the extent they can be done in the FSM at present before we attempt to go further
with the programme, and we shall do so here for the most urgent tests which can
immediately be implemented.

In view of the present limitations of the FSM and our specific aim of testing
just the change in mixing from the SM to the FSM, we have devised the following
procedure specially tailored for the purpose:

• **(A)** At present, one is not in a position to calculate loop corrections in general
  in the FSM, not having yet developed the tools for doing so. However, to
  check with the data at present accuracy, tree-level results are not adequate.
  We propose therefore to adopt the following as a test criterion. It is widely
  accepted, and we will thus take for granted, that the loop-corrected predictions
  of the SM are in agreement with experiment within present errors. Then,
  assuming that the difference in loop corrections between the SM and FSM is
  of higher order in smallness, we consider that the difference between their tree-
  level results should be a good estimate already of the loop-corrected difference.
  Hence, in what follows, if it is found that the tree-level prediction of FSM for a
  certain quantity deviates from the tree-level prediction of the SM by less than
  the present experimental error, we would consider that the FSM has passed
  the test, regardless of whether the tree-level predictions themselves of either
  the SM or the FSM are within errors of the experimentally measured values.

• **(B)** At tree level, the $W^\pm$ bosons are not affected by the change in mixing of
  the neutral partners of the FSM from the SM. We may thus choose to regard
  the $W$ mass and width as given and determine from them the two parameters
  $g_2$ and $\zeta_W$. Then together with $e$ for the photon, and the colour coupling $g_3$,
  independently measured from $Z$ decays and perturbative QCD, we have all
  the parameters needed for the above mixing schemes (both of the SM and the
  FSM), except for $\zeta_S$ for which there is only a very crude order of magnitude
  estimate [2 3]. We may thus choose to formulate our test of the FSM mixing
  scheme as follows. We calculate with these parameters the tree-level properties
  of the $Z$ as predicted by respectively the SM and FSM mixing schemes, and
  compare the results. And if the two answers for each quantity differ by less
  than the experimental error, then by **(A)** above, we shall consider that the
  FSM has passed the test. This may not be the usual procedure taken, which
  tends more to start with the $Z$, for which errors are generally smaller, to predict
  the $W$. But since we fix the relevant parameters from the $W$ data, this makes
  the logic clearer, and is equivalent.
Taking then from the PDG tables \([5]\)

\[
\begin{align*}
\zeta_W &= 246 \text{ GeV} \\
m_W &= 80.385 \text{ GeV,}
\end{align*}
\]

and determining \(g_2\), one has then, together with \(e\) and \(g_3\) from the PDG tables and \(\zeta_S\) from \([3]\), the following list of parameters taken at the scale of the \(Z\) mass:

\[
\begin{align*}
e^2 &= 0.098175 (= 4\pi/128) \\
g_2^2 &= 0.4271 \\
g_3^2 &= 1.4828 \\
\zeta_S &\sim \text{order TeV.}
\end{align*}
\]  

(31)

We recall that the values of \(g_1\) differ in the two mixing schemes. We shall reserve the symbol \(g_1\) for FSM and denote the SM value as \(g_1^{\text{SM}}\).

\[
\begin{align*}
g_1^2 &= 0.1440 \\
(g_1^{\text{SM}})^2 &= 0.1275
\end{align*}
\]  

(32)

With these parameters, we shall calculate:

- the \(Z\) mass,
- the decay widths of \(Z\) into \(\ell^+\ell^-\) and \(q\bar{q}\) pairs.

and show that provided one chooses \(\zeta_S \geq 2\) TeV, the difference between the two mixing schemes is within the present experimental errors. By (A) then, we conclude that the FSM has so far passed the test.

We note that the values quoted in (31), (32) and (33) are all just central values each with an error which has not been displayed. Thus all predictions deduced from them on the \(Z\) mass and decay widths will inherit an error from them and these errors will have to be accounted for when applying criterion (A). This is particularly relevant here when deducing \(Z\) properties from the \(W\) since the experimental errors on the \(W\) are generally bigger than on the \(Z\). We shall leave for later the details how this point will be accounted for in each case.
3 The $Z$ mass

3.1 Exact and approximate formulae

The tree-level mass of the $Z$ is given by the smaller of the two roots of (24):

$$m_Z^2 = \frac{1}{2} \left( (\ell g_2^2 + \frac{k}{4} g_3^2 + (\ell + \frac{k}{3}) g_1^2) 
- \sqrt{((\ell g_2^2 + \frac{k}{4} g_3^2 + (\ell + \frac{k}{3}) g_1^2)^2 - k\ell (g_1^2 g_3^2 + g_2^2 g_3^2 + \frac{3}{4} g_1^2 g_2^2)} \right)$$

(34)

For comparison later with the SM, we expand this in powers of $(\ell/k)$, which according to (32) is a small parameter. To zeroth order, one has:

$$m_Z' = \frac{3}{4} g_1^2 g_3^2 + g_1^2 g_2^2 + \frac{3}{4} g_2^2 g_3^2 \quad \ell = \frac{\ell A}{B}$$

(35)

There is no first order term, which will be seen to be significant. Besides, it means that to this order, $m_Z$ is independent of $k$ so that we do not need to know the value of $\zeta_S$. In the last equality of (35) $A$ was defined above and $B = g_1^2 + \frac{3}{4} g_2^2$, which will be useful later.

3.2 Comparison with SM

From the relations (13) and (27) for respectively the SM and FSM, we deduce the relation:

$$\frac{1}{(g_1^{SM})^2} = \frac{1}{g_1^2} + \frac{1}{\frac{3}{4} g_3^2} = g_1^2 + \frac{3}{4} g_3^2.$$ (36)

Using (36) we can then easily work out that

$$m_Z' = \ell ((g_1^{SM})^2 + g_2^2) = m_Z^{SM}. \quad \text{(37)}$$

This means first that, provided $\zeta_W^2/\zeta_3^2$ is small, the FSM mass is a good approximation of the SM mass, both evaluated at tree level. Secondly, because the first order $\ell/k$ term vanishes, the difference between the FSM and SM values is not of order $\ell/k \sim 10^{-2}$ which would have been disastrous, but of order $(\ell/k)^2 \sim 10^{-4}$, for our benchmark value of $\zeta_S \sim 2$ TeV, which would bring it to about the present experimental error. That this is indeed the case will be shown explicitly in what follows.
3.3 Numerical results and comparison with experiment

We recall that before any mixing scheme is implemented, whether in the SM or the FSM, the $Z$ and the $W$ are degenerate in mass. What mixing does is to introduce a shift in the masses $m_{\text{shift}} = m_Z - m_W$, and it is precisely this quantity the models predict which should be compared to experiment.

The PDG [5] gives

$$m_{\text{shift}} = m_Z - m_W = 10.803 \pm 0.015 \text{ GeV}, \quad (38)$$

with an impressively small error:

$$\Delta m_{\text{shift}}^{\text{exp}} = 15 \text{ MeV}. \quad (39)$$

According to criterion (A), this is to be compared to the difference in mass shifts obtained respectively in the SM and FSM: $m_{\text{shift}}^{\text{SM}} - m_{\text{shift}} = \Delta m_{\text{shift}}$. Now $m_W$ being the same in SM and FSM, we have

$$\Delta m_{\text{shift}} = \Delta m_Z = m_Z^{\text{SM}} - m_Z. \quad (40)$$

Using the values in (32) and (33), and the expressions (34), and (35), we obtain for our benchmark value of 2 TeV for $\zeta_S$:

$$m_Z = 91.5884 \text{ GeV}$$
$$m_Z^{\text{SM}} = 91.5988 \text{ GeV}, \quad (41)$$

giving

$$\Delta m_{\text{shift}} = \Delta m_Z = 10.4 \text{ MeV}. \quad (42)$$

We thus indeed have:

$$\Delta m_{\text{shift}} = 10.4 \text{ MeV} < 15 \text{ MeV} = \Delta m_{\text{shift}}^{\text{exp}}, \quad (43)$$

as criterion (A) requires for the FSM to pass the test.

Notice that both the tree-level predictions for the $Z$ mass of the two models (41) lie outside present experimental bounds

$$m_Z^{\text{exp}} = 91.1876 \pm 0.0021 \text{ GeV}, \quad (44)$$

which shows the inadequacy of tree-level approximations at present experimental accuracy, and hence the relevance of criterion (A) for testing the FSM at tree level. Note also that in comparing $m_{\text{shift}}$ to experiment instead of $m_Z$ directly, we have folded in the error on $m_W$ which at present is larger than the error on $m_Z$ itself.

It will be shown in the last section that the difference between the predicted values decreases with increasing $\zeta_S$ so that criterion (A) remains satisfied for all $\zeta_S$ above 2 TeV.
4  Z decay into a fermion-antifermion pair

4.1  Exact and approximate formulae

From equation (28) we can write

\[ Z_\mu = -X_- A_\mu + Y_- \tilde{B}_\mu^3 - W_- \tilde{C}_\mu^8. \]  (45)

The last term does not contribute since \( \tilde{C}_\mu \) does not couple directly to quarks and leptons, and we shall ignore it henceforth. The \( su(2) \) part of the \( Z \) couples to only the left-handed fermion, while the \( u(1) \) part of \( Z \) couples to both left and right handed fermions, though differently.

We can now write the current for \( Z \) coupled to the fermion field \( f \):

\[ j_\mu = -\frac{1}{2} g_1 X_- \left[ \left( \bar{f}_L (2Q - 2I_3) \gamma_\mu f_L \right) + \left( \bar{f}_R (2Q) \gamma_\mu f_R \right) \right] + g_2 Y_- \left( \bar{f}_L I_3 \gamma_\mu f_L \right) \]

\[ = \bar{f} \gamma_\mu \left[ -g_1 X_- (Q - I_3) + g_2 Y_- I_3 \right] \frac{1 - \gamma_5}{2} f - \bar{f} \left[ g_1 X_- Q \right] \frac{1 + \gamma_5}{2} f \]

\[ = \frac{1}{2} \bar{f} \gamma_\mu \left[ -2g_1 X_- Q + (g_1 X_- + g_2 Y_-) I_3 - (g_1 X_- + g_2 Y_-) I_3 \gamma_5 \right] f. \]  (46)

We define the vector and axial vector couplings \( c_V \) and \( c_A \) in the standard form:

\[ c_V = g_1 X_-(2Q + I_3) + g_2 Y_- I_3 \]

\[ c_A = g_1 X_- I_3 + g_2 Y_- I_3 \]  (47)

so that

\[ j_\mu = \frac{1}{2} \bar{f} \gamma_\mu \left( c_V - c_A \gamma_5 \right) f. \]  (48)

Hence the \( Z \) vertex factor is \( -i \frac{1}{2} \gamma_\mu (c_V - c_A \gamma_5) \), from which using standard formulae [7], in the case where the fermion masses can be neglected, we can write the partial width as:

\[ \Gamma(Z \rightarrow f \bar{f}) = \frac{m_Z}{48\pi} (c_V^2 + c_A^2). \]  (49)

This formula is often re-expressed in terms of the very accurately determined Fermi constant \( G_F \), with an accuracy of 500 ppb [5]:

\[ \Gamma(Z \rightarrow f \bar{f}) = \frac{G_F m_Z^3 (c_V^2 + c_A^2)}{6\sqrt{2} \pi g_Z^2}, \quad G_F = 1.166378 \times 10^{-5} \text{ GeV}^{-2}, \]  (50)

where \( g_Z^2 = (g_{1}^{SM})^2 + g_Z^2 = A/B \). This is the formula we shall use.
From this, again for later comparison with the SM, we next derive approximate formulae, by expanding in powers of \((\ell/k)\). First, dropping all terms depending on \((\ell/k)\) we obtain from equation (29) the zeroth order approximations:

\[
X' = \frac{k A g_1}{N_2 N_-} - \frac{k g_1^2 g_2 \sqrt{A}}{N_3 N_-} = \frac{k g_1}{N_2 N_-} (A - g_1^2 g_2)
\]

\[
Y' = \frac{k A g_2}{N_2 N_-} + \frac{k g_1^3 \sqrt{A}}{N_3 N_-} = \frac{k g_2}{N_2 N_-} (A + g_1^4)
\]

\[
N_-^2 = k^2 A (A + g_1^4) = k^2 A B (g_1^2 + g_2^2),
\]

giving

\[
X'_- = \frac{g_1}{\sqrt{A B}} \left( \frac{3}{4} g_3^2 \right), \quad Y'_- = g_2 \sqrt{B/A}.
\] (51)

Substituting into (47), we obtain

\[
c'_V = -\frac{3}{2} g_1^2 g_3^2 Q + AI_3 \sqrt{A B}
\]

\[
c'_A = \sqrt{\frac{A}{B}} I_3
\] (52)

Hence the zeroth order partial width is

\[
\Gamma'(Z \rightarrow f \bar{f}) = \frac{G_F m_Z^3}{6 \sqrt{2} \pi} \left( c_V^2 + c_A^2 \right) \frac{g_3^2}{g_Z^2}.
\] (53)

Next we look at order \(\ell/k\), and find that

\[
X_- = \frac{g_1}{\sqrt{A B}} \left( \frac{3}{4} g_3^2 - \frac{3 A}{B^2} g_1^2 \left( \frac{\ell}{k} \right) \right) + \mathcal{O} \left( \frac{\ell^2}{k^2} \right),
\] (54)

which has an \(\ell/k\) term multiplied by \(g_1^4\), while \(Y_-\) has no \(\ell/k\) term. Now in both \(c_V\) and \(c_A\), we have \(X_-\) multiplied again by \(g_1\) so that the first order term in \(\ell/k\) in the decay width is in fact multiplied by \(g_1^4\) which according to (33) is approximately 0.02. This is comparable to another order of \(\ell/k\), meaning that the decay width has effectively no \(\ell/k\) term. As in the \(Z\) mass case, this fact is of significance when comparing with the SM and with experiment.
4.2 Comparison with SM

The expression in SM for the partial width into a fermion pair is given by:

$$\Gamma_{SM}(Z \rightarrow f \bar{f}) = \frac{g_z^2 m_Z}{48\pi}((c'_{V})^2 + (c'_{A})^2),$$

(55)

and the vector and axial vector couplings in SM are given by

$$c'_{V} = -2Q \sin^2 \theta_W + I_3$$
$$c'_{A} = I_3$$

(56)

Substituting in \(\sin \theta_W = e/g_2\) we can easily work out that, to zeroth order in \((\ell/k)\),

$$c_{V} \rightarrow c'_{V} = g_z c'_{V}$$
$$c_{A} \rightarrow c'_{A} = g_z c'_{A},$$

(57)

(58)

which means that, comparing with equation (49),

$$\Gamma'(Z \rightarrow f \bar{f}) = \Gamma_{SM}'(Z \rightarrow f \bar{f}).$$

(59)

So once again, as is the case for \(m_Z\), the \(Z\) decay widths in the FSM in the limit \(\ell/k \sim \zeta_W^2/\zeta_S^2 \rightarrow 0\) is identical to the SM prediction. We may ask why it should be so, since in the FSM the \(Z\) has a component in \(su(3)\), which introduces two new parameters into the problem, namely, besides the strong vacuum expectation \(\zeta_S\), here effectively put to \(\infty\), also the colour coupling \(g_3\). There is perhaps a deep reason for this that we have not yet fathomed, but we can observe that in \(m'_Z\) and \(\Gamma'\) the couplings \(g_1\) and \(g_3\) always come in the combination (36) so that the two together can be replaced by \(g_{1,SM}'\). This does not in itself prove that the expressions must be equal, but it does at least provide an indication how it can occur.

Recalling next from 4.1 that the \(\ell/k\) term in the decay width \(\Gamma\) is multiplied by \(g_4^4 \sim 0.02\), we conclude that the FSM prediction for the width will differ from that of the SM only by order \(g_4^4(\ell/k) \sim 10^{-4}\), that is, similar to the deviation in the predictions for the \(Z\) mass. This will explain why numerically the deviations of the FSM from the SM are found to remain within the stringent experimental bounds, as shown below.
4.3 Numerical results and comparison with experiment

Using the parameter values (32) and (33) with the benchmark value $\zeta_S = 2$ TeV, we obtain the mixing matrix (28) as:

$$
\begin{bmatrix}
0.8257 & 0.4794 & 0.2971 \\
-0.4507 & 0.8776 & -0.1635 \\
-0.3392 & 0.0011 & 0.9407
\end{bmatrix}.
$$

Using the numerical values of $X_-$ and $Y_-$ above, we can evaluate the different partial widths into fermion pairs\(^3\), as shown in Table 1.

| Decay               | $\Gamma^{\exp}$ | $\Gamma$ | $\Gamma^{\text{SM}}$ | $\Delta\Gamma$ | $\Delta\Gamma^{\exp}$ |
|---------------------|------------------|----------|------------------------|-----------------|-------------------------|
| $Z \rightarrow e^+e^-$ | 83.91 ± 0.12     | 83.452   | 83.480                 | 0.028           | 0.12                    |
| $Z \rightarrow u\bar{u}$ | 286.060        | 286.100  | 286.504                | 0.040           |                         |
| $Z \rightarrow d\bar{d}$ | 368.417        | 368.504  | 0.087                  |                 |                         |
| $Z \rightarrow \text{hadrons}$ | 1744.4 ± 2.0   | 1677.371 | 1677.712               | 0.341           | 2.0                     |

Table 1: Partial widths $\Gamma \rightarrow f\bar{f}$ in MeV. Note that in the last two columns $\Delta\Gamma < \Delta\Gamma^{\exp}$ thus satisfying criterion (A) for the FSM to pass the test, despite the tree-level predictions $\Gamma$ and $\Gamma^{\text{SM}}$ being both outside experimental bounds.

Again as for the case of the mass,

$$
\Delta\Gamma = \Gamma^{\text{SM}} - \Gamma
$$

represents the difference between the FSM and SM predictions, assuming again that the loop corrections are of higher order. This difference is in every case less than the experimental error, and following then the criterion (A) we consider that the FSM has passed the test also in this case. It is worth noting that the accuracy of the approximate formula for the partial width far exceeds the estimated value of the neglected parameter $\ell/k$, as explained at the end of the last subsection. Since the differences between the SM and the FSM in the decay widths are already within the

\(^3\)The formulae (49), (50), (53) and (55) hold in the limit of zero fermion masses, which should be a good enough approximation for present purposes. Hence they are identical for all charged leptons ($e, \mu, \tau$), identical for the $U$-type quarks ($u, c$), and also identical for all $D$-type quarks ($d, s, b$). For definiteness, we quote for comparison only the experimental results for the decays $Z \rightarrow e^+e^-$, and $Z \rightarrow \text{hadrons}$. 
smaller experimental error in $Z$ decay, there is no need to fold in the larger error from the $W$, as we did for the mass test.

And again, as will be shown in the last section, the difference between the SM and FSM decreases with increasing $\zeta_S$, remaining thus within present experimental errors for all $\zeta_S \geq 2$ TeV.

5 New physics and the heavy partner $G$

The explicit formulae being known, it is straightforward to evaluate the above deviations of FSM from SM for any value of the parameter $\zeta_S$. In Figure 1, we show, from top to bottom, (i) the mass shift of the $Z$ boson $m_{\text{shift}} = m_Z - m_W$, (ii) the partial width $\Gamma(Z \rightarrow e^+e^-)$, (iii) the partial width $\Gamma(Z \rightarrow \text{hadrons})$. We notice that all three deviations decrease with increasing $\zeta_S$ as anticipated, so that having satisfied ourselves that they stay within present experimental bounds at $\zeta_S = 2$ TeV, we can conclude that they will do the same for all higher values of $\zeta_S$, as claimed.

We can take a positive view and regard these deviations as new physics to be looked for in experiment. In connection with this, we note for possible interest the following experimental fact. We can deduce from equations (41), (40), (38) that

\[ (m_Z - m_W) < (m_{Z}^{SM} - m_W), \]

(62)
a conclusion which has been checked analytically by expanding $m_Z$ in (34) to order $(l/k)$. Experimentally, $m_Z$ is much better measured (error $\sim 2$ MeV) than $m_W$ (error $\sim 15$ MeV), so measured values of $m_W$ are customarily compared to the SM value as predicted by taking $m_Z$ as input. It is curious to note that Figure 2, taken from [8], shows that measurements at LEP, Tevatron, and LHC so far actually all give central values for $m_W$ larger than that predicted by the standard model, though each by only 1-2 $\sigma$. Now, however, that $m_W$ in these plot should be bigger than the SM prediction is what is predicted by (62) from the FSM. So if the error can in future be reduced and the apparent departure for the SM persists, it could be interpreted as a point in favour of FSM.

For the decay widths also, the FSM value in Table 1 is smaller than the SM value because of the extra mixing, as has again been checked analytically by expanding to order $(l/k)$, but here the deviations, being smaller, may require greater effort for their detection.
Figure 1: Various plots showing $\zeta_S$ dependence—see also text
In any case, there being only the one parameter $\zeta_S$, these predicted deviations are all correlated, so that in Figure 1, if one draws a vertical line through the $x$-axes, one will obtain all the three different deviations at that particular value of $\zeta_S$. In other words, once any one deviation is detected, the other two deviations will be predicted by FMS in absolute terms.

Of all the predicted new physics, the most attractive to look for presumably is the $G$ boson itself. By virtue of mixing, it has a component in the standard model $Z$, and can thus be produced in any reaction which produces the $Z$ and also decay into any final states that the $Z$ does. Hence the $G$ boson can be searched for as a $\ell^+\ell^-$ anomaly at the LHC in the few TeV region, for which purpose, we would wish to know its properties.

What has been done for $Z$ in the last two sections can also be done for this third heavy partner $G$ in the mixing complex. Thus, the $G$ boson has a mass given by the larger root of the quadratic equation (24), and for our benchmark value $\zeta_s = 2$ TeV, it is

$$m_G \approx 1057 \text{ GeV}. \quad (63)$$

Since the running of the couplings $g_1, g_2,$ and $g_3$ is small in this range, it does not, for the present purpose, make any appreciable numerical difference at which scale we do the calculation.

The dependence of the $G$ mass on $\zeta_S$ is rather simple. If one takes the form of the discriminant given in (25) and neglects the last term proportional to $g_1^4 \sim 0.02$, one sees that the $G$ mass is linear in $\zeta_S$ to a very good approximation

$$m_G \approx \frac{\sqrt{2}}{3} \zeta_S \sqrt{(g_1^2 + \frac{3}{4} g_3^2)}, \quad (64)$$

as is borne out in the Figure 1.

For the decay of $G$ into a fermion-antifermion pair we can extract a neutral current from the electroweak Lagrangian, just as for the $Z$, and the formulae (47), (49) and (50) are analogous, except that we replace $X_-, Y_-$ by $X_+, Y_+$, according to the mixing matrix (28); and also we have $m_G$ instead of $m_Z$. In other words, we get the formulae:

$$\Gamma(G \rightarrow f \bar{f}) = \frac{m_G}{48\pi} (c_V^2 + c_A^2). \quad (65)$$

where

$$c_V = g_1 X_+ (-2Q + I_3) + g_2 Y_+ I_3$$

$$c_A = g_1 X_+ I_3 + g_2 Y_+ I_3 \quad (66)$$
Figure 2: The ATLAS measurement of the W boson mass and the combined values measured at the LEP and Tevatron colliders compared to the Standard Model prediction (mauve) and the FSM predictions (green) at $\zeta_S = 2.0$ TeV (left) and $\zeta_S = 1.5$ TeV (right).

Again for our benchmark value $\zeta_s = 2$ TeV, we get:

$$\Gamma(G \to e^+ e^-) \sim 290 \text{ MeV}. \quad (67)$$

To facilitate the search for the $G$ at the LHC, we would wish to have estimates also for its production cross section, and its total width. Both look possible, given that the couplings of $G$ to $q\bar{q}$ and even to the “hidden sector” are governed by the colour gauge coupling $g_3$ already known from perturbative QCD. Whether it works out or not, however, is still under investigation.

The search for $G$ will be of interest not only for its own sake. According to [3], $G$ is in the FSM one of only two known portals into the “hidden sector”. Being itself mostly in the hidden sector, it will, once produced, decay preferentially into particles which are hitherto unknown to us, including in particular some eligible dark matter.
candidates. And, any information one can gather in that direction at present will be of paramount significance.

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