Rogue waves in quantum lattices with correlated disorder

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We investigate the outbreak of anomalous quantum wavefunction amplitudes in a one-dimensional tight-binding lattice featuring correlated diagonal disorder. Such rogue-wave-like behavior is fostered by a competition between localization and mobility. The effective correlation length of the disorder is ultimately responsible for bringing the local disorder strength to a minimum, fueling the occurrence of extreme events of much higher amplitudes, specially when compared to the case of uncorrelated disorder. Our findings are valid for a class of discrete one-dimensional systems and reveal profound aspects of the role of randomness in rogue-wave generation.

I. INTRODUCTION

Rare and unpredictable events carrying huge impact are widespread in nature, from stock markets to physical sciences. Outliers may change the course of things more often than we are prone to think, sometimes leading to hazardous consequences. One example is the emergence of rogue waves in the ocean. The famous Draupner wave recorded in 1995 at a gas platform in Norway was twice as big as the significant wave height of the area. This happened to be the first scientific evidence of a rogue wave [1] and led to a burst of interest in the field as studies began to suggest that these extreme events would occur more frequently than assumed from ordinary Gaussian statistics [2]. About a decade later, Solli et al. introduced rogue waves in optics based on observations made on fibre supercontinuum generation in the presence of noise [3].

The analogy drawn between oceanic rogue waves and extreme instabilities in optics associated to long-tailed statistics set the stage for a number of theories aimed to explain the physical mechanisms behind rogue-wave generation. Much of the effort has been directed toward establishing whether and which linear or nonlinear processes play the biggest roles [4–11]. Oceanic rogue waves, for instance, may result from various mechanisms in action, such as constructive interference of random fields, modulational instability, and soliton modes, depending on sea and wind conditions [12–13]. Even though there is no definitive consensus on that matter, neither robust ways to predict where and when rogue waves will occur, noise and randomness seem to be key ingredients for their occurrence [11,14,15] (deterministic rogue waves are also discussed in Refs. [4,5]).

Some degree of disorder is paramount for generating rogue waves via linear mechanisms in particular [11,14–25]. Very recently, this was addressed in the context of quantum walks [25]. Therein, the authors primarily sought to explore the (hitherto elusive) relationship between Anderson localization and rogue wave manifestation. They reported a minimal disorder threshold $\alpha \propto N^{-1/2}$ ($N$ being the number of sites) above which rogue waves were created, whereas intermediate disorder levels would maximize the chances of seeing one due to a proper balance between trapping and mobility (see also Ref. [21]). Such results, besides bringing the rich subject of rogue waves up to the realm of quantum transport, add important elements to the issue of the actual role played by randomness.

With those ideas in mind, here we set out to track the dynamics of rogue waves on a single quantum particle propagating in a lattice featuring correlated disorder. Anderson localization theory settles that all single-particle eigenstates are exponentially localized for any amount of uncorrelated disorder in one and two dimensions [26]. This can be violated, however, when the disorder displays intrinsic correlations [27,29]. Scale-free correlations, for instance, are known to support a metal-insulator transition with sharp mobility edges [27].

In this article, our goal is to investigate the development of sudden, anomalous quantum amplitudes due to the interplay between localized and extended states. Indeed, long-range correlated fluctuations in the random input phases was recently shown to produce rogue waves way above the threshold in an experiment on linear light diffraction in 1D [22].

We look at a particular kind of correlated disorder in which a single parameter is able to control the typical correlation length and, in turn, the local disorder strength. The latter is found to be a crucial factor underlying the generation of the rogue waves for it sets up the right amount of wavefunction mobility. This boosts not only the number of occurrences, but also the average rogue wave amplitude.

II. MODEL

We consider a single quantum particle propagating in a one-dimensional array described by the tight-binding Hamiltonian

$$H = J \sum_{n=1}^{N} (|n\rangle\langle n+1| + |n+1\rangle\langle n|) + \sum_{n} \epsilon_{n} |n\rangle\langle n|, \quad (1)$$
where $J$ is the nearest-neighbor hopping strength, $\varepsilon_n$ is the on-site potential, and states $|n\rangle$ represent the location of the particle and span the whole Hilbert space. As such, an arbitrary quantum state can be written as $|\Psi\rangle = \sum_n \psi_n |n\rangle$, with the normalization condition $\sum_n |\psi_n|^2 = 1$. The time evolution of the wave function $\psi_n$ is given by the Schrödinger equation

$$i\hbar \frac{d}{dt} \psi_n = \psi_{n+1} + \psi_{n-1} + \varepsilon_n \psi_n,$$

where $\hbar = J = 1$ without loss of generality.

Here we go beyond the standard case of uncorrelated disorder and consider that the local potentials $\varepsilon_n$ are embedded with correlations, as given by

$$\varepsilon_n = \sum_m Z_m \frac{d_{n,m}}{(1 + d_{n,m}/A)^2},$$

where $Z_m$ is a random number in $[-1,1]$, $d_{n,m}$ is the Euclidean distance between sites $n$ and $m$, and $A$ controls the correlation length of the series. We impose periodic boundary conditions, $|N + 1\rangle = |1\rangle$, such that each site $n$ can be identified through angular and Cartesian coordinates, $\theta_n = (2\pi/N)n$ and $(x_n = R \sin \theta_n, y_n = R \cos \theta_n)$, respectively, rendering $d_{n,m} = \sqrt{(x_n - x_m)^2 + (y_n - y_m)^2}$. The disordered sequence is further normalized to have zero mean and unit variance.

To see how the above disorder configuration play out with the correlation parameter $A$, in Fig. 1(a) we plot the autocorrelation function $C(r) = \text{cov}(\varepsilon_i, \varepsilon_{i+r}) = \sum_{i=1}^{N-r} \varepsilon_i \varepsilon_{i+r}/(N-r)$. Upon increasing $A$, $C(r)$ exhibits a slower decay, as expected from Eq. (3). We are then able to set an effective correlation length $L_c$ by fitting $C(r) \propto e^{-r/L_c}$. Figure 1(b) shows that $L_c \propto A$ thereby affirming what the latter stands for.

### III. RESULTS

We are ready to search for the occurrence of rogue waves and address the role of the correlated disorder. In all simulations below, the set of equations given in Eq. (2) is numerically solved by employing a high-order Taylor expansion of the time evolution operator:

$$U(\Delta t) = \exp(iH\Delta t) = 1 + \sum_{l=1}^{n_0} \frac{(iH\Delta t)^l}{l!},$$

with time step $\Delta t = 0.01J^{-1}$ and $n_0 = 20$, which is enough to produce smooth outcomes and keep the norm conserved during the whole time interval. In order to avoid ambiguity between a rogue wave event and trapping of the wavefunction due to Anderson localization we set $\psi_n(t=0) = 1/\sqrt{N}$ for all $n$.

Figure 2 shows typical scenarios of rogue waves, as told by the probability amplitude of the particle wavefunction $|\psi_n(t)|^2$, for different values of the correlation strength $A$. For weak correlations the evolution is characterized by sparse, low-amplitude waves that eventually add up to produce the rogue events, as shown in Fig. 2(a) around $t = 3000/J$. As the correlation strength is increased, the background becomes more inhomogeneous – a fundamental trait in the generation of rogue waves via linear processes. There sets in distinct amplitude domains in space, indicating that $A$ is pushing for weaker local fluctuations in the disorder distribution (to be addressed in a moment). As a consequence, rogue waves of exceptionally higher amplitudes are likely to occur [cf. Fig. 2(d)].

We will not be using here any particular rogue-wave criteria. One measure employed in various contexts is the significant wave height, commonly defined as twice the average of largest one third of values in a data set. An event is thus considered a rogue wave whenever it beats that level. Considering our initial state and the results obtained in Ref. 25, that deals with a similar class of problem, such threshold would be of the order of $1/N$. As our following analysis is built on extreme-value statistics, the data is heavily loaded with amplitudes well above that mark.

Let us now obtain one of the limiting extreme-value distributions, according to the Fisher-Tippett-Gnedenko theorem. In order to do so, we pick the maximum wavefunction amplitude in space at each time
step for several independent realizations of the disorder. The stacked outcomes are seen in the probability density functions (PDFs) shown in Fig. 3 for different degrees of correlation $A$. Note that this effective correlation length indeed brings forth higher amplitudes. However, there is a threshold value for $A$ above which the right tail of the distribution begins to deflate (rare, extreme rogue waves can still develop). To learn more about this non-monotonic relationship between the rogue-wave maximum amplitudes and the correlation parameter $A$, we portray in Fig. 4 the maximum amplitudes $|\psi|^2_{max}$ allowed at a probability level just above $10^{-7}$ (see Fig. 3). This is done in order to avoid extremely rare outcomes. The scaling with $N$ in Fig. 4 is employed so we can filter out finite-size effects and focus on the role of the correlation degree only. Compared with what one would obtain from uncorrelated series of the potential $\varepsilon_n$ (cf. dashed curve in Fig. 4), the rogue waves can reach almost twice as large amplitudes when supported by the correlated disorder.

We also find that all the skewed PDFs shown in Fig. 3 belong to the Gumbel class of extreme value distributions, fitted by $P(x) \propto \exp[-(\alpha x + \beta \exp(-\alpha x))]$. This is expected since the tail of the parent function $p(|\psi|^2)$ decays exponentially (slower than that of a Gaussian distribution, which is another signature of rogue waves). Similar behavior is found in quantum walks featuring uncorrelated disorder [25], meaning that the wavefunction fluctuations are well described by processes involving independent and identically distributed random variables.

The regularity of rogue wave events occurring on systems featuring static disorder can be predicted to some extent based on the energy resonance conditions across the lattice. Either very weak or very strong levels of disorder should suppress the onset of anomalous wavefunction fluctuations. Some studies report that an optimal balance between localization and mobility can maximize their chances to happen [21, 25]. Here, this balance is effectively imposed by the correlation level $A$. To show
FIG. 4. Maximum wavefunction probability amplitude $|\psi|^2_{\text{max}}$ scaled with $N$ (excluding exceptionally rare events) for a range of $A/N$ values.

FIG. 5. Mean number of sites actively involved in the generation of rogue waves $\Delta n_{\text{RW}}$ versus correlation strength $A$, averaged over 500 independent realizations of disorder, for times up to $t = 15000/J$ and $N = 100$. Note that the values of $A$ that give highest $\Delta n_{\text{RW}}$ also leads to outbreak of much intense rogue waves.

there is indeed a coordinated dynamics taking place, let us track how many sites, for a given disordered sample, are mostly involved in the those extreme events. Defining $\pi(n)$ as the probability of a rogue occurring at site $n$, we compute the mean number of participating sites as $\Delta n_{\text{RW}} = 1/\left[\sum n \pi(n)^2\right]$, which ranges from 1 (fully biased) to $N$ (equally distributed). Figure 5 displays this quantity against the correlation parameter $A$. Again, the non-monotonic behavior is clear and indicates that intermediate values of $A$ implies in more sites actively participating in the generation of rogue waves. In turn, waves of higher amplitudes becomes more likely (cf. Figs. 3 and 4).

Last, we complement the above discussion by taking a look back at the potential series given by Eq. (3). Figure 6(a) shows how it typically sets up along the lattice. For $A = 1$, the series features a white noise (almost uncorrelated) profile. Then, at $A = 25$ local fluctuations are drastically reduced. As we further increase the effective correlation length, a rougher landscape is obtained, but still carrying a predominant harmonic component.

To put all that together, let us compute the local disorder strength in terms of the local standard deviation $\sigma_{L_0} = \left(\sum_{k=1}^{M} \sigma_{k,L_0}\right)/M$ within a segment with $L_0$ sites, where

$$\sigma_{k,L_0} = \left[\frac{\sum_{n=(k-1)L_0+1}^{kL_0} \varepsilon_n^2}{L_0} - \left(\frac{\sum_{n=(k-1)L_0+1}^{kL_0} \varepsilon_n}{L_0}\right)^2\right]^{1/2}$$

Results are shown in Fig. 6(b), where we are able to confirm once for all that intermediate values of $A$ makes for minimum local fluctuations, which is consistent with the smooth potential landscape seen in Fig. 6(a).
IV. CONCLUDING REMARKS

We explored the intrinsic role played by correlated disorder on the emergence of rogue waves in a simple quantum tight-binding model. The amplitude of the particle wave function was found to exhibit strong anomalous fluctuations inherently unpredictable in time and space. An approach based on extreme-value statistics revealed that such fluctuations follow a Gumbel distribution, belonging to the same class as those reported in a quantum walk featuring uncorrelated disorder [25].

We learned that intermediate values of the correlation parameter $A$, acting here as an effective correlation length, brings down the local disorder strength. This properly enhances the mobility of the wavefunction (by spanning eigenstates with larger localization lengths) and so the number of sites on which rogue-wave events take place, what in turn amplifies their characteristic amplitude (almost twice as large when compared to uncorrelated scenarios [21, 22, 23]).

While disorder must be present to promote random fluctuations of the wavefunction, the underlying single-particle eigenstates must be wide enough in order to allow the linear superposition of the many components needed to produce localized waves in space and time. A general result we can establish is that rogue waves are expected to occur more often in the regime of weak disorder. Our findings are general to a class of 1D discrete disordered systems and bring about another perspective on their dynamics as well as reveal fundamental aspects of the role of randomness in the generation of rogue waves.

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