ABSTRACT. We compute the Fukaya category of the symplectic blowup of a compact rational symplectic manifold at a point in the following sense: Suppose a collection of Lagrangian branes satisfy Abouzaid’s criterion [Abo10] for split-generation of a bulk-deformed Fukaya category of cleanly-intersecting Lagrangian branes. We show (Theorem 1.1) that for a small blow-up parameter, their inverse images in the blowup together with a collection of branes near the exceptional locus split-generate the Fukaya category of the blowup. This categorifies a result on quantum cohomology by Bayer [Bay04] and is an example of a more general conjectural description of the behavior of the Fukaya category under transitions occurring in the minimal model program, namely that mmp transitions generate additional summands.

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1. Introduction

In this paper we study the Fukaya category of a symplectic manifold obtained by a symplectic blowup at a point. In particular we show that given a collection of branes satisfying Abouzaid’s criterion for split-generation [Abo10], the Fukaya category of the blowup is split-generated by the image of an embedding of the Fukaya category of the original manifold (with bulk deformation) together with a collection of branes near the exceptional locus. This is a symplectic analog of Orlov’s blowup formula [Orl93] that gives a semi-orthogonal decomposition of the derived category of a blowup. We also show (conditional on a generalization of a result of Ganatra [Gan12]) that in this situation the quantum cohomology isomorphic to the Hochschild cohomology of the Fukaya category, verifying an expectation of Kontsevich [Kon94, p.18].

The main result may be stated as follows. Let \( X \) be a compact symplectic manifold and \( QH(X, b) \) its quantum cohomology ring at bulk deformation \( b \in QH(X) \). To keep technicalities at a minimum, we assume that the cohomology class \( [\omega] \in H^2(X, \mathbb{R}) \) of the symplectic form \( \omega \) is rational and define a simplified version \( \text{Fuk}_L(X, b) \) of the Fukaya category whose objects \( L \) are components of a single rational immersed brane, and whose composition maps count holomorphic disks with boundary in \( L \), perturbed using the Cieliebak-Mohnke scheme [CM07]. There are natural open-closed and closed-open maps

\[
HH_\bullet(\text{Fuk}_L(X, b), \text{Fuk}_L(X, b)) \xrightarrow{OC(b)} QH^\bullet(X, b) \\
\xrightarrow{CO(b)} HH^\bullet(\text{Fuk}_L(X, b), \text{Fuk}_L(X, b)).
\]

A criterion for the closed-open map to be an isomorphism is provided by results of Abouzaid [Abo10] and Ganatra [Gan12]. Given a subset \( G \subset L \) let \( \text{Fuk}_G(X, b) \) denote the sub Fukaya category with objects \( G \). Write

\[
QH_G(X, b) = (OC(b))(HH_\bullet(\text{Fuk}_G(X, b), \text{Fuk}_G(X, b)))
\]

for the image of \( HH_\bullet(\text{Fuk}_G(X, b), \text{Fuk}_G(X, b)) \) under the open-closed map. We say \( QH^\bullet(X; b) \) is generated by \( G \) iff \( QH_G(X, b) = QH^\bullet(X, b) \). In this situation, \( G \) split-generates \( \text{Fuk}_L(X, b) \). For technical reasons, we assume that \( L \) has the property that the union of image of branes in \( L \) is strongly rational. (This condition can
always be achieved by Hamiltonian perturbation, if each $G \in \mathcal{G}$ is rational, by [PW, Lemma 4.2], but we do not discuss such perturbations in this paper.)

We blow-up the given symplectic manifold at a point. Suppose that $\epsilon > 0$ is a small real number and $\tilde{X}$ is the $\epsilon$-blowup of $X$ at a point $p \in X$. Let $\mathcal{E}$ be the collection of Clifford-type Lagrangians near the exceptional locus $E \subset \tilde{X}$ constructed in [CW], roughly speaking obtained by thickening the Clifford torus in the exceptional locus and equipping the resulting torus with a brane structure so that the Floer cohomology is non-trivial. Let $\tilde{b}$ be the preimage of $b$ in $\tilde{X}$. The main result relates the Fukaya category of $\tilde{X}$ at a bulk deformation $\tilde{b}$ with the shifted bulk deformation of $X$ given by $b + q^{-p}$.

**Theorem 1.1.** Suppose that for $\epsilon \in \mathbb{Q}_+$ sufficiently small, $G \subset \mathcal{L}$ generates $QH^\bullet(X, b + q^{-p})$. Then $\pi^{-1}(\mathcal{G}) \cup \mathcal{E}$ generates $\tilde{QH}^\bullet(\tilde{X}, \tilde{b})$. In particular for any collection of branes $\tilde{\mathcal{L}} \subset \pi^{-1}(\mathcal{L}) \cup \mathcal{E}$ containing $\pi^{-1}(G) \cup \mathcal{E}$ the open-closed map

$$HH_\bullet(Fuk_{\tilde{\mathcal{L}}}(\tilde{X}, \tilde{b}), Fuk_{\tilde{\mathcal{L}}}(\tilde{X}, \tilde{b})) \to \tilde{QH}^\bullet(\tilde{X}, \tilde{b})$$

is a surjection, $\tilde{\mathcal{G}}$ split-generates the Fukaya category $Fuk_{\tilde{\mathcal{L}}}(\tilde{X}, \tilde{b})$ of $\tilde{X}$ with bulk deformation $\tilde{b}$ and (conditional on the extension of Ganatra [Gan12] to the compact case) there are isomorphisms

$$HH_{\dim(X)-\bullet}(Fuk_{\tilde{\mathcal{L}}}(\tilde{X}, \tilde{b}), Fuk_{\tilde{\mathcal{L}}}(\tilde{X}, \tilde{b})) \to \tilde{QH}^\bullet(\tilde{X}, \tilde{b})$$

$$\to HH^\bullet(Fuk_{\tilde{\mathcal{L}}}(\tilde{X}, \tilde{b}), Fuk_{\tilde{\mathcal{L}}}(\tilde{X}, \tilde{b})).$$

The theorem confirms in this case Kontsevich’s expectation that Hochschild cohomology of the Fukaya category is isomorphic to the quantum cohomology [Kon94, p.18]. K. Ono communicated to us that he also proved results in this direction, and some special cases are proved in Sanda [San]. Pedroza [Ped] studied the effect of blowups on the Floer cohomology of Lagrangians disjoint from the blowup point, in the monotone case. Fukaya categories of certain blow-ups of toric varieties are studied from the viewpoint of the Strominger-Yau-Zaslow conjecture in Abouzaid-Auroux-Katzarkov [AAK16]. The theorem also slightly generalizes a result for small quantum cohomology of Bayer [Bay04], who proved that semi-simplicity of quantum cohomology is preserved under point blowups. The result here is slightly stronger than Bayer’s stated result [Bay04]: Let $D^\pi Fuk_{\tilde{\mathcal{L}}}(\tilde{X}, \tilde{b})$ denote the sum of the idempotent-completed derived categories $D^\pi Fuk_{\tilde{\mathcal{L}}}(\tilde{X}, \tilde{b})_w$ as $w$ ranges over possible values of the curvature, see for example Sheridan [She16].

**Corollary 1.2.** Let $n = \dim(X)$. Under the hypotheses in Theorem 1.1, there exists a collection of objects

$$\mathcal{E} = \{\phi(1), \ldots, \phi(n-1)\} \subset \text{Ob}(Fuk_{\tilde{\mathcal{L}}}(\tilde{X}, \tilde{b}))$$

supported near the exceptional locus and an equivalence of idempotent-completed derived categories

$$D^\pi Fuk_{\tilde{\mathcal{L}}}(\tilde{X}, \tilde{b}) \cong D^\pi Fuk_{\tilde{\mathcal{L}}}(X, b + q^{-p}) \oplus D^\pi Fuk_{\tilde{\mathcal{L}}}(\tilde{X}, \tilde{b})$$
as well as an isomorphism of quantum cohomology rings
\[ QH(\tilde{X}, \tilde{b}) \cong QH(X, b + q^{-\epsilon}p) \oplus QH(pt)^{\oplus n-1}. \]

See also González-Woodward [GW] and Iritani [Iri, Theorem 1.3]. We expect a similar result to hold for flips. The decomposition in Corollary 1.2 is also expected to be only semi-orthogonal with respect to some categorical analog of the quantum connection.

To explain the technical setup in more detail, recall that the Fukaya category is meant to be a categorification of the quantum cohomology whose composition maps count holomorphic disks with Lagrangian boundary conditions. Let
\[ \Lambda = \left\{ \sum_{i=1}^{\infty} c_i q^{d_i}, \ c_i \in \mathbb{C}, \ d_i \in \mathbb{R}, \ \lim_{i \to \infty} d_i = \infty \right\} \]
denote the universal Novikov field. The valuation by powers of \( q \)
\[ \text{val}_q : \Lambda - \{0\} \to \mathbb{R}, \ \sum_{i=1}^{\infty} c_i q^{d_i} \mapsto \min_{c_i \neq 0} (d_i) \]
is well-defined. Denote the subsets with non-negative resp. positive valuation
\[ \Lambda_{\geq 0} = \{ f \in \Lambda \mid \text{val}_q(f) \geq 0 \}, \ \text{resp.} \ \Lambda_{>0} = \{ f \in \Lambda \mid \text{val}_q(f) > 0 \}. \]
Let \( \Lambda^\times \subset \Lambda \) be the subgroup with zero \( q \)-valuation.

We allow as objects of the Fukaya category compact immersed self-transverse Lagrangian branes. Let \( X \) be a compact symplectic manifold with symplectic form \( \omega \) with rational symplectic class \([\omega] \in H^2(X, \mathbb{Q})\). A Lagrangian brane is a compact immersed Lagrangian equipped with a \( \Lambda^\times \)-valued local system
\[ \phi : L \to X, \ y : \pi_1(\phi(L)) \to \Lambda^\times \]
on \( \phi(L) \), a relative spin structure, and a grading. Given a rational collection of pairwise-clealy-intersecting and self-transversally-intersecting immersed branes \( \mathcal{L} \) and a \( \Lambda_{\geq 0} \)-valued pseudocycle \( b \) denote by \( \text{Fuk}_\mathcal{L}(X, b) \) the Fukaya \( A_\infty \) category of \( X \) supported on \( \mathcal{L} \) with bulk deformation \( b \). Objects
\[ \text{Ob}(\text{Fuk}_\mathcal{L}(X)) = \mathcal{L} \]
are the given branes and morphisms are Floer cochains
\[ \text{Hom}(\phi_-, \phi_+) = CF(\phi_-, \phi_+), \ \phi_+, \phi_- \in \mathcal{L}. \]
In the Morse model used here, Floer cochains are formal combinations of critical points of a Morse function
\[ F(\phi_-, \phi_+) : (\phi_- \times \phi_+)^{-1}(\Delta) \to \mathbb{R} \]
chosen on the preimage of the diagonal
\[ (\phi_- \times \phi_+)^{-1}(\Delta) \subset L_- \times L_+ \]
assumed to be cut out cleanly. The composition maps
\[ m_d : \Hom(\phi_0, \phi_1) \otimes \ldots \otimes \Hom(\phi_{d-1}, \phi_d) \to \Hom(\phi_0, \phi_d)[d-2], \quad d \geq 0 \]
count treed holomorphic disks \( u : C \to X \) with some markings mapping to the bulk deformation \( b \in CF(X) \); these are maps from combinations \( C = S \cup T \) of disks \( S_v \subset S \) and segments \( T_e \subset T \) that satisfy Gromov’s pseudoholomorphicity conditions on the disks \( S_v \) and the gradient flow equation on the segments \( T_e \).

We take as a convenient regularization scheme the Cieliebak-Mohnke method of Donaldson hypersurfaces [CM07]. To stabilize domains of holomorphic disks we choose a Donaldson hypersurface
\[ D(L) \subset X, \quad [D(L)] = k[\omega], \quad k \gg 0 \]
so that the \( \phi(L) \) is exact in the complement of \( D(L) \). For a suitably chosen almost complex structure, any holomorphic sphere \( u : \mathbb{P}^1 \to X \) intersects \( D(L) \) at least three times [CM07, 8.17]. Furthermore, each holomorphic disk \( u : \mathbb{D} := \{ \| z \| \leq 1 \} \to X \) bounding \( \phi(L) \) intersects \( D(L) \) at least once, by the exactness condition. For any holomorphic curve \( C \to X \) with Lagrangian boundary in \( \phi \), the intersections of any treed disk \( u : C \to X \) with the divisors \( D(L) \) then stabilize the domain \( C \) except possibly for disk components \( S_v \subset C \) with two special points. The first part of the paper is taken up with showing the following: For any combinatorial type \( \Gamma \) with \( d \) incoming leaves denote by
\[ \mathcal{M}_\Gamma(\phi, D(L)) = \{(u : C \to X, (\partial C)_i : (\partial u)_i \to L_i, i = 0, \ldots d)\}/\sim \]
the moduli space of holomorphic treed disks bounding \( \phi \) with interior leaves corresponding to intersections of the map with a Donaldson hypersurface \( D(L) \), modulo the equivalence \( \sim \) defined by isomorphism of domains. A type is uncrowded if at most one interior leaf of a given label lies on each ghost subtree. As in [CW], there exist coherent perturbations
\[ P = (P_\Gamma), \quad P_\Gamma = (J_\Gamma, H_\Gamma, F_\Gamma, E_\Gamma) \]
consisting domain-dependent almost complex structures \( J_\Gamma \), Hamiltonian perturbations\(^1\) \( H_\Gamma \) supported away from the divisors \( D(L) \), perturbed Morse functions \( F_\Gamma \) on the Lagrangians, and perturbations \( E_\Gamma \) of the evaluation maps at the interior leaves so that all moduli spaces \( \mathcal{M}_\Gamma(\phi, D(L)) \) of holomorphic treed disks with uncrowded type \( \Gamma \) of expected dimension at most one are regular of expected dimension and counts of zero-dimensional moduli spaces define a strictly unital \( A_\infty \) category \( \Fuk_\Lambda(X, b) \).

Having defined a simplified version of the Fukaya category, we investigate the Hochschild homology and cohomology of the factors in its spectral decomposition. For any element \( b \in \Hom(\phi, \phi) \) with positive \( q \)-valuation define the Maurer-Cartan
\[ ^1\text{Hamiltonian perturbations are needed only for immersed Lagrangians, to perturb away constant disks mapping to self-intersection points.} \]
Fukaya categories of blowups

\[ m(b) := \sum_{d \geq 0} m_d(b, \ldots, b). \]

Following Fukaya-Oh-Ohta-Ono [FOOO09] for any \((\phi : L \to X) \in \mathcal{L}\) denote by \(MC(\phi)\) the space of solutions to the weak Maurer-Cartan equation

\[ MC(\phi) := \{ b \in \text{Hom}^{\text{odd}}(\phi, \phi) \mid m(b) \in \Lambda_1 \}. \]

For each \(w \in \Lambda\), one denotes by \(\text{Fuk}_L(X, b, w)\) the flat \(A_\infty\) category whose objects are pairs

\[ \text{Ob}(\text{Fuk}_L(X, b, w)) = \{ (\phi, b) \mid \phi \in \mathcal{L}, m(b) = w_{1\phi} \} \]

consisting of an object in \(\text{Fuk}_L(X, b)\) and a weakly bounding cochain \(b \in MC(\phi)\).

The morphisms are Floer cochains

\[ \text{Hom}(\phi_-, \phi_+) = CF(\phi_-, \phi_+). \]

The composition maps for \(d \geq 1\) are

\[ m^w_d : \text{Hom}(\phi_0, \phi_1) \otimes \cdots \otimes \text{Hom}(\phi_{d-1}, \phi_d) \to \text{Hom}(\phi_0, \phi_d)[d - 2], \]

\[ (c_1, \ldots, c_d) \mapsto \sum_{k_0, \ldots, k_d} m_{d+k_0+\cdots+k_d}(b_0, \ldots, b_{k_0}, c_1, b_1, \ldots, b_{k_1}, \ldots, c_d, b_d, \ldots, b_{k_d}). \]

By definition, the elements

\[ 0 = m^w_0(1) \in \text{Hom}(\phi, \phi) \]

vanish; one checks that since \(b \in MC(\phi)\), the \(A_\infty\) axiom

\[ 0 = \sum_{d_1, d_2 \geq 0 \atop d_1 + d_2 \leq d} (-1)^{k^w_{d_1, d_2}} m^w_{d_1, d_2+1}(x_1, \ldots, x_{d_1}, \ldots, x_{d_1+d_2}, x_{d_1+d_2+1}, \ldots, x_d) \]

holds for any \(x_1, \ldots, x_d \in I(\phi_-, \phi_+)\) where

\[ k^w_f = \sum_{i \leq j \leq k} \|x_j\| \]

and

\[ \|x_j\| = |x_j| + 1 \]

is the reduced degree. In this way, one obtains a family of flat \(A_\infty\) categories \(\text{Fuk}_L(X, b, w)\) indexed by values of the potential \(w \in \Lambda\) and bulk deformation \(b\). In particular, the operator \(m^w_1\) squares to zero for any objects \((\phi_\pm, b_\pm)\) with the same \(w\) and we denote by

\[ HF^\bullet(\phi_-, \phi_+; b) = H(\text{Hom}((\phi_-, b_-), (\phi_+, b_+))) = \frac{\ker(m^w_1)}{\text{im}(m^w_1)} \]

its Floer cohomology.

The open-closed and closed-open maps relate the Hochschild (co)homology with quantum cohomology. By definition the Hochschild cohomology is the cohomology
of the endomorphism algebra of the identity: For any bulk deformation $b$ the closed-open map

$$CO(b) : QH^\bullet(X, b) \rightarrow HH^\bullet(Fuk_L(X, b), Fuk_L(X, b))$$

is a unital ring homomorphism. On the other hand, the open-closed map sends the Hochschild homology to the quantum cohomology $QH^{\dim(X)-\bullet}(X, b)$. By definition Hochschild homology for each $w \in \Lambda$ is the homology of a differential on

$$(8) \quad CC_\bullet(Fuk_L(X, b, w), Fuk_L(X, b, w)) = \bigoplus_{k=0}^{\infty} \bigoplus_{i_1, \ldots, i_k} \Hom(\phi_{i_0}, \phi_{i_1}) \otimes \cdots \otimes \Hom(\phi_{i_{k-1}}, \phi_{i_k}) \otimes \Hom(\phi_{i_k}, \phi_0)$$

obtained by a signed sum possible contractions using the $A_\infty$ composition maps on $Fuk_L(X, b, w)$, see (40) below. These groups combine via a direct sum into what we will call the Hochschild cohomology of the curved category $Fuk_L(X, b)$:

$$HH_\bullet(Fuk_L(X, b), Fuk_L(X, b)) = \bigoplus_{w \in \Lambda} HH_\bullet(Fuk_L(X, b, w), Fuk_L(X, b, w)).$$

We then have an open-closed map

$$CO(b) : HH^\bullet(Fuk_L(X, b), Fuk_L(X, b)) \rightarrow QH^\bullet(X, b).$$

A criterion for the closed-open map to be an isomorphism is provided by results of Abouzaid [Abo10] and Ganatra [Gan12].

**Definition 1.3.** Given a subset $G \subset L$ let $Fuk_G(X, b)$ denote the sub Fukaya category with objects $G$. Write

$$QH_G(X, b) = (OC(b))(HH_\bullet(Fuk_G(X, b), Fuk_G(X, b)))$$

for the image of $HH_\bullet(Fuk_G(X, b), Fuk_G(X, b))$ under the open-closed map. Say

$$QH^\bullet(X; b)$$

is generated by $G$ iff $QH_G(X, b) = QH^\bullet(X, b)$.

**Remark 1.4.** Because the open-closed map depends analytically on the bulk-deformation, if the open-closed map is surjective for at least one bulk-deformation $b$ then surjectivity holds for generic $b$. See Proposition 4.11 below.

**Theorem 1.5.** (Abouzaid [Abo10] in the exact case, extended to the compact case in Section 5 below) If $QH^\bullet(X, b)$ is generated by $G$ then for each $w \in \Lambda_{\geq 0}$ there is a subset of $G$ that split-generates $Fuk_L(X, b, w)$ for each $w \in \Lambda_{\geq 0}$.

**Remark 1.6.** Ganatra has shown in the exact setting [Gan12] that under these assumptions $b$-deformed Hochschild homology and cohomology of $Fuk_L(X, b)$ are isomorphic as vector spaces (after a degree shift):

$$HH_\bullet(Fuk_G(X, b), Fuk_G(X, b)) \cong HH^{\dim(X)-\bullet}(Fuk_G(X, b), Fuk_G(X, b))$$

and (in the compact setting here) are both isomorphic to the quantum cohomology $QH^{\dim(X)-\bullet}(X, b)$. Ganatra’s results [Gan12] are written up in detail only for the exact, undeformed case, and not completely published. We see no obstruction in extending his results to the compact, rational case here using Cieliebak-Mohnke
perturbations or the reader’s favorite regularization scheme, but the results on isomorphisms stated in (1) for the compact case could be more properly described as conditional on this extension.

Our main result applies the Abouzaid criterion Theorem 1.5 to blowups. Recall that the blowup of affine space $X = \mathbb{C}^n$ at $p = 0$ is

$$\text{Bl}(\mathbb{C}^n, 0) = \{(z, \ell) \in \mathbb{C}^n \times \mathbb{P}^{n-1} | z \in \ell\}$$

and is equipped with a natural holomorphic projection

$$\pi : \text{Bl}(\mathbb{C}^n, 0) \to \mathbb{C}^n, \quad (z, \ell) \mapsto z.$$  

The inverse image of the blowup point $E = \pi^{-1}(p)$, $E \cong \mathbb{P}^{n-1}$ is the exceptional locus of the blowup. A symplectic blowup $\tilde{X}$ of a symplectic manifold $X$ at a point $p$ is defined similarly using a Darboux chart $U \ni p$ and gluing in the local model of the previous paragraph:

$$\tilde{X} = ((X - \{p\}) \cup \pi^{-1}(U))/\sim.$$  

A natural family of symplectic forms $\tilde{\omega}_\epsilon$ on $\tilde{X}$ arises from the family of symplectic forms on $\text{Bl}(\mathbb{C}^n, 0)$ considered as a toric variety with moment polytope

$$\{(x_1, \ldots, x_n) \in \mathbb{R}_{\geq 0}^n | x_1 + \ldots + x_n \geq \epsilon\}.$$  

The resulting symplectic manifold $\tilde{X}$ is the $\epsilon$-blowup of $X$ at $p$; it seems possible that the symplectomorphism type of the blow-up depends on the choice of $\epsilon$ and Darboux chart $U$.

An embedding of the original Fukaya category into the Fukaya category of its blowup will be realized after a shift in bulk deformation given by homology classes. As explained in Zinger [Zin08] any integral homology class may be represented by a pseudocycle. Recall that a pseudocycle in a compact manifold $X$ is a smooth map $j : Z \to X$ whose boundary

$$\Omega_j(Z) := \bigcap_{K \subseteq Z \text{ compact}} \overline{j(Z - K)}$$

satisfies the following condition: There exists a smooth map $\iota : W \to X$ from a smooth manifold $W$ to $X$ with $\text{dim}(W) \leq \text{dim}(Z) - 2$ such that

$$\Omega_j(Z) \subseteq \iota(W).$$

Denote the set of formal combinations of pseudocycles $b_i$ in $X$ with coefficients in $\Lambda$

$$\Psi(X, \Lambda) = \left\{ b = \sum_{i=1}^m c_i b_i \left| c_i \in \Lambda, \ b_i \subset X \right. \right\}.$$  

For any $b \in \Psi(X, \Lambda)$ denote by

$$|b| = \bigcup_{i=1}^m b_i \subset X$$
the support of $b$ given as the union of geometric pseudocycles in $b$. Denote by
\[ \Psi_p(X, \Lambda) \subset \Psi(X, \Lambda) \]
the space of pseudocycles whose supports are disjoint from $p$. There is a canonical map of pseudocycles
\[ \Psi_p(X, \Lambda) \to \Psi(\tilde{X}, \Lambda), \quad b \mapsto \pi^{-1}(b) \]
obtained by inverse image:
\[ (b = \sum_{i=1}^d c_i b_i) \implies \left( \pi^{-1}(b) = \sum_{i=1}^d c_i \pi^{-1}(b)_i \right). \]

**Definition 1.7.** Given any Lagrangian brane $\phi : L \to X$ disjoint from $p$, denote by
$\tilde{\phi} : L \to \tilde{X}$ the unique Lagrangian brane in $\tilde{X}$ with $\pi \circ \tilde{\phi} = \phi$.

**Remark 1.8.** Disjoint-ness $\phi(L) \cap \{p\}$ always holds after a small Hamiltonian diffeomorphism of $X$.

**Theorem 1.9.** Suppose that $\dim(X) > 1$ and $L$ consists of branes disjoint from $p$. For $\epsilon > 0$ sufficiently small and suitable perturbation data $\mathcal{P} = (P_\epsilon)$, the structure maps of $\text{Fuk}_{\mathcal{L}}(X, b + q^{-\epsilon}p)$ are convergent and define an $A_\infty$ category with the following property: There exists a homotopy equivalence of $A_\infty$ categories
\[ \text{Fuk}_{\mathcal{L}}(X, b + q^{-\epsilon}p) \to \text{Fuk}_{\pi^{-1}(\mathcal{L})}(\tilde{X}, \tilde{b}). \]

**Remark 1.10.** The bulk deformation has possibly negative $q$-valuation $\text{val}_q(\tilde{b})$ and so is not of the type usually allowed. Indeed,
\[ b_\epsilon = b + q^{-\epsilon}p \in \Psi(X, \Lambda) \]
has coefficient $q^{-\epsilon}$ of $p$ with $q$-valuation $-\epsilon < 0$. In general bulk deformations $\tilde{b}$ with negative $q$-valuation $\text{val}_q(\tilde{b}) < 0$ may not define convergent $A_\infty$ structure maps (6). This is the case if, for example, $\dim(X) = 1$. In this case there are contributions to the deformed composition maps $m_{d(\mathcal{L})}(\tilde{b})$ for fixed $d(\mathcal{L})$ from a fixed holomorphic curve $u : C \to X$ with arbitrary numbers $d(\mathcal{L}, b)$ of interior leaves $T_\epsilon$ marked with the bulk deformation $\tilde{b}$. As this number $d(\mathcal{L}, b)$ tends to infinity, the $q$-valuation of their contribution to $m_d(b)$ tends to minus infinity. So the sum is not convergent in the Novikov field $\Lambda$.

**Remark 1.11.** One can rephrase Theorem 1.9 as follows: There exist perturbation data so that the $A_\infty$ category $\text{Fuk}_{\mathcal{L}}(X, b + q^{-\epsilon}p)$, defined a-priori only for $\epsilon > 0$, admits an analytic extension to $\epsilon < 0$ that agrees with the Fukaya category $\text{Fuk}_{\pi^{-1}(\mathcal{L})}(\tilde{X}, \tilde{b})$ for $\tilde{X}$ the $-\epsilon$-blowup of $X$.

The proof relies on a correspondence between pseudoholomorphic curves induced by the projection. Namely, given any holomorphic curve $\tilde{u} : C \to \tilde{X}$ one obtains a holomorphic curve in the original manifold $u : C \to X$ by projection $u = \pi \circ \tilde{u}$. This correspondence defines a map of moduli spaces
\[ \overline{\mathcal{M}}(\tilde{\phi}, \pi^{-1}(D(\mathcal{L}))) \to \overline{\mathcal{M}}(\phi, D(\mathcal{L})) \]
where $\mathcal{M}$ denotes the union over all combinatorial types $\mathcal{M}_\Gamma$. Let $I(u), I(\tilde{u}) \in 2\mathbb{Z}$ denote the Maslov indices, defined as the sum of the Maslov indices $I(u_\alpha)$ of the disk components and twice the Chern numbers of the sphere components. The projection (10) does not preserve the expected dimension of the moduli spaces for types given by the same tree $\Gamma$, determined by the Maslov indices $I(u) \neq I(\tilde{u})$ and combinatorial data attached to the type $\Gamma$, but does preserve expected dimension (and give a bijection $\mathcal{M}_\Gamma(\phi, D(L)) \to \mathcal{M}_{\tilde{\Gamma}}(\tilde{\phi}, \pi^{-1}(D(L)))$) if the map $u$ is considered as a constrained map of type $\tilde{\Gamma}$ with point bulk deformation. To prove the theorem it therefore suffices to show that perturbation data $\tilde{P}_\Gamma$ pulled back under the projection $\pi : \tilde{X} \to X$ make all moduli spaces of curves in $\tilde{X}$ regular; one may then simply compose with the projection to obtain the correspondence.

We wish to complete the collection of “old branes” to a collection of split-generators for the Fukaya category of the blowup. Recall from e.g. [Sei08b] that a collection of objects $\mathcal{G}$ split-generates an $A_\infty$ category if every object can be constructed from the generators $\mathcal{G}$ by repeatedly taking mapping cones and splitting off direct summands. In a previous paper [CW] Charest and the second author constructed a finite collection of Floer-non-trivial Lagrangian branes near the exceptional locus. The symplectic blowup can be obtained as a symplectic cut in the sense of Lerman [Ler95]

$$\text{Bl}_\epsilon(\mathbb{C}^n, 0) = \{z \in \mathbb{C}^n \mid |z| \geq \epsilon\} / \langle z \sim e^{i\theta}z, |z| = \epsilon \rangle.$$  

Denote by

$$L_\epsilon = (S^1)^n, \quad \phi_\epsilon : L_\epsilon \to \text{Bl}_\epsilon(\mathbb{C}^n, 0) = \{(z_1, \ldots, z_n) \mid |z_1| = \ldots = |z_n| = \epsilon\}$$

the inclusion in $\text{Bl}_\epsilon(\mathbb{C}^n, 0)$. By Charest-Woodward [CW] for certain collection of local systems

$$y_{(k)} \in \mathcal{R}(\phi), \quad k = 1, \ldots, n - 1$$

there exists a weakly bounding cochain $b_{(k)} \in MC(\phi_{(k)})$ so that with the resulting brane structure the Floer cohomology is isomorphic to the usual cohomology

$$HF^*(\phi_{(k)}, \phi_{(k)}) \cong H^*(L, \Lambda);$$

we add here the addendum Proposition 3.11 that as a ring is isomorphic to a non-degenerate Clifford algebra. Denote by

$$\mathcal{E} = \{\phi_{(1)}, \ldots, \phi_{(n-1)}\}$$

the resulting set of Lagrangian branes (all with the same image) in $\tilde{X}$, with the local systems corresponding to critical points of the Givental potential

$$W : \mathcal{R}(\phi) \to \Lambda, \quad (y_1, \ldots, y_n) \mapsto y_1 + \ldots + y_n + qy_1 \ldots y_n.$$  

Theorem 1.1 shows the Abouzaid criterion [Abo10], [Gan12] holds for the collection of Lagrangian branes so produced. We remark that more recent results of Ganatra [Gan] and Perutz-Sheridan [PS] offer alternative routes to showing that the Fukaya category is split-generated by this collection of branes. However, in this case the open-closed map is not hard to compute, at least to leading order, and the computation perhaps gives a feel for the mechanism.
We end with some miscellaneous remarks. The main result Theorem 1.1 is a categorical version of a result of A. Bayer [Bay04], who proves that blowup creates algebra summands in the quantum cohomology. In particular, if $QH^\bullet(X, \mathfrak{b})$ is semisimple for generic $\mathfrak{b}$ (with positive $q$-valuation), then so is the quantum cohomology $QH^\bullet(\tilde{X}, \tilde{\mathfrak{b}})$ of the reverse flip or blowup. We expect that these generalize to non-point blowups and reverse flips. In the case of non-point blowups, say the blowup $Bl_Z X$ of $X$ along a symplectic submanifold $Z$ the first issue that would have to be addressed is the possibility of thickening Lagrangians in $Z$ to Lagrangians in $L$; and then one should prove weak unobstructedness of these thickened Lagrangians.

2. Moduli spaces of treed disks

In this section we define the moduli spaces used in the definition of bulk-deformed Fukaya categories and regularize them using Cieliebak-Mohnke perturbations.

2.1. Trees. First we introduce terminology for trees. Given a tree $\Gamma$ the set of edges $\text{Edge}(\Gamma)$ is equipped with head and tail maps

$$h, t : \text{Edge}(\Gamma) \to \text{Vert}(\Gamma) \cup \{\infty\}.$$ 

The valence of any vertex $v \in \text{Vert}(\Gamma)$ is the number

$$|v| = \#\{e \in h^{-1}(v) \cup t^{-1}(v)\}$$

of edges meeting the vertex $v$. An edge $e \in \text{Edge}(\Gamma)$ is combinatorially finite if $\infty \notin \{h^{-1}(e), t^{-1}(e)\}$, semi-infinite if $\{h^{-1}(e), t^{-1}(e)\} = \{v, \infty\}$ for some $v \in \text{Vert}(\Gamma)$, and infinite if $h(e) = t(e) = \infty$. Denote

$$\text{Edge}_\text{finite}(\Gamma) \text{ resp. } \text{Edge}_\rightarrow(\Gamma) \subset \text{Edge}(\Gamma)$$

the set of finite resp. semi-infinite edges. We always assume trees are rooted which means that when $\text{Vert}(\Gamma) \neq \emptyset$ there is a distinguished vertex $v_{\text{root}} \in \text{Vert}(\Gamma)$ called the root and a distinguished semi-infinite edge $e_{\text{out}} \in \text{Edge}_\rightarrow(\Gamma)$ with $t(e_{\text{out}}) = v_{\text{root}}$ called the output. All edges are then oriented towards the output. Note that our trees do not necessarily have vertices: An infinite edge is a tree $\Gamma$ with $\text{Vert}(\Gamma) = \emptyset$ with one infinite edge. However we set $\text{Edge}_\rightarrow = \{e_{\text{in}}, e_{\text{out}}\}$, the incoming and the outgoing ends of the infinite edge.

A based tree is a tree equipped with a subtree $\Gamma$ corresponding to the disk vertices, which is equipped with a ribbon structure corresponding to the ordering of the edges around each vertex. A ribbon structure on a tree $\Gamma$ consists of a cyclic ordering $o_v : \{e \in \text{Edge}(\Gamma), e \ni v\} \to \{1, \ldots, |v|\}$ of the edges incident to each vertex $v \in \text{Vert}(\Gamma)$; a cyclic ordering is an equivalence class $[o_v]$ of orderings where two orderings $o_v, o'_v$ are equivalent if they are related by a cyclic permutation.

**Definition 2.1 (Subtree, based trees, and stability).** (a) A rooted subtree of a tree $\Gamma$, denoted by $\overline{\Gamma}$, consists of a subset of vertices $\text{Vert}(\Gamma)$ containing the
root $v_{\text{root}}$ of $\Gamma$, together with all finite edges connecting vertices in $\text{Vert}(\Gamma)$ and a subset of semi-infinite edges connected to vertices in $\text{Vert}(\Gamma)$ containing the output $e_{\text{out}}$ of $\Gamma$.

(b) A **based tree** is a tree $\Gamma$ together with a rooted subtree $\Gamma'$ with a ribbon structure on $\Gamma$.

(c) A based tree $\Gamma$ is **stable** if each $v \in \text{Vert}(\Gamma) \setminus \text{Vert}(\Gamma')$ has valence at least three and for each $v \in \text{Vert}(\Gamma')$ the number of edges in $\Gamma$ connected to $v$ plus twice of the number of edges not in $\Gamma$ connected to $v$ is at least three.

We denote by 
$$\text{Leaf}(\Gamma) := \text{Edge}_{\text{finite}}(\Gamma) \setminus \text{Edge}_{\text{finite}}(\Gamma')$$
the set of semi-infinite edges not in the base which are called *leaves*.

A moduli space of metric trees is obtained by allowing the finite edges on the base to acquire lengths.

**Definition 2.2.** Let $\Gamma$ be a based tree. A **metric** on $\Gamma$ is a function 
$$\ell : \text{Edge}_{\text{finite}}(\Gamma) \to [0, +\infty).$$
A **metric type** on $\Gamma$ is the associated decomposition 
$$\text{Edge}_{\text{finite}}(\Gamma) = \text{Edge}_{0}(\Gamma) \sqcup \text{Edge}_{+}(\Gamma)$$
corresponding to edges with zero or positive lengths. We denote the metric type by $\ell$ although it does not only depends on the metric $\ell$.

To compactify the set of gradient segments we allow the lengths of the edges to go to infinity and break. A **broken metric tree** is obtained from a finite collection of metric trees by gluing outputs with inputs as follows: given two metric trees $(\Gamma_1, \ell_1)$ and $(\Gamma_2, \ell_2)$ with specified leaves $e_1 \in \text{Edge}_{\to}(\Gamma_1)$ and $e_2 \in \text{Edge}_{\to}(\Gamma_2)$, let $\Gamma_1$ resp. $\Gamma_2$ denote the space obtained by adding a point $\infty_1$ resp. $\infty_2$ at the open end of $e_1$ resp. $e_2$. The space 
$$(13) \quad \Gamma := \Gamma_1 \cup_{\infty_1 \sim \infty_2} \Gamma_2$$
is a broken metric tree, the point $\infty_1 \sim \infty_2$ being called a *breaking*. To obtain a well-defined root for the glued tree we require that exactly one of $e_1$ and $e_2$ is the output. See Figure 1. In general, a broken metric tree $\Gamma$ are obtained from broken metric trees $\Gamma_1, \Gamma_2$ as in (13) in such a way that the resulting space $\Gamma$ is connected and has no non-contractible cycles, that is, $\pi_0(\Gamma)$ is a point and $\pi_1(\Gamma)$ is the trivial group. We think of the gluing points as breakings rather than vertices, so that there are no new vertices in the glued tree $\Gamma$.

---

2 When defining the open-closed map we will consider based trees whose root vertex is not in the base.

3 To define the Fukaya category we only need to consider metric on boundary edges. When we define the open-closed and closed open maps we need more general metric types.

4 Later when we consider treed annuli we will allow loops.
In order to obtain Fukaya algebras with strict units, we wish for our moduli spaces to obtain a certain forgetful map.

**Definition 2.3.** Consider an unbroken tree $\Gamma$.

(a) A weighting on $\Gamma$ is a map

$$w : \text{Edge}_\rightarrow(\Gamma) \rightarrow [0,1]$$

satisfying

$$w|_{\text{Leaf}(\Gamma)} \equiv 0$$

and

$$\prod_{e \in \text{Edge}_{\text{in}}(\Gamma)} w(e) = w(|e_{\text{out}}|)$$

(14)

The underlying decomposition

$$\text{Edge}_\rightarrow(\Gamma) = \text{Edge}^*(\Gamma) \sqcup \text{Edge}^*(\Gamma) \sqcup \text{Edge}^*(\Gamma) := w^{-1}(0) \sqcup w^{-1}((0,1)) \sqcup w^{-1}(1).$$

is called a weighting type. A tree $\Gamma$ with a weighting is called a weighted tree.

(b) If the output $e_{\text{out}}$ of $\Gamma$ is unweighted (forgettable or unforgettable) then an isomorphism $\psi : (\Gamma, w) \rightarrow (\Gamma', w')$ of weighted trees is an isomorphism of trees that preserves the weightings. If the output $e_{\text{out}}$ of $\Gamma$ is weighted (which implies $\Gamma$ has no interior incoming edge and all boundary incoming edges are weighted or forgettable), then an isomorphism $\psi : (\Gamma, w) \rightarrow (\Gamma', w')$ is an isomorphism of trees such that there is a positive number $\alpha$ such that

$$w(e) = w'(\psi(e))^{\alpha}, \forall e \in \text{Edge}_\rightarrow(\Gamma).$$

(c) If $\Gamma$ is broken, then a weighting on $\Gamma$ consists of weightings on all unbroken components that agree over breakings.

2.2. **Treed disks.** The domains of treed holomorphic disks are unions of disks, spheres, and line segments. A disk is a bordered Riemann surface biholomorphic to the complex unit disk

$$\mathbb{D} = \{ z \in \mathbb{C} \mid \|z\| \leq 1 \}.$$ 

The automorphism group of $\mathbb{D}$ is $\text{Aut}(\mathbb{D}) \cong PSL(2,\mathbb{R})$. A nodal disk with a single boundary node is a topological space $S$ obtained from a disjoint union of disks $S_1, S_2$ by identifying pairs of boundary points $w_{12} \in S_1, w_{21} \in S_2$ on the boundary of each component so that

(16) $$S = S_1 \cup_{w_{12} \sim w_{21}} S_2.$$
See Figure 2. The image of $w_{12}, w_{21}$ in the space $S$ is the nodal point. A nodal disk $S$ with multiple nodes $w_{ij}, i, j \in \{1, \ldots, k\}, i \neq j$ is obtained by repeating this construction (16) with $S_1, S_2$ nodal disks with fewer nodes, and $w_{12}, w_{21}$ distinct from the previous nodes. More generally we allow boundary and interior markings. For an integer $d \geq 0$ a nodal disk with $d + 1$ boundary markings is a nodal disk $S$ equipped with a finite ordered collection of points $\overline{z} = (x_0, \ldots, x_d)$ on the boundary $\partial S$, disjoint from the nodes, in counterclockwise cyclic order around the boundary $\partial S$. A $(d + 1)$-marked nodal disk $(S, \overline{z})$ is stable if each component $S_v$ has at least three special (nodal or marked) points, or equivalently the group $\text{Aut}(S, \overline{z})$ of automorphisms of $S$ leaving $\overline{z}$ pointwise fixed is trivial. The moduli space of $(d + 1)$-marked stable disks $[(S, \overline{z})]$ forms a compact cell complex, isomorphic as a cell complex to the associahedron from Stasheff [Sta63, Sta70].

More complicated configurations involve spherical components. A sphere is a complex surface biholomorphic to the projective line $S^2 \cong \mathbb{P}^1$. We allow sphere components $\mathbb{P}^1 \cong S_v \subset S$ and interior markings $z_1, \ldots, z_d \in \text{int}(S)$ in the definition of marked nodal disks $S$. A nodal disk $S$ with a single interior node $w \in S$ is defined similarly to that of a boundary node by using the construction (16), except in this case $S$ is obtained by gluing together a nodal disk $S_1$ with a marked sphere $S_2$ with $w_{12}, w_{21}$ points in the interior $\text{int}(S)$.

General treed disks are defined as in Oh [Oh93], Cornea-Lalonde [CL06], Biran-Cornea [BC07, BC09], and Seidel [Sei11].

**Definition 2.4** (Treed disks, combinatorial types, moduli spaces). (a) A combinatorial type for treed disks is a based tree $\Gamma$ together with a metric type $\ell$ (see Definition 2.2) and a weighting type $w$ (see Definition 2.3).

(b) A treed disk $C$ of type $(\Gamma, \ell, w)$ consists of the surface part $S = (S_v, \overline{z}_v, \overline{z}_v)_{v \in \text{Vert}(\Gamma)}$ (where $\overline{z}_v$, resp. $\overline{z}_v$ denotes the ordered set of boundary resp. interior markings), a tree part $T = (T_e)_{e \in \text{Edge}(\Gamma)}$, (where $T_e$ is a finite interval of a certain length $\ell(e)$ if $e$ is combinatorially finite, a semi-infinite interval $[0, +\infty)$ or $(-\infty, 0]$ if $e$ is semi-infinite, so that $(\Gamma, \ell)$ becomes a metric tree whose underlying metric type agrees with $\ell^5$),

\[^{5}\text{If } e \text{ is an infinite edge, then regard } T_e \text{ as the real line } (-\infty, +\infty) \text{ which is also the union of two rays labelled by the input and the output.}\]
a weighting $w : \text{Edge}_+(\Gamma) \to [0, 1]$ whose underlying weighting type agrees with $w$, and \textit{nodal points}

$$z_{e,+} \in S_{h(e)}, \quad z_{e,-} \in S_{l(e)}, \quad \forall e \in \text{Edge}(\Gamma).$$

These data must satisfy the following conditions: for each vertex $v \in \text{Vert}(\Gamma)$, the set of \textit{special points}, i.e., the collection of boundary and interior markings and nodal points are distinct. See Figure 3 for a typical configuration of a treed disk.

(c) An \textit{isomorphism} of treed disks $\phi$ from $C = S \cup T$ to $C' = S' \cup T'$ consists of an isomorphism $\psi : (\Gamma, \ell, w) \to (\Gamma', \ell', w')$ of underlying weighted metric trees, a collection of isomorphisms $\phi_v : S_v \to S'_{\psi(v)}$ of disks or spheres preserving the markings and special points, and a collection of isomorphisms $\phi_e : T_e \to T'_{\psi(e)}$ of intervals.

(d) A treed disk is stable if its underlying combinatorial type is stable (see Definition 2.1).\footnote{As in the case of spheres, treed disk is stable if and only if its automorphism group is trivial.}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{treed_disk.png}
\caption{A treed disk with three disk components and one sphere component}
\end{figure}

\textbf{Remark 2.5.} There is a natural partial order among all stable combinatorial types of treed disks, denoted by $\Gamma' \preceq \Gamma$. Instead of giving the full definition, we only recall the typical situations. These typical situations include the case of bubbling off holomorphic spheres, bubbling off holomorphic disks, and breaking of gradient lines, in which $\Gamma'$ is obtained from $\Gamma$ by a change of the underlying tree. Moreover, when the length of an edge of $\Gamma$ changes from positive to zero, one obtains a different type $\Gamma' \prec \Gamma$ by changing the metric type; when the weighting of one or more semi-infinite edges of $\Gamma$ changes to zero or one, one also obtains a different type $\Gamma' \prec \Gamma$ by changing the weighting type accordingly. In general $\Gamma' \preceq \Gamma$ if $\Gamma'$ can be obtained from $\Gamma$ by finitely many such changes.

The moduli spaces of stable weighted treed disks are naturally cell complexes. Suppose $\Gamma$ is a stable combinatorial type with $k$ boundary inputs and $l$ interior inputs. Let $\mathcal{M}_\Gamma$ denote the set of all isomorphism classes of treed disks of type $\Gamma$,.
which has a natural topology. Then $\mathcal{M}_\Gamma$ is a manifold of dimension

$$\dim(\mathcal{M}_\Gamma) = k + 2l + \#\text{Edge}^*(\Gamma) - \#\text{Edge}_0(\Gamma) - 2\#\text{Edge}_{\text{interior}}(\Gamma)$$

$$+ \begin{cases} -2 & \text{if } e_{\text{out}} \notin \text{Edge}^*(\Gamma), \\ -4 & \text{if } e_{\text{out}} \in \text{Edge}^*(\Gamma). \end{cases}$$

Denote

$$\overline{\mathcal{M}}_\Gamma = \bigsqcup_{\substack{\Gamma' \leq \Gamma'\text{ stable}}} \mathcal{M}_{\Gamma'}.$$

As in the definition of Gromov convergence of pseudoholomorphic curves, there is a natural way to endow $\overline{\mathcal{M}}_\Gamma$ a compact Hausdorff topology that agrees on the manifold topology on each stratum $\mathcal{M}_{\Gamma'}$, so that $\overline{\mathcal{M}}_\Gamma$ is a cell complex with $\mathcal{M}_\Gamma$ being the top cell.

**Remark 2.6.** The moduli spaces of weighted treed disks are related to unweighted moduli spaces by taking products with intervals: If $\Gamma$ has at least one vertex and $\Gamma'$ denotes the combinatorial type of $\Gamma$ obtained by setting the weights $w(e)$ to zero and the output $e_{\text{out}}$ of $\Gamma$ is unweighted then

$$\mathcal{M}_\Gamma \cong \mathcal{M}_{\Gamma'} \times (0, 1)^{\left|\text{Edge}^*(\Gamma)\right|}.$$

If the outgoing edge $e_{\text{out}}$ is weighted then

$$\mathcal{M}_\Gamma \cong \mathcal{M}_{\Gamma'} \times (0, 1)^{\left|\text{Edge}^*(\Gamma)\right|-2}$$

because of the way we define isomorphism of weighted types (see Definition 2.3).

Figure 4 illustrates a one-dimensional moduli space with weighted output and its boundary strata.

![Figure 4](image.png)

**Figure 4.** A one-dimensional moduli space of weighted treed disks with all three semi-infinite edges being weighted.

In general moduli spaces of stable curves only admit universal curves in an orbifold sense. In the setting here orbifold singularities are absent and the moduli spaces of stable treed disks admit honest universal curves. For any combinatorial type $\Gamma$ let $\mathcal{U}_\Gamma$ denote the universal treed disk (or called the universal curve) consisting of isomorphism classes of pairs $(C, z)$ where $C$ is a treed disk of type $\Gamma$ and $z$ is a point
in $C$, possibly on a disk component $S_v \cong \{ |z| \leq 1 \}$, a sphere component $S_v \cong \mathbb{P}^1$, or one of the edges $e$ of the tree part $T \subset C$ (the infinite of semi-infinite edges are allowed). The map

$$\overline{U}_\Gamma \to \overline{M}_\Gamma, \ [C, z] \to [C]$$

is the universal projection. Because of the stability condition, there is a natural bijection

$$\overline{U}_\Gamma = \bigsqcup_{[C] \in \overline{M}_\Gamma} C.$$

In case $\Gamma$ has no vertices we define $\overline{U}_\Gamma$ to be the real line, considered as a fiber bundle over the point $\overline{M}_\Gamma$.

---

**Figure 5.** Treed disks with interior leaves

**Figure 6.** Boundary of a treed disk with $d = 2$ incoming edges

We introduce notation for particular subsets of the universal curves. First, for each vertex $v \in \text{Vert}(\Gamma)$, let

$$\overline{U}_{\Gamma, v} \subset \overline{U}_\Gamma$$
denote the closed subset corresponding to points on the surface component $S_v$. For each edge $e \in \text{Edge}(\Gamma)$, let
\[ \overline{U}_{\Gamma,e} \subset \overline{U}_{\Gamma} \]
the closed subset corresponding to points on the tree component $T_e$. Denote
\[ \overline{S}_\Gamma := \bigcup_{v \in \text{Vert}(\Gamma)} \overline{U}_{\Gamma,v} \]
and
\[ \overline{T}_\Gamma := \bigcup_{e \in \text{Edge}(\Gamma)} \overline{U}_{\Gamma,e}. \]
Moreover, for each subtree $\Pi \subset \Gamma$ (not necessarily containing the root), denote by
\[ \overline{U}_{\Gamma,\Pi} \subset \overline{U}_{\Gamma} \]
the set of points on components corresponding to vertices and edges of $\Pi$. There is a contraction map contracting edges not in $\Gamma$
\[ \overline{U}_{\Gamma,\Pi} \to \overline{U}_{\Pi}. \]
In particular, for the base $\Gamma$, one has
\[ \overline{U}_{\Gamma,\Gamma} \subset \overline{U}_{\Gamma}. \]
Lastly, for $\Pi \preceq \Gamma$, one has a boundary stratum
\[ \overline{U}_{\Pi} \subset \overline{U}_{\Gamma}. \]
The boundary is divided up into parts between the boundary inputs. Given a treed disk $C$ of type $\Gamma$ define a one-manifold $\partial C$ by gluing together the boundary of each disk $S_v, v \in \text{Vert}(\Gamma)$ minus the points $T_e \cap S, e \in \text{Edge}(\Gamma)$ where edges attach with two copies of each edge $T_e$ as in Figure 6. Denote by $(\partial C)_i$ the component of $\partial C$ between the $i$-th and $i + 1$-st leaves, in cyclic order. Similarly we can define the $i$-th boundary part of the universal curve
\[ \partial_i \overline{U}_{\Gamma} \subset \overline{U}_{\Gamma}. \]

### 2.3. Geometric input.
In this subsection we specify the geometric objects inside the symplectic manifold relevant to our construction. Let $(X, \omega)$ be a compact symplectic manifold. Fix a finite collection of Lagrangian immersions
\[ \mathcal{L} := (\phi : L_\phi \to X)_{\phi \in \mathcal{L}}. \]
We assume that each member of $\mathcal{L}$ is self-transverse and each pair of immersions intersect cleanly. Denote
\[ \mathcal{L} := \bigcup_{\phi \in \mathcal{L}} L_\phi, \quad |\mathcal{L}| := \bigcup_{\phi \in \mathcal{L}} \phi(L_\phi) \subset X. \]
For each pair $(\phi_-, \phi_+) \in \mathcal{L}^2$, the preimage of the diagonal $\Delta \subset X \times X$, denoted by
\[ L_{\phi_- \phi_+} := (\phi_- \times \phi_+)^{-1}(\Delta) \]
is a smooth manifold (of varying dimensions).
We impose the following constraints at interior leaves. Consider a general bulk deformation given as a combination of pseudocycles
\[ b = \sum_{i=1}^{N} c_i b_i, \quad c_i \in \Lambda. \]

Here each component \( b_i \) contains the datum of a smooth map
\[ j_i : Z_i \to X \]
(17)
together with a map
\[ \iota_i : W_i \to X \]
covering the boundary of \( j_i(Z_i) \) such that \( W_i \) has codimension at least two. Define the support of the bulk deformation
\[ |b| := \bigcup_{i=1}^{N} j_i(Z_i) \cup \bigcup_{i=1}^{N} \iota_i(W_i). \]

In practice, our bulk deformations will be multiples of the point \( p \in X \) where the blowup will be performed. We assume the following conditions on the bulk deformation.

**Definition 2.7.** The bulk deformation is admissible if \( p \) is not in \( |\mathcal{L}| \cup |b| \) and each \( Z_i \) has an even and positive codimension.

We will adopt the stabilizing divisor technique introduced by Cieliebak-Mohnke [CM07]. For this we need to recall the rationality assumption.

**Definition 2.8.** A symplectic manifold \( X \) is rational if \([\omega] \in H^2(X, \mathbb{R}) \) lies in the image of \( H^2(X, \mathbb{Q}) \), or equivalently, if there exists a line bundle with connection \( \hat{X} \to X \) whose curvature (up to a factor of \( i/2\pi \)) is \( k\omega \) for some \( k \in \mathbb{N} \).

**Definition 2.9** (Donaldson hypersurface). Given a rational symplectic manifold \((X, \omega)\) a Donaldson hypersurface is a compact codimension two symplectic submanifold \( D \subset X \) whose Poincaré dual is a multiple of \( k[\omega] \). The positive integer \( k \) is called the degree of the Donaldson hypersurface.

**Definition 2.10** (Rational Lagrangian). Let \((X, \omega)\) be a compact rational symplectic manifold. The collection of Lagrangian immersions \( \mathcal{L} \) is called rational if there exists a line bundle with connection \( \hat{X} \to X \) whose curvature is a positive integral multiple \( k\omega \) of \( \omega \) and there exists a nonzero section
\[ s \in \Gamma(\hat{X}|_{|\mathcal{L}|}) \]
whose pullback via each \( \phi : L \to X \) is flat with respect to the connection on \( \hat{X} \). The collection \( \mathcal{L} \) is called exact in an open subset \( U \subset X \) if
\begin{enumerate}
  \item \( \phi(L_\phi) \subset U \) for all \( \phi \in \mathcal{L} \);
  \item there is a 1-form \( \theta \in \Omega^1(U) \) such that \( \omega|_U = d\theta \);
\end{enumerate}
(c) there exists a function
\[ f : |\mathcal{L}| \to \mathbb{R} \]
such that for each \( \phi \in \mathcal{L} \) there holds
\[ \phi^* \theta = d\phi^* f. \]

From now on we assume that \((X, \omega)\) is rational and \(\mathcal{L}\) is strongly rational.

**Lemma 2.11.** (c.f. Charest-Woodward [CW17, Section 3.1], [CM07, Lemma 8.7]) Choose a compatible almost complex structure \( J \in J_{\text{comp}}(X) \) such that all Lagrangian immersions in the collection \( \mathcal{L} \) are totally real. Then for \( l \in \mathbb{N} \) sufficiently large there exists a degree \( l \) Donaldson hypersurface \( D(\mathcal{L}) \subset X \) disjoint from \( |\mathcal{L}| \) with the properties:

1. \( \mathcal{L} \) is exact in the complement \( X - D(\mathcal{L}) \);
2. \( D(\mathcal{L}) \) is \( \theta \)-approximately \( J \)-holomorphic and
3. there is a tamed almost complex structure \( J_0 \in J_{\text{tame}}(X, \omega) \) making \( D(\mathcal{L}) \) almost complex such that all nonconstant \( J_0 \)-holomorphic spheres in \( X \) intersect \( D(\mathcal{L}) \) at finite but at least three points.
4. \( D(\mathcal{L}) \) is disjoint from \( p \).

**Proof.** The construction is an extension of the original construction of Donaldson [Don96] (see also [Aur97] and [AGM01]). Given a sufficiently generic approximately-holomorphic section
\[ s : X \to \hat{X}^l \]
of some tensor power \( \hat{X}^l \) of a line bundle \( \hat{X} \to X \) whose curvature is the symplectic form (up to a factor of \( 2\pi/i \)) and whose value on \( |\mathcal{L}| \) is close to the given flat section on \( \phi(L) \) for all \( \phi \in \mathcal{L} \), one obtains a symplectic hypersurface as the zero-set:
\[ D(\mathcal{L}) = s^{-1}(0). \]

By Cieliebak-Mohnke [CM07, Corollary 8.16], for sufficiently generic tamed almost complex structures, each non-constant pseudoholomorphic sphere \( u : \mathbb{P}^1 \to D(\mathcal{L}) \) intersects \( D(\mathcal{L}) \) in at least three points:
\[ \#u^{-1}(D(\mathcal{L})) \geq 3. \]

On the other hand, since \( \mathcal{L} \) is exact in the complement of \( D(\mathcal{L}) \), each nonconstant pseudoholomorphic disk \( u : \mathbb{D} \to X \) with boundary in \( \mathcal{L} \) intersects \( D(\mathcal{L}) \) in at least one interior point \( z \in C, u(z) \in D(\mathcal{L}) \). Transversality to \( p \) and to the components of the bulk deformation \( b \) follows from the stabilization property in [CM07, Theorem 8.1]. Then [CM07, Theorem 8.1] and the modification in [CW17, Theorem 3.6] make the Lagrangians exact in the complement. \( \square \)
2.4. Perturbations. We consider domain-dependent perturbation data defined on the universal curves. We first define an important condition called the locality, which should be satisfied by the perturbation data. This condition plays an important role in Cieliebak–Mohnke’s approach [CM07]. For each spherical vertex \( v \in \text{Vert}_{\text{sphere}}(\Gamma) \), let

\[ \Gamma(v) = \bigcup_{e \ni v} e \]

denote the subtree of \( \Gamma \) consisting of the vertex \( v \) and all edges \( e \) of \( \Gamma \) meeting \( v \). Let \( \Gamma \) denote the combinatorial type obtained from \( \Gamma \) by collapsing all spherical components. Let

\[
\pi : \pi_1 \times \pi_2 : \mathcal{U}_\Gamma \to \mathcal{M}_\Gamma \times \mathcal{U}_{\Gamma(v)}
\]

be the product of maps where \( \pi_1 \) is given by projection followed by the forgetful morphism and \( \pi_2 \) is the contraction \( C \to S_v \).

**Definition 2.12 (Locality).** Let \( Z \) be a set. A map \( f : \mathcal{U}_\Gamma \to Z \) is called local if the following conditions are satisfied.

(a) For each \( v \in \text{Vert}_{\text{sphere}}(\Gamma) \), the restriction of \( f \) to \( \mathcal{U}_{\Gamma,v} \) is equal to the pull-back of a function \( f_v : \mathcal{M}_\Gamma \times \mathcal{U}_{\Gamma(v)} \to Z \).

(b) The restriction of \( f \) to \( \mathcal{U}_{\Gamma,\Gamma} \) is equal to the pull-back of a function \( f_\Gamma : \mathcal{U}_\Gamma \to Z \).

(c) The restriction of \( f \) to any boundary stratum \( \mathcal{U}_\Pi \) for \( \Pi \prec \Gamma \) is a local map.

We would like to specify open sets where the perturbations should vanish.

**Lemma 2.13.** For all stable combinatorial types \( \Gamma \), there exist three collection of open subsets

\[
S_\Gamma,J \subset S_\Gamma \subset \mathcal{U}_\Gamma, \quad S_\Gamma,H \subset S_\Gamma \subset \mathcal{U}_\Gamma, \quad T_\Gamma,F \subset T_\Gamma \subset \mathcal{U}_\Gamma
\]

satisfying the following properties.

(a) The characteristic functions of \( S_\Gamma,J, S_\Gamma,H, T_\Gamma,F \), viewed as maps from \( \mathcal{U}_\Gamma \) to \( \{0,1\} \), are local maps.

(b) The open set \( S_{\Gamma,J} \) intersected with any fiber \( C = S \cup T \subset \mathcal{U}_\Gamma \) is a neighborhood of all special points on the surface part so that for all \( v \in \text{Vert}(\Gamma) \), the complement \( S_{\Gamma,J} \setminus U_{\Gamma,J} \) has non-empty intersection with \( S_v \).

(c) The open set \( S_{\Gamma,H} \) is a neighborhood of all nodes \( w_e, e \in \text{Edge}(\Gamma) \) and the complement \( S_\Gamma \setminus S_{\Gamma,H} \) is contained in the base \( S_{\Gamma,\Gamma} \). Moreover, for each disk component \( v \in \text{Vert}(\Gamma) \), \( S_v \) has non-empty intersection with the complement of \( S_{\Gamma,H} \) and the intersection

\[ S_v \cap (S_\Gamma \setminus S_{\Gamma,H}) \]

is contained in the union of all strip-like ends.

(d) The open set \( T_{\Gamma,F} \) is a neighborhood of the locus corresponding to infinities of semi-infinite edges in all degenerations \( \Pi \prec \Gamma \).
(e) If $\Gamma$ is separated by a breaking into two subtrees $\Gamma_1$ and $\Gamma_2$, then $\mathfrak{S}_{\Gamma,J}$ resp. $\mathfrak{T}_{\Gamma,F}$ is the product

$$ \mathfrak{S}_{\Gamma_1,J_1} \boxtimes \mathfrak{S}_{\Gamma_2,J_2} \text{ resp. } \mathfrak{S}_{\Gamma_1,H_1} \boxtimes \mathfrak{S}_{\Gamma_2,H_2} \text{ resp. } \mathfrak{T}_{\Gamma_1,F_1} \boxtimes \mathfrak{T}_{\Gamma_2,F_2} $$

where

$$ \mathfrak{S}_{\Gamma_1,J_1} \boxtimes \mathfrak{S}_{\Gamma_2,J_2} = \pi_1^{-1}(\mathfrak{S}_{\Gamma_1,J_1}) \times \pi_2^{-1}(\mathfrak{S}_{\Gamma_2,J_2}) $$

etc.

(f) There exist positive numbers $a_{\Gamma} > 0$ for all stable types $\Gamma$ that only depends on the base $\Gamma$ such that

(i) For each $C = S \cup T \subset U_{\Gamma}$, the area of $S \setminus \mathfrak{S}_{\Gamma,H}$ (with respect to flat metric over the strip-like end) is less than $a_{\Gamma}$.

(ii) If $\Gamma$ is separated by a breaking into $\Gamma_1, \Gamma_2$, then $a_{\Gamma} = a_{\Gamma_1} + a_{\Gamma_2}$.

The proof is left to the reader. We need to specify certain Banach space norms on perturbation data. Notice that after taking away the open sets $\mathfrak{S}_{\Gamma,J}$, $\mathfrak{S}_{\Gamma,H}$, and $\mathfrak{T}_{\Gamma,F}$, the surface part and the tree part of the universal curve $U_{\Gamma}$,

$$ \mathfrak{S}_{\Gamma} \setminus \mathfrak{S}_{\Gamma,J}, \mathfrak{S}_{\Gamma} \setminus \mathfrak{S}_{\Gamma,H}, \mathfrak{T}_{\Gamma} \setminus \mathfrak{T}_{\Gamma,F} $$

are smooth manifolds. To measure the norms of smooth functions we choose Riemannian metrics on these complements in a way that the metrics are local functions on the universal curve and respect degeneration of curves. We omit the details of such conditions. Then choose a sequence of positive numbers $(\epsilon_i)_{i=1}^{\infty}$ converging to zero such that Floer’s $C^\epsilon$-norm

$$ \|f\|_\epsilon := \sum_{i=0}^{\infty} \epsilon_i \|\nabla^i f\|_{C^0} $$

is complete and in each dimension the space of smooth functions with finite $C^\epsilon$-norms contains bumped functions of arbitrary small supports (see [Flo88]).

**Remark 2.14.** The use of Hamiltonian perturbation is to deal with the possible loss of transversality caused by trivial polygons mapped into intersections of Lagrangians. We first specify the following space of Hamiltonian functions on the target manifold. Recall that $|L|$ is disjoint from $D(\mathcal{L})$. Choose a neighborhood $U(D)$ of $D(\mathcal{L})$ that is still disjoint from $|L|$. For $\lambda > 0$ define

$$ \text{Ham}^\lambda(X, U(D)) = \{ h \in C^\infty(X) \mid h|_{U(D)} = 0, \|h\|_{C^0(X)} < \lambda \}.$$

For $\lambda$ sufficiently small, one can guarantee that for any one-parameter family of Hamiltonian perturbations $H_t \in \text{Ham}^\lambda(X, U(D))$ and the corresponding time-1 map $\psi$ there holds

$$ \psi(|L|) \cap \overline{U(D)} = \emptyset. $$

With Hamiltonian perturbations the energy of perturbed holomorphic maps are no longer topological. We need an a priori energy bound for these objects. Suppose we have a domain-dependent almost complex structure

$$ J_\Gamma : U_\Gamma \to \mathcal{J}_{\text{tame}}(X, \omega) $$

where

$$ J_\Gamma = J_{\text{tame}} $$

and

$$ J_{\text{tame}} = \mathcal{J}_{\text{tame}}(X, \omega) $$
and a domain-dependent Hamiltonian perturbation
\[ H_\Gamma \in \Omega^1(S_\Gamma \setminus S_{\Gamma,H}, \text{Ham}^\lambda(X,U(D))). \]

Consider a fibre \( C = S \cup T \subset U_\Gamma \) and a \((J_\Gamma, H_\Gamma)\)-holomorphic map \( u : S \to X \) with boundary lying in certain Lagrangians in the collection \( \mathcal{L} \). The perturbed energy-area relation \([\text{MS04}, 8.1.9]\) reads
\[ E(u) = \int_S u^* \omega + \int_S F_{H_\Gamma}(u) \]
where \( F_{H_\Gamma} \in \Omega^2(S, \mathcal{H}_X) \) is the curvature of the Hamiltonian perturbation. Then
\[ E(u) \leq \int_S u^* \omega + \lambda a_\Gamma, \]
where \( a_\Gamma \) is from 2.13 and \( \lambda \) is from (19). Since the first term is proportional to the expected intersection number between \( u \) and the stabilizing divisor \( D \) of degree \( k \), one can define
\[ E(\Gamma) = \frac{\#\text{Edge}(\Gamma)}{k} + \lambda a_\Gamma \]
which is an \textit{a priori} combinatorial bound of the energy.

We would like to specify suitable spaces of almost complex structures that do not allow holomorphic spheres in the Donaldson hypersurface. Recall that we have fixed a tamed almost complex structure \( J_0 \in J_{\text{tame}}(X,\omega) \) (which we will call the \textit{base} almost complex structure).

\textbf{Lemma 2.15.} \([\text{CM07}, \text{Corollary 8.16}]\) \textit{For any} \( E > 0 \), \textit{there exists an open neighborhood} \( J_{\text{tame}}(X,\omega) \subset J_{\text{tame}}(X,\omega) \) \textit{satisfying: for every} \( J \in J_{\text{tame}}(X,\omega) \), \textit{all non-constant} \( J \)-holomorphic spheres \textit{with energy at most} \( E \) \textit{intersect} \( D(\mathcal{L}) \) \textit{at finite but at least three points}.

We now perturb the given sphere-free almost complex structure, staying with the open neighborhood in Lemma 2.15 to obtain regularity for treed holomorphic disks, without allowing sphere bubbling or disk bubbling.

\textbf{Definition 2.16.} A \textit{perturbation datum} for a stable combinatorial type \( \Gamma \) of treed disks is a collection \( P_\Gamma \) consisting of
\begin{enumerate}
\item[(a)] A \textit{domain-dependent almost complex structure}
\[ J_\Gamma : \mathfrak{S}_\Gamma \to J_{\text{tame}}^E(\Gamma), \]
that is equal to the base almost complex structure \( J_0 \) over the open set \( \mathfrak{S}_{\Gamma,J} \) and in a fixed neighborhood of \( D(\mathcal{L}) \). Here \( E(\Gamma) \) is the energy bound defined by (20).
\item[(b)] A \textit{domain-dependent Hamiltonian perturbation}
\[ H_\Gamma \in \Omega^1(\mathfrak{S}_\Gamma, \text{Ham}^\lambda(X,U(D))). \]
\end{enumerate}
on $\mathcal{S}_\Gamma$ with values in the space $\text{Ham}^\lambda(X, U(D))$ of smooth functions that vanishes over the open set $\mathcal{S}_{\Gamma,H}$.

(c) For each edge $e \in \text{Edge}(\Gamma)$ a collection **domain-dependent smooth functions**

$$F_e : \mathcal{T}_\Gamma \times \left( X \sqcup \bigsqcup_{(\phi_-, \phi_+)} L_{\phi_-, \phi_+} \right) \to \mathbb{R}$$

on both the target manifold $X$ and all possible Lagrangian intersections.

(d) A **domain-dependent perturbation of the evaluation map** which is a collection of continuous maps for the interior inputs

$$E_{\Gamma,e} : \mathcal{M}_\Gamma \to \text{Diff}(X) \ \forall e \in \text{Leaf}(\Gamma)$$

that are smooth in the interior $\mathcal{M}_\Gamma$ (with respect to the manifold structure of $\mathcal{M}_\Gamma$). Each $E_{\Gamma,e}$ can be viewed as a map from the universal curve by pullback via $U_\Gamma \to \mathcal{M}_\Gamma$.

Moreover, the triple $P_{\Gamma} = (J_{\Gamma}, H_{\Gamma}, F_{\Gamma}, E_{\Gamma})$ can be viewed as a map from the universal curve $U_\Gamma$ to a certain set. We require that this map is a local map (see Definition 2.12).

There is a base perturbation for each stable $\Gamma$ in which $J_{\Gamma}$ is the base almost complex structure $J_0$ specified by Lemma 2.11, $H_{\Gamma} = 0$, and $F_{\Gamma} = 0$, and $E_{\Gamma} = \text{Id}_X$. We will only consider perturbations in a small $C^\epsilon$-neighborhood of this base perturbation. The tangent space of $\mathcal{J}_{\text{tame}}(X, \omega)$ at $J_0$ is

$$T_{J_0} \mathcal{J}_{\text{tame}}(X, \omega) = \{ \xi \in \text{End}(TX) \mid J_0 \xi + \xi J_0 = 0 \}.$$  

For $\delta > 0$ sufficiently small we identify the $\delta$-neighborhood of $J_0$ in $\mathcal{J}_{\text{tame}}(X, \omega)$ with respect to the $C^0$-norm with the $\delta$-ball of the tangent space $T_{J_0} \mathcal{J}_{\text{tame}}(X, \omega)$, denoted by $\mathcal{J}_{\text{tame}}(X, \omega)$. Then a domain-dependent almost complex structure $J_{\Gamma} : \mathcal{S}_\Gamma \to \mathcal{J}_{\text{tame}}(X, \omega)$ that is $C^0$-close to the base $J_0$ can be viewed as a vector in the linear space $C^\infty(\mathcal{S}_\Gamma \setminus \mathcal{S}_{\Gamma,H}, T_{J_0} \mathcal{J}_{\text{tame}}(X, \omega))$ so one can measure its norms. Similarly, a domain-dependent diffeomorphism $E_{\Gamma,e} : X \to X$ that is $C^0$-close to the identity can be identified with a $C^0$-small vector field on $X$, denoted by $E_{\Gamma,e} - \text{Id}_X$. On the other hand, the terms $H_{\Gamma}$ and $F_{\Gamma}$ are naturally in a vector space. Then for each stable $\Gamma$, define

$$P_{\Gamma} := \left\{ P_{\Gamma} = (J_{\Gamma}, H_{\Gamma}, F_{\Gamma}, E_{\Gamma}) \mid \|J_{\Gamma} - J_0\|_{C^\epsilon} + \|H_{\Gamma}\|_{C^\epsilon} + \|F_{\Gamma}\|_{C^\epsilon} + \|E_{\Gamma} - \text{Id}_X\|_{C^\epsilon} < \infty \right\}.$$  

This set with the $C^\epsilon$-norm is a separable Banach manifold (in fact an open set of a separable Banach space). $C^\epsilon$-spaces (for two different sequences $\{\epsilon_i\}$).

Once a perturbation datum $P_{\Gamma}$ for a stable domain type is fixed we obtain perturbations for not-necessarily-stable types as follows. Let $C$ be a holomorphic treed disk of type $\Gamma$ not necessarily stable, and $f(C)$ its stabilization, naturally identified with a fiber of the universal curve $U_f(\Gamma)$ for the type $f(\Gamma)$. Via the stabilization map $C \to f(C)$ the perturbation data $P_{f(\Gamma)}$ pulls back to perturbation data $P_{\Gamma}$ for $\Gamma$. 

2.5. Holomorphic treed disks. Holomorphic treed disks are combinations of holomorphic disks and gradient flow segments. Let \((X, \omega)\) be a compact symplectic manifold. Let \(\mathcal{L}\) be a finite collection of self-transverse Lagrangian branes \(\phi : L \rightarrow X\) that are pair-wise cleanly intersecting. For each pair \((\phi_-, \phi_+), (\phi_- = \phi_+))\) let

\[ F_{\phi_-, \phi_+} : (\phi_- \times \phi_+)^{-1}(\Delta) \rightarrow \mathbb{R}, \]

be a Morse function on the clean intersection. Its critical points will be asymptotic constrains for gradient rays. In order to obtain strict units, we expand the set of critical points as follows. For each pair \((\phi_-, \phi_+)), define

\[
I_{\phi_-, \phi_+} = \{ \text{crit}(F_{\phi_-, \phi_+}), \phi_- \neq \phi_+, \text{crit}(F_{\phi_-, \phi_+}) \cup I_{hu}^{\phi_-}, \phi_- = \phi_+ = \phi. \}
\]

Namely, for each connected component \(c\) of \(L_{\phi}\) (the domain of the immersion \(\phi : L_{\phi} \rightarrow X\)) we add two additional constraints \(1_{\phi,c}^\nu, 1_{\phi,c}^\psi\).

Interior labelling data provide constraints of maps at interior markings. Recall that one has specified components of the bulk deformation \(b\) which are smooth maps \(j_i : Z_i \rightarrow X\) together with their “boundaries” \(\iota_i : W_i \rightarrow X\). On the other hand, one has the stabilizing divisor \(D(\mathcal{L}) \subset X\) which intersects each \(\iota_i\) transversely. Denote

\[
I_X := \{X\} \cup \{(D(\mathcal{L}), m) | m \geq 1\} \cup \{j_i(Z_i) | 1 \leq i \leq N\} \cup \{\iota_i(W_i) | 1 \leq i \leq N\} \cup \{D(\mathcal{L}) \cap j_i(Z_i) | 1 \leq i \leq N\} \cup \{j_i(Z_i) \cap j_{i'}(Z_{i'}) | 1 \leq i, i' \leq N\}
\]

which will be used to label all possibly interior constraints. The notation \((D(\mathcal{L}), m)\) will indicate tangency order of \(m\) to the stabilizing divisor.

**Definition 2.17** (Map types). Given a combinatorial type \(\Gamma\) of treed disks, a **map type** consists of

(a) A boundary constraint datum given by a sequence of Lagrangian branes

\[ \hat{\phi} = (\phi_0, \phi_1, \ldots, \phi_d) \]

labelling the boundary components of treed disks.

(b) A corner constraint datum given by a sequence of elements

\[ \mathcal{E} := (x_0 \in I_{\phi_0, \phi_d}, x_1 \in I_{\phi_0, \phi_1}, \ldots, x_d \in I_{\phi_d, \phi_d}) \]

satisfying the following requirement regarding the weighting types. The \(i\)-th leaf \(e_i\) is forgettable resp. weighted if and only if \(x_i = 1_{\phi,c}^\nu\) resp. \(x_i = 1_{\phi,c}^\psi\) for certain \(\phi \in \mathcal{L}\) and a component \(c \in \pi_0(L_{\phi, \phi})\). For each boundary edge \(e \in \text{Edge}(\Gamma)\) there is then an unordered pair \((\phi_{e,-}, \phi_{e,+})\) of branes given by \(\mathcal{E}\). Abbreviate

\[ L_e := L_{\phi_{e,-}, \phi_{e,+}} = (\phi_{e,-}, \phi_{e,+})^{-1}(\Delta). \]
A homology datum which is a map
\[ \beta : \text{Vert}(\Gamma) \to H_2(X, \phi) := H_2(X, \bigcup_{\phi_i \in \phi} (\phi_i(L_i)); \mathbb{Z}). \]

(d) An interior constraint datum which is a map
\[ \diamond : \text{Edge}_{\text{interior}, \text{input}}(\Gamma) \to I_X. \]

A map type is denoted by \( \Gamma = (\Gamma, x, \beta, \diamond). \)

Perturbed treed holomorphic disks are defined by allowing the almost complex structure, Hamiltonian perturbation, and Morse function to vary in the domain. Let \( \Gamma \) be a combinatorial type of treed disks (not necessarily stable). Let \( \Gamma^{st} \) be the stabilization of \( \Gamma \) (which is not empty). Let \( C \) be a treed disk of type \( \Gamma \) and \( C^{st} \) its stabilization which is of type \( \Gamma^{st} \). Suppose we are given a perturbation datum \( P_{\Gamma^{st}} \) for type \( \Gamma^{st} \). Then on each surface part \( S_v \) of \( C \), \( P_{\Gamma^{st}} \) induces a domain-dependent almost complex structure \( J_v \) and a domain-dependent Hamiltonian perturbation \( H_v \); on each tree part \( T_e \) of \( C \), \( P_{\Gamma^{st}} \) induces a domain-dependent function \( F_e : T_e \times \bigcup_{\phi_-, \phi_+} L_{\phi_-, \phi_+} \to \mathbb{R} \).

These data allows one to define the equation componentwise. For each surface component \( S_v \) and a smooth map \( u_v : S_v \to X \), define
\[ d_{H_v} u_v = du_v - H_v(u_v) \in \Omega^1(S_v, u_v^*TX) \]
and
\[ \overline{\partial}_{J_v, H_v} u_v = (d_{H_v} u_v)^{0,1} = \frac{1}{2} (J_v \circ d_{H_v} u_v - d_{H_v} u_v \circ j_v) \in \Omega^{0,1}(S_v, u_v^*TX). \]

We say that \( u_v \) is \((J_v, H_v)\)-holomorphic if \( \overline{\partial}_{J_v, H_v} u_v = 0 \). For each tree component \( T_e \) and a smooth map
\[ u_e : T_e \to \bigcup_{\phi_-, \phi_+} L_{\phi_-, \phi_+} \]
we say that \( u_e \) is a perturbed negative gradient segment if
\[ u_e'(s) + \nabla F_e(s, (u_e(s))) = 0. \]

**Definition 2.18.** Let \( \Gamma = (\Gamma, x, \beta, \diamond) \) be a map type with underlying combinatorial type \( \Gamma \) of treed disks. Let \( C = S \cup T \) be a treed disk of type \( \Gamma \). Let \( \Gamma^{st} \) be the stabilization of \( \Gamma \) and \( P_{\Gamma^{st}} \) be a perturbation datum on \( U_{\Gamma^{st}} \). Then a \( P_{\Gamma^{st}} \)-perturbed adapted treed holomorphic map from \( C \) to \( X \) of map type \( \Gamma \) is a collection of continuous maps
\[ u : C \to X, \quad \partial u : \partial C \to \bigcup_{\phi \in \mathcal{L}} L_{\phi} \]
satisfying the following conditions (using notations specified before this definition).
(a) The restriction of $u$ to the surface component $S_v$, denoted by $u_v : S_v \to X$, is $(J_v, H_v)$-holomorphic, namely
\[ \overline{\partial} J_v H_v u_v = 0. \]

(b) The restriction of $u$ to the tree component $T_e$ is contained in $L_e$, denoted by $u_e : T_e \to L_e$, is a perturbed negative gradient segment, namely
\[ u'_e(s) + \nabla F_e(s,(u_e(s))) = 0. \]

(c) The restriction of $u$ to the $\partial C$ agrees with the pull-back of $\partial u$, namely
\[ u|_{\partial C} = \phi \circ \partial u. \]

(d) For each semi-infinite edge $e$, the map $u_e$ converges to the limit specified by the datum $x$.

(e) For each interior leaf $e$ attached to a vertex $v_e$, if the interior constraint is not $(D(L), m)$, then
\[ E_{\Gamma,e}(u_{ve}(z_e)) \in X_e \subset X. \]
where $X_e$ is one of the components $j_i(Z_i)$ of the bulk deformation (17). Here $E_{\Gamma,e} : X \to X$ is the diffeomorphism contained in the perturbation datum. If the interior constraint is $(D(L), m)$, then $u_{ve}(z_e) \in D(L)$ and if $u_{ve}$ is not a constant map, then the tangency order of $u_{ve}$ with $D(L)$ is $m$.

The triple $(C,u,\partial u)$ is called an (adapted) treed holomorphic disk of map type $\Gamma$.

Isomorphisms of perturbed treed holomorphic disks are defined in a way similar to that for stable pseudoholomorphic maps. A perturbed treed holomorphic disk is called stable if its automorphism group is finite, or equivalently

(a) every sphere component $u_v : S_v \to X$ with $d_{H_v}u_v \equiv 0$ has at least three special points, and

(b) every disk component $u_v : S_v \to X$ with $d_{H_v}u_v \equiv 0$ either has at least three boundary special points, or one boundary special point and one interior special point, or at least two interior special points.

(c) over each infinite edge $T_e \subset C$ the map $u_e : T_e \to L_e$ is nonconstant.

Given a map type $\Gamma = (\Gamma, x, \phi, \diamond)$, denote by
\[ \mathcal{M}_\Gamma(P_{\Gamma^{st}}) \]
the set of isomorphism classes of stable $P_{\Gamma^{st}}$-perturbed adapted treed holomorphic disks. One can also define a Gromov topology and compactify the moduli spaces (we omit the details). We only consider the compactification for the case $\Gamma$ being stable. In this case, the Gromov compactification is
\[ \overline{\mathcal{M}}_\Gamma(P_\Gamma) := \bigcup_{\Pi \preceq \Gamma} \mathcal{M}_\Pi(P_\Gamma|_{\Pi_{\Gamma^{st}}}). \]

Here the partial order $\Pi \preceq \Gamma$ naturally extends the partial order $\Pi \preceq \Gamma$ among combinatorial types of domains.
Definition 2.19 (Partial order among map types). Let $\Gamma = (\Gamma, x, \beta, \diamond)$ and $\Gamma' = (\Gamma', x', \beta', \diamond')$ be two map types. We denote $\Gamma' \preceq \Gamma$ if $\Gamma' \preceq \Gamma$ (which induces a morphism $\psi : \Gamma' \to \Gamma$), $x = x'$ and

$$\beta(v) = \sum_{v' \in \psi^{-1}(v)} \beta'(v');$$

moreover, for each interior leaf $e$ of $\Gamma$ with the corresponding leaf $e'$ of $\Gamma'$, either $X_e = X_{e'}$, or

$$(X_e, X_{e'}) \in \{(j_i(Z_i), \iota_i(W_i)) \mid 1 \leq i \leq N\} \cup \{(D(L) \cap j_i(Z_i), \iota_i(W_i)) \mid 1 \leq i \leq N\} \cup \{(j_i(Z_i) \cap j_i'(Z_i'), \iota_i(W_i)), (j_i(Z_i) \cap j_i'(Z_i'), \iota_i'(W_i')) \mid 1 \leq i, i' \leq N\}.$$  

The composition laws of Fukaya algebras rely on the following relation among perturbation data.

Definition 2.20 (Coherent perturbations). A collection of perturbation data $P := (P_\Gamma)^\Gamma$ for all stable domain types are called coherent if the following conditions are satisfied.

(a) (Cutting-edges axiom) If a breaking separates $\Gamma$ into $\Gamma_1$ and $\Gamma_2$ then $P_\Gamma$ is the product of the perturbations $P_{\Gamma_1}, P_{\Gamma_2}$ under the isomorphism $U_\Gamma \simeq \pi_1^* U_{\Gamma_1} \cup \pi_2^* U_{\Gamma_2}$.

(b) (Degeneration axiom) If $\Gamma' \prec \Gamma$, then the restriction $P_\Gamma$ to $U_{\Gamma'}$ is equal to $P_{\Gamma'}$.

(c) (Forgetful axiom) For a forgettable boundary input $e$ of $\Gamma$, let $\Gamma_e$ be the domain type obtained from $\Gamma$ by forgetting $e$ and stabilizing. Then $P_\Gamma$ is equal to the pullback of $P_{\Gamma_e}$ via the contraction $U_\Gamma \to U_{\Gamma_e}$.

2.6. Transversality. In this subsection we regularize the moduli spaces used in our construction. We first review very briefly the Fredholm theory associated to treed holomorphic maps. Let $\Gamma$ be a map type. We specify a pair $(k, p)$ with $k \in \mathbb{N}, p > 2$.

The set $B^{k,p}(C, \Gamma)$ of maps of type $\Gamma$ has the structure of a Banach manifold. An element $(u, \partial u) \in B^{k,p}(C, \Gamma)$ is defined as in Definition 2.18 without requiring the holomorphic curve and gradient flow equations and instead requiring $(u, \partial u)$ to be of class $W^{k,p}$ over each surface or tree component. Tangency conditions for a maximal order $m$ as in (24) are defined for $k$ sufficiently large. Choose a perturbation datum $P_\Gamma = (J_\Gamma, H_\Gamma, F_\Gamma, E_\Gamma)$. Over this Banach manifold there is a Banach vector bundle $E^{k,p}(C, \Gamma)$ so that the defining equations of Definition 2.18 provides a section $F : B^{k,p}(C, \Gamma) \to E^{k,p}(C, \Gamma)$ combining the Cauchy-Riemann operators on the surface parts and gradient flow operators on the edges. Since the Lagrangians are always totally real with respect to
the domain-dependent almost complex structures, the section is a Fredholm section. Its index can be easily calculated, which provides the expected dimension of the moduli space.

To include the variations of the domains, one takes an open neighborhood $\mathcal{M}_\Gamma \subset \mathcal{M}_F$ of $[C]$ over which the universal curve $\mathcal{U}_\Gamma$ has a trivialization 

$$\mathcal{U}_\Gamma|_{\mathcal{M}_\Gamma} \cong \mathcal{M}_\Gamma \times C.$$ 

The linearization of a $P_\Gamma$-perturbed treed holomorphic disk $(C,u,\partial u)$ is then a Fredholm map

$$D_u : T_{(u,\partial u)}B^{k,p}(C,\Gamma) \times T_{[C]}\mathcal{M}_\Gamma \to E^{k,p}(C,\Gamma)|_{(u,\partial u)}.$$ 

The expected dimension of the moduli space is given by

$$\dim \mathcal{M}_\Gamma(P_\Gamma) = \dim \mathcal{M}_\Gamma + \mu(\beta) + i(\bar{x}) - i(\diamond)$$

where $\mu(\beta)$ is the total Maslov index of the disk class, $i(\bar{x})$ is the sum of Morse indices of asymptotic constraints, and $i(\diamond)$ is the effect of interior constraints. For example, $\Gamma$ has $k$ interior inputs all of which are labelled by $(D(L),1)$, then $i(\diamond) = 2k$.

Following Cieliebak-Mohnke [CM07], we introduce the following types that we will regularize.

**Definition 2.21.** A map type $\Gamma = (\Gamma, x, \phi, \diamond)$ is called *uncrowded* if each ghost sphere bubble tree contains at most one interior input $e$ whose interior constraint is in

$$\{(D(L),m) \mid m = 1,\ldots\}.$$ 

Otherwise $\Gamma$ is called *crowded*.

**Remark 2.22.** Cieliebak-Mohnke perturbations can never make crowded configurations $u : C \to X$ transversally cut out, since one can replace an interior leaf $T_e$ with a given label $D$ and replace it with a sphere bubble $S_v$ with two interior leaves $T_{e_1}, T_{e_2}$ attached with the same label $D$, which reduces the expected dimension of a stratum by two. This produces (eventually after repeating) a non-empty stratum $\mathcal{M}_{\Gamma'}(\phi, D)$ of negative expected dimension, a contradiction if the perturbations are regular.

We will need certain forgetful maps to treat crowded configurations. Let $\Gamma$ be a stable domain type and choose a subset

$$W \subset \text{Vert}_{\text{sphere}}(\Gamma).$$ 

Define $\Gamma_W$ to be the domain type obtained by the following operation: For each connected component $W_i \subset W$, remove all interior leaves except the one with the largest labelling on $W_i$, and stabilize the remaining configuration. The set $W$ descends to a (possibly empty) subset $W' \in \text{Vert}_{\text{sphere}}(\Gamma_W)$. An important consequence of the locality condition on the perturbation data is that each $P_\Gamma \in \mathcal{P}_\Gamma$ descends to a perturbation datum $P_{\Gamma'} \in \mathcal{P}_{\Gamma_W}$ whose restriction to surface components $S_v$ for $v \in W'$
equals to the base almost complex structure $J_0$.\footnote{The descent $P_{\Gamma W}$ may not agree with a member of any prechosen coherent collection of perturbation data.} Let
\[ P_{\Gamma W, W'} \subset P_{\Gamma W} \]
be the subset of perturbations that agree with the base almost complex structure $J_0$ over surface components corresponding to vertices in $W'$. Then this forgetful construction leads to a smooth map of Banach manifolds
\[ P_{\Gamma} \to P_{\Gamma W, W'}. \]
Indeed this is essentially a surjective linear map, hence admits a smooth right inverse.

**Definition 2.23.** Let $\Gamma$ be a stable domain type. A perturbation $P_\Gamma \in P_\Gamma$ is called \textit{regular} if all uncrowded maps of type $\Gamma$ with underlying domain type being $\Gamma$ are regular. The perturbation $P_\Gamma$ is called \textit{strongly regular} if for any subset $W \subset \text{Vert}_{\text{sphere}}(\Gamma)$ and for any uncrowded map type $\Gamma_W$ whose homology classes on surface components corresponding to vertices in $W'$ are zero, every map of type $\Gamma_W$ is regular.

The main result of this section is the regularity of moduli spaces for uncrowded map types and the selection of a coherent collection of perturbation data.

**Theorem 2.24.** There exist a coherent collection of perturbation data $P = (P_\Gamma)$ whose members are all strongly regular.

**Proof.** The proof is an induction on the possible types according to the partial order introduced above. First we introduce an equivalence relation among stable domain types. We denote $\Gamma \sim \Pi$ if roughly they have isomorphic base. More precisely, this is the equivalence relation by the relation $\Pi \preceq \Gamma$ and $\rho_\Pi : \Pi \to \Gamma$ is an isomorphism.

Let $[\Gamma]$ denote the equivalence class of $\Gamma$. The partial order relation among domain types descends to an equivalence relation among their equivalence classes.

The inductive step is the following. Fix an equivalence class $[\Gamma]$. Suppose we have chosen strongly regular perturbation data $P_\Pi$ for all stable domain types $\Pi$ with $[\Pi] \prec [\Gamma]$ as well as domain types with strictly fewer boundary inputs or the same number of boundary inputs but strictly fewer interior inputs, such that the chosen collection is coherent in the sense of Definition 2.20. For each $\Gamma$ in this class $[\Gamma]$, there is a closed Banach submanifold
\[ P^*_\Gamma \subset P_\Gamma \]
consisting of perturbation data whose values over all lower strata $U_\Pi$ with $\Pi \prec \Gamma$ and $[\Pi] \prec [\Gamma]$ agree with the prechosen on $P_\Pi$.

We prove the following sublemma.

**Sublemma.** There is a comeager subset $P^*_{\Gamma, \text{reg}} \subset P^*_\Gamma$ whose elements are regular.
Proof of the sublemma. Let $M_i \Gamma$ be a subset of $M \Gamma$ over which the universal curve $U \Gamma$ is trivial, and $U_i \Gamma$ the restriction of $U \Gamma$ to $M_i \Gamma$. For each uncrowded map type $\Gamma$ with underlying domain type $\Gamma$, consider the universal moduli space

$$M_{i, \text{univ}}(P \ast \Gamma) = \{([u : C \to X], P \Gamma) | P \Gamma \in P \ast \Gamma, C \subset U_i \Gamma, [u] \in M \Gamma(P \Gamma)\}.$$ 

of maps with domain in $U_i \Gamma$ together with a perturbation datum $P \Gamma$. By Sard-Smale theorem this sublemma can be proved once we show the regularity of the local universal moduli space. Suppose this is not the case, so that for some $(u, P \Gamma)$ the linearization of the defining equation of the universal moduli is not surjective. By elliptic regularity, there exists a nonzero smooth section

$$\eta \in \mathcal{E}^{k,p}(C, \Gamma)$$

which is in the $L^2$-orthogonal complement of the image of the linearization. We will derive a contradiction by showing that each component of $\eta$ vanishes identically on that component.

First, consider a nonconstant sphere component $u_v : S_v \to X$. Since the support of the perturbation $J_{\Gamma,v}$ has nonzero intersection with $S_v$, the restriction $\eta_v$ of $\eta$ to $S_v$ must vanish over a nonempty open set of $S_v$. The unique continuation principle for first order elliptic equation implies that $\eta_v$ vanishes identically. Second, for any disk component $u_v : S_v \to X$, since the domain is stable, the support of the Hamiltonian perturbation $H_{\Gamma,v}$ is nonempty. If $d_{H_v}u_v$ is not identically zero, then it is nonzero over an open and dense set. Since $\eta$ must be orthogonal to the images of deformations of $J_{\Gamma,v}$ under the linearized operator, $\eta_v \equiv 0$ (notice that this component is not contained in $D(\mathcal{L})$). If $d_{H_v}u_v \equiv 0$, then by the property of the space of Hamiltonian perturbations $\text{Ham}^\lambda(X, U(D))$ in Remark 2.14, the image of the support of the Hamiltonian perturbation has no intersection with $\overline{U(D)}$, on which the Hamiltonian vanishes. Hence using the Hamiltonian perturbation one can also show $\eta_v$ is identically on this component. Third, for an edge $T_e$ with positive or infinite length, if the gradient segment $u_e : T_e \to L_e$ is mapped into a positive dimensional target $L_e$, then since the support of the perturbation $F_e$ is nonempty, it also follows that the restriction $\eta_e$ to $T_e$ vanishes identically. If $L_e$ is zero-dimensional, then by definition $\eta_e \equiv 0$.

It remains to consider the linearization over a constant sphere component $u_v : \mathbb{P}^1 \to X$ mapped to a point $x_v \in X$ where deformations of $J_{\Gamma}$ are in the kernel of the linearization. For any domain-dependent almost complex structure, the linear map

$$\overline{\partial}J_v : \Omega^0(\mathbb{P}^1, T_{x_v}X) \to \Omega^{0,1}(\mathbb{P}^1, T_{x_v}X)$$

is surjective with kernel being the finite dimensional subspace of constant vector fields. However, there might be constraints coming from special points on this component. For this we use the uncrowdedness condition. Consider a maximal ghost sphere tree $W \subset \text{Vert}_{\text{sphere}}(\Gamma)$. There is at most one special point on $W$ which is constrained by $(D(\mathcal{L}), m)$; this puts a two-dimensional constraint on the constant vector field. For any other interior marking $e$, one can use the deformation of the diffeomorphism $E_{\Gamma,e}$ to free the constraints at that marking. For any node connecting
W to a nonconstant component, the constraints can be freed by using deformations on the adjacent nonconstant component. Then \( \eta \) vanishes on components in \( W \).

End of the proof of the sublemma.

Next we would like to find a comeager subset \( P_{\Gamma_{W}}^{s, \text{reg}} \) of strongly regular perturbations. Indeed, for any subset \( W \subset \text{Vert}_{\text{sphere}}(\Gamma) \), consider the domain type \( \Gamma_{W} \) with a descent subset \( W' \subset \text{Vert}_{\text{sphere}}(\Gamma_{W}) \). Notice that \( \Gamma_{W} \) and \( \Gamma \) has isomorphic base. Hence the prechosen perturbations provides a subset \( P_{\Gamma_{W}, W'}^{s, \text{reg}} \subset P_{\Gamma_{W}, W'}^{s} \) consisting perturbations whose values are fixed precisely over strata \( \Pi' \) with \( [\Pi'] \prec [\Gamma_{W}] \).

Moreover, the forgetful map (26) restricts to a forgetful map \( \pi_{W} : P_{\Gamma}^{s, \text{reg}} \to P_{\Gamma_{W}, W'}^{s, \text{reg}} \) which has a right inverse given by pullback. By the same argument as the proof of the above sublemma, there is a comeager subset \( P_{\Gamma_{W}}^{s, \text{reg}} \) consisting of perturbations \( P_{\Gamma_{W}}^{s} \) that regularize moduli spaces \( M_{\Gamma_{W}}(P_{\Gamma_{W}}) \) for map types \( \Gamma_{W} \) that are ghost on surface components corresponding to vertices in \( W' \). Then define

\[
P_{\Gamma_{W}}^{s, \text{reg}} := \bigcap_{W \subset \text{Vert}_{\text{sphere}}(\Gamma)} \pi_{W}^{-1}(P_{\Gamma_{W}, W'}^{s, \text{reg}}).
\]

This is still a comeager subset and all its elements are strongly regular.

Lastly we choose \( P_{\Gamma} \) extending the prechosen perturbations on lower-dimensional strata. We define smaller comeager subsets \( P_{\Gamma}^{s**, \text{reg}} \) inductively as follows. If \( \Gamma \) is a smallest element of the equivalence class \( [\Gamma] \), then define \( P_{\Gamma}^{s**, \text{reg}} := P_{\Gamma}^{s, \text{reg}} \). Suppose for a general \( \Gamma \) in \( [\Gamma] \) one has defined \( P_{\Gamma'}^{s**, \text{reg}} \) for all \( \Gamma' \prec \Gamma \) with \( [\Gamma'] = [\Gamma] \). Then define

\[
P_{\Gamma}^{s**, \text{reg}} := P_{\Gamma}^{s, \text{reg}} \cap \bigcap_{\Gamma' \prec \Gamma, [\Gamma'] = [\Gamma]} \pi_{\Gamma, \Gamma'}^{-1}(P_{\Gamma'}^{s**, \text{reg}}).
\]

Here \( \pi_{\Gamma, \Gamma'} : P_{\Gamma} \to P_{\Gamma'} \) is the map defined by restricting to boundary strata. Then we have defined \( P_{\Gamma}^{s**, \text{reg}} \) for all \( \Gamma \) in this equivalence class. Notice that \( [\Gamma] \) has a unique maximal element \( \Gamma_{\text{max}} \). Choose an arbitrary perturbation \( P_{\Gamma_{\text{max}}} \in P_{\Gamma_{\text{max}}}^{s**, \text{reg}} \). By boundary restriction this choice induces \( P_{\Gamma} \) for all \( \Gamma \) in this equivalence class. By construction, all these \( P_{\Gamma} \) extend the existing perturbations on lower-dimensional strata. By induction one obtains the claimed collection \( P \).

\[ \Box \]

2.7. Essential moduli spaces. We consider moduli spaces of expected dimension zero or one. Fix a coherent collection of perturbation data \( \mathcal{P} = (P_{\Gamma}) \) and abbreviate all moduli spaces \( M_{\Gamma}(P_{\Gamma}) \) by \( M_{\Gamma} \).

**Definition 2.25.**

(a) A map type \( \Gamma \) is called essential if it has no breaking, no edges of length zero or infinity, no spherical components, if all interior constraints are either \( (D(L), 1) \) or \( j_{i}(Z_{i}) \), and for each disk vertex \( v \in \text{Vert}_{\text{disk}}(\Gamma) \), the number of interior leaves labelled by \( (D(L), 1) \) is equal to \( k\omega(\beta_{v}) \) where \( k \) is the degree of the Donaldson hypersurface.

(b) Given asymptotic data \( x \) (as in Definition 2.17), let

\[
M(x)_{0} \quad \text{resp.} \quad M(x)_{1}
\]
be the union of moduli spaces $\mathcal{M}_\Gamma$ for essential map types of expected dimension zero resp. one whose asymptotic data is $x$.

**Remark 2.26.** As in [WW] the determinant lines of the linearized operators become equipped with orientations induced by relative spin structures. In particular, if all strata of $\mathcal{M}(x)_0$ are regular then there is a map
\[ \epsilon : \mathcal{M}(x)_0 \rightarrow \{\pm 1\}. \]

The following lemma classifies types of topological boundaries of one-dimensional moduli spaces.

**Lemma 2.27.** Suppose $P = (P_\Gamma)$ is a coherent and regular collection of perturbations. For an essential combinatorial type $\Gamma$ of expected dimension zero, the moduli space $\mathcal{M}_\Gamma(P_\Gamma)$ is compact. For a combinatorial type $\Gamma$ of expected dimension one, the boundary of the compactified one-dimensional cell moduli $\overline{\mathcal{M}}_\Gamma(P_\Gamma)$ is the disjoint union of moduli spaces $\mathcal{M}_\Pi(P_\Pi)$ where $\Pi$ is a combinatorial type related to $\Gamma$ by exactly one of the following operations.

(a) Collapsing an edge $e \in \text{Edge}(\Pi)$ of length zero.
(b) Setting a finite edge $e \in \text{Edge}(\Pi)$ to have length $\ell(e)$ zero or infinity.
(c) In the case when the output edge $e_0 \in \text{Edge}(\Gamma)$ is not weighted, setting weight $\rho(e)$ of exactly one weighted input $e \in \text{Edge}(\Gamma)$ to be zero or one.
(d) In the case when the output $e_0 \in \text{Edge}(\Gamma)$ is weighted, the weight $\rho(e)$ of exactly one weighted input $e \in \text{Edge}(\Gamma)$ becomes one.

**Sketch of proof.** It suffices to check sequential compactness. Let $(C_\nu, u_\nu, \partial u_\nu)$ be a sequence of treed holomorphic disks representing a sequence of points in $\mathcal{M}_\Gamma$. By the general compactness result for treed holomorphic disks, there is a subsequence (still indexed by $\nu$) that converges to a limiting treed holomorphic disk $(C_\infty, u_\infty, \partial u_\infty)$ of certain map type $\Pi$. We first claim that the domain type $\Pi$ is stable. Suppose on the contrary it is not the case. Then there is either an unstable disk component $u_\infty : \mathbb{D} \rightarrow X$ or an unstable sphere component $u_\infty : \mathbb{P}^1 \rightarrow X$. By the stability condition, they must be nonconstant maps. Moreover, they must be pseudoholomorphic with respect to a constant tamed almost complex structure $J$ on $X$. By the definition of perturbation data, one has
\[ J \in \mathcal{J}^{E(\Gamma)}_{\text{tame}}(X, \omega). \]
Since the convergence preserves the total energy, the disk or the sphere has energy at most $E(\Gamma)$, hence the $J$-holomorphic sphere $u_\infty : \mathbb{P}^1 \rightarrow X$ is not contained in $D(\mathcal{L})$ and must intersect $D(\mathcal{L})$ at at least three points. Then the convergence implies that for $\nu$ sufficiently large, $u_\nu$ intersects with $D(\mathcal{L})$ at at least three nearby points. Since the type $\Gamma$ is essential, all these intersection points are marked points labelled by $(D(\mathcal{L}), 1)$. Then the convergence implies that the intersection points of $u_\infty$ with $D(\mathcal{L})$ must all be marked points, contradicting the assumption that the domain of the sphere is unstable. Similarly, by the exactness of $\mathcal{L}$ in $X \setminus D(\mathcal{L})$, the disk $u_\infty : \mathbb{D} \rightarrow X$ must intersect $D(\mathcal{L})$ at some point which must also be a
marked point, contradicting the assumption in the same way. Therefore the domain type $\Pi$ is stable. Since $\Pi \preceq \Gamma$, the perturbation datum $P_\Gamma$ induces by restriction a perturbation datum $P_{\Pi}$. Moreover,

$$[C_\infty, u_\infty, \partial u_\infty] \in \mathcal{M}_{\Pi}(P_\Pi).$$

In particular, there is no nonconstant spheres contained in $D(\mathcal{L})$.

Next we show that type of the limit constructed in the previous paragraph is uncrowded. Suppose this is not the case, then let $W \subset \text{Vert}_{\text{sphere}}(\Pi)$ be the (nonempty) set of ghost sphere components. By the locality property, the perturbation data $P_{\Pi}$ descends to a perturbation $P_{\Pi_W}$ which is equal to $J_0$ over $W'$. The limiting configuration $[C_\infty, u_\infty, \partial u_\infty]$ then descends to an element

$$[C', u', \partial u'] \in \mathcal{M}_{\Pi_W}(P_{\Gamma_W}).$$

Since $P_{\Pi}$ is uncrowded, the above moduli space is regular and nonempty. However, similar to the argument of [CM07], the reduction drops the expected dimension by at least two. This contradiction shows that $\Pi$ must be uncrowded.

Finally, we claim $\Pi$ has no sphere components. This is because each sphere component will drop the dimension of the domain moduli space by two and $\Gamma$ has no sphere components. It follows from the dimension formula for $\mathcal{M}_\Gamma$ that when $\dim \mathcal{M}_\Gamma = 0$, $\Pi$ must be identical to $\Gamma$ and hence $\mathcal{M}_\Gamma$ is compact. When $\dim \mathcal{M}_\Gamma = 1$, the only possibly types of $\Pi$ are described in the above list. \qed

Moreover, we distinguish the boundary strata as either true or fake boundary components. The true boundaries are those corresponding to edge breaking and weight changing to zero or one while the fake boundaries are those corresponding to disk bubbling or edges shrinking to zero. For one-dimensional moduli strata $\mathcal{M}(\underline{x})_1$, define

$$\overline{\mathcal{M}}(\underline{x})_1 = \bigcup \overline{\mathcal{M}}_{\Gamma}$$

to be the union of all compactified moduli space of expected dimension one while identifying fake boundaries. Standard gluing constructions (gluing disks or gradient lines) show that $\overline{\mathcal{M}}(\underline{x})_1$ a topological 1-manifold with boundary and its cutoff at any level (indexed by the number of interior markings) is compact. The boundaries are strata corresponding to edge breaking and weight changing to zero or one.

2.8. Disks bounding the exceptional branes. In this section, we describe perturbation data on a blow-up that is standard near the exceptional divisor. We make explicit computations involving holomorphic disks whose boundary maps to exceptional branes – these are certain Lagrangian tori in the neighborhood of the exceptional divisor. The perturbation data we consider are multivalued – this is necessary to achieve some symmetry properties of the composition maps.

We recall some geometric details about the neighborhood of the exceptional divisor needed for the construction of our perturbation data. Let $p \in X$ be the blowup point. Recall that the bulk deformation, the collection of Lagrangian branes, and
the Donaldson hypersurface are all disjoint from \( p \), hence disjoint from a Darboux chart \( U \ni p \). Let \( \tilde{U} \subset \tilde{X} \) be the preimage of \( U \) under the projection \( \tilde{X} \to X \). Fix the Darboux coordinate in \( U \). Let \( J_{\tilde{U}} \) be the integrable almost complex structure on \( \tilde{U} \) which is the pullback from the standard complex structure with respect to the Darboux coordinates in \( U \). The exceptional branes in \( E \) are all supported on an embedded Lagrangian \( \tilde{L} \subset \tilde{U} \).

First we describe the local model as a symplectic quotient. Let \( \tilde{X} = \text{Bl}_0 (\mathbb{C}^n) \) be the blowup of \( X = \mathbb{C}^n \) at 0. Let \( \phi : L \cong (S^1)^n \to X, \ z \mapsto z \) be a Lagrangian torus orbit of some \( x \in \tilde{X} \). We realize \( \tilde{X} \) as the symplectic quotient of \( \hat{X} = X \times \mathbb{C} \cong \mathbb{C}^{n+1} \) by the diagonal action of \( \mathbb{C} \times \) with at moment value \( \epsilon \). The Lagrangian \( L \) lifts to a Lagrangian \( \hat{L} \cong (S^1)^{n+1} \) in \( \hat{X} \) given by

\[
\hat{L} = \left\{ (z_1, \ldots, z_n) \big| \left| z_i \right|^2 = \frac{\epsilon_i}{\pi}, \ i = 1, \ldots, n+1 \right\} \subset \hat{X}
\]

for some \( \epsilon_1, \ldots, \epsilon_{n+1} \in \mathbb{R}_{>0} \).

**Definition 2.28.** A Blaschke product of degree \((d_1, \ldots, d_k)\) is a map from the disk \( D \) to \( \hat{X} \) prescribed by coefficients

\[
\left| \zeta_i \right| = 1, \quad a_{i,j} \in \mathbb{C}, \quad |a_{i,j}| < 1, \quad i \leq n + 1, \quad j \leq d_i:
\]

\[
u : D \to \mathbb{C}^{n+1}, \quad z \mapsto \left( \zeta_i \prod_{j=1}^{d_i} \frac{z - a_{i,j}}{1 - z \overline{a_{i,j}}} \right)_{i=1, \ldots, n+1}.
\]

We include the following proposition from Cho-Oh [CO06] for completeness:

**Lemma 2.29.** The index of any Blaschke product \( u : D \to \mathbb{C}^{n+1} \) given by (27) is

\[
I(u) = \sum_{i=1}^{n+1} 2d_i.
\]

If \( \hat{L} = \left\{ \left| z_i \right|^2 = \frac{\epsilon_i}{2\pi} \right\} \) is the lift of \( L \) and \( u \) takes bound in \( L \) then the area of \( u \) is

\[
A(u) = \sum_{i=1}^{n+1} d_i \epsilon_i.
\]

In particular, if \( \epsilon_i = \epsilon \) for all \( i, j \) then \( L \) is monotone.

**Proof.** As in Cho-Oh [CO06], the products (27) are a complete description of holomorphic disks with boundary in \( \hat{L} \). Since the image of \( \tilde{u}(z) \) is disjoint from the semistable locus, the Blaschke products descend to disks \( u : (D, \partial D) \to (X, L) \). We compute their Maslov index using the splitting

\[
(\tilde{u}^*TV, \tilde{u}^*TV_\mathbb{R}) \cong (u^*TX(\partial u)^*TL) \oplus (g_C, g)
\]
where \( g_C, g \) denotes the trivial bundle and real boundary condition with fiber \( g_C \) resp. \( g \) the Lie algebras of the complex resp. real torus acting on \( X \). We write

\[
I(E, F) \in \mathbb{Z}
\]

for the Maslov index of a pair \((E, F)\) consisting of a complex vector bundle \( E \) on the disk \( \mathbb{D} \) and a totally real sub-bundle \( F \) over the boundary \( \partial\mathbb{D} \). Since the Maslov index of bundle pairs is additive,

\[
I(\tilde{u}^*TV, (\partial\tilde{u})^*TV_R) = I(u^*TX, (\partial u)^*TL) + I(u^*g_C, (\partial u)^*g).
\]

The second factor has Maslov index \( I(u^*g_C, (\partial u)^*g) = 0 \), as a trivial bundle. So the Maslov index of the disk \( u \) is given by

\[
I(u) = I(u^*TX, (\partial u)^*TL) = \sum_{i=1}^{n+1} 2d_i = 2\#u^{-1}\left(\sum_{i=1}^{k}[D_i]\right);
\]

that is, \( I(u) \) is twice the sum of the intersection number with the anticanonical divisor

\[
[K^{-1}] = \sum_{i=1}^{k}[D_i] \in H^2(X, \mathbb{Z})
\]

which is the disjoint union of the prime invariant divisors

\[
D_i = [z_i = 0] \subset \mathbb{C}^{n+1}/\mathbb{C}, i = 1, \ldots, k.
\]

In particular the disks of index two are those maps \( u_i : \mathbb{D} \to X \) with lifts of the form

\[
\hat{u}_i : \mathbb{D} \to \hat{X}, \quad z \mapsto (b_1, \ldots, b_{i-1}, b_iz, b_{i+1}, \ldots, b_{n+1}).
\]

The area of each such disk is

\[
A(u_i) = A(\hat{u}_i) = \epsilon
\]

since

\[
\int \hat{u}_i^*\hat{\omega} = \int_{r=2}^{r=2=\epsilon/2\pi} rdrd\theta = \epsilon.
\]

The homology class of higher index Maslov disks \( u : C \to X, I(u) > 2 \) is a weighted sum

\[
[u] = \sum d_i[u_i]
\]

of homology classes of primitive disks \( u_i, i = 1, \ldots, n+1 \). So the area \( A(u) \in \mathbb{R} \) of such a disk \( u \) is the weighted sum

\[
A(u) = \sum d_iA(u_i)
\]

of the areas \( A(u_i) \) of disks \( u_j \) of index \( I(u_j) = 2 \).

\(\square\)

**Corollary 2.30.**

(a) \((\tilde{U}, L, \tilde{\omega}|_\tilde{E})\) is monotone with minimal Maslov index two.

(b) The moduli space of \( J_0 \)-holomorphic disks \( \mathcal{M}_{d,1}(\tilde{U}, L, J_0) \) in \( \tilde{U} \) with boundary in \( \tilde{L} \), with one boundary marking and \( d \) interior markings mapped to the exceptional divisor \( \tilde{Y} \) is regular and the evaluation map \( ev : \mathcal{M}_{d,1}(\tilde{U}, L, J_0) \to \tilde{L} \) is a submersion.
(c) All nonconstant $\tilde{J}_0$-holomorphic spheres in $\tilde{U}$ have positive Chern numbers and are contained in the exceptional divisor $\tilde{Y}$. Moreover, the moduli space of these spheres with one marking is regular (as maps into $\tilde{Y}$) and the evaluation map at the marking is a submersion onto $\tilde{Y}$.

Proof. The first two items follow from Lemma 2.29. For the third, note that any holomorphic sphere $u : \mathbb{P}^1 \to \tilde{U}$ defines a holomorphic sphere in $\tilde{Y}$ by projection, necessarily of degree $d$, together with a section of the pull-back of the normal bundle, necessarily a line bundle of degree $-d$. Since such bundles have no sections, $u$ has image in the exceptional divisor. The claim follows from homogeneity of $\tilde{Y}$, and the fact that the Chern number of any degree $d$ map to $\tilde{Y}$ is $d(n-1)$. □

To regularize the moduli spaces we take a transform of a Donaldson hypersurface under the blowup. Given a Donaldson hypersurface $D \subset X$ disjoint from $p$ that is a stabilizer in the sense of Lemma 2.11 for $J$-holomorphic spheres and disks we obtain a hypersurface $\tilde{D}(\mathcal{L}) \subset \tilde{X}$ by

$$\tilde{D}(\mathcal{L}) = \pi^{-1}(D).$$

Remark 2.31. The submanifold $\tilde{D}(\mathcal{L})$ represents the cohomology class that is a multiple of $\pi^* [\omega]$ rather than $[\tilde{\omega}]$. So $\tilde{D}(\mathcal{L})$ is not a Donaldson hypersurface. Nevertheless, if $J$ is a tamed almost complex structure on $\tilde{X}$ for which the projection $\pi : \tilde{X} \to X$ is $(\tilde{J}, J)$-holomorphic then any $\tilde{J}$-holomorphic sphere or disk in $\tilde{X}$ not contained in $E$ projects to a non-trivial sphere or disk in $X$, while spheres in $E$ are positive index, so $\tilde{D}(\mathcal{L})$ is a disk and sphere-stabilizer in the sense of Lemma 2.11.

To regularize the moduli spaces of treed maps which have spherical components mapped to the exceptional divisor one has to use a different notion of regularity, as the normal direction to the exceptional divisor may bring in obstructions in the usual sense. Let $\Gamma$ be a domain type, $u : C \to \tilde{X}$ be a treed holomorphic disk of type $\Gamma$. Consider maximal spherical subtrees of $\Gamma$ whose energy is positive and whose images are contained in the exceptional divisor $\tilde{Y}$ (such a subtree may have ghost components). Let $R$ be the union of these subtrees. Such a map is said to be of type $(\Gamma, R)$. In order to obtain compactness, we require that $J_\Gamma$ takes values in the open subset of almost complex structures $J$ such that there is no non-constant $J$-holomorphic sphere $u$ contained in $\tilde{D}(\mathcal{L})$ such that the energy of $\pi \circ u$ is at most $E(\Gamma)$ from (20). In order to obtain a compactness result, the notion of strong regularity of Definition 2.23 needs the following modification.

Definition 2.32. A perturbation $P_\Gamma = (J_\Gamma, H_\Gamma, F_\Gamma, E_\Gamma)$ is called exceptionally regular if the following conditions are satisfied. For each subset $R \subset \text{Vert}_{\text{sph}}(\Gamma)$, an uncrowded treed holomorphic disk $u : C \to \tilde{X}$ of type $(\Gamma, R)$ is regular as a map of type $(\Gamma, R)$. Namely, we treat components in $R$ as maps to $\tilde{Y}$ but not to $\tilde{X}$.

---

8 A ghost spherical tree mapped into $\tilde{Y}$ with all neighboring components not mapped into $\tilde{Y}$ is not contained in $R$. 

Remark 2.33. The exceptional regularity implies regularity for the following maps obtained by the forgetful construction. Let $u : C \to \tilde{X}$ be a treed holomorphic disk of type $(\Gamma, R)$. Let $C'$ be the (possibly disconnected) treed disk obtained by removing all spherical components labelled by vertices in $R$, and $u' : C' \to \tilde{X}$ the induced map which has no nonconstant sphere components mapped into $\tilde{D}$. Let $\Gamma'$ be the domain type (possibly disconnected) corresponding to $C'$. Notice that $C'$ has new markings corresponding to nodes connecting $R$ and its complement. (See Figure 7.) By the locality property of the perturbation data (see Definition 2.12), $P_\Gamma$ induces a perturbation $P_{\Gamma'}$ and $u'$ is $P_{\Gamma'}$-holomorphic. Since nonconstant spheres in $\tilde{Y} \cong \mathbb{CP}^{n-1} \subset \mathcal{O}(-1)$ have obstructions to be deformed out of $\tilde{Y}$, the transversality at nodes connecting components in $R$ and not in $R$ implies the evaluation map at the new markings from the moduli space of $P_{\Gamma'}$-holomorphic treed disks is transversal to $(\tilde{Y})^l$ at the point represented by $u'$.

One can first construct perturbations needed for the Fukaya algebra of the exceptional torus. The construction can be done without referring to other Lagrangians away from the exceptional divisor and be continued with induction. The construction of $\text{Fuk}(\tilde{L})$ has been done in [CW].

Proposition 2.34. One can choose collection of exceptionally regular perturbations $P$ for the exceptional torus $\tilde{L}$ such that for all stable type $\Gamma$, $J_\Gamma$ is equal to $\tilde{J}_0$ inside $\tilde{U}$ and $H_\Gamma = 0$ inside $\tilde{U}$.

Proof. We need to show that the argument using the Sard–Smale theorem in the proof of Theorem 2.24 still works if $J_\Gamma = \tilde{J}_0$ within $\tilde{U}$. Indeed, for a disk vertex $v \in$
Vert(Γ), if \( u_v : S_v \to \tilde{X} \) is contained in \( \tilde{U}_0 \), then the regularity holds automatically; if \( u_v : S_v \to \tilde{X} \) is not contained in \( \tilde{U}_0 \), then the universal linearization is still surjective if one includes variations of the almost complex structure or the Hamiltonian. It is a similar situation for spheres. □

**Proposition 2.35.** There exists \( \epsilon_0 \) such that for all \( \epsilon \in (0, \epsilon_0] \cap \mathbb{Q} \), all smooth domain dependent almost complex structure \( J : D \to J_{\text{tame}}(\tilde{X}_c, \tilde{\omega}_c) \) with \( J|_{\tilde{U}} = \tilde{J}_0 \), all \( J \)-holomorphic disk \( u : D \to \tilde{X}_c \) bounding \( \tilde{L}_c \) with energy at most \( \epsilon \) are contained in \( \tilde{U} \), and hence are the standard Blaschke product of Maslov index two.

**Proof.** This is a consequence of the monotonicity property of pseudoholomorphic curves. Suppose the statement is not the case, then for all \( \epsilon \) there is a certain domain-dependent almost complex structure \( J \) and a holomorphic map \( u : D \to \tilde{X}_c \) with area at most \( \epsilon \) but not contained in the neighborhood \( \tilde{U} \). Let \( \tilde{U}'' \subset \tilde{U}' \subset \tilde{U} \) be smaller neighborhoods of the exceptional divisor. Then \( u(\partial D) \subset \tilde{U}'' \). Let \( S \subset \tilde{U} \) be the closure of \( u(D) \cap (\tilde{U}' \setminus \tilde{U}'') \), which is a compact minimal surface with boundary. Notice that the geometry between \( \tilde{U}'' \) and \( \tilde{U} \) is independent of \( \epsilon \). Then by the monotonicity property of minimal surfaces (see [Law74, 3.15]) there is a constant \( \delta_0 > 0 \) which is independent of \( \epsilon \) such that for all non-constant compact minimal surface \( \Sigma \) with nonempty boundary in the interior of \( \tilde{U}' \setminus \tilde{U}'' \) and \( \delta < \delta_0 \) there holds

\[
\begin{align*}
x \in \Sigma, \; \partial \Sigma \cap B(x, \delta) = \emptyset \implies \text{Area}(\Sigma) \geq c\delta^2.
\end{align*}
\]

Applying the monotonicity property to \( S \) one sees that the holomorphic map \( u \) has an area lower bound which is independent of \( \epsilon \), a contradiction. □

Next we introduce multivalued perturbations that are needed to establish a weak version of the divisor equation for the Fukaya algebra of the exceptional torus. Given a stable domain type \( \Gamma \), a **multivalued perturbation** is a formal linear combination of perturbations

\[
P_\Gamma = p_1 P_{\Gamma_1} + \ldots + p_k P_{\Gamma_k}
\]

for real numbers \( p_1, \ldots, p_k \geq 0 \). We can still define restrictions of \( P_\Gamma \) onto particular components or particular lower strata in the obvious way. So we can have the notion of coherent collection of multivalued perturbation data for all stable domain types. Given a multivalued perturbation \( P_\Gamma \) we write

\[
\mathcal{M}(x; P_\Gamma)_d = \bigcup_{i=1}^{k} \mathcal{M}_{\Gamma_i}(x; P_{\Gamma_i})_d.
\]

If regular, we consider it as weighted manifold with weights given by the coefficients \( p_1, \ldots, p_k \). We call each \( \mathcal{M}_{\Gamma_i}(x; P_{\Gamma_i})_d \) a branch of \( \mathcal{M}_{\Gamma}(x; P_\Gamma)_d \). A multivalued perturbation is (strongly) regular if its all branches are (strongly) regular.

**Remark 2.36.** In this paper, we will only consider multivalued perturbations \( P_\Gamma = (J_\Gamma, H_\Gamma, F_\Gamma, E_\Gamma) \) such that \( J_\Gamma, H_\Gamma, \) and \( E_\Gamma \) are all single valued.
Definition 2.37. A perturbation datum $P_\Gamma$ is semi-invariant if for any disk vertex $v \in \text{Vert}(\Gamma)$ with no interior leaves and exactly two incoming boundary leaves $e', e''$, the perturbation $P_\Gamma$ is equal to the perturbation pulled back by the maps $T_{e'} \to T_{e''}$ and vice versa.

Lemma 2.38. Suppose that the base almost complex structure $J$ has the following property: There exists $E > 0$ such that every non-constant holomorphic disk or sphere $u_0 : S_v \to X$ of energy at most $E$ is regular with Maslov index equal to two. Then there exist multivalued coherent perturbations so that, in addition to the sphere-free, local, and coherent conditions above, the perturbations are semi-invariant for maps of energy at most $E$.

Proof. We choose perturbations first on the "core" of the universal curve and extend in a symmetric way. We may assume we have chosen perturbations $P_{\Gamma_0}$ for types $\Gamma_0$ with no incoming edges so that the disks $u_0$ contributing to $m_{0,\beta}$ are regular, and with the additional property that the Morse function on the segments is unperturbed. Indeed, the condition on the index of spheres and disks implies that the only non-empty moduli spaces are those counting treed disks of index two, and the stability condition implies that at most one disk (constant or non-constant) appears in each configuration.

Given a treed disk $C$ with $d(\delta)$ incoming leaves with type $\Gamma$, let $C_0$ denote the disk obtained by forgetting the incoming leaves and stabilizing with type $\Gamma_0$. Thus $C$ is the union of a treed disk $C_0$ with no incoming leaves and a collection of treed disks $C_1, \ldots, C_k$ with no interior leaves (since each disk in the configuration $C_i, i > 0$ must have zero energy.) Perturbations $P_{\Gamma_0}^{\text{pre}}$ for types $\Gamma$ of disks with $d$ inputs may be chosen to agree with the perturbations $P_{\Gamma_0}$ on $C_0$ and perturbing only the Morse functions on the leaves. Indeed, if $C_i$ has $d_i$ incoming leaves then the configuration is defined by $d_i - 1$ matching conditions at the (constant) disks and an extra matching condition at the output where it connects to $C_0$; if the disk has $v - 1$ incoming edges then a perturbation on $v - 2$ of the corresponding leaves is enough to make the matching conditions transverse. By induction, all matching conditions are cut out transversely.

We average over the perturbations on the incoming leaves to obtain a multivalued perturbation that is semi-invariant: That is, let

$$P_\Gamma = (d!)^{-1} \sum_\sigma \sigma^* P_{\Gamma_0}^{\text{pre}}$$

where $\sigma \in \Sigma_d$ ranges over permutations that exchange edges attached to the same disk.

Finally we show that a version of Gromov compactness holds for the perturbations constructed as above, with complex structure standard in a neighborhood of the exceptional locus.
Proposition 2.39 (Improved compactness). For a coherent collection of exceptionally regular perturbations, sequential compactness for moduli spaces of essential types of index at most one (exactly the same statement as Theorem 2.27) holds.

Proof. We extend the proof of Theorem 2.27 to the case of sphere bubbling in the exceptional divisor, which is ruled out by an index argument. Consider an essential map type $\Gamma$ with index at most one and consider a sequence of treed holomorphic disks $u_i : C_i \to \tilde{X}$ representing a sequence of points in $\mathcal{M}_\Gamma(P_\Gamma)$. By the general compactness result, a subsequence converges to a limiting treed holomorphic disk $u : C \to \tilde{X}$. First one can as in the proof of Theorem 2.27 (also the argument of Cieliebak–Mohnke [CM07]) remove crowded ghost components, and so assume that the map type of $u$ is uncrowded. Second, if one can rule out the possibility of a non-constant sphere mapped into the exceptional divisor, then the theorem follows from the same argument of the proof of Theorem 2.27 as the exceptional regularity agrees with the regularity.

Suppose on the contrary that there are non-constant spherical components of $u$ mapped into the exceptional divisor. We would like to derive a contradiction using the two types of regularity conditions of Definition 2.32 and Remark 2.33. Let $\Pi$ be the domain type of $u$. Consider maximal sphere bubble trees in $\Pi$ whose energy is positive and whose images are mapped into $\tilde{Y}$. Let $R \subset \text{Vert}_{\text{sphere}}(\Pi)$ be the union of these bubble trees, and let $\Gamma'$ be the domain type obtained from $\Pi$ by forgetting $R$. Suppose $R$ has $m$ connected components $R_1, \ldots, R_m$ with positive degrees $d_1, \ldots, d_m$. Suppose $\Gamma'$ have $k + 1$ connected components $S_0, S_1, \ldots, S_k$ where $S_0$ has boundary and $S_1, \ldots, S_k$ are spherical trees. Suppose the homology class of $S_i$ is $\beta_i$ and $S_i$ has $l_i$ new markings. Suppose the component $S_0$ has map type $\Gamma_0$ and $l_0$ new markings. Then there holds the equality

$$l := l_0 + l_1 + \cdots + l_k = k + m.$$  

Furthermore, to simplify the computation of the indices, without loss of generality, assume all the spherical trees $R_i$ or $S_j$ have single vertices and the disk components have no bubbling of disks or breaking of edges, because otherwise the index will be dropped even lower. Then the index of $u$ as a $(\Pi, R)$-type map (see Definition 2.32) is (here $2n$ is the dimension of $X$ and $d = d_1 + \cdots + d_m$ is the total degree of spheres in the exceptional divisor)

$$\sum_{i=1}^{k} \left[ 2n + m(\beta_i) + 2l_i - 6 + \text{ind}(\Gamma_0) + 2l_0 + (2n - 2)m + 2nd + 2l - 6m - \right] \frac{2nl}{\text{matching condition}}$$

$$= (2n - 6)k - (2n - 4)l + (2n - 8)m + \text{ind}(\Gamma) + 2d$$

$$= 2d - 2k - 4m + \text{ind}(\Gamma) \geq 0$$

Hence

$$d \geq k + 2m.$$  

(29)
On the other hand, consider the induced object of type \( \Gamma' \). The index is
\[
\sum_{i=1}^{k} \left( 2n + m(\beta_i) + 2l_i - 6 + \text{ind}(\Gamma_0) + 2l_0 - \text{index of } S_i \right) \sum_{i=1}^{2n+2} \left( \text{index of } S_0 \right) - \text{constraints at new markings} = (2n - 6)k + \text{ind}(\Gamma) - 2(n - 1)d \geq 0.
\]
Hence
\[
d \leq k - \frac{2k}{n-1}.
\]
This contradicts (29). Hence in the limit there cannot be any non-constant spherical components mapped into the exceptional divisor. \( \square \)

3. Bulk-deformed Fukaya categories

In this section we introduce bulk-deformed Fukaya categories associated to a given strongly rational finite collection of Lagrangian immersions. Given the regularization of the moduli spaces in the previous section, the material in this section is fairly straightforward adaption of that in Fukaya-Oh-Ohta-Ono [FOOO09] and Akaho-Joyce [AJ10].

3.1. Composition maps. In this section we apply the transversality results of the previous section to construct immersed Floer theory. In the Morse model the generators of the immersed Floer cochains are critical points of a Morse function on the Lagrangian together with ordered self-intersection points of the immersion. Recall for each pair \( (\phi_-, \phi_+) \in L^2 \) there is a smooth manifold (of varying dimensions)
\[
L_{\phi_-, \phi_+} := (\phi_+ \times \phi_+)^{-1}(\Delta).
\]
We have chosen a Morse function
\[
F : \bigcup_{(\phi_-, \phi_+) \in L^2} L_{\phi_-, \phi_+} \to \mathbb{R}.
\]
For each pair \( (\phi_-, \phi_+) \), expand the set of critical points as
\[
\mathcal{I}(\phi_-, \phi_+) := \left\{ \begin{array}{ll}
c\text{rit}F_{\phi_-, \phi_+}, & \phi_- \neq \phi_+, \\
\text{crit}F_{\phi_-, \phi_+} \cup \mathcal{I}_{\text{hu}}(\phi), & \phi_- = \phi_+ = \phi \end{array} \right.
\]
where
\[
\mathcal{I}_{\text{hu}}(\phi) := \bigcup_{c \in \pi_0(L_\phi)} \{ 1_{\phi, c}^\ast, 1_{\phi, c}^\circ \}
\]
The degrees of the extra generators are assigned as
\[
|1_{\phi, c}^\ast| = -1, \quad |1_{\phi, c}^\circ| = 0.
\]
In order to obtain graded Floer cohomology groups a grading on the set of generators is defined as follows. Let \( N \in \mathbb{Z} \) be an even integer and \( \text{Lag}^N(X) \to \text{Lag}(X) \) an \( N \)-fold Maslov cover of the bundle of Lagrangian subspaces as in Seidel [Sei00];
we always assume that the induced 2-fold cover \( \text{Lag}^2(X) \to \text{Lag}(X) \) is the bundle of oriented Lagrangian subspaces. A grading of \( \phi : L \to X \) is a lift
\[
\phi^N : L \to \text{Lag}^N(X)
\]
of the natural map
\[
L \to \text{Lag}(X), \quad x \mapsto \text{Im}(D_x\phi).
\]
Given such a grading, there is a natural \( \mathbb{Z}_N \)-valued map
\[
I(\phi_-, \phi_+) \to \mathbb{Z}_N, \quad x \mapsto |x|
\]
obtained by assigning to any critical point the index mod \( N \) of the path from \( \phi^N_-(T_{x_-}(L_-)) \) to \( \phi^N_+(T_{x_+}(L_+)) \). Denote by \( I^k(\phi) \) the subset of \( x \in I(\phi) \) with \( |x| = k \). We suppose that each brane \( \phi : L \to X \) is equipped with a flat \( \Lambda \times \) -line bundle, and denote for any holomorphic treed disk \( u : C \to X \) with boundary in \( \phi(L) \) the holonomy of the local systems around the boundary of the disks components in \( C \) by \( y(u) \in \Lambda^x \).

The moduli space of holomorphic disks is non-compact, and to remedy this the structure maps of the Fukaya algebra are defined over Novikov rings in a formal variable. The Floer cochain space is the free module over generators given by Morse critical points, self-intersection points, and the two additional generators from (30) necessary to achieve strict units. Let
\[
CF^\bullet(\phi_-, \phi_+) = \bigoplus_{x \in I(\phi_-, \phi_+)} \Lambda x.
\]
The space of Floer cochains is naturally \( \mathbb{Z}_N \)-graded by
\[
CF^\bullet(\phi_-, \phi_+) = \bigoplus_{k \in \mathbb{Z}_N} CF^k(\phi_-, \phi_+), \quad CF^k(\phi_-, \phi_+) = \bigoplus_{x \in I^k(\phi_-, \phi_+)} \Lambda x.
\]
Put
\[
1_\phi := \sum_{c \in \pi_0(L_{\phi}, \phi)} 1_{\phi,c} \in CF(\phi_-, \phi_+).
\]
The \( q \)-valuation on \( \Lambda \) extends naturally to \( CF(\phi_-, \phi_+) \):
\[
\text{val}_q : CF(\phi_-, \phi_+) - \{0\} \to \mathbb{R}, \quad \sum_{x \in I(\phi_-, \phi_+)} c(x)x \mapsto \min(\text{val}_q(c(x))).
\]

**Definition 3.1.** Fix a coherent collection of strongly regular perturbation data
\[
P = (P_\Gamma)_\Gamma
\]
(whose existence is provided by Theorem 2.24). Define higher composition maps
\[
m_d : CF(\phi_0, \phi_1) \otimes \ldots \otimes CF(\phi_{d-1}, \phi_d) \to CF(\phi_0, \phi_d)[2 - d]
\]
on generators by the weighted count of treed disks
\[
m_d(x_1, \ldots, x_d) = \sum_{x_0, u \in M(\phi)_{0}} (-1)^\nu w(u)x_0
\]
where the weightings
\begin{equation}
    w(u) := c(u, b)p(u)y(u)q(u)A(u)o(u)d(u)^{-1}
\end{equation}
are defined as follows:

- the denominator
\begin{equation}
    d(u) = \prod_\partial d(\partial, \partial)! 
\end{equation}
is the product of factorials corresponding to the number of reorderings of
\(d(\partial, \partial)\) interior leaves \(e \in \text{Edge}_e(\Gamma)\) labelled with the deformation \(\partial(e) = \partial\);
- the coefficient \(c(u, b)\) is a product of coefficients \(c_i\) of the bulk deformation
as defined in (9), with product taken over interior leaves mapping to \(b\),
- the coefficient \(p(u)\) is the coefficient \(p_i\) of the multivalued perturbation \(P_{1}\) of
(28) evaluated at the branch containing \(u\), and
- the exponent \(A(u)\) is the symplectic area of the map \(u\).
- the sign \(o(u)\) arises from the choice of coherent orientations and the overall
sign \(\heartsuit\) is given by
\begin{equation}
    \heartsuit = \sum_{i=1}^{d} i|x_i|.
\end{equation}

**Theorem 3.2.** For any strongly regular coherent perturbation system \(P = (P_{1})\) the
maps \((m_d)_{d \geq 0}\) satisfy the axioms of a (possibly curved) \(A_{\infty}\) category \(\mathrm{Fuk}_{b}^\sim(X, b)\) with
strict units \(1_{\phi} \in CF(\phi, \phi; b)\).

**Proof.** We must show that the composition maps \(m_d, d \geq 0\) satisfy the \(A_{\infty}\)-associativity equations (7). Up to sign the relation (7) follows from the description
of the boundary in Lemma 2.27 of the one-dimensional components, while the sign
computation in [CW] is independent of whether the Lagrangian is immersed or em-
bedded. The strict unit axiom follows in the same way as in [CW], by noting that
by definition for any edge \(e \in \text{Edge}_e(\Gamma)\) the perturbation data is pulled back under
the morphism of universal moduli spaces forgetting \(e\) and stabilizing (whenever such
a map exists). \(\square\)

**Remark 3.3.** The \(A_{\infty}\) homotopy type of \(\mathrm{Fuk}_{b}^\sim(X, b)\) (as a curved \(A_{\infty}\) algebra with
curvature with positive \(q\)-valuation over the Novikov ring \(\Lambda_{\geq 0}\) should be independent
of the choice of almost complex structures, perturbations, stabilizing divisors,
and depend only on the isotopy class of bulk deformation. Let \(J_{t, x}\) be an isotopy of
almost complex structures, \(D_{t}(\phi)\) an isotopy of Donaldson hypersurfaces, and let \(b_t\)
be an isotopy of pseudocycles \(b_0\) to \(b_1\). Requiring that the markings map to \(b_t\), the
nodal disks are \(J_{t, x}\) holomorphic and adapted to \(D_t(\phi)\) on components at distance
\(1/(1 - t) - 1/t\) produces a moduli space \(\tilde{M}(\phi, D)\) with a one-dimensional components
whose ends include \(\tilde{M}(\phi, D_0, b_0)\) and \(\tilde{M}(\phi, D_1, b_1)\) and also configurations in which
a boundary edge has achieved infinite length \( \ell(e) \). The operator defined by

\[
\phi_d(x_1, \ldots, x_d) = \sum_{x_0, u \in \mathcal{M}(\phi, D, x_0)} (-1)^{\mathcal{O}} w(u)x_0
\]

with weightings from (31) is then a homotopy equivalence between \( \text{Fuk}_\infty^\sim (X, b_0) \) and \( \text{Fuk}_\infty^\sim (X, b_1) \). We leave for future work (by ourselves or others) the question of invariance under Hamiltonian isotopy and the relation to the Fukaya categories defined by other regularization schemes, and dependence only on the cobordism class of pseudocycle \( b \), and so the homology class \([b]\).

**Remark 3.4 (Floer cohomology).** The Floer cohomology is defined for projective solutions to the Maurer-Cartan equation. The element \( m_0(1) \in CF(\phi, \phi) \) is the curvature of the Fukaya algebra \( CF(\phi, \phi) \) and has positive \( q \)-valuation

\[
\text{val}_q(m_0(1)) \in \mathbb{R}_{>0}.
\]

The Fukaya algebra \( CF(\phi, \phi) \) is flat if \( m_0(1) \) vanishes and projectively flat if \( m_0(1) \) is a multiple of the identity \( 1_\phi \). Consider the sub-space of \( CF(\phi, \phi) \) consisting of elements with positive \( q \)-valuation with notation from (4):

\[
CF(\phi, \phi)_+ = \bigoplus_{x \in \mathcal{I}(\phi, \phi)} \Lambda_{>0} x.
\]

Define the **Maurer-Cartan map**

\[
m : CF^{\text{odd}}(\phi, \phi)_+ \to CF(\phi, \phi), \quad b \mapsto m_0(1) + m_1(b) + m_2(b, b) + \ldots
\]

Let \( MC(\phi) \) denote the space of weak solutions to the Maurer-Cartan space

\[
MC(\phi) = \{ b \in CF^{\text{odd}}(\phi, \phi) \mid \mu(b) = W(b)1_\phi, \ W(b) \in \Lambda \}.
\]

The value \( W(b) \) of \( \mu(b) \) for \( b \in MC(\phi) \) defines the disk potential

\[
W : MC(\phi) \to \Lambda.
\]

For \( b \in MC(\phi) \) define \( HF^*((\phi, b), (\phi, b)) \) as the homology of \( m^b_1 \) of (6).

We wish to choose perturbations so that the composition maps satisfy a version of the **divisor equation** as in Cho-Oh [CO06, Section 6]. Let

\[
\underline{x} = \sum_{i=1}^{k} n_i x_i \in CF(\phi, \phi), n_i \in \mathbb{Q}
\]

be a Morse cycle (that is, \( m_1(\underline{x})_{q=0} = 0 \)) of classical degree 1 and \( \underline{x} \in H^1(L) \) its homology class. For any homology class \( \beta \in H_2(\phi) \) denote by \( \langle \underline{x}, \partial \beta \rangle \in \mathbb{Q} \) the pairing with the class \( \partial \beta \in H_1(L) \) and \( m_{d, \beta} \) denote the contribution to \( m_d \) from disks of class \( \beta \). For \( \underline{x} = (x_1, \ldots, x_d) \) a collection of cocycles and \( (x_1, \ldots, x_d) \) with each \( x_i \) appearing in \( x_i \) with coefficient \( n(x_i) \) write

\[
n(\underline{x}) = n(x_1)n(x_2)\ldots n(x_d)
\]
for the product of coefficients and
\[ M_{d(\phi_d)}(\phi, D, \mathbf{x})_0 = \bigcup_{\mathbf{x}} n(\mathbf{x}) M_{d(\phi_d)}(\phi, D, \mathbf{x})_0 \]
the weighted union over components of \( \mathbf{x}_i \) for each \( i = 0, \ldots, d \).

**Lemma 3.5.** Suppose that the base almost complex structure \( J \) has the following property: There exists \( E > 0 \) such that every non-constant holomorphic disk or sphere \( u_0 : S_\nu \to X \) of energy at most \( E \) is regular with index two. Then there exists multivalued coherent perturbations so that, in addition to the sphere-free, local, and coherent conditions above, for any cocycles \( \mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_d) \) of degree one and cochain \( \mathbf{x}_0 \) the union of forgetful maps on moduli spaces of energy at most \( E \)
\[ f_{d(\phi)} : \bigcup_{\sigma \in S_{d(\phi)}} M^{<E}_{d(\phi)}(\phi, D, \sigma(\mathbf{x}))_0 \to M^{<E}_{0}(\phi, D, \mathbf{x}_0)_0 \]
where \( \sigma(\mathbf{x}) \) denotes the reordering of \( \mathbf{x} \), is a finite-to-one map with fiber over \( u_0 \in M_0(\phi, D, \mathbf{x}_0)_0 \) having signed order the intersection number
\[ \# f_{d(\phi)}^{-1}(u_0) = \prod_{i=1}^{d(\phi)} (|\partial u_0|, [\mathbf{x}_i]). \]

**Proof.** We choose the perturbations to be semi-invariant in the sense of Lemma 2.38. Given a configuration \( u \in M_{d(\phi)}(\phi, D, \sigma(\mathbf{x}))_0 \) in with \( e', e'' \) attached to a constant disk, say with labels \( x_{i'}, x_{i''} \), we obtain a new configuration \( \sigma(u) \in M_{d(\phi)}(\phi, D, \sigma(\mathbf{x}))_0 \) with the labels \( x_{i'}, x_{i''} \) reversed. In the case that \( x_{i'}, x_{i''} \) are degree one, the definition of signs implies that the orientation of \( \sigma(u) \) is opposite that of \( u \). Thus the contributions of configurations \( u \) where some \( C_i \) has more than one incoming leaf (or equivalently, has non-empty surface part \( S_i \subset C_i \)) vanishes.

The configurations contribution to (35) are those where each incoming leaf in \( C \) is attached to a disk or segment in \( C_0 \). That is, \( C_0 \) is obtained from \( C \) by forgetting all incoming leaves, with no collapse of disk components. Since the number of intersections of the boundary of each disk \( u_0 \) of class \( \beta \) with \( \mathbf{x}_i \) is \( \langle \mathbf{x}_i, \partial \beta \rangle \), for each disk \( u_0 \) contributing to \( m_{0, \beta}(1) \) there are \( \prod_{i=1}^{d(\phi)} |\mathbf{x}_i|, \partial \beta \) points in the fiber, with order accounted for in the sum over orderings in (34).

**Corollary 3.6.** If perturbations are chosen as in Lemma 3.5 then the following (restricted) divisor equation holds for any cocycles \( \mathbf{x}_1, \ldots, \mathbf{x}_d \) of degree one for any homology class \( \beta \) with area \( \omega(\beta) < E \):
\[ \sum_{\sigma \in \Sigma_d} m_{d, \beta}(\mathbf{x}_{\sigma(1)}, \ldots, \mathbf{x}_{\sigma(d)}) = \prod_{i=1}^{d} (|\mathbf{x}_i|, \partial \beta) m_{0, \beta}(1) \]
where the sum is over permutations \( \sigma \in \Sigma_d \) of \( \{1, \ldots, d\} \). In particular, for any Morse cocycles \( \mathbf{x}_1, \mathbf{x}_2 \) of degree one and class \( \beta \in H_2(\phi) \) with \( \omega(\beta) < E \) we have
\[ m_{2, \beta}(\mathbf{x}_1, \mathbf{x}_2) + m_{2, \beta}(\mathbf{x}_2, \mathbf{x}_1) = \langle \mathbf{x}_1, \partial \beta \rangle \langle \mathbf{x}_2, \partial \beta \rangle m_{0, \beta}(1). \]
3.2. **Spectral decomposition.** Given a curved $A_\infty$ category, flat $A_\infty$ categories are obtained by restricting to particular values of the curvature.

**Definition 3.7.** For any $w \in \Lambda$ let $\text{Fuk}_L^w(X, b, w)$ denote the category whose

(a) objects are pairs $(\phi, b)$ consisting of a Lagrangian brane $(\phi : L \to X) \in L$ and an element $b \in MC(\phi)$ with $W(b) = w$,

(b) morphisms are Floer cochains, that is,

$$\text{Hom}((\phi_-, b_-), (\phi_+, b_+)) = CF(\phi_-, \phi_+).$$

(c) Higher composition maps for $d \geq 1$ are deformed composition maps for $b_i \in MC(\phi_i), i = 0, \ldots, d$,

$$m_d(x_1, \ldots, x_d) = \sum_{k_0, \ldots, k_d \geq 0} m_{d+k_0+\ldots+k_d}(b_0, \ldots, b_0, x_1, b_1, \ldots, b_1, \ldots, x_d, b_d, \ldots, b_d),$$

and the curvature maps $m_0(1) \in \text{Hom}(\phi, \phi)$ vanish.

**Proposition 3.8.** For any $w \in \Lambda$, the category $\text{Fuk}_L^w(X, b, w)$ is a flat $A_\infty$ category.

**Proof.** Flatness $m_0(1) = 0$ holds by definition. The $A_\infty$ relation

$$(37) \quad 0 = \sum_i m_{d(\phi)-i+1}(b_0, \ldots, b_i, x_{i_1}, \ldots, b_1, \ldots, b_1, \ldots, b_{j+i}, x_{j+i+1}, \ldots, x_{d(\phi)}, b_{d(\phi)}),$$

follows from the $A_\infty$ relation for $\text{Fuk}_L^w(X, b, w)$, the strict identity relation, and the inclusion

$$\sum_{i \geq 0} m_i(b_j, \ldots, b_j) \in \text{span}(1_{\phi_j}) \quad \forall j = 0, \ldots, d(\phi).$$

□

**Definition 3.9.** Define a flat $A_\infty$ category

$$\text{Fuk}_L(X, b) := \bigcup_{w \in \Lambda} \text{Fuk}_L^w(X, b; w)$$

whose set of objects is the disjoint union of all objects in the eigen-subcategories, and the space of morphisms between objects in different eigen-subcategories is the zero vector space.
3.3. Hochschild (co)homology. Hochschild homology of a category is the homology of a contraction operator on the space of all composable sequences of morphisms. In the case of curved $A_\infty$ categories, there seems to be no good definition at the moment, although we understand from Abouzaid that he and Varolgunes and Groman are developing such a theory. For our purposes it suffices to use the Hochschild theory for flat categories in combination with a spectral decomposition. We first recall the definition from, for example, [Sei08a, Section 2].

**Definition 3.10.** Let $\mathcal{F}$ be a flat $A_\infty$ category.

(a) As in Seidel [Sei08a, Section 2] an $A_\infty$ bimodule $\mathcal{M}$ over $(\mathcal{F}, \mathcal{F})$ consists of

(i) a map assigning to any pair of objects $\phi_-, \phi_+$ a vector space $\mathcal{M}(\phi_-, \phi_+)$ and

(ii) multiplication maps for integers $d_+, d_- \geq 0$ and objects $\phi_0, \ldots, \phi_d, \pm \in \text{Ob}(\mathcal{F})$

\begin{align*}
&\quad (38) \quad m_{d_-, d_+} : \text{Hom}(\phi_{d_-}, \phi_{d_- - 1, -}) \otimes \cdots \otimes \text{Hom}(\phi_{1, -}, \phi_{0, -}) \otimes \mathcal{M}(\phi_{0, -}, \phi_{0, +}) \otimes \\
&\quad \text{Hom}(\phi_{0, +}, \phi_{1, +}) \otimes \cdots \otimes \text{Hom}(\phi_{d_+, - 1, +}, \phi_{d_+, +}) \rightarrow \mathcal{M}(\phi_{d_-, -}, \phi_{d_+, +})
\end{align*}

satisfying the $A_\infty$ bimodule axiom, see [Sei08a, Section 2].

(b) Given an $A_\infty$ bimodule $\mathcal{M}$ over $(\mathcal{F}, \mathcal{F})$, the space of Hochschild chains with values in $\mathcal{M}$ is the direct sum

\begin{align*}
&\quad (39) \quad CC_\bullet(\mathcal{F}, \mathcal{M}) = \bigoplus_{\phi_0, \ldots, \phi_d \in \text{Ob}(\mathcal{F})} \text{Hom}(\phi_0, \phi_1) \otimes \text{Hom}(\phi_1, \phi_2) \otimes \\
&\quad \cdots \otimes \text{Hom}(\phi_{i-1}, \phi_i) \otimes \mathcal{M}(\phi_i, \phi_{i+1}) \otimes \text{Hom}(\phi_{i+1}, \phi_{i+2}) \otimes \\
&\quad \cdots \otimes \text{Hom}(\phi_{d-1}, \phi_d) \otimes \text{Hom}(\phi_d, \phi_0).
\end{align*}

(c) In particular $\mathcal{F}$ is itself a bimodule over $(\mathcal{F}, \mathcal{F})$, called the diagonal bimodule. The boundary operator in the case $\mathcal{M} = \mathcal{F}$ is defined by summing over all possible contractions,

\begin{align*}
&\quad (40) \quad \delta : a_1 \otimes \cdots \otimes a_d \mapsto \sum_{i+j \leq d} (-1)^{\xi} a_i \otimes \cdots \otimes a_{i+j} \otimes m_{d-j-1}(a_{i+j+1} \otimes \cdots \otimes a_{i-1}) \\
&\quad + \sum_{i+j \leq d} (-1)^{\xi} a_1 \otimes \cdots \otimes a_{i-1} \otimes m_{j+1}(a_i \otimes \cdots \otimes a_{j+i}) \otimes a_{j+i+1} \otimes \cdots \otimes a_d
\end{align*}

where

\begin{align*}
&\quad \xi = \xi_{i-1} \cdot (1 + \xi_d) + \xi_{d-1} + 1.
\end{align*}

For $\mathcal{F}$ flat as above denote by

\begin{align*}
&\quad HH_\bullet(\mathcal{F}, \mathcal{F}) = \frac{\ker(\delta)}{\text{im}(\delta)}
\end{align*}

the homology of $\delta$.

(d) For a curved $A_\infty$ category $\mathcal{F}^\sim$ let

\begin{align*}
&\quad \mathcal{F} := \bigsqcup_{w \in \Lambda} \mathcal{F}_w^\sim
\end{align*}
the flat $A_\infty$ category obtained via the spectral decomposition. Denote by

$$HH_\bullet(F^\sim, F^\sim) = HH_\bullet(F, F)$$

the direct sum over possible values $w$ of the potential of the Hochschild homologies of the flat categories obtained by fixing the value of the curvature:

$$HH_\bullet(F^\sim, F^\sim) := HH_\bullet(F, F) := \bigoplus_{w \in \Lambda} HH_\bullet(F_w^\sim, F_w^\sim).$$

The Hochschild cohomology is defined for a flat $A_\infty$ category as follows. A Hochschild cochain $\tau$ on a $A_\infty$ category $F$ valued in $F$ is a collection

$$\tau := (\tau_{\psi,d})_{\psi \in \text{Obj}(F), d \geq 0}$$

where $\tau_{\psi,d}$ is a linear map

$$\tau_{\psi,d} : \bigoplus_{\phi_1, \ldots, \phi_d} \text{Hom}(\psi, \phi_1) \otimes \ldots \otimes \text{Hom}(\phi_{d-1}, \phi_d) \to \text{Hom}(\psi, \phi_d).$$

The space of Hochschild cochains valued in $F$, denoted by $CC^\bullet(F, F)$, has the structure of an $A_\infty$ algebra by viewing Hochschild cochains as natural transformations of the identity functor on $F$. We recall from [Sei08b, 1d] that the space of functors between two $A_\infty$ categories $F_0, F_1$, denoted by $\text{Func}(F_0, F_1)$, is itself an $A_\infty$ category, whose objects are functors, and morphisms are natural transformations. In particular $CC^\bullet(F, F)$ is a $A_\infty$ algebra whose composition maps are

\begin{align*}
(41) \quad (m_1^{CC})(a_1, \ldots, a_d) &= \sum_{i,j} (-1)^{\tilde{\tau}} m^{d-j+1}_F(a_1, \ldots, a_i, a_{i+1}, \\
&\ldots, a_{i+j}, a_{i+j+1}, \ldots, a_d) \quad \text{where} \quad \tilde{\tau} = (|\tau| - 1)(|a_1| + \ldots + |a_{i_1+\ldots+i_{k-1}}| - i_1 - \ldots - i_{k-1}), \quad \clubsuit := i + \sum_{j=1}^i |a_j| + |\tau| - 1,
\end{align*}

and for $e \geq 2$

\begin{align*}
(42) \quad (m_e^{CC}(\tau_1, \ldots, \tau_e))^{d_j}(a_1, \ldots, a_d) &= \sum_{i_1, \ldots, i_e, j_1, \ldots, j_e} (-1)^{\circ} m^{d-\sum j_k}_F(a_1, \ldots, a_{i_1}, \tau^{j_1}_{i_1+1}, \\
&\ldots, a_{i_1+j_1}, a_{i_1+j_1+1}, \ldots, a_{i_e}, \tau^{j_e}_{i_e+j_e+1}, \ldots, a_d), \quad \circ := e \sum_{j=1}^e \sum_{k=1}^{i_j} (|\tau_j| - 1)(|a_k| - 1).
\end{align*}
Assuming $\tau_{\psi,d} = 0$ (that is, $\tau_{\psi,d}$ is itself flat) the boundary operator $m^1_{CC^*}$ squares to zero, and we denote by $HH^\bullet(\mathcal{F}, \mathcal{F})$ the Hochschild cohomology of $\mathcal{F}$ valued in $\mathcal{F}$.

The Hochschild cohomology is equipped with a natural identity. Suppose the $A_\infty$ category is strictly unital. Consider the cochain $1_F \in CC_0(\mathcal{F}, \mathcal{F})$ defined by

$$1_F, d(x_1 \otimes \cdots \otimes x_d) = \begin{cases} 0, & d > 0, \\ 1_\psi, & d = 0. \end{cases}$$

Then $m^1_{CC}(1_F) = 0$ and the cohomology class of $1_F$ is the identity of the Hochschild cohomology ring.

The Hochschild cohomology restricts to the cohomology of objects. Let $\psi$ be an object of the (flat) category $\mathcal{F}$. Restricting cochains to $\text{Hom}(\psi, \psi)$ induces a map $HH^\bullet(\mathcal{F}, \mathcal{F}) \rightarrow H^\bullet(\psi)$.

3.4. The case of a blowup. We compute the Hochschild cohomology of exceptional Lagrangian branes in blowups, based on the identification of the Floer cohomology groups as Clifford algebras. The local computation relies on the classification of disks in toric varieties with boundary on a toric moment fiber in Cho-Oh [CO06]. Let $\tilde{X}$ be the blowup of $X$ at $p$ and $\phi : L \rightarrow \tilde{X}$ the embedding of a toric moment fiber in the local model in the previous paragraph. By the main result of [CW] the curvature of the Fukaya algebra $\text{Fuk}(\phi)$ is given in terms of the local system

$$y = (y_1, \ldots, y_n) \in \mathcal{R}(\phi) \cong (\Lambda^\times)^n$$

by

$$m_0(1) = q^\phi(y_1 + \ldots + y_n + y_1y_2\ldots y_n + \text{h.o.t.})1_\phi := W_\phi(y_1, \ldots, y_n)1_\phi$$

where h.o.t. denotes higher order terms measured by $q$-valuation. By [CW, Proposition 4.34], for each critical point of $W_\phi : \mathcal{R}(\phi) \rightarrow \Lambda_{\geq 0}$ there exists a Maurer-Cartan element

$$b = b(y) \in MC(\phi) = MC(\phi(y))$$

so that

$$HF((\phi, b, y), (\phi, b, y)) \cong H(L, \Lambda) \neq \{0\}.$$ 

Each local system

$$y(k) = (y_{(k),1}, \ldots, y_{(k),n}) \in \mathcal{R}(\phi)$$

for which the Floer cohomology $HF(\phi, \phi)$ is unobstructed and non-trivial is a solution to

$$dq^\phi(y_{(k),1} + \ldots + y_{(k),n} + y_{(k),1}y_{(k),2}\ldots y_{(k),n}) = 0.$$ 

We solve

$$y_{(k),i} + y_{(k),1}y_{(k),2}\ldots y_{(k),n} = 0, \quad \forall i = 1, \ldots, n$$

Hence

$$y_{(k),1} = \ldots = y_{(k),n}, \quad y_{(k),1}^{n-1} = 1.$$
This implies

\[ y_{(k),i} \in \{1, \mathbb{N}, \mathbb{N}^2, \ldots, \mathbb{N}^{n-2}\} \]

where \( \mathbb{N} \) is an \( n - 1 \)-st root of unity:

\[ \mathbb{N} = 1, \exp(2\pi i/(n - 1)), \ldots, \exp((n - 2)2\pi i/(n - 1)). \]

Similar to the computation in Cho-Oh of the ring structure on toric moment fibers [CO06] we have the following. Let

\[ W_0 : \mathcal{R}(\phi) \to \Lambda^\times, \quad (y_1, \ldots, y_n) \mapsto y_1 + \ldots + y_n + q'y_1 \ldots y_n \]

be the leading order part of the disk potential \( W = W(b(y), y) \) in [CW, Section 4], defined by counting disks of smallest possible non-zero energy bounding \( \phi \). Denote by \( \phi_{(k)} \) the immersion \( \phi \) equipped with the local system \( y_{(k)} \) and weakly bounding cochain \( b_{(k)} \).

**Proposition 3.11.** Each \( HF^\bullet(\phi_{(k)}, \phi_{(k)}) \) is isomorphic to a Clifford algebra corresponding to a non-degenerate quadratic form whose leading order is the Hessian

\[ \partial^2 W : H^1(L, \Lambda) \times H^1(L, \Lambda) \to \Lambda. \]

**Proof.** Let \( \phi \) be one of the branes \( \phi_{(k)} \). Let \( E > 0 \) be a constant greater than the area \( A(u) \) of the smallest-area non-constant holomorphic disks but smaller than the area of any other tree disk of area greater than \( A(u) \). By the main result of [CW] the curvature \( m_0(1) \) of the Fukaya algebra has leading order terms

\[ m_0(1)^{<E} = \sum_{u \in \mathcal{M}_d(\phi), A(u) = E} (-1)^{\phi(w(u))} w(u)x_0 \]

with weightings from (31). Consider \( m_0(1) \) a function of

\[ y \in \mathcal{R}(\phi) \cong \text{Hom}(H_1(L), \Lambda^\times) \]

and write

\[ y = \exp(x), \quad x \in H_1(L, \Lambda). \]

The holomorphic disks \( u : \mathbb{D} \to X \) of lowest non-zero area \( A(u) > 0 \) are given by Blaschke products (27) and by transversality persist under perturbation. Let \( x_1, x_2 \in \mathcal{I}(\phi) \) represent codimension one cycles \( W_{x_1} \cong (S^1)^{n-1}, W_{x_2} \cong (S^1)^{n-1} \) in \( L \). In the neck-stretching limit for the blow-up constructed in [CW], all of the non-constant holomorphic disks and spheres in a small neighborhood of the exceptional locus have positive index. By (36), we may assume that the perturbations are chosen so that (c.f. Cho-Oh [CO06, Section 5]) for \( \epsilon \) small the terms \( m_2^{<E}, m_0^{<E} \) counting treed disks of energy less than \( E \) are related by

\[ m_2^{<E}(x_1, x_2) + m_2^{<E}(x_2, x_1) = \sum_{\beta \in H_2(\phi)} (x_1, \partial \beta)(x_2, \partial \beta)m_0^{<E}(1) \]

\[ = \partial_{x_1} \partial_{x_2} m_0^{<E}(1). \]

Here we are viewing \( m_0(1) \) as a function of the local system \( y \) and taking the second derivative with respect to \( y \). By the computation of the potential \( m_0(1) \) in (43)
\[ m_0 < E (1) \text{ has a non-degenerate critical point at each } y_{(k)}. \] It follows that \( HF^\bullet (\phi, \phi) \) is a deformation of the Clifford algebra of a non-degenerate quadratic form, and since such Clifford algebras are rigid (by Corollary 3.12, \( HH^2 \) vanishes and so there are no deformations) \( HF^\bullet (\phi, \phi) \) is itself a Clifford algebra. \( \square \)

**Corollary 3.12.** The Hochschild (co)homology
\[
HH^\bullet (\phi(k), \phi(k)) \cong HH^{\dim(X) - \bullet} (\phi(k), \phi(k))
\]
of each brane \( \phi(k) \) is one-dimensional and generated by a top-degree class in \( HF(\phi(k), \phi(k)) \).

**Proof.** By Proposition 3.11 \( HF(\phi(k), \phi(k)) \) is a Clifford algebra. The graded Hochschild homology of Clifford algebras is computed in Kassel [Kas86, Proposition 1]. The graded Hochschild homology \( HH^\bullet (A) \) of a Clifford algebra \( A \cong k^n \) over a field \( k \) is one-dimensional and generated by the class in \( HH_0 (A) \) corresponding to the class in \( A \) corresponding to the product of generators \( e_1 \ldots e_n \). To relate this to the Hochschild homology of the \( A_\infty \) category, note that the spectral sequence for the length filtration on the space of Hochschild chains \( CC^\bullet (CF(\phi(k), \phi(k))) \) has \( E_1 \) term the Hochschild complex of the algebra \( HF(\phi(k), \phi(k)) \). Since Clifford algebras have no deformations, being semisimple, we may ignore contributions to the chain complex from higher \( A_\infty \) products, see [ST01, Proposition 4.7]. Hence the spectral sequence collapses at the \( E_2 \) page, which is the Hochschild homology of \( HF(\phi(k), \phi(k)) \), see [GJ90, Proof of 5.3]. \( \square \)

4. **Open-closed and closed-open maps**

In this section we define open-closed maps from Hochschild homology to quantum cohomology and closed-open maps from quantum cohomology to the Hochschild cohomology as in, for example, Abouzaid [Abo10], Ganatra [Gan12], [Gan], and Ritter-Smith [RS17].

**4.1. Quantum cohomology.** To incorporate the construction of the quantum cohomology with the Fukaya category, we extend the terminology of trees and treed disks used in previous discussions.

**Definition 4.1.** (a) A **domain type of treed spheres** consists of a rooted tree \( \Gamma \) with empty base and a decomposition
\[
\text{Leaf}(\Gamma) = \text{Leaf}_{\text{long}}(\Gamma) \sqcup \text{Leaf}_{\text{short}}(\Gamma)
\]
of the set of leaves into subsets of **long** and **short** leaves (which eventually will correspond to leaves that map to gradient trajectories in \( X \), or leaves that map to the stabilizing divisor or bulk deformation.) A domain type of treed spheres is **stable** if the valence of each vertex is at least three.\(^9\)

\(^9\)To define the quantum multiplication we do not need to allow finite edges to acquire length. However to prove the associativity of the quantum multiplication finite edges with positive lengths are necessary.
(b) A treed sphere $C = S \cup T$ of $\Gamma$ is obtained from a nodal sphere whose combinatorial type is described by $\Gamma$ by attaching an interval $(-\infty, 0]$ for each long leaf and an interval $[0, +\infty)$ for the output. The surface part $S$ is the union of spherical components labelled by vertices $v \in \text{Vert}(\Gamma)$ while the tree part $T$ is the union of these semi-infinite intervals.

(c) Given a stable domain type of treed spheres $\Gamma$, the universal curve $\mathcal{U}_\Gamma$ is formally the disjoint union

$$\mathcal{U}_\Gamma = \bigsqcup_{[C] \in \mathcal{M}_\Gamma} C.$$ 

A natural partial order can be defined in a similar way as Section 2. We would like to impose similar domain-dependent perturbations on the universal curves. We define the locality condition for treed spheres.

**Definition 4.2.** Let $\Gamma$ be a stable domain type of treed spheres and $Z$ be a set. A map $f : \mathcal{U}_\Gamma \to Z$ is called local if

(a) For each $v \in \text{Vert}(\Gamma)$, the restriction of $f$ to $\mathcal{U}_{\Gamma \setminus (v)}$ is equal to the pullback of a map from $\mathcal{U}_{\Gamma \setminus (v)}$ to $Z$.

(b) Let $\Gamma'$ be the type obtained from $\Gamma$ by forgetting all short leaves and stabilizing. The restriction of $f$ to edge components is equal to the pullback of a map from $\mathcal{U}_{\Gamma'}$ to $Z$.

The open-closed and closed-open maps will be defined by counting treed disks where a distinguished leaf is a Morse trajectory of a function on the ambient symplectic manifold. Choose a Morse function

$$F_X : X \to \mathbb{R}.$$  

Choose a Riemannian metric $h_X$ on $X$ so that the gradient flow of $F_X$ with respect to $h_X$ satisfies the Morse–Smale condition. The pair $(F_X, h_X)$ is called a Morse–Smale pair. Signed counts of isolated negative gradient trajectories define a cochain complex

$$(CF^* (X), \delta_{\text{Morse}})$$

whose cohomology is isomorphic to the cohomology of $X$. The PSS construction [PSS96] allows one to define a ring structure on the cohomology which can be identified with the quantum cohomology. We repeat the construction here using our perturbation scheme.

To define quantum cohomology, consider perturbed treed holomorphic spheres of types $\Gamma$ whose equation on surface components is the pseudoholomorphic curve equation with domain-dependent almost complex structures (without Hamiltonian perturbations) and whose equation on long tree components is the negative gradient flow equation of $F_X$ (with domain-dependent perturbation of the function). The matching conditions at short leaves (markings not labelled by critical points of $F_X$) are also taken after a generic diffeomorphism on $X$. We will also need a coherent, local collection of perturbations $P_\Gamma = (J_\Gamma, F_\Gamma, E_\Gamma)$ for all stable types $\Gamma$. 

...
The product is defined by domain types with one spherical component, two long inputs and one long output. Given the regularity of the perturbation data, we see that for a one-dimensional moduli space of such type $\Gamma$, the boundary strata of $\mathcal{M}_\Gamma$ only contain objects with a single broken edge. Hence this induces a bilinear map

$$\star_b : H^\bullet(CF^\bullet(X), \delta_{\text{Morse}}) \otimes H^\bullet(CF^\bullet(X), \delta_{\text{Morse}}) \to H^\bullet(CF^\bullet(X), \delta_{\text{Morse}}).$$

With appropriate extension of the perturbation to treed spheres with three long leaves one can show that $\star_b$ is associative. The resulting graded ring, which we call the $b$-deformed quantum cohomology, is denoted by

$$QH^\bullet(X; b).$$

**Remark 4.3.** It is well-known that for two different Morse–Smale pairs $(F_X, h_X), (F'_X, h'_X)$, their cohomology are canonically isomorphic (via the continuation map). In defining the quantum multiplication, we can use the perturbation so that on different long leaves (including the output) one has different generic Morse–Smale pairs which do not vary with the domain. In particular, this feature implies that the perturbations respect certain forgetful maps which forget long leaves. Moreover, the resulting multiplication on the cohomology can be proved to be independent of the choices of these Morse–Smale pairs.

The quantum cohomology has a natural identity element defined as follows. Since $X$ is connected, one can choose $F_X$ such that it has a unique critical point $1_X$ of maximal Morse index. It is clearly a cochain and the fact that is cohomology class is the identity follows from the fact that the Morse–Smale pairs on long leaves are fixed and the perturbation respects forgetting long leaves.

**4.2. Open-closed maps.** The open-closed maps, roughly speaking, are defined via counts of treed holomorphic disks where the inputs are on the boundary (generators of morphisms spaces of the Fukaya category) and the outputs are critical points in the ambient symplectic manifolds. We need to combine the combinatorial structures used previously. Consider rooted based trees $\Gamma$ whose root is not necessarily contained in the base $\Gamma$. Moreover, the output $e_{\text{out}}$ is not an semi-infinite edge of the base. Therefore, all semi-infinite edges on the base are regarded as inputs. We call such a tree a spiked tree. A metric on a spiked tree is a map

$$\ell : \text{Edge}_{\text{finite}}(\Gamma) \to [0, +\infty).$$

A weighting on a spiked tree is a map

$$w : \text{Edge}_{\text{e}}(\Gamma) \to [0, 1]$$

which is zero on all interior semi-infinite edges. We do not require here the relation (14). The discrete datum underlying $\ell$ resp. $w$ is called a metric type resp. weighting type and denoted by $\ell$ resp. $w$.

**Definition 4.4.** An open-closed domain type consists of a spiked tree $\Gamma$ together with a metric type $\ell$ and a weighting type $w$. 
Open-closed domain types describe treed disks with an interior output. So the stability condition can be defined in the usual way. There is also a natural partial order among open-closed domain types. Notice that a broken open-closed domain type may have unbroken components which are domain types of treed disks or domain types of treed spheres. So it is natural to choose perturbation data for open-closed domain types which extend the existing perturbation data chosen for defining the Fukaya category and the quantum cohomology. An open-closed map type then include an extra labelling on the interior input by a critical point of $F_X$ from (44). For any perturbation datum $P_\Gamma$ and a map type $\Gamma$ one can consider the moduli space

$$\mathcal{M}_\Gamma(P_\Gamma).$$

We can define similar notion of uncrowdedness and achieve transversality in the same way as proving Lemma 2.24. A general compactness theorem can be proved so that the compactification is

$$\overline{\mathcal{M}}_\Gamma(P_\Gamma) = \bigsqcup_{\Pi \preceq \Gamma} \mathcal{M}_{\Pi}(P_{\Pi|_{\Pi_{\text{ext}}}})$$

where lower strata also include (arbitrarily many) breakings in the distinguished interior semi-infinite edge.

**Definition 4.5.** An open-closed map type $\Gamma$ is called essential if it has no spherical components nor edges of length zero, all interior markings are either $(D(L), 1)$ or $j_i(Z_i)$, and for each disk component $v$, the number of interior markings labelled by $(D(L), 1)$ is equal to $k\omega(\beta_v)$ where $k$ is the degree of the Donaldson hypersurface.

The following lemma can be proved in the same way as Lemma 2.27.

**Lemma 4.6.** Let $\Gamma$ be an essential open-closed map type. If the expected dimension of $\Gamma$ is zero, then $\mathcal{M}_\Gamma$ is compact. If the expected dimension is one, then $\overline{\mathcal{M}}_\Gamma$ is a compact topological 1-manifold with boundary where the boundary strata consist of moduli spaces $\mathcal{M}_\Pi$ where $\Pi$ is either obtained from $\Gamma$ by one of the operations listed in Lemma 2.27, or obtained from $\Gamma$ by breaking the interior semi-infinite edge once.

Counting holomorphic treed disks with an interior edge considered outgoing defines the open-closed map; see Figure 8 for an illustration.

**Definition 4.7 (Open-closed map).** Write for simplicity

$$CC_\bullet(X, L, b) := CC_\bullet(Fuk_L(X, b), Fuk_L(X, b)).$$

Define the bulk-deformed open-closed map

$$OC_d(b) : CC_d(X; b) \to CF(X)$$

(45) $$(x_1, \ldots, x_d) =: \mathbf{x} \mapsto \sum_{[u] \in \mathcal{M}(\mathbf{2}; \mathbf{x})_0} \sum_{x \in x_d} (-1)^{\partial + |x_d|} w(u)\mathbf{x}$$
with weightings from (31). The chain-level open-closed map $OC(b)$ is the direct sum $OC_d$ deformed by the Maurer-Cartan data on each Lagrangian brane:

$$OC(b) : CC(X;b) \rightarrow CF(X),$$

$$x_1 \otimes \ldots \otimes x_d \mapsto \sum_{j_1,...,j_d \geq 0} OC_{i_1+\ldots+i_d}(x_1, b_1, \ldots, x_d, b_d),$$

where $x_i \in CF((\phi_{i-1}, b_{i-1}), (\phi_i, b_i))$.

**Proposition 4.8.** The open-closed map $OC(b) : CC_\bullet(X;b) \rightarrow CF(X)$ is a chain map, that is,

$$OC(b) \circ \delta_{CC_\bullet} = m_1 \circ OC(b)$$

where $\delta_{CC_\bullet}$ is the Hochschild differential on $CC(X;b)$ and $m_1$ is the Floer (equal to the Morse) differential on $CF(X) = CF(\Delta, \Delta; b)$.

**Sketch of proof.** Follows from the description of the boundary strata of open-closed moduli spaces Lemma 4.6 with verification of signs. \hfill \Box

**Corollary 4.9.** The open-closed map $OC(b) = (OC_d(b))$ induces a map in (co)homology

$$[OC(b)] : HH_\bullet(X;b) \rightarrow QH^\bullet(X;b).$$

**Remark 4.10.** We do not prove here the independence of the open-closed maps from the choice of perturbation data and representative $b$ of $[b] \in QH^\bullet(X)$.

The assumptions of the main result Theorem 1.1 require knowledge that the open-closed map is surjective for a particular bulk deformation before the blowup occurs. The following shows that this assumption is often satisfied, if it holds for at least one bulk-deformation before the blowup:

**Proposition 4.11.** If the open-closed map $OC(b_0)$ is surjective for some bulk deformations $b_0, b_1, \ldots, b_k$ then the open-closed map $OC(b)$ for $b$ lying in some Zariski open subset of $\text{span}_C(b_0, \ldots, b_k)$.
Proof. Suppose that $OC(b_0)$ is surjective for some $b_0$. Choose cycles 

$$h_1, \ldots, h_m \in CC_\bullet(X)$$

so that the restriction of $OC(b_0)$ to the span of $h_1, \ldots, h_m$ is an isomorphism onto $QH^\bullet(X; b_0)$. Surjectivity of $OC(b_0)|_{\text{span}(h_1, \ldots, h_m)}$ is equivalent to the non-vanishing of a determinant $\det(OC(b_0))$ valued in $\Lambda$. Given $E > 0$ write 

$$\det(OC(b_0)) = \det(OC(b_0))_{< E} + \det(OC(b_0))_{> E}$$

where $\det(OC(b_0))_{< E}$ resp. $\det(OC(b_0))_{> E}$ has only terms of the form $cq^e$ with $q$-valuation $e$ less than resp. greater than $E$. Since $OC(b_0)|_{\text{span}(h_1, \ldots, h_m)}$ is an isomorphism, we may choose $E$ so that $\det(OC(b_0))_{< E}$ is non-vanishing. Then $\det(OC(b))_{E}$ is an algebraic function of the bulk-deformation $t_0b_0 + \ldots + t_kb_k$ and is non-vanishing for a Zariski open subset of $\mathbb{C}^{k+1} = \{(t_0, \ldots, t_k)\}$. □

4.3. Closed-open maps. The closed-open map takes as input a quantum cohomology class and its output is an element of Hochschild cohomology. In the monotone situation, this map is a special case of the functor described in the work of Ma'u, Wehrheim, and the second author [MWW18]. For monotone symplectic manifolds $X_0, X_1$, [MWW18] defines an $A_\infty$ functor 

$$\Phi : \text{Fuk}(X_0^{-} \times X_1) \to \text{Func}(\text{Fuk}(X_0), \text{Fuk}(X_1)).$$

If $X_0 := X_1 := X$, $\Phi$ maps the diagonal $\Delta \subset X^{-} \times X$ to the identity functor $\text{Id}_{\text{Fuk}(X)}$ on $\text{Fuk}(X)$, and $\Phi$ restricts to an $A_\infty$ map from the Fukaya algebra of the diagonal to the space of natural transformations on the identity functor, i.e., the space of Hochschild cochains.

In this paper we only consider the closed-open map up to the cohomology level. We first describe the combinatorics of the domains. A closed-open domain type consists of a based tree $\Gamma$ with the output in the base and with exactly one long leaf, a metric type $\ell$ and a weighting type $w$. We require that all interior finite edges have zero length. A closed-open map type consists of a closed-open domain type $\Gamma$ (which have $d$ boundary inputs and one long leaf), a collection $x = (x_0, x_1, \ldots, x_d) \in I(\phi_d, \phi_0) \times I(\phi_0, \phi_1) \times \cdots \times I(\phi_{d-1}, \phi_d)$ of critical points corresponding to a sequence of Lagrangians 

$$\phi = (\phi_0, \ldots, \phi_d),$$

a collection 

$$\beta = (\beta_v)_{v \in \text{Vert}(\Gamma)}$$

of homology classes, a critical point $x$ of the Morse function $F_X : X \to \mathbb{R}$ from (44), and additional interior labelling data $\diamond$. A treed holomorphic disk of type $\Gamma$ is defined in a similar way as the case for the open-closed maps. Given a perturbation $P_\Gamma$, let 

$$\mathcal{M}_\Gamma(P_\Gamma)$$
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denote the moduli space of stable holomorphic treed disks of map type $\Gamma$. Regularization of these moduli spaces can be achieved using Donaldson hypersurfaces constructed in the same way as in Theorem 2.24.

Again, a closed-open map type is essential if it has no spherical components, all boundary edges have positive lengths, all short leaves are labelled by top strata and the number of markings labelled by the Donaldson hypersurface on each surface component is equal to the expected numbers. For a collection of boundary inputs $\underline{x} = (x_0, x_1, \ldots, x_d)$ and a critical point $\underline{x}$ of $F_X$, let

$$\mathcal{M}(\underline{x}, 0) := \bigsqcup_{\Gamma} \mathcal{M}_{\Gamma}(P_{\Gamma})$$

denote the union of the moduli spaces of maps of essential map type, whose expected dimension is zero, whose long leaf is labelled $\underline{x}$, and whose boundary insertions are $x_0, x_1, \ldots, x_d$. Define a collection of maps

$$CO_d(\underline{x})(\underline{x} \otimes \cdots \otimes x_d) = \sum_{x_0 \in I(\phi_0, \phi_d)} \sum_{u \in \mathcal{M}(\underline{x}, 0)} (-1)^\psi w(u) x_0$$

with weightings from (31), extended linearly over $\Lambda$.

**Definition 4.12.** The chain level closed-open map from the $b$-deformed quantum cohomology of $X$ to the flat $A_\infty$ category $\text{Fuk}_L(X; b)$ is a map

$$CO(\underline{x}) : CF^*(X) \to CC^*_{\underline{x}}(X; b)$$

defined as follows. For each $\underline{x}$, $CO(\underline{x})(\underline{x})$ is the cochain that maps

$$x_1 \otimes \cdots \otimes x_d \in \text{Hom}((\phi_0, b_0), (\phi_1, b_1)) \otimes \cdots \otimes \text{Hom}((\phi_d, b_d-1), (\phi_d, b_d))$$

to

$$\sum_{j_0, j_1, \ldots, j_d \geq 0} CO_d+j_1+\cdots+j_d(\underline{x})(b_0, \ldots, b_0, x_1, \ldots, x_d, b_d, \ldots, b_d).$$

See Figure 9 for an illustration of a typical configuration possibly contributing to the closed-open map.

First we show the closed-open map induces a map from the quantum cohomology to the Hochschild cohomology.

**Theorem 4.13.** The map $CO(\underline{x}) : CF^*(X) \to CC^*_{\underline{x}}(X; b)$ defined by Definition 4.12 has the following properties.

(a) $CO(\underline{x})$ is a cochain map.
(b) For all $d \geq 1$ and $x \in CF^\bullet(X)$ and $i = 1, \ldots, d$, if $x_i = 1$ then
\[ CO_{d, x}(b)(x_1 \otimes \cdots \otimes x_d) = 0. \]

Proof. For (a), that $CO(b)$ is a chain map, consider one-dimensional moduli spaces for fixed labelling data $x$ and $x$. A refined compactness theorem similar to Lemma 2.27 shows that the boundary of such a moduli space consists of once-broken configurations. The breaking could be on the boundary, which corresponds to the differential of the Hochschild cochain complex, or on the interior long leaf, which corresponds to the Morse differential on $X$.

For property (b), notice that the perturbation data still respect the forgetful operation which forgets forgettable boundary inputs. Hence Property (b) holds in the same way as the unitality of the Fukaya category. \hfill \square

Lastly we show that the map on the cohomology level intertwines with the ring structures.

**Theorem 4.14.** The map
\[ [CO(b)] : QH^\bullet(X; b) \to HH^\bullet_2(X; b) \]
induced from the chain-level map intertwines with the ring structures.

The central ingredient of the proof is the notion of a special type of configurations similar to the notion of *quilted disks*. Consider a marked disk $S \simeq \mathbb{D}$ with two interior markings $z_1, z_2$ and a boundary marking $z_0$. We say that the marked disk $(S, z_1, z_2, z_0)$ is *balanced* if $z_1, z_2, z_0$ are on the same circle inside $\mathbb{D}$. Notice that this condition is invariant under $PSL(2; \mathbb{R}) \cong \text{Aut}(\mathbb{D})$. We make the following generalization to treed disks.

**Definition 4.15.** Consider a treed disk $C = S \cup T$ of domain type $\Gamma$ that has two long leaves and one boundary output. We say that $C$ is *balanced* if the following conditions are satisfied. Let $v_1, v_2 \in \text{Vert}(\Gamma)$ be the two vertices in the base that are closest to the two long leaves $e_1$ and $e_2$ respectively.
(a) If \( v_1 \neq v_2 \), then for the (unique) path \( e_1, e_2, \ldots, e_k \) in \( \Gamma \) connecting \( v_1 \) and \( v_2 \), require
\[
\sum_{i=1}^{k} \pm \ell(e_i) = 0
\]
where the signs depend on whether the direction of the path is towards the root or against the root.

(b) If \( v_1 = v_2 = v \), then let \( z_1, z_2 \in S_v \simeq \mathbb{D} \) be the node corresponding to them and let \( z_0 \in \partial S_v \) be the node towards the output. Then we require that the marked disk \((S_v, z_1, z_2, z_0)\) is balanced.

Notice that for any stable domain type \( \Gamma \) of treed disks with two long leaves, inside the moduli space \( M_\Gamma \) of stable treed disks the locus of balanced treed disks, denoted by
\[
M_\Gamma^b \subset M_\Gamma
\]
is a real codimension one submanifold. See Figure 10 for an illustration of a compactified moduli space of balanced treed disks with two long leaves.

![Figure 10. The compactified moduli space of balanced treed disks with two long leaves (interior markings) and one output. This moduli space is one-dimensional and has two fake boundary strata.](image)

For any map type \( \Gamma \) one can consider the moduli space
\[
M_\Gamma^b(P_\Gamma).
\]
The argument of achieving transversality can be extended to guarantee that \( M_\Gamma^b(P_\Gamma) \) is cut out transversely as long as \( \Gamma \) is uncrowded.

**Proof of Theorem 4.14.** Fix \( \mathfrak{x}_1, \mathfrak{x}_2 \in \text{crit} F_X \). We would like to show that the difference
\[
CO(b)(\mathfrak{x}_1 \ast_b \mathfrak{x}_2) - CO(b)(\mathfrak{x}_1) \ast CO(b)(\mathfrak{x}_2)
\]
is a coboundary in the Hochschild cochain complex. We first construct the coboundary. Fix a value of the potential function \( w \). For any sequence of unobstructed branes
\[
(\phi_0, b_0), \ldots, (\phi_d, b_d)
\]
with potential function having value \( w \), and for generators \( \underline{x} = (x_0, \ldots, x_d) \) where
\[
x_1 \in CF^*(\phi_0, \phi_1), \ldots, x_d \in CF^*(\phi_{d-1}, \phi_d), x_0 \in CF^*(\phi_0, \phi_d).
\]
Fix \( j_0, j_1, \ldots, j_d \geq 0 \) and consider balanced domain types \( \Gamma \) with two long leaves, \( d + j_0 + \cdots + j_d \) boundary inputs and essential map types \( \Gamma \) of expected dimension.
zero whose long leaves are labelled by $x_1, x_2$ and whose boundary inputs are labelled by $b_0, \ldots, b_0, x_1, b_1, \ldots, b_1, x_2, \ldots, x_d, b_d, \ldots, b_d, x_0$ (in counterclockwise orientation, the last one is the output). For each such moduli space $\mathcal{M}^b_\Gamma$ the count of rigid elements defines an element
\[
\tau(x_1, x_2) \in \Lambda.
\]
The linear span of these $\tau(x_1, x_2)$ then defines a cochain
\[
\tau(x_1, x_2) \in \text{CC}^*_\text{L}(X; b).
\]
We claim that
\[
\text{CO}(b)(x_1 \star_b x_2) - \text{CO}(b)(x_1) \star \text{CO}(b)(x_2) = m_1^\text{CC}(\tau(x_1, x_2)).
\]
To show this relation, consider a one-dimensional balanced moduli space with long leaves labelled by $x_1, x_2$ and any number of boundary inputs. There are three types of true boundary strata. The first type includes those that have an interior breaking. The second type includes those that has two boundary breakings on a path connecting the two disk components having the two long leaves. The third type includes those that have one boundary breaking that is not in the path connecting the disk components having the two long leaves. These types correspond to the three terms in (46).

Remark 4.16. Using quilted objects one can prove as in [MWW18] that the chain-level closed-open map extends to an $A_\infty$ map from the Fukaya algebra of the diagonal in $X \times X$ (whose cohomology is the quantum cohomology) to the $A_\infty$ algebra $\text{CC}^*_\text{L}(X; b)$. This construction requires a more sophisticated discussion of perturbations on the universal curve over the moduli space of quilts needed to regularize the moduli space of quilted spheres with arbitrary number of long leaves. The moduli space of balanced treed disks we used is not exactly the same as the moduli of quilted disks, but their difference does not appear on the cohomology level.

4.4. OC/CO for projective spaces. As a warm-up for the blow-up situation, we compute the open-closed and closed-open maps for Clifford tori in projective spaces using the Blaschke classification (27) (or, rather, its generalization to arbitrary toric manifolds in Cho-Oh [CO06].)

Theorem 4.17. Let $X = \mathbb{P}^m$, let $\mathcal{L} = \{\phi(1), \ldots, \phi(n+1)\}$ denote the collection of brane structures on the embedding $\phi : (S^1)^n \to X$ with image the Clifford torus and brane structures
\[
y_{(k)} = (N^k, \ldots, N^k) \in \mathcal{R}(\phi) \cong (\Lambda^X)^n
\]
for $\Lambda$ an $n+1$-st root of unity as in Lemma (3.12) with image and $b = 0$ the trivial bulk deformation. Then with notation from Proposition 3.11:
(a) There exists an isomorphism

\[ HH_\bullet(\text{Fuk}_L(X,0), \text{Fuk}_L(X,0)) \cong \Lambda^{\oplus(n+1)} \]

mapping the collection of point classes

\[ [\text{pt}_k] \in HF(\phi(k), \phi(k)), k = 1, \ldots, n + 1 \]

to the standard basis;

(b) The open-closed map \( OC(0) : HH_\bullet(X,0) \to QH_\bullet(X,0) \) is a \( q \)-deformation of the finite Fourier transform and in particular surjective (and so an isomorphism, conditional on Ganatra’s theorem in Remark 1.6.)

(c) The closed-open maps \( CO(0) : QH_\bullet(X,0) \to HF(\phi(k), \phi(k)) \) are given by

\[ [\mathbb{P}^2] \mapsto y_{(k),1} \cdots y_{(k),n-\ell}^{(n-\ell)/(n+1)}, \quad k, \ell = 0, \ldots, n. \]

To prove the theorem we compute the open-closed map explicitly using the Blaschke classification of disks (27). Choose a basis of cycles \( Z_0, \ldots, Z_n \subset X \) given by

\[ Z_k = \{ [z_0, \ldots, z_k, 0, \ldots, 0] \} \subset \mathbb{P}^n. \]

We write the finite Fourier transform in terms of a primitive \( n \)-st root of unity \( \zeta, \zeta^n = 1 \)

\[ \text{FFT} : \Lambda^{n+1} \to \Lambda^{n+1}, \quad (\lambda_1, \ldots, \lambda_{n+1}) \mapsto \left( \sum_{a=1}^{n+1} \zeta^{a-b} \lambda_a \right)^n_{b=0} \]

Introduce a quantization of the finite Fourier transform: For \( \epsilon > 0 \)

\[ \text{FFT}_q : \Lambda^{n+1} \to \Lambda^{n+1}, \quad (\lambda_1, \ldots, \lambda_{n+1}) \mapsto \left( q^{b/(n+1)} \sum_{a=1}^{n+1} \zeta^{a-b} \lambda_a \right)^n_{b=0}. \]

In particular

\[ \text{FFT}_q |_{q=1} = \text{FFT}. \]

Consider the embedding of the Clifford torus

\[ \phi : L \cong (S^1)^n \to \mathbb{P}^n. \]

As in Cho-Oh [CO06], there is one disk of the form (27) of Maslov index two associated to each prime divisor, each with area \( 1/(n+1) \) assuming the symplectic form is normalized so that its integral over the standard generator of \( H_2 \) is 1. As a result, the disk potential as a function of local system \( y \in \mathcal{R}(\phi) \) is given by

\[ W : \mathcal{R}(\phi) \to \Lambda, \quad (y_1, \ldots, y_n) \mapsto q^{1/(n+1)}(y_1 + \ldots + y_n + (y_1 \ldots y_n)^{-1}). \]

As a result, local systems that are critical points of the potential are solutions to

\[ y_1 = \ldots = y_n, \quad y_{k+1} = 1, \quad \forall k = 1, \ldots, n. \]

Let \( \mathcal{L} \) denote the resulting collection of Lagrangian branes given by \( \phi \) with the local system \( y \in \mathcal{R}(\phi) \).
Lemma 4.18. The open-closed map $OC(0)|HH_\bullet(X,0)$ has matrix with respect to the bases
\[
\{[\text{pt}_k] \in HF(\phi(k),\phi(k)), k = 0, \ldots, n]\subset HH_\bullet(X,0)
\]
\[
\{[Z_k], k = 0, \ldots, n\} \subset QH(X,0)
\]
(see (47)) that is given by the finite Fourier transform
\[
[OC(0)|HH_\bullet(X,0)] = \text{FFT}_q.
\]

Proof. We give an explicit computation of the open-closed map using the Blaschke classification. Let $\phi : L \to X$ be one of the Lagrangian branes in the collection $\{\phi(k)\}$. By Lemma 3.12 $\phi(k)$ has one-dimensional Hochschild homology which proves (a). For (b), recall from 3.11 that the Floer cohomology $HF(\phi,\phi)$ a non-degenerate Clifford algebra corresponding to the Hessian $\partial^2_\phi \partial^\phi W(y)$ of the potential $W(y)$. By Corollary 3.12 the Hochschild homology $HH_\bullet(HF(\phi,\phi))$ is generated by the point class $[\text{pt}] \in HF_n(\phi,\phi)$. Via the Blaschke classification (27) there is a unique disk $u : D \to X$ of index $I(u) = 2k$ with an interior point $z \in D$ mapping to $Z_k$ and boundary on $\phi(L)$. Let $\gamma_1, \ldots, \gamma_n \in \pi_1((S^1)^n)$ be the standard set of generators for $\pi_1((S^1)^n)$. By (27) again, the leading order contributions in the open-closed map $OC(0)$ arise from disks $u : C \to X$ with a single point constraint $u(T_e) \subset Z_k$ on the boundary. It follows that the open-closed map $OC(0)$ is given as a function of the local system $y$ on the point class $[\text{pt}] \in HF(\phi,\phi)$ by
\[
[OC(0)]([\text{pt}]) = (y(\gamma_1), y(\gamma_1 \gamma_2), \ldots, y(\gamma_1 \ldots \gamma_n))
\]
As a result, the point class in the brane with local system corresponding to $k = \exp(2\pi i k/(n+1))$ is mapped under the open-closed map $OC(0)$ to
\[
[\text{pt}_k] \to (q^{1/(n+1)}Z_1 + q^{2/(n+1)}Z_2 + \ldots + q^{n/(n+1)}Z_n).
\]
In the basis given by $[Z_1], \ldots, [Z_n]$ the open-closed map is therefore the finite Fourier transform $\text{FFT}_q$. \hfill \Box

Proof of Theorem 4.17. The claim (a) on the Hochschild homology follows from Lemma 3.12. The determinant of the matrix for $OC(0)$ with respect to the given basis
\[
[\text{pt}_{(k)}] \in HF(\phi(k),\phi(k)), k = 1, \ldots, n+1
\]
is the determinant of the finite Fourier transform $\text{FFT}$ times a power of $q$. Since $\text{FFT}$ is surjective, so is $OC(0)$ hence (b). For (c) note that for each cycle $P_\ell$, the Blaschke products mapping $0$ to $P_\ell$ with index $2(n-\ell)$ are those with the first $n-\ell$ components
\[
(u_1, \ldots, u_{n-\ell})(z) = u : D \to \mathbb{C}^{n+1}, \quad z \mapsto \left(\zeta_i \frac{z-a}{1-\overline{z}a}\right)_{i=1, \ldots, n-\ell}.
\]
are non-vanishing with a common root at some $a \in D$. Hence the moduli space $\mathcal{M}_{1,1}(\phi, D)$ is non-empty only if the output is a point constraint. In the case of a
point constraint there is a single disk with an interior point mapping to \( Z_k \) and the contribution is \( y_{(k),1} \ldots y_{(k),n-\ell} \varphi_{(k)} \in HF(\varphi_{(k)}, \varphi_{(k)}) \).

\[ \square \]

**Remark 4.19.** The closed-open map is a ring homomorphism as predicted by Theorem 4.14. For example, in quantum cohomology we have \([\mathbb{P}^{n-1}]^{n+1} = q\) while in Hochschild cohomology

\[
(CO([\mathbb{P}^{n-1}])^{n+1} = (q^{1/(n+1)}y_{(k),1})^{n+1} = q
\]

for any of the branes \( \varphi_{(k)} \) in question.

We also show that the bulk-deformed open-closed map is an isomorphism, so that the set of examples to which the hypotheses of the main results are satisfied is non-empty.

**Corollary 4.20.** The bulk-deformed open-closed map \( OC(b + q^{-\epsilon}p) \) is an isomorphism for \( \epsilon < 1 \).

**Proof.** We compute the disks with bulk insertions at a point. By homogeneity we may take the point \( p = [0,0,\ldots,0,1] \in \mathbb{P}^n \) to be a toric fixed point disjoint from the cycles \( Z_1, \ldots, Z_{n-1} \). The requirement that the disk passes through \( p \) forces \( n \) additional roots in the Blaschke product (27). So the total number of roots in any Blaschke disk with a point constraint at the point, also contributing to the open-closed map for \( Z_1, \ldots, Z_{n-1} \), is at least \( n + 1 \). Thus \( OC(b + q^{-\epsilon}p) \) is the finite Fourier transform \( FFT_q \) plus terms with \( q \)-valuation at least \( 1 - \epsilon \) greater. For \( \epsilon < 1 \), the higher order terms in \( OC(b + q^{-\epsilon}p) \), that is, the difference \( OC(b + q^{-\epsilon}p) - FFT_q \), do not affect the leading order determinant in the open-closed map. So \( OC(b + q^{-\epsilon}p) \) still has non-vanishing determinant in the basis above. \( \square \)

5. **Abouzaid’s criterion**

In this section we adapt Abouzaid’s criterion [Abo10] for the split-generation of the Fukaya category to the non-exact case in which the \( A_\infty \) composition maps are defined by counts of treed disks.

5.1. **The Cardy diagram.** The idea of Abouzaid’s construction is to produce the maps necessary for writing a Lagrangian as mapping cone by degenerating holomorphic annuli to pairs of disks. Given a collection \( \mathcal{G} \) of objects \( \varphi \) of \( \text{Fuk}_\mathcal{L}(X, b) \), we wish to show that any object \( \psi \) of \( \text{Fuk}_\mathcal{L}(X, b) \) is split-generated by the objects of \( \mathcal{G} \). For example, we might hope to show that \( \psi \) is a sub-object of some object \( \varphi \) of \( \mathcal{G} \); to show this we want maps

\[
\alpha \in \text{Hom}(\psi, \varphi), \quad \beta \in \text{Hom}(\varphi, \psi)
\]

such that

\[
m_2(\alpha, \beta) = 1_\psi \in \text{Hom}(\psi, \psi).
\]

Naturally one hopes that the chains \( \alpha, \beta \) can be produced geometrically as a count of holomorphic disks with two outputs. If this is the case, one can glue to obtain
holomorphic annuli with an output labelled by the identity \(1_\psi\). A degeneration of the annulus to “infinite length” shows that a count of holomorphic disks with a single output must be non-trivial, see Figure 11.

The result, Abouzaid’s criterion Theorem 1.5, gives a factorization of the open-closed and closed-open maps through the tensor product of Yoneda modules. Given an object \(\phi\) of \(\text{Fuk}_L(X,b)\) denote by \(\mathcal{Y}_\psi^r\) resp. \(\mathcal{Y}_\psi^l\) the right resp. left Yoneda module over \(\text{Fuk}_L(X,b)\) defined on objects by
\[
\mathcal{Y}_\psi^r(\phi) = \text{Hom}(\psi, \phi), \quad \mathcal{Y}_\psi^l(\phi) = \text{Hom}(\phi, \psi).
\]
The tensor product of Yoneda modules is an \(A_\infty\) bimodule over \(\text{Fuk}_L(X,b)\) denoted \(\mathcal{Y}_\psi^r \otimes \mathcal{Y}_\psi^l\). The Hochschild homology
\[
HH_*(\text{Fuk}_L(X,b), \mathcal{Y}_\psi^r \otimes \mathcal{Y}_\psi^l) = H_*(\mathcal{Y}_\psi^r \otimes_{\text{Fuk}_L(X,b)} \mathcal{Y}_\psi^l)
\]
is computed by the bar complex
\[
B(\mathcal{Y}_\psi^r \otimes_{\text{Fuk}_L(X,b)} \mathcal{Y}_\psi^l) = \bigoplus_{k=0}^{\infty} \bigoplus_{\phi_1, \ldots, \phi_k \in \text{Ob}(\text{Fuk}_L(X,b))} \text{Hom}(\psi, \phi_1) \otimes \text{Hom}(\phi_1, \phi_2) \otimes \cdots \otimes \text{Hom}(\phi_k, \psi)
\]
with differential \(\partial_{\mathcal{Y}_\psi^r \otimes_{\text{Fuk}_L(X,b)} \mathcal{Y}_\psi^l}\) given by the possible ways of collapsing. Define the collapsing morphism \(\mu\)
\[
\mu : B(\mathcal{Y}_\psi^r \otimes_{\text{Fuk}_L(X,b)} \mathcal{Y}_\psi^l) \to \text{Hom}(\psi, \psi)
\]
by composing all factors in (49):
\[
\mu : x_- \otimes x_1 \otimes \cdots \otimes x_k \otimes x_+ \mapsto (-1)^\diamond m_{k+2}(x_-, x_1, \ldots, x_k, x_+)
\]
where \(\diamond\) is the Koszul sign
\[
|x_-| + \sum_{j=1}^{k} \|x_j\|.
\]
There is a natural \(A_\infty\) coproduct \(\delta\) given by a morphism of bimodules
\[
\delta : \text{Fuk}_L(X,b) \to \mathcal{Y}_\psi^l \otimes \mathcal{Y}_\psi^r.
\]
By definition such a morphism consists of a collection of maps \(\{\delta_{r|1,s}\}_{r,s \geq 0}\)
\[
\delta_{r|1,s} : CF(\phi_{r-1}, \phi_r) \otimes \cdots CF(\phi_0, \phi_1) \otimes CF(\phi_0, \phi_0') \otimes CF(\phi_0, \phi_1') \otimes \cdots \otimes CF(\phi_{s-1}, \phi_s) \to CF(\psi, \phi_r) \otimes CF(\phi_s, \psi),
\]
satisfying an \(A_\infty\) axiom. The morphism is defined by counting holomorphic disks
\[
(x_r, \ldots, x_1, x_0, x'_1, \ldots, x'_s) \overset{\delta_{r|1,s}}{\longrightarrow} \sum_{x,x', u \in M_T(\phi_r, D, \pm b)} (-1)^4 w(u) x \otimes x'
\]
where the sum is over rigid maps $u$ with two adjacent output leaves and one distinguished input (in this case $x_0$) among a list of input leaves. The sign $‡$ is given as in Abouzaid [Abo10, 4.17] by
\[
\sum_{j=1}^{s} (s - j + 1) |x_j'| + s |x_0| + \sum_{j=1}^{r} (j + s) |x_j|.
\]
The coproduct $\delta$ induces a map on the cyclic bar complex
\[
CC_d(\delta) : CC_d(Fuk(X, b)) \to \mathcal{Y}_\psi^d \otimes \mathcal{Y}_\psi^d
\]
\[(x_1 \otimes \ldots \otimes x_d) = \sum (-1)^{\circ} x_{r+1} \otimes \ldots x_{d-s-1} \otimes \mathcal{T}(\delta_{r+1}|d)(x_{d-s} \otimes \ldots \otimes x_r))\]
where $\mathcal{T}$ is the map that reorders the factors
\[
\mathcal{T}(a_{r+1} \otimes \ldots a_{d-s-1} \otimes q \otimes p) = (-1)^{\circ} q \otimes a_{r+1} \otimes \ldots \otimes a_{d-s-1} \otimes p
\]
and the signs are given by the formulae
\[
\circ = \mathbf{x}_1^d (1 + \mathbf{x}^d_{r+1}) + \text{dim}(X) \mathbf{x}_{r+1}^{d-s-1}
\]
and
\[
\circ = \deg(q)(\deg(p) + \mathbf{x}_{r+1}^{d-s-1}).
\]
Alternatively, we may define
\[
CC_d(\delta)(x_1 \otimes \ldots \otimes x_d) = \sum_{r,s \geq 0, p, q, \quad u \in M_d(\phi, x, p, q)} (-1)^{\circ + \circ} w(u)(p \otimes x_{r+1} \otimes \ldots x_{d-s-1} \otimes q),
\]
with weightings from (31), where the sum is over $p \in CF(\phi_r, \psi)$, $q \in CF(\psi, \phi_{d-s-1})$, and rigid maps $u$ of the form
\[
\begin{tikzpicture}
\draw (0,0) circle (1cm);
\node (u) at (0,0) {$u$};
\draw (1.5,0) -- (2,0);
\draw (-1.5,0) -- (-2,0);
\draw (0,1.5) -- (0,2);
\draw (0,-1.5) -- (0,-2);
\node (x1) at (-1.5,0) {$x_1$};
\node (x2) at (-2,0) {$x_2$};
\node (x3) at (2,0) {$x_r$};
\node (x4) at (1.5,0) {$x_d$};
\node (x5) at (0,2) {$x$};
\node (x6) at (0,1.5) {$x_{d-1}$};
\node (x7) at (0,-1.5) {$x_{d-s}$};
\node (x8) at (0,-2) {$q$};
\end{tikzpicture}
\]
The chain level coproduct map is defined as the direct sum of the maps $CC_d(\delta)$ deformed by the Maurer-Cartan data on each Lagrangian brane:
\[
CC_\bullet(\delta)(x_1 \otimes \ldots \otimes x_d)
\]
\[= \sum_{j_1, \ldots, j_d \geq 0} OC_{d+|\sum_{i=1}^{d} j_i|}(x_1, b_1, \ldots, b_1, x_2, b_2, \ldots, b_2, x_3, \ldots, x_d, b_d, \ldots, b_d),
\]
where $x_i \in CF((\phi_{i-1}, b_{i-1}), (\phi_i, b_i))$, and in the above expression $b_i$ is repeated $j_i$ times.

The resulting map, collapse map $\mu$, and open-closed and closed-open maps $CO(b), OC(b)$ fit into a commutative-up-to-sign Cardy diagram:
Theorem 5.1. (Abouzaid [Abo10] in the exact, embedded case; see also Ganatra [Gan12]) For any collection $G \subset L$ any any object $\psi \in L$, there is a Cardy diagram

\[
HH_* (\text{Fuk}_G(X, b), \text{Fuk}_G(X, b)) \to HH_* (Y_\psi \otimes \text{Fuk}_G(X, b), Y_\psi)
\]

that commutes up to an overall sign of $(-1)^{\dim(X)(\dim(X)+1)/2}$. The left vertical arrow is the open-closed map $[OC(b)]$, the bottom arrow is the closed-open map $[CO(b)]$, the right-hand map is the collapse map $[\mu]$, and the top map is the Hochschild homology of the diagonal morphism.

5.2. Holomorphic treed annuli. To prove the Cardy diagram Theorem 5.1 in the version of the Fukaya category considered here, we begin with some preliminaries. Given sequences of Lagrangians $\phi_0, \ldots, \phi_k \in L$ the composition of maps on either direction of the diagram consist of maps

\[
\text{CF}(\phi_0, \phi_1) \otimes \ldots \otimes \text{CF}(\phi_k, \phi_0) \to \text{CF}(\psi, \psi)
\]

counting certain degenerate treed holomorphic annuli which we now define.

Definition 5.2. An \textit{annulus} is a complex curve with boundary biholomorphic to a standard annulus

\[
A_{\rho_1, \rho_2} = \{ z \in \mathbb{C} \mid \rho_1 \leq |z| \leq \rho_2 \}
\]

The boundary components are denoted by

\[
\partial_- A_{\rho_1, \rho_2} := \{ z : |z| = \rho_1 \}, \quad \partial_+ A_{\rho_1, \rho_2} := \{ z : |z| = \rho_2 \}.
\]

Definition 5.3. (Stable treed annuli)

(a) (Marked annulus) For $d(\circ) = (d(\circ)_-, d(\circ)_+)$ a pair of positive integers and $d(\bullet) \in \mathbb{Z}_{\geq 0}$ a $(d(\circ), d(\bullet))$-marked annulus consists of the following data: an inner and outer radius $\rho_1 < \rho_2$, and a collection of interior marked points

\[
z_{*, i} \in \text{int}(A_{\rho_1, \rho_2}), \quad 1 \leq i \leq d(\bullet)
\]

and a collection of boundary marked points

\[
z_{\circ, i}^\pm \in \partial_{\pm} A_{\rho_1, \rho_2}, \quad 1 \leq i \leq d(\circ).
\]

(b) (Treed annulus) If there is at least one marking $z_i$ there is a compactification of the moduli space of marked annuli by allowing stable nodal annuli: nodal annuli $S$ with no non-trivial infinitesimal automorphisms $\varphi \in \text{aut}(S, z)$. As in the case of stable marked disks, a combinatorial type underlying a stable annulus is a graph $\Gamma$. A treed annulus $C$ is obtained from a nodal annuli by replacing each boundary node $w_e, e \in \text{Edge}_{\circ, -}(\Gamma)$ by a (possibly
broken) tree segment $T_e$ equipped with a length $\ell(e) \in [0, \infty]$, and attaching a semi-infinite treed segment $T_e$ at each boundary marking $z_e$, $e \in \text{Edge}_{\partial, \infty}(\Gamma)$.

(c) (Additional features) We consider treed annuli with some additional features to prove the Cardy relation and an orthogonality relation in Section 5.4.

(i) (Distinguished leaves and Balanced lengths) The leaves $z_{d(c)+}^+$ and $z_{d(c)-}^-$ are distinguished leaves and are constrained to have an angle offset of $\pi$:

$$\exists \theta : z_{d(c)+}^+ = \rho_2 e^{i\theta}, \quad z_{d(c)-}^- = \rho_1 e^{i(\theta + \pi)}.$$  (Angle offset)  

The lengths of treed segments are subject to a balancing condition:

The path of edges $\gamma_+$ resp. $\gamma_-$ connecting the component component $S_{v+}$ containing $z_{d(c)+}^+$ to the component $S_{v-}$ containing $z_{d(c)-}^-$ have the same total length:

$$\sum_{e \in \gamma_-} \ell(e) = \sum_{e \in \gamma_+} \ell(e);$$  (Balanced)  

this is an analog of the balancing condition in Definition 4.15. See Figure 11 where the paths $\gamma_{\pm}$ are the vertical paths in the two left diagrams.

(ii) (Treed segment at an interior node) An interior node that disconnects $z_{d(c)+}^+$ from $z_{d(c)-}^-$ is called a path node. We allow path nodes to be replaced by treed segments.

We introduce a moduli space of stable treed annuli with fixed angle offset as follows. Denote by $M_{ann_{d(c),d(\bullet)}}$ the moduli space of stable treed annuli whose distinguished boundary marked points have an angle offset of $\pi$ (as in (57)) and that satisfy the balancing condition (58) for treed segments at path edges. Standard arguments show that the moduli space $M_{ann_{d(c),d(\bullet)}}$ is compact and Hausdorff. The subspace of $M_{ann_{d(c),d(\bullet)}}$ that parametrizes curves with at most one path node is a topological manifold of dimension

$$\dim(M_{ann_{d(c),d(\bullet)}}) = |d_+(\circ)| + |d_-(\circ)| + 2|d(\bullet)| - 1.$$  

The moduli space is equipped with a universal curve $\overline{U}_{ann_{d(c),d(\bullet)}}$ which decomposes into a surface part $\overline{S}_{ann_{d(c),d(\bullet)}}$ and tree part $\overline{T}_{ann_{d(c),d(\bullet)}}$. There is a forgetful map

$$f : \overline{M}_{ann_{d(c),d(\bullet)}} \to \overline{M}_{ann_{(1,1),0}}$$  (59)  

that forgets all non-distinguished markings.

Remark 5.4. In the moduli space of treed annuli, we fixed the angle offset between distinguished boundary markings as $\Phi := \pi$. This choice is arbitrary. In fact, choosing any non-zero angle offset $\Phi \in (0, 2\pi)$ produces a homeomorphic moduli space. The angle offset zero $\Phi = 0$ produces a different moduli space, which we will use in Section 5.4.
Example 5.5. We describe the moduli space of isomorphism classes of annuli with one inner boundary leaf and one outer boundary leaf (with an angle offset of $\pi$) as follows: There is a homeomorphism
\begin{equation}
\rho : \mathcal{M}_{(1,1),0}^{\text{ann}} \cong [-\infty, \infty]
\end{equation}
defined as follows: For configurations containing an annulus component with inner radius $\rho_1$ and outer radius $\rho_2$ we define
\[
\rho_1(C) = \frac{\rho_1 \rho_2^{-1}}{1 + \rho_1 \rho_2^{-1}}.
\]
In the case that $C$ consists of two disks connected by an interior tree segment $T_e$, $e \in \text{Edge}_{\bullet, p}$ of length $\ell(e)$ define
\[
\rho_1(C) = -\ell(e).
\]
In the case $C$ consists of a single disk joined to itself by an edge $e$ of length $\ell(e)$ define
\[
\rho_1(C) = \ell(e) + 1.
\]
See Figure 11. The second component $\rho_2(C) \in S^1$ is the angle offset between the first inner and first outer boundary leaf. The map
\[
C \mapsto e^{-\rho_1(C)} \rho_2(C) \in \mathbb{D}
\]
defines the desired homeomorphism, ending the Example.

The description of the one inner-and-out marking moduli space in the previous paragraph leads to the following natural defined functions on moduli spaces with higher numbers of inner and outer markings: Composing the homeomorphism $\rho$ in (60) with the forgetful map (59), we obtain a map
\begin{equation}
f : \mathcal{M}_{d(\circ), d(\bullet)}^{\text{ann}} \to [-\infty, \infty].
\end{equation}
For any $\rho \in [-\infty, \infty]$ the fiber $f^{-1}(\rho)$ is the moduli space of annuli with a fixed ratio of inner and outer radii, and is denoted by
\[
\mathcal{M}_{d(\circ), d(\bullet)}^{\text{ann}}(\rho) \subset \mathcal{M}_{d(\circ), d(\bullet)}^{\text{ann}}.
\]

Remark 5.6. The moduli space of treed annuli admits an orientation induced from choices of orientations on nodal annuli induced from the positions of the interior and boundary markings: We identify each element of $\mathcal{M}_{d(\circ), d(\bullet), \rho}^{\text{ann}}$ with a single surface component (necessarily $0 < \rho < 1$) with a fixed annulus $A$; recording the attaching points of the leaves gives a map
\[
\mathcal{M}_{d(\circ), d(\bullet), \rho}^{\text{ann}} \subset \int A^{d(\bullet)} \times (\partial A)^{d(\circ) - d(\bullet) + 1 + 1} / S^1
\]
and orient the boundary circles $(\partial A) \cong S^1 \sqcup S^1$ in the same direction. The orientations on this stratum extends to a global orientation on the manifold with boundary $\mathcal{M}_{d(\circ), d(\bullet)}^{\text{ann}}$. The boundary of $\mathcal{M}_{d(\circ), d(\bullet)}^{\text{ann}}$ consists of configurations where the ratio $\rho$ is equal to $\infty$, configurations where the ratio $\rho$ is equal to $-\infty$ (in the sense that the lengths of the paths $\gamma_\pm$ above are infinite) and configurations where a collection
of leaves $T_e$ have bubbled onto disks $S_v, v \in \text{Vert}(\Gamma)$ attached to the outer boundary, and configurations where leaves $T_e$ have bubbled onto disks $S_v$ on the inner boundary. The latter two types of boundary strata $\mathcal{M}_\Gamma, \Gamma = \Gamma_1 \# \Gamma_2$ have opposite orientations compared to the product orientation on $\mathcal{M}_{\Gamma_1} \times \mathcal{M}_{\Gamma_2}$.

Regularizing families of holomorphic maps from treed annuli requires regularization of holomorphic disks, strips, and spheres as before. Given a collection of branes $\phi$, an \textit{adapted} holomorphic treed annulus with boundary in $\phi$ is a pair of maps

$$(u : C \to X, \partial u : \partial C \to L)$$

where $C$ is a treed annulus, $L$ denotes the union of domains of $\phi$ the map $u$ is holomorphic on the surface components $S_v, v \in \text{Vert}(\Gamma)$, and in which any leaf $e \in \text{Edge}(\Gamma)$ labelled

$$\diamond \in \mathcal{L}^2 \cup \mathcal{L} \cup \{D(X), X, \mathfrak{b}\}$$

is required to correspond to a gradient trajectory $u|_{T_e} : T_e \to X$ of the corresponding function $F(\diamond)$; if $e \in \mathcal{L} \cup \{D(X)\}$ then these functions are constant and so the maps $u|_{T_e}$ are constant. Denote by $\overline{\mathcal{M}}_{d(o), d(\bullet)}^{\text{ann}}(\phi, D)$ the moduli space of perturbed stable treed holomorphic annuli and $\overline{\mathcal{M}}_{d(o), d(\bullet)}^{\text{ann}}(\phi, D)_d$ the locus of expected dimension $d$ and a fixed angle offset $\Theta = \pi$ between distinguished boundary markings, see (57). For generic choices of coherent perturbation data $P = (P_\Gamma)$ the moduli spaces of treed holomorphic annuli of dimension at most one become regular and compact, by the same arguments as for treed disks. After fixing the perturbations on a neighborhood of the nodes and ends of the segments, for a comeager subset of perturbations the moduli spaces $\overline{\mathcal{M}}_{d(o), d(\bullet)}^{\text{ann}}(\phi, D)$ are regular with true boundary corresponding to formation of a broken Morse trajectory $T_e = T^1_e \# T^2_e$ connecting a disk and an annulus, or degeneration of the annulus $S_0$ to one of the possibilities corresponding to the ratio of inner and outer radii approaching zero or infinity shown.
in Figure 11:

\[ \partial \mathcal{M}^\text{ann}_{d(\circ),1,d(\bullet)}(\phi, D)_1 = \mathcal{M}^\text{ann}_{d(\circ),d(\bullet), \rho = \infty}(\phi, D)_0 \cup \mathcal{M}^\text{ann}_{d(\circ),d(\bullet), \rho = -\infty}(\phi, D)_0 \]

\[ \cup \bigcup_{i,k \geq 0, i+k \leq d(\circ)_+} \mathcal{M}^\text{ann}_{((d(\circ)_+-k-1,d(\circ)_-),d(\bullet)_1)}(\phi_1, D)_0 \times \mathcal{I}(\phi_i, \phi_{i+k}) \mathcal{M}_{k,d(\bullet)_2}(\phi_2, D)_0 \]

\[ \cup \bigcup_{i,k \geq 0, i+k \leq d(\circ)_-} \mathcal{M}^\text{ann}_{((d(\circ)_+-k+1,d(\circ)_-),d(\bullet)_1)}(\phi_1, D)_0 \times \mathcal{I}(\phi_i, \phi_{i+k}) \mathcal{M}_{k,d(\bullet)_2}(\phi_2, D)_0 \]

where \( \phi_1, \phi_2 \) are the labels on disk components separated by an edge with infinite length.

5.3. **Commutativity.** Using the moduli spaces of holomorphic treed annuli, we define a homotopy operator relating the composition of the maps around the two sides of the diagram in Theorem (5.1). This shows that the diagram in Theorem 5.1 is commutative. Define a map

\[ \mathcal{H}: CC_\bullet(\text{Fuk}_L(X, b), \text{Fuk}_L(X, b)) \rightarrow CF(\psi, \psi) \]

by a count of holomorphic treed annuli. Omitting the bulk deformation \( b \), \( \mathcal{H} \) is a collection of maps

\[ \mathcal{H} = (\mathcal{H}_d)_{d \geq 0}, \quad \mathcal{H}_d: CF(\phi_0, \phi_1) \otimes \cdots \otimes CF(\phi_{d-1}, \phi_d) \times CF(\phi_d, \phi_0) \rightarrow CF(\psi, \psi)[1-d] \]

defined as

\[ (x_0, \ldots, x_d) \mapsto \sum_{u \in \mathcal{M}^\text{ann}_{d(\circ),d(\bullet),x_0,b}_0} (-1)^\delta w(u)x'_0. \]

c.f. Abouzaid [Abo10, Equation 6.22].

**Figure 12.** Cardy relation: End-points of a one-dimensional moduli space of holomorphic treed annuli, \( x_0, x_d \) are distinguished leaves.

**Theorem 5.7.** The operator \( \mathcal{H} \) is a homotopy operator relating the two sides of the Cardy diagram:

\[ \partial \mathcal{H} + \mathcal{H} \partial = (-1)^{\dim(X)(\dim(X)+1)/2} OC \circ CO - \mu \circ \delta. \]
Proof. It follows from the description of the boundary (see Figure 12) in (62) and
the sign computation in [Abo10], which we will not repeat here. \qed

Abouzaid’s generation criterion Theorem 1.1 follows from Theorem 5.1 in the
same way as in [Abo10].

5.4. Orthogonality for disjoint Lagrangians. We prove two results about the
orthogonal of images under the open-closed map. The first concerns the image under
the open-closed map of Floer classes from disjoint Lagrangians.

Theorem 5.8. Suppose that \( L_-, L_+ \subset L \) are disjoint collections of Lagrangians in
\( X \), that is, for every \( \phi_- \in L_- \) and \( \phi_+ \in L_+ \), we have \( \phi_-(L_-) \cap \phi_+(L_+) = \emptyset \). The
images of elements
\[
\alpha_- \in HH_\bullet(Fuk_{L_-}(X, b), Fuk_{L_-}(X, b)), \quad \alpha_+ \in HH_\bullet(Fuk_{L_+}(X, b), Fuk_{L_+}(X, b))
\]
are orthogonal with respect to the natural Frobenius pairing on \( \mathbb{Q}H^\bullet(X, b) \):
\[
\langle (OC(b))(\alpha_-), (OC(b))(\alpha_) \rangle = \{0\},
\]
The proof of Theorem 5.8 is a study of the ends of the moduli space of treed
annuli, with a fixed underlying annulus. Fix a ratio \( \rho \in [-\infty, \infty] \) and consider the
moduli space of holomorphic treed annuli (as in Definition 5.3) with ratio \( \rho \) between
the inner and outer annuli and angle offset \( \pi \) between the first inner leaf and first
outer leaf:
\[
\mathcal{M}^{\text{ann}}_{d(o), d(\bigcdot), \rho}(\phi, D); \quad \mathcal{M}^{\text{ann}}_{d(o), d(\bigcdot), \rho}(\phi, D) \to \mathcal{M}^{\text{ann}}_{1, 0} \cong [-\infty, \infty]
\]
where all leaves except for the distinguished leaves on both the boundary components
are forgotten, see (60). Define an invariant by counting elements of
\( \mathcal{M}^{\text{ann}}_{d(o), d(\bigcdot), \rho}(\phi, D, x_- x_+, b) \):

Definition 5.9. For \( x_- \in CF(\phi_-)^{d(o)}-, x_+ \in CF(\phi_+)^{d(o)}+ \) define
\[
\langle x_- x_+ \rangle = \sum_{u \in \mathcal{M}^{\text{ann}}_{d(o), d(\bigcdot), \rho}(\phi, D, x_- x_+, b)} (-1)^\circ w(u).
\]

Lemma 5.10. The topological boundary of the moduli space \( \mathcal{M}^{\text{ann}}_{d(o), d(\bigcdot), \rho}(\phi, D, x_- x_+, b) \)
consists of strata \( \mathcal{M}^{\text{ann}}_{1, 0}(\phi, D, x_- x_+, b) \) of infinite length \( \ell(e) = \infty \) that separates an annulus from a disk.

Proof. As in the case of treed disks in Theorem 2.24 and Lemma 2.27. \qed

Corollary 5.11. The pairing \( \langle \cdot, \cdot \rangle_\rho \) induces a pairing on cohomology where \( L = L_- \cup L_+ \)
\[
HH_\bullet(X, b) \otimes HH_\bullet(X, b) \to \Lambda, \quad (\alpha_-, \alpha_+) \mapsto \langle \alpha_-, \alpha_+ \rangle_\rho.
\]
Proof. It follows from Lemma 5.10 that (64) is a chain map, with the trivial boundary operator on $\Lambda$. 

\textbf{Lemma 5.12.} For any two values $\rho', \rho$ there exists a compact oriented cobordism between 

$$\mathcal{M}_{d(\circ),d(\bullet),\rho}^{\text{ann}}(\phi, D, \underline{x}_-, \underline{x}_+)_0 \cup (\mathcal{M}_{d(\circ),d(\bullet),\rho'}^{\text{ann}}(\phi, D, \underline{x}_-, \underline{x}_+)_0)^-$$

(with superscript $-$ indicating a reversal of orientation) and the collection of configurations with ratio $\rho'' \in (\rho, \rho')$ and a boundary edge of infinite length:

$$\bigcup_{* \pm k=0} d(\circ)_* \mathcal{M}_{d(\circ)-k+1,d(\bullet)-\rho''}^{\text{ann}}(\phi_-, D)_0 \times \Gamma(\phi,\phi_{\pm k}) \mathcal{M}_{k,d(\bullet)_+}(\phi_+, D)_0,$$

where $k_+ = (k,0)$ and $k_- = (0,k)$.

Proof. This is a standard cobordism argument using a parametrized moduli space of tree annuli with parameter $\rho''$ as $\rho''$ varies from $\rho$ to $\rho'$.

\textbf{Corollary 5.13.} The weighted count of elements in $\mathcal{M}_{d(\circ),d(\bullet),\rho}^{\text{ann}}(\phi, D)$ is independent of the ratio $\rho$ and equal to the pairing between the elements $(\text{OC}(b))(\alpha_-)$, $(\text{OC}(b))(\alpha_+)$ for $\alpha_\pm \in HH_*(X, b)$ is

$$\langle (\text{OC}(b))(\alpha_-), (\text{OC}(b))(\alpha_+) \rangle = \langle \alpha_-, \alpha_+ \rangle_\rho.$$

Proof. For $\rho = -\infty$, the moduli space is equal to the fiber product

$$\mathcal{M}_{d(\circ),d(\bullet),-\rho}^{\text{ann}}(\phi, D, \underline{x}_-)_0 \times \Gamma(X) \mathcal{M}_{d(\circ),d(\bullet),\rho}^{\text{ann}}(\phi, D, \underline{x}_+)_0.$$

Hence the signed count $\langle \alpha_-, \alpha_+, \rho \rangle$ is equal to the pairing $\langle \text{OC}(\alpha_-), \text{OC}(\alpha_+) \rangle$.

Proof of Theorem 5.8. We degenerate the annulus to a pair of disks. As in the Cardy relation, consider a degeneration of the annulus $S$ to a nodal curve $S_0 = S_+ \cup S_-$ with two connecting nodes $\{w_1, w_2\} = S_+ \cap S_-$, in which the first boundary leaf $T_{e_1, \pm}, e_1, \pm \in \text{Edge}_\circ(\Gamma)$ not labelled $b_\pm$ on the inner boundary has limit in $S_\pm$. If $\phi_-(L_-)$ is disjoint from $\phi_+(L_+)$ for every $\phi_\pm \in L_\pm$ then the moduli space, hence pairing, is zero.

5.5. Orthogonality for Lagrangians with different disk potentials. The second orthogonality result concerns Lagrangians with different values of the disk potential, which we learned from Abouzaid-Fukaya-Oh-Ohta-Ono in personal communication. We assume that $\mathcal{G}_\pm \subset L$ are subsets of weakly unobstructed Lagrangian branes with weakly bounding cochains with curvature $w_\pm$.

\textbf{Theorem 5.14.} If $w_- \neq w_+ \in \Lambda$ then the images of

$$\alpha_+ \in HH_*(\text{Fuk}_{\mathcal{G}_+}(X, b, w_+)), \text{Fuk}_{\mathcal{G}_+}(X, b, w_+))$$

$$\alpha_- \in HH_*(\text{Fuk}_{\mathcal{G}_-}(X, b, w_-)), \text{Fuk}_{\mathcal{G}_-}(X, b, w_-))$$

are orthogonal in $QH(X, b)$:

$$\langle \text{OC}(\alpha_-), \text{OC}(\alpha_+) \rangle = 0.$$
The first step of the proof is to show that the pairing $\langle OC(\alpha_-), OC(\alpha_+) \rangle$ can be realized as the count of treed holomorphic one-node annuli, as in Figure 13 and defined as follows:

**Definition 5.15.** (a) (Treed annuli with zero angle offset) A treed annuli with zero angle offset is same as the treed annuli in definition ..., and thus is equipped with a distinguished leaf $z_{d(c)_\pm}$ on the inner and outer boundaries. The angle offset between the distinguished boundary leaves is zero:

\[
\exists \theta : \quad z_{d(c)_+}^+ = \rho_2 e^{i\theta}, \quad z_{d(c)_-}^- = \rho_1 e^{i\theta}.
\]

The moduli space of treed annuli with $d(c)_+$ resp. $d(c)_-$ leaves on the outer resp. inner boundary and $d(\bullet)$ interior leaves and zero angle offset between distinguished boundary leaves $z_{d(c)_\pm}^\pm$ is denoted by $\mathcal{M}^{\text{ann}, s}_{d(c), d(\bullet)}$.

There is no balancing condition for the annuli with zero angle offset, because the distinguished leaves $z_{d(c)_\pm}^\pm$ either lie in the same component, or there is a unique path of edges connecting the curve components containing the two distinguished leaves. As in the case of treed annuli with non-zero angle offset, there is a map

\[
f : \mathcal{M}^{\text{ann}, s}_{d(c), d(\bullet)} \to \mathcal{M}^{\text{ann}, s}_{(1,1),0} \simeq [-\infty, \infty]
\]

that forgets all non-distinguished leaves, see Figure 13.

(b) (One-node annuli) The nodal annuli occuring in the inverse image $f^{-1}(-\infty)$ are called one-node annuli. The moduli space of one-node annuli is denoted by $\mathcal{M}^{\text{ann}, 1}_{d(c), d(\bullet)} := \mathcal{M}^{\text{ann}, s}_{d(c), d(\bullet), \rho=-\infty} \subset \mathcal{M}^{\text{ann}, s}_{d(c), d(\bullet)}$.

After choosing coherent perturbation data, we denote by $\mathcal{M}^{\text{ann}, 1}_{d(c), d(\bullet), \phi, D, \mathcal{X}_-, \mathcal{X}_+}$ the moduli space of holomorphic treed one-node annuli, that consists of maps $u : C \to X$ where $C$ is a treed one-node annulus, and the map $u$ is holomorphic and adapted. A similar argument to Lemma 5.12 implies that on cohomology the pairing
\( \langle \text{OC}(\alpha_+), \text{OC}(\alpha_-) \rangle \) is equal to the pairing

\[
\langle \alpha_+ - \alpha_-, \alpha_+ - \alpha_- \rangle_1 = \sum_{u \in \mathcal{M}_{\text{ann},1}(\phi, D, \overline{x_-}, \overline{x_+}, b) \cap \mathcal{M}_{\text{ann},2}(\phi, D, \overline{x_-}, \overline{x_+}, b)} (-1)^\nabla w(u).
\]

defined by counting one-node holomorphic treed annuli \( u \in \mathcal{M}_{\text{ann},1}(\phi, D, \overline{x_-}, \overline{x_+}, b) \cap \mathcal{M}_{\text{ann},2}(\phi, D, \overline{x_-}, \overline{x_+}, b) \).

This leads to the following Lemma.

**Lemma 5.16.** For any \( \alpha_\pm \in HH_* (X, b) \),

\[
\langle \text{OC}(\alpha_+), \text{OC}(\alpha_-) \rangle = \langle \alpha_+, \alpha_- \rangle_1.
\]

**Proof.** The proof is similar to the proof of Lemma 5.12, and follows from the fact that a one-node annulus is connected by a path of annuli with finite lengths \( \ell(e) < \infty \) to an annulus with a single node, and then annuli with non-zero width \( \rho > 1 \) and zero angle offset, see Figure 13.

The theorem is proved using counts of holomorphic annuli with two nodes. Let

\[
\overline{\mathcal{M}}_{d(c),d(\bullet)}^{\text{ann},2}(\phi) \subset \overline{\mathcal{M}}_{d(c),d(\bullet)}^{\text{ann},1}(\phi)
\]

denote the subset of \( \overline{\mathcal{M}}_{d(c),d(\bullet)}^{\text{ann},1}(\phi) \) that consists of configurations whose domain nodal annulus contains two disks joined to each other by two edges of infinite length as in Figure 14. Such a configuration is called a *two-node annulus.*

![Figure 14. Two-node annulus](image)

As in the previous regularization of holomorphic annuli in Section 5.2, the moduli spaces \( \overline{\mathcal{M}}_{d(c),d(\bullet)}^{\text{ann},1}(\phi) \) and \( \overline{\mathcal{M}}_{d(c),d(\bullet)}^{\text{ann},2}(\phi) \) admit regularizations by requiring that interior leaves to map to the Donaldson hypersurface \( D \). In particular \( \overline{\mathcal{M}}_{d(\bullet),d(c)}^{\text{ann},2}(\phi, D) \) has one-dimensional component with smooth one-dimensional cells and is compact for any given energy bound. We will not give the details of the construction, since it is similar to the regularization of the moduli spaces considered previously.
Remark 5.17. The locus of two-node treed annuli is a codimension one subset of the moduli space of one-node treed annuli, but it is a “fake boundary component”, since we can glue at either of the nodes to obtain a nodal annulus with a single node. We illustrate this fact using the moduli space $M_{\text{ann},1}^{(1,1)}$ of one-node annuli with a distinguished leaf each on the inner and outer boundary with zero angle offset, and an interior marking. The compactified moduli space $\overline{M}_{\text{ann},1}^{(1,1)}$ is homeomorphic to an annulus, and the subset $\overline{M}_{\text{ann},2}^{(1,1)} \subset \overline{M}_{\text{ann},1}^{(1,1)}$ is a radial line in the annulus.

The count of rigid elements in $\overline{M}_{\text{ann},2}^{d(\phi),d(D)}$ defines a “two-node” pairing similar to the “one-node” pairing (66)

$$\langle x_-, x_+ \rangle_2 = \sum_{i_0, \pm, \ldots, i_{d(D)} \geq 0} \sum_{u \in \overline{M}_{\text{ann},2}^{d(\phi),d(D),a}} (-1)^{\bigtriangledown_+ + \bigtriangledown_-} w(u)$$

where $i_0, \ldots, i_{d(D)} \geq 0$ are number of some additional boundary insertions labelled $b_0, \pm, \ldots, b_{d(D)} \in \text{MC}(\phi_k, \pm)$ and $d'(\circ) = (d'(-), d'(\circ) \geq d(\circ)$ is the total number of boundary insertions for the various Lagrangians and $\bigtriangledown_\pm$ are the signs for the inputs on the inner and outer boundaries.

Proposition 5.18. With notation as above, the relation holds in the Novikov field $\Lambda$

$$\langle w_+ - w_-, \delta(x_-), x_+ \rangle_1 = \langle \delta(x_-), x_+ \rangle_2 - \langle x_-, \delta(x_+) \rangle_2$$

Proof. We prove the Proposition assuming that the Lagrangians are unobstructed and therefore, Maurer-Cartan boundary insertions are zero. The proof of the general case is analogous. The true boundary of the moduli space $\overline{M}_{\text{ann},2}^{d(\phi),d(D)}$ consists of configurations where a separating edge length $\ell(e)$ has become infinite. Indeed, formation of a non-separating infinite edge length is a fake boundary component of the moduli space by an argument as in Remark 5.17. The identity (68) is obtained by a signed count of the true boundary points of one-dimensional components of the moduli space of holomorphic two-node annuli: Suppose a map $u : C \to X$ of type $\Gamma$ is a true boundary point. An edge $e$ of infinite length separates $u$ into a treed annulus $u_1 : C_1 \to X$ of combinatorial type $\Gamma_1$ and a treed disk $u_2 : C_2 \to X$ of type $\Gamma_2$. The map $u$ contributes to the right-hand-side of (68) if $\Gamma_2$ contains an input edge, since the boundary operator $\delta_{CC}$ of the Hochschild chain complex collapses at least one input. If the type $\Gamma_2$ has no inputs, by unobstructedness the output of $\Gamma_2$ is $1_{\phi_+}$ for some $\phi_+ \in \phi_-$. Suppose the infinite edge $e$ is incident on a vertex $v$ in $\Gamma_1$. Forgetting the attaching point on $\Gamma_1$ and collapsing unstable components leads to a configuration in a moduli space of negative expected dimension except if $v$ has
a valence of three, and the map is constant on $S_v$ and on the treed segments incident on $S_v$. The only such configurations are those in Figure 15, and they contribute to the left-hand-side of (68).

\[
\begin{align*}
(x_0^+ & \rightarrow x_0^- \rightarrow 1_o^+ x) \\
(x_0^+ & \rightarrow x_0^- \rightarrow 1_o^+ x)
\end{align*}
\]

**Figure 15.** Configurations in the boundary of the moduli space of two-node annuli that contribute to the left-hand-side of (68)

**Proof of Theorem 5.14.** Suppose that $\alpha_\pm$ are closed in Hochschild homology so that

\[
[\alpha_\pm] \in HH_\bullet(Fuk_{G_\pm}(X, b, w_\pm), Fuk_{G_\pm}(X, b, w_\pm)).
\]

The right-hand-side of (68) vanishes and we obtain

\[
(w_+ - w_-) \langle \Omega_-, \Omega_+ \rangle_1 = 0.
\]

Since $w_+ \neq w_-$ we have $\langle \Omega_-, \Omega_+ \rangle_1 = 0$ as desired. □

**Remark 5.19.** The same argument shows that if $\psi$ is a brane with potential $w_+$ and $\alpha_- \in HH_\bullet(Fuk_{G_-}(X, b, w_-), Fuk_{G_-}(X, b, w_-))$ then

\[
CO(b) \circ OC(b)(\alpha_-) = 0 \in HF(\psi, \psi).
\]

Indeed as explained in the last section the composition $CO(b) \circ OC(b)$ is computed by a count of holomorphic annuli of any fixed radius, and the relation above gives

\[
(w_+ - w_-)(CO(b) \circ OC(b)(\alpha_-), \alpha_+) = \langle \delta(b)(\alpha_-), \alpha_+ \rangle_2 + \langle \alpha_-, m_1 b_+ (\alpha_+) \rangle_2
\]

where $\delta(b)$ is the bulk-deformed Hochschild differential. Since the right-hand-side vanishes for cycles and the pairing on the left is non-degenerate on cohomology, we must have $CO(b) \circ OC(b)(\alpha_-) = 0$.

**Corollary 5.20.** (c.f. Ganatra [Gan]) Suppose $(\psi, b)$ is an object of $Fuk_L(X, b, w)$ with the property that $1 \in HF(\psi, \psi)$ is in the image of $HH_\bullet(Fuk_G(X, b), Fuk_G(X, b))$. The pair $(\psi, b)$ is in the sub-category of $Fuk_L(X, b, w)$ split-generated by $Fuk_G(X, b, w)$, that is, by branes in $G$ with curvature $w$.

**Proof.** By Remark 5.19, the images of Hochschild classes under the composed open-closed and closed-open maps

\[
HH_\bullet(Fuk_G(X, b, w'), Fuk_G(X, b, w')) \to HF(\psi, \psi)
\]
are trivial if \( w' \neq w \). So the identity \( 1_{\psi} \in HF(\psi, \psi) \) must be in the image of \( HH_0(F_{\mathfrak U}(X, b, w), F_{\mathfrak U}(X, b, w)) \). The appendix of Abouzaid [Abo10] then applies.

\[ \square \]

6. Split-generation for blowups

6.1. Embedding of the downstairs Fukaya category. In preparation for the split-generation result we introduce two collections of Lagrangian branes in the blowup: those Lagrangians obtained by inverse image from the original symplectic manifold and those studied in [CW] near the exceptional locus. Recall that \( \tilde{X} \) is an \( \epsilon \)-blowup of \( X \) at a point \( p \in X \).

We introduce the following notation for moduli spaces with insertions at the exceptional locus or blowup point. Let \( b \) be a bulk deformation disjoint from \( p \) and \( \tilde{b} = \pi^{-1}(b) \) its preimage in \( \tilde{X} \). Let \( \pi: \tilde{X} \to X \) denote the projection. For \( \tilde{\Gamma} \) be a combinatorial type obtained from \( \tilde{\Gamma}' \) by forgetting \( d \) interior leaves labelled \( \tilde{D} \). Let \( \Gamma \) be the corresponding combinatorial type of treed disk in \( X \) obtained by replacing the decorations \( \tilde{d}(v) \in H_2(\tilde{\phi}) \) with their projections \( d(v) \in H_2(\phi) \). Denote by

\[ \begin{align*}
\mathcal{M}_{\tilde{\Gamma}}(\tilde{\phi}, \tilde{D}) & \quad \text{the moduli space of holomorphic treed disks of type } \tilde{\Gamma} \text{ and } u(T_e) \in \tilde{D}(\mathcal{L}) \text{ for each edge } e \in \text{Edge}(\Gamma) \text{ labelled } \tilde{D}; \\
\mathcal{M}_{\tilde{\Gamma}}(\tilde{\phi}, \tilde{D}, E, \ldots, E) & \quad \text{the moduli space of configurations with, in addition, } u(T_e) \in E \text{ for all forgotten edges } e \in \text{Edge}(\Gamma) \text{ and } d \text{ interior leaves labelled } \tilde{b}; \\
\mathcal{M}_{\Gamma}(\phi, D, p, \ldots, p) & \quad \text{the moduli space of configurations } u(T_e) \in \{p\} \text{ for all forgotten edges } e \in \text{Edge}(\Gamma) \text{ and } d \text{ interior leaves labelled } b. \\
\end{align*} \]

Let \( \mathcal{M}^{(u, E)}_{\Gamma}(\tilde{\phi}, \tilde{D}) \) denote the moduli space of configurations with intersection number \( (u, E) = d \).

**Theorem 6.1.** Suppose we have chosen a perturbation datum \( P = (J_{\Gamma}, H_{\Gamma}, F_{\Gamma}, E_{\Gamma}) \) with the property that for any type \( \Gamma \) of treed disk to \( X \), the almost complex structure \( J_{\Gamma} \) is standard in a neighborhood of \( p \), that is, given by the standard complex structure in the Darboux chart used for the blowup and \( H_{\Gamma} \) vanishes in a neighborhood of \( p \). The projection \( \pi: \tilde{X} \to X \) induces a bijection between treed disks

\[ \begin{align*}
\mathcal{M}^{(u, E)}_{\Gamma}(\tilde{\phi}, \tilde{D})_0 & \cong \bigcup_{\tilde{\Gamma}' \to \Gamma} \mathcal{M}_{\tilde{\Gamma}'}(\tilde{\phi}, \tilde{D}, E, \ldots, E)_{d}, \\
\mathcal{M}_{\Gamma}(\phi, D, p, \ldots, p) & \cong \bigcup_{\tilde{\Gamma}' \to \Gamma} \mathcal{M}_{\tilde{\Gamma}'}(\tilde{\phi}, \tilde{D}, p, \ldots, p)_{d}, \\
\tilde{u} & \mapsto u = \pi \circ \tilde{u}. \\
\end{align*} \]

**Proof.** By Theorem 2.24, we may choose \( J_{\Gamma} \) to be the standard complex structure on any contractible open neighborhood \( U \) of \( p \), and the Hamiltonian perturbation
Proof of Theorem 1.9 from the Introduction. The bijection
\[ \mathcal{M}(X, \phi, b + q^{-\epsilon}p) \to \mathcal{M}(\tilde{X}, \tilde{\phi}, b) \]
constructed in Theorem 6.1 preserves orientations \( o(u) \), number of interior leaves \( d(\bullet, \phi) \), and (after the adjustment by \( q^{-\epsilon} \) in the bulk insertion \( p \)) symplectic areas in the sense that
\[ A(\tilde{u}) = A(u) - \epsilon([\tilde{u}],[E]) \].
Indeed, any pseudoholomorphic curve in \( \tilde{X} \) projects to a curve in \( X \), with intersections \( \tilde{u}^{-1}(E) \) with the exception locus \( E \) mapping to intersections \( u^{-1}(p) \) with the blowup point \( p \).

Regarding orientations, after capping off the the strip like ends as in [WW] we may assume that the boundary condition is given by a single totally real subbundle \((\partial u)^*TL\). Any deformation of the Lagrangian \((\partial u)^*TL\) to a trivial one for \( \phi \) induces a similar isotopy for \( \tilde{\phi} \). The pullback \( \tilde{u}^*TX \) of the tangent bundle of \( \tilde{X} \) around an intersection with the exceptional divisor \( E \) has a natural trivialization away from \( \tilde{u}^{-1}(E) \). The projection \( \pi \) naturally identifies sections of \( \tilde{u}^{-1}(E) \) locally with sections of the \( u^*TX \) vanishing at 0. The orientations on moduli spaces of disks constructed in [FOOO09] are defined by pinching off sphere bubbles on which the linearized operator has a complex kernel and cokernel, preserving the complex structure. It follows that the induced orientations on the determinant lines for \( u \) and \( \tilde{u} \) are equal.
6.2. Proof of split-generation. In this section we examine the open-closed maps on the collections of branes in the blowup constructed above. Let \( \mathcal{L} \) be a collection of Lagrangian branes in \( X \) disjoint from \( p \) and pairwise cleanly intersecting. Let \( \tilde{\mathcal{L}} \) denote the union of the images of \( \mathcal{L} \) in \( \tilde{X} \) and the branes supported on the Clifford torus near the exception locus. Denote by

\[
Z_k = \{ [z_0, \ldots, z_k, 0, \ldots, 0] \} \cong \mathbb{P}^k \subset E, \quad k = 1, \ldots, n-1
\]

the standard collection of cycles generating the positive-degree homology \( H^{>0}(E) \) of the exceptional locus \( E \):

\[
H^{>0}(E) \cong \text{span}([Z_1], \ldots, [Z_{n-1}]).
\]

We fix the following notations. Let \( \ell \in L \) be a base point. We consider the unit disk \( \mathbb{D} \subset \mathbb{C} \) equipped with the distinguished points \( 0 \in \text{int}(\mathbb{D}) \) and \( 1 \in \partial \mathbb{D} \).

**Corollary 6.2.** For each \( k = 1, \ldots, n-1 \) there is a unique map \( u_k : \mathbb{D} \to \tilde{X}, \) of index \( 2(n-k) \) bounding \( L \) and satisfying

\[
u_k(0) \in Z_k, \quad u_k(1) = \ell, \quad A(u_k) = (n-k)\epsilon.
\]

The map \( u_k \) is regular as a map with these constraints (that is, the linearized operator restricted to sections vanishing at 0 and 1 is surjective) and there are no other stable disks with these properties.

**Proof.** The requirement in (72) is that in the Blaschke classification the degree of any such map \( u : \mathbb{D} \to X \) in the last \( n-k \) components is at least one, \( \text{deg}(u_i) \geq 1, i \geq n-k+1 \) and the maps \( u_1, \ldots u_{n+1} \) have a common zero

\[
z \in \mathbb{D}, u_1(z) = \ldots = u_{n+1}(z) = 0.
\]

Thus \( a_{k+1,1} = \ldots = a_{n+1,1}. \) In particular the index of \( u \) is

\[
I(u) = 2(n-k).
\]

Regularity of \( u \) follows from regularity of the lift \( \hat{u} \) to \( \tilde{X} \), as explained in Cho-Oh [CO06], and the constraints are cut out transversally since varying the parameters \( a_{k+1,j} \) produces an arbitrary variation in the evaluation map \( u(0) \), and variation in the parameters \( \zeta_i \) in (27) produces an arbitrary variation in the evaluation \( u(1) \).

Since \( \tilde{X} \) is Fano, any nodal disk \( u : S \to X \) consists of a Blaschke product \( u_{v_0} : S_{v_0} \to X \) together with a collection of nodal spheres \( u_{v_i}, i = 1, \ldots, m \) of positive index \( I(u_{v_i}) > 0 \); the Blaschke product of \( u_{v_0} \) would have index less than \( 2(n-k) \) and therefore could not meet \( Z_k \) as required. \( \square \)

**Proposition 6.3.** The composition of \( OC(\tilde{b})|HH_\bullet(Fuk_{\pi-1}(\mathcal{L}))(\tilde{X}, \tilde{b}), Fuk_{\pi-1}(\mathcal{L})(\tilde{X}, \tilde{b}) \) with the pull-back \( QH^\bullet(X, b + q^{-p}p) \to QH^\bullet(\tilde{X}, \tilde{b}) \) is equal to the restriction of \( OC(b+q^{-p}) \) to \( HH_\bullet(Fuk_{\mathcal{L}}(X, b+q^{-p}), Fuk_{\mathcal{L}}(X, b+q^{-p})) \). In particular, the image of \( HH_\bullet(Fuk_{\pi-1}(\mathcal{L}))(\tilde{X}, \tilde{b}), Fuk_{\pi-1}(\mathcal{L})(\tilde{X}, \tilde{b}) \) is dimension at least that of \( HH_\bullet(Fuk_{\mathcal{L}}(X, b+q^{-p}), Fuk_{\mathcal{L}}(X, b+q^{-p})) \).

**Proof.** This follows immediately from Theorem 6.1. \( \square \)
Next we relate the open-closed map on the Lagrangians near the exceptional locus to the finite Fourier transform. Let
\[ \zeta \in \mathbb{C}, \quad \zeta^{n-1} = 1 \]
be a primitive \( n - 1 \)-st root of unity. Define
\[ \text{FFT} : \Lambda^{n-1} \to \Lambda^{n-1}, \quad (\lambda_1, \ldots, \lambda_{n-1}) \mapsto \left( \sum_{a=1}^{n-1} \zeta^{ab} \lambda_a \right)_{b=1}^{n-1}. \]

Introduce a quantization of the finite Fourier transform: For \( \epsilon > 0 \)
\[ \text{FFT}_q : \Lambda^{n-1} \to \Lambda^{n-1}, \quad (\lambda_1, \ldots, \lambda_{n-1}) \mapsto \left( \epsilon^b \sum_{a=1}^{n-1} \zeta^{ab} \lambda_a \right)_{b=1}^{n-1}. \]

Thus in particular
\[ \text{FFT}_q |_{q=1} = \text{FFT}. \]

Write \( QH^\bullet(\tilde{X}, \tilde{b}) \) as the direct sum of the image of \( QH^\bullet(X, b) \) and a collection of cycle classes \([Z_1], \ldots, [Z_{n-1}]\), supported on the exceptional divisor \( E \simeq \mathbb{P}^{n-1} \) with each \( Z_k \) diffeomorphic to a complex projective space \( \mathbb{P}^{k} \). Thus we have an isomorphism of vector spaces
\[ QH^\bullet(\tilde{X}, \tilde{b}) \cong QH^\bullet(X, b) \oplus QH^\bullet(\mathbb{P}^{n-1})/\Lambda \cong QH^\bullet(X, b) \oplus \Lambda^{n-1}. \]

Recall the definition of the exceptional collection \( \mathcal{E} \) from (12).

**Lemma 6.4.** The leading order term in the restriction of the open-closed map
\[ OC(0)|HH_\ast(\text{Fuk}_C(\tilde{X}), \text{Fuk}_C(\tilde{X})) \]
composed with projection
\[ QH^\bullet(\tilde{X}, \tilde{b}) \to QH^\bullet(\tilde{X}, \tilde{b})/\pi^\ast QH^\bullet(X, b + q^{-\epsilon}p) \cong \text{span}([Z_1], \ldots, [Z_{n-1}]) \]
is of the form
\[ OC(\tilde{b})|HH_\ast(\text{Fuk}_C(\tilde{X}, \tilde{b}), \text{Fuk}_C(\tilde{X}, \tilde{b})) \text{ mod } QH^\bullet(X, b + q^{-\epsilon}p) = \text{FFT}_q \text{ mod } q^\delta \]
where \( \text{FFT}_q |_{q=1} \) is the finite Fourier transform and \( \delta \) is a constant independent of the choice of \( \epsilon \). As a result, for \( \epsilon \) sufficiently small \( OC(\tilde{b})|HH_\ast(\text{Fuk}_C(\tilde{X}, \tilde{b}), \text{Fuk}_C(\tilde{X}, \tilde{b})) \) surjects onto \( QH^\bullet(\tilde{X}, \tilde{b})/\pi^\ast QH^\bullet(X, b + q^{-\epsilon}p) \).

**Proof.** The proof is similar to the proof surjectivity for the Clifford torus in Theorem 4.17. Via the Blaschke classification (27) there is a unique disk of index 2\( k \) with an interior point mapping to \( Z_k \) and boundary on \( \mathcal{E} \). Let
\[ \gamma_1, \ldots, \gamma_n \in \pi_1((S^1)^n) \]
be the standard set of generators for \( \pi_1((S^1)^n) \). The leading order contributions in the open-closed map \( OC(\tilde{b}) \) arise from disks \( u : \mathbb{D} \to X \) with a single point constraint \( u(z) = x \) on the boundary \( z \in \partial \mathbb{D} \). It follows that the open-closed map \( OC(\tilde{b}) \) sends the point class \([pt] \in HF(\phi, \phi)\) to
\[ (OC(\tilde{b}))(\{pt\}) = (y(\gamma_1), y(\gamma_1 \gamma_2), \ldots, y(\gamma_1 \ldots \gamma_n)) + \text{h.o.t.} \]
similar to the terms in (48). As a result, the point class for the brane \( \phi_{(k)} \) of Proposition 3.11 with local system corresponding to

\[
\mathbb{N} = \exp(2\pi ik/(n-1))
\]

is mapped under the open-closed map \([OC(\tilde{b})]\) to

\[
OC(0)([pt]) = (q^1\mathbb{N}[Z_1] + q^2\mathbb{N}^2[Z_2] + \ldots + q^{(n-1)\mathbb{N}^{n-1}[Z_{n-1}]} + \text{h.o.t.}) \in QH(\tilde{X})
\]

plus higher order terms in \( q \). In the basis given by \([Z_1], \ldots, [Z_{n-1}]\) the open-closed map \( OC(\tilde{b}) \) composed with projection onto the span of \( Z_1, \ldots, Z_{n-1} \) is therefore the finite Fourier transform \( \text{FFT}_q \) to leading order. It follows that the composition of \( OC(b)\rvert_{HH^\bullet(Fuk_L(\tilde{X}, \tilde{b}), Fuk_E(\tilde{X}, \tilde{b}))} \) with projection is an isomorphism onto \( QH^\bullet(\tilde{X}, \tilde{b})/\pi^*QH^\bullet(X, b + q^{-\epsilon}p) \) for \( \epsilon \) sufficiently small.

\[\square\]

Proof of Theorem 1.1. For sufficiently small \( \epsilon \), the Lagrangians \( \pi^{-1}(L) \) and \( E \) are disjoint. By Theorem 5.8 the images of \( HH^\bullet(Fuk_L(\tilde{X}, \tilde{b}), Fuk_L(\tilde{X}, \tilde{b})) \) and \( HH^\bullet(Fuk_E(\tilde{X}, \tilde{b}), Fuk_E(\tilde{X}, \tilde{b})) \) in \( QH^\bullet(\tilde{X}, \tilde{b}) \) under the open-closed map \( OC(\tilde{b}) \) are orthogonal. Therefore it suffices to show that their images have complementary dimension in \( QH^\bullet(\tilde{X}, \tilde{b}) \). By Corollary 3.12, we have

\[
\dim HH^\bullet(Fuk_E(\tilde{X}, \tilde{b}), Fuk_E(\tilde{X}, \tilde{b})) = n - 1
\]

and it injects into \( QH(\tilde{X}) \) by Lemma 6.4. On the other hand, by Theorem 1.9

\[
\dim HH^\bullet(Fuk_{\pi^{-1}(L)}(\tilde{X}, \tilde{b}), Fuk_{\pi^{-1}(L)}(\tilde{X}, \tilde{b})) = \dim QH^\bullet(X, b)
\]

and it injects into \( QH(\tilde{X}) \) by Proposition 6.3. The claim now follows.

\[\square\]

Proof of Corollary 1.2. Equation (3) is an immediate consequence of the split generation statement and the fact that the exceptional and unexceptional Lagrangians are disjoint. For the splitting of quantum cohomology, consider the decomposition of the Hochschild cohomology

\[
HH_{\dim(X)-\bullet}(Fuk_L(\tilde{X}, \tilde{b}), Fuk_L(\tilde{X}, \tilde{b})) \cong QH^\bullet(\tilde{X}, \tilde{b})
\]

according to subspaces generated the collections \( \pi^{-1}(L), E \). Since these two collections of branes are disjoint and thus have orthogonal image we obtain

\[
QH(\tilde{X}, \tilde{b}) \cong QH(X, b + q^{-\epsilon}p) \oplus QH(pt)^{\oplus n-1}
\]

as desired.

\[\square\]

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