Wave-Particle Duality and the Objectiveness of “True” and “False”

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Received: 14 September 2020 / Accepted: 7 June 2021 / Published online: 8 July 2021
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Abstract
The traditional analysis of the basic version of the double-slit experiment leads to the conclusion that wave-particle duality is a fundamental fact of nature. However, such a conclusion means to imply that we are not only required to have two contradictory pictures of reality but also compelled to abandon the objectiveness of the truth values, “true” and “false”. Yet, even if we could accept wave-like behavior of quantum particles as the best explanation for the build-up of an interference pattern in the double-slit experiment, without the objectivity of the truth values we would never have certainty regarding any statement about the world. The present paper discusses ways to reconcile the correct description of the double-slit experiment with the objectiveness of “true” and “false”.

Keywords Truth values · Double-slit experiment · Propositions · Mathematical statements · Truth value assignment · Wigner’s friend

1 Introduction

Wave-particle duality is the concept of quantum mechanics, which holds that every quantum particle can be described as either a particle or a wave [1]. This dualism is a direct consequence of Niels Bohr’s principle of complementarity which states that having precise knowledge of one of two experimental outcomes complementing each other prevents us from obtaining complete information about the remaining one [2]. Though wave-particle dualism was proposed long ago, an explanation of its meaning has not been agreed yet, and it persists in being perplexing to the mind. As an illustration of an unfading interest to the question of “particle” versus “wave” and its relevancy in our time, see papers [3] and [4], just to name a few.
To make plain the perplexing character of wave-particle duality, let us consider the basic version of the Young’s double-slit experiment: Emitted one at a time, quantum particles (like photons or electrons) hit a plate pierced by two slits (labeled 1 and 2), which are located along the x-axis at $x = 0$ and $x = d$, respectively, and are afterwards observed on a screen behind the plate.

Let us examine the following statement: «In the double-slit experiment, the quantum particle passes through one or the other slit, but not both» (to set statements off from the rest of the text, in this paper they are inserted in the double angle quotation marks). For brevity, let the above statement be denoted by the capital letter $C$.

On the word of Feynman [5], if one has a piece of apparatus able to determine whether a quantum particle goes through the slit 1 or the slit 2 (called a which-way detector), then one can say that the particle goes through either the slit 1 or the slit 2 and so one can say that the statement $C$ is true. But, when both slits are open and there is no which-way detector, an interference pattern builds up slowly on the screen as more particles go through the slits. In that case, one may not say that each quantum particle passes through either the slit 1 or the slit 2, and, correspondingly, one may not say that the statement $C$ is true. Providing $C$ may be either true or false, this last means that the statement $C$ is false.

In this way, the concept of wave-particle duality brings in the dual valuation for the statement $C$. Namely, in case the quantum particle is described as a classically defined particle, $C$ is true, but if the quantum particle is described as a classically defined wave, $C$ becomes false.

However, the problem is that the character of the statement $C$ is quite different from that of a contingent statement which may be true in one instance but false in another. As a matter of fact, if a which-way detector is present in the experiment, the statement $C$ is a tautology, i.e., $C$ is true in every instance (here “instance” is understood as a recording of the particle’s position on the screen). Put differently, $C$ is true regardless of the truths and falsities of the contingent statements «The quantum particle passes through the slit 1» denoted $P_1$ and «The quantum particle passes through the slit 2» denoted $P_2$. But as soon as the which-way detector is out (and both slits are open), the statement $C$ (that can be written as «$P_1$ or $P_2$, but not, $P_1$ and $P_2$») makes a transition from a tautology to a contradiction since then there are no instances wherein $C$ could fail to be false.

A transformation of a tautology into a contradiction (and vice versa) would be possible if “true” and “false” were to be not objective (absolute) but relative to experience. To be sure, suppose that “true” and “false” were dependent on the act of observing the quantum particle in the double-slit experiment. Then, the type of observation would determine which of the properties of the quantum particle—pertaining to a particle or to a wave—would show up in the experiment and, hence, what kind of a logical statement would be $C$—a tautology or a contradiction.

More to the point, let both slits be open and suppose that a which-way detector is brought in the experiment sometime. Then, one finds that even if the statement $C$ initially is a contradiction, it may nonetheless turn out to be a tautology at a later time. This would be equivalent to saying that “false” can be converted into “true” when the observation is made or when “false” is subject to empirical testing. Clearly, this could happen only if the truth values were subjective.
On the other hand, if “true” and “false” were to be nonabsolute, we would never have certainty regarding any statement about the world—including statements grounded in meanings, independent of matters of fact. Take, for example, the mathematical statement «2+2=5». It is false solely by definition, that is, we need not consult experience to determine whether «2+2=5» is false or not. Moreover, even though this statement can be falsified by experience, it is not grounded in experience. This means that the aforesaid statement is necessarily false and so we can be certain that it was false in the past and will remain false at any future moment, regardless of the limits of our present knowledge or our powers of theoretical understanding. However, were “false” to be subjective, the statement «2+2=5» would be empirically false, that is, the falsity of «2+2=5» would be contingent on the observation of facts. As a result, we could not rule out an occasion when this statement was verified by experience.

Hence, the wave-particle duality appears to lead to the conclusion that the truths and falsities of logic and mathematics need confirmation by observations.

Since this conclusion is controvertible at best, the puzzle is, then, how to bring together the successful description of the double-slit experiment with the objectiveness of “true” and “false”. The present paper seeks the answer to this puzzle.

2 The Motivation

The attentive reader may have noticed that the validity of dual valuation for the statement C, pointed out in the previous section, is based on the following two conditions:

(a) a truth value of the complex statement «P₁ or P₂, but not, P₁ and P₂» entirely depends on truth values of its constituent statements, i.e., P₁ and P₂,

(b) a relation between the set of statements and the set of the truth values (that has just two members, “true” and “false”) is a total function (as a result, every statement is either true or false but not both, and it is not the case that a statement is not true and not false).

So, to dismiss dual valuation for C (and thus preserve the objectiveness of “true” and “false”), either one (or all) of the above conditions must be revised.

As is well known, it is bivalent truth-functional propositional logic (usually called classical logic) which takes for granted that truth values of complex statements are defined by truth values of their constituent statements. Therefore, a revision of the condition (a) entails nothing less than a modification of classical logic.

Potentially, the laws of classical logic can be modified in various ways. One of such modifications (that replaces the classical logical connectives by different ones inspired by the lattice-oriented operations) is quantum logic proposed by Birkhoff and von Neumann [6] in 1936. Since that time, other forms of quantum logics have been developed, a few of them are: a dynamic quantum logic [7], exogenous
quantum propositional logic [8], and a categorical quantum logic [9] (more can be found in [10]).

Another modification of classical logic is the calculus of partial propositional functions introduced in [11] and systematically studied in [12]. This modification takes into account that arbitrary statements may be pairwise “incompatible” and thus non-connectable. According to the calculus of partial propositional functions, the complex statements «X or not-X» and «X and not-X» are always true and false, respectively, even though their constituent statements may lack a truth value. What is more, this lack of a truth value can be considered responsible for the interference pattern in the double-slit experiment [13].

On the other hand, no modification can change the fact that propositional logic—as a mathematical model that allows us to reason about the truth or falsity of expressions constructed from simple statements (i.e., ones that are not linked by logical connectives)—cannot see inside those statements. For example, consider two simple statements: «In the double-slit experiment, both slits are open» and «In the double-slit experiment, only one slit is open». Let the letters $S_{both}$ and $S_{one}$ denote the said statements in the order given. Propositional logic sees $S_{both}$ and $S_{one}$ as indivisible entities, and from that reason, the parts of $S_{both}$ and $S_{one}$ concerning the number of open slits cannot be taken into consideration. This implies that it is impossible to distinguish «$S_{both}$ and $P_n$» from «$S_{one}$ and $P_n$» (where $n$ is 1 or 2) and, hence, to decide which of these complex statements may lack a truth value by remaining exclusively within the frame of propositional logic (or for that matter, any of its modifications including the calculus of partial propositional functions and whichever form of quantum logic).

But rather than exploring more complicated branches of logic to address this problem, one may prefer to keep up the laws of classical logic and use an alternative approach, in which an assignment of truth values to statements about quantum systems directly accesses the mathematical formalism of quantum theory. This approach will be presented in the next sections.

### 3 Truth Values of Statements About the Double-Slit Experiment

Let us start by recalling that a simple statement of a mathematical relation (as equality or inequality) between meaningful expressions (i.e., symbols or combinations of symbols representing a value, a function, an object or the like) is called an atomic mathematical statement [14, 15]. A statement like this can be classified as analytic since its truth and falsity depend solely on the meaning of its terms. A simple synthetic (i.e., not analytic) statement, which affirms or denies something meaningful about the world and is capable of being true or false, is called an atomic proposition [16].

More complex mathematical statements (or propositions) are called molecular ones. They are built up out of atomic components via logical connectives, for example, $\sqcup$ (logical disjunction), $\sqcap$ (logical conjunction) and $\neg$ (logical negation).

In the paper, an atomic mathematical statement is denoted by a lowercase letter (which may contain sub- and superscript characters), for example, «$3 \leq 4 \rangle = \alpha_1$$,$ while
an atomic proposition is denoted by an uppercase letter (which also may contain sub- and superscript characters), e.g., «Today is Sunday» = \(A\), where the combination of «...» and = stands for “...is denoted by ...”.

Correspondingly, the atomic proposition asserting that in the double-slit experiment the quantum particle passes through the particular slit – 1 or 2 – can be presented in the following way:

\[
\begin{align*}
n \in \{1, 2\} : \ & \text{ «The quantum particle passes through the slit } n \text{»} = P_n. \tag{1}
\end{align*}
\]

Suppose that at the moment when the quantum particle comes out from the double-slit plate, its state is described by \(\Psi(x)\), the complex scalar function of \(x\) (whose codomain contains more than the value 0). Providing \(\Psi(x)\) is given, one can consider the atomic mathematical statement posed as

\[
\begin{align*}
n \in \{1, 2\} : \ & \text{ «} \Psi(x) \in \{c_n \phi_n(x) \mid c_n \in \mathbb{C}, c_n \neq 0\} \text{»} = s_n, \tag{2}
\end{align*}
\]

where \(\phi_1(x)\) and \(\phi_2(x)\) are spatially separated complex functions of \(x\) localized at \(x = 0\) and \(x = d\), in that order. The statement \(s_n\) is true if \(\Psi(x)\) is equal to the function \(\phi_n(x)\) multiplied by some (non-zero) complex number \(c_n\), otherwise \(s_n\) is false.

Due to their spatial separation, the functions \(\phi_1(x)\) and \(\phi_2(x)\) can be made orthonormal over the interval \(-\infty \leq x \leq +\infty\), to be exact,

\[
\begin{align*}
n, l \in \{1, 2\} : \ & \langle \phi_n(x) | \phi_l(x) \rangle = \int_{-\infty}^{+\infty} \phi^*_n(x) \phi_l(x) \, dx = \delta_{nl}. \tag{3}
\end{align*}
\]

As a result, the statements \(s_1\) and \(s_2\) cannot be true together, i.e.,

\[
\begin{align*}
n \in \{1, 2\}, m = n - (-1)^n : \ & (s_n \cap s_m) \equiv \bot, \tag{4}
\end{align*}
\]

where the connective \(\equiv\) corresponds to the expression “is equivalent to” and \(\bot\) stands for an arbitrary contradiction. In other words, the truth of \(s_1\) means the falsity of \(s_2\), and the truth of \(s_2\) means the falsity of \(s_1\).

However, it may be the case that \(s_1\) and \(s_2\) are false together, to be exact,

\[
\begin{align*}
(s_n \cup s_m) \equiv \top, \tag{5}
\end{align*}
\]

where \(\equiv\) stands for “not equivalent to” and \(\top\) signifies an arbitrary tautology. For example, assume that the state of the quantum particle, just as it emerges from the double-slit plate, is described by a superposition of the functions \(\phi_1(x)\) and \(\phi_2(x)\), namely,

\[
\begin{align*}
\Psi(x) = c_1 \phi_1(x) + c_2 \phi_2(x). \tag{6}
\end{align*}
\]

Then, \(\Psi(x)\) is not an element of \(\{c_1 \phi_1(x) \mid c_1 \in \mathbb{C}, c_1 \neq 0\}\), and neither it is an element of \(\{c_2 \phi_2(x) \mid c_2 \in \mathbb{C}, c_2 \neq 0\}\), that is, both \(s_1\) and \(s_2\) are false.

Suppose that at the moment \(t\) the particle reaches the screen behind the double-slit plate and at that time its state is given by

\[
\begin{align*}
\Psi(x, t) = c_1 \phi_1(x, t) + c_2 \phi_2(x, t). \tag{7}
\end{align*}
\]
where $\langle \phi_n(x,t)|\phi(x,t)\rangle = \delta_{nl}$. Understanding that the function $\Psi(x,t)$ is orthonormal over the interval $-\infty \leq x \leq +\infty$, the following holds

$$
\langle \Psi(x,t)|\Psi(x,t)\rangle = \int_{-\infty}^{+\infty} |\Psi(x,t)|^2 \, dx = 1, \quad (8)
$$

where, in accordance with (7),

$$
|\Psi(x,t)|^2 = |c_1 \phi_1(x,t)|^2 + |c_2 \phi_2(x,t)|^2 + c_1^* c_2 \phi_1^*(x,t) \phi_2(x,t) + c_2^* c_1 \phi_2^*(x,t) \phi_1(x,t). \quad (9)
$$

The cross terms in the above expression represent interference.

Those terms will drop off if a which-way detector is placed at the double-slit plate. To demonstrate this, assume that $|d_1\rangle$ and $|d_2\rangle$ are two possible quantum states of a which-way detector such that

$$
\langle d_1|d_1\rangle = \delta_{nl}. \quad (10)
$$

In the presence of this detector, the quantum states of the paths $|\phi_1\rangle$ and $|\phi_2\rangle$ given by

$$
|\phi_n\rangle = \int_{-\infty}^{+\infty} \phi_n(x)|x\rangle \, dx \approx \int_{x_n-\Delta x}^{x_n+\Delta x} \phi_n(x)|x\rangle \, dx, \quad (11)
$$

where $x_1 = 0$, $x_2 = d$ and $\Delta x$ is some positive value, get entangled with the corresponding states of the detector $|d_1\rangle$ and $|d_2\rangle$, so that immediately after the particle emerges from the double slit plate (equipped with the detector), the total quantum state of the experiment $|\Psi_{\text{exp}}\rangle$ is

$$
|\Psi_{\text{exp}}\rangle = \int_{-\infty}^{+\infty} \Psi_{\text{exp}}(x)|x\rangle \, dx, \quad (12)
$$

where

$$
\Psi_{\text{exp}}(x) = c_1 \phi_1(x)|d_1\rangle + c_2 \phi_2(x)|d_2\rangle. \quad (13)
$$

Due to orthogonality of $|d_1\rangle$ and $|d_2\rangle$, the cross terms will be missing in the square modulus of $\Psi_{\text{exp}}(x,t)$:

$$
|\Psi_{\text{exp}}(x,t)|^2 = |c_1 \phi_1(x,t)|^2 + |c_2 \phi_2(x,t)|^2. \quad (14)
$$

Moreover, caused by the macroscopic nature of the which-way detector, the state $|\Psi_{\text{exp}}\rangle$ will evolve over some (short) period $\tau$ into one of the entangled states, meaning that the quantum particle will be reported by the macroscopic detector at exactly one slit. In symbols,

$$
c_1 \phi_1(x)|d_1\rangle + c_2 \phi_2(x)|d_2\rangle \xrightarrow{\tau} \text{either } c_1 \phi_1(x, \tau)|d_1\rangle \text{ or } c_2 \phi_2(x, \tau)|d_2\rangle. \quad (15)
$$

Therefore, the atomic mathematical statements
\( n \in \{1, 2\} : \ \langle \Psi_{\text{exp}}(x, \tau) \in \{ c_n \phi_n(x, \tau)|d_n| \mid c_n \in \mathbb{C}, c_n \neq 0 \rangle = s'_n \) \hspace{1cm} (16)

will be neither true together nor false together, that is, \((s'_1 \cap s'_2) \Leftrightarrow \bot\) and \((s'_1 \cup s'_2) \Leftrightarrow \top\).

In contrast to propositional logic, assume that the notions of truth and falsity of a statement cannot be regarded as primitive; rather, a proof must be provided in order to accept that the statement is true or false. For a mathematical statement, such a proof can be either constructive or non-constructive. A non-constructive proof confirms the validity of a mathematical relation between expressions constituting a statement without providing an instance of those expressions. Whereas a constructive proof demonstrates that the mathematical relation is valid by creating an instance of the expressions. Take the mathematical statement such as \(\langle a^b \in \mathbb{Q}\rangle\) where the symbols \(a\) and \(b\) denote certain irrational numbers: \(a, b \in \mathbb{R} \setminus \mathbb{Q}\). To confirm the validity of the relation \(\in\) between \(a^b\) and the set of rational numbers, \(\mathbb{Q}\), the non-constructive proof does not provide an instance of \(a\) and \(b\) but shows that the said relation is possible; whilst on the contrary, the constructive proof gives such an example: \((\sqrt{2})^{\log \sqrt{2}} = 3\).

As to a proposition, its proof can be provided by the truth of the relating mathematical statements. Specifically, the truth of the mathematical statement \(s_n\) (or \(s'_n\)) can be taken as positive evidence witnessing the truth of the proposition \(P_n\). By the same token, the truth of \(s_m\) (or \(s'_m\)) can serve as negative evidence demonstrating the falsity of \(P_n\).

The problem is how to define a truth value of the proposition \(P_n\) if neither evidence exists, that is, if both \(s_n\) and \(s_m\) are false and no which-way detector is present.

To state this problem formally, let us use the double-bracket notation \([\cdot]\) to express a truth value of a mathematical statement or a proposition.

Due to its nature, a mathematical statement—atomic and molecular alike—cannot be both true and false as well as neither true nor false. Hence, the relation between the set of mathematical statements and the set of truth values is a total surjective-only function, i.e.,

\[ v : \mathbb{S} \rightarrow \mathbb{B}_2, \] \hspace{1cm} (17)

where \(\mathbb{S}\) denotes the set of mathematical statements and \(\mathbb{B}_2\) stands for the set of two truth values, \(T\) ("true") and \(F\) ("false"), which can be interpreted as integers 1 and 0, respectively. The image of a mathematical statement, for example, \(s\), under this function can be denoted by \([s] = v(s)\).

Let us introduce a total bijective (i.e., both injective and surjective) function \(b\) that takes truth values of two (different) mathematical statements to elements of \(\mathbb{B}_2\):

\[ b : \mathbb{B}_2 \times \mathbb{B}_2 \rightarrow \mathbb{B}_2. \] \hspace{1cm} (18)

Using this function, the truth value of the proposition \(P_n\) can be considered as the image of ordered pair \((\[s_n\], \[s_m\])\) under \(b\), that is,

\[ \[P_n\] = b(\[s_n\], \[s_m\]). \] \hspace{1cm} (19)
Explicitly, the function $b$ will return 1 if $s_n$ is true, and $b$ will return 0 if $s_m$ is true; in symbols, $b(1,0) = 1$ and $b(0,1) = 0$.

Then again, both $s_n$ and $s_m$ can be false, so, there is one more pair, $(0, 0)$. Given three different objects—i.e., three ordered pairs $(1, 0)$, $(0, 1)$ and $(0, 0)$—but only two elements of $\mathbb{P}_2$ to map them onto, one has a problem (which can be called the problem of an extra object): What is the image of the pair $(0, 0)$ under the function $b$? In other words, what truth value does the proposition $P_n$ have if $\Psi(x)$ is a superposition of the states $\phi_1(x)$ and $\phi_2(x)$? Symbolically, this problem can be presented as follows:

$$
\|P_n\| = b(\|s_n\|, \|s_m\|) = \begin{cases} 
1, & s_n = 1 \\
0, & s_m = 1 \\
?, & s_n = s_m = 0 
\end{cases}.
$$

(20)

4 Truth Values of a Propositional Formula

The problem of an extra object also concerns an assignment of the truth values to a propositional formula (i.e., an expression involving finitely many logical connectives and propositions).

To ascertain this, let us turn to projection operators, i.e., self-adjoint operators with spectrum contained in the two-element set $\{0, 1\}$. Such operators are in one-to-one correspondence with the closed linear subspaces of a Hilbert space $\mathcal{H}$ (i.e., a complex vector space upon which an inner or scalar product is defined) [17].

Let $\hat{P}$ be a projection operator; then, every unit vector $|\Psi\rangle \in \mathcal{H}$ can be decomposed uniquely as $|\Psi\rangle = |\psi\rangle + |\phi\rangle$ with $|\psi\rangle = \hat{P}|\Psi\rangle$ and $|\phi\rangle = -\hat{P}|\Psi\rangle$, where $-\hat{P}$ is the projection operator corresponding to the negation of $\hat{P}$ which can be expressed through the identical operator $\hat{1}$ and $\hat{P}$ as $-\hat{P} = \hat{1} - \hat{P}$, $|\psi\rangle$ belongs to the closed linear subspace $\mathcal{H}_{\hat{P}}$, while $|\phi\rangle$ lies in $\mathcal{H}_{\hat{P}}^\perp$, i.e., the closed linear subspace orthogonal to $\mathcal{H}_{\hat{P}}$. Therefore, $\hat{P}$ breaks the Hilbert space $\mathcal{H}$ into two orthogonal subspaces,

$$
\mathcal{H} = \mathcal{H}_{\hat{P}} \oplus \mathcal{H}_{\hat{P}}^\perp,
$$

(21)

such that $\hat{P}$ leaves any vector in $\mathcal{H}_{\hat{P}}$ invariant but annihilates any vector in $\mathcal{H}_{\hat{P}}^\perp$, namely,

$$
\mathcal{H}_{\hat{P}} = \{ |\psi\rangle \in \mathcal{H} : \hat{P}|\psi\rangle = |\psi\rangle \},
$$

(22)

$$
\mathcal{H}_{\hat{P}}^\perp = \{ |\phi\rangle \in \mathcal{H} : \hat{P}|\phi\rangle = 0 \}.
$$

(23)

For the Hermitian operator $A$ with a discrete orthonormal basis in each eigenspace (i.e., a subspace containing eigenvectors $|a_n\rangle$ of a given eigenvalue $a_n$), the projection operator $\hat{P}$ can be presented as
\[ \hat{P}_n = |a_n\rangle\langle a_n|, \quad (24) \]

on condition that the eigenvalue \( a_n \) is nondegenerate (that is, the eigenspace is 1-dimensional). Accordingly, two projection operators \( \hat{P}_n \) and \( \hat{P}_m \) of the same Hermitian operator \( A \) satisfy

\[ \hat{P}_n \hat{P}_m = \hat{P}_m \hat{P}_n = \delta_{nm} \hat{P}_n. \quad (25) \]

In the case of the position operator \( X \) (whose spectrum is continuous), the projection operators \( \hat{P}_n \) are associated with the corresponding intervals \([x_n - \Delta x, x_n + \Delta x]\), specifically,

\[ \hat{P}_n = \int_{x_n - \Delta x}^{x_n + \Delta x} |x\rangle\langle x| \, dx, \quad (26) \]

so that \( \hat{P}_1 \) and \( \hat{P}_2 \) are orthogonal if these intervals do not intersect (i.e., if \( 2\Delta x < d \)). The resolution of identity related to the given case is

\[ \int_{-\infty}^{+\infty} |x\rangle\langle x| \, dx = \hat{P}_1 + \hat{P}_2 + \sum_{k=1}^{3} \int_{A_k}^{B_k} |x\rangle\langle x| \, dx = 1, \quad (27) \]

where \( A_k \) and \( B_k \) are the endpoints of the intervals \( I_k^{\text{plate}} \) containing \( x \)-coordinates situated on the side of the plate opposite to the source emitting quantum particles, i.e., behind the slits:

\[ I_k^{\text{plate}} = \begin{cases} 
( -\infty, x_1 - \Delta x ] , & k = 1 \\
[ x_1 + \Delta x, x_2 - \Delta x ] , & k = 2 \\
[ x_2 + \Delta x, +\infty ) , & k = 3 
\end{cases} \quad (28) \]

The eigenkets of the projection operators \( \hat{P}_k^{\text{plate}} = \int_{A_k}^{B_k} |x\rangle\langle x| \, dx \) associated with these intervals, i.e., the vectors that meet the condition \( \hat{P}_k^{\text{plate}} |\phi_k^{\text{plate}}\rangle = |\phi_k^{\text{plate}}\rangle \), can be written as

\[ |\phi_k^{\text{plate}}\rangle = \int_{A_k}^{B_k} \phi_k^{\text{plate}}(x) |x\rangle \, dx. \quad (29) \]

Provided that the plate is impenetrable to the particles (and non-classical paths of the particles, which cause higher-order corrections to the interference pattern \[18\], are absent), each function \( \phi_k^{\text{plate}}(x) \) must be equal to zero in the interval \( I_k^{\text{plate}} \) and so each eigenket \( |\phi_k^{\text{plate}}\rangle \) must be the zero vector \( |0\rangle \) (belonging to the zero subspace \{0\}). For this reason, every \( \hat{P}_k^{\text{plate}} \) can be regarded as the zero operator \( \hat{0} \) (that takes any vector to the zero vector). Accordingly, in this case the completeness relation has the form:

\[ \hat{P}_1 + \hat{P}_2 = \hat{1}. \quad (30) \]
Let $L(\mathcal{H})$ denote the set of the closed linear subspaces of the Hilbert space $\mathcal{H}$. As stated by [19], the pair of elements in $L(\mathcal{H})$ that represent the distinct propositions $Q$ and $P$ are the subspaces $\mathcal{H}_Q$ and $\mathcal{H}_P$, respectively. These subspaces can be called related or comparable in case the mathematical statement $z = z_1 \cup z_2$ is true, that is, if the following holds
\[
[z] = \max \{ [[z_1]], [[z_2]] \} = 1,
\]
where $z_1$ and $z_2$ stand in for the mathematical statements that affirm the subset relation $\subseteq$ among $\mathcal{H}_Q$ and $\mathcal{H}_P$. Explicitly,
\[
\langle \mathcal{H}_Q \subseteq \mathcal{H}_P \rangle = z_1,
\]
\[
\langle \mathcal{H}_P \subseteq \mathcal{H}_Q \rangle = z_2.
\]
Note that when $z$ is true, the projection operators $\hat{Q}$ and $\hat{P}$ commute (are compatible), explicitly, $\hat{Q}\hat{P} = \hat{P}\hat{Q}$.

Contrastively, the subspaces $\mathcal{H}_Q$ and $\mathcal{H}_P$ can be called orthogonal if the mathematical statement $w = w_1 \cup w_2$ is true, i.e., if
\[
[w] = \max \{ [[w_1]], [[w_2]] \} = 1,
\]
where $w_1$ and $w_2$ substitute for the mathematical statements asserting the orthogonality relation among $\mathcal{H}_Q$ and $\mathcal{H}_P$:
\[
\langle \mathcal{H}_Q \subseteq \mathcal{H}_P^\perp \rangle = w_1,
\]
\[
\langle \mathcal{H}_P \subseteq \mathcal{H}_Q^\perp \rangle = w_2.
\]
Note that in case $w$ is true, the projection operators $\hat{Q}$ and $\hat{P}$ are compatible and orthogonal, i.e., $\hat{Q}\hat{P} = \hat{P}\hat{Q} = 0$.

Clearly, if the subspaces $\mathcal{H}_Q$ and $\mathcal{H}_P$ are orthogonal, they are incomparable, and if they are comparable, they are not orthogonal. To be exact, except for the subspaces $\mathcal{H}_0 = \{0\}$ and $\mathcal{H}_1 = \mathcal{H}$, the statements $w$ and $z$ cannot be true together:
\[
(z \cap w) \Leftrightarrow \perp.
\]
Even so, $z$ and $w$ may be false together:
\[
(z \cup w) \Leftrightarrow T.
\]
In particular, if the projection operators $\hat{Q}$ and $\hat{P}$ are not compatible, i.e., $\hat{Q}\hat{P} \neq \hat{P}\hat{Q}$, then $\mathcal{H}_Q \nsubseteq \mathcal{H}_P$ and $\mathcal{H}_P \nsubseteq \mathcal{H}_Q$, as well as $\mathcal{H}_Q \nsubseteq \mathcal{H}_P^\perp$ and $\mathcal{H}_P \nsubseteq \mathcal{H}_Q^\perp$.

Let the inequality $Q \leq P$ be expressible as $Q \cap P \Leftrightarrow Q$ or as $Q \cup P \Leftrightarrow P$. Also, let the inequality $Q \geq P$ be expressible as $Q \cap P \Leftrightarrow P$ or as $Q \cup P \Leftrightarrow Q$. Then, one can construct the propositional formula $Q \leq P$ meaning...
\[ Q \lesssim P \Leftrightarrow (Q \leq P) \sqcup (Q \geq P). \]  

(39)

It can be phrased as the following statement: «Two distinct propositions about the same quantum system can be true together». Given that for the above formula the equivalences \([z] = 1\) and \([w] = 1\) act respectively as positive and negative evidence, its truth value can be considered as the image of the ordered pair \(([z], [w])\) under the map (18), i.e.,

\[ [Q \lesssim P] = b([z], [w]). \]

(40)

Specifically, \(Q\) and \(P\) can be true together if \(\mathcal{H}_Q\) and \(\mathcal{H}_P\) are comparable and, hence, not orthogonal; on the contrary, \(Q\) and \(P\) cannot be true together if \(\mathcal{H}_Q\) and \(\mathcal{H}_P\) are orthogonal, therefore, incomparable. That is, \([Q \lesssim P]\) is equal to 1 on the pair \((1, 0)\) and is equal to 0 on the pair \((0, 1)\).

Again, similar to the situation with the atomic proposition \(P_n\), the third pair exists, \((0, 0)\), which gives rise to the problem of an extra object. The problem is this: How should a truth value of \(Q \lesssim P\) be defined in case that neither evidence exists for this propositional formula, that is, if the subspaces \(\mathcal{H}_Q\) and \(\mathcal{H}_P\) representing \(Q\) and \(P\) are neither comparable nor orthogonal? Symbolically,

\[ [Q \lesssim P] = b([z], [w]) = \begin{cases} 
1, & [z] = 1 \\
0, & [w] = 1 \\
?, & [z] = [w] = 0 
\end{cases}. \]

(41)

5 Three-Valued Semantics

An apparently justified solution to the problem of an extra object is to abandon the principle, according to which the set of the truth values must be limited to only two elements. Particularly, one can suggest adding a third element to the set \(\{T, F\}\), for example, \(U\) (i.e., the “undefined” or “undetermined” truth value, as it is assumed in Kleene’s “(strong) logic of indeterminacy” [20] and Priest’s “logic of paradox” [21], respectively). This element can be interpreted as a real number lying between 0 and 1, say, \(\frac{1}{2}\), in agreement with Łukasiewicz logic \(L_3\) [22]. The nature of the added element is supposed to be the same as the one of the truth values \(T\) and \(F\) (otherwise, it would be hard to guarantee the consistency and unambiguity of what constitutes and what does not constitute a set of the truth values [23]).

As a result, the simple resolution of the extra object problem immediately follows. Indeed, using the total bijective function

\[ b : \mathbb{B}_2 \times \mathbb{B}_2 \rightarrow \mathbb{B}_3 \]

(42)

whose codomain \(\mathbb{B}_3\) is the collection of three elements, \(\{T, U, F\}\) or \(\{1, \frac{1}{2}, 0\}\), a truth value of the proposition \(P_n\) can be defined by
Attractive as this proposal (initially introduced by Reichenbach [24]) might seem, it is open to serious objection.

First, why are there two different sets of the truth values—that is, the set \{T, F\} for statements of mathematical relations (i.e., analytic statements) and the set \{T, U, F\} for propositions (i.e., synthetic statements)? Since the existence of a precisely cut distinction between analytic and synthetic statements is doubtful [25], the presence of two sets of the truth values appears unlikely.

Second, if the intermediate truth value U is assigned to at least some proposition(s) in a propositional formula, how is one to determine a truth value thereof? It is known to be a major problem for any multi-valued semantics since the act of ascertaining such a value is arbitrary by its very nature [26]. For example, let us take the truth function of conjunction, \(\sqcap\), and suppose that its operands have the truth values U and F. Then, according to Kleene’s “(strong) logic of indeterminacy” [27], the said function must return F. However, in accordance with Bochvar’s “internal” three-valued logic [28], the same function must produce U.

Third, let’s assume that if both slits are open and so the truth value U is assigned to both \(P_1\) and \(P_2\) in accordance with (43), the propositional formulas \(P_1 \sqcup P_2\) and \(P_1 \sqcap P_2\) have equal truth values, namely, U. Now, suppose that at some time, a which-way detector is placed at the double-slit plate. After that, \(P_1 \sqcup P_2\) and \(P_1 \sqcap P_2\) transform respectively into a tautology and a contradiction. Hence, unlike the end-point truth values T and F, the intermediate truth value U gets destroyed as soon as the observation is made. Recalling that U belongs to the same class of the objects as the truth values T and F do, and so it must survive the observation like T and F do, one comes to absurdity.

To avoid the above arguments, the problem of an extra object must be solved using a semantics in which propositions may only have two possible truth values, T and F.

Such solutions will be examined in the next sections of the paper.

### 6 Birkhoff and von Neumann’s Proposal

A solution proposed by Birkhoff and von Neumann [6, 29] is to assume that the function (18) is a total surjective but not injective function. This function is not injective because it associates two elements of \(\mathbb{B}_2 \times \mathbb{B}_2\) with one and the same element of \(\mathbb{B}_2\). As a result, the image of the pair \((0, 0)\) under \(b\) may be equal to either 1 together with the pair \((1, 0)\), or 0 together with the pair \((0, 1)\). In either case,

\[
b(0, 0) \in \mathbb{B}_2.
\]
On the other hand, using truth tables it is straightforward to demonstrate that for any two mathematical statements, say, \( r_1 \) and \( r_2 \), such that \( (r_1 \cap r_2) \iff \bot \) and \( (r_1 \cup r_2) \iff \top \), the logical biconditional holds:

\[
 r_n \cup (\neg r_1 \cap \neg r_2) \iff \neg r_m. \tag{45}
\]

Hence, the application of the proposal (44) to (20) makes a truth value of the atomic proposition \( P_n \) subject to a truth value of either the statement \( s_n \) or the statement \( s_m \) alone, but not both.

Concretely, in case \( b(0, 0) = 1 \), a truth value of \( P_n \) is determined by

\[
\llbracket P_n \rrbracket = 1 - \llbracket s_n \cdot (-1)^r \rrbracket. \tag{46}
\]

In contrast, if \( b(0, 0) = 0 \), then the assignment of a truth value to \( P_n \) is given by

\[
\llbracket P_n \rrbracket = \llbracket s_n \rrbracket. \tag{47}
\]

In both cases, if either \( s_1 \) or \( s_2 \) is true, then \( \llbracket P_n \rrbracket = 1 \) while \( \llbracket P_m \rrbracket = 0 \). Providing truth values of the propositional formulas \( P_1 \cap P_2 \) and \( P_1 \cup P_2 \) are determined by truth values of its components, namely,

\[
\llbracket P_1 \cap P_2 \rrbracket = \min \{ \llbracket P_1 \rrbracket, \llbracket P_2 \rrbracket \}, \tag{48}
\]

\[
\llbracket P_1 \cup P_2 \rrbracket = \max \{ \llbracket P_1 \rrbracket, \llbracket P_2 \rrbracket \}, \tag{49}
\]

one finds that \( \llbracket P_1 \cap P_2 \rrbracket = 0 \) but \( \llbracket P_1 \cup P_2 \rrbracket = 1 \). That is, if either evidence for \( P_n \) is present (e.g., either slit is open, or a which-way detector reports the quantum particle at either slit), the complex statement «\( P_1 \) or \( P_2 \), but not, \( P_1 \) and \( P_2 \)» written down as exclusive disjunction \( P_1 \cup P_2 \), namely,

\[
P_1 \cup P_2 \iff (P_1 \cup P_2) \cap \neg (P_1 \cap P_2), \tag{50}
\]

is a tautology. The truthfulness of the statement \( P_1 \cup P_2 \) suggests that the quantum particle behaves as a classically defined particle.

Now, consider the case where both \( s_1 \) and \( s_2 \) are false (that is, the case where state of the quantum particle at the instant it passes the double-slit plate is described by a superposition). Assume that \( b(0, 0) = 0 \). This assumption brings on the equivalence \( \llbracket P_1 \rrbracket = \llbracket P_2 \rrbracket = 0 \), which means that before being recorded on the screen the quantum particle went through neither slit. This does not seem to make much physical sense. Hence, so long as \( b \) is a total non-injective surjective function, the image of the pair \( (0, 0) \) under this function should be 1. In symbols,

\[
b : \mathbb{B}_2 \times \mathbb{B}_2 \to \mathbb{B}_2 \\
(0, 0) \mapsto 1, \tag{51}
\]

where the second part is read: “\( (0, 0) \) maps onto 1”.

Applying the above to (41) results in the following assignment:
Since the mathematical statement \( w \) is either true or false, any two distinct propositions about the same quantum system are either able or unable to be true together. This fact implies that every pair of elements in \( L(\mathcal{H}) \) is either ordered or not ordered by the subset relation \( \subseteq \). In this way, \( L(\mathcal{H}) \) proves to be a set with a partial order (a poset).

Being elements of the poset, any two subspaces in \( L(\mathcal{H}) \), say \( \mathcal{H}_{\hat{Q}} \) and \( \mathcal{H}_{\hat{P}} \), may have a meet (denoted \( \mathcal{H}_{\hat{Q}} \land \mathcal{H}_{\hat{P}} \)) and a join (denoted \( \mathcal{H}_{\hat{Q}} \lor \mathcal{H}_{\hat{P}} \)), regardless of compatibility between the projection operators \( \hat{Q} \) and \( \hat{P} \) that correspond to those subspaces. Particularly, since for any set of subsets, the set-intersection \( \cap \) interprets meet \( \land \), the meet operation on elements of the poset \( L(\mathcal{H}) \) can be defined as follows

\[
\mathcal{H}_{\hat{Q}} \land \mathcal{H}_{\hat{P}} = \mathcal{H}_{\hat{Q}} \cap \mathcal{H}_{\hat{P}}.
\]

Furthermore, because the subspace \( \mathcal{H}_{\neg \hat{Q}} \) is the set of all vectors of \( \mathcal{H} \) that are not in \( \mathcal{H}_{\hat{Q}} \), except the zero subspace, \( \{0\} \), namely,

\[
\mathcal{H}_{\neg \hat{Q}} = \{ |\psi\rangle \in \mathcal{H} : (\hat{1} - \hat{Q}) |\psi\rangle = |\psi\rangle \},
\]

it holds \( \mathcal{H}_{\neg \hat{Q}} = \mathcal{H}_{\hat{Q}}^\perp \). Similarly, \( \mathcal{H}_{\neg \hat{P}} = \mathcal{H}_{\hat{P}}^\perp \). Thus, the join \( \mathcal{H}_{\hat{Q}} \lor \mathcal{H}_{\hat{P}} \) can be derived from the meet operation using De Morgan’s law [30], i.e.,

\[
\mathcal{H}_{\hat{Q}} \lor \mathcal{H}_{\hat{P}} = (\mathcal{H}_{\neg \hat{Q}} \cap \mathcal{H}_{\neg \hat{P}})^\perp.
\]

In those circumstances, the poset \( L(\mathcal{H}) \) can be held as a complete lattice (usually called a Hilbert lattice).

Provided that \( Q \cap P \) and \( Q \cup P \) are represented by the meet and join of the subspaces \( \mathcal{H}_{\hat{Q}} \) and \( \mathcal{H}_{\hat{P}} \), truth values of these propositional formulas are supposed to be assigned after the valuation (46). Specifically,

\[
[[Q \cap P]] = 1 - [[|\Psi\rangle \in (\mathcal{H}_{\hat{Q}} \land \mathcal{H}_{\hat{P}})^\perp]]
\]

\[
[[Q \cup P]] = 1 - [[|\Psi\rangle \in (\mathcal{H}_{\hat{Q}} \lor \mathcal{H}_{\hat{P}})^\perp]]
\]

where \(|\Psi\rangle\) is the vector of \( \mathcal{H} \) describing the state of the quantum particle when it comes out from the double-slit plate:

\[
|\Psi\rangle = \int \Psi(x) |x\rangle \, dx.
\]

If \( \Psi(x) \) is a superposition \( c_1 \phi_1(x) + c_2 \phi_2(x) \), then \(|\Psi\rangle\) is a sum of vectors

\[
|\Psi\rangle = c_1 |\phi_1\rangle + c_2 |\phi_2\rangle
\]

whose constituents are in the subspaces \( \mathcal{H}_{\hat{P}_n} \) corresponding to the projection operators \( \hat{P}_n \).
\[ \mathcal{H}_{\hat{P}_n} = \{ |\phi_n\rangle \in \mathcal{H} : \hat{P}_n |\phi_n\rangle = |\phi_n\rangle \}. \tag{60} \]

In that case, the mathematical statement «\((c_1|\phi_1\rangle + c_2|\phi_1\rangle) \in \{ c_{n-(-1)^r}|\phi_{n-(-1)^r}\rangle \}\)» is false, and so—in accordance with (46)—one gets

\[ \|P_n\| = 1 - \|\langle c_1|\phi_1\rangle + c_2|\phi_2\rangle \in \{ c_{n-(-1)^r}|\phi_{n-(-1)^r}\rangle \}\| = 1. \tag{61} \]

Recalling that the propositional formulas \(P_1 \cap P_2\) and \(P_1 \cup P_2\) are truth-functional, this entails \(\|P_1 \cap P_2\| = \|P_1 \cup P_2\| = 1\). That is, if two slits are open (and no which-way detector is present), the statement \(P_1 \cup P_2\) whose truth value is defined by

\[ \|P_1 \cup P_2\| = \min \{ \|P_1 \cup P_2\|, 1 - \|P_1 \cap P_2\| \} \tag{62} \]

is a contradiction.

However, this may not suggest that the quantum particle is wave-like. The falsity of \(P_1 \cup P_2\) can be explained away by non-distributivity of the logical operators \(\cap\) and \(\cup\) over each other caused by non-distributiveness of the lattice \(L(\mathcal{H})\). To demonstrate this, let us suppose that the vector \(|\Psi\rangle = c_1|\phi_1\rangle + c_2|\phi_2\rangle\) belongs to the linear subspace \(\mathcal{H}_{\hat{Q}}\), which is in one-one correspondence with the projection operator \(\hat{Q}\) representing some proposition \(Q\).

Consider the propositional formula \(\neg Q \cup (P_1 \cap P_2)\), where \(\neg Q\) is the negation of \(Q\). Because \(\neg Q\) is false in the state \(|\Psi\rangle \in \mathcal{H}_{\hat{Q}}\), the truth value of the said formula therein can be defined as

\[ \|\neg Q \cup (P_1 \cap P_2)\| = \max \{ \|\neg Q\|, \|P_1 \cap P_2\| \} = \|P_1 \cap P_2\|. \tag{63} \]

Since \(\mathcal{H}_{\hat{P}_1} \wedge \mathcal{H}_{\hat{P}_2}\) is the zero subspace, \(\{0\}\), orthogonal to the identical subspace, \(\mathcal{H}\), the mathematical statement «\(|\Psi\rangle \in (\mathcal{H}_{\hat{P}_1} \wedge \mathcal{H}_{\hat{P}_2})^\perp\)» is true for any vector \(|\Psi\rangle\) in \(\mathcal{H}\). Consequently,

\[ \|P_1 \cap P_2\| = 1 - \|\langle \Psi\rangle \in (\mathcal{H}_{\hat{P}_1} \wedge \mathcal{H}_{\hat{P}_2})^\perp\| = 0. \tag{64} \]

By the same token, as the mathematical statement «\(|\Psi\rangle \in (\mathcal{H}_{\hat{P}_1} \vee \mathcal{H}_{\hat{P}_2})^\perp\)» equivalent to «\(|\Psi\rangle \in \{0\}\rangle is false for any nonzero vector \(|\Psi\rangle\) in \(\mathcal{H}\), one gets

\[ \|P_1 \cup P_2\| = 1 - \|\langle \Psi\rangle \in (\mathcal{H}_{\hat{P}_1} \vee \mathcal{H}_{\hat{P}_2})^\perp\| = 1. \tag{65} \]

Now, take the propositional formula \((\neg Q \cup P_1) \cap (\neg Q \cup P_2)\). In the state \((c_1|\phi_1\rangle + c_2|\phi_1\rangle) \in \mathcal{H}_{\hat{Q}}\), the truth value of this formula is defined by

\[ \|((\neg Q \cup P_1) \cap (\neg Q \cup P_2))\| = \min \{ \|\neg Q \cup P_1\|, \|\neg Q \cup P_2\| \} = \min \{ \|P_1\|, \|P_2\| \}. \tag{66} \]

Then again, taking into consideration that the subspace \(\mathcal{H}_{\hat{Q}}\) intersects (meets) the subspace \(\mathcal{H}_{\hat{P}_n}\) at \(\{0\}\), that is,

\[ \mathcal{H}_{\hat{Q}} \cap \mathcal{H}_{\hat{P}_n} = \{ |\psi\rangle \in \mathcal{H} : \hat{Q}|\psi\rangle = |\psi\rangle \text{ and } \hat{P}_n|\psi\rangle = |\psi\rangle \} = \{0\}, \tag{67} \]
the statement \(\langle c_1 | \phi_1 \rangle + c_2 | \phi_1 \rangle \in (\mathcal{H}_{\neg \hat{Q}} \vee \mathcal{H}_{\hat{P}_n})^\perp\). is false for any nonzero \(c_1\) and \(c_2\). This bring
\[
\lll \neg Q \cup P_n \lll = 1 - \lll \langle c_1 | \phi_1 \rangle + c_2 | \phi_1 \rangle \in (\mathcal{H}_{\neg \hat{Q}} \vee \mathcal{H}_{\hat{P}_n})^\perp \lll = 1,
\]
therefore, the valuation (66) returns \(1 = \min \{|P_1|, |P_2|\}\) meaning \(|P_1 \cap P_2| = 1\).

As it turns out, the solution of the problem of an extra object discussed in [6, 29] causes nondistributiveness of the lattice \(L(\mathcal{H})\), namely,
\[
\mathcal{H}_{\neg \hat{Q}} \lor (\mathcal{H}_{\hat{P}_1} \land \mathcal{H}_{\hat{P}_2}) \neq \bigwedge_{n \in \{1,2\}} \left( \mathcal{H}_{\neg \hat{Q}} \lor \mathcal{H}_{\hat{P}_n} \right).
\]
which can be taken to be responsible for duality in the truth assignment for the propositional formula \(P_1 \cap P_2\).

One can assume from the above that even if both slits are open (and no which-way detector is placed behind them), the quantum particle will pass through just one slit, and so the exclusive disjunction \(P_1 \cup P_2\) will always be true. Regarding the valuation \(|P_1 \cap P_2| = 1\), one can blame it on the use of the distributivity law that does not hold generally in the Hilbert lattice.

From this assumption the ensuing hypothesis can be construed: A non-distributive logic (i.e., one where the statement \(\neg Q \cup (P_1 \cap P_2) \Rightarrow (\neg Q \cup P_1) \cap (\neg Q \cup P_2)\) does not need to be valid) underlies quantum phenomena and, for this reason, should be regarded as the correct logic for reasoning about the microscopic world [31]. In such a view, classical logic is merely a limiting case of a non-distributive logic [32]. It is obvious that the minute the said hypothesis is accepted, the conundrum of wave-particle duality will cease to exist.

However, the aforesaid hypothesis gives rise to another problem, which can be posed as the following question: Given that the true logic is non-distributive, how may it be the case that the logical connectives \(\cap\) and \(\cup\) still distribute one over the other on some occasions?

The essence of this problem stated previously in the Section 2 (and discussed briefly in [33]) is that it is impossible to determine—based solely on a propositional formula—in what circumstances the logical operators \(\cap\) and \(\cup\) will distribute. To do so, the meaning of propositions involved in the formula, specifically, the fact that the propositions have a classical content, must be taken into consideration. However, in propositional logic (including its modification such as a non-distributive logic), a propositional formula is concerned with the rules used for constructing an expression from atomic propositions and has nothing to do with an interpretation or meaning given to these propositions.
7 The Proposal of a Partial Bivaluation

The alternative proposal allowing to overcome the problem of an extra object is to assume that the function (18) is a partial bijective function that associates elements of $\mathbb{B}_2$ with some elements of $\mathbb{B}_2 \times \mathbb{B}_2$. Specifically, this function denotes the following:

$$b : \mathbb{B}_2 \times \mathbb{B}_2 \rightarrow \mathbb{B}_2$$

$$(0, 0) \mapsto \tau \in \mathbb{B}_2,$$

(70)

where the first part can be read as: “$b$ is a partial function from $\mathbb{B}_2 \times \mathbb{B}_2$ to $\mathbb{B}_2$”, while the second part is read: “$(0, 0)$ does not map onto anything in $\mathbb{B}_2$”. That is, $b$ is only defined on the pairs $(1, 0)$ and $(0, 1)$ whereas $b(0, 0)$ stays undefined.

The application of this proposal (which can be called the proposal of a partial bivaluation) to (20) yields the following truth assignment:

$$n \in \{1, 2\} : \llbracket P_n \rrbracket = b(\llbracket s_n \rrbracket, \llbracket s_{n-(−1)^n} \rrbracket) = \begin{cases} 1 & \llbracket s_n \rrbracket = 1 \\ 0 & \llbracket s_{n-(−1)^n} \rrbracket = 1 \\ \text{undefined} & \llbracket s_n \rrbracket = \llbracket s_{n-(−1)^n} \rrbracket = 0 \end{cases}.$$  

(71)

As this assignment indicates, the lack of negative evidence (that is, the equivalence $\llbracket s_{n-(−1)^n} \rrbracket = 0$) does not guarantee that the proposition $P_n$ is true; even more so, when neither evidence is given (i.e., both $\llbracket s_n \rrbracket$ and $\llbracket s_{n-(−1)^n} \rrbracket$ are zero), this proposition has no truth value at all. Hence, the above assignment may be viewed as one done constructively, i.e., using a semantics which only admits constructive proofs.

It is worthy of notice that a truth-value gap—i.e., lack of a truth value—does not stand for an intermediate truth value, akin to the value $U$ (“undefined” or “undetermined”) in a three-valued semantics.

To see this, let us interpret the truth-value gap as the indeterminate form $\frac{0}{0}$. In a loose manner of speaking, $\frac{0}{0}$ can take on the values 0, 1, or $\infty$. That is, the expression $\frac{0}{0}$ does not provide sufficient information to determine its value; in other words, it is undefined [34].

Hence, it is impossible to say whether the form $\frac{0}{0}$ is greater than or equal to 0, or whether $\frac{0}{0}$ is less than or equal to 1. Likewise, it is impossible to say whether a truth-value gap is “truer” than or identical to $F$ or whether it is “falser” than or identical to $T$. This means that a truth-value gap cannot be an intermediate truth value. Correspondingly, a semantics allowing a truth-value gap has no interpretation as a three-valued semantics.

Pondering upon the relation between a truth-value gap and the epistemic predicates “verified” and “falsified”, it is possible to state the following: Seeing that “verified” and “falsified” are not identified with “true” and “false” (it is sufficient to say that a statement may be verified at one time and unverified at another, however, it may be true even at times when it was not verified; likewise, it may be false without being falsified [35]), a statement may be verified or falsified at a certain time without being previously either true or false (as a result, from that
time onward this statement would be thought to be either a true one or a false
one). For example, providing reproducibility of the double-slit experiment, the
proposition \( P_n \), which has a truth-value gap when both slits are open, can be veri-
fied in one instance and falsified in another by positioning a which-way detector
at the slits. With that, the occurrence of the proposition \( P_n \) being verified/falsified
would be subject to variations due to chance.

This gives a reason to assign a probability to one or the other of the epistemic
predicates “verified” and “falsified”. Suppose that at some time, a proposition is
verified by an experience (e.g., an observation or experiment). Understandably,
such an outcome of the experience would be either certain or impossible if the
proposition where to have a definite truth value, that is, either “true” or “false”
respectively. On the other hand, where the proposition to have no truth value at
all, the outcome “verified” would be neither certain nor impossible. Accordingly,
the probability of being verified must be dispersion-free, i.e., either 1 or 0, for
a proposition having a definite truth value and different from both 1 and 0 for a
proposition having a truth-value gap. The above can be presented symbolically as
follows

\[
\Pr[\text{«}P \text{ is verified} \rangle] = \begin{cases} 
1 & , \quad [P] = 1 \\
0 & , \quad [P] = 0 \\
x \in (0, 1) & , \quad [P] \text{ is undefined}
\end{cases}, \quad (72)
\]

where \( \Pr[\text{«}P \text{ is verified} \rangle] \) denotes the probability that a proposition \( P \) will be verified
by the experience sometime.

The application of the proposal (70) to (41) produces

\[
[[Q \preceq P]] = b([[z]], [[w]]) = \begin{cases} 
1 & , \quad [z] = 1 \\
0 & , \quad [w] = 1 \\
\text{undefined} & , \quad [z] = [w] = 0
\end{cases}. \quad (73)
\]

As it follows, if the mathematical statements \( z \) and \( w \) are false together, the proposi-
tion \( Q \preceq P \) has no truth value. That is, in case the projection operators \( \hat{Q} \) and \( \hat{P} \) do not
commute, the propositions \( Q \) and \( P \) are neither able nor unable to be true together.
Consequently, one has no permission to say that the subspaces \( \mathcal{H}_Q \) and \( \mathcal{H}_P \) repre-
senting \( Q \) and \( P \) are either ordered or not ordered by the subset relation \( \subseteq \). Otherwise
stated, one may not say that \( L(\mathcal{H}) \) is a poset.

Nevertheless, consider a Boolean block, that is, a subset of \( L(\mathcal{H}) \), in which any
two elements, say, the subspaces \( \mathcal{H}_{\hat{Q}} \) and \( \mathcal{H}_{\hat{P}} \), correspond to mutually compatible
projection operators, \( \hat{Q} \) and \( \hat{P} \), respectively. Inside this block, either \( [z] \) or \( [w] \) is
equal to 1; thus, the proposition \( Q \preceq P \) is either true or false. From this it is pos-
possible to deduce that each Boolean block is a partially ordered subset of \( L(\mathcal{H}) \). So,
the meet \( \mathcal{H}_{\hat{Q}} \cap \mathcal{H}_{\hat{P}} \) and the join \( \mathcal{H}_{\hat{Q}} \vee \mathcal{H}_{\hat{P}} \) can be defined within every Boolean
block. E.g.,
Outside Boolean blocks, i.e., in case $\hat{Q} \hat{P} \neq \hat{P} \hat{Q}$, the above operations are not defined. One can infer from this fact that propositional formulas constructed from propositions associated with noncompatible projection operators are not defined either.

If the proposition $Q \preceq P$ has a truth value (i.e., if it happens inside a Boolean block), the mathematical statements

$$
\langle |\Psi\rangle \in (\mathcal{H}_Q \wedge \mathcal{H}_P) = s_\wedge
$$

which serve as positive and negative evidence, respectively, for the propositional formula $Q \cap P$, cannot be true together. Explicitly, for any nonzero vector $|\Psi\rangle$ of a Hilbert space $\mathcal{H}$, one finds that $(s_\wedge \cap s_\vee) \Leftrightarrow \bot$. Similarly, the statements

$$
\langle |\Psi\rangle \in (\mathcal{H}_Q \vee \mathcal{H}_P) = s_\vee
$$

which are positive and negative evidence, in that order, for the propositional formula $Q \cup P$, do not admit each other, i.e., $(s_\vee \cap s_\wedge) \Leftrightarrow \bot$, if $|\Psi\rangle \in \mathcal{H}$ and $|\Psi\rangle \neq 0$. Hence, in line with the proposal (70), $Q \cap P$ and $Q \cup P$ may be valuated using only positive evidence, namely,

$$
\hat{Q} \hat{P} = \hat{P} \hat{Q} : \quad [Q \cap P] = [s_\wedge] \quad [Q \cup P] = [s_\vee].
$$

Thus, allowing the statement $Q \preceq P$ to be partially valuated is equivalent to *giving up the lattice condition*: As a result of doing this, the logical connectives $Q \cap P$ and $Q \cup P$ turn out to exist only for countable sets of propositions $Q$ and $P$ that are represented by pairwise orthogonal or colinear subspaces of the Hilbert space of the system (by contrast, in the works [36] and [37], the abandonment of the lattice condition is suggested as an original assumption).

Recall that as per the formula (26), the projection operators $\hat{P}_1$ and $\hat{P}_2$ representing the atomic propositions $P_1$ and $P_2$ in the double-slit experiment are compatible and orthogonal. Moreover, according to the completeness relation $\hat{P}_1 + \hat{P}_2 = \hat{1}$, the sum of their corresponding subspaces $\mathcal{H}_{P_1}$ and $\mathcal{H}_{P_2}$ is the identical subspace $\mathcal{H}$; in symbols, $\mathcal{H}_{\hat{P}_1} + \mathcal{H}_{\hat{P}_2} = \mathcal{H}$. So, the meet and join of the subspaces $\mathcal{H}_{\hat{P}_1}$ and $\mathcal{H}_{\hat{P}_2}$ can be defined by

$$
\hat{P}_1 \hat{P}_2 = \hat{P}_2 \hat{P}_1 = 0 : \quad \mathcal{H}_{\hat{P}_1} \wedge \mathcal{H}_{\hat{P}_2} = \{0\}
$$

$$
\mathcal{H}_{\hat{P}_1} \vee \mathcal{H}_{\hat{P}_2} = (\mathcal{H}_{\hat{P}_2} \wedge \mathcal{H}_{\hat{P}_1})^\perp = \mathcal{H}.
$$

(78)
From the above one finds that the mathematical statements «\(\langle \Psi \rangle \in (H_{\hat{P}_1} \land H_{\hat{P}_2})\)» and «\(\langle \Psi \rangle \in (H_{\hat{P}_1} \lor H_{\hat{P}_2})\)» are false and true, respectively, for any non-zero vector \(\langle \Psi \rangle\) in the Hilbert space \(H\). Hence, at one with (77), the following must hold

\[
\forall \langle \Psi \rangle \in H \setminus \{0\} : \quad \begin{cases} [P_1 \cap P_2] = [\langle \Psi \rangle \in (H_{\hat{P}_1} \land H_{\hat{P}_2})] = 0 \\ [P_1 \cup P_2] = [\langle \Psi \rangle \in (H_{\hat{P}_1} \lor H_{\hat{P}_2})] = 1 \end{cases} .
\]

(79)

The projection operator \(\hat{0}\), which is in the one-to-one correspondence with the subspace \(H_{\hat{P}_1} \land H_{\hat{P}_2}\) representing \(P_1 \cap P_2\), is compatible with any projection operator; thus, the propositional formula \(\neg Q \cup (P_1 \cap P_2)\) is defined and has the same truth value as the negation \(\neg Q\) does. To be sure,

\[
(\hat{1} - \hat{Q})\hat{0} = \hat{0}(\hat{1} - \hat{Q}) : \quad [\neg Q \cup (P_1 \cap P_2)] = \begin{cases} 1 , & \langle \Psi \rangle \in H_{\neg \hat{Q}} \rangle = 1 \\ 0 , & \langle \Psi \rangle \in H_{\hat{Q}} \rangle = 1 \end{cases} .
\]

(80)

On the other hand, the projection operator \(\hat{Q}\) corresponding to the subspace \(H_{\hat{Q}}\) (which contains the vector \(\langle \Psi \rangle = c_1 \langle \phi_1 \rangle + c_2 \langle \phi_2 \rangle\) where \(\langle \phi_1 \rangle \in H_{\hat{P}_1}\) and \(\langle \phi_2 \rangle \in H_{\hat{P}_2}\) is compatible with neither \(\hat{P}_1\) nor \(\hat{P}_2\). The same holds for \(\neg \hat{Q}\) and \(\hat{P}_n\):

\[
(\hat{1} - \hat{Q})\hat{P}_n = \hat{P}_n((\hat{1} - \hat{Q})\hat{0}) .
\]

Given that the subspaces \(H_{\neg \hat{Q}}\) and \(H_{\hat{P}_n}\) cannot be in one Boolean block, the operation \(H_{\neg \hat{Q}} \land H_{\hat{P}_n}\) is not defined and therefrom the propositional formula \((\neg Q \cup P_1) \cap (\neg Q \cup P_2)\) is not defined either.

Note the difference between the false statement \(\neg Q \cup (P_1 \cap P_2) \Leftrightarrow (\neg Q \cup P_1) \cap (\neg Q \cup P_2) \Leftrightarrow \text{n.d.f.}\), where \(\text{n.d.f.}\) stays for “not defined formula”. The negation of the former yields \(Q \cap (P_2 \cup P_1) \Rightarrow (Q \cup P_2) \cup (Q \cap P_1)\), that is, \(Q \Rightarrow \perp\), which is true, whereas the negation of the latter returns \(Q \Rightarrow \perp\), which is meaningless once again. The said difference can be interpreted as that unlike the proposal suggested by Birkhoff and von Neumann, the one that assumes a partial bivaluation does not entail the failure of distributivity.

Let either of slits be open (or let a which-way detector register the particle at one or the other slit). Then, in accordance with (71), one of the propositions \(P_n\) is true while the other is false, so that the statement \(P_1 \cup P_2\) is a tautology. This implies that in the given case the logical connectives \(\cap\) and \(\cup\) are truth-functional.

Now, suppose that both slits are open, and no which-way detector is placed at the double-slit plate. In that case, the operators of logical conjunction and disjunction cannot be said to be truth-functional. In fact, even though both \(P_1\) and \(P_2\) happen to have a truth-value gap when both slits are open, \(P_1 \cup P_2\) and \(P_1 \cap P_2\) continue to be true and false, respectively, and the statement \(P_1 \cup P_2\) remains necessary true, i.e., it stays such that no instance exists in which this statement could fail to be true:

\[
\forall \Psi \in H \setminus \{0\} : \quad [P_1 \cup P_2] = \min \{[\langle \Psi \rangle \in H], 1 - [\langle \Psi \rangle \in \{0\}]\} = 1 .
\]

(81)

Hence, the proposal based on the assumption of a partial bivaluation does not bring about the wave-particle duality. As stated by this proposal, the quantum particle can
always be described as a particle-like thing, regardless of determination of a measuring device in the double-slit experiment.

8 Concluding Remarks

Birkhoff and von Neumann’s proposal assumes that without negative evidence—i.e., when the mathematical statement $s_n - (-1)^n$ is false—the proposition $P_n$ should be accepted as a true one. In other words, according to this proposal, the absence of a demonstration that a proposition is false guarantees that the proposition is true. Clearly, such an assumption would be correct if every proposition were to have a definite truth value regardless of proof.

On the other hand, in a semantics validating classical logic, every proposition is conceived as possessing a determinate truth value independently of whether we know it or have at our disposal the means to prove it [38]. One can conclude from this that a semantics validating Birkhoff and von Neumann’s proposal is like a semantics that bears out classical logic.

But here lies the irony: Birkhoff and von Neumann’s proposal, which was intended to be a replacement for classical logic in the domain of quantum mechanics, has at its core a semantics of classical logic that underlies the principles of classical mechanics. This may explain why Birkhoff and von Neumann’s proposal for quantum logic has not made a great deal of progress in a solution of the quantum conceptual difficulties.

Contrastively, a semantics validating the proposal of a partial bivaluation differs from any semantics that classical logic might have (at least in some crucial respect). Namely, in the said semantics (called supervaluationism), a propositional formula can possess a definite truth value even if its constituent propositions do not [39–41]. Hence, supervaluation semantics may offer a different approach to the quantum conceptual problems.

For example, according to the textbook interpretation of quantum mechanics [42], our choice of what to observe in the double-slit experiment determines the properties of a quantum particle therein. Accordingly, the quantum particle stops behaving like a wave and becomes a particle-like entity when an observation of a particle’s path takes place.

But according to the proposal of a partial bivaluation, the quantum particle’s properties are not contingent upon observation. To be exact, in the double-slit experiment, the quantum particle always behaves like a particle-like entity—i.e., one that goes along one path or the other but not both—irrespective of the observation. Together with all that, a statement affirming that the quantum particle follows a certain path has no truth value at all. One can only consider the probability that this statement will be verified if the actual particle’s path is observed. Because of that, behavior of a quantum particle emerging from the double-slit plate is impossible to convey in the classical concepts—i.e., ones that are based on classical logic.
The similar conclusion is reached in [43]: The statements «The particle passed through the slit 1» and «The particle passed through the slit 2» are completely indeterminate if we do not measure where the particle passed through. However, the premise, from which this conclusion has been inferred, is totally different from that stated in the present paper. Here, the truth or the falsity of any statement is assumed to be not primitive but derivative of evidence; as a result, a statement has no truth value if no evidence exists attesting that the statement is true or false. By way of contrast, in [43] the indeterminate status of a statement happens on account of the special definition of the conditional probability for measurement outcomes (this probability is defined in a such way that the distributive law need not be valid).

Furthermore, the application of the standard interpretation of quantum mechanics to any sentient creature leads to a paradox known as Schrödinger’s cat. The paradox involves a cat which—in agreement with the aforesaid interpretation—may be dead and alive at the same time. It could remain in such an inconceivable and absurd state for an arbitrarily long period of time, until the observer opens an opaque box enclosing the cat, at which point the animal is either dead or alive.

In contrast, according to the proposal of a partial bivaluation, the cat (together with a decaying radioactive atom on which its fate depends) is in either one state or another but never both, regardless of a “conscious” or “unconscious” observation. That is, for the cat, the premise of “macroscopic realism” [44] (declaring that a macroscopic system is in one or other of two macroscopically distinct states available to it but not in both) is always true. Therewithal, the proposition asserting that the cat is in a certain state (e.g., dead) prior to the observation has absolutely no truth value (because neither evidence exists for this proposition before the observation). Still, this proposition may be verified/falsified (with some probability) by opening the box.

Giving the great conceptual value of Wigner’s thought experiment [45], it is worth discussing—even in brief—how the proposal of a partial bivaluation explains Wigner’s conundrum of the friend.

Let us consider a slightly modified version of the standard account of the Wigner-friend thought experiment. This version posits an inside observer, a friend of Wigner, who has been asked to perform the double-slit experiment in a completely closed laboratory (so that an outside observer, Wigner, cannot be aware of anything happening in it until its door is open). Suppose that before the moment when the laboratory is closed, both Wigner and his friend agree that the statement $P_1 \sqcup P_2$ is a contradiction. Also suppose that the afterwards the inside observer introduces a which-way detector at the double-slit plate. Thenceforth, for this observer the statement $P_1 \sqcup P_2$ is a tautology. However, for the period that the door of the laboratory keeps on being closed, the same statement $P_1 \sqcup P_2$ will continue to be a contradiction for the outside observer. Denoting truth values, which Wigner’s friend and Wigner assign to an arbitrary statement, as $[[\cdot]]_F$ and $[[\cdot]]_W$, respectively, one finds

$$[[P_1 \sqcup P_2]]_F \neq [[P_1 \sqcup P_2]]_W.$$ (82)
In this way, the application of the instrumentalist description of quantum mechanics (i.e., the description which merely relates the mathematical formalism of quantum theory to data and prediction) alongside classical logic appears to show that the statement $P_1 \sqcup P_2$ is paradoxical: At one and the same time, it is true for one observer and false for the other.

To make this paradox even more dramatic, in Frauchiger and Renner’s thought experiment [46], Wigner uses two different methods to assign a truth value to the statement $P_1 \sqcup P_2$. When he uses the standard quantum formalism (i.e., the instrumentalist description of quantum mechanics in conjunction with classical logic), he gets $\|P_1 \sqcup P_2\|_W = 0$ (as stated above). But when he reasons about what truth value his friend might have assigned to this statement, Wigner may decide $\|P_1 \sqcup P_2\|_W = 1$ or $\|P_1 \sqcup P_2\|_W = 0$. Whatever the case may be, when the second method is used, $\|P_1 \sqcup P_2\|_W$ cannot be a certain value, in direct contradiction to the unconditional status of the statement $P_1 \sqcup P_2$. As Frauchiger and Renner argue, such a contradiction indicates that quantum theory cannot be extrapolated to complex systems.

Wigner’s puzzle takes us back to the central question of the present paper: Are the truth values “true” and “false” absolute, or relative to observers?

Based on what is proposed in [47], the inequality (82) could be dissolved if the inside observer were regarded as a rational agent (i.e., an entity like a team of scientists sharing notebooks, calculations, observations, etc., who may freely take actions on parts of the world external to themselves [48]). In that case, Wigner’s friend might believe that assigning truth values to the statement $P_1 \sqcup P_2$ conditioned solely to events happening inside the laboratory would not be rational, and consequentially the valuation $\|P_1 \sqcup P_2\|_F = 1$ would not be correct.

But according to both quantum Bayesianism (abbreviated QBism) [49] and the relational quantum mechanics (abbreviated RQM) [50, 51], the paradoxicality of the truth assignment for $P_1 \sqcup P_2$ disappears when “true” and “false” are identified with the “experiences” of different observers. In both QBism and RQM, “true” and “false” are not objective, rather “facts relative to the observers”—in other words, they are observers’ personal judgements [52]. In a slogan: ‘One’s “true” might be someone else’s “false”’.

This by no means indicates that QBism and RQM are self-contradictory: The moment the door of the laboratory is open for a second time, Wigner and his friend (along with the whole laboratory) will combine into one system, and so Wigner’s “false” will be converted into friend’s “true”. Consequently, Wigner and his friend will never be able to prove a disagreement between their truth assignments.

Though the case be such, not only does the dependency of the truth values upon an agent amount to giving up the absolute nature of facts, but most importantly it has the value of forsaking the objectivity of mathematics. That is, if “true” and “false” are subject to agent’s judgements, then the truth of every mathematical statement must be an agent’s belief, “supremely strong, but nonetheless a belief”.

The question regarding the objectivity of mathematics, such as whether mathematical truth is objective or subjective, is perhaps one of the oldest and hardest questions in Western philosophy.
On the one hand, seeing as mathematics is a free activity of the mind, one may consider mathematical truth subjective. In agreement with Putnam [35], if the only method allowed in mathematics had consisted in deriving conclusions from axioms, which have been fixed permanently and for all possible agents (using mathematics), then the truth of any mathematical statement would have been objective, i.e., independent of agents’ personal judgements. But it did not. To prove a mathematical statement an agent may use empirical and probabilistic arguments seemed plausible for this agent but not acceptable for the other. As a result, the mathematical statement in question would be true for the former and false for the latter. For example, until a well-defined meaning was given to the Dirac delta function, \( \delta(x) \), the computations made using this function appeared to most mathematicians as nonsense [53]. Consequently, even though a mathematical statement involving this function such as «\((x\delta(x) = 0) \land (\int_{-\infty}^{+\infty} \delta(x)dx = 1)\)», might be considered true by some mathematicians, it would be regarded as false by the rest.

Despite this, if a mathematical statement is true because it has a constructive proof, such a statement will be true absolutely. Indeed, in that case there is a constructive witness to the truthfulness of the statement, that is, an actual example proving it true. And because this example exists in fact and not merely potential or possible, it is the fact for all agents. Hence, the said mathematical statement will be true for every agent, that is, true absolutely.

From the above reasoning it follows that by accepting the relativity of the truth values QBism and RQM merely reject constructivist philosophy. That is quite different from the proposal of a partial bivaluation. To be sure, let us consider the mathematical statement «\(|\Psi\rangle \in \mathcal{H} \setminus \{0\}\)». Any physically meaningful state, i.e., every state describable by non-zero vector \(|\Psi\rangle\) of a finite-dimensional Hilbert space \(\mathcal{H}\), is computable. Therefore, as far as a physical meaning is concerned, there exists computational evidence witnessing the truth of «\(|\Psi\rangle \in \mathcal{H} \setminus \{0\}\)». What is more, because computability of \(|\Psi\rangle\) implies a constructive mechanism generating \(|\Psi\rangle\), one may say that «\(|\Psi\rangle \in \mathcal{H} \setminus \{0\}\)». is constructively true.

This entails the absolute truth of \(P_{1} \sqcup P_{2}\) whose evidence, in accordance with (81), is provided by the truthfulness of «\(|\Psi\rangle \in \mathcal{H} \setminus \{0\}\)». Consequently, the truth of the unconditional statement «In the double-slit experiment, the quantum particle passes through one or the other slit, but not both» must be shared alike by Wigner’s friend and Wigner:

\[
\|P_{1} \sqcup P_{2}\|_{F} = \|P_{1} \sqcup P_{2}\|_{W} = 1, \quad (83)
\]
i.e., it must be an observer-independent fact.

Using the terminology of Brukner’s no-go theorem [54], one can declare that the above equality implies the compatibility of “universal validity of quantum theory” (that is, the applicability of the mathematical formalism of the theory to any physical system without restriction), “locality” (i.e., the independence of one observer’s measurement settings and other observer’s outcomes) and “freedom of choice” (meaning that Wigner’s friend can freely decide to introduce or not to introduce a which-way detector) with the assumption of “observer-independent facts” (under this assumption, \(\|P_{1} \sqcup P_{2}\|_{F}\) and \(\|P_{1} \sqcup P_{2}\|_{W}\) may be defined together).
Now, let us consider the valuation of the atomic propositions $P_n$. For Wigner, these propositions have no truth values at all, accordingly, the set of $[[P_n]]_W$ is $\emptyset$, the set with no elements. At the same time, $P_n$ are verified/falsified by his friend and, as a result, the set of $[[P_n]]_F$ is not $\emptyset$. In symbols:

$$\left\{ n \in \{1, 2\} : [[P_n]]_F \right\} \neq \left\{ n \in \{1, 2\} : [[P_n]]_W \right\}.$$  \hfill (84)

This means that one cannot define $[[P_n]]_F$ and $[[P_n]]_W$ together. Moreover, one cannot define a joint probability $p_{\text{joint}}$ such that

$$p_{\text{joint}} = \Pr\left[ \left. \text{«}[[P_n]]_F \in \{0, 1\} \text{ and } [[P_n]]_W \in \{0, 1\} \text{»} \right| \right],$$  \hfill (85)

and neither can one define a truth-valued function corresponding to a binary connective (say, conjunction) that joins the proposition $P_n$ evaluated by Wigner’s friend with the proposition $P_n$ evaluated by Wigner. In this way, the truth or falsity of the statement «The quantum particle passes through the slit $n$» is a fact only for a particular observer.

Looking at (83) and (84), one can infer that quantum theory treats facts in a dual manner: On the one hand, in this theory statements known to be true or false from an experience of an observer are facts relative to this observer. But on the other hand, quantum theory also grants entrance to observer-independent facts, i.e., statements which are true or false for all observers without regard for their experiences or beliefs. Such a mode of action can be taken as an indication or sign of the dual—i.e., objective-subjective—nature of facts in quantum theory.

In the view of that nature, Brukner’s no-go theorem, which asserts that in quantum theory one can only define facts relative to an observation and an observer, might be applicable only in part. The same holds for a new strong no-go theorem by Bong et al [55] built on assumptions strictly weaker than those of Brukner’s no-go theorem.

As to the recent attempt by Proietti et al [56] to perform an extended Wigner’s friend thought experiment (with four observers), the following may be remarked in passing. Performing a thought experiment is not a question of technical capabilities or ingenuity. Even if it could be possible to perform a thought experiment, there need not be an intention (a motive or a purpose) to perform it. This is so because in thought experiments one gains new information not by observing or measuring events or experiences but by reorganizing or rearranging already known experimental data in a new way with the aim of drawing new inference from them [57].

So, regarding the Wigner’s friend thought experiment (the original version or an extended modification of it), one already has all the necessary empirical data (provided by almost 100 years of quantum mechanics studies) allowing one to formulate statements about physical systems’ states after the measurement. Furthermore, the purpose of the said experiment is to challenge (or even refute) the Copenhagen interpretation of quantum mechanics (using the device of the imagination known as reductio ad absurdum). Given that the Copenhagen interpretation is unfalsifiable (no
evidence that comes to life can contradict it), any attempt to perform the Wigner’s friend experiment in real life is merely fruitless.

Acknowledgements The author wishes to thank the anonymous referee for the inspiring remarks and constructive comments which helped him enrich and deepen this paper.

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