Extremal Graph Theory for Metric Dimension and Girth

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Abstract

A set \( W \subseteq V(G) \) is called a resolving set for \( G \), if for each two distinct vertices \( u, v \in V(G) \) there exists \( w \in W \) such that \( d(u, w) \neq d(v, w) \), where \( d(x, y) \) is the distance between the vertices \( x \) and \( y \). The minimum cardinality of a resolving set for \( G \) is called the metric dimension of \( G \), and denoted by \( \beta(G) \). In this paper, it is proved that in a connected graph \( G \) of order \( n \) which has a cycle, \( \beta(G) \leq n - g(G) + 2 \), where \( g(G) \) is the length of a shortest cycle in \( G \), and the equality holds if and only if \( G \) is a cycle, a complete graph or a complete bipartite graph \( K_{s,t} \), \( s, t \geq 2 \).

Keywords: Resolving set; Metric dimension; Girth; 2-connected graph.

1 Introduction

Throughout the paper, \( G \) is a finite, simple, and connected graph of order \( n \) with vertex set \( V(G) \) and edge set \( E(G) \). The distance between two vertices \( u \) and \( v \), denoted by \( d(u, v) \), is the length of a shortest path between \( u \) and \( v \) in \( G \). The diameter of \( G \), denoted by \( \text{diam}(G) \) is \( \max\{d(u, v) \mid u, v \in V\} \). The degree of a vertex \( v \), \( \deg(v) \), is the number of its neighbors. The notations \( u \sim v \) and \( u \not\sim v \) denote the adjacency and non-adjacency relations between \( u \) and \( v \), respectively. A cycle of order \( n \), \( C_n \), is denoted by \( (v_1, v_2, \ldots, v_n, v_1) \). The number of edges in a cycle is its length. If \( G \) has a cycle, then the length of a shortest cycle in \( G \) is called the girth of \( G \) and denoted by \( g(G) \). For a subset \( S \) of \( V(G) \), \( G \setminus S \) is the induced subgraph \( \langle V(G) \setminus S \rangle \) by \( V(G) \setminus S \) of \( G \). A vertex \( v \in V(G) \) is a cut vertex in \( G \) if \( G \setminus \{v\} \) has at least two components. If \( G \neq K_n \) has no cut vertex, then \( G \) is called a 2-connected graph.

For an ordered set \( W = \{w_1, w_2, \ldots, w_k\} \subseteq V(G) \) and a vertex \( v \) of \( G \), the \( k \)-vector

\[
    r(v|W) := (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))
\]

is called the metric representation of \( v \) with respect to \( W \). The set \( W \) is called a resolving set for \( G \) if distinct vertices have different representations. A resolving set for \( G \) with minimum cardinality is called a metric basis, and its cardinality is the metric dimension of \( G \), denoted by \( \beta(G) \). It is obvious that to see whether a given set \( W \) is a resolving set,
it is sufficient to consider the vertices in \( V(G) \setminus W \), because \( w \in W \) is the unique vertex of \( G \) for which \( d(w, w) = 0 \).

In [13], Slater introduced the idea of a resolving set and used a *locating set* and the *location number* for a resolving set and the metric dimension, respectively. He described the usefulness of these concepts when working with U.S. Sonar and Coast Guard Loran stations. Independently, Harary and Melter [7] discovered the concept of the location number as well and called it the metric dimension. For more results related to these concepts see [2, 3, 4, 6, 9, 10]. The concept of a resolving set has various applications in diverse areas including coin weighing problems [12], network discovery and verification [1], robot navigation [10], mastermind game [3], problems of pattern recognition and image processing [11], and combinatorial search and optimization [12]. Chartrand et al. [5] obtained the following bound for metric dimension in terms of order and diameter.

**Theorem A.** [5] If \( G \) is a connected graph of order \( n \), then \( \beta(G) \leq n - \text{diam}(G) \).

Ten years later, Hernando et al. [8] characterized all graphs \( G \) of order \( n \) and metric dimension \( n - \text{diam}(G) \). The main goal of this paper is to prove that for a connected graph \( G \) of order \( n \) and girth \( g(G) \)

\[
\beta(G) \leq n - g(G) + 2
\]

and characterize all graphs such that this bound is tight for them. In fact, it is proved that cycles, complete and complete bipartite graphs are all graphs with \( \beta(G) = n - g(G) + 2 \). To prove the main results the following known results are needed. It is obvious that for a graph \( G \) of order \( n \), \( 1 \leq \beta(G) \leq n - 1 \). Chartrand et al. [5] characterized all graphs of order \( n \) and metric dimension \( n - 1 \).

**Theorem B.** [5] Let \( G \) be a graph of order \( n \). Then \( \beta(G) = n - 1 \) if and only if \( G = K_n \).

They also characterized all graphs of order \( n \) and metric dimension \( n - 2 \).

**Theorem C.** [5] Let \( G \) be a graph of order \( n \geq 4 \). Then \( \beta(G) = n - 2 \) if and only if \( G = K_{s,t} \) \((s, t \geq 1)\), \( G = K_s \vee \overline{K}_t \) \((s \geq 1, t \geq 2)\), or \( G = K_s \vee (K_t \cup K_1) \) \((s, t \geq 1)\).

The following definition is needed to state some results in the next section.

**Definition 1.** An ear of a graph \( G \) is a maximal path whose internal vertices are of degree 2 in \( G \). An ear decomposition of \( G \) is a decomposition \( G_0, G_1, \ldots, G_k \) such that \( G_0 \) is a cycle and for each \( i, 1 \leq i \leq k \), \( G_i \) is an ear of \( G_0 \cup G_1 \cup \ldots \cup G_i \).

Whitney [14] proved the following important result for 2-connected graphs.

**Theorem D.** [14] A graph is 2-connected if and only if it has an ear decomposition. Moreover, every cycle in a 2-connected graph is the initial cycle in some ear decomposition.
2 Main Results

The aim of this section is to find a bound for metric dimension in terms of order and girth of a graph and characterize all graphs which attend this bound. This bound is presented in the next theorem.

**Theorem 1.** Let $G$ be a graph of order $n$. If $G$ has a cycle, then $\beta(G) \leq n - g(G) + 2$.

**Proof.** Let $g(G) = g$ and $C_g = (v_1, v_2, \ldots, v_g, v_1)$ be a shortest cycle in $G$. Since $\{v_1, v_2\}$ is a metric basis of $C_g$, $V(G) \setminus \{v_3, \ldots, v_g\}$ is a resolving set for $G$ of size $n - g(G) + 2$. Therefore, $\beta(G) \leq n - g(G) + 2$. \qed

Note that, $\beta(K_n) = n - 1 = n - g(K_n) + 2$, $\beta(C_n) = 2 = n - g(C_n) + 2$, and for $r, s \geq 2$, $\beta(K_{r,s}) = r + s - 2 = n - g(K_{r,s}) + 2$. Therefore, the bound in Theorem 1 is tight for these graphs. In the remainder of this section, it is proved that these are all graphs such that this bound is tight for them. First some required results are presented.

**Proposition 1.** Let $v$ be a cut vertex in a graph $G$. Then each resolving set for $G$ can be disjoint from at most one component of $G \setminus \{v\}$. Moreover, if $W$ is a resolving set for $G$ which is not disjoint from two components of $G \setminus \{v\}$, then $W \setminus \{v\}$ is a resolving set for $G$.

**Proof.** Let $W$ be a resolving set for $G$ and $H$ and $K$ be two components of $G \setminus \{v\}$. If $W \cap V(H) = W \cap V(K) = \emptyset$, then let $x \in V(H)$ and $y \in V(K)$ such that $x \sim v$ and $y \sim v$. Therefore, for each $w \in W$, $d(x, w) = d(x, v) + d(v, w) = 1 + d(v, w) = d(y, v) + d(v, w) = d(y, w)$, which contradicts the assumption that $W$ is a resolving set for $G$. Thus $W$ can be disjoint from at most one component of $G \setminus \{v\}$.

Now let $W \cap V(H) \neq \emptyset$, $W \cap V(K) \neq \emptyset$, $x \in W \cap V(H)$, and $y \in W \cap V(K)$. If $W \setminus \{v\}$ is not a resolving set for $G$, then there exist vertices $a, b \in V(G)$ such that $d(a, v) \neq d(b, v)$ and for each $w \in W \setminus \{v\}$, $d(a, w) = d(b, w)$. If $a, b \notin V(H)$, then $d(a, x) = d(a, v) + d(v, x) \neq d(b, v) + d(v, x) = d(b, x)$. This gives $d(a, x) \neq d(b, x)$, which is impossible. Hence, $a, b \in V(H)$. Therefore, $d(a, y) = d(a, v) + d(v, y) \neq d(b, v) + d(v, y) = d(b, y)$. That is $d(a, y) \neq d(b, y)$. This contradiction implies that $W \setminus \{v\}$ is a resolving set for $G$. \qed

**Corollary 1.** Let $u$ be a vertex of degree 1 in a graph $G$ and $v$ be the neighbor of $u$. If $W$ is a resolving set for $G$, then $(W \cup \{u\}) \setminus \{v\}$ is also a resolving set for $G$. 


Proof. Let $W$ be a resolving set for $G$. Clearly $W \cup \{u\}$ is also a resolving set for $G$. Note that $v$ is a cut vertex of $G$ and $\langle \{u\}\rangle$ is a component of $G \setminus \{v\}$. If $W \cap (V(G) \setminus \{u, v\}) \neq \emptyset$, then by Proposition 1, $(W \cup \{u\}) \setminus \{v\}$ is also a resolving set for $G$. If $W \subseteq \{u, v\}$, then by Proposition $1$, $G \setminus \{v\}$ has exactly two components. On the other hand, for each $x \in V(G) \setminus \{u\}$, $r(x|\{u, v\}) = (a, a - 1)$, for some integer $a \geq 1$. Since $\{u, v\}$ is a resolving set for $G$, the first coordinate of the metric representation of all vertices in $V(G) \setminus \{u\}$ are different from each other. Therefore, $\{u\} \cup \{x\}$ is also a resolving set for $G$. ■

The following proposition states that all graphs $G$ of order $n$ with metric dimension $n - g(G) + 2$ is 2-connected.

**Proposition 2.** Let $G$ be a graph of order $n$ which has a cycle. If $\beta(G) = n - g(G) + 2$, then $G$ is 2-connected.

Proof. Suppose, on the contrary, that $v$ is a cut vertex of $G$. Let $C_g = (v_1, v_2, \ldots, v_g, v_1)$ be a shortest cycle in $G$. Since $v$ is a cut vertex, there exists a component $H$ of $G \setminus \{V\}$, such that $V(C_g) \subseteq V(H) \cup \{v\}$. By the assumption, $V(G) \setminus \{v_3, \ldots, v_g\}$ is a basis of $G$, which is not disjoint from at least two components of $G \setminus \{v\}$. Therefore, by Proposition $1$, $v \in \{v_3, \ldots, v_g\}$, say $v = v_i$, $3 \leq i \leq g$. But $B = V(G) \setminus \{v_1, v_2, v_{i-2}, v_{i+2}, \ldots, v_g\}$ is a basis of $G$ which contains $v = v_i$, since $B$ is not disjoint from at least two components of $G \setminus \{v\}$. Thus, by Proposition $1$, $B \setminus \{v\}$ is a resolving set for $G$ of size smaller than $\beta(G)$. This contradiction implies that $G$ is a 2-connected graph. ■

**Theorem 2.** Let $G$ be a graph of order $n$ which has a cycle. Then $\beta(G) = n - g(G) + 2$ if and only if $G$ is a cycle $C_n$, complete graph $K_n$, $n \geq 3$, or complete bipartite graph, $K_{r,s}$, $r, s \geq 2$.

Proof. It is easy to see that if $G$ is a cycle $C_n$, complete graph $K_n$, $n \geq 3$, or complete bipartite graph, $K_{r,s}$, $r, s \geq 2$, then $\beta(G) = n - g(G) + 2$. Now let $\beta(G) = n - g(G) + 2$ and $C_g = (v_1, v_2, \ldots, v_g, v_1)$ be a shortest cycle in $G$. By Proposition $2$, $G$ is a 2-connected graph. Therefore, By Theorem $1$, $G$ has an ear decomposition with initial cycle $C_g$. Assume that $C_g, G_1, G_2, \ldots, G_k$ be an ear decomposition of $G$ with initial cycle $C_g$. If $G = C_g$, then $G$ is a cycle as it is desired. Now let $G \neq C_g$ and $G_1 = (x_0, x_1, \ldots, x_t)$. Thus $x_0, x_t \in V(C_g)$. Without loss of generality one can assume that $x_0 = v_i$ and $x_t = v_j$, where $1 \leq i < j \leq g$. Since $C_g$ is a shortest cycle in $G$, $j - i < t$ and $g + i - j \leq t$. If $t \geq 3$, then the set

$$W = V(G) \setminus \{x_2, v_1, v_2, \ldots, v_{i-1}, v_{i+2}, \ldots, v_g\}$$

is not a resolving set for $G$, because $|W| = \beta(G) - 1$. Therefore, there exist vertices $a, b \in V(G) \setminus W$ such that $r(a|W) = r(b|W)$. Since $\{v_i, v_{i+1}\}$ is a basis for $C_g$ and the distances in $C_g$ and $G$ are the same, $W \cap V(C_g)$ resolves $C_g$. Consequently, $x_2 \in \{a, b\}$, say $x_2 = a$. Then, $b \in V(C_g) \setminus \{v_i, v_{i+1}\}$. Hence, $d(b, x_1) = d(a, x_1) = 1$, because $x_1 \in W$. 

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Note that $x_1$ is an internal vertex of the ear $G_1$ of the graph $C_g \cup G_1$ and hence $x_1$ is of degree 2 in $C_g \cup G_1$. But $x_1$ is adjacent to vertices $x_0, x_2$ and $b$. Since $b \notin \{x_0, x_2\}$, $x_1$ is of degree at least 3 in $C_g \cup G_1$. This contradiction implies that $r \leq 2$. Therefore, $j - i \leq 2$ and $g + i - j \leq 2$. Thus, $g \leq 4$. If $g = 3$, then $\beta(G) = n - 3 + 2 = n - 1$ and by Theorem [B] $G = K_n$. If $g = 4$, then $\beta(G) = n - 4 + 2 = n - 2$ and by Theorem [C] $G$ is $K_{r,s}$, $K_r \vee K_s$, or $K_r \vee (K_s \cup K_1)$. But $K_{r,s}$, $r, s \geq 2$ is the only graph among these graphs whose girth is 4. ■

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