Multiple algebraisations of an elliptic Calogero-Sutherland model

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Abstract

Recently, Gómez-Ullate et al. [1] have studied a particular N-particle quantum problem with an elliptic function potential supplemented by an external field. They have shown that the Hamiltonian operator preserves a finite dimensional space of functions and as such is quasi exactly solvable (QES). In this paper we show that other types of invariant function spaces exist, which are in close relation to the algebraic properties of the elliptic functions. Accordingly, series of new algebraic eigenfunctions can be constructed.

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I. INTRODUCTION

The first example of a non-trivial, integrable quantum many-body Hamiltonian was found by Calogero [2]. It describes a system of N particles in one dimension interacting pairwise by means of an inverse square potential. The similar model endowed with an inverse sine-square potential is also integrable as shown by Sutherland [3]. In fact these two potentials are particular cases of a two parameter-family of potentials defined by the Weierstrass function [4,5]. A detailed analysis of these models generalising the Calogero-Sutherland (CS) quantum models was reported in [6]. Their classical counterparts are discussed e.g. in [7].

While integrable models (classical or quantum) can be studied because of their mathematical interest, it became apparent in recent years that the CS models can be applied to a large number of fields of physics. These range from condensed matter (quantum Hall liquids, quantum spin chains, ..) [8] to gauge theories [9], soliton theory [10] as well as recently to questions related to black holes and (Anti)-deSitter space [11,12]. In particular it was shown in [12] that the asymptotic dynamics of 2-dimensional gravity in Anti-deSitter and deSitter space respectively can be described by a generalised two-body CS model.

The property of a model to be integrable (i.e. to have a complete set of commuting constants of motion) does not necessarily imply that the spectrum and the eigenfunctions of the corresponding Hamiltonian can be constructed explicitly. The models which have this property are called solvable. From the beginning the CS models were known to be solvable, while further properties of their spectrum were obtained only recently, see e.g. [13,14]. However, the explicit form of the spectrum is still missing as far as the full Weierstrass-function potential is considered for generic values of N.

A step forward in the construction of solvable N-body problems interacting via a Weierstrass function was achieved in [1]. The authors indeed showed that, when the Weierstrass potential is supplemented by a suitable external potential, a finite number of eigenvectors can be computed explicitly in terms of special functions. Stated differently, the model is quasi exactly solvable (QES) according to the definition of [15]. In fact, the kind of interaction considered in [1] and in the present paper generalises a potential first introduced in [16].

Following the ideas of [15], the QES property holds when the Hamiltonian operator possesses a finite-dimensional invariant vector space of functions. Such a vector space was indeed constructed in [1] for the Hamiltonian considered. The purpose of this paper is to demonstrate that this Hamiltonian possesses alternative invariant finite-dimensional vector spaces of functions. The way these new vector spaces are constructed is very reminiscent to the multiple algebraisations of the Lamé equations (see e.g. [17]), which occur due to the properties of the Jacobi elliptic functions.

The Hamiltonian is presented in Sect. II. In this section we also give the transformation putting the Hamiltonian in a Lie-algebraic form which reveals its QES property. The new invariant vector spaces are constructed in Sect. III and the Hamiltonian is studied for particular values of the parameters. The results are summarized in Sect. IV.

II. AN ELLIPTIC CALOGERO-SUTHERLAND MODEL

The quantum Hamiltonian proposed recently by Gómez-Ullate et al. [1] is given by:
\[ H_N(x) = -\sum_{k=1}^{N} \frac{\partial^2}{\partial x_k^2} + V_N(x) , \quad x = (x_1, x_2, \ldots, x_N) . \]  

(1)

It describes \( N \) particles on a line interacting through the potential

\[ V_N(x) = c_m \sum_{k=1}^{N} \mathcal{P}(x_k + i\beta) + 4b(b-1) \sum_{k=1}^{N} \mathcal{P}(2x_k) + a(a-1) \sum_{j,k=1 \atop j \neq k}^{N} [\mathcal{P}(x_j + x_k) + \mathcal{P}(x_j - x_k)] . \]

(2)

Here \( \mathcal{P}(z) \equiv \mathcal{P}(z; g_2, g_3) \) denotes the Weierstrass function with invariants \( g_2, g_3 \). The constants \( a, b \) are real and positive, \( c_m \) is real. The term proportional to \( c_m \) can be interpreted as the potential of an external field.

The Hamiltonian (1) was shown to admit an invariant, finite dimensional vector space of functions \([1]\). Restricting the operator to this vector space, the eigenvalue equation \( H_N\psi = E\psi \) is reduced to a matrix equation and, accordingly, a finite number of eigenvectors can be determined algebraically. Following the definition of \([15]\) the operator \( H_N \) is called Quasi Exactly Solvable (QES).

To reveal this property \( H_N \) has to be transformed appropriately. The authors of Ref. \([1]\) introduced the function (called “gauge factor”)

\[ \mu(x) = \prod_{j<k} [\mathcal{P}(x_j + i\beta) - \mathcal{P}(x_k + i\beta)]^a \prod_{k} [\mathcal{P}'(x_k + i\beta)]^b \]  

(3)

and the new variables

\[ z_k = \mathcal{P}(x_k + i\beta) , \quad k = 1, \ldots, N . \]  

(4)

Then a Hamiltonian \( \tilde{H}_N \) -spectrally equivalent to \( H_N \) is constructed according to \( \tilde{H}_N(z) = \mu^{-1}(z)H_N(x)\mu(z) \). If the coupling constant \( c_m \) is chosen according to

\[ c_m = [2m + 2a(N-1) + 4b][2m + 1 + 2a(N-1) + 2b] , \quad m \in \mathbb{N} \]  

(5)

\( \tilde{H}_N(z) \) preserves the finite dimensional polynomial space \([18]\)

\[ \mathcal{M}_m = \text{span}\{\tau_1^{l_1}\tau_2^{l_2}\cdots\tau_N^{l_N} ; \sum_{i=1}^{N} l_i \leq m\} \]  

(6)

with the \( k \)-th elementary symmetric function

\[ \tau_k \equiv \sum_{i_1<i_2<\ldots<i_k} z_{i_1}z_{i_2}\cdots z_{i_k} , \quad 1 \leq k \leq N . \]  

(7)

A lengthy calculation leads to

\[ \tilde{H}_N(z) = -\sum_{k=1}^{N} p_k \frac{\partial^2}{\partial z_k^2} - 2a \sum_{k,l=1 \atop k \neq l}^{N} \frac{p_k}{z_k - z_l} \frac{\partial}{\partial z_k} - \left( b + \frac{1}{2} \right) \sum_{k=1}^{N} p'_k \frac{\partial}{\partial z_k} + \tilde{V}_N(z) , \]  

(8)
where $p_k \equiv p(z_k)$ and $p'_k \equiv p'(z_k)$ with

$$p(z) = 4z^3 - g_2z - g_3, \quad p'(z) \equiv \frac{dp}{dz} = 12z^2 - g_2. \quad (9)$$

In the following we will use the roots, $e_i, i = 1, 2, 3,$ of $p(z)$:

$$p(z) = 4(z - e_1)(z - e_2)(z - e_3) = 4z^3 - g_2z - g_3. \quad (10)$$

These numbers are equal to the values of the Weierstrass function at its half-periods.

The potential $\bar{V}_N$ in (8) is given by:

$$\bar{V}_N(z) = m(12b + 8a(N - 1) + 4m + 2)\tau_1, \quad \tau_1 = \sum_{k=1}^{N} z_k. \quad (11)$$

The crucial observation is that the Hamiltonian $\bar{H}_N(z)$ can be written as a quadratic polynomial of the differential operators

$$D_k = \frac{\partial}{\partial \tau_k}, \quad N_{jk} = \tau_j \frac{\partial}{\partial \tau_k}, \quad U_k = \tau_k(r - \sum_{i=1}^{N} \tau_i \frac{\partial}{\partial \tau_i}), \quad j, k = 1, 2, \ldots, N \quad (12)$$

with $r = m$. These operators form a representation of the Lie algebra $sl(N + 1)$ for generic value of the real parameter $r$, for $r = m$ they preserve the vector space $M_m$ and the representation is finite dimensional.

Denoting by $\bar{H}_N^{(+)}$ the part of $\bar{H}_N$ which increases the degree of elements of $M_m$, we find:

$$\bar{H}_N^{(+)} = -4\tau_1(N - m)(N + m + \frac{1}{2} + 2a(N - 1) + 3b), \quad N \equiv \sum_{k=1}^{N} N_{kk}. \quad (13)$$

Obviously, the factor $(N - m)$ leads to the annihilation of all the monomials in $M_m$ which have overall degree $m$. Therefore, we find: $\bar{H}_N M_m \subseteq M_m$. As a consequence, eigenvectors of $\bar{H}_N$ (and therefore also of $H_N$) can be constructed in $M_m$. In the following we will refer to this property as to an “algebrasion” of $H_N$.

### III. ADDITIONAL GAUGE FACTORS

Inspired by the construction of the Lamé polynomials (see e.g. [17]), we introduce one further transformation of the Hamiltonian $H_N$:

$$\bar{H}_N \rightarrow \tilde{H}_N = \tilde{\mu}^{-1}\bar{H}_N \tilde{\mu} \quad (14)$$

with the gauge factor $\tilde{\mu}$ of the form:

$$\tilde{\mu}(z) = \prod_{k=1}^{N}(z_k - e_1)^{\nu_1}(z_k - e_2)^{\nu_2}(z_k - e_3)^{\nu_3}. \quad (15)$$
The choice $\nu_1 = \nu_2 = \nu_3 = 0$ obviously corresponds to [1]. After a calculation, we find that for each value of the form

$$\nu_i = 0 \text{ or } \nu_i = \frac{1}{2} - b, \quad i = 1, 2, 3$$  

(16)

the Hamiltonian $\tilde{H}_N$ can be expressed as a quadratic combination of the operators (12) with suitable values (depending on the values of $\nu_i$'s) of the parameter $r$. We then found eight gauge factors (15) leading to algebrasisations of the initial operator $H_N$. Let us now investigate the relations between $r$ and the different parameters involved in the equations.

We find that the degree-increasing part, say $\tilde{H}_N^{(+)}$, of $\tilde{H}_N$ is given by:

$$\tilde{H}_N^{(+)} = -4\tau_1 \left( N - (m + bn_f - \frac{1}{2} n_f) \right) \left( N + m + 2a(N-1) + (3-n_f)b + \frac{1}{2}(1+n_f) \right).$$  

(17)

Here, $n_f$ denotes the number of non-zero exponents $\nu_i$, $i = 1, 2, 3$ in (15), i.e. is either 0, 1, 2 or 3. Note that for $n_f = 1$ and $n_f = 2$ three different algebrasations are available.

If we allow $m$ to be a non-integer and require instead that $\tilde{m}$ with

$$\tilde{m} = m + bn_f - \frac{1}{2} n_f$$  

(18)

is an integer, we conclude that now

$$\tilde{H}_N \mathcal{M}_{\tilde{m}} \subseteq \mathcal{M}_{\tilde{m}}.$$  

(19)

In the special case $b = 0$, we can distinguish two different cases: 1) both $m$ and $\tilde{m}$ are integers and 2) only $\tilde{m}$ is an integer. For 1) we find a quadruple algebrasisation of the Hamiltonian $\tilde{H}_N$ (one algebrasisation for $n_f = 0$ and three for $n_f = 2$):

$$\tilde{H}_N \mathcal{M}_m \subseteq \mathcal{M}_m \quad \text{for } n_f = 0,$$  

(20a)

$$\tilde{H}_N \mathcal{M}_{m-1} \subseteq \mathcal{M}_{m-1} \quad \text{for } n_f = 2.$$  

(20b)

Similarly, for 2) we find

$$\tilde{H}_N \mathcal{M}_{m-\frac{1}{2}} \subseteq \mathcal{M}_{m-\frac{1}{2}} \quad \text{for } n_f = 1,$$  

(21a)

$$\tilde{H}_N \mathcal{M}_{m-\frac{3}{2}} \subseteq \mathcal{M}_{m-\frac{3}{2}} \quad \text{for } n_f = 3.$$  

(21b)

Now, $\tilde{m} = m - \frac{1}{2}$ should be an integer. Again, this is a quadruple algebrasisation of the Hamiltonian $\tilde{H}_N$ (one algebrasisation for $n_f = 3$ and three for $n_f = 1$).
A. $a = b = 0$: Relation between the Hamiltonian $H_N$ and the Lamé operators

In order to understand the pattern of the algebraic solutions obtained for the model (1), (2), it is useful to study the limit $a = b = 0$. Using the relation

$$P(x + i\beta) = e_3 + (e_2 - e_3)\text{sn}^2(\sqrt{e_1 - e_3}x, k), \quad k^2 \equiv \frac{e_2 - e_3}{e_1 - e_3},$$

(22)

it is easy to see that for $a = b = 0$ the operator (1) takes the form

$$H_N(u) = (e_1 - e_3) \sum_{j=1}^{N} \left\{-\frac{\partial^2}{\partial u_j^2} + 2m(2m + 1)k^2\text{sn}^2(u_j, k)\right\} + 2m(2m + 1)e_3N$$

(23)

with $u_j \equiv \sqrt{e_1 - e_3}x_j$. The operator inside the brackets $\{\}$ of (23) constitutes $N$ decoupled copies of the Lamé operator $L(u)$:

$$L(u) = -\frac{d^2}{du^2} + 2m(2m + 1)k^2\text{sn}^2(u, k), \quad 0 \leq k \leq 1,$$

(24)

which admits $(4m + 1)$ algebraic eigenvalues if $m$ is an integer or a half integer.

If $m$ is an integer $(m + 1)$ eigenvectors of $L(u)$ are of the form $p_m(\text{sn}^2)$ and $(3m)$ eigenvectors are of the form $\text{cn} \cdot p_{m-1}(\text{sn}^2), \text{sn} \cdot p_{m-1}(\text{sn}^2), \text{dn} \cdot p_{m-1}(\text{sn}^2)$. $\text{sn}$, $\text{cn}$, $\text{dn}$ are abbreviations for the Jacobi elliptic functions $\text{sn}(u, k), \text{cn}(u, k), \text{dn}(u, k)$ and $p_n$ denotes a polynomial of degree $n$ in its argument. If $m$ is a half integer $3(m + 1/2)$ eigenvectors of $L(u)$ are of the form $\text{sn} \cdot \text{cn} \cdot p_{m+1/2}(\text{sn}^2), \text{sn} \cdot \text{dn} \cdot p_{m+1/2}(\text{sn}^2), \text{cn} \cdot \text{dn} \cdot p_{m+1/2}(\text{sn}^2)$ and $(m - 1/2)$ eigenvectors are of the form $\text{sn} \cdot \text{cn} \cdot \text{dn} \cdot p_{m-1/2}(\text{sn}^2)$.

Therefore, a total number of $(4m + 1)^N$ algebraic eigenvectors of the Hamiltonian (23) can be constructed. However, not all of them are completely symmetric under the permutations of the coordinates. Since the procedure of algebraisation is crucially related to the symmetrized variables $\tau_k$ (see (7)), only the completely symmetric solutions can be hoped to be recovered in the generic case for which $a \neq 0$ and/or $b \neq 0$.

Studying the solutions of the operator (23) and the structure of the eigenfunctions of the Lamé operator, it is not difficult to see that the number of completely symmetric solutions is given by:

$$C_{m+N}^N + 3C_{m+N-1}^N$$

(25a)

if $m$ is an integer and

$$3C_{m+N}^{m'} + C_{m+N-1}^{m'}, \quad m' \equiv m + \frac{1}{2}$$

(25b)

if $m$ is a half integer, respectively. $C_q^p$ denotes the usual combinatoric symbol.

We find that for $b = 0$ the number of algebraic solutions available by applying the method described here agrees nicely with these above numbers. Moreover, we checked for several particular cases that, indeed, the relevant Lamé solutions are reproduced in the limit $a \to 0$. Note that in [1] only $C_{m+N}^N$ solutions were found for integer values of $m$. Our supplementary factorisations therefore complete the pattern.
B. The case $N = m = 2$, $b = 0$

For the choice $N = m = 2$, (20a) leads to a $6 \times 6$ matrix with respect to the basis 
\{1, $\tau_1, \tau_2, \tau_1^2, \tau_1 \tau_2, \tau_2^2$\} [1]:

\[
\begin{pmatrix}
0 & g_2(2a + b + 1) & -2ag_3 & 4g_3 & 0 & 0 \\
16a + 24b + 20 & 0 & g_2(b + \frac{1}{2}) & 4g_2(a + b + 1) & 2g_3(1 - a) & 0 \\
0 & 8a + 24b + 12 & 0 & 0 & g_2(2a + 2b + 5) & -4g_3(a + 1) \\
0 & 8a + 12b + 14 & 0 & 0 & g_2(b + \frac{1}{2}) & 2g_3 \\
0 & 0 & 8a + 12b + 14 & 16(a + 3b + 3) & 0 & g_2(2b + 3) \\
0 & 0 & 0 & 8a + 24b + 28 & 0 & 0
\end{pmatrix}
\]

(26)

For (20b) we obtain 3 different $3 \times 3$ matrices with respect to the basis \{1, $\tau_1, \tau_2$\}:

\[
h_i = \begin{pmatrix}
(6 + 4a)e_i & g_2(2a + 1) + 8e_i^2 & -2ag_3 \\
14 + 8a & (10 + 4a)e_i & g_2/2 + 4e_i^2 \\
0 & 28 + 8a & (14 + 4a)e_i
\end{pmatrix}, \ i = 1, 2, 3.
\]

(27)

We thus obtain fifteen algebraic solutions, i.e. an additional nine to the ones obtained in [1].

In FIGs. 1a and 1b we show the energy eigenvalues as functions of $\epsilon$ for

\[
e_1 = 2, \ e_2 = -1 + \epsilon, \ e_3 = -1 - \epsilon
\]

(28)

and $a = 0$ and $a = 5.0$, respectively. FIG. 1a corresponds to two decoupled Lamé operators.

The limit $\epsilon = 0$ further corresponds to the completely integrable case of two decoupled oscillators ($e_2 = e_3$, so $k = 0$ and the potential vanishes in (22), (23)). The eigenvalues of this system are of the form $3(j_1^2 + j_2^2) - 40$ where $j_1, j_2$ are integers. The set of algebraic eigenvalues obtained with our factorisation (15) represents just the completely symmetric case, i.e. $j_1 + j_2 = 2n, n = 0, 1, 2, ...$ in this limit. This can be checked in FIG. 1a. In FIG. 1b the effect of an interaction potential on the energy eigenvalues is demonstrated for $a = 5.0$.

The case for which two of the numbers $e_1, e_2, e_3$ are equal is in itself special, since the fifteen eigenvalues can be expressed as linear functions of $a$ and the system is highly degenerated, irrespectively of $a$. E.g. for $e_1 = 2, e_2 = e_3 = -1$ three eigenvalues of the $6 \times 6$ matrix are not degenerate:

\[
-8(5 + 4a), \ -4(7 + 2a), \ 8(1 + 2a),
\]

(29a)

the other three eigenvalues of the $6 \times 6$ matrix coincide with those of the $3 \times 3$ matrix $h_1$:

\[
-8(2 + a), \ 4(5 + 4a), \ 8(7 + 2a),
\]

(29b)

and finally the eigenvalues of the $3 \times 3$ matrices $h_2$ and $h_3$ coincide and read:

\[
-2(17 + 10a), \ -2(5 - 2a), \ 2(7 + 2a).
\]

(29c)

This is clearly shown in FIGs. 1a and 1b, where at $\epsilon = 0$ three of the dotted curves, which correspond to three of the eigenvalues of the $6 \times 6$ matrix, and the three dashed
curves, which correspond to the three eigenvalues of $h_1$, cross both for $a = 0$ and $a = 5.0$, respectively. Similarly, the three solid lines and the three dotted-dashed lines, which correspond to the three eigenvalues of the matrices $h_2$ and $h_3$, respectively, cross at $\epsilon = 0$. How these degeneracies disappear for a generic choice of $e_i, i = 1, 2, 3$ is also shown in these figures.

Finally, in FIG. 2 we demonstrate the dependence of the eigenvalues on the parameter $a$ for the special choice $e_1 = 2, e_2 = -3/2, e_3 = -1/2$.

**IV. SUMMARY**

The construction of integrable models of Calogero-Sutherland (CS) type has recently received a lot of attention in relation to new applications related to different domains of theoretical physics. The class of N-body integrable models remains however very tiny and several generalisations are worth considering. The construction of quasi exactly solvable Hamiltonians describing N degrees of freedom appears to be a possible extension of the notion of integrable systems. As seen in [1,16] the potential can be more general than those related to the root system of a Lie algebra (typically of the type $A_N$ for potentials depending on the differences of the particles’ coordinates).

In this paper, we reconsidered such a QES model proposed recently in [1]. It depends on four parameters: two coupling constant $a,b$ and the two periods of the Weierstrass function $\mathcal{P}$, parametrized by $g_2,g_3$. More popular models are recovered for special limits of these constants: an Inozemtsev model for $b = 0$, a system of N decoupled Lamé equations if $a = b = 0$ and a system of N decoupled oscillators if, in addition, $e_2 = e_3$ (or equivalently $g_2^3 = 27 g_3^2$). We have seen that the case $b = 0$ possesses a particularly rich algebraic spectrum.

By investigation of the spectrum available in these limits, it appears that the solutions constructed in [1] do not constitute the full set of completely symmetric algebraic eigenfunctions of the initial Hamiltonian (1). Following closely the construction of the Lamé polynomials we have found additional algebraisations of the operator $H_N$. The set of algebraic eigenfunctions obtained in this way coincides exactly with the number of possible algebraic functions. We assume that an extension of the type of Hamiltonian considered here to $2 \times 2$ matrix valued operators [19] might be possible, but leave this construction as a future project [20].

**Note Added**

After the paper was finished several papers appeared dealing with the same topic. These are e.g. K. Takemura : math.QA/0205274 and O. Chalykh et al. : math.QA/0212029.

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REFERENCES

[1] D. Gomez-Ullate, A. Gonzalez-Lopez and M.A. Rodriguez, Phys. Lett. B511 (2001) 112.
[2] F. Calogero, J. Math. Phys. 12 (1971) 419.
[3] B. Sutherland, Phys. Rev. A 4 (1971) 2019.
[4] F. Calogero, Lett. Nuovo Cim. 13 (1975), 411;
    A. M. Perelomov, Lett. Math. Phys. 1 (1977), 531.
[5] I. M. Krichever, Funct. Anal. Appl. 14 (1980), 282.
[6] M. A. Olshanetsky and A. M. Perelomov, Phys. Rep. 94 (1983) 313.
[7] A. M. Perelomov, *Integrable systems of classical mechanics and Lie algebra*, Birkhäuser, 1990.
[8] F. D. M. Haldane, Phys. Rev. Lett. 60 (1980), 635;
    H. Azuma and S. Iso, Phys. Lett. B331 (1994) 107.
[9] A. Gorskiii and N. Nekrasov, Nucl. Phys. B 414 (1994), 317;
    J. A. Minaian and A. P. Polychronakos, Phys. Lett. B326 (1994), 288.
[10] A. P. Polychronakos, Phys. Rev. Lett. 74 (1995), 5153.
[11] G. W. Gibbons and P. K. Townsend, Phys. Lett. B454 (1999), 187.
[12] M. Brigante, S. Cacciatori, D. Klemm and D. Zanon, JHEP 0203 (2002), 005.
[13] R. Stanley, Adv. Math. 77 (1988), 76.
[14] L. Lapointe and L. Vinet, Comm. Math. Phys. 178 (1996), 425.
[15] A. Turbiner, Comm. Math. Phys. 118 (1988), 467.
[16] V. I. Inozemtsev, Lett. Math. Phys. 17 (1989) 11.
[17] Y. Brihaye and M. Godard, J. Math. Phys. 34 (1993) 5283.
[18] W. Rühl and A. Turbiner, Mod. Phys. Lett. 10 (1995) 2213.
[19] Y. Brihaye and B. Hartmann, Mod. Phys. Lett. 16 (2001) 1895.
[20] Y. Brihaye and B. Hartmann, “Quasi exactly solvable matrix N-body Hamiltonians”,
    in preparation
FIG. 1a. The energy eigenvalues of the $6 \times 6$ matrix (dotted) and of the $3 \times 3$ matrices $h_i$, ($i = 1$ dashed, $i = 2$ solid, $i = 3$ dotted-dashed), which correspond to the choice $N = m = 2$, are shown for $a = b = 0$ as a function of $\epsilon$, where $e_1 = 2$, $e_2 = -1 + \epsilon$ and $e_3 = -1 - \epsilon$.

FIG. 1b. Same as Fig.1a, but for $a = 5.0$. 
FIG. 2. The energy eigenvalues of the $6 \times 6$ matrix (dotted) and of the $3 \times 3$ matrices $h_i$, ($i = 1$ dashed, $i = 2$ solid, $i = 3$ dotted-dashed), which correspond to the choice $N = m = 2$, are shown for $b = 0$ and $e_1 = 2, e_2 = -3/2, e_3 = -1/3$ as a function of $a$. 

\begin{align*}
e_1 &= 2, \\
e_2 &= -3/2, \\
e_3 &= -1/2
\end{align*}