Computational complexity of guarding of proximity graphs

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Abstract

Computational complexity is studied for the problem of stabbing set of straight line segments with the smallest cardinality set of disks of fixed radii $r > 0$ where the set of segments forms straight line drawing of planar graph. This problem and its relatives can arise in physical network security analysis for telecommunication, wireless and road networks represented by geometric graphs based on euclidean distances between their vertices (proximity graphs). Among those proximity graphs are Delaunay triangulations, its subgraphs and half-$\theta_6$ graphs which admit efficient geometric routing. Being of particular interest computational complexity of this problem did not receive much attention in the literature. In this paper we claim strong NP-hardness of the problem over the classes of 4-connected (i.e. Hamiltonian) plane triangulations of bounded vertex degree as well as of 4-connected plane half-$\theta_6$ graphs for small $r$. It remains strongly NP-hard over the classes of Delaunay triangulations and some of their connected subgraphs (Gabriel and relative neighbourhood graphs) for values of $r$ of the same scale as minimum graph edge length whereas for large $r$ the problem becomes polynomially solvable over connected plane graphs. In both hard cases of the problem edge length variance is $O(n)$ where $n$ is the number of vertices.

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1 Introduction

Guarding boundaries of geometric objects or complexes of plane embeddings of planar graphs is widely studied class of problems in computational geometry, see e.g. [9], [12] and [23]. Usually in such problems one needs to find the smallest cardinality set $C$ of guards (e.g. points on the plane) having bounded (in some sense) visibility area such that each piece of the boundary or of the graph complex (e.g. edge or face) is within visibility area of some guard from $C$. Designing exact and approximate algorithms for these problems finds its applications in security, sensor placement, lighting and robotics. In this paper the computational complexity of the following problem ($\text{IPGD}$) is studied: 

\textit{given some straight line drawing of an arbitrary simple (i.e. without loops and parallel edges) planar graph $G = (V,E)$ without edge crossings and a constant $r > 0$ find the smallest cardinality set $C \subset \mathbb{R}^2$ of points (disk centers) such that each edge $e \in E$ is within (euclidean) distance $r$ from some point $c = c(e) \in C$ (i.e. disk of radius $r$ centered at $c$ intersects $e$).}

In $\text{IPGD}$ problem each guard has circular visibility area which does not depend on the obstacles in contrast to known Guarding Art Gallery problem [23] where the coverage area of video camera is affected by gallery walls.

There are two basic applications of studying $\text{IPGD}$ problem. In the first setting given network of physical devices distributed in some geographical area and communicating with each other through physical links (e.g. optical fibers) the problem is to monitor its security (e.g. connectivity) by locating the smallest number of sensors having circular scope area such that each network link is within the scope of some sensor. Here network nodes are modelled by points on the plane while its links are given in the form of straight line segments. The second application is again from network

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2stands for Intersecting Plane Graph with Disks
security ones [1, 2, 3]. Given network of physical devices one needs to evaluate its vulnerability to simultaneous technical failures caused by natural (floods, fire, electromagnetic pulses) or human sources. Catastrophic event (threat) is usually localized in a particular geographical area and is modelled [2] by a disk for simplicity. It is also assumed that these disks are all of the same radius but the other model assumptions about threats can also be adopted [3]. Threat impacts network link when the corresponding disk and line segment intersect. Evaluation of network vulnerability may be formulated as finding the minimum number of threats along with their positions that cause all network links to break and its solution reports those parts of the network that need additional protection. Thus both aforementioned applications bring us to IPGD problem as usually network links are geographically non-overlapping.

IPGD is closely related to several well known combinatorial optimization problems. Obviously IPGD problem is the special case of geometric Hitting Set problem: given the set \( N = \{ N_r(e) \}_{e \in E} \) of euclidean \( r \)-neighbourhoods of edges from \( E \) find the smallest cardinality set \( C \subset \mathbb{R}^2 \) such that \( N_r(e) \cap C \neq \emptyset \) for every \( e \in E \). Following common terminology the set \( C \) forms a hitting set for the set of objects \( N \). Known Vertex Cover (VC) problem for planar graphs can be obtained from IPGD by setting \( r = 0 \) because it makes one choose disk centers at the graph vertices. For the case of IPGD where \( G \) consists of isolated vertices (i.e. segments from \( E \) are all of zero length) we have continuous Disk Cover (CDC) problem.

Most closely related work to IPGD problem we found is from [2] where a different setting is considered whose computational complexity is not studied though: given straight line drawing of planar graph and positive integer \( k \) we are to find the locations of centers of \( k \) disks of radii \( r \) whose union intersects the largest number of the graph edges. Some of research is focused on the case of finding a single disk that stabs all segments, see e.g. [7], [11]. Strong NP-hardness is known for CDC and VC problems as well as for Hitting Set problem over axis-parallel rectangles.

In this paper computational complexity of IPGD problem is studied for different values of \( r \) over classes of plane graphs with the special emphasis on those classes of embeddings that are based on euclidean distances between vertices (proximity graphs). Narrowing the scope to proximity graphs is motivated mainly by network analysis applications. Plane graph notion is used to denote simple planar graph along with its straight line drawing on the plane without edge crossings. More specifically a triple \( G = (V, E, F) \) is called a plane graph if \( V \subset \mathbb{R}^2 \), set \( E \) consists of nondegenerate straight line segments crossing only at their endpoints which are from \( V \) and \( F \) denotes the set of all open (in \( \mathbb{R}^2 \)) regions bounded by segments from \( E \) where each \( f \in F \) does not intersect with any segment from \( E \). Our first main result is about computational complexity of IPGD problem over highly connected plane triangulations as well as over their specific (euclidean distance based) straight line embeddings on the plane. Triangulations can denote somewhat different graph classes in the literature. In this paper they are used in the sense that is common for planar graph theory. Plane graph whose faces are triangles except for possibly outer one is referred to as general plane triangulation. A general plane triangulation having triangular outer face is called simply plane triangulation.

Consider a set \( S \) of \( n \) points in general position on the plane. We call a plane graph \( G = (V, E, F) \) a Delaunay triangulation if \( V = S \), outer face of \( G \) is defined by the boundary of \( \text{conv} \ V \) and each inner face \( f \in F \) is a triangle that admits circumscribing disk \( c(f) \) where the only points of \( S \) that are contained in \( c(f) \) are the vertices of \( f \). Let us define also another form of Delaunay triangulation. Suppose a division of the plane is given which is induced by 6 distinct rays originating from a common point \( u \) where one of these rays is codirectional to \( Ox \) axis whereas all cones bounded by consecutive rays are identical of angular measure \( \pi/3 \). Positive cone \( K \) is exactly one which either contains \( Oy \)-codirectional ray originated at \( u \) as its bisector or has one
of two symmetric Ox-parallel rays on its boundary bd K being below it. Consider a finite point
set S such that no its pair of points defines the straight line which is parallel to aforementioned
rays. A plane graph \( G = (V, E, F) \) with \( V = S \) is called half-\( \theta_6 \)-graph iff for each \( e = [u, v] \in E \)
either point \( v \) (or \( u \)) is contained in some positive cone \( K \) induced by \( u \) (respectively by \( v \)) and
for \( w = u \) and \( z = v \) (respectively for \( w = v \) and \( z = u \)) we have \( |w - z'|_2 \leq |w - h'|_2 \) for every
\( h \in S \cap K \) with \( h \neq z \) where \( z' \) and \( h' \) are orthogonal projections of points \( z \) and \( h \) respectively
on the bisector of \( K \). Delaunay triangulations and half-\( \theta_6 \)-graphs admit efficient local geometric
routing algorithms [8] and therefore could represent real world network topologies.

Using connections with maximum independent set problem (MIS) the strong NP-hardness of both
IPGD and VC problems (IPGD and VC become equivalent for small disk radius \( r \)) is reported over the subclass of 4-connected (i.e. Hamiltonian) plane triangulations of bounded
type degree. Graphs of this subclass are graph isomorphic to Delaunay triangulations due to [15]
which supports the conjecture that both VC and IPGD are NP-hard over 4-connected Delaunay
ones. In this paper we get weaker result on the NP-hardness of IPGD over the subclass of 4-
connected half-\( \theta_6 \)-graphs having \( O(n) \) variance of graph edge lengths (i.e. ratio of the longest and
shortest edge lengths) where \( n \) is the number of vertices. In fact our result is mostly combinatorial
and contributes into the complexity analysis of VC and MIS problems. It extends one from [12]
that claims NP-hardness of VC for graphs obtained after a finite number ofstellations of 3-
connected 3-regular plane graphs where a single stellation consists in a series of elementary ones
(applied for pairwise edge non-intersecting faces) which themselves are just adding an additional
vertex inside some face of (3-connected) plane graph along with adding edges connecting it with
the set of vertices of that face. Such a stellated graphs from [12] are not plane triangulations in
general.

A graph class that contains every induced subgraph (or every subgraph, or every minor) along
with every its graph is referred to as hereditary class. In the recent relevant literature NP-hardness
of VC and MIS problems is given over hereditary classes of planar graphs of vertex degree of
at most 3 [4]. It is easy to give an example witnessing that 4-connected plane triangulations do
not form hereditary class even with respect to edge contractions (without multiple edges) which
prohibits direct application of MIS complexity analysis over those classes. Other works claim
NP-hardness of VC over 2-connected 3-regular planar graphs [22], over stellated 3-connected 3-
regular ones [12] and over 3-regular planar Hamiltonian graphs [17]. It contrasts to our result
for plane triangulations which have high connectivity as well as the maximum number of edges
(roughly twice as much as for 3-regular planar graphs) within the class of connected plane graphs.

Let \( S \) be set of \( n \) points in general position on the plane. There are several classes of connected
subgraphs of Delaunay triangulations. A plane graph \( G = (V, E, F) \) with \( V = S \) is called Gabriel
graph where \( [u, v] \in E \) iff the disk having \([u, v]\) as its diameter does not contain any points
of \( S \) distinct from \( u \) and \( v \). A relative neighbourhood graph is the plane graph \( G \) with \( V = S \)
for which \([u, v] \in E \) iff there is no any other point \( w \in S \) distinct from \( u \) and \( v \) such that
max\{\(|u - w|_2, |v - w|_2\} < |u - v|_2 \). Finally a plane graph is called euclidean minimum spanning
tree if it is the minimum weight spanning tree of weighted complete graph \( K_{|S|} \) having its vertices
at \( S \) where its edge weight is given by euclidean distance between the edge endpoints. Gabriel and
relative neighbourhood graphs arise in modelling wide variety of real world networks including
road ones. Our second main result claims strong NP-hardness of IPGD over classes of Delaunay
triangulations, half-\( \theta_6 \) graphs as well as over classes of their connected subgraphs such as Gabriel,
relative neighbourhood graphs and minimum euclidean spanning trees for \( r \) of the same scale as
the smallest graph edge length where the variance of edge lengths is again \( O(n) \).

Let \( R(E) \) be the smallest radius of the disk that intersects all segments from \( E \) which can
be found in time $O(|E|)$ [1]. We finish our complexity analysis of IPGD problem by considering the setting where $r$ is at the same scale as $R(E)$ i.e. is within some multiplicative constant $0 < \theta_0 < 1$ from $R(E)$. This setting looks natural because in practice we choose $r$ to be adequate to network size. In this case IPGD problem admits polynomial (in time and space) algorithm whose complexity depends exponentially on $\theta_0$ as having $r$ scaled in this way implies the upper bound on the problem optimum.

2 Preliminaries

We call graph isomorphism $\sigma$ defined by a pair of plane embeddings $H = (V, E, F)$ and $H' = (V', E', F')$ of the same planar graph $G$ a combinatorial isomorphism [14] iff it extends to bijection $\sigma : V \cup E \cup F \to V' \cup E' \cup F'$ that keeps not only incidence between vertices and edges but also incidence of vertices and edges with faces: more formally if $x \in V \cup E$ lies on the boundary of some face $f \in F$ then $\sigma(x)$ lies on the boundary of the face $\sigma(f) \in F'$. We say that planar graph admits unique plane drawing if isomorphism induced by an arbitrary pair of its plane embeddings (including one that just renames graph vertices according to any graph automorphism) extends to combinatorial isomorphism.

Computational complexity of Facial MIS. Given graph $G = (V, E)$ a subset $V' \subseteq V$ is called independent set if no pair of vertices from $V'$ is adjacent. Consider the maximum independent set problem (MIS): given graph $G = (V, E)$ find the maximum cardinality independent subset $V' \subseteq V$.

To prove our first main result (theorem 3) we use one on the NP-completeness of dual form of MIS problem considered over plane embeddings of (possibly non-simple) planar graphs which is to find the maximum cardinality set of non-incident (i.e. having no common edges) faces instead of set of vertices. Moreover to simplify our proof we need to pick out a specific class of hard instances of that dual. It is done by taking a series of reductions [22] starting from special 3-satisfiability problem. Namely in [19] a special class of strongly NP-complete 3-satisfiability problems is considered called 3-connected PLANAR 3-SAT. Let $\Phi$ be a boolean formula represented in conjunctive normal form, $C$ be its set of clauses and $X$ be its set of variables. In $\Phi$ each clause from $C$ contains exactly 3 distinct literals (each either variable from $X$ itself or its negation) and the bipartite graph $G_\Phi$ is 3-connected planar graph over the set of vertices $V = X \cup C$ with the set of edges $E : e = \{x, c\} \in E$ iff either $x$ or $\neg x$ is contained in $c$.

A cycle $C$ of length $k$ of graph $G$ is called its $k$-cycle. In [22] 3-connected PLANAR 3-SAT is reduced to VC (and MIS) problem (in its decision form) over the special class of simple 2-connected 3-regular planar graphs which is build as follows. In each graph $G_\Phi$ each vertex $x \in X$ is replaced by $2k$-cycle $C_x$ whereas each $c \in C$ is replaced by triangle $T_c$ where $k$ is the number of occurrences of both $x$ or $\neg x$ in clauses from $C$. In $C_x$ vertices are ordered as follows: first comes pair $(u, v)$ of vertices that corresponds to the first occurrence of $x$ in $\Phi$ where $u$ denotes occurrence of $x$ whereas $v$ matches to occurrence of $\neg x$; then comes pair of vertices for the second occurrence of $x$ and so on. Vertices of triangle $T_c$ correspond to variables that are involved in the clause $c$. An $u$-vertex (respectively $v$-vertex) of $C_x$ is joined by an edge with the vertex of $T_c$ iff $x$ (respectively $\neg x$) occurs in the clause $c$. Obviously vertices of $T_c$ and $C_x$ can be arranged in such a way that the graph $\hat{G}_\Phi$ thus obtained is 2-connected planar. Then each vertex $v$ of degree 2 of $\hat{G}_\Phi$ is replaced by two vertices $w_1$ and $w_2$ connected by the following gadget $\Delta_v$ : two triangles that have the only common edge are connected to $w_1$ and $w_2$ by their non-common vertices. Denote the resulting graph by $G'_\Phi$ which is 2-connected 3-regular planar.

Let $C_0$ be subclass of so constructed graphs $G'_\Phi$ over all boolean formulas $\Phi$ involved in 3-
connected PLANAR 3-SAT problem. It is shown that both VC and MIS are strongly NP-complete over \( C_0 \). Let \( \mathcal{T}_0 \) be a class of dual planar embeddings to ones from the class \( C_0 \). It is obvious that the faces of every graph from \( \mathcal{T}_0 \) are all triangles that have at most one edge in common. Let us elaborate a remark from [22] that for every pair of distinct plane embeddings of any graph from \( \mathcal{T}_0 \) there exists a combinatorial isomorphism between them. The parallel edges of each graph from \( \mathcal{T}_0 \) have endpoints which correspond (under duality) to cycles \( C_x \). Being a subdivision of 3-connected planar graph \( \tilde{G}_\Phi \) admits unique plane drawing [16]. The only triangulation that could be obtained from the dual of any such subdivision is the one where the same gadget is placed “inside” each face bounded by a pair of parallel edges. Therefore every graph from \( \mathcal{T}_0 \) admits the only distinct plane embeddings that are induced by transposing pairs of gadget vertices (i.e. by some graph automorphism). This means that all these distinct embeddings are combinatorially isomorphic.

Finally let us formulate Facial MIS problem: given plane embedding of (possibly non-simple) graph \( G = (V, E, F) \in \mathcal{T}_0 \) one needs to find the largest cardinality set \( F' \subset F \) such that each pair of faces from \( F' \) does not have an edge from \( E \) in common. This set of faces is called maximum independent set on the face set \( F \). The following theorem holds [22].

**Theorem 1** Facial MIS problem (in its decision form) is strongly NP-complete over \( \mathcal{T}_0 \).

### Computational complexity of CDC problem.

To prove our second main result we exploit tough connection between IPGD and CDC problems. To get sharper one we also single out a specific class of hard instances of CDC problem. In doing this we take a series of reductions starting from NP-complete minimum dominating set problem where given simple planar graph \( G_0 = (V_0, E_0) \) of vertex degree at most 3 (i.e. subcubic graph) one needs to find the smallest cardinality subset of vertices \( V'_0 \subseteq V_0 \) such that for each \( u \in V_0 \setminus V'_0 \) there exists \( v = v(u) \in V'_0 \) which is adjacent to \( u \).

A plane orthogonal drawing of planar graph \( G_0 \) is the drawing where vertices from \( V_0 \) are represented by points on the plane whereas edges of \( E_0 \) are given in the form of polylines formed by sequences of connected axis-parallel straight line segments of the form \([p_1, p_2], [p_2, p_3], \ldots, [p_{k-1}, p_k]\) and intersecting only at edge endpoints. In [21] NP-completeness of CDC is given by reduction from minimum dominating set problem. This reduction involves using plane orthogonal drawing of planar subcubic graph \( G_0 \) on the bounded integer grid. More specifically in this reduction a set \( D \) is build on that grid with \( V_0 \subset D \). Resulting hard instance of CDC problem is over the set \( D \) for some integer (constant) radius \( r_0 \geq 1 \). Let us observe here that due to theorem 2 from [25] \( G_0 \) admits orthogonal drawing (which could be computed in polynomial time and space) on the grid of size \( \Theta(|V_0|) \times \Theta(|V_0|) \). Proof of NP-completeness of CDC could be taken taking into account this observation. We can formulate the following

**Theorem 2** CDC problem is strongly NP-complete for constant integer radius \( r_0 \) over point sets \( D \) constructed in [21] on the integer grid of size \( O(|D|) \times O(|D|) \).

### 3 Computational complexity of IPGD problem over proximity graphs

Computational complexity of IPGD for classes of plane graphs is studied under additional assumptions both on radius \( r \) and (geometric) plane graph size. More specifically we have the parameter

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\(^3\)under conventional geometric duality transform over plane embeddings of planar graphs
\[ \lambda = \lambda(G) = \frac{r}{\mu(G)} \] bounded from below while keeping another parameter \( \mu = \mu(G) = \frac{d_{\max}}{d_{\min}} \) bounded from above by some polynomially computable functions of the input length where \( d_{\max} \) and \( d_{\min} \) are euclidean lengths of the longest and shortest edges of \( G \) respectively. It is easy to note that given some \( r > 0 \) we can scale plane graph in such a way that makes \( r \) extremely small or large with respect to \( R(E) \). Therefore the first parameter defines the scale of the problem while the second serves as a measure of variance of edge lengths of the graph \( G \). Fixing these two parameters is natural because in practice value of \( r \) is chosen adequate to network size.

**IPGD** problem over connected plane graph can be transformed in polynomial time and space (in \( |E| \)) to equivalent one for the same graph where disk centers are chosen in some finite set \( D_r(G) \) which is referred to as discrete **IPGD** problem in the sequel. Indeed consider an arbitrary maximum (with respect to inclusion) subsystem (MFS) of objects (i.e. \( r \)-neighbourhoods of graph edges) from \( \mathcal{N} \) having nonempty intersection. W.l.o.g. we can assume that an optimal solution to **IPGD** problem consists of centers of radius \( r \) disks each of which lies at the intersection of some MFS. The set \( M \) of points at the intersection of an arbitrary MFS is bounded, closed and convex. Moreover \( M \) has a boundary (denote it by \( \text{bd} M \)) which is composed of pieces of boundaries of objects from \( \mathcal{N} \). Due to connectedness of \( G \) set \( M \) is contained at the intersection of at least two objects from \( \mathcal{N} \). Consider degenerate case where two edges \( e_1 \) and \( e_2 \) from \( E \) are parallel and have their \( r \)-neighbourhoods touching in such a way that touching points form a segment. In this case obviously \( |\text{bd} N_r(e_1) \cap \text{bd} N_r(e_2)| = \infty \). It is also possible when edges \( e_1 \) and \( e_2 \) intersect at their common vertex. In both cases \( \text{bd} N_r(e_1) \cap \text{bd} N_r(e_2) \) has at most 2 extreme points of intersection. In the other (non-degenerate) ones \( |\text{bd} N_r(e_1) \cap \text{bd} N_r(e_2)| \) is obviously bounded by some absolute constant taking into account the fact that each \( \text{bd} N_r(e), e \in E \) is compound of a pair of parallel straight line segments and of a pair of half-circles. Set

\[ D_r(G) = \{ u \in \mathbb{R}^2 : u \in \text{extr} (\text{bd} N_r(e_1) \cap \text{bd} N_r(e_2)), e_1, e_2 \in E \}, \]

where \( \text{extr} N \) means the set of (extreme in two aforementioned degenerate cases) points of one-dimensional set \( N \). It is easy to note that \( M \) has at least one vertex which is just an (possibly extreme) intersection point of \( \text{bd} N_r(e_1) \cap \text{bd} N_r(e_2) \) for some distinct edges \( e_1, e_2 \in E \). W.l.o.g. we can restrict each point of feasible solution to **IPGD** problem to lie in the set \( D_r(G) \) of cardinality of the order \( O(|E|^2) \) which can be found in polynomial (in \( |E| \)) time. Let us observe that unlike general **IPGD** problem each feasible solution to discrete **IPGD** one has cardinality which is bounded by \( O(|E|^2) \).

### 3.1 NP-hardness over highly connected triangulations for small \( r \)

Below complexity of discrete **IPGD** will be studied over highly connected triangulations where \( r \) is small with respect to \( R(E) \). In fact both in **VC** (over plane graphs) and discrete **IPGD** problems cardinalities of hitting sets are minimized for finite sets of straight line segments and their \( r \)-neighbourhoods respectively. We can prove that for small \( r \) these two problems become equivalent in the following sense being defined on the same plane graphs:

1. every feasible solution to **VC** can be converted in polynomial time and space in \( |E| \) to some feasible solution of discrete **IPGD** keeping its cardinality;

2. every feasible solution to discrete **IPGD** can be converted in polynomial time and space to some feasible one for **VC** with the same cardinality counting multiplicity.

The following lemma reduces **IPGD** problem to **VC** for small \( r \).
Lemma 1 Let $G = (V, E, F)$ be plane graph with $V \subset \mathbb{Z}^2$. Discrete IPGD problem over graph $G$ is equivalent to VC for the same graph where $r = \Theta \left( \frac{1}{\text{diam} V} \right)$.

Schnyder embedding [24] is polynomial (in time and space) algorithm that gives straight line drawing of a simple planar graph with $n \geq 3$ vertices on the integer grid of size $n - 2 \times n - 2$. Obviously $\mu = O(n)$ for such drawing. Due to lemma 1, VC problem on the $n$-vertex plane graph $G$ having its vertices on the integer grid $n - 2 \times n - 2$ is equivalent to IPGD for the same graph if $r = \Theta \left( \frac{1}{n} \right)$. Thus IPGD problem inherits VC problem complexity status being considered over respective plane graph classes on the integer grid where $\lambda = \Omega \left( \frac{1}{n^2} \right)$. Now our first main result comes.

Theorem 3 For $\lambda = \Theta \left( \frac{1}{n^2} \right)$ and $\mu = O(n)$ discrete IPGD (and VC) problem is strongly NP-hard over the class of 4-connected (Hamiltonian) plane triangulations whose vertices have degree not exceeding $\log n$ where $n$ is the number of vertices of triangulation.

Proof. Given positive integer $k$ it can be checked in polynomial time and space that an arbitrary sequence of points from $D_k(G)$ gives feasible solution of cardinality not exceeding $k$ to discrete IPGD problem. Therefore discrete IPGD is in NP.

To prove the theorem it is sufficient to build the reduction from NP-complete problem to VC one over the class of 4-connected plane triangulations having bounded (by $\log n$) vertex degrees. Let $T$ be the class of plane triangulations. We give reduction to VC from Facial MIS which is NP-complete by theorem 1. For each graph $G_0 = (V_0, E_0, F_0) \in T_0$ we construct a graph $G = (V, E, F) \in T$ as follows. We replace each face $f_0 = v_1(f_0)v_2(f_0)v_3(f_0) \in F_0\setminus\{f_0, \infty\}$ by a graph $G(f_0)$ shown on the fig. 1 (symbol $f_0$ is omitted in denotations of vertices of $G(f_0)$ in the sequel) where $f_0, \infty$ is the outer face of $G_0$. It is easy to see that the only minimum vertex cover for graph $G(f_0)$ does not contain $v_{10}$ and coincides with $U_{f_0} = \{v_4, v_5, v_6, v_7, v_8, v_9\}$. More specifically this follows from the fact that each triangle from the triple $\Delta v_1v_5v_7$, $\Delta v_2v_6v_8$ and $\Delta v_3v_4v_9$ (see fig. 1) requires two vertices to cover. Considering all possible cases we get that $|V_{f_0}^\prime| \geq 7$ for every vertex cover $V_{f_0}^\prime$ if $v_{10} \in V_{f_0}^\prime$. Also we have that $W_{f_0} = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$ is vertex cover for $G(f_0)$.

Graph $G_0$ could be embedded on the plane in polynomial time and space starting from straight line drawing of dual of the corresponding 3-connected planar graph obtained using Schnyder embedding [24]. Place inside $f_0$ a graph $G(f_0)$ such that sides of generally curvilinear triangle $v_1v_2v_3$ become parallel to sides of face $f_0$ e.g. by scaling the face by given rational factor. Connect

![Figure 1: Graph $G(f_0)$ corresponding to face $f_0$ of $G_0$](image-url)
graphs $G(f_{01})$ and $G(f_{02})$ that correspond to faces $f_{01}$ and $f_{02}$ from $F_0 \setminus \{f_{0,\infty}\}$ having common edge $e_0 \in E_0$ using connector graph $G(e_0)$ as shown in the fig. Figure 2. Graph $G(e_0)$ has a pair of common edges with each of two graphs $G(f_{01})$ and $G(f_{02})$. Graph $G(f_{0,\infty})$ to which face $f_{0,\infty}$ corresponds is isomorphic to graph $G(f_0)$ for arbitrary $f_0 \in F_0 \setminus \{f_{0,\infty}\}$. It is shown in the fig. Figure 3 together with respective connector graphs $G_i$, $i = 1, 2, 3$. (graphs $G_i$ are depicted schematically without some edges and vertices). Edges of graphs $G(f_0)$, $f_0 \in F_0$ and $G(e_0)$, $e_0 \in E_0$ are embedded in such a way that edges of graphs $G(e_{01})$ and $G(e_{02})$ intersect at most at their common endpoint (if exists) for every $e_{01}$ and $e_{02}$ from $E_0$. For every $v_0 \in V_0$ each set $E_0(v_0) = \{e_0 \in E_0 : v_0 \in e_0\}$ corresponds to the set $Q(v_0) = \{G(e_0) : e_0 \in E_0(v_0)\}$ of $\deg(v_0)$ connector graphs where $\deg(v_0) = |E_0(v_0)|$. Vertex $v_0$ is in the interior of some generally curvilinear simple $2\deg(v_0)$-gon $M(v_0)$ bounded by edges of connectors from $Q(v_0)$ (see fig. Figure 4). Let $U(v_0) = \{u_1, \ldots, u_{2\deg(v_0)}\}$ be clockwise ordered sequence of vertices of $M(v_0)$ where $u_1$ coincides with one of two vertices $v_{14}$ or $v_{15}$ of graph connectors from $Q(v_0)$. Triangulate $M(v_0)$ as follows (and denote the graph thus obtained as $G(v_0)$) : add series of edges $\{u_1, u_3\}$, $\{u_3, u_5\}, \ldots$ then series of edges $\{u_1, u_3\}$, $\{u_5, u_9\}, \ldots$ continuing doing so until $M(v_0)$ becomes triangulated. Set

$$G := \bigcup_{f_0 \in F_0} G(f_0) \cup \bigcup_{v_0 \in V_0} G(v_0) \cup \bigcup_{e_0 \in E_0} G(e_0).$$

As number of graph connectors is $|E_0|$ and degree of each vertex from $G(v_0)$ is of the order $O(\log |E_0|)$ the complexity of construction of graph $G$ for graph $G_0$ is of the order $O(|F_0| + \ldots$
Moreover the number of vertices, edges and faces of $G$ is of the order $O(|F_0| + |E_0|)$. Length of representation of vertices of $G$ differs only by a constant factor from one for graph $G_0$. Having that $G$ is simple triangulation (we “subdivided” all parallel edges of $G_0$) we get 3-connectedness of $G$. Graph $G$ does not have any edges that connect vertices from $G(f_0)\setminus G(e_0)$ with those from $G(e_0)\setminus G(f_0)$; the same holds for $G(f_0)$ and $G(v_0)$ as well as for $G(e_0)$ and $G(v_0)$. 

Brute-force check gives that for every graph $G(f_0)$, $f_0 \in F_0$ and $G(e_0)$, $e_0 \in E_0$ each its 3-cycle coincides with its face. In the process of triangulating $M(v_0)$ the only triangles that we have are faces by construction. Therefore each 3-cycle of $G$ is its face. Using lemma 2.3 from [13] we get 4-connectedness of $G$.

Let us show that each independent set of faces $F'_0 \subset F_0$ with $|F'_0| \geq k$ can be converted into the vertex cover $V'$ for $G$ where $|V'| \leq 7|F_0| - k + 4|E_0|$. Set $V'_{f_0} := U_{f_0}$ for every $f_0 \in F'_0$; for each $f_0 \in F_0 \setminus F'_0$ set $V'_{f_0} := W_{f_0}$. To get vertex cover $V'_{e_0}$ of graph $G(e_0)$ that connects graphs $G(f_0)$ and $G(f_0)$ for $f_0, f_0 \in F_0$, take vertices $v_{14}$ and $v_{15}$ as well as one vertex from pairs $\{v_{16}, v_{19}\}$ and $\{v_{17}, v_{18}\}$; we choose vertex from each such pair depending on whether $f_0 \in F'_0$ and $f_0 \in F'_0$: more specifically if $f_0 \in F'_0$ and $f_0 \notin F'_0$ vertices $v_{17}$ and $v_{18}$ are chosen; for the case $f_0, f_0 \notin F'_0$ choice of vertices from these pairs is arbitrary. Obviously $V' = \bigcup_{f_0 \in F_0} V'_{f_0} \cup \bigcup_{e_0 \in E_0} V'_{e_0}$ covers edges of $G$ and $|V'| \leq 7|F_0| - k + 4|E_0|$. 

Conversely let $V'$ be minimum cardinality vertex cover for $G$ and $|V'| \leq 7|F_0| - k + 4|E_0|$. Set $V'_{f_0} = V' \cap V(f_0)$ for each $f_0 \in F_0$ where $V(f_0)$ is vertex set of the graph $G(f_0)$. At least two vertices are required to cover edges that belong to both $G(f_0)$ and $G(e_0)$ or to both $G(f_0)$ and $G(e_0)$ where $e_0$ is common edge of faces $f_0$ and $f_0$. At least four vertices are required to cover edges of triangles over vertices $V_1(e_0) = \{v_{15}, v_{16}, v_{19}\}$ and $V_2(e_0) = \{v_{14}, v_{17}, v_{18}\}$. If $|V' \cap V_1(e_0)| = 2$ or $|V' \cap V_2(e_0)| = 2$ then $|V'_{f_0}| \geq 7$ in case of $|V'_{f_0}| = 6$. In view of minimality of $V'$ the case when $|V' \cap V_1(e_0)| = |V' \cap V_2(e_0)| = 3$ and $|V'_{f_0}| = |V'_{f_0}| = 6$ hold simultaneously is void. Therefore face set $F'_0 = \{f \in F_0 : |V'_{f_0}| = 6\}$ is independent. As $|V' \cap (V_1(e_0) \cup V_2(e_0))| \geq 4$ we have $6|F'_0| + 7(|F_0| - |F'_0|) + 4|E_0| \leq |V'|$ so $|F'_0| \geq k$.

Remark 1 In fact strong NP-hardness of discrete IPGD is given for $r/d_{min} = \Omega \left( \frac{1}{n} \right)$ i.e. where $r$ is small with respect to the smallest edge length of triangulation.

Moreover we note that due to [24] every $n$-vertex plane triangulation can be embedded on the integer grid of size $\Theta(n) \times \Theta(n)$ as a plane half-$\theta_6$-graph. Therefore we get

Corollary 1 For $\lambda = \Omega \left( \frac{1}{n} \right)$ and $\mu = O(n)$ discrete IPGD (and VC) problem is strongly NP-hard over the class of 4-connected (Hamiltonian) half-$\theta_6$-graphs whose vertices have degree not exceeding $\log n$. 
In spite of the fact that any 4-connected plane triangulation is isomorphic (in usual graph sense) to some Delaunay triangulation \cite{15} it is still an open (possibly NP-hard) problem of getting combinatorially equivalent Delaunay triangulation for a given 4-connected plane one (see e.g. a survey \cite{6}). Nevertheless we hope it is possible to do in polynomial time and space for the subclass of 4-connected plane triangulations from the proof of the theorem \footnote{3}. 

**Conjecture 1** IPGD (and VC) problem is NP-hard even over 4-connected Delaunay triangulations whose vertices have degrees not exceeding \( \log n \) where \( n \) is the number of vertices in triangulation.

Consider an arbitrary planar graph and its simple cycle. Graph edge is called a chord if it connects nonconsecutive vertices of that cycle. The length of the longest graph cycle without chords is called graph chordality. In the recent complexity research for VC problem its polynomial solvability is given over plane graph classes of bounded chordality \cite{4}, \cite{18}. Obviously due to lemma 2.1 from \cite{13} any class of 3-connected general plane triangulations having chordality bounded (uniformly within the class) by \( k \) contains only those triangulations whose outer facial cycle has length not exceeding \( k \). Possibly this could not be the case when considering classes of real-world networks. Therefore it may be of interest whether unbounded chordality witnesses NP-hardness of VC problem over subclasses of general plane triangulations. Below an example is given of the subclass of 3-connected general plane triangulations of infinite chordality for which VC problem admits polynomial time algorithm.

**Example 1** Let \( p \geq 2 \). Applying a sequence of elementary stellations to an axis-parallel square mesh of size \( 2p \times 2p \) on the integer grid we obtain general triangulation and the corresponding class \( \mathcal{GT}_0 \) of such ones over all positive integers \( p \). The class \( \mathcal{GT}_0 \) has infinite chordality but VC problem for every graph from \( \mathcal{GT}_0 \) can be solved in time linear with respect to the number of vertices. Vertices of squares form a minimum vertex cover of cardinality \( 4p^2 + 4p + 1 \). Minimality follows by induction on \( p \). For \( p = 2 \) it is proved by tedious check. Assume that for the stellated square mesh of size \( 2(p-1) \times 2(p-1) \) the minimum cardinality of vertex cover is \( 4(p-1)^2 + 4(p-1) + 1 \). The edges of the last \((2p-1)\)th and \(2p\)th stellated rows and columns require at least \( 8p \) additional vertices to be covered which sums up to \( 4p^2 + 4p + 1 \).

### 3.2 NP-hardness over Delaunay triangulations and their subgraphs

To prove NP-hardness of IPGD problem over claimed proximity graph classes we build reduction from CDC problem considered over point sets given in preliminary section. We exploit a simple idea that disk covers a set of points \( D \) iff a slightly larger disk covers respective straight line segments each of which is close to some point of \( D \) and has small length with respect to distances between points of \( D \). In fact proof idea of theorem \footnote{4} below consists in doubling each point of \( D \) to a pair of close points. As there are many proximity graph classes which contain edges that connect pairs of close points this technique gives NP-hardness proof of IPGD over such classes which involves CDC problem.

Let us give the following technical

**Lemma 2** Let \( X \subset \mathbb{Z}^2 \), \( r \geq 1 \) be some integer and \( \rho(u;v,w) \) be the minimum of two distances from \( u \in X \) to circles of radius \( r \) passing through distinct points \( v \) and \( w \) from \( X \) with \( |v-w|_2 \leq 2r \). Then

\[
\min_{u \in C(v,w), v \neq w, u, v, w \in X} \frac{1}{100r^4}
\]
where $C(v, w)$ is the union of two radius $r$ circles through $v$ and $w$.

Now we are ready to prove the following

**Theorem 4** Discrete IPGD problem is strongly NP-hard for $\lambda = \Omega \left( \frac{1}{n} \right)$ and $\mu = O(n)$ over the class of Delaunay triangulations where $n$ is the number of vertices in triangulation.

**Proof.** Let us use notations of preliminary section. For any hard instance of CDC problem given in the theorem $2$ we build discrete IPGD problem instance with $r = r_0 + \delta$ as follows where $\delta = \frac{1}{400 \sqrt{2} r_0}$. For each point $u \in D$ points $u_0$ and $v_0$ are chosen such that $|u - u_0|_\infty \leq \delta/4$ and $|u - v_0|_\infty \leq \delta/4$ where $I_u = [u_0, v_0]$ has length at least $\delta/4$. More specifically set $I_D = \{I_u = [u_0, v_0] : u \in D\}$. Endpoints of segments from $I_D$ are defined in sequential manner in polynomial time and space by defining new segment $I_u$ to provide generality position for the set of endpoints of the set $I_{D'} \cup \{I_u\}$, $D' \subset D$, where segments of $I_{D'}$ are already defined. Here endpoints of $I_u$ are chosen in the rational grid containing $u$ whose elementary square size is $\frac{n^2}{m^2} \times \frac{n^2}{m^2}$ for some small absolute rational constant $c_1$. Assuming $u = (u_x, u_y)$ point $u_0$ is chosen in the lower part of the grid with $y$-coordinates less than $u_y - \delta/8$ whereas $v_0$ is taken from the upper one where $y$-coordinates exceed $u_y + \delta/8$.

Let $S$ be the set of endpoints of segments from $I_D$. Every disk having $I_u$ as its diameter does not contain any points of $S$ distinct from endpoints of $I_u$. Let $G = (V, E, F)$ be Delaunay triangulation for $S$ which can be computed in polynomial time and space in $|D|$. Obviously each segment $I_u$ coincides with some edge from $E$. We have $\mu = O(n)$ and $\lambda = \Omega \left( \frac{1}{n} \right)$ where $n = |S|$. Moreover representation length for vertices of $V$ is polynomial with respect to representation length for points of $D$.

Let $k$ be a positive integer. Obviously centers of at most $k$ disks of radius $r_0$ whose union covers $D$ give centers of radius $r \geq r_0$ disks whose union is intersected with each segment from $E$. Furthermore using only polynomial time and space resulting disk centers could be chosen to lie in $D_r(G)$. Conversely let $H$ be a disk of radius $r_0 + \delta$ that intersects a subset $I_{D'} = \{I_u : u \in D'\}$ of segments where $D' \subseteq D$. When $|D'| = 1$ it is easy to transform $H$ to one which contains segment $I_{D'}$. As points of $D$ have integer coordinates squared (euclidean) distance between each pair of points of the subset $D'$ does not exceed $(2r_0 + 4\delta)^2 = 4r_0^2 + 16r_0\delta + 16\delta^2$. Therefore points from $D'$ are located within the distance $2r_0$ from each other. Let us use Helly theorem. Minimum radius $R$ of the disk containing any triple $u_1, u_2$ and $u_3$ from $D'$ does not exceed $r_0 + 2\delta$ where say $u_1$ and $u_2$ are on the disk boundary. Let us show that the case $R > r_0$ is void. If we slightly shift center of that disk (along the midperpendicular to $[u_1, u_2]$) to have $u_1$ and $u_2$ at the distance $r_0$ from the shifted center $O'$ then the distance from $u_3$ to the circle centered at $O'$ of radius $r_0$ does not exceed $2\sqrt{r_0\delta + \delta^2} + 2\delta < \frac{1}{100\sqrt{2} r_0}$. By lemma $2$ we have $R \leq r_0$. Thus $D'$ is contained in some disk of radius $r_0$. As minimum radius disk covering given set of points can be found in polynomial time and space we can convert any set of at most $k$ disks of radius $r$ whose union is intersected with each segment from $E$ to some set of at most $k$ disks of radius $r_0$ whose union covers $D$.

**Remark 2** Strong NP-hardness of discrete IPGD is given where $r/d_{\min} = \Theta(1)$.

Proof of theorem $4$ is obviously extendable for classes of Gabriel and relative neighbourhood graphs. It is easy to observe that segments $I_u, u \in D$, form the subset of edges of any minimum euclidean spanning tree. In contrary suppose a spanning tree $T$ does not contain $I_u$ but contains edges incident to both endpoints $u_0$ and $v_0$ of $I_u$ for some $u \in D$. Let us remove edge $e = [u_0, w_0]$ from $T$ and add edge $I_u$ to it where $w_0$ is the parent of $u_0$ in $T$. Obviously the resulting tree has smaller weight, a contradiction.
Corollary 2 IPGD is strongly NP-hard for \( \lambda = \Omega \left( \frac{1}{n} \right) \) and \( \mu = O(n) \) over classes of Gabriel, relative neighbourhood, half-\( \theta_6 \) graphs and of minimum euclidean spanning trees.

Let us point out that the absence of fully polynomial time approximation scheme for IPGD over aforementioned proximity graph classes implied by our results is likely extendable for constant \( r \) to the absence of efficient polynomial time approximation scheme (EPTAS) of the complexity \( O \left( f(\varepsilon)|V|^c \right) \) for any function \( f(\cdot) \) and constant \( c > 0 \) due to our proof technique from theorem \[4\] and strong results of \[20\] in fact claiming \( W[1] \)-hardness of CDC problem in its parameterized setting. Whereas for IPGD problem with \( r = o \left( \frac{1}{\text{diam}V} \right) \) (i.e. when it is equivalent to VC problem) such EPTAS \[5\] exists.

4 Polynomial solvability of IPGD problem over connected plane graphs for large \( r \)

Let us say a few words about computational complexity of IPGD problem for large \( r \). First it is obvious that IPGD is solvable over connected planar graphs for \( \lambda \geq 1 \) in time \( O(|E|) \) \[7\]. Let us consider the case of IPGD over this graph class where \( 0 < \lambda < 1 \). In view of the fact that each disk of radius \( r \) contains an axis-parallel rectangle whose side is \( r \sqrt{2} \) roughly at most \( k = k(\lambda) = \left[ \frac{\sqrt{2}R(E)}{r} \right]^2 = \left[ \frac{\sqrt{2}}{\lambda} \right]^2 \) disks are needed to intersect all segments from \( E \). Therefore the brute-force search algorithm could be applied that just sequentially tries each subset of \( D_r(G) \) of cardinality of at most \( k \) which amounts roughly to \( O \left( k^2|E|^{2k+1} \right) \) time complexity. Therefore having \( \lambda \geq \theta_0 \) for some absolute constant \( 0 < \theta_0 < 1 \) we arrive at the polynomial time and space algorithm whose complexity depends exponentially on \( \theta_0 \).

5 Conclusion

Computational complexity is studied for the problem of intersecting a structured set of straight line segments with the smallest number of disks of radii \( r > 0 \) where a structural information about segments is given in the form of set of edges of connected proximity graph. It is shown that the problem remains strongly NP-hard over classes of \( n \)-vertex Delaunay triangulations, some of their connected subgraphs as well as of half-\( \theta_6 \) graphs having \( O(n) \) edge length variance at different scales of \( r \) with respect to minimum radius \( R(E) \) of the disk that stabs all segments. Moreover for \( r = cR(E) \) and some absolute constant \( c \) the problem becomes polynomially solvable by brute-force search. Our results bring need for designing efficient approximation algorithms for this problem as well as of studying its complexity status over proximity graph classes for which the variance of graph edge lengths is small.

A Proof of Lemma 1

Proof. Set \( r := r_0 = \frac{1}{4 \text{diam}V} \). First note that getting feasible solution of discrete IPGD from one (denote it by \( C \)) for VC could be done in polynomial time as follows: for each point \( c \in C \) we find the set \( E_c \subseteq E \) of all edges of \( G \) containing \( c \) and running throughout the set \( D_r(G) \) we find its element that belongs to each of the corresponding \( r_0 \)-neighbourhoods of edges from \( E_c \).

Now we prove the second part of equivalence definition. Suppose the system \( \mathcal{N}^0 \) of \( r_0 \)-neighbourhoods for segments of some subsystem \( S' \subseteq S \) has nonempty intersection. We apply
Helly theorem. Euclidean distance between two nonintersecting line segments coincides with one between the pair of points of these two segments one of which is exactly their endpoint. Using lemma 1 from [10] we have that in each subsystem of $\mathcal{N}'$ of cardinality 3 every pair of corresponding line segments from $\mathcal{S}'$ has nonempty intersection. Denote arbitrary triple of segments from $\mathcal{S}'$ as $S_1, S_2$ and $S_3$. It is sufficient to show that the case $S_1 \cap S_2 \cap S_3 = \emptyset$ is impossible. Let us use the following fact from basic geometry: in every triangle the radius of the inscribed circle is larger than the minimum height divided by 3. Therefore the radius of the inscribed circle for the triangle bounded by segments $\{S_i\}_{i=1,2,3}$ exceeds $\frac{1}{3 \text{diam}\mathcal{V}}$. From the other hand it can not exceed $r_0$. Finally we have $S_1 \cap S_2 \cap S_3 \neq \emptyset$ which means that the system $\mathcal{S}'$ has nonempty intersection. Obviously an intersection point for segments from $\mathcal{S}'$ can be found in polynomial (in $|E|$) time.

B Proof of Lemma 2

Proof. Let $u = (x, y), v = (x_1, y_1)$ and $w = (x_2, y_2)$ be distinct points of $X$. Consider an arbitrary circle of radius $r$ among a pair of ones through $v$ and $w$ (denote its center by $O$). We are to bound the distance $\pi(u, v, w)$ from that circle to the point $u \notin C(v, w)$ from below.

Let us denote $\Delta = |v - w|_2$, $\lambda = \sqrt{r^2 - \frac{\Delta^2}{4}}$, $a = (u - v, u - w)$ and $b = (u - v, (v - w)^\perp)$ where $(v - w)^\perp = \pm(y_1 - y_2, -x_1 + x_2)$. The distance $\pi$ could be written in the form:

$$\pi = \pi(u, v, w) = \left| \frac{a + \frac{2\lambda b}{\Delta}}{\sqrt{a + \frac{2\lambda b}{\Delta} + r^2 + r}} \right|.$$ 

Consider the case where $u$ lies inside the disk of radius $2r$ centered at $O$. Indeed otherwise $\pi \geq r \geq \frac{1}{r}$. Let us bound denominator of fraction $\pi$ taking into account that $\Delta \leq 2r$, $|u - v|_2 \leq 3r$ and $b/\Delta \leq 3r$:

$$\sqrt{a + \frac{2\lambda b}{\Delta} + r^2 + r} \leq 5r.$$ 

As points of $X$ have integer coordinates then $a$ and $b$ are integer and $\pi > 0$. When $\Delta^2 \leq 4r^2 - 1$ let us show that

$$\left| a + \frac{2\lambda b}{\Delta} \right| \geq \frac{1}{20r^3}$$

in each of two cases where $\frac{2\lambda b}{\Delta}$ is either integer or has nonzero fractional part. For integer $\frac{2\lambda b}{\Delta}$ we arrive at the bound $\pi \geq \frac{1}{5r}$.

Assume that $\frac{2\lambda b}{\Delta}$ has nonzero fractional part. Let $q = \{\frac{2\lambda}{\Delta}\}$ and $k = \lceil\frac{2\lambda}{\Delta}\rceil$ where $\{\cdot\}$ and $\lceil\cdot\rceil$ denote fractional and integer parts of real number respectively. Consider the case where $\frac{4k^2}{\Delta^2}$ is not integer. We have $2kq + q^2 \geq \{2kq + q^2\} \geq \left\{ \frac{4k^2}{\Delta^2} \right\}$ whence

$$q \geq \sqrt{k^2 + \left\{ \frac{4k^2}{\Delta^2} \right\}} - k \geq \left\{ \frac{4k^2}{\Delta^2} \right\} - \frac{1}{5r^3} \geq \frac{1}{20r^3}$$

and $\pi \geq \frac{1}{100r}$. Now suppose that $\frac{4k^2}{\Delta^2}$ is integer. Denote by $m$ the smallest integer with $m^2 < \frac{4k^2}{\Delta^2} < (m + 1)^2$. As $q > 0$ we have $\{\frac{2\lambda}{\Delta}\} \geq \{\sqrt{m^2 + 1}\}$. Due to concavity of square root we can write:

$$\left\{ \sqrt{m^2 + 1} \right\} \geq \left\{ m + \frac{1}{2m + 1} \right\} = \frac{1}{2m + 1} \geq \frac{1}{\frac{4\lambda}{\Delta} + 1} \geq \frac{1}{5r}.$$
which gives $\pi \geq \frac{1}{25r^2}$.

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