Mirror Symmetries for Brane Configurations  
and Branes at Singularities

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Abstract

We study local mirror symmetry on non-compact Calabi-Yau manifolds (conifold type of singularities) in the presence of D3 brane probes. Using an intermediate brane setup of NS 5-branes ‘probed’ by D4 resp. D5 branes, we can explicitly T-dualize three isometry directions to relate a non-compact Calabi-Yau manifold to its local mirror. The intermediate brane setup is the one that is best suited to read off the gauge theory on the probe. Both intervals and boxes of NS 5-branes appear as brane setups. Going from one to the other is equivalent to performing a conifold transition in the dual geometry. One result of our investigation is that the brane box rules as they have been discussed so far should be generalized. Our new rules do not need diagonal fields localized at the intersection. The old rules reappear on baryonic branches of the theory.
1 Introduction

Mirror symmetry is a symmetry which relates topologically distinct pairs of (complex) \(d\)-dimensional Calabi-Yau manifolds to each other \([1, 2]\). In the past, mirror symmetry was mainly discussed for compact Calabi-Yau manifolds. If \(\mathcal{M}\) and \(\mathcal{W}\) constitute a Calabi-Yau mirror pair, it follows that

\[
h^{1,1}(\mathcal{M}) = h^{1,d-1}(\mathcal{W}), \quad h^{1,d-1}(\mathcal{M}) = h^{1,1}(\mathcal{W}),
\]

(1)

where \(h^{p,q}\) denotes the dimension of the cohomology of \(p\)-holomorphic and \(q\)-anti-holomorphic forms. This observation leads to very powerful predictions, namely it identifies the classical moduli space of complex structures of \(\mathcal{M}\) (\(\mathcal{W}\)) with the quantum moduli space of the complexified Kähler classes of \(\mathcal{W}\) (\(\mathcal{M}\)), which includes quantum corrections from holomorphic curves.

The Calabi-Yau mirror symmetry was originally discovered in the context of perturbative string compactifications on Calabi-Yau three-folds, where the mirror operation just corresponds to a sign flip of the charges of the \(U(1)\) currents in the underlying superconformal \(n = 2\) algebra \([3, 4]\). It implies that the perturbative heterotic string is invariant under the mirror symmetry, whereas the perturbative type IIA and IIB superstrings on Calabi-Yau three-folds are mapped onto each other by the mirror operation. More recently, assuming that mirror symmetry extends to a symmetry of the full non-perturbative string theory, authors of \([5]\) provided a geometric interpretation to the mirror map. They showed that \(\mathcal{M}\) has a quantum mirror provided it is a \(T^d\) fibration over a \(d\)-dimensional base \(B\), where the fibers \(T^d\) are Lagrangian submanifolds relative to the Kähler form. Mirror symmetry is than the \(T\)-duality transformation with respect to the volume of \(T^d\), i.e. it inverts all radii of \(T^d\).

In this paper we discuss T-duality and mirror symmetry for type II string theory on a particular class of Calabi-Yau spaces. In our point of view, the geometry will serve as a background, and we will study the gauge theories living on D-branes probing the manifolds.

Consider first a D3 brane probing a Calabi-Yau manifold \(\mathcal{M}\). At a smooth point in \(\mathcal{M}\) the tangent space is \(\mathbb{R}^6\) and the D3 brane will have \(N = 4\) supersymmetry on the world volume. To get something more interesting, we have to consider Calabi-Yau manifolds with singularities. Since we are only interested in the local physics near the singularity, our manifolds will all be non-compact Calabi-Yau spaces which, if one desires to, can be viewed as having a completion to compact Calabi-Yau manifold.
Some of the Calabi Yau manifolds we will study have hyperquotient singularities that can be obtained as orbifolds of the well known conifold singularity $C$, and so are of the form $C/\Gamma$, where $\Gamma$ is a discrete symmetry group a conifold admits. Recently the gauge theory of D3 brane probing a conifold singularity was derived in [6]. The theory of D3 brane on $C/\Gamma$ is then defined as a quotient of the theory on $C$ by $\Gamma$.\textsuperscript{1}

We actually need to be a little bit more precise about the meaning of singularities in string theory. The Calabi-Yau singularities often have topologically distinct resolutions, the conifold singularity being the simplest example. In that case, we could either deform the complex structure of the conifold or its Kähler structure to obtain a smooth manifold. Now, when we discuss the conifold in string theory we have to specify the means of smoothing the singularity. This is because on the Kähler side of the conifold it suffices to turn on the $B$ field flux to obtain theory isomorphic to that on the smooth space. The D3 brane theory constructed in [6] is the theory on the resolution of the conifold. Taking a quotient of this theory by $\Gamma$ the resulting theories should be viewed as coming from the Kähler side of the singularity.

Locally, complex and Kähler structure moduli spaces decouple. Thus, if we are interested in a neighborhood in which $\mathcal{M}$ develops a singularity through degeneration of its Kähler structure, we can take the complex structure to be nice and smooth, and therefore trivial. Canonically mirror symmetry acts by exchange of complex and Kähler structure. If $(\mathcal{M}, \mathcal{W})$ form a mirror pair, it is the complex structure of $\mathcal{W}$ that will be interesting.

The mirror geometries of the singularities will be constructed precisely in the spirit of [5], namely by performing T-duality transformations around three isometric directions of the geometric singularity. The singularities we are interested in have a toric description so we can equivalently [10] apply the local mirror map in the toric language [11]. The first point of view will be more useful for us, since it will allow us to follow the action of mirror

\textsuperscript{1}The discussion of D3 branes on 6-dimensional singularities is very closely related to the by now famous AdS/CFT correspondence. Superconformal $N = 2$ or $N = 1$ (and also $N = 0$) gauge theories can be constructed as the duals of supergravity on $AdS_5 \times X^5$, where $X^5$ is a certain five-dimensional (Einstein) manifold. First, for the case of D3 branes on six-dimensional orbifold singularities $\mathcal{O} = \mathbb{R}^6/\Gamma$, where $\Gamma$ is a discrete group, $X^5$ is given by $S^5/\Gamma$, as discussed in [7, 8]. The corresponding orbifold gauge theory can be calculated using string perturbation theory. The conifold singularities were later obtained in [6], where for the simplest conifold the corresponding Einstein space $X^5$ is the homogeneous space $T^{1,1} = (SU(2) \times SU(2))/U(1)$. Further conifold type of singularities were recently discussed in [9].
symmetry on D-branes. Since the mirror symmetry acts in the space transverse to the D3 branes, the IIB gauge theory of D3 branes probing the space $\mathcal{M}$ will be mirror to an identical gauge theory but now due to IIA D6 branes wrapping a 3-cycles in $\mathcal{W}$. “Mirror” of the D3 brane at a smooth point will be a D6 brane wrapping $T^3$. What will be the mirror of a D3 brane at the singular point? The mirror D6 brane will wrap a three cycle which is still a special Lagrangian, but is now a degenerate three cycle which is homologous to the fiber at a generic point.

As it is known already for some time [12, 13], the geometric orbifold or conifold singularities are T-dual to a certain number of Neveu-Schwarz (NS) 5-branes. This T-duality can be used [14, 15, 16, 9, 17, 18] to transform the D3 branes probing a singularity into a pure brane configuration of intersecting NS branes and D branes of the Hanany-Witten type [19, 20]. It is this fact that we will systematically explore here.

In our case manifold $\mathcal{M}$ has three isometries on which T-duality $T_{\text{mirror}}$ can be performed to obtain $\mathcal{W}$. We can write the mirror transform $T_{\text{mirror}}$ as a composition of two dualities $T_U$ and $T_V$, such that starting with a singularity $\mathcal{M}$ and acting with $T_{\text{mirror}}$ on that space, we will first dualize to a certain brane configuration and subsequently further to the mirror geometry $\mathcal{W}$. From the brane point of view (taking NS5 branes to have $x^{0,1,2,3}$ as common directions and extend along $x^{4,5}$ and $x^{8,9}$) so we will call $T_U = T_6$ and $T_V = T_{48}$. From these two differently oriented NS branes we can build boxes or intervals, respectively, each giving rise to a pair of $(T_U, T_V)$ dual mirror geometries. As is well known, one can suspend D4 branes on the intervals, and D5 branes on the boxes to obtain four dimensional gauge theories on the D brane world volumes. T-duality will map these to either probe D3 branes or the D6 branes wrapping three-cycles of the mirror geometry. The resulting field theory should be the same in all the T-dual realizations.

Using these relations we can derive the rules that govern which gauge theory is encoded in a given brane setup.

The paper is organized as follows. In the next section we will introduce the relevant geometries, namely the conifold singularities and the orbifold singularities and their generalizations. In the third section we will discuss the gauge theories that appear on the D3 brane probes. In section four we will introduce the T-dual brane setups – T-duality by $T_U$ or $T_V$ respectively – and will discuss the T-duality without the probe. Putting together the two T-duality transformations we will see the mirror geometries emerging. In section five we than incorporate the D3 brane probes. We will find that the
brane box is the natural dual of the blowup of the orbifolded conifold and of the deformed generalized conifold. In order to incorporate this result we have to modify the brane box rules of Hanany and Zaffaroni [20]. Their gauge theories reappear in a special corner of moduli space. Our new construction makes some aspects of the box rules more transparent. In section six we will wrap up by considering some related issues. We will show that by the same transformation T_{468} mirror symmetry can be defined for brane setups as well, turning 2-cycles into 3-cycles. We will show how to put both, the box and the interval together in one picture. This way we obtain a domain wall in an N = 1 4d gauge theory that lifts to M-theory via a G_2 3-cycle as in [21].

2 The geometries

2.1 Conifold

The simplest isolated singularity a three dimensional Calabi Yau manifold can develop is the conifold:

\[ C : \quad xy - uv = 0 \]  

(2)

The singularity is located at \( x = y = u = v = 0 \) where the manifold fails to be transverse: \( f = xy - uv = 0, \partial_i f = 0 \) have a common solution there. There are two ways of smoothing the singularity, resulting in topologically distinct spaces.

- The so called small resolution – replacing the singular point by a \( \mathbb{CP}^1 \), thereby changing the Kähler structure. The resulting space has \( h^{1,1} = 1, h^{2,1} = 0 \).

- By deformation of the defining equation, thereby changing the complex structure. After the deformation, \( h^{1,1} = 0, h^{2,1} = 1 \).

Small Resolution

There are many ways in which one can exhibit the small resolution of the conifold. The one particularly well suited for our purposes is as follows. One can solve equation (2) by simply putting

\[ x = A_1B_1, \quad y = A_2B_2, \quad u = A_1B_2, \quad v = A_2B_1, \]  

(3)
where \( A_i, B_j \in \mathbb{C}^4 \). There clearly is a redundancy in this identification, since for any \( \lambda \in \mathbb{C}^* \), taking \( A_i \rightarrow \lambda A_i, B_j \rightarrow \lambda^{-1}B_j \) maps to the same point of the conifold. We can remedy this as follows. We will think about \( \mathbb{C}^* \) as \( \mathbb{R}_+ \times S^1 \), that is we will put \( \lambda = Re^{i\theta} \), with \( R > 0 \). Take a quotient by \( \mathbb{R}_+ \) first, by picking \( R \) to set
\[
|A_1|^2 + |A_2|^2 - |B_1|^2 - |B_2|^2 = 0.
\] (4)

To obtain a space isomorphic to the conifold we started with we must still divide by the \( S^1 = U(1) \).

One can obtain a more physical interpretation of what we have done, which stems from observation that the description of the conifold we have come up with above is precisely that of a Higgs branch of a particular linear sigma model. It corresponds to a theory with four real supercharges, gauge group \( U(1) \) with four matter fields \( A_i, B_j \) with charges +1 and −1, respectively and no superpotential. The D-flatness conditions are then given by equation (4). This is of course not a new construction [22, 6].

Turning on the FI parameter \( r \) will modify the D-flatness conditions to
\[
|A_1|^2 + |A_2|^2 - |B_1|^2 - |B_2|^2 = r.
\] (5)

We have three cases to consider here.

a.) For \( r = 0 \) we have a singular manifold the conifold.

b.) For \( r > 0 \), the origin \( A_i = 0 = B_j \) of the conifold is replaced by a sphere of size \( |A_1|^2 + |A_2|^2 = r \). From the point of view of geometry, turning on the FI parameter [22] is naturally interpreted as blowing up a sphere of size \( r \).
c.) For \( r < 0 \), from the point of view of b) the Kähler class is negative. We do still have a smooth manifold, because now the origin is replaced by \( |B_1|^2 + |B_2|^2 = r \).

The manifolds in b.) and c.) are topologically distinct – they are related by a flop transition (see Fig.1).

**Deformation**

In addition to the smoothings we discussed above, conifold singularity can be smoothed out by keeping the Kähler structure fixed but modifying the defining equation. For this it suffices to change the complex structure to:

\[
xy - uv = \epsilon.
\]

As long as \( \epsilon \neq 0 \), the conifold singularity has been removed. By examining the equation in detail, one can show that the origin was replaced by an \( S^3 \).

### 2.2 More General Singularities

We are now more or less in place to introduce toric geometry, as a tool for treating more complicated singularities.

We will use the language of linear sigma models to put the discussion on a more physical basis [22, 23]. We are constructing a linear sigma model whose moduli space will be a Calabi-Yau manifold \( \mathcal{M} \). First, the number of independent FI parameters, or equivalently the number of \( U(1) \) factors, will equal \( h^{1,1}(\mathcal{M}) \) (unless stated otherwise, by \( \mathcal{M} \) we mean the manifold obtained by smoothing out the singularity). It is this number, and the charges of various matter multiplets that toric geometry must encode.

A toric diagram consists of \( d + n \) vectors \( \{\vec{v}_i\} \) in a lattice \( \mathbb{N} = \mathbb{Z}^d \). Every vector \( \vec{v}_i \) corresponds to a matter multiplet in our sigma model which we will call \( x_i \). To describe a toric variety homeomorphic to other than flat space we need \( n > 0 \). Since \( \mathbb{N} \) is \( d \)-dimensional, there are \( n \) relations between the \( d + n \) vectors which we will write in the form

\[
\sum_{i=1}^{d+n} Q_i^a \vec{v}_i = 0, \quad a = 1, \ldots, n.
\]
It is clear that $Q$’s should be interpreted as the charges of the matter fields under the $n$ $U(1)$’s. As a consequence the D-flatness conditions will read,

$$\sum_{i=1}^{d+n} Q_i^a |x_i|^2 = r_a, \quad a = 1, \ldots, n. \tag{7}$$

$\mathcal{M}$ is a space of solutions to (7), up to the identifications imposed by gauge symmetry. Or, instead of setting D-terms to zero and dividing by the gauge group, we could have taken a quotient by the complexified gauge group $x_i \to \lambda Q_i^a x_i, \quad a = 1, \ldots, n$, where $\lambda \in \mathbb{C}^*$ and express the moduli space as the space of gauge invariant polynomials in $x$’s, modulo any relations between them. This is the language of eq.(2).

There is one slight simplification that occurs when $\mathcal{M}$ is a (non-compact) Calabi-Yau manifold. Namely, $\mathcal{M}$ is a Calabi-Yau if and only if there exists a vector $\vec{h} \in \mathbb{M}$, where $\mathbb{M}$ is the dual lattice of $\mathbb{N}$, such that

$$< \vec{h}, \vec{v}_i > = 1, \quad \forall \vec{v}_i$$

i.e. if and only if all the vectors $\vec{v}_i$ live on a hyperplane a unit distance away from the origin of $\mathbb{N}$. Therefore in all of our examples toric singularities can be described by planar diagrams, only.

Hyperquotient Singularities

As is well known, one can obtain more complicated geometries by taking a quotient of the simpler ones by a properly chosen group action. Dividing $\mathbb{C}^n$ by a discrete symmetry group $\Gamma$ we obtain orbifolds with quotient singularities. Taking a quotient of a hypersurface singularity like $\mathbb{C}$ we obtain what are called hyperquotient singularities. Both can be treated easily in the language of toric geometry. First however, we must find appropriate symmetry group of our manifold. Clearly, any action $x_i \to \lambda_i x_i, |\lambda| = 1$ leaves the manifold invariant. The symmetry group is $U(1)^{n+d}/U(1)^n = \exp(2\pi i \mathbb{Z}^d)$. More precisely, the toric variety $\mathcal{M}$ will contain the torus $T^d$ as a dense open subset. There is a natural action of $U(1)^d = T^d$ on the toric variety given as follows. To any element $\vec{n} \in \mathbb{Z}^d$, we can associate an element of $U(1)^d$ via $x_i \to e^{in_i \theta} x_i$, where $\vec{n} = \sum n_i \vec{v}_i$ defined up to $\sum Q_i^a \vec{v}_i = 0$.

So far, our lattice was integral. Now suppose we refine the lattice by adding a vector in $\mathbb{Q}^d$ for example $\vec{q} = \frac{1}{\pi}(a_1, \ldots, a_d)$. For as long as the lattice was integral the torus action was well defined. Now, it will be so only
if we induce additional identifications on the $x_i$'s, namely writing $\vec{q} = \sum q_i \vec{v}_i$, (mod $\sum Q_i \vec{v}_i$), we should identify
\[ x_i \sim e^{2\pi i q_i/r} x_i. \]

Perhaps a better way to express the action of the quotient, is in terms of gauge invariant monomials. Clearly, any $\mathbb{C}^*$ invariant monomial is of the form
\[ x^m = \prod x_i^{<\vec{v}_i, \vec{m}>}, \]
so the space of $\mathbb{C}^*$ invariant monomials is just the dual lattice $M$. Actually we want a bit less, since a) only the positive powers should appear, so we only want those $\vec{m}$'s that satisfy $<\vec{m}, \vec{v}_i> \geq 0, \forall i$, and b) we only want the independent ones. Then, the identification induced on the monomials is
\[ x^m \sim e^{2\pi i <\vec{q}, \vec{m}>} x^m. \]

In any event, we should now be ready to produce new spaces. We are up to producing orbifolds of the conifold, $\mathcal{C}/\Gamma$. Let us take $\Gamma = \mathbb{Z}_k \times \mathbb{Z}_l$. So, start with our conifold $\mathcal{C}$, defined by four vectors $\vec{v}_{1,2,3,4} \in \mathbb{N}$ as before, but refine the lattice to $\mathbb{N}'$ by adding two vectors, $\vec{e}_k = (\frac{1}{k}, 0, 0)$, and $\vec{e}_l = (0, \frac{1}{l}, 0)$. The resulting toric diagram (cf. Fig.2) “looks” the same as that for the conifold $\mathcal{C}$, except for the fact that it lives in a finer lattice. This, as explained above, results in the following identifications:
\[ \vec{e}_k = \frac{1}{k}(\vec{v}_2 - \vec{v}_1), \]
we find that the quotient acts by
\[ A_1 \sim e^{-\frac{2\pi i}{k}} A_1, \quad B_1 \sim e^{\frac{2\pi i}{k}} B_1, \quad A_2 \sim A_2, \quad B_2 \sim B_2, \]
and similarly for $\vec{e}_l$,
\[ \vec{e}_l = \frac{1}{l}(\vec{v}_3 - \vec{v}_1), \]
the quotient is by
\[ A_1 \sim e^{-\frac{2\pi i}{l}} A_1, \quad B_1 \sim B_1, \quad A_2 \sim A_2, \quad B_2 \sim e^{\frac{2\pi i}{l}} B_2. \]
Equivalently on $xy = uv$, we identify $x \sim x, y \sim y, u \sim e^{-\frac{2\pi i}{k}} u, v \sim e^{\frac{2\pi i}{k}} v$, and $x \sim e^{-\frac{2\pi i}{l}} x, y \sim e^{\frac{2\pi i}{l}} y, u \sim u, v \sim v$. In terms of $\Gamma$ invariant coordinates
The defining equation of the conifold becomes simply $z = w$. Taking into account that not all the invariant monomials are independent, the $\Gamma = \mathbb{Z}_k \times \mathbb{Z}_l$ orbifolded conifold, after obvious renaming of variables becomes:

$$C_{k,l}: \quad xy = z^l, \quad uv = z^k. \quad (8)$$

\textbf{Blowing Up}

Toric geometry has equipped us with a means of blowing up the singularity. First let’s look at the orbifold $C_{k,l}$. There are still only four vectors defining the diagram which were inherited from the conifold. There is a single relation between them, and thus a single Kähler class but this is insufficient to smooth out $C_{k,l}$. However, due to the fact that the lattice is finer, there exist lattice points within the rectangle, these are: all the points $\overline{\nu}_{i,j} = (i, j, 1)$, $0 \leq i \leq k$, $0 \leq j \leq l$. We can add these points to the toric diagram. In the language of linear sigma models, the effect is to add more matter fields, but also more $U(1)$ factors, and thus more FI parameters. Clearly, the resolved manifold will have $h^{1,1}(C_{k,l}) = (k + 1)(l + 1) - 3$, which is the total number of linearly dependent vectors within the diagram. (Points outside the diagram can be added as well. However they will not contribute to the resolution of the singularity, but only modify it by irrelevant pieces.) We will not try to specify precise region in the Kähler structure moduli space where the resolution lives, which would correspond to picking a triangulation of the toric diagram, because we will not need this piece of information. It is clear there will be very many different such regions, and they are all related by flops.
Finally, starting from the orbifolded conifold $C_{k,l}$, with $k$, $l$ sufficiently large, by performing partial resolutions we can obtain essentially any other toric singularity\footnote{These singularities have been introduced in the physics literature for the description of gauge theories in [24].}. The basic fact to note that adding or subtracting one of the boundary points of the diagram changes $h^{1,1} \rightarrow h^{1,1} - 1$. The right interpretation of this is that we are probing the region of the Kähler structure moduli space where the four cycle associated to this point in the toric diagram becomes large enough, that it in fact becomes irrelevant to the local physics – the associated vectors can be dropped altogether.

We will provide some more examples of the spaces we will explicitly use in this paper and introduce some terminology.

Starting from an orbifold of $C$ by $\mathbb{Z}_k$,

$$C_k : \ xy = z^k, \ uv = z, \quad (9)$$

or equivalently $xy = (uv)^k$, which has $h^{1,1}(C_k) = 2k - 1$, by partial resolution we can obtain the generalization of a conifold,

$$G_{kl} : \ xy = u^kv^l \quad (10)$$

with only $k + l - 1$ Kähler structure deformations. Clearly, in our notation $C_k \equiv G_{kk}$.

\[ (0,0) \quad (0,k) \quad (1,l) \quad (0,k) \]

**Fig.3.** (Blowup of) generalized conifold $G_{kl}$.

\[ (0,k) \quad (1,0) \]

**Fig.4.** Toric diagram of the $\mathbb{Z}_k \times \mathbb{Z}_l$-orbifold $O_{kl}$. 

The conventional orbifold $O_{kl} = \mathbb{C}^3/\mathbb{Z}_k \times \mathbb{Z}_l$ can be found in the Kähler structure moduli space of the orbifolded conifold $C_{kl}$, the toric diagram of the orbifold being contained in that of the orbifolded conifold. One way to see this is indeed a toric diagram corresponding to $\mathbb{C}^3/\mathbb{Z}_k \times \mathbb{Z}_l$ is to use the fact that the diagram can be obtained starting from a toric diagram
containing just three vectors \( \vec{v}_1 = (0, 0, 1), \vec{v}_2 = (1, 0, 1), \vec{v}_3 = (0, 1, 1) \) in an integral lattice \( \mathbb{N} \), which gives a toric variety homeomorphic to flat space, and then refine the lattice to \( \mathbb{N}' \), as in Fig.4. The map from the toric variety in \( \mathbb{N} \) to the one living in \( \mathbb{N}' \) is one to one provided one includes discrete identifications on the three matter fields \( A_i, i = 1, 2, 3, \)

\[
A_1 \sim e^{-2\pi i/l} A_1, \quad A_2 \sim e^{2\pi i/l} A_2, \quad A_3 \sim A_3,
\]

and

\[
A_1 \sim e^{-2\pi i/k} A_1, \quad A_2 \sim A_2, \quad A_3 \sim e^{2\pi i/k} A_3.
\]

As before, number of Kähler structure deformations is just the number of independent points in the toric diagram, and this number will clearly depend on whether \((k, l)\) are coprime or not, since the number of points on the diagonal is \(\gcd(k, l) + 1\).

**Deformations**

- The orbifolded conifold \( C_{kl} \) : \( xy = z^k, \ uv = z' \) can be deformed to a smooth space by modifying the defining equation as:

\[
xy = \prod_{i=1}^{k}(z - w_i) \quad uv = \prod_{j=1}^{l}(z - w'_j).
\] (11)

One of these parameters can be set to 1 by shifting \( z \), so we are left with \( k + l - 1 \) parameters. This gives \( h^{2,1}(C_{kl}) = k + l - 1 \).

- The generalized conifold \( G_{kl} \) : \( xy = u^k v^l \) can be deformed into

\[
xy = \sum_{i,j=0}^{k,l} m_{ij} u^i v^j.
\] (12)

This time we see \( h^{2,1}(G_{kl}) = (k + 1)(l + 1) - 3 \) complex structure deformations \( m_{ij} \) : by shifting \( u, v \), we can eliminate two of the parameters, and another one by rescaling the defining equation.
Mirror Symmetry

Toric geometry is well adopted to discussing mirror symmetry as well. We will review it here very briefly, only. Mirror symmetry exchanges the Kähler structure parameters with the complex structure parameters. Now, to understand the mirror map, we first need to know something about the complex structure moduli space. How is the complex structure encoded in the equation of the manifold? The answer is as follows: the coefficients of the monomials appearing in the defining equation are coordinates on the complex structure moduli space. What they parameterize are the “sizes” of various three-cycles (i.e. the periods of the holomorphic three form) and the metric on the moduli space. The periods, (and therefore the metric – the moduli space has special geometry structure) can be derived directly as a solution to a system of differential equations. The main point is that the differential equations depend solely on the relationships between the monomials in the defining equation and nothing else.

Given the toric manifold, relations between the vectors in the toric diagram of \( M \) (we are assuming a completely smooth space here, with all the possible blowups performed),

\[
\sum_{i=1}^{n+d} Q_i^a \vec{v}_i, \quad a = 1, \ldots n
\]

map to relationships between the monomials in the defining equation of \( W \), the mirror of \( M \), given by

\[
W: \sum_i a_i m_i = 0, \quad (13)
\]

where \( a_i \) are coefficients, and \( m_i \) monomials, the monomials must satisfy

\[
\prod_{i=1}^{n+d} m_i^{Q_a} = 1, \quad a = 1, \ldots n. \quad (14)
\]

Any solution to these equations (and in general there are more than one) will represent the same complex structure (by decoupling of complex and Kähler moduli spaces). Note that there are \( n + d \) monomials with \( n \) relations between them. Together with the hypersurface equation, this gives a \( d - 1 \) dimensional manifold, but the homogeneity of the monomial relations will
allow us to remove one more. The mirror will naively have \(d - 2\) dimensions. This is not a problem, rather an artifact of the fact that local mirror symmetry is encoding all the information about the complex structure of the mirror, and nothing but. One can fix the “dimensionality” of the local mirror by adding quadratic pieces, as this will not influence the complex structure moduli space.

Let us briefly show how this works for the two examples we are going to be concerned with in this work \(^*\), \(G_{kl}\) and \(C_{kl}\). Consider first the blowup of \(C_{kl}\). We want to interpret the same diagram Fig.2 as defining the complex structure of the mirror. Assigning the vector \((i, j, 1)\) to a monomial \(u^i v^j\) clearly eq.(14) is satisfied for all the relations. The defining equation for the mirror of \(C_{kl}\) hence becomes according to eq.(13):

\[
\sum_{i,j=0}^{k,l} m_{ij} u^i v^j = 0
\]

or after adding the irrelevant quadratic piece \(xy\)

\[
xy = \sum m_{ij} u^i v^j
\]

which is nothing but the deformation of \(G_{kl}\).

Having established that the deformation of \(G_{kl}\) is mirror to the blowup of \(C_{kl}\) we can find another dual pair by following our geometries through a conifold transition. We should find that the blowup of \(G_{kl}\) is the mirror of the deformation of \(C_{kl}\). Let us see how this works. As above we read of the mirror to be

\[
\prod_{i=1}^{k} (z - w_i) + t \prod_{j=1}^{l} (z - w'_j) = 0.
\]

Because \(t\) appears only linearly this encodes the same complex structure as

\[
\prod_{i=1}^{k} (z - w_i) = uv, \quad \prod_{j=1}^{l} (z - w'_j) = xy
\]

which is indeed the deformation of \(C_{kl}\) as presented in eq.(11).

\(^*\)These examples and many more along these lines have been recently analyzed in great detail in [25].
3 The gauge theories

Having introduced the geometric background spaces, we will now discuss the corresponding gauge theories if one adds $M$ D3 branes with world volume transverse to the non-compact manifolds. The corresponding gauge group for the orbifolded conifold $\mathcal{C}_{kl}$, eq.(8), is given by the following $N = 1$ supersymmetric gauge theory:

$$SU(M)^{kl} \times SU(M)^{kl}$$

(15)

with matter fields $(A_1)_{i+1,j+1,I,J}$, $(A_2)_{i,j+1,I,J}$, $(B_1)_{i,j+1,I,J+1}$, $(B_2)_{i,j+1,I,J+1}$. We label the gauge groups with $i, I = 1 \ldots k$ and $j, J = 1 \ldots l$. All the matter fields are bifundamental under the gauge groups indicated by the indices. The just $\mathbb{Z}_k$ orbifolded conifold arises as the special case $l = 1$. In addition there will be a quartic superpotential

$$W = \sum_{i,j} (A_1)_{i+1,j+1,I,J}(B_1)_{i,j+1,I,J+1}(A_2)_{i,j+1,I,J+1}(B_2)_{i,j+1,I,J+1}$$

$$- \sum_{i,j} (A_1)_{i+1,j+1,I,J}(B_1)_{i,j+1,I,J+1}(A_2)_{i+1,j+1,I,J+1}(B_2)_{i+1,j+1,I,J+1}.$$  

(16)

The other singularity, the generalized conifold eq.(10) corresponds to

$$SU(M)^{k+l}$$

with bifundamental matter according to Uranga’s rules [9] and quartic superpotentials.

Finally consider $M$ D3 branes on a transversal orbifold singularity $\mathcal{O}_{kl}$. They give rise to an

$$SU(M)^{kl}$$

(17)

gauge theory with 3 types of chiral bifundamental multiplets $H_{i,j+1,j}$, $V_{i,j+1,j}$ and $D_{i+1,j+1,j}$ in each gauge group and a cubic superpotential

$$W = \sum_{i,j} H_{i,j+1,j}V_{i+1,j+1,j+1}D_{i+1,j+1,j} -$$

$$\sum_{i,j} V_{i,j+1,j}H_{i+1,j+1,j+1}D_{i+1,j+1,j}.$$  

(18)
This way the orbifold gauge theories will have $3M$ matter fields per gauge group and cubic superpotentials, leaving us with a finite theory. The conifold gauge theories have $2M$ matter fields per gauge group and quartic superpotentials. These theories are non-finite but flow to a fixed line parameterized by a marginal operator in the IR.

4 The T-dual brane setups and mirror symmetry

In this section we would like to discuss the brane configurations which are T-dual to the singularities introduced in section 2. Specifically, we are interested in two different T-duality transformations: the first one was recently discussed by Uranga [9] and by Dasgupta and Mukhi [17]. The dual brane picture consists of NS and rotated NS' 5-branes we will henceforth refer to it as $T_U$. The D3 branes which we want to study in the next section become D4 branes after the $T_6$ duality transformation which live on the compact interval in $x^6$.

Second we perform a T-duality along the compact directions $x^4$ and $x^8$, $T_{48} = T_4T_8$. This maps the singularities again to NS and NS' branes, where now the D3 probes become D5 branes which fill the compact brane box in the $x^4 - x^8$ spatial directions. This T-duality was first introduced in [13] and for a special point in moduli space used by [15] to study D3 branes on orbifold singularities. We will henceforth refer to it as $T_V$.

These T-dualities are very useful in the sense that they allow us to read off the gauge groups on the D3 brane world volume according to some very intuitive graphic rules encoded in the brane configuration. While for the orbifold a perturbative string calculation is also available to get the gauge group, for the more general singularities discussed here, one would have to rely on a technique as in [6].

Combining the two, that is doing $T_{468}$ we actually performed a local mirror symmetry transformation. We will see explicitly, that $T_{\text{mirror}}$ takes a geometry $W$ into its mirror geometry $\mathcal{M}$. The gauge theory of a D3 brane probing $W$ has to be identical to that on a D6 brane wrapping a 3-cycle in $\mathcal{M}$.

This should correspond to the mirror transformations for Calabi-Yau spaces, which are the compact counterparts of our non-compact singular-
ities. These compact CY’s are assumed $T^3$-fibrations (with $T^3$ a special Lagrangian submanifold of the CY) and the mirror transformations acts as the inversion of the volume of the $T^3$. Obviously, this $T^3$ corresponds to our three directions $x^4, x^6$ and $x^8$, on which the mirror symmetry acts.

4.1 The brane setup

Before we embark on the discussion let us briefly recall the basic brane setup. There are two configurations we are going to consider, for one the standard HW [19, 26] type of brane setup, where D4 branes are stretched in between NS and NS’ branes, former living along 012345 and latter along 012389, the rotation being necessary in order to break SUSY down from 8 to 4 supercharges. In order to have a supersymmetric theory from D4 branes on the interval all the NS and NS’ branes have to be at the same position in the 7 direction. Separations along the 7 direction would be interpreted as FI terms or baryonic branches in the gauge theory and effectively leads to a breaking of the gauge group we want to see. Similarly we should require all the NS branes to have the same position in 89 and all the NS’ branes to have the same position in 45 space. They are separated along the 6 direction building the intervals, along which the D4 branes (living in 01236) stretch.

The second kind of brane setup we are going to consider are the so called brane boxes [20], which are a straight forward generalization of the interval theories. The brane box is a rectangle bounded by NS and NS’ branes with a D5 brane suspended on it. This can be achieved by the same NS and NS’ branes as above but now all branes have to be located at the same 67 position, closing the intervals. We can open up the boxes by separating the NS and NS’ branes along their 48 directions (unfortunately this way we differ from the notation in [20], where the boxes were taken to live in the 46 space. This is necessary, since it is crucial for us, that box and interval can be realized by the same set of NS and NS’ branes). We still want to keep the 5 and 9 positions equal in order to preserve supersymmetry of the suspended probes. Deformations along these directions are again FI terms in the gauge theory, which are reinterpreted as baryonic branches after freezing out the diagonal $U(1)$s.
4.2 Deformations and blowups

As mentioned above, it is important to distinguish whether we want to study
the deformation or the blowup of the singularity under investigation. The
corresponding parameters should have an interpretation in the brane picture
as well. If the dual is ‘pure brane’, i.e. consists only of branes in flat space,
this interpretation will be solely in terms of NS brane positions and, as will
be established later, on brane shapes. Otherwise some of the parameters
encode blowups of the non-trivial background geometry. Even though latter
description of the probe may still be useful, e.g. in order to read off gauge
group and matter content, we would like to focus in the rest of our discussion
on the case, where the dual is ‘pure brane’.

Let us forget for a moment about the D brane probes altogether. That
is, we want to study the map of the singular geometry into a configuration
of NS branes, as pioneered in [13]. Actually it turns out to be easier to start
with the NS brane configurations, where it is clear what we mean by the 4, 6
and 8 direction. Performing $T_{48}$ and $T_6$ respectively we will find two different
geometries, which have to be the local mirrors of each other. By construction
these are precisely the geometries that have a pure brane dual (we started out
with a pure brane setup!). We will find the following relations, as indicated
in Fig.8 in the summary at the end of this paper:

- The blowup of the generalized conifold is $T_U$ dual to NS branes sepa-
  rated along 67 (the interval). These are in turn $T_V$ dual to the mirror,
  the deformation of the orbifolded conifold.

- Similarly the blowup of the orbifolded conifold will $T_V$ dualize into a box
  and then $T_U$ dualize in the mirror, the deformation of the generalized
  conifold.

Indeed these two transformations are related by a conifold transition, that
is bringing together the NS branes on the interval and then separating them
along 4589 instead corresponds to blowing down the 2-cycles and opening
up the 3-cycles of the deformed conifold (and vice versa for the orbifolded
conifold).
4.3 The brane box, blowup of the orbifolded and deformation of the generalized conifold

Let \( m_i = (x^8, x^9) \), \( m'_j = (x^4, x^5) \) positions of the \( k \) NS and \( l \) NS’ branes respectively in \( x^{4,5,8,9} \), and \( w_i = (x^6, x^7) \), \( w'_j = (x^6, x^7) \) the positions in the other two directions.

Let us start with a “brane box”, that is we set all the \( w_i \) and \( w'_j \) to zero. T-dualizing the brane box along \( x^4, x^8 \) we obtain a manifold we call \( \mathcal{M} \) and T-dualizing along \( x^6 \) we obtain \( \mathcal{W} \). The resulting geometries are related by \( T_{468} = T_{\text{mirror}} \).

- \( T_V = T_{48} \): The T-dual space \( \mathcal{M} \) is a \( \mathbb{Z}_k \times \mathbb{Z}_l \) orbifolded conifold

\[
\mathcal{C}_{k,l} : \quad xy = z^l, \quad uv = z^k
\]

as in (8), where \( k, l \) are numbers of NS and NS’ branes. This is a double \( \mathbb{C}^* \) fibration over the \( z \) plane, that is the space has 2 \( U(1) \) isometries used in T duality. The \( x^4, x^8 \) separations of the branes must map into B-fluxes through 2-cycles of the T-dual space. We must therefore identify \( m_i, m'_j \) as deformations of the Kähler structure. Deformations of the Kähler structure cannot change the complex structure, so the \( m_i \) and \( m'_j \) will not be visible in the defining equations. Having identified \( m_i, m'_j \) as the Kähler structure parameters, \( w_i \) and \( w'_j \) are identified as complex structure parameters. But they are frozen, since turning them on would destroy the box structure.

For definiteness take IIB theory on \( \mathcal{C}_{kl} \). \( T_V \) duality takes us back to type IIB with NS branes. In this case Kähler structure parameters, that is the 2-sphere sizes, sit in hypermultiplets. The other 3 scalars in this multiplet are the NS-NS B-flux the RR B-flux and the RR 4-form-flux through the sphere. Latter is a 2-form in 4d, which can be dualized into a scalar. The 2-sphere size and the NS-NS B-flux are the complexified Kähler parameter which map to \( m_i \) and \( m'_j \) under \( T_V \). In the brane box the two other scalars come from Wilson lines of the NS-world volume gauge fields in 45 and 89 which pair up in hypermultiplets with \( m_i \) and \( m'_j \) respectively.

Note however that we have a puzzle. The orbifolded conifold \( \mathcal{C}_{kl} \) has, as we have found from the toric description,

\[
(k + 1)(l + 1) - 3 = kl + k + l - 2
\]
Kähler structure parameters $m_{ij}$ which can be turned on to smooth out $C_{kl}$. Only $k + l - 2$ have been realized in terms of the (relative) brane positions $m_i$ and $m'_j$.

So where are the $kl$ hypermultiplets in the brane box skeleton? They sit at the $kl$ intersections! Strings stretching from NS to NS’ give rise to precisely these hypermultiplets $^\ast 4$.

Turning on vevs for the two scalars corresponding to 2-sphere sizes and NS-NS fluxes resolves the intersection of the NS and NS’ into a smooth object, a little ‘diamond’. For non-zero B-fields this diamond will open up in the 48 plane, for 2-sphere sizes in the 59 plane. This interpretation will become more suggestive after discussing $T_U$ on this configuration and once we start discussing the D3 brane probes.

In the geometry the 2-spheres give rise to strings from wrapping D3 branes around them. How do we see them in the NS5 box skeleton? The D3 branes on the $k + l - 2$ spheres from the curves of singularities correspond to (fractional) D3 branes living in the boxes (or better in whole stripes). The additional $kl$ strings must now correspond to D3 branes in the diamonds. We will indeed see that the diamonds allow for such a configuration.

Of course the same story can be repeated in type IIA. Here the diamonds will correspond to matter on the intersection of type IIA NS5 branes, this time sitting in a vector multiplet. Again the 2 scalars correspond to the $kl$ sizes and B-fluxes of the corresponding 2-spheres. Instead of the two additional scalars in the hypermultiplet we this time see a vector from the RR 3-form on the sphere. In the brane language the Wilson lines of the NS5 gauge field have to be substituted by Wilson lines of the (2,0) 2-form field, again giving rise to vectors.

$\bullet \; T_U = T_6$, T-duality to $W$. What happens now is as follows. Since we did a $T_6$ duality, $x^6$ separations will become the B-fields. Thus, now the $w_i$ (which had to be put to zero since we are discussing a box) parameters are Kähler structure deformations, while the non-zero $m_{ij}$ now should show up as complex structure deformations.

$^\ast 4$ They are Strominger’s D3 brane on the vanishing 3-sphere in the geometry (remember that we only consider blowups, so the 3-spheres are fixed at zero size).
The dual geometry should be a single $\mathbb{C}^*$ fibration. This will be described by an equation whose parameters, the complex structure deformations, must be $m_{ij}$. Let us first study the situation where the vevs of the hypers living at the intersections are zero. In this case the $\mathbb{C}^*$ fibration must degenerate over the NS and NS’ positions $m_i, m'_j$, but in an independent way, since the branes are orthogonal – it must contain two curves of singularities $A_{m-1}$, and $A_{n-1}$ corresponding to NS and NS’ branes. There is one such equation for generic values of $m_i$'s

$$W : \quad uv = \prod_{i=1}^{k} (z - m_i) \prod_{j=1}^{l} (w - m'_j)$$

The curve contains $kl$ conifold singularities located at $z = m_i$ and $w = m'_j$ corresponding to the fact that all the hypermultiplets at the intersections where turned off.

Let us jump ahead and let us realize $W$ directly as the mirror of $\mathcal{M}$. Performing the local mirror map we obtain:

$$W : \quad uv = \sum_{i=0}^{k} \sum_{j=0}^{l} m_{ij} z^i w^j.$$ 

By now the T-dual interpretation of this more general space should be clear. It describes a single NS brane wrapping a curve

$$\Sigma : \quad 0 = \sum_{i=0}^{k} \sum_{j=0}^{l} m_{ij} z^i w^j.$$ 

The smoothing out of the intersections corresponds to the diamonds. For example one intersecting NS and NS’ brane is described by $zw = 0$. Turning on the hypermultiplet corresponds to smoothing this out to $zw = m_{00}$, as e.g discussed in [27] for the related case of intersecting D7 branes. Indeed the resulting smooth curve has a non vanishing circle of radius $(m_{00})^{1/2}$ as can be seen by writing it as $x^2 + y^2 = m_{00}$ and restrict oneself to the real section thereof, for example$^*$. This is precisely what we need: we can suspend a D3 brane as a soap bubble on the NS skeleton, its boundary being given by the circle. The tension

$^*$We are very grateful to M. Bershadsky for very helpful discussions on this point.
of the resulting string is given by the area of the disk and hence is proportional to \( m_{00} \) as expected from the dual geometry \( \mathcal{M} \) (where the size of the 2-sphere was also proportional to \( m \)). In \( \mathcal{W} \) the same string will be given by a D4 brane on the vanishing 3-sphere.

In the same way we can T-dualize any singularity that can be represented as a toric variety into a generalized box of NS branes, with a certain amount of diamonds frozen.

### 4.4 Going to the interval: the conifold transition

We can derive a second T-dual triple of geometry T-dual brane setup and mirror geometry by studying \( T_U \) and \( T_V \) on the interval theory. Note that the interval theory can be directly obtained from the box by brane motions. First we move all the NS and NS' branes on top of each other, setting all \( m_{ij} \) to zero, closing all the boxes and diamonds. This is the conifold point. Now we see that we have the choice to open up the intervals, by turning on the \( w_i \) and \( w'_j \).

We can follow this transition in the geometry as well. Let us see what it does to \( \mathcal{M} \). For one we have shrunk all the 2-spheres to zero size, putting us at the most singular point of the geometry. In addition we have put all the B-fields to zero. So we are really sitting at the real codimension 2 locus of Kähler moduli space, where the closed string CFT description goes bad [22]. This is once more the conifold point. From there we can deform the singularity by turning on 3-spheres to obtain \( \mathcal{M}_T \) and this is precisely what corresponds to turning on the \( w_i \) and \( w'_j \) in the brane picture. This is a (non-abelian) conifold transition [28]. We went from the blowup of the orbifolded conifold \( C_{kl} \) to its deformation. Let us see that \( T_V \) still works. The \( w_i, w'_i \) must now be identified with complex structure deformations. The geometry has to have a \( \mathbb{C}^* \times \mathbb{C}^* \) fibration which degenerates over those points. This leads us to

\[
xy = \prod_{i=1}^{k}(z - w_i)
\]

\[
u \nu = \prod_{j=1}^{l}(z - w'_j)
\]

as the T-dual geometry, indeed.
Last but not least we can study the effect on $\mathcal{W}$. In going to $\mathcal{W}_T$, the mirror of $\mathcal{M}_T$, we this time send all the 3-spheres to zero size and then turn on blowup modes, taking us from the deformed generalized conifold $\mathcal{G}_{kl}$ to its blowup.

5 Probing the mirror geometries

5.1 Introducing the probe: elliptical models

As a next step we want to introduce $M$ D3 brane probes on top of our geometry. This way we break the supersymmetry down to 4 supercharges and get interesting $N = 1$ 4d gauge theory. The deformation parameters of the singularity appear as parameters in the gauge theory, the moduli space of the gauge theory describes the motion of $M$ D3 branes on the singular space. These probe theories have received a lot of attention recently. They give rise to conformal field theories and have a dual $AdS$ description.

In principle we could take any of the four geometries we introduced, compactify type IIB on it and then put a D3 brane probe on top of the singularity. The two situations we are going to study are $M$ D3 branes on the blowup of the generalized conifold $\mathcal{G}_{kl}$ (on $\mathcal{W}_T$) and $M$ D3 branes on the blowup of the orbifolded conifold $\mathcal{C}_{kl}$ (on $\mathcal{M}$).

Performing our two T-dualities $T_U$ and $T_V$ we will find two different realizations of each of the probe theories. The background geometry will transform precisely as we discussed in the last section. This way

- $M$ D3 brane probes of the blowup of the generalized conifold $\mathcal{W}_T$ are $T_U$ dual to D4 branes on an interval defined by $w_i$ and $w'_j$ and $T_{\text{mirror}}$ to D6 branes wrapping 3-cycles in $\mathcal{M}_T$

- $M$ D3 brane probes of the blowup of the orbifolded conifold $\mathcal{M}$ are $T_V$ dual to D5 branes on a box defined by $m_{ij}$ and $T_{\text{mirror}}$ to D6 branes wrapping 3-cycles in $\mathcal{W}$.

We will have to deal with what is usually referred to as elliptical models in the literature [29, 15]. That is the 6 direction of the interval or the 48 direction of the box are actually compact, leaving no room for semi-infinite branes. All D-brane groups will actually be gauged.
5.2 The generalized conifold and the interval

First we would like to consider the gauge theory on the world volume of \( M \) \( D3 \) brane probes on the blowup of a generalized conifold singularity\(^*\). This gauge theory is given e.g. in [9] and can be read off most easily in the dual brane setup we are about to describe. In the last section we showed that this geometry is \( T_U \) dual to NS and NS' branes on a circle, forming intervals with 67 separations given by \( w_i \) and \( w'_j \), all the \( m_{ij} \) being zero. As utilized in [9, 17] this means that the \( M \) \( D3 \) brane probes turn into an elliptical model with \( M \) \( D4 \) branes wrapping the circle. It is straightforward to read off the gauge theory from this according to the standard HW rules. Of course it agrees perfectly with the one obtained from applying a standard orbifold procedure directly on the conifold gauge theory of [6].

There is yet another realization of the same gauge theory. Performing the whole \( T_{\text{mirror}} = T_{468} \) we can turn \( \mathcal{W}_T \), the blowup of the generalized conifold on which we originally put the \( D3 \) brane probes, into \( \mathcal{M}_T \), the deformation of the orbifolded conifold. The theory with which we have to compare is that on the mirror of the \( D3 \) probe, that is a \( D6 \) brane wrapping SUSY 3-cycles in \( \mathcal{M}_T \). But this is precisely the situation discussed in [31]. The parameters \( w_i \) and \( w'_i \) in \( \mathcal{M}_T \), given by (11) determine the loci in the \( z \) plane where the \( \mathbb{C}^* \times \mathbb{C}^* \) fibration degenerates. As found in [31] in order to have a BPS state the \( w_i \) and \( w'_i \) have to align along a line in the \( z \) plane. Since the \( S^1 \times S^1 \) fibration degenerates over \( w_i \) and \( w'_i \), we can regard this fibration over the interval between neighboring \( w_i \) and \( w'_i \) as a 3-cycle. In [31] it was shown that this 3-cycle is \( S^3 \) and \( S^2 \times S^1 \) respectively, depending on whether neighboring points are a \( w, w' \) pair or both \( w \) (both \( w' \)). In the former case one obtains a quartic superpotential, in the latter case an \( N = 2 \) like setup. Obviously this yields the same gauge theory as the \( D3 \) probe on \( \mathcal{W}_T \) and the \( D4 \) brane on the interval.

5.3 D5 branes on the box: the modified box rules

The second theory we would like to consider are \( M \) \( D3 \) branes on an \( \mathbb{Z}_k \times \mathbb{Z}_l \) orbifolded conifold. As shown above, the geometry dualizes under \( T_U \) into brane boxes where the NS5 brane skeleton wraps the curve \( \sum_{i,j=0}^{k,l} m_{ij} z^i w^j \), \( k + l - 2 \) of the \( m_{ij} \) parameters can be associated to brane positions, while the other \( kl \) parameters correspond to diamonds, that is the hypermultiplets

\(^*\)Similar setups have been discussed recently in [30].
sitting at the NS NS’ intersections, whose vev smoothes out the singular intersections.

The probe D3 branes turn into D5 branes living on these boxes and diamonds. Again this should in principle be a very useful duality in the sense that we can read off the associated gauge theories by using some analogue of the HW rules. In addition some information about the corresponding quantum gauge theory should be obtainable by lifting the setup to M-theory. In order to understand our rules it is best to start with the easiest example, the conifold $C$, eq.(2), itself. The dual description just is that of a single NS and NS’ brane on a square torus, as depicted in the upper left corner of Fig.5. The conifold has one blowup parameter, corresponding to the one

![Diagrams of boxes and diamonds](image)

**Fig.5.** Upper left: the box with generic B-value; Upper right: maximal B-value; Lower left: Taking B to 0 sending one gauge coupling to infinity.

diamond sitting at the intersection. As long as we keep the size of the 2-sphere zero, the B-flux through the sphere will correspond to the size of the diamond. As we have argued in the last section, the curve describing the diamond actually supports a non-trivial $S^1$ on which the D5 brane can end, so the gauge theory will have two group factors, $SU(M) \times SU(M)$. The inverse gauge couplings are proportional to the area of the corresponding faces. There is a special point, when the diamond has the same area as the
other gauge group, that is the diamond occupies half of the torus. In this case we know that we have to recover the standard conifold gauge theory of [6]. This can easily be implemented using the simple brane rules specified in the upper right corner. We have to demand, that half of the matter multiplets we would naively expect are projected out. The orientation of the arrows seems quite arbitrary. Indeed we will see that the orientation can be changed and that this corresponds to performing flop transitions in the dual geometry. Indeed one can easily establish that these rules also are capable of realizing more complicated setups. Generically, the gauge theory on the $\mathbb{Z}_k \times \mathbb{Z}_l$ orbifolded conifold has a $SU(M)^{kl} \times SU(M)^{kl}$ gauge group. In our picture the gauge group factors will correspond to the $kl$ diamonds and the $kl$ boxes respectively. Again it is easiest to compare at the point, where all gauge couplings are equal. In this case, both the diamonds as well as the boxes degenerate to rhombes, as pictured in Fig.6, where we denoted them as filled and unfilled boxes. Generalizing our $A$ and $B$ fields from above we will find that the matter fields transform as (where the two sets of $kl$ gauge groups are indexed by small and capital letters respectively)

![Fig.6. The diamond rules at the point of maximal B-fields.](image)
which are exactly the rules expected [9]. This proposal can also easily deal with the situation of non-trivial identifications on the torus as discussed in [15]. In addition there will be quartic superpotential for every closed rectangle, the relative sign being given by the orientation

\[
W = \sum_{i,j}(A_1)_{i+1,j+1;I,J}(B_1)_{I,J;i,j+1}(A_2)_{i,j+1;I,J+1}(B_2)_{I,J+1;i,j+1} - \sum_{i,j}(A_1)_{i+1,j+1;I,J}(B_2)_{I,J;i+1,j}(A_2)_{i+1,j;I,J+1}(B_2)_{I+1,J;i+1,j+1}
\]

We do not expect that this picture changes when we take the sizes of box and diamond to differ. We will still see the A and B fields. Only the relative couplings will change and no new fields or interactions appear, since they certainly don’t in the dual geometry. The singular conifold points correspond to the situations where diamonds close. From the field theory point of view this just means that we take the corresponding gauge coupling to infinity. As in the standard HW situation with only parallel NS branes this corresponds to a strong coupling fixed point with possibly enhanced global symmetry if several NS branes coincide.

Another interesting question to consider is to ask ourselves what happens when we blow up the spheres to finite size. This now should correspond to some mode of the diamond that “rotates” it away out of the 48 plane into the 59 plane. According to common lore this should correspond to a FI term in the gauge theory. We will no longer be able to support a D5 brane stretched inside the diamonds in a supersymmetric fashion, independent of their size (that is the B-field)**. Since we expect that the center of mass

**This is very similar to what happens on the interval: blowing up a sphere corresponds to moving off an NS brane in the 7 direction. Since in order to preserve supersymmetry branes are only allowed to stretch along the 6 direction this effectively reduces the number of gauge groups (the number of intervals) by one. The 6 position of the brane we moved away (the B-field on the blown up sphere) does not affect the massless matter content anymore.
$U(1)$s are frozen out as in [29], the FI term will be reinterpreted as usual as a baryonic branch. Especially there should exist a baryonic branch along which we reduce to the orbifold gauge theory.

Indeed as shown in [9] the gauge theories we described here do have such a baryonic branch. Giving a vev to (say) all the $A_2$ fields will break each $SU(M)_{ij} \times SU(M)_{IJ}$ pair down to its diagonal $SU(M)_{ab}$ subgroup. The remaining massless fields after the Higgs mechanism are

\[
\begin{align*}
D_{a+1,b+1;a,b} &= (A_1)_{a+1,b+1;A,B} (\Box_{a+1,b+1}, \Box_{a,b}) \\
H_{a,b;a,b+1} &= (B_1)_{a,B;a,b+1} (\Box_{a,B}, \Box_{a,b+1}) \\
V_{a,b;a+1,b} &= (B_2)_{a,B;a+1,b} (\Box_{a,B}, \Box_{a+1,b})
\end{align*}
\]

with the remaining superpotential:

\[
W \sim \sum_{a,b} D_{a+1,b+1;a,b} H_{a,b;a,b+1} V_{a,b+1;a+1,b} - \sum_{a,b} D_{a+1,b+1;a,b} V_{a,b;a+1,j} H_{a+1,b;a+1,b+1}
\]

which are precisely the box rules of [20], as claimed. Note that the diagonal $D$ fields are not special at all, they arise just from the fundamental $A, B$ degrees of freedom of the generalized box.

A small complication arises once we consider situations that are more involved than the conifold. For simplicity let us study the case of the $\mathbb{Z}_2$ orbifolded conifold. Since this can as well be thought of as the $\mathcal{G}_{22}$ generalized conifold, it has an interval dual as well as a box dual. Both of them are displayed in Fig.7 for various values of the $B$-fields. The gauge group is $SU(M)^4$. We should see 3 $B$-fields governing the relative sizes of the gauge couplings. According to our scenario this will correspond to one relative brane position $B$ and the sizes of two diamonds $b_1$ and $b_2$. In the interval picture $b_{1,2}$ will be the distance between NS$_{1,2}$ and NS$'_{1,2}$ while $B$ is the distance between the center of masses of the two NS NS' pairs, denoted as circles in Fig.7. Take the circle to have circumference 2 and the torus to have sides 2 and 1. Since $B$-fields (=inverse gauge couplings) are length on the interval and areas on the torus, in these units the area of a given gauge group on the torus should have the same numerical value as the corresponding length on the circle (total area=total length=2). The third picture in Fig.7 shows $B = 1$ $b_1 = b_2 = 1/2$ Both sides have 4 gauge groups of size 1/2.
It is easy to identify in both theories the point where all gauge couplings are equal, the point where all B-fields are zero (the most singular point) and the point where the setup looks like two separated conifolds. Similarly for all positive values of the $b_i$ and of $B$ we can read off the gauge theory from the diamonds, just using the standard $A$ and $B$ fields, representing the diamonds as rhombes of area $b_i$. However from the interval it is clear, that we can also pass an NS’ brane through an NS brane, performing Seiberg duality on the gauge theory and simultaneously changing the sign of one of the $b_i$ fields [9, 32]. If we set $b_1 = b_2 = -1/2$ the picture looks the same as for $b_1 = b_2 = 1/2$. The overall sign does not matter. However the sixth picture of Fig.7 shows a setup where the signs of the $b_i$ differ. We should assign our diamonds an orientation in order to be able to address this issue. This orientation assigns whether the $A$ or the $B$ fields point outward or inward, the other doing the opposite. The rules we have introduced are valid for the case that all orientations are equal. The situation with opposite orientations is slightly more complicated. The rules can be determined by comparing with the interval. Whenever the arrows point around the closed rectangle we write
down a quartic superpotential. If diamonds with different orientation touch, we will have to introduce additional ‘meson’ fields with cubic superpotential (see the 6th picture in Fig. 7). Since this inversion of orientation should correspond to Seiberg duality in the field theory, we basically found this way a realization of $N = 1$ dualities in the box and diamond picture! It would be clearly interesting to pursue this point further, for example by studying theories with orientifolds. This may give us a hint of a brane realization of Pouliot like dualities [33] and spinors, since it is easy to realize the magnetic side of these theories in the box and diamond picture using orientifolds.

Last but not least we should be able to see the same gauge groups in the third T-dual realization as well, that is from D6 branes wrapping the 3-cycles of the deformed generalized conifold geometry (12)

$$xy = \sum_{i,j=1}^{k,l} m_{ij} u^i v^j$$

in the same spirit as above following [31]. It would be interesting to work this out and see if some properties of the gauge theory can be better understood in this language.

6 Mirror branes and Domain Walls

6.1 The mirror branes

The D3 brane probe we have been considering so far maps to a D4 brane on the interval and a D5 brane in the box respectively. We identified the corresponding gauge theories above. For a special subclass of models we were considering we can actually perform both. These geometries are those whose toric diagram is given by two rows of $k$ points. Viewing them as $\mathbb{Z}_k$ orbifolded conifolds $\mathcal{C}_k$, they (or better their blowup) turn into a box with 1 NS’ and $k$ NS under $T_V$. We can as well describe them as a $\mathcal{G}_{kk}$ generalized conifold and hence $T_U$ dualize them into an interval with $k$ NS and $k$ NS’ branes. According to our philosophy these two ways of realizing the gauge theory should actually be mirror to each other! We turned one HW setup into its ‘mirror branes’.

Now we can try to solve these gauge theories via the lift to M-theory. Interestingly enough, the intervals lift via SUSY 2-cycles in $\mathbb{R}^6$ while the
boxes lift via SUSY 3-cycles \([34]\) in \(\mathbb{R}^6\). So for every 3-cycle we should find a dual 2-cycle encoding the same information and vice versa.

### 6.2 Putting together intervals and boxes

Above we obtained an \(N = 1\) \(d = 4\) gauge theory from intervals in type IIA and boxes in type IIB setups respectively. Of course we can as well build a box in type IIA or an interval in type IIB in order to obtain odd dimensional gauge theories with 4 supercharges. The singular point should correspond to having all NS branes coinciding.

We can do both together, that is put branes on the box and the interval simultaneously, provided we put in enough NS branes so that we can open up both, a box and an interval. From the dual geometry point of view this corresponds to consider manifolds with both complex and Kähler deformations turned on simultaneously. An interesting example is type IIA with NS 012345, NS' 012389, D4 01236, D4 01248. It is easy to convince oneself, that this now lifts to M-theory via a SUSY 3-cycle in \(G_2\). That is we now break another half of the SUSY, leaving us with 2 unbroken supercharges, or \(N = 1\) in \(d = 3\). Note that this gauge theory actually only lives on the boxes, since the interval theory is 4d while the box theory is 3d. Things become more interesting if we compactify the \(x^3\) direction. In this case both the interval and the box give 3d gauge theories.

These brane setups fit nicely into the framework of brane cubes. These also lead to 2 supercharges. They lift via \(G_2\) and \(SU(4)\) 4-cycle respectively and are dual to probes on \(SU(4)\) and \(G_2\) orbifold. Now we have a 3rd kind of brane setup in this league, which lifts via \(G_2\) 3-cycle and should probably also be dual to probes on a \(G_2\) singularity.

Note that from the point of view of the four dimensional theory on the D4 branes on the interval, the D4 branes on the box look like domain walls (they are localized in \(x^3\)). This is nice, since Witten argued \([21]\) before that domain walls in \(d = 4\), \(N = 1\) gauge theory should be associated to M5 branes on \(G_2\) 3-cycles.

### 7 Summary

Let us briefly summarize the main results of the paper. For two classes of non-compact (complex 3-dimensional) Calabi-Yau spaces we constructed the
T-dual NS brane configurations. Specifically blowups (resp. deformations) of orbifolded conifold singularities, denoted by \( C_{kl} \), are \( T_V \) dual to boxes (resp. intervals) of NS branes, whereas blowups (resp. deformations) of generalized conifold singularities, called \( G_{kl} \), are \( T_U \) dual to intervals (resp. boxes) of NS branes. Since the composition of \( T_U \) and \( T_V \) corresponds to a T-duality with respect to three isometrical \( U(1) \) directions of \( M \) resp. \( W \), it should not come as a surprise that \( C_{kl} \) and \( G_{kl} \) are actually mirror pairs. The Kähler resp. complex structure parameters of the geometric singularities correspond to positions of the NS branes in the dual brane picture. Moreover the conifold transition for the non-compact Calabi-Yau spaces \( C_{kl} \) or \( G_{kl} \) via shrinking 2-cycles and growing up 3-cycles precisely corresponds to the transition between the box and interval theory or vice versa, by first moving all NS branes on top of each other and then removing them into different directions. All this is summarized in Fig.8 below.

![Diagram](image)

**Fig.8.** The proposed picture.

Constructing gauge theories from branes, the geometric singularities as well as the NS brane configurations serve as backgrounds, which are probed by a certain number of D branes. We have seen that the “mirror map” does not change the corresponding gauge theories. At the conifold point some of the gauge couplings go to infinity.

In order to establish the duality between conifold singularities and brane
boxes we had to generalize the concept of brane boxes by also including brane diamonds. We formulate rules for deriving the matter content of the gauge theories living on boxes and diamonds. Along a baryonic branch of the gauge theory, which corresponds to partially resolving the conifolds $C_{kl}$ to the orbifold singularities $O_{kl}$ we recover the orbifold gauge theories from our general rules.

Blowups (or deformations) of certain geometries, namely $C_{1k} \equiv G_{kk}$, allow both for a dual brane box as well as for a dual interval description. It follows that the corresponding gauge theory on the interval and on the brane box are mirror to each other. This observation could be useful for the investigation of the non-perturbative quantum dynamics of these kind of $N = 1$ gauge theories: namely for every supersymmetric 2-cycle which describes the dynamics of the interval theory embedded in M-theory, there should exist a mirror supersymmetric 3-cycle for the brane box theory also embedded in M-theory. It would be interesting to work out this mirror map between 2- and 3-cycles explicitly. Moreover one could expect that due to quantum corrections the physics of the gauge theories at the conifold point is not as singular as in the classical description we have discussed. Finally, it would be also interesting to relate the brane constructions of $N = 1$ supersymmetric gauge theories, considered here, to the geometric engineering approach, where various branes are wrapped around non-trivial cycles of Calabi-Yau 4-folds or manifolds of $G_2$ holonomy.

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