Broken Ergodicity in a Stochastic Model with Condensation

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We introduce a variant of the asymmetric random average process with continuous state variables where the maximal transport is restricted by a cutoff. For periodic boundary conditions, we show the existence of a phase transition between a pure high flow phase and a mixed phase, whereby the latter consists of a homogeneous high flow and a condensed low flow substate without translation invariance. The finite system alternates between these substates which both have diverging lifetimes in the thermodynamic limit, so ergodicity is broken in the infinite system. However, the scaling behaviour of the lifetimes in dependence of the system size is different due to different underlying flipping mechanisms.

\[ P(m) = \prod_i P(m_i) \quad \text{with} \quad P(m_i) = \frac{4m_i}{\rho^2} e^{-2m_i/\rho} \]

is the most relevant for our work. Eq. (2) has been obtained in the context of the q model [3] that shares many properties with the ARAP. Further results for the ARAP can be found in [1,2,8–10].

In the ARAP the mass \( r_i m_i \) transferred from site \( i \) to site \( i+1 \) is in principle unbounded. This is different in the truncated ARAP that we introduce here. In the TARAP all transfers of masses \( r_i m_i \) larger than a cutoff \( \Delta > 0 \) are rejected. We like to focus on the simplest model only, the truncated free ARAP. The corresponding fraction density is then given by \( \phi = \prod_i \phi(r_i, m_i) \) with

\[ \phi(r_i, m_i) = [1 - R(m_i)] \delta(r_i) + \Theta(R(m_i) - r_i) , \]

where \( \Theta \) is the Heaviside step-function and

\[ R(m_i) \equiv \min \left( 1, \frac{\Delta}{m_i} \right) \]

represents the maximum possible fraction. Note that \( \phi \) has become locally state dependent, i.e. the pdf depends on the mass \( m_i \) at the corresponding site explicitly.

Without loss of generality we set \( \rho = 1 \) for the rest of the letter because every TARAP defined by \( (\rho, \Delta) \) can be mapped onto a \( (1, \hat{\Delta}) \)-system (Fig. 1). Furthermore we introduce the rescaled cutoff

\[ \hat{\Delta} = 2L^{-\frac{1}{2}} \Delta \]

which ensures \( L \)-independence of the critical point.

We begin by investigating the relation between the steady state current \( J \), defined by the average mass transfer per site, and the cutoff parameter \( \Delta \) (Fig. 1). For
$\Delta \to 0$ the flow vanishes because the transferred mass per site is always smaller than $\Delta$ while for $\Delta \to \infty$ the system behaves like a free ARAP and the value of $J$ tends to its maximum $J_{\text{max}}^\text{high} \equiv \frac{1}{2}$.

If we study the process in more detail we discover the system to exist in two phases whereby the transition point $\tilde{\Delta}_c = 1$ has been determined numerically and by analytical approximations.

For rescaled cutoffs $\tilde{\Delta} > \tilde{\Delta}_c$ the steady state mass distribution is nearly identical to the one of the free ARAP and approximately given by (2). The flow is independent of $\Delta$ and corresponds to the maximum current $J_{\text{max}}^\text{high} \equiv J_{\text{max}}^\text{max}$. Therefore we refer to this parameter range as the high flow phase.

For rescaled cutoffs smaller than $\tilde{\Delta}_c$ the steady state of the finite system is a composition of two different substates: the system can either exist in a high flow state (with properties as described above) or a state given by a macroscopic condensate, i.e. a finite fraction of the total mass $M$ resides on one randomly chosen site (even in the thermodynamic limit). So the mass aggregation is proportional to $L$, but not extensive in space. The remaining mass is distributed equally with an algebraically decaying mass distribution. The current of the condensate state depends on $\tilde{\Delta}$ but does not exceed $J_{\text{low}}^\text{max} \equiv \frac{1}{4}$. Therefore we call this state low flow state and denote the parameter region $\tilde{\Delta} < \tilde{\Delta}_c$ as mixed phase.

In a finite system the transition probabilities between low and high flow states are small but nonzero. The system switches between these states while evolving in time and an alternating current-time relation is obtained (Fig. 2). Note that the switching time between the two states is much smaller than their lifetimes.

This alternating behaviour is very similar to systems with spontaneously broken symmetry [6,7]. However, in general the flipping is between states of broken symmetry. For example in [6] an ASEP with two particles (+) and (−) is introduced and in regimes of spontaneously broken symmetry the system switches between states dominated by (+) or (−) particles. In case of the TARAP the flipping occurs between the symmetric high flow state which is translation invariant and a low flow state in which this symmetry is broken.

In the thermodynamic limit the average lifetimes $\tau_H$ and $\tau_L$ of the high and low flow states diverge (Figs. 3 and 4). This implies that the steady state in the mixed phase is not unique. Ergodicity is broken and the steady state can either be in the high flow state or the low flow state depending on the initial condition. However, the lifetimes of the substates do not scale equally with size $L$ (see next paragraph).

Although Monte Carlo simulations are difficult in the mixed phase since the lifetimes of the substates are very large it is safe to say that both the average lifetimes of low and high flow states diverge in the thermodynamic limit. Fig. 3 indicates that $\tau_H \sim \exp(\alpha(\tilde{\Delta})L)$ we obtain that

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**FIG. 1.** Current-cutoff diagram for system size $L = 100$ and different densities $\rho$ ($\Diamond = 0$, $\bigcirc = 1$, $\blacksquare = 2$).

**FIG. 2.** Current-time diagram of the mixed flow phase. Parameters used: $L = 100$ and $\tilde{\Delta} = 0.846$. Each current value is averaged over $10^3$ time steps.

**FIG. 3.** Linear-logarithmic plot of the lifetime $\tau_H$ of the high flow state in dependence of the system size $L$ for several rescaled cutoffs $\tilde{\Delta} = 0.76(\Diamond), 0.80(\bigcirc), 0.84(\blacksquare)$ and $0.88(\triangle)$.
in dependence of the system size $L$. This shows that the average mass shift $J$ for $m \to \infty$ is an increasing function of $\Delta$. Note that in the pure $\alpha$ one-stick configurations, are very stable. On the other hand, homogeneous configurations maximize the current. We will exemplify this in the following paragraphs.

First we study the low flow state using the approximation

$$
J(m) \equiv \langle r \rangle_{\phi(r,m)} = \begin{cases} m & \text{for } 0 \leq m < \Delta \\
\frac{1}{2m} & \text{for } \Delta \leq m < \infty. \end{cases} \quad (6)
$$

This shows that the average mass shift $J(m)$ tends to zero for $m \to \infty$ and $m \to 0$. So high (low) columns shrink (grow) very slowly and accordingly low flow states, resp. one-stick configurations, are very stable. On the other

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described best by the one-condensate picture, especially in the vicinity of the critical point. This is confirmed by numerical investigations that have shown that configurations with two or more aggregates are not stable. The pre-condensates struggle for masses until only one stick is left. This mechanism of competition is different to coarsening where aggregates emerge at several sites and then coalesce [4].

Focussing on the high flow state, resp. phase, every site carries the same mass $\rho$ on average. For large $L$ we approximate, motivated by the numerical results, the mass distribution by the product measure solution of the free ARAP (2) and calculate by the help of (6) the high flow state current

$$J_{\text{high}} = \max_{\text{high}} \left\{ 1 - \left(1 + \frac{L^2}{\Delta} \right) e^{-L^2/\Delta} \right\} \rightarrow J_{\text{high}}^{\max}$$

for all rescaled cutoffs. So the high flow state exists both in the mixed phase and in the high flow phase. Equation (10) also predicts that the current in the high flow state differs from $J_{\text{high}}^{\max}$ only for small absolute cutoffs $\Delta$ in finite systems. We have verified this by Monte Carlo simulations although it is no longer possible to distinguish the states by their flows which are nearly identical in this parameter range. So we used the appearance of a macroscopic condensate as a criterion.

A model that is strongly related to the TARAP is studied in [5,14] where a similar kind of phase separation as in the mixed domain has been observed. The underlying process is the continuum limit (Krauss model) of the Nagel-Schreckenberg cellular automaton model [13] that is defined by continuous velocities and spatial coordinates. Although this traffic model is given by more complex dynamics than the TARAP, both processes have a common feature: moves may be rejected if a uniformly distributed random variable exceeds a given threshold. So the TARAP can be viewed as a toy model that catches some of the fundamental physics behind the Krauss model. We have also rewritten a simplified version of the Krauss model in terms of fraction densities to point out the similarities with the TARAP explicitly [11].

In the Krauss model an additional congested phase has been observed [5]. An analogous phase, corresponding to a pure condensed phase, could be expected for the TARAP in the limit of small cutoffs. However, in that regime the lifetime of the low flow state is much larger than the corresponding lifetime of the high flow state, in particular for small systems. Therefore, in Monte Carlo simulations it is difficult to distinguish between a pure condensed phase and the mixed regime. Furthermore, our analytical calculations have not shown any evidence for a transition point separating a (new) pure low flow phase and the mixed phase.

What are the essential ingredients of the TARAP leading to the observed phenomena? First truncation and an unbounded local state space allow for an unlimited condensation. The formation of infinite aggregates can also be found in a model of aggregation and fragmentation [4,15] or a special zero-range process [16], both equipped with unbounded but discrete state variables. However, in these examples the condensed phase is unique and not associated with a high flow counterpart, i.e. high flow phase and congested phase are separated by a phase transition.

Therefore we believe that in case of the TARAP the continuous state variables allow for the coexistence of high and low flow state in the mixed phase, yielding nonsymmetric ergodicity breaking in the thermodynamic limit. The TARAP is free of an intrinsic mass scale in contrast to models defined on integer state space, equipped with a smallest mass unit 1. Thus, in the TARAP we obtain a trivial dependence on the density $\rho$ which is also reflected in the scaling law $(\rho, \Delta) \leftrightarrow (1, \Delta/\rho)$. However, in discrete systems physics may change in the low density regime, e.g. an ARAP with total mass $M = 1$ is nothing else than one-particle ASEP. This difference may indicate why in the continuous variant the homogeneous high flow state can persist over the whole parameter range, resulting in the coexistence phenomenon.

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