Resonance identity and multiplicity of non-contractible closed geodesics on Finsler $\mathbb{R}P^n$

Hui Liu$^1$,∗ Yuming Xiao$^2$,†

1 School of Mathematics and Statistics, Wuhan University, Wuhan 430072, Hubei, People’s Republic of China
2 School of Mathematics, Sichuan University, Chengdu 610064, People’s Republic of China

Abstract

In this paper, we establish first the resonance identity for non-contractible homologically visible prime closed geodesics on Finsler $n$-dimensional real projective space $(\mathbb{R}P^n, F)$ when there exist only finitely many distinct non-contractible closed geodesics on $(\mathbb{R}P^n, F)$, where the integer $n \geq 2$. Then as an application of this resonance identity, we prove the existence of at least two distinct non-contractible closed geodesics on $\mathbb{R}P^n$ with a bumpy and irreversible Finsler metric. Together with two previous results on bumpy and reversible Finsler metrics in [14] and [39], it yields that every $\mathbb{R}P^n$ with a bumpy Finsler metric possesses at least two distinct non-contractible closed geodesics.

Key words: Non-contractible closed geodesics; Resonance identity; Non-simply connected manifolds; Morse theory; Index iteration theory; Systems of irrational numbers; Kronecker’s approximation theorem

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1 Introduction

In this paper, we are concerned with the multiplicity of closed geodesics on $n$-dimensional real projective space $\mathbb{R}P^n$ with a Finsler metric $F$, which is the typically non-simply connected manifold

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E-mail: huiliu@ustc.edu.cn.
†Supported by the Scientific Research Funds for Young Teachers of Sichuan University, Grant 2012SCU11083.
e-mail: yumingxiao@scu.edu.cn.
with the fundamental group $\mathbb{Z}_2$. One of the main ingredients is a new resonance identity of non-contractible homologically visible prime closed geodesics on $(\mathbb{RP}^n, F)$ when there exist only finitely many distinct non-contractible closed geodesics on $(\mathbb{RP}^n, F)$. The second one is the precise iteration formulae of Morse indices for non-orientable closed geodesics which can be seen as a complement of the index iteration theory for the orientable case. The third one is the application of Kronecker’s approximation theorem in Number theory to the multiplicity of non-contractible closed geodesics on $(\mathbb{RP}^n, F)$.

A closed curve on a Finsler manifold is a closed geodesic if it is locally the shortest path connecting any two nearby points on this curve. As usual, on any Finsler manifold $(M, F)$, a closed geodesic $c : S^1 = \mathbb{R}/\mathbb{Z} \to M$ is prime if it is not a multiple covering (i.e., iteration) of any other closed geodesics. Here the $m$-th iteration $c^m$ of $c$ is defined by $c^m(t) = c(mt)$. The inverse curve $c^{-1}$ of $c$ is defined by $c^{-1}(t) = c(1 - t)$ for $t \in \mathbb{R}$. Note that unlike Riemannian manifold, the inverse curve $c^{-1}$ of a closed geodesic $c$ on a irreversible Finsler manifold need not be a geodesic. We call two prime closed geodesics $c$ and $d$ distinct if there is no $\theta \in (0, 1)$ such that $c(t) = d(t + \theta)$ for all $t \in \mathbb{R}$. For a closed geodesic $c$ on $(M, F)$, denote by $P_c$ the linearized Poincaré map of $c$. Recall that a Finsler metric $F$ is bumpy if all the closed geodesics on $(M, F)$ are non-degenerate, i.e., $1 \notin \sigma(P_c)$ for any closed geodesic $c$.

Let $\Lambda M$ be the free loop space on $M$ defined by
\begin{equation}
\Lambda M = \left\{ \gamma : S^1 \to M \mid \gamma \text{ is absolutely continuous and } \int_0^1 F(\gamma, \dot{\gamma})^2 dt < +\infty \right\}, \quad (1.1)
\end{equation}
endowed with a natural structure of Riemannian Hilbert manifold on which the group $S^1 = \mathbb{R}/\mathbb{Z}$ acts continuously by isometries (cf. Shen [37]).

It is well known (cf. Chapter 1 of Klingenberg [22]) that $c$ is a closed geodesic or a constant curve on $(M, F)$ if and only if $c$ is a critical point of the energy functional
\begin{equation}
E(\gamma) = \frac{1}{2} \int_0^1 F(\gamma, \dot{\gamma})^2 dt. \quad (1.2)
\end{equation}
Based on it, many important results on this subject have been obtained (cf. [1], [4], [15], [18]-[19], [33]-[34]). In particular, in 1969 Gromoll and Meyer [17] used Morse theory and Bott’s index iteration formulae [8] to establish the existence of infinitely many distinct closed geodesics on $M$, when the Betti number sequence $\{\beta_k(\Lambda M; \mathbb{Q})\}_{k \in \mathbb{Z}}$ is unbounded. Then Vigué-Poirrier and Sullivan [43] further proved in 1976 that for a compact simply connected manifold $M$, the Gromoll-Meyer condition holds if and only if $H^*(M; \mathbb{Q})$ is generated by more than one element. Here the Gromoll-Meyer theorem is valid actually for any field $\mathbb{F}$. Note that it can not be applied to the compact
rank one symmetric spaces

\[ S^n, \mathbb{R}P^n, \mathbb{C}P^n, \mathbb{H}P^n \text{ and } \mathbb{CaP}^2, \]

since \( \{ \beta_k(\Lambda M, F) \}_{k \in \mathbb{Z}} \) with \( M \) in (1.3) is bounded with respect to any field \( F \) (cf. Ziller [44]). In fact, each of them endowed with Katok metrics possesses only finitely many distinct prime closed geodesics (cf. Katok [21], also Ziller [45]).

In 2005, Bangert and Long [7] (published in 2010) showed the existence of at least two distinct closed geodesics on every Finsler \( S^2 \). Subsequently, such a multiplicity result for \( S^n \) with a bumpy Finsler metric, i.e., on which all closed geodesics are non-degenerate, was proved by Duan and Long [10] and Rademacher [36] independently. In recent years, more interesting results on this problem have been obtained, such as [11]-[13], [20], [30], [35], [40]-[41]. We refer the readers to the survey papers of Long [29], Taimanov [38] and Oancea [32] for more studies on this subject.

Besides many works on closed geodesics in the literature which study closed geodesics on simply connected manifolds, we are aware of not many papers on the multiplicity of closed geodesics on non-simply connected ones published before 2015, at least when they are endowed with Finsler metrics. For example, Ballman, Thorbergsson and Ziller [3] of 1981 and Bangert and Hingston [5] of 1984 dealt with the non-simply connected manifolds with a finite/infinite cyclic fundamental group respectively by the min-max principle.

In order to apply Morse theory to the multiplicity of closed geodesics on \( \mathbb{R}P^n \), motivated by the studies on the simply connected manifolds, in particular, the resonance identity proved by Rademacher [33], Xiao and Long [42] in 2015 investigated the topological structure of the non-contractible loop space and established the resonance identity for the non-contractible closed geodesics on \( \mathbb{R}P^{2n+1} \) by using \( \mathbb{Z}_2 \) coefficient homology. As an application, Duan, Long and Xiao [14] proved the existence of at least two distinct non-contractible closed geodesics on \( \mathbb{R}P^3 \) endowed with a bumpy and irreversible Finsler metric. In a very recent paper [39], Taimanov studied the rational equivariant cohomology of the spaces of non-contractible loops in compact space forms and proved the existence of at least two distinct non-contractible closed geodesics on \( \mathbb{R}P^2 \) endowed with a bumpy and irreversible Finsler metric. Then Liu [25] combined Fadell-Rabinowitz index theory with Taimanov’s topological results to get multiplicity results of non-contractible closed geodesics on positively curved Finsler \( \mathbb{R}P^n \).

Motivated by [39] and [42], in section 2 of this paper we obtain the resonance identity for the non-contractible closed geodesics on \( \mathbb{R}P^n \) by using rational coefficient homology for any \( n \geq 2 \) regardless of whether \( n \) is odd or not.
Theorem 1.1 Suppose the Finsler manifold \( M = (\mathbb{RP}^n, F) \) possesses only finitely many distinct non-contractible prime closed geodesics, among which we denote the distinct non-contractible homologically visible prime closed geodesics by \( c_1, \ldots, c_r \) for some integer \( r > 0 \), where \( n \geq 2 \). Then we have

\[
\sum_{j=1}^{r} \hat{\chi}(c_j) = \hat{B}(\Lambda_g M; \mathbb{Q}) = \begin{cases} 
\frac{n+1}{2(n-1)}, & \text{if } n \in 2\mathbb{N} - 1, \\
\frac{n}{2(n-1)}, & \text{if } n \in 2\mathbb{N}.
\end{cases}
\]  

(1.4)

where \( \Lambda_g M \) is the non-contractible loop space of \( M \) and the mean Euler number \( \hat{\chi}(c_j) \) of \( c_j \) is defined by

\[
\hat{\chi}(c_j) = \frac{1}{n_j} \sum_{m=1}^{n_j} \sum_{l=0}^{2n-2} (-1)^l i(c_j) k_l(c_j^{2m-1}) \in \mathbb{Q},
\]

and \( n_j = n_{c_j} \) is the analytical period of \( c_j \), \( k_l(c_j^{2m-1}) \) is the local homological type number of \( c_j^{2m-1} \), \( i(c_j) \) and \( \hat{i}(c_j) \) are the Morse index and mean index of \( c_j \) respectively.

In particular, if the Finsler metric \( F \) on \( \mathbb{RP}^n \) is bumpy, then (1.4) has the following simple form

\[
\sum_{j=1}^{r} (-1)^{\hat{i}(c_j)} = \begin{cases} 
\frac{n+1}{n-1}, & \text{if } n \in 2\mathbb{N} - 1, \\
\frac{n}{n-1}, & \text{if } n \in 2\mathbb{N}.
\end{cases}
\]  

(1.5)

Based on Theorem 1.1, the precise iteration formulae of Morse indices for closed geodesics and Morse theory, especially the \( S^1 \)-equivariant Poincaré series of \( \Lambda_g M \) derived by Taimanov (cf. Lemma 2.3), and using some techniques in Number theory, we can prove the following multiplicity result of non-contractible closed geodesics on \( (\mathbb{RP}^n, F) \).

Theorem 1.2 Every \( \mathbb{RP}^n \) endowed with a bumpy and irreversible Finsler metric \( F \) has at least two distinct non-contractible closed geodesics, where \( n \geq 2 \).

Remark 1.1 For any compact simply-connected bumpy Finsler manifold, Duan, Long and Wang in [12] proved the same conclusion as Theorem 1.2. However, their method is not applicable to our problem. Indeed, one of the crucial facts in their proof is that if there is only one prime closed geodesic on such a manifold, its Morse index must be greater than or equal to some positive integer. But there is always a minimal point of the energy functional on \( \Lambda_g (\mathbb{RP}^n) \) with Morse index 0.

If \( F \) is a bumpy and reversible Finsler metric, the same conclusion as Theorem 1.2 has been proved in Theorem 1.2 of [14] and the remark behind Theorem 5 of [39]. As a combined outcome, we immediately get the desired result as follows.
Corollary 1.1 Every $\mathbb{R}P^n$ endowed with a bumpy Finsler metric has at least two distinct non-contractible closed geodesics, where $n \geq 2$.

This paper is organized as follows. In section 2, we use Morse theory to establish the resonance identity of Theorem 1.1. Then in section 3, we investigate the precise iteration formulae of Morse indices for closed geodesics on $\mathbb{R}P^n$ and build a bridge between their Morse indices and a division of an interval. In section 4, a special system of irrational numbers associated to our problem is carefully studied and a key result on it for our later proof of Theorem 1.2 is obtained. Finally in section 5, we draw support from the well known Kronecker’s approximation theorem in Number theory and give the proof of Theorem 1.2.

We close this introduction with some illustrations of notations in this paper. As usual, let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{Q}^c$ denote the sets of natural integers, integers, rational numbers and irrational numbers respectively. We also use notations $E(a) = \min\{k \in \mathbb{Z} | k \geq a\}$, $[a] = \max\{k \in \mathbb{Z} | k \leq a\}$, $\varphi(a) = E(a) - [a]$ and $\{a\} = a - [a]$ for any $a \in \mathbb{R}$. Throughout this paper, we use $\mathbb{Q}$ coefficients for all homological and cohomological modules.

2 Morse theory and resonance identity of non-contractible closed geodesics on $(\mathbb{R}P^n, F)$

Let $M = (M, F)$ be a compact Finsler manifold, the space $\Lambda = \Lambda M$ of $H^1$-maps $\gamma : S^1 \to M$ has a natural structure of Riemannian Hilbert manifolds on which the group $S^1 = \mathbb{R}/\mathbb{Z}$ acts continuously by isometries. This action is defined by $(s \cdot \gamma)(t) = \gamma(t + s)$ for all $\gamma \in \Lambda$ and $s, t \in S^1$. For any $\gamma \in \Lambda$, the energy functional is defined by

$$E(\gamma) = \frac{1}{2} \int_{S^1} F(\gamma(t), \dot{\gamma}(t))^2 dt.$$  \hspace{1cm} (2.1)

It is $C^{1,1}$ and invariant under the $S^1$-action. The critical points of $E$ of positive energies are precisely the closed geodesics $\gamma : S^1 \to M$. The index form of the functional $E$ is well defined along any closed geodesic $c$ on $M$, which we denote by $E''(c)$. As usual, we denote by $i(c)$ and $\nu(c) - 1$ the Morse index and nullity of $E$ at $c$. In the following, we denote by

$$\Lambda^{\kappa} = \{d \in \Lambda \mid E(d) \leq \kappa\}, \quad \Lambda^{\kappa-} = \{d \in \Lambda \mid E(d) < \kappa\}, \quad \forall \kappa \geq 0. \hspace{1cm} (2.2)$$

For a closed geodesic $c$ we set $\Lambda(c) = \{\gamma \in \Lambda \mid E(\gamma) < E(c)\}$.

For $m \in \mathbb{N}$ we denote the $m$-fold iteration map $\phi_m : \Lambda \to \Lambda$ by $\phi_m(\gamma)(t) = \gamma(mt)$, for all $\gamma \in \Lambda, t \in S^1$, as well as $\gamma^m = \phi_m(\gamma)$. If $\gamma \in \Lambda$ is not constant then the multiplicity $m(\gamma)$ of $\gamma$ is
the order of the isotropy group \( \{ s \in S^1 \mid s \cdot \gamma = \gamma \} \). For a closed geodesic \( c \), the mean index \( \hat{i}(c) \) is defined as usual by \( \hat{i}(c) = \lim_{m \to \infty} i(e^m)/m \). Using singular homology with rational coefficients we consider the following critical \( \mathbb{Q} \)-module of a closed geodesic \( c \in \Lambda \):

\[
\overline{C}_* (E, c) = H_* \left( (\Lambda(c) \cup S^1 \cdot c)/S^1, \Lambda(c)/S^1; \mathbb{Q} \right).
\]

In the following we let \( M = \mathbb{R}P^n \), where \( n \geq 2 \), it is well known that \( \pi_1(\mathbb{R}P^n) = \mathbb{Z}_2 = \{ e, g \} \) with \( e \) being the identity and \( g \) being the generator of \( \mathbb{Z}_2 \) satisfying \( g^2 = e \). Then the free loop space \( \Lambda M \) possesses a natural decomposition

\[
\Lambda M = \Lambda_e M \sqcup \Lambda_g M,
\]

where \( \Lambda_e M \) and \( \Lambda_g M \) are the two connected components of \( \Lambda M \) whose elements are homotopic to \( e \) and \( g \) respectively. We set \( \Lambda_g(c) = \{ \gamma \in \Lambda_g M \mid E(\gamma) < E(c) \} \). Note that for a non-contractible prime closed geodesic \( c \), \( e^m \in \Lambda_g M \) if and only if \( m \) is odd.

We call a non-contractible prime closed geodesic satisfying the isolation condition, if the following holds:

**(Iso)** For all \( m \in \mathbb{N} \) the orbit \( S^1 \cdot e^{2m-1} \) is an isolated critical orbit of \( E \).

Note that if the number of non-contractible prime closed geodesics on \( M = \mathbb{R}P^n \) is finite, then all the non-contractible prime closed geodesics satisfy (Iso).

If a non-contractible closed geodesic \( c \) has multiplicity \( 2m-1 \), then the subgroup \( \mathbb{Z}_{2m-1} = \{ \frac{l}{2m-1} \mid 0 \leq l < 2m-1 \} \) of \( S^1 \) acts on \( \overline{C}_* (E, c) \). As studied in p.59 of [34], for all \( m \in \mathbb{N} \), let \( H_*(X, A)_{\mathbb{Z}_{2m-1}} = \{ [\xi] \in H_*(X, A) \mid T_* [\xi] = \pm [\xi] \} \), where \( T \) is a generator of the \( \mathbb{Z}_{2m-1} \)-action. On \( S^1 \)-critical modules of \( e^{2m-1} \), the following lemma holds:

**Lemma 2.1** (cf. Satz 6.11 of [34] and [7]) Suppose \( c \) is a non-contractible prime closed geodesic on a Finsler manifold \( M = \mathbb{R}P^n \) satisfying (Iso). Then there exist \( U_{c^{2m-1}} \) and \( N_{c^{2m-1}} \), the so-called local negative disk and the local characteristic manifold at \( e^{2m-1} \) respectively, such that \( \nu(e^{2m-1}) = \dim N_{c^{2m-1}} \) and

\[
\overline{C}_q (E, e^{2m-1}) = H_q \left( (\Lambda_g(e^{2m-1}) \cup S^1 \cdot e^{2m-1})/S^1, \Lambda_g(e^{2m-1})/S^1 \right)
\]

\[
= \left( H_{i(q^{2m-1})}(U_{c^{2m-1}}^{-} \cup \{ e^{2m-1} \}, U_{c^{2m-1}}^{-}) \otimes H_{q-i(e^{2m-1})}(N_{c^{2m-1}}^{-} \cup \{ e^{2m-1} \}, N_{c^{2m-1}}^{-}) \right)^{\mathbb{Z}_{2m-1}},
\]

where \( U_{c^{2m-1}}^{-} = U_{c^{2m-1}} \cap \Lambda_g(e^{2m-1}) \) and \( N_{c^{2m-1}}^{-} = N_{c^{2m-1}} \cap \Lambda_g(e^{2m-1}) \).

(i) When \( \nu(e^{2m-1}) = 0 \), there holds

\[
\overline{C}_q (E, e^{2m-1}) = \begin{cases} \mathbb{Q}, & \text{if } q = i(e^{2m-1}), \\ 0, & \text{otherwise}, \end{cases}
\]
(ii) When $\nu(c^{2m-1}) > 0$, there holds
\[
\overline{C}_q(E, c^{2m-1}) = H_{q-i(c^{2m-1})}(N_{c^{2m-1}}^- \cup \{c^{2m-1}\}, N_{c^{2m-1}}^- + \mathbb{Z}_{2m-1}),
\]
where we have used the fact $i(c^{2m-1}) - i(c) \in 2\mathbb{Z}$.

As usual, for $m \in \mathbb{N}$ and $l \in \mathbb{Z}$ we define the local homological type numbers of $c^{2m-1}$ by
\[
k_l(c^{2m-1}) = \dim H_l(N_{c^{2m-1}}^- \cup \{c^{2m-1}\}, N_{c^{2m-1}}^- + \mathbb{Z}_{2m-1}). \tag{2.4}
\]

Based on works of Rademacher in [33], Long and Duan in [30] and [11], we define the analytical period $n_c$ of the closed geodesic $c$ by
\[
n_c = \min \{ j \in 2\mathbb{N} | \nu(c^j) = \max_{m \geq 1} \nu(c^m), \ \forall m \in 2\mathbb{N} - 1 \}. \tag{2.5}
\]

Note that here in order to simplify the study for non-contractible closed geodesics in $\mathbb{R}P^n$, we have slightly modified the definition in [30] and [11] by requiring the analytical period to be even. Then by the same proofs in [30] and [11], we have
\[
k_l(c^{2m-1+hn_c}) = k_l(c^{2m-1}), \quad \forall m, h \in \mathbb{N}, l \in \mathbb{Z}. \tag{2.6}
\]

For more detailed properties of the analytical period $n_c$ of a closed geodesic $c$, we refer readers to the two Section 3s in [30] and [11].

As in [6], we have

**Definition 2.1** Let $(M, F)$ be a compact Finsler manifold. A closed geodesic $c$ on $M$ is homologically visible, if there exists an integer $k \in \mathbb{Z}$ such that $\overline{C}_k(E, c) \neq 0$. We denote by $\text{CG}_{hv}(M, F)$ the set of all distinct homologically visible prime closed geodesics on $(M, F)$.

**Lemma 2.2** Suppose the Finsler manifold $M = (\mathbb{R}P^n, F)$ possesses only finitely many distinct non-contractible prime closed geodesics, among which we denote the distinct non-contractible homologically visible prime closed geodesics by $c_1, \ldots, c_r$ for some integer $r > 0$. Then we have
\[
\hat{i}(c_i) > 0, \quad \forall 1 \leq i \leq r. \tag{2.7}
\]

**Proof:** First, we claim that Theorem 3 in [6] for $M = \mathbb{R}P^n$ can be stated as:

“Let $c$ be a closed geodesic in $\Lambda_2 M$ such that $i(c^m) = 0$ for all $m \in \mathbb{N}$. Suppose $c$ is neither homologically invisible nor an absolute minimum of $E$ in $\Lambda_2 M$. Then there exist infinitely many closed geodesics in $\Lambda_2 M$.”
Indeed, one can focus the proofs of Theorem 3 in [6] on $\Lambda_g M$ with some obvious modifications. Assume by contradiction. Similarly as in [6], we can choose a different $c \in \Lambda_g M$, if necessary, and find $p \in \mathbb{N}$ such that $H_p(\Lambda_g(c) \cup S \cdot c, \Lambda_g(c)) \neq 0$ and $H_q(\Lambda_g(c) \cup S \cdot c, \Lambda_g(c)) = 0$ for every $q > p$ and every closed geodesic $d \in \Lambda_g M$ with $i(d^m) \equiv 0$.

Consider the following commutative diagram

$$
\begin{array}{ccc}
H_p(\Lambda_g(c) \cup S \cdot c, \Lambda_g(c)) & \xrightarrow{\psi^m} & H_p(\Lambda_g(c^m) \cup S \cdot c^m, \Lambda_g(c^m)) \\
\downarrow i_* & & \downarrow i_* \\
H_p(\Lambda_g M, \Lambda_g(c)) & \xrightarrow{\psi^m} & H_p(\Lambda_g M, \Lambda_g(c^m)),
\end{array}
$$

where $m$ is odd and $\psi^m : \Lambda_g M \to \Lambda_g M$ is the $m$-fold iteration map. By similar arguments as those in [6], there is $A > 0$ such that the map $i_* \circ \psi^m$ is one-to-one, if $E(c^m) > A$ and none of the $k_i \in K_0$ divides $m$ where

$$
K_0 = \{k_0, k_1, k_2, \ldots, k_s\},
$$

with $k_0 = 2$ and $k_1, k_2, \ldots, k_s$ therein. Here note that the required $m$ is odd and so $c^m \in \Lambda_g(M)$ for $c \in \Lambda_g M$.

On the other hand, we define

$$
K = \{m \geq 2 \mid E(c^m) \leq A\} \cup K_0.
$$

Then by Corollary 1 of [6], there exists $\tilde{m} \in \mathbb{N}\setminus\{1\}$ such that no $k \in K$ divides $\tilde{m}$ and $\psi^m \circ i_*$ vanishes. In particular, $E(c^m) > A$ and none of the $k_i \in K_0$ divides $\tilde{m}$. Due to $\psi^m \circ i_* = i_* \circ \psi^m$ in (2.8), this yields a contradiction. Hence there exist infinitely many closed geodesics in $\Lambda_g M$.

Accordingly, Corollary 2 in [6] for $M = \mathbb{R}^P$ can be stated as:

“Suppose there exists a closed geodesic $c \in \Lambda_g M$ such that $c^m$ is a local minimum of $E$ in $\Lambda_g M$ for infinitely many odd $m \in \mathbb{N}$. Then there exist infinitely many closed geodesics in $\Lambda_g M$.”

Based on the above two variants of Theorem 3 and Corollary 2 in [6], we can prove our Lemma 2.2 as follows.

It is well known that every closed geodesic $c$ on $M$ must have mean index $\hat{i}(c) \geq 0$.

Assume by contradiction that there is a non-contractible homologically visible prime closed geodesic $c$ on $M$ satisfying $\hat{i}(c) = 0$. Then $i(c^m) = 0$ for all $m \in \mathbb{N}$ by Bott iteration formula and $c$ must be an absolute minimum of $E$ in $\Lambda_g M$, since otherwise there would exist infinitely many distinct non-contractible closed geodesics on $M$ by the above variant of Theorem 3 on p.385 of [6].

On the other hand, by Lemma 7.1 of [34], there exists a $k(c) \in 2\mathbb{N}$ such that $\nu(c^{m+k(c)}) = \nu(c^m)$ for all $m \in \mathbb{N}$. Specially we obtain $\nu(c^{mk(c)+1}) = \nu(c)$ for all $m \in \mathbb{N}$ and then elements of
ker\(^{(E''(c^{mk(c)+1}))}\) are precisely \(mk(c)\) 1st iterates of elements of ker\((E''(c))\). Thus by the Gromoll-Meyer theorem in [16], the behavior of the restriction of \(E\) to ker\((E''(c^{mk(c)+1}))\) is the same as that of the restriction of \(E\) to ker\((E''(c))\). Then together with the fact \(i(c^m) = 0\) for all \(m \in \mathbb{N}\), we obtain that \(c^{mk(c)+1}\) is a local minimum of \(E\) in \(\Lambda_gM\) for every \(m \in \mathbb{N}\). Because \(M\) is compact and possessing finite fundamental group, there must exist infinitely many distinct non-contractible closed geodesics on \(M\) by the above variant of Corollary 2 on p.386 of [6]. Then it yields a contradiction and proves (2.7). \(\square\)

In [39], Taimanov calculated the rational equivariant cohomology of the spaces of non-contractible loops of \(\mathbb{R}P^n\) which is crucial for us to prove Theorem 1.1 and can be stated as follows.

**Lemma 2.3** (cf. Theorem 3 of [39] or Lemma 2.2 of [25]) For \(M = \mathbb{R}P^n\), we have

(i) When \(n = 2k + 1\) is odd, the \(S^1\)-cohomology ring of \(\Lambda_gM\) has the form

\[
H^{S^1,*}(\Lambda_gM; \mathbb{Q}) = \mathbb{Q}[w, z]/\{w^{k+1} = 0\}, \quad \deg(w) = 2, \quad \deg(z) = 2k
\]

Then the \(S^1\)-equivariant Poincaré series of \(\Lambda_gM\) is given by

\[
P^{S^1}(\Lambda_gM; \mathbb{Q})(t) = \frac{1 - t^{2k+2}}{(1 - t^2)(1 - t^{2k})} = \frac{1}{1 - t^2} + \frac{t^{2k}}{1 - t^{2k}} = (1 + t^2 + t^4 + \ldots + t^{2k} + \ldots) + (t^{2k} + t^{4k} + \ldots),
\]

which yields Betti numbers

\[
\tilde{\beta}_q = \text{rank}H_q^{S^1}(\Lambda_gM; \mathbb{Q}) = \begin{cases} 
2, & \text{if } q \in \{j(n-1) \mid j \in \mathbb{N}\}, \\
1, & \text{if } q \in (2\mathbb{N} \cup \{0\}) \setminus \{j(n-1) \mid j \in \mathbb{N}\}, \\
0, & \text{otherwise}.
\end{cases}
\]  

(2.9)

and the average \(S^1\)-equivariant Betti number of \(\Lambda_gM\) satisfies

\[
\bar{B}(\Lambda_gM; \mathbb{Q}) = \lim_{q \to +\infty} \frac{1}{q} \sum_{k=0}^{q} (-1)^k \tilde{\beta}_k = \frac{n + 1}{2(n-1)}.
\]  

(2.10)

(ii) When \(n = 2k\) is even, the \(S^1\)-cohomology ring of \(\Lambda_gM\) has the form

\[
H^{S^1,*}(\Lambda_gM; \mathbb{Q}) = \mathbb{Q}[w, z]/\{w^{2k} = 0\}, \quad \deg(w) = 2, \quad \deg(z) = 4k - 2
\]

Then the \(S^1\)-equivariant Poincaré series of \(\Lambda_gM\) is given by

\[
P^{S^1}(\Lambda_gM; \mathbb{Q})(t) = \frac{1 - t^{4k}}{(1 - t^2)(1 - t^{4k-2})} = \frac{1}{1 - t^2} + \frac{t^{4k-2}}{1 - t^{4k-2}} = (1 + t^2 + t^4 + \ldots + t^{2k} + \ldots) + (t^{4k-2} + t^{2(4k-2)} + \ldots),
\]
which yields Betti numbers

\[ \tilde{\beta}_q = \text{rank} H^S_q(\Lambda g M; \mathbb{Q}) = \begin{cases} 
2, & \text{if } q \in \{2j(n - 1) \mid j \in \mathbb{N}\}, \\
1, & \text{if } q \in (2\mathbb{N} \cup \{0\}) \setminus \{2j(n - 1) \mid j \in \mathbb{N}\}, \\
0, & \text{otherwise}.
\] (2.11)

and the average \( S^1 \)-equivariant Betti number of \( \Lambda g M \) satisfies

\[ \bar{\beta}(\Lambda g M; \mathbb{Q}) \equiv \lim_{q \to +\infty} \frac{1}{q} \sum_{k=0}^{\infty} (-1)^k \tilde{\beta}_k = \frac{n}{2(n - 1)}. \] (2.12)

Remark 2.1 For the case of \( \mathbb{R}P^{2n+1} \), the same conclusions as (2.9) and (2.10) were obtained in [42] where the coefficient field \( \mathbb{Z}_2 \) was used and they are also effective to our problem since the multiplicity of every curve on \( \Lambda g M \) is odd.

Now we give the proof of the resonance identity in Theorem 1.1.

Proof of Theorem 1.1. Recall that we denote the non-contractible homologically visible prime closed geodesics by \( \text{CG}_{hv}(M) = \{c_1, \ldots, c_r\} \) for some integer \( r > 0 \) when the number of distinct non-contractible prime closed geodesics on \( M = \mathbb{R}P^n \) is finite. Note also that by Lemma 2.2 we have \( \hat{i}(c_j) > 0 \) for all \( 1 \leq j \leq r \).

Let

\[ m_q \equiv M_q(\Lambda g M) = \sum_{1 \leq j \leq r, \, m \geq 1} \text{dim} \overline{C}_q(E, c_j^{2m-1}), \quad q \in \mathbb{Z}. \]

The Morse series of \( \Lambda g M \) is defined by

\[ M(t) = \sum_{h=0}^{+\infty} m_h t^h. \] (2.13)

Claim 1. \( \{m_h\} \) is a bounded sequence.

In fact, by (2.6), we have

\[ m_h = \sum_{j=1}^{r} \sum_{m=1}^{n_j/2} \sum_{l=0}^{2n-2} k_l(c_j^{2m-1}) \# \left\{ s \in \mathbb{N} \cup \{0\} \mid h - \hat{i}(c_j^{2m-1+sn_j}) = l \right\}, \] (2.14)

and by Theorem 9.2.1, Theorems 10.1.2 of [27], and Lemmas 3.1-3.2 below, we have \( |\hat{i}(c_j^{2m-1+sn_j}) - \)

\[ \]
(2m − 1 + sn_j)i(c_j) ≤ 2n − 2, then

\[
\# \left\{ s \in \mathbb{N} \cup \{0\} \mid h - i(c_j^{2m-1+sn_j}) = l \right\} \\
= \# \left\{ s \in \mathbb{N} \cup \{0\} \mid l + i(c_j^{2m-1+sn_j}) = h, \ |i(c_j^{2m-1+sn_j}) - (2m - 1 + sn_j)i(c_j)| \leq 2n - 2 \right\} \\
\leq \# \left\{ s \in \mathbb{N} \cup \{0\} \mid 2n - 2 \geq |h - l - (2m - 1 + sn_j)i(c_j)| \right\} \\
= \# \left\{ s \in \mathbb{N} \cup \{0\} \mid \frac{h - l - 2n + 2 - (2m - 1)i(c_j)}{n_ji(c_j)} \leq s \leq \frac{h - l + 2n - 2 - (2m - 1)i(c_j)}{n_ji(c_j)} \right\} \\
\leq \frac{4n - 4}{n_ji(c_j)} + 1.
\]

Hence Claim 1 follows by (2.14) and (2.15).

We now use the method in the proof of Theorem 5.4 of [31] to estimate

\[ M^q(-1) = \sum_{h=0}^{q} m_h(-1)^h. \]

By (2.13) and (2.6) we obtain

\[ M^q(-1) = \sum_{h=0}^{q} m_h(-1)^h \\
= \sum_{j=1}^{r} \sum_{m=1}^{n_j/2} \sum_{l=0}^{2n-2} \sum_{h=0}^{q} (-1)^h k_l(c_j^{2m-1}) \# \left\{ s \in \mathbb{N} \cup \{0\} \mid h - i(c_j^{2m-1+sn_j}) = l \right\} \\
= \sum_{j=1}^{r} \sum_{m=1}^{n_j/2} \sum_{l=0}^{2n-2} (-1)^{l+i(c_j)} k_l(c_j^{2m-1}) \# \left\{ s \in \mathbb{N} \cup \{0\} \mid l + i(c_j^{2m-1+sn_j}) \leq q \right\}. \]

On the one hand, we have

\[
\# \left\{ s \in \mathbb{N} \cup \{0\} \mid l + i(c_j^{2m-1+sn_j}) \leq q \right\} \\
= \# \left\{ s \in \mathbb{N} \cup \{0\} \mid l + i(c_j^{2m-1+sn_j}) \leq q, \ |i(c_j^{2m-1+sn_j}) - (2m - 1 + sn_j)i(c_j)| \leq 2n - 2 \right\} \\
\leq \# \left\{ s \in \mathbb{N} \cup \{0\} \mid 0 \leq (2m - 1 + sn_j)i(c_j) \leq q - l + 2n - 2 \right\} \\
= \# \left\{ s \in \mathbb{N} \cup \{0\} \mid 0 \leq s \leq \frac{q - l + 2n - 2 - (2m - 1)i(c_j)}{n_ji(c_j)} \right\} \\
\leq \frac{q - l + 2n - 2}{n_ji(c_j)} + 1.
\]
On the other hand, we have

\[
\# \left\{ s \in \mathbb{N} \cup \{0\} \mid l + i(c_j)^{2m-1+sn_j} \leq q \right\} \\
= \# \left\{ s \in \mathbb{N} \cup \{0\} \mid l + i(c_j)^{2m-1+sn_j} \leq q, \ |i(c_j)^{2m-1+sn_j} - (2m - 1 + sn_j)i(c_j)| \leq 2n - 2 \right\} \\
\geq \# \left\{ s \in \mathbb{N} \cup \{0\} \mid i(c_j)^{2m-1+sn_j} \leq (2m - 1 + sn_j)i(c_j) + 2n - 2 \leq q - l \right\} \\
\geq \# \left\{ s \in \mathbb{N} \cup \{0\} \mid 0 \leq s \leq \frac{q - l - 2n + 2 - (2m - 1)i(c_j)}{n_ji(c_j)} \right\} \\
\geq \frac{q - l - 2n + 2}{n_ji(c_j)} - 1.
\]

Thus we obtain

\[
\lim_{q \to +\infty} \frac{1}{q} M^q(-1) = \sum_{j=1}^{r} \sum_{m=1}^{n_j/2} \sum_{l=0}^{2n-2} (-1)^{l+i(c_j)} k_l(c_j)^{2m-1} \frac{1}{n_ji(c_j)} = \sum_{j=1}^{r} \hat{\chi}(c_j) \frac{1}{i(c_j)}.
\]

Since \(m_h\) is bounded, we then obtain

\[
\lim_{q \to +\infty} \frac{1}{q} M^q(-1) = \lim_{q \to +\infty} \frac{1}{q} P_{\mathbb{A}_g(q)}^q(\Lambda_g M; \mathbb{Q})(-1) = \lim_{q \to +\infty} \frac{1}{q} \sum_{k=0}^{q} (-1)^k \beta_k = \bar{B}(\Lambda_g M; \mathbb{Q}),
\]

where \(P_{\mathbb{A}_g(q)}^q(\Lambda_g M; \mathbb{Q})(t)\) is the truncated polynomial of \(P_{\mathbb{A}_g(q)}^q(\Lambda_g M; \mathbb{Q})(t)\) with terms of degree less than or equal to \(q\). Thus by (2.10) and (2.12) we get

\[
\sum_{j=1}^{r} \hat{\chi}(c_j) \frac{1}{i(c_j)} = \begin{cases} 
\frac{n+1}{2(n-1)}, & \text{if } n \in 2\mathbb{N} - 1, \\
\frac{n}{2(n-1)}, & \text{if } n \in 2\mathbb{N}.
\end{cases}
\]

which proves (1.4) of Theorem 1.1. For the special case when each \(c_j^{2m-1}\) is non-degenerate with \(1 \leq j \leq r\) and \(m \in \mathbb{N}\), we have \(n_j = 2\) and \(k_l(c_j) = 1\) when \(l = 0\), and \(k_l(c_j) = 0\) for all other \(l \in \mathbb{Z}\).

Then (1.4) has the following simple form

\[
\sum_{j=1}^{r} (-1)^{i(c_j)} \frac{1}{i(c_j)} = \begin{cases} 
\frac{n+1}{n-1}, & \text{if } n \in 2\mathbb{N} - 1, \\
\frac{n}{n-1}, & \text{if } n \in 2\mathbb{N}.
\end{cases}
\]

which proves (1.5) of Theorem 1.1.

\[\square\]

3 Index iteration theory for closed geodesics

3.1 Index iteration formulae for closed geodesics

In [26] of 1999, Y. Long established the basic normal form decomposition of symplectic matrices. Based on it, he further established the precise iteration formulae of Maslov \(\omega\)-indices for symplectic
paths in [27], which can be related to Morse indices of either orientable or non-orientable closed geodesics in a slightly different way (cf. [23] and Chap. 12 of [28]). Roughly speaking, the orientable (resp. non-orientable) case corresponds to $i_1$ (resp. $i_{-1}$) index, where $i_1$ and $i_{-1}$ denote the cases of $\omega$-index with $\omega = 1$ and $\omega = -1$ respectively (cf. Chap. 5 of [28]). Although we are concerned with $\mathbb{R}^n$ in this paper, we will state such a relation precisely in a general form due to its independent interest. Throughout this section we denote the Morse index of a closed geodesic $c$ by $\text{ind}(c)$ in stead of $i(c)$ to avoid confusion of notations and write $i_1(\gamma)$ as $i(\gamma)$ for short.

For the reader’s convenience, we briefly review some basic materials in Long’s book [28].

Let $P$ be a symplectic matrix in $\text{Sp}(2N - 2)$ and $\Omega^0(P)$ be the path connected component of its homotopy set $\Omega(P)$ which contains $P$. Then there is a path $f \in C([0, 1], \Omega^0(P))$ such that $f(0) = P$ and

$$f(1) = N_1(1, 1)^{\omega p} \circ I_{2p_0} \circ N_1(1, -1)^{\omega p}$$

$$\circ N_1(-1, 1)^{\omega q} \circ (-I_{2q_0}) \circ N_1(-1, -1)^{\omega q}$$

$$\circ R(\theta_1) \circ \cdots \circ R(\theta_{r'}) \circ R(\theta_{r'+1}) \circ \cdots \circ R(\theta_r)$$

$$\circ N_2(e^{i\alpha_1}, A_1) \circ \cdots \circ N_2(e^{i\alpha_{r'}}, A_{r'})$$

$$\circ N_2(e^{i\beta_1}, B_1) \circ \cdots \circ N_2(e^{i\beta_{r'}}, B_{r'})$$

$$\circ H(\pm 2)^{\omega h},$$

where $N_1(\lambda, \chi) = \begin{pmatrix} \lambda & \chi \\ 0 & \lambda \end{pmatrix}$ with $\lambda = \pm 1$ and $\chi = 0, \pm 1$; $H(b) = \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}$ with $b = \pm 2$;

$$R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$ with $\theta \in (0, 2\pi) \setminus \{\pi\}$ and we suppose that $\pi < \theta_j < 2\pi$ iff $1 \leq j \leq r'$;

$$N_2(e^{i\alpha_j}, A_j) = \begin{pmatrix} R(\alpha_j) & A_j \\ 0 & R(\alpha_j) \end{pmatrix}$$ and $N_2(e^{i\beta_j}, B_j) = \begin{pmatrix} R(\beta_j) & B_j \\ 0 & R(\beta_j) \end{pmatrix}$

with $\alpha_j, \beta_j \in (0, 2\pi) \setminus \{\pi\}$ are non-trivial and trivial basic normal forms respectively.

Let $\gamma_0$ and $\gamma_1$ be two symplectic paths in $\text{Sp}(2N - 2)$ connecting the identity matrix $I$ to $P$ and $f(1)$ satisfying $\gamma_0 \sim_\omega \gamma_1$. Then it has been shown that $i_\omega(\gamma_0^m) = i_\omega(\gamma_1^m)$ for any $\omega \in S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Based on this fact, we always assume without loss of generality that each $P_c$ appearing in the sequel has the form (3.1).

**Lemma 3.1** (cf. Theorem 8.3.1 and Chap. 12 of [28]) Let $c$ be an orientable closed geodesic on an $N$-dimensional Finsler manifold with its Poincaré map $P_c$. Then, there exists a continuous
symplectic path $\gamma$ with $\gamma(0) = I$ and $\gamma(1) = P_c$ such that

$$\text{ind}(c^m) = i(\gamma^m) = m(i(\gamma) + p_+ + p_0 - r) - (p_+ + p_0 + r) - \frac{1 + (-1)^m}{2}(q_0 + q_+),$$

$$+ 2 \sum_{j=1}^{r} E\left(\frac{m\theta_j}{2\pi}\right) + 2 \sum_{j=1}^{r*} \varphi\left(\frac{ma_j}{2\pi}\right) - 2\tau, \quad (3.2)$$

and

$$\text{null}(c^m) = \nu(\gamma^m) = \nu(\gamma) + \frac{1 + (-1)^m}{2}(q_+ + 2q_0 + q_+) + 2\varsigma(c, m), \quad (3.3)$$

where we denote by

$$\varsigma(c, m) = \left(\frac{1}{r} - \sum_{j=1}^{r} \varphi\left(\frac{m\theta_j}{2\pi}\right)\right) + \left(\frac{1}{r*} - \sum_{j=1}^{r*} \varphi\left(\frac{ma_j}{2\pi}\right)\right) + \left(\frac{1}{r_0} - \sum_{j=1}^{r_0} \varphi\left(\frac{m\beta_j}{2\pi}\right)\right).$$

From now on, we focus on the non-orientable case.

**Lemma 3.2** Let $c$ be a non-orientable closed geodesic on a $d$-dimensional Finsler manifold with its linear Poincaré map $P_c$. Then, the following two claims hold.

(i) If $d$ is even, there is a symplectic path $\gamma$ in $Sp(2d-2)$ with $\gamma(0) = I$ and $\gamma(1) = P_c$ satisfying

$$\text{ind}(c^m, \text{null}(c^m)) = \begin{cases} (i_{-1}(\gamma^m), \nu_{-1}(\gamma^m)), & \text{if } m \text{ is odd,} \\ (i(\gamma^m), \nu(\gamma^m)), & \text{if } m \text{ is even.} \end{cases}$$

(ii) If $d$ is odd, there is a symplectic path $\tilde{\gamma}$ in $Sp(2d)$ with $\tilde{\gamma}(0) = I$ and $\tilde{\gamma}(1) = N_1(1, 1) \circ P_c$ satisfying

$$\text{ind}(c^m, \text{null}(c^m)) = \begin{cases} (i_{-1}(\tilde{\gamma}^m), \nu_{-1}(\tilde{\gamma}^m) - 1), & \text{if } m \text{ is odd,} \\ (i(\tilde{\gamma}^m), \nu(\tilde{\gamma}^m) - 1), & \text{if } m \text{ is even.} \end{cases}$$

**Proof:** For the case of $m = 1$, such a conclusion has been obtained by Theorem 1.1 of [23]. Based on it, Lemma 3.2 is a direct application of Bott formulae (cf. Theorem 9.2.1 in [28]). □

For any $m \in \mathbb{N}$, we define

$$E_m(a) = E\left(a - \frac{1 - (-1)^m}{4}\right), \quad \varphi_m(a) = \varphi\left(a - \frac{1 - (-1)^m}{4}\right), \quad \forall a \in \mathbb{R}.$$ 

By Lemmas 3.1 and 3.2, we now derive the precise iteration formulae of Morse indices for a non-orientable closed geodesic on a Finsler manifold.
Theorem 3.1 Let \( c \) be a non-orientable closed geodesic on a \( d \)-dimensional Finsler Manifold \( M \) with its linear Poincaré map \( P_c \). Then for every \( m \in \mathbb{N} \), we have

\[
\text{ind}(c^m) = m(\text{ind}(c) + q_0 + q_+ - 2r') - (q_0 + q_+) - \frac{1 + (-1)^m}{2} \left( r + p_+ + p_0 + \frac{1 - (-1)^d}{2} \right) \\
+ 2 \sum_{j=1}^{r} E_j \left( \frac{m \theta_j}{2\pi} \right) + 2 \sum_{j=1}^{r_*} \varphi_{m} \left( \frac{m \alpha_j}{2\pi} \right) - 2r_* ,
\]

and

\[
\text{null}(c^m) = \text{null}(c) + \frac{1 + (-1)^m}{2} \left( p_+ + 2p_0 + p_+ + \frac{1 - (-1)^d}{2} \right) + 2 \tilde{\varsigma}(c, m),
\]

where we denote by

\[
\tilde{\varsigma}(c, m) = \left( r - \sum_{j=1}^{r} \varphi_{m} \left( \frac{m \theta_j}{2\pi} \right) \right) + \left( r_* - \sum_{j=1}^{r_*} \varphi_{m} \left( \frac{m \alpha_j}{2\pi} \right) \right) + \left( r_0 - \sum_{j=1}^{r_0} \varphi_{m} \left( \frac{m \beta_j}{2\pi} \right) \right).
\]

**Proof:** We only prove the case when \( d \) is even and \( m \) is odd, since it is just the case we encounter in this paper and the other cases can be proved similarly. By Lemma 3.2, there exists a symplectic path \( \gamma \) in \( \text{Sp}(2d-2) \) with \( \gamma(0) = I \) and \( \gamma(1) = P_c \) such that

\[
(\text{ind}(c^m), \text{null}(c^m)) = (i_{-1}(\gamma^m), \nu_{-1}(\gamma^m)), \forall \ m \in 2\mathbb{N} - 1.
\]

It together with the Bott-type formulae (cf. Theorem 9.2.1 of [28]) and Lemma 3.1 gives

\[
i_{-1}(\gamma^m) = i(\gamma^{2m}) - i(\gamma^m) \\
= m(i(\gamma) + p_+ + p_0 - r) - (q_0 + q_+) \\
+ 2 \sum_{j=1}^{r} E \left( \frac{m \theta_j}{\pi} \right) - E \left( \frac{m \theta_j}{2\pi} \right) + 2 \sum_{j=1}^{r_*} \varphi \left( \frac{m \alpha_j}{2\pi} \right) - \varphi \left( \frac{m \alpha_j}{2\pi} \right) \\
= m(i(\gamma) + p_+ + p_0 - r) - (q_0 + q_+) \\
+ 2 \sum_{j=1}^{r} E \left( \frac{m \theta_j}{2\pi} - \frac{1}{2} \right) + 2 \sum_{j=1}^{r_*} \varphi \left( \frac{m \alpha_j}{2\pi} - \frac{1}{2} \right) - 2r_* ,
\]

\[
i(\gamma) = i_{-1}(\gamma) + (q_0 + q_+) + (r - 2r') - (p_0 + p_-),
\]

which is a result of direct computation on splitting numbers based on Theorem 12.2.3 of [28].
Observing by definition \( \nu^{-1}(\gamma) = q_- + 2q_0 + q_+ \), we obtain similarly as above that

\[
\nu^{-1}(\gamma^m) = \nu(\gamma^{2m}) - \nu(\gamma^m) \\
= (q_- + 2q_0 + q_+) - 2 \sum_{j=1}^{r} \left[ \varphi \left( \frac{m\theta_j}{\pi} \right) - \varphi \left( \frac{m\theta_j}{2\pi} \right) \right] \\
- 2 \sum_{j=1}^{r_*} \left[ \varphi \left( \frac{m\alpha_j}{\pi} \right) - \varphi \left( \frac{m\alpha_j}{2\pi} \right) \right] - 2 \sum_{j=1}^{r_0} \left[ \varphi \left( \frac{m\beta_j}{\pi} \right) - \varphi \left( \frac{m\beta_j}{2\pi} \right) \right] \\
= \nu^{-1}(\gamma) + 2 \left( r - \sum_{j=1}^{r} \varphi \left( \frac{m\theta_j}{2\pi} - \frac{1}{2} \right) \right) \\
+ 2 \left( r_* - \sum_{j=1}^{r_*} \varphi \left( \frac{m\alpha_j}{2\pi} - \frac{1}{2} \right) \right) + 2 \left( r_0 - \sum_{j=1}^{r_0} \varphi \left( \frac{m\beta_j}{2\pi} - \frac{1}{2} \right) \right). \tag{3.8}
\]

Thus (3.4) and (3.5) immediately follow from (3.6), (3.7) and (3.8). \( \square \)

### 3.2 A variant of Precise index iteration formulae

In this section, we give a variant of the precise index iteration formulae in section 3.1 which makes them more intuitive and enables us to apply the Kronecker’s approximation theorem to study the multiplicity of non-contractible closed geodesics on \( \mathbb{R}P^n \).

To prove Theorem 1.2, we always assume that there exists only one non-contractible prime closed geodesic \( c \) on \( M = \mathbb{R}P^n \) with a bumpy metric \( F \), which is then just the well known minimal point of the energy functional \( E \) on \( \Lambda_g M \) satisfying \( \text{ind}(c) = 0 \). Now the Morse-type number is given by

\[
m_q = M_q(\Lambda_g M) = \sum_{m \geq 1} \dim C_q(E, c^{2m-1}), \quad \forall q \in \mathbb{N} \cup \{0\}.
\]

Then by Lemma 2.1(i), Lemma 2.3 and Morse inequality, we have the following conclusion.

**Lemma 3.3** (cf. Lemma 3.1 of [14]) *Assuming the existence of only one non-contractible prime closed geodesic \( c \) on \( \mathbb{R}P^n \) with a bumpy metric \( F \), there hold

\[
m_{2q+1} = \tilde{\beta}_{2q+1} = 0 \quad \text{and} \quad m_{2q} = \tilde{\beta}_{2q}, \quad \forall \ q \in \mathbb{N} \cup \{0\}. \tag{3.9}
\]

We consider two cases according to the parity of dimension of the real projective space. First we study the case of \( \mathbb{R}P^{2n+1} \). Note that the other one behaves similarly.
Lemma 3.4 Suppose \( c \) is the only one non-contractible prime closed geodesic \( c \) on \( (\mathbb{RP}^{2n+1}, F) \) with a bumpy metric \( F \). Then there exist \( \hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_k \) in \( \mathbb{Q}^c \) with \( 2 \leq k \leq 2n \) such that

\[
\sum_{j=1}^{k} \theta_j = \frac{1}{2} \left( k + \frac{n}{n+1} \right), \quad (3.10)
\]

\[
\text{ind}(c^m) = m \left( \frac{n}{n+1} \right) + k - 2 \sum_{j=1}^{k} \{ m\theta_j \}, \quad \forall m \geq 1. \quad (3.11)
\]

**Proof:** See (3.6), (3.7) and (3.8) in [14]. Also compare the proof of Lemma 3.6. \( \square \)

Now we give a variant of the precise index iteration formulae (3.11) specially for our purpose. Let \( m = 2(n+1)l + 2L + 1 \) with \( l \in \mathbb{N} \) and \( L \in \mathbb{Z} \). By (3.10) and (3.11) we obtain

\[
\text{ind}(c^m) = 2nl + k + (2L + 1) \frac{n}{n+1}
\]

\[
-2 \left( \left\{ \frac{k}{2} + \frac{(2L+1)n}{2(n+1)} \right\} - \sum_{j=2}^{k} \{ m\theta_j \} \right) + \sum_{j=2}^{k} \{ m\hat{\theta}_j \}
\]

\[
= 2nl + 2 \left[ \frac{k}{2} + \frac{(2L+1)n}{2(n+1)} \right] + 2 \left\{ \frac{k}{2} + \frac{(2L+1)n}{2(n+1)} \right\}
\]

\[
-2 \left( \left\{ \frac{k}{2} + \frac{(2L+1)n}{2(n+1)} \right\} - \sum_{j=2}^{k} \{ m\theta_j \} \right) + \sum_{j=2}^{k} \{ m\hat{\theta}_j \}
\]

\[
= 2nl + 2\lfloor Q_L \rfloor + 2 \{ Q_L \} - 2 \left( \left\{ Q_L \right\} - \sum_{j=2}^{k} \{ m\hat{\theta}_j \} \right) + \sum_{j=2}^{k} \{ m\hat{\theta}_j \}, \quad (3.12)
\]

where in the last identity for notational simplicity, we denote by \( Q_L = \frac{k}{2} + \frac{(2L+1)n}{2(n+1)} \).

Since \( \sum_{j=2}^{k} \{ m\theta_j \} \in \mathbb{Q}^c \), we obtain by (3.12) that for \( 1 \leq i \leq k-2 \),

\[
\text{ind}(c^m) = \begin{cases} 
2nl + 2\lfloor Q_L \rfloor, & \text{if } \sum_{j=2}^{k} \{ m\theta_j \} \in (0, \{ Q_L \}), \\
2nl + 2\lfloor Q_L \rfloor - 2i, & \text{if } \sum_{j=2}^{k} \{ m\theta_j \} \in (i - 1 + \{ Q_L \}, i + \{ Q_L \}), \\
2nl + 2\lfloor Q_L \rfloor - 2(k-1), & \text{if } \sum_{j=2}^{k} \{ m\theta_j \} \in (k - 2 + \{ Q_L \}, k-1).
\end{cases} \quad (3.13)
\]

Let \( I_0(L) = (0, \{ Q_L \}), \) \( I_{k-1}(L) = (k-2 + \{ Q_L \}, k-1) \), and

\[
I_i(L) = (i - 1 + \{ Q_L \}, i + \{ Q_L \}) \quad \text{for} \quad 1 \leq i \leq k-2.
\]

Then, (3.13) can be stated in short as that for any integers \( m = 2(n+1)l + 2L + 1 \) and \( 0 \leq i \leq k-1 \),

\[
\text{ind}(c^m) = 2nl + 2\lfloor Q_L \rfloor - 2i \quad \text{if and only if} \quad \sum_{j=2}^{k} \{ m\theta_j \} \in I_i(L). \quad (3.14)
\]

**Remark 3.1** Let \( (\tau(1), \tau(2), \ldots, \tau(k)) \) be an arbitrary permutation of \( (1, 2, \ldots, k) \). Then, the same conclusion as (3.14) with \( j \) ranging in \( \{ \tau(1), \tau(2), \ldots, \tau(k-1) \} \) instead is still valid.
The following lemma will be also needed in the proof of Theorem 1.2 for $\mathbb{R}P^{2n+1}$ in Section 5.

**Lemma 3.5** Under the assumption of Lemma 3.4, for any positive integers $l$ and $m$, we have

$$|\text{ind}(c^m) - 2nl| > 2n \quad \text{holds whenever} \quad |m - 2(n+1)l| > 4(n+1).$$

**Proof:** From (3.11), we have

$$\text{ind}(c^m) = 2nl + (m - 2(n+1)l) \cdot \frac{n}{n+1} + k - 2 \sum_{j=1}^{k} \{m\hat{\theta}_j\},$$

which yields immediately that

$$|\text{ind}(c^m) - 2nl| \geq |m - 2(n+1)l| \cdot \frac{n}{n+1} - |k - 2 \sum_{j=1}^{k} \{m\hat{\theta}_j\}|$$

$$> 4n - k \geq 4n - 2n = 2n,$$

where the fact $k \leq 2n$ is used. \hfill $\Box$

For the case of $\mathbb{R}P^{2n}$, similar to Lemma 3.4, we have

**Lemma 3.6** Suppose $c$ is the only one non-contractible prime closed geodesic $c$ on $(\mathbb{R}P^{2n}, F)$ with a bumpy metric $F$. Then, there exist $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_{2r}$ in $\mathbb{Q}^c$ with $2 \leq r \leq 2n - 1$ such that

$$\sum_{j=1}^{2r} \hat{\theta}_j = \frac{1}{2} \left( 2r + \frac{2n-1}{2n} \right), \quad (3.15)$$

$$\text{ind}(c^{2m-1}) = (2m-1) \left( \frac{2n-1}{2n} \right) + 2r - 2 \sum_{j=1}^{2r} \{ (2m-1)\hat{\theta}_j \}, \quad \forall \ m \geq 1. \quad (3.16)$$

**Proof:** Since the Finsler metric $F$ is bumpy, it follows $\text{null}(c^m) = 0$ for every $m \in \mathbb{N}$. In particular, $\text{null}(c) = \nu_{-1}(\gamma) = q_- + 2q_0 + q_+ = 0$, which implies $q_- = q_0 = q_+ = 0$. In addition by (3.5), $\text{null}(c^2) = 0$ then yields $p_- = p_0 = p_+ = 0$. As a result, we get

$$\zeta(c, m) = 0, \ \forall m \in \mathbb{N},$$

and so $\frac{\theta_1'}{2\pi}$s, $\frac{\alpha_1'}{2\pi}$s and $\frac{\beta_1'}{2\pi}$s are all in $\mathbb{Q}^c \cap (0, 1)$. 

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Due to $\text{ind}(c) = 0$, by (3.4) in Theorem 3.1 we obtain
\[
\text{ind}(c^{2m-1}) = -2(2m-1)r' + \sum_{j=1}^{r} E \left( \frac{(2m-1)\theta_j}{2\pi} - \frac{1}{2} \right)
\]
\[
= -2(2m-1)r' + 2 \sum_{j=1}^{r} \left( \left( \frac{(2m-1)\theta_j}{2\pi} - \frac{1}{2} \right) - \left( \frac{(2m-1)\theta_j}{2\pi} - \frac{1}{2} \right) + 1 \right)
\]
\[
= (2m-1) \left( -2r' + \sum_{j=1}^{r} \frac{\theta_j}{\pi} \right) - 2 \sum_{j=1}^{r} \left( \left( \frac{(2m-1)\theta_j}{2\pi} - \frac{1}{2} \right) - \frac{1}{2} \right)
\]
\[
= (2m-1) \left( -2r' + \sum_{j=1}^{r} \frac{\theta_j}{\pi} \right) + 2r - 2 \sum_{j=1}^{r} \left( \left( \frac{(2m-1)\theta_j}{\pi} \right) + \left\{ \frac{-(2m-1)\theta_j}{2\pi} \right\} \right)
\]
which implies $\hat{i}(c) = -2r' + \sum_{j=1}^{r} \frac{\theta_j}{\pi}$. It together with (1.5) of Theorem 1.1 yields
\[
\sum_{j=1}^{r} \frac{\theta_j}{2\pi} = \frac{1}{2} \left( 2r' + \frac{2n-1}{2n} \right).
\]
(3.18)

Let $\hat{\theta}_j = \frac{\theta_j}{\pi} - \left\{ \frac{\theta_j}{\pi} \right\} + 1$ for $1 \leq j \leq r$ and $\hat{\theta}_j = -\frac{\theta_j}{2\pi}$ for $r+1 \leq j \leq 2r$, then (3.15) follows from (3.18), and (3.16) follows from (3.17) and (3.18).

**Remark 3.2** If we replace $2n-1$ and $2r$ in Lemma 3.6 by $n$ and $k$ respectively, (3.15) and (3.16) are just the same form as (3.10) and (3.11) respectively. Hence (3.12)-(3.14) also hold when we replace $n$ and $k$ by $2n-1$ and $2r$ respectively.

For the case of $\mathbb{RP}^{2n}$, similar to Lemma 3.5, we have

**Lemma 3.7** Under the assumption of Lemma 3.6, for any positive integers $l$ and $m \in 2\mathbb{N} - 1$, we have
\[
|\text{ind}(c^m) - 2(2n-1)l| > 2(2n-1) \quad \text{holds whenever} \quad |m - 4nl| > 8n.
\]

### 4 The system of irrational numbers

Let $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ be a set of $m$ irrational numbers. As usual, we have

**Definition 4.1** The set $\alpha$ of irrational numbers is linearly independent over $\mathbb{Q}$, if there do not exist $c_1, c_2, \ldots, c_m$ in $\mathbb{Q}$ such that $\sum_{j=1}^{m} |c_j| > 0$ and
\[
\sum_{j=1}^{m} c_j \alpha_j \in \mathbb{Q},
\]
(4.1)
and is linearly dependent over \( \mathbb{Q} \) otherwise. The rank of \( \alpha \) is defined to be the number of elements in a maximal linearly independent subset of \( \alpha \), which we denote by \( \text{rank}(\alpha) \).

**Lemma 4.1** Let \( r = \text{rank}(\alpha) \). Then there exist \( p_{jl} \in \mathbb{Z} \), \( \beta_l \in \mathbb{Q}^c \) and \( \xi_j \in \mathbb{Q} \) for \( 1 \leq j \leq r \) and \( 1 \leq l \leq m \) such that

\[
\alpha_j = \sum_{l=1}^{r} p_{jl} \beta_l + \xi_j, \quad \forall 1 \leq j \leq m. \tag{4.2}
\]

**Proof:** Let \( \alpha' = \{\alpha_{m_1}, \alpha_{m_2}, \ldots, \alpha_{m_r}\} \) be a maximal linearly independent subset of \( \alpha \). Then there exist \( c_{jl} \in \mathbb{Q} \) and \( \xi_j \in \mathbb{Q} \) such that

\[
\alpha_j = \sum_{l=1}^{r} c_{jl} \alpha_{ml} + \xi_j, \quad \forall 1 \leq j \leq m. \tag{4.3}
\]

For every \( 1 \leq l \leq r \), we define \( J_l = \{1 \leq j \leq m \mid c_{jl} \neq 0\} \) and then for \( j \in J_l \) let \( c_{jl} = \frac{r_{jl}}{q_{jl}} \) with \( r_{jl} \) prime to \( q_{jl} \). Define \( q_l = \prod_{j \in J_l} q_{jl} \) and

\[
\beta_l = \frac{\alpha_{ml}}{q_l} \in \mathbb{Q}^c \text{ and } p_{jl} = q_l c_{jl} \in \mathbb{Z}, \quad \forall 1 \leq j \leq m \text{ and } 1 \leq l \leq r.
\]

Then, (4.2) follows. \( \square \)

In order to study the multiplicity of closed geodesics on \((\mathbb{R}P^{2n+1}, F)\) with a bumpy Finsler metric \( F \), we are particularly interested in the irrational system \( \{\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_k\} \) with rank 1 satisfying (3.10). Then by Lemma 4.1, it can be reduced to the following system

\[
\hat{\theta}_j = p_j \theta + \xi_j, \quad \forall 1 \leq j \leq k, \tag{4.4}
\]

with \( \theta \in \mathbb{Q}^c \), \( p_j \in \mathbb{Z} \setminus \{0\} \), \( \xi_j \in \mathbb{Q} \cap [0, 1) \) satisfying

\[
p_1 + p_2 + \cdots + p_k = 0, \tag{4.5}
\]

\[
\{\xi_1 + \xi_2 + \cdots + \xi_k\} \in (0, 1) \setminus \{1/2\}, \tag{4.6}
\]

where to get \( \xi_j \in [0, 1) \), if necessary, we can replace \( \hat{\theta}_j \) and \( \xi_j \) by \( \hat{\theta}_j = \hat{\theta}_j - [\xi_j] \) and \( \tilde{\xi}_j = \{\xi_j\} \).

Take arbitrarily \( \eta \in \mathbb{Q} \) and make the following natural \( \eta \)-action to the system (4.4):

\[
\eta(\theta) = \theta + \eta, \quad \eta(\hat{\theta}_j) = \hat{\theta}_j - [\xi_j - p_j \eta] \text{ and } \eta(\xi_j) = \{\xi_j - p_j \eta\}, \quad \forall 1 \leq j \leq k, \tag{4.7}
\]

which is obviously induced by the transformation \( \eta(\theta) = \theta + \eta \). Then, we get a new system

\[
\eta(\hat{\theta}_j) = p_j \eta(\theta) + \eta(\xi_j), \quad \forall 1 \leq j \leq k, \tag{4.8}
\]
with
\[
\begin{align*}
\{\eta(\xi_1) + \eta(\xi_2) + \cdots + \eta(\xi_k)\} &= \{(\xi_1 - p_1\eta) + (\xi_2 - p_2\eta) + \cdots + (\xi_k - p_k\eta)\} \\
&= \{\xi_1 + \xi_2 + \cdots + \xi_k - (p_1 + p_2 + \cdots + p_k)\eta\} \\
&= \{\xi_1 + \xi_2 + \cdots + \xi_k\},
\end{align*}
\]
where the third equality we have used the condition (4.5). For simplicity of writing, we also denote the new system (4.8) by (4.4) meaning that it comes from (4.4) by an \(\eta\)-action.

For the system (4.4) \(\eta\) with \(\eta \in \mathbb{Q}\), we divide the set \(\{1 \leq j \leq k\}\) into the following three parts:
\[
\begin{align*}
K_0^+(\eta) &= \{1 \leq j \leq k \mid \eta(\xi_j) = 0, \ p_j > 0\}, \\
K_0^-(\eta) &= \{1 \leq j \leq k \mid \eta(\xi_j) = 0, \ p_j < 0\}, \\
K_1(\eta) &= \{1 \leq j \leq k \mid \eta(\xi_j) \neq 0\}.
\end{align*}
\]
Denote by \(k_0^+(\eta)\), \(k_0^-(\eta)\) and \(k_1(\eta)\) the numbers \#\(K_0^+(\eta)\), \#\(K_0^-(\eta)\) and \#\(K_1(\eta)\) respectively. For the case of \(\eta = 0\), we write them for short as \(k_0^+, k_0^-\) and \(k_1\). It follows immediately that
\[
k_0^+(\eta) + k_0^-(\eta) + k_1(\eta) = k.
\]
By (4.6) and (4.9), it is obvious that \(k_1(\eta) \geq 1\) for every \(\eta \in \mathbb{Q}\).

**Definition 4.2** For every \(\eta \in \mathbb{Q}\), the **absolute difference number** of (4.4) \(\eta\) is defined to be the non-negative number \(|k_0^+(\eta) - k_0^-(\eta)|\). The **effective difference number** of (4.4) is defined by
\[
\max\{|k_0^+(\eta) - k_0^-(\eta)| \mid \eta \in \mathbb{Q}\}.
\]
Two systems of irrational numbers with rank 1 are called to be **equivalent**, if their effective difference numbers are the same one.

**Remark 4.1** By the definition of an \(\eta\)-action in (4.7), it can be checked directly that \(\eta_1 \circ \eta_2 = \eta_1 + \eta_2\) for every \(\eta_1\) and \(\eta_2\) in \(\mathbb{Q}\). So every system of irrational numbers with rank 1 is equivalent to the one which comes from itself by an \(\eta\)-action.

We have first the following simple equivalent pairs.

**Lemma 4.2** Assume that
\[
\begin{align*}
\hat{\theta}_j &= p_j\theta + \xi_j, \quad \forall 1 \leq j \leq k - 1, \\
\hat{\theta}_k &= p_k\theta,
\end{align*}
\]with \(\sum_{j=1}^k p_k = 0\) and \(\left\{\sum_{j=1}^{k-1} \xi_k\right\} \in (0, 1)\setminus\{1/2\}\).
Then, (4.11) is equivalent to

\[
\begin{cases}
\hat{\theta}_j = p_j \theta + \xi_j, & \forall \ 1 \leq j \leq k - 1, \\
\hat{\theta}_{k,l} = \text{sgn}(p_k) \theta + \frac{l}{|p_k|}, & \forall \ 0 \leq l \leq |p_k| - 1,
\end{cases}
\]

(4.12)

where as usual we define \(\text{sgn}(a) = \pm 1\) for \(a \in \mathbb{R} \setminus \{0\}\) when \(\pm a > 0\).

Proof: Take \(\eta \in \mathbb{Q}\) arbitrarily and recall the definition of \(\eta\)-action in (4.7). Then the equation \(\hat{\theta}_k = p_k \theta\) contributes \(\text{sgn}(p_k)\) to the absolute difference number of (4.11) \(\eta\) if and only if

\[
\eta(0) = \{0 - p_k \eta\} = \{-p_k \eta\} = 0,
\]

that is \(\eta \in \mathbb{Z}_{|p_k|}\), which is also the sufficient and necessary condition such that the equations

\[
\hat{\theta}_{k,l} = \text{sgn}(p_k) \theta + \frac{l}{|p_k|}, \quad \forall \ 0 \leq l \leq |p_k| - 1,
\]

contribute \(\text{sgn}(p_k)\) to the absolute difference number of (4.12) \(\eta\). Since the other equations with \(1 \leq j \leq k - 1\) in (4.11) and (4.12) are the same, so do their contributions to the absolute difference numbers of (4.11) \(\eta\) and (4.12) \(\eta\). As a result, the absolute difference numbers of (4.11) \(\eta\) and (4.12) \(\eta\) are equal for any \(\eta \in \mathbb{Q}\) which yields that the effective difference numbers of (4.11) and (4.12) are the same and so they are equivalent. \(\square\)

Remark 4.2 For the system (4.12), we have

\[
\left\{ \sum_{j=1}^{k-1} \xi_j + \sum_{l=0}^{|p_k| - 1} \frac{l}{|p_k|} \right\} = \left\{ \begin{cases} \sum_{j=1}^{k-1} \xi_j, & \text{if } p_k \text{ is odd,} \\
\sum_{j=1}^{k-1} \xi_j + \frac{1}{2}, & \text{if } p_k \text{ is even.} \end{cases} \right.
\]

By the assumption of \(\left\{ \sum_{j=1}^{k-1} \xi_j \right\} \in (0,1) \setminus \{1/2\}\), it follows that

\[
\left\{ \sum_{j=1}^{k-1} \xi_j + \sum_{l=0}^{|p_k| - 1} \frac{l}{|p_k|} \right\} \in (0,1) \setminus \{1/2\}.
\]

Lemma 4.3 If there exist \(1 \leq j' < j'' \leq k\) satisfying that \(p_{j'} \cdot p_{j''} = -1\) and \(\{\xi_{j'} + \xi_{j''}\} = 0\) in

\[
\hat{\theta}_j = p_j \theta + \xi_j, \quad \forall 1 \leq j \leq k,
\]

(4.13)

then (4.13) is equivalent to the system

\[
\hat{\theta}_j = p_j \theta + \xi_j, \quad \forall j \in \{1, 2, \ldots, k\} \setminus \{j', j''\}.
\]

(4.14)
As we will see, such a phenomenon does not occur if the condition (4.6) holds. For instance, we consider the system

Thus, \( \eta \) it then follows immediately that (4.13) is equivalent to (4.14).

Theorem 1.2 in Section 5.

Remark 4.3 The condition (4.6) can not be replaced by the weaker condition

Thus, \( \eta(\xi_{j'}) = 0 \) if and only if \( \eta(\xi_{j''}) = 0 \), that is, \( j' \in K_{\bar{Q}}^+ (\eta) \) if and only if \( j'' \in K_{\bar{Q}}^+ (\eta) \). As a result, \( p_{j'} \) and \( p_{j''} \) together contribute nothing to the absolute difference number of (4.13) for any \( \eta \in \mathbb{Q} \).

It then follows immediately that (4.13) is equivalent to (4.14). \( \square \)

The following theorem is our main result of this section which is concerned with the lower estimate on the effective difference number of (4.4) and will play a crucial role in our proof of Theorem 1.2 in Section 5.

**Theorem 4.1** For every system of irrational numbers (4.4) satisfying the conditions (4.5) and (4.6), it holds that

\[
\max \{|k_0^+(\eta) - k_0^-(\eta)| \mid \eta \in \mathbb{Q} \} \geq 1. \tag{4.15}
\]

**Remark 4.3** The condition (4.6) can not be replaced by the weaker condition

\[
\{ \xi_1 + \xi_2 + \cdots + \xi_k \} \in (0,1). \tag{4.16}
\]

For instance, we consider the system \( \hat{\theta}_1 = -\theta + \frac{1}{2}, \hat{\theta}_2 = -\theta, \hat{\theta}_3 = 2\theta \), which satisfies the conditions (4.5) and (4.16) but (4.6). However, one can check directly that \( |k_0^+(\eta) - k_0^-(\eta)| = 0 \) for any \( \eta \in \mathbb{Q} \). As we will see, such a phenomenon does not occur if the condition (4.6) holds.

**Proof of Theorem 4.1:** We carry out the proof with two steps.

**Step 1:** First, letting \( \eta_k = \frac{\hat{\theta}_k}{p_k} \) and making \( \eta_k \)-action to the original system (4.4), we obtain by (4.7) that

\[
\begin{align*}
\eta_k(\hat{\theta}_j) &= p_j \eta_k(\theta) + \eta_k(\xi_j), \quad \forall 1 \leq j \leq k - 1, \\
\eta_k(\hat{\theta}_k) &= p_k \eta_k(\theta).
\end{align*} \tag{4.17}
\]

Then by Lemma 4.2, the system (4.17) is equivalent to

\[
\begin{align*}
\eta_k(\hat{\theta}_j) &= p_j \eta_k(\theta) + \eta_k(\xi_j), \quad \forall 1 \leq j \leq k - 1, \\
\hat{\theta}_{k,l'} &= \text{sgn}(p_k) \eta_k(\theta) + \frac{l'}{|p_k|}, \quad \forall 0 \leq l' \leq |p_k| - 1, \tag{4.18}
\end{align*}
\]

Secondly, taking \( \eta_{k-1} \in \mathbb{Q} \) such that \( \eta_{k-1} \circ \eta_k(\xi_{k-1}) = 0 \) and making \( \eta_{k-1} \)-action to the system (4.18), we get

\[
\begin{align*}
\eta_{k-1} \circ \eta_k(\hat{\theta}_j) &= p_j \eta_{k-1} \circ \eta_k(\theta) + \eta_{k-1} \circ \eta_k(\xi_j), \quad \forall 1 \leq j \leq k - 2, \\
\eta_{k-1} \circ \eta_k(\hat{\theta}_{k-1}) &= p_{k-1} \eta_{k-1} \circ \eta_k(\theta), \\
\eta_{k-1} \circ (\hat{\theta}_{k,l'}) &= \text{sgn}(p_k) \eta_{k-1} \circ \eta_k(\theta) + \eta_{k-1}\left(\frac{l'}{|p_k|}\right), \quad \forall 0 \leq l' \leq |p_k| - 1. \tag{4.19}
\end{align*}
\]
Again by Lemma 4.2, the system (4.19) is equivalent to
\[
\begin{align*}
\eta_{k-1} \circ \eta_k(\hat{\theta}_j) &= p_j \eta_{k-1} \circ \eta_k(\theta) + \eta_{k-1} \circ \eta_k(\xi_j), \quad \forall \ 1 \leq j \leq k - 2, \\
\hat{\theta}_{k-1,l''} &= \text{sgn}(p_{k-1}) \eta_{k-1} \circ \eta_k(\theta) + \frac{l''}{|p_{k-1}|}, \quad \forall \ 0 \leq l'' \leq |p_{k-1}| - 1, \\
\eta_{k-1}(\hat{\theta}_{k,l'}) &= \text{sgn}(p_k) \eta_{k-1} \circ \eta_k(\theta) + \eta_{k-1}(\frac{l'}{|p_k|}), \quad \forall \ 0 \leq l' \leq |p_k| - 1,
\end{align*}
\]
(4.20)

Repeating the above procedure for the rest equations with \( j = k - 2, k - 3, \ldots, 2, 1 \) one at a time in order, we can finally get a system equivalent to the original system (4.4) which can be written in a simple form such as
\[
\hat{\alpha}_{jl} = \text{sgn}(p_j) \alpha + \xi_{jl}, \quad \forall \ 1 \leq j \leq k \text{ and } 0 \leq l \leq |p_j| - 1,
\]
(4.21)
with \( \alpha \in \mathbb{Q}^c \) and \( \xi_{jl} \in \mathbb{Q} \cap [0,1) \). Moreover, by (4.9) and Remark 4.2 we have
\[
\left\{ \sum_{j=1}^{k} \sum_{l=0}^{|p_j| - 1} \xi_{jl} \right\} \in (0, 1) \setminus \{1/2\}.
\]
(4.22)

**Step 2:** We can cut off all the superfluous equations of the system (4.21), if there are such pairs as that in Lemma 4.3. That is, (4.21) is equivalent to some a system
\[
\hat{\theta}'_i = p'_i \alpha + \xi'_i, \quad \forall \ 1 \leq i \leq \tilde{k},
\]
(4.23)
with \( |p'_i| = 1, \sum_{i=1}^{\tilde{k}} p'_i = 0 \) and
\[
\left\{ \sum_{i=1}^{\tilde{k}} \xi'_i \right\} \in (0, 1) \setminus \{1/2\}.
\]
(4.24)

Here notice that \( \tilde{k} \geq 1 \) is ensured by the condition (4.24).

Since all the superfluous equations are cut off, it follows that \( \tilde{k}^+_0 \cdot \tilde{k}^-_0 = 0 \). Assume without loss of generality that \( \tilde{k}^+_0 = \tilde{k}^-_0 = 0 \), otherwise we have nothing to do. Since \( \sum_{i=1}^{\tilde{k}} p'_i = 0 \), we get
\[
\#\{1 \leq i \leq \tilde{k} \mid p'_i = 1\} = \#\{1 \leq i \leq \tilde{k} \mid p'_i = -1\}.
\]

Take arbitrarily out \( i_1 \in \{1 \leq i \leq \tilde{k} \mid p'_i = 1\} \). Let \( \tilde{\eta} = \xi'_{i_1} \) and make the \( \tilde{\eta} \)-action to (4.23). Then it follows immediately that \( \tilde{k}^+_0(\tilde{\eta}) \geq 1 \). Recalling again that all the superfluous equations have been cut off at the beginning of Step 2, we obtain \( \tilde{\eta}(\xi'_i) = \{\xi'_{i_1} + \xi'_i\} \neq 0 \) for every \( i \in \{1 \leq i \leq \tilde{k} \mid p'_i = -1\} \) which yields \( \tilde{k}^-_0(\tilde{\eta}) = 0 \). As a result, we get
\[
\max\{|\tilde{k}^+_0(\eta) - \tilde{k}^-_0(\eta) | \eta \in \mathbb{Q}\} \geq |\tilde{k}^+_0(\tilde{\eta}) - \tilde{k}^-_0(\tilde{\eta})| = \tilde{k}^+_0(\tilde{\eta}) \geq 1.
\]

Since the original system (4.4) is equivalent to (4.23), the estimate (4.15) follows immediately. \( \square \)

The proof of Theorem 4.4 can be illuminated by the concrete example below.
Example 4.1 Consider the irrational system

\[
\hat{\theta}_1 = -\theta + \frac{5}{6}, \quad \hat{\theta}_2 = -2\theta + \frac{1}{3}, \quad \hat{\theta}_3 = 3\theta + \frac{1}{2}.
\]  

(4.25)

One can check directly that the system (4.25) is a special case of (4.4) satisfying the conditions (4.5) and (4.6).

We now come to solve its effective difference number.

**Step 1:** First, we make the change of \( \alpha = \theta + \frac{1}{6} \) to transform (4.25) to

\[
\hat{\alpha}_1 = -\alpha, \quad \hat{\alpha}_2 = -2\alpha + \frac{2}{3}, \quad \hat{\alpha}_3 = 3\alpha.
\]

(4.26)

By Lemma 4.2, (4.26) is equivalent to

\[
\hat{\alpha}_1 = -\alpha, \quad \hat{\alpha}_2 = -2\alpha + \frac{2}{3}, \quad \hat{\alpha}_31 = \alpha, \quad \hat{\alpha}_32 = \alpha + \frac{1}{3}, \quad \hat{\alpha}_33 = \alpha + \frac{2}{3}.
\]

(4.27)

Secondly, we make the change of \( \beta = \alpha - \frac{1}{3} \) to transform (4.27) to

\[
\hat{\beta}_1 = -\beta + \frac{2}{3}, \quad \hat{\beta}_2 = -2\beta, \quad \hat{\beta}_31 = \beta + \frac{1}{3}, \quad \hat{\beta}_32 = \beta + \frac{2}{3}, \quad \hat{\beta}_33 = \beta.
\]

(4.28)

Again by Lemma 4.2, (4.28) is equivalent to

\[
\hat{\beta}_1 = -\beta + \frac{2}{3}, \quad \hat{\beta}_21 = -\beta, \quad \hat{\beta}_22 = -\beta + \frac{1}{2}, \quad \hat{\beta}_31 = \beta + \frac{1}{3}, \quad \hat{\beta}_32 = \beta + \frac{2}{3}, \quad \hat{\beta}_33 = \beta.
\]

(4.29)

**Step 2:** By Lemma 4.3, we can cut off the following superfluous pairs in (4.29):

\[
\hat{\beta}_1 = -\beta + \frac{2}{3} \quad \text{&} \quad \hat{\beta}_31 = \beta + \frac{1}{3}, \quad \text{and} \quad \hat{\beta}_21 = -\beta \quad \text{&} \quad \hat{\beta}_33 = \beta.
\]

That is, (4.29) is equivalent to

\[
\hat{\beta}_22 = -\beta + \frac{1}{2}, \quad \hat{\beta}_32 = \beta + \frac{2}{3}.
\]

(4.30)

Finally, we make the change of \( \gamma = \beta + \frac{2}{3} \) to transform (4.30) to

\[
\hat{\gamma}_22 = -\gamma + \frac{1}{6}, \quad \hat{\gamma}_32 = \gamma.
\]

(4.31)

It is obvious that the effective difference number of (4.31) is 1 and so the system (4.25) does. \( \Box \)
5 Proof of Theorem 1.2

In this section, we prove our main Theorem 1.2. Firstly we give a proof of Theorem 1.2 for $(\mathbb{R}^{2n+1}, F)$ which is involved in the irrational system $\{\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_k\}$ with $2 \leq k \leq 2n$ satisfying (3.10). For sake of readability, we divide it into two cases according to whether $\text{rank}(\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_k) = 1$ or not. We will give in details the proof for the first case. Based on the well known Kronecker’s approximation theorem in Number theory, the second one can be then proved quite similarly and so we only sketch it. While for $(\mathbb{R}^{2n}, F)$, the proof is similar and will be explained at the end of this section.

**Proof of Theorem 1.2 for $(\mathbb{R}^{2n+1}, F)$:** We carry out the proof into two cases.

**Case 1:** $r = \text{rank}(\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_k) = 1$.

As we have mentioned in Section 4, the irrational system (3.10) with $r = 1$ can be seen as a special case of (4.4) satisfying (4.5) and (4.6).

Since any $\eta$-action with $\eta \in \mathbb{Q}$ to (4.4), if necessary, does no substantive effect on our following arguments, by Theorem 4.1 and Remark 3.1 we can assume without loss of generality that $|k_0^+ - k_0^-| \geq 1$ and $\mathcal{K}_1 = \{1, 2, \ldots, k_1\}$, with $k_1 \geq 1$ due to (4.6), and denote by $\xi_j = \frac{p_j}{q_j}$ for $1 \leq j \leq k_1$.

Let $\bar{q} = q_1 q_2 \cdots q_{k_1}$ and $m_l = 2(n + 1) \bar{q} l + 1$ with $l \in \mathbb{N}$. Then, by (4.4) we have

$$
\sum_{j=2}^{k} \left\{ m_l \hat{\theta}_j \right\} = \sum_{j=2}^{k_1} \left\{ m_l \hat{\theta}_j \right\} + \sum_{j=k_1+1}^{k} \left\{ m_l \hat{\theta}_j \right\} = \sum_{j=2}^{k_1} \left\{ p_j \{m_l \theta\} + \xi_j \right\} + \sum_{j=k_1+1}^{k} \left\{ p_j \{m_l \theta\} \right\}. (5.1)
$$

Due to $\theta \in \mathbb{Q}^c$, the set $\{\{m_l \theta\} \mid l \in \mathbb{N}\}$ is dense in $[0, 1]$. For every $L \in \mathbb{Z}$, we introduce the auxiliary function

$$
f_L(x) = \sum_{j=2}^{k_1} \left\{ p_j x + \xi_j \right\} + 2L \hat{\theta}_j + \sum_{j=k_1+1}^{k} \left\{ p_j x + 2L \hat{\theta}_j \right\}, \forall x \in [0, 1], (5.2)
$$

and denote for simplicity by $f = f_0$.

Let $a$ and $b$ in $(0, 1)$ be two real numbers sufficiently close to 0 and 1 respectively. Then,

$$
f(a) = \sum_{j=2}^{k_1} \left\{ p_j a + \xi_j \right\} + \sum_{j=k_1+1}^{k} \left\{ p_j a \right\} = \sum_{j=2}^{k_1} \left\{ p_j a + \xi_j \right\} + \sum_{j \in \mathcal{K}_0^+} p_j a + \sum_{j \in \mathcal{K}_0^-} \left(1 + p_j a\right) = k_0^- + \sum_{j=2}^{k_1} p_j a + \sum_{j=2}^{k_1} \xi_j, (5.3)
$$

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and by similar computation,

\[ f(b) = k_0^+ + \sum_{j=2}^{k} p_j(b - 1) + \sum_{j=2}^{k_1} \xi_j. \]  

It follows by (5.3) and (5.4) that

\[ |f(b) - f(a)| = |k_0^+ - k_0^- + \sum_{j=2}^{k} p_j(b - 1 - a)| = |k_0^+ - k_0^- + p_1(-b + 1 + a)|, \]  

where the second identity we have used \( \sum_{j=1}^{k} p_j = 0 \).

**Lemma 5.1** If \( a \) and \( b \) in \((0, 1)\) are sufficiently close to 0 and 1 respectively, then

(i) \( f(a) \) and \( f(b) \) lie in different intervals of (3.14) with \( L = 0 \),

(ii) \( f_L(a) \) and \( f_L(b) \) lie in the same interval of (3.14) for any \( 1 \leq |L| \leq \tilde{N} \) with \( \tilde{N} \in \mathbb{N} \) prior fixed, including \( f_L(0) \).

**Proof:** (i) By (5.5) and the assumption, \( |f(b) - f(a)| \approx |k_0^+ - k_0^-| \). Here and later, we use \( X \approx Y \) as usual to mean that \( X \) is sufficiently close to \( Y \) in the context of writing. Since the length of each interval in (3.14) with \( L = 0 \) is less than or equal to 1, so \( f(a) \) and \( f(b) \) must lie in different ones, provided that \( |k_0^+ - k_0^-| \geq 2 \).

If \( |k_0^+ - k_0^-| = 1 \), then \( |f(b) - f(a)| \approx 1 \). For the case of \( k = 2 \), since the length of each interval of (3.14) with \( L = 0 \) is less than 1, (i) follows immediately. The rest case is \( k \geq 3 \), which still contains three subcases.

1° If \( k_1 \geq 2 \), we get by (5.3) that

\[ \{f(a)\} \approx \left\{ k_0^- + \sum_{j=2}^{k_1} \xi_j \right\} = \left\{ -\xi_1 + \sum_{j=1}^{k_1} \xi_j \right\} = \left\{ \sum_{j=1}^{k_1} \xi_j \right\} - \xi_1 = \{Q_0\} - \xi_1 \].

Notice that the dividing points of the intervals in (3.14) with \( L = 0 \) are

\[ 0, \{Q_0\}, 1 + \{Q_0\}, 2 + \{Q_0\}, \ldots, k - 2 + \{Q_0\}, k - 1. \]

Therefore, \( \{k_0^- + \sum_{j=2}^{k_1} \xi_j\} = \{\{Q_0\} - \xi_1\} \) must be an interior point of these intervals, so does \( f(a) \).

It then yields that \( f(a) \) and \( f(b) \) must lie in two different intervals.

2° If \( k_1 = 1 \) and \( k_0^- \geq 1 \), then \( f(a) \approx k_0^- \) is also an interior point and (i) follows.

3° If \( k_1 = 1 \) and \( k_0^- = 0 \), then \( f(a) = \sum_{j=2}^{k} p_j a \) lies in the first interval whose length is \( \{Q_0\} < 1 \) and so \( f(b) \) must lie in another one.

(ii) It can be checked directly that \( \lim_{a \to 0} f_L(a) = \lim_{b \to 1} f_L(b) = f_L(0) \in \mathbb{Q}^c \), since \( \xi_j \in \mathbb{Q} \) for \( 1 \leq j \leq k \) and \( \sum_{j=2}^{k} 2L\hat{\theta}_j \in \mathbb{Q}^c \). But the dividing points of these intervals in (3.14) with
1 \leq |L| \leq \tilde{N} \text{ are finitely many rational numbers, so } f_L(0) \text{ is an interior point of these intervals and (ii) follows.} \quad \square

Notice that $f$ is almost continuous on $(0, 1)$. Without loss of generality, we assume $a$ and $b$ to be two points of continuity of $f$ and choose $l_1, l_2 \in \mathbb{N}$ with $l_2 - l_1$ sufficiently large such that \{m_i \theta \} \approx a \text{ and } \{m_i \theta \} \approx b. \text{ Then by (5.1), (5.2) and (i) of Lemma 5.1, we get } \sum_{j=2}^k \{m_i \theta_j\} \text{ and } \sum_{j=2}^k \{m_i \hat{\theta}_j\} \text{ lie in different intervals of (3.14) with } L = 0. \text{ Suppose that}

\[ \sum_{j=2}^k \{m_i \theta_j\} \in I_{i'} \text{ and } \sum_{j=2}^k \{m_i \hat{\theta}_j\} \in I_{i''}, \]

with \{i', i''\} \subseteq \{0, 1, 2, \ldots, k - 1\} \text{ and } i' \neq i''. \text{ By (3.14) we have } i(c_{m_1}) = 2nq_l + 2\lfloor Q_0 \rfloor - 2i' \text{ and } i(c_{m_2}) = 2nq_l + 2\lfloor Q_0 \rfloor - 2i''. \quad (5.6)

Since $2n \mid (2nq_l + 2\lfloor Q_0 \rfloor - 2i'')$ if and only if $2n \mid (2nq_l + 2\lfloor Q_0 \rfloor - 2i'')$, we get by (2.9) that

\[ \beta_{2nq_l + 2\lfloor Q_0 \rfloor - 2i''} = \beta_{2nq_l + 2\lfloor Q_0 \rfloor - 2i''} \equiv \beta. \]

Take $\tilde{N} > 4(n + 1)$ in (ii) of Lemma 5.1 and observe $|2\lfloor Q_0 \rfloor - 2i''| \leq k \leq 2n. \text{ By Lemma 3.3 and Lemma 3.5, there exist } L_i \in \mathbb{Z} \text{ with } 1 \leq |L_i| \leq \tilde{N} \text{ and } 1 \leq i \leq \beta \text{ such that}

\[ i(c_{m_1 + 2L_i}) = 2nq_l + 2\lfloor Q_0 \rfloor - 2i''. \]

Since $\sum_{j=2}^k \{(m_l + 2L_i)\hat{\theta}_j\}$ and $\sum_{j=2}^k \{(m_i + 2L_i)\hat{\theta}_j\}$ are in the same interval of (3.14) with $1 \leq |L_i| \leq \tilde{N}$ by (ii) of Lemma 5.1, we get again by (3.14) that

\[ i(c_{m_2 + 2L_i}) = 2nq_l + 2\lfloor Q_0 \rfloor - 2i'', \forall 1 \leq i \leq \beta. \quad (5.7)\]

By (5.6) and (5.7), it yields $\beta \equiv \beta_{2nq_l + 2\lfloor Q_0 \rfloor - 2i''} = \beta + 1$ which is obviously absurd.

**Case 2:** $r = \text{rank}(\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_k) \geq 2.$

By Lemma 4.1, there are $p_{jl} \in \mathbb{Z}$, $\theta_{k_l} \in \mathbb{Q}^c$ \text{ and } $\xi_j \in \mathbb{Q}$ with $1 \leq l \leq r$ and $1 \leq j \leq k$ such that

\[ \hat{\theta}_j = \sum_{l=1}^r p_{jl} \theta_{k_l} + \xi_j, \forall 1 \leq j \leq k. \quad (5.8)\]

Moreover, $\theta_{k_1}, \theta_{k_2}, \ldots, \theta_{k_r}$ are linearly independent over $\mathbb{Q}$. Due to (3.10), it follows

\[ \sum_{j=1}^k p_{jl} = 0, \forall 1 \leq l \leq r. \quad (5.9)\]
Our basic idea for proving Case 2 is to construct an irrational system with rank 1 associated to (5.8), which plays the essential role in our sequel arguments due to the following result.

**Kronecker’s approximation theorem** (cf. Theorem 7.10 in [2]): If \( \theta_1, \theta_2, \ldots, \theta_r \) are linearly independent over \( \mathbb{Q} \), then the set \( \{(m\theta_1,m\theta_2,\ldots,m\theta_r) \mid m \in \mathbb{N}\} \) is dense in
\[
[0,1]_r = [0,1] \times [0,1] \times \cdots \times [0,1].
\]

**Lemma 5.2** There are \( s_2, s_3, \ldots, s_r \in \mathbb{Z} \) such that
\[
p_{j1} + \sum_{l=2}^{r} s_l p_{jl} \in \mathbb{Z}\{0\}, \quad \forall 1 \leq j \leq k,
\]
(5.10)

**Proof:** Let \( J_0 = \{ 1 \leq j \leq k \mid p_{j1} = 0 \} \). If \( J_0 = \emptyset \), we need only take \( s_2 = s_3 = \cdots = s_r = 0 \). If \( J_0 \neq \emptyset \), we claim that \( (p_{j2},p_{j3},\ldots,p_{jr}) \neq (0,0,\ldots,0) \) for each \( j \in J_0 \). Otherwise, then (5.8) yields that \( \hat{\theta}_j = \xi_j \in \mathbb{Q} \), which contradicts to \( \hat{\theta}_j \in \mathbb{Q}^c \). So the set
\[
X_j \equiv \{(x_2,x_3,\ldots,x_r) \mid p_{j2}x_2 + p_{j3}x_3 + \cdots + p_{jr}x_r = 0 \},
\]
is a subspace of dimension \( r-2 \) in \( \mathbb{R}^{r-1} \) which yields that \( X = \cup_{j \in J_0} X_j \) is a proper subset of \( \mathbb{R}^{r-1} \). Take arbitrarily out an integral point \( (\bar{s}_2,\bar{s}_3,\ldots,\bar{s}_r) \in \mathbb{R}^{r-1}\setminus X \). Then for every \( \bar{N} \in \mathbb{N} \) we have
\[
|p_{j1} + \sum_{l=2}^{r} \bar{N} \bar{s}_l p_{jl}| = \begin{cases} 
\bar{N}|\sum_{l=2}^{r} \bar{s}_l p_{jl}| \neq 0, & \text{if } j \in J_0, \\
|p_{j1}| \neq 0, & \text{if } j \notin J_0 \text{ and } \sum_{l=2}^{r} \bar{s}_l p_{jl} = 0, \\
|p_{j1} + \bar{N} \sum_{l=2}^{r} \bar{s}_l p_{jl}|, & \text{if } j \notin J_0 \text{ and } \sum_{l=2}^{r} \bar{s}_l p_{jl} \neq 0.
\end{cases}
\]
(5.11)

For the third case in the righthand side of (5.11), we can take \( \bar{N} \in \mathbb{N} \) sufficiently large so that \( |p_{j1} + \bar{N} \sum_{l=2}^{r} \bar{s}_l p_{jl}| \neq 0 \) for all these \( j \)'s therein. Finally let \( s_l = \bar{N} \bar{s}_l \) and (5.10) follows. \( \square \)

By Lemma 5.2, we can make the change of variables \( \hat{\theta}_{k_1} = \theta_{k_1} \) and \( \hat{\theta}_{k_l} = \theta_{k_l} - s_l \theta_{k_1} \) for \( 2 \leq l \leq r \). Then the system (5.8) is transformed to
\[
\hat{\theta}_j = \sum_{l=1}^{r} \bar{p}_{jl} \bar{\theta}_{k_1} + \xi_j, \quad \forall 1 \leq j \leq k,
\]
(5.12)

with \( \bar{p}_{j1} = p_{j1} + \sum_{l=2}^{r} s_l p_{jl} \in \mathbb{Z}\{0\} \), and by (5.9) we have
\[
\sum_{j=1}^{k} \bar{p}_{j1} = \sum_{j=1}^{k} p_{j1} + \sum_{j=1}^{k} \sum_{l=2}^{r} s_l p_{jl} = 0 + \sum_{l=2}^{r} s_l \left( \sum_{j=1}^{k} p_{jl} \right) = 0.
\]
Since \( \theta_{k_1}, \theta_{k_2}, \ldots, \theta_{k_r} \) are linearly independent over \( \mathbb{Q} \), so do \( \bar{\theta}_{k_1}, \bar{\theta}_{k_2}, \ldots, \bar{\theta}_{k_r} \).

Consider the following irrational system with rank 1 associated to (5.12)
\[
\hat{\alpha}_j = \bar{p}_{j1} \bar{\theta}_{k_1} + \xi_j, \quad \forall 1 \leq j \leq k.
\]
(5.13)
By Theorem 4.1, without loss of generality we can assume for \((5.13)\) that \(|\tilde{k}_0^- - \tilde{k}_0^+| \geq 1\) and \(\tilde{k}_1 = \{1, 2, \ldots, \tilde{k}_1\}\), and denote by \(\xi_j = \frac{r_j}{q_j}\) for \(1 \leq j \leq \tilde{k}_1\).

Let \(\tilde{q} = q_1 q_2 \cdots q_{\tilde{k}_1}\) and \(\tilde{m}_l = 2(n + 1)\tilde{q}l + 1\) for \(l \in \mathbb{N}\). Then, we get by \((5.12)\) that

\[
\sum_{j=2}^{\tilde{k}_1} \left\{ n_j\tilde{\theta}_j \right\} = \sum_{j=2}^{\tilde{k}_1} \left\{ \tilde{m}_j\tilde{\theta}_j \right\} + \sum_{j=\tilde{k}_1+1}^{\tilde{k}_1} \left\{ n_j\tilde{\theta}_j \right\}
\]

\[
= \sum_{j=2}^{\tilde{k}_1} \left\{ \sum_{l=1}^{r} \tilde{p}_{jl}\tilde{m}_l\tilde{\theta}_k + \xi_j \right\} + \sum_{j=\tilde{k}_1+1}^{\tilde{k}_1} \left\{ \sum_{l=1}^{r} \tilde{p}_{jl}\tilde{m}_l\tilde{\theta}_k \right\}
\]

\[
= \sum_{j=2}^{\tilde{k}_1} \left\{ \sum_{l=1}^{r} \tilde{p}_{jl}\tilde{m}_l\tilde{\theta}_k + \xi_j + 2L\tilde{\theta}_j \right\} + \sum_{j=\tilde{k}_1+1}^{\tilde{k}_1} \left\{ \sum_{l=1}^{r} \tilde{p}_{jl}\tilde{m}_l\tilde{\theta}_k \right\}
\]

By Kronecker’s approximation theorem, the set \(\{\{\tilde{m}_l\tilde{\theta}_k\}, \{\tilde{m}_l\tilde{\theta}_k\}, \ldots, \{\tilde{m}_l\tilde{\theta}_k\\} \mid l \in \mathbb{N}\}\) is dense in \([0, 1]^r\). For every \(L \in \mathbb{Z}\), we can introduce the auxiliary multi-variable function on \([0, 1]^r\),

\[
g_L(x_1, x_2, \ldots, x_r) = \sum_{j=2}^{\tilde{k}_1} \left\{ \sum_{l=1}^{r} \tilde{p}_{jl}x_l + \xi_j + 2L\tilde{\theta}_j \right\} + \sum_{j=\tilde{k}_1+1}^{\tilde{k}_1} \left\{ \sum_{l=1}^{r} \tilde{p}_{jl}x_l + 2L\tilde{\theta}_j \right\},
\]

and denote for simplicity by \(g = g_0\). Similarly as before, we have

**Lemma 5.3** If \((a_1, a_2, \ldots, a_r)\) and \((b_1, b_2, \ldots, b_r)\) in \((0, 1)^r\) are sufficiently close to \((0, 0, 0, \ldots, 0)\) and \((1, 0, 0, \ldots, 0)\) respectively by a suitable means, then

(i) \(g(a_1, a_2, \ldots, a_r)\) and \(g(b_1, b_2, \ldots, b_r)\) lie in different intervals of \((3.14)\) with \(L = 0\).

(ii) \(g_L(a_1, a_2, \ldots, a_r)\) and \(g_L(b_1, b_2, \ldots, b_r)\) lie in the same interval of \((3.14)\) for any \(1 \leq |L| \leq \bar{N}\) with \(\bar{N} \in \mathbb{N}\) prior fixed, including \(g_L(0, 0, \ldots, 0)\).

**Proof:** (i) Since \(a_1, a_2, \ldots, a_r\) (resp. \(b_1, b_2, \ldots, b_r\)) are independent, we can select them by such a way that the decimal functions in \(g(a_1, a_2, \ldots, a_r)\) and \(g(b_1, b_2, \ldots, b_r)\) are mainly determined by \(a_1\) and \(b_1\) respectively. For instance, this can be realized by requiring \(a_l\) (resp. \(b_l\)) with \(2 \leq l \leq r\) to be much smaller than \(a_1\) (resp. \(1 - b_1\)). The rest proof is then similar as that in Lemma 5.1-(i), with \(g\) in stead of \(f\) therein.

(ii) follows the same line as Lemma 5.1-(ii) and do not need such a choice as above.

Due to Lemma 5.3, the rest proof is then almost word by word as that in Case 1 and so we omit the tedious details. \(\Box\)

**Remark 5.1** As for \(\mathbb{R}P^3\), we can give a more direct and easier proof. Indeed, we can make a reduction by \((3.10)\) (with \(n = 1\) and \(k = 2\)) so that only one irrational number is rest. The uniformly distribution mod one in Number theory then enables the authors in [14] to find some \(l \in \mathbb{N}\) such that the Betti number \(\tilde{\beta}_{2l} = 1\) which contradicts to the topological structure of the non-contractible loop space on \(\mathbb{R}P^3\) obtained in [42]. However when one tries to use such a means to deal
with higher dimensional $\mathbb{R}P^{2n+1}$, more irrational numbers are rest to be controlled simultaneously for larger $k$. What is even worse, those irrational numbers may be linearly dependent over $\mathbb{Q}$. These facts make the arguments in section 3.3 of [14] difficult to continue, even for $\mathbb{R}P^5$.

To overcome these difficulties, we discover a general character of the irrational systems (4.4) satisfying the conditions (4.5) and (4.6), which are closely associated to our problem. That is, the effective difference number of each of such irrational systems is larger than or equal to 1 (cf. Theorem 4.1). Based on it and the Kronecker’s approximation theorem, we can get the desired contradiction dynamically (quite different from the static way in [14]), provided that there is only one non-contractible closed geodesics.

**Proof of Theorem 1.2 for $(\mathbb{R}P^{2n}, F)$:**

For the case of even $n$, it shares the same essential properties with the odd case except for some quantitative differences, such as the resonance identity, the precise index iteration formulae and the irrational systems. Hence we only sketch its proof to avoid this paper being too long and tedious.

We now give some explanations to the proof of Theorem 1.2 for the case of $\mathbb{R}P^{2n}$.

Note that in the proof of Theorem 1.2 for the case of $\mathbb{R}P^{2n+1}$, only (2.9), (3.9)-(3.14), Lemmas 3.3-3.5 and Theorem 4.1 are used. As for the case of $\mathbb{R}P^{2n}$, Lemma 3.3 and Theorem 4.1 still hold, thus by using (2.11) instead of (2.9), Lemma 3.6 instead of Lemma 3.4, Lemma 3.7 instead of Lemma 3.5, and noticing Remark 3.2, we can go through the proof of Theorem 1.2 for the case of $\mathbb{R}P^{2n}$ word by word as that of the case of $\mathbb{R}P^{2n+1}$. We complete the proof of Theorem 1.2. □

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