On some properties of the Attractor Equations

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ABSTRACT

We discuss the Attractor Equations of $N = 2$, $d = 4$ supergravity in an extremal black hole background with arbitrary electric and magnetic fluxes (charges) for field-strength two-forms.

The effective one-dimensional Lagrangian in the radial (evolution) variable exhibits features of a spontaneously broken supergravity theory. Indeed, non-BPS Attractor solutions correspond to the vanishing determinant of a (fermionic) gaugino mass matrix. The stability of these solutions is controlled by the data of the underlying Special Kähler Geometry of the vector multiplets’ moduli space.

Finally, after analyzing the 1-modulus case more in detail, we briefly comment on the choice of the Kähler gauge and its relevance for the recently discussed entropic functional.
1 Introduction

It is known that the Attractor Equations for stationary, spherically symmetric, asymptotically flat black holes are described by an effective one-dimensional Lagrangian ([1], [2]), with an effective scalar potential and effective fermionic “mass terms” terms controlled by the field-strength fluxes, i.e. electric and magnetic charges

\[ e_\Lambda \equiv \int_{S^2_\infty} G^\Lambda, \quad m^\Lambda \equiv \int_{S^2_\infty} F^\Lambda, \quad \Lambda = 0, 1, ..., n_V, \]

where, in the case of \( N = 2, d = 4 \) supergravity, \( n_V \) denotes the number of Abelian vector supermultiplets. Here \( F^\Lambda = dA^\Lambda \) and \( G^\Lambda \) is the “dual” field-strength two-form ([3], [4]).

The “apparent” gravitino mass is given by the central charge function \( Z(z, \bar{z}; e, m) \), where \( z^i (i = 1, ..., n_V) \) are the complex coordinates of the Special Kähler-Hodge manifold \( \mathcal{V} (\dim \mathcal{V} = n_V) \) of vector supermultiplets. The gaugino mass matrix \( \Lambda_{ij} \) is given by [5]

\[ \Lambda_{ij} = C_{ijk} g^{k\ell} D_\ell Z, \]

where \( C_{ijk} \) is the (Kähler-covariantly holomorphic) completely symmetric, rank-3 tensor of Special Kähler Geometry (SKG), and it should also be recalled that \( \nabla^\ell Z = 0 \). Finally, the supersymmetry order parameter, related to the mixed gravitino-gaugino coupling, is given by \( D_i Z = (\partial_i + \frac{1}{2} \partial \partial K) Z \), with \( K \) being the Kähler potential in \( \mathcal{V} \).

In this paper we intend to discuss some general features of the Attractor Equations [1], relating extremal non-BPS black holes exhibiting an Attractor behavior to a degenerate matrix. Such a matrix must have vanishing determinant in order for a non-BPS, extremal black hole with Attractor behavior to exist.

The plan of the paper is as follows. In Sect. 2 we recall the Attractor potential for a generic \( N = 2, d = 4 \) supergravity theory and its BPS Attractor solutions ([6], [7], [8]). Then, in Sect. 3 we discuss the non-BPS case and the related eigenvalue problem, with an explicit application to the one-modulus case. In Sect. 4 we give the Hessian matrix which controls the stability of the non-BPS extrema and discuss some properties, especially on the one-modulus case. Finally, in Sect. 5 we try to connect our results to the entropic functional recently introduced in [9] in connection with the conjectured relation between the black hole entropy and the topological string partition function [10].

2 Black Hole Potential and Supersymmetric Attractors

In \( N = 2, d = 4 \) supergravity the general form of the “effective black hole potential” function reads ([1], [4], [8])

\[ V_{BH} (z, \bar{z}; e, m) = |Z|^2 + g\tilde{s} (D_i Z) (\nabla^\ell Z). \]
The aim of the present work is to discuss the extrema of $V_{BH}$ in a general fashion. First of all, we recall that, as shown in [8], for BPS extremal black holes it holds that

$$D_i W = (\partial_i + \partial_i K) W = 0 \quad \forall i = 1, \ldots, n_V,$$  \hspace{1cm} (2.2)

where $W$ is the holomorphic prepotential [4]

$$W(z) = e_A X^A(z) - m^A F_A(z), \quad \bar{\partial}_i W(z) = 0, \forall i = 1, \ldots, n_V. \hspace{1cm} (2.3)$$

Here $(X^A(z), F_A(z))$ are the holomorphic sections of SKG. Clearly, Eq. (2.2) can also be rewritten as

$$D_i Z = \left( \partial_i + \frac{1}{2} \partial_i K \right) Z = 0 \quad \forall i = 1, \ldots, n_V, \hspace{1cm} (2.4)$$

with

$$Z(z, \bar{z}; e, m) = e_A L^A(z, \bar{z}) - m^A M_A(z, \bar{z}), \quad \bar{\partial}_i Z = 0, \forall i = 1, \ldots, n_V, \hspace{1cm} (2.5)$$

where $(L^A(z, \bar{z}), M_A(z, \bar{z})) = e^{K(z, \bar{z})/2} (X^A(z), F_A(z))$ are the Kähler-covariantly holomorphic sections of the $U(1)$ Hodge bundle of SKG.

Since

$$D_i V_{BH} = \partial_i V_{BH} = 2 \text{Re} \, D_i Z + g^{i\bar{j}} (D_i D_j Z) \bar{D}_j Z,$$  \hspace{1cm} (2.6)

it is obvious that $D_i Z = 0$ is an extremum of the potential. Moreover, such an extremum is stable, because the Hessian of $V_{BH}$ evaluated at this point is (strictly) positive definite [11]:

$$\begin{align*}
(D_i D_j V_{BH})_{BPS-extr.} &= (\partial_i \partial_j V_{BH})_{BPS-extr.} = 0, \\
(D_i \bar{\partial}_j V_{BH})_{BPS-extr.} &= (\partial_i \bar{\partial}_j V_{BH})_{BPS-extr.} = 2 (g_{i\bar{j}} V_{BH})_{BPS-extr.} = 2 g_{i\bar{j}} |Z|_{BPS-extr.}^2 > 0,
\end{align*} \hspace{1cm} (2.7)$$

where the notation “$> 0$” (“$< 0$”) is clearly understood as strict positive (negative) definiteness throughout all the present work. Notice that the Hermiticity and (strict) positive definiteness of the Hessian of $V_{BH}$ at the critical supersymmetric points $D_i Z = 0$ are due to the Hermiticity and (strict) positive definiteness of the metric $g_{\bar{i} j}$ of the moduli space. If such a manifold is endowed with SKG, it holds that

$$D_i D_j Z = i C_{ijkl} g^{k\bar{l}} \bar{D}_k Z, \hspace{1cm} (2.8)$$

trivially vanishing at $D_i Z = 0$. If we consider the $N = 2$ Lagrangian in a black hole background [3] and we denote by $\psi$ the gravitino and by $\lambda^i$ the gaugino fields, it is easy
to see that such a Lagrangian contains terms of the type

\[ Z \psi \psi; \]

\[ C_{ijk} g^{jk} (D_k Z) \lambda^i \lambda^j; \]

\[ (D_i Z) \lambda^i \psi, \]

(2.9)

in such a way that at the BPS critical points the gaugino mass term and the \( \lambda \psi \) term vanish, while the gravitino “apparent mass” term is proportional to the value of \( Z \). This is of course a consequence of the fact that the horizon geometry at the BPS points is Bertotti-Robinson \( AdS_2 \times S^2 \) ([11], [12], [13]).

We also remark that at the BPS supersymmetric Attractor points we can also express the black hole charges \((e_\Lambda, m_\Lambda)\) in terms of the SKG data by the so-called BPS supersymmetric Attractor Equations \[8\]

\[
\begin{align*}
  e_\Lambda &= 2 \text{Re} \left( -i Z \overline{M}_\Lambda \right); \\
  m_\Lambda &= 2 \text{Re} \left( -i Z \overline{L}_\Lambda \right),
\end{align*}
\]

(2.10)

where the right-hand sides are evaluated at \( D_i Z = 0 \).

### 3 Non-Supersymmetric Attractors

In [1] the potential \( V_{BH} \) was also considered at more general Attractor extrema, defined by the criticity condition

\[
\partial_i V_{BH} = 2Z D_i Z + g^{ij} (D_i D_j Z) D_j Z = 0 \quad \forall i = 1, ..., n_V
\]

(3.1)

and by the condition of positive definiteness of the Hessian of \( V_{BH} \), which we will, in a shorthand notation, denote as\(^1\)

\[
(\partial_i \partial_j V_{BH})_{\partial V = 0} > 0.
\]

(3.2)

It can be immediately realized that such Attractors do not necessarily satisfy the BPS condition \( D_i Z = 0 \) \( \forall i = 1, ..., n_V \). In particular, the critical points satisfying Eqs. (3.1) and (3.2) but with \( D_i Z \neq 0 \) for at least one index \( i \) are non-supersymmetric Attractor points, corresponding to an extremal, non-BPS black hole background with squared mass

\(^1\)By construction, \( V_{BH} \) is positive definite for a (not necessarily strictly) positive definite metric \( g_{ij} \) of the moduli space. Consequently, the stable Attractors will necessarily be minima of such a potential. Eqs. (3.1) and (3.2) are the general definitions of Attractor points in the considered framework (as also recently pointed out by [14]). In the following treatment a geometry will be named regular if the related metric tensor is strictly positive definite.
\[ M_{BH}^2 = V_{BH}|_{non-BPS-extr.} = \]
\[ = |Z|_{non-BPS-extr.}^2 + \left[ g^i (D_i Z) (D_{\overline{z}} Z) \right]_{non-BPS-extr.} > |Z|_{non-BPS-extr.}^2. \]  
\[ (3.3) \]

Now, by substituting the SKG result \( (2.8) \) in the general criticity condition \( (3.1) \), one obtains \[ \partial_i V_{BH} = 0 \iff 2ZD_i Z + \sum_{ijk} g^{ik} (D_k Z) (\overline{D_j Z}) = 0. \]  
\[ (3.4) \]

Thus, we may observe that if \( D_i Z \neq 0 \) for at least one index \( i \) and \( Z \neq 0 \), then \( C_{ijk} \neq 0 \), i.e. the SKG rank-3 symmetric tensor will for sure have some non-vanishing components in order for Eq. \( (3.4) \) to be satisfied.

By reconsidering the general criticality condition \( (3.1) \) (which is independent of the eventually special Hodge-Kähler geometric structure of the moduli space), it is easy to see that it is nothing but the general form of the Ward identity relating the gravitino mass \( Z \), the gaugino masses \( D_i D_j Z \) and the supersymmetry-breaking parameter \( D_i Z \) in a generic spontaneously broken supergravity theory \[ \text{[18].} \]

It can be recast in the form

\[ (M_{ij} h^j)_{\partial V = 0} = 0, \]  
\[ (3.5) \]

with

\[ M_{ij} \equiv D_i D_j Z + 2 \frac{Z}{[g^{ik} (D_k Z) (\overline{D_j Z})]} (D_i Z) (D_j Z) \]  
\[ (3.6) \]

and

\[ h^j \equiv g^{jk} (\overline{D_j Z}). \]  
\[ (3.7) \]

In order for Eq. \( (3.5) \) to have a solution for non-vanishing \( h^j \), the \( n_V \times n_V \) complex matrix \( M_{ij} \) must have vanishing determinant, so actually it has at most \( n_V - 1 \) non-vanishing eigenvalues.

It is immediate to see that for \( n_V = 1 \), i.e. in the one-modulus case (\[ \text{[14], [19]} \]), the condition

\[ M_{11} = 0 \]  
\[ (3.8) \]

is the same as the condition \( \partial_z V_{BH} = 0 \) (for \( D_z Z \neq 0 \)). By considering the SKG framework, for \( n_V = 1 \) Eq. \( (3.8) \) becomes (we introduce the scalar functions \( C_{111} \equiv C (z, \overline{z}) \in \mathbb{C}_0 \) and \( g_{i\overline{z}} \equiv g (z, \overline{z}) \in \mathbb{R}_0^+ \))

\[ \partial_z V_{BH} = 0 \iff 2ZD_z Z + iC g^{-2} (\overline{D_z Z})^2 = 0, \]  
\[ (3.9) \]

which also implies the following relation at the non-supersymmetric critical points:

\[ |D_z Z|_{non-BPS-extr.}^2 = 4 \left[ g^4 |z|^2 \right]_{non-BPS-extr.}. \]  
\[ (3.10) \]
Thus, in SKG for $n_V = 1$ Eq. (3.3) may be written as

$$V_{BH|_{non-BPS-extr.}} = |Z|_{non-BPS-extr.}^2 + g^{-1} |D_z Z|_{non-BPS-extr.}^2 =$$

$$= |Z|_{non-BPS-extr.}^2 \left[ 1 + 4 \left( \frac{g^3}{|C|^2} \right)_{non-BPS-extr.} \right] > |Z|_{non-BPS-extr.}^2. \quad (3.11)$$

Thus, we see that a general feature of the one-modulus case in SKG is that the entropy $S_{BH,non-BPS-extr.} = \pi V_{BH|_{non-BPS-extr.}}$ at the non-BPS, non-supersymmetric Attractors gets a multiplicative “renormalization”

$$S_{BH,non-BPS-extr.} = \pi \gamma |Z|_{non-BPS-extr.}^2, \quad (3.12)$$

with

$$\gamma - 1 = 4 \left( \frac{g^3}{|C|^2} \right)_{non-BPS-extr.} > 0. \quad (3.13)$$

Consequently, the strict positivity of $\gamma - 1$ and the multiplicative “renormalization” of the black hole entropy yield that (at least in the considered framework) the BPS and non-BPS cases of extremal black hole Attractors are “discretely disjoint” one from the other.

4 The Hessian Matrix

Another interesting issue we intend to address here is the study of the condition

$$\frac{\partial^2 V_{BH}}{\partial \varphi^a \partial \varphi^b} > 0 \quad \text{at} \quad \frac{\partial V_{BH}}{\partial \varphi^a} = 0, \quad (4.1)$$

namely the condition of (strict) positive definiteness of the Hessian matrix of $V_{BH}$ (which is nothing but the squared mass matrix of the moduli) at the critical points of $V_{BH}$. As previously pointed out, since $V_{BH}$ is positive definite, a stable critical point (i.e. an Attractor) is necessarily a minimum, satisfying Eq. (4.1). The $\varphi^a$’s ($a = 1, ..., 2n_V$) are real moduli, whose complexification may be given by the unitary transformation

$$z^k \equiv \varphi^{2k-1} + i \varphi^{2k} / \sqrt{2}, \quad k = 1, ..., n_V. \quad (4.2)$$

From the (strict) positive definiteness of the metric tensor $g_{ij}$ in a regular moduli space, we have seen above that the condition (4.1) is automatically satisfied at the BPS supersymmetric Attractor points (defined by $D_i Z = 0 \ \forall i$). Of course, for $D_i Z \neq 0$ for at least some index $i$, this is no longer true, and a more detailed analysis of (4.1) is needed.
In the complex basis, the $2n_V \times 2n_V$ Hessian of $V_{BH}$ reads
\begin{equation}
H_{ij}^{V_{BH}} \equiv \begin{pmatrix}
D_i D_j V_{BH} & D_i \overline{D_j} V_{BH} \\
D_j \overline{D_i} V_{BH} & \overline{D_i} \overline{D_j} V_{BH}
\end{pmatrix} = \begin{pmatrix}
\mathcal{M}_{ij} & \mathcal{N}_{i\bar{j}} \\
\overline{\mathcal{N}_{j\bar{i}}} & \overline{\mathcal{M}_{i\bar{j}}}
\end{pmatrix},
\end{equation}
where the hatted indices $i$ and $j$ may be holomorphic or anti-holomorphic.

In the SKG framework, one gets
\begin{equation}
\mathcal{M}_{ij} \equiv D_i D_j V_{BH} = D_j D_i V_{BH} = 4iZ C_{ijk} g^{k\bar{k}} \overline{(D_{k}Z)} + i (D_j C_{ikl}) g^{k\bar{k}} g^{l\bar{l}} (D_{k}Z) (D_{l} \overline{Z});
\end{equation}
\begin{equation}
\mathcal{N}_{i\bar{j}} \equiv D_i \overline{D_j} V_{BH} = \overline{D_j} D_i V_{BH} = 2 \left[ g_{\bar{i}j} |Z|^2 + (D_i Z) (D_{\bar{j}} \overline{Z}) + g^{\bar{m}n} C_{ikl} \overline{C}_{\bar{j}mn} g^{k\bar{k}} g^{l\bar{l}} (D_{k}Z) (D_{l} \overline{Z}) \right].
\end{equation}

Thus, $\mathcal{M}^T = \mathcal{M}$ and $\mathcal{N}^\dagger = \mathcal{N}$. Here we just note that for symmetric special Kähler-Hodge moduli spaces $D_j C_{ikl} = 0$, and that the tensor $g^{\bar{m}n} C_{ikl} \overline{C}_{\bar{j}mn} g^{k\bar{k}} g^{l\bar{l}} (D_{k}Z) (D_{l} \overline{Z})$ can be rewritten in terms of the Riemann-Christoffel tensor by using the so-called “SKG constraints” (see e.g. [4]):
\begin{equation}
g^{\bar{m}n} C_{ikl} \overline{C}_{\bar{j}mn} = g_{\bar{i}j} g_{km} + g_{im} g_{j\bar{k}} - R_{\bar{i}jkm}.
\end{equation}

Note that the given expression for $\mathcal{M}_{ij}$ is actually symmetric, since through the constraints (4.6) the Bianchi identities for $R_{\bar{i}jkm}$ imply $D_{[j} C_{i]kl} = 0$, with square brackets denoting antisymmetrization of enclosed indices.

At the critical points of $V_{BH}$, the Kähler-covariant derivatives may be substituted by the flat derivatives, and the Hessian becomes
\begin{equation}
H_{ij}^{V_{BH}} \bigg|_{\partial V = 0} \equiv \begin{pmatrix}
\partial_i \partial_j V_{BH} & \partial_i \overline{\partial_j} V_{BH} \\
\partial_j \overline{\partial_i} V_{BH} & \overline{\partial_i} \overline{\partial_j} V_{BH}
\end{pmatrix} = \begin{pmatrix}
\mathcal{M} & \mathcal{N} \\
\overline{\mathcal{N}} & \overline{\mathcal{M}}
\end{pmatrix},
\end{equation}
instead, in the real basis the Hessian is a $2n_V \times 2n_V$ real symmetric matrix, which may be decomposed as follows:
\begin{equation}
\frac{\partial^2 V_{BH}}{\partial \varphi^a \partial \varphi^b} = \begin{pmatrix}
\mathcal{A} & \mathcal{C} \\
\mathcal{C}^T & \mathcal{B}
\end{pmatrix},
\end{equation}
with $\mathcal{A}$ and $\mathcal{B}$ being $n_V \times n_V$ real symmetric matrices ($\mathcal{A}^T = \mathcal{A}$, $\mathcal{B}^T = \mathcal{B}$). By relating the real and complex basis of the special Kähler moduli space $\mathcal{V}$ through the unitary transformation [12], the relation between the complex matrices $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{A}$, $\mathcal{B}$ is given by
\begin{equation}
\begin{cases}
\mathcal{M} = (\mathcal{A} - \mathcal{B}) + i (\mathcal{C} + \mathcal{C}^T); \\
\mathcal{N} = (\mathcal{A} + \mathcal{B}) + i (\mathcal{C}^T - \mathcal{C}),
\end{cases}
\end{equation}
or its inverse

\[
\begin{align*}
A &= \frac{1}{2} (\text{Re} M + \text{Re} N); \\
B &= \frac{1}{2} (\text{Re} N - \text{Re} M); \\
C &= \frac{1}{2} (\text{Im} M - \text{Im} N).
\end{align*}
\] (4.10)

It is worth noticing that the unitary transformation (4.2) may be represented by a unitary
\((U(2n_V))\) matrix \(U\), acting as follows \((U^{-1} = U^\dagger)\):

\[
H_{R}^{\text{BH}} = U^{-1} H_{C}^{\text{BH}} (U^T)^{-1},
\] (4.11)

or equivalently

\[
H_{C}^{\text{BH}} = U H_{R}^{\text{BH}} U^T,
\] (4.12)

where the subscripts “\(\mathbb{R}\)” and “\(\mathbb{C}\)” denote the real and complex parametrization of the
Hessian \(H^{\text{BH}}\), respectively.

Now, it should be recalled that the properties of SKG yield that at a generic point in
the Hodge-Kähler moduli space \(\mathcal{V}\), the following relation holds true [20]:

\[
\mathcal{P} - i \Omega M(N) \mathcal{P} = -2i Z \nabla - 2i G^{\bar{j}j} (D_j \bar{Z}) (D_j V),
\] (4.13)

where \(\mathcal{P}\) and \(V\) respectively are the \(Sp(2n_V)\)-covariant vectors of black hole charges and
of the Kähler-covariantly holomorphic sections of the SKG:

\[
\mathcal{P} \equiv \begin{pmatrix} m^\Lambda \\ e_\Lambda \end{pmatrix}, \quad V(z, \bar{z}) \equiv \begin{pmatrix} L^\Lambda(z, \bar{z}) \\ M^\Lambda(z, \bar{z}) \end{pmatrix}, \quad \overline{D}_j V(z, \bar{z}) = 0,
\] (4.14)

and \(\Omega\) is the \((2n_V + 2)\)-dim. symplectic metric:

\[
\Omega \equiv \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}
\] (4.15)

(\(\mathbb{I}\) stands for the \((n_V + 1)\)-dim. unit matrix). \(M(N)\) is a real \((2n_V + 2)\)-dim. square
matrix, defined as \([\mathbb{I}, \mathbb{N}]\)

\[
M(\text{Re} (N), \text{Im} (N)) \equiv
\]

\[
\equiv \begin{pmatrix}
\text{Im} (N) + \text{Re} (N) (\text{Im} (N))^{-1} \text{Re} (N) & -\text{Re} (N) (\text{Im} (N))^{-1} \\
-(\text{Im} (N))^{-1} \text{Re} (N) & (\text{Im} (N))^{-1}
\end{pmatrix},
\] (4.16)
where $N_{\Lambda \Sigma}$ is a $(n_V + 1) \times (n_V + 1)$ moduli-dependent complex symmetric matrix defined by the fundamental Ansätze of SKG (see e.g. [4]):

$$M_{\Lambda}(z, \bar{z}) \equiv N_{\Lambda \Sigma}(z, \bar{z}) L^\Sigma(z, \bar{z});$$

(4.17)

$$D_i M_{\Lambda}(z, \bar{z}) \equiv N_{\Lambda \Sigma}(z, \bar{z}) D_i L^\Sigma(z, \bar{z}).$$

(4.18)

By taking the real part $^2$ of the SKG identity (4.13) and evaluating it at the critical points of $V_{BH}$ (satisfying Eq. (3.4)), one gets the general form of the extremal black hole Attractor Eqs. ([21], [15], [16]; see also [17]) :

$$\mathcal{P} = 2 \left\{ Re \left[ -iZV + \frac{1}{2Z} \overline{C} \eta^{\alpha \beta} g^{jk} (D_j Z)(D_k Z)(D_i V) \right] \right\}_{\partial V=0}. \quad (4.19)$$

Notice that at the BPS supersymmetric extremal black hole Attractors (satisfying Eq. (2.2) or equivalently Eq. (2.4)) Eqs. (4.19) consistently reduce to the well-known supersymmetric extremal black hole Attractor Eqs. (2.10).

Let us now reconsider the one-modulus case. For $n_V = 1$, the moduli-dependent matrices $A, B, C, M$ and $N$ introduced above are simply scalar functions. In particular, $N$ is real, since $C$ trivially satisfies $C = C^T$. The stability condition (4.1) in this case reads

$$\frac{\partial^2 V_{BH}}{\partial \varphi^a \partial \varphi^b} > 0 \text{ at } \frac{\partial V_{BH}}{\partial \varphi^a} = 0, \quad (4.20)$$

with $a, b = 1, 2$, and it may be rewritten as

$$N|_{\partial V=0} > |M|_{\partial V=0}, \quad (4.21)$$

where

$$N \equiv D_z \overline{D}_z V_{BH} = \overline{D}_z D_z V_{BH} = 2 \left[ g |Z|^2 + |D_z Z|^2 + |C|^2 g^{-3} |D_z Z|^2 \right]; \quad (4.22)$$

$$M \equiv D_z D_z V_{BH} = 4iZC g^{-1} \left( \overline{D}_z Z \right) + i (D_z C) g^{-2} \left( \overline{D}_z Z \right)^2. \quad (4.23)$$

Once again, the 1-modulus stability condition (4.21) is automatically satisfied for $D_z Z = 0$, i.e. at BPS supersymmetric Attractors. Instead, by evaluating the functions $N$ and $M$ at the non-BPS, non-supersymmetric critical points of $V_{BH}$ and using the relation (3.10), one finally gets

$$N|_{non-BPS-extr.} = 2 \left[ |D_z Z|^2 \left( 1 + \frac{5}{4} |C|^2 g^{-3} \right) \right]_{non-BPS-extr}; \quad (4.24)$$

$^2$Actually, the imaginary part of the SKG relation is a new identity satisfied by the central charge $Z$.
\[ |\mathcal{M}|^2_{\text{non-BPS-extr.}} = 4 \left[ |D_z Z|^4 |C|^4 g^{-6} + \frac{1}{4} g^{-4} |D_z C|^2 + 2 g^{-3} \text{Re} \left[ C (\overline{D_z C}) (\overline{D_z \ln Z}) \right] \right]_{\text{non-BPS-extr.}}. \] (4.25)

Therefore, Eq. (4.21) yields that the stability condition for non-BPS, non-supersymmetric extremal black hole attractors in the \( n_V = 1 \) SKG framework reads

\[ \mathcal{N}|_{\text{non-BPS-extr.}} > |\mathcal{M}|_{\text{non-BPS-extr.}} ; \]

\[ 1 + \frac{5}{4} (|C|^2 g^{-3})_{\text{non-BPS-extr.}} > \]

\[ > \sqrt{[|C|^4 g^{-6} + \frac{1}{4} g^{-4} |D_z C|^2 + 2 g^{-3} \text{Re} \left[ C (\overline{D_z C}) (\overline{D_z \ln Z}) \right] ]_{\text{non-BPS-extr.}}}. \] (4.27)

One may immediately notice that Eq. (4.27) is surely satisfied for \( D_z C = 0 \), corresponding to 1-dimensional complex symmetric special Kähler spaces [22], but of course this is not the only possibility. In the general case, a new fundamental data, \( D_z C \) turns out to play a crucial role in determining the stability of non-supersymmetric attractors.

5 Kähler Gauges and Entropic Functional

In this last Section we discuss some properties of the “black hole effective potential” function \( V_{BH} \) given by Eq. (2.1).

First of all, we observe that the “central charge” function \( Z \) given by Eq. (2.5) is a section of the \( U(1) \) Hodge bundle over the special Kähler manifold \( \mathcal{V} \); indeed, under a general Kähler gauge transformation

\[ K (z, \overline{z}) \rightarrow K (z, \overline{z}) + f (z) + \overline{f} (\overline{z}) \] (5.1)

the holomorphic sections (having Kähler weights \((2, 0)\)) transform as

\[ X^A (z) \rightarrow X^A (z) e^{-f(z)}, \quad F^A (z) \rightarrow F^A (z) e^{-f(z)}, \] (5.2)

and therefore, Eq. (2.5) yields that \( Z \) transforms by a pure phase:

\[ Z (z; \overline{z}; e, m) \rightarrow Z (z; \overline{z}; e, m) e^{-\frac{1}{2} (f(z) - \overline{f(\overline{z})})} = Z (z; \overline{z}; e, m) e^{-i m f(z)}. \] (5.3)

By recalling Eq. (2.3), it is instead clear that the “holomorphic central charge” function \( W (z; e, m) \), like the holomorphic sections \( X^A (z) \) and \( F^A (z) \), is a section of the line (Hodge) bundle [5], since it transforms as

\[ W (z; e, m) \rightarrow W (z; e, m) e^{-f(z)}. \] (5.4)
If we introduce the Kähler gauge-invariant real function \[23\] (well defined if \(|W| (z, \bar{z}; e, m) \neq 0)\)
\[
G (z, \bar{z}; e, m) \equiv K (z, \bar{z}) + \ln \left|W\right|^2 (z, \bar{z}; e, m) \] (5.5)
\[
\downarrow
\]
\[
e^G (z, \bar{z}; e, m) = e^K (z, \bar{z}) \left|W\right|^2 (z, \bar{z}; e, m) = \left|Z\right|^2 (z, \bar{z}; e, m), \] (5.6)
it is easy to check that the potential \(V_{BH}\) given by Eq. (2.1) may be rewritten as
\[
V_{BH} (z, \bar{z}; e, m) = e^G (z, \bar{z}; e, m) \left\{ 1 + g^{i j} (z, \bar{z}) \partial_i G (z, \bar{z}; e, m) \partial_j G (z, \bar{z}; e, m) \right\}. \] (5.7)
Thus, for a given black hole charge configuration \((e_\Lambda, m^\Lambda)\) and non-vanishing \(Z\), an equivalent expression of the definitions (2.2) and (2.4) of the BPS, supersymmetric Attractor points reads as follows:
\[
\partial_i G (z, \bar{z}; e, m) = 0, \quad \forall i = 1, ..., n_V; \tag{5.8}
\]
at such points \(V_{BH}\) reduces to
\[
V_{BH} |_{\partial G = 0} = e^G |_{\partial G = 0} = \left|Z\right|^2 |_{\partial G = 0}. \tag{5.9}
\]
Moreover, at supersymmetric Attractors Eq. (2.7) yields
\[
(D_i D_j V_{BH}) |_{\partial G = 0} = (\partial_i \partial_j V_{BH}) |_{\partial G = 0} = 0,
\]
\[
(D_i \overline{D_j} V_{BH}) |_{\partial G = 0} = (\partial_i \overline{\partial_j} V_{BH}) |_{\partial G = 0} = 2 (g^{i j} V_{BH}) |_{\partial G = 0} =
\]
\[
= 2 (g^{i j} e^G) |_{\partial G = 0}. \tag{5.10}
\]
Thus, one obtains once again that for a positive definite metric tensor \(g^{i j} |_{\partial G = 0}\) (as it should be in regular SKG) all the critical points of \(V_{BH}\) satisfying the condition (5.8) are actually minima of \(V_{BH}\), corresponding to BPS Attractors.

Let us now consider the relevance of the above results in relation to the entropic functional given by Gukov, Saraikin and Vafa (GSV) in Eq. (3.1) of [9]. It reads
\[
S_{GSV} = \frac{\pi}{4} e^{-K}. \tag{5.11}
\]
Since it holds that
\[
e^{-K} = e^{-G} |_{W = 1} = \left|Z\right|^{-2} |_{W = 1}, \tag{5.12}
\]
it is clear that \(S_{GSV}\) is \(\frac{\pi}{4}\) times \(e^{-G}\) in a Kähler gauge in which \(W = 1:\)
\[
S_{GSV} = \frac{\pi}{4} e^{-K} = \frac{\pi}{4} e^{-G} |_{W = 1} = \frac{\pi}{4} \left|Z\right|^{-2} |_{W = 1}. \tag{5.13}
\]
Therefore, due to the Kähler gauge-invariance of \( G \), in a regular SKG framework with non-vanishing \( Z \), the second Kähler-covariant derivatives of \( S_{GSV} \) at the BPS Attractors read as follows:

\[
(D_iD_jS_{GSV})_{G=0} = (D_iD_je^{-G})_{G=0} = (\partial_i\partial_je^{-G})_{G=0} =
\]

\[
= -(\partial_i\partial_j G)_{G=0} e^{-G}|_{G=0} = 0; \quad (5.14)
\]

\[
(D_iD_jS_{GSV})_{G=0} = (D_iD_je^{-G})_{G=0} = (\overline{\partial}_i\overline{\partial}_j e^{-G})_{G=0} =
\]

\[
= -(\overline{\partial}_i\overline{\partial}_j G)_{G=0} e^{-G}|_{G=0} < 0, \quad (5.15)
\]

where the results of [1], concerning the strict positive definiteness of the Hessians of \( V_{BH} \) and \( |Z| \) at the supersymmetric Attractors, have been used; they imply that, in the assumptions made above, at \( \partial_i G = 0 \; \forall \; i = 1, \ldots, n_V \) the Hessian of the Kähler gauge-invariant function \( G \) is strictly positive definite.

Summarizing, for a given black hole charge configuration \((e_A, m^A)\), in the regular SKG and for non-vanishing \( Z \), Eqs. (5.14) and (5.15) yield that all the moduli configurations satisfying the BPS condition (5.8) are maxima of the GSV entropic functional \( S_{GSV} \) or of its rigorous Kähler gauge-invariant completion \( e^{-G} \). Such a result confirms the recent observation of [24].

Finally, it is worth pointing out that it is hard to give a meaning to the Hessian of \( S_{GSV} \) or of its rigorous Kähler gauge-invariant completion \( e^{-G} \) at non-BPS, non-supersymmetric Attractor points. Indeed, such points are extrema of the whole “black hole potential” function \( V_{BH} \) given by Eqs. (2.1) and (5.7), but not of (the inverse of) its BPS, supersymmetric part, proportional to \( S_{GSV} \) and \( e^{-G} \).

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