A semiclassical formulation of the spin Hall effect for physical systems satisfying Dirac-like equation is introduced. We demonstrated that the main contribution to the spin Hall conductivity is given by the spin Chern number whether the spin is conserved or not at the quantum level. We illustrated the formulation within the Kane-Mele model of graphene in the absence and in the presence of the Rashba spin-orbit coupling term.

1 Introduction

In response to the electric field in ferromagnets a spontaneous Hall current can be generated. A semiclassical formulation of this anomalous Hall effect was given in [1] within the Fermi liquid theory. There the anomalous Hall conductivity was calculated considering the equations of motion in the presence of the Berry gauge fields derived from the Bloch wave function. When this system is subjected to an external magnetic field definition of the particle density and the electric current should be done appropriately. Nevertheless the computed value of the anomalous Hall conductivity remains unaltered [2]. Hall currents without a magnetic field can be generated also in fermionic systems described by Dirac-like Hamiltonians [3]. Taking into account the spin of electrons these systems yield Hall currents due to the spin transport which is known as the spin Hall effect [4] or Chern insulator. We would like to present a semiclassical formulation of the spin Hall conductivity using a differential form formalism for fermions which are described by Dirac-like Hamiltonians.

In semiclassical kinetic theory the spin degrees of freedom can be considered by treating them as dynamical variables. However to calculate the spin Hall conductivities it would be more appropriate to keep the Hamiltonian and the related Berry gauge fields as matrices in “spin indices”. In this respect a differential form formalism was presented in [5]. Dynamical variables in this semiclassical formalism are the usual space coordinates and momentum but the symplectic form is matrix valued. We will show that this formalism is suitable to calculate the spin Hall conductivity for Dirac-like systems. We deal with electrons, so that without loss of generality we consider the third component of spin

\[ E-mail \text{ addresses: dayi@itu.edu.tr, yunt@itu.edu.tr} \]
denoted by $S_z$, whose explicit form depends on the details of the underlying Dirac-like Hamiltonian. When the third component of spin is conserved at the quantum level, constructing spin current is not intriguing. However spin Hall effect can persist even if the third component of spin is not conserved. In the latter case semiclassical definition of the spin Hall conductivity is not very clear. Within the Kane-Mele model of graphene (2 + 1 dimensional topological insulator) [4] it was argued that one cannot anymore use the Berry curvature to obtain the main contribution to the spin Hall effect when the spin nonconserving Rashba term is present [6]. We will show that even for the systems where the spin is not a good quantum number it is always possible to establish the leading contribution to the spin Hall effect in terms of the Berry field strength derived in the appropriate basis. Moreover, we will demonstrate that it is always given in terms of the spin Chern number which is defined to be one half the difference of the Chern numbers of spin-up and spin-down sectors [7].

The formulation will be illustrated within the Kane-Mele model of graphene: When only the intrinsic spin-orbit coupling is present the third component of the spin is a good quantum number and the spin Hall conductivity can be acquired straightforwardly in terms of the Berry curvature [8]. When the Rashba term is switched on the third component of spin ceases to be conserved. Nevertheless, we will show that by choosing the correct basis one can still establish the leading contribution to the spin Hall conductivity by the Berry curvature. It is given by the spin Chern number calculated in [9].

The starting point of the method is the matrix valued symplectic form [5, 10]. We will show in Appendix A that it can be obtained in terms of the wave packets formed by the positive energy solutions of Dirac-like equations adapting the formalism of [11, 12].

The formalism of deriving the velocities of phase space variables in terms of the phase space variables themselves will be presented in Section 2. It leads to the anomalous Hall effect straightforwardly as we will discuss briefly in Section 3. Definition of the spin current is presented in Section 4. It is shown that if one adopts the correct definition of the spin current in two space dimensions the essential part of the spin Hall conductivity is always given by the spin Chern number. We will illustrate the method by applying it to the Kane-Mele model first in the absence and then in the presence of Rashba coupling in Section 5. The last section is devoted to discussions of the results obtained.

2 Semiclassical Formalism

We deal with electrons which effectively satisfy Dirac-like equations. In Appendix A we presented the semiclassical theory established in terms of the wave packet composed of positive energy solutions. It yields a semiclassical description of the system whose dynamics is governed by gauge fields which are matrices labeled by “spin indices”. It is so called because the basis of the wave packets are solutions of a Dirac-like Hamiltonian. Obviously range of this index depends on the spacetime dimension as well as on the intrinsic properties of the system considered.

In d+1 dimensions, we deal with the following matrix valued one-form,

$$\eta_H = p_a dx_a + [ea^a_{\text{ext}}(x,t) + a_a(x,p)]dx_a + A_a(x,p)dp_a - H(x,p)dt.$$ 

$(x_a, p_a); a = 1, 2, ..., d,$ denote the classical phase space variables and $e > 0$, is the electron charge. $a_a(x,p,t)$ and $A_a(x,p,t)$ are the matrix-valued Berry gauge potentials. $H(x,p) = H_0(p) - ea^a_{\text{ext}}(x)$ comprises of $H_0$ which is the diagonalized Dirac-like free Hamiltonian projected on positive energies and the electromagnetic scalar field $a^a_{\text{ext}}$. We suppress the unit matrices. The related symplectic
two-form is defined by
\[ w_H = d\eta_H - i\eta_H \wedge \eta_H = dp_a \wedge dx_a + F + G + M - \left( e \frac{\partial a^e_{ext}}{\partial t} + \frac{\partial H}{\partial x_a} + i[H, a_a] \right) dx_a \wedge dt \] (2.1)

\[- \left( \frac{\partial H}{\partial p_a} + i[H, A_a] \right) dp_a \wedge dt.\]

For \( a_a = 0 \), this coincides with the matrix-valued two form considered in [5]. \( F = \frac{1}{2} F_{ab} dx_a \wedge dx_b, \) \( G = \frac{1}{2} G_{ab} dp_a \wedge dp_b, \) and \( M = \frac{1}{2} M_{ab} dp_a \wedge dx_b \) are the two-forms with the following components,

\[ F_{ab} = \frac{\partial a_b}{\partial x_a} - \frac{\partial a_a}{\partial x_b} - i[a_a, a_b] + e \left( \frac{\partial a^e_{ext}}{\partial x_a} - \frac{\partial a^e_{ext}}{\partial x_b} \right), \]

\[ M_{ab} = \frac{\partial A_b}{\partial p_a} - \frac{\partial A_a}{\partial x_b} - i[A_a, a_b], \]

\[ G_{ab} = \frac{\partial A_b}{\partial p_a} - \frac{\partial A_a}{\partial p_b} - i[A_a, A_b]. \]

In order to obtain the equations of motion, we introduce the matrix valued vector field
\[ \tilde{v} = \frac{\partial}{\partial t} + \dot{x}_a \frac{\partial}{\partial x_a} + \dot{p}_a \frac{\partial}{\partial p_a}. \] (2.2)

Here, \((\dot{x}_a, \dot{p}_a)\) are the matrix-valued time evolutions of the phase space variables \((x_a, p_a)\). This is analogous to the situation in the canonical formulation of the Dirac particle where the velocities are matrices though the phase space variables are ordinary vectors. The equations of motion are derived by demanding that the interior product of \( w_H, (2.1) \), with the matrix-valued vector field \( \tilde{v}, (2.2) \), vanish:

\[ i_{\tilde{v}} w_H = 0. \]

The resulting equations are

\[ \dot{p}_a = \dot{x}_c F_{ac} + e_a - M_{ca} \dot{p}_c, \]

\[ \dot{x}_a = G_{ca} \dot{p}_c - f_a - \dot{x}_c M_{ac}, \]

where in terms of the external electric field \( E_a = \frac{\partial a^e_{0} / \partial x_a - \partial a^e_{ext} / \partial t, \) we defined

\[ e_a \equiv e E_a + i[H_0, a_a], \]

\[ f_a \equiv -\frac{\partial H_0}{\partial p_a} + i[H_0, A_a]. \]

The Lie derivative of the volume form \( \Omega_{d+1} = (-1)^{d+1} w_H^d \wedge dt \) can be used to attain the matrix-valued velocities \((\dot{x}_a, \dot{p}_a)\) in terms of the phase space variables \((x_a, p_a)\). We will illustrate it for \( d = 2 \), due to the fact that basically we are interested in \( 2 + 1 \) spacetime dimensional Dirac-like systems. In \( 2 + 1 \) dimensions, where the extended phase space is 5 dimensional, the volume form reads

\[ \Omega_{2+1} = -\frac{1}{2} w_H \wedge w_H \wedge dt. \] (2.3)

We express it through the canonical volume element of the phase space \( dV \), as

\[ \Omega_{2+1} = \tilde{w}_{1/2} dV \wedge dt, \] (2.4)
where \( \tilde{w}_{1/2} \) is the Pfaffian of the following \( 4 \times 4 \) matrix,

\[
\begin{pmatrix}
F_{ij} & -\delta_{ij} - M_{ij} \\
\delta_{ij} + M_{ij} & -G_{ij}
\end{pmatrix}.
\]

We do not treat the spin indices on the same footing with the phase space indices \((x_i, p_i); \ i = 1, 2\). Thus the Pfaffian \( \tilde{w}_{1/2} \) is a matrix in spin indices. The Lie derivative associated with the vector field \((2.2)\) of the volume form \((2.4)\) can be expressed formally as

\[
L_v \Omega_{2+1} = (i_vd + d_i v)(\tilde{w}_{1/2}dV \wedge dt) = \left( \frac{\partial}{\partial t} \tilde{w}_{1/2} + \frac{\partial}{\partial x_i}(\dot{x}_i \tilde{w}_{1/2}) + \frac{\partial}{\partial p_i}(\tilde{w}_{1/2} \dot{p}_i) \right) dV \wedge dt. \tag{2.5}
\]

Actually, to obtain it explicitly one should employ the definition \((2.3)\) yielding

\[
L_v \Omega_{2+1} = -\frac{1}{2} d\tilde{w}^2_H.
\]

Comparing the exterior derivative of

\[
w_H \wedge \tilde{w}_H = dp_i \wedge dx_i \wedge dp_j \wedge dx_j + 2M \wedge dp_i \wedge dx_i + 2e_j dx_j \wedge dt \wedge dp_j \wedge dx_j
\]

\[
+ \frac{1}{2} (F f_i + f_i F) \wedge dp_i \wedge dt + (M e_i + e_i M) \wedge dx_i \wedge dt
\]

\[
+ 2f_i dp_i \wedge dt \wedge dp_j \wedge dx_j + F \wedge G + G \wedge F + (M f_i + f_i M) \wedge dp_i \wedge dt
\]

\[
+ \frac{1}{2} (G e_i + e_i G) \wedge dx_i \wedge dt,
\]

with the formal expression \((2.5)\) one obtains the solutions

\[
\tilde{w}_{1/2} = 1 - M_{ii} - \frac{1}{4} (F_{ij} G_{ij} + G_{ij} F_{ij}), \tag{2.6}
\]

\[
\dot{x}_i \tilde{w}_{1/2} = -f_i + (M_{ij} f_j + f_j M_{ij}) - (M_{jj} f_i + f_i M_{jj}) - \frac{1}{2} (G_{ij} e_j + e_j G_{ij}), \tag{2.7}
\]

\[
\tilde{w}_{1/2} \dot{p}_i = e_i - (M_{ji} e_j + e_j M_{ji}) + (M_{jj} e_i + e_i M_{jj}) + \frac{1}{2} (F_{ji} f_j + f_j F_{ji}). \tag{2.8}
\]

These solutions are useful even for Schrödinger type Hamiltonian systems where the origin of the Berry gauge fields will be different. Indeed, to illustrate the power of the differential form method in general we would like to deal briefly with the anomalous Hall effect in two dimensions.

### 3 Anomalous Hall Effect

The intrinsic anomalous Hall effect in ferromagnetic materials arise from the Berry curvature in the crystal momentum space of Bloch electrons either in the absence or in the presence of an external magnetic field \([1, 2]\). In the latter case one should define the electric current by taking corrections to the path integral measure into account. The anomalous Hall conductivity can be derived within the formalism of Section \([2]\). Obviously, in this case the Berry gauge fields are derived from the occupied Bloch states. Consider the electrons which are constrained to move in the \(xy\)-plane in the presence of the constant magnetic field in the \(z\)-direction \(F_{xy} = B\), and the Berry curvature \(G_{xy}\). The equations of motion \((2.6)-(2.8)\) become

\[
\sqrt{w} = 1 - BG_{xy}, \tag{3.1}
\]

\[
\sqrt{w} \dot{x}_i = \frac{\partial H}{\partial p_i} - e \epsilon_{ij} \mathcal{E}_j G_{xy}, \tag{3.2}
\]

\[
\sqrt{w} \dot{p}_i = e \mathcal{E}_i + \epsilon_{ij} \frac{\partial H}{\partial p_j} B.
\]
(3.1) is the correction to the path integral measure. Hence, the correct definition of electric current is

\[ j_i = e \int \frac{d^2p}{(2\pi \hbar)^2} \sqrt{w\dot{x}_i}f(x, p, t), \]

where \( f(x, p, t) \) is the ground state distribution (occupation) function. Plugging (3.2) into this definition one obtains the total electric current as

\[ j_i = e \int \frac{d^2p}{(2\pi \hbar)^2} \left( \frac{\partial H}{\partial p_i} - e\epsilon_{ij}E_jG_{xy} \right) f(x, p, t). \]

The term proportional to the electric field yields the anomalous Hall current

\[ j_i^{AH} = -e^2\epsilon_{ij}E_j \int \frac{d^2p}{(2\pi \hbar)^2} G_{xy}f(x, p, t) \equiv \sigma_{AH}\epsilon_{ij}E_j, \]

where \( \sigma_{AH} \) denotes the anomalous Hall conductivity. For electrons obeying Fermi-Dirac distribution at zero temperature, the ground state distribution is given by the theta-function at the Fermi energy \( E_F \) : \( f = \theta(E - E_F) \). Thus, the anomalous Hall conductivity reads

\[ \sigma_{AH} = -e^2 \int_{E > E_F} \frac{d^2p}{(2\pi \hbar)^2} G_{xy}. \]

On the other hand the first Chern number, which is a topological invariant, is defined by

\[ N_1 = \frac{1}{2\pi \hbar} \int d^2pG_{xy}. \]

Therefore, one concludes that the anomalous Hall conductivity

\[ \sigma_{AH} = -\frac{e^2}{2\pi \hbar} N_1, \]

is a topological invariant.

4 Spin Chern Number vs Spin Hall Conductivity

The semiclassical currents of the electrons obeying Dirac-like equations should be defined in terms of the velocity which is weighted with the correct measure \( \dot{x}_a \tilde{w}_1/2 \). Recall that it is a matrix in spin indices. We only deal with the spin current generated by the third component of spin \( S_z \), though any spin component can be studied similarly. The most convenient representation is

\[ S_z = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \text{(4.1)} \]

where the dimension of the unit matrix \( I \) depends on the system considered. To define the spin current one also needs to introduce the ground state distribution functions \( f^\dagger(x, p, t) \) and \( f^\dagger(x, p, t) \) for the electrons with spin-up and spin-down. In the representation (4.1) we can define the distribution matrix by

\[ f = \begin{pmatrix} f^\dagger & 0 \\ 0 & f^\dagger \end{pmatrix}, \]
where the unit matrix $I$ is suppressed. Now, the appropriate choice for the semiclassical spin current seems to be
\[ j^z_a = \frac{\hbar}{2} \int \frac{d^d p}{(2\pi\hbar)^d} \text{Tr} \left[ S_z \dot{x}_a \tilde{w}_{1/2} f \right]. \tag{4.2} \]

Basis of the matrix representation are the positive energy solutions of the underlying Dirac-like equation (see Appendix A). If they are not eigenfunctions of the spin matrix $S_z$ simultaneously, definition (4.2) does not make sense. Hence, to adopt (4.2) as the definition of the spin current we should choose the basis functions with a definite spin. Once this is done we can set the ground state distribution functions to unity by restricting our integrals to energies higher than the ground state energy. However, this is already the case because we deal with the wave packet composed of the positive energy solutions. Now, in $d = 2$, let us consider the spin Hall current which results from the last term in (2.7):
\[ j^{SH}_i = -e \mathcal{E}_j \frac{\hbar}{2} \int \frac{d^2 p}{(2\pi\hbar)^2} \text{Tr} \left[ S_z G_{ij} f \right] \equiv \sigma^{SH}_{ij} \mathcal{E}_j. \]

We are obliged to choose the basis which are spin eigenvalues so that spin Hall conductivity can be expressed as
\[ \sigma^{SH} = -\frac{e}{2\pi} C_s, \]
where the spin Chern number
\[ C_s = \frac{1}{2} (N^\uparrow - N^\downarrow), \tag{4.3} \]
is one half the difference of the spin-up and spin-down first Chern numbers defined by
\[ N^{\uparrow,\downarrow} = \frac{1}{2\pi \hbar} \int d^2 p \ \text{Tr} G_{xy}^{\uparrow,\downarrow}. \]

We demonstrated that the spin Hall conductivity is given by the spin Chern number (4.3), which is a topological invariant characterizing the spin Hall effect. Hence, it will be the main contribution to the spin Hall conductivity if the spin Hall phase exists. This is the main conclusion of this work. In the following section we will illustrate this formalism by applying it to the Kane-Mele model of graphene which is also known as Chern insulator in $2 + 1$ dimensions.

### 5 Kane-Mele Model

Time reversal invariant $2 + 1$ dimensional topological insulator can be formulated as the spin Hall effect in graphene within the Kane-Mele model described by the Hamiltonian
\[ H = v_F \sigma_x \tau_z p_x + v_F \sigma_y p_y + \Delta_{SO} \sigma_z \tau_z s_z + \lambda_R (\sigma_x \tau_z s_y - \sigma_y s_x). \tag{5.1} \]

It is the effective theory of electrons on graphene with the Fermi velocity $v_F$. The intrinsic and Rashba spin-orbit coupling constants are denoted by $\Delta_{SO}$ and $\lambda_R$, respectively. $\sigma_{x,y,z}$ are the Pauli matrices in the representation $\sigma_z = \text{diag}(1,-1)$, which act on the states of sublattices. $\tau_z = \text{diag}(1,-1)$, labels the states at the Dirac points (valleys) $K$ and $K'$, and the Pauli matrices, $s_{x,y,z}$ act on the real spin space in the representation where the third component is diagonal $s_z = \text{diag}(1,-1)$.

The main difference between the Kane-Mele model with and without the Rashba spin-orbit coupling term lies in whether the third component of spin is a good quantum number or not. In the former case $s_z$ is conserved and application of the semiclassical approach is straightforward. However, also in the latter case the spin Hall conductivity is non-vanishing with the condition $\Delta_{SO} > \lambda_R$. We will illustrate how the semiclassical formulation can be applied in both cases and demonstrate that main contribution to the spin Hall conductivity is always given by the spin Chern number defined in [7].
5.1 The $\lambda_R = 0$ Case

In this case the Hamiltonian is

$$H^{SO} = v_F \sigma_x \tau_z p_x + v_F \sigma_y p_y + \Delta_{SO} \sigma_z \tau_z s_z. \quad (5.2)$$

The Foldy-Wouthuysen transformation [8]

$$U = \frac{\sigma_z \tau_z s_z H^{SO} + E}{\sqrt{2E(E + \Delta_{SO})}},$$

diagonalizes the Hamiltonian (5.2):

$$\mathcal{H}^{SO} = U H^{SO} U^\dagger = E \sigma_z \tau_z s_z,$$

where $E = \sqrt{v_F^2 p^2 + \Delta_{SO}^2}$. $U$ can be employed to acquire the eigenfunctions of (5.2) as

$$u^{(\alpha)}(p) = U^\dagger v^{(\alpha)},$$

where $v^{(1)} = (1, 0, \ldots, 0)^T, \ldots, v^{(8)} = (0, 0, \ldots, 8)^T$. The Hamiltonian projected on positive energy eigenstates in the presence of the external electric field $\mathcal{E}$ is

$$H_0^{SO} = (E + e\mathcal{E} \cdot x) 1_\tau 1_s.$$

In the rest of this section we will keep the unit matrices explicit. The Berry gauge field can be shown to be

$$A_i = \frac{\hbar v_F^2}{2E(E + \Delta_{SO})} \epsilon_{ij} p_j 1_\tau s_z.$$

Hence, the corresponding Berry curvature, $G_{xy} = (\partial_{p_x} A_y - \partial_{p_y} A_x)$, is

$$G_{xy} = -\frac{\hbar v_F^2 \Delta_{SO}^2}{2E^3} 1_\tau s_z.$$

In the absence of a magnetic field the phase space measure (2.6) is trivial: $\tilde{w}_{1/2} = 1$. Thus, the equations of motion (2.7)-(2.8) yield

$$\dot{x}_i = -\frac{v_F^2 p_i}{E} \tau_z s_z - e\epsilon_{ij} \mathcal{E}_j G_{xy},$$
$$\dot{p}_i = e\mathcal{E}_i 1_\tau 1_s.$$

In the representation which we adopted the third component of spin becomes

$$S_z = 1_\tau s_z. \quad (5.3)$$

Note that $u^{(\alpha)}$ are also the eigenstates of the spin matrix (5.3). Therefore, the distribution matrix $f = 1_\tau \text{diag}(f^\dagger, f^\dagger)$ is adequate to define the spin current by

$$j_i^{S} = \frac{\hbar}{2} \int \frac{d^2 p}{(2\pi\hbar)^2} \text{Tr} [S_z \sqrt{w} \dot{\tilde{x}}_i f].$$

It yields the spin Hall current $j_i^{SH} = \sigma_{S,H} \epsilon_{ij} \mathcal{E}_j$, where the spin Hall conductivity is given by

$$\sigma_{S,H} = -\frac{e\hbar}{2} \int \frac{d^2 p}{(2\pi\hbar)^2} \text{Tr} [S_z G_{xy}]. \quad (5.4)$$
Let us decompose (5.4) such that the contributions arising from spin subspace and \(K, K'\) valleys become apparent. One can easily observe that

\[
\sigma_{SH} = -\frac{e\hbar}{2} \int \frac{d^2p}{(2\pi\hbar)^2} (G_{xy}^K - G_{xy}^{K'} + G_{xy}^{K} - G_{xy}^{K'})
\]

\[
= -\frac{e}{4\pi} (N_1^{\uparrow K} - N_1^{\downarrow K} + N_1^{\uparrow K'} - N_1^{\downarrow K'}).
\]

Each contribution is associated with the first Chern number of the related subspace. This has been observed in [8] where the related Chern numbers were calculated. We conclude that the spin Hall conductivity is proportional to the sum of the spin Chern number of the \(K\) valley, \(C_s^K\) and the spin Chern number of the \(K'\) valley, \(C_s^{K'}\):

\[
\sigma_{SH} = -\frac{e}{2\pi} (C_s^K + C_s^{K'}) = -\frac{e}{2\pi} C_s = -\frac{e}{2\pi}.
\]

In the absence of Rashba term we defined the spin current straightforwardly since the Hamiltonian (5.2) commutes with \(s_z\).

### 5.2 The \(\lambda_R \neq 0\) Case

Although, in the presence of the Rashba term \(s_z\) does not commute with the Hamiltonian (5.1), the spin Hall effect still exists for \(\Delta_{SO} > \lambda_R\) [4] [14] [15]. However, the semiclassical calculation is not clear as we discussed in Section 4. There we also discussed the correct definition of spin current. Nevertheless, before proceeding as indicated in Section 4 let us carry on with the computation of the Berry gauge field naively using the positive energy eigenfunctions of (5.1).

The \(K\) and \(K'\) subspaces corresponding to \(\tau_z = \pm 1\) yield the same energy eigenvalues and eigenstates which are presented in Appendix B. Thus, it is sufficient to consider only the \(4 \times 4\) Hamiltonian in \(K\) subspace denoted by \(H_K\):

\[
H_K \Phi_\alpha = E_\alpha \Phi_\alpha,
\]

where \(\alpha = 1, ..., 4\) and the energy eigenvalues \(E_\alpha\) are

\[
E_1 = \lambda + \sqrt{(\Delta_{SO} + \lambda)^2 + v_F^2 p^2}, \quad E_2 = -\lambda + \sqrt{(\Delta_{SO} - \lambda)^2 + v_F^2 p^2},
\]

\[
E_3 = \lambda - \sqrt{(\Delta_{SO} + \lambda)^2 + v_F^2 p^2}, \quad E_4 = -\lambda - \sqrt{(\Delta_{SO} - \lambda)^2 + v_F^2 p^2}.
\]

We deal with the coupling constants satisfying \(\Delta_{SO} > 2\lambda_R\), so that \(E_1, E_2\) and \(E_3, E_4\) are positive and negative, respectively.

The diagonalized Hamiltonian is \(H^\Phi_K = \text{diag}(E_1, E_2, E_3, E_4)\). When we project on the positive energy eigenstates and take both of the contributions coming from the \(K\) and \(K'\) subspaces, the Hamiltonian becomes

\[
H_0^\Phi = 1_\tau \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} + e\mathbf{E} \cdot \mathbf{x} 1_\tau 1_s.
\]

The Berry gauge field turns out to be Abelian:

\[
A^\Phi_t = \hbar \epsilon_{ij} \frac{p_J}{p^J} 1_\tau \begin{pmatrix} -1 & 2N_1 N_2 \\ 2N_1 N_2 & -1 \end{pmatrix}.
\]

The corresponding Berry curvature \(G^\Phi_{xy} = (\partial_x A^\Phi_y - \partial_y A^\Phi_x)\), can easily be computed as

\[
G^\Phi_{xy} = 1_\tau \begin{pmatrix} 0 & -2\hbar \partial_y (N_1 N_2) \\ -2\hbar \partial_y (N_1 N_2) & 0 \end{pmatrix}.
\]
According to (2.6)-(2.8) the equations of motion calculated in the energy eigenfunction basis are
\[ \dot{\tilde{x}}_i = -\frac{1}{\tau} v^2 p_i \begin{pmatrix} \frac{1}{E_1 - \lambda} & 0 \\ 0 & \frac{1}{E_2 + \lambda} \end{pmatrix} + 2 N_1 N_2 \frac{p_j}{p^2} \epsilon_{ij} p_j (E_1 - E_2) 1_\tau s_y - e \epsilon_{ij} \epsilon_j 1_\tau G_{xy}^\Phi, \]
\[ \dot{\tilde{p}}_i = e E_i 1_\tau 1_s, \]
where we set \( \tilde{w}_{1/2} = 1 \).

The spin current cannot be defined by (4.2) with a diagonal \( f \). Choosing it diagonal would lead to a vanishing spin Hall current due to the fact that
\[ \text{Tr} \left[ S_z G_{xy}^\Phi \right] = 0. \]
The difficulty stems from the fact that energy eigenfunctions are not simultaneously eigenstates of the spin operator \( S_z = \text{diag}(s_z, s_z) \). In \( K \) subspace eigenstates of the spin operator \( S_z \), constructed from the energy eigenstates \( \Phi_\alpha \), are
\[ \Psi_1 = \frac{1}{\sqrt{2}} (\Phi_1 + \Phi_2), \Psi_2 = \frac{1}{\sqrt{2}} (\Phi_1 - \Phi_2), \]
\[ \Psi_3 = \frac{1}{\sqrt{2}} (\Phi_3 + \Phi_4), \Psi_4 = \frac{1}{\sqrt{2}} (\Phi_3 - \Phi_4). \]
\( \Psi_1, \Psi_2 \) and \( \Psi_3, \Psi_4 \) correspond to positive and negative energy sectors, respectively. They satisfy
\[ S_z \Psi_1,3 = \Psi_1,3, \quad S_z \Psi_2,4 = -\Psi_2,4. \]
The Hamiltonian in \( \Psi_\alpha \) basis is obtained by the transformation
\[ \mathcal{H}_K^\Psi = U_\Psi H_K U_\Psi^\dagger, \]
where \( U_\Psi^\dagger = (\Psi_1 \quad \Psi_2 \quad \Psi_3 \quad \Psi_4) \). Notice that \( U_\Psi \) is related to the unitary transformation that diagonalizes \( H_K \), denoted by \( U_\Phi \), via \( U_\Psi = RU_\Phi \), where \( R = \begin{pmatrix} \tilde{R} & 0 \\ 0 & \tilde{R} \end{pmatrix} \) with
\[ \tilde{R} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \]
Thus we acquired
\[ \mathcal{H}_K^\Psi = (1_\tau R) \mathcal{H}_K^\Phi (R 1_\tau) \]
\[ = \frac{1}{2} \begin{pmatrix} E_1 + E_2 & E_1 - E_2 & 0 & 0 \\ E_1 - E_2 & E_1 + E_2 & 0 & 0 \\ 0 & 0 & E_3 + E_4 & E_3 - E_4 \\ 0 & 0 & E_3 - E_4 & E_3 + E_4 \end{pmatrix} 1_\tau. \]
The Hamiltonian in \( \Psi_\alpha \) basis projected on positive energy eigenstates in the presence of external electric field \( \mathcal{E} \), is
\[ H_0^\Psi = (E_1 + E_2) 1_\tau 1_s + (E_1 - E_2) 1_\tau s_x + e \mathcal{E} \cdot x 1_\tau 1_s. \]
The basis transformation \( \tilde{R} \) sustains the connection between \( A_\Psi \) and \( A_\Phi \) via the relation \( A_\Psi = (1_\tau \tilde{R}) A_\Phi (\tilde{R} 1_\tau) \), so that
\[ A_{i}^\Psi = \hbar \epsilon_{ij} \frac{p_j}{p^2} 1_\tau \begin{pmatrix} -1 + 2 N_1 N_2 & 0 \\ 0 & -1 + 2 N_1 N_2 \end{pmatrix} \equiv 1_\tau \begin{pmatrix} A_i^\uparrow & 0 \\ 0 & A_i^\downarrow \end{pmatrix}. \]
The corresponding Berry curvature $G_{xy}^\Psi = (1_\tau \hat{R})G_{xy}^{\Phi}(\hat{R}1_\tau)$ is calculated as

$$G_{xy}^\Psi = -\frac{2\hbar}{p} \frac{\partial}{\partial p} \left[ N_1 N_2 \right] \frac{1}{\tau s_z \equiv \frac{1}{\tau s_z}} ( G_{xy}^\uparrow 0 0 \ G_{xy}^\downarrow ) .$$

Hence, the equations of motion are

$$\dot{x}_i = -\tau v^2 F_{pi} ( 1_E - \lambda + \frac{1}{E_1 - \lambda} - \frac{1}{E_2 + \lambda} ) - 2 \frac{N_1 N_2}{p^2} \epsilon_{ij} p_j (E_1 - E_2) 1_\tau s_y - e \epsilon_{ij} E_j G_{xy}^\Psi ,$$

$$\dot{p}_i = e E_i 1_\tau .$$

The spin Hall current can now be written as

$$j_i^{SH} = -\frac{e \hbar}{2} \epsilon_{ij} \epsilon_j \int \frac{d^2 p}{(2\pi \hbar)^2} \text{Tr}[S_z G_{xy}^\Psi f] ,$$

where $f = 1_\tau \text{diag}(f^\uparrow, f^\downarrow)$. Therefore, $f$ restricts the integral to positive energies and the spin Hall conductivity becomes

$$\sigma_{SH} = -\frac{e}{2\pi C_s} \left( C_s^K + C_s^K' \right) = -\frac{e}{2\pi} C_s .$$

In [13] this spin Chern number is calculated as $C_s = 1$. Therefore, we conclude that

$$\sigma_{SH} = -\frac{e}{2\pi} .$$

Indeed, in [4] it was argued that the value of the spin Hall conductivity slightly differs from this value which is confirmed either in terms of numerical methods [14] or deriving the related effective theory [15].

6 Discussion

In 2+1 dimensions we established the anomalous Hall conductivity as well as the spin Hall conductivity from the term linear in the electric field and the Berry curvature in $\dot{x}_i \hat{w}_{1/2}$. This anomalous velocity term survives in any $d + 1$ spacetime dimension:

Independent of the spacetime dimension and the origin of the Berry curvature in the time evolution of the coordinates there is always a term which is linear in both electric field and the Berry field strength,

$$\frac{\partial}{\partial \mathcal{E}_a} (\hat{w}_{1/2} \dot{x}_b) |_{\mathcal{E} = 0} \propto G_{ab} .$$

In the basis where a certain component of spin is diagonal this term will be diagonal. Therefore, procedure of calculating the spin Hall conductivity can be generalized to any dimension. However, topological origin of this conductivity should be discussed within the underlying physical system.
Appendix A

Dirac equation possesses negative and positive energy solutions. Obviously one can form a wave packet by superposing only positive energy solutions. However, relativistic invariance of the Dirac theory demands to superpose both positive and negative solutions. Nevertheless by ignoring the relativistic momenta one can only deal with a wave packet composed of positive energy solutions. Indeed this is the starting point of the semiclassical approximation. We denote the spinor corresponding to a positive energy solution of Dirac equation by $u^{(α)}(p, x_c)$, which is a function of the momentum $p$, and the position of the wave packet center in coordinate space $x_c$:

$$H_0(p)u^{(α)}(p, x_c) = E_α u^{(α)}(p, x_c); \quad E_α > 0.$$ 

The normalization is $u_+^{(α)}(p, x_c)u_+^{(β)}(p, x_c) = δ_{αβ}$. Let us consider the following wave packet obtained by superposing only positive energy solutions labeled by the superscript $α$,

$$\Psi_x ≡ \Psi_x(p_c, x_c) = \int [dp] |a(p, t)| e^{-iγ(p, t)} ∫ α ξ_α ψ_x^{(α)}(p, x_c), \quad (A.1)$$

where $[dp]$ denotes the measure of the $d$ dimensional momentum space. The distribution $|a(p, t)| e^{-iγ(p, t)}$ has a peak at the wave packet center $p_c$, and satisfies $∫ |a|^2 dp = 1$. The expansion coefficients $ξ_α$ are also normalized, $∑_α |ξ_α|^2 = 1$. $ψ_x^{(α)}(p, x_c)$ is composed of two parts

$$ψ_x^{(α)}(p, x_c) = u^{(α)}(p, x_c)φ_x(p), \quad (A.2)$$

with

$$φ_x(p) = \frac{1}{(2π)^{d/2}} e^{ip·x}.$$ 

The normalization is

$$∫ [dp] φ_x^+(p)φ_y(p) = δ(x − y).$$

When the position operator, $\hat{x}$ acts on $φ_x(p)$ we get

$$\hat{x}φ_x(p) = -i ∂pφ_x(p) = x φ_x(p).$$

The completeness relation is $∫ [dx] φ_x^+(p)φ_x(q) = δ(p − q)$. As a result of these definitions, (A.2) has the following normalization

$$∫ [dx] ψ_x^{(α)}(p, x_c)ψ_x^{(β)}(q, x_c) = δ_{αβ}δ(p − q). \quad (A.3)$$

We would like to calculate the expectation value of the position operator over the wave packet (A.1), which is equivalent to $x_c = ∫ [dx] |Ψ_x|^2 x$. The calculation proceeds as follows; we first calculate $\hat{x}Ψ_x$, in which we use

$$\hat{x}ψ_x^{(α)} = u^{(α)}(p, x_c)\hat{x}φ_x(p) = u^{(α)}(p, x_c)x φ_x(p) = u^{(α)}(p, x_c)(-i ∂p φ_x(p).$$

Now, integrating by parts we obtain

$$\hat{x}Ψ_x = i ∫ [dp] \frac{∂|a(p, t)|}{∂p} e^{-iγ(p, t)} ∫ α ξ_α ψ_x^{(α)}(p, x_c)φ_x(p) + ∫ [dp]|a(p, t)| e^{-iγ(p, t)} ∫ α ξ_α u^{(α)}(p, x_c)φ_x(p) + i ∫ [dp]|a(p, t)| e^{-iγ(p, t)} ∫ α ξ_α \frac{∂u^{(α)}(p, x_c)}{∂p} φ_x(p).$$
Then we reach the following result
\[
\int [dx] \Psi_x^\dagger \dot{\Psi}_x = i \int [dp] |a(p, t)| \frac{\partial |a(p, t)|}{\partial p} + \int [dp] |a(p, t)|^2 \frac{\partial \gamma(p, t)}{\partial p} + \int [dp] |a(p, t)|^2 \sum_{\alpha, \beta} \xi_{\beta}^* u^{\dagger(\beta)}(p, x_c) \frac{\partial u^{(\alpha)}(p, x_c)}{\partial p} \xi_\alpha.
\] (A.4)

The first term vanishes since \( \int |a|^2 dp = 1 \). The second and the third terms are obtained using (A.3).

The distribution has the mean momentum, \( p_c \), defined through the integral
\[
p_c = \int [dp] p |a(p, t)|^2.
\]

Moreover, for any function \( f(p) \), we get
\[
f(p_c) = \int [dp] f(p) |a(p, t)|^2.
\] (A.5)

Using the definition (A.5) in (A.4), we obtain
\[
x_c = \frac{\partial \gamma_c}{\partial p_c} + \sum_{\alpha, \beta} \xi_{\beta}^* u^{\dagger(\beta)}(p_c, x_c) \frac{\partial}{\partial p_c} u^{(\alpha)}(p_c, x_c) \xi_\alpha,
\] (A.6)

where \( \gamma_c \equiv \gamma(p_c, t) \). We would like to obtain the one-form \( \eta \), which is defined through \( dS \):
\[
dS \equiv \int [dx] \Psi_x^\dagger (id - H_0 dt) \Psi_x = d\gamma_c + \sum_{\alpha} \xi_{\alpha} \eta^{\alpha \beta} \xi_\beta.
\]

We start by computing
\[
d\Psi_x = dt \frac{\partial \Psi_x}{\partial t} + dx_c \frac{\partial \Psi_x}{\partial x_c}
\]
\[
= dt \int [dp] \frac{\partial a(p, t)}{\partial t} e^{-i\gamma(p, t)} \sum_{\alpha} \xi_{\alpha} u^{(\alpha)}(p, x_c) \phi_x(p)
- idt \int [dp] a(p, t) \frac{\partial \gamma(p, t)}{\partial t} e^{-i\gamma(p, t)} \sum_{\alpha} \xi_{\alpha} u^{(\alpha)}(p, x_c) \phi_x(p)
+ dx_c \int [dp] a(p, t) e^{-i\gamma(p, t)} \sum_{\alpha} \xi_{\alpha} \frac{\partial}{\partial x_c} u^{(\alpha)}(p, x_c) \phi_x(p).
\]

So that we obtain
\[
\int [dx] \Psi_x^\dagger id\Psi_x = dt \frac{\partial \gamma_c}{\partial t} + dx_c \sum_{\alpha, \beta} \xi_{\beta}^* u^{\dagger(\beta)}(p_c, x_c) \frac{\partial}{\partial x_c} u^{(\alpha)}(p_c, x_c) \xi_\alpha.
\]

To transform the first term, we use \( d\gamma_c = dt \frac{\partial \gamma_c}{\partial t} + dp_c \frac{\partial \gamma_c}{\partial p_c} \) and (A.6):
\[
\int [dx] \Psi_x^\dagger id\Psi_x = d\gamma_c - dp_c \cdot x_c - idp_c \sum_{\alpha, \beta} \xi_{\beta}^* u^{\dagger(\beta)}(p_c, x_c) \frac{\partial}{\partial x_c} u^{(\alpha)}(p_c, x_c) \xi_\alpha + idx_c \sum_{\alpha, \beta} \xi_{\beta}^* u^{\dagger(\beta)}(p_c, x_c) \frac{\partial}{\partial p_c} u^{(\alpha)}(p_c, x_c) \xi_\alpha.
\]
Here is a convenient point to define the following matrix valued Berry gauge fields

\[ u^{(\alpha)}(p_c, x_c) \frac{\partial}{\partial x_c} u^{(\beta)}(p_c, x_c) = A^{\alpha\beta}, \]

\[ u^{(\alpha)}(p_c, x_c) \frac{\partial}{\partial p_c} u^{(\beta)}(p_c, x_c) = A^{\alpha\beta}. \]

The Dirac-like free Hamiltonian only depends on the derivatives with respect to \( x \), so that we get

\[ \int [dx] \Psi_x^\dagger H_0 \frac{\partial}{\partial x} \Psi_x = \sum_{\alpha\beta} \xi_\alpha E_\alpha(p_c) \delta^{\alpha\beta} \xi_\beta. \]

Thus, by defining \( H_0^{\alpha\beta} = E_\alpha \delta^{\alpha\beta} \), we obtain

\[ dS = \int [dx] \Psi_x^\dagger (id - H_0 dt) \Psi_x = d\gamma_c - \sum_{\alpha\beta} \xi_\alpha^* \left( dp_c \cdot x_c \delta^{\alpha\beta} - dp_c A^{\alpha\beta} + dx_c A^{\alpha\beta} - H_0^{\alpha\beta} dt \right) \xi_\beta. \]

Therefore, we can define the matrix valued one-form \( \eta^{\alpha\beta} \) by,

\[ \eta^{\alpha\beta} = \delta^{\alpha\beta} x_c \cdot dp_c - A^{\alpha\beta} \cdot dp_c + A^{\alpha\beta} \cdot dx_c - H_0^{\alpha\beta} dt. \]

It governs the dynamics of the wave-packet. Note that under the unitary transformation of the basis \( \hat{u}^{(\alpha)} = U_{\beta\alpha} u^{(\beta)} \), the one-form \( \eta \) transforms as \( \tilde{\eta} = U \eta U^\dagger \).

**Appendix B**

The Hamiltonian for the \( K \) subspace is obtained from (5.1) by setting \( \tau_z = 1 \):

\[ H_K = \begin{pmatrix} \Delta_{SO} s_z & 1_s v_p(p_x + i p_y) + \lambda_R(s_y + i s_x) \\ 1_s v_p(p_x - i p_y) - \Delta_{SO} s_z & \end{pmatrix}. \]

The eigenstates of \( H_K \) corresponding to the energy eigenvalues (5.3) can be shown to be

\[ \Phi_1 = N_1 \begin{pmatrix} \frac{i p_x - i p_y}{v_p(p_x + i p_y)} \\ 1 \end{pmatrix}, \Phi_2 = N_2 \begin{pmatrix} \frac{i p_x - i p_y}{v_p(p_x + i p_y)} \\ -\frac{i p_x - i p_y}{v_p(p_x + i p_y)} \end{pmatrix}, \Phi_3 = N_3 \begin{pmatrix} \frac{i p_x - i p_y}{v_p(p_x + i p_y)} \\ \frac{i p_x - i p_y}{v_p(p_x + i p_y)} \end{pmatrix}, \Phi_4 = N_4 \begin{pmatrix} \frac{i p_x - i p_y}{v_p(p_x + i p_y)} \\ \frac{i p_x - i p_y}{v_p(p_x + i p_y)} \end{pmatrix}, \]

where the normalizations are \( N_\alpha(p) = \frac{v_p}{\sqrt{2(v_p^2 p^2 + (E_\alpha - \Delta_{SO})^2)}} \).

When \( \tau_z = -1 \) is taken in (5.1), the Hamiltonian for the \( K' \) valley is obtained:

\[ H_{K'} = \begin{pmatrix} -\Delta_{SO} s_z & -1_s v_p(p_x - i p_y) - \lambda_R(s_y - i s_x) \\ 1_s v_p(p_x + i p_y) + \lambda_R(s_y + i s_x) & \end{pmatrix}. \]

The eigenstates of \( H_{K'} \) are as follows,

\[ \Phi_5 = N_1 \begin{pmatrix} \frac{i E_1 - \Delta_{SO}}{v_p(p_x + i p_y)} \\ \frac{1}{v_p(p_x + i p_y)} \end{pmatrix}, \Phi_6 = N_2 \begin{pmatrix} \frac{i E_2 - \Delta_{SO}}{v_p(p_x + i p_y)} \\ \frac{-i p_x - i p_y}{v_p(p_x + i p_y)} \end{pmatrix}, \Phi_7 = N_3 \begin{pmatrix} \frac{i E_3 - \Delta_{SO}}{v_p(p_x + i p_y)} \\ \frac{i p_x - i p_y}{v_p(p_x + i p_y)} \end{pmatrix}, \Phi_8 = N_4 \begin{pmatrix} \frac{i E_4 - \Delta_{SO}}{v_p(p_x + i p_y)} \\ \frac{-i p_x - i p_y}{v_p(p_x + i p_y)} \end{pmatrix}. \]

The corresponding energy eigenvalues are given by (5.5) since \( E_5 = E_1, E_6 = E_2, E_7 = E_3, E_8 = E_4 \).
References

[1] F.D.M. Haldane, Phys. Rev. Lett. 93 (2004) 206602.

[2] D. Xiao, J. Shi and Q. Niu, Phys. Rev. Lett. 95 (2005) 137204; C. Duval, Z. Horváth, P. A. Horváthy, L. Martina, and P. Stichel, Phys. Rev. Lett. 96 (2006) 099701; D. Xiao, J. Shi and Q. Niu, Phys. Rev. Lett. 96 (2006) 099701.

[3] F. D. M. Haldane, Phys. Rev. Lett. 61 (1988) 2015.

[4] C. L. Kane and E. J. Mele, Phys. Rev. Lett. 95 (2005) 226801.

[5] Ö. F. Dayi, M. Elbistan, A Semiclassical Formulation of the Chiral Magnetic Effect and Chiral Anomaly in Even d+1 Dimensions, arXiv:1402.4727

[6] D. Culcer, J. Sinova, N. A. Sinititsyn, T. Jungwirth, A. H. MacDonald and Q. Niu, Phys. Rev. Lett. 93 (2004) 046602.

[7] E. Prodan, Phy. Rev. B 80 (2009) 125327.

[8] Ö. F. Dayi, E. Yunt, Phys. Lett. A 375 (2011) 2484.

[9] Y. Yang, Z. Xu, L. Sheng, B. Wang, D. Y. Xing and D. N. Sheng, Phys. Rev. Lett. 107 (2011) 066602.

[10] Ö. F. Dayi, J. Phys. A: Math. Theor. 41 (2008) 315204.

[11] G. Sundaram, Q. Niu, Phys. Rev. B 59, (1999) 14915.

[12] D. Culcer, Y. Yao, Q. Niu, Phys. Rev. B 72, (2005) 085110.

[13] Y. Yang et al, Phys. Rev. Lett. 107, (2011) 066602.

[14] L. Sheng, D. N. Sheng, C. S. Ting and F. D. M. Haldane, Phys. Rev. Lett. 95 (2005) 136602.

[15] Ö. F. Dayi, M. Elbistan, J. Phys. A: Math. Theor. 46 (2013) 435001.