Spaces of real polynomials with common roots

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Let $RX^l_{k,n}$ be the space consisting of all $(n+1)$–tuples $(p_0(z), \ldots, p_n(z))$ of monic polynomials over $\mathbb{R}$ of degree $k$ and such that there are at most $l$ roots common to all $p_i(z)$. In this paper, we prove a stable splitting of $RX^l_{k,n}$.

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1 Introduction

Let $\text{Rat}_k(\mathbb{C}P^n)$ denote the space of based holomorphic maps of degree $k$ from the Riemannian sphere $S^2 = \mathbb{C} \cup \infty$ to the complex projective space $\mathbb{C}P^n$. The basepoint condition we assume is that $f(\infty) = [1, \ldots, 1]$. Such holomorphic maps are given by rational functions:

$\text{Rat}_k(\mathbb{C}P^n) = \{(p_0(z), \ldots, p_n(z)) : \text{each } p_i(z) \text{ is a monic polynomial over } \mathbb{C} \text{ of degree } k \text{ and such that there are no roots common to all } p_i(z)\}$. 

There is an inclusion $\text{Rat}_k(\mathbb{C}P^n) \hookrightarrow \Omega^2_2 \mathbb{C}P^n \simeq \Omega^2 S^{2n+1}$. Segal [6] proved that the inclusion is a homotopy equivalence up to dimension $k(2n - 1)$. Later, the stable homotopy type of $\text{Rat}_k(\mathbb{C}P^n)$ was described by Cohen et al [2; 3] as follows. Let $\Omega^2 S^{2n+1} \simeq \bigvee_{1 \leq q \leq k} D_q(S^{2n-1})$ be Snaith’s stable splitting of $\Omega^2 S^{2n+1}$. Then

\begin{equation}
\text{Rat}_k(\mathbb{C}P^n) \simeq \bigvee_{q=1}^{k} D_q(S^{2n-1}).
\end{equation}

In Kamiyama [4], (1–1) was generalized as follows. We set

$X^l_{k,n} = \{(p_0(z), \ldots, p_n(z)) : \text{each } p_i(z) \text{ is a monic polynomial over } \mathbb{C} \text{ of degree } k \text{ and such that there are at most } l \text{ roots common to all } p_i(z)\}$. 

In particular, $X^0_{k,n} = \text{Rat}_k(\mathbb{C}P^n)$. Let

$f^l(S^{2n}) \simeq S^{2n} \cup e^{4n} \cup e^{6n} \cup \ldots \cup e^{2ln} \subset \Omega S^{2n+1}$
be the $l$th stage of the James filtration of $\Omega S^{2n+1}$, and let $W^l(S^{2n})$ be the homotopy theoretic fiber of the inclusion $J^l(S^{2n}) \hookrightarrow \Omega S^{2n+1}$. We generalize Snaith’s stable splitting of $\Omega^2 S^{2n+1}$ as follows:

$$W^l(S^{2n}) \simeq \bigvee_{1 \leq q} D_q \xi^l(S^{2n}).$$

Then we have a stable splitting

$$X^l_{k,n} \simeq \bigvee_{q=1}^k D_q \xi^l(S^{2n}).$$

The purpose of this paper is to study the real part $RX^l_{k,n}$ of $X^l_{k,n}$ and prove a stable splitting of this. More precisely, let $RX^l_{k,n}$ be the subspace of $X^l_{k,n}$ consisting of elements $(p_0(z), \ldots, p_n(z))$ such that each $p_i(z)$ has real coefficients. Our main results will be stated in Section 2. Here we give a theorem which generalizes (1–1).

Since the homotopy type of $RX^0_{k,1}$ is known (see Example 2.1 (iii)), we assume $n \geq 2$.

In this case, there is an inclusion

$$RX^0_{k,n} \hookrightarrow \Omega S^n \times \Omega^2 S^{2n+1}.$$

(See Lemma 3.1.)

**Theorem 1.1** For $n \geq 2$, we define the weight of stable summands in $\Omega S^n$ as usual, but those in $\Omega^2 S^{2n+1}$ we define as being twice the usual one. Then $RX^0_{k,n}$ is stably homotopy equivalent to the collection of stable summands in $\Omega S^n \times \Omega^2 S^{2n+1}$ of weight $\leq k$. Hence,

$$RX^0_{k,n} \simeq \bigvee_{p+2q \leq k} \sum_{p=0}^{p(n-1)} D_q(S^{2n-1}) \vee \bigvee_{p=1}^k S^{p(n-1)}.$$

This paper is organized as follows. In Section 2 we state the main results. We give a stable splitting of $RX^l_{k,n}$ in Theorem A and Theorem B. In order to prove these theorems, we also consider a space $Y^l_{k,n}$, which is an open set of $RX^l_{k,n}$. We give a stable splitting of $Y^l_{k,n}$ in Proposition C. In Section 3 we prove Proposition C. In Section 4 we prove Theorem A and Theorem B.
2 Main results

We set
\[ Y^l_{k,n} = \{(p_0(z), \ldots, p_n(z)) \in RX^l_{k,n} : \text{there are no real roots common to all } p_i(z)\} \]

The spaces \( Y^l_{k,n} \) and \( RX^l_{k,n} \) are in the following relation:
\[
\begin{align*}
\bigcap \ Y^k_{k,n} & \supset \bigcap \ Y^{k-1}_{k,n} \supset \cdots \supset \bigcap \ Y^1_{k,n} \supset \bigcap \ Y^0_{k,n} = Y^0_{k,n} \\
\bigcap \ RX^k_{k,n} & \supset \bigcap \ RX^{k-1}_{k,n} \supset \cdots \supset \bigcap \ RX^1_{k,n} \supset \bigcap \ RX^0_{k,n} \supset \bigcap \RX^0_{k,n}
\end{align*}
\]
where each subset is an open set. Moreover, \( Y^{2i+1}_{k,n} = Y^{2i}_{k,n} \). In fact, if \( \alpha \in H_+ \) (where \( H_+ \) is the open upper half-plane) is a root of a real polynomial, then so is \( \overline{\alpha} \in H_- \).

We have the following examples.

Example 2.1

(i) It is proved by Mostovoy [5] that \( Y^k_{k,1} \) consists of \( k + 1 \) contractible connected components.

(ii) The following result is proved by Vassiliev [7]. For \( n \geq 3 \), there is a homotopy equivalence \( Y^k_{k,n} \simeq J^k(S^{n-1}) \), where \( J^k(S^{n-1}) \) is as above the \( k \)th stage of the James filtration of \( \Omega S^n \). For \( n = 2 \), these spaces are stably homotopy equivalent.

(iii) It is proved by Segal [6] that
\[ RX^k_{k,n} \simeq \bigcap_{q=0}^{k} \text{Rat}_{\min(q,k-q)}(\mathbb{C}P_1). \]

(iv) \( RX^{k-1}_{k,n} \cong \mathbb{R}^k \times (\mathbb{R}^k)^* \) and \( RX^k_{k,n} \cong \mathbb{R}^k \times (\mathbb{R}^k)^* \).

In fact, \( (p_0(z), \ldots, p_n(z)) \in RX^k_{k,n} \) is an element of \( RX^{k-1}_{k,n} \) if and only if \( p_i(z) \neq p_j(z) \) for some \( i, j \). Hence, the first homeomorphism holds.

Now we state our main results.

**Theorem A** For \( n \geq 1 \) and \( i \geq 0 \), there is a homotopy equivalence
\[ RX^{2i+1}_{k,n} \simeq X^i_{\left[\frac{k}{2}\right], n}, \]
where \( \left[\frac{k}{2}\right] \) denotes as usual the largest integer \( \leq \frac{k}{2} \).
Theorem B  For \( n \geq 1 \) and \( i \geq 0 \), there is a stable homotopy equivalence

\[
RX_{k,n}^{2i} \simeq X_i \left[ \frac{k}{2} \right]_n \vee \sum_{1 \leq p}^{2i} \left( \vee_{p+2q \leq k-2i} \sum_{q=1}^{p(n-1)} D_q(S^{2n-1}) \right) \vee \sum_{p=1}^{k-2i} S^{p(n-1)}.
\]

We study \( RX_{k,n}^l \) by induction with making \( l \) larger. Hence, the induction starts from \( RX_{k,n}^0 = Y_{k,n}^0 \). We study \( Y_{k,n}^l \) by induction with making \( l \) smaller, where the initial condition is given in Example 2.1 (ii). In fact, we have the following proposition.

Proposition C

(i) For \( n \geq 2 \), we define the weight of stable summands in \( \Omega S^n \) as usual, but those in \( W^i(S^{2n}) \) we define as being twice the usual one. Then \( Y_{k,n}^{2i} \) is stably homotopy equivalent to the collection of stable summands in \( \Omega S^n \times W^i(S^{2n}) \) of weight \( \leq k \). Hence,

\[
Y_{k,n}^{2i} \simeq \sum_{p+2q \leq k} \sum_{q=1}^{p(n-1)} D_q(S^{2n}) \vee \sum_{p=1}^{k-2i} S^{p(n-1)}.
\]

(ii) When \( n = 1 \), there is a homotopy equivalence

\[
Y_{k,1}^{2i} \simeq \prod_{q=0}^{k} X_{\min(q,k-q),1}^i.
\]

Note that Proposition C (ii) is a generalization of Example 2.1 (i) and (iii).

3 Proof of Proposition C

We study the space of continuous maps which contains \( Y_{k,n}^k \) or \( RX_{k,n}^0 \). For simplicity, we assume \( n \geq 2 \). (The case for \( n = 1 \) can be obtained by slight modifications.) Each \( f \in Y_{k,n}^k \) defines a map \( f : S^1 \to \mathbb{R} P^n \), where \( S^1 = \mathbb{R} \cup \infty \). Hence, there is a natural map

\[
Y_{k,n}^k \to \Omega k_{\text{mod} 2} \mathbb{R} P^n \simeq \Omega S^n.
\]

Example 2.1 (ii) implies that \( Y_{k,n}^k \) is the \( k(n-1) \)-skeleton of \( \Omega S^n \).
On the other hand, let $\text{Map}^T_k(\mathbb{C}P^1, \mathbb{C}P^n)$ be the space of continuous basepoint-preserving conjugation-equivariant maps of degree $k$ from $\mathbb{C}P^1$ to $\mathbb{C}P^n$. Then there is an inclusion

$$RX^0_{k,n} \hookrightarrow \text{Map}^T_k(\mathbb{C}P^1, \mathbb{C}P^n).$$

**Lemma 3.1** For $n \geq 2$, $\text{Map}^T_k(\mathbb{C}P^1, \mathbb{C}P^n) \simeq \Omega S^n \times \Omega^2 S^{2n+1}$.

**Proof** It is easy to see that

$$\text{Map}^T_k(\mathbb{C}P^1, \mathbb{C}P^n) \simeq \text{Map}^T_0(\mathbb{C}P^1, \mathbb{C}P^n).$$

Since $\text{Map}^T_0(\mathbb{C}P^1, \mathbb{C}P^n)$ can be thought as the space of maps

$$(D^2, S^1, *) \to (\mathbb{C}P^n, \mathbb{R}P^n, *)$$

of degree 0, there is a fibration

$$\Omega^2 S^{2n+1} \to \text{Map}^T_0(\mathbb{C}P^1, \mathbb{C}P^n) \to \Omega S^n.$$
Proof In [4, Propositions 4.5 and 5.4], we constructed a similar long exact sequence from the fact that
\[ X^i_{k,n} - X^{i-1}_{k,n} = \mathbb{C}^l \times \text{Rat}_{k-1}(CP^n), \]
where \( \mathbb{C}^l \times \text{Rat}_{k-1}(CP^n) \) corresponds to the subspace of \( X^i_{k,n} \) consisting of elements \( (p_0(z), \ldots, p_n(z)) \) such that there are exactly \( l \) roots common to all \( p_i(z) \). The proposition is proved similarly using the fact that
\[ Y^{2i}_{k,n} - Y^{2i-2}_{k,n} \cong \text{SP}^i(H^+) \times RX^0_{k-2i,n}, \]
where \( \text{SP}^i(H^+) \) denotes the \( i \)th symmetric product of \( H^+ \).

Proof of Proposition 3.2 In order to prove Proposition 3.2 by induction, we introduce the following total order to \( Y^{2i}_{k,n} \) for \( k \geq 1 \) and \( i \geq 0 \):
\( Y^{2i}_{k,n} < Y^{2i}_{k',n} \) if and only if
(i) \( k < k' \), or
(ii) \( k = k' \) and \( i > i' \).

By Example 2.1 (ii), Proposition 3.2 holds for \( Y^k_{k,n} \). Assuming that Proposition 3.2 holds for \( Y^{2i}_{k,n} \) and \( RX^0_{k-2i,n} \), we prove for \( Y^{2i}_{k,n} \). We have the following long exact sequence:
\[
\begin{align*}
\cdots & \to H_*(X^{i-1}_{[\frac{k}{2}],n}) \to H_*(X^i_{[\frac{k}{2}],n}) \\
& \to \psi H_{*-2in}(\text{Rat}_{[\frac{k}{2}]-l}(CP^n)) \to H_{*-1}(X^{i-1}_{[\frac{k}{2}],n}) \to \cdots
\end{align*}
\]
For \( n \geq 2 \), we consider the homomorphism
\[ 1 \otimes \psi: H_*(\Omega S^n) \otimes H_*(X^i_{[\frac{k}{2}],n}) \to H_*(\Omega S^n) \otimes H_{*-2in}(\text{Rat}_{[\frac{k}{2}]-l}(CP^n)). \]
Restricting the domain to \( H_*(Y^{2i}_{k,n}) \), we obtain the homomorphism \( \phi \) in Lemma 3.3. Now it is easy to prove Proposition 3.2. 

Proof of Proposition C (i) We construct a stable map from the right-hand side of Proposition C (i) to \( Y^{2i}_{k,n} \). Since our constructions are similar, we construct a stable map \( g_{p,q,i,n}: \Sigma^{p(n-1)}D_q\xi^i(S^{2n}) \to Y^{2i}_{k,n} \). First, using the fact that \( RX^0_{1,n} \cong S^{n-1} \) (see Example 2.1 (iv)), there is a stable map \( f_{p,n}: S^{p(n-1)} \to RX^0_{p,n} \). Second, there is a stable section \( e_{q,i,n}: D_q\xi^i(S^{2n}) \to X^i_{q,n} \). Third, there is an inclusion
\[
\eta_{q,i,n}: X^i_{q,n} \hookrightarrow Y^{2i}_{2q,n}.
\]
Spaces of real polynomials with common roots

To construct this, we fix a homeomorphism \( h : \mathbb{C} \rightarrow H_+ \). For \( (p_0(z), \ldots, p_n(z)) \in X_{q,n}^2 \), we write \( p_j(z) = \prod_{s=1}^{q}(z - \alpha_{s,j}) \). Then we set

\[
\eta_{q,i,n}(p_0(z), \ldots, p_n(z)) = \left( \prod_{s=1}^{q}(z - h(\alpha_{s,0}))(z - \overline{h(\alpha_{s,0})}) \right) \ldots \left( \prod_{s=1}^{q}(z - h(\alpha_{s,n}))(z - \overline{h(\alpha_{s,n})}) \right).
\]

Now consider the following composite of maps

\[
(3-3) \quad S^{p(n-1)} \times D_{q,i}^n(S^{2n})f_{p,n}^{\alpha_{q,i,n}} \xrightarrow{\mu} RX_{p,n}^0 \times Y_{2q,n}^{2i} \xrightarrow{\mu} Y_{p+2q,n}^{2i} \hookrightarrow Y_{k,n}^{2i},
\]

where \( \mu \) is a loop sum which is constructed in the same way as in the loop sum \( \text{Rat}_k(\mathbb{C} P^n) \times \text{Rat}_l(\mathbb{C} P^n) \rightarrow \text{Rat}_{k+l}(\mathbb{C} P^n) \) in Boyer–Mann [1]. We can construct \( g_{p,q,i,n} \) from (3–3).

Note that the stable map for Proposition C (i) is compatible with the homology splitting by weights. Using Proposition 3.2, it is easy to show that this map induces an isomorphism in homology, hence is a stable homotopy equivalence. This completes the proof of Proposition C (i).

**Proof of Proposition C (ii)** By a similar argument to the proof of Proposition 3.2, we can calculate \( H_*(Y_{k,1}^{2i}) \). Then we can construct an unstable map from the right-hand side of Proposition C (ii) to \( Y_{k,1}^{2i} \) in the same way as in Proposition C (i). \( \square \)

### 4 Proof of Theorem A and Theorem B

**Proposition 4.1** The homologies of the both sides of Theorem A or Theorem B are isomorphic.

**Proof** We prove the proposition about \( RX_{k,n}^l \) by induction with making \( l \) larger. As in Lemma 3.3, there is a long exact sequence

\[
\cdots \rightarrow H_*(RX_{k,n}^l) \rightarrow H_*(RX_{k,n}^{l+1}) \rightarrow H_{*-l+1}(RX_{k-(l+1),n}) \xrightarrow{\Theta} H_{*+l}(RX_{k,n}) \rightarrow \cdots.
\]

This sequence is constructed from the following decomposition as sets

\[
RX_{k,n}^{l+1} - RX_{k,n}^l = \coprod_{a+2b=l+1} SP^a(\mathbb{R}) \times SP^b(H_+) \times RX_{k-(l+1),n}^0.
\]
and the fact that $H_c^*(\text{SP}^a(\mathbb{R})) = 0$ for $a \geq 2$, where $H_c^*$ is the cohomology with compact supports.

Assuming that the proposition holds for $l \leq 2i + 1$, we determine $H_*(RX_{k,n}^{2i+2})$. The homomorphism $\Theta$ is given as follows. Note that Theorem B is equivalent to

$$RX_{k,n}^{2i} \cong X_{[\frac{1}{2}],n}^i \vee (2i+1)n^{-1} (RX_{k-2i-1,n}^0 \vee S^0).$$

From inductive hypothesis, we have

$$H_{*(2i+2)}(RX_{k-2i-2,n}^0) \cong H_{*(2i+2)}(\text{Rat}_{[\frac{1}{2}]}(\mathbb{C}P^n)) \oplus H_{*(2i+2)}(\Sigma^{n-1} RX_{k-2i-3,n}^0 \vee S^{n-1})$$

and

$$H_{*+1}(RX_{k,n}^{2i+1}) \cong H_{*+1}(X_{[\frac{1}{2}],n}^i).$$

Recall the homomorphism $\theta$ in (3–1) with $i$ replaced by $i+1$. Then $\Theta$: (4–2) $\rightarrow$ (4–3) is given by mapping the first summand by $\theta$ and the second summand by 0. Hence, $H_*(RX_{k,n}^{2i+2})$ is isomorphic to the homology of the right-hand side of (4–1) with $i$ replaced by $i+1$.

By a similar argument, we can determine $H_*(RX_{k,n}^{2i+1})$ inductively by assuming the truth of the proposition for $l \leq 2i$. This completes the proof of Proposition 4.1. □

Finally, we construct an unstable map (resp. a stable map) from the right-hand side of Theorem A (resp. (4–1)) to $RX_{k,n}^{2i+1}$ (resp. $RX_{k,n}^{2i}$). First, the unstable map from the right-hand side of Theorem A or the first stable summand in (4–1) is essentially the inclusion

$$X_{q,n}^i \xrightarrow{\eta_{q,i,n}} Y_{2q,n}^{2i} \subset RX_{2q,n}^{2i},$$

where $\eta_{q,i,n}$ is defined in (3–2). Next, the stable map from the second stable summand in (4–1) is constructed in the same way as in $g_{p,q,i,n}$ (see (3–3)) using the fact that $RX_{2i+1,n}^{2i} \cong S^{(2i+1)n-1}$ (see Example 2.1 (iv)). This completes the proofs of Theorem A and Theorem B. □

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Geometry & Topology Monographs, Volume 10 (2007)
Spaces of real polynomials with common roots

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