Contact Structures and Periodic Fundamental Groups

Hansjörg Geiges and Charles B. Thomas

1 Introduction

This paper is concerned with the existence of contact structures on (connected, closed, orientable) 5-manifolds with certain finite fundamental groups. As such, it constitutes a sequel to [6] (which gave corresponding existence results for highly connected manifolds of arbitrary (odd) dimension and some ad hoc results for finite fundamental groups) and our joint paper [7], where we showed that every 5-manifold $M$ with fundamental group $\pi_1(M) = \mathbb{Z}_2$ and universal cover $\tilde{M}$ a spin manifold can be obtained from one of ten ‘model manifolds’ by surgery along a link of 2-spheres and, as an application of this structure theorem, that every manifold of this kind admits a contact structure.

In the present paper we combine the ideas of [7] with those of the extensive literature on the existence of positive scalar curvature (psc) metrics – in particular [10, 13, 14, 15] (see also [8] and [16] for more recent surveys on this literature) – to arrive at the following existence result.

**Theorem 1** Let $\pi$ be a finite group of odd order $|\pi|$ and finite cohomological period. Furthermore, assume that $|\pi|$ is not divisible by 9. Then every closed 5-dimensional spin manifold $M$ with fundamental group $\pi_1(M) \cong \pi$ admits a contact structure.

Finite groups of odd order and finite cohomological period are metacyclic with presentation

$$\{x, y | x^n = y^n = 1, yxy^{-1} = x^r, \gcd((r - 1)n, m) = 1, r^n \equiv 1 \mod m\}.$$  

This is an old result of Burnside, and it includes the class of cyclic groups ($m = 1$). Geometrically, these groups are characterized (among groups of odd order) by the property of acting freely and smoothly on some homotopy sphere. The case when $n = 3$ and $r$ is a primitive cube root of 1 mod $m$ is of special
interest for 5-manifold topology, because the corresponding groups act freely (but not linearly) on $S^5$. These groups were discussed in [6], and Theorem 27 of that paper is a corollary of Theorem 1 above. More about the connection with that previous paper will be said at the end of Section 6, where we take a more geometric view at some of the algebraic arguments in Sections 4 and 5.

The assumption that $|\pi|$ be odd and not divisible by 9 would seem to be a defect of the proof rather than a defect of nature. In fact, a large portion of the theory developed in this paper extends to arbitrary groups of finite cohomological period. But the case when $|\pi|$ is even (which, as regards the existence of psc metrics, has been tackled successfully in [1]) seems to present difficulties of a different order.

For further motivation and some historical comments we refer the reader to the introduction of [6]. To avoid undue repetition, we assume the reader to be familiar with the basic definitions of contact geometry and the fundamental results of contact surgery, due to Eliashberg [4] and Weinstein [19], as expounded in Sections 2 and 3 of [6] or the corresponding sections of [7]. On the other hand, while (equivariant) cobordism arguments have become standard fare in the literature on psc metrics, this is certainly only true to a much smaller extent in the contact geometric world, so to make this paper reasonably self-contained we have chosen to include some arguments which for anyone familiar with the cited references will certainly cause a sensation of *déjà vu*. We shall allow ourselves, however, to quote liberally from the standard treatise on periodic maps by Conner and Floyd [3].

We now briefly recall the main features of contact surgery on which we shall rely later on (all details can be found in the beginning sections of our two earlier papers). In particular, we wish to emphasize the minor but nonetheless important differences when comparing this with the surgical arguments for manifolds with psc metrics. Whereas for the latter any surgery is permitted up to codimension 3, the restrictions on contact surgeries are:

- Contact surgery is only possible up to the middle dimension.
- The sphere along which surgery is performed has to be isotropic, i.e. tangent to the contact structure, and it must have trivial conformal symplectic normal bundle.
- The framing of the surgery is fixed up to a change in trivializing the conformal symplectic normal bundle (CSN).

Because of our restriction to dimension 5, the first point does not entail any differences between the two theories (contact structures or psc metrics). The second condition is controlled by an $h$-principle and can be guaranteed by
requiring the given contact structure to have first Chern class $c_1$ evaluating to zero on 2-spheres. As regards the third condition, for surgeries along 1-spheres the rank of the CSN is high enough to allow the realization of any topologically possible framing, and for 2-surgeries we have no choice of framing because of $\pi_2(SO_3) = 0$. This may serve as an indication that corresponding existence results for contact structures in higher dimensions will be much harder to come by.

2 Periodic fundamental groups

A finite group $\pi$ is said to have periodic cohomology (or simply to be periodic) if there is some $d > 0$ such that $H^n(\pi) \cong H^{n+d}(\pi)$ for all $n > 0$, and the least such $d$ is called the period of $\pi$.

We shall use two well-known facts about periodic groups (cf. [2, VI.9]):

1. Each Sylow subgroup of $\pi$ is cyclic or a generalized quaternion group (so only the former happens if $|\pi|$ is odd). Indeed, this statement is equivalent to $\pi$ having periodic cohomology.

2. $H_2(\pi) = 0$ (in fact, $H_n(\pi) = 0$ for $n$ even, $n \geq 2$).

Recall from [8] that a contact structure $\xi = \ker \alpha$ on a 5-manifold $M$ (where $\alpha$ is a 1-form with $\alpha \wedge (d\alpha)^2 \neq 0$) induces a reduction of the structure group of the tangent bundle $TM$ to $U(2) \times 1$. On an orientable 5-manifold $M$ such a reduction exists if and only if the third integral Stiefel-Whitney class $W_3(M) = \beta w_2(M) \in H^3(M; \mathbb{Z})$ vanishes (where $\beta$ denotes the Bockstein operator of the coefficient sequence $\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2$), or equivalently, if the second Stiefel-Whitney class $w_2(M) \in H^2(M; \mathbb{Z}_2)$ admits an integral lift $c_1 \in H^2(M; \mathbb{Z})$ (Given $\xi$, such an integral lift is provided by the first Chern class of the conformally symplectic bundle $(\xi, d\alpha) \subset TM$).

The following simple observation shows that we need not be concerned with this topological obstruction if $\tilde{M}$ is spin and $\pi_1(M)$ periodic (and $M$ orientable).

**Lemma 2** Let $\pi$ be a group with periodic cohomology and $M$ a manifold with $\pi_1(M) \cong \pi$ and universal cover $\tilde{M}$ a spin manifold. Then $W_3(M) = 0$.

**Remark.** This lemma is only included for completeness and future reference (and to indicate what the optimal statement subsuming Theorem 1 might be). When we restrict attention to fundamental groups of odd order $|\pi|$, then $H_2(\tilde{M}; \mathbb{Z}_2) \to H_2(M; \mathbb{Z}_2)$ is surjective (because any 2-cycle in $M$ admits a $|\pi|$-fold covering by a 2-cycle in $\tilde{M}$), and hence $w_2(M) = 0$ if and only if $w_2(\tilde{M}) = 0$. The arguments

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in the present paper require that $M$ be spin, but in Theorem 1 is proved for $\pi_1(M) = \mathbb{Z}_2$ under the weaker assumption that $\tilde{M}$ be spin.

Proof. By a theorem of Hopf (cf. [2]) there is an exact sequence
\[
\pi_2(M) \to H_2(M) \to H_2(\pi),
\]
where the first map is the Hurewicz homomorphism. From statement (2) above we deduce that the Hurewicz homomorphism is surjective for periodic fundamental groups. Furthermore, the universal covering map $\tilde{M} \to M$ induces an isomorphism $\pi_2(\tilde{M}) \to \pi_2(M)$.

Combining this with the assumption $w_2(\tilde{M}) = 0$, we find that $w_2(M)$ maps to zero under the natural homomorphism
\[
H^2(M; \mathbb{Z}_2) \to \text{Hom}(H_2(M), \mathbb{Z}_2).
\]

Now consider the commutative diagram built from the Bockstein exact sequence and the universal coefficient theorem:
\[
\begin{array}{ccc}
\text{Ext}(H_1(M), \mathbb{Z}) & \to & H^2(M; \mathbb{Z}) \\
\downarrow & & \downarrow \\
\text{Ext}(H_1(M), \mathbb{Z}_2) & \to & H^2(M; \mathbb{Z}_2) \to \text{Hom}(H_2(M), \mathbb{Z}_2) \\
& & \downarrow_{\beta} \\
& & H^3(M; \mathbb{Z})
\end{array}
\]

By the right exactness of $\text{Ext}(G, -)$, the homomorphism between the Ext groups in this diagram is surjective. Then a simple diagram chase allows to conclude that $W_3(M) = \beta w_2(M) = 0$. 

3 Contact groups and a reduction theorem

For any finite group $\pi$ let $\Omega^\text{Spin}_5(B\pi)$ be the 5-dimensional spin bordism group of $\pi$. In other words, elements of this group are equivalence classes of pairs $(f : V \to B\pi, \sigma)$, where $(V, \sigma)$ is a closed 5-dimensional spin manifold with spin structure $\sigma$ and $f$ is a continuous map into the classifying space of $\pi$, and spin bordant pairs are regarded as equivalent. Define $\text{Cont}^5(\pi) \subset \Omega^\text{Spin}_5(B\pi)$ as the set of all classes with representatives of the form $(f : V \to B\pi, \sigma)$, where $V$ admits a contact structure defining the orientation given by $\sigma$ and with first Chern class $c_1 = 0$ on the image of $\pi_2(V)$ in $H_2(V)$.

Changing from a contact structure $\xi = \ker \alpha$ to $\xi = \ker(-\alpha)$, which amounts to changing the coorientation of $\xi$, changes the orientation determined by the volume form $\alpha \wedge (da)^2$. Thus, if $V$ admits a spin and a contact structure, it does
so for either orientation, which allows to take inverses in Cont\(_5(\pi)\). The sum operation in \(\Omega^5_{\text{Spin}}(B\pi)\) is given by disjoint union, and Cont\(_5(\pi)\) always contains the zero element of \(\Omega^5_{\text{Spin}}(B\pi)\), represented by \(S^5\) and the constant map into \(B\pi\), say, so Cont\(_5(\pi)\) is actually a subgroup of \(\Omega^5_{\text{Spin}}(B\pi)\).

**Theorem 3** Let \((M, \sigma)\) be a connected, closed 5-dimensional spin manifold with fundamental group \(\pi\) and let \(f : M \to B\pi\) be the classifying map of the universal cover \(\tilde{M} \to M\). If \((f : M \to B\pi, \sigma)\) represents an element in Cont\(_5(\pi)\), then \(M\) admits a contact structure.

The following statement is an immediate consequence of this theorem and the fact that Cont\(_5(\pi)\) always contains the zero element.

**Corollary 4** If \((f : M \to B\pi, \sigma)\) as in the theorem represents the zero element in \(\Omega^5_{\text{Spin}}(B\pi)\), that is, if \(M = \partial W\) with \(W\) a compact spin manifold and \(f\) extends over \(W\), then \(M\) admits a contact structure.

Because of \(\Omega^5_{\text{Spin}} = 0\), this corollary includes the result that every simply connected 5-dimensional spin manifold admits a contact structure (see [6] for a stronger theorem in this simply connected case).

In view of Theorem 3 we call \(\pi\) a contact group if

\[
\text{Cont}_5(\pi) = \Omega^5_{\text{Spin}}(B\pi).
\]

Thus, for contact groups the conclusion of Theorem 1 holds. Conversely, the result of [6] implies that \(\mathbb{Z}_2\) is a contact group, if one observes that any class in \(\Omega^5_{\text{Spin}}(B\mathbb{Z}_2)\) can be represented by a manifold with fundamental group \(\mathbb{Z}_2\) (see Section 4 for the corresponding statement for \(\mathbb{Z}_p, p\) an odd prime).

It might seem more attractive to require, in the definition of Cont\(_5(\pi)\), that \(f\) be the classifying map for the universal cover of \(V\). Part of the argument for proving Theorem 3 as it stands could then be used to prove that Cont\(_5(\pi)\) is still a subgroup, and the proof of Theorem 3 with the alternative definition of Cont\(_5(\pi)\) would simplify correspondingly. In some sense, this would be the approach analogous to the one taken by Rosenberg in [14]. The present approach is analogous to that of Kwasik and Schultz [10] and has the advantage that we get similar naturality properties for Cont\(_5(\pi)\) as they get for a corresponding subgroup Pos\(_5(\pi)\) \(\subset\) \(\Omega^5_{\text{Spin}}(B\pi)\).

Before proving Theorem 3 we continue with the general set-up for the proof of Theorem 1.

Given a group homomorphism \(h : \pi \to \pi'\) we have an induced homomorphism

\[
(Bh)_* : \quad \Omega^5_{\text{Spin}}(B\pi) \longrightarrow \Omega^5_{\text{Spin}}(B\pi').
\]

\[\quad (f : V \to B\pi, \sigma) \longmapsto ((Bh) \circ f : V \to B\pi', \sigma).\]
If $h$ is an inclusion, there is a transfer homomorphism

$$(Bh)^! : \Omega^{\text{Spin}}_5(B\pi') \rightarrow \Omega^{\text{Spin}}_5(B\pi),$$

which is defined geometrically as follows: Given

$$(f' : V' \rightarrow B\pi', \sigma') \in \Omega^{\text{Spin}}_5(B\pi'),$$

let $\hat{V} \rightarrow V'$ be the principal $\pi'$-bundle defined by $f'$. Then the subgroup $h(\pi) \equiv \pi$ of $\pi'$ also acts on $\hat{V}$. Set $V = \hat{V}/\pi$, let $f : V \rightarrow B\pi$ be the classifying map of the covering $\hat{V} \rightarrow V$, and lift the spin structure $\sigma'$ on $V'$ to a spin structure $\sigma$ on $V$ via the covering $V \rightarrow V'$. Then define

$$(Bh)^!(f' : V' \rightarrow B\pi', \sigma') = (f : V \rightarrow B\pi, \sigma).$$

We have the following naturality properties of $\text{Cont}_5(\pi)$ with respect to these homomorphisms.

**Lemma 5**

(i) $(Bh)_*$ sends $\text{Cont}_5(\pi)$ to $\text{Cont}_5(\pi')$.

(ii) If $h$ is an inclusion, $(Bh)^!$ sends $\text{Cont}_5(\pi')$ to $\text{Cont}_5(\pi)$.

*Proof.* The first statement is obvious from the construction, and for the second statement we only need to observe that a contact structure on $V'$ with $c_1 = 0$ on 2-spheres lifts to such a structure on $V$. \qed

The following reduction theorem is the direct analogue of Proposition 1.5 in [10].

**Theorem 6** Let $\pi$ be a finite group of odd order, let $p$ be a prime dividing $|\pi|$, and let $j_p : \pi_p \rightarrow \pi$ be the inclusion of a Sylow $p$-subgroup. Then a class $\alpha \in \Omega^{\text{Spin}}_5(B\pi)$ lies in $\text{Cont}_5(\pi)$ if and only if the images $(Bj_p)^!\alpha \in \Omega^{\text{Spin}}_5(B\pi_p)$ under the transfer homomorphism of $j_p$ lie in $\text{Cont}_5(\pi_p)$ for all $p$.

The proof of this theorem can in principle be taken word for word from the cited paper. For the reader’s convenience we reproduce this proof in Section 6, including additional details of the ‘standard’ transfer arguments used by Kwasik and Schultz. For our computations in the subsequent sections we have to discuss the Atiyah-Hirzebruch bordism spectral sequence, and with details about this spectral sequence at hand the mentioned transfer arguments become quite transparent.

Using property (1) of periodic groups, we see that it suffices now to prove Theorem 1 for cyclic groups $\pi = \mathbb{Z}_p^k$ with $p$ an odd prime (and $k = 1$ only for $p = 3$). With Theorem 3 in mind we see that we are left with showing that
these cyclic groups are contact groups. This will be done in the following two sections.

**Proof of Theorem 3.** Write \( M_0 = M, f_0 = f \). By assumption, there is a closed (but not necessarily connected) 5-dimensional spin manifold \( M_1 \) admitting a contact structure with \( c_1 = 0 \) on 2-spheres, and a map \( f_1 : M_1 \to B\pi \) spin bordant to \( f_0 \). That is, we have a 6-dimensional compact spin manifold \( W \) (which we may assume to be connected) with boundary \( \partial W = M_1 - M_0 \), inducing the given spin structures on \( M_0, M_1 \), and a map \( F : W \to B\pi \) restricting to \( f_i : M_i \to B\pi \) on the boundary components. Write \( j_i \) for the inclusion of \( M_i \) in \( W \) and denote by subscript ‘\#’ induced homomorphisms on homotopy groups. We have the sequence of homomorphisms

\[
\pi_1(M_0) \xrightarrow{j_0\#} \pi_1(W) \xrightarrow{F_\#} \pi, \]

where the composition

\[
F_\# \circ j_0\# = (F \circ j_0)_\# = f_0\#
\]

is an isomorphism by our hypotheses. We thus obtain a split exact sequence

\[
1 \to \ker F_\# \to \pi_1(W) \xrightarrow{F_\#} \pi \to 1.
\]

The group \( \ker F_\# \) is generated by embedded copies of \( S^1 \) in \( W \) not meeting the boundary, and performing surgery along these circles will kill \( \ker F_\# \). The choice of framing lies in \( \pi_1(SO_5) \cong \mathbb{Z}_2 \), and for one of the two framings the surgery will preserve the spin structure.

So we may assume that \( F_\# \) and \( j_0\# \) are isomorphisms. Then the homotopy exact sequence of the pair \( (W, M_0) \) becomes

\[
\pi_2(M_0) \xrightarrow{j_0\#} \pi_2(W) \to \pi_2(W, M_0) \to 0.
\]

Represent a set of elements of \( \pi_2(W) \) generating

\[
\pi_2(W, M_0) \cong \pi_2(W)/j_0\#\pi_2(M_0)
\]

by smoothly embedded 2-spheres which do not meet the boundary (which is possible by the Whitney embedding theorem). Since \( W \) is a spin manifold, these spheres have trivial normal bundle, and surgery along these 2-spheres will kill \( \pi_2(W, M_0) \) and preserve fundamental group and spin structure.

We have thus reduced the problem to the case where \( (W, M_0) \) is 2-connected. A result of Wall [18, Theorem 3] says that homotopical connectivity implies geometrical connectivity in codimension \( \geq 4 \), so \( (W, M_0) \) is actually *geometrically*
2-connected. This means that \( W \), viewed as a cobordism on \( M_0 \), contains only handles of index \( \geq 3 \), and thus \( M_0 \) is obtained from \( M_1 \) by surgery in dimension less than or equal to 2.

It remains to be checked that all these surgeries can be performed as contact surgeries. Clearly there is no problem with 0-surgeries. The choice of framing of contact 1-surgeries lies in \( \pi_1(U_1) \cong \mathbb{Z} \) (the conformal symplectic normal bundle of an \( S^1 \) in a contact 5-manifold has rank 2). The homomorphism \( \pi_1(U_1) \to \pi_1(SO_4) = \mathbb{Z}_2 \) induced by inclusion is surjective, so any topological framing can be realized by a contact surgery. Furthermore, the framing in \( \pi_1(U_1) \) determines \( c_1 \) of the resulting contact manifold, and since all surgeries preserve the spin structure, we can actually ensure that the property \( c_1|\pi_2 = 0 \) is preserved. Then the remaining surgeries along a link of 2-spheres can be performed as contact surgeries as well. \( \square \)

4 Cyclic groups of prime order

In this section we consider the case \( \pi \cong \mathbb{Z}_p \) with \( p \) an odd prime. The fact that all these groups are contact groups is a consequence of the following proposition, since every lens space \( L^5_p \) (indeed, any quotient of \( S^{2n+1} \) under a discrete group acting freely and linearly, cf. [6]) admits a contact structure, and any such structure trivially has \( c_1|\pi_2 = 0 \), since \( \pi_2(L^5_p) = 0 \).

**Proposition 7** We have \( \Omega'_{5\text{Spin}}(B\mathbb{Z}_3) \cong \mathbb{Z}_9 \) and \( \Omega'_{5\text{Spin}}(B\mathbb{Z}_p) \cong \mathbb{Z}_p \oplus \mathbb{Z}_p \) for \( p \) a prime \( \geq 5 \). All these groups are generated by 5-dimensional lens spaces.

This proposition is essentially due to Conner and Floyd as far as the computation of cobordism groups is concerned, and the observation about lens spaces as generators was made by Rosenberg [13]. We have not been able to infer this observation from the reference he quotes, though, and therefore provide our own proof, which actually yields a slightly stronger result (see the statement before Lemma [8]).

**Proof.** Write \( \Omega'_k \) for \( \Omega'_{5\text{Spin}} \) or \( \Omega_k \) and \( \Omega'_k(B\pi) \) for the kernel of the homomorphism

\[
\Omega'_k(B\pi) \longrightarrow \Omega'_k(\{\ast\}) = \Omega'_k
\]

induced by the constant map. Since \( \Omega'_{5\text{Spin}} = 0 \) we have \( \Omega'_{5\text{Spin}}(B\pi) = \Omega'_{5\text{Spin}}(B\pi) \) of course, but for determining this group it is more convenient to work with reduced bordism groups.

There is an Atiyah-Hirzebruch spectral sequence for \( \Omega'_s \) of the form

\[
E^2_{r,s} = \tilde{H}_r(B\pi;\Omega'_s) \Rightarrow \Omega'_s(B\pi)
\]
We have $\tilde{H}_s(B\mathbb{Z}_p) \cong \mathbb{Z}_p$ in positive odd dimensions $r$ and $0$ otherwise, and hence $\tilde{H}_s(B\mathbb{Z}_p; \mathbb{Z}_2) = 0$. Now $\Omega'_1$ has only 2-torsion (cf. [12]). In fact, the relevant groups for us are

$$\Omega'_0 \cong \Omega'_4 \cong \mathbb{Z}, \quad \Omega'_1^{\text{Spin}} \cong \Omega'_2^{\text{Spin}} \cong \mathbb{Z}_2, \quad \text{and} \quad \Omega'_1 = \Omega'_2 = \Omega'_3^{\text{Spin}} = 0.$$ 

Thus the spectral sequence collapses and $E_{r,s}^\infty = E_{r,s}^2$. So we obtain the short exact sequence

$$0 \to H_1(B\mathbb{Z}_p) \to \tilde{\Omega}_5'(B\mathbb{Z}_p) \to H_5(B\mathbb{Z}_p) \to 0,$$

that is,

$$0 \to \mathbb{Z}_p \to \tilde{\Omega}_5'(B\mathbb{Z}_p) \to \mathbb{Z}_p \to 0,$$

where by [3, (7.2)] the homomorphism $\mu$ is given by

$$(f : M \to B\mathbb{Z}_p) \mapsto f_*(M).$$

The map $\Omega^\text{Spin}_s \otimes \mathbb{Z}_p \to \Omega^s \otimes \mathbb{Z}_p$ given by forgetting the spin structure is an isomorphism (we only need this in dimension 4, where it follows from explicit calculations, cf. [3]). Then the 5-lemma applied to the two short exact sequences above (for $\Omega_5$ and $\Omega'_5^{\text{Spin}}$) shows that $\tilde{\Omega}_5^{\text{Spin}}(B\mathbb{Z}_p) \to \tilde{\Omega}_5(B\mathbb{Z}_p)$ is an isomorphism (indeed, this is again true in all dimensions, cf. [4]).

We notice in particular that $\tilde{\Omega}_5(B\mathbb{Z}_p)$ has order $p^2$. For $p = 3$, 5-dimensional lens spaces have order 9 in $\tilde{\Omega}_5(B\mathbb{Z}_3)$ according to [3, (36.1)], hence $\tilde{\Omega}_5(B\mathbb{Z}_3) \cong \mathbb{Z}_9$. For $p \geq 5$, that same theorem states that lens spaces have order $p$. So we can define a splitting for $\mu$ by sending a suitable generator of $H_5(B\mathbb{Z}_p)$ to the class of some 5-dimensional lens space in $\tilde{\Omega}_5(B\mathbb{Z}_p)$, and we see that $\tilde{\Omega}_5(B\mathbb{Z}_p) \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$.

In order to prove that $\Omega_5(B\mathbb{Z}_p)$ is generated by lens spaces also for $p \geq 5$ we appeal to (34.5) of [3], which states that an element in $\tilde{\Omega}_5(B\mathbb{Z}_p)$ is zero if and only if all its mod $p$ Pontrjagin numbers are zero and thus implies that it suffices to find two lens spaces $L^5_p$ (for each $p$) whose pairs of mod $p$ Pontrjagin numbers are linearly independent over $\mathbb{Z}_p$.

We briefly recall the definition of mod $p$ Pontrjagin numbers, cf. [3, (34.4)]. Choose a generator $d_1$ of $H^1(B\mathbb{Z}_p; \mathbb{Z}_p)$ and let $d_2 \in H^2(B\mathbb{Z}_p; \mathbb{Z}_p)$ be the image of $d_1$ under the Bockstein operator of the coefficient sequence $\mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}_p$, followed by mod $p$ reduction. Then $d_1d_2^2$ is a generator of $H^3(B\mathbb{Z}_p; \mathbb{Z}_p)$. Given a 5-dimensional lens space $L^5_p$, let $f : L^5_p \to B\mathbb{Z}_p$ be a classifying map for its universal covering. Specifying a generator for $\pi_1(L^5_p) \cong \mathbb{Z}_p$ amounts to choosing a homotopy class of classifying maps $f : L^5_p \to B\mathbb{Z}_p$. Continue to write $d_i$ for $f^*d_i$, $i = 1, 2$. Further, let $p_1 \in H^4(L^5_p; \mathbb{Z}_p)$ be the mod $p$ reduction of the
first Pontrjagin class of $L^5_p$. Then the mod $p$ Pontrjagin numbers of $L^5_p$ are the integers mod $p$
\[ \beta_0 = (d_1d_2^2, [L^5_p]) \text{ and } \beta_1 = (p_1d_1, [L^5_p]) , \]
where $[L^5_p]$ is the fundamental cycle of $L^5_p$ and $\langle -,- \rangle$ the Kronecker product. Here $\beta_0$ is always nonzero.

The Pontrjagin classes of lens spaces have been computed by Folkman [3], cf. [12]. For the quotient of $S^5 \subset \mathbb{C}^3$ under the action of $\mathbb{Z}_p$ generated by
\[ T: (z_1, z_2, z_3) \mapsto (\alpha_1 z_1, \alpha_2 z_2, \alpha_3 z_3) \]
with $\alpha_j = \exp(2\pi i q_j/p)$ we have
\[ p_1 = (q_1^2 + q_2^2 + q_3^2)d_2^2 \]
(the choice of a generator $T$ determines $d_1$ and hence $d_2$). Replacing $T$ by $T^m$ with $m$ coprime to $p$ amounts to replacing $q_j$ by $mq_j$ ($j = 1, 2, 3$) and $d_i$ by $kd_i$ ($i = 1, 2$) with $mk \equiv 1 \mod p$. So the mod $p$ Pontrjagin numbers of $L^5_p(q_1, q_2, q_3)$ modulo the choice of classifying map $f : L^5_p \to B\mathbb{Z}_p$ are
\[ (\beta_0, \beta_1) = (q_1^2 + q_2^2 + q_3^2)\beta_0 \]
modulo the equivalence relation
\[ (\beta_0, \beta_1) \sim (k^3 \beta_0, k\beta_1) \]
for $k$ not divisible by $p$.

The proof of Proposition 7 is therefore completed with the following lemma, which proves more than we really need, namely, that it is possible to find two lens spaces $L^{5}_{p,1}$ and $L^{5}_{p,2}$ such that $[L^{5}_{p,1}, f_1]$ and $[L^{5}_{p,2}, f_2]$ generate $\tilde{\Omega}_5(B\mathbb{Z}_p)$ for any choice of classifying maps $f_i : L^5_{p,i} \to B\mathbb{Z}_p$ of their universal coverings. In this lemma we write
\[ Q = q_1^2 + q_2^2 + q_3^2 \text{ and } R = r_1^2 + r_2^2 + r_3^2 . \]

\[ \text{Lemma 8 } \text{ For any prime } p \geq 5 \text{ there are triples } (q_1, q_2, q_3) \text{ and } (r_1, r_2, r_3) \text{ of integers mod } p \text{ such that the equation } \]
\[ a(k^3 \beta_0, kQ\beta_0) + b(l^3 \beta'_0, lR\beta'_0) \equiv (0, 0) \mod p \]
has no solution $\beta_0, \beta'_0, a, b, k, l$ (coprime to $p$).
Proof. The pair of equations in the lemma yields
\[ b(Rk^2 - Ql^2)lβ'_0 \equiv 0. \]
Since we are assuming \( l \) to be coprime to \( p \) we can divide mod \( p \) by \( l^2 \) and obtain, by neglecting the factors coprime to \( p \) and replacing \( k^2/l^2 \) by \( k^2 \),
\[ Rk^2 - Q \equiv 0. \]
We begin with \((q_1, q_2, q_3) = (1, 1, 1)\) and \((r_1, r_2, r_3) = (1, 1, 2)\), that is, \( Q = 3 \) and \( R = 6 \). This yields the equation \( 6k^2 - 3 \equiv 0 \), and hence \( 2k^2 - 1 \equiv 0 \), since \( p \geq 5 \). Rewriting this as \( 2k^2 - 1 = (2n + 1)p \) we get
\[ k^2 = np + \frac{p + 1}{2}. \]
So if \((p + 1)/2\) is not a quadratic residue mod \( p \) (e.g. if \( p = 5 \)), we are done. Assume, on the contrary, that it is. Then we may take \( Q = 3 \) and
\[ R = \frac{p + 1}{2} + \frac{p + 1}{2} + 1^2 \equiv 2. \]
This gives \( 2k^2 - 3 \equiv 0 \), and hence
\[ k^2 \equiv \frac{p + 3}{2}. \]
Again, if \((p + 3)/2\) is not a quadratic residue mod \( p \), we are done. But, since we are assuming that \((p + 1)/2\) is a quadratic residue mod \( p \), \( Q \) can also take the value
\[ Q = \frac{p + 1}{2} + \frac{p + 1}{2} + 2^2 \equiv 5, \]
and repeating the argument sufficiently many times (always with \( R = 2 \)) we either find an equation for \( k^2 \) without any solution, or we can realize \( Q \equiv p \equiv 0 \) as a sum of three squares mod \( p \). But then the equation
\[ Rk^2 \equiv Rk^2 - Q \equiv 0 \]
does not have any solution \( k \) coprime to \( p \) if we choose \( R \not\equiv 0 \mod p \), as was desired. \( \square \)

5 Cyclic groups of prime power order

We now show that \( \mathbb{Z}_{p^k} \) is also a contact group, at least for primes \( p \geq 5 \), by the same method as in the previous section.
Proposition 9 Write \( h = h_{k,l} \) for the inclusion \( \mathbb{Z}_p^k \to \mathbb{Z}_p^l, \, k \leq l \). For \( p \geq 5 \) there is a short exact sequence

\[
0 \longrightarrow \Omega^\text{Spin}_5(B\mathbb{Z}_p^{k-1}) \xrightarrow{(Bh)_*} \Omega^\text{Spin}_5(B\mathbb{Z}_p^k) \xrightarrow{(Bh)_!} \Omega^\text{Spin}_5(B\mathbb{Z}_p) \longrightarrow 0,
\]

and \( \Omega^\text{Spin}_5(B\mathbb{Z}_p^k) \) is generated by lens spaces.

Proof. A spectral sequence argument as in the preceding section shows that \( \Omega^\text{Spin}_5(B\mathbb{Z}_p^k) \cong \tilde{\Omega}_5(B\mathbb{Z}_p^k) \) has order \( p^{2k} \). The inclusion homomorphism \( (Bh)_* \) is injective because the corresponding homomorphism on homology is injective and the bordism spectral sequence collapses at the \( E^2 \)-page, cf. \[3, (37.2)\]. Furthermore, the transfer homomorphism \( (Bh)_! \) is surjective, for we have shown that \( \Omega^\text{Spin}_5(B\mathbb{Z}_p) \) is generated by \( \mathbb{Z}_p \)-lens spaces, and every free linear \( \mathbb{Z}_p \)-action on \( S^5 \) extends to a free linear \( \mathbb{Z}_p \)-action. Finally, the composition

\[
(Bh_{k-1,k-1})_!(Bh_{k-1,k})_* : \Omega^\text{Spin}_5(B\mathbb{Z}_p^{k-1}) \longrightarrow \Omega^\text{Spin}_5(B\mathbb{Z}_p^k)
\]

is multiplication by \( p \), the index of \( \mathbb{Z}_p^{k-1} \) in \( \mathbb{Z}_p^k \) (see \[3, (20.2)\]), this is a general statement about the composition of inclusion and transfer for central subgroups. Therefore the composition

\[
\Omega^\text{Spin}_5(B\mathbb{Z}_p^{k-1}) \longrightarrow \Omega^\text{Spin}_5(B\mathbb{Z}_p^k) \longrightarrow \Omega^\text{Spin}_5(B\mathbb{Z}_p^{k-1}) \longrightarrow \Omega^\text{Spin}_5(B\mathbb{Z}_p)
\]

is the zero map, because every element in \( \Omega^\text{Spin}_5(B\mathbb{Z}_p) \) has order \( p \) (here the argument fails for \( p = 3 \)). This proves that the sequence in the proposition is exact, since the order of the middle group is the product of the order of the two outer groups.

Arguing inductively, we assume that \( \Omega^\text{Spin}_5(B\mathbb{Z}_p^{k-1}) \) is generated by lens spaces. Given \( u \in \Omega^\text{Spin}_5(B\mathbb{Z}_p^k) \), we know that \( (Bh)_!(u) \in \Omega^\text{Spin}_5(B\mathbb{Z}_p) \) can be represented by a sum of \( \mathbb{Z}_p \)-lens spaces. Lifting these \( \mathbb{Z}_p \)-actions to \( \mathbb{Z}_p \)-actions, we get a sum \( u_0 \) of \( \mathbb{Z}_p \)-lens spaces such that \( (Bh)_!(u) = (Bh)_!(u_0) \). Notice, however, that the order of a \( \mathbb{Z}_p \)-lens space in \( \Omega^\text{Spin}_5(B\mathbb{Z}_p^k) \) is \( p^k \) \[3, (37.9)\], so the short exact sequence is not split. Then \( u - u_0 = (Bh)_*(u_1) \) with \( u_1 \) represented by a sum of lens spaces by the induction assumption. This proves the proposition. \( \square \)

6 Proof of the reduction theorem

As mentioned earlier, our proof of Theorem \[10\] differs from the corresponding proof in \[10\] only insofar as we include some additional details, and that our situation is a bit simpler because of the restriction to dimension five and to odd order groups.
Proof of Theorem. One direction of the theorem is the content of Lemma. For the converse, we now assume that we are given \( \alpha \in \Omega_{\text{Spin}}^5(B\pi) \) with \((Bj_p)_!\alpha \in \text{Cont}_5(\pi)\) for all primes \( p \) dividing \(|\pi|\), and we need to show that \( \alpha \in \text{Cont}_5(\pi) \).

Write \( T_p \) for the composition \((Bj_p)_! (Bj_p)^!\). While the composition of inclusion and transfer (in this order) can be computed, at least for normal subgroups (we used this in the proof of Proposition), this is not true, in general, for a composition of transfer and inclusion. We circumvent this problem by reducing the computation of \( T_p \) on bordism groups to that of the corresponding homomorphism on homology groups.

First we reproduce an elementary algebraic lemma of [10].

Lemma 10 Let \( R \) be a Noetherian ring, \( \Omega \) a finitely generated \( R \)-module, and \( T \) an automorphism of \( \Omega \). If \( P \) is a submodule of \( \Omega \) such that \( T(P) \subset P \), then \( T(P) = P \).

Proof. The ascending chain of submodules

\[ P \subset T^{-1}(P) \subset T^{-2}(P) \subset \ldots \]

must terminate, since \( R \) is Noetherian. Thus \( T^{-m}(P) = T^{-m-1}(P) \) for some \( m \), which on applying \( T^{m+1} \) yields \( T(P) = P \). \( \square \)

In the next lemma, \( \mathbb{Z}(p) \) denotes the integers localized at \( p \) and \( \Omega_{\text{Spin}}^5(B\pi)(p) \) the \( p \)-primary component of \( \Omega_{\text{Spin}}^5(B\pi) \).

Lemma 11 For any prime \( p \) dividing \(|\pi|\), the homomorphism \( T_p \otimes \mathbb{Z}(p) \) is an isomorphism of \( \Omega_{\text{Spin}}^5(B\pi)(p) \).

Proof. The cobordism spectral sequence yields the following commutative diagram with exact rows (except for the commutativity this follows from the argument in the proof of Proposition since \( H_*(B\pi) \) admits a \( p \)-primary decomposition with \( p \) ranging over the primes dividing \(|\pi|\), cf. [2, III.10.2]).

\[
\begin{array}{ccc}
H_1(B\pi) & \to & \Omega_{\text{Spin}}^5(B\pi) \\
\downarrow & & \downarrow \\
H_1(B\pi_p) & \to & \Omega_{\text{Spin}}^5(B\pi_p) \\
\downarrow & & \downarrow \\
H_1(B\pi) & \to & \Omega_{\text{Spin}}^5(B\pi) \\
\end{array}
\]

The vertical arrows at the top denote the transfer homomorphism \((Bj_p)^!\), those at the bottom the inclusion homomorphism \((Bj_p)_!\). Commutativity of the squares on the right is proved in [3] (20.3)]. Commutativity of the squares on the left follows similarly by considering the isomorphism \( \mu : \Omega_1(B\pi) \to H_1(B\pi) \).
and the inclusion of $\tilde{\Omega}_1(B\pi)$ in $\tilde{\Omega}_5(B\pi)$ by tensoring with $\Omega_4$ (and the same for $\pi_p$). Alternatively, this can be seen directly from the geometric definitions of the maps in question.

On homology the composition $T_p = (Bj_p)_*(Bj_p)^!$ is multiplication by the index of $\pi_p$ in $\pi$ (cf. [4, III.9.5]). Thus, $T_p \otimes \mathbb{Z}(\pi)$ is an isomorphism on homology localized at $p$, and by the five-lemma applied to the $p$-primary part of the diagram above it is also an isomorphism on $\Omega^{\text{Spin}}_5(B\pi)(p)$. This proves the lemma.

By assumption we have $(Bj_p)^!\alpha \in \text{Cont}_5(\pi_p)$. Then by Lemma 5 we have $T_p\alpha \in \text{Cont}_5(\pi)$. So $(T_p \otimes \mathbb{Z}(\pi))(\alpha(\pi)) \in \text{Cont}_5(\pi)(\pi)$, and $\text{Cont}_5(\pi)(\pi)$ is $(T_p \otimes \mathbb{Z}(\pi))-invariant$ by that lemma. Then it follows from Lemmas 10 and 11 that $\alpha(\pi) \in \text{Cont}_5(\pi)$. Thus, the proof of the reduction theorem.

**Remark.** Even though the bordism spectral sequence no longer collapses for groups of even order, the reduction theorem still holds (by essentially the same argument). Combining this with the fact that $\mathbb{Z}_2$ is a contact group as proved in [7], we see that in Theorem 1 we may actually allow that $|\pi|$ contains a single prime factor 2.

In some instances one can be more specific about the contact manifolds which generate $\Omega^{\text{Spin}}_5(B\pi)$. To illustrate this, we briefly return to the metacyclic groups of the introduction with $m = p^k$ for $p$ some prime greater than or equal to five, $n = 3$, and $r$ a primitive cube root of 1 mod $p^k$. Write $D_{p^k,3}$ for these groups.

As shown by Madsen [11, Theorem 4.13], there is a smooth 5-dimensional spherical space form $M_{p^k,3}$ with fundamental group isomorphic to $D_{p^k,3}$ which is covered by lens spaces $L^5_3 \to M$ and $L^5_{p^k} \to M$ (Madsen’s result is in fact more general). In other words, with $\alpha \in \Omega^{\text{Spin}}_5(BD_{p^k,3})$ denoting the class of $M_{p^k,3}$ and the classifying map of its universal covering, both $(Bj_3)^!\alpha$ and $(Bj_p)^!\alpha$ are represented by lens spaces. By Theorem 3 $\alpha$ lies in $\text{Cont}_5(D_{p^k,3})$, and by Theorem 6 we know that $M_{p^k,3}$ admits a contact structure. By comparison, Theorem 27 of [6] only guarantees the existence of a contact structure on some special 5-dimensional space form with fundamental group $D_{p^k,3}$, obtained via a construction of Petrie.

More can be said, however. The same spectral sequence argument as in the proof of Proposition 7 shows that $\Omega^{\text{Spin}}_5(BD_{p^k,3})$ has order $9p^k$. Now $(Bj_3)^!\alpha$ is represented by a $\mathbb{Z}_3$-lens space and thus has order 9, whereas $(Bj_p)^!\alpha$ has order 9$p^k$. This factor suggests that the contact structure on $M_{p^k,3}$ behaves in a particularly simple way, and we believe that a detailed study of these manifolds could provide further insights into the nature of contact structures in this family. However, this remains an open question.
order \( p^k \). So \( \alpha \) is an element of order (at least) \( 9p^k \) and therefore \( \Omega^{\text{Spin}}_5 (BD_{p^k},3) \) is a cyclic group \( \mathbb{Z}_{9p^k} \), generated by \( \alpha \). We have thus found a 5-dimensional spherical space form with fundamental group \( D_{p^k,3} \) which carries a contact structure and generates \( \Omega^{\text{Spin}}_5 (BD_{p^k},3) \).

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