An Information-Theoretic Proof of a Bound on the Number of Independent Sets in Bipartite Graphs

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Abstract

The present paper provides an information-theoretic proof of Kahn’s conjecture (2001) for a tight upper bound on the number of independent sets in a graph, where our proof applies to bipartite graphs that are regular on one side (the other side may be irregular). It extends the entropy-based proof for regular bipartite graphs (Kahn, 2001). This conjecture has been recently proved for general graphs by a group at MIT (2019), utilizing an interesting approach which is unrelated to information theory.

Keywords: Shannon entropy, Shearer’s lemma, counting, independent sets, graphs.

I. INTRODUCTION

The Shannon entropy and other classical information measures serve as a powerful tool in various combinatorial and graph-theoretic applications (see, e.g., [2], [3], [4], [5], [6], [7], [8], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21]) such as the method of types, applications of Shearer’s lemma, Moore bound for irregular graphs, sub and supermodularity properties of several information measures and their applications, an entropy-based proof of Bregman’s conjecture on the permanent of matrices, and so on.

The enumeration of discrete structures that satisfy certain local constraints, and particularly the enumeration of independent sets in graphs is of interest in discrete mathematics. Many important structures can be modeled by independent sets in a graph, i.e., subsets of vertices in a graph where none of them are connected by an edge. For example, if a graph models some kind of incompatibility, then an independent set in this graph represents a mutually compatible collection. Upper bounding the number of independent sets in a regular graph was motivated in [1] by a conjecture which has several applications in combinatorial group theory. A survey paper on upper bounding the number of independent sets in graphs, along with some of their applications, is provided in [24]. The problem of counting independent sets in graphs received, in general, significant attention in the literature of discrete mathematics over the last three decades, and also in the information theory literature ([15], [16]).

A tight upper bound on the number of independent sets in finite and undirected general graphs was proved in the special setting of bipartite regular graphs in [13], and it was conjectured there to hold for general (irregular) graphs (2001, see Conjecture 4.2 in [13]). A decade later (2010), it was extended in [25] to regular graphs (that are not necessarily bipartite); a year later (2011), it was proved in [9] for graphs with small degrees (up to 5). Finally, this conjecture was recently (2019) proved in general [22], by utilizing a new approach. The reader is referred to [23] for an announcement on the solution of this conjecture as a frustrating combinatorial problem for...
two decades, along with some history of this problem and a description by the authors of their work in [22].

The recently-introduced proof of the conjecture for general graphs [22] uses an induction on the number of vertices in a graph, and it obtains a recurrence inequality whose derivation involves some judicious applications of Hölder’s inequality (see [22] Sections 2 and 4) and [23]). It settled for the first time Conjecture 4.2 in [13] by an interesting approach, which is unrelated to information theory; the possibility of generalizing the information-theoretic proof in [13] to the irregular setting was left in [22] as an open issue.

The initiative in the present work is providing an information-theoretic proof which generalizes Kahn’s proof for bipartite regular graphs [13], and which also confirms his conjecture for an extended family of bipartite graphs (in comparison to [13]) that are regular on one side of the graph.

The structure of the paper is as follows: Section II provides preliminaries and notation that are essential for the presentation of our proof. Section III explains (in more details) the scientific merit and contributions of the present work; for the sake of causal presentation, we provide these explanations after Section II. Sections IV and V are the core of this work, providing information-theoretic proofs.

II. PRELIMINARIES AND NOTATION

We provide in this section the notation and preliminary material which is essential for the presentation in this paper.

A. Notation

The following notation is used in the present paper:

- Let $X$ be a discrete random variable that takes its values on a set $\mathcal{X}$, and let $P_X$ be the probability mass function (PMF) of $X$. The Shannon entropy of $X$ is given by
  \[ H(X) \triangleq - \sum_{x \in \mathcal{X}} P_X(x) \log P_X(x), \]  
  where throughout this paper, logarithms are taken on base 2, and entropies are expressed in bits.

- For $p \in [0, 1]$,
  \[ H_b(p) \triangleq -p \log p - (1 - p) \log(1 - p), \]  
  where $H_b(\cdot)$ is the binary entropy function. By continuous extension, the convention $0 \log 0 = 0$ is used.

- $\mathbb{N} \triangleq \{1, 2, \ldots\}$ denotes the set of natural numbers.

- $X^n \triangleq (X_1, \ldots, X_n)$ denotes an $n$-dimensional random vector of discrete random variables, having a joint probability mass function (PMF) that is denoted by $P_{X^n}$.

- For every $n \in \mathbb{N}$, $[1 : n] \triangleq \{1, \ldots, n\}$.

- $X_S \triangleq (X_i)_{i \in S}$ is a random vector for an arbitrary nonempty subset $S \subseteq [1 : n]$; if $S = \emptyset$, then conditioning on $X_S$ is void.

- $\mathbb{1}\{E\}$ denotes the indicator of an event $E$; i.e., it is equal to 1 if this event is satisfied, and it is zero otherwise.
B. Shearer’s Lemma

Shearer’s lemma extends the subadditivity property of the entropy, where

\[ H(X^n) \leq \sum_{i=1}^{n} H(X_i), \]  

(3)

with equality in (3) if and only if \( X_1, \ldots, X_n \) are independent random variables (though they need not be identically distributed).

Due to its simplicity and usefulness in this paper (and elsewhere), we state and prove it here.

Proposition 1 (Shearer’s Lemma, [3]): Let \( X_1, \ldots, X_n \) be discrete random variables, and let the sets \( S_1, \ldots, S_m \subseteq [1 : n] \) include every element \( i \in [1 : n] \) in at least \( k \geq 1 \) of these subsets. Then,

\[ k H(X^n) \leq \sum_{j=1}^{m} H(X_{S_j}). \]  

(4)

As a special case of (4), setting \( S_i \triangleq \{i\} \) as singletons for all \( i \in [1 : n] \) gives (3) by having \( k = 1 \) and \( m = n \).

Proof: Let \( S = \{i_1, \ldots, i_\ell\} \) with \( 1 \leq i_1 < \ldots < i_\ell \leq n \). By invoking the chain rule in this order,

\[
H(X_S) = H(X_{i_1}) + H(X_{i_2} | X_{i_1}) + \ldots + H(X_{i_\ell} | X_{i_1}, \ldots, X_{i_{\ell-1}}) \\
\geq \sum_{i \in S} H(X_i | X^{i-1}) \\
= \sum_{i=1}^{n} \mathbb{1}\{i \in S\} H(X_i | X^{i-1}),
\]  

(5)

where the last inequality holds since additional conditioning cannot increase the entropy. By assumption, for \( i \in [1 : n] \),

\[ \sum_{j=1}^{m} \mathbb{1}\{i \in S_j\} \geq k, \]  

(6)

since the number of subsets \( \{S_j\}_{j=1}^{m} \) that include \( i \) as an element is at least \( k \). Consequently, it follows that

\[
\sum_{j=1}^{m} H(X_{S_j}) \geq \sum_{j=1}^{m} \sum_{i=1}^{n} \mathbb{1}\{i \in S_j\} H(X_i | X^{i-1}) \\
= \sum_{i=1}^{n} \left\{ \sum_{j=1}^{m} \mathbb{1}\{i \in S_j\} H(X_i | X^{i-1}) \right\} \\
\geq k \sum_{i=1}^{n} H(X_i | X^{i-1}) \\
= k H(X^n),
\]  

(7)

where (7) follows from (5); (8) holds by interchanging the order of summation; (9) holds by (6); and (10) holds by the chain rule for the Shannon entropy.

Shearer’s lemma and some of its variants (see [8] and [11]) have been successfully applied in various occasions (see, e.g., [8], [10], [11], [13], [14], [21]). Shearer’s lemma is also instrumental in this paper.
C. Graphs, Independent Sets, and Tensor Products

Let $G$ be an undirected graph, and let $V(G)$ and $E(G)$ denote respectively the sets of vertices and edges in $G$.

A graph $G$ is called $d$-regular if the degree of all vertices in $V(G)$ is equal to $d$. Otherwise, if the graph $G$ is not $d$-regular for some $d \in \mathbb{N}$, then $G$ is an irregular graph.

A graph is called bipartite if it has two types of vertices, and an edge cannot connect vertices of the same type; we refer to the vertices of a bipartite graph $G$ as left and right vertices.

A graph $G$ is called complete if every vertex $v \in V(G)$ is connected to all the other vertices in $V(G) \setminus \{v\}$ (and not to itself); similarly, a bipartite graph is called complete if every vertex is connected to all the vertices of the other type in the graph. A complete graph of the same type; we refer to the vertices of a bipartite graph $G$ and edges in $d$

Likewise, a complete bipartite graph is denoted by $K_{d,d}$, having a number of vertices $|V(K_{d,d})| = 2d$ (i.e., $d$ vertices of each of the two types), and a number of edges $|E(K_{d,d})| = d^2$.

An independent set of an undirected graph $G$ is a subset of its vertices such that none of the vertices in these subset are adjacent (i.e., none of them are joined by an edge). Let $I(G)$ denote the set of all the independent sets in $G$, and let $|I(G)|$ denote the number of independent sets in $G$. Similarly to [1], [9], [10], [13], [15], [16], [21], [22], [24], [25] (and references therein), our work considers the question of how many independent sets can $G$ have.

The tensor product $G \times H$ of two graphs $G$ and $H$ is a graph such that the following holds:

- The vertex set of $G \times H$ is the Cartesian product $V(G) \times V(H)$;
- Two vertices $(g, h), (g', h') \in V(G \times H)$ are adjacent if and only if $g$ is adjacent to $g'$, and $h$ is adjacent to $h'$, i.e., if $(g, g') \in E(G)$ and $(h, h') \in E(H)$. This is denoted by $((g, h), (g', h')) \in E(G \times H)$.

In general, the following identities hold:

\[
|V(G \times H)| = |V(G)| \cdot |V(H)|, \quad \text{(11)}
\]

\[
|E(G \times H)| = 2 \cdot |E(G)| \cdot |E(H)|. \quad \text{(12)}
\]

Following the definition of a complete $d$-regular graph $K_d$, the graph $K_2$ is specialized to two vertices that are connected by an edge. Let us label the two vertices in $K_2$ by 0 and 1. For a graph $G$, the tensor product $G \times K_2$ is a bipartite graph, called the bipartite double cover of $G$, where the set of vertices in $G \times K_2$ is given by

\[
V(G \times K_2) = \{ (v, i) : v \in V(G), i \in \{0, 1\} \}, \quad \text{(13)}
\]

and its set of edges is given by

\[
E(G \times K_2) = \{ ((u, 0), (v, 1)) : (u, v) \in E(G) \}. \quad \text{(14)}
\]

Hence, every edge $e = (u, v) \in E(G)$ is mapped into the two edges $((u, 0), (v, 1)) \in E(G \times K_2)$ and $((v, 0), (u, 1)) \in E(G \times K_2)$ (since the graph $G$ is undirected). This implies that the numbers of vertices and edges in $G \times K_2$ are doubled in comparison to their respective numbers in $G$; moreover, every edge in $G$, which connects a pair of vertices of specified degrees, is mapped into two edges in $G \times K_2$ where each of these two edges connects a pair of vertices of the same specified degrees.
D. Upper Bounds on the Number of Independent Sets

The present subsection introduces the relevant results to this paper. The following theorem provides a tight upper bound on the number of independent sets in bipartite regular graphs, and its derivation in [13] makes a clever use of Shearer’s lemma (Proposition 1).

Theorem 1 (Kahn 2001, [13]): If $G$ is a bipartite $d$-regular graph with $n$ vertices, then

$$|\mathcal{I}(G)| \leq (2^{d+1} - 1)^{\frac{n}{2d}}.$$  \hspace{1cm} (15)

Furthermore, if $n$ is an even multiple of $d$, then the upper bound in the right side of (15) is tight, and it is obtained by a disjoint union of $\frac{n}{2d}$ complete $d$-regular bipartite graphs $(K_{d,d})$.

Kahn’s result was later extended by Zhao [25] for a general $d$-regular graph via a brilliant combinatorial reduction to the setting of $d$-regular bipartite graphs, which proved [13, Conjecture 4.1].

Theorem 2 (Zhao 2010, [25]): The upper bound on the number of independent sets in (15) continues to hold for all $d$-regular graphs with $n$ vertices.

Recently, Sah et al. [22] proved Kahn’s conjecture in [13, Conjecture 4.2] (made eighteen years earlier) for an upper bound on the number of independent sets in a general undirected graph with no isolated vertices. The proof in [22] is combinatorial, and it extends the result in Theorem 2 as follows.

Theorem 3 (Sah et al. 2019, [22]): Let $G$ be an undirected graph without isolated vertices or multiple edges connecting any pair of vertices. Let $d_v$ denote the degree of a vertex $v \in V(G)$. Then,

$$|\mathcal{I}(G)| \leq \prod_{(u,v) \in E(G)} (2^{d_u} + 2^{d_v} - 1)^{\frac{1}{d_u d_v}}$$  \hspace{1cm} (16)

with an equality if $G$ is a disjoint union of complete bipartite graphs.

Let $K_{d_u,d_v}$ be a complete bipartite graph where the degrees of its left and right vertices are equal to $d_u$ and $d_v$, respectively. Then, the number of independent sets in such a complete bipartite graph is equal to

$$|\mathcal{I}(K_{d_u,d_v})| = 2^{d_u} + 2^{d_v} - 1$$  \hspace{1cm} (17)

since every subset of the left vertices, as well as every subset of the right vertices, forms an independent set of $K_{d_u,d_v}$; on the other hand, any subset which contains both left and right vertices is not an independent set (since the bipartite graph $K_{d_u,d_v}$ is complete). Note that the subtraction by 1 in the right side of (17) is because, in the counting of the number of subsets of left vertices ($2^{d_u}$) or right vertices ($2^{d_v}$), the empty set is counted twice. Hence, (16) can be rewritten in an equivalent form as

$$|\mathcal{I}(G)| \leq \prod_{(u,v) \in E(G)} |\mathcal{I}(K_{d_u,d_v})|^{\frac{1}{d_u d_v}}.$$  \hspace{1cm} (18)

Since $E(K_{d_u,d_v}) = d_u d_v$, it follows that the bound in (16) (or (18)) is achieved for the complete bipartite graph $K_{d_u,d_v}$. More generally, it is achieved with equality for a disjoint finite union of such complete bipartite graphs since the number of independent sets in a disjoint union of graphs is equal to the product of the number of independent sets in each of the component graphs.
For the extension of the validity of Theorem 1 to Theorem 2, obtained by relaxing the requirement that the graph is bipartite, the following inequality was introduced by Zhao for every finite graph $G$ \cite[Lemma 2.1]{25}:

\[ |\mathcal{I}(G)|^2 \leq |\mathcal{I}(G \times K_2)|, \] (19)

which relates the number of independent sets in a graph to the number of independent sets in the bipartite double cover of this graph.

The transition from Theorem 1 to Theorem 2, as introduced in \cite{25}, is a one-line proof. Let $G$ be a $d$-regular graph with $n$ vertices, then $G \times K_2$ is $d$-regular bipartite graph with $2n$ vertices. Hence, (15) and (19) give that

\[ |\mathcal{I}(G)|^2 \leq |\mathcal{I}(G \times K_2)| \leq (2^{d+1} - 1)^{2n}, \] (20)

and squaring the leftmost and rightmost sides of (20) implies that (15) continues to hold even when the regular graph $G$ is not necessarily bipartite.

III. SCIENTIFIC MERIT AND CONTRIBUTIONS OF THE PRESENT WORK

After introducing Shearer’s lemma (Proposition 1) and Theorems 1–3, we address the scientific merit and contributions of the present work in more details (in comparison to the introduction in Section I).

Theorem 3 was recently proved in \cite{22} (see also \cite{23}) for general graphs, without relying on information theory. Our work is motivated by the following sentences from \cite{22}, p. 174:

Kahn’s proof \cite{13} of the bipartite case of Theorem 1 made clever use of Shearer’s entropy inequality \cite{3}. It remains unclear how to apply Shearer’s inequality in a lossless way in the irregular case, despite previous attempts to do so, e.g., \cite[Section 3]{15} and \cite[Section 5.C]{16}.

The present paper gives an information-theoretic proof of Theorem \cite{3} in a setting where the bipartite graph is regular on one side (i.e., the vertices on the other side of the bipartite graph can be irregular, and have arbitrary degrees). Its contributions are as follows:

- Section IV provides a (non-trivial) extension of the proof of (15), from regular bipartite graphs \cite{13} to general bipartite graphs. It leads to the upper bound on the number of independent sets that is given in (65), which is in general loosen than the bound in (16) (or its equivalent form in (18)). However, for the family of bipartite graphs that are regular on one side of the graph, the bound in (65) coincides with the bound in (16). This result extends the entropy-based proof in (13), which applies to regular bipartite graphs, leading to a tight bound that is satisfied for the extended family of bipartite graphs that are regular on one side of the graph.
- This work also provides an entropy-based proof of Zhao’s inequality (19), which is a variant of the proof in \cite[Section 2]{25} (see Section V here).

It is interesting to note that the observation that (16) can be extended from (undirected) bipartite graphs to general graphs, by utilizing (19), was made in \cite[Lemma 3]{9}. However, a computer-assisted proof of (16) was restricted there to graphs whose maximal degrees are at most 5 (see \cite[Theorem 2]{9}).
IV. AN INFORMATION-THEORETIC PROOF OF THEOREM 3 FOR A FAMILY OF BIPARTITE GRAPHS

The core of the proof of Theorem 3 is proving (16) for an undirected bipartite graph. In this section, we provide an extension of the entropy-based proof by Kahn [13] from bipartite \(d\)-regular graphs to general bipartite graphs, and then we prove (16) for the family of bipartite graphs that are regular on one side.

Consider first a general bipartite graph \(G\) with a number of vertices \(|V(G)| = n\), and where none of its vertices is isolated. Label them by the elements of \([1 : n]\). Let \(L\) and \(R\) be the vertices of the two types in \(V(G)\) (called, respectively, the left and right vertices in \(G\)), so \(V(G) = L \cup R\) is a disjoint union. Let \(D_L\) and \(D_R\) be, respectively, the sets of all possible degrees \(d\) of vertices in \(L\) and \(R\). For all \(d \in D_L\), let \(L_d\) be the set of vertices in \(L\) having degree \(d\), and let \(R_d\) be the set of vertices in \(R\) that are adjacent to vertices in \(L_d\) (note that the vertices in \(R_d\) are not necessarily those vertices in \(R\) having degree \(d\), so the definitions of \(L_d\) and \(R_d\) differ, i.e., they are not similar up to a replacement of left vertices of degree \(d\) with right vertices of the same degree). Then,

\[
L = \bigcup_{d \in D_L} L_d, \quad R = \bigcup_{d \in D_L} R_d,
\]

where the first equality in (21) is (by definition) a union of pairwise disjoint sets.

Let \(S \in \mathcal{I}(G)\) be an independent set in \(G\), which is selected uniformly at random from \(\mathcal{I}(G)\), and let \(X^n \triangleq (X_1, \ldots, X_n)\) be given by

\[
X_i \triangleq 1\{i \in S\}, \quad i \in [1 : n],
\]

so the binary random vector \(X^n\) indicates which of the \(n\) vertices in \(V(G)\) belongs to the randomly selected independent set \(S\). Since \(S\) is equiprobable in \(\mathcal{I}(G)\), we have

\[
H(X^n) = \log |\mathcal{I}(G)|.
\]

Let \(X_L = (X_i)_{i \in L}\) and \(X_R = (X_i)_{i \in R}\) be used as a shorthand. Then,

\[
H(X^n) = H(X_L, X_R)
\]

\[
= H(X_L) + H(X_R | X_L)
\]

\[
\leq \sum_{d \in D_L} H(X_{L_d}) + H(X_R | X_{L_d})
\]

\[
\leq \sum_{d \in D_L} H(X_{L_d}) + \sum_{d \in D_L} H(X_{R_d} | X_{L_d})
\]

\[
= \sum_{d \in D_L} \{H(X_{L_d}) + H(X_{R_d} | X_{L_d})\},
\]

where inequalities (26) and (27) hold by the subadditivity of the entropy, and due to (21). It should be noted that although the first summand in the right side of (28) is an entropy of \(X_{L_d}\), the conditioning on \(X_L\) (rather than just on \(X_{L_d}\)) in the second term leads to a stronger upper bound on \(H(X^n)\) (since \(L_d \subseteq L\), and conditioning reduces the entropy). This is essential for the continuation of the proof (see (30)).

We next upper bound the two summands in the right side of (28), starting with the conditional entropy. By invoking the subadditivity property of the entropy, for every \(d \in D_L\),

\[
H(X_{R_d} | X_L) \leq \sum_{r \in R_d} H(X_r | X_L).
\]
For every \( r \in \mathcal{R}_d \), let \( \mathcal{N}(r) \) be the set of all the vertices that are adjacent to the vertex \( r \). Since the graph \( G \) is bipartite, we have \( \mathcal{N}(r) \subseteq \mathcal{L} \) (but, in general, \( \mathcal{N}(r) \not\subseteq \mathcal{L}_d \)), and consequently
\[
H(X_r | X_L) \leq H(X_r | X_{\mathcal{N}(r)}). \tag{30}
\]
Combining (29) and (30) gives that
\[
H(X_{\mathcal{R}_d} | X_L) \leq \sum_{r \in \mathcal{R}_d} H(X_r | X_{\mathcal{N}(r)}). \tag{31}
\]
For \( r \in \mathcal{R}_d \), let
\[
Q_r \triangleq \mathbb{1}(S \cap \mathcal{N}(r) = \emptyset) \tag{32}
\]
be the indicator function of the event where none of the vertices that are adjacent (in \( G \)) to the vertex \( r \) are included in the (randomly selected) independent set \( S \). Then,
\[
H(X_r | X_{\mathcal{N}(r)}) \leq H(X_r | Q_r) \tag{33}
\]
since the random vector \( X_{\mathcal{N}(r)} \) indicates which of the indices \( i \in \mathcal{N}(r) \) are included in \( S \), whereas the binary random variable \( Q_r \) only indicates if there is such an index. Consequently, (31) and (33) imply that
\[
H(X_{\mathcal{R}_d} | X_L) \leq \sum_{r \in \mathcal{R}_d} H(X_r | Q_r). \tag{34}
\]
For the binary random variable \( Q_r \), let
\[
q_r \triangleq P[Q_r = 1]. \tag{35}
\]
By (32), \( Q_r = 0 \) if and only if \( S \cap \mathcal{N}(r) \neq \emptyset \), which implies that \( r \not\in \mathcal{S} \) since there is a vertex in \( \mathcal{N}(r) \) that belongs to the independent set \( S \). Therefore, if \( Q_r = 0 \), then \( X_r = 0 \) (see (22)), so
\[
H(X_r | Q_r = 0) = 0. \tag{36}
\]
If \( Q_r = 1 \), then \( X_r \in \{0, 1\} \), and
\[
H(X_r | Q_r = 1) \leq 1. \tag{37}
\]
Hence, from (35–37),
\[
H(X_r | Q_r) = q_r \cdot H(X_r | Q_r = 1) + (1 - q_r) \cdot H(X_r | Q_r = 0) \leq q_r, \tag{38}
\]
and the combination of (34) and (38) yields
\[
H(X_{\mathcal{R}_d} | X_L) \leq \sum_{r \in \mathcal{R}_d} q_r. \tag{39}
\]
We next upper bound \( H(X_{\mathcal{L}_d}) \), which is the first summand in the right side of (28), and here Shearer’s lemma (see Proposition 1) comes into the picture. Since, by definition, \( \mathcal{R}_d \) is the set of the vertices that are connected to the subset \( \mathcal{L}_d \) of the degree-\( d \) vertices in \( \mathcal{L} \), and \( \mathcal{N}(r) \) is the set of vertices in \( \mathcal{L} \) that are connected to a vertex \( r \in \mathcal{R}_d \) in the bipartite graph \( G \), then it follows that every vertex in \( \mathcal{L}_d \) belongs to at least \( d \) of the subsets \( \{\mathcal{N}(r)\}_{r \in \mathcal{R}_d} \). Hence, by Shearer’s lemma,
\[
H(X_{\mathcal{L}_d}) \leq \frac{1}{d} \sum_{r \in \mathcal{R}_d} H(X_{\mathcal{N}(r)}). \tag{40}
\]
The binary random variable $Q_r$ is a deterministic function of the random vector $X_{N(r)}$ since, from (22) and (32), $Q_r = 1$ if and only if all the entries of $X_{N(r)}$ are equal to 0. Consequently, for all $r \in \mathcal{R}_d$,

$$H(X_{N(r)}) = H(X_{N(r)}, Q_r)$$

$$= H(Q_r) + H(X_{N(r)}|Q_r)$$

$$= H_b(q_r) + H(X_{N(r)}|Q_r),$$

(41)

where the equality in (43) follows from (2) and (35). Next, from (35),

$$H(X_{N(r)}|Q_r) = q_r H(X_{N(r)}|Q_r = 1) + (1 - q_r) H(X_{N(r)}|Q_r = 0).$$

(44)

If $Q_r = 1$, then $X_{N(r)}$ is a vector of zeros, so

$$H(X_{N(r)}|Q_r = 1) = 0.$$ 

(45)

Otherwise, if $Q_r = 0$, then $X_i = 1$ for at least one element $i \in N(r)$; since $|N(r)| = d_r$ is the degree of the vertex $r$ (by assumption, there are no multiple edges connecting any pair of vertices), it follows that the vector $X_{N(r)} \in \{0, 1\}^{d_r}$ cannot be the zero vector, so

$$H(X_{N(r)}|Q_r = 0) \leq \log(2^{d_r} - 1).$$ 

(46)

Combining (41)–(46) gives

$$H(X_{N(r)}) \leq H_b(q_r) + (1 - q_r) \log(2^{d_r} - 1),$$

(47)

and, from (40) and (47), we get the following upper bound on the first summand in the right side of (28):

$$H(X_{L_d}) \leq \frac{1}{d} \sum_{r \in \mathcal{R}_d} \left\{ H_b(q_r) + (1 - q_r) \log(2^{d_r} - 1) \right\}. 

(48)

Consequently, combining (24)–(28), (39) and (48) implies that

$$H(X^n) \leq \sum_{d \in \mathcal{D}_L} \left\{ H(X_{L_d}) + H(X_{R_d}|X_{L_d}) \right\},$$

(49)

$$\leq \sum_{d \in \mathcal{D}_L} \left\{ \sum_{r \in \mathcal{R}_d} q_r + \frac{1}{d} \sum_{r \in \mathcal{R}_d} \left\{ H_b(q_r) + (1 - q_r) \log(2^{d_r} - 1) \right\} \right\},$$

(50)

$$= \sum_{d \in \mathcal{D}_L} \left\{ \frac{1}{d} \sum_{r \in \mathcal{R}_d} \left\{ H_b(q_r) + (1 - q_r) \log(2^{d_r} - 1) + q_r \log(2^d) \right\} \right\},$$

(51)

$$= \sum_{d \in \mathcal{D}_L} \left\{ \frac{1}{d} \sum_{r \in \mathcal{R}_d} \left\{ H_b(q_r) + q_r \log \left( \frac{2^d}{2^{d_r} - 1} \right) + \log(2^{d_r} - 1) \right\} \right\}. 

(52)

Since $q_r \in [0, 1]$ for $r \in \mathcal{R}_d$, we next maximize an auxiliary function $f_r : [0, 1] \rightarrow \mathbb{R}$, defined as

$$f_r(x) \triangleq H_b(x) + x \log \left( \frac{2^d}{2^{d_r} - 1} \right), \quad x \in [0, 1],$$

(53)

in order to obtain an upper bound on the right side of (52) which is independent of $\{q_r\}$. By (2), setting the derivative of $f_r(\cdot)$ to zero gives the equation

$$\log \left( \frac{1 - x}{x} \right) + \log \left( \frac{2^d}{2^{d_r} - 1} \right) = 0,$$

(54)
whose solution is given by

\[ x = \frac{2^d}{2^d + 2^d - 1}. \]  

(55)

Consequently, it follows from (49)–(53) and (55) that

\[ H(X^n) \leq \sum_{d \in D} \left\{ \frac{1}{d} \sum_{r \in R} \left\{ f_r(q_r) + \log(2^{d_r} - 1) \right\} \right\} \]  

(56)

\[ \leq \sum_{d \in D} \left\{ \frac{1}{d} \sum_{r \in R} \left\{ f_r \left( \frac{2^d}{2^d + 2^d - 1} \right) + \log(2^{d_r} - 1) \right\} \right\} \]  

(57)

and the calculation of the term in the inner sum in the right side of (57) gives

\[ f_r \left( \frac{2^d}{2^d + 2^d - 1} \right) + \log(2^{d_r} - 1) \]

\[ = H_b \left( \frac{2^d}{2^d + 2^d - 1} \right) + \left( \frac{2^d}{2^d + 2^d - 1} \right) \log \left( \frac{2^d}{2^d - 1} \right) + \log(2^{d_r} - 1) \]  

(58)

\[ = - \left( \frac{2^d}{2^d + 2^d - 1} \right) \log \left( \frac{2^d}{2^d + 2^d - 1} \right) - \left( \frac{2^d - 1}{2^d + 2^d - 1} \right) \log \left( \frac{2^d - 1}{2^d + 2^d - 1} \right) \]

\[ + \left( \frac{2^d}{2^d + 2^d - 1} \right) \log \left( \frac{2^d - 1}{2^d - 1} \right) + \log(2^{d_r} - 1) \]  

(59)

\[ = - \left( \frac{2^d}{2^d + 2^d - 1} \right) \log \left( \frac{2^d}{2^d + 2^d - 1} \right) - \log \left( \frac{2^d - 1}{2^d + 2^d - 1} \right) + \log(2^{d_r} - 1) \]  

(60)

\[ = - \left( \frac{2^d}{2^d + 2^d - 1} \right) \log \left( \frac{2^d - 1}{2^d + 2^d - 1} \right) - \left( \frac{2^d - 1}{2^d + 2^d - 1} \right) \log \left( \frac{2^d - 1}{2^d + 2^d - 1} \right) \]

\[ + \log(2^{d_r} - 1) \]  

(61)

\[ = - \log \left( \frac{2^d - 1}{2^d + 2^d - 1} \right) + \log(2^{d_r} - 1) \]  

(62)

\[ = \log(2^d + 2^d r - 1), \]  

(63)

where (58) and (59) hold, respectively, by (53) and (2). Substituting the equality in (63) into the upper bound on the entropy in the right side of (57), together with (23), gives

\[ \log |\mathcal{I}(G)| \leq \sum_{d \in D} \left\{ \frac{1}{d} \sum_{r \in R} \log(2^d + 2^d r - 1) \right\}, \]  

(64)

which, by exponentiation of both sides of (64), gives

\[ |\mathcal{I}(G)| \leq \prod_{d \in D} \prod_{r \in R_d} (2^d + 2^d r - 1)^{\frac{1}{2}}. \]  

(65)

The upper bound in the right side of (65) is in general looser than the bound in Theorem 3. However, there is an interesting case where they both coincide.
Let \( G \) be a bipartite graph that is \( d \)-regular on one side (i.e., one type of its vertices have a fixed degree \( d \), and the other type of vertices can be irregular with arbitrary degrees). Without any loss of generality, one can assume that the left vertices are \( d \)-regular (as otherwise, the graph can be flipped without affecting its independent sets, and also the bound in Theorem 3 is symmetric in the degrees \( d_u \) and \( d_v \)).

We next wish to show that for such a bipartite graph, where all the left vertices are of degree \( d \), both bounds in (65) and Theorem 3 are identical. Indeed, in this setting, we have \( L_d = L \) and \( R_d = R \) (recall that, by assumption, there are no isolated vertices). Consequently, the right side of (65) is specialized to

\[
|I(G)| \leq \prod_{r \in R} (2^d + 2^{d_r} - 1)^{\frac{1}{d_r}}.
\]

(66)

Since there are exactly \( d_r \) edges connecting each vertex \( r \in R \) with vertices in \( L \), and (by the latter assumption) all of the left vertices in \( L \) are of a fixed degree \( d \), it follows that in this setting, the right side of (66) can be rewritten in the form

\[
\prod_{r \in R} (2^d + 2^{d_r} - 1)^{\frac{1}{d_r}} = \prod_{(u,v) \in E(G)} \left(2^{d_u} + 2^{d_v} - 1\right)^{\frac{1}{d_u d_v}},
\]

(67)

which then coincides with the right side of (16) for bipartite graphs that are regular on one side of the graph (without any restriction on the other side).

V. AN ENTROPY-BASED PROOF OF ZHAO’S INEQUALITY (19)

The present section suggests an entropy-based proof of (19) that forms a variant of the proof in [25, Lemma 2.1].

Let \( G \) be a finite set, and let \( |V(G)| = n \). Label the vertices in the left and right sides of the bipartite graph \( G \times K_2 \) (i.e., the bipartite double cover of \( G \)) by \( \{(i, 0)\}_{i=1}^n \) and \( \{(i, 1)\}_{i=1}^n \), respectively.

Choose independently and uniformly at random two independent sets \( S_0, S_1 \in I(G) \). For \( i \in [1 : n] \), let \( X_i, Y_i \in \{0, 1\} \) be random variables defined as \( X_i = 1 \) if and only if \( i \in S_0 \), and \( Y_i = 1 \) if and only if \( i \in S_1 \). Then, by the statistical independence and equiprobable selection of the two independent sets from \( I(G) \), we have

\[
H(X^n, Y^n) = H(X^n) + H(Y^n)
\]

(69)

\[
= 2 \log |I(G)|,
\]

(70)

where (69) holds since \( X^n = (X_1, \ldots, X_n) \) and \( Y^n = (Y_1, \ldots, Y_n) \) are statistically independent (by construction), and (70) holds since they both have an equiprobable distribution over a set whose cardinality is \( |I(G)| \).

Consider the following set of vertices in \( G \times K_2 \):

\[
S \triangleq \{S_0 \times \{0\}\} \cup \{S_1 \times \{1\}\}
\]

(71)

\[
= \bigcup_{i \in S_0, j \in S_1} \{(i, 0), (j, 1)\}.
\]

(72)
The set $S$ is not necessarily an independent set in $G \times K_2$ since $(i, 0, (j, 1)) \in E(G \times K_2)$ for all $i \in S_0$ and $j \in S_1$ for which $(i, j) \in E(G)$ (see (14)). We next consider all $(i, j) \in E(G)$ such that $X_i = Y_j = 1$. To that end, fix an ordering of all the $2^n$ subsets of $V(G)$, and let $T \in V(G)$ be the first subset in this particular ordering that includes exactly one endpoint of each edge $(i, j) \in E(G)$ for which $X_i = Y_j = 1$. Consequently,

- if $(i, 0) \in S$ and $i \in T$, then $(i, 0)$ is replaced by $(i, 1)$;
- likewise, if $(j, 1) \in S$ and $j \in T$, then $(j, 1)$ is replaced by $(j, 0)$.

In view of these possible replacements, let $\tilde{S}$ be the new set of vertices which is a subset of $V(G \times K_2)$. Then, $\tilde{S} \in I(G \times K_2)$ since all adjacent vertices in $S$ are no longer connected in $\tilde{S}$. Similarly to the way $X^n, Y^n \in \{0, 1\}^n$ were defined, let $\tilde{X}^n, \tilde{Y}^n \in \{0, 1\}^n$ be defined such that, for all $i \in [1 : n]$, $\tilde{X}_i = 1$ if and only if $(i, 0) \in \tilde{S}$, and $\tilde{Y}_i = 1$ if and only if $(i, 1) \in \tilde{S}$.

The mapping from $(X^n, Y^n)$ to $(\tilde{X}^n, \tilde{Y}^n)$ is injective. Indeed, it is shown to be injective by finding all indices $(i, j) \in E(G)$ such that $\tilde{X}_i = \tilde{X}_j = 1$ or $\tilde{Y}_i = \tilde{Y}_j = 1$, finding the first subset $T \in V(G)$ according to our previous fixed ordering of the $2^n$ subsets of $V(G)$ that includes exactly one endpoint of each such edge $(i, j) \in E(G)$, and performing the opposite operation to return back to $X^n$ and $Y^n$ (e.g., if $(i, j) \in E(G)$, $\tilde{X}_i = \tilde{X}_j = 1$ and $i \in T$ while $j \not\in T$, then $\tilde{X}_i = 1$ is transformed back to $Y_i = 1$, and $\tilde{X}_j = 1$ is transformed back to $X_j = 1$). Consequently, we get

$$H(X^n, Y^n) = H(\tilde{X}^n, \tilde{Y}^n) \leq \log |I(G \times K_2)|,$$

(73)

(74)

where (73) holds by the injectivity of the mapping from $(X^n, Y^n)$ to $(\tilde{X}^n, \tilde{Y}^n)$, and (74) holds since $\tilde{S}$ is an independent set in $G \times K_2$, which implies that $(\tilde{X}^n, \tilde{Y}^n)$ can get at most $|I(G \times K_2)|$ possible values (by definition, there is a one-to-one correspondence between $\tilde{S}$ and $(\tilde{X}^n, \tilde{Y}^n)$).

Combining (69), (70), (73) and (74) gives

$$2 \log |I(G)| \leq \log |I(G \times K_2)|,$$

(75)

which gives (19) by exponentiation of both sides of (75).

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