CANONICAL BASIS, KLR-ALGEBRAS AND PARITY SHEAVES

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Abstract. We give a construction of a basis of the positive part of the Drinfeld-Jimbo quantum enveloping algebra associated with a Dynkin quiver in terms of parity sheaves.

Contents
1. Introduction 2
2. Geometric construction of KLR-algebras 6
2.1. Quivers 6
2.2. KLR-algebra 6
2.3. A faithful representation of $R_{\nu,A}$ 6
2.4. Reminder on Borel-Moore homology 6
2.5. Algebra $Z_{V,A}$ 6
2.6. Algebras $P_{V,A}$ and $S_{V,A}$ 6
2.7. Stratification on $F_{V} \times F_{V}$ 6
2.8. Stratification on $Z_{V}$ 6
2.9. Generators of $Z_{V,A}$ 6
2.10. The $A$-algebra structure on $F_{V,A}$ 7
2.11. KLR-algebra and Borel-Moore homology 7
2.12. Stratified varieties 9
2.13. Lusztig category $Q_{V}$ 9
2.14. Local systems 9
2.15. KLR-algebra and Yoneda algebras 10
2.16. Functor $Y'$ 10
2.17. Induction and restriction 12
2.18. Projective $R_{\nu,k}$-modules 12
2.19. The algebra $f$ 13
2.20. A new $Z$-basis in $A f$ 13
2.21. Indecomposable objects in $Q_{V}$ 14
3. Parity sheaves 15
3.1. Parity sheaves 15
3.2. Parity sheaves on quiver varieties 16
3.3. Extensions of parity complexes 17
3.4. Nakajima quiver varieties 18
3.5. Resolutions via Nakajima quiver varieties 20
3.6. Restriction diagrams 21
3.7. Restriction of Nakajima sheaves 22
3.8. Restriction of Lusztig sheaves 23
3.9. Coalgebra structure on $K(Par)$ 24
3.10. Even quivers 26
3.11. Type $A$ 27
3.12. Quiver Schur algebras 28
Acknowledgements 32
References 32
1. Introduction

By a variety we will always mean a reduced and quasi-projective scheme of finite type over $\mathbb{C}$.

Let $\mathbf{k}$ be a field. Let $\Gamma$ be a Dynkin quiver with the set of vertices $I$ and let $\nu$ be a dimension vector. Let $E\nu V$ be the space of representations of $\Gamma$ on a $\nu$-dimensional $I$-graded vector space $V$. Let $G_{\nu} V$ be the group of graded linear automorphisms of $V$. Let $Y_{\nu}$ be the set of all pairs $y = (i, a)$ where $i = (i_1, i_2, \ldots, i_m)$ is a sequence of elements of $I$ and $a = (a_1, a_2, \ldots, a_m)$ is a sequence of positive integers such that $\sum_{l=1}^m a_i i_l = \nu$. Let $I^{\nu} \subseteq Y_{\nu}$ be the set of all pairs $y = (i, a)$ such that $a = 1$ for all $i$. We will abbreviate $i$ for $y = (i, a)$ if $y \in I^{\nu}$. For each $y \in Y_{\nu}$ let $\pi_y : F_y \to E\nu V$ be the Lusztig analogue of the Springer map. Let $L_{V,k} = \bigoplus_{i \in I^{\nu}} \pi_{i,k}F_i$ be the Lusztig complex. Set $\delta L_{V,k} = \bigoplus_{i \in I^{\nu}} \pi_{i,k}F_i[\dim_\mathbb{C} F_i]$. Let $f$ be the positive part of the Drinfeld-Jimbo quantized enveloping algebra associated with the quiver $\Gamma$. Set $A = \mathbb{Z}[q, q^{-1}]$. Denote by $\delta A$ the Lusztig’s integral form of $f$.

The KLR-algebras are recently introduced by Khovanov and Lauda [11] and by Rouquier [21]. They yield categorification of quantum groups. Denote by $R_{\nu,k}$ the KLR-algebra over $\mathbf{k}$ associated with the quiver $\Gamma$ and the dimension vector $\nu$. We denote by $D_{G_{\nu} V}(E\nu V, \mathbf{k})$ the bounded $G_{\nu}$-equivariant derived category of sheaves of $\mathbf{k}$-vector spaces on $E\nu V$ constructible with respect to the stratification of $E\nu V$ by $G_{\nu}$-orbits. Let $\mathcal{Q}_V$ be the full additive subcategory of $D_{G_{\nu} V}(E\nu V, \mathbf{k})$ such that the objects of $\mathcal{Q}_V$ are the direct sums of shifts of direct factors of $L_{V,k}$. Denote by $\text{proj}(R_{\nu,k})$ the category of $\mathbb{Z}$-graded projective finitely generated modules over $R_{\nu,k}$.

The goal of Section 2 is to give a geometric construction of KLR-algebras in arbitrary characteristic. The zero characteristic case was done in [22] and [25]. The KLR-algebras in positive characteristic were studied in [12]. More precisely, we prove the following in Theorems 2.20, 2.21.

Theorem 1.1.

1. There exists a graded $\mathbf{k}$-algebra isomorphism $R_{\nu,k} = \text{Ext}^*_A (\delta L_{V,k}, \delta L_{V,k})$.

2. The functor $\mathcal{Q}_V \to \text{proj}(R_{\nu,k})$, $\mathcal{F} \mapsto \text{Ext}^*_A (\delta L_{V,k}, \delta L_{V,k})$ yields an equivalence of categories from the opposite category $\mathcal{Q}_V^{\text{op}}$ of $\mathcal{Q}_V$ to the category $\text{proj}(R_{\nu,k})$.

Let us denote by $K(R_{\nu,k})$ the split Grothendieck group of the additive category $\text{proj}(R_{\nu,k})$. Set $R_{\nu,k} = \bigoplus_{\nu \in \mathbb{N}^I} R_{\nu,k}$. $K(R_{\nu,k}) = \bigoplus_{\nu \in \mathbb{N}^I} K(R_{\nu,k})$. Consider also the split Grothendieck group $K(Q_V)$ of the additive category $\mathcal{Q}_V$ and set $Q = \bigoplus_{\nu \in \mathbb{N}^I} Q_V$, $K(Q) = \bigoplus_{\nu \in \mathbb{N}^I} K(Q_V)$, where the sum is taken over the isomorphism classes of $I$-graded finite dimensional $\mathbb{C}$-vector spaces. The $\mathbb{Z}$-modules $K(Q)$ and $K(R_{\nu,k})$ have the $A$-module structures, where $q$ acts by the shift of grading. The induction and restriction functors yield $A$-algebra and $A$-coalgebra structures on $K(R_{\nu,k})$. Theorem 1.1 (2) yields an $A$-linear isomorphism $K(Q) \to K(R_{\nu,k})$. So we can transfer the algebra structure from $K(R_{\nu,k})$ to $K(Q_H)$. It happens that this algebra structure coincides with the algebra structure given by Lusztig’s induction and restriction functors, see Section 2.20. This yields an $A$-algebra isomorphism $\lambda_Q : K(Q) \to A$, see Theorem 2.22. However, there is no geometric construction of a coproduct on $K(Q)$ via the geometry associated with $Q$.

In Section 3 we study parity sheaves on $E\nu V$. Let $\text{Par}_{G_{\nu} V}(E\nu V)$ be the full additive subcategory of parity complexes in $D_{G_{\nu} V}(E\nu V, \mathbf{k})$. Let $K(\text{Par}_{G_{\nu} V}(E\nu V))$ be its split...
Grothendieck group. We set
\[
\text{Par} = \bigoplus_V \text{Par}_{G_V}(E_V), \quad K(\text{Par}) = \bigoplus_V K(\text{Par}_{G_V}(E_V)),
\]
where the sum is taken over the isomorphism classes of \(I\)-graded finite dimensional \(\mathbb{C}\)-vector spaces. The \(\mathbb{Z}\)-module \(K(\text{Par})\) has the \(\mathcal{A}\)-module structure, where \(q\) acts by the shift of grading. The Lusztig’s restriction functor \(\text{Res}\) yields an \(\mathcal{A}\)-coalgebra structure on \(K(\text{Par})\). However, a priori we have no geometric construction of an algebra structure on \(K(\text{Par})\). In Section 3 we get the following result.

**Theorem 1.2.** There exists an \(\mathcal{A}\)-coalgebra isomorphism \(\beta_\mathcal{A} : K(\text{Par}) \to \mathcal{A}_f\).

Theorem 1.2 yields an \(\mathcal{A}\)-basis in \(\mathcal{A}_f\) in terms of parity sheaves. We say that the quiver \(\Gamma\) is \(k\)-even if the complexes \(\pi_y E_{\mathcal{F}_y}\) are even for each \(y \in Y_\nu\). We say that \(\Gamma\) is even if it is \(k\)-even for each field \(k\). If the quiver \(\Gamma\) is \(k\)-even then the categories \(\mathcal{Q}_V\) and \(\text{Par}_{G_V}(E_V)\) coincide and we have a bialgebra isomorphism \(\lambda_\mathcal{A} : K(Q) \to \mathcal{A}_f\), see Section 3.10. Note that this is the case if the characteristic of \(k\) is zero. However we have no example of a Dynkin quiver that is not \(k\)-even for some field \(k\) of positive characteristic. This suggests the following conjecture.

**Conjecture 1.3.** Each Dynkin quiver is even.

To prove Conjecture 1.3 it is enough to verify that the fibers of the morphisms \(\pi_y, y \in Y_\nu\) have no odd cohomology group over each field, see Lemma 3.7.

Let \(\Lambda_\nu\) be the set of \(G_\nu\)-orbits in \(E_V\). For \(\lambda \in \Lambda_\nu\) we will write \(\mathcal{O}_\lambda\) for the corresponding orbit. The parity sheaves in \(D_{G_V}(E_V, k)\) and the indecomposable objects in \(\mathcal{Q}_V\) modulo shifts are parametrized by the the set \(\Lambda_\nu\), see Section 2.21 and Remark 3.19 (a). We denote them respectively \(\mathcal{E}(\lambda)\) and \(R_\lambda\) with \(\lambda \in \Lambda_\nu\). The classes of the complexes \(R_\lambda\), with \(\lambda \in \Lambda_\nu\), form an \(\mathcal{A}\)-basis in \(K(\mathcal{Q}_V)\). Thus, the elements \(\lambda_\mathcal{A}(\{R_\lambda\})\) with \(\lambda \in \Lambda_\nu\) and all \(V\) form an \(\mathcal{A}\)-basis in \(\mathcal{A}_f\). On the other hand the classes of the complexes \(\mathcal{E}(\lambda)\) with \(\lambda \in \Lambda_\nu\) form an \(\mathcal{A}\)-basis in \(K(\text{Par}_{G_V}(E_V))\) and thus the elements \(\beta_\mathcal{A}(\{\mathcal{E}(\lambda)\})\) with \(\lambda \in \Lambda_\nu\) and all \(V\) form an \(\mathcal{A}\)-basis in \(\mathcal{A}_f\). It is natural to compare these bases. We expect that the following result holds.

**Conjecture 1.4.** For each finite dimensional \(I\)-graded \(\mathbb{C}\)-vector space \(V\) and each \(\lambda \in \Lambda_\nu\) we have \(\lambda_\mathcal{A}(\{R_\lambda\}) = \beta_\mathcal{A}(\{\mathcal{E}(\lambda)\})\).

Note that Conjecture 1.4 follows directly from Conjecture 1.3 because if \(\Gamma\) is \(k\)-even we have \(R_\lambda = \mathcal{E}(\lambda)\) for each \(\lambda \in \Lambda_\nu\) and each \(V\), see Remark 3.19 (b).

## 2. Geometric construction of KLR-algebras

In this paper \(A\) will always denote a commutative ring of finite global dimension.

### 2.1. Quivers

Let \(\Gamma\) be a quiver without loops. We denote by \(I\) and \(H\) the sets of its vertices and arrows respectively. For an arrow \(h \in H\) we will write \(h^i\) and \(h^n\) for its source and target respectively. For a dimension vector \(\nu = \sum_{i \in I} \nu_i \cdot i \in NI\) we set
\[
E_V = \bigoplus_{h \in H} \text{Hom}(V_{h^n}, V_{h^i}), \quad |\nu| = \sum_{i \in I} \nu_i,
\]
where \(V_i\) is a \(\mathbb{C}\)-vector space of dimension \(\nu_i\) for every \(i \in I\). If \(\Gamma\) is a Dynkin quiver, the natural action of \(G_V = \prod_{i \in I} GL(V_i)\) on \(E_V\) has finitely many orbits. This defines a stratification \(\prod_{\lambda \in \Lambda_\nu} \mathcal{O}_\lambda\) on \(E_V\), where \(\Lambda_\nu\) labels all orbits.

Let us introduce some notation for a later use. Let \(h_{i,j}\) be the number of arrows from \(i\) to \(j\) in \(\Gamma\) and set
\[
i \cdot j = -h_{i,j} - h_{j,i}, \quad i \cdot i = 2, \quad i \neq j.
\]
Let $Y_\nu$ be the set of all pairs $y = (i, a)$ where $i = (i_1, i_2, \cdots, i_m)$ is a sequence of elements of $I$ and $a = (a_1, a_2, \cdots, a_m)$ is a sequence of positive integers such that $\sum_{i=1}^m a_i i = \nu$. We will write $y = (i_1^{a_1}, \cdots, i_m^{a_m})$. Let $I^\nu \subset Y_\nu$ be the set of all pairs $y = (i, a)$ such that $a_i = 1$ for all $i$. We will abbreviate $i$ for $y = (i, a)$ if $y \in I^\nu$. We denote by $F_\nu$ the variety of all flags

$$\phi = \{ (0) = V^0 \subset V^1 \subset \cdots \subset V^m = V \}$$

in $V = \bigoplus_{i \in I} V_i$ such that the $I$-graded vector space $V^r/V^{r-1}$ has graded dimension $a_r \cdot i_r$ for $r \in [1, m]$. We denote by $\bar{F}_\nu$ the variety of pairs $(x, \phi) \in E_V \times F_\nu$ such that $x(V^r) \subset V^r$ for $r \in \{0, 1, \cdots, m\}$. Let $\pi_y$ be the natural projection from $\bar{F}_\nu$ to $E_V$, i.e., $\pi_y : \bar{F}_\nu \to E_V$, $(x, \phi) \mapsto x$. For $y_1, y_2 \in Y_\nu$ we denote by $Z_{y_1, y_2}$ the variety of triples $(x, \phi_1, \phi_2) \in E_V \times F_{y_1} \times F_{y_2}$ such that $x$ preserves $\phi_1$ and $\phi_2$. For $i \in I^\nu$ and $l = 1, 2, \cdots, |\nu|$, we define $O_{F_{\nu}}(l)$ to be the $G_V$-equivariant line bundle over $\bar{F}_\nu$ whose fiber at the flag $\phi$ is equal to $V^l/V^{l-1}$.

2.2. **KLR-algebra.** Set $m = |\nu|$. The group $\mathfrak{S}_m$ acts on $I^\nu$ by $w \cdot (i_1, \ldots, i_m) = (i_{w^{-1}(1)}, \ldots, i_{w^{-1}(m)})$. We denote by $s_l$ the transposition $(l, l+1) \in \mathfrak{S}_m$ for $l \in [1, m-1]$. For each sequence $i = (i_1, i_2, \ldots, i_m) \in I^\nu$ and each integers $k, l \in [1, m]$, $l \neq m$ we abbreviate

$$h_i(l) = h_{i, l+1} \text{ if } s_l(i) \neq i, \quad h_i(l) = -1 \text{ if } s_l(i) = i.$$

$$a_i(l) = h_i(l) + h_{s_l(i)}(l) = -i \cdot i_{l+1}.$$

**Definition 2.1.** The **KLR-algebra** $R_{\nu, A}$ (or simply $R_{\nu}$ if the ring of coefficients is clear) over the ring $A$ associated with $I$ and the dimension vector $\nu$ is the $A$-algebra generated by $1_I$, $x_i(k)$, $\tau_i(l)$ with $i \in I^\nu$, $1 \leq k, l \leq m$ and $l \neq m$, modulo the following defining relations:

- $l_l^I = \delta_l^I 1_I$,
- $\tau_i(l) = 1_{s_l(i)}(l) 1_i$,
- $x_i(k) = 1_i x_i(k) 1_i$,
- $x_i (k) x_i(k') = x_i(k') x_i(k)$,
- $\tau_s(i)(l) \tau_i(l') = Q_{l, l'}(x_i(l), x_i(l+1))$,
- $\tau_{s_l(i)}(l') \tau_s(i)(l) = \tau_{s_l(i)}(l') \tau_s(i)(l)$ if $|l-l'| > 1$,
- $\tau_{s_l s_{l+1}}(l+1) \tau_s(i)(l) 1_{s_l(i)}(l+1) = (Q_{l+1}(x_i(l+1), x_i(l+1)) - Q_{l+1}(x_i(l), x_i(l+1))) (x_i(l+2) - x_i(l+1))^{-1}$ if $s_l l+2 = 1$ and 0 else,
- $\tau_i(l) x_i(k) - x_i(s_l(i) s_l(k)) 1_{s_l(i)}(l) = \begin{cases} -1_i & \text{if } k = l, \ s_l(i) = i, \\ 1_i & \text{if } k = l + 1, \ s_l(i) = i, \\ 0 & \text{else}. \end{cases}$

Here we have set $s_{l, l+2} = s_l s_{l+1} s_l$ if $l \neq m - 1, m$ and

$$Q_{l, l'}(u, v) = \begin{cases} (-1)^{h_i(l)} (u - v)^a_i(l) & \text{if } s_l(i) \neq i, \\ 0 & \text{else}. \end{cases}$$

Note that the fraction

$$(Q_{l, l'}(x_i(l+2), x_i(l+1)) - Q_{l, l'}(x_i(l), x_i(l+1)) (x_i(l+2) - x_i(l))^{-1}$$

is indeed an element of $A[x_i(1), \cdots x_i(m)]$. Note also that $R_{\nu, A} = \bigoplus_{l \in I^\nu} R_{\nu, A}^l$, where $R_{\nu, A}^l = 1_I R_{\nu, A} 1_I$.

The KLR-algebra $R_{\nu, A}$ has a natural $\mathbb{Z}$-grading such that $\deg 1_I = 0$, $\deg x_i(k) = 2$ and $\deg \tau_i(l) = a_i(l)$. The relations in Definition 2.1 are homogeneous with respect to this grading.
2.3. A faithful representation of $R_{\nu,A}$. Now, we consider the representation of $R_{\nu,A}$ on the space 

$$\mathcal{P}ol_{\nu,A} = \bigoplus_{i \in I^\nu} \mathcal{P}ol_{i,A}, \quad \mathcal{P}ol_{i,A} = A[x_i(1), \ldots, x_i(m)],$$

which is given by:

- the element $1_i \in R_{\nu,A}$ acts on $\mathcal{P}ol_{\nu,A}$ by the projection on $\mathcal{P}ol_{i,A}$,
- the element $x_i(k) \in R_{\nu,A}$ acts on $\mathcal{P}ol_{i,A}$ by 0 if $i \neq j$ and by multiplication by $x_i(1)$ if $i = j$,
- the element $\tau_i(l)$ acts on $\mathcal{P}ol_{i,j,A}$ by 0 if $i \neq j$ and it sends $f \in \mathcal{P}ol_{i,A}$ to 
  - $-(x_i(l) - x_i(l + 1))$ if $s_i(i) = i$,
  - $(x_{s_i(i)}(l) - x_{s_i(i)}(l + 1))h_i(l)1_i$ if $s_i(i) \neq i$.

This representation is faithful, see [21, Sec. 3.2.2].

2.4. Reminder on Borel-Moore homology. In this section we suppose $G$ is an algebraic group which acts on a variety $X$. Denote by $D_G(X, A)$ the $G$-equivariant derived category of sheaves of $A$-modules on $X$. Let $\mathcal{D}_{X,A}$ be the $G$-equivariant dualizing complex on $X$, see [10, Def. 3.1.16] in the non-equivariant case and [2, Def. 3.5.1] in the equivariant case. The space of $G$-equivariant Borel-Moore homology is $H^G_*(X, A) = H^*_G(X, \mathcal{D}_{X,A})$. Note also that $\mathcal{D}_{X,A} = \mathcal{D}_X[2d]$ if $X$ is a smooth $G$-variety of pure dimension $d$, where we denote by $\mathcal{D}_X$ the $G$-equivariant constant sheaf on $X$, see [3, Prop. 8.3.4]. In this case we have

$$H^G_*(X, A) = H^*_G(X, \mathcal{D}_X[2d]).$$

So we can identify $H^G_*(X, A)$ with $H^*_G(X, A)[2d]$. The functor

$$\mathcal{D} = \mathcal{D}_X : D_G(X, A) \rightarrow D_G(X, A), \quad \mathcal{E} \mapsto \mathcal{E} \otimes \mathcal{D}_{X,A}$$

has the following properties, see [2, Thm. 3.5.2]:

- $(\mathcal{D}_X)^2 = \text{Id}_{D_G(X,A)}$,
- $\mathcal{D}_Y f_* = f_* \mathcal{D}_X$, $\forall f : X \rightarrow Y$ continuous $G$-map,
- $\mathcal{D}_X f^* = f^* \mathcal{D}_Y$, $\forall f : X \rightarrow Y$ continuous $G$-map.

2.5. Algebra $Z_{V,A}$. Set $G = GL(V)$. Then $G_V$ is a closed reductive subgroup of $G$. Consider the varieties $Z_V = \bigcap_{i \in I^\nu} Z_{i,j}$ and $\tilde{F}_V = \bigcap_{i \in I^\nu} \tilde{F}_i$, see Section [27]. For $i, j \in I^\nu$ we set

$$Z_{V,A} = H^G_{*V}(Z_{V,A}), \quad Z_{i,j,A} = H^G_{*V}(Z_{i,j,A}),$$

$$F_{V,A} = H^G_{*V}(\bar{F}_V, A), \quad F_{i,A} = H^G_{*V}(\bar{F}_i, A).$$

We equip $Z_{V,A}$ with the convolution product $\ast$ relative to the closed embedding $Z_V \subset \tilde{F}_V \times \tilde{F}_V$. See [4, Sec. 2.7] for details. In the same way we have the $Z_{V,A}$-module structure on $F_{V,A}$. We will denote it also by $\ast$.

Denote by $F$ the variety of all complete flags in $V$. Denote by $F_V$ the variety of all complete flags in $V$ that agree with the grading of $V$, i.e., $F_V = \bigcap_{i \in I^\nu} F_i$. Then $F_V \subset F$. Let $T_V$ be a maximal torus in $G_V$. Denote by $\mathfrak{S}_V$ and $\mathfrak{S}$ the Weyl groups of $(G_V, T_V)$ and $(G, T_V)$ respectively. Fix once and for all a $T_V$-stable flag $\phi_V$ in $F_V$. Write

$$\phi_V = (0 \subset D_1 \subset D_1 \oplus D_2 \subset \cdots \subset D_1 \oplus D_2 \oplus \cdots \oplus D_{m-1} \subset D_1 \oplus D_2 \oplus \cdots \oplus D_m = V),$$

where $D_1, \ldots, D_m$ are $T_V$-stable one-dimensional vector spaces in $V$. The action of $\mathfrak{S}_V$ on the set $\{D_1, D_2, \ldots, D_m\}$ identifies $\mathfrak{S}_V$ with the symmetric group $S_m$. Note also that the action of $\mathfrak{S}$ on the set $\{D_1, D_2, \ldots, D_m\}$ induces the action of $\mathfrak{S}$ on the set of $T_V$-stable flags in $F_V$. Set $\phi_{V,w} = w(\phi_V)$ for $w \in \mathfrak{S}$. Denote by $B$ and
Let $\Pi$ be the set of simple reflections. Let $\chi_1, \chi_2, \cdots, \chi_m \in \mathfrak{t}_V^*$ be the weights of the lines $D_1, D_2, \cdots, D_m$ respectively, where $\mathfrak{t}_V = \text{Lie}(T_V)$. The set of simple roots in $\Delta^+$ is $\{\chi_1 - \chi_{l+1}; l = 1, 2, \cdots, m - 1\}$. Let $s_l \in \Pi$ denote the reflection with respect to the simple root $\chi_l - \chi_{l+1}$. Under the identification $\mathfrak{g} = \mathfrak{g}_m$ the reflection $s_l$ is a transposition $(l, l+1)$.

2.6. Algebras $P_{V,A}$ and $S_{V,A}$. Denote by $P_{V,A}$ the $A$-algebra $H^*_T \{\bullet, A\}$. The $A$-algebra $P_{V,A}$ is isomorphic to the algebra of polynomials on $\mathfrak{t}_V$:

$$P_{V,A} = A[\chi_1, \chi_2, \cdots, \chi_m]$$

where $\chi_1, \cdots, \chi_m$ are as in Section 2.6. Let $S_{V,A}$ be the $A$-algebra $H^*_G \{\bullet, A\}$. For each positive integer $n$ the fundamental group of $GL_n(\mathbb{C})$ is equal to $\mathbb{Z}$. Thus, the fundamental group of $G_V$ does not contain elements of finite order. Hence, by [23, Lem. 1] the torsion index of $G_V$ is equal to 1 (see [8, Sec. 2] for the definition of a torsion index of a Lie group). So by [8, Cor. 2.3] the algebra $S_{V,A}$ is isomorphic to the algebra $P_{V,A}^{S_{V,A}}$ of $S_{V,A}$-invariant polynomials on $\mathfrak{t}_V$.

2.7. Stratification on $F_V \times F_V$. The group $G$ acts diagonally on $F \times F$. The action of the subgroup $G_V$ preserves the subset $F_V \times F_V$. For $x \in \mathfrak{g}$ let $O^x_V$ be the set of all pairs of flags in $F_V \times F_V$ which are in relative position $x$. More precisely, write $\phi_{V,w',w} = (\phi_{V,w'}, \phi_{V,w})$, $\forall w, w' \in \mathfrak{g}$. Then we set $O^x_V = (F_V \times F_V) \cap G\phi_{V,w',w}$.

2.8. Stratification on $Z_V$. Let $Z^x_V$ be the Zariski closure of the locally closed subset $q^{-1}(O^x_V)$, where $q$ is a natural map

$$q : Z_V \to F_V \times F_V, (x, F_1, F_2) \mapsto (F_1, F_2).$$

Put

$$Z^x_V = \bigcup_{w \in S^x_V} Z^w_V, \quad Z^x_{V,A} = H^*_sG^s_G(Z^x_V, A), \quad Z^x_{V,A} = H^*_sG^s_G(Z^x_V, A),$$

where $\leq$ is the Bruhat order.

2.9. Generators of $Z^x_{V,A}$. Denote by $l_i$ the fundamental class of $Z_{1,i}$ in $H^*_G(Z_{1,i}, A)$ regarded as an element of $H^*_G(Z_V, A)$. Fix a simple reflection $s = s_l \in \Pi$. The fundamental class of $Z^x_V$ yields an elements $\sigma(l) \in Z^{\leq s}_{V,A}$. For $i, i' \in I'$ we write

$$\sigma_{i,i'}(l) = 1_{i'} * \sigma(l) * 1_i \in Z^{\leq s}_{V,A}.$$ 

Next, suppose $k \in [1, m]$. The pull-back of the first Chern class of the line bundle $\bigoplus_i \mathcal{O}_{\tilde{F}_i}(k)$ by the first projection

$$Z^x_V \subset \tilde{F}_V \times \tilde{F}_V \xrightarrow{pr_1} \tilde{F}_V$$

belongs to $H^*_G(Z^x_V, A)$, here $\mathcal{O}_{\tilde{F}_i}(k)$ is as in Section 2.6. It yields an element $x(k) \in Z^x_{V,A}$. We write

$$x_{i,i'}(k) = 1_{i'} * x(k) * 1_i \in Z^x_{V,A}.$$ 

We will need the following standard lemma, see [4, Chap. 6].
Proposition 2.2. \( \text{The direct image by the closed embedding } Z^x_V \subset Z_V \text{ gives an injective } A\text{-module homomorphism } Z^x_V \rightarrow Z_{V,A}. \)

(b) For \( x, y \in \mathfrak{S} \) such that \( l(xy) = l(x) + l(y) \) the convolution product in \( Z_{V,A} \) yields an inclusion \( Z^x_V \cdot Z^y_V \subset Z^x_V \). In particular \( Z_{V,A} \) is a subalgebra of \( Z_{V,A} \).

Proof. All the statements are obvious except, perhaps, the injectivity in (a). It follows from the existence of a decomposition

\[
Z^x_V = \prod_{w \leq x} q^{-1}(O_w).
\]

We need the following obvious lemma.

Lemma 2.3. For each \( G\)-orbit \( O \) in \( F_V \times F_V \) the map \( p : q^{-1}(O) \rightarrow F_i \) such that \( (x, \phi_1, \phi_2) \mapsto \phi_2 \) for each \( x \in E_V \), \( \phi_1, \phi_2 \in F_V \) is an affine \( G\)-equivariant fibration, where \( i \) is the unique element of \( I^\nu \) such that \( O \subset F_V \times F_i \).

Now we can conclude applying the arguments of [11, Sec. 6.2] and the exact sequence in equivariant Borel-Moore homology. We should just replace [4, Lem. 6.2.5] by Lemma 2.3.

\[ \square \]

So we can consider \( x_{VA}(k) \) and \( \sigma_{VA}(l) \) as elements of \( Z_{VA} \).

2.10. The \( A\)-algebra structure on \( F_{VA} \). Let \( i \in I^\nu \). Assigning to a formal variable \( x_i(l) \) of degree 2 the first Chern class of \( G\)-equivariant line bundle \( O_{F_i}(l) \) for \( l = 1, 2, \ldots, m \) yields a graded \( A \)-algebra isomorphism

\[ A[x_i(1), x_i(2), \ldots, x_i(m)] = F_{IA}. \]

The multiplication of polynomials equips each \( F_{IA} \) with an obvious graded \( A \)-algebra structure. Thus, \( F_{VA} = \bigoplus_{l \in I^\nu} F_{IA} \) is also a graded \( A \)-algebra.

Now we can consider \( Z_{VA} \) as a \( F_{VA} \)-module such that \( f(x_i(1), \ldots, x_i(m)) \in F_{IA} \)

acts on \( Z_{VA} \) by the operator \( z \mapsto f(x_i(1), \ldots, x_i(m)) \ast z \). We can interpret this \( F_{VA} \)-action on \( Z_{VA} \) as follows. Note that

\[ \tilde{F}_V = \{ (x, \phi) \in E_V \times F_V; x \text{ preserves } \phi \} = Z^x_V. \]

Then \( F_{VA} = Z_{VA} \). So the \( F_{VA} \)-action on \( Z_{VA} \) is the action of the subalgebra \( Z_{VA} \) on \( Z_{VA} \).

2.11. KLR-algebra and Borel-Moore homology. For each sequence \( i = (i_1, i_2, \ldots, i_m) \in I^\nu \) and for each integers \( k, l \in [1, m] \), \( l \neq m \) we consider the following elements\n
\[ x_i(k), \sigma_i(l) \in Z_{VA} \]

Theorem 2.4. We have an \( A \)-algebra isomorphism \( R_{VA} \approx Z_{VA} \).

Proof. First, we claim that there is an \( A \)-algebra homomorphism \( \Phi_A : R_{VA} \rightarrow Z_{VA} \) given by

\[ \Phi_A(1) = 1, \quad \Phi_A(x_i(k)) = x_i(k), \quad \Phi_A(\sigma_i(l)) = (-1)^{b_{i,l}} \sigma_i(l). \]

To prove this we must check that the relations from Definition 2.1 hold for the elements \( 1, x_i(k), (-1)^{b_{i,l}} \sigma_i(l) \in Z_{VA} \). Now according to [23, Thm. 3.6], this holds for \( A = \mathbb{C} \). We want to deduce from this that the relations hold for any \( A \). To do this, we first construct an \( F_{VA} \)-basis of \( Z_{VA} \) using the following lemma, which can be proved in the same way as Lemma 2.2...
Lemma 2.5. We have $Z_{V,A}^{\leq x} = \bigoplus_{w \leq x} F_{V,A} \ast [Z^w]_V$ and $Z_{V,A}^x = \bigoplus_{w < x} F_{V,A} \ast [Z^w]_V$ for $w \in \mathcal{S}$. In particular $Z_{V,A}$ is a free graded $F_{V,A}$-module of rank $m!$.

By Lemma 2.5 the $\mathbb{Z}$-module $Z_{V,Z}$ is free. So by the universal coefficient theorem we have an inclusion of rings $i_A : Z_{V,Z} \to Z_{V,A}$. By [25, Thm. 2.5] the $\mathbb{Z}$-module $R_{v,Z}$ is free. So we have an inclusion $i_A : R_{v,Z} \to R_{v,A}$. Since $j_{\mathbb{C}}$ takes the elements $l_1, x_i(k), (-1)^{h_i(l)} \sigma_i(l)$ in $Z_{V,Z}$ to the corresponding elements in $Z_{V,C}$, the defining relations in $R_{v,Z}$ are satisfied by the generators of $Z_{V,Z}$. Thus, the ring homomorphism $\Phi_Z : R_{v,Z} \to Z_{V,Z}$ is well-defined and we have the following commutative diagram

$$
R_{v,C} \xrightarrow{\Phi_C} Z_{V,C} \\
\uparrow_{i_C} \quad \uparrow_{j_C} \\
R_{v,Z} \xrightarrow{\Phi_Z} Z_{V,Z}.
$$

Now we must prove that $\Phi_Z$ is surjective. Note that the map $\Phi_Z$ is $F_{V,Z}$-linear. Further, the morphism $\Phi_Z$ is injective because $i_C$ and $\Phi_C$ are. Let us prove that $\Phi_Z$ is surjective. We will need the following lemma.

Lemma 2.6. The following relations hold in $Z_{V,Z}$:

(a) $\sigma_{i,j}(l) = 0$ unless $i \in \{1, s_j(j)\}$,

(b) $\sigma_{i,i}(l) = (x_i(l + 1) - x_i(l))^{h_{\mu \nu} \rho + 1}$ if $s_i(l) \neq i$.

Proof. By [25, Prop. 2.22] and [25, Lem. 1.8 (a)] these relations hold in $Z_{V,C}$. Thus, they also hold in $Z_{V,Z}$ by injectivity of the map $j_C$. □

The image of $\Phi_Z$ contains the elements $x_i(k)$ and $l_i$ for $i \in I^\nu$ and $k \in [1, m]$. So it contains $F_{V,Z} \ast [Z^*_V]$ because

$$[Z^*_V] = \sum_{i \in I^\nu} l_i, \quad F_{V,Z} = \bigoplus_{i \in I^\nu} \mathbb{Z}[x_i(1), x_i(2), \ldots, x_i(m)].$$

Next, we prove that the image of $\Phi_Z$ contains $[Z^*_V]$ for $l \in [1, m - 1]$. By Lemma 2.6 (a) we have

$$[Z^*_V] = \sigma(l) = \sum_{i \in I^\nu} \sigma_{i,j}(l) = \sum_{i \in I^\nu} \sigma_{s_i(i),i}(l) + \sum_{i \in I^\nu, s_i(i) \neq i} \sigma_{i,i}(l) = \sum_{i \in I^\nu} \sigma_i(1) + \sum_{i \in I^\nu, s_i(i) \neq i} \sigma_{i,i}(l).$$

We know that $\sigma_i(l) = \Phi_Z((-1)^{h_i(l)} \eta_i(l))$. If $s_i(l) \neq i$ then by Lemma 2.6 (b) we have

$$\sigma_i(l) = (x_i(l+1) - x_i(l))^{h_{\mu \nu} \rho + 1} = \Phi_Z((x_i(l+1) - x_i(l))^{h_{\mu \nu} \rho + 1}).$$

So $[Z^*_V]$ is in the image of $\Phi_Z$. To conclude we use Lemma 2.6 and the following lemma, see [25, Lem. 3.7].

Lemma 2.7. If $l(s_w) = l(w) + 1$ we have $[Z^*_V] \ast [Z^*_V] = [Z^{s_w}_{V}]$ in $Z^{s_w}_{V}/Z^{s_w}_{V}$. This proves that $\Phi_Z$ is bijective. Thus $\Phi_A$ is well-defined and bijective for arbitrary commutative ring $A$ because $R_{v,A} = R_{v,Z} \otimes A$ and $Z_{V,A} = Z_{V,Z} \otimes A$. □
2.12. **Stratified varieties.** Let $X$ be a $G$-variety for some connected algebraic group $G$. We fix a stratification $X = \bigsqcup_{\lambda \in \Lambda} X_{\lambda}$ of $X$ into smooth connected locally closed $G$-stable subsets such that the closure of each stratum is a union of strata. Let $D_G(X, A)$ be the bounded $G$-equivariant derived category of sheaves of $A$-modules on $X$ constructible with respect to the stratification $X = \bigsqcup_{\lambda \in \Lambda} X_{\lambda}$.

**Lemma 2.8.** Suppose that $A$ is either a complete discrete valuation ring or a field. The category $D_G(X, A)$ is a Krull-Schmidt category.

**Proof.** By [13] Thm., the category $D_G(X, A)$ is Karoubian because its $t$-structure is bounded, see [2] Def. 2.2.4, Prop. 2.5.2. Let $k$ be the field of $A$ by its unique maximal ideal. We need to prove that the endomorphism ring $\text{End}(\mathcal{F})$ of each indecomposable object $\mathcal{F}$ in $D_G(X, A)$ is local. Note that under the assumption on the ring $A$ the endomorphism ring $\text{End}(\mathcal{F})$ is local if and only if the ring $k \otimes A \text{End}(\mathcal{F})$ is local.

Now, suppose that the ring $k \otimes A \text{End}(\mathcal{F})$ is not local. The endomorphism ring of every object of $D_G(X, A)$ is of finite type over $A$, see the construction of the bounded equivariant derived category in [2] Sec. 2.1-2.2. Thus the ring $k \otimes A \text{End}(\mathcal{F})$ is a finite dimensional $k$-algebra. In particular the ring $k \otimes A \text{End}(\mathcal{F})$ isartinian. Then the identity of $k \otimes A \text{End}(\mathcal{F})$ can be written as a sum of two orthogonal idempotents $e_1$ and $e_2$ because $k \otimes A \text{End}(\mathcal{F})$ is not local. Moreover, by [10] Ex. 6.16 we can lift these idempotents to the idempotents $\tilde{e}_1$, $\tilde{e}_2$ of $\text{End}(\mathcal{F})$ such that $\tilde{e}_1 + \tilde{e}_2 = 1_{\mathcal{F}}$. Thus by Karoubianness the object $\mathcal{F}$ is not indecomposable.

For each $\lambda \in \Lambda$ we denote by $i_\lambda$ the inclusion $X_\lambda \to X$ and by $d_\lambda$ the complex dimension of $X_\lambda$. For each $x \in X$ we denote by $i_x$ the inclusion $\{x\} \to X$. For all $\lambda \in \Lambda$, let $\mathcal{A}_\lambda$ be the $G$-equivariant constant sheaf on $X_\lambda$. For each $\lambda$ let $\text{Loc}_G(X_\lambda, A)$ be the category of $G$-equivariant local systems of free $A$-modules on $X_\lambda$. Let $\mathcal{A}_X$ be the $G$-equivariant constant sheaf on $X$.

2.13. **Lusztig category** $Q_V$. From now on we suppose that $\Gamma$ is a Dynkin quiver. We have the stratification $E_V = \bigsqcup_{\lambda \in \Lambda_V} O_{\lambda}$ on $E_V$. Here $\Lambda_V$ is the set of $G_V$-orbits in $E_V$. For $\lambda \in \Lambda_V$ we will use the notation $i_\lambda$ and $d_\lambda$ as in Section 2.12, i.e., the map $i_\lambda$ is the inclusion $O_{\lambda} \to E_V$ and $d_\lambda$ is the complex dimension of $O_{\lambda}$. Consider the following complex in $D_{G_V}(E_V, A)$:

$$\mathcal{L}_{V, A} = \bigoplus_{i \in I'}.\mathcal{L}_i,$$

$$\mathcal{L}_i = \pi_i \mathcal{A}_{e_i}, \ i \in I'.$$

Consider also the complex in $D_{G_V}(E_V, A)$:

$$\delta \mathcal{L}_{V, A} = \bigoplus_{i \in I'} \delta \mathcal{L}_i, \quad \delta \mathcal{L}_i = \mathcal{L}_i[d_\lambda], \ i \in I'.$$

**Definition 2.9.** Let $Q_V$ be the full additive subcategory of $D_{G_V}(E_V, A)$ whose objects are the finite direct sums of shifts of direct factors of $\mathcal{L}_{V, A}$.

2.14. **Local systems.** The following lemma is proved in [3] Prop. 2.2.1.

**Lemma 2.10.** The reductive part of the stabilizer in $G_V$ of an element in $E_V$ is isomorphic to $\prod_i GL_{n_i}(\mathbb{C})$ for some positive integers $n_i$.

We get the following result from Lemma 2.10 and [4] Lem. 8.4.11.

**Corollary 2.11.** Let $O$ be a $G_V$-orbit in $E_V$. Each local system in $\text{Loc}_{G_V}(O)$ is a multiple of the trivial $G_V$-equivariant local system $\mathcal{A}_O$ on $O$. 


2.15. **KLR-algebra and Yoneda algebras.** Consider the $G_V$-equivariant extension group $\text{Ext}_{G_V}^* (\mathcal{L}_{V,A}, \mathcal{L}_{V,A})$ with the $A$-algebra structure given by the Yoneda product.

**Theorem 2.12.** We have the following $A$-algebra isomorphism $R_{\nu,A} = \text{Ext}_{G_V}^* (\mathcal{L}_{V,A}, \mathcal{L}_{V,A})$.

**Proof.** We have proved in Theorem 2.4 that there is an $A$-algebra isomorphism $R_{\nu,A} = H^G_{\nu} (Z_V, A)$.

Now it suffice to show that there exists as $A$-algebra isomorphism $H^G_{\nu} (Z_V, A) = \text{Ext}_{G_V}^* (\mathcal{L}_{V,A}, \mathcal{L}_{V,A})$.

For each $i,j \in I'$ consider the following commutative diagram

\[
\begin{array}{ccc}
Z_{ij} & \xrightarrow{p_2} & \tilde{F}_j \\
p_1 \downarrow & & \downarrow \pi_j \\
F_i & \xrightarrow{\pi_i} & E_V,
\end{array}
\]

where $p_1$ and $p_2$ are the natural projections. For each $n \in \mathbb{N}$ we have

\[
\text{Ext}_{G_V}^n ((\pi_1), (\pi_j), (\pi_j), (\pi_j)) = \text{Ext}_{G_V}^n (\tilde{A}_{\tilde{F}_i}, (p_1)^{\ast} (\pi_j), (p_2)^{\ast} (\pi_j))
\]

\[
= \text{Ext}_{G_V}^n (\tilde{A}_{\tilde{F}_i}, (p_1), (p_2)^{\ast} (\pi_j))
\]

\[
= \text{Ext}_{G_V}^n ((p_1)^{\ast} \tilde{A}_{\tilde{F}_i}, (p_2)^{\ast} (\pi_j))
\]

\[
= \text{Ext}_{G_V}^n (\tilde{A}_{\tilde{F}_i}, D_{Z_{\tilde{F}_i}} [-2 \dim C \tilde{F}_j])
\]

\[
= H^n_{G_V} (Z_{\tilde{F}_i}, D_{Z_{\tilde{F}_i}})
\]

\[
= H^n_{G_V} (Z_{\tilde{F}_i}, D_{Z_{\tilde{F}_i}})
\]

Thus we have a set bijection between $H^G_{\nu} (Z_V, A)$ and $\text{Ext}_{G_V}^* (\mathcal{L}_{V,A}, \mathcal{L}_{V,A})$. Now we need to show that the convolution product on $H^G_{\nu} (Z_V, A)$ coincides with the Yoneda product on $\text{Ext}_{G_V}^* (\mathcal{L}_{V,A}, \mathcal{L}_{V,A})$. At first note that this follows from [H Prop. 8.6.35] for $A = \mathbb{C}$. Now we show this for $A = \mathbb{Z}$. We have a commutative diagram

\[
\begin{array}{ccc}
H^G_{\nu} (Z_V, \mathbb{C}) & \xrightarrow{\text{Ext}_{G_V}^*} & \text{Ext}_{G_V}^* (\mathcal{L}_{V,C}, \mathcal{L}_{V,C}) \\
\uparrow & & \uparrow \\
H^G_{\nu} (Z_V, \mathbb{Z}) & \xrightarrow{\text{Ext}_{G_V}^*} & \text{Ext}_{G_V}^* (\mathcal{L}_{V,Z}, \mathcal{L}_{V,Z}),
\end{array}
\]

where the left vertical map is an algebra inclusion, the right vertical map is an algebra homomorphism, the top horizontal map is an algebra isomorphism and the bottom horizontal map is a set bijection. Thus the right vertical map is automatically an algebra inclusion and the bottom horizontal map is automatically an algebra isomorphism. The general case follows from the case $A = \mathbb{Z}$ by extension of scalars.

**Remark 2.13.** The results of this section are also true for each quiver $\Gamma$ without loops.

2.16. **Functor $Y'$.** We consider two different gradings on the algebra $R_{\nu,A}$:

1. the grading introduced in Section 2.2
2. the grading $\bigoplus_{n \in \mathbb{N}} \text{Ext}_{G_V}^n (\mathcal{L}_{V,A}, \mathcal{L}_{V,A})$, see Theorem 2.12.
Denote by \( \text{mod}(R_{\nu,A}) \) (resp. \( \text{mod}'(R_{\nu,A}) \)) the category of finitely generated \( \mathbb{Z} \)-graded \( R_{\nu,A} \)-modules with respect to the first (resp. second) grading. Denote by \( \text{proj}(R_{\nu,A}) \) (resp. \( \text{proj}'(R_{\nu,A}) \)) the category of projective finitely generated \( \mathbb{Z} \)-graded \( R_{\nu,A} \)-modules with respect to the first (resp. second) grading. Consider the following contravariant functor:

\[
Y': D_{G_{V}}(E_{V}, A) \to \text{mod}'(R_{\nu,A}), \mathcal{L} \mapsto \text{Ext}^*_G_{V}(\mathcal{L}, L_{V,A}).
\]

**Theorem 2.14.** The restriction of the functor \( Y' \) to \( Q_V \) yields an equivalence of categories \( Q^\text{op}_V \to \text{proj}'(R_{\nu,A}) \).

**Proof.** Suppose that \( \mathcal{F} \) is a direct factor of the complex \( L_{V,A} \) in \( D_{G_{V}}(E_{V}, A) \). We have \( L_{V,A} = \mathcal{F} \oplus \mathcal{G} \) for some complex \( \mathcal{G} \) in \( D_{G_{V}}(E_{V}, A) \). We have

\[
R_{\nu,A} = \text{Ext}^*_G_{V}(L_{V,A}, L_{V,A}) = \text{Ext}^*_G_{V}(\mathcal{F}, L_{V,A}) \oplus \text{Ext}^*_G_{V}(\mathcal{G}, L_{V,A}) = Y'(\mathcal{F}) \oplus Y'(\mathcal{G}).
\]

Thus \( Y'(\mathcal{F}) \) is a projective and finitely generated \( R_{\nu,A} \)-module. From this we see that for every \( X \in Q_V \) the \( R_{\nu,A} \)-module \( Y'(X) \) is projective and finitely generated. The \( R_{\nu,A} \)-module \( Y'(X) \) is graded by \( Y'(X)_r = \text{Ext}^*_G_{V}(X, L_{V,A}) \) for each \( r \in \mathbb{Z} \). Now we prove that the functor \( Y': Q^\text{op}_V \to \text{proj}'R_{\nu,A} \) is an isomorphism on morphisms. We need to show that for \( R, S \in Q_V \) the morphism

\[
Y_{R,S}: \text{Hom}_{G_{V}}(R, S) \to \text{Hom}_{\text{proj}'(R_{\nu,A})}(\text{Ext}^*_G_{V}(S, L_{V,A}), \text{Ext}^*_G_{V}(R, L_{V,A}))
\]

is an isomorphism. First, we consider the case \( R = S = L_{V,A} \). Then we have

\[
\text{Hom}_{\text{proj}'(R_{\nu,A})}(\text{Ext}^*_G_{V}(S, L_{V,A}), \text{Ext}^*_G_{V}(R, L_{V,A})) = \text{Hom}_{\text{proj}'(R_{\nu,A})}(R_{\nu,A}, R_{\nu,A}) = \text{Hom}_{Q_V}(L_{V,A}, L_{V,A}).
\]

Here \( R_{\nu,A}^0 \) means the 0-graded component of \( R_{\nu,A} \) with respect to the second grading. So for \( R = S = L_{V,A} \) the statement is true. Then it is also true if \( R \) and \( S \) are direct factors of \( L_{V,A} \). Then it is also true if \( R \) and \( S \) are shifts of direct factors of \( L_{V,A} \). Finally we see that the statement is true for arbitrary \( R, S \in Q_V \).

To conclude we need to show that \( Y': Q^\text{op}_V \to \text{proj}'(R_{\nu,A}) \) is surjective on isomorphism classes. Let \( P \) be a module in \( \text{proj}'(R_{\nu,A}) \). A choice of a finite number of homogeneous generators of \( P \) yields an epimorphism

\[
\pi: \bigoplus_{i \in J} R_{\nu,A}[n_i] \to P
\]

for some finite set \( J \) and some integers \( n_i \). By projectivity this epimorphism splits. Consider the complex \( \mathcal{L} = \bigoplus_{i \in J} L_{V,A}[-n_i] \). We have

\[
Y'(\mathcal{L}) = \text{Hom}(\mathcal{L}, L_{V,A}) = \bigoplus_{i \in J} R_{\nu,A}[n_i].
\]

The composition of the natural projection and inclusion

\[
Y'(\mathcal{L}) \xrightarrow{\pi} P \xrightarrow{\iota_u} Y'(\mathcal{L})
\]

yields an idempotent \( e \in \text{End}_{\text{proj}'(R_{\nu,A})}(Y'(\mathcal{L})) \). The functor \( Y': Q^\text{op}_V \to \text{proj}'(R_{\nu,A}) \) is an isomorphism on morphisms. So we have an idempotent \( u \in \text{End}_{Q_V}(\mathcal{L}) \) such that \( Y'(u) = e \). The idempotents \( u \) and \( 1 - u \) split in \( D_{G_{V}}(E_{V}, A) \) yielding a decomposition \( \mathcal{L} = X \oplus Y \), where \( X, Y \in D_{G_{V}}(E_{V}, A) \) and \( u \) is a composition of the natural projection and the natural inclusion \( \mathcal{L} \xrightarrow{\pi} X \xrightarrow{\iota_u} \mathcal{L} \). By definition of \( Q_V \) the objects \( X, Y \) belong to \( Q_V \). Now we verify that the map \( \pi_e \circ Y'(\iota_u): Y'(X) \to P \)
is an isomorphism. Let us prove that the map $Y'(\pi_u) \circ i_e : P \to Y'(X)$ is its inverse. We have

$$
\pi_e \circ Y'(i_u) \circ Y'(\pi_u) \circ i_e = \pi_e \circ Y'(u) \circ i_e \\
= \pi_e \circ e \circ i_e \\
= (\pi_e \circ i_e) \circ (\pi_e \circ i_e) \\
= \text{Id}_P
$$

and

$$
Y'(\pi_u) \circ i_e \circ \pi_e \circ Y'(i_u) = Y'(\pi_u) \circ e \circ Y'(i_u) \\
= Y'(\pi_u) \circ Y'(u) \circ Y'(i_u) \\
= Y'(\pi_u \circ u \circ i_u) \\
= Y'(\pi_u \circ i_u) \circ (\pi_u \circ i_u) \\
= Y'(\text{Id}_X) \\
= \text{Id}_{Y'(X)}.
$$

\[\square\]

2.17. **Induction and restriction.** Let $k$ be a field.

In this section we describe the induction and the restriction functors for representations of KLR-algebras. See [11, Sec. 2.6] for more details. Consider $\nu_1, \nu_2 \in \mathbb{N}$ and set $\nu = \nu_1 + \nu_2$. We set

$$
|\nu_1| = m_1, \quad |\nu_2| = m_2, \quad |\nu| = m.
$$

There is a unique inclusion of graded $k$-algebras $R_{\nu_1, k} \otimes R_{\nu_2, k} \subset R_{\nu, k}$ such that

$$
1_1 \otimes 1_j \mapsto 1_k, \quad x_1(k) \otimes 1_j \mapsto x_k(k), \quad 1_1 \otimes x_1(k) \mapsto x_k(m_1 + k), \\
\tau_1(l) \otimes 1_j \mapsto \tau_k(l), \quad 1_1 \otimes \tau_1(l) \mapsto \tau_k(m_1 + l)
$$

for each $k, l, i, j$. Let $1_{\nu_1, \nu_2}$ be the image of the identity element by this inclusion. We get the following functors:

$$
\text{Ind}_{\nu_1, \nu_2} : \text{mod}(R_{\nu_1, k}) \times \text{mod}(R_{\nu_2, k}) \to \text{mod}(R_{\nu, k}), \quad (M_1, M_2) \mapsto R_{\nu, k}1_{\nu_1, \nu_2} \otimes (M_1 \otimes M_2),
$$

$$
\text{Res}_{\nu_1, \nu_2} : \text{mod}(R_{\nu, k}) \rightarrow \text{mod}(R_{\nu_1, k} \otimes R_{\nu_2, k}), \quad M \mapsto 1_{\nu_1, \nu_2}M.
$$

These functors take projective modules to projective modules.

2.18. **Projective $R_{\nu, k}$-modules.** Denote by $A$ the ring $\mathbb{Z}[q, q^{-1}]$. For $m \in \mathbb{N}$, $m > 0$ consider the following elements of $A$

$$
[m] = \sum_{i=1}^{m} q^{m+1-2i} = \frac{q^m - q^{-m}}{q - q^{-1}}, \quad [m]! = \prod_{i=1}^{m} [i], \quad l_m = m(m - 1)/2.
$$

We also set $[0]! = 1$, $l_0 = 0$. For $m = (m_1, \cdots, m_k), m_i \in \mathbb{N}$ set

$$
[m]! = \prod_{i=1}^{k} [m_i]!, \quad l_m = \sum_{i=1}^{k} l_{m_i}.
$$

Given a pair $y = (i, a) \in Y_\nu$ we define a projective $R_{\nu, k}$-module $R_y$ as follows

- If $I = \{i\}, \nu = mi, i = (i, i, \cdots, i)$ and $y = (i, m)$ we set

$$
P_y = P_{i, m} = \mathcal{P}_{\nu_1, A}[l_m].
$$

$R_{\nu, k} \simeq \oplus_{w \in \Sigma_\nu} P_{i, m}[2l(w) - l_m].$

So $P_{i, m}$ is a direct summand of $R_{\nu, k}[l_m]$. We choose once and for all an idempotent $1_{i, m} \in R_{\nu, k}$ such that

$$
P_{i, m} = (R_{\nu, k} \cdot 1_{i, m})[l_m].$$
Lemma 2.15. We have the following lemma, see [11, Sec. 2.5-2.6].

We have the following lemma, see [11, Sec. 2.5-2.6].

\[ P_y = (R_{i,k} \cdot 1_y)[/a], \]

We have the following lemma, see [11, Sec. 2.5-2.6].

Lemma 2.15. We have the following graded projective \( R_{i,k} \)-module isomorphisms

\[
\begin{align*}
(a) \quad P_1 & \simeq [a]! P_y, \quad \text{where } y = (i_1^{(a_1)} \cdots i_k^{(a_k)}) \in Y_\nu, \ i = (i_1^{(a_1)} \cdots i_k^{(a_k)}) \in I^\nu, \ a = (a_1, \cdots, a_k), \\
(b) \quad \text{Ind}_{\nu_1, \nu_2}(P_{y_1}, P_{y_2}) & \simeq P_{y_1 y_2}, \quad \text{where } \nu_1, \nu_2 \in \mathbb{N} I, \ y_1 \in Y_{\nu_1}, y_2 \in Y_{\nu_2} \quad \text{and} \quad y_1 y_2 \in Y_{\nu_1 + \nu_2} \text{ is the concatenation of } y_1 \text{ and } y_2.
\end{align*}
\]

2.19. The algebra \( f \). We recall the definition and general properties of the negative part \( f \) of the Drinfeld-Jimbo quantized enveloping algebra associated with the quiver \( \Gamma \). See [14] for more details.

Definition 2.16. The algebra \( f \) is the \( \mathbb{Q}(q) \)-algebra generated by the elements \( \theta_i \), \( i \in I \) with the relations

\[
\sum_{a+b=1-i-j} (-1)^a \theta_i^{(a)} \theta_j^{(b)}, \quad i \neq j, \ a \geq 0, \ b \geq 0,
\]

where \( \theta_i^{(a)} = \theta_i^a / [a]! \).

The assignment \( \deg \theta_i = i \) for \( i \in I \) defines a \( \mathbb{N} I \)-grading \( f = \bigoplus_{\nu \in \mathbb{N} I} f_\nu \). For each homogeneous element \( x \in f_\nu \), we set \( |x| = \sum_i \nu_i \). The tensor product (over \( \mathbb{Q}(q) \)) \( f \otimes f \) has a \( \mathbb{Q}(q) \)-algebra structure defined by

\[
(x_1 \otimes x_2)(x_1' \otimes x_2') = q^{-|x_2|x_1'}x_1 \otimes x_2 x_2'
\]

where \( x_1, x_2, x_1', x_2' \in f \) are homogeneous. There exists a unique coproduct \( r : f \to f \otimes f \) such that

- \( r(\theta_i) = \theta_i \otimes 1 + 1 \otimes \theta_i \) for \( i \in I \),
- \( r \) is a \( \mathbb{Q}(q) \)-algebra homomorphism.

Definition 2.17. Let \( \mathcal{A} f \) be the \( \mathcal{A} \)-subalgebra of \( f \) generated by the elements \( \theta_i^{(a)} \) with \( i \in I, a \in \mathbb{N} \).

2.20. A new \( \mathcal{Z} \)-basis in \( \mathcal{A} f \). The construction of this section is very similar to the construction given in Section 2.19. Let \( K(Q) \) be the split Grothendieck group of the additive category \( Q \), i.e., \( K(Q) \) is the Abelian group with one generator \( [L] \) for each isomorphism class of objects of \( Q \) and with relations \( [L'] + [L''] = [L] \) whenever \( L \) is isomorphic to \( L' \oplus L'' \). Set

\[
\mathcal{Q} = \bigoplus_V Q_V, \quad K(\mathcal{Q}) = \bigoplus_V K(Q_V),
\]

where \( V \) runs over the isomorphism classes of finite dimensional \( I \)-graded \( k \)-vector spaces and

\[
R_k = \bigoplus_{\nu \in \mathbb{N} I} R_{\nu,k}, \quad K(R_k) = \bigoplus_{\nu \in \mathbb{N} I} K(R_{\nu,k}).
\]

The functors \( \text{Ind}_{\nu_1, \nu_2} \) induce an associative unital \( \mathcal{A} \)-algebra structure on \( K(R_k) \). The functors \( \text{Res}_{\nu_1, \nu_2} \) induce a coassociative counital \( \mathcal{A} \)-coalgebra structure on \( K(R_k) \). We will denote the multiplication and the comultiplication on \( K(R_k) \) by \( \text{Ind} \) and \( \text{Res} \) respectively.
Remark 2.18. There exists a non-degenerate $A$-bilinear form $(\cdot, \cdot): K(R_k) \times K(R_k) \to \mathbb{Q}(q)$ such that

$$(x, \text{Ind}(y_1, y_2)) = (\text{Res}(x), y_1 \otimes y_2), \quad \forall x, y_1, y_2 \in K(R_k),$$

see [11] Sec. 2.5.

The following theorem is proved in [11] Prop. 3.4, Sec. 3.2.

**Theorem 2.19.** There is an $A$-bialgebra isomorphism $\gamma_A: \mathcal{A} \to K(R_k)$ that takes $\theta_y$ to $[P_y]$ for each $y \in Y_v$.

The grading of $R_{v,k}$ from Section 2.2 is different from the Ext-grading, see Section 2.10. To avoid this we must modify a bit the complex $L_{Y,k}$. We get the following theorem, see [25] Thm. 3.6. Recall the complex $L_{Y,k}$ from Section 2.13.

**Theorem 2.20.** We have the following graded $k$-algebra isomorphism

$$R_{v,k} \to \text{Ext}^*_{\mathcal{G}_V}(\delta L_{Y,k}, \delta L_{Y,k}).$$

The proof of the following theorem is the same as the proof of Theorem 2.14.

**Theorem 2.21.** We have the following equivalence of categories.

$$Y': \mathcal{Q}_{V}^{op} \to \text{proj}(R_{v,k}), \quad \mathcal{L} \mapsto \text{Ext}^*_{\mathcal{G}_V}(\Omega, \delta L_{Y,k}).$$

So we have an $A$-linear map $Y: K(\mathcal{Q}) \to K(R_k)$ such that $[\mathcal{F}] \mapsto [Y(\mathcal{F})]$. Moreover, it is clear from the definition of $Y$ that $Y(\delta L_i) = P_i$ for $i \in I^\circ$. If $i \in I^\circ$ is the expansion of the element $y = (j,a)$ in $Y_v$ then we have $\delta L_i = [a]^d \mathcal{L}_y$ (see [25] (4.7)) and $[P_i] = [a][P_y]$ (see Lemma 2.15). Thus, $Y(\delta L_y) = P_y$ for each $y \in Y_v$.

So we have $\gamma_A^{-1} \circ Y(\delta L_y) = \theta_y$ for $y \in Y_v$, where $\gamma_A$ is as in Theorem 2.19.

Consider $\nu_1, \nu_2 \in NI$. Let $V_1, V_2$ be $I$-graded $\mathbb{C}$-vector spaces with graded dimensions $\nu_1, \nu_2$ respectively. Analogically with the case when the characteristic of $k$ is zero, see [13] Sec. 9.2.7], we have a multiplication

$$\circ: \mathcal{Q}_{V_1} \times \mathcal{Q}_{V_2} \to \mathcal{Q}_{V_1 \oplus V_2},$$

such that $\delta L_{y_1} \circ \delta L_{y_2} = \delta L_{y_1 \oplus y_2}$ for $y_1 \in Y_{\nu_1}, y_2 \in Y_{\nu_2}$. Thus, $\gamma_A^{-1} \circ Y$ is an algebra homomorphism. We get the following theorem.

**Theorem 2.22.** There exists an algebra isomorphism $\lambda_A: K(\mathcal{Q}) \to \mathcal{A}$ such that for each $y \in Y_v$, $\nu \in NI$ we have $\lambda_A([\delta L_y]) = \theta_y$.

For each $I$-graded finite dimensional $\mathbb{C}$-vector space $V$ the indecomposable complexes in $\mathcal{Q}_V$ form a $\mathbb{Z}$-basis in $K(\mathcal{Q}_V)$. Combining this for all $V$ we get a $\mathbb{Z}$-basis in $K(\mathcal{Q})$. The homomorphism $\lambda_A$ takes this $\mathbb{Z}$-basis to a $\mathbb{Z}$-basis of $\mathcal{A}$.

### 2.21. Indecomposable objects in $\mathcal{Q}_V$

For each $\lambda \in \Lambda_V$ there is $y_\lambda \in Y_v$ such that $\tilde{F}_{y_\lambda} \to \mathcal{O}_\lambda$ is a resolution of the orbit closure $\overline{\mathcal{O}_\lambda}$, see [20] Thm. 2.2. Let $i_\lambda$ and $\tilde{i}_\lambda$ be the inclusions $i_\lambda: \mathcal{O}_\lambda \to E_V$ and $\tilde{i}_\lambda: (\pi_{y_\lambda})^{-1}(\mathcal{O}_\lambda) \to \tilde{F}_{y_\lambda}$. Denote also $\pi'_{y_\lambda}$ the morphism $\pi_{y_\lambda}: (\pi_{y_\lambda})^{-1}(\mathcal{O}_\lambda) \to \mathcal{O}_\lambda$ induced by $\pi_{y_\lambda}$. By the base change theorem we have

$$i'_{\lambda} \mathcal{L}_{y_\lambda} \cdot k_{\mathcal{F}_{y_\lambda}} = \pi'_{y_\lambda} \cdot \tilde{i}_\lambda \mathcal{L}_{y_\lambda} = \mathcal{L}_{\lambda}.$$

So the complex $\mathcal{L}_{y_\lambda} = \pi_{y_\lambda} \cdot k_{\mathcal{F}_{y_\lambda}}$ has a unique indecomposable factor supported on all $\overline{\mathcal{O}_\lambda}$. Denote by $R_{y_\lambda}$ the shift by $d_\lambda$ of this direct factor. We have $i'_{\lambda} R_{y_\lambda} = k_{\lambda}[d_\lambda]$. Note that $R_{y_\lambda}$ is a shift of a direct factor of $\mathcal{L}_i$ for some $i \in I^\circ$, because if $i = (i_{(a_1)} \cdots i_{(a_k)})$ is the expansion of $y = (i_{(a_1)} \cdots i_{(a_k)})$ then we have $\delta L_{i_1} = [a]^d \mathcal{L}_y$, where $a = (a_1, \ldots, a_k)$, see Section 2.20. Thus, $R_{y_\lambda}$ belongs to $\mathcal{Q}_V$. For $\lambda \in \Lambda_V$ we denote by $IC(\mathcal{O}_\lambda)$ the $G_V$-equivariant intersection cohomology complex on $E_V$ associated with $\mathcal{O}_\lambda$ and the $G_V$-equivariant local system $k_{\lambda}$, see [3] Sec. 8.4].
Lemma 2.23. The complex $R_{y_\lambda}$ does not depend on the choice of the element $y_\lambda \in Y_\nu$ such that $\pi_{y_\lambda}$ is a resolution of the orbit closure $\overline{O_\lambda}$.

Proof. Fix an element $y_\lambda$ as above for each $\lambda \in \Lambda_V$. Suppose that $k$ is a field of characteristic zero. Then by the decomposition theorem [1, Thm. 6.2.5] we have $R_{y_\lambda} = IC(O_\lambda)$. Thus, the complex $IC(O_\lambda)$ belongs to $Q^V$. Hence, in view of Corollary 2.11 the complexes $R_{y_\lambda} = IC(O_\lambda)$ with $\lambda \in \Lambda_V$, are the representatives of the isomorphism classes of indecomposable objects in $Q^V$ modulo shifts, and they form an $A$-basis in $K(Q^V)$. This yields the equality

$$\dim_A(\Lambda^\nu_\lambda) = \#(\Lambda_V) \quad (2.2)$$

Now let $k$ be an arbitrary field. The number of indecomposable objects modulo shifts in $Q^V$ is equal to $\dim_A K(Q^V) = \dim_A(\Lambda^\nu_\lambda)$, see Theorem 2.22 and the complexes $R_{y_\lambda}$, $\lambda \in \Lambda_V$ are among them. Thus, (2.2) shows that $R_{y_\lambda}$, $\lambda \in \Lambda_V$ are the representatives of isomorphism classes of indecomposable objects in $Q^V$ modulo shifts. Hence, each indecomposable object in $Q^V$ with support equal to $\overline{O_\lambda}$ is equal to a shift of $R_{y_\lambda}$. From this we see that $R_{y_\lambda}$ does not depend on $y_\lambda$. □

Definition 2.24. Let us write $R_\lambda$ instead of $R_{y_\lambda}$.

The following lemma follows directly from the proof of Lemma 2.23

Lemma 2.25. The indecomposable objects in $Q^V$ are exactly the complexes $R_\lambda$, $\lambda \in \Lambda_V$ modulo shifts.

3. Parity sheaves

Let $A$ be a complete discrete valuation ring or a field.

3.1. Parity sheaves. First, we recall some basic facts about parity sheaves. See [2] for more details. Let $G$, $X$ be as in Section 2.12. Here we will use the notation introduced in Section 2.12. We make the following assumption

$$H^0_G(X_\lambda, \mathcal{L}) = 0, \ H^i_G(X_\lambda, \mathcal{L}) \text{ is a free } A\text{-module} \quad \forall \mathcal{L} \in Loc_G(X_\lambda), \forall \lambda \in \Lambda. \quad (3.1)$$

Definition 3.1. A complex $\mathcal{F} \in D_G(X, A)$ is even (resp. odd) if $H^r(\mathcal{F}) = H^r(\mathcal{D}_r\mathcal{F}) = 0$ for each odd (resp. even) positive integer $r$. A complex $\mathcal{F}$ is parity if it is a sum of an even complex and an odd complex.

Lemma 3.2. Suppose $\mathcal{F} \in D_G(X, A)$. The following conditions are equivalent.

(a) $\mathcal{F} \in D_G(X, A)$ is even,

(b) $H^k(i^*_x\mathcal{F}) = H^k(i^*_x\mathcal{D}_r\mathcal{F}) = 0$ for each odd integer $k$ and each $\lambda \in \Lambda$,

(c) $H^k(i^*_x\mathcal{F}) = H^k(i^*_x\mathcal{D}_r\mathcal{F}) = 0$ for each odd integer $k$ and each $x \in X$.

Proof. First we show that (a)$\iff$(c) By [10] Rem. 2.6.9 we have

$$i^*_x H^k(\mathcal{F}) = H^k(i^*_x\mathcal{F}).$$

Now $H^k(\mathcal{F})$ is zero iff $i^*_x H^k(\mathcal{F})$ is zero for each $x \in X$. Thus, $H^k(\mathcal{F}) = 0$ iff $H^k(i^*_x\mathcal{F}) = 0$ for all $x \in X$. In the same way we prove that $H^k(\mathcal{D}_r\mathcal{F}) = 0$ iff $H^k(i^*_x\mathcal{D}_r\mathcal{F}) = 0$ for all $x \in X$. Thus, (a)$\iff$(c). To see that (b)$\iff$(c) we prove in the same way that $H^k(i^*_x\mathcal{F}) = 0$ iff $H^k(i^*_x\mathcal{D}_r\mathcal{F}) = 0$ for all $x \in X_\lambda$ and $H^k(i^*_x\mathcal{F}) = 0$ iff $H^k(i^*_x\mathcal{D}_r\mathcal{F}) = 0$ for all $x \in X_\lambda$. □

Lemma 3.3. Assume that (3.1) holds. Let $\mathcal{F}$ be an indecomposable parity complex. Then

1. the support of $\mathcal{F}$ is of the form $X_\lambda$ for some $\lambda \in \Lambda$,
2. the restriction $i^*_x\mathcal{F}$ is isomorphic to $\mathcal{L}[m]$ for some indecomposable object $\mathcal{L}$ in $Loc_G(X_\lambda)$ and some integer $m$,
(3) any indecomposable parity complex supported on $X_\lambda$ which extends $\mathcal{L}[m]$ is isomorphic to $\mathcal{F}$, where $\mathcal{L}$ is as in (2).

Idea of proof. The lemma is proved in [9, Thm. 2.12]. This proof uses the following.

Lemma 3.4. Assume that (3.1) holds. Let $\mathcal{F}, \mathcal{G} \in D_G(X, A)$. Suppose that $H^\text{odd}(\mathcal{F}) = 0$ and $H^\text{odd}(\mathcal{F} \mathcal{G}) = 0$. Then we have

$$\text{Hom}(\mathcal{F}, \mathcal{G}) \simeq \bigoplus_{\lambda \in \Lambda} \text{Hom}(i^*_\lambda \mathcal{F}, i^*_\lambda \mathcal{G}).$$

Definition 3.5. A parity sheaf is an indecomposable parity complex supported on $X_\lambda$ extending $\mathcal{L}[d_\lambda]$ for some indecomposable $\mathcal{L} \in \text{Loc}_G(X_\lambda, A)$ and some $\lambda \in \Lambda$. If such a complex exists, we will denote it by $\mathcal{E}(\lambda, \mathcal{L})$. If $\mathcal{L}$ is the constant sheaf $\Delta$, we will write $\mathcal{E} = \mathcal{E}(\lambda, \mathcal{L})$.

3.2. Parity sheaves on quiver varieties. We still suppose that $\Gamma$ is a Dynkin quiver. We want to study $G_V$-equivariant parity sheaves on $E_V$ with respect to the stratification $\coprod_{\lambda \in \Lambda_v} O_\lambda$, see Section 2.13. We must check that the $G_V$-variety $X = E_V$ satisfy the condition (3.1). This is true by the following lemma, see Corollary 2.11.

Lemma 3.6. For every $G_V$-orbit $\mathcal{O}$ in $E_V$ we have $H^\text{odd}_{G_V}(\mathcal{O}, \Delta_{\mathcal{O}}) = 0$ and $H^*_G(\mathcal{O}, \Delta_{\mathcal{O}})$ is a free $A$-module.

Proof. Let $x \in \mathcal{O}$. Denote by $H$ the stabilizer of $x$ in $G_V$. Let $H_\text{red}$ be the reductive part of $H$. We have

$$H^*_G(\mathcal{O}, \Delta_{\mathcal{O}}) = H^*_G(\mathcal{O}, \Delta_{\mathcal{O}})|_{G_V/H} = H^*_H(\bullet, A) = H^*_H(\bullet, A) = H^*(BH_{\text{red}}, A).$$

Note that for each positive integer $n$ the fundamental group of $GL_n(\mathbb{C})$ is equal to $\mathbb{Z}$. Then the fundamental group of $H_{\text{red}}$ does not contain elements of finite order, see Lemma 2.10. Moreover, the reductive group $H_{\text{red}}$ is of type $A$. Then by [24, Lem. 1] the torsion index of $H_{\text{red}}$ is equal to 1, see Section 2.6. Thus, the classifying space $BH_{\text{red}}$ has no odd cohomology and its cohomology is free over $A$, see [8, Cor. 2.3].

The following lemma is helpful to prove the parity of some complexes on $E_V$.

Lemma 3.7. Suppose that there exists a smooth $G_V$-variety $Y$ and a $G_V$-equivariant proper morphism $\pi : Y \to E_V$. Then the complex $\pi_* \Delta_Y$ is even if and only if $H^\text{odd}(\pi^{-1}(x), A) = 0$ for each $x \in E_V$.

Proof. Note that the complex $\pi_* \Delta_Y$ is constructible with respect to the stratification of $E_V$ because it is $G_V$-equivariant and the strata are $G_V$-orbits. By Lemma 3.2 the complex $\pi_* \Delta_Y$ is even if and only if for every inclusion $j : \{x\} \to E_V$ we have $H^\text{odd}(j^* \pi_* \Delta_Y) = 0$ and $H^\text{odd}(j^* \mathcal{D} \pi_* \Delta_Y) = 0$. Let $d$ be the complex dimension of $Y$. Note that

$$\mathcal{D} \pi_* \Delta_Y = \pi_* \mathcal{D} \Delta_Y,$$

$$\mathcal{D} \Delta_Y = \Delta_Y[2d],$$

(3.2) (3.3)
because the morphism \( \pi \) is proper. We set \( F = \pi^{-1}(x) \). We have the following commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{\pi} & E_V \\
\uparrow i & & \uparrow j \\
F & \xrightarrow{\pi} & \{x\},
\end{array}
\]

where \( i \) is the inclusion of \( F \) to \( Y \). We denote the restriction of \( \pi \) to \( F \) again by \( \pi \).

By the base change theorem we have

\[
j^* \pi_* \mathcal{A}_V = \pi_* i^* \mathcal{A}_V, \tag{3.4}
\]

\[
j^* \pi_* \mathcal{D}_V = \pi_* i^* \mathcal{D}_V. \tag{3.5}
\]

Now, from (3.2), (3.3), (3.4) and (3.5) we get

\[
H^*(\{x\}, j^* \pi_* \mathcal{A}_V) = H^*(F, A),
\]

\[
H^*(\{x\}, j^* \pi_* \mathcal{D}_V) = H^{*+2d}(F, A).
\]

This completes the proof. \( \square \)

3.3. Extensions of parity complexes. Let \( \mathcal{F} \) be a parity complex in \( DG_r(E_V, A) \). For each \( \lambda \in \Lambda_V \) and \( n \in \mathbb{N} \) the sheaf \( H^n(i^* \pi_* \mathcal{A}_V) \) is a direct sum of copies of \( \mathcal{A}_\lambda \), see Corollary 2.11. Let us denote the rank of the sheaf \( H^n(i^* \pi_* \mathcal{A}_V) \) by \( d_{\lambda, n}(\mathcal{F}) \).

Lemma 3.8. Let \( \mathcal{F}, \mathcal{G} \) be parity complexes in \( DG_r(E_V, A) \). Suppose that we have

\[
d_{\lambda, n}(\mathcal{F}) = d_{\lambda, n}(\mathcal{G}) \quad \text{for all } \lambda \in \Lambda_V, n \in \mathbb{N}. \]

Then \( \mathcal{F} \simeq \mathcal{G} \).

Proof. Let us decompose \( \mathcal{F} \) and \( \mathcal{G} \) into direct sums of shifts of parity sheaves

\[
\mathcal{F} = \bigoplus_{\lambda \in \Lambda_V, k \in \mathbb{Z}} \mathcal{E}(\lambda)[k]^{\oplus f_{\lambda, k}}, \quad \mathcal{G} = \bigoplus_{\lambda \in \Lambda_V, k \in \mathbb{Z}} \mathcal{E}(\lambda)[k]^{\oplus g_{\lambda, k}}.
\]

We will prove the statement by induction on the number of indecomposable factors of \( \mathcal{F} \). If \( \mathcal{F} = 0 \) then for each \( \lambda \in \Lambda_V, n \in \mathbb{Z} \) we have \( d_{\lambda, n}(\mathcal{F}) = d_{\lambda, n}(\mathcal{G}) = 0 \). Thus, \( \mathcal{F} = \mathcal{G} = 0 \). Now suppose that \( \mathcal{F} \neq 0 \). Define a partial order \( \preceq \) on \( \Lambda_V \) by setting \( \mu \preceq \lambda \) if and only if \( O_\mu \subset O_\lambda \). Fix a total order \( \preceq \) on \( \Lambda_V \) that refines \( \preceq \). Let \( \lambda_0 \) be the largest element of \( \Lambda_V \) with respect to \( \preceq \) such that \( f_{\lambda_0, k} \neq 0 \) for some integer \( k \). Then the decomposition of \( \mathcal{F} \) does not contain any shift of the parity sheaves \( \mathcal{E}(\mu) \) with \( \lambda_0 < \mu \). If for some \( \mu, \lambda \in \Lambda_V \) and \( n \in \mathbb{N} \) we have \( H^n(i^* \pi_* \mathcal{E}(\lambda)) \neq 0 \) then \( \mu \preceq \lambda \). Thus, we have \( d_{\mu, n}(\mathcal{F}) = 0 \) for each \( \mu > \lambda_0, n \in \mathbb{Z} \). Now, the equality \( d_{\mu, n}(\mathcal{G}) = 0 \) for each \( \mu > \lambda_0, n \in \mathbb{Z} \) assures that the decomposition of \( \mathcal{G} \) does not contain any shift of the parity sheaves \( \mathcal{E}(\mu) \) with \( \lambda_0 < \mu \). Choose \( n_0 \) such that \( d_{\lambda_0, n_0}(\mathcal{F}) = d_{\lambda_0, n_0}(\mathcal{G}) \neq 0 \). We have

\[
H^{n_0}(i^* \pi_* \mathcal{F}) \simeq \mathcal{A}_{\lambda_0}^{\oplus f_{\lambda_0, k_0}}, \quad H^{n_0}(i^* \pi_* \mathcal{G}) \simeq \mathcal{A}_{\lambda_0}^{\oplus g_{\lambda_0, k_0}}
\]

for \( k_0 = -d_{\lambda_0} - n_0 \). This shows that

\[
f_{\lambda_0, k_0} = d_{\lambda_0, n_0}(\mathcal{F}) = d_{\lambda_0, n_0}(\mathcal{G}) = g_{\lambda_0, k_0}.
\]

Thus, we have

\[
\mathcal{F} = \mathcal{F}' \oplus \mathcal{E}(\lambda_0)[k_0]^{\oplus f_{\lambda_0, k_0}}, \quad \mathcal{G} = \mathcal{G}' \oplus \mathcal{E}(\lambda_0)[k_0]^{\oplus f_{\lambda_0, k_0}}.
\]

Thus, we have \( d_{\lambda, n}(\mathcal{F}') = d_{\lambda, n}(\mathcal{G}') \) for each \( \lambda \in \Lambda_V, n \in \mathbb{Z} \). Thus, by induction hypothesis we have \( \mathcal{F}' \simeq \mathcal{G}' \). Hence, \( \mathcal{F} \simeq \mathcal{G} \). \( \square \)

Lemma 3.9. Let \( A \to B \to C \to A \) be a distinguished triangle in \( DG_r(E_V, A) \). Suppose that the complexes \( A \) and \( C \) are even. Then we have \( B \simeq A \oplus C \). In particular the complex \( B \) is also even.
Hence, we have
for each odd integer
for each
\(\lambda\)
Nakajima quiver varieties.

\[\mathbb{D}C \to \mathbb{D}B \to \mathbb{D}A \xrightarrow{\pm 1}.\]

The long exact sequences in cohomology are
\[\to \mathcal{H}^{n-1}(C) \to \mathcal{H}^n(A) \to \mathcal{H}^n(B) \to \mathcal{H}^n(C) \to \mathcal{H}^{n+1}(A) \to,\]
\[\to \mathcal{H}^{n-1}(\mathbb{D}A) \to \mathcal{H}^n(\mathbb{D}B) \to \mathcal{H}^n(\mathbb{D}C) \to \mathcal{H}^{n+1}(\mathbb{D}C) \to .\]

For each odd integer \(n\) we have
\[\mathcal{H}^n(A) = \mathcal{H}^n(\mathbb{D}A) = \mathcal{H}^n(C) = \mathcal{H}^n(\mathbb{D}C) = 0\]
and thus \(\mathcal{H}^n(B) = \mathcal{H}^n(\mathbb{D}B) = 0\). So the complex \(B\) is also even. For each \(\lambda \in \Lambda_V\) the distinguished triangle
\[i_{\lambda}^*A \to i_{\lambda}^*B \to i_{\lambda}^*C \xrightarrow{\pm 1}\]
yields a long exact sequence in cohomology
\[\to \mathcal{H}^{n-1}(i_{\lambda}^*C) \to \mathcal{H}^n(i_{\lambda}^*A) \to \mathcal{H}^n(i_{\lambda}^*B) \to \mathcal{H}^n(i_{\lambda}^*C) \to \mathcal{H}^{n+1}(i_{\lambda}^*A) \to .\]

If \(n\) is an even integer then \(\mathcal{H}^{n-1}(i_{\lambda}^*C) = \mathcal{H}^{n+1}(i_{\lambda}^*A) = 0\), see Lemma 3.8. Thus, we have a short exact sequence
\[0 \to \mathcal{H}^n(i_{\lambda}^*A) \to \mathcal{H}^n(i_{\lambda}^*B) \to \mathcal{H}^n(i_{\lambda}^*C) \to 0.\]

Hence, we have
\[d_{\lambda,n}(B) = d_{\lambda,n}(A) + d_{\lambda,n}(C) = d_{\lambda,n}(A \oplus C)\]
for each \(\lambda \in \Lambda_V\) and each even integer \(n\) and
\[d_{\lambda,n}(A) = d_{\lambda,n}(B) = d_{\lambda,n}(C) = 0\]
for each odd integer \(n\). Thus, by Lemma 3.2 we have \(B \simeq A \oplus C\).

3.4. Nakajima quiver varieties. Let \(k\) be a field. From now all sheaves and cohomology groups are to be understood with coefficients in \(k\)-vector spaces.

We recall the notion of a (graded) Nakajima quiver variety. We keep the notation of Section 3.1 and we still suppose that the quiver \(\Gamma\) is a Dynkin quiver. The notation \(i \sim j\) means that there is an arrow \(i \to j\) or \(j \to i\) in \(\Gamma\). Fix a function \(\xi: I \to \mathbb{Z}, i \mapsto \xi_i\) such that \(\xi_i = \xi_j + 1\) if there is an arrow from \(i\) to \(j\). It is unique modulo a shift by an integer constant. Consider the sets
\[\tilde{I} = \{(i, n) \in I \times \mathbb{Z}; \xi_i - n \in 2\mathbb{Z}\},\]
\[\tilde{J} = \{(i, n) \in I \times \mathbb{Z}; (i, n - 1) \in \tilde{I}\}.

To the quiver \(\Gamma\) we will associate the quiver \(\hat{\Gamma}\) as follows:

- the quiver \(\hat{\Gamma}\) contains two types of vertices:
  - a vertex \(w_i(n)\) for each \((i, n) \in \tilde{I}\), we will call them \(w\)-vertices,
  - a vertex \(t_i(n)\) for each \((i, n) \in \tilde{J}\), we will call them \(t\)-vertices,
- the quiver \(\hat{\Gamma}\) contains three types of edges:
  - an edge \(w_i(n) \to t_i(n - 1)\) for each \((i, n) \in \tilde{I}\),
  - an edge \(t_i(n) \to w_i(n - 1)\) for each \((i, n) \in \tilde{J}\),
  - an edge \(t_i(n) \to t_j(n - 1)\) for each \((i, n), (j, n - 1) \in \tilde{J}\) such that \(i \sim j\).
Consider an \( \hat{J} \)-graded finite dimensional \( \mathbb{C} \)-vector space \( W = \bigoplus_{(i,n) \in \hat{J}} W_i(n) \) and a \( \hat{J} \)-graded finite dimensional \( \mathbb{C} \)-vector space \( T = \bigoplus_{(i,n) \in \hat{J}} T_i(n) \). Set
\[
L^\bullet(T, W) = \bigoplus_{(i,n) \in \hat{J}} \text{Hom}(T_i(n), W_i(n-1)),
\]
\[
L^\bullet(W, T) = \bigoplus_{(i,n) \in \hat{J}} \text{Hom}(W_i(n), T_i(n-1)),
\]
\[
E^\bullet(T) = \bigoplus_{(i,n) \in \hat{J}, i \sim j} \text{Hom}(T_i(n), T_j(n-1)),
\]
and put
\[
M^\bullet(T, W) = E^\bullet(T) \oplus L^\bullet(W, T) \oplus L^\bullet(T, W).
\]
Note that \( M^\bullet(T, W) \) is just the variety of representations of \( \hat{J} \) in the \( \hat{J} \)-graded vector space \( T \oplus W \). We will write \( (B, \alpha, \beta) \) for an element of \( M^\bullet(T, W) \). The components of \( B, \alpha, \beta \) are
\[
B_{ij}(n) \in \text{Hom}(T_i(n), T_j(n-1)),
\]
\[
\alpha_i(n) \in \text{Hom}(W_i(n), T_i(n-1)),
\]
\[
\beta_i(n) \in \text{Hom}(T_i(n), W_i(n-1)).
\]
Let \( \Lambda^\bullet(T, W) \) be the subvariety of the affine space \( M^\bullet(T, W) \) defined by the equations
\[
\alpha_i(n-1)\beta_i(n) + \sum_{i \sim j} \varepsilon(i, j)B_{ji}(n-1)B_{ij}(n) = 0, \quad \forall (i, n) \in \hat{J},
\]
where \( \varepsilon(i, j) = 1 \) (resp. \( \varepsilon(i, j) = -1 \)) if \( i \to j \) is an arrow of \( \Gamma \) (resp. \( i \to j \) is not an arrow of \( \Gamma \)).

The algebraic group \( G_T = \prod_{(i,n) \in \hat{J}} GL_{T_i(n)} \) acts on \( M^\bullet(T, W) \) by \( g \cdot (B, \alpha, \beta) = (B', \alpha', \beta') \), where
\[
B'_{ij}(n) = g_j(n-1)B_{ij}(n)g_i(n)^{-1}, \quad \alpha'_i(n) = g_i(n-1)\alpha_i(n), \quad \beta'_i(n) = \beta_i(n)g_i(n)^{-1}.
\]
Consider the categorical quotients
\[
\mathfrak{M}^\bullet_\ast(T, W) = \Lambda^\bullet(T, W)/G_T,
\]
i.e., the coordinate ring of \( \mathfrak{M}^\bullet_\ast(T, W) \) is the ring of \( G_T \)-invariant functions on \( \Lambda^\bullet(T, W) \). The closed points of \( \mathfrak{M}^\bullet_\ast(T, W) \) parametrize the closed \( G_T \)-orbits in \( \Lambda^\bullet(T, W) \). We will denote by \( [B, \alpha, \beta] \) the element of \( \mathfrak{M}^\bullet_\ast(T, W) \) represented by \( (B, \alpha, \beta) \in \Lambda^\bullet(T, W) \). We say that a point \( (B, \alpha, \beta) \in \Lambda^\bullet(T, W) \) is stable if the following condition holds: if a \( \hat{J} \)-graded subspace \( T' \) of \( T \) is \( B \)-invariant and contained in \( \text{Ker} \beta \), then \( T' = 0 \). Let \( \Lambda^\bullet_\ast(T, W) \) be the set of stable points in \( \Lambda^\bullet(T, W) \). Consider the set theoretical quotient
\[
\mathfrak{M}^\bullet_\ast(T, W) = \Lambda^\bullet_\ast(T, W)/G_T.
\]
This quotient coincides with a quotient in the geometric invariant theory (see [15 Sec. 3]). There exists a projective morphism
\[
\pi_{T, W} : \mathfrak{M}^\bullet_\ast(T, W) \to \mathfrak{M}^\bullet_\ast(T, W),
\]
see [15 Sec. 3.18]. For each \( \hat{J} \)-graded subspace \( T' \) of \( T \) we have a closed embedding
\[
\mathfrak{M}^\bullet_\ast(T', W) \subset \mathfrak{M}^\bullet_\ast(T, W).
\]
This allows to define the variety
\[
\mathfrak{M}^\bullet_\ast(\infty, W) = \bigcup_T \mathfrak{M}^\bullet_\ast(T, W).
\]
Let $\mathfrak{M}_0^\ast(T,W)$ be the open subset of $\mathfrak{M}_0^\ast(T,W)$ parametrizing the closed free $G\tau$-orbits in $\Lambda^\ast(T,W)$. Denote by $S(W)$ the set of isomorphism classes of finite dimensional $J$-graded vector spaces $T$ such that $\mathfrak{M}_0^\ast(T,W) \neq \emptyset$. For each $W$ as above we have a decomposition
\[
\mathfrak{M}_0^\ast(\infty,W) = \bigsqcup_{T \in S(W)} \mathfrak{M}_0^\ast(T,W),
\]
see [17] (4.5).

3.5. Resolutions via Nakajima quiver varieties.

Definition 3.10. Suppose $(i,n),(j,k)$ are in $\hat{I}$. A path from $(i,n)$ to $(j,k)$ is a path in the quiver $\hat{\Gamma}$ from $w_i(n)$ to $w_j(k)$ that does not contain another $w$-vertex. To a path $p$ from $(i,n)$ to $(j,k)$ and an element $(B,\alpha,\beta) \in \Lambda^\ast(T,W)$ we can associate an element $A_p(B,\alpha,\beta) \in \text{Hom}(W_i(n),W_j(k))$ as follows. If
\[
p = (w_i(n) \rightarrow t_i(n-1) \rightarrow t_i(n-2) \rightarrow \cdots \rightarrow t_i(n-r) \rightarrow w_j(k)),
\]
where $r = n - k - 1$, then we set
\[
A_p(B,\alpha,\beta) = \beta_j(n-r)B_{i,n-i}(n-(r-1)) \cdots B_{i,1}(n-1)\alpha_i(n): W_i(n) \rightarrow W_j(k).
\]

The following theorem is proved in [2] Prop. 9.4, Thm. 9.11.

Theorem 3.11. There exists an injective map $\tau: I \rightarrow \hat{I}$ (depending on the orientation of $\Gamma$) such that for each arrow $i \rightarrow j$ in $\Gamma$ there exists a path $i \rightsquigarrow j$ from $\tau(i)$ to $\tau(j)$ such that the following property holds: to each finite dimensional $\check{I}$-graded $C$-vector space $V$ we assign the $\check{I}$-graded $C$-vector space $W$ by setting
\[
W_{\tau(i)} = V_i, \quad \forall i \in I, \quad W_i = 0 \quad \forall i \in \hat{I} \backslash \tau(I).
\]

We have an isomorphism
\[
\mathfrak{M}_0^\ast(\infty,W) \rightarrow E_V, \quad [B,\alpha,\beta] \mapsto \bigoplus_{i \rightarrow j} A_{i \rightarrow j}(B,\alpha,\beta).
\]

From now for each arrow $i \rightarrow j$ in $\Gamma$ we fix a path $i \rightsquigarrow j$ from $\tau(i)$ to $\tau(j)$ as in Theorem 3.11.

Remark 3.12. The isomorphism in Theorem 3.11 depends on the choice of the path $i \rightsquigarrow j$ from $\tau(i)$ to $\tau(j)$ for each arrow $i \rightarrow j$ in $\Gamma$. Given another path $p$ from $\tau(i)$ to $\tau(j)$, the morphisms
\[
A_{i \rightarrow j}: \mathfrak{M}_0^\ast(\infty,W) \rightarrow \text{Hom}(V_i,V_j), \quad A_p: \mathfrak{M}_0^\ast(\infty,W) \rightarrow \text{Hom}(V_i,V_j)
\]
are such that $A_p = \lambda A_{i \rightarrow j}$ for some $\lambda \in \mathbb{C}$. This follows from two following lemmas.

For each arrow $i \rightarrow j$ in $\Gamma$ we equip the variety $\text{Hom}(V_i,V_j)$ with the $G_V$-action by $g \cdot \phi = g_j \phi g_i^{-1}$ for $g \in G_V$, $\phi \in \text{Hom}(V_i,V_j)$.

Lemma 3.13. The map
\[
A_p: E_V = \mathfrak{M}_0^\ast(\infty,W) \rightarrow \text{Hom}(V_i,V_j)
\]
is linear and $G_V$-equivariant.

Proof. The statement is obvious. \hfill \Box

Lemma 3.14. Let $h_0 = (i \rightarrow j)$ be an arrow of $\Gamma$. Then each linear $G_V$-equivariant morphism $f: E_V \rightarrow \text{Hom}(V_i,V_j)$ is of the form $x \mapsto \lambda x_{h_0}$ for some $\lambda \in \mathbb{C}$.
Proof. We view $E_V$ and $\text{Hom}(V_i,V_j)$ as linear representations of $G_V$. The representation $E_V$ is a direct sum of irreducible representations $E_V = \bigoplus_{h \in H} E_h$, where $E_h = \text{Hom}(V_h,V_{h'})$ and $h',h''$ are as in Section 3.4. In particular, we have $E_{h_0} = \text{Hom}(V_0,V_j)$. Now, by Schur’s lemma we have

$$\text{Hom}_{G_V}(E_h,E_{h_0}) = \begin{cases} 0 & \text{if } h \neq h_0, \\ C \cdot \text{Id}_{E_{h_0}} & \text{if } h = h_0. \end{cases}$$

\qed

The following lemma is proved in [16] Thm. 7.4.1.

**Lemma 3.15.** Let $T$ and $W$ be as in Section 3.4. Then the fibers of $\pi_{T,W}$ have no odd cohomology groups over $\mathbb{Z}$ and their cohomology groups over $\mathbb{Z}$ are torsion free.

Let us associate $W$ to $V$ as in Theorem 3.11. We will identify $E_V$ with $\mathfrak{M}^0\Lambda(V,W)$ as in Theorem 3.11. The following lemma is proved in [7, Prop. 9.8]. It interprets the stratification on $E_V$ in terms of Nakajima quiver varieties.

**Lemma 3.16.** The decomposition $\mathfrak{M}^0\Lambda(V,W) = \bigsqcup_{T \in \mathcal{S}(W)} \mathfrak{M}^0_{\text{reg}}(T,W)$, coincides with the decomposition of $E_V$ into $G_V$-orbits.

Let $T$ be a finite dimensional $\mathfrak{g}$-graded $\mathbb{C}$-vector space. The morphism $\pi_{T,W}$ defined in Section 3.4 together with the inclusion $\mathfrak{M}^*_0(T,W) \subset \mathfrak{M}^*_0(\infty,W)$ yields a morphism

$$\pi_{T,W} : \mathfrak{M}^*_0(T,W) \to \mathfrak{M}^*_0(\infty,W) \cong E_V.$$ 

We can consider the following complex in $D_{G_V}(E_V,k)$.

$$\mathcal{L}_{T,W} = \pi_{T,W} \ast k_{\text{sr}}(T,W)$$

Lemmas 3.7, 3.15 and 3.16 yield the following result.

**Corollary 3.17.** The complex $\mathcal{L}_{T,W}$ is an even complex in $D_{G_V}(E_V,k)$.

The following lemma assures the existence of the parity sheaves on $E_V$, see Definition 3.5.

**Lemma 3.18.** For each $\lambda \in \Lambda_V$ there exists a parity sheaf $\mathcal{E}(\lambda)$ on $E_V = \bigsqcup_{\lambda \in \Lambda_V} \mathcal{O}_\lambda$.

**Proof.** Let $T$ be in $\mathcal{S}(W)$, see Section 3.4. By [15] Prop. 3.24] the morphism

$$\pi_{T,W} : \mathfrak{M}^*_0(T,W) \to \mathfrak{M}^*_0(\infty,W)$$

is an isomorphism over $\mathfrak{M}^*_0(\text{reg})(T,W)$ because $\mathfrak{M}^*_0(T,W)$ is smooth and connected by [16] Thm. 5.5.6] and $\mathfrak{M}^*_0(\text{reg})(T,W)$ is a dense open subset of $\mathfrak{M}^*(T,W)$. Moreover, the complex $\mathcal{L}_{T,W}$ is even by Corollary 3.17. Thus, the base change argument in Section 2.21 shows each parity sheaf $\mathcal{E}(\lambda)$, $\lambda \in \Lambda_V$ is well-defined and appears as a direct factor of some $\mathcal{L}_{T,W}$ for some $T$. \qed

**Remark 3.19.**

(a) Lemma 3.18 is enough to insure the existence of $G_V$-equivariant parity sheaves on $E_V$, see Corollary 2.11. Note that in the same way we can prove the existence of $G_V$-equivariant parity sheaves on $E_V$ over a complete discrete valuation ring.

(b) We have $i^*_x R_\lambda = i^*_y \mathcal{E}(\lambda) = k_{\lambda}(d\lambda)$, see Section 2.24. It is natural to ask whether the indecomposable complexes $\mathcal{E}(\lambda)$ and $R_\lambda$ are isomorphic. To get this, it is enough to prove that $R_\lambda$ is a parity complex, see Lemma 3.3.
(c) For $\lambda \in \Lambda_V$ set $\tilde{\mathcal{E}}(\lambda) = \mathcal{E}(\lambda)[-d_\lambda]$. Let $T \in \mathcal{S}(W)$ be the $\tilde{T}$-graded $\mathbb{C}$-vector space such that $\mathfrak{M}^{\text{res}}_0(T, W) \simeq \mathcal{O}_\lambda$. Suppose that the characteristic of $k$ is zero. Then by decomposition theorem [1, Thm. 6.2.5] the unique indecomposable factor of $\mathcal{E}_{T,W}$ supported on all $\mathcal{O}_\lambda$ is $\mathcal{O}(\mathcal{O}_\lambda)[-d_\lambda]$, because we have $i_T^* \mathcal{E}_{T,W} = i_T^* \tilde{\mathcal{E}}(\lambda) = \mathfrak{k}_W$. On the other hand $i_T^* \mathcal{E}_{T,W}$ is the sum of shifts of parity sheaves and thus the same argument shows that this factor is equal to $\tilde{\mathcal{E}}(\lambda)$. Thus, we have $\mathcal{O}(\mathcal{O}_\lambda) \simeq \mathcal{E}(\lambda)$. In particular, we have already seen in the proof of Lemma 2.23 that $R_\lambda \simeq \mathcal{O}(\mathcal{O}_\lambda)$. Thus, in zero characteristic we have $\mathcal{E}(\lambda) \simeq R_\lambda \simeq \mathcal{O}(\mathcal{O}_\lambda)$.

3.6. Restriction diagrams. In this section we recall the construction of the restriction diagrams for Nakajima quiver varieties and Lusztig quiver varieties and we compare them.

Let $W$ be a finite dimensional $\tilde{T}$-graded $\mathbb{C}$-vector space. Let $W_2$ be an $\tilde{T}$-graded subspace of $W$ and set $W_1 = W/W_2$. Consider the subvariety of $\mathfrak{M}^\bullet_0(\infty, W)$ given by

$$\mathfrak{M}^\bullet_0(W_1, W_2) = \{[B, \alpha, \beta] \in \mathfrak{M}^\bullet_0(\infty, W); \beta B^k \alpha(W_2) \subset W_2 \forall k \in \mathbb{N}\},$$

where $[B, \alpha, \beta]$ denotes the element of $\mathfrak{M}^\bullet_0(\infty, W)$ represented by $(B, \alpha, \beta) \in \Lambda^\bullet(V, W)$ for some $V$. Consider the following diagram, see [20, Sec. 3.5] and [13, Sec. 3.5].

$$\mathfrak{M}^\bullet_0(\infty, W_1) \times \mathfrak{M}^\bullet_0(\infty, W_2) \xleftarrow{\iota'} \mathfrak{M}^\bullet_0(W_1, W_2) \xrightarrow{\iota} \mathfrak{M}^\bullet_0(\infty, W),$$

where $\iota'$ is the inclusion and $\iota'$ is the induced map defined as follows. Fix the element $[B, \alpha, \beta] \in \mathfrak{M}^\bullet_0(W_1, W_2)$ represented by the tuple $(B, \alpha, \beta) \in \Lambda^\bullet(T, W)$ for some $T$. Let $T'$ be the maximal ($\tilde{T}$-graded) subspace of $T$ stable by $B$ such that $\beta(T') \subset W_2$. We have $\alpha(W_2) \subset T'$ by definition of $\mathfrak{M}^\bullet_0(W_1, W_2)$. Then $(B, \alpha, \beta)$ defines elements of $\Lambda^\bullet(T/T', W_1)$ and $\Lambda^\bullet(T', W_2)$. These elements yield elements of $\mathfrak{M}^\bullet(\infty, W_1)$ and $\mathfrak{M}^\bullet_0(\infty, W_2)$.

Now let $V$ be an $I$-graded finite dimensional $\mathbb{C}$-vector space. Let $V_2$ be an $I$-graded subspace of $V$ and set $V_1 = V/V_2$. Consider the subvariety $F$ of $E_V$ consisting of the representations preserving $V_2$. We consider the following diagram, see [14, Sec. 9.2].

$$E_{V_1} \times E_{V_2} \xleftarrow{\iota} F \xrightarrow{\iota} E_V,$$

where $\iota$ is the inclusion and $\iota$ assigns to an elements of $F$ its quotient and restriction. Now assign $W$ to $V$ as in Theorem 3.11. The subspace $V_2 \subset V$ yields a subspace $W_2 \subset W$ and the quotient $W_1 = W/W_2$ would correspond to $V_1 = V/V_2$.

**Theorem 3.20.** We have an isomorphism of diagrams

$$\mathfrak{M}^\bullet_0(\infty, W_1) \times \mathfrak{M}^\bullet_0(\infty, W_2) \xleftarrow{\iota'} \mathfrak{M}^\bullet_0(W_1, W_2) \xrightarrow{\iota} \mathfrak{M}^\bullet_0(\infty, W),$$

where the leftmost and the rightmost isomorphisms are as in Theorem 3.11.

**Proof.** Consider the element $[B, \alpha, \beta] \in \mathfrak{M}^\bullet_0(\infty, W)$ represented by the tuple $(B, \alpha, \beta) \in \Lambda^\bullet(T, W)$ for some $T$. Let $i \to j$ be an arrow in $\Gamma$. Recall that the map $V_i \to V_j$ associated with $[B, \alpha, \beta]$ is

$$\beta_{i_k}(n_k)B_{i_{k-1}, i_k}(n_{k-1}) \cdots B_{i_0, i_1}(n_0)\alpha_{i_0}(n_0 + 1).$$

So in view of Remark 3.12 the conditions that $[B, \alpha, \beta] \in \mathfrak{M}^\bullet_0(W_1, W_2)$ means that the corresponding element of $E_V$ preserves $V_2$. So we have an isomorphism
\( \mathfrak{M}(W_1, W_2) \simeq F \) making the right part of the statement to be commutative. The left part is also commutative because both \( \kappa' \) and \( \kappa \) were defined as the product of the quotient and the restriction.

The restriction diagram for Lusztig quiver varieties yields a functor
\[
\Res_{V_1,V_2}^\kappa = \kappa^* : D_{GV}(E_V,k) \to D_{G_{V_1} \times G_{V_2}}(E_{V_1} \times E_{V_2},k).
\]
Consider also its modification
\[
\Res_{V_1,V_2} = \Res_{V_1,V_2}[M_{V_1,V_2}],
\]
where
\[
M_{V_1,V_2} = \sum_{h \in H} \dim(V_1)_h \dim(V_2)_{h'} - \sum_{i \in I} \dim(V_1)_i \dim(V_2)_i.
\]

### 3.7. Restriction of Nakajima sheaves

Let \( T \) be a \( \tilde{J} \)-graded finite dimensional \( \mathbb{C} \)-vector space. Let \( \mathcal{L}_{T,W} \) be the shift \( \mathcal{L}_{T,W} = \mathcal{L}_{T,W}[\dim \mathcal{M}^*(T,W)] \), where \( \mathcal{L}_{T,W} \) is as in Section 3.5. We identify the restriction diagrams for Nakajima quiver varieties and Lusztig quiver varieties as in Theorem 3.20. We will write \( \nu = \nu' \), \( \kappa = \kappa' \) to simplify the notation. Set
\[
\tilde{\mathfrak{M}}^*(T,W_1,W_2) = (\pi_{T,W})^{-1}(\mathfrak{M}^*(W_1,W_2)).
\]
We have the commutative diagram
\[
\begin{array}{ccc}
\tilde{\mathfrak{M}}^*(T,W_1,W_2) & \xrightarrow{\tilde{i}} & \mathfrak{M}^*(T,W) \\
\downarrow p & & \downarrow \pi_{T,W} \\
\tilde{\mathfrak{M}}^*(W_1,W_2) & \xrightarrow{i} & \mathfrak{M}^*(\infty,W),
\end{array}
\]
where \( \tilde{i} \) is the inclusion, \( p \) is the restriction of \( \pi_{T,W} \).

Set
\[
U(T) = \{(\omega_1, \omega_2) \in \mathbb{N}\tilde{J} \times \mathbb{N}\tilde{J}; \omega_1 + \omega_2 = \text{grdim } T \}.
\]
For each pair \((\omega_1, \omega_2) \in U(T)\) set
\[
\tilde{F}(\omega_1, \omega_2) = \{[B, \alpha, \beta] \in \tilde{\mathfrak{M}}^*(T,W_1,W_2); \text{grdim } T/T' = \omega_1, \text{grdim } T' = \omega_2 \},
\]
where the subset \( T' \subset T \) is associated with \([B, \alpha, \beta] \) as in the definition of the restriction diagram for Nakajima quiver varieties (see the definition of the morphism \( \kappa' \)). For each \((\omega_1, \omega_2) \in U(T)\) we have a morphism
\[
\alpha_{\omega_1, \omega_2} : \tilde{F}(\omega_1, \omega_2) \to \mathfrak{M}^*(T_1,W_1) \times \mathfrak{M}^*(T_2,W_2),
\]
where \( T_1, T_2 \) are \( \tilde{J} \)-graded \( \mathbb{C} \)-vector spaces with dimension vectors \( \omega_1, \omega_2 \) respectively. The morphism \( \alpha_{\omega_1, \omega_2} \) is a vector bundle by [19, Prop. 3.8]. Let \( d_{T_1,T_2} \) be its rank. The following Lemma is a positive characteristic analogue of [20, Lem. 4.1].

#### Lemma 3.21
We have
\[
\Res_{V_1,V_2}(\mathcal{L}_{T,W}) = \bigoplus_{T_1 \oplus T_2 \simeq T} \mathcal{L}_{T_1,W_1} \otimes \mathcal{L}_{T_2,W_2}[-2d_{T_1,T_2}],
\]
where the sum is taken over the isomorphism classes of \( \tilde{J} \)-graded \( \mathbb{C} \)-vector spaces \( T_1, T_2 \) such that \( T_1 \oplus T_2 \simeq T \).

**Proof.** For each \((\omega_1, \omega_2) \in U(T)\) and each \( \tilde{J} \)-graded \( \mathbb{C} \)-vector spaces \( T_1, T_2 \) of graded dimensions \( \omega_1, \omega_2 \) we have a commutative diagram
\[
\begin{array}{ccc}
\tilde{F}(\omega_1, \omega_2) & \xrightarrow{} & \tilde{\mathfrak{M}}^*(T,W_1,W_2) \\
\downarrow \alpha_{\omega_1, \omega_2} & & \downarrow \kappa_p \\
\mathfrak{M}^*(T_1,W_1) \times \mathfrak{M}^*(T_2,W_2) & \xrightarrow{} & \mathfrak{M}^*(\infty,W_1) \times \mathfrak{M}^*(\infty,W_2).
\end{array}
\]
The upper horizontal arrow is the obvious inclusion and the lower one is \( \pi_{T_1, W_1} \times \pi_{T_2, W_2} \). The varieties \( \tilde{F}(\omega_1, \omega_2) \) with \( (\omega_1, \omega_2) \in U(T) \) form a locally closed partition of \( \tilde{M}_0(T, W_1, W_2) \). Let us enumerate them \( \tilde{F}_1, \cdots, \tilde{F}_r \) in such a way that for \( j \in [1, r] \) the subvariety \( \tilde{F}_{\leq j} = \bigcup_{j' \leq j} \tilde{F}_{j'} \) is closed in \( \tilde{M}_0(T, W_1, W_2) \). Let \( f_j \) and \( f_{\leq j} \) be the restrictions of \( \Lambda \)

\[
f_j : \tilde{F}_j \to M_0^*(\infty, W_1) \times M_0^*(\infty, W_2), \quad f_{\leq j} : \tilde{F}_{\leq j} \to M_0^*(\infty, W_1) \times M_0^*(\infty, W_2).
\]

We also set \( \tilde{F}_{< j} = \tilde{F}_{\leq j-1}, f_{< j} = f_{\leq j-1} \) for \( j \in [2, r] \). Let \( a_j \) and \( b_j \) be the inclusions

\[
a_j : \tilde{F}_j \to \tilde{F}_{\leq j}, \quad b_j : \tilde{F}_{\leq j} \to \tilde{F}_{< j}.
\]

By \([1]\) Sec. 1.4.3, we have a distinguished triangle

\[
a_j a_j^* k_{\tilde{F}_{\leq j}} \to k_{\tilde{F}_{\leq j}} \to b_j b_j^* k_{\tilde{F}_{\leq j}} \to 0. \tag{3.6}
\]

Applying the functor \((f_{\leq j})_!\) to this triangle we get the distinguished triangle

\[
(f_j)_! k_{\tilde{F}_j} \to (f_{\leq j})_! k_{\tilde{F}_{\leq j}} \to (f_{< j})_! k_{\tilde{F}_{< j}} \to 0.
\]

Now if \( \tilde{F}_j = \tilde{F}(\omega_1, \omega_2) \) and \( T_1, T_2 \) are \( \tilde{J} \)-graded \( \mathbb{C} \)-vector spaces with graded dimensions \( \omega_1, \omega_2 \) respectively, we have

\[
\alpha_{\omega_1, \omega_2} k_{\tilde{F}_j} = \chi_{\tilde{M}_0^*(T_1, W_1) \times \tilde{M}_0^*(T_2, W_2)}[-2d_{T_1, T_2}]
\]

because \( \alpha_{\omega_1, \omega_2} \) is a vector bundle of rank \( d_{T_1, T_2} \). Thus,

\[
f_j k_{\tilde{F}_j} = \tilde{D}_{T_1, W_1} \otimes \tilde{D}_{T_2, W_2}[-2d_{T_1, T_2}].
\]

In particular it is an even complex, see Corollary 3.17. Thus, from (3.6) and Lemma 3.9 we have by induction

\[
\kappa_{T} k_{\tilde{M}_0^*(T, W_1, W_2)} = \bigoplus_{T_1 \oplus T_2 = T} \tilde{D}_{T_1, W_1} \otimes \tilde{D}_{T_2, W_2}[-2d_{T_1, T_2}].
\]

This finishes the proof because by the base change theorem we have

\[
\kappa_{T}^* \tilde{D}_T^* = \bigoplus_V Par_G(E_V), \quad K(Par) = \bigoplus_V K(Par_G(E_V)).
\]

Let \( S(W) \) be as in Section 3.4

**Lemma 3.22.** The complexes \( \tilde{D}_{T, W}, T \in S(W) \) form an \( \mathcal{A} \)-basis in \( K(Par_G(E_V)) \).

**Proof.** Note that \( i_\lambda^* \tilde{E}(\lambda) = k_{\lambda} \), where \( \tilde{E}(\lambda) \) is as in Remark 3.19. Let \( \leq \) be the total order on \( A_V \) as in the proof of Lemma 3.8. Denote by \( T_\lambda \) the \( \tilde{J} \)-graded finite dimensional \( \mathbb{C} \)-vector space such that \( M_0^*(T_\lambda, W) \simeq O_\lambda \), see Lemma 3.16. We have \( \{ T_\lambda; \lambda \in A_V \} = S(W) \). The complex \( \tilde{D}_{T, W} \) is even, see Corollary 3.17. By the base change theorem we have \( i_\lambda^* \tilde{D}_{T, W} = k_{\lambda} \). Thus,

\[
\tilde{D}_{T, W} = \tilde{E}(\lambda) \oplus \bigoplus_{\mu < \lambda, d \in \mathbb{Z}} \tilde{E}(\mu)[2d]^\oplus_{a_{\mu, d}} \quad a_{\mu, d} \in \mathbb{N}.
\]
Hence, the classes $[\tilde{F}_{\lambda_1\lambda_2}]$, $\lambda \in \Lambda_V$ form an $A$-basis in $K(\text{Par}_{G_V}(E_V))$ because the classes $[\tilde{E}(\lambda)]$, $\lambda \in \Lambda_V$ form an $A$-basis in $K(\text{Par}_{G_V}(E_V))$. \hfill\qed

By Lemmas \ref{feynman} and \ref{feynman2}, the functor $\text{Res}_{V_1,V_2}$ yields an $A$-module homomorphism

$$K(\text{Par}_{G_V}(E_V)) \to K(\text{Par}_{G_{V_1} \times G_{V_2}}(E_{V_1} \times E_{V_2})).$$

where $\text{Par}_{G_{V_1} \times G_{V_2}}(E_{V_1} \times E_{V_2})$ is defined as $\text{Par}_{G_V}(E_V)$ using the quiver $\Gamma \coprod \Gamma$. A $G_{V_1} \times G_{V_2}$-orbit in $E_{V_1} \times E_{V_2}$ is of the form $\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2}$. So we identify $\Lambda_V$ with $\Lambda_{V_1} \times \Lambda_{V_2}$. The complex $\tilde{E}(\lambda_1) \times \tilde{E}(\lambda_2)$ is indecomposable because its endomorphism ring is a tensor product of endomorphism rings of complexes $\tilde{E}(\lambda_1)$ and $\tilde{E}(\lambda_2)$, thus it is local. It is even, supported on $\mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2}$, and we have

$$i_{\lambda_1 \times \lambda_2}^!(\tilde{E}(\lambda_1) \otimes \tilde{E}(\lambda_2)) = \mathbb{k}_{\lambda_1 \times \lambda_2}.
$$

Thus, by Lemma \ref{feynman3} we have

$$\tilde{E}(\lambda_1) \otimes \tilde{E}(\lambda_2) \simeq \tilde{E}(\lambda_1 \times \lambda_2).
$$

Thus, we have

$$K(\text{Par}_{G_{V_1} \times G_{V_2}}(E_{V_1} \times E_{V_2})) \to K(\text{Par}_{G_{V_1}}(E_{V_1})) \otimes_A K(\text{Par}_{G_{V_2}}(E_{V_2})).$$

Finally, the functors $\text{Res}_{V_1,V_2}$ yield a comultiplication on

$$K(\text{Par}) = \bigoplus_V K(\text{Par}_{G_V}(E_V)),$
$$

where the sum is taken by isomorphism classes of $I$-graded finite dimensional $\mathbb{C}$-vector spaces. Thus, the modified restriction functors $\text{Res}_{V_1,V_2}$ yield a coproduct $\text{Res}$ on $K(\text{Par})$. From now we will always consider the coalgebra structure on $K(\text{Par})$ given by $\text{Res}$.

3.8. Restriction of Lusztig sheaves. Now we study the restriction of Lusztig sheaves. It is similar to the constructions of Nakajima sheaves in Section \ref{feynman}.

Let $E_{V_1} \times E_{V_2} \xrightarrow{\tilde{F}} F \xrightarrow{\pi_y} E_V$

be the restriction diagram as in Section \ref{feynman6}. Fix a sequence $(a_1, \ldots, a_r)$. Set $y = (i_1^{(a_1)} \cdots i_r^{(a_r)})$, an element in $Y_v$. Set also

$$U(y) = \{(y_1, y_2) \in Y_{v_1} \times Y_{v_2}; \ y_1 = (i_1^{(a_1')} \cdots i_r^{(a_r')}), \ y_2 = (i_1^{(a_1'')} \cdots i_r^{(a_r''})},
$$

$$a_r', a_r'' \in \mathbb{N}, \ a_r' + a_r'' = a_r, \forall r \in [1, k].$$

We have a commutative diagram

$$\begin{array}{ccc}
\tilde{F} & \longrightarrow & \tilde{F}_y \\
\downarrow & & \downarrow \pi_y \\
F & \longrightarrow & E_V,
\end{array}$$

where $\tilde{F} = \pi_y^{-1}(F)$. The variety $\tilde{F}$ consists of the pairs $(x, \phi) \in E_V \times F_v$ such that $x$ preserves $V_2$ and $x$ preserves the flag $\phi = \{0\} = V^0 \subset V^1 \subset V^2 \subset \cdots \subset V^k = V$. For $(y_1, y_2)$, denote by $\tilde{F}(y_1, y_2)$ the subset of $\tilde{F}$ consisting of pairs $(x, \phi) \in \tilde{F}$ such that the flag $(V^0/(V^0 \cap V_2) \subset V^1/(V^1 \cap V_2) \subset \cdots \subset V^r/(V^r \cap V_2))$ has type $y_1$, and the flag $(V^0 \cap V_2 \subset V^1 \cap V_2 \subset \cdots \subset V^r \cap V_2)$ has type $y_2$. We have a vector bundle

$$\tilde{F}(y_1, y_2) \to \tilde{F}_{y_1} \times \tilde{F}_{y_2},$$

see \cite{feynman} Sec. 9.2.4. Denote its rank by $m_{y_1,y_2}$. 


Lemma 3.23. Suppose that for each \((y_1, y_2) \in U(y)\) the complexes \(L_{y_1}, L_{y_2}\) are even. Then we have
\[
\text{Res}_{V_1, V_2}(L_y) = \bigoplus_{(y_1, y_2) \in U(y)} L_{y_1} \otimes L_{y_2}[-2m_{y_1, y_2}].
\]

Proof. The proof is the same as the proof of Lemma 3.21. See also [13 Sec. 9.2]. \(\square\)

Lemma 3.24. For each \(\lambda \in \Lambda_V\) let \(y_\lambda \in Y_\nu\) be as in Section 2.27. The classes \([L_{y_\lambda}], \lambda \in \Lambda_V\) form an \(A\)-basis in \(K(Q_V)\). The first statement follows from Remark 3.19 (b). The second statement is necessary to apply Lemma 3.23.

Proof. The proof is the same as the proof of Lemma 3.22. \(\square\)

3.9. Coalgebra structure on \(K(\text{Par})\). The following Lemma follows directly from Lemma 3.8.

Lemma 3.25. Let \(A[\Lambda_V \times \mathbb{Z}]\) be a free \(A\)-module with basis \(1_{\lambda, n}, \lambda \in \Lambda_V, n \in \mathbb{Z}\). Then there is an \(A\)-module inclusion
\[
K(\text{Par}_{G_V}(E_V)) \hookrightarrow A[\Lambda_V \times \mathbb{Z}], \quad F \mapsto \sum_{\lambda, n} d_{\lambda, n}(F)1_{\lambda, n},
\]
where \(d_{\lambda, n}\) are as in Section 3.8.

Theorem 3.26. There exists an \(A\)-coalgebra isomorphism \(\beta_A: A\mathfrak{f} \rightarrow K(\text{Par})\).

Proof. At first suppose \(k = \mathbb{C}\). In this case the statement is already known, see [14 Thm. 13.2.11]. Now let \(k\) be an arbitrary field. To avoid confusion we will indicate the field \(k\) or \(\mathbb{C}\) in the rest of the proof. By Lemma 3.13 the numbers \(d_{\lambda, n}(\hat{T}_{T, W, k})\) do not depend on the field \(k\), where \(\lambda \in \Lambda_V, n \in \mathbb{N}, T\) is a finite dimensional \(\hat{J}\)-graded \(\mathbb{C}\)-vector space. Thus, the image of the inclusion in Lemma 3.25 does not depend on the field either. Hence, there exists a linear isomorphism
\[
K(\text{Par}_{G_V}(E_V), k) \rightarrow K(\text{Par}_{G_V}(E_V), \mathbb{C})
\]
sending \(\hat{T}_{T, W, k}\) to \(\hat{T}_{T, W, \mathbb{C}}\) for each \(T\). To conclude we need only to note that this isomorphism preserves the coproduct because the constants \(d_{T_1, T_2}\) from Lemma 3.21 are independent of the field. \(\square\)

Remark 3.27. There is an \(A\)-basis in \(K(\text{Par})\) given by parity sheaves. It corresponds to some \(A\)-basis of \(A\mathfrak{f}\) by \(\beta_A\). So Theorem 3.26 yields an \(A\)-basis in \(A\mathfrak{f}\) in terms of parity sheaves.

3.10. Even quivers.

Definition 3.28. The quiver \(\Gamma\) is \(k\)-even if the complex \(L_y\) over \(k\) is even for each \(y \in Y_\nu\) and each \(\nu \in N_I\). The quiver \(\Gamma\) is even if it is \(k\)-even for each field \(k\).

In this section we suppose that our Dynkin quiver \(\Gamma\) is even. This hypothesis is necessary to apply Lemma 3.23.

Theorem 3.29. Suppose that \(\Gamma\) is an even Dynkin quiver. Let \(V\) be as above. Then the full subcategories \(Q_V\) and \(\text{Par}_{G_V}(E_V)\) of \(D_{G_V}(E_V, k)\) coincide. Moreover, there exists an isomorphism of bialgebras \(K(Q) \cong A\mathfrak{f}\), where the algebra structure on \(K(Q)\) is as in Section 2.20 and the coproduct is given by \(\text{Res}\).

Proof. The first statement follows from Remark 3.19 (b). The second statement is already known if the characteristic of \(k\) is zero, see [14 Thm. 13.2.11]. In positive characteristic we already have the algebra isomorphism
\[
\lambda_A: A\mathfrak{f} \rightarrow K(Q)
\]
by Theorem 2.22. We need only to verify that \(\lambda_A\) preserves the coproduct. This follows from the zero characteristic case and Lemma 3.23 because the constants \(m_{y_1, y_2}\) is Lemma 3.23 do not depend on the field. \(\square\)
3.11. **Type A.** We say that a quiver is of type $A$ if each of its connected components has type $A_n$ for some positive integer $n$. Now we show that the quivers of type $A$ are even. We start by proving some helpful lemmas.

**Definition 3.30.** Suppose $n, k, s \in \mathbb{N}, k \leq n, s > 0$. Suppose $v$ and $d$ are sequences of nonnegative integers $v = (v_1, \ldots, v_s), d = (d_1, \ldots, d_s)$ such that $v_1 + \cdots + v_s = n$. Let $W$ be an $n$-dimensional $\mathbb{C}$-vector space and let

$$\phi = \{\{0\} = W_0 \subset W_1 \subset \cdots \subset W_s = W)$$

be a flag of type $v$ in $W$, i.e., $\dim W_a/W_{a-1} = v_a$ for each $a$ in $[1, s]$. Set

$$X_{v,d,k} = \{U \in \text{Gr}_k(W); \dim U \cap W_a = d_a, \forall a \in [1, s]\}.$$

**Lemma 3.31.** The set $X_{v,d,k}$ is either empty or a variety with an affine cell decomposition.

**Proof.** Suppose that $X_{v,d,k}$ is not empty. Let $P$ be the parabolic subgroup in $\text{GL}(W)$ preserving the flag $\phi$. The variety $X_{v,d,k}$ is isomorphic to a $P$-orbit in $\text{Gr}_k(W)$. Let $B$ be a Borel subgroup contained in $P$. A $P$-orbit in $\text{Gr}_k(W)$ is a disjoint union of $B$-orbits. Each of these $B$-orbits is isomorphic to an affine space. 

**Definition 3.32.** Let $W$ be a finite dimensional $\mathbb{C}$-vector space. Suppose $s, t \in \mathbb{N}, s > 0, t > 0$. Let

$$\phi = \{\{0\} = W_0 \subset W_1 \subset \cdots \subset W_s = W), \psi = \{\{0\} = V_0 \subset V_1 \subset \cdots \subset V_t = W)$$

be two partial flags in $W$. Let $v$ be a sequence of nonnegative integers $v = (v_1, \ldots, v_s)$ and let $D = (d_{ij})$ be an $s \times t$-matrix of integers. Set

$$Y_{\phi,\psi,v,D} = \{(\{0\} = U_0 \subset U_1 \subset \cdots \subset U_s = W); \dim U_a/U_{a-1} = v_a, \dim U_a \cap V_b = d_{a,b}, W_a \subset U_a, \forall a \in [1, s], b \in [1, t]\}.$$

**Lemma 3.33.** The set $Y_{\phi,\psi,v,D}$ is either empty or a variety with an affine cell decomposition.

**Proof.** The proof is by induction on $s$. The case $s = 1$ follows from Lemma 3.31. Suppose $s > 1$. Set

$$\phi' = \{\{0\} = W_0 \subset W_2 \subset W_3 \subset \cdots \subset W_s = W), \psi' = (v_1 + v_2, v_3, \ldots, v_s).$$

Let $D'$ be the matrix that we get from $D$ by erasing the first row. Forgetting the $U_1$-component of the flag yields the morphism $Y_{\phi,\psi,v,D} \to Y_{\phi',\psi',v',D'}$. This is a fibration with the fibre of the form $X_{\star}$, see Definition 3.59. So $Y_{\phi,\psi,v,D}$ has a decomposition into affine cells by induction hypothesis and Lemma 3.31.

Suppose that the quiver $\Gamma$ is of type $A_n$. We enumerate its vertices $I = \{i_1, \ldots, i_n\}$ such that there is an arrow in some direction between $i_a$ and $i_{a+1}$ for $a \in [1, n-1]$. Let $V$ and $Y$ be as in Section 2.1. Suppose $y = (j_1^{(a_1)}, \ldots, j_m^{(a_m)}) \in Y_v$. Let

$$\{0\} = W_0 \subset W_1 \subset \cdots \subset W_k = V_a$$

be a partial flag in $V_{i_a}$. For each $m \times k$-matrix $D = (d_{ij})$ of integers we set

$$F_D = \{(\{0\} = V^0 \subset V^1 \subset \cdots \subset V^m = V) \in \pi_y^{-1}(x); \dim V^r \cap W_s = d_{r,s}, \forall r \in [1, m], s \in [1, k]\}.$$

We have the decomposition $\pi_y^{-1}(x) = \bigsqcup_D F_D$.

**Lemma 3.34.** The set $F_D$ is either empty or a variety with an affine cell decomposition.
Proof. The proof is by induction on \( n \). For \( n = 1 \) the statement follows from Lemma \( \text{Lemma} \) \( \text{Lemma} \). Now suppose \( n \geq 2 \). Suppose that the quiver \( \Gamma \) contains the arrow \( i_{n-1} \rightarrow i_n \). We denote this arrow by \( h_0 \). Consider the following flag in \( V_{i_{n-1}} \)

\[
\{0\} = W_0^1 \subset W_1^1 \subset \cdots \subset W_{k+1} = V_{i_{n-1}}, \quad W_r = x_{h_0} (W_{r-1}), \quad r \in [1, k],
\]

where \( x_{h_0} \) is the \( h_0 \)-component of \( x \). Let \( \Gamma' \) be the quiver that we get from \( \Gamma \) by deleting the vertex \( i_n \). Set \( V' = \bigoplus_{r=1}^{n-1} V_{i_r} \), \( \nu' = \nu - \nu_i \cdot i_n \). As before we denote by \( H \) the set of arrows in \( \Gamma \) and for an arrow \( h \in H \) we write \( h' \) and \( h'' \) for its source and target respectively. Set \( H' = H \setminus h_0 \). Denote by \( E_{V'} \) the variety of representations of \( \Gamma' \) in \( V' \), i.e., \( E_{V'} = \bigoplus_{h \in H'} \text{Hom}(V_{h'}, V_{h'}) \). Set \( x' = \bigoplus_{h \in H'} x_h \in E_{V'} \), where \( x_h \) is the \( h \)-component of \( x \in E_{V'} \). Set \( y' = (y_1, \ldots, y_m) \in Y_{\nu'} \), where

\[
a'_t = \begin{cases} 
a, & \text{if } j_t \neq i_n, \\ 0, & \text{if } j_t = i_n, \end{cases} \quad \forall t \in [1, m].
\]

We leave the zero terms in \( y' \) to simplify the notation. For each \( m \times (k+1) \)-matrix of integers \( D' = (d'_{s,t}) \) we set

\[
F'_{D'} = \{ \{0\} = V'^0 \subset V'^1 \subset \cdots \subset V'^m = V' \} \in \pi^{-1}_{y'} (x') : \\
\dim V'^r \cap W'_s = d'_{s,t}, \forall r \in [1, m], s \in [1, k+1].
\]

We have the morphism \( e : \pi^{-1}_{y'} (x) \rightarrow \pi^{-1}_{y'} (x') \) given by the intersection of each component of the flag with \( V' \). Set

\[
F_{D,D'} = F_D \cap e^{-1}(F'_{D'}).
\]

We have the decomposition \( F_D = \coprod_{D'} F_{D,D'} \). So it is enough to construct a decomposition into affine cells for each \( F_{D,D'} \). Let us study the following restriction of \( e \)

\[
e_{D,D'} : F_{D,D'} \rightarrow F'_{D'}.
\]

We want to show that it is a fibration. Let \( c_1 < \cdots < c_t \) be the integers in \([1, m]\) such that

\[
j_{c_1} = j_{c_2} = \cdots = j_{c_t} = i_n, \quad j_s \neq i_n \quad \forall s \in [1, m] \setminus \{c_1, \ldots, c_t\}.
\]

Set also \( c_0 = 0 \). Let

\[
\phi' = \{ \{0\} = V'^0 \subset V'^1 \subset \cdots \subset V'^m = V' \}
\]

be a flag in \( F'_{D'} \). We have

\[
e^{-1}_{D,D'} (\phi') = \{ \{0\} = V^0 \subset V^1 \subset \cdots \subset V^m = V \} \in F_D:
\]

\[
V^s \cap V'^r, \forall s \in [1, m]\}
\]

\[
\{ \{0\} = U^0 \subset U^1 \subset \cdots \subset U^m = V_{i_n} \}:
\]

\[
\{ \{0\} = V'^0 \oplus U^0 \subset V'^1 \oplus U^1 \subset \cdots \subset V'^m \oplus U^m = V \} \in F_D \}
\]

\[
\{ \{0\} = U^0 \subset U^1 \subset \cdots \subset U^m = V_{i_n} \}:
\]

\[
U^s = U^{s-1}, \forall s \in [1, m] \setminus \{c_1, \ldots, c_t\},
\]

\[
\dim U^s / U^{s-1} = a_s, \forall s \in \{c_1, \ldots, c_t\},
\]

\[
x_{h_0} (V'^m \cap V_{i_{n-1}}) \subset U^r, \forall r \in [1, m],
\]

\[
\dim U^r \cap W_p = d_{r,p}, \forall r \in [1, m], p \in [1, k].
\]

\[
Y_{\phi, \psi, \nu, D,}
\]

where

\[
\phi = (x_{h_0} (V'^0 \cap V_{i_{n-1}}) \subset x_{h_0} (V'^1 \cap V_{i_{n-1}}) \subset \cdots \subset x_{h_0} (V'^{m-1} \cap V_{i_{n-1}}) \subset V_{i_n}),
\]

\[
\psi = (\{0\} = W_0 \subset W_1 \subset \cdots \subset W_k = V_{i_n}), \quad \nu = (a_1 - a_1', \ldots, a_m - a_m'),
\]
see Definition 3.32. Moreover, this fibre does not depend on the choice of \( \phi' \in F'_{\mu} \) because the types of the flags \( \phi, \psi \) and their relative position do not depend on \( \phi' \).

Really,

\[
\dim x_{h0}(V'^{\alpha} \cap V_{i_{n-1}}) \cap W_b = \dim x_{h0}(V'^{\alpha} \cap V_{i_{n-1}} \cap x_{h0}^{-1}(W_b)) = \dim x_{h0}(V'^{\alpha} \cap W_{b+1}^{\alpha}) = \dim V'^{\alpha} \cap W_{b+1}^{\alpha} - \dim V'^{\alpha} \cap W_{b+1} \cap \text{Ker} x_{h0} = \dim V'^{\alpha} \cap W_{b+1} - \dim V'^{\alpha} \cap W_1^{\alpha} = d'_{a,b+1} - d'_{a,1}, \quad \forall a \in [1, m-1], b \in [1, k],
\]

\[
\dim x_{h0}(V'^{\alpha} \cap V_{i_{n-1}}) = \dim x_{h0}(V'^{\alpha} \cap V_{i_{n-1}}) \cap W_k = d'_{a,k+1} - d'_{a,1}, \quad \forall a \in [1, m-1].
\]

So \( F_{D,D'} \) has a decomposition into affine cells by induction hypothesis and Lemma 3.33.

Now suppose that the quiver \( \Gamma \) contains the arrow \( i_n \to i_{n-1} \). Let \( \Gamma^{\text{op}} \) be the quiver that we get from \( \Gamma \) by inverting the orientation of each arrow. Consider the \( I \)-graded \( \mathbb{C} \)-vector space \( V^* = \bigoplus_{i \in I} V_i^* \) dual to \( V \). Consider the representation \( x^* \in E_{V^*} \) dual to \( x \). Let \( y^{\text{op}} \) be the element \( Y_{\mu} \) that we get from \( y \) by reversing the order of its components. Then we have the variety isomorphism

\[
\pi^{-1}_y(x) \to \pi^{-1}_{y^{\text{op}}}(x^*),
\]

\( \{0\} = V^0 \subset V^1 \subset \cdots \subset V^m = V \to \{0\} = (V^m)^{\perp} \subset \cdots \subset (V^1)^{\perp} \subset (V^0)^{\perp} = V^*, \)

Here \( \cdot^{\perp} \) denotes the annihilator in the dual space \( V^* \). Moreover, this isomorphism preserves the decompositions

\[
\pi^{-1}_y(x) = \prod_D F_D, \quad \pi^{-1}_{y^{\text{op}}}(x^*) = \prod_D F_D
\]

(with some permutation of indices). Here \( F_D \) is defined analogically to \( F_D \) with respect to the flag

\[
\{0\} = (W_k)^{\perp} \subset \cdots \subset (W_1)^{\perp} \subset (W_0)^{\perp} = V_{i_{n-1}}^*,
\]

where \( \cdot^{\perp} \) denotes the annihilator in \( V_{i_{n-1}}^* \) (not in \( V^* \)). So we reduce the case \( i_n \to i_{n-1} \) to the case \( i_{n-1} \to i_n \).

**Theorem 3.35.** Let \( \Gamma \) be a quiver of type A (maybe not connected). Suppose \( y \in Y_{\mu}, x \in E_V \). Then the fibre \( \pi^{-1}_y(x) \) is either empty or has a decomposition into affine cells.

**Proof.** It is enough to prove the statement in the case when the quiver \( \Gamma \) is connected. In this case the statement follows from Lemma 3.34.

**Corollary 3.36.** A quiver of type A is even.

### 3.12. Quiver Schur algebras

In this section, we apply a similar approach to the quiver Schur algebras in [23]. Let \( e \) be an integer, \( e > 1 \). Let \( \Gamma \) be a quiver with the set of vertices \( I = \mathbb{Z}/e\mathbb{Z} \) and the set of arrows \( H = \{ i \to i + 1; \ i \in I \} \).

We keep the notation of Section 2.1. Let \( E^0_V \subset E_V \) be the subvariety of nilpotent representations.

Let us change slightly the notation. Let \( V\text{Comp}_e(\nu) \) be the set of tuples \( \mu = (\mu^{(1)}, \ldots, \mu^{(k)}) \) of nonzero elements of \( \mathbb{N}I \) such that \( \mu^{(1)} + \mu^{(2)} + \cdots + \mu^{(k)} = \nu \). We will call an element of \( V\text{Comp}_e(\nu) \) a vector composition. For \( \mu \) as above we denote by \( F_\mu \) the variety of all \( I \)-graded flags

\[
\phi = (0 = V^0 \subset V^1 \subset \cdots \subset V^k = V)
\]
in the $I$-graded vector space $V = \bigoplus_{i \in I} V_i$ such that the $I$-graded vector space $V^k/V^{k-1}$ has graded dimension $\mu^{(k)}$ for each $r \in \{1, 2, \cdots, k\}$. Let $\tilde{F}_\mu$ be the variety of pairs $(x, \phi) \in E_V \times F_\mu$ that are compatible, i.e., we have $x(V^r) \subset V^{r-1}$ for $r \in [1, k]$. Let $\pi_\mu$ be the natural projection from $\tilde{F}_\mu$ to $E_V$, i.e.,

$$\pi_\mu : \tilde{F}_\mu \to E_V, \ (x, \phi) \mapsto x.$$

Let $F$ be the set of all flags in $V$ (not necessarily completely). Let $F_V \subset F$ be the subset of $I$-graded flags. Set

$$\tilde{F} = \{(\phi = (V^0 \subset \cdots \subset V^k = V), x) \in F \times N; \ x(V^r) \subset V^{r-1} \forall r \in [1, k]\},$$

$$\tilde{F}_V = \{(\phi = (V^0 \subset \cdots \subset V^k = V), x) \in F_V \times E^\vee_V; \ x(V^r) \subset V^{r-1} \forall r \in [1, k]\}.$$ 

Let $\pi : \tilde{F} \to N$, $\pi_V : \tilde{F}_V \to E^\vee_V$ be natural projections. Set $G = GL(V)$, $g = \mathfrak{gl}(V) = \text{Lie}(G)$. Fix a primitive $\eta$th root of unity $\xi \in \mathbb{C}$. Consider the element $s \in G$ preserving each $V_i$ and acting by $\xi^i$ on $V_i$, $i \in I$. Note that the group $G \times \mathbb{C}^\ast$ acts on $\mathfrak{gl}(V)$ with $G$ acting by the adjoint action and $\mathbb{C}^\ast$ acting by the multiplication by scalars. Let $N \subset g$ be the nilpotent cone. The $G \times \mathbb{C}^\ast$-action on $g$ yields a $G \times \mathbb{C}^\ast$-action on $N$. Set $\tilde{s} = (s, \xi^{-1}) \in G \times \mathbb{C}^\ast$. We have $\tilde{s}^2 \simeq E_V$ and $\tilde{s}^2 \simeq E^\vee_V$. Here the top index $\tilde{s}$ always means the set of $\tilde{s}$-stable points.

Similarly, the $G$-action on $F$ yields a $G \times \mathbb{C}^\ast$-action on $F$, where $\mathbb{C}^\ast$ acts trivially. The $G \times \mathbb{C}^\ast$-action on $F$ and $N$ yields a $G \times \mathbb{C}^\ast$-action on $\tilde{F}$. We have $F^\tilde{s} \simeq F_V$ and $F^\tilde{s} \simeq \tilde{F}_V$. The following lemma is an analogue of Conjecture \ref{L3}.

**Lemma 3.37.** For each $\mu \in \text{VComp}_h(\nu)$ and each $x \in E^\vee_V$, we have $H^{od}(I^\mu_\nu)^{-1}(x), \mathbb{Z}) = 0$.

**Proof.** The restriction of $\pi$ to the set of $\tilde{s}$-stable points yields a morphism $\tilde{\pi} : \tilde{F}^\tilde{s} \to N^\tilde{s}$. We have the following commutative diagram

$$
\begin{array}{ccc}
\tilde{F}_V & \xrightarrow{\pi_V} & E^\vee_V \\
\downarrow & & \downarrow \\
\tilde{F}^\tilde{s} & \xrightarrow{\tilde{\pi}} & N^\tilde{s}.
\end{array}
$$

Then for each $x \in E^\vee_V$ we can identify the fibre $\pi_\mu^{-1}(x)$ with $(\tilde{\pi}^{-1}(x))^\tilde{s} = (\tilde{s}^{-1}(x))^\tilde{s}$. By \cite[Thm. 3.9]{5} we have $H^{od}((\tilde{s}^{-1}(x))^\tilde{s}, \mathbb{Z}) = 0$. This implies the statement because for each $\mu \in \text{VComp}_h(\nu)$ the variety $\tilde{F}_V$ contains a connected component isomorphic to $\tilde{F}_\mu$ and the restriction of $\pi_V$ to this component coincides with $\pi_\mu$. \hfill $\square$

It is well-known that the set of $G_V$-orbits in $E^\vee_V$ is finite. This observation together with Lemma \ref{L3} motivates us to study the parity sheaves on $E^\vee_V$. As before, we fix an arbitrary field $k$. Let $E^\vee_V = \coprod_{\lambda \in \Lambda^\vee_V} \mathcal{O}_\lambda$ be the stratification by $G_V$-orbits.

**Lemma 3.38.** There are no nontrivial $G_V$-equivariant local systems on $\mathcal{O}_\lambda$ for each $\lambda \in \Lambda^\vee_V$.

**Proof.** The proof is the same as the proof of Corollary \ref{2.11} \hfill $\square$

We can now prove that the stratified variety $E^\vee_V$ satisfies the condition (3.1) in the same way as in Lemma \ref{L3}.

Consider the following complexes in $D_{G_V}(E^\vee_V)$:

$$\delta \mathcal{L}_\mu = (\pi_\mu)_! k_{F_\mu}[\dim F_\mu], \quad \delta \mathcal{L}_V^\lambda = \bigoplus_{\mu \in \text{VComp}_h(\nu)} \delta \mathcal{L}_\mu.$$

Let $Q_V$ be the full additive subcategory of all direct sums of shifts of direct summands of $\delta \mathcal{L}_V^\lambda$ in $D_{G_V}(E^\vee_V)$.
Lemma 3.39. For each $\lambda \in \Lambda^n_V$, the parity sheaf $\mathcal{E}(\lambda)$ exists. It is contained in $\mathcal{Q}_V$.

Proof. Consider the $G_V$-orbit $\mathcal{O}_\lambda$. Let $\mathcal{O}^\lambda$ be the $G$-orbit in $E_V$ that contains $\mathcal{O}_\lambda$. We can find a flag type $d = (d_1, \ldots, d_k)$ with $d_1, \ldots, d_k \in \mathbb{N}$ and $\sum_{i=1}^k d_i = \dim V$, such that the restriction of $\pi$ to the connected component of $\tilde{F}$ corresponding to $d$ yields a resolution of singularities of $\overline{\mathcal{O}^\lambda}$. After passing to $\tilde{F}$-stable points in this resolution, we get a morphism of the form

$$\prod_{\mu \in C} \pi_{\mu} : \prod_{\mu \in C} \tilde{F}_\mu \to \overline{\mathcal{O}^\lambda} \subset E^n_V$$

for some subset $C \subset V\text{Comp}_n(\nu)$. Thus, there is a unique $\mu \in C \subset V\text{Comp}_n(\nu)$ such that $\pi_{\mu}$ induces an isomorphism $\pi_{\mu}^{-1}(\mathcal{O}_\lambda) \to \mathcal{O}_\lambda$. Moreover, $\pi_{\mu}$ has even fibres, see Lemma 3.37. Thus the complex $(\pi_{\mu})_{|\tilde{F}_\mu}$ is even, see Lemma 3.4. To show that it contains a direct factor isomorphic to a shift of $\mathcal{E}(\lambda)$ it is enough to prove that $\overline{\mathcal{O}^{\lambda'}}$ can not contain a $G_V$-orbit $\mathcal{O}_{\lambda'}$ such that $\mathcal{O}_\lambda \subset \mathcal{O}_{\lambda'}$ and $\mathcal{O}_{\lambda'} \neq \mathcal{O}_\lambda$. Suppose that $\overline{\mathcal{O}^{\lambda'}}$ contains such a $G_V$-orbit $\mathcal{O}_{\lambda'}$. Then $\mathcal{O}^{\lambda'}$ implies $\mathcal{O}_{\lambda'} \subset \overline{\mathcal{O}^{\lambda'}}$. Moreover, we have $\mathcal{O}_{\lambda'} \neq \mathcal{O}_{\lambda'}$ because $\mathcal{O}_{\lambda}$ and $\mathcal{O}_{\lambda'}$ are not in the same $G$-orbit. (Otherwise, $\mathcal{O}_\lambda$ and $\mathcal{O}_{\lambda'}$ must have equal dimensions. This is not possible because $\mathcal{O}_\lambda \subset \mathcal{O}_{\lambda'} \setminus \mathcal{O}_{\lambda'}$. Now we see that $\overline{\mathcal{O}^{\lambda'}}$ does not contain the $G$-orbit $\mathcal{O}^{\lambda'}$. Thus $\overline{\mathcal{O}^{\lambda'}}$ can not contain the $G_V$-orbit $\mathcal{O}_{\lambda'}$.

Thus the parity sheaf $\mathcal{E}(\lambda)$ exists and belongs to $\mathcal{Q}_V$. \hfill $\Box$

Corollary 3.40. The subcategories $\mathcal{Q}_V$ and $\text{Par}_{G_V}(E^n_V)$ of $D_{G_V}(E^n_V)$ coincide.

Definition 3.41. For a given graded dimension vector $\nu \in \mathcal{N}$ the quiver Schur algebra is the graded algebra $A_{\nu,k} = \text{Ext}^*_G(\delta \mathcal{L}_{V,k}, \delta \mathcal{L}_{V,k})$, where the $n$th graded component is given by the $n$th extension group.

Remark 3.42. It is clear from the definition that, for each $\mu \in V\text{Comp}_n(\nu)$, the algebra $A_{\nu,k}$ contains an idempotent of degree zero $1_\mu$ such that $1_\mu A_{\nu,k} 1_\mu = \text{Ext}^*_G(\delta \mathcal{L}_{\nu_1}, \delta \mathcal{L}_{\nu_1})$.

Definition 3.43. For each $\mu \in V\text{Comp}_n(\nu)$, let $P_\mu$ be the graded projective $A_{\nu,k}$-module $A_{\nu,k} 1_\mu$.

Definition 3.44. Let $U^-_{\nu,Z}$ be the integral form of the generic nilpotent Hall algebra of the quiver $\Gamma$, see [23, Sec. 2.5]. For each $\nu \in \mathcal{N}$, let $f_\nu \in U^-_{\nu,Z}$ be the characteristic function of the semi-simple representation of $\Gamma$ with graded dimension vector $\nu$.

Let $\text{proj}(A_{\nu,k})$ be the category of graded projectively finitely generated $A_{\nu,k}$-modules. Let $K(A_{\nu,k})$ be the split Grothendieck group of $\text{proj}(A_{\nu,k})$ viewed as an $A$-module. Set $K(A_{\nu,k}) = \bigoplus_{\nu \in \mathcal{N}} K(A_{\nu,k})$ and $K(Q) = \bigoplus_{\nu} K(Q_{\nu})$. For each $\nu_1, \nu_2 \in \mathcal{N}$, we have an algebra homomorphism $A_{\nu_1,k} \otimes A_{\nu_2,k} \to A_{\nu_1+\nu_2,k}$, see [23, Def. 2.8]. It yields an algebra structure on $K(A_{\nu,k})$. The following theorem is proved in [23, Prop. 2.12].

Theorem 3.45. There is an $A$-algebra isomorphism $K(A_{\nu,k}) \to U^-_{\nu,Z}$ such that $[P_\nu] \mapsto f_\nu$, for each $\nu \in \mathcal{N}$, where $\nu$ is viewed as a trivial vector multicomposition in $V\text{Comp}_n(\nu)$.

Analogously to Theorem 2.21, we get the following result.

Theorem 3.46. The functor $Y : Q^n_V \to \text{proj}(A_{\nu,k})$ such that $\mathcal{F} \mapsto \text{Ext}^*_G(\mathcal{F}, \delta \mathcal{L}_{V,k})$ is an equivalence of categories.
For \( \nu_1, \nu_2 \in \mathbb{N} \), \( \mu_1 \in \text{VComp}_e(\nu_1) \), \( \mu_2 \in \text{VComp}_e(\nu_2) \) let \( \mu_1 \cup \mu_2 \in \text{VComp}_e(\nu_1 + \nu_2) \) denote the concatenation of \( \mu_1 \) and \( \mu_2 \). We can also define an algebra structure on \( K(\mathbb{Q}) \) as in Section 2.20 such that \( \delta L_{\mu_1} \circ \delta L_{\mu_2} = \delta L_{\mu_1 \cup \mu_2} \). We get the following result.

**Corollary 3.47.** There is an \( \mathcal{A} \)-algebra isomorphism \( K(\mathbb{Q}) \rightarrow U_{\mathbb{C}, \mathbb{Z}}^- \) such that \( \delta L_\nu \mapsto f_\nu \) for each \( \nu \in \mathbb{N} \), where \( \nu \) is viewed as a trivial vector multicomposition in \( \text{VComp}_e(\mu) \).

**Proof.** We need only to verify that the algebra structures on \( K(\mathbb{Q}) \) agrees with the algebra structure on \( U_{\mathbb{C}, \mathbb{Z}}^- \). This is true because \( Y(\delta L_\mu) = P_\mu, [P_{\mu_1}][P_{\mu_2}] = [P_{\mu_1 \cup \mu_2}] \) (see [23, Prop. 2.9]) and the elements \( f_\nu \) for \( \nu \in \mathbb{N} \) generate \( U_{\mathbb{C}, \mathbb{Z}}^- \) as an \( \mathcal{A} \)-algebra.

In particular Corollary 3.47 yields an \( \mathcal{A} \)-basis of \( U_{\mathbb{C}, \mathbb{Z}}^- \) in terms of parity sheaves.

**Acknowledgements**

I would like to thank Eric Vasserot for his guidance and helpful discussions during my work on this paper. I would also like to thank Geordie Williamson for his comments on the paper and useful discussions. I am also grateful to Catharina Stroppel for the suggestion to study the quiver Schur algebras in this paper.

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