Duality for actions of weak Kac algebras and crossed product inclusions of II$_1$ factors

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Abstract

We show that indecomposable weak Kac algebras are free over their Cartan subalgebras and prove a duality theorem for their actions. Using this result, for any biconnected weak Kac algebra we construct a minimal action on the hyperfinite II$_1$ factor. The corresponding crossed product inclusion of II$_1$ factors has depth 2 and an integer index. Its first relative commutant is, in general, non-trivial, so we derive some arithmetic properties of weak Kac algebras from considering reduced subfactors.

1 Introduction

It is well understood now that Kac algebras (Hopf C*-algebras) are closely related with the subfactors theory: it was announced by Ocneanu and proved in [23], [12], [3], [1] that irreducible depth 2 inclusions of type II factors come from crossed products with Kac algebras. This result was recently extended to the case of general (i.e., not necessarily irreducible) finite index depth 2 subfactors in [17], where it was shown that if $N \subset M \subset M_1 \subset M_2 \subset \ldots$ is the Jones tower constructed from such a subfactor $N \subset M$, then $K = M' \cap M_2$ has a natural structure of a finite-dimensional weak Kac algebra (if the index $[M : N]$ is integer) or a weak Hopf C*-algebra (if $[M : N]$ is non-integer) and there is a minimal action of $K$ on $M_1$ such that

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$M$ is the fixed point subalgebra of $M_1$ and $M_2$ is isomorphic to the crossed product of $M_1$ and $K$. This result establishes an injective correspondence between finite index depth 2 subfactors of a given II$_1$ factor and weak $C^*$-Hopf algebras.

It is natural to ask if this correspondence is one-to-one in the case of the hyperfinite II$_1$ factor. Note that in [25] Yamanouchi constructed an outer action of any finite dimensional Kac algebra $K$ on the hyperfinite II$_1$ factor $R$, here the outerness means that $R' \cap R \bowtie K = \mathbb{C}$, i.e. the first relative commutant of the crossed product inclusion $R \subset R \bowtie K$ is minimal. His construction used the Takesaki duality for actions of Kac algebras [3].

In this work we extend this result to weak Kac algebras, i.e., we show that any weak Kac algebra has a minimal action on $R$. Finite dimensional weak Kac algebras generalize both finite groupoid algebras and usual Kac algebras. Note that weak Kac algebra is a special case of a weak Hopf $C^*$-algebra introduced in [3] and [13], which is characterized by the property $S^2 = \text{id}$. It was shown in [16] that the category of weak Kac algebras is equivalent to the categories of generalized Kac algebras of T. Yamanouchi [24] and Kac bimodules (an algebraic version of Hopf bimodules [8]). Compared with these objects, the advantage of the language of weak Kac algebras is that their definition is transparently self-dual, so it is easy to work with both weak Kac algebra and its dual simultaneously.

The paper is organized as follows.

In Section 2 we collect the necessary definitions and facts about weak Kac algebras, their actions and crossed-products, and their Cartan subalgebras; we also give a brief description of the basic construction for $*$-algebras.

In Section 3 we introduce a $\lambda$-Markov condition for weak Kac algebras. A weak Kac algebra $K$ satisfies the $\lambda$-Markov condition if the normalized Haar trace on $K$ is the $\lambda$-Markov trace for the inclusion $K_s \subset K$, where $K_s$ is the source Cartan subalgebra of $K$. This condition is automatically satisfied if $K$ is indecomposable, i.e., not isomorphic to the direct sum of two weak Kac algebras. Theorem 3.5 shows that being $\lambda$-Markov is equivalent to the freeness of $K$ over its Cartan subalgebras; in particular, $\lambda^{-1}$ must be a positive integer. As a corollary, we obtain that indecomposable weak Kac algebras of prime dimension are group algebras of cyclic groups, which extends the well-known result of Kac [14].

Also in this section we introduce and study basic properties of connected and biconnected weak Kac algebras, i.e., those for which the inclusion $K_s \subset K$ is connected (resp. inclusions $K_s \subset K$ and $K_s^* \subset K^*$ are
connected). The latest class of indecomposable weak Kac algebras is the most important for the applications to subfactors in Section 5, so we describe a way of constructing biconnected weak Kac algebras from two-sided crossed-products introduced in [13].

The central result of Section 4 is a duality theorem for actions of weak Kac algebras. This theorem is an analogue of the well known duality results for actions of locally compact groups [14], Kac algebras [4], and Hopf algebras [2]. It states that if $K$ satisfies the $\lambda$-Markov condition and acts on a $C^*$-algebra (von Neumann algebra) $A$, then the dual crossed product algebra $(A\rtimes K)^*\rtimes K$ is isomorphic to $A \otimes M_{\lambda^{-1}}(\mathbb{C})$. Let us note that a similar result for depth 2 inclusions of von Neumann algebras was proved in [8].

In Section 5 for any biconnected weak Kac algebra $K$ we construct a minimal action on the hyperfinite II$_1$ factor $R$ (where the minimality means that the relative commutant $R' \cap R\rtimes K$ is minimal). The resulting crossed product inclusion $R \subset R\rtimes K$ of II$_1$ factors has depth 2 and an integer index $\lambda^{-1}$. We compute the standard invariant of this inclusion, and show, in particular, that the first relative commutant is isomorphic to the Cartan subalgebra of $K$: $R' \cap R\rtimes K \cong K_s$.

Finally, in Section 6 we construct examples of irreducible subfactors reducing the inclusion $R \subset R\rtimes K$ by the minimal projection in $K_s = R' \cap R\rtimes K$. In this way we can associate an irreducible finite depth subfactor of $R$ with every irreducible representation of $K$ or $K_s$. This allows us to derive certain arithmetic properties of biconnected weak Kac algebras.

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2 Preliminaries

2.1 Weak Kac algebras [8], [16].

Throughout this paper all weak Kac algebras are supposed to be finite-dimensional.

The notion of weak Kac algebra [16] is a special case of a more general concept of weak $C^*$-Hopf algebra introduced in [8]; see [16] for a discussion on equivalence of weak Kac algebras with other algebraic versions of quan-
tum groupoid (generalized Kac algebras of T. Yamanouchi [24] and Kac bimodules).

A weak Kac algebra $K$ is a finite dimensional $C^*$-algebra equipped with the following linear maps

(a) comultiplication $\Delta : K \to K \otimes K$,
(b) counit $\varepsilon : K \to \mathbb{C}$,
(c) antipode $S : K \to K$,

where $\Delta$ is a (not necessarily unital) homomorphism of $C^*$-algebras, $\varepsilon$ is a positive (not necessarily multiplicative) functional, $S$ is a $*$-preserving anti-multiplicative and anti-comultiplicative involution (i.e., $S^2 = \text{id}$) such that the following identities hold (we denote $\varepsilon(x) = (\text{id} \otimes \varepsilon)((1 \otimes x)\Delta(1))$ and $\varepsilon_t(x) = (\varepsilon \otimes \text{id})(\Delta(1)(x \otimes 1))$):

(1) $(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta$, $\varepsilon \otimes \text{id})\Delta = \text{id} = (\text{id} \otimes \varepsilon)\Delta$,
i.e., $K$ is a coalgebra,

(2) $\varepsilon_s(xy) = (\text{id} \otimes \varepsilon)((1 \otimes x)\Delta(y))$,

(3) $(\varepsilon \otimes \text{id})\Delta(x) = (1 \otimes x)\Delta(1)$,

(4) $m(S \otimes \text{id})\Delta(x) = \varepsilon_s(x)$, $x, y \in K$,

where $m$ denotes multiplication.

The following identities are equivalent to the above axioms (2)–(4) respectively.

(2') $x\varepsilon_t(y) = (\varepsilon \otimes \text{id})(\Delta(x)(y \otimes 1))$,

(3') $(\text{id} \otimes \varepsilon_t)\Delta(x) = \Delta(1)(x \otimes 1)$,

(4') $m(\text{id} \otimes S)\Delta(x) = \varepsilon_t(x)$, $x, y \in K$.

The dual vector space $K^*$ has a natural structure of a weak Kac algebra given by dualizing the structure operations of $K$:

$\langle \phi \psi, x \rangle = \langle \phi \otimes \psi, \Delta(x) \rangle$ (multiplication),
$\langle \Delta(\phi), x \otimes y \rangle = \langle \phi, xy \rangle$ (comultiplication),
$\langle S(\phi), x \rangle = \langle \phi, S(x) \rangle$ (antipode),
$\langle \phi^*, x \rangle = \langle \phi, S(x^*) \rangle$ ($*$-operation),
for all $\phi, \psi \in K^*$, $x, y \in K$. Unit is given by $\varepsilon$ and counit is by $\phi \mapsto \langle \phi, 1 \rangle$.

Below we collect the most important results of the theory of weak Kac algebras. The proofs can be found in [16].

The maps $\varepsilon_s$ and $\varepsilon_t$ are called source and target counital maps respectively, we have $\varepsilon_s^2 = \varepsilon_s$ and $\varepsilon_t^2 = \varepsilon_t$. Their images are unital $C^*$-subalgebras, called Cartan subalgebras of $K$:

\[ K_s = \{ x \in K \mid \varepsilon_s(x) = x \} = \{ x \in K \mid \Delta(x) = 1 \otimes x_2 = 1 \otimes 1 \cdot x \}, \]
\[ K_t = \{ x \in K \mid \varepsilon_t(x) = x \} = \{ x \in K \mid \Delta(x) = x_1 \otimes 1 = 1 \cdot x \otimes 1 \}. \]

Cartan subalgebras commute: $[K_s, K_t] = 0$; also we have $S \circ \varepsilon_s = \varepsilon_t \circ S$ and $S(K_s) = K_t$.

Like usual finite-dimensional Kac algebras (= Hopf $C^*$-algebras), weak Kac algebras have integrals in the following sense.

There exists a unique projection $p_\varepsilon \in K$, called a Haar projection, such that for all $x \in K$:

\[ p_\varepsilon x = p_\varepsilon \varepsilon_s(x), \quad xp_\varepsilon = \varepsilon_t(x)p_\varepsilon, \quad \varepsilon_s(p_\varepsilon) = \varepsilon_t(p_\varepsilon) = 1. \]

There exists a unique faithful trace $\tau$ on $K$, called a normalized Haar trace, such that

\[ (\tau \otimes \text{id})\Delta = (\tau \otimes \varepsilon_s)\Delta, \quad (\text{id} \otimes \tau)\Delta = (\varepsilon_t \otimes \tau)\Delta, \quad \tau \circ \varepsilon_s = \tau \circ \varepsilon_t = \varepsilon. \]

Note that normalized Haar projection and trace are unimodular, i.e. $S(p_\varepsilon) = p_\varepsilon$ and $\tau \circ S = \tau$. By duality, $\tau$ is the normalized Haar projection for the dual weak Kac algebra $K^*$.

The maps

\[ E_s : K \to K_s : E_s(x) = (\tau \otimes \text{id})\Delta(x), \]
\[ E_t : K \to K_t : E_t(x) = (\text{id} \otimes \tau)\Delta(x) \]

define $\tau$-preserving conditional expectations (see 2.3. for the definition) from $K$ to Cartan subalgebras.

To fix the notation in what follows, let

\[ K \cong \bigoplus_{i=1}^N M_{d_i}(\mathbb{C}), \quad K_s \cong K_t \cong \bigoplus_{\alpha=1}^L M_{m_\alpha}(\mathbb{C}), \]

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and let $\{e^{(i)}_{kl}\} (i = 1 \ldots N; k,l = 1,\ldots,d_i)$ be a system of matrix units in $K$, $\{f^{(\alpha)}_{rs}\} \in K_s$, and $\{g^{(\alpha)}_{rs}\} \in K_t$ ($\alpha = 1,\ldots,L; r,s = 1,\ldots,m_\alpha$). By [16] we have:

$$\Delta(p_\varepsilon) = \sum_i \frac{1}{d_i} \sum_{kl} e^{(i)}_{kl} \otimes S(e^{(i)}_{ik}),$$

$$\Delta(1) = \sum_\alpha \frac{1}{m_\alpha} \sum_{rs} f^{(\alpha)}_{rs} \otimes S(f^{(\alpha)}_{sr}) = \sum_\alpha \frac{1}{m_\alpha} \sum_{rs} S(g^{(\alpha)}_{sr}) \otimes g^{(\alpha)}_{rs}.$$

In particular, $p_\varepsilon$ is cocommutative, i.e., $\Delta(p_\varepsilon) = \varsigma \Delta(p_\varepsilon)$, where $\varsigma$ is the flip on the tensor product $K \otimes K$.

Also we denote $\Lambda = (\Lambda_{ij})$ the $(L \times N)$ inclusion matrix [9] of $K_s$ (or $K_t$) into $K$.

### 2.2 Actions, dual actions, and crossed-products [15].

By a $\ast$-algebra we understood an associative algebra over $\mathbb{C}$ equipped with an antilinear anti-automorphism of order 2 (involution), $x \mapsto x^\ast$.

Action of a weak $C^\ast$-Hopf algebra on a $\ast$-algebra was defined in [15]. We slightly modify that definition, since we consider only those actions for which the map $x \mapsto (x \triangleright 1)$ ($x \mapsto (1 \triangleleft x)$) is injective on a Cartan subalgebra. We need definitions of left and right actions.

A left (resp. right) action of a weak Kac algebra $K$ on a unital $\ast$-algebra $A$ is a linear map

$$K \otimes A \ni h \otimes a \mapsto (h \triangleright a) \in A \quad \text{(resp. } A \otimes K \ni a \otimes h \mapsto (a \triangleleft h) \in A),$$

defining a structure of a left (resp. right) $K$-module on $A$ such that:

1. $h \triangleright ab = (h_{(1)} \triangleright a)(h_{(2)} \triangleright b)$ (resp. $ab \triangleleft h = (a \triangleleft h_{(1)})(b \triangleleft h_{(2)})$),
2. $(h \triangleright a)^\ast = Sh^\ast \triangleright a^\ast$ (resp. $(a \triangleleft h)^\ast = a^\ast \triangleleft Sh^\ast$),
3. $h \triangleright 1 = \varepsilon_t(h) \triangleright 1$, and $h \triangleright 1 = 0$ iff $\varepsilon_t(h) = 0$
   (resp. $1 \triangleleft h = 1 \triangleleft \varepsilon_s(h)$, and $1 \triangleleft h = 0$ iff $\varepsilon_s(h) = 0$).

If $A$ is a $C^\ast$-algebra or a von Neumann algebra then we also require that the map $a \mapsto (h \triangleright a)$ (resp. $a \mapsto (a \triangleleft h)$) to be norm continuous or weakly continuous for all $h \in K$.
Note that the map $z \mapsto (z \triangleright 1)$ (resp. $z \mapsto (1 \triangleleft z)$) defines an injective *-homomorphism from $K_t$ (resp. $K_s$) to $A$. Thus, $A$ must contain a *-subalgebra isomorphic to a Cartan subalgebra of $K$.

A *trivial* left (resp. right) action of $K$ on $K_t$ (resp. $K_s$) is given by

$$h \triangleright a = \varepsilon_t(ha) \quad \text{(resp. } a \triangleleft h = \varepsilon_s(ah)), \quad h \in K, \ a \in K_t \text{(resp. } K_s).$$

A *dual* left (resp. right) action of $K^*$ on $K$ is given by

$$\phi \triangleright h = h_{(1)} \langle \phi, h_{(2)} \rangle \quad \text{(resp. } h \triangleleft \phi = \langle \phi, h_{(1)} \rangle h_{(2)}), \quad \phi \in K^*, \ h \in K.$$

Given a left (resp. right) action of $K$ on a *-algebra $A$, there is a left (resp. right) *crossed product* *-algebra* $A\rtimes K$ (resp. $K\triangleright A$) constructed as follows. As a $\mathbb{C}$-vector space it is $A \otimes_{K_t} K$ (resp. $K \otimes_{K_s} A$), where $K$ is a *-algebra* (von Neumann algebra). There is a left $K_t$-module (resp. right $K_s$-module) via multiplication and $A$ is a right $K_t$-module (resp. left $K_s$-module) via multiplication by image of $K_t$ (resp. $K_s$) under $z \mapsto (z \triangleright 1)$ (resp. $z \mapsto (1 \triangleleft z)$); that is, we identify

$$a(z \triangleright 1) \otimes h \equiv a \otimes zh \quad \text{(resp. } hz \otimes a \equiv h \otimes (1 \triangleleft z)a),$$

for all $a \in A, h \in K, z \in K_t(K_s)$. Let $[a \otimes h]$ (resp. $[h \otimes a]$) denote the class of $a \otimes h$ (resp. $h \otimes a$). A *-algebra* structure is defined by

$$[a \otimes h][b \otimes k] = [a(h_{(1)} \triangleright b) \otimes h_{(2)} k], \quad [a \otimes h]^* = [(h_{(1)}^* \triangleright a^*) \otimes h_{(2)}^*]$$

for all $a, b \in A, h, k \in K$. The maps $i_A : a \mapsto [a \otimes 1_K]$ (resp. $a \mapsto [1_K \otimes a]$) and $i_K : h \mapsto [1_A \otimes h]$ (resp. $h \mapsto [h \otimes 1_A]$) are inclusion of *-algebras such that* $A \rtimes K = i_A(A) i_K(K)$ (resp. $K \triangleright A = i_K(K) i_A(A)$). Moreover, if $A$ is a $C^*$-algebra (von Neumann algebra), then the crossed product is naturally *-isomorphic to a norm closed (weakly closed) *-algebra of operators on some Hilbert space, i.e., it becomes a $C^*$-algebra (von Neumann algebra).

For the crossed products constructed from the trivial actions of $K$ on Cartan subalgebras we have

$$K_t \rtimes K \cong K \quad \text{and} \quad K \triangleright K_s \cong K.$$

A left (resp. right) *dual action* of $K^*$ on the crossed product $A \rtimes K$ (resp. $K \triangleright A$) is defined as

$$\phi \triangleright [a \otimes h] = [a \otimes (\phi \triangleright h)] \quad \text{(resp. } [h \otimes a] \triangleleft \phi = [(h \triangleleft \phi) \otimes a]).$$
for all \( a \in A, h \in K, \phi \in K^* \). The action of \( K^* \) on \( K \) defined above is dual to the trivial action of \( K \) on a Cartan subalgebra.

### 2.3 The basic construction for \(*\)-algebras [23].

Let \( B \) be a unital \(*\)-algebra, \( A \) be its \(*\)-subalgebra containing the unit of \( B \). A conditional expectation \( E : B \rightarrow A \) is a faithful (i.e., such that \( E(Bb) = 0 \) implies \( b = 0 \), for \( b \in B \)) linear \(*\)-preserving map satisfying

\[
E(ab) = aE(b), \quad E(ba) = E(b)a, \quad \text{and} \quad E(a) = a,
\]

for all \( a \in A, b \in B \). A finite family \( \{ u_1, \ldots, u_n \} \subset B \) is called a quasi-basis for \( E \) if

\[
b = \sum_i u_i E(u_i^* b) \quad \text{for all} \quad b \in B.
\]

It is called a basis if “coefficients” \( E(u_i^* b) \) are unique, i.e., if \( \sum_i u_i a_i = 0, a_i \in A \Leftrightarrow a_i = 0 (\forall i) \). A conditional expectation \( E \) is of finite-index type if there exists a quasi-basis for \( E \). In this case index of \( E \) is defined as

\[
\text{Index } E = \sum_i u_i u_i^* \in B.
\]

Index \( E \) belongs to the center of \( B \) and does not depend on the choice of quasi-basis.

The basic construction for \( E \) is a \(*\)-algebra \( B \otimes_A B \) with multiplication and involution given by

\[
(b_1 \otimes b_2)(b_3 \otimes b_4) = b_1 E(b_2 b_3) \otimes b_4, \quad (b_1 \otimes b_2)^* = b_2^* \otimes b_1^*,
\]

for all \( b_1, b_2, b_3, b_4 \in B \). Note that the unit of this algebra is \( \sum_i u_i \otimes u_i^* \), where \( \{u_i\} \) is the quasi basis for \( E \).

In what follows we consider only conditional expectations of finite-index type for which a basis (not just a quasi-basis) exists.

In this case \( B \otimes_A B \) is canonically isomorphic to \( \text{End}_A^r(B) \), the algebra of endomorphisms of \( B \) viewed as a right \( A \)-module; the isomorphism \( \phi : B \otimes_A B \rightarrow \text{End}_A^r(B) \) is given by

\[
\phi(b_1 \otimes b_2)(b) = b_1 E(b_2 b), \quad b, b_1, b_2 \in B.
\]

\( B \) is canonically identified with the subalgebra of left multiplication operators \( (x \mapsto bx) \) for \( x \in B \) in \( \text{End}_A^r(B) \). Clearly, \( \text{End}_A^r(B) \cong M_n(A) \), the \(*\)-algebra of \( (n \times n) \) matrices over \( A \), since \( B \) is free of rank \( n \) over \( A \).

Note that \( e_A = E \in \text{End}_A^r(B) \) is a projection such that
(1) \( e_A be_A = E(b)e_A \) for all \( b \in B \),
(2) the map \( A \ni a \mapsto ae_A \in \text{End}^*_A(B) \) is injective,
and \( \text{End}^*_A(B) \) is generated by \( B \) and \( e_A \).

Conversely, if \( C \) is a \( * \)-algebra containing \( B \) as a unital \( * \)-subalgebra and generated by \( B \) and some projection \( e_A \) satisfying properties (1) and (2) above then \( C \) is canonically \( * \)-isomorphic to \( B \otimes_A B \), the isomorphism is given by \( b_1e_A b_2 \mapsto b_1 \otimes b_2 \).

Due to this fact, we will denote the basic construction for \( E \) by \( \langle B, e_A \rangle \).

When \( A \) and \( B \) are \( C^* \)-algebras (von Neumann algebras) then \( \langle B, e_A \rangle \) naturally becomes a \( C^* \)-algebra (von Neumann algebra).

3 \( \lambda \)-Markov Condition and Connected Weak Kac Algebras

We show that indecomposable weak Kac algebras have the Markov property (i.e., \( E_s(p_c) \) is a scalar). We prove that this property is equivalent to existence of a basis for \( E_s \), i.e., to freeness of \( K \) over the Cartan subalgebra \( K_s \), and can be expressed in terms of the inclusion matrix of \( K_s \subset K \).

We also introduce notions of connected and biconnected weak Kac algebras which are important for applications to the theory of subfactors.

**Definition 3.1** A weak Kac algebra \( K \) is **decomposable** if it is isomorphic to the direct sum of two weak Kac algebras, \( K \cong K_1 \oplus K_2 \); otherwise \( K \) is **indecomposable**.

Clearly, decomposable weak Kac algebras are of little interest in themselves. Indecomposability can be expressed in terms of the algebra \( K_s \cap K_t \cap Z(K) \), the **hypercenter** of \( K \) \([\ref{hypercenter} \text{ }] \) (here \( Z(K) \) is the center of \( K \)).

**Proposition 3.2** \( K \) is indecomposable iff the hypercenter of \( K \) is trivial, i.e. \( K_s \cap K_t \cap Z(K) = \mathbb{C} \).

**Proof.** If \( q \) is a projection in \( K_s \cap K_t \cap Z(K) \), then \( S(q) = q \), \( \Delta(q) = (q \otimes q)\Delta(1) \), and \( \Delta(qh) = (q \otimes q)\Delta(h) \) for all \( h \in K \). Therefore, \( qK \) and \( (1-q)K \) are weak Kac algebras and \( K = qK \oplus (1-q)K \). Conversely, if \( K \) is decomposable and \( K_1 \) is its direct summand, then \( \varepsilon_s(1_{K_1}) = \varepsilon_t(1_{K_1}) = 1_{K_1} \) and the unit of \( K_1 \) belongs to the hypercenter of \( K \).

It turns out that indecomposable weak Kac algebras satisfy a crucial \( \lambda \)-Markov condition.
**Definition 3.3** A weak Kac algebra $K$ satisfies a $\lambda$-Markov condition if

$$E_s(p_\varepsilon) = E_t(p_\varepsilon) = \lambda$$

for some $\lambda > 0$ (note that since $p_\varepsilon$ is cocommutative, we always have $E_s(p_\varepsilon) = E_t(p_\varepsilon)$).

**Proposition 3.4** If $K$ is indecomposable, then it satisfies the $\lambda$-Markov condition for some $\lambda$.

**Proof.** Note that

$$E_s(p_\varepsilon) = E_t(p_\varepsilon) = \sum_i \frac{1}{d_i} \sum_k e_{kk}^{(i)} \tau(e_{kk}^{(i)}) = \sum_i \tau_i \sum_k e_{kk}^{(i)} \in Z(K),$$

where $\tau_i = \tau(e_{kk}^{(i)})$ does not depend on $k$. Therefore, if $E_s(p_\varepsilon) \neq \lambda$, then the hypercenter is non-trivial and $K$ is decomposable by Proposition 3.2.

The following theorem describes the $\lambda$-Markov condition in several equivalent ways.

**Theorem 3.5** The following conditions are equivalent:

(i) $K$ satisfies the $\lambda$-Markov condition,

(ii) $\tau = \lambda \text{Tr}$ where $\text{Tr}$ is the trace of the left regular representation of $K$ on itself,

(iii) $(\Lambda\Lambda^t)\vec{m} = \lambda \vec{m}$, where $\vec{m} = (m_1 \ldots m_L)$ is the dimension vector of a Cartan subalgebra, and $\Lambda$ is the $L \times M$ inclusion matrix of $K_s \subset K$.

(iv) $n = \lambda^{-1}$ is an integer and there is a basis $\{x_\nu\}_{\nu=1 \ldots n}$ for $E_s$, i.e. a basis of $K$ over $K_s$ such that $x = \sum_\nu x_\nu E_s(x_\nu^* x)$ for all $x \in K$,

(v) $n = \lambda^{-1}$ is an integer and there is a basis $\{y_\nu\}_{\nu=1 \ldots n}$ for $E_t$, i.e. a basis of $K$ over $K_t$ such that $y = \sum_\nu y_\nu E_t(y_\nu^* y)$ for all $y \in K$.

**Proof.** (i)$\iff$(ii). As we have seen in the proof of Proposition 3.4, $E_s(p_\varepsilon) = \lambda$ iff there is $\lambda$ such that $\tau(e_{kk}^{(i)}) = \lambda d_i$, i.e., $\tau = \lambda \text{Tr}$.

(i)$\iff$(iii). It suffices to prove that (iii) holds true if and only if $\text{Tr}$ is normalized by conditions $(\text{Tr} \otimes \text{id})\Delta(1) = (\text{id} \otimes \text{Tr})\Delta(1) = \lambda^{-1}$ (it was
shown in [16] that \((\text{Tr} \otimes \text{id}) \Delta = (\text{Tr} \otimes \varepsilon_s) \Delta\) and \((\text{id} \otimes \text{Tr}) \Delta = (\varepsilon_t \otimes \text{Tr}) \Delta\).

Since \(\text{Tr} \circ S = \text{Tr}\), we have

\[
(\text{Tr} \otimes \text{id}) \Delta(1) = \sum_{\alpha=1}^{K} \frac{1}{m_\alpha} \sum_r \text{Tr}(g_{rr}^{(\alpha)}) g_{rr}^{(\alpha)} = \sum_{\alpha=1}^{K} \frac{1}{m_\alpha} \sum_i \Lambda_{\alpha i} d_i \sum_r g_{rr}^{(\alpha)}
\]

\[
(\text{id} \otimes \text{Tr}) \Delta(1) = \sum_{\alpha=1}^{K} \frac{1}{m_\alpha} \sum_r \varepsilon_t(f_{rr}^{(\alpha)}) \text{Tr}(f_{rr}^{(\alpha)}) = \sum_{\alpha=1}^{K} \frac{1}{m_\alpha} \sum_i \Lambda_{\alpha i} d_i \sum_r f_{rr}^{(\alpha)}.
\]

This shows that (ii) is equivalent to the following condition :

\[
\sum_i \Lambda_{\alpha i} d_i = \lambda^{-1} m_\alpha, \quad \alpha = 1, \ldots, L.
\]

But \(d_i = \sum_{\beta=1}^{L} \Lambda_{\beta i} m_\beta\) since the inclusion \(K_s \subset K\) is unital. Hence, we can rewrite the last condition as

\[
\sum_{i=1}^{N} \sum_{\beta=1}^{L} \Lambda_{\alpha i} \Lambda_{\beta i} m_\beta = \lambda^{-1} m_\alpha, \quad \alpha = 1 \ldots L,
\]

which means precisely that \((\Lambda \Lambda^t) \vec{m} = \lambda^{-1} \vec{m}\).

(iii)⇒(iv). It is clear that \(\lambda^{-1}\) is a positive rational number since all entries of \((\Lambda \Lambda^t)\) and \(\vec{m}\) are positive integers. On the other hand, \(\lambda^{-1}\) is an algebraic integer, since it is an eigenvector of the integer matrix \((\Lambda \Lambda^t)\), therefore, \(\lambda^{-1}\) is an integer.

For all \(\alpha = 1, \ldots, L\) and \(r = 1, \ldots, m_\alpha\) define \(K_{\alpha r} = K f_{rr}^{(\alpha)}\). We have

\[
\dim(K_{\alpha r}) = \text{Tr}(f_{rr}^{(\alpha)}) = \sum_i a_{\alpha i} d_i = nm_\alpha.
\]

For all \(y, z \in K_{\alpha r}\) :

\[
E_s(y^* z) = f_{rr}^{(\alpha)} E_s(y^* z) f_{rr}^{(\alpha)} = (y, z) f_{rr}^{(\alpha)},
\]

where \((y, z)\) is a scalar since \(f_{rr}^{(\alpha)}\) is minimal in \(K_s\). Clearly, \((\cdot, \cdot)\) defines an inner product in \(K_{\alpha r}\), which is non-degenerate since \(E_s\) is faithful. Let us choose an orthonormal bases \(\{x_{\mu r}^{\alpha}\}, (\mu = 1, \ldots, nm_\alpha)\) in \(K_{\alpha r}\), \(\alpha = 1, \ldots L, r = 1, \ldots, m_\alpha\) in such a way that

\[
x_{\mu t}^{\alpha t} = x_{\mu r}^{\alpha r} f_{rr}^{(\alpha)} \quad \text{for all} \quad t, r = 1 \ldots m_\alpha, \quad \mu = 1 \ldots nm_\alpha.
\]

Then we have the following relation

\[
E_s((x_{\mu r}^{\alpha r})^* x_{\mu r'}^{\alpha' r'}) = \delta_{\alpha \alpha'} \delta_{\mu \mu'} f_{rr}^{(\alpha)} \quad \text{for all} \quad \alpha, \alpha'; \quad \mu, \mu'; \quad r, r'.
\]
We claim that
\[ x_\nu = \sum_\alpha \sum_{rs} \frac{1}{\sqrt{m_\alpha}} \exp \left( \frac{2sr\pi i}{m_\alpha} \right) x_{\nu+ (s-1)n}^{\alpha_r} \quad \nu = 1, \ldots, n, \]
is a basis of \( K \) over \( K_s \). Indeed:
\[
\sum_\nu x_\nu E_s(x_\nu^* x_\nu^\beta) = \sum_\nu \sum_{\alpha rs} \frac{1}{m_\alpha} x_{\nu+ (s-1)n}^{\alpha_r} E_s((x_{\nu+ (s-1)n}^{\alpha_r})^* x_\nu^{\beta})
= \frac{1}{m_\beta} \sum_r x_\nu^{\beta r} f_r^{(\beta)} = x_\nu^{\beta}
\]
for all \( \beta = 1 \ldots K, \ t = 1 \ldots m_\beta, \ \mu = 1 \ldots nm_\beta \). Next,
\[
E_s(x_\nu^* x_\nu^\beta) = \sum_{\alpha r' s'} \frac{1}{m_\alpha} \exp \left( \frac{2(s r' - s' r') \pi i}{m_\alpha} \right) E_s((x_{\nu+ (s-1)n}^{\alpha_r})^* x_{\nu+ (s'-1)n}^{\beta r'})
= \delta_{\nu k} \sum_{\alpha r' s} \frac{1}{m_\alpha} \exp \left( \frac{2s (r - r') \pi i}{m_\alpha} \right) f_{rr'}^{(\alpha)}
= \delta_{\nu k} \sum_{\alpha r} f_{rr}^{(\alpha)} = \delta_{\nu k}.
\]
Since ‘\( E_s \)-orthogonality’ implies linear independence over \( K_s \), we conclude that \( \{x_\nu\} \) is a basis for \( E_s \).

(iv)⇒(iii). If there is a basis for \( E_s : K \to K_s \) then the basic construction \( \langle K, e_{K_s} \rangle \) is isomorphic to \( M_n(K_s) \). This means that that the inclusion matrix \( B \) of the inclusion \( K_s \subset \langle K, e_{K_s} \rangle \) satisfies \( B \bar{m} = \lambda^{-1} \bar{m} \). But \( B = \Lambda \Lambda^t \).

(iv)⇔(v). We will prove (iv)⇒(v), the converse implication is completely analogous. If \( x = \sum_\nu x_\nu E_s(x_\nu^* x) \) then \( Sx^* = \sum_\nu Sx_\nu^* E_t(Sx_\nu Sx^*) \), since \( E_t = S \circ E_s \circ S \), and we can take \( y_\nu = Sx_\nu^* \), \( \nu = 1 \ldots \lambda \) as a basis for \( E_t \).

**Corollary 3.6** If the equivalent conditions of Theorem 3.5 are satisfied then \( \tau \) is a \( \lambda \)-Markov trace for the inclusion \( K_t \subset K \) (\( K_s \subset K \)).

**Proof.** We need to show that \( \Lambda^t \Lambda \bar{t} = \lambda^{-1} \bar{t} \), where \( \bar{t} \) is the ‘trace-vector’ corresponding to \( \tau \) (\( \mathbb{1} \), 3.2.3(ii)). Since \( \tau = \lambda \Tr \), we have \( \bar{t} = \lambda \bar{d} \), where \( \bar{d} = (d_1 \ldots d_N) \) is the ‘dimension-vector’ of \( K \). Using Theorem 3.5(iii) we compute
\[
\Lambda^t \Lambda \bar{t} = \lambda \Lambda^t \Lambda \bar{d} = \lambda \Lambda^t \Lambda \bar{m} = \Lambda \bar{m} = \bar{d} = \lambda^{-1} \bar{t}.
\]
Remark 3.7  (i) Proposition 3.2 says that $K$ is indecomposable iff the matrix $\Lambda$ is indecomposable in the sense of [9]. In this case Theorem 3.5(iii) implies that $\vec{m}$ is the Perron-Frobenius eigenvector of the matrix $(\Lambda \Lambda^t)$. It is well-known that in this case the corresponding eigenvalue $\lambda^{-1}$ is equal to the spectral radius of $(\Lambda \Lambda^t)$, so

$$\lambda^{-1} = ||\Lambda \Lambda^t|| = ||\Lambda||^2.$$  

(ii) Theorem 3.5(iv) and (v) show that an indecomposable weak Kac algebra $K$ is free over its Cartan subalgebras $K_s$ and $K_t$. In particular, $\dim K_s$ divides $\dim K$ and

$$\lambda^{-1} = \frac{\dim K}{\dim K_s}.$$  

(iii) Conditional expectations $E_s$ and $E_t$ are of index-finite type and their index is an integer scalar: $\text{Index } E_s = \text{Index } E_t = \lambda^{-1}$.

Corollary 3.8 If $K$ is indecomposable and $\dim K = p$, where $p$ is a prime, then $K \cong \mathbb{C}Z_p$, a group algebra of a simple abelian group.

Proof. Remark 3.7(ii) implies that Cartan subalgebras of $K$ must be 1-dimensional, so $K$ is a Kac algebra. But in this case the result is well known [11].

The $\lambda$-Markov condition is invariant under duality.

Proposition 3.9 $K$ satisfies the $\lambda$-Markov condition iff $K^*$ satisfies the $\lambda$-Markov condition (with the same $\lambda$).

Proof. Since $K$ satisfies the $\lambda$-Markov condition iff every its indecomposable component does, it sufficed to prove this statement in the case when $K$ is indecomposable. But this is trivial by Proposition 3.4 and Remark 3.7(ii), since $\dim K_s = \dim K^*_s$.

Connected weak Kac algebras (i.e., those with connected Bratteli diagram of the inclusion $K_s \subset K$) form a subclass of indecomposable weak Kac algebras important for the applications to subfactors in section 5.

Definition 3.10 A weak Kac algebra $K$ is connected if the inclusion $K_s \subset K$ is connected, i.e., $K_s \cap Z(K) = \mathbb{C}$ (or, equivalently, $K_t \cap Z(K) = \mathbb{C}$), where $Z(\cdot)$ denotes the center of an algebra. $K$ is biconnected if both $K$ and $K^*$ are connected.
Proposition 3.11 (cf. [15]) The following conditions are equivalent:

(i) $K$ is connected,

(ii) $K_*^a \cap K_*^t = \mathbb{C}$,

(iii) $p_\varepsilon$ is a minimal projection in $K$ (i.e., the counital representation of $K$ (cf. [14], Section 2.2) is irreducible).

Proof. (i)⇒(ii). Suppose that there is $\beta \in K_*^a \cap K_*^t$, $\beta \notin \mathbb{C}$. Since the Cartan subalgebras commute, $\beta$ must belong to $Z(K_*^a)$, the center of $K_*^a$. Consider the element $b \in K \cong K^{**}$ defined as $\langle b, \phi \rangle = \langle 1, \beta \phi \rangle$ for all $\phi \in K^*$. We can compute:

$$
\langle b, \phi(1)\phi(2) \rangle = \langle 1, \beta \phi(1)\phi(2) \rangle = \langle 1, \beta(1)\phi(1)\beta(2)\phi(2) \rangle = \beta \phi,
$$

$$
\phi(1)\langle b, \phi(2) \rangle = \phi(1)\langle 1, \beta \phi(2) \rangle = \varepsilon(1)\phi(1, \beta \varepsilon(2)) = \beta \phi,
$$

therefore, $b \in Z(K)$. Also, for all $\phi \in K^*$ we have

$$
\langle \varepsilon_s(b), \phi \rangle = \langle b, \varepsilon_s(\phi) \rangle = \langle 1, \beta \varepsilon_s(\phi) \rangle = \langle 1, \beta \phi \rangle = \langle b, \phi \rangle,
$$

therefore $\varepsilon_s(b) = b$ and $b \in K_s$. Thus, $Z(K) \cap K_s \neq \mathbb{C}1$, so $K$ is not connected.

(ii)⇒(i). If $K$ is not connected, then there exists $b \in Z(K) \cap K_t$, $b \notin \mathbb{C}$. Define $\beta \in K^*$ by $\beta : x \mapsto \varepsilon(bx)$. We have, for all $x \in K$:

$$
\langle \beta, \varepsilon_s(x) \rangle = \varepsilon(b(1))\varepsilon(x(1)) = \varepsilon(x(b(2))) = \varepsilon(xb),
$$

$$
\langle \beta, \varepsilon_t(x) \rangle = \varepsilon(b(2))\varepsilon(1(1)) = \varepsilon(bx) = \varepsilon(xb),
$$

from where $\varepsilon_s(\beta) = \beta = \varepsilon_t(\beta)$ and $K_*^a \cap K_*^t \neq \mathbb{C} \varepsilon$.

(i)⇒(iii). If there is a proper subprojection $q$ of $p_\varepsilon$ then from the formula for $\Delta(p_\varepsilon)$ we get $\varepsilon_s(q) \neq 1$ and $\varepsilon_s(q) \in Z(K)$, so $K$ is not connected.

(iii)⇒(i). Let $P_\varepsilon$ be the central support of $p_\varepsilon$. It was shown in [14] that the quotient map $K \mapsto P_\varepsilon K$ (which is a homomorphism of weak Kac algebras) is one-to-one on the Cartan subalgebras. Therefore, $K_s \cap Z(K)$ is contained in $Z(P_\varepsilon K)$, and $K_s \cap Z(K) = \mathbb{C}$ when $p_\varepsilon$ is minimal.

A very general method of constructing connected weak Kac algebras from two-sided crossed-products was introduced in [13]. Let $H$ be a usual finite-dimensional Kac algebra (i.e., finite-dimensional Hopf $C^*$-algebra) acting on a finite-dimensional $C^*$-algebra $A$ on the left via $h \otimes a \mapsto (h \triangleright a)$ and on the right via $a \otimes h \mapsto (a \triangleleft h)$, $a \in A$, $h \in H$. 

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Definition 3.12 A two-sided crossed-product $C^*$-algebra $A \rtimes H \ltimes A$ is defined as a vector space $A \otimes H \otimes A$ with multiplication and involution given by

\[(b \otimes h \otimes a)(b' \otimes h' \otimes a') = b(h(1) \triangleright b') \otimes h(2)h'(1) \otimes (a \triangleleft h'(2))a',\]

\[(b \otimes h \otimes a)^* = (h^*_1 \triangleright b^*) \otimes h^*_2 \otimes (a^* \triangleleft h^*_2),\]

for all $a,a',b,b' \in A, h,h' \in H$.

Let $\{f^\alpha_{rs}\}$ be a system of matrix units in $A = \bigoplus \alpha M_{m\alpha}(\mathbb{C})$ and $\tau$ be the trace of the left regular representation of $A$ on itself. Given a $*$-anti-automorphism $S_A$ of $A$ define

\[1 \otimes 1 = \sum_{\alpha rs} \frac{1}{m\alpha} f^\alpha_{rs} \otimes S_A(f^\alpha_{sr})\]

(any projection $P \in A \otimes A$ with the properties $\mu(id \otimes S_A^{-1})P = \mu(S_A \otimes id)P = 1$ and $(id \otimes \tau)P = (\tau \otimes id)P = 1$, where $\mu$ denotes the multiplication in $A$, has such a form, cf. [16], Lemma 2.3.6).

Suppose that the above actions of $H$ on $A$ satisfy

\[1 \otimes (h \triangleright 1) = (1 \otimes h) \otimes 1 \quad \forall h \in H.\]

Proposition 3.13 ([13]) There is a structure of a weak Kac algebra on $K = A \rtimes H \ltimes A$ defined by

\[\Delta(b \otimes h \otimes a) = (b \otimes h(1) \otimes 1_{(1)}) \otimes ((h(2) \triangleright 1_{(2)}) \otimes h(3) \otimes a)\]

\[\varepsilon(b \otimes h \otimes a) = \tau((S_H(h) \triangleright b)a)\]

\[S(b \otimes h \otimes a) = S_A(a) \otimes S_H(h) \otimes S_A^{-1}(b)\]

The source and target Cartan subalgebras of $K$ are

\[K_s = \{1 \otimes 1 \otimes a \mid a \in A\}, \quad K_t = \{b \otimes 1 \otimes 1 \mid b \in A\}.\]

Clearly, $K_s \cap K_t = \mathbb{C}$, so $K^*$ is connected by Proposition 3.11. It is easy to check that if the fixed points algebras

\[A^H = \{b \in A \mid h \triangleright b = \varepsilon(h)b \quad \forall h \in H\},\]

\[H_A = \{a \in A \mid a \triangleleft h = \varepsilon(h)a \quad \forall h \in H\}\]

are trivial, then $K_s \cap Z(K) = \mathbb{C}$, and $K$ is biconnected.

In the special case when $H = \mathbb{C}$ acts trivially on $A$, $K^*$ is isomorphic to the full matrix algebra $M_d(\mathbb{C})$, $d = \dim A$. Such weak Kac algebras were classified in [16].
Example 3.14 Let $A$ be a Kac subalgebra of $H^*$ and actions of $H$ on $A$ be induced by the dual actions of $H$ on $H^*$:

$$h \triangleright a = a_{(1)} \langle h, a_{(2)} \rangle, \quad a \triangleleft h = \langle h, a_{(1)} \rangle a_{(2)}.$$ 

Define $1_{(1)} \otimes 1_{(2)} = \Delta(p)$, where $p$ is the Haar projection in $A$. Since $p$ is cocommutative, the condition $1_{(1)} \otimes (h \triangleright 1_{(2)}) = (1_{(1)} \triangleleft h) \otimes 1_{(2)}$ is satisfied for all $h \in H$ and $K = A \triangleright H \triangleleft A$ is a biconnected weak Kac algebra with $\lambda^{-1} = (\dim H)(\dim A)$.

In Section 6 we derive some arithmetic properties of biconnected weak Kac algebras from the existence of a minimal action of any such algebra on the hyperfinite II$_1$ factor.

4 Duality for actions

In this section $K$ is a weak Kac algebra satisfying the $\lambda$-Markov condition (e.g., indecomposable) and acting on a $C^*$-algebra $A$. Left actions are assumed everywhere; the right counterparts of the results below can be obtained similarly and are left to the reader.

Lemma 4.1 For all $a \in A$,

$$(n \triangleright a) = a(n \triangleright 1), \quad n \in K_s \quad (n \triangleright a) = (n \triangleright 1)a, \quad n \in K_t.$$

Proof. For all $n \in K_s$ we have

$$n \triangleright a = (n_{(1)} \triangleright a)(n_{(2)} \triangleright 1) = (1_{(1)} \triangleright a)(1_{(2)} n \triangleright 1) = a(n \triangleright 1),$$

and similarly the second statement.

Proposition 4.2 The map $E_A: A \triangleright K \to A$, defined as

$$E_A([a \otimes h]) = a(E_t(h) \triangleright 1), \quad a \in A, \ h \in K,$$

is a faithful conditional expectation. If \{y_\nu\}_{\nu=1}^n is a basis for $E_t$ as in Theorem 3.5(v), then \{[1 \otimes y_\nu]\}_{\nu=1}^n is a basis for $E_A$. 

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Proof. For all $z \in K$, we compute

$$E_A([a \otimes zh]) = a(E_t(zh) \triangleright 1) = a(zE_t(h) \triangleright 1)$$

$$= a(z \triangleright 1)(E_t(h) \triangleright 1) = E_A([a(z \triangleright 1) \otimes h]),$$

therefore $E_A$ is well-defined on $A \rtimes K$. Clearly, $E_A|_A = id_A$. Let us check other properties (using Lemma 4.1): 

$$E_A([a \otimes [b \otimes h][c \otimes 1]]) = E_A([ab(h(1) \triangleright c) \otimes h(2)])$$

$$= ab(h(1) \triangleright c)(E_t(h(2)) \triangleright 1) = ab(E_t(h) \triangleright c)$$

$$= ab(E_t(h) \triangleright 1)c = aE_A([b \otimes h])c,$$

for all $a, b, c \in A$ and $h \in K$, so $E_A$ is a conditional expectation. We have $h = \Sigma_{\nu} y_{\nu} E_t(y_{\nu}'h) = \Sigma_{\nu} E_t(h y_{\nu}) y_{\nu}'$ for all $h \in K$ by Theorem 3.5(v), so

$$[a \otimes h] = \Sigma_{\nu} [a \otimes E_t(h y_{\nu}) y_{\nu}'] = \Sigma_{\nu} [a(E_t(h y_{\nu}) \triangleright 1) \otimes 1][1 \otimes y_{\nu}']$$

$$= \Sigma_{\nu} [E_A([a \otimes h][1 \otimes y_{\nu}']) \otimes 1][1 \otimes y_{\nu}'],$$

applying the involution we get

$$[a \otimes h] = \Sigma_{\nu} [1 \otimes y_{\nu}][E_A([1 \otimes y_{\nu}'][a \otimes h]) \otimes 1], \quad a \in A, \ h \in K.$$ 

Therefore, every $x \in A \rtimes K$ can be written as $x = \Sigma_{\nu} [1 \otimes y_{\nu}][a_{\nu} \otimes 1]$ for some $a_{\nu}$, $\nu = 1 \ldots n$. Since $E_A([1 \otimes y_{\nu}y_{\kappa}]) = \delta_{\nu,\kappa}$, we have

$$E_A(x^{*}x) = \Sigma_{\nu, \kappa} E_A([a_{\nu}^{*} \otimes 1][1 \otimes y_{\nu}'][1 \otimes y_{\kappa}][a_{\kappa} \otimes 1] = \Sigma_{\nu} a_{\nu}^{*}a_{\nu},$$

and $x = 0$ iff $E_A(x^{*}x) = 0$ iff $a_{\nu} = 0$ ($\forall \nu$). This proves that $E_A$ is faithful and $\{(1 \otimes y_{\nu})\}_{\nu=1 \ldots n}$ is a basis for $E_A$.

Remark 4.3 Index $E_A = Index E_t = \lambda^{-1}$.

In what follows we consider $C^*$-algebras $A, K, K^*$, $A \rtimes K$, and $K \rtimes K^*$ as subalgebras of $(A \rtimes K) \rtimes K^*$ in an obvious way with inclusion maps denoted by $i_A, i_K$ etc.

Lemma 4.4 Let $e_A = i_{K^*}(\tau) \in (A \rtimes K) \rtimes K^*$. Then

(i) $e_A i_A \rtimes K(x) e_A = i_A(E_A(x)) e_A$ for all $x \in A \rtimes K$,

(ii) the map $A \ni a \mapsto i_A(a)e_A \in (A \rtimes K) \rtimes K^*$ is injective.
Moreover, \( E_{A \rtimes K}(e_A) = \lambda \).

Proof. For all \( a \in A, \ h \in K \) we compute
\[
e_A i_{A \rtimes K}([a \otimes h]) e_A = \bigl[ \tau(1) \triangleright [a \otimes h] \otimes \tau(2) \bigr]
= [\tau(1) \triangleright [a \otimes h] \otimes \varepsilon(\tau(2))] [1_{A \rtimes K} \otimes \tau]
= [\tau \triangleright [a \otimes h] \otimes \varepsilon][1_{A \rtimes K} \otimes \tau] = i_A(E_a([a \otimes h])) e_A,
\]
which proves (i). Next, we compute
\[
E_{A \rtimes K}(i_A(a) e_A) = E_{A \rtimes K}([a \otimes 1] \otimes \tau) = [a \otimes 1]\{\lambda \varepsilon \triangleright [1 \otimes 1]\} = \lambda i_A(a),
\]
thus proving that the map \( a \mapsto i_A(a) e_A \) is injective. Taking \( a = 1 \) in the last formula, we obtain \( E_{A \rtimes K}(e_A) = \lambda \).

**Proposition 4.5** \( (A \rtimes K) \rtimes K^* = (A \rtimes K) e_A (A \rtimes K) \).

Proof. Observe that for all \( a \in A, \ g, h \in K \)
\[
i_{A \rtimes K}([a \otimes h]) e_A i_K(g) = i_A(a) (i_K(h) e_A i_K(g)).
\]
Since \( (A \rtimes K) \rtimes K^* = \text{span}\{i_A(a) i_{K \rtimes K^*}(x) \mid a \in A, \ x \in K \rtimes K^*\} \) it suffices to show that \( K \rtimes K^* = KeK \) (here \( e_K = [1_0 \otimes \tau] \in K \rtimes K^* \).

For this purpose, we need to show that every element of \( K \rtimes K^* \) can be written as a linear combination of elements \( i_K(h) e_K i_K(g) \), \( h, g \in K \).

Let \( \{\phi_{ij}^\gamma\} \) be a system of matrix units in \( K^* \). Since \( \tau \) is the normalized Haar projection in \( K^* \), we have
\[
\Delta(\tau) = \sum_{\gamma} \frac{1}{c_{\gamma}} \sum_{ij} \phi_{ij}^\gamma \otimes S(\phi_{ji}^\gamma),
\]
for some integers \( c_{\gamma} \). Let \( \{v_{ij}^\gamma\} \) be the system of comatrix units in \( K \) dual to \( \{\phi_{ij}^\gamma\} : \Delta(v_{ij}^\gamma) = \sum_k v_{ik}^\gamma \otimes v_{kj}^\gamma, \varepsilon(v_{ij}^\gamma) = \delta_{ij} \).

Fix \( x \in K \) and let \( h_k = x S(v_{ik}^\gamma), \ g_k = c_{\gamma} v_{kl}^\gamma \) (\( x \in K \)) for some \( \gamma, p, l \) (\( k = 1 \ldots m_{\gamma} \)). Then
\[
\sum_k i_K(h_k) e_K i_K(g_k) = \sum_{kijm} [x S(v_{ik}^\gamma) v_{km}^\gamma \otimes \langle \phi_{ij}^\gamma, v_{ml}^\gamma \rangle S(\phi_{ji}^\gamma)]
= \sum_{km} [x S(v_{ik}^\gamma) v_{km}^\gamma \otimes S(\phi_{lm}^\gamma)]
= \sum_{m} [x \varepsilon_s(v_{pm}^\gamma) \otimes S(\phi_{lm}^\gamma)]
= [x \otimes \sum_m \langle \varepsilon(1), v_{pm}^\gamma \rangle \varepsilon(2) S(\phi_{lm}^\gamma)].
\]
Since \( x \in K \) is arbitrary, it remains to show that elements of the form 
\[
\psi^\gamma_{lp} = \sum_m \langle \varepsilon^{(1)}, v^\gamma_{pm} \rangle \varepsilon^{(2)} S(\phi^\gamma_{lm})
\]
form a linear basis for \( K^* \). We have
\[
\langle S(\psi^\gamma_{lp}), v^\beta_{pq} \rangle = \sum_m \langle \phi^\gamma_{lm}, v^\beta_{pq} \rangle \langle \varepsilon^{(1)}, v^\beta_{pm} \rangle S(\varepsilon^{(2)}) S(\phi^\gamma_{lm}) = \delta_{\gamma \beta} \delta_{lp} \sum_m \langle \varepsilon, v^\gamma_{mq} S(v^\gamma_{pm}) \rangle = \delta_{\gamma \beta} \delta_{lp} \delta_{pq},
\]
therefore, \( \psi^\gamma_{lp} = S(\phi^\gamma_{lp}) \).

**Corollary 4.6** \((A \rhd K) \rhd K^* \cong (A \rhd K, e_A)\), i.e., \((A \rhd K) \rhd K^*\) is the basic construction for the conditional expectation \( E_A \).

**Proof.** Propositions 4.2 and 4.5 show that \((A \rhd K) \rhd K^*\) is generated by \(A \rhd K\) and projection \( e_A \), in the way characterizing the basic construction (see 2.3).

The following result is an analogue of the Takesaki duality theorem for actions of Kac algebras 3 and Hopf algebras 2.

**Theorem 4.7 (Duality for actions)** Let \( K \) be a weak Kac algebra satisfying the \( \lambda \)-Markov condition, acting on a \( C^* \)-algebra \( A \). Then
\[
(A \rhd K) \rhd K^* \cong A \otimes M_n(\mathbb{C}), \quad \text{where } n = \lambda^{-1}.
\]

**Proof.** By Proposition 4.2 there is a basis for \( E_A \), therefore \((A \rhd K, e_A) \cong A \otimes M_n(\mathbb{C})\), and the result follows from Corollary 4.6.

**Lemma 4.8** Let \( K \) be a weak Kac algebra acting on the right on a \( * \)-algebra \( A \). Then \( K_t \subset A' \cap K \rhd A \).

**Proof.** If \( z \in K_t \), then
\[
i_A(a)i_K(z) = [z^{(1)} \otimes (a \rhd z^{(2)})] = [z1^{(1)} \otimes (a \rhd 1^{(2)})]
\]
\[
= [z \otimes a] = i_K(z)i_A(a),
\]
thus, \( K_t \subset A' \cap K \rhd A \).

**Definition 4.9** A right action of \( K \) on \( A \) is minimal if \( K_t = A' \cap K \rhd A \).
5 Construction of a minimal action of a biconnected weak Kac algebra on the hyperfinite II$_1$ factor.

In this section we assume that $K$ is a biconnected weak Kac algebra, in particular it satisfies the $\lambda$-Markov condition for some $\lambda = n^{-1}$.

**Lemma 5.1** Let $K$ act on a finite-dimensional $C^*$-algebra $A$. Suppose that $tr$ is a trace on $A\rtimes K$, and $E_A$ from Proposition 4.2 is the $tr$-preserving conditional expectation. Then $tr_1 = tr \circ E_{A\rtimes K}$ is a trace on $(A\rtimes K, e_A)$, extending $tr$ and satisfying $E_{A\rtimes K}(e_A) = \lambda$. In other words, if $tr$ is a trace on $A\rtimes K$ such that $E_A$ preserves it, then $tr$ is a $\lambda$-Markov trace for the inclusion $A \subset A\rtimes K$, and $tr_1$ is its $\lambda$-Markov extension to $(A\rtimes K, e_A)$.

**Proof.** Clearly, $tr_1$ is a positive functional on $(A\rtimes K, e_A)$ extending $tr$. Let us show that $tr_1$ is a trace. By Lemma 4.4, $E_{A\rtimes K}(e_A) = \lambda$, therefore

$$tr_1((x_1e_Ay_1)(x_2e_Ay_2)) = tr_1((x_1E_A(y_1x_2)e_Ay_2)) = \lambda tr(E_A(y_1x_2)E_A(y_1x_2)) = tr_1((x_2e_Ay_2)(x_1e_Ay_1)),$$

for all $x_1, y_1, x_2, y_2 \in A\rtimes K$. Since $(A\rtimes K, e_A)$ is spanned by elements of the form $xe_Ay$, $(x, y \in A\rtimes K)$ the result follows from [10], 3.2.5.

**Remark 5.2** In conditions of Lemma 5.1, $e_A$ is the Jones projection for the inclusion $i_K(K) \subset A\rtimes K$ with respect to the Markov trace $tr$ and $E_{A\rtimes K} : (A\rtimes K, e_A) \rightarrow A\rtimes K$ is the tr-preserving conditional expectation.

Note that the map $\phi \mapsto (\phi \triangleright 1)$ gives an isomorphism between $K^*_t$ and $K_s$ in the crossed product algebra $K\rtimes K^*$.

**Proposition 5.3** Suppose that $K$ is connected and let $tr$ be the unique Markov trace for the inclusion $i_K(K) \subset K\rtimes K^*$. Then

$$i_K(K) \cup i_K(K_s) \equiv i_K^*(K_t^*) \cup i_K^*(K_s^*) \subset i_K^*(K^*)$$

is a symmetric commuting square with respect to $tr$. 


Proof. By Corollary 3.6, \( \tau \) is a Markov trace for the inclusion \( K_t \subset K \), and \( E_t \) is the \( \tau \)-preserving conditional expectation. Since \( K \bowtie K^* = (K_t \bowtie K) \bowtie K^* \), it follows from Lemma 5.1 that \( \text{tr} \) extends \( \tau \) and \( E_K : K \bowtie K^* \to K \) is the \( \tau \)-preserving conditional expectation. We have

\[
E_K(i_{K^*}(\phi)) = E_K([1 \otimes \phi]) = i_K(\phi \bowtie 1) \in i_K(K_s),
\]

for all \( \phi \in K^* \). This proves that the square is commuting. It is symmetric since \( K_\bowtie K^* = i_K(K) i_{K^*}(K^*) \).

Corollary 4.6 implies that the sequence

\[
K_t \subset K \subset K_\bowtie K^* \subset K_\bowtie K^* \bowtie K \subset \cdots \subset M
\]

is the Jones tower for the inclusion \( K_t \subset K \) on \( K \). When \( K \) is connected, all the inclusions in this sequence are connected and the union of these \( C^* \)-algebras admits a unique tracial state, consequently, its von Neumann algebra completion \( M \) with respect to this trace is a copy of the hyperfinite \( \text{II}_1 \) factor. Using the standard procedure of iterating the basic construction we can construct a von Neumann subalgebra \( N \subset M \) from the above symmetric commuting square.

Proposition 5.4 The lattice of \( C^* \)-algebras obtained by iterating the basic construction (in the horizontal direction) for the symmetric commuting square from Proposition 5.3 is given by two sequences of alternating crossed products with \( K \) and \( K^* \):

\[
K \subset K \bowtie K^* \subset K \bowtie K^* \bowtie K \subset \cdots \subset M
\]

where we identify all \( C^* \)-subalgebras with their images in \( M \).

Proof. Identities \( K^* \bowtie K = \langle K, e_K \rangle \), \( K^* \bowtie K \bowtie K^* = \langle K^* \bowtie K, e_K \bowtie K^* \rangle \) etc. follow immediately from Proposition 4.5.

Proposition 5.5 There is a *-isomorphism between finite dimensional \( C^* \)-algebras

\[
A' = \underbrace{K \bowtie K^* \bowtie \cdots \bowtie K \bowtie K^*}_{2r \text{ factors}} \quad \text{and} \quad B' = \underbrace{K \bowtie K^* \bowti K \bowtie K^*}_{2r \text{ factors}}.
\]

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given by the ‘identity’ map
\[ [h^1 \otimes \phi^1 \otimes \cdots \otimes h^r \otimes \phi^r] \mapsto [h^1 \otimes \phi^1 \otimes \cdots \otimes h^r \otimes \phi^r], \]
where \( h^i \in K, \phi^i \in K^\ast \).

Proof. By the definition of crossed product, the above algebras are isomorphic to
\[ K \otimes_{K_i=K_i^s} K^\ast \otimes_{K_i=K_i^s} \cdots \otimes_{K_i=K_i^s} K^\ast \]
as vector spaces. By Theorem 4.7 we know that these algebras are isomorphic to
\[ M_{n^r}(\mathbb{C}) \otimes K, \]
where \( n = \lambda^{-1} \). To see that the ‘identity’ map defines a *-algebra isomorphism, it suffices to note that
\[ [h^1 \otimes \phi^1 \otimes \cdots \otimes h^r \otimes \phi^r] \cdot A [g^1 \otimes \psi^1 \otimes \cdots \otimes g^r \otimes \psi^r] = [h^1 (\phi^1 \triangleright g^1) \otimes \cdots \otimes h^r (\phi^r \triangleright g^r) \otimes \phi^r] \]
for all \( h^i, g^i \in K, \phi^i, \psi^i \in K^\ast, i = 1 \ldots r \), i.e. multiplications in \( A^r \) and \( B^r \) are the same.

Corollary 5.6 The lattice of algebras from Proposition 5.4 is isomorphic to
\[ K \subset K \triangleright K^\ast \subset K \triangleright K^\ast \triangleright K \subset \cdots \subset M \]
\[ K_s \subset K^\ast \subset K^\ast \triangleright K \subset \cdots \subset N. \]

Proof. Clearly, the isomorphisms constructed in Proposition 5.5 are compatible with all inclusions of the lattice from Proposition 5.4.

Our next goal is to show that there is a right action of \( K \) on \( N \) such that \( M \cong K \triangleleft N \).

Proposition 5.7 Let \( i_K : h \mapsto [h \otimes \varepsilon \otimes 1 \otimes \cdots] \) be the inclusion of \( K \) in \( M \), \( E_N : M \to N \) be the trace preserving conditional expectation. Then the map
\[ x \triangleleft h = \lambda^{-1} E_N(i_K (p_x) x i_K (h)), \quad x \in N, h \in K \]
defines a right action of \( K \) on \( N \) such that \( M = K \triangleleft N \) (cf. [22], Sect. 5).
Proof. There is a right action of \( K \) on the \(*\)-subalgebra given by the union of the generating sequence of \( C^* \)-algebras of \( N \):

\[
[\phi \otimes g \otimes \cdots] \triangleright h = [(\phi \triangleleft h) \otimes g \otimes \cdots], \quad h, g \in K, \phi \in K^*.
\]

We have

\[
[\phi \otimes g \otimes \cdots] \triangleright h = \lambda^{-1}[(\varepsilon \triangleleft E_s(p_{\varepsilon}))(\phi \triangleleft h) \otimes g \otimes \cdots] = \lambda^{-1}E_N([p_{\varepsilon} \otimes (\phi \triangleleft h) \otimes g \otimes \cdots])
\]

to a weakly continuous action of \( K \) on \( N \). Clearly, \( K \triangleright N = i_k(K)N = M \).

Corollary 5.8 \([M : N] = \lambda^{-1}\).

Proof. Follows from Remark 4.3 and Proposition 5.1.9 in [10].

Let us compute the higher relative commutants of the inclusion \( N \subset M \).

Lemma 5.9 Let \( K \) act on the left on a \( C^* \)-algebra \( A \) then

\[
i_{K^*}(K^*)' \cap i_{A \otimes K}(A \otimes K) \cap (A \otimes K) \otimes K^* = i_A(A).
\]

Proof. Let \( C = i_{K^*}(K^*)' \cap i_{A \otimes K}(A \otimes K) \cap (A \otimes K) \otimes K^* \) and \( x \in C \).

Recall that \( e_A = i_{K^*}(\tau) \). Then \( xe_A = e_Axe_A = E_A(x)e_A \) and since the map \( A \otimes K \ni x \mapsto i_{A \otimes K}(x)e_A \) is injective (Lemma 4.4), it follows that \( x \in i_A(A) \) and \( C \subset i_A(A) \).

Conversely, for all \( a \in A, \phi \in K^* \) we have

\[
i_{K^*}(\phi)i_A(a) = [i_{A \otimes K} \otimes \phi][[\phi_{(1)} \triangleright [a \otimes 1] \otimes \varepsilon]] = [(\phi_{(1)} \triangleright [a \otimes 1]) \otimes \phi_{(2)}] = [[a \otimes 1] \triangleright [\phi_{(1)} \triangleright [1 \otimes 1] \otimes \phi_{(2)}] = [[a \otimes 1] \triangleright \phi] = i_A(a)i_{K^*}(\phi),
\]

therefore \( i_A(A) = C \).

Proposition 5.10 Let \( N \subset M = M_0 \subset M_1 \subset M_2 \cdots \) be the Jones tower constructed from the inclusion \( N \subset M \). Then

\[
N' \cap M_n \cong \cdots \otimes K \otimes K^* \otimes K_t, \quad n \geq 0
\]

\[
M' \cap M_n \cong \cdots \otimes K^* \otimes K \otimes K_t^*, \quad n \geq 1.
\]

In particular, the action of \( K \) is minimal.
Proof. Iterating the basic construction for the commuting square from Proposition 5.3 in the vertical direction and using Proposition 5.5 we get the lattice

\[
\begin{array}{c}
\cdots \quad \cdots \\
\cup \\
K^*\triangleleft K \\ \\
K \\
\cup \\
K_t \equiv K_s \\
\cup \\
K^* \\
\end{array}
\]

The Ocneanu compactness argument ([10], 5.7) and Lemma 5.9 imply that

\[
N' \cap M = K_t, \quad N' \cap M_1 = K^*, \quad N' \cap M_2 = K \triangleright\triangleleft K^* \ldots
\]

Similarly, one computes the relative commutants for \( M \).

**Corollary 5.11** ([15]) *The inclusion \( N \subset M \) is of depth 2.*

Proof. We have seen in Section 4 that \( K \triangleright\triangleleft K^* \cong K_t \otimes M_n(\mathbb{C}) \), where \( n = \lambda^{-1} \). Therefore, \( \dim Z(N' \cap M) = \dim Z(N' \cap M_2) \), and \( N \subset M \) is of depth 2.

**Corollary 5.12** *The \( \lambda \)-lattice of higher relative commutants [18] of the inclusion \( N \subset M \) is given by*

\[
\begin{array}{c}
\mathbb{C} \subset K_t \equiv K_s \subset K \subset K^* \triangleleft K \subset K \triangleright\triangleleft K^* \subset \cdots \\
\cup \\
K_t \equiv K_s \\
\cup \\
K^* \\
\end{array}
\]

**Remark 5.13** In a similar way one can construct a left minimal action of a biconnected weak Kac algebra on the hyperfinite \( \text{II}_1 \) factor.

6 Examples of subfactors and arithmetic properties of biconnected weak Kac algebras

Let \( K \) be a biconnected weak Kac algebra. Recall the notation

\[
K \cong \bigoplus_{i=1}^{N} M_{d_i}(\mathbb{C}), \quad K_s \cong K_t \cong \bigoplus_{\alpha=1}^{L} M_{m_{\alpha}}(\mathbb{C}),
\]

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from Section 2.1. Let us also denote $d = \dim K$. We have $\dim K = d\lambda^{-1}$.

Reducing the inclusion $N \subset M = K\rhd N$ constructed in Section 5 by a minimal projection $q \in N' \cap M = K_t$ we get an irreducible inclusion $qN \subset qMq$ of hyperfinite $II_1$ factors with index $[qMq : qN] = \tau(q)^2\lambda^{-1}$, where $\tau$ is the normalized trace on $M$ ( $qN \subset qMq$ is of finite depth [1], and therefore extremal, see [19, 1.3.6]). But $\tau(q) = \frac{m_\alpha}{d}$, when $q \in M_{Ma}(\mathbb{C})$, therefore

$$[qMq : qN] = \frac{m_\alpha^2 \lambda^{-1}}{d^2}.$$ 

Note that since $qN \subset qMq$ has a finite depth, its index is an algebraic integer. But by Theorem 3.3, $\lambda^{-1}$ is an integer, so $[qMq : qN]$ is rational. Therefore, $[qMq : qN]$ is an integer.

**Corollary 6.1** $d^2$ divides $m_\alpha^2 \lambda^{-1}$ for all $\alpha$.

**Corollary 6.2** If $\lambda^{-1} = p$ is a prime, then $K \cong \mathbb{CZ}_p$.

*Proof.* By the previous corollary we must have $d = 1$, so $\dim K = d\lambda^{-1} = p$ and the result follows from Corollary 3.8.

Next, reducing the inclusion $M \subset M_2$ by a minimal projection $q$ from the relative commutant $M' \cap M_2 = K$ we get an irreducible inclusion $qM \subset qM_2q$. Clearly, this inclusion depends only on the equivalence class of $q$, so inclusions of the above type are in one-to-one correspondence with irreducible representations of $K$. The index is

$$[qM_2q : qM] = \tau(q)^2[M_2 : M] = \tau(q)^2\lambda^{-2} = \left(\frac{d_i}{d}\right)^2,$$

whenever $q \in M_{d_i}(\mathbb{C})$. Again, the index must be an integer, so we get the following arithmetic property of biconnected weak Kac algebras.

**Corollary 6.3** The dimension of a Cartan subalgebra of $K$ divides the degree of any irreducible representation of $K$, i.e. $d$ divides $d_i$ for all $i$. In particular, $d^2$ divides $\dim K$, and $d$ divides $\lambda^{-1} = [M : N]$.

Finally, let us remark that considering the biconnected weak Kac algebra $K = H \bowtie H^* \bowtie H$ constructed from a Kac algebra $H$ as in Example 3.14, we can associate an irreducible subfactor with any irreducible representation of $H$ (since we have $K_t = H$ in this case).
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