On Secure Distributed Linearly Separable Computation

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Abstract

Distributed linearly separable computation, where a user asks some distributed servers to compute a linearly separable function, was recently formulated by the same authors and aims to alleviate the bottlenecks of stragglers and communication cost in distributed computation. For this purpose, the data center assigns a subset of input datasets to each server, and each server computes some coded packets on the assigned datasets, which are then sent to the user. The user should recover the task function from the answers of a subset of servers, such the effect of stragglers could be tolerated.

In this paper, we formulate a novel secure framework for this distributed linearly separable computation, where we aim to let the user only retrieve the desired task function without obtaining any other information about the input datasets, even if it receives the answers of all servers. In order to preserve the security of the input datasets, some common randomness variable independent of the datasets should be introduced into the transmission.

We show that any non-secure linear-coding based computing scheme for the original distributed linearly separable computation problem, can be made secure without increasing the communication cost (number of symbols the user should receive). Then we focus on the case where the computation cost of each server (number of datasets assigned to each server) is minimum and aim to minimize the size of the randomness variable (i.e., randomness size) introduced in the system while achieving the optimal communication cost. We first propose an information theoretic converse bound on the randomness size. We then propose secure computing schemes based on two well-known data assignments, namely

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fractional repetition assignment and cyclic assignment. These schemes are optimal subject to using these assignments. Motivated by the observation of the general limitation of these two schemes on the randomness size, we propose a computing scheme with novel assignment, which strictly outperforms the above two schemes. Some additional optimality results are also obtained.

Index Terms

Distributed computation; linearly separable function; security

I. INTRODUCTION

Distributed linearly separable computation, which is a generalization of many existing distributed computing problems such as distributed gradient coding [1] and distributed linear transform [2], was originally proposed in [3] considering two important bottlenecks in the distributed computation systems: communication cost and stragglers. In this computation scenario, a user aims to compute a function of $K$ datasets $(D_1, \ldots, D_K)$ on a finite field $\mathbb{F}_q$ through $N$ distributed servers. The task function can be seen as $K_c$ linear combinations of $K$ intermediate messages (the $n^{th}$ intermediate message $W_n$ is a function of dataset $D_n$ and contains $L$ symbols). The problem contains three phases, assignment, computing, decoding. During the assignment phase, the data center with access to the $K$ datasets assigns $M$ datasets to each server, where $M$ represents the computation cost of each server. During the computing phase, each server first computes the intermediate message of each dataset assigned to it, and then transmits a coded packet of the computed intermediate messages to the user. During the decoding phase, from the answers of any $N_r$ servers, the user should recover the task function such that the system can tolerate $N - N_r$ stragglers. The worst-case number of symbols (normalized by $L$) needed to be received is defined as the communication cost. The objective is to minimize the communication cost for each given computation cost. The optimality results for some cases have been founded in the literature and are summarized below:

- $K_c = 1$. The computation problem reduces to the distributed gradient coding problem in [1]. When the computation cost is minimum (i.e., $M = \frac{K}{K_c}(N - N_r + 1)$), the gradient coding scheme in [1] achieves the optimal communication cost (equal to $N_r$) as proved in [3]. Then some extended gradient coding schemes were proposed in [4], [5] which characterize the optimal communication cost under the constraint of linear coding, for each possible computation cost.
• **Minimum computation cost** $M = \frac{K}{N}(N - N_r + 1)$. The optimal communication cost with the cyclic assignment (an assignment widely used in the related distributed computing problems) was characterized in [3], when the computation cost is minimum.

• For the general case, [6] proposed a computing scheme under some parameter regimes, which is order optimal within a factor of 2 under the constraint of the cyclic assignment.

In this paper, we consider a novel secure framework for this distributed linearly separable computation problem, where we aim to let the user only retrieve the desired task function without obtaining any other information about the $K$ datasets. We notice that this security model has been widely used in the literature in the context of secure multiparty computation [7], [8] and secure aggregation for federated learning [9], [10].

Let us focus on a small but instructive example in Fig. 1, where $K = N = 3$, $N_r = 2$, $K_c = 1$, $M = 2$, and the task function is $W_1 + W_2 + W_3$. Assume the field is $F_3$. We use the cyclic assignment in [1] to assign $D_1$ to servers 1, 3; assign $D_2$ to servers 1, 2; assign $D_3$ to servers 2, 3. In addition, for the sake of secure computation, the data center generates a randomness variable $Q$ uniformly over $[F_3]_L$, which is independent of the datasets, and assigns $Q$ to each server. In the computing phase of the novel proposed scheme, server 1 computes $2W_1 + W_2 + Q$; server 2 computes $W_3 + 2W_2 - Q$; server 3 computes $W_1 - W_3 + Q$. It can be seen that from the answers of any two servers, the user can recover the task function $W_1 + W_2 + W_3$. Moreover, even if the user receives the answers of all servers, it cannot get any other information about the messages (nor the datasets) because $Q$ is unknown to it. Notice that the communication cost in this example is 2, which is the same as the gradient coding scheme in [1].

The above example shows that it is possible to preserve the security of the datasets (except the task function) from the user. The main questions we ask in this paper are (i) do we need additional communication cost to satisfy this security constraint? (ii) how much randomness is required to guarantee security?

Compared to the existing works on coded distributed secure computation on matrix multiplication in [11]–[18], the main differences of the consider secure problem are as follows: (i) in the above existing works, the data center assigns the coded version of all input datasets to the distributed servers, while in the considered problem the assignment phase is uncoded; (ii) the above existing works aim to preserve the security of the input datasets from the servers, where each distributed server can only access the datasets assigned to it while in the considered problem we aim to preserve the security of the input datasets (except the task function) from
Our scheme: After receiving any two, the user recovers \( W_1 + W_2 + W_3 \).

Fig. 1: Secure distributed linearly separable computation with \( K = N = 3 \), \( N_r = 2 \), \( K_c = 1 \), and \( M = 2 \).

the user who may receive all answers of the servers.

Contributions

In this paper, we formulate the secure distributed linearly separable computation problem. We first show that any non-secure linear-coding based computing scheme for the original distributed linearly separable computation problem, can be made secure without increasing the communication cost. Then we focus on the secure distributed linearly separable computation problem where \( K_c = 1 \) and \( M = \frac{K}{N}(N - N_r + 1) \) (i.e., the computation cost is minimum), and aim to minimize the randomness size\(^1\) while achieving the optimal communication cost \( N_r \). Our contributions on this objective are as follows:

- For each possible assignment, we propose an information theoretic converse bound on the randomness size, which is also a converse bound on the randomness size while achieving the optimal communication cost.
- When \( N - N_r + 1 \) divides \( N \), we propose a secure computing scheme with the fractional repetition assignment in [1], which coincides with the proposed converse bound on the randomness size.

\(^1\) This randomness should be broadcasted from the data center to the servers and stored at the servers; thus reducing the randomness size can reduce the communication cost from the data center and the the storage cost at the servers.
• Under the constraint of the widely used cyclic assignment \cite{1, 3–6, 19, 20}, we propose an optimal secure computing scheme in the sense that minimum randomness size is achieved.

• Motivated by the observation that the computing scheme with the fractional repetition assignment can only work for the case where \( N - N_r + 1 \) divides \( N \) and that the computing scheme with the cyclic assignment is highly sub-optimal in terms of the randomness size, we propose a new computing scheme with novel assignment strategies. The novel computing scheme can cover the optimality results of the computing scheme with the fractional repetition assignment; in general it needs a lower randomness size while achieving the optimal communication cost than that of the computing scheme with the cyclic assignment. We also prove that it is optimal when \( \frac{N - N_r + 1}{\text{GCD}(N, N - N_r + 1)} \leq 4 \).

**Paper Organization**

The rest of this paper is organized as follows. Section \[ \text{II} \] introduces the secure distributed linearly separable computation problem. Section \[ \text{III} \] provides the main results in this paper and some numerical evaluations. Section \[ \text{IV} \] presents the proposed distributed computing schemes. Section \[ \text{V} \] concludes the paper and some of the proofs are given in the Appendices.

**Notation Convention**

Calligraphic symbols denote sets, bold lower-case letters denote vector, bold upper-case letters denote matrices, and sans-serif symbols denote system parameters. We use \(| \cdot |\) to represent the cardinality of a set or the length of a vector; \([a : b] := \{a, a + 1, \ldots, b\}\); \([n] := [1 : n]\); \(\mathbb{F}_q\) represents a finite field with order \( q \); \(M^T\) and \(M^{-1}\) represent the transpose and the inverse of matrix \(M\), respectively; \(I_n\) represents the identity matrix with dimension \( n \times n \); \(0_{m \times n}\) represents the zero matrix with dimension \( m \times n \); the matrix \([a; b]\) is written in a Matlab form, representing \([a, b]^T\); \((M)_{m \times n}\) represents the dimension of matrix \(M\) is \( m \times n \); \(M^{(S)}_r\) represents the sub-matrix of \(M\) which is composed of the rows of \(M\) with indices in \(S\) (here \(r\) represents ‘rows’); \(M^{(S)}_c\) represents the sub-matrix of \(M\) which is composed of the columns of \(M\) with indices in \(S\) (here \(c\) represents ‘columns’); \(\text{Mod}(b, a)\) represents the modulo operation on \(b\) with integer divisor \(a\) and in this paper we let \(\text{Mod}(b, a) \in \{1, \ldots, a\}\) (i.e., we let \(\text{Mod}(b, a) = a\) if \(a\) divides \(b\)); \(\text{GCD}(b, a)\) represents the Greatest Common Divisor of integers \(b\) and \(a\); we let \((x)_y = 0\) if \(x < 0\) or \(y < 0\) or \(x < y\). In this paper, for each set of integers \(S\), we sort the elements in \(S\) in an increasing order and denote the \(i^{th}\) smallest element by \(S(i)\), i.e., \(S(1) < \ldots < S(|S|)\).
II. SYSTEM MODEL

We formulate a \((K, N, N_r, K_c, M)\) secure linearly separable computation problem over the canonical user-server distributed system, as illustrated in Fig. 1. Compared to the distributed computing framework in [3], an additional security constraint will be added. The detailed system model is as follows.

The user wants to compute a function

\[
f(D_1, \ldots, D_K)
\]
on \(K\) independent datasets \(D_1, \ldots, D_K\). As the data sizes are large, the computing task function is distributed over a group of \(N\) servers. For distributed computation to be possible, we assume that the function is linearly separable with respect to the datasets, i.e., that there exist functions \(f_1, \ldots, f_K\) such that \(f(\cdot)\) can be written as

\[
f(D_1, \ldots, D_K) = g(f_1(D_1), \ldots, f_K(D_K))
\]

\[
= G \begin{bmatrix} W_1; & \ldots; & W_K \end{bmatrix},
\]

where \(G\) is a \(K_c \times K\) matrix and we model \(f_k(D_k), k \in [K]\) as the \(k\)-th message \(W_k\) and \(f_k(\cdot)\) is an arbitrary function. Notice that when \(K_c = 1\), without loss of generality, we assume that

\[
f(D_1, \ldots, D_K) = G \begin{bmatrix} W_1; & \ldots; & W_K \end{bmatrix} = W_1 + \cdots + W_K.
\]

In this paper, we assume that \(f(D_1, \ldots, D_K)\) contains one linear combination of the \(K\) messages, and that the \(K\) messages are independent. Each message is composed of \(L\) uniformly i.i.d. symbols over a finite field \(\mathbb{F}_q\) for some large enough prime-power \(q\). As in [3] we assume that \(\frac{K}{N}\) is an integer.

A computation scheme for our problem contains three phases, data assignment, computing, and decoding.

**Data assignment phase:** The data center/global server assigns each dataset \(D_k\) where \(k \in [K]\) to a subset of the \(N\) servers in an uncoded manner. The set of datasets assigned to server \(n \in [N]\) is denoted by \(Z_n\), where \(Z_n \subseteq [K]\). The assignment constraint is that

\[
|Z_n| \leq M.
\]

The assignment for all servers is denoted by \(Z = (Z_1, \ldots, Z_N)\).

As an additional problem constraint, we impose that the user learns no further information
about \((D_1, \ldots, D_K)\) other than the task function \(f(D_1, \ldots, D_K)\). To this purpose, the data center also generates a randomness variable \(Q \in \mathcal{Q}\), and assign \(Q\) to each server \(k \in [K]\). Notice that
\[
I(Q; D_1, \ldots, D_K) = I(Q; W_1, \ldots, W_K) = 0. \tag{4}
\]

The randomness size \(\eta\) measures the amount of randomness, i.e.,
\[
\eta = \frac{H(Q)}{L}. \tag{5}
\]

**Computing phase:** Each server \(n \in [N]\) first computes the message \(W_k = f_k(D_k)\) for each \(k \in \mathcal{Z}_n\). Then it generates
\[
X_n = \psi_n(\{W_k : k \in \mathcal{Z}_n\}, Q) \tag{6}
\]
where the encoding function \(\psi_n\) is such that
\[
\psi_n : [\mathbb{F}_q]^{|\mathcal{Z}_n|L} \times |Q| \to [\mathbb{F}_q]^{T_n}, \tag{7}
\]
and \(T_n\) represents the length of \(X_n\). Finally, server \(n\) sends \(X_n\) to the user.

**Decoding phase:** The computation scheme should tolerate \(N - N_s\) stragglers. As the user does not know a priori which servers are stragglers, the computation scheme should be designed so that from the answers of any \(N_s\) servers, the user can recover \(G[W_1; \ldots; W_K]\). Hence, for any subset of servers \(\mathcal{A} \subseteq [N]\) where \(|\mathcal{A}| = N_s\), with the definition
\[
X_S := \{X_n : n \in \mathcal{S}\}, \tag{8}
\]
for any set \(\mathcal{S} \subseteq [N]\), there exists a decoding function \(\phi_{\mathcal{A}}\) such that
\[
\phi_{\mathcal{A}}(X_{\mathcal{A}}) = G[W_1; \ldots; W_K], \tag{9a}
\]
\[
\phi_{\mathcal{A}} : [\mathbb{F}_q]^{|\sum_{n \in \mathcal{A}} T_n|} \to [\mathbb{F}_q]^{KcL}. \tag{9b}
\]
Notice that \(Q\) is unknown to the user, and thus \(Q\) cannot be used in the decoding procedure in (9). In order to protect the security, even if receiving the answers of all servers in \([N]\), the user cannot learn any information about the messages except the desired task function; it should satisfy that\(^2\)
\[
I(W_1, \ldots, W_K; X_{[N]}|G[W_1; \ldots; W_K]) = 0. \tag{10}
\]

\(^2\)Notice \(X_{[N]}\) is a function of \((W_1, \ldots, W_N)\) and \(Q\). By the data processing inequality, the security constraint in (10) is equivalent to
\[
I(D_1, \ldots, D_K; X_{[N]}|G[W_1; \ldots; W_K]) = 0.
\]
We denote the communication cost by,

\[ R := \max_{A \subseteq [N]:|A|=N_r} \frac{\sum_{n \in A} T_n}{L}, \tag{11} \]

representing the maximum normalized number of symbols received by the user from any \( N_r \) responding servers.

When the computation cost is minimum, it was proved in [3, Lemma 1] that each dataset is assigned to \( N - N_r + 1 \) servers and each server obtains \( M \) datasets, where

\[ M = |Z_1| = \cdots = |Z_N| = \frac{K}{N}(N - N_i + 1). \]

In this paper, we mainly focus on the case where \( K_c = 1 \) and the computation cost is minimum and search for the minimum communication cost \( R^* \). In addition, with the optimal communication cost, we aim to search the minimum randomness size \( \eta^* \) necessary to achieve the security constraint \((10)\).

### III. MAIN RESULTS

In this section, we present our main results.

In the following, we show that compared to the distributed linearly separable computation problem in [3], the optimal communication cost does not change when the security constraint in \((10)\) is added.

**Theorem 1.** For the \((K, N, N_r, K_c, M)\) secure distributed linearly separable computation problem with \( M = \frac{K}{N}(N - N_i + 1) \) and \( K_c = 1 \), the optimal communication cost is \( R^* = N_r \). \( \square \)

**Proof:**

Converse: Obviously, the converse bound for the distributed linearly separable computation problem in [3] which is without the security constraint in \((10)\) is also a converse bound for the considered secure distributed linearly separable computation problem. Hence, from [3] \((16a)\) we have

\[ R^* \geq N_r. \tag{12} \]

Achievability: We can use an extension of the distributed computing scheme in [3] for the case \( K_c = 1 \) and \( M = \frac{K}{R}(N - N_i + 1) \).

**Assignment phase.** The cyclic assignment is used, which was widely used in the existing works on the distributed computing problems [1], [3]–[6], [19], [20]. More precisely, we divide all the
datasets into $N$ non-overlapping and equal-length groups, where the $i^{th}$ group for each $i \in [N]$ is $\mathcal{G}_i = \{k \in [K] : \text{Mod}(k, N) = i\}$ containing $\frac{K}{N}$ datasets. We assign all datasets in $\mathcal{G}_i$ to the servers in

$$\mathcal{H}_i = \{\text{Mod}(i, N), \text{Mod}(i - 1, N), \ldots, \text{Mod}(i - N + N_r, N)\}. \tag{13}$$

Thus the set of groups assigned to server $n \in [N]$ is

$$\mathcal{Z}_n' = \{\text{Mod}(n, N), \text{Mod}(n + 1, N), \ldots, \text{Mod}(n + N - N_r, N)\} \tag{14}$$

with cardinality $N - N_r + 1$.

**Computing phase.** For each $i \in [N]$, we define a merged message as $W_i' = \sum_{k \in [K]: \text{Mod}(k, N) = i} W_k$. All datasets in $\mathcal{G}_i$ are assigned to each server in $\mathcal{H}_i$, which can compute $W_i'$. We then introduce $Q$ as a set of $N_r - 1$ independent randomness variables $Q_1, \ldots, Q_{N_r - 1}$, where $Q_j, j \in [N_r - 1]$ is uniformly i.i.d. over $[\mathbb{F}_q]^L$, and we assign $Q$ to each server.

In the computing phase, we let each server transmit one linear combination of merged messages and randomness variables, such that the user can receive $N_r$ linear combinations of merged messages from any set of $N_r$ responding servers, and then recover $\mathbf{F}'[W_1'; \ldots; W_N'; Q_1; \ldots; Q_{N_r - 1}]$ where

$$\mathbf{F}' = \begin{bmatrix}
1 & \cdots & 1 & 0 & \cdots & 0 \\
* & \cdots & * & + & \cdots & + \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
* & \cdots & * & + & \cdots & + \\
(S')_{(N_r \times N)} & (S')_{(N_r \times (N_r - 1))}
\end{bmatrix} \tag{15}$$

Notice that each ‘*’ represents a symbol uniformly i.i.d over $\mathbb{F}_q$, and ‘+’ represents the generic element of the matrix (i.e., $S'$ can be any full-rank matrix over $[\mathbb{F}_q]^{(N_r - 1) \times (N_r - 1)}$). The next step is to determine the transmission vector of each server $n \in [N]$, denoted by $s_n$ where the transmitted linear combination by server $n$ is

$$X_n = s_n \mathbf{F}' [W_1'; \ldots; W_N'; Q_1; \ldots; Q_{N_r - 1}]. \tag{16}$$

Notice that the number of merged messages which server $n$ cannot compute is $N_r - 1$ and that $Q_1, \ldots, Q_{N_r - 1}$ have been assigned to server $n$. The sub-matrix of $\mathbf{F}'$ including the columns with the indices in $[N] \setminus \mathcal{Z}_n'$ has the dimension $N_r \times (N_r - 1)$. As each ‘*’ is uniformly i.i.d. over $\mathbb{F}_q$,

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3 Recall that by convention, we let $\text{Mod}(b, a) = a$ if $a$ divides $b$.

4 Recall that $(\mathbf{M})_{m \times n}$ represents the dimension of matrix $\mathbf{M}$ is $m \times n$. 
a vector basis for the left-side null space of this sub-matrix contains one linearly independent vectors with high probability. Hence, we let \( s_n \) be this left-side null space vector, such that in the linear combination (16) the coefficients of the merged messages which server \( n \) cannot compute are 0. It was proved in [3] that for each set \( A \subseteq [N] \) where \( |A| = N_r \), the vectors \( s_n \) where \( n \in A \) are linearly independent with high probability. Hence, the user can recover \( F'[W_1'; \ldots; W_N'; Q_1; \ldots; Q_{N_r-1}] \) from the answer of workers in \( A \), which contains the desired task function.

For the security, it can be seen that from the answers of all server, the user can only recover \( F'[W_1'; \ldots; W_N'; Q_1; \ldots; Q_{N_r-1}] \) containing \( N_r \) linearly independent combinations. In addition, \( S' \) is full-rank (with rank equal to \( N_r - 1 \)). Hence, the user can only recover \( W_1 + \cdots + W_K \) without \( Q_1, \ldots, Q_{N_r-1} \), i.e., \( H(X|W_1, \ldots, W_K) = H(X|W_1 + \cdots + W_K) = (N - 1)L \), and thus the security constraint in (10) holds.

It can be seen that the communication cost of the proposed scheme is \( N_r \) and the size of randomness is \( N_r - 1 \).

From Theorem 1, it can be seen that the additional security constraint does not increase the communication cost. More interestingly, in Appendix A we will show the following theorem.

**Theorem 2.** Any linear-coding based computing scheme for the \((K, N, N_r, M, K_c)\) non-secure distributed linearly separable computation problem where \( K_c \in [K] \), \( M = \frac{K}{N}(N - N_r + m) \) and \( m \in [N_r] \), can be made secure without increasing the communication cost. ■

From Theorem 2 we can add the security into the distributed computing schemes in [4], [5] for the case that \( K_c = 1 \), and also add the security into the distributed computing scheme in [3] for the case that \( M = \frac{K}{N}(N - N_r + 1) \), without increasing the communication cost.

In the rest of this paper, we focus on the \((K, N, N_r, K_c, M)\) secure distributed linearly separable computation problem with \( M = \frac{K}{N}(N - N_r + 1) \) and \( K_c = 1 \), and aim to minimize the randomness size \( \eta \) while achieving the optimal communication cost \( R^* = N_r \).

We first introduce a novel converse bound on \( \eta \) for a fixed assignment, whose proof can be found in Appendix B

\[\text{注意到如果选择每个'•'都均匀独立地随机选择} \quad \text{F}_q, \quad \text{用户可以恢复所需的任务函数。} \]

\[\text{因此，我们可以选择'•'s的值，使得方案是可解的。}\]
**Theorem 3.** For the \((K, N, N_r, K_c, M)\) secure distributed linearly separable computation problem with \(M = \frac{K}{N}(N - N_r + 1)\) and \(K_c = 1\), for a fixed assignment \(Z = (Z_1, \ldots, Z_N)\), if there exists an ordered set of servers in \([N]\) denoted by \(s = (s_1, \ldots, s|s|)\), such that

\[
Z_{s_i} \setminus (Z_{s_1} \cup \cdots Z_{s_{i-1}}) \neq \emptyset, \quad \forall i \in [|s|],
\]

it must hold that

\[
\eta \geq |s| - 1.
\]

Notice that while deriving the converse bound in Theorem 3, we do not use the constraint that communication cost is minimum. Hence, it is a converse on the randomness size, which is also a converse bound on the randomness size while achieving the optimal communication cost.

A general converse bound over all possible assignments can be directly obtained from Theorem 3.

**Corollary 1.** For the \((K, N, N_r, K_c, M)\) secure distributed linearly separable computation problem with \(M = \frac{K}{N}(N - N_r + 1)\) and \(K_c = 1\), it must hold that

\[
\eta^* \geq \min_Z \max_{s: Z_{s_i} \setminus (Z_{s_1} \cup \cdots Z_{s_{i-1}}) \neq \emptyset, \forall i \in [|s|]} |s| - 1.
\]

To solve the min-max optimization problem in (19) is highly combinatorial and becomes a part of ongoing works. For some specific cases, this optimization problem has been solved in this paper (see Theorems 4 and 7). In the following, we provide a generally loosen version of the converse bound in (19).

**Corollary 2.** For the \((K, N, N_r, K_c, M)\) secure distributed linearly separable computation problem with \(M = \frac{K}{N}(N - N_r + 1)\) and \(K_c = 1\), it must hold that

\[
\eta^* \geq \left\lceil \frac{N}{N - N_r + 1} \right\rceil - 1.
\]

**Proof:** By definition, there are \(K\) datasets in the library and we assign \(\frac{K}{N}(N - N_r + 1)\) datasets
to each server. Hence, for any possible assignment, we can find

$$\left\lceil \frac{K}{M} \right\rceil = \left\lceil \frac{N}{N-N_r+1} \right\rceil$$

servers, where each server has some dataset which is not assigned to other \( \left\lceil \frac{N}{N-N_r+1} \right\rceil - 1 \) servers.

By Theorem 3, we have \( \eta^* \geq \left\lceil \frac{N}{N-N_r+1} \right\rceil - 1 \).

We then characterize the optimal randomness size for the case where \( N-N_r+1 \) divides \( N \).

**Theorem 4.** For the \((K, N, N_r, K_c, M)\) secure distributed linearly separable computation problem where \( M = \frac{K}{N}(N-N_r+1) \), \( K_c = 1 \), and \( N-N_r+1 \) divides \( N \), to achieve the optimal communication cost, the minimum randomness size is

$$\eta^* = \frac{N}{N-N_r+1} - 1.$$  \hspace{1cm} (21)

**Proof:** The converse part of (21) directly comes from Corollary 2. We then describe the achievable scheme, which is based on the fractional repetition assignment in [1].

We define that \( n := \frac{N}{N-N_r+1} \) which is a positive integer. We divide the \( K \) datasets into \( n \) groups, where the \( i \)-th group is \( D_i = \left[(i-1)\frac{K}{n} + 1 : i\frac{K}{n}\right] \) for each \( i \in [n] \). We assign all datasets in \( D_i \) to servers in \([(i-1)(N-N_r+1) + 1 : i(N-N_r+1)]\).

In the computing phase, we introduce \( n-1 \) independent randomness variables \( Q_1, \ldots, Q_{n-1} \), where \( Q_j \) is uniformly i.i.d. over \([\mathbb{F}_q]^L\).

We let each server in \([N-N_r+1]\) compute

$$A_1 = Q_1 + \sum_{k \in D_1} W_k;$$  \hspace{1cm} (22)

for each \( i \in [2 : n-1] \), we let each server in \([(i-1)(N-N_r+1) + 1 : i(N-N_r+1)]\) compute

$$A_i = -Q_{i-1} + Q_i + \sum_{k \in D_i} W_k;$$ \hspace{1cm} (23)

finally, we let each server in \([N_r : N]\) compute

$$A_n = -Q_{n-1} + \sum_{k \in D_n} W_k.$$ \hspace{1cm} (24)

Recall that each group has \( N-N_r+1 \) servers. For the decodability, from the answers of any \( N_r \) responding servers (i.e., there are \( N_r \) stragglers), the user always receives \( A_1, A_2, \ldots, A_n \). By summing \( A_1, A_2, \ldots, A_n \), the user recovers \( W_1 + \cdots + W_k \).
For the security, from the answers of all servers, the user receives $A_1, A_2, \ldots, A_n$, totally $n$ linear combinations. In the linear space of these linear combinations, there is $W_1 + \cdots + W_K$. The projection of this $n$-dimensional linear space on $[Q_1; \ldots; Q_{n-1}]$, has the dimension equal to $n - 1$. Hence, the user can only recover $W_1 + \cdots + W_K$ without $Q_1, \ldots, Q_{n-1}$.

In the following theorem, we focus on the cyclic assignment.

**Theorem 5.** For the $(K, N, N_r, K_c, M)$ secure distributed linearly separable computation problem with $M = \frac{K}{N}(N - N_r + 1)$ and $K_c = 1$, to achieve the optimal communication cost, the minimum randomness size under the constraint of the cyclic assignment is

$$\eta_{cyc}^* = N_r - 1.$$ (25)

**Proof:** The achievability part was described in the proof of Theorem 1. In the following, we prove the converse part.

If the cyclic assignment is used, let us focus on an ordered set of $N_r$ neighbouring servers

$$s = (N_r, N_r - 1, \ldots, 1).$$ (26)

For each $n \in [N_r]$, dataset $D_n$ is assigned to servers in $\{n, \text{Mod}(n - 1, N), \ldots, \text{Mod}(n - N + N_r, N)\}$; thus servers in $\{N_r, N_r - 1, \ldots, n + 1\}$ do not know $D_n$. Hence, the ordered set $s$ in (26) satisfies the constraint in (17), and we have $\eta_{cyc}^* \geq |s| - 1 = N_r - 1$, which proves (25).

Comparing Theorems 4 and 5, it can be seen that the computing scheme with the cyclic assignment is highly sub-optimal where the multiplicative gap to the optimality could be unbounded.\(^6\) However, when $N - N_r + 1$ does not divide $N$, the fractional repetition assignment in [1] cannot be used. On the observation that most of existing works on this distributed linearly separable computing problem (without security) are either based on the cyclic assignment (such as [1], [3]–[6], [20], [21]) or the fractional repetition assignment (such as [1], [22]), we need to design new assignments for the considered secure computation problem.

For the ease of notation, we define that

$$M' := N - N_r + 1.$$ (27)

\(^6\) For example, when $N = 2(N - N_r + 1)$ and $N$ is very large, the optimal randomness size is 1 as shown in (21), while the needed randomness size of the computing scheme with the cyclic assignment is $N_r - 1 = \frac{N}{2}$. 


In Section IV we will propose five novel achievable schemes for different ranges of system parameters. The performance of the combined scheme given in the following theorem is based on a recursive algorithm illustrated in Fig. 2, which will be explained in Remark 1.

**Theorem 6.** For the \((K, N, N_r, K_c, M)\) secure distributed linearly separable computation problem with \(M = \frac{K}{N} M'\) and \(K_c = 1\), to achieve the optimal communication cost, the randomness size \(\eta = h(N, M') - 1\) is achievable, where the function \(h(\cdot, \cdot)\) has the following properties:

- By directly using the scheme with the fractional repetition assignment for Theorem 4, we have
  \[
  h(N, 1) = N. \tag{28}
  \]

- By Scheme 1 described in Section IV-A we have
  \[
  h(N, M') = h\left(\frac{N}{\text{GCD}(N, M')}, \frac{M'}{\text{GCD}(N, M')}\right). \tag{29}
  \]

- For the case where \(N > 2M'\), by Scheme 2 described in Section IV-B we have
  \[
  h(N, M') = h(N - \lfloor N/M' - 1 \rfloor M', M') + \lfloor N/M' - 1 \rfloor. \tag{30}
  \]

- For the case where \(1.5M' \leq N < 2M'\) and \(M'\) is even, by Scheme 3 described in Section IV-C we have
  \[
  h(N, M') = h\left(N - M', \frac{M'}{2}\right) + 1. \tag{31}
  \]

- For the case where \(1.5M' \leq N < 2M'\) and \(M'\) is odd, by Scheme 4 described in Section IV-D we have
  \[
  h(N, M') = N - \frac{3M' - 5}{2}; \tag{32}
  \]

- For the case where \(M' < N < 1.5M'\), by Scheme 5 described in Section IV-E we have
  \[
  h(N, M') = h(M', 2M' - N). \tag{33}
  \]

Notice that it can be seen that, the needed randomness size of the combined scheme for Theorem 6 is
\[
h(N, M') - 1 \leq \frac{K}{N}(N - M' + 1) - 1 = N_r - 1,
\]
where $N_r - 1$ is the needed randomness size of the computing scheme with the cyclic assignment for Theorem 5. In addition, the multiplicative gap between the needed randomness sizes of the computing scheme with the cyclic assignment for Theorem 5 and the combined scheme for Theorem 6 could be unbounded. We provide some examples: (i) let us focus on the example where $K = N = nM + 1$ and $M$ does not divide $N$. By Scheme 2, $h(N, M') = h(N, M) = h(M + 1, M) + n - 1$; by Scheme 5, $h(M + 1, M) = 2$. Hence, the needed randomness size is $h(N, M') - 1 = n$, while that of the computing scheme with the cyclic assignment is $N_r - 1 = N - M = (n - 1)M + 1$. (ii) We then focus on the example where $K = N = 1.5M$. By Scheme 3, $h(N, M') = h(N, M) = h(0.5M, 0.5M) + 1$; by Scheme 1, $h(0.5M, 0.5M) = 1$. Hence,
the needed randomness size is \( h(N, M') - 1 = 1 \), while that of the computing scheme with the cyclic assignment is \( N_r - 1 = N - M = 0.5M \). (iii) Finally, we focus on the example where \( K = N = \frac{3M+1}{2} \) and \( M \) is an odd not dividing \( N \). By Scheme 4, \( h(N, M') = h(N, M) = 3 \). Hence, the needed randomness size is \( h(N, M') - 1 = 2 \), while that of the computing scheme with the cyclic assignment is \( N_r - 1 = N - M = \frac{M+1}{2} \).

**Remark 1** (High-level ideas for Theorem 6). We divide the \( K \) datasets into \( N \) non-overlapping and equal-length groups, where the \( i^{\text{th}} \) group denoted by \( G_i = \{ k \in [K] : \text{Mod}(k, N) = i \} \) contains \( \frac{K}{N} \) datasets, for each \( i \in [N] \). Group \( G_i \) is assigned to \( M' = N - N_r + 1 \) servers, each of which can compute the merged message \( W'_i \). Hence, we treat the \((K, N, N_r, 1, M')\) secure distributed linearly separable computation problem as the \((N, N_r, 1, M')\) secure distributed linearly separable computation problem.

As in Appendix A the design on the computing phase contains two stages.

- In the first stage, we do not consider the security constraint in (10). We let each server send one linear combination of merged messages which it can compute, such that from any set of \( N_r \) responding servers, the user can recover \( W'_1 + \cdots + W'_{N} \). Assume that from the answers of all servers, the user can recover \( F[W'_1; \ldots; W'_{N}] \) where the dimension of \( F \) is \( \lambda \times N \) and \( \lambda \) represents the number of totally transmitted linearly independent combinations of merged messages. Thus the transmission of server \( n \in [N] \) can be expressed as \( s_nF[W'_1; \ldots; W'_{N}] \), where \( s_n \) represents the transmission vector of server \( n \).

- In the second stage, we take the security constraint in (10) into consideration. We introduce \( \lambda - 1 \) independent randomness variables \( Q_1, \ldots, Q_{\lambda-1}, \) where \( Q_i, i \in [\lambda - 1] \) is uniformly i.i.d. over \([F_q]^{L} \). We then generate the matrix \( F' = [\mathbf{(F)}_{\lambda \times N}, (S)_{\lambda \times (\lambda-1)}] \), where \( S = [0_{1 \times (\lambda-1)}; S'] \) and \( S' \) is full-rank with dimension \((\lambda - 1) \times (\lambda - 1)\).

We let each server \( n \in [N] \) transmit \( s_nF'[W'_{1,1}; \ldots; W'_{K}; Q_1; \ldots; Q_{\lambda-1}] \). It is proved in Appendix A that the resulting scheme is decodable and secure. The needed randomness size \( \eta \) is equal to \( \lambda - 1 \).

The second stage can be immediately obtained once the first stage is fixed. Hence, now we only need to focus on the first stage where we aim to minimize the number of totally transmitted linearly independent combinations (i.e., \( \lambda \)) for the \((N, N_r, 1, M')\) non-secure distributed linearly separable computation problem (for the sake of simplicity, we will call it \((N, M')\) non-secure problem since \( N_r = N - M' + 1 \)). Notice that if in the first stage \( F \) is chosen as that in (15) where
all elements outside the first line are chosen i.i.d. over $\mathbb{F}_q$, the computing scheme becomes the scheme with the cyclic assignment for Theorem 5. To reduce the number of totally transmitted linearly independent combinations, in the combined scheme for Theorem 6 we design more structured $F$.

The flow diagram of the combined scheme for Theorem 6 where we have $\lambda = h(N, M')$, is given in Fig. 2. The procedure in the flow diagram is finished when either $M' = 1$ or $1.5M' \leq N < 2M'$ and $M'$ is odd. There must exist an output for each input case because when none of the above two constraints are satisfied, $M'$ will be further reduced. □

Comparing the achievable scheme in Theorem 6 with the proposed converse bounds in Corollaries 1 and 2, we can characterize the following optimality result, whose proof could be found in Appendix C.

**Theorem 7.** For the $(K, N, N_r, K_c, M)$ secure distributed linearly separable computation problem with $M = \frac{K}{N} M'$, $K_c = 1$, and $\frac{M'}{\gcd(N, M')} \leq 4$, to achieve the optimal communication cost, the minimum randomness size is $h(N, M') - 1$, where $h(\cdot, \cdot)$ is defined in Theorem 6 □

Notice that when $\frac{M'}{\gcd(N, M')} = 1$, we have that $M'$ divides $N$; in this case Theorem 7 reduces to Theorem 4.

At the end of this section, we provide some numerical evaluations to compare the needed randomness sizes of the computing scheme with the cyclic assignment for Theorem 5 (equal to $N_r - 1$) and the combined scheme for Theorem 6 (equal to $h(N, M') - 1$), while achieving the optimal communication cost. We consider the $(K, N, N_r, K_c, M)$ secure distributed linearly separable computation problem with $K = N$, $M = \frac{K}{N} M'$, and $K_c = 1$. In Fig. 3a, we fix $N = 22$ and plot the tradeoffs between $M$ and $\eta$. In Fig. 3b, we fix $M = 8$ and plot the tradeoffs between $N$ and $\eta$. In Fig. 3c, we plot the $(N, M, \eta)$ tradeoffs for the case where $16 \leq N \leq 30$ and $5 \leq M \leq 15$. From all figures, it can be seen that the combined scheme for Theorem 6 needs a much lower randomness size than that in Theorem 5.

**IV. NOVEL ACHIEVABLE SCHEMES FOR THEOREM 6**

As explained in Remark 1, by a grouping strategy, we treat the $(K, N, N_r, 1, M)$ secure distributed linearly separable computation problem as the $(N, N, N_r, 1, M')$ secure distributed linearly separable computation problem. For the ease of notation, in this section we directly consider the case where $K = N$; thus we also have $M = M'$. 
Computing scheme with the cyclic assignment for Theorem 5
Combined scheme for Theorem 6
Converse bound for Corollary 2

(a) \((M, \eta)\) tradeoff with \(N = 22\).
(b) \((N, \eta)\) tradeoff with \(M = 8\).

(c) \((N, M, \eta)\) tradeoff.

Fig. 3: Numerical evaluations for the considered secure distributed linearly separable computation problem.

The proposed schemes for Theorem 6 contain two stages, where in the first stage we consider a \((N, N, N_r, 1, M)\) non-secure distributed linearly separable computation problem (a.k.a., \((N, M)\) non-secure problem), and aim to minimize the number of totally transmitted linear combinations of messages \(\lambda\) while achieving the optimal communication cost \(N_r\); then the second stage can be immediately obtained by introducing \(\lambda - 1\) independent randomness variables such that the security is guaranteed. Because the second stage is unified for each proposed scheme, we only present the first stage (i.e., the \((N, M)\) non-secure problem) in the rest of this section.
A. Scheme 1 for (29)

We consider the \((N, M)\) non-secure problem where \(\text{GCD}(N, M) > 1\), and aim to construct a scheme (Scheme 1) to prove (29). Intuitively, we want to consider a set of \(\text{GCD}(N, M)\) messages as a single message and a set of \(\text{GCD}(N, M)\) servers as a single server. Thus Scheme 1 is a recursive scheme which is based on the proposed scheme for the \(\left(\frac{N}{\text{GCD}(N, M)}, \frac{M}{\text{GCD}(N, M)}\right)\) non-secure problem. We assume that the latter scheme has been designed before, whose number of totally transmitted linearly independent combinations of messages is \(h \left(\frac{N}{\text{GCD}(N, M)}, \frac{M}{\text{GCD}(N, M)}\right)\).

We first partition the \(N\) datasets into \(\frac{N}{\text{GCD}(N, M)}\) groups, where the \(i\)th group is

\[K_i = \{(i - 1)\text{GCD}(N, M) + 1 : i \text{ GCD}(N, M)\}\]

for each \(i \in \left[\frac{N}{\text{GCD}(N, M)}\right]\). In addition, we let \(M_i = \sum_{k \in K_i} W_k\); thus the task function could be expressed as \(W_1 + \cdots + W_N = M_1 + \cdots + M_{\frac{N}{\text{GCD}(N, M)}}\).

We also partition the \(N\) servers into \(\frac{N}{\text{GCD}(N, M)}\) groups, where the \(i\)th group of servers is

\[U_i = \{(i - 1)\text{GCD}(N, M) + 1 : i \text{ GCD}(N, M)\}\]

for each \(i \in \left[\frac{N}{\text{GCD}(N, M)}\right]\).

We now prove that the the proposed scheme for the \(\left(\frac{N}{\text{GCD}(N, M)}, \frac{M}{\text{GCD}(N, M)}\right)\) non-secure problem can be directly applied to the \((N, M)\) non-secure problem.

In the proposed scheme for the \(\left(\frac{N}{\text{GCD}(N, M)}, \frac{M}{\text{GCD}(N, M)}\right)\) non-secure problem, we assume that the set of assigned datasets to each server \(n' \in \left[\frac{N}{\text{GCD}(N, M)}\right]\) is \(Z'_n' \subseteq \left[\frac{N}{\text{GCD}(N, M)}\right]\); obviously, \(\left|Z'_n'\right| = \frac{M}{\text{GCD}(N, M)}\). In the computing phase, server \(n'\) computes a linear combination of the \(\frac{N}{\text{GCD}(N, M)}\) messages, where the coefficients of the messages with indices in \(\left[\frac{N}{\text{GCD}(N, M)}\right] \setminus Z'_n'\) are 0. We assume that the vector of the coefficients in this linear combination is \(v_{n'}\), containing \(\frac{N}{\text{GCD}(N, M)}\) elements. From the answers of any \(\frac{N}{\text{GCD}(N, M)} - \frac{M}{\text{GCD}(N, M)} + 1\) servers, the user can recover the sum of the \(\frac{N}{\text{GCD}(N, M)}\) messages.

We then apply the above scheme to the \((N, M)\) non-secure problem.

**Assignment phase.** For each \(i \in \left[\frac{N}{\text{GCD}(N, M)}\right]\), we assign all datasets in group \(K_i\) to each server in group \(U_j\) where \(j \in \left[\frac{N}{\text{GCD}(N, M)}\right]\) and \(i \in Z'_n\). As each group of servers contains \(\text{GCD}(N, M)\) servers, each dataset is assigned to \(\text{GCD}(N, M)\) \(= M\) servers; as each group of datasets contains \(\text{GCD}(N, M)\), the number of datasets assigned to each server is \(\text{GCD}(N, M)\) \(= M\). Thus the assignment constraints are satisfied.
Computing phase. For each \( i \in \left\lceil \frac{N}{\text{GCD}(N,M)} \right\rceil \), we let each server in group \( \mathcal{U}_i \) compute
\[
\mathbf{v}_n \left[ M_1; \ldots; M_{\frac{N}{\text{GCD}(N,M)}} \right],
\]
where \( \mathbf{v}_n \) represents the vector of the coefficients in the linear combination sent by server \( n \) in the \( \left( \frac{N}{\text{GCD}(N,M)}, \frac{M}{\text{GCD}(N,M)} \right) \) non-secure problem.

Decoding phase. Following the original scheme for the \( \left( \frac{N}{\text{GCD}(N,M)}, \frac{M}{\text{GCD}(N,M)} \right) \) non-secure problem, for any set \( \mathcal{A} \subseteq \left[ \frac{N}{\text{GCD}(N,M)} \right] \) where \( |\mathcal{A}| = \frac{N}{\text{GCD}(N,M)} - \frac{M}{\text{GCD}(N,M)} + 1 \), if the user receives the answers of the servers in \( \mathcal{A} \), it can recover the task function.

Let us go back to the \((N, M)\) non-secure problem. The user can receive the answers of \( N - M + 1 \) servers. As each group of servers contains \( \text{GCD}(N, M) \) servers, it can be seen that these \( N - M + 1 \) servers are from at least \( \left\lceil \frac{N - M + 1}{\text{GCD}(N,M)} \right\rceil = \frac{N}{\text{GCD}(N,M)} - \frac{M}{\text{GCD}(N,M)} + 1 \) groups. Hence, the user recovers the task function.

In conclusion, we proved \( h(N, M) = h \left( \frac{N}{\text{GCD}(N,M)}, \frac{M}{\text{GCD}(N,M)} \right) \), coinciding with (29).

B. Scheme 2 for (30)

We will start with an example to illustrate the main idea.

**Example 1.** We consider the \((N, M) = (5, 2)\) non-secure problem. It can be seen that in this example \( N_r = N - M + 1 = 4 \). For the sake of simplicity, while illustrating the proposed schemes through examples, we assume that the field is a large enough prime field. It will be proved that in general this assumption is not necessary in our proposed schemes.

Assignment phase. We assign the datasets as follows.

| Server 1 | Server 2 | Server 3 | Server 4 | Server 5 |
|----------|----------|----------|----------|----------|
| \( D_1 \) | \( D_1 \) | \( D_3 \) | \( D_4 \) | \( D_5 \) |
| \( D_2 \) | \( D_2 \) | \( D_4 \) | \( D_5 \) | \( D_3 \) |

Computing phase. We let servers 1 and 2 compute \( W_1 + W_2 \).

We then focus on servers 3, 4, 5. It can be seen that datasets \( D_4, D_5, D_6 \) are assigned to servers 3, 4, 5 in a cyclic way. Hence, as the computing scheme illustrated in the Introduction, we let server 3 compute \( 2W_3 + W_4 \); let server 4 compute \( W_4 + 2W_5 \); let server 5 compute \( W_3 - W_5 \).

Decoding phase. Among the answers of any \( N_r = 4 \) servers, there must exist \( W_1 + W_2 \) and two answers of servers 3, 4, 5. From any two answers of servers 3, 4, 5, the user can recover \( W_3 + W_4 + W_5 \). Together with \( W_1 + W_2 \), the user can recover \( W_1 + \cdots + W_5 \).
It can be seen that the number of linearly independent combinations transmitted by servers 3, 4, 5 is two. Hence, the number of totally transmitted linearly independent combinations is \( h(5, 2) = 3 \), which is equal to \( h(3, 2) + 1 \) coinciding with (30). □

We now consider the \((N, M)\) non-secure problem where \(N > 2M\), and aim to construct a scheme (Scheme 2) to prove (30). Scheme 2 is a recursive scheme which is based on the proposed scheme for the \((N - \lfloor N/M - 1 \rfloor M, M)\) non-secure problem. We assume that the latter scheme has been designed before, whose number of totally transmitted linearly independent combinations of messages is \( h(N - \lfloor N/M - 1 \rfloor M, M) \).

Assignment phase. We divide the whole system into \([N/M]\) blocks. For each \(i \in [\lfloor N/M \rfloor]\), the \(i^{th}\) block contains datasets \(\{D_k : k \in B_i\}\) and servers in \(B_i\), where

\[
B_i = \begin{cases} 
(i - 1)M + 1 : iM, & \text{if } i \in [\lfloor N/M - 1 \rfloor]; \\
[N/M - 1]M + 1 : N, & \text{if } i = \lfloor N/M \rfloor.
\end{cases}
\]

(34)

The datasets in one block are only assigned to the servers in the same block. More precisely, for each \(i \in [\lfloor N/M \rfloor]\),

- if \(i \in [\lfloor N/M - 1 \rfloor]\), we assign all datasets in \(\{D_k : k \in B_i\}\) to each server in \(B_i\).
- if \(i = \lfloor N/M \rfloor\), the block contains \(N - \lfloor N/M - 1 \rfloor M\) servers and \(N - \lfloor N/M - 1 \rfloor M\) datasets, where each dataset should be assigned to \(M\) servers and each server should obtain \(M\) datasets.

Hence, we can apply the assignment phase of the proposed scheme for the \((N - \lfloor N/M - 1 \rfloor M, M)\) non-secure problem, to assign the datasets \(\{D_k : k \in B_i\}\) to the servers in \(B_i\).

Computing phase. For each \(i \in [\lfloor N/M - 1 \rfloor]\), we let the servers in the \(i^{th}\) block compute

\[
\sum_{k \in B_i} W_k.
\]

We then focus on the \(i^{th}\) block where \(i = \lfloor N/M \rfloor\) (i.e., the last block), to which we apply the computing phase of the proposed scheme for the \((N - \lfloor N/M - 1 \rfloor M, M)\) non-secure problem. In the proposed scheme for the \((N - \lfloor N/M - 1 \rfloor M, M)\) non-secure problem, server \(n' \in [N - \lfloor N/M - 1 \rfloor M]\) computes a linear combination of the \(N - \lfloor N/M - 1 \rfloor M\) messages, where the coefficients of the messages it cannot compute are 0. We assume that the vector of the coefficients in this linear combination is \(v_{n'}\), containing \(N - \lfloor N/M - 1 \rfloor M\) elements.

Go back to the \(i^{th}\) block where \(i = \lfloor N/M \rfloor\) of the \((N, M)\) non-secure problem. For each
$j \in \{1, \ldots, |B_i|\}$, we let server $B_i(j)$ compute
\[
v_j \left[ W_{\left\lfloor \frac{N}{M} - 1 \right\rfloor M + 1}; W_{\left\lfloor \frac{N}{M} - 1 \right\rfloor M + 2}; \ldots; W_N \right],
\]
where $v_j$ represents the vector of the coefficients in the linear combination sent by server $j$ in the $(N - \left\lfloor \frac{N}{M} - 1 \right\rfloor M, M)$ non-secure problem.

Decoding phase. The user receives the answers of $N_r = N - M + 1$ servers. In other words, the user does not receive the answers of $M - 1$ servers. Recall that in each of the first $\left\lfloor \frac{N}{M} - 1 \right\rfloor$ blocks there are $M$ servers; in the last block there are $N - \left\lfloor \frac{N}{M} - 1 \right\rfloor M$ servers. Hence, among these $N_r = N - M + 1$ responding servers, there must be at least one server in each of the first $\left\lfloor \frac{N}{M} - 1 \right\rfloor$ blocks, and at least $N - \left\lfloor \frac{N}{M} - 1 \right\rfloor M - M + 1$ servers in the last block. By construction, from the answers of any $N - \left\lfloor \frac{N}{M} - 1 \right\rfloor M - M + 1$ servers in the last block, the user can recover $\sum_{k \in \left\lfloor \frac{N}{M} - 1 \right\rfloor M + 1:N} W_k$. Together with the transmissions of the first $\left\lfloor \frac{N}{M} - 1 \right\rfloor$ blocks, the user can recover $W_1 + \cdots + W_N$.

In conclusion, we proved $h(N, M) = \left\lfloor \frac{N}{M} - 1 \right\rfloor + h(N - \left\lfloor \frac{N}{M} - 1 \right\rfloor M, M)$, coinciding with (30). In addition, it can be seen that $M < N - \left\lfloor \frac{N}{M} - 1 \right\rfloor M < 2M$ if $N > 2M$ and $M$ does not divide $N$.

C. Scheme 3 for (31)

We first provide an example to illustrate the main idea.

Example 2. We consider the $(N, M) = (7, 4)$ non-secure problem. Notice that $N_r = N - M + 1 = 4$.

Assignment phase. We assign the datasets as follows.

| Server 1 | Server 2 | Server 3 | Server 4 | Server 5 | Server 6 | Server 7 |
|----------|----------|----------|----------|----------|----------|----------|
| $D_1$    | $D_1$    | $D_1$    | $D_1$    | $D_2$    | $D_3$    | $D_4$    |
| $D_2$    | $D_2$    | $D_5$    | $D_5$    | $D_5$    | $D_6$    | $D_7$    |
| $D_3$    | $D_3$    | $D_6$    | $D_6$    | $D_3$    | $D_4$    | $D_2$    |
| $D_4$    | $D_4$    | $D_7$    | $D_7$    | $D_6$    | $D_7$    | $D_5$    |

Computing phase. We let servers 1, 2 compute a same linear combination of messages, assumed to be $A_1$. Similarly, we let servers 3, 4 compute a same linear combination of messages, assumed to be $A_2$. Recall that $N_r = 4$. Thus from $A_1$ and $A_2$, the user should recover the task function.

\[\text{Recall that } B_i(j) \text{ represents the } j^{th} \text{ element in } B(i).\]
We construct \( A_1 \) and \( A_2 \) such that from \( A_1 \) and \( A_2 \), we can recover the following two linear combinations,

\[
F_1 = W_1 + \cdots + W_7;
\]

\[
F_2 = W_2 + W_3 + W_4 + 2(W_5 + W_6 + W_7).
\]

This can be done by letting \( A_1 = 2F_1 - F_2 = W_1 + W_2 + W_3 + W_4 \), which can be computed by servers 1, 2, and letting \( A_2 = F_2 - F_1 = -W_1 + W_5 + W_6 + W_7 \), which can be computed by servers 3, 4.

We then focus on servers 5, 6, 7. The assignment for servers 5, 6, 7 can be expressed as follows.

We divide the datasets in \([2 : 7]\) into three pairs, \( P_1 = \{2, 5\} \), \( P_2 = \{3, 6\} \), \( P_3 = \{4, 7\} \). The three pairs of datasets are assigned to servers 5, 6, 7 in a cyclic way. We also let 

\[
A_3 = P_1 + P_2 \]

\[
A_4 = P_2 + 2P_3 \]

\[
A_5 = P_1 - P_3.
\]

Decoding phase. As shown before, if the set of \( N_r = 4 \) responding servers contains one server in \([2]\) and one server in \([3, 4]\), the user can recover the task function from \( A_1 \) and \( A_2 \).

We then consider the case where from the answers of the responding servers, the user can only receive one of \( A_1 \) and \( A_2 \). In this case, the set of responding servers must contain at least two servers in \([5 : 7]\). By construction, from the answers of any two servers in \([5 : 7]\), the user can recover \( F_2 \). Together with \( A_1 = 2F_1 - F_2 \) or with \( A_2 = F_2 - F_1 \), the user can recover \( F_1 \), which is the task function.

The number of totally transmitted linearly independent combinations is \( h(7, 4) = 3 \), which is equal to \( h(3, 2) + 1 \) coinciding with (31). \( \square \)

We now consider the \((N, M)\) non-secure problem where \( 1.5M \leq N < 2M \) and \( M \) is even, and aim to construct a scheme (Scheme 3) to prove (31). Scheme 3 is a recursive scheme which is based on the proposed scheme for the \((N - M, \frac{M}{2})\) non-secure problem. We assume that the latter scheme has been designed before, whose number of totally transmitted linearly independent combinations of messages is \( h (N - M, \frac{M}{2}) \).

We define that \( N = 2M - y \). In this case, we have \( y \leq M/2 \) and \( N_r = N - M + 1 = M - y + 1 \leq M. \)
Assignment phase. We first focus on the assignment for the servers in \([M]\), which is as follows.

| Server 1 | \(\cdots\) | Server \(\frac{M}{2}\) | Server \(\frac{M}{2} + 1\) | \(\cdots\) | Server \(M\) |
|-----------|-------------|----------------|----------------|-------------|----------------|
| \(D_1\)   | \(\cdots\) | \(D_1\)       | \(D_1\)       | \(\cdots\) | \(D_1\)       |
| \(\cdots\)| \(\cdots\) | \(\cdots\)    | \(\cdots\)    | \(\cdots\)  | \(\cdots\)    |
| \(D_y\)   | \(\cdots\) | \(D_y\)       | \(D_y\)       | \(\cdots\) | \(D_y\)       |
| \(D_{y+1}\)| \(\cdots\)| \(D_{y+1}\)  | \(D_{M+1}\)  | \(\cdots\) | \(D_{M+1}\)  |
| \(\cdots\)| \(\cdots\) | \(\cdots\)    | \(\cdots\)    | \(\cdots\)  | \(\cdots\)    |
| \(D_M\)   | \(\cdots\) | \(D_M\)       | \(D_N\)       | \(\cdots\) | \(D_N\)       |

It can be seen that we assign \(D_1, \ldots, D_y\) to all servers in \([M]\), and assign each dataset \(D_k\) where \(k \in [y+1:N]\) to \(\frac{M}{2}\) servers in \([M]\).

We then focus on the assignment for the servers in \([N-M]\). We need to assign \(N-y = 2(N-M)\) datasets (which are in \([y+1:N]\)) to totally \(N-M\) servers, where each dataset is assigned to \(\frac{M}{2}\) servers and each server obtains \(M\) datasets. We divide datasets in \([y+1:N]\) into \(\frac{N-y}{2} = N-M\) pairs, where the \(i\)th pair is \(P_i = \{y+i, M+i\}\) for each \(i \in [N-M]\). Hence, we can apply the assignment phase of the proposed scheme for the \((N-M, \frac{M}{2})\) non-secure problem, to assign \(\frac{N-y}{2} = N-M\) pairs to \(N-M\) servers where each pair is assigned \(\frac{M}{2}\) servers and each server obtains \(\frac{M}{2}\) pairs.

Computing phase. We first focus on the servers in \([M]\). We let the servers in \([\frac{M}{2}]\) with the same datasets compute a same linear combination of messages, which is denoted by \(A_1\). Similarly, we let the servers in \([\frac{M}{2}+1:M]\) with the same datasets compute a same linear combination of messages, which is denoted by \(A_2\). We construct \(A_1\) and \(A_2\) such that from \(A_1\) and \(A_2\), we can recover the following two linear combinations

\[
F_1 = W_1 + \cdots + W_N; \tag{35a}
\]

\[
F_2 = W_{y+1} + \cdots + W_M + 2(W_{M+1} + \cdots + W_N). \tag{35b}
\]

This can be done by letting

\[
A_1 = 2F_1 - F_2 = 2(W_1 + \cdots + W_y) + W_{y+1} + \cdots + W_M
\]

which can be computed by servers in \([\frac{M}{2}]\), and letting

\[
A_2 = F_2 - F_1 = -(W_1 + \cdots + W_y) + W_{M+1} + \cdots + W_N
\]

which can be computed by servers in \([\frac{M}{2}+1:M]\).
We then focus on the servers in \([M+1:N]\). For each pair of datasets \(P_i = \{y+i, M+i\}\) where \(i \in [N-M]\), we let \(P_i = W_{y+i} + 2W_{M+i}\). Hence, we can express \(F_2\) in (35b) as \(P_1 + \cdots + P_{N-M}\). Next we apply the computing phase of the proposed scheme for the \((N-M, \frac{M}{2})\) non-secure problem. In the proposed scheme for the \((N-M, \frac{M}{2})\) non-secure problem, server \(n' \in [N-M]\) computes a linear combination of the \(N-M\) messages, where the coefficients of the messages that server \(n'\) cannot compute are 0. We assume that the vector of the coefficients in this linear combination is \(v_{n'}\), containing \(N-M\) elements.

Go back to the \((N,M)\) non-secure problem. We let each server \(n \in [M+1:N]\) compute

\[
A_{n-M+2} = v_{n-M} \left[ P_1; \ldots; P_{N-M} \right],
\]

where \(v_{n-M}\) represents the vector of the coefficients in the linear combination sent by server \(n-M\) in the \((N-M, \frac{M}{2})\) non-secure problem.

Decoding phase. If the set of \(N_r = N-M+1\) responding servers contains one server in \([\frac{M}{2}]\) and one server in \([\frac{M}{2}+1:M]\), from \(A_1\) and \(A_2\) the user can recover the task function.

We then consider the case where from the answers of the responding servers, the user can only receive one of \(A_1\) and \(A_2\). In this case, the set of \(N_r\) responding servers contains at least

\[
N_r - \frac{M}{2} = N - \frac{3M}{2} + 1
\]

servers in \([M+1:N]\). Notice that in the \((N-M, \frac{M}{2})\) non-secure problem, the answers of any \(N-M - \frac{M}{2} + 1 = N - \frac{3M}{2} + 1\) servers can re-construct the task function. Hence, in the \((N,M)\) non-secure problem, the answers of any \(N - \frac{3M}{2} + 1\) servers in \([M+1:N]\) can re-construct \(P_1 + \cdots + P_{N-M} = F_2\). Together with \(A_1 = 2F_1 - F_2\) or with \(A_2 = F_2 - F_1\), the user can recover the task function \(F_1\).

It can be seen that the number of linearly independent combinations transmitted by servers in \([M+1:N]\) is \(h(N-M, \frac{M}{2})\), the linear space of which contains \(F_2\). In addition, the number of linearly independent combinations transmitted by servers in \([M]\) is two, the linear space of which also contains \(F_2\). Hence, the number of totally transmitted linearly independent combinations is \(h(N,M) = 2 + h(N-M, \frac{M}{2}) - 1 = h(N-M, \frac{M}{2}) + 1\), coinciding with (31).

D. Scheme 4 for (32)

We first provide an example to illustrate the main idea.
Example 3. We consider the \((N, M) = (8, 5)\) non-secure problem. It can be seen that in this example \(N_r = N - M + 1 = 4\).

Assignment phase. We assign the datasets as follows.

| Server 1 | Server 2 | Server 3 | Server 4 | Server 5 | Server 6 | Server 7 | Server 8 |
|----------|----------|----------|----------|----------|----------|----------|----------|
| \(D_1\)  | \(D_1\)  | \(D_1\)  | \(D_1\)  | \(D_1\)  | \(D_3\)  | \(D_3\)  | \(D_3\)  |
| \(D_2\)  | \(D_2\)  | \(D_2\)  | \(D_2\)  | \(D_2\)  | \(D_4\)  | \(D_4\)  | \(D_4\)  |
| \(D_3\)  | \(D_3\)  | \(D_6\)  | \(D_6\)  | \(D_6\)  | \(D_5\)  | \(D_5\)  | \(D_5\)  |
| \(D_4\)  | \(D_4\)  | \(D_7\)  | \(D_7\)  | \(D_7\)  | \(D_6\)  | \(D_7\)  | \(D_8\)  |
| \(D_5\)  | \(D_5\)  | \(D_8\)  | \(D_8\)  | \(D_8\)  | \(D_7\)  | \(D_8\)  | \(D_6\)  |

Computing phase. We let each user send one linear combination of messages, such that the user can recover \(F[W_1; \ldots; W_N]\) from the answers of any \(N_r\) responding servers, where

\[
F = \begin{bmatrix}
    f_1 \\
    f_2 \\
    f_3
\end{bmatrix} = \begin{bmatrix}
    1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
    0 & 0 & 2 & 2 & 2 & 1 & 1 & 1 \\
    0 & 0 & 0 & 0 & 0 & * & * & *
\end{bmatrix} = \begin{bmatrix}
    1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
    0 & 0 & 2 & 2 & 2 & 1 & 1 & 1 \\
    0 & 0 & 0 & 0 & 0 & 1 & 2 & 3
\end{bmatrix}.
\]

(36)

and each ‘*’ is uniformly i.i.d. over \(\mathbb{F}_q\) and in this example we assume that the last three ‘*’ in \(f_3\) are \((1, 2, 3)\). We also define that \([F_1; F_2; F_3] = F[W_1; \ldots; W_N]\).

We let servers 1, 2 with datasets \(D_1, \ldots, D_5\) compute

\[X_1 = X_2 = F_1 - F_2 = W_1 + W_2 - W_3 - W_4 - W_5.\]

For servers in \([3 : 5]\) with datasets \(D_1, D_2, D_6, D_7, D_8\), we construct their transmissions such that from the answers of any two of them we can recover

\[2F_1 - F_2 = 2W_1 + 2W_2 + W_6 + W_7 + W_8;\]

\[F_3 = 2W_6 + W_7.\]

Notice that both of \(2F_1 - F_2\) and \(F_3\) can be computed by each server in \([3 : 5]\). Hence, we each server in \([3 : 5]\) compute a random linear combination of \((2F_1 - F_2)\) and \(F_3\). For example, we let servers 3, 4, 5 compute \(X_3, X_4, X_5\), respectively, where

\[X_3 = (2F_1 - F_2) + F_3;\]

\[X_4 = (2F_1 - F_2) + 2F_3;\]

\[X_5 = (2F_1 - F_2) + 4F_3.\]
For servers in \([6 : 8]\), we construct their transmissions such that from the answers of any two of them we can recover \(F_2\) and \(F_3\). This can be done by letting servers 6, 7, 8 compute \(X_6, X_7, X_8\), respectively, where

\[
X_6 = 3F_2 - F_3 = 6W_3 + 6W_4 + 6W_5 + 2W_6 + W_7; \\
X_7 = F_2 - F_3 = 2W_3 + 2W_4 + 2W_5 - W_7 - 2W_8; \\
X_8 = 2F_2 - F_3 = 4W_3 + 4W_4 + 4W_5 + W_6 - W_8.
\]

Decoding phase. For any set of \(N_r = 4\) servers, denoted by \(A\), we are in one of the following three cases:

- **Case 1:** \(A\) contains at least two servers in \([3 : 5]\). From the answers of any two servers in \([3 : 5]\), the user can recover \(2F_1 - F_2\) and \(F_3\). Besides, \(A\) contains at least either one server in \([2]\) or one server in \([6 : 8]\). It can be seen that each of \(X_1, X_2, X_6, X_7, X_8\) is linearly independent of \(2F_1 - F_2\) and \(F_3\). Hence, the user then recovers \(F_1\).

- **Case 2:** \(A\) contains at least two servers in \([6 : 8]\). From the answers of any two servers in \([6 : 8]\), the user can recover \(F_2\) and \(F_3\). Besides, \(A\) contains at least one server in \([5]\). It can be seen that in the transmitted linear combination of each server in \([5]\) contains \(F_1\). Hence, the user then recovers \(F_1\).

- **Case 3:** \(A\) contains servers 1, 2, one server in \([3 : 5]\), and one server in \([6 : 8]\). In this case, we can also check that the user receives three independent linear combinations in \(F_1, F_2, F_3\), such that it can recover \(F_1\).

It can be seen that the number of totally transmitted linearly independent combinations is \(h(8, 5) = 3\), coinciding with \((32)\). \(\Box\)

We now consider the \((N, M)\) non-secure problem where \(1.5M \leq N < 2M\) and \(M\) is odd, and aim to construct a scheme (Scheme 4) to prove \((32)\). We also define that \(N = 2M - y\).
Assignment phase. The assignment is as follows.

| Server 1 | ⋯ | Server y | Server y + 1 | Server y + 2 | ⋯ | Server M | Server M+1 | Server M+2 | ⋯ | Server N |
|----------|---|----------|--------------|--------------|---|----------|------------|-----------|---|----------|
| $D_1$    | ⋯ | $D_1$    | $D_1$        | $D_1$        | ⋯ | $D_1$    | $D_{M-1}+1$ | $D_{M-1}+1$ | ⋯ | $D_{M-1}+1$ |
| ⋯        | ⋯ | ⋯        | ⋯            | ⋯            | ⋯ | ⋯        | ⋯          | ⋯         | ⋯ | ⋯        |
| $D_{M-1}$| ⋯ | $D_{M-1}$| $D_{M-1}$    | $D_{M-1}$    | ⋯ | $D_{M-1}$| $D_M$      | $D_M$     | ⋯ | $D_M$    |
| $D_{M-1}+1$| ⋯ | $D_{M-1}+1$| $D_{M-1}+1$ | $D_{M-1}+1$ | ⋯ | $D_{M-1}+1$| $D_{N}$   | $D_{M+1}$ | ⋯ | $D_{N}$ |
| ⋯        | ⋯ | ⋯        | ⋯            | ⋯            | ⋯ | ⋯        | ⋯          | ⋯         | ⋯ | ⋯        |
| $D_M$    | ⋯ | $D_M$    | $D_{M-1}+1$ | $D_{M-1}+1$ | ⋯ | $D_{M-1}+1$| $D_{M+1}$ | $D_{M+1}$ | ⋯ | $D_{M+1}$ |

In the assignment, we divide the $N$ datasets into three parts, where the first part contains $D_1, \ldots, D_t$ (later we will explain the reason to choose $t = \frac{M-1}{2}$) which are all assigned to servers in $[M]$; the second part contains $D_{t+1}, \ldots, D_M$ which are all assigned to servers in $[y] \cup [M+1 : N]$; the third part contains $D_{M+1}, \ldots, D_N$, which are assigned to servers in $[y+1 : M]$ in a cyclic way where each server obtains $M - t$ neighbouring datasets in $[M : M-1 + 1 : N]$. The datasets $D_{M+1}, \ldots, D_N$ are also assigned to servers in $[M+1 : N]$ in a cyclic way where each server obtains $t$ neighbouring datasets in $[M-1 : M]$.

As we assign the datasets in $[M+1 : N]$ to the servers in $[y+1 : M]$ in a cyclic way where each server obtains $M - t$ datasets, we can choose $N - M - (M - t) + 1 = N - 2M + t + 1$ neighbouring servers in $[y+1 : M]$ satisfying the constraint in (17); in addition, server 1 has $D_{t+1}, \ldots, D_M$, which are not assigned to the servers in $[y+1 : M]$. Hence, the ordered set of the above $N - 2M + t + 2$ servers satisfies the constraint in (17).

Similarly, we assign the datasets in $[M+1 : N]$ to the servers in $[M+1 : N]$ in a cyclic way where each server obtains $t$ datasets, we can choose $N - M - t + 1$ neighbouring servers in $[M+1 : N]$ satisfying the constraint in (17); in addition, server 1 has $D_1, \ldots, D_t$, which are not assigned to the servers in $[M+1 : N]$. Hence, the ordered set of the above $N - M - t + 2$ servers satisfies the constraint in (17).

Similar to the derivation of (49d), by the chain rule of entropy, under the above assignment we have

$$H(X[N]) / L \geq \max \{N - 2M + t + 2, N - M - t + 2\} \cdot \frac{t - \frac{M-1}{2}}{2} N - M - \frac{M - 1}{2} + 2 = \frac{M + 5}{2} - y.$$

Hence, we let $t = \frac{M-1}{2}$.

Computing phase. We design the computing phase such that the total number of independent
transmitted linear combinations of messages by all servers is $\frac{M+5}{2} - y$. These linear combinations are in $\mathbb{F} [W_1; \ldots; W_N]$ where

$$F = \begin{bmatrix} f_1 \\ \vdots \\ f_{\frac{M+5}{2} - y} \end{bmatrix} = \begin{bmatrix} 1, \ldots, 1 \\ 0, \ldots, 0 \\ 0, \ldots, 0 \end{bmatrix}, \begin{bmatrix} 1, \ldots, 1 \\ a, \ldots, a \\ 0, \ldots, 0 \end{bmatrix}, \begin{bmatrix} 1, \ldots, 1 \\ \ast, \ldots, \ast \end{bmatrix}. \tag{37}$$

Notice that $a$ represents a symbol uniformly i.i.d. over $\mathbb{F}_q \setminus \{0, 1\}$, and ‘$\ast$’ represents a symbol uniformly i.i.d. over $\mathbb{F}_q$. We divide matrix $F$ into three column-wise sub-matrices, $F_1$ with dimension $(\frac{M+5}{2} - y) \times \frac{M-1}{2}$ which corresponds to the messages in $[\frac{M-1}{2}]$, $F_2$ with dimension $(\frac{M+5}{2} - y) \times \frac{M+1}{2}$ which corresponds to the messages in $[\frac{M+1}{2}]$, and $F_3$ with dimension $(\frac{M+5}{2} - y) \times (N - M)$ which corresponds to the messages in $[M + 1 : N]$. We also define that $F_i = f_i[W_1; \ldots; W_N]$ for each $i \in [\frac{M+5}{2} - y]$. Thus the transmission of each server could be expressed as a linear combination of $[F_1; \ldots; F_{\frac{M+5}{2} - y}]$.

As each server $n \in [y]$ cannot compute $W_{M+1}, \ldots, W_N$, we let it compute

$$s_n \ F \ [W_1; \ldots; W_N] = [1, -1, 0, \ldots, 0] \ F \ [W_1; \ldots; W_N] \ \tag{38a}$$

and

$$= W_1 + \cdots + W_{\frac{M}{2} - 1} + (1 - a)(W_{\frac{M}{2} + 1} + \cdots + W_M), \tag{38b}$$

such that the coefficients of $W_{M+1}, \ldots, W_N$ which it cannot compute are 0.

For the servers in $[y + 1 : M]$, we construct their transmissions such that from the answers of any $\frac{M+3}{2} - y$ servers in $[y + 1 : M]$, the user can recover $aF_1 - F_2, F_3, \ldots, F_{\frac{M+5}{2} - y}$. More precisely, we let server $n \in [y + 1 : M]$ compute

$$s_n \begin{bmatrix} a f_1 - f_2; f_3; \ldots; f_{\frac{M+5}{2} - y} \end{bmatrix} [W_1; \ldots; W_N], \tag{39a}$$

where

$$\begin{bmatrix} a f_1 - f_2; f_3; \ldots; f_{\frac{M+5}{2} - y} \end{bmatrix} = \begin{bmatrix} a \ldots a & 0 & \ldots & 0 \\ 0 & \ldots & 0 \end{bmatrix}, \begin{bmatrix} a - 1 \ldots a - 1 \end{bmatrix} \begin{bmatrix} \ast & \ldots & \ast \end{bmatrix}. \tag{39b}$$

We design $s_n$ as follows. Notice that $W_1, \ldots, W_{\frac{M}{2} - 1}$ can be computed by server $n$; and that in the linear combination $\tag{39a}$ the coefficients of $W_{\frac{M}{2} + 1}, \ldots, W_M$ are 0. Hence, in order to guarantee that in $\tag{39a}$ the coefficients of the messages which server $n$ cannot compute are
0, we only need to consider the messages in $W_{M+1}, \ldots, W_N$, whose related columns are in $F_3'$. Server $n$ cannot compute $N - M - \frac{M+1}{2} = \frac{M-1}{2} - y$ messages in $W_{M+1}, \ldots, W_N$; thus the column-wise sub-matrix of $F_3'$ corresponding to these $\frac{M-1}{2} - y$ messages has the dimension $\left(\frac{M+3}{2} - y\right) \times \left(\frac{M-1}{2} - y\right)$. In addition, each ‘*’ is uniformly i.i.d. over $F_q$. Hence, the left-hand side nullspace of this sub-matrix contains $\frac{M+3}{2} - y - \left(\frac{M-1}{2} - y\right) = 2$ vectors, each of which has $\frac{M+3}{2} - y$ elements. We let $s_n$ be a random linear combination of these two vectors, where each of the two coefficients is uniformly i.i.d over $F_q$.

The following lemma will be proved in Appendix D-A.

**Lemma 1.** From any $\frac{M+3}{2} - y$ answers of servers in $[y+1:M]$, the user can recover $aF_1 - F_2, F_3, \ldots, F_{\frac{M+3}{2}-y}$ with high probability. $\square$

Finally, we focus on the servers in $[M+1:N]$. We construct their transmissions such that from any the answers of any $\frac{M+3}{2} - y$ servers in $[M+1:N]$, the user can recover $F_2, F_3, \ldots, F_{\frac{M+3}{2}-y}$.

More precisely, we let server $n \in [M+1:N]$ compute

$$s_n \left[ f_2; f_3; \ldots; f_{\frac{M+3}{2}-y} \right] \left[ W_1; \ldots; W_N \right]. \tag{40}$$

We design $s_n$ as follows. Notice that in the linear combination (40) the coefficients of $W_1, \ldots, W_{\frac{M+1}{2}}$ are 0; and that $W_{\frac{M+1}{2}}, \ldots, W_M$ can be computed by server $n$. Hence, in order to guarantee that in (40) the coefficients of the messages which server $n$ cannot compute are 0, we only need to consider the messages in $W_{M+1}, \ldots, W_N$, whose related columns are in $F_3^{|[2:\frac{M+3}{2}-y]|}$. Server $n$ cannot compute $N - M - \frac{M-1}{2} = \frac{M+1}{2} - y$ messages in $W_{M+1}, \ldots, W_N$; thus the column-wise sub-matrix of $F_3^{|[2:\frac{M+3}{2}-y]|}$ corresponding to these $\frac{M+1}{2} - y$ messages has the dimension $\left(\frac{M+3}{2} - y\right) \times \left(\frac{M+1}{2} - y\right)$. In addition, each ‘*’ is uniformly i.i.d. over $F_q$. Hence, the left-hand side nullspace of this sub-matrix contains $\frac{M+3}{2} - y - \left(\frac{M+1}{2} - y\right) = 1$ vector, which has $\frac{M+3}{2} - y$ elements. We let $s_n$ be this vector.

It can be seen that the choice of $s_n$ for the servers in $[M+1:N]$ is from the computing scheme with the cyclic assignment in [3, Section IV-B] for the $(N-M, \frac{M-1}{2})$ non-secure problem. Hence, as proved in [3, Section IV-B], from any $N - M - \frac{M-1}{2} + 1 = \frac{M+3}{2} - y$ answers of servers in $[M+1:N]$, the user can recover $F_2, \ldots, F_{\frac{M+3}{2}-y}$ with high probability.

---

8 Recall that $M^{(S)}_i$ represents the sub-matrix of $M$ which is composed of the rows of $M$ with indices in $S$. 
Decoding phase. Now we analyse each possible set of \( N_t = N - M + 1 = M - y + 1 \) responding servers, assumed to be \( \mathcal{A} \).

- **First case:** \( \mathcal{A} \) contains at least \( \frac{M+3}{2} - y \) servers in \( [y+1 : M] \). Here we consider the worst case, where \( \mathcal{A} \) contains all servers in \( [y+1 : M] \). From Lemma 1, the user can recover \( aF_1 - F_2, F_3, \ldots, F_{\frac{M+5}{2} - y} \) with high probability.

In \( \mathcal{A} \), there remains \( N_t - (M - y) = 1 \) server outside \( [y+1 : M] \). If this server is in \( [y] \), it computes \( F_1 - F_2 \). The user can recover \( F_1 \) from \( F_1 - F_2 \) and \( aF_1 - F_2 \) because \( a \notin \{0, 1\} \). If this server is in \( [M+1 : N] \), it can be seen from (37) that each ‘\(*\)’ in \( F^{[2, \frac{M+5}{2} - y]} \) is generated uniformly i.i.d over \( \mathbb{F}_q \), and thus the transmission of this server can be expressed as \( a_1 F_2 + a_2 F_3 + \cdots + a_{\frac{M+5}{2} - y} F_{\frac{M+5}{2} - y} \), where \( a_1 \) is not zero with high probability (from the proof in [3, Appendix C]). Hence, the user then recovers \( F_1 \) with high probability.

- **Second case:** \( \mathcal{A} \) contains at least \( \frac{M+3}{2} - y \) servers in \( [M+1 : N] \). Here we consider the worst case, where \( \mathcal{A} \) contains all servers in \( [M+1 : N] \). By construction, the user can recover \( F_2, \ldots, F_{\frac{M+5}{2} - y} \) with high probability.

In \( \mathcal{A} \), there remains \( N_t - (N - M) = 1 \) server outside \( [M+1 : N] \). Similar to Case 1 described above, with the answer from any other server outside \( [M+1 : N] \), the user then recovers \( F_1 \) with high probability.

- **Third case:** \( \mathcal{A} \) contains \( \frac{M+1}{2} - y \) servers in \( [y+1 : M] \), \( \frac{M+1}{2} - y \) servers in \( [M+1 : N] \), and \( y \) servers in \( [y] \). Notice that the servers in \( [y] \) compute \( F_1 - F_2 \), and that the servers in \( [M+1 : N] \) compute linear combinations of \( F_2, \ldots, F_{\frac{M+5}{2} - y} \). Hence, the union of the answers of servers in \( [y] \) and the \( \frac{M+1}{2} - y \) servers in \( [M+1 : N] \), contains \( \frac{M+1}{2} - y + 1 \) linearly independent combinations of \( F_1, \ldots, F_{\frac{M+5}{2} - y} \). We denote the set of these linear combinations by \( \mathcal{L}_1 \). Moreover, it can be seen the coefficients in the linear combinations in \( \mathcal{L}_1 \) are independent of the value of \( a \). This is because the answer of the servers in \( [y] \) is \( F_1 - F_2 \), and the answer of each server in \( [M+1 : N] \) is a linear combination of \( F_2, \ldots, F_{\frac{M+5}{2} - y} \) whose coefficients are determined by \( F^{[2, \frac{M+5}{2} - y]}_3 \), independent of \( a \).

We then introduce the following lemma which will be proved in Appendix D-B

**Lemma 2.** Among the answers of any \( \frac{M+1}{2} - y \) servers in \( [y+1 : M] \), with high probability there exists some linear combination which is independent of the linear combinations in \( \mathcal{L}_1 \). \hfill \Box

By Lemma 2, the user can totally obtain \( \frac{M+1}{2} - y + 1 + 1 = \frac{M+5}{2} - y \) linearly independent
combinations of $F_1, \ldots, F_{\frac{M+5}{2} - y}$ with high probability, and thus it can recover its desired task function $F_1$.

We have proved that if $a$ is generated uniformly over $\mathbb{F}_q \setminus \{0, 1\}$ and each $\ast$ in $F_3^{([\frac{3M+5}{2} - y]_r)}$ is generated uniformly i.i.d. over $\mathbb{F}_q$, the user can recover the task function with high probability. Hence, we only need to pick one realization of $F_3^{([\frac{3M+5}{2} - y]_r)}$ and $a$, such that we can guarantee the successful decoding.

By the above scheme, the number of linearly independent transmissions by all servers is equal to the number of rows in $F$, i.e., $\frac{M+5}{2} - y = N - \frac{3M-5}{2}$, coinciding with (32).

**E. Scheme 5 for (33)**

Finally, we consider the case where $M < N < 1.5M$, and aim to construct a scheme (Scheme 5) to prove (33). Scheme 5 is a recursive scheme which is based on the proposed scheme for the $(M, 2M - N)$ non-secure problem. We assume that the latter scheme has been designed before, whose number of totally transmitted linearly independent combinations of messages is $h(M, 2M - N)$.

**Assignment phase.** We first assign datasets $D_1, \ldots, D_{N - M}$ to each server in $[M]$. Then we assign datasets $D_{N - M + 1}, \ldots, D_N$ to each server in $[M + 1 : N]$. So far, each server in $[M + 1 : N]$ has obtained $M$ datasets, while each server in $[M]$ has obtained $N - M < M$ datasets. In addition, each dataset in $[N - M + 1 : N]$ has been assigned to $N - M < M$ servers. Hence, in the next step we should assign each dataset $D_k$ where $k \in [N - M + 1 : N]$ to $M - (N - M) = 2M - N$ servers in $[M]$, such that each server in $[M]$ obtains $M - (N - M) = 2M - N$ datasets in $[N - M + 1 : N]$. Thus we can apply the assignment phase of the proposed scheme for the $(M, 2M - N)$ non-secure problem, to assign datasets $D_{N - M + 1}, \ldots, D_N$ to servers in $[M]$.

**Computing phase.** Let us first focus on the $(M, 2M - N)$ non-secure problem, where the $M$ messages are assumed to be $W''_1, \ldots, W''_M$. In the proposed scheme for the $(M, 2M - N)$ non-secure problem, each server computes a linear combination of the $M$ messages. Considering the transmitted linear combinations by all servers, the number of linearly independent combinations is denoted by $h(M, 2M - N)$ and these $h(M, 2M - N)$ linear combinations can be expressed as

$$F_4 [W''_1; \ldots; W''_M].$$

The transmission of each server $n' \in [M]$ can be expressed as

$$s_{n'} F_4 [W''_1; \ldots; W''_M].$$
Let us then go back to the \((N,M)\) non-secure problem. We construct the answer of the \(N\) servers, such that the transmissions of all servers totally contain \(h(M, 2M-N)\) linearly independent combinations and these \(h(M, 2M-N)\) linear combinations can be expressed as \(F[W_1; \ldots; W_N]\), where (each \(a_i\) where \(i \in [h(M, 2M-N)-1]\) represents a symbol uniformly i.i.d. over \(\mathbb{F}_q\))

\[
F = \begin{bmatrix}
1 & \cdots & 1 \\
a_1 & \cdots & a_1 \\
\vdots & & \ddots \\
a_{h(M, 2M-N)-1} & \cdots & a_{h(M, 2M-N)-1} \\
\end{bmatrix}.
\]

(42)

Each ‘+’ represents an element of \(F_4\) in (41). Notice that the dimension of \(F_5\) is \((h(M, 2M-N)-1) \times (N-M)\) and the dimension of \(F_4\) is \((h(M, 2M-N)-1) \times M\).

For each server \(n \in [M]\), by the construction of the assignment phase, datasets \(D_1, \ldots, D_{N-M}\) are assigned to it and the assignment on the datasets \(D_{N-M+1}, \ldots, D_N\) is from the assignment phase of the proposed scheme for the \((M, 2M-N)\) non-secure problem. Hence, we let server \(n\) compute \(s_n F[W_1; \ldots; W_N]\), where \(s_n\) is the same as the transmission vector of server \(n\) in the \((M, 2M-N)\) non-secure problem.

For each server \(n \in [M+1 : N]\), it cannot compute \(W_1, \ldots, W_{N-M}\), which correspond to the column-wise sub-matrix \(F_5\), whose rank is 1. So the right-hand side null space of \(F_5\) contains \(h(M, 2M-N)-1\) linearly independent vectors, each of which has \(h(M, 2M-N)\) elements. We let the transmission vector of server \(n\), denoted by \(s_n\), be a random linear combination of these \(h(M, 2M-N)-1\) linearly independent vectors, where the \(h(M, 2M-N)-1\) coefficients are uniformly i.i.d. over \(\mathbb{F}_q\); in other words, server \(n\) computes \(s_n F[W_1; \ldots; W_N]\).

**Decoding phase.** Assume the set of responding servers is \(\mathcal{A}\), where \(\mathcal{A} \subseteq [N]\) and \(|\mathcal{A}| = N_r\). We also define that \(\mathcal{A}_1 = \mathcal{A} \cap [M]\) and \(\mathcal{A}_2 = \mathcal{A} \setminus [M]\). By definition, \(|\mathcal{A}_2| \leq N - M = N_r - 1\).

As \(M \geq N - M + 1 = N_r\), it can be seen that \(|\mathcal{A}_2|\) could be 0. If \(|\mathcal{A}_2| = 0\), from the answers of the servers in \(\mathcal{A}_1\), the user can recover \(f_1[W_1; \ldots; W_N]\), where \(f_1\) is the first row of \(F\). This is from the decodability of the proposed scheme for the \((M, 2M-N)\) non-secure problem.

We then consider the case where \(0 < |\mathcal{A}_2| \leq N - M\) and from the answers of \(\mathcal{A}_1\) the user cannot recover the task function. Assume the number of linearly independent combinations of messages transmitted by the servers in \(\mathcal{A}_1\) is \(\lambda_1\). Besides the answers of the servers in \(\mathcal{A}_1\), if the user obtains any \(h(M, 2M-N) - \lambda_1\) linear combinations in \(F[W_1; \ldots; W_N]\), which are
all independent of the answers of the servers in $A_1$, the user can recover $F[W_1; \ldots; W_N]$. In Appendix E, we will prove the following lemma.

**Lemma 3.** The answers in servers $A_2$ contains $h(M, 2M - N) - \lambda_1$ linear combinations, independent of the answers of the servers in $A_1$ with high probability.

By Lemma 3, from the answers of servers in $A$, the user can recover $F[W_1; \ldots; W_N]$.

In conclusion, we have $h(N, M) = h(M, 2M - N)$, which coincides with (33). We have proved that if $a_j$ where $j \in [h(M, 2M - N) - 1]$ is generated uniformly i.i.d. over $F_q$, the user can recover the task function with high probability. Hence, we only need to pick one realization of them, such that we can guarantee the successful decoding.

**V. Conclusions**

In this paper, we formulated the secure distributed linearly separable computation problem, where the user should only recover the desired task function without retrieving any other information about the datasets. It is interesting to see that to preserve this security, we need not to increase the communication cost if the computing scheme is based on linear coding. We then focused on the problem where the computation cost is minimum. In this case, while achieving the optimal communication cost, we aim to minimize the size of the randomness variable which is independent of the datasets and is introduced in the system to preserve the security. For this purpose, we proposed an information theoretic converse bound on the randomness size for each possible assignment. We then proposed a secure computing scheme with novel assignment strategies, which outperforms the optimal computing schemes with the well-know fractional repetition assignment and cyclic assignment in terms of the randomness size. Exact optimality results have been obtained from the proposed computing scheme under some system parameters. Ongoing work includes deriving tighter converse bounds over all possible assignments and minimizing the needed randomness size for more general case where the computation cost is not minimum and the user requests multiple linear combinations of messages.

---

9 The reason is as follows. In each of the proposed schemes (except a special case in Scheme 3), from the answers of any $N_r$ responding servers, the user can recover the transmissions by all servers. The only special case is in Scheme 3 when the set of responding servers contains one server in $[M / 2]$ and one server in $[M / 2 + 1 : M]$. However, if $A_1$ contains such two servers, the user can directly recover the task function; otherwise, we can find $N_r - |A_1|$ servers in $[M + 1 : N]$, such that from the answers of the total $N_r$ servers, the user can recover the transmissions by all servers.
APPENDIX A

PROOF OF THEOREM 2

We consider the distributed linearly separable computation problem in [6] where \( M = \frac{K}{N} (N - N_r + m) \) for \( m \in [N_r] \) and the user requests \( K_c \in [K] \) linearly independent combinations of messages. We now describe on a general linear coding computing scheme. In this scheme, we divide each message \( W_k \) where \( k \in [K] \) into \( \ell \) non-overlapping and equal-length sub-messages, \( W_k = \{ W_{k,i} : i \in [\ell] \} \). Server \( n \in [N] \) sends \( \frac{\ell T_n}{\ell} \) linearly independent combinations of \( W_{1,1}, W_{1,2}, \ldots, W_{K,\ell} \).

Considering the transmitted linear combinations by all servers, the number of linearly independent combinations is denoted by \( \lambda \) and these \( \lambda \) linear combinations can be expressed as

\[
F[ W_{1,1}; W_{1,2}; \ldots; W_{K,\ell} ]
\]

where

\[
F = \begin{bmatrix}
    f_1 \\
    \vdots \\
    f_\lambda
\end{bmatrix}
= \begin{bmatrix}
    f_{1,1} & \cdots & f_{1,K} \\
    \vdots & \ddots & \vdots \\
    f_{\lambda,1} & \cdots & f_{\lambda,K}
\end{bmatrix}.
\]

(43)

Notice that any linear scheme can be transformed in the above manner. Among the linear combinations in (43), \( f_i[ W_{1,1}; W_{1,2}; \ldots; W_{K,\ell} ] \) where \( i \in [\ell K_c] \) represent the desired task function of the user. The transmission of each server \( n \in [N] \) can be express as

\[
S_n F[ W_{1,1}; W_{1,2}; \ldots; W_{K,\ell} ],
\]

(44)

where the dimension of \( S_n \) is \( \frac{\ell T_n}{\ell} \times \lambda \).

For any set of \( N_r \) responding servers (denoted by \( A = \{ A(1), \ldots, A(N_r) \} \)), the user receives

\[
S_A F[ W_{1,1}; W_{1,2}; \ldots; W_{K,\ell} ]
\]

where \( S_A \) represents the row-wise sub-matrix of \( [ S_{A(1)}; \ldots; S_{A(N_r)} ] \) which has the same rank as \( [ S_{A(1)}; \ldots; S_{A(N_r)} ] \). Assume \( S_A \) contains \( r_A \) rows. In the decoding phase, the user multiply \( S_A F[ W_{1,1}; W_{1,2}; \ldots; W_{K,\ell} ] \) by \( D_A \), where the dimension of \( D_A \) is \( \ell K_c \times r_A \) and

\[
D_A S_A = [ I_{\ell K_c}, 0_{\ell K_c \times (r_A - \ell K_c)} ];
\]

(45)

where \( I_n \) represents the identity matrix with dimension \( n \times n \) and \( 0_{m \times n} \) represents the zero matrix with dimension \( m \times n \). Hence, the user can recover the desired task function from

\[
D_A S_A F[ W_{1,1}; W_{1,2}; \ldots; W_{K,\ell} ].
\]

Now we take the security constraint (10) into consideration, and extend the above general
linear coding scheme. We introduce $\lambda - \ell K_c$ independent randomness variables $Q_1, \ldots, Q_{\lambda - \ell K_c}$, where $Q_i, i \in [\lambda - \ell K_c]$ is uniformly i.i.d. over $[\mathbb{F}_q]^{\frac{1}{2}}$. We then let $F' = [F, S]$, where $S = [0_{\ell K_c \times (\lambda - \ell K_c)}; S']$ and $S'$ is full-rank with dimension $(\lambda - \ell K_c) \times (\lambda - \ell K_c)$.

We let each server $n \in [N]$ transmit

$$S_n F' [W_{1,1}; W_{1,2}; \ldots; W_{K,\ell}; Q_1; \ldots; Q_{\lambda - \ell K_c}],$$

where $S_n$ is the same as that in (44). It can be seen that in the transmitted linear combinations (46), the coefficients of the sub-messages which server $n$ cannot compute are still 0 as the original non-secure scheme.

For any set of $N_r$ responding servers $A$, the user receives

$$S_A F' [W_{1,1}; W_{1,2}; \ldots; W_{K,\ell}],$$

and recovers its desired task function from $D_A S_A F' [W_{1,1}; W_{1,2}; \ldots; W_{K,\ell}; Q_1; \ldots; Q_{\lambda - \ell K_c}]$, since (45) holds.

Finally, we will prove that the above scheme is secure, i.e., the security constraint (10) holds. From the answers of all servers, the user can only recover totally $\lambda$ linearly independent combinations, which are

$$F' [W_{1,1}; W_{1,2}; \ldots; W_{K,\ell}; Q_1; \ldots; Q_{\lambda - \ell K_c}].$$

In addition, $S'$ is full-rank (with rank equal to $\lambda - \ell K_c$). Hence, the user can only recover the desired task function (i.e., the first $\ell K_c$ linear combinations in (47)) without $Q_1, \ldots, Q_{\lambda - \ell K_c}$, and thus the above scheme is secure.

**APPENDIX B**

**PROOF OF THEOREM 3**

By the security constraint in (10), the user can only obtain $W_1 + \cdots + W_K$ without knowing any other information about the messages after receiving the answers of all servers. Recall that $X_S = \{X_n : n \in S\}$. Intuitively by [23], we need a key with length at least $H(X_{[N]}) - H(W_1 + \cdots + W_K)$ such that except $W_1 + \cdots + W_K$, the other information about $W_1, \ldots, W_K$ transmitted in $X_{[N]}$ is hidden. More precisely, from (10) we have

$$0 = I(W_1, \ldots, W_K; X_{[N]} | W_1 + W_2 + \cdots + W_K)$$

$$= H(X_{[N]} | W_1 + W_2 + \cdots + W_K) - H(X_{[N]} | W_1, \ldots, W_K)$$

(48a)

(48b)
\[ \geq H(X_{[N]}|W_1 + W_2 + \cdots + W_K) - H(Q, W_1, \ldots, W_K|W_1, \ldots, W_K) \quad \text{(48c)} \]
\[ = H(X_{[N]}|W_1 + W_2 + \cdots + W_K) - H(Q) \quad \text{(48d)} \]
\[ = H(X_{[N]}) - I(X_{[N]}; W_1 + W_2 + \cdots + W_K) - H(Q) \quad \text{(48e)} \]
\[ \geq H(X_{[N]}) - H(W_1 + W_2 + \cdots + W_K) - H(Q) \quad \text{(48f)} \]

where (48c) comes from that the \( X_{[N]} \) is a function of \( Q, W_1, \ldots, W_K \), (48d) comes from that \( Q \) is independent of \( W_1, \ldots, W_K \). Hence, from (48f) and we have
\[ \eta L \geq H(Q) \geq H(X_{[N]}) - H(W_1 + \cdots + W_K) \quad \text{(49a)} \]
\[ \geq H(X_{[N]}) - L \quad \text{(49b)} \]
\[ \geq H(X_{s_1}, \ldots, X_{s_v}) - L \quad \text{(49c)} \]
\[ = H(X_{s_1}) + H(X_{s_2}|X_{s_1}) + \cdots + H(X_{s_v}|X_{s_1}, \ldots, X_{s_{v-1}}) - L, \quad \text{(49d)} \]

where (49b) comes from that the \( K \) messages are independent and each message is uniformly i.i.d. over \([F_q]^L\), (49d) comes from the chain rule of entropy.

Let us then focus on each entropy term in (49d), \( H(X_{s_i}|X_{s_1}, \ldots, X_{s_{i-1}}) \) where \( i \in [v] \). Recall that server \( s_i \) can compute some message (assumed to be message \( W_j \)) which cannot be computed by servers \( s_1, \ldots, s_{i-1} \), and that each message cannot computed by \( N_r - 1 \) servers. We assume that the set of servers which cannot compute \( W_j \) is \( \overline{A}_j \). Obviously, \( \{s_1, \ldots, s_{i-1}\} \subseteq \overline{A}_j \). Now consider that the set of responding servers is \( \overline{A}_j \cup \{s_i\} \), totally containing \( N_r \) servers. As \( W_j \) can only computed by server \( s_i \) among the servers in \( \overline{A}_j \cup \{s_i\} \), and from the answers of servers in \( \overline{A}_j \cup \{s_i\} \) the user should recover \( W_1 + \cdots + W_K \), we have
\[ H(X_{s_i}|X_{s_1}, \ldots, X_{s_{i-1}}) \geq H(X_{s_i}|X_{s_k} : k \in \overline{A}_j) \geq L. \quad \text{(50)} \]

Hence, we take (50) into (49d) to obtain,
\[ \eta L \geq \nu L - L, \quad \text{(51)} \]
which proves (18).

**Appendix C**

**Proof of Theorem 7**

We first introduce the following lemma which will be proved in Appendix C-D and will be used in the proof of Theorem 7.
Lemma 4. For the \((K, N, K_r, K_c, M)\) secure distributed linearly separable computation problem with \(M = \frac{K}{N}M'\), \(K_c = 1\), \(\frac{M'}{\text{GCD}(N,M')} \geq 3\), and \(\text{Mod}\left(\frac{N}{\text{GCD}(N,M')}, \frac{M'}{\text{GCD}(N,M')}\right) = \frac{M'}{\text{GCD}(N,M')} - 1\), to achieve the optimal communication cost, it must hold that

\[
\eta^* \geq \left\lfloor \frac{N}{M'} \right\rfloor.
\]

(52)

□

We then start to prove Theorem 7 and focus on the \((K, N, K_r, K_c, M)\) secure distributed linearly separable computation problem where \(M = \frac{K}{N}M'\), \(K_c = 1\), and \(\frac{M'}{\text{GCD}(N,M')} \leq 4\). Notice that when \(\frac{M'}{\text{GCD}(N,M')} = 1\) (i.e., \(M'\) divides \(N\)), the optimality directly comes from Theorem 4. Hence, in the following, we consider the case where \(\frac{M'}{\text{GCD}(N,M')} \in [2 : 4]\).

A. \(\frac{M'}{\text{GCD}(N,M')} = 2\)

When \(\frac{M'}{\text{GCD}(N,M')} = 2\), it can be seen that \(\text{Mod}\left(\frac{N}{\text{GCD}(N,M')}, \frac{M'}{\text{GCD}(N,M')}\right) = 1\). Hence, by the proposed scheme for Theorem 6, the needed randomness size is

\[
h(N, M') - 1 = h\left(\frac{N}{\text{GCD}(N, M')}, \frac{M'}{\text{GCD}(N, M')}\right) - 1
\]

(53a)

\[
= \left\lfloor \frac{N}{M'} - 1 \right\rfloor + h(3, 2) - 1
\]

(53b)

\[
= \left\lfloor \frac{N}{M'} - 1 \right\rfloor + 2 - 1 = \left\lceil \frac{N}{M'} \right\rceil - 1,
\]

(53c)

where (53a) comes from (29), (53b) comes from (30), (53c) comes from that \(h(3, 2) = 2\) and that \(\left\lfloor \frac{N}{M'} + 1 \right\rfloor = \left\lceil \frac{N}{M'} \right\rceil\) since \(M'\) does not divide \(N\). It can be seen that the needed randomness size (53c) coincides with the converse bound in Corollary 2, and thus is optimal.

B. \(\frac{M'}{\text{GCD}(N,M')} = 3\)

When \(\frac{M'}{\text{GCD}(N,M')} = 3\), it can be seen that \(\text{Mod}\left(\frac{N}{\text{GCD}(N,M')}, \frac{M'}{\text{GCD}(N,M')}\right) \in [2]\).

For the case where \(\text{Mod}\left(\frac{N}{\text{GCD}(N,M')}, \frac{M'}{\text{GCD}(N,M')}\right) = 1\), by the proposed scheme for Theorem 6, the needed randomness size is

\[
h(N, M') - 1 = h\left(\frac{N}{\text{GCD}(N, M')}, \frac{M'}{\text{GCD}(N, M')}\right) - 1
\]

(54a)

\[
= \left\lfloor \frac{N}{M'} - 1 \right\rfloor + h(4, 3) - 1
\]

(54b)

\[
= \left\lfloor \frac{N}{M'} - 1 \right\rfloor + 2 - 1 = \left\lceil \frac{N}{M'} \right\rceil - 1,
\]

(54c)
where (54a) comes from (29), (54b) comes from (30), and (54c) comes from that \( h(4, 3) = 2 \).
The needed randomness size in (54c) coincides with the converse bound in Corollary 2, and thus is optimal.

For the case where \( \text{Mod} \left( \frac{N}{\text{GCD}(N, M')}, \frac{M'}{\text{GCD}(N, M')} \right) = 2 \), by the proposed scheme for Theorem 6, the needed randomness size is

\[
\begin{align*}
\text{h}(N, M') - 1 & = h \left( \frac{N}{\text{GCD}(N, M')}, \frac{M'}{\text{GCD}(N, M')} \right) - 1 \\
& = \lfloor N/M' - 1 \rfloor + h(5, 3) - 1 \\
& = \lfloor N/M' - 1 \rfloor + 3 - 1 = \lceil N/M' \rceil,
\end{align*}
\]

(55a)

(55b)

(55c)

where (55a) comes from (29), (55b) comes from (30), and (55c) comes from that \( h(5, 3) = 3 \).
The needed randomness size in (55c) coincides with the converse bound in Lemma 4, and thus is optimal.

C. \( \frac{M'}{\text{GCD}(N, M')} = 4 \)

When \( \frac{M'}{\text{GCD}(N, M')} = 4 \), it can be seen that \( \text{Mod} \left( \frac{N}{\text{GCD}(N, M')}, \frac{M'}{\text{GCD}(N, M')} \right) \in \{1, 3\} \).

For the case where \( \text{Mod} \left( \frac{N}{\text{GCD}(N, M')}, \frac{M'}{\text{GCD}(N, M')} \right) = 1 \), by the proposed scheme for Theorem 6, the needed randomness size is

\[
\begin{align*}
\text{h}(N, M') - 1 & = h \left( \frac{N}{\text{GCD}(N, M')}, \frac{M'}{\text{GCD}(N, M')} \right) - 1 \\
& = \lfloor N/M' - 1 \rfloor + h(5, 4) - 1 \\
& = \lfloor N/M' - 1 \rfloor + 2 - 1 = \lceil N/M' \rceil - 1,
\end{align*}
\]

(56a)

(56b)

(56c)

where (56a) comes from (29), (56b) comes from (30), and (56c) comes from that \( h(5, 4) = 2 \).
The needed randomness size in (56c) coincides with the converse bound in Corollary 2, and thus is optimal.

For the case where \( \text{Mod} \left( \frac{N}{\text{GCD}(N, M')}, \frac{M'}{\text{GCD}(N, M')} \right) = 3 \), by the proposed scheme for Theorem 6, the needed randomness size is

\[
\begin{align*}
\text{h}(N, M') - 1 & = h \left( \frac{N}{\text{GCD}(N, M')}, \frac{M'}{\text{GCD}(N, M')} \right) - 1 \\
& = \lfloor N/M' - 1 \rfloor + h(7, 4) - 1 \\
& = \lfloor N/M' - 1 \rfloor + 3 - 1 = \lceil N/M' \rceil,
\end{align*}
\]

(57a)

(57b)

(57c)
where (57a) comes from (29), (57b) comes from (30), and (57c) comes from that $h(7, 4) = 3$. The needed randomness size in (57c) coincides with the converse bound in Lemma 4 and thus is optimal.

**D. Proof of Lemma 4**

Recall that $M = \frac{K}{N} M'$, and that $\text{Mod} \left( \frac{N}{\text{GCD}(N, M')} \cdot \frac{M'}{\text{GCD}(N, M')} \right) = \frac{M'}{\text{GCD}(N, M')} - 1$. For the ease of notation, we let $a := \frac{M'}{\text{GCD}(N, M')}$. By the constraints in Lemma 4, we have $a \geq 3$.

We assume that $N = (2a - 1 + ab)\text{GCD}(N, M')$, where $b$ is a non-negative integer. To prove (52), it is equivalent to prove

$$\eta \geq \left\lceil \frac{N}{M'} \right\rceil = 2 + b - \frac{1}{a} = b + 2.$$  

In the following we use the induction method to prove that for any assignment, there must exist some ordered set of $b + 3$ servers satisfying the constraint in (17).

**Proof step 1.** We first consider the case where $b = 0$, and aim to prove that for any possible assignment, we can always find an ordered set of three servers $s = (s_1, s_2, s_3)$ such that server $s_2$ has some dataset not assigned to server $s_1$ and server $s_3$ has some dataset not assigned to servers $s_1, s_2$.

In this case, $N = 2M' - \text{GCD}(N, M')$. For any possible assignment, each dataset is assigned to $M'$ servers, and each server has (recall that $a := \frac{M'}{\text{GCD}(N, M')} \geq 3$)

$$M = \frac{K}{N} M' = \frac{M'K}{2M' - \text{GCD}(N, M')} = \frac{aK}{2a - 1}$$

datasets. It can be seen that $\frac{K}{2} < M < \frac{2}{3}K$. We can prove that there exist two servers (assumed to be server $s_1, s_2$) such that $\mathcal{Z}_{s_1} \neq \mathcal{Z}_{s_2}$ and $\mathcal{Z}_{s_1} \cup \mathcal{Z}_{s_2} \neq [K]$.\(^{10}\) Assumed that dataset $D_k$ is not assigned to server $s_1, s_2$. We then pick one server which has dataset $D_k$ (assumed to be server $s_3$). It can be see that the ordered set of servers $(s_1, s_2, s_3)$ satisfies the constraint in (17).

**Proof step 2.** We then focus on the case where $b = 1$. In this case,

$$N = (3a - 1)\text{GCD}(N, M') = 3M' - \text{GCD}(N, M'),$$

$$K = \frac{K}{N} M' = \frac{K}{N} \text{GCD}(N, M') = 3M - \frac{K}{N} \text{GCD}(N, M').$$

\(^{10}\) As $\frac{K}{2} < M < \frac{2}{3}K$, there must be three servers with three different sets of obtained datasets. We denote these three servers by servers $n_1, n_2, n_3$. We then prove by contradiction that there exist two servers in $\{n_1, n_2, n_3\}$, where some dataset is not assigned to any of them. Assume that $\mathcal{Z}_{n_1} \cup \mathcal{Z}_{n_2} = [K]$, $\mathcal{Z}_{n_1} \cup \mathcal{Z}_{n_3} = [K]$, and $\mathcal{Z}_{n_2} \cup \mathcal{Z}_{n_3} = [K]$. Hence, each dataset must be assigned to at least two servers in $\{n_1, n_2, n_3\}$, and thus there are at least $2K/3$ datasets assigned to each server, which contradicts $M < \frac{2}{3}K$. 

Hence, $K - M = 2M - \frac{K}{N}\text{GCD}(N, M') > 2M - \frac{K}{N}M' = M$. WLOG, we assume that server 1 has datasets $D_1, \ldots, D_M$. Let us then focus on the datasets $D_{M+1}, \ldots, D_K$, and consider the following two cases:

- **Case 1:** among $Z_2 \cap [M + 1 : K], \ldots, Z_N \cap [M + 1 : K]$, there are at least three different non-empty sets. As shown before, there must exist an ordered set of three servers in $[2 : N]$, which is assumed to be $(s_1, s_2, s_3)$, satisfying the constraint in (17). In addition, server 1 does not have any datasets in $[M + 1 : K]$. Hence, the ordered set of servers $(s_1, s_2, s_3)$ also satisfies the constraint in (17).

- **Case 2:** among $Z_2 \cap [M + 1 : K], \ldots, Z_N \cap [M + 1 : K]$, there are only two different non-empty sets. In this case, there are $M'$ servers (assumed to be $V_1$) whose obtained datasets in $[M + 1 : K]$ are non-empty and the same; there are other $M'$ servers (assumed to be $V_2$) whose obtained datasets in $[M + 1 : K]$ are non-empty and the same. We then consider the sets $Z_n$ where $n \in (V_1 \cup V_2)$. It is not possible that there are only two different sets among them, because the datasets in $[M]$ are only assigned to the servers in $[N] \setminus (V_1 \cup V_2)$ where $|[N] \setminus (V_1 \cup V_2)| < M'$. Hence, there must exist at least three different sets among $Z_n$ where $n \in (V_1 \cup V_2)$. Thus there must be an ordered set of three servers in $(V_1 \cup V_2)$ satisfying the constraint in (17), which is assumed to be $(s_1, s_2, s_3)$. In addition, in the union set of the assigned datasets to these three servers, there are at most $2M - (K - M) < M$ datasets in $[M]$.

Hence, the ordered set of servers $(s_1, s_2, s_3, 1)$ also satisfies the constraint in Theorem 3.

**Proof step 3.** Finally, we assume when $b \in [x]$, there exists some ordered set of servers $(s_1, \ldots, s_{b+3})$ satisfying the constraint in Theorem 3. We will show that when $b = x + 1$, there exists some ordered set of servers $(s_1, \ldots, s_{x+4})$ satisfying the constraint in Theorem 3.

In this case,

$$N = ((3 + x)a - 1)\text{GCD}(N, M') = (3 + x)M' - \text{GCD}(N, M'),$$

$$K = \frac{K}{N}N = (3 + x)\frac{K}{N}M' - \frac{K}{N}\text{GCD}(N, M') = (3 + x)M - \frac{K}{N}\text{GCD}(N, M').$$

Hence, we have $(x + 2)M < K < (x + 3)M$. As there are $K$ datasets, each of which is assigned to $M'$ servers, and the number of datasets assigned to each server is $M$, we can find $x + 1$ servers where each server has some dataset not assigned to other $x$ servers.

If there exists some server $j \in [x + 1]$ where at least $\frac{K}{N}\text{GCD}(N, M') + 1$ datasets assigned to server $j$ have also been assigned to some other server in $[x + 1] \setminus \{j\}$, it can be seen that there
are at least
\[ K - (x + 1)M + \frac{K}{N} \text{GCD}(N, M') + 1 = 2M + 1 \]
datasets not assigned to \([x + 1]\); thus we can find three servers which has some dataset not assigned to the servers in \([x + 1]\) nor the other two servers. Hence, we can find an ordered set of \(x + 4\) servers satisfying the constraint in (17).

Hence, in the following we consider that at most \(\frac{K}{N} \text{GCD}(N, M')\) datasets assigned to each server in \([x + 1]\) have also been assigned to some other server in \([x + 1]\). There are totally at most \((x + 1)M\) different datasets to these \(x + 1\) servers, and thus there remains at least \(K - (x + 1)M = 2M - \frac{K}{N} \text{GCD}(N, M')\) datasets which are not assigned to these \(x + 1\) servers.

WLOG, we assume that the \(x + 1\) servers are in \([x + 1]\) and these \(2M - \frac{K}{N} \text{GCD}(N, M')\) datasets are \(D_{(x+1)M+1}, \ldots, D_K\).

We then consider two cases:

- **Case 1:** among \(Z_{x+2} \cap [(x + 1)M + 1 : K], \ldots, Z_N \cap [(x + 1)M + 1 : K]\), there are at least three different non-empty sets. As shown before, there must exist an ordered set of three servers in \([x + 2 : N]\), which is assumed to be \((s_1, s_2, s_3)\), satisfying the constraint in (17). In addition, servers in \([x + 1]\) do not have any datasets in \([M + 1 : K]\). Hence, the ordered set of servers \((1, 2, \ldots, x + 1, s_1, s_2, s_3)\) also satisfies the constraint in (17).

- **Case 2:** among \(Z_{x+2} \cap [(x + 1)M + 1 : K], \ldots, Z_N \cap [(x + 1)M + 1 : K]\), there are only two different non-empty sets. In this case, there are \(M'\) servers (assumed to be \(\mathcal{V}_1\)) whose obtained datasets in \([(x + 1)M + 1 : K]\) are non-empty and the same; there are other \(M'\) servers (assumed to be \(\mathcal{V}_2\)) whose obtained datasets in \([(x + 1)M + 1 : K]\) are non-empty and the same. We then consider the sets \(Z_n\) where \(n \in (\mathcal{V}_1 \cup \mathcal{V}_2)\).

  - If there are two different sets among them, we have completely assigned \(2M\) datasets to \(2M'\) servers, each of which has \(M\) datasets. Hence, if we focus on the assignment for the servers in \([N] \setminus (\mathcal{V}_1 \cup \mathcal{V}_2)\), the assignment is equivalent to the problem where we assign \(K_1 = K - 2M\) datasets to \(N_1 = N - 2M' = (2a - 1 + (x - 1)a) \text{GCD}(N, M')\) servers, where each dataset is assigned to \(M'\) servers and the number of assigned datasets to each server is \(M\). Therefore, we can use the induction assumption to find an ordered set of \((x - 1) + 3 = x + 2\) servers satisfying the constraint in (17), which are assumed to be servers \((s_1, \ldots, s_{x+2})\). In addition, we can pick two server in \(\mathcal{V}_1 \cup \mathcal{V}_2\) with different sets of datasets, which are assumed to be servers \(s_{x+3}, s_{x+4}\). In summary, the ordered
set of servers \((s_1, \ldots, s_{x+4})\) also satisfies the constraint in \((17)\).

- If there are at least three different sets among them, we can find an ordered set of three servers in \((V_1 \cup V_2)\) satisfying the constraint in \((17)\), which is assumed to be \((s_1, s_2, s_3)\). In addition, in the union set of the assigned datasets to these three servers, there are at most \(2M - (K - (x + 1)M) = \frac{N}{N'} \text{GCD}(N, M')\) datasets in \([x+1]M\). In addition, each server in \([x+1]\) has \(M \geq 3\frac{N}{N'} \text{GCD}(N, M')\) datasets and at most \(\frac{K}{N} \text{GCD}(N, M')\) datasets assigned to each server in \([x+1]\) have also been assigned to some other server in \([x+1]\). As a result, the ordered set of servers \((s_1, s_2, s_3, 1, 2, \ldots, x+1)\) satisfies the constraint in \((17)\).

In conclusion, by the induction method, we proved Lemma 4.

APPENDIX D

PROOF OF DECODABILITY OF SCHEME 4

A. Proof of Lemma

Recall that for each server \(n \in [y + 1 : M]\), it computes a linear combination \(s_n \left[ a f_1 - f_2; f_3; \ldots; f_{M+5-y} \right]\), where \(s_n\) is a random linear combination of the two linearly independent vectors in the left-hand side nullspace of \(\frac{M-1}{2} - y\) neighbouring columns in \(F'_3\). We aim to prove that for any set \(V \subseteq [y + 1 : M]\) where \(|V| = \frac{M+3}{2} - y\), the vectors \(s_n\) where \(n \in V\) are linearly independent.

As the field size \(q\) is large enough, following the decodability proof in [3, Appendix C] (which also proves that the transmission vectors by a set of servers are linearly independent with high probability) based on the Schwartz-Zippel lemma [24]–[26], we only need to find out one specific realization of

\[
F'_3 = \begin{bmatrix}
  a & a & \ldots & a \\
  * & * & \ldots & * \\
  \vdots & \vdots & \ddots & \vdots \\
  * & * & \ldots & * 
\end{bmatrix},
\]

such that the vectors \(s_n\) where \(n \in V\) are linearly independent.

We sort the servers in \(V\) in an increasing order, \(V = \{v_1, \ldots, v_{\frac{M+3}{2} - y}\}\), where \(v_i < v_j\) if \(i < j\). For each \(i \in \left[\frac{M+1}{2} - y\right]\), we let the \((i+1)\)th row of \(F'_3\) be

\[
[*, *, \ldots, *, 0, 0, \ldots, 0, *, *, \ldots, *],
\]

where each 0 corresponds to one distinct message which server \(v_i\) cannot compute.
By the construction of $F'_3$, we let the transmission vector $s_{v_i}$ of each server $v_i$ where $i \in \left[ \frac{M+1}{2} - y \right]$ be as follows:\textsuperscript{11}

\[ s_{v_i} = [0, \ldots, 0, 1, 0, \ldots, 0], \]

where $s_{v_i}$ has $\frac{M+3}{2} - y$ elements and the $(i+1)^{th}$ element is 1.

Let us then design the transmission vector $s_{v_{\frac{M+3}{2} - y}}$ of server $v_{\frac{M+3}{2} - y}$ by letting its first element be 1 and second element be 0. By the proof in [3, Appendix D], the remaining elements in $s_{v_{\frac{M+3}{2} - y}}$ are obtained by solving linear equations, which are all not necessarily be all 0. Hence, by construction $s_{v_1}, \ldots, s_{v_{\frac{M+3}{2} - y}}$ are linearly independent.

\textbf{B. Proof of Lemma 2}\textsuperscript{2}

Recall that $L_1$ contains $\frac{M+1}{2} - y$ linearly independent combinations of $F_2, \ldots, F_{\frac{M+5}{2} - y}$, and one linear combination $F_1 - F_2$. The coefficients in these linear combinations are independent of $a$.

Let us focus on the servers in $[y+1 : M]$ and the servers in $[M + 1 : N]$. Recall that the transmission of each server $n \in [y+1 : M]$ is

\[ s_n \left[ a f_1 - f_2; f_3; \ldots; f_{\frac{M+1}{2} - y} \right] [W_1; \ldots; W_N]. \]

Server $n$ can compute $\frac{M+1}{2}$ neighbouring messages in $W_{M+1}, \ldots, W_N$, which are

\[ W_{\text{Mod}(n-y,N-M)+M}; W_{\text{Mod}(n-y+1,N-M)+M}; \ldots; W_{\text{Mod}(n-y+\frac{M-1}{2},N-M)+M}. \tag{58} \]

The transmission vector $s_n$ of server $n$ is in the left-hand side null space of the column-wise sub-matrix of $F'_3$, including the columns corresponding to the $N - M - \frac{M+1}{2} = \frac{M-1}{2} - y$ messages in $W_{M+1}, \ldots, W_N$ which server $n$ cannot compute. Notice that this null space contains $\frac{M+1}{2} - y + 1 - \frac{M-1}{2} - y = 2$ linearly independent vectors, and $s_n$ is a random combination of them.

It can be seen that the number of servers in $[y+1 : M]$ is the same as the number of servers in $[M + 1 : N]$, which are both equal to $N - M = M - y$. In addition, for each $n \in [y+1 : M]$, server $n + (M - y)$ (which is in $[M + 1 : N]$) can compute the messages

\[ W_{\text{Mod}(n-y,N-M)+M}; W_{\text{Mod}(n-y+1,N-M)+M}; \ldots; W_{\text{Mod}(n-y+\frac{M-1}{2} - 1,N-M)+M}. \]

\textsuperscript{11} Notice that $F'_3$ contains $\frac{M+3}{2} - y$ rows, and server $v_i$ cannot compute $\frac{M+1}{2} - y$ messages in $W_{M+1}, \ldots, W_N$; thus we need to fix two positions in $s_{v_i}$ and then to determine the other elements $s_{v_i}$ by solving linear equations. More precisely, we let the first element be 0 and the $(i+1)^{th}$ element be 1. With the above construction, by the proof in [3, Appendix D], the remaining elements in $s_{v_i}$ are obtained by solving linear equations, which are all 0.
among the messages in \([M + 1 : N]\). Hence, compared to the set of messages in \([M + 1 : N]\) that server \(n + (M - y)\) can compute, it can be seen from (58) that server \(n\) additionally has \(W_{\text{Mod}}(n - y + M + 1, N - M) + M\).

Let us then prove Lemma 2 by the Schwartz-Zippel lemma \([24]–[26]\), as in \([3\) Appendix C\]. More precisely, for any set of \(M + 1 - y\) servers in \([M + 1 : N]\) (denoted by \(A_1\)) and any set of \(M + 1 - y\) servers in \([y + 1 : M]\) (denoted by \(A_2\)), we aim to find one specific realization of \(a\) and \(F_{3}^{[3;M+5-2-y]}\), such that there exists one server in \(A_2\) whose transmission is linearly independent of the linear combinations in \(L_1\).

We sort the servers in \(A_1\) in an increasing order, \(A_1(1) < \cdots < A_1\left(\frac{M + 1}{2} - y\right)\). For each \(i \in \left[\frac{M + 1}{2} - y\right]\), we let the \(i\)th row of \(F_{3}^{[3;M+5-2-y]}\), be

\[ [*, *, \cdots, *, 0, 0, \cdots, 0, *, *, \cdots, *], \]

where each 0 corresponds to one distinct message which server \(A_1(i)\) cannot compute. After determining such \(F_{3}^{[3;M+5-2-y]}\), as shown in \([3\) Appendix D\], the transmission of server \(A_1(i)\) is \(s_{A_1(i)} [f_2; \ldots; f_{M+5-2-y}] [W_1; \ldots; W_N]\), where

\[ s_{A_1(i)} = [0, \cdots, 0, 1, 0, \cdots, 0]. \]

Notice that \(s_{A_1(i)}\) contains \(\frac{M + 3}{2} - y\) elements and 1 is located at the \((i + 1)\)th position.

In other words, it can be seen that the transmissions of the \(\frac{M + 1}{2} - y\) servers in \([M + 1 : N]\) are \(F_3, \ldots, F_{M+5-2-y}\). In addition, the transmission of each server in \([y]\) is \(F_1 - F_2\).

Let us then determine the transmissions of the servers in \(A_2 \subseteq [y + 1 : M]\). It can be easily proved that among the servers in \(A_2\), there must exist one server (assumed to be \(n'\)) the set of whose available messages in \(W_{M+1}, \ldots, W_N\), is a super set of the available messages in \(W_{M+1}, \ldots, W_N\) to some server (assumed to be \(n''\)) in \([M + 1 : N] \setminus A_1\).\(^{12}\) The transmission of \(n''\) is assumed to be

\[ s_{n''} [f_2; \ldots; f_{M+5-2-y}] [W_1; \ldots; W_N], \]

where \(s_{n''}\) is the vector in the left-hand side null space (which contains two independent vectors) of the column-wise sub-matrix of \(F_{3}^{[3;M+5-2-y]}\), corresponding the unavailable messages in \([M + 1 : N]\).

\(^{12}\) The proof is as follows. For any \(N - M - |A_1| = \frac{M - 1}{2}\) servers in \([M + 1 : N]\), there are at least \(\frac{M - 1}{2} + 2\) servers in \([y + 1 : M]\) the set of whose obtained datasets is a super set of the set of the obtained datasets by some of these \(\frac{M - 1}{2}\) servers. Hence, \(A_2\) contains \(\frac{M - 1}{2} - y\) servers in \([y + 1 : M]\), which must contain some of these \(\frac{M - 1}{2} + 2\) servers because \(\frac{M - 1}{2} - y + \frac{M - 1}{2} + 2 = M - y + 2 > M - y\).
N] to server \( n'' \). By the proof in \([3\) Appendix C\], the first element of \( s_{n''} \) is not 0 with high probability; thus \( s_{n''} \) is linearly independent of the transmissions of the servers in \( A_1 \) with high probability. Recall that the transmission vector of server \( n' \) is a random vector in the left-hand side null space (which contains two independent vectors) of the column-wise sub-matrix of \( F_3' \) corresponding the unavailable messages in \([M + 1 : N]\) to server \( n' \). In addition, the set of available messages in \([M + 1 : N]\) to server \( n' \) is a super set of that to server \( n'' \). Hence, \( s_{n''} \) can be also the transmission vector of server \( n' \); that is, server \( n' \) transmits

\[
  s_{n'} \begin{bmatrix}
    a f_1 - f_2; f_3; \ldots; f_{M+5-y}
  \end{bmatrix} \begin{bmatrix}
    W_1; \ldots; W_N,
  \end{bmatrix},
\]

where \( s_{n'} \) is obtained by replacing the first element of \( s_{n''} \) (assumed to be \( s \)) by \( \frac{s}{a-1} \).

Recall that \( \mathcal{L}_1 \) contains \( F_3, \ldots, F_{M+5-y} \) and \( F_1 - F_2 \). As \( a \) is uniformly over \( F_q \setminus \{0, 1\} \), it can be seen that the transmission in (59) is independent of the linear combinations in \( \mathcal{L}_1 \) with high probability.

**APPENDIX E**

**PROOF OF LEMMA 5**

We will prove Lemma 5 by the Schwartz-Zippel lemma \([24]-[26]\). Hence, we need to find one specific realization of \( a_1, \ldots, a_{h(M,2M-N)-1} \), such that the answers of any \( h(M, 2M-N) - \lambda_1 \) servers in \( A_2 \) are linearly independent of the answers of the servers in \( A_1 \).

Without loss of generality, we assume a possible set of \( h(M, 2M-N) - \lambda_1 \) linearly independent transmission vectors is,

\[
  \left[ (S'_1)(h(M,2M-N)-\lambda_1) \times \lambda_1; I_{h(M,2M-N)-\lambda_1} \right],
\]

where the \( h(M, 2M-N) - \lambda_1 \) linear combinations in \([S'_1, I_{h(M,2M-N)-\lambda_1}] F[W_1; \ldots; W_N]\) are linearly independent of the answers of the servers in \( A_1 \).

We now prove that there exist a matrix \([1; a_1; \ldots; a_{h(M,2M-N)-1}]\), whose left-hand side null space contains all row vectors in (60). This can be proved by randomly choosing the values of

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13 It holds that \(|A_2| \geq h(M, 2M-N) - \lambda_1 \). This is because, as shown in Footnote 9, if from \( A_1 \) the user cannot recover the task function, then there must exist \( N_s = |A_1| + |A_2| \) servers in \([M]\) containing the servers in \( A_1 \), such that the answers of these \( N_s \) servers contain \( h(M, 2M-N) \) linearly independent combinations. Recall that the number of linearly independent combinations transmitted by the servers in \( A_1 \) is \( \lambda \). Hence, we have \(|A_2| \geq h(M, 2M-N) - \lambda_1 \).
and then determining the values in \([a_{\lambda_1}; \ldots; a_{h(M,2M-N)-1}]\) as follows,

\[

\begin{bmatrix}
  a_{\lambda_1} \\
  \vdots \\
  a_{h(M,2M-N)-1}
\end{bmatrix}
= -S_1'\begin{bmatrix}
  1 \\
  a_1 \\
  \vdots \\
  a_{\lambda_1-1}
\end{bmatrix}

\]

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