ABSTRACT

We use normal coordinate methods to obtain the expansion with respect to fermionic coordinates of the 11-dimensional supermembrane action in a supergravity background. Likewise, expansions for various branes in other dimensions can be obtained. These methods allow a systematic and unambiguous expansion of the vielbein to any order in the fermionic coordinates and avoid the complications encountered in the gauge completion approach.

PACS: 04.65.+e,11.90.+t,12.60.Jv

Keywords: Superspace, Supergravity, Membranes.
1 Introduction

In the past few years, in the course of investigations of properties of branes in supergravity backgrounds, several authors [1, 2, 3, 4, 5, 6, 7] have performed expansions of supermembrane actions with respect to the fermionic coordinates $\theta^\alpha$. For such expansions, which involve in particular expanding the vielbein $E^A(x, \theta)$, they have employed an abomination known as “gauge completion” (pace friends), a procedure which manages to cast component results in a superspace language by laboriously comparing order-by-order component supersymmetry transformations with superspace coordinate transformations. This procedure is complicated, ambiguous, unmanageable at higher orders, and just plain ugly. It may be unavoidable in some cases where the appropriate superspace formalism does not exist or is not sufficiently developed. (We note that for some specific backgrounds compact explicit form of the vielbein can be obtained. This is the case for AdS$_5 \times$S$^5$ [8], as well as AdS$_4 \times$S$^7$ and AdS$_7 \times$S$^4$ [9, 2]; see also ref. [10].) However, in all cases where a geometric description is available in terms of superspace covariant derivatives with torsions and curvatures satisfying suitable constraints a much more elegant and efficient method exists, based on normal coordinate expansions of the vielbein and all other quantities [11, 12, 13, 14, 15, 16]. This method has been used primarily for the expansion of the Green-Schwarz superstring action [12, 13, 14], or for the derivation of the superspace density formula [15, 16], but it can be readily applied to the expansion of various brane actions. In this paper we demonstrate its use for the case of the 11-dimensional supermembrane. In a later publication we shall consider various branes in 10-dimensional supergravity backgrounds.

In the following sections we first summarize the basic elements of 11-dimensional supergravity and the description of the supermembrane. We then give the general form of the normal coordinate expansion and specialize to the present case, writing down the expansion of the vielbein components $E_M^A$ through the first three orders in the fermionic coordinates. We identify the component fields (graviton, gravitino and three-form field strength, as well as their (component) covariant derivatives) that appear in the expansion, and finally give the low-order expansion of the membrane, specializing eventually to a bosonic background.

2 11-dimensional supergravity

The theory is described in a superspace with coordinates $Z^M = (x^m, \theta^\mu)$ by the vielbein $E^A(x, \theta) = dZ^M E_M^A$ and three-form $B = (1/3!) E^C E^B E^A B_{ABC}$ satisfying torsion constraints and field-strength constraints respectively, as follows [17, 18]:
\[\begin{align*}
(a) \quad T_{\alpha\beta}^\gamma &= -i (\Gamma^c)_{\alpha\beta} \\
(b) \quad T_{\alpha\beta}^\gamma &= T_{ab}^c = T_{ab}^c = 0 \\
(c) \quad H_{\alpha\beta\gamma\delta} &= H_{\alpha\beta\gamma\delta} = H_{abcd} = 0 \\
(d) \quad H_{\alpha\beta\gamma\delta} &= i (\Gamma_{cd})_{\alpha\beta} \\
\end{align*}\]

with \( H = dB = (1/4!) E^D E^C E^B E^A H_{ABCD} \) and

\[H_{ABCD} = \sum_{(ABCD)} \nabla_A B_{BCD} + T_{AB}^E B_{ECD}\] (2.1)

(We use real Majorana gamma–matrices \( \Gamma^a \), and \( \Gamma^{abc} \) antisymmetrized with unit strength.) These constraints put the theory on shell.

From the Bianchi identities \( DT^A = E^B R_B^A, DR_A^B = 0 \) and \( dH = 0 \), or

\[\begin{align*}
\sum_{(ABC)} (R_{ABC}^D - \nabla_A T_{BC}^D - T_{AB}^E T_{EC}^D) &= 0 \\
\sum_{(ABCD)} (\nabla_A R_{BCD}^E + T_{AB}^F R_{FCD}^E) &= 0 \\
\sum_{(ABCD)} (\nabla_A H_{BCDE} + T_{AB}^F H_{FCDE}) &= 0
\end{align*}\] (2.3)

one derives expressions for the remaining components of the torsion and the components of the curvature:

\[\begin{align*}
(e) \quad T_{\alpha\beta}^\gamma &= \frac{1}{36} (\delta^b_a \Gamma_{bcde}^\gamma + \frac{1}{8} \Gamma_{a b c d e}^\gamma) H_{bcde} \\
(f) \quad T_{ab}^c &= \frac{i}{42} (\Gamma_{cd}^\gamma)^{\alpha\beta} \nabla_\beta H_{abcd} \\
(g) \quad (\Gamma^{abc})_{\alpha\beta} T_{bc}^\gamma &= 0 \\
(h) \quad R_{ab,\gamma}^\delta &= \nabla_a T_{b\gamma}^\delta - \nabla_b T_{a\gamma}^\delta + \nabla_\gamma T_{ab}^\delta + T_{a\gamma}^\epsilon T_{bc}^\delta - T_{b\gamma}^\epsilon T_{a\delta}^\gamma \\
(i) \quad R_{ab,cd} &= \frac{i}{2} [(\Gamma_b)_{\alpha\beta} T_{cd}^\beta - (\Gamma_c)_{\alpha\beta} T_{db}^\beta + (\Gamma_d)_{\alpha\beta} T_{cb}^\beta] \] (2.4)

\[\begin{align*}
(j) \quad R_{\alpha\beta,ab} &= -\frac{i}{6} \left[(\Gamma_{cd})_{\alpha\beta} H_{abcd} + \frac{i}{24} (\Gamma_{abcdef})_{\alpha\beta} H_{cdef} \right]
\end{align*}\]

with

\[ R_{AB\gamma}^\delta = \frac{1}{4} R_{ABcd} (\Gamma_{cd})_{\gamma}^\delta \] (2.5)

\[ R_{AB\gamma}^\delta = \frac{1}{4} R_{ABcd} (\Gamma_{cd})_{\gamma}^\delta \] (2.6)

\[ R_{AB\gamma}^\delta = \frac{1}{4} R_{ABcd} (\Gamma_{cd})_{\gamma}^\delta \] (2.6)

\[ R_{AB\gamma}^\delta = \frac{1}{4} R_{ABcd} (\Gamma_{cd})_{\gamma}^\delta \] (2.6)

\[ R_{AB\gamma}^\delta = \frac{1}{4} R_{ABcd} (\Gamma_{cd})_{\gamma}^\delta \] (2.6)

\[ R_{AB\gamma}^\delta = \frac{1}{4} R_{ABcd} (\Gamma_{cd})_{\gamma}^\delta \] (2.6)

In eq. (19) of the second paper of ref. [17] the factor of 1/3 should be replaced by 1/24. We thank P. Howe for a communication concerning this.
We will need the following additional consequence of the Bianchi identities:

\[(k)\quad \nabla_\alpha H_{bcde} = -6i(\Gamma_{[bc})\alpha\beta T_{de]}\beta \quad (2.7)\]

\[(l)\quad \nabla_\alpha R_{bc,de} = \nabla_b R_{c\alpha,de} - \nabla_c R_{b\alpha,de} + T_{b\alpha}^\gamma R_{\gamma c,de} - T_{c\alpha}^\gamma R_{\gamma b,de} - T_{bc}^\beta R_{\beta \alpha,de}\]

which can be used to relate higher components (in \(\theta\)) of field strengths and curvatures to lower components.

We also note the three-form equation of motion (a consequence of the constraints)

\[
\nabla^a H_{abcd} = -\frac{1}{1728} \varepsilon_{bcde1\cdots e8} H^{e1\cdots e4} H^{e5\cdots e8} \quad (2.8)
\]

### 3 The Supermembrane

The supermembrane action is written in terms of superspace embedding coordinates \(Z^M(\zeta) = (x^m(\zeta), \theta^\mu(\zeta))\), functions of the world-volume coordinates \(\zeta^i\) \((i = 0, 1, 2)\):

\[
S(Z) = \int d^3\zeta \left[ -\sqrt{-\det G(Z)} - \frac{1}{6} \varepsilon^{ijk} \Pi_i^A \Pi_j^B \Pi_k^C B_{CBA}(Z) \right] \quad (3.1)
\]

with \(\Pi_i^A = (\partial Z^M/\partial \zeta^i) E_M{}^A(Z)\) and \(G_{ij} = \Pi_i^a \Pi_j^b \eta_{ab}\). It is \(\kappa\) invariant when the background satisfies the supergravity constraints.

For the applications we have in mind, one wants to expand the action with respect to the fermionic coordinates \(\theta\) and identify the coefficients with various supergravity component fields: the component vielbein, the gravitino, etc. Aside from some technical complications, this expansion is not difficult for tensors such as the three-form, but is highly nontrivial for the vielbein if one proceeds in a brute force manner. As we have indicated in the introduction, previous workers have used gauge completion to achieve this task, but it is fairly clear that this procedure becomes quickly unmanageable if one tries to proceed beyond second order on the way to the full expansion at 32nd order.

Our approach, which is based on applying standard normal coordinate expansions to this case, is systematic, recursive, covariant, and in principle could be carried out all the way to the 32nd order - not that we recommend it. In this approach the fermionic variables are normal coordinates on the curved superspace manifold in directions “perpendicular” to the bosonic base manifold, and the expansion automatically produces coefficients which are covariant objects, torsions, curvatures and field strengths. In general, the normal coordinates \(y^a\) are related by field redefinitions to the ordinary \(\theta^a\) used in the gauge completion procedure, but are equally suitable for all applications.
4 Normal Coordinate Expansions

We shall rely primarily on ref. [15] (see also ref. [13]) and we refer the reader to that reference for an explanation of the procedure and some of the details. Basically, one chooses a point \( Z^M = (x^m, \theta^\mu = 0) \) on the superspace manifold as the origin of normal coordinates, and parametrizes its neighborhood by coordinates along the tangent hyperplane, \( y^A = (y^a, y^\alpha) \). Subsequently, by setting \( y^a = 0 \) one thus parametrizes the manifold by \( (x^m, y^\alpha) \). One introduces an operation \( \delta \) whose repeated iteration gives the successive terms in the Taylor expansion

\[
E^A(z; y) \equiv dZ^M E_M^A = E^A(Z) + \delta E^A + \frac{1}{2!} \delta^2 E^A + \frac{1}{3!} \delta^3 E^A + \ldots
\]  

(4.1)

This operator acts as follows:

\[
\delta E^A = Dy^A + y^C E^B T_{BC}^A
\]  

(4.2)

where the covariant differential is

\[
Dy^A \equiv E^B \nabla_B y^A = dy^A - y^B \omega_B^A = dy^A - y^B E^C \omega_{CB}^A
\]  

(4.3)

Also

\[
\delta Dy^A = -y^B E^C y^D R_{DCB}^A
\]  

(4.4)

and for any tensor

\[
\delta T = y^A \nabla_A T
\]  

(4.5)

We record here the first few terms in the expansion of the vielbein:

\[
\delta^2 E^A = -y^B E^C y^D R_{DCB}^A + y^C E^B y^D \nabla_D T_{BC}^A + y^C (Dy^B + y^E T_{DE}^B) T_{BC}^A
\]  

(4.6)

\[
\delta^3 E^A = -y^D (Dy^B + y^F E^G T_{GF}^B) y^C R_{CBD}^A - y^D E^B y^C y^F \nabla_F R_{CBD}^A + 2y^C (Dy^B + y^E E^F T_{FE}^B) T_{DG}^B T_{BC}^A + y^C Dy^B y^D \nabla_D T_{BC}^A + y^C y^D E^F y^E R_{EF}^B T_{BC}^A + y^C y^F y^D y^E (\nabla_E T_{DF}^B) T_{BC}^A
\]  

(4.7)

The next term is:

\[
\delta^4 E^A = +3y^C y^F (Dy^E + y^G E^H T_{HG}^E) T_{EF}^B y^D \nabla_D T_{BC}^A + 3y^C (Dy^B + y^F E^G T_{GF}^B) y^D y^E \nabla_E T_{BC}^A + y^C y^D (Dy^F + y^G E^H T_{HG}^F) y^E R_{EF}^B T_{BC}^A + 2y^C y^F (Dy^D + y^E E^G T_{GE}^B) y^H (\nabla_H T_{DF}^B) T_{BC}^A
\]  

(4.8)
Higher order terms can be obtained by applying the $\delta$ operation iteratively. Lest the reader be frightened by the large number of terms one generates, let us point out that the supergravity constraints render many of the terms in the expansion zero, as we shall see in the next section.

The pullbacks $\Pi^A_i = \partial_i Z^M E_M^A$ have an equivalent expansion\footnote{With our conventions, there is a sign difference from these references.} \[ (4.9) \text{ with } \delta \Pi^A_i = D_i y^A + y^B \Pi^C_i T_{CB}^A \]

and

\[ (4.10) \]

so that the successive terms can be read immediately from \( (4.2, 4.6, 4.8) \).

In the above formulas $\omega_{BC}^A$ is the supergravity Lorentz connection; this non-covariant term will drop out in any expression which is Lorentz invariant. Furthermore, it would appear that in the expansion of the three-form, according to \( (4.5) \) non-gauge-invariant supergravity covariant derivatives $\nabla_{AB}^{BCD}$ would appear. However, it is not difficult to check that in the expansion of the Wess-Zumino term in the action, from the first variation of the vielbein factors, one generates torsion factors so that in fact

\[ \delta \varepsilon^{ijk} \Pi^A_i \Pi^B_j \Pi^C_k B_{CBA}(Z) = \varepsilon^{ijk} y^A \Pi^B_j \Pi^C_k H_{DCBA} \]

up to total derivatives. Subsequent terms in the expansion involve only the gauge-invariant field strength.

For the actual application we have in mind, the expansion of the supermembrane action in powers of the fermionic variables, the following simplifications occur, in addition to those due to the supergravity constraints:

- $y^A = (0, y^n)$
• All quantities in the expansion are evaluated at $Z^M = (x^m, 0)$ so that they just involve the $\theta = 0$, first components, of the superfields and their derivatives. In particular we have (in Wess-Zumino gauge but in fact our expansion is completely supergravity gauge-covariant), with $E^A_M| = E^A_M(x, 0)$

\[
E_m^a| = e_m^a(x) \\
E_m^\alpha| = \psi_m^\alpha(x) \\
E_\mu^a| = 0 \\
E_\mu^\alpha| = \delta^\alpha_\mu
\]

as well as

\[
T_{cd}^\alpha| = \hat{\psi}_{cd}^\alpha
\]

the supercovariantized gravitino field strength. We also note that the choice of Wess-Zumino gauge implies that the spinor covariant derivative connection vanishes at $\theta = 0$, (see, for example, *Superspace*, eq. (5.6.8) [19].)

\[
\omega_{\alpha \beta \gamma} = 0 \quad (4.14)
\]

• One is often interested in bosonic backgrounds, and in that case any individual object containing an odd number of fermionic indices can be set to zero.

## 5 The Expansion of the Vielbein

We illustrate the remarks at the end of the previous section by examining the first few terms in the expansion. To begin with, we consider the bosonic $E^a$ and take advantage of the constraints. We find, for the lowest orders

\[
\delta E^a = y^\beta E^C T_{C\beta}^a = -iy^\beta E^\gamma (\Gamma^a)_{\beta \gamma} = -idx^m(y \Gamma^a \psi_m) \quad (5.1)
\]

\[
\delta^2 E^a = -y^\beta E^C y^\delta R_{\delta C \beta}^a + y^\gamma E^B y^\delta \nabla_\delta T_B^a + y^\gamma (Dy^\beta \delta_B + y^\gamma E^D T_{DE}^B T_B^a) T_B^a \\
= -iy^\gamma Dy^\beta (\Gamma^a)_{\beta \gamma} - iy^\gamma E^d y^\gamma T_{de}^\beta (\Gamma^a)_{\beta \gamma} \\
= -i(y \Gamma^a Dy) - idx^m e_m^d y^\gamma T_{de}^\beta (\Gamma^a)_{\beta \gamma} \quad (5.2)
\]

\[
\delta^3 E^a = iy^\gamma y^\delta dx^m \psi^\phi_m y^\epsilon R_{\epsilon \phi \delta}^\beta (\Gamma^a)_{\beta \gamma} + iy^\gamma y^\delta dx^m e_m^f y^\epsilon R_{\epsilon \phi \delta}^\beta (\Gamma^a)_{\beta \gamma} \\
- y^\gamma y^\delta dx^m \psi^\phi_m (\Gamma^d)_{\phi \epsilon} T_{de}^\beta (\Gamma^a)_{\beta \gamma} - iy^\gamma y^\delta dx^m e_m^d y^\epsilon (\nabla_\epsilon T_{de}^\beta (\Gamma^a)_{\beta \gamma}) \quad (5.3)
\]
In a similar manner, for the fermionic $E^\alpha$,

$$
\delta E^\alpha = D y^\alpha + y^\gamma dx^m e_m^b T_{b\gamma}^\alpha \\
$$

(5.4)

$$
\delta^2 E^\alpha = -y^3 dx^m e_m^c y^\delta R_{\delta c b}^\alpha - y^3 dx^m \psi_m^\gamma y^\delta R_{\delta \gamma b}^\alpha \\
+ y^\gamma dx^m e_m^b y^\delta \nabla_\delta T_{b\gamma}^\alpha - i y^\gamma dx^m (y\Gamma_b^c \psi^c) T_{b\gamma}^\alpha \\
$$

(5.5)

$$
\delta^3 E^\alpha = -y^3 D y^\delta y^\gamma R_{\gamma b}^\alpha - y^3 y^\delta dx^m (e_m^c T_{e\phi}^\beta y^\gamma R_{\gamma b}^\alpha - i \psi_m^\gamma (\Gamma_b^c)_{\kappa \phi} y^\gamma R_{\gamma b}^\alpha) \\
- y^3 dx^m (\psi_m^\gamma y^\delta \nabla_\phi R_{\gamma b\delta}^\alpha + e_m^c y^\gamma y^\delta \nabla_\phi R_{\gamma b\delta}^\alpha) \\
- i y^\gamma y^\delta dx^m \psi_m^\gamma (\Gamma_b^c)_{\kappa \phi} y^\gamma \nabla_\delta T_{b\gamma}^\alpha \\
+ y^\gamma dx^m e_m^b y^\delta \nabla_\delta T_{b\gamma}^\alpha - i y^\gamma y^\delta D y^\delta (\Gamma_b^c)_{\delta \kappa} T_{b\gamma}^\alpha \\
- i y^\gamma y^\delta y^\gamma dx^m f T_{f e}^\delta (\Gamma_b^c)_{\delta \kappa} T_{b\gamma}^\alpha - i y^\gamma y^\delta dx^m \psi_m^\delta (\Gamma_b^c)_{\delta \kappa} y^\gamma \nabla_\delta T_{b\gamma}^\alpha \\
$$

(5.6)

etc. All the quantities on the right-hand side are evaluated at $\theta = 0$.

We have used the constraints, and also the structure of the tangent space Lorentz group which implies $R_{AB\gamma}^a = 0$. The torsion $T_{\alpha b \gamma}^\delta$ is given by the constraints (2.4) in terms of the three-form field strength evaluated at $\theta = 0$, i.e. the component field strength. We note the vanishing of $\nabla_\delta T_{b\gamma}^\alpha$. We also note the appearance of the component vielbein and of the gravitino. As we show in the next section the other quantities, all evaluated at $\theta = 0$, can be expressed, via the solution of the Bianchi identities, in terms of component quantities.

The expansion of the individual vielbein components, $E_M^A$ can be read from the above expansion of $E^A = E_m^A dx^m + E_\mu^A dy^\mu$ (in WZ gauge $y^\alpha = \delta^\alpha_\mu y^\mu$). Thus, to read off the expansion of $E_\mu^A$ at a given order in $y$, one has to go one order higher for $E^A$. Although in our approach these components are not needed for the expansion of the membrane, we give the first few orders. We obtain, from (5.1-3) (and the $Dy$ part of (1.8))

$$
E_m^a = e_m^a(x) - i(y\Gamma^a \psi_m) \\
- \frac{i}{2} [y^y f^e m^d T_{d e}^\beta (\Gamma^a)_{\beta \gamma} + y^a (\Gamma^a)_{\alpha \beta} y^\gamma e_m^c \omega_{c \gamma}^\beta] \\
+ \frac{1}{3} [i y^\gamma y^\delta y^\phi e_m^f T_{e d c}^\beta (\Gamma^a)_{\beta \gamma} + i y^\gamma y^\delta e_m^f y^\phi T_{e f d}^\beta (\Gamma^a)_{\beta \gamma} - y^\gamma y^\delta y^\phi e_m^c T_{e d c}^\beta (\Gamma^a)_{\beta \gamma}] + O(y^4)
$$

(5.7)

$$
E_\mu^a = -\frac{i}{2} y^\gamma (\Gamma^a)_{\gamma \beta} \delta^\beta_\mu \\
+ \frac{1}{4} \delta^\beta_\mu \left[ y^\gamma y^\delta y^\kappa f T_{e d c}^\phi (\Gamma^a)_{\phi \gamma} + i y^\gamma y^\delta y^\kappa R_{e \delta c}^\phi (\Gamma^a)_{\phi \gamma}\right] + O(y^5)
$$

(5.8)
and from (5.4-6)

\[
E_m^\alpha = \psi_m^\alpha - iy^\beta e_m^\gamma \omega_{\gamma \beta}^\alpha + y^\gamma e_m^b T_{b \gamma}^\alpha \\
+ \frac{1}{2} \left[ -y^\beta e_m^\gamma \delta R_{c \delta}^\gamma \alpha - y^\beta \psi_m^\gamma \delta R_{b \gamma}^\alpha + y^\gamma e_m^b \delta \nabla_\delta T_{b \gamma}^\alpha - iy^\gamma (y^\Gamma_b \psi_m) T_{b \gamma}^\alpha \right] \\
+ \frac{1}{3!} \left[ y^\delta y^\epsilon e_m^\gamma \omega_{\epsilon \delta}^\gamma \beta R_{\gamma \beta}^\alpha - y^\delta y^\phi (e_m^c \nabla_\phi \gamma R_{\gamma \beta}^\alpha - iy^\gamma e_m^c \psi_m^\gamma (\Gamma_b^\phi y^\gamma R_{\gamma \beta}^\alpha) \\
- y^\gamma e_m^b \delta \nabla_\epsilon T_{b \gamma}^\alpha + iy^\gamma e_m^\beta \omega_{c \beta}^\gamma \delta (\Gamma_b^\delta \nabla_\delta T_{b \gamma}^\alpha) \\
+ iy^\gamma y^\epsilon e_m^b T_{f \epsilon}^\delta (\Gamma_b^\epsilon \nabla_\delta T_{b \gamma}^\alpha - iy^\gamma y^\phi e_m^c \omega_{c \beta}^\gamma \delta (\Gamma_b^\delta \nabla_\delta T_{b \gamma}^\alpha) \right] + O(y^4) \quad (5.9)
\]

\[
E_\mu^\alpha = \delta_\mu^\alpha + \frac{1}{2} \left[ \delta_\mu^\gamma \delta y^\gamma (\Gamma_b^\delta \nabla_\delta T_{b \gamma}^\alpha - iy^\gamma y^\phi \psi_m^\gamma (\Gamma_b^\delta \nabla_\delta T_{b \gamma}^\alpha) \right] + O(y^4) \quad (5.10)
\]

We describe in the next section how the various superspace tensors appearing above, which are evaluated at \( \theta = 0 \), are to be identified with component fields.

The first two orders of the expansion can be compared with, for example, (4.15, 5.1) of ref. [1], although the authors of that reference, using gauge completion, were not able to obtain all the \( \mathcal{O}(\theta^2) \) terms.

6 The Component Fields

In the general expansion we encounter superspace fields and their spinor derivatives evaluated at \( \theta = 0 \) and we must express these quantities in terms of the fields of component supergravity.

From the relations in (2.4, 2.5) it is clear that all torsions and two of the curvatures can be expressed in terms of the field strength \( H_{abcd}(x, \theta) \). We identify

\[
H_{abcd} = \hat{h}_{abcd}(x) \quad (6.1)
\]

where \( \hat{h}_{abcd} \) is the supercovariantized component field strength. Similarly, for the curvature

\[
R_{ab\gamma}^\delta = \hat{r}_{ab\gamma}(x) \quad (6.2)
\]

the supercovariant component curvature tensor.

We will need the \( \theta = 0 \) spinor derivatives of some of these quantities. To begin with, from (1) in eq. (2.7)

\[
\nabla_\alpha H_{abcd} = 6i(\Gamma_{[ab]}^\alpha \beta T_{cd]b}) = 6i(\Gamma_{[ab]}^\alpha \beta \hat{\psi}_{cd]}^\beta(x) \quad (6.3)
\]
in terms of the supercovariantized gravitino field strength. Taking a second spinor derivative,
\[ \nabla_\gamma \nabla_\alpha H_{abcd} = 6i(\Gamma_{[ab]}^\beta \nabla_\gamma T_{cd]\beta} \] (6.4)
and at \( \theta = 0 \) the spinor derivative of the torsion can be expressed, by means of (h) in (2.5) in terms of the component curvature, \( \nabla_a T_{b}\gamma \delta \sim \nabla_a H_{bcde} (\nabla_a H_{bcde} = \hat{\nabla}_a \hat{h}_{bcde} \) is a supercovariantized space-time derivative), and the products of two \( H \). Clearly this procedure can be repeated so that at every stage spinor derivatives of \( H_{abcd} \) can be expressed in terms of the component curvature, gravitino field strength, \( \hat{h}_{abcd} \), and space-time derivatives thereof.

Turning to the curvatures, a similar procedure applies. Relations (i) and (j) in (2.5) allow us to express curvatures with some spinor indices in terms of the component quantities above. The relation (l) in (2.7) gives us the first spinor derivative of the curvature with vector indices. A second spinor derivative will lead to terms such as \( \nabla_\beta \nabla_\gamma R_{\alpha\epsilon\delta} \sim \nabla_\beta \nabla_\gamma R_{\alpha\epsilon\delta} + [\nabla_\gamma, \nabla_\beta] R_{\alpha\epsilon\delta} \) and the commutator can be expressed in terms of known torsions and curvatures.

The procedure we have outlined above provides a systematic and unambiguous way of obtaining the component expansions to any order. Obviously higher orders will get more and more complicated, but they don’t require any ingenuity, just straightforward application of the rules we have described.

7 Expansions in bosonic backgrounds

The higher order expansions simplify considerably if we consider purely bosonic backgrounds - the case often of interest. The main simplification comes about because we can set to zero all quantities with an odd number of spinor indices. In particular, it is evident that odd terms in the expansion of \( E^a \) and even terms in the expansion of \( E^a \) vanish. It is then a simple matter to examine the general expansion and write down the actual form for this case. Thus we find, from the general expansion in Section 4

\[ \delta E^a = 0 \] (7.1)
\[ \delta^2 E^a = -i(y\Gamma^a Dy) - iy^\gamma y^\epsilon e^d T_{de} \beta (\Gamma^a)_{\beta\gamma} \] (7.2)
\[ \delta^3 E^a = 0 \] (7.3)
\[ \delta^4 E^a = iy^\gamma y^\delta e^f y^\epsilon (\nabla_a R_{\epsilon\phi\delta\beta})(\Gamma^a)_{\beta\gamma} - y^\gamma y^\epsilon y^f D y^\phi (\Gamma^d)_{\phi\epsilon} T_{de} \beta (\Gamma^a)_{\beta\gamma} \]
\[ -y^\gamma y^\epsilon y^\lambda e^h T_{h\lambda} \phi (\Gamma^a)_{\beta\gamma} - iy^\gamma y^\delta e^f y^\epsilon (\nabla_\nu \nabla_\delta T_{f\phi} \beta)(\Gamma^a)_{\beta\gamma} \] (7.4)
\[ \delta E^\alpha = Dy^\alpha + y^\gamma e^b T_{b\gamma}^\alpha \]  
(7.5)

\[ \delta^2 E^\alpha = 0 \]  
(7.6)

\[
\begin{align*}
\delta^3 E^\alpha &= -y^\delta D y^\beta y^\gamma [R_{\gamma\beta\delta}^\alpha + i (\Gamma_b)^{\beta\delta} T_{b\gamma}^\alpha] \\
 &- y^\delta y^\gamma e^b T_{b\gamma}^\beta R_{\gamma\delta}^\alpha - \nabla_{\gamma} R_{\phi\delta}^\alpha - \nabla_{\delta} \nabla_{\gamma} T_{b\gamma}^\alpha + i T_{b\gamma}^\alpha (\Gamma^b)_{\kappa\phi} T_{b\kappa}^\alpha
\end{align*}
\]  
(7.7)

In these expressions several quantities appear (evaluated at \( \theta = 0 \)) which must be worked out using the Bianchi identities and their consequences, as explained earlier. We proceed in the following manner:

The torsion \( T_{a\beta\gamma} \) is expressed directly, from (2.4) in terms of the field strength \( H_{abcd} \) which at \( \theta = 0 \) is the component field strength.

The spinor derivative of the spinor-vector curvature is obtained as follows:

\[
\begin{align*}
\nabla_{\phi} R_{\gamma b\delta}^\alpha &= \frac{1}{4} \nabla_{\phi} R_{\gamma b c d} (\Gamma^{cd})_{\delta}^\alpha \\
&= \frac{i}{8} (\Gamma^{cd})_{\delta}^\alpha [ (\Gamma_b)^{\gamma\beta} \nabla_{\phi} T_{c\beta}^d - (\Gamma_c)^{\gamma\beta} \nabla_{\phi} T_{d\beta}^c + (\Gamma_d)^{\gamma\beta} \nabla_{\phi} T_{b\beta}^c] \\
&= \frac{i}{8} (\Gamma^{cd})_{\delta}^\alpha (\Gamma_b)^{\gamma\beta} R_{c d \phi}^\beta - 2 \nabla_{\phi} T_{d \phi}^\beta - 2 T_{d \phi}^\beta T_{b \beta}^c - 2 (\Gamma_c)^{\gamma\beta} (R_{d \phi}^\beta - \nabla_d T_{b \phi}^\beta + \nabla_b T_{d \phi}^\beta - T_{d \phi}^\gamma T_{b \gamma}^\beta + T_{d \phi}^\delta T_{b \delta}^\beta)
\end{align*}
\]  
(7.8)

where we have used the relations (2.7) and the solution (h) to the Bianchi identities to determine objects such as \( \nabla_{\phi} T_{c d \beta} \). In the final expression all quantities, evaluated at \( \theta = 0 \) involve the component curvature, the component four-form field strength and space-derivatives thereof.

In a similar fashion we have

\[
\begin{align*}
\nabla_{\epsilon} \nabla_{\delta} T_{b\gamma}^\alpha &= \frac{1}{36} (\delta^a_b \Gamma^{cde} + \frac{1}{8} \Gamma_{b \ acde})_\gamma^{\alpha} \nabla_{\epsilon} \nabla_{\delta} H_{acde} \\
&= -\frac{i}{6} (\delta^a_b \Gamma^{cde} + \frac{1}{8} \Gamma_{b \ acde})_\gamma^{\alpha} (\Gamma_{[ac]} \delta \beta \nabla_{\epsilon} T_{d \epsilon}^\beta) \\
&= -\frac{i}{6} (\delta^a_b \Gamma^{cde} + \frac{1}{8} \Gamma_{b \ acde})_\gamma^{\alpha} (\Gamma_{[ac]} \delta \beta \\
&\cdot [R_{d \epsilon}^\beta - \nabla_d T_{c \epsilon}^\beta + \nabla_c T_{d \epsilon}^\beta - T_{d \epsilon}^\gamma T_{c \gamma}^\beta + T_{c \epsilon}^\gamma T_{d \gamma}^\beta]
\end{align*}
\]  
(7.9)

and again, at \( \theta = 0 \), the quantities in the last line are recognizable component curvatures, field strengths, and space-derivatives thereof.
8 The expansion of the brane action

The first ingredient one needs is the expansion of the first term in the membrane action, $S_\sigma = \sqrt{-\det G_{ij}} = \sqrt{-\det(\Pi_i^a\Pi_i^a)}$. The general expansion is straightforward though lengthy. We quote the first few terms

$$\delta S_\sigma = \frac{1}{2} S_\sigma (\text{tr}G^{-1}\delta G)^2$$  \hspace{1cm} (8.1)

$$\delta^2 S_\sigma = \frac{1}{4} S_\sigma (\text{tr}G^{-1}\delta G)^2 - \frac{1}{2} S_\sigma (\text{tr}G^{-1}\delta GG^{-1}\delta G - \text{tr}G^{-1}\delta^2 G)$$ \hspace{1cm} (8.2)

$$\delta^3 S_\sigma = \frac{1}{8} S_\sigma (\text{tr}G^{-1}\delta G)^3 + \frac{3}{4} S_\sigma (\text{tr}G^{-1}\delta G)(\text{tr}G^{-1}\delta^2 G) - \frac{3}{4} S_\sigma (\text{tr}G^{-1}\delta GG^{-1}\delta G) + S_\sigma (\text{tr}G^{-1}\delta GG^{-1}\delta GG^{-1}\delta G)$$ \hspace{1cm} (8.3)

with $\delta G_{ij} = \delta \Pi_i^a \Pi_j^a + \Pi_i^a \delta \Pi_j^a$. The next term in the expansion can be obtained straightforwardly.

Next we consider the expansion of the Wess-Zumino term. Starting with the first order variation $\delta(WZ)$ one applies the $\delta$ operation repeatedly. Thus one finds, evaluating at $\theta = 0$, with

$$\Pi_i^a = \partial_i x^m e_m^a(x) = \pi_i^a$$ \hspace{1cm} (8.4)

$$G_{ij} = \Pi_i^a \Pi_j^a = \pi_i^a \pi_j^a \equiv g_{ij}(x)$$

the following:

$$\delta(WZ) = \frac{i}{2} \epsilon^{ijk} y^a \partial_i x^m \pi_j^c \pi_k^d \psi_m^a (\Gamma_{dc})_{\beta\alpha}$$ \hspace{1cm} (8.5)

$$\delta^2(WZ) = \frac{1}{6} \epsilon^{ijk} y^a [3(\delta \Pi_i^a) \Pi_j^c \Pi_k^d H_{DCBa} + \Pi_i^c \Pi_j^d H_{DCBa} y^\lambda \nabla_\lambda H_{DCBa}]$$ \hspace{1cm} (8.6)

$$\delta^3(WZ) = \frac{1}{6} \epsilon^{ijk} y^a [3(\delta^2 \Pi_i^a) \Pi_j^c \Pi_k^d H_{DCBa} + 6(\delta \Pi_i^a) \delta \Pi_j^c \Pi_k^d H_{DCBa}$$

$$+ 6(\delta \Pi_i^a) \Pi_j^c \Pi_k^d y^\lambda \nabla_\lambda H_{DCBa} + \Pi_i^c \Pi_j^d \Pi_k^a y^\lambda \nabla_\lambda \nabla_\mu H_{DCBa}]$$

$$= \frac{1}{2} \epsilon^{ijk} y^a [(\delta^2 \Pi_i^a) \pi_j^c \pi_k^d (\Gamma_{dc})_{\beta\alpha} + 2(\delta \Pi_i^a) \pi_j^c \partial_k x^m \psi_m^d (\Gamma_{cb})_{\delta\alpha}$$

$$+ 4(\delta \Pi_i^a) (\delta \Pi_j^c) \pi_k^d (\Gamma_{dc})_{\beta\alpha} + 2(\delta \Pi_i^a) (\delta \Pi_j^c) \partial_k x^m \psi_m^d (\Gamma_{cb})_{\delta\alpha}]$$ \hspace{1cm} (8.7)

etc. The $\delta^a \Pi_i^A$ can be read from the $\delta^a E^A$ in Section 5.
8.1 The membrane up to second order

It is now a straightforward matter to determine the low-order expansion of the brane action. We have

\[ S^{(0)}(x; y) = \int d^3 \zeta \left[ -\sqrt{-g(x)} - \frac{1}{6} \varepsilon^{ijk} \pi^a_i \pi^b_j \pi^c_k b_{cba}(x) \right] \]

(8.8)

\[ S^{(1)}(x; y) = \int d^3 \zeta \left[ -i \sqrt{-g} g^{ij} \pi^a_i \partial_j x^n (y \Gamma_a \psi_n) + \frac{1}{2} i \varepsilon^{ijk} \pi^a_i \pi^b_j \partial_k x^n (y \Gamma_{ab} \psi_n) \right] \]

(8.9)

\[ S^{(2)}(x; y) = \int d^3 \zeta \left[ \frac{1}{2} \sqrt{-g} \left\{ -g^{ij} \pi^a_i \partial_j x^n (y \Gamma_a \psi_m) + 2 g^{ij} g^{kl} \pi^a_i \pi^b_j \partial_k x^n \partial_l x^n (y \Gamma_{a} \psi_m)(y \Gamma_{b} \psi_n) \right. \right. \]

\[ -g^{ij} \partial_j x^m (y \Gamma^a \psi_m) \partial_j x^n (y \Gamma_a \psi_n) - i g^{ij} \pi^a_i (y \Gamma_a D_j y) - i g^{ij} \pi^a_i \pi^b_j y^\gamma y^\delta T_{\gamma \delta}^{\beta} (\Gamma_{a} \beta) \gamma \delta \]

\[ -\frac{1}{4} i \varepsilon^{ijk} \left\{ \pi^a_i \pi^b_j (y \Gamma_{ba} D_k y) + \pi^a_i \pi^b_j \pi^c_k y^\gamma y^\delta T_{\gamma \delta}^{\beta} (\Gamma_{a} \beta) \gamma \delta \right. \]

\[ + 2 i \pi^a_i \partial_j x^m \partial_k x^n (y \Gamma^b \psi_m)(y \Gamma_{ab} \psi_n) \right\} \]

(8.10)

In these expressions we must substitute for the torsion \( T_{\gamma \delta}^{\beta} \) as given in (2.4).

The result above could be used for one-loop calculations and for identifying gravitino emission vertices \[4\].

8.2 The membrane in bosonic backgrounds

Matters simplify in a bosonic background where we can take advantage of the vanishing of odd variations of \( \Pi^a \) which implies \( \delta G = \delta^3 G = 0 \). One finds then

\[ \delta S_{\sigma} = 0 \]

\[ \delta^2 S_{\sigma} = \frac{1}{2} S_{\sigma}(\text{tr} G^{-1} \delta^2 G) \]

\[ \delta^3 S_{\sigma} = 0 \]

\[ \delta^4 S_{\sigma} = \frac{3}{4} S_{\sigma}(\text{tr} G^{-1} \delta^2 G)^2 - \frac{3}{2} S_{\sigma}(\text{tr} G^{-1} \delta^2 G \delta^2 G^{-1} \delta^2 G) + \frac{1}{2} S_{\sigma}(\text{tr} G^{-1} \delta^4 G) \]

(8.11)

where

\[ \delta^2 G_{ij} = \Pi^a_i \delta^2 \Pi^a_j + \delta^2 \Pi^a_i \Pi^a_j \]

(8.12)

\[ \delta^4 G_{ij} = \Pi^a_i \delta^4 \Pi^a_j + 2 \delta^2 \Pi^a_i \delta^2 \Pi^a_j + \delta^4 \Pi^a_i \Pi^a_j \]

(8.13)

For the Wess-Zumino term one finds a very simple result to this order:

\[ \delta(WZ) = 0 \]
\[\delta^2(WZ) = \frac{i}{2} \epsilon^{ijk} y^\alpha \delta \Pi_i^\beta \Pi_j^\gamma \Pi_k^d (\Gamma_{cd})_{\alpha \beta}\]

\[\delta^3(WZ) = 0\]

\[\delta^4(WZ) = \frac{i}{2} \epsilon^{ijk} y^\alpha [\delta^3 \Pi_i^\beta \Pi_j^\gamma \Pi_k^d (\Gamma_{cd})_{\alpha \beta} + \delta^2 \Pi_i^\beta \delta \Pi_j^\gamma \Pi_k^d (\Gamma_{cd})_{\alpha \beta}]\]

The second order expansion of the action can be read immediately from (8.10) by setting the gravitino to zero:

\[S^{(2)}(x; y) = \frac{1}{2} \int d^3 \zeta \left[ -i \sqrt{-g} g^{ij} \left\{ \pi_i^a (y \Gamma_a D_j y) + \pi_i^a \pi_j^b y^\gamma y^\nu T_{\nu \gamma}^\beta (\Gamma_\alpha)_{\beta \gamma} \right\} - \frac{1}{2} i \epsilon^{ijk} \left\{ \pi_i^a \pi_j^b (y \Gamma_{ba} D_k y) + \pi_i^a \pi_j^b \pi_k^c y^\gamma T_{\nu \gamma}^\beta (\Gamma_{cb})_{\beta \alpha} \right\} \right]\]

From (8.11-14) the fourth order expansion can be written as

\[S^{(4)}(x : y) = \frac{1}{4!} \int d^3 \zeta \left[ -\sqrt{-g} \left\{ 3 |\pi_i^a \pi_j^b - \pi_i^a \pi_j^b | + \frac{1}{3} g^{ij} \pi_i^a \pi_j^b \right\} \delta^2 \Pi_i^\beta \delta^2 \Pi_j^\beta \right.\]

\[\left. + g^{ij} \pi_i^a \pi_j^b \delta \Pi_i^\beta \Pi_j^\gamma \right\} - \frac{i}{2} \epsilon^{ijk} y^\alpha [\delta^3 \Pi_i^\beta \pi_j^\gamma + \delta \Pi_i^\beta \delta^2 \Pi_j^\beta \pi_k^d (\Gamma_{cd})_{\alpha \beta}]\]

In this expression the variations \(\delta^3 \Pi_i^\alpha \) must be substituted. As explained earlier, they can be read from the corresponding variations of the vielbein in Section 7. Explicitly,

\[\delta^2 \Pi_i^\alpha = -i(y \Gamma^\alpha D_i y) - iy^\gamma y^\nu \pi_i^a T_{de}^\beta (\Gamma_{cd})_{\alpha \gamma}\]

\[\delta^4 \Pi_i^\alpha = i y^\gamma y^\nu \pi_i^a \langle y^\nu \delta_{\nu \gamma} \rangle (\Gamma_{\alpha \beta \gamma}) - y^\gamma y^\nu y^\lambda h_{\lambda \gamma} (\Gamma_{\alpha \beta \gamma}) - iy^\gamma y^\nu \pi_i^a \langle y^\nu \delta_{\nu \gamma} \rangle (\Gamma_{\alpha \beta \gamma}) - i y^\gamma y^\nu \pi_i^a \langle y^\nu \delta_{\nu \gamma} \rangle (\Gamma_{\alpha \beta \gamma}) - i \delta \Pi_i^\beta \delta \Pi_i^\beta \]

\[\delta^3 \Pi_i^\alpha = \delta \Pi_i^\beta \delta \Pi_i^\beta \]

The torsions and curvatures that appear in these expressions, all evaluated at \(\theta = 0\), are given in eqs. (2.4, 7.8, 7.9) in terms of component curvatures, field strengths and space-derivatives thereof. We will refrain from carrying out the substitution, or giving the next order in the expansion.
9 Conclusions

We have presented the ingredients for expanding the 11-dimensional membrane action in powers of the superspace fermionic coordinates. In a future publication we intend to carry out similar work for various branes in 10-dimensional supergravity backgrounds. Our main tool has been the superspace normal coordinate expansion suitably adapted to our application. This expansion allows for a systematic and unambiguous method for proceeding to any order, using as its main ingredient the geometrical formulation of the background superspace, i.e. the constraints on torsions, curvatures, and 3-form field strength. We have used the standard constraints of refs. [17, 18] but the procedure is flexible enough to allow for modifications such as those considered by Cederwall et al. [20] (see also [7]).

The membrane action is invariant under the various superspace symmetries, as well as \(\kappa\)-symmetry. In refs. [13, 14], in the context of a background field expansion for the Green-Schwarz superstring, we have discussed how these symmetries are reflected in symmetries of the action after the normal coordinate expansion. What we wish to emphasize is that no additional checking of \(\kappa\)-invariance is necessary.

Our results could be specialized to specific AdS\(\times S\) backgrounds where the various curvatures and field strengths become very simple. This would allow comparison with the complete expansions obtained for such backgrounds in refs. [8, 9, 10]. We hope to pursue this, as well as other applications of our results, in a future publication.

Acknowledgments

Part of this work was done at the Aspen Center for Physics. We thank W. Siegel for reading the manuscript and for many useful suggestions. We also acknowledge conversations with B. de Wit, J. Plefka, and W. Taylor. MTG thanks the Physics Department of McGill University, where some of this work was done.

References

[1] B. de Wit, K. Peeters and J. Plefka, Nucl. Phys. B532 (1998) 99, [hep-th/9803209].

[2] B. de Wit, [hep-th/9902143].

[3] A. Dasgupta, H. Nicolai and J. Plefka, JHEP 0005 (2000) 007, [hep-th/0003280].

[4] J. Plefka, [hep-th/0009193].
[5] K. Millar, W. Taylor and M. van Raamsdonk, (hep-th/0007157)

[6] M. Cvetic, H. Lu, C.N. Pope and K.S. Stelle, Nucl. Phys. B573 (2000) 149, (hep-th/9907202)

[7] K. Peeters, P. Vanhove and A. Westerberg, (hep-th/0010167)

[8] R. Kallosh, J. Rahmfeld and A. Rajaraman, JHEP 9809 (1998) 002, (hep-th/9805217)

[9] B. de Wit, K. Peeters, J. Plefka and A. Sevrin Phys.Lett. B443 (1998) 153, (hep-th/9808052).

[10] R. Roiban and W. Siegel, (hep-th/0010104).

[11] I. McArthur, Class. Quant. Grav. 1 (1984) 233.

[12] J. Atick and A. Dhar, Nucl. Phys. B284 (1987) 131.

[13] M.T. Grisaru and D. Zanon, Nucl. Phys. B310 (1988) 57.

[14] M.T. Grisaru, H. Nishino and D. Zanon, Nucl. Phys. B314 (1989) 363.

[15] M.T. Grisaru, M.E. Knutt and W. Siegel, Nucl. Phys. B523 (1998) 663, (hep-th/9711120).

[16] S.J. Gates, M.T. Grisaru, M.E. Knutt and W. Siegel, Phys. Lett. B421 (1998) 203, (hep-th/9711151).

[17] L. Brink and P. Howe, Phys. Lett. 91B (1980) 384; P. Howe, Phys. Lett. B415 (1997) 149, (hep-th/9707184).

[18] E. Cremmer and S. Ferrara, Phys. Lett. 91B (1980) 61.

[19] S.J. Gates, M.T. Grisaru, M. Roček, and W. Siegel, *Superspace*, Addison-Wesley (1989).

[20] M. Cederwall, U. Gran, M. Nielsen and B.E.W. Nilsson, (hep-th/0010042).