Higher Derivative Gravity and Torsion from the Geometry of $C$-spaces

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Abstract

We start from a new theory (discussed earlier) in which the arena for physics is not spacetime, but its straightforward extension—the so called Clifford space ($C$-space), a manifold of points, lines, areas, etc.; physical quantities are Clifford algebra valued objects, called polyvectors. This provides a natural framework for description of supersymmetry, since spinors are just left or right minimal ideals of Clifford algebra. The geometry of curved $C$-space is investigated. It is shown that the curvature in $C$-space contains higher orders of the curvature in the underlying ordinary space. A $C$-space is parametrized not only by 1-vector coordinates $x^\mu$ but also by the 2-vector coordinates $\sigma^{\mu\nu}$, 3-vector coordinates $\sigma^{\mu\nu\rho}$, etc., called also holographic coordinates, since they describe the holographic projections of 1-lines, 2-loops, 3-loops, etc., onto the coordinate planes. A remarkable relation between the “area” derivative $\partial/\partial\sigma^{\mu\nu}$ and the curvature and torsion is found: if a scalar valued quantity depends on the coordinates $\sigma^{\mu\nu}$ this indicates the presence of torsion, and if a vector valued quantity depends so, this implies non vanishing curvature. We argue that such a deeper understanding of the $C$-space geometry is a prerequisite for a further development of this new theory which in our opinion will lead us towards a natural and elegant formulation of $M$-theory.

1 Introduction

A deeper understanding of geometry and its relation to algebra has always turned out very useful for the advancement of physical theories. Without analytical geometry New-
ton mechanics, and later special relativity, could not have acquired its full power in the
description of physical phenomena. Without development of the geometries of curved
spaces, general relativity could not have emerged. The role of geometry is nowadays be-
ing investigated also within the context of string theory, and especially in the searches
for M-theory. The need for suitable generalizations, such as non commutative geomet-
tries is being increasingly recognized. It was recognized long time ago [1] that Clifford
algebra provided a very useful tool for description of geometry and physics, and contains
a lot of room for important generalizations of the current physical theories. So it was
suggested in refs. [2]–[5] that every physical quantity is in fact a polyvector, that is, a
Clifford number or a Clifford aggregate. This has turned out to include [1, 6] also spinors
as the members of left or right minimal ideals of Clifford algebra [1] and thus provided
a framework for a description and a deeper understanding of supersymmetries, i.e., the
transformations that relate bosons and fermions. Moreover, it was shown that the well
known Fock-Stueckelberg theory of relativistic particle [7] can be embedded in the Clif-
ford algebra of spacetime [4, 5]. Many other fascinating aspects of Clifford algebra are
described in a recent book [5] and refs.[2, 3]. A recent overview of Clifford algebras and
their applications is to be found in a nice book [8].

Also there is a number of works which describe the collective dynamics of p-branes in
terms of area variables [9]. It has been observed [10] that this has connection to C-space,
and also to the branes with variable tension [11] and wiggly branes [12, 13, 14]. Moreover,
the bosonic p-brane propagator was obtained from these methods [14]. The logarithmic
corrections to the black hole entropy based on the geometry of Clifford space (shortly
C-space) have been furnished by Castro and Granik [15].

In previous publications it has been already shown that we are proposing a new phys-
ical theory in which the arena for physics is no longer the ordinary spacetime, but a
more general manifold of Clifford algebra valued objects—polyvectors. Such a manifold
has been called pan dimensional continuum [2] or Clifford space (shortly C-space) [3]. It
describes on a unifying footing the objects of various dimensionalities: not only points,
but also closed lines, surfaces, volumes,..., called 0-loops, 2-loops, 3-loops, etc.. Those
geometric objects may be taken to correspond to the well known physical objects, namely
closed p-branes. The ordinary spacetime is just a subspace of C-space. A “point” of
C-space corresponds to a p-loop in ordinary spacetime. Rotations in C-space transform
one point of $C$-space into another point of $C$-space, and this manifests in ordinary space as transformations from a $p$-loop into another $p'$-loop of different dimensionality $p'$. Technically those transformations are generalizations of Lorentz transformations to $C$-space. This is reviewed in Sec. 2 where some important physical implications of the theory so generalized are pointed out.

Instead of flat $C$-space we may consider a curved $C$-space. As the passage from flat Minkowski spacetime to a curved spacetime had provided us with a tremendous insight into the nature of one of the fundamental interactions, namely gravity, so we expect that introduction of a curved $C$-space will even further increase our understanding of the other fundamental interactions and their unification with gravity.

Motivated by these important developments and prospects we study in Sec. 3 the geometry of curved $C$-space and show a remarkable relation between the bivector (holographic) coordinates $\sigma^{\mu\nu}$ and the presence of curvature and/or torsion. We also demonstrate that the curvature in $C$-space contains the higher orders of the curvature in ordinary space. Higher derivative gravity is thus contained within the “usual” gravity (without higher derivatives) in $C$-space. Recently, Hawking [16] has studied the consequences of a higher derivative gravity in quantum gravity. In this paper we provide a deeper understanding of higher derivative gravity and its relation to a very prospective more general theory as the relativity in (curved) $C$-space certainly is. Unfortunately we cannot demonstrate in a short letter the full power of $C$-space physics and its relevance to many current trends in theoretical physics. In order to get a better insight that a really important new physics is in sight the reader is advised to look at ref. [3] where many aspects of $C$-space physics are discussed (see also ref. [2]–[5]).

2 Extending relativity from Minkowski spacetime to $C$-space

One can naturally generalize the notion of a spacetime interval in Minkowski space to $C$-space as follows:

$$dX^2 = d\Omega^2 + dx_\mu dx^\mu + dx_\mu dx_\nu^{\mu\nu} + ...$$ (1)

The Clifford number

$$X = \Omega \mathbb{1} + x^\mu \gamma_\mu + x^{\mu\nu} \gamma_\mu \wedge \gamma_\nu + ...$$ (2)
will be called the \textit{coordinate polyvector} and will denote position in a manifold, called \textit{Clifford space} or \textit{C-space} (see Sec. 3.2 for more details). If we take differential $dX$ of $X$ and compute the scalar product $dX \ast dX$ we obtain (1).

We have set the Planck scale to unity. A length parameter is needed in order to combine objects of different dimensionalities: 0-loops, 1-loops, ..., $p$-loops. (See Sec.3.2 and refs. [2]–[5].) Einstein introduced the speed of light as a universal absolute invariant in order to combine space with time in the Minkowski space interval:

$$ds^2 = c^2 dt^2 - dx_i dx^i$$  \hspace{1cm} (3)

A similar requirement is needed here to combine objects of different dimensions, such as $x^\mu$, $X^{\mu\nu}$, etc.. The Planck scale is another universal invariant in constructing a relativity theory in C-spaces [3].

The analog of Lorentz transformations in C-spaces transform a polyvector $X$ into another polyvector $X'$:

$$X' = RXR^{-1}$$  \hspace{1cm} (4)

with

$$R = \exp[i(\theta_1 + \theta_\mu \gamma_\mu + \theta^{\mu_1 \mu_2} \gamma_{\mu_1} \wedge \gamma_{\mu_2} ......)]$$  \hspace{1cm} (5)

and

$$R^{-1} = \exp[-i(\theta_1 + \theta^\nu \gamma_\nu + \theta^{\nu_1 \nu_2} \gamma_{\nu_1} \wedge \gamma_{\nu_2} ......)]$$  \hspace{1cm} (6)

where the theta parameters:

$$\theta; \theta_\mu; \theta^{\mu\nu}; ...$$  \hspace{1cm} (7)

are the C-space version of the Lorentz rotations/boosts parameters.

Since a Clifford algebra admits a matrix representation one can write the norm of polyvectors in terms of the trace operation as:

$$||X||^2 = \text{Trace } X^2$$  \hspace{1cm} (8)

Hence under C-space Lorentz transformations the norms of polyvectors behave like

$$\text{Trace } X'^2 = \text{Trace } [RX^2 R^{-1}] = \text{Trace } [RR^{-1} X^2] = \text{Trace } X^2$$  \hspace{1cm} (9)

Hence, the norms are invariant under C-space Lorentz transformations due to the cyclic property of the trace operation and $RR^{-1} = 1$. 4
Instead of a generic polyvector of the form (2) we can consider particular types of polyvectors which are elements of left or right minimal ideals of Clifford algebra. In other words, for particular choices of the coefficients $\Omega, \ x^\mu, \ x^{\mu\nu}, \ldots$ we obtain particular polyvectors which belong to left or right minimal ideals. It was observed long ago [3] that such particular Clifford numbers have the same properties as spinors. That is, spinors are just particular Clifford numbers: they belong to left or right minimal ideals.

A possible spinor basis is of the form [6]

\[
\begin{align*}
  u_0 &= \frac{1}{4}(1 - \gamma^0 + i\gamma^1\gamma^2 - i\gamma^0\gamma^1\gamma^2) \\
  u_1 &= \frac{1}{4}(-\gamma^1\gamma^3 + \gamma^0\gamma^1\gamma^3 + i\gamma^2\gamma^3 - i\gamma^0\gamma^2\gamma^3) \\
  u_2 &= \frac{1}{4}(-i\gamma^3 - i\gamma^3 + \gamma^1\gamma^2\gamma^3 + \gamma^0\gamma^1\gamma^2\gamma^3) \\
  u_3 &= \frac{1}{4}(-i\gamma^1 - \gamma^0\gamma^1 - \gamma^2 - \gamma^0\gamma^2)
\end{align*}
\] (10)

A generic spinor is a superposition of the basis spinors:

\[
\psi = \psi^0 u_0 + \psi^1 u_1 + \psi^2 u_2 + \psi^3 u_3 \equiv \psi^\alpha u_\alpha
\] (11)

A method of how to generate spinor representations in any dimensions in terms of $\gamma^\mu$ was recently systematically investigated by Mankoč and Nielsen [17].

If a spinor (belonging to a left minimal ideal) is multiplied from the left by an arbitrary Clifford number, it remains a spinor:

\[
\psi' = A\psi
\] (12)

But if it is multiplied from the right, it in general transforms into another Clifford number which is not necessarily a spinor. Scalars, vectors, bivectors, ..., and spinors can be reshuffled by the elements of Clifford algebra. In particular the latter elements could be of the form (3). By extending the theory from the ordinary spacetime to C-space we have obtained a theoretical framework in which scalar, vectors, etc., can be transformed into spinors, and vice versa. This is just a sort of generalized “supersymmetry”. So far we had a flat C-space, but we could generalize it to a curved C-space. This is discussed in Sec. 3. Here let us just mention our expectation that a curved C-space contains supergravity as a particular case.
On the other hand, we may consider strings and branes in $C$-space. Suppose now that we have a mapping from a set of parameters $\xi^a$, $a = 1, 2, \ldots, n$, to a point of $C$-space given by the coordinate polyvector (1):

$$X = \Omega(\xi^a)1 + X^\mu(\xi^a)\gamma_\mu + X^{\mu\nu}(\xi^a)\gamma_\mu \wedge \gamma_\nu + \ldots$$

(13)

This represents a polyvector valued extended object, a $p$-brane in $C$-space. In its description there occur not only the “bosonic” coordinate functions $X^\mu(\xi^a)$, but also the coordinate functions $\Omega(\xi^a)$, $X^{\mu\nu}(\xi^a)$, $X^{\mu\nu\rho}(\xi^a)$, ..., which altogether, according to (11) and (12), embed spinor functions $\psi^a(\xi^a)$. An extended object described by (13) is a brane in $C$-space, which from the point of view of the ordinary spacetime behaves as a generalized super $p$-brane, i.e., an object with a generalized spacetime (target space) supersymmetry.

A next logical step is to introduce polyvector coordinates $\xi$, $\xi^a$, $\xi^{ab}$, ... and corresponding basis vectors $1$, $e_a$, $e_a \wedge e_b$, ... in the $p$-brane’s world manifold and consider the mapping

$$x^\mu = X^\mu(\xi, \xi^a, \xi^{ab}, \ldots)$$

(14)

which describes a $p$-brane with generalized world manifold supersymmetry, and no target space supersymmetry.

Finally we may have [5]

$$X = \Omega(\xi, \xi^a, \xi^{ab}, \ldots) + X^\mu(\xi, \xi^a, \xi^{ab}, \ldots)\gamma_\mu + X^{\mu\nu}(\xi, \xi^a, \xi^{ab}, \ldots)\gamma_\mu \wedge \gamma_\nu + \ldots$$

(15)

which describes an extended object with generalized world manifold and target space supersymmetry.

It would be very important to explore whether the generalized $p$-branes (13)–(15) contain as a particular case the well known examples of super $p$-branes or spinning $p$-branes including superstrings, superparticles or spinning strings and spinning particles.

It is known that string theory in ordinary spacetime contains higher dimensional branes (D-branes). So we can start from string theory; the higher extended objects are automatically present. Spacetime in which a string lives is curved: this is determined by the string (quantum) dynamics. It has been observed that all different possible string theories are different sectors of a single theory, M-theory, whose low energy limit is 11-dimensional supergravity [18].
Now, following the preceding discussion we propose to explore in detail the C-space strings and branes. They automatically contain generalized target and world manifold C-space supersymmetries. Target C-space and the brane world manifold C-space are necessarily curved; so we have gravity in C-space. From the point of view of the ordinary spacetime, gravity in C-space looks like a generalized supergravity. A question occurs of whether such generalized supergravity which —according to the fact that spinors are automatically present in C-space— certainly exists has any relation to the well known supergravity (or perhaps contains it as a particular, or limiting, case).

We have a vision that the C-space strings and branes will lead us towards M-theory. Quantum fluctuations of the C-space string give gravity in target C-space whose (low energy?) limit is supergravity in eleven dimensions. The other limit of the C-space string/brane is superstring in ten dimensions.

As a preparation for such a task as to investigate the preceding vision, we shall explore in this paper the geometry of a curved C-space.

3 On the geometry of C-space

3.1 Ordinary space

Before going to a C-space let us first consider an ordinary curved space of arbitrary dimension \( n \). Let \( \gamma_\mu, \mu = 1,2,\ldots,n \) be a set of \( n \) independent basis vectors which are functions of positions \( x^\mu \) and satisfy the relation

\[
\partial_\mu \gamma_\nu = \Gamma^\rho_{\mu\nu} \gamma_\rho
\]

where \( \Gamma^\rho_{\mu\nu} \) is the connection and \( \partial_\mu \equiv \partial/\partial x^\mu \) the partial derivative.

Let us apply the partial derivative \( \partial_\mu \) to a vector \( a = a^\nu \gamma_\nu \). We obtain

\[
\partial_\mu (a^\nu \gamma_\nu) = \partial_\mu a^\nu \gamma_\nu + a^\nu \partial_\mu \gamma_\nu = (\partial_\mu a^\nu + \Gamma^\nu_{\mu\rho} a^\rho) \gamma_\rho \equiv D_\mu a^\nu \gamma_\nu
\]

where \( D_\mu \) is the covariant derivative.

Instead of the differential operator \( \partial_\mu \) which depends on the particular frame field \( \gamma_\mu \) it is convenient to define the vector derivative, called gradient,

\[
\partial \equiv \gamma^\mu \partial_\mu
\]
It is independent of any particular frame field.

Acting on a vector, the gradient gives
\[ \partial a = \gamma^\mu \partial_\mu (a^\nu \gamma_\nu) = \gamma^\mu \gamma_\mu D_\mu a^\nu = \gamma^\mu \gamma^\nu D_\mu a_\nu \] (19)

Using the following decomposition of the Clifford (“geometric”) product \[1\]
\[ \gamma^\mu \gamma^\nu = \gamma^\mu \cdot \gamma^\nu + \gamma^\mu \wedge \gamma^\nu \] (20)

where
\[ \gamma^\mu \cdot \gamma^\nu \equiv \frac{1}{2}(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) = g^{\mu\nu} \] (21)
is the inner product and
\[ \gamma^\mu \wedge \gamma^\nu \equiv \frac{1}{2}(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) \] (22)
the outer product, eq. (19) becomes
\[ \partial a = D_\mu a^\mu + \gamma^\mu \wedge \gamma^\nu D_\mu a_\nu = D_\mu a^\mu + \frac{1}{2} \gamma^\mu \wedge \gamma^\nu (D_\mu a_\nu - D_\nu a_\mu) \] (23)

Without employing the expansion in terms of \(\gamma^\mu\) we have simply
\[ \partial a = \partial \cdot a + \partial \wedge a \] (24)

Acting twice on a vector by the operator \(\partial\) we have\[1\]
\[ \partial^2 a = \gamma^\mu \partial_\mu (\gamma^\nu \partial_\nu)(a^\alpha \gamma_\alpha) = \gamma^\mu \gamma^\nu \gamma_\alpha D_\mu D_\nu a^\alpha \]
\[ = \gamma_\alpha D_\mu D^\mu a^\alpha + \frac{1}{2} (\gamma^\mu \wedge \gamma^\nu) \gamma_\alpha [D_\mu, D_\nu] a^\alpha \]
\[ = \gamma_\alpha D_\mu D^\mu a^\alpha + \gamma^\mu (R_{\mu\rho} a^\rho + K_{\mu\alpha} D_\rho a^\alpha) \]
\[ + \frac{1}{2} (\gamma^\mu \wedge \gamma^\nu \wedge \gamma_\alpha)(R_{\mu\nu\rho} a^\rho + K_{\mu\nu} D_\rho a^\alpha) \] (25)

We have used
\[ [D_\mu, D_\nu] a^\alpha = R_{\mu\nu\rho} a^\rho + K_{\mu\nu} D_\rho a^\alpha \] (26)
where
\[ K_{\mu\nu} = \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu} \] (27)
is torsion and \(R_{\mu\nu\rho} a^\alpha\) the curvature tensor. Using eq. (16) we find
\[ [\partial_\alpha, \partial_\beta] \gamma_\mu = R_{\alpha\beta\mu}^\nu \gamma_\nu \] (28)

\[ ^1\text{We use } (a \wedge b)c = (a \wedge b) \cdot c + a \wedge b \wedge c \] and \((a \wedge b) \cdot c = (b \cdot c)a - (a \cdot c)b.\]
from which we have

\[ R_{\alpha\beta\mu} = (\partial_\alpha \partial_\beta \gamma_\mu) \cdot \gamma^\nu \quad (29) \]

Thus in general the commutator of partial derivatives acting on a vector does not give zero, but is given by the curvature tensor.

In general, for an r-vector \( A = a^{\alpha_1 \ldots \alpha_r} \gamma_{\alpha_1} \gamma_{\alpha_2} \ldots \gamma_{\alpha_r} \) we have

\[
\partial \partial \ldots \partial A = (\gamma^{\mu_1} \partial_{\mu_1}) (\gamma^{\mu_2} \partial_{\mu_2}) \ldots (\gamma^{\mu_k} \partial_{\mu_k}) (a^{\alpha_1 \ldots \alpha_r} \gamma_{\alpha_1} \gamma_{\alpha_2} \ldots \gamma_{\alpha_r}) \\
= \gamma^{\mu_1} \gamma^{\mu_2} \ldots \gamma^{\mu_k} \gamma_{\alpha_1} \gamma_{\alpha_2} \ldots \gamma_{\alpha_r} D_{\mu_1} D_{\mu_2} \ldots D_{\mu_k} a^{\alpha_1 \ldots \alpha_r} \quad (30)
\]

At this point it is convenient to define the covariant derivative as acting not only on the scalar components such as \( a^\alpha \), \( a^{\alpha \beta} \), etc., but also on vectors, bivectors, etc.. In particular, when acting on a basis vector the covariant derivative — because of (16) — gives

\[ D_\mu \gamma_\nu = \partial_\mu \gamma_\nu - \Gamma^\rho_{\mu \nu} \gamma_\rho = 0 \quad (31) \]

Therefore, for instance, the partial derivative of a vector is equal to the covariant derivative:

\[ \partial_\mu a = \partial_\mu (a^\nu \gamma_\nu) = (D_\mu a^\nu) \gamma_\nu = D_\mu a \quad (32) \]

In general, for an arbitrary r-vector \( A \) we have

\[ \partial_\mu A = \partial_\mu (a^{\alpha_1 \ldots \alpha_r} \gamma_{\alpha_1} \gamma_{\alpha_2} \ldots \gamma_{\alpha_r}) = (D_\mu a^{\alpha_1 \ldots \alpha_r}) \gamma_{\alpha_1} \gamma_{\alpha_2} \ldots \gamma_{\alpha_r} = D_\mu A \quad (33) \]

For the commutator of partial derivatives acting on a vector \( a = a^\rho \gamma_\rho \) we have

\[ [\partial_\mu, \partial_\nu] a = a^\rho [\partial_\mu, \partial_\nu] \gamma_\rho = R_{\mu \nu \rho}^\sigma a^\rho \gamma_\sigma \quad (34) \]

whilst for the commutator of covariant derivatives we find

\[ [D_\mu, D_\nu] a = \gamma_\rho [D_\mu, D_\nu] a^\rho = (R_{\mu \nu \rho}^\sigma a^\rho + K_{\mu \nu}^\alpha D_\alpha a^\sigma) \gamma_\sigma \quad (35) \]

### 3.2 C-space

Let us now consider C-space. A basis in C-space is given by

\[ E_A = \{ \gamma_\mu, \gamma_\mu \wedge \gamma_\nu, \gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho, \ldots \} \]
where in an \( r \)-vector \( \gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge ... \wedge \gamma_{\mu_r} \) we take the indices so that \( \mu_1 < \mu_2 < ... < \mu_r \).

An element of \( C \)-space is a Clifford number, called also \textit{Polyvector} or \textit{Clifford aggregate} which we now write in the form

\[
X = X^A E_A = s 1 + x^\mu \gamma_\mu + \sigma^{\mu\nu} \gamma_\mu \wedge \gamma_\nu + ... \tag{37}
\]

A \( C \)-space is parametrized not only by 1-vector coordinates \( x^\mu \) but also by the 2-vector coordinates \( \sigma^{\mu\nu} \), 3-vector coordinates \( \sigma^{\mu\nu\alpha} \), etc., called also \textit{holographic coordinates}, since they describe the holographic projections of 1-lines, 2-loops, 3-loops, etc., onto the coordinate planes. By \( p \)-loop we mean a closed \( p \)-brane; in particular, a 1-loop is closed string.

In order to avoid using the powers of the Planck scale length parameter \( \Lambda \) in the expansion of the polyvector \( X \) we use the dilatationally invariant units \([5]\) in which \( \Lambda \) is set to 1. The dilation invariant physics was discussed from a different perspective also in refs. \([19, 20, 21]\).

In a flat \( C \)-space the basis vectors \( E^A \) are constants. In a curved \( C \)-space this is no longer true. Each \( E_A \) is a function of the \( C \)-space coordinates

\[
X^A = \{ s, x^\mu, \sigma^{\mu\nu}, ... \} \tag{38}
\]

which include scalar, vector, bivector,..., \( r \)-vector,..., coordinates.

Now we define the connection \( \tilde{\Gamma}^C_{AB} \) in \( C \)-space according to

\[
\partial_A E_B = \tilde{\Gamma}^C_{AB} E_C \tag{39}
\]

where \( \partial_A \equiv \partial/\partial X^A \) is the partial derivative in \( C \)-space. This definition is analogous to the one in ordinary space. Let us therefore define the \( C \)-space curvature as

\[
\mathcal{R}^{CD}_{ABC} = ([\partial_A, \partial_B] E_C) \star E^D \tag{40}
\]

which is a straightforward generalization of the relation \([29]\). The ‘star’ means the \textit{scalar product} between two polyvectors \( A \) and \( B \), defined as

\[
A \star B = \langle A B \rangle_S \tag{41}
\]

where ‘\( S \)’ means ‘the scalar part’ of the geometric product \( AB \).
In the following we shall explore the above relation for curvature and see how it is related to the curvature of the ordinary space. Before doing that we shall demonstrate that the derivative with respect to the bivector coordinate $\sigma^{\mu\nu}$ is equal to the commutator of the derivatives with respect to the vector coordinates $x^\mu$.

Returning now to eq.(39), the differential of a $C$-space basis vector is given by

$$dE_A = \frac{\partial E_A}{\partial X^B} dX^B$$

In particular, for $A = \mu$ and $E_A = \gamma_\mu$ we have

$$d\gamma_\mu = \frac{\partial \gamma_\mu}{\partial x^\nu} dx^\nu + \frac{\partial \gamma_\mu}{\partial \sigma^{\alpha\beta}} d\sigma^{\alpha\beta} + \cdots = \tilde{\Gamma}^A_{\nu\mu} E_A dx^\nu + \tilde{\Gamma}^A_{[\alpha\beta]\mu} E_A d\sigma^{\alpha\beta} + \cdots$$

$$= (\tilde{\Gamma}^\alpha_{\nu\mu} \gamma_\alpha + \tilde{\Gamma}^{[\rho\sigma]}_{\nu\mu} \gamma_\rho \wedge \gamma_\sigma + \cdots) dx^\nu$$

$$+ (\tilde{\Gamma}^\rho_{[\alpha\beta]\mu} \gamma_\rho + \tilde{\Gamma}^{[\rho\sigma]}_{[\alpha\beta]\mu} \gamma_\rho \wedge \gamma_\sigma + \cdots) d\sigma^{\alpha\beta} + \cdots$$

We see that the differential $d\gamma_\mu$ is in general a polyvector, i.e., a Clifford aggregate. In eq.(43) we have used

$$\frac{\partial \gamma_\mu}{\partial x^\nu} = \Gamma^\alpha_{\nu\mu} \gamma_\alpha$$

$$\frac{\partial \gamma_\mu}{\partial \sigma^{\alpha\beta}} = \tilde{\Gamma}^\rho_{[\alpha\beta]\mu} \gamma_\rho$$

Let us now consider a restricted space in which the derivatives of $\gamma_\mu$ with respect to $x^\nu$ and $\sigma^{\alpha\beta}$ do not contain higher rank multivectors. Then eqs. (44),(45) become

$$\frac{\partial \gamma_\mu}{\partial x^\nu} = \Gamma^\alpha_{\nu\mu} \gamma_\alpha$$

$$\frac{\partial \gamma_\mu}{\partial \sigma^{\alpha\beta}} = \tilde{\Gamma}^\rho_{[\alpha\beta]\mu} \gamma_\rho$$

Further we assume that

(i) the components $\tilde{\Gamma}^\alpha_{\nu\mu}$ of the $C$-space connection $\tilde{\Gamma}^C_{AB}$ coincide with the connection $\Gamma^\alpha_{\nu\mu}$ of an ordinary space.

(ii) the components $\tilde{\Gamma}^\rho_{[\alpha\beta]\mu}$ of the $C$-space connection coincide with the curvature tensor $R_{\alpha\beta\mu}^\rho$ of an ordinary space.

Hence, eqs.(46),(47) read

$$\frac{\partial \gamma_\mu}{\partial x^\nu} = \Gamma^\alpha_{\nu\mu} \gamma_\alpha$$

11
\[
\frac{\partial \gamma_\mu}{\partial \sigma^{\alpha\beta}} = R^{\rho}_{\alpha\beta\mu} \gamma_\rho
\]  
(49)

and the differential becomes
\[
d\gamma_\mu = (\Gamma_{\alpha\mu}^\rho dx^\alpha + \frac{1}{2} R_{\alpha\beta\mu}^\rho d\sigma^{\alpha\beta}) \gamma_\rho
\]  
(50)

The same relation was obtained by Pezzaglia [2] by using a different method, namely by considering how polyvectors change with position. The above relation demonstrates that a geodesic in C-space is not a geodesic in ordinary spacetime. Namely, in ordinary spacetime we obtain Papapetrou’s equation. This was previously pointed out by Pezzaglia [2].

Although a C-space connection does not transform like a C-space tensor, some of its components, i.e., those of eq. (47), may have the transformation properties of a tensor in an ordinary space.

Under a general coordinate transformation in C-space
\[
X^A \rightarrow X'^A = X'^A(X^B)
\]  
(51)

the connection transforms according to
\[
\bar{\Gamma}^{C}_{AB} = \frac{\partial X^C}{\partial X^E} \frac{\partial X^J}{\partial X'^A} \frac{\partial X^K}{\partial X'^B} \Gamma^{E}_{JK} + \frac{\partial X^C}{\partial X^J} \frac{\partial^2 X^J}{\partial X'^A \partial X'^B}
\]  
(52)

In particular, the components which contain the bivector index \( A = [\alpha\beta] \) transform as
\[
\bar{\Gamma}^{\rho}_{[\alpha\beta]\mu} = \frac{\partial X^{lp}}{\partial x^e} \frac{\partial X^J}{\partial \sigma^{\alpha\beta}} \frac{\partial X^K}{\partial x'^\mu} \Gamma_{EJK} + \frac{\partial x^{lp}}{\partial x^e} \frac{\partial^2 X^J}{\partial \sigma^{\alpha\beta} \partial x'^\mu}
\]  
(53)

Let us now consider a particular class of coordinate transformations in C-space such that
\[
\frac{\partial x^{lp}}{\partial \sigma^{\mu\nu}} = 0, \quad \frac{\partial \sigma^{\mu\nu}}{\partial x'^\alpha} = 0
\]  
(54)

Then the second term in eq. (53) vanishes and the transformation becomes
\[
\bar{\Gamma}^{\rho}_{[\alpha\beta]\mu} = \frac{\partial X^{lp}}{\partial x^e} \frac{\partial \sigma^{\rho\sigma}}{\partial \sigma^{\alpha\beta}} \frac{\partial x^{\gamma}}{\partial x'^\mu} \bar{\Gamma}^{\rho}_{\gamma}\]
(55)

\(^2\)This can be derived from the relation
\[
dE'_A = \frac{\partial E'_A}{\partial X'^B} dX'^B \quad \text{where} \quad E'_A = \frac{\partial X^D}{\partial X'^A} E_D \quad \text{and} \quad dX'^B = \frac{\partial X'^B}{\partial X^C} dX^C
\]
Now, for the bivector whose components are \( d \sigma^{\alpha \beta} \) we have

\[
d \sigma'^{\alpha \beta} \gamma'_\alpha \wedge \gamma'_\beta = d \sigma^{\alpha \beta} \gamma_\alpha \wedge \gamma_\beta
\] (56)

Taking into account that in our particular case (54) \( \gamma_\alpha \) transforms as a basis vector in an ordinary space

\[
\gamma'_\alpha = \frac{\partial x^\mu}{\partial x'^\alpha} \gamma_\mu
\] (57)

we find that (56) and (57) imply

\[
d \sigma'^{\alpha \beta} \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} = d \sigma^{\mu \nu}
\] (58)

which means that

\[
\frac{\partial \sigma^{\mu \nu}}{\partial \sigma^{\alpha \beta}} = \frac{1}{2} \left( \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} - \frac{\partial x^\nu}{\partial x'^\alpha} \frac{\partial x^\mu}{\partial x'^\beta} \right) \equiv \frac{\partial x^{[\mu}}{\partial x'^{\alpha}} \frac{\partial x^{\nu]}}{\partial x'^{\beta}}
\] (59)

The transformation of the bivector coordinate \( \sigma^{\mu \nu} \) is thus determined by the transformation of the vector coordinates \( x^\mu \). This is so because the basis bivectors are the wedge products of basis vectors \( \gamma_\mu \).

From (55) and (59) we see that \( \tilde{\Gamma}^{\rho}_{[\rho \sigma] \gamma} \) transforms like a 4th-rank tensor in an ordinary space.

Comparing eq. (55) with the relation (28) we find

\[
\frac{\partial \gamma_\mu}{\partial \sigma^{\alpha \beta}} = [\partial_\alpha, \partial_\beta] \gamma_\mu
\] (60)

The partial derivative of a basis vector with respect to the bivector coordinates \( \sigma^{\alpha \beta} \) is equal to the commutator of partial derivatives with respect to the vector coordinates \( x^\alpha \).

The above relation (60) holds for the basis vectors \( \gamma_\mu \). For an arbitrary polyvector

\[
A = A^A E_A = s + a^\alpha \gamma_\alpha + a^{\alpha \beta} \gamma_\alpha \wedge \gamma_\beta + ...
\] (61)

we have

\[
\frac{DA}{D \sigma^{\mu \nu}} = [D_\mu, D_\nu] A
\] (62)

where \( D/\sigma^{\mu \nu} \) is the covariant derivative, defined in analogous way as in eqs. (17), (31):

\[
\frac{DE^A}{DX^B} = 0, \quad \frac{DA^A}{DX^B} = \frac{\partial A^A}{\partial X^B} + \tilde{\Gamma}^A_{BC} A^C
\] (63)

In general, thus, we employ the commutator of the covariant derivatives.
From (61) and (62) we obtain, after using (35), that

\[
Ds + Da^\alpha + Da^{\alpha\beta} \gamma_\alpha \wedge \gamma_\beta + \ldots = [D_\mu, D_\nu] s + [D_\mu, D_\nu] a^\alpha \gamma_\alpha + [D_\mu, D_\nu] a^{\alpha\beta} \gamma_\alpha \wedge \gamma_\beta + \ldots
\]  

(64)

From the latter polyvector equation we obtain

\[
\frac{Ds}{\sigma^{\mu\nu}} = [D_\mu, D_\nu] s = K_{\mu\nu} \rho \partial_\rho s
\]  

(65)

\[
\frac{Da^\alpha}{\sigma^{\mu\nu}} = [D_\mu, D_\nu] a^\alpha = R_{\mu\nu\rho} \alpha a^\rho + K_{\mu\nu} \rho D_\rho a^\alpha
\]  

(66)

Using (63) we have that

\[
\frac{Ds}{\sigma^{\mu\nu}} = \frac{\partial s}{\partial \sigma^{\mu\nu}}
\]  

(67)

and

\[
\frac{Da^\alpha}{\sigma^{\mu\nu}} = \frac{\partial a^\alpha}{\partial \sigma^{\mu\nu}} + \tilde{\Gamma}_{[\mu\nu]} \rho a^\rho = \frac{\partial a^\alpha}{\partial \sigma^{\mu\nu}} + R_{\mu\nu\rho} \alpha a^\rho
\]  

(68)

where, according to (ii), \( \tilde{\Gamma}_{[\mu\nu]} \rho \) has been identified with curvature. So we obtain, after inserting (67), (68) into (65), (66) that

(a) the partial derivative of the coefficients \( s \) and \( a^\alpha \), which are Clifford scalars\(^3\), with respect to \( \sigma^{\mu\nu} \) are related to torsion:

\[
\frac{\partial s}{\partial \sigma^{\mu\nu}} = K_{\mu\nu} \rho \partial_\rho s
\]  

(69)

\[
\frac{\partial a^\alpha}{\partial \sigma^{\mu\nu}} = K_{\mu\nu} \rho D_\rho a^\alpha
\]  

(70)

(b) whilst those of the basis vectors are related to curvature:

\[
\frac{\partial \gamma_\alpha}{\partial \sigma^{\mu\nu}} = R_{\mu\nu\beta} \alpha \gamma_\beta
\]  

(71)

In other words, the dependence of coefficients \( s \) and \( a^\alpha \) on \( \sigma^{\mu\nu} \) indicates the presence of torsion. On the contrary, when basis vectors \( \gamma_\alpha \) depend on \( \sigma^{\mu\nu} \) this indicates that the corresponding vector space has non vanishing curvature.

\(^3\)In the geometric calculus based on Clifford algebra, the coefficients such as \( s, a^\alpha, a^{\alpha\beta}, \ldots \), are called scalars (although in tensor calculus they are called scalars, vectors and tensors, respectively), whilst the objects \( \gamma_\alpha, \gamma_\alpha \wedge \gamma_\beta, \ldots \), are called vectors, bivectors, etc.
3.3 On the relation between the curvature of \( C \)-space and the curvature of an ordinary space

Let us now consider the \( C \)-space curvature defined in eq.(40). The indices \( A, B, \) can be of vector, bivector, etc., type. It is instructive to consider a particular example.

\[ A = [\mu\nu], \hspace{1em} B = [\alpha\beta], \hspace{1em} C = \gamma, \hspace{1em} D = \delta \]

\[
\left( \frac{\partial}{\partial \sigma^\alpha}, \frac{\partial}{\partial \sigma^\beta} \right) \gamma_\gamma = \mathcal{R}_{[\mu\nu][\alpha\beta]\gamma}^\delta \]

(72)

Using (49) we have

\[
\frac{\partial}{\partial \sigma^\mu} \frac{\partial}{\partial \sigma^\nu} \gamma_\gamma = \frac{\partial}{\partial \sigma^\mu} (R_{\alpha\beta\gamma\rho} \gamma_\rho) = R_{\alpha\beta\gamma}^\rho R_{\mu\nu\rho}^\sigma \gamma_\sigma \]

(73)

where we have taken

\[
\frac{\partial}{\partial \sigma^\mu} R_{\alpha\beta\gamma}^\rho = 0 \]

(74)

which is true in the case of vanishing torsion (see also an explanation that follows after the next paragraph). Inserting (73) into (72) we find

\[
\mathcal{R}_{[\mu\nu][\alpha\beta]\gamma}^\delta = R_{\mu\nu\gamma}^\rho R_{\alpha\beta}^\rho \delta - R_{\alpha\beta\gamma}^\rho R_{\mu\nu\rho}^\delta \]

(75)

which is the product of two usual curvature tensors. We can proceed in analogous way to calculate the other components of \( \mathcal{R}_{ABC}^D \) such as \( \mathcal{R}_{[\alpha\beta\gamma\delta][\rho\sigma\epsilon]}^\mu, \mathcal{R}_{[\alpha\beta\gamma\delta][\rho\sigma\tau\kappa]}^{[\mu\nu]} \), etc.

These contain higher powers of the curvature in an ordinary space. All this is true in our restricted \( C \)-space given by eqs.(46),(47) and the assumptions (i),(ii) below those equations. By releasing those restrictions we would have arrived at an even more involved situation which is beyond the scope of the present paper.

After performing the contractions of (75) and the corresponding higher order relations we obtain the expansion of the form

\[
\mathcal{R} = R + \alpha_1 R^2 + \ldots \]

(76)

So we have shown that the \( C \)-space curvature can be expressed as the sum of the products of the ordinary space curvature. This bears resemblance to the string effective action in curved spacetimes.

Let us now show that for vanishing torsion the curvature is independent of the bivector coordinates \( \sigma^{\mu\nu} \), as it was taken in eq.(74). Consider the basic relation

\[
\gamma_\mu \cdot \gamma_\nu = g_{\mu\nu} \]

(77)
Differentiating with respect to $\sigma^{\alpha\beta}$ we have

$$\frac{\partial}{\partial \sigma^{\alpha\beta}}(\gamma_\mu \cdot \gamma_\nu) = \frac{\partial \gamma_\mu}{\partial \sigma^{\alpha\beta}} \cdot \gamma_\nu + \gamma_\mu \cdot \frac{\partial \gamma_\nu}{\partial \sigma^{\alpha\beta}} = R_{\alpha\beta\mu\nu} + R_{\alpha\beta\nu\mu} = 0 \quad (78)$$

This implies that

$$\frac{\partial g_{\mu\nu}}{\partial \sigma^{\alpha\beta}} = [\partial_\alpha, \partial_\beta]g_{\mu\nu} = 0 \quad (79)$$

Hence the metric, in this particular case, is independent of the holographic (bivector) coordinates. Since the curvature tensor — when torsion is zero — can be written in terms of the metric tensor and its derivatives, we conclude that not only the metric, but also the curvature is independent of $\sigma^{\mu\nu}$. In general, when the metric has a dependence on the holographic coordinates one expects further corrections to eq. (78) that would include torsion.

4 Conclusion

C-space, i.e., the space of Clifford numbers or Clifford aggregates, is a natural generalization of an ordinary space. An ordinary space consists of points, whilst a C-space consists also of 1-loops, 2-loops, etc. Its geometric structure is thus very rich and provides a lot of room for interesting new physics, for instance, a description of various branes, including the fermionic degrees of freedom. It might turn out that M-theory which is conjectured to provide a unified description of various string theories (including D-branes) could be naturally formulated within the framework of C-space. Moreover, the Clifford geometric product of basis vectors $\gamma_\mu$ reproduces automatically the standard symmetric metric $g_{\mu\nu}$ in addition to a nonsymmetric object $\gamma_\mu \wedge \gamma_\nu$. Hence, as it was shown in ref. [5], the action for strings moving in such C-space background has connection to strings moving in an antisymmetric background $B_{\mu\nu}$. A nice geometric interpretation of the Dirac-Born-Infeld action may be found. We expect that C-space description is related to Moffat’s nonsymmetric theory of gravity as well [22]. Furthermore, the formalism of C-space explains very naturally the behavior of the variable speed of light cosmologies and a variable fine structure constant [23].

In the present paper we have studied some basic properties of a curved C-space. We have found that the curvature of C-space can be expressed in terms of the products of the ordinary curvature of the underlying space over which the C-space is defined. The
Einstein gravity in $C$-space thus becomes the higher derivative gravity in ordinary space. Torsion is also present in a general case, when the metric $g_{\mu\nu}$, is assumed to depend on the holographic coordinates $\sigma^{\mu\nu}$.

In our opinion Clifford algebra and $C$-space will turn out to provide an elegant and natural way for the formulation of M-theory. This is our vision justified by a number of promising results, some of them quoted in the introduction and Sec.2, the most notorious being the fact that in $C$-space spinors (as left or right minimal ideals of Clifford algebra) and supersymmetry are automatically contained both at the classical and at the quantum level. The deep interrelationship between the “area” derivative $\partial/\partial \sigma^{\alpha\beta}$ and the curvature and torsion derived in this paper is one of the crucial results on which the further development of the theory will be based. Understanding of the geometry of $C$-space is a prerequisite for the progress of the theory based on $C$-space which —as we think— will lead us towards M-theory and towards the unified theory of the known fundamental interactions.

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