DYNAMICS OF LOCALLY LINEARIZABLE COMPLEX TWO
DIMENSIONAL CUBIC HAMILTONIAN SYSTEMS

YANGYOU PAN
Department of Mathematics and Computer Science, Chizhou University
Chizhou 274000, China

YUZHEN BAI
School of Mathematical Sciences, Qufu Normal University
Qufu 273165, China

XIANG ZHANG∗
School of Mathematical Sciences, and MOE–LSC, Shanghai Jiao Tong University
Shanghai 200240, China

Abstract. The aim of this paper is to characterize global dynamics of locally
linearizable complex two dimensional cubic Hamiltonian systems. By finding
invariants, we prove that their associated real phase space \( \mathbb{R}^4 \) is foliated by
two dimensional invariant surfaces, which could be either simple connected, or
double connected, or triple connected, or quadruple connected. On each of the
invariant surfaces all regular orbits are heteroclinic ones, which connect two
singularities, either both finite, or one finite and another at infinity, or both at
infinity, and all these situations are realizable.

1. Introduction and statement of the main results. The study on the prob-
lem of locally analytic linearizability of analytic differential systems at a singularity
has a long history, and it can be traced back to Poincaré. Along this direction there
have appeared lots of results, see e.g. [4, 16] and the references therein. For real
planar analytic systems at a singularity having a pair of pure imaginary eigenvalues
this problem is equivalent to characterize the existence of isochronous center at the
singularity, see for instance the survey paper [4]. In 2015 Llibre and Romanovski [10]
extended their study to complex two dimensional polynomial Hamiltonian systems
with Hamiltonians being of the form

\[
H = -xy + h(x, y),
\]

where \( h \) is a cubic homogeneous polynomial or consists of cubic and quartic homo-
geogeneous polynomials. In the former it was given in [10] the necessary and sufficient
conditions for the Hamiltonian system to be linearizable at the origin. In the latter
there provided five sufficient conditions for the Hamiltonian system to be lineariz-
able at the origin.
The main aim of this paper is to investigate the global dynamics of locally linearizable complex two dimensional cubic Hamiltonian systems with Hamiltonians of the form (1). Here we do not consider the complex quadratic Hamiltonian systems with Hamiltonians of the form (1) because it will be studied uniformly for general quadratic Hamiltonian system in [17].

We must say that on global dynamics of complex differential systems there are only a few known results, see for instance [1, 7, 8, 19] and the references therein. In fact, even for planar real Hamiltonian systems, there are few differential systems where the global dynamics has been completely characterized, see e.g. [3, 5]. Here by finding new invariants we complete the characterization of the global dynamics of the mentioned systems.

Consider complex two dimensional cubic Hamiltonian systems, i.e.

\[ \dot{x} = x - a_{10}x^2 - 2b_{01}xy - a_{12}y^2 - a_{20}x^3 - a_{11}x^2y - 3b_{02}xy^2 - a_{13}y^3, \]
\[ \dot{y} = -y + b_{21}x^2 + 2a_{10}xy + b_{01}y^2 + b_{31}x^3 + 3a_{20}x^2y + a_{11}xy^2 + b_{02}y^3, \] (2)

with the Hamiltonian

\[ H = - xy + \frac{b_{21}}{3}x^3 + a_{10}x^2y + b_{01}xy^2 + \frac{a_{12}}{3}y^3 + \frac{b_{31}}{4}x^4 + a_{20}x^3y + \frac{a_{11}}{2}x^2y^2 + b_{02}xy^3 + \frac{a_{13}}{4}y^4, \] (3)

where \( x, y \) are complex variables and \( a_{ij}, b_{ij} \) are complex coefficients. Llibre and Romanovski [10, Theorem 4] obtained the next result.

**Theorem A.** The Hamiltonian system (2) is linearizable at the origin if one of the following conditions holds:

(a) \( b_{31} = b_{21} = b_{02} = a_{20} = a_{11} = a_{10} = 0, \)
(b) \( b_{02} = b_{01} = a_{20} = a_{13} = a_{12} = a_{11} = 0, \)
(c) \( b_{31} = b_{21} = 2b_{01}^2 + 9b_{02} = a_{20} = 2a_{12}b_{02} - a_{13}b_{01} = 4a_{12}b_{01} + 9a_{13} = a_{11} = a_{10} = 0, \)
(d) \( a_{12} = a_{13} = b_{02} = b_{01} = a_{11} = 2a_{10}^2 + 9a_{20} = 2b_{21}a_{20} - b_{31}a_{10} = 4b_{21}a_{10} + 9b_{11} = 0, \)
(e) \( 27b_{01}^3 + a_{12}^2b_{21} = 9a_{10}b_{01} - a_{13}b_{21} = a_{10}a_{12} + 3b_{01}^2 = 3a_{10}^2 + b_{01}b_{21} = -\frac{4}{3}a_{10}b_{21} + b_{31} = -4b_{01}^2 + b_{02} = \frac{4}{3}b_{01}b_{21} + a_{20} = -\frac{4}{3}a_{12}b_{01} + a_{13} = \frac{4}{3}a_{12}b_{21} + a_{11} = 0. \)

We remark that in Theorem A the condition \( 2a_{12}b_{02} - a_{13}b_{01} = 0 \) in (e), \( 2b_{21}a_{20} - b_{31}a_{10} = 0 \) in (d), and the two conditions \( 9a_{10}b_{01} - a_{12}b_{21} = 3a_{10}^2 + b_{01}b_{21} = 0 \) in (e) are extra, which can be obtained from the other conditions in the same item.

Note from the proof of [10, Theorem 4] that except in the case (e) all the other cases have the linearization transformation defined only locally. So their global dynamics cannot be characterized by their locally linearized systems. Next we present global dynamics of the Hamiltonian system (2) under each of the five conditions. For doing so, we will use the real form of the complex Hamiltonian system (2). Set

\[ x = x_1 + ix_2, \quad y = y_1 + iy_2, \]
\[ a_{10} = a_1 + ia_2, \quad b_{01} = b_1 + ib_2, \]
\[ a_{12} = c_1 + ic_2, \quad a_{20} = a_1 + ia_2, \]
\[ a_{11} = e_1 + ie_2, \quad b_{02} = f_1 + if_2, \]
\[ a_{13} = g_1 + ig_2, \quad b_{21} = h_1 + ih_2, \]
\[ b_{31} = i_1 + i_2, \quad i = \sqrt{-1}. \]
Under the real expressions of the complex variables and of complex coefficients, the complex Hamiltonian system (2) can be represented as

\[\begin{align*}
\dot{x}_1 &= x_1 - a_1(x_1^2 - x_2^2) + 2a_2x_1x_2 - 2b_1(x_1y_1 - x_2y_2) + 2b_2(x_1y_2 + x_2y_1) \\
&\quad - c_1(y_1^2 - y_2^2) + 2c_2y_1y_2 - (d_1x_1^2 + d_2x_2^2) + 3x_1x_2(d_2x_1 + d_1x_2) \\
&\quad - (x_1^2 - x_2^2)(e_1y_1 - e_2y_2) + 2x_1x_2(e_2y_1 + e_1y_2) - 3(f_1x_1 - f_2x_2)(y_1^2 - y_2^2) \\
&\quad + 6(f_2x_1 + f_1x_2)y_1y_2 + 3(g_2y_1 + g_1y_2)y_1y_2 - (g_1y_1^3 + g_2y_2^3), \\
\dot{x}_2 &= x_2 - a_2(x_1^2 - x_2^2) - 2a_1x_1x_2 - 2(b_2x_1 + b_1x_2)y_1 - 2(h_1x_1 - h_2x_2)y_2 \\
&\quad - c_2(y_1^2 - y_2^2) - 2c_1y_1y_2 - d_1x_1^3 + d_2x_2^3 - 3x_1x_2(d_1x_1 - d_2x_2) \\
&\quad - (x_1^2 - x_2^2)(e_2y_1 + e_1y_2) - 2x_1x_2(e_1y_1 - e_2y_2) - 3(f_2x_1 + f_1x_2)(y_1^2 - y_2^2) \\
&\quad + 6(f_1x_1 - f_2x_2)y_1y_2 - 3(g_1y_1 - g_2y_2)y_1y_2 + g_2y_1^3 + g_1y_2^3, \\
\dot{y}_1 &= -y_1 + h_1(x_1^2 - x_2^2) - 2h_2x_1x_2 + 2(a_1x_1 - a_2x_2)y_1 - 2(a_2x_1 + a_1x_2)y_2 \\
&\quad + b_1(y_1^2 - y_2^2) - 2b_2y_1y_2 + i_1x_1^3 + i_2x_2^3 - 3x_1x_2(i_2x_1 + i_1x_2) \\
&\quad + 3(x_1^2 - x_2^2)(d_1y_1 - d_2y_2) - 6x_1x_2(d_2y_1 + d_1y_2) + (e_1x_1 - e_2x_2)(y_1^2 - y_2^2) \\
&\quad - 2(e_2x_1 + e_1x_2)y_1y_2 - 3(f_2y_1 + f_1y_2)y_1y_2 + f_1y_1^3 + f_2y_2^3, \\
\dot{y}_2 &= -y_2 + h_2x_1^2 - h_2x_2^2 + 2h_1x_1x_2 + 2(a_2x_1 + a_1x_2)y_1 + 2(a_1x_1 - a_2x_2)y_2 \\
&\quad + b_2(y_1^2 - y_2^2) + 2b_1y_1y_2 + i_2x_1^3 - i_1x_2^3 - 3x_1x_2(i_1x_1 - i_2x_2) \\
&\quad + 3(d_2y_1 + d_1y_2)(x_1^2 - x_2^2) - 6x_1x_2(d_1y_1 - d_2y_2) + (e_2x_1 + e_1x_2)(y_1^2 - y_2^2) \\
&\quad + 2(e_1x_1 - e_2x_2)y_1y_2 + 3(f_1y_1 - f_2y_2)y_1y_2 + f_2y_1^3 - f_1y_2^3.
\end{align*}\]

Correspondingly, the real and imaginary parts of the Hamiltonian (3) are respectively

\[\begin{align*}
H_R &= -x_1y_1 + x_2y_2 + \frac{h_1x_1^3 + h_2x_2^3}{3} - (h_1x_1 + h_2x_2)x_1x_2 \\
&\quad + (x_1^2 - x_2^2)(a_1y_1 - a_2y_2) - 2x_1x_2(a_1y_2 + a_2y_1) + (b_1x_1 - b_2x_2)(y_1^2 - y_2^2) \\
&\quad - 2(b_2x_1 + b_1x_2)y_1y_2 + \frac{c_1y_1^3 + c_2y_2^3}{3} - (c_2y_1 + c_1y_2)y_1y_2 \\
&\quad + \frac{i_1(x_1^4 - 6x_1^2x_2^2 + x_2^4)}{4} - i_2x_1x_2(x_1^2 - x_2^2) + x_1(x_1^2 - 3x_2^2)(d_1y_1 - d_2y_2) \\
&\quad - (3x_1^2 - x_2^2)x_2(d_2y_1 + d_1y_2) + \frac{e_1(x_1^2 - x_2^2)(y_1^2 - y_2^2)}{2} - c_2(x_1^2 - x_2^2)y_1y_2 \\
&\quad - c_2x_1x_2(y_1^2 - y_2^2) - 2c_1x_1x_2y_1 + (f_1x_1 - f_2x_2)y_1(y_1^2 - 3y_2^2) \\
&\quad - (f_2x_1 + f_1x_2)(d_1^2 - d_2^2)y_1y_2 - 2d_1y_1y_2(y_1^2 - y_2^2) + \frac{g_1(y_1^4 - 6y_1^2y_2^2 + y_2^4)}{4}, \\
H_I &= -x_1y_2 - x_2y_1 + \frac{h_2x_1^3 - h_1x_2^3}{3} + x_1x_2(h_1x_1 - h_2x_2) \\
&\quad + (x_1^2 - x_2^2)(a_2y_1 + a_1y_2) + 2x_1x_2(a_1y_1 - a_2y_2) + 2(b_1x_1 - b_2x_2)y_1y_2 \\
&\quad + (b_2x_1 + b_1x_2)(y_1^2 - y_2^2) + (c_1y_1 - c_2y_2)y_1y_2 + \frac{c_2y_1^3 - c_1y_2^3}{3} \\
&\quad + \frac{i_2(x_1^4 - 6x_1^2x_2^2 + x_2^4)}{4} + i_1x_1x_2(x_1^2 - x_2^2) + x_1^2(d_2y_1 + d_1y_2) \\
&\quad + \frac{c_1y_1^3 + c_2y_2^3}{3} - (c_2y_1 + c_1y_2)y_1y_2 \\
&\quad + \frac{i_1(x_1^4 - 6x_1^2x_2^2 + x_2^4)}{4} - i_2x_1x_2(x_1^2 - x_2^2) + x_1(x_1^2 - 3x_2^2)(d_1y_1 - d_2y_2) \\
&\quad - (3x_1^2 - x_2^2)x_2(d_2y_1 + d_1y_2) + \frac{e_1(x_1^2 - x_2^2)(y_1^2 - y_2^2)}{2} - c_2(x_1^2 - x_2^2)y_1y_2 \\
&\quad - c_2x_1x_2(y_1^2 - y_2^2) - 2c_1x_1x_2y_1 + (f_1x_1 - f_2x_2)y_1(y_1^2 - 3y_2^2) \\
&\quad - (f_2x_1 + f_1x_2)(d_1^2 - d_2^2)y_1y_2 - 2d_1y_1y_2(y_1^2 - y_2^2) + \frac{g_1(y_1^4 - 6y_1^2y_2^2 + y_2^4)}{4},
\end{align*}\]
of Theorem A.

statements hold.

obtained local topological structure of nonlinear ones near simple singularity. Lerman and Umanskiy [9] systems of 2–degrees of freedom, Arnold [2] presented global dynamics of linear (Hamiltonian or not, see e.g. [11, 12, 14, 15, 18]. For integrable real Hamiltonian are lots of known results on characterization of integrability of differential systems

\[
- x_2^3(d_1y_1 - d_2y_2) - 3(d_2x_1 + d_1x_2)x_1x_2y_2 + 3(d_1x_1 - d_2x_2)x_1x_2y_1 \\
+ \frac{e_2(y_1^2 - y_2^2)(x_1^2 - x_2^2)}{2} + e_1(x_1^2 - x_2^2)y_1y_2 + e_1x_1x_2(y_1^2 - y_2^2) \\
- 2e_2x_1x_2y_1y_2 + (f_2x_1 + f_1x_2)y_1(y_1^2 - 3y_2^2) + (f_1x_1 - f_2x_2)(3y_1^2 - y_2^2)y_2 \\
+ g_1y_1y_2(y_1^2 - y_2^2) + \frac{g_2(y_1^2 - 6y_1^2y_2^2 + y_2^4)}{4}.
\]

We can check that system (4) is Hamiltonian with the Hamiltonian \( H \) and the first integral \( H_I \) in involution. Indeed, set \( \xi = (x_1, y_2, x_2)^T \) with \( \tau \) the transpose of a matrix. One has

\[
\dot{\xi} = J\nabla_\xi H_R, \quad \text{where} \quad J = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.
\]

Here \( \nabla_\xi H_R \) denotes the gradient of \( H_R \) with respect to \( \xi \). As we knew, there are lots of known results on characterization of integrability of differential systems (Hamiltonian or not, see e.g. [11, 12, 14, 15, 18]. For integrable real Hamiltonian systems of 2–degrees of freedom, Arnold [2] presented global dynamics of linear Hamiltonian ones with an elliptic–elliptic singularity. Lerman and Umanskiy [9] obtained local topological structure of nonlinear ones near simple singularity.

Now we can state our main results of this paper depending on the five conditions of Theorem A.

**Theorem 1.1.** For system (2) under the condition (a) of Theorem A, the following statements hold.

(I) If \( b_{01} = 0 \), the associated real four dimensional phase space \( \mathbb{R}^4 \) is foliated by real 2–dimensional invariant surfaces, in which one is the plane \( y = 0 \), another one saying \( S_0 \) is diffeomorphic to the \( y \)–plane, all the others saying \( S_I \) are diffeomorphic to the \( y \)–plane minus the origin, and they approach the infinity of the \( x \)–plane when \( y \to 0 \).

- System (2) restricted to \( y = 0 \) is an unstable linear star node.
- System (2) restricted to the \( y \)–plane is equivalent to a stable linear star node.
- System (2) restricted to each \( S_I \) has its orbits being topological lines and positively going to the infinity of the \( x \)–plane.

(II) If \( b_{01} \neq 0 \), system (2) has exactly two singularities \( S_{a1}^* \) and \( S_{a2}^* \), and the associated real four dimensional phase space \( \mathbb{R}^4 \) is foliated by real 2–dimensional invariant surfaces, in which there are two invariant planes \( y = 0 \) and \( b_{01}y = 1 \), two invariant surfaces \( L_{a1} \) and \( L_{a2} \) passing respectively \( S_{a1}^* \) and \( S_{a2}^* \), and all the other invariant surfaces \( L_{a\ell} \) approaching both \( y = 0 \) and \( b_{01}y = 1 \) at their infinity.

- System (2) restricted to \( y = 0 \) is an unstable linear star node.
- System (2) restricted to \( b_{01}y = 1 \) is a stable linear star node.
- All orbits, except two, on \( L_{a1} \) positively approach \( S_{a1}^* \) and negatively go to infinity of \( b_{01}y = 1 \).
- All orbits, except two, on \( L_{a2} \) negatively approach \( S_{a2}^* \) and positively go to infinity of \( y = 0 \).
- All orbits, except two, on \( L_{a\ell} \) are heterolinic to infinity of both \( y = 0 \) and \( b_{01}y = 1 \).
The two exceptional orbits mentioned above are also heteroclinic with some special connection described in the proof of this theorem.

Under the condition \((b)\) of Theorem A, system (4) has the same dynamics as those in Theorem 1.1, which can be seen by exchanging the coordinates \((x_1, x_2)\) with \((y_1, y_2)\).

**Theorem 1.2.** For system (2) under the condition \((c)\) of Theorem A, the following statements hold.

(I) If \(b_{01} = 0\), system (2) has the same dynamics as those in statement (I) of Theorem 1.1.

(II) If \(b_{01} \neq 0\), system (2) has exactly three singularities \(S_{c_1}^*, \ S_{c_2}^*, \ S_{c_3}^*\), and the associated real four dimensional phase space \(\mathbb{R}^4\) is foliated by real two dimensional invariant surfaces, in which there are three invariant planes \(y = 0, 2b_{01}y = 3\) and \(b_{01}y = 3\), one invariant surface \(L_{c_1}\) passing both \(S_{c_1}^*\) and \(S_{c_3}^*\), one invariant surface \(L_{c_2}\) passing \(S_{c_2}^*\), and all the other invariant surfaces \(L_{c_\ell}\) having three boundaries approaching respectively \(y = 0, 2b_{01}y = 3\) and \(b_{01}y = 3\) at their infinities.

- System (2) restricted to any one of \(y = 0, 2b_{01}y = 3\) and \(b_{01}y = 3\) is linear, and has a star node, which is unstable, stable and unstable.
- \(L_{c_1}\) is the global stable manifold of both \(S_{c_1}^*\) and \(S_{c_3}^*\), and on it all orbits, except two, negatively approach infinity of \(2b_{01}y = 3\).
- \(L_{c_2}\) is the global unstable manifold of \(S_{c_2}^*\), and on it all orbits, except two, positively approach infinity of either \(y = 0\) or of \(b_{01}y = 3\).
- All orbits on \(L_{c_\ell}\) are heteroclinic, and except four of them, negatively go to infinity of \(2b_{01}y = 3\) and positively approach infinity of either \(y = 0\) or \(b_{01}y = 3\).
- The two or four exceptional orbits mentioned in the last three items have one end approaching infinity of one of \(y = 0, 2b_{01}y = 3\) and \(b_{01}y = 3\), and the other end tending to infinity of \(\mathbb{R}^4\) with finite slope.

Under the condition \((d)\) of Theorem A, system (4) has the same dynamics as those in Theorem 1.2, which can be obtained by exchanging the coordinates \((x_1, x_2)\) with \((y_1, y_2)\).

Under the condition \((e)\) of Theorem A, as shown in [10] by Llibre and Romanovski system (2) can be globally linearized via the symplectic transformation

\[
u = x - \frac{a_{12}}{3} y^2, \quad v = y,
\]

if \(b_{21} = 0\), or

\[
u = x + a_{10}x^2 + \frac{6a_{10}}{b_{21}} xy + \frac{9a_{10}^2}{b_{21}^2} y^2, \quad v = y - \frac{b_{21}}{3} x^2 - 2a_{10} xy - \frac{3a_{10}^2}{b_{21}} y^2,
\]

if \(b_{21} \neq 0\). So in this case the topological dynamics of system (2) is simple and is determined by the linear system \(\dot{x} = x, \quad \dot{y} = -y\).

The remaining of this paper is to prove Theorems 1.1 and 1.2.

2. **Proof of Theorem 1.1.** In the proof of Theorem 1.1 we will use the real expressions of system (2) together with their first integral \(H\). Now the condition
(a) of Theorem A written in real form are
\[ i_1 = 0, \quad i_2 = 0, \quad h_1 = 0, \quad h_2 = 0, \quad f_1 = 0, \quad f_2 = 0, \]
\[ d_1 = 0, \quad d_2 = 0, \quad e_1 = 0, \quad e_2 = 0, \quad a_1 = 0, \quad a_2 = 0. \]
Correspondingly, system (4) is reduced to
\[ \dot{x}_1 = x_1 - 2b_1(x_1y_1 - x_2y_2) + 2b_2(x_1y_2 + x_2y_1) - c_1(y_1^2 - y_2^2), \]
\[ + 2c_2y_1y_2 - g_1(y_1^3 - 3y_1y_2^2) + g_2(3y_1^2y_2 - y_2^3), \]
\[ \dot{x}_2 = x_2 - 2b_2(x_1y_1 - x_2y_2) - 2b_1(x_1y_2 + x_2y_1) - c_2(y_1^2 - y_2^2), \]
\[ - 2c_1y_1y_2 - g_2(y_1^3 - 3y_1y_2^2) - g_1(3y_1^2y_2 - y_2^3), \]
\[ \dot{y}_1 = - y_1 + b_1(y_1^2 - y_2^2) - 2b_2y_1y_2, \]
\[ \dot{y}_2 = - y_2 + b_2(y_1^2 - y_2^2) + 2b_1y_1y_2. \]
Moreover, we can check that
\[ \det \begin{pmatrix} \frac{\partial H_R}{\partial x_1} & \frac{\partial H_R}{\partial x_2} \\ \frac{\partial H_I}{\partial x_1} & \frac{\partial H_I}{\partial x_2} \end{pmatrix} = F_1(y_1, y_2)F_2(y_1, y_2), \]
where
\[ F_1 = y_1^2 + y_2^2, \]
\[ F_2 = 1 - 2(b_1y_1 - b_2y_2) + (b_1^2 + b_2^2)(y_1^2 + y_2^2). \]
This means that the two first integrals \( H_R \) and \( H_I \) of system (4) are functionally independent, and so the system is integrable in the Liouvillian sense. Let \( X_a \) be the vector field associated to system (4). Some calculations show that
\[ X_a(F_1) = K_1F_1, \quad K_1 = -2(1 - b_1y_1 + b_2y_2), \]
\[ X_a(F_2) = K_2F_2, \quad K_2 = 2(b_1y_1 - b_2y_2). \]
This verifies that \( F_1 = 0 \) is always an invariant plane of system (5), and so is \( F_2 = 0 \) provided that \( b_0 \neq 0 \), i.e. \( b_1^2 + b_2^2 \neq 0 \). Here we have used the fact that \( F_2 \) can be written as
\[ F_2(y_1, y_2) = \frac{((b_1^2 + b_2^2)y_1 - b_1)^2 + ((b_1^2 + b_2^2)y_2 + b_2)^2}{b_1^2 + b_2^2}. \]
According to the Darboux theory of integrability [13, 20], \( F_1 \) and \( F_2 \) are called Darboux polynomials of system (4), and \( K_1 \) and \( K_2 \) are factors of \( F_1 \) and \( F_2 \), respectively.
Recall from the Darboux theory of integrability [13, 20] that for a polynomial vector field \( Z \) defined in \( \mathbb{R}^n \) or \( \mathbb{C}^n \) with coordinates \( z \), a polynomial \( F \in \mathbb{C}[z] \) is a Darboux polynomial of the vector field \( Z \) if there exists a polynomial \( K \in \mathbb{C}[z] \) such that
\[ Z(F) = KF, \]
where \( K \) is cofactor of \( F \). If the vector field \( Z \) has \( p \) Darboux polynomials \( F_j \) with cofactors \( K_j \), \( j = 1, \ldots, p \) satisfying
\[ \sigma_1K_1 + \ldots + \sigma_pK_p = 0, \]
with \( \sigma_j \in \mathbb{C} \) constants not all vanishing, then \( F_1^{\sigma_1} \ldots F_p^{\sigma_p} \) is a first integral of the vector field \( Z \), which is called a Darboux first integral.
(I) \( b_{01} = 0 \), i.e. \( b_1 = b_2 = 0 \). By (6) and (7) the two first integrals \( H_R \) and \( H_I \) of system (5) have their level sets intersecting transversally everywhere except on the invariant plane \( F_1 = 0 \). So, it follows from the Liouville–Arnold integrability theorem (see e.g. [8, 19]) that the phase space \( \mathbb{R}^4 \) of system (5) when \( b_{01} = 0 \) is foliated by two dimensional invariant manifolds.

Note that the level set \( (H_R, H_I) = (0, 0) \) has two branches \( F_1 = 0 \) and \( S_0 \) with the expression given by

\[
\begin{align*}
x_1 &= -\frac{1}{12} (8c_2 y_1 y_2 + 3g_2(3y_1^2 y_2 - y_2^3) - 4c_1(y_1^2 - y_2^2) - 3g_1(y_1^3 - 3y_1 y_2^2)) , \\
x_2 &= \frac{1}{12} (8c_1 y_1 y_2 + 3g_2(y_1^3 - 3y_1 y_2^2) + 3g_1(3y_1 y_2 - y_2^3) + 4c_2(y_1^2 - y_2^2)) .
\end{align*}
\]

Clearly when restricted to \( F_1 = 0 \) system (5) is linear and has the origin as an unstable star node. \( S_0 \) is defined in the full \( \mathbb{R}^2 \) plane, smooth and passes the origin. Hence its dynamics is completely determined by the last two equations of system (5) with \( b_1 = b_2 = 0 \). Consequently, \( S_0 \) has the origin as a globally stable node.

The level sets \( (H_R, H_I) = (l_1, l_2) \) with \( \ell = (l_1, l_2) \neq (0, 0) \) have a unique branch \( S_\ell \) with the expression given by

\[
\begin{align*}
x_1 &= -\frac{l_1 y_1 + l_2 y_2}{y_1^2 + y_2^2} , \\
x_2 &= \frac{l_2 y_1 - l_1 y_2}{y_1^2 + y_2^2} + \frac{1}{12} (8c_1 y_1 y_2 + 3g_2(y_1^3 - 3y_1 y_2^2) + 3g_1(3y_1 y_2 - y_2^3) + 4c_2(y_1^2 - y_2^2)) ,
\end{align*}
\]

which is defined in \( \mathbb{R}^2 \setminus \{(0,0)\} \). Obviously, the surface \( S_\ell \) approaches the infinity of the \((x_1, x_2)\)-plane when either \( F_1 \to 0 \) or \( F_1 \to \infty \). Moreover, all orbits on \( S_\ell \) are topological lines, and are heteroclinic to the infinity of the \((x_1, x_2)\)-plane.

(II) \( b_{01} \neq 0 \), i.e. \( b_1^2 = b_2^2 \neq 0 \). Again by (6) and (7) the level sets of the two first integrals \( H_R \) and \( H_I \) of system (5) intersect transversally everywhere outside the invariant planes \( F_1 = 0 \) and \( F_2 = 0 \). Note from the expression (8) of \( F_2 \) with the Liouville–Arnold integrability theorem that the phase space \( \mathbb{R}^4 \) of system (5) when \( b_{01} \neq 0 \) is foliated by two dimensional invariant manifolds.

Now we have another Darboux polynomial

\[
F_3 = b_2 y_1 + b_1 y_2 ,
\]

with the cofactor \( K_3 = -1 + 2(b_1 y_1 - b_2 y_2) \), because \( \mathcal{X}_a(F_3) = K_3 F_3 \). So again by the Darboux theory of integrability [13, 20] system (5) has the rational first integral

\[
H_3 = F_1 F_2 F_3^{-2} = \frac{(y_1^2 + y_2^2)(((b_1^2 + b_2^2)y_1 - b_1)^2 + ((b_1^2 + b_2^2)y_2 + b_2)^2)}{(b_1^2 + b_2^2)(b_2 y_1 + b_1 y_2)^2} .
\]

Some calculations with the help of Mathematica shows that the gradients of \( H_R(\xi) \), \( H_I(\xi) \), \( H_3(\xi) \) are linearly dependent if and only if \( \xi \in \{ F_1 = 0 \} \cup \{ F_2 = 0 \} \cup \{ F_4 = 0 \} \), where

\[
F_4 = (b_1^2 + b_2^2)(y_1^2 + y_2^2) - b_1 y_1 + b_2 y_2 ,
\]
is also a Darboux polynomial with the cofactor \( K_4 = 2b_1y_1 - 2b_2y_2 - 1 \). Moreover, \( F_4 \) can be written in
\[
F_4 = \frac{(2(b_1^2 + b_2^2)y_1 - b_1)^2 + (2(b_1^2 + b_2^2)y_2 + b_2)^2 - (b_1^2 + b_2^2)}{4(b_1^2 + b_2^2)}.
\]

Note that the third and fourth equations of system (5) are independent of \( x_1, x_2 \), and they form a 2–dimensional system with exactly two singularities
\[
S_1 = (0, 0), \quad S_2 = \left( \frac{b_1}{b_1^2 + b_2^2}, -\frac{b_2}{b_1^2 + b_2^2} \right),
\]
which are respectively stable and unstable nodes. Easy calculations verify that \( F_4 = 0 \) is an invariant elliptic cylinder and contains the two singularities and two heteroclinic orbits connecting these two singularities. By computing the dynamics of this two dimensional system at the infinity of the \((y_1, y_2)\)–plane we get the phase portrait as shown in Fig. 1.

![Figure 1. Phase portrait of the last two equations of system (5)](image)

Lifting the invariant orbits and singularities of the 2–dimensional system to the phase space \( \mathbb{R}^4 \) of system (5), we get the two invariant planes \( F_1 = 0 \) and \( F_2 = 0 \) lifted from \( S_1 \) and \( S_2 \) respectively, and two families of 3–dimensional invariant cylinders, which are heteroclinic to \( F_1 = 0 \) and \( F_2 = 0 \).

System (5) restricted to \( F_1 = 0 \) is linear and has the origin as an unstable star node. System (5) restricted to \( F_2 = 0 \) is also linear and has a unique singularity, which is a stable star node.

We can check that system (5) has exactly two singularities
\[
S^*_a_1 = (0, 0, 0, 0), \quad S^*_a_2 = \left( \frac{q_1}{(b_1^2 + b_2^2)^3}, \frac{q_2}{(b_1^2 + b_2^2)^3}, \frac{b_1}{b_1^2 + b_2^2}, -\frac{b_2}{b_1^2 + b_2^2} \right),
\]
which are both saddles and have eigenvalues \((1, 1, -1, -1)\), where
\[
q_1 = -\frac{b_1^2 + b_2^2}{b_1^2 + b_2^2}c_1 - 2(b_1^2 + b_2^2)b_1b_2c_2 - b_1(b_1^2 - 3b_2^2)g_1 - (3b_1^2 - b_2^2)b_2g_2,
q_2 = -\frac{b_1^2 - b_2^2}{b_1^2 + b_2^2}c_2 + 2(b_1^2 + b_2^2)b_1b_2c_1 - b_1(b_1^2 - 3b_2^2)g_2 + (3b_1^2 - b_2^2)b_2g_1.
\]
Since \( F_1 = 0 \) passes through \( S^*_{a_1} \), by the dynamics on \( F_1 = 0 \) it follows that system (5) has a 2-dimensional stable analytic manifold tangent to the \((y_1, y_2)\)-plane. Note that \((H_R, H_I)|_{S^*_{a_1}} = (0, 0)\), and \((H_R, H_I) = (0, 0)\) has a branch with the expression given by

\[
x_i = A_{1i}(y_1, y_2) + \frac{B_{1i}(y_1, y_2)}{F_2}, \quad i = 1, 2,
\]

where \( A_{1i} \) and \( B_{1i} \) are polynomials of degrees two and one, respectively. Denote this branch by \( L_{a1} \). Then \( L_{a1} \) is the global stable analytic manifold, and it approaches the infinity of \( F_2 = 0 \) when \( F_2 \to 0 \). On \( L_{a1} \) all orbits except two ones positively tend to \( S^*_{a_1} \) and negatively go to the infinity of the invariant plane \( F_2 = 0 \). The exceptional two ones have one positively approaching \( S^*_{a_1} \) and negatively tending to the infinity of \((x_1, x_2)\)-plane when \( y_1^2 + y_2^2 \to \infty \); and another one negatively approaching the infinity of the invariant plane \( F_2 = 0 \) and positively going to the infinity of \((x_1, x_2)\)-plane when \( y_1^2 + y_2^2 \to \infty \).

Let \( (H_R, H_I)|_{S^*_{a_2}} = (v_{21}, v_{22}) \). We can show that \((H_R, H_I) = (v_{21}, v_{22})\) has two branches \( F_2 = 0 \) and \( L_{a2} \) with the latter the 2-dimensional global unstable analytic manifolds of \( S^*_{a_2} \), which approaches the infinity of the invariant plane \( F_2 = 0 \) when \( F_2 \to 0 \). Moreover, on \( L_{a2} \) all orbits except two ones negatively tend to \( S^*_{a_2} \) and positively go to infinity of the invariant plane \( F_1 = 0 \). The exceptional two ones have one negatively approaching \( S^*_{a_2} \) and positively tending to the infinity of \((x_1, x_2)\)-plane when \( y_1^2 + y_2^2 \to \infty \); and another one positively approaching the infinity of the invariant plane \( F_1 = 0 \) and negatively going to the infinity of \((x_1, x_2)\)-plane when \( y_1^2 + y_2^2 \to \infty \).

For \( \ell \not\in \{(0, 0), (v_{21}, v_{22})\} \), we get that the level set \((H_R, H_I) = \ell\), denoted by \( L_{a\ell} \), satisfies the following expressions

\[
x_1 = A_1(y_1, y_2)/(F_1F_2), \quad x_2 = B_1(y_1, y_2)/(F_1F_2),
\]

with \( A_1, B_1 \) polynomials of degree six in \( y_1, y_2 \), whose expressions will not be presented here, because they are tedious and could be obtained by direct calculations. Some further calculations show that \( L_{a\ell} \) is defined in the full \((y_1, y_2)\)-plane except at \( S_1 \) and \( S_2 \), where the 2-dimensional invariant manifold \( L_{a\ell} \) approximates the infinity of \( F_1 = 0 \) and \( F_2 = 0 \), respectively. Moreover, on \( L_{a\ell} \) all orbits are topological lines, and are heteroclinic to the infinity of the \((x_1, x_2)\)-plane, either along \( F_1 = 0 \), or along \( F_2 = 0 \), or during the process \( y_1^2 + y_2^2 \to \infty \).

This proves statement \((II)\) and consequently completes the proof of the theorem.

\( \square \)

3. **Proof of Theorem 1.2.** As in the proof of Theorem 1.1, we use the real form of system (2). Now the conditions (c) are the following

\[
\begin{align*}
  a_1 &= 0, \quad a_2 = 0, \quad d_1 = 0, \quad d_2 = 0, \quad c_1 = 0, \quad e_2 = 0, \quad f_1 &= -\frac{2}{9}(b_1^2 - b_2^2), \\
  f_2 &= -\frac{4}{9}b_1b_2, \quad g_1 = -\frac{4}{9}(b_1c_1 - b_2c_2), \quad g_2 = -\frac{4}{9}(b_2c_1 + b_1c_2), \\
  h_1 &= 0, \quad h_2 = 0, \quad i_1 = 0, \quad i_2 = 0.
\end{align*}
\]

System (4) is reduced to

\[
\begin{align*}
  \dot{x}_1 &= x_1 - 2b_1(x_1y_1 - x_2y_2) + 2b_2(x_1y_2 + x_2y_1) - c_1(y_1^2 - y_2^2) + 2c_2y_1y_2 \\
  &\quad + \frac{2}{3}(b_1^2 - b_2^2)(x_1(y_1^2 - y_2^2) - 2x_2y_1y_2) - \frac{4}{3}b_1b_2(x_2(y_1^2 - y_2^2) + 2x_1y_1y_2)
\end{align*}
\]
\[ x_2 = x_2 - 2b_2(x_1y_1 - x_2y_2) - b_1(x_1y_2 + x_2y_1) - c_2(y_1^2 - y_2^2) - 2c_1y_1y_2 \]
\[ + \frac{2}{3}(b_1^2 - b_2^2)(x_2(y_1^2 - y_2^2) + 2x_1y_1y_2) + \frac{4}{3}b_1b_2(x_1(y_1^2 - y_2^2) - 2x_2y_1y_2) \]
\[ + \frac{4}{9}(b_1c_2 + b_2c_1)(y_1^3 - 3y_1y_2^2) + \frac{4}{9}(b_1c_1 - b_2c_2)(3y_1^2y_2 - y_2^3), \]
\[ \dot{y}_1 = -y_1 + b_1(y_1^2 - y_2^2) - 2b_2y_1y_2 + \frac{2}{9}(b_1^2 - b_2^2)y_1^3 \]
\[ + \frac{2}{3}(b_1^2 - b_2^2)y_1y_2 + \frac{4}{9}b_1b_2(3y_1^2y_2 - y_2^3), \]
\[ \dot{y}_2 = -y_2 + b_2(y_1^2 - y_2^2) + 2b_1y_1y_2 - \frac{4}{9}b_1b_2(y_1^3 - 3y_1^2y_2^2) \]
\[ - \frac{2}{3}(b_1^2 - b_2^2)y_1y_2^2 + \frac{2}{9}(b_1^2 - b_2^2)y_2^3. \]
Correspondingly, the two first integrals are reduced to
\[ H_{xc} = -x_1y_1 + x_2y_2 + (b_1x_1 - b_2x_2)(y_1^2 - y_2^2) - 2(b_2x_1 + b_1x_2)y_1y_2 \]
\[ + \frac{1}{9}(3c_1 - 2(b_1^2 - b_2^2)x_1 + 4b_1b_2x_2)(y_1^3 - 3y_1y_2^2) \]
\[ - \frac{1}{9}(3c_2 - 2(b_1^2 - b_2^2)x_2 - 4b_1b_2x_1)(3y_1^2y_2 - y_2^3) \]
\[ + \frac{4}{9}(b_1c_2 + b_2c_1)y_1y_2(y_1^2 - y_2^2) - \frac{1}{9}(b_1c_1 - b_2c_2)(y_1^4 - 6y_1^2y_2^2 + y_2^4), \]
\[ H_{xc} = -x_1y_2 - x_2y_1 + (b_2x_1 + b_1x_2)(y_1^2 - y_2^2) + 2(b_1x_1 - b_2x_2)y_1y_2 \]
\[ + \frac{1}{9}(3c_2 - 4b_1b_2x_1 - 2(b_1^2 - b_2^2)x_2)(y_1^3 - 3y_1y_2^2) \]
\[ + \frac{1}{9}(3c_1 - 2(b_1^2 - b_2^2)x_1 + 4b_1b_2x_2)(3y_1^2y_2 - y_2^3) \]
\[ - \frac{4}{9}(b_1c_1 - b_2c_2)(y_1^4y_2 - y_1y_2^3) - \frac{1}{9}(b_1c_2 + b_2c_1)(y_1^4 - 6y_1^2y_2^2 + y_2^4). \]
Let $\mathcal{X}$ be the vector field associated to system (9). Set
\[ F_{c1} = y_1^2 + y_2^2, \]
\[ F_{c2} = 9 - 12(b_1y_1 - b_2y_2) + 4(b_1^2 + b_2^2)(y_1^2 + y_2^2), \]
\[ F_{c3} = 9 - 6(b_1y_1 - b_2y_2) + (b_1^2 + b_2^2)(y_1^2 + y_2^2). \]
Some calculations show that
\[ \mathcal{X}(F_{c1}) = K_{c1}F_{c1}, \]
\[ \mathcal{X}(F_{c2}) = K_{c2}F_{c2}, \]
\[ \mathcal{X}(F_{c3}) = K_{c3}F_{c3}, \]
where
\[ K_{c1} = -2 + 2(b_1y_1 - b_2y_2) - \frac{4}{9}(b_1^2 - b_2^2)(y_1^2 - y_2^2) + \frac{16}{9}b_1b_2y_1y_2, \]
\[ K_{c2} = \frac{4}{3}(b_1y_1 - b_2y_2) - \frac{4}{9}(b_1^2 - b_2^2)(y_1^2 - y_2^2) + \frac{16}{9}b_1b_2y_1y_2, \]
\[ K_{c3} = \frac{2}{3}(b_1y_1 - b_2y_2) - \frac{4}{9}(b_1^2 - b_2^2)(y_1^2 - y_2^2) + \frac{16}{9}b_1b_2y_1y_2. \]
Hence \( F_{c1} \) is always a Darboux polynomial of system (9), and so are \( F_{c2} \) and \( F_{c3} \) provided that \( b_1 \) and \( b_2 \) do not simultaneously vanish.

Moreover, we can check that the gradients of \( H_{Rc}(\xi) \) and \( H_{Ic}(\xi) \) are linearly independent in \( \mathbb{R}^4 \) if and only if \( \xi \) does not belong to the invariant sets \( \{ F_{c1} = 0 \} \cup \{ F_{c2} = 0 \} \cup \{ F_{c3} = 0 \} \).

(I) \( b_{01} = 0 \). Applying similar calculations as in the proof of Theorem 1.1 we get that system (9) has the same dynamics as those in statement (I) of Theorem 1.1.

(II) \( b_{01} \neq 0 \), i.e. \( b_1^2 + b_2^2 \neq 0 \). By the above calculations it follows that \( F_{c2} \) and \( F_{c3} \) are also Darboux polynomials of system (9). In addition, we can rewrite (9) in the form

\[
F_{c2} = \frac{1}{b_1^2 + b_2^2} \left( (2b_1^2 + b_2^2)y_1 - 3b_1^2 + (2b_1^2 + b_2^2)y_2 + 2b_2^2 \right),
\]

\[
F_{c3} = \frac{1}{b_1^2 + b_2^2} \left( (b_1^2 + b_2^2)y_1 - 3b_1^2 + (b_1^2 + b_2^2)y_2 + 2b_2^2 \right).
\]

Hence, \( F_{c1} = 0 \), \( F_{c2} = 0 \) and \( F_{c3} = 0 \) are three invariant planes, which contain respectively exactly one of the three singularities, saying \( S_{c1}^* \), \( S_{c2}^* \) and \( S_{c3}^* \), of system (9). Furthermore, we can check that

\[
F_{c4} = b_2y_1 + b_1y_2
\]

is also a Darboux polynomial of system (9). The invariant hyperplane \( F_{c4} = 0 \) passes through the three singularities \( S_{c1}^* \), \( S_{c2}^* \) and \( S_{c3}^* \).

On the invariant plane \( F_{c1} = 0 \) system (9) is reduced to

\[
\dot{x}_1 = x_1, \quad \dot{x}_2 = x_2.
\]

On the invariant plane \( F_{c2} = 0 \) system (9) is reduced to

\[
\dot{x}_1 = -\frac{3(b_1^2 - b_2^2)c_1 + 6b_1b_2c_2}{4(b_1^2 + b_2^2)} - \frac{x_1}{2}, \quad \dot{x}_2 = \frac{6b_1b_2c_1 - 3(b_1^2 - b_2^2)c_2}{4(b_1^2 + b_2^2)} - \frac{x_2}{2}.
\]

On the invariant plane \( F_{c3} = 0 \) system (9) is reduced to

\[
\dot{x}_1 = \frac{3(b_1^2 - b_2^2)c_1 + 6b_1b_2c_2}{(b_1^2 + b_2^2)^2} + x_1, \quad \dot{x}_2 = \frac{3(b_1^2 - b_2^2)c_2 - 6b_1b_2c_1}{(b_1^2 + b_2^2)^2} + x_2.
\]

Outside \( \{ F_{c1} = 0 \} \cup \{ F_{c2} = 0 \} \cup \{ F_{c3} = 0 \} \), as shown before \( (H_{Rc}, H_{Ic}) = \ell = (\ell_1, \ell_2) \) with \( (\ell_1, \ell_2) \notin \{(H_{Rc}, H_{Ic})|S_{c1}^*, (H_{Rc}, H_{Ic})|S_{c2}^*, (H_{Rc}, H_{Ic})|S_{c3}^*\} \), is a 2-dimensional surface, denoted by \( L_{c\ell} \), whose expression is

\[
x_1 = -\frac{3(b_1^2 - b_2^2)c_1 + 6b_1b_2c_2}{4(b_1^2 + b_2^2)^2} - \frac{(b_1c_1 + b_2c_2)y_1 - 2(b_1c_2 - b_2c_1)y_2}{2(b_1^2 + b_2^2)} - \frac{\gamma_1 y_1 + \gamma_2 y_2}{F_{c1}} - \frac{(b_1^2 + b_2^2)(\gamma_1 y_1 + \gamma_2 y_2) - 3(b_1\gamma_1 - b_2\gamma_2)}{4(b_1^2 + b_2^2)^2 F_{c2}},
\]

\[
x_2 = \frac{6b_1b_2c_1 - 3(b_1^2 - b_2^2)c_2}{4(b_1^2 + b_2^2)^2} - \frac{(b_1c_1 - b_2c_2)y_1 + (b_1c_1 + b_2c_2)y_2}{2(b_1^2 + b_2^2)} - \frac{\gamma_2 y_1 - \gamma_1 y_2}{F_{c1}} + \frac{3(b_1\gamma_2 + b_2\gamma_1) - (b_1^2 + b_2^2)(\gamma_2 y_1 - \gamma_1 y_2)}{4(b_1^2 + b_2^2)^2 F_{c2}} - \frac{A_{c2}(y_1, y_2)}{4(b_1^2 + b_2^2)^2 F_{c2}},
\]

(10)
where

\[ A_{c1} = 27(b_1^2 - b_2^2)c_1 + 54b_1b_2c_2 - 48(b_1^2 + b_2^2)2(b_1\gamma_1 - b_2\gamma_2) \]
\[ - 18(b_1^3 - 3b_1b_2^2)(c_1y_1 + c_2y_2) - 18(3b_1^2b_2 - b_2^3)(c_2y_1 - c_1y_2) \]
\[ + 32(b_1^2 + b_2^2)3(\gamma_1y_1 + \gamma_2y_2). \]
\[ A_{c2} = 27(b_1^2 - b_2^2)c_2 - 54b_1b_2c_1 - 48(b_1^2 + b_2^2)2(b_2\gamma_1 + b_1\gamma_2) \]
\[ - 18(b_1^3 - 3b_1b_2^2)(c_2y_1 - c_1y_2) + 18(3b_1^2b_2 - b_2^3)(c_1y_1 + c_2y_2) \]
\[ + 32(b_1^2 + b_2^2)3(\gamma_2y_1 - \gamma_1y_2). \]

Note that the last two equations of system (9) are independent of \( x_1, x_2, \) and have three singularities

\[ S_{c1} = (0, 0), \quad S_{c2} = \left( \frac{3b_1}{2(b_1^2 + b_2^2)}, -\frac{3b_2}{2(b_1^2 + b_2^2)} \right), \quad S_{c3} = \left( \frac{3b_1}{b_1^2 + b_2^2}, \frac{3b_2}{b_1^2 + b_2^2} \right), \]

which are stable, unstable and stable nodes, respectively. Further studying the dynamics of these two equations in the \((y_1, y_2)\)-plane using the Poincaré compactification [6] we get the phase portrait of the last two equations of system (9) in the Poincaré disc as shown in Fig. 2.

**Figure 2.** Phase portrait of the last two equations of system (9)

Lifting the orbits in Fig. 2 we get invariant planes and 3-dimensional invariant surfaces of system (9). The three singularities \( S_{c1}, S_{c2} \) and \( S_{c3} \) are lifted to the three invariant planes \( F_{c1} = 0, F_{c2} = 0 \) and \( F_{c3} = 0 \), all the other regular orbits which are heteroclinic are lifted to 3-dimensional invariant surfaces of system (9).

We can check that for \( \ell = (\ell_1, \ell_2) = (H_{Rc}, H_{Ic})|_{S_{c1}} = (H_{Rc}, H_{Ic})|_{S_{c2}^*} = (0, 0) \), the expression (10) is defined in \( \mathbb{R}^2 \setminus \{ S_{c2} \} \), and the level surface \( L_{c1} := \{ \xi \in \mathbb{R}^4| (H_{Rc}(\xi), H_{Ic}(\xi)) = (0, 0) \} \) passes through the singularities \( S_{c1}^* \) and \( S_{c3}^* \) and approaches the infinity of the invariant plane \( F_{c2} = 0 \).

For \( \ell = (\ell_1, \ell_2) = (H_{Rc}, H_{Ic})|_{S_{c2}^*} \), we have \( A_{c1}(y_1, y_2) = A_{c2}(y_1, y_2) \equiv 0 \). So the expression (10) is defined in \( \mathbb{R}^2 \setminus \{ S_{c1}, S_{c3} \} \), and the level surface \( L_{c2} := \{ \xi \in \mathbb{R}^4| (H_{Rc}(\xi), H_{Ic}(\xi)) = (H_{Rc}, H_{Ic})|_{S_{c2}^*} \} \) passes through the singularity \( S_{c2}^* \) and approaches the infinity of both the invariant planes \( F_{c1} = 0 \) and \( F_{c3} \).

For \( \ell = (\ell_1, \ell_2) \not\in \{(H_{Rc}, H_{Ic})|_{S_{c1}^*}, (H_{Rc}, H_{Ic})|_{S_{c2}^*}, (H_{Rc}, H_{Ic})|_{S_{c3}^*}\} \), the expression (10) is defined only in \( \mathbb{R}^2 \setminus \{ S_{c1}, S_{c2}, S_{c3} \} \), and the surface \( L_{c3} \) approaches the infinity of all the three invariant planes \( F_{c1} = 0, F_{c2} \) and \( F_{c3} \). Furthermore, each
of these surfaces $L_{cl}$ transversally intersects the 3–dimensional invariant manifolds lifted by the regular orbits in Fig. 2. Using these information together with Fig. 2 and working in a similar way as in the proof of Theorem 1.1, we can complete the proof of statement (II), and consequently of Theorem 1.2.

4. Conclusions. Complex 2–dimensional polynomial Hamiltonian systems whose dynamics were characterized are very few. Even for complex planar quadratic and cubic Hamiltonian systems, it is also open on the characterization of their global dynamics. As a first step to understand the global dynamics of this kind of systems, here we focus on some special systems, for instance the locally linearizable complex 2–dimensional Hamiltonian systems. By finding new invariants together with qualitative theory of dynamical systems we completely characterize the global dynamics of the mentioned systems.

Acknowledgments. We would like to thank the referees for their comments on presentations of this paper.

REFERENCES

[1] M. J. Alvarez, A. Gasull and R. Prohens, Topological classification of polynomial complex differential equations with all the critical points of centre type, J. Difference Equ. Appl., 16 (2010), 411–423.
[2] V. I. Arnold, Ordinary Differential Equations, 3rd edition, Springer-Verlag, Berlin, 1992.
[3] J. C. Artés and J. Llibre, Quadratic Hamiltonian vector fields, J. Differential Equations, 107 (1994), 80–95.
[4] J. Chavarriga and M. Sabatini, A survey of isochronous centers, Qual. Theory Dyn. Syst., 1 (1999), 1–70.
[5] A. Cima and J. Llibre, Bounded polynomial vector fields, Trans. Amer. Math. Soc., 318 (1990), 557–579.
[6] F. Dumortier, J. Llibre and J. C. Artés, Qualitative Theory of Planar Differential Systems, Springer, Berlin, 2006.
[7] A. García, A. Gasull and X. Jarque, Local and global phase portrait of equation $z' = f(z)$, Discrete Contin. Dyn. Syst., 17 (2007), 309–329.
[8] A. Gasull, J. Llibre and X. Zhang, One–dimensional quaternion homogeneous polynomial differential equations, J. Math. Phys., 50 (2009), 082705, 17 pp.
[9] L. M. Lerman and Ya. L. Umanskiy, Four–dimensional Integrable Hamiltonian Systems with Simple Singular Points (Topological Aspects), Transl. Math. Monographs, American Math. Soc., Providence, Rhode Island, 1998.
[10] J. Llibre and V. G. Romanovski, Isochronicity and linearizability of planar polynomial Hamiltonian systems, J. Differential Equations, 259 (2015), 1649–1662.
[11] J. Llibre and C. Valls, Darboux integrability of 2–dimensional Hamiltonian systems with homogeneous potentials of degree 3, J. Math. Phys., 55 (2014), 033507, 12 pp.
[12] J. Llibre and C. Valls, Liouvillian first integrals for a class of generalized Liénard polynomial differential systems, Proc. Roy. Soc. Edinburgh Sect. A, 146 (2016), 1195–1210.
[13] J. Llibre and X. Zhang, On the Darboux integrability of polynomial differential systems, Qual. Theory Dyn. Syst., 11 (2012), 129–144.
[14] A. J. Maciejewski, M. Przybylska and H. Yoshida, Necessary conditions for the existence of additional first integrals for Hamiltonian systems with homogeneous potential, Nonlinearity, 25 (2012), 255–277.
[15] Y. P. Martinez and C. Vidal, Classification of global phase portraits and bifurcation diagrams of Hamiltonian systems with rational potential, J. Differential Equations, 261 (2016), 5923–5948.
[16] V. G. Romanovski and D. S. Shafer, The Center and Cyclicity Problems: A Computational Algebra Approach, Birkhäuser, Boston, 2009.
[17] H. Shi, X. Zhang and Y. Zhang, Linearization and dynamics of complex planar Hamiltonian systems, Preprint.
[18] C. Valls, Rikitake system: Analytic and Darbouxian integrals, *Proc. Roy. Soc. Edinburgh Sect. A*, 135 (2005), 1309–1326.

[19] X. Zhang, Global structure of quaternion polynomial differential equations, *Comm. Math. Phys.*, 303 (2011), 301–316.

[20] X. Zhang, *Integrability of Dynamical Systems: Algebra and Analysis*, Springer, Singapore, 2017.

Received September 2017; revised December 2017.

E-mail address: ppy1984@yeah.net
E-mail address: baiyu99@126.com
E-mail address: xzhang@sjtu.edu.cn