PARTIAL REGULARITY OF SOLUTIONS TO THE FRACTIONAL NAVIER-STOKES EQUATIONS

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Abstract. We study the partial regularity of suitable weak solutions to the Navier-Stokes equations with fractional dissipation $\sqrt{-\Delta^s}$ in the critical case of $s = \frac{3}{2}$. We show that the two dimensional Hausdorff measure of space-time singular set of these solutions is zero.

1. Introduction. In this paper, we are interested in the partial regularity theory of suitable weak solutions of the following 3D incompressible fractional Navier-Stokes equations

$$\begin{cases}
    u_t + u \cdot \nabla u + \sqrt{-\Delta^s} u + \nabla p = 0, & \text{in } \mathbb{R}^3 \times (0,T), \\
    \nabla \cdot u = 0, & \text{in } \mathbb{R}^3 \times (0,T),
\end{cases}$$

where $u$ represents the velocity field and $p$ the scalar pressure.

When $s = 2$, system (1) is the well-known 3D classical Navier-Stokes equation. The existence of global weak solutions was proved by Leray [11] in whole space $\mathbb{R}^3$ and by Hopf [7] in bounded domains. The global regularity of the Leray-Hopf weak solutions is still a challenging problem. In [15, 16, 18], Scheffer started the program of studying the Hausdorff measure of the singular set of the weak solutions to the 3D Navier-Stokes equations. In this context, the best result was obtained by Caffarelli, Kohn and Nirenberg [2], in which they introduced the concept of suitable weak solutions and proved that the 1D Hausdorff measure of space-time singular set of these solutions is zero. Lin in [12] simplified Caffarelli, Kohn and Nirenberg’s proof, especially the way of estimating pressure $p$. See further various versions by Ladyzhenskay and Seregin [10], Tian and Xin [24], Vasseur [26], Hou and Lei [8] and [17, 4, 27, 5, 6] for the study on higher dimensional Navier-Stokes equations. These results all rely on the so-called generalized energy inequality.

For the fractional Navier-Stokes equations with hyperdissipative laplacian, Lions proved the global existence of classical solutions to (1) with $s \geq \frac{5}{2}$ in [13]. Later on, Tao [22] improved Lions’s result by a gain of a half of log derivative. Barbato, Morandin and Romito [1] further improved Tao’s result by a gain of a log of derivative. See [23, 28] for further results. Using the Littlewood-Paley theory, Katz and
Pavlović in [9] studied the partial regularity of solutions to the fractional Navier-Stokes equations at the first blow-up time when $2 < s < \frac{5}{2}$. Recently, Tang and Yu [20, 21] studied the partial regularity of solutions to the fractional Navier-Stokes equations in the case of $\frac{5}{4} < s < 2$. The method of Tang and Yu depends on the characterizations for fractional Laplacian established by Caffarelli and Silvestre [3]. However, the case when $s = \frac{3}{2}$ is left open in [20, 21].

To take a glance of the criticality of $s = \frac{3}{2}$, let us first recall the following imbedding property

$$L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; \dot{H}^{\frac{3}{2}}(\mathbb{R}^3)) \hookrightarrow L^{\frac{2(3+s)}{3}}(\mathbb{R}^3 \times (0, T)),$$

which is an easy consequence of standard Sobolev imbedding theorem and interpolation inequality. On the other hand, to control the right hand side in the generalized energy inequality (7), we need the space-time $L^3$ estimate of the velocity field $u$. So to control the space-time $L^3$ norm of $u$ by the natural localized energy, a necessary condition would be $\frac{2(3+s)}{3} \geq 3$, i.e. $s \geq \frac{3}{2}$.

In this paper, we are interested in the partial regularity theory of the fractional Navier-Stokes equations in the critical case of $s = \frac{3}{2}$. Let $T > 0$ and denote $Q_T = \mathbb{R}^3 \times (0, T)$. Our main result is the following theorem:

**Theorem 1.1.** Let $s = \frac{3}{2}$. If $(u, p)$ is a suitable weak solution of the fractional Navier-Stokes equations (1) on $Q_T$, then the two dimensional Hausdorff measure of space-time singular set of $u$ is zero.

Following the method in the paper of Caffarelli, Kohn and Nirenberg [2], to estimate the Hausdorff dimension and measure of singular set, we only need to prove the following theorem. The notations used below will be introduced in Section 2.

**Theorem 1.2.** Suppose that $s = \frac{3}{2}$ and $(u, p)$ is a suitable weak solution of the fractional Navier-Stokes equations (1) on $Q_T$. There exists a positive constant $\epsilon_0$ such that if $z_0 = (x_0, t_0) \in Q_T$ and

$$\limsup_{r \to 0^+} r^{-2} \int_{Q_r(z_0)} |\tilde{\nabla} \tilde{u}|^2 y^{-\frac{1}{2}} dydz < \epsilon_0,$$

then $z_0$ is a regular point of $u$. More precisely, $u$ is essentially bounded in a neighborhood (parabolic ball) of $z_0$.

The definition of suitable weak solutions to fractional Navier-Stokes equations (see [20, 21]) is based on the characterizations for fractional powers of Laplacian operator obtained by Caffarelli and Silvestre [3]. These characterizations shown in [3] are proved by the well-known harmonic extensional idea. Such kind of higher dimensional extensional method for the fractional Laplacian can also be found in [14]. For convenience, we use $\nabla$ and $\tilde{\nabla}$ to denote the gradient operator defined on $\mathbb{R}^3$ and its extension on $\mathbb{R}^4_+$, and the associated divergence operators are denoted by div and Div. More precisely, for any $u \in \dot{H}^{\frac{3}{2}}(\mathbb{R}^3)$, we define $\tilde{u}$ on $\mathbb{R}^4_+$ by solving the following extension problem

$$\begin{cases}
\text{Div}(y^{-\frac{1}{2}} \tilde{\nabla} \tilde{u}) = 0, & \text{for } (x, y) \in \mathbb{R}^4_+, \\
\tilde{u}(x, 0) = u(x), & \text{for } x \in \mathbb{R}^3.
\end{cases}$$

According to [3], we have

$$C_1 \int_{\mathbb{R}^3_+} |\tilde{\nabla} \tilde{u}|^2 y^{-\frac{1}{2}} dx dy = \int_{\mathbb{R}^3} |\xi|^{\frac{3}{2}} |\tilde{u}|^2 d\xi,$$
\[- C_1 \lim_{y \to 0^+} y^{-\frac{1}{2}} \tilde{u}_y = \sqrt{-\Delta} \tilde{u}, \quad (5)\]

where $C_1$ is a positive constant. Moreover, $\tilde{u}$ defined in (3) minimizes the following functional

\[ J(v) = \int_{\mathbb{R}^3} |\nabla v|^2 y^{-\frac{1}{2}} dx dy. \quad (6) \]

We follow [20, 21] to define the suitable weak solutions of fractional Navier-Stokes equations (1) on $Q_T$, if

- $u \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^{\frac{3}{2}}(\mathbb{R}^3))$ and $p \in L^2(Q_T)$.
- $(u, p)$ satisfies equations (1) in the sense of distribution.
- For any nonnegative smooth function $\phi(x, y, t)$ with compact support in $\mathbb{R}^3 \times [0, \infty)$, the following generalized energy inequality holds:

\[ \int_{\mathbb{R}^3 \times \{t\}} |u|^2 \phi_0 + 2C_1 \int_0^t \int_{\mathbb{R}^4} |\nabla \tilde{u}|^2 y^{-\frac{1}{2}} \phi \leq \int_0^t \int_{\mathbb{R}^3} (u \cdot \nabla \phi_0)(2p + |u|^2) + C_1 \int_0^t \int_{\mathbb{R}^4} |\tilde{u}|^2 \text{Div}(y^{-\frac{1}{2}} \nabla \phi) \]

\[ + \int_0^t \int_{\mathbb{R}^3} |u|^2 \{ \partial_t \phi_0 + C_1 \lim_{y \to 0^+} (y^{-\frac{1}{2}} \partial_y \phi) \}, \quad (7) \]

where $\phi_0(x, t) = \phi(x, 0, t)$ and $C_1$ is a positive constant given in (4).

**Remark 1.** Formally, for smooth solutions, using (5), we have the following computations:

\[ \int_{\mathbb{R}^3} \sqrt{-\Delta}^\frac{3}{2} u \cdot u \phi_0 dx = - C_1 \int_{\mathbb{R}^3} \lim_{y \to 0^+} y^{-\frac{1}{2}} \partial_y \tilde{u} \cdot \tilde{u} \phi_0 dx = C_1 \int_{\mathbb{R}^4} \text{Div}(y^{-\frac{1}{2}} \nabla \tilde{u} \cdot \tilde{u}) dx dy. \]

Then by (3), the above equals to

\[ C_1 \int_{\mathbb{R}^4} y^{-\frac{1}{2}} |\nabla \tilde{u}|^2 \phi dx + C_1 \int_{\mathbb{R}^4} y^{-\frac{1}{2}} \nabla \tilde{u} \cdot \tilde{u} \nabla \phi dx dy. \]

The first term above corresponds to the viscous term in (7) and the second term above, by integration by parts, becomes

\[ - \frac{C_1}{2} \int_{\mathbb{R}^4} |\tilde{u}|^2 \text{Div}(y^{-\frac{1}{2}} \nabla \phi) dx dy - \frac{C_1}{2} \int_{\mathbb{R}^3} |u|^2 \lim_{y \to 0^+} y^{-\frac{1}{2}} \partial_y \phi dx, \]

which correspond to the second line and the second part of last term in (7).

To prove Theorem 1.2, we follow the strategy in [20, 21]. At first, we need to get the smallness of more dimensionless quantities in some smaller domains under the assumption (2) (see Lemma 3.4). Then, we use inductive arguments to improve the decay rate, which is crucial for the higher regularity estimate (see Theorem 4.1). The approach depends on the generalized energy inequality (7) with some appropriate test functions. Besides the usual Hölder and interpolation inequalities, the key step is to deal with the nonlinear term $pu$ on the right hand side of (7).
To control the term $pu$, we study the quantities $C$ and $D$ (the notations will be introduced in Section 2) for velocity field $u$ and pressure $p$, respectively. Note that, the main difference between the case of $s = \frac{3}{2}$ and the case of $\frac{3}{2} < s < 2$ is that we could not obtain the optimal decay rate of $D$. This means that we are not able to get the estimate of the decay rate or smallness of quantity $D$ defined in a smaller ball by itself in a larger ball directly. However, under the smallness assumption (2), we can obtain the optimal control on quantities $C$ and $D$ by introducing another two quantities $A$ and $F$ for the velocity field and the pressure, respectively. The reason we introduce the quantity $F$ is that it has better decay rate than $D$. This comes from the observation that the norm of pressure $p$ with larger time integrable index has better decay rate. Furthermore, the “bad” quantity $D$ which has a less decay rate can be controlled by the “good” quantity $F$ which has a better decay rate, see Lemma 3.2 for more details. Our approach mainly depends on the selection of the appropriate norm on the pressure $p$, which is very important for the final inductive arguments. Finally, we emphasize that in Section 3 we also generalize the restriction of the time integrable index of pressure $p$ in [20, 21].

The paper is organized as follows: Some notations and definitions of dimensionless quantities are introduced in Section 2. The basic properties of $p$, $u$ and its extension $\tilde{u}$ are also given there. Then in Section 3 we prove some preliminary but important estimates and lemmas. Section 4 is devoted to the proof of our main result.

2. Preliminaries. Let us first introduce several notations. Spacial and space-time balls in $\mathbb{R}^3$, $\mathbb{R}_+^4$ and $\mathbb{R}^3 \times (0, T)$, $\mathbb{R}_+^4 \times (0, T)$ are defined as follows:

\[
\begin{align*}
\mathbf{z}_0 &= (x_0, t_0),
\mathbf{Q}_r(z_0) &= B_r(x_0) \times (t_0 - r^{\frac{3}{2}}, t_0),
\mathbf{\tilde{B}}_r(x_0) &= B_r(x_0) \times (0, r),
\mathbf{\tilde{Q}}_r(z_0) &= \mathbf{\tilde{B}}_r(x_0) \times (t_0 - r^{\frac{3}{2}}, t_0),
[v]_{x_0, r} &= \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} v dx.
\end{align*}
\]

Following [2, 12] and [20, 21], we will need the following dimensionless quantities:

\[
\begin{align*}
A_r(z_0) &= \text{ess sup}_{t_0 - r^{\frac{3}{2}} < t < t_0} r^{-2} \int_{B_r(x_0)} |u|^2 dx, \\
C_r(z_0) &= r^{-3} \int_{Q_r(z_0)} |u|^3 dz, \\
E_r(z_0) &= r^{-2} \int_{\mathbf{\tilde{Q}}_r(z_0)} |\nabla \tilde{u}|^2 y^{-\frac{3}{2}} dy dz, \\
F_r(z_0) &= r^{m-\frac{3m}{4} - \frac{3}{2}} \int_{t_0 - r^{\frac{3}{2}}}^{t_0} \left( \int_{B_r(x_0)} |p|^q \right)^{\frac{m}{q}} dt,
\end{align*}
\]

where $m$ and $q$ are determined later. We shall need an extra quantity involving pressure which is also used by Lin in [12]:

\[
D_r(z_0) = r^{-3} \int_{Q_r(z_0)} |p|^{\frac{3}{2}} dz.
\]
Besides, in order to show \( u \) is essentially bounded in a neighborhood of \( z_0 \), we study some quantities corresponding to the higher regularity of \( u \) and \( p \) defined as follows:

\[
\bar{A}_r(z_0) = r^{-1} A_r(z_0), \quad \bar{C}_r(z_0) = r^{-\frac{3}{2}} C_r(z_0), \\
\bar{D}_r(z_0) = r^{-1} D_r(z_0), \quad \bar{E}_r(z_0) = r^{-1} E_r(z_0).
\]

For ease of notations, we denote \( B_r = B_r(0), A_r = A_r(0, 0) \) and the similar notations for other various balls (spacial, space-time) and quantities.

When estimating the pressure \( p \), we will constantly use the following decomposition

\[
p = p_{1,0,r}^1 + p_{2,0,r}^2,
\]

where

\[
p_{1,0,r} = \frac{1}{4\pi} \int_{\mathbb{R}^3} \nabla^2_{xy} \frac{1}{|x-y|} (u^i - [u^i]_{x_0,2r})(u^j - [u^j]_{x_0,2r}) \eta dy, \\
p_{2,0,r} = \frac{1}{4\pi} \int_{\mathbb{R}^3} \nabla^2_{xy} \frac{1}{|x-y|} (u^i - [u^i]_{x_0,2r})(u^j - [u^j]_{x_0,2r})(1 - \eta) dy.
\]

Here \( 0 \leq \eta \leq 1 \) is a smooth cut-off function supported in \( B_r(x_0) \) and \( \eta = 1 \) in \( B_{2\bar{r}}(x_0) \). Then \( p_{2,0,r}^2 \) is harmonic in \( B_{2\bar{r}}(x_0) \). In what follows, we simply denote \( p_{1,0,r} \) by \( p_1 \) and the similar to \( p_{2,0,r} \).

At the end of this section, we present the following proposition for the extension function \( \tilde{u} \), which plays an important role in the whole paper. This is essentially proved in \([20, 21]\), where \( s \) is required to satisfy \( \frac{3}{4} < s < 2 \). There is no difficulty to apply the proof in \([20, 21]\) to \( s = \frac{3}{2} \). For completeness, we include its proof below.

**Proposition 1.** Suppose that \( u \in \dot{H}^{\frac{3}{2}}(\mathbb{R}^3) \) and the extension function \( \tilde{u} \) satisfies (3). Then the following inequalities hold:

\[
\int_{\bar{B}_r(x_0)} |\tilde{u}|^2 y^{-\frac{1}{2}} dxdy \lesssim r^2 \int_{\bar{B}_r(x_0)} |u|^2 dx + r^2 \int_{\bar{B}_r(x_0)} |\nabla \tilde{u}|^2 y^{-\frac{1}{2}} dxdy, \\
\left( \int_{B_2(x_0)} |u|^4 dx \right)^{\frac{1}{2}} \lesssim \left( r^{-\frac{1}{2}} \int_{B_r(x_0)} |u|^2 dx \right)^{\frac{1}{2}} + \left( \int_{B_r(x_0)} |\nabla \tilde{u}|^2 y^{-\frac{1}{2}} dxdy \right)^{\frac{1}{2}}, \\
\left( \int_{B_2(x_0)} |u - [u]_{x_0,r}|^4 dx \right)^{\frac{1}{2}} \lesssim \left( \int_{B_r(x_0)} |\nabla \tilde{u}|^2 y^{-\frac{1}{2}} dxdy \right)^{\frac{1}{2}}.
\]

**Proof.** By translation and scaling, we only need to prove the proposition in the case of \( x_0 = 0, r = 1 \). Using the identity \( \tilde{u}(x, y) = u(x) + \int_0^y \partial_w \tilde{u} dw \), one has

\[
\int_{\bar{B}_1} |\tilde{u}|^2 y^{-\frac{1}{2}} dxdy \lesssim \int_{B_1} |u|^2 dx + \int_{B_1} |\nabla \tilde{u}|^2 y^{-\frac{1}{2}} dxdy.
\]

This gives (9).

Now let \( 0 \leq \eta \leq 1 \) be a smooth cut-off function in \( \mathbb{R}^3 \times [0, \infty) \) and satisfy

\[
\eta(x, y) = \begin{cases} 
1, & (x, y) \in B_1 \times [0, \frac{1}{2}], \\
0, & (x, y) \notin B_1 \times [0, 1].
\end{cases}
\]
Lemma 3.1. For \( 4r \leq \rho \), we have

\[
C_r \lesssim \left( \frac{r}{\rho} \right)^{\frac{3}{2}} A_{\rho}^\frac{3}{2} \right) + \left( \frac{\rho}{r} \right)^{\frac{3}{2}} A_r^\frac{3}{2} E_\rho.
\]
Proof. By interpolation inequality and Proposition 1, we have
\[ \|u\|_{L^2(B_r)} \lesssim \|u\|_{L^2(\Omega)}^{\frac{1}{2}} \|u\|_{L^4(\Omega)}^{\frac{1}{2}} \]
\[ \lesssim \|u\|_{L^2(B_r)}^{\frac{1}{2}} (r^{-\frac{1}{2}} \|u\|_{L^2(B_{2r})} + \|\nabla u\|_{L^2(B_{2r}; y^{-\frac{1}{2}})}) \]
(12)
\[ \lesssim r^{-\frac{1}{2}} \|u\|_{L^2(B_r)} + \|u\|_{L^2(B_{2r})} \|\nabla u\|_{L^2(B_{2r}; y^{-\frac{1}{2}})}. \]

By Hölder inequality and Proposition 1,
\[ \int_{B_{2r}} |u|^2 = \int_{B_{2r}} |u|^2 - |[u]|^2 + \int_{B_{2r}} |[u]|^2 \]
\[ \lesssim \|u - [u]\|_{L^2(B_{2r})}^2 + \left( \frac{C}{r^3} \right)^3 \int_{B_r} |u|^2 \]
\[ \lesssim r^3 \|u\|_{L^2(B_r)} \|u - [u]\|_{L^2(B_{2r})} + \left( \frac{C}{r^3} \right)^3 \int_{B_r} |u|^2 \]
(13)

Combining (12) with (13), we have
\[ \int_{Q_r} |u|^3 \lesssim \int_{-r^2}^0 r^{-\frac{1}{2}} \left( \frac{C}{r^3} \right)^\frac{3}{2} \|u\|_{L^2(B_r)}^3 \]
\[ + r^{-\frac{1}{2}} \|u\|_{L^2(B_r)} \|\nabla u\|_{L^2(B_{2r}; y^{-\frac{1}{2}})}^3 + \|u\|_{L^2(B_r)} \|\nabla u\|_{L^2(B_{2r}; y^{-\frac{1}{2}})}^3 \ dt. \]

By Young inequality, it holds
\[ \int_{Q_r} |u|^3 \lesssim \int_{-r^2}^0 r^{-\frac{1}{2}} \left( \frac{C}{r^3} \right)^\frac{3}{2} \|u\|_{L^2(B_r)}^3 + \left( \frac{C}{r^3} \right)^\frac{3}{2} \|u\|_{L^2(B_{2r})} \|\nabla u\|_{L^2(B_{2r}; y^{-\frac{1}{2}})}^3 \ dt \]
\[ \lesssim r^3 \left( \frac{C}{r^3} \right)^\frac{3}{2} A_m^\frac{3}{2} + \rho^3 \left( \frac{C}{r^3} \right)^\frac{3}{2} A_m^\frac{3}{2} E^\frac{3}{2}. \]

Thus, the proof is completed.

The following lemma corresponds to the pressure \( p \). It gives the estimates of quantities \( D \) and \( F \).

Lemma 3.2. Suppose that \( (m, q) \) satisfies \( 1 < q \leq \frac{3}{2}, \ m \geq \frac{3}{2} \) and \( \frac{1}{m} + \frac{2}{q} \geq 2 \), we have for \( 3r \leq \rho \)
\[ F_\rho \lesssim \left( \frac{C}{r^3} \right)^m A_m^m + \left( \frac{C}{r^3} \right)^{2m} A_m^{2m} \]
(14)
\[ D_\rho \lesssim \left( \frac{C}{r^3} \right)^{-2m} A_m^{-2m} E^2 + \left( \frac{C}{r^3} \right)^{-3m} E^3 + \left( \frac{C}{r^3} \right)^3 E^3. \]
(15)

Remark 2. From (14), we see that \( F \) has more decay rate provided that \( m \) is large enough. This fact is crucial for the inductive arguments in Section 4. What’s more, the quantity \( D \) used to bound the nonlinear term \( pu \) on the right hand side of (7) satisfies (14) when \( m = q = \frac{3}{2} \). However, we could not get any decay rate at all. Fortunately, (15) shows that we can estimate \( D \) by a “good” quantity \( F \).

Proof. Let \( \eta \) be a smooth cut-off function on \( B_2^\frac{3}{2} \). According to the decomposition (8), by interpolation inequality and Calderon-Zygmund estimate [19], for \( 1 < q \leq \frac{3}{2} \),
we have
\[ \|p_1^2\|_{L^6(B_r)} \lesssim |u| \|u\|_{L^2(B_{\frac{3}{2}})} \lesssim |u| \|u\|_{L^2(B_{\frac{3}{2}})} \lesssim |u| \|u\|_{L^2(B_{\frac{3}{2}})} \lesssim |u| \|u\|_{L^2(B_{\frac{3}{2}})} \] (16)

Since \( \frac{1}{m} + \frac{2}{q} \geq 2 \), we have by integrating the above over time
\[ \|p_1^2\|_{L^m(Q_r)} \lesssim \rho^{\frac{3}{4} + \frac{3}{m} - 1} E_\rho^{\frac{3}{2} - \frac{3}{4}}. \] (17)

Due to the fact that \( p_2^2 \) is harmonic in \( B_{\frac{3}{2}} \), it holds
\[ \|p_2^2\|_{L^m(B_r)} \lesssim r^{\frac{3}{4}} \|p_2^2\|_{L^\infty(B_r)} \lesssim (\frac{r}{\rho})^{\frac{3}{4}} \|p_2^2\|_{L^m(B_{\frac{3}{2}})} \lesssim (\frac{r}{\rho})^{\frac{3}{4}} (\|p\|_{L^m(B_{\frac{3}{2}})} + \|p_1^2\|_{L^m(B_{\frac{3}{2}})}). \]

Similarly, integrating the above over time, we obtain
\[ \|p_2^2\|_{L^m(L^\infty(D_t; Q_r))} \lesssim \rho^{\frac{3}{4} + \frac{3}{m} - 1} A_{\rho}^{\frac{3}{2} - \frac{3}{4}} E_\rho^{\frac{3}{2} - \frac{3}{4}}. \] (18)

Replacing \( r \) by \( \frac{r}{\rho} \) in (16), we get
\[ \|p_1^2\|_{L^m(L^\infty(D_t; Q_r))} \lesssim \rho^{\frac{3}{4} + \frac{3}{m} - 1} A_{\rho}^{\frac{3}{2} - \frac{3}{4}} E_\rho^{\frac{3}{2} - \frac{3}{4}}. \] (19)

By (17)-(19), we immediately get (14).

The proof of (15) follows in a similar manner. In fact, from (17) with \( m = q = \frac{3}{2} \), we have
\[ \|p_1^2\|_{L^\frac{3}{2}(Q_r)} \lesssim \rho^2 A_{\rho}^\frac{1}{2} E_\rho^\frac{2}{3}. \] (20)

For the harmonic part of \( p \), it holds that
\[ \|p_2^2\|_{L^\frac{3}{2}(B_r)} \lesssim r^{\frac{3}{2}} \|p_2^2\|_{L^\infty(B_r)} \lesssim r^{\frac{3}{2} - \frac{3}{4}} \|p_2^2\|_{L^m(B_{\frac{3}{2}})} \lesssim r^{\frac{3}{2} - \frac{3}{4}} (\|p\|_{L^m(B_{\frac{3}{2}})} + \|p_1^2\|_{L^m(B_{\frac{3}{2}})}). \]

Since \( m > \frac{3}{2} \), by integrating the above with respect to time \( t \) and (19)
\[ \|p_2^2\|_{L^\frac{3}{2}(Q_r)} \lesssim r^{3 - \frac{3}{4} - \frac{3}{m} - 1} \rho^{-\frac{3}{4}} (\|p\|_{L^m(L^\infty(D_t; Q_r))} + \|p_1^2\|_{L^m(L^\infty(D_t; Q_r))}) \lesssim r^{3 - \frac{3}{4} - \frac{3}{m} - 1} \rho^{-\frac{3}{4}} (\|p\|_{L^m(L^\infty(D_t; Q_r))} + \|p_1^2\|_{L^m(L^\infty(D_t; Q_r))}) \lesssim r^{3 - \frac{3}{4} - \frac{3}{m} - 1} \rho^{-\frac{3}{4}} (\|p\|_{L^m(L^\infty(D_t; Q_r))} + \|p_1^2\|_{L^m(L^\infty(D_t; Q_r))}). \] (21)

Combining (20) with (21), we obtain
\[ D_t \lesssim (\frac{r}{\rho})^{\frac{3}{4} - \frac{3}{m} - \frac{3}{2}} (F_{\rho}^{\frac{3}{2}} + A_{\rho}^{\frac{3}{2} - \frac{3}{4} E_{\rho}^{\frac{3}{2} - \frac{3}{4}}}) + (\frac{r}{\rho})^{\frac{3}{2}} A_{\rho}^{\frac{1}{2} E_{\rho}}. \]

Thus, the proof is completed.

Now, let us focus on the dimensionless quantity \( A \).
Lemma 3.3. Suppose that $6r \leq \rho$ and $(m,q)$ satisfies the assumptions in Lemma 3.2, then the following inequality holds

$$A_r \lesssim \left( \frac{r}{\rho} \right)^{5} A_{\rho} + \left( \frac{r}{\rho} \right)^{4} (A_{\rho}^{\frac{2}{3}} E_{\rho}^{\frac{2}{3}} + A_{\rho} E_{\rho}^{\frac{1}{2}}) + \left( \frac{r}{\rho} \right)^{2\frac{3}{4}} (F_{\rho}^{\frac{2}{m}} + A_{\rho}^{\frac{2}{3}} E_{\rho}^{\frac{4}{3}}).$$

(22)

Proof. Let $\eta(x,y,t) = \eta_1(x)\eta_2(y)\eta_3(t)$ be a smooth cut-off function in $Q_{\rho}$, where

$$0 \leq \eta_1(x) = \begin{cases} 1, & x \in B_r \\ 0, & x \notin B_{3r} \end{cases}$$

$$0 \leq \eta_2(y) = \begin{cases} 1, & y \in [0,r) \\ 0, & y \notin [0,\frac{3r}{2}) \end{cases}$$

$$0 \leq \eta_3(t) = \begin{cases} 1, & t \in (-r^\frac{2}{3},0) \\ 0, & t \notin (-\frac{2r}{3},\frac{2r}{3}) \end{cases}$$

And $\eta(x,y,t)$ satisfies

$$\sum_{i=1}^{2} r^i (|\partial^i_x \eta| + |\partial^i_y \eta|) + r^\frac{2}{3} |\partial_t \eta| \lesssim 1.$$

Taking $\eta$ as the test function in the generalized energy inequality (7), we have

$$\int_{B_r \times \{0\}} |u|^2 \leq I_1 + I_2 + I_3,$$

where

$$I_1 = \int_{Q_{\frac{3r}{2}}} (u \cdot \nabla \eta_0)(2p + |u|^2 - ||u||_{x,3r}^2)$$

$$\lesssim r^{-1} \int_{Q_{\frac{3r}{2}}} |u|(||u||^2 - ||u||_{x,3r}^2) + |pu|,$$

$$I_2 = C_1 \int_{Q_{\frac{3r}{2}}} |\tilde{u}|^2 \text{Div}(y^{-\frac{1}{2}} \tilde{\nabla} \eta)$$

$$\lesssim \int_{Q_{\frac{3r}{2}}} |\tilde{u}|^2 (|\partial_y (y^{-\frac{1}{2}} \partial_y \eta)| + |\partial_x (y^{-\frac{1}{2}} \partial_x \eta)|)$$

$$\lesssim r^{-2} \int_{Q_{\frac{3r}{2}}} |\tilde{u}|^2 y^{-\frac{1}{2}}$$

and

$$I_3 = \int_{Q_{\frac{3r}{2}}} |u|^2 \left\{ \partial_t \eta_0 + C_1 \lim_{y \to 0^+} (y^{-\frac{1}{2}} \partial_y \eta) \right\}$$

$$\lesssim r^{-\frac{2}{3}} \int_{Q_{\frac{3r}{2}}} |u|^2.$$

By Hölder inequality and Proposition 1, the terms $I_2$ and $I_3$ satisfy

$$I_2 + I_3 \lesssim r^{-\frac{2}{3}} \int_{Q_{\frac{3r}{2}}} |u|^2 + \int_{Q_{\frac{3r}{2}}} |\tilde{\nabla} \tilde{u}|^2 y^{-\frac{1}{2}}$$

By Hölder inequality and Proposition 1, the terms $I_2$ and $I_3$ satisfy
\[
\lesssim r^2 C_{r}^{\frac{3}{2}} + \rho^{2} E_{\rho}.
\]

By the estimate of the quantity \( C \) in Lemma 3.1, it holds that

\[
r^{-2} (I_2 + I_3) \lesssim \frac{r}{\rho} A_{\rho} + \left( \frac{\rho}{r} \right)^{3} A_{\rho}^{\frac{4}{3}} E_{\rho}^{\frac{4}{3}} + \left( \frac{\rho}{r} \right)^{2} E_{\rho} \]

\[
\lesssim \frac{r}{\rho} A_{\rho} + \left( \frac{\rho}{r} \right)^{5} E_{\rho}.
\]

The proof of (22) reduces to the estimate of the term \( I_1 \). Using Hölder inequality and Proposition 1, we have

\[
r^{-3} \int_{Q_{\frac{r}{2}}} |u| \|u\|^2 - \|u\|_{L^2(B_{r})}^2 \]

\[
\lesssim r^{-3} \int_{-\left(\frac{3}{2}r\right)^3}^{0} |u| |L^2(B_{3r})| |u| L^4(B_{3r}) |u - \|u\|_{L^2(B_{3r})}^2| \]

\[
\lesssim r^{-3} \int_{-\left(\frac{3}{2}r\right)^3}^{0} |u| |L^2(B_{3r})| \left( r^{-\frac{3}{2}} |u| |L^2(B_{3r})| + |\nabla \tilde{u}|_{L^2(B_{3r})} \right) \]

\[
\lesssim \left( \frac{\rho}{r} \right)^{5} (A_{\rho} E_{\rho}^{\frac{1}{4}} + A_{\rho}^{\frac{3}{2}} E_{\rho}^{\frac{1}{2}})
\]

and

\[
r^{-3} \int_{Q_{\frac{r}{2}}} |pu| \lesssim r^{-3} \|P\|_{L^2(Q_{\frac{r}{2}})} |u| \|L^3(Q_{\frac{r}{2}})\|
\]

\[
\lesssim C_{r}^{\frac{3}{2}} + D_{r}^{\frac{1}{2}}
\]

\[
\lesssim \frac{r}{\rho} A_{\rho} + \left( \frac{\rho}{r} \right)^{3} A_{\rho}^{\frac{1}{3}} E_{\rho}^{\frac{1}{3}} + \left( \frac{\rho}{r} \right)^{4} A_{\rho}^{\frac{2}{3}} E_{\rho}^{\frac{2}{3}} + \left( \frac{\rho}{r} \right)^{2} \tilde{F}_{\rho}^{\frac{4}{3}} (F_{\rho}^{\frac{2}{3}} + A_{\rho}^{\frac{3}{2}} E_{\rho}^{\frac{3}{2}})
\]

\[
\lesssim \frac{r}{\rho} A_{\rho} + \left( \frac{\rho}{r} \right)^{5} E_{\rho} + \left( \frac{\rho}{r} \right)^{4} A_{\rho}^{\frac{2}{3}} E_{\rho}^{\frac{2}{3}} + \left( \frac{\rho}{r} \right)^{2} \tilde{F}_{\rho}^{\frac{4}{3}} (F_{\rho}^{\frac{2}{3}} + A_{\rho}^{\frac{3}{2}} E_{\rho}^{\frac{3}{2}})
\]

Thus,

\[
r^{-2} I_1 \lesssim \frac{r}{\rho} A_{\rho} + \left( \frac{\rho}{r} \right)^{5} E_{\rho} + \left( \frac{\rho}{r} \right)^{4} (A_{\rho}^{\frac{3}{2}} E_{\rho}^{\frac{3}{2}} + A_{\rho} E_{\rho}^{\frac{3}{2}} + A_{\rho}^{\frac{3}{2}} E_{\rho})
\]

\[
+ \left( \frac{r}{\rho} \right)^{2} \tilde{F}_{\rho}^{\frac{4}{3}} (F_{\rho}^{\frac{2}{3}} + A_{\rho}^{\frac{3}{2}} E_{\rho}^{\frac{3}{2}}).
\]

Combining (23) with (24), we complete the proof of Lemma 3.3.

\[
\limsup_{r \to 0^+} E(z_0, r) < \epsilon_0,
\]

then there is a sufficiently small constant \( r_0 \) such that

\[
A_r(z_0) + E_r(z_0) + C_r(z_0) + D_r(z_0) + F_r(z_0) \leq \epsilon_1, \quad \forall \ r < r_0.
\]
Proof. Without loss of generality, we prove it for $z_0 = (0, 0)$. Let us assume initially that $\frac{1}{3} < q < \frac{5}{4}$ and set $K_r \triangleq A_r + F_3^{\frac{m}{q}}$. By Lemma 3.2, (22) and noting that $0 < \frac{4}{q} - 2 < 1$ for $\frac{1}{3} < q < \frac{5}{4}$, then utilizing Young inequality, we have for $6r \leq \rho$,

$$F_r \lesssim \left(\frac{r}{\rho}\right)^{2-\frac{3}{q}m} F_\rho^\frac{2}{q} + \left(\frac{r}{\rho}\right)^{2-\frac{3}{q}m} A_\rho^{\frac{3}{q}m-2} E_\rho^{\frac{2}{q}m-2}$$

$$\lesssim \left(\frac{r}{\rho}\right)^{2-\frac{3}{q}m} F_\rho^\frac{2}{q} + \left(\frac{r}{\rho}\right)^{2-\frac{3}{q}m} A_\rho + \left(\frac{r}{\rho}\right)^{\alpha_1(m,q)} E_\rho^{\beta_1(m,q)}$$

and

$$A_r \lesssim \left(\frac{r}{\rho}\right)^{\alpha_1(m,q)} E_\rho + \left(\frac{r}{\rho}\right)^{\alpha_1(m,q)} K_\rho + \left(\frac{r}{\rho}\right)^{\alpha_1(m,q)} E_\rho^{\beta_1(m,q)}$$

Here, $\alpha_1(m,q)$ and $\beta_1(m,q)$ are some positive constants.

To conclude this section, we introduce an important lemma which is crucial for the inductive arguments in next section. Combining (9) with an argument similar to the one used in [20, 21] (see Lemma 2.1), we can easily get this lemma. Thus, we omit the proof here.

Lemma 3.5. Let $r_k = \gamma^k \rho$, $0 < \gamma \leq \frac{1}{2}$. Then, we have for any $k \geq 1$

$$\tilde{A}_{r_k} + \tilde{E}_{r_k} \lesssim \tilde{A}_\rho + \tilde{E}_\rho + \gamma^{-4} \sum_{j=0}^{k-1} (r_j^\frac{2}{3} \tilde{C}_{r_j} + \gamma^\frac{4}{3} \tilde{C}_{r_j} \tilde{D}_{r_j})$$
4. The proof of main result. We will prove Theorem 1.2 by inductive arguments. Before the proof, as in [20, 21], we firstly set up the arguments for the inductive arguments. Note that the quantities $A$, $C$, $D$, $E$ and $F$ are scaling invariant under the natural scaling:

\[ u_\lambda(x, t) = \lambda^{\frac{2}{3}} u(\lambda x, \lambda^{\frac{2}{3}} t), \quad p_\lambda(x, t) = \lambda p(\lambda x, \lambda^{\frac{2}{3}} t). \]

The extension function $\tilde{u}_\lambda$ in [3] can be given by

\[ \tilde{u}_\lambda(x, y, t) = \lambda^{\frac{2}{3}} \tilde{u}(\lambda x, \lambda^{\frac{2}{3}} y, \lambda^{\frac{2}{3}} t). \]

By Lemma 3.4, Lebesgue differentiation Theorem and scaling technique, it is enough to prove Theorem 4.1 under the following condition:

**Condition.** 1) $(m, q)$ satisfies $1 < q < \frac{3}{2}$, $m > \frac{3}{2}$ and $\frac{1}{m} + \frac{2}{q} \geq 2$. 

2) There exists a sufficiently small positive constant $\epsilon$ such that

\[ A_1(z_0) + E_1(z_0) + C_1(z_0) + D_1(z_0) + F_1(z_0)^{\frac{3}{4}} \lesssim \epsilon. \]

Now, we are ready to prove the following theorem.

**Theorem 4.1.** Suppose that $(u, p)$ is a suitable weak solution to fractional Navier-Stokes equations (1) and satisfies the above condition with $m > \frac{9}{2}$. Then, there exists a sufficiently small positive constant $\gamma < \frac{1}{6}$ such that for any $z \in Q(z_0, \frac{1}{2})$ and $k \geq 1$, the following holds

\[ \mathcal{C}_{r_k}(z), \mathcal{D}_{r_k}(z) \leq \epsilon^\gamma, \quad \mathcal{A}_{r_k}(z), \mathcal{E}_{r_k}(z) \leq \epsilon^\frac{\gamma}{2} \quad \text{and} \quad r_k^{-1} F_{r_k}^{\frac{3}{2}}(z) \leq \epsilon^\frac{\gamma}{2}, \]

(28)

where $r_k = \gamma^k$.

**Proof.** Without loss of generality, we assume that $z = (0, 0)$. At first, it is easy to show that (28) holds for $k = 1$ provided $\epsilon$ is sufficiently small. Then, suppose that (28) holds for $1 \leq k < k_0$. By Lemma 3.2, as $\frac{1}{2} - \frac{9}{4m} > 0$, we have for $k = k_0$

\[
\begin{align*}
    r_k^{-1} F_{r_k}^{\frac{3}{2}} & \lesssim r_k^{-1} \gamma^\frac{3}{2} - \frac{9}{4m} F_{r_{k_0}}^{\frac{3}{2}} + r_k^{-1} \gamma^\frac{3}{2} - \frac{9}{4m} A_{r_{k_0}}^{\frac{3}{2}} - \frac{3}{2} E_{r_{k_0}} - \frac{3}{2} F_{r_{k_0}}^{-1} \\
    & \lesssim \gamma^\frac{3}{2} - \frac{9}{4m} F_{r_{k_0}}^{\frac{3}{2}} + \gamma^\frac{3}{2} - \frac{9}{4m} A_{r_{k_0}}^{\frac{3}{2}} - \frac{3}{2} E_{r_{k_0}} - \frac{3}{2} F_{r_{k_0}}^{-1} \\
    & \lesssim \gamma^\frac{3}{2} - \frac{9}{4m} \epsilon^{\frac{\gamma}{2}} + \gamma^\frac{3}{2} - \frac{9}{4m} \epsilon^{\frac{\gamma}{2}} \\
    & \lesssim \epsilon^{\frac{\gamma}{2}}
\end{align*}
\]

provided $\gamma$ and $\epsilon$ are small enough.

According to Lemma 3.5, the quantities $\tilde{A}_{r_{k_0}}$ and $\tilde{E}_{r_{k_0}}$ can be estimated as follows:

\[
\begin{align*}
    \tilde{A}_{r_{k_0}} + \tilde{E}_{r_{k_0}} & \lesssim \tilde{A}_1 + \tilde{E}_1 + \gamma^{-4} \sum_{j=0}^{k-1} (r_j^\frac{1}{4} \tilde{C}_{r_j} + r_j^\frac{1}{4} \tilde{C}_{r_j}^2 + D_{r_j}^2) \\
    & \lesssim \epsilon + \gamma^{-4} \sum_{j=0}^{k-1} (r_j^\frac{1}{4} + r_j^\frac{1}{4}) \epsilon^\frac{\gamma}{2}.
\end{align*}
\]

Note that $\sum_{j=0}^{k-1} (r_j^\frac{1}{4} + r_j^\frac{1}{4})$ can be bounded by a finite number which is independent of $k$. Thus, it holds

\[ \tilde{A}_{r_{k_0}} + \tilde{E}_{r_{k_0}} \leq \epsilon^\frac{\gamma}{2} \]

provided $\epsilon$ is sufficiently small.
For the quantity $\bar{C}_{r_{k_0}}$, by Lemma 3.1, we have
$$\bar{C}_{r_{k_0}} \lesssim A_{r_{k_0}}^{\frac{3}{2}} + \gamma^{-6} A_{r_{k_0}}^{\frac{7}{2}} E_{r_{k_0}}^{\frac{1}{2}}$$
$$\lesssim \epsilon^{\frac{1}{8}} + \gamma^{-6} \epsilon^{\frac{1}{8}}$$
$$\lesssim \epsilon^{\frac{1}{2}}.$$

Similarly, by Lemma 3.2, it holds
$$\bar{D}_{r_{k_0}} \lesssim \gamma^{-4} r_{k_0}^{-\frac{1}{2}} A_{r_{k_0}}^{\frac{3}{2}} E_{r_{k_0}} + \gamma^2 r_{k_0}^{-\frac{1}{2}} \bar{D}_{r_{k_0}}^{\frac{3}{2}} + \gamma^2 r_{k_0}^{-\frac{1}{2}} A_{r_{k_0}}^{\frac{3}{2}} E_{r_{k_0}}^{\frac{1}{2}}$$
$$\lesssim \epsilon^{\frac{1}{8}} + \gamma^2 \epsilon^{\frac{1}{8}}$$
$$\lesssim \epsilon^{\frac{1}{2}}.$$

Thus, the proof is completed.

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