The violation of the no-hair conjecture in four-dimensional ungauged supergravity

Pablo Bueno\(^1\) and C S Shahbazi\(^2\)

\(^1\) Instituto de Física Teórica UAM/CSIC C/ Nicolás Cabrera, 13–15, C.U. Cantoblanco, 28049 Madrid, Spain
\(^2\) Institut de Physique Théorique, CEA Saclay CNRS URA 2306, F-91191 Gif-sur-Yvette, France

E-mail: pab.bueno@estudiante.uam.es and carlos.shabazi@cea.fr

Received 23 April 2014, revised 2 June 2014
Accepted for publication 3 June 2014
Published 3 July 2014

Abstract

By choosing a particular, string-theory-inspired, special Kähler manifold, we are able to find an \(\mathcal{N} = 2\) four-dimensional ungauged supergravity model that contains supersymmetric black hole solutions that violate the folk uniqueness theorems that are expected to hold in ungauged supergravity. The black hole solutions are regular in the sense that they have a positive mass and a unique physical singularity hidden by an event horizon. In contrast to most examples already known in the literature, we find our solutions in a theory without scalar potential, gaugings, or higher-order curvature terms.

Keywords: black holes, supergravity, no-hair theorems

(Some figures may appear in colour only in the online journal)

Introductory remarks

The study of black holes in string theory is an extremely active field of research\(^3\). This is to be expected, since string theory is thought to be a consistent theory of quantum gravity, and black holes seem to be the perfect theoretical laboratories in which to study the already-old problem of gravitational field quantization.

One way to study black holes in string theory is by finding their classical description as solutions to the different supergravities that appear as the low-energy limit of string theory compactifications. Over the last decades, an enormous effort has been devoted to the study of

\(^3\) See [1–7] and references therein.
the huge space of such string-theory-embeddable black hole solutions—an effort which, after all, seems to be far from concluded.

From a purely classical point of view, black holes are already extremely interesting objects whose physical properties have continued to amaze us since the discovery of the Schwarzschild solution [12]. The existence of event horizons [13], the observation that certain black hole spacetimes contain completely causally disconnected regions [14, 15], and the discovery that black holes satisfy the laws of thermodynamics [16–18] are three good examples of such completely unexpected properties.

Another interesting feature of these gravitational wonders comes from their exclusiveness. Indeed, it has been known for a long time that all the stationary, asymptotically flat, black hole solutions to the Einstein-Maxwell theory, in a sort of general relativistic version of the Gauss law, are uniquely determined by a few parameters: their mass, their angular momentum, and their and their electric and magnetic charges [19–23].

The possible generalizations of these uniqueness (or no-hair) theorems to systems with more fields, such as scalars or non-Abelian vectors, has been an active area of research [25–27] since the proofs of the theorems for the simplest cases were carried out. On the other hand, the search for counterexamples to the corresponding conjectured uniqueness theorems in such scenarios has also attracted a lot of attention, and produced some interesting results. In particular, it is now known that the no-hair conjecture can be violated or circumvented in certain Einstein-Yang-Mills-Higgs systems (see [28–35] and references therein) and in higher-curvature theories of gravity [36, 37].

In this paper, we will construct a particular $N = 2, d = 4$ ungauged supergravity model that admits pairs of supersymmetric, static, spherically symmetric, and asymptotically flat black hole solutions sharing the same mass, charges, and asymptotic values of the scalar fields, providing, to the best of our knowledge, the first counterexample to the corresponding uniqueness conjecture in the context of an ungauged supergravity theory, and one of the first (some previous examples can be found in [32]) for a system without scalar potential, non-Abelian vector fields, or higher-order curvature corrections.

In [38] we obtained for the first time black hole solutions to a type-IIA string theory compactification on a Calabi-Yau manifold in the presence of nonperturbative corrections to the special Kähler geometry of the vector multiplet sector. These black holes were given in terms of harmonic functions on euclidean $\mathbb{R}^3$, as it must be for supersymmetric black hole solutions of ungauged four-dimensional supergravity [41, 42], but they also contained a special function called the Lambert function. As we argued in [38], the fact that the Lambert function is multivalued opened up the possibility of using its different branches to build inequivalent black hole solutions with the same conserved charges at infinity. However, this possibility was forbidden by the large volume compactification limit we assumed would hold through all the calculations. That limit only allowed us to consider solutions such that the argument of the Lambert function lied into a set of values for which the function was uniquely valued.

Inspired by this result, we will construct a particular supergravity model that can be analytically solved, and such that its supersymmetric black hole solutions share some of the

---

4 See [7–11] and references therein.
5 Thus, the only possible solution for a stationary, axisymmetric, and electrovacuum black hole is given by the well-known Kerr-Newmann spacetime [24].
6 Which, in the case under consideration, will be a particular instance of general relativity coupled to an arbitrary number of Abelian vector fields and scalars.
7 The effective theory of Type-IIA string theory on a Calabi-Yau manifold is four-dimensional $N = 2$ ungauged supergravity [39, 40].
8 see appendix A.
characteristics of those in [38], but without any approximation involved. In particular, we will be able to construct solutions whose metric and scalars will depend on the Lambert function. In this case, both branches will be available, and we will show how to construct a family of pairs of inequivalent solutions, providing a violation of the conjecture.

In order to illustrate the result, we will show an explicit example for a model with two scalar fields. We will find that both solutions are regular, in the sense that the only physical singularity of the space-time will be hidden by an event horizon of a nonzero positive area for each solution in the pair. However, we will also see that the special Kähler metric will not be positive definite (as happens in other counterexamples to the conjecture [36]) when evaluated on our solutions or, equivalently, that the energy-momentum tensor of at least one of the scalars will not satisfy in general the null energy condition (NEC). In this respect, and although the no-hair conjecture does not, in principle, make reference to stability issues, it is fair to say that the spirit of the conjecture seems to remain partially alive.

The structure of the paper is as follows: In section 1, we explain the structure of the bosonic sector of \( N = 2, d = 4 \) supergravity. In section 2, we briefly review the so called H-FGK formalism [43–47], which is essential for the construction of the black hole solutions. In section 3, we motivate the supergravity model under consideration, and we illustrate how we found it. In section 4, we explicitly construct the supersymmetric solution without making any approximation. In section 5, we explain how the family of pairs of solutions that we have constructed in section 4 can be used to violate the no-hair theorem, and finally, in section 6, we present an explicit pair of supersymmetric solutions, with the same mass, charges, and asymptotic value for the scalars at infinity.

1. \( N = 2, d = 4 \) supergravity

\( N = 2, d = 4 \) supergravity stands for any four-dimensional field theory invariant under the action of two independent local supersymmetry generators [40]. Due to the \( \mathbb{Z}_2 \) symmetry \( \phi_b \rightarrow \phi_b^\ast \) (bosonic fields), \( \phi_f \rightarrow - \phi_f^\ast \) (fermionic fields) present in any supergravity action, setting the fermions to zero is always a consistent truncation of the theory, which we will assume henceforth. We will also restrict ourselves to theories containing terms only up to two derivatives. In that case, the bosonic sector of any \( N = 2, d = 4 \) supergravity can be written as follows [39, 40]:

\[
S = \int d^4x \sqrt{|g|} \left\{ R + h_{\alpha\beta}(q) \partial^\alpha q^\ast \partial^\beta q^\ast + \mathcal{G}_\alpha(z, \bar{z}) \partial^\alpha \bar{z}^i \partial^i \bar{z}^j \right. \\
+ 2 \mathfrak{g}_{\alpha\beta}(z, \bar{z}) F^\alpha \wedge F^\beta - 2 \mathfrak{g}_{\alpha\beta}(z, \bar{z}) F^\alpha \wedge F^\beta \},
\]

where \( R \) denotes the scalar curvature of the Levi-Civita connection associated to the space-time metric \( g \); \( q^\ast \) \((u = 1, \ldots, 4n_h)\) denotes the hyperscalars, which parametrize a \( 4n_h \)-dimensional quaternionic manifold \( \mathcal{HM} \) with Riemannian metric \( h_{\alpha\beta}(q) \); \( z^i \) \((i = 1, \ldots, n_v)\) denotes the \( n_v \) complex scalar fields of the vector multiplets of the theory, which parametrize the \( n_v \)-dimensional base of a special Kähler bundle, \( \mathcal{SM} \otimes \mathcal{L} \), with structural group \( \text{Sp}(2n_v + 2, \mathbb{R}) \times U(1) \) and Riemannian metric \( \mathcal{G}_\alpha(z, \bar{z}) \); \( F^\alpha \) \((\alpha = 0, \ldots, n_v)\) denotes the field strengths of the \( \Lambda = 0, \ldots, n_v \) 1-form connections \( \Lambda^\alpha \) of the vector multiplets plus the graviphoton, \( \Lambda^\alpha \); and \( \mathfrak{g}_{\alpha\beta} \equiv \Im \mathcal{N}_{\alpha\beta}(z, \bar{z}) \), \( \Re \mathcal{N}_{\alpha\beta}(z, \bar{z}) \) stand for the imaginary

---

\(^9\) FGK stands for Ferrara, Gibbons, and Kallosh [48], (see section 2) whereas the H stands for Harmonic, Hyperbolic, etc., and refers to the fact that all supersymmetric and most extremal nonsupersymmetric solutions of \( N = 2, d = 4 \) ungauged supergravity can be written in terms of harmonic functions in this formalism, whereas most nonextremal solutions are in turn given by linear combinations of hyperbolic sines and cosines (see, e.g. [47]).
(negative definite) and real parts of the symplectic period matrix $\mathcal{N}$, which determines the couplings of the 1-form connections, $\Lambda^i$, to the scalars of the vector multiplets.

The hyperscalars $q^i$ are only coupled to themselves and of course to gravity, and, as a consequence, they can always be consistently fixed to constant values $q^i = q^i_0$. This simply means that the equations of motion for the hyperscalars $q^i$ always admit the constant solution, $q^i = q^i_0$. Of course, one may try to turn them on, but it has been argued that no regular\textsuperscript{10} black hole solutions with nontrivial hyperscalars can exist, since they would develop scalar hair.

Supersymmetry constrains the couplings and kinetic terms of all the fields of the theory in a very particular way, which is beautifully codified in the language of special geometry\textsuperscript{11} for the vector multiplet sector. Indeed, the bosonic action in the absence of hyperscalars is determined as soon as we choose a holomorphic section, $\Omega \in \mathcal{T}(S^Y)$, or, equivalently when it exists, a homogeneous function, $\mathcal{F}(\lambda)$, of degree 2, called prepotential, from which $\bar{\mathcal{F}}$ and $\mathcal{N}_{\Lambda \Sigma}$ can be obtained as

\[
\mathcal{G}_\theta = - \partial_j \partial_j \ln \left[ i \left( \bar{X}^i \partial_i \mathcal{F} - X^i \partial_i \bar{\mathcal{F}} \right) \right],
\]

\[
\mathcal{N}_{\Lambda \Sigma} = \partial_i \partial_\Sigma \bar{\mathcal{F}} + 2i \text{Im} \left( \partial_i \partial_\Sigma \mathcal{F} \right) X^i \text{Im} \left( \partial_i \partial_\Sigma \bar{\mathcal{F}} \right) X^\Sigma - \frac{\text{Im} \left( \partial_i \partial_\Sigma \mathcal{F} \right) X^i \text{Im} \left( \partial_i \partial_\Sigma \bar{\mathcal{F}} \right) X^\Sigma}{X^\Sigma},
\]

where $X^i$ are homogeneous coordinates on the scalar manifold, related to the $z^i$ by

\[
z^i = \frac{X^i}{X^n},
\]

and where we have used the notation $\partial_i \equiv \frac{\partial}{\partial z^i}$.

Therefore, it should be clear that if we choose a second-degree homogeneous function $\mathcal{F}$, we automatically fix an $\mathcal{N} = 2$, $d = 4$ supergravity theory coupled to vector multiplets. On the other hand, it is also reasonable to expect that not every election of prepotential will correspond to a supergravity theory susceptible of being embedded in string theory.

### 2. H-FGK formalism

The most general static and spherically symmetric solution to (1.1) takes the form [7, 44, 48]

\[
g = e^{\gamma \theta / (\tau)} dt \otimes dt - e^{-\gamma \theta / (\tau)} r^2 \, dx^2 \otimes dx^2,
\]

\[
\gamma \frac{d}{ds} dx^2 \otimes dx^2 = \frac{r_0^2}{\sinh \frac{r_0}{r_0} \tau} \left[ \frac{r_0^2}{\sinh \frac{r_0}{r_0} \tau} \, dt \otimes d\tau + h_{s^2} \right],
\]

\[
h_{s^2} = d\theta \otimes d\theta + \sin^2 d\phi \otimes d\phi,
\]

where $\tau$ is the radial coordinate and $r_0$ is the nonextremality parameter, which parametrizes how non-extremal the black hole is, when (2.1) does in fact correspond to a black hole spacetime. In such a case, the exterior of the event horizon is covered by $\tau \in (-\infty, 0)$, with the event horizon corresponding to $\tau \to -\infty$ and spacial infinity at $\tau \to 0^-$. The inner part of the Cauchy horizon is covered by $\tau \in (\tau_i, \infty)$, with the inner horizon at $\tau \to \infty$ and the singularity at $\tau = \tau_i$ for a certain positive and finite $\tau_i$ [43].

\textsuperscript{10} That is, with the black hole singularity hidden by an event horizon.

\textsuperscript{11} For an introduction to special geometry see [7, 40].
We assume a static and spherically symmetric spacetime, as well as exclusively radial dependence for all the fields of the theory. In this case, the Maxwell equations can be integrated explicitly, and the vector fields can be written as functions of $\tau$ and the symplectic vector $Q^M$ of electric $q_i$ and magnetic $p^i$ charges, $Q^M \equiv \left( p^i, q_i \right)$. Indeed, let $\Psi \equiv \left( \psi^i, \chi_i \right)$ be a symplectic vector whose components are the time components (which are the only nonvanishing ones) of the electric $A^i$ and magnetic $A_\lambda$ vector fields. Then, it can be shown that

$$\Psi^M = \int \frac{1}{2} e^{2\ell} M^{MN} Q_N d\tau \,,$$

where $M^{MN}$ is a symplectic and symmetric matrix constructed from the couplings of the scalars and the vector fields as

$$(M^{MN}(N)) \equiv \begin{pmatrix}
(3 + N i - i N)^{-1} & - (N i)^{1}\alpha
- (3 - i N)^{1}\alpha & (N - 1)^{1}\alpha
\end{pmatrix} .$$

The bosonic sector of the four-dimensional $\mathcal{N} = 2$ supergravity action coupled to vector multiplets can be shown to be equivalent, assuming the space-time background given by (2.1), to the one-dimensional effective FGK action [48] for the $(2n_\nu + 1)$ real fields $z'(\tau)$ and $U(\tau)$

$$I_{FGK} \left[ U, z' \right] = \int d\tau \left\{ \left( \dot{U} \right)^2 + G_{z^z} z^{z} + e^{2\ell} V_{bh} \left( z, \bar{z}, Q \right) \right\} ,$$

together with the Hamiltonian constraint, which encodes the explicit independence of the effective FGK Lagrangian with respect to $\tau$,

$$\left( \dot{U} \right)^2 + G_{z^z} z^{z} + e^{2\ell} V_{bh} \left( z, \bar{z}, Q \right) = r_0^2 .$$

In the previous expressions, $V_{bh}$ is the so-called black hole potential, which is defined by [48]

$$V_{bh} \left( z, \bar{z}, Q \right) \equiv \frac{1}{2} M^{MN}(N) Q^M Q^N .$$

As we have said, the nondependence of the effective FGK Lagrangian on $\tau$ makes the corresponding Hamiltonian constant. In fact, the dimensional reduction over (2.1) imposes such a constant to be precisely the square of the nonextremality parameter, $r_0^2$.

The H-FGK formalism [43–47] consists of a change of variables from $\left( U, z' \right)$ to a new set of $(2n_\nu + 2)$ variables, $H^M(\tau)$, which transform under a symplectic, linear representation of the U-duality group of the theory and become harmonic functions in $\mathcal{R}^3$ in the supersymmetric case. The equations of motion in the new variables, $H^M(\tau)$, read

$$g_{MN} \dot{H}^N + \left( \partial_S s_{PM} - \frac{1}{2} \partial_M g_{SN} \right) H^N \dot{H}^P + \partial_M V = 0 ,$$

together with the Hamiltonian constraint

$$\frac{1}{2} g_{MN} H^M H^N + V + r_0^2 = 0 ,$$

where the noninvertible metric, $g_{MN}(H)$, and the potential, $V(H)$, of the H-FGK effective action are given in terms of the so-called Hesse potential, $W(H)$, by

$\text{Symplectic indices } M, N, \ldots \text{ are raised and lowered with the symplectic metric } \Omega_{MN} \equiv \text{antidiag}(1, -1) ; \text{ so, for example, } H_0 = \partial_0 H^0 , H^0 = H_0 \Omega^M M \text{ and } \Omega^M \partial_0 H^M .}$
\[ g_{\mu\nu}(H) \equiv \partial_\mu \partial_\nu \log W - 2 \frac{H_\mu H_\nu}{W^2}, \] (2.9)

\[ V(H) \equiv \left\{ - \frac{1}{4} \partial_\mu \partial_\nu \log W + \frac{H_\mu H_\nu}{W^2} \right\} Q^\mu Q^\nu. \] (2.10)

The covariantly holomorphic symplectic section, \( \mathcal{V}^M \) of the special Kähler bundle, \( \mathcal{S}^V \), characterizing the vector multiplets sector of any \( \mathcal{N} = 2, d = 4 \) supergravity theory is related to the Hesse potential through

\[ \mathcal{W}(H) \equiv \tilde{H}_d(H) H^M = e^{-2\mathcal{U}}, \tilde{H}^M + iH^M = \mathcal{V}^M/X, \] (2.11)

where \( X \) is a complex variable with the same Kähler weight as \( \mathcal{V}^M \), making the quotient \( \mathcal{V}^M/X \) Kähler invariant. \( \tilde{H}^M(H) \) stands for the real part \( \tilde{H}^M(H) \) of \( \mathcal{V}^M/X \) written as a function of the imaginary part \( H^M \), something that can always be done by solving the so-called stabilization equations.

The effective theory is now expressed in terms of \( 2(n_i + 1) \) variables \( H^M \) and depends on \( 2(n_i + 1) + 1 \) parameters: \( 2(n_i + 1) \) charges \( Q^\mu = (p^\mu, q^\mu) \) and the non extremality parameter \( \tau_0 \), from which it is possible to reconstruct the solution in terms of the original fields of the theory (that is, the space-time metric, the scalars, and the 1-form connections).

### 3. A stringy motivation for the model

The purpose of this paper is to study the supersymmetric black hole solutions of a particular \( \mathcal{N} = 2 \) four-dimensional ungauged supergravity coupled to vector multiplets, which will we find to violate the folk uniqueness theorems that are supposed to hold in ungauged four-dimensional supergravity. Of course, such a model did not appear out of the blue, but it has its seed and motivation in our previous paper [38]. In [38], we obtained a new class of supersymmetric black hole solutions of type-IIA string theory compactified to four dimensions on a Calabi-Yau manifold in the presence of nonperturbative stringy corrections. The supersymmetric solution was given by

\[ e^{-2\mathcal{U}} = \tilde{H}^i H^i, \quad z^i = i \frac{H^i}{H^0}, \] (3.1)

where

\[ \tilde{H}^0 = \frac{\pi d H^i}{W \left( \sqrt{\frac{3n (d_H H^i)^3}{2\kappa_H H H^i H^j}} \right)}, \] (3.2)

\[ \tilde{H}^i = \frac{1}{2} \kappa_{ij} H^i H^j W \left( \sqrt{\frac{3n (d_H H^i)^3}{2\kappa_{ij} H^i H^j H^k}} \right), \] (3.3)

and

\[ H^i = a^i - \frac{p^i}{\sqrt{2} \tau}, \] (3.4)

Notice that the H-FGK formalism introduces an extra degree of freedom. As a consequence, the H-FGK action enjoys a gauge symmetry which, by gauge fixing, allows us to get rid of it [46].
where $W(x)(a = 0, -1)$ was any of the two real branches of Lambert’s $W$ function, $s_a = \pm 1^{14}$; $\kappa_{ijk}$ were the classical intersection numbers of the Calabi-Yau manifold; and $n$ and $d_i$ are certain constants associated to the most relevant holomorphic mapping of the genus 0 world-sheet instanton onto the $h^{1,1}$ two-cycles of the Calabi-Yau (see [38] for details) in the large volume limit.

In order to solve the involved stabilization equations and obtain (3.2), (3.3), and (3.4), we were forced to consider the large volume limit, $\Im z' \to \infty$, of the compactification, where certain simplifications could be made. As a consequence, the approximation, $\Im z' \to \infty$, also had to be imposed on the solution. As explained in [38], only one of the two real branches of the $W$ function (the one with $a = 0$) was consistent with such a condition, which also implied the argument of $W_0(x(\tau))$ to be positive. We argued how, had this condition not been present, we could have tried to build two different solutions solving the same equations of motion, by choosing $W_0(x(\tau))$ or $W_{-1}(x(\tau))$ through (3.2), (3.3), and (3.4).

In fact, we could have assigned, through a suitable election of the parameters available in the solution, the near horizon ($\tau \to -\infty$) and asymptotic ($\tau \to 0$) limits of the argument $\tau_W(x(\tau))$ to any pair of values chosen at will. In particular, we could have selected $\tau_W(x(\tau)) = -\frac{1}{e}$ and $\lim_{\tau \to -\infty} \tau_W(x(\tau)) = \beta$, $\beta \in (-\frac{1}{e}, 0)$, and then the solution built with $W_0(x(\tau))$ and the one constructed with $W_{-1}(x(\tau))$ would have had exactly the same asymptotic behaviour, but different profiles away from infinity. Note also (appendix A) that $W_0$ and $W_{-1}$ are not even symmetric, in contradistinction to the branches of other real multivalued functions such as the inverse trigonometric functions. That is, we would have been dealing with two completely different regular solutions with the same mass, $M$, charges, and asymptotic values of the scalar fields, in contradiction to the aforementioned conjecture.

Let us state that when we write regular, we mean a black hole solution with positive mass, $M$, such that there is a unique physical singularity in the space-time that is hidden by an event horizon with a nonzero, positive area.

In order to accomplish the construction of our solutions, we are going to somewhat forget about string theory and propose a prepotential which we can solve exactly, such that the corresponding supersymmetric solutions enjoy the same desirable properties as the string-theory-forbidden ones of [38]. In particular, we will use the same truncation in the $H$-variables, to wit,

$$H^0 = H_0 = H_i = 0, \quad p^0 = q_0 = q_i = 0.$$  

(3.5)

In addition, we want the Lambert function to appear when solving the corresponding 0-electric component of the stabilization equations. We have found that the following prepotential fulfils the required conditions:

$$F(X) = n \left[ d_a X^{d_i} \right] \left[ e^{X_a} \frac{d_i}{X^{d_i}} - 2i \left[ e^{X_a} Ei \left[ 2id_i \frac{X_i}{X_0} \right] - d_i \frac{X_i}{X_0} X_i X_0^{-l} \right] \right],$$  

(3.6)

where $Ei(z)$ is the exponential integral function$^{15}$, and $d_{i\beta} = d_{(i\beta)}, n$, and $d_i^{16}$ are now arbitrary constants not constrained by any string theory requirement, since we are considering a purely supergravity model.

In the next section, we will obtain the supersymmetric black hole solutions corresponding to the four-dimensional $N = 2$ supergravity theory defined by (3.6), assuming the truncation (3.5).

$^{14}$ see appendix A.

$^{15}$ see appendix B.

$^{16}$ The indices $i, j, k, l ...$ run from 1 to a fixed arbitrary positive integer, $n$. 

7
4. The supersymmetric solution

In the H-FGK formalism, it is trivial to see that any $N = 2, d = 4$ supergravity model admits a solution for the $H^M$ variables given by

$$H^M = A^M - \frac{Q^M}{\sqrt{2} \tau},$$  \hspace{1cm} (4.1)

which turns out to correspond to a supersymmetric black hole [41, 49–51]. Using the truncation (3.5), we have

$$H' = d' - \frac{\rho^i}{\sqrt{2}} \tau, \hspace{0.5cm} H^M = 0, \hspace{0.5cm} M \neq i.$$  \hspace{1cm} (4.2)

For the prepotential under consideration (3.6) and the truncation (3.5), it is easy to see [38] that the corresponding stabilization equation for $\hat{H}^0$ is

$$\frac{\partial F (X)}{\partial X^0} = d_{ab} X^a X^b + n \left[ d_a X^a \right] e^{2 d_a X^a / X^0} = 0,$$  \hspace{1cm} (4.3)

which is solved by

$$\hat{H}^0 = \frac{d_a H^a}{W_a \left( s_{a b} \frac{e^{2 d_b H^b / H^0}}{d_a H^a H^b} \right)}.$$  \hspace{1cm} (4.4)

This is precisely the same result that we found for the $\hat{H}^0$ of the solution in the string theory case [38], and it incorporates the Lambert function, as we wanted. The remaining stabilization equation to be solved is

$$\hat{H}_i = \frac{\partial F (X) e^{X^i / X^0}}{\partial X^i},$$  \hspace{1cm} (4.5)

and its solution reads

$$\hat{H}_i = \frac{3 d_{ab} H^a H^b}{H^0} + n d_i \left[ e^{2 d_i H^i / H^0} \hat{H}^0 + \left[ 4 d_{mb} H^m \right] Ei \left[ - \frac{2 d_i H^i}{H^0} \right] \right].$$  \hspace{1cm} (4.6)

$\hat{H}_i$ becomes an explicit function of the $H^i$ once we substitute (4.4) into (4.6). In any case, this result is different from the corresponding result in the string theory solution, which is to be expected since the model, although sharing some general characteristics, is different. The metric warp factor is hence given by

$$e^{-2\mu} = n d_a H^a \left[ e^{\frac{2 d_{ab} H^a H^b}{H^0}} \hat{H}^0 + \left[ 4 d_{mb} H^m \right] Ei \left[ - \frac{2 d_i H^i}{H^0} \right] \right] + \frac{3 d_{ab} H^a H^b H^i}{H^0},$$  \hspace{1cm} (4.7)

whereas the scalars read

$$\varepsilon^i = \frac{X^i}{X^0} = i \frac{H^i}{d_a H^a} W_a \left( s_{a b} \frac{n \left( d_x H^x \right)^3}{d_a H^a H^b H^c} \right).$$  \hspace{1cm} (4.8)

This completes the general construction of the supersymmetric solution. Of course, in order to have a regular solution, now we have to require several conditions, which will now be studied.
4.1. Regularity conditions

In order to have a regular solution, the following requirements must be satisfied:

1. The warp factor must be nonzero, namely
   \[ e^{2U} > 0 , \quad \forall \tau \in (-\infty, 0) . \quad (4.9) \]

2. The mass, \( M \), of the solution must be positive and finite:
   \[ M \equiv \dot{U}(\tau \to 0) > 0 . \quad (4.10) \]
   This requires a bit more explanation. Indeed, it turns out that the definition of the black hole mass involves derivatives of the Lambert function evaluated at \( x(0) \), which will appear multiplicatively in the different factors of \( \dot{U}(\tau) \). As we have sketched already and will explain in the next section, in order to jeopardize the no-hair conjecture, we want to fix the parameters of our solution so that the argument of the Lambert function evaluated at spatial infinity (\( \tau = 0 \)) takes the value \(-e^{-1/e}\), where the two branches of \( W \) make contact. However, it turns out that \( W'(x) \) diverges as \( x \to -1/e \), as explained in appendix A. Fortunately, it is not difficult to cure this behaviour and get a positive and finite mass by choosing the parameters of the solution so that \( \dot{x}(\tau) \to 0 \) faster than \( W'(0) \to \pm \infty \). For instance, we can impose that the coefficient of order, \( \tau_0^0 \), in the numerator of \( \dot{x}(\tau) \) vanishes. As we will see in the explicit examples in section 6, this suffices to obtain a finite and positive mass for our pairs of inequivalent black holes.

3. The Kähler potential must be consistently defined. That is,
   \[ e^{-K} = i \sum_{ijkl} \Omega_{ij} \Omega_{kl} \quad (4.11) \]
   must be positive. For the prepotential (3.6), the Kähler potential is given by
   \[ e^{-K} = i \sum_{ijkl} (z - z') (z - z')^i (z + z')^j \left( e^{2idz^i} - e^{-2idz'}^j \right) + 4n \left[ d_z z^i \left( Ei \left[ 2idz^j \right] + Ei \left[ -2idz^j \right] \right) \right]. \quad (4.12) \]
   Since the supersymmetric solution that we have constructed has purely imaginary scalars, we can use \( \dot{z} = -z' \) to simplify this expression:
   \[ \frac{e^{-K}}{8} = i \sum_{ijkl} \dot{z} z' \dot{z}' z' + n \left[ d_z z^i \left( Ei \left[ 2idz^j \right] \right) \right]. \quad (4.13) \]

To summarize, if we obtain a solution such that the metric factor, the Kähler potential, and the mass are definite positive, we will have a regular black hole solution with a physical singularity hidden by an event horizon, and no other space-time singularities.

5. The violation of the no-hair conjecture

The resolution of the stabilization equations given in section gives us the opportunity to build the supersymmetric solution either using \( W_0 \) (the solution which we will denote by \( S_0 \)) or \( W_{-1} \) (the solution which we will denote by \( S_{-1} \)). Therefore, in order to prepare the setup for the violation of the uniqueness conjecture, we need to construct a solution such that the argument of \( W \), which we denote by \( x(\tau) \), lies entirely in the interval \((-1/e, 0)\), only
touching the value $-1/e$ when $\tau = 0$—that is, at spatial infinity. Notice that if we want the argument $x(\tau)$ to be negative, we have to choose $s_0 = s_{-1} = -1$, which we will assume henceforth. This way, we will be able to construct two different black hole solutions that solve the same equations of motion and have the same mass, charges, and moduli at infinity; however, they are different, since the profiles of $W_0$ and $W_{-1}$ are different and asymmetric when evaluated in $(-1/e, 0)$. Hence, we need to impose

$$x(0) = -\frac{n (d_i a^i)}{d_{ji} a^j a^k} = -\frac{1}{e},$$

(5.1)

and

$$x(\tau) \in \left(\frac{-1}{e}, 0\right), \quad \forall \, \tau \in (-\infty, 0).$$

(5.2)

Of course, as explained in the previous section, in order to have a regular solution, we need to impose $M > 0$ and $e^{-2U}, e^{-X} > 0$ for $\tau \in (-\infty, 0)$. Assuming that (4.10), the discussion under (4.10), and (5.1) hold, the value of the scalars at infinity as well as the mass for both solutions $S_0$ and $S_{-1}$ will be given by

$$M = U (\tau \to 0), \quad \exists \varepsilon_\infty = -\frac{a^j}{d_i a^i}.$$  

(5.3)

In order to show that it is both possible and easy to choose the parameters available in the model so that we can obey all the conditions (regularity plus (5.1) and (5.2)), in the next section we will explicitly construct a pair of solutions that satisfy the required properties for a particular model with two scalar fields.

Another issue, related to the stability of the solution, is the positive definiteness of the scalar metric $G_{ij}$ evaluated on the solution. Such a condition, which is related to the fulfillment of the NEC associated to the energy-momentum tensor of all the scalar fields in our solution, turns out to be difficult to satisfy. In particular, for the simple models in which we have worked out the explicit construction of pairs of solutions with the same masses, charges, and asymptotic values of the moduli (like the one in section 6), the scalar metric turns out to have both positive and negative eigenvalues for both solutions in each pair, meaning that some of the scalars in our solutions fail to satisfy the NEC, just like in other counterexamples to the no-hair conjecture [36]. At this point, it is not clear to us whether this is a feature shared by all the possible solutions eluding the no-hair conjecture susceptible of being constructed in our model (for any number of scalar fields), or not. This is an open question which could be addressed by different approaches [52]. On one hand, one could always try to map (brute force-wise) the parameter space for models with different numbers of scalars, looking for a solution that satisfies all the requirements but with a positive definite scalar metric. It would also be possible to consider other prepotentials giving rise to stabilization equations whose solutions involve multivalued functions, and study the situation therein. On the other hand, it might just be that our procedure of placing the spatial infinity at the branch point of the Lambert function necessarily implies some unstable behaviour for the corresponding solutions, which is not incompatible with their regularity. This could be related to the structure of the attractor flows associated with each pair of solutions. Let us see how this works.
5.1. Attractors

Although both solutions $S_0$ and $-S_{-1}$ have exactly the same asymptotic limit, $\tau \to 0$, since the flow is different, one should expect that the corresponding attractors, $z_0$ and $-z_{-1}$, are different. This is indeed the case; they are given by

$$z_i^j = \frac{p^j}{d_j p^j} W_a \left[ -\sqrt[3]{n (d_j p^j)^3} \right],$$  \hfill (5.4)

This can be understood in the context of the basins of attractions [53]. Let us suppose that we impose $x(0) = -\frac{1}{e}$, instead of $x(0) = -\frac{1}{e}$. Then $S_0$ and $-S_{-1}$ have different asymptotic limits at spatial infinity. In particular, the asymptotic value of the scalars at infinity is different for $S_0$ and $S_{-1}$. Therefore, we have two basins of attraction, $B_0$ and $B_1$, such that the solution $S_0$ corresponds to $B_0$ and $S_{-1}$ corresponds to $B_{-1}$. When we impose

$$x(0) = -\frac{1}{e},$$  \hfill (5.6)

we precisely choose a point which lies in $B_0$ and $B_{-1}$; that is, we choose a point in the common border of the two basins of attraction. As a result, we end up with two different solutions, with different attractors, which however have the same mass, charges, and asymptotic values of the scalars at infinity.

This standpoint suggests that there could be, in fact, some instability associated with our election of the Lambert’s function argument at the branch point. If this were the case, it would simply mean that, just as it appears in other counterexamples available in the literature (usually in theories with scalar potential, gaugings or higher-order curvature corrections), the no-hair conjecture remains robust when stability issues are considered.

6. An explicit example

Let us consider a model with two scalar fields, $z^1$ and $z^2$. The warp factor of the spacetime metric and the scalars can be read off directly from (4.7) and (4.8) with $H^1 = a^1 - \frac{p^1}{\tau^2}, H^2 = a^2 - \frac{p^2}{\tau^2}$. Imposing the regularity conditions, the correct asymptotic behaviour of the metric ($e^{1/6 - 1/6}$), and choosing the parameters in the argument of the two branches of the Lambert function in the way explained in the previous section (and such that (5.1) and (5.2) hold), it is not difficult to construct solutions with the required properties, and which, in all the examples constructed automatically, satisfy the condition $e^{-\tau} > 0 \forall \tau \in (-\infty, 0)$. Let us choose a particular model with $d_1 = d_2 = 1$, $d_{12} = 0$, $d_{22} \approx -0.270$, $d_{21} \approx 0.320$, $d_{41} \approx -2.040$, $n \approx -0.011$, and with the following constants for our solutions: $\Im z_0^1 = -1/3$, $\Im z_{-1}^1 = -2/3$, and $p^1 = p^2 = 1$. The explicit dependence on $\tau$ of the warp factor and the imaginary parts of our scalars for the examples at hand is, in general, very messy, so instead of reproducing it here, let us have a look at the corresponding plots for this particular example, for which the mass turns out to be $M = 2/3$ (figures 1, 2).
As we can see, both solutions are completely regular, and share the same mass, charges, and asymptotic values of the scalars.

Acknowledgments

We wish to thank Tomás Ortín for long and illuminating discussions, comments, and for suggesting the interpretation of the solutions in the context of the basins of attraction. We are also thankful to Nick E. Mavromatos for useful comments related to the stability issue. PB wishes to thank the CERN Theory Division for its hospitality. This work has been supported in part by the Spanish Ministry of Science and Education Grant FPA2012-35043-C02-01, the Comunidad de Madrid Grant HEPHACOS S2009ESP-1473, and the Spanish Consolider-Ingenio 2010 program CPAN CSD2007-00042. The work of PB has been supported by the JAE-predoc Grant JAEPre 2011 00452. The work of CSS has been supported by the JAEPre 2010 00613 grant and the ERC Starting Grant 259133. The authors acknowledge the support of the Spanish MINECO’s Centro de Excelencia Severo Ochoa Programme under Grant SEV-2012-0249.

As we can see, both solutions are completely regular, and share the same mass, charges, and asymptotic values of the scalars.

Acknowledgments

We wish to thank Tomás Ortín for long and illuminating discussions, comments, and for suggesting the interpretation of the solutions in the context of the basins of attraction. We are also thankful to Nick E. Mavromatos for useful comments related to the stability issue. PB wishes to thank the CERN Theory Division for its hospitality. This work has been supported in part by the Spanish Ministry of Science and Education Grant FPA2012-35043-C02-01, the Comunidad de Madrid Grant HEPHACOS S2009ESP-1473, and the Spanish Consolider-Ingenio 2010 program CPAN CSD2007-00042. The work of PB has been supported by the JAE-predoc Grant JAEPre 2011 00452. The work of CSS has been supported by the JAEPre 2010 00613 grant and the ERC Starting Grant 259133. The authors acknowledge the support of the Spanish MINECO’s Centro de Excelencia Severo Ochoa Programme under Grant SEV-2012-0249.
Appendix A. The Lambert W function

The Lambert W function, \( W(z) \), was first introduced by Johann Heinrich Lambert in 1758 [54]. Throughout its history, it has found numerous applications in different areas of physics, mostly during the 20th century [55–65].

\( W(z) \) is defined (implicitly) through the equation

\[
W(z) = \frac{W(z)}{z} e^{W(z)}, \quad \forall z \in \mathbb{C} . \tag{A.1}
\]

Since \( f(z) = ze^z \) is not injective, \( W(z) \) is not uniquely defined, and \( W(z) \) stands for the whole set of branches solving (A.1). For \( W : \mathbb{R} \to \mathbb{R} \), \( W(x) \) has two branches, \( W_0(x) \) and \( W_{-1}(x) \) defined respectively in the intervals \( x \in (-1/e, \infty) \) and \( x \in (-1/e, 0) \), as seen in figure A1. Both functions coincide in the branching point, \( x = -1/e \), where \( W_0(-1/e) = W_{-1}(-1/e) = -1 \). Therefore, the defining equation, \( x = W(x)e^{W(x)} \), admits two different solutions in the interval \( x \in [-1/e, 0) \).

The derivative of \( W(z) \) reads

\[
\frac{dW(z)}{dz} = \frac{W(z)}{z(1 + W(z))}, \quad \forall z \notin \{0, -1/e\}; \quad \left. \frac{dW(z)}{dz} \right|_{z=0} = 1 , \tag{A.2}
\]

and is not defined for \( z = -1/e \), because the function is not differentiable there. At that point, one finds

\[
\lim_{z \to -1/e} \frac{dW_0(x)}{dx} = \infty, \quad \lim_{z \to -1/e} \frac{dW_{-1}(x)}{dx} = -\infty . \tag{A.3}
\]

Appendix B. The exponential integral function

The exponential integral \( Ei[z] \) (see figure B1), \( z \in \mathbb{C} \) is a special function on the complex plane. For real nonzero values \( x \), it is defined as follows:

\[
Ei(x) = - \int_{-x}^{\infty} \frac{e^{-t}}{t} dt . \tag{B.1}
\]
We only need the exponential integral function evaluated in real numbers since in our solutions it appears only with a real argument, although in the definition of the prepotential (3.6), it appears with an argument that can, in general, be complex. $Ei(x)$ is negative for $x \in (-\infty, c)$, where $c \sim 0, 375$, zero in $x = c$ and positive for $x > c$. In addition, $\lim_{x\to 0}Ei(x) = -\infty$.

References

[1] Maldacena J M 1996 Black holes in string theory arXiv:hep-th/9607235
[2] Skenderis K 2000 Black holes and branes in string theory Lect. Notes Phys. 541 325–64
[3] Peet A W 2000 TASI lectures on black holes in string theory arXiv:hep-th/0008241
[4] Mohaupt T 2001 Black hole entropy, special geometry and strings Fortsch. Phys. 49 3–161
[5] Das S R and Mathur S 2000 The quantum physics of black holes: results from string theory Ann. Rev. Nucl. Part. Sci. 50 153–206
[6] Myers R C 2003 Black holes and string theory Rev. Mex. Fis. 49S1 14–18
[7] Shahbazi C 2013 Black Holes in supergravity with applications to string theory arXiv:1307.3064
[8] Andrianopoli L, D’Auria R, Ferrara S and Trigiante M 2008 Extremal black holes in supergravity Lect. Notes Phys. 737 661–727
[9] Bellucci S, Ferrara S and Marrani A 2008 Attractors in Black Fortsch. Phys. 56 761–85
[10] Dall’Agata G 2013 Black holes in supergravity: flow equations and duality Springer Proc. Phys 142 1–46
[11] Ortin T 2010 Supersymmetric solutions of 4-dimensional supergravities AIP Conf. Proc. 1318 175–86
[12] Schwarzschild K 1916 On the gravitational field of a sphere of incompressible fluid according to Einstein’s theory Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) 1916 424–34
[13] Hawking S and Ellis G 1973 The large scale structure of space-time (Cambridge: Cambridge University Press)
[14] Einstein A and Rosen N 1935 The Particle Problem in the General Theory of Relativity Phys. Rev. 48 73–77
[15] Kruskal M 1960 Maximal extension of Schwarzschild metric Phys. Rev. 119 1743–5
[16] Bekenstein J D 1973 Black holes and entropy Phys. Rev D 7 2333–46
[17] Hawking S 1971 Gravitational radiation from colliding black holes Phys. Rev. Lett. 26 1344–6
[18] Bardeen J M, Carter B and Hawking S 1973 The Four laws of black hole mechanics Commun. Math. Phys. 31 161–70
[19] Jebsen Ark J 1921 Mat. Ast. Fys. (Stockholm) 15 18
[20] Birkhoff G 1923 Relativity and Modern Physics (Cambridge: University Press) p 253
[21] Israel W 1968 Event horizons in static electrovac space-times Commun. Math. Phys. 8 245–60

Figure B1. The exponential integral function on the real axis.
[22] Carter B and Hole Has Axisymmetric Black 1971 Only Two Degrees of Freedom Phys. Rev. Lett. 26 331–3
[23] Mazur P O 2000 Black hole uniqueness theorems arXiv:hep-th/0101012
[24] Mazur P 1982 Proof of uniqueness the Kerr-Newman black hole solution J. Phys. A 15 3173–80
[25] Bekenstein J D 1972 Nonexistence of baryon number for static black holes Phys. Rev. D 5 1239–46
[26] Coleman S R, Preskill J and Wilczek F 1992 Quantum hair on black holes Nucl. Phys B 378 175–246
[27] Bekenstein J 1995 Novel ‘no scalar hair’ theorem for black holes Phys. Rev D 51 6608–11
[28] Greene B R, Mathur S D and O’Neill C M 1993 Eluding the no hair conjecture: black holes in spontaneously broken gauge theories Phys. Rev. D 47 2242–59
[29] Kleihaus B, Kunz J, Radu E and Sabugo B 2013 Axially symmetric static solitons and black holes with scalar hair Phys. Lett. B 725 489–94
[30] Anabalon A 2012 Exact hairy black holes arXiv:1211.2765
[31] Anabalon A 2012 Exact Black Holes and Universality in the Backreaction of non-linear Sigma Models with a potential in (AdS)4 J. High Energy Phys. JHEP06(2012)127
[32] Anabalon A, Astefanesei D and Mann R 2013 Exact asymptotically flat charged hairy black holes with a dilaton potential arXiv:1308.1693
[33] Bueno P, Canfora F, Giacomini A and Oliva J 2012 Black holes with primary hair in gauged N = 8 supergravity J. High Energy Phys. JHEP06(2012)010
[34] Winstanley E and Mavromatos N 1995 Instability of hairy black holes in spontaneously broken Einstein-Yang-Mills Higgs systems Phys. Lett. B 352 242–6
[35] García-Compeán H, Louiza-Brito O, Martínez-Merino A and Santos-Silva R 2014 Half-fl
[36] Mavromatos N 1996 Eluding the no hair conjecture for black holes arXiv:gr-qc/9606008
[37] Kanti P, Mavromatos N, Rizos J, Tamvakis K and Winstanley E 1996 Dilatonic black holes in higher curvature string gravity Phys. Rev D 54 5049–58
[38] Bueno P and Shahbazi C 2014 Non-perturbative black holes in Type-IIA String Theory versus the No-Hair conjecture Class. Quantum Grav. 31 015023 (arXiv:1304.8079)
[39] de Wit B, Lauwers P, Philippe R, Su S and van Proeyen A 1984 Gauge and Matter Fields Coupled to N = 2 Supergravity Phys. Lett. B 134 37
[40] Andrianopoli L et al 1997 N = 2 supergravity and N = 2 superYang-Mills theory on general scalar manifolds: Symplectic covariance, gaugings and the momentum map J. Geom. Phys. 23 111–89
[41] Huebscher M, Meessen P and Ortin T 2006 Supersymmetric solutions of N = 2 D = 4 sugra: The Whole ungauged shebang Nucl. Phys B 759 228–48
[42] Meessen P, Ortin T and Vaula S 2010 All the timelike supersymmetric solutions of all ungauged d = 4 supergravities J. High Energy Phys. JHEP11(2010)072
[43] Galli P, Ortin T, Perez J and Shahbazi C S 2011 Non-extremal black holes of N = 2, d = 4 supergravity J. High Energy Phys. JHEP07(2011)041
[44] Meessen P, Ortin T, Perez J and Shahbazi C 2012 H-FGK formalism for black-hole solutions of N = 2, d = 4 and d = 5 supergravity Phys. Lett. B 709 260–5
[45] Mohaupt T and Vaughan O 2012 The Hesse potential, the c-map and black hole solutions J. High Energy Phys. JHEP07(2012)163
[46] Galli P, Meessen P and Ortin T 2013 The Freudenthal gauge symmetry of the black holes of N = 2, d = 4 supergravity J. High Energy Phys. JHEP05(2013)011
[47] Bueno P, Galli P, Meessen P and Ortin T 2014 Black holes and equivariant charge vectors in N = 2, d = 4 supergravity J. High Energy Phys. JHEP09(2013)010
[48] Ferrara S, Gibbons G W and Kallosh R 1997 Black holes and critical points in moduli space Nucl. Phys. B 500 75–93
[49] Tod K 1983 All Metrics Admitting Supercovariantly Constant Spinors Phys. Lett. B 121 241–4
[50] Gauntlett J P, Gutowski J B, Hull C M, Pakis S and Reall H S 2003 All supersymmetric solutions of minimal supergravity in five- dimensions Class. Quantum Grav. 20 4587–634
[51] Meessen P and Ortin T 2006 The Supersymmetric configurations of N = 2, D = 4 supergravity coupled to vector supermultiplets Nucl. Phys. B 749 291–324
[52] Bueno P and Shahbazi C S Multivalued attractors to appear
[53] Kallosh R, Linde A D and Shamkova M 1999 Supersymmetric multiple basin attractors J. High Energy Phys. JHEP11(1999)010
[54] Lambert J H 1758 Observationes variar in mathesin puram Acta Helveticae physico-mathematico-anatomico-botanico-medica 128 168
[55] Corless D J R M and Valluri S R 2000 lambert w function to physics Can. J. Phys. 78 823–31
[56] Caillol J-M 2003 Some applications of the Lambert W function to classical statistical mechanics Journal of Physics A Mathematical General 36 10431–42
[57] Gardi E, Grunberg G and Karliner M 1998 Can the QCD running coupling have a causal analyticity structure? J. High Energy Phys. JHEP07(1998)007
[58] Magradze B 1999 The Gluon propagator in analytic perturbation theory Conf. Proc. C980518 158–69
[59] Nesterenko A 2003 Analytic invariant charge in QCD Int. J. Mod. Phys. A 18 5475–520
[60] Cvetic G and Kondrashuk I 2011 Explicit solutions for effective four- and five-loop QCD running coupling J. High Energy Phys. JHEP12(2011)019
[61] Sonoda H 2013 Solving RG equations with the Lambert W function Phys. Rev. D 87 085023
[62] Ashoorioon A, Kempf A and Mann R B 2005 Minimum length cutoff in inflation and uniqueness of the action Phys. Rev. D 71 023503
[63] Scott T C and Mann R B 2006 General relativity and quantum mechanics: towards a generalization of the Lambert W function Phys. Rev. D 71 023503
[64] Mann R B and Ohta T 1997 Exact solution for the metric and the motion of two bodies in (1+1)-dimensional gravity Phys. Rev. D 55 4723–47
[65] Belyakova Y and Nesterenko A 2011 A nonperturbative model for the strong running coupling within potential approach Int. J. Mod. Phys. A 26 981–93