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A Lagrangian scheme with the preservation of symmetry and conservation in cylindrical geometry: preliminary study

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Abstract
We report our preliminary results in developing a new cell-centered control volume Lagrangian scheme for solving Euler equations of compressible gas dynamics in cylindrical coordinates. Based on a local coordinate transform strategy, the scheme can preserve one-dimensional spherical symmetry in a two-dimensional cylindrical geometry when computed on an equal-angle-zoned initial grid. Unlike many previous area weighted schemes that possess the spherical symmetry property, our scheme is discretized on the true volume and it can preserve the conservation property for all the conserved variables including density, momentum and total energy. Preliminary two dimensional numerical examples in cylindrical coordinates are presented to demonstrate the performance of the scheme in terms of symmetry, accuracy and non-oscillatory properties.

Keywords: Lagrangian scheme; spherical symmetry; conservative; cell-centered; compressible Euler equations; cylindrical geometry

1. Introduction
In numerical simulations of multidimensional fluid flow, there are two typical choices: a Lagrangian framework, in which the mesh moves with the local fluid velocity, and an Eulerian framework, in which the fluid flows through a grid fixed in space. More generally, the motion of the grid can also be chosen arbitrarily, resulting in the so-called Arbitrary Lagrangian-Eulerian method (ALE; see, e.g. [11]).

The Lagrangian method is widely used in many fields for multi-material flow simulations such as astrophysics and computational fluid dynamics (CFD), due to its distinguished advantage in capturing material interfaces sharply. In the past years, many efforts have been made to develop Lagrangian methods. Some algorithms are built on a staggered
discretization in which velocity (momentum) is stored at vertices, while density and internal energy are stored at cell centers. The density / internal energy and velocity are solved on two different control volumes, see, e.g. [16, 1, 3]. The dissipation of kinetic energy into internal energy through shock waves is ensured by an artificial viscosity term [16, 5, 6]. Other algorithms start from the cell-centered discretization in which density, momentum and energy are all centered within cells and evolved on the same control volume. Advantages for the cell-centered schemes such as [9, 15, 13, 7, 8] include that this kind of methods is easy to remap when necessary and it does not require the addition of an explicit artificial viscosity for shock capturing. Numerical diffusion is added implicitly in the Riemann solution in a very natural way.

A difficult problem in flow simulation is the exact preservation of a one-dimensional symmetry (for example, cylindrical or spherical) in a coordinate system distinct from that symmetry. Concerning the critical issue related to spherical symmetry preservation many works have been done in the framework of staggered-grid hydrodynamics. The most widely used method that preserves symmetry exactly on an equal-angle-zoned grid is the area-weighted method [21, 18, 20, 4, 14]. In this approach one uses a Cartesian form of the momentum equation in the cylindrical coordinate system, hence integration is performed on area rather than on the true volume in cylindrical coordinates. However, these area-weighted schemes might violate strict momentum and total energy conservation. This shortcoming of the schemes has been partially corrected in [3] by constructing a compatible area-weighted scheme which can preserve the conservation of total energy. However it seems that their flaw in the loss of momentum conservation has not been solved so far. Differently from the area-weighted scheme, Browne [2] presented a Lagrangian scheme termed “integrated total average” which is discretized on the true control volume in cylindrical coordinates. The scheme has been proven to be able to preserve the desired spherical symmetry in the two dimensional cylindrical geometry for equal-angle zoning. Unfortunately, this scheme also can not keep the momentum and total energy conservation.

In this short paper, we report our preliminary results in proposing and testing a new cell-centered Lagrangian scheme on quadrilateral grids for solving compressible Euler equations in cylindrical coordinates. The scheme is based on a genuine volume discretization formulation. It preserves the conservation of mass, momentum and total energy. By using a local coordinate transform strategy, the scheme can preserve one dimensional spherical symmetry in a two dimensional cylindrical geometry for equal-angle-zoned initial grids.

An outline of the rest of this paper is as follows. In Section 2, we describe the individual steps of the Lagrangian scheme with the symmetry and conservation properties for Euler equations in cylindrical coordinates. In Section 3, preliminary numerical examples are given to demonstrate the performance of the new Lagrangian method. In Section 4 we will give concluding remarks. More details not included here will be given in a future full version of this paper.

2. A cell-centered conservative Lagrangian scheme with the spherical symmetry property

2.1. The compressible Euler equations in a Lagrangian formulation in cylindrical coordinates

The compressible inviscid flow is governed by the Euler equations which have the following integral form in the Lagrangian formulation

\[
\begin{align*}
\frac{d}{dt} \int_{\Omega} \rho dV &= 0, \\
\frac{d}{dt} \int_{\Omega} M dV &= -\int_{\Gamma} p \mathbf{n} ds, \\
\frac{d}{dt} \int_{\Omega} E dV &= -\int_{\Gamma} \mathbf{u} \cdot \mathbf{n} dS,
\end{align*}
\]

where \(\rho\) is the density, \(p\) is the pressure, \(M\) and \(\mathbf{u}\) are the momentum and velocity vectors respectively, and \(E\) is the total energy.

In this paper, we seek to study the axisymmetric compressible Euler system. Its specific form in the cylindrical coordinates is as follows

\[
\begin{align*}
\frac{d}{dt} \int_{\Omega} \rho r dr dz &= 0, \\
\frac{d}{dt} \int_{\Omega} M_r r dr dz &= -\int_{\Gamma} p n_r rd\ell, \\
\frac{d}{dt} \int_{\Omega} M_z r dr dz &= -\int_{\Gamma} p n_z rd\ell + \int_{\Omega} p r dz, \\
\frac{d}{dt} \int_{\Omega} \eta r dr dz &= -\int_{\Gamma} \eta u_r rd\ell,
\end{align*}
\]

where \(z\) and \(r\) are the axial and radial directions respectively, \(M = (M_z, M_r, \mathbf{u} = (u, v)\), where \(M_z, u, M_r, v\) are the momentum and velocity components in the \(z\) and \(r\) directions respectively, \(\mathbf{n} = (n_z, n_r)\) is the unit outward normal to the boundary \(\Gamma(t)\) in the \(z-r\) coordinates, and \(u_r = (u, v) \cdot \mathbf{n}\) is the normal velocity.
2.2. The construction of the scheme

Figure 1 indicates the cylindrical geometry \((z, r)\) we are concerned with. The 2D spatial domain \(\Omega\) is discretized into \(M \times N\) computational cells. Figure 1 shows an equal angular grid with logical \(l\)-lines radially outward and logical \(k\)-lines in the angular direction. \(I_{k+1/2,l+1/2}\) is a quadrilateral cell constructed by the four vertices \([(z_k, r_l), (z_{k+1}, r_l), (z_{k+1}, r_{l+1}), (z_k, r_{l+1})]\). \(S_{k+1/2,l+1/2}\) and \(V_{k+1/2,l+1/2}\) denote the area and volume of the cell \(I_{k+1/2,l+1/2}\) with \(k = 1, \ldots, M, l = 1, \ldots, N\) respectively. For a given cell \(I_{k+1/2,l+1/2}\), the location of the cell center is denoted by \((z_{k+1/2}, r_{l+1/2})\). The fluid velocity \((u_{k+1/2}, v_{k+1/2})\) is defined at the vertex of the grid.

In this paper, in order to obtain a scheme with both the spherical symmetry and conservation properties in a 2D cylindrical geometry, we will discretize the momentum equation in (2) by first transforming the cylindrical \((z, r)\) coordinates to the local polar coordinates \(\xi, \theta\) at each cell, where \(\xi\) is the radial direction passing through the cell center and \(\theta\) is the angular direction which is orthogonal to \(\xi\). To perform this local polar coordinate transformation, in each cell, the Euler equations in cylindrical coordinates are rewritten in the following form

\[
\begin{aligned}
\frac{\partial}{\partial z} \int_{I_{k+1/2,l+1/2}} \rho r dr dz &= 0 \\
\frac{\partial}{\partial l} \int_{I_{k+1/2,l+1/2}} M_\theta r dr dz &= -\int_{I_{k+1/2,l+1/2}} m_\theta r dr l + \int_{\partial I_{k+1/2,l+1/2}} p \sin(\theta + \theta_{k+1/2,l+1/2}) d\theta dz \\
\frac{\partial}{\partial z} \int_{I_{k+1/2,l+1/2}} M_\xi r dr dz &= -\int_{I_{k+1/2,l+1/2}} m_\xi r dr l + \int_{\partial I_{k+1/2,l+1/2}} p \cos(\theta + \theta_{k+1/2,l+1/2}) d\theta dz
\end{aligned}
\]

(3)

where \(n = (n_x, n_y)\) is the outward unit normal of the cell boundary \(\partial I_{k+1/2,l+1/2}\), \(\theta_{k+1/2,l+1/2}\) is the angle between the axial coordinate \(z\) and the local \(\xi\) coordinate. Thus we have \(z = \xi \cos(\theta + \theta_{k+1/2,l+1/2}), r = \xi \sin(\theta + \theta_{k+1/2,l+1/2})\).

For the cell-centered scheme, all the variables except velocity are stored at the cell center of \(I_{k+1/2,l+1/2}\) in the form of cell averages. For example, the values of the cell averages for the cell \(I_{k+1/2,l+1/2}\), denoted by \(\bar{\rho}_{k+1/2,l+1/2}, \bar{M}_{k+1/2,l+1/2}\) and \(\bar{E}_{k+1/2,l+1/2}\), are defined as follows

\[
\begin{aligned}
\bar{\rho}_{k+1/2,l+1/2} &= \frac{1}{V_{k+1/2,l+1/2}} \int_{I_{k+1/2,l+1/2}} \rho r dr dz, \\
\bar{M}_{k+1/2,l+1/2} &= \frac{1}{V_{k+1/2,l+1/2}} \int_{I_{k+1/2,l+1/2}} M_\theta r dr dz, \\
\bar{M}_{k+1/2,l+1/2} &= \frac{1}{V_{k+1/2,l+1/2}} \int_{I_{k+1/2,l+1/2}} M_\xi r dr dz.
\end{aligned}
\]
Ek+1/2,l+1/2 = 1
Vk+1/2,l+1/2
∫∫ Ik+1/2,l+1/2
Erdrdz
where Vk+1/2,l+1/2 ... center of Cell Ik+1/2,l+1/2 in the local ξ-θ coordinates. pc is defined as follows,

\[ pc = \frac{1}{2}(pk_{+1/2,l} + pk_{+1/2,l+1}) \]  

2.2.1. Spatial discretization

We first formulate the semi-discrete finite volume scheme of the governing equations (3) as

\[
\begin{align*}
\frac{d}{dt} & \left( \begin{array}{c}
M^p_{k+1/2,l+1/2} V_{k+1/2,l+1/2} \\
M^p_{k+1/2,l+1/2} V_{k+1/2,l+1/2} + M^e_{k+1/2,l+1/2} V_{k+1/2,l+1/2}
\end{array} \right) \\
= & - \int_{\partial h_{k+1/2,l+1/2}} \hat{F} dl + \left( \begin{array}{c}
p_r \sin(\theta) + \theta \sin(\theta) \frac{V_{k+1/2,l+1/2}}{V_{k+1/2,l+1/2} + V_{k+1/2,l+1/2}} \\
p_r \cos(\theta) + \theta \cos(\theta) \frac{V_{k+1/2,l+1/2}}{V_{k+1/2,l+1/2} + V_{k+1/2,l+1/2}}
\end{array} \right)
\end{align*}
\]

where

\[
\int_{\partial h_{k+1/2,l+1/2}} \hat{F} dl = \int_{\partial h_{k+1/2,l+1/2}} \left( \begin{array}{c}
\hat{f}^p(U^r_n, U^r_n) \\
\hat{f}^e(U^r_n, U^r_n) \\
\hat{f}^m(U^r_n, U^r_n) \\
\hat{f}^d(U^r_n, U^r_n)
\end{array} \right) dl
\]

and

\[
\begin{align*}
\hat{f}^p(U_n, U_n) & = 0, \\
\hat{f}^e(U_n, U_n) & = pn_r, \\
\hat{f}^m(U_n, U_n) & = pn_r, \\
\hat{f}^d(U_n, U_n) & = pu_r.
\end{align*}
\]

\(\hat{f}^p, \hat{f}^e, \hat{f}^m, \) and \(\hat{f}^d\) are the numerical fluxes for mass, \(\xi\)-momentum, \(\theta\)-momentum and total energy across the cell boundary respectively. \(U^p = (\rho, M_r^p, M_\theta^p, E^p)\) are the values of mass, \(\xi\)-momentum, \(\theta\)-momentum and total energy at two sides of the cell boundary. \(U^m = (\rho, M_r^m, M_\theta^m, E^m)\), where \(M_r^m\) and \(M_\theta^m\) are the left and right component values of the momentum which is normal to the cell boundary, i.e., \(M_r^m = (M_r^p, M_r^p) \cdot n\).

The first step for establishing the scheme is to determine the line integral term on the right side of Eq. (4). Suppose the cell boundary \(\partial h_{k+1/2,l+1/2}\) consists of \(M\) edges. The line integral concerned with the flux in Eq. (4) is discretized by the following formula,

\[
\int_{\partial h_{k+1/2,l+1/2}} \hat{F} dl = \sum_{m=1}^{M} \hat{F}(U^m_n, U^m_n) \Delta l_m
\]

where \(\Delta l_m\) is the length of the cell edge \(m\). Here \(\hat{F}(U^m_n, U^m_n)\) is a numerical flux at the edge \(m\).

To determine the fluxes \(\hat{F} = (\hat{f}^p, \hat{f}^e, \hat{f}^m, \hat{f}^d)\) at the cell edge, we first need to identify the values of the primitive variables at each side of the edge, that is, \(U^m_n, m = 1, \ldots, M\). Here we use the information of the left and right cell average values of the conserved variables to obtain \(U^m_n, m = 1, \ldots, M\).

Next, we will compute the fluxes given the primitive states \(U^m_n, m = 1, \ldots, M\) at each side of the cell’s boundary. Here we use the Dukowicz numerical flux [10] to obtain the values of \(U_n\) at the cell boundary. Thus by now, we have finished the determination of the flux term of Eq. (4).

For the source term in Eq. (4), \(\theta_c\) is set to be the \(\theta\) coordinate of the center of Cell \(I_{k+1/2,l+1/2}\) in the local \(\xi-\theta\) coordinates. \(p_c\) is defined as follows,

\[ p_c = \frac{1}{2}(pk_{+1/2,l} + pk_{+1/2,l+1}) \]
where \( p_{k+1/2,l} \) and \( p_{k+1/2,l+1} \) are the values of pressure at Edges 1 and 3 of Cell \( I_{k+1/2,l+1} \) (see Fig. 1) obtained in the above flux determination procedure.

Based on the above manipulations, the scheme can preserve the conservation for all the conserved variables, since it is discretized on the true volume and the numerical flux across each cell boundary is single-valued for the update of its two neighboring cells. Meanwhile, we can also prove that the scheme can keep the spherical symmetry property if equal-angular polar initial grids are used. The specific choices in the discretization of the fluxes and source terms in (4), as outlined above, are responsible for the exact maintenance of spherical symmetry. The proof itself is omitted in this short paper and will be provided in our upcoming full paper.

The vertex velocity of the mesh is determined in the same way as in [7].

2.2.2. Time discretization

The time marching for the semi-discrete scheme (4) can be accomplished by the Euler forward method. Thus the fully discretized scheme can be written as follows

\[
\begin{align*}
\left( \begin{array}{c}
p_{k+1/2,l+1}^{n+1} \\
M_{k+1/2,l+1}^{n+1} \\
M_{k+1/2,l+1}^{n+1} \\
E_{k+1/2,l+1}^{n+1}
\end{array} \right) & = \Delta t \sum_{m=1}^{M} \hat{F}(U_m^n) \Delta_t e_k^n + \left( \begin{array}{c}
0 \\
p_e^n \sin(\theta_l + \theta_{k+1/2,l+1/2}) c_{k+1/2,l+1/2}^n \\
p_e^n \cos(\theta_l + \theta_{k+1/2,l+1/2}) c_{k+1/2,l+1/2}^n \\
0
\end{array} \right).
\end{align*}
\]

Here the variables with the superscripts \( n \) and \( n+1 \) represent the values of the corresponding variables at the \( n \)-th and \((n+1)\)-th time steps respectively.

The time step \( \Delta t^e \) is chosen as follows

\[
\Delta t^e = \lambda \min_{k=1,\ldots,M, l=1,\ldots,N} (\Delta t_{k+1/2,l+1/2,0}^e / c_{k+1/2,l+1/2}^e)
\]

where \( \Delta t_{k+1/2,l+1/2}^e \) is the shortest edge length of the cell \( I_{k+1/2,l+1/2} \), and \( c_{k+1/2,l+1/2}^e \) is the sound speed within this cell. The Courant number \( \lambda \) in the following tests is set to be 0.5 unless otherwise stated.

3. Numerical results in the cylindrical coordinates

In this section, we perform some preliminary numerical experiments in two dimensional cylindrical coordinates. Purely Lagrangian computation and the ideal gas with \( \gamma = 5/3 \) are used in the following tests unless otherwise stated.

Example 1 (The Noh problem in a cylindrical coordinate system [17]).

We test the Noh problem which is a well known test problem widely used to validate Lagrangian scheme in the regime of strong shock waves. In this test case, the initial state of the fluid is uniform with \( (\rho, e) = (1, 10^{-5}) \) and an inward radial velocity of magnitude 1. The equi-angular polar grid is applied in the \( \frac{1}{2}\)-circle computational domain defined in a polar coordinates by \([0, 1] \times [0, \pi/2] \). The shock is generated in a perfect gas by bringing the cold gas to rest at a rigid wall \( (r = 0) \). The analytical post shock density is 64 and the shock speed is 1/3. Figure 2 shows the computational grids at the initial and final time and the density result at \( t = 0.6 \) with \( 100 \times 30 \) grids. In the plot of grids, we observe the symmetry is perfectly preserved. In the plot of density result, we observe no spurious oscillation near the discontinuity region. The shock location and the shock magnitude agree quite well with the exact solution. These results demonstrate the good performance of our scheme in symmetry and conservation preservation and non-oscillatory property.

Example 2 (The Sedov problem in a cylindrical coordinate system with polar grids [19]).
Figure 2: The result of the Noh problem. Top: grids. Left: initial grid; Right: final grid at $t = 0.6$. Bottom: density vs radial radius at $t = 0.6$. Solid line: the exact solution; dashed line: computational solution.
We present the result of the Sedov blast wave in a cylindrical coordinate system as an example of a diverging shock wave. The initial computational domain is a $\frac{1}{4}$-circle region defined in a polar coordinates by $[0, 1.125] \times [0, \pi/2]$ consisting of $100 \times 30$ equal angular cells. The initial density is unity and the initial velocity is zero. The internal energy is zero except in the cells connecting the origin where they share a total value of 0.2468. The analytical solution is a shock at radius unity at time unity with a peak density of 4. Figure 3 shows the results of our scheme by a purely Lagrangian computation. These results demonstrate that the Lagrangian scheme can produce symmetrical and accurate results for the Sedov problem in the cylindrical coordinates.

**Example 3** (implosion problem of Lazarus). The implosion problem of Lazarus [12] is a problem with self-similar solution. A sphere of unit initial radius with zero specific internal energy and unit density is driven by an inward radial velocity given in good approximation by

$$v_r(t) = \frac{-\alpha f}{(1 - ft)^{\alpha}}$$

where $\alpha = 0.6883545$, $f = 1 - \varepsilon t - \delta t^3$, $\varepsilon = 0.185$, $\delta = 0.28.$

We use a grid of $200 \times 30$ equal-angle uniform cells in the initial computational domain $[0, 1] \times [0, \pi/2]$ in a polar coordinates. Figure 4 shows the result of the grid at $t = 0.8$ and the density as a radial radius function at the typical time $t = 0.74, 0.75, 0.8$. Symmetry is preserved quite well as expected for the scheme. In the plot of density, we can see the shock wave arrives at the origin just after $t = 0.75$, and at this time the density is flat. At $t = 0.8$, the shock wave has reflected from the center of convergence and is moving outward into the already shocked medium. This result and the magnitude of the shock are consistent with the analytical solution shown in [4].
Figure 4: The result of the Lazarus problem. Top: grids. Left: initial grid; Right: final grid at $t = 0.8$. Bottom: density vs radial radius at $t = 0.74, 0.75, 0.8$. 
4. Concluding remarks

In this short paper we have briefly described a new cell-centered Lagrangian scheme for solving Euler equations in cylindrical coordinates. The scheme can preserve the symmetry property for one dimensional spherical problem computed on an equal-angle-zoned initial grid. Compared with many current Lagrangian type schemes, our scheme has the distinguished feature in being able to guarantee both symmetry and conservation properties for all conserved variables including mass, momentum and total energy. Preliminary two dimensional examples in the cylindrical coordinates have been presented which demonstrate the good performance of the scheme. The extension of the scheme to high order accuracy constitutes future work.

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