Capacity of Some Index Coding Problems with Symmetric Neighboring Interference

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Abstract—A single unicast index coding problem (SUICP) with symmetric neighboring interference (SNI) has equal number of K messages and K receivers, the kth receiver $R_k$ wanting the kth message $x_k$ and having the side-information $K_k = (I_k \cup x_k)^c$, where $I_k = \{x_{k-U}, \ldots, x_{k-2}, x_{k-1}\} \cup \{x_{k+1}, x_{k+2}, \ldots, x_{k+D}\}$ is the interference with $D$ messages after and $U$ messages before its desired message. Maleki, Cadambe and Jafar obtained the capacity of this symmetric neighboring interference single unicast index coding problem (SNI-SUICP) with $(K)$ tending to infinity and Blasiak, Kleinberg and Lubetzky for the special case of $(D = U = 1)$ with $K$ being finite. In this work, for any finite $K$ and arbitrary $D$ we obtain the capacity for the case $U = \gcd(K, D+1) - 1$. Our proof is constructive, i.e., we give an explicit construction of a linear index code achieving the capacity.

I. INTRODUCTION AND BACKGROUND

An index coding problem, comprises a transmitter that has a set of $K$ independent messages, $X = \{x_0, x_1, \ldots, x_{K-1}\}$, and a set of $M$ receivers, $R = \{R_0, R_1, \ldots, R_{M-1}\}$. Each receiver, $R_k$ is a set of messages, $K_k \subseteq X$, called its Known-set or the side-information, and demands to know another subset of messages, $W_k \subseteq K_k^c$, called its Want-set or Demand-set. A naive technique would be to broadcast all the messages in $K$ time slots. Instead, the transmitter can take cognizance of the side-information of the receivers and broadcast coded messages, called the index code, over a noiseless channel. The objective is to minimize the number of coded transmissions, called the length of the index code, such that each receiver can decode its demanded message using its side-information and the coded messages.

The problem of index coding with side-information was introduced by Birk and Kol [3]. Ong and Ho [4] classified the binary index coding problem depending on the demands and the side-information possessed by the receivers. An index coding problem is unicast if the demand sets of the receivers are disjoint. An indexing coding problem is single unicast if the demand sets of the receivers are disjoint and the cardinality of demand set of every receiver is one. Any unicast index problem can be converted into a single unicast index coding problem. A single unicast index coding problem (SUICP) can be described as follows: Let $\{x_0, x_1, \ldots, x_{K-1}\}$ be the $K$ messages, $\{R_0, R_1, \ldots, R_{K-1}\}$ are $K$ receivers and $x_k \in A$ for some alphabet $A$ and $k = 0, 1, \ldots, K - 1$. Receiver $R_k$ is interested in the message $x_k$ and knows a subset of messages in $\{x_0, x_1, \ldots, x_{K-1}\}$ as side-information.

A solution (includes both linear and nonlinear) of the index coding problem must specify a finite alphabet $A_P$ to be used by the transmitter, and an encoding scheme $\varepsilon : A^t \rightarrow A_P$ such that every receiver is able to decode the wanted message from the $\varepsilon(x_0, x_1, \ldots, x_{K-1})$ and the known information. The minimum encoding length $l = \lceil \log_2 |A_P| \rceil$ for messages that are $t$ bit long ($|A| = 2^t$) is denoted by $\beta_t(G)$. The broadcast rate of the index coding problem with side-information graph $G$ is defined [5] as,

$$\beta(G) \triangleq \inf_{t} \frac{\beta_t(G)}{t}.$$ 

If $t = 1$, it is called scalar broadcast rate. For a given index coding problem, the broadcast rate $\beta_t(G)$ is the minimum number of index code symbols required to transmit to satisfy the demands of all the receivers. The capacity $C(G)$ for the index coding problem is defined as the maximum number of message symbols transmitted per index code symbol such that every receiver gets its wanted message symbols and all the receivers get equal number of wanted message symbols. The broadcast rate and capacity are related as

$$C(G) = \frac{1}{\beta_t(G)}.$$ 

Instead of one transmitter and $K$ receivers, the SUICP can also be viewed as $K$ source-receiver pairs with all $K$ sources connected with all $K$ receivers through a common finite capacity channel and all source-receiver pairs connected with either zero of infinite capacity channels. This problem is called multiple unicast index coding problem in [1].

In a symmetric neighboring interference single unicast index coding problem (SNI-SUICP) with equal number of $K$ messages and receivers, each receiver has interfering messages, corresponding to the $D$ messages after and $U$ messages before its desired message. In this setting, the $kth$ receiver $R_k$ demands the message $x_k$ having the interference

$$I_k = \{x_{k-U}, \ldots, x_{k-2}, x_{k-1}\} \cup \{x_{k+1}, x_{k+2}, \ldots, x_{k+D}\}.$$ 

The side-information of this setting is given by

$$K_k = (I_k \cup x_k)^c.$$ 

Maleki et al. [1] found the capacity of SNI-SUICP with $K \rightarrow \infty$ to be

$$C = \frac{1}{D + 1}$$

per message.
Also, it was shown in \( D \) that the outer bound for the capacity of SNI-SUICP for finite \( K \) and is given by

\[
C \leq \frac{1}{D + 1}. \tag{4}
\]

Blasiak et al. \([5]\) found the capacity of SNI-SUICP with \( U = D = 1 \) by using linear programming bounds to be \( \frac{4}{5} \).

Jafar \([2]\) established the relation between index coding problem and topological interference management problem. The SNI-SUICP is motivated by topological interference management problems. The capacity and optimal coding results in index coding can be used in corresponding topological interference management problems.

A. Contributions

The contributions of this paper are summarized below:

- We derive the capacity of SNI-SUICP with \( D \) interfering messages after and \( U = \gcd(K, D + 1) - 1 \) interfering messages before the desired message.
- We show that AIR matrices of size \( K \times (D + 1) \) can be used as an encoding matrix to generate optimal index code for every field.

All the subscripts in this paper are to be considered modulo \( K \). In the remaining paper, we refer SNI-SUICP with \( D \) interfering messages after and \( U = \gcd(K, D + 1) - 1 \) interfering messages before the desired message as SNI-SUICP.

The remaining part of this paper is organized as follows. In Section II we define and review the properties of Adjacent Row Independent (AIR) matrices which is already discussed in detail in \([7]\) in the context of optimal index codes with symmetric, neighboring consecutive side-information. Except for the proof of Lemma \([1]\) this section is a slightly modified version available in \([8]\) and is present only for the sake of being self-contained. In \([1]\) we show that AIR matrix can be used as an encoding matrix to generate optimal index code for SNI-SUICP. We conclude the paper in Section IV.

II. REVIEW OF AIR MATRICES

In \([7]\), we gave the construction of AIR matrix and we used AIR matrices to give optimal length index codes for one-sided symmetric neighboring and consecutive side-information index coding problems (SNC-SUICP). In \([6]\), we constructed optimal vector linear index codes for two-sided SNC-SUICP. In \([8]\), we gave a low-complexity decoding for SNC-SUICP with AIR matrix as encoding matrix. The low complexity decoding method helps to identify a reduced set of side-information for each user with which the decoding can be carried out. By this method every receiver is able to decode its wanted message symbol by simply adding some index code symbols (broadcast symbols).

Given \( K \) and \( D \) the \( K \times (D + 1) \) matrix obtained by Algorithm I is called the \((K, D)\) AIR matrix and it is denoted by \( I_{K \times (D + 1)} \). The general form of the \((K, D)\) AIR matrix is shown in Fig. 1. It consists of several submatrices (rectangular boxes) of different sizes as shown in Fig 1. The location and sizes of these submatrices are used subsequently to prove the main results in the following section Theorems \([1]\) and \([2]\).

The description of the submatrices are as follows: Let \( m \) and \( n \) be two positive integers and \( n \) divides \( m \). The following matrix denoted by \( I_{m \times n} \) is a rectangular matrix.

\[
I_{m \times n} = \begin{bmatrix}
I_n \\
I_n \\
\vdots \\
I_n
\end{bmatrix}
\]

and \( I_{n \times m} \) is the transpose of \( I_{m \times n} \). We will call the \( I_{m \times n} \) matrix the \((m \times n)\) identity matrix.

**Algorithm 1. Algorithm to construct the AIR matrix \( L \) of size \((D + 1) \times (D + 1)\)**

Given \( K \) and \( D \) let \( L = K \times (D + 1) \) blank unfilled matrix.

**Step 1**

1.1: Let \( K = q(D + 1) + r \) for \( r < D + 1 \).

1.2: Use \( I_{q(D+1) \times (D+1)} \) to fill the first \( q(D + 1) \) rows of the unfilled part of \( L \).

1.3: If \( r = 0 \), Go to Step 3.

**Step 2**

2.1: Let \( D + 1 = q^r r' + r'' \) for \( r' < r \).

2.2: Use \( I_{q^r r' \times r} \) to fill the first \( q^r r' \) columns of the unfilled part of \( L \).

2.3: If \( r' = 0 \), go to Step 3.

2.4: \( K \leftarrow r \) and \( D + 1 \leftarrow r'' \).

2.5: Go to Step 1.

**Step 3. Exit.**

Towards explaining the other quantities shown in the AIR matrix shown in Fig. 1 for a given \( K \) and \( D \), let \( \lambda_{-1} = D + 1, \lambda_0 = K - D - 1 \) and

\[
D + 1 = \beta_0 \lambda_0 + \lambda_1,
\]

\[
\lambda_0 = \beta_1 \lambda_1 + \lambda_2,
\]

\[
\lambda_1 = \beta_2 \lambda_2 + \lambda_3,
\]

\[
\lambda_2 = \beta_3 \lambda_3 + \lambda_4,
\]

\[
\vdots
\]

\[
\lambda_l = \beta_{l+1} \lambda_{l+1} + \lambda_{l+2},
\]

\[
\vdots
\]

\[
\lambda_{l-1} = \beta_l \lambda_l.
\tag{6}
\]

where \( \lambda_{l+1} = 0 \) for some integer \( l, \lambda_i, \beta_i \) are positive integers and \( \lambda_i < \lambda_{i-1} \) for \( i = 1, 2, \ldots, l \). The number of submatrices in the AIR matrix is \( l + 2 \) and the size of each submatrix is shown using \( \lambda_i, \beta_i, i \in [0 : l] \). The submatrices are classified in to the following three types.

- The first submatrix is the \( I_{(D+1) \times (D+1)} \) matrix at the top of Fig. 1 which is independent of \( \lambda_i, \beta_i, i \in [0 : l] \). This will be referred as the \( I_{(D+1)} \) matrix henceforth.
The set of matrices of the form \( I_{\lambda_i \times \beta_i \lambda_i} \) for \( i = 0, 2, 4, \ldots \) (for all \( i \) even) will be referred as the set of even-submatrices.

The set of matrices of the form \( I_{\beta_i \lambda_i \times \lambda_i} \) for \( i = 1, 3, 5, \ldots \) (for all \( i \) odd) will be referred as the set of odd-submatrices.

Note that the odd-submatrices are always “fat” and the even-submatrices are always “tall” including square matrices in both the sets. By the \( i \)-th submatrix is meant either an odd-submatrix or an even-submatrix for \( 0 \leq i \leq l \). Also whenever \( \beta_0 = 0 \), the corresponding submatrix will not exist in the AIR matrix. To prove the main result in the following section the location of both the odd- and even-submatrices within the AIR matrix need to be identified. Towards this end, we define the following intervals. Let \( R_0, R_1, R_2, \ldots, R_{\lceil \frac{l}{2} \rceil} \) be the intervals that will identify the rows of the submatrices as given below:

- \( R_0 = [0 : K - \lambda_0 - 1] \)
- \( R_1 = [K - \lambda_0 : K - \lambda_2 - 1] \)
- \( R_2 = [K - \lambda_2 : K - \lambda_4 - 1] \)
- \( R_i = [K - \lambda_{2(i-1)} : K - \lambda_{2i} - 1] \)
- \( R_{\lceil \frac{l}{2} \rceil} = [K - \lambda_{2(\lceil \frac{l}{2} \rceil - 1)} : K - \lambda_{2\lceil \frac{l}{2} \rceil} - 1] \)

Also whenever \( \beta_0 = 0 \), the corresponding submatrix will not exist in the AIR matrix. Then, for a given \( L \), the \( (j_R, k_R) \) be the (row-column) indices of \( L(j, k) \) within the submatrix in which \( L(j, k) \) is present. Then, for a given \( L(j, k) \), the indices \( j_R \) and \( k_R \) are as given below:

- If \( L(j, k) \) is present in \( I_{D+1} \), then \( j_R = j \) and \( k_R = k \).
- If \( L(j, k) \) is present in \( I_{\beta_i \lambda_i \times \lambda_i} \) for \( i \in [0 : \lceil \frac{l}{2} \rceil - 1] \) or \( I_{\beta_i \lambda_i \times \lambda_i} \) for \( i \in [\lceil \frac{l}{2} \rceil : \lceil \frac{\lambda_i}{2} \rceil] \). Let \( (j_R, k_R) \) be the (row-column) indices of \( L(j, k) \) within the submatrix in which \( L(j, k) \) is present.

\[ S = I_{\lambda_i \times \beta_i \lambda_i} \text{ if } l \text{ is even and } S = I_{\beta_i \lambda_i \times \lambda_i} \text{ otherwise.} \]

![AIR Matrix](image_url)

Fig. 1. AIR matrix of size \( K \times (D + 1) \).
Let $d_{\text{down}}(k) = K - D - 1 + (\beta_{2i} - 1)\lambda_{2i}$. The down distance is given by

$$d_{\text{down}}(k) = K - D + 1 + \beta_{2i} - 1 - c\lambda_{2i}. \quad (7)$$

Proof. Proof is given in Appendix A. \qed

Lemma 2. The up-distance of $L(j, k)$ is as given below.

- If $L(j, k) \in \mathbf{I}_{2i+1 \times 2i+1}$ for $i \in \left[ \frac{r}{2} \right] - 1$, then $d_{\text{up}}(j, k)$ is $\lambda_{2i+1}$.
- If $L(j, k) \in \mathbf{I}_{2i+1 \times 2i+1}$ for $i \in \left[ \frac{r}{2} \right]$ and $k_R = c\lambda_{2i} + d$ for some positive integer $c, d (d < \lambda_{2i})$, then $d_{\text{up}}(j, k)$ is $\lambda_{2i+1} - \lambda_{2i}c$.

Proof. Proof is available in Appendix C of [3]. \qed

Lemma 3. The right-distance of $L(j, k)$ is as given below.

- If $k_R \in \left[ \frac{r}{2} \right] \lambda_{2i+1}$ for $i \in \left[ \frac{r}{2} \right] - 1$, then $d_{\text{right}}(j, k)$ is $\lambda_{2i}$.
- If $k_R \in \left[ \frac{r}{2} \right] \lambda_{2i+1}$ for $i \in \left[ \frac{r}{2} \right]$ and $d_{\text{right}}(j, k)$ depends on $j_R$. If $j_R = c\lambda_{2i+1} + d$ for some positive integers $c, d (d < \lambda_{2i+1})$, then $d_{\text{right}}(j, k)$ is $\lambda_{2i} - c\lambda_{2i+1}$.

Proof. Proof is available in Appendix D of [3]. \qed

From Euclid algorithm and (6), we can write

$$\lambda_l = \gcd(K, D + 1). \quad (8)$$

From (6), we have

$$\lambda_0 > \lambda_1 > \ldots > \lambda_{2i} > \ldots > \lambda_l = \gcd(K, D + 1) \quad (9)$$

for $i \in \left[ \frac{r}{2} \right]$ and $c \leq \beta_{2i}$.

Define

$$\tilde{C}_i = \{x + \lambda_0 : \forall x \in C_i\} \quad \text{for } i \in \left[ \frac{r}{2} \right] - 1.$$

That is,

$$\tilde{C}_i = \{n - \lambda_{2i+1} : \lambda_{2i+1} \leq n \leq \lambda_{2i} - 1\}. \quad (10)$$

We have $C_0 \cup C_1 \cup \ldots \cup C_{\left[ \frac{r}{2} \right]} = \{n : D - 1\}$, hence

$$\tilde{C}_0 \cup \tilde{C}_1 \cup \ldots \cup \tilde{C}_{\left[ \frac{r}{2} \right]} = \{n : K - 1\}. \quad (11)$$

III. OPTIMAL INDEX CODING FOR SNI-SUICP BY USING AIR MATRICES

A scalar linear index code of length $D + 1$ generated by an AIR matrix of size $K \times (D + 1)$ is given by

$$[c_0 \ c_1 \ \ldots \ c_D] = [x_0 \ x_1 \ \ldots \ x_{K-1}]L = \sum_{k=0}^{K-1} x_k L_k \quad (11)$$

where $L_k$ is the $k$th row of $L$ for $k \in [0 : K - 1]$. We prove that for $k \in [0 : K - 1]$, every receiver $R_k$ decodes its wanted message $x_k$ by using $[c_0 \ c_1 \ \ldots \ c_D]$ and its side-information.

In this section we show that the AIR matrix with parameter $K$ and $D + 1$ is an encoding matrix for the optimal length code for our SNI-SUICP.
Theorem 1. Let \( L \) be the AIR matrix of size \( K \times (D + 1) \). The matrix \( L \) can be used as an encoding matrix for the SNI-SUICP with \( K \) messages, \( D \) interfering messages after and \( U = \gcd(K, D + 1) - 1 \) interfering messages before the desired message.

Proof. Proof is given in Appendix B. \( \square \)

Theorem 2. The capacity of SNI-SUICP with \( K \) messages, \( D \) interfering messages after and \( U = \gcd(K, D + 1) - 1 \) interfering messages before the desired message is \( \frac{1}{D+1} \).

Proof. In Theorem 1, we proved that AIR of size \( K \times (D + 1) \) can be used as an encoding matrix for this SNI-SUICP. The rate achieved by using AIR matrix is \( \frac{1}{D+1} \). From (4), the rate of SNI-SUICP is always greater than or equal to \( \frac{1}{D+1} \). Hence, the capacity of SNI-SUICP with \( K \) messages, \( D \) interfering messages after and \( U = \gcd(K, D + 1) - 1 \) interfering messages before the desired message is \( \frac{1}{D+1} \). \( \square \)

Remark 1. Let \( \tau_k \) be the set of broadcast symbols used by receiver \( R_k \) to decode \( x_k \). The number of broadcast symbols used by receiver \( R_k \) by using AIRM as encoding matrix is given below:

- If \( k \in [0, \lambda_0 - 1] \), then \( |\tau_k| = 1 \).
- If \( k \in \mathcal{D}_i \) for \( i \in [0, \left\lceil \frac{1}{2} \right\rceil] \), then \( |\tau_k| = 2 \).
- If \( k \in \mathcal{E}_i \) for \( i \in [0, \left\lceil \frac{1}{2} \right\rceil - 1] \), then \( |\tau_k| = p_{k'} + 2 \), where \( k' = k - \lambda_0 \) and \( p_{k'} \) is the number of 1s below \( L(k + d_{down}(k'), k' + d_{right}(k' + d_{down}(k'), k')) \) in AIR matrix.
- If \( k \in \mathcal{E}_i \) for \( i \in \left[ \left\lceil \frac{1}{2} \right\rceil \right] \), then \( |\tau_k| = 1 \).

Remark 2. Let \( \gamma_k \) be the set of side-information used by receiver \( R_k \) to decode \( x_k \). Let \( N_k \) be the number of message symbols present in \( c_k \) for \( k \in [0, D] \). The number of side-information used by receiver \( R_k \) by using AIRM as encoding matrix is given below:

- If \( k \in [0, \lambda_0 - 1] \), then \( |\gamma_k| = N_k \mod (D+1) - 1 \).
- If \( k \in \mathcal{D}_i \) for \( i \in [0, \left\lceil \frac{1}{2} \right\rceil] \), then \( |\gamma_k| = N_k + N_{k'} + \mu_{k'} - 3 \), where \( k' = k - \lambda_0 \).
- If \( k \in \mathcal{E}_i \) for \( i \in [0, \left\lceil \frac{1}{2} \right\rceil - 1] \), then \( |\gamma_k| = N_k + N_{k'} + \mu_{k'} + \sum_{j=1}^{\lambda_{k'}-1} N_{k'} + \lambda_{k'} - 2pk' - 3 \).
- If \( k \in \mathcal{E}_i \) for \( i \in \left[ \left\lceil \frac{1}{2} \right\rceil \right] \), then \( |\gamma_k| = N_k - 1 \).

Example 1. Consider a SNI-SUICP with \( K = 12, D = 7, U = 3 \). The capacity of this SNI-SUICP is \( \frac{1}{8} \). AIRM of size \( 12 \times 8 \) can be used as an optimal length encoding matrix for this SNI-SUICP. The encoding matrix \( L_{12 \times 8} \) is given below. The code symbols and side-information used by each receiver to decode its wanted message is given in Table 1.

| \( R_k \) | \( W_k \) | \( D_{\text{max}}(k) \) | \( \mu_k \) | \( \mu_{k'} \) | \( t_{k', 1} \) | \( \tau_k \) | \( \gamma_k \) |
|------|--------|----------------|--------|--------|-------------|--------|--------|
| \( R_1 \) | \( x_0 \) | 9 | 4 | - | - | \( a_1 \) | \( x_8 \) |
| \( R_2 \) | \( x_1 \) | 8 | 4 | - | - | \( a_2 \) | \( x_9 \) |
| \( R_2 \) | \( x_2 \) | 8 | 4 | - | - | \( a_2 \) | \( x_10 \) |
| \( R_3 \) | \( x_3 \) | 8 | 4 | - | - | \( a_3 \) | \( x_{11} \) |
| \( R_4 \) | \( x_4 \) | 4 | 4 | - | - | \( a_0, c_4 \) | \( x_{20} \) |
| \( R_5 \) | \( x_5 \) | 4 | 4 | - | - | \( a_1, c_5 \) | \( x_{21} \) |
| \( R_6 \) | \( x_6 \) | 4 | 4 | - | - | \( a_2, c_6 \) | \( x_{22} \) |
| \( R_6 \) | \( x_7 \) | 4 | 4 | - | - | \( a_3, c_7 \) | \( x_{23} \) |
| \( R_6 \) | \( x_8 \) | - | - | - | - | \( a_4 \) | \( x_{24} \) |
| \( R_6 \) | \( x_9 \) | - | - | - | - | \( a_5 \) | \( x_{25} \) |
| \( R_6 \) | \( x_{10} \) | - | - | - | - | \( a_0, c_9 \) | \( x_{26} \) |
| \( R_6 \) | \( x_{11} \) | - | - | - | - | \( a_{11} \) | \( x_{27} \) |

Example 2. Consider a SNI-SUICP with \( K = 33, D = 20, U = 2 \). The capacity of this SNI-SUICP is \( \frac{1}{D+1} \). AIRM of size \( 33 \times 21 \) can be used as an optimal length encoding matrix for this SNI-SUICP. For this SNI-SUICP, \( D + 1 = 21, \lambda_1 = 9, \lambda_2 = 3, \beta_0 = 1, \beta_1 = 1, \beta_2 = 3, \) and \( l = 2 \). The encoding matrix for this SNI-SUICP is shown in Fig. 1. The code symbols and side-information used by each receiver to decode its wanted message is given in Table 1.

Example 3. Consider a SNI-SUICP with \( K = 432, D = 175, U = 15 \). The capacity of this SNI-SUICP is \( \frac{1}{D+1} \). For this SNI-SUICP, we have \( D + 1 = 176, \lambda_1 = 176, \lambda_2 = 80, \lambda_3 = 16, \beta = 0, \beta_1 = 1, \beta_2 = 2, \beta_3 = 5 \) and \( l = 3 \). AIRM of size \( 432 \times 176 \) given in Fig. 4 can be used as an optimal length encoding matrix for this SNI-SUICP.

Example 4. Consider a SNI-SUICP with \( K = 432, D = 255, U = 15 \). The capacity of this SNI-SUICP according to Theorem 2 is \( \frac{1}{D+1} \). For this SNI-SUICP, we have \( D + 1 = 256, \lambda_1 = 80, \lambda_2 = 16, \beta_0 = 1, \beta_1 = 2, \beta_2 = 5 \) and \( l = 2 \). AIRM of size \( 432 \times 256 \) given in Fig. 5 can be used as an optimal length encoding matrix for this SNI-SUICP.

IV. Discussion

In this paper, we derived the capacity of SNI-SUICP and proposed optimal length coding scheme to achieve the capacity. Some of the interesting directions of future research are as follows:

- The capacity and optimal coding for SNI-SUICP with arbitrary \( U \) and \( D \) is a challenging open problem.
Maleki et al. [11] proved the capacity of X network setting with local connectivity and $ML$ number of messages when the number of source-receiver pairs ($M$) tends to infinity. The capacity of this network is $\frac{2}{L(L+1)}$ per message. However, for finite $M$, the capacity is upper bounded by $\frac{2}{L(L+1)}$, but unknown.

APPENDIX A

Proof of Lemma [7]

*Case (i):* $l$ is even and $k \in C_i$ for $i \in [0 : \lfloor \frac{l}{2} \rfloor]$ or $l$ is odd and $k \in C_i$ for $i \in [0 : \lfloor \frac{l}{2} \rfloor + 1]$. In this case, from the definition of down distance, we have $L(k + d_{\text{down}}(k), k) \in I_{(\lambda_2 \times \lambda_2)}$. Let $k \mod (D + 1 - \lambda_{2i-1}) = c\lambda_{2i} + d$ for some positive integers $c$ and $d$ ($d < \lambda_{2i}$). From Figure 6 we have

\begin{equation}
    d_{\text{down}}(k) = d_1 + d_2 + d_3,
\end{equation}

and

\begin{align}
    d_1 &= (D + 1) - k, \\
    d_2 &= K - D - 1 - \lambda_{2i}, \\
    d_3 &= k - (D + 1 - \lambda_{2i-1}) - c\lambda_{2i}.
\end{align}
By using (12) and (13), we have

\[ d_{\text{down}}(k) = d_1 + d_2 + d_3 \]

\[ = (D + 1) - k + K - D - 1 - \lambda_{2i} \]

\[ + k - (D + 1 - \lambda_{2i-1}) - c\lambda_{2i} \]

\[ = K - D - 1 + \lambda_{2i-1} - (c + 1)\lambda_{2i}. \tag{14} \]

By replacing \( \lambda_{2i-1} \) with \( \beta_{2i}\lambda_{2i} + \lambda_{2i+1} \) in (14), we get

\[ d_{\text{down}}(k) = K - D - 1 + \lambda_{2i+1} + (\beta_{2i} - 1 - c)\lambda_{2i}. \]

Case (ii): \( l \) is odd and \( k \in C_l \). In this case, from the definition of down distance, we have

\[ L(k + d_{\text{down}}(k), k) \in I_{\beta_1\lambda_1 \times \lambda_1}. \]

From Figure 7, we have

\[ d_{\text{down}}(k) = d_1 + d_2 + d_3, \tag{15} \]

and

\[ d_1 = (D + 1) - k, \]

\[ d_2 = K - D - 1 - \beta_1\lambda_1, \]

\[ d_3 = \beta_1\lambda_1 - d_3. \tag{16} \]

We have \( L(k, k) \in I_{D+1} \) and \( L(k + d_{\text{down}}(k), k) \in I_{\lambda_1} \) of \( I_{\beta_1\lambda_1 \times \lambda_1} \), as shown in Figure 7. Hence, we have \( d_1 = d_4 \) and \( d_4 = d_5 \). By using (15) and (16), we have

\[ d_{\text{down}}(k) = d_1 + d_2 + d_3 \]

\[ = d_1 + K - D - 1 - \beta_1\lambda_1 + \beta_1\lambda_1 - d_3 = K - D - 1. \]

For \( i \in \left[ \frac{1}{2} \right] \), we have \( \lambda_{2i} \left[ \frac{1}{2} \right] = \lambda_{2i+1} \left[ \frac{1}{2} \right] = 0 \). We can write \( d_{\text{down}}(k) = K - D - 1 + \lambda_{2i+1} + (\beta_{2i} - 1 - c)\lambda_{2i} \).

Hence

\[ d_{\text{down}}(k) = K - D - 1 + \lambda_{2i+1} + (\beta_{2i} - 1 - c)\lambda_{2i}. \]

\[ \text{APPENDIX B} \]

It turns out that the interval \( \tilde{C}_i \) defined in (10) for \( i \in [0 : \left[ \frac{1}{2} \right] \] needs to be partitioned into two as \( \tilde{C}_i = \tilde{D}_i \cup \tilde{E}_i \) as given below to prove the main result Theorem 4. Let

\[ \tilde{D}_i = [K - \lambda_{2i-1} : K - \lambda_{2i-1} + (\beta_{2i-1} - 1)\lambda_{2i} - 1] \tag{17} \]

\[ \tilde{E}_i = [K - \lambda_{2i-1} + (\beta_{2i-1} - 1)\lambda_{2i} : K - \lambda_{2i+1} - 1]. \tag{18} \]

for \( i \in [0 : \left[ \frac{1}{2} \right] \].

\[ \text{Proof of Theorem 1} \]

A scalar linear index code of length \( D + 1 \) generated by an AIR matrix of size \( K \times (D + 1) \) is given by

\[ [c_0 \ c_1 \ ... \ c_D] = [x_0 \ x_1 \ ... \ x_{K-1}]L = \sum_{k=0}^{K-1} x_k L_k \tag{19} \]
Fig. 6. Maximum-down distance calculation

Fig. 7. Maximum-down distance calculation
where $L_k$ is the $k$th row of $L$ for $k \in [0 : K - 1]$. We prove that for $k \in [0 : K - 1]$, every receiver $R_k$ decodes its wanted message $x_k$ by using $[c_0 \ c_1 \ldots \ c_D]$ and its side-information.

Case (i): $k \in [0 : \lfloor \frac{c}{2} \rfloor - 1]$. From Lemma 2, we have $c_k = x_k + x_{k+D} + 1$. In $c_k$, the message symbol $x_{k+D} + 1$ is in the side-information of receiver $R_k$. Hence, $R_k$ can decode its wanted message symbol $x_k$ from $c_k$.

If $K - D - 1 < \lfloor \frac{c}{2} \rfloor$, the broadcast symbol $c_k$ is given by $c_k = x_k + x_{k+D} + 1$. In $c_k$, the message symbol $x_{k+D} + 1$ is in the side-information of receiver $R_k$. Hence, $R_k$ can decode its wanted message symbol $x_k$ from $c_k$.

If $K - D - 1 \geq \lfloor \frac{c}{2} \rfloor$, we show that $R_k$ can decode $x_k$ from $c_k \mod (D + 1)$. In this case, from (6), we have $\beta_0 = 0$ and $\lambda_1 = D + 1$.

If $k \leq D (k \mod (D + 1) = k)$, From Lemma 2 we have
\[
d_{\text{up}}(k + D + 1, k) = D + 1.
\]

This indicates that $x_{k+1}, x_{k+2}, \ldots, x_{k+D}$ are not present in $c_k$.

From Lemma 1 we have
\[
d_{\text{down}}(k) = K - D - 1 + \lambda_{2i+1} + (\beta_{2i} - 1 - c)\lambda_{2i} \\
\leq K - \lambda_1 = K - \gcd(K, D + 1).
\]

This indicates that $x_{k-gcd(K,D+1)}, \ldots, x_{k-1}$ are not present in $c_k$. Hence, every message symbol in $c_k$ is in the side-information of $R_k$ excluding the message symbol $x_k$ and $R_k$ can decode $x_k$.

If $k \in [D + 1 : \lambda_0 - 1]$, From Lemma 2 we have
\[
d_{\text{up}}(k, k \mod (D + 1)) = d_{\text{up}}(k + D + 1, k \mod (D + 1)) = D + 1.
\]

Hence, $c_k \mod (D + 1)$ does not contain message symbols from the set $\{x_{k-D}, \ldots, x_{k-1}\} \cup \{x_{k+1}, \ldots, x_{k+D}\}$ and $R_k$ can decode $x_k$ from $c_k \mod (D + 1)$.

Case (ii): $k \in \hat{D}_i$ for $i \in [0 : \lfloor \frac{c}{2} \rfloor]$. Let $k' = k - \lambda_0$. In this case, we have $k'_R \in [0 : (\beta_{2i} - 1)\lambda_{2i} - 1]$, where $k'_R = c\lambda_{2i} + d$ for some positive integers $c$ and $d$ and $d \leq \lambda_{2i}$. From Lemma 3 we have $\mu_{k'} = \lambda_{2i}$, from Definition 1 we have $t_{k',r} = 0$ for $r \in [1 : p_k]$. From Lemma 4 we have
\[
d_{\text{down}}(k') = K - D - 1 + \lambda_{2i+1} + (\beta_{2i} - 1 - c)\lambda_{2i} \\
= K - D - 1 + \lambda_{2i+1} - (c + 1)\lambda_{2i}.
\]

From Lemma 2 we have
\[
d_{\text{up}}(k', d_{\text{down}}(k'), k') = \lambda_{2i-1} - c\lambda_{2i} \\
d_{\text{up}}(k', d_{\text{down}}(k'), k' + \mu_{k'}) = \lambda_{2i-1} - (c + 1)\lambda_{2i}.
\]

From (22) and (23)
\[
d_{\text{down}}(k') - d_{\text{up}}(k', d_{\text{down}}(k'), k' + \mu_{k'}) = K - D - 1.
\]

This indicates that $x_k$ is present in the code symbol $c_{k'} + \mu_{k'}$ and among $D$ interfering messages after $x_k$ ($x_{k+1}, x_{k+2}, \ldots, x_{k+D}$), only $x_{k'} + d_{\text{down}}(k')$ is present in $c_{k'} + \mu_{k'}$. Fig. 8 and 9 illustrate this. From (23),
\[
d_{\text{up}}(k', d_{\text{down}}(k'), k') = \lambda_{2i} \geq \gcd(K, D + 1).
\]

This along with (24) indicates that every message symbol in $c_{k'}$ is in the side-information of $R_k$ except $x_{k' + d_{\text{down}}(k')}$. Fig. 8 and 9 illustrate this. Hence, every message symbol in $c_{k'} + \mu_{k'}$ is in the side-information of $R_k$ and $R_k$ decodes $x_k$.

Case (iii): $k \in \hat{E}_i$ for $i \in [0 : \lfloor \frac{c}{2} \rfloor - 1]$. Let $k' = k - \lambda_0$. In this case, we have $k'_R \in (\beta_{2i} - 1)\lambda_{2i}$. From (24), we have $k'_R = (\beta_{2i} - 1)\lambda_{2i} + c\lambda_{2i+1} + d$ for some positive integers $d$ and $d < \lambda_{2i+1}$. We have $k' = D - 1 + \lambda_{2i+1} + k'_R$. From Lemma 1 we have
\[
d_{\text{down}}(k') = K - D - 1 + \lambda_{2i+1}.
\]

From Lemma 5 we have
\[
\mu_{k'} = d_{\text{right}}(k' + d_{\text{down}}(k'), k') \\
= d_{\text{right}}(D - 1 + \lambda_{2i+1} + k'_R) \\
= d_{\text{right}}(K - \lambda_{2i} + c\lambda_{2i+1} + d, k') \\
= \lambda_{2i} + c\lambda_{2i+1} + d.
\]

From Lemma 2 we have
\[
d_{\text{up}}(k' + d_{\text{down}}(k'), k' + \mu_{k'}) = \lambda_{2i+1}.
\]

From (22) and (23)
\[
d_{\text{down}}(k') - d_{\text{up}}(k' + d_{\text{down}}(k'), k') = K - D - 1.
\]

This indicates that $x_k$ is present in the code symbol $c_{k'} + \mu_{k'}$ and among $D$ interfering messages after $x_k$ ($x_{k+1}, x_{k+2}, \ldots, x_{k+D}$), the interfering messages $x_{k' + d_{\text{down}}(k')}$ and $x_{k' + t_{k',r} + d_{\text{down}}(k')}$ for $r \in [1 : p_k]$ are present in $c_{k'} + \mu_{k'}$. Fig. 10 is useful to understand this.

From Lemma 1 and Definition 1 $k' + d_{\text{down}}(k') + t_{k',r}$ is always less than the number of rows in the matrix $L$. That is, $k' + t_{k',r} + d_{\text{down}}(k') < K$. Hence, we have
\[
t_{k',r} < K - k' - d_{\text{down}}(k') \\
= K - (D - 1 + \lambda_{2i+1} + (\beta_{2i} - 1)\lambda_{2i} + c\lambda_{2i+1} + d) - (K - D - 1 + \lambda_{2i+1}) \\
= \lambda_{2i} - c\lambda_{2i+1} - d
\]

From (22) and (30)
\[
t_{k',r} < \mu_{k'} - d.
\]

From (30), we have
\[
k'_R + t_{k',r} < k'_R + \lambda_{2i} - c\lambda_{2i+1} - d \\
= (\beta_{2i} - 1)\lambda_{2i} + c\lambda_{2i+1} + d + \lambda_{2i} - c\lambda_{2i+1} - d \\
= \beta_{2i}\lambda_{2i}.
\]

Hence,
\[
k'_R + t_{k',r} \in [(\beta_{2i} - 1)\lambda_{2i}, \beta_{2i}\lambda_{2i} - 1].
\]
and
\[ \mathbf{L}(k', k') = \mathbf{L}(k' + t_{k', r} + d_{down}(k' + t_{k', r}), k' + t_{k', r}) \in \mathbf{I}_{\lambda_2l, \beta_2l, \lambda_2i} \]  
for \( r \in [1 : p_{k'}]. \) We have
\[ d_{down}(k') = d_{down}(k' + t_{k', r}) \]  
for \( r \in [1 : p_{k'}]. \)

From Lemma 2 for \( \mathbf{L}(k' + t_{k', r} + d_{down}(k' + t_{k', r}), k' + t_{k', r}) \) for \( i \in [0 : \left\lfloor \frac{1}{2} \right\rfloor] \)
\[ d_{up}(k' + t_{k', r} + d_{down}(k' + t_{k', r}), k' + t_{k', r}) = \lambda_2i + \lambda_2i+1. \]  
From (28) and (34), we have
\[ d_{up}(k' + t_{k', r} + d_{down}(k' + t_{k', r}), k' + t_{k', r}) = \lambda_2i \geq gcd(K, D + 1), \] (35)
\[ d_{up}(k' + d_{down}(k'), k' + \mu k') = \lambda_2i \geq gcd(K, D + 1), \] (36)
this along with (29) indicates that every message symbol in \( \mathbf{c}_{k'} \) is in the side-information of \( \mathbf{R}_k \) except \( x_{k' + d_{down}(k')}. \)

From (29), (35) and (37), the interfering message symbol \( x_{k' + t_{k', r} + d_{down}(k')} \) can be canceled by adding the index code symbol \( c_{k'} + d_{down}(k') \) for \( r \in [1 : p_{k'}]. \)

Hence, receiver \( \mathbf{R}_k \) decodes the message symbol \( x_k \) by adding the index code symbols \( c_{k'}, c_{k'} + \mu k', \) and \( c_{k'} + t_{k', r} \) for \( r \in [1 : p_{k'}]. \)

Case (iv): \( k \in [K - \lambda_l : K - 1] = \tilde{E}_i \) for \( i = \left\lfloor \frac{i}{2} \right\rfloor \). Let \( k' = k - \lambda_0 \). In this case, from Lemma 1 we have
\[ d_{down}(k') = K - D - 1 = \lambda_0. \] (38)

This indicates that \( x_k \) is present in \( c_{k'} \) and \( x_{k+1}, x_{k+2}, \ldots, x_{k+D} \) are not present in \( c_{k'}. \) From Lemma 3 we have
\[ d_{up}(k) = \lambda_l = gcd(K, D + 1). \]

This indicates that \( x_{k - gcd(K, D + 1) + 1}, \ldots, x_{k-1} \) are not present in \( c_{k'}. \) Hence, every message symbol in \( c_{k'} \) is in the side-information of \( \mathbf{R}_k \) excluding the message symbol \( x_k \) and \( \mathbf{R}_k \) can decode \( x_k. \) From (10) and (17), case (i), case (ii), case (iii) and case (iv) span \( k \in [0 : K - 1]. \) This completes the proof.
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