Cobweb posets as noncommutative prefabs

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Summary
A class of new type graded infinite posets with minimal element is introduced. These so called cobweb posets proposed recently by the present author constitute a wide range of new noncommutative and nonassociative prefab combinatorial schemes' examples with characteristic graded sub-posets as primes. These schemes are defined here via relaxing commutativity and associativity requirements imposed on the composition in prefabs by the fathers of this fertile concept. The construction and the very first basic properties of cobweb prefabs are disclosed. An another new type prefab example with single valued commutative and associative composition is provided. "En passant" though not by accident - we discover new combinatorial interpretation of all classical \( F \)-nomial coefficients hence specifically incidence coefficients of reduced incidence algebras of full binomial type are given a new cobweb combinatorial interpretation also.

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1 Introduction
The concept of prefab (with associative and commutative composition) was introduced in [1], see also [2,3]. Here we shall deliver a class of similar combinatorial structure of new type based on the so called cobweb posets. For the sake of completeness we recall in Section 1. the definition of a cobweb poset as well as a combinatorial interpretation of its characteristic binomial-type coefficients (for example- fibonomial ones) [4,5].

In Section 2. after relaxing associativity and commutativity requirements imposed on the composition in prefabs by the authors of this concept [1] we observe that the vast family of all cobweb posets becomes by construction a new type of nonassociative noncommutative prefabs' subclass. The very first basic properties of these cobweb prefabs are shown up. As a result a class of new type of graded infinite posets with minimal element are employed here as an enveloping framework for the completely new class of combinatorial prefab structures with noncommutative and nonassociative composition (synthesis) of its objects since now on called prefabiants. Cobweb infinite posets \( P \) are designated uniquely by any cobweb admissible sequence of integers \( F = \{n_F\}_{n \geq 0} \) and are by construction endowed with self-similarity property. Namely at each graded level vertex a family of infinite cobweb sub-posets isomorphic to \( P \) may be rooted.

The number of finite characteristic sub-posets (prime prefabiants) of \( P \) in between levels of graded Hasse digraph of \( P \) is given by \( F \)-nomial coefficients. These include: incidence coefficients such as binomial or \( q \)-Gaussian ones for finite geometries or fibonomial coefficients [4,5] which are not incidence coefficients. Here these \( F \)-nomial numbers are introduced also via \( c_2 \) axiom in the Definition 1 from [1]:

\[
|a \circ b| = \frac{f(a \circ b)}{f(a)f(b)} = \binom{n}{k}_F .
\]

We notice with emphasis and not only occasionally, that \( c_2 \) axiom in Definition 1 from [1] is equivalent to the fundamental Theorem 1 from [1]. More then that - in Section 3 we shall see that all objects from Equation 1 gain specific uniform...
2 Cobweb posets - presentation and their combinatorial interpretation

Given any sequence \( \{F_n\}_{n \geq 0} \) of nonzero reals one defines its corresponding binominal-like \( F - nomial \) coefficients in the spirit of Ward’s Calculus of sequences \( [6] \)(reals may be replaced for example by any field of characteristic zero) as follows

**Definition 1**

\[
\binom{n}{k}_F = \frac{F_n!}{F_k!F_{n-k}!} \equiv \frac{n_F^k}{k_F!} \quad n_F \equiv F_n \neq 0, n \geq 0
\]

where we make an analogy driven identifications in the spirit of Ward’s Calculus of sequences (0\( F \equiv 0 \)):

\[n_F! \equiv n_F(n-1)_F(n-2)_F(n-3)_F \ldots 2_F 1_F;\]

\[0_F! = 1; \quad n_F^k \equiv n_F(n-1)_F(n-2)_F \ldots (n-k+1)_F.\]

This is just the adaptation of the notation for the purpose Fibonomial Calculus case (see Example 2.1 in \([7]\)). Given any such sequence \( \{F_n\}_{n \geq 0} \equiv \{n_F\}_{n \geq 0} \) of now nonzero integers we define following \([4,5]\) the partially ordered graded infinite set \( P \) - called afterwards a cobweb poset - as follows. Its vertices are labelled by pairs of co-ordinates: \( \langle i,j \rangle \in N \times N_0 \) where \( N_0 \) denotes the nonnegative integers. Vertices show up in layers (“generations”) of \( N \times N_0 \) grid along the recurrently emerging subsequent \( s - th \) levels \( \Phi_s \) where \( s \in N_0 \) i.e.

**Definition 2**

\[\Phi_s = \{\langle j,s \rangle 1 \leq j \leq s_F \}, s \in N_0.\]

We shall refer to \( \Phi_s \) as to the set of vertices at the \( s - th \) level. The population of the \( k - th \) level (“generation”) counts \( k_F \) different member vertices for \( k > 0 \) and one for \( k = 0 \). Here down a disposal of vertices on \( \Phi_k \) levels is visualized for the case of Fibonacci sequence (the subtlety of \( F_0 = 0 \) is manageable)

![Figure 1. The s-th levels in N x N0, N0 - nonnegative integers](image)

**Figure 1.** The \( s - th \) levels in \( N \times N_0 \), \( N_0 \) - nonnegative integers
Accompanying the set $E$ of edges to the set $V$ of vertices - we obtain the Hasse diagram where here down $p, q, s \in N_0$. (Convention: Edges stay for arrows directed - say - upwards) Namely:

**Definition 3**

$$P = (V, E), \quad V = \bigcup_{0 \leq p} \Phi_p, \quad E = \{\{(j, p), (q, (p + 1))\}\} \bigcup \{\{(1, 0), (1, 1)\}\},$$

where $1 \leq j \leq p_F, 1 \leq q \leq (p + 1)_F$.

**Definition 4** The finite cobweb sub-poset $P_m = \bigcup_{0 \leq s \leq m} \Phi_s$ is called the prime cobweb poset.

In reference [3,4] a partially ordered infinite set $P$ was introduced via descriptive picture of its Hasse diagram. Indeed, we may picture out the partially ordered infinite set $P$ from the Definition 3 with help of the sub-poset $P_m$ (rooted at $F_0$ level of the poset) to be continued then ad infinitum in now obvious way as seen from the figures Fig.1 – Fig.5 of $P_m$ cobweb posets below. These look like the Fibonacci rabbits’ way generated tree with a specific “cobweb”[4,5,8]. This is an example of acyclic directed graphs (DAG) [9] cobweb subclass.

![Diagram](image-url)
Fig. 2. Display of Even Natural numbers' cobweb poset.

Fig. 4. Display of divisible by 3 natural numbers' cobweb poset.
Fig. 5. Display of Fibonacci numbers' cobweb poset.

Compare with the bottom 6 levels of a Young-Fibonacci lattice, introduced by Richard Stanley in Curtis Greene's gallery of posets: www.haverford.edu/math/cgreene/posets/posetgallery.html.

As seen above - for example the Fig. 5. displays the rule of the construction of the Fibonacci "cobweb" poset. It is being visualized clearly while defining this cobweb poset \( P \) with help of its incidence matrix. The incidence \( \zeta \) function matrix representing uniquely just this cobweb poset \( P \) has the staircase structure correspondent with "cobwebbed" Fibonacci Tree i.e. a Hasse diagram of the particular partial order relation under consideration. This is seen below on the Fig. 6 [8]:

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{bmatrix}
\]

Figure 6. The staircase structure of incidence matrix \( \zeta \) for the Fibonacci cobweb poset case
Note The knowledge of $\zeta$ matrix explicit form enables one to construct (count) via standard algorithms [10] the Möbius matrix $\mu = \zeta^{-1}$ and other typical elements of incidence algebra perfectly suitable for calculating number of chains, of maximal chains etc. in finite sub-posets of $P$. All elements of the corresponding incidence algebra are then given by a matrix of the Fig.5 with 1’s replaced by any reals (or ring elements in more general cases).

Right from the definition of $P$ via its Hasse diagram here now follow quite obvious and important observations. They lead us to a combinatorial interpretation of cobweb poset’s characteristic binomial-like coefficients (for example - fibonacci ones [4,5]). Here they are with the first obvious observation at the start.

Observation 1
The number of maximal chains starting from The Root (level 0) to reach any point at the $n$–th level with $n_F$ vertices is equal to $n_F!$.

Observation 2 $(k > 0)$
The number of maximal chains rooted in any vertex at the $k$–th level reaching the $n$–th level with $n_F$ vertices is equal to $n_k^{m_F}$, where $m + k = n$.

Indeed. Denote the number of ways to get along maximal chains from any point in $\Phi_k$ to $\Rightarrow \Phi_n, n > k$ with the symbol $[\Phi_k \Rightarrow \Phi_n]$ then obviously we have:

$$[\Phi_0 \Rightarrow \Phi_n] = n_F!$$

and

$$[\Phi_0 \Rightarrow \Phi_k] \times [\Phi_k \Rightarrow \Phi_n] = [\Phi_0 \Rightarrow \Phi_n].$$

In order to formulate the combinatorial interpretation of $F$–sequence–nomial coefficients ($F$-nomial - in short) [6,4,5,8] let us consider all finite "max-disjoint" sub-posets rooted at the $k$–th level at any fixed vertex $(r,k), 1 \leq r \leq k_F$ and ending at corresponding number of vertices at the $n$–th level $(n = k + m)$ where the "max-disjoint" sub-posets are defined below.

Definition 5 Two isomorphic copies of $P_m$ are said to be max-disjoint if being considered as sets of maximal chains they are disjoint i.e they have no maximal chain in common. All of $P_m$’s constitute from now on a family of prime [1] prefabiants.

Definition 6 We denote the number of all max-disjoint isomorphic copies of $P_m$ rooted at any vertex $(j,k), 1 \leq j \leq k_F$ of $k$–th level with the symbol

$$\binom{n}{k}_F.$$

We use the accustomed to practical convention: $\binom{0}{0}_F = 1$.

Naturally the above definition make sense not for arbitrary $F$ sequences as $F$–nomial coefficients should be nonnegative integers.

Definition 7 A sequence $F = \{n_F\}_{n \geq 0}$ is called cobweb-admissible iff

$$\binom{n}{k}_F \in N \cup \{0\} \quad \text{for} \quad k, n \in N \cup \{0\} \equiv Z_\geq.$$
Recall now that the number of ways to reach an upper level from a lower one along any of maximal chains i.e. the number of all maximal chains from the level $\Phi_k$ to $\Phi_n$, $n > k$ is equal to

$$[\Phi_k \to \Phi_n] = \frac{n^m}{m!}.$$

Naturally then we have

$$(2) \quad \left( \begin{array}{c} n \\ k \end{array} \right)_F \times [\Phi_0 \to \Phi_m] = [\Phi_k \to \Phi_n] = \frac{n^m}{m!}$$

where $[\Phi_0 \to \Phi_m] = mF!$ counts the number of maximal chains in any copy of the $P_m$.

With this in mind we see that the following holds.

**Observation 3 (n,k ≥ 0)**

Let $n = k + m$. The number of max-disjoint sub-posets isomorphic to $P_m$ (max-disjoint isomorphic copies of prime prefabiants) rooted at the $k$-th level and ending at the $n$-th level is equal to

$$\frac{n^m}{m!} = \left( \begin{array}{c} n \\ m \end{array} \right)_F$$

$$= \left( \begin{array}{c} n \\ k \end{array} \right)_F = \frac{n^m}{kF!}.$$

**Note** The Observation 3 provides us with the new combinatorial interpretation of the class of all classical $F$-nomial coefficients including distinguished binomial or distinguished Gauss $q$-binomial ones or Konvalina generalized binomial coefficients of the first and of the second kind [11]- which include Stirling numbers too. The vast family of Ward-like [6] admissible by $\psi = \frac{1}{n^m}, n \geq 0$-extensions $F$-sequences [7,12] includes also those desired here which shall be called ”GCD-morphic” sequences. This means that $GCD[F_n, F_m] = F_{GCD[n,m]}$ where $GCD$ stays for Greatest Common Divisor operator. The Fibonacci sequence is a much nontrivial and guiding famous example of GCD-morphic sequence. Naturally incidence coefficients of any reduced incidence algebra of full binomial type [13] are GCD-morphic sequences therefore they are now independently given a new cobweb combinatorial interpretation via Observation 3. More on that - see the next section where prefab combinatorial description is being served. Before that - on the way - let us formulate the following problem (open?).

**Problem 1** Find effective characterizations of the cobweb admissible sequence i.e. find all examples.

### 3 Cobweb posets as prefabs with nonassociative and noncommutative composition

Finite cobweb sub-posets i.e. isomorphic copies of $P_m, m \geq 0$ constitute connected acyclic digraphs as well as the Hasse diagram of the infinite cobweb poset $P$ is. Directed acyclic graphs are denoted as DAG’s [9]. Hence one might call connected DAG’s - directed trees. As for the recent development on acyclic digraphs we refer to [14] and references therein. As in [14] one considers here digraphs on labeled vertices and a ”digraph” means a simple graph with at most one edge directed from vertex to vertex. Loops and cycles of length two are permitted in general, but parallel edges are forbidden. ”Acyclic” means that there are no cycles of any length. Apart from Theorem 1 there note in [14] also Bibliographic remarks on acyclic digraphs referring
to Robinson and Stanley and then to Bender et al. and Gessel. Because of an easy
access to Plotnikov's paper [9] we shall take other definitions from there - if needed -
for granted. These are temporarily used just for the guiding observation relating cob-
web prefabs' digraphs to [9]. Namely - in terminology of [9] - we make rather obvious
observation.

**Observation 4** The Hasse (here upward oriented) diagram of any prime cobweb poset
or P is an oDAG.

For the sake of explanation we quote after [9]: A poset P is of the dimension 2;
dim P = 2 if there exist two chains L1 and L2 such that P = L1 \cap L2.
A digraph G is called the orderable digraph (oDAG) if there exists a dim 2 poset such
that its Hasse diagram coincides with the digraph G.

We shall pass over now to the brief presentations of a cobweb prefab combinatorial
structure [1] in which each object (prefabiant) is uniquely representable by construc-
tion as a synthesis (composition) of powers of prime objects where here these are the
cobweb sub-posets Pn of P that are to be identified with prime prefabiants. Since now
on we shall adhere to the notation and terminology of [1]. We assume the acquaintance
of [1] which is justly considered as famous as important.
The definition of prefab combinatorial structure (S, \( \circ \), f) here is assumed to be given
by Definition 1 from [1] except for associativity requirement a1 and commutativity
requirement a2), which are postponed until stated otherwise. In general a \( \circ \) b ≠
b \( \circ \) a, a, b ∈ S - already for prime objects. The definition of weighted (not necessarily
associative, commutative) prefab and enumerator g(A), A ⊆ S are then Definitions 2
and 3 from [1] correspondingly. We shall now formulate Observation 5 (to be checked
by careful examination)- observation of distinguished importance for the combinatorial
interpretation of the property c2) from the Definition 1 [1] of the prefab. The
property c2) postulate from [1] is

\[ |a \circ b| = \frac{f(a \circ b)}{f(a)f(b)} \quad a, b \in S, \]

whenever a and b have no common factor different from identity prefabiant i [1] where
here |A| denotes here the number of max-disjoint isomorphic copies of prime prefabiants
in the set A = a \( \circ \) b. The function f satisfies the requirement c1 - of course.

**Observation 5** Let the enumerator or generating function for prefab subsets be defined
as indicated above. Then the set of requirements Prefab_{c(2)} = \{a_3, b_1, b_2, c_1, c_2\} is
equivalent to set of requirements Prefab_{(Tn,1)} = \{a_3, b_1, b_2, c_1, Theorem.1) , where
Theorem 1 means Theorem 1 from [1].

Both sets of requirements define on S the same prefab structure (not necessarily commu-
tative and associative ) where requirements b1), b2) are to be understood as rewritten
in an order and brackets being taken into account fashion.

Now comes the example of the class of weighted prefabs (S, \( \circ \), f, \( \omega \)) with noncommu-
tative, nonassociative synthesis (composition) \( \circ \). We shall call this binary multivalued
operation' [1] analogue case here a "coopt-synthesis" ⊙.

**Cobweb prefab combinatorial structure.** The family S of combinatorial objects
(prefabiants) consists of all layers (\( \Phi_k \rightarrow \Phi_n \)), \( k < n \), \( k, n \in N \cup \{0\} \equiv Z_\geq \)
and an empty prefabiant i. Layer is considered here to be the set of all max-disjoint
isomorphic copies (iso-copies) of P_{n-k}. The set \( \varphi \) of prime objects consists of all sub-
posets (\( \Phi_0 \rightarrow \Phi_m \)) i.e. all P_m’s m \( \in N \cup \{0\} \equiv Z_\geq \) constitute from now on a family
of prime prefabiants [1]. The Z_\geq grading preserving \( \circ \) coopt-synthesis for prime
prefabiants P_k \( \circ \) P_m = (\( \Phi_k \rightarrow \Phi_m \)), n = k + m means: consider the leaves of P_k to be
the roots of all max-disjoint isomorphic copies (iso-copies) of P_m. Run through all the
leaves (now - roots). The Z_\geq grading preserving \( \circ \) synthesis of (not necessarily prime)
prefabiants - accordingly means the same procedure with the requirement added (see: Example 5 in [1]). If this algorithm applied to subsequent prime elements of the second prefab gives rise to a layer of max-disjoint prefabs more then one way - keep only one copy of it. As a result we have:

\[(\Phi_k \rightarrow \Phi_n) \odot P_s = (\Phi_n \rightarrow \Phi_{n+s}), \quad k \in Z, s \in N, n > k.\]

Accordingly the \(Z \times Z\) grading of \(S\) preserving \(\odot\) synthesis (\(\odot\) coopt-synthesis) is defined for arbitrary elements of \(S\) as simply as follows:

\[(\Phi_k \rightarrow \Phi_n) \odot (\Phi_t \rightarrow \Phi_{t+s}) = (\Phi_n \rightarrow \Phi_{n+s}); \quad t, k \in Z, n > k, s > 0.\]

In order to satisfy the requirement \(a_3\) we postulate for an empty prefabiant \(i\) that

\[(\Phi_k \rightarrow \Phi_n) \odot i = i \odot (\Phi_k \rightarrow \Phi_n) = (\Phi_k \rightarrow \Phi_n), \quad k \in Z, n > k.\]

The appropriately adjusted requirements \(b_1), b_2\) are satisfied by construction as

\[(\Phi_k \rightarrow \Phi_n) = P_k \odot P_{n-k}, \quad k \in Z, n > k.\]

The coopt-synthesis \(\odot\) is nonassociative by construction as

\[((\Phi_k \rightarrow \Phi_n) \odot (\Phi_t \rightarrow \Phi_{t+s})) \odot (\Phi_p \rightarrow \Phi_{p+q}) = P_{n+s} \odot P_{q}; \quad p, t, k \in Z, n > k, s \geq 0, q \geq 0\]

while

\[(\Phi_k \rightarrow \Phi_n) \odot ((\Phi_t \rightarrow \Phi_{t+s}) \odot (\Phi_p \rightarrow \Phi_{p+q})) = P_n \odot P_{q}; \quad p, t, k \in Z, n > k, s > 0, q > 0.\]

As for the size functions let us aid with an analogy (see: Example 5 in [1]).

**Analogy:**

**Graphs**........................**Cobweb - prefabs** I..............**Cobweb - prefabs** II

*vertices*........................**max - chains**..............................**leafs of** \(P_n\)’s

*connected \(G_n\) on \([n]\) .........\(P_n\) cobweb..............................\(P_n\) cobweb

\[\text{size}(G_n) = n..............................\text{size}_1(P_n) = n!..............................\text{size}_2(P_n) = n\]

\[f(G_n) = n!..............................f(P_n) = n!..............................f(P_n) = n! .\]

Recall now the Equation 3. We may draw now from all the above the following conclusion.

**Conclusion I**

In the finite cobweb posets setting the \(f\) function may be chosen so as to be the \(size_1\) of a prime prefabiant with \(n_F\) leafs or so as to be factorial of the \(size_2\) of a prime prefab with \(n_F\) leafs. This gives:

\[f(P_n) = n_F!, \quad f(P_n^{km}) = (km)!\]

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and the Equation 1 gets the required, expected combinatorial interpretation for any
cobweb prefab structure determined by the choice of any sequence of natural numbers
from the countless family of cobweb admissible sequences. Thus we are equipped with
the cobweb prefab’s uniform combinatorial interpretation of all of them at once.

Conclusion II
Naturally the Corollary 1 from [1] also holds in our case. Choosing now the weight
function to be of the form
\[ \omega(a) = x^n, \quad n = \text{size}_2(a), \quad a \in S \]
we have the weighted cobweb prefab and consequently (see: Examples 5,10 in [1]) the
formula for the cobweb weighted prefab enumerator reads
\[ g(S) = \exp\{g(\wp)\}, \]
where
\[ g(\wp) = \exp_F\{x\} - 1, \]
while
\[ \exp_F\{x\} = \sum_{n \geq 0} \frac{x^n}{n!} \]
\[ \exp_F \] function [6,12,7,15] - is the primary object of extended finite operator calculus
being recently developed in [7,12,15]. There it serves to define a central object of
extended umbral calculus i.e. the generalized translation operator \( E^a(\partial_F) \) where the
linear difference operator \( \partial_F : \partial_F x^n = n_F x^{n-1}; \quad n \geq 0 \) is known under the name of the \( F \)-derivative [6,7,14,15]. here comes the example (11) from [1] interpreted in the
language of \( \psi \) - extensions [6] in their operator form [12,15,7].

Bender - Goldman - prefab example Let the ”prefabian” \( \hat{q} \)-Bell numbers \( B^{\text{pref}}_{\gamma} \)
be defined as sums over \( k \) of \( \hat{S}_{\gamma}(n,k) \) Stirling numbers equal of the number of unordered
direct sums decompositions of the \( n \)-dimensional vector space \( V_{\hat{q},n} \) over \( GF(q) \equiv F_q \)
with \( k \) summands. Then the Bender-Goldman exponential formula (17 ) from [1] in
\( \psi \)-extensions' notation [12,15,7] reads
\[ B^{\text{pref}}_{\gamma}(x) = \sum_{n \geq 0} B^{\text{pref}}_{\gamma} \frac{x^n}{n!} = \exp\{\exp_{\gamma}(x) - 1\}. \quad (\gamma - \text{e.g.f.}) \]
Here
\[ n_{\gamma}! = (q^n - 1)(q^n - q^1)…(q^n - q^{n-1}) = |GL_n(F_q)|, \]
\( D_0(q) = 1 \) by convention while \( D_n(q) \equiv B^{\text{pref}}_{\gamma} \) = number of all unordered direct
sums decompositions of the vector space \( V_{\hat{q},n} \).

The natural hint
The appealing analogy of the above schema and example just presented give rise to
questions on their eventual correspondents as \( \text{Stirling}_F \) numbers of the second kind,
\( \exp_F \)-polynomial polynomials and \( F\)-Dobinski like formulas. Such extensions are more
or less implicit in some papers . for example - see Wagner’s (1.15) formula in [16] which
formally becomes of the \( (\gamma - \text{e.g.f.}) \) formula form from above with now almost arbitrary
\( \gamma = \langle \frac{1}{n_1} \rangle_{n \geq 0} \) sequence (see also [17] and references therein). These questions are to
be considered elsewhere. As for the related (determined by \( F - \text{nomial's} \)) extended
umbral calculi in its operator form one may contact also very recent review [18].
4 Cobweb posets as prefabs with associative and commutative composition

Here another single valued commutative and associative composition case is presented in brief. The definition of the next prefab combinatorial structure with the single valued composition \((S, \circ, f)\) is assumed to given here by the Definition 1 from [1] including associativity requirement \(a_1\) and commutativity requirement \(a_2\).

The family \(S\) of combinatorial objects (prefabiants) consists now of all layers \(\langle \Phi_k \rightarrow \Phi_n \rangle\), \(k < n\), \(k, n \in N \cup \{0\} \equiv Z_\geq\) and an empty prefabiant \(i\) to be interpreted as the name or representant of all ”empty layers” \(\langle \Phi_m \rightarrow \Phi_m \rangle\).

The family \(S\) of combinatorial objects \((\Phi_k \rightarrow \Phi_n)\), \(k < n\), \(k, n \in N \cup \{0\} \equiv Z_\geq\) constitutes from now on a family of prime prefabiants [1].

Category III

In this finite cobweb posets setting with associative and commutative composition \(\circ\) being single valued the \(f\) function may be chosen as constant equal to 1 function (note the other possibilities: \(f(\langle \Phi_k \rightarrow \Phi_n \rangle) = \alpha^{n-k}\), \(\alpha \neq 0, n > k\)).

This implies validity of the Corollary 2 and the Corollary 3 from [1] also in the cobweb ”\(\circ\) - case” - providing us with direct efficient analogy to the cases of unlabelled graphs, unordered partitions or factorizations of integers (see Examples 1, 2, 3 in [1]).

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