On bounds for norms of reparameterized ReLU artificial neural network parameters: sums of fractional powers of the Lipschitz norm control the network parameter vector

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Abstract

It is an elementary fact in the scientific literature that the Lipschitz norm of the realization function of a feedforward fully-connected rectified linear unit (ReLU) artificial neural network (ANN) can, up to a multiplicative constant, be bounded from above by sums of powers of the norm of the ANN parameter vector. Roughly speaking, in this work we reveal in the case of shallow ANNs that the converse inequality is also true. More formally, we prove that the norm of the equivalence class of ANN parameter vectors with the same realization function is, up to a multiplicative constant, bounded from above by the sum of powers of the Lipschitz norm of the ANN realization function (with the exponents $1/2$ and 1).

Moreover, we prove that this upper bound only holds when employing the Lipschitz norm but does neither hold for Hölder norms nor for Sobolev-Slobodeckij norms. Furthermore, we prove that this upper bound only holds for sums of powers of the Lipschitz norm with the exponents $1/2$ and 1 but does not hold for the Lipschitz norm alone.

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1 Introduction

In recent years, artificial neural networks (ANNs) have become an extremely powerful tool for tackling a wide variety of complex tasks, such as recognizing natural language, handwritten text, or objects in images, as well as controlling motor vehicles or robotic devices in general. Although gradient descent (GD) optimization schemes have often turned out to be highly performant for training ANNs in practice, it remains a fundamental open problem of research to rigorously prove under which conditions GD optimization schemes converge or diverge. However, there are several promising mathematical analysis approaches in the scientific literature which provide a step in this area of research and prove the convergence of various optimization schemes under suitable assumptions. In the following, we want to briefly outline some of the findings in a selection of these works and we refer to the below mentioned references for details and further references.

One of the most well-known and fundamental results in the field of time-continuous GD optimization methods goes back to Lojasiewicz [14], in which it was shown that a non-divergent solution of a gradient flow (GF) associated to a real analytic risk function (which is often referred to as energy function in the context of GFs) converges to a single limit point. The basic idea is to prove that for real analytic risk functions the so-called Lojasiewicz inequality holds and, using this, to control the length of non-divergent GF trajectories around their limit points (see also Absil et al. [1], Section 2). This argument was extended, for example, in Bolte et al. [6] to a broad class of nonsmooth risk functions by replacing the differential by a subdifferential, so that the convergence of bounded GF trajectories of corresponding subgradient dynamical systems could be shown.

Furthermore, there are several results in the scientific literature which employ Lojasiewicz’s original idea and analyze time-discrete descent methods. In particular, in Attouch & Bolte [2, Theorem 1] it was shown that every bounded sequence generated by a proximal algorithm, applied to a risk function that satisfies the Lojasiewicz inequality around its generalized critical points, converges to a generalized critical point. This abstract convergence result was further extended in Attouch et al. [3, Theorems 3.2, 4.2, 4.3, 5.1, 5.3, 5.6, and 6.2] to achieve various convergence results for bounded sequences of descent methods such as inexact gradient methods, inexact proximal algorithms, forward-backward splitting algorithms, gradient projection methods, and regularized Gauss-Seidel methods satisfying sufficient decrease assumptions and allowing a relative error tolerance. In addition, in Absil et al. [1] there are abstract convergence results for analytic risk functions and non-divergent sequences generated by general time-discrete descent methods.

Several convergence results can be applied to the training of ANNs using GD optimization schemes. Specifically, under suitable assumptions, in the context of training ANNs with finitely many training data, it was shown that every limit point of a bounded sequence generated by the stochastic subgradient method is a critical point of the risk function and that the risk function values converge (see Davis et al. [8, Corollary 5.11]). Moreover, in Dereich & Kassing [9] the convergence of bounded stochastic gradient descent schemes was studied, in particular, in the case of deep ANNs with an analytic activation function, compactly supported input data, and compactly supported output data. In addition, in Jentzen & Riekert [13, Theorem 1.3] (cf. Eberle et al. [11, Theorem 1.2]) it was recently proved that every non-divergent GF trajectory in the training of deep ANNs with rectified linear unit (ReLU) activation, under the assumption that the unnormalized probability density function and the target function are piecewise polynomial, converges with a strictly positive rate of convergence to a generalized critical point in the sense of the limiting Fréchet subdifferential. In the case of constant target functions in the training of deep ANNs with ReLU activation, the boundedness and convergence for stochastic gradient descent (SGD) processes was demonstrated (see Hutzenthaler et al. [12] and the references mentioned therein). We also want to mention results in the area of inertial Breg-
man proximal gradient methods and block coordinate descent methods with a possibly variable metric (cf., e.g., Mukkamala et al. [16], Ochs [17], Xu & Yin [20], and Zeng et al. [21]). For additional references on GD optimization schemes we refer, for example, to the overview articles Bottou et al. [7], E et al. [10], and Ruder [19].

In view of these scientific findings and the frequently made assumption that the sequence of ANN parameter vectors generated by the optimization method is bounded, it is a key contribution of this article to discover a new relationship between norms of ANN parameter vectors and sums of powers of the Lipschitz norm of the ANN realization function. More formally, it is an elementary fact in the scientific literature that the Lipschitz norm of the realization function of a deep rectified linear unit ANN with $L \in \mathbb{N}$ many affine linear transformations can, up to a multiplicative constant, be bounded from above by sums of powers of the norm of the ANN parameter vector with the exponents 1 and $L$ (cf., e.g., Beck et al. [4, Corollary 2.37] and Miyato et al. [15, Section 2.1]). Roughly speaking, in this work we reveal in the case of shallow ANNs that the converse inequality is also true (but with the exponents $1/2$ and 1 instead of 1 and 2). While the inequality that the Lipschitz norm of the realization function of shallow ANNs can be controlled by sums of powers of the norm of the ANN parameter vector is an elementary fact (see Lemma 2.12 below) the converse inequality (see (2) below) is non-trivial with a much more involved proof. To illustrate this converse inequality in a more accurate form, we now present the first main result of our article, Theorem 1.1 below, and we refer to Subsection 2.4 below for more explicit estimates.

![Graphical illustration of the considered shallow ANN architecture in Theorem 1.1 and Theorem 1.3 in the special case of an ANN with $d = 3$ neurons on the input layer and $h = 5$ neurons on the hidden layer. In this situation there are $dh = 15$ real weight parameters and $h = 5$ real bias parameters for the first affine linear transformation from the three-dimensional input layer to the five-dimensional hidden layer and there are $h = 5$ real weight parameters and 1 real bias parameter for the second affine linear transformation from the five-dimensional hidden layer to the one-dimensional output layer. The total number of parameters of this ANN thus satisfies $d = dh + 2h + 1 = 26$. We have that for every ANN parameter vector $\theta = (\theta_1, \ldots, \theta_h) \in \mathbb{R}^h = \mathbb{R}^{26}$ the associated realization function $\mathbb{R}^3 \ni x \mapsto N^\theta(x) \in \mathbb{R}$ maps the three-dimensional input vector $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ to the scalar output $N^\theta(x) = \theta_3 + \sum_{i=1}^5 \theta_{dh+i} + \sum_{j=1}^3 \theta_{(i-1)d+j} x_j, 0 \in \mathbb{R}$.](3)
**Theorem 1.1.** Let $d, h, d \in \mathbb{N}$, $\alpha, \sigma \in \mathbb{R}$, $\theta \in (\alpha, \infty)$ satisfy $\mathcal{D} = d h + 2 h + 1$, for every $\theta = (\theta_1, \ldots, \theta_b) \in \mathbb{R}^b$ let $\mathcal{N}^\theta \in C([\alpha, \theta]^{d}, \mathbb{R})$ satisfy for all $x = (x_1, \ldots, x_d) \in [\alpha, \theta]^{d}$ that

$$\mathcal{N}^\theta(x) = \theta_b + \sum_{i=1}^{b} \theta_{d_h+i} \max\{\theta_{d_h+i} + \sum_{j=1}^{d} \theta_{(i-1)d+j} x_j, 0\},$$

for every $n \in \mathbb{N}$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ let $\|x\| = (\sum_{j=1}^{n} |x_j|^2)^{1/2} \in \mathbb{R}$, let $z \in [\alpha, \theta]^{d}$, and for every $f : [\alpha, \theta]^{d} \rightarrow \mathbb{R}$ let $\|f\| = |f(z)| \sup_{x, y \in [\alpha, \theta]^{d}, x \neq y} |f(x) - f(y)|/\|x - y\| \in [0, \infty]$. Then there exist $\vartheta, \mathcal{C} \in \mathbb{R}$ such that for all $\theta \in \mathbb{R}^b$ there exists $\vartheta \in \mathbb{R}^b$ such that $\mathcal{N}^\theta = \mathcal{N}^\vartheta$ and

$$\|\vartheta\| \leq c\left(\|\mathcal{N}^\vartheta\|^{1/2} + \|\mathcal{N}^\theta\|\right) \leq \mathcal{C} \left(\|\vartheta\|^{1/2} + \|\vartheta\|^2\right).$$

Theorem 1.1 is an immediate consequence of Corollary 2.13 in Subsection 2.4 below combined with the fact that for all $x, y \in [0, \infty)$ it holds that $\max\{x, y\} \leq x + y \leq 2 \max\{x, y\}$. Corollary 2.13 follows from Corollary 2.9 in Subsection 2.3 below which, in turn, builds on Theorem 2.7 in Subsection 2.3. In the following we add some explanatory comments regarding the mathematical objects which appear in Theorem 1.1 above.

The natural number $d \in \mathbb{N} = \{1, 2, 3, \ldots\}$ in Theorem 1.1 specifies the number of neurons on the input layer, whereas the natural number $h \in \mathbb{N}$ specifies the number of neurons on the hidden layer. There are $d h$ real weight parameters and $h$ real bias parameters for the first affine linear transformation from the $d$-dimension input layer to the $h$-dimensional hidden layer and there are $h$ real weight parameters and $1$ real bias parameter for the second affine linear transformation from the $h$-dimensional hidden layer to the one-dimensional output layer (cf. also Figure 1 above for a graphical illustration of the considered shallow ANN architecture). The total number of parameters, specified by the natural number $d \in \mathbb{N}$ in Theorem 1.1, thus satisfies

$$d = (d h + h) + (h + 1) = d h + 2 h + 1.$$  

The range of the permissible input data of the ANNs considered in Theorem 1.1 is described by the real parameters $\alpha \in \mathbb{R}$ and $\theta \in (\alpha, \infty)$. Note that for every ANN parameter vector $\theta \in \mathbb{R}^b$ we have that the function

$$[\alpha, \theta]^{d} \ni x \mapsto \mathcal{N}^\theta(x) \in \mathbb{R}$$

in Theorem 1.1 constitutes the realization function associated to the ANN parameter vector $\theta$. Moreover, in Theorem 1.1, for a fixed point $z \in [\alpha, \theta]^{d}$ we have that for every function $f : [\alpha, \theta]^{d} \rightarrow \mathbb{R}$ the extended real number

$$\|f\| = |f(z)| \sup_{x, y \in [\alpha, \theta]^{d}, x \neq y} |f(x) - f(y)|/\|x - y\| \in [0, \infty]$$

specifies the Lipschitz norm of $f$. We note that there are several definitions of the Lipschitz norm in the scientific literature, however, all of the Lipschitz norms are equivalent.

Under these conditions, Theorem 1.1 establishes that there exist real numbers $c, \mathcal{C} \in \mathbb{R}$ such that for every ANN parameter vector $\theta \in \mathbb{R}^b$ there exists an ANN parameter vector $\vartheta \in \mathbb{R}^b$ such that $\mathcal{N}^\theta = \mathcal{N}^\vartheta$ and

$$\|\vartheta\| \leq c\left(\|\mathcal{N}^\vartheta\|^{1/2} + \|\mathcal{N}^\theta\|\right) \leq \mathcal{C} \left(\|\vartheta\|^{1/2} + \|\vartheta\|^2\right).$$

Thus, for every ANN parameter vector there exists a reparameterization, by which we mean an ANN parameter vector with the same realization function, such that the standard norm of the parameters is bounded, up to a multiplicative constant, by the sum of powers of the Lipschitz norm of the realization function with the exponents $1/2$ and $1$. Note, however, due to the fact that all norms on $\mathbb{R}^d$ are equivalent (can up to a multiplicative constant be estimated against each other), we have that the statement of Theorem 1.1 with the standard norm replaced by another norm is also true.
Furthermore, observe that the right inequality in (6) is elementary and follows from the well-known fact that there exists a real number $c \in \mathbb{R}$ such that for all ANN parameter vectors $\theta \in \mathbb{R}^d$ it holds that
\[
\|\Lambda^{\theta}\| \leq c(\|\theta\| + \|\theta\|^2).
\]  
(7)
The left inequality, on the other hand, or a reparameterization bound comparable in kind, is to the best of our knowledge not known in the scientific literature and is one of the key contributions of this article. We emphasize that a reparameterization of the ANNs is mandatory for the left inequality to hold, since parts of the parameters of every ANN can be chosen arbitrary large without changing its realization function, for example, by scaling the input weights and output weights of hidden neurons. Both inequalities combined roughly give a kind of equivalence for the class of ANN parameter vectors with the same realization function and the Lipschitz norm of the ANN realization function.

The upper bounds for the reparameterized network parameters from Theorem 1.1 and its more general version in Theorem 2.7, respectively, are also relevant in other aspects. For example, in the training of ANNs with one hidden layer and ReLU activation in a supervised learning problem the position of global minima of the underlying risk function can be specified in more detail. Specifically, in Corollary 1.2 below we show in the special situation where there is only one neuron on the input layer (corresponding to the case $d = 1$ in Theorem 1.1) and where the target function $f : [a, b] \to \mathbb{R}$ is Lipschitz continuous that there exists a global minimum of the risk function within an area that depends on the permissible input domain specified by $a, b \in \mathbb{R}$, the network width $h \in \mathbb{N}$, the Lipschitz constant of the target function, and the supremum norm of the target function. We now present the precise statement of Corollary 1.2.

**Corollary 1.2.** Let $h, d \in \mathbb{N}$, $a, L, \mathbb{E} \in \mathbb{R}$, $b \in (a, \infty)$ satisfy $d = 3h + 1$, let $f : [a, b] \to \mathbb{R}$ satisfy for all $x, y \in [a, b]$ that $|f(x) - f(y)| \leq L|x - y|$ and
\[
\mathbb{E} \geq \max\{\max\{2, |a|, |b|\}^1|\mathbb{E}|^{1/2}, (2(b - a)h^2 + h)L + \sup_{z \in [a, b]}|f(z)|\},
\]  
(8)
let $\mu : \mathcal{B}([a, b]) \to [0, \infty]$ be a measure, and let $\mathcal{L} : \mathbb{R}^d \to \mathbb{R}$ satisfy for all $\theta = (\theta_1, \ldots, \theta_h) \in \mathbb{R}^h$ that
\[
\mathcal{L}(\theta) = f^\mathbb{E}_d(f(z) - \theta_0 - \sum_{i=1}^h \theta_{2h+i} \max\{\theta_{h+i} + \theta_i x, 0\})^2 \mu(dx).
\]  
(9)
Then there exists $\bar{\theta} \in [-\mathbb{E}, \mathbb{E}]^{h}$ such that $\mathcal{L}(\bar{\theta}) = \inf_{\theta \in \mathbb{R}^h} \mathcal{L}(\theta)$.

Corollary 1.2 is a direct consequence of [13, Proposition 2.26] combined with Theorem 2.7 in Subsection 2.3 below. Note that Corollary 1.2, for example, ensures in the special situation in the training of an ANN with $h = 5$ neurons on the hidden layer and where there are $m \in \mathbb{N}$ many input-output data pairs given by the input data $x_1, x_2, \ldots, x_m \in [0, 1]$ and the output data $y_1, y_2, \ldots, y_m \in [-1, 1]$, which satisfy that for all $i, j \in \{1, 2, \ldots, m\}$ with $i \neq j$ it holds that $|y_i - y_j| \leq |x_i - x_j|$, there exists a global minimum point $\theta = (\theta_1, \ldots, \theta_h) \in \mathbb{R}^h = \mathbb{R}^{16}$ of the mean squared error risk function
\[
\mathbb{R}^h \ni \bar{\theta} = (\bar{\theta}_1, \ldots, \bar{\theta}_5) \mapsto \mathcal{L}(\bar{\theta}) = \frac{1}{m} \sum_{i=1}^m |y_i - \bar{\theta}_0 - \sum_{j=1}^h \bar{\theta}_{2h+j} \max\{\bar{\theta}_{h+j} + \bar{\theta}_j x_i, 0\}|^2 \in \mathbb{R},
\]  
(10)
which satisfies $\max_{i \in \{1, 2, \ldots, 5\}} |\bar{\theta}_i| \leq 56$.

We also want to mention the relationship of Theorem 1.1 to the concept of the so-called inverse stability of the realization map. This concept deals with the question under which circumstances ANNs with similar realization functions can be reparameterized so that their new representatives are close together (cf. Berner et al. [5, Definition 1.1]). In Petersen et al. [18, Section 4] it was shown that the inverse stability of the realization map for deep ANNs with non affine-linear Lipschitz continuous activation functions fails with respect to the uniform norm. Berner et al. [5], on the other side, demonstrates that the inverse stability does hold on a restricted parametrization space for shallow ANNs with ReLU activation without biases with respect to the Sobolev semi-norm.
The second main result of our article, Theorem 1.3 below, addresses the optimality of (2) in Theorem 1.1. In the following we show, on the one hand, that the Lipschitz norm in (2) cannot be replaced by Hölder norms and, on the other hand, that the range of the exponents of the powers of the Lipschitz norm cannot be attenuated. For the precise statement, we now present Theorem 1.3.

**Theorem 1.3.** Let \( d, h, δ \in \mathbb{N}, a \in \mathbb{R}, \delta \in (a, \infty) \) satisfy \( δ = dh + 2h + 1 \), for every \( θ = (θ_1, \ldots, θ_δ) \in \mathbb{R}^δ \) let \( N^θ \in C([a, δ]^d, \mathbb{R}) \) satisfy for all \( x = (x_1, \ldots, x_d) \in [a, δ]^d \) that
\[
N^θ(x) = θ_θ + \sum_{i=1}^h θ_{dh+i} + \max\{θ_{dh+i} + \sum_{j=1}^h θ_{(i-1)d+j}x_j, 0\},
\]
for every \( n \in \mathbb{N}, x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) let \( \|x\| = (\sum_{j=1}^n |x_j|^2)^{1/2} \in \mathbb{R} \), for every \( f : [a, δ]^d \to \mathbb{R} \) and every \( γ \in [0, 1] \) let \( \|f\|_γ = \sup_{x,y \in [a, δ]^d} |f(x)| + \sup_{x,y \in [a, δ]^d, x \neq y} |f(x) - f(y)|/∥x - y∥_γ \in [0, ∞] \), and let \( n \in \mathbb{N}, γ_1, γ_2, \ldots, γ_n \in [0, 1], δ_1, δ_2, \ldots, δ_n \in [0, ∞) \) satisfy \( \max\{γ_1, γ_2, \ldots, γ_n\} \{0,1\}(\min\{γ_1, γ_2, \ldots, γ_n\}) < 1 \). Then the following two statements are equivalent:

(i) There exists \( c \in \mathbb{R} \) such that for all \( θ \in \mathbb{R}^\delta \) there exists \( θ \in \mathbb{R}^\delta \) such that \( N^θ = N^θ \) and
\[
\|θ\| ≤ c(\|N^θ\|_{γ_1} + \ldots + \|N^θ\|_{γ_n}).
\]

(ii) There exist \( i, j \in \{1, 2, \ldots, n\} \) such that \( γ_i = γ_j = 1 \), \( δ_i ≤ 1/2 \), and \( δ_j ≥ 1 \).

Theorem 1.3 is a direct consequence of Corollary 2.10 in Subsection 2.3 below, Corollary 3.3 in Subsection 3.2 below, and Corollary 4.8 in Subsection 4.2 below. We have that for every function \( f : [a, δ]^d \to \mathbb{R} \) and every \( γ \in [0, 1] \) the extended real number
\[
\|f\|_γ = \sup_{x \in [a, δ]^d} |f(x)| + \sup_{x,y \in [a, δ]^d, x \neq y} |f(x) - f(y)|/∥x - y∥_γ \in [0, ∞]
\]
in Theorem 1.3 above specifies the Hölder norm of \( f \). Observe that, in the case that \( γ = 1 \), the Hölder norm in (13) is equivalent to the Lipschitz norm in (5) and can therefore be considered as the Lipschitz norm. In the following we want to explain Theorem 1.3 in more detail.

Note that Theorem 1.3, in the case that \( \min\{γ_1, γ_2, \ldots, γ_n\} = 1 \), shows that the range of the exponents of the powers of the Lipschitz norm of the ANN realization function must extend at least from 1/2 to 1 for the upper bound of the reparameterized network parameters to hold. In particular, this implies that the upper bound for the reparameterized network parameters in (2) only holds for sums of powers of the Lipschitz norm with the exponents 1/2 and 1 but does not hold for the Lipschitz norm alone. Moreover, Theorem 1.3 above, in the case that \( \min\{γ_1, γ_2, \ldots, γ_n\} < 1 \), demonstrates that it is not possible to control the network parameters of reparameterized ANNs using sums of powers of the Hölder norm of the realization function with arbitrary exponents. In Corollary 4.8 in Subsection 4.2 we show that this does also hold for Sobolev-Slobodeckij norms. Specifically, the realization map for shallow ANNs with ReLU activation is not inverse stable with respect to Hölder norms and Sobolev-Slobodeckij norms.

The remainder of this article is organized in the following way. In Section 2 we establish upper bounds for norms of reparameterized ANNs using Lipschitz norms. In Section 3 we address the optimality of the bounds from Section 2 and prove lower bounds for norms of reparameterized ANNs using Lipschitz norms. Finally, in Section 4 we consider different norms for the realization function and establish lower bounds for norms of reparameterized ANNs using Hölder norms and Sobolev-Slobodeckij norms.
2 Upper bounds for norms of reparameterized artificial neural networks (ANNs) using Lipschitz norms

In this section we establish in Corollary 2.9 in Subsection 2.3 below upper bounds for norms of reparameterized ANN parameter vectors using Lipschitz norms. In particular, we show that every ANN parameter vector \( \theta \in \mathbb{R}^3 = \mathbb{R}^{dh+2h+1} \) can be reparameterized by an ANN \( \vartheta \in \mathbb{R}^3 \) such that the maximum norm of \( \vartheta \) is, up to a multiplicative constant, bounded by the maximum of powers of the Lipschitz norm of the realization function \( N^\vartheta : [\alpha, \delta]^d \to \mathbb{R} \) with the exponents \( 1/2 \) and \( 1 \). The proof of Corollary 2.9 uses our main result of this section, the upper bounds for norms of reparameterized ANNs in Theorem 2.7 in Subsection 2.3. Theorem 2.7, in turn, builds on the well-known properties of tessellations of compact convex polytopes established in Lemma 2.2 in Subsection 2.1 below, on the well-known properties of affine hyperplanes in compact cubes established in Lemma 2.3 and Lemma 2.4 in Subsection 2.2 below, and on the essentially well-known ability to isolate points of affine linear hyperplanes in compact cubes presented in Lemma 2.5 in Subsection 2.2. In Corollary 2.13 in Subsection 2.4 below we combine Corollary 2.9 and the well-known upper bounds of the Lipschitz constant and the Lipschitz norm of the realization function of an ANN established in Lemma 2.11 and Lemma 2.12 in Subsection 2.4, respectively, to obtain a kind of equivalence for the class of ANN parameter vectors with the same realization function and the Lipschitz norm of the ANN realization function.

In Setting 2.6 in Subsection 2.3 we describe our mathematical setup to introduce the architecture of the considered shallow ANNs, specified by the number of input neurons \( d \in \mathbb{N} \) and the number of hidden neurons \( h \in \mathbb{N} \), the dimension of the parameter space \( d = dh + 2h + 1 \in \mathbb{N} \), and the realization function \( N^\vartheta : [\alpha, \delta]^d \to \mathbb{R} \) associated to every ANN parameter vector \( \theta \in \mathbb{R}^3 \). For the convenience of the reader we recall the notions of the standard scalar product and of the standard norm in Definition 2.1 in Subsection 2.1 and for every \( A \subseteq [\alpha, \delta]^d \) with \( A \neq \emptyset \) and every \( f : [\alpha, \delta]^d \to \mathbb{R} \) we introduce in Definition 2.8 in Subsection 2.3 the extended real number \( \|f\|_A \in [0, \infty) \), which corresponds to the Lipschitz norm of \( f \) in the case that \( A = \{z\} \) for a fixed point \( z \in [\alpha, \delta]^d \).

2.1 Properties of tessellations of compact convex polytopes

**Definition 2.1.** We denote by \( \langle \cdot, \cdot \rangle : (\bigcup_{d \in \mathbb{N}} (\mathbb{R}^d \times \mathbb{R}^d)) \to \mathbb{R} \) and \( \|\cdot\| : (\bigcup_{d \in \mathbb{N}} \mathbb{R}^d) \to \mathbb{R} \) the functions which satisfy for all \( d \in \mathbb{N} \), \( x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d) \in \mathbb{R}^d \) that \( \langle x, y \rangle = \sum_{i=1}^d x_i y_i \) and \( \|x\| = (\sum_{i=1}^d |x_i|^2)^{1/2} \).

**Lemma 2.2.** Let \( d, N \in \mathbb{N} \), \( \alpha, \beta \in (\alpha, \infty) \), \( w_1, w_2, \ldots, w_N \in \mathbb{R}^d \), \( b_1, b_2, \ldots, b_N \in \mathbb{R} \), and for every \( i \in \{1, 2, \ldots, N\} \), \( \ell \in \{0, 1\} \) let \( H_i^{\ell} \subseteq [\alpha, \beta]^d \) satisfy \( H_i^{\ell} = \{ x \in [\alpha, \beta]^d : (-1)^\ell (b_i + \langle w_i, x \rangle) \leq 0 \} \) (cf. Definition 2.1). Then for all \( x \in [\alpha, \beta]^d \) there exist \( y \in [\alpha, \beta]^d \), \( \ell_1, \ell_2, \ldots, \ell_N \in \{0, 1\} \), \( \varepsilon \in (0, \infty) \) such that

\[
x \in \left( \bigcap_{i=1}^N H_i^{\ell_i} \right) \quad \text{and} \quad \{ u \in [\alpha, \beta]^d : \|y - u\| \leq \varepsilon \} \subseteq \left( \bigcap_{i=1}^N H_i^{\ell_i} \right).
\]

**Proof of Lemma 2.2.** Throughout this proof let \( \mu : B([\alpha, \beta]^d) \to [0, \infty) \) be the Lebesgue measure, let \( P_\ell \subseteq [\alpha, \beta]^d \), \( \ell \in \{0, 1\}^N \), satisfy for all \( \ell = (\ell_1, \ldots, \ell_N) \in \{0, 1\}^N \) that \( P_\ell = \bigcap_{i=1}^N H_i^{\ell_i} \), and let \( A_\varepsilon \subseteq \{0, 1\}^N \), \( x \in [\alpha, \beta]^d \) satisfy for all \( \ell \in [\alpha, \beta]^d \) that

\[
A_\varepsilon = \{ \ell \in \{0, 1\}^N : x \in P_\ell \}.
\]

Observe that the fact that for all \( i \in \{1, 2, \ldots, N\} \), \( \ell \in \{0, 1\} \) it holds that \( H_i^{\ell} \subseteq [\alpha, \beta]^d \) is closed ensures that for all \( \ell \in \{0, 1\}^N \) it holds that \( P_\ell \subseteq [\alpha, \beta]^d \) is closed. Therefore, we obtain that for all \( x \in [\alpha, \beta]^d \), \( \ell \in \{0, 1\}^N \setminus A_\varepsilon \) there exists \( \varepsilon \in (0, \infty) \) such that

\[
\{ u \in [\alpha, \beta]^d : \|x - u\| \leq \varepsilon \} \subseteq [\alpha, \beta]^d \setminus P_\ell.
\]
This implies that for all \( x \in [a, \sigma]^d \) there exists \( \varepsilon \in (0, \infty) \) such that it holds that
\[
\{ u \in [a, \sigma]^d : \| x - u \| \leq \varepsilon \} \cap (\bigcup_{\ell \in \{0,1\}^N \setminus A_x} P_\ell) = \emptyset.
\]  
(17)

The fact that \([a, \sigma]^d = \bigcup_{\ell \in \{0,1\}^N} P_\ell\) hence shows that for all \( x \in [a, \sigma]^d \) there exists \( \varepsilon \in (0, \infty) \) such that it holds that
\[
\{ u \in [a, \sigma]^d : \| x - u \| \leq \varepsilon \} \subseteq (\bigcup_{\ell \in A_x} P_\ell).
\]  
(18)

Therefore, we obtain that for all \( x \in [a, \sigma]^d \) there exists \( \varepsilon \in (0, \infty) \) such that \( \mu(\bigcup_{\ell \in A_x} P_\ell) \geq \mu(\{ u \in [a, \sigma]^d : \| x - u \| \leq \varepsilon \}) > 0 \). This proves that for all \( x \in [a, \sigma]^d \) there exists \( \ell \in A_x \) such that \( \mu(P_\ell) > 0 \). Hence, we obtain that for all \( x \in [a, \sigma]^d \) there exist \( y \in [a, \sigma]^d \), \( \ell_1, \ell_2, \ldots, \ell_N \in \{0,1\} \), \( \varepsilon \in (0, \infty) \) such that
\[
x \in \left( \bigcap_{i=1}^N H_{\ell_i}^{d_i} \right) \quad \text{and} \quad \{ u \in [a, \sigma]^d : \| x - u \| \leq \varepsilon \} \subseteq \left( \bigcap_{i=1}^N H_{\ell_i}^{d_i} \right).
\]  
(19)

The proof of Lemma 2.2 is thus complete. \( \square \)

### 2.2 Properties of affine hyperplanes in compact cubes

**Lemma 2.3.** Let \( d \in \mathbb{N}, a \in \mathbb{R}, \sigma \in (a, \infty), z \in [a, \sigma]^d \), \( w_1, w_2 \in \mathbb{R}^d \), \( b_1, b_2 \in \mathbb{R} \), assume \( \{ x \in [a, \sigma]^d : b_1 + \langle w_1, x \rangle = 0 \} = \{ x \in [a, \sigma]^d : b_2 + \langle w_2, x \rangle = 0 \} \), assume \( \{ x \in [a, \sigma]^d : b_1 + \langle w_1, x \rangle \geq 0 \} \neq \emptyset \), and assume for all \( i \in \{1, 2\} \) that \( b_i + \langle w_i, z \rangle < 0 \) (cf. Definition 2.1). Then \( \| w_1 \|^2 = \| w_2 \| = \| w_1 \|^2 = \| w_2 \|^2 \) holds.

**Proof of Lemma 2.3.** Throughout this proof let \( \varphi_x : [0,1] \to \mathbb{R}, x \in \mathbb{R}^d \), satisfy for all \( x \in \mathbb{R}^d \), \( t \in [0,1] \) that \( \varphi_x(t) = b_1 + \langle w_1, (1-t)x + tz \rangle \) and let \( A = (A_1, A_2) \in \mathbb{R}^{2 \times d} \) satisfy that
\[
A_1 = w_1 \quad \text{and} \quad A_2 = w_2.
\]  
(20)

Note that the assumption that \( \{ x \in [a, \sigma]^d : b_1 + \langle w_1, x \rangle \geq 0 \} \neq \emptyset \) demonstrates that there exists \( v \in (a, \sigma)^d \) which satisfies that
\[
b_1 + \langle w_1, v \rangle \geq 0.
\]  
(21)

Observe that (21) and the assumption that \( b_1 + \langle w_1, z \rangle < 0 \) establish that
\[
\varphi_v(0) = b_1 + \langle w_1, v \rangle \geq 0 \quad \text{and} \quad \varphi_v(1) = b_1 + \langle w_1, z \rangle < 0.
\]  
(22)

This and the fact that \( \varphi_v \in C([0,1], \mathbb{R}) \) ensure that there exists \( t \in [0,1] \) such that \( \varphi_v(t) = 0 \). Therefore, we obtain that there exist \( \{ x \in [a, \sigma]^d : \varepsilon \in (0, \infty) \} \) which satisfy that
\[
b_1 + \langle w_1, u \rangle = 0 \quad \text{and} \quad \{ x \in \mathbb{R}^d : \| x - u \| \leq \varepsilon \} \subseteq [a, \sigma]^d.
\]  
(23)

Note that (23) and the assumption that \( \{ x \in [a, \sigma]^d : b_1 + \langle w_1, x \rangle = 0 \} = \{ x \in [a, \sigma]^d : b_2 + \langle w_2, x \rangle = 0 \} \) imply that for all \( i \in \{1, 2\} \) it holds that
\[
b_i + \langle w_i, u \rangle = 0.
\]  
(24)

Hence, we obtain that for all \( i \in \{1, 2\} \), \( x \in \mathbb{R}^d \setminus \{ u \} : b_i + \langle w_i, y \rangle = 0 \) it holds that
\[
b_i + \langle w_i, u + \frac{\varepsilon}{\| x-u \|} (x-u) \rangle = b_i + \langle w_i, u \rangle + \frac{\varepsilon}{\| x-u \|} \langle w_i, x-u \rangle = \frac{\varepsilon}{\| x-u \|} (b_i + \langle w_i, x \rangle - \langle w_i, u \rangle) = \frac{\varepsilon}{\| x-u \|} (b_i + \langle w_i, x \rangle) - \frac{\varepsilon}{\| x-u \|} (b_i + \langle w_i, u \rangle) = 0.
\]  
(25)
Combining this, (23), (24), and the assumption that \( \{x \in [a, b]^d : b_1 + \langle w_1, x \rangle = 0 \} = \{x \in [a, b]^d : b_2 + \langle w_2, x \rangle = 0 \} \) shows that for all \( i, j \in \{1, 2\}, x \in \{y \in \mathbb{R}^d \setminus \{u\} : b_1 + \langle w_1, y \rangle = 0 \} \) it holds that

\[
b_j + \langle w_j, x \rangle = \left[ b_j + \langle w_j, x \rangle \right] - \left[ b_j + \langle w_j, u \rangle \right] = \langle w_j, x - u \rangle = \frac{\|x - u\|}{\varepsilon} \left( b_j + \langle w_j, u \rangle + \frac{\varepsilon}{\|x - u\|} \langle w_j, x - u \rangle \right) = 0.
\]

(26)

This and (24) prove that for all \( i, j \in \{1, 2\}, x \in \{y \in \mathbb{R}^d : b_1 + \langle w_1, x \rangle = 0 \} \) it holds that \( b_j + \langle w_j, x \rangle = 0 \). Therefore, we obtain that

\[
\{x \in \mathbb{R}^d : b_1 + \langle w_1, x \rangle = 0 \} = \{x \in \mathbb{R}^d : b_2 + \langle w_2, x \rangle = 0 \}.
\]

(27)

Furthermore, observe that (24) demonstrates that for all \( i \in \{1, 2\}, x \in \{y \in \mathbb{R}^d : \langle w_i, y \rangle = 0 \} \) it holds that \( b_1 + \langle w_i, u + x \rangle = b_1 + \langle w_i, u \rangle + \langle w_i, x \rangle = 0 \). Combining this with (24) and (27) establishes that for all \( i, j \in \{1, 2\}, x \in \{y \in \mathbb{R}^d : \langle w_i, x \rangle = 0 \} \) it holds that \( \langle w_j, x \rangle = b_j + \langle w_j, u \rangle + \langle w_j, x \rangle = b_j + \langle w_j, u + x \rangle = 0 \). Hence, we obtain that

\[
\{x \in \mathbb{R}^d : \langle w_1, x \rangle = 0 \} = \{x \in \mathbb{R}^d : \langle w_2, x \rangle = 0 \}.
\]

(28)

This ensures that \( \ker(A) = \{x \in \mathbb{R}^d : \langle w_1, x \rangle = 0 \} \cap \{x \in \mathbb{R}^d : \langle w_2, x \rangle = 0 \} = \{x \in \mathbb{R}^d : \langle w_1, x \rangle = 0 \} \). The rank-nullity theorem therefore implies that

\[
\text{rank}(A) = d - \dim\ker(A) = d - \dim\{x \in \mathbb{R}^d : \langle w_1, x \rangle = 0 \} = d - (d - 1) = 1.
\]

(29)

Hence, we obtain that there exists \( \lambda \in \mathbb{R} \setminus \{0\} \) which satisfies that

\[
w_1 = \lambda w_2.
\]

(30)

Note that (24), (30), and the assumption that \( b_1 + \langle w_1, z \rangle < 0 \) show that

\[
0 > b_1 + \langle w_1, z \rangle = [b_1 + \langle w_1, z \rangle] - [b_1 + \langle w_1, u \rangle] = \langle w_1, z - u \rangle = \lambda \langle w_2, z - u \rangle
\]

\[
= \lambda \left( [b_2 + \langle w_2, z \rangle] - [b_2 + \langle w_2, u \rangle] \right) = \lambda \langle b_2 + \langle w_2, z \rangle \rangle.
\]

(31)

The assumption that \( b_2 + \langle w_2, z \rangle < 0 \) therefore proves that \( \lambda > 0 \). Combining this with (30) demonstrates that \( \|w_1\| = \|\lambda w_2\| = \|w_2\| \). This and the fact that \( \min\{\lambda, \|w_1\|\} > 0 \) establish that

\[
\lambda = \|w_1\|/\|w_2\|.
\]

(32)

Moreover, observe that (24) and (30) ensure that \( b_1 = -\langle w_1, u \rangle = -\lambda \langle w_2, u \rangle = \lambda b_2 \). Combining this with (30) and (32) implies that \( \|w_1\|w_2 = \|w_2\|w_1 \) and \( \|w_1\|b_2 = \|w_2\|b_1 \). The proof of Lemma 2.3 is thus complete. \( \square \)

**Lemma 2.4.** Let \( d \in \mathbb{N}, a, b \in \mathbb{R}, \sigma \in (a, \infty), z \in [a, b]^d, w_1, w_2 \in \mathbb{R}^d, b_1, b_2 \in \mathbb{R} \), assume that \( \{x \in [a, b]^d : b_1 + \langle w_1, x \rangle = 0 \} \neq \{x \in [a, b]^d : b_2 + \langle w_2, x \rangle = 0 \} \), and assume for all \( i \in \{1, 2\} \) that \( b_i + \langle w_i, z \rangle = 0 \) (cf. Definition 2.1). Then for all \( \lambda \in \mathbb{R} \setminus \{0\} \) it holds that

\[
w_1 \neq \lambda w_2.
\]

(33)

**Proof of Lemma 2.4.** We show (33) by contradiction. In the following we thus assume that there exists \( \lambda \in \mathbb{R} \setminus \{0\} \) which satisfies that

\[
w_1 = \lambda w_2.
\]

(34)
Note that (34) and the assumption that for all $i \in \{1, 2\}$ it holds that $b_i + \langle w_i, z \rangle = 0$ prove that for all $i, j \in \{1, 2\}, x \in \{y \in [\alpha, \delta]^d : b_i + \langle w_i, y \rangle = 0\}$ there exists $\epsilon \in \{\lambda, \lambda^{-1}\}$ such that

$$b_j + \langle w_j, x \rangle = [b_j + \langle w_j, x \rangle] = \epsilon(b_j + \langle w_j, x \rangle) = \epsilon(\langle b_j, x \rangle) = \epsilon(b_j + \langle w_j, x \rangle) = 0. \quad (35)$$

Hence, we obtain that $\{x \in [\alpha, \delta]^d : b_1 + \langle w_1, x \rangle = 0\} = \{x \in [\alpha, \delta]^d : b_2 + \langle w_2, x \rangle = 0\}$. This contradiction demonstrates (33). The proof of Lemma 2.4 is thus complete. □

**Lemma 2.5.** Let $d, N \in \mathbb{N}, \alpha \in \mathbb{R}, \delta \in (\alpha, \infty), w_1, w_2, \ldots, w_N \in \mathbb{R}^d, b_1, b_2, \ldots, b_N \in \mathbb{R},$ for every $i \in \{1, 2, \ldots, N\} = \{0, 1\}$ let $H_i^1 \subseteq [\alpha^\delta]^d$ satisfy $H_i^1 = \{x \in [\alpha, \delta]^d : (-1)^\ell(b_i + \langle w_i, x \rangle) \leq 0\}$, assume for all $i \in \{1, 2, \ldots, N\}$ that $H_i^1 \neq [\alpha, \delta]^d$ and $H_i^1 \cap (\alpha, \delta)^d \neq \emptyset,$ and assume for all $i, j \in \{1, 2, \ldots, N\}$ with $i \neq j$ that $(H_i^1 \cap H_j^1) \neq (H_i^1 \cap H_j^1)^1$ (cf. Definition 2.1). Then there exist $p_1, p_2, \ldots, p_N \in (\alpha, \delta)^d \in (0, \infty)$ which satisfy for all $i \in \{1, 2, \ldots, N\}$ that $p_i \in (H_i^0 \cap H_i^1)$ and $\{x \in [\alpha, \delta]^d : \|x - p_i\| \leq \varepsilon\} \subset \bigcup_{\ell \in \{1, 2, \ldots, N\}} \bigcup_{i=1}^N \bigcup_{t=0}^N (H_0^0 \cap H_i^1) = \emptyset. \quad (36)$

**Proof of Lemma 2.5.** Throughout this proof let $\varphi^{x,y}_i : [0, 1] \rightarrow \mathbb{R}, i \in \{1, 2, \ldots, N\}, x, y \in \mathbb{R}^d$, satisfy for all $i \in \{1, 2, \ldots, N\}, x, y \in \mathbb{R}^d, t \in [0, 1]$ that $\varphi^{x,y}_i(t) = b_i + \langle w_i, (1-t)x + ty \rangle$ and let $A^{i,j} = (A^{ij}_1, A^{ij}_2) \in \mathbb{R}^{2 \times d}$ satisfy for all $i, j \in \{1, 2, \ldots, N\}$ that

$$A^{i,j}_1 = w_i \quad \text{and} \quad A^{i,j}_2 = w_j. \quad (37)$$

Observe that the assumption that for all $i \in \{1, 2, \ldots, N\}$ it holds that $H_i^1 \neq [\alpha, \delta]^d$ and $H_i^1 \cap (\alpha, \delta)^d \neq \emptyset$ establishes that there exist $u_1, u_2, \ldots, u_N \in [\alpha, \delta]^d, v_1, v_2, \ldots, v_N \in (\alpha, \delta)^d$ which satisfy for all $i \in \{1, 2, \ldots, N\}$ that

$$b_i + \langle w_i, u_i \rangle < 0 \quad \text{and} \quad b_i + \langle w_i, v_i \rangle \geq 0. \quad (38)$$

Note that (38) ensures that for all $i \in \{1, 2, \ldots, N\}$ it holds that

$$\varphi^{u_i, v_i}_i(0) = b_i + \langle w_i, u_i \rangle < 0 \quad \text{and} \quad \varphi^{u_i, v_i}_i(1) = b_i + \langle w_i, v_i \rangle \geq 0. \quad (39)$$

This and the fact that for all $i \in \{1, 2, \ldots, N\}$ it holds that $\varphi^{u_i, v_i}_i \in C([0, 1], \mathbb{R})$ imply that for all $i \in \{1, 2, \ldots, N\}$ there exists $t \in (0, 1)$ such that $\varphi^{u_i, v_i}_i(t) = 0$. Therefore, we obtain that there exist $q_1, q_2, \ldots, q_N \in (\alpha, \delta)^d, \delta \in (0, \infty)$ which satisfy for all $i \in \{1, 2, \ldots, N\}$ that

$$b_i + \langle w_i, q_i \rangle = 0 \quad \text{and} \quad \{x \in \mathbb{R}^d : \|x - q_i\| \leq \delta\} \subseteq (\alpha, \delta)^d. \quad (40)$$

Let $M_i \subseteq \{1, 2, \ldots, N\}, i \in \{1, 2, \ldots, N\}$, satisfy for all $i \in \{1, 2, \ldots, N\}$ that

$$M_i = \{j \in \{1, 2, \ldots, N\} : b_j + \langle w_j, q_i \rangle = 0\}. \quad (41)$$

Observe that the fact that for all $i \in \{1, 2, \ldots, N\}$ it holds that $\mathbb{R}^d \ni x \mapsto b_i + \langle w_i, x \rangle \in \mathbb{R}$ is continuous shows that for all $i \in \{1, 2, \ldots, N\}, j \in \{1, 2, \ldots, N\} \setminus M_i$ there exists $\eta \in (0, \infty)$ such that for all $x \in \{y \in \mathbb{R}^d : \|x - q_i\| \leq \delta\}$ it holds that $b_j + \langle w_j, q_i \rangle > 0$. Hence, we obtain that there exists $\eta \in (0, \delta]$ which satisfies for all $i \in \{1, 2, \ldots, N\}, j \in \{1, 2, \ldots, N\} \setminus M_i, x \in \{y \in [\alpha, \delta]^d : \|x - q_i\| \leq \eta\}$ that

$$b_j + \langle w_j, x \rangle > 0. \quad (42)$$

In the following we distinguish between the case $d = 1$ and the case $d > 1$. We first prove (36) in the case

$$d = 1. \quad (43)$$
Moreover, note that \((H_0^0 \cap H_1^1) \neq \{0\}\) and \(\{x \in [a, \delta]^d : \|x - q_i\| \leq \eta\} \cap \left(\bigcup_{j \in \{1, 2, \ldots, N\}} (H_0^0 \cap H_1^1)\right) = \emptyset.\) \(\tag{44} \)

This ensures \((36)\) in the case \(d = 1.\) In the next step we prove \((36)\) in the case \(d > 1.\) \(\tag{45} \)

Let \(\mu : \mathcal{B}([a, \delta]^d) \to [0, \infty)\) be the Lebesgue measure and let \(G_i \subseteq [a, \delta]^d, i \in \{1, 2, \ldots, N\},\) satisfy for all \(i \in \{1, 2, \ldots, N\}\) that \(G_i = \{x \in [a, \delta]^d : \langle w_i, x \rangle = 0\}.\) Observe that Lemma 2.4 (applied for every \(i \in \{1, 2, \ldots, N\}, j \in M_i \backslash \{i\}\) with \(d \cap d, a \cap a, \delta \cap \delta, z \cap q_i, w_1 \cap w_i, w_2 \cap w_j, b_1 \cap b_i, b_2 \cap b_j\) in the notation of Lemma 2.4) implies that for all \(i \in \{1, 2, \ldots, N\}, j \in M_i \backslash \{i\}, \lambda \in \mathbb{R}\backslash \{0\}\) it holds that \(w_i \neq \lambda w_j.\) \(\tag{46} \)

Furthermore, note that the rank-nullity theorem and the fact that for all \(i \in \{1, 2, \ldots, N\}\) it holds that \(w_i \neq 0\) show that for all \(i \in \{1, 2, \ldots, N\}\) it holds that

\[
\dim_{\mathbb{R}}(G_i) = \dim_{\mathbb{R}}(\{x \in \mathbb{R}^d : \langle w_i, x \rangle = 0\}) = d - \dim_{\mathbb{R}}(\{y \in \mathbb{R} : \exists x \in \mathbb{R}^d : \langle w_i, x \rangle = y\}) = d - 1.
\]

Therefore, we obtain that there exist \(f_i : G_i \to \mathbb{R}^{d-1}, i \in \{1, 2, \ldots, N\},\) which satisfy for all \(i \in \{1, 2, \ldots, N\}, x, y \in G_i, \lambda \in \mathbb{R}\) that

\[
f_i(\lambda x) = \lambda f_i(x), \quad f_i(x + y) = f_i(x) + f_i(y), \quad \ker(f_i) = \{0\}, \quad \text{and} \quad f_i(G_i) = \mathbb{R}^{d-1}. \tag{48} \]

Observe that \((46), (48),\) the fact that for all \(i, j \in \{1, 2, \ldots, N\}\) it holds that \(G_i \cap G_j = \{x \in \mathbb{R}^d : \langle w_i, x \rangle = 0\} \cap \{x \in \mathbb{R}^d : \langle w_j, x \rangle = 0\} = \ker(A^{i,j}),\) and the rank-nullity theorem demonstrate that for all \(i \in \{1, 2, \ldots, N\}, j \in M_i \backslash \{i\}\) it holds that

\[
\dim_{\mathbb{R}}(f_i(G_i \cap G_j)) = \dim_{\mathbb{R}}(G_i \cap G_j) = \dim_{\mathbb{R}}(\ker(A^{i,j})) = d - 2. \tag{49} \]

This establishes that for all \(i \in \{1, 2, \ldots, N\}, j \in M_i \backslash \{i\}\) it holds that \(\mu(f_i(G_i \cap G_j)) = 0.\) Hence, we obtain that for all \(i \in \{1, 2, \ldots, N\}\) it holds that

\[
0 \leq \mu(f_i(G_i \cap (\bigcup_{j \in M_i \backslash \{i\}} G_j))) = \mu(\bigcup_{j \in M_i \backslash \{i\}} f_i(G_i \cap G_j)) \leq \sum_{j \in M_i \backslash \{i\}} \mu(f_i(G_i \cap G_j)) = 0. \tag{50} \]

Moreover, note that \((48)\) ensures that for all \(i \in \{1, 2, \ldots, N\}\) it holds that \(\mu(f_i(G_i)) = \mu(\mathbb{R}^{d-1}) = \infty.\) Combining this with \((50)\) proves that \(G_i \not\subseteq \bigcup_{j \in M_i \backslash \{i\}} G_j.\) Therefore, we obtain that there exist \(m_1, m_2, \ldots, m_N \in \mathbb{R}^d\backslash \{0\}\) which satisfy for all \(i \in \{1, 2, \ldots, N\}, j \in M_i \backslash \{i\}\) that

\[
\langle w_i, m_i \rangle = 0, \quad \|\langle w_j, m_i \rangle\| > 0, \quad \text{and} \quad \|m_i\| \leq \eta/2. \tag{51} \]

Observe that \((51)\) and the fact that for all \(i \in \{1, 2, \ldots, N\}\) it holds that \(\mathbb{R}^d \ni x \mapsto b_i + \langle w_i, x \rangle \in \mathbb{R}\) is continuous imply that for all \(i \in \{1, 2, \ldots, N\}, j \in M_i \backslash \{i\}\) there exists \(\varepsilon \in (0, \infty)\) such that for all \(x \in \{y \in \mathbb{R}^d : \|y - m_i\| \leq \varepsilon\}\) it holds that \(\|\langle w_j, x \rangle\| > 0.\) Hence, we obtain that there exists \(\varepsilon \in (0, \eta/2)\) which satisfies for all \(i \in \{1, 2, \ldots, N\}, j \in M_i \backslash \{i\}, x \in \{y \in \mathbb{R}^d : \|y - m_i\| \leq \varepsilon\}\) that

\[
\|\langle w_j, x \rangle\| > 0. \tag{52} \]
Note that (40) and (52) show that for all \( i \in \{1, 2, \ldots, N\} \), \( j \in M_i \setminus \{i\} \), \( x \in \{y \in \mathbb{R}^d: \|x - (q_i + m_i)\| \leq \varepsilon\} \) it holds that

\[
|b_j + \langle w_j, x \rangle| = |\langle w_j, x \rangle| > 0. \tag{53}
\]

In addition, observe that (51) demonstrates that for \( i \in \{1, 2, \ldots, N\} \), \( x \in \{y \in \mathbb{R}^d: \|x - (q_i + m_i)\| \leq \varepsilon\} \) it holds that

\[
\|x - q_i\| \leq \|x - (q_i + m_i)\| + \|m_i\| \leq \varepsilon + \eta/2 \leq \eta. \tag{54}
\]

Combining this with (42) and (53) establishes that for all \( i \in \{1, 2, \ldots, N\} \) it holds that

\[
\{x \in [\alpha, \theta]^d: \|x - (q_i + m_i)\| \leq \varepsilon\} \cap \left(\bigcup_{j \in \{1, 2, \ldots, N\} \setminus \{i\}} (H_j^0 \cap H_j^1)\right) = \emptyset. \tag{55}
\]

Furthermore, note that (40) and (51) ensure that for all \( i \in \{1, 2, \ldots, N\} \) it holds that

\[
b_i + \langle w_i, q_i + m_i \rangle = b_i + \langle w_i, q_i \rangle + \langle w_i, m_i \rangle = 0 \quad \text{and} \quad q_i + m_i \in ([\alpha, \theta])^d. \tag{56}
\]

This and (55) prove (36) in the case \( d > 1 \). The proof of Lemma 2.5 is thus complete. \( \square \)

### 2.3 Upper bounds for norms of reparameterized ANNs using Lipschitz norms

**Setting 2.6.** Let \( d, h, \theta \in \mathbb{N} \), \( \alpha \in \mathbb{R} \), \( \theta \in ([\alpha, \theta])^d \) satisfy \( d = dh + 2h + 1 \) and for every \( \theta = (\theta_1, \ldots, \theta_3) \in \mathbb{R}^3 \) let \( \mathcal{N}^{\theta} \in C([\alpha, \theta]^d, \mathbb{R}) \) satisfy for all \( x = (x_1, \ldots, x_d) \in [\alpha, \theta]^d \) that

\[
\mathcal{N}^{\theta}(x) = \theta_3 + \sum_{i=1}^h \theta_{ih} + \max\{\theta_{ih} + \sum_{j=1}^d \theta_{(i-1)d+j, x_j}, 0\}.
\]

**Theorem 2.7.** Assume Setting 2.6 and let \( \theta \in \mathbb{R}^3 \). Then there exists \( \vartheta = (\vartheta_1, \ldots, \vartheta_3) \in \mathbb{R}^3 \) such that \( \mathcal{N}^{\vartheta} = \mathcal{N}^{\theta} \) and

\[
\max_{i \in \{1, 2, \ldots, N\}} |\vartheta_i| \leq \max\left\{\max_{2, |x| \sqrt{d}, |\vartheta| \sqrt{d}} \left[\sup_{x, y \in ([\alpha, \theta]^d, x \neq y} \frac{\mathcal{N}^{\theta}(x) - \mathcal{N}^{\theta}(y)}{\|x - y\|}\right]^{1/2}, \right. \\
\left. \inf_{x \in ([\alpha, \theta]^d)} [\mathcal{N}^{\theta}(x)] \right\} + 2h(\theta - \alpha) \sqrt{d} \left[\sup_{x, y \in ([\alpha, \theta]^d, x \neq y} \frac{\mathcal{N}^{\theta}(x) - \mathcal{N}^{\theta}(y)}{\|x - y\|}\right]^{1/2}. \tag{57}
\]

(cf. Definition 2.1).

**Proof of Theorem 2.7.** Throughout this proof let \( \theta_1, \theta_2, \ldots, \theta_3, L \in \mathbb{R} \) satisfy \( \theta = (\theta_1, \ldots, \theta_3) \) and

\[
L = \sup_{x, y \in ([\alpha, \theta]^d, x \neq y} \frac{\mathcal{N}^{\theta}(x) - \mathcal{N}^{\theta}(y)}{\|x - y\|}, \tag{58}
\]

let \( w = (w_1, \ldots, w_h) = (w_{i, j})_{(i, j) \in \{1, 2, \ldots, h\} \times \{1, 2, \ldots, d\}} \in \mathbb{R}^{h \times d} \), \( b = (b_1, \ldots, b_h) \), \( v = (v_1, \ldots, v_h) \in \mathbb{R}^h \) satisfy for all \( i \in \{1, 2, \ldots, h\} \), \( j \in \{1, 2, \ldots, d\} \) that

\[
w_{i, j} = \theta_{(i-1)d+j}, \quad b_i = \theta_{ih}, \quad \text{and} \quad v_i = \theta_{ih} \tag{59}
\]

for every \( i \in \{1, 2, \ldots, h\} \), \( \ell \in \{0, 1\} \) let \( H^{i}_{\ell} \subseteq ([\alpha, \theta]^d, x \neq y} \sup_{x, y \in ([\alpha, \theta]^d, x \neq y} \frac{\mathcal{N}^{\theta}(x) - \mathcal{N}^{\theta}(y)}{\|x - y\|}\right]^{1/2}. \tag{57}
\]

for every \( i \in \{1, 2, \ldots, h\} \), \( \ell \in \{0, 1\} \) let \( H^{i}_{\ell} \subseteq ([\alpha, \theta]^d, x \neq y} \sup_{x, y \in ([\alpha, \theta]^d, x \neq y} \frac{\mathcal{N}^{\theta}(x) - \mathcal{N}^{\theta}(y)}{\|x - y\|}\right]^{1/2}. \tag{57}
\]

for every \( i \in \{1, 2, \ldots, h\} \), \( \ell \in \{0, 1\} \) let \( H^{i}_{\ell} \subseteq ([\alpha, \theta]^d, x \neq y} \sup_{x, y \in ([\alpha, \theta]^d, x \neq y} \frac{\mathcal{N}^{\theta}(x) - \mathcal{N}^{\theta}(y)}{\|x - y\|}\right]^{1/2}. \tag{57}
\]

for every \( i \in \{1, 2, \ldots, h\} \), \( \ell \in \{0, 1\} \) let \( H^{i}_{\ell} \subseteq ([\alpha, \theta]^d, x \neq y} \sup_{x, y \in ([\alpha, \theta]^d, x \neq y} \frac{\mathcal{N}^{\theta}(x) - \mathcal{N}^{\theta}(y)}{\|x - y\|}\right]^{1/2}. \tag{57}
\]

for every \( i \in \{1, 2, \ldots, h\} \), \( \ell \in \{0, 1\} \) let \( H^{i}_{\ell} \subseteq ([\alpha, \theta]^d, x \neq y} \sup_{x, y \in ([\alpha, \theta]^d, x \neq y} \frac{\mathcal{N}^{\theta}(x) - \mathcal{N}^{\theta}(y)}{\|x - y\|}\right]^{1/2}. \tag{57}
\]

for every \( i \in \{1, 2, \ldots, h\} \), \( \ell \in \{0, 1\} \) let \( H^{i}_{\ell} \subseteq ([\alpha, \theta]^d, x \neq y} \sup_{x, y \in ([\alpha, \theta]^d, x \neq y} \frac{\mathcal{N}^{\theta}(x) - \mathcal{N}^{\theta}(y)}{\|x - y\|}\right]^{1/2}. \tag{57}
\]

for every \( i \in \{1, 2, \ldots, h\} \), \( \ell \in \{0, 1\} \) let \( H^{i}_{\ell} \subseteq ([\alpha, \theta]^d, x \neq y} \sup_{x, y \in ([\alpha, \theta]^d, x \neq y} \frac{\mathcal{N}^{\theta}(x) - \mathcal{N}^{\theta}(y)}{\|x - y\|}\right]^{1/2}. \tag{57}
\]

for every \( i \in \{1, 2, \ldots, h\} \), \( \ell \in \{0, 1\} \) let \( H^{i}_{\ell} \subseteq ([\alpha, \theta]^d, x \neq y} \sup_{x, y \in ([\alpha, \theta]^d, x \neq y} \frac{\mathcal{N}^{\theta}(x) - \mathcal{N}^{\theta}(y)}{\|x - y\|}\right]^{1/2}. \tag{57}
\]
Note that Lemma 2.2 (applied with \( d \cap d, N \cap b, a \cap e, \theta \cap \theta, (w_i)_{i \in \{1, \ldots, N\}} \cap (w_i)_{i \in \{1, \ldots, b\}}, (b_i)_{i \in \{1, \ldots, N\}} \cap (b_i)_{i \in \{1, \ldots, b\}}, x \cap z \) in the notation of Lemma 2.2) shows that there exist \( z = (x_1, \ldots, x_d) \in [a, \theta]^d \), \( L_1, L_2, \ldots, L_b \in \{0, 1\}, \varepsilon \in (0, \infty) \) which satisfy that

\[
z \in \left( \bigcap_{i=1}^b H_i^L \right) \quad \text{and} \quad \{x \in [a, \theta]^d : \| x - z \| \leq \varepsilon \} \subseteq \left( \bigcap_{i=1}^b H_i^L \right).
\] (62)

Furthermore, observe that (60) demonstrates that for all \( i, j \in \{1, 2, 3\} \) with \( i \neq j \) it holds that

\[
\{1, 2, \ldots, b\} = A_1 \cup A_2 \cup A_3 \quad \text{and} \quad A_i \cap A_j = \emptyset.
\] (63)

In the following we distinguish between the case \( L = 0 \), the case \([L \in (0, \infty)] \cap [N = 0] \), the case \([L \in (0, \infty)] \cap [0 < N < b] \), and the case \([L \in (0, \infty)] \cap [N = b] \). We first prove (57) in the case \( L = 0 \).

Let \( \vartheta = (\vartheta_1, \ldots, \vartheta_b) \in \mathbb{R}^b \) satisfy for all \( i \in \{1, 2, \ldots, d - 1\} \) that

\[
\vartheta_i = 0 \quad \text{and} \quad \vartheta_b = \mathcal{N}^\vartheta(z).
\] (65)

Note that (64) establishes that for all \( x \in [a, \theta]^d \) it holds that \( \mathcal{N}^\vartheta(x) = \mathcal{N}^\vartheta(z) \). This and (65) ensure that for all \( x \in [a, \theta]^d \) it holds that

\[
\mathcal{N}^\vartheta(x) = \vartheta_b + \sum_{i=1}^b \vartheta_i h_i + \max\{\vartheta_i h_i + \sum_{j=1}^d \vartheta_{i-1} h_i h_j, 0\} = \vartheta_b = \mathcal{N}^\vartheta(z) = \mathcal{N}^\vartheta(x).
\] (66)

Moreover, observe that (61) and (65) imply that

\[
|\vartheta_b| = |\mathcal{N}^\vartheta(z)| = \inf_{x \in [a, \theta]^d} |\mathcal{N}^\vartheta(x)|.
\] (67)

The fact that for all \( i \in \{1, 2, \ldots, d - 1\} \) it holds that \( \vartheta_i = 0 \) therefore shows that

\[
\max_{i \in \{1, \ldots, b\}} |\vartheta_i| \leq \max\{2, |a| \sqrt{d}, |\theta| \sqrt{d}, \sqrt{\mathcal{N}^\vartheta(x)} \}, \mathcal{N}^\vartheta(x) \} = 2h L (\vartheta - \theta) \sqrt{d} \}. \] (68)

Combining this with (66) demonstrates (57) in the case \( L = 0 \). In the next step we prove (57) in the case

\[
[L \in (0, \infty)] \cap [N = 0].
\] (69)

Let \( u = (u_1, \ldots, u_d), u = (u_1, \ldots, u_d) \in \mathbb{R}^d \) satisfy for all \( j \in \{1, 2, \ldots, d\} \) that

\[
u_j = \sum_{i \in A_j} v_i w_{i,j}, \quad \delta \leq \gamma_{\max\{1, \|u\|\}} \quad \text{and} \quad u = \begin{cases} \sqrt{\mathcal{L}} / \|u\| & : \|u\| > 0 \\ 0 & : \|u\| = 0. \end{cases}
\] (70)

Note that the fact that \([a, \theta]^d \) \( \ni x \mapsto \langle u, x \rangle \in \mathbb{R} \) is continuous establishes that there exists \( q \in [a, \theta]^d \) which satisfies that

\[
\langle u, q \rangle = \inf_{x \in [a, \theta]^d} \langle u, x \rangle.
\] (71)

Let \( \vartheta = (\vartheta_1, \ldots, \vartheta_b) \in \mathbb{R}^b \) satisfy for all \( i \in \{2, 3, \ldots, b\} \), \( j \in \{1, 2, \ldots, d\} \) that

\[
\vartheta_j = u_j, \quad \vartheta_{db+1} = -\langle u, q \rangle, \quad \vartheta_{db+b+1} = \|u\| / \sqrt{\mathcal{L}}, \quad \vartheta_b = \mathcal{N}^\vartheta(z) + \langle u, q - z \rangle, \quad \text{and} \quad \vartheta_{(i-1)d+j} = \vartheta_{db+i} = \vartheta_{db+b+i} = 0.
\] (72)

Observe that (72) ensures that for all \( x = (x_1, \ldots, x_d) \in [a, \theta]^d \) it holds that

\[
\mathcal{N}^\vartheta(x) = \vartheta_b + \sum_{i=1}^b \vartheta_{db+b+i} \max\{\vartheta_{db+i} + \sum_{j=1}^d \vartheta_{(i-1)d+j} x_j, 0\}
= \vartheta_b + \vartheta_{db+b+i} \max\{\vartheta_{db+i} + \sum_{j=1}^d \vartheta_j x_j, 0\}
= \vartheta_b + \|u\| / \sqrt{\mathcal{L}} \max\{\langle u, x \rangle - \langle u, q \rangle, 0\}.
\] (73)
In addition, note that (71) implies that for all \(x \in [\alpha, \theta]^d\) it holds that
\[
\langle u, x \rangle \geq \inf_{x \in [\alpha, \theta]^d} \langle u, x \rangle = \langle u, q \rangle.
\] (74)
Combining this, (70), (72), and the fact that \(\|u\| = \sqrt{L}u\) shows that for all \(x \in [\alpha, \theta]^d\) it holds that
\[
\vartheta_b + \frac{\|u\|}{\sqrt{L}} \max \{\langle u, x \rangle - \langle u, q \rangle, 0\} = \mathcal{N}^\theta(z) + \langle u, q - z \rangle + \frac{\|u\|}{\sqrt{L}} \langle u, x \rangle - \langle u, q \rangle
\]
\[
= \mathcal{N}^\theta(z) + \langle u, q \rangle - \langle u, z \rangle + \langle u, x \rangle - \langle u, q \rangle
\]
\[
= \mathcal{N}^\theta(z) - \langle u, z \rangle + \langle u, x \rangle
\]
\[
= \mathcal{N}^\theta(z) - \sum_{i \in A_1} v_i \langle w_i, z \rangle + \sum_{i \in A_1} v_i \langle w_i, x \rangle.
\] (75)
Furthermore, observe that (69) demonstrates that \(A_2 = \emptyset\). The fact that for all \(x \in [\alpha, \theta]^d\), \(i \in A_1, j \in A_3\) it holds that \(b_i + \langle w_i, x \rangle \geq 0\) and \(b_j + \langle w_j, x \rangle \leq 0\), (59), and (63) hence prove that for all \(x \in [\alpha, \theta]^d\) it holds that
\[
\mathcal{N}^\theta(x) = \vartheta_b + \sum_{i = 1}^b \theta_{d_b + b + 1} \max \{\theta_{d_b + i} + \sum_{j = 1}^d \theta_{(i-1)d+j} x_j, 0\}
\]
\[
= \vartheta_b + \sum_{i = 1}^b v_i \max \{b_i + \langle w_i, x \rangle, 0\} = \vartheta_b + \sum_{i \in A_1} v_i (b_i + \langle w_i, x \rangle)
\] (76)
Combining this, (73), and (75) establishes that for all \(x \in [\alpha, \theta]^d\) it holds that
\[
\mathcal{N}^\theta(x) = \vartheta_b + \frac{\|u\|}{\sqrt{L}} \max \{\langle u, x \rangle - \langle u, q \rangle, 0\}
\]
\[
= \mathcal{N}^\theta(z) - \sum_{i \in A_1} v_i \langle w_i, z \rangle + \sum_{i \in A_1} v_i \langle w_i, x \rangle
\] (77)
Next note that (72), the fact that \(\|u\| \leq \sqrt{L}\), and the Cauchy Schwarz inequality ensure that for all \(j \in \{1, 2, \ldots, d\}\) it holds that
\[
|\vartheta_j| = |u_j| \leq \|u\| \leq \sqrt{L} \quad \text{and} \quad |\vartheta_{d_b + 1}| = |\langle u, q \rangle| \leq \|u\||\|q\| \leq \sqrt{dL} \max \{|\alpha|, |\theta|\}. \] (78)
Moreover, observe that (70), (76), and the fact that \(z + \delta u \in [\alpha, \theta]^d\) imply that
\[
|\mathcal{N}^\theta(z + \delta u) - \mathcal{N}^\theta(z)| = |\sum_{i \in A_1} v_i \langle w_i, z + \delta u \rangle - \sum_{i \in A_1} v_i \langle w_i, z \rangle| = |\delta| \sum_{i \in A_1} v_i \langle w_i, u \rangle|
\]
\[
= |\langle u, u \rangle| = |\delta||u|^2 = |\|u\|| |(z + \delta u) - z|. \] (79)
Therefore, we obtain that
\[
\|u\| \leq \sup_{x, y \in [\alpha, \theta]^d, x \neq y} \frac{|\mathcal{N}^\theta(x) - \mathcal{N}^\theta(y)|}{\|x - y\|} = L. \] (80)
Combining this with (61), (72), and the Cauchy Schwarz inequality shows that
\[
|\vartheta_b| = |\mathcal{N}^\theta(z) + \langle u, q - z \rangle| \leq |\mathcal{N}^\theta(z)| + \|u\|||q - z|| \leq \inf_{x \in [\alpha, \theta]^d} |\mathcal{N}^\theta(x)| + L(\theta - \alpha)\sqrt{d} \] (81)
and \(|\vartheta_{d_b + b + 1}| = \|u\|/\sqrt{L} \leq \sqrt{L}\). The fact that for all \(i \in \{2, 3, \ldots, b\}, j \in \{1, 2, \ldots, d\}\) it holds that \(\vartheta_{(i-1)d+j} = \vartheta_{d_b + i} = \vartheta_{d_b + b + i} = 0\) and (78) hence demonstrate that
\[
\max_{i \in \{1, \ldots, b\}} |\vartheta_i| \leq \max \{2, |\alpha|\sqrt{d}, |\theta|\sqrt{d}\} \sqrt{L}, \inf_{x \in [\alpha, \theta]^d} |\mathcal{N}^\theta(x)| + 2bL(\theta - \alpha)\sqrt{d}. \] (82)
Combining this with (77) proves (57) in the case \([L \in (0, \infty)] \land |N = 0\]. Next we prove (57) in the case
\[
[L \in (0, \infty)] \land |0 < N < b|.
\] (83)
Let $m_1, m_2, \ldots, m_N \in A_2$ satisfy for all $s, t \in \{1, 2, \ldots, N\}$ with $s \neq t$ that
\[
(H_{m_s}^0 \cap H_{m_t}^1) \neq (H_{m_t}^0 \cap H_{m_s}^1),
\]
(84)
let $D_s^l \subseteq \mathbb{N}$, $s \in \{1, 2, \ldots, N\}$, $\ell \in \{0, 1\}$, satisfy for all $s \in \{1, 2, \ldots, N\}$, $\ell \in \{0, 1\}$ that
\[
D_s^l = \{i \in A_2 : [(H_{m_s}^0 \cap H_{m_t}^1) = (H_{m_t}^0 \cap H_{m_s}^1)], z \in H_{l}^1\},
\]
(85)
and let $u = (u_1, \ldots, u_d), u = (u_1, \ldots, u_d) \in \mathbb{R}^d$ satisfy for all $j \in \{1, 2, \ldots, d\}$ that
\[
u_j = \sum_{i \in A_1} v_i w_{i,j} + \sum_{s=1}^{N} \sum_{i \in D_s^l} v_i w_{i,j}, \quad \text{and} \quad u = \begin{cases} \frac{\sqrt{T}}{\|u\|} u & : \|u\| > 0 \\ 0 & : \|u\| = 0. \end{cases}
\]
(86)
Note that the fact that $[\alpha, \beta]^d \ni x \mapsto \langle u, x \rangle \in \mathbb{R}$ is continuous establishes that there exists $q \in [\alpha, \beta]^d$ which satisfies that
\[
\langle u, q \rangle = \inf_{x \in [\alpha, \beta]^d} \langle u, x \rangle.
\]
(87)
In addition, observe that (60) ensures that for all $s \in \{1, 2, \ldots, N\}$ it holds that $\|w_{ms}\| > 0$. This and the fact that for all $s \in \{1, 2, \ldots, N\}$ it holds that $z \in H_{m_s}^0 \Delta H_{m_s}^1$ imply that there exist $\omega = (\omega_1, \ldots, \omega_N) = (\omega_{s,j})_{(s,j) \in \{1, 2, \ldots, N\} \times \{1, 2, \ldots, d\} \in \mathbb{R}^N \times d}$, $b = (b_1, \ldots, b_N) \in \mathbb{R}^N$ which satisfy for all $s \in \{1, 2, \ldots, N\}$ that
\[
\omega_s = \begin{cases} \frac{\sqrt{T}}{\|w_{ms}\|} w_{ms} & : \zeta \in H_{m_s}^0 \\ -\frac{\sqrt{T}}{\|w_{ms}\|} w_{ms} & : \zeta \in H_{m_s}^1 \end{cases} \quad \text{and} \quad b_s = \begin{cases} \frac{\sqrt{T}}{\|w_{ms}\|} b_{ms} & : \zeta \in H_{m_s}^0 \\ -\frac{\sqrt{T}}{\|w_{ms}\|} b_{ms} & : \zeta \in H_{m_s}^1 \end{cases}
\]
(88)
Let $H_s^l \subseteq [\alpha, \beta]^d$, $s \in \{1, 2, \ldots, N\}$, $\ell \in \{0, 1\}$, satisfy for all $s \in \{1, 2, \ldots, N\}$, $\ell \in \{0, 1\}$ that
\[
H_s^l = \{x \in [\alpha, \beta]^d : (-1)^{\ell}(b_s + \langle \omega_s, x \rangle) \leq 0\}.
\]
(89)
Note that (83) shows that there exists $\vartheta = (\vartheta_1, \ldots, \vartheta_b) \in \mathbb{R}^b$ which satisfies for all $s \in \{1, 2, \ldots, N\}$, $j \in \{1, 2, \ldots, d\}$, $t \in \{N+2, N+3, \ldots, b\}$ that
\[
\vartheta_{(s-1)d+j} = \omega_{s,j}, \quad \vartheta_{dh+s} = b_s, \quad \vartheta_{dh+b+s} = \sum_{i \in D_s^l \cup D_s^l} v_i \|w_i\|/\sqrt{T}, \\
\vartheta_{N+d+j} = u_j, \quad \vartheta_{dh+N+1} = \langle u, q \rangle, \quad \vartheta_{dh+b+N+1} = \|u\|/\sqrt{T}, \\
\vartheta_b = \mathcal{N}^0(z) + \langle u, q - z \rangle \quad \text{and} \quad \vartheta_{(t-1)d+j} = \vartheta_{dh+t} = \vartheta_{dh+b+t} = 0.
\]
(90)
Observe that (90) demonstrates that for all $x = (x_1, \ldots, x_d) \in [\alpha, \beta]^d$ it holds that
\[
\mathcal{N}^0(x) = \vartheta_0 + \sum_{i=1}^{b} \vartheta_{dh+b+i} \max \{\vartheta_{dh+i} + \sum_{j=1}^{d} \vartheta_{(i-1)d+j} x_j, 0\} = \vartheta_0 + \sum_{s=1}^{N+1} \vartheta_{dh+b+s} \max \{\vartheta_{dh+s} + \sum_{j=1}^{d} \vartheta_{(s-1)d+j} x_j, 0\} = \vartheta_0 + \|u\|/\sqrt{T} \max \{\langle u, x \rangle - \langle u, q \rangle, 0\} + \sum_{s=1}^{N} \vartheta_{dh+b+s} \max \{b_s + \langle \omega_s, x \rangle, 0\}.
\]
Furthermore, note that (87) proves that for all $x \in [\alpha, \beta]^d$ it holds that
\[
\langle u, x \rangle \geq \inf_{x \in [\alpha, \beta]^d} \langle u, x \rangle = \langle u, q \rangle.
\]
(92)
Combining this with (91) and the fact that $\|u\|u = \sqrt{T}u$ establishes that for all $x \in [\alpha, \beta]^d$ it holds that
\[
\mathcal{N}^0(x) = \vartheta_0 + \|u\|/\sqrt{T} \max \{\langle u, x \rangle - \langle u, q \rangle, 0\} + \sum_{s=1}^{N} \vartheta_{dh+b+s} \max \{b_s + \langle \omega_s, x \rangle, 0\} \]
(93)
\[
= \vartheta_0 + \|u\|/\sqrt{T} \langle u, x \rangle - \langle u, q \rangle + \sum_{s=1}^{N} \vartheta_{dh+b+s} \max \{b_s + \langle \omega_s, x \rangle, 0\}.
\]
Moreover, observe that (86) and (90) ensure that for all $x \in [\alpha, \beta]^d$ it holds that
\[
\vartheta_0 + \langle u, x \rangle - \langle u, q \rangle = N^\vartheta(z) + \langle u, q - z \rangle + \langle u, x \rangle - \langle u, q \rangle
\]
\[
= N^\vartheta(z) - \langle u, z \rangle + \langle u, x \rangle
\]
\[
= N^\vartheta(z) - \sum_{i \in A_1} v_i(w_i, z) - \sum_{s=1}^N \sum_{i \in D_s^1} v_i(w_i, z) + \sum_{s=1}^N \sum_{i \in D_s^1} v_i(w_i, x).
\] (94)

In addition, note that Lemma 2.3 (applied for every $s \in \{1, 2, \ldots, N\}$, $i \in D_s^0$ with $d \cap d$, $\alpha \cap \alpha$, $\beta \cap \beta$, $z \cap z$, $w_1 \cap w_i$, $w_2 \cap w_x$, $b_1 \cap b_i$, $b_2 \cap b_s$ in the notation of Lemma 2.3) and the fact that for all $s \in \{1, 2, \ldots, N\}$ it holds that $\|w_s\| = \sqrt{\mathcal{L}}$ imply that for all $s \in \{1, 2, \ldots, N\}$, $i \in D_s^0$ it holds that
\[
\|w_i\| = \|w_s\| = \|w_i\| w_i = \sqrt{\mathcal{L}} w_i \quad \text{and} \quad \|w_i\| b_s = \|w_s\| b_i = \sqrt{\mathcal{L}} b_i.
\] (95)

Furthermore, observe that Lemma 2.3 (applied for every $s \in \{1, 2, \ldots, N\}$, $i \in D_s^1$ with $d \cap d$, $\alpha \cap \alpha$, $\beta \cap \beta$, $z \cap z$, $w_1 \cap -w_i$, $w_2 \cap w_x$, $b_1 \cap -b_i$, $b_2 \cap b_s$ in the notation of Lemma 2.3) and the fact that for all $s \in \{1, 2, \ldots, N\}$ it holds that $\|w_s\| = \sqrt{\mathcal{L}}$ show that for all $s \in \{1, 2, \ldots, N\}$, $i \in D_s^1$ it holds that
\[
\|w_i\| = -\|w_s\| = -\sqrt{\mathcal{L}} w_i \quad \text{and} \quad \|w_i\| b_s = -\|w_s\| b_i = -\sqrt{\mathcal{L}} b_i.
\] (96)

Combining this and (95) demonstrates that for all $s \in \{1, 2, \ldots, N\}$, $x \in \mathcal{H}_s^0$, $y \in \mathcal{H}_s^1$, $i \in D_s^0$, $j \in D_s^1$ it holds that
\[
\langle w_i, x \rangle + b_i = \frac{\|w_i\|}{\sqrt{\mathcal{L}}} (b_s + \langle w_s, x \rangle) \leq 0,
\]
\[
\langle w_j, x \rangle + b_j = -\frac{\|w_j\|}{\sqrt{\mathcal{L}}} (b_s + \langle w_s, x \rangle) \geq 0,
\]
\[
\langle w_i, y \rangle + b_i = \frac{\|w_i\|}{\sqrt{\mathcal{L}}} (b_s + \langle w_s, y \rangle) \geq 0, \quad \text{and} \quad (97)
\]
\[
\langle w_j, y \rangle + b_j = -\frac{\|w_j\|}{\sqrt{\mathcal{L}}} (b_s + \langle w_s, y \rangle) \leq 0.
\]

The fact that for all $x \in [\alpha, \beta]^d$, $i \in A_1$, $j \in A_3$ it holds that $b_i + \langle w_i, x \rangle \geq 0$ and $b_j + \langle w_j, x \rangle \leq 0$, the fact that $A_2 = \bigcup_{s=1}^N (D_s^0 \cup D_s^1)$, the fact that for all $s, t \in \{1, 2, \ldots, N\}$ with $s \neq t$ it holds that $D_s^0 \cap D_t^1 = \emptyset$, $D_s^1 \cap D_t^0 = \emptyset$, and (63), and (97) therefore prove that for all $\ell_1, \ell_2, \ldots, \ell_N \in \{0, 1\}$, $x = (x_1, \ldots, x_d) \in (\bigcap_{s=1}^N \mathcal{H}_s^d)$ it holds that
\[
N^\vartheta(x) = \vartheta_0 + \sum_{i=1}^d \max\{\theta_{db_i + \mathbf{1}: i \neq 0}\}
\]
\[
= \vartheta_0 + \sum_{i=1}^d v_i \max\{b_i + \langle w_i, x \rangle, 0\}
\]
\[
= \vartheta_0 + \sum_{i \in A_1} v_i (b_i + \langle w_i, x \rangle) + \sum_{s=1}^N \sum_{i \in A_i} v_i (b_i + \langle w_i, x \rangle) + \sum_{s=1}^N \sum_{i \in D_s^1} v_i (b_i + \langle w_i, x \rangle)
\] (98)

This and the fact that $z \in (\bigcap_{s=1}^N \mathcal{H}_s^d)$ establish that for all $x \in [\alpha, \beta]^d$ it holds that
\[
N^\vartheta(z) - \sum_{i \in A_1} v_i (w_i, z) - \sum_{s=1}^N \sum_{i \in D_s^1} v_i (w_i, z) + \sum_{s=1}^N \sum_{i \in A_i} v_i (w_i, x) + \sum_{s=1}^N \sum_{i \in D_s^1} v_i (w_i, x)
\]
\[
= \vartheta_0 + \sum_{i \in A_1} v_i (b_i + \langle w_i, z \rangle) + \sum_{s=1}^N \sum_{i \in D_s^1} v_i (b_i + \langle w_i, z \rangle) - \sum_{s=1}^N \sum_{i \in A_i} v_i (w_i, z)
\]
\[
- \sum_{s=1}^N \sum_{i \in D_s^1} v_i (w_i, z) + \sum_{i \in A_1} v_i (w_i, x) + \sum_{s=1}^N \sum_{i \in D_s^1} v_i (w_i, x)
\]
\[
= \vartheta_0 + \sum_{i \in A_1} v_i b_i + \sum_{s=1}^N \sum_{i \in D_s^1} v_i b_i + \sum_{i \in A_1} v_i (w_i, x) + \sum_{s=1}^N \sum_{i \in D_s^1} v_i (w_i, x)
\]
\[
= \vartheta_0 + \sum_{i \in A_1} v_i (b_i + \langle w_i, x \rangle) + \sum_{s=1}^N \sum_{i \in D_s^1} v_i (b_i + \langle w_i, x \rangle).
\] (99)
Moreover, note that the fact that for all \( s \in \{1, 2, \ldots, N\} \), \( x \in \mathcal{H}_s^0 \), \( y \in \mathcal{H}_s^1 \) it holds that \( b_s + \langle w_s, x \rangle \leq 0 \) and \( b_s + \langle w_s, y \rangle \geq 0 \), the fact that for all \( s \in \{1, 2, \ldots, N\} \) it holds that \( D_s^0 \cap D_s^1 = \emptyset \), (95), and (96) ensure that for all \( \ell_1, \ell_2, \ldots, \ell_N \in \{0, 1\} \), \( x \in \bigcap_{s=1}^N \mathcal{H}_s^\ell_s \) it holds that

\[
\sum_{s=1}^N \partial_{db+b+s} \max \{b_s + \langle w_s, x \rangle, 0\} = \sum_{s=1}^N \partial_{db+b+s} \max \{b_s + \langle w_s, x \rangle \}
\]

(100)

Combining this, (93), (94), (98), and (99) implies that for all \( \ell_1, \ell_2, \ldots, \ell_N \in \{0, 1\} \), \( x \in \bigcap_{s=1}^N \mathcal{H}_s^\ell_s \) it holds that

\[
\mathcal{N}^\theta(x) = \partial_\theta + \langle u, x \rangle - \langle u, q \rangle + \sum_{s=1}^N \partial_{db+b+s} \max \{b_s + \langle w_s, x \rangle, 0\}
\]

\[
= \mathcal{N}^\theta(z) - \sum_{i \in A_1} v_i(w_i, z) - \sum_{s=1}^N \sum_{i \in D_s^0} v_i(w_i, z) + \sum_{i \in A_1} v_i(w_i, x) + \sum_{s=1}^N \partial_{db+b+s} \max \{b_s + \langle w_s, x \rangle, 0\}
\]

\[
= \partial_\theta + \sum_{i \in A_1} v_i(b_i + \langle w_i, x \rangle) + \sum_{s=1}^N \sum_{i \in D_s^0} v_i(b_i + \langle w_i, x \rangle)
\]

(101)

\[
= \partial_\theta + \sum_{i \in A_1} v_i(b_i + \langle w_i, x \rangle) + \sum_{s=1}^N \sum_{i \in D_s^0} v_i(b_i + \langle w_i, x \rangle)
\]

\[
= \mathcal{N}^\theta(x).
\]

The fact that \([\alpha, \beta]^d = \bigcup_{\ell_1, \ell_2, \ldots, \ell_N \in \{0, 1\}} \bigcap_{s=1}^N \mathcal{H}_s^\ell_s\) hence shows that for all \( x \in [\alpha, \beta]^d \) it holds that

\[
\mathcal{N}^\theta(x) = \mathcal{N}^\theta(x).
\]

(102)

In addition, observe that the fact that for all \( s \in \{1, 2, \ldots, N\} \) it holds that \( \|w_s\| = \sqrt{L} \), the fact that \( \|u\| \leq \sqrt{L} \), and (90) demonstrate that for all \( s \in \{1, 2, \ldots, N\} \), \( j \in \{1, 2, \ldots, d\} \) it holds that

\[
|\langle v_j, x \rangle| = \|w_{sj}\| \leq \sqrt{L} \quad \text{and} \quad |\langle v_{Nj}, x \rangle| = |v_j| \leq \|u\| \leq \sqrt{L}.
\]

(103)

Furthermore, note that Lemma 2.5 (applied with \( d \land d, N \land N, \alpha \land \alpha, \beta \land \beta \), \( (w_i)_{i \in \{1, 2, \ldots, N\}} \) \( \land \) \( (w_s)_{s \in \{1, 2, \ldots, N\}} \), \( (b_i)_{i \in \{1, 2, \ldots, N\}} \) \( \land \) \( (b_s)_{s \in \{1, 2, \ldots, N\}} \) in the notation of Lemma 2.5) proves that there exist \( p_1, p_2, \ldots, p_N \in [\alpha, \beta]^d \), \( \delta \in (0, \varepsilon/\max(1, \|u\|)) \) which satisfy that

(i) it holds for all \( s \in \{1, 2, \ldots, N\} \) that \( p_s \in (\mathcal{H}_s^0 \cap \mathcal{H}_s^1) \),

(ii) it holds for all \( s \in \{1, 2, \ldots, N\} \) that \( \{x \in \mathbb{R}^d : \|x - p_s\| \leq \delta\} \subseteq [\alpha, \beta]^d \), and

(iii) it holds for all \( s \in \{1, 2, \ldots, N\} \) that \( \{x \in \mathbb{R}^d : \|x - p_s\| \leq \delta\} \cap (\bigcup_{s=1}^N \mathcal{H}_{\ell_s}^0 \cap \mathcal{H}_{\ell_s}^1) = \emptyset \).

Observe that item (i) establishes that for all \( s \in \{1, 2, \ldots, N\} \) it holds that \( b_s + \langle w_s, p_s \rangle = 0 \). The fact that for all \( s \in \{1, 2, \ldots, N\} \) it holds that \( \|w_s\| = \sqrt{L} \), the fact that \( \|u\| \leq \sqrt{L} \), the Cauchy Schwarz inequality, and (90) therefore ensure that for all \( s \in \{1, 2, \ldots, N\} \), \( j \in \{1, 2, \ldots, d\} \) it holds that

\[
|\langle v_j, x \rangle| = |\langle w_{sj}, p_s \rangle| \leq \|w_s\| \|p_s\| \leq \sqrt{dL} \max\{|\alpha|, |\beta|\}
\]

(104)
and
\[ |\vartheta_{dh+N+1}| = |\langle u, q \rangle| \leq \|u\|\|q\| \leq \sqrt{dL} \max\{|e|, \|\sigma\|\}. \]  

(105)

Moreover, note that (93) and the fact that for all \( s \in \{1, 2, \ldots, N\}, x \in \mathcal{H}_s^0, y \in \mathcal{H}_s^1 \) it holds that
\[ b_s + \langle w_s, x \rangle \leq 0 \quad \text{and} \quad b_s + \langle w_s, y \rangle \geq 0 \]
implies that for all \( \ell_1, \ell_2, \ldots, \ell_N \in \{0, 1\}, x, y \in (\bigcap_{s=1}^N \mathcal{H}_s^0) \) it holds that
\[
\mathcal{N}^\vartheta(x) - \mathcal{N}^\vartheta(y) = \left[ \vartheta_b + \langle u, x \rangle - \langle u, q \rangle + \sum_{s=1}^N \vartheta_{dh+b+s} \max\{b_s + \langle w_s, x \rangle, 0\} \right] \\
- \left[ \vartheta_b + \langle u, y \rangle - \langle u, q \rangle + \sum_{s=1}^N \vartheta_{dh+b+s} \max\{b_s + \langle w_s, y \rangle, 0\} \right] \\
= \langle u, x - y \rangle + \sum_{s \in \{1, 2, \ldots, N\}, \ell_s = 1} \vartheta_{dh+b+s} \left( b_s + \langle w_s, x \rangle \right) \\
- \sum_{s \in \{1, 2, \ldots, N\}, \ell_s = 1} \vartheta_{dh+b+s} \left( b_s + \langle w_s, y \rangle \right) \\
= \langle u, x - y \rangle + \sum_{s \in \{1, 2, \ldots, N\}, \ell_s = 1} \vartheta_{dh+b+s} \left( w_s, x - y \right) \\
(106)

In addition, observe that the fact that \([e, \sigma]^d = \bigcup_{\ell_1, \ell_2, \ldots, \ell_N \in \{0, 1\}} (\bigcap_{s=1}^N \mathcal{H}_s^{\ell_s}) \) and item (i) show that for all \( s \in \{1, 2, \ldots, N\} \) there exist \( \ell_1, \ell_2, \ldots, \ell_N \in \{0, 1\} \) such that \( p_s \in (\bigcap_{s=1}^N \mathcal{H}_s^{\ell_s}) \) and \( \ell_s = 1 \). Combining this, items (ii) and (iii), (106), and the fact that for all \( s \in \{1, 2, \ldots, N\} \) it holds that \( \|w_s\| = \sqrt{\mathcal{L}} \) demonstrates that for all \( s \in \{1, 2, \ldots, N\} \) there exists \( \ell_1, \ell_2, \ldots, \ell_N \in \{0, 1\} \) such that
\[
\mathcal{N}^\vartheta(p_s + \frac{\delta}{\|w_s\|} w_s) - \mathcal{N}^\vartheta(p_s) \\
= \langle u, \frac{\delta}{\|w_s\|} w_s \rangle + \sum_{\ell_s = 1} \vartheta_{dh+b+s} \langle \frac{\delta}{\|w_s\|} w_s, w_s \rangle \\
= \langle u, \frac{\delta}{\|w_s\|} w_s \rangle + \sum_{\ell_s = 1} \vartheta_{dh+b+s} \langle \frac{\delta}{\|w_s\|} w_s, w_s \rangle \\
= \mathcal{N}^\vartheta(p_s) - \mathcal{N}^\vartheta(p_s - \frac{\delta}{\|w_s\|} w_s) + \vartheta_{dh+b+s} \sqrt{\mathcal{L}}. \\
(107)

This and (102) prove that for all \( s \in \{1, 2, \ldots, N\} \) it holds that
\[
|\vartheta_{dh+b+s}| \leq \frac{1}{\sqrt{\mathcal{L}}} \left[ \frac{1}{2} |\mathcal{N}^\vartheta(p_s + \frac{\delta}{\|w_s\|} w_s) - \mathcal{N}^\vartheta(p_s)| + \frac{1}{2} |\mathcal{N}^\vartheta(p_s) - \mathcal{N}^\vartheta(p_s - \frac{\delta}{\|w_s\|} w_s)| \right] \\
= \frac{1}{\sqrt{\mathcal{L}}} \left[ \frac{1}{2} |\mathcal{N}^\vartheta(p_s + \frac{\delta}{\|w_s\|} w_s) - \mathcal{N}^\vartheta(p_s)| + \frac{1}{2} |\mathcal{N}^\vartheta(p_s) - \mathcal{N}^\vartheta(p_s - \frac{\delta}{\|w_s\|} w_s)| \right] \\
\leq \frac{2}{\sqrt{\mathcal{L}}} \left( \sup_{x, y \in [e, \sigma]^d, x \neq y} \frac{|\mathcal{N}^\vartheta(x) - \mathcal{N}^\vartheta(y)|}{\|x - y\|} \right) = 2\sqrt{\mathcal{L}}. \\
(108)

Furthermore, note that (86), (102), (106), the fact that \( x \in (\bigcap_{s=1}^N \mathcal{H}_s^0) \), and the fact that \( x + \delta u \in (\bigcap_{s=1}^N \mathcal{H}_s^0) \) establish that
\[
|\mathcal{N}^\vartheta(x + \delta u) - \mathcal{N}^\vartheta(x)| = |\mathcal{N}^\vartheta(x) - \mathcal{N}^\vartheta(x)| = |\langle u, \delta u \rangle| = \delta \|u\|^2 = \|u\|\|(x + \delta u) - x\|. \\
(109)

Hence, we obtain that
\[
\|u\| \leq \sup_{x, y \in [e, \sigma]^d, x \neq y} \frac{|\mathcal{N}^\vartheta(x) - \mathcal{N}^\vartheta(y)|}{\|x - y\|} = \mathcal{L}. \\
(110)

Combining this with (61), (90), and the Cauchy Schwarz inequality ensures that
\[
|\vartheta_b| = |\mathcal{N}^\vartheta(z) + \langle u, q - z \rangle| \leq |\mathcal{N}^\vartheta(z)| + |\langle u, q - z \rangle| \\
\leq (\inf_{x \in [e, \sigma]^d} |\mathcal{N}^\vartheta(x)|) + \|u\|\|q - z\| \leq (\inf_{x \in [e, \sigma]^d} |\mathcal{N}^\vartheta(x)|) + L(\vartheta - \alpha) \sqrt{d} \\
\]
and \( |\vartheta_{dh+b+N+1}| = \|u\|/\sqrt{\mathcal{L}} \leq \sqrt{\mathcal{L}}. \) This, (103), (104), (105), (108), and the fact that for all \( t \in \{N+2; N+3, \ldots, b\}, j \in \{1, 2, \ldots, d\} \) it holds that \( \vartheta_{(t-1)d+j} = \vartheta_{dh+t} = \vartheta_{dh+b+t} = 0 \) imply that
\[
\max_{i \in \{1, \ldots, d\}} |\vartheta_i| \leq \max\{2, |\vartheta|, |\vartheta|, \sqrt{d}, |\vartheta|, |\vartheta|, \sqrt{d}, \sqrt{d}, |\vartheta|, |\vartheta|, \sqrt{d}, |\vartheta|, |\vartheta|, \sqrt{d}, |\vartheta|, |\vartheta|, \sqrt{d}, |\vartheta|, |\vartheta|, \sqrt{d}, |\vartheta|, |\vartheta|, \sqrt{d}, |\vartheta|, |\vartheta|, \sqrt{d}, |\vartheta|, \sqrt{d}, \} \leq 2bL(\vartheta - \alpha) \sqrt{d}. \\
(112)
Combining this with (102) shows (57) in the case \([L \in (0, \infty)] \land [N < h]\). In the last step we prove (57) in the case
\[
[L \in (0, \infty)] \land [N = h].
\] (113)
Observe that (60) and (113) demonstrate that for all \(i \in \{1, 2, \ldots, h\}\) it holds that \(\|w_i\| > 0\). Therefore, we obtain that there exists \(\vartheta = (\vartheta_1, \ldots, \vartheta_h) \in \mathbb{R}^h\) which satisfies for all \(i \in \{1, 2, \ldots, h\}, j \in \{1, 2, \ldots, d\}\) that
\[
\vartheta_{(i-1)d+j} = \frac{\sqrt{T_{w_{i-1}}}}{\|w_{i-1}\|}, \quad \vartheta_{dh+i} = \frac{\sqrt{T_d}}{\|w_{h}\|}, \quad \vartheta_{dh+h+i} = \frac{v_i\|w_i\|}{\sqrt{h}}, \quad \text{and} \quad \vartheta_{b} = \vartheta_0. \tag{114}
\]
Note that (59) and (114) establish that for all \(x = (x_1, \ldots, x_d) \in [\alpha, \beta]^d\) it holds that
\[
\mathcal{N}^\delta(x) = \vartheta_0 + \sum_{i=1}^{h} \vartheta_{dh+b+i} \max \left\{ \vartheta_{dh+i} + \sum_{j=1}^{d} \vartheta_{(i-1)d+j} x_j, 0 \right\} = \vartheta_0 + \sum_{i=1}^{h} v_i \|w_i\| \max \left\{ b_i + \sum_{j=1}^{d} w_{i,j} x_j, 0 \right\}
= \vartheta_0 + \sum_{i=1}^{h} v_i \max \{ b_i + \sum_{j=1}^{d} w_{i,j} x_j, 0 \}
= \vartheta_0 + \sum_{i=1}^{h} \vartheta_{dh+b+i} \max \{ \vartheta_{dh+i} + \sum_{j=1}^{d} \vartheta_{(i-1)d+j} x_j, 0 \} = \mathcal{N}^\theta(x).
\] (115)
Moreover, observe that (114) ensures that for all \(i \in \{1, 2, \ldots, h\}, j \in \{1, 2, \ldots, d\}\) it holds that
\[
|\vartheta_{(i-1)d+j}| = \frac{\sqrt{T_{w_{i-1}}}}{\|w_{i-1}\|} \leq \sqrt{T}.
\] (116)
In addition, note that Lemma 2.5 (applied with \(d \land n, \alpha \land \beta, \theta \land \delta, (w_i)_{i \in \{1, 2, \ldots, N\}} \land (w_i)_{i \in \{1, 2, \ldots, h\}} (b_i)_{i \in \{1, 2, \ldots, N\}} \land (b_i)_{i \in \{1, 2, \ldots, h\}}\) in the notation of Lemma 2.5) implies that there exist \(p_1, p_2, \ldots, p_h \in [\alpha, \beta]^d, \delta \in (0, \infty)\) which satisfy that
(i) it holds for all \(i \in \{1, 2, \ldots, h\}\) that \(p_i \in (H^0 \cap H^1_i)\),
(ii) it holds for all \(i \in \{1, 2, \ldots, h\}\) that \(\{x \in \mathbb{R}^d : \|x - p_i\| \leq \delta\} \subseteq [\alpha, \beta]^d\), and
(iii) it holds for all \(i \in \{1, 2, \ldots, h\}\) that \(\{x \in \mathbb{R}^d : \|x - p_i\| \leq \delta\} \cap (\bigcup_{j \in \{1, \ldots, h\} \setminus \{i\}} (H^0 \cap H^1_j)) = \emptyset\).
Observe that item (i) shows that for all \(i \in \{1, 2, \ldots, h\}\) it holds that \(b_i + \langle w_i, p_i \rangle = 0\). Combining this with (114) and the Cauchy Schwarz inequality proves that for all \(i \in \{1, 2, \ldots, h\}\) it holds that
\[
|\vartheta_{dh+i}| = \frac{\sqrt{T_{w_{i}}}}{\|w_{i}\|} |b_i| = \frac{\sqrt{T_{w_{i}}}}{\|w_{i}\|} \|w_{i}\| |p_i| \leq \sqrt{T} \max \{ |\alpha|, |\beta| \}.
\] (117)
Next note that the fact that for all \(i \in \{1, 2, \ldots, h\}, x \in H^0_i, y \in H^1_i\) it holds that \(b_i + \langle w_i, x \rangle \leq 0\) and \(b_i + \langle w_i, y \rangle \geq 0\) demonstrates that for all \(\ell_1, \ell_2, \ldots, \ell_h \in \{0, 1\}, x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d) \in (\bigcap_{i=1}^{h} H^i_1)\) it holds that
\[
\mathcal{N}^\theta(x) - \mathcal{N}^\theta(y) = \left[ \vartheta_0 + \sum_{i=1}^{h} \vartheta_{dh+b+i} \max \left\{ \vartheta_{dh} + \sum_{j=1}^{d} \vartheta_{(i-1)d+j} x_j, 0 \right\} \right] - \left[ \vartheta_0 + \sum_{i=1}^{h} \vartheta_{dh+b+i} \max \left\{ \vartheta_{dh} + \sum_{j=1}^{d} \vartheta_{(i-1)d+j} y_j, 0 \right\} \right]
= \sum_{i=1}^{h} v_i \max \{ b_i + \langle w_i, x \rangle, 0 \} - \sum_{i=1}^{h} v_i \max \{ b_i + \langle w_i, y \rangle, 0 \}
= \sum_{i=1}^{h} v_i \max \{ b_i + \langle w_i, x \rangle, 0 \} - \sum_{i=1}^{h} v_i \max \{ b_i + \langle w_i, y \rangle, 0 \}
= \sum_{i \in \{1, 2, \ldots, h\}, \ell_i = 1} v_i (b_i + \langle w_i, x \rangle - [b_i + \langle w_i, y \rangle])
= \sum_{i \in \{1, 2, \ldots, h\}, \ell_i = 1} v_i \langle w_i, x - y \rangle.
\] (118)
Furthermore, observe that the fact that \([\alpha, \beta]^d = \bigcup_{\ell_1, \ell_2, \ldots, \ell_h \in \{0, 1\}} (\bigcap_{i=1}^{h} H^i_j)\) and item (i) establish that for all \(i \in \{1, 2, \ldots, h\}\) there exist \(\ell_1, \ell_2, \ldots, \ell_h \in \{0, 1\}\) such that \(p_i \in (\bigcap_{j=1}^{h} H^i_j)\) and...
\( \ell_i = 1 \). Combining this, items (ii) and (iii), and (118) ensures that for all \( i \in \{1, 2, \ldots, h\} \) there exist \( \ell_1, \ell_2, \ldots, \ell_h \in [0, 1] \) such that for all \( i \in \{1, 2, \ldots, h\} \) it holds that

\[
\mathcal{N}_\theta(p_i + \frac{\delta}{\|w_i\|} w_i) - \mathcal{N}_\theta(p_i) = \sum_{j \in \{1, 2, \ldots, h\} \setminus \{i\}, \ell_j = 1} v_j \langle w_j, \frac{\delta}{\|w_i\|} w_i \rangle + v_i \frac{\delta}{\|w_i\|} \langle w_i, w_i \rangle
\]

\[
= \sum_{j \in \{1, 2, \ldots, h\} \setminus \{i\}, \ell_j = 1} v_j \langle w_j, \frac{\delta}{\|w_i\|} w_i \rangle + \delta v_i \|w_i\|
\]

\[
= \mathcal{N}_\theta(p_i) - \mathcal{N}_\theta(p_i - \frac{\delta}{\|w_i\|} w_i) + \delta v_i \|w_i\|.
\]

(119)

Hence, we obtain that for all \( i \in \{1, 2, \ldots, h\} \) it holds that

\[
|\theta_{db+b+i}| = \frac{|v_i||w_i|}{\sqrt{L}} \leq \frac{1}{\sqrt{L}} \left[ \frac{1}{\delta} |\mathcal{N}_\theta(p_i + \frac{\delta}{\|w_i\|} w_i) - \mathcal{N}_\theta(p_i)| + \frac{1}{\delta} |\mathcal{N}_\theta(p_i) - \mathcal{N}_\theta(p_i - \frac{\delta}{\|w_i\|} w_i)| \right]
\]

\[
\leq \frac{2L}{\sqrt{L}} \left( \sup_{x, y \in [\alpha, \beta]^d, x \neq y} \frac{|\mathcal{N}_\theta(x) - \mathcal{N}_\theta(y)|}{\|x - y\|} \right) = 2\sqrt{L}.
\]

(120)

The fact that for all \( i \in \{1, 2, \ldots, h\} \) it holds that \( b_i + \langle w_i, p_i \rangle = 0 \), the fact that for all \( x \in \mathbb{R} \) it holds that \( |\max \{x, 0\}| \leq |x| \), and the Cauchy Schwarz inequality therefore imply that for all \( i \in \{1, 2, \ldots, h\} \) it holds that

\[
|v_i |\max \{b_i + \langle w_i, z \rangle\}| = |v_i |\max \{\langle w_i, z - p_i \rangle\}| \leq |v_i|\|w_i\|\|z - p_i\|
\]

\[
\leq \sqrt{\frac{\mathcal{N}}{\|w_i\|}} |\theta_{db+b+i}|\|w_i\|(|\beta - \alpha|\sqrt{d}) \leq 2L(|\beta - \alpha|\sqrt{d}).
\]

(121)

This, (59), (61), and (114) show that

\[
|\theta_{b} - |\theta_{b} - |\mathcal{N}_\theta(z) - \sum_{i=1}^{k} v_i \max \{b_i + \langle w_i, z_i \rangle, 0\}|\]

\[
\leq |\mathcal{N}_\theta(z)| + |\sum_{i=1}^{k} v_i \max \{b_i + \langle w_i, z_i \rangle, 0\}|
\]

\[
\leq (\inf_{x \in [\alpha, \beta]} |\mathcal{N}_\theta(x)|) + 2hL(\beta - \alpha)\sqrt{d}.
\]

(122)

Combining this, (116), (117), and (120) proves that

\[
\max_{i \in \{1, \ldots, h\}} |\theta_i| \leq \max \{2, |\alpha|\sqrt{d}, |\beta|\sqrt{d}, \sqrt{L}, [\inf_{x \in [\alpha, \beta]} |\mathcal{N}_\theta(x)|] + 2hL(\beta - \alpha)\sqrt{d} \}.
\]

(123)

This and (115) demonstrate (57) in the case \([L \in (0, \infty)] \land [N = h]\). The proof of Theorem 2.7 is thus complete.

**Definition 2.8.** Let \( d \in \mathbb{N}, \alpha \in \mathbb{R}, \beta \in (\alpha, \infty), A \subseteq [\alpha, \beta]^d \) satisfy \( A \neq \emptyset \) and let \( f : [\alpha, \beta]^d \rightarrow \mathbb{R} \) be a function. Then we denote by \( \|f\|_A \in [0, \infty] \) the extended real number given by

\[
\|f\|_A = \inf_{x \in A} |f(x)| + \sup_{x, y \in [\alpha, \beta]^d, x \neq y} \frac{|f(x) - f(y)|}{\|x - y\|}
\]

(124)

(cf. Definition 2.1).

**Corollary 2.9.** Assume Setting 2.6 and let \( \theta \in \mathbb{R}^d, A \subseteq [\alpha, \beta]^d \) satisfy \( A \neq \emptyset \). Then there exists \( \vartheta = (\vartheta_1, \ldots, \vartheta_d) \in \mathbb{R}^d \) such that \( \mathcal{N}_\theta = \mathcal{N}_{\vartheta} \) and

\[
\max_{i \in \{1, 2, \ldots, h\}} |\theta_i| \leq \max \{2, |\alpha|\sqrt{d}, |\beta|\sqrt{d}, 2h(\beta - \alpha)\sqrt{d} \} \max \{|\mathcal{N}_\theta\|_A^{1/2}, |\mathcal{N}_\theta\|_A\}.
\]

(125)

(cf. Definition 2.8).

**Proof of Corollary 2.9.** Throughout this proof let \( L \in [0, \infty) \) satisfy

\[
L = \sup_{x, y \in [\alpha, \beta]^d, x \neq y} \frac{|\mathcal{N}_\theta(x) - \mathcal{N}_\theta(y)|}{\|x - y\|}
\]

(126)
(cf. Definition 2.1). Note that Theorem 2.7 establishes that there exists $\vartheta = (\vartheta_1, \ldots, \vartheta_\delta) \in \mathbb{R}^\delta$ which satisfies that $\mathcal{N}^\vartheta = \mathcal{N}^{\vartheta_0}$ and
\[
\max_{i \in \{1, \ldots, \delta\}} |\vartheta_i| \leq \max\{\max\{2, |\vartheta_1|\sqrt{d_1} |\vartheta_2| \sqrt{L}, |\inf_{x \in [a, \vartheta] \cap \mathcal{N}^{\vartheta_0}(x)| \} + 2hL(\vartheta - \vartheta_0) \sqrt{d} \}\}. \quad (127)
\]
Furthermore, observe that the fact that $L \leq \|\mathcal{N}^{\vartheta_0}\|_A$ ensures that
\[
\max\{2, |\vartheta_1|\sqrt{d_1} |\vartheta_2| \sqrt{L}, |\inf_{x \in [a, \vartheta] \cap \mathcal{N}^{\vartheta_0}(x)| \} + 2hL(\vartheta - \vartheta_0) \sqrt{d} \}\leq \max\{2, |\vartheta_1|\sqrt{d_1} |\vartheta_2| \sqrt{L}, 2h(\vartheta - \vartheta_0) \sqrt{d} \}\|\mathcal{N}^{\vartheta_0}\|_A^{1/2} \quad (128)
\]
(cf. Definition 2.8). Moreover, note that the fact that $\inf_{x \in [a, \vartheta] \cap \mathcal{N}^{\vartheta_0}(x)| \} \leq \inf_{x \in A} |\mathcal{N}^{\vartheta_0}(x)|$ implies that
\[
|\inf_{x \in [a, \vartheta] \cap \mathcal{N}^{\vartheta_0}(x)| \} + 2hL(\vartheta - \vartheta_0) \sqrt{d} \leq \inf_{x \in A} |\mathcal{N}^{\vartheta_0}(x)| + 2hL(\vartheta - \vartheta_0) \sqrt{d} \leq \max\{2, |\vartheta_1|\sqrt{d_1} |\vartheta_2| \sqrt{L}, 2h(\vartheta - \vartheta_0) \sqrt{d} \}\|\mathcal{N}^{\vartheta_0}\|_A \quad (129)
\]
Combining this with (127) and (128) shows that
\[
\max_{i \in \{1, \ldots, \delta\}} |\vartheta_i| \leq \max\{2, |\vartheta_1|\sqrt{d_1} |\vartheta_2| \sqrt{L}, |\inf_{x \in [a, \vartheta] \cap \mathcal{N}^{\vartheta_0}(x)| \} + 2hL(\vartheta - \vartheta_0) \sqrt{d} \}\leq \max\{2, |\vartheta_1|\sqrt{d_1} |\vartheta_2| \sqrt{L}, 2h(\vartheta - \vartheta_0) \sqrt{d} \}\max\{\|\mathcal{N}^{\vartheta_0}\|_A^{1/2}, \|\mathcal{N}^{\vartheta_0}\|_A \}. \quad (130)
\]
The proof of Corollary 2.9 is thus complete.

**Corollary 2.10.** Assume Setting 2.6 and let $n \in \mathbb{N}$, $\delta_1, \delta_2, \ldots, \delta_n \in [0, \infty)$, $A \subseteq [a, \vartheta]^d$ satisfy $\min\{\delta_1, \delta_2, \ldots, \delta_n\} \leq \delta_1$ and $\frac{n}{\delta_1} \geq \frac{1}{\delta_1}$. Then there exists $c \in \mathbb{R}$ such that for all $\vartheta \in \mathbb{R}^\delta$ there exists $\hat{\vartheta} \in \mathbb{R}^\delta$ such that $\mathcal{N}^{\vartheta} = \mathcal{N}^{\hat{\vartheta}}$ and
\[
\|\vartheta\| \leq c \left(\sum_{i=1}^{\delta} \|\mathcal{N}^{\vartheta}\|_A^{\delta_i} \right) \quad (131)
\]
(cf. Definitions 2.1 and 2.8).

**Proof of Corollary 2.10.** Observe that the assumption that $\min\{\delta_1, \delta_2, \ldots, \delta_n\} \leq \delta_1$ and the assumption that $\max\{\delta_1, \delta_2, \ldots, \delta_n\} \geq \delta_1$ prove that there exist $i, j \in \{1, 2, \ldots, n\}$ which satisfy that
\[
\delta_i \leq \delta_1 \quad \text{and} \quad \delta_j \geq \delta_1. \quad (132)
\]
Note that Corollary 2.9 and (132) demonstrate that there exists $c \in \mathbb{R}$ such that for all $\vartheta \in \mathbb{R}^\delta$ there exists $\hat{\vartheta} \in \mathbb{R}^\delta$ such that $\mathcal{N}^{\vartheta} = \mathcal{N}^{\hat{\vartheta}}$ and
\[
\|\vartheta\| = \left[\sum_{j=1}^{\delta} |\vartheta_j|^2 \right]^{1/2} \leq \left[\sum_{j=1}^{\delta} \max_{i \in \{1, 2, \ldots, \delta\}} |\vartheta_i|^2 \right]^{1/2} = \sqrt{\delta} \max_{i \in \{1, 2, \ldots, \delta\}} |\vartheta_i| \leq \sqrt{\delta} c \max\{\|\mathcal{N}^{\vartheta_0}\|_A^{\delta_1}, \|\mathcal{N}^{\vartheta_0}\|_A^{\delta_2} \} \quad (133)
\]
(cf. Definitions 2.1 and 2.8). The proof of Corollary 2.10 is thus complete.
2.4 Equivalence of norms of reparameterized ANNs and Lipschitz norms

Lemma 2.11. Assume Setting 2.6 and let $\theta \in \mathbb{R}^3$. Then
\[
\sup_{x,y \in [a, \delta]^d, x \neq y} \frac{|N^\theta(x) - N^\theta(y)|}{\|x - y\|} \leq \|\theta\|^2.
\] (cf. Definition 2.1).

Proof of Lemma 2.11. Observe that the fact that for all $x, y \in \mathbb{R}$ it holds that $|\max\{x, 0\} - \max\{y, 0\}| \leq |x - y|$ establishes that for all $x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d) \in [a, \delta]^d$ it holds that
\[
|N^\theta(x) - N^\theta(y)| = |\theta_0 + \sum_{i=1}^{\delta} \theta_{d_h + h + i} \max\{\theta_{d_h + i} + \sum_{j=1}^{d_d} \theta_{(i-1)d+j} x_j, 0\} - \theta_0 - \sum_{i=1}^{\delta} \theta_{d_h + h + i} \max\{\theta_{d_h + i} + \sum_{j=1}^{d_d} \theta_{(i-1)d+j} y_j, 0\}|
\leq \sum_{i=1}^{\delta} \theta_{d_h + h + i} \sum_{j=1}^{d_d} \theta_{(i-1)d+j} (x_j - y_j).
\]
Furthermore, note that the Cauchy-Schwarz inequality ensures that for all $x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d) \in [a, \delta]^d$ it holds that
\[
\sum_{i=1}^{\delta} \theta_{d_h + h + i} \sum_{j=1}^{d_d} \theta_{(i-1)d+j} (x_j - y_j)
\leq \|x - y\| \sum_{i=1}^{\delta} \theta_{d_h + h + i} \left(\sum_{j=1}^{d_d} \theta_{(i-1)d+j}^2\right)^{1/2}
\leq \|x - y\| \|\theta\|^2.
\] (cf. Definition 2.1). Combining this with (135) implies that for all $x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d) \in [a, \delta]^d$ it holds that
\[
|N^\theta(x) - N^\theta(y)| \leq \sum_{i=1}^{\delta} \theta_{d_h + h + i} \sum_{j=1}^{d_d} \theta_{(i-1)d+j} (x_j - y_j) \leq \|x - y\| \|\theta\|^2.
\] Hence, we obtain that
\[
\sup_{x,y \in [a, \delta]^d, x \neq y} \frac{|N^\theta(x) - N^\theta(y)|}{\|x - y\|} \leq \|\theta\|^2.
\] The proof of Lemma 2.11 is thus complete. \qed

Lemma 2.12. Assume Setting 2.6 and let $\theta \in \mathbb{R}^3$, $A \subseteq [a, \delta]^d$ satisfy $A \neq \emptyset$. Then
\[
\|N^\theta\|_A \leq \|\theta\| + (2 + \inf_{x \in A} \|x\|) \|\theta\|^2 \leq \|\theta\| + (2 + \max\{|a|, |\delta|\} \sqrt{d}) \|\theta\|^2
\] (cf. Definitions 2.1 and 2.8).

Proof of Lemma 2.12. Observe that the fact that for all $x \in \mathbb{R}$ it holds that $|\max\{x, 0\} \leq |x|$ shows that for all $y = (y_1, \ldots, y_d) \in A$ it holds that
\[
\inf_{x \in A} |N^\theta(x)| \leq |N^\theta(y)| = \|\theta_0 + \sum_{i=1}^{\delta} \theta_{d_h + h + i} \max\{\theta_{d_h + i} + \sum_{j=1}^{d_d} \theta_{(i-1)d+j} y_j, 0\}|
\leq \|\theta_0\| + \sum_{i=1}^{\delta} \theta_{d_h + h + i} \max\{\theta_{d_h + i} + \sum_{j=1}^{d_d} \theta_{(i-1)d+j} y_j, 0\}
\leq \|\theta\| + \sum_{i=1}^{\delta} \theta_{d_h + h + i} \sum_{j=1}^{d_d} \theta_{(i-1)d+j} y_j
\leq \|\theta\| + \sum_{i=1}^{\delta} \theta_{d_h + h + i} \sum_{j=1}^{d_d} \theta_{(i-1)d+j} y_j
\]
(140)
Furthermore, note that the Cauchy Schwarz inequality proves that for all \( y = (y_1, \ldots, y_d) \in A \) it holds that
\[
\sum_{i=1}^{b} |\theta_{dh+b+i}| + \sum_{j=1}^{d} |\theta_{(i-1)d+j}| \leq \left[ \sum_{i=1}^{b} \|y\| \sum_{i=1}^{b} |\theta_{dh+b+i}| \sum_{j=1}^{d} |\theta_{(i-1)d+j}| \right]^{1/2}.
\]
If \( \theta \) and \( y \) are such that
\[
\inf_{x \in A} |\mathcal{N}^\theta(x)| \leq \|\theta\| + (1 + \inf_{x \in A} \|x\|) \|\theta\|^2.
\]
Combining this with (40) demonstrates that for all \( y = (y_1, \ldots, y_d) \in A \) it holds that
\[
\inf_{x \in A} |\mathcal{N}^\theta(x)| \leq \|\theta\| + (1 + \inf_{x \in A} \|x\|) \|\theta\|^2.
\]
Therefore, we obtain that
\[
\inf_{x \in A} |\mathcal{N}^\theta(x)| \leq \|\theta\| + (1 + \inf_{x \in A} \|x\|) \|\theta\|^2.
\]
Combining this, Lemma 2.11, and the fact that for all \( x \in A \) it holds that \( \|x\| \leq \max\{\|\alpha\|, \|\beta\|\} \sqrt{d} \) establishes that
\[
\|\mathcal{N}^\theta \|_A = \inf_{x \in A} |\mathcal{N}^\theta(x)| + \sup_{x,y \in A} \|\mathcal{N}^\theta(x) - \mathcal{N}^\theta(y)\|_{x,y}
\]
\[
\leq \|\theta\| + (1 + \inf_{x \in A} \|x\|) \|\theta\|^2 = \|\theta\| + (2 + \inf_{x \in A} \|x\|) \|\theta\|^2.
\]
Corollary 2.13. Assume Setting 2.6 and let \( \theta \in \mathbb{R}^d \), \( A \subseteq [\alpha, \beta]^d \) satisfy \( A \neq \emptyset \). Then for all \( \theta \in \mathbb{R}^d \) there exists \( \psi \in \mathbb{R}^3 \) such that \( \mathcal{N}^\theta = \mathcal{N}^\psi \) and
\[
\|\theta\| \leq \sqrt{d} \max\{2, \|\alpha\| \sqrt{d}, \|\beta\| \sqrt{d}, 2h(\beta - \alpha) \sqrt{d}\} \|\mathcal{N}^\theta \|_A \|\mathcal{N}^\theta \|_A
\]
\[
\leq 8 \sqrt{d} \max\{2, \|\alpha\| \sqrt{d}, \|\beta\| \sqrt{d}, 2h(\beta - \alpha) \sqrt{d}\}^2 \max\{\|\theta\|^{1/2}, \|\theta\|^2\}.
\]
Furthermore, note that Corollary 2.9 shows that for all $\theta \in \mathbb{R}^3$ there exists $\vartheta = (\vartheta_1, \ldots, \vartheta_3) \in \mathbb{R}^3$ such that $\mathcal{N}^\vartheta = \mathcal{N}^\theta$ and

$$\sqrt{3} \max_{i \in \{1, 2, \ldots, 3\}} |\vartheta_i| \leq \sqrt{3} \max \left\{ 2, |a| \sqrt{d}, |\vartheta| \sqrt{d}, 2\theta(\vartheta - a) \sqrt{d} \right\} \max \left\{ \| \mathcal{N}^\vartheta \|_{A}^{1/2}, \| \mathcal{N}^\theta \|_{A} \right\}$$  \hspace{1cm} (149)

Combining this with (148) proves that for all $\theta \in \mathbb{R}^3$ there exists $\vartheta = (\vartheta_1, \ldots, \vartheta_3) \in \mathbb{R}^3$ such that $\mathcal{N}^\vartheta = \mathcal{N}^\theta$ and

$$\| \vartheta \| = \left[ \sum_{j=1}^{3} |\vartheta_j|^2 \right]^{1/2} \leq \sqrt{3} \max_{i \in \{1, 2, \ldots, 3\}} |\vartheta_i| \leq \sqrt{3} \max \left\{ 2, |a| \sqrt{d}, |\vartheta| \sqrt{d}, 2\theta(\vartheta - a) \sqrt{d} \right\} \max \left\{ \| \mathcal{N}^\vartheta \|_{A}^{1/2}, \| \mathcal{N}^\theta \|_{A} \right\}$$ \hspace{1cm} (150)

The proof of Corollary 2.13 is thus complete.

**Corollary 2.14.** Assume Setting 2.6 and let $\theta \in \mathbb{R}^3$, $A \subseteq [a, \vartheta]^d$ satisfy $A \neq \emptyset$. Then there exist $a, \vartheta \in \mathbb{R}$ such that for all $\theta \in \mathbb{R}^3$ there exists $\vartheta \in \mathbb{R}^3$ such that $\mathcal{N}^\vartheta = \mathcal{N}^\theta$ and

$$\max\{1, \| \vartheta \|\} \leq e \max\{1, \| \mathcal{N}^\vartheta \|_{A}\} \leq \vartheta \max\{1, \| \vartheta \|^{2}\}. \hspace{1cm} (151)$$

(cf. Definitions 2.1 and 2.8).

**Proof of Corollary 2.14.** Observe that Corollary 2.13 demonstrates that there exist $a, \vartheta \in [2, \infty)$ which satisfy that for all $\theta \in \mathbb{R}^3$ there exists $\vartheta \in \mathbb{R}^3$ such that $\mathcal{N}^\vartheta = \mathcal{N}^\theta$ and

$$\| \vartheta \| \leq e \max\{\| \mathcal{N}^\vartheta \|_{A}^{1/2}, \| \mathcal{N}^\theta \|_{A}\} \leq \vartheta \max\{\| \vartheta \|^{1/2}, \| \vartheta \|^{2}\}$$ \hspace{1cm} (152)

(cf. Definitions 2.1 and 2.8). Note that (152) establishes that for all $\theta \in \mathbb{R}^3$ there exists $\vartheta \in \mathbb{R}^3$ such that $\mathcal{N}^\vartheta = \mathcal{N}^\theta$ and

$$\max\{1, \| \vartheta \|\} \leq \max\{1, e \max\{\| \mathcal{N}^\vartheta \|_{A}^{1/2}, \| \mathcal{N}^\theta \|_{A}\}\} \leq \max\{1, e \max\{1, \| \mathcal{N}^\vartheta \|_{A}\}\} \leq e \max\{1, \| \mathcal{N}^\vartheta \|_{A}\} = \max\{e, \vartheta \max\{\| \vartheta \|^{1/2}, \| \vartheta \|^{2}\}\} \leq \max\{e, \vartheta \max\{1, \| \vartheta \|^{2}\}\} \leq \max\{e, \vartheta \max\{1, \| \vartheta \|^{2}\}\} \leq \max\{e, \vartheta \max\{1, \| \vartheta \|^{2}\}\}. \hspace{1cm} (153)$$

The proof of Corollary 2.14 is thus complete.

**3 Lower bounds for norms of reparameterized ANNs using Lipschitz norms**

This section addresses the optimality of the upper bounds from Section 2 with regard to the exponents $1/2$ and $1$ of the powers of the Lipschitz norm of the realization function and is devoted to establish lower bounds for norms of reparameterized ANN parameter vectors using Lipschitz norms. In Corollary 3.3 in Subsection 3.2 below we show that it is not possible to bound reparameterized ANN parameter vectors from above by sums of powers of the Lipschitz norm of the realization function if the range of the exponents does not extend from $1/2$ to $1$. Our proof of Corollary 3.3 uses the lower bounds for reparameterized ANNs established in Theorem 3.2 in Subsection 3.2, which, in turn, is based on the result for output biases of ANNs with a maximum number of different kinks shown in Lemma 3.1 in Subsection 3.1 below.
3.1 Output biases of ANNs with a maximum number of different kinks

Lemma 3.1. Assume Setting 2.6 and let \( c \in \mathbb{R} \), \( \theta = (\theta_1, \ldots, \theta_b) \in \mathbb{R}^b \) satisfy for all \( x = (x_1, \ldots, x_d) \in [\alpha, \theta]^d \) that \( N^\theta(x) = c + \sum_{i=1}^b \max\{x_i - \alpha - \frac{i(\delta - \alpha)}{b+1}, 0\} \). Then \( \theta_0 = c \).

Proof of Lemma 3.1. Throughout this proof let \( u = (1, 0, 0, \ldots, 0) \in \mathbb{R}^d \), \( w = (w_1, \ldots, w_b) = (w_{i,j})_{(i,j) \in \{1,2,\ldots,b\} \times \{1,2,\ldots,d\}} \in \mathbb{R}^{b \times d} \), \( b = (b_1, \ldots, b_b) \in \mathbb{R}^b \) satisfy for all \( i \in \{1,2,\ldots,b\} \), \( j \in \{1,2,\ldots,d\} \) that
\[
w_{i,j} = \theta_{i-1} d + j, \quad b_i = \theta_{i-d+i}, \quad \text{and} \quad v_i = \theta_{d+b+i},
\]
for every \( i \in \{1,2,\ldots,b\} \), \( \ell \in \{0,1\} \), let \( H^\ell_i \subseteq [\alpha, \theta]^d \) satisfy \( H^\ell_i = \{x \in [\alpha, \theta]^d : (-1)^\ell(b_i + \langle w_i, x \rangle) \leq 0\} \), let \( A_k \subseteq \mathbb{N} \), \( k \in \{1,2,3\} \), satisfy
\[
A_1 = \{i \in \{1,2,\ldots,b\} : [H^1_i = [\alpha, \theta]^d]\},
\]
\[
A_2 = \{i \in \{1,2,\ldots,b\} : ([H^1_i \neq [\alpha, \theta]^d] \land [H^1_i \cap (\alpha, \theta)^d \neq \emptyset])\},
\]
and
\[
A_3 = \{i \in \{1,2,\ldots,b\} : [H^1_i \cap (\alpha, \theta)^d = \emptyset]\},
\]
let \( N \in \mathbb{R} \) satisfy \( N = \#(\bigcup_{i \in A_2} (H^1_i \cap H^1_j)) \), let \( A_4 \subseteq A_2 \) satisfy for all \( i, j \in A_4 \) with \( i \neq j \) that \( (H^1_i \cap H^1_j) \neq (H^1_j \cap H^1_i) \) and \#\( A_4 = N \), and let \( q_1, q_2, \ldots, q_b \in [\alpha, \theta]^d \), \( \varepsilon \in (0, \infty) \) satisfy for all \( i \in \{1,2,\ldots,b\} \) that
\[
q_i = \left(\alpha + \frac{i(\delta - \alpha)}{b+1}, \alpha, \ldots, \alpha\right) \quad \text{and} \quad \varepsilon < \frac{\delta - \alpha}{b+1}.
\]
(cf. Definition 2.1). Observe (155) ensures that for all \( i, j \in \{1,2,3\} \) with \( i \neq j \) it holds that
\[
\{1,2,\ldots,b\} = A_1 \cup A_2 \cup A_3 \quad \text{and} \quad A_i \cap A_j = \emptyset.
\]
We now imply by contradiction that for all \( i \in \{1,2,\ldots,b\} \) there exists \( j \in \{1,2,\ldots,b\} \) such that
\[
(H^0_j \cap H^1_j) = \{x \in [\alpha, \theta]^d : \langle u, x \rangle - \alpha - \frac{i(\delta - \alpha)}{b+1} = 0\}.
\]
In the following we thus assume that there exists \( i \in \{1,2,\ldots,b\} \) which satisfies that for all \( j \in \{1,2,\ldots,b\} \) it holds that
\[
(H^0_j \cap H^1_j) \neq \{x \in [\alpha, \theta]^d : \langle u, x \rangle - \alpha - \frac{i(\delta - \alpha)}{b+1} = 0\}.
\]
Note that Lemma 2.5 (applied with \( d = d, N \cap N + 1, a = a, \theta = \theta, (w_{ij}) \in \{1,2,\ldots, N - 1\} \cap (w_{ij}) \in A_1, w_N \cap u, (b_{ij}) \in \{1,2,\ldots, N - 1\} \cap (b_{ij}) \in A_4, b_N \cap a - i(\delta - \alpha)[b + 1]^{-1} \) in the notation of Lemma 2.5) and (159) show that there exist \( p \in [\alpha, \theta]^d \), \( \delta \in (0, \frac{\delta - \alpha}{b+1}) \) which satisfy that
\begin{enumerate}
  \item[(i)] it holds that \( \langle u, p \rangle - \alpha - \frac{i(\delta - \alpha)}{b+1} = 0 \),
  \item[(ii)] it holds that \( \{x \in \mathbb{R}^d : \|x - p\| \leq \delta\} \subseteq [\alpha, \theta]^d \), and
  \item[(iii)] it holds that \( \{x \in \mathbb{R}^d : \|x - p\| \leq \delta\} \cap (\bigcup_{j \in A_2} (H^0_j \cap H^1_j)) = \emptyset \).
\end{enumerate}

Observe that items (i) and (ii) prove that for all \( x \in \{y \in \mathbb{R}^d : \|y\| \leq \delta\} \) it holds that
\[
N^\theta(p + x) = c + \sum_{j=1}^b \max\{\langle u, p + x \rangle - \alpha - \frac{i(\delta - \alpha)}{b+1}, 0\}
\]
\[
= c + \sum_{j=1}^b \max\{\langle u, p \rangle - \alpha - \frac{i(\delta - \alpha)}{b+1} + \frac{(j-1)(\delta - \alpha)}{b+1} + \langle u, x \rangle, 0\}
\]
\[
= c + \sum_{j=1}^b \max\{\frac{(j-i)(\delta - \alpha)}{b+1} + \langle u, x \rangle, 0\}.
\]
Hence, we obtain that
\[ N^\theta(p + \delta u) - 2N^\theta(p) + N^\theta(p - \delta u) \]
\[ = \sum_{j=1}^{b} \max \left\{ \frac{(i-j)(\delta - \alpha)}{b+1} + \delta, 0 \right\} - 2 \sum_{j=1}^{b} \max \left\{ \frac{(i-j)(\delta - \alpha)}{b+1}, 0 \right\} \]
\[ = \sum_{j=1}^{b} \max \left\{ \frac{(i-j)(\delta - \alpha)}{b+1} - \delta, 0 \right\} - 2 \sum_{j=1}^{b} \max \left\{ \frac{(i-j)(\delta - \alpha)}{b+1}, 0 \right\} \]
\[ = \sum_{j=1}^{b} \left\{ \frac{(i-j)(\delta - \alpha)}{b+1} + \delta \right\} - \sum_{j=1}^{b} \left\{ \frac{(i-j)(\delta - \alpha)}{b+1} - \delta \right\} = \delta. \tag{161} \]

Furthermore, note that items (ii) and (iii) demonstrate that for all \( x \in \{ y \in \mathbb{R}^d : \| y - p \| \leq \delta \} \) it holds that
\[ \{ j \in A_2 : p \in H_j \} = \{ j \in A_2 : p + x \in H_j \}. \tag{162} \]
Combining this, (157), and the fact that for all \( j \in A_3 \), \( x \in [\alpha, \delta]^d \) it holds that \( b_j + \langle w_j, x \rangle \leq 0 \) establishes that for all \( x \in \{ y \in \mathbb{R}^d : \| y - p \| \leq \delta \} \) it holds that
\[ N^\theta(x) = \theta_0 + \sum_{j=1}^{b} v_j \max \left\{ b_j + \langle w_j, x \rangle, 0 \right\} \]
\[ = \theta_0 + \sum_{j \in A_1} v_j \langle b_j + \langle w_j, x \rangle \rangle + \sum_{j \in A_2} v_j \max \left\{ b_j + \langle w_j, x \rangle, 0 \right\} \]
\[ = \theta_0 + \sum_{j \in A_1} v_j \langle b_j + \langle w_j, x \rangle \rangle + \sum_{j \in A_2, p \in H_j} v_j \langle b_j + \langle w_j, x \rangle \rangle \]
\[ = \theta_0 + \sum_{j \in A_1 \cup A_2, p \in H_j} v_j \langle b_j + \langle w_j, x \rangle \rangle \]
\[ = \theta_0 + \sum_{j \in A_1 \cup A_2, p \in H_j} v_j b_j + \sum_{j \in A_1 \cup A_2, p \in H_j} v_j \langle w_j, x \rangle. \tag{163} \]
This and (161) ensure that
\[ \delta = N^\theta(p + \delta u) - 2N^\theta(p) + N^\theta(p - \delta u) \]
\[ = \sum_{j \in A_1 \cup A_2, p \in H_j} v_j \langle w_j, p + \delta u \rangle - 2 \sum_{j \in A_1 \cup A_2, p \in H_j} v_j \langle w_j, p \rangle \]
\[ = \sum_{j \in A_1 \cup A_2, p \in H_j} v_j \langle w_j, \delta u \rangle - \sum_{j \in A_1 \cup A_2, p \in H_j} v_j \langle w_j, \delta u \rangle = 0. \tag{164} \]
This is a contradiction to the assumption that \( \delta \in (0, \infty) \). Therefore, we obtain that for all \( i \in \{ 1, 2, \ldots, b \} \) there exists \( j \in \{ 1, 2, \ldots, b \} \) such that
\[ (H_j^i \cap H_j) = \left\{ x \in \{ \alpha, \delta \}^d : \langle u, x \rangle - \alpha = \frac{i(\delta - \alpha)}{b+1} = 0 \right\}. \tag{165} \]
This implies that there exists a bijective function \( \varphi : \{ 1, 2, \ldots, b \} \rightarrow \{ 1, 2, \ldots, b \} \) which satisfies that for all \( i \in \{ 1, 2, \ldots, b \} \) it holds that
\[ (H_{\varphi(i)}^0 \cap H_{\varphi(i)}^1) = \left\{ x \in \{ \alpha, \delta \}^d : \langle u, x \rangle - \alpha = \frac{i(\delta - \alpha)}{b+1} = 0 \right\}. \tag{166} \]
Observe that (156) shows that for all \( i \in \{ 1, 2, \ldots, b \} \), \( t \in [-\varepsilon, \varepsilon] \) it holds that \( q_i + tu \in [\alpha, \delta]^d \) and
\[ N^\theta(q_i + tu) = c + \sum_{j=1}^{b} \max \left\{ \langle u, q_i + tu \rangle - \alpha - \frac{j(\delta - \alpha)}{b+1}, 0 \right\} \]
\[ = c + \sum_{j=1}^{b} \max \left\{ \langle u, q_i \rangle + t\langle u, u \rangle - \alpha - \frac{j(\delta - \alpha)}{b+1}, 0 \right\} \]
\[ = c + \sum_{j=1}^{b} \max \left\{ \frac{(i-j)(\delta - \alpha)}{b+1} + t, 0 \right\}. \tag{167} \]
Hence, we obtain that for all \( i \in \{ 1, 2, \ldots, b \} \) it holds that
\[ N^\theta(q_i + \varepsilon u) - 2N^\theta(q_i) + N^\theta(q_i - \varepsilon u) \]
\[ = \sum_{j=1}^{b} \max \left\{ \frac{(i-j)(\delta - \alpha)}{b+1} + \varepsilon, 0 \right\} - 2 \sum_{j=1}^{b} \max \left\{ \frac{(i-j)(\delta - \alpha)}{b+1}, 0 \right\} \]
\[ = \sum_{j=1}^{b} \max \left\{ \frac{(i-j)(\delta - \alpha)}{b+1} - \varepsilon, 0 \right\} - 2 \sum_{j=1}^{b} \max \left\{ \frac{(i-j)(\delta - \alpha)}{b+1}, 0 \right\} \]
\[ = \sum_{j=1}^{b} \left\{ \frac{(i-j)(\delta - \alpha)}{b+1} + \varepsilon \right\} - 2 \sum_{j=1}^{b} \left\{ \frac{(i-j)(\delta - \alpha)}{b+1} \right\} + \sum_{j=1}^{b} \left\{ \frac{(i-j)(\delta - \alpha)}{b+1} - \varepsilon \right\} = \varepsilon. \tag{168} \]

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Moreover, note that (166) and the fact that for all \(i \in \{1, 2, \ldots, h\}\) it holds that \(q_i \in (H_{\varphi(i)}^0 \cap H_{\varphi(i)}^1)\) prove that for all \(i \in \{1, 2, \ldots, h\}\), \(t \in [-\varepsilon, \varepsilon]\) it holds that

\[
\{j \in \{1, 2, \ldots, h\} : q_i \in H_j^1\} = \{j \in \{1, 2, \ldots, h\} \setminus \{\varphi(i)\} : q_i + tu \in H_j^1\}. \tag{169}
\]

This and the fact that \(A_1 \cup A_3 = \emptyset\) demonstrate that for all \(i \in \{1, 2, \ldots, h\}\), \(t \in [-\varepsilon, \varepsilon]\) it holds that

\[
\mathcal{N}^\vartheta(q_i + tu) = \theta_0 + \sum_{j=1}^{h} v_j \max\{b_j + \langle w_i, q_i + tu \rangle, 0\} = \theta_0 + \sum_{j \in A_2 \setminus \{\varphi(i)\}} v_j \max\{b_j + \langle w_i, q_i + tu \rangle, 0\} + v_{\varphi(i)} \max\{b_{\varphi(i)} + \langle w_{\varphi(i)}, q_i + tu \rangle, 0\}
\]

\[= \theta_0 + \sum_{j \in A_2 \setminus \{\varphi(i)\}, q_i \in H_j^1} v_j \langle b_j + \langle w_i, q_i + tu \rangle, 0\rangle + v_{\varphi(i)} \max\{\langle t(w_{\varphi(i)}, u), 0\rangle\}
\]

\[
= \theta_0 + \sum_{j \in A_2 \setminus \{\varphi(i)\}, q_i \in H_j^1} v_j \langle b_j + \langle w_i, q_i \rangle, 0\rangle + \sum_{j \in A_2 \setminus \{\varphi(i)\}, q_i \in H_j^1} v_j \langle t(w_j, u), 0\rangle + v_{\varphi(i)} \max\{\langle t(w_{\varphi(i)}, u), 0\rangle\}
\]

\[= \theta_0 + \sum_{j \in A_2 \setminus \{\varphi(i)\}, q_i \in H_j^1} v_j \langle b_j + \langle w_i, q_i \rangle, 0\rangle + \sum_{j \in A_2 \setminus \{\varphi(i)\}, q_i \in H_j^1} v_j \langle t(w_j, u), 0\rangle
\]

\[
= \varepsilon v_{\varphi(i)} \max\{\langle w_{\varphi(i)}, u \rangle, 0\rangle + \max\{\langle -w_{\varphi(i)}, u \rangle, 0\} \}
\]

The fact that for all \(x \in \mathbb{R}\) it holds that \(\max\{x, 0\} + \max\{-x, 0\} = |x|\) and (168) therefore establish that for all \(i \in \{1, 2, \ldots, h\}\) it holds that

\[
\varepsilon = \mathcal{N}^\vartheta(q_i + \varepsilon u) - 2\mathcal{N}^\vartheta(q_i) + \mathcal{N}^\vartheta(q_i - \varepsilon u)
\]

\[= \sum_{j \in A_2 \setminus \{\varphi(i)\}, q_i \in H_j^1} \varepsilon v_j \langle w_j, u \rangle + v_{\varphi(i)} \max\{\varepsilon \langle w_{\varphi(i)}, u \rangle, 0\}
\]

\[
- \sum_{j \in A_2 \setminus \{\varphi(i)\}, q_i \in H_j^1} \varepsilon v_j \langle w_j, u \rangle + v_{\varphi(i)} \max\{-\varepsilon \langle w_{\varphi(i)}, u \rangle, 0\}
\]

\[
= \varepsilon v_{\varphi(i)} \max\{\varepsilon \langle w_{\varphi(i)}, u \rangle, 0\} + \max\{-\varepsilon \langle w_{\varphi(i)}, u \rangle, 0\}
\]

Combining this and the fact that \(\varphi\) is bijective ensures that for all \(i \in \{1, 2, \ldots, h\}\) it holds that

\[
v_i \langle w_i, u \rangle = 1. \tag{172}
\]

In addition, observe that (167) implies that it holds that

\[
\mathcal{N}^\vartheta(q_h + \varepsilon u) - \mathcal{N}^\vartheta(q_h + \frac{1}{2}\varepsilon u) = \sum_{i=1}^{h} \max\left\{\frac{(b_i(\varepsilon - \delta))}{b_i + 1} + \varepsilon, 0\right\} - \sum_{i=1}^{h} \max\left\{\frac{(b_i(\varepsilon - \delta))}{b_i + 1} + \frac{1}{2}\varepsilon, 0\right\}
\]

\[= \sum_{i=1}^{h} \left[\frac{(b_i(\varepsilon - \delta))}{b_i + 1} + \varepsilon\right] - \sum_{i=1}^{h} \left[\frac{(b_i(\varepsilon - \delta))}{b_i + 1} + \frac{1}{2}\varepsilon\right] = \varepsilon h - \frac{1}{2}\varepsilon h = \frac{1}{2}\varepsilon h. \tag{173}
\]

Furthermore, note that (166) shows that

\[
\{i \in \{1, 2, \ldots, h\} : q_h + \varepsilon u \in H_i^1\} = \{i \in \{1, 2, \ldots, h\} : q_h + \frac{1}{2}\varepsilon u \in H_i^1\}. \tag{174}
\]

Combining this, (173), and the fact that \(A_1 \cup A_3 = \emptyset\) proves that

\[
\frac{1}{2}\varepsilon h = \mathcal{N}^\vartheta(q_h + \varepsilon u) - \mathcal{N}^\vartheta(q_h + \frac{1}{2}\varepsilon u)
\]

\[= \sum_{i=1}^{h} v_i \max\{b_i + \langle w_i, q_h + \varepsilon u \rangle, 0\} - \sum_{i=1}^{h} v_i \max\{b_i + \langle w_i, q_h + \frac{1}{2}\varepsilon u \rangle, 0\}
\]

\[= \sum_{i \in A_2, q_h + u \in H_i^1} v_i (b_i + \langle w_i, q_h + \varepsilon u \rangle) - \sum_{i \in A_2, q_h + u \in H_i^1} v_i (b_i + \langle w_i, q_h + \frac{1}{2}\varepsilon u \rangle)
\]

\[= \sum_{i \in A_2, q_h + u \in H_i^1} v_i (\langle w_i, u \rangle, u) - \sum_{i \in A_2, q_h + u \in H_i^1} v_i (\langle w_i, \frac{1}{2}\varepsilon u \rangle, u).
\]

Moreover, observe that (172) demonstrates that for all \(i \in \{1, 2, \ldots, h\}\) it holds that \(v_i > 0\). This, (172), and (175) establish that

\[
\frac{h}{2} \geq \mathcal{N}^\vartheta(q_h + \varepsilon u) - \mathcal{N}^\vartheta(q_h + \frac{1}{2}\varepsilon u)
\]

\[= \sum_{i \in A_2, q_h + u \in H_i^1} v_i \langle w_i, u \rangle \leq \sum_{i \in A_2, q_h + u \in H_i^1} v_i \langle w_i, u \rangle
\]

\[= \#\{i \in A_2 : q_h + \varepsilon u \in H_i^1\} \leq h. \tag{176}
\]
Hence, we obtain that for all $i \in \{1, 2, \ldots, h\}$ it holds that $q_b + \epsilon u \in H^1_i$. This ensures that for all $i \in \{1, 2, \ldots, h\}$ it holds that $q_1 - \epsilon u \in H^0_i$. Combining this with (156) implies that

$$
\theta_0 = \theta_b + \sum_{i=1}^{h} v_i \max\{b_i + \langle w_i, q_1 - \epsilon u \rangle, 0\} = N^\theta (q_1 - \epsilon u)
$$

$$
eq c + \sum_{i=1}^{h} \max\{\langle u, q_1 - \epsilon u \rangle - \epsilon - \frac{\epsilon - \theta}{b+1}, 0\}
$$

$$
eq c + \sum_{i=1}^{h} \max\{\frac{(1-i)(\epsilon - \theta)}{b+1} - \epsilon, 0\} = c
$$

(177)

The proof of Lemma 3.1 is thus complete. 

\[ \square \]

### 3.2 Lower bounds for norms of reparameterized ANNs using Lipschitz norms

**Theorem 3.2.** Assume Setting 2.6 and let $\epsilon \in [0, \infty)$, $\delta \in [\epsilon, \infty)$, $A \subseteq [\eta, \delta]^d$ satisfy $|\epsilon > 1/2 \lor |\delta < 1$ and $A \neq \emptyset$. Then for all $c \in \mathbb{R}$ there exists $\theta \in \mathbb{R}^d$ such that for all $\theta \in \{\eta \in \mathbb{R}^d : N^n = N^\theta\}$ it holds that

$$
\|\theta\| > c \max\{\|N^\theta\|_{A}, \|N^\theta\|_{A}^\delta\}. 
$$

(178)

(cf. Definitions 2.1 and 2.8).

**Proof of Theorem 3.2.** Throughout this proof let $u = (1, 0, 0, \ldots, 0) \in \mathbb{R}^d$ and for every $\theta = (\theta_1, \ldots, \theta_b) \in \mathbb{R}^b$, $i \in \{1, 2, \ldots, h\}$ let $H^n_i \subseteq [\eta, \delta]^d$ satisfy

$$
H^n_i = \{x = (x_1, \ldots, x_d) \in [\eta, \delta]^d : \theta_{db+i} + \sum_{j=1}^{d} \theta_{(i-1)d+j} x_j = 0\}
$$

(179)

In the following we distinguish between the case $\epsilon > 1/2$ and the case $\delta < 1$. We first prove (178) in the case

$$
\epsilon > 1/2.
$$

(180)

Let $\theta = (\theta_1, \ldots, \theta_b) : \mathbb{N} \rightarrow \mathbb{R}^b$ satisfy for all $n \in \mathbb{N}$, $i \in \{2, 3, \ldots, h\}$, $j \in \{1, 2, \ldots, d\}$ that

$$
\theta_1(n) = 1, \quad \theta_{db+i}(n) = -\alpha, \quad \theta_{db+b+1}(n) = n^{-1},
$$

(181)

and $\theta_i(n) = \theta_{(i-1)d+j}(n) = n^{-1} \max\{\theta_{db+i}(n) + \sum_{j=1}^{d} \theta_{(i-1)d+j}(n)x_j, 0\}$

Therefore, we obtain that for all $n \in \mathbb{N}$, $x = (x_1, \ldots, x_d) \in [\eta, \delta]^d$ it holds that

$$
|N^\theta(x)| = n^{-1} \max\{|u, x - \alpha, 0\}| = n^{-1} \max\{|u, x - \alpha, 0\} \leq n^{-1} (\delta - \alpha).
$$

(183)

Furthermore, observe that (182), the fact that for all $x, y \in \mathbb{R}$ it holds that $|\max\{x, 0\} - \max\{y, 0\}| \leq |x - y|$, and the Cauchy Schwarz inequality demonstrate that for all $n \in \mathbb{N}$, $x, y \in [\eta, \delta]^d$ it holds that

$$
|N^\theta(x) - N^\theta(y)| = n^{-1} \max\{|u, x - \alpha, 0\} - n^{-1} \max\{|u, y - \alpha, 0\} = n^{-1} \max\{|u, x - \alpha, 0\} - n^{-1} \max\{|u, y - \alpha, 0\} \leq n^{-1}\|u\| \|x - y\| = n^{-1}\|x - y\|.
$$

(184)

Combining this and (183) establishes that for all $n \in \mathbb{N}$ it holds that

$$
\|N^\theta\|_{A} = \inf_{x \in A} |N^\theta(x)| + \sup_{x, y \in [\eta, \delta]^d, x \neq y} \frac{|N^\theta(x) - N^\theta(y)|}{\|x - y\|} \leq n^{-1} (\delta - \alpha) + n^{-1} (\delta - \alpha + 1) n^{-1}.
$$

(185)
Moreover, note that Lemma 2.11 and (182) ensure that for all \( n \in \mathbb{N}, \vartheta \in \{ \eta \in \mathbb{R}^d : \mathcal{N}^\eta = \mathcal{N}^{\eta(n)} \} \) it holds that
\[
\|\vartheta\|^2 \geq \sup_{x,y \in [\epsilon, \delta]^{d}, x \neq y} \frac{|\mathcal{N}^{\eta}(x) - \mathcal{N}^{\eta}(y)|}{|x - y|} \geq \frac{1}{\epsilon - \delta} \left| \mathcal{N}^{\eta}(\delta, \delta, \ldots, \delta) - \mathcal{N}^{\eta}(\epsilon, \epsilon, \ldots, \epsilon) \right| = \frac{1}{\epsilon - \delta} n^{-1} (\delta - \epsilon) = n^{-1}.
\] (186)

The fact that \( \lim_{n \to \infty} n^{\epsilon - 1/2} = \infty \) and (185) hence imply that for all \( \epsilon \in [0, \infty) \) there exists \( n \in \mathbb{N} \) such that for all \( \vartheta \in \{ \eta \in \mathbb{R}^d : \mathcal{N}^\eta = \mathcal{N}^{\eta(n)} \} \) it holds that \( \|\mathcal{N}^\eta\|_A \leq 1 \) and
\[
\|\vartheta\| \geq n^{-1/2} = n^{\epsilon - 1/2} \epsilon^{-1} > \epsilon (\delta - \epsilon + 1)^{\epsilon} n^{\epsilon} = \epsilon (\delta - \epsilon + 1)^{\epsilon} n^{-1/2} n^{\epsilon} \geq \epsilon \|\mathcal{N}^{\eta(n)}\|_A = \epsilon \max\{\|\mathcal{N}^{\eta(n)}\|_A, \|\mathcal{N}^{\eta(n)}\|_A^{\delta/\delta} \}.
\] (187)

Therefore, we obtain that for all \( \epsilon \in \mathbb{R} \) there exists \( \theta \in \mathbb{R}^d \) such that for all \( \vartheta \in \{ \eta \in \mathbb{R}^d : \mathcal{N}^\eta = \mathcal{N}^{\eta(n)} \} \) it holds that
\[
\|\vartheta\| > \epsilon \max\{\|\mathcal{N}^{\eta}\|_A, \|\mathcal{N}^{\eta(n)}\|_A^{\delta/\delta} \}.
\] (188)

This proves (178) in the case \( \epsilon > 1/2 \). In the next step we will prove (178) in the case \( \delta < 1 \).

Let \( \theta = (\theta_1, \ldots, \theta_d) : \mathbb{N} \to \mathbb{R}^d \) satisfy for all \( n \in \mathbb{N}, i \in \{1, 2, \ldots, b\} \), \( j \in \{2, 3, \ldots, d\} \) that
\[
\theta_{(i-1)d+j}(n) = 1, \quad \theta_{(i-1)d+j}(n) = 0, \quad \theta_{db+i}(n) = -\delta - \frac{i(\delta - \epsilon)}{b+1},
\] (190)

Observe that (190) shows that for all \( n \in \mathbb{N}, x = (x_1, \ldots, x_d) \in [\epsilon, \delta]^d \) it holds that
\[
\mathcal{N}^{\eta(n)}(x) = \theta_{b}(n) + \sum_{i=1}^{b} \theta_{dh+b+i}(n) \max\{\theta_{db+i}(n) + \sum_{j=1}^{d} \theta_{(i-1)d+j}(n)x_j, 0\}
\] (191)

Hence, we obtain that for all \( n \in \mathbb{N}, x = (x_1, \ldots, x_d) \in [\epsilon, \delta]^d \) it holds that
\[
|\mathcal{N}^{\eta(n)}(x)| = |n + \sum_{i=1}^{b} \max\{\langle u, x \rangle - \delta - \frac{i(\delta - \epsilon)}{b+1}, 0\}|
\] (192)

In addition, note that (191), the fact that for all \( x, y \in \mathbb{R} \) it holds that \( |\max\{x, 0\} - \max\{y, 0\}| \leq |x - y| \), and the Cauchy Schwarz inequality demonstrate that for all \( n \in \mathbb{N}, x, y \in [\epsilon, \delta]^d \) it holds that
\[
|\mathcal{N}^{\eta(n)}(x) - \mathcal{N}^{\eta(n)}(y)| = \left| \sum_{i=1}^{b} \max\{\langle u, x \rangle - \delta - \frac{i(\delta - \epsilon)}{b+1}, 0\} - \max\{\langle u, y \rangle - \delta - \frac{i(\delta - \epsilon)}{b+1}, 0\} \right|
\] (193)

This and (192) establish that for all \( n \in \mathbb{N} \) it holds that
\[
\|\mathcal{N}^{\eta(n)}\|_A = \inf_{\vartheta \in A} |\mathcal{N}^{\eta(n)}(x)| + \sup_{x,y \in [\epsilon, \delta]^d, x \neq y} \frac{|\mathcal{N}^{\eta(n)}(x) - \mathcal{N}^{\eta(n)}(y)|}{|x - y|} \leq n + \delta + b = n + b(\delta - \epsilon + 1).
\] (194)
Furthermore, observe that (191) and Lemma 3.1 (applied for every \( n \in \mathbb{N} \), \( \theta \in \{ \eta \in \mathbb{R}^d : \mathcal{N}^\eta = \mathcal{N}^{\eta(n)} \} \) with \( c \cap n, \theta \cap \theta \) in the notation of Lemma 3.1) ensure that for all \( n \in \mathbb{N} \), \( \theta = (\theta_1, \ldots, \theta_\nu) \in \{ \eta \in \mathbb{R}^d : \mathcal{N}^\eta = \mathcal{N}^{\eta(\nu)} \} \) it holds that

\[
\|\theta\| \geq |\theta_3| = n. \tag{195}
\]

The fact that for all \( \epsilon \in \mathbb{R} \) it holds that \( \lim_{n \to \infty} n(1 - \epsilon n^{\delta - 1}) = \infty \), the fact that for all \( x, y \in [0, \infty) \) it holds that \( (x + y)^\delta \leq x^\delta + y^\delta \), and (194) therefore imply that for all \( \epsilon \in [0, \infty) \) there exists \( n \in \mathbb{N} \) such that for all \( \theta \in \{ \eta \in \mathbb{R}^d : \mathcal{N}^\eta = \mathcal{N}^{\eta(n)} \} \) it holds that \( \|\mathcal{N}^{\eta(n)}\|_A \geq 1 \) and

\[
\|\theta\| \geq n = c n^\delta + n(1 - c n^{\delta - 1}) \geq c n^\delta + c \|h(\theta - \epsilon + 1)\| \geq c n^\delta + c \|h(\theta - \epsilon + 1)\|^\delta.
\]

Hence, we obtain that for all \( \epsilon \in \mathbb{R} \) there exists \( \theta \in \mathbb{R}^d \) such that for all \( \theta \in \{ \eta \in \mathbb{R}^d : \mathcal{N}^\eta = \mathcal{N}^{\theta} \} \) it holds that

\[
\|\theta\| > c \max\{\|\mathcal{N}^{\theta}\|_A, \|\mathcal{N}^\theta\|_A\}. \tag{197}
\]

This proves (178) in the case \( \delta < 1 \). The proof of Theorem 3.2 is thus complete.

**Corollary 3.3.** Assume Setting 2.6 and let \( n \in \mathbb{N} \), \( \delta_1, \delta_2, \ldots, \delta_n \in [0, \infty) \), \( A \subseteq [a, b]^d \) satisfy

\[
\min\{\delta_1, \delta_2, \ldots, \delta_n\} > 1/2 \vee \max\{\delta_1, \delta_2, \ldots, \delta_n\} < 1
\]

and \( A \neq \emptyset \). Then for all \( \epsilon \in \mathbb{R} \) there exists \( \theta \in \mathbb{R}^d \) such that for all \( \theta \in \{ \eta \in \mathbb{R}^d : \mathcal{N}^\eta = \mathcal{N}^{\theta} \} \) it holds that

\[
\|\theta\| > c \left( \sum_{k=1}^n \|\mathcal{N}^{\theta}\|_A^\delta \right). \tag{198}
\]

**Proof of Corollary 3.3.** Throughout this proof let \( i, j, \in \{1, 2, \ldots, n\} \) satisfy that

\[
\delta_i = \min\{\delta_1, \delta_2, \ldots, \delta_n\} \quad \text{and} \quad \delta_j = \max\{\delta_1, \delta_2, \ldots, \delta_n\}. \tag{199}
\]

Note that the assumption that \( \min\{\delta_1, \delta_2, \ldots, \delta_n\} > 1/2 \vee \max\{\delta_1, \delta_2, \ldots, \delta_n\} < 1 \) shows that \( \delta_i > 1/2 \vee \delta_j < 1 \). Theorem 3.2 therefore demonstrates that for all \( \epsilon \in [0, \infty) \) there exists \( \theta \in \mathbb{R}^d \) such that for all \( \theta \in \{ \eta \in \mathbb{R}^d : \mathcal{N}^\eta = \mathcal{N}^{\theta} \} \) it holds that

\[
c (\sum_{k=1}^n \|\mathcal{N}^{\theta}\|_A^\delta) \leq c n \max_{k \in \{1, 2, \ldots, n\}} \|\mathcal{N}^{\theta}\|_A^{\delta_k} \leq c n \max_{k \in \{1, 2, \ldots, n\}} \|\mathcal{N}^{\theta}\|_A^{\delta_k} < \|\theta\|. \tag{200}
\]

This establishes that for all \( \epsilon \in \mathbb{R} \) there exists \( \theta \in \mathbb{R}^d \) such that for all \( \theta \in \{ \eta \in \mathbb{R}^d : \mathcal{N}^\eta = \mathcal{N}^{\theta} \} \) it holds that

\[
\|\theta\| > c \left( \sum_{k=1}^n \|\mathcal{N}^{\theta}\|_A^\delta \right). \tag{201}
\]

The proof of Corollary 3.3 is thus complete.

## 4 Lower bounds for norms of reparameterized ANNs using Hölder norms and Sobolev-Slobodeckij norms

In this section we consider different norms for the realization function than Lipschitz norms and we prove, in Corollary 4.8 in Subsection 4.2 below, lower bounds for norms of reparameterized ANN parameter vectors using Hölder norms and Sobolev-Slobodeckij norms. As a consequence, Corollary 4.8 implies that it is not possible to control the norm of reparameterized ANNs using sums of powers of Hölder norms of the realization function or sums of powers of Sobolev-Slobodeckij norms of the realization function with arbitrary exponents. The proof of Corollary 4.8 employs the lower bounds for reparameterized ANNs demonstrated in Theorem 4.7 in Subsection 4.2 and the well-known relationships between different Hölder norms and different Sobolev-Slobodeckij norms established in Lemma 4.3 and Lemma 4.6 in Subsection 4.1.
below, respectively. Only for completeness we also include the detailed proofs of Lemma 4.3 and Lemma 4.6. Moreover, we note that our proof of Lemma 4.6 makes use of the elementary integral results presented in Lemma 4.4 and Lemma 4.5 in Subsection 4.4.

For the convenience of the reader we recall the notions of H"older norms and Sobolev-Slobodeckij norms in Definition 4.1 and Definition 4.2 in Subsection 4.1.

4.1 H"older norms and Sobolev-Slobodeckij norms

Definition 4.1. Let \( d \in \mathbb{N}, \alpha \in \mathbb{R}, \beta \in (\alpha, \infty), \gamma \in [0, 1], \nu \in [\alpha, \beta] \) and let \( f : [\alpha, \beta]^d \to \mathbb{R} \) be a function. Then we denote by \( \{f\}_{\gamma, \nu} \in [0, \infty] \) the extended real number given by

\[
\{f\}_{\gamma, \nu} = \sup_{x \in [\alpha, \beta]^d} |f(x)| + \sup_{x, y \in [\alpha, \beta]^d, x \neq y} \frac{|f(x)-f(y)|}{\|x-y\|^\gamma}
\]  

(cf. Definition 2.1).

Definition 4.2. Let \( d \in \mathbb{N}, \alpha \in \mathbb{R}, \beta \in (\alpha, \infty), \gamma \in [0, 1], p \in [1, \infty) \) and let \( f : [\alpha, \beta]^d \to \mathbb{R} \) be measurable. Then we denote by \( \|f\|_{\gamma, p} \in [0, \infty] \) the extended real number given by

\[
\|f\|_{\gamma, p} = \left[ \int_{[\alpha, \beta]^d} |f(x)|^p \, dx \right]^{1/p} + \left[ \int_{[\alpha, \beta]^d} \int_{[\alpha, \beta]^d} \frac{|f(x)-f(y)|^p}{\|x-y\|^{\gamma p}} \, dx \, dy \right]^{1/p}
\]  

(cf. Definition 2.1).

Lemma 4.3. Let \( d \in \mathbb{N}, \alpha \in \mathbb{R}, \beta \in (\alpha, \infty), \gamma, \lambda \in [0, 1], \nu \in [\alpha, \beta] \) satisfy \( \gamma \leq \lambda \) and \( \nu \leq w \). Then for all functions \( f : [\alpha, \beta]^d \to \mathbb{R} \) it holds that

\[
\{f\}_{\gamma, \nu} \leq \max\{1, [d^{1/2}(\beta - \alpha)]^{\lambda-\gamma}\} \{f\}_{\lambda, w}
\]  

(cf. Definition 4.1).

Proof of Lemma 4.3. Observe that the fact that for all \( x, y \in [\alpha, \beta]^d \) it holds that \( \|x - y\| \leq d^{1/2}(\beta - \alpha) \) and the assumption that \( \gamma \leq \lambda \) and \( \nu \leq w \) ensure that for all functions \( f : [\alpha, \beta]^d \to \mathbb{R} \) it holds that

\[
\{f\}_{\gamma, \nu} = \sup_{x \in [\alpha, \beta]^d} |f(x)| + \sup_{x, y \in [\alpha, \beta]^d, x \neq y} \frac{|f(x)-f(y)|}{\|x-y\|^\gamma} \
= \sup_{x \in [\alpha, \beta]^d} |f(x)| + \sup_{x, y \in [\alpha, \beta]^d, x \neq y} \frac{|f(x)-f(y)|}{\|x-y\|^\lambda} |x-y|^{\lambda-\gamma} \
\leq \sup_{x \in [\alpha, \beta]^d} |f(x)| + [d^{1/2}(\beta - \alpha)]^{\lambda-\gamma} \sup_{x, y \in [\alpha, \beta]^d, x \neq y} \frac{|f(x)-f(y)|}{\|x-y\|^\lambda} \
\leq \max\{1, [d^{1/2}(\beta - \alpha)]^{\lambda-\gamma}\} \{f\}_{\lambda, w}
\]  

(cf. Definitions 2.1 and 4.1). The proof of Lemma 4.3 is thus complete.

Lemma 4.4. Let \( d \in \mathbb{N}, r \in (0, \infty), \gamma \in (-d, \infty) \) and let \( \Gamma : (0, \infty) \to \mathbb{R} \) satisfy for all \( x \in (0, \infty) \) that \( \Gamma(x) = \int_0^x t^{-d-1} e^{-t} \, dt \). Then

\[
\int_{\{y \in \mathbb{R}^d : \|y\| \leq r\}} \|x\|^\gamma \, dx = \frac{2\pi^{d/2}}{(d+\gamma)\Gamma(\gamma/2)} r^{d+\gamma}
\]  

(cf. Definition 2.1).

Proof of Lemma 4.4. Throughout this proof let \( S : \mathcal{B}(\mathbb{R}^d) \to [0, \infty] \) be the \((d-1)\)-dimensional spherical measure. Note that the coarea formula and the fact that for all \( t \in (0, \infty) \) it holds
that \( S(\{y \in \mathbb{R}^d : \|y\| = t\}) = 2^{d/2} \Gamma(d/2)^{-1} t^{d-1} \) imply that

\[
\int_{\{y \in \mathbb{R}^d : \|y\| \leq r\}} ||x||^r \, dx = \int_{\mathbb{R}^d} ||x||^r \mathbb{1}_{[0,r]}(||x||) \, dx
\]
\[
= \int_0^\infty \int_{\{y \in \mathbb{R}^d : \|y\| = t\}} ||x||^r \mathbb{1}_{[0,r]}(||x||) S(dx) \, dt
\]
\[
= \int_0^\infty \int_{\mathbb{R}^d} t^r \mathbb{1}_{[0,r]}(t) S(dx) \, dt
\]
\[
= \int_0^\infty t^r S(\{y \in \mathbb{R}^d : \|y\| = t\}) \, dt = \frac{2^d}{\Gamma(d/2)} \int_0^\infty t^{d+\gamma-1} \, dt
\]
\[
= \frac{2^d}{\Gamma(d/2)} \frac{t^{d+\gamma}}{d+\gamma} \bigg|_0^\infty = \frac{2^d}{(d+\gamma) \Gamma(d/2)} t^{d+\gamma}
\]

(cf. Definition 2.1). The proof of Lemma 4.4 is thus complete. □

**Lemma 4.5.** Let \( d \in \mathbb{N}, \ a \in \mathbb{R}, \ \delta \in (a, \infty), \ \gamma \in (-d, \infty) \) and let \( \Gamma : (0, \infty) \to \mathbb{R} \) satisfy for all \( x \in (0, \infty) \) that \( \Gamma(x) = \int_0^x t^{\gamma-1} e^{-t} \, dt \). Then

\[
\int_{[a,\delta]^d} \int_{[a,\delta]^d} ||x-y||^\gamma \, dx \, dy \leq \frac{2^d}{(d+\gamma) \Gamma(d/2)} \]

(cf. Definition 2.1).

**Proof of Lemma 4.5.** Observe that the fact that for all \( x, y \in [a, \delta]^d \) it holds that \( ||x-y|| \leq d^{1/2}(\delta-a) \) and Lemma 4.4 (applied with \( d \to d, \ r \to d^{1/2}(\delta-a), \ \gamma \to \gamma \) in the notation of Lemma 4.4) prove that for all \( y \in [a, \delta]^d \) it holds that

\[
\int_{[a,\delta]^d} ||x-y||^\gamma \, dx = \int_{x \in [a,\delta]^d} ||x-y|| \, dx \leq \int_{x \in \mathbb{R}^d : ||x|| \leq d^{1/2}(\delta-a)} ||x||^\gamma \, dx
\]
\[
= \frac{2^d}{(d+\gamma) \Gamma(d/2)} \left[ d^{1/2}(\delta-a) \right]^{d+\gamma} \frac{d/d}{(d+\gamma) \Gamma(d/2)}
\]

(cf. Definition 2.1). Hence, we obtain that

\[
\int_{[a,\delta]^d} \int_{[a,\delta]^d} ||x-y||^\gamma \, dx \, dy \leq \int_{[a,\delta]^d} \frac{2^d}{(d+\gamma) \Gamma(d/2)} \, dy = \frac{2^d}{(d+\gamma) \Gamma(d/2)}.
\]

The proof of Lemma 4.5 is thus complete. □

**Lemma 4.6.** Let \( d \in \mathbb{N}, \ a \in \mathbb{R}, \ \delta \in (a, \infty), \ \gamma, \lambda \in [0,1], \ p, q \in [1, \infty) \) satisfy \( \gamma < \lambda \) and \( p < q \) and let \( \Gamma : (0, \infty) \to \mathbb{R} \) satisfy for all \( x \in (0, \infty) \) that \( \Gamma(x) = \int_0^x t^{\gamma-1} e^{-t} \, dt \). Then for all measurable functions \( f : [a, \delta]^d \to \mathbb{R} \) it holds that

\[
\|f\|_{\gamma,p} \leq \left[ \max \left\{ (\delta-a)^d, \frac{2^d \Gamma(d/2)^{\gamma/p}}{(d+\gamma) \Gamma(d/2)} \right\} \right]^{1-n/p} \|f\|_{\lambda,q} \]

(cf. Definition 4.2).

**Proof of Lemma 4.6.** Note that the Hölder inequality and the assumption that \( p < q \) show that for all measurable functions \( f : [a, \delta]^d \to \mathbb{R} \) it holds that

\[
\left[ \int_{[a,\delta]^d} |f(x)|^p \, dx \right]^{1/p} \leq \left[ \left( \int_{[a,\delta]^d} \left[ |f(x)|^p \right]^{q/p} \, dx \right)^{q/p} \left[ \int_{[a,\delta]^d} \left[ 1 \right]^{q/(q-p)} \, dx \right]^{(q-p)/q} \right]^{1/p}
\]
\[
= \left[ \int_{[a,\delta]^d} |f(x)|^q \, dx \right]^{1/q} \left[ (\delta-a)^d \right]^{(q-p)/q}
\]
\[
= (\delta-a)^{d(q-p)/q} \left[ \int_{[a,\delta]^d} |f(x)|^q \, dx \right]^{1/q}.
\]
Furthermore, observe that Lemma 4.5 (applied with \(d \cap d, \alpha \cap \beta, \gamma \cap \lambda - \gamma \)) and the assumption that \(\gamma < \lambda\) and \(p < q\) demonstrate that
\[
\int_{[\alpha, \beta]^d} \int_{[\alpha, \beta]^d} |x - y|^{(\lambda - \gamma)qp/(q-p) - d} \, dx \, dy \leq \frac{2\pi^{d/2}(\lambda - \gamma)qp/(q-p)\, d(\lambda - \gamma)qp/(q-p)}{(\lambda - \gamma)qp/d} \tag{213}
\]
(c.f. Definition 2.1). Combining this, the Hölder inequality, and the assumption that \(p < q\) establishes that for all measurable functions \(f : [\alpha, \beta]^d \to \mathbb{R}\) it holds that
\[
\left[ \int_{[\alpha, \beta]^d} \int_{[\alpha, \beta]^d} \frac{|f(x) - f(y)|^p}{|x - y|^{\lambda p - \gamma p + dp/q - d}} \, dx \, dy \right]^{1/p} \leq \left( \int_{[\alpha, \beta]^d} \int_{[\alpha, \beta]^d} \|x - y\|^{\lambda p - \gamma p + dp/q - d} \, dx \, dy \right)^{1/p} \left( \int_{[\alpha, \beta]^d} \int_{[\alpha, \beta]^d} \|x - y\|^{(\lambda - \gamma)qp/(q-p) - d} \, dx \, dy \right)^{(q-p)/qp} \tag{214}
\]
This and (212) ensure that for all measurable functions \(f : [\alpha, \beta]^d \to \mathbb{R}\) it holds that
\[
\|f\|_{\gamma, p} = \left[ \int_{[\alpha, \beta]^d} \int_{[\alpha, \beta]^d} \frac{|f(x)|^p}{|x - y|^{\gamma p + d}} \, dx \, dy \right]^{1/p} + \left[ \int_{[\alpha, \beta]^d} \int_{[\alpha, \beta]^d} \frac{|f(x) - f(y)|^p}{|x - y|^{\gamma p + d}} \, dx \, dy \right]^{1/p} \leq (\beta - \alpha)^d \left( \int_{[\alpha, \beta]^d} |f(x)|^q \, dx \right)^{1/q} + \left( \int_{[\alpha, \beta]^d} \int_{[\alpha, \beta]^d} \frac{|f(x) - f(y)|^q}{|x - y|^{\gamma p + d}} \, dx \, dy \right)^{1/q} \leq \max \left\{ (\beta - \alpha)^d \frac{(\lambda - \gamma)qp/(q-p)}{\lambda qp/d^2} \right\} \|f\|_{\lambda, q} \leq \max \left\{ (\beta - \alpha)^d, 2^{\pi^{d/2}(\lambda - \gamma)qp/(q-p)\, d(\lambda - \gamma)qp/(q-p)} \right\} \left( \frac{\lambda}{\lambda - \gamma} \right)^{qp/qp} \|f\|_{\lambda, q} \tag{215}
\]
(c.f. Definition 4.2). The proof of Lemma 4.6 is thus complete.

\[\square\]

4.2 Lower bounds for norms of reparameterized ANNs using Hölder norms and Sobolev-Slobodeckij norms

**Theorem 4.7.** Assume Setting 2.6 and let \(\gamma \in [0, 1), v \in [\alpha, \beta], p \in [1, \infty)\). Then for all \(e \in (0, \infty)\) there exists \(\theta \in \mathbb{R}^d\) such that for all \(\vartheta \in \mathbb{R}^d : N^{\gamma} = N^{\theta}\) it holds that
\[
\|\vartheta\| > e \quad \text{and} \quad \max \left\{ \|N^{\theta}\|_{\gamma, p}, \|N^{\theta}\|_{\gamma, p} \right\} < e^{-1}. \tag{216}
\]
(c.f. Definitions 2.1, 4.1, and 4.2).

**Proof of Theorem 4.7.** Throughout this proof let \(q \in (0, \infty)\) satisfy
\[
\gamma q < 1 - \gamma \quad \text{and} \quad (p - d)q < d, \tag{217}
\]
let \( \theta = (\theta_1, \ldots, \theta_b) : \mathbb{N} \rightarrow \mathbb{R}^{p} \) satisfy for all \( n \in \mathbb{N}, i \in \{2, 3, \ldots, b\}, j \in \{1, 2, \ldots, d\} \) that
\[
\theta_j(n) = n^q, \quad \theta_{dn+1}(n) = n^{-1} - n^q d, \quad \theta_{dn+b+1}(n) = 1, \tag{218}
\]
and \( \theta_{(i-1)d+j}(n) = \theta_{di+j}(n) = \theta_{di+b+j}(n) = \theta_b(n) = 0 \), let \( u = (1, 1, \ldots, 1) \), \( v = (\vartheta, \vartheta, \ldots, \vartheta) \in \mathbb{R}^d \), let \( \varepsilon_n \in (0, \infty) \), \( n \in \mathbb{N} \), and \( A_n \subseteq [\vartheta, \vartheta]^d \), \( n \in \mathbb{N} \), satisfy for all \( n \in \mathbb{N} \) that
\[
\varepsilon_n = \min\{n^{-q}d^{-1}, \vartheta - \vartheta\} \quad \text{and} \quad A_n = \{x \in [\vartheta, \vartheta]^d : n^q \langle u, x \rangle + n^{-1} - n^q d \geq 0\}, \tag{219}
\]
and let \( \Gamma : (0, \infty) \rightarrow \mathbb{R} \) satisfy for all \( x \in (0, \infty) \) that \( \Gamma(x) = \int_0^x t^{x-1}e^{-t} \, dt \) (cf. Definition 2.1). Note that (218) implies that for all \( n \in \mathbb{N} \), \( x = (x_1, \ldots, x_d) \in [\vartheta, \vartheta]^d \) it holds that
\[
N^{\theta(n)}(x) = \theta_b(n) + \sum_{i=1}^b \theta_{di+b+i}(n) \max\{\theta_{di+b+i}(n) + \sum_{j=1}^d \theta_{(i-1)d+j}(n) x_j, 0\}
= \max\{n^{-1} - n^q d + \sum_{j=1}^d n^q x_j, 0\} = \max\{n^q \langle u, x \rangle + n^{-1} - n^q d, 0\}. \tag{220}
\]
Furthermore, observe that for all \( n \in \mathbb{N} \) it holds that
\[
n^q \langle u, v - \varepsilon_n u \rangle + n^{-1} - n^q d \vartheta = n^q \langle u, v \rangle - \varepsilon_n n^q \langle u, u \rangle + n^{-1} - n^q d \vartheta
= n^{-1} - \varepsilon_n n^q d \geq n^{-1} - n^{-1} = 0. \tag{221}
\]
The fact that for all \( n \in \mathbb{N} \) it holds that \( \vartheta - \varepsilon_n \in [\vartheta, \vartheta] \) therefore proves that for all \( n \in \mathbb{N} \) it holds that \( v - \varepsilon_n u \in A_n \). Combining this with Lemma 2.11 and (220) shows that for all \( n \in \mathbb{N} \), \( \vartheta \in \{\eta \in \mathbb{R}^p : \mathcal{N}^\eta = \mathcal{N}^{\theta(n)}\} \) it holds that
\[
\|\vartheta\| \geq n^{q/2} d^{1/4}. \tag{223}
\]
Next note that (220) and the fact that for all \( x \in [\vartheta, \vartheta]^d \) it holds that \( \langle u, x \rangle \leq d \vartheta \) demonstrate that for all \( n \in \mathbb{N} \), \( x \in [\vartheta, \vartheta]^d \) it holds that
\[
|N^{\theta(n)}(x)| = |\max\{n^q \langle u, x \rangle + n^{-1} - n^q d, 0\}| \leq |\max\{n^{-1}, 0\}| = n^{-1}. \tag{224}
\]
Moreover, observe that (220), the fact that for all \( x, y \in \mathbb{R} \) it holds that \( |\max\{x, 0\} - \max\{y, 0\}| \leq |x - y| \), and the Cauchy Schwarz inequality establish that for all \( n \in \mathbb{N} \), \( x, y \in [\vartheta, \vartheta]^d \) it holds that
\[
N^{\theta(n)}(x) - N^{\theta(n)}(y)
= |\max\{n^q \langle u, x \rangle + n^{-1} - n^q d, 0\} - \max\{n^q \langle u, y \rangle + n^{-1} - n^q d, 0\}|
\leq |n^q \langle u, x \rangle + n^{-1} - n^q d| - |n^q \langle u, y \rangle + n^{-1} - n^q d|
= |n^q \langle u, x \rangle - n^q \langle u, y \rangle| \leq n^q \|u\| \|x - y\| = n^q d^{1/2} \|x - y\|. \tag{225}
\]
In addition, note that (219) ensures that for all \( n \in \mathbb{N} \), \( x = (x_1, \ldots, x_d) \in A_n \) it holds that
\[
\|x - v\| = \left[\sum_{j=1}^d (\vartheta - x_j)^2\right]^{1/2} \leq \left[\sum_{j=1}^d (\vartheta - x_j) \sum_{i=1}^d (\vartheta - x_i)\right]^{1/2} = \sum_{j=1}^d |x_i - \vartheta|
= \sum_{j=1}^d (\vartheta - x_i) = d \vartheta - \langle u, x \rangle = -n^{-q} (n^q \langle u, x \rangle - n^q d \vartheta) \leq -n^{-q} n^{-1} = n^{-1-q}. \tag{226}
\]
Combining this and (225) implies that for all \( n \in \mathbb{N} \), \( x, y \in A_n \) with \( x \neq y \) it holds that
\[
\frac{|N^{\theta(n)}(x) - N^{\theta(n)}(y)|}{\|x - y\|^\gamma} \leq n^q d^{1/2} n^{-1} \|x - y\|^{1-\gamma} = n^q d^{1/2} [2n - q]^{1-\gamma} = 2^{1-\gamma} d^{1/2} n^{q + 1 - \gamma} \leq 2^{1-\gamma} d^{1/2} n^{q + \gamma - 1} \leq \max \{2^{1-\gamma} d^{1/2}, d^{1/2}\} n^{q + \gamma - 1}.\] (227)

Furthermore, observe that (219) and the Cauchy Schwarz inequality prove that for all \( x \in A_n \), \( y \in [\alpha, \beta]^d \backslash A_n \) it holds that
\[
\sum_{n=1}^{\infty} \frac{n^{-q} d^{-1/2} (n^q(u,x) + n^{-1} - n^q d\theta)}{\|x - y\|^\gamma} \leq n^{-q} d^{-1/2} \left( \left[n^q(u,x) + n^{-1} - n^q d\theta\right] - \left[n^q(u,y) + n^{-1} - n^q d\theta\right] \right) \leq n^{-q} d^{-1/2} n^{-1} n^q d\theta = 2 n^{-q} d^{1/2} n^{1-\gamma} \leq 2 n^{-q} d^{1/2} n^{q + \gamma - 1}. (228)
\]

This, (227), and the fact that for all \( n \in \mathbb{N} \), \( x, y \in [\alpha, \beta]^d \backslash A_n \) it holds that \( |N^{\theta(n)}(x) - N^{\theta(n)}(y)| = 0 \) demonstrate that for all \( n \in \mathbb{N} \), \( x, y \in [\alpha, \beta]^d \) with \( x \neq y \) it holds that
\[
\frac{|N^{\theta(n)}(x) - N^{\theta(n)}(y)|}{\|x - y\|^\gamma} \leq \max \{2^{1-\gamma} d^{1/2}, d^{1/2}\} n^{q + \gamma - 1}. (230)
\]

Combining this with (224) establishes that for all \( n \in \mathbb{N} \) it holds that
\[
\left\{ N^{\theta(n)} \right\}_{\gamma, u} = \sup_{x \in [\alpha, \beta]^d} |N^{\theta(n)}(x)| + \sup_{x,y \in [\alpha, \beta]^d, x \neq y} \frac{|N^{\theta(n)}(x) - N^{\theta(n)}(y)|}{\|x - y\|^\gamma} \leq n^{-1} + \max \{2^{1-\gamma} d^{1/2}, d^{1/2}\} n^{q + \gamma - 1}. (231)
\]

(cf. Definition 4.1). The assumption that \( \gamma q < 1 - \gamma \) therefore ensures that \( \gamma q + \gamma - 1 < 0 \) and
\[
\lim_{n \to \infty} |N^{\theta(n)}|_{\gamma, u} = 0. (232)
\]

Moreover, note that (224) implies that for all \( n \in \mathbb{N} \) it holds that
\[
\left[ \int_{[\alpha, \beta]^d} |N^{\theta(n)}(x)|^p \, dx \right]^{1/p} \leq \left[ \int_{[\alpha, \beta]^d} |n^{-1}|^p \, dx \right]^{1/p} = n^{-p} (\beta - \alpha)^d \right]^{1/p} = n^{-1} (\beta - \alpha)^{d/p}. (233)
\]

In addition, observe that Fubini’s theorem and (220) prove that for all \( n \in \mathbb{N} \) it holds that
\[
\int_{[\alpha, \beta]^d} \int_{[\alpha, \beta]^d} \frac{|N^{\theta(n)}(x) - N^{\theta(n)}(y)|^p}{\|x - y\|^{\gamma p + d}} \, dx \, dy = \int_{A_n} \int_{A_n} \frac{|N^{\theta(n)}(x) - N^{\theta(n)}(y)|^p}{\|x - y\|^{\gamma p + d}} \, dx \, dy + \int_{[\alpha, \beta]^d \backslash A_n} \int_{A_n} \frac{|N^{\theta(n)}(x) - N^{\theta(n)}(y)|^p}{\|x - y\|^{\gamma p + d}} \, dx \, dy
\]
\[
+ \int_{[\alpha, \beta]^d \backslash A_n} \int_{[\alpha, \beta]^d \backslash A_n} \frac{|N^{\theta(n)}(x) - N^{\theta(n)}(y)|^p}{\|x - y\|^{\gamma p + d}} \, dx \, dy
\]
\[
+ \int_{[\alpha, \beta]^d \backslash A_n} \int_{[\alpha, \beta]^d \backslash A_n} \frac{|N^{\theta(n)}(x) - N^{\theta(n)}(y)|^p}{\|x - y\|^{\gamma p + d}} \, dx \, dy
\]
\[
= \int_{A_n} \int_{A_n} \frac{|N^{\theta(n)}(x) - N^{\theta(n)}(y)|^p}{\|x - y\|^{\gamma p + d}} \, dx \, dy + 2 \int_{A_n} \int_{[\alpha, \beta]^d \backslash A_n} \frac{|N^{\theta(n)}(x) - N^{\theta(n)}(y)|^p}{\|x - y\|^{\gamma p + d}} \, dx \, dy
\]
\[
\leq 3 \int_{A_n} \int_{[\alpha, \beta]^d} \frac{|N^{\theta(n)}(x) - N^{\theta(n)}(y)|^p}{\|x - y\|^{\gamma p + d}} \, dx \, dy.
\]
Furthermore, note that the fact that for all \( x, y \in [\alpha, \beta]^d \) it holds that \( \| x - y \| \leq d^{1/2}(\beta - \alpha) \) and Lemma 4.4 (applied with \( d \cap d, r \cap d^{1/2}(\beta - \alpha), \gamma \cap (1 - \gamma)p - d \) in the notation of Lemma 4.4) show that for all \( n \in \mathbb{N}, y \in [\alpha, \beta]^d \) it holds that

\[
\int_{[\alpha, \beta]^d} \| x - y \|^{(1 - \gamma)p - d} \, dx = \int_{x \in [\alpha, \beta]^d} \| x - y \|^{(1 - \gamma)p - d} \, dz \\
\leq \int_{\{x \in \mathbb{R}^d : \| x \| \leq d^{1/2}(\beta - \alpha) \}} \| x \|^{(1 - \gamma)p - d} \, dz = \frac{2^{\eta/2}}{n^{(1 - \gamma)p / (1/2)}} \frac{d^{1/2}(\beta - \alpha)^{(1 - \gamma)p}}{(1 - \gamma)p^t (1/2)}.
\]

Moreover, observe that (226) demonstrates that for all \( n \in \mathbb{N} \) it holds that \( A_n \subseteq \{ x \in \mathbb{R}^d : \| x - v \| \leq n^{-1 - \eta} \} \). Lemma 4.4 (applied for every \( n \in \mathbb{N} \) with \( d \cap d, r \cap n^{-1 - \eta}, \gamma \cap 0 \) in the notation of Lemma 4.4) hence establishes that for all \( n \in \mathbb{N} \) it holds that

\[
\int_{A_n} 1 \, dy \leq \int_{\{x \in \mathbb{R}^d : \| x - v \| \leq n^{-1 - \eta} \}} 1 \, dy = \int_{\{x \in \mathbb{R}^d : \| x \| \leq n^{-1 - \eta} \}} 1 \, dy = \frac{2^{\eta/2}}{n^{(1 - \gamma)p - d - d}}.
\]

Combining this with (225) and (235) ensures that for all \( n \in \mathbb{N} \) it holds that

\[
3 \int_{A_n} \int_{[\alpha, \beta]^d} \frac{\| x - y \|^{(1 - \gamma)p - d}}{\| x - y \|^{(\gamma + d)}} \, dx \, dy \leq 3 \int_{A_n} \int_{[\alpha, \beta]^d} \frac{n^{\eta/2}\| x - y \|^{p}}{\| x - y \|^{(\gamma + d)}} \, dx \, dy \\
= 3n \int_{A_n} \int_{[\alpha, \beta]^d} \frac{\| x - y \|^{(1 - \gamma)p - d}}{\| x - y \|^{(\gamma + d)}} \, dx \, dy \leq 3n \int_{A_n} \int_{[\alpha, \beta]^d} \frac{2^{\eta/2}(\beta - \alpha)^{(1 - \gamma)p}}{(1 - \gamma)p^t (1/2)} \, dx \, dy \\
= 3n \int_{A_n} \int_{[\alpha, \beta]^d} \frac{2\eta/2(\beta - \alpha)^{(1 - \gamma)p}}{(1 - \gamma)p^t (1/2)} \, dx \, dy \\
= \frac{12n \eta/2(\beta - \alpha)^{(1 - \gamma)p}}{(1 - \gamma)p^t (1/2)} \int_{A_n} \int_{[\alpha, \beta]^d} \frac{1}{n^{(p - d)q - d}} \, dx \, dy
\]

This and (234) imply that for all \( n \in \mathbb{N} \) it holds that

\[
\left[ \int_{[\alpha, \beta]^d} \int_{[\alpha, \beta]^d} \frac{\| x - y \|^{(1 - \gamma)p - d}}{\| x - y \|^{(\gamma + d)}} \, dx \, dy \right]^{1/p} \leq \left[ 3 \int_{A_n} \int_{[\alpha, \beta]^d} \frac{\| x - y \|^{(1 - \gamma)p - d}}{\| x - y \|^{(\gamma + d)}} \, dx \, dy \right]^{1/p}
\]

Combining this and (233) proves that for all \( n \in \mathbb{N} \) it holds that

\[
\| N^{(n)} \|_{\gamma, p} = \left[ \int_{[\alpha, \beta]^d} \int_{[\alpha, \beta]^d} \frac{\| x - y \|^{(1 - \gamma)p - d}}{\| x - y \|^{(\gamma + d)}} \, dx \, dy \right]^{1/p} \leq \left[ \frac{12n \eta/2(\beta - \alpha)^{(1 - \gamma)p}}{(1 - \gamma)p^t (1/2)} \right]^{1/p} \int_{A_n} \int_{[\alpha, \beta]^d} \frac{1}{n^{(p - d)q - d}} \, dx \, dy
\]

(cf. Definition 4.2). The assumption that \( p - d < q < d \) therefore shows that \( \frac{(p - d)q - d}{p} < 0 \) and

\[
\lim_{n \to \infty} \| N^{(n)} \|_{\gamma, p} = 0.
\]

Combining this, (232), and (231) demonstrates that for all \( \epsilon \in (0, \infty) \) there exists \( n \in \mathbb{N} \) such that for all \( \tilde{\eta} \in \{ \eta \in \mathbb{R}^3 : N^\eta = N^{\tilde{\eta}(n)} \} \) it holds that

\[
| \tilde{\eta} | \geq n^{\eta/2} d^{1/4} > \epsilon \quad \text{and} \quad \max \{ N^{\tilde{\eta}(n)} \} \| N^{\tilde{\eta}(n)} \|_{\gamma, p} < \epsilon^{-1}.
\]

Hence, we obtain that for all \( \epsilon \in (0, \infty) \) there exists \( \theta \in \mathbb{R}^3 \) such that for all \( \tilde{\theta} \in \{ \eta \in \mathbb{R}^3 : N^\eta = N^{\tilde{\theta}} \} \) it holds that

\[
| \tilde{\theta} | > \epsilon \quad \text{and} \quad \max \{ N^{\tilde{\theta}} \} \| N^{\tilde{\theta}} \|_{\gamma, p} < \epsilon^{-1}.
\]

The proof of Theorem 4.7 is thus complete.

\[\square\]
Corollary 4.8. Assume Setting 2.6 and let \( n \in \mathbb{N} \), \( \gamma_1, \gamma_2, \ldots, \gamma_n \in [0, 1) \), \( v_1, v_2, \ldots, v_n \in [\alpha, \beta] \), \( p_1, p_2, \ldots, p_n \in [1, \infty) \), \( \delta_1, \delta_2, \ldots, \delta_n \in [0, \infty) \). Then for all \( c \in \mathbb{R} \) there exists \( \theta \in \mathbb{R}^3 \) such that for all \( \theta \in \{ \eta \in \mathbb{R}^3 : N^\eta = N^\theta \} \) it holds that
\[
\| \vartheta \| > c \left( \sum_{i=1}^n \| \mathcal{N}^\theta \|_{\tau_i, v_i} \right) \quad \text{and} \quad \| \vartheta \| > c \left( \sum_{i=1}^n \| \mathcal{N}^\theta \|_{\tau_i, p_i} \right)
\] (cf. Definitions 2.1, 4.1, and 4.2).

Proof of Corollary 4.8. Throughout this proof let \( \lambda \in [0, 1) \), \( q \in [1, \infty) \) satisfy
\[
\lambda = \frac{1 + \max\{\gamma_1, \gamma_2, \ldots, \gamma_n\}}{2} \quad \text{and} \quad q = 1 + \max\{p_1, p_2, \ldots, p_n\}.
\] (444)

Note that Lemma 4.3 (applied for every \( i \in \{1, 2, \ldots, n\} \) with \( d \wedge d, \alpha \wedge \alpha, \beta \wedge \beta, \gamma \wedge \gamma_i, \lambda \wedge \lambda, v \wedge v_i, w \wedge \theta \) in the notation of Lemma 4.3) and Lemma 4.6 (applied for every \( i \in \{1, 2, \ldots, n\} \) with \( d \wedge d, \alpha \wedge \alpha, \beta \wedge \beta, \gamma \wedge \gamma_i, \lambda \wedge \lambda, p \wedge p_i, q \wedge q \) in the notation of Lemma 4.6) establish that for all \( i \in \{1, 2, \ldots, n\} \) there exist \( c, \mathcal{C} \in (0, \infty) \) such that for all \( \theta \in \mathbb{R}^3 \) it holds that
\[
|\mathcal{N}^\theta|_{\tau_i, v_i} \leq c |\mathcal{N}^\theta|_{\lambda, \theta} \quad \text{and} \quad |\mathcal{N}^\theta|_{\tau_i, p_i} \leq \mathcal{C} |\mathcal{N}^\theta|_{\lambda, \theta}.
\] (445)

Furthermore, observe that that Theorem 4.7 (applied for every \( c \in \mathbb{R} \) with \( \alpha \wedge \max\{c \mathcal{E}, \mathcal{C}\} \), \( \gamma \wedge \lambda, p \wedge q \) in the notation of Theorem 4.7) ensures that for all \( c \in \mathbb{R} \) there exists \( \theta \in \mathbb{R}^3 \) such that for all \( \theta \in \{ \eta \in \mathbb{R}^3 : N^\eta = N^\theta \} \) it holds that
\[
\| \vartheta \| > \max\{c \mathcal{E}, \mathcal{C}\} \geq c \mathcal{E} \quad \text{and} \quad \max\{|\mathcal{N}^\theta|_{\lambda, \theta}, |\mathcal{N}^\theta|_{\lambda, \theta}\} < \left[ \max\{c \mathcal{E}, \mathcal{C}\}^{-1} \right] \mathcal{C}^{-1}.
\] (446)

Combining this and (445) implies that for all \( c \in [0, \infty) \) there exists \( \theta \in \mathbb{R}^3 \) such that for all \( \theta \in \{ \eta \in \mathbb{R}^3 : N^\eta = N^\theta \} \) it holds that
\[
eq c \left( \sum_{i=1}^n |\mathcal{N}^\theta|_{\tau_i, v_i} \right) \leq c \left( \sum_{i=1}^n \mathcal{E} |\mathcal{N}^\theta|_{\lambda, \theta} \right) \leq c \left( \sum_{i=1}^n 1 \right) = c \mathcal{E} < \| \vartheta \|.
\] (447)

Moreover, observe that that Theorem 4.7 (applied for every \( c \in \mathbb{R} \) with \( \alpha \wedge \max\{c \mathcal{E}, \mathcal{C}\} \), \( \gamma \wedge \lambda, p \wedge q \) in the notation of Theorem 4.7) ensures that for all \( c \in \mathbb{R} \) there exists \( \theta \in \mathbb{R}^3 \) such that for all \( \theta \in \{ \eta \in \mathbb{R}^3 : N^\eta = N^\theta \} \) it holds that
\[
\| \vartheta \| > \max\{c \mathcal{E}, \mathcal{C}\} \geq c \mathcal{E} \quad \text{and} \quad \max\{|\mathcal{N}^\theta|_{\lambda, \theta}, |\mathcal{N}^\theta|_{\lambda, \theta}\} < \left[ \max\{c \mathcal{E}, \mathcal{C}\}^{-1} \right] \mathcal{C}^{-1}.
\] (448)

Hence, we obtain that for all \( c \in \mathbb{R} \) there exists \( \theta \in \mathbb{R}^3 \) such that for all \( \theta \in \{ \eta \in \mathbb{R}^3 : N^\eta = N^\theta \} \) it holds that
\[
\| \vartheta \| > c \left( \sum_{i=1}^n |\mathcal{N}^\theta|_{\tau_i, v_i} \right) \quad \text{and} \quad \| \vartheta \| > c \left( \sum_{i=1}^n |\mathcal{N}^\theta|_{\tau_i, p_i} \right).
\] (449)

The proof of Corollary 4.8 is thus complete. □

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