A Note on the Spectral Area of Toeplitz Operators

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Abstract. In this note, we show that for hyponormal Toeplitz operators, there exists a lower bound for the area of the spectrum. This extends the known estimate for the spectral area of Toeplitz operators with an analytic symbol.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane. Let $L^2$ denote the Lebesgue space of square integrable functions on the unit circle $\partial \mathbb{D}$. The Hardy space $H^2$ is the subspace of $L^2$ of analytic functions on $\mathbb{D}$. Let $P$ be the orthogonal projection from $L^2$ to $H^2$. For $f \in L^\infty$, the space of bounded Lebesgue measurable functions on $\partial \mathbb{D}$, the Toeplitz operator $T_f$ and the Hankel operator $H_f$ with symbol $f$ are defined on $H^2$ by

$$T_fh = P(fh),$$

and

$$H_fh = U(I - P)(fh),$$

for $h \in H^2$. Here $U$ is the unitary operator on $L^2$ defined by

$$Uh(z) = \overline{z}h(\overline{z}).$$

Recall that the spectrum of a linear operator $T$, denoted as $sp(T)$, is the set of complex numbers $\lambda$ such that $T - \lambda I$ is not invertible; here $I$ denotes the identity operator. Let $[T^*, T]$ denote the operator $T^*T - TT^*$, called the self-commutator of $T$. An operator $T$ is called hyponormal if $[T^*, T]$ is positive. Hyponormal operators satisfy the celebrated Putnam inequality [11]

**Theorem 1.1.** If $T$ is a hyponormal operator, then

$$\| [T^*, T] \| \leq \frac{\text{Area}(sp(T))}{\pi}.$$
Notice that a Toeplitz operator with analytic symbol $\phi$ is hyponormal, and by the Spectral Mapping Theorem (cf. [12, p. 263]), $sp(T_\phi) = \phi(\mathbb{D})$. The lower bounds of the area of $sp(T_\phi)$ were obtained in [9] (see [2],[1] [13] and [14] for generalizations to uniform algebras and further discussions). Together with Putnam’s inequality such lower bounds were used to prove the isoperimetric inequality (see [4],[5] and the references there). Recently, there has been revived interest in the context of analytic Topelitz operators on the Bergman space (cf. [3], [10] and [7]). Together with Putnam’s inequality, the latter lower bounds have provided an alternative proof of the celebrated St. Venant’s inequality for torsional rigidity.

In the general case, Harold Widom [15] proved the following theorem for arbitrary symbols.

**Theorem 1.2.** Every Toeplitz operator has a connected spectrum.

The main purpose of this note is to show that for a rather large class of Topelitz operators on $H^2$, hyponormal operators with a harmonic symbol, there is still a lower bound for the area of the spectrum, similar to the lower bound obtained in [9] in the context of uniform algebras.

We shall use the following characterization of the hyponormal Toeplitz operators given by Cowen in [6]

**Theorem 1.3.** Let $\varphi \in L^\infty(\partial \mathbb{D})$, where $\varphi = f + \bar{g}$ for $f$ and $g$ in $H^2$. Then $T_\varphi$ is hyponormal if and only if

$$g = c + T_h f,$$

for some constant $c$ and $h \in H^\infty$ with $\|h\|_\infty \leq 1$.

**2. Main Results**

In this section, we obtain the lower bound for the area of the spectrum for hyponormal Toeplitz operators by estimating the self-commutators.

**Theorem 2.1.** If

$$\varphi = f + T_h f,$$

for $f, h \in H^\infty$, $\|h\|_\infty \leq 1$ and $h(0) = 0$. Then

$$\| [T_\varphi^*, T_\varphi] \| \geq \int |f - f(0)|^2 \frac{d\theta}{2\pi} = ||P(\varphi) - \varphi(0)||_2^2.$$

**Proof.** Let

(2.1)

$$g = T_h f.$$

For every $p$ in $H^2$,

$$\langle [T_\varphi^*, T_\varphi]p, p \rangle = (T_\varphi p, T_\varphi p) - (T_\varphi^* p, T_\varphi^* p)$$

$$= (fp + P(gp), fp + P(gp)) - (gp + P(f\bar{p}), gp + P(f\bar{p}))$$

$$= ||fp||^2 - ||P(f\bar{p})||^2 - ||gp||^2 + ||P(gp)||^2$$

$$= ||f\bar{p}||^2 - ||P(f\bar{p})||^2 - ||gp||^2 + ||P(gp)||^2$$

$$= ||H_f p||^2 - ||H_g p||^2,$$
where $|| \cdot ||$ means the $|| \cdot ||_{L^2(\mathbb{D})}$. The third equality holds because

$$\langle fp, P(\bar{g}p) \rangle = \langle fp, \bar{g}p \rangle = \langle gp, \bar{f}p \rangle = \langle gp, P(\bar{f}p) \rangle.$$

By the computation in [6, p. 4], (2.1) implies

$$H_{\beta} = T_k H_f,$$

where $k(z) = \overline{h(z)}$. Thus

$$(2.2) \quad \langle [T_k^*, T_\varphi]p, p \rangle = ||H_f p||^2 - ||T_k H_f p||^2,$$

for $k \in H^\infty$, $||k||_\infty \leq 1$ and $k(0) = 0$.

First, we assume $k$ is a Blaschke product vanishing at 0. Then

$$|k| = 1 \text{ on } \partial \mathbb{D}.$$ 

Let $u = H_f p \in H^2$. By (2.2) we have

$$(2.3) \quad \langle [T_k^*, T_\varphi]p, p \rangle = ||u||^2 - ||T_k u||^2 = ||u||^2 - ||\bar{k} u||^2 + ||H_k u||^2 = ||H_k u||^2.$$

Then

$$||H_k u|| = ||(I - P) \overline{k u}|| = ||\overline{k u} - \overline{P(k u)}|| \geq \sup_{m(0) = 0} \frac{|(\overline{\overline{k u} - \overline{P(k u)}}, m)|}{||m||} \geq \sup_{m(0) = 0} \frac{1}{||m||} \int |\overline{\overline{k u} m}| \frac{d\theta}{2\pi}.$$

The last equality holds because $m(0) = 0$ implies that $\overline{m}$ is orthogonal to $H^2$. Since $k(0) = 0$, taking $m = k$, we find

$$(2.4) \quad ||H_k u|| \geq \int |u(0)| d\theta = ||u||.$$ 

Next, suppose $k$ is a convex linear combination of Blaschke products vanishing at 0, i.e.

$$k = \alpha_1 B_1 + \alpha_2 B_2 + ... + \alpha_l B_l,$$

where $B_j$’s are Blaschke products with $B_j(0) = 0$, $\alpha_j \in [0, 1]$ and $\sum_{j=1}^{l} \alpha_j = 1$.

By (2.3) and (2.4), for each $j$

$$||u||^2 - ||T_{B_j} u||^2 = ||H_{B_j} u||^2 \geq ||u(0)||^2 \Rightarrow ||T_{B_j} u|| \leq \sqrt{||u||^2 - ||u(0)||^2} = ||u - u(0)||.$$

Then

$$(2.5) \quad ||u||^2 - ||T_k u||^2 = ||u||^2 - \left|\alpha_1 T_{B_1} u + \alpha_2 T_{B_2} u + ... + \alpha_l T_{B_l} u \right|^2 \geq ||u||^2 - \left( \alpha_1 ||T_{B_1} u|| + \alpha_2 ||T_{B_2} u|| + ... + \alpha_l ||T_{B_l} u|| \right)^2 \geq ||u||^2 - ||u - u(0)||^2 = ||u(0)||^2.$$
In general, for \( k \) in the closed unit ball of \( H^\infty \), vanishing at 0, by Carathéodory’s Theorem (cf. [8, p. 6]), there exists a sequence \( \{B_n\} \) of finite Blaschke products such that
\[
B_n \longrightarrow k \quad \text{pointwise on } D.
\]
Since \( B_n \)'s are bounded by 1 in \( H^2 \), passing to a subsequence we may assume
\[
B_n \longrightarrow k \quad \text{weakly in } H^2.
\]
Then by [12, Theorem 3.13], there is a sequence \( \{k_n\} \) of convex linear combinations of Blaschke products such that
\[
k_n \longrightarrow k \quad \text{in } H^2.
\]
Since \( k(0) = 0 \), we can let those \( k_n \)'s be convex linear combinations of Blaschke products vanishing at 0.

Then
\[
||T_{k_n^*} u - T_k u|| = ||P(\tilde{k}_n u - \tilde{k} u)|| \leq ||k_n - k|| \cdot ||u|| \rightarrow 0.
\]
Since (2.5) holds for every \( k_n \), we have
\[
\langle [T_{\varphi^*}, T_{\varphi}] p, p \rangle = ||u||^2 - ||T_k u||^2 = \lim_{n \to \infty} (||u||^2 - ||T_k u||^2) \geq ||u(0)||^2 = ||(H_{f^*} p)(0)||^2.
\]
By the definition of Hankel operator (1.1),
\[
||(H_{f^*} p)(0)|| = ||\langle \bar{p} \bar{f}, \bar{z} \rangle|| = \left| \int \bar{f} \bar{z} p \frac{d\theta}{2\pi} \right|.
\]
From the standard duality argument (cf. [8, Chapter IV]), we have
\[
\sup_{||p|| = 1, p \in H^2} \left| \int \bar{f} \bar{z} p \frac{d\theta}{2\pi} \right| = \sup \left\{ \left| \int \bar{f} \bar{p} \frac{d\theta}{2\pi} \right| : p \in H^2, ||p|| = 1, p(0) = 0 \right\} = \text{dist}(\bar{f}, H^2) = ||f - f(0)||.
\]
Hence
\[
||[T_{\varphi^*}, T_{\varphi}]|| = \sup_{||p|| = 1, p \in H^2} ||\langle [T_{\varphi^*}, T_{\varphi}] p, p \rangle|| \geq ||f - f(0)||^2.
\]

Remark 2.1. For arbitrary \( h \) in the closed unit ball of \( H^\infty \), it follows directly from (2.2) that \( T_{\varphi} \) is normal if and only if \( h \) is a unimodular constant. So we made the assumption that \( h(0) = 0 \) to avoid these trivial cases. Of course, Theorem 2.1 implies right away that \( T_{\varphi} \) is normal if and only if \( f = f(0) \), i.e., when \( \varphi \) is a constant, but under more restrictive hypothesis that \( h(0) = 0 \).

Applying Theorem 1.1 and 1.3, we have
Corollary 2.1. If
\[ \varphi = f + T_{\overline{h}} f, \]
for \( f, h \in H^\infty, \|h\|_\infty \leq 1 \) and \( h(0) = 0 \). Then
\[ \text{Area}(\text{sp}(T_\varphi)) \geq \pi \|P(\varphi) - \varphi(0)\|_2^2. \]

Remark 2.2. Thus, the lower bound for the spectral area of a general hyponormal Toeplitz operator \( T_\varphi \) on \( \partial D \) still reduces to the \( H^2 \) norm of the analytic part of \( \varphi \). For analytic symbols this is encoded in [9, Theorem 2] in the context of Banach algebras. In other words, allowing more general symbols does not reduce the area of the spectrum.

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