GENERALIZATION OF FROBENIUS’ THEOREM FOR GROUP DETERMINANTS

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ABSTRACT. Frobenius built a representation theory of finite groups in the process of obtaining the irreducible factorization of the group determinant.

Here, we give a generalization of Frobenius’ theorem. The generalization leads to a corollary on irreducible representations of finite groups.

1. INTRODUCTION

In this paper, we give a generalization of Frobenius’ theorem. In addition, the generalization leads to a corollary on irreducible representations of finite groups.

Let $G$ be a finite group, $\hat{G}$ a complete set of irreducible representations of $G$ over $\mathbb{C}$, and $R = \mathbb{C}[x_g] = \mathbb{C}[x_g; g \in G]$ the polynomial ring in $\{x_g \mid g \in G\}$ with coefficients in $\mathbb{C}$. The group determinant $\Theta(G) \in R$ is the determinant of a matrix whose elements are independent variables $x_g$ corresponding to $g \in G$. Frobenius proved the following theorem about the irreducible factorization of the group determinant.

**Theorem 1 (Frobenius [2]).** Let $G$ be a finite group, for which we have the irreducible factorization,

$$\Theta(G) = \prod_{\varphi \in \hat{G}} \det \left( \sum_{g \in G} \varphi(g)x_g \right)^{\deg \varphi}.$$

Frobenius built a representation theory of finite groups in the process of obtaining Theorem 1. Here, we give a generalization of Theorem 1, i.e., a generalization of Frobenius’ theorem. The theorem is as follows.

**Theorem 2 (Generalization of Forbenius’ theorem).** Let $G$ be a finite group, $H$ a subgroup of $G$, $L$ a left regular representation from $RG$ to $\text{Mat}([G : H], RH)$, $\alpha = \sum_{g \in G} x_g g \in RG$, and $L(\alpha) = \sum_{h \in H} C_h h$, where $C_h \in \text{Mat}([G : H], R)$. Then, we have

$$\Theta(G) = \prod_{\psi \in H} \det \left( \sum_{h \in H} \psi(h) \otimes C_h^{(G : H)} \right)^{\deg \psi}.$$

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Corollary 3. Let $G$ be a finite group and $H$ a subgroup of $G$. For all $\varphi \in \hat{G}$, we have

$$\deg \varphi \leq |G : H| \times \max \left\{ \deg \psi \mid \psi \in \hat{H} \right\}.$$

Theorem 2 is obtained by using left regular representations of the group algebra. In Section 3, we review the left regular representation and properties of the left regular representation needed for proving Theorem 2. The last section proves a generalization of Theorem 1.

2. Group determinant

Let $G$ be a finite group, $\{x_g \mid g \in G\}$ be independent commuting variables, and $R = \mathbb{C}[x_g] = \mathbb{C}[x_g; g \in G]$ the polynomial ring in $\{x_g \mid g \in G\}$ with coefficients in $\mathbb{C}$. The group determinant $\Theta(G)$ is the determinant of the $|G| \times |G|$ matrix $(x_g h)_g \in G$, where $x_g h = x_{g^{-1}} h$, for $g, h \in G$, and it is thus a homogeneous polynomial of degree $|G|$ in $x_g$. Frobenius proved the following theorem about the factorization of the group determinant.

Theorem 4 (Frobenius [2]). Let $G$ be a finite group, for which we have the irreducible factorization,

$$\Theta(G) = \prod_{\varphi \in \hat{G}} \det \left( \sum_{g \in G} \varphi(g)x_g \right)^{\deg \varphi}.$$  

The above equation holds from the following theorem.

Theorem 5 ([7 Theorem 4.4.4]). Let $G$ be a finite group, $\{\varphi_1, \varphi_2, \ldots, \varphi_s\}$ a complete set of inequivalent irreducible representations of $G$, $d_i = \deg \varphi_i$, and $L_G$ the left regular representation of $G$. Then,

$$L_G \sim d_1 \varphi_1 \oplus d_2 \varphi_2 \oplus \cdots \oplus d_s \varphi_s.$$  

3. Prepare for the main result

Here, we review the left regular representation of the group algebra and describe some of the properties of the left regular representation that will be needed later.

Let $R$ be a commutative ring, $G$ a group, $H$ a subgroup of $G$ of finite index, and $RG$ the group algebra of $G$ over $R$ whose elements are all possible finite sums of the form $\sum_{g \in G} a_g g$, where $a_g \in R$. We take a complete set $T = \{t_1, t_2, \ldots, t_{|G:H|}\}$ of left coset representatives of $H$ in $G$, where $|G:H|$ is the index of $H$ in $G$.

Definition 6 (Left regular representation). For all $A \in \operatorname{Mat}(m, RG)$, there exists a unique $L_T(A) \in \operatorname{Mat}(m|G:H|, RH)$ such that

$$A(t_1 I_m t_2 I_m \cdots t_{|G:H|} I_m) = (t_1 I_m t_2 I_m \cdots t_{|G:H|} I_m)L_T(A).$$  

We call the map $L_T : \operatorname{Mat}(m, RG) \ni A \mapsto L_T(A) \in \operatorname{Mat}(m|G:H|, RH)$ the left regular representation from $\operatorname{Mat}(m, RG)$ to $\operatorname{Mat}(m|G:H|, RH)$ with respect to $T$.

Obviously, $L_T$ is an injective $R$-algebra homomorphism.

To give an expression for $L_T$ when $H$ is a normal subgroup of $G$, we will use the Kronecker product. Let $A = (a_{ij})_{1 \leq i \leq m_1, 1 \leq j \leq n_1}$ be an $m_1 \times n_1$ matrix and
Let $B = (b_{ij})_{1 \leq i \leq m_2, 1 \leq j \leq n_2}$ an $m_2 \times n_2$ matrix. The Kronecker product $A \otimes B$ is the $(m_1 m_2) \times (n_1 n_2)$ matrix

$$A \otimes B = \begin{bmatrix} a_{11} B & a_{12} B & \cdots & a_{1n_1} B \\ a_{21} B & a_{22} B & \cdots & a_{2n_1} B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m_1} B & a_{m_1} B & \cdots & a_{m_1 n_1} B \end{bmatrix}.$$ 

Let

$$P = \begin{bmatrix} t_1 I_m \\ t_2 I_m \\ \vdots \\ t_{[G:H]} I_m \end{bmatrix}.$$ 

Now, we have the following lemma.

**Lemma 7 ([12] Lemma 12).** Let $H$ be a normal subgroup of $G$, $L_T$ the left regular representation from $	ext{Mat}(m, RG)$ to $\text{Mat}(m[G : H], RH)$ with respect to $T$, $L_{G/H}$ the left regular representation from $R(G/H)$ to $\text{Mat}([G/H], R\{e\})$ with respect to $G/H$, and $A = \sum_{t \in T} tA_t \in \text{Mat}(m, RG)$, where $A_t \in \text{Mat}(m, RH)$. Accordingly, we have

$$L_T(A) = P^{-1} \left( \sum_{t \in T} L_{G/H}(tH) \otimes tA_t \right) P.$$ 

Let $K \subset H \subset G$ be a sequence of groups, $H = u_1 K \cup u_2 K \cup \cdots \cup u_{[H,K]}$ and $U = \{u_1, u_2, \ldots, u_{[H,K]}\}$. We can now prove the following theorem.

**Lemma 8 ([12] Lemma 13).** Let $L_T : \text{Mat}(m, RG) \rightarrow \text{Mat}(m[G : H], RH)$ the representation with respect of $T$ and $L_U : \text{Mat}(m[G : H], RH) \rightarrow \text{Mat}(m[G : K], RK)$ the representation with respect of $U$. Then there exists a unique representation $L_V$ from $\text{Mat}(m, RG)$ to $\text{Mat}(m[G : K], R\{e\})$ with respect to $V$ such that

$$L_V = L_U \circ L_T$$

where $V = \{v_1, v_2, \ldots, v_{[G : K]}\}$ is a complete set of left coset representatives of $K$ in $G$.

The following lemma connects the left regular representation with the group determinant.

**Lemma 9 ([12] Lemma 24).** Let $G$ be a finite group, $\Theta(G)$ the group determinant of $G$, $\alpha = \sum_{g \in G} x_g g \in RG$, and $L : RG \rightarrow \text{Mat}([G], R\{e\})$ a left regular representation. We have

$$(\det \circ L)(\alpha) = \Theta(G)e.$$ 

4. Generalization of Frobenius' theorem

Here, we prove the generalization of Frobenius' theorem. In addition, the proof leads to a corollary on irreducible representations of finite groups.

We define $F_m : \text{Mat}(m, RG) \rightarrow \text{Mat}(m, R)$ by

$$F_m \left( \sum_{g \in G} x_{ij}(g)g \right)_{1 \leq i \leq m, 1 \leq j \leq m} = \left( \sum_{g \in G} x_{ij}(g) \right)_{1 \leq i \leq m, 1 \leq j \leq m}.$$
where \( x_{ij}(g) \in R \). We denote \( F_m(A) \) by \( A^{F_m} \) for all \( A \in \text{Mat}(m, RG) \). Then, \( F_m \) is an \( R \)-algebra homomorphism that satisfies with \( \det \circ F_m = F_1 \circ \det \).

Let \( G \) be a finite group and \( K \) a normal subgroup of \( G \) and \( H \). The lemmas will be needed later.

**Lemma 10.** Let \( H \) be a normal subgroup of \( G \), \( L \) a left regular representation from \( \text{Mat}(m, RG) \to \text{Mat}(m[G : H], RH) \), and \( A = \sum_{t \in T} tA_t \), where \( A_t \in \text{Mat}(m, RH) \).

We have

\[
(\text{Det} A)^{F_1} = (\det \circ F_m[G : H] \circ L)(A) = \prod_{\varphi \in \hat{G}/H} \det \left( \sum_{t \in T} \varphi(tH) \otimes A_t^{F_m} \right).
\]

**Proof.** Let \( L \sim \varphi'_1 \oplus \varphi'_2 \oplus \cdots \oplus \varphi'_{s'} \) where \( \varphi'_i \) is an irreducible representation of \( G \). From Lemma 7 and Theorem 5 we find that

\[
(\det \circ F_m[G : H] \circ L)(A)
= \det \left( P^{-1} \left( \sum_{t \in T} L_{G/H}(tH) \otimes tA_t \right) P \right)^{F_m[G : H]}
= \det \left( \sum_{t \in T} \begin{bmatrix} \varphi'_1(tH) & \varphi'_2(tH) & \cdots & \varphi'_{s'}(tH) \end{bmatrix} \otimes A_t^{F_m} \right)
= \prod_{\varphi \in \hat{G}/H} \det \left( \sum_{t \in T} \varphi(tH) \otimes A_t^{F_m} \right)^{\deg \varphi}.
\]

This completes the proof. \( \square \)

**Lemma 11.** Let \( L : \text{Mat}(m, RG) \to \text{Mat}(m[G : H], RH) \) be a left regular representation, \( A = \sum_{v \in V} vB_v \), and \( \Lambda(A) = \sum_{u \in U} uC_u \), where \( B_v \in \text{Mat}(m, RK) \) and \( C_u \in \text{Mat}(m[G : H], RK) \). We have

\[
\prod_{\varphi \in \hat{G}/R} \det \left( \sum_{v \in V} \varphi(vK) \otimes B_v^{F_m} \right)^{\deg \varphi} = \prod_{\psi \in \hat{H}/K} \det \left( \sum_{u \in U} \psi(uK) \otimes C_u^{F_m[G : H]} \right)^{\deg \psi}.
\]
Proof. From Lemma 8 and 10, we have
\[
\prod_{\varphi \in \hat{G}/K} \det \left( \sum_{v \in V} \varphi(vK) \otimes B_v^{F_1} \right)^{\deg \varphi} = \left( \det \circ F_{m[G:K]} \circ L_V \right)(A)
\]
\[
= \left( \det \circ F_{m[G:K]} \circ L_U \circ L_T \right)(A)
\]
\[
= \left( \det \circ F_{m[G:K]} \circ L_U \circ L \right)(A)
\]
\[
= \left( \det \circ F_{m[G:K]} \circ L_U \right) \left( \sum_{u \in U} uC_u \right)
\]
\[
= \prod_{\varphi \in \hat{H}/K} \det \left( \sum_{u \in U} \psi(uK) \otimes C_u^{F_{m[G:H]}} \right)^{\deg \psi}
\]
This completes the proof. \[\square\]

The following is the proof of the generalization of Frobenius’ theorem.

**Theorem 12.** Let \( G \) be a finite group, \( \Theta(G) \) the group determinant of \( G \), \( H \) a subgroup of \( G \), \( L \) a left regular representation from \( RG \) to \( \text{Mat}(\mathbb{Z}[G:H], RH) \), \( \alpha = \sum_{g \in G} xg \in RG \), and \( L(\alpha) = \sum_{h \in H} C_h h \), where \( C_h \in \text{Mat}(\mathbb{Z}[G:H], R) \). We have
\[
\Theta(G) = \prod_{\psi \in \hat{H}} \det \left( \sum_{h \in H} \psi(h) \otimes C_h^{F_{m[G:H]}} \right)^{\deg \psi}.
\]

Proof. For all \( v \in V \), there exists \( B_v \in \text{Mat}(m, R\{e\}) \) such that
\[
\Theta(G) = (\Theta(G)e)^{F_1}
\]
\[
= \prod_{\varphi \in \hat{G}/\{e\}} \det \left( \sum_{v \in V} \varphi(vK) \otimes B_v^{F_1} \right)^{\deg \varphi}
\]
\[
= \prod_{\psi \in \hat{H}/\{e\}} \det \left( \sum_{u \in U} \psi(u\{e\}) \otimes C_u^{F_{m[G:H]}} \right)^{\deg \psi}
\]
\[
= \prod_{\varphi \in \hat{H}} \det \left( \sum_{h \in H} \psi(h) \otimes C_h^{F_{m[G:H]}} \right)^{\deg \psi}
\]
from Lemma 9, 10, and 11 \[\square\]

The polynomial ring \( \mathbb{C}[x_g] \) is a unique factorization domain. Therefore, we have the following corollary from Theorem 4 and 12.

**Corollary 13.** Let \( G \) be a finite group and \( H \) a subgroup of \( G \). For all \( \varphi \in \hat{G} \), we have
\[
\deg \varphi \leq [G : H] \times \max \left\{ \deg \psi \mid \psi \in \hat{H} \right\}.
\]
Proof. We have
\[
\deg \varphi = \deg \left( \det \left( \sum_{g \in G} \varphi(g)x_g \right) \right)
\leq \max \left\{ \deg \left( \det \left( \sum_{h \in H} \psi(h) \otimes C_u^{F[G,H]} \right) \right) \mid \psi \in \hat{H} \right\}
= \max \left\{ \deg \psi \times |G : H| \mid \psi \in \hat{H} \right\}
= |G : H| \times \max \left\{ \deg \psi \mid \psi \in \hat{H} \right\}.
\]
This completes the proof. □

Remark that Corollary 13 follows from Frobenius reciprocity[6].
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