A new type of Ramsey-Turán problems

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Abstract

We introduce and study a new type of Ramsey-Turán problems, a typical example of which is the following one:

Let \( \varepsilon > 0 \) and \( G \) be a graph of sufficiently large order \( n \) with minimum degree \( \delta(G) > 3n/4 \). If the edges of \( G \) are colored in blue or red, then for all \( k \in [4, \lceil (1/8 - \varepsilon)n \rceil] \), there exists a monochromatic cycle of length \( k \).

Keywords: Ramsey-Turán problems; minimum degree; monochromatic cycles.

Introduction and main results

In notation we follow [2]. Given a graph \( G \) with edge set \( E(G) \), a 2-coloring of \( G \) is a partition \( E(G) = E(R) \cup E(B) \), where \( R \) and \( B \) are subgraphs of \( G \) with \( V(R) = V(B) = V(G) \).

It is well-known (see [3,10]) that if \( n > 3 \) and the edges of the complete graph \( K_{2n-1} \) are colored in two colors, there is a monochromatic cycle of length \( k \) for every \( k \in [3, n] \). This is best possible since the complete bipartite graph \( K_{n-1,n-1} \) and its complement contain no \( n \)-cycle when \( n \) is odd; yet the statement can be improved further: as proved in [9], essentially the same conclusion follows from weaker premises:

Let \( G \) be a graph of sufficiently large order \( 2n - 1 \), with \( \delta(G) \geq (2 - 10^{-6})n \). If \( E(G) = E(R) \cup E(B) \) is a 2-coloring, then \( C_k \subset R \) for all \( k \in [3, n] \) or \( C_k \subset B \) for all \( k \in [3, n] \).

This assertion suggests a new class of Ramsey-Turán extremal problems, which somewhat do not fit the traditional framework of this area as given, e.g., in [11]. In this note, we will be mainly concerned with the following conjecture.

Conjecture 1 Let \( n \geq 4 \) and let \( G \) be a graph of order \( n \) with \( \delta(G) > 3n/4 \). If \( E(G) = E(R) \cup E(B) \) is a 2-coloring, then \( C_k \subset R \) or \( C_k \subset B \) for all \( k \in [4, \lceil n/2 \rceil] \).

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If true, this conjecture is tight as shown by the following example: let \( n = 4p \), color the edges of the complete bipartite graph \( K_{2p,2p} \) in blue, and insert a red \( K_{p,p} \) in each of its vertex classes. The union of the blue and red edges gives a graph \( G \) with \( \delta(G) = 3n/4 \), but clearly the coloring produces no monochromatic odd cycle. In fact, in the concluding remarks we describe a more sophisticated coloring scheme that gives \( 2^{n^2/8-O(n \log n)} \) non-isomorphic colorings of \( G \) with no monochromatic odd cycle.

Note that under the premises of Conjecture 1 we don’t necessarily get a monochromatic triangle: indeed, it is long known (see \cite{12}) that splitting \( K_5 \) into a blue and a red 5-cycle and then blowing-up this coloring, we obtain a graph \( G \) with \( \delta(G) = 4n/5 \) with no monochromatic triangle.

The abundance of nonisomorphic extremal colorings suggests that the complete proof of Conjecture 1 might be difficult. In this note we give a partial solution of the conjecture, stated in the theorem below.

**Theorem 2** Let \( \varepsilon > 0 \), let \( G \) be a graph of sufficient large order \( n \), with \( \delta(G) > 3n/4 \). If \( E(G) = E(R) \cup E(B) \) is a 2-coloring, then \( C_k \subset R \) or \( C_k \subset B \) for all \( k \in [4, \lceil (1/8 - \varepsilon) n \rceil] \).

We have to admit that even the proof of this weaker statement needs quite a bit of work. Before proving the theorem we introduce some notation and a number of supporting results.

**Some graph notation**

Given a graph \( G \), we write:
- \( G[U] \) for the graph induced by a set \( U \subset V(G) \);
- \( \Gamma_G(u) \) for the set of neighbors of a vertex \( u \in V(G) \) and \( d_G(u) \) for \( |\Gamma_G(u)| \);
- \( \Delta(G) \) for the maximum degree of \( G \);
- \( ec(G) \) and \( oc(G) \) for the lengths of the longest even and odd cycles in \( G \).

**Preliminary results**

Here we state several known results that will be needed throughout our proof.

**Theorem A (Bondy \cite{3})** If \( G \) is a graph of order \( n \), with \( \delta(G) > n/2 \), then \( G \) contains \( C_k \) for all \( k \in [3,n] \).

In \cite{1}, p. 150, Bollobás gave the following size version of Theorem A.

**Theorem B (Bollobás \cite{1})** If \( G \) is a graph of order \( n \), with \( e(G) > n^2/4 \), then \( G \) contains \( C_k \) for all \( k \in [3,\lceil n/2 \rceil] \).

**Theorem C (Häggkvist \cite{7})** Let \( k \geq 2 \) be a n integer and let \( G \) be a nonbipartite, 2-connected graph of sufficiently large order \( n \). If

\[
\delta(G) > \frac{2}{2k + 1} n,
\]
then $G$ contains a $C_{2k-1}$.

Theorem D (Gould, Haxell, Scott [6]) For all $c > 0$, there exists $K = K(c)$ such that if $G$ is a graph of order $n > K$, with $\delta(G) \geq cn$, then $C_k \subset G$ for all even $k \in [4, ec(G) - K]$ and all odd $k \in [K, oc(G) - K]$.

Theorem E (Erdős, Gallai [4]) Let $k \geq 1$. If $e(G) > k |G|/2$, then $G$ contains a path and a cycle of length at least $k + 1$.

Proof of Theorem 2

Let $G$ be a graph of sufficiently large order $n$ with minimum degree $\delta > 3n/4$, and let $V = V(G)$ be the vertex set of $G$. Suppose $E(G) = E(R) \cup E(B)$ is a 2-coloring, and let $\varepsilon > 0$. Clearly it is enough to prove the theorem for $\varepsilon$ sufficiently small, so we shall assume that $\varepsilon$ is as small as needed.

Assume that $n$ is sufficiently large and that there is no monochromatic $k$-cycle for some $k \in [4, \lfloor (1/8 - \varepsilon)n \rfloor]$. For convenience we shall refer to this assumption as the main assumption. In the following claims, we establish a number of properties of $R$ and $B$ that follow from the main assumption.

Claim 3 Both graphs $R$ and $B$ are nonbipartite.

Proof Indeed, suppose that, say, $B$ is bipartite and let $U$ and $W$ be its vertex classes. By symmetry, assume that $|U| \geq |W|$ and let $R_1 = R[ U ]$. Then for every $u \in U$,

$$d_{R_1}(u) \geq \delta(G) - |W| > \frac{3}{4}n - n + |U| \geq \frac{|U|}{2}.$$

Now, Theorem A implies that $R_1$ contains $C_k$ for all $k \in [3, |U|]$. Since $|U| \geq n/2$, this inequality contradicts the main assumption, completing the proof of Claim 3.

Claim 4 $e(R) > n^2/8$ and $e(B) > n^2/8$.

Proof Assume, by symmetry, that $e(R) \leq n^2/8$. Then

$$e(B) \geq \frac{1}{2} \delta n - e(R) > \frac{3}{8} n^2 - \frac{1}{8} n^2 = \frac{1}{4} n^2.$$

Theorem B now implies that $B$ contains a $k$-cycle for all $k \in [3, \lceil n/2 \rceil]$, contradicting the main assumption and proving the claim.

Claim 5 $\delta(R) > 2\varepsilon n$ and $\delta(B) > 2\varepsilon n$.
Proof Assume, by symmetry, that there is a vertex $u$ such that $d_R(u) \leq 2\varepsilon n$. Then
\[
    d_B(u) = \delta(G) - d_R(u) > \left(\frac{3}{4} - 2\varepsilon\right)n. \tag{1}
\]
Let $U = \Gamma_B(u)$. Clearly $\Gamma_G(v) \cap U = \Gamma_G(v) \setminus (V \setminus U)$, and so
\[
    d_{R[U]}(v) + d_{B[U]}(v) = |\Gamma_G(v) \cap U| \geq |\Gamma_G(v)| - |V \setminus U| > \frac{3}{4}n - n + |U| = |U| - \frac{1}{4}n.
\]
Therefore,
\[
    e(R[U]) + e(B[U]) > \frac{1}{2} \left(|U| - \frac{1}{4}n\right)|U|.
\]
If $e(R[U]) > |U|^2/4$, then by Theorem B, $R$ contains $C_k$ for all $k \in [3, \lfloor |U|/2 \rfloor]$, and for sufficiently small $\varepsilon$, this contradicts the main assumption. Thus $e(R[U]) \leq |U|^2/4$. On the other hand, if $B[U]$ contains a path $P$ of order $\lceil(1/8 - \varepsilon)n\rceil$, we again reach a contradiction, since $P$, together with the vertex $u$, gives a $k$-cycle for all $k \in [3, \lfloor(1/8 - \varepsilon)n\rfloor]$. Therefore, Theorem E implies that $e(R[U]) \leq (1/8 - \varepsilon)n|U|/2$. In summary,
\[
    \frac{1}{2} \left(|U| - \frac{1}{4}n\right)|U| < e(R[U]) + e(B[U]) \leq \frac{1}{4}|U|^2 + \frac{1}{2} \left(\frac{1}{8} - \varepsilon\right)n|U|,
\]
implies that
\[
    |U| - \frac{1}{4}n < \frac{1}{2}|U| + \left(\frac{1}{8} - \varepsilon\right)n
\]
and so,
\[
    |U| < \left(\frac{3}{4} - 2\varepsilon\right)n,
\]
a contradiction with (1), completing the proof of the claim. \qed

Claim 6 At least one of the graphs $R$ or $B$ is 2-connected.

Proof Assume for a contradiction that both $R$ and $B$ are at most 1-connected. That is to say, we can remove a vertex $v$ such that $R - v$ is disconnected and a vertex $u$ such that $B - u$ is disconnected. Note that the components of $R - v$ and $B - u$ are at least of size $\varepsilon n$, as follows from the previous claim. Letting $S = \{u, v\}$ and $W = V \setminus S$, we see that we can remove a set $S$ of at most two vertices so that the graphs $R' = R[W]$ and $B' = B[W]$ are disconnected. Hence $R' = R_1 \cup R_2$ and $B' = B_1 \cup B_2$, where $R_1$ and $R_2$ are vertex disjoint graphs, and so are $B_1$ and $B_2$. Set
\[
    V_1 = R_1 \cap B_1, \ V_2 = R_1 \cap B_2, \ V_3 = R_2 \cap B_1, \ V_4 = R_2 \cap B_2.
\]
Clearly, $W = \bigcup_{i=1}^{4} V_i$ is a partition of $W$. Note also that there are no cross edges in $G$ between $V_1$ and $V_4$, and also between $V_2$ and $V_3$. We thus have

$$d_{G[W]}(w) \leq \begin{cases} n - |S| - 1 - |V_4| & \text{if } w \in V_1 \\ n - |S| - 1 - |V_1| & \text{if } w \in V_4 \\ n - |S| - 1 - |V_2| & \text{if } w \in V_3 \\ n - |S| - 1 - |V_3| & \text{if } w \in V_2 \end{cases}$$

This immediately implies that

$$3n - 3|S| - 4 \geq 4\delta(G[W]) \geq 4(\delta(G) - |S|) > 3n - 4|S|,$$

a contradiction, completing the proof of the claim. \qed

**Existence of monochromatic short odd cycles**

Below we shall prove that, under the premises of the theorem, there is a monochromatic $(2k + 1)$-cycle for every $k \geq 2$, provided $n > n_0(k)$. Unfortunately this cannot give the full proof since $n_0(k)$ grows faster than linear in $k$. In the claim below and the subsequent short argument, we dispose of all odd cycles that are longer than 5.

**Claim 7** If $\Delta(B) \geq n/2 + 4k$, then $B$ contains a $C_{2k+1}$.

**Proof** Suppose that $\Delta(B) \geq n/2 + 4k$, choose a vertex $u$ with $d_B(u) = \Delta(B)$, and write $U$ for the set of its neighbors. If $B[U]$ contains a path of order $2k$, this path, together with $u$, gives a cycle $C_{2k+1}$, so the claim is proved in this case. On the other hand, $e(R[U]) \leq |U|^2/4$, for otherwise Theorem B implies that $R[U]$ has cycles of all lengths from 3 to $|U|/2$, and since $|U|/2 \geq n/4 + 2k > n/8$, this contradicts the main assumption. Hence,

$$e(R[U]) + e(B[U]) \leq \frac{1}{4} |U|^2 + k|U|,$$

On the other hand, as in the proof of Claim 5 we see that

$$e(R[U]) + e(B[U]) \geq \frac{1}{2} (\delta(G) - (n - |U|)) |U| > \frac{1}{2} \left( |U| - \frac{1}{4} n \right) |U|,$$

and so

$$\frac{1}{2} \left( |U| - \frac{1}{4} n \right) |U| < \frac{1}{4} |U|^2 + k|U|,$$

implying in turn $|U| < n/2 + 4k$, contrary to our selection of $U$. This completes the proof. \qed

In view of Claim 7 to prove the existence of monochromatic $(2k + 1)$-cycle we can assume that $\Delta(B) < n/2 + 4k$, and consequently, $\delta(R) > n/4 - 4k$. Note that $n/4 - 4k >
To say, $U \subset C_{2k-1}$ for every $k \geq 4$ provided $n$ is sufficiently large. Then, since $R$ is 2-connected, Theorem C implies that $R$ contains $C_{2k-1}$ for every $k \geq 4$ provided $n$ is sufficiently large.

It remains to prove that there is a monochromatic $C_5$; this is the bulk of our effort. We shall assume that $\delta (R) > n/4 - 16$, and for a contradiction let us assume that there is no monochromatic $C_5$. The proof is split in three cases.

(i) $R$ contains no triangle

Theorem C implies that there is a 7-cycle $C$ in $R$, and since $C_3 \not\subseteq R$ and $C_5 \not\subseteq R$, the cycle $C$ must be induced. Let $v_1, \ldots, v_7$ be its vertices listed consecutively along the cycle, and note that $v_1, v_3, v_5, v_7, v_2, v_4, v_6$ is also a 7-cycle in $B$; we shall refer to it by $C'$. Note that no vertex of $V$ can be joined in $R$ to three vertices of $C$, for otherwise we have either $C_3 \subset R$ or $C_5 \subset R$. Hence

$$\sum_{i=1}^{7} d_R(v_i) \leq 2n.$$ That is to say,

$$\sum_{i=1}^{7} d_B(v_i) > 7 \cdot \frac{3n}{4} - \sum_{i=1}^{7} d_R(v_i) \geq \left( 7 \cdot \frac{3}{4} - 2 \right) n \geq \frac{13n}{4}.$$ Therefore, there is a vertex in $V$, joined to 4 vertices of $C'$ in $B$. An easy check shows that $C_5 \subset B$, a contradiction. Thus, $R$ must contain a triangle.

(ii) $R$ contains triangles, but no two triangles share an edge

Let $v_1, v_2, v_3$ be the vertices of a triangle, and $U_1, U_2, U_3$ be their neighborhoods in $R$. Clearly our premise implies that

$$|U_1 \cap U_2| = |U_2 \cap U_3| = |U_1 \cap U_3| = 1.$$ Also letting

$$U'_1 = U_1 \setminus \{ u_2, u_3 \}, \quad U'_2 = U_2 \setminus \{ u_1, u_3 \}, \quad U'_3 = U_3 \setminus \{ u_1, u_2 \},$$

we see that there are no red cross edges between $U'_i$ and $U'_j$, $(1 \leq i < j \leq 3)$, for otherwise $C_5 \subset R$. Since $C_5 \not\subseteq R$, none of the graphs $R[U'_i]$ contains a path of length 3; thus, Theorem E implies that $e (R[U'_i]) \leq |U'_i|$. That is to say, for every $i = 1, 2, 3$, we can choose a vertex $v_i \in U'_i$ such that

$$|\Gamma_R (v_i) \setminus U'_i| \geq d_R(v_i) - \delta (R[U'_i]) > d_R(v_i) - 2 > n/4 - 18.$$ Now, for every $i = 1, 2, 3$, set $W_i = \Gamma_R (v_i) \setminus U'_i$, and note that $W_i \cap U'_j = \emptyset$ and $W_i \cap W_j = \emptyset$ for all $1 \leq i < j \leq 3$. Indeed, $W_i \cap U_j = \emptyset$ as there are no red cross-edges between $U'_i$ and $U'_j$. Likewise if, say $w \in W_1 \cap W_2$, then $w, v_1, u_1, u_2, v_2$ is a 5-cycle since $w \neq u_1, u_2$. 6
Now we see that the 6 sets $U_i', W_i$, $(i = 1, 2, 3)$ are pairwise disjoint and so,

$$n \geq \sum |U_i'| + |W_i| \geq 6 \left(\frac{n}{4} - 18\right),$$

which is a contradiction for $n$ large enough, completing the proof in this case. It remains to consider the last possibility:

(iii) $R$ contains two triangles sharing an edge

Let $v_1, v_2, v_4$ and $v_2, v_3, v_4$ be the vertices of these triangles, and let and $U_1, U_2, U_3$ be the neighborhoods of $v_1, v_2, v_3$ in $R$. Clearly since $C_5 \not\subseteq R$, then

$$|U_1 \cap U_2| \leq 2, \quad |U_2 \cap U_3| \leq 2, \quad |U_1 \cap U_3| = 2.$$

Also letting

$$U'_1 = U_1 \setminus \{u_2, u_4, u_3\}, \quad U'_2 = U_2 \setminus \{u_1, u_3, u_4\}, \quad U'_3 = U_3 \setminus \{u_1, u_2, u_4\},$$

we see that there are no cross edges in $R$ between $U'_i$ and $U'_j$, $(1 \leq i < j \leq 3)$, for otherwise $C_5 \subseteq R$.

Again, since $C_5 \not\subseteq R$, none of the graphs $R[U'_i]$ contains a path of length 3; thus, Theorem E implies that $e(R[U'_i]) \leq |U'_i|$. That is to say, for every $i = 1, 2, 3$, we can choose a vertex $v_i \in U'_i$ such that

$$|\Gamma_R(v_i) \setminus U'_i| \geq d_R(v_i) - \delta(R[U'_i]) \geq d_R(v_i) - 2 > \frac{n}{4} - 18.$$

Now, for every $i = 1, 2, 3$, set $W_i = \Gamma_R(v_i) \setminus U'_i$, and note that $W_i \cap U'_i = \emptyset$ and $W_i \cap W_j = \emptyset$ for all $1 \leq i < j \leq 3$. Indeed, $W_i \cap U_j = \emptyset$, as there are no red cross-edges between $U'_i$ and $U'_j$. Likewise if, say $w \in W_i \cap W_2$, then $w, v_1, u_1, u_2, v_2$ is a 5-cycle since $w \neq u_1, u_2$.

Now we see that the 6 sets $U'_i, W_i$, $(i = 1, 2, 3)$ are pairwise disjoint and so

$$n \geq \sum |U'_i| + |W_i| \geq 3 \left(\frac{n}{4} - 19\right) + 3 \left(\frac{n}{4} - 18\right),$$

which is a contradiction for $n$ large enough, completing the proof of the existence of monochromatic short odd cycles.

**Existence of all cycles**

At this point we know that $R$ is a 2-connected nonbipartite graph, with $e(R) > n^2/8$ and $\delta(R) \geq 2\varepsilon n$. Let $K = K(2\varepsilon)$, where $K(\cdot)$ is the function from Theorem D and set

$L = 2 \left\lceil \frac{1}{2\varepsilon} \right\rceil + K$. Based on the above properties of $R$ we shall prove that $C_k \subseteq R$ for all even $k \in [4, n/8 - L]$ and all odd $k \in [K, n/8 - L]$. Our main tool will be Theorem D, but to apply it we have to prove the following claim.

**Claim 8** $ec(R) > n/8 - 2k$ and $oc(R) > n/8 - 2k$. 

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Proof In view of
\[
\delta (R) > 2\varepsilon n = \frac{2}{2/ (2\varepsilon)^n} > \frac{2}{2 (\lceil 1/2\varepsilon \rceil)} + 1^n,
\]
Theorem C implies that \( R \) contains a \((2r - 1)\)-cycle \( C' \), where \( r = \lceil 1/2\varepsilon \rceil \); let say \( v_1, \ldots, v_{2r-1} \) be the vertices of \( C' \) listed consecutively along the cycle. Write \( R' \) for the graph obtained from \( R \) by omitting the set \( \{v_1, \ldots, v_{2r-1}\} \). We have
\[
e (R') > e (R) - (2r - 1)(n - 1) > \frac{n^2}{8} - (2r - 1)(n - 1) \geq \left( \frac{n}{8} - 2r \right)(n - 2r + 1),
\]
the last inequality holding for \( n \) sufficiently large. Now, Theorem E implies that \( R' \) contains an \( l \)-cycle \( C'' \) for some \( l > n/4 - 4r \); let say \( u_1, \ldots, u_l \) be its vertices listed consecutively along the cycle. Since \( R \) is 2-connected, there exist two vertex disjoint paths \( P' \) and \( P'' \) joining two vertices \( v_i, v_j \) of \( C'' \) to two vertices \( u_s, u_t \) of \( C'' \) and having no other vertices in common with either \( C'' \) or \( C'' \). The vertices \( u_s, u_t \) split \( C'' \) into two paths with common ends \( u_s \) and \( u_t \). Taking the longer of these paths, together with \( P' \) and \( P'' \), we obtain a path \( P \) joining \( v_i \) and \( v_j \) having no other vertices in common with \( C'' \). Note that the length of \( P \) is at least \( \lceil 1/2 \rceil + 2 > n/8 - 2r \). Finally note that \( v_i \) and \( v_j \) split \( C'' \) into two paths joining \( v_i \) and \( v_j \) - one of even length and one of odd length; these paths, together with \( P \), give one odd and one even cycle, each longer than \( n/8 - 2r \). \( \Box \)

Applying Theorem D with \( c = 2\varepsilon \), and letting \( K = K (2\varepsilon) \), we see that \( C_k \subset R \) for all even \( k \in [4, ec (G) - K] \) and all odd \( k \in [K, oc (G) - K] \). In view of Claim \( \Box \) we see that \( C_k \subset R \) for all even \( k \in [4, n/8 - L] \) and all odd \( k \in [K, n/8 - L] \), as stated.

Now, to complete the proof of Theorem \( \Box \) recall that for all \( k \geq 3 \) and \( n \) sufficiently large, either \( C_{2k-1} \subset R \) or \( C_{2k-1} \subset B \). Therefore, for all \( k \in [4, n/8 - L] \), either \( C_k \subset R \) or \( C_k \subset B \), a contradiction with the main assumption, completing the proof of the theorem. \( \Box \)

Concluding remarks

Let \( U_1, U_2, U_3, U_4 \) be sets of size \( p \). Let \( G \) be the graph with vertex set \( \bigcup_{i=1}^{4} U_i \), and let the edges of \( G \) be all \( U_i - U_j \) edges for \( i \neq j \). Setting \( n = 4p \), we shall show that there exist \( 2^{n^2/8 - O(n \log n)} \) non-isomorphic edge colorings of \( G \) which do not produce any monochromatic cycle. Indeed, color all \( U_1 - U_2 \) and \( U_3 - U_4 \) edges in blue; color all \( U_1 - U_3 \) and \( U_2 - U_4 \) edges in red; color all \( U_1 - U_4 \) and \( U_2 - U_3 \) edges arbitrarily in blue or red. It is easy to check that both the red and blue graphs are bipartite, and so there is no monochromatic odd cycle. Since the \( U_1 - U_4 \) and \( U_2 - U_3 \) edges can be colored in \( 2^{2p^2} \) ways, we obtain at least \( 2^{n^2/8}/n! = 2^{n^2/8 - O(n \log n)} \) different colorings of \( G \) all of which avoid a monochromatic odd cycle.

We conclude with another conjecture, which seems a bit easier than Conjecture \( \Box \)

**Conjecture 9** Let \( 0 < c < 1 \) and \( G \) be a graph of sufficiently large order \( n \). If \( \delta (G) > cn \), then for every 2-coloring of \( E (G) \), there is a monochromatic \( C_k \) for some \( k \geq cn \).
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