Analytical solution for longitudinal and transverse loading of elastic-plastic beam

E Elenitskiy

Head of Research and Calculation Department of “Globaltanksengineering” LTD, Samara, Russia

Abstract. In this article, based on equations of equilibrium of forces applied in a rectangular cross section, subjected to elastic-plastic deformations differential equations were obtained and a new exact solution for a single-span straight beam under the action of longitudinal and transverse loads was built. All the components of the stress-strain state, borders of elastic-plastic zones, as well as expressions for the unit cross-section stiffness were explicitly defined.

In solving physically nonlinear problems of structural mechanics, the accuracy of approximate methods usually is defined by means of other approximate procedures. For example, in monograph [1] results of the calculation of beam by iterative methods of elastic solutions [2], variable parameters of elasticity [3], the Newton-Kantorovich [4] and others [5-6] are evaluated by comparing with the similar results received by method of finite differences [7]. Checking numerical calculations by using accurate nonlinear analytical ratios in most cases is not possible, because their set is very limited. In order to expand the set of exact solutions and to obtain explicitly calculated proportions of high precision this article regards the boundary value problem for a beam of rectangular cross-section from ideal elastic-plastic material. The first analytical solution of this problem was obtained by A.R. Rzhanitsin provided only lateral force action for the simplest cases of a single-span beam loading [8]. A generalization of this closed decision was made in [9] for a square distribution law of bending moment along the statically determined beam on two supports acted upon only by the lateral load. A simpler solution without integration of differential equations in a pure bending beam is presented in the article [10]. Thus obtained dependences were effectively used to identify micro-level mechanical properties of materials in the problems of micro-engineering. Similarly rectilinear beam tensions for various models of elastic-plastic behavior [11] and the curved beam for various inhomogeneous materials [12] were analytically studied.

The case of longitudinal-transverse bending of a rectangular beam, which is a generalization of the study of its pure bending [2, 13], was considered in [14]. The solution is getting much more complicated, because the stress distribution on the sectional height becomes asymmetric even for symmetrical cross-sectional shape. However, the results presented in [14] are approximate, since the beam is modeled by a set of short sections with constant parameters of stiffness along the length and the linearization is performed by iterative procedure on the basis of the relationship between the fictitious load and deformation of the fixed sections. Analogical problem for the longitudinal-transverse loading cantilever beam made of elastic-plastic hardening material was investigated by finite element method [15]. The iterative approach adjusting the unit rigidity of each area of the beam using the secant modulus of elasticity was implemented in [16]. The problem of determining the boundaries of elastic-plastic zones in the longitudinal section of the beam was precisely solved in [8, 14, 17, 18].
Below there is a statement and an integration of a physically nonlinear boundary value problem for longitudinal-transverse loading of a single-span and rectangular cross-section beam with the boundary conditions that ensure statically determination (Figure 1a, b). At the same time all the components of the stress-strain state of the beam and the elastic-plastic border zones are clearly defined. Isotropic material is characterized by a linear module of elasticity $E$, and its deformation is subject to Prandtl diagram with limit values of deformation $\Delta_p$ and stresses $\sigma_p$. Like in other similar studies, the hypothesis of incompressible planar cross-sections is used, which accurately describes the deformation of the structure, regardless of the properties material.

![Diagram of beam loading and deformation](image)

**Figure 1.** Design models and results of beams calculation

1. **Statement and integration of the boundary problem**
Let’s consider first a cantilever scheme of beam $l$ in length, for which at the free end in the plane of the main a force with components $P_x$, $P_y$ is applied. These forces cause plastic deformations noted in Figure 1a by hatching. The span of beam with constant height and width $h$, $b$ of cross-section can be divided into three distinct zones, each of which corresponds to a particular scheme of arrangement of the elastic core of the section. In the zone I the upper and lower extreme fibers experience plastic
deformation, in the zone II plastic deformation occurs on only one cross-section side, in the zone III the entire material behaves elastically. The boundaries of the zones are characterized by dimensions $l_1$, $l_2$ to be determined in the $xy$ coordinate system.

The stress distribution by cross-section height and the deformation of the elementary section of length $dx$ for zone I are shown in Figure 2a, b. The dimensions and position of the elastic core are defined by the formula:

$$z_{1,2} = -0.5n \pm \sqrt{0.75(1-m-n^2)},$$

(1)
derived from the equations of equilibrium of external and internal efforts (Figure 2a).

![Figure 2. Elastic-plastic deformation of beams sections](image)

The equation (1) uses expression for dimensionless borders $z_{1,2} = Z_{1,2}/h$, dimensionless efforts $n, m$, as well as $N_p, M_p$ efforts, providing independently a transfer of the entire section to plastic condition [14], as follows:

$$n = N / N_p \quad m = M / M_p \quad N_p = bh \sigma_p \quad M_p = bh^2 \sigma_p / 4$$

(2)

$$N = P_x \quad M = P_x + M_o$$

(3)
Formulas (3) contain expressions regarding longitudinal force \( N \) and the bending moment \( M \) in the cross-section, and for cantilever beam (Figure 1a):

\[
M_0 = 0 \quad P = P_y
\]  

(4)

The main parameters of the middle line of the section of the zone I are deformations \( \varepsilon_0 \) and curvature determined according to the scheme of deformation (Figure 2b.) by the formulas:

\[
\varepsilon_0 = \varepsilon_x (z_2 + z_1) (z_2 - z_1)^{-1} \quad 1 / \rho = 2 \varepsilon_x [ (z_2 - z_1) h ]^{-1}
\]  

(5)

On the basis of equality of the second derivative of a displacements function \( w(x) \) to the curvature of the middle line of the beam \( \frac{1}{\rho} \), taking into account (5), we have:

\[
d^2 \! w / dx^2 = 2 \varepsilon_x [(z_2 - z_1) h]^{-1}
\]

(6)

In case of (1) the differential equation (6) takes the form:

\[
d^2 \! w / dx^2 = 2 \lambda [3(k_3 - k_1 x)]^{-0.5}
\]

(7)

which, taking into account the equalities (2) indicates

\[
k_i = P / M_p \quad k_3 = M_0 / M_p \quad k_i = 1 - n^2 - k_3 \quad \lambda = \sigma_{xx} / Eh
\]

(8)

Integrating (7), we arrive, within a precision up to constants \( C_{11}, C_{12} \), at the following expressions for the rotation angles \( \psi_1 \) and normal deflections \( w_1 \) in the zone I:

\[
\psi_1 = \frac{-4 \lambda}{\sqrt{3} k_1} \sqrt{k_3 - k_1} x + C_{11} \quad w_1 = \frac{8 \lambda}{3 \sqrt{3} k_1} (k_3 - k_1 x)^{1.5} + C_{11} x + C_{12}
\]  

(9)

The parameters of the elastic core boundaries derived from the equations of equilibrium of forces in the section of the beam in the zone II (Figure 2c) are defined by the relations:

\[
z_1 = 1 - 3m (1 - n)^{-1} / 4 \quad \alpha = 4(1 - n)^2 (1 - n - 0.5m)^{-1} / 3
\]

(10)

Considering the longitudinal and flexural deformations of the elementary section of the beam for the studied zone (Figure 2d), we represent them as follows:

\[
\varepsilon_0 = \varepsilon_x (0.5 - \alpha z_4) (0.5 + z_4)^{-1} \quad 1 / \rho = \varepsilon_x (1 + \alpha) [(0.5 + z_4) h]^{-1}
\]

(11)

The equilibrium equation taking into account the second dependence (11) and the relations (10) takes the form:

\[
d^2 \! w / dx^2 = 8 \lambda (1 - n)^2 [0.25 m^2 (1 - n)^{-1} + 1 - m - n]^{-1} / 9
\]

(12)

Integrating (12), we get within a precision up to the constants \( C_{21}, C_{22} \) the following expression for the angular and linear displacements \( \psi_2, w_2 \) of the beam in the zone II:

\[
\psi_2(x) = -32 \lambda (1 - n)^3 \left[ 9 k_1 k_3 (k_1 x + k_2 - 2 l - n) \right]^{-1} + C_{21}
\]

\[
w_2(x) = -32 \lambda k_1^2 (1 - n)^3 \ln [ 2 (1 - n) - k_1 x - k_2 ] / 9 + C_{21} x + C_{22}
\]

(13)

Elastic nature of the deformation of the beam across the whole height of the section in the zone III allows to represent in a conventional manner [16] its angular \( \psi_3 \) and linear displacement \( w_3 \) within a precision up to the constants \( C_{31}, C_{32} \).

The boundaries between the zones I-II and II-III are characterized by conditions: \( z_2 = -0.5 \) for the formula (1) (Figure 2a), and \( z_1 = 0.5 \) for the first formula (10) (Figure 2c). Then the zone borders can be
established from the following equations:

for \( x = l_1 \) \( m = 2(1 - 2n^2 + n) / 3 \) for \( x = l_2 \) \( m = 2(1 - n) / 3 \)  

(14)

Dependencies (14), supplemented by the condition of education in the cross-section of full plastic hinge [10]:

\[ m \leq 1 - n^2 \]

(15)

and presented graphically in Figure 1c determine the scope of the solutions for different schemes of the elastic core position. At the same time the zone IV corresponds to a loss of the bearing capacity of the section.

Considering equations (2), (3), (8) from (14) it follows:

\[ l_1 = 2k_1^{-4} (1 - 2n^2 + n) / 3 - k_2 / k_1 \]

\[ l_2 = 2k_1^{-4} (1 - n) / 3 - k_2 / k_1 \]

(16)

Six arbitrary constants of integration of differential equations are determined from six obvious boundary conditions having the following shape:

\[ \psi (l_1) = \psi (l_2) \]

\[ w_1 (l_1) = w_2 (l_2) \]

(17)

\[ \psi (l) = w_1 (l) = 0 \]

(18)

\[ \psi (l) = w_2 (l) = 0 \]

(19)

where the first group of conditions (17), (18) refers to the internal borders, and the second group of conditions (19) provides cantilevers fixation of the beam.

Proceeding to the hinged beam, we notice (Figure 1b) that received calculated ratios do not change except for formulas (3), (19), which, taking into account the symmetry of the design scheme should be as follows:

\[ M_0 = P_y l \]

\[ P = -P_y \]

(20)

\[ \psi (0) = 0 \]

\[ w_1 (l) = 0 \]

(21)

In the process of calculating the nonlinear beam systems [19, 20, 21] it is necessary to obtain unit stiffness sections subject to elastic-plastic deformation. As a rule [16], these parameters are determined by presenting a set of sufficiently narrow cross-section bands. Stiffness characteristics of these bands are evaluated on the basis on their distance from the neutral axis and the secant modulus of elasticity which is specified on each step of the iterative calculation. In contrast to this approximate method, we obtain below the following explicit formulas for the longitudinal \( G_1 \) and flexural \( G_2 \) strengths of the whole rectangular section, using the following obvious relations:

\[ N = G_1 \epsilon \]

\[ M = G_2 / \rho \]

(22)

In view of (5), (11) the expressions for the unknown parameters take the form:

\[ G_1 = E bh (z_1 - z_2) \]

\[ G_2 = m E bh / 8 \]

(23)

\[ G_1 = n E bh (0.5 + z_1) / (0.5 - \alpha z_1) \]

\[ G_2 = 0.25 m E bh (0.5 + z_1) / (1 + \alpha) \]

(24)

where the first and second line correspond to sections in the zones I, II.

The solution will be numerically illustrated with the example of the initial data: \( l = 3 \) m, \( b = 0.1 \) m, \( h = 0.15 \) m, \( E = 206000 \) MPa, \( \sigma_p = 240 \) MPa, \( P_x = 400 \) kN, \( P_y = 44 \) kN. The Figure 1d shows the boundary of plastic deformations of a cantilever beam obtained by formulas (1), (10), if \( P_y = 0 \) (the dotted line) and \( P_x \)
#0 (solid line). In the first case we have $l_1 = l_2 = 2.045$ m and a height of the elastic core of the section adjacent to the support of $Z = Z_1 - Z_2 = 0.0387$ m. The presence of the longitudinal force changes the shape of the area of inelastic deformation, so its dimensions are characterized by the parameters $l_1 = 2.222$ m, $l_2 = 1.818$ m, $Z = 0.0258$ m. The Figure 1e presents the results for the deflection of a cantilever beam made of elastic material (curve 1), elastic-plastic material for $P_p = 0$ (curve 2), elastic-plastic material for $P_p ≠ 0$ (curve 3). Displacements at the free end of the beam for the calculation of these options amounted to 68.35 mm, 83.34 mm, 89.41 mm. Therefore, the presence of a longitudinal force resulted in an increase in the maximum deflections of 7.3%. A calculation of the hinged supported beam under the action shown on the Figure 1b load demonstrated that all results on a half-flight coincide with the results of calculation of a cantilever beam (Figure 1a). With the help of the exact solution presented above the ANSYS computing system using a finite element Beam189 was tested. The maximum deflection was 89.07 mm. The difference between the results of precise and finite element calculation does not exceed 0.38%.

2. Conclusion

High-precision calculated ratios to determine the stress-strain state of statically determined beam of rectangular section in elastic-plastic stage of the material work were established. In contrast to the known solutions, the effect of not only bending moment, but the longitudinal force was considered.

The sizes of zones with different elastic core arrangement schemes were explicitly established. An automatic fullfilling of conditions of zones connections can reduce the number of arbitrary constants from six to two.

Numerical analysis of the results showed that when a certain level of load the longitudinal forces have a significant influence on the deflection of the beam.

Testing of finite element of ANSYS Beam189 computer system in conjunction with a model of the ideal elastic-plastic material showed his good working capacity.

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