Convolution of Ultradistributions and 
Field Theory *

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Abstract

In this work, a general definition of Convolution between two arbitrary Tempered Ultradistributions is given. When one of the Tempered Ultradistributions is rapidly decreasing this definition coincides with the definition of J. Sebastiao e Silva.

The product of two arbitrary distributions of exponential type is defined via the Convolution of its corresponding Fourier Transforms.

Several examples of Convolution of two Tempered Ultradistributions and singular products are given. In particular, we reproduce the results obtained by A. Gonzales Dominguez and A. Bredimas.

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1 Introduction

In physics, it is sometimes necessary to work with functions that grow exponentially in space or time. For those cases Schwartz Space of Tempered Distributions (ref.[1]) is too restrictive. On the other hand the space of test functions with bounded support allows the distributions to blow-up more rapidly than any exponential. In this sense they should be considered to be too “permissive” for physical applications. What is needed is an equilibrium between the necessities in x-space and the possibility to work in the Fourier transformed space (p-space) with propagators. The latter are from a mathematical point of view analytic functionals defined on a space of entire test functions.

We shall see that a point of equilibrium is achieved by working with Tempered Ultradistributions (see below). They also have the advantage of being representable by means of analytic functions. So that, in general, they are easier to work with and have interesting properties. One of those properties (as we shall see) is the possibility of defining a convolution product which is general enough to be valid for any two Tempered Ultradistributions, and of course, this automatically provided a definition for the product of Distributions of the Exponential Type in x-space.

In sections 2 and 3 we define the Distributions of Exponential Type and the Fourier transformed Tempered Ultradistributions. Each of them is part of a Guelfand’s Triplet (or Rigged Hilbert Space[2]) together with their respectively duals and a “middle term” Hilbert space.

In section 4 we give a general expression for the convolution. We also state and prove some existence theorems.

In section 5 we present several examples. Some of them imply singular products.

Finally, section 6 is reserved for a discussion of the principal results. For the benefit of the reader an Appendix is added containing some formulas utilized in the text.

2 Distributions of Exponential Type

For the sake of the reader we shall present a brief description of the principal properties of Tempered Ultradistributions.
The space \( H \) of test functions such that \( e^{p|x|} |D^n \phi(x)| \) is bounded for any \( p \) and \( q \) is defined (ref.\[3\]) by means of the countably set of norms:

\[
\| \hat{\phi} \|''_p = \sup_{0 \leq q \leq p} e^{p|x|} \left| D^q \hat{\phi}(x) \right| , \quad p = 0, 1, 2, ...
\]  

(2.1)

According to reference\[4\] \( H \) is a \( \mathcal{K}\{M_p\} \) space with:

\[
M_p(x) = e^{(p-1)|x|} , \quad p = 1, 2, ...
\]

(2.2)

\( \mathcal{K}\{e^{(p-1)|x|}\} \) satisfies condition (\( N \)) of Guelfand (ref.\[3\]). It is a countable Hilbert and nuclear space:

\[
\mathcal{K}\{e^{(p-1)|x|}\} = H = \bigcap_{p=1}^{\infty} H_p
\]

(2.3)

where \( H_p \) is obtained by completing \( H \) with the norm induced by the scalar product:

\[
< \hat{\phi}, \hat{\psi} >_p = \int_{-\infty}^{\infty} e^{2(p-1)|x|} \sum_{q=0}^{p} D^q \hat{\phi}(x) D^q \hat{\psi}(x) \, dx \quad ; \quad p = 1, 2, ...
\]

(2.4)

If we take the usual scalar product:

\[
< \hat{\phi}, \hat{\psi} > = \int_{-\infty}^{\infty} \bar{\phi}(x) \hat{\psi}(x) \, dx
\]

(2.5)

then \( H \), completed with (2.5), is the Hilbert space \( \mathcal{H} \) of square integrable functions.

The space of continuous linear functionals defined on \( H \) is the space \( \Lambda_\infty \) of the distributions of the exponential type (ref.\[3\]).

The “nested space”

\[
( H, \mathcal{H}, \Lambda_\infty )
\]

(2.6)

is a Guelfand’s triplet (or a Rigged Hilbert space \[3\]).

Any Guelfand’s triplet \( (\mathcal{A}, \mathcal{H}, \mathcal{A}') \) has the fundamental property that a linear and symmetric operator on \( \mathcal{A} \), admitting an extension to a self-adjoint operator in \( \mathcal{H} \), has a complete set of generalized eigen-functions in \( \mathcal{A}' \) with real eigenvalues.
3 Tempered Ultradistributions

The Fourier transform of a function $\hat{\phi} \in H$ is

$$\phi(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \ e^{izx} \hat{\phi}(x)$$ (3.1)

$\phi(z)$ is entire analytic and rapidly decreasing on straight lines parallel to the real axis. We shall call $h$ the set of all such functions.

$$h = \mathcal{F} \{ H \}$$ (3.2)

It is a $\mathcal{Z}\{M_p\}$ space ( ref.[5] ), countably normed and complete, with:

$$M_p(z) = (1 + |z|)^p$$ (3.3)

$h$ is also a nuclear space with norms:

$$\|\phi\|_{M_n} = \sup_{|ln(z)| \leq n} (1 + |z|)^p |\phi(z)|$$ (3.4)

We can define the usual scalar product:

$$< \phi(z), \psi(z) > = \int_{-\infty}^{\infty} \phi(z) \psi_1(z) dz = \int_{-\infty}^{\infty} \hat{\phi}(x) \hat{\psi}(x) dx$$ (3.5)

where:

$$\psi_1(z) = \int_{-\infty}^{\infty} dx \ e^{-izx} \hat{\psi}(x)$$

By completing $h$ with the norm induced by (3.5) we get the Hilbert space of square integrable functions.

The dual of $h$ is the space $\mathcal{U}$ of tempered ultradistributions ( ref.[3] ). In other words, a tempered ultradistribution is a continuous linear functional defined on the space $h$ of entire functions rapidly decreasing on straight lines parallel to the real axis.

The set $(h, \mathcal{H}, \mathcal{U})$ is also a Guelfand’s triplet.

$\mathcal{U}$ can also be characterized in the following way ( ref.[3] ): let $\mathcal{A}$ be the space of all functions $F(z)$ such that:

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I. $F(z)$ is analytic for $\{z \in \mathbb{C} : |Im(z)| > p\}$.

II. $F(z)/z^p$ is bounded continuous in $\{z \in \mathbb{C} : |Im(z)| \geq p\}$, where $p = 0, 1, 2, ...$ depends on $F(z)$.

Let $\Pi$ be the set of all $z$-dependent polynomials, $z \in \mathbb{C}$. Then $\mathcal{U}$ is the quotient space:

$$\mathcal{U} = \mathcal{A}/\Pi$$

Due to these properties it is possible to represent any ultradistribution as (ref.[3]):

$$F(\phi) = \langle F(z), \phi(z) \rangle = \oint_{\Gamma} dz \ F(z) \phi(z) \quad (3.6)$$

where the path $\Gamma$ runs parallel to the real axis from $-\infty$ to $\infty$ for $Im(z) > \rho$, $\rho > p$ and back from $\infty$ to $-\infty$ for $Im(z) < -\rho$, $-\rho < -p$. ( $\Gamma$ lies outside a horizontal band of width $2p$ containing all the singularities of $F(z)$).

Formula (3.6) will be our fundamental representation for a tempered ultradistribution. Sometimes use will be made of “Dirac formula” for ultradistributions (ref.[6]):

$$F(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dt \ f(t) \frac{1}{t - z} \quad (3.7)$$

where the “density” $f(t)$ is such that

$$\oint_{\Gamma} dz \ F(z) \phi(z) = \int_{-\infty}^{\infty} dt \ f(t) \phi(t) \quad (3.8)$$

While $F(z)$ is analytic on $\Gamma$, the density $f(t)$ is in general singular, so that the r.h.s. of (3.8) should be interpreted in the sense of distribution theory.

Another important property of the analytic representation is the fact that on $\Gamma$, $F(z)$ is bounded by a power of $z$ (ref.[3]):

$$|F(z)| \leq C|z|^p \quad (3.9)$$

where $C$ and $p$ depend on $F$.

The representation (3.6) makes evident that the addition of a polynomial $P(z)$ to $F(z)$ do not alter the ultradistribution:

$$\oint_{\Gamma} dz \ \{F(z) + P(z)\} \phi(z) = \oint_{\Gamma} dz \ F(z) \phi(z) + \oint_{\Gamma} dz \ P(z) \phi(z)$$
But:

\[
\oint_{\Gamma} dz P(z) \phi(z) = 0
\]

as \(P(z) \phi(z)\) is entire analytic (and rapidly decreasing),

\[\therefore \oint_{\Gamma} dz \{F(z) + P(z)\} \phi(z) = \oint_{\Gamma} dz F(z) \phi(z) \quad (3.10)\]

## 4 The Convolution

If we try to define the convolution product by means of the natural formula:

\[
(F \ast G)\{\phi\} = \oint_{\Gamma_1} \oint_{\Gamma_2} dk_1 \ dk_2 \ F(k_1) G(k_2) \phi(k_1 + k_2) \quad (4.1)
\]

we will soon discover that it is not always defined. The reason is simple. The result of:

\[
\oint_{\Gamma} dk \ F(k) \phi(k + k') = \chi(k')
\]

does not, in general, belong to \(h\). However, if at least one of the ultradistributions \(F\) and \(G\) is rapidly decreasing (say \(G\)), then a convolution can be defined (ref.\[6\]) by:

\[
H(k) = \int_{-\infty}^{\infty} dt \ f(t) G(k - t) \quad (4.2)
\]

where \(f(t)\) is the density associated to \(F(k)\) (cf.(3.7)).

In order to eliminate the test function from (4.1) use can be made of the complex \(\delta\)-function, which is an ultradistribution (Cauchy’s theorem):

\[
\delta_{z'}\{\phi\} = -\frac{1}{2\pi i} \oint_{\Gamma} dz \ \frac{\phi(z)}{z - z'} = \phi(z') \quad (4.3)
\]

where the point \(z'\) is enclosed by \(\Gamma\). (This procedure was previously used in ref.\[7\]). We can then write (4.1) as:

\[
(F \ast G)\{\phi\} = -\frac{1}{2\pi i} \oint_{\Gamma} dz \oint_{\Gamma_1} \oint_{\Gamma_2} dk_1 \ dk_2 \ F(k_1) G(k_2) \frac{F(k_1)G(k_2)}{z - k_1 - k_2} \phi(z) \quad (4.4)
\]
The path $\Gamma$ must have:

$$|\text{Im}(z)| > |\text{Im}(k_1)| + |\text{Im}(k_2)|$$  \hspace{1cm} (4.5)

in order to embrace the point $k_1 + k_2$, ( $k_1 \in \Gamma_1$, $k_2 \in \Gamma_2$).

Equation (4.4) leads to:

$$F * G = H = -\frac{1}{2\pi i} \oint_{\Gamma_1} \oint_{\Gamma_2} dk_1 \, dk_2 \, \frac{F(k_1)G(k_2)}{z - k_1 - k_2}$$  \hspace{1cm} (4.6)

However, we do not expect (4.6) to define a tempered ultradistribution for every pair $F, G$. Note that in (4.1) $F$ and $G$ operate on $\phi(k)$ which is rapidly decreasing, while in (4.6) they act on $(z-k)^{-1} (k = k_1 + k_2)$. Furthermore, due to (4.5) and the fact that $\Gamma_1$ and $\Gamma_2$ run outside a horizontal band containing all the singularities of $F$ and $G$, the integrand in (4.6) is analytic at every point of the integration paths. Taking into account the property (3.9) of tempered ultradistributions, we come to the conclusion that the integrations in (4.6) have at most, a tempered singularity for $k \to \infty$. In order to control this possible singularity we introduce a regulator (see ref. [8]).

We define:

$$H_\lambda(z) = i \frac{1}{2\pi} \oint_{\Gamma_1} \oint_{\Gamma_2} dk_1 \, dk_2 \, \frac{k_1^\lambda F(k_1)k_2^\lambda G(k_2)}{z - k_1 - k_2}$$  \hspace{1cm} (4.7)

Now, if we have the bounds:

$$|F(k_1)| \leq C_1 |k_1|^m \quad ; \quad |G(k_2)| \leq C_2 |k_2|^n$$  \hspace{1cm} (4.8)

Then (4.7) is convergent for

$$\text{Re}(\lambda) < -l - 1 \quad ; \quad l = \max\{m, n\}$$  \hspace{1cm} (4.9)

It is also analytic in the region (4.9) of the $\lambda$ plane. As the derivative with respect to $\lambda$ merely multiplies by a logarithmic factor the integrand of (4.7) without spoiling the convergence.

According to the method of ref. [8], $H_\lambda$ can be analytically continued to other parts of the $\lambda$ plane. In particular near the origin we have the Laurent (or Taylor) expansion:

$$H_\lambda = \sum_n H^{(n)}(z) \lambda^n$$  \hspace{1cm} (4.10)
where the sum might have terms with negative \( n \). We now define the convolution product as the \( \lambda \)-independent term of \((4.10)\):

\[
H(z) = H^{(0)}(z) \tag{4.11}
\]

Note that the derivatives of \( H_\lambda(z) \) with respect to \( z \) can be obtained from \((4.7)\) by taking different powers of the denominator:

\[
\frac{d^p H_\lambda(z)}{dz^p} = (-1)^p p! \frac{i}{2\pi} \oint_{\Gamma_1} \oint_{\Gamma_2} dk_1 dk_2 \frac{k_1^\lambda F(k_1)k_2^\lambda G(k_2)}{(z - k_1 - k_2)^{p+1}} \tag{4.12}
\]

The convergence of \((4.7)\) also ensures that of \((4.12)\). Therefore, also ensure analyticity in \( z \), outside the horizontal band defined by \((4.5)\). We will now show that \(|H_\lambda(z)|\) is bounded by a power of \(|z|\) (cf.\((3.9)\)).

To that aim we take:

\[
\text{Im}(\lambda) = 0 \quad \lambda < -l - 1 \quad z = x + iy
\]

\[
k_i = \kappa_i \pm i\sigma_i \quad \sigma_i > 0 \quad dk_i = d\kappa_i
\]

The integrals along \( \Gamma_1 \), can be expressed as integrals on \( d\kappa_i \) between \( 0 \to \infty \). Then we have:

\[
|H_\lambda| = \frac{1}{2\pi} \left| \oint_{\Gamma_1} \oint_{\Gamma_2} dk_1 dk_2 \frac{k_1^\lambda F(k_1)k_2^\lambda G(k_2)}{z - k_1 - k_2} \right| \leq \frac{C_1 C_2}{2\pi} \int_0^\infty \int_0^\infty d\kappa_1 d\kappa_2 \left( \kappa_1^2 + \sigma_1^2 \right)^{\frac{l-m}{2}} \left( \kappa_2^2 + \sigma_2^2 \right)^{\frac{l+n}{2}} \tag{4.13}
\]

We make now the change of variables \( w_i = \kappa_i^2 \) and obtain:

\[
(4.13) = \frac{2C_1 C_2}{\pi} \int_0^\infty dw_1 w_1^{-\frac{l}{2}} \left( w_1 + \sigma_1^2 \right)^{\frac{l-m}{2}} \int_0^\infty dw_2 w_2^{-\frac{l}{2}} \left( w_2 + \sigma_2^2 \right)^{\frac{l+m}{2}} = \tag{4.14}
\]
\[
\frac{2C_1 C_2}{\pi} B\left(\frac{1}{2}, -\lambda + m + 1\right) B\left(\frac{1}{2}, -\lambda + n + 1\right) \sigma_{1}^{\lambda + m + 1} \sigma_{2}^{\lambda + n + 1} \leq C(\lambda, m, n) |z|^{|\lambda + m + n + 1|} \tag{4.15}
\]

where \( B(x, y) \) is Gauss beta function.

It is to be noted that if \( G(k) \) is a rapidly decreasing ultradistribution, then \( H_\lambda(z) \) ( eq.(4.7) ) coincides with \( H_0(z) \):

\[
H_0(z) = \frac{i}{2\pi} \oint_{\Gamma_1} dk_1 F(k_1) \oint_{\Gamma_2} dk_2 \frac{G(k_2)}{z - k_1 - k_2} \tag{4.16}
\]

In fact, near \( \lambda = 0 \) we have ( \(|k| > 1\) ):

\[
|k^\lambda - 1| \leq \lambda(2\pi + |\ln|k||)|k|^\lambda \tag{4.17}
\]

\[
H_\lambda - H_0(z) = \frac{i}{2\pi} \oint_{\Gamma_1} dk_1 k_1^\lambda F(k_1) \oint_{\Gamma_2} dk_2 (k_2^\lambda - 1) \frac{G(k_2)}{z - k_1 - k_2} + \frac{i}{2\pi} \oint_{\Gamma_1} dk_1 (k_1^\lambda - 1) F(k_1) \oint_{\Gamma_2} dk_2 \frac{G(k_2)}{z - k_1 - k_2} \tag{4.18}
\]

In eq.(4.18) the integrals are convergent, as \( G(k) \) and \( k^\lambda G(k) \) are both rapidly decreasing. Furthermore, due to (4.17) the difference \( H_\lambda - H_0 \) is proportional to \( \lambda \). So that

\[
\lim_{\lambda \to 0} [H_\lambda - H_0] = 0 \tag{4.19}
\]

Again, when \( G(k) \) is rapidly decreasing, the convolution defined in ref.[6]:

\[
H(z) = \int_{-\infty}^{\infty} dt \ f(t)G(z - t) \tag{4.20}
\]

(where \( f(t) \) is given by (3.7),(3.8)), also coincides with (4.16). To show that (4.16) implies (4.20), we use (3.8) in (4.16)

\[
H_0(z) = \frac{i}{2\pi} \int_{-\infty}^{\infty} dt \ f(t) \oint_{\Gamma_2} dk_2 \frac{G(k_2)}{z - k_1 - k_2}
\]
But, if \( G(t) \) is the density associated to \( G(z) \), then

\[
\frac{i}{2\pi} \oint_{\Gamma_2} dk_2 \frac{G(k_2)}{z - t - k_2} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dt_2 \frac{g(t_2)}{t_2 - (z - t)} = G(z - t)
\]

I.e.:

\[
H_0(z) = H(z)
\]

(4.21)

5 Examples

In this section we are going to use definition (4.7) to evaluate convolution of tempered ultradistributions and, indirectly, product of distributions \( \in \Lambda_\infty \) cf.§2).

The convolution theorem tells that:

\[
\mathcal{F} \{ f_1(x) f_2(x) \} = \frac{1}{2\pi} \hat{f}_1(k) \ast \hat{f}_2(k)
\]

(5.1)

where

\[
\hat{f} = \mathcal{F} \{ f(x) \} (k)
\]

I).-As a first example we shall take the distribution \( x^\alpha_{\pm} \) ( ref.[8], ch.1, §3.2, also ref.[9], ch.4 ) whose Fourier transform we write

\[
\hat{x}^\alpha_{\pm} = ie^{\mp i\frac{1}{2} \alpha} \Gamma(\alpha + 1)k^{-\alpha - 1}\Theta[\mp \epsilon(k)]
\]

(5.2)

where \( \Theta(x) \) is Heaviside’s step function and \( \epsilon(k) = sgn \, Im(k) \).

The ultradistribution (5.2) has a line of singularities ( a discontinuity ) on the real axis. Then the path \( \Gamma \) ( cf. (2.6) ) runs parallel to the real axis at a distance as small as we please.

\[
\mathcal{F} \{ x^\alpha_{\pm} x^\beta_{\pm} \} = \frac{i}{4\pi^2} \oint_{\Gamma_1} dk_1 \oint_{\Gamma_2} dk_2 \frac{\hat{x}^\alpha_{\pm} \hat{x}^\beta_{\pm}}{z - k_1 - k_2} =
\]

\[
\left[ \frac{i}{4\pi^2} ie^{-i\frac{\pi}{2} \alpha} \Gamma(\alpha + 1)ie^{-i\frac{\pi}{2} \beta} \Gamma(\beta + 1) \right] \times
\]

\[
\oint_{\Gamma_1} dk_1 k_1^{-\alpha - 1} \Theta[-\epsilon(k_1)] \oint_{\Gamma_2} dk_2 k_2^{-\beta - 1} \Theta[-\epsilon(k_2)]
\]

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The functions $\Theta[\epsilon(k_1)]$ and $\Theta[\epsilon(k_2)]$ eliminate the branches of $\Gamma_1$ and $\Gamma_2$ (resp.) on the lower half plane of $k_1$ and $k_2$. By taking the remaining integration arbitrarily close to the real axis, we get:

$$\mathcal{F}\left\{x_+^{\alpha}x_+^{\beta}\right\} = -\int d k_1 k_1^{-\alpha-1}\Theta[-\epsilon(k_1)] \int_0^\infty dy \frac{(y-i0)^{-\beta-1}}{z-k_1-y} =$$

$$-\int d k_1 k_1^{-\alpha-1}\Theta[-\epsilon(k_1)] \int_0^\infty dy \frac{y^{-\beta-1} + e^{-i\pi(-\beta-1)}y^{-\beta-1}}{z-k_1-y} =$$

$$-\int d k_1 k_1^{-\alpha-1}\Theta[-\epsilon(k_1)] \int_0^\infty dy \frac{\Gamma(-\beta)\Gamma(1+\beta)}{(z-k_1)^{\beta+1}} \times$$

$$e^{-i\pi(-\beta-1)} - e^{-i\pi(z)(-\beta-1)} =$$

$$2i[\Theta[-\epsilon(z)]\Gamma(-\beta)\Gamma(1+\beta)\sin\pi(-\beta-1) \times$$

$$\int d k_1 \frac{k_1^{-\alpha-1}}{(z-k_1)^{\beta+1}}\Theta[-\epsilon(k_1)] =$$

$$2i\pi\Theta[-\epsilon(z)][\int_0^\infty dx \frac{x^{-\alpha-1} + e^{-i\pi(-\alpha-1)}x^{-\alpha-1}}{(z-x)^{\beta+1}} =$$

$$2i\pi\Theta[-\epsilon(z)][\mathcal{B}(-\alpha, \beta + \alpha + 1) \left[e^{i\pi\epsilon(z)\alpha} - e^{i\pi\alpha}\right] z^{-\alpha-\beta-1} =$$

$$2i\pi\left\{\Theta[-\epsilon(z)]\right\}^2 \left[\frac{\Gamma(-\alpha)\Gamma(\beta + \alpha + 1)}{\Gamma(\beta + 1)}\right] 2i\sin\pi(-\alpha)z^{-\alpha-\beta-1} =$$

$$ie^{-i\pi\epsilon(z)\alpha}\Gamma(\alpha + \beta + 1)z^{-\alpha-\beta-1}\Theta[-\epsilon(z)] =$$

$$\hat{x}_+^{\alpha+\beta} = \mathcal{F}\left\{x_+^{\alpha+\beta}\right\} = \mathcal{F}\left\{x_+^{\alpha}x_+^{\beta}\right\} \quad \text{(5.3)}$$

Where use has been made of equation (A.4) of the Appendix.

For the evaluation of the convolution $\hat{x}_+^{\alpha} * \hat{x}_-^{\beta}$ the procedure is entirely similar. However, in this case one of the integrations gives rise to a factor $\Theta[-\epsilon(z)]$ and the other to a factor $\Theta[\epsilon(z)]$. So that, instead of $\left\{\Theta[-\epsilon(z)]\right\}^2 = \Theta[-\epsilon(z)]$, we get $\Theta[-\epsilon(z)]\Theta[\epsilon(z)] = 0$. I.e.

$$\hat{x}_+^{\alpha} * \hat{x}_-^{\beta} \equiv 0 \quad \therefore \quad x_+^{\alpha} \cdot x_-^{\beta} = 0 \quad \text{(5.4)}$$
II).-As a second example we consider Dirac's $\delta$-functions, whose Fourier transform is:

$$\tilde{\delta}(m) = i^{m} k^{m} \frac{\epsilon(k)}{2} \quad (5.5)$$

For the convolution (4.7) we have:

$$\tilde{\delta}(m) * \tilde{\delta}(n) = \frac{i}{4\pi} \int_{\Gamma_1} dk_1 i^{m} k_1^{\lambda+m} \frac{\epsilon(k_1)}{2} \int_{\Gamma_2} dk_2 i^{n} k_2^{\lambda+n} \frac{\epsilon(k_2)}{2} \frac{z - k_1 - k_2}{z - k_1 - k_2} =$$

In this case, the factors $\epsilon_1$ and $\epsilon_2$ change the sign of integrations of the lower half plane of $k_1$ and $k_2$.

$$\tilde{\delta}(m) * \tilde{\delta}(n) = \frac{i}{4\pi} \int_{\Gamma_1} dk_1 i^{m} k_1^{\lambda+m} \frac{\epsilon(k_1)}{2} \int_{-\infty}^{\infty} dy \frac{(y + i0)^{\lambda+n} + (y - i0)^{\lambda+n}}{z - k_1 - y} =$$

$$\frac{i^{m+n+1}}{2\pi} \int_{\Gamma_1} dk_1 k_1^{\lambda+m} \frac{\epsilon(k_1)}{2} \int_{-\infty}^{\infty} dy \frac{y^{\lambda+n} + \cos \pi(\lambda + n)y^{\lambda+n}}{z - k_1 - y} =$$

$$\frac{i^{m+n+1}}{2\pi} \int_{\Gamma_1} dk_1 k_1^{\lambda+m} \frac{\epsilon(k_1)}{2} \Gamma(\lambda + n + 1)\Gamma(-\lambda - n) \left[ \cos \pi(\lambda + n) - e^{-i\pi \epsilon(z)(\lambda+n)} \right] =$$

$$-\frac{i\pi \epsilon(z)}{2\pi} \int_{-\infty}^{\infty} dx \frac{x^{\lambda+m} + \cos \pi(\lambda + m)x^{\lambda+m}}{(z - x)^{-\lambda-n}} =$$

$$\frac{\epsilon(z)}{2} i^{m+n+1} \frac{\Gamma(\lambda + m + 1)\Gamma(-2\lambda - m - n - 1)}{\Gamma(-\lambda - n)} z^{2\lambda+m+n+1} \times$$

$$\left[ e^{-i\pi \epsilon(z)(\lambda+m+1)} + \cos \pi(\lambda + m) \right] =$$

$$\frac{\epsilon(z)^2}{2} i^{m+n+1} \frac{\Gamma(\lambda + m + 1)\Gamma(-2\lambda - m - n - 1)}{\Gamma(-\lambda - n)} \sin \pi(\lambda + m) z^{2\lambda+m+n+1} =$$

$$\lambda \to 0 \rightarrow 0 = \tilde{\delta}(m) * \tilde{\delta}(n) \quad (5.6)$$

There are two reasons for this null result. The $\Gamma$ functions have simple poles when their arguments are negative integer ( or zero ). So that the quotient of $\Gamma$ functions has a finite limit. However they are multiplied by $\sin \pi(\lambda + m)_{\lambda \to 0} \rightarrow 0$. 

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Furthermore, \([\epsilon(z)]^2 = 1\), and
\[
z^{2\lambda+m+n+1} \xrightarrow{\lambda \to 0} z^{m+n+1}
\]
Then we can put (\(C = \text{arbitrary constant}\))
\[
\delta^{(m)} * \delta^{(n)} = Cz^{m+n+1}
\]
But due to the property III, \(\S 3\), the ultradistribution \((5.7)\) is equivalent to zero.
We have then:
\[
\delta^{(m)}(x) \cdot \delta^{(n)}(x) = 0
\]
This result was previously obtained in ref.[10] and can be summarized in general as:
“\(\text{The product of two distributions with point support is zero}^\dagger\).”
III)-We can combine examples I and II, to find the product \(\delta^{(m)} \cdot \hat{x}_\alpha\).

\[
\frac{1}{2\pi} \delta^{(m)} * \hat{x}_\alpha = \left[\frac{i}{4\pi^2} i^m ie^{-i\frac{m}{2}\alpha} \Gamma(\alpha + 1)\right] \times
\]
\[
\int_1 \int_2 dk_1 \frac{k_1^{\lambda+m} \epsilon(k_1)}{2} \int_2 \int_1 dk_2 \frac{k_2^{-\alpha} \Theta[-\epsilon(k_2)]}{z - k_1 - k_2} =
\]
\[
2\pi i \Theta[-\epsilon(z)] \int_{-\infty}^{\infty} dx \frac{x^{\lambda+m} + \cos \pi (\lambda + m)x^{\lambda+m}}{(z - x)^{\alpha+1}} =
\]
\[
2\pi i \Theta[-\epsilon(z)] \Gamma(\lambda + m + 1)\Gamma(\alpha - \lambda - m)_{z^{\lambda+m-\alpha}} \times
\]
\[
\left[e^{-i\pi \epsilon(z)(\lambda+m+1)} + \cos \pi (\lambda + m)\right] =
\]
\[
2\pi i \Theta[-\epsilon(z)] \left(-\frac{i^m}{4\pi^2} e^{-i\frac{m}{2}\alpha}\right) \Gamma(\lambda + m + 1)\Gamma(\alpha - \lambda - m) \times
\]
\[
i \sin \pi (\lambda + m) \epsilon(z)_{z^{\lambda+m-s}} \xrightarrow{0_{\lambda \to 0}},
\]
if \(\alpha\) is not an integer \(s\) such that \(s \leq m\).
When \(0 \leq \alpha = s \leq m\):
\[
\frac{1}{2\pi} \delta^{(m)} * \hat{x}_s = -2i\pi \Theta[-\epsilon(z)] \frac{i^m}{4\pi^2} (-i)^s \epsilon(z) z^{\lambda+m-s} \times
\]
\[
\frac{\Gamma(\lambda + m + 1)}{\Gamma(\lambda + m + 1 - s)} \sin \pi(\lambda + m) = \\
\frac{i^m}{2} (-i)^s \frac{\Gamma(\lambda + m + 1)}{\Gamma(\lambda + m + 1 - s)} \sin \pi(\lambda + m - s) \Theta[-\epsilon(z)] \epsilon(z) z^{\lambda + m - s}
\]
\[
\lambda \to 0 \quad \rightarrow \quad (-1)^s \frac{i^m}{2} \frac{m!}{(m-s)!} \epsilon(z) z^{m-s} = \\
\frac{(-1)^s}{2} \frac{m!}{(m-s)!} \delta^{(m-s)}
\]

(5.10)

In particular, for \( s = 0 \) we get:

\[
\delta^{(m)}(x) \Theta(x) = \frac{1}{2} \delta^{(m)}(x)
\]

(5.11)

If \( \alpha = s \) = negative number = \(-n\) we must be careful as \( x_+^{\alpha} \) has a pole for \( \alpha = -n \). We shall deal with this case by the replacement \( \alpha = -n - \lambda \) in (5.9).

\[
\Gamma(\alpha - \lambda - m) \to \Gamma(-2\lambda - m - n) = \\
\pi \frac{\Gamma(2\lambda + m + n + 1)}{\sin \pi(2\lambda + m + n)}
\]

And by taking the limit \( \lambda \to 0 \):

\[
\frac{1}{2\pi} \delta^{(m)} * x_+^{-n} = \frac{i^{m+n}}{2} \frac{m!}{(m+n)!} \frac{(-1)^n \epsilon(z)}{2} z^{m+n} = \\
\frac{(-1)^n}{4} \frac{m!}{(m+n)!} \delta^{(m+n)}
\]

(5.12)

In eqs. (5.10) and (5.12) we have used:

\[
\Theta[-\epsilon(z)] \epsilon(z) = -\Theta[-\epsilon(z)] = \frac{1}{2} (\epsilon(z) - 1) = \frac{\epsilon(z)}{2} - \frac{1}{2}
\]

\[
\Theta[-\epsilon(z)] \epsilon(z) z^s = \frac{\epsilon(z)}{2} z^s - \frac{1}{2} z^s \approx \frac{\epsilon(z)}{2} z^s
\]

As \( Cz^s \) is equivalent to zero ( cf. (5.7) ).

There are also similar expressions which originates in the use of \( \check{x}_+^{\alpha} \) in (5.9). In particular, if we use

\[
\check{x}^{-n} = \check{x}_+^{-n} + (-1)^n \check{x}_-^{-n}
\]

(5.13)
then we easily find
\[
\frac{1}{2\pi} \tilde{\delta}^{(m)} \ast \tilde{x}^{-n} = (-1)^n \frac{m!}{(m+n)!} \tilde{\delta}^{(m+n)}
\] (5.14)

The case \( m = 0, n = 1 \), was first published in ref. [11]. For \( m = n \) eq. (5.14) coincides with ref. [12].

IV).-To illustrate the use of (4.10) and (4.11), we are now going to examine an interesting example.

Let us take the ultradistribution (5.13), which is found to be:
\[
\tilde{x}^{-n} = \frac{(-i)^n \pi}{(n-1)!} \left[ -\frac{1}{\pi i} \ln(k) + \frac{\epsilon(k)}{2} \right] k^{n-1}
\] (5.15)

The convolution product is now:
\[
\tilde{x}^{-m} \ast \tilde{x}^{-n} = -\frac{(-i)^{m+n+1}}{4(m-1)!(n-1)!} \oint_{\Gamma_1} \oint_{\Gamma_2} dk_1 \, dk_2
\]
\[
\left\{ \begin{array}{l}
-1 \frac{1}{\pi^2} \frac{k_1^{\lambda+m-1} \ln(k_1) k_2^{\lambda+n-1} \ln(k_2)}{z - k_1 - k_2} - \frac{1}{2\pi} \frac{k_1^{\lambda+m-1} \ln(k_1) k_2^{\lambda+n-1} \epsilon(k_2)}{z - k_1 - k_2} \\
- \frac{1}{2\pi i} \frac{k_1^{\lambda+m-1} \epsilon(k_1) k_2^{\lambda+n-1} \ln(k_2)}{z - k_1 - k_2} + \frac{1}{4} \frac{k_1^{\lambda+m-1} \epsilon(k_1) k_2^{\lambda+n-1} \epsilon(k_2)}{z - k_1 - k_2}
\end{array} \right\}
\] (5.16)

The last term of (5.16) is null according to example II). We analyze now the first term. We shall use the identity
\[
k_\lambda^{\lambda+m-1} \ln(k) = D_\alpha k^{\alpha+m-1} \ ; \ D_\alpha = \frac{\partial}{\partial \alpha}|_{\alpha=\lambda}
\]

Then we have:
\[
\frac{i}{4\pi^2} \frac{(-i)^{m+n}}{(m-1)!(n-1)!} \oint_{\Gamma_1} \oint_{\Gamma_2} dk_1 \, dk_2 \frac{k_1^{\lambda+m-1} \ln(k_1) k_2^{\lambda+n-1} \ln(k_2)}{z - k_1 - k_2}
\]
\[
= \frac{i}{4\pi^2} \frac{(-i)^{m+n}}{(m-1)!(n-1)!} \oint_{\Gamma_1} dk_1 \, D_\alpha k^{\alpha+m-1} \oint_{\Gamma_2} dk_2 \frac{D_\beta k_2^{\beta+n-1} \ln(k_2)}{z - k_1 - k_2}
\]
\[ [D_\alpha D_\beta \oint_{\Gamma_1} \frac{k_1^{\alpha+m-1}}{(z-k_1)^{1-\beta-n}} 2\sin\pi(\beta + n - 1) \Gamma(\beta + n) \Gamma(1 - \beta - n) = ] \]

\[ 2\pi i [D_\alpha D_\beta \oint_{\Gamma_1} \frac{k_1^{\alpha+m-1}}{(z-k_1)^{1-\beta-n}} = ] \]

\[ 2\pi i [D_\alpha D_\beta \frac{\Gamma(\alpha+m) \Gamma(1-\alpha-m-\beta-n)}{\Gamma(1-\beta-n)} 2\sin\pi(\alpha+m-1) \ z^{\alpha+\beta+m+n-1} = ] \]

\[ 4\pi [D_\alpha D_\beta \frac{\Gamma(\alpha+m) \Gamma(\beta + n) \sin\pi\alpha \sin\pi\beta}{\Gamma(\alpha + \beta + m + n) \sin\pi(\alpha + \beta)} z^{\alpha+\beta+m+n-1} = ] \]

\[ -\frac{1}{\pi (m+n-1)!} D_\alpha D_\beta \left\{ \frac{\sin\pi\alpha \sin\pi\beta}{\sin\pi(\alpha + \beta)} z^{\alpha+\beta+m+n-1} \right\} \]

(5.17)

where we have used the fact that any derivative, \( D_\alpha \) or \( D_\beta \) acting on a \( \Gamma \) function will lead to a null result of (5.17) through the substitutions \( \alpha = \lambda, \beta = \lambda, \lambda \to 0 \). Now the derivatives in (5.17) give raise essentially to two types of terms. The two derivatives acting on the trigonometric functions give rise to a pole term (in \( \lambda \)). If one takes a derivative of the trigonometric functions and a derivative of \( z^{\alpha+\beta} \), a constant term is obtained. For the term \( D_\alpha D_\beta z^{\alpha+\beta} \), the limit \( \lambda \to 0 \) of the trigonometric functions is zero. Thus we get:

\[ (5.17) = -\frac{(-i)^{m+n-1}}{(m+n-1)!} z^{m+n-1} \left\{ \frac{1}{4 \lambda} z^{2\lambda} + \frac{1}{2} \ln(z) \right\} \]

The second and third terms of (5.16) have the same contribution, and can be evaluated by a similar procedure. This contribution is:

\[ \frac{1}{8\pi (m-1)! (n-1)!} \oint_{\Gamma_1} \oint_{\Gamma_2} \frac{k_1^{\lambda+m-1} \ln(k_1) k_2^{\lambda+n-1} \epsilon(k_2)}{z - k_1 - k_2} = \]

\[ \frac{(-i)^{m+n}}{(m+n-1)!} \frac{\pi}{4} \epsilon(z) z^{m+n-1} \]

(5.18)

According to (5.17) and (5.18), we finally get:

\[ (5.16) = \frac{(-i)^{m+n}}{(m+n-1)!} z^{m+n-1} \left\{ \frac{i}{4 \lambda} z^{2\lambda} + \frac{i}{2} \ln(z) + \frac{\pi}{2} \epsilon(z) \right\} = \]

16
\[
\frac{(-i)^{m+n}}{(m+n-1)!} z^{m+n-1} \left\{ i \frac{1}{4 \lambda} \left(1 + 2 \lambda \ln(z)\right) + \frac{i}{2} \ln(z) + \frac{\pi}{2} \epsilon(z) \right\} = \\
\frac{(-i)^{m+n} \pi}{(m+n-1)!} z^{m+n-1} \left\{ -\frac{1}{\pi i} \ln(z) + \frac{1}{2} \epsilon(z) \right\} 
\]

(5.19)

The \( \lambda \)-independent term is recognized to be \( \tilde{x}^{-m-n} \) (cf.5.15). The pole term is equivalent to zero, according to §3 III.

\[ V \]

Finally, we give a physical example. We consider a massless scalar \( \frac{\lambda}{\pi} \phi^4(x) \) theory in four dimensions. For this theory we shall evaluate the self-energy Green function.

The propagator for the field \( \phi(x) \) is (ref.[9]):

\[
\Delta(x) = \left[-4 \pi^2(u^2 - i0)\right]^{-1} 
\]

(5.20)

According to eq.(A.5)-(A.10) of Appendix we can write:

\[
(u^2 - i0)^{-1} = (2k_0)^{-1} \left[ (k_0 - r)^{-1} + (k_0 + r)^{-1} \right] + \\
(2r)^{-1} \left[ \delta(k_0 - r) + \delta(k_0 + r) \right] + C \delta(k_0 - r) \delta(k_0 + r) 
\]

(5.21)

( where \( C \) is an arbitrary constant appearing in the definition of some distributions, ref.[9], 8.8, 8.9 (See also Appendix)).

And using the results of I) to IV) it is easy show that:

\[
(u^2 - i0)^{-1}(u^2 - i0)^{-1} = (u^2 - i0)^{-2} 
\]

Then, we have for the self-energy

\[
\Sigma(x) = (\Delta(x))^2 = \frac{1}{16 \pi^4} (u^2 - i0)^{-2} 
\]

(5.22)

Where \( (u^2 - i0)^{-2} \) is defined in ref.[9], 8.8, 8.9.

6 Discussion

When we use the perturbative development in Quantum Field Theory, we have to deal with products of distributions in configuration space, or else, with convolutions in the Fourier transformed p-space. Unfortunately, products or convolutions (of distributions) are in general ill-defined quantities. However, in physical applications one introduces some “regularization”
scheme, which allows us to give sense to divergent integrals. Among these procedures we would like to mention the dimensional regularization method (ref. \[14, 15\]). Essentially, the method consists in the separation of the volume element \( \frac{d^\nu p}{p^\nu - 1} \) into an angular factor \( d\Omega \) and a radial factor \( p^{\nu - 1} dp \). First the angular integration is carried out and then the number of dimensions \( \nu \) is taken as a free parameter. It can be adjusted to give a convergent integral, which is an analytic function of \( \nu \).

Our formula (4.7) is similar to the expression one obtains with dimensional regularization. However, the parameter \( \lambda \) is completely independent of any dimensional interpretation.

All ultradistributions provide integrands (in (4.7)) that are analytic functions along the integration path. The parameter \( \lambda \) permits us to control the possible tempered asymptotic behavior (cf. eq. (3.9)). The existence of a region of analyticity for \( \lambda \), and a subsequent continuation to the point of interest (ref. [8]), defines the convolution product.

Those properties show that tempered ultradistributions provide an appropriate framework for applications to physics. Furthermore, they can “absorb” arbitrary polynomials, thanks to eq. (3.10). A property that is interesting for renormalization theory. (See for example the elimination of the pole term in (5.19)). Consequently, we began this paper with a summary of the main characteristics of tempered ultradistributions and their Fourier transformed distributions of the exponential type.
Appendix

Definitions

From ref. [8] we quote the formula:

\[ B(\lambda, \mu) = \frac{1}{2} \int_0^1 dx \, x^{\lambda-1} \left[ (1 - x)^{\mu-1} - \sum_{r=0}^{k-1} (-1)^r \frac{\Gamma(\mu) x^r}{r! \Gamma(\mu - r)} \right] + \]

\[ \int_{1/2}^1 dx \, (1 - x)^{\mu-1} \left[ x^{\lambda-1} - \sum_{r=0}^{s-1} (-1)^r \frac{\Gamma(\lambda)(1 - x)^r}{r! \Gamma(\lambda - r)} \right] + \]

\[ \sum_{r=0}^{k-1} \frac{(-1)^r \Gamma(\mu)}{2^{r+\lambda} r! \Gamma(\mu - r)(r + \lambda)} + \sum_{r=0}^{s-1} \frac{(-1)^r \Gamma(\lambda)}{2^{r+\mu} r! \Gamma(\lambda - r)(r + \mu)} \]  \hspace{1cm} (A. 1)

valid for \( Re \lambda > -k, \ Re \mu > -s \), where \( k \) and \( s \) are positive integers.

From ref. [13] we get:

\[ B(\lambda, \mu) = \frac{\Gamma(\lambda) \Gamma(\mu)}{\Gamma(\lambda + \mu)} \] \hspace{1cm} (A. 2)

\[ \Gamma(\lambda) = \int_0^\infty dt \, t^{\lambda-1} e^{-t} \] \hspace{1cm} (A. 3)

\[ \Gamma(\lambda) \Gamma(1 - \lambda) = \frac{\pi}{\sin \pi \lambda} \] \hspace{1cm} (A. 4)

From ref. [9] we have:

\[ \delta^{(m)}(u^2) = \delta^{(m)}(x_0 + r)(x_0 - r)^{-m-1} sgn(x_0 - r) + \]

\[ \delta^{(m)}(x_0 - r)(x_0 + r)^{-m-1} sgn(x_0 + r) \] \hspace{1cm} (A. 5)

where:

\[ u^2 = x_0^2 - x_1^2 - \cdots - x_{n-1}^2 \] \hspace{1cm} (A. 6)

\[ r^2 = x_1^2 + x_2^2 + \cdots + x_{n-1}^2 \] \hspace{1cm} (A. 7)

\[ (u^2 \pm i0)^{-m} = u^{-2m} \pm \frac{(-1)^m}{(m-1)!} i \pi \delta^{(m-1)}(u^2) \] \hspace{1cm} (A. 8)

\[ x^{-m} sgn(x) = \frac{(-1)^{m-1}}{(m-1)!} \{ |x|^{-1} \}^{(m-1)} \] \hspace{1cm} (A. 9)

\[ |x|^{-1} = \{ sgn(x) \ln |x| \}' + C \delta(x) \] \hspace{1cm} (A. 10)

where \( C \) is an arbitrary constant.


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