PROOF OF A CONJECTURE OF COLLIOIT-THÉLÈNE

JAN DENEF

Abstract. We prove a conjecture of Colliot-Thélène that implies the Ax-Kochen Theorem on $p$-adic forms. We obtain it as an easy consequence of a diophantine purity theorem whose proof forms the body of the present paper.

1. Introduction

In this paper we prove the following conjecture of Colliot-Thélène [5].

1.1. Colliot-Thélène’s Conjecture. Let $f : X \to Y$ be a dominant morphism of nonsingular proper geometrically integral varieties over $\mathbb{Q}$, with geometrically integral generic fibre. Assume that for any nontrivial discrete valuation on the function field $K$ of $Y$, with valuation ring $A \supset \mathbb{Q}$, there exists an integral regular $A$-scheme $\mathcal{X}$, flat and proper over $A$, with generic fibre $K$-isomorphic to the generic fibre of $f$, and special fibre having an irreducible component of multiplicity 1 which is geometrically integral. Then the map $X(\mathbb{Q}_p) \to Y(\mathbb{Q}_p)$, induced by $f$, is surjective for almost all primes $p$.

Here $\mathbb{Q}_p$ denotes the field of $p$-adic numbers, and with “almost all primes” we mean “all but a finite number of primes”.

Actually we prove a stronger result, namely:

Main Theorem 1.2. Let $f : X \to Y$ be a dominant morphism of nonsingular proper geometrically integral varieties over $\mathbb{Q}$, with geometrically integral generic fibre. Assume for any modification $f' : X' \to Y'$ of $f$, with same generic fibre as $f$, and $X', Y'$ nonsingular, and for any irreducible divisor $D'$ on $Y'$, the following: the divisor $f'^{-1}(D')$ on $X'$ has an irreducible component $C'$ with multiplicity 1 and geometrically integral generic fibre over $D'$ (i.e. the morphism $C' \to D'$, induced by $f'$, has geometrically integral generic fibre). Then the map $X(\mathbb{Q}_p) \to Y(\mathbb{Q}_p)$, induced by $f$, is surjective for almost all primes $p$.

Date: April 28, 2013.
We say that \( f' \) is a modification of \( f \) if \( f' \) fits into a commutative square of morphisms of varieties, with vertical arrows \( f, f' \), and horizontal arrows birational proper morphisms \( X' \rightarrow X, \ Y' \rightarrow Y \), see Definition 2.1.

The conjecture of Colliot-Thélène is a direct consequence of Theorem 1.2, because any \( D' \) as in the theorem induces a discrete valuation on the function field of \( Y' \), which equals the function field of \( Y \). Moreover, if there exists an \( A \)-scheme \( \mathcal{X} \) as in the conjecture, then the special fibre of any integral regular proper flat \( A \)-scheme \( \mathcal{X}' \), with same generic fibre as \( \mathcal{X} \), has an irreducible component of multiplicity 1 which is geometrically integral. Indeed this is Proposition 3.9.(b) in Colliot-Thélène’s lecture notes [6].

Note that Theorem 1.2 is substantially stronger than the conjecture, because it requires the assumption in the conjecture only for divisorial discrete valuations on the function field of \( Y \).

Colliot-Thélène [5] proved the following: if \( f : X \rightarrow Y \) is the universal family over \( \mathbb{Q} \) of all projective hypersurfaces of degree \( d \) in projective \( n \)-space, with \( n \geq d^2 \), then \( f \) satisfies the hypotheses of the conjecture and also the hypotheses of Theorem 1.2. Since our proof of Theorem 1.2 is purely algebraic geometric, this yields a new proof of the theorem of Ax and Kochen [3] on \( p \)-adic forms, that does not rely on methods from mathematical logic. The theorem of Ax and Kochen states that for each \( d \in \mathbb{N} \) there exists \( N \in \mathbb{N} \) such that for all primes \( p > N \), each hypersurface of degree \( d \) in projective \( n \)-space over \( \mathbb{Q}_p \), with \( n \geq d^2 \), has a \( \mathbb{Q}_p \)-rational point.

One of the motivations of Colliot-Thélène in formulating his conjecture was to obtain an algebraic geometric proof of the Ax-Kochen Theorem that, unlike all previous ones, does not rely on methods from mathematical logic. At the same time, the author of the present paper also found another purely algebraic geometric proof of the Ax-Kochen Theorem, see [7]. Both proofs are based on the Tameness Theorem (see section 4), which is proved in [7] using the Weak Toroidalization Theorem of Abramovich an Karu [1] (extended to non-closed fields [2]).

We prove the Main Theorem 1.2 in section 3 as an easy consequence of what we call a diophantine purity theorem. The proof of this Purity Theorem 2.2 forms the body of the present paper and is contained in section 2. It depends on the Tameness Theorem 1.1 which is treated in section 4. Using mathematical logic one can give a simpler proof of Colliot-Thélène’s Conjecture 1.1. However we don’t see how to extend this to prove the stronger Theorem 1.2 or the Purity Theorem 2.2. This alternative proof is given in section 3.3.
1.3. Notation and conventions.

For any prime $p$ we denote the ring of $p$-adic integers by $\mathbb{Z}_p$, and the field with $p$ elements by $\mathbb{F}_p$. The $p$-adic valuation on $\mathbb{Q}_p$ is denoted by $\text{ord}_p$. For any integral domain $A$ we denote its fraction field by $\text{Frac}(A)$. With a variety over an integral domain $R$ we mean an integral separated scheme of finite type over $R$. With a morphism of varieties over $R$ we mean an $R$-morphism of schemes over $R$.

For ease of notation we work with the completions $\mathbb{Q}_p$ of $\mathbb{Q}$, but all results in the present paper remain true replacing $\mathbb{Q}$ by any number field $K$ and $\mathbb{Q}_p$ by the non-archimedean completions of $K$.

Acknowledgements. We thank J.-L. Colliot-Thélène, O. Gabber, and O. Wittenberg for a mutual conversation that resulted in the alternative proof 3.3. We also thank D. Abramovich, R. Cluckers, J.-L. Colliot-Thélène, S. D. Cutkosky, K. Karu, and O. Wittenberg for stimulating conversations and useful information.

2. The Purity Theorem

Definition 2.1. Let $R$ be a noetherian integral domain, and $X$ a variety over $R$. A modification of $X$ is a proper birational morphism $X' \to X$ of varieties over $R$.

Let $f : X \to Y$ be a dominant morphism of varieties over $R$. A modification of $f$ is a morphism $f' : X' \to Y'$ of varieties over $R$, which fits into a commutative diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{\alpha} & X \\
f' \downarrow & & \downarrow f \\
Y' & \xrightarrow{\beta} & Y
\end{array}
\]

with $\alpha$ a modification of $X$, and $\beta$ a modification of $Y$. This implies that $f'$ is dominant. Clearly, if $f$ is proper, then also $f'$ is proper.

When $f : X \to Y$ is a dominant morphism of varieties over $R$, and $\beta : Y' \to Y$ is a modification of $Y$, then there exists a unique irreducible component $X'$ of the fibre product $Y' \times_Y X$ that dominates $X$. Let $f'$ and $\alpha$ be the restrictions to $X'$ of the projections $Y' \times_Y X \to Y'$, and $Y' \times_Y X \to X$. Then $\alpha$ is a modification of $X$, and we call $f'$ the strict transform of $f$ with respect to $\beta$. Clearly $f'$ is a modification of $f$. Such a modification is called a strict modification of $f$. Note that any strict modification of $f'$ is also a strict modification of $f$. 

Purity Theorem 2.2. Let $f : X \to Y$ be a proper dominant morphism of varieties over $\mathbb{Z}$, with $Y \otimes \mathbb{Q}$ nonsingular. Assume that for each strict modification $f' : X' \to Y'$ of $f$, with $Y' \otimes \mathbb{Q}$ nonsingular, there exists a closed subscheme $S'$ of $Y'$, of codimension $\geq 2$, such that for almost all primes $p$ we have

$$\{ y \in Y'(\mathbb{Z}_p) \mid y \mod p \not\in S'(\mathbb{F}_p) \} \subset f'(X'(\mathbb{Z}_p)).$$

Then the map $X(\mathbb{Z}_p) \to Y(\mathbb{Z}_p)$, induced by $f$, is surjective for almost all primes $p$.

We prove the Purity Theorem at the end of the present section, after some lemma’s. But first we mention some observations whose proofs are straightforward.

2.3. Observations

(a) Let $f : X \to Y$ be a morphism of schemes of finite type over an excellent henselian discrete valuation ring $R$, with $Y$ smooth over the fraction field of $R$. Let $S$ be a closed subscheme of $Y$, containing no irreducible component of $Y$. If $Y(R) \setminus S(R) \subset f(X(R))$, then $Y(R) \subset f(X(R))$. Indeed this follows from Greenberg’s theorem [9], because $Y(R) \setminus S(R)$ is dense in $Y(R)$ with respect to the adic topology on $Y(R)$, since $Y$ is smooth over the fraction field of $R$.

(b) Let $f : X \to Y$ be a dominant morphism of varieties over $\mathbb{Z}$, and let $f' : X' \to Y'$ be a strict modification of $f$. Assume that $Y \otimes \mathbb{Q}$ and $Y' \otimes \mathbb{Q}$ are nonsingular, and let $p$ be a prime number. Then, the map $X(\mathbb{Z}_p) \to Y(\mathbb{Z}_p)$, induced by $f$, is surjective, if and only if the map $X'(\mathbb{Z}_p) \to Y'(\mathbb{Z}_p)$, induced by $f'$, is surjective. This remains true when $f'$ is a modification of $f$ which is not strict, if we assume that also $X \otimes \mathbb{Q}$ is nonsingular. These claims follow directly from (a).

(c) Let $f : X \to Y$ be a proper dominant morphism of varieties over $\mathbb{Z}$, with $Y \otimes \mathbb{Q}$ nonsingular, satisfying the assumption of the Purity Theorem. If $U$ is a nonempty open subscheme of $Y$, then also the morphism $f^{-1}(U) \to U$, induced by $f$, satisfies the assumption of the Purity Theorem.

Proof of the observation (c). It suffices to show that any modification $\beta_0 : U' \to U$ of $U$, with $U' \otimes \mathbb{Q}$ nonsingular, factors as an open immersion $j : U' \to Y'$ composed with a modification $\beta : Y' \to Y$ of $Y$, with $Y' \otimes \mathbb{Q}$ nonsingular, and $j(U') = \beta^{-1}(U)$. To achieve this, let $\beta_1 : U' \to Y$ be the composition of $\beta_0$ with the inclusion $U \subset Y$. Apply Nagata’s compactification theorem (see e.g. [12]) to factorize $\beta_1$ as an open immersion $j_2 : U' \to Y''$ composed with a proper morphism $\beta_2 : Y'' \to Y$ of $\mathbb{Z}$-varieties. This implies that $U'' := j_2(U') = \beta_2^{-1}(U)$ and that $\beta_2$ is a modification of $Y$. There exists a resolution of $Y'' \otimes \mathbb{Q}$
which is a composition of blowups with nonsingular centers \( C_1, \ldots, C_r \) that lie above \((Y'' \setminus U'') \otimes \mathbb{Q}\). Denote by \( C_1' \) the closure of \( C_1 \) in \( Y'' \). Denote by \( C_2' \) the closure of \( C_2 \) in the blowup of \( Y'' \) with center \( C_1' \), and so on. Let \( \pi : Y' \to Y'' \) be the modification of \( Y'' \) obtained by composing the blowups with centers \( C_1, \ldots, C_r \). Then \( Y' \otimes \mathbb{Q} \) is nonsingular. Moreover, \( \pi \) is an isomorphism above \( U'' \). This yields an open immersion \( j_1 : U'' \to Y' \). Clearly \( j := j_1 \circ j_2 \) and \( \beta := \beta_2 \circ \pi \) satisfy the required properties. \(\Box\)

(d) Let \( f : X \to Y \) be a proper dominant morphism of varieties over \( \mathbb{Z} \), with \( Y \otimes \mathbb{Q} \) nonsingular, satisfying the assumption of the Purity Theorem. Let \( f_1 : X_1 \to Y_1 \) be a strict modification of \( f \), with \( Y_1 \otimes \mathbb{Q} \) nonsingular. Then \( f_1 \) satisfies the assumption of the Purity Theorem. This follows directly from the fact that any strict modification of \( f_1 \) is also a strict modification of \( f \).

Remark. We will often use (without mentioning) the following well known facts. Any morphism \( f_0 : X_0 \to Y_0 \) of varieties over \( \mathbb{Q} \) has a model \( f \) over \( \mathbb{Z} \). This means that \( f \) is a morphism \( f : X \to Y \) of varieties over \( \mathbb{Z} \) whose base change to \( \mathbb{Q} \) is isomorphic to \( f_0 \). Combining this with Nagata’s compactification theorem (see e.g. [12]), we see that we can choose \( f \) to be proper, when \( f_0 \) is proper. Two models of \( f_0 \) over \( \mathbb{Z} \) become isomorphic after base change to \( \mathbb{Z}[1/N] \), for \( N \in \mathbb{N} \) large enough. Hence, if \( f_0 \) is proper then any model \( f \) of \( f_0 \) over \( \mathbb{Z} \) becomes proper after base change to \( \mathbb{Z}[1/N] \), for \( N \in \mathbb{N} \) large enough.

Lemma 2.4. Let \( f : X \to Y \) be a proper dominant morphism of varieties over \( \mathbb{Z} \), with \( Y \) smooth over \( \mathbb{Z} \), satisfying the assumption of the Purity Theorem. Let \( h : Y \to \mathbb{A}_\mathbb{Z}^1 \) be a smooth morphism. Then for almost all primes \( p \) we have the following: for each \( b \in Y(\mathbb{Z}_p) \) there exists an \( a \in X(\mathbb{Z}_p) \) such that \( f(a) \equiv b \mod p \) and \( h(f(a)) = h(b) \).

Proof. By noetherian induction, it suffices to show that for any integral closed subscheme \( W \) of \( Y \), there exists a nonempty open subscheme \( W_0 \) of \( W \), such that, for almost all \( p \), the assertion of the lemma holds for all \( b \in Y(\mathbb{Z}_p) \) satisfying \( b \mod p \in W_0 \). Clearly, we may assume that \( W \otimes \mathbb{Q} \) is nonempty, and that \( W \not\subseteq Y \). Indeed, if \( W = Y \) then we can directly apply the assumption of the Purity Theorem, with \( f' = f \), to find \( W_0 \). By observation 2.3 (c), we may also assume that \( W \) is smooth over \( \mathbb{Z} \).

Let \( \beta : Y' \to Y \) be the blowup of \( Y \) with center \( W \), and let \( f' : X' \to Y' \) be the strict transform of \( f \) with respect to \( \beta \). Because the assumption of the Purity Theorem is assumed, there exists a closed subscheme \( S' \) of \( Y' \), of codimension \( \geq 2 \), such that for almost all primes
To start, we take $W_0$ equal to $W$, but later on we will replace $W_0$ by a smaller nonempty open subscheme of $W$ if necessary.

When the restriction of $h$ to $W$ is dominant, then making $W_0$ smaller if necessary, we may suppose that the restriction of $h \circ \beta$ to $\beta^{-1}(W_0)$ is smooth, whence $h \circ \beta$ is smooth at each point of $\beta^{-1}(W_0)$.

When the restriction of $h$ to $W$ is not dominant, then $h \otimes \mathbb{Q}$ is constant with value say $v$. Because $h$ is smooth, the multiplicity of $\beta^{-1}(W)$, in the divisor of $(h \circ \beta)-v$, equals 1. Let $C'$ be the intersection of $\beta^{-1}(W)$ with the union of the other irreducible components of this divisor. Clearly $C'$ has codimension $\geq 2$ in $Y'$, and $h \circ \beta$ is smooth at each point of $\beta^{-1}(W) \setminus C'$. Enlarging $S'$ if necessary, we may suppose that $C' \subset S'$.

Thus in either case, we can suppose that $h \circ \beta$ is smooth at each point of $\beta^{-1}(W_0) \setminus S'$.

Since the image under $\beta$ of $\beta^{-1}(W) \setminus S'$ is a constructible subset of $W$, and since $S'$ has codimension $\geq 2$, we may suppose by replacing $W_0$ by a smaller open subscheme of $W$, that $W_0 \subset \beta(Y' \setminus S')$.

Let $p$ be a big enough prime, and consider any $b \in Y'(\mathbb{Z}_p)$ satisfying $\bar{b} := b \mod p \in W_0$. Because the scheme-theoretic fibre of $\beta$ over $\bar{b}$ is not contained in $S'$, and isomorphic to a projective space over $\mathbb{F}_p$, there exists a $\mathbb{F}_p$-rational point $\bar{b}'$ on $Y' \setminus S'$ with $\beta(\bar{b}') = \bar{b}$. Because $h \circ \beta$ is smooth at $\bar{b}'$, this point lifts to a point $b' \in Y'(\mathbb{Z}_p)$ with $b' \mod p \not\in S'$, $\beta(b') \equiv b \mod p$, and $(h \circ \beta)(b') = h(b)$. Hence there exists $a' \in X'(\mathbb{Z}_p)$ with $f'(a') = b'$. Let $a \in X(\mathbb{Z}_p)$ be the image of $a'$ under the natural morphism $X' \rightarrow X$. Then $f(a) = \beta(b') \equiv b \mod p$, and $h(f(a)) = h(\beta(b')) = h(b)$. This terminates the proof of the lemma.

\[\square\]

**Remark 2.5.** The previous Lemma 2.4 also holds when there is no $h$ involved, if we drop the requirement that $h(f(a)) = h(b)$. This follows formally from this lemma, using observation 2.3(c), by covering $Y$ by finitely many small enough open subschemes on which there exists a smooth morphism to $\mathbb{A}^1_\mathbb{Z}$.

**Definition 2.6.** Let $R$ be a noetherian integral domain, and $X$ a variety over $R$. Let $A$ be a local $R$-algebra without zero divisors, and $m$ its maximal ideal.

Let $z, z' \in \text{Frac}(A)$. The elements $z, z'$ have *same multiplicative residue* if

$$z' \in z(1 + m).$$
Let \( a, a' \in X(A) \) and \( x_1, \ldots, x_r \) rational functions on \( X \). The points \( a, a' \) have same residues with respect to \( x_1, \ldots, x_r \), if for \( i = 1, \ldots, r 
 
1. \ a \mod m = a' \mod m, 
2. \ x_i(a) \) is defined as element of Frac(\( A \)) if and only if \( x_i(a') \) is defined, 
3. \( x_i(a), x_i(a') \) have same multiplicative residue, if both are defined.

**Lemma 2.7.** Let \( f : X \to Y \) be a proper dominant morphism of varieties over \( \mathbb{Z} \), with \( Y \otimes \mathbb{Q} \) nonsingular, satisfying the assumption of the Purity Theorem. Let \( y_1, \ldots, y_s \) be rational functions on \( Y \), and \( M \in \mathbb{N} \). Then for almost all primes \( p \) we have the following. For each \( b \in Y(\mathbb{Z}_p) \), with \( y_1(b), \ldots, y_s(b) \) defined, as elements of \( \mathbb{Q}_p \), and having \( \text{ord}_p(b) \leq M \), there exists \( a \in X(\mathbb{Z}_p) \) such that \( f(a) \) and \( b \) have same residues with respect to \( y_1, \ldots, y_s \).

**Proof.** Let \( D_\infty \subset Y \otimes \mathbb{Q} \) be the union of the polar loci of the rational functions \( y_1, \ldots, y_s \) restricted to \( Y \otimes \mathbb{Q} \). Using elimination of indeterminacy of the rational functions \( y_1, \ldots, y_s \) restricted to \( Y \otimes \mathbb{Q} \), modifying \( Y \), without changing \( Y \otimes \mathbb{Q} \setminus D_\infty \), and inverting a finite number of primes, we may suppose that \( y_1, \ldots, y_s \) induce morphisms from \( Y \) to \( \mathbb{P}^1_{\mathbb{Z}} \). Indeed, we can invert a finite number of primes because of observation [2.3] (c). Let \( D \subset Y \otimes \mathbb{Q} \) be the union of the zero loci and polar loci of the rational functions \( y_1, \ldots, y_s \), restricted to \( Y \otimes \mathbb{Q} \). Using embedded resolution of singularities of \( D \subset Y \otimes \mathbb{Q} \), modifying \( Y \), without changing \( Y \otimes \mathbb{Q} \setminus D \), inverting a finite number of primes, and again using observation [2.3] (c), we may assume the following. The variety \( Y \) is smooth over \( \mathbb{Z} \), and affine, and each \( y_i \) is a monomial, or the reciprocal of a monomial, in uniformizing parameters on \( Y \) over \( \mathbb{Z} \), i.e. regular functions on \( Y \) that induce an etale morphism to an affine space over \( \mathbb{Z} \). Hence, replacing the \( y_i \), we can moreover assume that \( y_1, \ldots, y_s \) are part of a set of uniformizing parameters on \( Y \) over \( \mathbb{Z} \).

It remains now to prove the lemma in the special case that \( Y \) is smooth over \( \mathbb{Z} \), and affine, say \( Y = \text{Spec}(A) \), and that \( y_1, \ldots, y_s \) are part of a set of uniformizing parameters on \( Y \) over \( \mathbb{Z} \). We prove this special case by induction on \( M \). Let \( p \) be a prime, big enough with respect to \( M \) and all data except \( b \), and let \( b \in Y(\mathbb{Z}_p) \) be any point with \( \text{ord}_p(y_i(b)) \leq M \) for all \( i = 1, \ldots, s \). If \( M = 0 \), then, in order to prove the lemma, it suffices to find \( a \in X(\mathbb{Z}_p) \) with \( f(a) \equiv b \mod p \). The existence of such an \( a \) follows from Remark [2.3]. Thus we may suppose that \( M > 0 \) and that

\[
I_0 := \{ i \in \mathbb{N} \mid \text{ord}_p(y_i(b)) > 0, 1 \leq i \leq s \} \neq \emptyset.
\]
Choose $i_0 \in I_0$ such that $\text{ord}_p(y_{i_0}(b)) = \text{Min}_{i \in I_0} \text{ord}_p(y_i(b))$.

Let $\pi : Y' \to Y$ be the blowup of the ideal sheaf on $Y$ generated by all the $y_i$ with $i \in I_0$. Consider the chart $U$ on $Y'$, defined as follows:

$$U := \text{Spec}(A[(y_i/y_{i_0})_{i \in I_0}]) \xrightarrow{\pi} \text{Spec}(A) = Y.$$

There exists a unique $b' \in U(\mathbb{Z}_p)$ with $\pi(b') = b$. Set $y'_i = y_i/y_{i_0}$ for $i \in I_0 \setminus \{i_0\}$, and $y'_i = y_i$ for the other $i \in \{1, \ldots, s\}$. Note that $y'_1, \ldots, y'_s$ are part of a set of uniformizing parameters on $U$ over $\mathbb{Z}$.

Clearly, either $0 \leq \text{ord}_p(y'_i(b')) < M$, for all $i$, or $\text{ord}_p(y'_i(b')) = 0$, for all $i \neq i_0$. We call these respectively the first case and the second case.

Let $f' : X' \to Y'$ be the strict transform of $f$ with respect to the blowup $\pi : Y' \to Y$. In the first case, we apply the induction hypothesis to the morphism $f'^{-1}(U) \to U$ induced by $f'$, and the rational functions $y'_1, \ldots, y'_s$, to find $a' \in X'(\mathbb{Z}_p)$, with $f'(a') \in U(\mathbb{Z}_p)$, such that $f'(a')$ and $b'$ have the same residues with respect to $y'_1, \ldots, y'_s$. In the second case, we apply Lemma 2.4 to the morphism $f'^{-1}(U) \to U$ induced by $f'$, and the morphism $U \to \mathbb{A}^1_{\mathbb{Z}}$ induced by $y_{i_0}$, to find $a' \in X'(\mathbb{Z}_p)$, with $f'(a') \in U(\mathbb{Z}_p)$, such that $f'(a') \equiv b' \mod p$ and $y_{i_0}(f'(a')) = y_{i_0}(b')$. Hence, also in the second case, $f'(a')$ and $b'$ have the same residues with respect to $y'_1, \ldots, y'_s$.

Denote by $a$ the image of $a'$ under the natural map $X'(\mathbb{Z}_p) \to X(\mathbb{Z}_p)$. Then the points $f(a) = \pi(f'(a'))$ and $b = \pi(b')$ have the same residues with respect to $y_1, \ldots, y_s$. This terminates the proof of the lemma. □

2.8. Proof of the Purity Theorem 2.2

The Purity Theorem 2.2 is a direct consequence of the above Lemma 2.4 and the Surjectivity Criterion 4.2. □

3. Proof of the Main Theorem 1.2

In this section we show that the Main Theorem 1.2 is an easy consequence of the Purity Theorem 2.2 and the following lemma whose proof is rather straightforward.

**Lemma 3.1.** Let $f : X \to Y$ be a proper dominant morphism of smooth varieties over $\mathbb{Z}$, with geometrically integral generic fibre. Assume for each $\mathbb{Z}$-flat irreducible divisor $D$ on $Y$, that the divisor $f^{-1}(D)$ on $X$ has an irreducible component $C$ with multiplicity 1 and geometrically integral generic fibre over $D$ (i.e. the morphism $C \to D$, induced by $f$, has geometrically integral generic fibre). Then there exists a closed subscheme $S$ of $Y$, of codimension $\geq 2$, such that for almost all primes $p$ we have

$$\{y \in Y(\mathbb{Z}_p) | y \mod p \notin S(\mathbb{F}_p)\} \subset f(X(\mathbb{Z}_p)).$$
Proof. By Théorème 9.7.7 of [10], there exists a reduced closed subscheme $E \subset Y$, of pure codimension 1, such that over the complement of $E$, the morphism $f$ is smooth with geometrically integral fibres. Hence, for almost all primes $p$, any $y \in Y(\mathbb{Z}_p)$, with $y \mod p \notin E(\mathbb{F}_p)$, belongs to $f(X(\mathbb{Z}_p))$. Indeed this follows from Hensel’s Lemma and the Lang-Weil bound [11].

For each irreducible component $D$ of $E$ we reason as follows. If $D$ is not flat over $\text{Spec}(\mathbb{Z})$, then $D(\mathbb{F}_p)$ is empty for almost all primes $p$. Suppose now that $D$ is flat over $\text{Spec}(\mathbb{Z})$. By assumption, the divisor $f^{-1}(D)$ on $X$ has an irreducible component $C$ with multiplicity 1 and geometrically integral generic fibre over $D$. In particular, $C$ dominates $D$. Hence there exists a reduced closed subscheme $S$ of $D$, of codimension $\geq 1$ in $D$, such that, over the complement of $S$, all fibres of $C \xrightarrow{f} D$ are geometrically integral and intersect the smooth locus of $f : X \to Y$. Indeed, $f$ is smooth at the generic point of $C$, because $C$ has multiplicity 1 in the divisor $f^{-1}(D)$. Again by Hensel’s Lemma and the Lang-Weil bound [11], we conclude for almost all primes $p$ that any $y \in Y(\mathbb{Z}_p)$, with $y \mod p \notin S(\mathbb{F}_p)$, belongs to $f(X(\mathbb{Z}_p))$.

Taking the union of the subschemes $S$, obtained as above for each $\mathbb{Z}$-flat irreducible component $D$ of $E$, we obtain a closed subscheme of $Y$, of codimension $\geq 2$, that satisfies the conclusion of the lemma. □

3.2. Proof of the Main Theorem [1, 2]

Let $f : X \to Y$ be a dominant morphism of nonsingular proper geometrically integral varieties over $\mathbb{Q}$, which satisfies the hypotheses of the Main Theorem. Choose a proper dominant morphism $\tilde{f} : \tilde{X} \to \tilde{Y}$ of smooth varieties over $\mathbb{Z}$, whose base change to $\mathbb{Q}$ is isomorphic to $f$. Because $X$ and $Y$ are proper, it suffices to prove that the map $X(\mathbb{Z}_p) \to Y(\mathbb{Z}_p)$, induced by $\tilde{f}$, is surjective for almost all primes $p$. By the Purity Theorem [2, 2] it suffices to prove that the morphism $\tilde{f}$ satisfies the assumption in the Purity Theorem, with $f$ replaced by $\tilde{f}$.

Let $\tilde{f}' : \tilde{X}' \to \tilde{Y}'$ be any strict modification of $\tilde{f}$, with $\tilde{Y}' \otimes \mathbb{Q}$ nonsingular. We have to prove that there exists a closed subscheme $S$ of $\tilde{Y}'$, of codimension $\geq 2$, such that for almost all primes $p$ we have

$$\{y \in \tilde{Y}'(\mathbb{Z}_p) \mid y \mod p \notin S(\mathbb{F}_p)\} \subset \tilde{f}'(\tilde{X}'(\mathbb{Z}_p)).$$

Composing $\tilde{f}'$ with a morphism whose base change to $\mathbb{Q}$ resolves the singularities of $\tilde{X}' \otimes \mathbb{Q}$, and inverting a finite number of primes, we see that in order to prove the above, we may assume the following. The varieties $\tilde{X}'$ and $\tilde{Y}'$ are smooth over $\mathbb{Z}$, and $\tilde{f}'$ is a modification of $\tilde{f}$,
with same generic fibre as $\tilde{f}$. But now $\tilde{f}'$ is not necessarily a strict modification of $\tilde{f}$ anymore.

Because, by assumption, $f$ satisfies the hypotheses of the Main Theorem, it is easy to verify that $\tilde{f}'$ satisfies the hypotheses of Lemma 3.1 (with $f$ replaced by $\tilde{f}'$). Hence this lemma implies the existence of a closed subscheme $S$ of $Y'$ with the required properties. This terminates the proof of the Main Theorem. □

3.3. An alternative proof of Colliot-Thêlène’s Conjecture.

Using model theory (mathematical logic) one can give a much simpler proof of Colliot-Thêlène’s Conjecture 1.1. However we don’t see how to extend this to prove the stronger Theorem 1.2 or the Purity Theorem 2.2. Moreover one of the motivations of Colliot-Thêlène was to obtain a new proof of the Ax-Kochen Theorem which does not rely on methods from mathematical logic. We briefly sketch this simpler proof of Colliot-Thêlène’s Conjecture.

Assume the notation and hypotheses in the formulation of Conjecture 1.1. Using a suitable ultraproduct of $p$-adic fields and the Ax-Kochen-Eršov Principle [3, 8], we see that in order to prove the conjecture, it suffices to show that the map $X(F[[t]]) \to Y(F[[t]])$, induced by $f$, is surjective for any pseudo algebraically closed field $F$ of characteristic zero. Let $y \in Y(F[[t]])$. We have to show that $y \in f(X(F[[t]]))$. Let $s$ be the closed point of $\text{Spec}(F[[t]])$. By slightly moving $y$ and using Greenberg’s Theorem [9], we may assume that the homomorphism $\mathcal{O}_{Y,y(s)} \to F[[t]]$ induced by $y$ is injective. Composing this homomorphism with the standard valuation on $F[[t]]$, induces a discrete valuation $\nu$ on the function field $K$ of $Y$, with valuation ring say $A$.

If $\nu$ is trivial, then $y(s)$ is the generic point of $Y$. Hence $f$ is smooth at each point in the fibre of $y(s)$, and this fibre is geometrically integral. This implies that $y$ lifts to a $F[[t]]$-rational point $x$ on $X$, by Hensel’s Lemma and the assumption that $F$ is pseudo algebraically closed.

Thus we may assume that the discrete valuation $\nu$ is not trivial. Hence there exists an integral regular $A$-scheme $\mathfrak{X}$ as in the formulation of Conjecture 1.1. Note that $y$ induces a $F[[t]]$-rational point $\tilde{y}$ on $\text{Spec}(A)$, and a homomorphism $K \to F((t))$. Using the hypothesis about the special fibre of $\mathfrak{X}$, Hensel’s Lemma, and the assumption that $F$ is pseudo algebraically closed, one easily verifies that $\tilde{y}$ lifts to a $F[[t]]$-rational point $\tilde{x}$ on $\mathfrak{X}$. Because the generic fibre of $\mathfrak{X}$ is $K$-isomorphic to the generic fibre of $f$, and because $\tilde{x}$ extends to a $F((t))$-rational point on $\mathfrak{X} \otimes K$, we find a $F((t))$-rational point on $X$, and hence, by the properness of $X$, also a $F[[t]]$-rational point $x$ on $X$ with $f(x) = y$. □
4. TAMENESS AND THE SURJECTIVITY CRITERION

The following result is a special case of the Tameness Theorem of [7].

**Tameness Theorem 4.1.** Let $f : X \to Y$ be a morphism of varieties over $\mathbb{Z}$. Given rational functions $x_1, \ldots, x_r$ on $X$, there exist rational functions $y_1, \ldots, y_s$ on $Y$, such that for almost all primes $p$ we have the following. Any $b \in Y(\mathbb{Z}_p)$ having same residues with respect to $y_1, \ldots, y_s$ as an image $f(a')$, with $a' \in X(\mathbb{Z}_p)$, is itself an image of an $a \in X(\mathbb{Z}_p)$ with same residues as $a'$ with respect to $x_1, \ldots, x_r$.

This special case, and the more general result in [7], can be proved easily by using Basarab’s theorem [4] on elimination of quantifiers. The special case itself is also an easy consequence of the theorem of Pas [13] on uniform $p$-adic quantifier elimination. The works of Pas and Basarab are based on methods from mathematical logic. However in [7] we gave a purely algebraic geometric proof of the Tameness Theorem which is based on the Weak Toroidalization Theorem of Abramovich and Karu [1] (extended to non-closed fields [2]).

We briefly sketch the geometric proof of the Tameness Theorem of [7]. Using (weak) toroidalization of the morphism $f \otimes \mathbb{Q}$, and induction on the dimension of $X \otimes \mathbb{Q}$ (to take care of the exceptional loci of the modifications used to obtain a toroidalization), one easily reduces to the following case. The morphism $f \otimes \mathbb{Q}$ is toroidal, $X \otimes \mathbb{Q}$ and $Y \otimes \mathbb{Q}$ are nonsingular, and the zero loci and polar loci of $x_1, \ldots, x_r$, restricted to $X \otimes \mathbb{Q}$, are contained in the support of the toroidal divisor on $X \otimes \mathbb{Q}$. Then $f \otimes \mathbb{Q}$ is log-smooth with respect to the toroidal divisors. In that case the Tameness Theorem follows directly from a logarithmic version of Hensel’s lemma. We refer to [7] for the details.

The following Surjectivity Criterium is based on the Tameness Theorem 4.1 and is essential for the proof of the Purity Theorem 2.2.

**4.2. Surjectivity Criterium.** Let $f : X \to Y$ be a morphism of varieties over $\mathbb{Z}$, with $Y \otimes \mathbb{Q}$ smooth. Suppose that, given any rational functions $y_1, \ldots, y_s$ on $Y$ and $M \in \mathbb{N}$, we have the following for almost all primes $p$. For each $b \in Y(\mathbb{Z}_p)$, with $y_1(b), \ldots, y_s(b)$ defined, as elements of $\mathbb{Q}_p$, and of $|p$-adic valuation$| \leq M$, there exists $a \in X(\mathbb{Z}_p)$ such that $f(a)$ and $b$ have same residues with respect to $y_1, \ldots, y_s$. If this condition is satisfied, then the map $X(\mathbb{Z}_p) \to Y(\mathbb{Z}_p)$, induced by $f$, is surjective for almost all primes $p$.

**Proof.** If we strengthen the hypothesis by imposing no bound on the $p$-adic valuations, then the Surjectivity Criterium is a trivial consequence of the Tameness Theorem 4.1 and observation 2.3(a). Otherwise, it follows easily from these two last facts, together with Lemma 4.4 below.
applied to $X, y_i \circ f$ and to $Y, y_i$. Indeed choose a point in each stratum of a common partition, and choose $M$ bigger than the $|p$-adic valuation$| of the coordinates of these points. We leave the details to the reader. □

**Definition 4.3.** Let $X$ be a variety over $\mathbb{Z}$, and $x_1, \ldots, x_r$ rational functions on $X$. Let $z = (z_1, \ldots, z_r) \in \mathbb{Q}_p^r$. We say that the multiplicative residue of $z$ is realizable with respect to $x_1, \ldots, x_r$ if there exists $a \in X(\mathbb{Z}_p)$ such that for each $i$ we have that $x_i(a)$ is defined, as element of $\mathbb{Q}_p$, and $x_i(a), z_i$ have same multiplicative residue.

For any $w \in \mathbb{Q}_p$, the angular component modulo $p$ of $w$ is defined as

$$\text{ac}(w) := wp^{-\ord_p(w)} \mod p \in \mathbb{F}_p,$$

with the convention that $\text{ac}(0) := 0$.

The following lemma is a direct consequence of the theorem of Pas [13] on uniform $p$-adic quantifier elimination. The work of Pas is based on methods from mathematical logic. Below we give a purely algebraic geometric proof of this lemma which is based on embedded resolution of singularities.

**Lemma 4.4. (Realizability of Residues)** Let $X$ be a variety over $\mathbb{Z}$, and $x_1, \ldots, x_r$ rational functions on $X$. There exists a finite partition of $\mathbb{Z}_p^r$ such that for almost all primes $p$ we have the following. Let $z, z' \in \mathbb{Q}_p^r$. Assume that the $p$-adic valuations of $z$ and $z'$ are in a same stratum of the partition, and that $\text{ac}(z_i) = \text{ac}(z'_i)$ for each $i$. Then the multiplicative residue of $z$ is realizable with respect to $x_1, \ldots, x_r$, if and only if the same holds for $z'$.

**Proof.** Let $D \subset X \otimes \mathbb{Q}$ be the union of the zero loci and polar loci of the rational functions $x_1, \ldots, x_r$ restricted to $X \otimes \mathbb{Q}$. Using embedded resolution of singularities of $D \subset X \otimes \mathbb{Q}$, modifying $X$, without changing $X \otimes \mathbb{Q} \setminus D$, and inverting a finite number of primes, we may assume the following. The variety $X$ is smooth over $\mathbb{Z}$, and affine, and each $x_i$ is a Laurent monomial (i.e. a monomial with exponents in $\mathbb{Z}$) in uniformizing parameters $y_1, \ldots, y_n$ on $X$ over $\mathbb{Z}$, multiplied with a unit (i.e. a regular function on $X$ with empty zero locus). This means that $y_1, \ldots, y_n$ induce an etale morphism to affine $n$-space over $\mathbb{Z}$. Let $E$ be the matrix over $\mathbb{Z}$ consisting of the exponents of these Laurent monomials, and let $\Delta$ be the linear map $\mathbb{Z}^n \to \mathbb{Z}^r$ determined by the matrix $E$.

For each subset $S \subset \{1, \ldots, n\}$, set

$$\Gamma_S := \{ (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \mid \forall j : \alpha_j = 0 \iff j \in S \}.$$
Choose a finite partition of \( \mathbb{Z}^r \) such that each \( \Delta(\Gamma_S) \) is a union of strata. Let \( z, z' \in \mathbb{Q}_p^r \) be as in the lemma and assume that the multiplicative residue of \( z \) is realizable with respect to \( x_1, \ldots, x_r \) by an element \( a \in X(\mathbb{Z}_p) \). We have to show that \( z' \) is also realizable. By slightly moving \( a \), we may suppose that \( y_j(a) \neq 0 \) for all \( j = 1, \ldots, n \). Let \( S \subset \{1, \ldots, n\} \) be such that \( (\text{ord}_p y_1(a), \ldots, \text{ord}_p y_n(a)) \in \Gamma_S \). Hence \( z = (\text{ord}_p x_1(a), \ldots, \text{ord}_p x_r(a)) \in \Delta(\Gamma_S) \). Because the \( p \)-adic valuations of \( z \) and \( z' \) are in a same stratum, there exists \( \alpha' = (\alpha'_1, \ldots, \alpha'_n) \in \Gamma_S \) with \( \Delta(\alpha') = \text{ord}_p(z') \). Note that \( \alpha'_j = 0 \) if and only if \( \text{ord}_p(y_j(a)) = 0 \), because \( \alpha' \in \Gamma_S \). By Hensel’s Lemma, applied to the etale morphism induced by \( y_1, \ldots, y_n \), there exists \( a' \in X(\mathbb{Z}_p) \) with \( a' \mod p = a \mod p \) and \( \text{ord}_p(y_j(a')) = \alpha'_j \) and \( \text{ac}(y_j(a')) = \text{ac}(y_j(a)) \). But this implies that \( \text{ac}(x_i(a')) = \text{ac}(x_i(a)) = \text{ac}(z_i) = \text{ac}(z'_i) \) and \( \text{ord}_p(x_i(a')) = (\Delta(\alpha'))_i = \text{ord}_p(z'_i) \). Thus the multiplicative residue of \( z' \) with respect to \( x_1, \ldots, x_r \) is realized by \( a' \). This terminates the proof of the lemma. □

References

[1] D. Abramovich and K. Karu. Weak semistable reduction in characteristic 0. Invent. Math., 139(2):241–273, 2000.
[2] Dan Abramovich, Jan Denef, and Kalle Karu. Weak toroidalization over non-closed fields. Preprint, ArXiv: 1010.6171, 2010.
[3] James Ax and Simon Kochen. Diophantine problems over local fields. I. Amer. J. Math., 87:605–630, 1965.
[4] Şerban A. Basarab. Relative elimination of quantifiers for Henselian valued fields. Ann. Pure Appl. Logic, 53(1):51–74, 1991.
[5] Jean-Louis Colliot-Thélène. Fibre spéciale des hypersurfaces de petit degré. C. R. Math. Acad. Sci. Paris, 346(1-2):63–65, 2008.
[6] Jean-Louis Colliot-Thélène. Variétés presque rationnelles, leurs points rationnels et leurs dégénérescences. Preprint, ArXiv:0809.1386v1, 2008.
[7] Jan Denef. Algebraic geometric proof of theorems of Ax-Kochen and Ershov. (In preparation).
[8] Ju. L. Eršov. On elementary theories of local fields. Algebra i Logika Sem., 4(2):5–30, 1965.
[9] Marvin J. Greenberg. Rational points in Henselian discrete valuation rings. Inst. Hautes Études Sci. Publ. Math., (31):59–64, 1966.
[10] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III. Inst. Hautes Études Sci. Publ. Math., (28):255, 1966.
[11] Serge Lang and André Weil. Number of points of varieties in finite fields. Amer. J. Math., 76:819–827, 1954.
[12] W. Lütkebohmert. On compactification of schemes. Manuscripta Math., 80(1):95–111, 1993.
[13] Johan Pas. Uniform \( p \)-adic cell decomposition and local zeta functions. J. Reine Angew. Math., 399:137–172, 1989.
(Denef) University of Leuven, Department of Mathematics, Celestijnenlaan 200 B, B-3001 Leuven (Heverlee), Belgium.

E-mail address: jan.denef@wis.kuleuven.be