Geometric Stability of Brane-worlds

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Abstract

The stability conditions for coordinate gauge independent perturbations of brane-worlds are analyzed. It is shown that, these conditions lead to the Einstein-Hilbert dynamics and to a confined gauge potential, independently of models and metric ansatzes. The size of the extra dimensions are estimated without assuming a fixed topology. The quantum modes corresponding to high frequency gravitational waves are defined through a canonical structure.

pacs 11.10.Kk, 04.50.+h, 04.60.+n

I. INTRODUCTION

The "brane-worlds" program proposes a unification of all interactions at the TeV scale, with the supposition that there are large (as compared to Planck’s length) extra dimensions. The hierarchy problem is solved by the assumption that the four-dimensional space-time geometry undergoes quantum fluctuations along the extra dimensions, but the gauge interactions of the standard model remain confined within the space-time. This is implemented by the introduction of a fundamental Planck scale in the higher dimensional bulk, while the effective four-dimensional Planck mass is adjusted in accordance with the geometry of the higher dimensional space. The geometrical scenario compatible with these assumptions is that of four-dimensional space-times, the brane-worlds, are dynamically embedded in a higher dimensional solution Einstein’s equations.

In contrast with Kaluza-Klein and Superstring theories the brane-world unification in principle may be experimentally verified by high-energy collisions at hundreds of TeV. Another implication that may soon be checked is the modification of Newton’s law of gravitation at sub millimeter scale.

There is not yet a theory of brane-worlds in the sense that the fundamental principles are well established. Some earlier publications, not always acknowledged, have proposed some basic concepts and formal tools. For example, the confinement of gauge interactions to space-time behaving as a potential well in a higher dimensional space has been suggested in the early eighties. Applications of space-time embeddings to the quantization of the geometry, to generate internal symmetries and as an alternative to compactification, have been around for a considerable time. Recent reviews and additional references can be found in.

Presently there are two conceptually distinct approaches to brane-worlds. One of them is based on a higher dimensional space with product topology $M_4 \times B_N$, $N \geq 2$, where $B_N$ is the a finite volume internal space, similar to Kaluza-Klein theory. The other, is defined on a 5-dimensional anti-deSitter geometry, with brane-like boundary conditions. Current problems include the stability of the brane-world structure; the lack of a proper explanation of the confinement of the gauge fields and a consistent definition of the quantum states.

The purpose of this paper is to show that the stability of the embedding is a central property of brane-worlds compatible with the aforementioned assumptions, independently of models and metric ansatzes. More specifically, we show in sections II and III that the stability conditions for the classical perturbations of a brane-world, regardless of the size, number and signature of the extra dimensions, provide the dynamics of the brane-world evolution. The cases with just one extra dimensions, studied in section IV, are found to have limited properties, including the need for confinement mechanisms which are independent and may interfere with the stability conditions. The quantum fluctuations of the brane-worlds described in section V, induce topological changes which are incompatible with a fixed topology. We show that such topological changes result from the multiparameter quantum fluctuations. In section VI it is shown that the differentiable structure of the brane-world also implies in the existence a confined gauge field structure. Finally, the concluding section also discusses on the number and size of the extra coordinates.
II. BRANE-WORLDS PERTURBATIONS

In very general terms, a brane-world may be described as a locally embedded space-time, capable of quantum fluctuations along the extra dimensions, but retaining the gauge interactions confined within. These conditions impose dynamical conditions on the embedding, producing what could be called a dynamical embedding of space-times as opposed to the more common analytic embedding.

It is well known, that any space-time $\tilde{V}_n$ may be locally embedded into a manifold $V_D$, with sufficiently large dimension $D$. The number of extra dimensions $N = D - n$ depends on the geometries of $\tilde{V}_n$ and of $V_D$, and of course of the type of embedding map, which may be local, non-local, isometric, and conformal among other possibilities.

The simplest examples are those of space-times isometrically embedded in a flat space $M_D$, where the embedding is given by analytic functions $\tilde{x}^i$ [22,23]. The analytic assumption greatly simplifies the embedding and it implies that $D$ is at most 10. In brane-worlds it is not obvious that the analyticity condition holds under the assumed conditions and in special under the quantum fluctuations. However, it seems reasonable to assume that the embedding remains differentiable, determined by the classic Gauss-Codazzi-Ricci equations. In this case, the limiting dimension for flat embeddings rises to $D = 14$, with a wide range of compatible signatures [20]. We will see that the differentiable hypothesis is consistent with the Einstein-Hilbert dynamical principle, which lies at the basis of the mentioned dynamical embedding.

For generality, we consider an n-dimensional submanifold with arbitrary metric signature embedded in a $D$-dimensional space, which is a solution of higher dimensional Einstein’s equations [24].

The embedding of the background $\tilde{V}_n$, with metric $\tilde{g}_{ij}$, is given by local the map $\tilde{x}^i : \tilde{V}_n \rightarrow V_D$ such that

$$\tilde{X}_i^{\alpha} \delta_{\alpha \beta} g_{\mu \nu} = \tilde{g}_{ij}, \tilde{X}_i^{\alpha} \eta^{A}_{\alpha} g_{\mu \nu} = 0, \eta^{\alpha}_{A} \eta^{B}_{\beta} g_{\mu \nu} = g_{AB}$$

where we have denoted by $g_{\mu \nu}$ the metric of $V_D$ in arbitrary coordinates and $g_{AB}$ denotes the components of the metric of the complementary space $B_N$ in the basis $\{\eta^A\}$.

One way to generate a brane-world is to deform the background space-time $\tilde{V}_n$, in such a way that it remains compatible with the gauge confinement and the quantum fluctuations.

The perturbations of an embedded geometry with respect to a small parameter $s$ along an arbitrary transverse direction $\zeta$ in $V_D$ has been defined long ago [24,25], starting with the perturbation of the embedding map

$$\tilde{X}^\mu(x^i, s) = \tilde{X}^\mu + s \zeta \tilde{X}^\mu = \bar{X}^\mu + s[\zeta, X]^\mu$$

The presence of a component of $\zeta$ tangent to $V_n$ is a cause for concern because it induces coordinate gauges. That is, a perturbation could be altered by a mere coordinate transformation. In general relativity, this problem is aggravated by the diffeomorphism invariance of the theory. There, the traditional solutions consist in imposing specific conditions on the metric, tetrads or even on the sources [22,23]. Another solution, consists in choosing a hypersurface orthogonal perturbation (or equivalently, using the ADM language, to eliminate the shift function), with the obvious limitations resulting from the use of special coordinates [24].

In the case of brane-worlds, the extra dimensions do not share the same diffeomorphism invariance of $V_n$. Consequently, the limitations imposed by the choice of a hypersurface orthogonal vector does not apply. An additional simplification is obtained by taking the norm of this vector to be $\pm 1$. This is equivalent to normalize the lapse function to one in the ADM case, and using the extra coordinates $s^A$ to play the role of the “lapses”.

With these precautions, the perturbations of the embedding map along an orthogonal direction $\tilde{\eta}_A$, for some fixed value of $A$, gives

$$\tilde{Z}^\mu(x, s^A) = \tilde{X}^\mu + s^A \tilde{\eta}^\mu_A(x).$$

On the other hand, since $\tilde{\eta}^A(x^i)$ are independent vectors, depending only of $x^i$, they remain unperturbed

$$\eta^\mu_A(x^i) = \tilde{\eta}^\mu_A + s^B [\tilde{\eta}^B_A, \tilde{\eta}^\mu_A] = \tilde{\eta}^\mu_A$$

The metric of the perturbed manifold may be written in generic coordinates as

$$g_{ij} = \bar{g}_{ij} + f_{ij}(x^i, s^A)$$

Taking $s^A$ small as compared to one, the Taylor expansion of $f_{ij}$ in terms of $s^A$, gives the four-dimensional gravitational field in the vicinity of the brane-world. For a specified direction $\eta_A$, the linear perturbation assumes the form

$$g_{ij} = \bar{g}_{ij} + s^A \gamma_{ijA}(x^i)$$

Applying the de Donder gauge condition to the linearized Einstein’s equations, we obtain the homogeneous
gravitational wave equations with respect to the extra dimension $\eta_A$. For a vacuum space-time the resulting wave equation reads as

$$\square_{kl}^s \Psi_s^A(x, s) = 0 \quad (6)$$

where $\Psi_s^A = \gamma_{ij} - 1/2\gamma_{ijkl}$, $\gamma = \tilde{g}^{mn}\gamma_{mn}$ and where we have denoted the generalized topological Dalambertian (de Rahm) wave operator by

$$\square_{kl}^s = \tilde{g}^{ij}\nabla_k \nabla_l + 2\tilde{R}_k^s + 2\tilde{R}_s^{ik}\delta_j^i$$

We interpret the solutions of (6) as describing gravitational waves over the curved background $\tilde{V}_n$, in response to the quantum fluctuations of the brane-world. As such, these gravitational waves belong to the so-called high frequency limit, with a wavelength which is small as compared with the characteristic length of the background geometry. To define this property we may use the Gaussian reference frame based on the perturbed submanifold and the normal $\eta_A$. The embedding equations for the perturbed geometry are

$$Z_i^\mu Z_j^\mu G_{\mu\nu} = g_{ij}, \quad Z_i^\mu \eta_A^\mu G_{\mu\nu} = g_{iA}, \quad \eta_A^\mu \tilde{G}_{\mu\nu} = g_{AB} \quad (7)$$

where now we have the mixed metric components $g_{iA} = s^M A_i MA$, with

$$A_{AB} = \eta_{B^i}^\mu \tilde{G}_{\mu\nu} = \tilde{G}_{AB} = \tilde{A}_{iA} \quad (8)$$

The extrinsic curvatures are

$$k_{ij} = -Z_i^\mu \eta_{A^j}^\mu \tilde{G}_{\mu\nu} \quad (9)$$

Replacing (3) in (7), we obtain the perturbation of the metric

$$g_{ij} = \tilde{g}_{ij} - 2s^A k_{ijA} + s^A B (\tilde{g}_{imn}\tilde{k}_{mnA} \tilde{k}_{jnB} + g_{MN} A_iMA_jNB) \quad (10)$$

and the perturbed extrinsic curvature in the same Gaussian frame

$$k_{ijA} = k_{ijA} - s^B (\tilde{g}_{imn}\tilde{k}_{mniA} \tilde{k}_{jnB} + g_{MN} A_iMA_jNB)$$

Equations (3), (8) and (10) give the evolution of the three basic geometrical attributes of the perturbed geometry. Comparing (10) and the derivative of (11) we obtain

$$\frac{\partial g_{ij}}{\partial s^A} = -2k_{ijA} \quad (11)$$

which generalizes York’s relation used in the ADM formalism. Similar expressions were applied to geometric perturbations in [27,28].

The curvature radii of the background $\tilde{V}_n$ are defined as the solutions of the homogeneous equation

$$(\tilde{g}_{ij} - s^A k_{ijA})dx^i = 0, \quad A \text{ fixed.} \quad (12)$$

From $\det(\tilde{g}_{ij} - s^A k_{ijA}) = 0$ we obtain $n \times N$ distinct solutions, $s = \rho_i^A$, one for each principal direction $dx^i$ and for each normal $\eta_A$. These are local invariant properties of $\tilde{V}_n$ and as such they are independent of the chosen Gaussian system [38]. Since (10) can be written as

$$g_{ij} = \tilde{g}_{mn}(\tilde{g}_{imn} - s^A k_{imA})(\tilde{g}_{jn} - s^B \tilde{k}_{jnB}) + g_{MN} A_iMA_jNB$$

it follows that the components

$$\tilde{g}_{ij} = \tilde{g}_{mn}(\tilde{g}_{imn} - s^A k_{imA})(\tilde{g}_{jn} - s^B \tilde{k}_{jnB}) \quad (13)$$

becomes singular at the curvature centers. Therefore, the fluctuations of the brane-world which are compatible with the regularity of the embedding should not reach these points. Since the smaller solutions $\rho_i^A$ contribute more significantly to the overall curvature of $\tilde{V}_n$, the characteristic length of the background geometry suitable for comparing with the wavelength is

$$\frac{1}{\rho} = \sqrt{\tilde{g}^{ij} g_{AB} \frac{1}{\rho_A} \frac{1}{\rho_B}} \quad (14)$$

which represents a classical limitation of the perturbations.

### III. DYNAMICS

To obtain the dynamical properties of the fluctuations which are compatible with the embedding, we apply the integrability conditions for the embedding, the Gauss, Codazzi and Ricci equations respectively

$$R_{ijkl} = 2g_{MN} k_{ijkl} + R_{\mu\nu\rho\sigma} Z_{ij}^\mu Z_{kl}^\nu Z_{k\sigma}^\rho \tilde{Z}_{l\mu}^\omega \quad (15)$$

These equations represent the conditions for a perturbation of $\tilde{V}_n$ to be stable as a submanifold (and in particular a brane-world) [38,40].

The relevance of the above equations to the stability of brane-worlds has been pointed out in some particular situations, using the conformal flatness properties of [38,40]. In this section we show that that the implications of (15) to the dynamics of brane-worlds is quite general, independently of models or any additional assumption on the geometries of $\tilde{V}_D$ and $\tilde{V}_n$. For this, purpose, consider the expression $\xi_{\mu\nu} G_{\mu\nu} = g^{ij} \tilde{Z}_{ij}^\mu \tilde{Z}_{ij}^\nu$. From (11) it follows that

$$\xi_{\mu\nu} G_{\mu\nu} = n \quad \xi_{\mu\nu} \eta_A^\mu \tilde{G}_{\mu\nu} = 0$$

so that $\xi_{\mu\nu}$ cannot be proportional to $G_{\mu\nu}$. Writing $\xi_{\mu\nu} = \tilde{G}_{\mu\nu} + \zeta_{\mu\nu}$, we find that $\xi_{\mu\nu}$ must satisfy

$$G_{\mu\nu} \zeta_{\mu\nu} = -N \quad \zeta_{\mu\nu} \eta_A^\mu \eta_B^\nu = -g_{AB}$$
The solution of these algebraic equations, compatible with (17), is \( \zeta^{\mu \nu} = g^{AB} \eta_A \eta_B \), so that

\[
g^{ij} Z^\mu_i Z^\nu_j = G^{\mu \nu} - g^{AB} \eta_A \eta_B
\]

(16)

Applying this result in the contractions of the first equation \([4]\), we obtain the Ricci scalar for the perturbed geometry

\[
R = (\kappa^2 - h^2) + \mathcal{R} - 2g^{MN} R_{\mu \nu} \eta^\mu_A \eta^\nu_N
- g^{AB} g^{MN} R_{\mu \nu \rho \sigma} \eta_A \eta_B \eta^\rho_M \eta^\sigma_N
\]

(17)

where we have denoted \( \kappa^2 = \kappa_{ij} A \kappa^{ij} A \) and the mean curvatures \( h_A = g^{ij} \kappa_{ij} A \) with norm \( h^2 = g^{AB} h_A h_B \). Using the Gaussian frame we see that the last term vanishes and that

\[
g^{AB} R_{\mu \nu} \eta^\mu_A \eta^\nu_B = -g^{AB} \partial h_A / \partial s_B + \kappa^2
\]

Therefore, (17) reduces to

\[
R = \mathcal{R} - (\kappa^2 + h^2) - 2g^{AB} \partial h_A / \partial s_B
\]

To obtain the canonical structure associated with the fluctuations we may write the Einstein-Hilbert Lagrangian for \( V_D \), after discarding the derivative terms \( \partial h_A / \partial s_B \) as surface terms.\(^3\)

\[
\mathcal{L}(g, g_{ij}, g_{ij}) = \mathcal{R} \sqrt{g} = \left[ \mathcal{R} + (\kappa^2 + h^2) \right] \sqrt{g}
\]

(18)

The components of the momentum canonically conjugated to \( \mathcal{G}_{\alpha \beta} \) relative to the normal direction \( \eta_A \) are

\[
p^{\alpha \beta}_{(A)} = \frac{\partial \mathcal{L}}{\partial \left( \partial \mathcal{G}_{\alpha \beta} / \partial s_A \right)}
\]

In particular, using \([11]\) we obtain the tangent components

\[
p^{ij}_{(A)} = -(\kappa^{ij} A + h_A g^{ij}) \sqrt{g}
\]

(19)

which describe the momentum the metric geometry of \( V_n \), when propagated along the extra dimensions.

On the other hand, the perturbation does not prescribe the evolution of \( G_{ij} A \) and \( G_{AB} B \). The corresponding momenta are set as constraints:

\[
p^{iA}_{(B)} = -2 \frac{\partial R_{\alpha \beta} \eta^\alpha \eta^\beta}{\partial \mathcal{G}_{ij}} \sqrt{g} = 0,
\]

(20)

\[
p^{AB}_{(C)} = -2 \frac{\partial R_{\alpha \beta} \eta^\alpha \eta^\beta}{\partial \mathcal{G}_{ij}} \sqrt{g} = 0
\]

(21)

\(^3\)To allow for different signatures, we have denoted \( g = \text{det}(\mathcal{G}_{\alpha \beta}) \) module the signature of the extra dimensions.

The constraint \( [27] \) correspond to our previous choice of zero shift function, while \( [27] \) corresponds to the normalization of the lapse.

The Hamiltonian for each single direction \( \eta^A \), is defined by the Legendre transformation

\[
\mathcal{H}_A(g, p) = p^{ij}_{(A)} g_{ij, A} - \mathcal{L} =
- R \sqrt{g} - \frac{1}{\sqrt{g}} \left( p_A^{2} \frac{n+1}{n+1} \right) - p_{ij(A)}^{ij}
\]

(22)

leading to Hamilton’s equations,

\[
\frac{dp_{ij}}{ds^A} = - \frac{\delta \mathcal{H}_A}{\delta g_{ij}} = - \frac{2}{\sqrt{g}} \left( g_{ij} p_A + p_{ij(A)} \right) \frac{2p_A p_{ij(A)} g_{ij}}{n+1}
\]

(23)

\[
\frac{dp_{ij}}{ds^A} = - \frac{\delta \mathcal{H}_A}{\delta g_{ij}} = G_{ij} \sqrt{g} + \frac{1}{\sqrt{g}} \left[ 2p_A p_{ij(A)} g_{ij} \right]
\]

(24)

The first of these equations coincides with \( [11] \) expressed in terms of \( p_{ij(A)} \). The second equation gives the evolution of the extrinsic curvature of the perturbation in terms of momentum. Consequently, the stability conditions for the perturbations, represented by \( [13] \) are consistent with the Einstein-Hilbert dynamics for \( V_D \).

\[\text{IV. HYPERSURFACE BRANE-WORLDS}\]

When \( D = n+1 \), the brane-worlds are hypersurfaces of \( V_D \). All expressions can be derived from the general case by setting \( A, B, \ldots = n+1 \) and \( g_{AB} = g_{n+1 n+1} = \varepsilon = \pm 1 \). Since in this case the “twisting vector” \( A_{iAB} \) vanishes, the integrability conditions lose Ricci’s equation. The remaining Gauss-Codazzi equations may be applied to derive the dynamics of the hypersurface for a general \( V_{n+1} \). Expression \( [10] \) in this case becomes

\[
g^{ij} Z^\mu_i Z^\nu_j = G^{\mu \nu} - \frac{1}{\varepsilon} \eta^\mu \eta^\nu
\]

(25)

As before, replacing in Gauss’ equation, after removing the total derivative terms the Lagrangian \( [18] \) reduces to

\[
\mathcal{L} = \left[ R + \frac{1}{\varepsilon} (\kappa^2 + h^2) \right] \sqrt{g}
\]

(26)

where now we have denoted \( h = g^{ij} k_{ij} \) and \( k^2 = k^{ij} k_{ij} \).

The momentum canonically conjugated to the metric \( \mathcal{G}_{\alpha \beta} \), with respect to the perturbation parameter \( s \) is (here the dot means derivation with respect to \( s \)) \( p^{\alpha \beta} = \partial \mathcal{L} / \partial (\mathcal{G}_{\alpha \beta}) \), with components

\[
p^{ij} = - \frac{1}{\varepsilon} (k^{ij} + h g^{ij}) \sqrt{g}
\]

\[
p^{i n+1} = -2 \frac{\partial R_{\alpha \beta} \eta^\alpha \eta^\beta}{\partial \mathcal{G}_{i n+1}} \sqrt{g} = 0
\]

\[
p^{n+1 i} = -2 \frac{\partial R_{\alpha \beta} \eta^\alpha \eta^\beta}{\partial \mathcal{G}_{n+1 i}} \sqrt{g} = 0
\]
The Hamiltonian corresponding to the one parameter perturbation is
\[
\mathcal{H} = p^{\alpha \beta} \dot{g}_{\alpha \beta} - \mathcal{L} = -R \sqrt{\mathcal{G}} - \frac{\varepsilon}{\mathcal{G}} \left( \frac{p^2}{n+1} - \mathcal{G}_{ij} \mathcal{G}^{ij} \right) \sqrt{\mathcal{G}} \quad (27)
\]
where \( p = \mathcal{G}_{\alpha \beta} p^{\alpha \beta} \). This describes the same perturbations of an arbitrary metric with some limitations to be noted:

Since \( A_{iAB} = 0 \), the perturbed metric in the Gaussian frame becomes simply
\[
g_{ij} = \bar{g}_{ij} = \bar{g}^{mn} (\bar{g}_{im} - s \bar{g}_{im})(\bar{g}_{jn} - s \bar{g}_{jn})
\]
which is singular at the curvature centers of \( V_{n+1} \) defined by \( \text{det}(\bar{g}_{ij} - s \bar{g}_{ij}) = 0 \). The solutions of this equation are the curvature radii \( \rho_i \), one for each principal direction \( dx^i \). The characteristic length has more significant contributions from the smaller \( \rho_i \), in accordance with \([14]\),
\[
\frac{1}{\bar{\rho}} = \sum \sqrt{\varepsilon} \frac{\bar{g}_{ij}}{\rho_i \rho_j}
\]
This is again an invariant property of the embedded background geometry. As it was already commented, the properties of this characteristic length have implications on the perturbations of brane-world geometry. In particular, a known result states that if \( V_n \) has more than two finite curvature radii \( \rho_i \), then the hypersurface becomes indeformable \([8]\).

A particularly interesting example is given by a constant curvature space \( V_{(n+1)} \). More specifically, consider the (non-compact) anti-De Sitter space \( AdS_{(n+1)} \), with \( \varepsilon = -1 \):
\[
R_{\mu \nu \rho \sigma} = -\Lambda \left( \mathcal{G}_{\mu \nu} \mathcal{G}_{\rho \sigma} - \mathcal{G}_{\mu \rho} \mathcal{G}_{\nu \sigma} \right) \quad (28)
\]
We obtain, \( R_{\mu \nu} = n \Lambda \mathcal{G}_{\mu \nu} \), so that the constraints \([27]\), \([27]\) become identities and \( \Lambda \) can be interpreted as a bulk cosmological constant.

In this particular case, the integrability conditions are considerably simpler:
\[
R_{ijkl} = -\frac{2}{\varepsilon} \kappa_{ijkl} - \Lambda \left( g_{ik} g_{jl} - g_{il} g_{jk} \right)
\]
\[
\kappa_{ijkl} = 0
\]
where we notice that the Riemann tensor of \( V_{(n+1)} \) is entirely projected into the \( n \)-dimensional hypersurface. Therefore, it is more natural to derive Einstein’s equations for \( V_n \). Using \([24]\), we obtain
\[
G_{ij} = \frac{1}{\varepsilon} t_{ij}(k) + \Lambda \left( \frac{n}{2} - 1 \right) (n - 1) g_{ij} = 8 \pi G T_{ij} \quad (29)
\]
where we have denoted
\[
t_{ij}(k) = \kappa^n_{ik} \kappa_{mij} - h k_{ij} - \frac{1}{2} (k^2 - h^2) g_{ij}
\]
and \( T_{ij} \) denotes the energy-momentum tensor of the four-dimensional sources. Since the last equality in \([29]\) is not a differentiable equation on \( g_{ij} \), it implies that this energy-momentum tensor becomes algebraically related to the extrinsic curvature \( k_{ij} \) which represents a serious limitation for the brane-world. In this respect, it has been noted that when \( \Lambda < 0 \), the null energy condition for \( T_{ij} \) is not compatible with the embedding \([11][2]\).

We will see in section VI that the condition \( A_{iAB} = 0 \) means that the gauge structure does not arise from the integrability conditions. Consequently it has to be imposed over the geometrical structure, requiring additional conditions to ensure the stability of the hypersurface brane-worlds.

V. QUANTUM STATES

Klein’s compactification of the extra dimension was introduced to make Kaluza’s theory compatible with quantum mechanics. Specifically, the normal modes of the harmonic expansion with respect to the internal parameters (of Planck’s length size) were set in correspondence with the quantum modes \([4]\). This eventually led to a major problem, namely the inability to generate light chiral fermions at the electroweak sector of the theory. More specifically, the strong curvature of the internal space contributed to large mass fermion states, which would necessarily be observable at the electroweak scale.

On the other hand, in brane-worlds the extra dimensions are macroscopic, so that the harmonic expansion of physical fields over \( V_D \) would not lead to the correct quantum phenomenology. In its place, we look at the linear gravitational wave equation \([4]\) in the de Donder gauge. However, two points must be observed: Firstly, the quantum fluctuations of the geometry should be independent of the classical approximation order on \( s^4 \). Furthermore, the classical waves correspond to the quantum fluctuations of the geometry. This suggests that the definition of the quantum states should precede the linear approximation of the metric.

One way to define the quantum states relative to the extra dimensions is to use the canonical formalism associated with the perturbations. In fact, the Poisson bracket structure for each extra dimension \( \eta_A \) is defined by
\[
[\mathcal{F}, \mathcal{H}_A] = \frac{\delta \mathcal{F}}{\delta g_{ij}} \frac{\delta \mathcal{H}_A}{\delta p^{ij}(A)} - \frac{\delta \mathcal{F}}{\delta p^{ij}(A)} \frac{\delta \mathcal{H}_A}{\delta g_{ij}}
\]
can be derived consistently with \([22]\), \([23]\) and \([24]\), with the commutator between two independent perturbations given by \([\mathcal{H}_A, \mathcal{H}_B] \). Therefore the quantum fluctuations may be defined for each independent extra dimension \( \eta_A \), and the superposition principle applied afterwards.

As one example, consider Schrödinger’s equation
\[
-i \hbar \frac{d \Psi_{ij(A)}}{d A} = \mathcal{H}_A \Psi_{ij(A)}
\]

5
where the operator $\hat{H}_A$ is constructed with the perturbation Hamiltonian (22). The probability of a brane-world to be in a state $\Psi_{ij(A)}$ is given by the Hilbert norm
\[ <\Psi_{ij(A)}, \Psi_{ij(A)} > = \int \Psi_{ij(A)}^* \Psi_{ij(A)} \, dv \]
where the integral extends over a volume in $V_D$ with a base on a compact region of the background and a finite extension of the extra coordinates $s^A$. Considering two independent directions $\eta_A$ and $\eta_B$, the transition probability between the corresponding states is given by the integral $<\Psi_{ij(A)}, \Psi_{kl(B)} >$.

Topological changes are expected to occur in any quantum theory of space-times [14–17]. Therefore, we cannot use a fixed product topology for $V_D$, as in [18]. With multiple evolution parameters we may have a more complex topological variation than those expected in a single time theories. For example, if $\eta_A$ and $\eta_B$ are both space-like, then $<\Psi_{ij(A)}, \Psi_{kl(B)} >$ corresponds to a space-like handle. On the other hand, if $\eta_A$ and $\eta_B$ have both time-like signatures, then the classical limit of the transition probability corresponds to a classical loop involving two internal time parameters, suggesting a multidimensional time machine. Finally, if $\eta_A$ and $\eta_B$ have different signatures, then the transition probability $<\Psi_{ij(A)}, \Psi_{kl(B)} >$ corresponds to a signature change.

An example of the last case is given by the Kruskal brane-world regarded as a perturbation of the embedded Schwarzschild space-time, in such a way that the latter becomes geodesically complete. These spaces are both embedded in six dimensional flat spaces, with signatures (5, 1) and (4, 2) respectively [19]. The Schwarzschild space-time is a subset of Kruskal space-time, but they do not belong to the same fixed embedding space. However, they may be considered as classical limits of distinct quantum states of the dynamically embedded Kruskal brane-world, with a signature transition at the horizon.

As a final remark we add that due to the extrinsic nature of the parameters $s^A$, equation (30) may be understood as first quantization of the brane-world geometry. However, it does not exclude the quantization of the metric as an effective field theory in four dimensions, taking for example $V_D$ as the space of all deformed metrics [19, 50].

**VI. CONFINEMENT**

The confinement hypothesis for gauge fields implies that regardless of the electromagnetic, weak and strong interactions taking place inside the brane-world, its differentiable structure remains intact. Since, those interactions are comonotonic with the quantum fluctuations of the geometry, it follows that equations (13) must also be compatible with the confinement of the gauge fields.

The basic field variables in (13) are $g_{ij}$, $\kappa_{ijA}$ and $A_{iAB}$. The first two vary with the perturbation and they are related by (11). On the other hand, from (8) it follows that $A_{iAB}$ remain confined in the sense that they remain unchanged, independently of the quantum fluctuations of the brane-world. The relevant but so far little explored fact, is that those components transform as the components of a gauge potential under the group of isometries of the complementary space $B_N$. This follows directly from the transformation of the mixed component of the metric tensor, under a local infinitesimal coordinate transformation of $B_N$

\[ s^A = s^A + \xi^A \text{ with } \xi^\perp = 0, \text{ and } \xi^A a = \Theta^A_M(x^i)s^M \]

where $\Theta^A_M$ are the infinitesimal parameters. Denoting generic coordinates in $V_D$ by $\{x^\mu\} = \{x^s, s^A\}$, it follows that

\[ g'_{iA} = g_{iA} + g_{i\nu} \xi^\nu_A + g_{A\mu} \xi^\mu_i + \xi^{ij} \frac{\partial g_{iA}}{\partial x^\nu} + 0(\xi^2) \]

The transformation of $A_{iAB}$ follows from

\[ A'_{iAB} = \Theta^A_B(x^i), \xi^A_i = \Theta^A_B s^B \text{ and } \xi^{AB} h = 0 \text{ we obtain} \]

\[ A'_{iAB} = A_{iAB} - 2g^{MN} A_{iMA}\Theta^M_{BN} + g_{MB}\Theta^M_{A,i} \tag{31} \]

which shows that indeed $A_{iAB}$ transforms as a gauge potential. Once the group of isometries of $B_N$ has been characterized, the above transformation may also be written in terms of its structure constants [50].

To conclude the characterization of $A_{iAB}$, we notice that the metric of $V_D$ written in the Gaussian frame has separate has components

\[ g_{ij} = Z_{ij} \tilde{Z}^{ij} \tilde{g}_{\mu\nu} = \tilde{g}_{ij} - 2s^A \tilde{k}_{ijA} + s^A s^B \left( g^{MN} k_{ijM} A_{iMN} + g^{MN} A_{iMA} A_{jNB} \right) \]

\[ g_{ij} = Z_{ij} \tilde{Z}^{ij} \tilde{g}_{\mu\nu} = s^A \tilde{A}_{iMA} \]

\[ g_{AB} = \eta^A_{i\nu} \eta^B_{j\mu} \tilde{g}_{ij} \]

or, in matrix notation,

\[ G_{\alpha\beta} = \left( \tilde{g}_{ij} + g^{MN} A_{iMA} A_{jNB} \right) \frac{A_{iA}}{A_{jB}} \tag{32} \]

where $\tilde{g}_{ij}$ is given by (13) and

\[ A_{iA} = s^M A_{iMA} \tag{33} \]

The metric (12) has the same appearance as the Kaluza-Klein metric ansatz, with the exception that $\tilde{g}_{ij}$ is not the background metric but rather an untwisted perturbation of it, given by (14).

The Einstein-Hilbert Lagrangian derived directly from (32) is

\[ \mathcal{L} = R \sqrt{\tilde{g}} - R \sqrt{\tilde{g}} + 4 \tilde{F}^2 \sqrt{\tilde{g}} \tag{34} \]
where we have denoted $\epsilon = \det(g_{\alpha \beta})$, $F^2 = F_{ij}F^{ij}$ and $F_{ij} = [D_i, D_j]$, $D_i = \partial_i + A_i$. Considering the gauge group as the group of isometries of $B_N$, the gauge connection $A_i$ can be expressed in the Killing basis $\{K^{AB}\}$ of the corresponding Lie algebra as

$$A_i = A_{iAB}K^{AB}$$

From (14), we see that the bulk gravitational field described by $g_{\alpha \beta}$ decomposes in a four dimensional gravitational interaction represented by $\tilde{g}_{ij}$, plus the gauge interactions represented by $A_{iAB}$, much in the sense of Kaluza-Klein theory.

**VII. CONCLUSION**

We have taken an approach to brane-world theory that is independent of the two known models proposed in [12]. Starting with the classic perturbation analysis of submanifolds, the geometric stability conditions (13) for the perturbation are assumed to hold throughout. The conclusion is that those equations in fact determine the brane-world dynamics in the general case.

Considering the geometric stability (13) for submanifolds, the geometric stability conditions (15) for perturbation become singular at the curvature centers of the background characterized by $\rho^A_i$, so that $s^A$ must remain in the open intervals $[0, \pm \rho^A]$. This means that as long as the curvature radii $\rho^A_i$ remain finite, the volume of the complementary space available for the graviton probes is finite. In this case, noting that the right hand side of (13), depends only on $x^i$, we may evaluate the action integral in that region

$$\int \mathcal{L}\sqrt{\tilde{g}}d^{n+N}v = \int \left( \int [R - (k^2 + h^2)]\sqrt{\tilde{g}}\sqrt{\epsilon}d^n v \right) d^N v$$

which leads to an argument similar to that in [1], but without assuming the product topology:

$$\frac{1}{M_{2+N}^2} = \frac{1}{M_{pl}^2}\left(1 - K\right) \mathcal{V}$$

where we have denoted $K = \int (k^2 + h^2)\sqrt{\epsilon} d^nv$, $\mathcal{V}$ is the finite volume of the region in the complementary space and $M_s$ is the Planck's mass in the bulk.

The volume $\mathcal{V}$ depends on the curvature of $\tilde{V}_n$ and it is detailed by the different values of $\rho^A$ as given by (14). Therefore, defining an internal spherical space with radius $\bar{\rho}$, we have $\mathcal{V} \approx \bar{\rho}^N$, so that

$$\bar{\rho}^N \approx \frac{M_{pl}^2}{M_{2+N}^2}\left(\frac{1}{(1 - K)}\right), \quad K \neq 1$$

This gives the same estimates as in [1] when $K$ is small as compared to one. The case $N = 1$ has been ruled out in our analysis because of the limitations imposed on the brane-world fluctuations. The actual distances probed by gravitons depend on further analysis of the quantum states.

**ACKNOWLEDGMENTS**

The authors wish to acknowledge the stimulating discussions on the subject with Drs. P. Caldas and Vanda Silveira.

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