The Faulty GPS Problem: Shortest Time Paths in Networks with Unreliable Directions

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Abstract

This paper optimizes motion planning when there is a known risk that the road choice suggested by a Satnav (GPS) is not on a shortest path. At every branch node of a network Q, a Satnav (GPS) points to the arc leading to the destination, or home node, H - but only with a high known probability p. Always trusting the Satnav’s suggestion may lead to an infinite cycle. If one wishes to reach H in least expected time, with what probability q = q(Q, p) should one trust the pointer (if not, one chooses randomly among the other arcs)? We call this the Faulty Satnav (GPS) Problem. We also consider versions where the trust probability q can depend on the degree of the current node and a ‘treasure hunt’ where two searchers try to reach H first. The agent searching for H need not be a car, that is just a familiar example – it could equally be a UAV receiving unreliable GPS information.

This problem has its origin not in driver frustration but in the work of Fonio et al (2017) on ant navigation, where the pointers correspond to pheromone markers pointing to the nest. Neither the driver or ant will know the exact process by which a choice (arc) is suggested, which puts the problem into the domain of how much to trust an option suggested by AI.

1 Introduction

A satellite navigation system (called Satnav, or GPS) suggests a road to take at every intersection. More abstractly, it suggests an arc of the traffic
network $Q$ to take from any branch node. This arc is supposed to lie on the shortest path to the destination, or Home node $H$. Of course it is well known that errors occur, so we model this by assuming that at every branch node other than $H$ there is a pointer (to one of the incident arcs) which is correct (goes along a shortest path) with a known probability $p$ called the reliability - otherwise it points to a random incorrect arc. The set of pointers are fixed throughout the journey, so if a node is encountered several times, the pointer will always suggest the same arc. If one always follows the pointer, one may cycle infinitely and never reach the destination. More generally, always following the pointer may not minimize the total travel time to $H$.

There are many ways that a real life driver deals with this problem. She might remember that taking a particular arc at an earlier occasion at the current node led back to it, so she might try something different the second time. We will not consider this or other advanced techniques that an alert driver with a good memory might use. Rather, we adopt a simple model of the driver (or autonomous vehicle navigation program). We assume a simple trust probability $q$ with which to follow the pointer. The question we consider is to how to optimize the trust $q$ to minimize time to $H$, given the initial and final nodes $I$ and $H$, the network $Q$, and the reliability $p$. With a probability equal to the trust value $q$, the arc indicated by the pointer is taken; otherwise one of the other arcs is chosen randomly. We call the question of optimizing trust The Faulty GPS Problem. After formalizing the problem in Section 2, we present in Section 3 a slow method of solving three particular networks: a triangle, a circle-with-spike and a simple tree. We then develop a general theory for (i) stars (Section 4), (ii) networks with bridges (Section 5) and (iii) trees (Section 6). In Section 7 we determine how long it takes to cross a line graph with varying length arcs. In Section 8 we consider briefly small cycle graphs of odd and even lengths. In Section 9 we consider a game theoretic treasure hunt version of the problem, in which two drivers with the same GPS system try to be the first one to reach the destination $H$. We solve this game on a very simple line graph, in the cases where the drivers start at the same or different node. Section 10 concludes.

It is worth mentioning that our Satnav metaphor is only that, a metaphor to simply describe the problem of shortest paths with unreliable directions. Real Satnav errors are not likely to be random as assumed here, but rather only suboptimal and generally still pointing in a good direction. We are not recommending our strategies to drivers! In fact the real life problem that motivated this paper comes from biology, as described in the next paragraph.
This problem has its origin not in a driver GPS setting, but in a study of how a species of ants navigates back to their nest, carried out by Fonio et al. (2016). They settled a long standing question by chemical analysis of pheromones laid by individual ants, showing that these deposits formed a decentralized system of pointers towards the nest. There and in Boczkowski, Korman, A and Rodeh, Y. (2018) a deep computer science analysis of query complexity and move complexity is carried out on unit tree networks (all arcs have unit length). It should be observed that for deterministic shortest path problems, a solution for unit networks could be easily applied to general ones by the insertion of additional degree two nodes at regular intervals. However in the Faulty GPS Problem such additional nodes greatly increase travel times, since we have not precluded backtracking. So considering networks of general arc lengths (as well as cycles) is required.

In our problem, the driver (searcher) does not see the whole network, only the node he currently occupies and its incident arcs. In this respect the problem is similar to the maze problem of Gal and Anderson (1990). There also, the searcher adopts a randomized strategy for leaving the current node. However instead of a pointer, the searcher has available markings he is allowed to make on earlier visits to that node. Allowing such marking in our problem is an interesting variation for future work, as it corresponds to driver memory alluded to above. More generally, the problem of finding the destination node $H$ could be seen as a network search problem. If Nature is viewed as antagonistic, the game models from Gal (1979) up to the discrete arc-choice model of Alpern (2017) could be seen as related. If the problem of inaccurate directions at a node can be thought of as a sort of search cost at the node, then the model of Baston and Kikuta (2013) is related. The game theoretic analysis of Section 9 follows the first-to-find paradigm of Nakai (1986) and Hohzaki (2013) and Duvoelle et al. (2017) and is similar to the winner-take-all game of Alpern and Howard (2017).

The problem on the line graph treated in Section 8 has similarities with what is known as dichotomous, or high-low, search on the line. After each move of arbitrary size along the line, the searcher is told the direction but not the distance to the target location $H$, which the searcher wishes to find in the least number of moves or some related efficiency measure. Unlike the current version, going past the target does not solve the problem. In some applications, the target is the demand for a product, as in the newsboy problem. Some of the original papers in this area are Baston and Bostock (1985), Alpern (1985), Alpern and Snower (1988) and Reyniers (1990), as
surveyed in Hassin and Sarid (2018). Computer scientists have worked on related problems from a different point of view, for example Miller and Pelc (2015).

From a more abstract AI perspective, the problem addressed in specific form here is how much to trust a course of action suggested by a process such as GPS planning, when the exact algorithm underlying the process is not known.

## 2 The Satnav (GPS) Problem

This section formalizes the Satnav Problem on a network $Q$. The network $Q$ has a node set $\mathcal{N}$ and a distinguished home node $H \in \mathcal{N}$ which represents the nest (for the ant problem) or the destination (in the satnav interpretation). The arcs $e$ of $Q$ have given lengths $\lambda(e)$. The branch nodes $\mathcal{N}_B$ of $\mathcal{N}$ are the nodes of degree at least 2 other than the destination $H$ itself. At these nodes the agent who wants to get home must make a decision as to which arc to take next. A direction vector $d : \mathcal{N}_B \rightarrow \mathcal{N}$ tells the agent which arc to take. So for each branch node $i \in \mathcal{N}_B$, the arc $d(i)$ is is specified by giving a node $j = d(i)$ which is adjacent to $i$. Alternatively we can specify the arc incident to node $i$. (For example if two arcs lead to the same node we must specify the arc, but this is unusual, and we can exclude multiple arcs if we wish.) So we think of $d(i)$ as an arc incident to node $i$, as the agent doesn’t know which node it leads to. The set of direction vectors is denoted by $\mathcal{D}$, and a measure $\mu$ on $\mathcal{D}$ is defined as follows ($\mu(d)$ is the probability that a Satnav with reliability $p$ chooses the direction vector $d$): Each arc $d(i)$ is chosen independently: with a given probability $p$ (called the reliability) an arc on a shortest path to $H$ is chosen randomly (generically such an arc is unique); with complementary probability $1 - p$ one of the other arcs is randomly chosen. A simple strategy for the Searcher is always to choose the arc $d(i)$ with a fixed probability $q$, called the trust (or trust probability). Following such a strategy, the expected time to reach the home node $H$ from the initial node $I$ is denoted by $T$. To indicate that $T$ is the travel time from one node to another, we also may write $T = T(A,B)$ for the time from $A$ to $B$. When the home node $H$ is fixed, we can write $T_A = T(A,H)$ for simplicity of notation. The Satnav Problem is to minimize $T$ by choosing the optimal trust probability (or just trust) $\hat{q} = \hat{q}(p)$. For a given direction vector $d$, the trust $q$ determines a Markov chain on the nodes $\mathcal{N}$ of $Q$, with absorbing node
and has an expected hitting time $T^d (A)$ from every possible starting node $A$. The time $T$ is an average time over all direction vectors

$$T (A, H) = T_A = \sum_{d \in D} \mu (d) \ T^d (A).$$

(1)

Note that $T (A, H)$ is a function of $p$ and $q$, as $\mu (d)$ is a function of $p$ and $T^d (A, H)$ is a function of $q$.

We will make the general assumption that $Q$ has no loops or multiple arcs and shortest paths are unique. Actually we can deal with the last two in some cases. We can also assume that $H$ is not a cut node.

We also consider a counting agent variation. This assumes that the searching agent, on reaching a node, can count how many arcs are incident at the node (he knows the degree of the node), and can choose to follow the direction at a node with a probability $q_k$ where $k$ is the degree of the node. In this variation the choice variable is the vector $(q_k)$ where $k$ varies over the degrees of the branch nodes of the network. Sometimes we consider the optimization problem for the trust at a single node, when trusts at all other nodes are fixed.

Our model is illustrated in Figure 1, where we show a network with a destination node $H$ and a direction vector (solid arrows at branch points). We add a dashed arrow at the upper left leaf node to indicate that one always reflects from that node. The correct pointers (leading to shortest paths to $H$) are in green, the incorrect ones in red. (These colors are for the reader, not for the searcher.) Note that if $q = 1$ and one gets to the (top left) leaf arc, then one never leaves it. Similarly, if one never follows the arrows, $q = 0$, then the leaf arc at $H$ is never taken (this is a simple case of the argument in Lemma 4).
3 Examples

Before developing any theory, we first introduce three simple examples which show how the GPS problem can be solved by a ‘slow method’. In some cases we will see later how the analysis can be simplified by more general theory developed later. Our main interest is how, in each example, the optimal trust probability \( \hat{q} \) depends on the reliability \( p \). In some cases we also consider the counting searcher problem.

3.1 A triangle network

Consider the network \( Q_1 \) pictured below in Figure 2. We will generally take the length \( x \) of the third side to be 3 to give exact values, but giving an arbitrary length shows the effect of arc length on the solution to this problem.
We consider the above triangle with \( I = A \) as the starting node and \( H = C \) the home, or destination, node. Let + and − denote clockwise and anti-clockwise directions for pointers. Take \( x > 2 \) so that the correct directions (shortest paths to \( C \)) are + at A and + at B. There are four possible direction vectors, which we label as \( d^1 = (+, +) \) (both correct), \( d^2 = (-, -) \) (both wrong), \( d^3 = (+, -) \) (correct at A, wrong at B) and \( d^4 = (-, +) \) (wrong at A, correct at B). Their respective probabilities are given by \( p^2, (1 - p)^2, p (1 - p) \), and \( (1 - p) p \). Let \( a_i = T^{d_i} (A, C) \) and \( b_i = T^{d_i} (B, C) \) denote the expected times to \( C \) from A and from B for trust probability \( q \) and direction vector \( d_i \). Note that \( p \) does not yet come into these probabilities.

When \( d = d^1 \) we have the equations

\[
a_1 = q (1 + b_1) + (1 - q) (x), \quad b_1 = q (1) + (1 - q) (1 + a_1), \quad \text{so}
\]

\[
a_1 = \frac{2q + x - qx}{q^2 - q + 1} \quad \text{and} \quad b_1 = \frac{q - (q - 1)(q - x(q - 1) + 1)}{q^2 - q + 1}.
\]

Similarly we have the formulæ

\[
a_2 = \frac{-2q + qx + 2}{-q + q^2 + 1}, \quad b_2 = \frac{q + q^2x - q^2 + 1}{-q + q^2 + 1},
\]

\[
a_3 = \frac{2q + x - qx}{(1 + q)(1 - q)}, \quad b_3 = \frac{-q^2x + qx + q^2 + 1}{(1 + q)(1 - q)}, \quad \text{and}
\]

\[
a_4 = \frac{-2q + qx + 2}{q(2 - q)}, \quad b_4 = \frac{-2q - q^2x + qx + q^2 + 2}{q(2 - q)}.
\]

Thus the expected time from A to \( C = H \) is given by \( T_A = T (A, C) \).

\[
T (A, C) = p^2 a_1 (q) + (1 - p)^2 a_2 (q) + p (1 - p) a_3 (q) + p (1 - p) a_4 (q),
\]
with a similar formula for $T(B,C)$. We can see in Figures 3 and 4 how $T_A$ and $T_B$ vary with $q$ when $x = 3$ and $p = 3/4 = .75$ and $p = 0.96$. For $p = .75$, starting from $A$ the optimal $q$ is about 0.68 and from $B$ it is about 0.72, while for $p = .96$ the optimal $q$ is about .885 starting at $A$ and about .879 starting at $B$. Thus the order has reversed. We numerically calculate a value of $\tilde{p} \approx 0.925$ when there is a uniformly optimal trust value $\hat{q} \approx 0.84$ (optimal for any start). For $p < \tilde{p}$, we have $\hat{q}(A)$ (starting at $A$) $< \hat{q}(B)$, while for $p > \tilde{p}$ we have $\hat{q}(A) > \hat{q}(B)$.

Figure 3: $T_A$ (top), $T_B$, $p = 3/4$. Figure 4: $T_A$ (top right), $T_B$, $p = 0.96$.

Note that $a_3$ has the factor $(1-q)$ in its denominator and hence goes to infinity when $q$ goes to 1; similarly $a_4$ has the factor $q$ in its denominator and hence goes to infinity when $q$ goes to 0. This observation can also be based on the cycle $ABAB...$ which will go on for a long time in these cases. A more generalizable argument is based on the observation that $C$ can be reached only by traversing one of the arcs $AC$ or $BC$. If both of these are directed (by pointers) at $A$ and $B$ away from $C$ (the pointer vector we called $d^3$) then if $q = 1$ the Home node $H = C$ cannot be reached. A similar argument works for $d^4$ with $q = 0$. So this is a good place to state the following easy generalization.

**Theorem 1** Fix $Q, I, H, p \in (0,1)$. and let $T(q)$ denote the expected time to reach $H$ from $I$ with trust $q$. Then we have $T(q) \to \infty$ as $q \to 0$ or 1. Hence $T(q)$ has an interior minimum $\hat{q} \in (0,1)$.

**Proof.** Suppose the direction vector $d^*$ is such that at every vertex adjacent to $H$, it points to $H$. When reaching such a vertex (or if starting there), one
has to follow the pointer eventually to reach $H$. The expected number of times this takes is $1/q$, so the expected time $T$ is at least $L/q$, where $L$ is the smallest edge length. So $T \geq \mu(d^*) L/q$, which goes to $\infty$ as $q \to 0$. For $q \to 1$ as similar result holds for any direction vector $d^{**}$ in which at every vertex adjacent to $H$, it doesn’t point to $H$. Note that for any fixed $p \in (0, 1)$ and direction vector $d$, $T^d(q)$ is a family of Markov chains with the same absorbing state $H$ and hence the hitting time of $H$ is continuous in $q$. It follows that $T$ has an interior minimum $\hat{q} \in (0, 1)$. 

3.2 The circle with spike network

We now apply the slow method to the circle-with-spike network shown in Figure 5. This graph has multiple edges but that does not give us any problems. There is six direction vectors (two choices at $A$, three at $X$). We apply the same ‘slow method’ as for the previous example, leaving out the details.

![Figure 5: Circle-with-spike networks.](image)

We note that the shortest path from $A$ to $H$ goes along the arc of length 1 and from $X$ it goes along the arc to $H$. First consider the ‘counting searcher’ version mentioned in the Introduction, where the trust probability $q_j$ is allowed to depend on the degree $j$ of the current node. Let $r = q_2$ denote the trust probability at $A$ and $s = q_3$ denote the trust probability at $X$. Using the same simultaneous equation method as in the last subsection, and averaging over the six direction vectors, we find

$$T(A, X) = 1 + p + (1 - 2p)r \quad \text{and also} \quad T(X, H) \quad \text{is given by}$$

$$2p^2(-1 + 2r)(-1 + 3s) + s(7 + 3s + 2r(1 + s)) - p(-5 + 13s - 2s^2 + 2r(-1 + 5s + 2s^2)))$$

$$2(1 - s)s$$
From $A$, since both arcs lead to $X$, one should take the one most likely to be the short arc. So the optimum $r = q_2$ is 1 when $p > 1/2$ and 0 when $p < 1/2$, which can also be seen from (2). It is easily calculated that for $p = 3/4$ the counting agent problem for $T(X, H)$ is minimized at about 5.056 with $r = q_2 = 1$ (this is true more generally for $p > 1/2$) and $s = q_3 \simeq 0.55051$, starting at either node. Later we will show how the counting problem can be solved more easily by considering an associated star network and applying the theory for stars developed in the next section. For the original (non-counting searcher) the solution starting from $X$ has $r = s = q$ minimized at 5.38, with $q \simeq 0.56$. Starting from $A$, the time $T(A, H)$ to $H$ is minimized at 6.85, with $q \simeq 0.57108$.

### 3.3 A tree with two branch nodes

Consider the tree network drawn in Figure 6, which has two leaf nodes 1 and 2, two branch nodes $A$ and $B$, and a destination node $H$. All arcs have unit length. The pointing directions are indicated for clarity (for example to $A,1$ or $H$ at node $B$). For the basic problem (single trust probability for all nodes), we let $q_A$ denote the trust everywhere when starting at $I = A$, so $\hat{q}_A$ minimizes $T_A = T(A, H)$ and similarly for $q_B$. For the counting searcher problem, let $q_2$ and $q_3$ denote the trust at $A$ (degree 2) and at $B$ (degree 3) when these are allowed to be different.

![Figure 6: A simple tree with two branch nodes.](image)

We begin by considering the problem with a ‘counting’ agent’ taking $r = q_2$ (trust at $A$) and $s = q_3$ (trust at $B$). For the ‘slow method’ of Section 3.1 we have to first consider the six direction vectors in $\{2, B\} \times \{A, 1, H\}$, the two directions at $A$ and the three at $B$. For the direction vector $d^* = (B, 1)$,
the travel times $T^d_r = T^d_r(A,H)$ and $T^d_r = T^d_r(B,H)$ satisfy

$$T^d_r = r \left(1 + T^d_r(B)\right) + (1-r) \left(2 + T^d_r(A)\right), \quad T^d_r = \frac{1-s}{2} \left(1 + (1 + T^d_r(A))\right) + s \left(2 + T^d_r(B)\right),$$

so

$$T^d_r = \frac{-4s + 4rs + 4}{r - rs}$$

and $T^d_r = \frac{r - 2s + 3rs + 2}{r - rs}.$

Using analogous methods for the five other direction vectors $d$, and then averaging them with weights $\mu(d)$, we obtain the formulae for $T(A,H)$ (top line) and $T(B,H)$ (bottom line) as follows

$$\frac{(2r + 3s - 6rs - 1)p^2 + (r^2 - 3r^2s - 2rs^2 + 12rs - 2r + s^2 - 3s)p + (r^2s^2 + r^2s - 4rs)}{rs(1-r)(s-1)}$$

and

$$\frac{(6rs - 3s - 2r + 1)p^2 + (3r^2s - r^2 - 2rs^2 - 8rs + 2r + s^2 + s)p + (rs^2 - 2r^2s + 3rs)}{rs(r - 1)(s - 1)}$$

For the original non-counting problem we set $q = r = s$ to minimize $T_A = T(A,H)$ and then $T_B = T(B,H)$ over $q$. In Figure 7 we fix reliability at $p = 3/4$ and plot the expected times to reach $H$ from $A$ (lowered by 2.78 to fit in picture) and from $B$. It can be seen that the optimal trust (at all nodes) when starting at $A$ is approximately .59, which is higher than the optimal trust of about .57 when starting at $B$. We mention this, because later we shall show that for the line graph the optimal trust probability does not depend on the starting node, there is a uniformly optimal trust. At these respective optimal trusts, we have $\hat{T}(A,H) \simeq 8.05$ and $\hat{T}(B,H) \simeq 5.28$.

Figure 7: Plots of time to $H$ from A,B for $p = 3/4.$
More generally, we set $r = s = q$ and calculate

\[
\frac{\partial T_A}{\partial q} = \frac{(3 - 5p)q^4 + (23p - 12p^2 - 7)q^3 + (15p^2 - 15p)q^2 + (5p - 9p^2)q + 2p^2}{(-1 + q)^3 q^3},
\]

\[
\frac{\partial T_B}{\partial q} = \frac{(1 - p)q^4 + (15p - 12p^2 - 5)q^3 + (15p^2 - 9p)q^2 + (3p - 9p^2)q + 2p^2}{(-1 + q)^3 q^3}.
\]

Setting fourth degree polynomials in the numerators to zero, we obtain implicit functions for $\hat{q}_A, \hat{q}_B$ as functions of $p$, which we plot as the two middle curves in Figure 8. We see that $\hat{q}_A > \hat{q}_B$ for all $p, 0 < p < 1$.

Finally, we consider the counting agent problem, where we can jointly minimize $T_A$ and $T_B$ with $r = q_2$ and $s = q_3$ ($A$ has degree 2, $B$ has degree 3). For our comparison base $p = 3/4$, the optimal values of trust are as follows: when the search agent can count the degree of a node, he can reach $H$ in expected time about 7.96 from $A$ and 5.23 from $B$; in both cases adopting trust $\hat{q}_2 = \frac{3}{2} - \frac{1}{2}\sqrt{3} \simeq 0.634$ when at $A$ and $\hat{q}_3 = 3 - \sqrt{6} \simeq 0.551$ when at $B$. In the main case, where he cannot count and must trust equally at all branch nodes, he trusts with probability $q_A \simeq 0.590$ at both nodes when starting at $A$, reaching $H$ in expected time 8.057. When starting from $B$, he reaches $H$ in expected time 5.283, trusting with probability $\bar{q}_B \simeq 0.573$ at both nodes. The four trust probabilities for this network are shown below in Figure 8. The important thing to note, probably with general applicability, is that when counting degree the searcher can use more extreme trust values, but when trust has to be the same at all nodes, less extreme trust values must be adopted.
4 Star Networks

We now consider a star network $Q^n$ where one of the $n$ rays (leaf arcs) leads to the home node $H$ and the start node $I$ is the central node. It turns out that the optimal trust probability $\hat{q} = \hat{q}_n$ (for the single branch node $I$) depends only on $p$ and the degree $n$ of the central node. The lengths of the rays do not matter, though of course they affect the optimal travel time. Our analysis of the star will have implications for other networks, because locally every node is a star.

**Theorem 2** Let $Q$ be a star network with a single branch node $I$ (the center node) of degree $n$. Assume that the home node $H$ is one of the leaf nodes, with a leaf arc of length $c$. The other $n - 1$ rays (arcs) $i = 1, \ldots, n - 1$, have lengths denoted by $\alpha_i$, whose sum is denoted $\alpha = \sum_{i=1}^{n-1} \alpha_i$. The expected time $T$ to get to $H$ from $I$ is given by

$$T = c + \frac{(2p - 4q + 2q^2 + 2nq - 2npq)}{q(1 - q)(n - 1)} \alpha,$$  \hfill (3)

![Figure 8: Plots of optimal trusts $q_2, q_A, q_B, q_3$.](image)
which is minimized by taking \( q = \hat{q}_n \) to be
\[
\hat{q}_n = \bar{q}_n(p) \equiv \frac{p - \sqrt{n-1} \sqrt{p(1-p)}}{1 - n(1-p)}, \quad \text{for } p \neq \frac{n-1}{n} \quad \text{and} \quad (4)
\]
\[
\hat{q}_n = \bar{q}_n(1/2) \equiv \frac{1}{2}, \quad \text{for } p = \frac{n-1}{n}, \quad (5)
\]

independent of the lengths of the rays.

**Proof.** Since there is a single branch node \( I \), the direction vector has a single element which we call just \( d \). If \( d = h \) (points to \( H \)), we calculate the time \( T^h \) to reach \( H \), using
\[
T^h = q(c) + \frac{1-q}{n-1} \sum_{i=1}^{n-1} (2\alpha_i + T^h)
= qc + (1-q) \left( T^h + 2\alpha/(n-1) \right), \quad \text{or}
T^h = \frac{1}{q} \left( cq - 2\frac{\alpha}{n-1} (q-1) \right).
\]

If \( d = i \) points along one of the other rays \( i = 1, \ldots, n-1 \), then the time \( T^d = T^i \) to reach \( H \) satisfies the equation
\[
T^i = q(2\alpha_i + T_i) + (1-q) (1/(n-1)) \left( c + \sum_{j \neq i} (T^i + 2\alpha_j) \right), \quad (7)
= q \left( 2\alpha_i + T^i \right) + \left( (1-q) / (n-1) \right) \left( c + (n-2) T^i + 2 (\alpha - \alpha_i) \right), \quad \text{or}
T^i = \frac{2\alpha - 2\alpha_i + c - 2\alpha q - cq + 2\alpha_i nq}{1-q}.
\]

So the overall time to reach \( H \) is given by \( T = pT^h \sum_{i=1}^{n-1} \frac{(1-p)T^i}{n-1} \), which simplifies to (3), as claimed. To find the optimal trust, it is enough to solve the first order condition
\[
2p - 4q + 2q^2 + 2nq - 2npq = 0
\]
which gives the optimal trust \( \hat{q} = \bar{q}_n \) of (4). In the case where the denominator of (4) is zero, the limiting value of 1/2 is obtained by L’Hospital’s rule. ■
Some values of $\hat{q}(n,p)$ for our standard reliability of $p = 3/4$ are given in Table 1. Note that the value for $n = 3$ (given to more places) is the same as we found using the slow method for the degree three node $X$ in the circle-with-spike graph of Figure 5, which is not a star. An exact analysis of the circle-with-spike network will be given in the next section.

| $n$ | 2   | 3   | 4   | 5   | 6   | 7   |
|-----|-----|-----|-----|-----|-----|-----|
| $\bar{q}(n,p)$ | 0.634 | 0.55051 | 0.500 | 0.464 | 0.436 | 0.414 |

Table 1. Trust values $\bar{q}_n = \bar{q}(n,p)$, $p = 3/4$, $n = 2, \ldots, 7$.

5 Graphs With Bridges

In the Star Theorem (Theorem 2), a searcher leaving the central node $I$ via an arc not leading to $H$ will come back immediately from the corresponding leaf node to $I$. It turns out that the same method of analysis works as long as when leaving $I$ by such an arc, the searcher returns to $I$ before reaching $H$. This property can be ensured by specifying that $IH$ is a bridge arc. Note that the circle-with-spike network of Figure 5 has this property for $I = X$. A more general version of the Star Theorem, which for example applies immediately to that network, can then be stated as follows.

**Theorem 3 (Bridge Theorem)** Let $XH$ be a bridge (disconnecting) arc of length $c$ of a network $Q$, with the degree of $X$ equal to $n$. Assume that the reliability $p$ and the trust probabilities $q(j)$ for all branch nodes $j$ other than $X$ are fixed arbitrarily. Label the arcs out of $X$ other than $XH$ as $i = 1, 2, \ldots, n - 1$. Let $\beta_i$ denote the expected time to return to $X$ when leaving $X$ via arc $i$. Then the expected time $T(X,H)$ to reach $H$ from $X$ is the same as for the star at $X$ with rays of length $\beta_i/2$. Regardless of the values of $q(j)$ and the arc lengths $\beta_i$, the value of $q = q(X)$ which minimizes $T(X,H)$ is given by $q = \bar{q}_n(p)$ as in (4) and the least expected time to reach $H$ is given by

$$
\hat{T}_H = \hat{T}(X,H) = c + M\beta, \text{ where } \beta = \sum_{i=1}^{n-1} \beta_i \text{ and } \quad (8)
$$

$$
M = M(n,p,q) = \frac{(p - 2q + q^2 + nq - npq)}{q(1 - q)(n - 1)}. \quad (9)
$$
Proof. The same derivation used for Theorem 2 holds in this situation, with $\alpha_i$ replaced by $\beta_i/2$. If $d$ points towards $H$, we have

$$T = qc + \frac{(1-q)}{n} \sum_{i=1}^{n-1} (T + \beta_i),$$

which is the same as (6) with $\beta_i$ replacing $2\alpha_i$. The same replacement holds for the equation (7) giving $T^i$, so the rest of the analysis follows in an identical fashion. Thus $M$ is half the constant given in (3).

It is worth noting that the Star Theorem is a special case of the Bridge Theorem with returns times twice the lengths of the leaf arcs. We can use the Bridge Theorem to give a simpler solution of the circle-with-spike network of Figure 5. Optimal trust at $I = X$ (namely $\overline{q}_3 \approx 0.55051$) now follows from the Bridge Theorem as $\overline{q}_3$. Since there is a unique node sequence $AXH$ to the home node, we have $T(A, H) = T(A, X) + T(X, H)$ (see Lemma 4 below). Suppose $p > 1/2$ for simplicity. Then a simple argument given earlier shows that the optimal trust at node $A$ is 1, which implies that the expected return time to $X$ when leaving via the arc of length $\lambda = 1, 2$ is given by $\beta_\lambda = \lambda + p(1) + (1 - p)(2) = \lambda - p + 2$ with sum $\beta = 7 - 2p$. We can also obtain the time $\hat{T}(X, H)$ evaluated as 5.056 in Section 3.2 by the slow method using (8) as

$$\hat{T}(X, H) = 1 + \frac{(p + \overline{q}_3 + \overline{q}_3^2 - 3p\overline{q}_3)}{2 \overline{q}_3 (1 - \overline{q}_3)} (7 - 2p) \approx 5.056, \text{ for } p = 3/4.$$

The nice thing about the Bridge Theorem is that the optimal trust probability at $X$ only depends on $p$ and $n$. Therefore, if there are multiple nodes connected to a bridge on different parts of the network of the same degree, they have the same optimal trust probability.

We conclude this section with a formalization of the claim about $T(A, H) = T(A, X) + T(X, H)$ mentioned in the previous paragraph.

Lemma 4 Let $A, B, C$ be nodes of $Q$ such that every path from $A$ to $C$ passes through $B$. Suppose reliability $p$ is fixed as well as the trust probabilities in $(0, 1)$ at every node of $Q$. Then

$$T(A, C) = T(A, B) + T(B, C).$$

Proof. From the assumptions, there is a finite state Markov chain on the nodes of $Q$, with $C$ as an absorbing state. Almost every sample path starting
at $A$ reaches $C$ and the expected hitting time is finite. For every sample path there are times $t^d_1$ from $A$ to $B$ and $t^d_2$ from first arrival at $B$ to $C$, with total time from $A$ to $C$ given by

$$t^d = t^d_1 + t^d_2.$$ 

Since expectation respects summation we have

$$T^d(A, C) = T^d(A, B) + T^d(B, C).$$

Taking expectations with respect to the finite space of direction vectors $d$ and the measure $\mu$, we similarly have (10).

Note that the analog of (10) for optimal times $\hat{T}$ (where $q$ might not be the same in the different terms) may be false; we might have that $\hat{T}(A, C) > \hat{T}(A, B) + \hat{T}(B, C)$ in the event that the last two times are minimized for different values of $q$. As an example, consider the tree of Figure 6, with $p = 3/4$. We showed in our earlier analysis that $\hat{T}(A, H) \approx 8.05$ and $\hat{T}(B, H) \approx 5.23$. We can now use the Bridge Theorem to determine $\hat{T}(A, B)$. We take $X = A$ and $H = B$, $n = 2$, $c = 1$ and $\beta = 2$. For $p = 3/4$, this gives $M = .866$ by (9). By (8) we have $\hat{T}(A, B) = c + M\beta = 1 + 2(.866) = 2.732$. So $\hat{T}(A, B) + \hat{T}(B, H) = 2.732 + 5.23 = 7.962 < \hat{T}(A, H) \approx 8.05$. Of course if in the larger time $T(A, C)$ we are allowed to choose the optimal trust at every nod, this cannot occur.

6 Trees

In the previous section, when applying results on the star to the circle-withspike network of Figure 5, we used the fact that certain arcs were bridges and certain nodes were cuts. These ideas work very well on trees, where all branch nodes are cuts and all arcs are bridges. For a tree $Q$, we choose to view the home (destination) node $H$ as the root. The following definitions apply to trees. For each node $i \neq H$ there is a unique shortest path to $H$. By relabeling the nodes, we can write this path as $j = 0, 1, \ldots, m$, with $A$ labeled $0$ and $H$ labeled $m$, with $j + 1 = s(j)$. Since all nodes are cuts, we can write by repeated application of (10), the
expected time from $A$ to $H$, with the notation $S(j) = T(j, j + 1)$, as
\[
T(A, H) = T(0, m) = \sum_{j=0}^{m-1} T(j, j + 1) = \sum_{j=0}^{m-1} S(j). \tag{11}
\]

For any node $i$, since the arc $i, s(i)$ is a bridge, we use Theorem 3 with $H = s(i)$ to write that
\[
S(i) = T(i, s(i)) = \lambda(i, s(i)) + M\beta
= \lambda(i, s(i)) + 2M \sum_{j \in a(i)} (\lambda(j, s(i)) + S(j)), \tag{12}
\]

taking $n = k(i)$, the degree of $i$, in the definition (9) of $M = M(n, p, q)$ and recalling that $\lambda(i, j)$ is the length of that arc. If we are considering the counting searcher problem we take trust $\hat{q}_k(i)$ for each node $i$, and the recursion (12) solves the problem, starting with leaf nodes and increasing the depth. Thus we have shown the following.

Theorem 5 Let $Q$ be a tree network. The counting searcher problem is solved by taking $\hat{q}_k$ equal to $\bar{q}_k$ as defined for the star in (4). For an $n$-ary tree, where all branch nodes have $n$ antecedents (and degree $n+1$) the general solution is $\hat{q} = \bar{q}_{n+1}$ (at all nodes). So for a binary tree the solution to the Satnav problem is $\hat{q} = \bar{q}_3$ and for the line graph the solution is similarly $\hat{q} = \bar{q}_2$.

7 Time to Cross a Line

We now consider the case on a line graph with nodes $0, 1, \ldots, n$ (or even on a one sided infinite line graph). We calculate the optimal time taken from a node to a larger node. We do this first with variable length arcs and then specialize to unit length arcs (graphs). We show that for $p \neq 1/2$ the time to reach $n$ is linear in $n$, but if $p = 1/2$ it is quadratic.

Theorem 6 Let $Q$ be a line graph on nodes $0, 1, \ldots$ where the length $\lambda(i, i + 1)$ of the arc between $i$ and $i + 1$ is denoted $a_i$ and $b_j \equiv \sum_{i=0}^{j-1} a_i$. Let $p \neq 1/2$. Then the least expected time $S(j) = T(j, j + 1)$ from node $j$ to node $j + 1$ is...
Given by

\[ S(j) = a_j + 2 \sum_{i=1}^{j} a_{j-i} z^i, \quad j \geq 0 \text{ and } S(0) = a_0, \quad \text{where} \]
\[ z = q^2 - 2pq + p \]
\[ q (1 - q). \quad \] (13)

The least expected time \( T(0,j) \) to reach node \( j \) from the leaf node 0 is given by

\[ T(0,j) = b_j + 2 \sum_{i=1}^{j-1} b_{j-i} z^i. \quad \] (15)

All of these times are minimized by taking the trust probability \( q \) to be \( \hat{q}_2(p) = \frac{p - \sqrt{p(1-p)}}{2p-1} \), which makes \( z = 2\sqrt{p(1-p)} \to 0 \) as \( p \to 1 \) (or to 0). Hence as the reliability goes to 1, the time to cross a line converges to its length \( b_j \).

We have

\[ T(0,j) < b_j \left( 1 + 2 \sum_{i=1}^{j} z^i \right) = b_j \left( 1 + \frac{2(z - z^{j+1})}{1 - z} \right), \]

so the crossing time is linear in the length of the line.

If \( p = 1/2 \) the optimal \( q = \hat{q} = 1/2 \), giving a random walk on the line, where the crossing time is quadratic.

**Proof.** To obtain the formula \([13]\) for the incremental times \( S(j) \) we note that \( S(0) = a_0 \) because 0 is a leaf node. To obtain a formula for \( S(j) \) in terms of \( S(j-1) \), we apply the Bridge Theorem, Theorem 3, with \( H = j+1 \) and \( X = j, c = a_j \). This gives \( q = \hat{q}_2 \) and gives \( S(j) = T(X,H) \) as

\[ T(X,H) = S(j) = a_j + \beta M. \]

The expected return time \( \beta \) when leaving node \( j \) by the arc \((j-1,j)\) is given by \( a_{j-1} \) plus the expected time to return to \( j \) from \( j-1 \), which by definition \( S(j-1) \). Finally, the general formula for \( M \) in \([9]\) simplifies to the number \( z \) in \([14]\) when \( n = 2 \). Thus we have the recursion

\[ S(j) = a_j + (a_{j-1} + S(j-1)) z, \quad \text{with } S(0) = a_0. \quad \] (16)
To check the formula inductively, we write

\[(a_{j-1} + S(j-1)) z = a_{j-1}z + \left(a_{j-1} + 2 \sum_{i=0}^{j-1} a_{j-i} z^i\right) z\]

\[= a_{j-1}z + \left(a_{j-1}z + 2 \sum_{i=0}^{j} a_{j-i} z^i\right)\]

\[= S(j) - a_j.

The formula for $T(0,j)$ given in (15) now follows from the Cut Lemma (Lemma 4) because every node $i > 0$ is a cut node. \[\Box\]

If all the arcs have unit length $a_i = 1$ then we have $b_j = j$ and then the formula (15) can be simplified.

**Corollary 7** If $Q$ is a line network with unit length arcs then, for $p \neq 1/2$, the optimal time to cross it is given by

\[T(0,j) = \frac{j - jz^2 + 2z(z^j - 1)}{(1-z)^2}\] (17)

If $p = 1/2$, the optimal time is the expected time for a random walk to reach node $j$ from node 0, that is, $j^2$. Obviously this is quadratic rather than linear in the length $j$ of the line. (For $p \neq 1/2$ we showed this time is linear in $j$, in a more general context.)

We plot below in Figure 9 the optimal expected travel times from the left leaf node 0 to node $j$. For $p = 0$ or 1, $T(0,j) = j$, a direct path can always be taken. Note that the expected travel times $\bar{S}(j) = T(j-1,j)$ between consecutive nodes is increasing.
Figure 9: Plots of $T(0, j)$, $j = 1$ to 6, gaps $S(j)$.

Note that it takes longer to traverse consecutive nodes $j - 1$ and $j$ as $j$ increases. It is interesting to notice the asymmetry of travel times, with this not present in traditional shortest path problems. Observe that for $i < j$ we have

$$T(0, i) + T(i, j) = T(0, j), \text{ or } T(i, j) = T(0, j) - T(0, i). \quad (18)$$

For example when $p = 3/4$ we have $T(3, 5) \approx 12.187$, using (18) and (17). Note that this doesn’t depend on the total length of the line graph, as 5 becomes an absorbing state. However if $i > j$ then the time for $T(i, j)$ (going left) does depend on the length $0, 1, \ldots, n$ of the line graph. Looking at it so that node $n$ is on the left, our earlier analysis shows that for $n = 7$ we have

$$T(5, 3) = T(n - 5, n - 3) = T(2, 4) \approx 9.763 \quad (19)$$

This is clearer if we take an extreme situation with $Q$ having nodes 0 to 100. If we want to go from 1 to 2, at most we can backtrack to 0. If we want to go from 2 to 1, if we are unlucky we may travel very far to the right before reaching 1. Note: Travel times from the leaf node 0 are greater than those
from node 1 by one. It is a matter of taste whether to give a formula for $T(1,j)$ or for $T(0,j)$. We plot below the optimal expected travel times from the left leaf node 0 to node $j$. For $p = 0$ or 1, $T(0,j) = j$, a direct path can always be taken. Note that the expected travel times $\bar{S}(j) = T(j - 1,1)$ between consecutive nodes is increasing.

8 Cycle Graphs

In this section we analyze the Satnav Problem on the cycle graphs $C_3$ and $C_4$ of Figure 10. We believe these represent the cases where there are an odd or even number of nodes. In the latter case there is an antipodal node (called $C$) to the home node $H$. Note that $C_4$ is an example with non-unique shortest paths to $H$. So the direction at $C$ is equiprobable. The quick general methods used on line graphs do not appear to help the analysis for cycles, so this section is really just an introduction to the general problem. The Satnav problem on $C_n$ is identical to the destination set problem on $L_{n+1}$, the graph 0-1-2-...-n where the problem is to reach the set \{0, n\} from a given node $i$, $0 < i < n$. However we find significant qualitative differences in the solution, for example on $C_4$ there is no uniformly optimal trust probability, this depends on the starting node. This is in contrast with the Satnav Problem on $L_n$, where we found a uniformly optimal trust probability $\hat{q}$. On the other hand, optimal travel times on a cycle are clearly symmetric, unlike the situation found for the line at the end of Section 7.

Figure 10: The cycle networks $C_3$ and $C_4$. 
8.1 The cycle $C_3$

We adopt the ‘slow method’ used in Section 3 for the case $x = 1$. This involves, for each of the four direction vectors $d^i$ on the two branch nodes, construction the two simultaneous equations for the expected time $a$ and $b$ to reach $H$ starting from $A$ and $B$, respectively. Compared with the solution for the triangle with $x > 2$ given in Section 3.1, the doubly correct vector is now $d^4 = (-, +)$ which occurs with probability $p^2$. Recalculating the time $a_i = T_{A}^{d_i} = T_{d_i}(A, H)$ and averaging over the probabilities $\mu(d_i)$, we get

\[
T_A = p^2 a_4 + p(1 - p) a_2 + (1 - p)p a_1 + (1 - p)^2 a_3, \text{ giving} \quad (20)
\]

\[
T_A = \frac{p^2}{q} - \frac{3p(p-1)}{q^2 - q + 1} + \frac{(1 - p)^2}{1 - q} \text{ for } x = 1. \quad (21)
\]

We obtain an implicit function of $\hat{q}$ as a function of $p$ by simply setting the partial derivative of $T_A$ with respect to $q$ equal to zero. We plot this implicitly in Figure 11. The symmetry of $A$ and $B$ means that this is also the optimal trust when starting at $B$. So there is a uniformly optimal trust function. In particular for our standard reliability $p = 3/4$, the optimal trust is approximately $0.78676$ as seen in Figure 11. Note the difference from the case $x = 3$ of Section 3.1.

![Figure 11: Optimal trust $\hat{q}$.](image1)

![Figure 12: Plot of $T_A(3/4, q)$.](image2)
8.2 The cycle $C_4$

The slow method for solving the Satnav Problem on $C_4$ has equation systems for each of the eight direction vectors on the branch nodes $A, C, B$. Note that regardless of the reliability $p$, the direction at $C$ is equally likely towards $A$ or $B$. For example, when all pointers are in the clockwise direction, $d = (+, +, +)$, we have the system (where $a$ is the expected time from $A$ to $H$, same for $b$ and $c$)

\[
\begin{align*}
    a &= (1 - q) (1) + q (1 + b) \\
    c &= (1 - q) (a + 1) + q (b + 1) \\
    b &= (1 - q) (1 + c) + q (1)
\end{align*}
\]

Using the same methods as for $C_3$, we can implicitly plot (see Figure 13) the optimal trust $\hat{q}$ at all nodes when starting at $A$ (or $B$), the lower red curve, and when starting at $C$ (the higher green curve). The important observation is that the cycle $C_4$, unlike the line graphs or the odd cycle $C_3$, does not have a uniform trust solution, the optimal trust depends on the starting node.

Figure 13: Circle $C_4 : \hat{q}(p)$ from $A$ (red, lower), $C$ (green).
9 First To Nest Wins (Treasure Hunting)

We consider a two-person constant sum game where the first player to reach the Home node \( H \) wins, and if they reach at the same time the winner is determined by a fair coin toss. The payoff is the probability that Player I wins. Since game problems are much harder than individual optimization, with take the simplest nontrivial network, the line with three nodes, 0, 1, 2, with \( H = 2 \). We consider both the symmetric game where both players start at node 1 and the asymmetric game where they start at 0 and 1. Note that this network is also the star with three nodes. So the individual time minimization problem has been solved earlier in two ways (star and line).

We note that this is a winner-take-all game in that each player gets a score (the hitting time to \( H \)) and the lowest score wins. Such games have been analyzed in Alpern and Howard (2018), but this version is not covered by any theory in that paper. Both players have the same satnav (the same pointer at node 1), which is correct with probability \( p \). Player I trusts with prob \( q \), II with prob \( r \). An alternative model, not analyzed here, is for the two players to have different Satnavs, with independent errors. In this case the game fits exactly into the Alpern-Howard scenario.

9.1 Symmetric Start

Here we assume that both players start at node 1, so we know the value (assuming it exists - it does) must be 1/2. If \( d = + \), pointer correctly points to 2, the payoff \( v^+ \) satisfies the following, recalling that a tie in reaching node 2 has payoff 1/2.

\[
v^+ = qr \left( \frac{1}{2} \right) + q \left( 1 - r \right) \left( 1 \right) + (1 - q) r \left( 0 \right) + (1 - r) \left( 1 - q \right) \left( v^+ \right), \text{ so}
\]

\[
v^+ = \frac{2q - qr}{2q + 2r - 2qr} \quad \text{and similarly} \quad v^- = \frac{1 - q + r - qr}{2 \left( 1 - qr \right)}.
\]

It follows that

\[
v(p, q, r) = p \frac{2q - qr}{2q + 2r - 2qr} + (1 - p) \frac{1 - q + r - qr}{2 \left( 1 - qr \right)}.
\]

Solving the equation \( \frac{\partial v(p, q, r)}{\partial r} = 0 \) to obtain \( \hat{r}(p, q) \) and solving \( q = \hat{r}(p, q) \) gives

\[
\hat{q}_{sym}(p) = \frac{-1 + p + \sqrt{1 - 3p + 3p^2}}{2p - 1}.
\]
Thus we have shown the following.

**Theorem 8** The optimal trust in the symmetric game on the line graph \( \{0, 1, 2 = H\} \) where both players start at node 1 is given by \( \hat{q}_{\text{sym}}(p) \) as in (22).

Figure 14 shows the intersection of the optimal response curves when \( p = 2/3 \) at \( \hat{q}_{\text{sym}}(2/3) = \sqrt{3} - 1 = 0.73205 \) for both players.

![Figure 14: Plots of \( q = \hat{r}(r) \) (red), \( r = \hat{r}(q) \) (blue).](image)

#### 9.2 Asymmetric Start

We now consider the scenario where Player I starts at node 1 (at time 1) and Player II starts at node 0. Note that as long as the game is being played, this will be the position at all odd times, and at all even times Player II will be at node 1 and Player I will be at node 0. There cannot be a tie. We take \( q \) as I’s trust and \( r \) as II’s trust.

**Theorem 9** Consider the game on the line with node set \( \{0, 1, 2 = H\} \) where first to \( H \) wins. Suppose player I starts at node 1 and player II starts at node 0. It is optimal for player II to follow a random walk, that is, trust \( r = 1/2 \).

For player I there are three cases.

1. If the reliability satisfies \( p \geq 4/5 \), I’s optimal trust is \( \hat{q} = 1 \), so the value is \( v = p \). (Either I goes immediately to \( H \) and wins or he oscillates between 1 and 0 and loses.)

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2. If $1/2 < p \leq 4/5$, then $\hat{q} = Q(p) = \left(1 + p - 3\sqrt{p(1-p)}\right)/(2p - 1)$. Player I wins with probability (value)

$$v = (4/3) \left(1 - \sqrt{p(1-p)}\right).$$

3. If $p = 1/2$ then both player I also optimally follows a random walk and wins with probability

$$v = \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \cdots = \frac{1/2}{1 - 1/4} = \frac{2}{3}.$$

**Proof.** Let $v = v(p, q, r)$ denote the payoff (probability I wins) when player I is at 1 and $w$ denote the payoff (probability that I wins) when player II is at 1. As above, I trusts with probability $q$, II with probability $r$. If $d = +$ (pointer at node 1 is correct, to right) then we have

$$v^+ = q(1) + (1 - q)(w^+)\quad w^+ = r(0) + (1 - r)(v^+) \text{, so}$$

$$v^+ = \frac{q}{q + r - qr} \quad \text{and} \quad w^+ = \frac{q - qr}{q + r - qr}.$$

Similarly if $d = -$ (points to left, to 0), we have

$$v^- = q(w^-) + (1 - q)(1)\quad w^- = r(v^-) + (1 - r)(0)$$

$$v^- = \frac{1 - q}{1 - qr}, \quad w^- = \frac{r - qr}{1 - qr}.$$

This gives the payoff (winning probability) for Player I, when starting at node 1, as

$$v(p, q, r) = p \frac{q}{q + r - qr} + (1 - p) \frac{(1 - q)}{1 - qr}. \quad (23)$$

For those preferring a more probabilistic coin tossing derivation of equation (23), consider that I and II have coins which come up heads with respective probabilities $a$ and $b$, and (starting with I) they alternate tossing until one of them gets heads and wins. The probability that I wins on the $2i + 1$th toss is $((1 - a)(1 - b))^i a$. So the probability that I wins is given by

$$a \sum_{i=1}^{\infty} \left( (1 - a)(1 - b) \right)^i = \frac{a}{1 - (1 - a)(1 - b)}.$$
If the pointer is correct, +, then the probabilities of going to node 2 = H and winning for I and II when at node 1 are given by \( a^+ = q, b^+ = r \) and when pointer is incorrect, they are \( a^+ = 1 - q, b^+ = 1 - r \). So the probability that I wins is given by

\[
p \left( \frac{a^+}{1 - (1 - a^+) (1 - b^+)} + (1 - p) \frac{a^-}{1 - (1 - a^-) (1 - b^-)} \right),
\]

which simplifies to (23). We now prove the three assertions.

1. Since \( q = 1 \) guarantees player I wins with probability \( p \), it is enough to show that a random walk \( r = 1/2 \) for player II guarantees that I wins with probability \( \leq p \). We calculate

\[
\frac{\partial v (p, q, 1/2)}{\partial q} = \frac{2 f (p, q)}{\left( q - 2 \right)^2 \left( q + 1 \right)^2}, \text{ where}
\]

\[
f (p, q) = -1 + 5p - 2q - 2pq - q^2 + 2pq^2.
\]

Since \( f (p, q) \) is positive on \( 4/5 < p \leq 1, 0 \leq q \leq 1 \), it follows that \( v (p, q, 1/2) \) is increasing in \( q \) in this range, so that the best response of player I to \( r = 1/2 \) is \( q = 1 \). Thus playing randomly for player II keeps the probability that I wins no more than \( p \).

2. In this region of \( p \), the first order equation \( f (p, q, 1/2) = 0 \) has the unique probability solution \( \hat{q} = Q (p) \) given in the statement. So \( Q (p) \) is the optimal response to \( r = 1/2 \). The optimal response function for Player II is obtained by the first order condition

\[
\frac{\partial v (p, q, r)}{\partial r} = 0, \text{ so the optimal response } \hat{r} = \hat{r} (p, q) \text{ is given by}
\]

\[
\frac{2q - 2q^2 + 2pq^2 - \sqrt{(-2q + 2q^2 - 2pq^2)^2 - 4(p - q^2 + pq^2)(-1 + p + 2q - 2pq - q^2 + 2pq^2)}}{2 (-1 + p + 2q - 2pq - q^2 + 2pq^2)}
\]

Now fix \( p \) and consider Player II’s best response to \( Q (p) \) for Player I. We find that

\[
\hat{r} (p, Q (p)) = 1/2.
\]

This means that the best response is \( r = 1/2 \), so \( Q (p) \) and 1/2 form an equilibrium.
3. The statement of the Theorem shows an easy way to compute the value of the game, given that both players adopt a random walk. The optimality of a trust of $1/2$, the random walk, can be obtained by continuity from part 2.

The alert reader will note that we have avoided the computation of the optimal response $\hat{q}(r)$ to a Player II strategy of $r$. In fact we have derived this response function and we plot the two curves in Figure 15, for $p = 2/3$, with an intersection at $q = Q(2/3) = 0.75736$ and $r = 1/2$.

![Figure 15: Response curves $\hat{q}(r)$ (red) and $\hat{r}(q)$, $p = 2/3$.](image)

![Figure 16: Optimal $q$ for asymmetric game (top), symmetric game, individual (bottom).](image)
Figure 16 summarizes the optimal trusts for the symmetric and asymmetric games, compared with an individual who wants to minimize the expected time to reach node 2 from node 1. It shows the optimal trust for (top) the player starting at node 1 in the asymmetric game, (middle) the symmetric game and (bottom) an individual using $\bar{q}_2(p)$ minimizing to minimize the expected time to get to node 2 from node 1. The optimal trust is $1/2$ for Player II in the asymmetric game.

Before leaving the asymmetric game, it is worth giving an intuitive but false idea for the solution. Note that when Player II is considering his choice of trust $r$, he realizes that this value will only be used if and when he gets to node 1, in which case Player I will be at node 0 (if the game has not ended). So in a sense he is in the same position as Player 1 was in at the start of the game. Consequently, at an equilibrium $r$ should be the same as $q$. We have shown this is false, but we leave it up to the reader to find a flaw in this argument.

10 Conclusion

This paper presents a very simple model of finding shortest time paths in networks with unreliable directional information. We give a simple but slow method which works on any network and derive some theory which gives quick solutions for some families of networks. Our model of the search agent is very simple. He trusts the pointer direction with a chosen probability, possibly dependent on the degree of the node he is at. More sophisticated agents might be modeled in the future. For example, it seems reasonable to assume that, in addition to counting the degree, he can remember which arc he has just arrived on. Then he can also choose that arc (that is, backtrack) with a different probability (likely smaller) than the other incident arcs.

We also considered a treasure hunt, where two agents try to be the first to reach the home node, and to find the treasure. Here, we modeled this problem in a scenario where both agents (players) have the same pointers, possibly because they use the same brand of Satnav (GPS). An alternative model which seems to present interesting facets is to assume they have different brands, and independent pointers. Additionally, the two brands might have different reliabilities. Or more generally, the players could have different targets. This is also a model of what are called ‘patent races’.

For future work, the model could be modified. For example, instead of a
searcher who seeks a fixed home node, we could have the home node viewed as another mobile searcher, as in the rendezvous problem of Ozsoyeller et al (2019). Or the searcher might want to visit a sequence of nodes (rather than just one) in an effort to patrol the graph against intruders as in Basilico et al (2017). A similar approach might be taken to deal with other recommendation systems provided by black box AI processes which are known to be faulty.

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