Parameter estimation for Galactic binaries by LISA

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ABSTRACT

We calculate how accurately parameters of the short-period binaries ($10^{-4} \lesssim f \lesssim 10^{-2} \text{Hz}$) will be determined from the gravitational waves by LISA. In our analysis the chirp signal $\dot{f}$ is newly included as a fitting parameter and dependence on observational period or wave frequency is studied in detail. Implications for gravitational wave astronomy are also discussed quantitatively.

Subject headings: gravitational waves – gravitation – binaries

1. Introduction

The Laser Interferometer space Antenna (LISA), a joint project of NASA and ESA (European Space Agency) would establish gravitational wave astronomy at low frequency band ($10^{-4} \text{Hz} \lesssim f \lesssim 10^{-1} \text{Hz}$). It would bring us essentially new information of the Universe (Bender et al. 1998). For example gravitational waves from merging super massive black holes (SMBHs) would be detected with significant signal-to-noise ratio ($SNR \gtrsim 10^3$), though event rate of such merging is highly unknown (e.g. Haehnelt 1994, Vecchio 1997). Galactic binaries are promising sources of LISA (Mironowksi 1965, Evans, Iben & Smarr 1987, Hils, Bender & Webbink 1991, Webbink & Han 1998). Gravitational waves from some known binaries (e.g. X-ray binary 4U1820-30) would be detected with $SNR > 5$ by one year integration (Bender et al. 1998). In addition more than thousands of close white dwarf binaries (CWDBs) are expected to exist in LISA band (for recent studies see Yungelson et al 2001, Nelemans et al. 2001, Napiwotzki et al. 2002). Our target in this article is these Galactic binaries. We examine how accurately information of binaries can be extracted from gravitational waves observed by LISA.

Cutler (1998) studied the estimation errors for binary parameters with special attention to angular variables such as direction and orientation of binaries (see also Peterseim et al.
He used approximation that emitted gravitational waves would be monochromatic, namely neglected the effects of the chirp signal \( \dot{f} \). But the wave frequency or the chirp signal are fundamental quantities for gravitational wave astronomy. From the measured chirp signal \( \dot{f} \) we can obtain the so called chirp mass \( M_c = M_1^{3/5} M_2^{3/5} (M_1 + M_2)^{-1/5} \) \( (M_1, M_2: \text{masses of two stars}) \) for a binary whose orbital evolution is determined by gravitational radiation reaction. Furthermore the distance to the binary could be estimated from the chirp signal \( \dot{f} \) and the amplitude of the wave signal \( (\text{Schutz 1986}) \). The frequency \( f \) itself also contains important information. One of the authors (Seto 2001) pointed out that signature of the periastron advance could be detected in gravitational waves from an eccentric binary by measuring its wave frequencies precisely. If this method works well, we can estimate the total mass \( M_{\text{total}} = M_1 + M_2 \) of the binary beside the chirp mass, and consequently each mass of the binary is obtained separately.

Estimation errors for fitting parameters correlate complicatedly to each other and depend largely on observational situations. For example longer observational periods would improve not only signal to noise ratio but also resolution of the frequency space. Note that the latter is crucial for reducing Galactic binary confusion noise, as the number of resolved binaries increases with decrease of the frequency bin \( \propto T_{\text{obs}}^{-1} \) \( (\text{see Seto 2002 for details}) \). At LISA age it would become an interesting observational challenge to optically identify the binaries whose gravitational waves are detected by LISA. We also discuss impacts of these observational efforts on estimation of binary parameters.

The magnitude of the chirp signal \( \dot{f} \) has strong dependence on the wave frequency \( f \) as \( \dot{f} \propto M_c^{5/3} f^{11/3} \). At lower frequencies the estimation error \( \Delta \dot{f} \) would be larger than the signal \( \dot{f} \) itself. Then it would be better to remove \( \dot{f} \) from fitting parameters and simply put \( \dot{f} = 0 \) from the beginning. By evaluating the estimation error \( \Delta \dot{f} \) we can quantitatively discuss these prescriptions for signal analysis.

In this article we only study parameter estimation errors on the assumption that (i) signal has been detected with high SNR, and (ii) noises are Gaussian distributed. We evaluate the estimation errors using the Fisher information matrix for maximum likelihood method. For low SNR our simple analysis would not be valid, as the probability distribution function of the fitting parameters could become highly complicated \( (\text{e.g. multimodal}) \) \( (\text{Balasubramanian & Dhurandhar 1998}) \). Thus our result should be regarded as a lower bound of the estimation errors. Non-Gaussian nature of noises would farther increases errors. This is a very important problem, but detailed quantitative analysis would be a formidable task.

This article is organized as follows. In \( \S 2 \) we briefly discuss the gravitational waveforms of chirping binaries and the data stream obtained by LISA. Then we mention the parame-
ter estimation based on the Matched filtering analysis. In §3 we numerically evaluate the parameter estimation errors, and discuss its dependence on observational period or wave frequency in detail. §4 is devoted to summary and discussions.

2. Gravitational Waveforms and Parameter Extraction

2.1. Gravitational Wave Measurement with LISA

LISA consists of three spacecrafts forming an equilateral triangle, and orbits around the Sun, trailing 20° behind the Earth. The sides of the triangle are \( L = 5 \times 10^6 \) km in length, and the plane of triangle is inclined at 60° with respect to the ecliptic. The triangle rotates annually. The gravitational wave signal is reconstructed from the three data streams that effectively correspond to three time-varying armlength data, \((\delta L_1, \delta L_2, \delta L_3)\) for gravitational waves with \( \lambda \lesssim L \). We basically analyze two data streams given by \( s_I(t) = (\delta L_1(t) - \delta L_2(t))/L \) and \( s_{II}(t) = (\delta L_1(t) + \delta L_2(t) - 2\delta L_3(t))/\sqrt{3}L \). These data can be regarded as the response of two 90°-interferometers rotated by 45° to one another (Cutler 1998). The data \( s_{I,II}(t) \) contain both gravitational waves signal \( h_{I,II}(t) \) to be fitted by matched filtering and the noise \( n_{I,II}(t) \). The latter is constituted by the detectors noise and the binary confusion noise. As in Cutler (1998) we assume that the noises are stationary, Gaussian and uncorrelated with each other.

The gravitational wave signals \( h_{I,II}(t) \) from a binary are written as

\[
h_{I,II}(t) = \frac{\sqrt{3}}{2} \left[ F_{I,II}^+(t)h_+(t) + F_{I,II}^\times(t)h_\times(t) \right],
\]

where \( F_{I,II}^{+,\times}(t) \) are the pattern functions which depend on the source’s angular position of the binary \((\bar{\theta}_S, \bar{\phi}_S)\), its orientation \((\bar{\theta}_L, \bar{\phi}_L)\) and detector’s configuration. The angular variables with bars are defined in a fixed barycenter frame of the solar system, and the direction and orientation of the binary are assumed to be constant during the observation in this frame. The quantities \( h_+, h_\times(t) \) are the two polarization modes of gravitational radiation from the binary. We can estimate both \((\bar{\theta}_S, \bar{\phi}_S)\) and \((\bar{\theta}_L, \bar{\phi}_L)\) from the time profiles of the two signals due to LISA’s annual rotation and revolution. Further discussion and details about the pattern functions are seen in Cutler (1998). We basically use his formulation but newly include the effects of the chirp signal \( \dot{f} \).
2.2. Gravitational Waveforms

We study short-period \(10^{-4} \text{ Hz} \lesssim f \lesssim 10^{-2} \text{ Hz}\) binaries such as the close white dwarf binaries (CWDBs) or the neutron star binaries (NBs) in our Galaxy (Hils, Bender & Webbink 1991). We only discuss binaries with circular orbits. This is an excellent approximation for CWDBs, as their orbits are circularized by strong tidal interaction in their earlier evolutionary stages. At LISA band some NBs or neutron star-white dwarf binaries would have non-negligible eccentricities \(e \sim 0.1\) induced at their supernova explosions (see e.g. Brown, Lee, Zwart, & Bethe 2001). But extension to these eccentric binaries is straightforward.

The chirping gravitational waveform is given by the quadrupole approximation (Peters 1964) as

\[
\begin{align*}
h_+ (t) &= A \cos \left[ 2\pi \left( f + \frac{1}{2} \dot{f} t \right) t + \phi_D(t) + \phi_0 \right] \times \left[ 1 + \left( \hat{L} \cdot \hat{n} \right)^2 \right], \\
h_\times (t) &= -2A \sin \left[ 2\pi \left( f + \frac{1}{2} \dot{f} t \right) t + \phi_D(t) + \phi_0 \right] \times \left( \hat{L} \cdot \hat{n} \right),
\end{align*}
\]  

(2)

where \(\hat{L}\) (given by \(\vec{\theta}_L, \vec{\phi}_L\)) is the unit vector in the direction of the binary’s orbital angular momentum, \(\hat{n}\) (given by \(\vec{\theta}_S, \vec{\phi}_S\)) is the unit vector toward the binary and \(\phi_0\) is an integral constant. We regard the frequency \(f\) and its time variation \(\dot{f}\) as constants in the above equations. A purely monochromatic waveform has \(\dot{f} = 0\). This is the case studied by Cutler (1998). When gravitational radiation reaction dominates evolution of the binary as in the case of CWDBs or NBs, the chirp signal \(\dot{f}\) is given as \(\dot{f} = \left( \frac{96\pi^{8/3}}{5} \right) f^{11/3} M_c^{5/3} / c\) with the chirp mass \(M_c\). The perturbative expansion for the intrinsic evolution of the wave frequency in equation (2) are valid under the condition \(\dot{f} T_{\text{obs}} \ll f\), where \(T_{\text{obs}}\) is the observational period. This condition is expressed as

\[
f \ll 0.14 \left( \frac{T_{\text{obs}}}{10 \text{ yr}} \right)^{-3/8} \left( \frac{M_c}{1 M_\odot} \right)^{-5/8} \text{ Hz}.
\]  

(3)

The amplitude \(A\) in equation (2) is given in terms of the wave frequency \(f\), chirp signal \(\dot{f}\) and the distance \(D\) as

\[
A = \frac{5}{96\pi^2 f^3 D}.
\]  

(4)

Thus we could determine the distance \(D\), if we could measure three observables \(f\), \(\dot{f}\) and \(A\) (Schutz 1986). This is an important aspect of gravitational wave astronomy.

The term \(\phi_D(t)\) in equation (2) is caused by revolution of LISA around the Sun and called the Doppler phase. Its explicit form is given by

\[
\phi_D(t) = 2\pi f R \sin \vec{\theta}_S \cos \left[ \vec{\phi}(t) - \vec{\phi}_S \right],
\]  

(5)
where \( R = 1 \text{ AU} \) and \( \bar{\phi}(t) = 2\pi t/T \) \((T = 1\text{ yr})\) is the direction of LISA in the fixed barycenter frame.

### 2.3. Parameter Extraction

Let us briefly discuss the matched filtering analysis and the parameter estimation errors (Finn 1992; Cutler & Flanagan 1994). We assume that the signal \( h_\alpha(t) \) is characterized by some unknown parameters \( \gamma_i \) (eight parameters in the present case: \((A, f, \dot{f}, \phi_0, \bar{\theta}_S, \bar{\phi}_S, \bar{\theta}_L, \bar{\phi}_L)\)). In the matched filtering analysis the variance-covariance matrix of the parameter estimation error \( \Delta \gamma_i \) is given by inverse of the Fisher information matrix \( \Gamma_{ij} \) as

\[
\langle \Delta \gamma_i \Delta \gamma_j \rangle = (\Gamma^{-1})_{ij}.
\]

For a quasi-monochromatic binary \((\dot{f}T_{\text{obs}} \ll f)\) the noise spectrum \( S_n(f) \) is nearly constant in the frequency region swept by the binary and the Fisher matrix simply becomes (Cutler 1998)

\[
\Gamma_{ij} = \frac{2}{S_n(f)} \sum_{\alpha=I,II} \int_0^{T_{\text{obs}}} dt \frac{\partial h_\alpha(t)}{\partial \gamma_i} \frac{\partial h_\alpha(t)}{\partial \gamma_j}.
\]

The error boxes for the angular parameters \( \hat{L} \) and \( \hat{n} \) become ellipses in the celestial sphere due to the correlation of two parameters \( \theta \) and \( \phi \). In this article we represent the estimation errors for direction and orientation of binaries in the form defined in Cutler (1998) as follows

\[
\Delta \Omega = 2\pi \sqrt{\langle \Delta \mu^2 \rangle \langle \Delta \phi^2 \rangle - \langle \Delta \mu \Delta \phi \rangle^2},
\]

where we have defined \( \mu = \cos \theta \). In the same manner the signal to noise ratio \((\text{SNR})\) is given by

\[
(\text{SNR})^2 = \frac{2}{S_n(f)} \sum_{\alpha=I,II} \int_0^{T_{\text{obs}}} dt \ h_\alpha(t)h_\alpha(t).
\]

From equations (6) and (8) it is apparent that the expressions for the estimation errors \( \langle \Delta \gamma_i \Delta \gamma_j \rangle \) do not depend on the noise spectrum \( S_n(f) \) when they are normalized by the signal-to-noise ratio (Cutler 1998). In this article we extensively use this normalization method. For example the parameter estimation errors for a simple wave form \( h(t) = A \sin[2\pi(f + \dot{f}t/2)t + \phi_0] \) with four fitting parameters \((A, f, \dot{f}, \phi_0)\) is easily evaluated as in Seto (2002). The explicit forms for \( \Delta A, \Delta f \) and \( \Delta \dot{f} \) are given by

\[
\Delta A = A \cdot 0.1 \left( \frac{\text{SNR}}{10} \right)^{-1},
\]

\[
\Delta f = \frac{4\sqrt{3}}{\pi} \frac{T_{\text{obs}}^{-1}}{\text{SNR}} = 0.22 \left( \frac{\text{SNR}}{10} \right)^{-1} T_{\text{obs}}^{-1},
\]

\[
(\text{SNR})^2 = \frac{2}{S_n(f)} \sum_{\alpha=I,II} \int_0^{T_{\text{obs}}} dt \ h_\alpha(t)h_\alpha(t).
\]
\[ \Delta \dot{f} = \frac{6\sqrt{5}}{\pi} \frac{T_{\text{obs}}^{-2}}{SNR} = 0.43 \left( \frac{SNR}{10} \right)^{-1} T_{\text{obs}}^{-2}. \]  

This simple analysis does not include information of the angular parameters, but would be helpful to understand more detailed numerical analysis in the following section.

3. Results

3.1. General Behavior

We have numerically evaluated the uncertainties of the estimated parameters for various quasi-monochromatic binaries. In this subsection we show results for a typical example with a fixed set of angular parameters at \( \cos \bar{\theta}_S = 0.3, \bar{\phi}_S = 5.0, \cos \bar{\theta}_L = -0.2 \) and \( \bar{\phi}_L = 4.0 \). In Table 1 we show LISA’s measurement accuracy for parameters \((A, f, \dot{f}, \Omega_S, \Omega_L)\) at frequencies \( f = 10^{-4}, 10^{-3} \) and \( 10^{-2} \) Hz. We present our results for two observational periods, \( T_{\text{obs}} = 1 \) and \( 10 \) yr. All results are normalized by \( SNR = 10 \) after integration over each observational period \( T_{\text{obs}} \). These results simply scale as \((SNR/10)^{-1}\) for errors \( \Delta A, \Delta f \) and \( \Delta \dot{f} \) and \((SNR/10)^{-2}\) for error ellipses \( \Delta \Omega_{S,L} \). The first row for each observational period \( T_{\text{obs}} \) represents the case when all the eight parameters \((A, f, \dot{f}, \Phi_0, \bar{\theta}_S, \bar{\phi}_S, \bar{\theta}_L, \bar{\phi}_L)\) are included in the matched filtering analysis. The second row corresponds to the case without the chirp signal \( \dot{f} \). Results for \( T_{\text{obs}} = 1 \) yr are obtained under the same condition with Table 1 (case A) in Cutler (1998). Our numerical values completely coincide with his ones. The third row is the case when direction of the binary \((\bar{\theta}_S, \bar{\phi}_S)\) is given exactly by other method (e.g. optical identification of the binary) and removed from the fitting parameters.

Figs 1 and 2 show clearly dependence of the orbital parameters on wave frequency and observational period \( T_{\text{obs}} \). In Fig.1 we show LISA’s measurement accuracy as a function of observational period \( T_{\text{obs}} \) at given frequencies \( f = 10^{-4}, 10^{-3} \) and \( 10^{-2} \) Hz. All results are normalized by \( SNR = 10 \) at integration period \( T_{\text{obs}} = 1 \) yr. The solid lines are results for fitting all the eight parameters \((A, f, \dot{f}, \Phi_0, \bar{\theta}_S, \bar{\phi}_S, \bar{\theta}_L, \bar{\phi}_L)\). The dotted lines represent the case when the angular position \((\bar{\theta}_S, \bar{\phi}_S)\) are removed from the fitting parameters (see also Hughes 2002). The dashed lines are results with fitting only the angular position \((\bar{\theta}_S, \bar{\phi}_S)\). For observational period \( T_{\text{obs}} \gtrsim 2 \) yr, the difference between the solid and the dotted lines is very small especially for \( \Delta A, \Delta f \) and \( \Delta \dot{f} \) irrespective of the frequency. Thus optical determination of the source direction \( \Omega_S \) would only slightly reduce the estimation errors of other parameters. Asymptotic behaviors of errors \( \Delta A, \Delta f \) and \( \Delta \dot{f} \) are given by

\[ \frac{\Delta A}{A} = 0.20 \left( \frac{SNR}{10} \right)^{-1}, \]  

(12)
\[
\Delta f = 0.22 \left( \frac{SNR}{10} \right)^{-1} T_{obs}^{-1},
\]
(13)

\[
\Delta \dot{f} = 0.43 \left( \frac{SNR}{10} \right)^{-1} T_{obs}^{-2},
\]
(14)

(see also Table.1), and the asymptotic time dependence is given as \( \Delta A \propto T_{obs}^{-1/2} \), \( \Delta f \propto T_{obs}^{-3/2} \), and \( \Delta \dot{f} \propto T_{obs}^{-5/2} \), since we have \( SNR \propto T_{obs}^{1/2} \) from equation (8). We have also \( \Delta \Omega_{S,L} \propto SNR^{-2} \propto T_{obs}^{-1} \) for directions and orientations of binaries. Numerical results for \( \Delta f \) and \( \Delta \dot{f} \) in equations (13) and (14) are almost identical to analytical ones in equations (10) and (11). Hence we can expect that the information of the angular parameters (the direction and orientation) does not affect the accuracy of the estimation for the frequency \( f \) and the chirp signal \( \dot{f} \) for observational period \( T_{obs} \gtrsim 2 \) yr. But for the amplitude \( A \), the above result (in [12]) is two times as large as equation (9). This is due to the fact that the amplitude \( A \) is tied with the angular parameters (the inclination \( \hat{L} \cdot \hat{n} \)) in equation (2) and the estimation error \( \Delta A \) strongly depends on the error in \( \hat{L} \cdot \hat{n} \) (see the detailed discussion in the next subsection).

Fig.2 is same as Fig.1 but given as a function of frequency \( f \) with fixed observational period \( T_{obs} = 1 \) and 10 yr. We find that for \( T_{obs} = 10 \) yr the errors \( \Delta A, \Delta f \) and \( \Delta \dot{f} \) do not depend on the frequency, as given in equations (12)-(14). But the angular resolution \( \Delta \Omega_{S} \) depends on the frequency and is given by

\[
\Delta \Omega_{S} = 4.8 \times 10^{-4} \left( \frac{f}{10^{-2} \text{Hz}} \right)^{-2} \left( \frac{SNR}{10} \right)^{-2} \text{sr},
\]
(15)

for higher frequencies \( f \gtrsim 10^{-3} \) Hz, and nearly constant for lower frequency \( f \lesssim 10^{-3} \) Hz.

Now we discuss correlation between Fisher matrix elements for \( T \gtrsim 2 \) yr. As expected from Figs 1 and 2, the source direction \( \Omega_{S} \) has almost no correlation with other parameters at higher frequencies \( f > 10^{-3} \) Hz \(^1\) and weakly correlates with source orientation \( \Omega_{L} \) at lower frequencies \( f < 10^{-3} \) Hz. This dependence seems reasonable considering the information used to determine the source direction \( \Omega_{S} \), namely, the Doppler phase at \( f > 10^{-3} \) Hz and amplitude modulation at \( f < 10^{-3} \) Hz. The orientation \( \Omega_{L} \) is strongly related to the latter. As a result angular variables \( \Omega_{S} \) and \( \Omega_{L} \) correlate at lower frequencies. The detected waveform \( h_{\alpha} \) is modulated by annual revolution and rotation of the detectors, and its phase

\(^1\)In other words, Fisher information matrix \( \Gamma_{ij} \) in equation (6) becomes diagonalized to the two parts; \( \Gamma \approx \begin{pmatrix} \Gamma_{A} & 0 \\ 0 & \Gamma_{S} \end{pmatrix} \), where \( \Gamma_{A} \) represents a 6 \times 6 matrix which corresponds to \( (A, f, \dot{f}, \phi_{0}, \bar{\theta}_{L}, \bar{\phi}_{L}) \) components and \( \Gamma_{S} \) represent a 2 \times 2 matrix which corresponds to \( (\bar{\theta}_{S}, \bar{\phi}_{S}) \) components.
has oscillating part due to the modulation that is more prominent at higher frequencies. Amplitudes of the derivatives $\partial h/\partial \Omega_s$ also show this oscillation. In the Fisher matrix their cross terms with other derivatives are significantly canceled with long time integration. As a result the estimation errors for the angular direction $\Omega_s$ show very weak correlation with other errors.

In the same manner we can also understand the asymptotic frequency dependence of the angular resolution $\Omega_s$. In the derivatives $\partial h/\partial \gamma_i$ for the Fisher matrix elements in equation (6) the frequency $f$ appears only through the Doppler phase $\phi_D(t)$ in equation (5). At lower frequencies $f < 10^{-3}\text{Hz}$ terms from the Doppler phase ($\propto f^1$) is smaller than the terms from the amplitude modulation ($\propto f^0$), and consequently the estimation error $\Delta\Omega_S$ does not depend on the frequency $f$. At higher frequencies $f > 10^{-3}\text{Hz}$ the Doppler phase term becomes dominant and the Fisher matrix are diagonalized. Thus the error $\Delta\Omega_S$ depends on the frequency $f$ as $\Delta\Omega_S \propto f^{-2}$ (Cutler & Vecchio 1998, Moore & Hellings 2002).

Finally we analytically investigate how the estimation errors improve with the observational period $T_{\text{obs}}$. We reanalyze the simple toy model $h(t) = A \sin[2\pi(f + \dot{f}t/2)t + \phi_0]$ (see the sentences before equation (9)) adding the information of the source direction ($\bar{\theta}_S, \bar{\phi}_S$) by the Doppler phase $\phi_D(t)$. This waveform is given as

$$h(t) = A \sin[2\pi(f + \dot{f}t/2)t + \phi_D(t) + \phi_0],$$

(16)

with six fitting parameters $\gamma_i = (A, f, \dot{f}, \phi_0, \bar{\theta}_S, \bar{\phi}_S)$. We evaluate the magnitude of the variance-covariance matrix $\langle \Delta \gamma_i \Delta \gamma_j \rangle$ by its determinant as $\det\langle \Delta \gamma_i \Delta \gamma_j \rangle = (\det \Gamma)^{-1}$. After some algebra the time dependence of $\det \Gamma$ is given analytically from equations (6) and (16) as,

$$\det \Gamma \propto T_{\text{obs}}^{12} \left(1 - \frac{6}{\pi^2 (T_{\text{obs}}/1\text{yr})^2} \right) \left(1 - \frac{90}{\pi^4 (T_{\text{obs}}/1\text{yr})^4} \right),$$

(17)

where we assume that the observational period $T_{\text{obs}}(\gg f^{-1})$ is a integer in units of year. In the above expression the quantity $(\det \Gamma)^{-1}$ formally diverges at $T_{\text{obs}} = \sqrt{\pi/90} = 0.98$ [yr] which is very close to 1, thus the estimation errors have large value at $T_{\text{obs}} = 1\text{yr}$. Fig.3 shows the inverse of the determinant of the Fisher matrix in the form $T_{\text{obs}}^{12}(\det \Gamma)^{-1}$ as a function of the observational period $T_{\text{obs}}$. From Fig.3, $T_{\text{obs}}^{12}(\det \Gamma)^{-1}$ rapidly converges at $T_{\text{obs}} \sim 2\text{yr}$. In this case we also obtain the estimation errors as follows;

$$\frac{\Delta A}{A} = \frac{1}{SNR},$$

(18)

$$\Delta f = \frac{4\sqrt{3}}{\pi} \frac{T_{\text{obs}}^{-1}}{SNR} \times \sqrt{\frac{1 - (45/8) x^2(1 + x^2)}{(1 - 6x^2)(1 - 90x^4)}},$$

(19)

$$\Delta \dot{f} = \frac{6\sqrt{5}}{\pi} \frac{T_{\text{obs}}^{-2}}{SNR} \times \frac{1}{\sqrt{1 - 90x^4}},$$

(20)
\[
\Delta \Omega_S = \frac{1}{\pi f^2 R^2 \cos \theta_S} \frac{1}{SNR^2} \times \frac{1}{\sqrt{(1 - 6x^2)(1 - 90x^4)}}
\]

\[
\approx 4.3 \times 10^{-4} \left(\frac{\cos \theta_S}{0.3}\right)^{-1} \left(\frac{f}{10^{-2}\text{Hz}}\right)^{-2} \left(\frac{SNR}{10}\right)^{-2} \text{sr},
\]

where \(x = \pi^{-1}(T_{\text{obs}}/1\text{yr})^{-1}\) and equation (22) is valid for \(T_{\text{obs}} \gtrsim 2\ \text{yr}\) and is similar to equation (15). The above results for \(\Delta f, \Delta \dot{f}\) and \(\Delta \Omega_S\) are about 40% smaller than the results in Fig.1 at \(T_{\text{obs}} = 1\ \text{yr}\), because we do not include the information of the source orientation. \(\Delta f\) and \(\Delta \dot{f}\) are asymptotically same as equations (10) and (11) for \(T_{\text{obs}} \gtrsim 2\ \text{yr}\).

### 3.2. Statistical Analysis

So far we have studied a specific set of the angular parameters \(\hat{n}\) and \(\hat{L}\). In this subsection we present statistical results for their various combinations at the asymptotic region \(T_{\text{obs}} \gtrsim 2\ \text{yr}\).

We have made 100 realizations of \(\hat{n}\) and \(\hat{L}\) that are distributed randomly on celestial spheres. Then we calculate the estimation errors \(\Delta f\) and \(\Delta \dot{f}\) for each binary normalized by \(SNR = 10\) with \(T_{\text{obs}} = 10\ \text{yr}\). We find that these errors depend very weakly (less than 10% scatter) on the directions \(\hat{n}\) and \(\hat{L}\) in contrast to the results for \(T_{\text{obs}} = 1\ \text{yr}\) (scattering typically factor \(2^{\pm 1}\) around the average). We calculate their mean values and obtain results given in the following forms

\[
\Delta f = 0.22 \left(\frac{SNR}{10}\right)^{-1} T_{\text{obs}}^{-1},
\]

\[
\Delta \dot{f} = 0.43 \left(\frac{SNR}{10}\right)^{-1} T_{\text{obs}}^{-2}.
\]

Note that these results do not depend on the frequency \(f\) and would be useful for quantitative analysis of quasi-monochromatic binaries with \(T_{\text{obs}} \gtrsim 2\ \text{yr}\). As expected from the previous subsection the simple analytical estimations given in equations (10) and (11) are very close to equations (23) and (24).

We have also studied the estimation errors \(\Delta \Omega_S, \Delta \Omega_L\) and \(\Delta A/A\). Distributions of the latter two have very large scatters. This is because (i) the Fisher matrix elements relating to the orbital orientation \(\hat{L}\) become singular at the highly symmetric face-on configuration \(\hat{n} = \pm \hat{L}\) and (ii) the estimation of the amplitude \(A\) is closely related to the inclination. For majority of realizations with \(|\hat{n} \cdot \hat{L}| \lesssim 0.8\) we have typically \(\Delta A/A \sim 0.2(SNR/10)^{-1}\). To determine the source direction \(\hat{n}\) we can use the information of the Doppler phase in addition
to the annual amplitude modulation caused by LISA’s rotation. Thus the error $\Delta \Omega_S$ does not show such a bad behavior. For the above 100 realization at $f = 0.01\text{Hz}$ with $SNR = 10$ and $T_{\text{obs}} = 10\text{yr}$ the errors $\Delta \Omega_S$ are distributed as $1.3 \times 10^{-4}\text{sr} \leq \Delta \Omega_S \leq 3.7 \times 10^{-3}\text{sr}$ with the mean value $\Delta \Omega_S = 7.1 \times 10^{-4}\text{sr}$. Therefore the following relation roughly gives the estimation error for the source direction with $T_{\text{obs}} \geq 2\text{yr}$

$$\Delta \Omega_S \sim 7.1 \times 10^{-4} \left(\frac{SNR}{10}\right)^{-2} \left(\frac{f}{10^{-2}\text{Hz}}\right)^{-2}\text{sr}, \quad (25)$$

at $f \gtrsim 2 \times 10^{-3}\text{Hz}$ where the Doppler phase becomes more important than the annual amplitude due to the rotation of LISA.

Now let us discuss issues concerned with the chirp signal $\dot{f}$ based on equation (24). We assume that the chirp signal is dominated by gravitational radiation reaction and introduce a parameter $R \equiv \Delta \dot{f}/\dot{f}$ that represents relative accuracy of the measured signal $\dot{f}$. For a given threshold $R$ we can solve the corresponding frequency $f_R$ as

$$f_R = 9.2 \times 10^{-4} \left(\frac{M_c}{1M_\odot}\right)^{-5/11} \left(\frac{SNR}{10}\right)^{-3/11} \left(\frac{T_{\text{obs}}}{10\text{yr}}\right)^{-6/11} \left(\frac{R}{1.0}\right)^{-3/11}\text{Hz} \quad (26)$$

for observational period $T_{\text{obs}} \gtrsim 2\text{yr}$. The frequency $f_{R=1}$ can be regarded as the critical frequency for treatment of the chirp signal $\dot{f}$. At higher frequencies $f \gtrsim f_{R=1}$ the parameter $\dot{f}$ should be included in the matched filtering, but the simple prescription $\dot{f} = 0$ would be better at $f \lesssim f_{R=1}$ since the expected signal would be completely buried in error. If the chirp signal is measured with accuracy $R(\ll 1)$, the chirp mass can be estimated with relative accuracy $\Delta M_c/M_c \simeq 3R/5$ from relation $\dot{f} \propto M_c^{5/3}$. The chirp signal $\dot{f}$ is also essential to determine the distance $D$ to the binary. From the simple relation $A = 5\dot{f}/96\pi^2f^3D$, the estimation error for distance $D$ is roughly evaluated as

$$\frac{\Delta D}{D} \simeq \frac{\Delta A}{A} + \frac{\Delta \dot{f}}{\dot{f}}, \quad (27)$$

$$\simeq \text{Max} \left\{ \frac{\Delta A}{A}, \frac{\Delta \dot{f}}{\dot{f}} \right\}, \quad (28)$$

$$\simeq 0.2 \left(\frac{SNR}{10}\right)^{-1} \max\left\{1, \left(\frac{f}{1.4 \times 10^{-3}\text{Hz}}\right)^{-11/3} \left(\frac{M_c}{1M_\odot}\right)^{-5/3} \left(\frac{T_{\text{obs}}}{10\text{yr}}\right)^{-2}\right\} \quad (29)$$

where we have used the typical error for the amplitude estimation $\Delta A/A \sim 0.2(SNR/10)^{-1}$. For compact binaries with chirp mass $M_c \sim 1M_\odot$ such as NBs or CWDBs the chirp signal $\dot{f}$ is the dominant source of the error at lower frequencies $f \lesssim 10^{-3}\text{Hz}$ and observational period $T_{\text{obs}} \sim 10\text{yr}$. The amplitude $A$ becomes dominant one at $f \gtrsim 10^{-3}\text{Hz}$. 
Finally, we calculate the estimation error for the three dimensional position of the Galactic binaries. The signal to noise ratio is calculated from equation (8) as $SNR \sim 380 \left( \frac{f}{5 \times 10^{-3}\text{Hz}} \right)^{2/3} \left( \frac{T_{\text{obs}}/10\text{yr}}{1/2} \right) \left( \frac{M_c/1M_\odot}{5/3} \right) \left( \frac{\sqrt{S_n}/8 \times 10^{-21}\text{Hz}^{-1/2}}{1/2} \right)^{1/2}$. The noise spectrum $S_n(f)$ (in units of Hz$^{-1}$) is nearly constant for $3 \times 10^{-3}$ Hz $\lesssim f \lesssim 10^{-2}$ Hz (see Fig.5 in Cutler 1998). Hence one could determine both the angular position and the distance to the binary with the accuracy of

$$\Delta \theta_S \sim 2 \left( \frac{f}{5 \times 10^{-3}\text{Hz}} \right)^{-5/3} \left( \frac{D}{10\text{kpc}} \right) \left( \frac{T_{\text{obs}}}{10\text{yr}} \right)^{-1/2} \times \left( \frac{M_c}{1M_\odot} \right)^{-5/3} \left( \frac{\sqrt{S_n}}{8 \times 10^{-21}\text{Hz}^{-1/2}} \right)^{1/2} \text{arcmin}, \quad (30)$$

$$\frac{\Delta D}{D} \sim 5 \times 10^{-3} \left( \frac{f}{5 \times 10^{-3}\text{Hz}} \right)^{-2/3} \left( \frac{D}{10\text{kpc}} \right) \left( \frac{T_{\text{obs}}}{10\text{yr}} \right)^{-1/2} \times \left( \frac{M_c}{1M_\odot} \right)^{-5/3} \left( \frac{\sqrt{S_n}}{8 \times 10^{-21}\text{Hz}^{-1/2}} \right)^{1/2}. \quad (31)$$

4. Conclusion

We have calculated LISA’s measurement accuracy for short-period binaries ($10^{-4}$ Hz $\lesssim f \lesssim 10^{-1}$ Hz) in our Galaxy, including the effects of chirp signal $\dot{f}$ and dependence on observational period $T_{\text{obs}}$. We find that the measurement accuracy rapidly improves for $T_{\text{obs}} \gtrsim 2\text{yr}$ comparing with $T_{\text{obs}} \sim 1\text{yr}$. This might be an important element for discussing operation period of LISA.

At observational period $T_{\text{obs}} \gtrsim 2$ the errors for quantities $f$ and $\dot{f}$ is independent on the information of the positions and orientations of binaries in contrast to $\Delta A$ and $\Delta \Omega_{S,L}$. It is also found that the estimation errors $\Delta A$, $\Delta f$ and $\Delta \dot{f}$ are almost independent on the frequency $f$. The fitting formulae of the estimation errors for the source parameters (such as frequency $f$, chirp signal $\dot{f}$, amplitude $A$, angular position $\Omega_S$ and distance $D$) are given as functions of $f$ and $T_{\text{obs}}$. We expect these would be powerful tools for quantitatively studying possibility gravitational wave astronomy.

The relative motion between the Sun and a binary could affect the frequency $f$ by the Doppler factor, $(1 + v_R)$ where $v_R$ is the relative radial velocity. The tangential velocity $v_T$ or accelerating motion $\dot{v}$ would affect the time variation of the frequency, $(\dot{f}/f)_{\text{gal}} \sim v_T^2/cD$ or $\sim \dot{v}/c \sim v_{\text{rot}}^2/cR_g$ respectively where $v_{\text{rot}}$ is the rotation velocity ($\sim 200\text{km/s}$) of the Galaxy, $R_g$ is the Galactic radius $\sim 10\text{kpc}$ and $D$ is the distance to the binary (see e.g. Damour & Taylor 1991). For most Galactic binaries in LISA band this should be negligibly small
compared with $(\dot{f}/f)_{GW}$ due to gravitational radiation reaction. This condition is expressed as $f \gg 3 \times 10^{-5}(M_c/1M_\odot)^{-5/8}(v_{rot}/200\text{km})^{3/4}(D/10\text{kpc})^{-3/8}$ Hz. For binaries very close to us $D \lesssim 100$ pc these effects would be important. But note that we might measure the proper motion of such binaries by optical observation.

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| $T_{\text{obs}}$ (yr) | $\Delta A/A$ | $T_{\text{obs}} \Delta f$ | $T_{\text{obs}}^2 \Delta \dot{f}$ | $\Delta \Omega_S$ (sr) | $\Delta \Omega_L$ (sr) |
|---------------------|-------------|----------------|-----------------|-----------------|-----------------|
| 1yr                 | 0.204       | 0.31           | 0.59            | $8.47 \times 10^{-2}$ | 0.154           |
|                     | 0.204       | 0.31           | 0.58            |                 |                 |
| 10yr                | 0.204       | 0.22           | 0.43            | $6.78 \times 10^{-2}$ | 0.185           |
|                     | 0.204       | 0.22           | 0.43            |                 |                 |

Table 1: LISA’s measurement accuracy for binaries with angular parameters ($\cos \tilde{\theta}_S = 0.3, \tilde{\phi}_S = 5.0, \cos \tilde{\theta}_L = -0.2, \tilde{\phi}_L = 4.0$). Results are normalized by $SNR = 10$. Errors scale as $(SNR/10)^{-1}$ for $\Delta A, \Delta f$ and $\Delta \dot{f}$, and $(SNR/10)^{-2}$ for $\Delta \Omega_{S,L}$. The second lines in each observational period $T_{\text{obs}}$ represent the case with removing the chirp signal $\dot{f}$ from fitting parameters and the third lines with removing the direction of the source $(\tilde{\theta}_S, \tilde{\phi}_S)$. 


Fig. 1.— LISA’s measurement accuracy for the binaries as a function of the observational period $T_{\text{obs}}$ with angular parameters $(\cos \bar{\theta}_S = 0.3, \bar{\phi}_S = 5.0, \cos \bar{\theta}_L = -0.2, \bar{\phi}_L = 4.0)$. The solid lines correspond to $\Delta \dot{f}, \Delta f, \Delta A, \Delta \Omega_L$ and $\Delta \Omega_S$ from top to bottom. The dotted lines represent the case source positions $(\bar{\theta}_S, \bar{\phi}_S)$ are removed from the fitting parameters, and the dashed lines represent the case the only source positions are included in the fitting parameters. The accuracies are normalized by $SNR = 10$ at 1yr observation.
Fig. 2.— Same as Fig. 1, but as a function of the frequency $f$ with the observational period $T_{\text{obs}} = 1, 10$ yr. The accuracies are normalized by $SNR = 10$.

Fig. 3.— The inverse of the determinant of the Fisher matrix for the simple waveform given in equation (16). The filled circles represent with integer $T_{\text{obs}}$ in units of year. We normalize the overall scale by unity at $T_{\text{obs}} = +\infty$. 