Phase diagram in the one-dimensional civil violence model

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The civil violence model is agent-based simulates a social protest process where the police force restores public order. The interactions of cops and citizens produce dynamics that do not yet have any analysis from the sociophysics approach. We present numerical simulations to characterize the properties of the one-dimensional civil violence model on stationary-state. To do this, we consider short- and long-range interactions on a Moore neighborhood and a random neighborhood, a Potts-like energy function, and construct the phase diagram. We find order-disorder phases like those observed in the opinion models. These results are a first approach to studying this model in other dimensions and topologies and considering other complexities of social protest dynamics.

I. INTRODUCTION

In recent years, the sociophysics or statistical physics of social dynamics has described different social phenomena as collective effects of the interaction between individuals [1, 2]. In particular, the study of opinion dynamics has generated a wide variety of models describing consensus, agreement, or uniformity under the paradigm of phase transitions [3–5]. Recent efforts aim at describing these models in their multi-state variations, such as, for example, the majority voting model [6], the multi-state voter model [7], multichoice opinion dynamics models [8], or the multi-state noisy q-voter model [9].

In parallel, social scientists have used agent-based models to reproduce emerging social phenomena [10, 11], such as the Schelling model of urban segregation [12] and the Axelrod model of cultural dissemination [13]. These models have attracted the attention of physicists, who have described the Schelling model as interacting physical particles [14] and as an Ising-like model [15]. Furthermore, they have characterized the static and dynamic properties in one and two dimensions [16] and their different behaviors using a phase diagram [17]. More recently, they have used a similar energy function to characterize both the Schelling and Sakoda models [18, 19]. On the other hand, physicists described the Axelrod model in two dimensions showing order-disorder phase transitions [20]. Then, they described the one-dimensional Axelrod model as a starting point for its description in more complex topologies [21]. In addition, they described the role of dimensionality on the order-disorder phase transitions [22] and the stability model using Lyapunov functions [23, 24].

Recently, Epstein presented a novel model of civil violence [25]. This model simulates a social protest process where the central authority responds by using police force to restore public order. Different scientists have modified this model to describe various social conflicts, such as workers’ protests [26], the spread of criminal activity [27], or civil war cases between ethnic groups [28]. In addition, some variations include legitimacy with endogenous feedback [29] or the influence of the distribution of money on the dynamics [30]. Despite these modifications, nobody characterized this model using concepts and tools of statistical physics in its original form. Understanding the dynamics of this model from the perspective of physics could help in the future to study the dynamics of social protests in other dimensions and topologies to approach the complexity of this social phenomenon.

This paper aims to characterize the one-dimensional civil violence model on stationary-state. To do this, we perform numerical simulations of the Epstein model with and without cops, short- and long-range interactions in a Moore neighborhood and a random neighborhood. We use a Potts-like energy function and construct the phase diagram to identify different behaviors. We find order-disorder phases like those observed in the opinion models. The paper is organized as follows: In Sec. II, we introduce the civil violence model and the global parameters used to describe the model’s behavior to reach the stationary state. The simulations for the model without and with cops and their respective phase diagrams are presented in Sec. III. Our Concluding remarks are in Sec. IV.

II. THE MODEL AND GLOBAL PARAMETERS

A. The Epstein Model

The civil violence model has two types of agents: citizens and cops. Citizens can be active when they participate in social protest, passive when they do not participate, or jailed when the cop catches them. Citizens can switch from one state to another depending on their neighborhood, internal parameters, and the global parameters of the system. On the other hand, a cop agent represents the central authority’s force. They are responsible for restoring order by capturing the active agents in their neighborhood. The neighborhood for all agents can be a Von Neumann neighborhood as used by Epstein [25] or a Moore neighborhood as in other works [27, 30].

The system’s dynamics emerge by relating the legi-
imacy of the authority and the grievance of the population, i.e., it depends on the relationship between the global parameters and the agents’ parameters. The global parameters are the same for all agents: legitimacy \(L\), a state change threshold \(T\), the maximum jail term \(J_{\text{max}}\), and the agents’ vision \(v\). The vision in the original model determines the size of the neighborhood, similarly to the rule radius in cellular automata [31] and range in other opinion models [32, 33]. In this work, we consider an one-dimensional lattice with periodic border conditions. Here, the vision represents the number of pairs of agents to consider to evaluate a switch agent’s state. For example, when \(v = 1\), the neighborhood of each agent consists of its nearest neighbors. Thus each agent considers two sites, one to their left and one to their right. When \(v = 7\), the neighborhood of the agents considers seven sites on the left and seven sites on the right, with fourteen agents in total. On the other hand, the internal agent parameters are hardship \(H\) and risk aversion \(R\). Both parameters are random values between zero and one uniformly distributed among all agents.

The rules that determine the agents’ actions are as follows:

1. **State change rule**: each agent will decide whether or not to join the protest, evaluating the equation \(G - N > T\), where \(G = H(1 - L)\) symbolizes the grievance and \(N = RP\) the net risk. The arrest probability equation \(P = 1 - \exp[-k(C/A)_i]\) depends on the active agents and cops in their neighborhood. The value of \(k\) is 2.3, and as reported by Epstein, it fulfills the function of ensuring plausible values [25]. In the complete form of the state switch equation,

\[
H(1 - L) - R(1 - \exp[-k(C/A)_i]) > T, \tag{1}
\]

we notice that the first element on the left depends on a combination of local and global values, and the other depends on the neighborhood conditions. When the agents’ state variables exceed the threshold, they switch from passive to active; otherwise, they remain passive agents.

2. **Capture rule**: cops randomly capture an active agent from their neighborhood. If there are no active agents, they do nothing. A jailed agent stops participating in the dynamics according to the jail parameter assigned at the beginning of the simulation. The assigned value is uniformly distributed randomly, with values between zero and the maximum determined at the beginning of the simulation. We used the same value used by Epstein, 30 iterations [25].

After setting global and local parameters of the model, we placed all the agents in random positions in the lattice to start the simulations. At each iteration time, all agents evaluate the dynamics rules asynchronously [25]. We show a schematic visualization of the agents’ states’ changes due to the interaction rules in Fig. 1. We have not used Epstein’s motion rule from the original model because, in this work, the agents occupy the whole lattice. With these considerations, the one-dimensional system can enter a stationary regime where the number of agents in any state remains constant on average. Thus, we are interested in understanding the asymptotic properties of the dynamics and characterizing the evolution towards the steady-state.

### B. Global parameters

1. **Energy.** Due to the features of the model, we have used the energy function like which defined by Potts [34]

\[
E[s] = - \sum_{\langle i,j \rangle} J_{ij} \delta(s_i, s_j). \tag{2}
\]

Here the symbol \( \sum_{\langle i,j \rangle} \) means the sum over all neighbors \( j \) of the agent \( i \). For all \( i \) and \( j \) \( J_{ij} = -1 \) and \( \delta(s_i, s_j) \) is a Dirac delta function, i.e., \( \delta(s_i, s_j) = 1 \) if \( s_i = s_j \) and zero for all \( s_i \neq s_j \).

2. **Concentration of agents.** To see the predominant state in the system, we define:

\[
C_\alpha = \frac{N_\alpha}{N}, \tag{3}
\]

where \( N \) is the size of the linear lattice, \( N_\alpha \) is the agents in the active, passive or jailed state. As usually used in opinion dynamics models [9, 35], \( \sum C_\alpha = 1 \) and we distinguish the following phases:

(i) The disordered phase, when all agent states are of a similar concentration in the system.

(ii) The ordered phase, when one agent state is majority over the others. A particular case is when all agents have the same state, so the system reaches a consensus state.
To obtain a first idea of the model dynamics, we study the concentration and Energy variations for different threshold fixed values in Fig. 2. We observe a variation of the concentration of agents when the legitimacy increases. With low values to legitimacy, the active agents are predominant. Then as legitimacy increases, the passive agents are predominant. When \( T = 0.10 \), the system’s predominant state changes, as shown in Fig. 2(a). Note that when \( L < 0.80 \), the active agents predominate, when \( L = 0.80 \), the concentration for two states are similar, and when \( L > 0.80 \), the passive agents are dominant. For \( L = 0.90 \), all agents of the system are in the passive state. We can see a translation of the point of concentrations similarity and the point when all agents of the system are passives states, when the threshold increases in 2(a), 2(b), 2(d), and 2(e) figures. The translations of these points indicate transitions in the system that we can see in the phase diagram. It is important to note that for all types of interactions, neighborhoods, and chain dimensions, the results showed collapse on the same curves. This behavior is because the state switch equation (4) is independent of the neighborhood and indicates that the system dynamic depending on the threshold and the initial simulation’s conditions.

We can see the energy variation as legitimacy increases with threshold fixed for short and long-range interactions, in figures 2(c) and 2(f), respectively. When
Phase I and II are ordered phases with a majority, where one state is predominant. Active agents are predominant in phase I and passives in phase II. The dashed line between these two phases shows when the system is with similar concentrations, therefore disordered. Phase III is a consensus, a particular ordered phase when all of the agents are passives. The solid line shows a transition from the order with a majority to an order with consensus. Every point in this diagram corresponds to an $L$ and $T$ value when the concentrations of active and passive agents are similar or when the system reaches a consensus. The points for all kinds of interactions, neighborhoods, and chain dimensions collapse in the same curves.

In summary, in the system without cops, we observe order-disorder transitions. Phase I and II are ordered phases with a majority, where one state is predominant. Active agents are predominant in phase I and passives in phase II. As a result of crossing the dashed line between these two phases, we observe a disordered phase with similarly active and passive agents concentrations. Phase III is a consensus, a particular ordered phase when all of the agents are passives. The solid line shows a transition from the order with a majority to order with consensus.

To observe the system transition, we study the stationary probability density function of the concentration of agents for a system without cops and Moore neighborhood with short-range interactions. Occur two transitions when the system increasing legitimacy and threshold fixed. A continuous transition from phase I to phase II across a disordered phase in $L = 0.80$, as seen in figures (a), (b), and (c). Then, a continuous transition from phase II to phase III when the systems reach a consensus, as see in figures (d), (e), and (f). These behaviors are the same for all kinds of interactions, neighborhoods, and chain dimensions.

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To observe the system transition, we study the stationary probability density function of the agents’ concentration. We show distributions for a system with Moore neighborhood with short-range interactions and $T = 0.10$ in Fig. 4 because we observe the same behavior for all simulations in figures 2 and 3. We can see the system transition from the order with active agents majority in $L = 0.75$ to a disordered phase in $L = 0.80$, then a transition to order with passive agent majority in $L = 0.85$, in figures 4(a), 4(b), and 4(c). As for legitimacy increases, we can observe a transition to a consensus in $L = 0.90$ in figures 4(d), 4(e), and 4(f).

In summary, in the system without cops, we observe order-disorder transitions like observed in opinion models. We denote Phase I and II as order with majority
Phases. When the agent concentrations are similar, we observe a disordered phase. We note that the system reaches a consensus when all agents are in Phase III. We study this transition with the stationary probability density function of the agents’ concentration and observe a continuous transition characteristic scenario. Finally, we confirm the dependence of the initial conditions of the system dynamics because, for all types of interactions, neighborhoods, and chain dimensions, the results collapse on the same lines.

B. System with cops

To study the model with cops, we used legitimacy values and the concentration of cops from 0.00 to 0.99 with a step of 0.01 for both variables. We vary the concentrations of cops as an initial condition in the original model because its value determines the system’s dynamics. The cop’s role is to dissuade a social protest, preventing citizen agents from becoming active agents and capturing active agents in the system. Besides, the cops’ action depends on the agents’ vision, so we study the effects of the short and long-range interactions separately. The cops’ inclusion makes jailed agents, and the state switch of agents depends on the state parameters and the neighborhood conditions, as we can see in equation (1). We used a fixed threshold value in 0.10 for these simulations because its role is to determine a limit value to the state switch equation. Furthermore, from the results we obtained in the simulations for a system without cops, we observe that the minimum energy obtained is more pronounced for this threshold value. It is important to note that the energy definition only considers the sum of the active and passive agents because the jailed agents do not participate in the model’s dynamics.

Figure (5) show the results for Moore and random neighborhoods with short-range interactions. When the interactions occur in a Moore neighborhood, We observe a variation of the concentration of agents when the legitimacy increases in figures 5(a), 5(b), and 5(c). With low values to legitimacy, the active agents are predominant. Then as legitimacy increases, the passive agents are predominant. The jailed agents’ concentration depends on the cops’ concentration. Then their variations only occur as cops’ concentration increases and produce a change of active agents concentration. With interactions in the random neighborhood, the jailed agent concentrations have predominant values for low legitimacy, as we can see in figures 5(e), 5(f), and 5(g). Then, passive agents are predominant as legitimacy increases. The active agent concentrations depend on the cops’ concentrations and decrease as increase the number of cops in the system.

FIG. 5. Concentration and Energy variations in a system with cops for different values of cops concentrations, legitimacy, and threshold fixed. All agents in the system have short-range interactions. The vertical solid line depends on the fixed threshold and indicates when the system reaches a consensus. The segmented lines indicate when the concentration of agents is similar in their different states. When the interactions occur in a Moore neighborhood, the predominance of active agents decreases when the cops’ concentrations increase because of increase jailed agents. As a result, we can see a translation of the concentration similarity points in figures (a), (b), and (c). These translations suggest predominant state changes and a point when the three states of the systems are similar, so a phase change occurs. The energy reaches a minimum for low cops concentrations and increases when the system reaches a consensus in figure (d). However, as it increases the cops’ concentration, it is possible to observe that the energy maintains a constant value before reach a consensus of passive agents. In a Random neighborhood, we can see the same dynamics of translation of the concentration similarity point in figures (e), (f), and (g). However, the cops can capture more active agents because of the constant neighborhood change. Thus, the predominance of jailed agents results till when the system reaches a high legitimacy. The energy for this neighborhood has a constant value and increase as legitimacy increase, as we can see in figures (h).
We can see the energy variation as legitimacy increases to cops concentrations fixed for Moore and random neighborhoods in figures 5(d) and 5(h). For Moore’s neighborhood and a low cops concentration, we can observe a similar energy behavior for a system without cops. The energy starts with a high value, then reaches a minimum when similar active and passive concentrations, and their value reaches a maximum when the system only has passive agents. However, as cops concentrations increase, the initial energy value decreases and remains constant until it reaches a more significant value when legitimacy has a high value. For random neighborhoods, we observe the same behavior as cops’ concentration increases. Due to energy’s definition only considers actives and passives agents, constant values only indicate similar concentrations for these states. The energy increment as legitimacy increases because all agents in the system switch to the passive state, including jailed agents.

As we noticed in the results for a system without cops in Fig. 2, in the systems with cops, there are also points where the state concentrations are similar. Their positions move as the cops’ concentrations increase. These translations suggest a change in the state predominant in the system. Furthermore, there is a point when the three states of the systems are similar, configuring the order-disorder transitions like opinions models.

To verify this idea, we search for phase boundaries defined by the points in which the agents’ concentrations are similar and when all agents reach a passive state. Every point corresponds to cops’ concentration and legitimacy value and depends on a pair of similar agent states. Thus, the $C_{ap}$ coordinate (●) is when the concentrations of active and passive agents are similar. The $C_{ja}$ coordinate (●) indicates similarity in the jailed and passive state, and the $C_{jp}$ point (●) when jailed and active agents are similar concentration. Each point formed a curve defining different regions on the phase diagram shown in Fig. 6. For both the Moore neighborhood in Fig. 6(a) and the random neighborhood in Fig. 6(b), we observe phases classified according to the transitions described for the system without cops. There are six ordered phases with a majority state. Each one has a label indicating the order of the predominant state. For example, the PAJ phase has predominant passives agents, followed by active and the jailed agents, and so on for the other phases. The system reaches a consensus in the passive phase when all agents are passives and legitimacy equal 0.90. This value is determined by the threshold value selected. The black region indicates when there are only cops in the system.

The system reaches a disordered phase labeled a triple point when the active, passive, and jailed states are in
similar concentrations. The position of this point and the regions of the phases depends on the neighborhood. Most phases have an observable region for the Moore neighborhood. The areas where jailed agents predominate are smaller because cops can only capture active agents among their nearest neighbors. In contrast, the cops are more likely to catch an active agent in a random neighborhood. Thus, we can observe a translation of the triple point and an increase in the regions’ size with predominant jailed agents and a decrease in the areas where active agents are dominant. It is important to note that now it is more difficult to see the order-disorder transition. In contrast to the system without cops, that we can observe this transition for all parameters. When there are cops in the system, the disordered phase is a point in this diagram.

We can observe the system transition with the stationary probability density function of the agents’ concentration in Fig. 7. As for legitimacy increases, the system change from the disordered phase to an ordered phase with a passive state majority and then reaches a consensus phase in a Moore neighborhood in figures 6(a), 6(b), and 6(c). For a random neighborhood, the transition from the ordered phase with the active state to a disordered phase, then then the consensus phase, as we see in figures 6(d), 6(e), and 6(f). The final passive agents’ concentration depends on the cop’s concentration fixed to observe the transition. So, in the Moore neighborhood, the final concentration is 0.6 and, in a random neighborhood, the passive agents’ concentration is 0.9.

Figure (8) show the results for Moore and random neighborhoods with long-range interactions. When the interactions occur in a Moore neighborhood, we can see a variation of the concentration of agents when the legitimacy increases in figures 8(a), 8(b), and 8(c). With low values to legitimacy, the active agents exist, but jailed agents are predominant. Then as legitimacy increases, the passive agents are predominant. As a cop’s concentration increases, the active agents’ concentrations minimize, passive agents increases, and the jailed agents decrease constantly.

With interactions in the random neighborhood, the jailed agent concentrations have predominant values for low legitimacy, as we can see in figures 8(e), 8(f), and 8(g). Furthermore, the active agent’s concentrations minimize at the beginning, and Passive agents are predominant as legitimacy increases. As cops’ concentrations increase, the active agent’s concentrations disappear, passives agents increase constantly, and the jailed agents decrease. This behavior indicates a change in the importance of the cops’ role to dissuade a protest. When the agents have long-range interactions, the relevant cops’ role is to prevent citizen agents from becoming active and prevent the emergence of a protest.

The energy variation as legitimacy increases, for both Moore and random neighborhoods in Fig. 8(d) and 8(h), is different from what we observed when the agents have short-range interactions. For these cases, the energy only depends on the passive agents’ concentrations because increasing the cops’ activity minimizes the active agents’ concentrations. However, the energy begins with a low value, increases as legitimacy increases, and reaches a maximum value for two neighborhoods when all the agents are passives states.

For low cops concentrations, points where the state concentrations are similar exist, suggesting changes in the predominant state in the system. To search for a point where the system shows an order-disorder transition, we built a phase diagram. Every point corresponds to cops’ concentration and legitimacy value and depends on a pair of similar agent states. Then, we can observe the phases formed for the system with cops and long-range interactions in Fig. 9. We see the same six phases observed in the system for the Moore neighborhood with short-range interactions in Fig. 9(a). The region’s size for every phase changes notably because of long-range interactions on the cops’ activity. As a result, we note that the regions with the predominance of passives agents are more significant than the others. Besides, we can observe a triple point where the disordered phase occurs, the region where the system only has cops, and a consensus phase with only passive agents. For the random neighborhood, the effect of the long-range interactions is
more significant, as shown in Fig. 9(b). Although the size of the regions dominated by passive agents is similar to those of the Moore neighborhood, we can notice that the JAP phase and the point at which all concentrations are similar disappears. As a result, we observe that there is no order-disorder transition. However, the system change between different order phases with a majority state depending on the cops’ concentration and legitimacy values. It reaches a consensus when all agents are in the passive state at the value of legitimacy is 0.90 because this depends on the threshold value fixed at the beginning of simulations.

We show the stationary probability density function of the agents’ concentration in Fig. 10 to observe the system transition. For the Moore neighborhood, we selected the cops’ concentrations 0.06 and varied the legitimacy. We can see, as legitimacy increases, the system change from the disordered phase to an ordered phase with a passive state majority in figures 10(a) and 10(b). Then reaches a consensus phase in Fig. 10(c). We selected a lower value of cops concentrations for the random neighborhood to observe the possibility of finding similar concentrations for the three states as legitimacy increases. Nevertheless, only find order with majority phases, as shown in figures 10(d), 10(e), and the consensus phase in Fig. 10(f).

In summary, in the system with cops, the dynamics depend on the interactions. When the system has short-range interactions, we observe order-disorder transitions like observed in opinion models. There are six ordered phases with a majority state and a disordered phase at a particular point when there are similar concentrations for the three states. The system reaches a consensus in the passive phase when all agents are passive. The position of the disorder point and the size regions of the phases depends on the neighborhood. For this interaction, we note that the capture of active agents is the most relevant cop’s role in dissuading a protest.

When the interactions are with long-range interactions, the cops’ activity increases, producing a change of the system dynamics. Now in the Moore neighborhood, the jailed agents are predominant over the active agents. As for legitimacy increases, passives agents are predominant. This effect produces order-disorder transitions. We can observe six ordered phases with a majority, a point of disorder phase, and the consensus phase. In contrast with the short-range interactions, the size region phases are different, and the point of the disorder requires a low cops concentration. Nevertheless, there is no order-disorder transition when considering a random neighborhood because the long-range interactions allow to cops increases their effect over the switch state of agents. Now, the relevant cops’ role is to prevent citizen agents from becoming active and prevent the emergence of a protest. As a result, we observe that one phase disappears, and the agents’ concentration never has similar values.
IV. CONCLUDING REMARKS

This paper studied the one-dimensional civil violence model with the whole lattice occupied to characterize their evolution on the stationary state. To do this, we performed extensive numerical simulations of the model with and without cops, considering short and long-range on Moore and random neighborhoods. We used the agent state concentration and introduced a Potts-like energy function to characterize the model.

In the system without cops, the phase diagram shows us order-disorder transitions like observed in opinion models. We denote Phase I and II as order phases with a majority state, active a passive, respectively. Crossing between these phases, find a disordered phase when the agent concentrations are similar. Phase III is a particular ordered phase when all agents are passive. We study the transitions with the stationary probability density function of the agents’ concentration and observe a continuous transition characteristic scenario. Finally, we confirm the dependence of the initial conditions of the system dynamics because, for all types of interactions, neighborhoods, and chain dimensions, the results collapse on the same lines. These results allow us to identify a threshold value when the system reaches the lowest minimum energy, which coincides with Epstein’s reported value in the original model.

In the system with cops, the dynamics depend on the interactions. When the system has short-range interactions, the phase diagram shows order-disorder transitions observed without cops. Now the system has three agent conditions: passive, active, and jailed. Each phase transition is characterized by a unique energy function and a specific neighborhood structure.
states. As a result, we observed six ordered phases with a majority state and a disordered phase at a particular point when there are similar concentrations for the three states. The system reaches a consensus in the passive phase when all agents are passives. The position of the disorder point and the size regions of the phases change with the neighborhood. When the interactions are with long-range interactions, the cops’ activity increases, producing a change of the system dynamics. While we can still observe order-disorder transitions in the Moore neighborhood, we do not observe these transitions in the random neighborhood. These results allow us to determine the relevance of the cops’ actions in different scenarios. For short-range interactions, the capture of active agents is the most relevant cop’s action to dissuading a protest. However, with long-range interactions, the relevant cops’ action prevents citizen agents from becoming active and prevents the emergence of a protest.

These results are the first approximation to generate a study framework for future work with other dimensions and topologies. Moreover, working with agents moving on the lattice generates scenarios closer to the original idea of Epstein’s model. Study this model from the perspective of opinion dynamics allows us to ask for more complex real scenarios. This article shows, like many others, that statistical physics tools are helpful to address the complexities of social phenomena, such as, in this case, the dynamics of social protest.

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