UNCONDITIONAL WELL-POSEDNESS FOR SOME NONLINEAR PERIODIC ONE-DIMENSIONAL DISPERSIVE EQUATIONS

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Abstract. We consider the Cauchy problem for one-dimensional dispersive equations with a general nonlinearity in the periodic setting. Our main hypotheses are both that the dispersive operator behaves for high frequencies as a Fourier multiplier by $i|\xi|^\alpha \xi$, with $1 \leq \alpha \leq 2$, and that the nonlinear term is of the form $\partial_x f(u)$ where $f$ is the sum of an entire series with infinite radius of convergence. Under these conditions, we prove the unconditional local well-posedness of the Cauchy problem in $H^s(\mathbb{T})$ for $s \geq 1 - \frac{\alpha}{2(\alpha + 1)}$. This leads to some global existence results in the energy space $H^{\alpha/2}(\mathbb{T})$, for $\alpha \in [\sqrt{2}, 2]$.

1. Introduction

Our goal is to establish low regularity well-posedness results for quite general one-dimensional nonlinear dispersive equations of KdV type in the periodic setting. Our class contains in particular the generalized KdV equations with fractional dispersion

$$\partial_t u - \partial_x D_x^\alpha u + \partial_x (f(u)) = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{T}, \quad (1.1)$$

$$u(0, x) = u_0(x), \quad x \in \mathbb{T}, \quad (1.2)$$

where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, $f : \mathbb{R} \to \mathbb{R}$ is the sum of an entire series, $\alpha \in [1, 2]$ and $D_x^\alpha$ is the Fourier multiplier by $|\xi|^\alpha$.

Famous equations in this class are the generalized Korteweg-de Vries equation (gKdV)

$$\partial_t u + \partial_x^3 u + \partial_x (f(u)) = 0$$

that corresponds to $\alpha = 2$ as well as the generalized Benjamin-Ono equation (gBO)

$$\partial_t u - \mathcal{H} \partial_x^2 u + \partial_x (f(u)) = 0,$$

where $\mathcal{H}$ is the Hilbert transform (Fourier multiplier by $-i\text{sgn}(\xi)$), that corresponds to $\alpha = 1$.

The Cauchy problem for this type of dispersive equations has been extensively studied since more than thirty years starting with the work of Kato [13] (see also...)
It is worth noticing that the first methods employed did not make use of the dispersive effects and the local well-posedness was restricted to Sobolev spaces with index $s > 3/2$. At the end of the eighties Kenig, Ponce and Vega succeeded in exploiting dispersive effects of the linear part of these equations in order to lower the required Sobolev regularity on the initial data (see for instance [16], [36]). They obtained in particular the well-posedness of the KdV equation in the energy space and were able in [17] to apply a contraction mapping argument on the integral formulation for some $k$-generalized KdV equations ($f(x) = x^k$). Then in the early nineties, Bourgain introduced the now so-called Bourgain’s spaces that take into account the localization of the space-time Fourier transform of the function and where one can solve by a fixed point argument a wide class of dispersive equations on $\mathbb{R}$ or $\mathbb{T}$ with very rough initial data ([3]). At this stage, let us underline that since the nonlinearity of these equations is in general algebraic, the fixed point argument ensures the real analyticity of the solution-map.

In the early 2000’s, Saut, Tzvetkov and the first author [27] noticed that a large class of “weakly” dispersive equations posed on the real line, including in particular (1.1) with $\alpha < 2$ and $f(x) = x^2$, cannot be solved by a fixed point argument for initial data in any Sobolev spaces $H^s(\mathbb{R})$. Since then, except in the case of integrability of the equation where methods linked to this remarkable property have been recently developed (see for instance [12], [6] in the periodic setting), mainly two types of approaches have been developed to solve this class of equation in low regularity Sobolev spaces. The first one consists in applying a gauge transform (see [41] for the Benjamin-Ono equation and [10] for the dispersion generalized KdV equation) to eliminate the worst nonlinear interactions that are some high-low frequency interactions and then to solve the equation after the gauge transform by using dispersive tools as Strichartz’s and Bourgain’s type estimates. The second one consists in enhancing the energy method (or modified energy method) with a priori estimates that takes the dispersion into account. For this one possibility is to improve the dispersive estimates by localizing a solution in space frequency depending time intervals and then mixing them with classical energy estimates. This type of methods was first introduced by Koch and Tzvetkov [23] (see also [18] for some improvements) in the framework of Strichartz’s spaces and then by Koch and Tataru [22] (see also [11]) in the framework of Bourgain’s spaces. Another method that does not need to localize in space frequency depending time intervals has been recently introduced in [30]. Note that in [31], [32] it is shown that in some cases this last method can be improved by adding some a priori estimates obtained
by localizing in space frequency depending time intervals. We implement this last
approach enhanced with some other tools in this paper.

1.1. Presentation of the main results. In this work we consider roughly the same
type of dispersion term as in [7] and [30] but with a much more general nonlinear
term. More precisely we consider dispersive equation of the form

$$\partial_t u + L_{\alpha+1} u + \partial_x (f(u)) = 0 \quad (1.3)$$

under the two following hypotheses on $L_{\alpha+1}$ and $f$.

**Hypothesis 1.** $L_{\alpha+1}$ is the Fourier multiplier operator by $-ip_{\alpha+1}$ where $p_{\alpha+1} \in C^1(\mathbb{R}) \cap C^2(\mathbb{R}\setminus\{0\})$ is a real-valued odd function satisfying, for some $\xi_0 > 0$, $p'_{\alpha+1}(\xi) \sim \xi^\alpha$ and $p''_{\alpha+1}(\xi) \sim \xi^{\alpha-1}$ for all $\xi \geq \xi_0$.

**Hypothesis 2.** $f : \mathbb{R} \rightarrow \mathbb{R}$ is the sum of an entire series with infinite radius of convergence.

*Remark 1.1.* We choose a sign in Hypothesis 1 only to fix in an easy way the
defocussing case that leads to a global existence result ((2) of Theorem 1.3 below).
Of course for all other results the sign will play no role here.

*Remark 1.2.* Hypothesis 1 is fulfilled by the purely dispersive operators $L := \partial_x D_x^\alpha$ with $\alpha > 0$, the linear Intermediate Long Wave operator $L := \partial_x D_x \coth(D_x)$ and some perturbations of the Benjamin-Ono equation as the Smith operator $L := \partial_x (D_x^2 + 1)^{1/2}$ (see [39]) for which $\alpha = 1$.

*Remark 1.3.* The hypothesis on $f$ ensures that $f$ is of class $C^\infty$ and is at each point
of $\mathbb{R}$ the sum of its Taylor expansion at the origin, i.e.,

$$f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n, \quad \forall x \in \mathbb{R}.$$  \hspace{.5cm} \text{(1.4)}

Of course any polynomial function enters in this class but also the exponential
functions as $e^x$, $\sin(x)$, $\cos(x)$ and their products and compositions.

Here we concentrate ourself on the periodic setting. Actually we notice that in
the real line case, following [23], one can exploit the classical Strichartz estimates
in $L_t^4 L_x^\infty$ to get the following improved Strichartz estimates on smooth solutions to
(1.1) in the purely dispersive case $L_{\alpha+1} = \partial_x D_x^\alpha$:

$$\|D_x^{1/2} u\|_{L_t^4 L_x^\infty} \lesssim C(\|u\|_{L_t^\infty H_x^{1-\alpha/(4+)}}).$$
With this estimate in hand, the same method as the one developed in this work but with a clearly more simple decomposition, lead to the unconditional local well-posedness of \((1.1)\) in \(H^{(1-\alpha/4)+}(\mathbb{R})\) (see \[35\] where the KdV case \(\alpha = 2\) is treated with initial data that may be non vanishing at infinity).

Our approach is based on the method introduced in \[30\] enhanced with some arguments that can be found in \[32\], \[19\] and \[23\]. As noticed in \[30\], this method is particularly well-adapted to solve the Cauchy problem associated with one-dimensional dispersive wave equations in the Sobolev spaces \(H^s\) for \(s > 1/2\). It combines classical energy estimates with Bourgain’s type estimates that measure the localization of the space-time Fourier transform of the solution around the curve given by the Fourier symbol of the associated linear dispersive equation. In our approach we combine this with improved Strichartz estimates and symmetrization arguments. The main strategy is to use symmetry arguments to distribute the lost derivative to several functions and then to recover it by either improved Strichartz estimates or Bourgain’s type estimates depending on whether the nonlinear interactions are resonant. Since the equation for the difference of two solutions enjoys less symmetries, this difference will be estimated in a lower regularity space than the solution itself and we will use the frequency envelope approach to recover the continuity result with respect to initial data.

We emphasize that our work was motivated by \[19\] where the local well-posedness of the \(k\)-gBO equation \((L_2 = \partial_x D_x \text{ and } f(x) = x^k, k \geq 2)\) is proven in \(H^{3/4}(\mathbb{T})\) by using another method that is short time Fourier transform restriction method combined with modified energy estimates. It is interesting to underline that our local well-posedness result in this case is exactly the same (except that we also get the unconditional uniqueness).\footnote{To be precise, they also deduced a priori estimate at the regularity \(s > 1/2\).} Both methods have the advantage to not use a gauge transformation and thus to be more flexible with respect to perturbations of the equation. However we believe that our approach is easier to implement to solve \(1.3\).

We are not aware of any low regularity results in the literature on the Cauchy problem associated with \((1.1)\) with a general nonlinearity in the periodic setting. However we notice that, in \[15\], the local well-posedness of the \(k\)-gKdV equation was proven to be well-posed in \(H^{1/2}(\mathbb{T})\) by a contraction mapping argument in Bourgain’s spaces. This suggests that the LWP of the gKdV equation (at least with \(f\) satisfying hypothesis \[2\]) should be locally well-posed in \(H^{1/2+}(\mathbb{T})\) whereas we only get the unconditional LWP of the gKdV equation in \(H^{2/3}(\mathbb{T})\) in this work. On the
other hand, in the case $\alpha < 2$, we do not know any local well-posedness results for (1.1) with $f(x) = x^k$, $k \geq 2$, in Sobolev spaces with lower regularity than the ones obtained in Theorem 1.1 below. Actually the only results we know are [19], mentioned above, and [29] where Ribaud and the first author proved the local well-posedness of the $k$-gBO equation in $H^1(\mathbb{T})$ by using gauge transform and Strichartz estimates. We believe that this approach could be adapted to quite general nonlinearity $\partial_x(f(u))$ to get the LWP of the gBO equation in $H^1(\mathbb{T})$. However it is not clear how to adapt such approach for more general dispersive term as it is done here.

Before stating our main result, let us recall our notion of solutions:

**Definition 1.** Let $s > 1/2$. We will say that $u \in L^\infty([0,T]; H^s(\mathbb{T}))$ is a solution to (1.3) associated with the initial datum $u_0 \in H^s(\mathbb{T})$ if $u$ satisfies (1.3) in the distributional sense, i.e. for any test function $\phi \in C_0^\infty([-T,T[\times \mathbb{T})$, it holds

$$
\int_0^\infty \int_\mathbb{T} \left[ (\phi_t + L_{\alpha+1}\phi)u + \phi_x f(u) \right] dx \, dt + \int_\mathbb{T} \phi(0,\cdot)u_0 \, dx = 0 \quad (1.5)
$$

**Remark 1.4.** Note that for $u \in L^\infty([0,T]; H^s(\mathbb{T}))$, with $s > 1/2$, $f(u)$ is well-defined and belongs to $L^\infty([0,T]; H^s(\mathbb{T}))$. Moreover, Hypothesis (1) forces

$$
L_{\alpha+1}u \in L^\infty([0,T]; H^{s-\alpha-1}(\mathbb{T})).
$$

Therefore $u_t \in L^\infty([0,T]; H^{s-\alpha-1}(\mathbb{T}))$ and (1.5) ensures that (1.3) is satisfied in $L^\infty([0,T]; H^{s-\alpha-1}(\mathbb{T}))$. In particular, $u \in C([0,T]; H^{s-\alpha-1}(\mathbb{T}))$ and (1.5) forces the initial condition $u(0) = u_0$. Note that this ensures that $u \in C_w([0,T]; H^s(\mathbb{T}))$ and thus $\|u_0\|_{H^s} \leq \|u\|_{L^\infty_t H^s}$. Finally, we notice that this also ensures that $u$ satisfies the Duhamel formula associated with (1.3).

Finally, let us recall the notion of unconditional well-posedness that was introduced by Kato [14], which is, roughly speaking, the local well-posedness with uniqueness of solutions in $L^\infty([0,T]; H^s(\mathbb{T}))$.

**Definition 2.** We will say that the Cauchy problem associated with (1.3) is unconditionally locally well-posed in $H^s(\mathbb{T})$ if for any initial data $u_0 \in H^s(\mathbb{T})$ there exists $T = T(\|u_0\|_{H^s}) > 0$ and a solution $u \in C([0,T]; H^s(\mathbb{T}))$ to (1.3) emanating from $u_0$. Moreover, $u$ is the unique solution to (1.3) associated with $u_0$ that belongs to $L^\infty([0,T]; H^s(\mathbb{T}))$. Finally, for any $R > 0$, the solution-map $u_0 \mapsto u$ is continuous from the ball of $H^s(\mathbb{T})$ with radius $R$ centered at the origin into $C([0,T(R)]; H^s(\mathbb{T}))$.

We note that Babin, Ilyin and Titi [2] proved the unconditional uniqueness of the KdV equation in $L^2(\mathbb{T})$ by integration by parts in time. This method, which is
actually a normal form reduction, has been now successfully applied to a variety of dispersive equations (see for instance [8, 20, 21, 24, 25, 33] and references therein).

Our main result is the following one:

**Theorem 1.1** (Unconditional well-posedness). Assume that Hypotheses 1-2 are satisfied with $1 \leq \alpha \leq 2$. Then for any $s \geq s(\alpha) = 1 - \frac{\alpha}{2(\alpha+1)}$ the Cauchy problem associated with (1.3) is unconditionally locally well-posed in $H^s(T)$ with a maximal time of existence $T \geq g(\|u_0\|_{H^s(T)}) > 0$ where $g$ is a smooth decreasing function depending only on $L^2\alpha+1$ and $f$.

**Remark 1.5.** The limitation $s \geq s(\alpha) = 1 - \frac{\alpha}{2(\alpha+1)}$ in the above theorem is due to the following nonlinear interactions: there are 3 input high frequencies of the same order than the output frequency whereas all other input frequencies are much lower.

In this configuration the nonlinear term can be resonant and to recover the lost derivative we put the 4 terms with high frequencies in $L^4_{t,x}$ (whereas the other ones are put in $L^\infty_{t,x}$) and we use the improved Strichartz estimates in Proposition 3.5 to control their norms in this space. This is the lost of derivatives in these Strichartz estimates that leads to the value of $s(\alpha)$. More precisely, in Proposition 3.5, it is proven that any solution $u \in L^\infty_t(0,T;H^s(T))$ to (1.3) satisfies $D^{-\beta}_x u \in L^4_tL^4_x$ with $\beta(\alpha) = \frac{1}{4(\alpha+1)}$. Then, in order to recover the lost derivative in this configuration, direct calculations lead to $4(s - \beta(\alpha)) \geq 2s + 1 \iff s \geq 1/2 + 2\beta(\alpha) \iff s \geq s(\alpha)$.

Equation (1.3) enjoys the following conservation laws at the $L^2$ and at the $H^{\alpha/2}$-level:

$$M(u) = \int_T u^2$$
$$E(u) = \frac{1}{2} \int_T u \partial_x^{-1} L_{\alpha+1} u + \int_T F(u)$$

where $\partial_x^{-1} L_{\alpha+1}$ is the Fourier multiplier by $\frac{p_{\alpha+1}(k)}{k}1_{k \neq 0}$ and

$$F(x) := \int_0^x f(y) dy . \quad (1.6)$$

At this stage it is worth noticing that Hypothesis II ensures that the restriction of the quadratic part of the energy $E$ to high frequencies behaves as the $H^{\alpha/2}(T)$-norm whereas its restriction to the low frequencies can be controlled by the $L^2$-norm. Therefore, gathering these conservation laws with the above local well-posedness result we are able to obtain the two following global existence results:

**Theorem 1.2** (Global existence for small initial data). Assume that Hypotheses 1-2 are satisfied with $\alpha \in [\sqrt{2}, 2]$. Then there exists $A = A(L_{\alpha+1}, f) > 0$ such that for any initial data $u_0 \in H^s(T)$ with $s \geq \alpha/2$ such that $\|u_0\|_{H^{\alpha/2}} \leq A$, the solution
constructed in Theorem 1.1 can be extended for all times. Moreover its trajectory is bounded in $H^{\alpha/2}(\mathbb{T})$.

**Theorem 1.3** (Global existence for arbitrary large initial data).
Assume that Hypotheses 1-2 are satisfied with $\alpha \in [\sqrt{2}, 2]$. Then the solution constructed in Theorem 1.1 can be extended for all times if the function $F$ defined in (1.6) satisfies one of the following conditions:

1. There exists $C > 0$ such that $|F(x)| \leq C(1 + |x|^{p+1})$ for some $0 < p < 2\alpha + 1$.
2. There exists $B > 0$ such that $F(x) \leq B$, $\forall x \in \mathbb{R}$.

Moreover its trajectory is bounded in $H^{\alpha/2}(\mathbb{T})$.

**Remark 1.6.** Typical examples for the case (1) are
- $f(x)$ is a polynomial function of degree strictly less than $2\alpha + 1$.
- $f(x)$ is a polynomial function of $\sin(x)$ and $\cos(x)$.

Whereas typical example for the case (2) are
- $f(x)$ is a polynomial function of odd degree with $\lim_{x \to +\infty} f(x) = -\infty$.
- $f(x) = -\exp(x)$ or $f(x) = -\sinh(x)$.

### 2. Notation, function spaces and basic estimates

#### 2.1. Notation.
Throughout this paper, $\mathbb{N}$ denotes the set of non-negative integers.

For any positive numbers $a$ and $b$, we write $a \lesssim b$ when there exists a positive constant $C$ such that $a \leq Cb$. We also write $a \sim b$ when $a \lesssim b$ and $b \lesssim a$ hold. Moreover, we denote $a \ll b$ if the estimate $b \lesssim a$ does not hold. For two non-negative numbers $a, b$, we denote $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$. We also write $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$.

For $u = u(t, x)$, $\mathcal{F}u = \hat{u}$ denotes its space-time Fourier transform, whereas $\mathcal{F}_x u = \hat{u}$ (resp. $\mathcal{F}_t u$) denotes its Fourier transform in space (resp. time). We define the Riesz potentials by $D^s_x g := \mathcal{F}_x^{-1}(|\xi|^s \mathcal{F}_x g)$. We also denote the unitary group associated to the linear part of (1.3) by $U_\alpha(t) = e^{-it\Lambda_{\alpha+1}}$, i.e.,

$$U_\alpha(t)u = \mathcal{F}_x^{-1}(e^{it\Lambda_{\alpha+1}}(\xi) \mathcal{F}_x u).$$

In the present paper, we fix a smooth cutoff function $\chi$: let $\chi \in C_0^\infty(\mathbb{R})$ satisfy

$$0 \leq \chi \leq 1, \quad \chi|_{-1,1} = 1 \quad \text{and} \quad \text{supp} \chi \subset [-2, 2].$$

We set $\phi(\xi) := \chi(\xi) - \chi(2\xi)$. For any $l \in \mathbb{N}$, we define

$$\phi_{2^l}(\xi) := \phi(2^{-l}\xi), \quad \psi_{2^l}(\tau, \xi) := \phi_{2^l}(\tau - p_\alpha(\xi)),$$
where \( i p_{\alpha+1}(\xi) \) is the Fourier symbol of \( L_{\alpha+1} \). By convention, we also denote

\[
\phi_0(\xi) = \chi(2\xi) \quad \text{and} \quad \psi_0(\tau, \xi) = \chi(2(\tau - p_{\alpha+1}(\xi))).
\]

Any summations over capitalized variables such as \( K, L, M \) or \( N \) are presumed to be dyadic. We work with non-homogeneous dyadic decompositions, i.e., these variables ranges over numbers of the form \( \{2^k; k \in \mathbb{N}\} \cup \{0\} \). We call those numbers nonhomogeneous dyadic numbers. It is worth pointing out that \( \sum_N \phi_N(\xi) = 1 \) for any \( \xi \in \mathbb{Z} \),

\[
\text{supp}(\phi_N) \subset \{ N/2 \leq |\xi| \leq 2N \}, \quad N \geq 1, \quad \text{and} \quad \text{supp}(\phi_0) \subset \{|\xi| \leq 1\}.
\]

Finally, we define the Littlewood–Paley multipliers \( P_N \) and \( Q_L \) by

\[
P_N u = \mathcal{F}^{-1}(\phi_N \mathcal{F} u) \quad \text{and} \quad Q_L u = \mathcal{F}^{-1}(\psi_L \mathcal{F} u).
\]

We also set \( P_{\geq N} := \sum_{K \geq N} P_K \), \( P_{\leq N} := \sum_{K \leq N} P_K \), \( Q_{\geq N} := \sum_{K \geq N} Q_K \) and \( Q_{\leq N} := \sum_{K \leq N} Q_K \).

### 2.2. Function spaces.

For \( 1 \leq p \leq \infty \), \( L^p(\mathbb{T}) \) is the standard Lebesgue space with the norm \( \| \cdot \|_{L^p} \).

In this paper we will use the frequency envelope method (see for instance [41] and [23]) in order to show the continuity result with respect to initial data. To this aim, we have to slightly modulate the classical Sobolev spaces in the following way: for \( s \geq 0 \) and a dyadic sequence \( \{\omega_N\} \), we define \( H^s_{\omega}(\mathbb{T}) \) with the norm

\[
\|u\|_{H^s_{\omega}} := \left( \sum_N \omega_N^2 (1 \vee N)^{2s} \|P_N u\|_{L^2}^2 \right)^{1/2}.
\]

Note that \( H^s_{\omega}(\mathbb{T}) = H^s(\mathbb{T}) \) when we choose \( \omega_N \equiv 1 \). Here, \( H^s(\mathbb{T}) \) is the usual \( L^2 \)-based Sobolev space. If \( B_x \) is one of spaces defined above, for \( 1 \leq p \leq \infty \) and \( T > 0 \), we define the space–time spaces \( L^p_t B_x := L^p(\mathbb{R}; B_x) \) and \( L^p_T B_x := L^p([0,T]; B_x) \) equipped with the norms (with obvious modifications for \( p = \infty \))

\[
\|u\|_{L^p_t B_x} = \left( \int_{\mathbb{R}} \|u(t, \cdot)\|_{B_x}^p dt \right)^{1/p} \quad \text{and} \quad \|u\|_{L^p_T B_x} = \left( \int_0^T \|u(t, \cdot)\|_{B_x}^p dt \right)^{1/p},
\]

respectively. For \( s, b \in \mathbb{R} \), we introduce the Bourgain spaces \( X^{s,b} \) associated to the operator \( L_{\alpha+1} \) endowed with the norm

\[
\|u\|_{X^{s,b}} = \left( \sum_{\xi = -\infty}^{\infty} \int_{-\infty}^{\infty} \langle \xi \rangle^{2s} (\tau - p_{\alpha+1}(\xi))^{2b} |\tilde{u}(\tau, \xi)|^2 d\tau \right)^{1/2}.
\]
We also use a slightly stronger space $X_{\omega}^{s,b}$ with the norm
\[ \|u\|_{X_{\omega}^{s,b}} := \left( \sum_{N} \omega_{N}^{2}(1 \vee N)^{2s}\|P_{N}u\|_{X_{\omega,0}^{s,b}}^{2} \right)^{1/2}. \]

We define the function spaces $Z_{s}^{s}$ (resp. $Z_{\omega}^{s}$), with $s \in \mathbb{R}$, as $Z_{s}^{s} := L_{t}^{\infty}H_{\omega}^{s} \cap X_{\omega}^{s-1,1}$ (resp. $Z_{\omega}^{s} := L_{t}^{\infty}H_{\omega}^{s} \cap X_{\omega}^{s-1,1}$), endowed with the natural norm
\[ \|u\|_{Z_{s}^{s}} = \|u\|_{L_{t}^{\infty}H_{\omega}^{s}} + \|u\|_{X_{\omega}^{s-1,1}} \] (resp. \(\|u\|_{Z_{\omega}^{s}} = \|u\|_{L_{t}^{\infty}H_{\omega}^{s}} + \|u\|_{X_{\omega}^{s-1,1}}\)).

We also use the restriction in time versions of these spaces. Let $T > 0$ be a positive time and $B$ be a normed space of space-time functions. The restriction space $B_{T}$ will be the space of functions $u : ]0, T[ \to \mathbb{R}$ or $\mathbb{C}$ satisfying
\[ \|u\|_{B_{T}} := \inf \{\|\tilde{u}\|_{B} \mid \tilde{u} : \mathbb{R} \times \mathbb{T} \to \mathbb{R} \text{ or } \mathbb{C}, \tilde{u} = u \text{ on } ]0, T[ \times \mathbb{T} \} < \infty. \]

Finally, we introduce a bounded linear operator from $X_{\omega,T}^{s-1,1} \cap L_{t}^{\infty}H_{\omega}^{s}$ into $Z_{\omega}^{s}$ with a bound which does not depend on $s$ and $T$. The existence of this operator ensures that actually $Z_{\omega,T}^{s} = L_{t}^{\infty}H_{\omega}^{s} \cap X_{\omega,T}^{s-1,1}$. Following \cite{26}, we define $\rho_{T}$ as
\[ \rho_{T}(u)(t) := U_{a}(t)\chi(t)U_{a}(-\mu_{T}(t))u(\mu_{T}(t)), \quad (2.1) \]
where $\mu_{T}$ is the continuous piecewise affine function defined by
\[ \mu_{T}(t) = \begin{cases} \quad 0 & \text{for } t \notin ]0, 2T[, \\ \quad t & \text{for } t \in [0, T], \\ 2T - t & \text{for } t \in [T, 2T]. \end{cases} \quad (2.2) \]

**Lemma 2.1.** Let $\delta \geq 1$, and suppose that the dyadic sequence $\{\omega_{N}\}$ of positive numbers satisfies $\omega_{N} \leq \omega_{2N} \leq \delta \omega_{N}$ for $N \geq 1$. Let $0 < T \leq 1$ and $s \in \mathbb{R}$. Then,
\[ \rho_{T} : X_{\omega,T}^{s-1,1} \cap L_{T}^{\infty}H_{\omega}^{s} \to Z_{\omega}^{s} \]
\[ u \mapsto \rho_{T}(u) \]
is a bounded linear operator, i.e.,
\[ \|\rho_{T}(u)\|_{L_{T}^{\infty}H_{\omega}^{s}} + \|\rho_{T}(u)\|_{X_{\omega,T}^{s-1,1}} \lesssim \|u\|_{L_{T}^{\infty}H_{\omega}^{s}} + \|u\|_{X_{T,T}^{s-1,1}}, \quad (2.3) \]
for all $u \in X_{\omega,T}^{s-1,1} \cap L_{T}^{\infty}H_{\omega}^{s}$. Moreover, it holds that
\[ \|\rho_{T}(u)\|_{L_{T}^{\infty}H_{\omega}^{s}} \lesssim \|u\|_{L_{T}^{\infty}H_{\omega}^{s}} \quad (2.4) \]
for all $u \in L_{T}^{\infty}H_{\omega}^{s}$. Here, the implicit constants in (2.3) and (2.4) can be chosen independent of $0 < T \leq 1$ and $s \in \mathbb{R}$.

**Proof.** See Lemma 2.4 in \cite{32} for $\omega_{N} \equiv 1$ but it is obvious that the result does not depend on $\omega_{N}$. \(\Box\)
2.3. Basic estimates. In this subsection, we collect some fundamental estimates. Well-known estimates are adapted for our setting $H^s_ω(\mathbb{T})$ and $f(u)$.

Lemma 2.2. Let $s > 0$. Let $δ ≥ 1$, and suppose that the dyadic sequence $\{ω_N\}$ of positive numbers satisfies $ω_N ≤ ω_{2N} ≤ δω_N$ for $N ≥ 1$. Then we have the estimate

$$
||uv||_{H^s_ω} ≲ ||u||_{H^s_ω} ||v||_{L^∞} + ||u||_{L^∞} ||v||_{H^s_ω}.
$$

(2.5)

In particular for any fixed real entire function $f$ with $f(0) = 0$, there exists a real entire function $G = G[f]$ that is increasing non negative on $\mathbb{R}_+$ such that

$$
||f(u)||_{H^s_ω} ≲ G(||u||_{L^∞}) ||u||_{H^s_ω}.
$$

(2.6)

Proof. The proof of (2.5) is identical with that of Lemma A.8 in [42]. For reader’s convenience, we provide the proof.

The triangle inequality gives

$$
||uv||_{H^s_ω} ≲ \left( \sum_{N≤1} ω^2_N ||P_N(uv)||^2_{L^2} \right)^{1/2} + \left( \sum_{N>1} ω^2_N N^{2s} ||P_N(uv)||^2_{L^2} \right)^{1/2} =: A + B.
$$

It is clear that $A ≲ ||u||_{H^s_ω} ||v||_{L^∞}$ since $ω_{2N} ≤ δ^iω_1$. We further split

$$
||P_N(uv)||_{L^2} ≲ ||P_N((P_{≤N}u)v)||_{L^2} + \sum_{M>N} ||P_N((P_Mu)v)||_{L^2}.
$$

For the first term, by impossible frequency interactions, we see that

$$
||P_N((P_{≤N}u)v)||_{L^2} ≤ ||(P_{≤N}u)P_{≈N}v||_{L^2} \lesssim ||u||_{L^∞} \sum_{M>N} ||P_Mv||_{L^2}.
$$

So, the contribution of this term to $B$ is bounded by $C ||u||_{L^∞} ||v||_{H^s_ω}$. For the second term, we simply bound

$$
\sum_{M≥N} ||P_N((P_Mu)v)||_{L^2} \lesssim ||v||_{L^∞} \sum_{M≥N} ||P_Mu||_{L^2}
$$

and so we see from $2^sω_N N^s ≤ ω_{2N}(2N)^s$ and the Young inequality that

$$
\left( \sum_{N≥1} \left( \sum_{M≥N} ω_N N^s ||P_Mu||_{L^2} \right)^2 \right)^{1/2} ≤ \left( \sum_{k≥1} \left( \sum_{j≥1} 2^{-sj} ω_{2^k+j} 2^{s(j+k)} ||P_{2^k+j}u||_{L^2} \right)^2 \right)^{1/2}
$$

$$
\lesssim ||u||_{H^s_ω},
$$

which implies that $B ≲ ||u||_{H^s_ω} ||v||_{L^∞} + ||u||_{L^∞} ||v||_{H^s_ω}$.

Now by the Minkowski inequality and (2.5)

$$
||f(u)||_{H^s_ω} \leq \sum_{k≥1} \left| \frac{f^{(k)}(0)}{k!} \right| ||u^k||_{H^s_ω} \leq \sum_{k≥1} \left| \frac{f^{(k)}(0)}{k!} \right| C^{k-1} ||u||_{L^∞}^{k-1} ||u||_{H^s_ω}
$$

$$
\leq G(||u||_{L^∞}) ||u||_{H^s_ω},
$$
such that (2.7) can be found in [9], Lemma 3.4. To prove (2.8) it suffices to check that

\[ \| f(u) - f(v) \|_{H^{s-1}} \leq G(\| u \|_{H^{s}} + \| v \|_{H^{s}}) \| u - v \|_{H^{s-1}}. \]  

(2.8)

Proof. (2.7) can be found in [9], Lemma 3.4. To prove (2.8) it suffices to check that for \( s > 1/2, (s - 1) + s = 2s - 1 > 0 \) and \( s - 1 < s - 1/2 + (s - 1) \). Therefore

\[
\| f(u) - f(v) \|_{H^{s-1}} \leq \sum_{k \geq 1} \frac{|f^{(k)}(0)|}{k!} \| u^k - v^k \|_{H^{s-1}}
\]

\[
\leq \sum_{k \geq 1} \frac{|f^{(k)}(0)|}{k!} \sum_{j=0}^{k-1} \| u^j v^{k-1-j} (u - v) \|_{H^{s-1}}
\]

\[
\leq \sum_{k \geq 1} \frac{|f^{(k)}(0)|}{k!} C^{k-1}(\| u \|_{H^{s}} + \| v \|_{H^{s}})^{k-1} \| u - v \|_{H^{s-1}}
\]

\[
\leq G(\| u \|_{H^{s}} + \| v \|_{H^{s}}) \| u - v \|_{H^{s-1}}
\]

with \( G(x) := \sum_{k \geq 1} \frac{|f^{(k)}(0)|}{k!} C^{k-1} x^{k-1} \).

We will frequently use the following lemma, which can be seen as a variant of the integration by parts.

**Lemma 2.4.** Let \( N \in 2^\mathbb{N} \cup \{0\} \). Then,

\[
\left| \int_T \Pi(u, v)wdx \right| \lesssim \| u \|_{L^2} \| v \|_{L^2} \| \partial_x w \|_{L^\infty},
\]

where

\[
\Pi(u, v) := v \partial_x P_N^2 u + u \partial_x P_N^2 v.
\]

(2.9)

Proof. It suffices to show that

\[
\| [\partial_x P_N^2, w] v \|_{L^2} \lesssim \| \partial_x w \|_{L^\infty} \| v \|_{L^2}
\]

(2.10)

since the integration by parts shows

\[
\int_T \Pi(u, v)wdx = \int_T u \partial_x P_N^2 vwdx - \int_T u \partial_x P_N^2 (vw)dx = - \int_T u[\partial_x P_N^2, w]vdx.
\]
The Poisson summation formula and the mean value theorem imply that
\[ |(\partial_x P^2_N w) v(x)| = \left| \int_T \mathcal{F}_x^{-1}(\xi \phi_N^2(\xi))(x)\sum_{k \in \mathbb{Z}} \mathcal{F}_{x,\mathbb{R}}^{-1}(\xi \phi_N^2(\xi))(x - y + 2\pi k)(w(y) - w(x))v(y)dy \right| \]
\[ \leq \left| \int_\mathbb{R} \mathcal{F}_{x,\mathbb{R}}^{-1}(\xi \phi_N^2(\xi))(x - y)(w(y) - w(x))v(y)dy \right| \]
\[ \leq \| \partial_x w \|_{L^\infty}\int_\mathbb{R} \left| (x - y)\mathcal{F}_{x,\mathbb{R}}^{-1}(\xi \phi_N^2(\xi))(x - y)v(y) \right|dy, \]
where \( \mathcal{F}_{x,\mathbb{R}}^{-1} \) is the inverse Fourier transform on \( \mathbb{R} \). A direct calculation gives
\[ \| x\mathcal{F}_{x,\mathbb{R}}^{-1}(\xi \phi_N^2(\xi))(x) \|_{L^1(\mathbb{R})} = \| x\mathcal{F}_{x,\mathbb{R}}^{-1}(\xi \phi_N^2(\xi))(x) \|_{L^1(\mathbb{R})} < \infty. \]
This and the Minkowski inequality show (2.10), which completes the proof. \( \square \)

3. Linear and improved Strichartz estimates

In this section, we establish improved Strichartz estimates which play an important role in our estimates. Since we treat the general operator \( L_{\alpha+1} \), we first have to restate the standard Strichartz estimate.

**Definition 3.** For \( \alpha \in [1, 2] \), we define
\[ \beta(\alpha) := \frac{1}{4(\alpha + 1)}, \quad b(\alpha) := \beta(\alpha) + \frac{1}{4} \left( = \frac{\alpha + 2}{4(\alpha + 1)} \right). \]

We make use of the Strichartz estimate in the Bourgain space, which is originally obtained in [3].

**Lemma 3.1.** There exists \( C > 0 \) such that for any \( v \in X^{0,b(\alpha)} \),
\[ \| v \|_{L^4_{t,x}} \leq C\| v \|_{X^{0,b(\alpha)}}. \]

Lemma 3.1 immediately follows from the following lemma (see the Appendix in [28] for similar considerations).

**Lemma 3.2.** There exists \( C = C(\xi_0) > 0 \) such that for any real-valued functions \( u, v \in L^2(\mathbb{R}; l^2(\mathbb{Z})) \) and any \( L_1, L_2 \in 2^\mathbb{N} \),
\[ \left\| (\psi_{L_1} u) \ast_{\tau,\xi} (\psi_{L_2} v) \right\|_{L^4_{\tau,\xi} L^2_{t,\xi}} \leq C(L_1 \wedge L_2)^{1/2}(L_1 \vee L_2)^{2b(\alpha)}\| \psi_{L_1} u \|_{L^4_{\tau,\xi}}\| \psi_{L_2} v \|_{L^4_{\tau,\xi}}. \]

For the proof of Lemma 3.2 we use the following:
Lemma 3.3. Let $I$ and $J$ be two intervals on the real line and $g \in C^1(J; \mathbb{R})$. Then
\[ \# \{ x \in J \cap \mathbb{Z}; g(x) \in I \} \leq \frac{|I|}{\inf_{x \in J} |g'(x)|} + 1. \]

Proof. See Lemma 2 in [37].

Proof of Lemma 3.2. Following [3], we may assume that $\text{supp}_\xi \psi_{L_1} u, \text{supp}_\xi \psi_{L_2} v \subset \mathbb{N}$ since $u$ and $v$ are real-valued. Following the argument of Lemma 3.1 in [38], the Cauchy-Schwarz inequality gives
\[ \| (\psi_{L_1} u) \ast_{r, \xi} (\psi_{L_2} v) \|_{L^2_\xi}^2 \]
\[ = \sum_{\xi \geq 0} \int \sum_{\xi \geq \xi_1} \int_{\tau_1} \psi_{L_1}(\tau_1, \xi_1) u(\tau_1, \xi_1) \psi_{L_2}(\tau - \tau_1, \xi - \xi_1) v(\tau - \tau_1, \xi - \xi_1) d\tau_1 \]
\[ \lesssim \sup_{(\tau, \xi) \in \mathbb{R} \times \mathbb{N}} A(\tau, \xi) \| \psi_{L_1} u \|_{L^2_\xi}^2 \| \psi_{L_2} v \|_{L^2_\xi}^2, \]
where
\[ A(\tau, \xi) \lesssim \text{mes}\{ (\tau_1, \xi_1) \in \mathbb{R} \times \mathbb{N}; \xi - \xi_1 \geq 0, \langle \tau_1 - p_{\alpha + 1}(\xi_1) \rangle \sim L_1 \]
\[ \text{and } \langle \tau - \tau_1 - p_{\alpha + 1}(\xi - \xi_1) \rangle \sim L_2 \}
\[ \lesssim (L_1 \wedge L_2) \# B(\tau, \xi) \]
with
\[ B(\tau, \xi) = \{ \xi_1 \geq 0; \xi - \xi_1 \geq 0 \text{ and } \langle \tau - p_{\alpha + 1}(\xi_1) - p_{\alpha + 1}(\xi - \xi_1) \rangle \lesssim L_1 \vee L_2 \}. \]

For simplicity, we put $L := L_1 \vee L_2$. When $\xi \leq L^{1/(\alpha + 1)} + 2\xi_0 + 2$, it is clear that
\[ \# B(\tau, \xi) \lesssim L^{1/(\alpha + 1)} + 2\xi_0 + 2 \leq C(\xi_0) L^{1/(\alpha + 1)} \]
since $0 \leq \xi_1 \leq \xi$ and $L \geq 1$. On the other hand, when $\xi \geq L^{1/(\alpha + 1)} + 2\xi_0 + 2$, putting $g(\xi) := \tau - p_{\alpha + 1}(\xi_1) - p_{\alpha + 1}(\xi - \xi_1)$, we obtain
\[ \# B(\tau, \xi) \lesssim \# \{ \xi_1 \geq 0; \xi \geq 2\xi_1 \text{ and } |g(\xi_1)| \lesssim L \}
\[ \leq \lfloor \xi_0 \rfloor + 1 + \# \{ \xi_1 \geq 0; \xi \geq 2\xi_1, \xi_1 \geq \xi_0 \text{ and } |g(\xi_1)| \lesssim L \}
\[ \lesssim \# \{ \xi_1 \in [\xi_0, \xi/2]; |\xi - 2\xi_1|^{\alpha + 1} \geq L \text{ and } |g(\xi_1)| \lesssim L \} + \lfloor \xi_0 \rfloor + L^{1/(\alpha + 1)}, \]
where $[a]$ is the integral part of $a$. Here, we used the symmetry in the first inequality. It is worth noticing that the set in the right hand side is not empty since $(\xi - L^{1/(\alpha + 1)})/2 \geq \xi_0$ by the assumption. Lemma 3.3 implies that
\[ \# \{ \xi_1 \in [\xi_0, \xi/2]; |\xi - 2\xi_1|^{\alpha + 1} \geq L \text{ and } |g(\xi_1)| \lesssim L \} \lesssim L^{1/(\alpha + 1)}. \]
Indeed, we have
\[
|g'(\xi_1)| = \left| \int_{\xi_1}^{\xi_2} p''_{\alpha+1}(\theta)d\theta \right| \sim (\xi_2 - \xi_1)^\alpha - (\xi_1)^\alpha \geq (\xi_2 - 2\xi_1)^\alpha \geq L^{\alpha/(\alpha+1)}
\]
since for \( \theta \geq \xi_1 \), \( p''_{\alpha+1} \) does not change sign since \( |p''_{\alpha+1}(\theta)| \sim |\theta|^\alpha - 1 \) and \( p''_{\alpha+1} \) is continuous outside 0. This completes the proof. \( \square \)

Lemma 3.1 enables to establish the following Strichartz estimate:

**Lemma 3.4.** Let \( T > 0 \). Then there exists \( C > 0 \) such that for any \( u \in L^2(\mathbb{T}) \),
\[
\|U_\alpha(t)u\|_{L^4_T L^4_x} \leq C T^{1/2-b(\alpha)} \|u\|_{L^2_x}.
\]

**Proof.** See Lemma 2.1 in [29]. \( \square \)

We are now ready to prove our improved Strichartz estimates for solutions to (1.3). We point out that it is crucial to state estimates in \( l^4(\mathbb{N}) \)-form since we are not allowed to lose any derivatives in order to reach \( s = s(\alpha) \). For that purpose, how to choose \( c_{j,N} \) plays an important role in the proof below. This type of argument can be found in [Lemma 2.4, 34] for instance.

**Proposition 3.5.** Let \( \delta \geq 1 \), and suppose that the dyadic sequence \{\( \omega_N \)\} of positive numbers satisfies \( \omega_N \leq \omega_{2N} \leq \delta \omega_N \) for \( N \geq 1 \). Let \( s > 1/2 \), \( \alpha \in [1,2] \) and \( 0 < T < 1 \). Let \( u \in C([0,T]; H^s_w(\mathbb{T})) \) satisfy (1.3)–(1.2) with \( u_0 \in H^s_w(\mathbb{T}) \) on \([0,T]\). Then,
\[
\left( \sum_N \omega^4_N \|D_x^{s-\beta(\alpha)} P_N u\|_{L^4_T L^4_x}^4 \right)^{1/4} \lesssim T^{1/8} G(\|u\|_{L^\infty_T L^\infty_x}) \|u\|_{L^\infty_T H^\delta_w} (3.1)
\]
and
\[
\left( \sum_N \|D_x^{1/3} P_N u\|_{L^3_T L^\infty_x}^3 \right)^{1/3} \lesssim T^{5/24} G(\|u\|_{L^\infty_T L^\infty_x}) \|u\|_{L^\infty_T H^{7/12+\beta(\alpha)}_w}, (3.2)
\]
where \( G = G[f] \) is an entire function that is increasing and non negative on \( \mathbb{R}_+ \).

**Proof.** It suffices to consider the case \( N \gg 1 \). We divide the interval in small intervals of length \( \sim TN^{-1} \). In other words, we define \{\( I_{j,N} \)\} for \( J \in \mathbb{N} \) so that \( \bigcup_{j \in \mathbb{N}} I_{j,N} = [0,T], |I_{j,N}| \sim TN^{-1} \) and \#\(J_N \lesssim N \). By the hypothesis, we see that \( \|D_x^{s} P_N u(t)\|_{L^4_x}^4 \in C([0,T]) \). For \( j \in J_N \), we choose \( c_{j,N} \) at which \( D_x^{s} P_N u(t)\|_{L^4_x}^4 \) attains its minimum on \( I_{j,N} \). We see from (1.3)–(1.2) that
\[
P_N u(t) = e^{-(t-c_{j,N}) L_\alpha} P_N u(c_{j,N}) + \int_{c_{j,N}}^{t} e^{-(t-t') L_{\alpha}+1} P_N \partial_x(f(u))(t')dt'.
\]
for $t \in I_{j,N}$. Lemma 3.4 and the Bernstein inequality show that

\[
\left( \sum_{N \gg 1} \sum_{j} \omega_N^4 \left\| D_{x}^{s-\beta(\alpha)} e^{-(t-c_{j,N})} L_{\alpha+1} P_N u(c_{j,N}) \right\|_{L^4(I_{j,N}; L_x^2)} \right)^{1/4} \leq \left( \sum_{N \gg 1} \sum_{j} \omega_N^4 \left\| I_{j,N} \right\|_{L^4(I_{j,N}; L_x^2)} \right)^{1/4} \leq T^{-\beta(\alpha)} \left( \sum_{N \gg 1} \sum_{j} \omega_N^4 \left\| D_x^s P_N u(c_{j,N}) \right\|_{L_x^2} \right)^{1/4} \leq T^{-\beta(\alpha)} \left( \sum_{N \gg 1} \int_0^T \omega_N^4 \left\| D_x^s P_N u(t) \right\|_{L_x^2} dt \right)^{1/4} \lesssim T^{1/4-\beta(\alpha)} \| u \|_{L_x^\infty H_x^s}.
\]

In the last inequality, we used the fact $l^2(\mathbb{N}) \hookrightarrow l^4(\mathbb{N})$. Next, we estimate the contribution of the Duhamel term. To simplify the expressions we set $\tilde{f} = f - f(0)$. Lemma 3.4, the Hölder inequality in time and (2.6) imply that

\[
\left( \sum_{N \gg 1} \sum_{j} \omega_N^4 \left\| \int_{c_{j,N}}^t e^{-(t-t')} L_{\alpha+1} D_x^{s-\beta(\alpha)} P_N \partial_x (\tilde{f}(u)) (t') dt' \right\|_{L^4(I_{j,N}; L_x^2)} \right)^{1/4} \leq \left( \sum_{N \gg 1} \sum_{j} \omega_N^4 \left\| I_{j,N} \right\|_{L^4(I_{j,N}; L_x^2)} \right)^{1/4} \leq T^{-\beta(\alpha)} \left( \sum_{N \gg 1} \sum_{j} \omega_N^4 \int_{I_{j,N}} \left\| D_x^s P_N \tilde{f}(t') \right\|_{L_x^2} dt' \right)^{1/4} \lesssim T^{-1-\beta(\alpha)} \left( \sum_{N \gg 1} \sum_{j} \omega_N^4 \int_{I_{j,N}} \left\| D_x^s P_N \tilde{f}(t') \right\|_{L_x^2} dt' \right)^{1/4} \lesssim T^{-1/2(1-\beta(\alpha))} \| \tilde{f}(u) \|_{L_x^\infty H_x^s} \lesssim T^{1/4+\beta(\alpha)} F(\| u \|_{L_x^\infty}) \| u \|_{L_x^\infty H_x^s}.
\]

This concludes the proof of (3.1) by using that $\beta(\alpha) \leq 1/8$ for $1 \leq \alpha \leq 2$. To prove (3.2), we notice that the Bernstein inequality in space and the Hölder inequality in time give

\[
\left\| D_x^{1/3} P_N u \right\|_{L^4(I_{j,N}; L_x^\infty)} \lesssim \| I_{j,N} \|_{1/2} \left\| D_x^{7/12} P_N u \right\|_{L^4(I_{j,N}; L_x^2)} \lesssim T^{1/12} \left\| D_x^{1/2} P_N u \right\|_{L^4(I_{j,N}; L_x^2)},
\]

where $|I_{j,N}| \sim TN^{-1}$ and $\#J \sim N$. This implies that

\[
\left( \sum_{N \gg 1} \left\| D_x^{1/3} P_N u \right\|_{L_x^3 L^2}^3 \right)^{1/3} \lesssim T^{1/12} \left( \sum_{N \gg 1} \sum_{j} \left\| D_x^{1/2} P_N u \right\|_{L^3(I_{j,N}; L_x^2)}^3 \right)^{1/3}.
\]

We can then obtain (3.2) by the same way as (3.1) with $\omega_N = 1$. \qed
For the estimate for the difference, we shall use the following estimates.

**Corollary 3.6.** Assume that $s > 1/2$. Let $\alpha \in [1, 2]$ and $0 < T < 1$. Let $u, v \in C([0, T]; H^s(\mathbb{T}))$ satisfy (1.3)–(1.2) with $u_0, v_0 \in H^s(\mathbb{T})$ on $[0, T]$, respectively. Then,

$$
\left( \sum_N [(1 \lor N)^{s-1-\beta(\alpha)} \| P_N w \|_{L^4_T L^4_x}]^4 \right)^{1/4} \lesssim T^{1/8} G(K) \| w \|_{L^\infty_T H^s_x}^{-1} \tag{3.3}
$$

and

$$
\left( \sum_N [(1 \lor N)^{-5/12} \| P_N w \|_{L^3_T L^6_x}]^3 \right)^{1/3} \lesssim T^{5/24} G(K) \| w \|_{L^\infty_T H^{s-5/12+\beta(\alpha)}_x}, \tag{3.4}
$$

where $w = u - v$, $K = \| u \|_{L^\infty_T H^s_x} + \| v \|_{L^\infty_T H^s_x}$ and $G = G[f]$ is an entire function that is increasing and non negative on $\mathbb{R}_+$.

**Proof.** The proof is the same as that of Proposition 3.5, but with using (2.8) instead of (2.5). \qed

**4. A priori estimate**

4.1. **Preliminary technical estimates.** Let us denote by $1_T$ the characteristic function of the interval $]0, T[$. As pointed out in [30], $1_T$ does not commute with $Q_L$. To avoid this difficulty, following [30], we further decompose $1_T$ as

$$
1_T = 1_{T,R}^{\text{low}} + 1_{T,R}^{\text{high}}, \quad \text{with} \quad \mathcal{F}_t(1_{T,R}^{\text{low}})(\tau) = \chi(\tau/R) \mathcal{F}_t(1_T)(\tau), \tag{4.1}
$$

for some $R > 0$ to be fixed later.

**Remark 4.1.** In the proofs of Propositions 4.8 and 5.1, since we integrate on an interval of time $]0, t[$ with $0 < t < T$, we will put the cutoff function $1_t$ on two functions of the integral. Actually, it is enough to put $1_t$ on one function if one merely seeks the commutativity of $Q_L$ with the low frequency part of $1_t$ such as in (4.3). The advantage of putting $1_t$ on two functions is that one can get a positive power of $T$ in the right hand side in (4.8) and (5.1), which enables us to bypass the scaling argument in the proof of the well-posedness. This is particularly comfortable here since (1.3) is in general not scaling invariant.

**Lemma 4.1** (Lemma 3.5 in [32]). Let $1 \leq p \leq \infty$. Let $L$ be a nonhomogeneous dyadic number. Then the operator $Q_{\leq L}$ is bounded in $L^p_t L^2_x$ uniformly in $L$. In other words,

$$
\| Q_{\leq L} u \|_{L^p_t L^2_x} \lesssim \| u \|_{L^p_t L^2_x}, \tag{4.2}
$$

for all $u \in L^p_t L^2_x$ and the implicit constant appearing in (4.2) does not depend on $L$. 


Lemma 4.2 (Lemma 3.6 in 32). For any $R > 0$ and $T > 0$, it holds
\[
\| \mathbf{1}_{T,R}^{\text{high}} \|_{L^1} \lesssim T \wedge R^{-1},
\] (4.3)
and
\[
\| \mathbf{1}_{T,R}^{\text{high}} \|_{L^\infty} + \| \mathbf{1}_{T,R}^{\text{low}} \|_{L^\infty} \lesssim 1.
\] (4.4)

Lemma 4.3 (Lemma 3.7 in 32). Assume that $T > 0$, $R > 0$, and $L \gg R$. Then, it holds
\[
\| Q_L(\mathbf{1}_{T,R}^{\text{low}}) \|_{L^2_t} \lesssim \| Q_{\sim L} u \|_{L^2_t},
\] (4.5)
for all $u \in L^2(\mathbb{R}_t \times \mathbb{T}_x)$.

It is well-known that the resonance function ($\Omega_{k+1}$ defined as below) with respect to higher order nonlinear terms such as $\partial_x (u^{k+1})$ can be resonant (i.e., very small). In what follows, we clarify non-resonant contributions in which we can recover the derivative loss by using a priori estimates in Bourgain’s spaces of solution to (1.3) proved in Lemma 4.7.

Definition 4. Let $j \in \mathbb{N}$. We define $\Omega_j(\xi_1, \ldots, \xi_{j+1}) : \mathbb{Z}^{j+1} \to \mathbb{R}$ as
\[
\Omega_j(\xi_1, \ldots, \xi_{j+1}) := \sum_{n=1}^{j+1} p_{\alpha+1}(\xi_n)
\]
for $(\xi_1, \ldots, \xi_{j+1}) \in \mathbb{Z}^{j+1}$, where $p_{\alpha+1}$ satisfies Hypothesis 1.

Lemma 4.4. Let $k \geq 1$ and $(\xi_1, \ldots, \xi_{k+2}) \in \mathbb{Z}^{k+2}$ satisfy $\sum_{j=1}^{k+2} \xi_j = 0$. Assume that $|\xi_1| \sim |\xi_2| \gtrsim |\xi_3|$ if $k = 1$ or $|\xi_1| \sim |\xi_2| \gtrsim |\xi_3| \gg k \max_{j \geq 4} |\xi_j|$ if $k \geq 2$. Then,
\[
|\Omega_{k+1}(\xi_1, \ldots, \xi_{k+2})| \gtrsim |\xi_3| |\xi_1|^\alpha
\]
for $|\xi_1| \gg (\max_{\xi \in [0,\xi_0]} |p'_{\alpha+1}(\xi)|)^{1/\alpha}$.

Proof. First we consider the case $k \geq 2$. We separate different cases:

Case 1: $|\xi_2| \gg |\xi_3|$. By hypothesis, it follows that $|\xi_2| \gg \xi_0$. The mean value theorem implies that there exists $\eta \in \mathbb{R}$ such that $|\eta| \sim |\xi_2|$ and that
\[
|p_{\alpha+1}(\xi_2 + \cdots + \xi_{k+2}) - p_{\alpha+1}(\xi_2)| = |\xi_3 + \cdots + \xi_{k+2}| p'_{\alpha+1}(\xi_2) \sim |\xi_3| |\xi_1|^\alpha.
\]
Here, we used $|\xi_1| \sim |\xi_2|$ and $|\xi_3| \gg k \max_{j \geq 4} |\xi_j|$. Now, if $|\xi_j| \leq \xi_0$ for $j \geq 4$, then
\[
|p_{\alpha+1}(\xi_j)| \leq |\xi_j| \max_{\xi \in [0,\xi_0]} |p'_{\alpha+1}(\xi)| \ll \frac{|\xi_3| |\xi_1|^\alpha}{k}
\]
since \( p_{\alpha+1}(0) = 0 \). On the other hand, if \(|\xi_j| \geq \xi_0 \) for \( j \geq 4 \), then
\[
|p_{\alpha+1}(\xi)| \leq |p_{\alpha+1}(\xi) - p_{\alpha+1}(0)| + |p_{\alpha+1}(0)| \\
\lesssim |\xi_0| \max_{\xi \in [0, \xi_0]} |p'_{\alpha+1}(\xi)| + |\xi|^{\alpha+1} \ll \frac{|\xi_3||\xi_1|^\alpha}{k}.
\]
Similarly, we can get \(|p_{\alpha+1}(\xi_3)| \ll |\xi_3||\xi_1|^\alpha\). Gathering these estimates leads to \(|\Omega_{k+1}| \gtrsim |\xi_3||\xi_1|^\alpha\).

**Case 2:** \(|\xi_2| \sim |\xi_3|\). Then we have \(|\xi_3| \gg \xi_0\). By impossible interactions, \(\xi_1, \xi_2\) and \(\xi_3\) do not have the same sign. By the symmetry and \(|\xi_1| \sim |\xi_2| \sim |\xi_3|\), it suffices to consider the case \(\xi_2, \xi_3 > 0\). We notice that
\[
\Omega_{k+1} = \int_{\xi_0}^{\xi_2} (p'_{\alpha+1}(\theta + \xi_3 + \cdots + \xi_{k+2}) - p'_{\alpha+1}(\theta))d\theta \\
+ p_{\alpha+1}(\xi_0 + \xi_3 + \cdots + \xi_{k+2}) - p_{\alpha+1}(\xi_3) - p_{\alpha+1}(\xi_0) - \sum_{j=4}^{k+2} p_{\alpha+1}(\xi_j)
\]
with
\[
|p_{\alpha+1}(\xi_0 + \xi_3 + \cdots + \xi_{k+2}) - p_{\alpha+1}(\xi_3)| \lesssim (|\xi_0| + k \max_{j \geq 4} |\xi_j|)|\xi_3|^\alpha \ll |\xi_3||\xi_1|^\alpha
\]
and
\[
p''_{\alpha+1}(\theta + \xi_3 + \cdots + \xi_{k+2}) - p'_{\alpha+1}(\theta) = \int_{0}^{\xi_3 + \cdots + \xi_{k+2}} p''_{\alpha+1}(\theta + \mu)d\mu.
\]
For \(\theta \geq |\xi_0, \xi_2|\), we get
\[
\int_{0}^{\xi_3 + \cdots + \xi_{k+2}} p''_{\alpha+1}(\theta + \mu)d\mu \sim \int_{0}^{\xi_3 + \cdots + \xi_{k+2}} (\theta + \mu)^{\alpha-1}d\mu \sim \xi_3^\alpha
\]
since \(\xi_3 \gg k \max_{j \geq 4} |\xi_4|\). Gathering these estimates, we obtain \(|\Omega_{k+1}| \gtrsim |\xi_3||\xi_1|^\alpha\).

For the case \(k = 1\), we can argue exactly as above. \(\square\)

**Lemma 4.5.** Let \(k \geq 2\) and \((\xi_1, \ldots, \xi_{k+2}) \in \mathbb{Z}^{k+2}\) satisfy \(\sum_{j=1}^{k+2} \xi_j = 0\). Assume that \(|\xi_1| \sim |\xi_2| \gg |\xi_3| \gtrsim |\xi_4|\) if \(k = 2\) or \(|\xi_1| \sim |\xi_2| \gg |\xi_3| \gtrsim |\xi_4|\) with \(|\xi_3 + \xi_4| \gg k \max_{j \geq 5} |\xi_j|\) if \(k \geq 3\). Then,
\[
|\Omega_{k+1}(\xi_1, \ldots, \xi_{k+2})| \gtrsim |\xi_3 + \xi_4||\xi_1|^\alpha
\]
for \(|\xi_1| \gg (\max_{\xi \in [0, \xi_0]} |p'_{\alpha+1}(\xi)|)^{1/\alpha}\).

**Proof.** First we consider the case \(k \geq 3\). We separate different cases:

**Case 1:** \(|\xi_3| \gg |\xi_4|\). Then we have \(|\xi_3| \gg k \max_{j \geq 5} |\xi_j|\) and \(|\xi_3| \sim |\xi_3 + \xi_4|\). Therefore, we can argue exactly as in Case 1 of Lemma 4.4 and obtain \(|\Omega_{k+1}| \gtrsim |\xi_3||\xi_1|^\alpha \sim |\xi_3 + \xi_4||\xi_1|^\alpha\).
Case 2: $|\xi_3| \sim |\xi_4|$. When $\xi_3 \xi_4 \geq 0$, it holds that $|\xi_3 + \xi_4| = |\xi_3| + |\xi_4|$. Then we have $|\xi_3|, |\xi_4| \gg k \max_{j \geq 5} |\xi_j|$. So, we can still argue as in Case 1 of Lemma 4.4 and obtain $|\Omega_{k+1}| \gtrsim |\xi_3 + \xi_4||\xi_1^\alpha|$. When $\xi_3 \xi_4 < 0$, by the mean value theorem, there exist $\eta_1, \eta_2 \in \mathbb{R}$ such that $|\eta_1| \sim |\xi_1|$, $|\xi_4| \lesssim |\eta_2| \lesssim |\xi_3|$ and

$$-\Omega_{k+1} = (\xi_3 + \xi_4 + \cdots + \xi_{k+2})p_{\alpha+1}'(\eta_1) - (\xi_3 + \xi_4)p_{\alpha+1}'(\eta_2) - \sum_{j=5}^{k+2} p_{\alpha+1}(\xi_j)$$

since $p_{\alpha+1}(\xi) = -p_{\alpha+1}(-\xi)$ for any $\xi \in \mathbb{R}$. Note that $|p_{\alpha+1}'(\eta_2)| \ll |\xi_1|^\alpha$. Indeed, $|p_{\alpha+1}'(\eta_2)| \leq \max_{\xi \in [0, \xi_0]} |p_{\alpha+1}'(\xi)| \ll |\xi_1|^\alpha$ when $|\eta_2| \leq \xi_0$, and $|p_{\alpha+1}'(\eta_2)| \sim |\eta_2|^\alpha \lesssim |\xi_3|^\alpha \ll |\xi_1|^\alpha$ when $|\eta_2| \geq \xi_0$. As in Case 1 of Lemma 4.4, we also have $k \max_{j \geq 5} |p_{\alpha+1}(\xi_j)| \ll |\xi_3 + \xi_4||\xi_1^\alpha|$. From these estimates, we obtain $|\Omega_{k+1}| \gtrsim |\xi_3 + \xi_4||\xi_1^\alpha|$. For the case $k = 2$, we can argue exactly as above.

Lemma 4.6. Let $\delta > 1$, and suppose that the dyadic sequence $\{\omega_N\}$ of positive numbers satisfies $\omega_N \leq \omega_{2N} \leq \delta \omega_N$ for $N \geq 1$ and $\omega_N \to \infty$ as $N \to \infty$. Then, for any $1 < \delta' < \delta$, there exists a dyadic sequence $\{\tilde{\omega}_N\}$ such that $\tilde{\omega}_N \leq \omega_N$, $\tilde{\omega}_N \leq \omega_{2N} \leq \delta' \tilde{\omega}_N$ for $N \geq 1$ and $\tilde{\omega}_N \to \infty$ as $N \to \infty$.

Proof. Let $1 < \delta' < \delta$. Set $\delta := \omega_{2N}/\omega_N$ for $N \geq 1$. Define $\{\tilde{\omega}_N\}$ so that $\tilde{\omega}_0 := \omega_0$, $\tilde{\omega}_1 := \omega_1$ and $\tilde{\omega}_{2N} := (\delta \wedge \delta')\tilde{\omega}_N$ for $N \geq 1$. Since $\delta, \delta' \geq 1$, it is clear that $\tilde{\omega}_N \leq \omega_{2N} \leq \delta' \omega_N$. We can show $\tilde{\omega}_N \leq \omega_N$ by the induction on $N$. Indeed, $\tilde{\omega}_0 = \omega_0$, $\tilde{\omega}_1 = \omega_1$ and

$$\tilde{\omega}_{2N} = (\delta \wedge \delta')\tilde{\omega}_N \leq \delta \omega_N \leq \delta' \omega_N \omega_N = \omega_{2N}$$

for $N \geq 1$. Now, we show $\tilde{\omega}_N \to \infty$ as $N \to \infty$.

Case 1: $\#\{N; \delta N > \delta'\} < \infty$. Put $N_0 := 1 \vee \max\{N; \delta N > \delta'\}$ and $C := \omega_{2N_0}/\omega_{2N_0}$. Then, we can deduce inductively that $\tilde{\omega}_N = C \omega_N$ for $N \geq 2N_0$, which shows $\tilde{\omega}_N \to \infty$ as $N \to \infty$ by the hypothesis.

Case 2: $\#\{N; \delta N > \delta'\} = \infty$. Define an increasing sequence $\{j_n\}_{n \geq 2} \subset \mathbb{N}$ so that $j_n := \#\{m \in \mathbb{N}; 1 \leq m \leq n - 1, \delta_{2m} > \delta'\}$ for $n \geq 2$. Then, we have $j_n \to \infty$ as $n \to \infty$. Observe that

$$\tilde{\omega}_{2n} = \prod_{m=0}^{n-1} (\delta_{2m} \wedge \delta')\tilde{\omega}_1 \geq (\delta')^{j_n} \omega_1 \to \infty$$

as $n \to \infty$ since $\delta' > 1$, which completes the proof.

Remark 4.2. For the given dyadic sequence $\{\omega_N\}$ of positive numbers, Lemma 4.6 enables us to assume $\delta \leq 2$, by defining a new dyadic sequence. We use this fact in
the proof of Proposition 4.8. Let $N, M$ be dyadic numbers satisfying $lN \geq M \geq 1$ for some $l \geq 2$. Using $\omega_{2N} \leq \delta \omega_N$, it holds

$$\frac{\omega_M}{\omega_N} \lesssim \delta^{\log_2 l} \lesssim l$$

which is uniformly in $\delta$.

4.2. Estimates on solutions to (1.3).

**Lemma 4.7.** Let $1 \leq \delta \leq 2$, and suppose that the dyadic sequence $\{\omega_n\}$ of positive numbers satisfies $\omega_n \leq \omega_{2n} \leq \delta \omega_n$ for $n \geq 1$. Let $0 < T < 1$, $s > 1/2$ and $u \in L^\infty \omega T^s H^s$ be a solution to (1.3) associated with an initial datum $u_0 \in H^s_\omega(\mathbb{T})$. Then $u \in Z^s_{\omega,T}$ and it holds

$$\|u\|_{Z^s_{\omega,T}} \lesssim \|u\|_{L^\infty T^s H^s} + G(\|u\|_{L^\infty T^s}) \|u\|_{L^\infty T^s H^s}.$$  \hspace{0.5cm} \text{(4.6)}

Moreover, for any couple $(u,v) \in (L^\infty T^s H^s)^2$ of solutions to (1.3) associated with a couple of initial data $(u_0,v_0) \in (H^s(\mathbb{T}))^2$ it holds

$$\|u - v\|_{Z^s_{\omega,T}} \lesssim \|u - v\|_{L^\infty T^s H^s} + G(\|u\|_{L^\infty T^s} + \|v\|_{L^\infty T^s}) \|u - v\|_{L^\infty T^s H^s}.$$  \hspace{0.5cm} \text{(4.7)}

**Proof.** According to the extension Lemma 2.1, it is clear that we only have to estimate the $X^s_{\omega,T}$-norm of $u$ to prove (4.6). As noticed in Remark 1.4, $u$ satisfies the Duhamel formula of (1.3) and $\|u_0\|_{H^s_\theta} \leq \|u\|_{L^\infty T^s H^s}$ for any $\theta \leq s$. Hence, standard linear estimates in Bourgain's spaces lead to

$$\|u\|_{X^s_{\omega,T}} \lesssim \|u_0\|_{H^s_\omega} + \|\partial_x (f(u))\|_{X^s_{\omega,T}} \|u\|_{L^\infty T^s} \|u\|_{L^\infty T^s H^s} \lesssim \|u\|_{L^\infty T^s H^s} + G(\|u\|_{L^\infty T^s}) \|u\|_{L^\infty T^s H^s},$$

by using (2.6).

In the same way, using this times (2.8), we get

$$\|u - v\|_{X^s_{\omega,T}} \lesssim \|u_0 - v_0\|_{H^s_\omega} + \|f(u) - f(v)\|_{L^2 T^s H^s} \lesssim \|u - v\|_{L^\infty T^s H^s} + G(\|u\|_{L^\infty T^s} + \|v\|_{L^\infty T^s}) \|u - v\|_{L^\infty T^s H^s},$$

which completes the proof. \hfill $\Box$

The following proposition is one of main estimates in the present paper.

**Proposition 4.8 (A priori estimate).** Let $\delta \geq 1$, and suppose that the dyadic sequence $\{\omega_n\}$ of positive numbers satisfies $\omega_n \leq \omega_{2n} \leq \delta \omega_n$ for $n \geq 1$. Let $0 < T < 1$, $s \in [1,2]$ and $2 \geq s \geq s(\alpha) := 1/2 + 2\beta(\alpha)$. Let $u_0 \in H^s(\mathbb{T})$ and
let $u \in L^\infty(0,T; H^s(\mathbb{T}))$ be a solution to \((1.3)-(1.2)\) on $[0,T]$. Then there exists an entire function $G = G[f]$ that is increasing and non-negative on $\mathbb{R}_+$ such that

$$
\|u\|_{L^\infty_T H^s_x}^2 \leq \|u_0\|_{H^s_x}^2 + T^{1/4}G(\|u\|_{Z^s_\omega})\|u\|_{Z^s_\omega} \|u\|_{L^\infty_T H^s_x}.
$$

(4.8)

Proof. First we notice that according to Lemma 4.7 it holds $u \in Z^s_\omega$. By using (1.3), we have

$$
\frac{d}{dt}\|P_N u(t,\cdot)\|_{L^2_x}^2 = -2 \int T P_N \partial_x(f(u)) P_N u dx.
$$

Fixing $t \in ]0,T[$ , integration in time between 0 and $t$, multiplication by $\omega_N^2(1 \vee N)^{2s}$ and summation over $N$ yield

$$
\|u(t)\|_{H^s_x}^2 = \sum_{N} \omega_N^2(1 \vee N)^{2s}\left\{\|P_N u_0\|_{L^2_x}^2 - 2 \int_0^t \int_T P_N \partial_x(f(u)) P_N u dx dt'\right\}
$$

$$
\leq \|u_0\|_{H^s_x}^2 + 2 \sum_{N} \omega_N^2(1 \vee N)^{2s}\left|\int_0^t \int_T P_N \partial_x(f(u) - f(0)) P_N u dx dt'\right|
$$

(4.9)

$$
\leq \|u_0\|_{H^s_x}^2 + 2 \sum_{N \geq 1} \omega_N^2 N^{2s}\left|\int_0^t \int_T (f(u) - f(0)) P_N^2 \partial_x u dx dt'\right|
$$

since $P_0 \partial_x u = 0$ and $\partial_x(f(u)) = \partial_x(f(u) - f(0))$. Now we rewrite $f(u) - f(0)$ as $\sum_{k \geq 1} \frac{f^{(k)}(0)}{k!} u^k$ and we notice that for any fixed $N \in 2\mathbb{N}$,

$$
\int_0^t \int_T (f(u) - f(0)) P_N^2 \partial_x u dx dt' = \sum_{k \geq 1} \frac{f^{(k)}(0)}{k!} \int_0^t \int_T u^k P_N^2 \partial_x u dx dt'.
$$

(4.10)

Indeed

$$
\sum_{k \geq 1} \frac{|f^{(k)}(0)|}{k!} \int_0^t \int_T u^k P_N^2 \partial_x u dx dt' \lesssim N \sum_{k \geq 1} \frac{|f^{(k)}(0)|}{k!} \int_0^t \|u^k\|_{L^2_x} \|u\|_{L^2_x} dx dt'
$$

$$
\lesssim N \sum_{k \geq 1} \frac{|f^{(k)}(0)|}{k!} \int_0^t \|u\|_{L^\infty_x} \|u\|_{L^2_x} \|u\|_{L^2_x} dx dt'
$$

$$
\lesssim NTG(\|u\|_{L^\infty_x}) \|u\|_{L^2_x} \|u\|_{L^2_x} < \infty,
$$

that proves (4.10) by Fubini-Lebesgue’s theorem. (4.10) together with Fubini-Tonelli’s theorem then ensure that

$$
\sum_{N \geq 1} \omega_N^2 N^{2s}\left|\int_0^t \int_T (f(u) - f(0)) P_N^2 \partial_x u dx dt'\right|
$$

$$
= \sum_{N \geq 1} \omega_N^2 N^{2s}\left|\sum_{k \geq 1} \frac{f^{(k)}(0)}{k!} \int_0^t \int_T u^k P_N^2 \partial_x u dx dt'\right|
$$

$$
\leq \sum_{N \geq 1} \sum_{k \geq 1} \omega_N^2 N^{2s}\left|\frac{f^{(k)}(0)}{k!} \int_0^t \int_T u^k P_N^2 \partial_x u dx dt'\right|
$$

(4.11)

$$
\leq \sum_{k \geq 1} \sum_{N \geq 1} \omega_N^2 N^{2s}\left|\frac{f^{(k)}(0)}{k!} \right| \int_0^t \int_T u^k P_N^2 \partial_x u dx dt' = \sum_{k \geq 1} \left|\frac{f^{(k)}(0)}{k!}\right| I_k,
$$
where

\[ I_k := \sum_{N \geq 1} \omega_N^2 N^{2s} \left| \int_0^t \int_T u^k P_N^2 \partial_x u \, dx \, dt' \right|. \]

By integration by parts it is easy to check that \( I_1 = 0 \). We set

\[ C_0 := \| u \|_{Z_T^{s(a)}}. \]

Let us now prove that for any \( k \geq 1 \) it holds

\[ I_k + 1 \leq C_k T^{1/4} G(C_0) C_0^k \left( \| u \|_{X_T^{s - 1, 1}} + \| u \|_{L_T^{\infty} H^s_x} \right) \| u \|_{L_T^{\infty} H^s_x}. \] (4.12)

that clearly leads (4.8) by taking (4.9) and (4.11) into account since

\[ \sum_{k \geq 1} \frac{|I_k(0)|}{(k + 1)!} C_k C_0^k < \infty. \]

In the sequel we fix \( k \geq 1 \). For simplicity, for any positive numbers \( a \) and \( b \), the notation \( a \lesssim_k b \) means there exists a positive constant \( C > 0 \) independent of \( k \) such that

\[ a \leq C^k b. \] (4.13)

Remark that \( a \leq k^m b \) for \( m \in \mathbb{N} \) can be expressed by \( a \lesssim_k b \) too since an elementary calculation shows \( k^m \leq m! e^k \) for \( m \in \mathbb{N} \). Here, \( e \) is Napier’s constant. The contribution of the sum over \( N \lesssim 1 \) in \( I_{k+1} \) is easily estimated by

\[ \sum_{N \lesssim 1} \omega_N^2 N^{2s} \left| \int_0^t \int_T u^{k+1} P_N^2 \partial_x u \, dx \, dt' \right| \leq T \sum_{N \lesssim 1} \| u \|_{L_T^{\infty} L_x^\infty} \| u \|_{L_T^{\infty} L_x^2} \| P_N^2 u \|_{L_T^{\infty} L_x^2} \lesssim C_0 \| u \|_{L_T^{\infty} H^s_x}^2. \] (4.14)

It thus remains to bound the contribution of the sum over \( N \gg 1 \) in \( I_{k+1} \). Putting

\[ A(\xi_1, \ldots, \xi_{k+2}) := \sum_{j=1}^{k+2} \phi_N^2(\xi_j) \xi_j, \]
\[ A_1(\xi_1, \xi_2) := \phi_N^2(\xi_1) \xi_1 + \phi_N^2(\xi_2) \xi_2, \]
\[ A_2(\xi_4, \ldots, \xi_{k+2}) := \sum_{j=4}^{k+2} \phi_N^2(\xi_j) \xi_j, \]
we see from the symmetry that

\[
\int_{\mathbb{T}} u^{k+1} P_N^2 \partial_x u \, dx = \frac{i}{k+2} \sum_{\xi_1 + \cdots + \xi_{k+2} = 0} A(\xi_1, \ldots, \xi_{k+2}) \prod_{j=1}^{k+2} \hat{u}(\xi_j)
\]

\[
= \frac{i}{k+2} \sum_{N_1, \ldots, N_{k+2}, \xi_1 + \cdots + \xi_{k+2} = 0} A(\xi_1, \ldots, \xi_{k+2}) \prod_{j=1}^{k+2} \phi_{N_j}(\xi_j) \hat{u}(\xi_j).
\]

(4.15)

By symmetry we can assume that \(N_1 \geq N_2 \geq N_3\) if \(k = 1\), \(N_1 \geq N_2 \geq N_3 \geq N_4\) if \(k = 2\) and \(N_1 \geq N_2 \geq N_3 \geq N_4 \geq N_5 = \max_{j \geq 5} N_j\) if \(k \geq 3\). We notice that the cost of this choice is a constant factor less than \((k+2)^4\). It is also worth noticing that the frequency projection operator \(P_N\) ensures that the contribution of any \(N_1 \leq N/4\) does cancel. We thus can assume that \(N_1 \geq N/4\) and that \(N_2 \geq N_1/k\) with \(N_2 \geq 1\).

First, we consider the contribution of \(A_2\). Note that we must have \(k \geq 2\) since otherwise \(A_2 = 0\). Also it suffices to consider the contribution of \((\phi_N(\xi_4))^2 \xi_4\) since the contributions of \((\phi_N(\xi_j))^2 \xi_j\) for \(j \geq 5\) are clearly simplest. Note that \(N_4 \sim N\) in this case. By the Bernstein inequality, we have

\[
\sum_K \| P_K u \|_{L^\infty_{T,x}} \lesssim \sum_K (1 + K^{1/2-s(\alpha)}) \| u \|_{L_T^\infty H_x^{s(\alpha)}} \lesssim \| u \|_{L_T^\infty H_x^{s(\alpha)}} \lesssim C_0.
\]

(4.16)
This together with Hölder’s and Young’s inequalities and (3.1) gives

\[
\sum_{N \gg 1} \sum_{N_1, \ldots, N_{k+2}} \omega^2_N N^{2s} \left| \int_0^t \int_T (\partial_x P_N^2 P_{N_4} u) \prod_{j=1, j \neq 4}^{k+2} P_{N_j} u dx dt' \right|
\]

\[
\lesssim_k \|u\|_{L^2_{T \rightarrow H^s_x}} \sum_{N_1 \geq N_2 \geq N_3 \geq N_4 \gg 1} \omega^2_{N_4} N^{2s+1} \prod_{j=1}^4 \|P_{N_j} u\|_{L^4_{T,x}}
\]

\[
\lesssim_k C_0^{k-2} \sum_{N_1 \geq N_2 \geq N_3 \geq N_4 \gg 1} \prod_{j=3}^4 \left( \frac{N_4}{N_j} \right)^{1/2+\beta(\alpha)} \|D_x^{1/2+\beta(\alpha)} P_{N_j} u\|_{L^4_{T,x}}
\]

\[
\lesssim_k C_0^{k-2} \left( \sum_{N_4} \|D_x^{1/2+\beta(\alpha)} P_{N_4} u\|_{L^4_{T,x}} \right)^{1/4}
\]

\[
\times \left( \sum_{N_4} \left( \sum_{N_3 \geq N_4} \left( \frac{N_4}{N_3} \right)^{1/2+\beta(\alpha)} \|D_x^{1/2+\beta(\alpha)} P_{N_3} u\|_{L^4_{T,x}} \right) \right)^{1/4}
\]

\[
\times \left( \sum_{N_4} \left( \sum_{K \geq N_4} \left( \frac{N_4}{K} \right)^{2(\beta(\alpha))} \omega^2_K \|D_x^{\beta(\alpha)} P_K u\|_{L^4_{T,x}} \right) \right)^{1/2}
\]

\[
\lesssim_k T^{1/2} C_0^k G(C_0) \|u\|_{L^2_{T \rightarrow H^s_x}}
\]

Next, we consider the contribution $A_1$. We notice that the frequency projector in $A_1$ ensures that either $N_1 \sim N$ or $N_2 \sim N$ and thus in any case $N \gtrsim N_3$. Moreover we can also assume that $N_3 \geq 1$ since otherwise the contribution of $A_1$ cancelled by integration by parts.

We divide the contribution $A_1$ into three cases: 1. $N_2 \lesssim N_3 \lesssim kN_4$, 2. $N_3 \gg kN_4$ or $k = 1$ and 3. $N_2 \gg N_3$. Set

\[
J_t := \sum_{N \gg 1} \sum_{N_1, \ldots, N_{k+2}} \omega^2_N N^{2s} \left| \int_0^t \int_T \Pi(P_{N_1} u, P_{N_2} u) \prod_{j=3}^{k+2} P_{N_j} u dx dt' \right|
\]

where $\Pi(f, g)$ is defined by (2.9). Note that $N \gg 1$ ensures that $N_1 \gg 1$.

**Case 1:** $N_2 \lesssim N_3 \lesssim kN_4$. Since $N \lesssim N_1 \lesssim kN_2 \lesssim kN_3 \lesssim k^2 N_4$, Hölder’s,
Bernstein’s and Young’s inequalities and (3.1) show that

\[ J_t \lesssim k^{2(2s+1)} \sum_{N_1, \ldots, N_k+2 \leq N_4} \omega_{N_1}^{2} N_4^{2s+1} \prod_{j=1}^{4} \| P_{N_j} u \|_{L_t^{4}} \prod_{j=5}^{k+2} \| P_{N_j} u \|_{L_t^{\infty}} \]

\[ \lesssim k \left( \frac{\omega_{N_1}}{\omega_{N_2}} \right) \sum_{N_1 \leq N_4, N_2 \geq N_4, N_3 \geq N_4} \frac{\omega_{N_1} \omega_{N_2}}{\omega_{N_2}} \left( \frac{N_4}{N_1} \right)^{s-\beta(\alpha)} \left( \frac{N_4}{N_2} \right)^{s-\beta(\alpha)} \left( \frac{N_4}{N_3} \right)^{\beta(\alpha)+1/2} \]

\[ \times \prod_{j=1}^{4} \left\| D_x^{s-\beta(\alpha)} P_{N_j} u \right\|_{L_t^{4}} \prod_{n=3}^{4} \left\| D_x^{\beta(\alpha)+1/2} P_{N_n} u \right\|_{L_t^{\infty}} \]

\[ \lesssim k \left( \sum_{\mathcal{K}} \omega_{\mathcal{K}}^{4} \left\| D_x^{s-\beta(\alpha)} P_{\mathcal{K}} u \right\|_{L_t^{4}}^{4} \right)^{1/2} \left( \sum_{\mathcal{K}} \left\| D_x^{\beta(\alpha)+1/2} P_{\mathcal{K}} u \right\|_{L_t^{\infty}}^{4} \right)^{1/2} \]

\[ \lesssim k \left( T^{1/2} C_0 G(C_0) \right) \left\| u \right\|_{L_t^{\infty} H_x^{\delta} }^{2} , \]

where we used that \( s \leq 2 \) so that \( k^{2(2s+1)} \leq k^{10} \) and that since \( N_1 \leq k N_2 \) we have \( \frac{\omega_{N_1}}{\omega_{N_2}} \leq k \) since \( \delta \leq 2 \).

**Case 2:** \( N_3 \gg k N_4 \) or \( k = 1 \). By impossible frequency interactions, we must have \( N \sim N_1 N_2 \). We take the extensions \( \tilde{u} = \rho_T(u) \) of \( u \) defined in (2.1). For simplicity, with a slight abuse of notation, we define the following functional:

\[ j_{\infty,1}^{(2)}(u_1, \ldots, u_{k+2}) := \sum_{N \geq 1} \sum_{N_1, \ldots, N_{k+2}} \omega_{N}^{2} N^{2s} \left\| \int_{\mathbb{R}} \int_{\mathbb{T}} \Pi(u_1, u_2) \prod_{j=3}^{k+2} u_j dx dt \right\| . \]

Setting \( R = N_1^{1/3} N_3^{4/3} \), we split \( J_t \) as

\[ J_t \leq J_{\infty}^{(2)}(P_{N_1}^{-1} \mathbb{1}_{t,R}^{\text{high}} \tilde{u}, P_{N_2}^{-1} \mathbb{1}_{t,R} \tilde{u}, P_{N_3}^{-1} \tilde{u}, \ldots, P_{N_{k+2}}^{-1} \tilde{u}) \]

\[ + J_{\infty}^{(2)}(P_{N_1}^{-1} \mathbb{1}_{t,R}^{\text{low}} \tilde{u}, P_{N_2}^{-1} \mathbb{1}_{t,R}^{\text{high}} \tilde{u}, P_{N_3}^{-1} \tilde{u}, \ldots, P_{N_{k+2}}^{-1} \tilde{u}) \]

\[ + J_{\infty}^{(2)}(P_{N_1}^{-1} \mathbb{1}_{t,R}^{\text{low}} \tilde{u}, P_{N_2}^{-1} \mathbb{1}_{t,R}^{\text{low}} \tilde{u}, P_{N_3}^{-1} \tilde{u}, \ldots, P_{N_{k+2}}^{-1} \tilde{u}) =: J_{\infty,1}^{(2)} + J_{\infty,2}^{(2)} + J_{\infty,3}^{(2)} . \]

For \( J_{\infty,1}^{(2)} \), we see from (4.3) that \( \| \mathbb{1}_{t,R}^{\text{high}} \|_{L^1} \lesssim T^{1/4} N_1^{-1/4} N_3^{-1} \), which gives

\[ J_{\infty,1}^{(2)} \lesssim \sum_{N_1, \ldots, N_{k+2}} \omega_{N_1}^{2} N^{2s} \| \mathbb{1}_{t,R}^{\text{high}} \|_{L_1}^{1} \| P_{N_1} \tilde{u} \|_{L_t^{\infty} L_x^{2}} \| P_{N_2} \tilde{u} \|_{L_t^{\infty} L_x^{2}} \| P_{N_3} \tilde{u} \|_{L_t^{\infty} L_x^{2}} \prod_{j=3}^{k+2} \| P_{N_j} \tilde{u} \|_{L_t^{\infty}} \]

\[ \lesssim k T^{1/4} \| u \|_{L_t^{\infty} H_x^{\delta} }^{k} \| u \|_{L_t^{\infty} H_x^{\delta} }^{2} \sum_{N_1} N_1^{-1/4} \lesssim k T^{1/4} C_0^{k} \| u \|_{L_t^{\infty} H_x^{\delta} }^{2} \]

since \( N \sim N_1 N_2 \). In the last inequality, we used (2.4). By (4.4), \( J_{\infty,2}^{(2)} \) can be estimated by the same bound as above. For \( J_{\infty,3}^{(2)} \), we see from Lemma 4.3 that
\[ |{\Omega}_{k+1}| \geq N_3 N_1^\alpha \gg R. \] Then, defining \( L := N_3 N_1^\alpha \), we decompose \( J_{\infty,3}^{(2)} \) as
\[
J_{\infty,3}^{(2)} \leq J_{\infty}^{(2)}(P_{N_1} Q_{\geq L}(1_{t,R} \tilde{u}), P_{N_2} 1_{t,R} \tilde{u}, P_{N_3} \tilde{u}, \ldots, P_{N_{k+2}} \tilde{u})
+ \sum_{j=4}^{k+2} J_{\infty}^{(2)}(P_{N_1} Q_{\leq L}(1_{t,R} \tilde{u}), P_{N_j} \tilde{u}, \ldots, P_{N_{k+2}} \tilde{u})
+ T^{1/2} \| (1_{t,R} \tilde{u}) \|_{L_t^\infty L_x^2} \leq T^{1/2} \| (1_{t,R} \tilde{u}) \|_{L_t^\infty L_x^2}.
\]
For \( J_{\infty,3,1}^{(2)} \), Lemmas 2.4 and 2.1 the Hölder inequality, (4.3) and (4.18) imply that
\[
J_{\infty,3,1}^{(2)} \lesssim \sum_{k=3}^{k+2} \| P_{N_j} \tilde{u} \|_{L_t^\infty L_x^2} \| P_{N_1} Q_{\geq L}(1_{t,R} \tilde{u}) \|_{L_t^2 L_x^1} \| P_{N_2} Q_{\leq L}(1_{t,R} \tilde{u}) \|_{L_t^2 L_x^2} \| P_{N_3} \tilde{u} \|_{L_t^\infty L_x^2} \lesssim T^{1/4} C_0^k \| \tilde{u} \|_{L_t^\infty H_x^1} \| \tilde{u} \|_{X_1^{s-1.1}} \lesssim_T T^{1/4} C_0^k \| \tilde{u} \|_{L_t^\infty H_x^1} \| \tilde{u} \|_{L_t^\infty L_x^2}.
\]
Here, we used \( N_1^{-\alpha} \leq N_1^{-1} \) since \( \alpha \in [1, 2] \). We can evaluate the contribution \( J_{\infty,3,2}^{(2)} \) by the same way with (4.18). Next, we consider the contribution \( J_{\infty,3,3}^{(2)} \). Lemmas 2.4 and 2.1 the Hölder inequality and (4.2) show
\[
J_{\infty,3,3}^{(2)} \lesssim \sum_{k=4}^{k+2} \| P_{N_1} Q_{\geq L}(1_{t,R} \tilde{u}) \|_{L_t^2 L_x^1} \| P_{N_2} Q_{\leq L}(1_{t,R} \tilde{u}) \|_{L_t^2 L_x^2} \times \| P_{N_3} \tilde{u} \|_{L_t^\infty L_x^2} \lesssim T^{1/2} \| (1_{t,R} \tilde{u}) \|_{L_t^\infty L_x^2} \lesssim_T T^{1/2} C_0^k \| \tilde{u} \|_{L_t^\infty H_x^1} \| \tilde{u} \|_{L_t^\infty L_x^2}.
\]
since \( s(\alpha) - 1 < 0 \). In a similar manner, we can evaluate the contribution \( J_{\infty,3,j}^{(2)} \) for \( j = 4, \ldots, k+2 \) by the same bound.
Case 3: $N_2 \gg N_3$. In this case, roughly speaking, we compare $|\xi_3 + \xi_4|$ and $|k|\xi_5|$ where $\xi_i$ is the $i$-th largest frequency. If $|\xi_3 + \xi_4| \gg k|\xi_5|$, we have a suitable non-resonance relation (Lemma 4.5) whereas otherwise we can share the lost derivative between three functions. We split $J_t$ as

$$J_t \leq \sum_{N \geq 1} \sum_{N_1, \ldots, N_{k+2}, kN_5 \leq M \leq N_3} \omega_N^2 N^{2s} \left| \int_0^t \int_\mathbb{T} \Pi(P_{N_1}u, P_{N_2}u) P_{\leq kN_3} \left( \prod_{j=3}^{k+2} P_{N_j} u \right) dx dt' \right|$$

$$+ \sum_{N \geq 1} \sum_{N_1, \ldots, N_{k+2}, kN_5 \leq M \leq N_3} \omega_N^2 N^{2s} \left| \int_0^t \int_\mathbb{T} \Pi(P_{N_1}u, P_{N_2}u) P_{M} \left( \prod_{j=3}^{k+2} P_{N_j} u \right) dx dt' \right|$$

$$=: I_t^{(1)} + J_t^{(1)}.$$

Remark that if $k = 2$, then the term $I_t^{(1)}$ does not appear, and it holds that $J_t \leq J_t^{(1)}$ with the summation $\sum_{M \leq N_3}$ instead of $\sum_{kN_5 \leq M \leq N_3}$. Note also that the contribution in respectively $I_t^{(1)}$ and $J_t^{(1)}$ of respectively $N_5 = 0$ and $N_5 = M = 0$ does vanish by integration by parts. Therefore we can always assume that $N_5 \geq 1$ in $I_t^{(1)}$ and that $M \geq 1$ in $J_t^{(1)}$. For $I_t^{(1)}$, since either $N_1 \sim N$ or $N_2 \sim N$, we see from the Young inequality and the assumption that $\delta \leq 2$ that

$$\sum_{N_1 \geq N_2, kN_2 \geq N_1} (\omega_{N_1}^2 N_1^{2s} + \omega_{N_2}^2 N_2^{2s}) \| P_{N_1} u \|_{L^s_x} \| P_{N_2} u \|_{L^s_x}$$

$$\leq k^s \sum_{kN_2 \geq N_1} \omega_{N_1} \omega_{N_2} \left( \frac{N_1}{kN_2} \right)^s \| D_x^s P_{N_1} u \|_{L^s_x} \| D_x^s P_{N_2} u \|_{L^s_x}$$

$$+ \sum_{N_2 \leq N_1} \omega_{N_1} \omega_{N_2} \left( \frac{N_2}{N_1} \right) \| D_x^s P_{N_1} u \|_{L^s_x} \| D_x^s P_{N_2} u \|_{L^s_x} \lesssim k^3 \| u \|_{H^3_x}.$$

This, Lemma 2.4 Hölder’s and Young’s inequalities and (3.2) show

$$I_t^{(1)} \lesssim \sum_{N_1 \geq N_2, kN_2 \geq N_1} \sum_{N_3, \ldots, N_{k+2}} (\omega_{N_1}^2 N_1^{2s} + \omega_{N_2}^2 N_2^{2s}) kN_5$$

$$\times \int_0^t \| P_{N_1} u \|_{L^s_x} \| P_{N_2} u \|_{L^s_x} \prod_{j=3}^{k+2} \| P_{N_j} u \|_{L^s_x} dt'$$

$$\lesssim k \| u \|_{L^{s(\alpha)} \cap H^3} \| u \|_{L^{s(\alpha)} \cap H^3} \sum_{N_1 \geq N_4 \geq N_5} \prod_{j=3}^5 \left( \frac{N_5}{N_j} \right)^{1/3} \| D_x^{1/3} P_{N_j} u \|_{L^{s(\alpha)} \cap H^3}$$

$$\lesssim k C_0^{k-3} \| u \|_{L^{s(\alpha)} \cap H^3} \sum_K \| D_x^{1/3} P_K u \|_{L^{s(\alpha)} \cap H^3} \lesssim k \delta^{5/8} C_0^k G(C_0) \| u \|_{L^{s(\alpha)} \cap H^3}$$

since $s(\alpha) = 1/2 + 2\beta(\alpha) \geq 7/12 + \beta(\alpha)$ for $\alpha \in [1/2]$. For $J_t^{(1)}$, we take the extensions $\tilde{u} = \rho_T(u)$ of $u$ defined in (2.1). Note that we have $N_1 \sim N_2 \sim N$. We
further decompose $J_t^{(3)}$ as in Case 2. In the same spirit as (4.17), with a slight abuse of notation, we define the functional for the sake of notation:

$$J_{\infty}^{(3)}(u_1, \ldots, u_{k+2}) := \sum_{N \geq 1} \sum_{N_1, \ldots, N_{k+2} \leq N} \sum_{k \leq M \leq N_3} \omega^2 N^{2s} \left| \int_{T} \Pi(u_1, u_2) P_M \left( \prod_{j=3}^{k+2} u_j \right) dx dt \right|.$$ 

Putting $R = N_1^{1/3} M^{4/3}$, we obtain

$$J_{\infty}^{(3)} \leq J_{\infty}^{(3)}(P_{N_1} 1_{t>R} \tilde{u}, P_{N_2} 1_{t>R} \tilde{u}, P_{N_3} \tilde{u}, \ldots, P_{N_{k+2}} \tilde{u})$$

$$+ J_{\infty}^{(3)}(P_{N_1} 1_{t>R} \tilde{u}, P_{N_2} 1_{t>R} \tilde{u}, P_{N_3} \tilde{u}, \ldots, P_{N_{k+2}} \tilde{u})$$

$$+ J_{\infty}^{(3)}(P_{N_1} 1_{t>R} \tilde{u}, P_{N_2} 1_{t>R} \tilde{u}, P_{N_3} \tilde{u}, \ldots, P_{N_{k+2}} \tilde{u}) =: J_{\infty, 1}^{(3)} + J_{\infty, 2}^{(3)} + J_{\infty, 3}^{(3)}.$$

Obviously, we have for any $\varepsilon > 0$ (for instance, $\varepsilon = \beta(\alpha)$)

$$\sum_{M \leq N_3} 1 \lesssim N_3^\varepsilon, \quad (4.19)$$

where the implicit constant does not depend on $N_3$. By (4.19), (2.24) and the same argument as that of $J_{\infty, 1}^{(2)}$, we obtain

$$J_{\infty, 1}^{(3)} + J_{\infty, 2}^{(3)} \lesssim k T^{1/4} \|\tilde{u}\|^{k} L_t^\infty H_x^{s(\alpha)} \|\tilde{u}\|^{2} L_t^\infty H_x^{s} \lesssim k T^{1/4} C_0^k \|\tilde{u}\|^{2} L_t^\infty H_x^{s}.$$

For $J_{\infty, 3}^{(3)}$, Lemma 4.3 implies that $|\Omega_{k+1}| \gtrsim MN_1^{\alpha} \sim |\xi_3 + \xi_4| N_1^{\alpha}$. So, defining $L := MN_1^{\alpha}$, we have

$$J_{\infty, 3}^{(3)} \leq J_{\infty}^{(3)}(P_{N_1} Q_{\geq L}(1_{t>R} \tilde{u}), P_{N_2} Q_{\geq L}(1_{t>R} \tilde{u}), P_{N_3} \tilde{u}, \ldots, P_{N_{k+2}} \tilde{u})$$

$$+ J_{\infty}^{(3)}(P_{N_1} Q_{\leq L}(1_{t>R} \tilde{u}), P_{N_2} Q_{\leq L}(1_{t>R} \tilde{u}), P_{N_3} \tilde{u}, \ldots, P_{N_{k+2}} \tilde{u})$$

$$+ J_{\infty}^{(3)}(P_{N_1} Q_{\leq L}(1_{t>R} \tilde{u}), P_{N_2} Q_{\leq L}(1_{t>R} \tilde{u}), P_{N_3} Q_{\leq L} \tilde{u}, \ldots, P_{N_{k+2}} \tilde{u})$$

$$+ \cdots + J_{\infty}^{(3)}(P_{N_1} Q_{\leq L}(1_{t>R} \tilde{u}), P_{N_2} Q_{\leq L}(1_{t>R} \tilde{u}), P_{N_3} Q_{\leq L} \tilde{u}, \ldots, P_{N_{k+2}} Q_{\geq L} \tilde{u})$$

$$=: J_{\infty, 3, 1}^{(3)} + \cdots + J_{\infty, 3, k+2}^{(3)}.$$

It is worth noting that $R \ll L = MN_1^{\alpha}$ since $N_1 \gg 1$. For $J_{\infty, 3, 1}^{(3)}$, we use the argument of $J_{\infty, 3, 1}^{(2)}$. Lemmas 2.3 and 2.4 the Hölder inequality, (4.15), (4.18) and
Let $\text{Proposition 5.1}$. Let $J^{(3)}_{\infty,3,1} \lesssim \sum_{N_1,\ldots,N_k+2} \sum_{M \leq N_3} \omega_{N_1}^2 N_1^{2s} \|P_{N_1} Q_{\leq L} (1^3_t R \tilde{u}))\|_{L^2_t L^2_x} \|P_{N_2} 1_t^\omega \tilde{u}\|_{L^2_t L^2_x} \prod_{j=3}^{k+2} \|P_{N_j} \tilde{u}\|_{L^\infty_t L^\infty_x} \lesssim_k \sum_{N_1,\ldots,N_k+2} \omega_{N_1}^2 N_1^{2s-\alpha} \|P_{N_1} \tilde{u}\|_{X^{0,1}} \|P_{\sim N_1} 1_t \tilde{u}\|_{L^2_t L^2_x} \prod_{j=3}^{k+2} \|P_{N_j} \tilde{u}\|_{L^\infty_t L^\infty_x} + T^{1/4} C_0^k \sum_{N_1 \leq N_3} \omega_{N_1}^2 N_1^{2s-\alpha-1/12} M^{-1/3} \|P_{N_1} \tilde{u}\|_{X^{0,1}} \|P_{\sim N_1} \tilde{u}\|_{L^\infty_t L^\infty_x} \lesssim_k T^{1/4} C_0^k \|\tilde{u}\|_{L^\infty_t H^2_x} \|\tilde{u}\|_{X^{s-1,1} \to X^{s-1,1}} \lesssim T^{1/4} C_0^k \|u\|_{L^\infty_t H^\alpha_x} \|u\|_{Z^\alpha, T}^2$.

We can estimate the contribution $J^{(3)}_{\infty,3,2}$ by the same way. For the contribution $J^{(3)}_{\infty,3,3}$, similarly to $J^{(2)}_{\infty,3,3}$, we see from (4.2), (4.19) and Lemma 2.1 that

$$J^{(3)}_{\infty,3,3} \lesssim_k T^{1/2} \|\tilde{u}\|_{L^\infty_t H^{\alpha}_x} \sum_{N_1 \geq N_3} \omega_{N_1}^2 N_1^{2s-\alpha} N_3^{\beta(\alpha)} \|P_{N_1} \tilde{u}\|_{L^\infty_t L^\infty_x} \|D_x^{1/2} P_{N_1} \tilde{u}\|_{X^{0,1}} \lesssim T^{1/2} C_0^k \|u\|_{L^\infty_t H^\alpha_x} \|u\|_{Z^\alpha, T}^2$$

In a similar manner, we can estimate the contribution $J^{(3)}_{\infty,3,j}$ for $j = 4, \ldots, k+2$ by the same bound.

Finally, we consider the contribution of $\phi^2_{N_1}(\xi_3) \xi_3$. We may assume that $N_3 \gg k N_4$. Otherwise the proof is the same as the contribution of $A_2$. When $N_3 \gg k N_4$, we can obtain the desired estimate as in Case 2. This completes the proof.

5. Estimate for the Difference

We provide the estimate (at the regularity $s-1$) for the difference $w$ of two solutions $u, v$ of (1.3). In this section, we do not use the frequency envelope, so we always argue on the standard Sobolev space $H^s(\mathbb{T})$.

**Proposition 5.1.** Let $0 < T < 1$, $\alpha \in [1,2]$ and $2 \geq s \geq s(\alpha) := 1/2 + 2\beta(\alpha)$. Let $u$ and $v$ be two solutions of (1.3) belonging to $Z^s_T$ associated with the initial data $u_0 \in H^s(\mathbb{T})$ and $v_0 \in H^s(\mathbb{T})$, respectively. Then there exists an entire function $G = G[f]$ that is increasing and non-negative on $\mathbb{R}_+$ such that

$$\|w\|_{L^\infty_t H^{s-1}_x} \leq \|u_0 - v_0\|_{H^{s-1}_x} + T^{1/4} G(\|u\|_{Z^s_T} + \|v\|_{Z^s_T}) \|w\|_{Z^{s-1}_T} \|w\|_{L^\infty_t H^{s-1}_x},$$

where we set $w = u - v$.

**Proof.** According to Lemma 4.17, we notice that $u, v \in Z^s_T$. Observe that $w$ satisfies

$$\partial_t w + L_{\alpha+1} w = -\partial_x (f(u) - f(v))$$

(5.2)
Rewriting $f(u) - f(v)$ as

$$f(u) - f(v) = \sum_{k \geq 1} \frac{f^{(k)}(0)}{k!} (u^k - v^k) = \sum_{k \geq 1} \frac{f^{(k)}(0)}{k!} w \sum_{i=0}^{k-1} u^i v^{k-1-i}$$

and arguing as in the proof of Proposition 4.8, we see from (5.2) that for $t \in [0, T]$ we have

$$\|w(t)\|^2_{H_t^{s-1}} \leq \|w_0\|^2_{H_t^{s-1}} + 2 \sum_{k \geq 1} \frac{|f^{(k)}(0)|}{(k-1)!} \max_{i \in \{0, ..., k-1\}} I_{k,i}^t,$$

where $w_0 = u_0 - v_0$ and

$$I_{k,i}^t := \sum_{N \geq 1} N^{2(s-1)} \left| \int_0^t \int_T u^i v^{k-1-i} w P_N^2 \partial_x w \, dx \, dt \right|.$$

It is clear that $I_{1,0}^t = 0$ by the integration by parts. Therefore we are reduced to estimate the contribution of

$$I_{k+1}^t = \sum_{N \geq 1} N^{2(s-1)} \left| \int_0^t \int_T \sum_{i=0}^{k} z^k w P_N^2 \partial_x w \, dx \, dt \right|$$

where $z^k$ stands for $u^i v^{k-i}$ for some $i \in \{0, ..., k\}$. We set

$$C_0 := \|u\|_{Z_t^1} + \|v\|_{Z_t^1}.$$

We claim that for any $k \geq 1$ it holds

$$I_{k+1}^t \leq C^k T^{1/4} C_0^k G(C_0) \|w\|_{Z_t^1} \|w\|_{L_T^\infty H_t^{s-1}}.$$  \hspace{1cm} (5.4)

that clearly leads (5.1) by taking (4.9) and (4.11) into account since $\sum_{k \geq 1} \frac{|f^{(k+1)}(0)|}{k!} C^k C_0^k < \infty$.

In the sequel we fix $k \geq 1$ and we estimate $I_k^t$. We also use the notation $a \lesssim_k b$ defined in (4.13). The contribution of the sum over $N \leq 1$ in (5.3) is easily estimated thanks to (2.5) by

$$\sum_{N \leq 1} (1 \lor N)^{2(s-1)} \left| \int_0^t \int_T \sum_{i=0}^{k} z^k w P_N^2 \partial_x w \, dx \, dt \right| \lesssim T \sum_{N \leq 1} \|w\|_{L_T^\infty H_t^{s-1}} \|z^k P_N^2 \partial_x w\|_{L_T^\infty H_t^{1-s}} \lesssim_k T C_0^k \|w\|_{L_T^\infty H_t^{s-1}}^2.$$  \hspace{1cm} (5.5)

In the last inequality, we used $1 - s < 1/2$. Therefore, in what follows, we can assume that $N \gg 1$. A similar argument to (4.15) yields

$$\sum_{N \gg 1} N^{2(s-1)} \left| \int_0^t \int_T \sum_{i=0}^{k} z^k w P_N^2 \partial_x w \, dx \, dt \right| \leq \sum_{N \gg 1} \sum_{N_1, ..., N_{k+2}} N^{2(s-1)} \left| \int_0^t \int_T \Pi(P_{N_1} w, P_{N_2} w) \prod_{j=3}^{k+2} P_{N_j} \partial_{x_j} w \, dx \, dt \right| =: J_t,$$
where $\Pi(f, g)$ is defined by \[ \text{(2.3)} \] and $z_i \in \{u, v\}$ for $i \in \{3, \ldots, k + 2\}$. By symmetry, we may assume that $N_1 \geq N_2$. Moreover, we may assume that $N_3 \geq N_4$ for $k = 2$ and $N_3 \geq N_4 \geq N_5 = \max_{j \geq 5} N_j$ for $k \geq 3$. Note again that the cost of this choice is a constant factor less than $(k + 2)^4$. It is also worth noticing that the frequency projectors in $\Pi(\cdot, \cdot)$ ensure that $N_1 \sim N$ or $N_2 \sim N$ and in particular $N_1 \gtrsim N$. We also remark that we can assume that $N_3 \geq 1$ since the contribution of $N_3 = 0$ does vanish by integration by parts. Finally we note that we can also assume that $N_2 \geq 1$ since in the case $N_2 = 0$ we must have $N_3 \gtrsim N_1/k$ and it is easy to check that

$$J_t \lesssim kT \|w\|_{L^\infty_t H^{s-1}_x} \|z_3\|_{L^\infty_t H^{s}_x} \|P_0 w\|_{L^\infty_T L^4_{x,t}} \prod_{j=4}^{k+2} \|z_j\|_{L^\infty_T H^{s}_x} \lesssim k \|w\|^2_{L^\infty_t H^{s-1}_x}.$$

We consider the contribution of $J_t$, dividing it into three cases:

- $N_1 \lesssim kN_4$ ($k \geq 2$),
- $N_1 \gg kN_4$ and $N_2 \gtrsim N_3$ (or $k = 1$ and $N_2 \gtrsim N_3$),
- $N_1 \gg kN_4$ and $N_2 \ll N_3$ (or $k = 1$ and $N_2 \ll N_3$).

**Case 1: $N_1 \lesssim kN_4$ ($k \geq 2$).**

Hölder’s and Young’s inequalities, \[ \text{(4.16)}, \ (3.1) \text{ and } (3.3) \] imply that

$$J_t \lesssim \sum_{N_1, \ldots, N_{k+2}} (N_1^{2s-1} + N_2^{2s-1}) \prod_{j=1}^2 \|P_{N_j} w\|_{L^4_{T,x}} \prod_{n=3}^4 \|P_{N_n} z_n\|_{L^4_{T,x}} \prod_{j=5}^{k+2} \|P_{N_j} z_j\|_{L^\infty_{T,x}}$$

$$\lesssim k C^{k-2}_0 \sum_{N_3 \geq N_4 \gtrsim k^{-1} N_1 \gtrsim k^{-1} N_2} N_1^{2s-1} \|P_{N_1} w\|_{L^4_{T,x}} \|P_{N_2} w\|_{L^4_{T,x}} \|P_{N_3} z_3\|_{L^4_{T,x}} \|P_{N_4} z_4\|_{L^4_{T,x}}$$

$$\lesssim k C^{k-2}_0 \sum_{N_3 \geq N_4 \gtrsim k^{-1} N_1 \gtrsim k^{-1} N_2} k^{2s-2\beta(\alpha)} \left( \frac{N_1}{k N_4} \right)^{2s-\beta(\alpha)} \left( \frac{N_2}{k N_4} \right)^{2s-\beta(\alpha)/2} \left( \frac{N_3}{N_4} \right)^{s-\beta(\alpha)} \left( \frac{N_4}{N_3} \right)^{1/2-\beta(\alpha)}$$

$$\times \prod_{j=1}^2 \|D_x^{-1/2+\beta(\alpha)} P_{N_j} w\|_{L^4_{T,x}} \prod_{n=3}^4 \|D_x^{-\beta(\alpha)} P_{N_n} z_n\|_{L^4_{T,x}}$$

$$\lesssim k T^{1/2} C^k_0 G(C_0) \|w\|^2_{L^\infty_t H^{s(\alpha)-1}}$$

since $2s - \beta(\alpha) - 1/2 > 0$ and $s - \beta(\alpha) > 1/2 - \beta(\alpha) > 0$. Note that in the last step we also used that $-1/2 + \beta(\alpha) = s(\alpha) - 1 - \beta(\alpha)$ since $s(\alpha) = 1/2 + 2\beta(\alpha)$.

**Case 2: $N_1 \gg kN_4$ and $N_2 \gtrsim N_3$ (or $k = 1$ and $N_2 \gtrsim N_3$).**

In this case, the contribution of $J_t$ in this case can be estimated by the same way as the contribution of $A_1$ in Proposition \[ \text{A.8} \] replacing $N^{2s}$, $P_{N_1} u$, $P_{N_2} u$, $P_{N_3} u$ for $j = 3, \ldots, k + 2$ by $N^{2(s-1)}$, $P_{N_1} u$, $P_{N_2} u$, $P_{N_3} z_j$ for $j = 3, \ldots, k + 2$, respectively.

**Case 3: $N_1 \gg kN_4$ and $N_2 \ll N_3$ (or $k = 1$ and $N_2 \ll N_3$).**
Note that in this case $N_1 \sim N_3 \sim N \gg N_2 \vee N_4$. We further divide the contribution of $J_t$ into three cases:

- $N_2 \gg k N_4$ or $k = 1$,
- $k N_4 \gtrsim N_2 \gtrsim N_5$ (or $k = 2$ and $N_4 \gtrsim N_2$),
- $N_2 \ll N_5$ ($k \geq 3$).

Subcase 3.1: $N_2 \gg k N_4$ or $k = 1$. Recall that we have $N_2 \geq 1$ and thus this subcase contains the case $N_4 = 0$. We take the extensions $\hat{w} = \rho_T(w)$ (resp. $\hat{z}_j = \rho_T(z_j)$ for $j \geq 3$) of $w$ (resp. $z_j$ for $j \geq 3$) defined in (2.1). For simplicity, with a slight abuse of notation, we shall use the following notation:

$$J_{\infty}^{(3,1)}(u_1, \ldots, u_{k+2}) := \sum_{N \geq 1} \sum_{N_1, \ldots, N_{k+2}} N^{2(s-1)} \left| \int_R \int_T \Pi(u_1, u_2) \prod_{j=3}^{k+2} u_j dx dt \right|.$$  

Setting $R = N_1^{1/3} N_2^{2/3}$, we divide $J_t$ as

$$J_t \leq J_{\infty}^{(3,1)}(P_{N_1} 1_{t,R}^{\text{high}} \hat{w}, P_{N_2} \hat{w}, P_{N_3} 1_{t,R}^{\text{high}} \hat{z}_3, P_{N_4} \hat{z}_4, \ldots, P_{N_{k+2}} \hat{z}_{k+2})$$
$$+ J_{\infty}^{(3,1)}(P_{N_1} 1_{t,R}^{\text{low}} \hat{w}, P_{N_2} \hat{w}, P_{N_3} 1_{t,R}^{\text{high}} \hat{z}_3, P_{N_4} \hat{z}_4, \ldots, P_{N_{k+2}} \hat{z}_{k+2})$$
$$+ J_{\infty}^{(3,1)}(P_{N_1} 1_{t,R}^{\text{low}} \hat{w}, P_{N_2} \hat{w}, P_{N_3} 1_{t,R}^{\text{low}} \hat{z}_3, P_{N_4} \hat{z}_4, \ldots, P_{N_{k+2}} \hat{z}_{k+2})$$
$$=: J_{\infty,1}^{(3,1)} + J_{\infty,2}^{(3,1)} + J_{\infty,3}^{(3,1)}.
$$

For $J_{\infty,1}^{(3,1)}$, we see from (4.3) that $\|1_{t,R}^{\text{high}}\|_{L^1} \lesssim T^{1/4}N_1^{-1/4}N_2^{-1}$, which gives

$$J_{\infty,1}^{(3,1)} \lesssim \sum_{N_1, \ldots, N_{k+2}} N_1^{2s-1} \|1_{t,R}^{\text{high}}\|_{L^1} \|P_{N_1} \hat{w}\|_{L^\infty_t L^2_x} \|P_{N_2} \hat{w}\|_{L^\infty_t L^2_x} \|P_{N_3} \hat{z}_3\|_{L^\infty_t L^2_x} \prod_{j=4}^{k+2} \|P_{N_j} \hat{z}_j\|_{L^\infty_t L^2_x} \lesssim_k T^{1/4} C_0^{k-1} \sum_{N_1, N_2} N_1^{2s-5/4} N_2^{2s-1} \|P_{N_1} \hat{w}\|_{L^\infty_t L^2_x} \|P_{N_2} \hat{w}\|_{L^\infty_t L^2_x} \|P_{N_3} \hat{z}_3\|_{L^\infty_t L^2_x} \lesssim_k T^{1/4} C_0^{k} \|\hat{w}\|_{L^\infty_t H^{s-1}}^2 \lesssim_k T^{1/4} C_0^{k} \|w\|_{L^\infty_t H^{s-1}}^2.$$

Here, in the first inequality, we used $N_2^{2s-1} \leq N_2^{2s-1}$ since $s \geq s(\alpha) > 1/2$. In the last inequality, we also used (2.4). In a similar manner, we can also estimate $J_{\infty,2}^{(3,1)}$ by the same bound as that of $J_{\infty,1}^{(3,1)}$. For $J_{\infty,3}^{(3,1)}$, we note that $|\Omega_{k+1}| \gtrsim N_2 N_1^\alpha =: L$ by Lemma 4.2. This enables us to decompose $J_{\infty,3}^{(3,1)}$ as

$$J_{\infty,3}^{(3,1)} \leq J_{\infty}^{(3,1)}(P_{N_1} Q_{L} 1_{t,R}^{\text{low}} \hat{w}, P_{N_2} \hat{w}, P_{N_3} 1_{t,R}^{\text{low}} \hat{z}_3, \ldots, P_{N_{k+2}} \hat{z}_{k+2})$$
$$+ J_{\infty}^{(3,1)}(P_{N_1} Q_{L} 1_{t,R}^{\text{low}} \hat{w}, P_{N_2} \hat{w}, P_{N_3} Q_{L} 1_{t,R}^{\text{low}} \hat{z}_3, \ldots, P_{N_{k+2}} \hat{z}_{k+2})$$
$$+ J_{\infty}^{(3,1)}(P_{N_1} Q_{L} 1_{t,R}^{\text{low}} \hat{w}, P_{N_2} Q_{L} \hat{w}, P_{N_3} Q_{L} 1_{t,R}^{\text{low}} \hat{z}_3, \ldots, P_{N_{k+2}} \hat{z}_{k+2}) + \cdots$$
$$J_{\infty}^{(3,1)}(P_{N_1} Q_{L} 1_{t,R}^{\text{low}} \hat{w}, P_{N_2} Q_{L} \hat{w}, P_{N_3} Q_{L} \hat{w}, P_{N_4} Q_{L} 1_{t,R}^{\text{low}} \hat{z}_3, \ldots, P_{N_{k+2}} Q_{L} \hat{z}_{k+2})$$
$$=: J_{\infty,3,1}^{(3,1)} + \cdots + J_{\infty,3,k+2}^{(3,1)}.
where $J_{\infty,3,n}^{(3,1)}$ for $4 \leq n \leq k + 2$ corresponds to the term in which $Q_{\geq L}$ lands on $P_{N_n \tilde{z}_n}$. It is worth noting that $R \ll L$. Then, the Hölder inequality, (4.5), (4.18) and Lemma 2.1 imply that

\[
J_{\infty,3,1}^{(3,1)} \lesssim \sum_{N_1, \ldots, N_{k+2}} N_1^{2s-1} \left\| P_{N_1} Q_{\geq L} 1_{t,R}^{\text{low}} \right\|_{L^1_t L^2_x} \left\| P_{N_2} \tilde{w} \right\|_{L^\infty_t L^2_x} \left\| P_{N_3} 1_{t,R}^{\text{low}} \tilde{z}_3 \right\|_{L^2_t L^2_x} \prod_{j=4}^{k+2} \left\| P_{N_j} \tilde{z}_j \right\|_{L^\infty_t L^2_x}.
\]

\[
\lesssim_k C_0^{k-1} \sum_{N_1, N_2} N_1^{2s-1-\alpha} \left\| P_{N_1} \tilde{w} \right\|_{X^{0,1}} \left\| P_{N_2} \tilde{w} \right\|_{L^{\infty_t} H_x^{-1/2}} \left\| P_{N_3} 1_{t,R}^{\text{low}} \tilde{z}_3 \right\|_{L^2_t L^2_x}.
\]

By the same way, we can obtain

\[
J_{\infty,3,2}^{(3,1)} \lesssim_k T^{1/4} C_0^{k-1} \left\| \tilde{w} \right\|_{L^\infty_t H_x^{-1}} \left\| \tilde{z}_3 \right\|_{X^{s-1,1}} \lesssim_k T^{1/4} C_0^k \left\| w \right\|_{Z_t^{s-1}} \left\| w \right\|_{L^\infty_t H_x^{s-1}}.
\]

Next, we consider the contribution of $J_{\infty,3,3}^{(3,1)}$. Lemma 2.1, the Hölder inequality and (4.2) show

\[
J_{\infty,3,3}^{(3,1)} \lesssim \sum_{N_1, \ldots, N_{k+2}} N_1^{2s-1} \left\| P_{N_1} 1_{t,R}^{\text{low}} \tilde{w} \right\|_{L^1_t L^2_x} \left\| P_{N_2} Q_{\geq L} \tilde{w} \right\|_{L^\infty_t L^2_x} \left\| P_{N_3} \tilde{z}_3 \right\|_{L^2_t L^2_x} \prod_{j=4}^{k+2} \left\| P_{N_j} \tilde{z}_j \right\|_{L^\infty_t L^2_x}.
\]

\[
\lesssim_k T^{1/2} C_0^{k-1} \sum_{N_1, N_2} N_1^{2s-1-\alpha} N_2^{-1} \left\| P_{N_1} \tilde{w} \right\|_{L^{\infty_t} L^2_x} \left\| D_x^{1/2} P_{N_2} \tilde{w} \right\|_{X^{0,1}} \left\| P_{N_3} \tilde{z}_3 \right\|_{L^\infty_t L^2_x}.
\]

\[
\lesssim_k T^{1/2} C_0^{k-1} \left\| \tilde{w} \right\|_{L^{\infty_t} H_x^{-1}} \left\| \tilde{z}_3 \right\|_{L^{\infty_t} H_x^s} \sum_{N_1, N_2} N_1^{-\alpha} N_2^{-1} \left\| P_{N_2} \tilde{w} \right\|_{X^{1/2,1}}.
\]

\[
\lesssim_k T^{1/2} C_0^k \left\| \tilde{w} \right\|_{X^{s-2,1}} \left\| \tilde{w} \right\|_{L^{\infty_t} H_x^{s-1}} \lesssim T^{1/2} C_0^k \left\| w \right\|_{Z_t^{s-1}} \left\| w \right\|_{L^\infty_t H_x^{s-1}}.
\]
since $s \geq s(\alpha) > 1/2$. Similarly, we obtain

$$J_{\infty,3}^{(3,1)}$$

$$\leq k T^{1/2} C_{0}^{k-2} \sum_{N_{1},N_{2},N_{4}} N_{1}^{2s-1} \left\| P_{N_{1}} \mathbf{1}_{l} \tilde{u} \right\|_{L_{x,t}^{\infty}} \left\| P_{N_{2}} \tilde{u} \right\|_{L_{x,t}^{\infty}} \left\| P_{N_{2}} \tilde{z}_{3} \right\|_{L_{x}^{\infty} L_{t}^{2}} \left\| P_{N_{4}} Q_{\geq l} \tilde{z}_{4} \right\|_{L_{x}^{2} L_{t}^{\infty}}$$

$$\leq k T^{1/2} C_{0}^{k-2} \sum_{N_{1},N_{2},N_{4}} N_{1}^{2s-\alpha} N_{2}^{-1} \left\| P_{N_{1}} \tilde{u} \right\|_{L_{x,t}^{\infty} L_{t}^{2}} \left\| P_{N_{2}} \tilde{u} \right\|_{L_{x,t}^{\infty} H_{x}^{1/2}}$$

$$\times \left\| P_{N_{4}} \tilde{z}_{4} \right\|_{L_{x}^{\infty} L_{t}^{2}} \left\| P_{N_{4}} \tilde{z}_{3} \right\|_{L_{x}^{\infty} H_{x}^{1/2}} \sum_{N_{1} \geq N_{4}} N_{1}^{-\alpha} \left\| P_{N_{4}} \tilde{z}_{4} \right\|_{X^{1/2,1}}$$

$$\lesssim T^{1/2} C_{0}^{k-1} \left\| \tilde{z}_{4} \right\|_{X^{s(\alpha)-1,1}}$$

$$\lesssim T^{1/2} C_{0}^{k-1} \left\| \tilde{z}_{4} \right\|_{X^{s(\alpha)-1,1}} \lesssim T^{1/2} C_{0}^{k} \left\| w \right\|_{L_{x,t}^{\infty} H_{x}^{1/2}}^{2}$$

We see from the above argument that $J_{\infty,3}^{(3,1)}$ for $5 \leq j \leq k + 2$ can be estimated by the same bound as that of $J_{\infty,3}^{(3,1)}$.

Subcase 3.2: $N_{5} \leq N_{2} \leq k N_{4}$ (or $k = 2$ and $N_{4} \geq N_{2}$). Note that the cases $N_{2} = 0$ or $N_{4} = 0$ have been already treated so that we can assume that $N_{2} \geq 1$ and $N_{4} \geq 1$. It suffices to consider the case $N_{5} \leq N_{2} \leq N_{4}$ since $k \geq 1$. First we set

$$J_{t} \leq \sum_{N \geq 1} \sum_{N_{1}, \ldots, N_{k+2}} N^{2(s-1)} \left| \int_{0}^{t} \int_{t} \partial_{t} \partial_{x} \sum_{N_{1}, \ldots, N_{k+2}} N^{2(s-1)} \left| \int_{0}^{t} \int_{T} \partial_{t} \partial_{x} \sum_{N_{1}, \ldots, N_{k+2}} N^{2(s-1)} \left| \int_{0}^{t} \int_{T} \partial_{t} \partial_{x} \sum_{N_{1}, \ldots, N_{k+2}} N^{2(s-1)} \left| \int_{0}^{t} \int_{T} \partial_{t} \partial_{x} \sum_{N_{1}, \ldots, N_{k+2}} N^{2(s-1)} \right| \right| \right| \right| \right| \right|$$

For $J_{t}^{(3,2)}$, we further divide it as in Case 3 in Proposition 4.8.

$$J_{t}^{(3,2)}$$

$$\leq \sum_{N \geq 1} \sum_{N_{1}, \ldots, N_{k+2}} N^{2(s-1)} \left| \int_{0}^{t} \int_{T} \partial_{t} \partial_{x} \sum_{N_{1}, \ldots, N_{k+2}} N^{2(s-1)} \left| \int_{0}^{t} \int_{T} \partial_{t} \partial_{x} \sum_{N_{1}, \ldots, N_{k+2}} N^{2(s-1)} \right| \right| \right| \right| \right|$$

where $\sum$ denotes $\sum_{N \geq 1} \sum_{N_{1}, \ldots, N_{k+2}}$ for the sake of simplicity. Remark that if $k = 2$, then the term $J_{t,1}^{(3,2)}$ does not appear, and it holds that $J_{t}^{(3,2)} \leq J_{t,2}^{(3,2)}$ with the summation $\sum_{M \leq N_{4}}$ instead of $\sum_{k N_{5} \leq 1 \leq M \leq N_{4}}$. For $J_{t,1}^{(3,2)}$, Hölder’s, Bernstein’s and
Minkowski’s inequalities, (3.2) and (3.4) give

\[ I_{t,1}^{(3,2)} \lesssim \sum_{N_1,\ldots,N_k} N_1^{2s-1} (k^{3/4} N_5^{3/4} \vee 1) \int_0^t \| P_{N_1} w \|_{L^2_t} \| P_{N_3} z_3 \|_{L^2_x} \| P_{N_2} w \|_{L^4_x} \times \| P_{N_4} z_4 \|_{L^1_t} \| P_{N_5} z_5 \|_{L^1_x} \prod_{j=6}^{k+2} \| P_{N_j} z_j \|_{L^\infty} dt' \]

\[ \lesssim_k C_0^{k-3} \| w \|_{L_T^\infty H_x^{s-1}} \| z_3 \|_{L_T^\infty H_x^s} \sum_{N_4 \geq N_5 \geq N_6} \left( \frac{N_2}{N_4} \right)^{5/12} \left( \frac{N_5 \vee 1}{N_4} \right)^{1/6} \times \| D_x^{-5/12} P_{N_2} w \|_{L_T^2 L_x^2} \| D_x^{7/12} P_{N_4} z_4 \|_{L_T^1 L_x^3} \| D_x^{7/12} P_{N_5} z_5 \|_{L_T^1 L_x^3} \lesssim_k T^{5/8} C_0^{k-2} G(C_0) \| w \|_{L_T^\infty H_x^{s-1}} \| w \|_{L_T^\infty H_x^s} \prod_{i=1}^5 \| z_i \|_{L_T^\infty H_x^{s+1}}^2 \lesssim_k T^{5/8} C_0^k G(C_0) \| w \|_{L_T^\infty H_x^{s-1}}^2 \]

since \( s(\alpha) \geq 7/12 + \beta(\alpha) \). In order to estimate \( I_{t,2}^{(3,2)} \), we further decompose it. We take the extensions \( \hat{w} = \rho_T(w) \) (resp. \( \hat{z}_j = \rho_T(z_j) \) for \( j \geq 3 \)) of \( w \) (resp. \( z_j \) for \( j \geq 3 \)) defined in (2.11). As in Subcase 3.1, with a slight abuse of notation, we shall use the follow notation:

\[ I_{\infty,2}^{(3,2)} (u_1, \ldots, u_{k+2}) := \sum_{N \gg 1} \sum_{N_1, \ldots, N_k+2} \sum_{N_5 \vee 1 \leq M \leq N_4} \sum_{j=4}^{k+2} N_2^{2(s-1)} \left| \int_R \int_T (\partial_x P_N^2 u_1) u_3 P_M \left( u_2 \prod_{j=4}^{k+2} u_j \right) dxdt' \right| \]

Putting \( R = N_1^{1/3} M^{4/3} \), we see that

\[ I_{t,2}^{(3,2)} \leq I_{\infty,2}^{(3,2)} (P_{N_1} 1_{t,R}^{\text{high}} \hat{w}, P_{N_2} \hat{w}, P_{N_3} 1_{t,R} \hat{z}_3, P_{N_4} \hat{z}_4, \ldots, P_{N_{k+2}} \hat{z}_{k+2}) + I_{\infty,2}^{(3,2)} (P_{N_1} 1_{t,R}^{\text{low}} \hat{w}, P_{N_2} \hat{w}, P_{N_3} 1_{t,R}^{\text{high}} \hat{z}_3, P_{N_4} \hat{z}_4, \ldots, P_{N_{k+2}} \hat{z}_{k+2}) + I_{\infty,2}^{(3,2)} (P_{N_1} 1_{t,R}^{\text{low}} \hat{w}, P_{N_2} \hat{w}, P_{N_3} 1_{t,R}^{\text{low}} \hat{z}_3, P_{N_4} \hat{z}_4, \ldots, P_{N_{k+2}} \hat{z}_{k+2}) =: I_{\infty,2,1}^{(3,2)} + I_{\infty,2,2}^{(3,2)} + I_{\infty,2,3}^{(3,2)}.
For $I^{(3,2)}_{\infty,2,1}$, we see from (4.3) that $\|1_{t,R}^{\text{high}}\|_{L^1} \lesssim T^{1/4}N_1^{-1/4}M^{-1}$. Then, we have

\[
I^{(3,2)}_{\infty,2,1} \lesssim \sum_{N_1, \ldots, N_{k+2}} \sum_{kN_1 \leq M \leq N_4} N_1^{2s-1} \|1_{t,R}^{\text{high}}\|_{L^1} \|P_{N_1} \tilde{w}\|_{L^\infty_t L^2_x} \|P_{N_3} \tilde{z}_3\|_{L^\infty_t L^2_x} \\
\times \|P_M \left( P_{N_2} \tilde{w} \prod_{j=4}^{k+2} P_{N_j} \tilde{z}_j \right) \|_{L^\infty_{t,x}} \lesssim T^{1/4} \sum_{N_1, \ldots, N_{k+2}} \sum_{M \leq N_4} N_1^{2s-5/4} \|P_{N_1} \tilde{w}\|_{L^\infty_t L^2_x} \|P_{N_3} \tilde{z}_3\|_{L^\infty_t L^2_x} \\
\times \|P_{N_2} \tilde{w} \prod_{j=4}^{k+2} P_{N_j} \tilde{z}_j \|_{L^\infty_t H_s^{-1/2}}.
\]

Note that (2.7) leads to

\[
\sum_{N_5, \ldots, N_{k+2}} \left\|P_{N_2} \tilde{w} \prod_{j=4}^{k+2} P_{N_j} \tilde{z}_j \right\|_{H_s^{-1/2}} \lesssim_k C_0^{k-2} \|P_{N_4} \tilde{z}_4\|_{H^{s_0}} \|P_{N_2} \tilde{w}\|_{H_s^{-1/2}}
\]  

with $1/2 < s_0 < s$. This together with (1.19) and (2.4) shows that

\[
I^{(3,2)}_{\infty,2,1} \lesssim_k T^{1/4} C_0^{k-2} \|w\|_{L^\infty_t H_s^{-1}} \|w\|_{L^\infty_t H_s^{1-s}} \|\tilde{z}\|_{L^\infty_t H_s^{1-s}} \|\tilde{z}_3\|_{L^\infty_t H_s^{1-s}} \|\tilde{z}_4\|_{L^\infty_t H_s^{1-s}} \sum_{N_1} N_1^{-1/4} \\
\lesssim_k T^{1/4} C_0^{k} \|w\|^2_{L^\infty_t H_s^{-1}}.
\]

Similarly, $I^{(3,2)}_{\infty,2,2}$ can be estimated by the same bound as above. For $I^{(3,2)}_{\infty,2,3}$, Lemma 4.5 shows $|\Omega_{k+1}| \gtrsim MN_1^{\alpha} \sim |\xi_2 + \xi_4|N_1^{\alpha}$. Therefore, setting $L := MN_1^{\alpha}$, we have

\[
I^{(3,2)}_{\infty,2,3} \leq \sum_{n=1}^{k+2} \left( P_{N_1} Q_{\geq L}(1_{t,R}^{\text{low}} \tilde{w}), P_{N_2} \tilde{w}, P_{N_3} 1_{t,R}^{\text{low}} \tilde{z}_3, \ldots, P_{N_{k+2}} \tilde{z}_{k+2} \right) \\
+ \sum_{n=1}^{k+2} \left( P_{N_1} Q_{\leq L}(1_{t,R}^{\text{low}} \tilde{w}), P_{N_2} \tilde{w}, P_{N_3} Q_{\geq L}(1_{t,R}^{\text{low}} \tilde{z}_3), \ldots, P_{N_{k+2}} \tilde{z}_{k+2} \right) \\
+ \sum_{n=1}^{k+2} \left( P_{N_1} Q_{\leq L}(1_{t,R}^{\text{low}} \tilde{w}), P_{N_2} Q_{\leq L} \tilde{w}, P_{N_3} Q_{\leq L}(1_{t,R}^{\text{low}} \tilde{z}_3), \ldots, P_{N_{k+2}} Q_{\leq L} \tilde{z}_{k+2} \right) + \cdots \\
+ \sum_{n=1}^{k+2} \left( P_{N_1} Q_{\leq L}(1_{t,R}^{\text{low}} \tilde{w}), P_{N_2} Q_{\leq L} \tilde{w}, P_{N_3} Q_{\leq L}(1_{t,R}^{\text{low}} \tilde{z}_3), \ldots, P_{N_{k+2}} Q_{\leq L} \tilde{z}_{k+2} \right) \\
=: I^{(3,2)}_{\infty,2,3,1} + \cdots + I^{(3,2)}_{\infty,2,3,k+2},
\]

where $I^{(3,2)}_{\infty,2,3,n}$ for $4 \leq n \leq k + 2$ corresponds to the term in which $Q_{\geq L}$ lands on $P_{N_n} \tilde{z}_n$. The contribution of $I^{(3,2)}_{\infty,2,3,1}$ is estimated, thanks to Lemma 2.4  (4.10), (4.5),
with $1/2 < s_0 < s$. Here, we used $N_1^{-\alpha} \leq N_1^{-1}$ since $\alpha \geq 1$. Similarly, we can estimate $I_{3,2,3,1}^{(3)}$ by

\[
I_{3,2,3,1}^{(3)} \lesssim_k T^{1/4} C_0^{k-2} \| \tilde{u} \|_{L^\infty_t H_x^{s-1}} \| \tilde{u} \|_{H_x^{s-1}} \| \tilde{\xi}_3 \|_{L^\infty_t H_x^{s}} \| \tilde{\eta}_4 \|_{L^\infty_t H_x^{s}} \lesssim_k T^{1/4} C_0^{k} \| w \|_{Z_x^{s-1}} \| w \|_{L_x^\infty H_x^{s-1}}.
\]

Next, for $I_{3,2,3,2}^{(3)}$, thanks to Lemma \[2.1 \ (4.3), \ (4.4), \ (4.2) \text{ and } (5.6)\], we have

\[
I_{3,2,3,2}^{(3)} \lesssim_k T^{1/2} \left\| \left( P_{N_2} Q_{\leq L} \tilde{u} \right) \prod_{j=4}^{k+2} P_{N_j} \tilde{\xi} \right\|_{L^2_t H_x^{s-1/2}} \lesssim_k T^{1/2} C_0^{k-2} \| \tilde{u} \|_{X^{s-2}} \| \tilde{\xi}_3 \|_{L^\infty_t H_x^{s}} \| \tilde{\eta}_4 \|_{L^\infty_t H_x^{s}} \lesssim_k T^{1/2} C_0^{k-2} \| \tilde{\xi}_3 \|_{L^\infty_t H_x^{s}} \| \tilde{\eta}_4 \|_{L^\infty_t H_x^{s}} \lesssim_k T^{1/2} C_0^{k} \| \tilde{u} \|_{Z_x^{s-1}} \| \tilde{\xi}_3 \|_{L^\infty_t H_x^{s}} \| \tilde{\eta}_4 \|_{L^\infty_t H_x^{s}} \lesssim_k T^{1/2} C_0^{k} \| \tilde{u} \|_{Z_x^{s-1}} \| \tilde{\xi}_3 \|_{L^\infty_t H_x^{s}} \| \tilde{\eta}_4 \|_{L^\infty_t H_x^{s}} \lesssim_k T^{1/2} C_0^{k} \| w \|_{Z_x^{s-1}} \| w \|_{L^\infty_t H_x^{s}}.
\]
with $1/2 < s_0 < s$. Here, we used $N_1^{-\alpha} \leq N_1^{-1}$. By a similar argument, we also have

$$I_{t,2}^{(3,2)} \leq \sum_{N_1, \ldots, N_{k+2}} N_1^{2s-1} M \left\| P_{N_1} \tilde{u} \right\|_{L_t^\infty L_x^2} \left\| P_{N_3} \tilde{z} \right\|_{L_t^\infty L_x^2}$$

\[ \times \left\| \left( P_{N_2} Q_{\leq M} N_1^2 \tilde{w} \right) \left( P_{N_4} Q_{\geq M} N_1^2 \tilde{z}_4 \right) \prod_{j=5}^{k+2} P_{N_j} \tilde{z}_j \right\|_{L_t^1 H_x^{s-1}} \]

\[ \lesssim_k T^{1/2} C^{k-2}_0 \left\| \tilde{w} \right\|_{L_t^\infty H_x^{s-1}} \sum_{N_1, \ldots, N_k} N_1^{2s-1} \left\| P_{N_1} \tilde{w} \right\|_{L_t^\infty L_x^2} \left\| P_{N_3} \tilde{z}_3 \right\|_{L_t^\infty L_x^2} \left\| P_{N_4} \tilde{z}_4 \right\|_{X^{s,0,1}} \]

\[ \lesssim_k T^{1/2} C^{k-2}_0 \left\| \tilde{w} \right\|_{L_t^\infty H_x^{s-1}} \left\| \tilde{z}_3 \right\|_{L_t^\infty H_x^{s-1}} \left\| \tilde{z}_4 \right\|_{X^{s,0,1}} \left\| \tilde{u} \right\|_{L_t^\infty H_x^{s-1}}^2 \lesssim_k T^{1/2} C^{k}_0 \left\| w \right\|_{L_t^\infty H_x^{s-1}}^2. \]

Here, we chose $s_0 \in (1/2, s(\alpha))$ and used $N_1^{-\alpha} \leq N_1^{-1}$. Similarly, we can estimate $I_{t,2}^{(3,2)}$ for $5 \leq n \leq k+2$ by the same bound as that of $I_{t,2}^{(3,2)}$, which completes the estimate of the contribution of $I_t^{(3,2)}$. On the other hand, the contribution of $J_t^{(3,2)}$ can be controlled by $I_t^{(3,2)}$ since $s > 1/2$ and $N_2^{2s-1} \leq N_2^{2s-1}$.

Subcase 3.3: $N_2 \ll N_5$ ($k \geq 3$). Recall that we can assume $N_2 \geq 1$ and note that we can also assume that $N_5 \geq 1$ since $N_5 = 0$ is included in Subcase 3.2: $N_2 \gtrsim N_5$. As in Subcase 3.2, we evaluate $I_t^{(3,2)}$ and $J_t^{(3,2)}$. Define $L_k$ for $k \geq 3$ so that $L_3 := N_2$ and $L_k := N_2 \vee k N_6$ for $k \geq 4$. We decompose

\[ I_t^{(3,2)} \leq \sum_{N_1, \ldots, N_{k+2}} N_1^{2(s-1)} \int_0^t \int_0^t \partial_x P_N^2 P_{N_1} w P_{N_3} z_3 P_{\leq L_k} \left( P_{N_2} w \prod_{j=4}^{k+2} P_{N_j} z_j \right) dx dt' \]

\[ + \sum_{L_k \ll M \leq N_{k+2}} N_1^{2(s-1)} \int_0^t \int_0^t \partial_x P_N^2 P_{N_1} w P_{N_3} z_3 P_M \left( P_{N_2} w \prod_{j=4}^{k+2} P_{N_j} z_j \right) dx dt' \]

\[ =: I_{t,1}^{(3,3)} + I_{t,2}^{(3,3)}, \]

where $\sum$ denotes $\sum_{N \gg 1} \sum_{N_1, \ldots, N_{k+2}}$ for simplicity. In what follows, we may assume that $k \geq 4$ since the case $k = 3$ can be treated by the same way as Subcase 3.2. For
\(I_{t,1}^{(3,3)}\), Hölder’s, Bernstein’s and Minkowski’s inequalities, (3.2) and (3.4) give

\[
\begin{aligned}
I_{t,1}^{(3,3)} & \lesssim \left( \sum_{\begin{smallmatrix} N_1, \ldots, N_k+2 \\ N_2 \geq k N_0 \end{smallmatrix}} N_1^{2s-1} N_2^{3/4} + \sum_{\begin{smallmatrix} N_1, \ldots, N_k+2 \\ N_2 < k N_0 \end{smallmatrix}} N_1^{2s-1} (k N_0)^{3/4} \right) \\
& \times \int_0^t \| P_{N_1} u \|_{L^2_t} \| P_{N_2} z \|_{L^3_t} \| P_{N_2} v \|_{L^4_t} \| P_{N_5} z_4 \|_{L^5_t} \| P_{N_6} z_5 \|_{L^6_t} \prod_{j=6}^{k+2} \| P_{N_j} z_j \|_{L^\infty_t} dt'
\end{aligned}
\]

\[
\lesssim_k C_0^{k-2} \| u \|_{L^\infty_t H^s} \sum_{N_4 \geq N_5 > N_2} \| D_x^{-5/12} P_{N_2} u \|_{L^4_t L^4} \prod_{i=4}^5 \left( \frac{N_3}{N_i} \right)^{7/12} \| D_x^{7/12} P_{N_3} z_i \|_{L^3_t L^3}
\]

\[
+ C_0^{k-3} \| u \|_{L^\infty_t H^s} \sum_{N_4 \geq N_5 > N_2 \vee N_6} \left( \frac{N_2}{N_4} \right)^{5/12} \| D_x^{-5/12} P_{N_2} u \|_{L^4_t L^4}
\]

\[
\times \left( \frac{N_6}{N_4} \right)^{1/6} \| D_x^{7/12} P_{N_4} z_4 \|_{L^3_t L^3} \left( \frac{N_6}{N_5} \right)^{7/12} \| D_x^{7/12} P_{N_5} z_5 \|_{L^3_t L^3} \| P_{N_6} z_6 \|_{L^\infty_t}
\]

\[
\lesssim_k T^{5/8} C_0^k G(C_0) \| u \|_{L^\infty_t H^s}^2
\]

since \( s(\alpha) \geq 7/12 + \beta(\alpha) \). On the other hand, \( I_{t,3}^{(3,3)} \) can be estimated by the same way as \( I_{t,2}^{(3,3)} \). We also notice that \( J_t^{(3,2)} \) is controlled by \( I_{t}^{(3,2)} \). This concludes the proof. \( \square \)

### 6. Local and Global-Well-Posedness

#### 6.1. Unconditional local well-posedness in \( H^s(\mathbb{T}) \) for \( s \geq s(\alpha) \).

**6.1.1. Unconditional uniqueness.** Let \( u_0 \in H^s(\mathbb{T}) \) with \( s \geq s(\alpha) \) and let \( u, v \) be two solutions to the Cauchy problem (1.3)-(1.2) that belong to \( L^\infty_t H^s \). According to Lemma 4.7, we know that \( u, v \in Z_t^\alpha \) and Proposition 5.1 together with (4.7) ensure that \( u \equiv v \) on \( [0, T_0] \) with \( 0 < T_0 \leq T \) that only depends on \( \| u \|_{Z_t^{\alpha}(\alpha)} + \| v \|_{Z_t^{\alpha}(\alpha)} \). Therefore \( u(T_0) = v(T_0) \) and we can reiterate the same argument on \( [T_0, T] \). This proves that \( u \equiv v \) on \( [0, T] \) after a finite number of iteration.

**6.1.2. Local existence.** We fix \( \alpha \in [1, 2] \). It is well known (see [11]) that the Cauchy problem (1.3)–(1.2), with \( L_{\alpha+1} \) satisfying Hypotheses [11] is locally well-posed in \( H^s(\mathbb{T}) \) for \( s > 3/2 \) with a minimal time of existence \( T = T(\| u_0 \|_{H^{3/2}_x}) \). Indeed, for any \( s \in \mathbb{R} \) and any smooth function \( g \) the \( H^s \)-scalar product \( \langle L_{\alpha+1} g, g \rangle_{H^s} \) vanishes, and the classical energy method (which consists of the parabolic regularization, the Kato–Ponce commutator estimate [15] and the Bona–Smith argument [4]) can be applied. So let \( u \in C([0, T_0]; H^\infty(\mathbb{T})) \) be a smooth solution to (1.3) emanating from
a smooth initial data $u_0 \in H^\infty(\mathbb{T})$. According to Lemma \[4.7\] and Proposition \[4.8\] with $w_N \equiv 1$, there exists an increasing entire function $G$ non negative on $\mathbb{R}_+$ such that

$$\|u\|_{L_T^\infty H^s}^2 \leq \|u_0\|_{H^s}^2 + T^{1/4}G(\|u\|_{L_T^\infty H^s})\|u\|_{L_T^\infty H^s}^2. \quad (6.1)$$

for any $0 < T \leq \min(1, T_0)$ and any $2 \geq s \geq s(\alpha)$.

Taking $s = s(\alpha)$ in \[(6.1)\] and setting $T_1 = G(2\|u_0\|_{H^s})^{-4}$, we deduce from a continuity argument that for all $0 < T < \min(T_0, T_1)$ it holds $\|u\|_{L_T^\infty H^s}^2 \leq 2\|u_0\|_{H^s}^2$. Re-injecting this in \( (6.1) \) we obtain that for any $2 \geq s \geq s(\alpha)$ and any $0 < T \leq \min(T_0, T_1)$ it holds $\|u\|_{L_T^\infty H^s}^2 \leq 2\|u_0\|_{H^s}^2$. Therefore the local well-posedness result in $H^s(\mathbb{T})$ for $s > 3/2$ ensures that $u$ does exist on $[0, T]$ with $u \in C([0, T_1]; H^\infty(\mathbb{T}))$.

Now let us fix $u_0 \in H^s(\mathbb{T})$ with $2 \geq s \geq s(\alpha)$. Setting $u_{0,n} = P_{\leq n} u_0$, it is clear that the sequence $\{u_{0,n}\}_{n \geq 1}$ belongs to $H^\infty(\mathbb{T})$ and converges to $u_0$ in $H^s(\mathbb{T})$. We deduce from above that the emanating sequence of solutions $\{u_n\}_{n \geq 1}$ to \[(1.3)\] is included in $C([0, T]; H^\infty(\mathbb{R}))$ with $T = G(2\|u_0\|_{H^s})^{-4}$ and satisfies $\|u_n\|_{L_T^\infty H^s}^2 \leq 2\|u_0\|_{H^s}^2$. Moreover, we infer from Propositon \[5.1\] together with \[4.7\] that $\{u_n\}_{n \geq 1}$ is a Cauchy sequence in $C([0, T]; H^{s-1}(\mathbb{T}))$ and thus converges to some function $u \in L^\infty([0, T]; H^s(\mathbb{T}))$ strongly in $C([0, T]; H^s(\mathbb{T}))$ for any $s' < s$ and in particular in $C([0, T] \times \mathbb{T})$. We can thus pass to the limit on the nonlinear term $f(u_n)$ and $u$ satisfies \[1.3\]-\[1.2\] at least in the distributional sense.

### 6.1.3. Strong continuity in $H^s(\mathbb{T})$.
Let $u$ be the solution emanating from $u_0 \in H^s(\mathbb{T})$. It suffices to check that $u(t)$ is continuous in $H^s(\mathbb{T})$ at $t = 0$ thanks to time translation invariance and reversibility in time (invariance by the change of variables $(t, x) \mapsto (-t, -x)$) of \[1.3\] together with the uniqueness of the solution. Since $u \in C([0, T]; H^{s'}(\mathbb{T})) \cap L^\infty([0, T]; H^{s'}(\mathbb{T}))$ for $s' < s$, the standard argument (see [Theorem 2.1, \[10\]] for instance) yields that $u(t)$ is weakly continuous in $H^s(\mathbb{T})$ on $[0, T]$. In particular, it holds that $\|u_0\|_{H^s} \leq \inf_{t \to 0} \|u(t)\|_{H^s}$. On the other hand, we see from \[4.8\] with $\omega_N \equiv 1$ that $\lim_{t \to 0} \|u(t)\|_{H^s} \leq \|u_0\|_{H^s}$ since $\|u(t)\|_{H^s} \leq \|u\|_{L_T^\infty H^s}$ for any $t \in [0, T]$ (see for instance [Lemma 2.4, \[32\]]). It thus follows that $\lim_{t \to 0} \|u(t)\|_{H^s} = \|u_0\|_{H^s}$. This together with the weak continuity shows that $u(t)$ is continuous in $H^s(\mathbb{T})$ at $t = 0$.

### 6.1.4. Continuity with respect to initial data.
Here, we make use of the frequency enveloppe $\omega$. Following \[23\], for any sequence $\{u_{0,k}\}_{k \geq 1}$ converging to $u_0$ in $H^s(\mathbb{T})$ there exists a dyadic sequence $\{\omega_N\}$ of positive numbers satisfies $\omega_N \leq \omega_{2N} \leq \delta \omega_N$
for $N \geq 1$ such that

$$\|u_0\|_{H^\omega} < \infty, \quad \sup_{k \geq 1} \|u_{0,k}\|_{H^\omega} < \infty \quad \text{and} \quad \omega_N \searrow +\infty. \quad (6.2)$$

By Remark 4.2, we may assume that $1 < \delta \leq 2$. Now let $\{u_{0,k}\}_{k \geq 1} \subset H^s(\mathbb{T})$, with $\sup_{k \geq 1} \|u_{0,k}\|_{H^s} \leq 2\|u_0\|_{H^s}$, that converges to $u_0$ in $H^s(\mathbb{T})$ and let $\{\omega_N\}$ satisfying (6.2). Applying Lemma 4.7 and Proposition 4.8 with $\{\omega_N\}$ we infer that the sequence of solution $\{u_k\}_{k \geq 1}$ emanating from $\{u_{0,k}\}_{k \geq 1}$ is bounded in $L^\infty_T H^s_{\omega}$ with $T = G(4\|u_0\|_{H^{s(\alpha)}})^{-4}$. This together with the Lipschitz bound (5.1) in $H^{s-1}(\mathbb{T})$ ensures $\{u_k\}_{k \geq 1}$ converges to $u$ in $C([0,T];H^s(\mathbb{T}))$.

### 7. Global existence

(1.3) is Hamiltonian and enjoys at least two conservation laws that correspond to the conservation of the mass and of the energy. The local-wellposedness result ensures that (1.3) is wellposed in the energy space $H^{\alpha/2}(\mathbb{T})$ for $\alpha \in [\sqrt{2}, 2]$. In this section we take advantage of this to prove some global existence result in this case. Note that since $p_{\alpha+1}$ is odd, $\frac{p_{\alpha+1}(\xi)}{\xi}$ is well-defined on $\mathbb{Z}$. We denote by $\partial_x^{-1}L_{\alpha+1}$ the Fourier multiplier by $\frac{p_{\alpha+1}(k)}{k}$.

#### 7.1. Conservation laws.

Assume that the $u_0 \in H^3(\mathbb{T})$ and let $u \in C([0,T];H^3(\mathbb{T}))$ be the associated solution to (1.3). Multiplying (1.3) by $u$ and integrating by parts, using that $p_{\alpha+1}$ is odd and $u$ is real valued, we infer that $M(u) = \int_T u^2(x) \, dx$ is an invariant of the motion. In the same way multiplying (1.3) by $\partial_x^{-1}L_{\alpha+1}u + f(u)$ and integrating by parts we infer that

$$E(u) = \frac{1}{2} \int_T u \partial_x^{-1}L_{\alpha+1}u + \int_T F(u)$$

where

$$F(x) = \sum_{k=0}^{+\infty} \frac{f(k)(0)}{(k + 1)!} x^{k+1}, \quad \text{i.e.} \quad F(x) = \int_0^x f(y) \, dy.$$  

is conserved along the flow. Now from Hypothesis 1 we infer that

$$p_{\alpha+1}(\xi) \sim \xi^{\alpha+1}, \quad \forall \xi \geq \xi_0. \quad (7.1)$$

Therefore $\frac{p_{\alpha+1}(\xi)}{\xi} > 0$ for $|\xi| \geq \xi_0$ and denoting by $\Lambda_{\alpha/2}$ the Fourier multiplier defined by

$$\Lambda_{\alpha/2}u(k) = \sqrt{\frac{p_{\alpha+1}(k)}{k}} \hat{u}(k) \mathbf{1}_{|k| \geq \xi_0}$$
we may thus rewrite \( \int_T u \partial_x^{-1} L_{\alpha+1} u \) as
\[
\int_T u \partial_x^{-1} L_{\alpha+1} u = \sum_{1 \leq |k| \leq \xi_0} \frac{p_{\alpha+1}(\xi)}{\xi} |\hat{u}(k)|^2 + \int_T |\Lambda_{\alpha/2} P_{\geq \xi_0} u|^2
\]
with
\[
\int_T |\Lambda_{\alpha/2} P_{\geq \xi_0} u|^2 \sim \|P_{\geq \xi_0} u\|_{H^{\alpha/2}}^2 \quad \text{and} \quad \left| \sum_{1 \leq |k| \leq \xi_0} \frac{p_{\alpha+1}(\xi)}{\xi} |\hat{u}(k)|^2 \right| \leq K_0 \|u\|_{L^2}^2 \quad (7.2)
\]
for some fixed \( K_0 > 0 \). In particular, \( u \mapsto \int_T u \partial_x^{-1} L_{\alpha+1} u \) is continuous from \( H^{\alpha/2}(\mathbb{T}) \) into \( \mathbb{R} \) which ensure that these conservation laws remain valid for \( u_0 \in H^{\alpha/2}(\mathbb{T}) \) thanks to the continuity with respect to initial data of the flow-map.

### 7.2. Arbitrary large initial data.
We assume that we are in one of the two following situations:

- **Case 1**: \( |F(x)| \lesssim (1 + |x|^{p+1}) \) for some \( 0 < p < 2\alpha + 1 \).
- **Case 2**: There exists \( B > 0 \) such that \( F(x) \leq B, \forall x \in \mathbb{R} \).

In Case 1, it holds
\[
\left| \int_T F(u) \right| \lesssim 1 + \|u\|_{L^{p+1}}^{p+1} \lesssim 1 + \|u\|_{H^{\alpha/2}}^{p+1} \lesssim 1 + \|u\|_{L^2}^{p+1} \|u\|_{H^{\alpha/2}}^{p-1} \|u\|_{H^{\alpha/2}}^{p-1}.
\]
Therefore since \( p < 2\alpha + 1 \) and thus \( \frac{p-1}{\alpha} < 2 \), the conservation of the \( L^2 \)-norm together with Young’s inequality and (7.2) lead to
\[
\left| \int_T F(u(t)) \right| \leq C_0 + C_1(\|u_0\|_{L^2}) \|u(t)\|_{H^{\alpha/2}}^{p-1} \leq C_0 + \frac{1}{4} \int_T |\Lambda_{\alpha/2} P_{\geq \xi_0} u|^2 + C_2(\|u_0\|_{L^2}).
\]

The conservation of the energy and (7.2) then ensure that the trajectory of \( u \) remains bounded in \( H^{\alpha/2}(\mathbb{T}) \). This, together with the local well-posedness result, leads to the global existence of the solution in \( H^s(\mathbb{T}) \) for \( s \geq \alpha/2 \). Typical examples of Case 1 are:

- \( f(x) \) is a polynomial function of degree strictly less than \( 2\alpha + 1 \).
- \( f(x) \) is a polynomial function of \( \sin(x) \) and \( \cos(x) \).

In Case 2, it holds \( \int_T F(u) \leq 2\pi B \) and the conservation of the energy together with (7.2) ensure that for any \( t \in [0, T] \)
\[
\|u(t)\|_{H^{\alpha/2}}^2 \lesssim E(u_0) + M(u_0) + \int_T F(u) \lesssim E(u_0) + M(u_0) + 2\pi B
\]
that leads to the global existence result. Typical examples of Case 2 are:

- \( f(x) \) is a polynomial function of odd degree with \( \lim_{x \to +\infty} f(x) = -\infty \).
- \( f(x) = -\exp(x) \) or \( f(x) = -\sinh(x) \).
7.3. Small initial data. We set
\[ \tilde{F}(x) = \sum_{k=2}^{+\infty} \frac{f^{(k)}(0)}{(k+1)!} x^{k+1} = F(x) - \frac{f'(0)}{2} x^2 \quad \text{and} \quad G(x) = \sum_{k=2}^{+\infty} |f^{(k)}(0)| x^{k-2}. \]
Then the conservations of \( M \) and \( E \) lead to
\[ E(u) + \left( K_0 + 1 - \frac{f'(0)}{2} \right) M(u) = \frac{1}{2} \int_{\mathbb{T}} \left( (K_0 + 1)|u|^2 + u \partial_x^{-1} L_{\alpha+1} u \right) + \int_{\mathbb{T}} \tilde{F}(u) \]
\[ = E(u_0) + \left( K_0 + 1 - \frac{f'(0)}{2} \right) M(u_0), \quad (7.3) \]
where \( K_0 > 0 \) is the constant interfering in (7.2).
We observe that \( \lim_{x \to 0} G(x) = |f'(0)/6| \) and thus there exists \( \delta_0 > 0 \) such that for \( \|u\|_{H^{\alpha/2}} < \delta_0 \) it holds \( |G(u)| \leq G(\|u\|_{L^\infty}) \leq |f'(0)| \). Under this restriction we thus have
\[ \left| \int_{\mathbb{T}} \tilde{F}(u) \right| \leq |f'(0)| \int_{\mathbb{T}} |u|^3 \leq C_0 |f'(0)| \|u\|_{H^{\alpha/2}}^3. \]
Therefore according to (7.2)-(7.3) there exists a positive constant \( C_1 > 0 \) that depends on \( L_{\alpha+1} \) such that
\[ C_1 \|u\|_{H^{\alpha/2}}^2 - C_0 |f'(0)| \|u\|_{H^{\alpha/2}}^3 \leq E(u) + \left( K_0 + 1 - \frac{f'(0)}{2} \right) M(u). \]
We set
\[ C = \max \left( \frac{C_0 |f'(0)|}{C_1}, \frac{1}{2\delta_0} \right) \quad \text{and} \quad p(x) = C_1 (x^2 - Cx^3). \]
It is easy to check that \( p(x) \leq C_1 x^2 - C_0 |f'(0)| x^3 \) on \( \mathbb{R}_+ \) and that \( p(\frac{1}{2C}) = \frac{C_1}{8} (\frac{C_1}{C})^2 > 0 \).

Let \( s \geq \alpha/2 \). We infer from above that for any \( u_0 \in H^s(\mathbb{T}) \) such that
\[ E(u_0) + \left( K_0 + 1 - \frac{f'(0)}{2} \right) M(u_0) < \frac{C_1}{8} \left( \frac{C_1}{C} \right)^2 \quad (7.4) \]
the solution \( u \in C([0,T];H^s(\mathbb{T})) \) emanating from \( u_0 \) satisfies
\[ p(\|u(t)\|_{H^{\alpha/2}}) < E(u_0) + \left( K_0 + 1 - \frac{f'(0)}{2} \right) M(u_0) < \frac{C_1}{8} \left( \frac{C_1}{C} \right)^2 \quad (7.5) \]
for any \( t \in [0,T] \) such that \( \|u(t)\|_{H^{\alpha/2}} < \delta_0 \). Since \( p(1/2C) = \frac{C_1}{8} (\frac{C_1}{C})^2 \) and \( 1/2C \leq \delta_0 \), a continuity argument ensures that for \( u_0 \in H^{\alpha/2}(\mathbb{T}) \) satisfying (7.4) with \( \|u_0\|_{H^{\alpha/2}} < \delta_0 \), the emanating solution \( u \) verifies (7.5) for all times and thus can be extended globally in times according to the LWP result. Finally since of course, by the continuity of the energy in \( H^{\alpha/2}(\mathbb{T}) \), (7.4) is satisfied for \( \|u_0\|_{H^{\alpha/2}} \) small enough, this leads to the global existence result in \( H^s(\mathbb{T}), s \geq \alpha/2 \), for initial data with \( H^{\alpha/2}\)-norm small enough.
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