A $(4+\epsilon)$-approximation for $k$-connected subgraphs

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Abstract. We obtain approximation ratio $2(2+\frac{1}{\nu})$ for the (undirected) $k$-Connected Subgraph problem, where $\nu \approx \frac{1}{2}(\log k n - 1)$ is the largest integer such that $2^{\nu-1}k^{2\nu+1} \leq n$. For large values of $n$ this improves the $6$-approximation of Cheriyan and Végh [4] when $n = \Omega(k^3)$, which is the case $\nu = 1$. For $k$ bounded by a constant we obtain ratio $4+\epsilon$. For large values of $n$ our ratio matches the best known ratio $4$ for the augmentation version of the problem [28,29], as well as the best known ratios for $k = 6,7$ [22]. Similar results are shown for the problem of covering an arbitrary crossing supermodular biset function.

1 Introduction

A graph is $k$-connected if it contains $k$ internally disjoint paths from every node to any other node. We consider the following problem on undirected graphs:

| $k$-Connected Subgraph |
|------------------------|
| **Input:** A graph $G = (V,E)$ with edge costs $\{c_e : e \in E\}$ and an integer $k$. |
| **Output:** A minimum cost $k$-connected spanning subgraph of $G$. |

For undirected graphs, the problem is NP-hard for $k = 2$ (the case $k = 1$ is the MST problem), even when all edges in $G$ have unit costs. This is since any feasible solution with $n = |V|$ edges is a Hamiltonian cycle. For directed graphs the problem is NP-hard already for $k = 1$, by a similar reduction. Let $\bar{\rho}(k,n)$ and $\rho(k,n)$ denote the best possible approximation ratios for the directed and undirected variants, respectively. A standard bi-direction reduction gives $\rho(k,n) \leq 2\bar{\rho}(k,n)$, while by [24] $\rho(k,n) \geq \bar{\rho}(k-n/2,n/2)$ for $k \geq n/2 + 1$. All in all we get that for $k \geq n/2 + 1$ the approximability of directed and undirected cases is the same, up to a 2 factor. This however does not exclude that the undirected case is easier when $n$ is much larger than $k$.

In the $k$-Connectivity Augmentation problem $\tilde{G}$ contains a spanning $(k-1)$-connected subgraph of cost 0. A feasible solution for the $k$-Connected Subgraph problem can be obtained by solving sequentially $\ell$-Connectivity Augmentation problems for $\ell = 1, \ldots k$, but this reduction usually invokes a factor of $H(k)$ in the ratio, where $H(k)$ denotes the $k$-th harmonic number. Several ratios for $k$-Connected Subgraph were obtained in this way, c.f. [5,23,8,28]. The currently best known ratios for the general and the augmentation version, for both directed and undirected graphs, are summarized in Table 1.
### General Augmentation

- **Undirected**
  - 6 if \( n \geq k^3\) (see also [14])
  - \(\lceil (k+1)/2 \rceil\) if \( k \leq 7 \) [29, 22]
  - \(k + 1\) if \( k \leq 6 \) [22]
- **Directed**
  - \(O\left(\ln(\frac{n}{n-k} \cdot \ln k)\right)\) [28]
  - \(O\left(\ln(\frac{n}{n-k} \cdot \ln n)\right)\) [28]

### Directed Augmentation

- **Undirected**
  - \(O\left(\ln(\frac{n}{n-k})\right)\) [28]
  - \(H(\mu) + 1\) [29]
  - \(H(\mu) + 3/2\) [29]
- **Directed**
  - \(O\left(\ln(n - k)\right)\) [28]
  - \(O\left(\ln(n - k)\right)\) [28]

| Table 1. Known approximation ratios for \(k\)-Connected Subgraph and \(k\)-Connectivity Augmentation problems. Here \(\mu = \left\lfloor \frac{n}{(n-k)/2} + 1 \right\rfloor\), and note that if \(\mu = 1\) then \(k \in \{1, 2\}\), and that if \(n \geq 3k - 2\) then \(\mu = 2\) and \(H(\mu) = 3/2\). |

Note that we consider the node-connectivity version of the problem, for which classic techniques like the primal dual method [17] and the iterative rounding method [20, 9] do not seem to be applicable directly. Ravi and Williamson [32] gave an example of a \(k\)-Connectivity Augmentation instance when the primal dual method has ratio \(\Omega(k)\). A related example of Aazami, Cheriyan and Laekhanukit [1] rules out the iterative rounding method. On the other hand, several works showed that the problem can be decomposed into a small number of "good" sub-problems. However, attempts to achieve a constant ratio for \(k = n - o(n)\) (e.g., for \(k = n - \Theta(\sqrt{n})\)) failed even in the easier augmentation case, thus Cheriyan and Végh [4] suggested the following question:

**What ratio can we achieve when \(n\) is lower bounded by a function of \(k\)?**

This essentially addresses the issue of "asymptotic approximability" – as a function of the single parameter \(k\), for all sufficiently large \(n\). For undirected graphs Cheriyan and Végh [4] gave an elegant 6 approximation when \(n \geq k^4\), and this bound was slightly improved to \(n \geq k^3\) in [16].

Note that the “asymptotic approximability” question seems almost settled for the augmentation version: by [28, 29] for both directed and undirected graphs we have a constant ratio unless \(k = n - o(n)\); furthermore, for undirected graphs we have ratio 4 for \(n \geq 3k - 2\) (ratio 3 for directed graphs) [29], and this is also the best known ratios when \(k = 6, 7\) for the general version [22].

From now and on we consider undirected graphs, unless stated otherwise. Note that 4 is a current “lower bound” on the “asymptotic approximability” of the problem, in the sense that no better ratio is known for much easier sub-problems. Our main result shows that this “lower bound” is (almost) achievable.

**Theorem 1.** \(k\)-Connected Subgraph admits approximation ratio \(2(2 + 1/\ell)\) where \(\ell\) is the largest integer such that \(n \geq k[(k^2 - 1)(2k^2 - 3k + 2)^{\ell-1} + 1]\).

Note that \(\ell \approx \frac{1}{2}(\log_k n - 1)\) and that Theorem [11] implies ratio 5 for \(n \geq 2k^5\) and ratio \(4 + \epsilon\) if \(k\) is bounded by a constant. In fact, we prove a generalization of Theorem [11] stated in biset function terms, see the next section.

We note that our result can be used to improve approximation ratios for the Min-Cost Degree Bounded \(k\)-Connected Subgraph problem, see [16].

We refer the reader to [31, 30] for surveys on approximation algorithms for node-connectivity problems, and to [11, 13] for a survey on polynomially solvable cases. Here we briefly mention the status of some restricted \(k\)-Connected Subgraph problems.
| Costs      | Undirected                                      | Directed                                                                 |
|------------|-------------------------------------------------|---------------------------------------------------------------------------|
| {0, 1}    | \(\min(2, 1 + \frac{k-1}{n})\) \[18\]         | in \(\mathcal{P}\) \[12\]                                               |
| {1, \(\infty\)} | \(1 - \frac{1}{n} + \frac{1}{2^k} \leq 1 + \frac{1}{2^k}\) \[13\] (see also \[27\]) \(1 - \frac{1}{n} + \frac{1}{2^k} \leq 1 + \frac{1}{2^k}\) \[13\] (see also \[27\]) | \(2 + k/n\) \[22\]                                                      |
| metric    | \(2 + (k - 1)/n\) \[22\]                     | \(2 + k/n\) \[22\]                                                      |

Table 2. Known approximation ratios of \(k\)-Connected Subgraph problems.

We may assume that the input graph \(\hat{G}\) is complete, by assigning infinite costs to “forbidden” edges. Under this assumption, except for general edge costs, three main types of costs are considered in the literature:

- \{0, 1\}-costs: Here we are given a graph \(G\), and the goal is to find a minimum size set \(J\) of new edges (any edge is allowed) such that \(G \cup J\) is \(k\)-connected.
- \{1, \(\infty\}\}-costs: Here we seek a \(k\)-connected spanning subgraph of \(\hat{G}\) with minimum number of edges.
- metric costs: The costs satisfy the triangle inequality \(c_{uv} \leq c_{uw} + c_{wv}\) for all \(u, w, v \in V\).

The currently best known approximation ratios for these costs types are summarized in Table 2, and we mention some additional results. For \{0, 1\}-costs the complexity status of the problem is not known for undirected graphs, but for any constant \(k\) an optimal solution can be computed in polynomial time \[19\]. When \(\hat{G}\) contains a spanning \((k-1)\)-connected subgraph of cost 0 the \{0, 1\}-costs case can be solved in polynomial time for any \(k\) \[33\]. In the case of \{1, \(\infty\}\}-costs, directed 1-Connected Subgraph admits ratio 3/2 \[34\]. In the case of metric costs 2-Connected Subgraph admit ratio 3/2 \[15\].

2 Biset functions and \(k\)-connectivity problems

While edge-cuts of a graph correspond to node subsets, a natural way to represent a node-cut of a graph is by a pair of sets called a “biset”.

Definition 1. An ordered pair \(\mathcal{A} = (A, A^+)\) of subsets of \(V\) with \(A \subseteq A^+\) is called a biset; \(A\) is the inner part and \(A^+\) is the outer part of \(\mathcal{A}\), and \(\partial \mathcal{A} = A^+ \setminus A\) is the boundary of \(\mathcal{A}\). The co-set of \(\mathcal{A}\) is \(A^* = V \setminus A^+\); the co-biset of \(\mathcal{A}\) is \(\mathcal{A}^* = (A^*, V \setminus A)\). We say that \(\mathcal{A}\) is void if \(A = \emptyset\), co-void if \(A^+ = V\) (namely, if \(A^* = \emptyset\)), and \(\mathcal{A}\) is proper otherwise.

A biset function assigns to every biset \(\mathcal{A}\) a real number; in our context, it will always be an integer (possibly negative).

Definition 2. An edge covers a biset \(\mathcal{A}\) if it goes from \(A\) to \(A^*\). For an edge-set/graph \(J\) let \(\delta_J(\mathcal{A})\) denote the set of edges in \(J\) covering \(\mathcal{A}\). The residual function of a biset function \(f\) w.r.t. \(J\) is defined by \(f^J(\mathcal{A}) = f(\mathcal{A}) - |\delta_J(\mathcal{A})|\). We say that \(J\) \(f\)-covers \(\mathcal{A}\) if \(|\delta_J(\mathcal{A})| \geq f(\mathcal{A})\), and we say that \(J\) covers \(f\) or that \(J\) is an \(f\)-cover if \(|\delta_J(\mathcal{A})| \geq f(\mathcal{A})\) for all \(\mathcal{A}\).
In biset terms, Menger’s Theorem says that the maximum number of internally disjoint $st$-paths in $G$ equals to $\min\{|\partial A| + |\delta_G(A)| : s \in A, t \in A^*\}$. Consequently, $G$ is $k$-connected if and only if $|\delta_G(A)| \geq k - |\partial A|$ for every proper biset $A$; note that non-proper bisets cannot and are not required to be covered. Thus $G$ is $k$-connected if and only if $G$ covers the $k$-connectivity biset function $f_{k-CS}$ defined by

$$f_{k-CS}(A) = \begin{cases} k - |\partial A| & \text{if } A \text{ is proper} \\ 0 & \text{otherwise} \end{cases}$$

We thus will consider the following generic problem:

**Biset-Function Edge-Cover**

*Input:* A graph $\hat{G} = (V, \hat{E})$ with edge costs and a biset function $f$ on $V$.

*Output:* A minimum cost edge-set $E \subseteq \hat{E}$ that covers $f$.

This LP is particularly useful if the biset function $f$ has good uncrossing/supermodularity properties. To state these properties, we need to define the intersection and the union of bisets.

**Definition 3.** The *intersection* and the *union* of two bisets $A, B$ are defined by $A \cap B = (A \cap B, A^+ \cap B^+)$ and $A \cup B = (A \cup B, A^+ \cup B^+)$. The biset $A \setminus B$ is defined by $A \setminus B = (A \setminus B, A^+ \setminus B)$. We say that $A, B$ intersect if $A \cap B \neq \emptyset$, and cross if $A \cap B \neq \emptyset$ and $A^+ \cup B^+ \neq V$.

The following properties of bisets are easy to verify.

**Fact 1** For any bisets $A, B$ the following holds. If $A$ is a directed/undirected edge $e$ covers one of $A \cap \hat{B}, A \cup \hat{B}$ then $e$ covers one of $A, B$; if $e$ is an undirected edge, then if $e$ covers one of $A \setminus \hat{B}, B \setminus A$, then $e$ covers one of $A, B$. Furthermore $|\partial A| + |\partial B| = |\partial(A \cap \hat{B})| + |\partial(A \cup \hat{B})| = |\partial(A \setminus \hat{B})| + |\partial(B \setminus A)|$.

For a biset function $f$ and bisets $A, B$ the supermodular inequality is

$$f(A \cap \hat{B}) + f(A \cup \hat{B}) \geq f(A) + f(B).$$

We say that a biset function $f$ is supermodular if the supermodular inequality holds for all $A, B$, and modular if the supermodular inequality holds as equality for all $A, B$: $f$ is symmetric if $f(A) = f(A^*)$ for all $A$. Using among others Fact 1 one can see the following.
– For any directed/undirected graph $G$ the function $-d_G(\cdot)$ is supermodular.
– The function $|\partial(\cdot)|$ is modular.
– For any $R \subseteq V$ the function $|A \cap R|$ is modular.

We say that a biset $A$ is $f$-positive if $f(A) > 0$. Some important types of biset functions are given in the following definition.

**Definition 4.** A biset function $f$ is intersecting/crossing supermodular if the supermodular inequality holds whenever $A, B$ intersect/cross; $f$ is positively intersecting supermodular if the supermodular inequality holds for any pair of intersecting $f$-positive bisets.

Biset-Function Edge-Cover with positively intersecting supermodular $f$ admits a polynomial time algorithm due to Frank [10] that for directed graphs computes an $f$-cover of cost $\tau(f)$ (this also can be deduced using the iterative rounding method); for undirected graphs the cost is at most $2\tau(f)$, by a standard bi-direction reduction. Note however that the function $f_{k-CS}$ that we want to cover is obtained by zeroing an intersecting supermodular function on co-void bisets, but $f_{k-CS}$ itself is not positively intersecting supermodular.

In general, changing a supermodular function on void bisets gives an intersecting supermodular function, while changing an intersecting supermodular function on co-void bisets gives a crossing supermodular function (not all crossing supermodular functions arise in this way – see [13]). In particular, zeroing a supermodular function on non-proper bisets gives a crossing supermodular function. For example, the $k$-connectivity function $f_{k-CS}$ is obtained in this way from the modular function $k - |\partial A|$; thus $f_{k-CS}$ is crossing supermodular.

A common way to find a “cheap” partial cover of $f_{k-CS}$ is to find a 2-approximate cover of the fan function $g_R$ obtained by zeroing the function $k - |\partial A| - |A \cap R|$ on void bisets, where $R \subseteq V$ with $|R| = k$. Note that $g_R$ is intersecting supermodular and that $g_R$ is non-positive on co-void bisets (e.g., $g_R((V, \emptyset)) = k - 0 - k = 0$). Fan functions were used in many previous works on $k$-CONNECTED SUBGRAPH problems starting from Khuller and Raghavachari [21], and also by Cheriyan and Végh [4]. In fact, covering $g_R$ is equivalent to the following connectivity problem. Let us say that a graph is $k$-in-connected to $r$ if it has $k$ internally disjoint $vr$ paths for every $v \in V$. Construct a graph $G_r$ by adding to $G$ a new node $r$ and a set $F_r$ of zero cost edge from each $v \in R$ to $r$; then $H = (V \cup \{r\}, J_r)$ is a $k$-in-connected to $r$ spanning subgraph of $G_r$ if and only if $J = J_r \setminus F_r$ covers $g_R$. The problem of finding an optimal $k$-in-connected spanning subgraph can be solved in strongly polynomial time for directed graphs [13] (see also [10]), and this implies a 2-approximation for undirected graphs.

Fan functions are considered as the “strongest” intersecting supermodular functions for the purpose of finding a partial cover of $f_{k-CS}$. However, an inclusion minimal directed cover $J$ of a fan function may be difficult to decompose, since $J$ may have directed edges with tail in $R$; this is so since a fan function requires to cover to some extent bisets $A$ with $A \cap R \neq \emptyset$. We therefore use a different type of functions defined below, that are “weaker” but have “better” decomposition properties.
For \( R \subseteq V \) the area function of \( f \) is defined by

\[
f_R(\mathcal{A}) = f(\mathcal{A}) - \max_{\mathcal{A}} f(\mathcal{A}) \cdot |A \cap R| .
\]

Note that \( f_R(\mathcal{A}) = f(\mathcal{A}) \) if \( A \subseteq V \setminus R \) and \( f_R(\mathcal{A}) \leq 0 \) otherwise, so \( f_R \) requires to \( f \)-cover only those bisets whose inner part is contained in the “area” \( V \setminus R \).

In the next two lemmas we give some properties of area functions. Let us denote

\[
k_f = 1 + \max\{|\partial \mathcal{A}| : f(\mathcal{A}) > 0\} .
\]

**Lemma 2.** If \( |R| \geq k_f \) then: \( f_R \) is non-positive on co-void bisets, \( f_R \) is intersecting supermodular if \( f \) is, and \( f_R \) is positively intersecting supermodular if \( f \) is crossing supermodular and \( |R| \geq 2k_f - 1 \).

**Proof.** The first two statements are easy, so we prove only the last statement. Let \( \mathcal{A}, \mathcal{B} \) be intersecting \( f_R \)-positive bisets. Then \( A \cap R = B \cap R = \emptyset \), and thus \((A \cap B) \cap R = (A \cup B) \cap R = \emptyset \). Consequently, \( f_R = f \) on the bisets \( \mathcal{A}, \mathcal{B}, \mathcal{A} \cap \mathcal{B}, \mathcal{A} \cup \mathcal{B} \). Moreover, \( A^* \cap B^* \cap R \neq \emptyset \), since \(|\partial \mathcal{A} \cup \partial \mathcal{B}| < 2(k_f - 1) + 1 \leq |R| \).

Thus \( \mathcal{A}, \mathcal{B} \) must cross, and since \( f \) is crossing supermodular

\[
f_R(\mathcal{A}) + f_R(\mathcal{B}) = f(\mathcal{A}) + f(\mathcal{B}) \leq f(\mathcal{A} \cap \mathcal{B}) + f(\mathcal{A} \cup \mathcal{B}) = f_R(\mathcal{A} \cap \mathcal{B}) + f_R(\mathcal{A} \cup \mathcal{B}) .
\]

Consequently, the supermodular inequality holds for \( \mathcal{A}, \mathcal{B} \) and \( f_R \). \( \square \)

For \( S \subseteq V \) let \( \gamma(S) \) denote the set of edges in \( \vec{E} \) with both end in \( S \). Consider the following algorithm for covering \( f_R \).

**Algorithm 1: Area-Cover(\( \vec{G}, c, f, R \))**

1. bidirect the edges in \( \gamma(V \setminus R) \) and direct into \( R \) the edges in \( \delta(R) \)
2. compute an optimal directed edge-cover \( I' \) of \( f_R \)
3. return the underlying undirected edge set \( I \) of \( I' \)

If \( f_R \) is positively intersecting supermodular, then step 2 in the algorithm can be implemented in polynomial time if the Biset-LP for \( f_R \) can be solved in polynomial time. In many specific cases strongly polynomial algorithms are available. E.g., if \( f \) is obtained by zeroing the function \( k - |\partial \mathcal{A}| \) on void bisets then we can use the Frank-Tardos algorithm [14] or the algorithm of Frank [10] for finding a directed min-cost \( k \)-in-connected subgraph – in the above reduction described for covering a fan function \( g_R \), the edge set \( F_r \) will have \( k = \max_{\mathcal{A}} f(\mathcal{A}) \) parallel directed edges from each \( v \in R \) to the root \( r \).

The following lemma relates the cost of the solution computed by Algorithm 1 to the Biset-LP value.

**Lemma 3.** Let \( f_R \) be positively intersecting supermodular and let \( x \) be a feasible Biset-LP solution for covering \( f_R \). Then Algorithm 1 returns an \( f_R \)-cover \( I \) of cost \( c(I) \leq \sum_{e \in \delta(R)} c_e x_e + 2 \sum_{e \in \gamma(V \setminus R)} c_e x_e \).
Proof. Edges in $\gamma(R)$ do not cover $f_R$-positive bisets, hence they can be removed. Let $E'$ be the bi-direction of $E$, where each undirected edge $e$ with ends $u,v$ is replaced by two arcs $uv, vu$ of cost $c_e$ and value $x_e$ each. Let $x'$ be obtained by zeroing the value of arcs leaving $R$; these arcs do not cover $f$-positive bisets.

We claim that:

$$c(I) \leq c(I') \leq \sum_{e \in E'} c_e x'_e = \sum_{e \in \delta(R)} c_e x_e + 2 \sum_{e \in \gamma(V \setminus R)} c_e x_e$$

The first inequality is obvious. The second inequality is since $f_R$ is positively intersecting supermodular and since $x'$ is a directed feasible Biset-LP solution for $f_R$ while $I'$ is an optimal one. The equality is by the construction. \qed

Assuming that for any residual function of $f^I$ of $f$, Algorithm\[1\] can be implemented in polynomial time whenever $f^I_R$ is positively intersecting supermodular, and that the Biset-LP for covering $f^I$ can be solved in polynomial time, we prove the following theorem that implies Theorem 1.

**Theorem 2.** Undirected Biset-Function Edge-Cover with symmetric crossing supermodular $f$ admits approximation ratio $2(2+1/\ell)$, where $\ell$ is the largest integer such that:

- $n \geq (2k_f - 1)(2k_f^2 - 3k_f + 2)^\ell + 1$ for symmetric crossing supermodular $f$.
- $n \geq k_g[(2k_g^2 - 3k_g + 2)^\ell + 1]$ if $f$ is obtained by zeroing an intersecting supermodular function $g$ on co-void bisets,
- $n \geq k[(k^2 - 1)(2k^2 - 3k + 2)^{\ell - 1} + 1]$ if $f = f_{k-CS}$.

3 Covering crossing supermodular functions (Theorem 2)

A biset function $f$ is **positively skew-supermodular** if the supermodular inequality or the co-supermodular inequality $f(A \setminus B) + f(B \setminus A) \geq f(A) + f(B)$ holds for $f$-positive bisets. The corresponding Biset-Function Edge-Cover problem, when $f$ is positively skew-supermodular, admits ratio 2 (assuming the Biset-LP can be solved in polynomial time) [9]; see also [11,10] for a simpler proof along the proof line of [25] for the set functions case.

We say that $A, B$ **co-cross** if $A \setminus B$ and $B \setminus A$ are both non-void, and that $A, B$ **independent** if they do not cross nor co-cross. One can verify that $A, B$ are independent if and only if at least one of the following holds: $A \subseteq \partial B$, or $A^* \subseteq \partial B$, or $B \subseteq \partial A$, or $B^* \subseteq \partial A$. A biset function $f$ is **independence-free** if no pair of $f$-positive bisets are independent. It is easy to see that if $f$ is symmetric and if $|A| \geq k_f$ holds for every $f$-positive biset $A$ then $f$ is independence-free.

**Lemma 4 ([19]).** Let $f$ be a symmetric crossing supermodular biset function. If $A, B$ are not independent then the supermodular or the co-supermodular inequality holds for $A, B$ and $f$. Thus if $f$ is independence-free then $f$ is positively skew-supermodular.
Proof. If $A, B$ cross then the supermodular inequality holds for $A, B$. Assume that $A, B$ co-cross. Then $A$ and $B^*$ cross, and thus the supermodular inequality holds for $A, B^*$ and $f$. Note that (i) $A \setminus B = A \cap B^*$; (ii) $A \cup B^*$ is the co-biset of $B \setminus A$, hence $f(A \cup B^*) = f(B \setminus A)$, by the symmetry of $f$. Thus we get $f(A \setminus B) + f(B \setminus A) = f(A \cap B^*) + f(A \cup B^*) \geq f(A) + f(B^*) = f(A) + f(B)$. \hfill \Box

This suggests a two phase strategy for covering an “almost” supermodular function $f$. First, find a “cheap” edge set $J$ such that the residual function $f^J$ will be independence-free so $f^J$ will have “good uncrossing properties”. Second, use some “known” algorithms to cover $f^J$. The idea is due to Jackson and Jordán \cite{19}, and it is also the basis of the algorithm of Cheriyan and Végh \cite{4} (see also \cite{26} where the same idea was used for a related problem). Specifically, if $f$ is crossing supermodular, we will seek a cheap $J$ that $f$-covers all bisets $A$ with $|A| \leq k_f - 1$; by Lemma \ref{lem:cheap-edges} the residual function $f^J$ will be positively skew-supermodular so we can use the 2-approximation algorithms of \cite{9} to cover $f^J$.

The algorithm of Cheriyan and Végh \cite{4} finds an edge $J$ as above of cost \leq 4\tau(f), by covering two fan functions. Our algorithm covers a pair of area functions. In fact, we will cover a sequence of $\ell \geq \ell$ pairs of area functions, and with the help of Lemma \ref{lem:sum-costs} will show that the sum of their costs is at most $2\tau(\ell + 1)$; we choose the cheapest pair cover that will have cost \leq 2\tau(1 + 1/\ell).

For an integer $p$ let $U(f, p) = \bigcup \{A : f(A) > 0, |A| \leq p\}$ be the union of inner parts of size \leq $p$ of the $f$-positive bisets. Note that if $R \subseteq V \setminus U(f, p)$ and if $I$ covers $f_R$ then $f^I(\partial R) \leq 0$ whenever $|A| \leq p$. Thus from Lemma \ref{lem:cheap-edges} we get:

**Corollary 1.** If $R \subseteq V \setminus U(f, k_f)$ and if $I'$ is an $f_R$-cover, then the residual function $f^{I'}$ of $f$ w.r.t. $I'$ is independence-free and thus is positively skew-supermodular.

Thus we just need to find $R \subseteq V \setminus U(f, k_f)$ with $|R| \geq k_f$ and compute a 2-approximate cover of $f_R$ – the residual function will be independence-free and thus positively skew-supermodular. However, such $R$ may not exist, e.g., for $f = f_{k-CS}$ we have $U(f, k) = V$. The idea of Cheriyan and Végh \cite{4} resolves this difficulty as follows: first find a “cheap” edge set $I$ such that $|U(f^I, k_f)| \leq n - k_f$ will hold for the residual function $f^I$, and only then compute for $f^I$ an edge set $I'$ as in Corollary \ref{cor:cheap-edges}. Then the function $f^{I \cup I'}$ is independence-free and thus is positively skew-supermodular.

Variants of the next lemma were proved in \cite{416} (our bound is just slightly better), and we use it to show that $I$ as above can be a cover of an area function, provided that $n$ is large enough. Let us say that a biset family $F$ is **weakly posi-uncrossable** if for any $A, B \in F$ such that both bisets $A \setminus B, B \setminus A$ are non-void, one of them is in $F$. If $f$ is crossing supermodular and symmetric then the family $F$ of $f$-positive bisets is weakly posi-uncrossable, see \cite{416}.

**Lemma 5 (\cite{416}).** Let $F$ be a weakly posi-uncrossable biset family, let $p = \max_{A \in F} |A|, q = \max_{A \in F} |\partial A|, U = \bigcup_{A \in F} A$, and let $\nu$ be the maximum number of pairwise inner part disjoint bisets in $F$. Then $|U| \leq \nu(2q(p - 1) + p)$.
Proof. Let \( \mathcal{F}' \) be an inclusion minimal subfamily of \( \mathcal{F} \) such that \( \bigcup_{\mathcal{A} \in \mathcal{F}'} \mathcal{A} = U \).

By the minimality of \( |\mathcal{F}'| \), for every \( \mathcal{A}_i \in \mathcal{F}' \) there is \( v_i \in A_i \) such that \( v_i \notin A_j \) for every \( j \neq i \). For every \( i \) let \( C_i \) be an inclusion minimal member of the family \( \{ \mathcal{C} \in \mathcal{F} : \mathcal{C} \subseteq \mathcal{A}_i, v_i \in \mathcal{C} \} \), where here \( A \subseteq B \) means that \( A \subseteq B \) and \( A^+ \subseteq B^+ \).

Since \( \mathcal{F} \) is weakly posi-uncrossable, the minimality of \( C_i \) implies that exactly one of the following holds for any \( i \neq j \):

- \( v_i \in \partial C_j \) or \( v_j \in \partial C_i \);
- \( C_i = C_i \setminus C_j \) or \( C_j = C_j \setminus C_i \).

Construct an auxiliary directed graph \( \mathcal{J} \) on node set \( \mathcal{C} = \{ C_i : \mathcal{A}_i \in \mathcal{F}' \} \). Add an arc \( C_i C_j \) if \( v_i \in \partial C_j \). The in-degree in \( \mathcal{J} \) of a node \( C_i \) is at most \( |\partial C_i| \leq q \). Thus every subgraph of the underlying graph of \( \mathcal{J} \) has a node of degree \( \leq 2q \).

A graph is \( d \)-degenerate if every subgraph of it has a node of degree \( \leq d \). It is known that any \( d \)-degenerate graph is \( (d + 1) \)-colorable. Hence \( \mathcal{J} \) is \( (2q + 1) \)-colorable, so its node set can be partitioned into at most \( 2q + 1 \) independent sets, say \( C_1, C_2, \ldots \), where the bisets in each independent set are pairwise inner part disjoint. W.l.o.g. we may assume that \( C_1 \) is a maximal subfamily in \( \mathcal{C} \) of pairwise inner part disjoint bisets, so any \( \mathcal{C} \in \mathcal{C} \setminus C_1 \) intersects some biset in \( C_1 \). Let \( \mathcal{F}'_1 \) be the subfamily of \( \mathcal{F}' \) that corresponds to \( C_1 \), so \( |\mathcal{F}'_1| = |C_1| \leq \nu \). Let \( U_i = \bigcup_{\mathcal{A} \in \mathcal{F}_1} \mathcal{A} \).

An easy argument shows that \( |U_1| \leq \nu p \) and that \( |U_i \setminus U_1| \leq \nu(p - 1) \) for \( i \geq 2 \). Consequently, \( |U| \leq \nu p + 2\nu \nu(p - 1) \), as claimed. \( \square \)

**Corollary 2.** If \( f \) is symmetric crossing supermodular and if \( I \) is a cover of \( f_R \) then \( |U(f^I, k_f) \cup R| \leq |R|(2k_f^2 - 3k_f + 2) \).

**Proof.** Denote \( r = |R|, r' = |U(f^I, k_f) \cap R| \), and \( k = k_f \). Substituting \( q + 1 = p = k_f \) and observing that \( \nu \leq r' \) in Lemma 5 we get

\[
|U(f^I, k_f) \cup R| \leq r'[(2(k - 1)^2 + k) + (r - r')] \leq r[(2(k - 1)^2 + k) = r(2k^2 - 3k + 2)
\]

as required. \( \square \)

Let us skip for a moment implementation details, and focus on bounding the cost of an edge set \( J \) computed by the following algorithm.

**Algorithm 2:** GROWING-COVER(\( \hat{G}, c, f \))

1. let \( \emptyset \neq R_1 \subset V \)
2. for \( i = 1 \) to \( \ell \) do
   3. \( I \leftarrow \text{AREA-COVER}(\hat{G}, c, f, R_i) \)
   4. \( R_{i+1} \leftarrow U(f^I, k_f) \cup R_i \)
   5. \( I' \leftarrow \text{AREA-COVER}(\hat{G}, c, f, V \setminus R_{i+1}) \)
   6. \( J_i \leftarrow I \cup I' \)
3. return the cheapest edge set \( J \) among the edge sets \( J_1, \ldots, J_\ell \) computed

Let us fix some optimal Biset-LP solution \( x \). For an edge set \( F \) the \( x \)-cost of \( F \) is defined as \( \sum_{e \in F} c_e x_e \). Let us use the following notation:
\( \tau = \sum_{e \in E} c_e x_e \) is the optimal solution value.

- \( \gamma_i \) is the \( x \)-cost of the edges with both ends in \( R_i \).
- \( \delta_i \) is the \( x \)-cost of the edges with one end in \( R_i \) and the other in \( V \setminus R_i \).
- \( \bar{\gamma}_i \) is the \( x \)-cost of the edges with both ends in \( V \setminus R_i \).

Clearly, for any \( i \) we have

\[
\tau = \gamma_i + \delta_i + \bar{\gamma}_i
\]

By Lemma 3 the cost of the covers \( I, I' \) computed at iteration \( i \) is bounded by

\[
c(I) \leq \delta_i + 2\bar{\gamma}_i \\
c(I') \leq \delta_{i+1} + 2\bar{\gamma}_{i+1}
\]

Thus we get

\[
c(J_i) \leq (\delta_i + 2\bar{\gamma}_i) + (\delta_{i+1} + 2\bar{\gamma}_{i+1}) = (\delta_i + \bar{\gamma}_i + \gamma_i) + (\delta_{i+1} + \bar{\gamma}_{i+1} + \gamma_i + 1) - (\bar{\gamma}_{i+1} - \gamma_{i+1}) = 2\tau + (\bar{\gamma}_i - \gamma_i) - (\bar{\gamma}_{i+1} - \gamma_{i+1})
\]

Summing this over \( \ell \) iterations and observing that the sum is telescopic we get

\[
\sum_{i=1}^{\ell} c(J_i) \leq 2\ell \tau + \sum_{i=1}^{\ell} [(\bar{\gamma}_i - \gamma_i) - (\bar{\gamma}_{i+1} - \gamma_{i+1})] = 2\ell \tau + (\bar{\gamma}_1 - \gamma_1) - (\bar{\gamma}_{\ell+1} - \gamma_{\ell+1}) = 2\tau (\ell + 1) - (2\gamma_1 + \delta_1 + \bar{\gamma}_{\ell+1} + \delta_{\ell+1})
\]

Thus there exists an index \( i \) such that

\[
c(J_i) \leq 2\tau (1 + 1/\ell)
\]

Note that if \( R_{i+1} = R_i \) for some \( i \) then \( c(J_i) \leq c(I) + c(I') \leq 2\delta_i + 2\bar{\gamma}_i + 2\bar{\gamma}_i = 2\tau \), hence in this case the algorithm can terminate with \( J = J_i \) and \( c(J) \leq 2\tau \).

Next we use Corollary 2 to lower bound \( n \) to ensure that the algorithm will have \( \ell \) iterations. Let \( r = |R_1| \), and \( r \) is also a lower bound on \( n - |R_\ell| \). To see the bounds on \( n \) in Theorem 2 note the following.

- In the case of intersecting supermodular \( f \) we choose \( r = 2k_f - 1 \) and need \( r(2k_f^2 - 3k_f + 2)^{\ell} \leq n - r \), namely, \( n \geq (2k_f - 1) [(2k_f^2 - 3k_f + 2)^\ell + 1] \).
- If \( f \) is obtained by zeroing an intersecting supermodular function \( g \) on covoid bisets we choose \( r = k_g \) and need \( r(2k_g^2 - 3k_g + 2)^{\ell} \leq n - r \), namely, \( n \geq k [(2k_g^2 - 3k_g + 2)^\ell + 1] \).
- When \( f = f_{k-\text{cs}} \), 10 shows a choice of \( R_1 \) such that \( |R_2| \leq k^3 - k \). We need \( (k^3 - k)(2k^2 - 3k + 2)^{\ell-1} \leq n - k \), namely \( n \geq k [(k^2 - 1)(2k^2 - 3k + 2)^{\ell-1} + 1] \).
To get a polynomial time implementation we need to find in step 4 the set $R_{i+1} = R_i \cup U(f^i, k_f)$ in polynomial time. We modify the algorithm by relaxing the step 4 condition $R_{i+1} = R_i \cup U(f^i, k_f)$ to $R_i \subseteq R_{i+1} \subseteq R_i \cup U(f^i, k_f)$ (so $R_1 \subseteq R_2 \subseteq \ldots$ will be a nested family), but require that for each $J_i = I \cup I'$ the algorithm will compute a cover $F_i$ of $f^{I_i}$ of cost $c(F_i) \leq 2 \tau(f^{I_i})$. This can be done in the same way as in [4], as follows.

The iterative rounding 2-approximation algorithm of [9] for covering a positively skew supermodular biset function, when applied on an arbitrary biset function $h$, either returns a 2-approximate cover $J$ of $h$, or a failure certificate: a pair $\mathcal{A}, \mathcal{B}$ of bisets with $h(\mathcal{A}) > 0$ and $h(\mathcal{B}) > 0$ for which both the supermodular and the co-supermodular inequality does not hold. In our case this can happen only if $\mathcal{A}, \mathcal{B}$ are independent, by Lemma 4.

Now consider some iteration $i$ of the algorithm. Since $f^{I' \cup I}$ is symmetric, then by interchanging the roles of $\mathcal{A}, \mathcal{A}^*, \mathcal{B}, \mathcal{B}^*$, we can assume w.l.o.g. that our failure certificate $\mathcal{A}, \mathcal{B}$ satisfies $\mathcal{A} \subseteq \partial \mathcal{B}$. We thus apply the following procedure. Start with $R_{i+1} = R_i$. Then iteratively, find $I'$ as in step 5 and apply the 2-approximation algorithm of [9] for covering $h = f^{I \cup I'}$: if the algorithm returns an edge set $F$ of cost $c(F) \leq 2 \tau(h)$, we keep the current $R_{i+1}$, set $J_i \leftarrow I \cup I'$ and $F_i \leftarrow F$, and continue to the next iteration. Else, we have a failure certificate pair $\mathcal{A}, \mathcal{B}$ of $h$-positive bisets with $\mathcal{A} \subseteq \partial \mathcal{B}$. Then $\mathcal{A} \subseteq U(f^I, k_f)$ and $\mathcal{A} \setminus R_{i+1} \neq \emptyset$ (since $I'$ $f$-covers bisets whose inner part is contained in $R_{i+1}$), and we can apply the same procedure with a larger candidate set $R_{i+1} \leftarrow R_{i+1} \cup \mathcal{A}$.

This concludes the proof of Theorem 2.

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