MORE ON CARDINAL INVARIANTS OF ANALYTIC P-IDEALS

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Abstract. Given an ideal $I$ on $\omega$ let $a(I)$ ($\bar{a}(I)$) be minimum of the cardinalities of infinite (uncountable) maximal $I$-almost disjoint subsets of $\omega^\omega$. We show that $a(I_h) > \omega$ if $I_h$ is a summable ideal; but $a(Z_{\vec{\mu}}) = \omega$ for any tall density ideal $Z_{\vec{\mu}}$ including the density zero ideal $Z$. On the other hand, you have $b \leq \bar{a}(I)$ for any analytic $P$-ideal $I$, and $\bar{a}(Z_{\vec{\mu}}) \leq a$ for each density ideal $Z_{\vec{\mu}}$.

For each ideal $I$ on $\omega$ denote $b_I$ and $d_I$ the unbounding and dominating numbers of $\langle \omega^\omega, \leq I \rangle$ where $f \leq I \forall x \in I \cap V^P \exists y \in I \cap V x \subseteq y$. $P$ is $I$-bounding iff $\forall x \in I \cap V^P \exists y \in I \cap V x \subseteq y$. $P$ is $I$-dominating iff $\exists y \in I \cap V \forall x \in I \cap V x \subseteq^* y$.

For each analytic $P$-ideal $I$ if a poset $P$ has the Sacks property then $P$ is $I$-bounding; moreover if $I$ is tall as well then the property $I$-bounding/$I$-dominating implies $\omega^\omega$-bounding/adding dominating reals, and the converses of these two implications are false.

For the density zero ideal $Z$ we can prove more: (i) a poset $P$ is $Z$-bounding iff it has the Sacks property, (ii) if $P$ adds a slalom capturing all ground model reals then $P$ is $Z$-dominating.

1. Introduction

In this paper we investigate some properties of some cardinal invariants associated with analytic $P$-ideals. Moreover we analyze related “bounding” and “dominating” properties of forcing notions.

Let us denote $FIN$ the Frechet ideal on $\omega$, i.e. $FIN = [\omega]^<\omega$. Further we always assume that if $I$ is an ideal on $\omega$ then the ideal is proper, i.e. $\omega \notin I$, and $\fin \subseteq I$, so especially $I$ is non-principal. Write $I^+ = P(\omega) \setminus I$ and $I^* = \{ \omega \setminus X : X \in I \}$.

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An ideal $\mathcal{I}$ on $\omega$ is analytic if $\mathcal{I} \subseteq \mathcal{P}(\omega) \simeq 2^\omega$ is an analytic set in the usual product topology. $\mathcal{I}$ is a $P$-ideal if for each countable $\mathcal{C} \subseteq \mathcal{I}$ there is an $X \in \mathcal{I}$ such that $Y \subseteq^* X$ for each $Y \in \mathcal{C}$, where $A \subseteq^* B$ iff $A \setminus B$ is finite. $\mathcal{I}$ is tall (or dense) if each infinite subset of $\omega$ contains an infinite element of $\mathcal{I}$.

A function $\varphi : \mathcal{P}(\omega) \to [0, \infty]$ is a submeasure on $\omega$ iff $\varphi(X) \leq \varphi(Y)$ for $X \subseteq Y \subseteq \omega$, $\varphi(X \cup Y) \leq \varphi(X) + \varphi(Y)$ for $X, Y \subseteq \omega$, and $\varphi(\{n\}) < \infty$ for $n \in \omega$. A submeasure $\varphi$ is lower semicontinuous iff $\varphi(X) = \lim_{n \to \infty} \varphi(X \cap n)$ for each $X \subseteq \omega$. A submeasure $\varphi$ is finite if $\varphi(\omega) < \infty$. Note that if $\varphi$ is a lower semicontinuous submeasure on $\omega$ then $\varphi(\bigcup_{n \in \omega} A_n) \leq \sum_{n \in \omega} \varphi(A_n)$ holds as well for $A_n \subseteq \omega$. We assign the exhaustive ideal $\text{Exh}(\varphi)$ to a submeasure $\varphi$ as follows

$$\text{Exh}(\varphi) = \{X \subseteq \omega : \lim_{n \to \infty} \varphi(X \setminus n) = 0\}.$$ 

Solecki, [So, Theorem 3.1], proved that an ideal $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is an analytic $P$-ideal or $\mathcal{I} = \mathcal{P}(\omega)$ iff $\mathcal{I} = \text{Exh}(\varphi)$ for some lower semicontinuous finite submeasure. Therefore each analytic $P$-ideal is $F_{\sigma\delta}$ (i.e. $\Pi^1_3$) so a Borel subset of $2^\omega$. It is straightforward to see that if $\varphi$ is a lower semicontinuous finite submeasure on $\omega$ then the ideal $\text{Exh}(\varphi)$ is tall iff $\lim_{n \to \infty} \varphi(\{n\}) = 0$.

Let $\mathcal{I}$ be an ideal on $\omega$. A family $A \subseteq \mathcal{I}^+$ is $\mathcal{I}$-almost-disjoint (I-AD in short), if $A \cap B \in \mathcal{I}$ for each $\{A, B\} \in [A]^2$. An I-AD family $A$ is an $\mathcal{I}$-MAD family if for each $X \in \mathcal{I}^+$ there exists an $A \in A$ such that $X \cap A \in \mathcal{I}^+$, i.e. $A$ is $\subseteq^*$-maximal among the $\mathcal{I}$-AD families.

Denote $\mathfrak{a}(\mathcal{I})$ the minimum of the cardinalities of infinite $\mathcal{I}$-MAD families. In Theorem 2.2 we show that $\mathfrak{a}(\mathcal{I}_h) > \omega$ if $\mathcal{I}_h$ is a summable ideal; but $\mathfrak{a}(\mathcal{Z}_\mu) = \omega$ for any tall density ideal $\mathcal{Z}_\mu$ including the density zero ideal

$$\mathcal{Z} = \left\{A \subseteq \omega : \lim_{n \to \infty} \frac{|A \cap n|}{n} = 0\right\}.$$ 

On the other hand, if you define $\mathfrak{a}(\mathcal{I})$ as minimum of the cardinalities of uncountable $\mathcal{I}$-MAD families then you have $\mathfrak{b} \leq \mathfrak{a}(\mathcal{I})$ for any analytic $P$-ideal $\mathcal{I}$, and $\mathfrak{a}(\mathcal{Z}_\mu) \leq \mathfrak{a}$ for each density ideal $\mathcal{Z}_\mu$ (see Theorems 2.6 and 2.8).

In Theorem 3.1 we prove under CH the existence of an uncountable Cohen-indestructible $\mathcal{I}$-MAD families for each analytic $P$-ideal $\mathcal{I}$.

A sequence $\langle A_\alpha : \alpha < \kappa \rangle \subseteq [\omega]^{\omega}$ is a tower if it is $\subseteq^*$-descending, i.e. $A_\beta \subseteq^* A_\alpha$ if $\alpha \leq \beta < \kappa$, and it has no pseudointersection, i.e. a set $X \subseteq [\omega]^{\omega}$ such that $X \subseteq^* A_\alpha$ for each $\alpha < \kappa$. In Section 4 we show it is consistent that the continuum is arbitrarily large and for each tall analytic $P$-ideal $\mathcal{I}$ there is towers of height $\omega_1$ whose elements are in $\mathcal{I}^*$. 
Given an ideal $\mathcal{I}$ on $\omega$ if $f, g \in \omega^\omega$ write $f \leq_I g$ if \{ $n \in \omega : f(n) > g(n)$ \} $\in \mathcal{I}$. As usual let $\leq^* = \leq_{\text{fin}}$. The unbounding and dominating numbers of the partially ordered set $\langle \omega^\omega, \leq_I \rangle$, denoted by $b_\mathcal{I}$ and $d_\mathcal{I}$ are defined in the natural way, i.e. $b_\mathcal{I}$ is the minimal size of a $\leq_I$-unbounded family, and $d_\mathcal{I}$ is the minimal size of a $\leq_I$-dominating family. By these notations $b = b_{\text{fin}}$ and $d = d_{\text{fin}}$. In Section 5 we show that $b_\mathcal{I} = b_{\text{fin}}$ and $d_\mathcal{I} = d_{\text{fin}}$. In Section 6 we introduce the $\mathcal{I}$-bounding and $\mathcal{I}$-dominating proper- ties of forcing notions for Borel ideals: $\mathbb{P}$ is $\mathcal{I}$-bounding iff any element of $\mathcal{I} \cap V^\mathbb{P}$ is contained in some element of $\mathcal{I} \cap V$; $\mathbb{P}$ is $\mathcal{I}$-dominating iff there is an element in $\mathcal{I} \cap V^\mathbb{P}$ which mod-finite contains all elements of $\mathcal{I} \cap V$.

In Theorem 6.2 we show that for each tall analytic P-ideal $\mathcal{I}$ if a forcing notion is $\mathcal{I}$-bounding then it is $\omega^\omega$-bounding, and if it is $\mathcal{I}$-dominating then it adds dominating reals. Since the random real forcing is not $\mathcal{I}$-bounding for each tall summable and tall density ideal $\mathcal{I}$ by Proposition 6.3 the converse of the first implication is false. Since a $\sigma$-centered forcing can not be $\mathcal{I}$-dominating for a tall analytic P-ideal $\mathcal{I}$ by Theorem 6.4 the standard dominating real forcing $\mathbb{D}$ witnesses that the converse of the second implication is also false.

We prove in Theorem 6.5 that the Sacks property implies the $\mathcal{I}$-bounding property for each analytic P-ideal $\mathcal{I}$.

Finally, based on a theorem of Fremlin we show that the $\mathcal{Z}$-bounding property is equivalent to the Sacks property.

2. AROUND THE ALMOST DISJOINTNESS NUMBER OF IDEALS

For any ideal $\mathcal{I}$ on $\omega$ denote $a(\mathcal{I})$ the minimum of the cardinalities of infinite $\mathcal{I}$-MAD families.

To start the investigation of this cardinal invariant we recall the definition of two special classes of analytic P-ideals: the density ideals and the summable ideals (see [Fa]).

**Definition 2.1.** Let $h : \omega \to \mathbb{R}^+$ be a function such that $\sum_{n \in \omega} h(n) = \infty$. The summable ideal corresponding to $h$ is

$$\mathcal{I}_h = \left\{ A \subseteq \omega : \sum_{n \in A} h(n) < \infty \right\}.$$
Let $\langle P_n : n < \omega \rangle$ be a decomposition of $\omega$ into pairwise disjoint nonempty finite sets and let $\vec{\mu} = \langle \mu_n : n \in \omega \rangle$ be a sequences of probability measures, $\mu_n : \mathcal{P}(P_n) \to [0, 1]$. The \textit{density ideal generated by} $\vec{\mu}$ is

$$\mathcal{Z}_{\vec{\mu}} = \{ A \subseteq \omega : \lim_{n \to \infty} \mu_n(A \cap P_n) = 0 \}.$$ 

A summable ideal $\mathcal{I}_h$ is tall iff $\lim_{n \to \infty} h(n) = 0$; and a density ideal $\mathcal{Z}_{\vec{\mu}}$ is tall iff

$$(\dagger) \quad \lim \max_{n \to \infty} \mu_n(\{i\}) = 0.$$ 

Clearly the density zero ideal $\mathcal{Z}$ is a tall density ideal, and the summable and the density ideals are proper ideals.

\textbf{Theorem 2.2.} (1) $a(\mathcal{I}_h) > \omega$ for any summable ideal $\mathcal{I}_h$. (2) $a(\mathcal{Z}_{\vec{\mu}}) = \omega$ for any tall density ideal $\mathcal{Z}_{\vec{\mu}}$.

\textit{Proof.} (1): We show that if $\{ A_n : n < \omega \} \subseteq \mathcal{I}_h$ is $\mathcal{I}$-AD then there is $B \in \mathcal{I}_h^+$ such that $B \cap A_n \in \mathcal{I}$ for $n \in \omega$.

For each $n \in \omega$ let $B_n \subseteq A_n \setminus \cup\{ A_m : m < n \}$ be finite such that

$$\sum_{i \in B_n} h(i) > 1,$$

and put

$$B = \cup\{ B_n : n \in \omega \}.$$ 

(2): Write $\vec{\mu} = \langle \mu_n : n \in \omega \rangle$ and $\mu_n$ concentrates on $P_n$. By (\dagger) we have $\lim_{n \to \infty} |P_n| = \infty$.

Now for each $n$ we can choose $k_n \in \omega$ and a partition $\{ P_{n,k} : k < k_n \}$ of $P_n$ such that

(a) $\lim_{n \to \infty} k_n = \infty$,

(b) if $k < k_n$ then $\mu_n(P_{n,k}) \geq \frac{1}{2^{k+1}}$.

Put $A_k = \cup\{ P_{n,k} : k < k_n \}$ for each $k \in \omega$. We show that $\{ A_k : k \in \omega \}$ is a $\mathcal{Z}_{\vec{\mu}}$-MAD family.

If $k_n > k$ then $\mu_n(A_k \cap P_n) = \mu_n(P_{n,k}) \geq \frac{1}{2^{k+1}}$. Since for an arbitrary $k$ for all but finitely many $n$ we have $k_n > k$ it follows that

$$\limsup_{n \to \infty} \mu_n(A_k \cap P_n) = \limsup_{n \to \infty} \mu_n(P_{n,k}) \geq \limsup_{n \to \infty} \frac{1}{2^{k+1}} = \frac{1}{2^{k+1}} > 0,$$

thus $A_k \in \mathcal{Z}_{\vec{\mu}}^+$. 

Assume that $X \in \mathcal{Z}_{\vec{\mu}}^+$ Pick $\varepsilon > 0$ with $\limsup_{n \to \infty} \mu_n(X \cap P_n) > \varepsilon$. For a large enough $k$ we have $\frac{1}{2^{k+1}} < \frac{\varepsilon}{2}$ so if $k < k_n$ then

$$\mu_n(P_n \setminus \cup\{ P_{n,i} : i \leq k \}) \leq \frac{1}{2^{k+1}} < \frac{\varepsilon}{2}.$$ 

So for each large enough $n$ there is $i_n \leq k$ such that $\mu_n(X \cap P_{n,i_n}) > \frac{\varepsilon}{2^{k+1}}$. Then $i_n = i$ for infinitely many $n$, so $\limsup_{n \to \infty} \mu_n(X \cap A_i) \geq \frac{\varepsilon}{2^{k+1}}$, and so $X \cap A_i \in \mathcal{Z}_{\vec{\mu}}^+$. □
This Theorem gives new proof of the following well-known fact:

**Corollary 2.3.** The density zero ideal \( Z \) is not a summable ideal.

Given two ideals \( \mathcal{I} \) and \( \mathcal{J} \) on \( \omega \) write \( \mathcal{I} \leq_{\text{RK}} \mathcal{J} \) (see \([\text{Ru}]\)) iff there is a function \( f : \omega \to \omega \) such that
\[
\mathcal{I} = \{ I \subseteq \omega : f^{-1}I \in \mathcal{J} \},
\]
and write \( \mathcal{I} \leq_{\text{RB}} \mathcal{J} \) (see \([\text{LaZh}]\)) iff there is a finite-to-one function \( f : \omega \to \omega \) such that
\[
\mathcal{I} = \{ I \subseteq \omega : f^{-1}I \in \mathcal{J} \}.
\]

The following Observations imply that there are \( \mathcal{I} \)-MAD families of cardinality \( \text{c} \) for each analytic P-ideal \( \mathcal{I} \).

**Observation 2.4.** Assume that \( \mathcal{I} \) and \( \mathcal{J} \) are ideals on \( \omega \), \( \mathcal{I} \leq_{\text{RK}} \mathcal{J} \) witnessed by a function \( f : \omega \to \omega \). If \( \mathcal{A} \) is an \( \mathcal{I} \)-AD family then
\[
\{ f^{-1}A : A \in \mathcal{A} \}
\]
is a \( \mathcal{J} \)-AD family.

**Observation 2.5.** \( \text{fin} \leq_{\text{RB}} \mathcal{I} \) for any analytic P-ideal \( \mathcal{I} \).

**Proof.** Let \( \mathcal{I} = \text{Exh}(\varphi) \) for some lower semicontinuous finite submeasure \( \varphi \) on \( \omega \). Since \( \omega \notin \mathcal{I} \) we have \( \lim_{n \to \infty} \varphi(\omega \setminus n) = \varepsilon > 0 \). Hence by the lower semicontinuous property of \( \varphi \) for each \( n > 0 \) there is \( m > n \) such that \( \varphi([n,m)) > \varepsilon/2 \).

So there is a partition \( \{ I_n : n < \omega \} \) of \( \omega \) into finite pieces such that \( \varphi(I_n) > \varepsilon/2 \) for each \( n \in \omega \). Define the function \( f : \omega \to \omega \) by the stipulation \( f''I_n = \{ n \} \). Then \( f \) witnesses \( \text{fin} \leq_{\text{RB}} \mathcal{I} \). \( \square \)

For any analytic P-ideal \( \mathcal{I} \) denote \( \bar{a}(\mathcal{I}) \) the minimum of the cardinalities of uncountable \( \mathcal{I} \)-MAD families.

Clearly \( a(\mathcal{I}) > \omega \) implies \( a(\mathcal{I}) = \bar{a}(\mathcal{I}) \), especially \( a(\mathcal{I}_h) = \bar{a}(\mathcal{I}_h) \) for summable ideals.

**Theorem 2.6.** \( \bar{a}(Z_{\vec{\mu}}) \leq a \) for each density ideal \( Z_{\vec{\mu}} \).

**Proof.** Let \( f : \omega \to \omega \) be the finite-to-one function defined by \( f^{-1}\{ n \} = P_n \) where \( \vec{\mu} = \langle \mu_n : n \in \omega \rangle \) and \( \mu_n : \mathcal{P}(P_n) \to [0,1] \). Specially \( f \) witnesses \( \text{fin} \leq_{\text{RB}} Z_{\vec{\mu}} \).

Let \( \mathcal{A} \) be an uncountable (fin-)MAD family. We show that \( f^{-1}[\mathcal{A}] = \{ f^{-1}A : A \in \mathcal{A} \} \) is a \( Z_{\vec{\mu}} \)-MAD family.

By Observation 2.4 \( f^{-1}[\mathcal{A}] \) is a \( Z_{\vec{\mu}} \)-AD family.

To show the maximality let \( X \in Z_{\vec{\mu}}^{+} \) be arbitrary, \( \limsup_{n \to \infty} \mu_n(X \cap P_n) = \varepsilon > 0 \). Thus
\[
J = \{ n \in \omega : \mu_n(X \cap P_n) > \varepsilon/2 \}
\]
is infinite. So there is \( A \in \mathcal{A} \) such that \( A \cap J \) is infinite.
Then \( f^{-1}A \in f^{-1}[A] \) and \( X \cap f^{-1}A \in \mathcal{Z}_{\bar{\nu}}^+ \) because there are infinitely many \( n \) such that we have \( P_n \subseteq f^{-1}A \) and \( \mu_n(X \cap P_n) > \varepsilon/2 \). \( \square \)

**Problem 2.7.** Does \( \bar{a}(I) \leq a \) hold for each analytic \( P \)-ideal \( I \)?

**Theorem 2.8.** \( b \leq \bar{a}(I) \) provided that \( I \) is an analytic \( P \)-ideal.

**Remark.** If \( \mathcal{X} \subset [\omega]^{\omega} \) is an infinite almost disjoint family then there is a tall ideal \( I \) such that \( \mathcal{X} \) is \( I \)-MAD. So the Theorem above does not hold for an arbitrary tall ideal on \( \omega \).

**Proof.** \( I = \text{Exh}(\varphi) \) for some lower semicontinuous finite submeasure \( \varphi \).

Let \( A \) be an uncountable \( I \)-AD family of cardinality smaller than \( b \). We show that \( A \) is not maximal.

There exists an \( \varepsilon > 0 \) such that the set \( A_{\varepsilon} = \{ A \in A : \lim_{n \to \infty} \varphi(A \setminus n) > \varepsilon \} \)

is uncountable. Let \( A' = \{ A_n : n \in \omega \} \subseteq A_{\varepsilon} \) be a set of pairwise distinct elements of \( A_{\varepsilon} \). We can assume that these sets are pairwise disjoint. For each \( A \in A \setminus A' \) choose a function \( f_A \in \omega^{\omega} \) such that

\[
\text{(\star_A)} \quad \varphi((A \cap A_n) \setminus f_A(n)) < 2^{-n} \text{ for each } n \in \omega.
\]

Using the assumption \( |A| < b \) there exists a strictly increasing function \( f \in \omega^\omega \) such that \( f_A \leq^* f \) for each \( A \in A \setminus A' \). For each \( n \) pick \( g(n) > f(n) \) such that \( \varphi(A_n \cap [f(n), g(n)]) > \varepsilon \), and let

\[
X = \bigcup_{n \in \omega} (A_n \cap [f(n), g(n)]).
\]

Clearly \( X \in \mathcal{Z}_{\bar{\nu}}^+ \) because for each \( n < \omega \) there is \( m \) such that \( A_m \cap [f(m), g(m)] \subseteq X \setminus n \) and so \( \varphi(X \setminus n) \geq \varphi(A_m \cap [f(m), g(m)]) > \varepsilon \), i.e. \( \lim_{n \to \infty} \varphi(X \setminus n) \geq \varepsilon \).

We have to show that \( X \cap A \in \mathcal{Z}_{\bar{\nu}}^+ \) for each \( A \in A \). If \( A = A_n \) for some \( n \) then \( X \cap A = X \cap A_n = A_n \cap [f(n), g(n)] \), i.e. the intersection is finite.

Assume now that \( A \in A \setminus A' \). Let \( \delta > 0 \). We show that if \( k \) is large enough then \( \varphi((A \cap X) \setminus k) < \delta \).

There is \( N \in \omega \) such that \( 2^{-N+1} < \delta \) and \( f_A(n) \leq f(n) \) for each \( n \geq N \).

Let \( k \) be so large that \( k \) contains the finite set \( \bigcup_{n < N} [f(n), g(n)] \).
Now \((X \cap A) \setminus k = \bigcup_{n \in \omega} (A_n \cap A \cap [f(n), g(n))] \setminus k\) and \((A_n \cap A \cap [f(n), g(n))] \setminus k = \emptyset\) if \(n < N\) so

\[
(X \cap A) \setminus k = \bigcup_{n \geq N} (A_n \cap A \cap [f(n), g(n))] \subseteq \bigcup_{n \geq N} ((A_n \cap A) \setminus f(n)) \subseteq \bigcup_{n \geq N} ((A_n \cap A) \setminus f_A(n)).
\]

Thus by \((\ast_A)\) we have

\[
\varphi((X \cap A) \setminus k) \leq \sum_{n \geq N} \varphi(A_n \cap A \setminus f_A(n)) \leq \sum_{n \geq N} \frac{1}{2^n} = 2^{-N+1} < \delta.
\]

\[
\square
\]

### 3. Cohen-indestructible \(\mathcal{I}\)-MAD families

If \(\varphi\) is a lower semicontinuous finite submeasure on \(\omega\) then clearly \(\varphi\) is determined by \(\varphi \upharpoonright [\omega]^{< \omega}\). Using this observation one can define forcing indestructibility of \(\mathcal{I}\)-MAD families for an analytic P-ideal \(\mathcal{I}\). The following theorem is a modification of Kunen’s proof for existence of Cohen-indestructible MAD family from CH (see [Ku] Ch. VIII Th. 2.3).

**Theorem 3.1.** Assume CH. For each analytic P-ideal \(\mathcal{I}\) then there is an uncountable Cohen-indestructible \(\mathcal{I}\)-MAD family.

**Proof.** We will define the uncountable Cohen-indestructible \(\mathcal{I}\)-MAD family \(\{A_\xi : \xi < \omega_1\} \subseteq \mathcal{I}^+\) by recursion on \(\xi \in \omega_1\). The family \(\{A_\xi : \xi < \omega_1\}\) will be fin-AD as well. Our main concern is that we do have \(a(\mathcal{I}) > \omega\) so it is not automatic that \(\{A_\eta : \eta < \xi\}\) is not maximal for \(\xi < \omega_1\).

Denote \(\mathbb{C}\) the Cohen forcing. Let \(\mathcal{I} = \text{Exh}(\varphi)\) be an analytic P-ideal. Let \(\{\langle p_\xi, \hat{X}_\xi, \delta_\xi \rangle : \omega \leq \xi < \omega_1\}\) be an enumeration of all triples \(\langle p, \hat{X}, \delta \rangle\) such that \(p \in \mathbb{C}, \hat{X}\) is a nice name for a subset of \(\omega\), and \(\delta\) is a positive rational number.

Write \(\varepsilon = \lim_{n \to \infty} \varphi(\omega \setminus n) > 0\). Partition \(\omega\) into infinite sets \(\{A_m : m < \omega\}\) such that \(\lim_{n \to \infty} \varphi(A_m \setminus n) = \varepsilon\) for each \(m < \omega\).

Assume \(\xi \geq \omega\) and we have \(A_\eta \in \mathcal{I}^+\) for \(\eta < \xi\) such that \(\{A_\eta : \eta < \xi\}\) is a fin-AD so especially an \(\mathcal{I}\)-AD family.

**Claim:** There is \(X \in \mathcal{I}^+\) such that \(|X \cap A_\xi| < \omega\) for \(\zeta < \xi\).

**Proof of the Claim.** Write \(\xi = \{\zeta_i : i < \omega\}\). Recursion on \(j \in \omega\) we can choose \(x_j \in [A_{\ell_j}]^{\leq \omega}\) for some \(\ell_j \in \omega\) such that

(i) \(\varphi(x_j) \geq \varepsilon/2\),
(ii) $x_j \cap (\cup_{i<j} A_{\xi_i}) = \emptyset$.

Assume that $\{x_i : i < j\}$ is chosen. Pick $\ell_j \in \omega \setminus \{\xi_i : i < j\}$. Let $m \in \omega$ such that $A_{\ell_j} \cap \cup \{A_{\xi_i} : i \leq j\} \subseteq m$. Since $\varphi(A_{\ell_j} \setminus m) \geq \varepsilon$ there is $x_j \in [A_{\ell_j} \setminus m]^{<\omega}$ with $\varphi(x_j) \geq \varepsilon/2$.

Let $X = \cup \{x_j : j < \omega\}$. Then $|A_{\xi} \cap X| < \omega$ for $\zeta < \xi$ and $\lim_{n \to \infty} (X \setminus n) \geq \varepsilon/2$. □

If $p_{\xi}$ does not force (a) and (b) below then let $A_{\xi}$ be $X$ from the Claim.

(a) $\lim_{n \to \infty} \varphi(\check{X}_\xi \setminus n) > \delta_\xi$,

(b) $\forall \eta < \xi \quad X_\xi \cap A_\eta \in I$.

Assume $p_{\xi} \Vdash (a) \land (b)$. Let $\{B^\xi_k : k \in \omega\} = \{A_\eta : \eta < \xi\}$ and $\{\ell_k : k \in \omega\} = \{p' \in C : p' \leq p_{\xi}\}$ be enumerations. Clearly for each $k \in \omega$ we have

$$p_{\xi} \Vdash \lim_{n \to \infty} \check{\varphi}((\check{X}_\xi \setminus \cup \{\check{B}^\xi_l : l \leq \check{k}\}) \setminus n) > \delta_\xi,$$

so we can choose a $q^\xi_k \leq p_{\xi}$ and a finite $a^\xi_k \subseteq \omega$ such that $\varphi(a^\xi_k) > \delta_\xi$ and $q^\xi_k \Vdash a^\xi_k \subseteq (\check{X}_\xi \setminus \cup \{\check{B}^\xi_l : l \leq \check{k}\}) \setminus \check{k}$. Let $A_{\xi} = \cup \{a^\xi_k : k \in \omega\}$. Clearly $A_{\xi} \in I^+$ and $\{A_\eta : \eta \leq \xi\}$ is a fin-AD family.

Thus $\mathcal{A} = \{A_{\xi} : \xi < \omega_1\} \subseteq I^+$ is a fin-AD family.

We show that $\mathcal{A}$ is a Cohen-indestructible $I$-MAD. Assume otherwise there is a $\xi$ such that $p_{\xi} \Vdash \lim_{n \to \infty} \varphi(\check{X}_\xi \setminus n) > \delta_\xi \land \forall \eta < \omega_1 \quad \check{X}_\xi \cap \check{A}_\eta \in I$, specially $p_{\xi} \Vdash (a) \land (b)$. There is a $q^\xi_k \leq p_{\xi}$ and an $N$ such that $p_{\xi}^\ell \Vdash \check{\varphi}((\check{X}_\xi \cap \check{A}_\xi) \setminus \check{N}) < \delta_\xi$. We can assume $k \geq N$, so $p_{\xi}^\ell \Vdash \check{\varphi}((\check{X}_\xi \cap \check{A}_\xi) \setminus \check{k}) < \delta_\xi$. By the choice of $q^\xi_k$ and $a^\xi_k$, we have $q^\xi_k \Vdash a^\xi_k \subseteq (\check{X}_\xi \cap \check{A}_\xi) \setminus \check{k}$, so $q^\xi_k \Vdash \check{\varphi}((\check{X}_\xi \cap \check{A}_\xi) \setminus \check{k}) > \delta_\xi$, contradiction. □

4. Towers in $I^*$

Let $I$ be an ideal on $\omega$. A $\subseteq^*$-decreasing sequence $\langle A_\alpha : \alpha < \kappa\rangle$ is a tower in $I^*$ if (a) it is a tower (i.e. there is no $X \in [\omega]^{<\omega}$ with $X \subseteq^* A_\alpha$ for $\alpha < \kappa$), and (b) $A_\alpha \in I^*$ for $\alpha < \kappa$. Under $\text{CH}$ it is straightforward to construct towers in $I^*$ for each tall analytic $P$-ideal $I$. The existence of such towers is consistent with $2^\omega > \omega_1$ as well by the Theorem 4.2 below. Denote $C_\alpha$ the standard forcing adding $\alpha$ Cohen reals by finite conditions.

Lemma 4.1. Let $I = \text{Exh}(\varphi)$ be a tall analytic $P$-ideal in the ground model $V$. Then there is a set $X \in V^{C_1} \cap I$ such that $|X \cap S| = \omega$ for each $S \in [\omega]^{<\omega} \setminus V$. 

Proof. Since $\mathcal{I}$ is tall we have $\lim_{n \to \infty} \varphi(\{n\}) = 0$. Fix a partition $\langle I_n : n \in \omega \rangle$ of $\omega$ into finite intervals such that $\varphi(\{x\}) < \frac{1}{2^n}$ for $x \in I_{n+1}$ (we can not say anything about $\varphi(\{x\})$ for $x \in I_0$). Then $X' \in \mathcal{I}$ whenever $|X' \cap I_n| \leq 1$ for each $n$.

Let $\{i^n_k : k < k_n\}$ be the increasing enumeration of $I_n$. Our forcing $\mathbb{C}$ adds a Cohen real $c \in \omega^\omega$ over $V$. Let

$$X_\alpha = \{i^n_k : c(n) \equiv k \text{ mod } k_n\} \in V^\mathbb{C} \cap \mathcal{I}.$$ 

A trivial density argument shows that $|X_\alpha \cap S| = \omega$ for each $S \in V \cap [\omega]^\omega$. □

Theorem 4.2. $\Vdash_{\mathbb{C}_\omega}$ "There exists a tower in $\mathcal{I}^*$ for each tall analytic $\mathcal{P}$-ideal $\mathcal{I}$." 

Proof. Let $V$ be a countable transitive model and $G$ be a $\mathbb{C}_\omega$-generic filter over $V$. Let $\mathcal{I} = \text{Exh}(\varphi)$ be a tall analytic $\mathcal{P}$-ideal in $V[G]$ with some lower semicontinuous finite submeasure $\varphi$ on $\omega$. There is a $\delta < \omega_1$ such that $\varphi \upharpoonright [\omega]^\omega \in V[G_\delta]$ where $G_\delta = G \cap \mathbb{C}_\delta$, so we can assume $\varphi \upharpoonright [\omega]^\omega \in V$.

Work in $V[G]$ recursion on $\omega_1$ we construct the tower $\bar{A} = \langle A_\alpha : \alpha < \omega_1 \rangle$ in $\mathcal{I}^*$ such that $\bar{A} \upharpoonright \alpha \in V[G_\alpha]$.

Because $\mathcal{I}$ contains infinite elements we can construct in $V$ a sequence $\langle A_n : n \in \omega \rangle$ in $\mathcal{I}^*$ which is strictly $\subseteq^*$-descending, i.e. $|A_n \setminus A_{n+1}| = \omega$ for $n \in \omega$. Assume $\langle A_\xi : \xi < \alpha \rangle$ are done.

Since $\mathcal{I}$ is a $\mathcal{P}$-ideal there is $A'_\alpha \in \mathcal{I}^*$ with $A'_\alpha \subseteq^* A_\beta$ for $\beta < \alpha$.

By lemma 4.4 there is a set $X_\alpha \in V[G_{\alpha+1}] \cap \mathcal{I}$ such that $X_\alpha \cap S \neq \emptyset$ for each $S \in [\omega]^\omega \cap V[G_\alpha]$.

Let $A_\alpha = A'_\alpha \setminus X_\alpha \in V[G_{\alpha+1}] \cap \mathcal{I}^*$ so $S \not\subseteq^* A_\alpha$ for any $S \in V[G_\alpha] \cap [\omega]^\omega$. Hence $V[G] \models \langle A_\alpha : \alpha < \omega_1 \rangle$ is a tower in $\mathcal{I}^*$.

□

Problem 4.3. Do there exist towers in $\mathcal{I}^*$ for some tall analytic $\mathcal{P}$-ideal $\mathcal{I}$ in ZFC?

5. Unbounding and Dominating Numbers of Ideals

A supported relation (see [Va]) is a triple $\mathcal{R} = (A,R,B)$ where $R \subseteq A \times B$, $\text{dom}(R) = A$, $\text{ran}(R) = B$, and we always assume that for each $b \in B$ there is an $a \in A$ such that $\langle a,b \rangle \notin R$.

The unbounding and dominating numbers of $\mathcal{R}$:

$$b(\mathcal{R}) = \min \{|A' : A' \subseteq A \land \forall b \in B \ A' \not\subseteq R^{-1}\{b\}|,$$

$$d(\mathcal{R}) = \min \{|B' : B' \subseteq B \land A = R^{-1}B'|.$$ 

For example $b_\mathcal{I} = b(\omega^\omega, \leq_\mathcal{I}, \omega^\omega)$ and $d_\mathcal{I} = d(\omega^\omega, \leq_\mathcal{I}, \omega^\omega)$. Note that $b(\mathcal{R})$ and $d(\mathcal{R})$ are defined for each $\mathcal{R}$, but in general $b(\mathcal{R}) \leq d(\mathcal{R})$ does not hold.
We recall the definition of Galois-Tukey connection of relations.

Definition 5.1. ([Vd]) Let \( R_1 = (A_1, R_1, B_1) \) and \( R_2 = (A_2, R_2, B_2) \) be supported relations. A pair of functions \( \phi : A_1 \to A_2, \psi : B_2 \to B_1 \) is a Galois-Tukey connection from \( R_1 \) to \( R_2 \), in notation \( (\phi, \psi) : R_1 \preceq R_2 \) if \( a_1 R_1 \psi(b_2) \) whenever \( \phi(a_1) R_2 b_2 \). In a diagram:

\[
\begin{array}{ccc}
\psi(b_2) & \in & B_1 \\
\downarrow & & \downarrow \\
R_1 & \iff & R_2 \\
& & \\
a_1 \in A_1 & \xrightarrow{\phi} & A_2 \ni \phi(a_1)
\end{array}
\]

We write \( R_1 \preceq R_2 \) if there is a Galois-Tukey connection from \( R_1 \) to \( R_2 \). If \( R_1 \preceq R_2 \) and \( R_2 \preceq R_1 \) also hold then we say \( R_1 \) and \( R_2 \) are Galois-Tukey equivalent, in notation \( R_1 \equiv R_2 \).

Fact 5.2. If \( R_1 \preceq R_2 \) then \( b(R_1) \geq b(R_2) \) and \( \delta(R_1) \leq \delta(R_2) \).

Theorem 5.3. If \( \mathcal{I} \preceq \mathcal{J} \) then \( (\omega^\omega, \leq_{\mathcal{I}}, \omega^\omega) \equiv (\omega^\omega, \leq_{\mathcal{J}}, \omega^\omega) \).

Proof. Fix a finite-to-one function \( f : \omega \to \omega \) witnessing \( \mathcal{I} \preceq \mathcal{J} \).

Define \( \phi, \psi : \omega^\omega \to \omega^\omega \) as follows:

\[
\phi(x)(i) = \max(x'' f^{-1}\{i\}), \\
\psi(y)(j) = y(f(j)).
\]

We prove two claims.

Claim 5.3.1. \((\phi, \psi) : (\omega^\omega, \leq_{\mathcal{I}}, \omega^\omega) \preceq (\omega^\omega, \leq_{\mathcal{J}}, \omega^\omega)\).

Proof of the claim. We show that if \( \phi(x) \leq_{\mathcal{I}} y \) then \( x \leq_{\mathcal{J}} \psi(y) \). Indeed, \( I = \{i : \phi(x)(i) > y(i)\} \in \mathcal{I} \). Assume that \( f(j) = i \notin I \). Then \( \phi(x)(i) = \max(x'' f^{-1}\{i\}) \leq y(i) \). Since \( y(i) = \psi(y)(j) \), so

\[
x(j) = \max(x'' f^{-1}\{f(j)\}) \leq y(f(j)) = \psi(y)(j)
\]

Since \( f^{-1}I \in \mathcal{J} \) this yields \( x \leq_{\mathcal{J}} \psi(y) \).

Claim 5.3.2. \((\psi, \phi) : (\omega^\omega, \leq_{\mathcal{J}}, \omega^\omega) \preceq (\omega^\omega, \leq_{\mathcal{I}}, \omega^\omega)\).

Proof of the claim. We show that if \( \psi(y) \leq_{\mathcal{J}} x \) then \( y \leq_{\mathcal{I}} \phi(x) \). Assume on the contrary that \( y \not\leq_{\mathcal{I}} \phi(x) \). Then \( A = \{i \in \omega : y(i) > \phi(x)(i)\} \in \mathcal{I}^+ \). By definition of \( \phi \), we have \( A = \{i : y(i) > \max(x'' f^{-1}\{i\})\} \).

Let \( B = f^{-1}A \in \mathcal{J}^+ \). For \( j \in B \) we have \( f(j) \in A \) and so

\[
\psi(y)(j) = y(f(j)) > \phi(x)(f(j)) = \max(x'' f^{-1}\{f(j)\}) \geq x(j).
\]

Hence \( \psi(y) \not\leq_{\mathcal{I}} x \), contradiction.
These claims prove the statement of the Theorem, so we are done. □

By Fact 5.2 we have:

**Corollary 5.4.** If $I \leq_{RB} J$ holds then $b_I = b_J$ and $d_I = d_J$.

By Observation 2.5 this yields:

**Corollary 5.5.** If $I$ is an analytic $P$-ideal then $(\omega^\omega, \leq^*, \omega^\omega) \equiv (\omega^\omega, \leq_J, \omega^\omega)$, and $b_I = b$ and $d_I = d$.

6. $I$-bounding and $I$-dominating forcing notions

**Definition 6.1.** Let $I$ be a Borel ideal on $\omega$. A forcing notion $\mathbb{P}$ is $I$-bounding if

$$\mathbb{P} \models \forall A \in I \exists B \in I \cap V A \subseteq B;$$

$\mathbb{P}$ is $I$-dominating if

$$\mathbb{P} \models \exists B \in I \forall A \in I \cap V A \subseteq^* B.$$

**Theorem 6.2.** Let $I$ be a tall analytic $P$-ideal. If $\mathbb{P}$ is $I$-bounding then $\mathbb{P}$ is $\omega^\omega$-bounding as well; if $\mathbb{P}$ is $I$-dominating then $\mathbb{P}$ adds dominating reals.

**Proof.** Assume that $I = \operatorname{Exh}(\varphi)$ for some lower semicontinuous finite submeasure $\varphi$. For $A \in I$ let

$$d_A(n) = \min\{k \in \omega : \varphi(A \setminus k) < 2^{-n}\}.$$

Clearly if $A \subseteq B \in I$ then $d_A \leq d_B$.

It is enough to show that $\{d_A : A \in I\}$ is cofinal in $(\omega^\omega, \leq^*)$. Let $f \in \omega^\omega$. Since $I$ is a tall ideal we have $\lim_{k \to \infty} \varphi(\{k\}) = 0$ but $\lim_{m \to \infty} (\omega \setminus m) = \varepsilon > 0$. Thus for all but finite $n \in \omega$ we can choose a finite set $A_n \subseteq \omega \setminus f(n)$ such that $2^{-n} \leq \varphi(A_n) < 2^{-n+1}$ so $A = \cup\{A_n : n \in \omega\} \in I$ and $f \leq^* d_A$.

Why? We can assume if $k \geq f(n)$ then $\varphi(\{k\}) < 2^{-n}$. Let $n$ be so large such that $2^{-n} < \varepsilon$. Now if there is no a suitable $A_n$ then $\varphi(\omega \setminus f(n)) \leq 2^{-n} < \varepsilon$, contradiction. □

The converse of the first implication of Theorem 6.2 is not true by the following Proposition.

**Proposition 6.3.** The random forcing is not $I$-bounding for any tall summable and tall density ideal $I$. 

Proof. Denote $\mathbb{B}$ the random forcing and $\lambda$ the Lebesgue-measure.

If $\mathcal{I} = \mathcal{I}_h$ is a tall summmable ideal then we can chose pairwise disjoint sets $H(n) \in [\omega]^\omega$ such that $\sum_{l \in H(n)} h(l) = 1$ and $\max \{ h(l) : l \in H(n) \} < 2^{-n}$ for each $n \in \omega$. Let $H(n) = \{l^n_k : k \in \omega \}$. For each $n$ fix a partition $\{ [B^n_k] : k \in \omega \}$ of $\mathbb{B}$ such that $\lambda(B^n_k) = h(l^n_k)$ for each $k \in \omega$. Let $\dot{X}$ be a $\mathbb{B}$-name such that $\models_\mathbb{B} \dot{X} = \{ l^n_k : [B^n_k] \in G \}$. Clearly $\models_\mathbb{B} \dot{X} \subseteq \mathcal{I}_h$. $\dot{X}$ shows that $\mathbb{B}$ is not $\mathcal{I}_h$-bounding.

Assume on the contrary that there is a $[B] \in \mathbb{B}$ and an $A \in \mathcal{I}_h$ such that $[B] \models \dot{X} \subseteq \dot{A}$. There is an $n \in \omega$ such that

$$\sum_{l^n_k \in A} \lambda(B^n_k) = \sum_{l^n_k \in A} h(l^n_k) < \lambda(B).$$

Choose a $k$ such that $l^n_k \notin A$ and $[B^n_k] \cap [B] \neq [\emptyset]$. We have $[B^n_k] \cap [B] \models \dot{X} \subseteq \dot{A}$. contradiction.

If $\mathcal{I} = \mathcal{Z}_\mu$ is a tall density ideal then for each $n$ fix a partition $\{ [B^n_k] : k \in P_n \}$ of $\mathbb{B}$ such that $\lambda(B^n_k) = \mu_n(\{k\})$ for each $k$. Let $\dot{X}$ be a $\mathbb{B}$-name such that $\models_\mathbb{B} \dot{X} = \{ \dot{k} : [\dot{B}^n_k] \in \dot{G} \}$. Clearly $\models_\mathbb{B} \dot{X} \in \mathcal{Z}_\mu$. $\dot{X}$ shows that $\mathbb{B}$ is not $\mathcal{Z}_\mu$-bounding.

Assume on the contrary that there is a $[B] \in \mathbb{B}$ and an $A \in \mathcal{Z}_\mu$ such that $[B] \models \dot{X} \subseteq \dot{A}$. There is an $n \in \omega$ such that

$$\sum_{k \in A \cap P_n} \lambda(B^n_k) = \mu_n(A \cap P_n) < \lambda(B).$$

Choose a $k \in P_n \setminus A$ such that $[B^n_k] \cap [B] \neq [\emptyset]$. We have $[B^n_k] \cap [B] \models \dot{X} \subseteq \dot{A}$, contradiction. \qed

The converse of the second implication of Theorem 6.2 is not true as well: the Hechler forcing is a counterexample according to the following Theorem.

**Theorem 6.4.** If $\mathbb{P}$ is $\sigma$-centered then $\mathbb{P}$ is not $\mathcal{I}$-dominating for any tall analytic $P$-ideal $\mathcal{I}$.

**Proof.** Assume that $\mathcal{I} = \text{Exh}(\varphi)$ for some lower semicontinuous finite submeasure $\varphi$. Let $\varepsilon = \lim_{n \to \infty} \varphi(\omega \setminus n) > 0$.

Let $\mathbb{P} = \cup \{ C_n : n \in \omega \}$ where $C_n$ is centered for each $n$. Assume on the contrary that $\models_\mathbb{P} \dot{X} \in \mathcal{I} \land \forall A \in \mathcal{I} \cap V A \subseteq^* \dot{X}$ for some $\mathbb{P}$-name $\dot{X}$.

For each $A \in \mathcal{I}$ choose a $p_A \in \mathbb{P}$ and a $k_A \in \omega$ such that

(0) \quad $p_A \models \dot{A} \setminus k_A \subseteq \dot{X} \land \varphi(\dot{X} \setminus k_A) < \varepsilon/2$. \quad
For each $n,k \in \omega$ let $C_{n,k} = \{ A \in \mathcal{I} : p_A \in C_n \land k_A = k \}$, and let $B_{n,k} = \bigcup C_{n,k}$. We show that for each $n$ and $k$

$$\varphi(B_{n,k} \setminus k) \leq \varepsilon/2.$$  

Assume indirectly $\varphi(B_{n,k} \setminus k) > \varepsilon/2$ for some $n$ and $k$. There is a $k'$ such that $\varphi(B_{n,k} \cap [k,k']) > \varepsilon/2$ and there is a finite $D \subseteq C_{n,k}$ such that $B_{n,k} \cap [k,k') = (\cup D) \cap [k,k')$. Choose a common extension $q$ of $\{ p_A : A \in D \}$. Now we have $q \models \cup \{ A \setminus k : A \in D \} \subseteq \check{X}$ and so

$$q \models \varepsilon/2 < \varphi(\check{B}_{n,k} \cap \check{k}, \check{k'}) = \varphi((\cup \check{D}) \cap \check{k}, \check{k'}) \leq \varphi(\check{X} \cap \check{k}, \check{k'}) \leq \varphi(\check{X} \setminus \check{k}),$$

which contradicts ($\check{\circ}$).

So for each $n$ and $k$ the set $\omega \setminus B_{n,k}$ is infinite, so $\omega \setminus B_{n,k}$ contains an infinite $D_{n,k} \in \mathcal{I}$. Let $D \in \mathcal{I}$ such that $D_{n,k} \subseteq^* D$ for each $n,k \in \omega$.

Then there is no $n,k$ such that $D \subseteq^* B_{n,k}$. Contradiction. \hfill \Box

By this Theorem an by Lemma 4.14 the Cohen forcing is neither $\mathcal{I}$-dominating nor $\mathcal{I}$-bounding for any tall analytic P-ideal $\mathcal{I}$.

Finally in the rest of the paper we compare the Sacks property and the $\mathcal{I}$-bounding property.

**Theorem 6.5.** If $\mathbb{P}$ has the Sacks property then $\mathbb{P}$ is $\mathcal{I}$-bounding for each analytic P-ideal $\mathcal{I}$.

**Proof.** Let $\mathcal{I} = \text{Exh}(\varphi)$. Assume $\models_{\mathbb{P}} \check{X} \in \mathcal{I}$. Let $d_X$ be a $\mathbb{P}$-name for an element of $\omega^{<\omega}$ such that $\models_{\mathbb{P}} d_X(\check{n}) = \min \{ k \in \omega : \varphi(\check{X} \setminus k) < 2^{-n} \}$. We know that $\mathbb{P}$ is $\omega^{<\omega}$-bounding. If $p \models d_X \leq \check{f}$ for some strictly increasing $f \in \omega^{<\omega}$ then by the Sacks property there is a $q \leq p$ and a slalom $S : \omega \rightarrow [\omega]^{<\omega}$, $|S(n)| \leq n$ such that

$$q \models \forall^{\infty} n \check{X} \cap [f(n), f(n+1)) \in S(n).$$

Now let

$$A = \bigcup_{n \in \omega} \{ D \in S(n) : \varphi(D) < 2^{-n} \}.$$ 

$A \in \mathcal{I}$ because $\varphi(A \setminus f(n)) \leq \sum_{k \geq n} \varphi(A \cap [f(k), f(k+1))) \leq \sum_{k \geq n} \frac{k}{2^k}$. Clearly $q \models \check{X} \subseteq^* \check{A}$. \hfill \Box

A supported relation $\mathcal{R} = (A,R,B)$ is called Borel-relation iff there is a Polish space $X$ such that $A,B \subseteq X$ and $R \subseteq X^2$ are Borel sets. Similarly a Galois-Tukey connection $(\phi, \psi) : \mathcal{R}_1 \leq \mathcal{R}_2$ between Borel-relations is called Borel GT-connection iff $\phi$ and $\psi$ are Borel functions. To be Borel-relation and Borel GT-connection is absolute for transitive models containing all relevant codes.

Some important Borel-relation:

(A): $(\mathcal{I}, \subseteq, \mathcal{I})$ and $(\mathcal{I}, \subseteq^*, \mathcal{I})$ for a Borel ideal $\mathcal{I}$. 
B. FARKAS AND L. SOUKUP

(B): Denote Slm the set of slaloms on \( \omega \), i.e. \( S \in \text{Slm} \) iff \( S : \omega \to [\omega]^{<\omega} \) and \( |S(n)| = 2^n \) for each \( n \). Let \( \sqsubseteq \) and \( \sqsubseteq^* \) be the following relations on \( \omega^\omega \times \text{Slm} \):

\[
f \sqsubseteq^{(*)} S \iff \forall^{(\infty)} n \in \omega \ f(n) \in S(n).
\]

The supported relations \((\omega^\omega, \sqsubseteq, \text{Slm})\) and \((\omega^\omega, \sqsubseteq^*, \text{Slm})\) are Borel-relations.

(C): Denote \( \ell_1^+ \) the set of positive summable series. Let \( \leq \) be the coordinate-wise and \( \leq^* \) the almost everywhere coordinate-wise ordering on \( \ell_1^+ \). \( (\ell_1^+, \leq, \ell_1^+) \) and \( (\ell_1^+, \leq^*, \ell_1^+) \) are Borel-relations.

**Definition 6.6.** Let \( \mathcal{R} = (A, R, B) \) be a Borel-relation. A forcing notion \( \mathbb{P} \) is \( \mathcal{R} \)-bounding if

\[
\Vdash_{\mathbb{P}} \forall a \in A \exists b \in B \cap V \text{aRb};
\]

\( \mathcal{R} \)-dominating if

\[
\Vdash_{\mathbb{P}} \exists b \in B \forall a \in A \cap V \text{aRb}.
\]

For example the property \( \mathcal{I} \)-bounding/dominating is the same as \( (\mathcal{I}, \subseteq^*, \mathcal{I}) \)-bounding/dominating.

We can reformulate some classical properties of forcing notions:

- \( \omega^\omega \)-bounding \( \equiv \) \( (\omega^\omega, \leq^{(*)}, \omega^\omega) \)-bounding
- Adding dominating reals \( \equiv \) \( (\omega^\omega, \leq^*, \omega^\omega) \)-dominating
- Sacks property \( \equiv \) \( (\omega^\omega, \sqsubseteq^{(*)}, \text{Slm}) \)-bounding
- Adding a slalom capturing \( \equiv \) \( (\omega^\omega, \sqsubseteq^*, \text{Slm}) \)-dominating
- All ground model reals

If \( \mathcal{R} = (A, R, B) \) is a supported relation then let \( \mathcal{R}^\perp = (B, \neg R^{-1}, A) \) where \( b(\neg R^{-1})a \) iff not \( aRb \). Clearly \( (\mathcal{R}^\perp)^\perp = \mathcal{R} \) and \( b(\mathcal{R}) = \emptyset(\mathcal{R}^\perp) \). Now if \( \mathcal{R} \) is a Borel-relation then \( \mathcal{R}^\perp \) is a Borel-relation too, and a forcing notion is \( \mathcal{R} \)-bounding iff it is not \( \mathcal{R}^\perp \)-dominating.

**Fact 6.7.** Assume \( \mathcal{R}_1 \preceq \mathcal{R}_2 \) are Borel-relations with Borel GT-connection and \( \mathbb{P} \) is a forcing notion. If \( \mathbb{P} \) is \( \mathcal{R}_2 \)-bounding/dominating then \( \mathbb{P} \) is \( \mathcal{R}_1 \)-bounding/dominating.

By Corollary 5.5 this yields

**Corollary 6.8.** For each analytic \( \mathcal{P} \)-ideal \( \mathcal{I} \) (1) a poset \( \mathbb{P} \) is \( \preceq_{\mathcal{I}} \)-bounding iff it is \( \omega^\omega \)-bounding, (2) forcing with a poset \( \mathbb{P} \) adds \( \preceq_{\mathcal{I}} \)-dominating reals iff this forcing adds dominating reals.

We will use the following Theorem.

**Theorem 6.9.** ([Fr] 526B, 524I) There are Borel GT-connections \((\mathcal{Z}, \subseteq, \mathcal{Z}) \preceq (\ell_1^+, \leq, \ell_1^+) \) and \((\ell_1^+, \leq^*, \ell_1^+) \equiv (\omega^\omega, \subseteq^*, \text{Slm}) \).
Note that there is no any Galois-Tukey connection from \((\ell^+_1, \leq, \ell^+_1)\) to \((\mathcal{Z}, \subseteq, \mathcal{Z})\) so they are not GT-equivalent (see [LoVe] Th. 7.).

**Corollary 6.10.** If \(\mathbb{P}\) adds a slalom capturing all ground model reals then \(\mathbb{P}\) is \(\mathcal{Z}\)-dominating.

**Proof.** By Fact 6.7 and Theorem 6.9 adding slalom is the same as \((\ell^+_1, \leq^*, \ell^+_1)\)-dominating. Let \(\dot{x}\) be a \(\mathbb{P}\)-name such that \(\| \mathbb{P} \| \dot{x} \in \ell^+_1 \land \forall y \in \ell^+_1 \cap V \ y \leq^* \dot{x}\). Moreover let \(\dot{X}\) be a \(\mathbb{P}\)-name such that \(\| \mathbb{P} \| \dot{X} = \{ z \in \ell^+_1 : |z\setminus\dot{x}| < \omega, \forall \ n \ (z(n) \neq \dot{x}(n) \Rightarrow z(n) \in \omega)\}\). Let \((\phi, \psi) : (\mathcal{Z}, \subseteq, \mathcal{Z}) \preceq (\ell^+_1, \leq, \ell^+_1)\) be a Borel GT-connection. Now if \(\dot{A}\) is a \(\mathbb{P}\)-name such that \(\| \mathbb{P} \| \forall z \in \dot{X} \ \psi(z) \subseteq \dot{A}\) then \(\dot{A}\) shows that \(\mathbb{P}\) is \(\mathcal{Z}\)-dominating. \(\square\)

Denote \(\mathbb{D}\) the dominating forcing and \(\mathbb{LOC}\) the Localization forcing.

**Observation 6.11.** If \(\mathcal{I}\) is an arbitrary analytic \(\mathbb{P}\)-ideal then two step iteration \(\mathbb{D} * \mathbb{LOC}\) is \(\mathcal{I}\)-dominating.

Indeed, let \(\mathcal{I} \in V \subseteq M \subseteq N\) be transitive models, \(d \in M \cap \omega^\omega\) be strictly increasing and dominating over \(V\), and \(S \in N, S : \omega \rightarrow [\omega]^{<\omega} \subseteq \omega, |S(n)| \leq n\) a slalom which captures all reals from \(M\). Now if

\[
    X_n = \cup \{ A \in S(n) \cap \mathcal{P}([d(n), d(n + 1)) : \varphi(A) < 2^{-n}\}
\]
then it is easy to see that \(Y \subseteq \cup X_n : n \in \omega\) \(\in \mathcal{I} \cap N\) for each \(Y \in V \cap \mathcal{I}\).

**Problem 6.12.** For which analytic \(\mathbb{P}\)-ideal \(\mathcal{I}\) does \((\mathcal{I}, \subseteq^{(e)}, \mathcal{I}) \preceq (\ell^+_1, \leq^{(e)}, \ell^+_1)\) hold, or “adding slaloms” imply \(\mathcal{I}\)-dominating, or at least \(\mathbb{LOC}\) is \(\mathcal{I}\)-dominating?

**Problem 6.13.** Does \(\mathcal{Z}\)-dominating (or \(\mathcal{I}\)-dominating) imply adding slaloms?

We will use the following deep result of Fremlin to prove Theorem 6.15.

**Theorem 6.14.** ([Fr] 526G) There is a family \(\{ P_f : f \in \omega^\omega\} \) of Borel subsets of \(\ell^+_1\) such that the following hold:

\[
\begin{align*}
    \text{(i) } & \ell^+_1 = \cup \{ P_f : f \in \omega^\omega\}, \\
    \text{(ii) } & \text{if } f \leq g \text{ then } P_f \subseteq P_g, \\
    \text{(iii) } & (P_f, \leq, \ell^+_1) \preceq (\mathcal{Z}, \subseteq, \mathcal{Z}) \text{ with a Borel GT-connection for each } f.
\end{align*}
\]

**Theorem 6.15.** \(\mathbb{P}\) is \(\mathcal{Z}\)-bounding iff \(\mathbb{P}\) has the Sacks property.
Proof. Let \( \{ P_f : f \in \omega^\omega \} \) be a family satisfying (i), (ii), and (iii) in Theorem 6.14 and fix Borel GT-connections \((\phi_f, \psi_f) : (P_f, \leq, \ell_1^+) \preceq (\mathcal{Z}, \subseteq, \mathcal{Z}) \) for each \( f \in \omega^\omega \). Assume \( \mathbb{P} \) is \( \mathcal{Z} \)-bounding and \( \models \mathbb{P} \dot{x} \in \ell_1^+ \). \( \mathbb{P} \) is \( \omega^\omega \)-bounding by Theorem 6.2 so using (ii) we have \( \models \mathbb{P} \dot{x} \in \ell_1^+ = \bigcup \{ P_f : f \in \omega^\omega \cap \mathbb{V} \} \). We can choose a \( \mathbb{P} \)-name \( \dot{f} \) for an element of \( \omega^\omega \cap \mathbb{V} \) such that \( \models \mathbb{P} \dot{x} \in P_f \). By \( \mathcal{Z} \)-bounding property of \( \mathbb{P} \) there is a \( \mathbb{P} \)-name \( \dot{A} \) for an element of \( \mathcal{Z} \cap \mathbb{V} \) such that \( \models \mathbb{P} \phi_f(\dot{x}) \subseteq \dot{A} \), so \( \models \mathbb{P} \dot{x} \leq \psi_f(\dot{A}) \in \ell_1^+ \cap \mathbb{V} \). So we have \( \mathbb{P} \) is \((\ell_1^+, \leq^{(e)}, \ell_1^+)\)-bounding. By Theorem 6.9 and Fact 6.7 \( \mathbb{P} \) has the Sacks property.

The converse implication was proved in Theorem 6.5.

\[ \square \]

Problem 6.16. Does the \( \mathcal{I} \)-bounding property imply the Sacks property for each tall analytic \( \mathbb{P} \)-ideal \( \mathcal{I} \)?

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