Shape versus Volume: Making Large Flat Extra Dimensions Invisible

Keith R. Dienes
Department of Physics, University of Arizona, Tucson, AZ 85721 USA
E-mail address: dienes@physics.arizona.edu
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Much recent attention has focused on theories with large extra compactified dimensions. However, while the phenomenological implications of the volume moduli associated with such compactifications are well understood, relatively little attention has been devoted to the shape moduli. In this paper, we show that the shape moduli have a dramatic effect on the corresponding Kaluza-Klein spectra: they change the mass gap, induce level crossings, and can even be used to interpolate between theories with different numbers of compactified dimensions. Furthermore, we show that in certain cases it is possible to maintain the ratio between the higher-dimensional and four-dimensional Planck scales while simultaneously increasing the Kaluza-Klein graviton mass gap by an arbitrarily large factor. This mechanism can therefore be used to alleviate (or perhaps even eliminate) many of the experimental bounds on theories with large extra spacetime dimensions.

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I. INTRODUCTION

Over the past several years, there has been an explosion of interest in theories with large extra spacetime dimensions. Much of this interest stems from the realization that large extra dimensions have the potential to lower the fundamental energy scales of physics, such as the Planck scale, the GUT scale, and the string scale. Indeed, as is well understood, the degree to which these scales may be lowered depends on the volume of compactification dimensions.

However, compactification manifolds are generally described by shape moduli (so-called “complex moduli”) as well as volume moduli (so-called “Kähler moduli”). This distinction has phenomenological relevance because the shape moduli also play a significant role in determining the experimental bounds on such scenarios. Unfortunately, in most previous discussions of extra dimensions, relatively little attention has been paid to the implications of these moduli.

In this paper, we shall discuss the phenomenological implications of the shape moduli by focusing on the simple case of a flat, two-dimensional toroidal compactification. In this case, the relevant shape modulus corresponds to the relative angle between the two directions of compactification. As we shall demonstrate, the corresponding Kaluza-Klein spectrum is strongly dependent on this angle, and exhibits level-crossing as well as a changing mass gap as this angle is varied. This indicates that shape moduli such as should not be ignored in phenomenological studies of large extra dimensions. Moreover, we shall see that such shape moduli even provide an interesting means of interpolation between theories with different numbers of extra spacetime dimensions. Finally, we shall show that under certain circumstances, it is possible to exploit shape moduli in order to increase the Kaluza-Klein mass gap by an arbitrarily large factor; this occurs even though the volume of compactification remains fixed. This surprising observation can therefore be used to alleviate (and perhaps even eliminate) many of the bounds that currently constrain such theories with large extra dimensions.

II. COMPACTIFICATION ON A TWO-TORUS WITH SHIFT ANGLE: KALUZA-KLEIN SPECTRUM

Since one-dimensional compactifications lack shape moduli, we begin the discussion by considering compactification on a general two-torus, as shown in Fig. 1. Such a torus is specified by three real parameters (the two radii of the torus as well as the shift angle ), and corresponds to identifying points which are related under the two coordinate transformations

\[
\begin{align*}
  y_1 &\rightarrow y_1 + 2\pi R_1 \\
  y_2 &\rightarrow y_2 \\
  y_1 &\rightarrow y_1 + 2\pi R_2 \cos \theta \\
  y_2 &\rightarrow y_2 + 2\pi R_2 \sin \theta.
\end{align*}
\]

(1)

Note that we are using orthogonal coordinates for the extra dimensions; likewise, since this is a toroidal compactification, the metric remains flat for all angles . As

![FIG. 1. General two-dimensional torus with shift angle .](image)
evident from Eq. (1), the physical significance of the angle \( \theta \) is that translations along the \( R_2 \) direction produce simultaneous translations along the \( R_1 \) direction. Note that tori with different angles \( \theta \) are topologically distinct (up to the modular transformations to be discussed below).

There are two “shape” parameters for such a torus: the ratio \( R_2/R_1 \) and the angle \( \theta \). While most previous discussions of large extra dimensions have focused on the volume of such tori (and even on the ratio \( R_2/R_1 \)), they have ignored the possibility of the shift \( \theta \), essentially fixing \( \theta = \pi/2 \). Our goal, therefore, is to understand the phenomenological implications of the angle \( \theta \).

Given the torus identifications in Eq. (1), it is straightforward to determine the corresponding Kaluza-Klein spectrum. The Kaluza-Klein eigenfunctions for such a torus are given by

\[
\exp\left[\frac{n_1}{R_1} (y_1 - \frac{y_2}{\tan \theta}) + i \frac{n_2}{R_2} \frac{y_2}{\sin \theta}\right],
\]

where \( n_i \in \mathbb{Z} \). Applying the \((\text{mass})^2\) operator \(-\left(\partial^2/\partial y_1^2 + \partial^2/\partial y_2^2\right)\), we thus obtain the corresponding Kaluza-Klein masses

\[
M_{n_1,n_2}^2 = \frac{1}{\sin^2 \theta} \left( \frac{n_1^2}{R_1^2} + \frac{n_2^2}{R_2^2} - 2 \frac{n_1 n_2}{R_1 R_2} \cos \theta \right).
\]

We see that while the Kaluza-Klein spectrum maintains its invariance under \((n_1,n_2) \to -(n_1,n_2)\), it is no longer invariant under \(n_1 \to -n_1\) or \(n_2 \to -n_2\) individually. The spectrum is, however, invariant under either of these shifts and the simultaneous shift \( \theta \to \pi - \theta \). We can therefore restrict our attention to tori with angles in the range \(0 < \theta \leq \pi/2\) without loss of generality.

It is clear from Eq. (3) that the Kaluza-Klein masses depend on \( \theta \) in a non-trivial, level-dependent way. In order to deduce the physics behind Eq. (3), let us first examine the case with \( R_1 = R_2 \equiv R \). We then find the results shown in Fig. 2. As guaranteed by Eq. (3), the ground state remains massless for all \( \theta \). However, as \( \theta \) is varied, we see from Fig. 2 that the excited Kaluza-Klein spectrum exhibits dramatic changes, with many light states becoming heavy and several heavier states becoming light.

Interestingly, as a result of this level-crossing, the identity of the lowest excited state is itself a function of \( \theta \), with the \((\pm 1,0)\) and \((0,\pm 1)\) states (four states total) serving as the lowest excitations for \( \pi/3 \leq \theta \leq \pi/2 \), and the \((\pm 1,1)\) states (two states total) filling this role for \( \theta \leq \pi/3 \). In general, we observe that both the mass gap \( \mu \) (defined as the splitting between the ground state and the first excited states) and the degeneracy \( \alpha \) of the first excited states are functions of the shape parameter \( \theta \).

This is particularly important in the case of Kaluza-Klein gravitons. In general, the presence of Kaluza-Klein gravitons induces deviations from Newtonian gravity, with the corresponding gravitational potential taking the form

\[
V(r) = -G_4 \frac{m_1 m_2}{r} (1 + \alpha e^{-\mu r} + ...)
\]

for \( r \gg 1/\mu \). Thus, in this simple case with \( R_1 = R_2 \), we see that the expected deviations from non-Newtonian gravity drop by a factor of two when \( \theta < \pi/3 \) — even though the radii are held fixed.

It is also possible to understand the behavior of the Kaluza-Klein spectrum as \( \theta \to 0 \). In this limit, the two cycles of the torus collapse onto each other. The resulting Kaluza-Klein spectrum therefore depends on whether the periodicities of the two cycles are commensurate. In the case with \( R_1 = R_2 \equiv R \), the two cycles are commensurate, and the torus identifications in Eq. (1) collapse to become the single identification corresponding to a circle of radius \( R \). This behavior is apparent in Fig. 2, the only Kaluza-Klein states which remain light as \( \theta \to 0 \) are those which effectively reproduce a one-dimensional circle-compactification with radius \( R \). Thus, we see that the shape parameter \( \theta \) allows us to smoothly *interpolate* between compactifications of different numbers of space-time dimensions. Note, in particular, that this method of interpolation is physically different from the standard method of interpolation in which a single radius is taken to infinity.
This interpolation behavior as $\theta \to 0$ is completely general, and arises for all rational values of $R_2/R_1$. In the limit $\theta \sim \epsilon \ll 1$, Eq. (3) becomes

$$M_{n_1,n_2}^2 \approx \frac{1}{e^2} \left( \frac{n_1}{R_1} - \frac{n_2}{R_2} \right)^2 + \frac{1}{3} \left( \frac{n_1^2}{R_1^2} + \frac{n_2^2}{R_2^2} + \frac{n_1 n_2}{R_1 R_2} \right) + O(e^2). \quad (5)$$

We thus see that $M_{n_1,n_2} \to \infty$ as $\epsilon \to 0$ for all $(n_1,n_2)$ unless $R_2/R_1$ is a rational number. In these cases, we may represent $R_2/R_1 = p/q$ where $p$ and $q$ are relatively prime. We then find that the radius of the resulting circle-compactification as $\theta \to 0$ is given by $R \equiv R_1/q = R_2/p$, with the torus modes $(n_1,n_2) = k(q,p)$ evolving to become the circle modes $M_k = k/R$ and all others becoming infinitely massive. Note, in particular, that when $p,q \neq 1$, the radius $R$ of the resulting circle compactification is generally smaller than either $R_1$ or $R_2$. This implies that the corresponding circle Kaluza-Klein states are heavier than the initial torus Kaluza-Klein states with which we started.

It is interesting to explore the case when $R_2/R_1$ is not a rational number. Indeed, unless there is some dynamics that fixes the radius moduli to have a rational ratio, this will be the generic situation. In such cases, all excited Kaluza-Klein states become infinitely massive as $\theta \to 0$: essentially the radius of the resulting circle-compactification is zero. This divergence of the Kaluza-Klein masses is ultimately a reflection of the incommensurate nature of the two torus periodicities, an incompatibility which grows increasingly severe as $\theta \to 0$. Note that the actual limit as $\theta \to 0$ is a singular one, corresponding to a degenerate compactification manifold. However, our point is that when $R_2/R_1$ is irrational, we can always make our excited Kaluza-Klein states arbitrarily heavy by choosing a sufficiently small value for $\theta$. Thus, for all intents and purposes, there always exists a (small, non-zero) value of $\theta$ for which we can make our extra dimensions truly “invisible” with respect to laboratory or observational constraints that rely on the presence of light Kaluza-Klein states.

### III. SHAPE VERSUS VOLUME

Thus far, we have shown that when $R_2/R_1$ is irrational, our excited Kaluza-Klein states become arbitrarily heavy as $\theta \to 0$. As such, these extra dimensions become “invisible”, even though the radii $R_1,R_2$ are held fixed. However, even though the radii are held fixed, the volume of the extra dimensions is falling to zero. Indeed, when $R_2/R_1$ is irrational, the compactification volume falls like $\sin \theta$ while the excited Kaluza-Klein masses diverge as $1/\sin \theta$. To what extent, then, can the compactification volume remain “large” while the corresponding Kaluza-Klein states become heavy? Indeed, what is the role of the shape moduli when the volume is held fixed? As we shall see, this issue is surprisingly subtle.

To address these issues, we now study the behavior of the Kaluza-Klein masses when the volume of the compactification manifold is held fixed. Towards this end, let us reparametrize the three torus moduli $(R_1,R_2,\theta)$ in terms of a single real volume modulus $V$ and a complex shape modulus $\tau$:

$$V \equiv 4\pi^2 R_1 R_2 \sin \theta, \quad \tau \equiv \frac{R_2}{R_1} e^{i \theta}. \quad (6)$$

We shall also define $\tau_1 \equiv \Re \tau$ and $\tau_2 \equiv \Im \tau$. Using these definitions, we can express $(R_1,R_2,\theta)$ in terms of $(V,\tau)$ via

$$\cos \theta = \tau_1/|\tau|, \quad \sin \theta = \tau_2/|\tau|, \quad R_1^2 = \frac{1}{4\pi^2 \tau_2} V, \quad R_2^2 = \frac{|\tau|^2}{4\pi^2 \tau_2} V. \quad (7)$$

The periodicities in Eq. (6) then take the form

![FIG. 3. The Kaluza-Klein spectrum as a function of the shape parameter $\theta$ for $R_1 = 4R_2$. Each state $(n_1,n_2)$ is two-fold degenerate with $-(n_1,n_2)$.
](image-url)
\[ z \rightarrow z + \sqrt{\frac{V}{\tau_2}}, \quad z \rightarrow z + \sqrt{\frac{V}{\tau_2}} \tau \] (8)

where \( z \equiv y_1 + iy_2 \), and the Kaluza-Klein wavefunctions in Eq. (2) take the form

\[ \exp \left\{ \frac{2\pi i}{\sqrt{V\tau_2}} \text{Im} \left[ \tau \left( n_1 \tau - n_2 \right) \right] \right\}. \] (9)

Operating with the (mass) \(^2\) operator \(-4\partial^2/\partial z\partial \overline{z}\) then yields the Kaluza-Klein masses

\[ M^2_{n_1,n_2} = \frac{4\pi^2}{V} \left| \frac{n_1}{\tau_2} \right|^2 \]

\[ = \frac{4\pi^2}{V} \left( (n_1 \tau_1 - n_2)^2 + n_1^2 \tau_2^2 \right). \] (10)

Note that although Eq. (10) is merely a rewriting of Eq. (3), we have now explicitly separated the effects of the volume modulus \( V \) from those of the shape modulus \( \tau \). Writing the remaining shape factors in terms of the original parameters \((R_1, R_2, \theta)\), we thus obtain

\[ \left( \frac{V}{4\pi^2} \right) M^2_{n_1,n_2} = \frac{1}{\sin \theta} \left[ \frac{n_2 R_2}{R_1} + n_1^2 \frac{R_1}{R_2} - 2n_1 n_2 \cos \theta \right]. \] (11)

Using this result, we can now consider the effects of the shape parameter \( \theta \) when the compactification volume is held fixed. In Fig. 4, we plot the Kaluza-Klein masses for the \( R_1 = R_2 \) and \( R_1 = 4R_2 \) cases considered earlier in Figs. and . Note that in order to keep the volume fixed as \( \theta \rightarrow 0 \), the radii are now forced to grow increasingly large (even though their ratio is held fixed).

This increase in the radii therefore provides an extra tendency towards lowering the Kaluza-Klein masses, as can be seen by comparing the masses plotted in Fig. to those plotted in Figs. and .

Despite this tendency towards smaller Kaluza-Klein masses, we see that the shape moduli can help to render the large extra dimensions effectively invisible even when the compactification volume is held fixed. For example, in the case with \( R_1 = 4R_2 \), we see that the mass gap increases by a factor of two near \( \sin \theta \approx 1/4 \). This occurs even though the ratio \( R_2/R_1 \) and the compactification volume are both being held fixed relative to their values at \( \theta = \pi/2 \). Indeed, these suppressions become even more pronounced for scenarios with \( R_2/R_1 = p/q \) with larger values of \((p, q)\). We stress that these features are possible only because of the introduction of non-trivial shape moduli.

Nevertheless, the utility of this mechanism is ultimately limited when \( R_2/R_1 \) is rational because of the appearance of an infinite tower of Kaluza-Klein states which become massless in the \( \theta \rightarrow 0 \) limit. Of course, these states are nothing but the circle-compactification states whose masses previously remained finite when the

FIG. 4. The Kaluza-Klein mass spectrum as a function of the shape parameter \( \theta \) for \( R_1 = R_2 \) (upper plot) and \( R_1 = 4R_2 \) (lower plot). In each case, the compactification volume is held fixed.
radii were held fixed in Figs. 2 and 3. As discussed above, these states now become massless when the volume is held fixed because the radii must now compensate by becoming infinitely large as \( \theta \to 0 \).

Given this, it is natural to wonder whether we may avoid the appearance of these dangerous massless states in the \( \theta \to 0 \) limit by considering the case when \( |\tau| \equiv R_2/R_1 \) is irrational. It is immediately clear that there are two opposite forces at play in such cases. First, there is the tendency towards masslessness which arises because of the expanding radii as \( \theta \to 0 \). However, this must compete against the opposite tendency, driven by the irrationality of the ratio \( R_2/R_1 \), which pushes the masses of the resulting “circle-compactified” states to infinity (or equivalently pushes the radius of the resulting “circle-compactification” to zero).

In order to determine the net effect on the Kaluza-Klein mass spectrum, let us return to Eq. (11) and consider the \( \theta \sim \epsilon \ll 1 \) limit:

\[
\left( \frac{V}{4\pi^2} \right) M^2_{n_1, n_2} = \frac{(n_2 - n_1 |\tau|)^2}{|\tau| \epsilon} + \left( \frac{n_2^2 + 4n_1 n_2 |\tau| + n_1^2 |\tau|^2}{6|\tau|} \right) \epsilon + O(\epsilon^3)
\tag{12}
\]

where \( |\tau| \equiv R_2/R_1 \) is irrational. This is the fixed-volume analogue of Eq. (11). Given this expression, we can immediately see the two tendencies at work. The first term on the right side of Eq. (12) generally diverges because \( n_2 - n_1 |\tau| \) never vanishes exactly. Thus, the general Kaluza-Klein state becomes infinitely heavy as \( \theta \to 0 \).

However, for any fixed chosen value of \( \epsilon \), we can always find special states \((n_1, n_2)\) for which this first term comes \textit{arbitrarily close} to cancelling; this simply requires choosing sufficiently large values of \((n_1, n_2)\). These special states with large \((n_1, n_2)\) are potentially massless. On the other hand, choosing such large values of \((n_1, n_2)\) drives the second term in Eq. (12) to larger and larger values. [Note that the third and higher terms are always suppressed relative to the second term in the \( \epsilon \to 0 \) limit, even as \((n_1, n_2)\) grow large.] Thus, because of the conflict between these two terms, it is not readily obvious whether these special, potentially massless states actually become massless in the \( \theta \to 0 \) limit.

The outcome of this competition between the first two terms in the Kaluza-Klein mass formula in Eq. (12) rests on the efficiency with which \( n_2 - n_1 |\tau| \) can be made to approach zero for integer \((n_1, n_2)\), as a function of \( n_2 \), given an arbitrary irrational number \( |\tau| \). Let us parametrize this efficiency in the form

\[
(n_2 - n_1 |\tau|)^2 \sim \frac{A^2}{n_1^{1+\gamma}} \quad \text{as} \quad (n_1, n_2) \to \infty \tag{13}
\]

for some constants \( A \) and \( \gamma \). We shall see shortly that this is indeed the most relevant parameterization for this asymptotic behavior. Thus, for these potentially massless states, Eq. (12) becomes

\[
\left( \frac{V}{4\pi^2} \right) M^2_{n_1, n_2} = \frac{A}{n_1} \left( y + \frac{1}{y} \right) + ... \tag{14}
\]

where \( y \equiv n_1^{2+\gamma} |\tau| \epsilon / A \).

Fortunately, \( y + y^{-1} \) is bounded from below for all values of \( y \). The issue therefore boils down to a simple number-theoretic question: what is the value of \( \gamma \)? Clearly, if \( \gamma > 0 \), we see from Eq. (14) that the lightest Kaluza-Klein states for irrational \(|\tau|\) become massless as \( \theta \to 0 \). By contrast, if \( \gamma \leq 0 \), then irrationality succeeds in preventing the appearance of massless states as \( \theta \to 0 \).

It turns out that the value of \( \gamma \) has been extensively investigated in the mathematical literature. Indeed, this is nothing but the ancient problem of Diophantine approximation, with \( 2(1 + \gamma) \) traditionally known as the “irrationality measure” or as the “Liouville-Roth constant”. The results are as follows [3]. According to an 1842 theorem by Dirichlet, for all irrational numbers \(|\tau|\) it is possible to find an infinite number of integer pairs \((n_1, n_2)\) such that Eq. (13) holds with \( \gamma \geq 0 \). However, for the case of \textit{algebraic} irrational numbers \(|\tau|\) (defined as irrational numbers which can be realized as solutions of non-zero polynomials with integral coefficients), a stronger 1955 theorem due to Roth [3] states that \( \gamma \leq 0 \). (No such stronger theorem has yet been proven for non-algebraic irrational numbers.) Combining these two results, we conclude that \( \gamma = 0 \) for the case of algebraic irrational numbers.

This result indicates that \textit{irrationality succeeds in preventing the appearance of massless Kaluza-Klein states as \( \theta \to 0 \), even if we hold the compactification volume fixed and the radii become infinitely large}. This result is rigorous for algebraic irrational values of \(|\tau| \equiv R_2/R_1\), and is likely (though unproven) to hold for certain non-algebraic (transcendental) irrational values as well. Thus, for algebraic irrational numbers \(|\tau|\), we see that these dangerous Kaluza-Klein states all have masses which are bounded from below:

\[
\left( \frac{V}{4\pi^2} \right) M^2_{n_1, n_2} \geq 2A \tag{15}
\]

In other words, the mass gap as \( \theta \to 0 \) is bounded from below according to Eq. (13). Moreover, for any given value of \( \theta \sim \epsilon \ll 1 \), the Kaluza-Klein state which comes closest to saturating this bound is simply the state for which \( y \approx 1 \) in Eq. (14). This is the state for which \( n_1^2 \approx A / (|\tau| \epsilon) \).

In order to measure the importance of the shape modulus \( \theta \), let us compare \( \mu' \), the mass gap in the \( \theta \to 0 \) limit, with \( \mu \), the original mass gap at \( \theta = \pi/2 \). According to Eq. (13), the original (mass)\(^2\) gap at \( \theta = \pi/2 \) is given by either \(|\tau|\) (if \(|\tau| \leq 1\)) or \(1/|\tau|\) (if \(|\tau| \geq 1\)). Let us
henceforth assume that $|\tau| \geq 1$ without loss of generality. Thus, in general, the (mass)$^2$ gap as $\theta \rightarrow 0$ is greater than the original (mass)$^2$ gap at $\theta = \pi/2$ by a factor

$$\left( \frac{\mu'}{\mu} \right)^2 = 2A|\tau|.$$  \hspace{1cm} (16)

Thus, if $2A|\tau| > 1$, we have an exponential suppression of the effects of the Kaluza-Klein states as $\theta \rightarrow 0$.

Let us give an explicit example by considering a compactification with $|\tau| \equiv R_2/R_1 = \frac{1}{2}(3 + \sqrt{5}) \approx 2.618$. (We shall see later that this example is motivated on both mathematical and physical grounds.) Our goal is to understand the corresponding spectrum of Kaluza-Klein states as we take $\theta \rightarrow 0$ while holding $|\tau|$ and the compactification volume fixed. As $\theta$ decreases, the above arguments indicate that the majority of Kaluza-Klein states become increasingly heavy. Indeed, the only states $(n_1, n_2)$ which have a tendency to become light are those which are approximately “circle-compactified”, satisfying $n_2 \approx |\tau|n_1$. In this example with $|\tau| \approx 2.618$, such low-lying states include $(1, 2), (1, 3), (2, 5)$, etc. The behavior of these states is illustrated in Fig. 5(a).

Once $\theta$ becomes sufficiently small, however, the lightest (and hence most dangerous) states are ultimately those which lie on the leading $\gamma = 0$ trajectory in Eq. (13). Which states are these? While methods exist [5] for determining these leading states unambiguously for any value of $|\tau|$, in this case with $|\tau| = \frac{1}{2}(3 + \sqrt{5})$ it turns out that these leading states can easily be determined from the famous Fibonacci sequence of integers $f_k = \{1, 2, 3, 5, 8, 13, ... \}$ defined by the recursion relation $f_k = f_{k-1} + f_{k-2}$ with $f_1 = 1$ and $f_2 = 2$. It is well known that the ratio of successive integers in this series rapidly approaches the “golden mean” $g \equiv \frac{1}{2}(1 + \sqrt{5})$ as $k \rightarrow \infty$. Thus, since $|\tau| = 1 + g$ in our example, the lightest Kaluza-Klein states as $\theta \rightarrow 0$ are simply the states $(n_1, n_2) = (f_k, f_{k+2})$ for increasingly large values of $k$. Note, in particular, that $f_{k+2}/f_k \rightarrow |\tau| = 1 + g$ as $k \rightarrow \infty$, as desired.

It is straightforward to verify that this set of Kaluza-Klein states satisfies Eq. (13) with $\gamma = 0$. This verifies that these states converge at the maximum possible rate — i.e., that these are indeed the lightest states as $\theta \rightarrow 0$. However, since $\gamma = 0$, the masses of these states are bounded from below, in accordance with Eq. (13). This behavior is illustrated in Fig. 5(b). Moreover, for each different value of $\theta$, we see from Fig. 5(b) that a different excited Kaluza-Klein state in the Fibonacci series has an enhanced tendency to become massless. This is precisely the behavior discussed below Eq. (15).

It turns out that $A = 1/\sqrt{5}$ in this example. Since $|\tau| = 1 + g = \frac{1}{2}(3 + \sqrt{5})$, the asymptotic mass gap ratio in this example is given by $(\mu'/\mu)^2 = 1 + 3/\sqrt{5} \approx 2.34$. This behavior is also shown in Fig. 5(b), where we have renormalized the overall Kaluza-Klein spectrum so that

![FIG. 5. The Kaluza-Klein mass spectrum with $|\tau| \equiv R_2/R_1 = \frac{1}{2}(3 + \sqrt{5})$. In the upper plot (a), we show the behavior of the (0, 1) state as well as the behavior of several other low-lying Kaluza-Klein states which initially tend towards masslessness as $\theta \rightarrow 0$. In the lower plot (b), we illustrate the behavior of these states as $\theta \rightarrow 0$ by plotting $\sin \theta$ on a logarithmic scale. In both plots, we have renormalized the overall Kaluza-Klein spectrum so that the mass gap at $\theta = \pi/2$ is set to 1. Note that the Kaluza-Klein mass gap is larger as $\theta \rightarrow 0$ than it is at $\theta = \pi/2$, even though the compactification volume $V$ and the radius ratio $|\tau| \equiv R_2/R_1$ are held fixed.](image)
the mass gap at $\theta = \pi/2$ is set to 1. Although not large in this example, this mass gap ratio exceeds 1 and thereby leads to a relative exponential suppression of the effects of the Kaluza-Klein states. Specifically, recall from Eq. (14) that in the case of Kaluza-Klein gravitons, the corresponding deviations from Newtonian gravity are given by $\Delta V(r_s) \sim \alpha e^{-\mu r_s}/r_s$ where $\alpha$ is the degeneracy of the first-excited Kaluza-Klein level, $\mu$ is the mass gap, and $r_s \gg \mu^{-1}$ is the distance scale associated with such measurements. A change in the mass gap from $\mu$ to $\mu'$ thereby suppresses these deviations by an exponential factor

$$\frac{\Delta V'}{\Delta V} = \frac{\alpha'}{\alpha} \exp \left[ - \left( \frac{\mu'}{\mu} - 1 \right) \mu r_s \right].$$

(17)

Taking $\mu r_s \approx 10$ as a reference value, we see that $\Delta V'/\Delta V \approx 5 \times 10^{-3}$ for this example. This can therefore be a significant factor contributing to the invisibility of the extra dimensions, even though we have held both the compactification volume and the radius ratio $|\tau| \equiv R_2/R_1$ fixed.

Thus far, we have shown that irrationality succeeds in preventing the appearance of massless Kaluza-Klein states as $\theta \to 0$. Indeed, we have seen that the Kaluza-Klein masses are always bounded from below, with the size of the mass gap as $\theta \to 0$ completely determined by the value of the parameter $A$ in Eq. (13), and the size of the mass gap ratio determined by the product $A|\tau|$ in Eq. (16). The next question, therefore, is to determine the sizes of $A$ and $A|\tau|$. How large can these parameters become, and for what values of $|\tau|$ are they maximized?

This issue has also been investigated in the mathematical literature, with the following results. It turns out that the set of all possible values of $A$ (the so-called “Lagrange spectrum”) is generally discrete, with only certain values allowed; moreover, the set of all algebraic irrational numbers $|\tau|$ may be sorted into equivalence classes depending on their corresponding values of $A$. According to a theorem of Hurwitz [1], the Lagrange spectrum is bounded from above by the value $1/\sqrt{5}$. Thus, our previous example already reaches the maximum possible value of $A$. However, the set of $|\tau|$ corresponding to each value of $A$ is countably infinite, and includes values of $|\tau|$ with ever-increasing magnitudes. Thus, we see that although $A \leq 1/\sqrt{5}$, the value of the mass gap ratio $2A|\tau|$ can be chosen to be arbitrarily large.

Once again, let us give an example. Rather than consider $|\tau| = 1+g$ as in our previous example, let us instead consider a more general compactification with the value $|\tau| = m + g$ where $m \in \mathbb{Z}^+$. Such values of $|\tau|$ are all in the same equivalence class as the golden mean $g$. It turns out that for any $m \in \mathbb{Z}^+$, the lightest Kaluza-Klein states as $\theta \to 0$ are those with $(n_1, n_2) = (f_k, f_k+1+m f_k)$ where $\{f_k\}$ are the same Fibonacci numbers as before. Indeed, it is easy to verify that these states continue to satisfy Eq. (13) with $\gamma = 0$ and $A = 1/\sqrt{5}$ for all $m$. However, while the value of $A$ does not change as a function of $m$, we can make $|\tau|$ arbitrarily large simply by choosing $m$ arbitrarily large! Thus, the mass gap ratio $2A|\tau|$ can be made to increase without bound. In other words, the phenomenological effects of taking $\theta \to 0$ become more and more significant as $m \to \infty$.

It should be noted that while the mass gap ratio $\mu'/\mu$ in Eq. (16) increases with increasing $m$, the mass gap $\mu$ at $\theta = \pi/2$ actually decreases. This is required by the fact that $A$ itself (and hence $\mu'$ itself) is bounded from above. However, the fact that the mass gap ratio can become arbitrarily large illustrates the fact that shape parameters can have a profound effect in altering the properties of the Kaluza-Klein spectrum. Moreover, while the original mass gap at $\theta = \pi/2$ depends strongly on $|\tau|$, we have seen that the mass gap as $\theta \to 0$ becomes independent of $|\tau|$. We shall see below that this has important phenomenological implications.

We also stress that none of these conclusions rely on choosing values of $|\tau|$ which are related to the golden mean $g = 1/\sqrt{5}$. Indeed, for any algebraic irrational number $\xi$, there is always a corresponding non-zero value $A_\xi$. We can then find a related set of irrational numbers $|\tau|$ in the same equivalence class as $\xi$ such that $2A_\xi|\tau| \to \infty$.

IV. DISCUSSION AND PHENOMENOLOGICAL IMPLICATIONS

In many situations, these observations can be exploited in order to dramatically weaken the experimental bounds on scenarios involving large extra dimensions.

To see this, let us consider the scenario of Ref. [1] in which large extra dimensions felt only by gravity are responsible for lowering the fundamental (higher-dimensional) Planck scale into the TeV range. In this scenario, the ratio between the four-dimensional and higher-dimensional Planck scales is set purely by the compactification volume $V$; the shape moduli are irrelevant in this regard. (This is evident from the usual Gauss-law arguments [1]; essentially the higher-dimensional limit is achieved by considering length scales so small that the precise shape of the compactification manifold becomes irrelevant.) We are therefore free to choose our shape moduli so as to avoid laboratory, astrophysical, and cosmological constraints.

It is precisely here that our observations come into play. Let us first consider the experimental bounds on extra dimensions that would normally apply in the case with $\theta = \pi/2$. In this case, we know from Eq. (10) that the lightest Kaluza-Klein states have masses

$$\left( \frac{V}{4\pi^2} \right) M_{n_1,n_2}^2 = \frac{1}{|\tau|}$$

(18)
where we have assumed $|\tau| \equiv R_2/R_1 \geq 1$ without loss of generality. However, no Kaluza-Klein states have ever been detected experimentally; thus the mass of the lightest Kaluza-Klein states must exceed some experimental limit $M_{\text{expt}}$. For example, in the case of extra dimensions felt only by gravity, the current bound $M_{\text{expt}} \sim (\text{mm})^{-1}$. Thus, demanding $M_{n_1,n_2} \geq M_{\text{expt}}$ in Eq. (18), we find

$$V|\tau| \leq 4\pi^2(M_{\text{expt}})^{-2}. \quad (19)$$

Note that since $V \equiv 4\pi^2R_1R_2$ and $|\tau| \equiv R_2/R_1 > 1$, this constraint equivalently takes the form

$$R_2^2 \leq (M_{\text{expt}})^{-2}. \quad (20)$$

Thus, we see that it is not the volume that is bounded when $\theta = \pi/2$ — strictly speaking, what is actually bounded is the size of the largest single radius.

By contrast, let us now consider the same situation as $\theta \to 0$. In this case, we have already seen that Eq. (15) is replaced by Eq. (13) where $A > 0$. Thus, again imposing the experimental constraint that the lightest Kaluza-Klein states have masses exceeding $M_{\text{expt}}$, we find that Eq. (10) is replaced by

$$V \leq 8\pi^2A(M_{\text{expt}})^{-2}. \quad (21)$$

Equivalently, multiplying both sides of Eq. (21) by $|\tau|$, we see that Eq. (20) is replaced by

$$R_2^2 \leq \frac{2A|\tau|}{\sin \theta}(M_{\text{expt}})^{-2}. \quad (22)$$

However, we have already demonstrated that it is possible to choose $|\tau|$ such that the product $A|\tau|$ becomes arbitrarily large. Moreover, in the limit we are considering, $\sin \theta \to 0$. Thus, in such cases, the experimental bounds when $\theta \to 0$ become infinitely weaker than they are when $\theta = \pi/2$.

To phrase this result somewhat differently, note that whereas Eq. (10) is really a bound on the single largest radius, Eq. (21) is actually a bound on the compactification volume. As such, it is completely insensitive to the size of the largest radius! As long as $|\tau| \equiv R_2/R_1$ is chosen within a fixed equivalence class of algebraic irrational numbers (so that the corresponding value of $A$ remains fixed), we can choose $R_2$ as large as we wish without running afoul of experimental constraints! The root of this result, of course, is our critical observation in Eq. (13) that the Kaluza-Klein mass gap becomes independent of $|\tau|$ in the $\theta \to 0$ limit.

This result is striking. After all, even in the usual case with $\theta = \pi/2$, we know that we retain the freedom to change the ratio $|\tau| \equiv R_2/R_1$ while keeping the volume fixed. However, this freedom can usually be exploited only up to a point: no single radius can exceed the size $(M_{\text{expt}})^{-1}$ set by the experimental constraints. By contrast, as $\theta \to 0$, we see that we can adjust this ratio $|\tau|$ at will, making either radius as large as we wish while holding the volume fixed. Indeed, as long as $|\tau|$ is chosen from within the same equivalence class of algebraic irrational numbers, there is no experimental limit on the size to which the single largest dimension can grow. Thus, in this sense, a large extra dimension can truly be rendered “invisible”, even when the compactification volume is held fixed.

This result implies that we must be extremely careful when interpreting the results of precision tests of non-Newtonian gravity. Indeed, we now see that such tests need not be interpreted as placing limits merely on the size of the largest extra dimension; in some cases they instead place limits directly on the compactification volume, leaving the compactification radii completely unconstrained. This observation also applies to astrophysical bounds that come from Kaluza-Klein graviton production (e.g., in supernovas), and likewise to bounds that may come from Kaluza-Klein neutrinos, axions, and other bulk fields.

One might worry that if this scenario is embedded into string theory, there might exist winding-mode states which become light when the larger radius becomes infinitely large and the smaller radius becomes correspondingly small. This would certainly be the case if the Kaluza-Klein masses were to become heavier than the fundamental string scale. However, as $\theta \to 0$, we have seen that the masses of the lightest Kaluza-Klein states do not grow arbitrarily large; instead, these masses are bounded from above because $A$ itself is bounded from above. Thus, as long as the Kaluza-Klein states remain light compared to the fundamental string scale, the winding-mode states remain correspondingly heavy.

Another potential worry concerns the existence of toroidal modular symmetries. Through modular transformations, it is always possible to redefine the values of $R_1, R_2$, and $\theta$ without changing any of the underlying physics. However, such modular transformations necessarily leave the Kaluza-Klein spectrum invariant. Thus, any effects which actually modify the Kaluza-Klein spectrum (such as the shape effects we have been studying) must represent physical effects which go beyond mere modular transformations. Indeed, the critical distinction between rational and irrational values of $|\tau|$ which we have observed as $\theta \to 0$ is not something that an $SL(2, \mathbb{Z})$ modular transformation (with its integer coefficients) can eliminate. This issue will be discussed further in Ref. [4].

Clearly, the scenario we have outlined in this paper rests upon the unique properties exhibited by the Kaluza-Klein spectrum that emerge as $\theta \to 0$ when $|\tau|$ is chosen to be an algebraic irrational number. While it may seem unnatural to take such small values of $\theta$, we stress that they are part of the allowed compactification moduli space as long as $\theta > 0$. Likewise, it may seem fine-
tuned to take $|\tau| \equiv R_2/R_1$ irrational. However, given that $(R_1, R_2)$ are a priori unconstrained, all values are equally likely, and indeed it is the rational values which represent fine-tuning.

In fact, given the modular symmetries of the torus, such moduli may even be preferred. As we have mentioned above, the Kaluza-Klein spectrum in Eq. (10) is invariant under the modular transformations $\tau \to \tau + 1$ [under which $(n_1, n_2) \to (n_1, n_2 - n_1)$] and $\tau \to -1/\tau$ [under which $(n_1, n_2) \to (-n_2, n_1)$]. Since these modular transformations do not change the underlying torus, they should be a symmetry of any dynamical effective potential $V_{\text{eff}}(\tau)$ which eventually stabilizes the moduli, with the extrema of the potential corresponding to fixed points under the modular transformations. (An example of this in the case of Casimir energies can be found in Ref. [11].) For $\tau_2 > 0$, it is well known that there are only two distinct fixed points: $\tau = i$ and $\tau = e^{\pi i/3}$. However, if we permit ourselves to consider the $\tau_2 \to 0$ limit, we find that there are a series of additional fixed points with

$$|\tau| = \tau_1 = \frac{1}{2} \left( p \pm \sqrt{p^2 - 4} \right), \quad p \in \mathbb{Z} \geq 2. \tag{23}$$

Indeed, for these points the modular transformation $\tau \to -1/\tau$ produces $\tau_1 \to \tau_1 + p$, which is identified with $\tau_1$ under the torus symmetries. Note that for $p \geq 2$, these values are all algebraic irrational numbers, as desired. Moreover, this series of fixed points includes the value $|\tau| = \frac{1}{2} (3 + \sqrt{5})$ which, as we have already seen, corresponds to the maximum possible value of $A$. Thus, even though such points with $\tau_2 = 0$ are at the “edge” of the allowed moduli space (and strictly speaking are not even within the fundamental domain of the modular group), we can imagine that the dynamics might cause the shape moduli to approach these fixed points given appropriate initial conditions. As such, these limiting cases could emerge as the result of non-perturbative string dynamics or via cosmological evolution.

In summary, then, we have shown that the shape moduli associated with large-radius compactifications can have a significant effect on the corresponding Kaluza-Klein spectrum and in turn on the resulting low-energy phenomenology. We investigated these ideas in the context of flat, two-dimensional toroidal compactifications, but similar effects are also likely to arise in more complicated higher-dimensional compactifications on more exotic manifolds [12], or even in “warped” compactifications [13]. Indeed, we have seen that in certain limiting cases, it is possible to make large extra dimensions essentially “invisible” with respect to experimental and observational constraints on light Kaluza-Klein states. The incorporation of shape moduli can therefore be used to significantly widen the allowed parameter space of higher-dimensional theories beyond what has previously been considered.

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[1] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. B 429 (1998) 263 [hep-ph/9803347]; I. Antoniadis et al., Phys. Lett. B 436 (1998) 257 [hep-ph/9804399].

[2] K.R. Dienes, E. Dudas and T. Gherghetta, Phys. Lett. B 436 (1998) 55 [hep-ph/9803464]; Nucl. Phys. B 537 (1999) 47 [hep-ph/9806292]; [hep-ph/9807522].

[3] E. Witten, Nucl. Phys. B 471 (1996) 135 [hep-th/9602070]; J.D. Lykken, Phys. Rev. D 54 (1996) 3693 [hep-th/9603133]; G. Shiu and S.-H.H. Tye, Phys. Rev. D 58 (1998) 106007 [hep-th/9805157]; Z. Kakushadze and S.-H.H. Tye, Nucl. Phys. B 548 (1999) 180 [hep-th/9809147].

[4] A. Kehagias and K. Sfetsos, Phys. Lett. B 472 (2000) 39 [hep-ph/9905417]; E.G. Floratos and G.K. Leontaris, Phys. Lett. B 465 (1999) 95 [hep-ph/9906238].

[5] See, e.g., G.H. Hardy and E.M. Wright, An Introduction to the Theory of Numbers, 4th Edition (Oxford University Press, 1959); M. Hindry and J.H. Silverman, Diophantine Geometry: An Introduction, Graduate Texts in Mathematics #201 (Springer-Verlag, 2000).

[6] K.F. Roth, Mathematika 2 (1955) 1.

[7] See, e.g., C.D. Hoyle et al., Phys. Rev. Lett. 86 (2001) 1418 [hep-ph/0011014].

[8] K.R. Dienes, E. Dudas and T. Gherghetta, Nucl. Phys. B 557 (1999) 25 [hep-ph/9811428]; N. Arkani-Hamed et al., [hep-ph/9811448].

[9] S. Chang, S. Tazawa and M. Yamaguchi, Phys. Rev. D 61 (2000) 084005 [hep-ph/9908515]; K.R. Dienes, E. Dudas and T. Gherghetta, Phys. Rev. D 62 (2000) 105023 [hep-ph/9912143]; L. Di Lella et al., Phys. Rev. D 62 (2000) 125011 [hep-ph/0006327].

[10] K.R. Dienes and A. Mafi, Compactification on Manifolds with Non-Trivial Shape Moduli, to appear.

[11] E. Pontón and E. Poppitz, JHEP 0106 (2001) 019 [hep-ph/0105021].

[12] See, e.g., N. Kaloper et al., Phys. Rev. Lett. 85 (2000) 928 [hep-ph/0002001].

[13] L. Randall and R. Sundrum, Phys. Rev. Lett. 83 (1999) 3370 [hep-ph/9905221]; Phys. Rev. Lett. 83 (1999) 4690 [hep-th/9906064].