Explicit Non-Abelian Gerbes with Connections

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Abstract

We derive the complete cocycle description for non-Abelian gerbes with connections whose structure 2-group is a 2-group with adjustment datum. We depart from the common fake-flat connections and employ adjusted connections, which is important for physical applications, especially in the context of supergravity. We give a number of explicit examples; in particular, we lift the principal bundle corresponding to an instanton–anti-instanton pair to a string 2-group bundle. We also outline how categorified forms of Bogomolny monopoles known as self-dual strings can be obtained via a Penrose–Ward transform of string bundles over twistor space.

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1. **Introduction and results**

Higher forms arise as gauge potentials in a number of contexts within physics. The prime example is the Kalb–Ramond $B$-field of string theory, which is then found again in the low-energy supergravity limits. A whole family of higher form gauge potentials arises in the tensor hierarchies of gauged supergravity theories and, consequentially, double and exceptional field theory. Mathematically, these gauge potentials are connections on higher or categorified principal bundles also known as gerbes. This perspective is evidently crucial when supergravity theories are to be considered on topologically non-trivial spaces.

In the Abelian case, the theory of gerbes is well-established and used in many contexts. There are a number of equivalent descriptions, such as the geometrically appealing bundle...
gerbes \([1, 2]\) or the computationally useful Hitchin–Chatterjee gerbes \([3, 4]\). Although the Kalb–Ramond \(B\)-field is locally an ordinary differential form, Abelian gerbes are not fully sufficient for its description. For example in the presence of gauge potential 1-forms, the two types of potentials mix non-trivially in the definition of curvatures and gauge potentials. This is the case, e.g. in gauged supergravity, and it was there that what are now called connections on string bundles had been observed for the first time \([5, 6]\). Similarly, in the context of T-duality, one often wants to describe string theory backgrounds which correspond to Abelian gerbes on the total space of principal torus bundles. These can be conveniently captured by non-Abelian generalisations of gerbes at the topological level \([7]\). A differential refinement, leading to the full Buscher rules is also possible \([8]\). Moreover, there is reason to believe that aspects of the dynamics of multiple M5-branes can be captured by non-Abelian gerbes \([9–11]\); see also \([12]\) for recent evidence that the string group \(\text{String}(3)\) naturally emerges on a single M5-brane.\(^1\) Finally, several kinds of non-Abelian gerbes have been used in a number of string-theory inspired twistor constructions \([14–16]\).

Non-Abelian gerbes have been introduced in various forms and all familiar definitions of principal bundles have found higher generalisations, cf. \([17]\) and references therein. One of the earliest forms is perhaps the differential cocycle descriptions given in \([18]\) and \([19]\), and because of its computational usefulness, we shall focus mostly on this picture in the following. This form of non-Abelian gerbes generalises the notion of a principal fibre bundle by lifting the cocycle relations for transition functions and local gauge patches to hold up to homotopies, which are then encoded in higher components of the cocycle.

While the topological description of non-Abelian gerbes is relatively straightforward, its differential refinement by a connection is subtle. An explicit cocycle description for such a differential refinement was obtained in \([18, 19]\), but these cocycles contain an additional local 2-form datum\(^2\) compared to the expected \(L_\infty\)-algebra-valued connections \([20]\). A detailed study of the evident notion of higher parallel transport \([21–23]\) then suggested that the additional datum is, in a sense, spurious, and that one should work instead with reduced cocycles \([21]\), which have subsequently been used throughout much of the literature. This, however, led to a second problem. Consistency of this parallel transport requires the 2-form part \(F\) of the total curvature of a non-Abelian gerbe with connection, which is also known as its fake flatness, to vanish \([22, 23]\). The fake-flatness condition, however, is problematic for two reasons: firstly, we know that interesting string backgrounds do not satisfy this condition, and secondly, it can be shown that locally, a fake-flat non-Abelian gerbe with

\(^1\)See also \([13]\) for a general review on higher structures in M-theory.

\(^2\)In \([18, 19]\), this 2-form is denoted by \(\delta_{ij}\).
connection is always isomorphic to an Abelian gerbe with connection, rendering it essentially useless for all but topological purposes [24, 10]. Another undesirable consequence is that after adding fake flatness to the cocycle conditions of non-Abelian gerbes with connection, ordinary principal bundles with non-flat connections are no longer trivially higher bundles. This is clearly undesirable from a category theoretical viewpoint.

The solution to these problems is provided by a third possible type of cocycles that are more general than the reduced ones of [21], but do not contain the additional 2-form datum of [18, 19]. These “adjusted” cocycles are obtained by deforming the definition of the 3-form part of the curvature by a term proportional to the fake curvature, which induces a deformation of the gauge transformation of the 2-form part of the connection. At a very explicit level and in one particular case, this had been observed already in [5, 6]. From a mathematical perspective, connections can locally be regarded as morphisms from the Weil algebra of the (higher) gauge algebra to the de Rham complex on the relevant coordinate patch. The necessary modifications of the curvature for string bundles were explained by a coordinate change on the Weil algebra in [25, 20]. The curvature modification was then dubbed an adjustment in [10], where also the adjustments for a number of other higher bundles were computed. In the latter paper, it was shown that contrary to the reduced cocycles of [21], adjusted infinitesimal gauge transformations close without the need for fake flatness. Shortly later, it was then shown that a consistent adjusted higher parallel transport that does not rely on fake flatness is indeed possible [26]. Moreover, the origin of the adjustment was traced back to a generalisation of the notion of higher gauge algebra in [27], where also adjustments for large classes of higher bundles were derived that are relevant in the context of gauged supergravity.

To our knowledge, the finite form of the adjusted cocycles for non-Abelian gerbes with connection has never been derived. This may be due to the fact that adjustments have to be discussed individually for each higher gauge group. In particular, certain higher gauge groups do not admit an adjustment, and an example can be found in [8]. It is the primary goal of this paper to fill this gap and to give explicit cocycle relations for adjusted connections with a focus on string bundles. A secondary goal is to construct an explicit example.

\footnote{The discussion in [26] makes it clear why the adjustment is only visible at the level of connections or parallel transport. Topologically, higher principal bundles are functors from the Čech groupoid to the delooping of the higher structure group, but the extension of the Čech groupoid to a higher groupoid is trivial in the sense that all higher morphisms are identities. Parallel transport, however, is defined as a functor from a higher path groupoid to the delooping of the higher structure group. In the former, the higher morphisms are non-trivial.}
Recall that a spin bundle over an \( n \)-dimensional manifold \( M \), is a lift of the frame bundle, a principal \( SO(n) \)-bundle, to a principal \( Spin(n) \)-bundle. The obstruction to the existence of such a lift is the second Stiefel–Whitney class of \( M \). One can now consider spin bundles over loop spaces, as suggested by string theory, and it turns out that the loop space \( LM \) of a manifold \( M \) carries a spin bundle if and only if spin bundles over \( M \) can be lifted to principal \( String(n) \)-bundles \([28]\). Here, \( String(n) \) is a topological group, defined only up to a large class of equivalence, which is a 3-connected cover of \( Spin(n) \), but otherwise shares the same homotopy groups as \( Spin(n) \).

It turns out that the group \( String(n) \) is conveniently described in terms of a 2-group model, cf. \([29]\), and from this perspective, string bundles are indeed non-Abelian gerbes. This is also the form in which string structures most naturally arise in supergravity and string theory. There are now essentially two extreme string 2-group models. Firstly, there is an infinite-dimensional one based on Kac–Moody central extensions of loop groups, which was developed in \([30]\). On the other hand, there is the finite-dimensional, but much more complicated and less explicit model \([31]\).\(^1\) For convenience, we shall be focusing on the infinite-dimensional model because it admits fully explicit formulas for all cocycles.

Explicitly, we have the following results to report:

(i) In Section 2.1, we show that each simply connected Lie group \( G \), when trivially regarded as a crossed module of Lie groups, is equivalent to a crossed module of Lie groups \( LG \).

(ii) In Section 2.3, we define the notion of an adjustment of a Lie 2-group and we develop the full cocycle description of principal \( LG \)-bundles with adjusted connection.

(iii) We show in the same section that any principal \( G \)-bundle with connection can be lifted to a fully equivalent principal \( LG \)-bundle with connection.

(iv) As an example, we give the equivalent formulation of the principal \( Spin(4) \)-bundle \( Spin(5) \to S^4 \cong Spin(5)/Spin(4) \), which describes an instanton–anti-instanton pair, as an \( LG \)-bundle in Section 3.3.

(v) Our main result is then the full cocycle description of principal \( String(G) \)-bundles with adjusted connections, which is given in Section 4.3.

(vi) We develop the description of higher coset spaces of string 2-groups in Section 5.1. As a simple example, we consider the space \( String(3)/String(2) \) and present an explicit cocycle description of this basic gerbe over \( S^3 \).

\(^1\)See \([32]\) for a description of (unadjusted) non-Abelian gerbes using this group.
(vii) Our formulas give the explicit finite gauge transformations for the non-Abelian self-dual strings introduced in [33], closing a significant gap. This is explained in Section 5.2.

(viii) As an example of a string 2-bundle, we extend our example of an $\text{LSpin}(4)$-bundle to a full and explicit $\text{String}(4)$-bundle in Section 5.3.\footnote{The case $\text{String}(4)$ is particularly interesting from an M-theory perspective, because it contains the Lie group $\text{Spin}(4) \cong \text{SU}(2) \times \text{SU}(2)$, which is important in the context of M2-brane models [34,35].}

(ix) Finally, in Section 6, we comment on an iterated Penrose–Ward transform that allows for a construction of solutions to the non-Abelian self-dual string equations of [33]. An additional minor result is our clarification of the relation between two group cocycles used in the literature to describe the Kac–Moody central extension of the loop group of a group at the beginning of Appendix C.

We have tried to be fairly self-contained and detailed in our presentation, mostly for two reasons. Firstly, we had to fix a number of sign inaccuracies in the literature we used, and we therefore wanted to present our computations in a verifiable way. Secondly, we would have found the detailed review part in this paper invaluable when we started this research project.

2. Principal fibre bundles with connection as higher bundles

To prepare our discussion of string bundles, we first consider the embedding of principal fibre bundles with connections into higher bundles. Whilst the flat case is trivial, the non-flat case requires to invoke adjusted curvatures [10].

2.1. Lie groups as crossed modules of Lie groups

Crossed modules of Lie groups. Recall that a crossed module of Lie groups is a pair of Lie groups $(G,H)$ together with an automorphism action $\triangleright$ of $G$ on $H$ as well as a morphism of Lie groups $t : H \rightarrow G$ such that

$$t(g \triangleright h_1) = gt(h_1)g^{-1} \quad \text{and} \quad t(h_1) \triangleright h_2 = h_1h_2h_1^{-1} \tag{2.1}$$

for all $g \in G$ and for all $h_1, h_2 \in H$. We usually write $\mathcal{G} := (H \xrightarrow{t} G, \triangleright)$ for a crossed module of Lie groups.

Crossed modules of Lie groups are categorified Lie groups and can be regarded as strict Lie 2-groups, cf. [36]. The most general equivalences are then given by smooth flippable butterflies; see [37] for the set theoretic version. In particular, given two crossed modules of
Lie groups, $G_1 = (H_1 \xrightarrow{t_1} G_1, \varepsilon_1)$ and $G_2 = (H_2 \xrightarrow{t_2} G_2, \varepsilon_2)$, a butterfly is a commutative diagram of Lie groups of the form

$$
\begin{array}{cccc}
H_1 & & H_2 \\
\lambda_1 & \vee & \lambda_2 \\
\gamma_1 & & \gamma_2 \\
G_1 & & G_2 \\
\end{array}
$$

(2.2)

where $E$ is a Lie group, $\lambda_{1,2}$ and $\gamma_{1,2}$ are morphisms of Lie groups, the NE–SW diagonal is a short exact sequence (i.e. a Lie group extension), and the NW–SE diagonal is a complex. The butterfly is called flippable whenever both diagonals are short exact sequences. Given such a flippable butterfly, we call $G_1$ and $G_2$ equivalent.

Path and loop groups. An example that is important for our later discussion is the following. Consider the based (parametrised) path space

$$
P_0G := \{ p \in \mathcal{C}^\infty([0,1], G) \mid p(0) = 1 \}
$$

(2.3a)

and the based (parametrised) loop space

$$
L_0G := \{ \ell \in \mathcal{C}^\infty([0,1], G) \mid \ell(0) = 1 = \ell(1) \}
$$

(2.3b)

of a simply connected Lie group $G$. Both $P_0G$ and $L_0G$ are Fréchet–Lie groups with the evident pointwise products. Furthermore, the endpoint evaluation map

$$
b : P_0G \to G, \quad p \mapsto p(1)
$$

(2.4)

is a morphism of Fréchet–Lie groups.

The based path and loop groups $P_0G$ and $L_0G$ can be arranged into the crossed module of Lie groups

$$
\mathcal{L}G := (L_0G \hookrightarrow P_0G, \text{Ad})
$$

(2.5a)

where Ad is the pointwise adjoint action of $L_0G$ on $P_0G$. On the other hand, we may trivially regard $G$ as the crossed module of Lie groups

$$
(1 \hookrightarrow G, \text{id})
$$

(2.5b)

1In our conventions for based path and loop groups, we follow [30] and we refer to this paper for further details. We shall ignore technicalities such as introducing sitting instances in our loop and path parametrisations, as these can be trivially incorporated into our constructions. For a detailed discussion of these issues, see e.g. [22]. The important point to keep in mind is that the concatenation of two elements in our path space with the same endpoint always forms an element in our loop space.
The following flippable butterfly shows that the two crossed modules (2.5) are equivalent:

![Diagram](image)

(2.6)

Note that $L_0G$ is, in fact, a normal subgroup of $P_0G$, and we have

$$P_0G \xrightarrow{b} P_0G/L_0G \xrightarrow{=} G.$$

(2.7)

The equivalence between the crossed modules (2.5) is a lift of the isomorphism between the quotient group $P_0G/L_0G$ and $G$.

**$L_\infty$-algebras and crossed modules of Lie algebras.** Differentiating\(^1\) a crossed module of Lie groups yields a **crossed module of Lie algebras**. Such a crossed module is given by a pair $(\mathfrak{g}, \mathfrak{h})$ of Lie algebras together with an automorphism action $\Rightarrow$ of $\mathfrak{g}$ on $\mathfrak{h}$ as well as morphism of Lie algebras $t : \mathfrak{h} \to \mathfrak{g}$ satisfying\(^2\)

$$t(X \Rightarrow Y_1) := [X, t(Y_1)] \quad \text{and} \quad t(Y_1) \Rightarrow Y_2 := [Y_1, Y_2]$$

(2.8)

for all $X \in \mathfrak{g}$ and for all $Y_1, Y_2 \in \mathfrak{h}$. We usually write $\text{Lie} (\mathcal{G}) := (\mathfrak{h} \xrightarrow{t} \mathfrak{g}, \Rightarrow)$ for a crossed module of Lie algebras.

Importantly, a crossed module of Lie algebras $(\mathfrak{h} \xrightarrow{t} \mathfrak{g}, \Rightarrow)$ can be viewed as a (strict 2-term) $L_\infty$-algebra, that is, a differential graded Lie algebra with underlying graded vector space

$$\mathfrak{L} := \bigoplus_{1 \leq i, j \leq \infty} \mathfrak{L}_{i-j} \quad \text{with differential}$$

$$\mu_1 : \mathfrak{L}_{-1} \to \mathfrak{L}_0,$$

$$\beta_1 \mapsto t(\beta_1),$$

(2.9b)

\(^1\)i.e. applying the tangent functor to

\(^2\)Note that we slightly abused notation here: the morphism $t$ and the action $\Rightarrow$ are, in fact, differentials of the morphism and action with the same labels of a crossed module of Lie groups.
and with Lie bracket
\[ \mu_2 : \mathfrak{L}_i \times \mathfrak{L}_j \to \mathfrak{L}_{i+j} , \]
\[ (\beta_1, \beta_2) \mapsto 0, \]
\[ (\alpha_1, \alpha_2) \mapsto [\alpha_1, \alpha_2] , \]
\[ (\alpha_1, \beta_1) \mapsto \alpha_1 \triangleright \beta_1 \] (2.9c)
for all \( \beta_1, \beta_2 \in \mathfrak{L}_{-1} \) and for all \( \alpha_1, \alpha_2 \in \mathfrak{L}_0 \) and with \( \mu_2 \) defined to be graded anti-symmetric.

The relations (2.8) are equivalent to the differential being a derivation of the Lie bracket and the graded Jacobi identity. Furthermore, the differentiated version of a flippable butterfly (2.2), which is the notion of equivalence for crossed modules of Lie algebras, becomes the notion of a quasi-isomorphism of \( L_\infty \)-algebras in homotopy algebraic language. For more details on \( L_\infty \)-algebras in our conventions, see e.g. [38,39].

**Path and loop algebras.** As a useful example, let us consider the crossed module of Lie algebras obtained by differentiating (2.5a). To this end, we define the based, parametrised path and loop algebras of the Lie algebra \( \mathfrak{g} \) as the Lie algebra of a simply connected Lie group in the evident way:

\[ P_0 \mathfrak{g} := \{ \gamma \in C^\infty([0,1], \mathfrak{g}) \mid \gamma(0) = 0 \} , \]
\[ L_0 \mathfrak{g} := \{ \gamma \in \mathcal{C}^\infty([0,1], \mathfrak{g}) \mid \gamma(0) = 0 = \gamma(1) \} \] (2.10)
with pointwise Lie brackets. The crossed module of Lie algebras \( \text{Lie}(\mathcal{L}G) \) is then

\[ \mathcal{L} \mathfrak{g} := (L_0 \mathfrak{g} \hookrightarrow P_0 \mathfrak{g}, \text{ad}) . \] (2.11)

Let us also consider the linearised version of the equivalence between the crossed modules of Lie groups (2.5) in the form of a quasi-isomorphism of \( L_\infty \)-algebras.\(^1\) Here, we have a pair of quasi-isomorphisms of \( L_\infty \)-algebras,

\[ \mathfrak{g} \xrightarrow{\phi} \mathcal{L} \mathfrak{g} \xrightarrow{\psi} \mathfrak{g} , \] (2.12a)

where \( \mathfrak{g} \) is trivially regarded as the crossed module \((0 \leftrightarrow \mathfrak{g}, 0)\). Recall that such a quasi-isomorphism is a morphism of \( L_\infty \)-algebras that induces an isomorphisms on the cohomologies of the underlying chain complexes. Such morphisms \( \phi \) are given by chain maps \( \phi_1 \) as well as a collection of higher maps \( \phi_i, i \in \mathbb{N}^+ \) of degree \( |\phi_i| = 1 - i \) that form intertwiners between the various brackets, see e.g. [38,39] for more details.

\(^1\)This quasi-isomorphism of \( L_\infty \)-algebras is a truncation of another quasi-isomorphism that we shall encounter in **Section 4.2** and which was first described in [30].
In the case at hand, we have \( \phi = (\phi_1, \phi_2, \phi_{i>2} = 0) \) and \( \psi = (\psi_1, \psi_{i>1} = 0) \). The chain maps are given by

\[
\begin{array}{c}
0 \xrightarrow{\phi_1} L_0 \mathfrak{g} \xrightarrow{\psi_1} 0 \\
\downarrow \phi_1 = \cdot \varphi \\
\downarrow \mathfrak{g} \xrightarrow{P_0} \mathfrak{g}
\end{array}
\]  

(2.12b)

where now \( b \) is the linearisation of (2.4) and \( \cdot \varphi : \mathfrak{g} \to P_0 \mathfrak{g} \) for a fixed map \( \varphi \in \mathcal{C}^\infty([0,1], \mathbb{R}) \) with \( \varphi(0) = 0 \) and \( \varphi(1) = 1 \) is the embedding defined by \( \mathfrak{g} \ni X \mapsto \varphi \cdot X \in P_0 \mathfrak{g} \). A simple example of the map \( \varphi \) is simply \( \varphi(t) = t \) for all \( t \in [0,1] \). The only non-vanishing component of \( \phi_2 \) is

\[
\phi_2 : \mathfrak{g} \times \mathfrak{g} \to L_0 \mathfrak{g} ,
\]

(2.12c)

\[
(X_1, X_2) \mapsto (\varphi - \varphi^2) \cdot [X_1, X_2] .
\]

Gauge potentials and \( L_{\infty} \)-algebras. Given an \( L_{\infty} \)-algebra \( (\mathfrak{L}, \mu_i) \), an element \( a \in \mathfrak{L}_1 \) is called a \textit{(generalised) gauge potential}, and its \textit{curvature} is given by

\[
f(a) := \mu_1(a) + \frac{1}{2!} \mu_2(a, a) + \frac{1}{3!} \mu_3(a, a, a) + \cdots \in \mathfrak{L}_2 .
\]

(2.13)

A \textit{Maurer–Cartan element} is a gauge potential \( a \) that satisfies the \textit{Maurer–Cartan equation} \( f(a) = 0 \). A morphism (and, in particular, a quasi-isomorphism) of \( L_{\infty} \)-algebras \( \phi = (\phi_i) : (\mathfrak{L}, \mu_i) \to (\tilde{\mathfrak{L}}, \tilde{\mu}_i) \) induces a map from gauge potentials \( a \in \mathfrak{L}_1 \) of \( (\mathfrak{L}, \mu_i) \) to gauge potentials \( \tilde{a} \in \tilde{\mathfrak{L}}_1 \) of \( (\tilde{\mathfrak{L}}, \tilde{\mu}_i) \) given by

\[
a \mapsto \tilde{a} := \phi_1(a) + \frac{1}{2!} \phi_2(a, a) + \frac{1}{3!} \phi_3(a, a, a) + \cdots ,
\]

(2.14)

which, in turn, induces the map

\[
\tilde{f}(\tilde{a}) = \phi_1(f(a)) + \phi_2(a, f(a)) + \frac{1}{2!} \phi_3(a, a, f(a)) + \cdots .
\]

(2.15)

In particular, Maurer–Cartan elements are mapped to Maurer–Cartan elements, see e.g. [38, 39] for a detailed review in our conventions.

As an example, consider a gauge potential on a topologically trivial (higher) principal fibre bundle over a manifold \( M \) whose higher structure group differentiates to the \( L_{\infty} \)-algebra \( \mathfrak{L} \). Such a gauge potential is a generalised gauge potential in the \( L_{\infty} \)-algebra \( \Omega^\ast(M) \otimes \mathfrak{L} \) with \( \Omega^\ast(M) \) the differential forms on \( M \), and a morphism \( \phi : \mathfrak{L} \to \tilde{\mathfrak{L}} \) of \( L_{\infty} \)-algebras then induces a morphism

\[
\Omega^\ast(M) \otimes \mathfrak{L} \xrightarrow{\text{id} \otimes \phi} \Omega^\ast(M) \otimes \tilde{\mathfrak{L}} .
\]

(2.16)

We shall make use of this induced morphism later; for a review of this construction, see again [38, 39].
2.2. Descent data for principal 1- and 2-bundles

In the following, let $M$ be a manifold.

**Covers and Čech groupoids.** Consider a surjective submersion $\sigma : Y \to M$ for another manifold $Y$ together with the fibre products

$$Y^{[n]} := \left( Y \times_M \cdots \times_M Y \right)^{n \text{ factors}} := \{(y_1, \ldots, y_n) \in Y^n | \sigma(y_1) = \cdots = \sigma(y_n)\} \quad (2.17)$$

for all $n \in \mathbb{N}^+$. We call such a map $\sigma : Y \to M$ a cover.¹ For example, we can consider an atlas $\{(U_i, \phi_i)\}_{i \in I}$ for $I$ some index set of the manifold $M$ with $\phi_i : U_i \to \phi_i(U_i) \subseteq \mathbb{R}^n$ homeomorphisms. Then, we can set $Y := \bigsqcup_{i \in I} \phi_i(U_i)$ and the inverses of the $\phi_i$ define the map $\sigma : Y \to M$. Moreover,

$$Y^{[1]} = \bigsqcup_{i \in I} \sigma^{-1}(U_i), \quad Y^{[2]} = \bigsqcup_{i,j \in I} \sigma^{-1}(U_i \cap U_j), \quad Y^{[3]} = \bigsqcup_{i,j,k \in I} \sigma^{-1}(U_i \cap U_j \cap U_k), \quad \ldots \quad (2.18)$$

are all the preimages of the coordinate patches, the double overlaps, the triple overlaps, and so on. To ensure that a cover is suitable for capturing all topological features of $M$, one can work with good covers for which all the $Y^{[n]}$ are simply connected. Note that any ordinary cover can be refined such that this condition is indeed satisfied.

We note that any surjective submersion $\sigma : Y \to M$ comes with an associated Čech groupoid given by

$$\tilde{\mathcal{C}}(Y \to M) := (Y^{[2]} \rightrightarrows Y) \quad \text{with} \quad y_1 \overset{(y_1, y_2)}{\longrightarrow} y_2 \quad (2.19)$$

with the evident concatenation and inverse. As an example, consider the Čech groupoid of the surjective submersion $\beta : P_0 G \to G$, in which case $^2 (P_0 G)^{[2]} \cong L_0 G$, and which is given by

$$\tilde{\mathcal{C}}(P_0 G \to G) \cong (L_0 G \rightrightarrows P_0 G) \quad \text{with} \quad t(\ell) \overset{\ell}{\longrightarrow} s(\ell), \quad (2.20)$$

where, for $\ell \in L_0 G$, the source and target maps $s$ and $t$ are the projections onto the (unique) path and the inverse path that compose to the loop in the path groupoid of $G$. Usually, we identify these with the loop segment $t \mapsto \ell(t)$ for all $t \in \left[0, \frac{1}{2}\right]$ and $t \mapsto \ell\left(\frac{3}{2} - t\right)$ for all

¹Note that $\sigma$ does not have to be a local homeomorphism.

²This identification is unique up to the definition of composition of paths in the path groupoid of $G$. 

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$t \in [\frac{1}{2}, 1]$, respectively. In particular, $\ell = \overline{t(\ell)} \circ s(\ell)$, where $\overline{t(\ell)}$ is the inverse path to $t(\ell)$ and $\circ$ denotes the composition of paths.

We note that in many cases, in particular in the cases involving the clutching construction of bundles over spheres, it can be more convenient to work with hypercovers, cf. [37], see also [40] for an application in the context of higher bundles similar to our goal. The extension of our formalism to hypercovers would require significantly more work, but it is in principle straightforward. In order to keep our discussion simple, we refrain from going into further detail.

**Principal $G$-bundles.** Let now $G$ be a Lie group. The cocycles describing the descent data of principal $G$-bundles with connection over $M$, also known as principal 1-bundles, subordinate to the cover $Y$ consist of transition functions $g$ and gauge potentials $A$,

$$g \in \mathcal{C}^\infty(Y^{[2]}, G) \quad \text{and} \quad A \in \Omega^1(Y^{[1]}, g)$$

such that

$$g_{ik} = g_{ij}g_{jk} \quad \text{and} \quad A_j = g^{-1}_{ij}A_ig_{ij} + g^{-1}_{ij}dg_{ij}$$

(2.21b)

for all appropriate $(i, j, \ldots) \in Y^{[n]}$.

Two such cocycles $(g, A)$ and $(\tilde{g}, \tilde{A})$ are said to be equivalent whenever they are related by a coboundary given by

$$a \in \Omega^0(Y^{[1]}, G)$$

(2.22a)

such that

$$\tilde{g}_{ij} = a_{i}^{-1}g_{ij}a_{j} \quad \text{and} \quad \tilde{A}_i = a_{i}^{-1}A_ia_{i} + a_{i}^{-1}da_{i}$$

(2.22b)

for all appropriate $(i, j, \ldots) \in Y^{[n]}$. Such bundle isomorphisms are usually known as gauge transformations.

**Principal $G$-bundles.** There is a categorified notion of principal $G$-bundles, which have crossed modules of Lie groups $\mathcal{G} = (H \xrightarrow{\triangleright} G, \triangleright)$ as their categorified structure groups. These are known as principal 2-bundles [18, 19], and we shall also use the term principal $G$-bundles. Let $(h \xrightarrow{\triangleright} g, \triangleright)$ be the structure crossed module of Lie algebras corresponding to $\mathcal{G}$. The cocycles describing the descent data of principal $G$-bundles over $M$ subordinate to the cover $Y$ consist of

$$h \in \mathcal{C}^\infty(Y^{[3]}, H)$$

$$\quad (g, \Lambda) \in \mathcal{C}^\infty(Y^{[2]}, G) \oplus \Omega^1(Y^{[2]}, \mathfrak{h})$$

$$\quad (A, B) \in \Omega^1(Y^{[1]}, g) \oplus \Omega^2(Y^{[1]}, \mathfrak{h})$$

(2.23a)
such that
\[ h_{ikl} h_{ijk} = h_{ijl}(g_{ij} \Rightarrow h_{jkl}) , \]
\[ g_{ik} = t(h_{ijk})g_{ij}g_{jk} , \]
\[ \Lambda_{ik} = \Lambda_{jk} + g_{jk}^{-1} \Lambda_{ij} - g_{ik}^{-1} (h_{ijk} \nabla_i h_{ijk}^{-1}) , \]
\[ A_j = g_{ij}^{-1} A_i g_{ij} + g_{ij}^{-1} d g_{ij} - t(\Lambda_{ij}) , \]
\[ B_j = g_{ij}^{-1} B_i + d \Lambda_{ij} + A_j \Rightarrow \Lambda_{ij} + \frac{1}{2} [\Lambda_{ij}, \Lambda_{ij}] , \]
\[ 0 = d A_i + \frac{1}{2} [A_i, A_i] + t(B_i) \]
for all appropriate \((i, j, \ldots) \in Y^{[n]}\). The data \((A, B, \Lambda)\) differentially refines the topological principal \(G\)-bundle, and we shall simply refer to this data as a connection. The total curvature of this connection is given by the 2-form \(\text{fake curvature}\) and the 3-form curvature that are given on \((i) \in Y^{[1]}\) as
\[ F_i := d A_i + \frac{1}{2} [A_i, A_i] + t(B_i) \quad \text{and} \quad H_i := d B_i + A_i \Rightarrow B_i . \]
(2.24)
The last condition in (2.23b) says that \(F_i = 0\), and this is known as \(\text{fake flatness}\).

Two principal \(G\)-bundles are said to be equivalent whenever the cocycles \((h, g, \Lambda, A, B)\) and \((\tilde{h}, \tilde{g}, \tilde{\Lambda}, \tilde{A}, \tilde{B})\) encoding their descent data are linked by a coboundary. Explicitly, such a coboundary consists of maps
\[ b \in C^\infty(Y^{[2]}, H) , \]
\[ (a, \lambda) \in C^\infty(Y^{[1]}, G) \oplus \Omega^1(Y^{[1]}, H) \]
which satisfy [18, 19]
\[ \tilde{h}_{ijk} = a_i^{-1} \Rightarrow (b_{ik} h_{ijk}(g_{ij} \Rightarrow b_{jk}^{-1}) b_{ij}^{-1}) , \]
\[ \tilde{g}_{ij} = a_i^{-1} t(b_{ij}) g_{ij} a_i , \]
\[ \tilde{\Lambda}_{ij} = a_j^{-1} \Rightarrow \Lambda_{ij} + \lambda_j - \tilde{g}_{ij}^{-1} \Rightarrow \lambda_i + (a_j^{-1} g_{ij}^{-1}) \Rightarrow (b_{ij}^{-1} \nabla_i b_{ij}) , \]
\[ \tilde{A}_i = a_i^{-1} A_i a_i + a_i^{-1} d a_i - t(\lambda_i) , \]
\[ \tilde{B}_i = a_i^{-1} \Rightarrow B_i + d \lambda_i + \tilde{A}_i \Rightarrow \lambda_i + \frac{1}{2} [\lambda_i, \lambda_i] \]
for all appropriate \((i, j, \ldots) \in Y^{[n]}\).

A new feature in the case of principal \(G\)-bundles when compared to principal \(G\)-bundles is that there are higher gauge transformations of bundle isomorphisms that link to gauge transformations or bundle isomorphisms
\[ (h, g, \Lambda, A, B) \xrightarrow{m} (\tilde{h}, \tilde{g}, \tilde{\Lambda}, \tilde{A}, \tilde{B}) , \]
(2.26)
where \( m \in C^\infty(Y^{[1]}, H) \) and the corresponding coboundary relation is
\[
\begin{align*}
\tilde{b}_{ij} &= m_i b_{ij} (g_{ij} \triangleright m_j^{-1}), \\
\tilde{a}_i &= t(m_i) a_i, \\
\tilde{\lambda}_i &= \lambda_i + a_i^{-1} \triangleright (m_i^{-1} (A_i \triangleright m_i) + m_i^{-1} dm_i)
\end{align*}
\] (2.27)
for all appropriate \((i,j, \ldots) \in Y^{[n]}\).

**Fake flatness.** As we have seen, the fake flatness condition \( F_i = 0 \) is part of our cocycle relations (2.23), and it is vital for their consistency. In particular, the consistency of gauge transformations and, correspondingly, the gluing conditions requires fake flatness, as does a consistent definition of parallel transport [22,23], cf. also the discussion in [26]. Explicitly, consider gluing two gauge transformations to a third one as follows:

\[
\begin{array}{c}
(A_1, B_1) \xleftarrow{(a_{12}, \lambda_{12})} (A_2, B_2) \xrightarrow{m_{123}} (A_{23}, B_{23}) \xleftarrow{(a_{23}, \lambda_{23})} (A_3, B_3) \\
(A_1, B_1) \xrightarrow{(a_{13}, \lambda_{13})} (A_3, B_3)
\end{array}
\] (2.28)

With the cocycle relations (2.23), this diagram is commutative if and only if
\[
(a_{23}^{-1} a_{12}^{-1}) \triangleright (m_{123}^{-1} (F_1 \triangleright m_{123})) = 0,
\] (2.29)
which, without further restrictions to the crossed module, requires us to impose the fake flatness condition \( F_1 = 0 \).

The consequences of having to impose the fake flatness are rather severe. First of all, fake-flat principal 2-bundles are locally isomorphic to abelian gerbes [24,10], which impedes their application in many physical situations. Secondly, one would hope for a homogeneous form of the cocycle relations, such as (2.23), that is independent of the crossed module of Lie groups chosen. This, however, is not possible as the following example shows. Recall that a Lie group \( G \) can be trivially regarded as a crossed module of Lie groups \((\mathbb{1} \hookrightarrow G, \text{id})\). Correspondingly, a topological principal bundle (i.e. one without connection) can be trivially regarded as a topological principal \( (\mathbb{1} \hookrightarrow G, \text{id}) \)-bundle. The extension to principal \( G \)-bundles with non-flat connections, however, requires us to remove the fake curvature condition from (2.23). The latter is consistent as in the case of the crossed module \((\mathbb{1} \hookrightarrow G, \text{id})\), equation (2.29) is automatically satisfied.

Local gauge potentials and their infinitesimal gauge transformations are described by generalised gauge potentials in an \( L_\infty \)-algebra, as explained above. At this level, it is well-known that, at least for certain \( L_\infty \)-algebras, there is a modification of the curvature...
(implying a modification of the gauge transformations) that lifts the infinitesimal version of the fake flatness condition. This modification was dubbed an adjustment in [10], and we shall develop its finite form below. The principal \( G \)-bundles defined above, as well as the cocycle and coboundary relations (2.23) and (2.25), will then be called unadjusted.

**Abelian gerbes.** We note that for the crossed module of Lie groups \( (G \to \mathbb{1}, \text{id}) \), the group \( G = \ker(t) \) is necessarily Abelian. A principal \( (G \to \mathbb{1}, \text{id}) \)-bundle is then an Abelian gerbe with connection in the sense of Hitchin–Chatterjee [3,4], and for these, the fake flatness issue is absent. Recall that any of the more general bundle gerbes [1] is stably isomorphic to a Hitchin–Chatterjee gerbe [2] and thus also, albeit indirectly, captured by our approach. To describe bundle gerbes more directly, one would have to work with hypercovers.

### 2.3. Adjusted descent data

To recast non-flat principal 1-bundles as principal 2-bundles, we need to deviate from the above unadjusted descent data to an adjusted form [10]. Before discussing all the details of the adjustment for the descent data, let us first develop the local and infinitesimal description for \( G \) a simply connected, compact, semi-simple gauge group.

**Local adjusted \( LG \)-connections.** We start from the local connection data as well as the infinitesimal gauge transformations, because they are already known; they are truncations of those for string structures, cf. [33,10]. Locally, on an open contractible cover \( Y \), an \( LG \)-connection is given by one- and 2-forms

\[
(A,B) \in \Omega^1(Y,P_0\mathfrak{g}) \oplus \Omega^2(Y,L_0\mathfrak{g}) \tag{2.30}
\]

with the associated curvature forms

\[
F := \text{d}A + \frac{1}{2}[A,A] + t(B) \in \Omega^2(Y,P_0\mathfrak{g}) ,
\]

\[
H := \text{d}B + A \gg B - \kappa(A,F) \in \Omega^3(Y,L_0\mathfrak{g}) ,
\]

where

\[
\kappa : \Omega^p(Y,P_0\mathfrak{g}) \times \Omega^q(Y,P_0\mathfrak{g}) \to \Omega^{p+q}(Y,L_0\mathfrak{g}) ,
\]

\[(\alpha_1,\alpha_2) \mapsto (\text{id} - \varphi \cdot \beta)([\alpha_1,\alpha_2]) \tag{2.31b}\]

for, as before, some fixed \( \varphi \in C^\infty([0,1],\mathbb{R}) \) with \( \varphi(0) = 0 \) and \( \varphi(1) = 1 \) and \( \beta \) is again the endpoint evaluation map (2.4). For book-keeping purposes, we have used here the generic symbols \( t \) and \( \gg \) to denote the morphism and the action (2.11) of \( L\mathfrak{g} \). Note that the function \( \varphi \) is now a crucial part of the adjustment datum \( \kappa \), and the curvatures are always defined
with respect to a specific choice of $\wp$. We shall discuss the equivalence of different choices of $\wp$ below; for now, we just note that the 3-form curvature $H$ can be written as

$$H = (\text{id} - \wp \cdot b)(dF).$$

(2.32)

It is not difficult to see that the above curvatures satisfy the Bianchi identities

$$dF + [A, F] = t(H + \kappa(A, F)) \quad \text{and} \quad dH = 0.$$  

(2.33)

The expressions for the curvature forms now determine the gauge transformations, cf. e.g. the discussion in [10]. Explicitly, infinitesimal gauge transformations are parametrised by $\alpha \in \mathcal{C}^\infty(Y, P_0\mathfrak{g})$ and $\lambda \in \Omega^1(Y, L_0\mathfrak{g})$, and the gauge potentials and curvatures behave under such transformations as

$$\begin{align*}
\delta A &\coloneqq d\alpha + [A, \alpha] - t(\lambda), \\
\delta B &\coloneqq -\alpha \triangleright B + d\lambda + A \triangleright \lambda + \kappa(\alpha, F) = d\lambda + A \triangleright \lambda + \kappa(\alpha, F - t(B)),
\end{align*}$$

(2.34a)

and

$$\begin{align*}
\delta F &= [F, \alpha] + t(\kappa(\alpha, F)) = \wp \cdot b([F, \alpha]) \quad \text{and} \quad \delta H = 0.
\end{align*}$$

(2.34b)

Likewise, higher gauge transformations are parametrised by $\vartheta \in \mathcal{C}^\infty(Y, L_0\mathfrak{g})$ and the gauge parameters transform as

$$\begin{align*}
\delta \alpha &\coloneqq t(\vartheta) \quad \text{and} \quad \delta \lambda := d\vartheta + A \triangleright \vartheta.
\end{align*}$$

(2.35)

**Local $G$- and $\mathcal{L}G$-connections.** Local $G$-connections $A$ can be regarded as local $\mathcal{L}G$-connections $(A^\circ, B^\circ)$. Explicitly, we use the equivalence (2.12) as well as the action (2.16) of morphisms of $L_{\infty}$-algebras on gauge potentials, and we define

$$\begin{align*}
\phi(A) &= (A^\circ, B^\circ) := (\wp \cdot A, \frac{1}{2}(\wp - \wp^2) \cdot [A, A]) \quad \text{and} \quad \psi(A^\circ, B^\circ) = A := b(A^\circ).
\end{align*}$$

(2.36)

Consequently, (2.15) together with (2.31) yield the curvature forms

$$\begin{align*}
\phi(F) &= (F^\circ, H^\circ) = (\wp \cdot F, 0) \quad \text{and} \quad \psi(F^\circ, H^\circ) = F = b(F^\circ).
\end{align*}$$

(2.37)

The form of the maps $\phi$ and $\psi$ make it now transparent that we would have to restrict $B^\circ$ further to establish an equivalence of the local connections. In the unadjusted case, this issue is resolved by the fake flatness condition. In the adjusted case, we would like the fake flatness condition to be satisfied only on the image of $t$. Generally, there is no natural projection onto this image, but the adjustment data, which includes in particular a choice
of the map \( \varphi \), yields such a map. We thus relax the fake flatness condition in (2.23) to the **adjusted fake flatness condition**

\[
(id - \varphi \cdot b)(F^\circ) = 0 .
\]  

(2.38)

This condition is consistent in the following sense. Firstly, because of (2.37), the gauge potentials \((A^\circ, B^\circ)\) given by \( \phi \) in (2.36) satisfy this condition. Secondly, this condition is a gauge invariant statement as follows directly from (2.34).

The adjusted fake flatness condition also fulfils our expectations that there are no additional degrees of freedom in \( B^\circ \): because of (2.32), we then have also \( H^\circ = 0 \), again a gauge invariant statement.\(^1\)

We note that the additional condition (2.38) has not appeared in the literature on higher principal bundles so far, because most of the explicit definitions employ the skeletal model of the string Lie 2-algebra, for which the complement to the cokernel of \( t \) is trivial.

**Adjusted finite gauge transformation.** Let us now develop the finite version of the gauge transformations (2.34). Adjusted finite gauge transformations are parametrised by \( a \in \mathcal{C}^\infty(Y,P_0G) \) and \( \lambda \in \Omega^1(Y,L_0g) \), and they act on the gauge potentials \((A,B)\) according to

\[
A \mapsto \tilde{A} := a^{-1} A a + a^{-1} da - t(\lambda) ,
\]

\[
B \mapsto \tilde{B} := a^{-1} \triangleright B + d\lambda + \tilde{A} \triangleright \lambda + \frac{1}{2}[\lambda, \lambda] - \kappa(a,F) = d\lambda + \tilde{A} \triangleright \lambda + \frac{1}{2}[\lambda, \lambda] - \kappa(a,F - t(B)) ,
\]

where \( F \) as given in (2.31) and

\[
\kappa(a_1, a_2) := (id - \varphi \cdot b)(a_1^{-1}a_2a_1 - a_2). \quad (2.39b)
\]

for all \( a_1 \in \mathcal{C}^\infty(Y,P_0G) \) and \( \alpha_2 \in \Omega^q(Y,P_0g) \). Evidently, the linearisation of these transformations yields (2.34). Furthermore, the induced transformations of the curvatures forms \((F,H)\) defined in (2.31) are

\[
F \mapsto \tilde{F} = F + \varphi \cdot b(a^{-1}Fa - F) \quad \text{and} \quad H \mapsto \tilde{H} = H , \quad (2.40)
\]

as a short calculation reveals. We note that the adjusted gauge transformations (2.39) can be written such that only the ordinary curvature 2-form \( F - t(B) \) appears, which confirms observations made earlier in the context of parallel transport [26], where the relevant curvature 2-form was also the ordinary one.

\(^1\)This is contrary to the unadjusted case; there \( H^\circ = 0 \) is not gauge invariant for generic \( F^\circ \).
It is now crucial to check that the above postulated gauge transformations glue together consistently, that is, we need the commutative diagram

\[
\begin{array}{ccc}
(A_1, B_1) & \xrightarrow{(a_{13}, \lambda_{13})} & (A_3, B_3) \\
\downarrow m_{123} & & \downarrow m_{123} \\
(A_2, B_2) & \xleftarrow{(a_{12}, \lambda_{12})} & (A_3, B_3)
\end{array}
\]  

(2.41)

We know from the case of unadjusted principal \( L G \)-bundles that the higher gauge transformation \( m_{123} \) is parametrised by an \( m_{123} \in \mathcal{C}^\infty(Y, L_0 G) \) that glues together the gauge transformations on the arrows according to

\[
a_{13} = t(m_{123})a_{12}a_{23} , \\
\lambda_{13} = \lambda_{23} + a_{23}^{-1} : \lambda_{12} - a_{13}^{-1} : (m_{123} \nabla_1 m_{123}^{-1}) .
\]  

(2.42)

The modification in the gauge transformations of \( B \) induces a modification of (2.29),

\[
(a_{23}^{-1} a_{12}^{-1}) : (m_{123}^{-1}(F_1 : m_{123})) = \kappa(a_{13}, F_1) - a_{23}^{-1} : \kappa(a_{12}, F_1) - \kappa(a_{23}, F_2) .
\]  

(2.43)

With our choice of adjustment (2.39b), this equation is identically satisfied. Contrary to the unadjusted case, there is no additional condition required for the consistency of the gauge transformations in the adjusted case.

**Adjustment for a general strict Lie 2-group.** Consider the equation (2.43) that describes the consistency of the gluing of gauge transformations, but for a general crossed module of Lie groups \( \mathcal{G} := \langle H \rightarrow \rightarrow G, \triangleright \rangle \). We can rewrite \( F_2 \) and \( a_{13} \) using the cocycle relations and replace the cocycle components with constant objects. Consequently, we must have

\[
(g_2^{-1} g_1^{-1}) : (h^{-1}(X : h)) + g_2^{-1} : \kappa(g_1, X) + \kappa(g_2, g_1^{-1} X g_1 - t(\kappa(g_1, X))) - \kappa(t(h) g_1 g_2, X) = 0
\]  

(2.44)

for all \( g_1, g_2 \in G \), for all \( h \in H \), and for all \( X \in \mathfrak{g} \). This makes it clear that an adjustment should be considered as an additional datum on the structure Lie 2-group. In particular, the adjustment is independent of the underlying topological gerbe and the base space. This is also in line with the observations made in [25,20], see also [10] and in particular [27], that at the linearised level, an adjustment is an additional datum on the structure \( L \mathcal{C} \)-algebra.

We thus call a crossed module of Lie groups \( \mathcal{G} := \langle H \rightarrow \rightarrow G, \triangleright \rangle \) together with a map

\[
\kappa : G \times \mathfrak{g} \rightarrow h
\]  

(2.45)
that is linear in \( g \) and satisfies (2.44) an \textit{adjusted crossed module} (of Lie groups). The tangent functor then linearises an adjusted crossed module of Lie groups to an adjusted crossed module of Lie algebras. As explained in [10] (see also [20] for a quasi-isomorphic case), the adjustment is here necessary for the consistent definition of invariant polynomials.

As we saw above, an example of an adjusted crossed module is the crossed module \( LG \) together with the map

\[
\kappa : P_0 G \times P_0 \mathfrak{g} \rightarrow L_0 \mathfrak{g} ,
\]

\[
(a_1, a_2) \mapsto (\mathrm{id} - \varphi \cdot \delta)(a_1^{-1}a_2a_1 - a_2) .
\]

\[
\text{(2.46)}
\]

\textbf{Adjusted descent data for adjusted crossed modules.} Consider an adjusted crossed module of Lie groups \( \mathcal{G} := (H \xrightarrow{\kappa} G, \triangleright, \kappa) \) with the corresponding crossed module of Lie algebras \( (\mathfrak{h} \xrightarrow{\kappa} \mathfrak{g}, \triangleright, \kappa) \). An adjusted cocycle for a principal \( G \) bundle is given by the data

\[
h \in \mathcal{C}^\infty(Y^3, H) ,
\]

\[
(g, A) \in \mathcal{C}^\infty(Y^2, G) \oplus \Omega^1(Y^2, \mathfrak{g}) ,
\]

\[
(A, B) \in \Omega^1(Y^1, \mathfrak{g}) \oplus \Omega^2(Y^1, \mathfrak{h}) ,
\]

\[
\text{(2.47a)}
\]

such that

\[
h_{i\kappa}h_{ij\kappa} = h_{ij\kappa}(g_{ij} \triangleright h_{jk\kappa}) \quad \text{and} \quad g_{ik} = t(h_{ijk})g_{ij}g_{jk} \quad \text{(2.47b)}
\]

hold for all appropriate \( (i, j, \ldots) \in Y^{[n]} \). In addition, we have the following, slightly modified cocycle conditions for the differential refinement

\[
\Lambda_{ik} = \Lambda_{jk} + g_{jk\}^{-1} \triangleright \Lambda_{ij} - g_{ik\}^{-1} \triangleright (h_{ijk} \nabla_i h_{ijk}^{-1}) ,
\]

\[
A_j = g_{ij\}^{-1}A_i g_{ij} + g_{ij\}^{-1}d g_{ij} - t(\Lambda_{ij}) ,
\]

\[
B_j = g_{ij\}^{-1} \triangleright B_i + d \Lambda_{ij} + A_j \triangleright \Lambda_{ij} + \frac{1}{2}[\Lambda_{ij}, \Lambda_{ij}] - \kappa(g_{ij}, F_i)
\]

\[
\text{(2.47c)}
\]

for all appropriate \( (i, j, \ldots) \in Y^{[n]} \), where \( \kappa \) was defined in (2.39b). The curvature forms are defined as

\[
F := dA + \frac{1}{2}[A, A] + t(B) \in \Omega^2(Y^1, \mathfrak{g}) ,
\]

\[
H := dB + A \triangleright B - \kappa(A, F) \in \Omega^3(Y^1, \mathfrak{h}) ,
\]

\[
\text{(2.47d)}
\]

where \( \kappa \) here is the appropriate linearisation of the adjustment datum also denoted by \( \kappa \). In order to ensure that equivalent 2-groups lead to equivalent principal 2-bundles with connections, we usually choose to impose an adjusted fake flatness condition\(^1\). In the case of \( LG \), this reads as

\[
0 = (\mathrm{id} - \varphi \cdot \delta)(F_i) \quad \text{for all} \quad (i) \in Y^{[1]} .
\]

\[
\text{(2.47e)}
\]

\(^1\)We stress that this is not a consistency requirement.
Likewise, a coboundary is given by

\[ b \in \mathcal{C}^\infty(Y^{[2]}, H), \]

\[ (a, \lambda) \in \mathcal{C}^\infty(Y^{[1]}, G) \oplus \Omega^1(Y^{[1]}, \mathfrak{h}) \]  

(2.48a)

and connects two cocycles \((h, g, \Lambda, A, B)\) and \((\tilde{h}, \tilde{g}, \tilde{\Lambda}, \tilde{A}, \tilde{B})\) as

\[
\begin{align*}
\tilde{h}_{ijk} & = a^{-1}_i \triangleright (b_{ik}h_{ijk}(g_{ij} \triangleright b^{-1}_{jk})b^{-1}_{ij}) , \\
\tilde{g}_{ij} & = a^{-1}_i t(b_{ij})g_{ij}a_j , \\
\tilde{\Lambda}_{ij} & = a^{-1}_j \triangleright \Lambda_{ij} + \lambda_j - \tilde{g}^{-1}_{ij} \triangleright \lambda_i + (a^{-1}_j g^{-1}_{ij}) \triangleright (b^{-1}_{ij} \nabla_i b_{ij}) , \\
\tilde{A}_i & = a^{-1}_i A_i + a^{-1}_i da_i - t(\lambda_i) , \\
\tilde{B}_i & = a^{-1}_i \triangleright B_i + d\lambda_i + \tilde{\Lambda}_i \triangleright \lambda_i + \frac{1}{2}[\lambda_i, \lambda_i] - \kappa(a_i, F_i)
\end{align*}
\]  

(2.48b)

for all appropriate \((i, j, \ldots) \in Y^{[n]}\). As in the case of an unadjusted principal \(G\)-bundle (2.27), we also have higher bundle isomorphisms parametrised by a higher coboundary \(m \in \mathcal{C}^\infty(Y^{[1]}, H)\) relating two coboundaries according to

\[
\begin{align*}
\tilde{b}_{ij} & = m_i b_{ij}(g_{ij} \triangleright m^{-1}_j) , \\
\tilde{a}_i & = t(m_i) a_i , \\
\tilde{\lambda}_i & = \lambda_i + a^{-1}_i \triangleright (m^{-1}_i(A_i \triangleright m_i) + m^{-1}_i dm_i)
\end{align*}
\]  

(2.49)

for all appropriate \((i, j, \ldots) \in Y^{[n]}\).

Comparing the adjusted cocycle conditions (2.47) with the unadjusted ones (2.23) as well as the adjusted coboundary conditions (2.48) with the unadjusted ones (2.25) as well as the higher coboundary transformations (2.27) and (2.49), we realise that the only modification in the adjusted descent data arises in the patching and gauge transformations of the gauge potential \(B\) as well as in the fake flatness condition.

**Equivalence of descent data.** Let us now explain the equivalence between principal \(G\)-bundles with connections and principal \(\mathcal{L}G\)-bundles with adjusted connections.

Firstly, we can construct the descent data \((h^\circ, g^\circ, \Lambda^\circ, A^\circ, B^\circ)\) for a principal \(\mathcal{L}G\)-bundle from the data \((g, A)\) of a principal \(G\)-bundle as follows. Let \(Y\) be a good cover such that in particular \(Y^{[2]}\) is contractible. Because the projection \(b : P_0 G \to G\) is a surjective submersion and \(Y^{[2]}\) is contractible\(^1\), there is a smooth lift \(g^\circ : Y^{[2]} \to \mathcal{C}^\infty(Y^{[2]}, P_0 G)\),

\[
\begin{array}{ccc}
P_0 G & \xrightarrow{g^\circ} & \mathcal{C}^\infty(Y^{[2]}, P_0 G) \\
\downarrow & & \downarrow \\
y^{[2]} & \xrightarrow{g} & G
\end{array}
\]

(2.50)

\(^1\)Because \(b\) is a surjective submersion, the pullback along \(b\) exists; this, together with the fact that \(Y^{[2]}\) is contractible, provides the lift.
This \( g^o \) trivially satisfies the cocycle condition for a principal \( P_0 G \)-bundle at the boundary. To extend this to a cocycle for a principal \( LG \)-bundle, we define \( h^o \in \mathcal{C}^\infty(Y^{[3]}, L_0 G) \) by

\[
h^o_{ijk} := g^o_{ik}(g^o_{jk})^{-1}(g^o_{ij})^{-1} \quad \text{for all } (i, j, k) \in Y^{[3]}.
\]  

(2.51a)

This indeed defines a loop in \( G \), that is, \( b(h^o_{ijk}) = 1 \), since \( g_{ij}g_{jk} = g_{ik} \). It is possible and convenient (but not necessary) to define the lift \( g^o \) in such a way that

\[
g^o_{ii} = 1 \quad \text{and} \quad g^o_{ij} = (g^o_{ij})^{-1}
\]

(2.51b)

with \( 1 \in P_0 G \) the constant path, which also implies the normalisations

\[
h^o_{iiij} = h^o_{ijji} = h^o_{iiji} = 1.
\]

(2.51c)

For the connection forms, we use the relation (2.36) and, following (2.36), we set

\[
A^o_i := \varphi \cdot A_i \quad \text{and} \quad B^o_i := \frac{1}{2} \left( \varphi - \varphi^2 \right) \cdot [A_i, A_i]
\]

(2.51d)

so that

\[
\Lambda^o_{ij} := (g^o_{ij})^{-1}A^o_i g^o_{ij} + (g^o_{ij})^{-1}dg^o_{ij} - A^o_j \quad \text{for all } (i, j) \in Y^{[2]}.
\]

(2.51e)

The right-hand side here is a loop in \( g \) due to the gluing relation (2.21) between \( A_i \) and \( A_j \). It is now not too difficult to verify that with all these definitions, all the cocycle conditions (2.47) including the adjusted fake flatness condition (2.47e) are satisfied.

Conversely, we can apply the endpoint evaluation (2.4) to project the descent data \( (h^o, g^o, \Lambda^o, A^o, B^o) \) for a principal \( LG \)-bundle with the adjusted fake flatness condition (2.47e) imposed to the data \( (g, A) \) of a principal \( G \)-bundle by means of

\[
g_{ij} := b(g^o_{ij}) \quad \text{and} \quad A_i := b(A^o_i)
\]

(2.52)

for all appropriate \( (i, j, \ldots) \in Y^{[n]} \). Again, it follows that this data fulfils the required cocycle conditions (2.21) as any homotopy lift vanishes at the endpoints.

What we have constructed above yields two morphisms between the descent data

\[
\begin{array}{ccc}
\text{Desc}(BG_{\text{conn}}) & \xrightarrow{\phi} & \text{Desc}(BLG_{\text{conn}}) \\
\downarrow{\psi} & & \downarrow{\phi} \\
\end{array}
\]

(2.53)

Clearly, \( \psi \circ \phi = \text{id} \), but we have to show that there is a bundle isomorphism \( \phi \circ \psi \Rightarrow \text{id} \).

Consider an element \( (h^o, g^o, \Lambda^o, A^o, B^o) \in \text{Desc}(BLG_{\text{conn}}) \) as well as

\[
(h^o, g^o, \Lambda^o, A^o, B^o) := \phi \circ \psi(h^o, g^o, \Lambda^o, A^o, B^o).
\]

(2.54)
We note that there is a coboundary \( (a^\circ, b^\circ, \lambda^\circ) \) with

\[
a^\circ_i := 1, \quad b_{ij}^\circ := g_{ij}^\circ(g_{ij}^\circ)^{-1}, \quad \text{and} \quad \lambda^\circ_i := (\text{id} - \varphi \cdot b)(A^\circ_i)
\]

(2.55)

for all appropriate \((i, j, \ldots) \in Y^{[n]}\) connecting \((h^\circ, g^\circ, \Lambda^\circ, A^\circ, B^\circ)\) with \((\bar{h}^\circ, \bar{g}^\circ, \bar{\Lambda}^\circ, \bar{A}^\circ, \bar{B}^\circ)\). In particular, the difference between \(g_{ij}^\circ\) and \(\bar{g}_{ij}^\circ\) is a loop and thus in the image of the bundle isomorphism with the given \(e_{ij}^\circ\). Furthermore, the maps \(\tilde{h}_{ijk}\) are fully determined by the maps \(\tilde{g}_{ij}^\circ\) because \(t\) is injective. Similarly, the difference between \(A_i^\circ\) and \(\bar{A}_i^\circ\) is a loop and thus in the image of the bundle isomorphism with the given \(\lambda_i^\circ\), which then fully determines the maps \(\lambda_{ij}^\circ\). The adjusted fake flatness condition (2.38) moreover fully determines \(\tilde{B}_i^\circ\) in terms of \(\bar{A}_i^\circ\) again because \(t\) is injective. Hence, for descent data satisfying the adjusted fake flatness condition (2.38), the coboundary data (2.55) indeed provide a bundle isomorphism \(\phi \circ \psi \Rightarrow \text{id}\).

It then follows that, in fact, the full gauge orbits are mapped to each other. Concretely, the equivalence (2.53) extends to

\[
\begin{array}{cccc}
(g, A) & \xrightarrow{\phi} & (h^\circ, g^\circ, \Lambda^\circ, A^\circ, B^\circ) \\
\downarrow{a} & & \downarrow{(a^\circ, b^\circ, \lambda^\circ)} & \xleftarrow{\psi} & (\bar{h}^\circ, \bar{g}^\circ, \bar{\Lambda}^\circ, \bar{A}^\circ, \bar{B}^\circ) \\
(\tilde{g}, \tilde{A}) & \xrightarrow{\phi} & (\tilde{h}^\circ, \tilde{g}^\circ, \tilde{\Lambda}^\circ, \tilde{A}^\circ, \tilde{B}^\circ) & \xleftarrow{\psi} & (\tilde{g}, \tilde{A})
\end{array}
\]

(2.56a)

Explicitly, \(a^\circ\) and \(\tilde{a}^\circ\) are lifts of \(a\) such that

\[
b(a^\circ_i) = a_i = b(\tilde{a}^\circ_i) \quad \text{for all} \quad (i) \in Y^{[1]}.
\]

(2.56b)

The maps \(b^\circ\) and \(\tilde{b}^\circ\) are then determined by the explicit form of \(\tilde{g}^\circ\),

\[
b_{ij}^\circ := a_i^\circ g_{ij}^\circ a_j^\circ a_j^\circ g_{ij}^\circ \quad \text{and} \quad \tilde{b}_{ij}^\circ := \tilde{a}_i^\circ \tilde{g}_{ij}^\circ \tilde{a}_j^\circ (\tilde{g}_{ij}^\circ)^{-1} \quad \text{for all} \quad (i, j) \in Y^{[2]},
\]

(2.56c)

where the expressions on the right-hand sides are indeed loops due to (2.56b). Furthermore, comparing \(A^\circ\) with \(\bar{A}^\circ\) fixes the maps \(\lambda^\circ\) and \(\tilde{\lambda}^\circ\),

\[
\lambda^\circ_i := (a^\circ_i)^{-1} A^\circ_i a^\circ_i + (a^\circ_i)^{-1} da^\circ_i - \bar{A}^\circ_i,
\]

\[
\tilde{\lambda}^\circ_i := (\tilde{a}^\circ_i)^{-1} \tilde{A}^\circ_i \tilde{a}^\circ_i + (\tilde{a}^\circ_i)^{-1} d\tilde{a}^\circ_i - \bar{\tilde{A}}^\circ_i
\]

(2.56d)

for all \((i) \in Y^{[1]}\). Having fixed the lifted gauge transformations, one can directly verify that the coboundary relations between the descent data \((h^\circ, g^\circ, \Lambda^\circ, A^\circ, B^\circ)\) and \((\bar{h}^\circ, \bar{g}^\circ, \bar{\Lambda}^\circ, \bar{A}^\circ, \bar{B}^\circ)\) hold.
It remains to check that the two constructed gauge transformations are linked by a higher gauge transformation (2.49). We put

\[ m_i^\circ := \tilde{a}_i^\circ (a_i^\circ)^{-1} \quad \text{for all} \quad (i) \in Y^{[1]} \quad (2.57) \]

which ensures that \( a^\circ \) and \( \tilde{a}^\circ \) are related appropriately. The corresponding relations between \( b^\circ \) and \( \tilde{b}^\circ \) as well as \( \lambda^\circ \) and \( \tilde{\lambda}^\circ \) are then straightforwardly checked.

3. Example: instanton–anti-instanton pair

Let us give a concrete and explicit example of how a principal \( G \)-bundle can be equivalently described as a principal \( LG \)-bundle. For physical applications, an interesting candidate is certainly the principal \( SU(2) \)-bundle over \( S^4 \) underlying the elementary instanton, which, from a geometric perspective, is very naturally extended to an instanton–anti-instanton pair. As explained in [33], self-dual gauge configurations are part of non-Abelian self-dual strings that naturally arise when M2-branes end on M5-branes. We shall discuss the extension of the \( LG \)-bundle description of the instanton–anti-instanton-pair to a principal \( \text{String}(4) \)-bundle in Section 5. Moreover, the quaternionic Hopf fibration underlying the elementary instanton also features prominently in the so-called hypothesis H, cf. e.g. [41], which states that the charge quantisation in M-theory happens in a particular cohomology theory.

3.1. Yang–Mills theory on reductive homogeneous spaces

The instanton–anti-instanton pair in which we are interested is most naturally obtained as the canonical connection on a reductive homogeneous space, and the same holds true for the principal \( \text{String}(4) \)-bundle we shall construct later in Section 5. We shall therefore review this construction in the following; see [42, Section II.11]. This construction is particularly appealing, as it results in solutions to the Yang–Mills equations [43,44] that are, in many cases, Bogomolny–Prasad–Sommerfield (BPS) solutions [43,45].

Reductive coset spaces. Let \( G \) be a compact semi-simple Lie group, \( H \) a closed Lie subgroup of \( G \), and let \( g \) and \( h \) be the corresponding Lie algebras. We further assume that \( G/H \) is a reductive homogeneous space, that is, there is a decomposition

\[ g \cong h \oplus m \quad (3.1) \]

For a recent discussion of instantons on coset spaces, see also [46].
with $m$ being $\text{Ad}(H)$-invariant. The latter condition implies that $[h, m] \subseteq m$, and the converse is true if $H$ is connected. Altogether, we have the relations

$$[h, h] \subseteq h, \quad [h, m] \subseteq m, \quad \text{and} \quad [m, m] \subseteq h \oplus m. \quad (3.2)$$

We denote the Cartan–Killing form on $g$ by $\langle -, - \rangle$, and assume that the decomposition (3.1) is orthogonal with respect to $\langle -, - \rangle$.

Let $\theta$ be the (left-invariant) Maurer–Cartan form on $G$. Because of the decomposition (3.1), we may write

$$\theta = A + m \quad \text{with} \quad A \in \Omega^1(G, h) \quad \text{and} \quad m \in \Omega^1(G, m). \quad (3.3)$$

Under the right $H$-action $g \mapsto gh$ on $G$ for all $g \in G$ and for all $h \in H$, we have

$$A \mapsto h^{-1}Ah + h^{-1}dh \quad \text{and} \quad m \mapsto h^{-1}mh. \quad (3.4)$$

Furthermore, the Maurer–Cartan equation $d\theta + \frac{1}{2}[\theta, \theta] = 0$ then decomposes as

$$dA + \frac{1}{2}[A, A] = -\frac{1}{2}[m, m]_h \quad \text{and} \quad \nabla_A m = -\frac{1}{2}[m, m]_m, \quad (3.5)$$

where $d$ denotes the exterior derivative on $G$ and the subscript indicates the projection onto the respective subspaces.

**Yang–Mills connection.** Recall that a principal $G$-bundle $\pi : P \to M$ over some manifold $M$ can conveniently be described subordinate to itself, because the pullback bundle $\pi^*P := P \times_M P = P[2]$ in

$$
\begin{array}{ccc}
P \times_M P & \longrightarrow & P \\
\downarrow & & \downarrow \pi \\
P & \longrightarrow & M
\end{array}
$$

is trivial. In particular, there is a global section $P \to P[2]$ given by $p \mapsto (p, p)$ for all $p \in P$. Hence, $P[2] \cong P \times G$ and, more generally, $P[n] \cong P \times G^{n-1}$. The Čech cocycles subordinate to the cover $\pi$ are then given by maps $(p_1, p_2) \mapsto g_{p_1p_2}$ for all $(p_1, p_2) \in P[2]$, where $g_{p_1p_2} \in G$ is uniquely defined by

$$p_2 =: p_1 g_{p_1p_2}, \quad (3.7)$$

and the cocycle condition $g_{p_1p_2g_{p_2p_3}} = g_{p_1p_3}$ is automatically satisfied for all $(p_1, p_2, p_3) \in P[3]$. In the case of the principal $H$-bundle $G \to G/H$, this yields the transition function

$$g : G[2] \to H, \quad (g_1, g_2) \mapsto h_{g_1g_2} := g_1^{-1}g_2. \quad (3.8)$$
Consider an atlas \( \{(U_i, \phi_i)\}_{i \in I} \) and define \( Y_i := \phi_i(U_i) \cong \mathbb{R}^n \). Furthermore, let \( s_i : U_i \to G \) be a local section. Using (3.5), we define the 1-forms

\[
A_{s_i} := (s_i \circ \phi_i^{-1})^* A \in \Omega^1(Y_i, \mathfrak{h}) \quad \text{and} \quad m_{s_i} := (s_i \circ \phi_i^{-1})^* m \in \Omega^1(Y_i, \mathfrak{m}) ,
\]

where now \( \mathfrak{m} \) denotes the exterior derivative on \( G \). The next, let \( \star \) be the Hodge star operator with respect to the metric on \( G/H \) that is given by the pullback via \( s_i \) of the left-invariant metric on \( G \) which is induced by the Cartan–Killing form \( \langle -, - \rangle \). It then follows that

\[
\nabla_{A_{s_i}} \star F_{s_i} = 0 \quad \text{with} \quad F_{s_i} := \frac{1}{2} [A_{s_i}, A_{s_i}] .
\]

To show this, one makes use of the Jacobi identity and the relations (3.2) and (3.10). Hence, \( A_{s_i} \) is a Yang–Mills gauge potential on \( Y_i \) for the gauge group \( \mathbb{H} \).

Finally, let \( (g_1, g_2) \in G^2 \) and let \( U_1 \) and \( U_2 \) be open neighbourhoods of \( \pi(g_1) = \pi(g_2) \). Furthermore, suppose that \( s_{1,2} : U_{1,2} \to G \) are two local sections such that \( (s_{1,2} \circ \pi)(g_{1,2}) = g_{1,2} \) and \( s_2 = s_1 h_{g_1, g_2} \). Therefore, with \( A_{s_{1,2}} := (s_{1,2} \circ \phi_{1,2}^{-1})^* A \in \Omega^1(Y_{1,2}, \mathfrak{h}) \), we obtain

\[
A_{s_2} = h_{g_1, g_2}^{-1} A_{s_1} h_{g_1, g_2} + h_{g_1, g_2}^{-1} dh_{g_1, g_2} .
\]

3.2. Instantons on \( S^4 \)

Recall that the elementary instanton solution on \( S^4 \), that is, an SU(2)-instanton of size and charge one, corresponds to the principal SU(2)-bundle that is given by the Hopf fibration \( S^3 \hookrightarrow S^7 \to S^4 \). This bundle has a non-vanishing second Chern class whose image in de Rham cohomology integrates to the unit instanton charge.

For our purposes, it will prove to be useful to extend this picture to the Spin(4) \( \cong \) SU(2) \( \times \) SU(2)-bundle

\[
S^4 \text{ as the coset Spin}(5)/\text{Spin}(4) .
\]

The explicit description of the fibration (3.13) is found e.g. in [47, Diagram 24.4], and we briefly review this description in the following. Recall that

\[
\text{Spin}(5) \cong \text{Sp}(2) = U(2, \mathbb{H}) := \{ g \in \text{Mat}(2, \mathbb{H}) \mid g^\dagger g = gg^\dagger = 1_2 \} ,
\]

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and thus, the elements of Spin(5) can be identified with matrices

\[ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{H}, \]

\[
\begin{align*}
|a|^2 + |b|^2 &= 1, & a\bar{c} + b\bar{d} &= 0, \\
|c|^2 + |d|^2 &= 1, & c\bar{a} + d\bar{b} &= 0, \\
|a|^2 + |c|^2 &= 1, & \bar{a}b + \bar{c}d &= 0, \\
|b|^2 + |d|^2 &= 1, & \bar{b}a + \bar{d}c &= 0.
\end{align*}
\]

(3.15)

Note that the above eight relations contain six independent conditions, so that Spin(5) is ten-dimensional. We further identify SU(2) with the unit quaternions \(\text{Sp}(1) \cong U(1, \mathbb{H})\), and consider the embedding

\[
\text{SU}(2) \times \text{SU}(2) \hookrightarrow \text{Spin}(5),
\]

(3.16)

We then have a canonical projection \(\text{Spin}(5) \to \text{Spin}(5)/\text{Spin}(4)\), where we identify

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} \quad \text{for all} \quad e, f \in \mathbb{H}.
\]

(3.17)

Given an element \(g \in \text{Spin}(5)\) of the form (3.15), we note that the product

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^\dagger = \begin{pmatrix} 2|a|^2 - 1 & 2a\bar{c} \\ 2c\bar{a} & 2|a|^2 - 1 \end{pmatrix}
\]

(3.18)

is invariant under the right-action (3.17) of Spin(4) on \(g\). Moreover, we can identify

\[
x^1 + ix^2 + jx^3 + kx^4 := 2a\bar{c} \quad \text{and} \quad x^5 := 2|a|^2 - 1,
\]

(3.19)

where \(i, j, k\) are the standard quaternionic units and \(x^1, \ldots, x^5 \in \mathbb{R}\). Because of the constraints in (3.15), we obtain \(\sum_{i=1}^{5} (x^i)^2 = 1\), that is, \(S^4 \hookrightarrow \mathbb{R}^5\). This yields a projection

\[
\pi : \text{Spin}(5) \to S^4,
\]

(3.20)

which we shall use below.

Similarly, we have a projection \(\text{Spin}(5) \to S^7 \cong \text{Spin}(5)/\text{SU}(2)\) that is given by

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a, c),
\]

(3.21)

where we regard \(S^7\) as the unit vectors in \(\mathbb{H}^2 \cong \mathbb{R}^8\) and the quotient group \(\text{SU}(2)\) is identified with the second factor in \(\text{Spin}(4) \cong \text{SU}(2) \times \text{SU}(2)\).
Instantons on $S^4$. We write elements of $\text{spin}(5)$ as quaternionic matrices

$$X = \begin{pmatrix} x & z \\ -\bar{z} & y \end{pmatrix} \quad \text{for all } x, y, z \in \mathbb{H} \quad \text{with } \text{Re}(x) = \text{Re}(y) = 0. \quad (3.22)$$

Here, $\text{spin}(4) \cong \text{su}(2) \oplus \text{su}(2)$ is identified with the two imaginary quaternions $x$ and $y$, which yields the decomposition

$$\text{spin}(5) \cong \text{spin}(4) \oplus \mathfrak{m}, \quad \begin{pmatrix} x & z \\ -\bar{z} & y \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} + \begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix} \quad (3.23a)$$

with the evident relations

$$[\text{spin}(4), \text{spin}(4)] \subseteq \text{spin}(4), \quad [\text{spin}(4), \mathfrak{m}] \subseteq \mathfrak{m}, \quad \text{and } [\mathfrak{m}, \mathfrak{m}] \subseteq \text{spin}(4). \quad (3.23b)$$

Our discussion in Section 3.1, then yields the canonical connection on the bundle $\text{Spin}(5) \to S^4$. As we shall see below, this connection is the direct sum of a fundamental instanton and a fundamental anti-instanton.

Clutching construction. Following our discussion in Section 3.1, we can describe the principal $\text{Spin}(4)$-bundle $P = \text{Spin}(5) \to S^4$ subordinate to the cover given by the bundle itself with transition function

$$g : (\text{Spin}(5))^{[2]} \to \text{SU}(2) \times \text{SU}(2), \quad (g_1, g_2) \mapsto g_1^{-1}g_2, \quad (3.24)$$

where we have again used the notation (2.17).

In order to make contact with the usual clutching construction of principal bundles over $S^4$, we embed $S^4$ into $\mathbb{R}^5$ and derive a cover $\sigma : Y := Y_+ \cup Y_- \to S^4$ from the stereographic projections

$$\pi_\pm : S^4 \setminus \{x^5 = \pm 1\} \to Y_\pm \cong \mathbb{H}, \quad (x^1, \ldots, x^5) \mapsto q_\pm := \frac{1}{1 \pm x^5} (x^1 \pm i x^2 \pm j x^3 \pm k x^4), \quad (3.25a)$$

where, as before, $i$, $j$, and $k$ are the standard quaternionic units. Consequently, $q_+q_- = 1$ on $Y^{[2]} = \sigma^{-1}(U_+ \cap U_-)$. We then consider the commutative diagram

$$\begin{array}{ccc}
    Y_+ \sqcup Y_- & \xrightarrow{\phi} & \text{Spin}(5) \\
    \downarrow & & \downarrow \\
    S^4 & & 
\end{array} \quad (3.26)$$
where \( \phi \) is given by

\[
\phi : Y_\pm \to \text{Spin}(5),
\]

\[
q_+ \mapsto \frac{1}{\sqrt{1 + |q_+|^2}} \begin{pmatrix} \bar{q}_+ & 1 \\ 1 & -q_+ \end{pmatrix}
\quad \text{and} \quad
q_- \mapsto \frac{1}{\sqrt{1 + |q_-|^2}} \begin{pmatrix} 1 & q_- \\ \bar{q}_- & -1 \end{pmatrix}.
\]  

(3.27)

The pullback \( g_{++} := \phi^* g \) of the transition function (3.24) to \( Y^2 \) then reads as

\[
g_{++} : Y^{[2]} \to \text{SU}(2) \times \text{SU}(2),
\]

\[
(q_+, q_-) \mapsto \left( \frac{q_+}{|q_+|}, \frac{\bar{q}_+}{|q_+|} \right) = \left( \frac{q_-}{|q_-|}, \frac{\bar{q}_-}{|q_-|} \right).
\]  

(3.28a)

This principal fibre bundle can be endowed with the connection that describes a pair of an elementary instanton with an elementary anti-instanton, which, in quaternionic notation, reads as:

\[
A_{\pm} := \frac{1}{2} \left( \frac{q_\pm d\bar{q}_\pm - dq_\pm \bar{q}_\pm}{1 + |q_\pm|^2}, \frac{\bar{q}_\pm dq_\pm - d\bar{q}_\pm q_\pm}{1 + |q_\pm|^2} \right) \quad \text{on} \quad Y_\pm,
\]  

(3.28b)

cf. [48]. In particular, we have

\[
A_- = g_{+-}^{-1} A_+ g_{+-} + g_{+-}^{-1} dg_{+-} \quad \text{on} \quad Y^{[2]},
\]  

(3.28c)

and the curvature reads as

\[
F_{\pm} = \left( \frac{dq_\pm \wedge d\bar{q}_\pm}{(1 + |q_\pm|^2)^2}, \frac{d\bar{q}_\pm \wedge dq_\pm}{(1 + |q_\pm|^2)^2} \right) \quad \text{on} \quad Y_\pm
\]  

(3.28d)

with the first component describing an instanton with self-dual field strength and the second component describing an anti-instanton with anti-self-dual field strength. The total first Pontryagin class (as is evident from its image in de Rham cohomology) vanishes, and this will become important later.

### 3.3. The instanton–anti-instanton pair as an \( \text{LSpin}(4) \)-bundle

In the following, we lift the cocycle data (3.28) of the above defined \( \text{Spin}(4) \)-bundle \( P \) to that of a principal \( \text{LSpin}(4) \)-bundle.

#### Choice of cover.

The cover \( Y = Y_+ \sqcup Y_- \) introduced above is not suitable for a lift of \( P \) to a principal \( \text{LSpin}(4) \)-bundle, because the transition function \( g_{++} \) in (3.28) does not factor through \( P_0 \text{Spin}(4) \). If it did, then the lift \( g_{+-}^0 \) of \( g_{+-} \) would trivialise,

\[
g_{+-}^0 = a_{++}^{-1} a_{--} \quad \text{with} \quad a_{\pm} : Y_\pm \to P_0 \text{Spin}(4)
\]  

(3.29)

\footnote{\textup{For a recent application of based loop groups as gauge groups in Yang–Mills theory, see also [49].}}
because $P_0\text{Spin}(4)$ is contractible. This would imply that $b(a_-)$ was a coboundary trivialising $g_{+-}$, 
\[
g_{+-} = b(g_{+-}^\circ) = a_+^{-1}a_- \quad \text{with} \quad a_- := b(a_-),
\]
and we know that this is not possible. Moreover, we also know that $Y^{[2]} = \sigma^{-1}(U_+ \cap U_-) \cong S^3 \times \mathbb{R}$, and therefore $P$ is topologically characterised by maps $S^3 \to \text{Spin}(4)$ and thus an element of the third homotopy group $\pi_3(\text{Spin}(4)) \cong \pi_3(S^3 \times S^3) \cong \mathbb{Z} \oplus \mathbb{Z}$. Because $P_0\text{Spin}(4)$ is contractible, the topologically non-trivial information of a principal $\mathcal{L}\text{Spin}(4)$-bundle subordinate to the cover $Y$ can only be contained in a map $h : Y^{[3]} \to L_0\text{Spin}(4)$. Since $Y^{[3]} \cong Y^{[2]} \cong S^3 \times \mathbb{R}$, the homotopy classes of this map are $\pi_3(L_0\text{Spin}(4)) \cong \pi_4(\text{Spin}(4)) \cong \pi_4(S^3 \times S^3) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, which is evidently not large enough.\footnote{The identification $\pi_n(G) \cong \pi_{n-1}(L_0G)$ is as follows. Let $M$ be a topological space and $p_0 \in M$ fixed. The \textit{reduced suspension} of $(M, p_0)$ is the quotient $\Sigma M := (M \times [0, 1])/{\sim}$ with the equivalence relation given by $(p, 0) \sim (p, 1) \sim (p_0, t)$ for all $p \in M$ and for all $t \in [0, 1]$. Then, for $N$ another topological space, there is the natural identification $\pi_n(\Sigma M, N) \cong \pi_{n-1}(M, L_0N)$ of continuous maps that preserve the base point. In the context of spheres, we have a homeomorphism between $\Sigma S^{n-1}$ and $S^n$. Indeed, we mark poles $x_0 \in S^{n-1}$ and $y_0 \in S^n$ and identify $x_0$ as well as all points $S^{n-1} \times \{0, 1\}$ with $y_0$. We then identify the remaining points $(0, 1)$ with $\mathbb{R}$ and map $S^{n-1}\setminus\{x_0\}$ to $\mathbb{R}^{n-1}$ by stereographic projection at $x_0$. The Cartesian product $\mathbb{R}^{n-1} \times \mathbb{R}$ is then identified with $S^{n}\setminus\{y_0\}$ by inverse stereographic projection at $y_0$.}

We can, however, construct a suitable 3-patch replacement cover $\tilde{Y}$ of the 2-patch cover $Y$ rather directly by hand. First of all, we note that $\pi_3(\text{Spin}(4)) \cong \pi_2(L_0\text{Spin}(4))$ so that we would like our triple overlaps to be homeomorphic to $S^2 \times \mathbb{R}^2$. We can reduce $Y^{[2]} \cong \mathbb{H}\setminus\{0\}$ to such a space by removing a straight line $L \cong \mathbb{R}$ through $0 \in \mathbb{H}$ since, topologically, $\mathbb{H}\setminus\mathbb{R} \cong S^2 \times \mathbb{R}^2$. In particular, let
\[
L_+ := \{ q \in \mathbb{H} \mid \text{Re}(q) > 0 \text{ and } \text{Im}(q) = 0 \},
\]
\[
L_- := \{ q \in \mathbb{H} \mid \text{Re}(q) < 0 \text{ and } \text{Im}(q) = 0 \},
\]
and set $L := L_- \cup \{0\} \cup L_+$; evidently $L \cong \mathbb{R}$. Next, we define a cover $\tilde{\sigma} : \tilde{Y} \to S^4$ by
\[
\tilde{Y}_1 := Y_+ \setminus L_+ \cong \mathbb{H} \setminus L_+ , \quad \tilde{Y}_2 := Y_+ \setminus L_- \cong \mathbb{H} \setminus L_- , \quad \tilde{Y}_3 := Y_- \cong \mathbb{H} ,
\]
and $\tilde{\sigma}$ is induced by $\sigma$. Note that $\tilde{Y}_1 \cup \tilde{Y}_2 = Y_+ \cong \mathbb{H}$, and $\tilde{Y}^{[3]} \cong \mathbb{H}\setminus L \cong S^2 \times \mathbb{R}^2$.

With respect to this cover, the bundle $P$ is described by transition functions
\[
g_{12} := -1 , \quad g_{13} := -g_{+-} , \quad g_{23} := g_{+-} ,
\]
where we inserted a constant change of frame between patches $Y_1$ and $Y_2$. This does not
affect the gauge potential, and we have

\[
A_{1,2} := \frac{1}{2} \left( \frac{q_{1,2}d\tilde{q}_{1,2} - d\tilde{q}_{1,2}q_{1,2}}{1 + |q_{1,2}|^2}, \frac{\tilde{q}_{1,2}d\tilde{q}_{1,2} - d\tilde{q}_{1,2}q_{1,2}}{1 + |q_{1,2}|^2} \right),
\]

\[
A_3 := \frac{1}{2} \left( \frac{q_3d\tilde{q}_3 - d\tilde{q}_3q_3}{1 + |q_3|^2}, \frac{\tilde{q}_3d\tilde{q}_3 - d\tilde{q}_3q_3}{1 + |q_3|^2} \right),
\]

(3.33b)

with \(q_i \in \tilde{Y}_i\), where \(q_{1,2}\) are restrictions of \(q_+\) and \(q_3 = q_-\).

**Lifted cocycle data.** The cocycle (3.33) can then be straightforwardly lifted, following the prescription in Section 2.3. Relative to the cover (3.32), we can define the following lift of the transition functions:

\[
g_{12}^\circ := (t \mapsto (e^{i\pi\varphi(t)}, e^{-i\pi\varphi(t)})) \quad \text{and} \quad g_{23}^\circ := (t \mapsto (u_i(t), \bar{u}_i(t)))
\]

(3.34a)

for \(i = 1, 2\) with

\[
u_i(t) := \frac{1 - \varphi(t) + (-1)^i\varphi(t)q_+}{1 - \varphi(t) + (-1)^i\varphi(t)q_+}
\]

(3.34b)

and for all \(t \in [0, 1]\), which induces

\[
h_{123}^\circ := g_{13}^\circ (g_{23}^\circ)^{-1}(g_{12}^\circ)^{-1}.
\]

(3.34c)

Furthermore, the lift of the connection reads as

\[
A_{12}^\circ := \varphi \cdot A_1, \quad A_{13}^\circ := \varphi \cdot A_2, \quad A_3^\circ := \varphi \cdot A_3,
\]

\[
B_1^\circ := \frac{1}{2}(\varphi - \varphi^2) \cdot [A_1, A_1], \quad B_2^\circ := \frac{1}{2}(\varphi - \varphi^2) \cdot [A_2, A_2],
\]

\[
B_3^\circ := \frac{1}{2}(\varphi - \varphi^2) \cdot [A_3, A_3],
\]

(3.34d)

and

\[
\Lambda_{12}^\circ := \left( t \mapsto \varphi(t) \cdot (e^{-i\pi\varphi(t)}, e^{i\pi\varphi(t)}) \left[ A_+, (e^{i\pi\varphi(t)}, e^{-i\pi\varphi(t)}) \right] \right),
\]

\[
\Lambda_{13}^\circ := \left( t \mapsto \frac{1}{1 + |q_+|^2} \left( \varphi(t)|q_+|^2 \bar{u}_i(t)u_i(1)d(\bar{u}_i(1)u_i(t)) + |q_+|^2(1 - \varphi(t))\bar{u}_i(t)du_i(t)
\right.
\]

\[
\left. + \bar{u}_i(t)du_i(t) - \varphi(t)\bar{u}_i(1)du_i(1), u_i(t) \leftrightarrow \bar{u}_i(t) \right) \right)
\]

(3.34e)

for \(i = 1, 2\) with \(u_i\) as given in (3.34b) and for all \(t \in [0, 1]\).

This is the complete cocycle data of a non-trivial and non-Abelian gerbe, albeit one which is equivalent to an ordinary principal bundle. The original cocycles are recovered by endpoint evaluation of the lifted cocycles as detailed in (2.52).
**Alternative lift.** The above lift has the disadvantage of being computationally rather involved. With a view towards our later discussion, we therefore also introduce a simplifying lift. To this end, we replace the 3-patch cover $\tilde{Y}$ by the 2-patch cover $\tilde{\sigma} : \tilde{Y} \to S^4$ with

$$\tilde{Y}_1 := Y_+(L_+ \cup L_-) \cong \mathbb{H} \setminus (L_+ \cup L_-), \quad \tilde{Y}_2 := Y_- \cong \mathbb{H}, \quad (3.35)$$

and $\tilde{\sigma}$ is a restriction of $\sigma$ to $\tilde{Y}$. Note that $\tilde{\sigma}(\tilde{Y}_1) \cup \tilde{\sigma}(\tilde{Y}_2) = S^4$, and $\tilde{Y}^{[3]} \cong \tilde{Y}^{[2]} \cong \mathbb{H} \setminus L \cong S^2 \times \mathbb{R}^2$.

With respect to this cover, the bundle $P$ is described by the transition function

$$g_{12} := g_{+-}, \quad (3.36a)$$

where we have restricted $g_{+-}$ from $Y^{[2]}$ to $Y^{[2]}$. This does not affect the gauge potential, and we have

$$A_{1,2} := \left( \frac{\text{Im}(q_{1,2}d\bar{q}_{1,2})}{1 + |q_{1,2}|^2}, \frac{\text{Im}(\bar{q}_{1,2}dq_{1,2})}{1 + |q_{1,2}|^2} \right), \quad (3.36b)$$

with $q_i \in \tilde{Y}_i$, where $q_1$ is a restriction of $q_+$ and $q_2 = q_-$. The cocycle (3.36) can then be straightforwardly lifted, following the prescription in Section 2.3. Relative to the cover (3.32), we can define the following lift of the transition functions:

$$g^{\circ}_{12} := (u_q, v_q) \quad \text{and} \quad g^{\circ}_{21} := (v_q, v_q), \quad (3.37a)$$

where $q = q_+$ and for all $t \in [0, 1]$

$$u_q(t) := \cos(\theta_q\varphi(t)) + \frac{\text{Im}(q)}{|\text{Im}(q)|} \sin(\theta_q\varphi(t)), \quad v_q(t) := \cos(\theta'_q\varphi(t)) + \frac{\text{Im}(q)}{|\text{Im}(q)|} \sin(\theta'_q\varphi(t)) \quad (3.37b)$$

with

$$\theta_q := \frac{\pi}{2} - \arctan\left( \frac{\text{Re}(q)}{|\text{Im}(q)|} \right) \quad \text{and} \quad \theta'_q := 2\pi - \theta_q. \quad (3.37c)$$

This induces

$$h^{\circ}_{121} := (g^{\circ}_{12}g^{\circ}_{21})^{-1} = \left( t \mapsto \left( \cos(2\pi\varphi(t)) + \frac{\text{Im}(q)}{|\text{Im}(q)|} \sin(2\pi\varphi(t)), \cos(2\pi\varphi(t)) + \frac{\text{Im}(q)}{|\text{Im}(q)|} \sin(2\pi\varphi(t)) \right) \right), \quad (3.37d)$$

where the last equality follows because $g^{\circ}_{12}$ and $g^{\circ}_{21}$ commute. We note that the above two expressions only depend on $\frac{\text{Im}(q)}{|\text{Im}(q)|}$, which describes the embeddings $S^2 \hookrightarrow \mathbb{R}^3 \hookrightarrow \mathbb{R}^4$, as expected from the abstract discussion involving reduced suspension mentioned previously.
The lift of the connection reads as

\[ A_1^\circ := \varphi \cdot A_1, \quad A_2^\circ := \varphi \cdot A_2, \]
\[ B_1^\circ := \frac{1}{2}(\varphi - \varphi^2) \cdot [A_1, A_1], \quad B_2^\circ := \frac{1}{2}(\varphi - \varphi^2) \cdot [A_2, A_2], \]

and

\[ A_{12}^\circ := \left( t \mapsto \frac{1}{1 + |q|^2} \left[ |q|^2 \varphi(t)\bar{u}_q(t)u_q(1)d(\bar{u}_q(1)u_q(t)) + |q|^2(1 - \varphi(t))\bar{u}_q(t)du_q(t) + \bar{u}_q(t)du_q(t) - \varphi(t)\bar{u}_q(1)du_q(1), u_q(t) \leftrightarrow \bar{u}_q(t) \right] \right). \]

Explicitly,

\[ A_{12}^\circ(t) = \frac{1}{1 + |q|^2} \left( Q_q(t) \frac{\text{Im}(\text{Im}(q)d\text{Im}(\bar{q}))}{2|\text{Im}(q)|^2}, Q_q(t) \frac{\text{Im}(\text{Im}(\bar{q})d\text{Im}(q))}{2|\text{Im}(q)|^2} \right) \]
\[ = \left( Q_q(t), Q_q(t) \right) \frac{\text{Im}(\text{Im}(q)d\text{Im}(q))}{2|\text{Im}(q)|^2(1 + |q|^2)}, \]

where

\[ Q_q(t) := \left( 1 + |q|^2 + \varphi(t)(q^2 - |q|^2) \right) \bar{u}_q^2(t) - \left( 1 + |q|^2 - \varphi(t) \left( 1 - \frac{\bar{q}^2}{|q|^2} \right) \right). \]

Similarly, we find

\[ A_{21}^\circ := \left( t \mapsto \frac{1}{1 + |q|^2} \left[ \varphi(t)\bar{v}_q(t)v_q(1)d(\bar{v}_q(1)v_q(t)) + (1 - \varphi(t))\bar{v}_q(t)dv_q(t) + |q|^2(\bar{v}_q(t)dv_q(t) - \varphi(t)\bar{v}_q(1)dv_q(1)), v_q(t) \leftrightarrow \bar{v}_q(t) \right] \right) \]
\[ = \left( t \mapsto \left( Q'_q(t), Q'_q(t) \right) \frac{\text{Im}(\text{Im}(q)d\text{Im}(q))}{2|\text{Im}(q)|^2(1 + |q|^2)} \right), \]

where

\[ Q'_q(t) := \left( 1 + |q|^2 - \varphi(t) \left( 1 - \frac{\bar{q}^2}{|q|^2} \right) \right) \bar{v}_q^2(t) - \left( 1 + |q|^2 + \varphi(t)(q^2 - |q|^2) \right). \]

The form of the lift of the gauge potentials \( A \) and \( B \) given in (3.37e) was convenient from the local, infinitesimal perspective, and it leads to the general lifting formulas (2.51). We suspect, however, that there is a more suitable lift of the gauge potentials that is tailored to the lift of \( g \), which would simplify the above formulas significantly.

4. String bundles with connections

We now turn to truly higher principal bundles with connection that are not mere reformulations of ordinary principal bundles with connections.
4.1. String structures

We start with a motivation of string structures, recalling some facts about $G$-structures.

**G-structures on H-bundles.** Consider a principal $H$-bundle $P$ over a manifold $M$ with structure group $H$. A reduction/lift of the structure group from $H$ to $G$ is a monomorphism/epimorphism of Lie groups $\phi : G \to H$, a principal $G$-bundle $Q$ over $M$, and a bundle isomorphism $\phi_Q : Q \times_G H \cong P$.

If we consider a vector bundle $E \to M$ of rank $n$, then its frame bundle is a principal $GL(n)$-bundle over $M$. A $G$-structure on $E$ is then a reduction of the structure group from $GL(n)$ to $G$ or the reduction from a $G$-structure to another $G$-structure.

Well-known examples of $G$-structures are obtained by considering the tangent bundle $E = TM$. An orientation on $M$ is the $G$-structure arising from a reduction of the structure group from $GL(n)$ to $GL^+(n)$ with $n = \dim(M)$. A Riemannian metric corresponds to a reduction of the structure group from $GL(n)$ to $O(n)$, which is always possible. A volume form on a Riemannian manifold corresponds to a further reduction of the structure group from $O(n)$ to $SO(n)$. Finally, a spin structure is a lift of the structure group from $SO(n)$ to $Spin(n)$. The obstructions to a manifold being orientable and carrying a volume form is governed by the first Stiefel–Whitney class, while the obstruction to a manifold being spin is governed by the second Stiefel–Whitney class.

**String structures.** If one is interested in the classical dynamics of strings, it is reasonable to consider the free loop space $LM$ of a manifold $M$. It can now be argued that $LM$ should be considered orientable, if $M$ is a spin manifold [50]; see also the discussions in [51,28]. As then shown in [28], the loop space $LM$ is spin, if $M$ is string, that is, a lift of the structure group of the frame bundle from $Spin(n)$ to $String(n)$. Here, $String(n)$ is a topological group that fits into the Whitehead tower

$$\cdots \to String(n) \to Spin(n) \to Spin(n) \to SO(n) \to O(n),$$

where the group homomorphisms are isomorphisms on all but the lowest homotopy groups. The string group $String(n)$ is thus a 3-connected cover of $Spin(n)$ with all higher homotopy groups isomorphic [52]. The obstruction for a $Spin(n)$-bundle $P$ to allow for a lift of the structure group to $String(n)$, and thus a string structure, is (for a simply connected manifold and $n \geq 5$) precisely $\frac{1}{2}p_1(P)$, where $p_1(P)$ is the first Pontryagin class of $P$ [28].

The string group is not uniquely defined in the sequence (4.1), and it is most conveniently described by an equivalent categorified or 2-group $String(n)$, cf. [29] and references therein. A string structure is then simply a topological principal $String(n)$-bundle. In our cases, we
shall work with a strict, but infinite-dimensional Lie 2-group $\text{String}(n)$; for the complications encountered when working with finite-dimensional models, see e.g. [32].

We recall from [52, Section 5] that if a principal bundle $P$ over a manifold $M$ admits a string structure due to $\frac{1}{2}p_1(P) = 0$, then the possible choices form a torsor for the group $H^3(M, \mathbb{Z})$.

Finally, we note that it was shown in [53] that isomorphism classes of string structures on a principal bundle $P$ are in bijection with isomorphism classes of the Chern–Simons 2-gerbe associated to $P$ by the construction of [54].

**Connective structures on string bundles.** The discussion of a differential refinement of string structures is slightly more subtle. Locally, the appropriate notion of connection data had first been identified in the context of supergravity [5, 6] and before the invention of string structures. It was then put into the context of higher gauge theory in [20, 53, 25].

As is clear from the general desire to lift the structure group of non-flat $\text{Spin}(n)$-bundles, we cannot work with fake-flat cocycles for non-Abelian gerbes but we have to implement an adjustment, which leads to a number of subtleties.

Important statements for string structures lift, as expected, to string structures with connections. In particular, isomorphism classes of string structures with connections on a principal bundle $P$ are in bijection with isomorphism classes of trivialisations of the Chern–Simons 2-gerbe with connection associated to $P$ [53], and this set is a torsor for the Deligne cohomology group $H^3_D(M, \mathbb{Z})$.

### 4.2. Strict Lie 2-group model

As mentioned above, the string group is defined up to $A_\infty$-equivalence, and various models exist. Here, we focus on the strict Lie 2-group model constructed in [30], see also [55, 56] as well as [57, Section 4] and [58, Section 4] for discussions of the involved central extension. Recall that a string 2-group model for a Lie group $G$ in the sense of [29] is a Lie 2-group $\mathcal{G}$ with a map $\pi : \mathcal{G} \to G$ such that the group of isomorphism classes of objects, usually denoted by $\pi_0\mathcal{G}$, is isomorphic to $G$, the group of automorphisms of $1 \in \mathcal{G}_0$, usually denoted by $\pi_1\mathcal{G}$, is isomorphic to $\text{U}(1)$, and $\pi$ is a 3-connected cover.

**String Lie 2-group.** Let $G$ be a simply connected compact semi-simple Lie group and consider the based path and loop groups defined in (2.3). Given the close relationship between string structures and spin structures on loop spaces and the fact that the spin group is a central extension of another group, it is perhaps not surprising that a 2-group
model $\text{String}(G)$ can be built from the central extension of 2-groups

$$\begin{align*}
1 & \longrightarrow \text{BU}(1) \longrightarrow \text{String}(G) \longrightarrow \mathcal{L}G \longrightarrow 1 ,
\end{align*}
$$

(4.2)

where $\text{BU}(1)$ is the crossed module $(\text{U}(1) \longrightarrow \ast, \text{id})$ and $\mathcal{L}G$ is defined in (2.5). Explicitly, our string 2-group model $\text{String}(G)$ is defined as the crossed module of Lie groups [30]

$$\text{String}(G) \coloneqq (\hat{L}_0G \xrightarrow{\iota} P_0G, \triangleright) ,$$

(4.3a)

where $\hat{L}_0G$ is the Kac-Moody central extension

$$\begin{align*}
1 & \longrightarrow \text{U}(1) \xrightarrow{\iota} \hat{L}_0G \xrightarrow{\pi} L_0G \longrightarrow 1 ,
\end{align*}
$$

(4.3b)

and $t$ is simply the concatenation of the projection $\pi$ and the embedding $L_0G \hookrightarrow P_3G$.

We shall also write $\text{String}(n)$ as a shorthand notation for $\text{String}(\text{Spin}(n))$.

Let us explain $\text{String}(G)$ in detail, reviewing and slightly expanding the discussion in [30]. In a first step, we construct the group product on $\hat{L}_0G$, following the discussion in [59], see also [60,61]. To this end, it is convenient to regard the trivialisation of the principal $\text{U}(1)$-bundle $\hat{L}_0G$ over the path space $P_0L_0G$, which leads to the commutative diagram

$$
\begin{array}{ccc}
P_0L_0G \times \text{U}(1) & \xrightarrow{\hat{b} \circ \hat{b}} & \hat{b}^* \hat{L}_0G \\
\downarrow & & \downarrow \\
P_0L_0G & \xrightarrow{b} & P_0L_0G/L_0L_0G \cong L_0G
\end{array}
$$

(4.4)

where $N$ is a normal subgroup of $P_0L_0G \times \text{U}(1)$ to be defined, the left horizontal maps are the canonical projections, and the lower right isomorphism is given by the generalisation of (2.7) to the present setting. Recall that the pullback bundle $b^*\hat{L}_0G$ is given by the fibre product

$$b^*\hat{L}_0G \coloneqq P_0L_0G \times_{L_0G} \hat{L}_0G ,$$

(4.5)

and the isomorphism $\hat{b}$ reads as

$$\hat{b}(f,z) \coloneqq (f,\hat{f}(1)z)$$

(4.6)

for all $(f,z) \in P_0L_0G \times \text{U}(1)$, where $\hat{f}$ is the horizontal lift of $f \in P_0L_0G$ to $\hat{L}_0G$ with $\hat{f}(0) = 1 \in \hat{L}_0G$.

An associative, unital product on $P_0L_0G \times \text{U}(1)$ can then be defined by

$$
(f_1,z_1)(f_2,z_2) \coloneqq (f_1f_2,z_1z_2c(f_1,f_2))
$$

(4.7)

---

1See Appendix A for a brief review of central extensions.

2All our definitions of based, parametrised path and loop spaces are the same as in Section 2.
for all \((f_1, z_1), (f_2, z_2) \in P_0 L_0 G \times U(1)\), where \(c\) is necessarily a group cocycle

\[
c : P_0 L_0 G \times P_0 L_0 G \to U(1) \quad \text{with} \quad c(f_1, f_2) c(f_1 f_2, f_3) = c(f_1, f_2 f_3) c(f_2, f_3)
\] (4.8)

for all \(f_1, f_2, f_3 \in P_0 L_0 G\). Let \(\mathfrak{g}\) be the Lie algebra of \(G\). The group cocycle \(c\) is the integrated form of the Kac–Moody 2-cocycle on the Lie algebra \(L_0 \mathfrak{g}\),

\[
\omega : L_0 \mathfrak{g} \times L_0 \mathfrak{g} \to \mathfrak{u}(1),
\]

\[
(\beta_1, \beta_2) \mapsto \frac{i}{2\pi} \int_0^1 \left( \beta_1(r) \frac{\partial \beta_2(r)}{\partial r} \right).
\] (4.9)

Here, \(\langle -, - \rangle\) is the inner product on \(\mathfrak{g}\), which is normalised in a standard fashion \([57]\).

The cocycle (4.9) also gives rise to the standard expression of the left-invariant 2-form curvature \(\omega \in \Omega^2(L_0 G, \mathfrak{u}(1))\) of the bundle \(\hat{L}_0 G \to L_0 G\) \([57]\),

\[
\omega_g(g \beta_1, g \beta_2) := \omega(\beta_1, \beta_2)
\] (4.10)

for all \(g \in L_0 G\) and for all \(g \beta_{1,2} \in T_g L_0 G\), or

\[
\omega_g = \frac{i}{4\pi} \int_0^1 \left( \theta_{g(r)} \frac{\partial \theta_{g(r)}}{\partial r} \right),
\] (4.11)

where \(\theta\) is the left-invariant Maurer–Cartan form on \(L_0 G\). This is a closed 2-form. The integrated cocycle is then obtained from the holonomy of \(\omega\). Explicitly, given paths \(f_{1,2} \in P_0 L_0 G\), we construct the loop given by the triangle

\[
\ell(f_{1,2}) := \overline{f_{1,2} \circ f_1(1) f_2} \in L_0 L_0 G,
\] (4.12)

where \(\overline{f}\) denotes the path \(f \in P_0 L_0 G\) with reversed orientation, that is, \(\overline{f}(t, r) := f(1-t, r)\). Let \(D_{\ell(f_{1,2})}\) be an arbitrary disc in \(L_0 G\) with boundary \(\ell(f_{1,2})\). Then, we have \([59]\)

\[
c(f_1, f_2) := \text{hol}(\ell(f_1, f_2))
\]

\[
:= \exp \left( - \int_{D_{\ell(f_{1,2})}} \omega \right)
\] (4.13)

\[
= \exp \left( - \frac{i}{2\pi} \int_0^1 \int_0^1 ds \int_0^s dt \left( f_1^{-1} \frac{\partial f_1}{\partial s} f_2 \left( \frac{\partial}{\partial r} \left( f_2^{-1} \frac{\partial f_2}{\partial t} \right) \right) f_2^{-1} \right) \right)
\]

for all \(f_1, f_2 \in P_0 L_0 G\) which we parametrised as \(f_1 = f_1(s, r)\) and \(f_2 = f_2(t, r)\) with \(s, t\) the path parameters and \(r\) the loop parameter.

---

1Explicitly, \(\langle h_\alpha, h_\alpha \rangle = 2\) when \(h_\alpha\) is the co-root corresponding to the highest root. In particular, we have \(\langle X, Y \rangle = -\text{tr}(XY)\) for \(\mathfrak{u}(n)\) for anti-Hermitian generators and \(\langle X, Y \rangle = -\frac{1}{2} \text{tr}(XY)\) for \(\mathfrak{so}(2n)\) with antisymmetric generators.
Using the identity
\begin{equation}
\frac{\partial}{\partial t} \left( \frac{\partial f_2(t, r)}{\partial r} f_2^{-1}(t, r) \right) = f_2(t, r) \left\{ \frac{\partial}{\partial r} \left( f_2^{-1}(t, r) \frac{\partial f_2(t, r)}{\partial t} \right) \right\} f_2^{-1}(t, r),
\end{equation}
we can simplify the expression (4.13) to [30]
\begin{equation}
c(f_1, f_2) = \exp \left( -\frac{i}{2\pi} \int_0^1 dr \int_0^1 ds \left\langle f_1^{-1}(s, r) \frac{\partial f_1(s, r)}{\partial s}, \frac{\partial f_2(s, r)}{\partial r} f_2^{-1}(s, r) \right\rangle \right),
\end{equation}
where, again, \( s \) and \( r \) are the path and loop parameters, respectively. The group cocycle condition (4.8) for \( c \) is then straightforwardly verified using the form (4.15).\(^1\)

We note that the left-invariance of the curvature \( \omega \) implies that
\begin{equation}
\text{hol}(\ell(f_1, f_2)) = \text{hol}(g\ell(f_1, f_2)) \quad (4.16)
\end{equation}
for all \( g \in L_0G \). The cocycle condition (4.8) then amounts to the integral of the curvature \( \omega \) over a tetrahedron vanishing, and the sides of this tetrahedron are given by the triangles
\begin{align}
(f_1 f_2, f_1(1)f_2, f_1), & \quad (f_1 f_2 f_3, f_1(1)f_2 f_3, f_1), \\
(f_1 f_2 f_3, (f_1 f_2)(1)f_3, f_1 f_2), & \quad (f_1(1)f_2 f_3, (f_1 f_2)(1)f_3, f_1(1)f_2),
\end{align}
\begin{equation}
(4.17)
\end{equation}

Next, we come to the normal subgroup \( \mathcal{N} \) appearing in (4.4), which is also defined in terms of holonomy,
\begin{equation}
\mathcal{N} := \left\{ (f, z) \in L_0L_0 \times \mathbf{U}(1) \mid z = \text{hol}^{-1}(f) = \exp \left( \int_{D_f} \omega \right) \right\} \quad (4.18)
\end{equation}

Here, \( D_f \) is a disc in \( L_0G \) with boundary \( f \). This is indeed a subgroup, because
\begin{equation}
\left( f_1, \exp \left( \int_{D_{f_1}} \omega \right) \right) \left( f_2, \exp \left( \int_{D_{f_2}} \omega \right) \right) = \left( f_1 f_2, \exp \left( \int_{D_{f_1} f_2} \omega \right) \right)
\end{equation}
\begin{equation}
(4.19)
\end{equation}
due to
\begin{equation}
c(f_1, f_2) = \exp \left( -\int_{D_{f_1}} \omega - \int_{D_{f_2}} \omega + \int_{D_{f_1} f_2} \omega \right),
\end{equation}
\begin{equation}
(4.20)
\end{equation}
which, in turn, is a direct consequence of the definition (4.13) specialised to loops \( f_{1,2} \in L_0L_0G \). Moreover, \( \mathcal{N} \) is normal because
\begin{align}
(f_1, z_1)(f_2, \text{hol}^{-1}(f_2))(f_1, z_1)^{-1} &= (f_1 f_2 f_1^{-1}, \text{hol}^{-1}(f_2) c(f_1, f_2) c(f_1 f_2, f_1^{-1}) c^{-1}(f_1, f_1^{-1})) \\
&= (f_1 f_2 f_1^{-1}, \text{hol}^{-1}(f_1 f_2 f_1^{-1}))
\end{align}
\begin{equation}
(4.21)
\end{equation}
\(^1\)See Appendix C for another form of the group cocycle.
for all $f_1 \in P_0L_0G$, for all $f_2 \in L_0L_0G$, and for all $z_1 \in U(1)$. Here, the last equality follows from combining the holonomy $\text{hol}^{-1}(f_2)$ with the holonomies in the cocycles, comparing the boundaries and invoking Stokes’ theorem, and using (4.16) once.

This completes the definition of the elements in (4.4), and we can turn to the definition (4.3a),

$$\text{String}(G) = \left( (P_0L_0G \times U(1))/N \xrightarrow{t} P_0G, \triangleright \right).$$

(4.22)

As mentioned, the map $t$ is the composition of the evident projection $(P_0L_0G \times U(1))/N \to P_0L_0G/L_0L_0G \to L_0G$ with the embedding $L_0G \hookrightarrow P_0G$. On elements $[(f, z)] \in (P_0L_0G \times U(1))/N$ this reads

$$t([(f, z)]) = b(f) \in L_0G,$$

(4.23)

which is indeed well-defined. In order to define the action $\triangleright$, we introduce the left-invariant 1-form $\xi_g \in \Omega^1(L_0G, u(1))$ given by (cf. [30] and [57, Section 4])

$$(\xi_g)_\ell := \frac{i}{2\pi} \int_0^1 dr \left\{ \theta_{\ell(r)}, g(r)^{-1} \frac{\partial g(r)}{\partial r} \right\}$$

(4.24)

for all $g \in P_0G$ and for all $\ell \in L_0G$. We have $(\text{Ad}_g)^* \omega = \omega + d\xi_g$, see Appendix C, which implies that for loops $h \in L_0L_0G$,

$$\exp \left( - \int_0^1 ds \xi_g \left( h^{-1}(s) \frac{\partial h(s)}{\partial s} \right) \right) = \text{hol}(ghg^{-1})\text{hol}^{-1}(h).$$

(4.25)

Here, $h(s) \in L_0G$ for all $s \in [0, 1]$. The definition of the action of $P_0G$ on $P_0L_0G \times U(1)$ is then [30]

$$g \triangleright (f, z) := (gf g^{-1}, z \exp \left( \int_0^1 ds \xi_g \left( f^{-1}(s) \frac{\partial f(s)}{\partial s} \right) \right))$$

$$= (gf g^{-1}, z \exp \left( \frac{i}{2\pi} \int_0^1 dr \int_0^1 ds \left\{ f^{-1}(s, r) \frac{\partial f(s, r)}{\partial s}, g^{-1}(r) \frac{\partial g(r)}{\partial r} \right\} \right))$$

(4.26)

for all $g \in P_0G$ and for all $(f, z) \in P_0L_0G \times U(1)$.

The action composes correctly and is indeed compatible with the product, which can be verified by direct computation, using the explicit expressions for $\xi_g$ and the group cocycle $c$.

Moreover, with (4.25), it is also clear that the action closes on $N$ and hence, it is well-defined on the quotient group $(P_0L_0G \times U(1))/N$. We also note that the action trivially satisfies $t(g \triangleright \hat{h}) = gt(\hat{h})g^{-1}$ for all $g \in P_0G$ and for all $\hat{h} \in P_0L_0G \times U(1)$. It only remains to show that $(t(\hat{h}_1) \triangleright \hat{h}_2)\hat{h}_1\hat{h}_2^{-1}\hat{h}_1^{-1} \in N$ for all $\hat{h}_{1,2} \in P_0L_0G \times U(1)$ as this implies the Peiffer identity, and the proof can be found in Appendix C.

In conclusion, we arrive at the 2-group model of the string group given by the crossed module (4.22).
**String Lie 2-algebra.** We identify the Lie algebra of $P_0 L_0 G \times U(1)$ with $P_0 L_0 \mathfrak{g} \oplus \mathfrak{u}(1)$. The Lie bracket is obtained in the usual way from the group commutator using (4.15),

$$
[(\gamma_1, q_1), (\gamma_2, q_2)] = \left( [\gamma_1, \gamma_2], -\frac{i}{2\pi} \int_0^1 ds \left\langle \frac{\partial \gamma_1(s, r)}{\partial s}, \frac{\partial \gamma_2(s, r)}{\partial r} \right\rangle \right),
$$

where we have integrated by parts in the second line.\(^1\)

Furthermore, because of

$$
\frac{d}{dt} \exp(t\gamma) = 0
$$

for all $\gamma \in L_0 \mathfrak{g}$, which we verify in Appendix C, the Lie algebra $\mathfrak{n}$ of $\mathcal{N}$ is just

$$
\mathfrak{n} = \{ (\gamma, 0) \in L_0 \mathfrak{g} \oplus \mathfrak{u}(1) \} \cong L_0 \mathfrak{g}.
$$

From (4.27) we immediately see that $\mathfrak{n}$ is indeed an ideal. Consequently,

$$
(P_0 L_0 \mathfrak{g} \oplus \mathfrak{u}(1))/\mathfrak{n} \cong L_0 \mathfrak{g} \oplus \mathfrak{u}(1).
$$

In particular, since the second component in the Lie algebra commutator depends only on the endpoint, we recover the Kac–Moody 2-cocycle (4.9) in the factor algebra. In addition, the linearisation of the action (4.26) is

$$
\alpha angle (\gamma, q) = \left( [\alpha, \gamma], \frac{i}{2\pi} \int_0^1 ds \left\langle \frac{\partial \alpha}{\partial t}, \gamma \right\rangle \right),
$$

for all $\alpha \in P_0 \mathfrak{g}$ and for all $(\gamma, q) \in L_0 \mathfrak{g} \oplus \mathfrak{u}(1)$. In summary, we thus have obtained the crossed module of Lie algebras [30]

$$
\text{string}(\mathfrak{g}) := (L_0 \mathfrak{g} \oplus \mathfrak{u}(1) \xrightarrow{t} P_0 \mathfrak{g}, \rangle)
$$

as a strict model of the string Lie 2-algebra. Here, $t$ is just the projection onto $L_0 \mathfrak{g}$ followed by the embedding into $P_0 \mathfrak{g}$.

\(^1\)Note that the extra minus sign is due to the fact that the difference between the horizontal lift of a commutator of vector fields and the commutator of horizontal lifts of vector fields is given by the curvature.

More explicitly, given a $\mathfrak{g}$-valued connection $\mu$ on a principal bundle whose curvature is $\omega$, we have for any vector fields $U, V$ on the base space the formula $\omega(U^h, V^h) = -\mu([U^h, V^h])$, where $^h$ denotes the horizontal lift. Upon identifying the Lie algebra $\mathfrak{g}$ with the space of vertical vector fields it generates, we can write this as $\omega(U^h, V^h) = [U, V]^h - [U^h, V^h]$, where we have used that $[U^h, V^h]_H = [U, V]^h$, where $H$ denotes the horizontal part of a vector (field). For the case of a central extension $\hat{\mathcal{L}}$ of the group $\mathcal{L}$ by $U(1)$, this formula at the identity gives the Lie algebra commutator $[([U, 0], [V, 0])] = ([U, V], 0) - (0, \omega(U^h, V^h)) = ([U, V], -\omega(U, V))$, where the horizontal lifts and the decomposition $\hat{\mathcal{L}} = \mathcal{L} \oplus \mathfrak{u}$ are defined by the connection.
Minimal model. We recall that a minimal model of the string Lie 2-algebra is given by the 2-term $L_{\mathcal{X}}$-algebra

$$\text{string}^\cdot(g) := \mathfrak{u}(1) \oplus g, \quad (4.33)$$

where $\text{string}^\cdot(g)_{-1} := \mathfrak{u}(1)$ and $\text{string}^\cdot(g)_0 := g$ with the non-trivial higher products

$$\mu_2 : g \times g \to g, \quad (X_1, X_2) \mapsto [X_1, X_2],$$

$$\mu_3 : g \times g \times g \to \mathfrak{u}(1), \quad (X_1, X_2, X_3) \mapsto i\langle X_1, [X_2, X_3] \rangle. \quad (4.34)$$

We can now extend the quasi-isomorphism (2.12) between the Lie algebra $g$, trivially regarded as a Lie 2-algebra, and the Lie 2-algebra $L_g$ to a quasi-isomorphism

$$\text{string}^\cdot(g) \xrightarrow{\phi} \text{string}(g) \xrightarrow{\psi} \text{string}^\cdot(g), \quad (4.35a)$$

and explicitly, we have the chain maps

$$\begin{array}{cccc}
\mathfrak{u}(1) & \xrightarrow{\phi_1} & L_0 g \oplus \mathfrak{u}(1) & \xrightarrow{\psi_1} & \mathfrak{u}(1) \\
\mathfrak{g} & \xrightarrow{\phi_2 := t} & P_0 g & \xrightarrow{\psi_2 := b} & \mathfrak{g}
\end{array} \quad (4.35b)$$

where in the top row $\phi_1$ and $\psi_1$ are the evident embedding and projection maps. In this case, both $\phi_2$ and $\psi_2$ are non-trivial,

$$\phi_2 : g \times g \to L_0 g, \quad (X_1, X_2) \mapsto (\phi - \phi^2) \cdot [X_1, X_2],$$

$$\psi_2 : P_0 g \times P_0 g \to \mathfrak{u}(1), \quad (X_1, X_2) \mapsto -\frac{i}{\pi} \int_0^1 dr \left( \left\langle \frac{\partial X_1}{\partial r}, X_2 \right\rangle - \left\langle X_1, \frac{\partial X_2}{\partial r} \right\rangle \right). \quad (4.35c)$$

4.3. Adjusted descent data

Let us now discuss adjustment for the descent data based on $\text{String}(G)$ for $G$ a simply connected compact semi-simple Lie group. As before, let us first work locally on an open and simply connected cover $\sigma : Y \to M$. The local kinematical data for higher gauge theory with gauge Lie 2-algebra $\text{string}(g)$ was already given in [33,10].
Local adjusted String(\mathbf{G})-connections. Consider the 2-group String(\mathbf{G}) defined in (4.22) and the associated Lie 2-algebra string(\mathbf{G}) defined in (4.32), respectively. Locally, on an open contractible cover Y, we have the connection

\[(A, \hat{B}) \in \Omega^1(Y, P_0\mathfrak{g}) \oplus \Omega^2(Y, L_0\mathfrak{g} \oplus \mathfrak{u}(1))\]  

(4.36)

together with the associated curvature forms

\[F := dA + \frac{1}{2}[A, A] + t(\hat{B}) \in \Omega^2(Y, P_0\mathfrak{g}) ,\]
\[\hat{H} := d\hat{B} + A \triangleright \hat{B} - \hat{\kappa}(A, F) = dB - \hat{\kappa}(A, F - t(\hat{B})) \in \Omega^3(Y, L_0\mathfrak{g} \oplus \mathfrak{u}(1)) ,\]  

(4.37a)

where

\[\hat{\kappa} : \Omega^p(Y, P_0\mathfrak{g}) \times \Omega^q(Y, P_0\mathfrak{g}) \to \Omega^{p+q}(Y, L_0\mathfrak{g} \oplus \mathfrak{u}(1)) ,\]
\[(\alpha_1, \alpha_2) \mapsto \left( \kappa(\alpha_1, \alpha_2), \frac{i}{2\pi} \int_0^1 dr \left< \hat{\partial}_\alpha \alpha_1, \alpha_2 \right> \right) ,\]  

(4.37b)

where \(\kappa\) was defined in (2.31b). Upon writing \(\hat{B} := (B, B') \in \Omega^2(Y, L_0\mathfrak{g} \oplus \mathfrak{u}(1))\) and \(\hat{H} := (H, H') \in \Omega^3(Y, L_0\mathfrak{g} \oplus \mathfrak{u}(1))\), the curvatures read as

\[F = dA + \frac{1}{2}[A, A] + t(B) ,\]
\[H = dB + A \triangleright B - \kappa(A, F) = \left( \text{id} - p \cdot b \right)(dF) ,\]
\[H' = dB' - \frac{1}{2\pi} \int_0^1 dr \left< \hat{\partial}_\alpha F, \omega - t(B) \right> = d(B' - \frac{1}{2}\omega(A, A)) - \frac{i}{2\pi} \text{cs}(b(A)) .\]

(4.38)

Here, ‘cs’ is the Chern–Simons form, and we have slightly abused notation and again used the symbols t and \(\triangleright\) to denote the morphism and the action (2.11) of \(\mathcal{L}_\mathfrak{g}\). Moreover, these curvatures satisfy the Bianchi identities

\[dF + [A, F] = t(H + \kappa(A, F)) ,\]
\[dH = 0 ,\]
\[dH' = -\frac{i}{2\pi} \left< \phi(F - t(B)), b(F - t(B)) \right> = -\frac{i}{2\pi} \left< \phi(F), b(F) \right> .\]

(4.39)

Next, infinitesimal gauge transformations are parametrised by \(\alpha \in \mathcal{C}^\infty(Y, P_0\mathfrak{g})\) and \(\hat{\lambda} := (\lambda, \lambda') \in \Omega^1(Y, L_0\mathfrak{g} \oplus \mathfrak{u}(1))\) and act on the gauge potentials as

\[\delta A \ := \ \text{d}\alpha + [A, \alpha] - t(\hat{\lambda}) \quad \text{and} \quad \delta \hat{B} \ := \ -\alpha \triangleright \hat{B} + d\hat{\lambda} + A \triangleright \hat{\lambda} + \hat{\kappa}(\alpha, F) \quad (4.40a)\]

or, explicitly,

\[\delta A \ := \ \text{d}\alpha + [A, \alpha] - t(\lambda) ,\]
\[\delta B \ := \ -\alpha \triangleright B + d\lambda + A \triangleright \lambda + \kappa(\alpha, F) = \text{d}\lambda + A \triangleright \lambda + \kappa(\alpha, F - t(B)) ,\]
\[\delta B' \ := \ \text{d}\lambda' + \frac{i}{2\pi} \int_0^1 dr \left\{ \left< \hat{\partial}_\alpha \lambda, t(\lambda) \right> + \left< \hat{\partial}_\alpha F, t(B) \right> \right\} .\]

(4.40b)
Consequently, the curvature forms transform as

\[
\delta F = [F, \alpha] + t(\kappa(\alpha, F)) = \varphi \cdot b([F, \alpha]) \quad \text{and} \quad \delta \hat{H} = 0. \tag{4.40c}
\]

Finally, higher gauge transformations are parametrised by \((\vartheta, \vartheta') \in C^{\infty}(Y, L_0G \oplus u(1))\), and the gauge parameters transform as

\[
\delta \alpha := t(\vartheta), \quad \delta \lambda := d\vartheta + A \Rightarrow \vartheta, \quad \text{and} \quad \delta \lambda' := d\vartheta' + \frac{i}{2\pi} \int_0^1 dr \left( \frac{\partial A}{\partial r}, t(\vartheta) \right). \tag{4.41}
\]

**Adjustment of String(G).** Given the infinitesimal version (4.37b), it is not too hard to derive a finite adjustment datum for String(G) that satisfies the crucial relation (2.44). We define

\[
\hat{\kappa} : P_0G \times P_0G \to L_0G \oplus u(1),
\]

\[
(g, X) \mapsto \left( \kappa(g, X), -\frac{i}{2\pi} \int_0^1 dr \left( \frac{\partial g}{\partial r} g^{-1}, X \right) \right). \tag{4.42}
\]

Here, \(\kappa\) was defined in (2.39b). The verification is lengthy but can be done by means of direct computation. We will verify below that this adjustment datum leads to consistent gluing of the connection part of the cocycle of a String(G)-bundle with connection.

**Adjusted finite gauge transformation.** Let us now specialise the general formulas (2.48b) to the local description of a connection. Adjusted finite gauge transformations are parametrised by \((a, \lambda) = (a, \lambda, \lambda') \in C^{\infty}(Y, P_0G) \oplus \Omega^1(Y, L_0G \oplus u(1))\), and they act on the gauge potentials \((A, B) = (A, B, B') \in \Omega^1(Y, P_0G) \oplus \Omega^2(Y, L_0G \oplus u(1))\) according to

\[
A \mapsto \tilde{A} := a^{-1} A a + a^{-1} d a - t(\lambda),
\]

\[
B \mapsto \tilde{B} := a^{-1} \Rightarrow B + d\lambda + \tilde{A} \Rightarrow \lambda + \frac{1}{2} [\lambda, \lambda] - \kappa(a, F)
\]

\[
= B + d\lambda + \tilde{A} \Rightarrow \lambda + \frac{1}{2} [\lambda, \lambda] - \kappa(a, F - t(B)),
\]

\[
B' \mapsto \tilde{B}' := B' + d\lambda'
\]

\[
+ \frac{i}{2\pi} \int_0^1 dr \left\{ \left( \frac{\partial \tilde{A}}{\partial r}, t(\lambda) \right) + \frac{1}{2} \left( \frac{\partial t(\lambda)}{\partial r}, t(\lambda) \right) + \left( \frac{\partial a}{\partial r} a^{-1}, F - t(B) \right) \right\}. \tag{4.43}
\]

Evidently, the linearisation of these transformations yields (4.40). Furthermore, as some algebraic manipulations reveal, the induced transformations of the curvatures forms \((F, \hat{H})\) defined in (4.37) are

\[
F \mapsto \tilde{F} = F + \varphi \cdot b(a^{-1} Fa - F) \quad \text{and} \quad \hat{H} \mapsto \tilde{H} = \hat{H}. \tag{4.44}
\]
As a consistency check, we can verify that the above postulated gauge transformations glue together consistently to the diagram

\[
\begin{array}{ccc}
(A_1, \hat{B}_1) & \xrightarrow{(a_{12}, \hat{\lambda}_{12})} & (A_2, \hat{B}_2) \\
\downarrow m_{123} & & \downarrow m_{123} \\
(A_3, \hat{B}_3) & \xrightarrow{(a_{13}, \hat{\lambda}_{13})} & (A_1, \hat{B}_1)
\end{array}
\]

(4.45)

with \( \hat{B}_a := (B_a, B'_a) \) and \( \hat{\lambda}_{ab} := (\lambda_{ab}, \lambda'_{ab}) \) for \( a, b = 1, 2, 3 \). Using (4.43), we compute

\[
A_3 = a_{23}^{-1}a_{12}^{-1}A_1a_{12}a_{23} + a_{23}^{-1}(a_{12}^{-1}da_{12} - t(\lambda_{12}))a_{23} + a_{23}^{-1}da_{23} - t(\lambda_{23})
\]

(4.46)

\[
\overset{1}{=} a_{13}^{-1}A_1a_{13} + a_{13}^{-1}da_{13} - t(\lambda_{13})
\]
as a consistency condition for the 1-form potentials. This condition is satisfied provided we introduce additional gauge parameters \( \hat{m}_{123} := \{(m_{123}, m'_{123})\} \in \mathcal{C}^\infty(Y, L_0G) \) that glue together the parameters \( a \) and \( \lambda \) over triangles according to

\[
a_{13} = t(\hat{m}_{123})a_{12}a_{23},
\]

\[
\lambda_{13} = \lambda_{23} + a_{23}^{-1}a_{13}^{-1}b(m_{123})\nabla_1b(m_{123})
\]

\[
\lambda'_{13} = \lambda'_{23} + \lambda'_{12} - m'_{123}d(m'_{123})
\]

\[
- \frac{i}{2\pi} \int_0^1 dr \left\langle \left( \frac{\partial a_{23}}{\partial r}, a_{23}^{-1}, \lambda_{12} \right) - \left( \frac{\partial a_{13}}{\partial r}, a_{13}^{-1}, b(m_{123})\nabla_1b(m_{123}) \right) \right)
\]

\[
+ \left\langle b(m_{123})\frac{\partial b(m_{123})}{\partial r}, A_1 \right) + \int_0^1 ds \left\langle \frac{\partial m_{123}}{\partial r}, m_{123}^{-1}, \frac{\partial (m_{123}d(m_{123})}{\partial s} \right) \right\rangle.
\]

(4.47)

We then obtain an additional consistency condition over tetrahedra formed by four such triangles,

\[
\hat{m}_{134}\hat{m}_{123} = \hat{m}_{124}(a_{12} \Rightarrow \hat{m}_{234}).
\]

(4.48)

Explicitly, this conditions reads as

\[
(m_{134}m_{123})^{-1}m_{124}(a_{12} \Rightarrow m_{234}) \in L_0L_0G,
\]

\[
m'_{134}m'_{123} = m'_{124}m'_{234} \exp \left( \frac{i}{2\pi} \int_0^1 ds \int_0^1 dr \left\langle a_{12}^{-1} \frac{\partial a_{12}}{\partial r}, m_{234}^{-1} \frac{\partial m_{234}}{\partial s} \right) \right) \exp \left( - \int \omega \right),
\]

(4.49)

where \( \square \) denotes the loop \( \overline{m_{134}} \circ (b(m_{134})m_{123}) \circ (b(m_{124})(a_{12} \Rightarrow m_{234})) \circ m_{124} \).

**Adjusted descent data for principal String(G)-bundles.** We can now discuss the specialisation of the cocycle, coboundary, and higher coboundary data for adjusted principal 2-bundles, (2.47), (2.48), and (2.49), to the case of principal String(G)-bundles and give some
explicit formulas. A cocycle is given by
\[
\hat{h} = [(h, h')] \in \mathcal{C}^\infty(Y^{[3]}, (P_0L_0G \times U(1))/\mathbb{N}) ,
\]
\[
(g, \hat{\lambda} = (\Lambda, \Lambda')) \in \mathcal{C}^\infty(Y^{[2]}, P_0G) \oplus \Omega^1(Y^{[2]}, L_0G \oplus u(1)) , \tag{4.50a}
\]
\[
(A, \hat{B} = (B, B')) \in \Omega^1(Y^{[1]}, P_0G) \oplus \Omega^2(Y^{[1]}, L_0G \oplus u(1)) ,
\]
such that for all appropriate \((i, j, \ldots) \in Y^{[n]}\), with \(Y^{[n]}\) given in (2.17), we have
\[
g_{ik} = t(h_{ijk})g_{ij}g_{jk} ,
\]
\[
(h_{ikl}h_{ijk}^{-1}h_{ijl}(g_{ij} \Rightarrow h_{kl}) \in L_0L_0G) , \tag{4.50b}
\]
\[
h_{ikl}^\prime h_{ijk}^\prime = h_{ijl}h_{ijkl}^\prime \exp \left( \frac{i}{2\pi} \int_0^1 ds \int_0^1 dr \left\langle \frac{\partial g_{ikl}}{\partial r} - \frac{\partial h_{ijkl}}{\partial s}, h_{ijl}^{-1}\frac{\partial h_{ijkl}}{\partial s} \right\rangle \right) \exp \left( - \int_0^\Box \right) ,
\]
where \(\Box\) denotes the loop \(\overline{h_{ikl} \circ (b(h_{ikl})h_{ikl}^{-1}) \circ (b(h_{ijl})(g_{ij} \Rightarrow h_{kl})) \circ h_{ijkl}}\). The cocycle conditions for the differential refinement specialise to
\[
\Lambda_{ik} = \Lambda_{jk} + g_{ij}^{-1} \Rightarrow \Lambda_{ij} - g_{ik}^{-1} \Rightarrow (b(h_{ikl}b(h_{ikl})^{-1}) ,
\]
\[
\Lambda_{ik}^\prime = \Lambda_{jk} + \Lambda_{ij} - h_{ijkl}^{-1} \Rightarrow 0 \Rightarrow \frac{1}{2\pi} \int_0^1 dr \left\langle \frac{\partial g_{ikl}}{\partial r} - \frac{\partial h_{ijkl}}{\partial s}, h_{ijl}^{-1}\frac{\partial h_{ijkl}}{\partial s} \right\rangle + \int_0^1 ds \left\langle \frac{\partial h_{ijkl}}{\partial r} - \frac{\partial h_{ijkl}}{\partial s}, h_{ijl}^{-1}\frac{\partial h_{ijkl}}{\partial s} \right\rangle ,
\]
\[
A_j = g_{ij}^{-1}A_ig_{ij} + g_{ij}^{-1}\partial g_{ij} - A_i ,
\]
\[
B_j = B_i + d\Lambda_{ij} + A_j \Rightarrow \Lambda_{ij} - \frac{1}{2} h_{ijkl} - \kappa(g_{ij}, F_i - B_i) ,
\]
\[
B_j^\prime = B_j^\prime + d\Lambda_{ij}^\prime + \frac{1}{2\pi} \int_0^1 dr \left\langle \frac{\partial A_j}{\partial r} - \frac{\partial \Lambda_{ij}}{\partial s} , \Lambda_{ij} \right\rangle + \frac{1}{2} \left\langle \frac{\partial \Lambda_{ij}}{\partial r} , \Lambda_{ij} \right\rangle + \left\langle \frac{\partial g_{ij}^{-1}}{\partial r} , F_i - B_i \right\rangle . \tag{4.50c}
\]
A coboundary is given by
\[
\hat{b} = [(b, b')] \in \mathcal{C}^\infty(Y^{[2]}, (P_0L_0G \times U(1))/\mathbb{N}) ,
\]
\[
(a, \hat{\lambda} = (\lambda, \lambda')) \in \mathcal{C}^\infty(Y^{[1]}, P_0G) \oplus \Omega^1(Y^{[1]}, L_0G \oplus u(1)) , \tag{4.51}
\]
and a higher coboundary is parametrised by \(\hat{m} = [(m, m')] \in \mathcal{C}^\infty(Y^{[1]}, L_0G)\). We refrain from giving the explicit specialisations of formulas (2.48), and (2.49) as their form is not particularly enlightening.

5. Examples

An evident candidate for an application of our above constructions is the non-Abelian self-dual string [33]. This is the lift of an \(G\)-instanton connection on \(\mathbb{R}^4\) to a connection on
a principal String($G$)-bundle and a Higgs field $\phi$ such that the $H = \ast \nabla \phi$ for $H$ the 3-form curvature. The non-Abelian self-dual string is a categorified version of a Bogomolny monopole, and as explained in [33], there is a straightforward dimensional reduction of the corresponding equations. Note that it does not make physical sense to consider a ‘compactified Bogomolny monopole’ on $S^3$ or, correspondingly, a non-Abelian self-dual string solution on $S^4$. Nevertheless, the lift of an instanton bundle to a higher bundle with a string 2-group as its structure group is certainly mathematically interesting, and it can serve as a useful non-trivial example of our constructions. Note that on $S^4$, the first Pontryagin class is a non-trivial obstruction, and we should thus consider an instanton–anti-instanton pair as done in Section 3.3.

The construction in Section 3.3 suggests an interpretation of the lifted string bundle as a higher coset space of categorified groups. We note that a similar construction, albeit without connection, is found in [40]. For simplicity, we first consider the purely Abelian case, where a corresponding higher coset construction arises.

5.1. Abelian self-dual strings from cosets of higher groups

Lie groupoids and categorified spaces. There is an evident generalisation of the construction reviewed in Section 3.1 to categorified spaces. For our purposes, it will be sufficient to work with Lie groupoids, i.e. groupoids internal to the category of smooth manifolds\(^1\). We note that any smooth manifold $M$ can be trivially regarded as the Lie groupoid

\[
M := (M \rightarrow M)
\]

with source and target maps being identities, and the morphisms also being identities. Lie groupoids naturally form a 2-category, but here, we can restrict ourselves to morphisms that are ordinary smooth functors.

Cosets of crossed modules. Recall that the categories of crossed modules of Lie groups and of strict Lie 2-groups are equivalent [36]. Explicitly, given a crossed module of Lie groups $G = (H \xrightarrow{i} G, \rhd)$, we associated to $G$ a strict Lie 2-group, denoted by $\tilde{G}$ given by the data

\[
\tilde{G} := (G \times H \rightarrow G) \quad \text{with} \quad \begin{array}{c}
g \\ (g_1, h_1) \circ (t(h_1^{-1})g_1, h_2) = (g_1, h_2 h_1) \\
(g_1, h_1) \otimes (g_2, h_2) := (g_2 g_1, (g_1^{-1} \rhd h_2) h_1) \end{array},
\]

\[
\text{\tiny (5.2)}
\]

\(^1\)Since this category does not have all pullbacks, one needs to refine this definition a bit, a technicality we suppress here.
for all \( g, g_1, g_2 \in G \) and for all \( h, h_1, h_2 \in H \). Note that our conventions here are those of [15], which are slightly different from [36].

By an embedding of a crossed module \( G := (H_1 \xrightarrow{\in} G_1, \cdot) \) into a crossed module \( G := (H_2 \xrightarrow{\in} G_2, \cdot) \), we mean a (strict) injective morphism of crossed modules \( e : G_1 \hookrightarrow G_2 \) which is given by embeddings of groups \( e_0 : G_1 \hookrightarrow G_2 \) and \( e_1 : H_1 \hookrightarrow H_2 \) such that

\[
e_0(t_1(h_1)) = t_2(e_1(h_1)) \quad \text{and} \quad e_0(g_1) \cdot_2 e_1(h_1) = e_1(g_1 \cdot_1 h_1). \tag{5.3}
\]

for all \( h_1 \in H_1 \) and for all \( g_1 \in G \). Such an embedding then induces an action of the morphisms of \( G_1 \) on the morphisms of \( G_2 \) by means of

\[
(g_1, h_1) \cdot (g_2, h_2) := (e_0(g_1), e_1(h_1)) \otimes (g_2, h_2) = (g_2 e_0(g_1), (e_0(g_1^{-1}) \cdot_2 h_2) e_1(h_1)) \tag{5.4}
\]

for all \( (g_1, h_1) \in G_1 \times H_1 \) and for all \( (g_2, h_2) \in G_2 \times H_2 \). Upon identifying morphisms under this action, we obtain the Lie groupoid

\[
\frac{G_2}{G_1} := \frac{G_2/G_1}{H_2} := \frac{(G_2 \times H_2)/(G_1 \times H_1)}{G_2/G_1}, \tag{5.5}
\]

where, again, the equivalence relation is

\[
g_2 \sim g_2 e_0(g_1) \quad \text{and} \quad (g_2, h_2) \sim (g_2 e_0(g_1), e_1(h_1) (e_0(g_1^{-1}) \cdot_2 h_2)) \tag{5.6}
\]

for all \( (g_1, h_1) \in G_1 \times H_1 \) and for all \( (g_2, h_2) \in G_2 \times H_2 \).

**Example.** A useful and simple example of the above construction for the purposes of our discussion is given by the crossed modules \( BU(1) = (U(1) \longrightarrow \ast, \text{id}) \) and \( \text{String}(G) \) as defined in (4.3a). We can embed \( BU(1) \) into \( \text{String}(G) \) as follows:

\[
\begin{array}{ccc}
U(1) & \xrightarrow{e_1} & \hat{L}_0G \\
\downarrow & & \downarrow \\
\ast & \xrightarrow{e_0} & P_0G
\end{array}
\]

\[e_1 : U(1) \ni z \mapsto (1, z) \in \hat{L}_0G, \quad e_0 : \ast \ni 1 \mapsto 1 \in P_0G. \tag{5.7a}\]

The resulting coset is

\[
\text{String}(G)/BU(1) \cong LG \tag{5.7b}
\]

with \( LG \) the crossed module defined in (2.5a). This is, in fact, the essence of the short exact sequence (4.2). Since \( G \) can be trivially identified with the crossed module \( (\ast \hookrightarrow G, \text{id}) \) and because of the latter’s identification with \( LG \) by virtue of (2.6), any Lie group \( G \) can thus be viewed as the coset \( \text{String}(G)/BU(1) \).

We note that the string 2-group \( \text{String}(1) \) can be identified with the group \( BU(1) \), so that the example (5.7) can be regarded as the coset space \( \text{String}(G)/\text{String}(1) \). We shall consider more general quotients of the string group later on.
Equivalence of Lie groupoids. Let us briefly mention a complication in identifying equivalent Lie groupoids. The identification $\mathcal{L}G$ with $G$ used above due to (2.6) does not persist if we forget the 2-group structure and compare $\mathcal{L}G$ as defined in (5.2) with $G$ as defined in (5.1). Recall that the appropriate notion of weak groupoid equivalence is an isomorphism in the 2-category that has Lie groupoids as objects, bibundles as morphisms, and bibundle maps as 2-morphisms, which is also known as Morita equivalence. Unfortunately, there is no bibundle equivalence between the Lie groupoids $\mathcal{L}G$ and $G$. The same problem arises when trying to identify the Čech groupoid $\check{C}(Y \to M)$ of a surjective submersion $Y \to M$ with the Lie groupoid $M$. This is a well-known deficiency of the 2-category of Lie groupoids: fully faithful and essentially surjective bibundle morphisms fail to be equivalences, cf. [31, Lemma 9]. In order to address this deficiency, we shall consider a Lie groupoid equivalent to its space of isomorphism classes of objects. For example, we consider $\check{C}(Y \to M)$ equivalent to $\pi_0((\check{C}(Y \to M)) \cong \pi_0(M) \cong M$ so that for the surjective submersion $\check{b} : P_0G \to G$ with $\check{C}(P_0G \to G) \cong (L_0G \rightrightarrows P_0G)$ as given in (2.20) we identify $(L_0G \rightrightarrows P_0G)$ with $\pi_0(L_0G \rightrightarrows P_0G) \cong \pi_0(G)$.

Equivalent formulations of $\mathcal{L}G$ and String$(G)$. Consider the strict Lie 2-group $\mathcal{L}G$ associated with $\mathcal{L}G$ by means of (5.2). Concretely, $\mathcal{L}G = (P_0G \ltimes L_0G \rightrightarrows L_0G)$. Furthermore, consider the Čech groupoid $\check{C}(P_0G \to G) \cong (L_0G \rightrightarrows P_0G)$ for the surjective submersion $\check{b} : P_0G \to G$ as defined in and discussed around (2.20). Together with the monoidal structure given by

$$\ell_1 \otimes \ell_2 := \ell_2 \ell_1$$

for all $\ell_1, \ell_2 \in L_0G$, the Lie groupoid $(L_0G \rightrightarrows P_0G)$ becomes a strict Lie 2-group. Then, there are isomorphisms of strict Lie 2-groups

$$\Phi : \mathcal{L}G \to (L_0G \rightrightarrows P_0G) \quad \text{and} \quad \Psi : (L_0G \rightrightarrows P_0G) \to \mathcal{L}G$$

(5.9a)

given by

$$
\begin{array}{ccc}
P_0G \times L_0G & \xrightarrow{\Phi_1} & L_0G \\
\downarrow & & \downarrow \\
P_0G & \xrightarrow{\text{id}} & P_0G
\end{array}
\quad
\begin{array}{ccc}
P_0G \times L_0G & \xrightarrow{\Phi_1} & L_0G \\
\downarrow & & \downarrow \\
P_0G & \xrightarrow{\text{id}} & P_0G
\end{array}
\quad
\begin{array}{ccc}
P_0G \times L_0G & \xrightarrow{\Phi_1} & L_0G \\
\downarrow & & \downarrow \\
P_0G & \xrightarrow{\text{id}} & P_0G
\end{array}

(5.9b)

with the only non-trivial group morphisms

$$\Phi_1(p, \ell) := \bar{p} \circ p \ell^{-1} \quad \text{and} \quad \Psi_1(\ell) := (t(\ell), s(\ell)^{-1} t(\ell))$$

(5.9c)

for all $p \in P_0G$ and for all $\ell \in L_0G$, where, as before, $\bar{p}$ is the inverse of the path $p$, $\circ$ denotes composition of paths, and the inverse and product are those in $P_0G$. We note that the Lie
2-group isomorphisms (5.9) $\Phi$ and $\Psi$ now lift to the evident Lie 2-group isomorphisms $\hat{\Phi}$ and $\hat{\Psi}$ between the groupoids in which $L_0G$ is replaced by the Kac–Moody central extension $\hat{L}_0G$. In particular, with (4.3a), we have

$$\text{String}(G) = \frac{P_0G \ltimes \hat{L}_0G}{\hat{\Phi}_1} \cong \frac{L_0G}{\hat{\Phi}} =: \text{String}(G)' , \quad (5.10a)$$

implying the equivalence of strict Lie 2-groups

$$\text{String}(G) \cong (\hat{L}_0G \rightarrow P_0G) . \quad (5.10b)$$

**Bundle gerbe of the Abelian self-dual string.** The elementary self-dual string is given by the fundamental Abelian gerbe over $S^3$, which is the generator of $H^3(S^3, \mathbb{Z})$. The total space of this gerbe can be identified with $\text{String}(3)$, cf. e.g. [62,33]. By (5.7b), have $\text{String}(3)/\text{BU}(1) \cong \mathcal{L}\text{Spin}(3) \cong \text{Spin}(3) \cong S^3$. In addition, we also have the diagram

$$\begin{array}{ccc}
L_0\text{Spin}(3) \\
\downarrow \\
L_0\text{Spin}(3) \longrightarrow P_0\text{Spin}(3) \\
\downarrow^b \\
S^3
\end{array} \quad (5.11)
$$

This is the standard bundle gerbe description of the basic gerbe over $S^3$, and the data is completed by a circle bundle isomorphism over $(P_0\text{Spin}(3))^{[3]}$ that relates the three different pullbacks of $L_0\text{Spin}(3) \rightarrow L_0\text{Spin}(3) \cong (P_0\text{Spin}(3))^{[2]}$ and that satisfies a cocycle condition over $(P_0\text{Spin}(3))^{[4]}$, cf. [1, 2]. The basic gerbe is thus the lifting bundle gerbe for the Kac–Moody central extension (4.3b), cf. [2].

**Hitchin–Chatterjee gerbe of the Abelian self-dual string.** We note that the picture (5.11) is very nice from the total space perspective on Abelian bundle gerbes, but it does not allow us to extract the corresponding Deligne cocycles directly. As explained e.g. in [2], Deligne cocycles describing Hitchin–Chatterjee gerbes are obtained directly if the principal fibre bundle appearing in the bundle gerbe is trivial. Because $L_0\text{Spin}(3) \rightarrow L_0\text{Spin}(3)$ is non-trivial, this is not the case here. We should therefore describe the bundle gerbe subordinate to a different surjective submersion.

For general Lie groups of sufficiently high rank, one can use the well-known construction of [63,64]; unfortunately, this construction does not work for the basic gerbe over $\text{SU}(2)$. 

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Since we have not been able to find an explicit cocycle description in the literature, we present it here.

We use the usual spherical coordinates \((\theta_1, \theta_2, \phi) \in \Theta := [0, \pi] \times [0, \pi] \times [0, 2\pi]\), and we have an embedding \(\Theta \to S^3 \hookrightarrow \mathbb{R}^4\) by

\[
x^1 := \sin \phi \sin \theta_1 \sin \theta_2, \quad x^3 := \cos \theta_1 \sin \theta_2, \\
x^2 := \cos \phi \sin \theta_1 \sin \theta_2, \quad x^4 := \cos \theta_2.
\]  

(5.12)

The volume element in spherical coordinates for \(S^3\) with volume \(2\pi\) is\(^1\)

\[
H = \frac{i}{\pi} \sin \theta_1 \sin^2 \theta_2 \, d\phi \wedge d\theta_1 \wedge d\theta_2.
\]

(5.13a)

It is well-known that this is the Dixmier–Douady class of the basic gerbe on \(S^3\). We introduce a 4-patch covering \(\sigma : Y := Y_1 \sqcup \ldots \sqcup Y_4 \to S^3\),

\[
Y_1 := \Theta \setminus \{\theta_2 = \pi\}, \quad Y_3 := \Theta \setminus (\{\theta_1 = \pi\} \setminus \{\theta_2 = 0, \pi\}), \\
Y_2 := \Theta \setminus \{\theta_2 = 0\}, \quad Y_4 := \Theta \setminus (\{\theta_1 = 0\} \setminus \{\theta_2 = 0, \pi\}).
\]

(5.13b)

Note that \(Y_1 \cup Y_2 = \Theta\) and \(Y_3 \cup Y_4 = \Theta\). Furthermore, on these patches we define the 2-form gauge potentials \(B_i \in \Omega^2(Y^{[1]}, u(1))\) for the 3-form curvature (5.13a) as follows:

\[
B_1 := \frac{i}{2\pi} (\theta_2 - \cos \theta_2 \sin \theta_2) \sin \theta_1 \, d\phi \wedge d\theta_1, \\
B_2 := \frac{i}{2\pi} (\theta_2 - \pi - \cos \theta_2 \sin \theta_2) \sin \theta_1 \, d\phi \wedge d\theta_1, \\
B_3 := \frac{i}{\pi} (\cos \theta_1 - 1) \sin^2 \theta_2 \, d\phi \wedge d\theta_2, \\
B_4 := \frac{i}{\pi} (\cos \theta_1 + 1) \sin^2 \theta_2 \, d\phi \wedge d\theta_2.
\]

(5.13c)

On the preimages of the various overlaps \((Y_i \sqcup Y_j)^{[2]}\), we have the non-trivial 1-forms \(\Lambda_{ij} \in \Omega^1(Y^{[2]}, u(1))\) given by

\[
\Lambda_{12} := \frac{i}{2} (\cos \theta_1 - 1) \, d\phi =: -\Lambda_{21}, \\
\Lambda_{13} := \frac{i}{2\pi} (\cos \theta_1 - 1)(\theta_2 - \cos \theta_2 \sin \theta_2) \, d\phi =: -\Lambda_{31}, \\
\Lambda_{14} := \frac{i}{2\pi} (\cos \theta_1 + 1)(\theta_2 - \cos \theta_2 \sin \theta_2) \, d\phi =: -\Lambda_{41}, \\
\Lambda_{23} := \frac{i}{2\pi} (\cos \theta_1 - 1)(\theta_2 - \pi - \cos \theta_2 \sin \theta_2) \, d\phi =: -\Lambda_{32}, \\
\Lambda_{24} := \frac{i}{\pi} (\cos \theta_1 + 1)(\theta_2 - \pi - \cos \theta_2 \sin \theta_2) \, d\phi =: -\Lambda_{42}, \\
\Lambda_{34} := \frac{i}{\pi} (\theta_2 - \pi - \cos \theta_2 \sin \theta_2) \, d\phi =: -\Lambda_{43}.
\]

(5.13d)

\(^1\)In our conventions, an Abelian gerbe comes with a Dixmier–Douady class, which is an element in \(H^3(S^3, \mathbb{Z})\) and whose image in de Rham cohomology is an element of \(H^3(S^3, u(1))\). We identify \(u(1) \cong \mathbb{iR}\).
patching together the $B_i$, and the $\Lambda_{ij}$ are patched together by the non-trivial functions $h_{ijk} \in \mathcal{C}^\infty(Y^{[3]}, U(1))$ given by

\begin{align*}
h_{124} &= h_{241} = h_{412} = h_{214}^{-1} = h_{421}^{-1} = e^{i\phi}, \\
h_{134} &= h_{341} = h_{413} = h_{314}^{-1} = h_{413}^{-1} = e^{i\phi}.
\end{align*}

One can easily check that the cocycle conditions of a principal 2-bundle (2.23) for the special case of structure crossed module of Lie groups $BU = (U(1) \rightarrow \ast, \text{id})$ are satisfied. We thus arrive at an explicit example of an Abelian principal 2-bundle. Recall that for these, an adjustment is not necessary, as the fake flatness is trivial.

5.2. Non-Abelian self-dual strings

Recall that a Dirac or $U(1)$-monopole is a circle bundle with connection $A$ and curvature $F$ on $\mathbb{R}^3 \setminus \{0\}$ together with a Higgs field $\phi : \mathbb{R}^3 \setminus \{0\} \rightarrow u(1)$ such that $F = \ast d\phi$. In string theory, Dirac monopoles can be obtained from D1-branes ending on D3-branes. A categorified version of this is the Abelian self-dual string, which describes M2-branes ending on an M5-brane [65]. Mathematically, it is given by an Abelian gerbe with connection $B$ and curvature $H$ on $\mathbb{R}^3 \setminus \{0\}$ together with a Higgs field $\phi : \mathbb{R}^3 \setminus \{0\} \rightarrow u(1)$ such that $H = \ast d\phi$.

The generalisation of the Dirac monopole to a non-Abelian principal bundle is straightforward and usually called the Bogomolny monopole. This is a principal $G$-bundle with connection over $\mathbb{R}^3$ and a $g$-valued Higgs field $\phi$ satisfying $F = \ast \nabla \phi$.

The generalisation of the self-dual string to the non-Abelian case is not quite as straightforward [33]. Firstly, one has to work with adjusted curvatures on non-Abelian gerbes, and secondly, one has to provide an additional equation for the curvature 2-form $F$. As shown in [33], a natural choice is $F = \ast F$, because this implies a nice dimensional reduction of the non-Abelian self-dual string equation to the Bogomolny monopole equation. Moreover, these configurations are BPS equations of an interesting six-dimensional superconformal field theory [33], see also [66].

Altogether, a non-Abelian self-dual string in the sense of [33] is a non-Abelian principal $\text{String}(G)$-bundle with adjusted connection over $\mathbb{R}^4$ together with a Higgs field $\phi : \mathbb{R}^4 \rightarrow \mathcal{L}_{0g}$ such that

\begin{equation}
H = \ast \nabla \phi , \quad b(\phi) = 0 , \quad \text{and} \quad F = \ast F ,
\end{equation}

where $F$ and $H$ are the 2- and 3-form curvatures and $b$ is the endpoint evaluation map (2.4).

\footnote{In the non-Abelian case, non-trivial solutions exist for trivial principal bundles.}
We emphasise that in [33], only infinitesimal gauge transformations were available. With the techniques developed in Section 4, we now also have finite gauge transformations under full control.

5.3. String bundle lift of the instanton–anti-instanton pair

The non-Abelian self-dual string is certainly interesting in its own right, in particular because it is, as mentioned, the higher generalisation of the non-Abelian Bogomolny monopole. Mathematically, however, it is also natural to ask for the lift of the principal Spin(4)-bundle we considered in Section 3.3 to a principal String(4)-bundle. We shall develop this lift in the following.

Cosets of string groups. Consider two string groups String(H) and String(G) of the form (4.22) for some Lie groups H and G together with an embedding H ↪ G. Such an embedding induces an embedding of crossed modules String(H) ↪ String(G), and we have resulting group homomorphisms

\[ P_0 H \hookrightarrow P_0 G , \quad P_0 L_0 H \hookrightarrow P_0 L_0 G , \quad L_0 L_0 H \hookrightarrow L_0 L_0 G . \]

The U(1)-factor of String(H) trivially embeds into the U(1)-factor of String(G) and this implies that the normal subgroup \( N_H \) arising in the description of String(H) trivially embeds into \( N_{G} \), the normal subgroup arising in the description of String(G). The quotient Lie groupoid is then

\[ \text{String}(G)/\text{String}(H) = \left( \frac{P_0 G \times ((P_0 L_0 G \times U(1))/N_G)}{P_0 H \times ((P_0 L_0 H \times U(1))/N_H)} \right) \cong P_0 G / P_0 H , \]

and because the embedding of the normal subgroups and the U(1)-factors, this reduces to

\[ \text{String}(G)/\text{String}(H) \cong \left( \frac{P_0 G \times L_0 G}{P_0 H \times L_0 H} \right) \cong P_0 G / P_0 H = L G / L H , \]

and we have the Lie groupoid identifications

\[ L G / L H \cong (P_0(G/H) \times L_0(G/H) \rightrightarrows P_0(G/H)) \]
\[ \cong (L_0(G/H) \rightrightarrows P_0(G/H)) \]
\[ \cong \mathcal{G}(P_0(G/H) \to G/H) , \]

where the last identification follows from (2.20). Following our discussion in Section 5.1, we can further identify \( L G / L H \cong G/H \) as manifolds.
**Bundle gerbe description.** For a natural construction, we now use our previous construction of cosets of string groups and make explicit the String(4)-bundle

\[
\text{String}(5) \to S^4 \cong \text{String}(5)/\text{String}(4) .
\] (5.17)

We emphasise that similar higher cosets were considered in [40], where an explicit string bundle, albeit without connection, was constructed over \( S^5 \). Furthermore, by virtue of (5.16), we have

\[
\text{String}(5)/\text{String}(4) \cong (L_0S^4 \overset{P_0}{\to} P_0S^4) ,
\] (5.18)

where we have used that \( \text{Spin}(5)/\text{Spin}(4) \cong S^4 \).

We can now arrange the above spaces into the following diagram:

\[
\begin{array}{c}
L_0\text{Spin}(5) \\
\downarrow \\
L_0\text{Spin}(5) \overset{\sim}{\longrightarrow} P_0\text{Spin}(5) \\
\downarrow \\
L_0S^4 \overset{\sim}{\longrightarrow} P_0S^4 \\
\downarrow \\
S^4
\end{array}
\] (5.19)

This diagram underlies the description of the principal String(4)-bundle as a non-Abelian bundle gerbe, cf. [19,67] as well as [17].

**Cocycle description.** We note that, just as in the case of the Abelian self-dual string (5.11), the bundle gerbe picture is not suitable for the direct extraction of a cocycle description: the bundle \( L_0\text{Spin}(5) \) is a non-trivial principal 2-bundle over \( L_0S^4 \). We therefore need to switch to a different cover. We also note that the different lifts form a torsor for \( H^3(S^4,\mathbb{Z}) \). Because this group is trivial, the lift is unique up to isomorphism. This allows us to work with the same simple cover \( \bar{Y}_1 \cup \bar{Y}_2 \to S^4 \) defined in (3.35), which proved to be sufficient in the description of the coset space \( L\text{Spin}(5)/L\text{Spin}(4) \cong \text{Spin}(5)/\text{Spin}(4) \cong S^4 \) in Section 3.3.

The cocycles for this bundle will evidently be an extension of the cocycles introduced in (3.37). Because, as mentioned above, the lift to a string bundle is unique up to isomorphism, the additional cocycle data does not contain any new information and is unique up to bundle isomorphisms. We start from the cocycle (3.37) and we have to extend this cocycle to account for the central extension. We set

\[
\hat{h}_{ijk} : \bar{Y}^{[3]} \to (P_0L_0\text{Spin}(4) \times U(1))/N ,
\]

\[
q_+ \mapsto [(s,t) \mapsto h_{ijk}^0(q_+,st), h_{ijk}^1] ,
\] (5.20a)
as well as
\[
\begin{align*}
h'_{111} & := h'_{222} := h'_{112} := h'_{122} := h'_{121} := 1, \\
\Lambda'_{11} & := \Lambda'_{22} := \Lambda'_{12} := 0 \quad \text{and} \quad B'_{1} := 0.
\end{align*}
\] (5.20b)
and
\[
\begin{align*}
h'_{212} & = g_{21} \Rightarrow \hat{h}_{121}, \\
\hat{\Lambda}_{21} & = \hat{h}_{121} \nabla \hat{h}_{212}^{-1} - g_{21}^{-1} \Rightarrow \hat{\Lambda}_{12}, \\
\hat{B}_{2} & = \hat{B}_{1} + d\hat{\Lambda}_{12} + A_{2} \Rightarrow \hat{\Lambda}_{12} + \frac{1}{2} [\hat{\Lambda}_{12}, \hat{\Lambda}_{12}] - \hat{k}(g_{12}, F_{1} - B_{1})
\end{align*}
\] (5.20d)
thus providing the missing data. The additional components arising in the lift to a string bundle read as
\[
\begin{align*}
h'_{212} & = h_{121}, \\
\hat{h}_{212} & = \exp\left(\frac{i}{2\pi} \int_{0}^{1} dt \int_{0}^{1} ds \left\langle g_{21}^{-1}(q,t), \frac{\partial g_{21}^{\circ}(q,t)}{\partial t}, h_{121}^{-1}(q,s), \frac{\partial h_{121}^{\circ}(q,s)}{\partial s} \right\rangle\right) \\
& = \exp(i\theta_{q}) = \frac{\text{Re}(q)}{|q|} + i\frac{|\text{Im}(q)|}{|q|}, \\
\Lambda'_{21} & = -\frac{i}{2\pi} \int_{0}^{1} dr \left\{ \left\langle \frac{\partial g_{12}^{\circ}}{\partial r}, g_{12}^{\circ}, A_{1}^{\circ} \right\rangle + \left\langle \frac{\partial g_{21}^{\circ}}{\partial r}, g_{21}^{\circ}, A_{2}^{\circ} \right\rangle \right. \\
& \left. \quad - \left\langle \frac{\partial g_{12}^{\circ}}{\partial r}, g_{12}^{\circ}, A_{1}^{\circ} \right\rangle \right\} \\
B'_{2} & = \frac{i}{4\pi} \left( \left\langle g_{12}^{\circ} d g_{12}^{\circ} + g_{12}^{\circ} A_{1}^{\circ} g_{12}^{\circ}, A_{2}^{\circ} \right\rangle + \left\langle g_{12}^{\circ} A_{1}^{\circ} g_{12}^{\circ}, g_{12}^{\circ} d g_{12}^{\circ} \right\rangle \right)_{r=1} \\
& \quad + \frac{i}{2\pi} \int_{0}^{1} dr \left\langle \frac{\partial g_{12}^{\circ}}{\partial r}, g_{12}^{\circ}, A_{1}^{\circ} \right\rangle \\
& \quad + \frac{i}{4\pi} \int_{0}^{1} dr \left\langle \frac{\partial A_{1}^{\circ}}{\partial r}, A_{1}^{\circ} \right\rangle - \left\langle \frac{\partial A_{2}^{\circ}}{\partial r}, A_{2}^{\circ} \right\rangle + \left\langle \frac{\partial g_{12}^{\circ} d g_{12}^{\circ}}{\partial r}, g_{12}^{\circ} d g_{12}^{\circ} \right\rangle
\end{align*}
\] (5.20e)
We note again that we expect that the formulas for $\Lambda_{21}^{\prime}$ and $B_{2}^{\prime}$ can be simplified by choosing a more suitable lift of the gauge potentials $A$ and $B$.

6. Outline of the twistor space description of self-dual strings

We now outline the twistor space description of non-Abelian self-dual strings by means of a Penrose–Ward transform.

**Penrose–Ward transform.** Many important field equations can be recast as the partial flatness of the curvature of a gauge potential when restricted to certain subspaces of spacetime $M$. The moduli space of the relevant subspaces is then called *twistor space* and
commonly denoted by $Z$. Fibred over both is the \textit{correspondence space} $F$, the (disjoint) union of relevant subspaces containing a particular space-time point over all space-time points:

$$
\begin{array}{c}
Z \\
\pi_2
\end{array}
\begin{array}{c}
F \\
\pi_1
\end{array}
\begin{array}{c}
M \\
\pi_2
\end{array}
$$

(6.1)

The projection $\pi_1$ forgets the subspaces considered, while the projection $\pi_2$ projects forgets the contained space-time point. We thus have the \textit{geometric twistor correspondence},

$$Z \ni z \leftrightarrow \pi_1(\pi_2^{-1}(z)) \subseteq M \quad \text{and} \quad Z \ni \pi_2(\pi_1^{-1}(m)) \leftrightarrow m \in M . \quad (6.2)$$

Under suitable topological conditions on the double fibration (6.1), the \textit{Penrose–Ward transform} then identifies the gauge orbits of solutions to some field equations with isomorphism classes of certain principal bundle $P$ over the corresponding twistor space $Z$. Examples include self-dual Yang–Mills theory [69, 70] and its supersymmetric extensions [71–73] as well as Yang–Mills theory [74–79] and its supersymmetric extensions [75, 80, 81]. For instance, in the context of four-dimensional self-dual (supersymmetric) Yang–Mills theory, the relevant subspaces are self-dual (super) surfaces in (super) space-time whereas for four-dimensional $\mathcal{N} = 3$ supersymmetric Yang–Mills theory, the relevant subspaces are super null-geodesics in $\mathcal{N} = 3$ super space-time, respectively.

Roughly speaking, we can pull back the bundle $P$ along $\pi_2$ and perform a trivialising isomorphism $\pi_2^* P \rightarrow \hat{P}$. This will lead to a relative connection along $\pi_2$ on the bundle $\hat{P}$ over $F$, and this connection is relatively flat. It turns out that this connection is necessarily of a form that allows for a push-down along $\pi_1$, leading to a connection on a principal bundle $\pi_1^* \hat{P}$ on space-time $M$. Relative flatness of the connection on $\hat{P}$ now implies flatness of the connection along the subspaces $\pi_1(\pi_2^{-1}(z)) \subseteq M$ for all $z \in Z$, which, in turn, implies the desired field equations on space-time.

\textbf{Generalisation to higher gauge theory.} Starting from the Abelian setting in [89, 90], this construction was generalised to higher gauge theories in [14, 91, 15, 16, 92]. These generalisations even allow for replacing the manifolds appearing in the double fibration (6.1) by categorified spaces [16].

However, what all these generalisations have in common is that the fake-flatness condition for the connection is built in. The key advantage of using adjusted string bundles, as

\footnote{see e.g. [68]}

\footnote{We note that such constructions also exist for self-dual gravity [82, 83] and its supersymmetric extensions [84–86], as well as Einstein gravity [87] and its supersymmetric extensions [88].}

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discussed in the preceding sections, over unadjusted non-Abelian gerbes is clearly the fact that the fake flatness no longer has to be imposed for the equations to be consistent. This certainly suggests more freedom compared to the previous discussions of the categorified Penrose–Ward transform. We note, however, that there is now a contradiction between the fibres of $\pi_2 : F \to Z$ being large enough to produce the full information of a 2-form $B$ on space-time, together with a constraint on $H$, and them being small enough to allow for a non-flat relative connection with a non-trivial relative 2-form curvature along $\pi_2 : F \to Z$. The latter, however, is required for our generalisation of adjusted string bundles to be noticeable. We show in the following how this issue can be circumvented.

**Twistor space diagram.** Despite the above mentioned issues, we can proceed with developing a concrete twistor picture that will allow for a step-wise construction of solutions to the non-Abelian self-dual string equation (5.14). For convenience, we consider flat space-time which we complexify, and we also work with a complexified gauge group. We then have the following fibrations of twistor space:

\[
\begin{array}{ccc}
\mathbb{C}^4 \times \mathbb{C}P^1 \times \mathbb{C}P^1 & \to & \mathbb{C}^4 \\
\pi_2 & \downarrow & \\
Z_{\text{Ins}} \times \mathbb{C}P^1 & \to & Z_{\text{Ins}} \\
\pi_3 & \downarrow & \pi_4 \\
Z_{\text{Hyp}} & \to & \mathbb{C}^4
\end{array}
\] (6.3)

Here, $Z_{\text{Ins}}$ denotes the Penrose twistor space for an instanton, which is the total space of the holomorphic vector bundle $\mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathbb{C}P^1$ and $Z_{\text{Hyp}}$ is the hyperplane twistor space introduced in [89], which is the total space holomorphic vector bundle $\mathcal{O}(1,1) \to \mathbb{C}P^1 \times \mathbb{C}P^1$. Since $\text{Spin}(4) \cong \text{SU}(2) \times \text{SU}(2)$, we can use spinorial indices $\alpha, \beta, \ldots, \dot{\alpha}, \dot{\beta}, \ldots = 1, 2$ and, we take $x^{\alpha \dot{\beta}}$ are coordinates on $\mathbb{C}^4$, $\lambda_\dot{\alpha}$ and $\mu_\alpha$ are homogeneous coordinates on the two projective spaces $\mathbb{C}P^1$, and $y$ and $z^\alpha$ and $z$ are the fibre coordinates of $\mathcal{O}(1,1) \to \mathbb{C}P^1 \times \mathbb{C}P^1$ and $\mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathbb{C}P^1$, respectively. Then, the various projections are given as follows,

\[
\begin{align*}
\pi_1 : (x^{\alpha \dot{\beta}}, \mu_\alpha, \lambda_\dot{\alpha}) &\mapsto (x^{\alpha \dot{\beta}}), \\
\pi_2 : (x^{\alpha \dot{\beta}}, \mu_\alpha, \lambda_\dot{\alpha}) &\mapsto (z^\alpha, \mu_\alpha, \lambda_\alpha) := (x^{\alpha \dot{\beta}} \lambda_\dot{\beta}; \mu_\alpha, \lambda_\dot{\alpha}), \\
\pi_3 : (z^\alpha, \mu_\alpha, \lambda_\alpha) &\mapsto (y, \mu_\alpha, \lambda_\alpha) := (z^\alpha \mu_\alpha; \mu_\alpha, \lambda_\alpha), \\
\pi_4 : (z^\alpha, \mu_\alpha, \lambda_\alpha) &\mapsto (z^\alpha, \lambda_\alpha).
\end{align*}
\] (6.4)

For more details and the conventions we use, see e.g. [72, 93] as well as [89].
Iterative Penrose–Ward transform. Consider a smoothly trivial holomorphic principal $G$-bundle $P_0$ over $\mathbb{C}P^1 \cong (\pi_4 \circ \pi_2)(\pi_1^{-1}(x)) \subseteq \mathbb{C}P^1$ for all $x \in \mathbb{C}^4$. As is well-known [69], this bundle describes a solution to the self-dual Yang–Mills equations on space-time with gauge group $G$ via the Penrose–Ward transform along the span given by $\pi_4 \circ \pi_2$ and $\pi_1$. We define the pullback

$$P_1 := \pi_4^* P_0 \ ,$$

and this bundle can be lifted to a holomorphic principal $\text{String}(G)$-bundle $\mathcal{P}_1$ as the second Chern class of $P_1$ necessarily vanishes.

Next, consider a holomorphic Abelian gerbe $\mathcal{G}$ over $\mathcal{Z}_{\text{Hyp}}$. As shown in [89], this gerbe by itself describes an Abelian self-dual string on $\mathbb{C}^4$ by means of a Penrose–Ward transform along the span given by $\pi_4 \circ \pi_3$ and $\pi_1$. We can now tensor the principal $\text{String}(G)$-bundle $\mathcal{P}_1$ by the pull-back $\pi_3^* \mathcal{G}$, which yields the holomorphic principal $\text{String}(G)$-bundle

$$\mathcal{P}_2 := \mathcal{P}_1 \otimes \pi_3^* \mathcal{G} \ .$$

Whilst the lift of $P_1$ to $\mathcal{P}_1$ is not unique, the different choices are parametrised precisely by the choices of $\mathcal{G}$. Consequently, we have to provide an additional, non-canonical choice for defining the lift. Nevertheless, we are certain to cover the solution space with any such choice, because the possible self-dual strings form a torsor for the group of the pull-backs of the possible gerbes $\mathcal{G}$. The resulting holomorphic principal string bundle $\mathcal{P}_2$ can then be taken as input data for a straightforward higher Penrose–Ward transform as in [14], and the result is a solution to the non-Abelian self-dual string equations (5.14) for $\text{String}(G)$ over $\mathbb{C}^4$. We note that a supersymmetric extension is also readily implemented, cf. [14].

Appendices

A. Central extension and cocycles

Abstract setting. A **central extension** $E$ of a group $G$ by an Abelian group $A$ is a short exact sequence of groups

$$1 \rightarrow A \rightarrow E \xrightarrow{\pi} G \rightarrow 1 \quad (A.1)$$

such that $\text{im}(\iota)$ is contained in the centre of $E$. Let now $s : G \rightarrow E$ be a section of $\pi : E \rightarrow G$, that is, $\pi \circ s = \text{id}$. Then, for all $g_1, g_2 \in G$,

$$\pi(s(g_1)s(g_2)) = \pi(s(g_1))\pi(s(g_2)) = \pi(s(g_1g_2)) \ ,$$

(A.2)
and we parametrise the failure of $s$ to be a group homomorphism by a map $c : G \times G \to A$ according to
\[ s(g_1 g_2) \cdot (c(g_1, g_2)) := s(g_1) s(g_2). \] (A.3)

It then follows that $c$ satisfies the condition
\[ c(g_1, g_2) c(g_1 g_2, g_3) = c(g_1, g_2 g_3) c(g_2, g_3) \] (A.4)
for all $g_1, g_2, g_3 \in G$. This condition, when evaluated for $(g_1, g_2, g_3) = (g, \mathbb{1}, g)$ and $(g_1, g_2, g_3) = (\mathbb{1}, \mathbb{1}, g)$ for $g \in G$, yields
\[ c(\mathbb{1}, \mathbb{1}) = c(\mathbb{1}, g) = c(g, \mathbb{1}). \] (A.5)

Furthermore, we have a bijection of sets $\phi : G \times A \to E$ defined by $\phi(g, a) := s(g) \cdot c(a)$ for all $(g, a) \in G \times A$. It then straightforwardly follows that $\phi(g_1, a_1) \phi(g_1, a_1) = s(g_1 g_2) \cdot c(a_1 a_2 c(g_1, g_2))$ and hence, we obtain a product on $G \times A$ given by
\[ (g_1, a_1)(g_2, a_2) := (g_1 g_2, a_1 a_2 c(g_1, g_2)) \] (A.6)
for all $(g_1, a_1), (g_2, a_2) \in G \times A$. This product is associative if and only if (A.4) holds.

Evidently, the product (A.6) makes $\phi$ an isomorphism of groups. Moreover, note that $(\mathbb{1}, (c(\mathbb{1}, \mathbb{1}))^{-1})$ is the neutral element with respect to (A.6), and the inverse $(g, a)^{-1}$ of an element $(g, a) \in G \times A$ is $(g, a)^{-1} = (g^{-1}, (ac(\mathbb{1}, \mathbb{1})c(g, g^{-1}))^{-1})$.

If now $\tilde{s}$ is another section of $E$, from $\pi \circ \tilde{s} = \text{id} = \pi \circ s$ it then immediately follows that $\tilde{s}(g) = s(g) \cdot c(d(g))$ for all $g \in G$ and where $d : G \to A$. In turn, this yields
\[ c(g_1, g_2) = c(g_1, g_2)(d(g_1 g_2))^{-1} d(g_1) d(g_2) \] (A.7)
for all $g_1, g_2 \in G$. Using (A.7), (A.5) then implies that without loss of generality we can always normalise $c$ so that
\[ c(\mathbb{1}, \mathbb{1}) = c(\mathbb{1}, g) = c(g, \mathbb{1}) = \mathbb{1}. \] (A.8)

Consider now the nerve of the groupoid $BG$. That is, let $G^p := G \times \cdots \times G$ ($p$-copies) with $G^0 := \mathbb{1}$ the trivial group. We define (face) maps $m_i : G^{p+1} \to G^p$ for $i = 1, \ldots, p + 1$ by
\[
m_i(g_1, \ldots, g_{p+1}) := \begin{cases} 
(g_2, \ldots, g_{p+1}) & \text{for } i = 1 \\
(g_1, \ldots, g_{i-1}, g_i, g_{i+1}, \ldots, g_{p+1}) & \text{for } i = 2, \ldots, p \\
(g_1, \ldots, g_p) & \text{for } i = p + 1
\end{cases}
\] (A.9)
for all \( g_1, \ldots, g_{p+1} \in G \). This allows us to introduce the map \( \delta : \text{Map}(G^p, A) \to \text{Map}(G^{p+1}, A) \) by

\[
\delta(c)(g_1, \ldots, g_{p+1}) := \prod_{i=1}^{p+1} (c \circ m_{2i-1})(g_1, \ldots, g_{p+1})((c \circ m_{2i})(g_1, \ldots, g_{p+1}))^{-1} \quad (A.10)
\]

for all \( g_1, \ldots, g_{p+1} \in G \), and it is easy to see that this is a differential yielding the complex

\[
\begin{array}{cc}
\text{Map}(G^0, A) & \xrightarrow{\delta} \text{Map}(G^1, A) & \xrightarrow{\delta} \text{Map}(G^2, A) & \xrightarrow{\delta} \cdots .
\end{array} \quad (A.11)
\]

Furthermore, for \( a \in \text{Map}(G^2, A) \), we have that \((A.4)\) is equivalent to \( \delta(c) = 1 \) and for \( m \in \text{Map}(G^1, A) \), the condition \((A.7)\) can be written as \( \tilde{c}(g_1, g_2) = c(g_1, g_2)\delta(d)(g_1, g_2) \) for all \( g_1, g_2 \in G \). In summary, central extensions are classified by the second cohomology group \( H^2(G, A) \) associated with the complex \((A.11)\). In this context, we shall refer to \((A.4)\) as a group cocycle condition and to \((A.7)\) as a group coboundary condition, respectively. Likewise, the first cohomology group \( H^1(G, A) \) can be interpreted in terms of the automorphisms of the central extension. Indeed, we say that two automorphisms of the central extension \((A.1)\) are equivalent whenever they differ by the conjugation automorphism induced by an element of \( A \). Then, it can be shown that \( H^1(G, A) \) acts transitively on the set of all automorphisms of the central extension \((A.1)\) modulo this equivalence. See e.g. [94] and in particular [58, Section 4] for more details.

**Remark A.1.** We shall always assume that the cocycles in \( H^2(G, A) \) are normalised such that \((A.8)\) holds. This means that the neutral element with respect to the product \((A.6)\) is \((1, 1)\) and the inverse of an element \((g, a) \in G \times A\) is given by \((g, a)^{-1} = (g^{-1}, (ac(g, g^{-1}))^{-1})\). Note that the cocycle condition \((A.4)\) then also implies that

\[
(g_1, a_1)^{-1}(g_2, a_2)(g_1, a_1) = (g_1^{-1}g_2g_1, a_2c(g_1^{-1}g_2, g_1)(c(g_1, g^{-1}_1g_2))^{-1}) \quad (A.12)
\]

for all \((g_1, a_1), (g_2, a_2) \in G \times A\).

**Fréchet–Lie groups.** So far, we have not specified the type of groups for which we wish to discuss central extensions. As we are mainly concerned with Lie groups, not necessarily finite-dimensional, let now \( G \) be a Fréchet–Lie group and \( A \) an Abelian Fréchet–Lie group. We are interested in central extensions \((A.1)\) for which \( E \) carries a Fréchet–Lie group structure with respect to the product \((A.6)\) and \( \pi : E \to G \) is a principal \( A \)-bundle. If, in addition, \( G \) is also connected, then one can show that this is the case if and only if the cocycle \( c \in H^2(G, A) \) is smooth in a neighbourhood of \((1, 1) \in G \times G\). See [95, Prop. 3.11] for the finite-dimensional case and [96] for the infinite-dimensional case.
Principal A-bundle. A central extension of Fréchet–Lie groups (A.1) gives rise to a principal A-bundle. We shall develop the relation between the group cocycle and the corresponding Čech cocycle \( t \in H^2(G, \mathbb{Z}) \) in the following.\(^1\) Let \( s : G \to E \) be a section that is smooth in an open neighbourhood \( U \) of \( 1 \in G \). Then, the associated cocycle characterising the extension is given by (A.3), and it is smooth in the open neighbourhood \( U \times U \) of \((1,1) \in G \times G\). We can construct an open cover \( \{U_g\}_{g \in G} \) of \( G \) from the patches \( U_g := L_g(U) \), where \( L \) denotes left-multiplication. On each patch \( U_g \), we consider the section \( \sigma_g : U_g \to \pi^{-1}(U_g) \) of \( \pi : E \to G \) defined by

\[
\sigma_g := \ L_{s(g)} \circ s \circ L_{g^{-1}} . \tag{A.13}
\]

Explicitly, \( \sigma_g(h) = s(g)(s(g^{-1}h)) = s(h)c(c(g,g^{-1}h)) \) for all \( h \in U_g \). From \( s(g)(s(g^{-1}h)) \) and the smoothness of \( s \) on \( U \) it follows that \( \sigma_g \) is smooth on \( U_g \). On non-empty intersections of patches we introduce smooth maps \( t_{g_1g_2} : U_{g_1} \cap U_{g_2} \to A \) by

\[
\sigma_{g_1}(h)(\sigma_{g_2}(h))^{-1} = \iota(c(g_1, g_1^{-1}h)(c(g_2, g_2^{-1}h))^{-1}) =: \iota(t_{g_1g_2}(h)) \tag{A.14}
\]

for all \( h \in U_{g_1} \cap U_{g_2} \).\(^2\) Evidently, for all \( g_1, g_2, g_3 \in G \) with \( U_{g_1} \cap U_{g_2} \cap U_{g_3} \neq \emptyset \) we obtain

\[
t_{g_1g_2}(h)t_{g_2g_3}(h) = t_{g_1g_3}(h) \tag{A.15}
\]

for all \( h \in U_{g_1} \cap U_{g_2} \cap U_{g_3} \), that is, the maps \( t_{g_1g_2} : U_{g_1} \cap U_{g_2} \to A \) satisfy a Čech cocycle condition. Similarly, we introduce smooth maps \( t_g : U_g \to A \) given by \( \iota(t_g(h)) := d(g)d(g^{-1}h) \) for all \( h \in U_g \), and the group coboundary transformation (A.7) yield

\[
\tilde{t}_{g_1g_2}(h) = t_{g_1}(h)t_{g_1g_2}(h)(t_{g_2}(h))^{-1} , \tag{A.16}
\]

that is, Čech coboundary transformation. Altogether, the central extension is characterised by an element of the Čech cohomology group \( \check{H}^1(\{U_g\}_{g \in G}, \mathbb{A}) \) and, upon taking the direct limit, of \( \check{H}^1(G, \mathbb{A}) \cong \check{H}^2(G, \mathbb{Z}) \). It is important to realise, however, that the map \( H^1(G, \mathbb{A}) \to \check{H}^2(G, \mathbb{Z}) \) thus constructed is not a bijection but rather a many-to-one map. Indeed, the Čech cohomology only knows about the bundle structure of \( E \to G \) but not about the group structure. In fact, one can have non-trivial central extensions which admit global smooth sections but which are not morphisms of groups.

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\(^1\)We thank Paul Skerritt for discussions on these constructions.

\(^2\)Note that \( U_g \ni h \mapsto c(g, g^{-1}h) \in A \) is not smooth since \( c \) is only smooth on a neighbourhood of \((1,1) \in G \times G\).
B. Useful relations

In the following, we list a number of relations that we found useful. Let \( (H \xrightarrow{\cdot} G, \triangleright) \) and \((h \xrightarrow{\cdot} g, \triangleright)\) be a crossed module of Lie groups and the corresponding crossed module of Lie algebras, respectively. For simplicity, we assume that \( G \) and \( H \) are matrix groups.

Furthermore, consider the Lie group- and Lie algebra-valued functions and forms \( g, g_i \in \mathcal{C}^{\infty}(U, G) \), \( h, h_i \in \mathcal{C}^{\infty}(U, H) \), \( A, A_i \in \Omega^1(U, g) \), \( B \in \Omega^2(U, h) \), and \( \Lambda \in \Omega^1(U, h) \) on some contractible manifold \( U \). Then we have the following useful implications:

\[
(h^{-1}(g \triangleright h))^{-1} = (g \triangleright h^{-1})h \quad \Rightarrow \quad d(h_1 h_2 h_1^{-1}) = h_1 [h_1^{-1} dh_1, h_2] h_1^{-1} + h_1 dh_2 h_1^{-1} ,
\]

\[(B.1a)\]

\[
t(h^{-1}(g \triangleright h)) = h^{-1} ghg^{-1} \quad \Rightarrow \quad t(h^{-1}(A \triangleright h)) = t(h^{-1}) At(h) - A ,
\]

\[(B.1b)\]

\[
g_1 \triangleright (g_2 \triangleright h) = g_2 \triangleright ((g_2^{-1} g_1 g_2) \triangleright h) \quad \Rightarrow \quad A \triangleright (g \triangleright h) = g \triangleright ((g^{-1} A g) \triangleright h) ,
\]

\[(B.1c)\]

\[
(g_1 \triangleright g_2) \triangleright h = (g_1 g_2 g_1^{-1}) \triangleright h \quad \Rightarrow \quad (g \triangleright A) \triangleright \Lambda = g \triangleright (A \triangleright (g^{-1} \triangleright \Lambda)) ,
\]

\[(B.1d)\]

\[
t(h^{-1}(g \triangleright h)) \triangleright \Lambda = (t(h^{-1}) gt(h) g^{-1}) \triangleright \Lambda
\]

\[
\quad \Rightarrow \quad t(h^{-1}) \triangleright A \triangleright t(h) \triangleright \Lambda - A \triangleright \Lambda = [h^{-1}(A \triangleright h), \Lambda] ,
\]

\[(B.1e)\]

\[
g_1 \triangleright (h(g_2 \triangleright h^{-1})) = (g_1 \triangleright h) h^{-1} h((g_1 g_2 g_1^{-1}) \triangleright h^{-1})(g_1 g_2 g_1^{-1}) \triangleright (h(g_1 \triangleright h^{-1}))
\]

\[
\quad \Rightarrow \quad A_1 \triangleright (h(A_2 \triangleright h^{-1}))
\]

\[
\quad = -A_2 \triangleright (h(A_1 \triangleright h^{-1})) + h([A_1, A_2] \triangleright h^{-1}) - [h(A_1 \triangleright h^{-1}), h(A_2 \triangleright h^{-1})] .
\]

\[(B.1f)\]

We also note that

\[
t(h_1) \triangleright (h_2 (A \triangleright h_2^{-1})) = h_1 h_2 (A \triangleright (h_1 h_2)^{-1}) - h_1 (A \triangleright h_1^{-1}) ,
\]

\[
h(t(B) \triangleright h^{-1}) = t(h) \triangleright B - B ,
\]

\[
d(g \triangleright h) = g \triangleright ((g^{-1} dg) \triangleright h) + g \triangleright dh ,
\]

\[(B.2)\]

and the last relation implies

\[
d(A \triangleright h) = (dA) \triangleright h - A \triangleright dh ,
\]

\[
d(g \triangleright \Lambda) = g \triangleright ((g^{-1} dg) \triangleright \Lambda) + g \triangleright d\Lambda ,
\]

\[(B.3)\]

\[
d(h(A \triangleright h^{-1})) = -[h(A \triangleright h^{-1}), hdh^{-1}] - A \triangleright (hdh^{-1}) + h(dA \triangleright h^{-1}) .
\]
Finally, the adjustment datum \( \kappa \) for the Lie 2-group \( L_G \) given in (2.39b) satisfies
\[
\begin{align*}
\kappa(g, A) &= \kappa(g, dA) + \kappa(g^{-1}dg, g^{-1}Ag) , \\
\kappa(g_1g_2, F) &= \kappa(g_2, g_1^{-1}Fg_1) + \kappa(g_1, F) , \\
\kappa(g, [A, A]) &= \kappa(A, A) - \kappa(g^{-1}Ag, g^{-1}Ag) . 
\end{align*}
\] (B.4)

C. Proofs for Section 4

**Different group cocycles.** The starting point of our discussion around (4.13) is the group cocycle\(^1\)
\[
c(f_1, f_2) = \exp\left( -\frac{i}{2\pi} \int_0^1 dr \int_0^1 ds \int_0^s dt \left\langle f_1^{-1} \frac{\partial f_1}{\partial s}, f_2 \left\{ \frac{\partial}{\partial r} \left( f_2^{-1} \frac{\partial f_2}{\partial t} \right) \right\} f_2^{-1} \right\rangle \right) \quad (C.1)
\]
given by Murray [59] for all \( f_1, f_2 \in P_0L_0G \) parametrised as \( f_1 = f_1(s, r) \) and \( f_2 = f_2(t, r) \) with \( s, t \) the path parameters and \( r \) the loop parameter. Furthermore, with the help of the identity (4.14), Murray’s group cocycle becomes
\[
c(f_1, f_2) = \exp\left( -\frac{i}{2\pi} \int_0^1 dr \int_0^1 ds \left\langle f_1^{-1}(s, r) \frac{\partial f_1(s, r)}{\partial s}, \frac{\partial f_2(s, r)}{\partial r} f_2^{-1}(s, r) \right\rangle \right) , \quad (C.2)
\]
and this is the form of the group cocycle given in Baez–Stevenson–Crans–Schreiber [30].

Next, Mickelsson provides another group cocycle given by [60]
\[
\tilde{c}(f_1, f_2) := \exp\left( -\frac{i}{4\pi} \int_0^1 dr \int_0^1 ds \left\{ \left\langle f_1^{-1}(s, r) \frac{\partial f_1(s, r)}{\partial s}, \frac{\partial f_2(s, r)}{\partial r} f_2^{-1}(s, r) \right\rangle \right. \right.
\]
\[
\left. \left. - \left\langle f_1^{-1}(s, r) \frac{\partial f_1(s, r)}{\partial r}, \frac{\partial f_2(s, r)}{\partial s} f_2^{-1}(s, r) \right\rangle \right\} \right) . \quad (C.3)
\]
The group cocycles \( c \) and \( \tilde{c} \) are related by a coboundary transformation
\[
\tilde{c}(f_1, f_2) = c(f_1, f_2)(d(f_1f_2))^{-1}d(f_1)d(f_2) , \quad (C.4)
\]
cf. (A.7), with
\[
d : P_0L_0G \to U(1) , \quad f \mapsto \exp\left( -\frac{i}{4\pi} \int_0^1 dr \int_0^1 ds \left\langle f^{-1}(s, r) \frac{\partial f(s, r)}{\partial s}, f^{-1}(s, r) \frac{\partial f(s, r)}{\partial r} \right\rangle \right) . \quad (C.5)
\]
This answers the question raised in [59] as to what the relation between \( c \) and \( \tilde{c} \) is.

\(^1\)Note that in the following, we fix some sign errors appearing in the original literature.
A formula for \((\text{Ad}_g)^*\omega\). Consider the 2-form curvature \(\omega\) defined in (4.11) and the left-invariant 1-form \(\xi_g\) for fixed \(g \in G\) defined in (4.24). Then

\[
(\text{Ad}_g)^*\omega = \omega + d\xi_g,
\]

(cf. [57, Equation (4.6.6)]). In order to prove this relation, it is sufficient to establish this at the identity \(1 \in L_0G\) due to left-invariance. For any two Lie algebra elements \(X, Y \in L_0g\) we have

\[
\omega_1(\text{Ad}_gX, \text{Ad}_gY) = \frac{i}{2\pi} \int_0^1 dr \left\langle \text{Ad}_{g(r)}X(r), \frac{\partial}{\partial r} \text{Ad}_{g(r)}Y(r) \right\rangle = \frac{i}{2\pi} \int_0^1 dr \left\langle \text{Ad}_{g(r)}X(r), \text{Ad}_{g(r)} \left( \frac{\partial Y(r)}{\partial r} + \left[ g^{-1}(r) \frac{\partial g(r)}{\partial r}, Y(r) \right] \right) \right\rangle.
\]

(C.7)

Using the \(\text{Ad}\)-invariance of the inner product and Cartan’s formula \((d\xi_g|_1)(X, Y) = -\xi_g|_1([X, Y])\), we see that

\[
\omega_1(\text{Ad}_gX, \text{Ad}_gY) = \frac{i}{2\pi} \int_0^1 dr \left\langle X(r), \frac{\partial Y(r)}{\partial r} \right\rangle - \frac{i}{2\pi} \int_0^1 dr \left\langle [X(r), Y(r)], g^{-1}(r) \frac{\partial g(r)}{\partial r} \right\rangle = \omega_1(X, Y) + (d\xi_g|_1)(X, Y).
\]

(C.8)

This verifies the desired formula.

Peiffer identity. Next, we wish to show that \((t(\hat{h}_1) \triangleright \hat{h}_2)\hat{h}_1\hat{h}_2^{-1}\hat{h}_1^{-1} \in N\) for all \(\hat{h}_1, \hat{h}_2 \in P_0L_0G \times U(1)\). Note that for all \(\hat{n}_1, \hat{n}_2 \in N\), we have

\[
(t(\hat{h}_1) \triangleright \hat{h}_2)\hat{h}_1\hat{h}_2^{-1}\hat{h}_1^{-1} = (t(\hat{h}_1) \triangleright \hat{h}_2)(t(\hat{h}_1) \triangleright \hat{n}_2)\hat{h}_1\hat{n}\hat{h}_2^{-1}\hat{n}_2^{-1}\hat{h}_1^{-1} = (t(\hat{h}_1) \triangleright \hat{h}_2)\hat{h}_1\hat{h}_2^{-1}\hat{h}_1^{-1} \times \hat{h}_1\hat{h}_2\hat{h}_1^{-1}\hat{h}_2^{-1}\hat{h}_1^{-1} \times (t(\hat{h}_1) \triangleright \hat{n}_2)\hat{h}_1\hat{n}_2^{-1}\hat{n}_1\hat{h}_1^{-1} \times \hat{h}_1\hat{n}_2\hat{h}_1^{-1}\hat{n}_2^{-1}\hat{n}_1\hat{h}_1^{-1}
\]

(C.9)

where we have used (4.23), the normality of \(N\), and the fact that the action closes on \(N\). Consequently, \((t(\hat{h}_1) \triangleright \hat{h}_2)\hat{h}_1\hat{h}_2^{-1}\hat{h}_1^{-1} \in N\) implies the Peiffer identity \(t([\hat{h}_1]) \triangleright [\hat{h}_2] = [\hat{h}_1][\hat{h}_2][\hat{h}_1]^{-1}\) for all \([\hat{h}_1], [\hat{h}_2] \in (P_0L_0G \times U(1))/N\) for the morphism (4.23) and the action (4.26).
To verify that indeed \((t,\hat{h}_1) \Rightarrow \hat{h}_2)\hat{h}_1\hat{h}_2^{-1}\hat{h}_1^{-1} \in \mathbf{N}\), let us set \(\hat{h}_{1,2} = (f_{1,2}, z_{1,2})\). Then, using (A.12) together with (4.23) and (4.26), we find

\[
\tilde{f} := b(f_1)f_2b(f_1^{-1})f_1f_2^{-1}f_1^{-1} \quad \text{(C.10a)}
\]

for the \(P_0L_0\)-component of \((t,\hat{h}_1) \Rightarrow \hat{h}_2)\hat{h}_1\hat{h}_2^{-1}\hat{h}_1^{-1}\). Evidently, \(b(f) = 1 \in P_0L_0\) and so \(f \in L_0L_0\). Likewise, again using (A.12) together with (4.23) and (4.26), the \(U(1)\)-component of \((t,\hat{h}_1) \Rightarrow \hat{h}_2)\hat{h}_1\hat{h}_2^{-1}\hat{h}_1^{-1}\) is

\[
z := c(b(f_1)f_2b(f_1^{-1}), f_1f_2^{-1}f_1^{-1})e^{-1}(f_1f_2, f_1^{-1})
\times c(f_1^{-1}, f_1f_2c^{-1}(f_1f_2f_1^{-1}, f_1f_2^{-1}f_1^{-1}) \exp \left( \int_0^1 dt \xi_0(f_1) \left( f_2^{-1}\frac{\partial f_2}{\partial \ell} \right) \right) \quad \text{(C.10b)}
\]

with \(\xi_g\) defined in (4.24). Therefore, it remains to show

\[
\text{hol}^{-1}(f) = z. \quad \text{(C.11)}
\]

To this end, recall that the cocycle is given as an integral over a disk with a particular boundary (4.12). In the case of \(c(b(f_1)f_2b(f_1^{-1}), f_1f_2^{-1}f_1^{-1})\), this boundary is, in fact, formed by two loops joined at the identity. The integral over one of the two disks produces exactly \(\text{hol}^{-1}(f)\). We can combine the second integral with the remaining cocycle terms in (C.10b) by matching the boundaries and using (4.16). Because \(\omega\) defined in (4.11) is closed, we now use Stokes' theorem, and we are left with an integral along a square \(\square\) whose sides are given by

\[
\left( b(f_1)f_1^{-1}, b(f_1)f_2, b(f_1f_2f_1^{-1}), b(f_1)f_2b(f_1^{-1}) \right) \quad \text{(C.12)}
\]

with the orientation from right to left. We can then parametrise this square as

\[
(s, t) \mapsto (b(f_1))r f_2(s, r) f_1^{-1}(1-t, r) \quad \text{for all } s, t \in [0, 1], \quad \text{(C.13)}
\]

where \(r\) is the loop parameter, to obtain

\[
\int_{\square} \omega = \frac{i}{4\pi} \int_0^1 dr \int_0^1 ds \left\langle f_2^{-1}(s, r) \frac{\partial f_2(s, r)}{\partial s}, b(f_1^{-1}(s, r))(\frac{\partial f_1(s, r)}{\partial r}) \right\rangle. \quad \text{(C.14)}
\]

Here, we have used the left-invariance of \(\omega\) to remove \(b(f_1)\) in the square parametrisation and (4.14) to perform the \(t\) integral. Upon exponentiating this expression, we see that this term precisely cancels the last factor in (C.10b).

**Infinitesimal holonomy.** Let \(\gamma \in L_0L_0\). For \(\exp(t\gamma) \in L_0L_0\) with \(t\) sufficiently small, we can write the holonomy as

\[
\text{hol}(\exp(t\gamma)) = \exp \left( -\int_{D_{\exp(t\gamma)}} \omega \right) = \exp \left( -\int_{\hat{D}_{t\gamma}} \hat{\omega} \right), \quad \text{(C.15)}
\]

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where \( \tilde{D}_{t\gamma} \) is a disk in \( L_0\mathfrak{g} \) with boundary \( t\gamma \) such that \( \exp(\tilde{D}_{t\gamma}) = D_{\exp t\gamma} \), and

\[
\hat{\omega} := \exp^* \omega = \frac{i}{4\pi} \int_0^1 dr \left\langle \theta X(\cdot), \frac{\partial \theta X(\cdot)}{\partial r} \right\rangle \in \Omega^2(L_0\mathfrak{g}) ,
\]

(C.16)

where

\[
\theta_X := \frac{1 - e^{-\text{ad}_X}}{\text{ad}_X} dX
\]

(C.17)

and \( X \) the identity function on \( L_0\mathfrak{g} \). If we now expand (C.15) in \( t \) to the first non-vanishing order, we find that

\[
\text{hol}(\exp(t\gamma)) = 1 - \frac{i}{4\pi} \int_{\tilde{D}_{t\gamma}} \int_0^1 dr \left\langle X(r), \frac{\partial X(r)}{\partial r} \right\rangle + \mathcal{O}(t^3)
\]

\[
= 1 + \frac{i}{4\pi} \int_{t\gamma} \int_0^1 dr \left\langle dX(r), \frac{\partial X(r)}{\partial r} \right\rangle + \mathcal{O}(t^3)
\]

\[
= 1 + \frac{it^2}{4\pi} \int_0^1 ds \int_0^1 dr \left\langle \frac{\partial \gamma(s,r)}{\partial s}, \frac{\partial \gamma(s,r)}{\partial r} \right\rangle + \mathcal{O}(t^3) .
\]

(C.18)

In the second step, we have used Stokes' theorem and then parametrised the path in \( L_0\mathfrak{g} \) as \( s \mapsto t\gamma(s) \). As a corollary, we immediately obtain

\[
\frac{d}{dt} \bigg|_{t=0} \text{hol}(\exp(t\gamma)) = 0 ,
\]

(C.19)

which is (4.28).

Finally, we derive a useful formula from the corollary. Consider a path \( \ell \) in \( L_0L_0G \) parametrised by \( t \) and beginning at \( \ell_0 \). Then we have

\[
\text{hol}(\ell_0) \frac{d}{dt} \bigg|_{t=0} \text{hol}^{-1}(\ell(t))
\]

\[
= -\frac{i}{2\pi} \int_0^1 ds \int_0^1 dr \left( \frac{\partial}{\partial s} \left( \frac{d}{dt} \bigg|_{t=0} \ell(t,s,r) \ell_0^{-1}(s,r) \right) , \frac{\partial \ell_0(s,r)}{\partial r} \ell_0^{-1}(s,r) \right)
\]

\[
= -\frac{i}{2\pi} \int_0^1 ds \int_0^1 dr \left( \ell_0^{-1}(s,r) \frac{\partial \ell_0(s,r)}{\partial s} , \frac{\partial}{\partial r} \left( \ell_0^{-1}(s,r) \frac{d}{dt} \bigg|_{t=0} \ell(t,s,r) \right) \right) .
\]

(C.20)

This formula follows from

\[
\frac{d}{dt} \bigg|_{t=0} (\text{hol}(\ell_0) \text{hol}^{-1}(\ell(t)) c^{-1}(\ell_0, \ell_0^{-1}) c(\ell_0^{-1}, \ell(t))) = \frac{d}{dt} \bigg|_{t=0} \text{hol}^{-1}(\ell_0^{-1} \ell(t)) = 0 ,
\]

(C.21)

where the first equality is obtained using the fact \( (\ell_0, \text{hol}^{-1}(\ell_0))^{-1} \cdot (\ell(t), \text{hol}^{-1}(\ell(t))) \in \mathbb{N} \), and the last equality is just (C.19).
Connection and curvature on $\widehat{L_0G}$. We start by giving a formula for the connection 1-form on $P_0L_0G \times U(1)$ for the 2-form curvature $\omega$ as given in (4.11). As $\omega \in \Omega^2(L_0G, u(1))$ is closed and $P_0L_0G$ is contractible, we can use the contracting homotopy $\Phi_t : P_0L_0G \to P_0L_0G$ for all $t \in [0, \infty)$ given by $(\Phi_t f)(s, r) := f(e^{-t}s, r)$ for all $f \in P_0L_0G$. Then, we set

$$\hat{\mu} := \mu' - \int_0^{\infty} dt \Phi^* \xi (\pi_1^* b^* \omega) \in \Omega^1(P_0L_0G \times U(1), u(1)),$$

(C.22)

where $\mu'$ is the pull-back to $P_0L_0G \times U(1)$ of the (imaginary) left-invariant Maurer–Cartan form on $U(1)$, $\xi$ is the vector field of the flow $\Phi_t$, $\pi_1$ is the contraction onto the first component from $P_0L_0G \times U(1)$, and $b$ is the endpoint evaluation map $P_0L_0G \to L_0G$. Explicitly, for $(f, z) \in P_0L_0G \times U(1)$ we obtain

$$\hat{\mu}_{(f, z)} = z^{-1} dz + \frac{i}{2\pi} \int_0^1 ds \left[ f^{-1}(s, r) \frac{\partial f(s, r)}{\partial s}, f^{-1}(s, r)df(s, r) \right].$$

(C.23)

Using the identities

$$d \left( f^{-1}(s, r) \frac{\partial f(s, r)}{\partial s} \right) = \frac{\partial}{\partial s} (f^{-1}(s, r) df(s, r)) + \left[ f^{-1}(s, r) \frac{\partial f(s, r)}{\partial s}, f^{-1}(s, r) df(s, r) \right],$$

(C.24a)

$$d \left< f^{-1}(s, r) \frac{\partial f(s, r)}{\partial s}, \frac{\partial}{\partial r} (f^{-1}(s, r) df(s, r)) \right> = \left< \frac{\partial}{\partial s} (f^{-1}(s, r) df(s, r)), \frac{\partial}{\partial r} (f^{-1}(s, r) df(s, r)) \right>,$$

(C.24b)

and

$$\int_0^1 ds \int_0^1 dr \left< \frac{\partial}{\partial s} (f^{-1}(s, r) df(s, r)), \frac{\partial}{\partial r} (f^{-1}(s, r) df(s, r)) \right> = \frac{1}{2} \int_0^1 dr \left< f^{-1}(1, r) df(1, r), \frac{\partial}{\partial r} (f^{-1}(1, r) df(1, r)) \right>,$$

(C.24c)

we then find that

$$d\hat{\mu} = \pi_1^* b^* \omega.$$

(C.25)

Finally, we wish to verify that $\hat{\mu}$ descends to the quotient space $(P_0L_0G \times U(1))/N \cong \widehat{L_0G}$. This requires checking that it does not depend on elements in the subgroup $N$. Indeed, for all $\hat{h} \in P_0L_0G \times U(1)$ and for all $\hat{n} \in N$, it follows that

$$R^*_{\hat{n}} \hat{\mu}_{\hat{h}} = \hat{\mu}_{\hat{n}} = L^*_{\hat{n}} \hat{\mu}_{\hat{h}},$$

where $R_{\hat{n}}$ and $L_{\hat{n}}$ are the right and left multiplications by $\hat{n}$, respectively. To see this, one uses formula (C.20). Alternatively and more explicitly, since the Lie derivative generates
pullbacks\textsuperscript{1} it is enough to check that

\[ \mathcal{L}_{X^R, L} \hat{\mu} = 0, \]  

(C.27)

where \( X^R \) and \( X^L \) are the right and left fundamental vector fields corresponding to the action by right and left multiplications by elements in \( N \), respectively. Using Cartan’s formula \( \mathcal{L}_X = d\iota_X + \iota_X d \) and the fact that \( \iota_X d \hat{\mu} = \iota_X \pi^*_B \omega = 0 \), which follows from the verticality of \( X^{R, L} \) with respect to \( \mathfrak{b} \), we only need to check that

\[ \iota_X \hat{\mu} = 0. \]  

(C.28)

For this, consider a path with \( L(t) = (\ell(t), \text{hol}^{-1}(\ell(t))) \in N \) which begins at the identity \( L(0) = (1_{P_0L_0G}, 1_{U(1)}) \). This implies that \( \dot{\ell}(0) := \frac{d}{dt}|_{0} \ell(t) \in L_0 \mathfrak{g} \). At a point \( F = (f, z) \in P_0L_0G \), the fundamental vector field corresponding to right multiplication by \( \mathfrak{n} \) has the explicit form

\[ X^R_F = \left. \frac{d}{dt} \right|_0 F \cdot L(t) = \left( f \dot{\ell}(0), z \frac{d}{dt}|_0 c(f, \ell(t)) \right), \]  

(C.29)

where we have used (C.19) in the second equality. Contracting the vector field with (C.23), we find

\[ \hat{\mu}_F(X^R_F) = \left. \frac{d}{dt} \right|_0 c(f, \ell(t)) + \frac{i}{2\pi} \int_0^1 ds \int_0^1 dr \left\langle f^{-1}(s, r), \frac{\partial f(s, r)}{\partial s}, \frac{\partial \ell(0)}{\partial r} \right\rangle = 0, \]  

(C.30)

which follows from the explicit expression for the cocycle in (4.15) and identity (4.14). We can establish (C.28) for \( X^L \) analogously, or just argue by normality of \( N \).

Notice that (C.30) (and thus (C.28)), which is the necessary condition for \( \hat{\mu} \) to descend to \( \hat{L}_0 \mathfrak{g} \), relates the expression for the cocycle to that of the connection and through (C.25) to that of the curvature \( \omega \). In particular, the correct signs of individual terms in (C.23) are crucial for consistency.

**Maurer-Cartan form on \( \hat{L}_0 \mathfrak{g} \).** Next, we construct the (left and right) Maurer–Cartan form on \( \hat{L}_0 \mathfrak{g} \). We start with the Maurer–Cartan form on \( P_0L_0G \times U(1) \). Consider a path \( \hat{h}(t) = (f(t), z(t)) \) in \( P_0L_0G \times U(1) \) for all \( t \in [0, 1] \) such that \( \frac{d}{dt}|_0 \hat{h}(t) \in T_{\hat{h}(0)}(P_0L_0G \times U(1)) \) is its tangent vector at \( \hat{h}(0) \). The left-invariant Maurer–Cartan form \( \theta^L \) is then given by its action on \( \frac{d}{dt}|_0 \hat{h}(t) \),

\[ \theta^L_{\hat{h}(0)} \left( \frac{d}{dt}|_0 \hat{h}(t) \right) := \hat{h}^{-1}(0) \frac{d}{dt}|_0 \hat{h}(t), \]  

(C.31)

\textsuperscript{1}Recall that \( \Phi^*_t \alpha = e^{t \mathcal{L}_X} \alpha \) for \( \Phi_t \) a flow generated by some vector field \( X \) and a form \( \alpha \).
so that
\[ \hat{\theta}^L_{(f,z)} = (\theta^L_f, \hat{\mu}_{(f,z)}) \] (C.32)
for all \((f, z) \in P_0L_0G \times U(1)\) and where \(\hat{\mu}\) was given in \((C.23)\). Likewise, one can compute the right-invariant (imaginary) Maurer–Cartan form
\[ \hat{\theta}^R_{(f,z)} = \left( \theta^R_f, \hat{\mu}_{(f,z)} + \frac{i}{2\pi} \int_0^1 dr \left\langle f^{-1}(1, r) \frac{\partial f(1, r)}{\partial r}, \theta^L_{f(1, r)} \right\rangle \right), \] (C.33)
where \(\theta^R\) is the right-invariant Maurer–Cartan form on \(P_0L_0G\). Due to the identity \((C.25)\), it is easy to verify the Maurer–Cartan equations
\[ d\hat{\theta}^L,R = \frac{1}{2} [\hat{\theta}^L,R, \hat{\theta}^L,R]. \] (C.34)
As show previously, \(\hat{\mu}\) descends to \(\widehat{L_0G}\). This, in turn, implies that also \(\hat{\theta}^L\) and \(\hat{\theta}^R\) descend to \(\widehat{L_0G}\).

A formula for \(\hat{h}\nabla \hat{h}^{-1}\). For working with the cocycle relations, we also have to make sense of the expression
\[ \hat{h}\nabla \hat{h}^{-1} = \hat{h}d\hat{h}^{-1} + \hat{h}A \Rightarrow \hat{h}^{-1}, \] (C.35)
where \(\hat{h} = (f, z) \in P_0L_0G \times U(1)\) and \(A\) is a 1-form valued in \(P_0\mathfrak{g}\). We have already identified the first part as \(\hat{h}d\hat{h}^{-1} = -\hat{\theta}^R_h\) in the previous paragraph. The second part is obtained by differentiating the expression
\[ \hat{h}(g \Rightarrow \hat{h}^{-1}) \]
\[ = (f, z)(g \Rightarrow (f, z)^{-1}) \]
\[ = \left( fgf^{-1}g^{-1}, c^{-1}(f, f^{-1})c(f, gf^{-1}f^{-1}) \right) \] (C.36)
\[ \times \exp \left( -\frac{i}{2\pi} \int_0^1 ds \int_0^1 dr \left\langle g^{-1}(r) \frac{\partial g(r)}{\partial r}, \frac{\partial f(s, r)}{\partial s}f^{-1}(s, r) \right\rangle \right), \]
where \(g \in P_0\mathfrak{g}\) and which is derived by using \((4.26)\) and the group multiplication in \(P_0L_0G \times U(1)\), at the identity of \(P_0\mathfrak{g}\). Reformulating the result, using the fact that \(\frac{\partial}{\partial s}A = 0\), and dividing by the ideal \(\mathfrak{n}\), we find
\[ \hat{h}A \Rightarrow \hat{h}^{-1} = \left( b(f)[A, b(f^{-1})], \frac{i}{2\pi} \int_0^1 dr \left\langle b(f^{-1}(s, r)) \frac{\partial b(f(s, r))}{\partial r}, A(r) \right\rangle \right). \] (C.37)

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