Restoration of chiral symmetry in quark models with effective one gluon exchange\textsuperscript{1}

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Abstract

The restoration of chiral symmetry at finite density and/or temperature is investigated in a wide class of one-gluon exchange models in the instantaneous approximation. If the effective quark interaction is less divergent than $1/k^2$ for small momentum transfer $k$, we obtain Gaussian critical exponents for the chiral phase transitions at finite temperature and density, respectively. In the opposite case, for an interaction diverging faster than $1/k^2$ in the infrared region, a qualitative different behavior of the quark self-energy near the critical Fermi momentum $k_c$ and the critical temperature $T_c$, respectively, is observed. In the first scenario, we find $k_c \approx 2 \ln 2 \ T_c$, which compares well with recent data from QCD lattice simulations.

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1. Introduction

The properties of hadrons in a hot and/or dense medium are of general interest, since they can be experimentally tested in heavy ion collisions. These experiments might provide informations of the widely unknown low energy sector of QCD. Furthermore, a detailed understanding of the quark ground state at finite density is necessary for a wide range of applications, e.g. to describe compact star matter \[1\]. In these applications, the restoration of the spontaneously broken chiral symmetry is of particular interest. For medium values of the density compared with the critical density, a powerful argument was developed which predicts a scaling of certain hadronic observables with the density \[2\]. This so-called BR-scaling yields phenomenological reasonable results for densities up to the critical density. Close to the critical values, physics is described by another approach; it was argued long ago \[3\], that the chiral transition at finite temperature is governed by dimensional reduction and universality. In particular, four-dimensional QCD should belong to the universality class of the three-dimensional \(\sigma\)-model with the corresponding global symmetries \[3\]. Recently, Kocić and Kogut argued, that the presuppositions which support the universality argument, are not satisfied. In particular, they observe that the critical exponents of the chiral transition in the three-dimensional Gross-Neveu model are of Gaussian type, instead of those given by the Ising model, as predicted by the universality argument \[4\]. The chiral transition at finite density is less known, which is partly due to conceptual and practical difficulties of the numerical simulation at finite densities \[5\]. It is tacitly assumed that the critical value and indices of the chiral transition at finite density resemble those of the transition at finite temperature.

In this letter, we study the chiral transition at finite density and/or temperature in a wide class of constituent quark models with effective one-gluon exchange. We will find that Gaussian critical exponents are the generic case, and will discuss the circumstances which imply a correspondence of the chiral transition at finite temperature and finite density.

2. Model description

Starting point of the one-gluon exchange models is the QCD Dyson-Schwinger equation for the quark propagator \(S(p)\), which in Landau gauge and in Euclidean spacetime reads

\[
i\not{\partial} + m + \int \frac{d^4k}{(2\pi)^4} \Gamma_\mu(p, p - k) S(k) \gamma_\nu t^a D_{\mu\nu}(p - k) = S^{-1}(p) , \tag{1}\]

where the Euclidean quark propagator is of the form

\[
S(p) = \frac{i}{Z(p^2)\not{p} + i\Sigma(p^2)} . \tag{2}\]
Equation (1) must be supplemented with Ansätze for the full gluon propagator $D_{\mu \nu}^{ab}(p-k)$ and the quark-gluon vertex function $\Gamma_{\mu}^{a}(p, p-k)$. The variety of models [6, 7, 8, 9, 10]—and this is not a complete list—differ in the particular choice of $D_{\mu \nu}^{ab}$ and $\Gamma_{\mu}^{a}$. These choices must match the well known behavior of the perturbative Green’s functions at large momentum transfer, i.e.

$$D_{\mu \nu}^{ab}(k) = \delta^{ab} \frac{16\pi^2}{11 \ln k^2/\Lambda_{QCD}^2} \frac{1}{k^2} \left( \delta_{\mu \nu} - \frac{k_{\mu}k_{\nu}}{k^2} \right), \quad \text{for} \quad k^2 \gg \Lambda_{QCD}^2.$$

Sometimes, the low-energy behavior of the gluon-propagator is chosen as

$$D_{\mu \nu}^{ab}(k) \propto \delta^{ab} \frac{1}{k^4} \left( \delta_{\mu \nu} - \frac{k_{\mu}k_{\nu}}{k^2} \right) \quad \text{for} \quad k^2 \approx 0.$$

This choice produces a linear confining potential between static quarks. It was observed, that the ansatz (3) yields a model which is one-loop renormalizable and realizes confinement by infrared-slavery [4], which enforces the quark propagator to vanish. In any case, we will assume that the interaction under consideration is strong enough to break chiral symmetry spontaneously.

A different confining mechanism arises when one assumes that the Yang-Mills vacuum is realized by random color electro-magnetic fields [11], which induce a complex constituent quark mass. In this case, the quark propagator is finite, but quarks do not exist as asymptotic states, and furthermore the Landau pole of the corresponding perturbative gluon-propagator is screened due to the presence of a dynamical mass [12]. This implies that the behavior

$$D_{\mu \nu}^{ab}(k) \propto \delta^{ab} \frac{1}{k^2} \left( \delta_{\mu \nu} - \frac{k_{\mu}k_{\nu}}{k^2} \right) \quad \text{for} \quad k^2 \approx 0.$$

is consistent with phenomenology.

In this letter, we need not to specify the full gluon-propagator beyond the constraint (3), but will refer to the different specific scenarios (4) and (5) when needed.

In order to stress the generality of our main result, we investigate to different Ansätze for the vertex function $\Gamma_{\mu}^{a}$, i.e. the bare vertex

$$\Gamma_{\mu}^{a}(p, q) = i^a \gamma_{\mu},$$

which is the simplest choice, and a more sophisticated vertex, i.e.

$$\Gamma_{\mu}^{a}(p, k) = t^a \gamma_{\mu} T_1(p^2, k^2) + (p_{\mu} + k_{\mu})(\vec{k} + \vec{p})T_2(p^2, k^2)$$
$$+ \ (p_{\mu} + k_{\mu})(\vec{k} - \vec{p})T_3(p^2, k^2) - i(p_{\mu} + k_{\mu})T_4(p^2, k^2) \left[ \Sigma(k^2) - \Sigma(p^2) \right],$$

which is motivated from Taylor-Slavnov identities [9, 13]. The last term on the right hand side of (7), which is proportional to the self-energy, ensures that the
Ansatz does not explicitly break chiral symmetry. From dimensional arguments, one observes that the functions $T_{2,4}$ decrease like $1/p^2$ and $1/k^2$ for large values for $p^2$ and $k^2$, respectively.

Using the Ansatz (3), the Dyson-Schwinger equation decays into two equations for $Z(p^2)$ and the quark self-energy $\Sigma(p^2)$. The equation for the latter can be cast into the form

$$\Sigma(p^2) = m + \frac{1}{H(p^2)} \int \frac{d^4k}{(2\pi)^4} \frac{\Sigma(k^2)}{Z^2(k^2)k^2 + \Sigma^2(k^2)} D(k, p), \quad (8)$$

where for the case of the bare vertex (6) the functions $H$ and $D$ are

$$H(p^2) = 1, \quad (9)$$
$$D(p, k) = \frac{1}{2} D_{\mu\nu}^a(p - k), \quad (10)$$

while in the case of the more complicated vertex (7), these functions are

$$H(p^2) = 1 + \int \frac{d^4k}{(2\pi)^4} \frac{Z(k^2)}{Z^2(k^2)k^2 + \Sigma^2(k^2)} D_1(p, k), \quad (11)$$
$$D(p, k) = \left[ Z(k^2)T_4(p^2, k^2) + 2T_2(p^2, k^2) \right] p_\mu D_{\mu\nu}^a(p - k) + \frac{1}{2} T_1(p^2, k^2) D_{\mu\nu}^a, \quad (12)$$

where

$$D_1(p, k) = T_4(p^2, k^2) p_\mu D_{\mu\nu}^a(p - k) p_\nu. \quad (13)$$

Note that the function $H(p^2)$ in (11) is ultra-violet finite, since $D_1(p, k)$ decreases like $1/k^4$ for large $k$.

In order to address the renormalization of the Dyson-equation, we exploit the fact that the self-energy $\Sigma(k^2)$ and the gluon induced interaction $D$ match the perturbative behavior at large momentum transfer, i.e.

$$\Sigma(k^2) \approx \frac{\text{const.}}{\ln k^2/\Lambda_{QCD}^2} d_m; \quad D \approx \frac{\text{const.}}{k^2 \ln k^2/\Lambda_{QCD}^2}, \quad (14)$$

where $d_m > 0$ is the anomalous dimension of the quark mass. The generic behavior of $Z(k^2)$ turns out to be weakly momentum dependent and asymptotically approaches a constant for large $k^2$. In fact, $Z(\mu^2)$ is renormalized to 1 at some large subtraction point $\mu$ in perturbative QCD with mass shell renormalization [14]. This large momentum behavior of $Z(k^2)$ and $\Sigma(k^2)$ implies that the momentum integration in the Dyson-equation (8) is ultra-violate finite. If one nevertheless cuts off this momentum integration by an $O(4)$-invariant cutoff $\Lambda$, the behavior of a self-consistent solution of (8) enforces the bare quark mass $m$ to vanish for $\Lambda \to \infty$ as (see e.g. [15])

$$m(\Lambda^2) = \frac{1}{\ln \Lambda^2/\mu^2} d_m m_R(\mu), \quad (15)$$
where $\mu$ is the renormalization point and $m_R$ is the renormalized quark current mass. Here we are only interested in the chiral limit implying
\[
m_R(\mu) = 0 \rightarrow m \equiv 0
\] (16)
from the very beginning.

In the following, we will adopt further simplifications which we believe not to change our conclusions. We will assume that the function $Z(k^2)$ is weakly dependent on the momentum $k^2$ for the models which we consider here, implying that we may safely approximate $Z(k^2) = 1$. In fact, one realizes that $Z(k^2) = 1$ is a solution of the Dyson-equation, if one neglects vector-contributions to the Dyson-Schwinger equation after a Fierz rearrangement of the interaction.

Finally, we assume a separation of the fermionic and gluonic energy scales $E_f$ and $E_g$, i.e. $E_g \gg E_f$. This assumption is motivated by lattice results [16], which show that the lowest-lying excitations of the Yang-Mills vacuum, the glue balls, have energies larger than $\approx 1.5 \text{ GeV}$, while the energy scale of the lowest quark modes is set by the constituent quark mass of $\approx 300 \text{ MeV}$. The above presupposition has two immediate consequences; firstly, the contribution of the temperature dependent gluon exchange to the loop integration in the Dyson-Schwinger equation is of order $\exp{-E_g/T}$, which is small compared with the fermionic part $\exp{-E_f/T}$. This implies that we may neglect the temperature dependence of the interaction. In fact, this idea is supported by simulations of lattice QCD, where one observes that the gluon condensate is weakly temperature dependent throughout the chiral transition [1], suggesting that the chiral transition is solely driven by the temperature dependence of the quark loop. Secondly, the dominant contribution of the $k_0$ integration in the loop integral of the Dyson-Schwinger equation stems from the region $k_0 \approx E_f$. This implies that we may neglect the $k_0$ dependence in the gluon exchange, and we end up with an instantaneous approximation.

3. Restoration of chiral symmetry at finite density

In order to investigate density effects, we introduce a chemical potential $\mu$, which enforces a non-vanishing vacuum density $\rho = \langle \Omega | q^\dagger q | \Omega \rangle$. In instantaneous approximation, the quark self-energy $\Sigma$ only depends on the three momentum $\vec{k}$, implying that the $k_0$-integration in the Dyson-Schwinger equation (8) and in $H_f(p^2)$ (11) can be done, i.e.

\[
\Sigma(p^2) = \frac{1}{H_f(p^2)} \int_{k^2 \geq k_f^2} \frac{d^3k}{(2\pi)^3} \frac{\Sigma(k^2)}{2\sqrt{k^2 + \Sigma^2(k^2)}} D(k, \vec{p}) ,
\]

\[
H_f(p^2) = 1 + \int_{k^2 \geq k_f^2} \frac{d^3k}{(2\pi)^3} \frac{1}{2\sqrt{k^2 + \Sigma^2(k^2)}} D_1(\vec{p}, k) ,
\]

\[1\] This argument becomes obvious, if one performs this integration in the complex plane by Cauchy’s theorem.
where the subscript “ρ” indicates the finite density case. Furthermore, the Fermi momentum \(k_f\) is related to the density by \(k_f^3 = \pi^2 \rho\) for \(N_c = 3\) colors and \(N_f = 1\) flavor. In the case of the bare vertex (6), the function \(H_f\) is again \(H_f(p^2) = 1\) and thus independent of the Fermi momentum.

We first derive an equation for the critical Fermi momentum \(k_c\), at which chiral symmetry is restored. To this aim, we apply the so-called bifurcation method [17] and exploit the fact that at the critical density, a second solution to the Dyson-Schwinger equation (17) exists besides the trivial one \(\Sigma \equiv 0\). At the critical density, the second solution differs from the trivial one only by an tiny amount implying that its momentum dependence \(f(p^2)\), where \(\Sigma(p^2) \propto f(p^2)\) with \(f(0) = 1\), can be calculated from the linearized equation, i.e.

\[
f(p^2) = \frac{1}{H_0(p^2)} \int_{k^2 \geq k_c^2} \frac{d^3k}{(2\pi)^3} \frac{1}{2\sqrt{k^2}} D(k, p) f(k^2),
\]

where

\[
H_0(p^2) = 1 + \int_{k^2 \geq k_c^2} \frac{d^3k}{(2\pi)^3} \frac{1}{2\sqrt{k^2}} D_1(p, k).
\]

This equation is essentially the Bethe-Salpeter equation describing fluctuations around the trivial vacuum at the critical point. Since the linearized equation is homogeneous in \(\Sigma\), it does not fix the magnitude of the self-energy. The critical density is obtained from (19) by demanding that \(f(p^2)\) is an eigenvector of the integral-kernel on the right hand side of (19) with eigenvalue \(1\).

In order to fix the size of the self-energy for a Fermi-momentum close to \(k_c\), we study the next to leading order of the small amplitude expansion of the Dyson-Schwinger equation (17), i.e.

\[
\Sigma(p^2) = \frac{1}{H_0(p^2)} \int_{k^2 \geq k_f^2} \frac{d^3k}{(2\pi)^3} \frac{\Sigma(k^2)}{2\sqrt{k^2}} D(k, \tilde{p})
\]

\[
+ \frac{1}{2H_0^2(p^2)} \int_{k^2 \geq k_f^2} \frac{d^3k}{(2\pi)^3} \frac{\Sigma(k^2)}{2\sqrt{k^2}} D(k, \tilde{p}) \int_{q^2 \geq k_f^2} \frac{d^3q}{(2\pi)^3} \frac{\Sigma^2(q^2)}{2(q^2)^{3/2}} D_1(q, \tilde{p})
\]

\[
- \frac{1}{2H_0(p^2)} \int_{k^2 \geq k_f^2} \frac{d^3k}{(2\pi)^3} \frac{\Sigma^2(k^2)}{2(k^2)^{3/2}} D(k, \tilde{p}).
\]

We will solve this integral equation approximately. On first realizes that for large external momentum \(p^2\) the last two terms of (21) are irrelevant, since the functions \(D_1(\tilde{p}, \tilde{k})\) are strongly peaked for \(\tilde{k}^2 \approx \tilde{p}^2\) implying that the these last two terms are suppressed by a factor \(\Sigma^2/p^2\) compared with the first term in (21). We therefore conclude that for \(p^2 \gg k_c^2\), the momentum dependence of the solution \(\Sigma(p^2)\) of (21)
is given by \( f(\vec{p}^2) \) from (19). This motivates the approximation
\[
\Sigma(\vec{p}^2) \approx \Sigma(1)(k_f) f(\vec{p}^2), \tag{22}
\]
where \( \Sigma(1) \) must be calculated from (21). The procedure implies that the ansatz (22) is a good approximation to the full solution of (21) at least at small and at large momentum \( \vec{p}^2 \). Using the implicit expression (19) for \( k_c \), one obtains to leading order in \( k_c^2 - k_f^2 \) from (21) at \( p^2 = 0 \)
\[
\Sigma(1) \int_{k^2 \geq k_c^2} \frac{d^3k}{(2\pi)^3} \int_{q^2 \geq k_c^2} \frac{d^3q}{(2\pi)^3} \frac{\mathcal{D}(0, \vec{k})f^3(\vec{k}^2)}{2(k^2)^{3/2}} = \int_{k^2 \geq k_c^2} \frac{d^3k}{(2\pi)^3} \frac{f(\vec{k}^2)}{\sqrt{k^2}} \mathcal{D}(0, \vec{k}), \tag{23}
\]
where \( \epsilon = 0 \) for case of the bare vertex (3) and \( \epsilon = 1 \) for the case of the improved vertex. If one assumes that the function \( T_4(p^2, k^2) \) is not singular for \( p^2 \to 0 \) (which is the generic behavior [13]), the function \( \mathcal{D}_1(p^2 = 0, k^2) \) vanishes implying that our result (23) is widely independent of the ansatz for the vertex function. The only dependence on the particular choice enters via the function \( T_1(p^2, k^2) \) through the projected gluon propagator \( \mathcal{D}(p, k) \).

The integral at the right hand side of (23) is proportional to
\[
\int_{k_f^2}^{k_c^2} du f(u) \frac{1}{2} T_1(0, u) D_{\mu\nu}^{aa}(u). \tag{24}
\]
We therefore obtain finally
\[
\Sigma(1) = \left[ \frac{k_c^2 f(k_c^2) T_4(0, k_c^2) D_{aa}^{\mu\nu}(k_c^2)}{4\pi^2 \int_{k^2 \geq k_c^2} \frac{d^3k}{(2\pi)^3} T_1(0, k^2) D_{\mu\nu}^{aa}(k^2) f^3(k^2)} \right]^{1/2} \sqrt{1 - \frac{k_f^2}{k_c^2}}, \tag{25}
\]
which corresponds to a Gaussian critical exponent 1/2.

4. Restoration at finite temperature

The quark interaction of the one-gluon exchange model is motivated from the gluonic interaction of QCD and therefore temperature dependent. However, numerical simulations of lattice QCD indicate that the gluonic properties only vary weakly, if the quark sector passes the chiral transition [18]. We therefore neglect here any explicit temperature dependence of the interaction in (1), but study the temperatures effects in the quark sector, since those apparently play the dominant role at the chiral transition.
The standard method to calculate the impact of temperature on Green’s function is the imaginary-time formalism [19], i.e. one confines the configuration space of the fermionic fields to the configurations which are anti-periodic in the Euclidean time direction with a periodic length $1/T$ with $T$ being the temperature. At finite temperature, the integration over the zeroth component of the Euclidean momentum in the terms of (8) is replaced by a discrete sum over Matsubara frequencies. In the case of the instantaneous approximation, this sum can be explicitly evaluated and yields

$$\Sigma(\vec{p}^2) = 1 + \epsilon \int \frac{d^3k}{(2\pi)^3} \tanh\left(\frac{E}{2}\right) \frac{1}{2E(\vec{k}^2)} D_{1}(\vec{p}, \vec{k}),$$

(27)

where the subscript “$T$” indicates the finite temperature case, where $E(\vec{k}^2) = \sqrt{\vec{k}^2 + \Sigma(\vec{k}^2)}$ and again $\epsilon = 0$ for the case of the bare vertex (6) and $\epsilon = 1$ otherwise. The close correspondence of the density and temperature dependence of the model stems from the similarity of the equations (26-27) and (17-18) respectively; whereas the $\tanh$ in (26-27) rapidly approaches 1 for $E \gg T$, small momenta at $E \ll T$ are suppressed. The same behavior is observed in the kernel of the integral equations (17-18) in the case of finite density. The only difference is that in this case the screening of low momenta is done by a step function due to Pauli blocking, whereas low momenta in the temperature case are linearly suppressed.

In order to investigate the variation of the self-energy with the temperature near the critical temperature $T_c$, we adopt the same method as in the case of finite density in the previous section. We only sketch the calculation. The critical temperature and the momentum dependence of the self-energy is obtained from the linearized equation, i.e.

$$f_T(\vec{p}^2) = \frac{1}{H_0(\vec{p}^2)} \int \frac{d^3k}{(2\pi)^3} \tanh\left(\frac{\sqrt{\vec{k}^2}}{2T_c}\right) \frac{1}{2\sqrt{\vec{k}^2}} D(\vec{k}, \vec{p}) f_T(\vec{k}^2),$$

(28)

Analogous to the density case, we approximate the self-energy for temperatures close to $T_c$ by

$$\Sigma(\vec{p}^2) \approx \Sigma_{(1T)}(T_c) f_T(\vec{p}^2),$$

(29)

and obtain from the to next to leading order linearized Dyson-Schwinger equation
\[ \Sigma_{(1T)} = \left[ 4 \int \frac{d^3k}{(2\pi)^3} \frac{e^{-k/T_c}}{(1 + e^{-k/T_c})^2} f_T(k^2) T_1(0, k^2) D_{\mu\mu}^{aa}(k^2) \right]^{1/2} \]
\[ \times \left[ \int \frac{d^3k}{(2\pi)^3} \frac{T_c e^{-2k/T_c} - 2k e^{-k/T_c}}{k^3(1 + e^{-k/T_c})^2} T_1(0, k^2) D_{\mu\mu}^{aa}(k^2) f_T^2(k^2) \right]^{-1/2} \]
\[ \times \sqrt{1 - \frac{T}{T_c}}, \]

where \( k := \sqrt{k^2} \). We find a Gaussian critical exponent for the chiral phase transition at finite temperature.

5. Comparison of the phase transition at finite temperature and finite density

All momentum integrals which occur in the previous sections, are ultra-violet finite, which is merely a consequence of asymptotic freedom and therefore independent of the details of the models under investigations. However, we have tacitly assumed that the integrals are also infra-red finite. In the case of finite density (section 3), this is of course true due to Pauli blocking of momentum states below the Fermi-momentum \( k_f \). In contrast, the case of finite temperature (section 4) needs further studies.

Expanding the term in the second line of (30) with respect to small \( k \), i.e.

\[ \frac{T_c - T_c e^{-2k/T_c} - 2k e^{-k/T_c}}{k^3(1 + e^{-k/T_c})^2} = \frac{1}{12T_c^2} + \mathcal{O}(k^2), \]

one observes that both integrals in (30) are infra-red finite, provided the product of vertex function and gluon propagator \( T_1(0, k^2) D_{\mu\mu}^{aa}(k^2) \) does not diverge faster than \( 1/k^2 \). In the derivation of (30), we must exclude models, which incorporate quark confinement by the gluon propagator via infra-red slavery (see (4)). For the gluon-propagator (4), the infra-red divergence is screened by a tiny, but non-vanishing self-energy at temperatures somewhat below the critical one. This indicates that the restoration of the chiral symmetry is described by a different law than the simple one with critical exponent \( 1/2 \) in (30). Since the restoration of the chiral symmetry at finite density is completely insensitive to the low energy behavior of the interaction due to Pauli’s principle, we expect a completely different behavior of the self-energy at finite density and at finite temperature, respectively, close to the critical point, if the model’s interaction is infra-red sensitive. In contrast, we have shown that chiral transition at finite density is of the same nature as the transition at finite temperature for interactions which are "well behaved" in the infra-red region.

Finally, we search for a relation between the critical Fermi-momentum and the critical temperature. To this aim, we compare equations (19) and (28), which provide
the critical Fermi momentum and the critical temperature, respectively. Since in the
latter case the \( \tanh \frac{k}{2T_c} \) rapidly approaches 1 for \( k \gg T_c \), and since the integral
kernel in both equation strongly peaks at \( k \approx p \), one finds that the function \( f(p^2) \)
(13) coincides with \( f_T(p^2) \) (28) for \( p^2 \gg T_c^2 \). On the other hand, both functions are
normalized to 1 at \( p^2 = 0 \). We might therefore approximate
\[ f(p^2) \approx f_T(p^2) \] (32)
in the entire momentum range. Equating (19) and (28) at \( p^2 = 0 \) then yields
\[ \int_{k_c}^{\infty} dk \; k T_1(0, k^2) D^{aa}_{\mu\mu}(k^2) f(k^2) = \int_0^{\infty} dk \; k \tanh \left( \frac{k}{2T_c} \right) T_1(0, k^2) D^{aa}_{\mu\mu}(k^2) f(k^2). \] (33)
A relation between \( k_c \) and \( T_c \) can be obtained, if one assumes that the function
\( k T_1(0, k^2) D^{aa}_{\mu\mu}(k^2) f(k^2) \) is monotonic decreasing with \( k \). In this case, one finds that
\[ \int_{k_c}^{\infty} dk \; k T_1(0, k^2) D^{aa}_{\mu\mu}(k^2) f(k^2) \geq \int_{k_0}^{\infty} dk \; k T_1(0, k^2) D^{aa}_{\mu\mu}(k^2) f(k^2), \] (34)
where \( k_0 \) must be calculated from
\[ \int_0^{\infty} dk \left[ \tanh \left( \frac{k}{2T_c} \right) - \theta(k - k_0) \right] = 0 \] (35)
with \( \theta(k) \) the step-function. From this equation, we obtain \( k_0 = 2 \ln 2 T_c \) and therefore the final result
\[ k_c \leq 2 \ln 2 T_c \approx 1.386 T_c \] (36)
Its instructive to confront this inequality with lattice data. Recent lattice simulations
[18] yield \( T_c \approx 260 \text{ MeV} \). From (36), we find \( k_c \approx 360 \text{ MeV} \). This value is
in good agreement with the standard estimate [20] \( k_c \approx m_N/3 \), where \( m_N \) is the
nucleon mass.

6. Conclusions
We have studied a wide class of effective one-gluon exchange models. The basic
assumption is a separation of the fermionic and gluonic energy scales justifying
an instantaneous approximation. The quark interaction is constrained in order to
reproduce the well-know behavior of the perturbative Green’s functions at high
momentum transfer. With some further assumption, which we believe to be justified
for most of the one-gluon exchange models, we have been able to extract quite
detailed relations for the critical behavior of various quantities, like the quark self
energy \( \Sigma \), at the chiral phase transition. We find that \( \Sigma \) as function of the Fermi
momentum \( k_f \) behave like
\[ \Sigma(p^2 = 0) \propto \sqrt{1 - \frac{k_f^2}{k_c^2}}, \] (37)
where $k_c$ is the critical value of $k_f$, and that the restoration of the chiral symmetry is insensitive to the infra-red behavior of the interaction due to Pauli blocking. In the case of finite temperature, the analogous behavior, i.e.

$$\Sigma(p^2 = 0) \propto \sqrt{1 - \frac{T}{T_c}}, \quad (38)$$

is found, only if the interaction diverges weaker than $1/k^2$ at small momentum transfer $k$. In the opposite case of an "infra-red sensitive" interaction, a completely different behavior of the self-energy near the critical density and the critical temperature, respectively, is expected. In the former case, we obtain Gaussian critical exponents supporting the recent ideas of Kocić and Kogut [4]. Assuming further that the effective quark interaction is a monotonic decreasing function of the momentum transfer, we obtain the inequality

$$k_c \leq 2 \ln 2 \ T_c, \quad (39)$$

which is in good agreement with recent QCD lattice simulations.

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