CONVERGENCE RESULTS WITH NATURAL NORMS: STABILIZED LAGRANGE MULTIPLIER METHOD FOR ELLIPTIC INTERFACE PROBLEMS

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ABSTRACT. A stabilized Lagrange multiplier method for second order elliptic interface problems is presented in the framework of mortar method. The requirement of LBB (Ladyzhenskaya-Babuška-Brezzi) condition for mortar method is alleviated by introducing penalty terms in the formulation. Optimal convergence results are established in natural norm which is independent of mesh. Numerical experiments are conducted in support of the theoretical derivations.

1. INTRODUCTION

The best part of considering Lagrange multiplier formulation is: it converts a constraint problem in to an unconstrained problem which is comparably an easy way for implementation (see [1]). Also, we can evaluate both primal and flux variable simultaneously. On the other hand, one major difficulty in considering the Lagrange multiplier method is: the finite dimensional problems have to obey the inf-sup condition (LBB condition) which rejects many natural choices for approximation. Fortunately this requirement has been alleviated by Barbosa and Hughes (see [5]). They proposed a stabilized multiplier method which is stable and optimally convergent with respect to a mesh-dependent norm. In [6] the convergence results of these methods are established with natural norms. Nitsche had introduced a penalty term on the boundary to derive optimal estimates for approximating elliptic problems with nonhomogeneous Dirichlet boundary condition without enforcing boundary condition on the finite element spaces in [14].

These ideas are extended to multi-domain problems with non-matching grids by Hansbo et al. in [13] and Becker et al. in [7]. Wherein, the optimal convergence results are established in mesh-dependent norm. In [13], a stabilization method has been proposed, which uses global polynomials as multipliers to avoid the cumbersome integration of products of unrelated mesh functions and derived the stability under the condition that the approximation space for the interface multiplier contains the constant. A Lagrange multiplier method with penalty for multi-domain problems with non-matching grid is discussed by Patel (see [15]) which is well-posed and stable but due to inconsistency there is loss of accuracy. Stenberg has pointed out a close connection between Nitsche’s method and stabilized schemes and proposed it as mortaring Nitsche method (see [16]). For a detail study, we refer to [7, 8, 9, 13, 16].

In this paper, we extend the ideas of [6] to multi-domain problems with non-matching grid and establish the optimal error estimates in natural norm. Here, the multipliers are simply the nodal basis functions restricted to the interface. Error estimates are obtained with an assumption that: the multiplier space satisfies the strong regularity property in

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the sense of Babuška (see [1]). We give some computational results in support of the theoretical results.

A brief outline is as follows. We recall some functional spaces and approximation results in Section 2. In Section 3 we define the stabilized Lagrange multiplier methods for an elliptic interface problem and derive the error estimates. We give a matrix formulation of the method in Section 4. In Section 5, some numerical experiments are given. Finally, we concluded in Section 6.

2. Preliminaries

Let \( \Omega \) be an open bounded polygonal domain in \( \mathbb{R}^2 \) with boundary \( \partial \Omega \). We define \( \alpha = (\alpha_1, \alpha_2) \) as a 2-tuple of non-negative integers \( \alpha_i, i = 1, 2 \) and with \( |\alpha| = \alpha_1 + \alpha_2 \) set

\[
D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}.
\]

The Sobolev space of order \( m \) (see [12]) over \( \Omega \) is defined as

\[
H^m(\Omega) = \left\{ v \in L^2(\Omega) : D^\alpha v \in L^2(\Omega), |\alpha| \leq m \right\}
\]
equipped with the norm and semi-norm

\[
\|v\|_{H^m(\Omega)} = \left( \sum_{|\alpha| \leq m} \int_{\Omega} |D^\alpha v|^2 \, dx \right)^{1/2},
\]

\[
|v|_{H^m(\Omega)} = \left( \sum_{|\alpha| = m} \int_{\Omega} |D^\alpha v|^2 \, dx \right)^{1/2},
\]

respectively. Let \( r = m + \sigma \) be a positive real number, where \( m \) and \( \sigma \) are the integral and fractional part of \( r \) respectively. The fractional Sobolev space \( H^r(\Omega) \) is defined as

\[
H^r(\Omega) = \left\{ v \in H^m(\Omega) : \int_{\Omega} \int_{\Omega} \frac{(D^\alpha v(x) - D^\alpha v(y))^2}{|x - y|^{2+2\sigma}} \, dx \, dy < \infty, |\alpha| = m \right\}
\]
with the norm

\[
\|v\|_{H^r(\Omega)} = \left( \|v\|_{H^m(\Omega)}^2 + \sum_{|\alpha| = m} \int_{\Omega} \int_{\Omega} \frac{(D^\alpha v(x) - D^\alpha v(y))^2}{|x - y|^{2+2\sigma}} \, dx \, dy \right)^{1/2}.
\]

We shall denote by \( H^{-1/2}(\partial \Omega) \) the space of traces \( v|_{\partial \Omega} \) over \( \partial \Omega \) of the functions \( v \in H^r(\Omega) \) equipped with the norm

\[
||v||_{H^{-1/2}(\partial \Omega)} = \inf_{v \in H^r(\Omega), v|_{\partial \Omega} = g} ||v||_{H^r(\Omega)}
\]
and

\[
H_0^1(\Omega) = \{ v \in H^1(\Omega) : v|_{\partial \Omega} = 0 \}.
\]

Let \( H^{-1/2}(\partial \Omega) \) be the dual space of \( H^1(\Omega) \) equipped with the norm

\[
\|\mu\|_{H^{-1/2}(\partial \Omega)} = \sup_{g \in H^{1/2}(\partial \Omega), g \neq 0} \frac{|\langle \mu, g \rangle_{-1/2, \partial \Omega}|}{\|g\|_{H^{1/2}(\partial \Omega)}},
\]
where \( \langle \cdot, \cdot \rangle_{-1/2, \partial \Omega} \) is the duality pairing between \( H^{-1/2}(\partial \Omega) \) and \( H^{1/2}(\partial \Omega) \). With \( \Gamma^+ \subset \partial \Omega \), let \( \tilde{v} \) be an extension of \( v \in H^{1/2}(\Gamma^+) \) by zero to all of \( \partial \Omega \). Then we set \( H_{00}^{1/2}(\Gamma^+) \), a subspace of \( H^{1/2}(\Gamma^+) \) as

\[
H_{00}^{1/2}(\Gamma^+) = \{ v \in H^{1/2}(\Gamma^+) : \tilde{v} \in H^{1/2}(\partial \Omega) \}.
\]
The norm in $H^{1/2}_0(\Gamma^*)$ is defined by:

$$\|g\|_{H^{1/2}_0(\Gamma^*)} = \inf_{v \in H^{1/2}_0(\Omega), v_{|\Gamma^*} = g} \|v\|_{H^1(\Omega)}.$$ 

Let $H^{-1/2}_0(\Gamma^*)$ be the dual space of $H^{1/2}_0(\Gamma^*)$. Also $\langle \cdot , \cdot \rangle_{00,\Gamma^*}$ denote the duality pairing between $H^{-1/2}_0(\Gamma^*)$ and $H^{1/2}_0(\Gamma^*)$ and let the norm on $H^{-1/2}_0(\Gamma^*)$ be defined by

$$\|\varphi\|_{H^{-1/2}_0(\Gamma^*)} = \sup_{\mu \in H^{1/2}_0(\Gamma^*), \mu \neq 0} \frac{|\langle \varphi, \mu \rangle_{00,\Gamma^*}|}{\|\mu\|_{H^{1/2}_0(\Gamma)}}.$$ 

Note that, $H^{-1/2}(\Gamma^*)$ is continuously embedded into $H^{-1/2}_0(\Gamma^*)$ (see [9]).

Let $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$ and $\nu_1$ and $\nu_2$ represent the outward normal components.

Let $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$ and $\Gamma$ be the common interface $\partial \Omega_1 \cap \partial \Omega_2$ (see Figure 1). We denote $\nu_1$ the unit outward normal oriented from $\Omega_1$ towards $\Omega_2$ and $\nu_2 = -\nu_1 = \nu$. For any function $v$ let $v_i = v|_{\Omega_i}$. Let $H^1_D(\Omega_i) = \{ v_i \in H^1(\Omega_i) : v_i|_{\partial \Omega \cap \partial \Omega_i} = 0 \}$. We define finite dimensional subspaces on each subdomain $\Omega_i$ as

$$X_{h_i} = \{ v_{h_i} \in C^0(\overline{\Omega_i}) : v_{h_i}|_K \in P_1(K) \text{ for } K \in T_{h_i}, \text{ } v_i = 0 \text{ on } \partial \Omega_i \cap \partial \Omega \}.$$
We set \( h = \max \{ h_1, h_2 \} \) and define the global space \( X_h \subset X \) defined as
\[
X_h = \{ v_h \in L^2(\Omega) : v_{h_i} \in X_{h_i} \text{ for } i = 1, 2 \}.
\]

Let \( W_h \) be the restriction of \( X_h \) to \( \Gamma_i = \partial \Omega_i \cap \Gamma \). Assume two different 1D triangulations on \( \Gamma_i \), \( T_{h_i}(\Gamma) \) and \( T_{h_2}(\Gamma) \) and correspondingly two different trace spaces \( W_{h_i} \) and \( W_{h_2} \). For our convenience we may choose the multiplier space \( W_h \) to be \( W_{h_2} \).

We recall the following approximation results (see [10, 15]).

**Lemma 2.1.** For all \( v_{h_i} \in X_{h_i} \), there exists a constant \( C > 0 \) independent of \( h \) such that
\[
||h^{1/2} \nabla v_{h_i} \cdot v_i||_{L^2(\Gamma)} \leq C ||\nabla v_{h_i}||_{L^2(\Omega_i)}.
\]

We assume \( C \) denotes a generic constant throughout the discussion.

**Lemma 2.2.** Let \( v_i \in H^l(\Omega_i) \) for \( l > 1 \) and \( i = 1, 2 \). Then there exist constants \( C > 0 \) independent of \( h \) and a sequence \( I_h v_i \in X_{h_i} \) such that for any \( 0 \leq l_i \leq l \)
\[
||v_i - I_h v_i||_{H^l(\Omega_i)} \leq C h^{l-i} ||v_i||_{H^l(\Omega_i)},
\]
\[
||v_i - I_h v_i||_{L^2(\Gamma_i)} \leq C h^{l-i/2} ||v_i||_{H^l(\Omega_i)}.
\]

For \( v \in X \) we set the interpolation \( I_h v \) as: \( I_h v \) equals \( I_h v_i \) on each \( \Omega_i \) for \( i = 1, 2 \). We define the \( L^2 \) projection \( \Pi M \from W_h \to X_h \) as below:
\[
\Pi M \phi = 0 \quad \forall \chi \in W_h.
\]

**Lemma 2.3.** [8] For any \( \sigma \geq 0 \), the following estimate holds: for all \( \phi \in H^{1/2+\sigma}(\Gamma) \) there exists a constant \( C > 0 \) independent of \( h \) such that
\[
h^{1/2}||\phi - \Pi \phi||_{L^2(\Gamma)} + ||\phi - \Pi \phi||_{H^{-1/2}(\Gamma)} \leq C h^{\eta+1} ||\phi||_{H^{1/2+\sigma}(\Gamma)}
\]
where \( \eta = \min(\sigma, 1) \).

### 3. Problem Formulation and Error Estimates

Consider a second order elliptic interface model problem: for \( i = 1, 2 \)
\[
\nabla \cdot (\beta_i(x) \nabla u_i) + a_i(x) u_i = f \quad \text{in } \Omega_i,
\]
\[
u_i = 0 \quad \text{on } \partial \Omega_i \cap \partial \Omega;
\]
\[
[u] = 0, \quad [\beta \nabla u \cdot n] = 0 \quad \text{along } \Gamma,
\]
where \( \beta \) is discontinuous along \( \Gamma \) but piecewise smooth in each subdomain, \( f \) is an appropriate smooth function, \( m_l \leq \beta_i(x) \leq m_u \) for some positive constants \( m_l \) and \( m_u \) and for all \( x \in \Omega_i \).

The mixed formulation of (3.1)-(3.3) is to seek a pair \((u, \lambda) \in X \times M\) such that
\[
a(u, v) + b(v, \lambda) + b(u, \mu) = \mathcal{F}(v) \quad \forall (v, \mu) \in X \times M,
\]
where
\[
a(v, w) = \sum_{i=1}^{2} \int_{\Omega_i} (\beta_i \nabla v_i \cdot \nabla w_i + a_i v_i w_i) \, dx,
\]
\[
\mathcal{F}(v) = \int_{\Omega} fv \, dx,
\]
\[
b(v, \mu) = \langle \mu, [v] \rangle_{00, \Gamma}
\]
and \( \lambda = \beta_1 \nabla u_1 \cdot n_1 = -\beta_2 \nabla u_2 \cdot n_2 \) is the Lagrange multiplier. We note that the bilinear forms \( a(\cdot, \cdot) \) and \( b(\cdot, \cdot) \) are continuous in \( X \times X \) and \( X \times M \) respectively.
The stabilized Nitsche’s mortaring approximation is: find \((u_h, \lambda_h) \in X_h \times W_h\) such that
\begin{equation}
A(u_h, \lambda_h; v_h, \mu_h) = F(v_h) \quad \text{for all } (v_h, \mu_h) \in X_h \times W_h,
\end{equation}
where
\begin{equation}
A(v, \mu; w, \Lambda) = a(v, w) + b(w, \mu) + S \int_{\Gamma} \gamma \mu \|\beta \nabla w \cdot \nu\| d\tau + b(v, \Lambda)
\end{equation}
and
\begin{equation}
- S \int_{\Gamma} \gamma \|\beta \nabla v \cdot \nu\| \|\beta \nabla w \cdot \nu\| d\tau + \int_{\Gamma} \gamma \|\beta \nabla v \cdot \nu\| \Lambda d\tau - \int_{\Gamma} \gamma \mu \Lambda d\tau.
\end{equation}
Here, \(S \in [0, 1]\) and \(\gamma\) is penalty parameter to be chosen later. When \(S = 1\) the above formulation is unsymmetric and for \(S = 1\), the formulation is symmetric.

Remark 3.1. In the paper [13] a stabilized Lagrange multiplier formulation to circumvent the inf-sup condition has been introduced where they used global polynomials over the interfaces as multipliers to avoid cumbersome integrations of functions from two different non-matching sides. We also propose a similar formulation but here we take the trace space as multiplier space, which involves integration of non-matched functions. But our aim here is to derive the error estimates in natural norms. These estimates can be extended for the case: using global polynomials as multipliers as in [13].

For any \(S \in [0, 1]\) it is easy to check that the problem (3.5) is consistent with the original problem (3.4) and hence the following lemma follows.

Lemma 3.2. The problem (3.5) is consistent with the original problem (3.4). Moreover, if \((u, \lambda)\) is the solution of (3.4) and \((u_h, \lambda_h)\) is the solution of (3.5), then
\begin{equation}
A(u - u_h, \lambda - \lambda_h; v_h, \mu_h) = 0 \quad \text{for all } (v_h, \mu_h) \in X_h \times W_h.
\end{equation}

Lemma 3.3. There exists \(\alpha > 0\) independent of \(h\) such that for all \((v_h, \mu_h) \in X_h \times W_h:\nA(v_h, \mu_h; v_h, -\mu_h) \geq \alpha \left( \sum_{i=1}^{2} \|v_h\|_{H^1(\Omega_i)}^2 + \|\gamma^{1/2} \mu_h\|_{L^2(\Gamma)}^2 \right) \quad \text{for } \gamma = \gamma_0 h \quad \text{with } 0 < \gamma_0 < \frac{m}{C_f m_u}, \quad C_f \text{ is a positive constant.}
\end{equation}

Proof. Taking \(v, w = v_h\) and \(\mu, \Lambda = \mu_h\) in (3.6), we arrive at
\begin{align}
A(v_h, \mu_h; v_h, -\mu_h) &= \sum_{i=1}^{2} \left( \beta_i \|\nabla v_h\|_{L^2(\Omega_i)}^2 + \alpha_i \|v_h\|_{L^2(\Omega_i)}^2 \right) \\
&\quad + (S - 1) \int_{\Gamma} \gamma \|\beta \nabla v_h \cdot \nu\| \mu_h d\tau - S \|\gamma^{1/2} \beta \nabla v_h \cdot \nu\|_{L^2(\Gamma)}^2 \\
&\quad + \|\gamma^{1/2} \mu_h\|_{L^2(\Gamma)}^2.
\end{align}

Using Cauchy-Schwarz inequality, Young’s inequality, Lemma 2.1 and using the bounds for \(\beta_i\), we find the third term in the right hand side of (3.8) as
\begin{equation}
\int_{\Gamma} \gamma \|\beta \nabla v_h \cdot \nu\| d\tau \leq \frac{C_f^2 \gamma_0 m_u^2}{2} \sum_{i=1}^{2} \|v_h\|_{H^1(\Omega_i)}^2 + \frac{1}{2} \|\gamma^{1/2} \mu_h\|_{L^2(\Gamma)}^2.
\end{equation}
Substituting (3.9) in (3.8) and using the bounds for $\beta_i$ and $a_i$, we find
\[
A(v_h, \mu_h; v_h, -\mu_h) \geq m_l \sum_{i=1}^{2} ||v_h||_{H^1(\Omega_i)}^2 + ||\gamma^{1/2} \mu_h||_{L^2(\Gamma)}^2 - \frac{SC^2 \gamma_m m_a^2}{2} \sum_{i=1}^{2} ||v_h||_{H^1(\Omega_i)}^2 \\
(S - 1) \left( \frac{C^2 \gamma_m m_a^2}{2} \sum_{i=1}^{2} ||v_h||_{H^1(\Omega_i)}^2 + \frac{1}{2} ||\gamma^{1/2} \mu_h||_{L^2(\Gamma)}^2 \right).
\]
Hence the result follows.

Note that the uniqueness of the solution of (3.5) is evident from the coercivity property (Lemma 3.3) of $A(\cdot, \cdot; \cdot, \cdot)$ which establish the existence of the solution.

**Theorem 3.4.** Let $(u, \lambda)$ and $(u_h, \lambda_h)$ be the solutions of (3.4) and (3.5) respectively with $u_i \in H^2(\Omega_i)$. Further, assume $W_h$ satisfies the strong regular property in the sense of Babuška (see [1]): there exists a constant $C > 0$ such that
\[
h^{1/2} ||\mu_h||_{L^2(\Gamma)} \leq C ||\mu_h||_{H^{-1/2}(\Gamma)} \quad \text{for all} \quad \mu_h \in W_h.
\]
Then for $\gamma = \gamma_0h$, there exists a positive constant $C$ independent of $h$ and $u$ such that the errors $e_u = u - u_h$ and $e_\lambda = \lambda - \lambda_h$ satisfies:
\[
||e_u||^2_X + ||e_\lambda||^2_{\lambda} \leq C \left( ||e_u||^2_X + ||e_\lambda||^2_{H^{-1/2}(\Gamma)} \right) \\
= C \|(e_u, e_\lambda)||^2 \leq C h^2 \sum_{i=1}^{2} ||u||^2_{H^2(\Omega_i)}.
\]

**Remark 3.5.** The $W_h$ space satisfying the strong regularity condition can be constructed using the technique of Hill functions, as in [2, 3] (see [1] page 186 for a discussion).

In order to prove the above theorem we require following results:

**Lemma 3.6.** There exist positive constants $C_1$ and $C_2$ such that for all $\mu_h \in W_h$,
\[
\sup_{0 \neq \mu_h \in X_h} \frac{b(v_h, \mu_h)}{||v_h||_X} \geq C_1 ||\mu_h||_{H^{-1/2}(\Gamma)} - C_2 \gamma_0 h^{1/2} ||\mu_h||_{L^2(\Gamma)}.
\]

**Proof.** For $i = 1, 2$, let $\bar{u}_i$ be the solution of the following mixed boundary value problem:
\[
-\Delta \bar{u}_i + \bar{u}_i = 0 \quad \text{on} \quad \partial \Omega_i \cap \partial \Omega_i, \quad \nabla \bar{u}_i \cdot \nu_i = (-1)^{i+1} \mu_h \quad \text{on} \quad \Gamma.
\]
Then there exist constants $C$ such that:
\[
||\bar{u}_i||_{H^{3/2}(\Omega_i)} \leq C ||\mu_h||_{L^2(\Gamma)}.
\]
Also from [1], we have
\[
||\bar{u}_i||_{H^{1}(\Omega_i)} \geq C ||\mu_h||_{H^{-1/2}(\Gamma)}.
\]
Let $\bar{u}_h$ be the Galerkin finite element approximation solution of (3.13), that is
\[
\int_{\Omega_i} \nabla \bar{u}_h \cdot \nabla \bar{u}_h \, dx + \int_{\partial \Omega_i} \bar{u}_h \cdot \bar{u}_h \, d\tau = \int_{\Gamma} (-1)^{i+1} \mu_h \bar{u}_h \, d\tau.
\]
Summing over $i = 1, 2$ we find:
\[
||\bar{u}_h||^2_X = - \int_{\Gamma} \mu_h [\bar{u}_h] \, d\tau = -b(\bar{u}_h, \mu_h)
\]
and

$$\|\bar{u} - \bar{u}_h\|_X \leq C \ h^{1/2} \sum_{i=1}^{2} \|\bar{u}_i\|_{H^{1/2}(\Omega_i)}.$$  

From (3.17), we arrive at

$$\sup_{0 \neq v_h \in X_h} \frac{b(v_h, \mu_h)}{\|v_h\|_X} \geq - \frac{b(\bar{u}_h, \mu_h)}{\|\bar{u}_h\|_X} = \|\bar{u}_h\|_X.$$  

Also, from triangle inequality, (3.14), (3.15) and (3.18), we find

$$\|\bar{u}_h\|_X \geq \|\bar{u}\|_X - \|\bar{u} - \bar{u}_h\|_X \geq C_1 \|\mu_h\|_{H^{-1/2}(\Gamma)} - C \ h^{1/2} \|\mu_h\|_{L^2(\Gamma)}.$$  

Hence, the Lemma follows from (3.19) and (3.20).

**Lemma 3.7.** If \( (v_h, \mu_h) \in X_h \times W_h \) then there exists \( C > 0 \) such that

$$A(u, \lambda; v_h, \mu_h) \leq C \left( \|u\|_X^2 + h \|\lambda\|_{L^2(\Gamma)}^2 \right)^{1/2} \|\lambda\|_{L^2(\Gamma)}$$

and

$$\sup_{(0,0) \neq (v_h, \mu_h) \in X_h \times W_h} \frac{A(w_h, \phi_h; v_h, \mu_h)}{\|v_h\|_X} \geq C \|A(w_h, \phi_h; v_h, \mu_h)\|.$$  

**Proof.** Applying Cauchy-Schwarz inequality and duality between \( H^{-1/2} \) and \( H^{1/2} \), we arrive at

$$A(u, \lambda; v_h, \mu_h) \leq \left[ \|u\|_X^2 + \|\lambda\|_{H^{-1/2}(\Gamma)}^2 \right]^{1/2} \|\lambda\|_{L^2(\Gamma)}$$

and

$$\sup_{(0,0) \neq (v_h, \mu_h) \in X_h \times W_h} \frac{A(w_h, \phi_h; v_h, \mu_h)}{\|v_h\|_X} \geq C \|A(w_h, \phi_h; v_h, \mu_h)\|.$$  

**Lemma 3.8.** There exists a constant \( C > 0 \) such that for \( (w_h, \phi_h) \in X_h \times W_h \):

$$\sup_{(0,0) \neq (v_h, \mu_h) \in X_h \times W_h} \frac{A(w_h, \phi_h; v_h, \mu_h)}{\|v_h\|_X} \geq C \|A(w_h, \phi_h; v_h, \mu_h)\|.$$  

**Proof.** Using (3.6), Lemma 3.7 and Young’s inequality, we find

$$A(w_h, \phi_h; -q_h, 0) = -A(w_h, 0; q_h, 0) - A(0, \phi_h; q_h, 0)$$

$$\geq -C \|w_h\|_X \|q_h\|_X - b(\phi_h, q_h) - S \int_{\Gamma} \gamma \phi_h \|\beta \nabla q_h \cdot \nu\| d\tau$$

$$\geq -C_1 \left( \frac{1}{c_1} \|w_h\|_X^2 + c_1 \|q_h\|_X^2 \right) - b(\phi_h, q_h)$$

$$\geq - \frac{C_1}{2} \left( \frac{1}{c_2} \|\phi_h\|_{L^2(\Gamma)}^2 + c_2 \|\nabla q_h \cdot \nu\|_{L^2(\Gamma)}^2 \right).$$

Now let \( q_h \in X_h \) be the function for which supremum occurs in condition (3.12) and assume that \( \|q_h\|_X = \|\phi_h\|_{H^{-1/2}(\Gamma)} \). Then using Lemma 2.1,

$$A(w_h, \phi_h; -q_h, 0) \geq - \frac{C}{2c_1} \|w_h\|_X^2 - (c_1 + c_2\gamma^2) \|\phi_h\|_{H^{-1/2}(\Gamma)}^2$$

$$- \frac{1}{2c_2} \gamma \|\phi_h\|_{L^2(\Gamma)}^2 + (c_4 \|\phi_h\|_{H^{-1/2}(\Gamma)} - c_5 \gamma \|\phi_h\|_{L^2(\Gamma)}) \|\phi_h\|_{H^{-1/2}(\Gamma)}$$

$$\geq -c_3 \|w_h\|_X^2 + c_4 \|\phi_h\|_{H^{-1/2}(\Gamma)}^2 - c_5 \gamma \|\phi_h\|_{L^2(\Gamma)}^2.$$
Let $0 < A_1 < \min(\alpha/c_1, \alpha/c_2)$. Considering $(v_h, \mu_h) = (w_h - \alpha_1 q_h, -\phi_h)$ and using the Lemma 3.3, we find

\[ A(w_h, \phi_h; v_h, \mu_h) = A(w_h, \phi_h; w_h - \alpha_1 q_h, -\phi_h) \]
\[ = A(w_h, \phi_h; w_h, -\phi_h) + \alpha_1 A(w_h, \phi_h; -q_h, 0) \]
\[ \geq (\alpha - \alpha_1 c_3) ||w_h||_X^2 + \alpha_1 c_4 ||\phi_h||_{H^{-1/2}(\Gamma)}^2 + (\alpha - \alpha_1 c_5) \gamma ||\phi_h||_{L^2(\Gamma)}^2 \]
(3.26)
\[ \geq C(||w_h||_X^2 + ||\phi_h||_{H^{-1/2}(\Gamma)}^2) = C(||(w, \phi)||^2. \]

Here, $c_1, c_2, c_3, c_4$ and $c_5$ are positive constants. Further,

\[ ||(v_h, \mu_h)||^2 \leq ||w_h||_X^2 + \alpha_1^2 ||q_h||_X^2 + ||\phi_h||_{H^{-1/2}(\Gamma)}^2 \]
(3.27)
\[ \leq C||(w_h, \phi_h)||^2. \]

Hence, (3.23) follows from (3.26) and (3.27). \hfill \Box

**Proof.** of Theorem 3.4: From Lemma 3.8, there exist a pair $(v_h, \mu_h) \in X_h \times W_h$ such that

(3.28)
\[ ||(v_h, \mu_h)|| < C \]

and that implies

(3.29)
\[ ||(I_h u - u_h, \Pi \lambda - \lambda_h)|| \leq A(I_h u - u_h, \Pi \lambda - \lambda_h; v_h, \mu_h). \]

From (3.29) and orthogonality (3.7) of $A(\cdot, \cdot, \cdot)$, we find

\[ ||(I_h u - u_h, \Pi \lambda - \lambda_h)|| \leq A(u - I_h u, \lambda - \Pi \lambda; v_h, \mu_h). \]

From (3.21) of Lemma 3.7,

\[ \frac{1}{2} ||(I_h u - u_h, \lambda - \Pi \lambda)|| \leq C \left[ \frac{1}{2} ||(u - I_h u, \lambda - \Pi \lambda)||^2 \right] \]
\[ + h ||\lambda - \Pi \lambda||_{L^2(\Gamma)}^{1/2} ||(v_h, \mu_h)||. \]
(3.30)

Note that

\[ ||(u - u_h, \lambda - \Pi \lambda)||^2 \leq 2 \left( ||(u - I_h u, \lambda - \Pi \lambda)||^2 + ||(I_h u - u_h, \Pi \lambda - \lambda_h)||^2 \right). \]

Hence, from (3.28), (3.30), Lemma 2.2 and Lemma 2.3, the (3.11) follows. \hfill \Box

For the $L^2$-error estimate, we appeal to the Aubin-Nitsche duality argument. Let $z_i = z_i|_{\Omega_i} \in H^2(\Omega_i) \cap H^1_0(\Omega_i), i = 1, 2$ be the solution of the interface problem

(3.31) \[-\nabla \cdot (\beta_i(x) \nabla z_i) + a_i z_i = u_i - u_h_i \text{ in } \Omega_i,\]
(3.32) \[z_i = 0 \text{ on } \partial \Omega \cap \partial \Omega_i,\]
(3.33) \[[z] = 0, [\beta \nabla z \cdot \nu] = 0 \text{ along } \Gamma,\]

which satisfies the regularity condition (see [4], [11])

(3.34) \[\sum_{i=1}^2 ||z_i||_{H^2(\Omega_i)} \leq c ||u - u_h||_{L^2(\Omega)}. \]

**Theorem 3.9.** Let $A_1(\cdot, \cdot, \cdot, \cdot)$ denotes the form $A(\cdot, \cdot, \cdot)$ with $S = 1$. Also let $(u, \lambda)$ and $(u_h, \lambda_h)$ be the solutions of respective equations as in Theorem 3.4 with $S = 1$. Then for $\gamma = \gamma_0 h$, $\gamma_0 > 0$, there exists a positive constant $C$ independent of $h$ and $u$ such that

(3.35) \[||u - u_h||_{L^2(\Omega)} \leq C h^2 \sum_{i=1}^2 ||u||_{H^2(\Omega_i)}. \]
Clearly, using trace inequality and Lemma 2.1, we find

\[ (3.36) \]

From (3.36) and (3.37) using Theorem 3.4, Lemma 2.2 and Lemma 2.3 with \( \gamma = \gamma_0 h \), we arrive at

\[ (3.38) \]
Hence, (3.35) follows by using the regularity condition (3.34).

Remark 3.10. For unsymmetric case, when $S \in [0, 1)$, the $L^2$-error estimate is of $O(h^{3/2})$.

However, with an additional assumption on the interpolants $I_k$ and $\Pi$ i.e.,

\begin{equation}
(h \left( \left\| \nabla I_h \nu \right\|_{L^2(\Gamma)}^2 + \left\| \Pi \lambda \right\|_{L^2(\Gamma)}^2 \right) \leq C h^2 \left( \left\| \nabla \nu \right\|_{H^{1/2}(\Gamma)}^2 \right)
\end{equation}

we can establish optimal order of $L^2$-estimate in a similar way as in Barbosa et al. (see [6]).

4. Matrix formulation

The stabilized Nitsche’s mortaring method (3.5) can be represented in matrix form by $AU = F$.

The stiffness matrix

\[ A = \begin{pmatrix}
A_{11}^1 & A_{12}^1 & 0 & 0 & 0 & Q_s + \frac{S_2}{2} R_s - \frac{S_2}{4} R_n^s \\
A_{12}^1 & A_{22}^1 & 0 & 0 & 0 & 0 \\
0 & 0 & A_{21}^2 & A_{22}^2 & -(Q_s)^T + \frac{S_2}{2} (R_m)^T - \frac{S_2}{4} (R_n^m)^T \\
0 & 0 & 0 & A_{22}^m & -\gamma Q_{mm}
\end{pmatrix}, \]

where, for $1 \leq l \leq 2$, 

\[
(A_{11})_{ij} = a(\phi_i^1, \phi_j^1), \quad (A_{12})_{ij} = a(\phi_i^1, \phi_j^2), \quad (A_{21})_{ij} = a(\phi_i^2, \phi_j^1), \quad (A_{22})_{ij} = a(\phi_i^2, \phi_j^2),
\]

\[
(Q_s)_{ij} = \int_{\Gamma_1} \phi_i^s \psi_j d\tau, \quad (Q_m)_{ij} = \int_{\Gamma_2} \psi_i^m \psi_j d\tau,
\]

\[
(R_n^s)_{ij} = \int_{\Gamma_2} \beta_1 \nabla \phi_i^s \cdot \nu_1 \psi_j d\tau, \quad (R_n^m)_{ij} = \int_{\Gamma_2} \beta_2 \nabla \phi_i^m \cdot \nu_2 \psi_j d\tau.
\]

Also, $U = (u_1, u_2, u_1^m, u_2^m, \lambda_m)^T$. The unknowns $u_i^1$ and $u_i^2$ are associated with the internal nodes in $\Omega_1$ and $\Omega_2$ respectively. Unknown $u_i^m$ are associated with $\Gamma_1$ and $\Gamma_2$ and $\lambda_m$ are the unknown Lagrange multipliers associated with $W_h$.

The load vector $F = (F_1^1, F_2^1, F_1^2, F_2^2, 0, 0)^T$, where,

\[
(F_1^1)_{i} = (f_i^1, \phi_i^1), \quad x_i \in \Omega_1, \quad (F_2^1)_{i} = (f_i^1, \phi_i^2), \quad x_i \in \Omega_1, \quad x_i \in \Gamma_1,
\]

\[
(F_1^2)_{i} = (f_i^2, \phi_i^m), \quad x_i \in \Omega_2, \quad (F_2^2)_{i} = (f_i^2, \phi_i^m), \quad x_i \in \Omega_2, \quad x_i \in \Gamma_2.
\]

Here, $\psi_i$ are the nodal basis functions for $W_h$. $\phi_i^s$ and $\phi_i^m$ are the basis functions for $W_{h_1}$ and $W_{h_2}$ respectively.

5. Numerical experiments

We choose problem (3.1)-(3.3) over the unit square domain $\Omega = (0, 1) \times (0, 1)$. We divide the domain $\Omega$ into two equal subdomains $\Omega_i$, $i = 1, 2$ (see Figure 2). Each subdomains further subdivided into linear triangular elements of different mesh size $h_i$. We choose the penalty parameter to be $\gamma = O(h)$. Set $a_1$ and $a_2$ to be zero. We choose $f$ such that the exact solution of the problem is $u(x, y) = \sin^2 \pi x \sin^2 \pi y$. 
The order of convergence ‘p’ for the error $||u - u_h||_{L^2(\Omega)}$ and the order of convergence ‘q’ for the error $||\lambda - \lambda_h||_{L^2(\Gamma)}$ with respect to the discretization parameter $h$ are computed by taking discontinuous coefficients pairs $(\beta_1, \beta_2) = (1, 10), (1, 10^7), (10^7, 10^{-7})$ in the subdomains $\Omega_1$ and $\Omega_2$, see Tables 1, 2, 3. Figure 3 (a) shows the computed order of convergence for $||u - u_h||_{L^2(\Omega)}$ with respect to $h$ in the log-log scale. Figure 3 (b) shows the convergence rate of the Lagrange multiplier with respect to $h$. Note that, since the exact solution is smooth, the convergence rates of error $||u - u_h||_{L^2(\Omega)}$ and $||\lambda - \lambda_h||_{L^2(\Gamma)}$ are computationally obtained as expected i.e., $O(h^2)$ and $O(h)$ respectively.

Table 1. Order of Convergence of $||u - u_h||_{L^2(\Omega)}$ and $||\lambda - \lambda_h||_{L^2(\Gamma)}$

| $(h_1, h_2)$   | $h$  | $||e_u||_{L^2(\Omega)}$ | $||e_\lambda||_{L^2(\Gamma)}$ | $p$           | $q$           |
|---------------|------|-----------------|-----------------|--------------|--------------|
| $(\frac{1}{4}, \frac{1}{16})$ | 1/4  | 0.062158        | 0.43687         |              |              |
| $(1/8, 1/16)$  | 1/8  | 0.016638        | 0.24098         | 1.901458062028300 | 0.858290623104806 |
| $(1/16, 1/24)$ | 1/16 | 0.0041882       | 0.11229         | 1.990079779863557 | 1.101683961685886 |
| $(1/32, 1/48)$ | 1/32 | 0.0010422       | 0.049426        | 2.006698177352890 | 1.183887393718836 |
| $(1/64, 1/96)$ | 1/64 | 0.00025934      | 0.021541        | 2.006715513479729 | 1.198184929427100 |
TABLE 2. Order of Convergence of $||u - u_h||_{L^2(\Omega)}$ and $||\lambda - \lambda_h||_{L^2(\Gamma)}$ with $\beta_1 = 1, \beta_2 = 10^7$

| $(h_1, h_2)$ | $h$ | $||e_u||_{L^2(\Omega)}$ | $||e_\lambda||_{L^2(\Gamma)}$ | $p$ | $q$ |
|-------------|-----|----------------------|----------------------|-----|-----|
| $(\frac{1}{4}, \frac{1}{4})$ | 1/4 | 0.061619             | 0.45209              |     |     |
| $(\frac{1}{8}, \frac{1}{8})$ | 1/8 | 0.016431             | 0.25338              | 1.906954983214551 | 0.83530735359031 |
| $(\frac{1}{16}, \frac{1}{16})$ | 1/16 | 0.0041218           | 0.11953              | 1.995073877668074  | 1.083929897366207 |
| $(\frac{1}{32}, \frac{1}{32})$ | 1/32 | 0.0010233           | 0.052978             | 2.010045342634421  | 1.173907469688573  |
| $(\frac{1}{64}, \frac{1}{64})$ | 1/64 | 0.00025429         | 0.023106             | 2.008682526770985  | 1.197125851836898  |

TABLE 3. Order of Convergence of $||u - u_h||_{L^2(\Omega)}$ and $||\lambda - \lambda_h||_{L^2(\Gamma)}$ with $\beta_1 = 10^{-7}, \beta_2 = 10^7$

| $(h_1, h_2)$ | $h$ | $||e_u||_{L^2(\Omega)}$ | $||e_\lambda||_{L^2(\Gamma)}$ | $p$ | $q$ |
|-------------|-----|----------------------|----------------------|-----|-----|
| $(\frac{1}{4}, \frac{1}{4})$ | 1/4 | 0.058294              | 1.8372e-07            |     |     |
| $(\frac{1}{8}, \frac{1}{8})$ | 1/8 | 0.015655             | 6.2439e-08            | 1.896723890978434 | 1.556989351729802  |
| $(\frac{1}{16}, \frac{1}{16})$ | 1/16 | 0.0040013         | 2.366e-08             | 1.968082803651483  | 1.399997358151351  |
| $(\frac{1}{32}, \frac{1}{32})$ | 1/32 | 0.0010066           | 1.032e-08             | 1.990978296765009  | 1.197007102916534  |
| $(\frac{1}{64}, \frac{1}{64})$ | 1/64 | 0.00025206         | 5.325e-09             | 1.997651406176591  | 0.954589540310056  |

(a) Order of Convergence of $||u - u_h||_{L^2(\Omega)}$ w.r.t. $h$.

(b) Order of Convergence of $||\lambda - \lambda_h||_{L^2(\Gamma)}$ w.r.t. $h$.

**FIGURE 3.** Order of Convergence with discontinuous coefficients. $\beta_1 = 1, \beta_2 = 10$

6. CONCLUSION

In order to alleviate the inf-sup condition in the mortar method with Lagrange multiplier, a stabilized method is presented and optimal error estimates are obtained in natural norm which is independent of mesh. Numerical experiments presented here depict the performance of the method and supports the theoretical error estimates. Here, the multipliers are simply the nodal basis functions restricted to the interface, one can consider global polynomials as multipliers as in [13] to avoid the cumbersome integration over unrelated meshes.
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