4D and 2D Evaporating Dilatonic Black Holes

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ABSTRACT

The picture of S-wave scattering from a 4D extremal dilatonic black hole is examined. Classically, a small matter shock wave will form a non-extremal black hole. In the “throat region” the $r - t$ geometry is exactly that of a collapsing 2D black hole. The 4D Hawking radiation (in this classical background) gives the 2D Hawking radiation exactly in the throat region. Inclusion of the back-reaction changes this picture: the 4D solution can then be matched to the 2D one only if the Hawking radiation is very small and only at the beginning of the radiation. We give (explicitly) that 4D solution. When the total radiating energy approaches the energy carried by the shock wave, the 4D picture breaks down. This happens even before an apparent horizon is formed, which suggests that the 4D semi-classical solution is quite different from the 2D one.

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1 Introduction

There is much current interest in both the classical and quantum aspects of dilatonic black holes in various dimensions. Let us review some relevant ideas. Consider the extremal 4D magnetically charged black hole which has two asymptotic spatial regions. One is far away from the “black hole”, and the other is “down the infinite throat” [1]. The 4D manifold down the infinite throat is \( \Sigma^4 = \Sigma^2 \times S^2 \), where \( S^2 \) is a 2-sphere of constant radius. By a Kaluza-Klein process one can get from it an effective 2D theory, defined on \( \Sigma^2 \) [2]. A general static solution of this 2D theory is a black hole [2,3]. Including matter fields, one can describe a collapse process, and semiclassical Hawking radiation [4]. The backreaction calculations can be handled in 2D. By extending the model [5-7], one can even solve exactly the one loop backreaction equations. The solutions describe formation and evaporation of a black hole. But there are still some open questions concerning the “end point” of the process [8,9] and the mass definition [10-13], so the information puzzle is still unresolved.

Is there a 4D interpretation of those 2D results? The standard picture is that the 2D process is the \( r - t \) part of a 4D S-wave scattering. In this paper we will examine this picture. It is very easy to find the relation between the 4D and 2D static black holes, and even to describe a classical collapsing star. Hawking radiation (in the classical background of a collapsing star) is well known in both 4D [14] and 2D [4], the relation between them being consistent with the S-wave picture, as we will see. When the back-reaction is taken into account, a simple S-wave scattering is consistent only with very small Hawking radiation, and only at the beginning of the radiation process, before the radiation carries away the total energy of the matter shock wave. In that case one can give an explicit 4D interpretation. Namely, one can
find a 4D solution (including the back-reaction) that will correspond to the
2D evaporating star. If the Hawking radiation is not very small, it is not
clear whether there is a 4D interpretation. It is not clear if one can find a 4D
solution (including the backreaction) that corresponds to the 2D evaporating
star. In that case the 4D “1 loop” Einstein equations, $G_{\mu\nu}^{(4D)} = < T_{\mu\nu}^{(4D)} >$
, are quite complicated, unlike the case of very small Hawking radiation, in
which those equations become very simple.

The structure of the paper is as follows: In section 2, we describe the
relation between the 4D and 2D static dilatonic black holes, and we will see
the explicit relation between the almost extremal 4D black hole and the 2D
one. In section 3, we will describe the classical shock wave collapsing star
(in 4D and 2D). In section 4, the Hawking radiation (in the classical back-
ground) is considered. In section 5, we will study the backreaction problem.
Concluding remarks are in section 6.

## 2 4D and 2D Static Dilatonic Black Holes

The 4D action describing dilaton gravity is [1]

$$S^{(4)} = \frac{1}{2\pi} \int d^4x \sqrt{-g} e^{-2\phi} \left( R^{(4)} + 4(\nabla \phi)^2 - \frac{1}{2} F^2 \right)$$  \hspace{1cm} (1)$$

The magnetically charge spherical symmetric static solution is

$$ds^2 = e^{2\phi_0} \left( \frac{(1 - r_H/r)}{(1 - r_s/r)} dt^2 + \frac{dr^2}{(1 - r_H/r)(1 - r_s/r)} + r^2 d\Omega_2^2 \right)$$

$$F_{\theta\phi} = Q \sin\theta$$

$$e^{-2\phi} = e^{-2\phi_0} \left( 1 - r_s/r \right).$$  \hspace{1cm} (2)$$

Here $M$ and $Q$ are the mass and charge of the black hole, $r_H$ is the horizon,
$r_H = 2M$ , and $r_s$ is the singularity, $r_s = Q^2 e^{2\phi_0}/M$. To prevent a naked
singularity, we require $r_H \geq r_s$ (or $2M^2 \geq Q^2 e^{2\phi_0}$, the equality defines the
extremal black hole). There are three regions in this space-time: 1) the exterior, \( r > r_H \). 2) \( r_H > r > r_s \), which is inside the black hole. 3) \( r_s > r > 0 \); this is an “extra region”, which includes a naked singularity. Region 3) is of course disconnected (and “irrelevant” to observations made in 1) and 2).). We will be interested only in regions 1) and 2), which describe the black hole, so the radial coordinate

\[ \bar{r} \equiv r - r_s \]

will be non-negative. It is convenient to define the deviation from the extremal solution

\[ \epsilon \equiv \frac{r_H}{r_s} - 1 \]

In terms of these, the metric becomes

\[ ds^2 = e^{2\phi_0} \left( -(1 - \epsilon r_s/\bar{r}) dt^2 + \frac{(\bar{r}/r_s + 1)^2 d\bar{r}^2}{(1 - \epsilon r_s/\bar{r})(\bar{r}/r_s)^2} + (\bar{r} + r_s)^2 d\Omega_2^2 \right) \]

and the dilaton field is

\[ e^{-2\phi} = e^{-2\phi_0} \frac{\bar{r}}{\bar{r} + r_s} \].

We can see from (5) and (6) that for \( \bar{r} << r_s \)

\[ ds^2 \rightarrow e^{2\phi_0} \left( -(1 - \epsilon r_s/\bar{r}) dt^2 + \frac{d\bar{r}^2}{(1 - \epsilon r_s/\bar{r})(\bar{r}/r_s)^2} + r_s^2 d\Omega_2^2 \right) \]
\[ e^{-2\phi} \rightarrow e^{-2\phi_0} \frac{\bar{r}}{r_s} \].

Thus, near \( r_s \), the 4D manifold reduces to \( \Sigma^4 = \Sigma^2 \times S^2 \), where \( S^2 \) is a 2-sphere, with constant radius \( R = e^{\phi_0} r_s \). The \( \Sigma^2 \) part represents a 2D black hole as long as \( \epsilon << 1 \). One can see that directly from (7), but we will see this explicitly in the following.

In the region \( \bar{r} << r_s \) we can use the standard Kaluza-Klein procedure to get (from (1)) the 2D action (on \( \Sigma^2 \))

\[ S^{(2)} = \frac{1}{2\pi} \int d^2x \sqrt{-g} e^{-2\phi} \left( R^{(2)} + 4(\nabla \phi)^2 + 4\lambda^2 \right) \]
where
\[ 4\lambda^2 = \frac{2}{R^2} - \frac{1}{2}F^2(r_s) = \frac{e^{-2\phi_0}}{r_s^2} - \frac{Q^2}{r_s^4} \] (10)

There is also a vector field term (from the Kaluza-Klein reduction) that we did not write explicitly in (9), because we will not consider excitations in it. Using conformal gauge, \( g_{++} = g_{--} = 0, \ g_{+-} = -\frac{1}{2}e^{2\phi} \), the static black hole solution (sometimes called Witten’s b.h.) of (9) is [3,4]
\[ e^{-2\phi} = e^{-2\rho} = \frac{m}{\lambda} - \lambda^2 x^+x^- \] (11)

where \( x^+, x^- \) are the light-cone coordinates for which the metric is \( g_{+-} = -\frac{1}{2}e^{2\rho} \), and \( m \) is the 2D ADM mass\(^2\). In the coordinates \((t, \hat{r})\)

\[ t \equiv \frac{1}{2}ln(-x^+/x^-) \] (12)

\[ 2\lambda^2 \hat{r} \equiv e^{-2\phi(x^+,x^-)} \] (13)

the 2D metric is
\[ ds^2 = \lambda^{-2} \left( -(1 - m/2\lambda^3 \hat{r})dt^2 + \frac{d\hat{r}^2}{(1 - m/2\lambda^3 \hat{r})(2\hat{r})^2} \right) \] (14)

We see that the \( \Sigma^2 \) part of the 4D solution (7),(8) and the 2D solution (13),(14) have the same form, but with different parameters. Before comparing them, we must remember that the 4D solution (2) is a three parameter solution \((M, Q, \phi_0)\), while the 2D solution (11) is a two parameter solution \((m, \lambda)\). The reason for that is that \( r_s \) (the radius of the 2-sphere), which is a function of the three 4D parameters, is a scale that cannot appear in the 2D solution. We therefore should fix this scale before relating the solutions. After choosing the coordinates (12),(13) it is convenient to take
\[ r_s = \frac{Q^2 e^{2\phi_0}}{M} = \frac{1}{2}. \] (15)

\(^2\)One should not confuse the 2D mass, \( m \), with the 4D mass, \( M \). The relation between them will be given later.
When this scale is fixed, we get from (10) and (15) that
\[ \lambda^2 = e^{-2\phi_0}(1 - \epsilon), \]
and the 4D solution (7),(8) become the 2-parameter set
\[ ds^2 \rightarrow \frac{(1 - \epsilon)}{\lambda^2} \left((-1 - \epsilon/2\tilde{r})dt^2 + \frac{d\tilde{r}^2}{(1 - \epsilon/2\tilde{r})(2\tilde{r})^2} + \frac{1}{4}d\Omega_2^2\right) \] (16)
\[ e^{-2\phi} \rightarrow \frac{2\lambda^2\tilde{r}}{1 - \epsilon} \] (17)

We can see from (13),(14) and (16),(17) that if we identify
\[ \hat{r} = \tilde{r}, \quad \epsilon = \frac{m}{\lambda^3} \] (18)
then for \( \epsilon << 1 \), \( \Sigma^2 \) is exactly the 2D Witten black hole.

At this point we have explicit relations describing the well known picture: the 2D black hole solution is the \( r - t \) part of a 4D almost extremal black hole \( (0 < \epsilon << 1) \) in the region “down the throat” \( (\tilde{r} << r_s) \). The 2D mass, \( m \), is really the mass deviation from the 4D extremal black hole (18). As was noticed by Witten [3], the 2D zero mass solution (the linear dilaton solution) represents an extremal 4D black hole and not a flat 4D space-time.

Outside the region \( \tilde{r} << r_s \), the 4D and 2D solutions are quite different. For example \( \tilde{r} \) but not \( \hat{r} \), is an asymptotically flat coordinate (see (5)). So if we want to give a 4D interpretation to the 2D results, we must restrict ourselves to \( \tilde{r} << r_s = 1/2 \) (or \( \hat{r} << 1 \)). Using (11) and (13) we see that this means that \( x^+x^- << 1 \). Let \( \tilde{r}_c = \hat{r}_c << 1 \) be the radius at which we “glue” the 4D solutions in the following sense: For \( \tilde{r} \leq \tilde{r}_c \) we will use the 2D results to describe the \( r - t \) part of the 4D space-time. But we cannot do that for \( \tilde{r} \geq \tilde{r}_c \); in that region one must solve the 4D equations. In the \( (x^+, x^-) \) plane, \( \hat{r} = \tilde{r}_c \) is a line \( x^+x^- = \text{const.} \) << 1.

The proper distance from \( \tilde{r} = \tilde{r}_c \) to the singularity \( (\tilde{r} = 0) \) goes like \( \ln(1 + c\tilde{r}_c/\epsilon) \) where \( c \) is some constant. So if we want a long throat, \( \tilde{r}_c/\epsilon \) must be much bigger then 1, which means \( \epsilon << \tilde{r}_c << 1 \). So from now on we can neglect \( \epsilon \) (but not \( \epsilon/\tilde{r}_c \) or \( \tilde{r}_c \)) relative to 1.
Classical Collapsing Star

Adding matter fields to (1) or (9) enables us to find solutions that describe a collapse process. A simple collapse process can be described by an “$f$-shock wave” (The $f$ fields are zero everywhere but at $x^+ = x^+_0$). The solution is the extremal black hole (m = 0 (or $\epsilon = 0$ )) for $x^+ < x^+_0$, and non-extremal black hole ($m > 0$) for $x^+ > x^+_0$. The energy carried by the $f$ fields is $m$.

First consider the 2D case [4]. The 2D classical action acquires a kinetic term for the matter fields $f_i$,

$$S^{(2)} = \frac{1}{2\pi} \int d^2x \sqrt{-g} \left[ e^{-2\phi} \left( R^{(2)} + 4(\nabla \phi)^2 + 4\lambda^2 \right) - \frac{1}{2} \sum_{i=1}^N (\nabla f_i)^2 \right]$$

The classical collapsing solution is

$$f_i = 0, \quad x^+ \neq x^+_0$$

$$e^{-2\phi} = e^{-2\rho} = -\lambda x^+ x^- - \frac{m}{\lambda x^+_0} (x^+ - x^+_0) \Theta(x^+ - x^+_0)$$

We see that this solution describes a linear dilaton solution for $x^+ < x^+_0$, and a 2D black hole (with mass $m$) for $x^+ > x^+_0$. The classical energy momentum tensor of the $f$ fields is $T^{(f)}_{\mu\nu} = \frac{m}{\lambda x^+_0} \delta(x^+ - x^+_0)$ (an incoming shock wave).

Is there a corresponding 4D solution? It should be the extremal black hole ($\epsilon = 0$) for $x^+ < x^+_0$, and non-extremal black hole ($\epsilon > 0$) for $x^+ > x^+_0$. It is very easy to see that this is indeed the case. For $\tilde{r} > \tilde{r}_c$ one must solve the 4D equations of motion, implied by (1) (with the matter fields)

$$e^{-2\phi}(R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \phi - F_{\mu\delta} F^\delta_{\nu}) + \pi T\!\!\!_C^{\mu\nu} = 0$$

$$4\nabla^2 \phi - 4(\nabla \phi)^2 + R - \frac{1}{2} F^2 = 0$$

$$\nabla_\mu (e^{-2\phi} F^{\mu\nu}) = 0$$

where $T\!\!\!_C^{\mu\nu}$ is the classical matter energy momentum tensor.
In our case $\frac{\epsilon}{\tilde{r}_c} << 1$, and we see from (5) that for $\tilde{r} \geq \tilde{r}_c$ the metric changes only slightly relative to the extremal metric ($\epsilon = 0$). So we can use the linearised equations. The background “vacuum” metric is the extremal black hole whose line element (see (5)) is

$$ds^2_{(0)} = g_{\mu\nu}^{(0)} dx^\mu dx^\nu = e^{2\phi_0} \left( -dt^2 + \left( 1 + \frac{r_s}{\tilde{r}} \right)^2 d\tilde{r}^2 + (\tilde{r} + r_s)^2 d\Omega_2^2 \right)$$

where $r^* = \tilde{r} + r_s \ln(\tilde{r})$. The linear deviations from the “vacuum” metric, dilaton and EM fields, are defined by

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + h_{\mu\nu}$$

$$h_{\mu\nu} = e^{2\phi_0} \text{diag}(\delta, \sigma, (\tilde{r} + r_s)^2 \eta, ((\tilde{r} + r_s)^2 \sin^2 \theta) \eta)$$

$$\phi(t, \tilde{r}) = \phi_0 - \frac{1}{2} \ln \left( \frac{\tilde{r}}{\tilde{r} + r_s} \right) + \gamma(t, \tilde{r})$$

$$F_{\theta\phi} = Q \sin \theta (1 + \rho)$$

where $\delta$, $\sigma$, $\eta$, $\gamma$ and $\rho$, are all much smaller then one. The linearised form of (23) using (25) leads to $\eta \simeq 0$, where in our approximation $\simeq 0$ means of order $\epsilon$ or $(\epsilon/\tilde{r}_c)^2$. Equation (22) is just the Bianchi identity obeyed by (21), so one should not consider it as an independent equation. The non-vanishing components of the linearised Einstein equations (21), are

$$(tt) \quad \frac{1}{2}(\dot{\sigma} + \delta'') + \frac{(\tilde{r} + r_s/2)}{(\tilde{r} + r_s)^2} \delta' - 2\dot{\gamma} = \pi e^{2\phi} T_{tt}^C$$

$$(rr) \quad -\frac{1}{2}(\dot{\sigma} + \delta'') - \frac{(\tilde{r} + r_s/2)}{(\tilde{r} + r_s)^2} \sigma' - 2\gamma'' = \pi e^{2\phi} T_{rr}^C$$

$$(tr) \quad -\frac{(\tilde{r} + r_s/2)}{(\tilde{r} + r_s)^2} \dot{\delta} - 2(\dot{\gamma})' = \pi e^{2\phi} T_{tr}^C$$

$$(\theta\theta) \quad \frac{\tilde{r}}{2}(\delta' + \sigma') - \frac{\tilde{r}r_s}{(\tilde{r} + r_s)^2} \sigma + \frac{r_s}{(\tilde{r} + r_s)^2} \rho = 0$$
where prime and dot denote differentiation with respect to $r^*$ and $t$ respectively. The 4D energy momentum tensor is that of the shock wave, and can be simply gotten from the 2D one: $(^{(4D)}T^{\mu\nu}_{\mu\nu} = (^{(2D)}T^{\mu\nu}_{\mu\nu})/4\pi(\tilde{r} + r_s)^2$, where $4\pi(\tilde{r} + r_s)^2$ is the surface factor relating the 2D and 4D densities. Using the 2D results, we find (dropping the 4D superscripts) $T^{C}_{tt} = T^{C}_{rr} = T^{C}_{tr} = T^{++}$, so

$$e^{2\phi}T^{C}_{tt} = e^{2\phi}T^{C}_{rr} = e^{2\phi}T^{C}_{tr} = \frac{x_0^+ e}{4\pi\tilde{r}(\tilde{r} + r_s)}\delta(x^+ - x_0^+)$$

(30)

where $x^\pm \equiv \pm\exp(\pm u^\pm) = \pm\exp[\pm(t \pm r^*)]$. The r.h.s of (26)-(29) is much smaller then 1, which is consistent with $\frac{\rho}{\rho_c} << 1$. We are using here the Eddington-Finkelstein coordinates, $(t, r^*)$, for which $\delta = -\sigma$. So we have from (29) $\rho = \tilde{r}\sigma$ and from (26) (or (27) using $T_{tt} = T_{rr}$)

$$\frac{1}{2}(\tilde{\sigma} - \sigma'') - \frac{(\tilde{r} + r_s/2)}{(\tilde{r} + r_s)^2}\sigma' - 2\tilde{\gamma} = \pi e^{2\phi}T^{C}_{tt}$$

(31)

The two equations (28) and (31) determine the two functions $\sigma$ and $\gamma$. There are two boundary conditions: the solution must vanish as $\tilde{r} \to \infty$, and coincide with the 2D one at $\tilde{r} = \tilde{r}_c$. The corresponding solution is

$$\delta = -\sigma \simeq \frac{\epsilon r_s}{\tilde{r}}\Theta(u^+ - u_0^+)$$

(32)

$$\gamma \simeq 0$$

(33)

$$\rho \simeq 0$$

(34)

which is the expected linearised form of the collapsing star.

### 4 Hawking Radiation in a Classical Background

Consider first the 2D case [4]. Using the trace anomaly one can calculate the Hawking radiation in the linear dilaton vacuum (the $m = 0$ vacuum)

$$T^{Q}_{++} = 0, \quad T^{Q}_{--} = \frac{\kappa \lambda^2}{4} \left[ \left( 1 - \frac{\epsilon}{2\tilde{r}} \right)^2 - \left( 1 - \frac{x_H^-}{x^-} \right)^2 \right]$$

(35)
where $x^- = x^-_H$ is the horizon, $x^-_H = \frac{m}{\lambda x_0}$, and $\kappa \sim \frac{N}{12}$ in the large $N$ limit \footnote{$\kappa$ depends on the quantization scheme that one uses [5-7,15]}. On $I^+_R$, $x^+ \to \infty$ ($\hat{r} \to \infty$), one gets

$$T^Q_{-+} \to \frac{\kappa \lambda^2}{4} \left(1 - \left(1 - \frac{x^-_H}{x^-}\right)^2\right)$$

and so for late times ($x^- \to x^-_H$), one gets the thermal Hawking radiation with the temperature $T_H = \lambda/2\pi$.

Notice that because $\frac{\hat{r}}{\hat{r}_c} << 1$, (35) and (36) are almost the same. This means that there is almost no redshifting\footnote{In 2D there is no surface factor between the energy and the energy density.} in $\hat{r} > \hat{r}_c$.

Because at $\hat{r} = \hat{r}_c$ the 2D is a good approximation, the 4D energy flux at $\hat{r}_c = \hat{r}_c$ will be (35) (multiply by a surface factor). For $\hat{r} > \hat{r}_c$, it would seem that one should solve the 4D equations. But as we are going to see now, it is unnecessary. The 4D radiation at $\hat{r} = \hat{r}_c$ will be almost the same as at infinity, so one can use the 2D radiation. Consider late time radiation ($x^- \to x^-_H$). The 4D radiation at $\hat{r} \to \infty$, is related to the Hawking temperature. As we know, this temperature goes like $M^{-1}$, where $M$ is the 4D mass; the 4D Hawking temperature might be quite different from the 2D one, and if this were the case, the 4D energy momentum tensor at $\hat{r}_c$ would be quite different than at infinity. But this is not the case. The 4D and 2D Hawking temperature are not very different, and in the $\epsilon \to 0$ “limit” they are exactly the same. There are several ways to calculate the Hawking temperature. One can use the $r-t$ part of the exact 4D solution (5), and calculate the Hawking radiation in the Israel-Hawking vacuum, or one can use the surface gravity $\mathcal{K}$,

$$T_H = \frac{1}{2\pi} \kappa = \frac{1}{2\pi} \left(\lim_{r \to \infty} g_{tt}^{-1/2}\right) \left((g_{tt}g_{rr})^{-1/2}\partial g_{tt}/\partial r\right)(r = r_H) \quad (37)$$
In the 2D case one get \( T_{H}^{(2D)} = \lambda / 2\pi \), and in 4D, \( T_{H}^{(4D)} = (8\pi M e^{\phi_0})^{-1} \). But using (4) and (15) we see that

\[
T_{H}^{(2D)} = (1 + \epsilon)(1 - \epsilon)^{1/2}T_{H}^{(4D)}
\]  

(38)

So (up to \( \epsilon \), which we neglect relative to 1) the two temperature are the same. This means that the 4D radiation at \( \tilde{r}_c \) is almost the same as at infinity, so (as in the 2D case) there is almost no redshifting in \( \tilde{r} > \tilde{r}_c \). This is consistent with (5), because if \( \epsilon/\tilde{r}_c << 1 \), the 4D metric for \( \tilde{r} > \tilde{r}_c \) is almost the “vacuum metric” (the extremal black hole metric).

All the above considerations assume a classical background metric (the classical collapsing star), without back-reaction; we next come to this.

5 4D and 2D back-reaction

In the 2D case, one can deal with the back-reaction calculations much more easily than in the 4D case. But even in this simple 2D world, there are no known exact solutions to the original CGHS model [4], though there are some numerical ones [8,9]. But one can extend the CGHS model and find exact results [5-7]. We will not describe the details here, but just say that those results describe formation and evaporation of a black hole, ending with a naked singularity. An interesting result is that the quantum energy momentum tensor (describing the Hawking radiation) is exactly the same as (36), the one that we get using the classical background.

Another thing is that the semiclassical approximation is valid as long as \( \frac{m}{\lambda} > \kappa \sim \epsilon = \frac{m}{\lambda^3} \) [16]. This means that \( \lambda^2 >> 1 \), or equivalently that the 4D coupling at infinity is small, \( G^{(4d)}(\tilde{r} \to \infty) \simeq \lambda^{-2} << 1 \). So although we have a very small 4D \( f \)-shock wave (\( \epsilon << 1 \)), the corresponding 2D mass
m, is not small in this weak coupling picture, unlike in [17], in which a very small 2D shock wave was studied.

The simple S-wave picture is that the 4D metric (for \( \tilde{r} > \tilde{r}_c \)) should not change much after the Hawking radiation is started, because if it changes, it is likely that there will be a big redshift, and the Hawking radiation at \( \tilde{r}_c \) will be much different than at infinity. In that case the only way to calculate it is to solve the complicated 4D equations (including the back-reaction). This is of course beyond the scope of this paper, and it is not clear at all if that 4D solution can be matched to the 2D one. Assuming small perturbations of the solution, we can still use the linearised equations. Assuming S-wave scattering we can use the perturbations (25), and the linearised form of (26)-(29), where \( T_{\mu\nu}^C \) should be replaced by \( T_{\mu\nu}^M = T_{\mu\nu}^C + T_{\mu\nu}^Q \). Using (30) and (36) we get

\[
e^{2\phi} T_{tt}^M = e^{2\phi} T_{rr}^M = \frac{1}{4\pi(\tilde{r} + r_s)} \left[ \frac{x_0^+ \epsilon}{\tilde{r}} \delta(x^+ - x_0^+) + \frac{\kappa}{4\tilde{r}} \left( \frac{2x_H^-}{x^-} - \left( \frac{x_H^-}{x^-} \right)^2 \right) \right]
\]

\[
e^{2\phi} T_{tr}^M = \frac{1}{4\pi(\tilde{r} + r_s)} \left[ \frac{x_0^+ \epsilon}{\tilde{r}} \delta(x^+ - x_0^+) - \frac{\kappa}{4\tilde{r}} \left( \frac{2x_H^-}{x^-} - \left( \frac{x_H^-}{x^-} \right)^2 \right) \right]
\]

(39)

The first part on the r.h.s of (39) is the classical shock wave part, and the second is the Hawking radiation part. As we saw in section 3, the shock wave part is much smaller than one, but what about the second part? It must also be much smaller than 1. This means that \( \kappa \sim \epsilon \ll 1 \). Very small \( \kappa \) means very small Hawking radiation. In that case the solution of (28) and (31) with (39), is

\[
\delta = -\sigma \simeq \frac{\epsilon r_s}{\tilde{r}} \Theta(u^+ - u_0^+) + \frac{\kappa}{4\tilde{r}} (x_H^- e^{-u} - \frac{1}{4} (x_H^-)^2 e^{-2u} + c)
\]

(40)

where \( c \) is a constant to be determined by the continuity condition at \( \tilde{r} = \tilde{r}_c \). Can (40) be continuously matched to the exact 2D solutions? Consider the
exact solution of [7]. The metric is \( g_{++} = g_{--} = 0 \) , \( g_{+-} = -\frac{1}{2} e^{2\rho} \), where 
\[ e^{-2\phi} = e^{-2\rho}, \]
and
\[ e^{-2\phi} + \frac{\kappa}{2} \phi = -\lambda^2 x^+ x^- - \frac{\kappa}{4} ln(-\lambda^2 x^+ x^-) \]
\[ -\frac{m}{\lambda x_0^+}(x^+ - x_0^+)\Theta(x^+ - x_0^+) \]  
(41)
The metric in the asymptotically flat coordinates is
\[ \tilde{g}_{+-} = -x^+(x^- - x_H^-)g_{+-} \]  
(42)
Using (13), (41) and (42) we get
\[ \tilde{g}_{tt} = \frac{1}{\lambda^2} \left( -1 + \left( \frac{\epsilon}{2\tilde{r}} + \frac{\kappa}{4\tilde{r}} ln(-2\tilde{r}/x^+ x^-) \right) \Theta(x^+ - x_0^+) \right) \]  
(43)
So
\[ \delta_{(2D)} = \left( \frac{\epsilon}{2\tilde{r}} + \frac{\kappa}{4\tilde{r}} ln(-2\tilde{r}/x^+ x^-) \right) \Theta(x^+ - x_0^+) \]  
(44)
The first term of (44) is exactly the one in (40), but what about the second term? Using (13), (41) and the fact that \( \kappa << 1 \), we get \( 2\tilde{r} \simeq -x^+(x^- - x_H^-) \), and
\[ \delta_{(2D)} \simeq \left( \frac{\epsilon}{2\tilde{r}} + \frac{\kappa}{4\tilde{r}} \ln \left( 1 + \frac{x_H^-}{x^-} \right) \right) \Theta(x^+ - x_0^+) \]  
(45)
Now we see that we can match (45) and (40) only if \( |x^-| >> |x_H^-| \) (and \( c = 0 \)).
Only at the beginning of the Hawking radiation can the 2D solution be matched to the linearised 4D one. When \( x^- \) approaches \( x_H^- \), the linearization breaks down, and one can no longer match the solutions.

In the case of [5,6], the linear dilaton is not a solution to the semiclassical equations even for \( x^+ < x_0^+ \), so the matching is more problematic. But still one can try to match the solutions at the beginning of the Hawking radiation process. The solution (of theory I [6]) for \( x^+ > x_0^+ \) is
\[ 2\Omega(\phi) = \frac{1}{\kappa} e^{u^+} (e^{-u^-} - m/\lambda) - \frac{1}{4} (u^+ - u^-) + T + \frac{m}{\lambda \kappa} + \frac{1}{2} ln(\kappa/4e) \]
\[ \rho = -ln(\lambda) - \frac{1}{\kappa} e^{-2\phi} + 2\Omega + \frac{1}{2} (u^+ - u^-) - \frac{1}{2} ln(\kappa/4e) \]  
(46)
where
\[
\Omega(\phi) = \frac{e^{-\phi}}{2\sqrt{\kappa}} \sqrt{\frac{e^{-2\phi}}{\kappa}} - 1 - \frac{1}{2} \ln \left( \frac{e^{-\phi}}{\sqrt{\kappa}} + \sqrt{\frac{e^{-2\phi}}{\kappa}} - 1 \right)
\]  \hspace{1cm} (47)

For \(\kappa \ll 1\), we get from (47) (and \(e^{-2\phi} = 2\tilde{r}\))
\[
\Omega(\phi) \approx \frac{e^{-2\phi}}{2\kappa} + \frac{1}{4} (\ln(\kappa/4) - 1) + \frac{\kappa}{8\tilde{r}} + \frac{\phi}{2}
\]  \hspace{1cm} (48)

and from (46) and (48) we get
\[
g_{tt} = -e^{2\rho} = -\frac{e^{(u^+ - u^-)}}{2\lambda^2\tilde{r}} \left( 1 + \frac{\kappa}{4\tilde{r}} \right)
\]  \hspace{1cm} (49)

For very small \(|x_H^-/x^-|\) we get
\[
g_{tt} \approx \frac{1}{\lambda^2} \left( -1 + \frac{\epsilon}{2\tilde{r}} - \frac{\kappa}{4\tilde{r}} \right)
\]  \hspace{1cm} (50)

So
\[
\delta_{(2D)} \approx \frac{\epsilon}{2\tilde{r}} - \frac{\kappa}{4\tilde{r}}
\]  \hspace{1cm} (51)

We see that (51) can be matched to (40) if \(c = 1\), and only for very very small \(|x_H^-/x^-|\). The reason that in this case the matching is to order zero in \(|x_H^-/x^-|\), while for (45) it was to first order, is that for \(x^+ < x^+_0\) the solution of [6] is the linear dilaton only to zero order. One can get similar results for their theory II.

At \(\tilde{r} = \tilde{r}_c\), and for \(x^+ > x^+_0\) the minimum value of \(|x_H^-/x^-|\) goes like \(\tilde{r}_c/\tilde{m}_X\), so the matching is possible only if \(\tilde{r}_c >> \tilde{m}_X\). This can be consistent with \(\frac{\tilde{m}_X}{\lambda} > \kappa\), because \(\tilde{r}_c >> \kappa\) (remember that we keep terms up to first order in \(\kappa/\tilde{r}_c\)). So in the case of [7], we can match the solutions (of course only at the beginning of the Hawking radiation process), but in the case of [5,6] it is again problematic.

According to our results it seems that indeed only a small 2D mass is consistent with a small 4D \(f\)-shock wave. In that case one should get the results of [17].
6 Conclusions

In this paper we studied the 4D interpretation of the 2D evaporating black holes. The 4D almost extremal black hole has the structure of $\Sigma^4 = \Sigma^2 \times S^2$ down the throat, where $\Sigma^2$ is the 2D (Witten) black hole. The 4D collapsing black hole, formed by a shock wave, has the same product structure (down the throat), where in that case $\Sigma^2$ is the collapsing 2D black hole. The 4D Hawking radiation in this classical background gives exactly the 2D Hawking radiation in $\Sigma^2$ [18].

When back-reaction is taken into account, the picture changes. A simple 4D S-wave scattering is consistent with the 2D solutions (down the throat) only if the Hawking radiation is very small\(^5\), $\kappa << 1$, and only at the beginning of the radiation. Just before an apparent horizon forms ($x^- = x^-_H$) the linear approximation breaks down, and the simple 4D S-wave scattering picture is no longer consistent. The amount of energy carried by the radiation at that point, is of order of $\kappa$, which is of the order of $\epsilon$, the energy carried by the shock wave. So exactly when the the problem of a positive define 2D mass arises [10-13], the 4D picture breaks down. Perhaps the 4D consideration may “save us” from the 2D problems. For example the 4D interpretation is consistent if $\kappa = 0$, and indeed the 2D ($\kappa = 0$) case is probably consistent [19].

The fact that the 4D picture breaks down even before an apparent horizon is formed means that the 4D picture could be quite different than the 2D evaporating picture. It seems reasonable to believe that the 4D black hole will radiate the energy of the shock wave and will return to extremality [16].

\(^5\)A small $\kappa$ can be consistent with a large $N$. If we define the measure of all the fields [15], with the metric $\hat{g} = exp(-2\alpha \phi)g$, then $\kappa = (1-\alpha)(N-24)/12$. In the limit $\alpha \rightarrow 1$, $\kappa$ will not vanish only if $N \rightarrow \infty$ (such that $(1-\alpha)N \rightarrow const.$).
Unlike the Reissner-Nordstrom case, in the dilatonic case, this process will not necessarily lead to information loss.

If a full 4D back-reaction calculation can be consistent with a throat region, and if these calculations will be free from positive energy problems, then the 2D solution must be consistent as well. The 2D theory that will give this solution is still (9). But the boundary conditions (at \( \hat{r} = \hat{r}_c \)) will be different. It could be interesting to find those boundary conditions (without solving the 4D theory), and to see if it corresponds to a reasonable 4D picture.

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