UNEXPECTED CURVES IN $\mathbb{P}^2$, LINE ARRANGEMENTS, AND MINIMAL DEGREE OF JACOBIAN RELATIONS

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Abstract. We reformulate a fundamental result due to Cook, Harbourne, Migliore and Nagel on the existence and irreducibility of unexpected plane curves of a set of points $Z$ in $\mathbb{P}^2$ using the minimal degree of a Jacobian syzygy of the defining equation for the dual line arrangement $A_Z$. Several applications of this new approach are given.

1. Introduction

Let $Z = \{p_1, p_2, \ldots, p_d\}$ be a finite set of $d$ points in $\mathbb{P}^2$. One says that $Z$ admits unexpected curves of degree $j \geq 2$ if

$$h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(j) \otimes \mathcal{I}(Z + (j-1)q)) > \max\left(0, h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(j) \otimes \mathcal{I}(Z)) - \left(\frac{j}{2}\right)\right),$$

where $q$ is a generic point in $\mathbb{P}^2$, the fat point scheme $kq$ is defined by the $k$-th power of the corresponding maximal ideal sheaf $\mathcal{I}(q)$, and hence $\mathcal{I}(Z + (j-1)q)$ is the ideal sheaf of functions vanishing on $Z$ and vanishing of order $(j-1)$ at $q$, see [6, 8, 23].

There is a more general definition, see [20, 23], but in this note we consider only the special case described above. Let $A_Z : f_Z = 0$ be the associated line arrangement in $\mathbb{P}^2$ as in [6, 8]. Let $(a_Z, b_Z)$ be the generic splitting type of the derivation bundle $E_Z$ associated to $A_Z$, and let $m(A_Z)$ be the maximal multiplicity of an intersection point in $A_Z$. It is well known that $a_Z + b_Z = d - 1$. For $i = 1, 2, \ldots, d$, let $Z_i = Z \setminus \{p_i\}$ be the set of $d - 1$ points obtained from $Z$ by forgetting the point $p_i$, and let $A_{Z_i} : f_{Z_i} = 0, (a_{Z_i}, b_{Z_i})$ and $m(A_{Z_i})$ be the corresponding objects associated with the set $Z_i$ as above. With this notation, the following fundamental result was established in [6, Theorem 1.2], [6, Lemma 3.5 (a)], [6, Corollary 5.5] and [6, Corollary 5.17], see also [8] for a discussion.

**Theorem 1.1.** The set of points $Z$ admits an unexpected curve if and only if

$$m(A_Z) \leq a_Z + 1 < \frac{d}{2}.$$

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If these conditions are fulfilled, then $Z$ admits an unexpected curve of degree $j$ if and only if
$$a_Z < j \leq d - a_Z - 2.$$  

The unexpected curve $C_q$ of minimal degree $j = a_Z + 1$ and having a point of multiplicity $a_Z$ at a generic point $q$ is unique. Moreover $C_q$ is irreducible if and only if $a_Z = a_{Z_i}$ for all $i = 1, 2, \ldots, d$.

For larger values of $j$, the corresponding unexpected curves of degree $j$ are obtained from $C_q$ by adding $j - a_Z - 1$ lines passing through $q$, see [6, Corollary 5.5]. The curve $C_q$ itself, if not irreducible, is the union of some lines through $q$ and an irreducible curve $C'_q$, having at $q$ a point of multiplicity $\deg(C'_q) - 1$.

Let $S = \mathbb{C}[x, y, z]$ be the polynomial ring in three variables $x, y, z$ with complex coefficients, and let $A : f = 0$ be an arrangement of $d$ lines in the complex projective plane $\mathbb{P}^2$. The minimal degree of a Jacobian syzygy for the polynomial $f$ is the integer $mdr(f)$ defined to be the smallest integer $r \geq 0$ such that there is a nontrivial relation
$$af_x + bf_y + cf_z = 0$$
among the partial derivatives $f_x, f_y$ and $f_z$ of $f$ with coefficients $a, b, c$ in $S_r$, the vector space of homogeneous polynomials in $S$ of degree $r$. The main result of this note is the following reformulation of Theorem 1.1.

**Theorem 1.2.** The set of points $Z$ admits an unexpected curve if and only if
$$m(A_Z) \leq mdr(f_Z) + 1 < \frac{d}{2}.$$  

If these conditions are fulfilled, then $Z$ admits an unexpected curve of degree $j$ if and only if
$$mdr(f_Z) < j \leq d - mdr(f_Z) - 2.$$  

Every unexpected curve $C_q$ of minimal degree $j = mdr(f_Z) + 1$ is irreducible if and only if $mdr(f_Z) = mdr(f_{Z_i})$ for all $i = 1, 2, \ldots, d$.

The advantage of having such a result comes from the wealth of information we have on the numerical invariant $mdr(f_Z)$, and on the relations between $mdr(f_Z)$ and $mdr(f_{Z_i})$ for various $i$, see [1, 10, 15]. Using these results, we prove in this note some new results, and also give shorter proofs for some known results. In particular, the results about the irreducibility of the curves $C_q$ of minimal degree seem to be easily proved using this new viewpoint.

In section 2 we recall some basic properties of the invariant $mdr(f)$, and show in Proposition 2.3 that a free line arrangement $A : f = 0$ with $mdr(f) = m(A) - 1$ is in fact supersolvable.

In section 3, we show first that $a_Z = mdr(f_Z)$ when the set $Z$ admits unexpected curves, see Theorem 3.1, and use this equality to prove Theorem 1.2 starting from
Theorem 1.1. As an application, we give in Corollary 3.3 a short proof for the fact that the set of points $Z$ dual to the monomial arrangement $\mathcal{A}_0^m$, for $m \geq 5$, has irreducible unexpected curves of minimal degree $m + 2$. This result was obtained first in [6, Proposition 6.12]. Then we prove in Proposition 3.4 a similar result for the set of points $Z$ dual to the full monomial arrangement $\mathcal{M}_m$, for $m \geq 4$. Note that the full monomial arrangement $\mathcal{M}_m$ is denoted by $\mathcal{A}_{3,m-2}^3$ in [23], and the claim in Proposition 3.4 is part of the claim in [23, Theorem 6]. However, the irreducible question does not seem to be addressed in [23]. Then we prove in Proposition 3.5 that a set $Z$ with $d = |Z| \leq 8$ never admits unexpected curves. Moreover, a set $Z$ with $d = |Z| = 9$ admits unexpected curves if and only if the associated line arrangement $\mathcal{A}_Z$ is projectively equivalent to the line arrangement $B_3$, see [18] where this result was proved first, as well as Proposition 3.7 where we give a shorter proof using Theorem 1.2. We also show that a set $Z$ in which at most 3 points are collinear does not admit unexpected curves, see Proposition 3.6.

In the final section we discuss several situations where we can add a new point $p'$ to $Z$ such that the new set $Z' = Z \cup \{p'\}$ also admits unexpected curves. First we discuss the arrangements $\mathcal{A}_1^1$ and $\mathcal{A}_2^2$, which interpolate between the arrangements $\mathcal{A}_0^m$ and $\mathcal{M}_{m+2} = \mathcal{A}_3^3$ discussed above. We prove in both cases that the corresponding dual set $Z$ has irreducible unexpected curves of minimal degree, claims that occur in [23, Theorem 6] without a proof of the irreducibility. In Proposition 4.3 we discuss what happens when we add a generic point $p'$ to $Z$, and in Proposition 4.4 we discuss what happens when we add a generic point $p'$ situated on a line in $\mathbb{P}^2$ which contains a maximal number of points in $Z$, namely $m(\mathcal{A}_Z)$ points. This gives examples of sets $Z$ having unexpected curves, without being duals of free line arrangements, see Example 4.5.

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2. Preliminaries

Let $S = \mathbb{C}[x, y, z]$ be the polynomial ring in three variables $x, y, z$ with complex coefficients, and let $\mathcal{A} : f = 0$ be an arrangement of $d$ lines in the complex projective plane $\mathbb{P}^2$. We denote by $n_j = n_j(\mathcal{A})$ the number of intersection points in $\mathcal{A}$ of multiplicity $j$. It is known that $\text{mdr}(f) = 0$ if and only if $n_d = 1$, hence $\mathcal{A}$ is a pencil of $d$ lines passing through one point. Moreover, $\text{mdr}(f) = 1$ if and only if $n_d = 0$ and $n_{d-1} = 1$, hence $\mathcal{A}$ is a near pencil, see for instance [12]. Let $AR(f) \subset S^3$ be the graded $S$-module such, for any integer $j$, the corresponding homogeneous component $AR(f)_j$ consists of all the triples $\rho = (a, b, c) \in S^3_j$ satisfying (1.1). Let $\alpha$ be the minimum of the Arnold exponents $\alpha_p$ (alias singularity indices or log canonical thresholds, see Theorem 9.5 in [22]) of the singular points $p$ of $\mathcal{A}$. The germ $(\mathcal{A}, p)$ is weighted homogeneous of type $(w_1, w_2; 1)$ with $w_1 = w_2 = \frac{1}{m_p}$, where
\( m_p \) is the multiplicity of \( \mathcal{A} \) at \( p \). It is known that

\[
\alpha_p = w_1 + w_2 = \frac{2}{m_p},
\]

see for instance [13, Formula (2.4.7)]. With this notation, [14, Theorem 9] can be restated in our setting as follows, see also [15, Theorem 2.1].

**Theorem 2.1.** Let \( \mathcal{A} : f = 0 \) be an arrangement of \( d \) lines in \( \mathbb{P}^2 \) and \( m = m(\mathcal{A}) \) be maximal multiplicity of an intersection point in \( \mathcal{A} \). Then \( AR(f)_k = 0 \) for all

\[
k < \frac{2}{m}d - 2.
\]

Equivalently, one has

\[
mdr(f) \geq \frac{2}{m}d - 2.
\]

**Remark 2.2.** Let \( \mathcal{A} : f = 0 \) be a line arrangement, and \( p = (1 : 0 : 0) \) an intersection point on \( \mathcal{A} \) of maximal multiplicity, say \( m = \text{mult}(\mathcal{A}, p) = m(\mathcal{A}) \). To this situation, one can associate a primitive Jacobian syzygy as explained in [10, Section 2.2]. We recall this construction here. Let \( g = 0 \) be the equation of the subarrangement of \( \mathcal{A} \) formed by the \( m \) lines in \( \mathcal{A} \) passing through \( p \) and note that \( g_x = 0 \). Then we can write \( f = gh \) for some polynomial \( h \in S \). The syzygy constructed as explained there is primitive and has degree \( r_p = d - m \), more precisely it is given by

\[
\rho_p = (a, b, c) = (xh_x - d \cdot h, yh_x, zh_x),
\]

where \( h_x \) denotes the partial derivative of \( h \) with respect to \( x \). As shown in [10, Theorem 1.2], the following cases are possible for \( r = mdr(f) \).

**Case A:** \( r = r_p = d - m \), in other words the constructed syzygy has minimal degree. If \( \mathcal{A} = \mathcal{A}_Z \), to have unexpected curves in this case we need

\[
m \leq d - m + 1 < \frac{d}{2}.
\]

These two inequalities cannot both hold, so in this case there are no unexpected curves.

**Case B:** \( r < r_p = d - m \), in other words the constructed syzygy has not minimal degree. Then the following two situations are possible.

**Subcase B1:** \( r = m - 1 \), and then \( 2m < d + 1 \) and \( \mathcal{A} \) is free with generic splitting type \( a = m - 1 < b = d - m \). This case occurs exactly for the supersolvable line arrangements. Indeed, if \( \mathcal{A} \) is supersolvable, with \( m = m(\mathcal{A}) \) satisfying \( 2m \leq d + 1 \), then \( m - 1 \leq d - m \), and hence \( r = m - 1 \), see [3, Equation (2.2)]. Conversely, a free line \( \mathcal{A} : f = 0 \) arrangement such that \( mdr(f) = m(\mathcal{A}) - 1 = m - 1 \) is supersolvable,
see Proposition 2.3 below. Note that unexpected curves occur in this case if and only if $d > 2m$, see [8, Theorem 3.7]. As an example, the full monomial arrangement

$$M_m : f = xyz(x^{m-2} - y^{m-2})(y^{m-2} - z^{m-2})(z^{m-2} - x^{m-2}) = 0,$$

is supersolvable, it has $d = |M_m| = 3m - 3$, $m(M_m) = m$ and hence the condition $d > 2m$ holds for any $m \geq 4$.

**Subcase B2:** $m \leq r \leq d - m - 1$, and then $2m < d$. One example of this case is provided by the Fermat arrangements, a.k.a. monomial arrangements

$$A^0_m : f_m = (x^{m-1} - y)(y^{m-1} - z)(z^{m-1} - x) = 0,$$

see [23] for more information. It is known that $m = m(A^0_m)$, $d = 3m$ and $r = mdr(f_m) = m + 1$ for $m \geq 3$. The unexpected curves occur in this case when $m \geq 5$ and are discussed in [6, Proposition 6.12]. In particular, it is shown there that the unexpected curves of minimal degree $m + 2$ are irreducible in this case. A new proof of this irreducibility is given below in Corollary 3.3.

**Proposition 2.3.** A free line $A : f = 0$ arrangement such that $mdr(f) = m(A) - 1$ is supersolvable.

**Proof.** This proof was communicated to us by Takuro Abe, and uses [1, Proposition 4.2], where line arrangements in $\mathbb{P}^2$ are regarded as central plane arrangements $\bar{A}$ in $\mathbb{C}^3$. Note that, since $\mathbb{C}^3 \setminus \bar{A} = (\mathbb{P}^2 \setminus A) \times \mathbb{C}^*$, one has

$$b_2(\mathbb{C}^3 \setminus \bar{A}) = b_1(\mathbb{P}^2 \setminus A) + b_2(\mathbb{P}^2 \setminus A) = (d - 1) + (m - 1)(d - m).$$

On the other hand, if we choose a flag

$$X_3 = \{0\} \subset X_2 = L \subset X_1 = P \subset X_0 = \mathbb{C}^3,$$

where the line $L$ corresponds to a point $p$ in $A$ of multiplicity $m$, and the plane $P$ corresponds to any line in $A$ containing $p$, then

$$\sum_{j=0}^{2} (|A_{X_{j+1}}| - |A_{X_j}|)|A_{X_j}| = 0 + (m - 1) \cdot 1 + (d - m) \cdot m = b_2(\mathbb{C}^3 \setminus \bar{A}).$$

This equality implies, via [1, Proposition 4.2], that the line arrangement $A$ is supersolvable. \qed

We end this section with a side remark on irreducible curves in $\mathbb{P}^2$, say of degree $d$ and having a point of multiplicity $d - 1$.

**Proposition 2.4.** Let $C : f = 0$ be an irreducible curve of degree $d$ in $\mathbb{P}^2$ having a singular point $p$ of multiplicity $d - 1 \geq 2$. Then the following hold.

1. $p$ is the only singular point of $C$;
2. $C$ is a rational curve;
3. the fundamental group $\pi_1(\mathbb{P}^2 \setminus C)$ is abelian;
(4) if the curve $C$ is cuspidal, i.e. if the singularity $(C, p)$ is irreducible, then $C$ is either free or nearly free.

**Proof.** The first two claims are well known. The third claim follows for instance from [9, Corollary 4.3.8]. The last claim follows from (3) using [16, Corollary 3.2]. □

**Example 2.5.** The $B_3$-arrangement is a special case of the full monomial arrangement $\mathcal{M}_m$, corresponding to $m = 4$. When $Z$ is the set of 9 points dual to the $B_3$-arrangement, the curve $C_q$ is an irreducible quartic with an ordinary triple point at $q$. This was one of the motivating examples in developing this theory, and it has occurred first in [7]. For more details, see Example 1.2 and Example 3.1 in [8] as well as the detailed study in [7] where the explicit equations of the unexpected curves $C_q$ in this case are given. We do not know whether an unexpected curve can ever be cuspidal. Note that 9 is the minimal value for $|Z|$ such that $Z$ admits an unexpected curve, in view of Proposition 3.5 below and for $|Z| = 9$, the set $Z$ is unique up-to projective equivalence, see [18] and Proposition 3.7 below.

3. The main results

We have the following relation between the invariants $a_Z$ and $mdr(f_Z)$.

**Theorem 3.1.** For any finite set $Z$, one has

$$a_Z \leq \min \left( mdr(f_Z), \left\lfloor \frac{d-1}{2} \right\rfloor \right).$$

Moreover, if $Z$ admits an unexpected curve, then

$$a_Z = mdr(f_Z).$$

**Proof.** The first claim follows from [2, Proposition 3.2 (1)]. The second claim follows from Theorem 1.1 which implies that $a_Z < (d-2)/2$ in this case, and from [2, Proposition 3.2 (2)]. □

3.2. **Proof of Theorem 1.2.** If $Z$ admits an unexpected curve, then $a_Z = mdr(f_Z)$ and the claims, except the last one, are clear. On the other hand, if

$$mdr(f_Z) + 1 < \frac{d}{2},$$

then it follows from [2, Proposition 3.2 (2)] that $a_Z = mdr(f_Z)$, and again the claims, except the last one, follow. For the last claim, note that $mdr(f_{Z_i}) \leq mdr(f_Z)$, see for instance [4, Proposition 2.12], and hence

$$mdr(f_{Z_i}) \leq mdr(f_Z) < \frac{d}{2} - 1 = \frac{(d-1)-1}{2}. $$
We claim that \( a_{Z_i} = \text{mdr}(f_{Z_i}) \) for any \( i \), which would complete the proof. Note that Theorem 3.1 implies \( a_{Z_i} \leq \text{mdr}(f_{Z_i}) \), hence it is enough to show that the inequality \( a_{Z_i} \leq \text{mdr}(f_{Z_i}) - 1 \) leads to a contradiction. Indeed, one has in this case
\[
a_{Z_i} \leq \text{mdr}(f_{Z_i}) - 1 < \frac{d}{2} - 2 = \frac{(d - 1) - 3}{2}.
\]
Using [2, Proposition 3.2 (2)] for the line arrangement \( A_{Z_i} \), we get \( a_{Z_i} = \text{mdr}(f_{Z_i}) \), hence a contradiction.

As a first application, we can give a shorter proof to the following known fact, see [6, Proposition 6.12].

**Corollary 3.3.** The dual set of points \( Z \) of the monomial arrangement
\[
A_0^m : f = (x^m - y^m)(y^m - z^m)(z^m - x^m) = 0,
\]
admits only irreducible unexpected curves of minimal degree \( m + 2 \), for \( m \geq 5 \).

**Proof.** If \( L \) is any line in \( A_0^m : f = 0 \), the number of intersection points on \( L \) is exactly \( m + 1 \). Recall that \( d = |A_0^m| = 3m \) and \( \text{mdr}(f) = m + 1 \). We apply now [4, Proposition 2.12] to determine \( \text{mdr}(f_L) \), where \( A_L : f_L = 0 \) is the line arrangement obtained from \( A_0^m \) by deleting the line \( L \). Since
\[
|A_0^m| - (m + 1) = 2m - 1 > m + 1 = \text{mdr}(f)
\]
for \( m \geq 3 \), it follows that \( \text{mdr}(f) = \text{mdr}(f_L) \). Theorem 1.2 implies that the minimal degree unexpected curves are irreducible. \( \square \)

One has also the following result, already stated in [23, Theorem 6].

**Proposition 3.4.** The dual set of points \( Z \) of the full monomial arrangement
\[
M_m : f = xyz(x^{m-2} - y^{m-2})(y^{m-2} - z^{m-2})(z^{m-2} - x^{m-2}) = 0,
\]
admits only irreducible unexpected curves of minimal degree \( m \), for \( m \geq 4 \).

**Proof.** We know that an unexpected curve for \( M_m \) has degree \( \geq \text{mdr}(f) + 1 = m \), and that \( m \geq 4 \) is a necessary and sufficient condition for the existence of such curves, see the discussion in Remark 2.2 Subcase B1. It remains to prove that such curves are irreducible, using Theorem 1.2. Note that if we remove any line from \( M_m \), the resulting arrangement \( A \) is still supersolvable, with \( d = |A| = 3m - 4 \) and \( m = m(A) \). Since \( m - 1 \leq d - m = 2m - 4 \) for \( m \geq 3 \), it follows that \( \text{mdr}(f_{Z_i}) = \text{mdr}(f) = m - 1 \). This completes the proof. \( \square \)

The following two results say that, if a set \( Z \) admits unexpected curves, then the associated line arrangement \( A_Z \) has to be rather complicated.

**Proposition 3.5.** A set of points \( Z \) with \( d = |Z| \leq 8 \) does not admit unexpected curves.
Proof. We prove only the case \(d = 8\), since the other cases are easier and can be treated in a completely similar way. Assume that \(Z\) has unexpected curves. Using Theorem 1.1 we get
\[
m(A_Z) \leq a_Z + 1 < 4
\]
and hence \(a_Z \leq 2\) and \(m(A_Z) \leq 3\). Using Theorem 2.1 we get that
\[
mdr(f_Z) \geq \frac{2}{3}8 - 2 = \frac{10}{3}.
\]
Hence \(mdr(f_Z) > 3\). On the other hand we know that \(a_Z = mdr(f_Z)\) by Theorem 3.1. This contradiction proves our claim. \(\square\)

Proposition 3.6. A set of points \(Z\) such that at most 3 points in \(Z\) are collinear does not admit unexpected curves. In other words, a set of points \(Z\) such \(m(A_Z) \leq 3\), does not admit unexpected curves

Note that the case \(m(A_Z) = 2\) was treated in \(\cite{6, Corollary 6.8}\), and a new, quick proof for this result can also be obtained using exactly the same argument as below.

Proof. It is enough, by the above remark, to treat the case \(m(A_Z) = 3\). Then Theorem 2.1 implies
\[
mdr(f_Z) \geq \frac{2}{3}d - 2.
\]
If \(Z\) admits unexpected curves, we have in addition by Theorem 1.2
\[
mdr(f_Z) + 1 < \frac{d}{2}.
\]
But
\[
\frac{2}{3}d - 2 < \frac{d}{2} - 1
\]
holds only for \(d \leq 5\), and in this range \(Z\) does not admit unexpected curves by Proposition 3.5. \(\square\)

The following result was first proved in \(\cite{18}\), but our proof seems shorter.

Proposition 3.7. A set of points \(Z\) with \(d = |Z| = 9\) admits unexpected curves if and only if \(Z\) is projectively equivalent to the set of 9 points dual to the \(B_3\)-arrangement described in Example 2.5 above.

Proof. As above, using Theorem 1.1 and Proposition 3.6 we get
\[
4 \leq m(A_Z) \leq a_Z + 1 < 4.5
\]
and hence \(a_Z = 3\) and \(m(A_Z) = 4\). Next Theorem 3.1 and Remark 2.2. Subcase B1 implies that the arrangement \(A_Z\) is free. The numbers \(n_k\), of the intersection
points of multiplicity \( k \) in a line arrangement \( \mathcal{A} \) with \( |\mathcal{A}| = d \), satisfy a number of relations. The easiest of them is the following.

\[
(3.1) \quad \sum_{k \geq 2} n_k \binom{k}{2} = \binom{d}{2},
\]

where \( d = |\mathcal{A}| \). For a line arrangement \( \mathcal{A} : f = 0 \), one has

\[
(3.2) \quad \tau(\mathcal{A}) = (d - 1)^2 - r(d - r - 1),
\]

where

\[
\tau(\mathcal{A}) = \sum_{k \geq 2} n_k (k - 1)^2
\]

and \( r = mdr(f) \), if and only if \( \mathcal{A} \) is free, see \[11][17\]. In our case, \( d = 9 \) and \( r = 3 \), so we get two equations

\[
2n_2 + 3n_3 + 6n_4 = 36 \quad \text{and} \quad 2n_2 + 4n_3 + 9n_4 = 49.
\]

The only solutions with non-negative integers \( n_j \) are the following four vectors

\[
(n_2, n_3, n_4) \in \{(9, 1, 4), (6, 4, 3), (3, 7, 2), (0, 10, 1)\}.
\]

A highly non-trivial restriction on these numbers is given by the Hirzebruch inequality, valid for non trivial line arrangements (i.e. for line arrangements not a pencil or a near pencil), see \[21\]:

\[
(3.3) \quad n_2 + \frac{3}{4}n_3 - d \geq \sum_{k > 4} (k - 4)n_k.
\]

Using \( (3.3) \), it follows that the vector \( (n_2, n_3, n_4) \) of our arrangement of 9 lines with unexpected curves can be only \( (9, 1, 4) \) and \( (6, 4, 3) \). Using Proposition \[2.3\], we see that the arrangement \( \mathcal{A}_Z \) is supersolvable, and using the classification of supersolvable arrangements with at least 3 modular points given in \[19\], our claim is proved.

\[\square\]

**Remark 3.8.** If one prefers not to use Proposition \[2.3\] to complete the proof above, then one can proceed as follows. Note that if a line \( L \in \mathcal{A} \) in a line arrangement \( \mathcal{A} \) contains at least 3 points of multiplicity 4, then clearly \( d = |\mathcal{A}| \geq 1 + 3 \times 3 = 10 \). Hence such a situation cannot occur for our line arrangement \( \mathcal{A} = \mathcal{A}_Z \). Let \( a \) (resp. \( b \)) be the number of lines in \( \mathcal{A} \) containing exactly two (resp. one) points of multiplicity 4 in \( \mathcal{A} \). Note that we have

\[
4n_4 = 2a + b,
\]

since when we count the lines containing at least one points of multiplicity 4 we get \( 4n_4 \), but the lines containing exactly two such points are counted twice.

If we assume \( (n_2, n_3, n_4) = (9, 1, 4) \), then we get \( 2a + b = 16 \) and hence

\[
d \geq a + b = 16 - a \geq 10
\]
since
\[ a \leq \binom{4}{2} = 6. \]

Hence this vector cannot occur when \( d = 9 \). If we assume \((n_2, n_3, n_4) = (6, 4, 3)\), then we get \(2a + b = 12\) and hence
\[ d \geq a + b = 12 - a \geq 9 \]
since
\[ a \leq \binom{3}{2} = 3. \]

Hence, when \( d = 9 \), we have \( a = 3 \) and \( b = 6 \). It follows that the 3 points of multiplicity 4, say \( p, q, r \), are connected by lines \( pq, qr, pr \) in \( A \), and through each of them pass 2 additional lines, say \( L_p \) and \( L'_p \) through \( p \) and so on. The 6 double points are the intersections of the 6 lines of type \( L_p, L'_p \) with the opposite line \( qr \). And the triple points are the intersection of lines of type \( L_p, L_q, L_r \). It follows that the arrangement \( A = A_Z \) is supersolvable, and hence it is projectively equivalent to the \( B_3 \)-arrangement using [19].

4. Adding a new point to \( Z \)

First we revisit some results stated in [23, Theorem 6]. Starting with the monomial arrangement \( A^0_m : f^0 = 0 \), denoted \( A^0_3(m) \) in [23], one can add the line \( L_x : x = 0 \) and get the new line arrangement
\[ A^1_m : f^1 = xf^0 = x(x^m - y^m)(y^m - z^m)(z^m - x^m) = 0. \]

**Proposition 4.1.** The line arrangement \( A^1_m \) is free with exponents \((m + 1, 2m - 1)\) and the corresponding dual set of points \( Z \) admits unexpected curves of degree \( j \) for \( m \geq 4 \) any integer \( j \) satisfying
\[ m + 2 \leq j \leq 2m - 2. \]

The unexpected curves of minimal degree \( j = m + 2 \) are all irreducible.

**Proof.** First we apply [4, Proposition 2.12] for the line arrangement \( A' = A^0_m \) and \( H = L_x \). Since the set \( I_H \) of intersection points of \( A^1_m \) on \( H \) has cardinal \( m + 2 \), and since
\[ |A'| - |I_H| = 3m - (m + 2) = 2m - 2 > m + 1 = mdr(f^0) \]
for \( m \geq 4 \), it follows that \( mdr(f^1) = m + 1 \). Using this equality, it is easy to check that \( A^1_m \) is free using the equation (3.2). The other claims, except the irreducibility claim, follow from Theorem [12]. Finally we address the irreducibility question. A line \( L \) in \( A^1_m \) has either \( m + 2 \) intersection points if \( L = L_x \), or just \( m + 1 \) intersection points when \( L \neq L_x \). We apply now [4, Proposition 2.12] to determine \( mdr(f_L) \),
where $\mathcal{A}_L : f_L = 0$ is the line arrangement obtained from $\mathcal{A}_m^1$ by deleting the line $L$. Since
\[
|\mathcal{A}_m^1| - |I_L| \geq (3m + 1) - (m + 2) = 2m - 1 > m + 1 = mdr(f^1)
\]
for $m \geq 3$, where $I_L$ denotes the set of intersection points of $\mathcal{A}_m^1$ situated on the line $L$. It follows that $mdr(f^1) = mdr(f_L)$. Theorem \[1.2\] implies that the minimal degree unexpected curves are irreducible.

Starting now with the monomial arrangement $\mathcal{A}_m^1 : f^1 = 0$, one can add the line $L_y : y = 0$ and get the new line arrangement
\[
\mathcal{A}_m^2 : f^2 = yf^1 = xy(x^m - y^m)(y^m - z^m)(z^m - x^m) = 0.
\]

**Proposition 4.2.** The line arrangement $\mathcal{A}_m^2$ in $\mathbb{P}^2$ is supersolvable with exponents $(m + 1, 2m)$ and the corresponding dual set of points $Z$ admits unexpected curves of degree $j$ for $m \geq 3$ and any integer $j$ satisfying
\[
m + 2 \leq j \leq 2m - 1.
\]

The unexpected curves of minimal degree $j = m + 2$ are all irreducible.

**Proof.** Since $\mathcal{A}_m^2$ is clearly supersolvable, with $(0 : 0 : 1)$ as modular point, all the claims except the claim about irreducibility are proved using Theorem \[1.2\]. As above, denote by $\mathcal{A}_L : f_L = 0$ is the line arrangement obtained from $\mathcal{A}_m^2$ by deleting the line $L$. It is enough to show that $mdf(f^2) = mdr(f_L)$. If $I_L$ denotes the set of intersection points of $\mathcal{A}_m^2$ situated on the line $L$, it is clear that $|I_L| \leq m + 2$. Since
\[
|\mathcal{A}_m^2| - |I_L| \geq (3m + 2) - (m + 2) = 2m > m + 1 = mdr(f^2)
\]
for $m \geq 2$, the result follows by \[4, Proposition 2.12\].

In the following two results, we add a point $p'$ to $Z$, and hence the corresponding dual line $L'$ to the arrangement $\mathcal{A}_Z$. In both cases, the unexpected curves of minimal degree are not irreducible, as follows using Theorem \[1.2\].

**Proposition 4.3.** Assume that the set of points $Z$ satisfies the stronger condition
\[
m(\mathcal{A}_Z) \leq mdr(f_Z) + \frac{3}{2} < \frac{d}{2}.
\]
Let $p'$ be a generic point in $\mathbb{P}^2$ and consider the new set $Z' = Z \cup \{p'\}$. Then $Z'$ admits an unexpected curve of degree $j$, for any integer $j$ such that
\[
mdr(f_Z) + 1 < j \leq d - mdr(f_Z) - 2.
\]

**Proof.** The point $p'$ gives by duality a generic line $L'$. Hence the arrangement $\mathcal{A}_{Z'}$ is given by adding a generic $L'$ to $\mathcal{A}_Z$. Using \[4, Proposition 4.11\], it follows that
\[
m(\mathcal{A}_{Z'}) = m(\mathcal{A}_Z), \ mdr(f_{Z'}) = mdr(f_Z) + 1 \text{ and } |\mathcal{A}_{Z'}| = |\mathcal{A}_Z| + 1.
\]
The claim follows using Corollary \[3.3\].
The point \( p' \) in Proposition 4.3 is generic if and only if \( p' \) is not situated on any line \( p_ip_j \) determined by two distinct points \( p_i, p_j \in Z \). Note also that the multiplicity \( m = m(A_Z) \) is exactly the maximal number of points in \( Z \) which are collinear. Let \( p_{i1}, p_{i2}, \ldots, p_{im} \) be a maximal set of collinear points in \( Z \) and let \( L \) be the line determined by these points. With this notation, we have the following result.

**Proposition 4.4.** Assume that the set of points \( Z \) satisfies the stronger condition

\[
m(A_Z) \leq \text{mdr}(f_Z) + \frac{3}{2} < \frac{d}{2}.
\]

Let \( p' \) be a generic point on the line \( L \) defined above and consider the new set \( Z' = Z \cup \{p'\} \). Then \( Z' \) admits an unexpected curve of degree \( j \), for any integer \( j \) such that

\[
\text{mdr}(f_Z) + 1 < j \leq d - \text{mdr}(f_Z) - 2.
\]

**Proof.** The point \( p' \) gives by duality a line \( L' \), which is generic in the pencil of lines passing through the common intersection point \( p_L \) of the lines \( L_j \), dual to the points \( p_{ij} \), for \( j = 1, \ldots, m \). In fact, \( p_L \) is the point dual to the line \( L \). The arrangement \( A_{Z'} \) is given by adding the line \( L' \) to \( A_Z \). Using [1, Proposition 4.10], it follows that

\[
m(A_{Z'}) = m(A_Z) + 1, \quad \text{mdr}(f_{Z'}) = \text{mdr}(f_Z) + 1 \quad \text{and} \quad |A_{Z'}| = |A_Z| + 1.
\]

Indeed, the case (3) in [1, Proposition 4.10] cannot occur, as explained in Remark 2.2 Case A. The claim follows using Corollary 3.3. \( \square \)

**Example 4.5.** When \( Z \) is the set of \( 3m \) points dual to the Fermat \( A_m^0 \)-arrangement considered in Remark 2.2 Subcase B2, the conditions in Proposition 4.3 are fulfilled for any \( m \geq 6 \). Note that the arrangement \( A_{Z'} = A_m \cup L' \) from Proposition 4.3 is far from being a free arrangement. Indeed, the global Tjurina number of the arrangement \( A_{Z'} = A_m \cup L' \) is given

\[
\tau(A_{Z'}) = \tau(A_m) + 3m = 7m^2 - 3m + 3.
\]

On the other hand, the global Tjurina number of a free arrangement \( B_m \) of \( 3m + 1 \) lines with \( \text{mdr}(B_m) = m + 2 \) is given by the formula (3.2) and hence

\[
\tau(B_m) = 9m^2 - (m + 2)(2m - 2) = 7m^2 - 2m + 4 > \tau(A_{Z'}).
\]

Hence \( A_{Z'} = A_m \cup L' \) gives rise to countable many examples of sets \( Z' \) admitting unexpected curves, and such that the corresponding arrangements \( A_{Z'} \) are not free, and in particular not supersolvable.

**References**

[1] T. Abe, Restrictions of free arrangements and the division theorem, in: "Perspectives in Lie Theory", Springer INdAM Series 19 (2017), 389–401.

[2] T. Abe, A. Dimca, On the splitting types of bundles of logarithmic vector fields along plane curves, Internat. J. Math. 29 (2018), no. 8, 1850055, 20 pp.
T. Abe, A. Dimca, On complex supersolvable line arrangements, arXiv: 1907.12497.

T. Abe, A. Dimca, G. Sticlaru, Addition-deletion results for the minimal degree of logarithmic derivations of arrangements, arXiv:1908.06885.

T. Bauer, G. Malara, T. Szemberg, J. Szpond, Quartic unexpected curves and surfaces, Manuscripta Math. (2018). https://doi.org/10.1007/s00229-018-1091-3.

D. Cook, B. Harbourne, J. Migliore, U. Nagel, Line arrangements and configurations of points with an unexpected geometric property. Compositio Math. 154(2018), 2150–2194.

T. Bauer, G. Malara, T. Szemberg, J. Szpond, Quartic unexpected curves and surfaces, Manuscripta Math. (2018). https://doi.org/10.1007/s00229-018-1091-3.

R. Di Gennaro, G. Ilardi, J. Valls, Singular hypersurfaces characterizing the Lefschetz properties. J. Lond. Math. Soc. (2) 89(2) (2014), 194–212.

M. Di Marca, G. Malara, A. Oneto, Unexpected curves arising from special line arrangements, Journal of Algebraic Combinatorics, https://doi.org/10.1007/s10801-019-00871-0.

A. Dimca, Singularities and Topology of Hypersurfaces, Universitext, Springer Verlag, New York, 1992.

A. Dimca, Curve arrangements, pencils, and Jacobian syzygies, Michigan Math. J. 66 (2017), 347–365.

A. Dimca, Freeness versus maximal global Tjurina number for plane curves, Math. Proc. Cambridge Phil. Soc. 163 (2017), 161–172.

A. Dimca, D. Ibadula, A. Măcinic, Numerical invariants and moduli spaces for line arrangements, arXiv:1609.06551, Osaka J. Math. (to appear).

A. Dimca and M. Saito: Some remarks on limit mixed Hodge structure and spectrum, An. Şt. Univ. Ovidius Constanţa 22(2) (2014), 69-78.

A. Dimca, M. Saito: Generalization of theorems of Griffiths and Steenbrink to hypersurfaces with ordinary double points, Bull. Math. Soc. Sci. Math. Roumanie, 60(108) (2017), 351–371.

A. Dimca, E. Sernesi, Syzygies and logarithmic vector fields along plane curves, Journal de l'École polytechnique-Mathématiques 1(2014), 247-267.

A. Dimca, G. Sticlaru, Free and nearly free curves vs. rational cuspidal plane curves, Publ. RIMS Kyoto Univ., 54 (2018), 163–179.

A.A. du Plessis, C.T.C. Wall, Application of the theory of the discriminant to highly singular plane curves, Math. Proc. Camb. Phil. Soc., 126 (1999), 259-266.

L. Farnik, F. Galuppi, L. Sodomaco, W. Trok, On the unique unexpected quartic in P², arXiv:1804.03590.

K. Hanumanthu, B. Harbourne, Real and complex supersolvable line arrangements in the projective plane, arXiv:1907.07712.

B. Harbourne, J. Migliore, U. Nagel, and Z. Teitler. Unexpected hypersurfaces and where to find them, arXiv:1805.10626, accepted for publication in Michigan Math. J.

F. Hirzebruch, Arrangements of lines and algebraic surfaces, Arithmetic and geometry, Vol. II, Progr. Math. 36, Birkhauser, Boston, Mass., 1983, 113–140.

J. Kollár: Singularities of pairs, Algebraic Geometry, Santa Cruz, 1995; Proceedings of Symposia in Pure Math. vol. 62, AMS, 1997, pages 221-287.

J. Szpond, Fermat-type arrangements, arXiv:1909.04089.