On the cohomology of orbit space of free $\mathbb{Z}_p$-actions on lens spaces

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Abstract. Let $G = \mathbb{Z}_p$, $p$ an odd prime, act freely on a finite-dimensional CW-complex $X$ with mod $p$ cohomology isomorphic to that of a lens space $L^{2m-1}(\xi_1, \ldots, \xi_m)$. In this paper, we determine the mod $p$ cohomology ring of the orbit space $X/G$, when $p^2 \nmid m$.

Keywords. Lens space; free action; cohomology algebra; spectral sequence.

1. Introduction

Let $p$ be an odd prime and $m > 1$ an integer. Consider the $(2m-1)$-sphere $S^{2m-1} \subset \mathbb{C} \times \cdots \times \mathbb{C}$ ($m$-times). Given integers $q_1, \ldots, q_m$ relatively prime to $p$, the map $(\xi_1, \ldots, \xi_m) \rightarrow (\xi_1^{q_1}, \xi_2^{q_2}, \ldots, \xi_m^{q_m})$, where $\xi = e^{2\pi i/p^2}$, defines a free action of $G = \langle \xi \rangle$ on $S^{2m-1}$. The orbit spaces of $G$ and the subgroup $N = \langle \xi^p \rangle$ are the lens spaces $L^{2m-1}(p^2; q_1, \ldots, q_m)$ and $L^{2m-1}(p; q_1, \ldots, q_m)$, respectively. Thus, we have a free action of $\mathbb{Z}_p$ on $L^{2m-1}(p; q_1, \ldots, q_m)$ with the orbit space $L^{2m-1}(p^2; q_1, \ldots, q_m)$. By a mod $p$ cohomology lens space, we mean a space $X$ whose Čech cohomology $H^*(X; \mathbb{Z}_p)$ is isomorphic to that of a lens space $L^{2m-1}(p; q_1, \ldots, q_m)$. We will write $X \sim_p L^{2m-1}(p; q_1, \ldots, q_m)$ to indicate this fact. If $G = \mathbb{Z}_p$ acts on a mod $p$ cohomology lens space $X$, then the fixed point set of $G$ on $X$ has been investigated by Su [4]. In this paper, we determine the cohomology ring (mod $p$) of the orbit space $X/G$, when $G$ acts freely on $X$. The following theorem is established.

Theorem. Let $G = \mathbb{Z}_p$ act freely on a finite-dimensional CW-complex $X \sim_p L^{2m-1}(p; q_1, \ldots, q_m)$. If $p^2 \nmid m$, then $H^*(X/G; \mathbb{Z}_p)$ is one of the following graded commutative algebras:

(i) $\mathbb{Z}_p[x,y,z,u_1,u_3,\ldots,u_{2p-3}]/I$, where $I$ is the homogeneous ideal

\[
\langle x^2, y^p, z^n, u_h - A_h y^{(h+1)/2}, u_h u_{2p-h}, u_h u_{h'} - B_{hh'} z u_{h+h'-1},
- C_{hh'} y^{(h+h')/2}, u_h u_{h'} - B'_{hh'} z u_{h+h'-2p-1} - C'_{hh'} y^{(h+h'-2p)/2}, \rangle,
\]

$m = np$, $\deg x = 1$, $\deg y = 2$, $\deg z = 2p$, $\deg u_h = h$, $\beta_p(x) = y$, and $0 = B_{hh'} = C_{hh'} = B'_{hh'} = C'_{hh'}$ when $h = h'$. ($\beta_p$ is the mod-$p$ Bockstein homomorphism associated with the sequence $0 \rightarrow \mathbb{Z}_p \rightarrow \mathbb{Z}_p^2 \rightarrow \mathbb{Z}_p \rightarrow 0$).

(ii) $\mathbb{Z}_p[x,z]/\langle x^2, z^m, \rangle$, where $\deg x = 1$ and $\deg z = 2$. 

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2. Preliminaries

In this section, we recall some known facts which will be used in the proof of our theorem. Given a $G$-space $X$, there is an associated fibration $X \xrightarrow{\iota} X_G \xrightarrow{\pi} B_G$, and a map $\eta: X_G \to X/G$, where $X_G = (E_G \times X)/G$ and $E_G \to B_G$ is the universal $G$-bundle. When $G$ acts freely on $X$, $\eta: X_G \to X/G$ is homotopy equivalence, so the cohomology rings $H^*(X_G)$ and $H^*(X/G)$ (with coefficients in a field) are isomorphic. To compute $H^*(X_G)$, we exploit the Leray–Serre spectral sequence of the map $\pi: X_G \to B_G$. The $E_2$-term of this spectral sequence is given by

$$E_2^{k,j} \cong H^k(B_G; \mathcal{A}^l(X))$$

(where $\mathcal{A}^l(X)$ is a locally constant sheaf with stalk $H^l(X)$ and group $G$) and it converges to $H^*(X_G)$, as an algebra. The cup product in $E_{r+1}$ is induced from that in $E_r$ via the isomorphism $E_{r+1} \cong H^r(X)$. When $\pi_1(B_G)$ operates trivially on $H^*(X)$, the system of local coefficients is simple (constant) so that

$$E_2^{k,j} \cong H^k(B_G) \otimes H^l(X).$$

In this case, the restriction of the product structure in the spectral sequence to the subalgebras $E_2^{0,j}$ and $E_2^{j,0}$ gives the cup products on $H^*(B_G)$ and $H^*(X)$ respectively. The edge homomorphisms

$$H^p(B_G) = E_2^{p,0} \to E_2^{p,0} \to \cdots \to E_2^{p,0} = E_2^{p,0} \subseteq H^p(X_G)$$

and

$$H^q(X_G) \to E_2^{0,q} = E_2^{0,q} \subseteq \cdots \subseteq E_2^{0,q} = H^q(X)$$

are the homomorphisms

$$\pi^*: H^p(B_G) \to H^p(X_G) \quad \text{and} \quad t^*: H^q(X_G) \to H^q(X),$$

respectively.

The above results about spectral sequences can be found in [3]. We also recall that

$$H^*(B_G; \mathbb{Z}_p) = \mathbb{Z}_p[s,t]/(s^2) = \Lambda(s) \otimes \mathbb{Z}_p[t],$$

where $\deg s = 1$, $\deg t = 2$ and $\beta_p(s) = t$.

The following fact will be used without mentioning it explicitly.

**PROPOSITION.**

*Suppose that $G = \mathbb{Z}_p$ acts on a finite-dimensional CW-complex space $X$ with the fixed point set $F$. If $H^j(X; \mathbb{Z}_p) = 0$ for $j > n$, then the inclusion map $F \to X$ induces an isomorphism

$$H^j(X_G; \mathbb{Z}_p) \to H^j(F_G; \mathbb{Z}_p)$$

for $j > n$ (see Theorem 1.5 in Chapter VII of [1]).*

3. Proof

The example of free action of $G = \mathbb{Z}_p$ on the lens space $L^{2m-1}(p; q_1, \ldots, q_m)$ described in the introduction is a test case for the general theorem. All cohomology groups in the
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proof should be considered to have coefficients in \( \mathbb{Z}_p \). Since \( \pi_1(B_G) = G \) acts trivially on \( H^*(X) \), the fibration \( X \xrightarrow{i} X_G \xrightarrow{\pi} B_G \) has a simple system of local coefficients on \( B_G \). So the spectral sequence has

\[
E_2^{k,l} \cong H^k(B_G) \otimes H^l(X).
\]

Let \( a \in H^1(X) \) and \( b \in H^2(X) \) be generators of the cohomology ring \( H^*(X) \). As there are no fixed points of \( G \) on \( X \), the spectral sequence does not collapse at the \( E_2 \)-term. Consequently, we have either \( d_2(1 \otimes a) = t \otimes 1 \) and \( d_2(1 \otimes b) = 0 \) or \( d_2(1 \otimes a) = 0 \) and \( d_2(1 \otimes b) = t \otimes a \).

If \( d_2(1 \otimes a) = 0 \) and \( d_2(1 \otimes b) = t \otimes a \), then we have \( d_2(1 \otimes b^q) = qt \otimes ab^q \) and \( d_2(1 \otimes ab^q) = 0 \) for \( 1 \leq q \leq m - 1 \). So \( 0 = d_2((1 \otimes b^{m-1}) \cup (1 \otimes b)) = mt \otimes ab^{m-1} \), which is true iff \( p \mid m \). Suppose that \( m = np \). Then

\[
d_2^p: E_2^{k,0} \rightarrow E_2^{k,0}.
\]

is an isomorphism if \( l \) is even and \( 2p \not| \ l \), and \( d_2 = 0 \) if \( l \) is odd or \( 2p \mid l \). So \( E_2^{k,l} = E_2^{k,l} \) for all \( k \) if \( l = 2ap \) or \( 2(q + 1)p - 1 \), where \( 0 \leq q < n \); \( k = 0, 1 \) if \( l \) is odd and \( 2p \not| \ (l + 1) \), and \( E_2^{k,l} = 0 \), otherwise. It is easily seen that \( d_3 = 0 \), for example, if \( u \in E_3^{0,2(q+1)p+1} \) and \( d_3(u) = A[st \otimes ab^{(q+1)p-1}](A \in \mathbb{Z}_p) \), then, for \( v = [t \otimes 1] \in E_3^{2,0} \), we have \( 0 = d_3(u \cup v) = A[st^2 \otimes ab^{(q+1)p-1}] \Rightarrow A = 0 \). A similar argument shows that the differentials \( d_4, \ldots, d_{2p-1} \) are all trivial. If

\[
d_2^{2p}: E_2^{0,2p-1} \rightarrow E_2^{2p,0}
\]

is also trivial, then

\[
d_2^{2p}: E_2^{k,l} \rightarrow E_2^{k+2p,l-2p+1}
\]

is trivial for every \( k \) and \( l \), because every element of \( E_2^{k,2l} \) can be written as the product of an element of \( E_2^{k,2p} \) by \( 1 \otimes ab^{p-1} \in E_2^{0,2p-1} \) and

\[
d_2^{2p}: E_2^{k,2l+p-1} \rightarrow E_2^{k+2p,2l+p+1}
\]

is obviously trivial. If \( n = 1 \), then \( E_\infty = E_2 \), where the top and bottom lines survive. This contradicts our hypothesis; so \( n > 1 \). If \( d_{2p+1}^{2p+1} = [st^p \otimes 1] \), then it can be easily verified that

\[
d_{2p+1}^{2p+1} = q[s^{2p} \otimes b^{(q+1)p-1}]
\]

and

\[
d_{2p+1}^{2p+1} = q[s^{2p} \otimes ab^{(q+1)p-1}]
\]

for \( 1 \leq q < n \). Consequently,

\[
0 = d_{2p+1}^{2p+1}((1 \otimes ab^{p-1}) \cup (1 \otimes b^p)) = n(st^p \otimes ab^{np-1}),
\]

which is not true for \( (n, p) = 1 \). On the other hand, if

\[
d_{2p+1}^{2p}: E_2^{0,2p} \rightarrow E_2^{2p+1,0}
\]
is trivial, then

\[ d_{2p+1}^{2p+1} : E_{2p}^{k,l} \rightarrow E_{2p}^{k+2p+1,l-2p} \]

is also trivial for every \( k \) and \( l \), as above. Now \( d_r = 0 \) for every \( r > 2p + 1 \), so several lines of the spectral sequence survive to infinity. This contradicts our hypothesis. Therefore,

\[ d_{2p} : E_{2p}^{0,2p-1} \rightarrow E_{2p}^{2p,0} \]

must be non-trivial. Assume that \( d_{2p}[1 \otimes ab^{p-1}] = [t \otimes 1] \). Then

\[ d_{2p} : E_{2p}^{k,l+2p-1} \rightarrow E_{2p}^{k+2p,l} \]

is an isomorphism for \( l = 2qp, 0 \leq q < n \), and is trivial homomorphism for other values of \( l \). Accordingly, we have \( E_n = E_{2p+1} \), and hence

\[ H^n(X_G) = \begin{cases} \mathbb{Z}_p, & j = 2qp, 2(q + 1)p - 1, 0 \leq q < n; \\ \mathbb{Z}_p \oplus \mathbb{Z}_p, & 2qp < j < 2(q + 1)p - 1, 0 \leq q < n; \\ 0, & j > 2np - 1. \end{cases} \]

The elements \( 1 \otimes b^{p} \in E_2^{0,2p} \) and \( 1 \otimes ab^{(h-1)/2} \in E_2^{0,h} \), for \( h = 1, 3, \ldots, 2p - 3 \) are permanent cocycles; so they determine elements \( z \in E_2^{0,2p} \) and \( w_h \in E_2^{0,h} \), respectively. Obviously, \( t^*(z) = b^{p}, z^0 = 0 \) and \( w_h w_{h'} = 0 \). Let \( x = \pi^*(s) \in E_1^{0,1} \) and \( y = \pi^*(t) \in E_1^{2,0} \). Then \( x^2 = 0, y^p = 0 \), and, by the naturality of the Bockstein homomorphism \( \beta_p \), we have \( \beta_p(x) = y \) and \( yw_h = 0 \) but \( xw_h \neq 0 \). It follows that the total complex \( \text{Tot} E_{\ast,n}^* \) is the graded commutative algebra

\[ \text{Tot} E_{\ast,n}^* = \mathbb{Z}_p[x, y, z, w_1, w_3, \ldots, w_{2p-3}] / \langle x^2, y^p, z^n, w_hw_{h'}, w_h y \rangle, \]

where \( h, h' = 1, 3, \ldots, 2p - 3 \). Choose \( u_h \in H^h(X_G) \) representing \( w_h \) for \( h = 1, 3, \ldots, 2p - 3 \). Then \( t^*(u_h) = ab^{(h-1)/2}, u_h^2 = 0 \) and \( u_h u_{2p-h} = 0 \). It follows that

\[ H^*(X_G) = \mathbb{Z}_p[x, y, z, u_1, u_3, \ldots, u_{2p-3}] / I, \]

where \( I \) is the ideal generated by the homogenous elements

\[ x^2, y^p, z^n, yu_h = A_h yx^{(h+1)/2}, u_h u_{2-p-h} = B_{h} xu_{h+h'-1} = C_{h} x y^{(h+h')/2} \]

and \( u_h u_{h'} = B_{h} x u_{h'+2p-1} = C_{h} x y^{(h+h'-2p)/2} \).

Here \( \deg x = 1, \deg y = 2, \deg z = 2p, \deg u_h = h \) and, when \( h = h' \), \( 0 = B_{h} = C_{h} \).

If \( p \nmid m \), then we must have \( d_2(1 \otimes a) = t \otimes 1, d_2(1 \otimes b) = 0 \). It can be easily observed that

\[ d_2 : E_2^{k,l} \rightarrow E_2^{k+2,l-1} \]

is a trivial homomorphism for \( l \) even and an isomorphism for \( l \) odd. It is now easy to see that \( d_r = 0 \) for every \( r > 2 \). So \( E_{\ast,n} = E_3^k = \mathbb{Z}_p \) for \( k < 2 \) and \( l = 0, 2, 4, \ldots, 2m - 2 \). Therefore, we have

\[ H^*(X_G) = \begin{cases} \mathbb{Z}_p, & 0 \leq j \leq 2m - 1; \\ 0, & \text{otherwise}. \end{cases} \]
If \( x \in H^1(X_G) \) is determined by \( s \otimes 1 \in E_2^{1,0} \), then \( x^2 \in E_2^{2,0} = 0 \). The multiplication by \( x \)

\[
x \cup (\cdot): E_\infty^{0,l} \to E_\infty^{1,l}
\]

is an isomorphism for \( l \) even. The element \( 1 \otimes b \in E_2^{0,2} \) is a permanent cocycle and determines an element \( z \in E_2^{0,2} = H^2(X_G) \). We have \( t^*(z) = b \) and \( z^m = 0 \). Therefore, the total complex \( \text{Tot } E_\infty^{*,*} \) is the graded commutative algebra

\[
\text{Tot } E_\infty^{*,*} = \mathbb{Z}_p [x, z]/\langle x^2, z^m \rangle.
\]

Notice that \( H^i(X_G) \) is \( E_\infty^{0,i} \) for \( j \) even and \( E_\infty^{1,j-1} \) for \( j \) odd. Hence,

\[
H^i(X_G) = \mathbb{Z}_p [x, z]/\langle x^2, z^m \rangle,
\]

where \( \deg x = 1 \) and \( \deg z = 2 \). This completes the proof. \( \square \)

4. Example

We realize here the second case of our theorem. Recall that \( G = \mathbb{Z}_p \) acts freely on \( L^{2m-1}(p^2; q_1, \ldots, q_m) \) with the orbit space \( L^{2m-1}(p^2; q_1, \ldots, q_m) / K \). We claim that

\[
H^i(K; \mathbb{Z}_p) = \mathbb{Z}_p [x, z]/\langle x^2, z^m \rangle,
\]

where \( \deg x = 1 \), \( \deg z = 2 \). It is known that \( K \) is a CW-complex with 1-cell of each dimension \( i = 0, 1, \ldots, 2m - 1 \) and the cellular chain complex of \( K \) is

\[
0 \to C_{2m-1} \xrightarrow{0} C_{2m-2} \times \mathbb{Z}^p \to C_{2m-3} \to \cdots \to C_2 \times \mathbb{Z}^p \xrightarrow{C_1 \to 0} C_0,
\]

where each \( C_i = \mathbb{Z} \). Accordingly, the co-chain complex of \( K \) with coefficients in \( \mathbb{Z}_p \) is

\[
0 \to \mathbb{Z}_p \to \mathbb{Z}_p \to \cdots \to \mathbb{Z}_p \to \mathbb{Z}_p \to 0,
\]

where each coboundary operator is the trivial homomorphism. Therefore

\[
H^i(K; \mathbb{Z}_p) = \begin{cases} 
\mathbb{Z}_p, & \text{for } 0 \leq j \leq 2m - 1; \\
0, & \text{for } j \geq 2m.
\end{cases}
\]

To determine the cup product in \( H^j(K; \mathbb{Z}_p) \), we first observe that the inclusion \( K^{(2i-1)} \to K^{(2i+1)} \) induces isomorphism \( H^j(K^{(2i-1)}; \mathbb{Z}_p) \cong H^j(K^{(2i+1)}; \mathbb{Z}_p) \) for \( j \leq 2i - 1 \) so that we can identify them. For \( j < 2i - 1 \), this follows from the cohomology exact sequence of the pair \((K^{(2i+1)}, K^{(2i-1)})\). The exact cohomology sequence of the pairs \((K^{(2i+1)}, K^{(2i-1)})\) and \((K^{(2i)}, K^{(2i-1)})\) show that \( H^{2i-1}(K^{(2i+1)}; \mathbb{Z}_p) \cong H^{2i-1}(K^{(2i)}; \mathbb{Z}_p) \)

and \( H^{2i}(K^{(2i+1)}; \mathbb{Z}_p) \cong H^{2i}(K^{(2i)}; \mathbb{Z}_p) \); the latter because the homomorphism \( H^j(K^{(2i)}; \mathbb{Z}_p) \to H^{2j}(K^{(2i)}; \mathbb{Z}_p) \) is surjective.

Now, we choose generators \( x \in H^1(K; \mathbb{Z}_p) \) and \( z \in H^2(K; \mathbb{Z}_p) \). Then obviously \( x^2 = 0 \) and \( z^m = 0 \). We can assume, by induction, that \( z^i \) and \( xz^i \) generate \( H^i(K; \mathbb{Z}_p) \) and \( H^{i+1}(K; \mathbb{Z}_p) \), respectively, for \( i \leq m - 2 \). Then, there is an element \( kxz^{m-2} \) such that \( z \cup kxz^{m-2} = kxz^{m-1} \) generates \( H^{2m-1}(K; \mathbb{Z}_p) \) (see Corollary 3.39 of [2]). We must have \( (k, p) = 1 \), otherwise the order of \( kxz^{m-1} \) would be less than \( p \). Thus \( xz^{m-1} \) generates \( H^{2m-1}(K; \mathbb{Z}_p) \), and this is true only if \( z^{m-1} \) generates \( H^{2m-2}(K; \mathbb{Z}_p) \). Hence our claim.
5. Remarks

(i) It is clear from the proof of the theorem that if \( p \nmid m \), then only the second possibility of the theorem holds. Furthermore, if \( X \sim_p L^{2m-1}(p; q_1, \ldots, q_m) \) and \( \pi_1(X) = \mathbb{Z}_p \), then there exists a simply connected space \( Y \) with a free action of \( \Delta = \mathbb{Z}_p \) such that \( Y \sim_p S^{2m-1} \) and \( Y/\Delta \approx X \) (Theorems 3.11 and 2.6 of \([4]\)). If \( G = \mathbb{Z}_p \) acts freely on \( X \), then the liftings of transformations (on \( X \)) induced by the elements of \( G \) form a group \( \Gamma \) of order \( p^2 \) which acts freely on \( Y \) and hence \( \Gamma \) must be cyclic. It is clear that \( \Gamma \) contains the group \( \Delta \) of deck transformations of the covering \( Y \rightarrow X \), and \( G = \Gamma/\Delta \). So \( X/G \approx Y/\Gamma \). Since \( Y \sim_p S^{2m-1} \), the mod \( p \) cohomology algebra of \( Y/\Gamma \) is a truncation of \( H^*(B\Gamma; \mathbb{Z}_p) \). Thus, in this case also, only the second possibility of the theorem holds irrespective of the condition whether or not \( p|m \).

(ii) We recall that a paracompact Hausdorff space \( X \) is called finitistic if every open covering of \( X \) has a finite dimensional open refinement (see p. 133 of \([1]\)). Our theorem and its proof go through for finitistic spaces.

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