Perturbative effects of spinning black holes with applications to recoil velocities

Hiroyuki Nakano, Manuela Campanelli, Carlos O. Lousto, Yosef Zlochower

Center for Computational Relativity and Gravitation, School of Mathematical Sciences, Rochester Institute of Technology, Rochester, New York 14623, USA

E-mail: nakano@astro.rit.edu, manuela@astro.rit.edu, lousto@astro.rit.edu, yosef@astro.rit.edu

Abstract. Recently, we proposed an enhancement of the Regge-Wheeler-Zerilli formalism for first-order perturbations about a Schwarzschild background that includes first-order corrections due to the background black-hole spin. Using this formalism, we investigate gravitational wave recoil effects from a spinning black-hole binary system analytically. This allows us to better understand the origin of the large recoils observed in full numerical simulation of spinning black hole binaries.

PACS numbers: 04.25.Nx, 04.70.Bw, 04.30.Db

Submitted to: Class. Quantum Grav.
1. Introduction

After the breakthroughs of 2005 [1, 2, 3] with the fully non-linear dynamical numerical simulation of the inspiral, merger and ringdown of black-hole binaries (BHBs), there were many important advances in the understanding of black-hole physics. Indeed, the discovery and modeling of very large recoil velocities [4, 5] acquired by the final remnant of spinning BHB mergers have attracted a lot of interest among astrophysicists.

Empirical formulae for the final remnant black hole recoil velocity (also mass and spin) from merging black-hole binaries were obtained in [6] (and references therein), where post-Newtonian results [7, 8] were used as guide to model the recoil dependence on the physical parameters of the progenitor BHB [6]. On the other hand, there is also a long history of recoil studies in black hole perturbation theory (e.g., see [9, 10]), and recently, an analytic treatment of the linear momentum flux of the plunge of a particle into a Kerr black hole has been considered in [11].

In [12], we extended the Regge-Wheeler-Zerilli (RWZ) equations [13, 14] for the Schwarzschild perturbations by including, perturbatively, a term linear in the spin of the larger black hole (SRWZ formulation). We have found good agreement in the full numerical and perturbative waveforms for intermediate mass ratio black-hole binaries, reaching 99.5% matching for the leading \((\ell, m) = (2, 2)\) mode [15, 12] and recently simulated a mass ratio 100 : 1 BHB merger [16]. This formalism can be considered as an extension of [17] in the close limit. Here, using the slow motion approximation, we derive the evolution of the linear momentum for binary systems in the SRWZ formalism analytically and compare to the post-Newtonian (PN) expansions.

2. Formulation

2.1. Spin as a perturbation

Our goal is to analytically model the waveforms from a particle with mass \(\mu\) orbiting around a spinning black hole with mass \(M\) (where \(\mu \ll M\)). In [12], we considered the Kerr metric up to \(O(a^1)\), where \(a\) denotes the spin of the BH which has the dimension of mass, and the spin direction is along the \(z\)-axis. In this paper, we set the spin along the \(x\)-axis, and the metric is given by

\[
ds^2 = - \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right) + \frac{4Ma}{r} dt \left(\sin \phi d\theta + \sin \theta \cos \theta \cos \phi d\phi\right) + O(a^2).
\]

in the Boyer-Lindquist radial coordinate. The last term in the right hand side of the above equation is treated as a perturbation in the Schwarzschild black hole background.

\[
g_{\mu\nu} = g_{\mu\nu}^{Sch} + h_{\mu\nu}^{(1, \text{spin})};
\]

\[
h_{t\theta}^{(1, \text{spin})} = h_{\theta t}^{(1, \text{spin})} = \frac{2S}{r} \sin \phi, \quad h_{t\phi}^{(1, \text{spin})} = h_{\phi t}^{(1, \text{spin})} = \frac{2S}{r} \sin \theta \cos \theta \cos \phi,
\]
Perturbative effects of spinning black holes

where \( S_x = Ma \). By using the expansion defined by the tensor harmonics given in [18], we find that the coefficients of the tensor harmonics are given by

\[
h_{011}^{(1,\text{spin})}(t, r) = -\sqrt{\frac{8\pi}{3}} \frac{S_x}{r}, \quad h_{01-1}^{(1,\text{spin})}(t, r) = \sqrt{\frac{8\pi}{3}} \frac{S_x}{r}.
\]

2.2. SRWZ formulation

We treat the coupling between the spin discussed above and first order metric perturbation (gravitational radiation) as a second order perturbation calculation. The Einstein equations up to the second perturbative order are formally written as

\[
G_{\mu\nu}^{(1)}[h^{(1)}] + G_{\mu\nu}^{(1)}[h^{(2)}] + G_{\mu\nu}^{(2)}[h^{(1)}, h^{(1)}] = 8\pi T_{\mu\nu}, \tag{4}
\]

where the energy-momentum tensor \( T_{\mu\nu} = T_{\mu\nu}^{(1)} + T_{\mu\nu}^{(2)} \), and \( h_{\mu\nu}^{(1)} \) and \( h_{\mu\nu}^{(2)} \) are the first and second order metric perturbations, respectively. We consider that the second order metric perturbation, \( h^{(2,\text{wave})} \) is created by the spin \( h_{\mu\nu}^{(1,\text{spin})} \) and radiation \( h_{\mu\nu}^{(1,\text{wave})} \) couplings. In this case, we solve

\[
G_{\mu\nu}^{(1)}[h^{(1,\text{wave})}] = 8\pi T_{\mu\nu}, \tag{5}
\]

\[
G_{\mu\nu}^{(1)}[h^{(2,\text{wave})}] = -G_{\mu\nu}^{(2)}[h^{(1,\text{wave})}, h^{(1,\text{spin})}], \tag{6}
\]

up to \( O(a^1) \). Here the square of the first order radiation has been ignored. We solve the above equations in the tensor harmonics expansion as in the RWZ formalism.

2.3. Gravitational radiation and linear momentum evolution

Gravitational wave modes in the RWZ formalism are given by the metric perturbation under an asymptotic flat (AF) gauge

\[
h^{(n)} = \sum_{\ell m} \left[ \frac{1}{2} \ell(\ell + 1)(\ell - 1)(\ell + 2) \right]^{1/2} G_{\ell m}^{(n)\text{AF}}(t, r) f_{\ell m} + \frac{2\ell(\ell + 1)(\ell - 1)(\ell + 2)}{2r^2} h_{2\ell m}^{(n)\text{AF}}(t, r) d_{\ell m}, \tag{7}
\]

where superscripts \((n)\) \((n = 1, 2)\) denote the perturbative order, and \( f_{\ell m} \) and \( d_{\ell m} \) are tensor harmonics. The even parity mode \( G_{\ell m}^{(n)\text{AF}} \) and odd parity mode \( h_{2\ell m}^{(n)\text{AF}} \) are expressed by the Regge-Wheeler-Zerilli functions \( \psi_{\ell m}^{(n)(\text{even})} \) and \( \psi_{\ell m}^{(n)(\text{odd})} \) as follows.

\[
G_{\ell m}^{(n)\text{AF}}(t, r) = \frac{1}{r} \psi_{\ell m}^{(n)(\text{even})}(t, r), \quad h_{2\ell m}^{(n)\text{AF}}(t, r) = i r \psi_{\ell m}^{(n)(\text{odd})}(t, r). \tag{8}
\]

(See the definition of the tensor harmonics expansion given in [18], and [12] for more precise discussion of the second perturbative order.)

Next, we discuss the time evolutions of the linear momentum of the binary system. This is obtained from the following expression (e.g. see Eq. (2.17) in [10]).

\[
\dot{P}_i = -\frac{1}{32\pi} \int d\Omega r^2 n_i \left\langle h^{\alpha\beta;\mu} h_{\alpha\beta;\mu} \right\rangle_{TT}, \tag{9}
\]
where \( n_i = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \). The subscript TT means the transverse traceless part and we may use the metric in Eq. (7) under the AF gauge. Then, the above equation is written as

\[
\ddot{P} = -\frac{1}{64\pi} \sum_{\ell m} \sum_{\ell' m'} \left[ r^2 \left( \dot{G}_{\ell m}^{(n)AF}(t, r) \dot{G}_{\ell' m'}^{(n)AF}(t, r) - \frac{1}{r^2} \dot{h}_{2 \ell m}^{(n)AF}(t, r) \dot{h}_{2 \ell' m'}^{(n)AF}(t, r) \right) P_{S-i}^{i}(\ell m \ell' m') \right.
\]

\[+ i \left( \dot{G}_{\ell m}^{(n)AF}(t, r) \dot{h}_{2 \ell m}^{(n)AF}(t, r) - \dot{h}_{2 \ell m}^{(n)AF}(t, r) \dot{G}_{\ell m}^{(n)AF}(t, r) \right) P_{C-i}^{i}(\ell m \ell' m') \right],
\]

(10)

where

\[ P_{S-i}^{i}(\ell m \ell' m') = \int d\Omega n_i \left( W_{\ell m} W_{\ell' m'} + \frac{1}{\sin^2 \theta} X_{\ell m} X_{\ell' m'} \right), \]

\[ P_{C-i}^{i}(\ell m \ell' m') = \int d\Omega n_i \frac{1}{\sin \theta} \left( W_{\ell m} X_{\ell' m'} - X_{\ell m} W_{\ell' m'} \right). \]

(11)

We note that \( P_{S-i}^{i} \) vanishes for \( \ell - \ell' = 2k \) (\( k: \) integer) because of the parity of the integration, while \( P_{C-i}^{i} \) vanishes for \( \ell - \ell' = 2k + 1 \) (\( k: \) integer).

In the following, we discuss only the leading order contribution of gravitational waves to the linear momentum evolution in the slow motion approximation. In practice, the combination of the first order \( \ell = 2, m = 0 \) and \( \ell = 3, m = 0 \) even parity modes produces the leading order contribution for the non-spinning case, and the combination of \( \ell = 2, m = 0 \) even parity and the \( \ell = 2, m = \pm 1 \) odd parity modes becomes the dominant contribution of spin in the situation discussed below.

3. Evolution of the linear momentum

3.1. Point particle’s motion

We consider a particle falling radially into a Schwarzschild black hole as the first order source. Assuming \( \Theta(\tau) = \Phi(\tau) = 0 \), where the particle’s location is given by \( \{T(\tau), R(\tau), \Theta(\tau), \Phi(\tau)\} \), the equation of motion of the particle is

\[
\left( \frac{dR}{dt} \right)^2 = - \left( 1 - \frac{2M}{R} \right)^3 \frac{1}{E^2} + \left( 1 - \frac{2M}{R} \right)^2,
\]

(12)

where \( R = R(t) \) is the location of the particle and the energy \( E \) is written by

\[ E = \left( 1 - \frac{2M}{R} \right) \frac{dT(\tau)}{d\tau}. \]

(13)

We also use

\[ \frac{d^2 R}{dt^2} = - \frac{3}{E^2} \left( 1 - \frac{2M}{R} \right)^2 \frac{M}{R^2} + 2 \left( 1 - \frac{2M}{R} \right) \frac{M}{R^2}, \]

(14)

to simplify the calculations below. In the slow motion approximation where we consider \( dR/dt \ll 1 \) and \( M/R \ll 1 \), we have \( E = 1 \) and \( d^2 R/dt^2 = -M/R^2 \).

\[ \dagger \] Although we write \( \Phi(\tau) = 0 \) here, \( \Phi(\tau) \) is arbitrary at \( \Theta(\tau) = 0 \).
The tensor harmonics coefficients of the first order stress-energy tensor become
\[
A^{(1)\ell m}(t, r) = \mu \frac{E R(t)}{R(t) - 2M} \left( \frac{dR}{dt} \right)^2 \frac{1}{(r - 2M)^2} \delta(r - R(t)) Y_{\ell m}^*(0, 0),
\]
\[
A^{(1)0\ell m}(t, r) = \mu \frac{E R(t)}{R(t) - 2M} \frac{(r - 2M)^2}{r^4} \delta(r - R(t)) Y_{\ell m}^*(0, 0),
\]
\[
A^{(1)1\ell m}(t, r) = \sqrt{2} i \mu \frac{E R(t)}{R(t) - 2M} \frac{dR}{dt} \frac{1}{r^2} \delta(r - R(t)) Y_{\ell m}^*(0, 0).
\]  
Otherwise the coefficients are zero. Here, only the \( m = 0 \) modes have a non-zero value.

We note that the black-hole spin effect in the equations of motion of the particle does not contribute to leading order in the slow motion approximation. Therefore, we do not consider any effect of the background spin on the particle’s trajectory in this paper.

3.2. First order \( \ell = 2, m = 0 \) and \( \ell = 3, m = 0 \), even parity perturbation

In order to calculate the radiative even parity modes, we use the Zerilli equation. For example, the Zerilli equation for the \( \ell = 2, m = 0 \) mode is given by
\[
\left[ -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^2} - \frac{6}{r^4} \frac{(r - 2M)(4r^3 + 4r^2M + 6rM^2 + 3M^3)}{r^4(2r + 3M)^2} \right] \psi^{(1)\text{even}}_{20}(t, r) = \left[ \frac{4\pi \mu}{R(t)} \delta(r - R(t)) + \frac{4\pi \mu}{3} \frac{d}{dr} \delta(r - R(t)) \right] Y_{20}^*(0, 0).
\]  
In the right hand side of the above equation, we used the slow motion approximation, i.e., the characteristic orbital velocity \( v \ll 1 \) where the particle’s velocity \( dR(t)/dt \sim v \) and the potential \( M/R(t) \sim v^2 \), and only consider leading-order terms in this approximation.

The Zerilli function is obtained by the Green’s function method. In practice, we used the Fourier transformation and prepared the Green’s function in the frequency domain. We consider only the outside Green’s function of the particle’s location in the \( M \rightarrow 0 \) limit. This is given by
\[
G(r, r') = i \omega j_2(\omega r) h_2^{(1)}(\omega r) \theta(r - r').
\]  
Here, we have asymptotic behaviors, \( j_2(\omega r) = (\omega r)^2/15 \) for small \( r \), and \( h_2^{(1)}(\omega r) = i \exp(i\omega r)/(\omega r) \) for large \( r \). Finally, we obtain
\[
\psi^{(1)\text{even}}_{20}(t, r) = \frac{16\pi}{15} \mu \left( \hat{R}^2 - \frac{M}{R} \right) Y_{20}^*(0, 0),
\]  
where the argument of \( R \) is the retarded time \( t - r \).

In the same way, we calculate the \( \ell = 3, m = 0 \) even parity mode. In the leading order of the slow motion approximation, the source term of the Zerilli equation becomes
\[
S^{(1)\text{even}}_{30} = \left[ -\frac{8\pi \mu}{5} \delta(r - R(t)) + \frac{4\pi \mu}{15} \frac{d}{dr} \delta(r - R(t)) \right] Y_{30}^*(0, 0).
\]  
Then, using the Green’s function method with asymptotic behaviors, \( j_3(\omega r) = (\omega r)^3/105 \) for small \( r \), and \( h_3^{(1)}(\omega r) = \exp(i\omega r)/(\omega r) \) for large \( r \). we obtain
\[
\psi^{(1)\text{even}}_{30}(t, r) = \frac{16\pi}{105} \mu \left( \hat{R}^3 - \frac{2M}{R} \hat{R} \right) Y_{30}^*(0, 0).
\]
3.3. First order dipole, even parity perturbation

Although the first order dipole ($\ell = 1$) even parity mode in the vacuum regions can be completely eliminated in the center of mass coordinate system, the metric perturbation is not globally pure gauge \[19\]. The coupling between this mode and the black-hole spin creates the leading order spin effect on the linear momentum evolution. We therefore have to include the $\ell = 1$ mode contributions.

For this mode, the metric perturbations are given by

$$ h^{(1)}_{10}(t, r) = \left(1 - \frac{2M}{r}\right) H^{(1)}_{010}(t, r) a_{10} - \sqrt{2} i H^{(1)}_{110}(t, r) a_{110} + \left(1 - \frac{2M}{r}\right)^{-1} H^{(1)}_{210}(t, r) a_{10} $$

$$ - \frac{2i}{r} h^{(1)(e)}_{010}(t, r) b_{010} + \frac{1}{2} h^{(1)(e)}_{110}(t, r) b_{110} + \sqrt{2} K^{(1)}_{10}(t, r) g_{10}. \quad (21) $$

The generators of the gauge transformation are by,

$$ x^\mu \rightarrow x^\mu + \xi^{(1)\mu}_{\ell=1} (x^\alpha); $$

$$ \xi^{(1)\mu}_{\ell=1} = \left\{ V_0^{(1)}(t, r) Y_1(t, \theta, \phi), V_1^{(1)}(t, r) Y_1(t, \theta, \phi), V_2^{(1)}(t, r) \frac{\partial_\theta Y_1(t, \theta, \phi)}{\sin^2 \theta} \right\}, \quad (22) $$

where $V_0^{(1)}$, $V_1^{(1)}$ and $V_2^{(1)}$ are three degrees of gauge freedom in the $\ell = 1$ mode. The metric perturbations transform under the above gauge transformation from a gauge ($G$) to a gauge ($G'$) as

$$ H^{(1)G'}_{010}(t, r) = H^{(1)G}_{010}(t, r) + 2 \frac{\partial}{\partial t} V^{(1)G \rightarrow G'}_0(t, r) + 2 \frac{M}{r(r - 2M)} V^{(1)G \rightarrow G'}_1(t, r), $$

$$ H^{(1)G'}_{110}(t, r) = H^{(1)G}_{110}(t, r) + \frac{r - 2M}{r} \frac{\partial}{\partial r} V^{(1)G \rightarrow G'}_0(t, r) - \frac{r}{r - 2M} \frac{\partial}{\partial t} V^{(1)G \rightarrow G'}_1(t, r), $$

$$ H^{(1)G'}_{210}(t, r) = H^{(1)G}_{210}(t, r) - 2 \frac{\partial}{\partial r} V^{(1)G \rightarrow G'}_1(t, r) + 2 \frac{M}{r(r - 2M)} V^{(1)G \rightarrow G'}_1(t, r), $$

$$ K^{(1)G'}_{(10)}(t, r) = K^{(1)G}_{(10)}(t, r) - \frac{2}{r} V^{(1)G \rightarrow G'}_1(t, r) + 2 V^{(1)G \rightarrow G'}_2(t, t), $$

$$ h^{(1)(e)G'}_{010}(t, r) = h^{(1)(e)G}_{010}(t, r) + \frac{r - 2M}{r} V^{(1)(e)G \rightarrow G'}_0(t, r) - r^2 \frac{\partial}{\partial t} V^{(1)(e)G \rightarrow G'}_2(t, r), $$

$$ h^{(1)(e)G'}_{110}(t, r) = h^{(1)(e)G}_{110}(t, r) - \frac{r}{r - 2M} V^{(1)(e)G \rightarrow G'}_1(t, r) - r^2 \frac{\partial}{\partial r} V^{(1)(e)G \rightarrow G'}_2(t, r). \quad (23) $$

When we choose the gauge so that $h^{(1)(e)Z}_{010} = h^{(1)(e)Z}_{110} = K^{(1)Z}_{10} = 0$, where the suffix $Z$ stands for the Zerilli gauge \[14\], we obtain the metric perturbations

$$ H^{(1)Z}_{010}(t, r) = \frac{8\pi \mu E}{3M(r - 2M)^2} \left( r^3 \frac{d^2 R(t)}{dt^2} + M(R(t) - 2M) \right) \theta(r - R(t)) Y^{*}_{10}(0, 0), $$

$$ H^{(1)Z}_{110}(t, r) = -\frac{8\pi \mu E}{r(r - 2M)^2} \frac{dR(t)}{dt} \theta(r - R(t)) Y^{*}_{10}(0, 0), $$

$$ H^{(1)Z}_{210}(t, r) = \frac{8\pi \mu E}{(r - 2M)^2} (R(t) - 2M) \theta(r - R(t)) Y^{*}_{10}(0, 0). \quad (24) $$
We note that the metric perturbations are not under the AF gauge. For the above metric perturbation, if we consider the gauge transformation,

$$V_0^{(1)D}(t, r) = -\frac{4\pi \mu E}{3M} \frac{r^3}{(r-2M)^2} \frac{dR(t)}{dt} \delta(r - R(t)) Y_{10}^*(0, 0),$$

$$V_1^{(1)D}(t, r) = -\frac{4\pi \mu E}{3M} \frac{r}{(r-2M)} (R(t) - 2M) \delta(r - R(t)) Y_{10}^*(0, 0),$$

$$V_2^{(1)D}(t, r) = -\frac{4\pi \mu E}{3M} \frac{1}{(r-2M)} (R(t) - 2M) \delta(r - R(t)) Y_{10}^*(0, 0),$$

we obtain the singular metric perturbation at the particle's location.

$$H_{010}^{(1)D}(t, r) = \frac{8\pi \mu E}{3M} \frac{R(t)^3}{(R(t) - 2M)^2} \left(\frac{dR(t)}{dt}\right)^2 \delta(r - R(t)) Y_{10}^*(0, 0),$$

$$H_{110}^{(1)D}(t, r) = -\frac{8\pi \mu E}{3M} \frac{R(t)^2}{R(t) - 2M} \frac{dR(t)}{dt} \delta(r - R(t)) Y_{10}^*(0, 0),$$

$$H_{210}^{(1)D}(t, r) = \frac{8\pi \mu E}{3M} R(t) \delta(r - R(t)) Y_{10}^*(0, 0),$$

$$h_{0010}^{(1)(e)D}(t, r) = -\frac{4\pi \mu E}{3M} R(t) \frac{dR(t)}{dt} \delta(r - R(t)) Y_{10}^*(0, 0),$$

$$h_{1010}^{(1)(e)D}(t, r) = \frac{4\pi \mu E}{3M} R(t)^2 \delta(r - R(t)) Y_{10}^*(0, 0),$$

$$K_{10}^{(1)D}(t, r) = 0.$$

The above means that although the metric perturbations vanish in the vacuum regions, there are some contributions at the location of the particle. This gauge choice has been discussed in [19] for circular orbit. The coordinate system under this gauge condition can be considered as the center of mass system which is suitable for the analysis of the evolution of the linear momentum. Therefore, we use this gauge and its metric perturbations here. It is noted that we also have another gauge choice where the metric perturbations become $C^0$ at the particle’s location [20].

### 3.4. Second order $\ell = 2, m = \pm 1$, odd parity perturbation

In the previous subsections, we focused on the first perturbative order. Here we treat the second perturbative order using the SRWZ formalism. We focus on the coupling between the first order $\ell = 1, m = 0$ even parity mode (with parity $(-1)^1$) and the spin effect of the central black hole in Eq. (3), which is given by the $\ell = 1, m = \pm 1$ odd parity modes (with parity $(-1)^{1+1}$). This coupling creates the second order $\ell = 2, m = \pm 1$ odd parity perturbation (with parity $(-1)^{2+1}$).

The Regge-Wheeler function for this odd parity perturbation satisfies

$$\left[ -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^2} - 6 \frac{(r-2M)(r-M)}{r^4} \right] \psi_{2\pm 1}^{(2)\text{(odd)}}(t, r) = S_{2\pm 1}^{(2)\text{RW}}(t, r).$$

The source term is derived from the effective stress-energy tensor,

$$T^{(2,\text{eff})}_{\mu\nu} = -\frac{1}{8\pi} G_{\mu\nu}^{(2)} [h_{(1, \text{dipole})}^{(1, \text{spin})}, h_{(1, \text{spin})}].$$
Perturbative effects of spinning black holes

by using the same tensor harmonics expansion as for the first perturbative order. When we consider the leading order in the slow motion approximation, the source term $S_{2±1}^{\text{RW}}$ becomes

$$
S_{2±1}^{(2)\text{RW}} = \pm \frac{\sqrt{10\pi \mu S_x}}{5M} Y_{10}(0,0) \times \left( \frac{2}{R(t)^2} \delta(r - R(t)) - \frac{4}{R(t)} \frac{d}{dr} \delta(r - R(t)) + \frac{d^2}{dr^2} \delta(r - R(t)) \right),
$$

if we use the first order $\ell = 1$, $m = 0$ even parity mode under the D gauge. From this source term, we obtain the Regge-Wheeler function

$$
\psi_{2±1}^{(2)(\text{odd})}(t, r) = \pm \frac{4\sqrt{10\pi}}{15} \mu S_2 R Y_{10}(0,0),
$$

by using the same Green’s function method discussed in Subsection 3.2. Here, we have written $S_2 = S_x$.

3.5. Spinning particle orbiting around a black hole

In the above subsection, we considered the black hole with mass $M$ that has a spin along the $x$-direction. Here, we introduce a particle’s spin which is parallel to the black-hole spin.

First, for simplicity, we consider a point particle with mass $\mu$ that has a spin vector $S^{(\mu)} = \{S_1, 0, 0\}$, and is located at $x_0 = \{0, 0, R\}$ in the Cartesian coordinates. When we discuss perturbations from the spinning particle, we use the following energy-momentum tensor.

$$
T^{\alpha\beta} = T^{\alpha\beta}_{\text{mass}} + T^{\alpha\beta}_{\text{spin}}; \\
T^{\alpha\beta}_{\text{mass}} = \mu \int d\tau u^\alpha u^\beta \frac{\delta^{(4)}(x - z(\tau))}{\sqrt{-g}}, \\
T^{\alpha\beta}_{\text{spin}} = -\nabla_{\gamma} \int d\tau S^{(\alpha)}_{(\mu)} u^\beta \frac{\delta^{(4)}(x - z(\tau))}{\sqrt{-g}},
$$

where we impose a spin supplementary condition (SSC) which determines the center of mass of the particle, $S^{(\alpha)}_{(\mu)} u^\beta = 0$. Since we focus on the leading order effect of the particle’s spin, we may consider only the contribution of $T^{\alpha\beta}_{\text{spin}}$. Furthermore, we can reduce this energy momentum tensor to the form

$$
T^{ij}_{\text{spin}} = \frac{1}{2} \int d\tau \frac{1}{\sqrt{-g}} \partial_\tau [S^{ij}_{(\mu)} \delta^{(4)}(x - z(\tau))] ,
$$

in the leading order of the slow motion approximation. The other components are higher order. Here, the spin tensor $S^{ij}_{(\mu)}$ is given by

$$
S^{r\theta} = -\frac{\sin \phi}{r} S_1, \\
S^{r\phi} = -\frac{\cos \theta \cos \phi}{r \sin \theta} S_1, \\
S^{\theta\phi} = \frac{\cos \phi}{r} S_1.
$$

The tensor harmonics coefficients of the stress-energy tensor are calculated as

$$
A_{1\ell m}^{\text{spin}} = \frac{i S_1}{\sqrt{2} r^{\ell + 1}} \left( \sin \Phi \partial_\theta Y_{\ell m}^*(\Theta, \Phi) + \frac{\cos \Theta \cos \Phi}{\sin \Theta} \partial_\phi Y_{\ell m}^*(\Theta, \Phi) \right) \delta(r - R) , \\
B_{0\ell m}^{\text{spin}} = \frac{i S_1}{\sqrt{2} (\ell + 1) r} \left( \sin \Phi \partial_\theta Y_{\ell m}^*(\Theta, \Phi) + \frac{\cos \Theta \cos \Phi}{\sin \Theta} \partial_\phi Y_{\ell m}^*(\Theta, \Phi) \right) \partial_r \left( \frac{\delta(r - R)}{r} \right) ,
$$

$$
C_{1\ell m}^{\text{spin}} = \frac{i S_1}{\sqrt{2} r^{\ell + 1}} \left( \sin \Phi \partial_\theta Y_{\ell m}^*(\Theta, \Phi) + \frac{\cos \Theta \cos \Phi}{\sin \Theta} \partial_\phi Y_{\ell m}^*(\Theta, \Phi) \right) \partial_\ell \left( \frac{\delta(r - R)}{r} \right) , \\
D_{0\ell m}^{\text{spin}} = \frac{i S_1}{\sqrt{2} (\ell + 1) r} \left( \sin \Phi \partial_\theta Y_{\ell m}^*(\Theta, \Phi) + \frac{\cos \Theta \cos \Phi}{\sin \Theta} \partial_\phi Y_{\ell m}^*(\Theta, \Phi) \right) \partial_r \partial_\ell \left( \frac{\delta(r - R)}{r} \right) .
$$
$Q_{0\ell m}^{\text{spin}} = -\frac{1}{\sqrt{2\ell(\ell + 1)}} \frac{S_1}{r} \left( \frac{\sin \Phi}{\sin \Theta} \partial_\Theta Y_{\ell m}^*(\Theta, \Phi) - \cos \Theta \cos \Phi \partial_\Phi Y_{\ell m}^*(\Theta, \Phi) \right) \times \partial_r \left( \frac{\delta(r - R)}{r} \right), \tag{34}$

where we set the location of the particle on the $z$-axis. It should be noted that only $\ell \geq 1$, $m = \pm 1$ modes have a non-zero value because of $Y_{\ell m}^*(\Theta, \Phi) \sim (\sin \Theta)^{|m|}$. And also, both the even and odd parity modes exist in the metric perturbations from the particle’s spin.

We evaluate the $\ell = 2$, $m = \pm 1$ odd parity perturbations from $T_{\text{spin}}^{\alpha\beta}$ in the first perturbative order calculation. These perturbations have the leading order effect of the particle’s spin. The tensor harmonics coefficient of the stress-energy tensor is given by

$$Q_{02\pm 1}^{\text{spin}} = \mp \sqrt{\frac{5}{32\pi}} \frac{S_1}{r} \partial_r \left( \frac{\delta(r - R)}{r} \right). \tag{35}$$

The wave function is obtained from the Regge-Wheeler equation,

$$\left[-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^2} - 6 \frac{(r - 2M)(r - M)}{r^4} \right] \psi_{2\pm 1}^{(1)(\text{odd})}(t, r) = S_{2\pm 1}^{\text{RW,spin}}(t, r), \tag{36}$$

and the source term in the leading order of the slow motion approximation is given by

$$S_{2\pm 1}^{\text{RW,spin}} = \mp \sqrt{\frac{5\pi}{6}} S_1 \left( -\frac{2}{R(t)} \frac{d}{dr} \delta(r - R(t)) + \frac{d^2}{dr^2} \delta(r - R(t)) \right), \tag{37}$$

where we have extended the interpretation of the particle’s location to a sequence of quasistatic locations $R = R(t)$. From the above source term, we obtain the Regge-Wheeler function

$$\psi_{2\pm 1}^{(1)(\text{odd})}(t, r) = \pm \frac{\sqrt{120\pi}}{15} S_1 \tilde{R}, \tag{38}$$

by using the same Green’s function discussed in Subsection 3.2. This wave function has the following relation with the black-hole spin effect in Eq. (30).

$$\psi_{2\pm 1}^{(1)(\text{odd})}(t, r) = -\frac{S_1 M}{S_2 \mu} \psi_{2\pm 1}^{(2)(\text{odd})}(t, r). \tag{39}$$

### 3.6. Evolution of the linear momentum

Using the results in the previous subsections and the approximated equation of motion in the slow motion approximation, $d^2 R/dt^2 = -M/R^2$, we can calculate the leading order evolution of the linear momentum. The contributions of the gravitational waveforms are summarized in Table I. The mass, velocity, orbital radius and spin dependence is estimated by

$$\frac{v^2}{R^2} \psi_{\ell m}^{(n)} \psi_{\ell m'}^{(n')},$$

from Eq. (10). Also, the direction can be derived from the angular integrations in Eq. (11).
Table 1. Leading order mode contributions to the evolution of the linear momentum in the head-on collision of spinning black holes.

| combination                              | dependence  | direction |
|------------------------------------------|-------------|-----------|
| $(\ell = 2, m = 0, \text{even}) \cdot (\ell = 3, m = 0, \text{even})$ | $\mu^2 v^7/R^2$ | $z$       |
| $(\ell = 2, m = 0, \text{even}) \cdot (\ell = 2, m = \pm 1, \text{odd})$ | $\mu S_1 v^6/R^3$ | $x-y$    |
| $(\ell = 2, m = 0, \text{even}) \cdot (\ell = 2, m = \pm 1, \text{odd})$ | $\mu^2 S_2 v^4/R^4$ | $x-y$    |

First, we find

$$\dot{P}_x = 0.$$ (40)

This result is also obtained by analyzing the symmetry of the system. In practice, we may use the symmetry between $m$ and $-m$ modes in the coefficients of the tensor harmonics of the metric perturbation $G_{\ell m}^{AF(i)}$ and $h_{2\ell m}^{AF(i)}$, and the integration $P_C^i$ and $P_S^i$.

Next, we discuss the spin independent contribution to the evolution of the linear momentum. This contribution arises in the $z$-direction as

$$\dot{P}_z = -\frac{16}{105} \mu^2 M^2 \left( \frac{\dot{R}^3 - 2M \dot{R}}{R^4} \right).$$ (41)

The above equation is derived from the combination of the first order $\ell = 2, m = 0$ and $\ell = 3, m = 0$ even parity modes in Eqs. (18) and (20).

Finally, for the $y$-direction, we have the leading order contribution of the spin effects for the evolution of the linear momentum.

$$\dot{P}_y = -\frac{16}{15} \mu^2 M^2 \frac{\dot{R}^2}{R^5} \left( \frac{S_2}{M} - \frac{S_1}{\mu} \right).$$ (42)

This is calculated from the combination of $\ell = 2, m = 0$ even parity and the $\ell = 2, m = \pm 1$ odd parity modes which include two different wave functions given in Eqs. (30) and (38).

These results are consistent with the calculation in the post-Newtonian approach [7]. Note that Kidder [7] derived the above results for general orbits. On the other hand, our calculation discussed here is limited to the head-on collision of spinning black holes given in Subsection 3.1.

4. Discussion

We have discussed gravitational wave recoil effects and analytically derived the leading order effects in the evolution of the linear momentum by using the SRWZ formalism. This formalism is an extension of the RWZ formalism with a perturbative spin of the background black hole.

In the appendix of [12], we applied the perturbative spin formalism to compute the corresponding quasi-normal modes and compare them with those obtained for the Kerr black hole for all values of the spin parameter. These results show that the SRWZ formalism provide reliable predictions for the spin parameter $a/M \leq 0.3$. 
From the analytic treatment of the gravitational radiation recoil, we confirm the leading \(q^2\) (where \(q = \mu/M\)) dependence of the large recoils out of the orbital plane \[21\]. In the black hole perturbation theory, this scaling with mass ratio is trivial, and in \[22\] the spin dependence has also been discussed in detail by solving Teukolsky equation \[23\] numerically.

The results of the recoil obtained here can be extended to higher PN order in the sense of the slow motion approximation where the characteristic orbital velocity \(v \ll 1\) (for example, see \[24\]). On the other hand, higher order spin effects are complicated. In this paper, we have focused only on the couplings between the first order perturbations about a Schwarzschild background and the black-hole spin. However, it is also necessary to treat the equations of motion with spin. We may use the Teukolsky formalism \[23\] for the spinning large black hole, i.e., use a Kerr background. Taking into account the spin of the particle adds a new degree of complication that makes it difficult to obtain an analytic expression of the gravitational radiation recoil for general orbits and we would need to perform a numerical calculation.

**Acknowledgments**

We gratefully acknowledge the NSF for financial support from Grants No. PHY-0722315, No. PHY-0653303, No. PHY-0714388, No. PHY-0722703, No. DMS-0820923, No. PHY-0929114, No. PHY-0969855, No. PHY-0903782, No. CDI-1028087; and NASA for financial support from NASA Grants No. 07-ATFP07-0158 and No. HST-AR-11763.

**References**

1. F. Pretorius, Phys. Rev. Lett. 95, 121101 (2005) \texttt{arXiv:gr-qc/0507014}.
2. M. Campanelli, C. O. Lousto, P. Marronetti and Y. Zlochower, Phys. Rev. Lett. 96, 111101 (2006) \texttt{arXiv:gr-qc/0511048}.
3. J. G. Baker, J. Centrella, D. I. Choi, M. Koppitz and J. van Meter, Phys. Rev. Lett. 96, 111102 (2006) \texttt{arXiv:gr-qc/0511103}.
4. M. Campanelli, C. O. Lousto, Y. Zlochower and D. Merritt, Astrophys. J. 659, L5 (2007) \texttt{arXiv:gr-qc/0701164}.
5. M. Campanelli, C. O. Lousto, Y. Zlochower and D. Merritt, Phys. Rev. Lett. 98, 231102 (2007) \texttt{arXiv:gr-qc/0702133}.
6. C. O. Lousto, M. Campanelli, Y. Zlochower and H. Nakano, Class. Quant. Grav. 27, 114006 (2010) \texttt{arXiv:0904.3541 [gr-qc]}.
7. L. E. Kidder, Phys. Rev. D 52, 821 (1995) \texttt{arXiv:gr-qc/9506022}.
8. E. Racine, A. Buonanno and L. E. Kidder, Phys. Rev. D 80, 044010 (2009) \texttt{arXiv:0812.4413 [gr-qc]}.
9. K. i. Oohara and T. Nakamura, Phys. Lett. A 94, 349 (1983).
10. M. J. Fitchett and S. L. Detweiler, Mon. Not. Roy. Astron. Soc. 211, 933 (1984).
11. Y. Mino and J. Brink, Phys. Rev. D 78, 124015 (2008) \texttt{arXiv:0809.2814 [gr-qc]}.
12. C. O. Lousto, H. Nakano, Y. Zlochower and M. Campanelli, \texttt{arXiv:1008.4360 [gr-qc]}.
13. T. Regge and J. A. Wheeler Phys. Rev. 108, 1063 (1957).
14. F. J. Zerilli, Phys. Rev. D 2, 2141 (1970).
[15] C. O. Lousto, H. Nakano, Y. Zlochower and M. Campanelli, Phys. Rev. Lett. 104, 211101 (2010) [arXiv:1001.2310 [gr-qc]].
[16] C. O. Lousto and Y. Zlochower, arXiv:1009.0292 [gr-qc].
[17] R. J. Gleiser and A. E. Dominguez, Phys. Rev. D 65, 064018 (2002) [arXiv:gr-qc/0109018].
[18] H. Nakano and K. Ioka, Phys. Rev. D 76, 084007 (2007) [arXiv:0708.0450 [gr-qc]].
[19] S. Detweiler and E. Poisson, Phys. Rev. D 69, 084019 (2004) [arXiv:gr-qc/0312010].
[20] C. O. Lousto and H. Nakano, Class. Quant. Grav. 26, 015007 (2009) [arXiv:0810.3824 [gr-qc]].
[21] C. O. Lousto and Y. Zlochower, Phys. Rev. D 79, 064018 (2009) [arXiv:0805.0159 [gr-qc]].
[22] P. A. Sundararajan, G. Khanna and S. A. Hughes, Phys. Rev. D 81, 104009 (2010) [arXiv:1003.0485 [gr-qc]].
[23] S. A. Teukolsky, Astrophys. J. 185, 635 (1973).
[24] M. Sasaki and H. Tagoshi, Living Rev. Rel. 6, 6 (2003) [arXiv:gr-qc/0306120].