BFKL, BK and the Infrared

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Abstract. The perturbative non-linear (NL) effects in the small-x evolution of the gluon densities depend crucially on the infrared (IR) regularization. The IR regulator, \( R_c \), is determined by the scale of the non-perturbative fluctuations of QCD vacuum. From the instanton models and from the lattice \( R_c \approx 0.3 \) fm. For perturbative gluons with the propagation length \( R_c = 0.26 \) fm the linear BFKL gives a good description of the proton structure function \( F_2(x, Q^2) \) in a wide range of \( x \) and \( Q^2 \). The NL effects turn out to be rather weak and amount to the 10% correction to \( F_2(x, Q^2) \) for \( x \lesssim 10^{-5} \). Much more pronounced NL effects were found in the non-linear model, described in the literature, with a very soft IR regularization corresponding to the IR cutoff at \( \Lambda_{\text{QCD}}^{-1} \). The latter issue is also commented below.

Keywords: small-x evolution, non-linear effects, infrared regularization

PACS: 13.15.+g 13.60.Hb

Introduction

The non-perturbative fluctuations of the QCD vacuum restrict the phase space for the perturbative (real and virtual) gluons of the BFKL cascade thus introducing a new scale: the correlation/propagation radius \( R_c \) of perturbative gluons. From the fits to lattice data on field strength correlators \( R_c \approx 0.3 \) fm. The effects of finite \( R_c \) are consistently incorporated by the color dipole (CD) BFKL equation. The perturbative gluons with short propagation length do not walk to large distances, where they supposedly fuse together. The fusion probability appears to be controlled by the dimensionless parameter \( R_c^2 / 8B \), where \( B \) stands for the diffraction cone slope.

In this communication we discuss the BFKL, BK phenomenology of DIS in presence of finite correlation length \( R_c \) with particular emphasis on a sharpened sensitivity of the non-linear effects to the infrared.

Color screening of BFKL gluons

The finite correlation length implies the Yukawa screened transverse chromo-electric field of the relativistic quark,

\[
\vec{E}(\vec{p}) \sim g_5(\rho)K_1(\rho/R_c)\vec{p}/\rho,
\]
where \( \rho \) is the \( q - g \) separation. The kernel \( \mathcal{K} \) of the CD BFKL equation for the color dipole cross section,
\[
\partial_\xi \sigma(\xi, r) = \mathcal{K} \otimes \sigma(\xi, r), \quad \xi = \log(1/x),
\]
is related to the flux of the Weizsäcker - Williams gluons \( |\vec{E}(\vec{\rho}) - \vec{E}(\vec{\rho} + \vec{r})|^2 \) radiated by the relativistic \( q\bar{q}\)-dipole \( \vec{r} \) \cite{4,5}. The asymptotic freedom dictates that \( \vec{E}(\vec{\rho}) \) must be calculated with the running QCD charge \( g_S(\rho) = \sqrt{4\pi \alpha_S(\rho)} \) and \( \alpha_S(\rho) = \frac{4\pi}{\beta_0 \log(C^2/\rho^2 \Lambda_{\text{QCD}}^2)} \), where \( C^2 = 1.5 \).

### DGLAP ordering of dipole sizes and the infrared

Eq.\((2)\), with the BK non-linearity \cite{6} included, greatly simplifies for the DGLAP ordering of dipole sizes, \( r^2 \ll \rho^2 \ll R_c^2 \),
\[
\partial_\xi \sigma(\xi, r) = \frac{C_F}{\pi} \alpha_S(\rho) r^2 \int_{\rho}^{R_c^2} \frac{d\rho^2}{\rho^4} \times \nonumber \]
\[\times \left[ 2\sigma(\xi, \rho) - \frac{\sigma(\xi, \rho)^2}{8\pi B} \right]. \tag{3}\]

Our definition of the profile function in the impact parameter \( b \)-space is
\[
\Gamma(\xi, r, b) = \frac{\sigma(\xi, r)}{4\pi B(\xi, r)} \exp \left[ -\frac{b^2}{2B} \right], \tag{4}\]
where \( B \) is the diffraction cone slope and \( d\sigma_{el}/dt \sim \exp[Bt] \). Eq.\((4)\) implies that the unitarity limit for \( \sigma \) is \( 8\pi B \).

The diffraction slope for the forward cone in the dipole-nucleon scattering was presented in \cite{9} in a very symmetric form
\[
B(\xi, r) = \frac{1}{2} \langle b^2 \rangle = \frac{1}{8} r^2 + \frac{1}{3} R_N^2 + 2\alpha'_p \xi. \tag{5}\]

The dynamical component of \( B \) is given by the last term in Eq.\((5)\) where \( \alpha'_p \) is the Pomeron trajectory slope evaluated first in \cite{10} (see also \cite{9}). Here we only cite the order of magnitude estimate \cite{9}
\[
\alpha'_p \sim \frac{3}{16\pi^2} \int d^2 \vec{r} \alpha_S(\rho) R_c^2 r^2 K_1^2(r/R_c) \sim \frac{3}{16\pi} \alpha_S(R_c) R_c^2, \tag{6}\]
which clearly shows the connection between the dimensionful \( \alpha'_p \) and the non-perturbative infrared parameter \( R_c \). In Eq.\((5)\) the gluon-probed radius of the proton is a phenomenological parameter to be determined from the experiment. The analysis of Ref. \cite{11} gives \( R_N^2 \approx 12 \text{GeV}^{-2} \).
The function $\rho^{-2}\sigma(\xi, \rho) \sim \alpha_S(\rho) G(x, \rho)$, where $G$ is the integrated gluon density, is flat in $\rho^2$ and the non-linear term in Eq.(3) is dominated by $\rho \sim R_c$:

$$\frac{1}{8B} \int_{r^2}^{R_c^2} \frac{d\rho^2}{\rho^4} \sigma(\xi, \rho)^2 \simeq \frac{R_c^2}{8B} \left( \frac{\pi^2}{N_c} \alpha_S(R_c) G(x, R_c) \right)^2.$$ 

Thus, the small parameter $R_c^2/8B$ enters the game. To see it in action a partial solution to Eq.(3) is needed.

**Partial solution to GLR-MQ**

The differential form of Eq.(3) for $G$ is the GLR-MQ equation [12]

$$\partial_\xi \partial_\eta G(\xi, \eta) = cG(\xi, \eta) - a(\eta) G^2(\xi, \eta),$$

where $c = 8C_F/\beta_0$, $\eta = \log [\alpha_S(R_c)/\alpha_S(\rho)]$,

$$a(\eta) = a(0) \exp[-\eta - \lambda(e^\eta - 1)],$$

and $a(0) = \alpha_S(R_c) \pi R_c^2/4\beta_0 B$ and $\lambda = 4\pi/\beta_0 \alpha_S(R_c)$. We solve Eq.(7) making use of the exact solution of

$$\partial_\xi \partial_\eta G(\xi, \eta) = cG(\xi, \eta)$$

as a boundary condition. To simplify Eq.(7) we substitute the steeply falling function $a(\eta)$ with $a(\eta) = a(0) \exp[-\gamma \eta]$. The partial solution of the simplified problem is found readily:

$$G(\xi, \eta) = \frac{\exp(\Delta \xi + \gamma \eta)}{\omega \exp(\Delta \xi) + \text{const}}.$$  

Here $\gamma = c/\Delta$ with asymptotic value $\Delta = 0.4$ and

$$\omega = \frac{a(0)}{c} = \frac{R_c^2}{8B} \cdot \frac{\pi \alpha_S(R_c)}{2N_c}.$$  

One can see that the gluon fusion mechanism tames the exponential $\xi$-growth of $G(\xi, \eta)$ but fails to stop the accumulation of large logarithms, $\eta = \log(1/\alpha_S(r))$, at very small $r^2 \ll R_c^2$.

For vanishing non-linearity, $\omega \to 0$, Eq.(9) matches the exact solution to the CD BFKL equation found in Ref.[13]

$$G \sim \exp(\Delta \xi + \gamma \eta) \sim \left( \frac{1}{x} \right)^{\Delta} \left[ \frac{1}{\alpha_S(r)} \right]^\gamma.$$  

In the limit $\xi \to \infty$ the gluon density saturates,

$$G \sim \omega^{-1} e^m,$$

and the corresponding dipole cross section reads

$$\sigma(r) = 8\pi B \cdot \frac{r^2}{R_c^2} \cdot 2 \left[ \frac{\alpha_S(R_c)}{\alpha_S(r)} \right]^\gamma.$$
Regime of the additive quark model

At large $r \sim R_c$ a sort of the additive quark model is recovered: the (anti)quark of the dipole $\vec{r}$ develops its own perturbative gluonic cloud and the pattern of the gluon fusion changes dramatically \[8\]. From Ref.\[8\] it follows that the non-linear correction to the dipole cross section is

$$\delta \sigma \sim R_c^{-2} \int_{R_c}^{R_c^2} d^2 \rho K_1^2(\rho/R_c) \frac{\sigma(\xi, \rho) \sigma(\xi, r)}{8\pi B} \sim \frac{\sigma(\xi, R_c) \sigma(\xi, r)}{8\pi B}.$$  \hspace{1cm} (14)

For $R_c$ much smaller than the nucleon size $\sigma(R_c) \propto R_c^2$. Therefore, the magnitude of non-linear effects is controlled, like in the case of small dipoles, by the ratio $R_c^2/8B$.

**Small $R_c$ - weak non-linearity**

In Ref.\[8\] we solved numerically the BFKL and BK equations with purely perturbative initial conditions and the IR regularization described above. Our finding is that the smallness of the ratio $R_c^2/8\pi B$ makes the non-linear effects rather weak even at the lowest Bjorken $x$ available at HERA. The linear BFKL with the running coupling and the infrared regulator $R_c = 0.26$ fm gives a good description of the proton structure function $F_2(x, Q^2)$ in a wide range of $x$ and $Q^2$ \[8\]. For the smallest available $x \sim 10^{-5}$ the 10\% NL correction improves the agreement with data, though.

**Soft IR regularization - fully developed non-linearity**

In Ref.\[14\] the non-linear BK-analysis of HERA data on $F_2(x, Q^2)$ was presented. The infrared cutoff for the purely perturbative BK-kernel is deep in the non-perturbative region,

$$r_{IR} = 2/Q_s \approx 4\text{GeV}^{-1} \approx \Lambda_{QCD}^{-1},$$  \hspace{1cm} (15)

and the non-linear effects are absolutely important to tame the rapid growth of the linear term in the BK equation. The running strong coupling $\alpha_S(r)$ at $r = r_{IR}$ in \[14\] appears to be surprisingly small, $\alpha_S(r_{IR}) \approx 0.45$, thus assuming an applicability of perturbative QCD to arbitrarily large distances $\sim \Lambda_{QCD}^{-1}$. However, it is well known that the non-perturbative fields form structures with sizes significantly smaller than $\Lambda_{QCD}^{-1}$ and local field strength much larger than $\Lambda_{QCD}^2$. Instantons are one of them \[1\]. Direct confirmation of this picture comes from the lattice \[3\]. Therefore, the approach developed in \[14\] may lead to a very good BK-description of the HERA data but does not agree with the current understanding of what is perturbative and non-perturbative in hadronic physics.
Summary

It is not surprising that introducing a small propagation length for perturbative gluons pushes the nonlinear effects to very small $x$. More surprising is that within the linear CD BFKL approach this small length, $R_c = 0.26$ fm, results in the correct $x$-dependence of $F_2(x, Q^2)$ in a wide range of $x$ and $Q^2$. The 10% non-linear BK-correction improves the agreement with HERA data at smallest available $x$[8].

The non-linear BK-description [14] of HERA data which extends perturbative QCD to distances $\sim \Lambda_{QCD}^{-1}$ contradicts to well established non-perturbative phenomena and apparently breaks the hierarchy of scales of soft and hard hadronic physics.

Acknowledgments

V.R. Z. thanks Alessandro Papa and Agustin Sabio Vera for the kind invitation to the workshop Diffraction 2012 and support. Thank are due to Wolfgang Schäfer and Bronislav Zakharov for useful discussions and to the Dipartimento di Fisica dell’Università della Calabria and the INFN - gruppo collegato di Cosenza for their warm hospitality while a part of this work was done. The work was supported in part by the Ministero Italiano dell’Istruzione, dell’Università e della Ricerca and by the RFBR grants 11-02-00441 and 12-02-00193.

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