LAMPERTI SEMI-DISCRETE METHOD

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Abstract. We study the numerical approximation of numerous processes, solutions of nonlinear stochastic differential equations, that appear in various applications such as financial mathematics and population dynamics. Between the investigated models are the CIR process, also known as the square root process, the constant elasticity of variance process CEV, the Heston 3/2-model, the Aït-Sahalia model and the Wright-Fisher model. We propose a version of the semi-discrete method, see \cite{[1]}, which we call Lamperti semi-discrete (LSD) method. The LSD method is domain preserving and seems to converge strongly to the solution process with order 1 and no extra restrictions on the parameters or the step-size.

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1. CIR Model

Let

\[ x_t = x_0 + \int_0^t (k_1 - k_2 x_s) ds + \int_0^t k_3 \sqrt{x_s} dW_s, \quad t \geq 0. \]  

SDE (1) is known as the CIR process, or square root process with a solution remaining in the positive axis, i.e. \( x_t > 0 \) a.s. when \( (k_3)^2 \leq 2k_1 \) and \( x_0 > 0 \), c.f. [2]. The Lamperti transformation of (1) is \( z = \frac{2}{k_3} \sqrt{x} \) and application of the Itô formula implies the following representation, see Appendix A

\[ z_t = z_0 + \int_0^t \left( \frac{2k_1}{(k_3)^2} (z_s)^{-1} - \frac{k_2}{2} + \frac{(k_3)^2}{8} \right) z_s \, ds + \int_0^t dW_s, \quad t \geq 0. \]  

(2)

To simplify notation, set \( a := \frac{2k_1}{(k_3)^2} \) and \( b := \frac{k_2}{2} + \frac{(k_3)^2}{8} \). The new coefficients \( a \) and \( b \) are positive. We consider three versions of the semi-discrete method for approximating (2). In the first two versions \( (y_t) \) and \( (\hat{y}_t) \), see Section 1.1, we use the semi-discrete method as originally proposed, see [3]; we discretize parts of the drift coefficient producing
new differential equation in each subinterval with known solution or a solution that is easily simulated or approximated. In the third version ($\hat{y}_t$) we examine a new modification of the semi-discrete method, where in each subinterval $[t_n, t_{n+1}]$ we do not need to solve a new differential equation, but only an algebraic equation.

1.1. Lamperti Semi-Discrete methods $z^1_n$ and $z^2_n$ for CIR. Rewrite (2) as

$$z_t = z_0 + \int_0^t (a(z_s)^{-1} - b z_s) \, ds + \int_0^t dW_s, \quad t \geq 0. \tag{3}$$

To approximate the constant diffusion SDE (3) we use the following two versions of the semi-discrete method. In the first version ($y_t$), we discretize the linear part of the drift coefficient and in the second ($\hat{y}_t$) we leave it as it since it produces a differential equation with known solution. Let $t \in (t_n, t_{n+1}]$, where we assume the length of each subinterval to be equal to $\Delta$ and consider

$$y_t = \Delta W_n + y_{t_n} - b y_{t_n} \Delta + \int_{t_n}^t a(y_s)^{-1} \, ds, \tag{4}$$

with $y_0 = z_0$ and

$$\hat{y}_t = \Delta W_n + \hat{y}_{t_n} + \int_{t_n}^t (a(\hat{y}_s)^{-1} - b \hat{y}_s) \, ds, \tag{5}$$

with $\hat{y}_0 = z_0$

(4) and (5) are Bernoulli type equations with solutions satisfying, see Appendix B,

$$y_t^2 = (\Delta W_n + (1 - b \Delta) y_{t_n})^2 + 2a(t - t_n) \tag{6}$$

and

$$\hat{y}_t^2 = (\Delta W_n + \hat{y}_{t_n})^2 e^{-2b(t-t_n)} + a \frac{1 - e^{-2b(t-t_n)}}{b} \tag{7},$$

respectively.

We propose the following versions of the semi-discrete method for the approximation of (2),

$$y_{t_{n+1}} = \sqrt{(\Delta W_n + (1 - b \Delta) y_{t_n})^2 + 2a \Delta} \tag{8}$$

and

$$\hat{y}_{t_{n+1}} = \sqrt{(\Delta W_n + \hat{y}_{t_n})^2 e^{-2b \Delta} + a \frac{1 - e^{-2b \Delta}}{b}} \tag{9},$$

respectively.
which suggests the versions of the Lamperti semi-discrete method \((\tilde{z}_n^1)_{n \in \mathbb{N}}, (\tilde{z}_n^2)_{n \in \mathbb{N}}\) for the approximation of (1)

\[
\tilde{z}_{t_{n+1}}^1 = \frac{(k_3)^2}{4} \left( (\Delta W_n + (1 - b\Delta)y_{t_n})^2 + 2a\Delta \right)
\]

and

\[
\tilde{z}_{t_{n+1}}^2 = \frac{(k_3)^2}{4} \left( (\Delta W_n + \hat{y}_{t_n})^2 e^{-2b\Delta} + a \frac{1 - e^{-2b\Delta}}{b} \right).
\]

### 1.2. Lamperti Semi-Discrete method \(\tilde{z}_n^3\) for CIR.

In this version of the semi-discrete method, \((\hat{y}_t)\) we examine a new modification of the semi-discrete method, where in each subinterval \((t_n, t_{n+1}]\) we do not need to solve a new differential equation, but only an algebraic equation. For \(t \in (t_n, t_{n+1}]\) consider

\[
\hat{y}_t = W_t - W_{t_n} + \hat{y}_{t_n} + a(\hat{y}_t)^{-1}\Delta - b\hat{y}_t\Delta.
\]

with \(\hat{y}_0 = z_0\). The solution of (12) satisfies

\[
(1 + b\Delta)(\hat{y}_t)^2 - (W_t - W_{t_n} + \hat{y}_{t_n})\hat{y}_t - a\Delta = 0.
\]

We propose the following version of the semi-discrete method for the approximation of (2),

\[
\hat{y}_{t_{n+1}} = \Delta W_n + \hat{y}_{t_n} + \sqrt{(\Delta W_n + \hat{y}_{t_n})^2 + 4(1 + b\Delta)a\Delta},
\]

which suggests the version of the Lamperti semi-discrete method \((\tilde{z}_n^3)_{n \in \mathbb{N}}\) for the approximation of (1)

\[
\tilde{z}_{t_{n+1}}^3 = \frac{(k_3)^2}{4} \left( \frac{(\Delta W_n + \hat{y}_{t_n} + \sqrt{(\Delta W_n + \hat{y}_{t_n})^2 + 4(1 + b\Delta)a\Delta})^2}{2(1 + b\Delta)} \right).
\]

### 1.3. Numerical experiment for CIR.

For a minimal numerical experiment we present simulation paths for the numerical approximation of (1) with \(x_0 = 4\) and compare with the SD method proposed in [4], which reads

\[
\tilde{y}_{t_{n+1}} = \left( \sqrt{\tilde{y}_{t_n} \left( 1 - \frac{k_2\Delta}{1 + k_2\theta\Delta} \right)} + \frac{\Delta}{1 + k_2\theta\Delta} \left( k_1 - \frac{(k_3)^2}{4(1 + k_2\theta\Delta)} \right) + \frac{k_3}{2(1 + k_2\theta\Delta)} \Delta W_n \right)^2,
\]
where $\theta$ represents the level of implicitness. The case $\theta = 0$ was studied in [3] where the idea of the semi-discrete method was originally presented. According to the results in [4] it is shown that SD method (16) is strongly convergent under some conditions on the coefficients $k_i$, the level of implicitness $\theta$ and the step-size $\Delta$. In particular, it strongly converges to the solution of (1) with a logarithmic rate if also $\mathbb{E}(x_0)^p < A$ for some $p \geq 2, (k_3)^2 \leq 4k_1(1 + k_2\theta\Delta)$ and $\Delta(1 - \theta) \leq (k_2)^{-1}$, while a polynomial rate of convergence is achieved with order at least $1/4$ for a smaller set of parameters, namely $(k_3)^2 \leq 2k_1$ and $(\frac{2k_1}{(k_3)^2} - 1)^2 > 16$ for $\theta = 0$, with $x_0 \in \mathbb{R}$. On the other hand, the LSD scheme (10) seems to work without any restriction on the step-size or on the parameters which is a very interesting result.

**Remark 1.** We would like to point out a mistake that escaped our attention. In the proof of the strong convergence properties of the SD scheme (16) proposed in [4] an auxiliary process $(h_t)$ appears, see [4, Rel. (2.3)]

\[
h_t = x_0 + \int_0^t \left( k_1 - k_2(1 - \theta)y_{\hat{s}} - k_2\theta y_{\tilde{s}} \right) ds + \int_0^t k_3 \sqrt{y_{\hat{s}}} dW_s,
\]

where $\hat{s} = t_j$ when $s \in (t_j, t_{j+1}]$, $j = 0, 1, \ldots, n$ and

\[
\tilde{s} = \begin{cases} 
 t_{j+1}, & \text{for } s \in [t_j, t_{j+1}], \\
 t, & \text{for } s \in [t_n, t]
\end{cases} \quad j = 0, \ldots, n - 1.
\]

The problem is that we can not apply directly the Itô formula on $(h_t)$ since $y_{\tilde{s}}$ is $\mathcal{F}_{t_{n+1}}$-measurable and not $\mathcal{F}_{t_n}$-measurable. Nevertheless, writing the drift of $(h_t)$ as $f_\theta(y_{\hat{s}}, y_{\tilde{s}})$ we can proceed in the same way. The remainder term $f_\theta(y_{\hat{s}}, y_{\tilde{s}}) - f_\theta(y_{\hat{s}}, y_{\tilde{s}}) = k_2\theta(y_{\hat{s}} - y_{\tilde{s}})$ can be easily bounded.

We also present the implicit scheme proposed in [5], which takes the following form

\[
\tilde{y}_{t_{n+1}} = \left( \frac{\sqrt{4(\tilde{y}_{t_n} + (k_1 - \frac{(k_3)^2}{2})\Delta)(1 + k_2\Delta) + (k_3)^2(\Delta W_n)^2 + k_3\Delta W_n}}{2(1 + k_3\Delta)} \right)^2,
\]

As a first graphical illustration we borrow the set of parameters from [4, Sec.4]; we take $k_1 = k_2 = 2, k_3 = 1, T = 1$ and $\theta = 1$ with various step-sizes $\Delta = 10^{-4}$ and $\Delta = 10^{-3}, \Delta = 10^{-2}$. We compare with the proposed two versions of LSD scheme (10) and (11) and the implicit
method ALF (17). Figure 1 shows that all the schemes perform in a similar way. We also give a presentation of the difference of the various SD approximations in Figure 2.

For a different configuration, we are able to compare with the exact solution. Note that by choosing $d = 4k_1/(k_3)^2 = 2$ then the solution of (1) is $x_t = (x_1(t))^2 + (x_2(t))^2$, where $x_j(t)$ is the solution of the Orstein-Uhlenbeck process

$$dx_j(t) = -\frac{1}{2}k_2x_j(t)dt + \frac{1}{2}k_3dW_t^{(j)}$$

and the Brownian motions $W_t^{(j)}$ are independent. For $t \in [t_n, t_{n+1}]$ the solution of (18) is

$$x_j(t) = e^{-\frac{1}{2}k_2(t-t_n)}x_j(t_n) + \frac{1}{2}k_3e^{-\frac{1}{2}k_2(t-t_n)}\int_{t_n}^{t}e^{\frac{1}{2}k_2(s-t_n)}dW_s^{(j)}.$$  

Actually, we approximate in each subinterval $[t_n, t_{n+1}]$ the last stochastic integral in (19) at $t_n$ producing the sequence

$$x_j(t_{n+1}) = e^{-\frac{1}{2}k_2\Delta}x_j(t_n) + \frac{k_3}{k_2}(1 - e^{-\frac{1}{2}k_2\Delta})\Delta W_n^{(j)}$$

and therefore the solution process of (1) at the grid points reads

$$x(t_{n+1}) = \left(e^{-\frac{1}{2}k_2\Delta}x_1(t_n) + \frac{k_3}{k_2}(1 - e^{-\frac{1}{2}k_2\Delta})\Delta W_n^{(1)}\right)^2$$

$$+ \left(e^{-\frac{1}{2}k_2\Delta}x_2(t_n) + \frac{k_3}{k_2}(1 - e^{-\frac{1}{2}k_2\Delta})\Delta W_n^{(2)}\right)^2.$$
We therefore choose $k_3 = 2$ for this second experiment, with all the other parameters unchanged, so that $d = 2$. Moreover $x_0 = (x_1(0))^2 + (x_2(0))^2$. We present in Figure 3 simulation paths of (10)-(17) and the exact solution (20) choosing as initial conditions $x_1(0) = \sqrt{mx(0)}$, $x_2(0) = \sqrt{(1 - m)x(0)}$ for different $0 < m < 1$. Moreover, the driving Wiener process in this case is produced in the following way

\begin{equation}
W(t) = \int_0^t x_1(u)dW_u^{(1)} + x_2(u)dW_u^{(2)} \sqrt{x(u)}.
\end{equation}
In practice the increments of the Wiener process we use for the derivation of the paths of all the approximation methods are

\[
\Delta W_n = \frac{x_1(t_n)\Delta W_n^{(1)} + x_2(t_n)\Delta W_n^{(2)}}{\sqrt{x(t_n)}}.
\]

In Figure 3 we cannot see the differences between the methods. By considering bigger step-sizes \( \Delta \), we take the picture in Figure 4 where again we see the relation between the schemes.

We note that the exact solution has a very similar behavior between different realizations of the Wiener processes. We need to take a bigger
\( \Delta = 0.1 \) to notice a small variation of the produced solution, see Figure 5.

We also examine numerically the order of strong convergence of the LSD method. The numerical results suggest that the Lamperti Semi-Discrete methods converge in the mean-square sense with order close to 1, see Figure 6.

Moreover, we perform one more numerical experiment to show the ability of the method to produce nonnegative solutions, outside the usual restrictions on the parameters, \((k_3)^2 \leq 2k_1\). The solution of the CIR process is nonnegative, with no extra restriction on the positive parameters \(k_i, i = 1, 2, 3\), i.e. \(x_t \geq 0\) a.s. when \(x_0 > 0\). Therefore, we change a bit the parameters, taking \(k_1 = 1, k_2 = 2\) and different values for \(k_3\) so that \((k_3)^2 > 2k_1\). In this case the proposed LSD methods (10), (11) and (15) work in the sense that they produce nonnegative values, whereas the implicit method (17) as well as the implicit method proposed in [6], which also takes an explicit representation in this case,
Figure 5. Trajectories of the processes $x_1$ and $x_2$ which give the exact solution \((21)\) with different step-sizes $\Delta$.

\[
\begin{align*}
\hat{y}_{t_{n+1}} &= \left( \sqrt{\left( \hat{y}_{t_n} + \frac{1}{2} k_3 \Delta W_n \right)^2 + \left( k_1 - \frac{(k_3)^2}{4} \right) \Delta + \hat{y}_{t_n} + \frac{1}{2} k_3 \Delta W_n } \right)^2, \\
\end{align*}
\]

with $\hat{y}_0 = \sqrt{x_0}$, do not even produce real values, see Figure \(4\) where for \(17\) and \(24\) the real parts of the solution is presented. Note that as $k_3$ increases the implicit methods \(17\) and \(24\) show an erratic behavior.

Finally, we present numerically the order of strong convergence of the LSD methods, see Figure \(8\) where we can once more see that it is close to 1.

2. CEV model

Let

\[
(25) \quad x_t = x_0 + \int_0^t (k_1 - k_2 x_s) ds + \int_0^t k_3 (x_s)^q dW_s, \quad t \geq 0.
\]

where $k_1, k_2, k_3$ are positive and $1/2 < q < 1$. SDE \(25\) is a mean-reverting constant elasticity of variance process (CEV) with $x_t > 0$ a.s.
Figure 6. Convergence of Lamperti Semi-Discrete methods (10), (11) and (15) for the approximation of (1) with different reference solution.

C.f. [7, App. A]. The Lamperti transformation of (25) is $z = \frac{1}{k_3(1-q)}x^{1-q}$ with dynamics,

(26)

$$z_t = z_0 + \int_0^t \left( k_1 k_3^{\frac{1-2q}{1-q}} (1-q)^{\frac{-q}{1-q}} (z_s)^{-\frac{q}{1-q}} - \frac{q}{2(1-q)}(z_s)^{-1} - k_2(1-q)z_s \right) ds + \int_0^t dW_s.$$  

Set $a = k_1 k_3^{\frac{1-2q}{1-q}} (1-q)^{\frac{-q}{1-q}}$, $b = q/(2-2q)$ and $c = k_2(1-q)$. We consider two versions of the semi-discrete method for approximating (26). In the first version ($y_t$), see Section 2.1, we use the semi-discrete method as originally proposed, and in the second version we examine the new version ($\hat{y}_t$), see Section 2.2, where in each subinterval $(t_n, t_{n+1}]$ we solve an algebraic equation.
2.1. Lamperti Semi-Discrete method $\tilde{z}_n$ for CEV. Rewrite (26) as

$$z_t = z_0 + \int_0^t \left( a(z_s)^{-\frac{q}{1-q}} - b(z_s)^{-1} - c z_s \right) ds + \int_0^t dW_s.$$
where $a, b, c$ positive. We consider the following semi-discrete method for approximating (26),

$$y_t = \Delta W_n + y_{tn} + \int_{t_n}^t \left( \frac{a}{(y_{ts})^{\frac{2q-1}{1-q}}} (y_s)^{-1} - \frac{b}{(y_{tn})^2} y_t - cy_s \right) ds$$

(28) \quad = \phi_\Delta(y_{tn}, \Delta W_n) + \int_{t_n}^t (B_n(y_s)^{-1} + C_n y_s) ds

with $y_0 = z_0$, where $\phi_\Delta(x, y) = \frac{x+y}{1+y_\Delta}$ and

$$B_n := \frac{a}{(y_{tn})^{\frac{2q-1}{1-q}}} + b(y_{tn})^{\frac{4q-3}{1-q}} \Delta, \quad C_n := \frac{1}{1 + \frac{b_\Delta}{(y_{tn})^2}}.$$
The Bernoulli equation \([28]\) has a solution satisfying, see Appendix 13,
\[
(y_t)^2 = \phi^2(y_{tn}, \Delta W_n) \exp \left\{ \frac{-2c}{1 + \frac{b\Delta}{(y_{tn})^2}} (t - t_n) \right\}
\]
\[
+ \frac{a}{c(y_{tn})^{2\alpha-1}} \left( 1 - \exp \left\{ \frac{-2c}{1 + \frac{b\Delta}{(y_{tn})^2}} (t - t_n) \right\} \right).
\]

(29)

We propose the following version of the semi-discrete method for the approximation of \([26]\),
\[
y_{tn+1} = \sqrt{\phi^2(y_{tn}, \Delta W_n)e^{-\frac{2c\Delta}{1 + \frac{b\Delta}{(y_{tn})^2}}} + \frac{a}{c(y_{tn})^{2\alpha-1}} \left( 1 - e^{-\frac{2c\Delta}{1 + \frac{b\Delta}{(y_{tn})^2}}} \right)},
\]

which suggests the versions of the Lamperti semi-discrete method \(\tilde{z}^1\) for the approximation of \([25]\) with \(\tilde{z}^1_n = (k_3(1-q)y_n)^{1/(1-q)}\) or
\[
(31) \quad \tilde{z}^1_{tn+1} = (k_3(1-q))^\frac{1}{1-q} \times \sqrt{\phi^2(y_{tn}, \Delta W_n)e^{-\frac{2c\Delta}{1 + \frac{b\Delta}{(y_{tn})^2}}} + \frac{a}{c(y_{tn})^{2\alpha-1}} \left( 1 - e^{-\frac{2c\Delta}{1 + \frac{b\Delta}{(y_{tn})^2}}} \right)}^{1/(2-2q)}.
\]

2.2. Lamperti Semi-Discrete methods \(\tilde{z}^2\), \(\tilde{z}^3\) for CEV. Let \(t \in (t_n, t_{n+1}]\) and consider the processes \((\tilde{y}_t)\) and \((\tilde{\gamma}_t)\) where
\[
(32) \quad \tilde{y}_t = W_t - W_{t_n} + \tilde{y}_{tn} + a(\tilde{y}_{tn})^{\frac{1-2q}{1-q}} \tilde{y}_t - b(\tilde{y}_{tn})^{-1} \Delta - c\tilde{y}_t \Delta,
\]
with \(\tilde{y}_0 = z_0\) and
\[
(33) \quad \tilde{\gamma}_t = W_t - W_{t_n} + \tilde{\gamma}_{tn} + a(\tilde{y}_{tn})^{\frac{1-2q}{1-q}} \Delta - b(\tilde{y}_{tn})^{-1} \Delta - (\tilde{y}_{tn})^{-1} \Delta - \tilde{\gamma}_t \Delta,
\]
with \(\tilde{\gamma}_0 = z_0\).

The solution of \((32)\) is such that
\[
(1 + c\Delta)(\tilde{y}_t)^2 - (W_t - W_{t_n} + \tilde{y}_{tn} - b(\tilde{y}_{tn})^{-1} \Delta) \tilde{y}_t - a(\tilde{y}_{tn})^{\frac{1-2q}{1-q}} \Delta = 0,
\]
while the solution of \((33)\) satisfies
\[
(1 + c\Delta)(\tilde{\gamma}_t)^2 - (W_t - W_{t_n} + \tilde{\gamma}_{tn} + a(\tilde{y}_{tn})^{\frac{1-2q}{1-q}} - b(\tilde{y}_{tn})^{-1} \Delta) \tilde{\gamma}_t - \Delta = 0.
\]
We propose the following versions of the semi-discrete method for the approximation of (26),
\begin{equation}
\hat{y}_{t_{n+1}} = \hat{\phi}_\Delta(\hat{y}_{t_n}, \Delta W_n) + \sqrt{\hat{\phi}_\Delta^2(\hat{y}_{t_n}, \Delta W_n) + 4a\Delta(1+c\Delta)(\hat{y}_{t_n})^{\frac{1-2q}{1-q}}},
\end{equation}
with \( \hat{\phi}_\Delta(x,y) = (y + x - bx^{-1}\Delta) \), and
\begin{equation}
\tilde{y}_{t_{n+1}} = \tilde{\phi}_\Delta(\tilde{y}_{t_n}, \Delta W_n) + \sqrt{\tilde{\phi}_\Delta^2(\tilde{y}_{t_n}, \Delta W_n) + 4\Delta(1+c\Delta)}
\end{equation}
with \( \tilde{\phi}_\Delta(x,y) = (y + x + ax^{-\frac{q}{1-q}}\Delta - bx^{-1}\Delta) \), which suggest the versions of the Lamperti semi-discrete method \((\tilde{z}^2_{t_n})_{n\in\mathbb{N}}\), \((\tilde{z}^3_{t_n})_{n\in\mathbb{N}}\) for the approximation of (25) with
\begin{equation}
\tilde{z}^2_{t_{n+1}} = (k_3(1-q))^{\frac{1}{1-q}}
\end{equation}
\begin{equation}
\times \left[ \frac{\hat{\phi}_\Delta(\tilde{y}_{t_n}, \Delta W_n) + \sqrt{\hat{\phi}_\Delta^2(\tilde{y}_{t_n}, \Delta W_n) + 4a\Delta(1+c\Delta)(\tilde{y}_{t_n})^{\frac{1-2q}{1-q}}}}{2(1+c\Delta)} \right]^{1/(1-q)},
\end{equation}
and
\begin{equation}
\tilde{z}^3_{t_{n+1}} = (k_3(1-q))^{\frac{1}{1-q}}
\end{equation}
\begin{equation}
\times \left[ \frac{\hat{\phi}_\Delta(\tilde{y}_{t_n}, \Delta W_n) + \sqrt{\hat{\phi}_\Delta^2(\tilde{y}_{t_n}, \Delta W_n) + 4\Delta(1+c\Delta)}}{2(1+c\Delta)} \right]^{1/(1-q)}.
\end{equation}

2.3. **Numerical experiment for CEV.** For a minimal numerical experiment we present simulation paths for the numerical approximation of (25) with \( x_0 = 1/16 \) and compare with the SD method proposed in [7], which reads
\begin{equation}
\tilde{y}_{t_{n+1}} = \left( \sqrt{\tilde{y}_{t_n} \left( 1 - \frac{k_2\Delta}{1+k_2\theta \Delta} \right)} + \frac{k_1\Delta}{1+k_2\theta \Delta} - \frac{(k_3)^2\Delta}{4(1+k_2\theta \Delta)^2} (\tilde{y}_{t_n})^{2q-1} \frac{k_3}{2(1+k_2\theta \Delta)} (\tilde{y}_{t_n})^{q-\frac{1}{2}} \Delta W_n \right)^2,
\end{equation}
where \( \theta \) represents the level of implicitness. In particular we choose the coefficients as in [7, Sec.6]; we take \( k_1 = \frac{1}{16}, k_2 = 1 \) and \( k_3 = 0.4 \) and \( q = 3/4 \) for the fully implicit SD scheme (40) with \( \theta = 1 \) and compare with the proposed versions of LSD scheme (31) and (38).
Remark 2. As in Remark [7], we note that in the proof of the strong convergence properties of the SD scheme (40) proposed in [7] an auxiliary process \((h_t)\) appears, see [7, Rel. (33)]. Following the same lines the results in [7] as well as in [8] where a more general CIR/CEV-type model is examined with delay, are true.

Moreover, we compare with the implicit method, see [6]. Set

\[ G(x) = x - (1 - q) \left( k_1 x^{q/(1-q)} - k_2 x - q(k_3)^2 x^{-1/2} \right) \Delta \]

and compute

\[ y_{n+1} = G^{-1}(y_n + k_3(1 - q)\Delta W_n) \]

and then transform back to get the following scheme

\[ y_{n+1}^{impl} = (y_{n+1})^{1/(1-q)}. \]

The simulation paths are presented in Figures 9 and 10.

![Simulation Paths](image)

(A) Trajectories with \( \Delta = 10^{-3} \). (B) Trajectories with \( \Delta = 10^{-2} \).

Figure 9. Trajectories of (31), (38), (39), (40) and (41) for the approximation of \((25)\) for different step-sizes.

We also examine numerically the order of strong convergence of the LSD methods. The numerical results suggest that the LSD schemes converge in the mean-square sense with order close to 1, see Figure 11.

3. Wright - Fisher model

Let

\[ x_t = x_0 + \int_0^t (k_1 - k_2 x_s)ds + k_3 \int_0^t \sqrt{x_s(1-x_s)}dW_s, \]

where \( k_i > 0, i = 1, 2, 3 \). If \( x_0 \in (0,1) \) and \( 2k_1 \geq (k_3)^2, 2(k_2 - k_1) \geq (k_3)^2 \), then \( 0 < x_t < 1 \) a.s., see [9]. SDE (42) appears in population...
Figure 10. Trajectories of the differences of the numerical methods for the approximation of (25) with various step-sizes.

dynamics to describe fluctuations in gene frequency of reproducing individuals among finite populations \[10\] and ion channel dynamics within cardiac and neuronal cells, (c.f. \[11\], \[12\], \[13\] and references therein).

The transformed process \((z_t)\) of \[42\] with \(z = 2 \arcsin(\sqrt{x})\) has dynamics,

\[
\begin{align*}
  z_t &= z_0 + \int_0^t \left( \left( k_1 - \frac{(k_3)^2}{4} \right) \cot\left( \frac{z_s}{2} \right) - \left( k_2 - k_1 - \frac{(k_3)^2}{4} \right) \tan\left( \frac{z_s}{2} \right) \right) ds \\
&\quad + k_3 \int_0^t dW_s.
\end{align*}
\]

Set \(a := k_1 - \frac{(k_3)^2}{4}\) and \(b := k_2 - k_1 - \frac{(k_3)^2}{4}\). The conditions on the parameters imply \(a > 0\) and \(b > 0\). We consider three versions of the semi-discrete method for approximating \[20\]. In the first version \((y_t)\), see Section 3.1 we use the standard semi-discrete method and in the
Figure 11. Convergence of (31), (38) and (39) for the approximation of (25) with different reference solutions.

other two versions ($\hat{y}_t$) and ($\tilde{y}_t$) we study the new versions see Section 3.2 where in each subinterval $(t_n, t_{n+1}]$ we solve an algebraic equation.

3.1. Lamperti Semi-Discrete method $\tilde{z}_n$ for Wright-Fisher. Rewrite (43) as

$$
(44) \quad z_t = z_0 + \int_0^t \left( a \cot \left( \frac{z_s}{2} \right) - b \tan \left( \frac{z_s}{2} \right) \right) ds + k_3 \int_0^t dW_s.
$$

For $t \in (t_n, t_{n+1}]$ consider the process $(y_t)$ where

$$
(45) \quad y_t = k_3 \Delta W_n + y_{t_n} - b \tan \left( \frac{y_{t_n}}{2} \right) \frac{y_t}{y_{t_n}} \Delta + \int_{t_n}^t \frac{a}{1 + \frac{b}{y_{t_n}} \tan \left( \frac{y_{t_n}}{2} \right)} \cot \left( \frac{y_s}{2} \right) ds,
$$
with $\hat{y}_0 = z_0$ and $\phi_{\Delta}(x, y) := \frac{k_3 y + x}{1 + \frac{b}{y} \tan(x/2) \Delta}$. Equation (45) has solution with the property, see Appendix C

\begin{equation}
|\cos(y_t/2)| = |\cos(\frac{\phi_{\Delta}(y_{t_n}, \Delta W_n)}{2})| \exp \left \{ -\frac{a/2}{1 + \frac{b}{y_{t_n}} \tan(\frac{y_{t_n}}{2}) \Delta} (t - t_n) \right \}.
\end{equation}

Note that $0 < z_t < \pi$ a.s therefore since $(y_t)$ converges to $(z_t)$ the process $(y_t)/2$ and belong to $(0, \pi/2)$. The proposed semi-discrete method $(y_t)$ for the approximation of (43) satisfies

\begin{equation}
\cos(\frac{y_{t_{n+1}}}{2}) = \cos(\frac{\phi_{\Delta}(y_{t_n}, \Delta W_n)}{2})| \exp \left \{ -\frac{a \Delta/2}{1 + \frac{b}{y_{t_n}} \tan(\frac{y_{t_n}}{2}) \Delta} \right \},
\end{equation}

which suggests the Lamperti semi-discrete method $(\tilde{z}_n^1)_{n \in \mathbb{N}}$ for the approximation of (42)

\begin{equation}
\tilde{z}_{t_{n+1}}^1 = 1 - \cos^2(\frac{\phi_{\Delta}(y_{t_n}, \Delta W_n)}{2}) \exp \left \{ -\frac{a \Delta}{1 + \frac{b}{y_{t_n}} \tan(\frac{y_{t_n}}{2}) \Delta} \right \}.
\end{equation}

Note that $(\tilde{z}_n^1)_{n \in \mathbb{N}} \in (0, 1)$ when $x_0 \in (0, 1)$.

3.2. Lamperti Semi-Discrete methods $\tilde{z}_n^2$, $\tilde{z}_n^3$ and $\tilde{z}_n^4$ for Wright-Fisher. For $t \in (t_n, t_{n+1}]$ consider the processes $(\hat{y}_t)$, $(\tilde{y}_t)$ and $(\bar{y}_t)$ where

\begin{equation}
\begin{aligned}
\hat{y}_t &= k_3(W_t - W_{t_n}) + \hat{y}_{t_n} + \hat{y}_t \left( \frac{a}{y_{t_n}} \cot(\frac{\hat{y}_{t_n}}{2}) - \frac{b}{y_{t_n}} \tan(\frac{\hat{y}_{t_n}}{2}) \right) \Delta \\
&= \frac{k_3(W_t - W_{t_n}) + \hat{y}_{t_n}}{1 - \frac{a}{y_{t_n}} \cot(\frac{\hat{y}_{t_n}}{2}) \Delta + \frac{b}{y_{t_n}} \tan(\frac{\hat{y}_{t_n}}{2}) \Delta};
\end{aligned}
\end{equation}

with $\hat{y}_0 = z_0$,

\begin{equation}
\begin{aligned}
\tilde{y}_t &= k_3(W_t - W_{t_n}) + \tilde{y}_{t_n} + a \cot(\frac{\tilde{y}_{t_n}}{2}) \Delta - \frac{b}{y_{t_n}} \tan(\frac{\tilde{y}_{t_n}}{2}) \Delta \tilde{y}_t \\
&= \frac{k_3(W_t - W_{t_n}) + \tilde{y}_{t_n} + a \cot(\frac{\tilde{y}_{t_n}}{2}) \Delta}{1 + \frac{b}{y_{t_n}} \tan(\frac{\tilde{y}_{t_n}}{2}) \Delta},
\end{aligned}
\end{equation}

with $\tilde{y}_0 = z_0$ and

\begin{equation}
\begin{aligned}
\bar{y}_t &= k_3(W_t - W_{t_n}) + \bar{y}_{t_n} + \left( a \cot(\frac{\bar{y}_{t_n}}{2}) - b \tan(\frac{\bar{y}_{t_n}}{2}) - \frac{1}{\bar{y}_{t_n}} \right) \Delta + \frac{\Delta}{\bar{y}_t}
\end{aligned}
\end{equation}
with \( \bar{y}_0 = z_0 \). The solution of \((\bar{y}_t)\) of the above equation satisfies

\[
(\bar{y}_t)^2 - \left( k_3(W_t - W_{t_n}) + \bar{y}_{t_n} - \Delta \sqrt{a \cot(\bar{y}_{t_n}/2) - b \tan(\bar{y}_{t_n}/2)} \right) (\bar{y}_t) - \Delta = 0.
\]

The proposed semi-discrete methods \((\hat{y}_t)\) and \((\tilde{y}_t)\) for the approximation of \((43)\) read

\[
\hat{y}_{t_n+1} = \frac{k_3 \Delta W_n + \hat{y}_{t_n}}{1 - \frac{a}{\hat{y}_{t_n}} \cot(\hat{y}_{t_n}/2) \Delta + \frac{b}{\hat{y}_{t_n}} \tan(\hat{y}_{t_n}/2) \Delta}
\]

and

\[
\tilde{y}_{t_n+1} = \frac{k_3 \Delta W_n + \tilde{y}_{t_n} + a \cot(\tilde{y}_{t_n}/2) \Delta}{1 + \frac{b}{\tilde{y}_{t_n}} \tan(\tilde{y}_{t_n}/2) \Delta},
\]

respectively, while for \((\bar{y}_t)\) we have that

\[
\bar{y}_{t_n+1} = \frac{\phi_\Delta(\bar{y}_{t_n}, \Delta W_n) + \sqrt{\phi_\Delta^2(\bar{y}_{t_n}, \Delta W_n) + 4 \Delta}}{2},
\]

with \( \phi_\Delta(x, y) = (k_3 y + x - \Delta x^{-1} + a (\cot(x/2) - b \tan(x/2)) \Delta \). Therefore, the versions of the Lamperti semi-discrete method \((\bar{z}^2_n)_{n \in \mathbb{N}}, (\bar{z}^3_n)_{n \in \mathbb{N}}\) and \((\bar{z}^4_n)_{n \in \mathbb{N}}\) for the approximation of \((42)\) are

\[
\bar{z}^2_{t_n+1} = \sin^2 \left( \frac{k_3 \Delta W_n + \bar{y}_{t_n}}{2 - \frac{2a}{\bar{y}_{t_n}} \cot(\bar{y}_{t_n}/2) \Delta + \frac{2b}{\bar{y}_{t_n}} \tan(\bar{y}_{t_n}/2) \Delta} \right),
\]

\[
\bar{z}^3_{t_n+1} = \sin^2 \left( \frac{k_3 \Delta W_n + \bar{y}_{t_n} + a \cot(\bar{y}_{t_n}/2) \Delta}{2 + \frac{2b}{\bar{y}_{t_n}} \tan(\bar{y}_{t_n}/2) \Delta} \right)
\]

and

\[
\bar{z}^4_{t_n+1} = \sin^2 \left( \frac{\phi_\Delta(\bar{y}_{t_n}, \Delta W_n) + \sqrt{\phi_\Delta^2(\bar{y}_{t_n}, \Delta W_n) + 4 \Delta}}{4} \right),
\]

respectively. Note that \((\bar{z}^j_{t_n})_{n \in \mathbb{N}} \in (0, 1), j = 2, 3, 4 \) when \( x_0 \in (0, 1) \).

### 3.3. Numerical experiment for Wright-Fisher.

The semi-discrete method we proposed in \([14]\) reads

\[
y_{t_n+1} = \sin^2 \left( \frac{k_3 \Delta W_n}{2} + \arcsin \left( \frac{y_{t_n} + \left( k_1 - \frac{(k_3)^2}{4} + y_{t_n} \left( \frac{(k_3)^2}{2} - k_2 \right) \right) \Delta}{\sqrt{y_{t_n} + \left( k_1 - \frac{(k_3)^2}{4} + y_{t_n} \left( \frac{(k_3)^2}{2} - k_2 \right) \right) \Delta}} \right) \right),
\]
which also possesses the qualitative property of domain preservation. Method (58) is well defined for all sufficiently small $\Delta$ such that $0 < y_n < 1$. Let $\beta := \frac{(k_3)^2}{2} - k_2$ with $\beta < 0$. We require $\Delta$ small enough so that $0 < y_n(1 + \beta \Delta) + a\Delta$. To simplify the conditions on $a, \beta, \Delta$, when necessary, we may adopt the strategy presented in [14] and consider the SD method

$$\bar{y}_{n+1} = \sin^2 \left( \frac{k_3}{2} \Delta W_n + \arcsin(\sqrt{y_n}) \right),$$

with

$$\bar{y}_n := y_n(1 + \beta \Delta) + a\Delta.$$

The numerical scheme (59) is mean square convergent when $(k_3)^2 < 2k_2$ for $\Delta < -1/\beta$, see [14, Prop. 2.5] where the order of strong convergence was not theoretically proved.

The Balance Implicit Split Step (BISS) method suggested in [11, (4.8)] reads

$$y_{BISS}^{n+1} = y_n + (k_1 - k_2 y_n)\Delta + \frac{C\sqrt{y_n(1 - y_n)}\Delta W_n}{1 + d^1(y_n)|\Delta W_n|} (1 - k_2 \Delta),$$

where $\Delta$ is the step-size of the equidistant discretization of the interval $[0, 1]$, the control function $d^1$ is given by

$$d^1(y) = \begin{cases} 
  k_3 \sqrt{(1 - \varepsilon)/\varepsilon} & \text{if } y < \varepsilon, \\
  k_3 \sqrt{(1 - y)/y} & \text{if } \varepsilon \leq y < 1/2, \\
  k_3 \sqrt{y/(1 - y)} & \text{if } 1/2 \leq y \leq 1 - \varepsilon, \\
  k_3 \sqrt{(1 - \varepsilon)/\varepsilon} & \text{if } y > 1 - \varepsilon,
\end{cases}$$

and

$$\varepsilon = \min\{k_1\Delta, (k_2 - k_1)\Delta, 1 - k_1\Delta, 1 - (k_2 - k_1)\Delta\}.$$ 

The hybrid (HYB) scheme as proposed in [15, (11)] is the result of a splitting method and reads

$$y_{HYB}^{n+1} = \frac{a}{\beta} (e^{\beta \Delta} - 1) + e^{\beta \Delta} \sin^2 \left( \frac{k_3}{2} \Delta W_n + \arcsin(\sqrt{y_n}) \right).$$

with the restriction that

$$\frac{k_1}{k_2} \in \left( \frac{(k_3)^2}{4k_2}, 1 - \frac{(k_3)^2}{4k_2} \right).$$

Moreover, we compare with the implicit method, see [6]. Set

$$G(x) = x - a \cot(x/2)\Delta - b(\tan(x/2))\Delta$$
and compute
\[ y_{n+1} = G^{-1}(y_n + k_3 \Delta W_n) \]
and then transform back to get the following scheme
\[ y_{n+1}^{impl} = \sin^2(y_{n+1}/2), \]

We use the set of parameters from [14, Sec.4] where all the methods work well, i.e. we take \((k_1, k_2, k_3) = (1, 2, 0.20101)\), with \(T = 1\) and various step-sizes and compare the proposed versions of LSD schemes (48), (55), (56) and (57) with the BISS, the HYB, the SD method (58) and the implicit method (62). The initial condition is chosen to be the steady state of the deterministic part, i.e. \(x_0 = k_1/k_2\).

Figure 12 shows paths for the proposed LSD and existing numerical methods for the Wright–Fisher model and Figure 13 shows a graphical estimation of the difference of the methods.

We also examine numerically the order of strong convergence of the LSD method. The numerical results suggest that the LSD is mean-square convergent with order close to 1, see Figure 14.

4. HESTON 3/2-MODEL

Let
\[ x_t = x_0 + \int_0^t (k_1 x_s - k_2 (x_s)^2) ds + \int_0^t k_3 (x_s)^{3/2} dW_s, \quad t \geq 0, \]
where the coefficients \(k_i, i = 1, 2, 3\) are positive and \(x_0 > 0\). SDE (63) is known as the Heston 3/2-model appearing in financial mathematics as a stochastic volatility process, see [16], and satisfies \(x_t > 0\) a.s.
Figure 13. Trajectories of the difference of numerical methods for the approximation of (42) with various ∆.

The Lamperti transformation of (63) is $z = \frac{2}{k_3} x^{-1/2}$ implying that, see Appendix A,

$$z_t = z_0 + \int_0^t \left( \frac{2k_2}{(k_3)^2} + 3 \right) (z_s)^{-1} - \frac{k_1}{2} z_s \right) - k_1 \int_0^t dW_s, \quad t \geq 0. \quad (64)$$

4.1. Lamperti Semi-Discrete method $\tilde{z}_{1n}$ and $\tilde{z}_{2n}$ for Heston 3/2-model. As in Section 1.1 we consider the following two versions of the semi-discrete method for approximating (64),

$$y_t = \Delta W_n + y_{tn} - \frac{k_1}{2} y_{tn} \Delta + \int_{t_n}^t \left( \frac{2k_2}{(k_3)^2} + 3 \right) (y_s)^{-1} ds, \quad t \in [t_n, t_{n+1}], \quad (65)$$
with \( y_0 = z_0 \) and

\[
\dot{y}_t = \Delta W_n + \dot{y}_{t_n} + \int_{t_n}^{t} \left( \frac{2k_2}{(k_3)^2 + 3} \right) \dot{y}_s^{-1} - \frac{k_1}{2} \dot{y}_s \, ds, \quad t \in (t_n, t_{n+1}].
\]
with \( \dot{y}_0 = z_0 \). (65) and (66) are Bernoulli type equations with solutions satisfying, see Appendix B,

\[ (67) \quad (y_t)^2 = (\Delta W_n + \left(1 - \frac{k_1 \Delta}{2}\right)y_t)^2 + \left(\frac{4k_2}{(k_3)^2} + 6\right)(t - t_n) \]

and

\[ (68) \quad (\dot{y}_t)^2 = (\Delta W_n + \dot{y}_n)^2 e^{-k_1(t-t_n)} + \left(\frac{4k_2}{(k_3)^2} + 6\right) \frac{1 - e^{-k_1(t-t_n)}}{k_1}, \]

respectively. We propose the following versions of the semi-discrete method for the approximation of (2),

\[ (69) \quad y_{t_{n+1}} = \sqrt{\left(\Delta W_n + \left(1 - \frac{k_1 \Delta}{2}\right)y_t\right)^2 + \left(\frac{4k_2}{(k_3)^2} + 6\right)\Delta} \]

and

\[ (70) \quad \dot{y}_{t_{n+1}} = \sqrt{\left(\Delta W_n + \dot{y}_n\right)^2 e^{-k_1\Delta} + \left(\frac{4k_2}{(k_3)^2} + 6\right) \frac{1 - e^{-k_1\Delta}}{k_1} }, \]

which suggests the versions of the Lamperti semi-discrete method \((\tilde{z}_n)_{n \in \mathbb{N}}\) for the approximation of (1)

\[ (71) \quad \tilde{z}^1_{t_{n+1}} = \frac{4}{(k_3)^2} \left(\Delta W_n + \left(1 - \frac{k_1 \Delta}{2}\right)y_t\right)^2 + \left(\frac{4k_2}{(k_3)^2} + 6\right)\Delta \]

\[ (72) \quad \tilde{z}^2_{t_{n+1}} = \frac{4}{(k_3)^2} \left(\Delta W_n + \dot{y}_n\right)^2 e^{-k_1\Delta} + \left(\frac{4k_2}{(k_3)^2} + 6\right) \frac{1 - e^{-k_1\Delta}}{k_1} \]

4.2. **Numerical experiment for Heston 3/2-model.** For a minimal numerical experiment we present simulation paths for the numerical approximation of (63) with \( x_0 = 1 \) and compare with the SD method proposed in [17, Sec. 5], which reads

\[ (73) \quad \tilde{y}_{t_{n+1}} = \tilde{y}_t \exp\left\{ \left(k_1 - k_2 \tilde{y}_n - \frac{(k_3)^2}{2} \tilde{y}_n\right) \Delta + k_3 \sqrt{\tilde{y}_n} \Delta W_n \right\}. \]

The semi-discrete method (73) is strongly converging and positivity preserving, see [17, Sec. 5].

Moreover, we compare with the implicit method proposed in [6]. Set

\[ G(x) = \left(1 + \frac{k_1 \Delta}{2}\right) x - \left(\frac{k_2}{2} + \frac{3(k_3)^2}{8}\right) \Delta x^{-1} \]

and compute

\[ y_{n+1} = G^{-1}\left(y_n - \frac{k_3}{2} \Delta W_n\right) \]
and then transform back to get the following scheme

\[ y_{n+1}^{impl} = (y_{n+1})^{-2}. \]

We use the set of parameters from [17, Sec.5]; we take \( k_1 = 0.1, k_2 = 70, k_3 = \sqrt{0.2}, T = 1 \) with various step-sizes. We compare the proposed two versions of LSD scheme (71) and (72) with the SD scheme (73) and the implicit method (74). Figure 15 shows that the pair of LSD1 and LSD2 and the pair of SD with the implicit method are almost identical for a step-size \( \Delta = 10^{-4} \). Moreover, the two pairs are getting very close. We give a presentation of the difference of the various approximations in Figure 16.

Finally, we examine numerically the order of strong convergence of the LSD method. The numerical results suggest that the LSD1 scheme as well as LSD2 converge in the mean-square sense with order close to 1, see Figure 17.

5. Aït-Sahalia model

Let

\[ x_t = x_0 + \int_0^t (k_1(x_s)^{-1} - k_0 + k_1 x_s - k_2 (x_s)^r) ds + \int_0^t k_3 (x_s) \rho dW_s, \quad t \geq 0. \]

where \( k_{-1}, k_0, k_1, k_2 \) and \( k_3 \) are positive constants with \( r > 1 \) and \( \rho > 1 \). SDE (75) is an Aït-Sahalia model with superlinear coefficients and the property \( x_t > 0 \) a.s. The Lamperti transformation of (75) is \( z = x^{1-\rho} \).
with dynamics, see Appendix A.

\[ z_t = z_0 + \int_0^t \left( k_{-1}(1 - \rho)(z_s)^{\frac{\rho+1}{\rho-1}} - k_0(1 - \rho)(z_s)^{\frac{\rho}{\rho-1}} + k_1(1 - \rho)z_s \right) ds + \int_0^t k_3(1 - \rho)dW_s. \]

(76)

Set \( K_i = k_i(\rho - 1), i = -1, \ldots, 3 \) and \( K_4 = \frac{\rho(\rho-1)(k_3)^2}{2} \). We examine the new version \((y_t)\), of the semi-discrete method for approximating \((70)\) see Section 5.1 where in each subinterval \((t_n, t_{n+1}]\) we solve an algebraic equation, producing a positive numerical scheme.
Rewrite (76) as

$$y_t = \frac{1}{2} \log \left| \frac{y_t - \bar{y}}{y_{t+1} - \bar{y}} \right|$$

5.1. Lamperti Semi-Discrete methods $\tilde{z}_n^1, \tilde{z}_n^2$ for Aït-Sahalia.

Rewrite (76) as

$$z_t = z_0 + \int_0^t \left( -K_1(z_s)^{\rho+1} + K_0(z_s)^{\rho-1} - K_1 z_s + K_2(z_s)^{\rho-r} + K_4(z_s)^{-1} \right) ds$$

(77) $- \int_0^t K_3 dW_s.$

Let $t \in (t_n, t_{n+1}]$ and

$$y_t = -K_3(W_t - W_{t_n}) + y_{t_n} - K_1(y_{t_n})^{\rho+1} y_t \Delta + K_0(y_{t_n})^{\rho-1} \Delta - K_1 y_t \Delta$$

(78) $+ K_2(y_{t_n})^{\rho-r} y_t \Delta + K_4(y_t)^{-1} \Delta$

with $y_0 = z_0$ and

$$\hat{y}_t = -K_3(W_t - W_{t_n}) + \hat{y}_{t_n} - K_1(\hat{y}_{t_n})^{\rho+1} \hat{y}_t \Delta + K_0(\hat{y}_{t_n})^{\rho-1} \Delta - K_1 \hat{y}_t \Delta$$

(79) $+ K_2(\hat{y}_{t_n})^{\rho-r} (\hat{y}_t)^{-1} \Delta + K_4(\hat{y}_t)^{-1} \Delta$

with $\hat{y}_0 = z_0$. The solutions of (78) and (79) are such that

$$(1 + K_1(y_{t_n})^{2\rho+1} \Delta + K_1 \Delta) (y_t)^2 - \left( -K_3(W_t - W_{t_n}) + y_{t_n} + K_0(y_{t_n})^{2\rho-1} \Delta \right) y_t$$

(80) $- \left( K_2(y_{t_n})^{2\rho-r-1} + K_4 \right) \Delta = 0$

and

$$(1 + K_1 \Delta) (\hat{y}_t)^2 - \left( -K_3(W_t - W_{t_n}) + \hat{y}_{t_n} + K_0(\hat{y}_{t_n})^{2\rho-1} \Delta - K_1(\hat{y}_{t_n})^{\rho+1} \Delta \right) \hat{y}_t$$

(81) $- \left( K_2(\hat{y}_{t_n})^{2\rho-r-1} + K_4 \right) \Delta = 0$
respectively.

We propose the following versions of the semi-discrete method for the approximation of (77),

\begin{equation}
y_{n+1} = \frac{\phi_\Delta(y_n, \Delta W_n) + \sqrt{\phi_\Delta^2(y_n, \Delta W_n) + 4C_1(y_n)C_2(y_n)}}{2C_1(y_n)},
\end{equation}

with \( \phi_\Delta(x, y) = -K_3y + x + K_0x^{\rho-1}\Delta, C_1(x) = (1 + K_1x^{\rho-1}\Delta + K_1\Delta) \)
and \( C_2(x) = \left(K_2x^{\frac{2(\rho-1)}{\rho-1}} + K_4\right)\Delta \) and

\begin{equation}
y_{n+1} = \frac{\hat{\phi}_\Delta(y_{n+1}, \Delta W_n) + \sqrt{\hat{\phi}_\Delta^2(y_{n+1}, \Delta W_n) + 4(1 + K_1\Delta)C_2(y_{n+1})}}{2(1 + K_1\Delta)}.
\end{equation}

with \( \hat{\phi}_\Delta(x, y) = \phi_\Delta(x, y) - K_1x^{\rho-1}\Delta \), which suggests the versions of the Lamperti semi-discrete method \((z^1_n)_{n\in\mathbb{N}}\) and \((z^2_n)_{n\in\mathbb{N}}\) for the approximation of (78) with \( z^1_n = (y_n)^{1/(1-\rho)} \), \( z^2_n = (y_n)^{1/(1-\rho)} \) or

\begin{equation}
z^1_{n+1} = \left. \frac{\phi_\Delta(y_n, \Delta W_n) + \sqrt{\phi_\Delta^2(y_n, \Delta W_n) + 4C_1(y_n)C_2(y_n)}}{2C_1(y_n)} \right|_n^{1/(1-\rho)}.
\end{equation}

\begin{equation}
z^2_{n+1} = \left. \frac{\hat{\phi}_\Delta(y_{n+1}, \Delta W_n) + \sqrt{\hat{\phi}_\Delta^2(y_{n+1}, \Delta W_n) + 4(1 + K_1\Delta)C_2(y_{n+1})}}{2(1 + K_1\Delta)} \right|_n^{1/(1-\rho)}.
\end{equation}

5.2. Numerical experiment for Aït-Sahalia. For a minimal numerical experiment we present simulation paths for the numerical approximation of (75) with \( x_0 = 4 \) and compare with the implicit method proposed in [6]. Set

\[ G(x) = x + \left(1 + K_1x^{\rho-1} - K_0x^{\rho-1} + K_1x - K_2x^{\rho-1} + K_4x^{-1}\right)\Delta \]

and compute

\[ y_{n+1} = G^{-1}(y_n - K_3\Delta W_n) \]

and then transform back to get the following scheme

\begin{equation}
y_{n+1}^\text{impl} = (y_{n+1})^{1/(1-\rho)}.
\end{equation}

We use a set of parameters so that (86) works; we take the coefficients \( k_{-1} = 2, k_0 = 3, k_1 = 4, k_2 = 6, k_3 = 1 \) the exponents \( r = 2 \) and \( \rho = 3/2 \) and \( T = 1 \). We compare the proposed versions of LSD schemes (84) and (85) with the implicit method (86). Figure 18 shows that the LSD1 and LSD2 are very close to the implicit method. We give a presentation of the difference of the two methods in Figure 19.
Finally, we examine numerically the order of strong convergence of the LSD method. The numerical results suggest that the LSD1 and LSD2 schemes converge in the mean-square sense with order close to 1, see Figure 20.

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Figure 19. Differences between (84), (85) and (86) for the approximation of (75) with various step-sizes.

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Figure 20. Convergence of LSD methods (84) and (85) for the approximation of (75) with different reference solutions.

Appendix A. Lamperti Transformation of (11), (63), (75)

Applying the Itô formula to the transformation \( z(x) = \frac{2}{k_3} \sqrt{x} \) of (11) we obtain

\[
\begin{align*}
dz_t &= \left( \frac{1}{k_3} (x_t)^{1/2} (k_1 - k_2 x_t) + \frac{1}{k_3} \frac{1}{2} \left( \frac{1}{2} \right) (x_t)^{-3/2} (k_3)^2 (x_t) \right) dt + \frac{1}{k_3} (x_t)^{-1/2} k_3 (x_t)^{1/2} dW_t \\
&= \left( \frac{k_1}{k_3} (x_t)^{-1/2} - \frac{k_2}{k_3} + \frac{1}{4} k_3 \sqrt{x_t} \right) dt + dW_t \\
&= \left( \frac{2k_1}{(k_3)^2} (z_t)^{-1} - \frac{k_2}{2} + \frac{(k_3)^2}{8} z_t \right) dt + dW_t,
\end{align*}
\]
or for $t \geq t_0$

\[ z_t = z_{t_0} + \int_{t_0}^{t} \left( \frac{2k_1}{(k_3)^2}(z_s)^{-1} - \left( \frac{k_2}{2} + \frac{(k_3)^2}{8} \right) z_s \right) ds + \int_{t_0}^{t} dW_s \]

\[ = z_{t_0} + \int_{t_0}^{t} \left( \frac{2k_1}{(k_3)^2}(z_s)^{-1} - \left( \frac{k_2}{2} + \frac{(k_3)^2}{8} \right) z_s \right) ds + W_t - W_{t_0}. \]

Analogously, the transformation $z(x) = \frac{2}{k_3}(x)^{-1/2}$ of (63) has the following dynamics,

\[ dz_t = \left( \frac{-1}{k_3}(x_t)^{-3/2}(k_1x_t - k_2(x_t)^2) + \frac{3}{4k_3}(x_t)^{-5/2}(k_3)^2(x_t)^3 \right) dt + \frac{-1}{k_3}(x_t)^{-3/2}k_3(x_t)^{3/2}dW_t \]

\[ = \left( \frac{-k_1}{k_3}(x_t)^{-1/2} + \frac{k_2}{k_3} + \frac{3}{2}k_3 \sqrt{x_t} \right) dt + dW_t \]

\[ = \left( \frac{2k_2}{(k_3)^2} + 3)(z_t)^{-1} - \frac{k_1}{2} z_t \right) dt + dW_t, \]

or for $t \geq t_0$,

\[ z_t = z_{t_0} + \int_{t_0}^{t} \left( \frac{2k_2}{(k_3)^2} + 3)(z_s)^{-1} - \frac{k_1}{2} z_s \right) ds + W_t - W_{t_0}. \]

Finally, the transformation $z(x) = x^{1-\rho}$ of (75) is such that

\[ dz_t = \left( (1 - \rho)(x_t)^{-\rho}(k_{-1}(x_t)^{-1} - k_0 + k_1 x_t - k_2(x_t)^{\rho}) - \frac{(1 - \rho)(k_3)^2}{2}(x_t)^{\rho+1} \right) dt \]

\[ + k_3(1 - \rho)(x_t)^{-\rho}dW_t \]

\[ = (1 - \rho) \left( k_{-1}(x_t)^{-\rho-1} - k_0(x_t)^{-\rho} + k_1(x_t)^{-\rho+1} - k_2(x_t)^{-\rho+\rho} - \frac{(1 - \rho)(k_3)^2}{2}(x_t)^{\rho+1} \right) dt \]

\[ + k_3(1 - \rho)dW_t \]

\[ = (1 - \rho) \left( k_{-1}(z_t)^{\frac{\rho+1}{\rho-1}} - k_0(z_t)^{\frac{\rho}{\rho-1}} + k_1 z_t - k_2(z_t)^{\frac{\rho-\rho+1}{\rho-1}} - \frac{(1 - \rho)(k_3)^2}{2}(z_t)^{-1} \right) dt \]

\[ + k_3(1 - \rho)dW_t, \]

or for $t \geq t_0$,

\[ z_t = z_{t_0} + \int_{t_0}^{t} \left( k_{-1}(1 - \rho)(z_s)^{\frac{\rho+1}{\rho-1}} - k_0(1 - \rho)(z_s)^{\frac{\rho}{\rho-1}} + k_1 (1 - \rho) z_s \right. \]

\[ - k_2(1 - \rho)(z_s)^{\frac{\rho-\rho+1}{\rho-1}} - \frac{(1 - \rho)(k_3)^2}{2}(z_s)^{-1} \right) ds + k_3(1 - \rho)(W_t - W_{t_0}). \]
Appendix B. Solution of Bernoulli equations (4), (5), (28)

Consider the following differential equation

\[ y_t = A_n + \int_{t_n}^{t} \left( B_n(y_s)^{-l} + C_n y_s \right) ds, \]

with \( l > 0 \). The dynamics for the transformation \( r = y^{1+l} \) are

\[ dr_t = (1 + l)B_n (1 + l)C_n r_t dt, \]

that is a linear equation with solution

\[
 r_t = \int_{t_n}^{t} e^{-(1+l)C_n(s-t_n)} (1 + l)B_n ds + (A_n)^{1+l} \\
 = (1 + l)B_n e^{(1+l)C_n} \int_{t_n}^{t} e^{-(1+l)C_n s} ds + (A_n)^{1+l} e^{(1+l)C_n(t-t_n)} \\
 = -(1 + l)B_n e^{(1+l)C_n t} e^{-(1+l)C_n t_n} - e^{-(1+l)C_n t_n} + (A_n)^{(1+l)} e^{(1+l)C_n(t-t_n)} \\
 = -\frac{B_n}{C_n} (1 - e^{(1+l)C_n(t-t_n)}) + (A_n)^{1+l} e^{(1+l)C_n(t-t_n)}, 
\]

where for the case \( C_n = 0 \) we read

\[ r_t = (1 + l)B_n(t - t_n) + (A_n)^{1+l}. \]

Appendix C. Solution of the equation (45)

We rewrite equation (45) as

\[ \frac{1}{\cot(y/2)} dy = A_n dt \]

and integrate between \([t_n, t]\) to get

\[
 \int_{t_n}^{t} \tan(y/2) d(y/2) = \frac{A_n}{2} (t - t_n) \\
 - \ln |\cos(y_t/2)| = \frac{A_n}{2} (t - t_n) + C \\
 |\cos(y_t/2)| = Ce^{-\frac{A_n}{2}(t-t_n)}. 
\]

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