Z\textsubscript{2} Symmetry Prediction for the Leptonic Dirac CP Phase

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Model-independent consequences of applying a generalized hidden horizontal Z\textsubscript{2} symmetry to the neutrino mass matrix are explored. The Dirac CP phase \(\delta_D\) can be expressed in terms of the three mixing angles as

\[
\cos\delta_D = (s_1^2 - c_1^2 s_2^2)(c_3^2 - s_3^2)
\]

where the \(s_i, c_i\) are sines and cosines of the atmospheric, solar, and reactor angles. This relation is independent of neutrino masses and whether neutrinos are Dirac- or Majorana-type. Given the present constraints on the angles, \(\delta_D\) is constrained to be almost maximal, a result which can be explored in experiments such as NO\(\nu\)A and T2K. The Majorana CP phases do not receive any constraint and are thus model-dependent. Also a distribution of \(\theta_x\) with a lower limit is obtained without specifying \(\delta_D\).

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\textit{Introduction} – The lepton sector has a quite different mixing pattern from that of the quarks. In the PMNS parameterization \cite{1}, the atmospheric mixing angle \(\theta_\odot \equiv \theta_{23}\) is almost maximal and the solar mixing angle \(\theta_\odot \equiv \theta_{12}\) is also large. The reactor mixing angle \(\theta_x \equiv \theta_{13}\) is small with an upper bound of about 10 degrees at the 1\(\sigma\) level. The current global fit is summarized in Table I.

| \(\sin^2 \theta_\odot (\theta_x)\) | \(\sin^2 \theta_\odot (\theta_\odot)\) | \(\sin^2 \theta_\odot (\theta_x)\) |
|---|---|---|
| CV | 0.312 (34.0°) | 0.466 (43.0°) | 0.016 (7.3°) |
| 1\(\sigma\) Range | 0.294-0.331 | 0.408-0.539 | 0.006-0.026 |
| \(32.8-35.1°\) | \(39.7-47.2°\) | \(4.4-9.3°\) |

\textbf{TABLE I: The global fit for the neutrino mixing angles}. The first row gives the central values.

Upcoming neutrino experiments (Daya Bay \cite{4}, Double CHOOZ \cite{5}, and Reno \cite{6}) will make precise measurements of the three mixing angles, especially \(\theta_x\), and the Dirac CP phase \(\delta_D\) which will be indirectly measured by experiments such as NO\(\nu\)A \cite{8} and T2K \cite{9}. A sizeable \(\theta_x\) is crucial for pinning down \(\delta_D\) since they always appear as the product \(\sin \theta_x e^{i\delta_D}\).

A model independent sign from experiment is \(\mu-\tau\) symmetry \cite{10} in the diagonal basis of the charged leptons \cite{11}. Under \(\mu-\tau\) symmetry \(\theta_x\) vanishes and \(\theta_\odot\) is maximal. This is incorporated in the widely accepted tribimaximal mixing \cite{12}.

The essential point of \(\mu-\tau\) symmetry is that it is a residual symmetry which directly determines the mixing pattern. Nevertheless, having one residual symmetry is not sufficient since \(\mu-\tau\) symmetry can determine just two of the three mixing angles. Another \(Z_2\) symmetry, which determines the solar mixing angle, has been proposed in \cite{13}.

However, deviations from the tribimaximal pattern are still allowed and need not be small \cite{2,14,15}. First, it is not accurate enough, especially for \(\theta_{13}\). It can serve as a zeroth-order approximation but should receive higher order corrections. Also, if tribimaximal mixing were exact, there would be no effect of the Dirac CP phase. This is not what we would prefer, especially from the perspective of leptogenesis. Thus, \(\mu-\tau\) symmetry should be abandoned. Secondly, the solar mixing angle deviates from the tribimaximal one by more than one degree. A generalization with \(\theta_\odot\) being set free is explored in \cite{16,17}. Also, in one model, \(\theta_\odot\) is expressed in terms of a golden ratio \cite{18} while another scheme is realized with dodeca-symmetry \cite{19}. We will explore the model-independent consequences of this generalized \(Z_2\) symmetry without assuming \(\mu-\tau\) symmetry.

\textit{Symmetry and Mixing} – The neutrino mixing matrix can be determined by two \(Z_2\) symmetries \cite{12} of which the solar mixing angle \(\theta_\odot\) is constrained by

\[
G_1(k) = \frac{1}{2 + k^2} \begin{pmatrix}
2 - k^2 & 2k & 2k \\
2k & k^2 & -2 \\
2k & -2 & -k^2
\end{pmatrix},
\]

the generator of the generalized \(Z_2^\ast\) symmetry \cite{16}. Tribimaximal mixing corresponds to \(k = -2\). Another choice

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is \( k = -3/\sqrt{2} \) with \( \theta_s \approx 33.7^\circ \) which fits the data better. The following discussion will concentrate on imposing only \( G_1 \).

In general, there are three possible horizontal symmetries \([12]\), \( G_i \), \( i = 1, 2, 3 \), which satisfy \( G_i U_\nu = U_\nu d_\nu^{(i)} \) where \( U_\nu \) is the mixing matrix of the neutrino mass matrix and the \( d_\nu^{(i)} \) are diagonal rephasing matrices. We can write this relation in two equivalent forms,

\[
U_\nu^\dagger G_i U_\nu = d_\nu^{(i)} \quad \iff \quad G_i = U_\nu d_\nu^{(i)} U_\nu^\dagger.
\]

For Majorana neutrinos, the diagonal elements of \( d_\nu^{(i)} \) are \( \pm 1 \) and thus there are only eight possibilities for \( d_\nu^{(i)} \) spanning a product group \( \mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2 \). But only \( \mathbb{Z}_2 \otimes \mathbb{Z}_2 \) is effective because the third \( \mathbb{Z}_2 \) just contributes an overall \( +1 \) or \( -1 \) factor. Since the \( G_i \) are similarity transformed representations of the corresponding \( d_\nu^{(i)} \) through \( U_\nu \), they are equivalent to \( d_\nu^{(1)} \). In other words, the effective residual symmetry of lepton mixing is \( \mathbb{Z}_2 \otimes \mathbb{Z}_2 \) for Majorana neutrinos. We call \( d_\nu^{(1)} \) the kernel of \( G_i^{(i)} \).

From the first form of \([2]\) we see that \( U_\nu \) is a diagonalization matrix of \( G_i \) with corresponding diagonalized matrix \( d_\nu^{(i)} \). Given a group representation, the mixing matrix can be obtained by diagonalizing the representation matrix without resorting to the mass matrix. This provides a very convenient and direct way of determining the mixing matrix. As no explicit mass matrix, and hence no mass eigenvalues, are involved, the relation between the horizontal symmetry and the mixing matrix is mass-independent \([20]\). Also, the second form tells us that the residual symmetries can be constructed in terms of the mixing matrix. In this sense, the symmetry can be determined phenomenologically.

For the generalized \( G_1(k) \), with \( d_\nu^{(1)} = \text{diag}(-1, 1, 1) \), there is a special form of the mixing matrix,

\[
U_k = \begin{pmatrix}
-\frac{k}{\sqrt{2+k^2}} & \frac{\sqrt{2}}{\sqrt{2+k^2}} & 0 \\
\frac{1}{\sqrt{2+k^2}} & -\frac{1}{\sqrt{2+k^2}} & -\frac{\sqrt{2}}{\sqrt{2+k^2}} \\
\frac{1}{\sqrt{2+k^2}} & \frac{1}{\sqrt{2+k^2}} & \frac{\sqrt{2}}{\sqrt{2+k^2}}
\end{pmatrix},
\]

with maximal \( \theta_s \) and vanishing \( \theta_x \).

Reconstruction of Mixing Matrix – Since there is a degeneracy between the eigenvalues of \( G_1 \), its diagonalization matrix is not unique. From \([2]\) and \([3]\) we get,

\[
G_1 = U_\nu d_\nu^{(1)} U_\nu^\dagger = U_k U_T d_\nu^{(1)} U_\nu^\dagger U_k^\dagger,
\]

where \( U_\nu \) denotes the physical neutrino mixing matrix. Thus the physical neutrino mixing matrix \( U_\nu \) can be expressed as \( U_\nu = U_k U_T \). The freedom of rotating between the degenerate eigenstates is represented by a 2–3 unitary rotation parameterized as,

\[
U_T = \begin{pmatrix}
1 & c_T & -s_T e^{i \beta_4} \\
0 & s_T e^{-i \beta_3} & c_T \\
0 & -s_T e^{i \beta_2} & e^{i \beta_3}
\end{pmatrix},
\]

with \( c_T = \cos \theta_T \) and \( s_T = \sin \theta_T \). The diagonal rephasing matrix \( d_\nu^{(1)} \) is invariant under \( U_T \), namely \( d_\nu^{(1)} = U_T d_\nu^{(1)} U_T^\dagger \).

Also, the physical mixing matrix \( U_\nu \) can be generally parameterized as \( P_\nu U_\nu Q_\nu \), where \( U_\nu \) is the standard parametrization of MNS matrix,

\[
U_\nu = \begin{pmatrix}
c_s c_x & -s_s c_x & -s_x e^{i \delta_D} \\
-s_s c_a - c_s s_a s_x e^{-i \delta_D} & c_s c_a + s_s s_a s_x e^{-i \delta_D} & -s_a c_x \\
-s_a s_a + c_s a_a s_x e^{-i \delta_D} & c_s a_a - s_s a_a s_x e^{-i \delta_D} & c_a c_x
\end{pmatrix},
\]

while \( P_\nu = \text{diag}(e^{i \alpha_1}, e^{i \alpha_2}, e^{i \alpha_3}) \) and \( Q_\nu = \text{diag}(e^{i \phi_1}, e^{i \phi_2}, e^{i \phi_3}) \) are two diagonal rephasing matrices \([16]\). The phases \( \phi_i \) in \( Q_\nu \) are Majorana CP phases while \( P_\nu \) is a diagonal rephasing matrix which does not have any direct physical significance.

These two expressions for the physical mixing matrix \( U_\nu \) must be equal,

\[
U_k U_T = P_\nu U_\nu Q_\nu.
\]

The Majorana CP phases \( \phi_i \) in \( Q_\nu \) could be absorbed by the phases \( \beta_i(i = 1, 2, 3) \) of \( U_T \).

Dirac CP Phase and Mixing Angles – We will first compare the elements of the first row of the two matrices on either side of Eq. \([7]\). They give

\[
\phi_1 - \beta_1 = -\alpha_1, \quad c_s c_x = \frac{-k}{\sqrt{2 + k^2}} ;
\]

\[
\phi_2 - \beta_2 = -\alpha_1, \quad s_s c_x = \frac{\sqrt{2} c_T}{\sqrt{2 + k^2}} ;
\]

\[
\phi_3 - \beta_3 = \beta_4 - \delta_D - \alpha_1, \quad s_x = \frac{-\sqrt{2} s_T}{\sqrt{2 + k^2}}.
\]

The matching of overall CP phases leaves some freedom which gives, at most, minus signs. These signs will not affect the final physical result so, for simplicity, we will omit them. From \([8]\) we can see that only the differences \( \phi_i - \beta_i \) are relevant. Majorana phases \( \phi_i \) can not be uniquely determined. The mixing angles, \( \theta_x \) and \( \theta_s \) can be expressed as functions of \( k \) and \( \theta_T \). Of the three relations for the mixing angles only two are independent because of unitarity. Conversely, \( k \) and \( \theta_T \) can be expressed as functions of mixing angles,

\[
k^2 = \frac{2 c_T^2 c_x^2}{1 - c_s^2 c_x^2}, \quad s_T^2 = \frac{s_x^2}{1 - c_s^2 c_x^2}.
\]
From (9) we can estimate the approximate value of the fitting parameters. Since $s_2^2$ is tiny, so is $s_2^2$, and $k^2 \approx 2\cot^2\theta_s$. We see that the value of $k$ is close to 2 but with some deviation. According to (9), $s_2^2$ is approximately $3s_2^2$. The central value of $\theta_s$ we used is about 7.3° rendering $\theta_T \approx 13^\circ$.

The (21) and (31) elements of Eq. (7) give the differences between $\alpha_i$ in terms of mixing angles and $\delta_D$,

\[ e^{i(\alpha_2 - \alpha_1)} = \frac{1}{\sqrt{2 + k^2} (s_2 c_a - c_2 s_a s_x e^{-i\delta_D})}, \quad (10a) \]

\[ e^{i(\alpha_3 - \alpha_1)} = \frac{1}{\sqrt{2 + k^2} (s_2 s_a + c_2 c_a s_x e^{-i\delta_D})}. \quad (10b) \]

The sum of the (23) and (33) elements gives,

\[ (e^{i\beta_3} c_a - e^{i\beta_2} s_a) c_x = \sqrt{\frac{2k^2}{2 + k^2}} \sin x e^{i\beta_4}, \quad (11) \]

with $\delta_i \equiv \alpha_i - \alpha_1 + \beta_4 - \delta_D$. The common $\beta_4$ cancels.

By comparing (11) with (9) we can eliminate $\theta_T$. Then (10a) and (10b) can be used to remove the Majorana phases $\alpha_i$. The parameter $k$ can be expressed in terms of the mixing angles through (8a). When all of these are combined a relation between $\delta_D$ and the three mixing angles emerges,

\[ \cos \delta_D = \frac{(s^2 - c^2 s_x^2)(c^2_a - s^2_a)}{4 c_a s_a c_s s_x}. \quad (12) \]

Note that this correlation between the mixing angles and $\delta_D$ is independent of unphysical parameters and thus it can be used to compare with experimental results directly. With vanishing $\delta_D$ (12) will reduce to the result of (21).

As $\theta_s$ is small and $\theta_a$ is nearly maximal, we can define $\theta_x \equiv 0^\circ + \delta_x$ and $\theta_a \equiv 45^\circ + \delta_a$. To leading order (12) reduces to,

\[ \delta_x \approx -\frac{\tan \theta_s}{\cos \delta_D}, \quad (13) \]

which is the main result of (12) obtained from the minimal seesaw model. In that model, both $\mu-\tau$ and $G_1$ symmetries are imposed at leading order and a soft mass term is used to break $\mu-\tau$ without affecting $G_1$. This is similar to our treatment here where only $G_1$ is applied to constrain the mixing angles and the Dirac CP phase without involving $\mu-\tau$ symmetry. The difference is that an expansion method was used in (16) while the discussion in the current work is more general and, most importantly, model-independent.

**Phenomenological Consequences** – The relation (12) can be used to put limits on $\theta_x$ from,

\[ \sin \theta_x = \left[ \mp \sqrt{c_D^2 + \cot^2 \theta_\text{a}} - c_D \right] \tan 2\theta_\text{a} \tan \theta_s, \quad (14) \]

with $c_D \equiv \cos \delta_D$. The result for the lower sign is shown in Fig. 1(a) where $\delta_D$ is assumed to be uniformly distributed in the range $[0, 2\pi]$ and normal distributions for $\theta_a$ and $\theta_s$ are used with central values and standard deviations given by Table I. Alternately we can use the limits $1 \geq \cos \delta_D \geq -1$ to get limits on $s_x$,

\[ \frac{s_x}{c_x} \frac{c_a + s_a}{|c_a - s_a|} \geq \sin \theta_x \geq \frac{s_x}{c_x} \frac{|c_a - s_a|}{c_a + s_a}. \quad (15) \]

The upper limit is of no use but the lower limit is clearly shown in Fig. 1(b).

Since the three mixing angles have been measured, a prediction of $\delta_D$ can be obtained using (12). This is plotted in Fig. 2(a) where there exists a corresponding distribution in the range of $(-180^\circ, 0^\circ)$. We can see that
\(\delta_D\) is constrained to be almost maximal which is a direct consequence of the fact that \(\theta_a\) is closer to its zeroth-order approximation than \(\theta_x\). From (12) we see that if \(\delta_a\) vanishes, namely \(c_a = s_a\), \(\cos \delta_D\) would also vanish. A similar result is obtained in (22) in a model-dependent way.

Experimentally, \(\delta_D\) is measured through the Jarlskog invariant \[ J_\nu \equiv c_a s_a c_x s_x c_a^* s_x \sin \delta_D \] whose distribution is shown in Fig. 2(b). There is an explicit upper limit around 0.04. The distribution peaks at \(|J_\nu| \approx 0.03\) which can be estimated with the best fit values listed in Table I. This will be tested by future experiments [3, 7].

Generalization to Dirac Type Neutrino – The direct relation between \(G_i\) and \(U_\nu\) shown in (2) is the same for Dirac- and Majorana-type fermions [13]. The only difference comes from the kernel \(\delta_D^{(i)}\). For Majorana neutrinos \(\delta_D^{(i)} \delta_D^{(i)\dagger} = I\), while for Dirac neutrinos the kernel can be complex, \(\delta_D^{(i)} \delta_D^{(i)\dagger} = I\).

As long as \(G_1\) is the same, its kernel \(\delta_D^{(1)}\) is still \((-1, 1, 1)\) for either Majorana or Dirac neutrinos. Thus, all the above results also apply for Dirac neutrinos. This result is quite general and has not been noticed before. In [13] the author considered the group \(S_4\) generated by subgroups \(F\) and \(G\) for the charged lepton and neutrino sectors respectively with the neutrinos being Majorana-type. The element \(G_i\) of the subgroup \(G\) has a kernel with diagonal elements being \(-1, +1, +1\) in some order. This satisfies the constraint on not only Majorana- but also Dirac-type kernels. Thus \(G\) is also true for the case of Dirac-type neutrinos. The generated \(S_4\) symmetry applies to both Dirac- and Majorana-type neutrinos, not just the Majorana-type as discussed in [13].

Conclusions – Model-independent consequences of a generalized \(G_1\) symmetry are explored. Due to degeneracy between the eigenvalues of \(G_1\), the mixing matrix cannot be uniquely determined. Nevertheless, \(G_1\) invariance gives a relation (12) between the mixing angles and the Dirac CP phase \(\delta_D\) for both Majorana and Dirac neutrinos. This can be used to predict \(\delta_\nu\), and consequently the Jarlskog invariant \(J_\nu\), in terms of the measured mixing angles leading to an almost maximal \(\delta_\nu\) as shown in Fig. 2. This appears to be the first time that a direct relation between a horizontal symmetry and the leptonic Dirac CP phase has been established in a model-independent way. These results will be tested by next generation of neutrino experiments [3, 7]. Comparison between this prediction of \(\delta_D\) and experiment can tell us whether or not there is a horizontal symmetry and, if so, what type of symmetry it is. Since the prediction is independent of model assignments, its conformation should be robust and conclusive. The Majorana CP phases are not constrained; they are model-dependent and must be studied case by case. In addition a clear lower limit on \(\theta_x\) is obtained as shown in Fig. 4.

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