Closed FRW model in Loop Quantum Cosmology

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Abstract

The basic idea of the LQC applies to every spatially homogeneous cosmological model, however only the spatially flat (so called \( k = 0 \)) case has been understood in detail in the literature thus far. In the closed (so called: \( k = 1 \)) case certain technical difficulties have been the obstacle that stopped the development. In this work the difficulties are overcome, and a new LQC model of the spatially closed, homogeneous, isotropic universe is constructed. The topology of the spacelike section of the universe is assumed to be that of SU(2) or SO(3). Surprisingly, according to the new results achieved in this paper, the two cases can be distinguished from each other just by the local properties of the quantum geometry of the universe! The quantum hamiltonian operator of the gravitational field takes the form of a difference operator, where the elementary step is the quantum of the 3-volume derived in the flat case by Ashtekar, Pawlowski and Singh. The mathematical properties of the operator are studied: it is essentially self-adjoint, bounded from above by 0, the 0 itself is not an eigenvalue, the eigenvectors form a basis. An estimate on the dimension of the spectral projection on any finite interval is provided.

1 Introduction

Loop Quantum Cosmology is the Loop Quantum Gravity [21, 20, 2] motivated approach to the quantization of the symmetric cosmological models. It was started by Martin Bojowald [8, 9, 10]. The simplification used in the quantum cosmological models framework consists in reducing the phase space of the full theory to the finite dimensional phase space of degrees of freedom of a given cosmological model.
idea had been applied long before the LQC [17, 18]. The new strategy introduced by Bojowald is to maintain in the construction of a quantum cosmological model the structure of the Loop Quantum Gravity [1]. The homogeneous, isotropic case is best understood these days. Already the first calculations [12, 11] provided qualitatively new insight into the quantum structure of spacetime near the event classically known as the Big-Bang singularity. The LQC evolution does not break down when the scale factor vanishes. However, these works were incomplete in that the physical sector of the theory had not been completed. This situation was rectified in [5, 4]. Furthermore, a more careful implementation of the physical considerations of full LQG [1, 4, 5] - in particular using the minimal quantum of 2-area [3] - finally led to the improved Hamiltonian constraint of Ashtekar, Pawlowski and Singh (APS) [6]. The key difference between this quantum dynamics and the one used previously is that each step in the resulting quantum evolution increases the 3-volume of the spacelike slice by a fixed, always the same, amount. This emergence of the elementary quantum of volume is a purely dynamical effect - the eigenvalues of the kinematical quantum volume operator take all the possible values from 0 to \( \infty \).

The outlined result of APS concerns the spatially flat case of the homogeneous, isotropic universe (so called k=0 case). Whereas the basic ideas of the (improved as well as unimproved) LQC approach easily apply to the closed (k = 1) or hyperbolic (k = −1) cases [14, 22], some technical difficulties were actual obstacle in completing the quantization task. In this work we overcome the difficulty and construct the LQC model of the spatially closed, homogeneous, isotropic universe. The topology of the spacelike section of the universe is assumed to be that of SU(2) or SO(3). The significant result is that the quantum hamiltonian operator of the gravitational field (that is the gravitational part of the scalar constraint) takes the form of the difference operator, where the elementary step is again the quantum of the 3-volume derived in the flat case by APS. Next we study the properties of the quantum hamiltonian operator defined in the kinematical Hilbert space of gravitational degrees of freedom. We show this operator is bounded from above, is essentially self-adjoint on its natural domain and its self-adjoint extension has trivial kernel. The Hilbert space admits decomposition into subspaces preserved by the operator, so called super selection sectors. In each of the super selection sectors the operator has a discrete spectrum. The difference between the SU(2) and SO(3) cases consists in different values of the parameter used through out the work. In the last section we point out the surprising local difference between the quantum geometries of the two globally different cases.

In this work we are concerned with the gravitational field degrees of freedom only. The full model should be defined in the tensor product of the Hilbert space we consider below and the Hilbert space of a given matter field excitations. Then, the full hamilto-

\[1\text{See review [13]}\]
nian is the gravitational hamiltonian plus the matter term. The results established in this paper still apply to the gravitational part of the hamiltonian operator. They will be used in the next paper [15] to provide exact analytic approach to the full quantum scalar constraint equation.

In the meantime, APS plus Vandersloot [7] have also constructed the LQC model for the spatially closed, homogeneous, isotropic universe. The gravitational parts of our models are equivalent (except that we are not sure whether or not [7] admits the SO(3) case, but that is not a big deal) including the factor ordering in the quantum hamiltonian operator. Moreover, the APS model includes the massless scalar field and their work contains the full quantum solution.

2 Kinematics

2.1 The Ashtekar connection variables

We consider now the Ashtekar phase space for spatially homogeneous, isotropic cosmologies. We will focus only on the case $k = 1$, where the symmetry group $S$ has the same Lie algebra as the isometry group of the round, 3-dimensional sphere $S_3$. We assume, the underlying manifold $G$ admits the structure of a Lie group, either SU(2) or SO(3). We fix on $G$ either of these Lie group structures. The symmetry group $S$ takes the form $S = G \times G$, where the left/right factor acts on $G$ by the left/right action.

The Ashtekar phase space $\Gamma_S^{grav}$ of the gravitational degrees of freedom of the cosmologies considered in this paper is a subspace of the space $\Gamma$ of pairs $(A, P)$ of fields on the 3-manifold $G$, where $A$ is an $\text{su}(2)$ valued 1-form and $P$ an $\text{su}(2)^*$ valued vector field of density weight 1. A pair $(A', P')$ on $\Sigma$ is said to be spatially homogeneous and isotropic or, for brevity, symmetric if for every $s \in S$ and for every $x \in \Sigma$ there exists a neighborhood $U_x$ and a gauge transformation $g : U_x \rightarrow SU(2)$, such that

$$ (s^*A', s^*P') = (g^{-1}A'g + g^{-1}dg, g^{-1}P'g), \quad (2.1) $$

on $U_x$. An example of a symmetric pair can be constructed as follows. Let

$$ ^oA = \omega_{MC} \quad (2.2) $$

where $\omega_{MC}$ (the Maurer-Cartan form) is a Lie algebra isomorphism which maps the Lie algebra of left invariant vector fields on $G$ into the Lie algebra $\text{su}(2)$. Let

$$ \epsilon_{MC} = ((\omega_{MC})^*)^{-1}, $$

\footnote{We thank Abhay Ashtekar for the hint}
hence $e_{MC}$ is a left invariant vector field on $G$ taking values in $\text{su}(2)$. To turn it into a density, let us fix an invariant scalar product $\eta$ in $\text{su}(2)$,

$$\eta(\xi, \zeta) = -2\text{Tr}(\xi\zeta).$$

It induces, in a natural way, a left and right invariant metric tensor $\gamma$ on $G$. We will use the density of the weight 1, namely $\sqrt{\det \gamma}$, to define

$$^oP = \sqrt{\det \gamma} e_{MC}. \quad (2.3)$$

An important geometric fact is, that for each symmetric pair $(A', P')$ there is a globally defined gauge transformation $g : \Sigma \to \text{SU}(2)$, such that the gauge transformed pair

$$(A, P) = (g^{-1}A'g + g^{-1}dg, g^{-1}P'g) \quad (2.4)$$

has the following simple form

$$A = c^oA, \quad P = p^oP, \quad (2.5)$$

where $c$ and $p$ are constants carrying the only non-trivial information contained in the pair $(A', P')$ (the under-bar will be removed after the suitable rescaling).

Define $\Gamma^S_{\text{grav}}$ to be the set of pairs $(2.5)$. The variables $(c, p)$ form a globally defined coordinate system. The symplectic form $\Omega^S_{\text{grav}}$ on $\Gamma^S_{\text{grav}}$ is given by the pullback of the symplectic form $\Omega_{\text{grav}}$

$$\Omega_{\text{grav}}(\delta_1, \delta_2) = \int_G \delta_1 A^a_i \wedge \delta_2 P^a_i - \delta_2 A^a_i \wedge \delta_1 P^a_i \quad (2.6)$$

of the full theory. The result is

$$\Omega^S_{\text{grav}} = 3V_0 \, dc \wedge dp, \quad (2.7)$$

where

$$V_0 = \int_G \sqrt{\det \gamma} \, d^3x = \begin{cases} 
16\pi^2 & \text{for } G = \text{SU}(2) \\
8\pi^2 & \text{for } G = \text{SO}(3) \end{cases} \quad (2.8)$$

is the volume of the 3-manifold $G$ with respect to the metric $\gamma$. The vector density $P$ defines the physical metric tensor $q$ on $G$. Let $\tau_1, \tau_2, \tau_3$ be a basis of $\text{su}(2)$ orthonormal with respect to the scalar product $\eta$, and $\tau^1, \tau^2, \tau^3$ the dual basis of $\text{su}(2)^*$. The orthonormal frame $e_1, e_2, e_3$ tangent to $G$, corresponding to $P = P_i \otimes \tau^i$ and $q$, is determined as follows

$$\sqrt{\det q} e_i = \kappa \gamma P_i, \quad (2.9)$$

where

$$\kappa = 8\pi G,$$
and $G$ is the Newton constant. Therefore, the physical meaning of the variable $\mathfrak{L}$ is

$$V = V_0(\kappa \gamma |\mathfrak{L}|)^{\frac{2}{3}},$$

where $V$ is the physical volume of $G$ defined by the physical metric tensor. It leads to the following rescaling

$$p = (V_0)^{\frac{2}{3}} \kappa \gamma \mathfrak{L},$$

upon which

$$|p| = V^{\frac{2}{3}},$$

and the sign of $p$ defines the orientation. The Poisson bracket between two functions $f, g \in C(\Gamma_{grav}^S)$ is:

$$\{f, g\} = \frac{\kappa \gamma}{3 \ell_0} (\partial_c f \partial_p g - \partial_p f \partial_c g), \quad \ell_0 = (V_0)^{\frac{1}{3}}.$$  \hspace{1cm} (2.11)

### 2.2 The loop quantization

In LQG the connection variable is replaced by the parallel transport variable. In the homogeneous case we consider the parallel transports along finite segments of geodesic curves in $G$, called edges. Let $\alpha : [0, \mu] \to G, \ \mu \in \mathbb{R}$, be an edge. There exists $\xi \in \mathfrak{su}(2)$, such that $\eta(\xi, \xi) = 1$, and

$$\alpha(t) = \alpha(0)e^{t \xi}.$$  \hspace{1cm} (2.12)

Along the edge $\alpha$ we consider the parallel transport defined by the connection $A = \omega_{MC}$,

$$h_\xi^{(\mu)} = \mathcal{P} \exp \left( - \int A \right) = e^{-\mu \xi} = \cos \frac{\mu \xi}{2} - 2 \sin \frac{\mu \xi}{2}.$$  \hspace{1cm} (2.13)

The entries $h_\xi^{(\mu)}_{\xi N}$ of the matrix $h_\xi^{(\mu)}$ are linear combinations of the functions $e^{\pm i \mu \xi}$.

The Poisson bracket between the variable $p$ and the loop variable given by (2.11) is:

$$\{p, e^{i \mu \xi} \} = -i \frac{8\pi G \gamma \mu}{6\ell_0} e^{i \mu \xi},$$

where $\mu \in ]-\infty, \infty[. \ \ \text{The quantum algebra of basic operators is defined by the commutator:}$

$$[e^{i \xi \frac{\mu}{\ell_0}}, \hat{p}] = -\frac{8\pi G \gamma}{6\ell_0} \mu e^{i \xi \frac{\mu}{\ell_0}}.$$  \hspace{1cm} (2.15)

For the kinematical Hilbert space we take $\mathcal{H}^{kin} = L^2(\mathbb{R}_{\text{Bohr}}, d\mu_{\text{Bohr}})$, where $\mathbb{R}_{\text{Bohr}}$ denotes the Bohr compactification of the real line. The space is the completion of the
vector space of the formal finite linear combinations of elements of the basis \( \{ |\mu\rangle : \mu \in \mathbb{R} \} \), endowed with the hermitian scalar product defined by

\[
\langle \mu_i | \mu_j \rangle = \delta_{\mu_i, \mu_j}.
\] (2.16)

The action of the fundamental operators is defined as follows:

\[
\hat{p}|\mu\rangle = \frac{8\pi l^2}{6\ell_0} \mu|\mu\rangle \quad \text{and} \quad e^{i\frac{\phi_c}{2}}|\mu\rangle = |\mu + \tilde{\mu}\rangle
\] (2.17)

3 Dynamics

3.1 Classical picture

The gravitational terms in the diffeomorphism and, respectively, Gauss constraints read:

\[
C_{G}^{ grav}(\Lambda) = \int G \Lambda D_a P^a_i, \quad C_{G}^{ grav}(\vec{N}) = \int G N^a F_{ab} P^b_i + C_{G}^{ grav}(\Lambda).
\]

where

\[
F = dA + A \wedge A, \quad \text{and} \quad D_a P^a_i = \partial_a P^a_i - P^a_j A^k c^j_{ki}.
\] (3.1)

Each pair \((A, P)\) defined in (2.5), satisfies automatically

\[
D_a P^a_i = 0, \quad \text{and} \quad F_{ab} P^b_i = 0.
\] (3.2)

Hence, the latter property (3.2) follows from the symmetry assumption only. The consequence is, that in the presence of matter, the corresponding matter terms of the Gauss and, respectively, diffeomorphism constraint have to vanish separately.

The gravitational part \(C_{sc}^{ grav}(N)\) of the scalar constraint (the time evolution generator) has the following general form

\[
C_{sc}^{ grav}(N) = \frac{1}{2} \sqrt{\frac{\gamma}{\kappa}} \int_G \frac{P^a_i P^b_j \epsilon^{ijk}}{|\det P|} \left( F_{ab}^k - (1 + \gamma^2) \epsilon^{k \, mn} K^m_{\ a} K^n_{\ b} \right)
\] (3.3)

Substituting the variables (2.5) and \(N = \text{const}\) one would get the following simple formula

\[
C_{sc}^{ grav}(N) = -N \left( \frac{3\sqrt{|p|}}{8\pi G \gamma^2 \ell_0^2} (-c^2 + c^2) + (1 + \gamma^2) \frac{3\ell_0^2 \sqrt{|p|}}{4 \cdot 8\pi G} \right).
\] (3.4)
3.2 Quantum picture: preparation

Our aim is to mimic the quantization scheme used in LQG. Therefore we go back to the LQG form of the scalar constraint (3.3). According to LQG, the curvature $F$ in the scalar constraint should be expressed by the holonomy map, the parallel transport functions along suitable closed loop. The functions are well defined operators in the kinematical Hilbert space. Next, in LQG, each loop is shrunk to a point. The limit of the corresponding scalar constraint operator exists in the dual space, the space of diffeomorphism invariant linear functionals. In LQC, on the other hand, the diffeomorphism constraint is solved on the classical level, the solutions are given in a fixed gauge (we choose one representative (2.5) from each diffeomorphism equivalence class). As the consequence, the quantum scalar constraint operator has to be defined directly in the kinematical Hilbert space. To preserve the analogy with LQG, in LQC we quantize the scalar constraint replacing the curvature components by suitably chosen parallel transport operators. However we stop shrinking the loops at the stage when they reach the minimal, non-zero area allowed by the LQG.

In what follows we use the following decomposition

$$\omega_{MC} = \alpha^i \tau_i, \quad \epsilon_{MC} = \epsilon^i \tau^i.$$ (3.5)

Specifically, to a component $F_{ab} e_0 e^a e^b_0$ we assign a two dimensional plane

$$S_{ij}^{(\mu)} = \{ e^{s \tau_i} e^{t \tau_j} : 0 \leq s, t \leq \mu \},$$

and the parallel transport $h_{ij}^{(\mu)}$ along its contour defined by the connection (2.5). The contour is a square, consisting of the four geodesic edges of equal length, each orthogonal to the neighbors. The parallel transports along the segments can be calculated using (2.13)\(^4\) The result is

$$h_{ij}^{(\mu)} = e^{-(1-c)\mu \tau_i} e^{\mu c \tau_i} e^{(1-c)\mu \tau_j} e^{-\mu c \tau_i}.$$ (3.6)

On the other hand, the curvature is

$$F_{ab} e_0 e^a e^b_0 = c_{ij}^k (-c + c^2),$$

and, as expected,

$$\lim_{\mu \to 0} \frac{2}{\mu^2} \text{Tr}(h_{ij}^{(\mu)} \tau_k) = F_{kab} e^a e^b_0.$$ (3.7)

\(^4\)Seemingly, one of the edges $\alpha_3(t) = e^{-c t} e^{\mu c t} e^{\mu c t}$ is an integral curve of a right invariant vector field, whereas (2.12) defines an integral line of a left invariant vector field, however we should remember that $e^{\xi} g = g e^{-1} \xi g$. 

7
However, as we have already mentioned, we are not going to shrink the loop to a point. Following [6] we will stop the shrinking at \( \mu = \bar{\mu} \) when the physical area of the loop reaches the minimal non-zero area eigenvalue \( a_0 \) [3], and replace the curvature component by

\[
\frac{2}{\mu^2} \text{Tr} h_{ij}^{(\mu)} \tau_k = \frac{1}{\mu^2} \sin((c - 1)\bar{\mu}) \sin(c\bar{\mu}) c_{kij}. \tag{3.8}
\]

To calculate the physical area of the surface \( S_{ij}^{(\mu)} \) note that its intrinsic geometry is flat. Indeed, consider the coordinate system \((s, t)\) defined on \( S_{ij}^{(\mu)} \) and the corresponding vector fields \( \partial_s \) and \( \partial_t \) tangent to \( S_{ij}^{(\mu)} \). They are commuting Killing vectors of the physical metric \( q \) defined on \( G \). Therefore the components of the induced metric tensor are constant on the surface due to the vanishing of the following Lie derivatives

\[
\mathcal{L}_{\partial_A} q(\partial_B, \partial_C) = 0, \quad A, B, C = t, s.
\]

In the point of \( G \) corresponding to the identity of the group structure, it is easy to compute

\[
q(\partial_A, \partial_B) = \frac{p}{\ell_0^2} \delta_{AB}.
\]

Therefore, the area of the surface \( S_{ij}^{(\mu)} \) is just

\[
A_r = \mu^2 \frac{|p|}{\ell_0^2}.
\]

This implies the following condition on the value \( \bar{\mu} \) of the parameter \( \mu \) is

\[
\frac{|p|}{\ell_0^2} \cdot \bar{\mu}^2 = a_0 = 2\sqrt{3\pi\gamma} l_{Pl}^2, \tag{3.9}
\]

hence

\[
\bar{\mu} = \sqrt{\frac{a_0}{|p|}} \ell_0 \tag{3.10}
\]

Alternatively to the square \( S_{ij}^{(\mu)} \), we could consider \( S_{ij}^{(\mu)} = \{e^{s\tau_j} e^{t\tau_i} : 0 \leq s, t \leq \mu\} \), but the resulting replacement for the curvature is not sensitive to that ambiguity.

Going back to the scalar constraint (3.3), we will still take advantage of the special form (2.5) of our variables \((A, P)\), namely, in the symmetric case we have the following extra identity

\[
2 K^i_{[a} K^j_{b]} = \frac{1}{\gamma^2} e^{ij} \epsilon^{k} F_{ab} + \frac{1}{2} \omega^{ij}_{[a} \omega^{i]b}, \tag{3.11}
\]

where the second term is a constant (independent of the dynamical variables).
Taking all that into account, we derive the following LQC modification of the gravitational term of the scalar constraint:

\[
C_{\text{grav}}^{\text{sc}}(N) = -N \frac{3\ell_0^2}{8\pi G_\gamma^2} \sqrt{|p|} \left( \frac{\sin^2 (\bar{\mu} (\frac{1}{2} - c))}{\bar{\mu}^2} - \frac{\sin^2 (\frac{1}{2} \bar{\mu})}{\bar{\mu}^2} + \frac{1}{4} (1 + \gamma^2) \right),
\]

(3.12)

with \(\bar{\mu}\) being defined in (3.10). Intentionally we wrote the latter expression in the way manifestly negative definite, and we will preserve that property by suitable quantization of the constraint.

We are done as far as expressing the curvature is concerned. In LQG we make one more trick: the factor

\[
\frac{P^a_i P^b_j}{\sqrt{\det P^i_j}}
\]

in (3.3) which requires the special care due to the denominator, is expressed by functions written in the form \(h^{-1}\{h, V\}\), where \(h\) is a parallel transport function (of the variable \(A\)) and \(V\) is the 3-volume function (of \(P^i_j\)).

In our case all that factor is proportional to \(\sqrt{|p|}\), hence it is well defined. However, for the sake of analogy with LQG, for the quantization we will write \(\sqrt{|p|}\) in the following exact form:

\[
sign(p)\sqrt{|p|} = \frac{4\ell_0}{3\kappa \mu} \sum_k \text{Tr} \left( h^{(\bar{\mu})^{-1}}(h^{(\bar{\mu})}, V) \tau_k \right).
\]

(3.13)

Consequently, \(\bar{\mu}\) is the one defined in (3.10).

### 3.3 Quantization

After that preparation we are in the position to define the quantum scalar constraint. However, as one could see, yet a new type of functions has emerged, namely the function \(e^{i\frac{\bar{\mu}c^2}{2}}\). The quantization is tricky because \(e^{i\frac{\bar{\mu}c^2}{2}}\) involves both variables \(c\) and \(p\). We begin with the observation that

\[
\exp(i\frac{k_c}{2}) |\mu\rangle = |\mu + k\rangle.
\]

Hence the operator \(\exp(i\frac{k_c}{2})\) is the pullback induced by the translation map

\[
\exp(k \frac{d}{d\mu}) : \mathbb{R} \ni \mu \mapsto \mu + k \in \mathbb{R}
\]

generated by the vector field \(k \frac{d}{d\mu}\). Now, by analogy, one defines the operator \(\exp(i\frac{\bar{\mu}c}{2})\) to be the pullback induced by the map \(\mathbb{R} \to \mathbb{R}\) generated by the vector field \(\bar{\mu}(\mu) \frac{d}{d\bar{\mu}}\), that is by \(\exp(\bar{\mu} \frac{d}{d\bar{\mu}})\). This operator can be expressed again as a translation, but in the different parametrization of \(\mathbb{R}\), namely \(\nu : \mathbb{R} \to \mathbb{R}\), such that

\[
\bar{\mu} \frac{d}{d\bar{\mu}} \nu = 1,
\]
for example by
\[ \nu = K \text{sgn}(\mu) |\mu|^{\frac{3}{2}}, \quad K = \frac{2\sqrt{2}}{\ell_0 \sqrt{\sqrt{6} \sqrt{3\sqrt{3}}} \sqrt{3\sqrt{3}} \sqrt{3\sqrt{3}}} \] (3.14)

To define the action of the operator \( \exp(i\hat{\mu}c^2/\hat{p}) \) we just need to relabel the basis \( |\mu\rangle \) in the following way
\[ |\nu\rangle := |\mu\rangle \bigg|_{\mu=\text{sgn}(\nu) |\nu|^{\frac{3}{2}}} \] (3.15)
and set
\[ e^{i\hat{\mu}c/\hat{p}} |\nu\rangle = |\nu + 1\rangle. \] (3.16)

Remarkably, the parameter \( \nu \) and the operator (3.16) have clear interpretation in terms of the physical 3-volume operator
\[ \hat{V} = |\hat{p}|^{3/2} \] (3.17)

Using (2.16) and (3.15) we get formula for eigenvectors and eigenvalues of the volume operator:
\[ \hat{V} |\nu\rangle = V_1 |\nu\rangle |\nu\rangle, \quad V_1 = \left( \frac{8\pi\gamma}{6\ell_0} \right)^{\frac{3}{2}} \mu \sqrt{\frac{1}{K}}. \] (3.17)

Hence, the operator \( \exp(i\hat{\mu}c/\hat{p}) \) is just the shift in the volume eigenvalue for the unit \( V_1 \), increasing or decreasing the volume depending on the orientation of the frame \( P \) (the sign of \( \nu \)).

We also extend our definition in the obvious way:
\[ U_b |\nu\rangle := e^{i\hat{\mu}c/\hat{p}} |\nu\rangle = |\nu + b\rangle \] (3.18)
where \( b \) is an arbitrary real number.

We are now in the position to define the operator corresponding to the factor \( \sqrt{|p|} \). Using (3.13) we get:
\[ \sqrt{|p|} |\nu\rangle = \text{sgn}(\nu) \frac{2\ell_0}{8\pi G \gamma h \mu} \left( e^{-i\hat{\mu}c/\hat{p}} \hat{V} e^{i\hat{\mu}c/\hat{p}} - e^{i\hat{\mu}c/\hat{p}} \hat{V} e^{-i\hat{\mu}c/\hat{p}} \right) |\nu\rangle. \] (3.19)

This operator turns out to be diagonal in the basis \( |\nu\rangle \). Equations (3.18) and (3.19) give us following formula:
\[ \sqrt{|p|} |\nu\rangle = A(\nu) |\nu\rangle, \quad A(\nu) = \frac{3\ell_0}{8\pi \gamma^3} \mu \left( \frac{8\pi\gamma}{6\ell_0} \right)^{\frac{3}{2}} \left| |\nu|^{1/3} |\nu + 1\rangle - |\nu - 1\rangle \right|. \] (3.20)
The next component of the future quantum scalar constraint operator corresponding to (3.12) is an operator corresponding to the factors \( \sin(\frac{\mu}{2} - c) / \mu \). With the due care, using the \( U \) operator (see 3.18) we define:

\[
\hat{\sin}(\frac{\mu}{2} - c) / \mu := \frac{1}{2i} \left( U_{-2} \hat{\exp}(\frac{i}{2} \mu) - U_{2} \hat{\exp}(-\frac{i}{2} \mu) \right)
\]

\[
\hat{\exp}(\frac{i}{2} \mu) |\nu\rangle := \hat{\exp}(\frac{i}{2} \mu(\nu)) |\nu\rangle.
\]

Since this operator does not satisfy the reality conditions, it is not symmetric. However it can be used to define the symmetric scalar constraint operator, as follows (we take \( N \) for the constant lapse function equal 1):

\[
\hat{C}_{\text{grav}}^{\text{sc}} = -\frac{3\ell_{0}^2}{8\pi G \gamma^2} \left( \sin(\frac{\mu}{2} - c) / \mu \right) \sqrt{|p|} \sin(\frac{\mu}{2} - c) / \mu \left( \sin(\frac{\mu}{2} - c) / \mu \right)^\dagger - \sqrt{|p|} \sin^2(\frac{\mu}{2}) / \mu^2 + \frac{1}{4}(1 + \gamma^2) \sqrt{|p|},
\]

(3.21)

4 Properties of the quantum scalar constraint operator

The (initial) domain. The quantum scalar constraint operator \( \hat{C}_{\text{grav}}^{\text{sc}} \) has been defined by (3.18, 3.20, 3.21, 3.15, 3.22) in the domain

\[
\mathcal{D} = \{ \Psi \in \mathcal{H}_{\text{kin}} : \Psi = \sum_{i=1}^{n} a_i |\nu_i\rangle, \ a_i \in \mathbb{C}, \ \nu_i \in \mathbb{R}, \ n \in \mathbb{N}, \}, \quad (4.1)
\]

where the elements of the basis \( \{|\mu\rangle : \mu \in \mathbb{R}\} \) (2.17) were relabeled using the parameter \( \nu : \mathbb{R} \to \mathbb{R} \) (3.15), proportional to the eigenvalues of the volume operator (3.17).

The action. The operator can be written in the form of the difference operator,

\[
\hat{C}_{\text{grav}}^{\text{sc}} = -\frac{3\ell_{0}^2}{32\pi G \gamma^2} (C_0 + U_4 C_4 + U_-4 C_-4),
\]

(4.2)

where \( C_{-4}, C_0, C_4 \) are some functions of the variable \( \nu \) acting by multiplication, that is \( |\nu\rangle \mapsto C_I(\nu)|\nu\rangle, \ I = -4, 0, 4 \) and \( U_{\pm 4} \) are the shift operators defined in (3.18). Specifically,

\[
\begin{align*}
C_4(\nu) &= -e^{-i\mu(\nu+2)} / \mu^2(\mu(\nu+2)) A(\nu+2) & C_{-4}(\nu) &= -e^{i\mu(\nu-2)} / \mu^2(\mu(\nu-2)) A(\nu-2) \\
C_0(\nu) &= A(\nu-2) / \mu^2(\mu(\nu-2)) + A(\nu+2) / \mu^2(\mu(\nu+2)) + (1 - 4 \sin^2(\mu(\nu)/2) / \mu^2(\mu(\nu)) + \gamma^2) A(\nu)
\end{align*}
\]

(4.3)
Similar properties of the action are well known from the \( k = 0 \) case. Due to them the evolution equation involving the gravitational part of the scalar constraint operator turns into a difference equation, with the step being a multiple of the volume difference \( 4V_1 \) [3.17] [14]. It is an important result, that those features can be generalized to the \( k = 1 \) case.

**The upper bounds.** The operator \( \hat{C}_{\text{sc}}^{\text{grav}} \) is manifestly negative definite,

\[ \hat{C}_{\text{sc}}^{\text{grav}} \leq 0. \]

Even stronger inequalities hold in the domain \( \mathcal{D} \), namely (see (3.20)):

\[ \hat{C}_{\text{sc}}^{\text{grav}} \leq -\frac{3\ell_0^2}{32\pi G} \sqrt{|p|}, \quad (4.4) \]

Indeed, the inequality follows from the obvious inequalities

\[ \frac{\sin(\tilde{\mu}(\frac{1}{2} - c))}{\tilde{\mu}} \sqrt{|p|} \frac{\sin(\tilde{\mu}(\frac{1}{2} - c))}{\tilde{\mu}} \geq 0, \]

\[ -\frac{\sin^2(\tilde{\mu}(\mu(\nu)))}{\tilde{\mu}^2(\mu(\nu))} + \frac{1}{4} \geq 0. \quad (4.5) \]

**The self-adjointness.** The operator \( \hat{C}_{\text{sc}}^{\text{grav}} \) has been defined as manifestly symmetric operator. Moreover:

**Proposition 1** The operator \( \hat{C}_{\text{sc}}^{\text{grav}} \) defined in the domain \( \mathcal{D} \) is essentially self adjoint.

We skip the proof to the end of this section. We denote by \( \mathcal{D}(\hat{C}_{\text{sc}}^{\text{grav}}) \) the self-adjoint extension of the domain \( \mathcal{D} \).

**Sharp negative definiteness of \( \hat{C}_{\text{sc}}^{\text{grav}} \).** The inequality (4.4) will be used in the spectral analysis of the operator \( \hat{C}_{\text{sc}}^{\text{grav}} \).

**Proposition 2** The equation

\[ \hat{C}_{\text{sc}}^{\text{grav}} \Psi = 0 \]

has no nontrivial solution for \( \Psi \in \mathcal{D}(\hat{C}_{\text{sc}}^{\text{grav}}) \). Therefore the scalar constraint is sharply negative operator on its extended domain \( \mathcal{D}(\hat{C}_{\text{sc}}^{\text{grav}}) \),

\[ \hat{C}_{\text{sc}}^{\text{grav}} < 0. \]
The decomposition into preserved subspaces. We will continue the analysis of the scalar constraint operator using decomposition of the Hilbert space into separable, preserved subspaces. The operator $\hat{C}_{\text{grav}}^{\text{sc}}$ preserves every subspace

$$H_\epsilon = \text{Span}\left( \{ \epsilon + 4n \in H^{\text{kin}} : n \in \mathbb{Z} \} \right),$$

where $\epsilon$ is an arbitrary real number. We have the following orthogonal decomposition into preserved subspaces:

$$H^{\text{kin}} = \bigoplus_\epsilon H_\epsilon. \quad (4.7)$$

Discreteness. Since physicists often mean some weaker definitions of the discreteness, let us be very precise here: given an essentially self-adjoint operator $X$ defined in some domain $D$ in the Hilbert space $H$, we say its spectrum is discrete whenever the following conditions are satisfied:

- there exists a basis of $H$ consisting of the eigenvectors of $X$,
- for each eigenvalue the corresponding eigenvectors span a finite dimensional subspace,
- for every finite interval $I$ of $\mathbb{R}$, the set of the eigenvalues of $X$ contained in $I$ is finite.

Going back to the case at hand:

**Proposition 3** For each of the subspaces $H_\epsilon$, the restricted operator $\hat{C}_{\text{grav}}^{\text{sc}} : H_\epsilon \to H_\epsilon$ considered as an essentially self-adjoint operator in the Hilbert space $H_\epsilon$ has a discrete spectrum.

Estimate on the number of eigenvectors. We need some more notation. Given a self-adjoint operator $X$ in a Hilbert space $H$, a number $\lambda \in \mathbb{R}$ and inequality relation

$$\iota = >, <, \leq, \geq$$

we will denote by

$$\mathcal{P}_{X\iota\lambda} : H \to H$$

the spectral projector of $X$ onto the interval $\{ x \in \mathbb{R} : x \iota \lambda \}$. The image will be denoted as follows

$$H_{X\iota\lambda} := \mathcal{P}_{X\iota\lambda}(H).$$

We are in the position to state our next result:
Proposition 4  For each of the subspaces $\mathcal{H}_e$, the restriction operator $\hat{C}_{\text{sc}}^{\text{grav}} : \mathcal{H}_e \to \mathcal{H}_e$ considered as an essentially self-adjoint operator in the Hilbert space $\mathcal{H}_e$ satisfies:

$$\dim \mathcal{H}_{C_{\text{sc}}>E} \leq \dim \mathcal{H}_{\tilde{A}>E} \leq \dim \mathcal{H}_{A'>E}$$

for arbitrary $E > 0$ where

$$A' := -\frac{3e_0^2}{32\pi G \sqrt{|p|}} : \mathcal{H}_e \to \mathcal{H}_e$$

$$\tilde{A} := -\frac{3e_0^2}{8\pi G \gamma^2} \left( -\sqrt{|p|} \frac{\sin^2(\frac{1}{2} \mu)}{\mu^2} + \frac{1}{4}(1 + \gamma^2)\sqrt{|p|} \right) : \mathcal{H}_e \to \mathcal{H}_e,$$

(4.8)

(4.9)

Since the operators $A'$ and $\tilde{A}$ act just by multiplication by functions (see (3.20), the numbers $\dim \mathcal{H}_{C_{\text{sc}}>E} \leq \dim \mathcal{H}_{A'>E}$ can be calculated in a straightforward way, for each $E > 0$.

**Proof of Proposition 1** To show the essentially self-adjointness of the operator $\hat{C}_{\text{sc}}^{\text{grav}}$ on the domain $\mathcal{D}$, we use (4.2) to present the operator in the form

$$\hat{C}_{\text{sc}}^{\text{grav}} = -\frac{3e_0^2}{32\pi G \gamma^2} (C_0 + H_1),$$

(4.10)

where the function operator $C_0$ is the same as in (4.2). Obviously, the operator $C_0$ is essentially self-adjoint in the domain $\mathcal{D}$. Therefore, it would be enough to show that

$$\|H_1\Psi\|^2 \leq \|C_0\Psi\|^2 + \beta\|\Psi\|^2,$$

(4.11)

for some constant $\beta$ and every $\Psi \in \mathcal{D}$, to conclude that also the operator $C_0 + H_1$ is essentially self-adjoint ([16] V.4.6). Applying the inequality

$$\|v + w\|^2 \leq 2\|v\|^2 + 2\|w\|^2,$$

true for arbitrary pair of vectors, elements of a Hilbert space, one can check that

$$\|H_1\Psi\|^2 = \|(U_4C_4 + U_-C_{-4})\Psi\|^2 \leq \langle \Psi, 2(|C_4|^2 + |C_{-4}|^2)\Psi \rangle,$$

(4.12)

where the function $|C_4|^2 + |C_{-4}|^2$ acts by multiplication as an operator in $\mathcal{D}$. However, it follows from (3.20, 3.10), that the function $(A(\nu)/\mu^2(\nu))$ is linear in $\nu$ for $\nu \leq -1$ as well as for $\nu \geq 1$. Therefore it is easy to see, that

$$C_0^2(\nu) = 2(|C_4|^2 + |C_{-4}|^2)(\nu) + f_0 + f_1(\nu) + f_2(\nu)$$
where \( f_0 \) is a constant, \( f_1 \) is some function of a compact support, and \( f_2(\nu) \geq 0 \) for every \( \nu \). That form is sufficient to conclude the condition (4.11).

**Proof of Proposition 2** Let us remind that \( \hat{C}_{\text{grav}}^{\text{sc}} \) is essentially self-adjoint on the domain \( \mathcal{D} \). For every \( \Psi_0 \in \mathcal{D}(\hat{C}_{\text{grav}}^{\text{sc}}) \), there exist sequence \( \Psi_n \in \mathcal{D} \),

\[
\Psi_n \to \Psi_0, \quad \hat{C}_{\text{grav}}^{\text{sc}} \Psi_n \to \hat{C}_{\text{grav}}^{\text{sc}} \Psi_0. \tag{4.13}
\]

**Remark:** This kind of convergence is referred as convergence in the graph norm

\[
\|\Psi\|^2_B = \|B\Psi\|^2 + \|\Psi\|^2, \quad \Psi \in \mathcal{D}(B),
\]

for operator \( B \).

Suppose \( \hat{C}_{\text{grav}}^{\text{sc}} \Psi_0 = 0 \). We will prove that the only possibility could be

\[
\Psi_0 = b|0), \quad b \in \mathbb{C}.
\]

But that possibility is ruled out by checking by inspection, that

\[
(0|\hat{C}_{\text{grav}}^{\text{sc}}|0) < 0.
\]

Let \( \Psi_n \in \mathcal{D}, n = 1, \ldots, \infty \) be a sequence such that (4.13). We have:

\[
\langle \Psi_n|\hat{C}_{\text{grav}}^{\text{sc}} \Psi_n \rangle \leq -\frac{3\ell_0^2}{32\pi G} \langle \Psi_n|\sqrt{|p|}\Psi_n \rangle \leq 0
\]

and we know that

\[
\lim_{n \to \infty} \langle \Psi_n|\hat{C}_{\text{grav}}^{\text{sc}} \Psi_n \rangle = 0.
\]

The comparison implies

\[
0 = \lim_{n \to \infty} \langle \Psi_n|\sqrt{|p|}\Psi_n \rangle = \lim_{n \to \infty} \|\sqrt{|p|}\Psi_n\|^2.
\]

The second equality, due to the closedness of the domain of \( \sqrt{|p|} \), implies that the vector \( \Psi_0 \) belongs to the domain of the self-adjoint extension, \( \Psi_0 \in \mathcal{D}(\sqrt{|p|}) \). In the consequence,

\[
\sqrt{|p|}\Psi_0 = 0.
\]

The only solution is

\[
\Psi_0 = b|0).
\]

That completes the proof.
Proofs of Proposition 3 and 4 The propositions are a consequence of a single lemma, we formulate and prove below (compare [19] XIII).

Lemma Let \((A, \mathcal{D}(A))\) and \((B, \mathcal{D}(B))\) be operators in a Hilbert space \(\mathcal{H}\) with their domains and \(\mathcal{D} \subset \mathcal{D}(A) \cap \mathcal{D}(B)\) be a dense subspace of \(\mathcal{H}\). Suppose the following conditions are satisfied:

- On the domain \(\mathcal{D}\) the following inequality holds
  \[ 0 \leq A \leq B, \]
- The operator \(B\) is essentially self-adjoint in \(\mathcal{D}\),
- \(A\), as an operator defined in \(\mathcal{D}(A)\), is self adjoint, positive and has discrete spectrum.

Then \(B\) is also positive and has discrete spectrum. Moreover, the following inequality holds for arbitrary \(\lambda \geq 0\)

\[ \dim \mathcal{H}_{B < \lambda} \leq \dim \mathcal{H}_{A \leq \lambda}. \]

Proof. Fix arbitrary \(\lambda \geq 0\) and the corresponding Hilbert space \(\mathcal{H}_{B < \lambda}\). Given any \(\delta > 0\) consider the projection

\[ P_{A \leq \lambda + \delta} : \mathcal{H}_{B < \lambda} \to \mathcal{H}_{A \leq \lambda + \delta}. \quad (4.14) \]

We will show that its kernel is trivial. Then it is clear that

\[ \dim \mathcal{H}_{B < \lambda} \leq \dim \mathcal{H}_{A \leq \lambda + \delta}. \]

Suppose \(\Psi_0\) belongs to the kernel of \((4.14)\) and \(\|\Psi_0\| = 1\). All the subspace \(\mathcal{H}_{B < \lambda}\) is contained in the domain of the self-adjoint extension of \(B\), that acts on this subspace as a bounded operator \(BP_{0 < B \leq \lambda - \delta}\) with the upper bound \(\lambda\). Hence,

\[ \lambda \geq \langle \Psi_0 | B \Psi_0 \rangle. \]

\(\Psi_0\) may not be in the common domain \(\mathcal{D}\). However, since \(B\) is essentially self-adjoint on \(\mathcal{D}\), there is a sequence of vectors \(\Psi_n\),

\[ \Psi_n \to \Psi_0, \quad B \Psi_n \to B \Psi_0 \quad \Psi_n \in \mathcal{D}. \]

It follows that

\[ \langle \Psi_n | B \Psi_n \rangle \to \langle \Psi_0 | B \Psi_0 \rangle, \quad \text{as} \quad n \to \infty. \]

\(^{4}\)We thank Jan Dereźniński for an important suggestion.
On the other hand, for every \( n \), we have
\[
\langle \Psi_n | B \Psi_n \rangle \geq \langle \Psi_n | A \Psi_n \rangle.
\]
On the right hand side of the inequality decompose
\[
\Psi_n = \Psi_n^{\leq} + \Psi_n^{>}, \quad \Psi_n^{\leq} \in \mathcal{H}_{A \leq \lambda + \delta}, \quad \Psi_n^{>} \in \mathcal{H}_{A > \lambda + \delta}.
\]
Note, that each \( \Psi_n^{\leq} \) belongs to the domain of the self-adjoint extension of \( A \), therefore so does \( \Psi_n^{>} \). Now,
\[
\langle \Psi_n | B \Psi_n \rangle \geq \langle \Psi_n^{\leq} | A \Psi_n^{\leq} \rangle + \langle \Psi_n^{>} | A \Psi_n^{>} \rangle \geq (\lambda + \delta) \| \Psi_n^{>} \|^2 \quad (4.15)
\]
But \( \| \Psi_n^{>} \|^2 \to 1 \) as \( n \to \infty \), since \( \Psi_0 = \Psi_0^{>} \), \( \| \Psi_0 \| = 1 \) and projection \( P_{A > \lambda + \delta} \) is continuous. Finally, taking the limit of (4.15) we find
\[
\lambda \geq \lim_{n \to \infty} \langle \Psi_n | B \Psi_n \rangle \geq \lambda + \delta.
\]
The contradiction shows the kernel of projection (4.14) is empty. However, since the spectrum of \( A \) is discrete, we have
\[
\mathcal{H}_{A \leq \lambda + \delta} = \mathcal{H}_{A \leq \lambda}
\]
for sufficiently small \( \delta > 0 \). That completes the proof of the inequality (4).

Proposition 2 follows from Lemma by fixing arbitrary \( \epsilon \in \mathbb{R} \) and defining \( \mathcal{H} \) to be the corresponding Hilbert space \( \mathcal{H}_\epsilon \) and the subspace \( \mathcal{D} := \mathcal{H}_\epsilon \), the domain of the following operators
\[
A = \sqrt{|p|} : \mathcal{H}_\epsilon \to \mathcal{H}_\epsilon \quad (4.16)
\]
\[
B = -\frac{32\pi}{3H_0^2} \hat{C}_{\text{grav}}^{\text{sc}} : \mathcal{H}_\epsilon \to \mathcal{H}_\epsilon. \quad (4.17)
\]
A similar choice implies Proposition 4.

5  Closing remarks and local difference between the SU(2) and SO(3) universes.

Perhaps we should explain what exactly the initial technical difficulty in the LQC closed FRW model was, that we solved in this paper. In the previous attempts, the loop used to replace the curvature component was somewhat complicated, that made calculations
unmanageable. We have found a neat analog of a flat square in SU(2), that fits the framework much better (see the contour $S^\mu_{ij}$ in Section 3.2).

We fixed a background metric tensor in SU(2) (SO(3)) corresponding to the sphere of radius 2. Unlike in the flat case, that metric can be naturally distinguished. It provides the fiducial volume $V_0$ and the parameter $\ell_0 = V_0^{1/3}$ present in our formulae. The value of $\ell_0$ labels the SU(2) and the SO(3) cases. Note, that the “quantum of volume” $4V_1$ (3.17) defined by the quantum hamiltonian evolution is $\ell_0$ invariant. So one could think that the two cases are locally not distinguishable by the quantum evolution. That conclusion would not be correct. In fact, a given eigenvalue of the volume operator (or, equivalently, the operator $\hat{p}$ (2.17) defines on SU(2) the geometry locally different than the geometry it defines on SO(3). Therefore the jump $4V_1$ in the total volume has different meanings from the point of view of the local geometry on SU(2) or SO(3) respectively. That shows, that even making local observations of the geometry of our universe we should be able to tell between the two cases.

The fact that 0 is not an element of the spectrum of the scalar constraint operator of the gravitational field for any of the super selection sectors, means that there are no quantum vacuum solutions (without the cosmological constant). That is in the agreement with the classical theory.

Our results on the properties of the gravitational field scalar constraint operator give a good insight for the analysis of the solutions of the full quantum scalar constraint of the gravitational field coupled with the matter, with or without the cosmological constant. We will demonstrate it in the coming paper.

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