A GENERAL CONSISTENT BGK MODEL FOR GAS MIXTURES

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Abstract. We propose a kinetic model of BGK type for a gas mixture of an arbitrary number of species with arbitrary collision law. The model features the same structure of the corresponding Boltzmann equations and fulfils all consistency requirements concerning conservation laws, equilibria, and H–theorem. Comparison is made to existing BGK models for mixtures, and the achieved improvements are commented on. Finally, possible application to the case of Coulomb interaction is briefly discussed.

1. Introduction. Kinetic models are very important for applications because of the complexity of nonlinear Boltzmann type kinetic equations. The most famous BGK–model of the Boltzmann equation introduced in [2] in 1954 was used since then in hundreds of important publications on rarefied gas dynamics and plasma physics. In applications, especially to plasma physics, we often have to deal not with a simple gas, but with a mixture of gases. At the first sight it may seem surprising that there is no obvious generalization of the BGK–model to the case of mixtures. We analyse this question below and discuss possible reasons. In some sense there are too many free parameters in the model even if we confine ourselves to the most natural class of BGK models for mixtures. Models of that class were first introduced in 1956 [14] and since then were modified and improved in many papers published by physicists mainly in 1960–70s (see e.g. [25, 24, 17, 10, 11]). A more mathematical approach to constructing such models begins probably in 2002 with the paper [1], where the questions of consistency of the model and its generality were put in the center of attention. The consistency means three basic properties: conservation laws, H–theorem and uniqueness of equilibrium solution. By generality we mean the validity

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of the model for any number of components of the mixture. The model proposed in [1] had all these properties, but also had some disadvantages. In particular, it was based on the assumption that the structure of each equation of the model must be exactly the same as the structure of BGK model for simple gas, i.e. with only one Maxwellian attractor. Thus the structure of that model differs from the one for Boltzmann equation. An important advantage of the model from [1] is that it gives exact equations of transfer of momentum and energy for Maxwell molecules (no other molecular models were considered in that paper). Here and below the words “Maxwell molecules” are used for the whole class of pseudo-Maxwell “particles”. The second consistent general BGK–model was published very recently [15, 16] with the aim of plasma applications. This interesting paper contains an exhaustive list of references on the subject on BGK models for mixtures. The model proposed there deals with an arbitrary number of (not necessarily Maxwell type) gases. It is also based on some assumptions on the parameters which are not entirely motivated by physics, though following a mathematical scheme for relaxation approximations of Boltzmann operators. This model is described in detail in [15]. It is not quite clear if it can give simultaneously exact equations of transfer of momentum and energy, but the authors propose some reasonable approximations of these equations.

To our knowledge, the two models mentioned above are the only consistent general BGK–models available for mixtures. For brevity we do not discuss here an interesting recent paper [18] for binary mixtures (see Section 5 below for that) and many important papers on more general kinetic models (with Maxwellians multiplied by polynomials, ellipsoidal attractors, etc.) [9, 7, 5]. Similarly, we do not discuss either relaxation models which have been proposed for the description of chemically reacting mixtures and relevant applications [13, 20, 12, 4], or for polyatomic gases [6, 3]. The main aim of our paper is to introduce an alternative consistent general model which is based on quite natural assumptions and hopefully will be useful for applications.

The paper is organized as follows. Section 2 is devoted to the statement of the problem. The formal description of the model is given in Section 3 (Proposition 1). The proof of consistency of the model is presented in Section 4, where main properties of the model are collected in Proposition 2. Section 5 is devoted to comparison with other published BGK–models for mixtures. We also discuss there the “indifferentiability” property and show that it can be fulfilled only in models of this class that do not allow collisions of identical particles (Proposition 3). Finally in Section 6 we discuss how to choose in reasonable way the free parameters (namely, collision frequencies) of our model. In this section main attention is paid to the non–Maxwell case, in particular to the Coulomb interactions.

2. Statement of the problem. We consider a mixture of rarefied gases described by Boltzmann–type kinetic equations for distribution functions \( \{f_i(x,v,t), \ i = 1,\ldots,N\} \), where \( x \in \mathbb{R}^3, v \in \mathbb{R}^3, t \in \mathbb{R}_+ \) denote respectively position, velocity and time variables. It is assumed that particles of \( i \)th sort have a mass \( m_i > 0 \) and interact with particles of \( k \)th sort with given potential \( U_{ik}(r) \), where \( r > 0 \) is a distance between interacting particles. The equations read

\[
D_t f_i = \left( \partial_t + v \cdot \partial_x + a_i \cdot \partial_v \right) f_i = Q_i, \quad i = 1,\ldots,N,
\]

where \( a_i(x,t) \in \mathbb{R}^3 \) denotes the acceleration caused by external or/and self–consistent force, \( Q_i \) denotes the collision term. We shall use below the Boltzmann collision
integrate for $Q_i$. Then the Landau integral [21] for Coulomb potential can be considered as the well-known limiting case. We obtain

$$Q_i = \sum_{k=1}^{N} Q_{ik}(f_i, f_k),$$

$$Q_{ik}(f_i, f_k) = \int_{\mathbb{R}^3 \times S^2} dw \ d\omega \ g_{ik}(|y|, \hat{y} \cdot \omega) \left[ f_i(v') f_k(w') - f_i(v) f_k(w) \right],$$

$$y = v - w, \quad \hat{y} = \frac{y}{|y|}, \quad \omega \in S^2,$n

$$v' = \mathbf{Y} + \frac{m_{ik}}{m_i} |y| \omega, \quad w' = \mathbf{Y} - \frac{m_{ik}}{m_k} |y| \omega,$$

$$\mathbf{Y} = \frac{m_i v + m_k w}{m_i + m_k}, \quad m_{ik} = \frac{m_i m_k}{m_i + m_k}; \quad i, k = 1, \ldots, N,$$

where irrelevant variables $(x, t)$ are omitted. Functions $g_{ik}(|y|, \mu), \mu \in [-1, 1]$, are considered as given non-negative functions. They depend also on reduced mass $m_{ik}$ and on the potential $U_{ik}(r)$. In particular, if

$$U_{ik}(r) = \frac{\alpha_{ik}}{r^p}, \quad p > 1, \quad \alpha_{ik} = \text{const},$$

then (see e.g. [19])

$$g_{ik}(|y|, \mu) = \left[ \frac{\alpha_{ik}}{m_{ik}} \right]^\frac{p}{2} |y|^{1 - \frac{p}{2}} g_p(\mu).$$

If particles of $i$th sort are hard spheres with radius $R_i$, then

$$g_{ik}(|y|, \mu) = \frac{1}{4} (R_i + R_k)^2 |y|.$$

We introduce the standard notations for partial densities $n_i$, average velocities $u_i$ and absolute temperatures $T_i$

$$n_i = \{f_i, 1\}, \quad n_i u_i = \{f_i, v\}, \quad 3 n_i T_i = m_i \{f_i, |v - u_i|^2\}; \quad i = 1, \ldots, N,$$

where

$$\{f, h\} = \int_{\mathbb{R}^3} dv \ f(v) h(v)$$

provided the integral is convergent. The Maxwell distribution function of particles with mass $m_i$ will be denoted by

$$f_i^{(M)} = n_i M_i, \quad M_i = M(\mathbf{v}, u_i, \frac{T_i}{m_i}),$$

where

$$M(\mathbf{v}; u, a) = (2\pi a)^{-3/2} \exp \left[ -\frac{1}{2a} |v - u|^2 \right], \quad u \in \mathbb{R}^3, \quad a > 0.$$

Note that $\{M, 1\} = 1$ independently of $u$ and $a$.

Basic properties of Eqs. (1)–(3) are: (A) conservation laws; (B) $H$–theorem; (C) uniqueness of equilibrium solutions. These properties are related to the following properties of collision terms $Q_i$:

(A) conservation laws:

$$\{Q_i, 1\} = 0, \quad i = 1, \ldots, N; \quad \sum_{i=1}^{N} n_i \{Q_i, v\} = 0; \quad \sum_{i=1}^{N} m_i \{Q_i, |v|^2\} = 0; \quad (10)$$
(B) $H$–theorem:

$$\sum_{i=1}^{N} (Q_i, \log f_i) \leq 0;$$

(C) uniqueness of equilibrium solutions:

$$Q_i = 0, \quad i = 1, \ldots, N \quad \Rightarrow \quad f_i = n_i \left( \mathbf{v} \cdot \mathbf{u}_i \frac{T_i}{m_i} \right), \quad i = 1, \ldots, N,$$

where $n_i$, $\mathbf{u}_i$, $T_i$ do not depend on $\mathbf{v}$.

A more detailed discussion of these properties can be found in any textbook on the Boltzmann equation (see e.g. [19], [8]).

The famous BGK model [2] of Eqs. (1)–(2) with $N = 1$ (simple gas) was introduced in 1954 in the form (1), where $N = 1$ and the collision term $Q_i$ is replaced by its model version

$$\bar{Q}_1 = \frac{n_1}{\sigma} \left( n_1 M_1 - f_1 \right)$$

in thenotations of Eqs. (8) with some velocity–independent positive parameter $\sigma$. In fact the authors of [2] used a simplified version of the Maxwellian $M_1$ with constant $T_1$ and $\mathbf{u}_1 = 0$ (see Eqs. (8), (9)). A similar model was independently introduced by Welander [26].

Obviously the main advantage of the BGK model is its simplicity. On the other hand all three basic properties (A), (B), (C) of the collision integral $Q_1$ are satisfied also for its model version $\bar{Q}_1$. Therefore we expect that the model kinetic equation can show a qualitatively correct behaviour of solutions to many physically interesting problems. This was confirmed by numerous publications (see e.g. [19], [8] and references there).

The situation with BGK–type models for mixtures is more complicated. There are relatively many publications on that subject, which propose different models. In particular, Gross and Krook [14] introduced in 1956 the following class of model collision terms $\bar{Q}_i$ in Eqs. (1):

$$\bar{Q}_i = \sum_{k=1}^{N} \bar{Q}_{ik}, \quad \bar{Q}_{ik} = \nu_{ik} \left( n_{ik} M_{ik} - f_i \right),$$

where

$$M_{ik} = M \left( \mathbf{v} \cdot \mathbf{u}_{ik} \frac{T_{ik}}{m_{ik}} \right); \quad i, k = 1, \ldots, N.$$  

Only the simplest case $N = 2$ was considered in [14] with a discussion of possible choice of free parameters $\{\nu_{ik}, n_{ik}, \mathbf{u}_{ik}, T_{ik}; \ i, k = 1, \ldots, N\}$. The generalization of the class of models to arbitrary $N$ is obvious, see Eqs. (14), (15). However, the choice of free parameters is far from obvious. Below we confine ourselves to the class of models described in Eqs. (14), (15). Then we have in realistic 3d case $6N^2$ free parameters and only $(N + 4)$ conservation laws. The number of free parameters can be slightly reduced under natural assumption that

$$n_{ii} = n_i, \quad \mathbf{u}_{ii} = \mathbf{u}_i, \quad T_{ii} = T_i, \quad i = 1, \ldots, N.$$

This means that collisions of identical particles are described by the usual BGK model. Still we have $(6N^2 - 5N)$ free parameters combined with $(N + 4)$ conservation laws for $N \geq 2$. Therefore it is not surprising that different authors obtained different models based on (sometimes artificial) extra conditions.
Note that the number of exact identities for collision terms \( Q_{ik}(f_i, f_k) \) is in fact greater than the number of conservation laws (10). Indeed we have for any pair \( i \neq k \):

\[
\langle Q_{ik}(f_i, f_k), 1 \rangle = 0, \quad (17)
\]

\[
m_i \langle Q_{ik}(f_i, f_k), v \rangle + m_k \langle Q_{ki}(f_k, f_i), v \rangle = 0, \quad (18)
\]

It is natural to impose similar conditions on model collision terms \( \tilde{Q}_{ik} \) in Eqs. (14). Then, in addition to (16), we obtain from Eq. (17):

\[
n_{ik} = n_i; \quad i, k = 1, \ldots, N. \quad (19)
\]

Hence the number \( K(N) \) of remaining free parameters \( \{\nu_{ik}, u_{ik}, T_{ik}\} \) and the number \( L(N) \) of Eqs. (18) are given by equalities

\[
K(N) = N(5N - 4), \quad L(N) = 2N(N - 1). \quad (20)
\]

We propose in the next section a model based on quite natural assumptions and prove in Section 4 that this model has all three basic properties of Eqs. (1)–(3).

3. **Construction of the model.** We fix any pair of indices \( i \neq k \) and consider collision terms \( Q_{ik}(f_i, f_k) \) and \( \tilde{Q}_{ik} \) given respectively in Eqs. (3) and (14). It is natural to assume that

\[
\langle Q_{ik} - \tilde{Q}_{ik}, 1 \rangle = \langle Q_{ik} - \tilde{Q}_{ik}, |v|^2 \rangle = 0, \quad (Q_{ik} - \tilde{Q}_{ik}, v) = 0. \quad (21)
\]

Of course equations (21) actually provide all the constraints that are needed, and the ones mentioned above in (17) and (18) will be satisfied as a consequence. Note that these equalities are automatically satisfied for \( i = k \) under conditions (16). We obtain from (14), (15)

\[
n_{ik} = n_i, \quad \nu_{ik} n_i \left( \frac{3 T_{ik} - T_i}{m_i} + |u_{ik}|^2 - |u_i|^2 \right) = \langle Q_{ik}, |v|^2 \rangle, \quad (22)
\]

Moments of \( Q_{ik} \) are computed in Appendix A. We obtain (see Eqs. (3))

\[
\langle Q_{ik}, v \rangle = - \frac{m_k}{m_i + m_k} \langle v \rangle_{ik},
\]

\[
\langle Q_{ik}, |v|^2 \rangle = - \frac{2 m_k}{(m_i + m_k)^2} \langle (m_i v + m_k w) \cdot (v - w) \rangle_{ik}, \quad (23)
\]

where

\[
\langle \psi(v, w) \rangle_{ik} = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} dv \, dw \, f_i(v) \, f_k(w) \, g^{(1)}_{ik}(|v-w|) \, \psi(v, w),
\]

\[
g^{(1)}_{ik}(|y|) = 2 \pi \int_{-1}^{1} d\mu \, (1 - \mu) \, g_{ik}(|y|, \mu), \quad y \in \mathbb{R}^3. \quad (24)
\]

We need to distinguish two different cases: (a) Maxwell molecules and (b) general case. Let us begin with the case (a). Then \( g_{ik}(|y|, \mu) \) is independent of \( |y| \). Hence,

\[
g^{(1)}_{ik} = \lambda_{ik} = \text{const}. \quad (25)
\]

We note that Eqs. (6) imply that

\[
\langle f_i, v \rangle = n_i u_i, \quad \langle f_i, |v|^2 \rangle = n_i \left( \frac{3 T_i}{m_i} + |u_i|^2 \right).
\]
Therefore
\[
\langle Q_{ik}, v \rangle = -\frac{m_k}{m_i + m_k} \lambda_{ik} n_i n_k (u_i - u_k)
\]
\[
\langle Q_{ik}, |v|^2 \rangle = -\frac{2m_k}{(m_i + m_k)^2} \lambda_{ik} n_i n_k \left[ 3(T_i - T_k) + (m_i u_i + m_k u_k) \cdot (u_i - u_k) \right].
\]
Substituting these formulas into Eqs. (21), we obtain after simple calculations
\[
\begin{align*}
\mathbf{u}_{ik} &= (1 - a_{ik}) \mathbf{u}_i + a_{ik} \mathbf{u}_k, \\
T_{ik} &= (1 - b_{ik}) T_i + b_{ik} T_k + \gamma_{ik} |u_i - u_k|^2,
\end{align*}
\]
where
\[
\begin{align*}
a_{ik} &= \frac{\lambda_{ik} n_k m_k}{\nu_i (m_i + m_k)}, & b_{ik} &= \frac{2a_{ik} m_i}{m_i + m_k}, \\
\gamma_{ik} &= \frac{m_i a_{ik}}{3} \left( \frac{2m_k}{m_i + m_k} - a_{ik} \right); \\
i, k &= 1, \ldots, N.
\end{align*}
\]
Eqs. (26), (27) define uniquely the parameters \{\mathbf{u}_{ik}, T_{ik}; i, k = 1, \ldots, N\} provided that free parameters \{\nu_{ik} > 0; i, k = 1, \ldots, N\} are given. This completes the construction of the model in the case (a) of Maxwell molecules.

What can be done in the general case (b)? Then the function \(g_{ik}^{(3)}(|v - w|)\) in Eqs. (24) does not allow to express integrals (23) in terms of \{\nu_i, \mathbf{u}_i, T_i; n_k, \mathbf{u}_k, T_k\}. On the other hand, the BGK model is a rather rough approximation of the Boltzmann equation. Therefore we can approximately replace \(g_{ik}^{(3)}(|v - w|)\) in the first Eq. (24) by its value in some typical point. In particular, there is a formula for the average square of relative velocity:
\[
\overline{|v_i - v_k|^2} = \frac{1}{n_i n_k} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} dv dw f_i(v) f_k(w) |v - w|^2 = 3 \left( \frac{T_i}{m_i} + \frac{T_k}{m_k} \right) + |u_i - u_k|^2.
\]
Then we roughly estimate integrals (23) by approximate formula
\[
\langle \psi(v, w) \rangle_{ik} \equiv \lambda_{ik} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} dv dw f_i(v) f_k(w) \psi(v, w),
\]
where
\[
\lambda_{ik} = g_{ik}^{(1)}(z_{ik}), \quad z_{ik} = \left[ 3 \left( \frac{T_i}{m_i} + \frac{T_k}{m_k} \right) + |u_i - u_k|^2 \right]^{1/2}
\]
in the notation of Eqs. (24). This approach obviously leads to exactly the same formulas (26), (27), with \(\lambda_{ik}\) given in Eqs. (29). Note that \(\lambda_{ik} = \lambda_{ki}\) in the general case, since \(g_{ik}(|y|, \mu) = g_{ki}(|y|, \mu)\) in Eqs. (3) and \(z_{ik} = z_{ki}\).

Hence, we obtain the following result for our model kinetic equations for mixtures. To simplify the presentation we formulate the result for the spatially homogeneous case (the generalization is straightforward).

**Proposition 1.** Under natural assumptions (16), (21) we obtain in the case (a) of Maxwell molecules the following model equations:
\[
\begin{align*}
\partial_t f_i(v, t) &= \mathcal{Q}_i = \sum_{k=1}^N \mathcal{Q}_{ik}, \\
\mathcal{Q}_{ik} &= \nu_{ik} \left[ n_i M_{ik} - f_i(v, t) \right], \\
M_{ik} &= M \left( \mathbf{v}; \mathbf{u}_i, \frac{T_{ik}}{m_i} \right); \\
i, k &= 1, \ldots, N,
\end{align*}
\]
where \(\nu_{ik} > 0\) are free parameters; parameters \(\mathbf{u}_i, T_{ik}\) of the Maxwellian \(M_{ik}\) are given in Eqs. (25), (26), (27). The model equations (30) can be also used for
the general case (b) of non–Maxwell molecules provided the parameters $u_{ik}$ and $T_{ik}$ are given in Eqs. (29), (26), (27).

We shall study main properties of these model equations in the next section.

4. Proof of consistency. Let us consider Eqs. (30), where $u_{ik}$ and $T_{ik}$ are given in Eqs. (26), (27). We assume that $\lambda_{ik} > 0$ in (27) is an arbitrary function of $T_i, T_k,$ and $|u_i - u_k|$ such that

$$\lambda_{ik}(T_i, T_k; |u_i - u_k|) = \lambda_{ki}(T_k, T_i; |u_k - u_i|), \quad i, k = 1, \ldots, N, \quad i \neq k.$$  \hspace{1cm} (31)

This assumption is obviously satisfied for functions $\lambda_{ik}$ from Eqs. (25), which are mentioned in Proposition 1. Our goal is to prove that Eqs. (30) have all main properties (10)-(12) of the Boltzmann-type kinetic equations.

First of all we need to check that all temperatures $T_{ik}$ from Eqs. (26) are positive. This condition is fulfilled by sufficiently large values of free parameters $\nu_{ik}$, as we shall see below.

Lemma 4.1. For any $i, k = 1, \ldots, N$, $i \neq k$, $T_{ik}$ from Eqs. (26) are strictly positive provided

$$T_i > 0, \quad T_k > 0, \quad \nu_{ik} \geq \frac{1}{2} \lambda_{ik} n_k.$$ \hspace{1cm} (32)

Proof. It is clear from Eqs. (26), (27) that the following two conditions are needed for any pair $(i, k), i \neq k$:

$$b_{ik} = \frac{2m_ia_{ik}}{m_i + m_k} \leq 1, \quad \frac{(m_i + m_k)a_{ik}}{2m_k} \leq 1.$$ \hspace{1cm} (33)

We denote $A_{ik} = \lambda_{ik} n_k / \nu_{ik}$ and obtain

$$2m_i m_k \frac{A_{ik}}{(m_i + m_k)^2} \leq 1, \quad \frac{1}{2} A_{ik} \leq 1.$$ \hspace{1cm} (34)

Hence the only essential condition is $A_{ik} \leq 2$ and this completes the proof. \hspace{1cm} \square

In fact it was proved that the conditions of Lemma 4.1 imply a stronger inequality $T_{ik} \geq \min(T_i, T_k)$, with the equality sign for $i = k$. Note that Eqs. (30) have an obvious probabilistic interpretation. They preserve positivity of $f_i(v, t), 1 \leq i \leq N,$ and satisfy conservation laws

$$n_i(t) = \text{const.}, \quad i = 1, \ldots, N, \quad \sum_{i=1}^{N} n_i m_i u_i(t) = \text{const.},$$

$$\sum_{i=1}^{N} n_i [3T_i(t) + m_i |u_i(t)|^2] = \text{const.}$$  \hspace{1cm} (35)

The conservation laws for momentum and energy follow from equations

$$\partial_t \sum_{i=1}^{N} (f_i, \psi_i(v)) = \sum_{i,k=1}^{N} \langle \tilde{Q}_{ik}, \psi_i(v) \rangle = 0$$ \hspace{1cm} (36)

with $\psi_i = m_i v$ and $\psi_i = m_i |v|^2$, $1 \leq i \leq N$. Indeed, by construction (see Sect. 3) of the model we obtain

$$\langle \tilde{Q}_{ik}, m_i v \rangle = -\frac{m_i m_k}{(m_i + m_k)} \lambda_{ik} n_i n_k (u_i - u_k); \quad i, k = 1, \ldots, N,$$

$$\langle \tilde{Q}_{ik}, m_i |v|^2 \rangle = -\frac{2m_i m_k}{(m_i + m_k)^2} \lambda_{ik} n_i n_k [3(T_i - T_k) + (m_i u_i + m_k u_k) \cdot (u_i - u_k)].$$ \hspace{1cm} (37)
It follows from conditions (31) that these scalar products are anti-symmetric with respect to indices \((i, k)\). Hence, we obtain Eqs. (34) for momentum and energy of the mixture. The conservation of mass is obvious, since \(\langle Q_{ik}, 1 \rangle = 0\) for any pair \((i, k)\). Hence, conservation laws (33) are proved under conditions (31).

It remains to prove H–theorem. The Boltzmann H–functional reads

\[
H^{(N)}(f_1, \ldots, f_N) = \sum_{i=1}^{N} H(f_i), \quad H(f) = \{ f, \log f \}.
\]

Then we obtain Eqs. (30)

\[
\partial_t H^{(N)} = \sum_{i,k=1}^{N} (\tilde{Q}_{ik}, \log f_i) = \sum_{i,k=1}^{N} \nu_{ik} \{ n_i M_{ik} - f_i, \log f_i \}. \tag{37}
\]

Our goal is to prove that \(\partial_t H^{(N)} \leq 0\). First, we use elementary inequality

\[
(y - x) \log x \leq y (\log y - 1) - x (\log x - 1), \quad x > 0, \quad y > 0.
\]

Since \(\langle M_{ik}, 1 \rangle = 1\) and \(\langle f_i, 1 \rangle = n_i\), we obtain

\[
\partial_t H^{(N)} \leq \sum_{i,k=1}^{N} \nu_{ik} \left[ H(n_i M_{ik}) - H(f_i) \right].
\]

Then we use the classical inequality

\[
H(f_i) \geq H(n_i M_{ii}), \quad M_{ii} = M_i = M \left( \mathbf{v} ; \mathbf{v}_i, \frac{T_i}{m_i} \right),
\]

and obtain

\[
\partial_t H^{(N)} \leq \sum_{i,k=1}^{N} \nu_{ik} \left[ H(n_i M_{ik}) - H(n_i M_i) \right] = -\frac{3}{2} \sum_{i,k=1}^{N} \nu_{ik} n_i \log \frac{T_{ik}}{T_i}
\]

where

\[
T_{ik} \geq (1 - b_{ik}) T_i + b_{ik} T_k
\]

in the notation of Eqs. (26), (27). It follows from convexity of \((- \log x)\) that

\[
\log[(1 - b)x + by] \geq (1 - b) \log x + b \log y, \quad 0 \leq b \leq 1, \quad x > 0, \quad y > 0.
\]

Therefore

\[
\log \frac{T_{ik}}{T_i} \geq b_{ik} \log \frac{T_k}{T_i},
\]

since \(0 < b_{ik} \leq 1\) under conditions of Lemma 4.1. Hence, we obtain

\[
\partial_t H^{(N)} \leq -\frac{3}{2} \sum_{i,k=1}^{N} \nu_{ik} n_i b_{ik} \log \frac{T_k}{T_i}.
\]

On the other hand, it follows from Eqs. (27) that

\[
c_{ik} = \nu_{ik} n_i b_{ik} = 2 \lambda_{ik} n_i n_k \frac{m_i m_k}{(m_i + m_k)^2} = c_{ki}; \quad i, k = 1, \ldots, N.
\]

Therefore

\[
\partial_t H^{(N)} \leq -\frac{3}{2} \sum_{i,k=1}^{N} c_{ik} \left( \log T_k - \log T_i \right) = 0.
\]

This completes the proof of H–theorem.
Hence, the solution of model equations (30) satisfies conservation laws and H–theorem provided conditions (31), (32) are fulfilled. Moreover it follows from Eqs. (35) that equations of transfer for momentum and energy have the following form:

\[
m_i n_i \frac{d}{dt} \mathbf{u}_i = \sum_{k=1}^{N} C_{ik} (\mathbf{u}_k - \mathbf{u}_i), \quad i = 1, \ldots, N,
\]

\[
n_i \frac{d}{dt} (3T_i + m_i |\mathbf{u}_i|^2) = \sum_{i=1}^{N} D_{ik} \left[3(T_k - T_i) + (m_i \mathbf{u}_i + m_k \mathbf{u}_k) \cdot (\mathbf{u}_k - \mathbf{u}_i)\right],
\]

\[
C_{ik} = \lambda_{ik} n_i n_k \frac{m_i m_k}{m_i + m_k}, \quad D_{ik} = \frac{2C_{ik}}{m_i + m_k},
\]

with \(C_{ik} = C_{ki}\) and \(D_{ik} = D_{ki}\). These equations with \(\lambda_{ik}\) given in Eqs. (25), (24) are exact in the case of Maxwell molecules. In the general case Eqs. (38) help to prove the following statement.

**Lemma 4.2.** Any nontrivial stationary solution \(f_i^{(eq)}(\mathbf{v})\), \(n_i = \{f_i^{(eq)}, 1\} > 0; i = 1, \ldots, N\), of Eqs. (30) under conditions (31), (32) is a Maxwellian with constant parameters \(\mathbf{u} \in \mathbb{R}^3\) and \(T > 0\), i.e.

\[
f_i^{(eq)}(\mathbf{v}) = n_i \left(\frac{2\pi T}{m_i}\right)^{-3/2} \exp \left[-\frac{m_i |\mathbf{v} - \mathbf{u}|^2}{2T}\right], \quad i = 1, \ldots, N.
\]

**Proof.** We begin with a simple observation. Consider a set of \(N(N - 1)\) positive numbers

\[G_{ik} = G_{ki} > 0, \quad i \neq k; \quad i, k = 1, \ldots, N,\]

and a set of \(N\) equations

\[\sum_{k=1}^{N} G_{ik} (x_k - x_i) = 0, \quad i = 1, \ldots, N,\]

for unknown \(\{x_i, i = 1, \ldots, N\}\). It is easy to see that these equations admit only trivial solution \(\{x_i = \text{const.}, i = 1, \ldots, N\}\). Indeed, we multiply the equations by \(x_i\) and sum over all \(1 \leq i \leq N\). Then we obtain

\[
\sum_{i,k=1}^{N} G_{ik} (x_i^2 - x_i x_k) = \frac{1}{2} \sum_{i,k=1}^{N} G_{ik} (x_i - x_k)^2 = 0.
\]

The result obviously follows. Similar considerations were used in [16].

Applying this observation to the stationary version of Eqs. (38) we obtain

\[\mathbf{u}_1 = \ldots = \mathbf{u}_N = \mathbf{u} = \text{const.}, \quad T_1 = \ldots = T_N = T = \text{const.}\]

Then we use Eqs. (26), (30) and obtain

\[\mathbf{u}_{ik} = \mathbf{u}, \quad T_{ik} = T, \quad M_{ik} = M \left(\mathbf{v}; \mathbf{u}, \frac{T}{m_i}\right).
\]

The stationary versions of Eqs. (30) read

\[R_i \left[n_i M \left(\mathbf{v}; \mathbf{u}, \frac{T}{m_i}\right) - f_i^{(eq)}\right] = 0, \quad R_i = \sum_{k=1}^{N} \nu_{ik}, \quad i = 1, \ldots, N.
\]
It follows from conditions of Lemma 4.2 that $R_i > 0$ for all $1 \leq i \leq N$. Therefore the only possible solution is given in Eqs. (39) with $T > 0$. The proof is completed.

Finally we formulate below the main result of this section.

**Proposition 2.** Eqs. (30) under conditions (31), (32) preserve positivity of solutions $f_i(v,t)$ and corresponding temperatures $T_i(t)$, $i = 1, \ldots, N$. They have all principal properties of Boltzmann equations (1)-(3):

(A) conservation laws (for mass, momentum and energy);
(B) H–theorem;
(C) uniqueness of equilibrium solution (see Eqs. (10)-(12)).

Eqs. (30) have $N^2$ free parameters $\{\nu_{ik} \geq 0, i, k = 1, \ldots, N\}$ satisfying conditions (32) for $i \neq k$. Corresponding equations of transfer for momentum and energy (38) are independent of these free parameters. Moreover these equations are exact in the case of Maxwell molecules.

Proposition 2 obviously follows from above considerations. We also note that Eqs. (30) were based (at least for Maxwell molecules) on two natural assumptions:

(1) general BGK- form of the model proposed by Gross and Krook [14];
(2) correct equations of transfer (38) for Maxwell molecules.

In the next section we will compare our model to some other models proposed by different authors.

5. **Comparison with other models.** For brevity we consider only models of BGK-form [14], for which the Boltzmann’s H–theorem is proved. For the sake of simplicity we discuss below just the spatially homogeneous case, i.e. Eqs. (30), where $\{\nu_{ik}, u_{ik}, T_{ik}; i, k = 1, \ldots, N\}$ are free parameters.

1. We begin with the well-known model by Andries, Aoki and Perthame [1] ($\text{A}^2\text{P}$-model). Historically it is the first consistent BGK-type model for $N$-component mixture of Maxwell gases. The $\text{A}^2\text{P}$-model has all properties of the Boltzmann equations mentioned in Proposition 2. In addition, it has a property called “indifferentiability”: for identical particles, the sum $f = f_1 + f_2 + \ldots + f_N$ satisfies the usual BGK-equation for a simple gas. However, this model is based on very strong artificial assumption:

$$u_{ik} = \bar{u}_i, \quad T_{ik} = \bar{T}_i; \quad i, k = 1, \ldots, N. \quad (40)$$

This assumption transforms Eqs. (30) to

$$\partial_t f_i = \nu_i (n_i \bar{M}_i - f_i), \quad (41)$$

where

$$\nu_i = \sum_{k=1}^{N} \nu_{ik}, \quad \bar{M}_i = M \left( \frac{\bar{u}_i}{\bar{m}_i} \right), \quad i = 1, \ldots, N. \quad (42)$$

The second assumption of the paper [1] is that equations of transfer must coincide with correct Eqs. (38) (only the case of Maxwell molecules is considered in [1]). This assumption allows to find parameters $\{\bar{u}_i, \bar{T}_i; i = 1, \ldots, N\}$. Thus, the $\text{A}^2\text{P}$-model has $N$ free parameters $\{\nu_i > 0; i = 1, \ldots, N\}$. These parameters were used to compute, in the continuum limit, the transport coefficients for diffusion velocities, viscous stress, and heat flux. In fact a natural requirement for any relaxation model of this type is that its results reproduce, even quantitatively (as much as possible), those of the actual Boltzmann equations. In addition, this allows to tune
free parameters for a best fit of experimental data in practical applications. This matter for the present model is under investigation, and will not be treated here.

An obvious disadvantage of the $A^2P$-model is that it does not correspond to the very structure of the Boltzmann collision integrals $Q_i$ as sums of $Q_{ik}$ (see Eqs. (2), (3)). This structure has a clear probabilistic interpretation and is preserved in our model. In any case, the $A^2P$ model has represented a substantial breakthrough in the development of BGK models for mixtures. It might also be regarded as an ellipsoidal statistical model with attractors determined by a suitable entropy principle [5].

2. The structure of collision integrals is also preserved in recently published model by Haack, Hauck and Murillo [15] ($H$-model). They considered Eqs. (30) with $N(5N - 4)$ free parameters $\{\nu_{ik}, u_{ik}, T_{ik}; i, k = 1, \ldots, N\}$ and use $2N(N - 1)$ identities (18) (see the last part of Section 2). These identities are considered as equations for $4N(N - 1)$ unknowns $\{u_{ik}, T_{ik}; i, k = 1, \ldots, N : i \neq k\}$. Since the number of unknowns is larger than the number of equations, the authors of [15] introduced an artificial assumption

$$u_{ik} = u_{ki}, \quad T_{ik} = T_{ki}, \quad i, k = 1, \ldots, N. \quad (43)$$

Thus they obtained explicit formulas for these parameters and consider $N^2$ numbers $\{\nu_{ik} \geq 0; i, k = 1, \ldots, N\}$ as free parameters. H-theorem is proved for this model in [15], where the model is investigated in detail. Certain disadvantage of this model is that it is difficult to satisfy simultaneously equations of transfer for both momentum and energy. Therefore the authors use two versions of the model, and that is not always convenient. Obviously, our model does not have this problem. At the same time there is a lot of interesting information in [15], in particular a list of references which contains more than 50 papers on the subject.

3. Another recent paper [18] presents more or less general consideration of all parameters in Eqs. (30) for binary mixture, i.e. the case $N = 2$. They make several assumptions, which look slightly less artificial than above assumptions (40) or (43):

$$u_{12} = (1 - a)u_1 + au_2, \quad T_{12} = (1 - b)T_1 + bT_2 + \gamma|u_1 - u_2|^2, \quad (44)$$

with some $0 \leq a, b \leq 1, \gamma \geq 0$. In other words they assume some properties, which we derive from the Boltzmann equations (see formulas (26)) under natural assumptions. The authors of [18] do not consider any connection with “true” kinetic equations and the case of multicomponent ($N \geq 3$) mixture. Nevertheless their results are very important since they show the most general class of BGK models of the form (30) with $N = 2$, which can be obtained by postulating (a) linear relations (44), (b) conservation laws and (c) H-theorem.

Finally we consider in this section an almost obvious generalization of $A^2P$-model (41) obtained by adding to the right hand side of Eqs. (41) the standard BGK terms for collisions of identical particles:

$$\partial_t f_i = \theta_i(n_i M_i - f_i) + \nu_i(n_i \dot{M}_i - f_i), \quad 1 \leq i \leq N, \quad (45)$$

where $\dot{M}_i$ is given in Eqs. (42) with parameters $\dot{u}_i$ and $\dot{T}_i$ from $A^2P$-model [1]. $M_i$ is the usual Maxwellian given in Eqs. (8), (9), $\theta_i \geq 0$ are free parameters, $i = 1, \ldots, N$. For any non-negative values of free parameters it is also a consistent model, satisfying conservation laws and H-theorem. In the simplest case $N = 2$ this model coincides with our Eqs. (30), (26), where

$$\nu_{11} = \theta_1, \quad \nu_{12} = \nu_1, \quad \nu_{21} = \nu_2, \quad \nu_{22} = \theta_2. \quad (46)$$
This explains a connection of our model with $A^2P$-model.

We shall see below that the generalized $A^2P$-model with $\theta_{1,2} > 0$ loses the indifferentiality property [1]. This property for arbitrary $N \geq 2$ means that in the case of identical particles having mass $m$ the sum $f = f_1 + f_2 + \ldots + f_N$ satisfies the usual BGK-equation

$$\partial_t f = \nu \left[ nM \left( \nu; u, \frac{T}{m} \right) - f \right],$$

(47)

with some $\nu > 0$ and

$$n = \sum_{i=1}^{N} n_i, \quad u = \frac{1}{n} \sum_{i=1}^{N} n_i u_i,$$

$$T = \frac{1}{n} \sum_{i=1}^{N} n_i \left[ T_i + \frac{m}{3} (|u_i|^2 - |u|^2) \right] = \frac{1}{n} \sum_{i=1}^{N} n_i T_i + \frac{m}{6} \frac{1}{n^2} \sum_{i=1, k=1 \atop (i \neq k)}^{N} n_i n_k |u_i - u_k|^2.$$  (48)

Hence, for the most general BGK–model with collisional terms (14) validity of the indifferentiality property implies the following identity for arbitrary (under obvious restrictions) functions $f_i(v)$, $1 \leq i \leq N$:

$$\nu(nM - f) = \sum_{i=1, k=1}^{N} \tilde{Q}_{ik} = \sum_{i=1, k=1}^{N} \nu_{ik} (n_{ik} M_{ik} - f_i)$$  (49)

in the notation of Eqs. (14), (15), (16), (47), (48).

Hence, we obtain

$$\sum_{i=1}^{N} f_i(v) \left( \nu - \sum_{k=1}^{N} \nu_{ik} \right) = \nu nM(v) - \sum_{i=1, k=1}^{N} \nu_{ik} n_{ik} M_{ik}(v), \quad v \in \mathbb{R}^3.$$  

Taking, for example, $f_i(v) = \exp[-i|v|]$, $1 \leq i \leq N$, we can conclude that

$$\sum_{k=1}^{N} \nu_{ik} = \nu, \quad i = 1, \ldots, N.$$  

Hence,

$$\nu nM(v) - \sum_{i=1, k=1}^{N} \nu_{ik} n_{ik} M_{ik}(v) = 0, \quad v \in \mathbb{R}^3.$$  (50)

By considering asymptotics for $|v| \to \infty$ we can easily show that this equality implies that

$$T_{ik} = T, \quad u_{ik} = u \quad \Rightarrow \quad M_{ik} = M, \quad i, k = 1, \ldots, N.$$  (51)

This, however, contradicts the assumption (14) for diagonal terms with $i = k$. Indeed the parameters

$$u_{ii} = u_i = \frac{1}{n_i} \{ f_i, v \}, \quad T_{ii} = T_i = \frac{m}{3 n_i} \{ f_i, |v - u_i|^2 \}, \quad i = 1, \ldots, N,$$  (52)

depend on arbitrary functions $f_i$, $1 \leq i \leq N$, and therefore the conditions (51) for $i = k$ can be easily violated. Hence, the terms with $i = k$ cannot be present in the sum in Eq. (50). The result can be formulated in the following way.

**Proposition 3.** The indifferentiality property (49) can be satisfied for the general BGK model (14), (15), (16) only if

$$\nu_{11} = \nu_{22} = \ldots = \nu_{NN} = 0$$

i.e. all collisions of identical particles are forbidden.
Proof. Without loss of generality we assume that \( \nu_{11} > 0 \). Then we need to satisfy conditions (51) for \( i = k = 1 \). By using Eqs. (48), (52) for \( i = 1 \) we obtain

\[
\mathbf{u}_{11} = \mathbf{u}_1 = \mathbf{u} = \sum_{i=1}^{N} n_i \mathbf{u}_i / \sum_{i=1}^{N} n_i.
\]

Since \( \{n_i > 0, \mathbf{u}_i \in \mathbb{R}^3, i = 1, \ldots, N\} \) are arbitrary, this equality can be easily violated. Hence, the assumption was wrong and therefore \( \nu_{11} = 0 \). This completes the proof.

Finally, we note that the indifferentiability property does not seem to be very important for plasma applications, where we always have ions and electrons with large mass ratio.

6. Choice of collision frequencies. To complete the construction of our model we need to choose \( N^2 \) free parameters \( \{\nu_{ik} > 0; i, k = 1, \ldots, N\} \). Note that the only restriction is the inequality (see Lemma 4.1)

\[
\nu_{ik} \geq \frac{1}{2} \lambda_{ik} n_k, \quad i \neq k.
\]

Any choice of the parameters, satisfying (53), leads (at least for Maxwell molecules) to correct equations of transfer for momentum and energy. We can make the choice based just on simplicity of resulting formulas (26)–(27). Let us consider two obvious examples, which apply to any given molecular interaction law.

1. Let \( \nu_{ik} = \lambda_{ik} n_k \), then Eqs. (26)–(27) imply that

\[
\mathbf{u}_{ik} = \frac{m_i \mathbf{u}_i + m_k \mathbf{u}_k}{m_i + m_k}, \quad T_{ik} = \frac{(m_i^2 + m_k^2) T_i + 2 m_i m_k T_k + m_i m_k^2 |\mathbf{u}_i - \mathbf{u}_k|^2}{(m_i + m_k)^2}; \quad i \neq k.
\]

2. Let \( \nu_{ik} = \frac{1}{2} \lambda_{ik} n_k \), then similarly we obtain

\[
\mathbf{u}_{ik} = \frac{m_i - m_k}{m_i + m_k} \mathbf{u}_i + \frac{2 m_k}{m_i + m_k} \mathbf{u}_k, \quad T_{ik} = \frac{(m_i - m_k)^2 T_i + 4 m_i m_k T_k}{(m_i + m_k)^2}; \quad i \neq k.
\]

Of course, these are merely illustrative examples. An actual test for the choice of relaxation parameters in practice could be the best fit of other properties, such as diffusion, viscosity, and thermal conductivity in the hydrodynamic limit. As already mentioned, this matter will be subject of a future work.

For brevity we do not discuss the choice of “diagonal” parameters \( \nu_{ii} \), since the corresponding BGK–models for simple gases are supposed to be known. In particular, we could use a similar (see Section 3) approach and choose

\[
\nu_{ii} = \frac{3}{4} g_{ii}^{(2)} \left( \sqrt{\frac{6 T_i}{m_i}} \right) n_i, \quad g_{ii}^{(2)}(|y|) = 2\pi \int_{-1}^{1} d\mu g_{ii}(|y|, \mu)(1 - \mu^2).
\]

It can be easily verified (all calculations are shown in Appendix B) that such choice of \( \nu_{ii} \) gives for Maxwell molecules all correct tensor moments of BGK–terms in (30) up to the second order for the “diagonal” collision operators, enhancing in this way the closeness to the Boltzmann equations beyond the requirement of capturing the correct momentum and temperature relaxations. Thus we obtain examples of a consistent BGK–model for mixture, where all parameters are completely defined.
We conclude the paper with some remarks on applications to general (non–Maxwell) mixtures. Our way of constructing the BGK–model can also be considered in the following way. First we “approximate” the Boltzmann integrals (3) by transformation to velocity–independent kernels

$$g_{ik}(|y|, \mu) \rightarrow g_{ik}(\langle |y| \rangle_{ik}, \mu)$$

with some “average” values $\langle |y| \rangle_{ik}$, see e.g. Eqs. (28)–(29). This leads to collision integrals of Maxwell type in the initial Boltzmann equations (1)–(3). At the second stage we naturally approximate these integrals by BGK–terms and obtain our model equations. Of course this is a very rough approximation at the “qualitative” level. We can hardly improve it by more complicated calculations of integrals with Maxwellian distributions (see [24, 15]). Therefore we do not try to do it in this paper.

Finally we briefly discuss how to treat the important Coulomb case. Then formally the function $g_{ik}(|y|, \mu)$ reads (see e.g. [22])

$$g_{ik}(|y|, \mu) = \left( \frac{q_i q_k}{m_{ik}} \right)^2 |y|^{-3} (1 - \mu)^{-2},$$

(55)

where $q_{i,k}$ are corresponding electric charges. Having in mind the Debye screening we cut the upper limit of integration in the second formula (24) at some value $\mu_0 = \cos \theta_0, \theta_0 \ll 1$. Then

$$g^{(1)}_{ii}(|y|) \approx 4\pi \left( \frac{q_i q_k}{m_{ik}} \right)^2 |y|^{-3} L, \quad L = \log \left[ \frac{1}{\sin(\theta_0/2)} \right],$$

(56)

where $L$ is the so–called Coulomb logarithm. The classical kinetic theory is valid only for $L \gg 1$ [23], moreover all quantities of order $O(1)$ can be neglected as compared with $L$ (logarithmic accuracy). Therefore we should consider $L$ in our model as a given large constant, its exact definition in classical and quantum cases can be found in the same textbook [23]. For collisions of identical particles (see Eq. (54)) we obtain

$$g^{(2)}_{ii}(|y|) \approx 2\pi \int_{-1}^{\mu_0} d\mu \ g_{ii}(\langle |y| \rangle, \mu) (1 - \mu^2) \approx 2 g^{(1)}_{ii}(|y|)$$

(57)

with the same logarithmic accuracy. The final step is to replace $g^{(1)}_{ik}(|y|)$ and $g^{(2)}_{ii}(|y|)$ with their velocity–independent values $g^{(1)}_{ik}(z_{ik})$ and $g^{(2)}_{ii}(z_{ii})$ in the notation of Eqs. (29) for all $i, k = 1, \ldots, N$. This completes the description of the model in the Coulomb case. Of course, integral (57) with the Rutherford cross section can be computed exactly, but this would hardly improve the accuracy of our model. The matter is that the Boltzmann (or Landau [21]) collision integral for Coulomb forces (with angular cutoff) is valid only with logarithmic accuracy, since it does not take into account some plasma effects described by the Balescu–Lennard integral (see e.g. the book [23] for details). Therefore the above formulas for integral cross sections $g^{1,2}_{i,k}$ are quite satisfactory for our rough BGK model.

Appendix A. By a standard change of variables we have

$$\langle Q_{ik}, v \rangle = \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} d\nu d\omega d\sigma \ v \ g_{ik}(\langle |y| \rangle, \hat{y} \cdot \omega) \left[ f_i(v') f_k(w') - f_i(v) f_k(w) \right]$$

$$= \int_{\mathbb{R}^3 \times \mathbb{R}^3} d\nu d\sigma \ f_i(v) f_k(w) \int_{S^2} d\omega \left( \nu' - \nu \right) g_{ik}(\langle |y| \rangle, \hat{y} \cdot \omega),$$

(58)
where $y$ is defined in equation (3), and then, on using (3) itself
\[
\langle Q_{ik}, \nu \rangle = -\frac{m_k}{m_i + m_k} \int_{\mathbb{R}^3 \times \mathbb{R}^3} d\mathbf{v} \, d\mathbf{w} \, f_i(\mathbf{v}) \, f_k(\mathbf{w}) \int_{S^2} d\omega \, g_{ik}(|y|, \dot{y} \cdot \omega) \, (y - |y|\omega)
\]
and, since
\[
\int_{S^2} d\omega \, g_{ik}(|y|, \dot{y} \cdot \omega) \, \omega = \dot{y} \, 2\pi \int_{-1}^{1} d\mu \, g_{ik}(|y|, \mu)
\]
we end up with
\[
\langle Q_{ik}, \nu \rangle = -\frac{m_k}{m_i + m_k} \int_{\mathbb{R}^3 \times \mathbb{R}^3} d\mathbf{v} \, d\mathbf{w} \, f_i(\mathbf{v}) \, f_k(\mathbf{w}) \, (\nu - |\nu|\omega) \, g_{ik}^{(1)}(|\nu - \mathbf{w}|)
\]

(59)

Analogously, since
\[
|\nu'|^2 - |\nu|^2 = \frac{2m_im_k}{(m_i + m_k)^2} |\nu|^2 + \frac{2m_k^2}{(m_i + m_k)^2} |\nu|^2 + \frac{2m_k(m_i - m_k)}{(m_i + m_k)^2} \nu \cdot \nu' + \frac{2m_k}{(m_i + m_k)^2} (m_i \nu + m_k \nu') \cdot |\nu|\omega
\]

\[
= -\frac{2m_k}{(m_i + m_k)^2} \left[ m_i |\nu|^2 - m_k |\nu|^2 + (m_k - m_i) \nu \cdot \nu' - (m_i \nu + m_k \nu') \cdot |\nu|\omega \right] - \frac{2m_k}{(m_i + m_k)^2} (m_i \nu + m_k \nu') \cdot (y - |y|\omega),
\]

we get, by the same steps as before,
\[
\langle Q_{ik}, |\nu|^2 \rangle = \int_{\mathbb{R}^3 \times \mathbb{R}^3} d\mathbf{v} \, d\mathbf{w} \, f_i(\mathbf{v}) \, f_k(\mathbf{w}) \int_{S^2} d\omega \, (|\nu'|^2 - |\nu|^2) \, g_{ik}(|y|, \dot{y} \cdot \omega)
\]

\[
= -\frac{2m_k}{(m_i + m_k)^2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} d\mathbf{v} \, d\mathbf{w} \, f_i(\mathbf{v}) \, f_k(\mathbf{w}) \left( m_i \nu + m_k \nu' \right) \cdot \left( \frac{\dot{y}}{|\nu|} \right) \cdot \left( \nu - |\nu|\omega \right)
\]

\[
= -\frac{2m_k}{(m_i + m_k)^2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} d\mathbf{v} \, d\mathbf{w} \, f_i(\mathbf{v}) \, f_k(\mathbf{w}) \left( m_i \nu + m_k \nu' \right) \cdot \left( \nu - |\nu|\omega \right) \, g_{ik}^{(1)}(|\nu - \mathbf{w}|)
\]

(60)

Appendix B. For Maxwellian molecules, setting, in the collision integral $Q_{ii}$,
\[
c = \nu - u_i \quad \Rightarrow \quad \int_{\mathbb{R}^3} d\mathbf{v} \, c_i f_i(\mathbf{v}) = 0
\]
and analogously $\tilde{c} = \nu - u_i$, and using the same transformation as in (58), we get for the general second order tensor moment
\[
\int_{\mathbb{R}^3} d\mathbf{v} \, c_j c_l Q_{ii} = \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} d\mathbf{v} d\mathbf{w} d\omega (c_i' c_l' - c_j c_l) g_{ii}(\dot{y} \cdot \omega) f_i(\mathbf{v}) f_l(\mathbf{w})
\]

(62)

where
\[
c_i' c_l' - c_j c_l = c_j \tilde{c}_l + \tilde{c}_j c_l + c_j \tilde{c}_l - 3c_j c_l - \frac{c_j + \tilde{c}_j}{4} |\nu|\omega_i - \frac{c_l + \tilde{c}_l}{4} |\nu|\omega_j + \frac{1}{4} |\nu|^2 \omega_i \omega_j.
\]

Performing the integration with respect to $\omega$ of the addend independent of $\omega$ yields
\[
-\frac{1}{2} g_{ii}^{(0)} \int_{\mathbb{R}^3 \times \mathbb{R}^3} d\mathbf{v} d\mathbf{w} \, c_j c_l f_i(\mathbf{v}) f_l(\mathbf{w}) \quad g_{ii}^{(0)} = 2\pi \int_{-1}^{1} d\mu \, g_{ii}(\mu).
\]
Performing analogous integration for the terms which are linear in the components of \( \mathbf{\omega} \), leads, on account of (59) and of the fact that \( y_{\hat{y}_i} = y_i = c_i - \hat{c}_i \), to the expression

\[
\frac{1}{2} (g_{i1}^{(0)} - g_i^{(1)}) \int_{\mathbb{R}^3 \times \mathbb{R}^3} d\mathbf{v} d\mathbf{w} \left( c_j c_l - \hat{c}_j \hat{c}_l \right) f_i(\mathbf{v}) f_j(\mathbf{w}),
\]

which vanishes by symmetry considerations. Finally, it is not difficult to verify that

\[
\int_{S^2} d\omega \omega_j \omega_l g_{il}(\hat{\mathbf{y}} \cdot \mathbf{\omega}) = \pi \int_{-1}^{1} d\mu (1 - \mu^2) g_{jl}(\mu) \delta_{jl} - \pi \int_{-1}^{1} d\mu (1 - 3\mu^2) g_{ij}(\mu) \hat{y}_j \hat{y}_l,
\]

so that, equating (65) to (64), we are led directly to the result (54).

Putting all terms together one is left with

\[
\int_{\mathbb{R}^3} d\mathbf{v} c_j c_l Q_{il} = \frac{1}{4} \delta_{jl} g_{ii}^{(2)} \int_{\mathbb{R}^3 \times \mathbb{R}^3} d\mathbf{v} d\mathbf{w} |c|^2 f_j(\mathbf{v}) f_i(\mathbf{w}) - \frac{3}{4} g_{ii}^{(2)} \int_{\mathbb{R}^3 \times \mathbb{R}^3} d\mathbf{v} d\mathbf{w} c_j c_l f_i(\mathbf{v}) f_j(\mathbf{w}),
\]

where \( g_{ii}^{(2)} \) (a constant here) is defined in Eq. (54) itself. In conclusion, since

\[
\int_{S^2} d\omega |\mathbf{v}|^2 f_j(\mathbf{v}) = n_i \frac{3T_i}{m_i}, \quad \int_{\mathbb{R}^3} d\mathbf{v} c_j c_l f_j(\mathbf{v}) = \frac{1}{m_i} P_{jl},
\]

where \( P \) denotes pressure tensor, the final result is

\[
\int_{\mathbb{R}^3} d\mathbf{v} c_j c_l Q_{il} = \frac{3}{4} g_{ii}^{(2)} \frac{n_i}{m_i} \left( n_i T_i \delta_{jl} - P_{jl} \right).
\]

For non-Maxwellian molecules, the same tensor moments are approximated by replacing the constant \( g_{ii}^{(2)} \) by \( g_{ii}^{(2)}(z_{ii}) \), with \( z_{ii} = \sqrt{6T_i/m_i} \) (see Eq. (29)).

On the other hand, by simple properties of Maxweilians, the same tensor moments of the BGK collision operator are

\[
\int_{\mathbb{R}^3} d\mathbf{v} c_j c_l \tilde{Q}_{il} = \nu_{ii} \frac{1}{m_i} \left( n_i T_i \delta_{jl} - P_{jl} \right),
\]

so that, equating (65) to (64), we are led directly to the result (54).

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**REFERENCES**

[1] P. Andries, K. Aoki and B. Perthame, A consistent BGK-type model for gas mixtures, *J. Stat. Phys.*, 106 (2002), 993–1018.

[2] P. L. Bhatnagar, E. P. Gross and K. Krook, A model for collision processes in gases, *Phys. Rev.*, 94 (1954), 511–524.

[3] M. Bisi and M. J. Cáceres, A BGK relaxation model for polyatomic gas mixtures, *Commun. Math. Sci.*, 14 (2016), 297–325.

[4] M. Bisi, M. Groppi and G. Spiga, Kinetic Bhatnagar–Gross–Krook model for fast reactive mixtures and its hydrodynamic limit, *Phys. Rev. E*, 81 (2010), 030327 (pp. 1–9).
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[5] S. Brull, An ellipsoidal statistical model for gas mixtures, *Commun. Math. Sci.*, 13 (2015), 1–13.
[6] S. Brull and J. Schneider, On the ellipsoidal statistical model for polyatomic gases, *Contin. Mech. Thermodyn.*, 20 (2009), 489–508.
[7] S. Brull, V. Pavan and J. Schneider, Derivation of a BGK model for mixtures, *Europ. J. Mech. B/Fluids*, 33 (2012), 74–86.
[8] C. Cercignani, *The Boltzmann Equation and its Applications*, Springer, New York, 1988.
[9] V. Garzó, A. Santos and J. J. Brey, A kinetic model for a multicomponent gas, *Phys. Fluids*, 1 (1989), 380–383.
[10] E. Goldman and L. Sirovich, Equations for gas mixtures, *Phys. Fluids*, 10 (1967), 1928–1940.
[11] J. M. Greene, Improved Bhatnagar–Gross–Krook model for electron–ion collisions, *Phys. Fluids*, 16 (1973), 2022–2023.
[12] M. Groppi, S. Rjasanow and G. Spiga, A kinetic relaxation approach to fast reactive mixtures: Shock wave structure, *J. Stat. Mech. - Theory Exp.*, 2009 (2009), P10010.
[13] M. Groppi and G. Spiga, A Bhatnagar–Gross–Krook-type approach for chemically reacting gas mixtures, *Phys. Fluids*, 16 (2004), 4273–4284.
[14] E. P. Gross and M. Krook, Model for collision processes in gases: Small–amplitude oscillations of charged two–component systems, *Phys. Rev.*, 102 (1956), 593–604.
[15] J. R. Haack, C. D. Hauck and M. S. Murillo, A conservative, entropic multispecies BGK model, *J. Stat. Phys.*, 168 (2017), 826–856.
[16] J. R. Haack, C. D. Hauck and M. S. Murillo, Interfacial mixing in high energy-density matter with a multiphysics kinetic model, *Phys. Rev. E*, 96 (2017), 063310 (pp. 1–14).
[17] B. B. Hamel, Kinetic model for binary gas mixtures, *Phys. Fluids*, 8 (1965), 418–425.
[18] C. Klingenberg, M. Pirner and G. Puppo, A consistent kinetic model for a two-component mixture with an application to plasma, *Kinet. Relat. Models*, 10 (2017), 445–465.
[19] M. N. Kogan, *Rarefied Gas Dynamics*, Plenum Press, New York, 1969.
[20] G. M. Kremer, M. Pandolfi Bianchi and A. J. Soares, A relaxation kinetic model for transport phenomena in a reactive flow, *Phys. Fluids*, 18 (2006), 037104, 15pp.
[21] L. D. Landau, Kinetic equation for the Coulomb interaction, *Phys. Z. Sowjetunion*, 10 (1936), 145–154.
[22] L. D. Landau and E. M. Lifshitz, *Mechanics*, Pergamon Press, Oxford, 1969.
[23] E. M. Lifshitz and L. P. Pitaevskii, *Physical Kinetics*, Butterworth–Heinemann, 1981.
[24] T. F. Morse, Kinetic model equations for a gas mixture, *Phys. Fluids*, 7 (1964), 2012–2013.
[25] L. Sirovich, Kinetic modeling of gas mixtures, *Phys. Fluids*, 5 (1962), 908–918.
[26] P. Welander, On the temperature jump in a rarefied gas, *Ark. Fys.*, 7 (1954), 507–533.

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