A FEYNMAN-KAC-ITÔ FORMULA FOR MAGNETIC SCHRÖDINGER OPERATORS ON GRAPHS

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Abstract. In this paper we prove a Feynman-Kac-Itô formula for magnetic Schrödinger operators on arbitrary weighted graphs. To do so, we have to provide a natural and general framework both on the operator theoretic and the probabilistic side of the equation. On the operator side, we start from regular Dirichlet forms on discrete sets and identify a very general class of potentials that allows the definition of magnetic Schrödinger operators. On the probabilistic side, we introduce an appropriate notion of stochastic line integrals with respect to magnetic potentials. Apart from linking the world of discrete magnetic operators with the probabilistic world through the Feynman-Kac-Itô formula, the insights from this paper gained on both sides should be of an independent interest. As applications of the Feynman-Kac-Itô formula, we prove a Kato inequality, a Golden-Thompson inequality and an explicit representation of the quadratic form domains corresponding to a large class of potentials.

1. Introduction

The conceptual importance of the classical Feynman-Kac formula stems from the fact that it links the world of operator theory (or partial differential equations) with that of probability. In particular, the semigroup of a Schrödinger operator of the form $-\Delta + v$ on $L^2(\mathbb{R}^n)$ is expressed in terms of an expectation value involving the Markov process of the free operator $-\Delta$, which is nothing but the Euclidean Brownian motion in this case. Now if one perturbs $-\Delta + v$ by a magnetic field with potential $\theta$, one has to deal with the magnetic Schrödinger operator $-\Delta_\theta + v$. In this case a very important extension of the Feynman-Kac formula is given by the Feynman-Kac-Itô formula. This formula again expresses the semigroup corresponding to the latter operator through Euclidean Brownian motion, where now one has to take the (Stratonovic) stochastic line integral of $\theta$ along the Brownian motion into account [37]. Such probabilistic representations have many important physical consequences through diamagnetism, e.g., one can easily deduce that switching on a magnetic field can only lead to an increase of the ground state energy of the systems.

Seeking for extensions of the above results to more general settings than the Euclidean $\mathbb{R}^n$, one will realize that the *Feynman-Kac* formula can be proven for locally compact regular Dirichlet spaces (see, e.g., [5]), where now one simply has to replace $-\Delta$ with the operator corresponding to the given Dirichlet form, and Brownian motion with the associated...
Markov process. However, it is not even clear how to formulate a Feynman-Kac-Itô formula in many situations. The reason for this is that a consistent theory of Schrödinger operators with local magnetic potentials in such a general setting as Dirichlet spaces is still missing. Although recently very promising progress into this direction has been made on the operator side \cite{2, 19, 18}, there still remains the issue of finding a reasonable way to define a proper notion of a stochastic line integral, which extends the above $\mathbb{R}^n$-theory in a consistent way.

The situation is fundamentally better for smooth Riemannian manifolds $M$. Here, magnetic potentials can be defined simply as real-valued 1-forms, and if $\theta$ is such a 1-form, then $-\Delta \theta + v$ can be defined invariantly in analogy to the Euclidean case (see for example \cite{35, 11} for details). Assuming some local control on $v_-$ (typically $L^1_{\text{loc}}$) and $\theta$ (typically smooth or $L^p_{\text{loc}}$), and a certain global control on $v_-$, the operator $-\Delta \theta + v$ will correspond to a well-defined self-adjoint semi-bounded operator in $L^2(M)$, and one can prove an analogue of the Feynman-Kac-Itô formula in this setting (replacing the Euclidean with the underlying Riemannian Brownian motion), without any further assumptions on $M$. An essential observation in this connection is that, as the underlying space now locally looks like the linear space $\mathbb{R}^n$, one can define the line integral of $\theta$ along the Riemannian Brownian motion by combining the corresponding definition from the Euclidean case above either with a patching procedure using charts \cite{21}, or equivalently, by embedding $M$ into some $\mathbb{R}^l$ with an appropriate $l \geq n$, as in \cite{7}. As a consequence of the Feynman-Kac-Itô formula in this setting and in analogy to the Euclidean $\mathbb{R}^n$, it becomes very easy to deduce several rigorous variants of the domination $"-\Delta \theta + v \geq -\Delta + v"$. Apart from physically relevant ones, these domination results also make it possible to transfer many important mathematical statements from zero magnetic potential to arbitrary magnetic potentials, such as essential self-adjointness results \cite{36, 11} or certain smoothing properties of the Schrödinger semigroups \cite{11, 11}.

On the other hand, going back to the fundamental papers \cite{15, 25, 39}, there is also a basic theory of magnetic Schrödinger operators for discrete graphs, and in the last years an extensive amount of research for these operators has been carried out into various directions. Let us only mention here that basic spectral properties and Kato’s inequality have been proven in \cite{6}, for a Hardy inequality see \cite{10}, for approximation results of spectral invariants see \cite{27, 28}, and for weak Bloch theory see \cite{17}. Recently there has been a strong focus on the question of essential self-adjointness of magnetic Schrödinger operators \cite{3, 10, 29, 30, 31, 40}.

Noticing now that the Markov processes corresponding to free Laplacians on discrete graphs are jump processes (which have very special path properties), and that in this setting magnetic potentials are typically defined as functions on the underlying set of edges, one might hope that it is possible to get a proper notion of line integrals in this setting, which produces a probabilistic representation of the underlying magnetic Schrödinger semigroups. The main result of this paper, a Feynman-Kac-Itô type formula for discrete graphs, precisely states that this is possible.
Unfortunately, so far all proposed settings for discrete magnetic Schrödinger operators are somewhat tailored to their specific applications and, thus, are often rather restrictive. In particular, a general and systematic treatment of the question, when the operators can actually be defined as genuine self-adjoint operators, seems to be missing. So the first question that arises is actually what a natural and sufficiently general framework might be in this context. Our starting point is the one of regular Dirichlet forms on discrete sets, following [23]. Next, we identify a class of potentials that is suitable to our cause. Having the goal of a Feynman-Kac-Itô formula in mind (where due to the presence of a magnetic potential one cannot expect to conclude exclusively with monotone convergence arguments), a natural assumption on the potential is that the corresponding non-magnetic quadratic form is semi-bounded from below on the functions with compact support. Remarkably, it turns out that the latter assumption is in fact all we need to get a closable semi-bounded form, and thus a self-adjoint semi-bounded operator, in the magnetic case. This is the content of Theorem 2.4. To the best of our knowledge, this result is even new in the setting of non-magnetic Schrödinger operators. We additionally give criteria for the above mentioned self-adjoint semi-bounded operator to be unique in an appropriate sense, which in turn also provides criteria for a certain uniqueness of the Markov processes.

Having established the necessary results on the operator side, we then give the definition of the stochastic line integral in terms of a sum along the path of the process. We establish our main result, the Feynman-Kac-Itô formula in Theorem 4.1. Let us stress that here we do not have to make any restrictions on the underlying geometry such as local finiteness of the graph. Furthermore, we do not require anything on the positive parts of the potentials, nor on the magnetic potentials. The only assumption we make on the negative part of the potential is the above one, namely, that the corresponding non-magnetic form is semi-bounded below on the functions with compact support. Compared to the manifold case this assumption is significantly weaker, obviously due to the discrete structure of our setting.

Finally, we remark that manifolds and graphs are essentially the most approachable and prominent non-trivial examples of local and non-local Dirichlet forms. So, having established a Feynman-Kac-Itô formula in both of these worlds appears to be a promising step towards a unified theory for all regular Dirichlet forms. Here, as we have already mentioned, the results of [2, 19, 18] should be very useful.

The paper is structured as follows: In Section 2, we introduce and establish all necessary operator theoretic results. In Section 3, we introduce the necessary probabilistic concepts (including the definition of the line integral in this setting). Section 4 is completely devoted to the presentation and the proof of our main result, the Feynman-Kac-Itô formula, Theorem 4.1, and finally, in Section 5, we have collected several applications such as semigroup formulas, Kato’s inequality, a Golden-Thompson inequality and a representation of the form domain for suitable potentials.
Note added: Let us mention the follow up work by one of the authors [12] which treats Hermitian vector bundles and connections over weighted graphs. This paper is heavily building on the results presented here.

2. Magnetic Schrödinger operators

In this section we introduce the set up in which we are going to prove the Feynman-Kac-Itô formula. While it is clear from earlier work how a magnetic Schrödinger operator should act, [15, 25, 39, 29], it is a non-trivial problem to determine when a self-adjoint semi-bounded operator can be defined. This starts with the problem that for general weighted graphs the formal operator does not necessarily map the compactly supported functions into $\ell^2$. Although the theory of quadratic forms provides a helpful tool, it raises the problem of determining whether the form, defined a priori on the compactly supported functions, is closable and semi-bounded from below. This, however, is a rather subtle issue which in general does not allow for a complete and applicable characterization. Here, we provide a rather general framework in which we give a sufficient condition (cf. Theorem 2.4 below) for the general magnetic case, which, remarkably, even turns out to be necessary in the non-magnetic case. Interestingly, we will actually already use a Feynman-Kac-Itô formula for potentials that are bounded below in order to derive the latter result.

After briefly reviewing the basic set up of weighted graph, we introduce the quadratic forms and the corresponding formal operators. Then, we give a sufficient criterion for closability and semi-boundedness of the forms. At the end, we discuss uniqueness of semi-bounded self-adjoint extensions/restrictions and present a result on semigroup convergence.

2.1. Weighted graphs. We essentially follow the setting of [23]. Let $(X, b)$ be a graph, that is, $X$ is a countable set and

$$ b : X \times X \rightarrow [0, \infty) $$

is a symmetric function with the properties $b(x, x) = 0$ and

$$ \sum_{y \in X} b(x, y) < \infty \quad \text{for all } x \in X. $$

Then, the elements of $X$ are called *vertices* and one says that $x, y \in X$ are *neighbors* or *connected by an edge*, if $b(x, y) > 0$, which is written as $x \sim y$. Therefore, $b(x, y)$ can be thought of being the edge weight between $x$ and $y$. The graph $X$ is called *locally finite*, if every vertex has only a finite number of neighbors. Furthermore, a *path* on the graph $X$ is a (finite or infinite) sequence of pairwise distinct vertices $(x_j)$ such that $x_j \sim x_{j+1}$ for all $j$, and $X$ is called *connected*, if for any $x, y \in X$ there is a path $(x_j)_{j=0}^n$ such that $x_0 = x$ and $x_n = y$.

For simplicity and without loss of generality, we will assume throughout the paper that the graph $X$ is connected.

We equip $X$ with the discrete topology, so that any function $m : X \rightarrow (0, \infty)$ gives rise to a Radon measure of full support on $X$ in the obvious way, and then the triple $(X, b, m)$ is
called a *weighted graph*. For \( x \in X \) we denote the *weighted vertex degree* by
\[
\deg_m(x) = \frac{1}{m(x)} \sum_{y \in X} b(x, y).
\]
This notion is motivated by the following observation: Whenever \( m \equiv 1 \) and \( b : X \times X \to \{0, 1\} \), the number \( \deg_m(x) = \deg_1(x) \) is equal to the number of edges emerging from a vertex \( x \).

2.2. **Quadratic forms.** Let \( C(X) \) be the linear space of all complex-valued functions on \( X \) and \( C_c(X) \) its subspace of functions with finite support. We denote the standard scalar product and norm on \( \ell^2(X, m) \) with \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \), respectively, that is,
\[
\langle f, g \rangle = \sum_{x \in X} f(x)g(x)m(x), \quad \|f\| = \langle f, f \rangle^{\frac{1}{2}}.
\]
Clearly, \( C_c(X) \) is dense in \( \ell^2(X, m) \). Let \( \delta_x \) be the function that takes the value \( 1/m(x) \) at \( x \) and 0 otherwise. By the discreteness of the underlying data, any linear operator \( A \) in \( \ell^2(X, m) \) with \( C_c(X) \subset D(A) \) has a unique integral kernel in the sense that the function
\[
A(\bullet, \bullet) : X \times X \to \mathbb{C}, \quad A(x, y) = \frac{1}{m(x)} \langle A\delta_x, \delta_y \rangle
\]
is the unique one such that
\[
Af(x) = \sum_{y \in X} A(y, x)f(y)m(y) \quad \text{for all } f \in D(A), x \in X.
\]
Let us introduce some further notions which will be useful in what follows: Following [3], we understand by a *magnetic potential* on the set \( X \) any function \( \theta : X \times X \to [-\pi, \pi] \) that satisfies \( \theta(x, y) = -\theta(y, x) \) for \( x, y \in X \). Any function \( \nu : X \to \mathbb{R} \) will be simply called a *potential*.

Throughout the paper, let \( \theta \) be an arbitrary magnetic potential, and if not further specified, then \( \nu \) denotes an arbitrary potential.

We define a symmetric sesqui-linear form on \( \ell^2(X, m) \) with domain of definition \( C_c(X) \) by
\[
Q^{(c)}_{\nu, \theta}(f, g) := \frac{1}{2} \sum_{x, y \in X} b(x, y)\left( f(x) - e^{i\theta(x,y)}f(y) \right)\overline{\left( g(x) - e^{i\theta(x,y)}g(y) \right)} + \sum_{x \in X} \nu(x)f(x)g(x)m(x).
\]
With
\[
\tilde{F}(X) := \left\{ f \in C(X) \left| \sum_{y \in X} b(x, y)|f(y)| < \infty \text{ for all } x \in X \right. \right\},
\]
we define the formal difference operator \( \tilde{L}_{\nu, \theta} : \tilde{F}(X) \to C(X) \) by
\[
\tilde{L}_{\nu, \theta}f(x) = \frac{1}{m(x)} \sum_{y \in X} b(x, y)\left( f(x) - e^{i\theta(x,y)}f(y) \right) + \nu(x)f(x).
\]
Note that if $X$ is locally finite, then one has $\tilde{F}(X) = C(X)$. However, in general, $\tilde{F}(X)$ does not even include $\ell^2(X, m)$.

The form $Q_{v,\theta}^{(c)}$ and the operator $\tilde{L}_{v,\theta}$ are related by Green’s formula.

**Lemma 2.1.** (Green’s formula) For all $f \in \tilde{F}(X)$, $g \in C_c(X)$, one has

$$
\sum_{x \in X} \tilde{L}_{v,\theta} f(x) \overline{g(x)} m(x) = \sum_{x \in X} f(x) \overline{\tilde{L}_{v,\theta} g(x)} m(x)
$$

$$
= \frac{1}{2} \sum_{x,y \in X} b(x, y) \left( f(x) - e^{i\theta(x,y)} f(y) \right) \left( g(x) - e^{i\theta(x,y)} g(y) \right) + \sum_{x \in X} v(x) f(x) \overline{g(x)} m(x).
$$

Moreover, if $\tilde{L}_{v,\theta}[C_c(X)] \subseteq \ell^2(X, m)$, then for all $f, g \in C_c(X)$ it holds that

$$
Q_{v,\theta}^{(c)}(f, g) = \langle \tilde{L}_{v,\theta} f, g \rangle = \langle f, \tilde{L}_{v,\theta} g \rangle.
$$

**Proof.** The statements follow from a direct computation, where absolute convergence of the sums is guaranteed by $g \in C_c(X)$ and $\sum_y b(x, y) |f(y)| < \infty$ as $f \in \tilde{F}(X)$ (cf. [14, Lemma 4.7]). □

An important consequence of Green’s formula will be that whenever $Q_{v,\theta}^{(c)}$ is bounded below, closable and the domain of the closure is contained in $\tilde{F}(X)$, the corresponding self-adjoint operator is a restriction of $\tilde{L}_{v,\theta}$.

2.3. **Potentials.** In the sequel, for a function $w : X \to \mathbb{R}$ we will write $w_\pm = (\pm w) \vee 0$. That is $w = w_+ - w_-$. Let $q_v$ be the symmetric sesquilinear form given by $v$, that is,

$$
D(q_v) := \ell^2(X, |v|m), \quad q_v(f, g) := \sum_{x \in X} v(x) f(x) g(x) \overline{g(x)} m(x).
$$

We consider the following classes of potentials

$$
\mathcal{A}_\theta := \left\{ w : X \to \mathbb{R} \mid \text{There is } C \geq 0 \text{ such that } q_{w_-}(f) \leq Q_{w_+,\theta}^{(c)}(f) + C\|f\|^2 \text{ for all } f \in C_c(X) \right\}
$$

and

$$
\mathcal{B}_\theta := \left\{ w : X \to \mathbb{R} \mid \text{There are } \varepsilon > 0 \text{ and } C \geq 0 \text{ such that } q_{w_-}(f) \leq (1 - \varepsilon)Q_{w_+,\theta}^{(c)}(f) + C\|f\|^2 \text{ for all } f \in C_c(X) \right\} \subseteq \mathcal{A}_\theta.
$$

Important examples of potentials in $\mathcal{A}_\theta$ are included in the following remark.

**Remark 2.2.** By [33, Theorem 3.1], the Kato class corresponding to $Q$ is contained in $\mathcal{A}_0$. Moreover, as can be seen from Theorem 2.1 below, the class of admissible potentials corresponding to the closure of $Q_{v,\theta}^{(c)}$, which has been introduced in [11, 42] includes $\mathcal{A}_0$. See [24] for further characterizations of the class of admissible potentials. The importance
of $\mathcal{B}_\theta$ is due to the fact, that for such potentials a lot more can be said about the closure of $Q_{v,\theta}^{(c)}$ and the corresponding self-adjoint operator, see Proposition 2.6 and Theorem 2.8 below.

2.4. Semi-boundedness, closability and associated operators. In this subsection we state the already mentioned result that for potentials in $\mathcal{A}_0$, the corresponding magnetic quadratic form is closable and semi-bounded, where of course by “semi-bounded” we will always understand semi-bounded from below in what follows.

By making suitable assumptions on the geometry of $(X,b)$ or on the negative part of the potential we can then actually compute the action of the operator which is associated with the closure of $Q_{v,\theta}^{(c)}$. It will turn out that in certain cases this operator is a restriction of $\tilde{L}_{v,\theta}$ (see Theorem 2.8).

To this end, for bounded below and closable $Q_{v,\theta}^{(c)}$, we denote its closure by $Q_{v,\theta}$ and the corresponding self-adjoint operator by $L_{v,\theta}$, see [34, Theorem VIII.15]. In what follows, we use the conventions $Q := Q_{0,0}$ and $L := L_{0,0}$.

Let us start with the following simple observation:

**Lemma 2.3.** The form $Q_{v,0}^{(c)}$ being semi-bounded from below is equivalent to $v \in \mathcal{A}_0$. Moreover, for any $v \in \mathcal{A}_0$ the form $Q_{v,\theta}^{(c)}$ is semi-bounded.

**Proof.** The first statement is straightforward from the definition. For the second statement note that for $f \in C_c(X)$, we obviously have $q_{v_-}(f) = q_{v_-}(|f|)$ and $Q_{v,0}^{(c)}(|f|) \leq Q_{v,\theta}^{(c)}(f)$.

In fact, one has:

**Theorem 2.4.** For every $v \in \mathcal{A}_0$ the form $Q_{v,\theta}^{(c)}$ is semi-bounded and closable.

**Corollary 2.5.** The form $Q_{v,0}^{(c)}$ is semi-bounded and closable if and only if $v \in \mathcal{A}_0$.

**Proof.** The ‘if’ follows directly from Theorem 2.4 and the ‘only’ if follows as $Q_{v,\theta}^{(c)}$ is not bounded below if $v$ is not in $\mathcal{A}_0$, by Lemma 2.3.

Theorem 2.4 will be proven by an approximation argument, cutting off the negative parts of the potentials and employing a Feynman-Kac-Itô formula for potentials that are bounded from below. The proof is given in Section 4.4. Here we will only prove a special case that is used in the proof later:

**Proposition 2.6.** For $v \in \mathcal{B}_\theta$ the form $Q_{v,\theta}^{(c)}$ is semi-bounded and closable. Its closure $Q_{v,\theta}$ is semi-bounded and given by $Q_{v,\theta} = Q_{v,\theta}^{(c)} - q_{v_-}$ with domain $D(Q_{v,\theta}) = D(Q_{v,\theta}^{(c)})$. Furthermore, for all $f, g \in D(Q_{v,\theta})$ one has

$$Q_{v,\theta}(f,g) = \frac{1}{2} \sum_{x,y \in X} \left( f(x) - e^{i\theta(x,y)} f(y) \right) \left( g(x) - e^{i\theta(x,y)} g(y) \right) + \sum_{x \in X} f(x) g(x) v(x) m(x).$$
Proof. We start by proving the closability for $v_+$. Define the form

$$Q_{v+,\vartheta}^{\max}: \ell^2(X, m) \longrightarrow [0, \infty]$$

by

$$Q_{v+,\vartheta}^{\max}(f) = \frac{1}{2} \sum_{x,y \in X} b(x, y)|f(x) - e^{i\vartheta(x,y)}f(y)|^2 + \sum_{x \in X} v_+(x)|f(x)|^2m(x).$$

In order to show that $Q_{v+,\vartheta}^{(c)}$ is closable, it suffices to demonstrate that $Q_{v+,\vartheta}^{\max}$ is lower semi-continuous. This however is a consequence of Fatou’s lemma and closability for $v_+$ is proven. Now as $v \in \mathcal{B}_\vartheta$, we obtain

$$\varepsilon Q_{v+,\vartheta}(f) + (C' - C)\|f\|^2 \leq Q_{v+,\vartheta}^{(c)}(f) + C'\|f\|^2 \leq Q_{v+,\vartheta}(f) + C\|f\|^2$$

for all $f \in C_c(X)$ with suitable constants $\varepsilon > 0$ and $C' > C > 0$. These inequalities and the closability of $Q_{v+,\vartheta}^{(c)}$ imply the closability of $Q_{v+,\vartheta}^{(c)}$ and $D(Q_{v+,\vartheta}) = D(Q_{v+,\vartheta}).$ For the statement on the action of the form, let us first note that

$$Q_{v+,\vartheta}^{\max}(f) = \frac{1}{2} \sum_{x,y \in X} b(x, y)|f(x) - e^{i\vartheta(x,y)}f(y)|^2 + \sum_{x \in X} v(x)|f(x)|^2m(x)$$

is well defined for all $f \in D(Q_{v,+\vartheta})$, i.e. $Q_{v+,\vartheta}^{\max}(f) < \infty$ and $q_{v_+}(f) < \infty$. To see this, pick a sequence of compactly supported functions $(f_n)$ converging to $f$ with respect to the form norm induced by $Q_{v_+}$. We then obtain by Fatou’s lemma and by $v \in \mathcal{B}_\vartheta$

$$Q_{v_+}^{\max}(f) \leq \liminf_{n \to \infty} Q_{v_+}(f_n) = Q_{v_+}(f)$$

and

$$q_{v_+}(f) \leq \liminf_{n \to \infty} q_{v_+}(f_n) \leq \liminf_{n \to \infty}(1 - \varepsilon)Q_{v_+}(f_n) + C\|f_n\|^2 = (1 - \varepsilon)Q_{v_+}(f) + C\|f\|^2.$$ 

Altogether, the above, Fatou’s lemma and $Q_{v_+}^{\max}$ being a quadratic form implies

$$|Q_{v_+}^{\max}(f) - Q_{v_+}(f)|^{1/2} = \lim_{n \to \infty} |Q_{v_+}^{\max}(f) - Q_{v_+}^{\max}(f_n)|^{1/2}$$

$$\leq \liminf_{n \to \infty} |Q_{v_+}^{\max}(f) - Q_{v_+}^{\max}(f_n)|^{1/2} + \liminf_{n \to \infty} |q_{v_+}(f) - q_{v_+}(f_n)|^{1/2}$$

$$\leq \liminf_{n \to \infty} Q_{v_+}^{\max}(f - f_n)^{1/2} + \liminf_{n \to \infty} q_{v_+}(f - f_n)^{1/2}$$

$$\leq \liminf_{n,m \to \infty} Q_{v_+}^{\max}(f_m - f_n)^{1/2}$$

$$+ \liminf_{n,m \to \infty} ((1 - \varepsilon)Q_{v_+}^{\max}(f_m - f_n) + C\|f_m - f_n\|^2)^{1/2}.$$ 

As $(f_n)$ is a Cauchy-sequence with respect to the form norm, these computations show the claim. \hfill \Box

Let us now put the mentioned condition $\tilde{L}_{v,\vartheta}[C_c(X)] \subseteq \ell^2(X, m)$ into perspective. The following was observed in [23] for the Dirichlet form case.

**Lemma 2.7.** The following assertions are equivalent:
(i) $\tilde{L}_{v,\theta}[C_c(X)] \subseteq \ell^2(X, m)$
(ii) For all $x \in X$ the function $X \to [0, \infty)$, $y \mapsto b(x, y)/m(y)$ belongs to $\ell^2(X, m)$.

If one of the above is satisfied, then $\ell^2(X, m) \subseteq \tilde{F}(X)$.

Proof. This follows from a straightforward computation, see Proposition 3.3 of [23] for details. □

We can now state the Theorem about the action of the operators.

**Theorem 2.8.** Suppose one of the following conditions holds.

(i) $v \in B_\theta$.
(ii) $\tilde{L}_{v,\theta}[C_c(X)] \subseteq \ell^2(X, m)$ and $v \in A_\theta$.
(iii) $(X, b)$ is locally finite and $v \in A_\theta$.

Then $Q^{(c)}_{v,\theta}$ is semibounded and closable and the corresponding operator is a restriction of $\tilde{L}_{v,\theta}$.

Proof. Clearly assumption (iii) implies (ii), hence it suffices to show the statement under assumption (i) and (ii). Let us assume (i). As seen in Proposition 2.6 the form $Q^{(c)}_{v,\theta}$ is semibounded, closable and satisfies $D(Q_{v,\theta}) = D(Q^{(c)}_{v,\theta})$. We will now show $D(Q^{(c)}_{v,\theta}) \subseteq \tilde{F}(X)$ since this, the action of $Q_{v,\theta}$ (see Proposition 2.6) and Green’s formula would imply

$$\langle L_{v,\theta} f, g \rangle = Q_{v,\theta}(f, g) = \sum_{x \in X} \tilde{L}_{v,\theta} f(x) g(x) m(x)$$

for all $f \in D(L_{v,\theta})$ and $g \in C_c(X)$, which would prove the claim. Thus, let $f \in D(Q_{v,\theta})$ be given. We then obtain

$$\sum_{y \in X} b(x, y)|f(y)| \leq \sum_{y \in X} b(x, y)|f(x)| - e^{i\theta(x, y)} f(y)| + \sum_{y \in X} b(x, y)|f(x)|$$

$$\leq \deg_1(x)^{1/2} \left( \sum_{y \in X} b(x, y)|f(x)| - e^{i\theta(x, y)} f(y)|^2 \right)^{1/2} + \deg_1(x)|f(x)|.$$ 

The above is finite by Proposition 2.6.

Now let us assume (ii) holds. Then $Q^{(c)}_{v,\theta}$ is semibounded and closable by the Friedrich’s extension theorem. Let $f \in D(L_{v,\theta})$ be given and $(f_n)$ be a sequence of compactly supported functions converging to $f$ in form norm. Then for all $g \in C_c(X)$ we obtain

$$\langle L_{v,\theta} f, g \rangle = Q_{v,\theta}(f, g) = \lim_{n \to \infty} Q^{(c)}_{v,\theta}(f_n, g)$$
$$= \lim_{n \to \infty} \sum_{x \in V} \tilde{L}_{v,\theta} f_n(x) g(x) m(x).$$
As $g$ is compactly supported, to prove the claim it suffices to show the pointwise convergence of $\tilde{L}_{v,\theta}f_n$ towards $\tilde{L}_{v,\theta}f$. For this it is sufficient to show the convergence
$$\sum_{y\in X} b(x, y)f_n(y)e^{i\theta(x, y)} \to \sum_{y\in X} b(x, y)f(y)e^{i\theta(x, y)}$$
for each $x \in X$, which can be deduced from
$$\sum_{y\in X} b(x, y)|f_n(y) - f(y)| \leq \left( \sum_{y\in X} b(x, y)^2 \right)^{1/2} \left( \sum_{y\in X} |f_n(y) - f(y)|^2 m(y) \right)^{1/2}$$
where the finiteness of the first factor of the right-hand side follows from Lemma 2.7.

2.5. **Uniqueness of semi-bounded self-adjoint restrictions.** Noting that the classical question of essential self-adjointness notion only makes sense if $\tilde{L}_{v,\theta}[C_c(X)] \subseteq \ell^2(X, m)$, we can nevertheless, in general, still ask for the uniqueness of semi-bounded self-adjoint restrictions of $\tilde{L}_{v,\theta}$ on $\ell^2(X, m)$ in an appropriate sense. (Precisely, we ask whether there is a unique dense subspace $D$ of $\ell^2(X, m)$ such that the restriction of $\tilde{L}_{v,\theta}$ to $D$ is a semi-bounded self-adjoint operator.) This is the content of this section. The latter uniqueness property of the operators directly leads to uniqueness of the semigroups, and thus also of the corresponding processes.

We now present two criteria for the uniqueness of self-adjoint restrictions. The first one, Theorem 2.10, essentially makes an assumption on the underlying weighted graph as a measure space, and the second one, Theorem 2.12, makes an assumption on the weighted graph as a metric space. Both criteria are based on the following abstract condition concerning uniqueness of the solutions of $(\tilde{L}_{v,\theta} - \lambda)u = 0$.

**Proposition 2.9.** Assume there exists some constant $C \in \mathbb{R}$ such that for all $\lambda < C$, every solution $u \in \tilde{F}(X) \cap \ell^2(X, m)$ of $(\tilde{L}_{v,\theta} - \lambda)u = 0$ satisfies $u \equiv 0$. Then $\tilde{L}_{v,\theta}$ has at most one semi-bounded self-adjoint restriction on $\ell^2(X, m)$. Furthermore, the following holds:

(i) If, additionally, $v \in \mathcal{B}_\theta$, then $\tilde{L}_{v,\theta}$ has a unique semi-bounded self-adjoint restriction.

(ii) If, additionally, $v \in \mathcal{A}_\theta$ and $\tilde{L}_{v,\theta}[C_c(X)] \subseteq \ell^2(X, m)$, then $\tilde{L}_{v,\theta}|_{C_c(X)}$ is essentially self-adjoint.

**Proof.** Suppose there are two such restrictions $L_1$ and $L_2$ on $\ell^2(X, m)$ which do not coincide. Let $C$ be a common lower bound of $L_1$ and $L_2$. Then, their resolvents $(L_1 - \lambda)^{-1}$ and $(L_2 - \lambda)^{-1}$ are different for $\lambda < C$. Hence, we infer
$$u = ((L_1 - \lambda)^{-1} - (L_2 - \lambda)^{-1})\varphi \neq 0$$
for some $\varphi \in C_c(X)$.

As $L_1$ and $L_2$ are both restrictions of $\tilde{L}_{v,\theta}$, we have $(\tilde{L}_{v,\theta} - \lambda)u = \varphi - \varphi = 0$ which proves the statement.

Under the additional assumption (i) the existence of a semi-bounded self-adjoint restrictions follows from Theorem 2.8.

For the statement under the assumptions of (ii) assume $\tilde{L}_{v,\theta}[C_c(X)] \subseteq \ell^2(X, m)$. Let $L_{\min} = \tilde{L}_{v,\theta}|_{C_c(X)}$ and $L_{\max} = L^*_{\min}$ its adjoint. It suffices to show that $L_{\max}$ is self-adjoint.
From Lemma 2.4 we infer \( \ell^2(X, m) \subseteq \tilde{F} \), which allows the application of Green’s formula, i.e.,

\[
\langle u, \tilde{L}_{v, \theta} f \rangle = \langle \tilde{L}_{v, \theta} u, f \rangle
\]

for all \( u \in \ell^2(X, m) \), such that \( \tilde{L}_{v, \theta} u \in \ell^2(X, m) \) and all \( f \in C_c(X) \). This shows that \( L_{\max} \) is a restriction of \( \tilde{L}_{v, \theta} \) with domain

\[
D(L_{\max}) = \{ u \in \ell^2(X, m) \mid \tilde{L}_{v, \theta} u \in \ell^2(X, m) \}.
\]

Now let \( L_{v, \theta} \) be the self-adjoint semi-bounded operator associated with the closure of \( Q_{v, \theta}^{(c)} \). By Theorem 2.8, \( L_{v, \theta} \) is a restriction of \( \tilde{L}_{v, \theta} \), satisfying \( D(L_{v, \theta}) \subseteq D(L_{\max}) \). Therefore, it suffices to show the other inclusion. Let \( u \in D(L_{\max}) \) be given and let \( w = (L_{v, \theta} - \lambda)^{-1}(\tilde{L}_{v, \theta} - \lambda)u \in D(L_{v, \theta}) \). We obtain \( (\tilde{L}_{v, \theta} - \lambda)(w - u) = 0 \), which implies \( u = w \in D(L_{v, \theta}) \) by our assumptions.

Now, the first criterion for uniqueness is based on a result from [23] for \( \theta = 0 \), which was generalized to locally finite magnetic operators in [10]. The result below stands somewhat skew to the one of [10], as there no assumption on the semi-boundedness of the quadratic form is made, where we do not assume local finiteness.

**Theorem 2.10.** (Uniqueness - measure space criterion) Assume that for some \( \alpha \in \mathbb{R} \) and all infinite paths \( (x_n)_{n=0}^{\infty} \) one has

\[
\sum_{n=1}^{\infty} m(x_n) \prod_{j=0}^{n-1} \left( 1 + \frac{v(x_j) - \alpha}{\deg_m(x_j)} \right)^2 = \infty.
\]

Then the following holds:

(i) If, additionally, \( v \in B_{\theta} \), then \( \tilde{L}_{v, \theta} \) has a unique semi-bounded self-adjoint restriction.

(ii) If, additionally, \( v \in A_{\theta} \) and \( \tilde{L}_{v, \theta}[C_c(X)] \subseteq \ell^2(X, m) \), then \( \tilde{L}_{v, \theta}|_{C_c(X)} \) is essentially self-adjoint.

**Proof.** As \( Q_{v, \theta}^{(c)} \) is bounded below by some constant \( C \) we infer \( \deg_m + v - \lambda > 0 \) for all \( \lambda < C \). Thus, if the sums in the assumption diverge for a particular \( \alpha \) then there is \( \lambda_0 < -(|C| + |\alpha|) \) such that these sums diverge for all \( \lambda < \lambda_0 \). Let \( u \in \ell^2(X, m) \cap \tilde{F}(X) \) be a solution to the equation \( (\tilde{L}_{v, \theta} - \lambda)u = 0 \) for some \( \lambda < \lambda_0 \). Then, one easily gets

\[
|u(x)| \leq \left( \frac{1}{m(x)} \sum_{y \in X} b(x, y)|u(y)| \right) |\deg_m(x) + v(x) - \lambda|^{-1},
\]

for all \( x \in V \). Suppose \( u \neq 0 \), i.e., there exists an \( x_0 \in X \) such that \( u(x_0) \neq 0 \). By the above inequality there is an \( x_1 \sim x_0 \) with

\[
|u(x_1)| \geq \left| 1 + \frac{v(x_0) - \lambda}{\deg_m(x_0)} \right| |u(x_0)|.
\]
Continuing this procedure, we may inductively choose an infinite path \((x_n)\) which satisfies
\[ |u(x_n)| \geq \prod_{i=0}^{n-1} \left| 1 + \frac{v(x_i) - \lambda}{\deg_m(x_i)} \right| |u(x_0)|. \]

Therefore, we obtain
\[ \|u\|^2 \geq \sum_{n=1}^{\infty} |u(x_n)|^2 m(x_n) \geq \sum_{n=1}^{\infty} m(x_n)|u(x_0)|^2 \prod_{j=0}^{n-1} \left| 1 + \frac{v(x_j) - \lambda}{\deg_m(x_j)} \right|, \]
which implies \(u(x_0) = 0\) by the assumption. As this contradicts \(u(x_0) \neq 0\), we can conclude \(u \equiv 0\) and the statement follows from Proposition 2.9.

As we have already remarked, the second criterion is going to deal with the completeness of the weighted graph with respect to some appropriate metric structure.

**Definition 2.11.** (a) A (pseudo) metric \(d\) on the underlying set \(X\) is called a path (pseudo) metric for the graph, if there is a map \(\sigma : X \times X \to [0, \infty)\) with the properties \(\{\sigma = 0\} \subseteq \{b = 0\}\) and
\[
d(x, y) = \inf_{x = x_0 \sim \ldots \sim x_n = y} \sum_{j=1}^{n} \sigma(x_{j-1}, x_j).
\]
(b) A (pseudo) metric \(d\) is called intrinsic with respect the underlying weighted graph, if
\[
\sum_{y \in X} b(x, y)d(x, y)^2 \leq m(x), \quad \text{for all } x \in X.
\]

We remark that the above definition of intrinsic metrics is adapted to our situation from the abstract Dirichlet space setting of [8]. Furthermore, any weighted graph admits an intrinsic metric. For example, one can take the path metric with weights \(\sigma(x, y) = (\deg_m(x) \wedge \deg_m(y))^{-\frac{1}{2}}\) for \(x \sim y\).

Now, we have the following result, which has also been suggested to us by O. Milatovic in a private communication.

**Theorem 2.12.** (Uniqueness - metric space criterion) Let \(d\) be an intrinsic pseudo metric on the underlying weighted graph.

(a) Assume that \(v \in B_\theta\) and that the metric balls with respect to \(d\) are all finite. Then the operator \(\tilde{L}_{v, \theta}\) has a unique semi-bounded self-adjoint restriction.

(b) Assume that \(v \in A_\theta\) and that the underlying graph is locally finite and \((X, d)\) is a complete path metric space. Then the operator \(\tilde{L}_{v, \theta}\big|_{C_c(X)}\) is essentially self-adjoint.

**Remark 2.13.** 1. Theorem 2.12 is a generalization of [20, Corollary 1 and Theorem 2] and [29, Theorem 1.5]. While the first reference does not allow magnetic fields and negative potentials, the second one assumes a uniformly bounded vertex degree, a condition that we will avoid by using the concept of intrinsic metrics. The proof works analogously to [29]. We refer also to [31] for results in this direction.
2. In view of the Kato class being contained in $\mathcal{A}_0$ (cf. [38, Theorem 3.1]), Theorem 2.12 can be considered in fact as a weighted-graph analogue of the corresponding result for geodesically complete Riemannian manifolds from [13]. The proof of Theorem 2.12 given below, is based on the following ground state transform: For any $f = f_1 + if_2$ with real-valued $f_1, f_2 \in \tilde{F}(X)$ we define

$$Q^{(f)}(g, h) = \frac{1}{2} \sum_{x, y \in X} b^{(f)}(x, y) \left( g(x) - g(y) \right) \left( h(x) - h(y) \right), \quad g, h \in C_c(X),$$

where $b^{(f)}(x, y)$ is defined for $x, y \in X$ as

$$b(x, y) \left( \cos(\theta(x, y))(f_1(x)f_1(y) + f_2(x)f_2(y)) + \sin(\theta(x, y))(f_1(y)f_2(x) - f_1(x)f_2(y)) \right).$$

**Proposition 2.14.** Assume $f \in \tilde{F}(X)$ and $\lambda \in \mathbb{R}$ are such that $(\tilde{L}_{v, \theta} - \lambda)f = 0$. Then, for all $g \in C_c(X)$, one has

$$Q^{(c)}(f, f) = Q^{(f)}(g, g) + \lambda \|fg\|^2.$$

**Proof.** The proof follows by direct calculation (cf. [29, Proposition 3.5] or [14, Proposition 3.2]). \qed

**Proof of Theorem 2.12.** Let $f \in \ell^2(X, m) \cap \tilde{F}(X)$ and $\lambda < C - 1$ be such that $(\tilde{L}_{v, \theta} - \lambda)f = 0$. We fix some $x_0 \in X$, denote the $R$-ball, $R > 0$, with respect to $d$ around $x_0$ by $B_R$ and let $\eta_R : X \to [0, 1]$, be given by

$$\eta_R(x) := 1 \wedge \left( \frac{2R - d(x, x_0)}{R} \right) \quad \text{for } x \in X.$$

By a Hopf-Rinow type theorem, [20, Theorem A1] (cf. [29, Section 6]), the balls are finite under the metric completeness assumption in (b). Hence, finiteness of the balls in (a) and (b) implies $\eta_R \in C_c(X)$. Then using $\eta_R|_{B_R} \equiv 1$, the semi-boundedness of $Q^{(c)}(\cdot, \cdot)$ by $\lambda + 1$, and Proposition 2.13 we obtain

$$\|f_1|_{B_R}\|^2 \leq \|\eta_R\|^2 \leq Q^{(c)}(f\eta_R, f\eta_R) - \lambda \|\eta_R\|^2 = Q^{(f)}(\eta_R, \eta_R).$$

Employing the inequalities $b^{(f)}(x, y) \leq b(x, y)(|f(x)|^2 + |f(y)|^2)$, $(\eta_R(x) - \eta_R(y)) \leq d(x, y)/R$ and the intrinsic metric property, yields

$$\ldots \leq \sum_{x \in X} |f(x)|^2 \sum_{y \in X} b(x, y)(\eta_R(x) - \eta_R(y))^2 \leq \sum_{x \in X} |f(x)|^2 \sum_{y \in X} b(x, y)d(x, y)^2 \leq \frac{1}{R^2} \|f\|^2.$$

Hence, letting $R \to \infty$ shows that $\|f\| = 0$. Thus, any solution $f$ in $\ell^2(X, m) \cap \tilde{F}(X)$ to $(\tilde{L}_{v, \theta} - \lambda)f = 0$ is trivial. Now, (a) follows directly by Proposition 2.9 while for (b) we additionally have to note that local finiteness implies $\tilde{L}_{v, \theta}[C_c(X)] \subseteq \ell^2(X, m).$ \qed

**Remark 2.15.** In the proofs of Theorem 2.10 and 2.12 the assumption $v \in \mathcal{A}_0$ is not needed to show uniqueness of solutions to the equation $(\tilde{L}_{v, \theta} - \lambda)u = 0$. Thus, $v \in \mathcal{A}_0$ can be replaced in both theorems by the more general assumption that $v$ is such that $Q^{(c)}_{v, \theta}$ is closable and semi-bounded.
2.6. **Semigroup convergence.** We close this section with a result on the convergence of certain geometrically defined restrictions of the semigroups \((e^{-tL_{v,\theta}})_{t \geq 0}\). This result will be central for the proof of the Feynman-Kac-Itô formula.

We start by introducing some notation that will be useful several times in the sequel: For any finite subset \(U \subseteq X\), we denote with slight abuse of notation the restriction of \(m\) to \(U\) also by \(m\) and we define \(Q^{(U)}_{v,\theta}\) to be the restriction of \(Q^{(c)}_{v,\theta}\) to 
\[\ell^2(U, m) = C_c(U) = C(U).\]

Here, it should be noticed that the finiteness of \(U\) implies that \(Q^{(U)}_{v,\theta}\) is automatically closed.

Let \(L^{(U)}_{v,\theta}\) be the operator corresponding to \(Q^{(U)}_{v,\theta}\). We have a canonical inclusion operator 
\[\iota_{U} : \ell^2(U, m) \hookrightarrow \ell^2(X, m)\]
which comes from extending functions to zero away from \(U\), and its adjoint will be denoted with \(\pi_{U} := \iota_{U}^*\).

**Definition 2.16.** A sequence \((X_n)_{n \in \mathbb{N}}\) of finite sets \(X_n \subseteq X\) is called an exhausting sequence for (the set) \(X\), if 
\[X_n \subseteq X_{n+1}\]
for all \(n\) and if \(X = \bigcup_{n \in \mathbb{N}} X_n\).

The following geometric approximation is based on the Mosco convergence of the quadratic forms.

**Proposition 2.17.** Suppose \(Q^{(c)}_{v,\theta}\) is semi-bounded and closable, and let \((X_n)_{n \in \mathbb{N}}\) be an exhausting sequence. Then for all \(t \geq 0\) one as 
\[\ell_{X_n} e^{-tL_{v,\theta}^{(X_n)}} \pi_{X_n} \to e^{-tL_{v,\theta}}\]
strongly in \(\ell^2(X, m)\) as \(n \to \infty\).

**Proof.** By Theorem [C.2] it suffices to show that the forms \(Q^{(X_n)}_{v,\theta}\) converge to \(Q_{v,\theta}\) as \(n \to \infty\) in the generalized Mosco sense. Part (i) of Definition [C.1] follows from the closedness of \(Q_{v,\theta}\) while part (ii) is due to the fact that \(C_c(X)\) is a core for \(Q_{v,\theta}\) by definition. \(\square\)

3. **Stochastic processes on discrete sets**

Let us introduce the necessary probabilistic framework. Recall that we set \(Q = Q_{0,0}\) and \(L = L_{0,0}\). We start by noting that \(Q\) is a regular Dirichlet form on \(\ell^2(X, m)\), cf. [23]. In particular, the semigroup \((e^{-tL})_{t \geq 0}\) defines a strongly continuous semigroup of bounded operators on \(\ell^2(X, m)\), which can be written as 
\[e^{-tL} f(x) = \sum_{y \in X} e^{-tL}(x, y) f(y) m(y).\]

We recall that the connectedness of the underlying graph implies the strict positivity \(e^{-tL}(x, y) > 0\) for all \(t > 0, x, y \in X\).

It follows from an abstract result on Dirichlet spaces that there is a bijection between quasi-regular Dirichlet forms on \(\ell^2(X, m)\) and \((m\)-equivalence classes of) right processes with state space \(X\) (see [26], Theorem 6.8 for this terminology). However, we prefer to be more concrete here and to recall a direct construction of a process which is associated to \(Q\).
To this end, we take a discrete time Markov chain \((Y_n)_{n \in \mathbb{N}}\) with state space \(X\) which satisfies
\[
\mathbb{P}(Y_n = x \mid Y_{n-1} = y) = \frac{b(x, y)}{\deg_1(y)} \quad \text{for all } n \geq 1,
\]
where in the following \((\Omega, \mathcal{F}, \mathbb{P})\) is some fixed probability space, \(\deg_1(x) = \sum_{y \in X} b(x, y), x \in X, \text{ and } \mathbb{N} = \{0, 1, 2, \ldots\}\). Let \((\xi_n)_{n \in \mathbb{N}}\) be a sequence of independent exponentially distributed random variables of parameter 1 which are also independent of \((Y_n)_{n \in \mathbb{N}}\). For \(n \geq 1\), we define the sequence of stopping times
\[
J_n := \frac{1}{\deg_1(Y_{n-1})} \xi_n, \quad \tau_n := J_1 + \cdots + J_n,
\]
with the convention \(\tau_0 := 0\). We, furthermore, define the predictable stopping time \(\tau := \sup_{n \in \mathbb{N}} \tau_n > 0\). With these preparations, one can define the jump process
\[
X : [0, \tau) \times \Omega \rightarrow X, \quad X|_{[\tau_n, \tau_{n+1}) \times \Omega} := Y_n \quad \text{for all } n \in \mathbb{N}.
\]
Note that \(X\) is maximally defined and that the \(\tau_n\)'s are precisely the jump times of \(X\). If \(\mathbb{P}_x := \mathbb{P}(\cdot \mid X_0 = x)\), and if \(\mathcal{F}_s\) denotes the filtration \(\mathcal{F}_t = \sigma(X_s \mid s \leq t), t \geq 0\), corresponding to \(X\), then the tuple
\[
(\Omega, \mathcal{F}, \mathcal{F}_s, X, (\mathbb{P}_x)_{x \in X})
\]
is a (reversible) strong Markov process (see for example Theorem 6.5.4 in [32] for a proof), which is associated to \(Q\) in the sense that for all \(f \in \ell^2(X, m)\) one has
\[
e^{-tL} f(x) = \sum_{y \in X} e^{-tL}(x, y)m(y) = \mathbb{E}_x \left[1_{\{t<\tau\}}f(X_t)\right].
\]
(1)

The latter formula is well-known (see for example [32]), in fact, this formula can also be consistently deduced from the Feynman-Kac-Itô formula in Theorem 4.1 below, as we are not going to use it in the proof of the latter result. Formula (1) has a simple but nevertheless important consequence, namely, one has
\[
e^{-tL}(x, y)m(y) = \mathbb{P}_x(X_t = y) \quad \text{for all } t > 0, x, y \in X,
\]
in particular, \(Q\) is stochastically complete, if and only if
\[
\mathbb{P}_x(t < \tau) = 1 \quad \text{for some/all } t \geq 0, x \in X.
\]
Let us denote the number of jumps of \(X\) until \(t\) by \(N(t)\), i.e.,
\[
N(t) = \sup\{n \in \mathbb{N} \mid \tau_n \leq t\}.
\]
The following definitions will be central for this paper. We define two random variables by
\[
\int_0^t \theta(dX_s) := \sum_{n=1}^{N(t)} \theta(X_{\tau_{n-1}}, X_{\tau_n}) : \{t < \tau\} \rightarrow \mathbb{R}
\]
and
\[
\mathcal{J}_t(v, \theta|X) := i \int_0^t \theta(dX_s) - \int_0^t v(X_s)ds : \{t < \tau\} \rightarrow \mathbb{C},
\]
so that, in particular, \( \mathcal{S}_t(v, 0|X) \) is the usual additive Feynman-Kac functional 
\[
\mathcal{S}_t(v, 0|X) = -\int_0^t v(X_s) ds.
\]
Here, the well-definedness of \( \int_0^t \theta(dX_s) \) and \( \int_0^t v(X_s) ds \) (and thus of \( \mathcal{S}_t(v, \theta|X) \)) follows from the simple observation \( \{N(t) < \infty\} = \{t < \tau\} \). Furthermore, it is easily seen that the processes 
\[
\int_0^\cdot \theta(dX_s) : [0, \tau) \times \Omega \to \mathbb{R}, \quad \mathcal{S}_\cdot(v, \theta|X) : [0, \tau) \times \Omega \to \mathbb{C}
\]
are \( \mathcal{F}_s \)-semimartingales under \( P_x \) with lifetime \( \tau \), which motivates the following definition.

Definition 3.1. The process \( \int_0^\cdot \theta(dX_s) \) is called the stochastic line integral of \( \theta \) along \( X \), and \( \mathcal{S}_\cdot(v, \theta|X) \) is called the Euclidean action corresponding to \( \theta \) and \( v \).

Here, the notions “line integral” and “Euclidean action” are both motivated from the manifold setting \[7\], where in the first case \( \theta \) is interpreted as a 1-form on the graph \( X \). We refer the reader to \[29\] for a justification of the latter geometric interpretation.

4. The Feynman-Kac-Itô Formula

4.1. Statement. The following theorem is the main result of this paper.

Theorem 4.1. (Feynman-Kac-Itô formula) Let \( v \in A_0 \). Then for any \( f \in \ell^2(X, m) \), \( t \geq 0 \) and \( x \in X \) one has 
\[
e^{-tL_{v,\theta}} f(x) = \mathbb{E}_x \left[ e^{\mathcal{S}_t(v, \theta|X)} f(X_t) \right].
\]

Remark 4.2. It should be noted that we make no assumptions on the underlying weighted graph, the magnetic potential \( \theta \) and the positive part \( v_+ \) of \( v \). The only assumption on \( v_- \) is semi-boundedness of the non-magnetic form. We believe that this setting should actually cover all possible applications.

The rest of the section is dedicated to the proof of Theorem 4.1. The proof is divided into several parts. First, we prove the formula for finite subgraphs. Secondly, we employ the latter fact combined with an exhaustion argument, using Proposition 2.17 to control the operator side. This yields the statement under the assumptions that the forms \( Q_{v,0}^{(c)} \) and \( Q_{v,0}^{(c)} \) are closable and semi-bounded. Thirdly, we use the formula for semi-bounded potentials to show closability of the quadratic forms \( Q_{v,\theta}^{(c)} \) for \( v \in A_0 \). Having this the statement follows by the second step.

4.2. Proof for finite subgraphs. For any finite subset \( U \subset X \) recall the notation from Section 2.6 and let 
\[
\tau_U := \inf\{ s \geq 0 | X_s \subset X \setminus U \}
\]
be the first exit time of \( X \) from \( U \), which is a \( \mathcal{F}_s \)-stopping time. The goal of this subsection is to prove the following proposition, which is the main tool in the proof of Theorem 4.1 but is in fact of an independent interest (see also the proof of Proposition 5.4 below). Here, it should again be noted that in view of the finiteness of \( U \), the potentials may be arbitrary.
Proposition 4.3. Let \( U \subseteq X \) be finite. Then for all \( f \in \ell^2(U, m) \), \( x \in U \), \( t \geq 0 \) one has
\[
e^{-tL(U)} f(x) = \mathbb{E}_x \left[ 1_{\{t < r_U\}} e^{S_t(v, \theta | X)} f(X_t) \right].
\]
The proof of the proposition above is based on three auxiliary lemmas.

Lemma 4.4. Let \( U \subseteq X \) be finite. Then, \( (T_t(v, \theta, U))_{t \geq 0} \) defined for \( f \in \ell^2(U, m) \) by
\[
T_t(v, \theta, U)f(x) := \mathbb{E}_x \left[ 1_{\{t < \tau_U\}} e^{S_t(v, \theta | X)} f(X_t) \right], \quad x \in U, t \geq 0,
\]
is a strongly continuous semigroup of bounded operators on \( \ell^2(U, m) \).

Proof. The asserted boundedness is trivial and the semigroup property follows from the Markov property of \( X \). By the semigroup property it is enough to check strong continuity at \( t = 0 \), which can be easily checked using the boundedness of the integrand and the right continuity of \( X \).

Lemma 4.5. Let \( f \in C^c_c(X) \), \( t > 0 \), and let the function \( \varphi_{t,f} : X \rightarrow \mathbb{C} \) be defined by
\[
\varphi_{t,f}(x) = \frac{1}{t} \mathbb{E}_x \left[ 1_{\{2 \leq N(t) < \infty\}} f(X_t) \right].
\]
Then, for all \( x \in X \), one has \( \varphi_{t,f}(x) \rightarrow 0 \) as \( t \searrow 0 \).

Proof. As \( f \) is bounded it suffices to show
\[
\frac{1}{t} \mathbb{P}_x(N(t) \geq 2) = \frac{1}{t} \mathbb{P}_x(N(t) = 0) - \frac{1}{t} \mathbb{P}_x(N(t) = 1) \rightarrow 0, \quad \text{as } t \searrow 0.
\]
From the considerations of Section 3 we derive that
\[
\mathbb{P}_x(N(t) = 0) = \mathbb{P}_x(t < \tau_1) = \mathbb{P}_x(\text{deg}_m(x)t < \xi_1) = e^{-\text{deg}_m(x)t}.
\]
Hence, the first summand of the right hand side of (3) tends to \( e^{-\text{deg}_m(x)t} \) as \( t \searrow 0 \). We write \( \text{deg} := \text{deg}_1 \), i.e., \( \text{deg}(z) = \sum_{y \in X} b(z, y) \), \( z \in X \). For determining the second summand, let us compute
\[
\mathbb{P}_x(N(t) = 1) = \sum_{y \in X} \mathbb{P}(N(t) = 1, X_{\tau_1} = y)
\]
\[
= \sum_{y \in X} \mathbb{P}_x(N(t) = 1|X_{\tau_1} = y) \mathbb{P}_x(X_{\tau_1} = y)
\]
\[
= \sum_{y \in X, \text{deg}_m(x) \neq \text{deg}_m(y)} \frac{\text{deg}_m(x)}{\text{deg}_m(x) - \text{deg}_m(y)} \left[ e^{-t\text{deg}_m(y)} - e^{-t\text{deg}_m(x)} \right] \frac{b(x, y)}{\text{deg}(x)}
\]
\[
+ \sum_{y \in X, \text{deg}(x) = \text{deg}(y) \neq \text{deg}_m(x)} \left[ t\text{deg}_m e^{-t\text{deg}_m(x)} \right] \frac{b(x, y)}{\text{deg}(x)}.
\]
The last equality follows from a simple computation using
\[
\{N(t) = 1\} = \{J_1 \leq t < J_1 + J_2\}
and that $J_1$ and $J_2$ are exponentially distributed under the conditions $X_0 = x$ and $X_{r_1} = y$. The above calculation and Lebesgue’s dominated convergence theorem imply
\[
\frac{\mathbb{P}_x(N(t) = 1)}{t} \to \frac{1}{m(x)} \sum_{y \in X} b(x, y) = \deg_{m}(x), \quad \text{as } t \searrow 0,
\]
showing our claim. \hfill \Box

Lemma 4.6. Let $U \subseteq X$ be finite. Then, for all $f \in \ell^2(U, m)$ and $x \in U$, one has
\[
\lim_{t \searrow 0} \frac{T_t(v, \theta, U)f(x) - f(x)}{t} = -L_{v, \theta} f(x).
\]

Proof. We fix an arbitrary $x \in U$ and compute
\[
\frac{T_t(v, \theta, U)f(x) - f(x)}{t}
\]
(4)
\[
= \mathbb{E}_x \left[ 1_{\{N(t) = 0\}} e^{-tv(x)} f(x) - f(x) \right] + \frac{\mathbb{E}_x \left[ 1_{\{N(t) = 1, X_{r_1} \in U\}} e^{\mathcal{J}(v, \theta)} f(X_t) \right]}{t} \psi_t(x).
\]
The error term $\psi_t(x)$ satisfies $|\psi_t(x)| \leq \varphi_{t, f}(x)$. Therefore, Lemma 4.5 implies $\psi_t(x) \to 0$ as $t \searrow 0$. For the first term of the right hand side of (4) we have
\[
\mathbb{E}_x \left[ 1_{\{N(t) = 0\}} e^{-tv(x)} f(x) \right] - f(x) = e^{-t(v(x) + \deg_{m}(x))} f(x) - f(x) \to -(v(x) + \deg_{m}(x)) f(x)
\]
as $t \searrow 0$. Now, let us turn to the second term of the right hand side of (4). We obtain
\[
\mathbb{E}_x \left[ 1_{\{N(t) = 1, X_{r_1} \in U\}} e^{\mathcal{J}(v, \theta)} f(X_t) \right]
\]
\[
= \sum_{y \in U} \mathbb{E}_x \left[ 1_{\{N(t) = 1, X_{r_1} = y\}} e^{i\theta(x, y)} \exp \left( -\tau_1 v(x) - (t - \tau_1) v(y) \right) f(y) \right]
\]
\[
= \sum_{y \in U} e^{i\theta(x, y)} f(y) \mathbb{E}_x \left[ 1_{\{N(t) = 1, X_{r_1} = y\}} \exp \left( -\tau_1 v(x) - (t - \tau_1) v(y) \right) \right] =: \rho_t(x, y)
\]
Setting
\[
C := 2 \max\{|v(x)| \mid x \in U\}
\]
and using $\tau_1 \leq t$ on $\{N(t) = 1\}$, a simple calculation yields
\[
e^{-tC} \mathbb{P}_x(N(t) = 1, X_{r_1} = y) \leq \rho_t(x, y) \leq e^{tC} \mathbb{P}_x(N(t) = 1, X_{r_1} = y).
\]
Hence, the same computation as in the proof of Lemma 4.5 shows that $\frac{1}{t} \rho_t(x, y) \to b(x, y)/m(x)$ as $t \searrow 0$ and that $\rho_t(x, y) = 0$ whenever $b(x, y) = 0$. These two facts and the fact $f \in C_c(U)$, as $U$ is finite, imply
\[
\frac{1}{t} \mathbb{E}_x \left[ 1_{\{N(t) = 1, X_{r_1} \in U\}} e^{\mathcal{J}(v, \theta)} f(X_t) \right] \to \frac{1}{m(x)} \sum_{y \in U} b(x, y) e^{i\theta(x, y)} f(y)
\]
as $t \searrow 0$, so, altogether we arrive at
\[
\frac{T_t(v, \theta, U)f(x) - f(x)}{t} \to -L_{v, \theta} f(x) \quad \text{as } t \searrow 0.
\]
With these preparations we can now prove Proposition 4.3.

**Proof of Proposition 4.3.** For finite $U \subseteq X$, we have $\ell^2(U,m) = C_c(U)$. In particular, $L_{v,\theta}^{(U)}$ is a finite dimensional operator and the convergence

$$-L_{v,\theta}^{(U)} = \lim_{t \to 0} \frac{1}{t} (T_t(v, \theta, U) - \text{id})$$

from Lemma 4.6 holds in the $\ell^2(U,m)$ sense. As $(T_t(v, \theta, U))_{t \geq 0}$ is a strongly continuous self-adjoint semigroup with generator $L_{v,\theta}^{(U)}$, it follows that $e^{-tL_{v,\theta}^{(U)}} = T_t(v, \theta, U)$ for all $t \geq 0$. □

4.3. Proof for closable forms.

**Proposition 4.7.** Let $v$ be such that $Q_{v,0}^{(c)}$ and $Q_{v,\theta}^{(c)}$ are closable and semi-bounded. Then for any $f \in \ell^2(X,m)$, $t \geq 0$ and $x \in X$ one has

$$e^{-tL_{v,\theta} f(x)} = \mathbb{E}_x \left[ \int_{t<\tau} e^{\mathcal{H}_t(v,\theta|X)} f(X_t) \right].$$

**Proof.** As we have already remarked, we are going to prove the asserted formula by using the approximation of $Q_{v,\theta}$ via its restrictions to finite sets. Let $(X_n)_{n \in \mathbb{N}}$ be an exhausting sequence in the sense of Definition 2.16. Then Proposition 2.17 states that

$$e^{-tL_{v,\theta} f(x)} = \lim_{n \to \infty} \tau_{X_n} e^{-tL_{v,\theta}^{(X_n)}} \pi_{X_n} f(x).$$

Combining this and Proposition 4.3, it remains to prove the equation

$$\lim_{n \to \infty} \mathbb{E}_x \left[ \int_{t<\tau_{X_n}} e^{\mathcal{H}_t(v,\theta|X)} \pi_{X_n} f(X_t) \right] = \mathbb{E}_x \left[ \int_{t<\tau} e^{\mathcal{H}_t(v,\theta|X)} f(X_t) \right].$$

This will be done in two steps:

1. $\theta = 0$ and $f \geq 0$: The sequence $\tau_{X_n}$ converges monotonously increasingly to $\tau$ and $\pi_{X_n} f(X_t)$ converges monotonously increasingly to $f(X_t)$. Hence the monotone convergence theorem for integrals yields the desired statement.
2. $\theta$ and $f$ arbitrary: By the assumption $Q_{v,0}^{(c)}$ gives rise to a self-adjoint semi-bounded operator $L_{v,0}$. Now, the first step implies

$$\mathbb{E}_x \left[ \int_{t<\tau} e^{\mathcal{H}_t(v,0|X)} |f|(X_t) \right] = e^{-tL_{v,0}} |f|(x) < \infty,$$

which allows us to use Lebesgue’s dominated convergence theorem to deduce the desired statement for general $\theta$ and $f$. □
4.4. Proof of closability of the forms. In this subsection we prove Theorem \ref{thm:Closability} which is stating that $Q_{v,\theta}^{(c)}$ is closable for all $v \in \mathcal{A}_0$.

Proof of Theorem \ref{thm:Closability}. Let $v \in \mathcal{A}_0$ be given. For $n \in \mathbb{N}$ set $v_n = v \lor (-n)$. By Proposition \ref{prop:Comparison} the forms $Q_{v_n,\theta}^{(c)}$ are closable, semi-bounded and $D(Q_{v_n,\theta}) = D(Q_{v+,\theta})$. Moreover, since $C_c(X)$ is a form core for all $Q_{v_n,\theta}$, there is some $C > -\infty$ such that $Q_{v_n,\theta} \geq C$ for all $n$, keeping $v \in \mathcal{A}_0$ in mind. Hence, $C \leq Q_{v_{n+1},\theta} \leq Q_{v_n,\theta}$ for all $n \in \mathbb{N}$. By monotone convergence of quadratic forms, \cite[Theorem S.16, p.373]{Davies}, we get that $e^{-tL_{v_n,\theta}} \to e^{-tS_{v,\theta}}$, $n \to \infty$, strongly, where $S_{v,\theta}$ denotes the operator corresponding to the form $s_{v,\theta}$ which is the closure of the largest closable quadratic form that is smaller than the limit form corresponding to $(Q_{v_n,\theta})_n$.

In order to show closability of $Q_{v,\theta}^{(c)}$, it remains to show that the form domain of $s_{v,\theta}$ includes $C_c(X)$ and $s_{v,\theta}$ coincides with $Q_{v,\theta}^{(c)}$ on $C_c(X)$.

We start by showing that $e^{-tS_{v,\theta}}$ allows for a Feynman-Kac-Itô representation:

Claim 1: For all $f \in L^2(X,m)$ and $x \in X$

$$e^{-tS_{v,\theta}} f(x) = \mathbb{E}_x \left[ 1_{\{t < \tau\}} e^{\mathcal{Z}_t(v,\theta|x)} f(X_t) \right].$$

By the strong convergence $e^{-tL_{v_n,\theta}} \to e^{-tS_{v,\theta}}$, $n \to \infty$, it suffices to show that

$$\lim_{n \to \infty} \mathbb{E}_x \left[ 1_{\{t < \tau\}} e^{\mathcal{Z}_t(v_n,\theta|x)} f(X_t) \right] = \mathbb{E}_x \left[ 1_{\{t < \tau\}} e^{\mathcal{Z}_t(v,\theta|x)} f(X_t) \right].$$

This, however, can be shown in two steps similar to the ones in the proof of Proposition \ref{prop:Regularity} above: We first employ the monotone convergence theorem for $\theta = 0$ and $f \geq 0$ in the first step and Lebesgue’s dominated convergence theorem in the second step. This proves the claim.

Next, we compute how the generator of $e^{-tS_{v,\theta}}$ acts:

Claim 2: For all $u \in C_c(X)$ and $x \in \text{supp} u$

$$\lim_{t \searrow 0} \frac{e^{-tS_{v,\theta}} u(x) - u(x)}{t} = -\mathcal{L}_{v,\theta} u(x).$$

Denote $U = \text{supp} u$. Recalling the definitions of $T_t(v,\theta,U)$ and $\varphi_{t,|u|}$ from above and using Claim 1 and Lemma \ref{lem:Differentiability} we obtain

$$\lim_{t \searrow 0} \frac{1}{t} \left| (T_t(v,\theta,U) - e^{-tS_{v,\theta}}) u(x) \right| \leq 2 \lim_{t \searrow 0} \varphi_{t,|u|}(x) = 0,$$

where the first inequality is readily seen by writing the semigroups in their Feynman-Kac-Itô representation and splitting up the expectation values into three parts corresponding to the events \{N(t) = 0\}, \{N(t) = 1\} and \{N(t) \geq 2\} as in the proof of Lemma \ref{lem:Regularity}. Then, one immediately sees that the terms for \{N(t) = 0\} and \{N(t) = 1\} coincide and the absolute value of each of the terms corresponding to \{N(t) \geq 2\} can be estimated by $\varphi_{t,|u|}(x)$. Having this, Lemma \ref{lem:Regularity} and the observation $\mathcal{L}_{v,\theta}^{(U)} u = \mathcal{L}_{v,\theta} u$ on $U$ yields the claim.
To finish the proof, we note that by Green’s formula (Lemma 2.1), Claim 2 and the semigroup characterization of $s_{v,\theta}$ (Lemma 1.3.4)

$$Q^{(c)}_{v,\theta}(u, u) = \langle u, \tilde{L}_{v,\theta}u \rangle = \lim_{t \searrow 0} \frac{1}{t} \langle u, u - e^{-tS_{v,\theta}}u \rangle = s_{v,\theta}(u, u),$$

where we also used $u \in C_c(X)$ in the first two equalities. In particular, this shows that $C_c(X) \subseteq D(s_{v,\theta})$. As $Q^{(c)}_{v,\theta}$ is a restriction of a closed form $s_{v,\theta}$, it is closable itself. Semi-boundedness follows as $C_c(X)$ is a form core and $v \in A_0$. The statement about the operator follows from Green’s formula, Lemma 2.1.

This readily gives the proof of the main theorem.

**Proof of Theorem 4.1.** By Theorem 2.4 the forms $Q^{(c)}_{v,0}$ and $Q^{(c)}_{v,\theta}$ are closable and semi-bounded for $v \in A_0$. Hence, the statement follows by Proposition 4.7.

5. Applications

We continue with several applications of Theorem 4.1. Remarkably, being equipped with the Feynman-Kac-(Itô) formula, all of the following partially highly nontrivial functional analytic results will be simple consequences of the trivial equality

$$ |e^{\mathcal{G}_t(v_1,\theta|X)}| \leq e^{\mathcal{G}_t(v_2,0|X)} \quad \text{in } \{ t < \tau \} \text{ for all } t \geq 0,$$

where $v_1 \geq v_2$ are potentials. This is the main advantage of the path integral formalism.

5.1. Semigroup formulas. We will start with the derivation of a probabilistic representation and applications thereof of the integral kernels corresponding to the perturbed magnetic semigroups. To this end, we define the probability measure $\mathbb{P}^t_{x,y}$ on $\{ t < \tau \}$ by

$$\mathbb{P}^t_{x,y} := \mathbb{P}_x(\cdot | X_t = y) \text{ for any } x, y \in X, t > 0,$$

and let $\mathbb{E}^t_{x,y}$ be the corresponding expected value. Clearly, (2) implies

$$\mathbb{P}_x(A) = \sum_{y \in X} \mathbb{P}^t_{x,y}(A)\mathbb{P}_x(X_t = y) = \sum_{y \in X} \mathbb{P}^t_{x,y}(A)e^{-tL}(x, y)m(y)$$

for any event $A \subset \{ t < \tau \}$, so that

$$L^1(\{ t < \tau \}, \mathbb{P}_x) \subset L^1(\{ t < \tau \}, \mathbb{P}^t_{x,y}).$$

**Theorem 5.1.** Let $v \in A_0$. Then for all $t > 0$, $x, y \in X$ one has

$$e^{-tL_{v,\theta}}(x, y) = \frac{1}{m(y)}\mathbb{P}_x(X_t = y)\mathbb{E}^t_{x,y}\left[e^{\mathcal{G}_t(v,\theta|X)}\right] = e^{-tL}(x, y)\mathbb{E}^t_{x,y}\left[e^{\mathcal{G}_t(v,\theta|X)}\right],$$

in particular,

$$\text{tr} \left[ e^{-tL_{v,\theta}} \right] = \sum_{x \in X} \mathbb{P}_x(X_t = x)\mathbb{E}^t_{x,x}\left[e^{\mathcal{G}_t(v,\theta|X)}\right] = \sum_{x \in X} e^{-tL}(x, x)\mathbb{E}^t_{x,x}\left[e^{\mathcal{G}_t(v,\theta|X)}\right] m(x) \in [0, \infty].$$
Proof. The Feynman-Kac-Itô formula in combination with (6) directly implies the first formula. It only remains to prove the formula for the trace. Clearly, by the semigroup property and self-adjointness, $\text{tr}\left[ e^{-\frac{t}{2}L_{v,\theta}} \right]$ is equal to the Hilbert-Schmidt norm of $e^{-\frac{t}{2}L_{v,\theta}} e^{-\frac{t}{2}L_{v,\theta}}$, which in view of the formula for $e^{-tL_{v,\theta}}(x,y)$ and the semigroup property and symmetry of the latter precisely has the asserted form. \hfill $\square$

5.2. Kato’s inequality. The following theorem includes a general version of Kato’s inequality and applications thereof. We refer the reader to [11] for probabilistic aspects of Kato’s inequality on noncompact Riemannian manifolds, and to [6] for a direct proof of Kato’s inequality on graphs (under stronger assumptions though). Moreover, some of the results below are also contained in [10] for locally finite graphs.

Theorem 5.2. (Kato’s inequality) Let $v_1 \geq v_2$ be potentials such that $v_1, v_2 \in A_0$. Then the following assertions hold:

(a) For all $t \geq 0$, $f \in \ell^2(X,m)$, $x \in X$ one has
\[ \left| e^{-tL_{v_1,\theta}} f(x) \right| \leq e^{-tL_{v_2,\theta}} |f|(x), \]
in particular, for all $x, y \in X$, $t > 0$ one has
\[ \left| e^{-tL_{v_1,\theta}}(x,y) \right| \leq e^{-tL_{v_2,\theta}}(x,y), \quad \text{tr} \left[ e^{-tL_{v_1,\theta}} \right] \leq \text{tr} \left[ e^{-tL_{v_2,\theta}} \right]. \]

(b) For any $h \in D(Q_{v_1,\theta})$, it holds that $|h| \in D(Q_{v_2,\theta})$ and $Q_{v_1,\theta}(h) \geq Q_{v_2,\theta}(|h|)$.

(c) One has $\min \sigma(L_{v_1,\theta}) \geq \min \sigma(L_{v_2,\theta})$.

(d) For any $f \in \ell^2(X,m)$, $\lambda \in \mathbb{C}$ with $\text{Re}(\lambda) > \min \sigma(L_{v_1,\theta})$, $x \in X$,
\[ \left| (L_{v_1,\theta} + \lambda)^{-1} f(x) \right| \leq (L_{v_2,\theta} + \lambda)^{-1} |f|(x). \]

(e) If $L_{v_2,\theta}$ has a compact resolvent, then $L_{v_1,\theta}$ has a compact resolvent.

Proof. Assertion (a) is implied by Theorem 5.1 together with (5). Statement (b) follows from (a) and the semigroup characterizations of $Q_{v_2,\theta}$ and $Q_{v_1,\theta}$, see [9, Lemma 1.3.4], and (c) follows from (b) and the variational characterization of the bottom of the spectrum, see [34], (or simply cf. [11, Theorem D.6] for both (b) and (c)). Statement (d) is a direct consequence of (a) and the Laplace’s formula for the resolvents. For (e) notice that the operators $e^{-tL_{v_2,\theta}}$ are positivity improving for all $t > 0$ by the Feynman-Kac formula and $(L_{v_2,\theta} + \lambda)^{-1}$ are positivity improving for all $\lambda \geq \max \sigma(L_{v_2,\theta})$ by the Laplace formula for resolvents. Thus, the statement of the theorem follows from (d) by using Pitt’s theorem (cf. Theorem A.1).

5.3. Golden-Thompson inequality. The following is a discrete analogue of the Golden-Thompson inequality.

Theorem 5.3. (Golden-Thompson inequality) Let $v_1 \geq v_2$ be potentials such that $v_1, v_2 \in A_0$. Then for any $t > 0$ one has
\[ \text{tr} \left[ e^{-tL_{v_1,\theta}} \right] \leq \sum_{x \in X} e^{-tL}(x,x)e^{-tv_2(x)} m(x) \leq C(t) \sum_{x \in X} e^{-tv_2(x)} \in [0, \infty], \]
where
\[ C(t) := \sup_{x \in X} e^{-tL}(x, x)m(x) \leq 1. \]

For the proof of the Golden-Thompson inequality, Theorem 5.3, we need the following monotonicity property of the trace, which should also be of independent interest as well.

**Proposition 5.4.** Let \( v \in A_0 \). Then for any exhausting sequence \( (X_n)_{n \in \mathbb{N}} \) one has
\[
\text{tr} \left[ e^{-tL_{v,0}}(X_n) \right] \nearrow \text{tr} \left[ e^{-tL_{v,0}} \right] \text{ as } n \to \infty \text{ for all } t > 0.
\]

**Proof.** Combining Proposition 4.3 with (6) easily implies
\[
\text{tr} \left[ e^{-tL_{v,0}}(X_n) \right] = \sum_{x \in X_n} e^{-tL}(x, x) \mathbb{E}_{x,x} \left[ 1_{\{t < \tau_{X_n}\}} e^{-\int_0^t v(x)ds} \right] m(x),
\]
which, using Theorem 5.1, tends to \( \text{tr} [ e^{-tL_{v,0}} ] \) in view of monotone convergence. \[ \square \]

**Proof of Theorem 5.3.** In view of Theorem 5.2 (a), we have \( \text{tr} \left[ e^{-tL_{v_1,0}} \right] \leq \text{tr} \left[ e^{-tL_{v_2,0}} \right] \). Let \( (X_n)_{n \in \mathbb{N}} \) be an exhausting sequence. Then applying the operator-version of Golden-Thompson inequality (Theorem 4.1) to \( q' = Q_{0,0}^{(X_n)} \), \( q'' = q_{v_2} \) in the Hilbert space \( \ell^2(X_n, m) \), where \( Q_{0,v_2}^{(X_n)} = Q_{0,0}^{(X_n)} + q_{v_2} \) is trivial in view of the finiteness of \( X_n \), we get the inequality in
\[
\text{tr} \left[ e^{-tL_{v_2,0}}(X_n) \right] \leq \text{tr} \left[ e^{-N t_{0,0}} \right] \leq \text{tr} \left[ e^{-N t_{0,0}} e^{-tL_{v_2,0}} e^{-tL_{v_2,0}} \right] = \sum_{x \in X_n} e^{-tL_{v_2}(x)} \sum_{y \in X_n} e^{-\frac{t}{2} L_{0,0}^{(x,y)}} e^{-\frac{t}{2} L_{0,0}^{(x,y)}} (y, x) m(y) m(x) (7)
\]

Here we have used self-adjointness and semigroup properties, as well as
\[
\left( e^{-\frac{t}{2} v_2} e^{-\frac{t}{2} L_{0,0}^{(X_n)}} \right) (x, y) = e^{-\frac{t}{2} v_2(x)} e^{-\frac{t}{2} L_{0,0}^{(X_n)}} (x, y) \text{ for all } (x, y) \in X_n \times X_n.
\]

Noting that
\[
1_{X_n \times X_n}(x, x) e^{-tL_{0,0}^{(X_n)}}(x, x) \nearrow e^{-tL}(x, x) \text{ for all } x \in X \text{ as } n \to \infty,
\]
monotone convergence implies that the right-hand side of (7) tends to the term in the middle of the asserted inequality as \( n \to \infty \). In view of Proposition 5.4, this completes the proof of the first inequality. For the second inequality we note that by (2), \( e^{-tL}(x, x)m(x) = \mathbb{P}_x(X_t = x) \leq 1 \).

**Remark 5.5.** We refer the reader to [37, Theorem 9.2] for an \( \mathbb{R}^m \)-version of the Golden-Thompson inequality, which uses a very different proof. Note that in this particular case, the Golden-Thompson inequality can be rewritten as a phase space bound. This has the
important physical consequence that the quantum mechanical partition function is always bounded from above by the corresponding classical partition function.

5.4. The form domain. Finally, we use the Feynman-Kac-Itô formula to derive an explicit description of the form domain of $Q_{v,\theta}$ under suitable assumptions on the potential.

**Theorem 5.6.** For any $v \in \mathcal{B}_0$, one has $Q_{v,\theta} = Q_{0,\theta} + q_v$, in particular, $D(Q_{v,\theta}) = D(Q_{0,\theta}) \cap \ell^2(X,|v|m)$.

**Proof.** By Proposition 2.6 it suffices to show the statement for $v \geq 0$. We prove a Feynman-Kac-Itô formula for $Q_{0,\theta} + q_v$ in order to conclude the assertion using Theorem 4.1. To this end, denote the operator arising from the form sum $Q_{0,\theta} + q_v$ by $L_{0,\theta} + v$.

With $v_n := v \wedge n \in \ell^\infty(X)$ we have $0 \leq v_n \not\nearrow v$ as $n \to \infty$ and it follows from monotone convergence for integrals that $Q_{v_n,\theta} = Q_{0,\theta} + q_{v_n} \not\nearrow Q_{0,\theta} + q_v$ as $n \to \infty$ in the sense of monotone convergence of quadratic forms. By [34, Theorem S.14, p.373] we have that $Q_{0,\theta} + q_v$ is closed and

$$
\lim_{n \to \infty} e^{-t(L_{0,\theta} + v_n)} f(x) = e^{-t(L_{0,\theta} + v)} f(x)
$$

for all $f \in \ell^2(X,m)$ and $x \in X$. Thus, in view of $Q_{v_n,\theta} = Q_{0,\theta} + q_{v_n}$ and $L_{v_n,\theta} = L_{0,\theta} + v_n$ (as $v_n$ is bounded) it only remains to prove

$$
\lim_{n \to \infty} \mathbb{E}_x \left[ 1_{\{t<\tau\}} e^{\mathcal{F}_t(v_n,\theta|X)} f(X_t) \right] = \mathbb{E}_x \left[ 1_{\{t<\tau\}} e^{\mathcal{F}_t(v,\theta|X)} f(X_t) \right].
$$

which, however, follows by Lebesgue’s dominated convergence. \qed

We finish with a corollary of the theorem above. For $v$ bounded below, recall the form $Q_{v,\theta}^{\max} : \ell^2(X,m) \to (-\infty,\infty]$ in the proof of Proposition 2.6, which is given by

$$
Q_{v,\theta}^{\max}(f) = \frac{1}{2} \sum_{x,y \in X} b(x,y)|f(x) - e^{iy(x,y)} f(y)|^2 + \sum_{x \in X} v(x)|f(x)|^2 m(x).
$$

It is bounded below and closed.

**Corollary 5.7.** If $Q_{0,\theta} = Q_{0,\theta}^{\max}$, then one has $Q_{v,\theta} = Q_{v,\theta}^{\max}$ for all $v$ bounded below.

**Proof.** As, obviously, $Q_{v,\theta}^{\max} = Q_{0,\theta}^{\max} + q_v$, the statement follows by the theorem above. \qed

**Appendix A. Pitt’s theorem**

**Theorem A.1.** Let $p_1 \in (1,\infty)$, $p_2 \in [1,\infty]$ and let $A, B : L^{p_1}(M,\mu) \to L^{p_2}(M,\mu)$ be bounded operators such that $A$ is positivity preserving and such that one has $|Bf| \leq A|f|$ for any $f \in L^{p_1}(M,\mu)$. Then $B$ is a compact operator, if $A$ is a compact operator.

This highly nontrivial fact on operator domination goes back to L.D. Pitt [33].
APPENDIX B. AN ABSTRACT GOLDEN-THOMPSON INEQUALITY

Theorem B.1. Let \( q', q'' \) be densely defined, closed, symmetric and semi-bounded sesquilinear forms on a common Hilbert space. Assume that \( q := q' + q'' \) is densely defined and denote the semigroups corresponding to \( q', q'' \) and \( q \) by \( (T_t')_{t \geq 0}, (T_t'')_{t \geq 0} \) and \( (T_t)_{t \geq 0} \), respectively. Then one has

\[
\text{tr}[T_t] \leq \text{tr} \left[ T_{t/2} T_t'' T_{t/2} \right] \quad \text{for all } t \geq 0.
\]

This result follows from Corollary 3.9 in [16]. Note that the above fact is even nontrivial for finite dimensional operators.

APPENDIX C. MOSCO-CONVERGENCE

Let \( (H_k, \langle \cdot, \cdot \rangle_k), k \in \mathbb{N}, \) and \( (H, \langle \cdot, \cdot \rangle) \) be Hilbert spaces with corresponding norms \( \| \cdot \|_k \) and \( \| \cdot \| \) respectively. Suppose \( (q_k, D(q_k)) \) and \( (q, D(q)) \) are densely defined closed symmetric sesquilinear forms on \( H_k \) and \( H \), respectively, which are bounded below by a constant \( C > -\infty \) which is uniform in \( k \). Each \( q_k \) is understood to be defined on the whole space \( H_k \) by the convention \( q_k(u) = \infty \) whenever \( u \in H_k \setminus D(q_k) \). Furthermore, we suppose that there exist bounded operators \( t_k : H_k \to H \) such that \( \pi_k := t_k^* \) is a left inverse of \( t_k \), that is

\[
\langle \pi_k f, k \rangle = \langle f, t_k k \rangle \quad \text{and} \quad \pi_k t_k f_k = f_k, \quad \text{for all } f \in H, f_k \in H_k.
\]

Moreover, we assume that \( \pi_k \) satisfies

\[
\sup_{k \in \mathbb{N}} \| \pi_k \| < \infty \quad \text{and} \quad \lim_{k \to \infty} \| \pi_k f \|_k = \| f \|.
\]

Definition C.1. In the above situation, we say that \( q_k \) is Mosco convergent to \( q \) as \( k \to \infty \) in the generalized sense, if the following conditions hold:

(i) If \( u_k \in H_k, u \in H \) and \( t_k u_k \to u \) weakly in \( H \), then

\[
\lim_{k \to \infty} \left( q_k(u_k) + C \| u_k \|^2_k \right) \geq q(u) + C \| u \|^2.
\]

(ii) For every \( u \in H \) there exist \( u_k \in H_k \), such that \( t_k u_k \to u \) in \( H \) and

\[
\limsup_{k \to \infty} \left( q_k(u_k) + C \| u_k \|^2_k \right) \leq q(u) + C \| u \|^2.
\]

Let \( (T_t^{(k)})_{t \geq 0} \) denote the semigroup associated with \( q_k \) and let \( (T_t)_{t \geq 0} \) be the semigroup of \( q \). For positive forms the following theorem which characterizes Mosco convergence can be found in the appendix of [4]. However, this result immediately extends to the situation of forms with uniform lower bound.

Theorem C.2. In the above situation, the following assertions are equivalent:

(a) \( q_k \) is Mosco convergent to \( q \) as \( k \to \infty \) in the generalized sense.

(b) One has \( t_k T_t^{(k)} \pi_k \to T_t \) as \( t \to \infty \) strongly and uniformly on any finite time interval.

Proof. Consider the positive quadratic forms \( \tilde{q}_k = q_k + C \| \cdot \|^2 \) and \( \tilde{q} = q + C \| \cdot \|^2 \). Obviously their semigroups \( \tilde{T}_t^{(k)} \) and \( \tilde{T}_t \) satisfy

\[
\tilde{T}_t^{(k)} = e^{-tC} T_t^{(k)} \quad \text{and} \quad \tilde{T}_t = e^{-tC} T_t.
\]
Combining this and the characterization of Mosco convergence for positive forms (Theorem 8.3 of [4]) we can deduce the result.

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