Three-body model for $K(1460)$ resonance

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The three-body $KK\bar{K}$ model for the $K(1460)$ resonance is developed on the basis of the Faddeev equations in configuration space. A single-channel approach is used with taking into account the difference of masses of neutral and charged kaons. It is demonstrated that a splitting of the mass of the $K(1460)$ resonance takes a place around 1460 MeV according to $K^0\bar{K}^0\bar{K}^0$, $K^0\bar{K}^+\bar{K}^-$ and $K^+\bar{K}^0\bar{K}^0$, $K^+\bar{K}^-\bar{K}^-$ neutral and charged particle configurations, respectively. The calculations are performed with two sets of $KK$ and $K\bar{K}$ phenomenological potentials, where the latter interaction is considered the same for the isospin singlet and triplet states. The effect of repulsion of the $KK$ interaction on the mass of the $KK\bar{K}$ system is studied and the effect of the mass polarization is evaluated. The first time the Coulomb interaction for description of the $K(1460)$ resonance is considered. The mass splitting in the $K(1460)$ resonances is evaluated to be in range of 10 MeV with taking into account the Coulomb force. The three-body model with the $K\bar{K}$ potential, which has the different strength of the isospin singlet and triplet parts that are related by the condition of obtaining a quasi-bound three-body state is also considered. Our results are in reasonable agreement with the experimental mass of the $K(1460)$ resonance.

I. INTRODUCTION

Since the early 1960s, when the quark model was developed it became clear that hadrons are not elementary particles, but composed of quarks and antiquarks. In the classical quark model, a baryon is composed of three quarks and a meson is composed of one quark and one antiquark. Today the internal structure of hadrons is a prominent topic of high energy physics [1,4]. Quarks and gluons are confined within the mesons and baryons. Thus, hadrons are composite objects of quarks and gluons governed by quantum chromodynamics (QCD), which has been established as the theory describing the strong interaction, but its application to low-energy hadron phenomenology is still relatively unexplored and there are open problems to be studied. The interpretation of hadronic states is one of the most important issues in hadronic physics, particularly for the exotic states which cannot be easily collected as quark-antiquarks or three quark states. In particular, some specific resonances cannot be simply explained by the quark model and may be of more complex structure. Common features for descriptions of such specific resonances are predictions for the existence of hadrons with substructures that are more complex than the standard quark-antiquark mesons and the three-quark baryons of the original quark model that provides a concise description of most of the low-mass hadrons [3].

In the low energy region, where perturbative QCD does not work, non-perturbative methods such as the QCD sum rule [5], lattice QCD [6,7], chiral perturbation theory [8–11], field correlator method (FCM) in QCD [12–15] and so on, are needed. We cite these works, but the recent literature on the subject is not limited by them.

The physics of three-body systems has received attention for decades. The general approach for solutions of the three-body problem at low energies is based on the use of modelless methods for studying the dynamics of three particles in discrete and continuum spectra. Currently among the most powerful approaches are the method of Faddeev equations in momentum or configuration spaces, the method of hyperspherical harmonics (HH), the variational method in the harmonic-oscillator basis and the variational method complemented with the use of explicitly correlated Gaussian basis functions. To investigate the three-body systems in hadron physics, one should solve the Faddeev equations [16,17]. The method of hyperspherical harmonics in momentum or configuration spaces is another method that is intensively used in a few-body physics, which despite its conceptual simplicity, offers great flexibility, high accuracy, and can be used to study diverse quantum systems, ranging from small atoms and molecules to light nuclei and hadrons. It is a very challenging task to solve the Faddeev equations exactly and we are usually introducing some reasonable approximations of the Faddeev equations, such as the use of separable potentials, energy-independent kernels, on-shell two-body scattering amplitudes, the fixed-center approximation (FCA), the Faddeev-type Alt-Grassberger-Sandhas equations’ [18], while an application of the HH for solution of three-body problem always relays to the reasonable convergence of the method.
Using the unitary extensions of chiral perturbation theory \(^1\) that is a good representation of QCD at low energies \(^2\) \(^3\), dynamically generated three-body resonances formed via the meson-meson and meson-baryon interactions were intensively studied using FCA for the Faddeev equations in Refs. \(^4\) \(^5\). Indeed, the use of chiral dynamics in the Faddeev equations within a fixed-center approximation allows the description of three-body resonances consisting from mesons and baryons. Thus, hadronic composite states are introduced as few-body systems in hadron physics \(^6\) and, therefore, interpretation of the states found in the system of mesons and baryons became one of the challenges in theoretical physics \(^7\).

The FCM is a promising formulation of the non-perturbative QCD that gives additional support to the quark model assumptions. Progress was made \(^8\) \(^9\) towards placing the computation of baryon masses within the FCM by describing a three particle system within the HH method. Using the HH approach the ground and \(p\)-wave excited states of \(nnn\), \(nn\), and \(ssn\) baryons can be obtained \(^9\) in the framework of the FCM \(^10\). It is interesting to consider a dynamical generation of \(K(1460)\) pseudoscalar resonance formed by a system of three kaons. In particular, noteworthy is the possibility of formation of the quasi-bound states of three kaons. The observation of pseudoscalar resonances is of fundamental importance towards the understanding of the meson spectrum. Let us go over a short history of \(K(1460)\) pseudoscalar resonance. \(K(1460)\) pseudoscalar was a subject of interest already several decades ago. The first evidence for a strangeness-one pseudoscalar meson with mass of \(\sim 1400\) MeV and a width of \(\sim 250\) MeV was reported via \(J^{P} = 0^-\) partial-wave analysis of the \(K\pi\pi\) system in the reaction \(K^{\pm}p \rightarrow K^{\pm}\pi^+\pi^-p\) \(^11\). The study of this process was carried out at SLAC, using a 13 GeV incident \(K^{\pm}\) beam. A few years later the diffractive process \(K^-p \rightarrow K^-\pi^+\pi^-p\) at 63 GeV was studied by ACCORD collaboration \(^12\) and the existence of a broad \(0^-\) resonance with a mass \(\sim 1460\) MeV may now be taken as established \(^1\) \(^2\). In the most recent study \(^3\) intermediate decays of the \(K(1460)\) meson are found to be roughly consistent with previous studies \(^4\) \(^5\), with approximately equal partial widths to \(\bar{K}\pi\pi\) and \(\pi\eta K\). The resonance energy and width were determined using two-body potentials that fit two-

\(K(1460)\) resonance in the \(K\pi\pi\) system was considered based on the coupled-channel complex-scaling method by introducing three channels \(KKK\), \(\pi\pi K\), and \(\pi\eta K\). The resonance energy and width were determined using two-body potentials that fit two-body scattering properties. The model potentials having the form of one-range Gaussians was proposed based on the experimental information about \(f_0\) and \(f_0\) resonances. In the model, the \(KK\) interaction depends on the pair isospin. In particular, the isospin triplet \(KK\) \((I = 1)\) interaction is essentially weaker than the isospin singlet \(KK - KK\) interaction in the channel \(\pi K - K\bar{K}(I = 0)\).

The aim of this paper is to systematically investigate the \(KK\bar{K}\) system in the framework of the new approach — the differential Faddeev equations. We try to answer the following questions: (i) What \(KK\bar{K}\) are deeply bound? (ii) Is there any strange structure peculiar to \(KK\bar{K}\) system? We present our study of the \(K(1460)\) resonance in the framework of a single-channel non-relativistic potential model using the Faddeev equations in configuration space and considering this resonance by means of a three-body kaonic system \(KK\bar{K}\). Such consideration allows to use \(KK\) and \(KK\) potentials for description of the \(KK\bar{K}\) system. In our approach these potentials are only inputs along with the masses of kaons. Following Ref. \(^1\) we study the \(KK\bar{K}\) system using effective phenomenological potentials but taking into account the difference in masses of \(K\) and \(\bar{K}\) kaons. The latter leads to splitting the mass of the \(K(1460)\) resonance according to the following neutral or charged particle configurations: \(K^0\bar{K}^0\bar{K}^0\), \(K^0\bar{K}^+\bar{K}^-\), \(K^+\bar{K}^0\bar{K}^0\), \(K^+\bar{K}^+\bar{K}^-\). We consider two cases for the \(KK\bar{K}\) system. In the first one, the strength of the isospin singlet and triplet parts of the \(K\bar{K}\) potential are the same. Such approach leads to a simplified version of the Faddeev equations in configuration space for three particle system. The second case is complicated by isospin dependence of the \(KK\) potential, when the strength of the isospin singlet and triplet parts of the potential are different and related by the condition of obtaining a quasi-bound three-body state. Results of our calculations are compared with the SLAC and ACCMOR collaboration experimental values for the mass of \(K(1460)\) resonance \(^11\) \(^12\) and the recent experimental study \(^13\). This paper is organized as follows. In Sec. II we present our theoretical model. The Faddeev equations in configuration space are formulated and we present the particle configurations in three-body kaonic system \(KK\bar{K}\). The results of calculations for the masses of different configurations, interpretation of the results including comparison
II.
THEORETICAL MODEL

A. Formalism

Configuration space methods are a valuable tool for the analysis of the three-body problem with short-range interactions [51, 52]. Considering the $KK\bar{K}$ system as three interacted bosons having positions $r_1, r_2$ and $r_3$, once the two-body interactions for the $KK$ and $K\bar{K}$ subsystems are defined, one can determine its wave function by solving the Faddeev equations. The bound state problem for the $KK\bar{K}$ system we formulate by using the Faddeev equations in configuration space [53] for bosonic $AAB$ system with two identical particles. The total wave function of the $KK\bar{K}$ system is decomposed into the sum of the Faddeev components $U$ and $W$ corresponding to the $(KK)\bar{K}$ and $(K\bar{K})K$ types of particles rearrangements: $\Psi = U + (I + P)W$, where $P$ is the permutation operator for two identical particles. For a three-body system, which includes two identical bosons, the Faddeev equations represent the set of two equations for the components $U$ and $W$ [54] that reads:

$$
\begin{align*}
(H^U_0 + V_{K\bar{K}} - E)U & = -V_{K\bar{K}}(W + PW), \\
(H^W_0 + V_{K\bar{K}} - E)W & = -V_{K\bar{K}}(U + PW),
\end{align*}
$$

(1)

where the potentials for $KK$ and $K\bar{K}$ pairs are defined as $V_{K\bar{K}}$ and $V_{K\bar{K}}$, respectively. In Eqs. (1) $H^U_0$ and $H^W_0$ are the kinetic energy operators of three particles written in the Jacobi coordinates (see Appendix A) corresponding to the $(KK)\bar{K}$ and $(K\bar{K})K$ types of the three particles rearrangements. The total isospin of the $KK\bar{K}$ system is considered to be $\frac{1}{2}$. The set of particles in the $KK\bar{K}$ system is defined by total isospin projections, which can be $-1/2$ or $1/2$. The possible isospin configurations with isospin $3/2$ are not taken into account in our calculations due to the smallness of corresponding contributions.

In general, we employ the $s$-wave isospin dependent $V_{K\bar{K}}$ and $V_{K\bar{K}}$ potentials having singlet and triplet components: $V_{K\bar{K}} = \text{diag}(v_{K\bar{K}}^s, v_{K\bar{K}}^t), V_{K\bar{K}} = \text{diag}(v_{K\bar{K}}^s, v_{K\bar{K}}^t)$. One should mention that due to Bose-Einstein statistics the strength of $s$-wave $KK$ interaction $v_{K\bar{K}}^s = 0$, because the isospin singlet wave function of the pair is antisymmetric. The corresponding interaction should be suppressed. The separation of isospin variables leads to the following form of the Faddeev equations:

$$
\begin{align*}
(H^U_0 + v_{K\bar{K}}^t - E)U & = -v_{K\bar{K}}^t(-\frac{1}{2}W^t + \frac{\sqrt{3}}{2}W^s - \frac{1}{2}pW^t + \frac{\sqrt{3}}{2}pW^s), \\
(H^W_0 + v_{K\bar{K}}^t - E)W & = -v_{K\bar{K}}^t((\frac{\sqrt{3}}{2}U + \frac{1}{2}pW^s - \frac{\sqrt{3}}{2}pW^t), \\
(H^W_0 + v_{K\bar{K}}^s - E)W & = -v_{K\bar{K}}^s((-\frac{1}{2}U - \frac{\sqrt{3}}{2}pW^s - \frac{1}{2}pW^t).
\end{align*}
$$

(2)

The singlet and triplet $W$ components of the wave function are noted by indexes $s$ and $t$, respectively. $U$ is the triplet component and the exchange operator $p$ acts on the particles’ coordinates only. Within $s$-wave approach the equation for the singlet component corresponding to the $KK$ singlet potential is omitted due to the isospin symmetry. The similar property is demonstrated for equations describing $AAB$ systems like $NN\bar{K}$ [55, 56] and $nnp$ [57].

For the description of the effective kaon-kaon interaction we use the potentials from Refs. [49, 47] that are written in one-range Gaussian form as $V_A(r) = \sum_{l=0,1} V_A^l \exp \left[ -\frac{(r/b)^2}{2} \right] I_A$, where $b$ is the range parameter having the same value as for the $\bar{K}N$ interaction, $P_A$ is the isospin projection operator and the index $A$ is related to the type of interaction $A \in KK, K\bar{K}$. The strength of strongly attractive in $s$-wave $\bar{K}K$ interaction in the isospin singlet and triplet states $v_{K\bar{K}}^s = v_{K\bar{K}}^t = -1155 - 283i$ MeV with $b = 0.47$ fm and $v_{K\bar{K}}^s = v_{K\bar{K}}^t = -630 - 210i$ MeV with $b = 0.66$ fm are considered the same for the isospins $I = 0$ and $I = 1$ [46, 47]. Considering that $KK$ and $K\bar{K}$ interactions are isospin invariant and there are no open decay channels for the $KK$ system, the $KK$ potential is a real. The

| TABLE I: Sets of parameters for the $KK$ and $K\bar{K}$ potentials. |
|---------------------------------------------------------------|
| $A$ ($b=0.47$ fm) | $B$ ($b=0.66$ fm) |
| Interaction | $v^s$, MeV | $v^t$, MeV | $v^s$, MeV | $v^t$, MeV |
| $KK$ | $-1155 - 283i$ | $-1155 - 283i$ | $-630 - 210i$ | $-630 - 210i$ |
| $K\bar{K}$ | 0 | 313 | 0 | 104 |
strength of the s-wave $KK$ interaction for $I = 0$ is $V_{KK}^{I=0} = v_{KK}^s = 0$ and for isospin $I = 1$ has a relatively weak repulsion that is considered as $V_{KK}^{I=1} = v_{KK}^t = 313$ MeV and $V_{KK}^{I=1} = v_{KK}^t = 104$ MeV for parameterizations with $b = 0.47$ fm and $b = 0.66$ fm, respectively. As mentioned above, the choice of the range parameters $b$ related to the description of the $KN$ interaction. The value $b = 0.66$ fm for the effective $KN$ interaction corresponds to the effective Akaishi-Yamazaki potential derived in Refs. [58, 59] phenomenologically by using $KN$ scattering and kaonic hydrogen data and reproducing the $\Lambda(1405)$ resonance as the $\bar{K}p$ bound state at 1405 MeV. This potential is energy independent. The value $b = 0.47$ fm for the effective $KN$ interaction corresponds to the potential obtained in Ref. [60] within the chiral SU(3) effective field theory and derived based on the chiral unitary approach for the s-wave scattering amplitude with the strangeness $S = -1$ and reproduces the total cross sections for the elastic and inelastic $\bar{K}p$ scattering, threshold branching ratios, and the $\pi\Sigma$ mass spectrum associated with the $\Lambda(1405)$. The strength of $\bar{K}K$ interaction was determined by fitting masses of the $f_0(980)$ and $a_0(980)$ resonances with the input width 60 MeV [17], while the strength of the $KK$ interaction was obtained in Ref. [47] to reproduce the $\bar{K}K$ scattering length given by a lattice QCD calculation in Ref. [61]. Following Ref. [47] we refer to the kaon-kaon interaction potential with $b = 0.47$ fm and $b = 0.66$ fm as A and B, respectively. The set of values of the potential strength $\tilde{c}$ for each interaction and two optimized values for the range parameter (set A and B, respectively) are given in Table I.

Taking into account the potentials have the same components in isospin single and triplet states, the Faddeev equations (2) are reduced by an algebraic transformation [55] defined by the diagonal matrix to the following form:

$$
\begin{align*}
(H_0^U + v_{KK}^t - E)U &= -v_{KK}^t(1 + p)\tilde{V}_2,
(H_0^W + v_{KK}^t - E)V &= -v_{KK}^t(U + p\tilde{W}).
\end{align*}
$$

The transformation related to $\bar{K}K$ potential is given as $v_{KK}^t = DV_{KK}DT$, where $V_{KK} = diag\{v_{KK}^s, v_{KK}^t\}$ and the diagonal matrix $D = (-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)$ defines the transformation. In Eqs. (3) is assumed that the strength of the isospin singlet and triplet components of the $\bar{K}K$ potential is the same $v_{KK}^t = v_{KK}^t$. The corresponding Faddeev components are $\tilde{V}^t = DW$, where $W = (W^s, W^t)^T$.

For the three-body system described by Eqs. (3) one can evaluate the mass polarization using the definition:

$$
\Delta = 2E_2 - E_3(V_{KK} = 0).
$$

Here, $E_2$ is $\bar{K}K$ two-body energy and $E_3(V_{KK} = 0)$ is the three-body energy, when the $KK$ interaction between identical particles is omitted. The value of $\Delta$ is positive and depends on the mass ratio of the particles [52, 54].

When in the system $KK\bar{K}$ at least two particles are charged the Coulomb interaction should be considered. The Coulomb potential can be included as a perturbation of the Hamiltonian in the left-hand side of Eq. (3). We present the structure of the set of Faddeev equations with the Coulomb interaction taken into account in Appendix A.

The complete isospin model is based on Eqs. (2) with splitting $\bar{K}K$ potential to two isospin channels $I=0$ and $I=1$, which have differed strength of the $\bar{K}K$ interaction. The splitting of the singlet and triplet components proposed in Ref. [50] can be expressed by a ratio of strength parameters for the components of potential, $V_{KK}^{I=1}/V_{KK}^{I=0}$. Eqs. (3) describe the case, when $v_{KK}^t/v_{KK}^s = 1$. This case corresponds to the $AAB$ system without spins and isospins (bosonic isospinless system) and the $KK\bar{K}$ demonstrates properties of such a system.

### B. Particle configurations in $KK\bar{K}$ system

One can consider different particle configurations in the $KK\bar{K}$ system. The configurations are differed by sets of masses and pair potentials. The Coulomb potential has to be included for a description of some configurations. To select the configuration, we used the difference between the masses of kaons presented in Table II. The configurations

| Particle(Anti-particle) | Quarks | Mass (MeV) | Isospin projection |
|------------------------|--------|-----------|--------------------|
| $K^+ (K^-)$            | us     | 493.7     | 1/2 (-1/2)        |
| $K^0 (\bar{K}^0)$     | ds     | 497.6     | -1/2 (1/2)        |

TABLE II: Kaons and anti-kaons with the mass deference and isospin projections.
are the following: \(K^0K^0\bar{K}^0\), \(K^0K^+K^-\), \(K^+K^0\bar{K}^0\), \(K^+K^+K^-\). Using the charge-isospin basis notations, the configurations can be identified as \(--+, --+, +++, +++, +++, ++, ++, +++.\) Thus, the first two configurations correspond to the states with projection of total isospin \(-1/2\) of the \(KK\bar{K}\) system, while the last two have the total isospin projection \(+1/2\).

Each configuration is represented as \(AAB\) system – the system with two identical particles and can be described by the Faddeev equations (2) and (3), considering the cases when the strength of isospin components of the \(KK\) potential is different and when \(v^{t}_{KK} = v^{s}_{KK} = 1\).

### III. RESULTS

Our interest is to examine the possibility of the existence of kaonic bound states in the \(KK\bar{K}\) system. For this purpose, we solve numerically differential Eqs. (2) and (3) in the case of the different strength for the isospin singlet and triplet components of \(KK\) potential and Eqs. (3), when \(v^{t}_{KK} = v^{s}_{KK}\). The differential Faddeev equations have been formulated in the pioneer work of Noyes and Fiedeldey [65] for the simplest case of \(s\)-wave three-particle scattering and have been generalized in Ref. [66]. Our numerical procedure for solution of the Faddeev equations in configuration space is based on the finite difference approximation with spline collocations [67, 68].

**Case when the strength of** \(v^{t}_{KK} = v^{s}_{KK}\). For this case results of our calculations for the binding energy and the width for the \(KK\bar{K}\) system are presented in Table III. In the same Table are given the results [45] obtained using the variational method for a single channel three-body potential model with two-body effective interactions, and a coupled-channel approach based on the solution of the Faddeev equations in momentum representation using as input on-shell parts of the two-body \(t\)-matrices that generate dynamically the \(f_0(980)\) and \(a_0(980)\) resonances. The total mass the \(KK\bar{K}\) system is ranged from 1463.8 to 1469.4 MeV, when we consider the same \(K\) meson mass \(m_K = 496\) MeV as in Refs. [45, 49]. The width falls into the \(41–49\) MeV range for all sets of the \(KK\) and \(KK\bar{K}\) interactions and coincides with results obtained within the HH method [49] and Faddeev equations in momentum representation [15]. The quasi-bound state for the \(KK\bar{K}\) with spin-parity 0– and total isospin 1/2 is found below the three-kaon threshold. The comparison of our results with those obtained with the variational method [45] shows that while binding energy obtained within the HH and variational calculations are close enough, the difference for the width is more than 50%. The alternative scenario is observed for the HH method and the Faddeev calculations in the momentum representation: the difference in masses is more than 40 MeV. Studying the various particle configurations of the \(KK\bar{K}\) system with the dependence of the particles masses, we are considering the systems: the mass \(m_K = 496\) MeV [45] corresponds to \(K^0K^0\bar{K}^0\) system, where \(m_K = (2m_K + m_{\bar{K}})/3\); the mass \(m_K = 493.7\) MeV corresponds to \(K^+K^-\) system; the mass \(m_K = 497.6\) MeV corresponds to \(K^0K^0\bar{K}^0\) system. For the \(K^0K^+K^-\) and \(K^+K^0\bar{K}^0\) configurations we used averaged mass of kaons for particle pair \(K^0\bar{K}^0\), \(m_K = 495.7\) MeV. The difference of the masses

| Resonance | System AAB | Particle masses | Model | \(E_3\) | \(E_3(V_{KK} = 0)\) | \(E_2\) | Mass polarization \(\delta (\%)\) | Mass upper bound |
|-----------|------------|----------------|-------|--------|----------------|--------|-------------------------------|----------------|
| \(K(1460)\) | \(KK\bar{K}\) | 496.0 [45] | A | -19.8 | -32.1 | -11.25 | 30.0 | 1468.2 | 1476.7 |
| \(K(1460)\) | \(KK\bar{K}\) | 495.7 | A | -19.7 | -31.9 | -11.18 | 29.9 | 1467.4 | 1475.9 |
| \(K(1460)\) | \(KK\bar{K}\) | B | -22.2 | -29.4 | -11.17 | 24.0 | 1464.9 | 1475.9 |
| \(K^0(1460)\) | \(K^0K^0\bar{K}^0\) | 497.6 | A | -20.4 | -33.0 | -11.61 | 29.6 | 1469.7 | 1481.2 |
| \(K^0(1460)\) | \(K^0K^0\bar{K}^0\) | B | -22.8 | -30.1 | -11.45 | 23.9 | 1467.3 | 1481.3 |
| \(K^0(1460)\) | \(K^0K^+K^-\) | \(m_{K^-} = 493.7\), \(A\) | -19.3 | -31.3 | -10.96 | 29.7 | 1465.8 | 1474.1 |
| \(K^0(1460)\) | \(K^0K^+K^-\) | \(m_{K^+} = 495.7\), \(B\) | -21.9 | -29.0 | -11.03 | 23.8 | 1463.2 | 1474.1 |
| \(K^+(1460)\) | \(K^+K^0\bar{K}^0\) | \(m_{\bar{K}^0} = 497.6\), \(A\) | -20.1 | -32.5 | -11.40 | 29.8 | 1468.9 | 1477.6 |
| \(K^+(1460)\) | \(K^+K^0\bar{K}^0\) | \(m_{K^+} = 495.7\), \(B\) | -22.5 | -29.8 | -11.34 | 23.9 | 1466.5 | 1477.7 |
| \(K^+(1460)\) | \(K^+K^+K^-\) | 493.7 | A | -18.9 | -30.9 | -10.74 | 29.5 | 1462.2 | 1470.4 |
| \(K^+(1460)\) | \(K^+K^+K^-\) | \(A_c\) | -20.9 | – | – | – | 1460.2 | – |
| \(K^+(1460)\) | \(K^+K^+K^-\) | \(B\) | -21.6 | -28.7 | -10.87 | 24.3 | 1459.5 | 1470.2 |
| \(K^+(1460)\) | \(K^+K^+K^-\) | \(B_c\) | -23.3 | – | – | – | 1457.8 | – |
for $K^0$ and $K^+$ violets the AAB model with two identical particles of $KK\bar{K}$ system. However, the approach with the averaged mass is completely satisfied to the AAB model due to the set of proposed potentials. The Coulomb potential acting in the $K^0K^0K^0$ system also violets the AAB symmetry and, therefore, we omitted the consideration of the Coulomb force for the $K^0K^0K^0$ configuration. The Coulomb potentials in the $K^+K^+K^-$ system was included in the calculations due to correspondence to the AAB symmetry. A brief description of the Faddeev equations in configuration space with the Coulomb force acting in the $KK\bar{K}$ is given in Appendix A.

The comparison of our energy $E = -21$ MeV with the result obtained in Ref. [45], shows some disagreement. We assume that the disagreement can be related to numerical method to solve the corresponding equations. In Ref. [45] is used variational method, which depends on a choice of initial basis functions. We use direct numerical method [68] for the solution of the Faddeev equations in configuration space. Here, the direct solution means a method of solution based on the finite-deferential approximation of the boundary problem for eigenvalues with discretization of [68] for the solution of the Faddeev equations in configuration space. Here, the direct solution means a method of solution based on the finite-deferential approximation of the boundary problem for eigenvalues with discretization of the coordinate space. The analysis of the method is performed in Appendix B, where two cross-check tests are given. The first is related to the test of our computer codes for the solution of a problem similar to the one considered for the $KK\bar{K}$ system with the comparison with results of others authors. In the second one, we proposed an alternative way to solve the Faddeev equations by the cluster reduction method [69] [70]. The both tests evidenced that the accuracy for the results listed in Table is reached in our calculations.

In Table [IV] we present the results of calculation for the $E_3(V_{KK} = 0)$ defined as energy of the quasi-bound state of the three-body system when the repulsive $KK$ interaction is omitted. In this case, the set of Eqs. (3) is reduced to the single equation for the $\bar{W}$ Faddeev component:

$$ (\hat{H}_0^{\bar{W}} + v_{KK} - E)\bar{W} = -v_{KK}\bar{p}\bar{W}. \quad (5) $$

The exchange term presented at the right-hand side of the equation provides the existence of a bound state with energy $E_3(V_{KK} = 0)$. In Table [IV] are shown the two-body energy of bound pair $E_2$ and three-body energy $E_3(V_{KK} = 0)$. Based on the analysis reformulated in Ref. [64] and according to Eq. (4), the relation between of $E_2$ and $E_3(V_{KK} = 0)$ can be rewritten as

$$ |E_3(V_{KK} = 0)/E_2| > 2. \quad (6) $$

The results of the calculations given in Table [IV] are in agreement with this relation. In nuclear physics, this relation is called "mass polarization effect" [63]. In terms of the Efimov physics [71] [72], the relation (4) is explained by the Efimov attraction as result of a mediated attraction between two particles by exchange of the third particle. Note that an expression, which is similar to Eq. (6), has been previously obtained in Ref. [74] for bosonic two-dimensional AAB systems to describe the relation of two-body and three-body energies. Interestingly enough to note that due to universality, Efimov physics applies to virtually any field of quantum physics, be it atomic and molecular physics, nuclear physics, condensed matter or even high-energy physics (see, for example, [73]).

The relation (6) agrees with so called "Efimov scenario" [72] defined for the model situation when pair potential is simple scaled by a multiplicative factor. To illustrate this fact, in Fig. [1] we present the results of calculations for the dependence of the ratio $E_3/E_2$ on two-body energy $|E_2|$ (right panel) and the value $1/\sqrt{|E_2|}$ (left panel). The $1/\sqrt{|E_2|}$ coincides with two-body scattering length $a_2$ due to approximation $|E_2| \approx \hbar^2/(m_Ka_2^2)$. These dependencies are obtained by introducing the scaled factor $\alpha$, which parameterizes the $KK$ potential and scales it as $v_{KK} \rightarrow \alpha v_{KK}$. Therefore, it is differed by multiplicative factor $\alpha$. These dependencies are parametric obtained. The parameter is the multiplicative factor $\alpha$ defining the scaled $KK$ potential. The region of the Efimov physics corresponds to small values of $|E_2|$ (large values of $1/\sqrt{|E_2|}$). Within this region, the ratio $E_3/E_2$ quickly increases and the possibility for an excited state is opening. In Fig. [2] we show the result for the A parameter set of the $KK$ potential ($\alpha = 1$). The corresponding state of the $K^0K^0K^0$ system is far from the Efimov states. The ratio $E_3/E_2$ asymptotically approaches 2. The repulsive $KK$ potential makes the $E_3/E_2$ ratio to be smaller than the $E_3(V_{KK} = 0)/E_2$ ratio. The strength of the $KK$ repulsion defines the difference. What will happen if the $KK$ interaction would be attractive? It is clear that an attractive $AA$ potential will make the ratio $E_3/E_2$ larger then $E_3(V_{AA} = 0)/E_2$. An example of a such situation with an attractive $AA$ potential is given in Appendix [11].

| Potentials/Kaon mass | $M$, MeV | $\Gamma$, MeV |
|----------------------|----------|-------------|
| A/497.6              | 1469.7   | 105         |
| A/496.0              | 1468.1 (1467) | 104 (110)  |
| B/497.6              | 1467.3   | 117         |

TABLE IV: The width and mass of the $K^0K^0K^0$ resonance using the A and B parameter sets for the $KK$ and $KK\bar{K}$ potentials. The results of Ref. [45] are shown in parentheses. The kaon masses are given in MeV.
Following Ref. [64] we evaluated the relative contribution of the mass polarization $\delta = \Delta / |E_3(V_{KK} = 0)|$ to the energy of the $KKK$ quasi-bound state for different physical particle configurations presented in Table III. The value depends on two factors: the mass ratio of kaons and the type of the $KK$ potential. For the considered systems, the mass ratio is approaching to one and the dependence of the mass polarization on the particle configuration is hidden. The second factor is more significant here. One can see the dependence by comparison of the results for the potentials of the A and B parameter sets. Summarizing the comparison, we conclude that the mass polarization effect for the potential of the A parameter set is about 30% and for the set B is about 24%. There is a correlation between of two-body scattering length $a_2$ obtained with the potential bounds non-identical particles and the relative contribution of the mass polarization $\delta$ [64]. The larger scattering length corresponds to the smaller mass polarization. For the $KKK$ system, the potentials of the B parameter set demonstrate larger scattering length and smaller mass polarization. This relation is shown in Fig. 2 with the results obtained in Ref. [64] for the $\alpha \Lambda \Lambda$ system with phenomenological potentials having different scattering parameters. The relatively small mass polarization in the $\alpha \Lambda \Lambda$ system is clarified by domination of the $\alpha$-particle mass in the system due to the ratio $m_\alpha / m_\lambda << 1$. The correlation between of the relative contribution of the mass polarization $\delta$ and two-body scattering length is only approximately linear, because the dependence of two-body parameters on the strength of a potential is more complex than the parametric dependence of a potential on the strength parameter considered above.

To show the difference between of the A and B parameter sets for the $KK$ potential, we averaged Eq. (5): $\langle H_0^W \rangle + \langle v_{KK} \rangle + \langle v_{KKP} \rangle = E_3(V_{KK} = 0)$. To evaluate the averaged kinetic energy $\langle H_0^W \rangle$, the method proposed in Ref. [75] was used. We considered Eq. (4) with scaled kaon masses by the factor $\gamma$ within the small vicinity of the point $\gamma = 1$. The energy becomes a function of the $\gamma = 1 \pm \Delta \gamma$ and $dE(\gamma)/d\gamma = -1/\gamma^2 \langle H_0^W \rangle$. The linear approximation for this derivation gives the evaluation of the averaged kinetic energy. The exchange term $\langle v_{KKP} \rangle$ depends on mass ratio and does not give a contribution to $\langle H_0^W \rangle$ as one has seen from the numerical results listed in Table III. Using the pattern $\langle H_0^W \rangle + \langle v_{KK} \rangle + \langle v_{KKP} \rangle = E_3(V_{KK} = 0)$, the results of averaging can be written as: $274 - 214 - 93 = -33$ and $182 - 142 - 70 = -30$ for the potentials of the A and B parameter sets, respectively. Here, all values are given in MeV. We see that the potentials with the set A are "more stronger", due to the fact that they act on shortest distances with the larger strength. We can assume that the $KKK$ system is more compact with the potentials of the A parameter set. One can assume that the contribution of the exchange term $\langle v_{KKP} \rangle$ could be corresponded to the value evaluated by Eq. (4) for the mass polarization term. We rewrite the expresses presented above as $274(1 - 214/274 - 93/274) = -33$ and $182(1 - 142/182 - 70/182) = -30$ for the potentials of the A and B parameter sets, correspondingly. The evaluation of $|\langle v_{KKP} \rangle|/|\langle H_0^W \rangle|$ gives 34% and 39% for the potentials of the set A and B, respectively. These values are in disagreement with the results for the mass polarization term from Table III. Note that the mass polarization is related to the kinetic energy operator in the Schrödinger equation [64]. By using the exchange term, one cannot directly separate this kinetic part. Thus, the $\delta$ more adequately evaluates the relative contribution of the mass polarization. At the same time, one can make sure that the relative contribution of the exchange term (Efimov attraction) increases with decreasing the strength of the potential according to the ’Efimov
FIG. 2: The correlation between the relative contribution of the mass polarization $\delta$ and two-body scattering length $a_2$ for the $\alpha\Lambda\Lambda$ and $KK\bar{K}$ systems calculated with different pair potentials. The mass ratios are $m_\Lambda/m_\alpha \approx 1/4$ and $m_K/m_{\bar{K}} \approx 1$, respectively.

scenario”.

In Table III we present the upper bounds for the mass of three-body quasi-bound state calculated as $\sum_{i=1,2,3} m_\gamma - |E_2|$, where $m_\gamma, i=1,2,3$ are the kaons masses. The values define a maximal value for the mass of the three-body resonance, when the quasi-bound state is approximately located on the two-body threshold. Obviously, the calculated mass of the three-body resonance is less the value of the upper bound. In Fig. 3 are presented the mass spectrum and the mass difference for different particle configurations of the $KK\bar{K}$ system. Note that the real part of

FIG. 3: Mass spectrum for different particle configurations of the $KK\bar{K}$ system calculated with the A and B parameter sets for pair potentials (left panel). The mass difference $M - M_{KK\bar{K}}$ for different particle configurations calculated with the set A for parameters of pair potentials (right panel).

the complex $K\bar{K}$ potentials dominates with the ratio for strengths of $\epsilon = 283/1155$ for the potential of the A
FIG. 4: The probability distribution of the particles in the $K^+ K^+ K^-$ system. The contour plot of the squared modulus of the Faddeev component $|U(x, y)|^2$ (upper panel) and $|W(x, y)|^2$ (lower panel) versus the corresponding Jacobi coordinates. The most probable spatial configuration of the particles is given as the inset. The $y$ coordinate is marked by the red color.

parameter set and $\epsilon = 210/630$ for the potential with set B. One can write the Hamiltonian of the $KK\bar{K}$ system as $H^R + i\text{Im} V_{KK}$, where $H^R = H_0 + \text{Re} V_{KK} + V_{KK}$. We have taken into account that the $KK$ potential has no an imaginary part. This complex-value expression for the Hamiltonian can be transformed to the real $2 \times 2$ matrix representation:

$$
\begin{pmatrix}
H^R & -\epsilon \text{Im} V_{KK} \\
\epsilon \text{Im} V_{KK} & H^R
\end{pmatrix}.
$$

The obtained matrix is a rotation-scaling matrix. The complex eigenvalues of the matrix are $H^R \pm i\epsilon \text{Im} V_{KK}$. The energy $E$ and width $\Gamma$ can be obtained by the averaging $E = \langle H^R \rangle \pm i\epsilon \langle V_{KK} \rangle = E^R \pm i\Gamma/2$. We have evaluated the averaged $KK$ potential energy as $\langle V_{KK} \rangle$ for the $K^0 K^0 K^0$ particle configuration. The corresponding results using potentials with the A and B parameter sets are presented in Table IV. A similar result for the A parameter set was obtained in Ref. [45], where the variational calculations have been performed. Taking into account the difference of our results and results [45], we test our codes, that is presented in Appendix B.

In the first test, we considered the system $npp$, which is described by Eqs. (3) as a bosonic isospinless model by direct solution of the Faddeev equations in configuration space. The second test is related to the alternative approach for solution of the Faddeev equations in configuration space for the $KK\bar{K}$ system obtained by using cluster reduction method (CRM). It is interesting to consider the density distribution of particles for the $KK\bar{K}$ system calculated in the framework of the Faddeev equations in configuration space. The spatial configuration of particles in the $KK\bar{K}$ system can be understood by plotting the spatial probability amplitudes, i.e. the squared modulus of the Faddeev components $|U(x, y)|^2$ and $|W(x, y)|^2$ in terms of the Jacobi coordinates $x$ and $y$. In Fig. 4, we present results of calculations of the probability distribution for the charged kaon resonance $K^+(1460)$ described as the $K^+ K^+ K^-$ system by using the potentials of the B parameter set. In these figures the contour plots of the squared modulus $|U(x, y)|^2$ and $|W(x, y)|^2$ in the frame of 5 fm×5 fm, as well as the related spatial configurations are presented. The
FIG. 5: The energy $E_3$ of quasi-resonance in the $K^+K^0\bar{K}^0$ system (potential with the A parameter set) for different values of the scaling parameter $\xi$ defined for the triplet $KK$ potential as $v_{KK} \rightarrow (1 - \xi)v_{KK}$ [50]. $E_2$ energy of subsystem $KK$ (isospin singlet ($s$) or isospin triplet ($t$)). The vertical line corresponds to the value of the ratio $v_{KK}/v_{\bar{K}K}$ proposed in [50].

TABLE V: The mass of the $K(1460)$ resonance for potentials with the parameter sets A and B (without the Coulomb force) for the scaling parameter $\xi=0.3$, $v_{KK}/v_{\bar{K}K}=0.7$ [50]. The notations are the same as in Table III. $\Gamma_3$ is width of three-body resonance.

| Resonance  | System | Model | $E_3$ | $\Gamma_3$ | $E_3(V_{KK}=0)$ | Mass [MeV] | Mass upper bound [MeV] |
|------------|--------|-------|-------|------------|----------------|------------|------------------------|
| $K^0(1460)$ | $K^0K^0\bar{K}^0$ | A      | -12.9 | 70         | -18.2         | -11.61     | 1479.9                | 1481.2              |
|            |        | B      | -14.7 | 78         | -18.8         | -11.45     | 1478.1                | 1481.3              |
| $K^0(1460)$ | $K^0K^+K^-$ | A      | -12.2 | 67         | -17.1         | -10.96     | 1472.9                | 1474.1              |
|            |        | B      | -14.0 | 76         | -18.0         | -11.03     | 1471.1                | 1474.1              |
| $K^+(1460)$ | $K^+K^0\bar{K}^0$ | A      | -12.7 | 69         | -17.8         | -11.40     | 1476.3                | 1477.6              |
|            |        | B      | -14.4 | 77         | -18.9         | -11.34     | 1474.6                | 1477.7              |
| $K^+(1460)$ | $K^+K^+K^-$ | A      | -11.9 | 69         | -16.7         | -10.74     | 1469.2                | 1470.4              |
|            |        | B      | -13.8 | 76         | -17.7         | -10.87     | 1467.3                | 1470.2              |

careful examination of the contour plots shows that the maximal values of the squared modulus of components are differed by two order and, therefore, the probabilities of the corresponding spatial configurations. The component $W(x,y)$ is dominant in the total wave function. The localization of the particles in the system corresponds to the most probability associated to the component $W(x,y)$ for the values of the coordinates $x$ and $y$. The modules of the coordinates in the most favorable position are approximately related to the ratio of $|y|/|x| \sim 1.5$. The squared modulus of the $W(x,y)$ component displays very large asymmetry, being strongly elongated in the $y$-direction. The spatial configuration presented in the inset reflects this ratio. Thus, the particles in the $K^+K^-K^-$ system are distributed along one line like a chain-like spatial configuration ($K^+-(K^-)-(K^+)$). The distance between $K^+K^-$ is 0.8 fm, while the distance between $K^+K^+$ is twice larger. The latter is not surprising, because the $K^+K^-$ interaction is strong and attractive, while the interaction between the identical particles $K^+K^+$ is weak and repulsive. The other spatial configuration has low probability and can be represented as a triangle spatial configuration with the basis side and hight of 0.8 fm, respectively.

**Case when the strength of $v_{KK}$ and $v_{\bar{K}K}$ is differed.** Let’s now focus on the dependence of three-body energy on the strength of isospin splitting of the $KK$ potential. To consider this case one should solve Eqs. (2). In Fig. 5 we present the results of calculation for $K^+\bar{K}^0K^0$ systems for the potentials of the A parameter set. The splitting means that the isospin triplet component of the potential is decreased as $v_{KK}^t = (1 - \xi)v_{KK}$ under the condition
that the isospin singlet potential is not changed and provides the two body threshold \( E_2 \) about 11 MeV (see Table IV). The value of 0.3 for the scaling parameter \( \xi \) corresponds to the proposed in Ref. 50 relation between the singlet and triplet components of the \( K\bar{K} \) potential. Our calculations show that the triplet bound state exists when the scale parameter \( \xi \) is less than 0.15. Thus, the proposed in [50] model assumes that the triplet state is not bound.

The quasi-bound state of the \( K^+K^0\bar{K}^0 \) system has the energy \(-12.4 \text{ MeV}\), that is near the two-body threshold. Due to the isospin splitting of the \( K\bar{K} \) interaction, the relation (3) is invalidated and the opposite relation takes place: \( |E_3(V_{KK} = 0)/E_2| < 2 \). Also the value of \( 2E_2 - E_3(V_{KK} = 0) \) becomes negative in contrast to the positive value of \( \Delta \) in Eq. (4).

IV. SUMMARY

In this work we developed a new framework for the Faddeev calculations in configuration space for the \( K(1460) \) dynamically generated resonance. Our three-body non-relativistic single channel model predicts a quasi-bound state for the \( KKK \) system of the mass around 1460 MeV. The calculations are performed using two sets of phenomenological \( KK \) and \( K\bar{K} \) potentials, when the strength of \( K\bar{K} \) interaction has no difference in the singlet and triplet isospin states and taken into account various particle configurations of the \( KKK \) system. Our study was extended to the more complicated case when the isospin singlet and triplet parts of the \( K\bar{K} \) potential are different and related by the condition of obtaining a quasi-bound three-body state.

In our study the mass difference between the kaons was taken into account to separate physical particle configurations of the \( KKK \) system: \( K^0K^0\bar{K}^0, K^0K^+\bar{K}^-, K^+K^0\bar{K}^0, K^+K^+\bar{K}^- \). These improvements enable us to investigate these kaonic configurations systematically, moreover, the first time the Coulomb interaction has been taken into account for description of the charged configurations. The mass splitting in the \( K(1460) \) resonances is evaluated to be in range of 10 MeV with taking into account the Coulomb force in the case of charged resonances. It is worth mentioning that a hypothetical chain-like spatial configuration \((-K^-)-(K^-)-(K^+)\) would constitute a favorable structure of the \( KKK \) system.

We considered the mass polarization effect in the \( KKK \) system and evaluated the effect of the repulsion strength of \( KK \) potential. The mass polarization term which is well separated in the Schrödinger equation as a part of the kinetic energy operator and the exchange term defined by the Faddeev equations is evaluated and discussed. This term is closely related to the "Efimov attraction". We have demonstrated that the model for the \( KKK \) system with \( KKK \) interaction having the same strength in the isospin singlet and triplet states is far from Efimov physics. The evaluation of the mass polarization in the \( KKK \) system in the framework of the Faddeev equations in configuration space allows us to understand, explain and interpret the contribution from the \( KK \) potential to the mass of the \( K(1460) \) as a dynamically generated resonance. It is shown that the contribution of mass polarization into the energy of the \( KKK \) system is large (up to 30%) and depends linearly on the \( KKK \) scattering length. Specifically the contribution is defined by the mass ratio of non-identical particles. As a result, the relative contributions can be the same for different systems.

We also studied the impact of isospin splitting of the \( K\bar{K} \) interaction on the energy of the \( KKK \) quasi-bound state. Generally, a model with the isospin dependence of a \( K\bar{K} \) potential leads to decreasing the binding energy of the system. In particular, we found that the \( K\bar{K} \) potential with an essential difference of isospin components produces a weak quasi-bound state. The comparison of our calculations with the recent experimental study 1482.40±3.58±15.22 MeV [43], where the first uncertainty is statistical and the second systematic, shows that the mass of the \( K(1460) \) resonance is in a satisfactory agreement with the mass upper bound calculated within our three-body model with isospin splitting \( K\bar{K} \) potential. Due to the experimental uncertainties in the relevant observable one can explore the possible range for the ratio of the strengths of isospin triplet and singlet components of the \( K\bar{K} \) interaction. On the other hand, our results obtained in the model with the same strength of the \( K\bar{K} \) interaction in the isospin singlet and triplet states are in reasonable agreement with the SLAC and ACCMOR collaboration experimental values of the mass of \( K(1460) \) resonance [40, 41].

It is worth noticing that despite its simplicity, the single-channel model is able to reproduce the mass of the \( K(1460) \) resonance. In our consideration there are no any fitting parameters and we are using \( s\)-wave \( K\bar{K} \) and \( KK \) two-body potentials as well as kaons masses only as the inputs in our model. The key ingredient of the model is the proper description of the isospin-dependent \( K\bar{K} \) interaction. Therefore, some refinements can be done, such as using more realistic two-body potentials, including \( p\)-wave components, and/or considering the coupled-channel approach. However, these will not affect dramatically the main conclusions obtained within the present approach.
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Appendix A: Coulomb potential

The general form of the Faddeev equations with Coulomb interactions reads as follows [53]:

\[
\{H_0 + V_\gamma^s(|x_\gamma|) - E\} \Psi_\gamma(x_\gamma, y_\gamma) = -V_\gamma(|x_\gamma|) \sum_{\beta \neq \gamma} \Psi_\beta(x_\beta, x_\beta),
\]

where \(V_\beta^{\text{Coul}}\) is the Coulomb potential between the particles belonging to the pair \(\beta\) and \(V_\gamma\) is the short-range pair potential in the channel \(\gamma\), \(\gamma = 1, 2, 3\). In (A1) \(H_0 = -\Delta x_\gamma - \Delta y_\gamma\) is the kinetic energy operator, \(E\) is the total energy, \(\Psi\) is the wave function of the three-body system given as a sum over three Faddeev components, \(\Psi = \sum_{\gamma=1}^{3} \Psi_\gamma\), \(x_\gamma\) and \(y_\gamma\) are the Jacobi coordinates for three particles with unequal masses \(m_1, m_2\) and \(m_3\) having positions \(r_1, r_2\) and \(r_3\) defined as

\[
x_i = \sqrt{\frac{m_j m_k}{m_j + m_k}} (r_j - r_k),
\]

\[
y_i = \sqrt{\frac{m_j (m_j + m_k)}{M}} \left( -r_i + \frac{m_j r_j + m_k r_k}{m_j + m_k} \right),
\]

\[
R = (m_1 r_1 + m_2 r_2 + m_3 r_3), \quad M = m_1 + m_2 + m_3, \quad i \neq j \neq k = 1, 2, 3.
\]

For a system with two identical particle (A1) is reduced to two equations. The system \(K^+K^+K^-\) has two types of the Coulomb potentials. The first one is repulsive and describes the interaction between two particles of the same charge and the second one is attractive and describes the interaction between two opposite charged particles. Each potential gives the contribution into each equation of the set. For example, the Coulomb potential of the first type is written as \(n_1/|x|\) for the first equation and \(n_2/|x'|\) for the second equation of the set [3], where \(x' = x/2 + y\) and \(n_k, k = 1, 2\) is reduced charge: \(n_k = e^2 m_k / \hbar^2\), \(m_k\) is a reduced mass of corresponding particle pair.

FIG. 6: The structure of the Coulomb force in the particle configurations of the \(KK\bar{K}\) system, a) \(K^0K^+K^-\), b) \(K^+K^+K^-\).

The particle configurations and corresponding Coulomb forces are schematically presented in Fig. 6. Note that the Coulomb potential of the particle configuration \(K^0K^+K^-\) violates the paradigm of \(AAB\) system. To describe this system with the Coulomb potential, one has to use the Faddeev equations (A1) for three non-identical particles.

Appendix B: Numerical solution of the Faddeev equations: the code testing

1. Bosonic model for \(nnp\) system with MT-V potential

Our calculation for \(KK\bar{K}\) system is tested by using the simple model for \(nnp\) system with the MT-V nucleon-nucleon potential [78]. The potential corresponds to a bosonic model for \(nnp\) system, when an isospin/spin independent \(s\)-wave potential is used. The MT-V bosonic model was motivated by spin averaging for the spin-dependent MT-III potential [77]. The configuration space Faddeev calculations for the model are based on Eqs. [3]. The MT-V potential is
defined as $V_{NN}(r) = \sum_{i=1,2} U_i \exp[-(\mu_i r)]/r$, where $U_i$ and $\mu_i$ are strength and range parameters, respectively. The range parameters are $\mu_1 = 1.55 \text{ fm}^{-1}$, $\mu_2 = 3.11 \text{ fm}^{-1}$. We used two sets for strength parameters of the potential known from literature [76]: (1) $U_1 = -570.316 \text{ MeV}$, $U_2 = 1438.4812 \text{ MeV}$ and (2) $U_1 = -578.098 \text{ MeV}$, $U_2 = 1458.047 \text{ MeV}. The results of the calculations are given in Table VI. Our results are in good agreement with the results obtained in Ref. [76], where the Faddeev equations in configuration space were also applied. Note, that the mass polarization evaluated by the value $\Delta/|E_3(V_{KK} = 0)|$ is the similar to the kaonic system $KK\bar{K}$ due to the similar mass ratio. For the bosonic model of the $nnp$ system, the mentioned above (Section III) correlations between two

| Potential | $E_3$ | $E_3(V_{nn} = 0)$ | $E_2$ | MP (%) |
|-----------|-------|------------------|-------|--------|
| MT-V(1)   | -7.54 | -1.01            | -0.35 | 30.7   |
| MT-V(2)   | -8.04 | -1.16            | -0.41 | 29.3   |

and three-body parameters takes place. In particular, the ratio $E_3/E_2$ with dependence on two-body energy $|E_2|$ presented in Fig. 7 shows the Efimov effect when two-body energy is close to three-body threshold. Here, the $nn$ potential is scaled by a factor $\alpha$ as $V_{nn} \rightarrow \alpha V_{nn}$ with the condition $\alpha > 0$. The strong attraction of the $nn$ pair makes the ratio $E_3/E_2$ to be more larger in comparison with the $E_3(V_{nn} = 0)/E_2$ case, when the $nn$ interaction is omitted. The results of our calculations for the $nnp$ system with two sets of the MT-V potential for the correlation between the relative contribution of the mass polarization $\delta$ and two-body scattering length $a$ are is represented in Fig. 7. A similar dependence is shown in Fig. 2 obtained in case of the $KK\bar{K}$ and $\alpha\Lambda\Lambda$ systems. The correlation between of $\delta$ and $a$ is represented by a linear dependence with the negative slope as one can see in Fig. 2.

### 2. Cluster reduction method versus direct numerical solution

The cluster reduction method [69, 70] was alternatively used for a numerical solution of the Faddeev equations (2). The method is based on the expansion of the components $U$ and $W$ in terms of the basis of the eigenfunctions of two-body Hamiltonian of the subsystems:

$$U(x, y) = \sum_{i \geq 1} \phi_i^U(x) F_i^U(y), \quad W(x, y) = \sum_{i \geq 1} \phi_i^W(x) F_i^W(y). \quad (B1)$$
Here, the functions $F^U_i$ and $F^W_i$, $i = 1, 2, \ldots, N$ describe the relative motion of "clusters" in each rearrangement channel ($KK\bar{K}$ and $K(K\bar{K})$, respectively. The functions $F^U_i$ ($F^W_i$) depend on the relative coordinate $y$. The solutions of the two-body Schrödinger equations form complete set of eigenfunctions in the box, $x \subset [0, R_x]$:

\[
\left(-\frac{\hbar^2}{2\mu} \frac{\partial^2}{\partial x^2} + V^{NN}_{NN}(x)\right) \phi^U_i(x) = \epsilon^U_i \phi^U_i(x),\left(-\frac{\hbar^2}{2\mu} \frac{\partial^2}{\partial x^2} + V_{N\bar{K}}(x)\right) \phi^W_i(x) = \epsilon^W_i \phi^W_i(x),
\]

where, $\mu^U$ (and $\mu^W$) is a reduced mass of the pairs and $\phi^U_i(0) = \phi^U_i(R_x) = 0$ ($\phi^W_i(0) = \phi^W_i(R_x) = 0$), $i = 1, 2, \ldots, N$. The parameter $R_x$ is chosen to be large enough to reproduce the pair binding energy. In our calculations $R_x = 35$ fm is used. The number $N$ is chosen by condition of total convergence of calculations results, when $N$ consequently increases.

The comparison of the CRM and direct solution is presented in Fig. 8. The results obtained using the both methods are in good agreement. The CRM calculations for the case of complete set of potentials and the case of restriction $V_{K^0K^0} = 0$ demonstrate the repulsive nature of the $KK$ potential. The convergence of the calculation results as a function of the number $N$ of the terms in Eq. (B1) is different for these cases. In the first case, the decreasing of binding energy is changed to increasing when the calculation becomes "more precise" with increasing the number $N$. For the second case, we have monotonic decreasing of the binding energy. Such behavior is related to consequent inclusion of the attraction for the $K\bar{K}$ pair and repulsion for the $KK$ pair.

![Fig 8](image)

**FIG. 8:** The binding energy of the $KK\bar{K}$ system ($K^0K^0\bar{K}^0$) calculated using CRM for different numbers of the terms in the expansion (B1). The case, when the $K\bar{K}$ potential is omitted, is also shown. The horizontal lines (solid and dashed) represent the results of the direct numerical solution of the Faddeev equations. The B parameter set of the potentials was applied.

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