Exact Solutions of G-invariant Chiral Equations
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Abstract

We give a methodology for solving the chiral equations $\left( \alpha g_{,z} g^{-1} \right)_{,z} + \left( \alpha g_{,\bar{z}} g^{-1} \right)_{,z} = 0$ where $g$ belongs to some Lie group $G$. The solutions are written in terms of Harmonic maps. The method could be used even for some infinite Lie groups.

One of the most important set of differential equations in mathematical physics is perhaps the chiral equations. The most popular of them are the SU(N)-invariant one. In general relativity chiral equations appear very often for the groups SU(1,1), SU(2,1), SL(2, R) etc. [1]. One of the first approaches for solving them is by means of the Inverse Scattering Method (ISM), given by Belinsky ans Zacharov [2]. They gave a Lax Pair representation of the chiral equations and solve them using the ISM. An equivalent aproach using the polinomial ansatz is given in ref [3]. In this last work we used the generalized Bäcklund transformations for finding exact solutions of the chiral equations.

Recently, Hussain [4] showed that self or antiself dual gravity reduces to chiral equation-like systems but the group remain here infinite. Algebra brackets are replaced by Poisson brackets and all of these methods mentioned above can not be applied in this case. In this paper we generalize the method of Harmonic maps for solving the chiral equations for all Lie groups. All the solutions obtained by the ISM or by Bäcklund transformations can be derived with this method choosing the Harmonic map conveniently. For example, all the solutions found by ISM in general relativity can also be derived by this method [10]. With the method of Harmonic maps the solutions appear classified into classes naturally. The solutions can be given in terms of arbitrary harmonic maps and the method can be used even for infinite groups.

Let $g$ be a mapping $g : \mathcal{C} \otimes \mathcal{C} \to G, \quad g = g(z, \bar{z}) \in G$, where $G$ is a paracompact Lie group.

The chiral equations for $g$

\[
\left( \alpha g_{,z} g^{-1} \right)_{,z} + \left( \alpha g_{,\bar{z}} g^{-1} \right)_{,z} = 0
\]  

$(\alpha^2 = \det g)$ are a set of non-linear, coupled, second order partial differential equations that appear in many topics in physics. Normally $g(z, \bar{z})$ is given in a representation of
One of the most important features of the chiral equations is that they can be derived from the Lagrangian.

\[ \mathcal{L} = \alpha tr (g_{zz}g^{-1}g_zg^{-1}). \] (2)

This Lagrangian represents a topological quantum field theory with gauge group \( G \), then it is of great interest to know explicit expression of the elements \( g \in G \) in terms of the local coordinates \( z \) and \( \overline{z} \).

Let \( G_c \) be a subgroup \( G_c \subset G \) such that \( c \in G_c \) implies \( c_{.z} = 0, \ c_{.\overline{z}} = 0 \). Then equation (1) is invariant under the left action \( L_c \) of \( G_c \) over \( G \). We say that the chiral equations (1) are invariant under this group.

**Proposition 1.** Let \( \beta \) be a complex function defined by

\[ \beta_{.z} = \frac{1}{4(\ln \alpha)_{.z}} tr (g_{zz}g^{-1})^2, \ g \in G \] (3)

and \( \beta_{.\overline{z}} \) with \( \overline{z} \) in place of \( z \). If \( g \) fulfills the chiral equations, then \( \beta \) is integrable.

Proof. \( \beta_{.\overline{z}} = \beta_{.\overline{z}} \).

Let \( \mathcal{G} \) be the corresponding Lie algebra of \( G \). The Maurer-Cartan form \( \omega_g \) of \( G \) defined by \( \omega_g = L_{g^{-1}} \omega_g \) is a one-form on \( G \) with value on \( \mathcal{G} \), \( \omega_g \in T^*_g G \otimes \mathcal{G} \) (\( T_x M \) represents the tangent space of the Manifold \( M \) at the point \( x \)). Let us define the mappings

\[ A_z : G \rightarrow \mathcal{G} \]
\[ g \mapsto A_z(g) = g_{.z}g^{-1} \]

\[ A_{\overline{z}} : G \rightarrow \mathcal{G} \]
\[ g \mapsto A_{\overline{z}}(g) = g_{.\overline{z}}g^{-1} \] (4)

If \( g \) is given in a representation of \( G \), then we can write the one-form \( \omega(g) = \omega_g \) as

\[ \omega = A_z dz + A_{\overline{z}} d\overline{z} \] (5)

We can now define a metric on \( \mathcal{G} \) in a standard manner. Since \( \omega_g \) can be written as in (50), the tensor

\[ l = tr (dgg^{-1} \otimes dgg^{-1}) \] (6)

on \( G \) defines a metric on the tangent bundle of \( G \).

**Theorem 1.** The submanifold of solutions of the chiral equations \( S \subset G \), is a symmetric manifold with metric (6).

This theorem was proved by Neugebauer and Kramer in reference [5] and we will only outline here the proof.

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Proof. We take a parametrization \( \lambda^a, a = 1, \cdots, n \), of \( G \). The set \( \{ \lambda^a \} \) is a local coordinate system of the \( n \)-dimensional differential manifold \( G \). In terms of this parametrization the Maurer-Cartan one form \( \omega \) can be written as

\[
\omega = A_a d\lambda^a,
\]

where \( A_a(g) = (\frac{\partial}{\partial \lambda^a} g) g^{-1} \). The chiral equations then read

\[
\nabla_b A_a(g) + \nabla_a A_b(g) = 0, \tag{8}
\]

with \( \nabla_a \) the covariant derivative defined by (6).

One infers the relation

\[
\nabla_b A_a(g) = \frac{1}{2} [A_a, A_b](g). \tag{9}
\]

With this, one can calculate the Riemannian curvature \( \mathcal{R} \) of (6). Their components read

\[
R_{abcd} = \frac{1}{4} tr(A_{[a} A_{b]} A_{[c} A_{d]}), \tag{10}
\]

([\( \cdot, \cdot \)] means index commutation) this can be done, because \( G \) is a paracompact manifold. It follows that \( \nabla \mathcal{R} = 0 \)

**Proposition 2.** The function \( \alpha = \det g \) is harmonic.

Proof. The trace of the chiral equations implies \( \alpha_{,zz} = 0 \).

In the following we will explain the method for calculating explicit chiral field.

Let \( V_p \) be a complete totally geodesic submanifold of \( G \) and let \( \{ \lambda^i \} \quad i = 1, \cdots, p \) be a set of local coordinates on \( V_p \). Because we know the Manifold \( G \), it is possible to know \( V_p \). In fact \( V_p \) is a subgroup of \( G \) and since \( V_p \) is a complete totally geodesic submanifold, \( V_p \) is also symmetric. The symmetries of \( G \) and \( V_p \) are in fact isometries, since both of them are paracompact manifolds, with Riemannian metrics (6) and \( i_* l \) respectively. (\( i \) is the restriction of \( V_p \) into \( G \)). Let us suppose that \( V_p \) possesses \( d \) isometries. Thus

\[
(\alpha \lambda^i_{,z})_{,z} + (\alpha \lambda^i_{,z})_{,z} + 2\alpha \sum_{ijk} \Gamma^i_{jk} \lambda^j_{,z} \lambda^k_{,z} = 0 \tag{11}
\]

\[
i, j, k = 1, \cdots, p
\]

where \( \Gamma^i_{jk} \) are the Christoffel symbols of \( i_* l \) and \( \lambda^i \) are the totally geodesic parameters on \( V_p \). In terms of the parameters \( \lambda^i \) the chiral equations read

\[
\nabla_i A_j(g) + \nabla_j A_i(g) = 0 \tag{12}
\]
where $\nabla_i$ is the covariant derivative of $V_p$. Equation (12) is the Killing equation on $V_p$ for the components of $A_i$. Since we know the manifold $V_p$, we know its isometries and therefore its Killing vector space. Let $\xi_s, s = 1, \cdots, d$, be a base of the Killing vector space of $V_p$ and $\Gamma^s$ be a base of the subalgebra corresponding to $V_p$. Then we can write

$$A_i(g) = \sum_s \xi^i_s \Gamma^s$$  \hspace{1cm} (13)

where $\xi^i_s = \sum_j \xi^j_s \frac{\partial}{\partial \lambda^j}$. The covariant derivative on $V_p$ is given by

$$\nabla_j A_i(g) = -\frac{1}{2} [A_i, A_j](g)$$  \hspace{1cm} (14)

where $A_i$ fulfills the integrability conditions

$$F_{ij} = \nabla_j A_i(g) - \nabla_i A_j(g) - [A_j, A_i](g) = 0$$  \hspace{1cm} (15)

i.e., $A_i$ has a pure gauge form.

The left action of $G_c$ over $G, L : G_c \times G \to G$ must be defined in a convenient manner in order to preserve the properties of the elements of $S$.

Knowing \{\xi_s\} and \{\Gamma^s\} one could integrate the elements of $S$, since $A_i(g) \in G_c$ can be mapped into the group by means of the exponential map. Nevertheless it is not possible to map all the elements one by one. Fortunately we have the following proposition.

Proposition 3. The relation $A^c_i r A_i$ iff there exist $ce G_c$ such that $A^c = A \circ L_c$, is an equivalence relation.

Proof. Trivial.

This equivalence relation separates the set \{\A_i\} into equivalence classes [\A_i]. Let $TB$ be a set of representatives of each class, $TB = \{[A_i]\}$. Now we map the elements of $TB \subset G$ into the group $S$ by means of the exponential map or by integration. Let us define $B$ as the set of elements of the group, mapped from each representative $B = \{g \in S| g = exp(A_i), A_i \in TB\} \subset G$. The elements of $B$ are also elements of $S$ because $A_i$ fulfills the chiral equations, i.e. $B \subset S$. For constructing all the set $S$ we have the following Theorem.

Theorem 2. $(S, B, \pi, G_c, L)$ is a principal fibre bundle with projection $\pi(L_c(g)) = g; \ L(c, g) = L_c(g)$.

Proof. The fibres of $G$ are the orbits of the group $G_c$ on $G, F_g = \{g' \in G|[g' = L_c(g)]\}$ for some $g \in B$. The topology of $B$ is its relative topology with respect to $G$. Let $\alpha_F$ be the bundle $\alpha_F = (G_c \times U_\alpha, U_\alpha, \pi)$, where $\{U_\alpha\}$ is an open covering of $B$. We have the following lemma.

Lemma 1. The bundle $\alpha_F$ and $\alpha = (\pi^{-1}(U_\alpha), U_\alpha, \pi|_{\alpha^{-1}(U_\alpha)})$ are isomorphic.

Proof. The mapping
\[ \psi_\alpha : \phi^{-1}(U_\alpha) = \{ g \in S | g' = L_c(g), g \in U_\alpha \} \subset G_c \rightarrow G_c \times U_\alpha \]

\[ g' \rightarrow \psi_\alpha(g') = (c, g) \]

is an homeomorphism and \( \pi|_{\pi^{-1}(U_\alpha)}(g') = g = \pi_2 \circ \psi_\alpha(g') \).

By lemma 1 the bundle \( \alpha \) is locally trivial. To end the proof of the Theorem it is sufficient to prove that the \( G_c \) spaces \((S, G_c, L)\) and \((G_c \times U_\alpha, G_c, \delta)\), are isomorphic, but that is so because

\[ \delta \circ id|_{G_c \times \psi_\alpha} = \psi_\alpha \circ L|_{G_c \times \pi^{-1}(U_\alpha)} \]

With this theorem it is now possible to explain the method.

a) Given the chiral equations (1), invariant under the group \( G \), chose a symmetric Riemannian space \( V_p \) with a \( d \) dimensional isometry group \( H \subset G \), \( p \leq n = \dim G \).

b) Look for a representation for the corresponding Lie Algebra \( G \) compatible with the commutating relations of the killing vectors, via equation (14).

c) Write the matrices \( A_i(g) \) explicitly in terms of the geodesic parameters of the symmetric space \( V_p \).

d) Use proposition 2 for finding the equivalence classes in \( \{A_i\} \) and choose a set of representatives.

e) Map the lie algebra representatives into the group.

The solutions can be constructed by means of the left action of the \( G_c \) group into \( G \).

Let us give an example. Suppose \( G = SL(2, \mathbb{R}) \).

a) We choose the one dimensional space \( V_1 \) with \( i_s l = d\lambda^2 \). It is Riemannian and symmetric with one killing vector

b) The algebra \( sl(2, \mathbb{R}) \) is the set of traceless and real \( 2 \times 2 \) matrices. The Killing equation reduces to the Laplace equation \((\alpha \lambda, z),_z + (\alpha \lambda, z),_z = 0 \).

c) Using (14) we obtain

\[ g, \lambda g^{-1} = A = \text{constant}. \]

d) The representative of the set of trasless and real constant \( 2 \times 2 \) matrices \( \{A_i\} \) is

\[ \{[A_i]\} = \left\{ \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix} \right\} \]
e) The mapping of the Lie algebra representatives can be done by integration of the matrix differential equation

\[ g, \lambda = [A_i] g. \]

The solutions depend on the characteristic polynomial of \([A_i]\). We obtain three cases: \(a > 0\), \(a < 0\), \(a = 0\). For each case the matrix \(g\) reads

- \(a > 0\): \( g = b \begin{pmatrix} 1 \ a \\ a \ a^2 \end{pmatrix} e^{a \lambda} + c \begin{pmatrix} 1 \ -a \\ -a \ a^2 \end{pmatrix} e^{-a \lambda}, \quad 4bca^2 = 1 \)
- \(a < 0\): \( g = b \begin{pmatrix} \cos(a \lambda + \psi_0) & -asin(a \lambda + \psi_0) \\ -asin(a \lambda + \psi_0) & -a^2 \cos(a \lambda + \psi_0) \end{pmatrix}, \quad a^2b^2 = -1 \)
- \(a = 0\): \( g = \begin{pmatrix} b\lambda + c & b \\ b & 0 \end{pmatrix}, \quad b^2 = -1 \)

where \(a, b, c\) and \(\psi_0\) are constants. So, for each solution \(\lambda\) of the Harmonic equation \((\alpha \lambda, \bar{z}) + (\alpha \lambda, \bar{z}) = 0\) we will have a new solution of the chiral one. The left action of \(SL_c(2, R)\) over \(SL(2, R)\) is represented as

\[ g' = CgD \]

where \(C, D \in SL_c(2, R)\). \(g'\) will be also a solution of the chiral equations.

If we choose a \(V_2\) manifold we can find another class of solutions. All \(V_2\) manifold is conformally flat, therefore the metric on \(V_2\) reads

\[ dl^2 = \frac{d\tau d\lambda}{(l + k \lambda \bar{z})^2} \]

But \(V_2\) symmetric implies \(k = \text{constant}\). A \(V_2\) manifold with constant curvature has three Killing vectors. Let

\[ \xi_1 = \frac{1}{2V^2} [(k \tau^2 + 1)d\lambda + (k \lambda + 1)d\tau] \]
\[ \xi_2 = \frac{1}{V^2} [-\tau d\lambda + \lambda d\tau] \quad V = 1 + k \lambda \tau \]
\[ \xi_3 = \frac{1}{2V^2} [(k \tau^2 - 1)d\lambda + (1 - k \lambda^2)d\tau] \]

be a base of the Killing vector space of \(V_2\). The correspondings commutation relations of the dual vectors are
\[ [\Gamma^1, \Gamma^2] = -4k\Gamma^3 \]
\[ [\Gamma^2, \Gamma^3] = 4k\Gamma^2 \]
\[ [\Gamma^3, \Gamma^1] = -4\Gamma^2 \]

We have to put \( k = -1 \) in order to have the commutation relations of \( sl(2, R) \). A representation of \( sl(2, R) \), compatible with (16) is

\[ \Gamma^1 = 2 \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Gamma^2 = \begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix}, \quad \Gamma^3 = \begin{pmatrix} 0 & -b \\ a & 0 \end{pmatrix} \quad ab = 1 \]

Then equation (13) reads

\[ g, \lambda g^{-1} = A_\lambda = \frac{1}{V^2} \begin{pmatrix} \tau^2 - 1 & b(1 - \tau)^2 \\ -a(1 + \tau)^2 & 1 - \tau^2 \end{pmatrix} \]
\[ g, \tau g^{-1} = A_\tau = \frac{1}{V^2} \begin{pmatrix} \lambda^2 - 1 & b(1 - \lambda)^2 \\ -a(1 + \lambda)^2 & 1 - \lambda^2 \end{pmatrix} \]

which integration is

\[ g = \frac{1}{1 - \lambda \tau} \begin{pmatrix} c(1 - \lambda)(1 - \tau) & e(\tau - \lambda) \\ e(\tau - \lambda) & d(1 + \lambda)(1 + \tau) \end{pmatrix} \]
\[ cd = -e^2, \quad a = \frac{e}{c}, \quad b = -\frac{e}{d} \]

The harmonic equation on \( V_2 \) reads

\[ (\alpha \lambda, z)_\tau + (\alpha \lambda, z)_\sigma + \frac{4\tau}{1 + \lambda \tau} \alpha \lambda, z, \tau = 0 \]
\[ (\alpha \tau, z)_\sigma + (\alpha \tau, z)_\lambda + \frac{4\lambda}{1 + \lambda \tau} \alpha \tau, z, \lambda = 0 \]

(18)

So, for each solution of (18) into (17) we find a new solution of the chiral equations for \( SL(2, R) \).

A general algorithm of integration even for the Lie Algebra \( sl(N, R) \), is given in [7]. The explicitly example for \( SL(4, R) \) is presented in ref. [8]. The application of the method for infinite groups will be published elsewhere [9].

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