COMMON TANGENTS TO POLYTOPES

FEDERICO CASTILLO, JOSEPH DOOLITTLE, AND JOSÉ A. SAMPER

ABSTRACT. It is well-known since the time of the Greeks that two disjoint circles in the plane have four common tangent lines. Cappell et al. proved a generalization of this fact for properly separated strictly convex bodies in higher dimensions. We have shown that the same generalization applies for polytopes. When the number of convex sets involved is equal to the dimension, we obtain an alternative combinatorial proof of Bisztriczky’s theorem on the number of common tangents to \( d \) separated convex bodies in \( \mathbb{R}^d \).

1. INTRODUCTION

It is known since the times of Euclid and Apollonius that two disjoint circles have four common tangents. In fact, they had explicit constructions with straightedge and compass to describe these lines. Note that the circles need to be disjoint for these four tangents to exist; the number of common tangents can be any integer less than four depending on whether the circles are internally/externally tangent, intersecting, or nested.

Our goal is to generalize this basic result in the context of convex geometry. The first step is to define an adequate analog to disjointedness for multiple sets. Let \( \mathcal{K}^d \) be the set of convex bodies (compact, convex, with non-empty interior) in \( \mathbb{R}^d \). We say that a family \( S = \{S_1, \ldots, S_m\} \subset \mathcal{K}^d \) is strongly separated if for every subset \( I \subseteq [m] \) there exists an affine hyperplane \( H \) such that \( \bigcup_{i \in I} S_i \subseteq H^- \) and \( \bigcup_{i \notin I} S_i \subseteq H^+ \). The set \( \mathcal{H}^d \) of all oriented hyperplanes in \( \mathbb{R}^d \) is parametrized by \( S^{d-1} \times \mathbb{R} \). Let \( \mathcal{T}(S) \subset \mathcal{H}^d \) be the set of oriented hyperplanes that are tangent to a family \( S \) and contain the family in its nonnegative side.

The main contribution of the present paper is a qualitative description of \( \mathcal{T}(P) \) when \( P \) consists of polytopes.

**Theorem A.** Let \( P = \{P_1, \ldots, P_m\} \subset \mathcal{K}^d \) be a family of strongly separated full dimensional polytopes in \( \mathbb{R}^d \) where \( m \leq d \). The set \( \mathcal{T}(P) \) is a polytopal complex combinatorially equivalent to the boundary of a \( d - m + 1 \) dimensional polytope, and thus homeomorphic to the sphere \( S^{d-m} \).
We say a hyperplane $H$ contained in one of the two closed halfspaces defined by $H_{u,\alpha}$ is tangent to the set $u_\alpha$ if $H$ is tangent to each element of $K$. Moreover, $H_{u,\alpha}$ is a hyperplane, and we often omit the subscripts $u$ and $\alpha$.

The core of this proof, as explained in [2], is to show that there exist exactly two common tangents to $d$ strongly separated convex bodies in $\mathbb{R}^d$ with every body on the same side of both tangents, so that informally these two hyperplanes sandwich the whole family. We prove this by approximating convex bodies with polytopes in the Hausdorff metric on $K$ and using polarity to remove the polytopal condition of Theorem A in the case $m = d$, see Proposition 4.3. We remark that [5] also generalizes their version of Theorem A to arbitrary convex bodies when $m = d$ in order to prove Theorem B.

There is another topological proof of Theorem B by Lewis, von Hohenbalken, and Klee [14] using Kakutani’s extension of Brouwer’s fixed point theorem. To highlight the subtlety of the arguments we note that in the introduction of [14] the authors mention that Bisztriczky had announced that his original proof of the Theorem was insufficient. In any case, there are now different proofs and also several generalizations of this theorem, see e.g. [1], [8], [9], and [10].

Finally, the subject of common tangents to multiple objects has been considered from an algebraic point of view; the circles in the original Greek problem are replaced by real quadrics or convex semialgebraic sets, and the lines are replaced by $k$-planes. See for example [3], [13], [19], and [20]. It would be interesting to know if the algebraic results for general $k$-planes extend to convex bodies.

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2. Preliminaries and notation

An (affine) hyperplane in $\mathbb{R}^d$ can be written as $H_{u,\alpha} := \{x \in \mathbb{R}^d : \langle x, u \rangle = \alpha\}$, where $u \in \mathbb{R}^d$ is a nonzero vector and $\alpha \in \mathbb{R}$ is any real scalar. Every affine hyperplane defines two open halfspaces

$$H^+_{u,\alpha} := \{x \in \mathbb{R}^d : \langle x, u \rangle > \alpha\}, \quad \text{and} \quad H^-_{u,\alpha} := \{x \in \mathbb{R}^d : \langle x, u \rangle < \alpha\}.$$ 

The positive and negative parts are exchanged if we replace $u$ by $-u$ and $\alpha$ by $-\alpha$. We denote their closures $H^\circ_{u,\alpha} = H^+_{u,\alpha} \cup H^-_{u,\alpha}$ and $H^\circ_{u,\alpha} = H^+_{u,\alpha} \cup H^-_{u,\alpha}$, and we often omit the subscripts $u$ and $\alpha$. We say a hyperplane $H$ is tangent (or supporting) to a set $S \subseteq \mathbb{R}^d$ if $S \cap H$ is nonempty and $S$ is contained in one of the two closed halfspaces defined by $H$.

A convex body $K \subseteq \mathbb{R}^d$ is a compact convex set with a nonempty interior. Two convex bodies $K_1$ and $K_2$ have strict separation if there exists an affine hyperplane $H$ such that $K_1 \subset H^+$ and $K_2 \subset H^-$. When the convex bodies are compact (the only case of interest for us) this property means that:

Theorem B (Bisztriczky’s theorem). Let $K = \{K_1, \ldots, K_d\} \in K^d$ be a family of strongly separated convex bodies of $\mathbb{R}^d$. For each set partition $A \sqcup B = [d]$ there exists exactly two affine hyperplanes $H$ such that:

- $H$ is tangent to each element of $K$.
- $\bigcup_{a \in A} K_a \subseteq H^+$.
- $\bigcup_{b \in B} K_b \subseteq H^-$.

These affine hyperplanes are all distinct and thus there are $2^d$ tangent affine hyperplanes to the family $K$.

In [5] Cappell, Goodman, Pach, Pollack, Sharir, and Wenger, proved a version of Theorem A with $m$ strictly convex bodies in $\mathbb{R}^d$ with $m \leq d$. Theorem A answers a question posed by I. Rivin in the Mathscinet Review [15] of [5]. As opposed to [5] our proof uses only basic topological facts instead of cobordism tools. More precisely, we devise a procedure to induct on $d$ by using polarity and Bruggesser and Mani’s approach to the shellability of polytope boundaries [4, Section 4].

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is equivalent to being disjoint. A convex body \( K \) is said to be strictly convex if every supporting hyperplane intersects \( K \) in a singleton. Unit balls are strictly convex, whereas polytopes are not.

In [2] and [5] they used a different definition of \textit{separated}, they call a family of subsets \( S = \{S_1, \ldots, S_m\} \subseteq K^d \) separated if for every \( n \)-dimensional affine subspace, with \( 1 \leq n \leq d - 2 \), intersects at most \( n + 1 \) members of \( S \). This notion of separation is equivalent to what we define as \textit{strong separation} (see [2] Lemma 1 for one direction).

Sometimes it is more useful to use vector configurations instead of point configuration. A set of vectors is said to be \textit{acyclic} if there is no linear dependence with all coefficients positive. See [21, Chapter 6.2]. Any point configuration \( A = \{p_1, \ldots, p_k\} \) in \( \mathbb{R}^d \) can be turned into an acyclic vector configuration in \( \mathbb{R}^{d+1} \) by transforming each point \( p \in A \) into the vector \( v = (p, 1) \). We call this construction the linearization of \( A \).

Conversely, for any acyclic vector configuration \( V = \{v_1, \ldots, v_k\} \) in \( \mathbb{R}^{d+1} \), there exists some vector \( v_h \) such that \( \langle v_h, v_i \rangle > 0 \) for all \( v_i \in V \). Intersecting the ray associated to each vector with the hyperplane \( H_{v_h, 1} \) gives a point configuration in \( \mathbb{R}^d \).

**Definition 2.1.** Let \( P \subseteq \mathbb{R}^d \) be a \( d \)-polytope and \( p \notin P \) an arbitrary point. We say that \( q \in P \) is \textit{visible from} \( p \) if \( p \cap \{q, p\} = \{q\} \). A facet \( F \) is \textit{visible from} \( p \) if one (or equivalently every) point in its relative interior is visible from \( p \). A nonempty set of facets \( \mathcal{F} \) of \( P \) is \textit{visible} if there exists a point \( p \) such that a facet \( F \) of \( P \) is visible from \( p \) if and only if \( F \in \mathcal{F} \). A nonempty set of facets of \( P \) is \textit{covisible} if the complement is visible.

3. PROOF OF THEOREM [A]

We approach Theorem [A] by considering the dual setting.

**Theorem C.** Let \( P \subseteq \mathbb{R}^d \) be a \( d \)-dimensional polytope such that its facets are colored with the set \( \{1, \ldots, m\} \), with \( 2 \leq m \leq d \), and for any proper subset of colors \( I \subset [d] \) the set \( \bigcup F \), union taken over all facets with colors in \( I \), is either visible or covisible. Then the set of all rainbow points is homeomorphic to \( S^{d-m} \).

**Proof.** We use induction on \( d \). For \( d = 2 \) the statement is clear. Now we assume \( d > 2 \).

If there is a covisible color, then the union of the complement is visible, so there can be at most one covisible color. Since \( m \geq 2 \) we can assume there is a visible color, say \( m \). This means there exists a point \( p \notin P \) such that \( p \) only sees the facets of \( P \) with color \( m \). Furthermore we can assume that \( p \notin \text{aff}(F) \) for any facet \( F \) with color different from \( m \).

We consider the polytope \( \hat{P} := \text{conv}(P \cup \{p\}) \) and take the vertex figure \( \hat{P}' \) at the vertex \( p \). The vertex figure [21, Chapter 2] is obtained by intersecting \( \hat{P} \) with an affine hyperplane \( H \) sufficiently close to \( p \). By [21, Proposition 2.4] there is a bijection between the proper faces of \( P' \) to the faces in the boundary of \( \bigcup_{\text{color}(F) = m} F \subseteq \partial P \). Facets in this boundary are ridges in \( P \), the intersection of exactly two facets: one of color \( m \) and another one of color \( j \neq m \). We color such a facet with the color \( j \). Thus facets of \( P' \) can be colored with the set \( \{1, \ldots, (d-1)\} \). Furthermore, the (co)visibility of any subsets of colors is preserved. For any \( I' \subset [d-1] \), the set of facets of \( P \) with colors \( I' \cup \{d\} \) is (co)visible, so there exists some point \( q \) as witness. The orthogonal projection \( q' \) of \( q \) onto \( H \) (the hyperplane containing the vertex figure \( P' \)) is then a witness for the (co)visibility of the facets with colors \( I' \in P' \).

By the induction hypothesis, the set of rainbow points in \( P' \) is homeomorphic to \( S^{(d-1)-(m-1)} = S^{d-m} \). The polytope \( P' \) and the polytopal complex \( \bigcup_{\text{color}(F) = d} F \) are homeomorphic via the projection from \( p \) and this homeomorphism identifies the corresponding set of rainbow points.

**Remark 3.1.** The point \( p \) required in the above proof can be constructed as follows. Each facet of \( P \) is the intersection of an affine linear hyperplane \( \{x \in \mathbb{R}^d : \langle a, x \rangle = b\} \) with \( P \). Furthermore we
can assume that \( \langle a, x \rangle \leq b \) for all points in \( P \). A point \( p \) outside of \( P \) that sees only the facets of color \( m \) is characterized by the following linear strict inequalities:

- \( \langle a, p \rangle < b \) if the corresponding facet is not of color \( m \).
- \( \langle a, p \rangle > b \) if the corresponding facet is of color \( m \).

To find such \( p \) we need to find a solution of a system of linear inequalities. This problem is equivalent to solving a linear program \([12] \text{ Theorem 10.4}\), so it can be solved efficiently using the simplex method or any other linear programming algorithm.

**Remark 3.2.** The case \( m = d \) of Theorem \( \mathbf{C} \) is reminiscent of the Knaster–Kuratowski–Mazurkiewicz Lemma \([11]\). For related results see an extension by Shapley \([18]\) with an alternative proof by Komiy \([12]\) and a recent generalization by Frick and Zerbib \([7]\).

**Proof of Theorem \( \mathbf{A} \)** Given a strongly separated family \( P \) of \( m \) full dimensional polytopes in \( \mathbb{R}^d \), let \( P = \text{conv}\{P_i \} \) be their convex hull. We think of vertices of the polytope \( P_j \) as being of color \( j \), so that the vertices of \( P \) are colored with the set \([m]\) and we are interested in the faces that contain a vertex of every color.

Let \( P^o \) be the polar of \( P \). We color each facet with the color of the corresponding vertex. A point in \( P^o \) is rainbow if it is contained in a facet of every color. The strong separation of the family \( P \) implies that every subset of colors in \( P^o \) is visible/covisible, so Theorem \( \mathbf{A} \) follows from Theorem \( \mathbf{C} \).

We can relax the condition that each polytope in the family is full dimensional with the slightly weaker condition that the whole family is affinely spanning, i.e., that their convex hull is full dimensional.

**Corollary 3.3.** Let \( P = \{P_1, \ldots, P_d \} \in \mathcal{P}^d \) be a family of strongly separated polytopes that is affinely spanning. Then there are two facets of \( P = \text{conv}\{P_1, \ldots, P_d \} \) which contain a vertex of every \( P_i \). The supporting hyperplanes for these facets are tangent to each \( P_i \), and each \( P_i \) lies on the same side of these hyperplanes.

**Proof.** To apply Theorem \( \mathbf{C} \) as in the proof of Theorem \( \mathbf{A} \) we only need the the family to be affinely spanning.

We briefly consider the situation where each individual polytope can be separated from the rest. The following example demonstrates that the result fails to hold.

**Example 3.4.** Let \( A = \{p_1, p_2, p_3, p_4 \} \subseteq \mathbb{R}^2 \) be a set the four vertices of a square oriented cyclically, so that the diagonals are \([p_1, p_3]\) and \([p_2, p_4]\).

Consider the family of polytopes \( P = \{P_1, P_2, P_3, P_4 \} \subseteq \mathcal{K}^4 \) where \( P_1 = p_1 \times [-1, 1]^2, P_2 = p_2 \times [-2, 2]^2, P_3 = p_3 \times [-1, 1]^2, \) and \( P_4 = p_4 \times [-2, 2]^2 \). Every color can be separated from the rest by a hyperplane, but not all subsets can be separated. No facet of \( \text{conv}\{P\} \) contains points from each polytope in the family.

We are still able to show a positive result in the singleton condition setting and \( m = d \).

**Corollary 3.5.** Let \( P \subseteq \mathbb{R}^d \) be a \( d \)-dimensional polytope such that its facets are colored with the set \([1, \ldots, d]\) and for any single color \( i \in [d] \) the set \( \bigcup F_i \) union taken over all facets with color \( i \), is either visible or covisible. Then there exists either two or zero vertices of \( P \) contained in a facet of every color.

**Proof.** This proceeds exactly as in the proof of Theorem \( \mathbf{A} \) The only difference is that in the passage to a lower dimensional case, a color may not appear. In this case, there will be no vertices contained in a facet of every color. Otherwise, the proof proceeds exactly the same way as the previous proof, and there are exactly two vertices of \( P \) contained in a facet of every color.
Proposition 4.1. Let $\mathcal{P} = \{P_1, \ldots, P_d\} \in \mathcal{P}^d$ be an affinely spanning family of strongly separated convex polytopes in $\mathbb{R}^d$. For each set partition $A \cup B = [d]$ there exists exactly two affine hyperplanes $H$ such that:

- $H$ is tangent to each element of $\mathcal{P}$.
- $\bigcup_{a \in A} P_a \subseteq H^\circ$.
- $\bigcup_{b \in B} P_b \subseteq H^\circ$.

Furthermore, if each polytope in the family is full dimensional, these affine hyperplanes are all different and thus there are $2^d$ tangent affine hyperplanes to the family $\mathcal{P}$.

To prove the proposition, we first move into the linear setting, where the negation of a set changes its position with respect to some hyperplane. After we find a desired hyperplane in the linear setting, we return to the affine setting to finish the proof.

Proof of Proposition 4.1. We linearize the $P_i$ to get a collection of cones $C_i$ which is linearly spanning and acyclic. Furthermore, the $C_i$ form a strongly separated family of cones. Let $\mathcal{C}$ be the collection of cones which replaces $C_i$ with $-C_i$ for each $i \in B$. Since the $C_i$ are a strongly separated family, there is a hyperplane $H_A$ which separates $A$ and $B$. The linearization

1We call a family of cones strongly separated if their relative interiors are.
of $H_A$ witnesses the acyclicity of $C$. Every element of $B$ has been negated, so every cone lies on the same side of the linearized hyperplane. For some generic partition $D \uplus E = [d]$, by the strong separation of the $P_i$, there is a hyperplane $H_D$ separating $D\Delta B$ from $E\Delta B$, where $\Delta$ represents the symmetric difference. The linearization of $H_D$ separates $D$ and $E$ in $C$, since each element of $B$ swapped parts within the partition. This shows that $C$ is strongly separated as well.

We apply the Corollary 3.3 to the affinization of $C$ to get two affine hyperplanes that are tangent to each color with all points of this affinization on one side of the hyperplanes. By linearization, we obtain linear hyperplanes which are again tangent to each colored cone, and all cones $\{C_i : i \in A\} \cup \{-C_i : i \in B\}$ are on the positive side. Finally, undoing the negation of the cones in $B$, and returning to the original affine setting, we have obtained two affine hyperplanes tangent to every $P_i$ and such that $C_A = \text{conv}\{v_{ij} : (i, j) \in A \times [k]\} \subseteq H^\circ$ and $C_B = \text{conv}\{v_{ij} : (i, j) \in B \times [k]\} \subseteq H^\circ$.

From the $2^{d-1}$ partitions of $[d]$ we obtain $2^d$ common tangent hyperplanes. We simply need to conclude these hyperplanes are distinct when each polytope is full dimensional. Given two hyperplanes gotten from different partitions, there is a pair of polytopes whose interiors are on a common side of one hyperplane, but separated by the other hyperplane, so no hyperplanes from different partitions can be the same. Since we already proved there are two distinct hyperplanes for each partition, there can be no repeated hyperplanes among the $2^d$ of them.

Example 4.2. In Figure 3 we illustrate an example of Proposition 4.1.

General convex sets. The set $K^d$ of compact convex sets in $\mathbb{R}^d$ can be made into a metric space via the Hausdorff metric:

$$d(K, S) := \max\{\sup_{x \in K} |x - y|, \sup_{y \in S} |x - y|\} \text{ for any } K, S \in K^d.$$

In this metric, polytopes are dense [16 Section 2.5]. More precisely, for each convex body $K$ there exists a convergent sequence of polytopes $\{P_n\}_{n \in \mathbb{N}}$ with $\lim_{n \to \infty} P_n = K$. We can further assume each $P_i$ in this sequence is simplicial and contained in the interior of $P_{i+1}$. This assumption further implies that all the polytopes of the sequence are in the interior of $K$.

The core component in generalizing our previous results to convex bodies is the following proposition, which is the extension of Corollary 3.3 to general convex bodies.

Proposition 4.3. Let $K = \{K_1, \ldots, K_d\} \subset K^d$ be a family of strongly separated convex bodies. Then there are two hyperplanes that are tangent to each $K_i$ with each $K_i$ on the same side of the hyperplane.

We start by bounding the number of tangent hyperplanes.

Lemma 4.4. There are at most two hyperplanes tangent to each $K_i$ with each $K_i$ on the same side of these hyperplanes.

Proof. If there are three (or more) distinct hyperplanes $H_1$, $H_2$ and $H_3$ such that $K_i \subseteq H_j^\circ$ for $1 \leq i \leq d$, $1 \leq j \leq 3$. Let $h_{i,j}$ be points of tangency between $H_j$ and $K_i$, i.e. $h_{i,j} \in H_j \cap K_i$. For each $i$, consider a full dimensional polytope $P_i \subset K_i$, so that $P_i$ contains $h_{i,j}$ for $j = 1, 2, 3$. These polytopes are strongly separated and tangent to each $H_1, H_2$ and $H_3$, contradicting Corollary 3.3.

To prove Proposition 4.3 we approximate the family $K = \{K_1, \ldots, K_d\}$ with a sequence of families of polytopes $P_n = \{P_{1,n}, \ldots, P_{d,n}\}$ and apply Corollary 3.3 to obtain a pair of tangent hyperplanes to $P_n$ for each $n$. We then verify that in the limit these two sequences of hyperplanes converge to different hyperplanes that are tangent to the family. To distinguish both limits we make use of a auxiliary $H$.

Proof of Proposition 4.3. We define a sequence of polytopes $\{P_{i,n}\}_{n \in \mathbb{N}}$ satisfying four conditions:
(1) For each \( i \in [d] \), \( \lim_{n \to \infty} P_{i,n} = K_i \).
(2) The convex hull \( Q_n := \text{conv}(P_{1,n} \cup P_{2,n} \cup \cdots \cup P_{d,n}) \) is a simplicial polytope.
(3) For each \( h \geq 1 \), \( 2\text{Vol}(P_{h,1}) > \text{Vol}(K_1) \), or equivalently \( \text{Vol}(P_{h,1}) > \text{Vol}(K_1 \setminus P_{h,1}) \).
(4) For each \( i = 1, \ldots, d \) and every \( n \geq 1 \), \( P_{i,n} \subset P_{i,n+1} \).

Since \( P_{i,n} \subset K_i \) for each \( i \), the family \( P_n = \{ P_{1,n}, \ldots, P_{d,n} \} \) is strongly separated for all \( n \in \mathbb{N} \).

By Corollary 3.3 each \( Q_n \) has exactly two facets that intersect each \( P_{1,n}, P_{2,n}, \ldots, P_{d,n} \). Since \( Q \) is simplicial, each of these facets has exactly one vertex in each of the \( P_{i,n} \).

Let \( V_n = \{ v_{1,n}, \ldots, v_{d,n} \} \) and \( W_n = \{ w_{1,n}, \ldots, w_{d,n} \} \) be the vertices in these facets of \( Q \) with \( v_{i,n}, w_{i,n} \in P_{i,n} \). For each \( \lambda \in [0, 1] \), let \( H(\lambda, n) \) be the oriented hyperplane passing through the points \( \lambda v_{i,n} + (1 - \lambda) w_{i,n} \) containing \( W_n \) in it non-negative side. The function \( f_n : [0, 1] \to \mathbb{R} \) such that \( f_n(\lambda) = \text{Vol}(K_1 \cap H(\lambda, n)^+) - \text{Vol}(K_1 \cap H(\lambda, n)^-) \) is continuous. Also, \( P_{1,n} \cap H(0, n)^+ \cap H(1, n)^- \), thus \( f_n(0) \geq \text{Vol}(P_{1,n}) - \text{Vol}(K_1 \setminus P_{1,n}) > 0 \) and \( f_n(1) < \text{Vol}(K_1 \setminus P_{1,n}) - \text{Vol}(K_1) < 0 \), which means that there is \( \lambda_n \in (0, 1) \) with \( f_n(\lambda_n) = 0 \).

By passing to a subsequence if necessary, we find sets \( V = \{ v_1, \ldots, v_d \} \), \( W = \{ w_1, \ldots, w_d \} \) and a real number \( \lambda \) determined by \( \lim_{n \to \infty} v_{i,n} = v_i \), \( \lim_{n \to \infty} w_{i,n} = w_i \) and \( \lim_{n \to \infty} \lambda_n = \lambda \). Notice that \( v_i, w_i \in K_i \) since the corresponding converging sequences are in \( K_i \) and the \( K_i \) are compact.

Claim: The hyperplane \( H \) passing through the points \( \lambda v_i + (1 - \lambda) w_i, 1 \leq i \leq d \), splits \( K_1 \) into two convex bodies with equal volume. To prove the claim consider the continuous function \( g : K_1 \times K_2 \times \cdots \times K_d \to \mathbb{R} \) that takes \( (p_1, \ldots, p_d) \) to \( |\text{Vol}(H^+ \cap K_1) - \text{Vol}(H^- \cap K_1)| \), where \( H \) is the unique hyperplane spanned by the points \( (p_1, \ldots, p_d) \) (strong separation implies that they do span a hyperplane). Since the function is continuous and it is equal to zero on the tuples given by \( p_{i,n} := \lambda_n v_{i,n} + (1 - \lambda_n) w_{i,n} \), it is also zero in the limit. This proves the claim.

To conclude we use II to show that there are two hyperplanes in the limit are different. Let \( F^+ \) and \( F^- \) be the unique hyperplanes passing through the points in \( V \) and \( W \) respectively. We will show first that they are tangent to each \( K_i \) and then show that they are distinct.

(1) **Tangency.** We prove the tangency for \( F^+ \) and note that the proof for \( F^- \) is analogous. Let \( i \in [d] \). Since \( v_i \in F^+ \cap K_i \) it suffices to show that \( F^+ \) does not intersect the interior of \( K_i \).

Let \( a \) be a point in the interior of \( K_i \) and let \( b \) be the orthogonal projection of \( a \) on \( F^+ \). We will show that \( a + b \) and hence \( a \notin F^+ \). For this let \( r_1, r_2, \ldots, r_d \) be the unique real numbers with \( r_1 + r_2 + \cdots + r_d = 1 \), such that \( b = r_1 v_1 + r_2 v_2 + \cdots + r_d v_d \). For each \( n > 0 \) consider the vector \( b_n = r_1 v_{1,n} + r_2 v_{2,n} + \cdots + r_d v_{d,n} \in H_n \). Since \( a \) is interior there is \( N > 0 \) such that \( a \in P_{N,i,j} \), which implies that is in the interior of \( P_{n,i,j} \) for all \( n > N \). Furthermore for \( n \geq N \), the distance from \( a \) to \( H_n \) is bounded below by the distance from \( a \) to \( rP_{N+1,i,j} \) call it \( C > 0 \) it follows that for \( n > N \), \( |b_n - a| \geq C \) and since \( \{ b_n \mid n \in N \} \) converges to \( b \), we get that \( |b - a| \geq C > 0 \), proving that \( a \notin F^+ \). Thus \( F^+ \) is tangent to \( K_i \).

(2) **Distinctness.** To show \( F^+ \) and \( F^- \) are distinct, instead assume that \( F^+ = F^- \) and notice that \( \lambda v_i + (1 - \lambda) w_i \in F^+ \), since both \( v_i \) and \( w_i \) are in \( F^+ \). By definition, \( \lambda v_i + (1 - \lambda) w_i \in H \). However, we know that \( H \) splits \( K_1 \) into two convex bodies of equal (positive) volume, so it cannot be tangent to \( K_1 \), contradicting tangency of \( F^+ \). Therefore \( F^+ \) and \( F^- \) are distinct.

This concludes the proof.

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**Proof of Theorem 3.3** We generalize the proof of Proposition 4.1 to convex sets, replacing Corollary 3.3 by Proposition 4.3.

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**Example 4.5.** Strong separation is crucial to the statement of Theorem since otherwise we can have an arbitrary number of tangents.

Consider an \( N \)-agon \( Q \) in \( \mathbb{R}^2 \) with vertices in the unit circle. Embed \( \mathbb{R}^2 \) in \( \mathbb{R}^3 \) by setting the last coordinate equal to zero. We define the following family of convex bodies. Let
Proposition 5.1. Let the complete family be affinely spanning.

Proof. Without loss of generality we assume focus on the case with \( d \). Any \( d \in \mathbb{R} \) element thickening ensures that the resulting family is affinely spanning in \( \mathbb{R} \). Any collection of partitions are?

Corollary 4.6. Let \( C = \{C_1, \ldots, C_d\} \in (\mathbb{R}^d)^d \) be a family of strongly separated compact sets, such that \( \dim(\operatorname{conv}(C_i)) = d \) for \( i = 1, \ldots, d \). Then there are exactly \( 2^d \) common tangents to \( C \).

If \( \{C_1, \ldots, C_d\} \) is a family of strongly separated compact sets, taking the convex hull of these sets is not strictly speaking sufficient to satisfy the conditions of the previous corollary. Specifically, the resulting convex bodies may not be full dimensional. However, the full dimensional condition is merely required to ensure the \( 2^d \) common tangents are distinct. As with Corollary 3.3, we can relax the condition that each member is full dimensional to the condition that the whole family is affinely spanning.

Corollary 4.7. Let \( \{C_1, \ldots, C_d\} \) be an affinely spanning family of strongly separated compact sets, and let \( A \cup B = [d] \) be a partition. There exist exactly two hyperplanes \( H \) such that for each \( i \), \( H \cap C_i \neq \emptyset \) and either \( C_i \subseteq H^\geq \) if \( i \in A \) or \( C_i \subseteq H^\leq \) if \( i \in B \).

We finish this section with an open question.

Question 4.8. Given two disjoint collections of partitions of \([n]\), when is there a family of \( n \) convex bodies in \( \mathbb{R}^n \), so that the first collection of partitions are all separated and none of the second collection of partitions are?

5. Two Applications

5.1. Collection of \( d + 1 \) convex bodies. We cannot have a strongly separated family with \( d + 2 \) or more convex sets in \( \mathbb{R}^d \) since strong separation implies that they are affinely independent. So we focus on the case with \( d + 1 \) elements. In this case being strongly separated automatically implies that the complete family is affinely spanning.

Proposition 5.1. Let \( K = \{K_1, \ldots, K_{d+1}\} \) be a family of strongly separated convex subsets of \( \mathbb{R}^d \) such that any \( d \)-subset is affinely spanning, and let \( A \cup B = [d+1] \) a set partition, together with a special element \( a \in A \). There exists a unique hyperplane \( H \) such that

1. \( H \) is tangent to \( S \setminus S_a \).
2. \( \bigcup_{i \in A} S_i \subseteq H^\geq \).
3. \( \bigcup_{i \in B} S_i \subseteq H^\leq \).
4. \( S_a \subseteq H^+ \).

Proof. Without loss of generality we assume \( a = d + 1 \). Embed the family \( K = \{K_1, \ldots, K_{d+1}\} \) in \( \mathbb{R}^{d+1} \) by using zero in the last coordinate, and additionally make a thickening of \( K_{d+1} \): replace it by its Minkowski sum with the ball \( B(\epsilon) \). If \( \epsilon \) is small enough the strong separation still holds. The thickening ensures that the resulting family is affinely spanning in \( \mathbb{R}^{d+1} \).

Now we apply Corollary 4.7 with the sets \( A, B \). We obtain two distinct hyperplanes \( L_1 \) and \( L_2 \) that are tangent to every set, including \( K_{d+1} + B(\epsilon) \). We go down to \( \mathbb{R}^d \) by intersecting with the hyperplane \( L = \{x \in \mathbb{R}^{d+1} : x_{d+1} = 0\} \), to obtain two hyperplanes \( H_1 = L \cap L_1, H_2 = L \cap L_2 \) in \( \mathbb{R}^d \) satisfying the conditions (1)–(4).
To conclude the proof we must prove that actually $H_1 = H_2$. We argue by contradiction and assume they are different. Running the same argument with $K_{d+1}$ on the $B$-side strictly we get at least one hyperplane $H_3$ in $\mathbb{R}^d$ satisfying (1)–(3) and (4) reversed. This hyperplane $H_3$ is necessarily different from $H_1$ and $H_2$ since the set $K_{d+1}$ lies on different sides with respect to the sets in $B$. But then the three hyperplanes $H_1$, $H_2$, and $H_3$ different and satisfy (2)–(3) with respect to $\{K_1, \ldots, K_d\}$ contradicting Corollary 4.7. This shows that $H_1 = H_2$ concluding the uniqueness of $H$.

5.2. A different separation condition. The motivation for this paper was certain conditions that arose in [6], which used a different definition of separation. We say a family $K = \{K_1, \ldots, K_{d+1}\}$ of convex sets in $\mathbb{R}^d$ is simplicially separated if

(*) The intersection of all simplices having a vertex on each set of the family is full dimensional.

Theorem [6, Theorem 5.9] states that if a family satisfy the condition, then intersection of all rainbow simplices is itself a simplex. The proof uses a version of Proposition 5.1 when $A$ is a singleton, but in that case the existence of the hyperplane is almost given by assumption and one need to check only uniqueness.

Proposition 5.2. Simplicial separation implies strong separation but the reverse is not true.

Proof. Simplicial separation implies that the intersection of all rainbow simplices is a simplex $S$. The simplex $S$ is described in [6] as follows. For each color $i$ there exists an affine hyperplane $H_i$ such that $H_i$ is tangent to $K_j$ for $j \neq i$, $\bigcup_{j \neq i} K_j \subseteq H_i^c$ whereas $K_i \subseteq H_i^+$. The $d+1$ hyperplanes $H_i$ define the facets of $S$. We denote $v_i$ the vertex of $S$ not contained in $H_i$.

Consider the affine hyperplane arrangement $\mathcal{H} = \{H_1, \ldots, H_{d+1}\}$ in $\mathbb{R}^d$. There is a unique bounded region, the simplex $S$, and $d + 1$ pointed cones, one opposite to each vertex of $S$. Each convex body $K_i$ is contained in the pointed cone opposite (with respect to $S$) to $v_i$.

Now consider any partition $A \cup B = [d+1]$. We have dim aff $\{v_a : a \in A\} +$ dim aff $\{v_b : b \in B\} = d - 1$, so their sum is an affine hyperplane $H$. If we translate $H$ so that it contains the
barycenter $b$ of $S$, then we obtain an affine hyperplane that do not intersect any of the pointy regions. This hyperplane is a strict separator for $A, B$.

On the other hand strong separation does not imply Property ($\star$), see for example Figure 4

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(F. Castillo) DEPARTAMENTO DE MATEMÁTICAS, PONTIFICIA UNIVERSIDAD CATÓLICA DE CHILE, SANTIAGO, CHILE

Email address: federico.castillo@mat.uc.cl

(J. Doolittle) INSTITUT FÜR GEOMETRIE, TECHNISCHE UNIVERSITÄT GRAZ, AUSTRIA

Email address: jdoollittle@tugraz.at

(J. Samper) DEPARTAMENTO DE MATEMÁTICAS, PONTIFICIA UNIVERSIDAD CATÓLICA DE CHILE, SANTIAGO, CHILE

Email address: jsamper@mat.uc.cl