On sequential maxima of exponential sample means, with an application to ruin probability

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Abstract

We obtain the distribution of the maximal average in a sequence of independent identically distributed exponential random variables. Surprisingly enough, it turns out that the inverse distribution admits a simple closed form. An application to ruin probability in a risk-theoretic model is also given.

Keywords: exponential distribution; maximal average; Lambert W function; ruin probability.

AMS MSC 2010: Primary 60E05, Secondary 60F05.

1 Introduction

Consider a sequence \((X_i)_{i \geq 1}\) of independent identically distributed (i.i.d.) random variables, each having exponential distribution with mean 1. For each \(i \in \mathbb{N}^+\) define the sample mean of the first \(i\) variables as \(\bar{X}_i := (X_1 + X_2 + \cdots + X_i)/i\). The supremum of this sequence,

\[ Z_\infty := \sup\{\bar{X}_i : i \in \mathbb{N}^+\}, \]

is finite because the sequence converges to 1 with probability 1.

In this note we compute the distribution function, \(F_\infty\), of \(Z_\infty\). In fact, what has nice form is the inverse of this distribution function. Our main result is the following.

Theorem 1.1. (a) \(Z_\infty\) has distribution function

\[ F_\infty(x) = 1 - \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} x^{k-1} e^{-kx} \]

for \(x > 0\), and density which is continuous on \(\mathbb{R}\setminus\{1\}\), positive on \((1, \infty)\), and zero on \((-\infty, 1)\).

(b) The restriction of \(F_\infty\) on \((1, \infty)\) is one to one and onto \((0, 1)\) with inverse

\[ F_{\infty}^{-1}(u) = \frac{-\log(1 - u)}{u} \quad \text{for all } u \in (0, 1). \]
Maxima of exponential sample means

Remark 1.2. (a) For $F_\infty$ we have the alternative expression

$$F_\infty(x) = 1 + \frac{1}{x}W_0(-xe^{-x})$$

where $W_0$ is the principal branch of the Lambert $W$ function, that is, the inverse function of $x \mapsto xe^x, x \geq 1$; see [3]. Indeed, the power series $\sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} y^k$ has interval of convergence $[-1/e, 1/e]$ and equals $-W_0(-y)$.

(b) Clearly, the results of the theorem extend immediately to the case that the $X_i$'s are i.i.d. and $X_1 = aY + b$ with $a > 0$, $b \in \mathbb{R}$ and $Y \sim \text{Exp}(1)$. However, we were not able to find an explicit formula for the distribution of $Z_\infty$ for any other distribution of the $X_i$'s.

(c) Although it is intuitively clear that $F_\infty(x) > 0$ for $x > 1$, it is not entirely obvious how to verify it by direct calculations. However, this fact is evident from Theorem 1.1.

(d) Formula (1.1) enables the explicit calculation of the percentiles of $F_\infty$. Therefore, the result is useful for the following kind of problems: Suppose that a quality control machine calculates subsequent averages, and alarms if some average $\bar{X}_n$ is greater than $c$, where $c$ is a predetermined constant such that the probability of false alarm is small, say $\alpha$. For $\alpha \in (0, 1)$, the upper percentage point of $F_\infty$ (that is, the point $c_\alpha$ with $F_\infty(c_\alpha) = 1 - \alpha$) is given by $c_\alpha = \frac{-\log \alpha}{-\log c}$, and thus the proper value of $c$ is $c = c_\alpha$.

If in the definition of $Z_\infty$ we discard the first $n - 1$ values of $\bar{X}_1$, we obtain the random variable

$$M_n := \sup \{\bar{X}_i : i \geq n\}$$

for which, however, (for $n \geq 2$) the distribution function is quite complicated even for the exponential case. For instance, the distribution of $M_2$ is given by (we omit the details)

$$F_{M_2}(x) = F_\infty(x) + e^{-2x} \frac{F_\infty(x)}{1 - F_\infty(x)}, \ x \geq 0.$$ 

What we can compute is the asymptotic distribution of $\sqrt{n}(M_n - 1)$ as $n \to \infty$. This distribution is the same for a large class of distributions of the $X_i$'s, as the following theorem shows.

Theorem 1.3. Assume that the $(X_i)_{i \geq 1}$ are i.i.d. with mean 0, variance 1, and there is $p > 2$ with $\mathbb{E}|X_1|^p < \infty$. Let $M_n := \sup \{\bar{X}_i : i \geq n\}$ for all $n \in \mathbb{N}^+$. Then,

$$\sqrt{n}M_n \Rightarrow |Z|$$

where $Z \sim \text{N}(0, 1)$ is a standard normal random variable.

It is easy to see that under the assumptions of Theorem 1.3, by the law of the iterated logarithm, it holds

$$\limsup_{n \to \infty} \frac{\sqrt{n}}{\sqrt{2 \log \log n}} M_n = 1.$$ 

2 Proofs

Proof of Theorem 1.1. (a) For each $n \in \mathbb{N}^+$ consider the random variable

$$Z_n := \max \{\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_n\}$$

and call $F_n$ its distribution function. Since the sequence $(Z_n)_{n \geq 1}$ is increasing and converges to $Z_\infty$, the distribution function of $Z_\infty$ at any $x \in \mathbb{R}$ equals

$$F_\infty(x) = \Pr(\cap_{n=1}^{\infty} \{Z_n \leq x\}) = \lim_{n \to \infty} F_n(x). \quad (2.1)$$
Maxima of exponential sample means

We will compute $F_n$ recursively. For $n \in \mathbb{N}^+$ and $x \geq 0$ we have

$$F_{n+1}(x) = \Pr[X_1 \leq x, X_1 + X_2 \leq 2x, \ldots, X_1 + X_2 + \cdots + X_{n+1} \leq (n+1)x]$$

$$= \int_0^x \int_0^{2x-y_1} \cdots \int_0^{(n+1)x-(y_1+y_2+\cdots+y_n)} e^{-(y_1+y_2+\cdots+y_{n+1})} dy_{n+1}$$

$$= \int_0^x \int_0^{2x-y_1} \cdots \int_0^{n+1} e^{-y_1-y_2-\cdots-y_{n+1}} e^{-(y_1+y_2+\cdots+y_n)} dy_n$$

$$= F_n(x) - e^{-(n+1)x} \text{Vol}(K_n(x))$$

where $dy_k = dy_k \cdots dy_2 dy_1$ and

$$K_n(x) := \{(y_1, y_2, \ldots, y_n) \in \mathbb{R}_+^n : 0 \leq y_1 + \cdots + y_i \leq ix, \ i = 1, 2, \ldots, n\}.$$

Note that $F_1(x) = 1 - e^{-x}$ and introduce the convention $\text{Vol}(K_0(x)) = 1$. It follows that $F_n(x) = 1 - \sum_{k=1}^n \text{Vol}(K_{k-1}(x))e^{-kx}$ and from Lemma 2.2, below, we get the explicit form

$$F_n(x) = 1 - \sum_{k=1}^n \frac{k-1}{k!} x^{k-1} e^{-kx}, \text{ for all } x \geq 0, n \in \mathbb{N}^+. \quad (2.2)$$

This implies the first formula for $F_\infty$. By the law of large numbers, we get that $F_\infty(x) = 0$ for all $x \in (-\infty, 1)$, and thus, the derivative of $F_\infty$ in $\mathbb{R} \setminus \{1\}$ is

$$f_\infty(x) := 1_{x > 1} \sum_{k=1}^\infty \frac{k-1}{k!} \left( k - \frac{k-1}{x} \right) x^{k-1} e^{-kx}. \quad (2.3)$$

Since $F_\infty$ is continuous in $\mathbb{R}$ and differentiable in $\mathbb{R} \setminus \{1\}$ with continuous derivative there, it follows that $f_\infty$ is a density for $Z_\infty$. The formula for $f_\infty$ shows that it is positive exactly at $1$.

(b) First we rewrite $F_\infty$ in a more convenient form. The fact that $F_\infty(x) = 0$ for $x \in [0, 1)$ implies the remarkable identity (see Fig. 1)

$$\sum_{k=1}^\infty \frac{k-1}{k!} x^{k-1} e^{-kx} = 1 \text{ for all } x \in [0, 1). \quad (2.4)$$

![Figure 1: The series (2.4) in the interval $0 \leq x \leq 4$.](http://www.imstat.org/ecp/)

Our aim is to compute the value of the series in the left hand side also for $x \geq 1$. The series converges uniformly for $x \in [0, \infty)$ because

$$\sup_{x \geq 0} \frac{k-1}{k!} x^{k-1} e^{-kx} = \frac{(k-1)k^{-1}}{k!} e^{-(k-1)} \sim \frac{1}{k^{3/2} \sqrt{2\pi}}.$$
which is summable in $k$. Thus, by continuity, (2.4) holds also for $x = 1$. Now we rewrite (2.4) in the form

$$
\sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} (xe^{-x})^k = x \text{ for all } x \in [0, 1].
$$

(2.5)

The power series $h(y) := \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} y^k$ is strictly increasing in $[0, e^{-1}]$ and thus (2.5) says that $h$ is the inverse function of the restriction, $g$, on $[0, 1]$ of the function $g : [0, \infty) \to [0, e^{-1}]$ with $g(x) = xe^{-x}$. The function $g$ is continuous, strictly increasing in $[0, 1]$, and strictly decreasing in $[1, \infty)$ with $g(0) = 0, g(1) = e^{-1}, g(\infty) = 0$. Thus, for each $x \in [1, \infty)$, there exists a unique $t = t(x) \in (0, 1]$ such that $g_r(t) = xe^{-x}$, i.e., $te^{-t} = xe^{-x}$; hence, we define

$$
t(x) := g_r^{-1}(xe^{-x}) = h(xe^{-x}), \quad x \geq 0.
$$

(2.6)

Since $t(x) = x$ for $x \in [0, 1]$, we have

$$
F_\infty(x) = \begin{cases} 
0, & x \leq 1, \\
1 - \frac{t(x)}{x}, & x \geq 1.
\end{cases}
$$

(2.7)

Now for any fixed $u \in (0, 1)$, the relation $F_\infty(x) = u$ gives $x = t(x) = xu$ so that $t(x) = (1 - u)x$. Consequently,

$$
e^{xu} = e^{-t(x)} = \frac{x}{t(x)} = \frac{1}{1 - u}.
$$

Thus, $x = -\log(1 - u)/u$, and the proof is complete. \hfill \Box

**Remark 2.1.** From the well-known relation $\mathbb{E} Z_\alpha^n = \alpha \int_0^\infty x^{\alpha-1} (1 - F_n(x)) \, dx$ for $\alpha > 0$ and formula (2.2), we obtain a simple expression for the moments:

$$
\mathbb{E} Z_\alpha^n = \alpha \sum_{k=1}^{n} \frac{\Gamma(\alpha + k - 1)}{k^n k!}.
$$

In particular,

$$
\mathbb{E} Z_\alpha = \sum_{k=1}^{n} \frac{1}{k^n}, \quad \mathbb{E} Z_\alpha^2 = 2 \sum_{k=1}^{n} \frac{1}{k^2}, \quad \mathbb{E} Z_\alpha^3 = 3 \sum_{k=1}^{n} \frac{1}{k^3} + 3 \sum_{k=1}^{n} \frac{1}{k^4}.
$$

Since $Z_\alpha \sim Z_\infty$ with probability one, the above relations combined with the monotone convergence theorem give the moments of $Z_\infty$ and in particular that it has mean $\frac{\alpha^2}{\alpha}$ and variance $\frac{\alpha^4}{\alpha}$.

The next lemma is a special case of Theorem 1 in [7] (see relation (7) in that paper), however, to keep the exposition self-contained, we provide a proof.

**Lemma 2.2.** For $x \geq 0, x + t \geq 0$, and $n \in \mathbb{N}^+$, define

$$
K_n(x, t) := \{(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n_+ : y_1 + \cdots + y_i \leq ix + t \text{ for all } i = 1, 2, \ldots, n\}.
$$

Then,

$$
\mathbb{V}_n(x, t) := \text{Vol}(K_n(x, t)) = \frac{1}{n!} (x + t)(n + 1)x^{n-1}, \quad n = 1, 2, \ldots, \quad (2.8)
$$

and, in particular, setting $t = 0$, Vol$(K_n(x)) = \frac{1}{n!} (n + 1)^{n-1}x^n$.

**Proof.** Clearly $V_1(x, t) = x + t$ and for $n \geq 1$

$$
\begin{align*}
V_{n+1}(x, t) &= \int_0^{x+t} \int_0^{x+t-y_1} \cdots \int_0^{x+t-(y_1+y_2+\cdots+y_n)} dy_{n+1} \\
&= \int_0^{x+t} \int_0^{x+(x+t-y_1)-y_2} \cdots \int_0^{x+(x+t-y_1)-(y_2+\cdots+y_n)} dy_{n+1} \\
&= \int_0^{x+t} V_n(x, x + t - y_1) dy_1.
\end{align*}
$$

(2.9)

The claim follows by induction on $n$. \hfill \Box
Maxima of exponential sample means

It is consistent with the recursion (2.9) for $V_n$ and (2.8) to define $V_0(x, t) := 1$ so that (2.8) holds for all $n \in \mathbb{N}^+ \cup \{0\}$. This agrees with the convention $\text{Vol}(K_0(x)) = 1$ we made in the proof of Theorem 1.1(a).

**Proof of Theorem 1.3.** By Theorem 2.2.4 in [4], we may assume that we can place $(X_i)_{i \geq 1}$ on the same probability space with a standard Brownian motion $(W_s)_{s \geq 0}$, so that, with probability 1, we have $|n \bar{X}_n - W_n|/n^{1/(\log n)^{1/2}} \to 0$ as $n \to \infty$. This implies that

$$
\lim_{n \to \infty} \sqrt{n} \left( M_n - \sup_{k \in \mathbb{N}, k \geq n} \frac{W_k}{k} \right) = 0
$$

with probability 1. On the other hand, with probability one, we have for all large $n$ the bound

$$
\sup_{s \in [n, n+1]} |W_s - W_n| \leq 2\sqrt{\log n},
$$

thus

$$
\lim_{n \to \infty} \sqrt{n} \sup_{s \geq n} \frac{W_s}{s} \stackrel{d}{=} \sup_{s \in [0, 1]} W_s \overset{d}{=} |W_1|,
$$

and the proof is complete. \qed

3 An application to ruin probability

Following the same steps as in the proof of Theorem 1.1(b), one can evaluate the distribution function, $F_{n; \lambda}$, of the random variable

$$Z_{n; \lambda} := \max \left\{ \frac{X_1}{1 + \lambda}, \frac{X_1 + X_2}{2 + \lambda}, \ldots, \frac{X_1 + X_2 + \cdots + X_n}{n + \lambda} \right\}
$$

for all $\lambda > -1$ and $n \in \mathbb{N}^+$. Indeed, using (2.8) and induction on $n$ it is easily verified that for all $x \geq 0$ we have

$$F_{n; \lambda}(x) = 1 - (1 + \lambda)e^{-\lambda x} \sum_{k=1}^{n} \frac{k(k + \lambda)^{k-2}}{k!} x^{k-1} e^{-kx}.
$$

Thus, the distribution function of $Z_{\infty; \lambda} := \lim_{n \to \infty} Z_{n; \lambda}$ equals

$$F_{\infty; \lambda}(x) = 1 - (1 + \lambda)e^{-\lambda x} \sum_{k=1}^{\infty} \frac{k(k + \lambda)^{k-2}}{k!} x^{k-1} e^{-kx} = 1 - \frac{t(x)}{x} e^{\lambda t(x) - x},
$$

where the function $t$ is defined by (2.6). To justify the equality (3.2), we use the same arguments that lead from (2.4) to (2.7). Similarly as in Theorem 1.1(b), we find that $F_{\infty; \lambda}$ is zero in $(-\infty, 1]$, strictly increasing in $[1, \infty)$ with range $[0, 1)$, and its distribution inverse is given by

$$F_{\infty; \lambda}^{-1}(u) = \frac{-\log(1 - u)}{1 - (1 - u)^{1/\lambda}} \times \frac{1}{\lambda + 1}, \quad 0 < u < 1.
$$
Maxima of exponential sample means

**Remark 3.1.** By the law of large numbers, the series in the right hand side of (3.1) equals to one for all \( x \in [0, 1] \). Therefore, setting \( x = \alpha, 1 + \lambda = \theta \) and \( k \to k + 1 \), the function

\[
p(k; \alpha, \theta) = \theta e^{-\alpha(\theta + k)} \frac{\alpha^k (k + \theta)^{k-1}}{k!}
\]

defines a probability mass function supported on \( \mathbb{N}^+ \cup \{0\} \), known (after a suitable re-parametrization) as *generalized Poisson distribution* with parameter \((\alpha, \theta) \in [0, 1] \times (0, \infty)\); see [2] and references therein.

Consider now the following risk model. Assume that the aggregate claim at time \( n \) is described by \( S_n := X_1 + \cdots + X_n \), where the \((X_i)_{i\geq 1}\) are i.i.d. with \( \mathbb{E}X_1 = 1 \), the premium rate (per time unit) is \( c = 1 + \theta > 0 \) (\( \theta \) is the safety loading of the insurance), and the initial capital is \( u > -(1 + \theta) \), where negative initial capital is allowed for technical reasons. The risk process is defined by

\[
U_n = u + cn - S_n, \quad n \in \mathbb{N}^+.
\]

Clearly, the ruin probability

\[
\psi(u) := \Pr(U_n < 0 \text{ for some } n \in \mathbb{N}^+)
\]

is of fundamental importance. Our explicit formulae are useful in computing the minimum initial capital needed to ensure that \( \psi(u) \) is small. In the following, we exclude the trivial case where the distribution of \( X_1 \) is concentrated at 1.

This particular problem (for general claims) has been studied in [6] under the name *discrete-time surplus-process model*, while the probability of ruin for more general models is studied in detail in the standard reference [1].

When \( c \leq 1 \), we have \( \psi(u) = 1 \) no matter how large \( u \) is. Indeed, when \( c < 1 \), the claim is a consequence of the strong law of large numbers, while when \( c = 1 \), since we have excluded the case \( \Pr(X_1 = 1) = 1 \), it follows from Theorems 4.1.2, 4.2.7 in [5] (which imply that \((n - S_n)_{n\geq 1}\) oscillates between \(-\infty \) and \( \infty \)). Hence, the problem is nontrivial only for \( c > 1 \), i.e., \( \theta > 0 \).

**Theorem 3.2.** Assume that the i.i.d. individual claims \((X_i)_{i\geq 1}\) are exponential random variables with mean 1, fix \( \alpha \in (0, 1) \) and \( \theta > 0 \), and set \( c = 1 + \theta \). Then,

(a) the ruin probability (3.4) is given by

\[
\psi(u) = \begin{cases} 
\frac{t(c)}{c} \exp \left(-u \left(1 - \frac{t(c)}{c}\right)\right), & \text{if } u > -c, \\
1 & \text{if } u \leq -c,
\end{cases}
\]

where the function \( t \) is given by (2.6);

(b) the minimum initial capital \( u = u(\alpha, \theta) \) needed to ensure that \( \psi(u) \leq \alpha \) is given by the unique root of the equation

\[
(1 + \theta + u) \left(1 - \alpha \frac{1 + \theta}{1 + \theta + u}\right) = - \log \alpha, \quad u > -(1 + \theta).
\]

**Proof.** (a) For \( u > -c \), we can use (3.2) to get

\[
\psi(u) = 1 - F_{\infty; u/c}(c) = \frac{t(c)}{c} e^{(u/c)(t(c)-c)}.
\]

which is (3.5). Then, the definition of \( t \) shows that \( \lim_{u \to -c+} \psi(u) = \frac{t(c)e^{-t(c)}}{ce^{-c}} = 1 \), and the monotonicity of \( \psi \) implies that \( \psi(u) = 1 \) for \( u \leq -c \).

(b) By the formula of part (a), the function \( \psi \) is strictly decreasing in the interval \((-c, \infty)\) and maps that interval to \((0, 1)\). Therefore, there is a unique \( u = u(\alpha, \theta) > -c \).
such that \( \psi(u) = \alpha \). Let \( \lambda := u/c \), which is greater than \(-1\). Then, using (3.3), we see that

\[
\psi(u) = \alpha \iff F_{\infty, \lambda}(c) = 1 - \alpha \iff c = F^{-1}_{\infty, \lambda}(1 - \alpha) = \frac{-\log \alpha}{(1 + \lambda)(1 - \alpha^{\frac{1}{1+\theta}})}.
\]

We substitute \( c = 1 + \theta, \lambda = u/(1 + \theta) \), and the above equivalences show that \( u \) is the unique solution of

\[
\left(1 + \frac{u}{1 + \theta}\right) \left(1 - \alpha^{\frac{1}{1+\theta}}\right) = -\frac{\log \alpha}{1 + \theta}.
\]

The exact values of \( u \) in (3.6) are in perfect agreement with the numerical approximations given in the last line of Table 1 in [6]. Notice that the initial capital \( u \) can be negative sometimes, e.g., \( u(0.5, 0.5) \simeq -0.3107 \).

References

[1] Asmussen, S. and Albrecher, H.: Ruin probabilities. Vol. 14. Singapore: World Scientific, 2010. MR-2766220

[2] Charalambides, C.: Abel series distributions with applications to fluctuations of sample functions of stochastic processes. Communications in Statistics – Theory & Methods, 19(1), (1990), 317–335. MR-1060414

[3] Corless, R.M, Gonnet G.H., Hare, D.E.G., Jeffrey D.J., and Knuth D.E.: On the Lambert \( W \) function. Advances in Computational Mathematics, 5(1), (1996), 329–359. MR-1414285

[4] Csörgö, M. and Révész, P.: Strong approximations in probability and statistics. Academic Press, 1981. MR-0666546

[5] Durrett, R.: Probability: theory and examples. 4th Edition. Cambridge University Press, 2010. MR-2722836

[6] Sattayatham, P., Sangaroon, K., and Klongdee, W.: Ruin probability-based initial capital of the discrete-time surplus process. Variance, Advancing the Science of Risk, 7(1), (2013), 74–81.

[7] Stanley, R. and Pitman, J.: A polytope related to empirical distributions, plane trees, parking functions, and the associahedron. Discrete & Computational Geometry, 27(4), (2002), 603–634. MR-1902680
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