Quantum Dilogarithm as a 6j-Symbol

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Abstract
The cyclic quantum dilogarithm is interpreted as a cyclic 6j-symbol of the Weyl algebra, considered as a Borel subalgebra $BU_q(sl(2))$. Using modified 6j-symbols, an invariant of triangulated links in triangulated 3-manifolds is constructed. Apparently, it is an ambient isotopy invariant of links.
Introduction
Solutions to the Yang-Baxter equation (YBE) \([Y, B1]\) play very important role both
in the theory of solvable two-dimensional models of statistical mechanics and field theory
(for a review see \([B2, F]\)), and in the theory of braid groups and links \([Jo]\). Algebraically
YBE is deeply related with quantum groups \([D1]\).

Recently, Rogers’ dilogarithm identity \([R]\) attracted much attention in connection
with solvable models and 3d mathematics, see \([DS]\) and references therein. In \([FK]\) quant-
umization of the dilogarithm function has been defined. Remarkably, quantized
dilogarithm identity, like YBE, is also connected with quantum groups. In this paper we
develop the result of paper \([FK]\), and show that the quantized Roger’s (pentagon) identity
for the cyclic quantum dilogarithm is a consequence of the co-associativity property of
the Weyl algebra, endowed with particular Hopf algebra structure (which is equivalent to
Borel subalgebra of \(U_q(sl(2))\)). Namely, quantum dilogarithm and cyclic 6j-symbols of the
algebra are closely related. This is the content of sections 1 and 2. In section 3 slightly
modified 6j-symbols are shown to acquire a natural interpretation in three dimensions. In
section 4 the invariant of triangulated links in 3-manifolds is constructed.

1. Cyclic Representations of Weyl Algebra
Consider unital Hopf algebra \(W\) generated by operators \(E\) and \(D\), satisfying Weyl permuta-
tion relations
\[
DE = \omega ED
\]
with \(\omega\) being a primitive \(N\) th root of unity, where odd \(N > 2\), and co-multiplications \([D2, Ji, FRT]\)
\[
\Delta(E) = E \otimes E, \quad \Delta(D) = 1 \otimes D + D \otimes E.
\]
Elements \(E^N\) and \(D^N\) are central, and their co-multiplications are
\[
\Delta(E^N) = E^N \otimes E^N, \quad \Delta(D^N) = 1 \otimes D^N + D^N \otimes E^N.
\]
That means we can consider a restricted class of representations with
\[
E^N = 1.
\]
If operator \(D\) is invertible, then each irreducible representation \(p\), called cyclic one, is
characterized by a non-zero complex number \(z_p\), the value of central element \(D^N\). Cyclic
representations of quantum groups \([DK, DJMM1, AC, BK]\) are related with the chiral
Potts model \([AMPY, MPTS, BPA]\) and its’ generalizations \([DJMM2, BKMS]\). If two rep-
resentations \(p\) and \(q\) are such that \(z_p + z_q \neq 0\), then, the tensor product of \(p\) and \(q\) can be
decomposed into direct sum of \(N\) copies of one and the same representation \(pq\) \((pq = qp)\)
given by:
\[
z_{pq} = z_p + z_q.
\]
“Inverse” representation \(p^{-1}\) and “complex conjugate” one \(p^*\) are defined by
\[
z_{p^{-1}} \equiv -z_p, \quad z_{p^*} \equiv z_p^*.
\]
Introduce the “standard” matrix realization of representation $p$:

$$p(D) = x_p X, \quad p(E) = Z^{-1}, \quad (1.6)$$

where $N$-by-$N$ matrices $Z$ and $X$ are defined by their matrix elements

$$\langle m|Z|n\rangle = \omega^m \delta_{m,n}, \quad \langle m|X|n\rangle = \delta_{m,n+1}, \quad m, n \in \mathbb{Z}_N, \quad (1.7)$$

$Z_N$ being identified with the set $\{0, 1, \ldots, N - 1\}$, and parameter $x_p$ is a particular $N$th root of $z_p$, chosen to be real for real $z_p$:

$$x_p = z_p^{1/N}, \quad z_p \in \mathbb{R} \Rightarrow x_p \in \mathbb{R}. \quad (1.8)$$

Let $V_p$ be an $N$ dimensional vector space where representation $p$ acts. Clebsh-Gordon (CG) operators $K_\alpha(p, q), \alpha \in \mathbb{Z}_N$, acting from $V_{pq}$ to $V_p \otimes V_q$, are linearly independent set of solutions for the intertwining equations

$$K_\alpha(p, q):V_{pq} \to V_p \otimes V_q, \quad K_\alpha(p, q)pq(.) = p \otimes q \circ \Delta(.)K_\alpha(p, q), \quad \alpha \in \mathbb{Z}_N, \quad (1.9)$$

where the dot symbolizes any element of $\mathcal{W}$. Dual CG operators $\overline{K}_\alpha^\alpha(p, q), \alpha \in \mathbb{Z}_N$, acting from $V_p \otimes V_q$ to $V_{pq}$, are defined similarly:

$$\overline{K}_\alpha^\alpha(p, q):V_p \otimes V_q \to V_{pq}, \quad \overline{K}_\alpha^\alpha(p, q)p \otimes q \circ \Delta(.) = pq(.)\overline{K}_\alpha^\alpha(p, q), \quad \alpha \in \mathbb{Z}_N. \quad (1.10)$$

Impose the following natural relations on these operators:

$$\overline{K}_\alpha^\alpha(p, q)K_\beta(p, q) = \delta^\alpha_\beta 1_{pq}, \quad \sum_{\alpha \in \mathbb{Z}_N} K_\alpha(p, q)\overline{K}_\alpha^\alpha(p, q) = 1_p \otimes 1_q, \quad (1.11)$$

where, e. g. $1_p$ is the identity matrix in $V_p$.

In what follows we choose $\omega^{1/2}$ as an $N$th root of 1:

$$\omega^{1/2} = \omega^{(N+1)/2}. \quad (1.12)$$

In particular basis explicit solutions to (1.9)–(1.11) have the form:

$$\langle i, j|K_\alpha(p, q)|k\rangle = \langle pq, q\rangle^{1/2} \omega^{\alpha j} w(x_q, x_p, x_{pq}|i, \alpha)\delta_{k,i+j}, \quad i, j, k, \alpha \in \mathbb{Z}_N, \quad (1.13)$$

$$\langle k|\overline{K}_\alpha^\alpha(p, q)|i, j\rangle = \frac{\langle pq, q\rangle^{1/2}}{\langle pq\rangle} \frac{\delta_{k,i+j}}{\omega^{\alpha j} w(x_q/\omega, x_p, x_{pq}|i, \alpha)}, \quad i, j, k, \alpha \in \mathbb{Z}_N \quad (1.14)$$

where for any representations $p$ and $q$ symbol $\langle p, q \rangle$ is defined as

$$\langle p, q \rangle \equiv N^{-1}(x_p^N - x_q^N)/(x_p - x_q), \quad \langle p \rangle \equiv \langle p, p \rangle = x_p^{N-1}, \quad (1.15)$$
and
\[ w(x, y, z|i, j) = w(x, y, z|i - j)\omega^{j^2/2}, \quad (1.16) \]
\[ w(x, y, z|i) = \prod_{j=1}^{i} \frac{y}{z - x\omega^j}, \quad (1.17) \]
for complex \(x, y, z\) such that
\[ x^N + y^N = z^N. \quad (1.18) \]

Using CG operators one can calculate in the standard way \(6j\)-symbols. The defining relations are as follows:
\[ K_{\alpha}(p, q)K_{\beta}(pq, r) = \sum_{\gamma, \delta} R(p, q, r)^{\gamma, \delta}_{\alpha, \beta} K_{\delta}(q, r)K_{\gamma}(p, qr). \quad (1.19) \]

These relations can be written in operator form, if one interprets \(R(p, q, r)^{\gamma, \delta}_{\alpha, \beta}\) as matrix elements of an linear operator \(R(p, q, r)\) in \(C^N \otimes C^N\), while CG operators, as operator valued vectors in \(C^N\):
\[ K_{1}(p, q)K_{2}(pq, r) = R_{12}(p, q, r)K_{2}(q, r)K_{1}(p, qr), \quad (1.20) \]
where indices 1 and 2 denote the two multipliers in the tensor product space \(C^N \otimes C^N\). The dual \(6j\)-symbols, \(R_{12}(p, q, r)^{\gamma, \delta}_{\alpha, \beta}\), are just matrix elements of the inverse operator \(R(p, q, r)^{-1}\) and they satisfy the relation:
\[ R_{12}(p, q, r)K_{1}(p, q)K_{2}(pq, r) = K_{2}(q, r)K_{1}(p, qr). \quad (1.21) \]

Relations (1.20), (1.21) are consistent, provided the \(6j\)-symbols satisfy the “pentagon” identity, which in our case looks like
\[ R_{12}(p, q, r)R_{13}(p, qr, s)R_{23}(q, r, s) = R_{23}(pq, r, s)R_{12}(p, q, rs), \quad (1.22) \]
and in terms of dual \(6j\)-symbols,
\[ R_{23}(q, r, s)R_{13}(p, qr, s)R_{12}(p, q, r) = R_{12}(p, q, rs)R_{23}(pq, r, s). \quad (1.23) \]

Such equations were derived in [M] using geometrical arguments. Formulas (1.13) and (1.14) lead to the following explicit expression for the \(6j\)-symbols:
\[ R(p, q, r)^{\gamma, \delta}_{\alpha, \beta} = \frac{\rho_{p, q, r}}{\langle qr \rangle} \omega^{\alpha\delta} w(x_{pq}, x_q, x_p x_r, x_{pq} x_{qr}|\gamma, \alpha)\delta_{\beta, \gamma + \delta}, \quad (1.24) \]
where
\[ \rho_{p, q, r} = \frac{(pq, q)^{1/2}(pq, r)^{1/2}(qr, r)^{1/2}}{\langle pqr, qr \rangle^{1/2}} f\left(\frac{x_r}{x_{pq}}, \frac{x_r}{\omega x_{qr}}, \frac{x_{pq} x_{qr}}{x_{pqr}}\right), \quad (1.25) \]
and function \( f(x, y|z) \) is defined as
\[
f(x, y|z) = \sum_{j=0}^{N-1} \frac{w(x|j)}{w(y|j)} z^j, \quad w(x|i) = \prod_{j=1}^{i} \frac{1}{1-x\omega^j},
\] (1.26)
for any \( x, y, z \) satisfying the equation
\[
\frac{1-x^N}{1-y^N} = z^N,
\] (1.27)
see (1.15)–(1.18) for other notations. Function \( f(x, y|z) \) has particular automorphic properties, described in [KMS2]. The dual 6j-symbols can be written in the form:
\[
\bar{R}^{\alpha,\beta}_{\gamma,\delta}(p, q, r) = \frac{\delta_{\beta,\gamma+\delta}}{\omega^{\alpha \delta} w(x_{pq}, x_q, x_{pq}, x_{qr} | \gamma, \alpha)},
\] (1.28)
where
\[
P_{p, q, r} = \frac{(pq, q)^{1/2}(pq, r)^{1/2}(qr, r)^{1/2}}{(pq, qr)^{1/2}} f \left( \frac{x_r}{x_{qr}}, \frac{x_r}{\omega x_{pq}, x_{pq}, x_{qr}} \right).
\] (1.29)

2. Quantum Dilogarithm as a 6j-Symbol

Operator \( R(p, q, r) \), given by (1.24), has the following algebraic properties:
\[
R_{12}(p, q, r)Z_1Y_2 = Z_1Y_2R_{12}(p, q, r),
Y_1Y_2R_{12}(p, q, r) = R_{12}(p, q, r)Y_1,
Z_2R_{12}(p, q, r) = R_{12}(p, q, r)Z_1Z_2,
\] (2.1)
where
\[
Y = \omega^{1/2} X Z,
\] (2.2)
while \( X \) and \( Z \) are defined in (1.7), and subscripts denote the multipliers in the tensor product space. Suppose we have some fixed invertible operator \( S \) in \( \mathbb{C}^N \otimes \mathbb{C}^N \), also satisfying relations (2.1):
\[
S_{12}Z_1Y_2 = Z_1Y_2S_{12}, \quad Y_1Y_2S_{12} = S_{12}Y_1, \quad Z_2S_{12} = S_{12}Z_1Z_2.
\] (2.3)
Then, combination of the form
\[
\Psi_{p, q, r} = S_{12}^{-1} R_{12}(p, q, r),
\] (2.4)
satisfies more simple relations
\[
\Psi_{p, q, r} Z_1Y_2 = Z_1Y_2 \Psi_{p, q, r}, \quad Y_1 \Psi_{p, q, r} = \Psi_{p, q, r} Y_1, \quad Z_1Z_2 \Psi_{p, q, r} = \Psi_{p, q, r} Z_1Z_2.
\] (2.5)
which imply that $\Psi_{p,q,r}$ as operator is a function of a particular combination of $Y$’s and $Z$’s:

$$\Psi_{p,q,r} = \Psi_{p,q,r}(-Y_1^{-1}Z_2^{-1}Y_2).$$

(2.6)

Let us choose operator $S_{12}$ as

$$S_{12} = N^{-1} \sum_{i,j \in \mathbb{Z}_N} \omega^{-ij}Z_1^iY_2^j.$$ 

(2.7)

In addition to relations (2.3), it commutes with operator $Z_1$:

$$S_{12}Z_1 = Z_1S_{12},$$

(2.8)

and satisfies the constant “pentagon” relation:

$$S_{12}S_{13}S_{23} = S_{23}S_{12}.$$ 

(2.9)

We can express from (2.4) $R(p,q,r)$ in terms of operators $S$ and $\Psi_{p,q,r}$, and substitute it into (1.22). By the use of (2.3) and (2.8) move now all $S$’s to the left, and drop them, using (2.9). Eventually, we end up with relation only in terms of $\Psi$’s:

$$\Psi_{p,q,r}(U)\Psi_{p,q,r,s}(V) = \Psi_{p,q,r,s}(V)\Psi_{p,q,r,s}(U),$$

(2.10)

where

$$U = -Y_1^{-1}Z_2^{-1}Y_2, \quad V = -Y_2^{-1}Z_3^{-1}Y_3.$$ 

(2.11)

Here operators $U$ and $V$ satisfy Weyl relation and have $N$th powers equal to $-1$:

$$UV = \omega VU, \quad U^N = V^N = -1.$$ 

(2.12)

Relation (2.10) coincides with quantum dilogarithm identity (3.9) of [FK]. It is equivalent to the restricted star-triangle relation of [BB2], playing the key role in the integrable 3d Baxter-Bazhanov model [BB1, KMS1, KMS2]. The latter is a multistate generalization of the Zamolodchikov model [Z].
3. Three-Dimensional Picture

Define a matrix $T_{12}(p, q, r|a, c)$, which unlike $R_{12}(p, q, r)$ depends on two additional $\mathbb{Z}_N$-arguments $a$ and $c$:

$$T_{12}(p, q, r|a, c) \equiv \omega^{ac/2} \langle qr \rangle Y_{1}^{-a} Z_{1}^{-c} R_{12}(p, q, r) Z_{2}^{c} Z_{2}^{-a}, \quad (3.1)$$

see sections 1 and 2 for the notations. The corresponding dual matrix $\overline{T}_{12}(p, q, r|a, c)$ is defined by the Hermitian conjugation of $T_{12}(p, q, r|a, c)$, combined with negation of matrix indices and complex conjugation of continuous arguments:

$$\overline{T}_{12}(p, q, r|a, c) \equiv C_{1}C_{2}(T_{12}(p^{*}, q^{*}, r^{*}|a, c))^{\dagger}C_{1}C_{2}, \quad (3.2)$$

where matrix $C$ is given by matrix elements of the form:

$$C_{\alpha,\beta} = \delta_{\alpha+\beta}, \quad \alpha, \beta \in \mathbb{Z}_N. \quad (3.3)$$

Formulas (3.1) and (3.2) generalize (1.24) and (1.28) in the sense that, if the “charges” $a$ and $c$ are zero, then the new definitions coincide with (1.24) and (1.28) up to scalar factors:

$$T_{12}(p, q, r|0, 0) = \langle qr \rangle R_{12}(p, q, r), \quad \overline{T}_{12}(p, q, r|0, 0) = \langle pq \rangle \overline{R}_{12}(p, q, r). \quad (3.4)$$

Define a pair of two-index tensors $G_{\alpha,\beta}(p, q)$ and $F_{\alpha,\beta}(p, q)$:

$$G_{\alpha,\beta} = \omega^{a^2/2} \delta_{\alpha+\beta}, \quad \alpha, \beta \in \mathbb{Z}_N, \quad (3.5)$$

and

$$F_{\alpha,\beta}(p, q) = \frac{(p, q^{-1}p)^{1/2}}{N(q, p^{-1}q)^{1/2}} f(0, \frac{x_p}{\omega x_q^{-1}p} \frac{x_p^{-1}q}{x_q}) \omega^{\alpha\beta}, \quad \alpha, \beta \in \mathbb{Z}_N. \quad (3.6)$$

The corresponding inverse tensors, $G^{\alpha,\beta}$ and $F^{\alpha,\beta}(p, q)$, are defined through the equations:

$$\sum_{\beta} G_{\alpha,\beta} G^{\beta,\gamma} = \delta_{\gamma}^{\alpha}, \quad \sum_{\beta} F_{\alpha,\beta}(p, q) F^{\beta,\gamma}(p, q) = \delta_{\gamma}^{\alpha}. \quad (3.7)$$

Using these definitions, one can prove the following formulas:

$$\sum_{\alpha,\gamma} T(p, q, r|a, c) G_{\gamma,\gamma}^{\alpha,\delta} G_{\gamma,\gamma}^{\alpha,\gamma} = \omega^{a/4} T(p^{-1}, q, r|a, 1/2 - a - c)^{\alpha,\delta,\gamma}, \quad (3.8)$$

$$\sum_{\alpha,\delta} T(p, q, r|a, c) G_{\alpha,\alpha}^{\delta,\delta} F^{\alpha,\alpha}(p, q) = \omega^{-c/4} T(p, q^{-1}, r|a, 1/2 - a - c)^{\alpha,\gamma,\delta}, \quad (3.9)$$

$$\sum_{\beta,\delta} T(p, q, r|a, c) F_{\delta,\delta}^{\beta,\beta}(q, r) F^{\beta,\beta}(p, q) = \omega^{a/4} T(p, q, r^{-1}|a, 1/2 - a - c)^{\gamma,\beta,\delta}. \quad (3.10)$$

A natural three-dimensional interpretation for these relations can be given. In what follows for a polyhedron $X$, considered as a collection of vertices, edges, triangles, and
tetrahedrons, we will use the standard notation \( \Lambda_i(X), i = 0, 1, 2, 3 \) for the sets of simplices of corresponding dimension.

First, consider a topological * tetrahedron \( T \) in \( \mathbb{R}^3 \). Order the vertices by fixing the bijective map \( u: \{0, 1, 2, 3\} \rightarrow \Lambda_0(T) \), \( i \mapsto u_i \). Put an arrow on each edge, pointing from a “larger” vertex (with respect to above ordering) to a “smaller” one. The tetrahedron itself has two possible orientations in the following sense. Let \( u_3 \) be the top of the tetrahedron. Let us look from it down at the vertices \( u_0, u_1, u_2 \). We will see two possible views: either \( u_0, u_1, u_2 \), in the order which they are written, go round in the counter-clockwise direction (the “right” orientation) or, in the clockwise one (“left” orientation). Introduce three maps:

\[
s: \Lambda_0(T) \rightarrow \mathbb{C}, \quad c: \Lambda_1(T) \rightarrow \mathbb{Z}_N, \quad \alpha: \Lambda_2(T) \rightarrow \mathbb{Z}_N,
\]

where \( s \) is injective, and \( c \) satisfies the following relations:

\[
\sum_{e \in \Lambda_1(T|v)} c(e) = 1/2, \quad v \in \Lambda_0(T), \quad \Lambda_1(T|v) \equiv \{ e \in \Lambda_1(T): v \in e \}.
\]

Let \( c_{ij} = c(u_i u_j) \) (\( u_i u_j \) is the edge having ends \( u_i \) and \( u_j \)), \( \alpha_i = \alpha(u_j u_k u_l) \), \( \{j, k, l\} = \{0, 1, 2, 3\} \setminus \{i\} \), and \( s_{ij} \) be a representation with \( z_{s_{ij}} = s(u_i) - s(u_j) \) *. Define the symbol associated with the tetrahedron \( T \):

\[
T_u(s, c, \alpha) = \begin{cases} T(s_{01}, s_{12}, s_{23}|c_{01}, c_{12})_{\alpha_2, \alpha_1}, & \text{right orientation;} \\ \overline{T}(s_{01}, s_{12}, s_{23}|c_{01}, c_{12})_{\alpha_2, \alpha_1}, & \text{left orientation,} \end{cases}
\]

Next, consider two-sided topological triangle \( F \) in \( \mathbb{R}^3 \), with one doubled edge, i.e. one pair of vertices is connected by two different edges. As above, fix the ordering map \( u: \{0, 1, 2\} \rightarrow \Lambda_0(F) \) in such a way, that \( u_0 \) does not belong to the doubled edge. Put arrows on single edges according to the same rule as for the tetrahedron above, while arrows on the double edges should point out to different vertices. There are two orientations of \( F \). Indeed, if we look from the vertex \( u_0 \) down at the double edges, then we will see again two possible views: either arrows on the double edges go round in the counter-clockwise direction (the “right” orientation), or in the clockwise one (“left” orientation). Fix three maps:

\[
s: \Lambda_0(F) \rightarrow \mathbb{C}, \quad c: \Lambda_1(F) \rightarrow \mathbb{Z}_N, \quad \alpha: \Lambda_2(F) \rightarrow \mathbb{Z}_N,
\]

where \( s \) is again injective, but \( c \) is just a zero map: \( c(e) = 0, \forall e \in \Lambda_1(F) \). Now associate with \( F \) a symbol:

\[
F_u(s, c, \alpha) = \begin{cases} F_{\alpha_1, \alpha_2}(s_{01}, s_{02}), & \text{right orientation;} \\ \overline{F}(\alpha_1, \alpha_2)(s_{01}, s_{02}), & \text{left orientation,} \end{cases}
\]

* the term “topological” here means that edges and faces of the tetrahedron can be curved

* Such a parametrization of the representations has been suggested to the author by V.V. Bazhanov, private communication

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where \( \{ \alpha_1, \alpha_2 \} = \alpha(\Lambda_2(F)) \), and the other notations are the same as in (3.13). Perform the same construction with another two-sided topological triangle \( G \) with one doubled edge, but the roles of the vertices \( u_0 \) and \( u_2 \) being exchanged, so the double edges connect now vertices \( u_0 \) and \( u_1 \). The corresponding symbol, associated with \( G \), has the form:

\[
G_u(s, c, \alpha) = \begin{cases} 
G_{\alpha_1, \alpha_2}, & \text{right orientation;} \\
G_{\alpha_1, \alpha_2}^{-1}, & \text{left orientation.}
\end{cases}
\]  

(3.16)

Now formulas (3.8), (3.9), and (3.10) (and their inverses) acquire the following geometrical interpretation. Take our topological tetrahedron \( T \), choose any edge, connecting nearest (in the sense of the defined ordering) vertices, and glue two two-sided topological triangles to two faces, sharing this edge (the doubled edges should match this chosen edge). Writing this operation in terms of symbols for these geometrical objects, one should use restrictions of the same maps \( u \), \( s \), and \( \alpha \) in all three symbols, and sum over \( Z_N \)-indices, corresponding to the glued faces. The result is the tetrahedron \( T \) with the integers, corresponding to the ends of the chosen edge, being exchanged. Namely, formula (3.8) is described by the change of the map \( u \) into \( u \circ \sigma_{01} \); (3.9), to \( u \circ \sigma_{12} \); and (3.10), to \( u \circ \sigma_{23} \), where \( \sigma_{ij} \) is an elementary permutation map of integers \( i \) and \( j \). These generate all tetrahedral group \( S_4 \). So, we conclude that symbol (3.13) is covariant under tetrahedral group up to \( N \)-th roots of unity.

To associate three-dimensional picture with a pentagon relation, generalizing (1.22), consider five points in \( \mathbb{R}^3 \), which are such that any four of them are non-coplanar. There is a convex polyhedron \( W \), having these points as its vertices. Fix the ordering map \( u: \{0, 1, 2, 3, 4\} \to \Lambda_0(W) \). The symmetry group of \( W \) is \( S_2 \times S_3 \). Group \( S_2 \) here acts nontrivially only on two vertices of \( W \), let they be \( \mathcal{O}_2 = \{u_1, u_3\} \), while group \( S_3 \) acts among remained three vertices, \( \mathcal{O}_3 = \{u_0, u_2, u_4\} \). To fix the orientation of \( W \), suppose that, looking from vertex \( u_3 \), we see three vertices \( u_0, u_2, u_4 \) running in the clockwise direction. \( W \) can be naturally splitted either into three tetrahedrons \( T^i = u_0u_1u_2u_3, T^2 = u_0u_1u_3u_4 \) and \( T^0 = u_1u_2u_3u_4 \), or only into two, \( T^1 = u_0u_2u_3u_4 \) and \( T^3 = u_0u_1u_2u_4 \). The pentagon relation equates these splittings on the level of symbols for the tetrahedrons. To make a precise statement, fix the maps, associated with \( W \),

\[
s: \Lambda_0(W) \to \mathbb{C}, \quad c: \Lambda_1(W) \to \mathbb{Z}_N, \quad \alpha: \Lambda_2(W) \to \mathbb{Z}_N,
\]  

(3.17)

where \( s \) is injective, and \( c \) satisfies the relations:

\[
\sum_{e \in \Lambda_1(W|v)} c(e) = \begin{cases} 
1/2, & v \in \mathcal{O}_2; \\
1, & v \in \mathcal{O}_3,
\end{cases}
\]  

(3.18)

see also (3.12), and the individual maps, associated with the tetrahedrons:

\[
s \big|_{T^i}: \Lambda_0(T^i) \to \mathbb{C}, \quad c_i: \Lambda_1(T^i) \to \mathbb{Z}_N, \quad \alpha_i: \Lambda_2(T^i) \to \mathbb{Z}_N, \quad i = 0, 1, 2, 3, 4.
\]  

(3.19)

Here \( c_i \)'s satisfy restrictions (3.12) as well as

\[
c = \sum_{i=0, 2, 4} c_i \bigg|_{\Lambda_1(W)} = \sum_{i=1, 3} c_i \bigg|_{\Lambda_1(W)},
\]  

(3.20)
where $c_i(e) = 0$ if $e \not\in \Lambda_1(T^i)$; and
\[
\alpha_i \bigg|_{\Lambda_2(W)} = \alpha, \quad \alpha_i \bigg|_{\Lambda_2(T^j)} = \alpha_j, \quad i, j = 0, 1, 2, 3, 4.
\]

(3.21)

The following pentagon relation holds
\[
\sum_{\alpha_0, \alpha_2, \alpha_4} \prod_{i=0,2,4} T_u^i(s, c_i, \alpha_i) = \langle s_{13} \rangle \sum_{\alpha_1, \alpha_3} \prod_{i=1,3} T_u^i(s, c_i, \alpha_i)
\]
provided
\[
\sum_{i=0,2,4} c_i(u_1 u_3) = 1.
\]

(3.22)

(3.23)

Applying consequently transformations (3.8)–(3.10), and their inverses to (3.22), we generate the whole set of identities, corresponding to arbitrary maps $u$.

Using non-degenerateness of the symbol (3.13), we can derive other relations having nice geometrical representations. First, define the symbol associated with a plain triangle. Let $D$ be a plain two-sided triangle (without doubled edges). Provide it with the ordering map $u$ and the standard set of maps $s$, $c$, and $\alpha$ (see the similar previous definitions above), $c$ being zero-map, and associate the symbol:
\[
D_u(s, c, \alpha) = \delta_{\alpha_1, \alpha_2},
\]
where $\{\alpha_1, \alpha_2\} = \alpha(\Lambda_2(D))$.

Tetrahedrons $T^0$ and $T^2$, entering the pentagon relation (3.22), themselves split another polyhedron $W'$ of the same type as $W$. One can use the corresponding pentagon identity to resplit $W'$ into three tetrahedrons, $T^1$, $T^3$, and $\overline{T^4}$ (the bar denotes the opposite orientation). Multiplying now both sides of (3.22) by inverse symbols of $T^1$ and $T^3$, we come to the “inversion” relation:
\[
\sum_{\alpha_0, \alpha_2} T_u(s, c, \alpha) \overline{T_u}(s, c', \alpha') = \langle s_{13} \rangle \langle s_{02} \rangle \delta_{\alpha_1, \alpha'_1} \delta_{\alpha_3, \alpha'_3},
\]
where $\alpha'_i = \alpha_i$ for $i = 0, 2$, and maps $c$ and $c'$ besides relations (3.12) satisfy also
\[
c(u_1 u_3) + c'(u_1 u_3) = 1, \quad c(u_0 u_1) + c'(u_0 u_1) = 0.
\]

(3.25)

(3.26)

Geometrically relation (3.25) equates, up to scalar factors, two tetrahedrons, glued along two common faces ($u_0 u_1 u_3$ and $u_1 u_2 u_3$), to two plain triangles, attached to each other along one edge. Putting in (3.25) $\alpha_1 = \alpha'_1$ and summing over $\alpha_1$, we get one more relation:
\[
\sum_{\alpha_0, \alpha_1, \alpha_2} T_u(s, c, \alpha) \overline{T_u}(s, c', \alpha') = N \langle s_{13} \rangle \langle s_{02} \rangle \delta_{\alpha_3, \alpha'_3},
\]
where $\alpha'_i = \alpha_i$ for $i = 0, 1, 2$, and maps $c$ and $c'$ satisfy (3.12) and (3.26). Relation (3.27) equates, again up to scalar factors, two tetrahedrons, glued along three faces (which share vertex $u_3$), to a plain triangle.

(3.27)
4. Invariant of Triangulated Links

Let $M$ be a finite triangulation of an oriented 3-dimensional manifold without boundary. Denote by $\Lambda_i(M)$ the set of $i$-simplices for $i = 0, 1, 2, 3$. Fix a subset of 1-simplices $L \subset \Lambda_1(M)$ in such a way that any 0-simplex belongs to exactly two elements from $L$, so $L$ is some triangulated link in $M$, passing through all vertices. Denote $I = \{0, 1, \ldots, K-1\}$, where $K$ is the number of vertices in $M$, and fix the following maps:

$$u: I \to \Lambda_0(M), \quad s: \Lambda_0(M) \to C, \quad c_L: \Lambda_3(M) \times \Lambda_1(M) \to \mathbb{Z}_N, \quad \alpha: \Lambda_2(M) \to \mathbb{Z}_N,$$

where $u$ is bijective, $s$, injective, and $c_L$ satisfies the restrictions:

$$e \not\in \Lambda_1(t), \quad t \in \Lambda_3(M) \Rightarrow c_L(t, e) = 0; \quad \sum_{e \in \Lambda_1(t|v)} c_L(t, e) = 1/2, \quad v \in \Lambda_0(t); \quad (4.2)$$

$$\sum_{t \in \Lambda_3(M)} c_L(t, e) = \begin{cases} 0, & e \in L; \\ 1, & \text{otherwise}, \end{cases} \quad (4.3)$$

see also (3.12) for the notations. Consider the following function:

$$\langle L \rangle_M = N^{2-K} \sum_{\alpha} \prod_{t \in \Lambda_3(M)} t_u(s, c_L, \alpha) \prod_{e \in \Lambda_1(M) \setminus L} (s(\partial e))^{-1}, \quad (4.4)$$

where representation $s(\partial e)$ is defined by $z_{s(\partial e)} = s(u_j) - s(u_i)$ for $e = u_iu_j$. 

**Theorem.** $Q(M, L) \equiv (\langle L \rangle_M)^N$ for fixed $M$ and $L$ depends on only $N$ and $\omega$. Moreover, $Q(M, L)$ depends on only an equivalence class of pairs $(M, L)$, which is defined as follows.

Call two pairs $(M, L)$ and $(M', L')$ equivalent, if one of them can be obtained from the other by a sequence of operations (together with inverse ones), described in Section 3 and which correspond to relations (3.22), (3.25), and (3.27)**.

The proof of the theorem is straightforward and uses the results of Section 3.

Our definition of equivalence can be considered as a simplicial analog of the ambient isotopy of links. In this sense we have got an ambient isotopy invariant of triangulated links. It is not clear, however, whether one can establish bijective correspondence between the classes of triangulated links just defined and ambient isotopy types of links. It seems plausible that such a bijection does exist, since, elementary local transformations of Section 3 (formulas (3.22), (3.25) and (3.27)) are the only natural ones which one can imagine. So we formulate the following conjecture: function $Q(M, L)$ is an ambient isotopy invariant of an unoriented link embedded into an oriented closed 3-manifold.

Consider some examples. Assuming our conjecture to be true, we will not concretize the used triangulations in calculations. The following formulas are written up to $N$-th roots of unity:

$$\langle \text{trivial knot} \rangle_{S^3} = 1, \quad (4.5)$$

** Note that such transformations lead to singular triangulations in intermediate steps, see [TV] for the definition
\langle \text{two unlinked trivial knots} \rangle_{S^3} = 0, \quad \langle \text{two-component Hopf link} \rangle_{S^3} = N, \quad (4.6)

\langle \text{trefoil} \rangle_{S^3} = \sum_{k=0}^{N-1} (\omega)_k, \quad \langle \text{figure-eight knot} \rangle_{S^3} = \sum_{k=0}^{N-1} |(\omega)_k|^2, \quad (4.7)

where

\[(\omega)_k = \prod_{j=1}^k (1 - \omega^j). \quad (4.8)\]

Note one general property of the obtained invariant:

\[Q(M, \text{mirror image of } L) = Q(M, L)^*, \quad (4.9)\]

which follows from the “unitarity” relation (3.2).

Summary

The cyclic quantum dilogarithm, introduced in \cite{FK}, is shown to be related with cyclic 6j-symbols of the Weyl algebra endowed with Hopf algebra structure (formulas (1.1), (1.2)).

The modified 6j-symbols (3.1), (3.2), (3.13) are associated with tetrahedrons with some data on vertices, edges and faces. They differ from the usual 6j-symbols in \(\mathbb{Z}_N\) -charges on edges of the corresponding tetrahedron. Formulas (3.8) – (3.10) describe particular spatial transformations of the tetrahedron, generating the whole tetrahedral group \(S_4\). These symbols satisfy generalized pentagon identity (3.22) as well as its’ important consequences (3.25) and (3.27). For a pair \((M, L)\) of triangulated 3-manifold \(M\) and triangulated link \(L\) in it (passing through the all vertices of \(M\)) function (4.4) appears to be invariant (up to roots of unity) under elementary moves (c.f. Alexander moves \cite{A}), implemented by relations (3.22), (3.25) and (3.27). Apparently, function (4.4) modulo \(N\)-th roots of unity is an ambient isotopy invariant of links. Unlike papers \cite{TV} and \cite{KMS}, we were not able to consider triangulated 3-manifolds themselves to construct their invariant. The reason is the above mentioned \(\mathbb{Z}_N\) charges.

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