Toward loop quantization of plane gravitational waves

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Abstract
The polarized Gowdy model in terms of Ashtekar–Barbero variables is reduced with an additional constraint derived from the Killing equations for plane gravitational waves with parallel rays. The new constraint is formulated in a diffeomorphism invariant manner and, when it is included in the model, the resulting constraint algebra is first class, in contrast to the prior work done in special coordinates. Using an earlier work by Banerjee and Date, the constraints are expressed in terms of classical quantities that have an operator equivalent in loop quantum gravity, making these plane gravitational wave spacetimes accessible to loop quantization techniques.

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1. Introduction

Despite the significant progress during the 75 years since the first in-depth work on quantum gravity by Bronstein [1], there is much uncertainty about the proper formulation of such a theory. This is in large part due to the paucity of observational evidence. But one area of such evidence, where there are both increasing levels of activity and precision, concerns constraints on violations of local Lorentz invariance (LLI). This paper on the classical formulation of plane gravitational waves—a special case of plane-fronted gravitational waves with parallel rays (pp-waves)—aims to provide a starting point for both testing loop quantum gravity (LQG) quantization techniques and studying the ties between specific structures in those techniques and LLI. Like other midi-superspace models, plane-wave spacetimes\textsuperscript{3}, being homogenous and isotropic on the wave fronts and inhomogeneous in the third direction, have a degree of difficulty lying between the more complicated full theory of general relativity and homogenous cosmological models amenable to loop quantization.

\textsuperscript{3} The pp-wave spacetimes admit a covariantly constant null vector field. The plane wave class of pp-wave spacetimes considered here has a five-dimensional group of isometries with an Abelian three-dimensional subgroup on null hypersurfaces [2, 3].
Similar models were quantized in the earlier work. In a series of papers, Neville considered the quantization of plane gravitational waves both within geometrodynamics and with complex connections [4]. Borissov [5] studied plane waves and the weave states. A model based on the observation that the symmetry reduction of non-compact toroidally symmetric spacetimes yields a system equivalent to a free massless scalar field on a fixed \((2 + 1)\)-dimensional background was quantized [7]. Using metric variables, Mena Marugán and Montejo reduced the model at the classical level using gauge choices and symmetry reduction [6]. Just as in the classic example in [28], the resulting model also depended on a single metric function. Avoiding Dirac brackets altogether, the authors introduced a non-local symplectic form and quantized the remaining model using Fock space techniques. The variables used for Fock space operators were rather complicated functions of connection variables. Thus the quantization techniques and ties between the resulting Lorentz invariance and LQG remain obscure. In this paper, we remain closer to the connection variables.

As is well known, the implementation of the algebra of constraints in LQG is one of the stumbling blocks preventing a full quantization of general relativity. The presence of a non-Lie Poisson algebra means that the Dirac procedure is difficult to complete. Indeed, the present model has structure functions (see (29)). Additionally, with quantum geometry kinematic states based on graphs, the diffeomorphism symmetry is naturally implemented through mapping the graphs to their orbits under diffeomorphisms. While in LQG the spatial diffeomorphism group is unitarily implemented without anomalies, there does not exist an operator of infinitesimal spatial diffeomorphisms. Since the Poisson bracket of two Hamiltonian constraints equals the diffeomorphism constraint—just this infinitesimal transformation—the corresponding quantum algebra cannot be well defined. For more discussion on this point see [9]. A recent implementation of the diffeomorphism constraint closer to the implementation of the Hamiltonian constraint holds promise for an anomaly-free dynamics for LQG [8]. In the present model the constraints, algebra and structure functions are simple enough so that these hurdles for quantization may be tested.

The second motivation of this work is to explore the effects of specific quantization choices in the Dirac quantization on the issue of LLI. This motivation arises from the abundance of investigations of conjectured Lorentz invariance violations, including an energy-dependent speed of light, deformations of special relativity and experimental bounds on such effects. These studies highlight the lack of \textit{ab initio} calculations of these effects from fundamental theories. Some of these conjectures are inspired by the granularity of space predicted by LQG [10–15], even though the full theory seems to satisfy local Lorentz covariance [16, 17]. To our knowledge, no models clearly reconcile the granularity of spatial quantum geometry with LLI.

Nonetheless, there is important work in this area. In a cosmological context Bojowald and Hossain [21] provided a derivation of gravitational wave dispersion in LQG using perturbative methods. Working with the linearized theory of tensor modes the authors found inverse-volume corrections, which were included in a phenomenological description. A constraint algebra analysis showed precisely how the propagation of light and gravitational perturbations were related. Sahlmann and Thiemann [22] outlined features of quantum field theory on curved spacetime in LQG by considering perturbations around semiclassical coherent states on a kinematical level. In a series of papers, Gambini et al studied spherical symmetric gravity in the new variables minimally coupled to a scalar field [18]. The model was set in the framework of the uniform discretization technique and was treated by minimizing the master constraint, the sum of the squares of the constraints. The quantization of the scalar field was treated without and with polymerization, and in the latter case the authors demonstrated that while zero likely does not lie in the spectrum of the master constraint, classical flat space
can be approximated well at small lattice spacing. Several intriguing points followed from this including regularization of the cosmological constant and the loss of LLI. The apparent smallness of the effects of Lorentz violation is surprising. In the context of effective field theory the presence of Lorentz violation at high mass dimension generically generates lower dimensional operators via radiative corrections and thus Lorentz violation may be ruled out. The authors argue that this naturalness problem may be evaded, due to the non-perturbative nature of the quantization and manifestly physical measurements [18]. For more discussion on this point, see [19, 20]. The idea in the present context is to implement the Dirac quantization procedure without resorting to the minimization of a master constraint, although since the system is first class the uniform discretization framework is expected to yield the same result as Dirac quantization. It would be interesting to implement the uniform discretization framework in the case of plane gravitational waves.

Of course a thorough study of a classically reduced system will not settle the question of LLI in LQG, but there is still much to learn. For in the context of a simple system, where there is considerable control on the analytic formulation, questions on specific structures of the quantization can be tied to specific results. For instance, if the wavelength-dependent modified dispersion relations are found, are these related to the scaling of a semiclassical state as in, e.g. [22]? If the dispersion relations are not modified, what aspect(s) of the quantization ensure that they are LLI? Plane gravitational waves appear to be simple enough for deducing a definitive answer—at least in the context of a model system—to the question of whether we should expect dispersion of these waves in the quantum gravity resulting from loop quantization techniques.

In this work we start from the polarized Gowdy model in a form prepared for loop quantization, thoroughly studied and presented by Banerjee and Date [23, 24]. Like spacetimes inhabited by plane waves, this model is homogenous in two dimensions. But the Gowdy model has a different global topology. It is compact, whereas our model has the global topology of Minkowski space. This will affect our quantization since, due to homogeneity, we can choose an arbitrary finite area from the plane wave fronts and, as we consider only finite wave packets, the global topology of the inhomogeneous direction is not relevant for our purpose. The essential step in the reduction from the polarized Gowdy model to plane waves is to single out spacetimes with waves traveling in one direction (unidirectional waves) and so to avoid colliding plane waves that lead to well-known complicated interaction processes and eventually to singularities [25]. One such reduction of the polarized Gowdy model to plane waves is carried out in [26].

In this previous paper, we used ‘Rosen’ coordinates introduced by Ehlers and Kundt [3] (see also [28]). This reduction involved second-class constraints and complicated Dirac bracket relations and resulted in a system of one canonical pair and one constraint. Although these brackets do not provide a comfortable point of departure for loop quantization, the Dirac brackets give analogues of the commutation relation of a linear self-dual field [27] with gravitational corrections that suggest corrections to $\hbar$ or the gravitational constant rather than a modified wave speed [26]. In the present approach, we formulate the reduction of the Gowdy model to plane waves by means of a set of first-class constraints, so that the system becomes accessible to loop quantum techniques. A Hamiltonian formulation of pp-waves of metric variables in light-like coordinates is in [29].

In this paper we review the formalism of Banerjee and Date in the following section. Then we show in section 3 that the existence of a null Killing vector introduces constraints on the canonical variables. The new set of constraints for the model are shown to be first class in section 4. Finally, initial steps toward quantization are carried out in section 5.
2. The polarized Gowdy model in Ashtekar–Barbero variables

The vacuum Gowdy models are characterized by closed spatial topologies and two spatial commuting Killing vectors. When the Killing vectors are orthogonal, then the model is ‘polarized’. For our purposes, it is convenient to employ the same formalism for plane waves as used for polarized Gowdy models in Ashtekar–Barbero variables as formulated by Banerjee and Date [23]. The model has to be adapted to the setting of plane waves. We replace the angular variable $\theta$ on $S^1$ of the Gowdy model by the variable $z$ on $\mathbb{R}$. We assume homogeneity in the $(x, y)$-plane and wave propagation in the $z$-direction. We choose adapted spatial triads with one leg in the $z$-direction and two arbitrary orthogonal vectors in the homogeneous $(x, y)$-plane. Densitized inverse triad variables are denoted by $E_{ai}$, $a = x, y$ and $i = 1, 2$. As in [23] we write $E^z = E_z$ (1)

and introduce ‘polar coordinates’ for the triad vectors in the $(x, y)$-plane,

$$E^x_1 = E^x \cos \eta, \quad E^x_2 = E^x \sin \eta,$$

$$E^y_1 = -E^y \sin \eta, \quad E^y_2 = E^y \cos \eta. \quad (2)$$

All variables depend only on $z$ and the time variable $t$. The gauge group $SU(2)$ of triad rotations has been reduced to $U(1)$ rotations in the $(x, y)$-plane, described by the dependence of the angle $\eta$ on $z$. In terms of these variables, the spatial metric is

$$ds^2 = h_{ab} dx^a dx^b = \xi E^y E^x dy^2 + E^x E^y d\xi^2 + E^z d\xi^2. \quad (3)$$

Like the polarized Gowdy model of [23], the plane wave spacetimes with either left or right moving waves are globally hyperbolic solutions to Einstein’s equations that have two independent, orthogonal, commuting spatial Killing vectors. Unlike this Gowdy model, however, the plane wave spacetimes are characterized by a null Killing field. To apply loop quantization techniques it is useful to introduce a hypersurface-orthogonal time coordinate and ask, under what conditions does the model describe plane wave spacetimes? The answer will be in the form of a system of first-class constraints in terms of the same phase space as the polarized Gowdy model.

In the polarized Gowdy model, the extrinsic curvature turns out to be diagonal, and its components are denoted by $K_x, K_y$, and $K_z$. The curvatures $K_x$ and $K_y$ turn out to be the corresponding diagonal elements of the Ashtekar–Barbero connection, divided by the Barbero–Immirzi parameter $\gamma$. As in [23], it is convenient to work with the eight-dimensional phase space $\{(K_x, E^x), (K_y, E^y), (\xi, \mathcal{A}), (\eta, P)\}$, where $\mathcal{A}$ is the component $A^z_{\mu}$ of the connection, divided by $\gamma$, and the momentum $P$ conjugate to $\eta$ is constructed from the connection. The quantities $E^x, A$ and $P$ transform as scalar densities, while the quantities $K_x, \xi$ and $\eta$ as scalars under diffeomorphisms along the $z$-axis. The fundamental Poisson brackets are

$$\{K_a(z), E^b(z')\} = \kappa \delta_a^b \delta(z - z'), \quad \{\mathcal{A}(z), \xi(z')\} = \kappa \gamma \delta(z - z'), \quad \{\eta(z), P(z')\} = \kappa \gamma \delta(z - z'). \quad (4)$$

where $\kappa = 8\pi G_{\text{Newton}}$ is the gravitational constant.

In terms of these variables we have the following set of first-class constraints for GR: the Gauß constraint

$$G = \frac{1}{\kappa \gamma} (\xi^2 + P), \quad (5)$$
(the prime denotes the derivative with respect to $z$), which generates rotations in the $(x, y)$-plane, the diffeomorphism constraint
\[
C = \frac{1}{\kappa} \left[ K'_x E^x + K'_y E^y - \mathcal{E}' A + \frac{\eta'}{\gamma} P \right], \tag{6}
\]
generating diffeomorphisms along the $z$-axis, and the Hamiltonian constraint
\[
H = -\frac{1}{\kappa \sqrt{\mathcal{E}' E^x E^y}} \left[ E'^x K_x E^y K_y + (E'^x K_x + E'^y K_y) \mathcal{E} \left( A + \frac{\eta'}{\gamma} \right) - \frac{1}{4} \mathcal{E}^{'2} - \mathcal{E} \mathcal{E}'' \right]
- \frac{1}{4} \mathcal{E}^2 \left( \left( \ln \frac{E'}{E^x} \right)' \right)^2 + \frac{1}{2} \mathcal{E} \mathcal{E}' (\ln E^x E^y)' - \frac{\kappa}{4 \sqrt{\mathcal{E}' E^x E^y}} G^2 - \gamma \left( \sqrt{\frac{\mathcal{E}}{E^x E^y}} G \right)' . \tag{7}
\]

In the following calculations it is convenient to use weakly equivalent forms of the last two constraints. Using the Gauß constraint the diffeomorphism constraint can be rewritten in the form
\[
\tilde{C} = \frac{1}{\kappa} \left[ K'_x E^x + K'_y E^y - \mathcal{E}' \left( A + \frac{\eta'}{\gamma} \right) \right] + \eta' G . \tag{8}
\]
We make use of the weakly equivalent constraint
\[
\tilde{C} := K'_x E^x + K'_y E^y - \mathcal{E}' \left( A + \frac{\eta'}{\gamma} \right) \approx \kappa C . \tag{9}
\]
This alternate form of the diffeomorphism constraint, $\tilde{C}$, is proportional to $C$ modulo $G$. Similarly we define a form of the Hamiltonian constraint $\tilde{H}$,
\[
\tilde{H} := -E'^x K_x E^y K_y - (E'^x K_x + E'^y K_y) \mathcal{E} \left( A + \frac{\eta'}{\gamma} \right) + \frac{1}{4} \mathcal{E}^{'2} + \mathcal{E} \mathcal{E}'' + \frac{1}{4} \mathcal{E} \left( \ln \left( \frac{E'}{E^x} \right) \right)' \tag{10}
- \frac{1}{2} \mathcal{E} \mathcal{E}' (\ln E^x E^y)'
\]
weakly equal to $\kappa \sqrt{\mathcal{E}' E^x E^y} H$.

3. Reduction to plane waves

We want to consider finite pulses of plane waves, traveling either in the positive or in the negative $z$-direction through flat space. Such waves are characterized by a null Killing vector field in the direction of propagation. In order to formulate the null Killing vector field we add an orthogonal time coordinate to the spatial manifold with the metric (3). Using a lapse function $N = N(t, z)$, we have
\[
d s^2 = -N^2 d t^2 + \mathcal{E} \frac{E^y}{E^x} d x^2 + \mathcal{E} \frac{E^x}{E^y} d y^2 + \frac{E^x E^y}{\mathcal{E}} d z^2 . \tag{11}
\]
The existence of the null Killing field satisfying $\nabla_{(\mu} k_{\nu)} = 0$ gives rise to constraints on the phase space variables. Using the relations worked out in appendix A, we find two new constraints
\[
U_x = E'^x K_x - \frac{1}{2} \mathcal{E}' - \frac{1}{2} \mathcal{E} \left( \frac{E'^y}{E^y} - \frac{E'^x}{E^x} \right) = 0 , \tag{12}
\]
\[
U_y = E'^y K_y - \frac{1}{2} \mathcal{E}' + \frac{1}{2} \mathcal{E} \left( \frac{E'^x}{E^x} - \frac{E'^y}{E^y} \right) = 0 , \tag{13}
\]
where the minus sign in $k^\mu$ in (A.1) was chosen. The two relations render the diffeomorphism constraint and the Hamiltonian constraint equivalent; rewriting $\bar{C}$ and $\bar{H}$ in terms of $U_x$ and $U_y$ gives

$$C \approx -\frac{1}{\xi} \bar{H} = \mathcal{E}'' + \frac{\mathcal{E}}{2} \left( \ln \frac{E^y}{E^x} \right)' - \frac{\mathcal{E}'}{2} \left( \ln E^x E^y \right)' - \mathcal{E}' \left( A + \eta' \gamma \right).$$

(14)

The weak equality in (14) implies that, modulo $U_x$, $U_y$, and the Gauß constraint,

$$C \approx -\sqrt{E^x E^y} H = -\sqrt{-g_{zz}} H.$$

This means that if we choose the lapse function $N = \sqrt{E^x E^y}$, then the time evolution generated by $H$ is equivalent to a spatial diffeomorphism generated by $C$, i.e. time derivatives are equal to minus $z$-derivatives and the variables depend only on $t - z$. The waves travel without dispersion.

In coordinates, such that $g_{zz} = 1$, i.e. $\mathcal{E} = E^x E^y$ and $A = \eta = 0$, the constraint equation $\bar{C} = 0$ reduces to $E^x E'' + E^y E'' = 0$ [26]. In the formulation by Ehlers and Kundt [3] (see also [28]), where $E^x = Le^{-\beta}$ and $E^y = Le^\beta$, this becomes the Einstein equation for plane waves

$$L'' + L(\beta')^2 = 0,$$

(15)

where the prime and the functional dependence are in terms of $t \pm z$.

4. The constraint algebra

We now investigate the possibility of imposing the unidirectionality conditions (12) and (13) as constraints, augmenting the GR constraints $G$, $C$ and $H$. It turns out that the whole set of constraints is reducible and that the resulting reduced system is first class.

The expressions $U_x$ and $U_y$, smeared out with test functions, are

$$U_{a}[f] := \int dz \, f(z) U_{a}(z).$$

(16)

The Poisson bracket structure may be summarized in the following matrix:

$$\left( \begin{array}{cc} [U_{x}[f], U_{x}[g]] & [U_{x}[f], U_{y}[g]] \\ [U_{y}[f], U_{x}[g]] & [U_{y}[f], U_{y}[g]] \end{array} \right) = \frac{1}{2} \left( \begin{array}{cc} 1 & 1 \\ -1 & 1 \end{array} \right) \int dz \, (f'g - fg') \mathcal{E}. $$

(17)

This matrix is diagonalized by the linear combinations

$$U_{+} := U_{x} + U_{y}, \quad U_{-} := U_{x} - U_{y},$$

(18)

and explicitly

$$U_{+} = E^x K_x + E^y K_y - \mathcal{E}'$$

and

$$U_{-} = E^x K_x - E^y K_y - \mathcal{E}' \left( \ln \frac{E^y}{E^x} \right)'.$$

(19)

(20)

The algebra of these constraints is

$$\{ U_{+}[f], U_{+}[g] \} = \{ U_{+}[f], U_{-}[g] \} = 0, \quad \text{and}$$

$$\{ U_{-}[f], U_{-}[g] \} = 2 \int dz \, (f'g - fg') \mathcal{E}. $$

(21)

(22)

4 One could of course choose the other sign to obtain left-moving waves. Since we will only be including waves moving in one direction, we can neglect these constraints.
In the local form, the non-vanishing Poisson brackets are
\[ \{ U_-(z), U_-(z') \} = -2(\mathcal{E}(z) + \mathcal{E}(z')) \delta(z - z'), \]
and accordingly the constraints \( U_-(z) \) are second class. But the full set of constraints \( G, C, H, U_+ \) and \( U_- \) are reducible. In appendix B we derive the relation
\[ U_-^2 = U_+^2 + 2\mathcal{E} \left[ \ln \frac{\mathcal{E}}{\mathcal{E}'\mathcal{E}''} \right]' + 2 \left( A + \frac{\eta'}{\eta} \right) U_+ + 4\mathcal{E}U'_+ + 4\mathcal{H} - 4\mathcal{E}\mathcal{C} \]
showing that \( U_- \) is identically zero on the \([C = 0, H = 0, U_+ = 0]-constraint surface. The constraint \( U_- \) depends nonlinearly on these constraints.

The constraint \( U_- \) generates no further secondary constraints since the Poisson brackets of \( U_- \) with \( G, C \) and \( H \) weakly or strongly vanish:
\[ \{ U_-[f], G[g] \} = 0, \quad \{ U_-[f], C[g] \} = -\frac{1}{\kappa} U_-[f'g] \approx 0, \]
\[ \{ U_-[f], H[g] \} = \frac{1}{\kappa} U_- \left[ \mathcal{E} \sqrt{E'E''} f'g + \frac{\mathcal{E}'fg}{\sqrt{E'E''}} \right] - \frac{1}{\kappa} U_+ \left[ \mathcal{E} \sqrt{E'E''} f'g \right] \approx 0; \]
\[ \{ U_+[f], G[g] \} = 0, \]
\[ \{ U_+[f], C[g] \} = -\frac{1}{\kappa} U_+[f'g] \approx 0, \]
\[ \{ U_+[f], H[g] \} = \frac{1}{\kappa} U_+ \left[ \mathcal{E} \sqrt{E'E''} f'g \right] - H[fg] \approx 0. \]

Thus, \( U_+ \) can be added as another first-class constraint. The model now has the enlarged Poisson bracket algebra of the first-class constraints \( G, C, H \) and \( U_+ \) that includes the standard algebra of constraints, adapted to the Gowdy model [23],
\[ \{ G[f], G[g] \} = \{ G[f], H[g] \} = 0, \quad \{ G[f], C[g] \} = -G[f'g]. \]
\[ \{ C[f], C[g] \} = C[f'g' - f'g], \quad \{ C[f], H[g] \} = H[fg']. \]
\[ \{ H[f], H[g] \} = C \left[ (f'g' - f'g) \mathcal{E} \right]/\mathcal{E}'\mathcal{E}''. \]

This sets the stage for quantization, as now the algebra is first class, unlike in the analysis of [26]. In that paper, the expressions for the constraints \( U_\alpha \) explicitly broke spatial diffeomorphism invariance. This resulted in a system of secondary constraints that would prove difficult to quantize. The above analysis shows that it is possible to express these ‘left-’ and ‘right-moving’ constraints in a manner that does not break diffeomorphism invariance and that the resulting algebra is first class. This removes the significant obstacle of second-class constraints. Of course there remain further obstacles to quantization including the existence of structure functions and operator ordering, discussed further in the following section.

The number of four first-class constraints is the maximum that can be imposed on a system with four canonical degrees of freedom. According to their nature as generators of gauge transformations, the three sets of standard constraints of GR reduce the number of
canonical degrees of freedom to 1 (two phase space functions). The constraints $U_+(z)$, on the other hand, reduce the two phase space degrees of freedom at every point—one field variable and its conjugate momentum—to one function, as in [6]. Although the constraint $U_+$ is first class, it does not generate gauge transformations.

Note that the kinematical constraint $U_+$ does not reduce the model to plane waves alone. We can see this by considering the Lie derivatives of $g_{11}$ and $g_{22}$:

$$L_k g_{11} = k_{1;1} = \frac{N k}{E} (U_+ - U_-),$$

$$L_k g_{22} = k_{2;2} = -\frac{N k}{E} (U_+ + U_-).$$

(30)

(31)

As these are not equal to zero when we merely set $U_+ = 0$, the flow in the null direction $k^\mu$ only becomes an isometry when the Hamiltonian and the diffeomorphism constraints are imposed, implying $U_- = 0$, as we can see from (24).

The quantity $E^x K_x + E^y K_y$ appearing in $U_+$ is the densitized trace of the extrinsic curvature of the wave fronts, when embedded into three-dimensional spacetime. It has a geometric interpretation as the expansion or contraction of a transverse area element under time evolution. Likewise, the difference $E^x K_x - E^y K_y$ in the constraint $U_-$ has an interpretation as shear of transverse area elements.

5. Preparation for quantization

For quantization it is important that $U_+$ can be given a meaning as a well-defined operator. Indeed, all the constraints can be formulated in a way that anticipates the construction of the corresponding loop quantum operators. Both $E^x K_x + E^y K_y$ and $E^\prime$ are scalar densities that can be naturally integrated along $z$. To construct an operator, we have to integrate them over some interval $I$ of the coordinate $z$:

$$U_+[I] = \int_I dz (E^x K_x + E^y K_y) - E_+ + E_-,$$

(32)

where $E_{\pm}$ are the values of $E$ at the endpoints of $I$. $E$ has a meaningful operator equivalent in the adapted LQG framework of [24]. In analogy with full LQG, the integral in (32) can be obtained as the Poisson bracket

$$\left\{ \int_I \frac{E^x K_x E^y K_y \sqrt{E E^\prime E^\prime}}{\sqrt{E E^\prime E^\prime}}, \int_I \frac{dE}{\sqrt{E E^\prime E^\prime}} \right\} = \frac{1}{2} \int_I dz (E^x K_x + E^y K_y).$$

(33)

The first expression in the Poisson bracket is the first term of the kinetic term in the Hamiltonian constraint of (28) in [24]. The second expression is the volume of a sandwich of space, constructed from a fiducial (unit) area in the $(x,y)$-plane as basis and an interval $I$ in the $z$-direction. Both expressions have operator equivalents in standard LQG; for the present case we find the corresponding operators in [24], equations (31) and (32). Now we are in a position to express $U_+$ and $H$ in terms of loop quantum operators, acting on one-dimensional spin network states, as demonstrated in [24].

Of course, as in full LQG, problems are to be expected in the quantization. Classically, the first-order property of the constraint ensures that unidirectional plane waves remain unidirectional under the evolution generated by the total Hamiltonian. In the Dirac quantization program, Poisson brackets are replaced by commutators and physical states are singled out by the condition $\hat{C} |\psi\rangle = 0$ for all constraint operators $\hat{C}$. In the realization of this program two problems arise, which we mention only briefly together with an additional comment, as work on them is ongoing.
(i) The problem of the domain of the constraint operators and of their solution space. In our case the $U_+$ operator is well defined on the kinematical Hilbert space, like the Hamiltonian constraint. If zero lies in the discrete part of the spectra of all constraints then the common solution is in one Hilbert space. As is well known, this is not the case in LQG and one employs other techniques such as refined algebraic quantization.

(ii) The problem of structure functions as operators. When the Poisson bracket in (28) is promoted to a commutator of two constraint operators, it depends on the factor ordering of $U_+$ and the structure function on the right-hand side. In this simple system, the effects of the ordering and quantization of the structure function may be explored.

(iii) Gauge constraints versus reduction constraints. When the application of the commutator $[C_a, C_b]$ of two constraints on a candidate physical state $|\psi\rangle$ is non-vanishing, a consistent implementation of all constraints requires the further condition $[C_a, C_b] |\psi\rangle = 0$. For gauge-generating constraints, such a quantum anomaly renders the dimension of the gauge-invariant Hilbert space smaller than the dimension of the corresponding classical solution space and the quantization would be inconsistent. With the constraint $U_+$ the situation is different; it is not a gauge constraint and an anomaly would not spoil the consistency of quantum gauge invariance. Here, the absence of such an anomaly means that there are physical states in the common kernel of all gauge constraints and the unidirectional constraint; dispersionless propagation of classical plane gravitational waves carries over to quantum theory. In contrast to an anomaly of gauge constraints, an anomaly in this case would mean that the model would not have common solutions to all the constraints in the Dirac quantization framework, suggesting dispersion of unidirectional plane waves (unless the system is inconsistent without the unidirectional constraint). The question of dispersion in the quantum theory is the subject of a subsequent paper.

6. Conclusion and outlook

In this paper, we showed how the polarized Gowdy model can be further reduced to describe plane gravitational waves. Surprisingly, the constraints from GR, augmented by those derived from the Killing equations, are reducible and form a first-class system amenable to LQG techniques. Much of the preparation of the quantization is carried out in [24]. Work is underway on a loop quantization of the model that may provide a framework to perform investigations on Lorentz violation and deformation for plane wave spacetimes.

In classical GR, the existence of a null Killing vector field in the direction of propagation of gravitational plane waves guarantees dispersion-free propagation of such waves at a constant speed. In this paper, we show that a linear combination of two Killing equations is implemented as a first-class constraint that describes the expansion of null geodesics. Including shear, so both $U_+$ and $U_-$, rendering the Hamiltonian and diffeomorphism constraints weakly equivalent results in a second-class system. More specifically, if we exchange the first-class set $G, C, H$ and $U_+$ for $G, C, U_+$ and $U_-$, to make $H$ equivalent to $C$ (see (14)), the system becomes second class. If we insist on implementing the full contents of the Killing equations as constraints at the kinematic level, we are led to handle the second-class constraints $U_-$ with the method of Dirac brackets and impose them strongly. This programme was carried out in detail in [26] in a non-diffeomorphism invariant manner. In order to be able to apply loop quantization techniques, we keep both $C$ and $H$ as independent constraints, and the classical equivalence of time derivatives to space derivatives is not manifested in the constraint algebra.

In the polarized Gowdy model, we start with vacuum GR being classically reduced by symmetry to effectively one dimension. It is a midi-superspace model with four continuous
degrees of freedom, i.e. its phase space is made up of four canonically conjugate pairs of functions on (a section of) the real line. The three sets of standard constraints of GR reduce the phase space to one canonical pair, when applied together with suitable gauge conditions. These functions represent the physical degrees of freedom of the Gowdy model, as well as of plane waves when the global topology is changed from $T^3$ to $\mathbb{R} \times \mathbb{R}^2$. From this, unidirectional waves are singled out by the constraints $U_+$, found in this paper. They are clearly not gauge constraints and not primary constraints; they restrict the physical degrees of freedom we are considering. When they are strongly imposed on the system, they reduce the degrees of freedom to one function. In [26] this was done in special coordinates leading to second-class constraints. As an inevitable side effect, the Dirac brackets of the remaining phase space function, taken at different places, become very nontrivial. In contrast to these non-local Dirac brackets, in the present paper we did not restrict the coordinates, but rather maintained one-dimensional diffeomorphism invariance. In this setting the constraints $U_+$ are first class, so that after transition to quantum theory they may be imposed as conditions on physical states by the Dirac quantization procedure. This carries over enough gravitational degrees of freedom to the quantum theory, so that we can study dispersion. This would have been lost had we imposed the unidirectional wave constraint prior to quantization. As the last step of symmetry reduction, namely the restriction from arbitrary left- and right-moving plane waves to unidirectional ones, is left for quantum theory, our resulting first-class system is a mixture of gauge and symmetry reducing constraints. As such, this classical model of gravitational plane waves provides a testing ground for quantization techniques and a starting point for an \textit{ab initio} investigation of Lorentz invariance in the quantum theory.

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Appendix A. Spacetime metric, Killing equation

In order to formulate a null Killing vector field we have to construct a spacetime manifold. To this purpose, we add an orthogonal time coordinate to the space-like manifold with the metric (3). The latter one becomes supplemented by a lapse function $N = N(t, z)$, so that the spacetime metric is given by (11).

The non-vanishing components of the Lévi-Civita connection are the following Christoffel symbols:

\[
\begin{align*}
\Gamma^0_{00} &= \frac{N}{N}, & \Gamma^1_{00} &= \frac{E^N E^y}{E^x E^z}, & \Gamma^0_{03} &= \Gamma^0_{30} = \frac{N'}{N}, \\
\Gamma^3_{33} &= \frac{1}{2N^2} \frac{\partial}{\partial t} \left( \frac{E^y E^y}{E} \right), & \Gamma^3_{33} &= \frac{1}{2} \frac{E}{E^x E^y} \left( \frac{E^y E^y}{E} \right)' , & \Gamma^3_{03} &= \Gamma^3_{30} = \frac{1}{2} \frac{E}{E^x E^y} \frac{\partial}{\partial t} \left( \frac{E^y E^y}{E} \right), \\
\Gamma^0_{11} &= \frac{1}{2N^2} \frac{\partial}{\partial x} \left( \frac{E^x E^y}{E} \right), & \Gamma^1_{01} &= \Gamma^1_{10} = \frac{1}{2} \frac{E^x}{E^y} \frac{\partial}{\partial t} \left( \frac{E^y E^y}{E} \right), \\
\Gamma^3_{11} &= \frac{1}{2} \frac{E}{E^x E^y} \left( \frac{E^x E^y}{E} \right)' , & \Gamma^1_{13} &= \Gamma^3_{31} = \frac{1}{2} \frac{E^x}{E^y} \left( \frac{E^y E^y}{E} \right)' , \\
\Gamma^0_{22} &= \frac{1}{2N^2} \frac{\partial}{\partial y} \left( \frac{E^x E^y}{E} \right), & \Gamma^2_{02} &= \Gamma^2_{20} = \frac{1}{2} \frac{E^x}{E^y} \frac{\partial}{\partial t} \left( \frac{E^y E^y}{E} \right),
\end{align*}
\]
\[ \Gamma_{22}^{3} = - \frac{1}{2} \frac{\partial}{\partial E^z} \left( \frac{E^x}{E^z} \right) ', \quad \Gamma_{23}^{3} = \Gamma_{32}^{3} = \frac{1}{2} \frac{E^y}{\partial E^z} \left( \frac{E^y}{E^z} \right) '. \]

With the above 4-metric a null vector field in the \( z \)-direction is of the form
\[ k^\mu = \left( \frac{E^z}{E} k, 0, 0, \pm Nk \right), \] (A.1)
with \( k \) being a function of \( t \) and \( z \). For \( k^\mu \) to be a Killing vector field, the Killing equations
\[ \nabla_i k^i = 0 \] in terms of covariant derivatives with respect to the Lévi-Civita connection must hold. From \( \nabla_i k^i = 0 \), we obtain the time derivative of \( k \):
\[ \dot{k} = - \left[ \frac{1}{2} \frac{\partial}{\partial E^z} \left( \frac{E^x E'}{E} \right) \pm \sqrt{\frac{E}{E^z}} \frac{N'}{N} \right] k, \] (A.2)
whereas from \( \nabla_i k^i = 0 \) we obtain the \( z \)-derivative
\[ k' = \left[ \pm \frac{1}{2N} \sqrt{\frac{E}{E^z}} \frac{\partial}{\partial E^z} \left( \frac{E^x E'}{E} \right) \pm \frac{1}{2} \frac{E^x E'}{E^z} \left( \frac{E^y E'}{E} \right)' + \frac{N'}{N} \right] k. \] (A.3)

With these derivatives, the equation \( \nabla_i k^i = 0 \) reduces to an identity.

Before imposing the remaining two non-trivial Killing equations \( \nabla_i k^i = 0 \), we introduce the extrinsic curvature. In the case of a vanishing shift vector it is
\[ K_{ab} = - \frac{1}{2N} g_{ab} \] (A.4)
so that we can express all time derivatives in the Christoffel symbols in terms of it. In the case of the polarized Gowdy model \[ 23 \] the diagonal components of the Ashtekar–Barbero connection, which are canonically conjugate to \( E^x \) and \( E^y \), are–modulo the Barbero–Immirzi parameter–equal to the extrinsic curvature components.

In the following, we will use the components \( K^i_a = e_b^i K^0_a \), converted with the un-densitized radial triad components:
\[ e_x^1 = \sqrt{\frac{E^z}{E}} \cos \frac{\eta}{2}, \quad e_x^2 = \sqrt{\frac{E^z}{E}} \sin \frac{\eta}{2} \]
\[ e_y^1 = - \sqrt{\frac{E^z}{E}} \sin \frac{\eta}{2}, \quad e_y^2 = \sqrt{\frac{E^z}{E}} \cos \frac{\eta}{2}, \quad e_z^3 = \sqrt{\frac{E^z}{E}} E', \] (A.5)

Corresponding to \( E^z \), \( E^y \) and \( E \), respectively. In terms of these components, we have
\[ K_z := \sqrt{(K_z^1)^2 + (K_z^2)^2} = \frac{1}{2N} \sqrt{\frac{E^z}{E^3}} \frac{\partial}{\partial E^z} \left( \frac{E^x E'}{E} \right) \] (A.6)
\[ K_y := \sqrt{(K_y^1)^2 + (K_y^2)^2} = \frac{1}{2N} \sqrt{\frac{E^y}{E^3}} \frac{\partial}{\partial E^y} \left( \frac{E^x E'}{E} \right) \] (A.7)
\[ K_x^3 = \frac{1}{2N} \sqrt{\frac{E^z}{E^3}} \frac{\partial}{\partial E^x} \left( \frac{E^x E'}{E} \right) = \frac{1}{2N} \left( \frac{\dot{E}^x}{E^x} + \frac{\dot{E}^y}{E^y} - \frac{\dot{E}^z}{E^z} \right) \] (A.8)

To express \( \nabla_i k^i = 0 \) in terms of canonical variables, we can use \( \Gamma_{11}^0 \) and \( \Gamma_{22}^0 \) in the form
\[ \Gamma_{11}^0 = K_k \frac{\sqrt{E^z}}{E^x} \text{ and } \Gamma_{22}^0 = K_y \frac{\sqrt{E^z}}{E^y}. \] (A.9)
The Killing equations are independent of \( k \):
\[ \mp E^i K_i = \frac{1}{2} E \left( \frac{\dot{E}^i}{E} + \frac{\dot{E}^i}{E^y} - \frac{\dot{E}^i}{E^z} \right) \] (A.10)
\[ \mp E^i K_i = \frac{1}{2} E \left( \frac{\dot{E}^i}{E} + \frac{\dot{E}^i}{E^y} + \frac{\dot{E}^i}{E^z} \right), \] (A.11)

Which are constraints (12) and (13).
Appendix B. The relation between $U_-, U_+, \bar{C}$ and $\bar{H}$

In this appendix, we show some steps of the straightforward demonstration of the dependence of $U_-, U_+, \bar{C}$ and $\bar{H}$. From the expressions for $U_-$ and $U_+$ in (19) and (20), we see that

\begin{equation}
E^o K_x = \frac{1}{2} \left[ U_+ + U_- + E' + E \left( \ln \frac{Ey}{Ex} \right) \right]
\end{equation}

and

\begin{equation}
E^o K_y = \frac{1}{2} \left[ U_+ - U_- + E' - E \left( \ln \frac{Ey}{Ex} \right) \right]
\end{equation}

give

\begin{equation}
E^o K_x + E^o K_y = U_+ + E'
\end{equation}

and

\begin{equation}
E^o K_x E^o K_y = \frac{1}{4} U_+^2 + \frac{1}{2} E' U_+ - \frac{1}{4} U_-^2 - \frac{1}{2} E \left( \ln \frac{Ey}{Ex} \right)' U_- + \frac{1}{4} (E')^2 - \frac{1}{4} E^2 \left[ \left( \ln \frac{Ey}{Ex} \right)' \right]^2.
\end{equation}

Using these results in $\bar{H}$ produces

\begin{equation}
\bar{H} = -\frac{1}{4} U_-^2 - \frac{1}{2} E' U_+ + \frac{1}{4} U_-^2 + \frac{1}{2} E \left( \ln \frac{Ey}{Ex} \right)' U_- - E \left( A + \frac{\eta}{\gamma} \right) U_+ + E E''
\end{equation}

\begin{equation}
+ \frac{1}{2} E^2 \left[ \left( \ln \frac{Ey}{Ex} \right)' \right]^2 \end{equation}
\begin{equation}
- \frac{1}{2} E E' \left( \ln \frac{Ey}{Ex} \right)' - E E' \left( A + \frac{\eta}{\gamma} \right).
\end{equation}

Differentiating $U_+$ and inserting into $\bar{C}$ gives

\begin{equation}
\bar{C} = U'_+ - \frac{1}{2} \left( \ln \frac{Ey}{Ex} \right)' U_+ + \frac{1}{2} E \left( \ln \frac{Ey}{Ex} \right)' U_- - \frac{1}{2} E' (\ln \frac{Ey}{Ex})' + E''
\end{equation}

\begin{equation}
+ \frac{1}{2} E \left[ \left( \ln \frac{Ey}{Ex} \right)' \right]^2 - E' \left( A + \frac{\eta}{\gamma} \right).
\end{equation}

From this we can eliminate $E E' \left( A + \frac{\eta}{\gamma} \right)$ from $\bar{H}$. This substitution is sufficient to express $U_-$ completely in terms of the first-class constraints $\bar{C}$, $\bar{H}$ and $U_+$, resulting in relation (24). Thus, we arrive at the quadratic equation (24) for $U_-$, the solutions of which are of course also zero modulo the first-class constraints.

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