PROPAGATION PHENOMENA FOR CNNS WITH ASYMMETRIC TEMPLATES AND DISTRIBUTED DELAYS

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Abstract. The aim of this work is to study propagation phenomena for monotone and nonmonotone cellular neural networks with the asymmetric templates and distributed delays. More precisely, for the monotone case, we establish the existence of the leftward \((c^*_-)\) and rightward \((c^*_+)\) spreading speeds for CNNs by appealing to the theory developed in [26, 27], and \(c^*_+ + c^*_- > 0\). Especially, if cells possess the symmetric templates and the same delayed interactions, then \(c^*_- = c^*_+ > 0\). Moreover, if the effect of the self-feedback interaction \(\alpha f'(0)\) is not less than 1, then both \(c^*_- > 0\) and \(c^*_+ > 0\). For the non-monotone case, the leftward and rightward spreading speeds are investigated by using the results of the spreading speed for the monotone case and squeezing the given output function between two appropriate nondecreasing functions. It turns out that the leftward and rightward spreading speeds are linearly determinate in these two cases. We further obtain the existence and nonexistence of travelling wave solutions under the weaker conditions than those in [46, 47] and show that the spreading speed coincides with the minimal wave speed.

1. Introduction. Classical digital computation methods have run into a serious speed bottleneck due to the kinds of their nature. In order to deal with this problem, based on some aspects of neurobiology and adapted to integrated circuits (see, e.g., [15, 16]), neural networks were well proposed. In the brain, the active medium is provided by a sheet-like array of massively interconnected excitable neurons whose energy comes from the burning of glucose with oxygen. In cellular neural networks, the active medium is provided by the local interconnections of active cells, whose building blocks include active nonlinear devices powered by the batteries.

One class of locally coupled neural networks, called Cellular Neural Networks (CNNS for short), were first introduced in 1988 by L.O.Chua and L.Yang [7, 8] as a novel class of information processing systems, which possesses some of the key features of neural networks (NNs) and which has important potential applications in such areas as image processing and pattern recognition (see, e.g., [6, 7, 8, 35]). Cellular neural networks share the best features of both worlds; its continuous time

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feature allows real-time signal processing found wanting in the digital domain and its local interconnection feature makes it tailor made for VLSI implementation. CNN is simply an analogue dynamic processor array, made of cells, which contain linear capacitors, linear resistors, linear and nonlinear controlled sources. This circuit has been used sometimes to test the circuit robustness as well as for implementing the simplest propagating template. The circuit model of a one-dimensional standard CNN without input terms is

$$\frac{dw_n(t)}{dt} = -w_n(t) + z + \sum_{k \in N_r(n)} A(n, k) f(w_k), \ n \in \mathbb{Z}. \quad (1)$$

Here the node voltage $w_n$ at $n$ is is called the state of the cell at $n$. The quantity $z$ is called a threshold or bias term and is related to independent voltage sources in electric circuits. The output function $f$ (a nonlinearity) is given by

$$f(x) = \frac{1}{2}(|x + 1| - |x - 1|). \quad (2)$$

For a positive integer $r$, the $r$-neighborhood $N_r(n)$ of a cell at $n$ is defined as

$$N_r(n) = \{k \in \mathbb{Z} : |k - n| \leq r\}.$$

If $r = 1$, then $N_r(n)$ becomes the nearest and the next-nearest neighbors of $n$. For each $n$ and $k \in N_r(n)$, $A(n, k)$ constitutes the so-called cloning template, which measures the coupling weights for the cell at $n$ from the cell at $k$ and specifies the interaction between each cell and all its neighbor cells in terms of their state and output variables. When the template is the space invariant, each cell is described by simple identical cloning template, i.e., $A(n, n + k) \equiv A(0, k) := a_k \ (k \in N_r(0))$ or $A(n, k) = A(n - k) \ (k \in N_r(n))$. Furthermore, the cloning template is called symmetric if $a_k = a_{-k} \ (k \in N_r(0))$. This symmetry notion represents symmetric coupling weights between cells. Otherwise, the cloning template is called asymmetric. If $r = 1$, letting $a_k := A(0, k) \ (k \in N_1(0))$, then these numbers can be arranged in a $1 \times 3$ matrix form $A := [a_{-1}, a_0, a_1]$ and (1) can be written by

$$\frac{dw_n(t)}{dt} = -w_n(t) + z + a_{-1} f(w_{n-1}) + a_0 f(w_n) + a_1 f(w_{n+1}), \ n \in \mathbb{Z}. \quad (3)$$

If $a_{-1} = a_1$, (3) can be described by the space-invariant symmetric A-template.

In modelling living neural networks, a typical construction is so-called interneuron. This means a time-delayed action, sometimes a delayed excitation, sometimes a delayed inhibition. The introduction of ideal delaying template elements in cellular neural networks (1) was motivated by this fact. Later, the synapse delay in general neural networks became widely used (see [9, 20, 35, 36]). Some theoretical results concerning the dynamic range, the steady-state behavior and some impressive and promising application of cellular neural networks with or without delays have been presented in [3, 6, 7, 8, 23, 32, 35, 36] and so on.

In CNNs, some experimental studies have revealed the propagation of traveling bursts of activity in slices of excitable neural tissue (see, e.g., [13, 14, 34]). The underlying mechanism for propagation of these waves (i.e., travelling waves) is thought to be synaptic in origin rather than diffusive as in the propagation of action potentials. CNNs of synaptically generated waves and with the space-invariant asymmetric A-template have been widely investigated, (see, e.g., [17, 18, 20, 21, 29, 41, 43, 44, 45, 46, 47]). The CNNs are a special kinds of Lattice differential equations, whose propagation of these waves has also been well studied in [4, 5, 19, 24, 33, 48].
Recently, Yu et al. [46] extended the existence results of wave propagations in [17, 18, 20, 21, 29, 41] to the time-delayed CNNs (DCNNs for short) with the space-invariant asymmetric A-templates and the general nonlinear output function

\[ w'_n(t) = -w_n(t) + \sum_{i=1}^{m} a_i \int_0^\tau J_i(y) f(w_{n-i}(t - y)) dy + \alpha \int_0^\tau J_{m+1}(y) f(w_n(t - y)) dy 
+ \sum_{j=1}^{l} \beta_j \int_0^\tau J_{m+1+j}(y) f(w_{n+j}(t - y)) dy, \]

(4)

for \( n \in \mathbb{Z}, m, l \in \mathbb{N} \). Furthermore, Yu et al. [47] studied propagation of these waves of (4) with the non-monotonic output function. More Recently, Wu and Hsu [43, 44] further studied the existence of entire solutions for (4). Letting \( J_i = \delta(y - \tau_i), i = 1, \ldots, m + l + 1 \), (4) can reduce to the following multiple discrete delays

\[ w'_n(t) = -w_n(t) + \sum_{i=1}^{m} a_i f(w_{n-i}(t - \tau_i)) + \alpha f(w_n(t - \tau_{m+1})) 
+ \sum_{j=1}^{l} \beta_j f(w_{n+j}(t - \tau_{m+1+j})). \]

(5)

Yu and Mei [45] investigated uniqueness and stability of travelling waves for (5) with the monotone output function. To the best of our knowledge, the existence, uniqueness and stability of travelling waves and the existence of entire solutions has been well studied, but the existence of the spreading speed and its coincidence with the minimal wave speed for DCNNs are still open.

The asymptotic speed of spread (spreading speed for short), as an important notion in biological invasions, was first introduced by Aronson and Weinberger [1, 2] for reaction-diffusion equations. Since then, lots of works have shown the coincidence of the spreading speed with the minimal speed for travelling waves under appropriate assumptions for various evolution systems. Weinberger [40] and Lui [31] established the theory of spreading speeds and monostable traveling waves for monotone (order-preserving) operators. This theory has been greatly developed recently in [10, 25, 26, 27, 28, 42] to monotone semiflows so that it can be applied to various discrete- and continuous-time evolution equations admitting the comparison principle. It is well known that many models with spatial structure are not monotone. The spreading speeds were also well obtained for some nonmonotone continuous-time integral equations, time-delayed reaction diffusion and lattice equations (see [11, 12, 22, 38, 39]).

Our main goals are to study the spreading behavior for monotone and nonmonotone DCNNs (4) with the asymmetric templates and the general output functions. More precisely, for the monotone case, we establish the existence of the leftward \((c^-)\) and rightward \((c^+)\) spreading speeds for CNNs by appealing to the theory developed in [26, 27], and \( c^- + c^+ > 0 \). Especially, if cells possess the symmetric space-invariant templates and the same delayed interactions, then \( c^- = c^+ > 0 \) (see Remark 2.7). Moreover, if the effect of the self-feedback interaction \( \alpha f'(0) \) is not less than 1, \( c^+ > 0 \) and \( c^- > 0 \) (see Remark 2.5). For the non-monotone case, the leftward and rightward spreading speeds are investigated by using the results of the spreading speed for the monotone case and squeezing the given output function between two appropriate nondecreasing functions. The leftward and rightward spreading speeds are linearly determinate no matter whether CNNs are monotone.
or non-monotone. We further obtain the nonexistence of travelling wave solutions with the weaker condition than that in [46, 47] (see Remark 2.2 (3)) and the spreading speed’s coincidence with the minimal wave speed. With the help of the theory of the spreading speed, we can also obtain the existence of travelling waves with the weaker condition (see Remark 2.2 (1)).

The remaining part of the paper is organized as follows. In section 2, we study the existence of spreading speed and its estimate for the monotone DCNNs via the theory in [26, 27]. In section 3, we investigate the spreading behavior and the existence of critical waves for the non-monotone DCNNs. Section 4 is devoted to some numerical simulations to illustrate our analytic results. For reader’s convenience, we present the abstract results of [26, 27] in the Appendix.

2. Spreading speeds: The monotone case. In this section, we investigate the existence of the left and right spreading speeds of DCNN model (4) with a monotone output function and further estimate it. We start with the definition of the left and right spreading speeds.

Definition 1. A number $c^+_*$ is called the rightward spreading speed for a function $w_n(t) \triangleq w(t,n) : \mathbb{R}_+ \times \mathbb{Z} \to \mathbb{R}_+$ if $\limsup_{t \to +\infty, n \geq ct} w_n(t) = 0$ for any $c > c^*_+$ and if there exists some $\rho > 0$ such that $\liminf_{t \to +\infty, n \leq ct} w_n(t) \geq \rho$ for any $c < c^*_+$. A number $c^-_*$ is called the leftward spreading speed for a function $w_n(t) \triangleq w(t,n) : \mathbb{R}_+ \times \mathbb{Z} \to \mathbb{R}_+$ if $\limsup_{t \to +\infty, n \leq ct} w_n(t) = 0$ for any $c > c^-_*$ and if there exists some $\rho > 0$ such that $\liminf_{t \to +\infty, n \geq ct} w_n(t) \geq \rho$ for any $c < c^-_*$.

We assume that

(A) (i) $\alpha > 0$, $a_1 > 0$, $a_i \geq 0$ ($i = 2, \ldots, m)$, $\beta_1 > 0$ and $\beta_j \geq 0$ ($j = 2, \ldots, l$).

$$a = \sum_{i=1}^{m} a_i \text{ and } \beta = \sum_{j=1}^{l} \beta_j.$$  (ii) $J_i : [0, \tau] \to [0, +\infty)$ is the piecewise continuous function satisfying

$$\int_{0}^{\tau} J_i(y) dy = 1 \text{ and } \int_{0}^{\tau} J_i(y) dy \neq 0 \text{ for any interval } [c, d] \subset [0, \tau] \text{ with } c < d,$$

where $0 < \tau < +\infty$ and $i = 1, \ldots, m + l + 1$.

(F1) $f$ is an odd function on $\mathbb{R}$, $f \in C([0, K], [0, K/\alpha + \beta])$, $f(0) = 0$, $(a + \alpha + \beta)f'(0) > 1$ and there is $K > 0$ such that

$$(a + \alpha + \beta)f(K) = K \quad \text{and} \quad |f(u) - f(v)| \leq f'(0)|u - v| \quad \text{for } u, v \in [0, K].$$

(F2) $f(u)$ is nondecreasing for $u \in (0, K]$ and $(a + \alpha + \beta)f(u) > u$ for $u \in (0, K)$.

Remark 2.1. It is easy to see that $f \in C([0, K], [0, K/\alpha + \beta])$, $f(0) = 0$ and $|f(u) - f(v)| \leq f'(0)|u - v|$, $\forall u, v \in [0, K]$, imply $f'(0)u \geq f(u)$ for $u \in [0, K]$.

Now we recall the existence of solution for (4) with the initial condition, its boundedness and comparison principle for the monotone output (see [43]).

Lemma 2.1. Assume that (A) and (F1)-(F2) hold. For any $w^0 = \{w^0_n\}_{n \in \mathbb{Z}} \in \mathcal{C}_K$, where $\mathcal{C}_K$ is defined in the Appendix, (4) has a unique global solution $w(t) = \{w_n(t, w^0_n)\}_{n \in \mathbb{Z}}$ through $w^0$ with $0 \leq w(t) \leq K$ for any $t \geq 0$.

Lemma 2.2 (Comparison principle). Assume that (A) and (F1)-(F2) hold. Let $\{\overline{w}_n(t)\}_{n \in \mathbb{Z}}$ and $\{\underline{w}_n(t)\}_{n \in \mathbb{Z}}$ be the solutions of (4) with the initial data $\{\overline{w}^0_n\}_{n \in \mathbb{Z}}$ and $\{\underline{w}^0_n\}_{n \in \mathbb{Z}}$, respectively, and let $w^0$ be an arbitrary solution of (4) with the initial data $\{w^0_n\}_{n \in \mathbb{Z}}$. Then $\underline{w}^0 \leq w(t) \leq \overline{w}^0$ for all $t \geq 0$.
and \( \{W_n^0(t)\}_{n \in \mathbb{Z}} \), respectively. If
\[
0 \leq W_n^0(s) \leq W_n^0(s) \leq K \quad \text{for } s \in [-\tau, 0], \ n \in \mathbb{Z},
\]
then
\[
0 \leq W_n(t) \leq W_n(t) \leq K \quad \text{for } t \in [0, +\infty), \ n \in \mathbb{Z}.
\]

2.1. Existence of spreading speeds. In order to obtain the existence of spreading speeds for CNNs with the monotone output \( f \), we appeal to the theory of spreading speeds developed in \cite{26, 27}.

Now we are concerned with the stability of an equilibrium of the spatially homogeneous system associated with (4)
\[
w'(t) = -w(t) + \sum_{i=1}^{m} a_i \int_{0}^{\tau} J_i(y) f(w(t - y))dy + \alpha \int_{0}^{\tau} J_{m+1}(y) f(w(t - y))dy
\]
\[+ \sum_{j=1}^{l} \beta_j \int_{0}^{\tau} J_{m+1+j}(y) f(w(t - y))dy.
\]

(6)
The linearized equation of (6) corresponding to the equilibrium 0 is
\[
w'(t) = -w(t) + f'(0) \sum_{i=1}^{m} a_i \int_{0}^{\tau} J_i(y) w(t - y)dy + \alpha f'(0) \int_{0}^{\tau} J_{m+1}(y) w(t - y)dy
\]
\[+ f'(0) \sum_{j=1}^{l} \beta_j \int_{0}^{\tau} J_{m+1+j}(y) w(t - y)dy =: Lw_t.
\]

(7)
The stability of the trivial solution of (7) is determined by the characteristic equation obtained by seeking solution of (7) of the form \( w(t) = e^{\lambda t} \). Then \( \lambda \) must be a root of
\[
\Delta(\lambda) := f'(0) \int_{0}^{\tau} J(s)e^{-\lambda s}ds - \lambda - 1 = 0,
\]
where
\[
J(s) := \sum_{i=1}^{m} a_i J_i(s) + \alpha J_{m+1}(s) + \sum_{j=1}^{l} \beta_j J_{m+1+j}(s).
\]
The stability modulus of \( L \) is defined as
\[
s(L) = \max\{\Re \lambda : \Delta(\lambda) = 0\}.
\]
Therefore, 0 is asymptotically stable if \( s(L) < 0 \) and unstable if \( s(L) > 0 \).

In order to obtain that the zero solution of (6) is unstable, we need to check that \( s(L) > 0 \). Indeed, consider an ordinary differential equation, which can be associated with (6) simply by ignoring delays, i.e.
\[
w'(t) = -w(t) + (a + \alpha + \beta) f(w(t)) := F(w(t)).
\]

(9)
It is obvious that (7) is cooperative and irreducible. According to Corollary 5.5.2 in \cite{37}, \( s(L) > 0 \) if and only if \( s(F'(0)) > 0 \). Since \( s(F'(0)) = (a + \alpha + \beta) f'(0) - 1 > 0 \), we have \( s(L) > 0 \).

Let \( Q_t \) be the solution map at time \( t \geq 0 \) of system (4), that is,
\[
Q_t(w^0)(\theta) = w(t + \theta, w^0), \quad \forall \theta \in [-\tau, 0], \ w^0 = \{w_n^0\}_{n \in \mathbb{Z}} \in \mathcal{C}_K.
\]
Then we have the following observation.
Lemma 2.3. Assume that (A) and (F1)-(F2) hold. Then for each \( t > 0 \), \( Q_t \) satisfies the assumptions (A1)-(A5) in the Appendix and \( \{ Q_t \}_{t \geq 0} \) is a semiflow on \( \mathcal{C}_K \).

Proof. It is easily checked that (A1) holds. Define the linear operator \( L(t) : \mathcal{C}_K \rightarrow \mathcal{C}_K, t \geq 0 \) by

\[
L(t)w^0(\theta) = \begin{cases} 
  w^0(t + \theta) - w^0(0), & t + \theta < 0, \\
  0, & t + \theta \geq 0, \quad -\tau \leq \theta \leq 0. 
\end{cases}
\]

It is obvious that \( L(t) = 0 \) for \( t \geq \tau \). Define \( S(t) := Q_t - L(t), t \geq 0 \). Thus, (A3) holds (see, [10, 27]).

According to Lemma 2.2, we easily see that \( Q_t(w^0) \) is monotone on \( w^0 \in \mathcal{C}_K \), that is, (A4) holds.

We easily check that \( Q : \tilde{\mathcal{C}}_K \rightarrow \tilde{\mathcal{C}}_K \) admits exactly two fixed points 0 and \( K \). Note that if \( w \) is a solution of (6), then \( w_n = w, n \in \mathbb{Z} \) is a solution of (4). Thus, \( Q : \tilde{\mathcal{C}}_K \rightarrow \tilde{\mathcal{C}}_K \) is monotone. Since 0 is unstable, and \( Q_t \) admits exactly two fixed points 0 and \( K \), it follows from the monotonicity of \( Q_t \) that (A5) holds.

Now we prove the condition (A2), that is, we need to verify that \( Q(t, w^0) = Q_t(w^0) \) is continuous in \((t, w^0) \in \mathbb{R}_+ \times \mathcal{C}_K \) with respect to the compact open topology.

Claim 1. For a given \( w^0 = \{ w^0_n \}_{n \in \mathbb{Z}} \in \mathcal{C}_K, Q(t, w^0) = Q_t(w^0) \) is continuous in \( t \in [0, \infty) \) with respect to the compact open topology.

Proof of Claim 1. We need to show that, for given \( w^0 = \{ w^0_n \}_{n \in \mathbb{Z}} \in \mathcal{C}_K \) and \( t \geq 0 \), and any \( \epsilon > 0 \), there is a \( \delta := \delta(\epsilon, w^0) > 0 \) such that \( d(Q_t(w^0), Q_{t_0}(w^0)) < \epsilon \) whenever \( |t - t_0| < \delta \).

Indeed, for any given \( w^0 = \{ w^0_n \}_{n \in \mathbb{Z}} \in \mathcal{C}_K \), it follows that \( w_n(t, w^0) \in \mathcal{C}_K \) and there is \( M \) (independent of \( n \in \mathbb{Z} \) and \( w^0 \)) such that \( \left| \frac{d w_n(t, w^0)}{d t} \right| \leq M \) for \( t \geq 0 \) and \( n \in \mathbb{Z} \). Thus, for any given \( t_0 \geq 0 \) and \( n \in \mathbb{N} \), \( w_n(t, w^0) \) is uniformly continuous for \( t \) on \([0, t_0 + 1]\). Since \( w^0 = \{ w^0_n \}_{n \in \mathbb{Z}} \in \mathcal{C}_K \), then for each \( n \in \mathbb{Z} \), \( w^0_n(t) \) is uniformly continuous for \( t \) on \([\tau, 0] \). Therefore, for each \( n \in \mathbb{Z} \), \( w_n(t, w^0) \) is uniformly continuous for \( t \) on \([\tau, t_0 + 1]\).

For any \( \epsilon > 0 \), Choose \( N = N(\epsilon) > 0 \) such that

\[
\sum_{k=N+1}^{\infty} \frac{K}{2k} \leq \frac{\epsilon}{2}.
\]

For such a \( \epsilon > 0 \) and any given \( t_0 \geq 0 \), there is a number \( \delta = \delta(\epsilon, t_0) \in (0, \min\{1, t_0 + \tau\}) \) such that

\[
|w_n(t + \theta) - w_n(t_0 + \theta)| < \frac{\epsilon}{2}
\]

whenever \( |t - t_0| < \delta \) and \( n \in [-N, N] \).
Then for the above given numbers $\epsilon > 0$, $t_0 \geq 0$ and $\delta \in (0, \min\{1, t_0 + \tau\})$, we have

\[
\|Q_t(w^0) - Q_{t_0}(w^0)\| = \|w_t(w^0) - w_{t_0}(w^0)\| \\
= \sum_{k=1}^{\infty} \max_{|n| \leq k, \theta \in [-\tau, 0]} |w_n(t + \theta, w^0) - w_n(t_0 + \theta, w^0)| 2^k \\
= \sum_{k=1}^{N} \max_{|n| \leq k, \theta \in [-\tau, 0]} |w_n(t + \theta, w^0) - w_n(t_0 + \theta, w^0)| 2^k \\
+ \sum_{k=N+1}^{\infty} \max_{|n| \leq k, \theta \in [-\tau, 0]} |w_n(t + \theta, w^0) - w_n(t_0 + \theta, w^0)| 2^k \\
\leq \sum_{k=1}^{N} \frac{\epsilon}{2^{k+1}} + \sum_{k=N+1}^{\infty} \frac{2K}{2^{k}} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\]

whenever $|t - t_0| < \delta$. This completes the proof of Claim 1. \hfill \Box

**Claim 2.** $Q_t(w^0)$ is uniformly continuous in $w^0 \in \mathcal{C}_K$ on any given interval $[0, t_0]$ with $t_0 > 0$ with respect to the compact open topology.

**Proof of Claim 2.** For the sake of convenience, we write the initial conditions $w^0(s) := \psi(s) = \{\psi_n(s)\}_{n \in \mathbb{N}}$ and $s \in [-\tau, 0]$. For any $t_0 > 0$, we only need to prove that

\[
Q_t(\psi^{(h)}) \rightarrow Q_t(\psi) \quad (h \rightarrow \infty)
\]

in $\mathcal{C}_K$ uniformly on the interval $[0, t_0]$ whenever $\psi^{(h)} \rightarrow \psi$ $(h \rightarrow \infty)$ in $\mathcal{C}_K$ and $h \in \mathbb{N}$.

**Step 1.** Prove that there is a convergent subsequence $\{Q_t(\psi^{(h^{(k)})})\}_{k=1}^{\infty}$ in $\mathcal{C}_K$ uniformly for $t \in [0, t_0]$. Let $\{w_n(t, \psi^{(h)})\}_{n \in \mathbb{Z}}$ be a solution of (4) with the initial condition $\psi^{(h)}(s) = \{\psi_n^{(h)}(s)\}_{n \in \mathbb{Z}}$ and $s \in [-\tau, 0]$, and $\psi^{(h)} \rightarrow \psi$ $(h \rightarrow \infty)$ in $\mathcal{C}_K$ and $h \in \mathbb{N}$. Set

\[
w(n, t, \psi^{(h)}) := \psi^{(h)}(s) \text{ for } n \in \mathbb{Z}
\]

For any given $t_0 > 0$ and each $n \in \mathbb{Z}$, we have $0 \leq w_n(t_1, \psi^{(h)}) \leq K$ and $|w_n(t_1, \psi^{(h)}) - w_n(t_2, \psi^{(h)})| \leq M|t_1 - t_2|$ for $t_1, t_2 \in [0, t_0]$ and $h \in \mathbb{N}$. It is easy to see that $\{w_n(t, \psi^{(h)})\}_{n=1}^{\infty}$ in $C([0, t_0], \mathbb{R})$ is bounded and equicontinuous. By Arzela-Ascoli Theorem, the numerical sequence $\{w_n(1, \psi^{(h)})\}_{n=1}^{\infty}$ has a convergent subsequence uniformly for $t \in [0, t_0]$, which we’ll write using double subscripts:

\[
\{w_n(1, \psi^{(h^{(1)}, k)})\}_{k=1}^{\infty} \quad \text{and} \quad \{w_n(1, \psi^{(h^{(1)}, k)})\}_{k=1}^{\infty}
\]

Also has a convergent subsequence $\{w_0(1, \psi^{(h^{(1)}, k)})\}_{k=1}^{\infty}$ and $\{h^{(1), k}\}_{k=1}^{\infty} \subset \{h^{(1), k}\}_{k=1}^{\infty}$. Similarly, $\{w_1(1, \psi^{(h^{(1)}, k)})\}_{k=1}^{\infty}$ has a convergent subsequence $\{w_0(1, \psi^{(h^{(1)}, k)})\}_{k=1}^{\infty}$ and $\{h^{(1), k}\}_{k=1}^{\infty} \subset \{h^{(1), k}\}_{k=1}^{\infty}$. In order to facilitate the mark, let $h^{(1), k} \triangleq \frac{1}{h_k^{(1)}}$. Therefore, all numerical sequences $\{w_n(1, \psi^{(h^{(1)}, k)})\}_{k=1}^{\infty}$, $\{w_0(1, \psi^{(h^{(1)}, k)})\}_{k=1}^{\infty}$ and $\{w_1(1, \psi^{(h^{(1)}, k)})\}_{k=1}^{\infty}$ converge uniformly for $t \in [0, t_0]$. Proceeding in this fashion, $\{w_n(2, \psi^{(h^{(1)}, k)})\}_{k=1}^{\infty}$ has a convergent subsequence $\{w_0(2, \psi^{(h^{(1)}, k)})\}_{k=1}^{\infty}$ and $\{h^{(1), k}\}_{k=1}^{\infty} \subset \{h^{(1), k}\}_{k=1}^{\infty}$, and $\{h^{(2), k}\}_{k=1}^{\infty} \subset \{h^{(1), k}\}_{k=1}^{\infty}$. For simplicity, denote $h^{(1), k} \triangleq h_k^{(2)}$. Thus, $\{w_n(t, \psi^{(h^{(2)}, k)})\}_{k=1}^{\infty}$ converges at the points $n = 0, \pm 1, \pm 2$ uniformly for $t \in [0, t_0]$.
Proceeding by the above way and by a standard diagonal argument, we can obtain the diagonal sequence \( \{w_n(t, \psi^{(h)})\}_{k=1}^\infty \) is a subsequence of the original sequence \( \{w_n(t, \psi^{(h)})\}_{k=1}^\infty \) that converges at each point \( n \in \mathbb{Z} \) uniformly for \( t \in [0, t_0] \). Since \( \psi^{(h)} \to \psi(h \to \infty) \) in \( C_K \), it follows that for each \( n \in \mathbb{Z} \), \( \psi^{(h)}_n \to \psi_n(h \to \infty) \) in \( C([-\tau, 0], \mathbb{R}) \). By a standard diagonal argument, \( \{\psi^{(h)}_n\}_{h=1}^\infty \) has also a convergent subsequence in \( C([-\tau, 0], \mathbb{R}) \), which we still write by \( \{\psi^{(h)}_{n_k}\}_{k=1}^\infty \) at each point \( n \in \mathbb{Z} \) in \( C([-\tau, 0], \mathbb{R}) \). Thus for each \( n \in \mathbb{Z} \), a subsequence \( \{w_n(t, \psi^{(h)}_{n_k})\}_{k=1}^\infty \) converges uniformly on \([-\tau, t_0] \). Then for each \( n \in \mathbb{Z} \), we obtain the subsequence \( \{w_n(t + \theta, \psi^{(h)}_{n_k})\}_{k=1}^\infty \) converges uniformly on \( \theta \in [-\tau, 0] \) and \( t \in [0, t_0] \), which implies that for each \( n \in \mathbb{Z} \), \( u^{(h)}_{n_k}(t + \cdot) = Q_t(\psi^{(h)}_{n_k})(n, \cdot) \) converges in \( C([-\tau, 0], \mathbb{R}) \) uniformly on \([0, t_0] \). Let \( u_n(t) \) is a limit point of \( u^{(h)}_{n_k}(t) \) in \( C([-\tau, 0], \mathbb{R}) \) uniformly on \([0, t_0] \), and then it yields \( u_n(t) \in C_K \).

For any \( \epsilon > 0 \), let \( N_1 = N_1(\epsilon) \in \mathbb{N} \) such that \( \sum_{i=N_1+1}^{\infty} \frac{K}{2^i} < \frac{\epsilon}{4} \). For such \( \epsilon > 0 \), there is a large enough number \( N_2 = N_2(\epsilon) \in \mathbb{N} \) such that

\[
|u_{n_k}^{(h)}(t + \theta) - u_n(t + \theta)| < \frac{\epsilon}{2}
\]

whenever \( k \geq N_2 \) for all \( t \in [0, t_0] \) and \( n \in [-N_1, N_1] \). Hence, when \( k \geq N_2 \), we have

\[
\|u^{(h)}_{n_k}(t) - u_n(t)\| = \|Q_t(\psi^{(h)}_{n_k}) - u_n(t)\| \\
\sum_{i=1}^{\infty} \max_{|n| \leq i, \theta \in [-\tau, 0]} |w_n(t + \theta, \psi^{(h)}_{n_k}) - u_n(t + \theta)| \\
\leq \sum_{i=1}^{N_1} \frac{\epsilon}{2^i} + \sum_{i=N_1+1}^{\infty} \frac{2K}{2^i} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

**Step 2.** If any subsequence \( Q_t(\psi^{(h_m)}) \) is convergent uniformly on \([0, t_0] \) in \( C_K \) then

\[
\lim_{m \to \infty} Q_t(\psi^{(h_m)}) = Q_t(\psi)
\]

uniformly on \([0, t_0] \).

Indeed, if \( u(t) = \{u_n(t)\}_{n \in \mathbb{Z}} \) is any limit point of \( Q_t(\psi^{(h_m)}) \) as \( m \to \infty \), it is easy to see that \( u(t) \) is a solution to (4) for any \( t \in [0, t_0] \), and hence \( u(t) = Q_t(\psi) \) for any \( t \in [0, t_0] \) in view of uniqueness of solutions and \( \psi^{(h)} \to \psi \) as \( h \to \infty \). Therefore, one concludes that \( \lim_{m \to \infty} Q_t(\psi^{(h_m)}) = Q_t(\psi) \) uniformly on \([0, t_0] \). This completes the proof of Step 2.

According to Steps 1-2, we obtain \( \lim_{h \to \infty} Q_t(\psi^{(h)}) = Q_t(\psi) \) in \( C_K \) uniformly on \([0, t_0] \) whenever \( \lim_{h \to \infty} \psi^{(h)} = \psi \) in \( C_K \). Consequently,

\[
\lim_{\psi \to \psi_0} Q_t(\psi) = Q_t(\psi_0)
\]

in \( C_K \) uniformly for \( t \in [0, t_0] \).
Lemma 2.4. Assume that (4) with the initial condition 0.

Proof. By Claims 1 and 2 and the inequality

\[ \|Q_t(\psi) - Q_{t_0}(\psi_0)\| \leq \|Q_t(\psi) - Q_t(\psi_0)\| + \|Q_t(\psi_0) - Q_{t_0}(\psi_0)\|, \]

we easily obtain that \(Q_t(w^0)\) is continuous in \((t, w^0) \in \mathbb{R}^+ \times \mathcal{C}_K\) with respect to the compact open topology, that is, (A2) holds.

It is easy to see that \(Q_0 = I\), and \(Q_{t_1 + t_2} = Q_{t_1} \circ Q_{t_2}\) for all \(t \geq 0\). Thus \(Q_t\) is a semiflow on \(\mathcal{C}_K\). This completes the proof.

Next, we give the strong positivity of the solution with the initial condition.

Lemma 2.4. Assume that (A) and (F1)-(F2) hold. Let \(w(t, w^0)\) be a solution of (4) with the initial condition \(w^0 \in \mathcal{C}_K\setminus\{0\}\). Then \(w_n(t, w^0) > 0\) for all \(t > 0\) and \(n \in \mathbb{Z}\).

Proof. Since \(w^0 \in \mathcal{C}_K\setminus\{0\}\), there exist \(n_0 \in \mathbb{Z}\) and \(s_0 \in [-\tau, 0]\) such that \(\|w_n(s_0)\| > 0\).

Case 1. \(s_0 = 0\). By (4), we easily obtain \(w_n(t) \geq e^{-\tau}w_n(0) > 0\) for \(t > 0\).

Next, prove \(w_{n+1}(t) > 0\) for \(t > 0\). Indeed, for any \(t > 0\), we have

\[ w_{n+1}(t) = e^{-\tau}w_{n+1}(0) + \int_0^t e^{-(t-s)} \sum_{i=1}^n a_i \int_0^s J_i(y)f(w_{n+1-i}(s-y))dy \]

\[ + \alpha \int_0^s J_m(s)dy \]

\[ + \sum_{j=1}^n \beta_j \int_0^s J_{m+j}(s)dy \]

\[ \geq a_1 \int_0^t e^{-(t-s)} \int_0^s J_1(y)f(w_n(s-y))dy \]

\[ = a_1 \int_0^t e^{-(t-s)} \int_{s-\tau}^s J_1(s-y)f(w_n(s-y))dy > 0. \]

Similarly, it follows from \(\beta_1 > 0\) that

\[ w_{n-1}(t) = \beta_1 \int_0^t e^{-(t-s)} \int_0^s J_{m+1}(s-y)f(w_n(s-y))dyds > 0 \]

\[ = \beta_1 \int_0^t e^{-(t-s)} \int_{s-\tau}^s J_{m+1}(s-y)f(w_n(s-y))dyds > 0. \]

Repeating these procedures, we obtain \(w_{n\pm k}(t) > 0\) for any \(t > 0\) and \(k \in \mathbb{N}\). Thus, \(w_n(t, w^0) > 0\) for all \(t > 0\) and \(n \in \mathbb{Z}\).

Case 2. \(s_0 \in [-\tau, 0]\). Since \(w_n(s_0) > 0\), there exists \(\delta \in (0, -s_0)\) such that \(w_n(y) > 0\) for \(y \in [s_0, s_0 + \delta]\). Thus, for any \(t > 0\), it holds

\[ w_n(t) \geq \alpha \int_0^t e^{-(t-s)} \int_0^s J_{m+1}(s-y)f(w_n(s-y))dyds \]

\[ = \alpha \int_0^t e^{-(t-s)} \int_{s-\tau}^s J_{m+1}(s-y)f(w_n(s-y))dyds \]

\[ > \alpha \int_0^t e^{-(t-s)} \int_{s_0+\delta}^s J_{m+1}(s-y)f(w_n(s-y))dyds > 0. \]
Similar to the process in Case 1, we easily verify that \( w_n(t, w^0) > 0 \) for all \( t > 0 \) and \( n \in \mathbb{Z} \). This completes the proof. \( \square \)

Note that in the statement of the general theorem on spreading speeds, it is often assumed that the initial data is larger than \( \sigma \) on a ball of radius \( r_\sigma \). Theorems A and C in the Appendix tell us that \( r_\sigma \) can be chosen to be independent of the positive real number \( \sigma \) in the case where the monotone map \( Q \) either is subhomogeneous or can be approximated from below by a sequence of linear operators. It is easy to see that the solution map \( Q \) does not possess subhomogeneous property. Thus, we need to the following proposition, whose proof will be given in the later.

**Proposition 1.** Assume that (A) and (F1)-(F2) hold. Then there is a sequence of linear operators \( M^{(k)} \) satisfying (C1)-(C7) in the Appendix such that the rightward and leftward spreading speed \( c_{k}^{\pm} \) of \( M^{(k)} \) converges to \( c_{+}^{\pm} \) (the rightward and leftward spreading speed of \( Q \)) as \( k \to \infty \) and that for each \( k \) there is \( \theta_k > 0 \) such that \( M^{(k)}[w^0] \leq Q[w^0] \) for any \( w^0 \in C_K \) with \( w^0_n \leq \sigma_k \).

According to Theorems A-C, we have the following results.

**Theorem 2.1.** Assume that (A) and (F1)-(F2) hold. Let \( w(t) \) be a solution of (4) with the initial condition \( w^0 \in C_K \). Then there exist \( c_{+}^{*} \) and \( c_{-}^{*} \) being the rightward and leftward spreading speeds of \( Q_1 \), respectively, such that the following statements are valid:

(i) For any \( c > c_{+}^{*} \) and \( c' > c_{-}^{*} \), if \( w^0 \in C_K \) with \( w^0_n(s) = 0 \) for \( s \in [-\tau, 0] \) and \( n \) outside a bounded interval, then \( \lim_{t \to \infty, n \geq c t} w_n(t) = 0 \) and \( \lim_{t \to \infty, n \leq -c' t} w_n(t) = 0 \).

(ii) For any \( c < c_{+}^{*} \) and \( c' < c_{-}^{*} \), if \( w^0 \in C_K \setminus \{0\} \), then \( \lim_{t \to \infty, -c' t \leq n \leq c t} w_n(t) = K \).

**Proof.** The statement (i) follows from Theorem A (i). Note that Theorem A (ii) needs the condition \( c_{+}^{*} + c_{-}^{*} > 0 \), which will be stated in the Remark 2.6.

According to Proposition 1, \( r_\sigma \) in Theorem A (ii) can be chosen to be independent of \( \sigma > 0 \). Let \( r_\sigma = r \). For any \( c < c_{+}^{*} \) and \( c' < c_{-}^{*} \), if \( w^0 \in C_K \) with \( w^0_n(\theta) > 0 \) for all \( \theta \in [-\tau, 0] \) and \( n \) on an interval \( I \) with the length \( 2r \), then there exists a positive number \( \sigma > 0 \) such that \( w^0_n(\theta) > \sigma \), for all \( \theta \in [-\tau, 0] \) and \( n \in I \), and hence, Theorem A (ii) implies that \( \lim_{t \to \infty, -c' t \leq n \leq c t} w_n(t) = K \). If \( w^0 \in C_K \setminus \{0\} \), Lemma 2.4 can tell us that \( w_n(t, w^0) > 0 \) for all \( t > \tau \) and \( n \in \mathbb{Z} \). Fixing \( t_0 > 0 \), it follows that \( w_n(t_0, w^0) > 0 \) for all \( n \in \mathbb{Z} \). By taking \( w_{n_0}(\cdot + \theta, w^0) \) for all \( \theta \in [-\tau, 0] \) and \( n \in \mathbb{Z} \) as a new initial data, we can obtain that statement (ii) holds. This completes the proof. \( \square \)

**Theorem 2.2.** Assume that (A) and (F1)-(F2) hold. Then we have the following results.

(i) For any \( c \geq c_{+}^{*} \), (4) has a travelling wave solution \( w_n(t) = \phi(n - c t) \) such that \( \phi(\xi) \) is continuous and nonincreasing on \( \xi \in \mathbb{R} \), and \( \phi(-\infty) = K \) and \( \phi(\infty) = 0 \). Moreover, for any \( c < c_{+}^{*} \), (4) has no travelling wave \( \phi(n - c t) \) connecting \( K \) and 0.

(ii) For any \( c \geq c_{-}^{*} \), (4) has a travelling wave solution \( w_n(t) = \psi(n + c t) \) such that \( \psi(\xi) \) is continuous and nondecreasing on \( \xi \in \mathbb{R} \), and \( \psi(-\infty) = 0 \) and \( \psi(\infty) = K \). Moreover, for any \( c < c_{-}^{*} \), (4) has no travelling wave \( \psi(n + c t) \) connecting 0 and \( K \).
Remark 2.2. (1) For the existence of travelling waves, Theorem 2.2 (i) and (ii) (in this case, we need to make a variable change, that is, \(c\) is replaced with \(-c\)) is consistent with Theorem 1.2 (i) in [46] and Theorem 1.1 (i) in [46], respectively. Moreover, the condition \((a + \alpha + \beta)f'(0) > 1\) in this paper is weaker than the conditions \((\alpha + \beta)f'(0) > 1\) in Theorem 1.1 [46] and \((a + \alpha)f'(0) > 1\) in Theorem 1.2 [46].

(2) According to Theorems 2.1 and 2.2, the minimal wave speed is consistent with the spreading speed. Moreover, the spreading speed is linearly determinate.

(3) Authors in [46] only obtained the non-existence of travelling wave with the condition of its exponential behavior for any \(c < c^*_\pm\). In this paper, we not only obtain the same results by the theory of spreading speed, but also the nonexistence of travelling wave without its exponential behavior.

Let \(C_{-K} := \{u \in C : -K \leq u \leq 0\}\). Since \(f(u)\) is an odd function, we easily obtain the following results.

**Theorem 2.3.** Assume that (A) and (F1)-(F2) hold. Let \(w(t)\) be a solution of (4) with the initial condition \(w^0 \in C_{-K}\). \(c^*_+\) and \(c^*_-\) are given in Theorem 2.1. Then the following statements are valid:

(i) For any \(c > c^*_+\) and \(c' > c^*_-,\) if \(w^0 \in C_{-K}\) with \(w^0(s) = 0\) for \(s \in [-\tau, 0]\) and \(n\) outside a bounded interval, then \(\lim_{t \to \infty, n \geq ct} w_n(t) = 0\) and \(\lim_{t \to \infty, n < -c't} w_n(t) = 0\).

(ii) For any \(c < c^*_+\) and \(c' < c^*_-,\) if \(w^0 \in C_{-K}\setminus\{0\}\), then \(\lim_{t \to \infty, -c't \leq n \leq ct} w_n(t) = -K\).

**Theorem 2.4.** Assume that (A) and (F1)-(F2) hold. Then we have the following results.

(i) For any \(c \geq c^*_+\), (4) has a travelling wave solution \(w_n(t) = \phi(n - ct)\) such that \(\phi(\xi)\) is continuous and nondecreasing on \(\xi \in \mathbb{R}\), and \(\phi(-\infty) = -K\) and \(\phi(\infty) = 0\). Moreover, for any \(c < c^*_+\), (4) has no travelling wave \(\phi(n - ct)\) connecting \(-K\) and 0.

(ii) For any \(c \geq c^*_-,\) (4) has a travelling wave solution \(w_n(t) = \psi(n + ct)\) such that \(\psi(\xi)\) is continuous and nonincreasing on \(\xi \in \mathbb{R}\), and \(\psi(-\infty) = 0\) and \(\psi(\infty) = -K\). Moreover, for any \(c < c^*_-,\) (4) has no travelling wave \(\psi(n + ct)\) connecting 0 and \(-K\).

**Remark 2.3.** Theorem 2.4 (i) and (ii) (in this case, we need to change a variable, that is, \(c\) is replaced with \(-c\)) is consistent with Theorem 1.2 (ii) in [46] and Theorem 1.1 (ii) in [46], respectively.

2.2. **The estimate of spreading speeds.** In this subsection, we estimate the rightward spreading speed \(c^*_+\) and the leftward spreading speed \(c^*_-\) in Theorem 4.

In order to estimate the asymptotic speeds of spread, we consider the linearized equation of (4) at the zero solution,

\[
w_n'(t) = -w_n(t) + \sum_{i=1}^{m} a_i f'(0) \int_0^t J_i(y) w_{n-i}(t-y) dy + \alpha f'(0) \int_0^t J_{m+1}(y) w_n(t-y) dy + \sum_{j=1}^{l} \beta_j f'(0) \int_0^t J_{m+1+j}(y) w_{n+j}(t-y) dy.
\]  

(10)
Let \( \{M_t\}_{t \geq 0} \) be the solution semiflow associated with (10). Thus, for each \( t > 0 \), the map \( M_t \) satisfies the assumptions (C1)-(C5). Note that \( f'(0)u \geq f(u) \) for \( u \in (0, K] \). Thus \( Q_t(w^0) \) is a lower solution of linear system (10) for \( t \in [0, +\infty) \) and it follows that

\[
Q_t(w^0) \leq M_t(w^0), \ \forall w^0 \in C_K, \quad t \geq 0.
\]

For each \( \phi \in C([\tau, 0], \mathbb{R}) \), Let \( \eta(t, \phi) \) be the unique solution of the linear delay equation

\[
\eta'(t) = -\eta(t) + \sum_{i=1}^{m} a_i f'(0) \int_0^T J_i(y)e^{\chi t} \eta(t-y)dy + \alpha f'(0) \int_0^T J_{m+1}(y) \eta(t-y)dy + \sum_{j=1}^{l} \beta_j f'(0) \int_0^T J_{m+1+j}(y)e^{-\chi j} \eta(t-y)dy
\]

with the initial condition \( \eta(\theta, \phi) = \phi(\theta), \ \forall \theta \in [\tau, 0] \). It is easily checked that \( w(t) = \{w_n(t)\}_{n=-\infty}^\infty \) with \( w_n = e^{-\lambda n} \eta(t, \phi) \) is a solution (10). Thus, we have

\[
B^t(\phi)(\theta) := M_t[\phi e^{-\lambda n}](\theta, 0) = \eta(t + \theta, \phi), \ \forall \theta \in [-\tau, 0]
\]

which implies that \( B^t \) is the solution map at time \( t \) of equation (11). It can be checked that \( B^{n_0} \) is a strongly positive linear operator and compact in \( \bar{C} \) for \( n_0 > \tau \). Therefore, (C6) holds for \( B^t \).

Since (11) is a cooperative and irreducible delay equation, we can obtain that its characteristic equation

\[
P(\lambda, \chi) := \lambda - \sum_{i=1}^{m} a_i f'(0) \int_0^\tau J_i(y)e^{\chi t}e^{-\lambda y}dy + \alpha f'(0) \int_0^\tau J_{m+1}(y)e^{-\lambda y}dy - \sum_{j=1}^{l} \beta_j f'(0) \int_0^\tau J_{m+1+j}(y)e^{-\chi j}e^{-\lambda y}dy = 0
\]

admits a real root \( \lambda = \lambda(\chi) \) that is greater than the real parts of all other roots.

Define \( \psi \in \bar{C} = C([\tau, 0], \mathbb{R}) \) by \( \psi(\theta) := e^{\lambda(\chi)\theta}, \ \forall \theta \in [-\tau, 0] \). It is obvious that \( \eta(t, \psi) = e^{\lambda(\chi)t}, \ \forall t \geq 0 \). Then we have

\[
B^t(\psi) = \eta(t + \cdot) = e^{\lambda(\chi)t}\psi, \ \forall t \geq 0.
\]

Thus, \( e^{\lambda(\chi)t} \) is the principle eigenvalue of \( B^t \) with the positive eigenfunction \( \psi \). Letting \( t = 1 \), \( e^{\lambda(\chi)} \) is the principle eigenvalue of \( B^1 := B_\chi \). Since (C6) holds, it follows from Lemma 3.7 in [26] that \( \lambda(\chi) \) is convex on \( \mathbb{R} \).

**Lemma 2.5.** Assume that (A) and (F1)-(F2) hold. The principle eigenvalue \( e^{\lambda(0)} \) of \( B_\chi \) at \( \chi = 0 \) is large than 1, that is, (C7) holds.

**Proof.** If \( \chi = 0 \), the characteristic function (12) can reduce to

\[
P(\lambda, 0) = f'(0) \int_0^\tau J(s)e^{-\lambda(0)s}ds - \lambda(0) - 1 = 0,
\]

where

\[
J(s) := \sum_{i=1}^{m} \alpha_i J_i(s) + a J_{m+1}(s) + \sum_{j=1}^{l} \beta_j J_{m+1+j}(s).
\]
We only need to prove
\[
P_0(\lambda) =: f'(0) \int_0^\tau J(s) e^{-\lambda s} ds - \lambda - 1 = 0
\]
has a unique positive root. Indeed, since \(P_0(0) = (a + \alpha + \beta)f'(0) - 1 > 0, P_0(+\infty) = -\infty\) and \(\frac{\partial P_0(\lambda)}{\partial \lambda} < 0\), then \(P_0(\lambda) = 0\) has a unique positive root. Since \(\lambda(0)\) is a root of \(P_0(\lambda) = 0\), we have \(\lambda(0) > 0\) and \(e^{\lambda(0)} > 1\). This completes the proof.

Define the function
\[
\Phi(\chi) := \frac{\lambda(\chi)}{\chi}.
\]

Thus, according to the convexity of \(\lambda(\chi)\) and \(\lambda(0) > 0\), and as a direct result of Lemma 3.8 in [26], we obtain the following result.

**Lemma 2.6.** Assume that \((A)\) and \((F1)-(F2)\) hold. The following statements are valid:

(i) \(\Phi(\chi) \to +\infty\) as \(\chi \to 0^+\).

(ii) \(\Phi(\chi)\) is decreasing near 0.

(iii) \(\Phi'(\chi)\) changes sign at most once on \((0, +\infty)\).

(iv) \(\lim_{\chi \to +\infty} \Phi(\chi)\) exists, where the limit may be infinite.

**Lemma 2.7.** Assume that \((A)\) and \((F1)-(F2)\) hold. \(\lim_{\chi \to +\infty} \Phi(\chi) = +\infty\).

**Proof.** Assume that the conclusion does not hold. According to Lemma 2.6 (iv), let \(\lim_{\chi \to +\infty} \Phi(\chi) := d < \infty\).

First, we claim that \(d > 0\). Indeed, assume that \(d < 0\) holds. It is obvious that \(\lim_{\chi \to +\infty} \lambda(\chi) = -\infty\). On the other hand, it follows from (12) that

\[
\lambda(\chi) \geq -1 + a_{i_0} f'(0) \int_0^\tau J_{i_0}(y) e^{\chi y} e^{-\lambda(\chi) y} dy > -1 + a_{i_0} f'(0) e^{\chi y_0},
\]

where some \(i_0\) \((1 \leq i_0 \leq m)\) such that \(a_{i_0} > 0\) and \(\chi\) is large enough. This implies that \(\lim_{\chi \to +\infty} \lambda(\chi) = +\infty\), which is a contradiction.

Now we assume \(d = 0\), then there is a large enough number \(X > 0\) such that \(\lambda(\chi) \tau / \chi < i_0 - \frac{1}{2}\) whenever \(\chi > X\) and

\[
\frac{\lambda(\chi)}{\chi} \geq \frac{-1}{\chi} + \frac{a_{i_0} f'(0) \int_0^\tau J_{i_0}(y) e^{\chi y} e^{-\lambda(\chi) y} dy}{\chi} \geq \frac{-1}{\chi} + \frac{a_{i_0} f'(0) e^{\chi y_0}}{\chi}
\]

which implies \(d = +\infty\). This is also a contradiction.

According to the above argument, we obtain \(d > 0\). Then there is a large enough number \(X > 0\) such that \(d/2 < \Phi(\chi) < 3d/2\) for all \(\chi > X\), and \(3d\chi/2 < \Phi(\chi)\chi = \lambda(\chi)\).
\[ \lambda(\chi) < d\chi/2 \text{ for all } \chi > X, \quad \text{and} \]
\[ \lambda(\chi) \geq -1 + a_i f'(0) \int_0^\tau J_i(y) e^{\chi y} e^{-\lambda(\chi)y} dy \]
\[ \geq -1 + a_i f'(0) \int_0^\tau J_i(y) e^{\chi y} e^{-(3d\chi)/2} dy \]
\[ = -1 + a_i f'(0) \left[ \int_0^{2i\theta/3d} J_i(y) e^{\chi y} e^{-(3d\chi)/2} dy + \int_{2i\theta/3d}^\tau J_i(y) e^{\chi y} e^{-(3d\chi/2)} dy \right]. \tag{13} \]

For \( \tau \geq 2i\theta/3d \), (13) implies that
\[ \lim_{\chi \to +\infty} \frac{\lambda(\chi)}{\chi} \geq \lim_{\chi \to +\infty} \frac{\int_0^{2i\theta/3d} J_i(y) e^{\chi y} e^{-(3d\chi)/2} dy}{\chi} = +\infty. \]
This is a contradiction. Similarly, for \( \tau < 2i\theta/3d \), \( \lim_{\chi \to +\infty} \frac{\lambda(\chi)}{\chi} = +\infty \) also holds according to the second inequality in (13). This completes the proof. \( \square \)

**Proposition 2.** Assume that (A) and (F1)-(F2) hold. Then we have the following conclusions:

1. There are a unique pair of numbers \( \chi^* \in (0, +\infty) \) and \( c_+^* \in \mathbb{R} \) such that
   \[ c_+^* = \inf_{\chi \in (0, +\infty)} \Phi(\chi) = \Phi(\chi^*). \]
2. \((c_+^*, \chi^*)\) is uniquely determined by \( \Delta_1(c, \chi) = 0 \) and \( \frac{\partial}{\partial \chi} \Delta_1(c, \chi) = 0 \), \( \chi > 0 \), where
   \[ \Delta_1(c, \chi) := c\chi + 1 - f'(0) \sum_{i=1}^m a_i \int_0^\tau J_i(y) e^{\chi(i-cy)} dy - f'(0) \alpha \int_0^\tau J_{m+1}(y) e^{-\chi cy} dy \]
   \[ - f'(0) \sum_{j=1}^l \beta_j \int_0^\tau J_{m+1+j}(y) e^{-\chi(j+cy)} dy. \tag{14} \]

**Proof.** (1). Since \( \lim_{\chi \to 0^+} \Phi(\chi) = +\infty \) and \( \lim_{\chi \to +\infty} \Phi(\chi) = +\infty \), by Lemma 2.6, there is the unique finite point \( \chi^* \in (0, +\infty) \) such that \( \Phi(\chi) \) has a minimum \( \Phi(\chi^*) \). Since \( Q_t(w^0) \leq M_t(w^0), \forall w^0 \in C_K, \quad t \geq 0 \), Theorem 3.10 (i) in [26] implies that
   \[ c_+^* \leq \inf_{\chi \in (0, +\infty)} \Phi(\chi) = \Phi(\chi^*). \]

On the other hand, let \( \epsilon \in (0, 1) \) and consider the linear system
\[ w_\epsilon'(t) = -w_\epsilon(t) + \sum_{i=1}^m a_i(1 - \epsilon)f'(0) \int_0^\tau J_i(y) w_{\epsilon-i}(t - y) dy \]
\[ + \alpha(1 - \epsilon)f'(0) \int_0^\tau J_{m+1}(y) w_\epsilon(t - y) dy \]
\[ + \sum_{j=1}^l \beta_j(1 - \epsilon)f'(0) \int_0^\tau J_{m+1+j}(y) w_{\epsilon+j}(t - y) dy. \tag{15} \]
Let \( M_t^\epsilon \) be the solution map of (15). We replace \( M_t \) by \( M_t^\epsilon \) and obtain a corresponding \( \Phi^\epsilon(\chi) \). For any \( \epsilon > 0 \), there exists a positive number \( \delta \) such that if \( 0 \leq u < \delta \), then \( f(u) > (1 - \epsilon)f'(0)u \). Let \( M_t^\epsilon \) be the solution map at time \( t \) of system (15).
It is easy to see that for any $\epsilon$, there exists a positive number $\delta'(\leq \delta)$ such that if $w^0 \in C$ with $w^0_n(\theta) < \delta'$ for any $n \in \mathbb{Z}$, $\theta \in [-\tau, 0]$, then $Q_t[w^0] \geq M^*_t[w^0]$ for all $t \in [0, 1]$. By Theorem D, it follows $c^*_+ \geq \inf_{\chi \in (0, +\infty)} \Phi(\chi).

According to the above argument, for any $\epsilon > 0$, we have

$$\inf_{\chi \in (0, +\infty)} \Phi(\chi) \leq c^*_+ \leq \inf_{\chi \in (0, +\infty)} \Phi(\chi).$$

Let $\epsilon \to 0$, we obtain the following equality

$$c^*_+ = \inf_{\chi \in (0, +\infty)} \Phi(\chi) = \Phi(\chi^*).$$

(2). We easily see that $c^*_+ = \Phi(\chi^*) = \frac{\lambda(\chi^*)}{\chi}$ and $\Phi(\chi^*) = 0$. Thus, we have

$$\lambda(\chi^*) = \chi^*c^*_+ \text{ and } \lambda(\chi^*) = \frac{\lambda(\chi^*)}{\chi} = c^*_+.$$

According to (12) and (16), it follows that

$$\chi^*c^*_+ = \lambda(\chi^*)$$

$$= -1 + \sum_{i=1}^{m} a_i f'(0) \int_0^\tau J_i(y)e^{\chi^* y} dy + M^*_t[y] e^{-\chi^* y} dy$$

$$+ \sum_{j=1}^{l} \beta_j f(0) \int_0^\tau J_{m+1+j}(y)e^{-\chi^* y} dy$$

and

$$c^*_+ = f'(0) \sum_{i=1}^{m} a_i \int_0^\tau (i - c^*_+) J_i(y) e^{\chi^*(i-c^*_+ - 1)} dy - f'(0) \alpha \int_0^\tau c^*_+ J_{m+1}(y)e^{-\chi^* c^*_+ y} dy$$

$$- f'(0) \sum_{j=1}^{l} \beta_j \int_0^\tau (j + c^*_+) J_{m+1+j}(y)e^{-\chi^*(j+c^*_+)} dy.$$ (17)

Thus, we obtain

$$\Delta_1(c^*_+, \chi^*) = 0 \text{ and } \frac{\partial \Delta_1}{\partial \chi}(c^*_+, \chi^*) = 0.$$

Therefore, $(c^*_+, \chi^*)$ can be uniquely determined by the equations

$$\Delta_1(c, \chi) = 0 \text{ and } \frac{\partial \Delta_1}{\partial c}(c, \chi) = 0, \chi > 0.$$

This completes the proof. \qed

Remark 2.4. If $(a + \alpha)f'(0) > 1$, then $c^*_+ > 0$. Indeed, we easily calculate that

$$\Delta_1(0, \chi) = 1 - f'(0) \sum_{i=1}^{m} a_i e^{\chi^* y} - f'(0) \alpha - f'(0) \sum_{j=1}^{l} \beta_j e^{-\chi^* y} < 1 - (a + \alpha)f'(0) < 0$$

and

$$\frac{\partial \Delta_1(c, \chi)}{\partial c} = \chi \left[ 1 + f'(0) \sum_{i=1}^{m} a_i \int_0^\tau y J_i(y) e^{\chi^*(i-c^*_+ - 1)} dy + f'(0) \alpha \int_0^\tau y J_{m+1}(y)e^{-\chi^* c^*_+ y} dy$$

$$+ f'(0) \sum_{j=1}^{l} \beta_j \int_0^\tau y J_{m+1+j}(y)e^{-\chi^*(j+c^*_+)} dy \right] > 0 \text{ for all } \chi > 0.$$
On the other hand, we have $\Delta_1(0, \chi^*) < 0 = \Delta_1(c_+^*, \chi^*)$. Thus, it follows that $c_+^* > 0$.

By a change of variable $v_n(t) = w_n(t)$, it follows that $c_+^*$ is the rightward spreading speed of the resulting equation for $v_n$:

$$v'_n(t) = -v_n(t) + \sum_{i=1}^{m} a_i \int_0^t J_i(y)f(v_{n+i}(t-y))dy + \alpha \int_0^t J_{m+1}(y)f(v_n(t-y))dy + \sum_{j=1}^{l} \beta_j \int_0^t J_{m+1+j}(y)f(v_{n-j}(t-y))dy. \quad (19)$$

Similarly, defining the function

$$\Psi(\chi) := \frac{\mu(\chi)}{\chi},$$

where $\mu(\chi)$ is the characteristic root of

$$\mu = -1 + \sum_{i=1}^{m} a_i f''(0) \int_0^t J_i(y)e^{-\chi y}e^{-\mu y}dy + \alpha f''(0) \int_0^t J_{m+1}(y)e^{-\mu y}dy + \sum_{j=1}^{l} \beta_j f''(0) \int_0^t J_{m+1+j}(y)e^{\chi j e^{-\mu y}}dy \quad (20)$$

and is greater than the real parts of all other roots. Similar to the above argument, Lemmas 2.5-2.7 hold. Thus, we have $\mu(0) > 0$ and $\Psi(+\infty) = +\infty$. Furthermore, we can obtain the similar result.

**Proposition 3.** Assume that (A) and (F1)-(F2) hold. Then we have the following conclusions:

1. There are a unique pair of numbers $\chi^* \in (0, +\infty)$ and $c_+^* \in \mathbb{R}$ such that

   $$c_+^* = \inf_{\chi \in (0, +\infty)} \Psi(\chi) = \Psi(\chi^*).$$

2. $(c_+^*, \chi^*)$ is uniquely determined by $\Delta_2(c, \chi) = 0$ and $\frac{\partial \Delta_2}{\partial \chi}(c, \chi) = 0$, $\chi > 0$, where

   $$\Delta_2(c, \chi) := c\chi + 1 - f'(0) \sum_{i=1}^{m} a_i \int_0^t J_i(y)e^{-\chi(i+c\chi)}dy - f'(0)\alpha \int_0^t J_{m+1}(y)e^{-\chi x y}dy$$

   $$- f'(0) \sum_{j=1}^{l} \beta_j \int_0^t J_{m+1+j}(y)e^{\chi(j-c\chi)}dy. \quad (21)$$

**Remark 2.5.** If $(\alpha + \beta) f''(0) > 1$, it follows $c_+^* > 0$. This proof is similar to that in Remark 2.4. If $\alpha f''(0) \geq 1$, it is easy to see that both $c_+^* > 0$ and $c_+^* > 0$.

**Remark 2.6.** According to the convex property of $\lambda(\chi)$ and $\mu(\chi) = \lambda(-\chi)$, it holds $c_+^* + c_+^* > 0$, (see, [30]).

**Proof of Proposition 1.** Letting $\epsilon = \frac{1}{k}$ in (15) and $M_t^{(k)}$ be the solution map of (15), we can verify that $M_t^{(k)}$ satisfies (C1)-(C7) and admits the rightward (leftward) spreading speeds $c_k^+$ ($c_k^-$). We replace $M_t$ (the solution semiflow of (10)) by $M_t^{(k)}$ and obtain a corresponding $\Phi^{(k)}(\chi)$. Similarly, we can verify that Lemmas 2.6 and 2.7 hold for $\Phi^{(k)}(\chi)$. By Proposition 3.9 in [26], we have $c_k^+ = \inf_{\chi \in (0, +\infty)} \Phi^{(k)}(\chi)$. 

For any $k > 1$ and $k \in \mathbb{N}$, there exists a positive number $\delta_k = \delta(k)$ such that if $0 \leq u < \delta_k$, then $f(u) > (1 - \frac{1}{k})f'(0)u$. Thus, there exists a positive number $\sigma_k < \delta_k$ such that if $w^0 \in \mathcal{C}_K$ with $w^0_n(\theta) \leq \sigma_k$ for any $n \in \mathbb{Z}$, $\theta \in [-\tau, 0]$, then $Q^t[w^0] \geq M^t[k][w^0]$ for all $t \in [0, 1]$. Therefore, we have

$$c^*_k = \inf_{\chi \in (0, \infty)} \Phi^{(k)}(\chi) \leq c^*_+ \leq \inf_{\chi \in (0, \infty)} \Phi(\chi),$$

which implies that $\lim_{k \to +\infty} c^*_k = c^*_+$. Similarly, it holds $\lim_{k \to +\infty} c^*_k = c^*_-$. This completes the proof. \hfill \square

**Remark 2.7.** Taking $m = l$, and $a_i = \beta_i$ and $J_i(y) = J_{m+1+j}(y) = J(y)\ (i = 1, \cdots, m, j = 1, \cdots, l)$, (4) reduces to CNNs with the symmetric template and the same delayed interactions

$$w_n'(t) = -w_n(t) + \sum_{i=1}^m a_i \int_0^\tau J(y)f(w_{n-i}(t-y))dy + \alpha \int_0^\tau J_{m+1}(y)f(w_n(t-y))dy$$

$$+ \sum_{i=1}^m a_i \int_0^\tau J(y)f(w_{n+i}(t-y))dy.$$ (22)

Thus, the solution semiflow of (22) satisfies the reflection, i.e. if $w_n(t) \ (n \in \mathbb{Z})$ is the solution of (22), so is $w_{-n}(t), \ (n \in \mathbb{Z})$. Then we can obtain $c^*_+ = c^*_- > 0$ and Theorems 2.1-2.4 hold.

3. The spreading speeds: Non-monotone case. Throughout this section, we assume that there exists a $b > 0$ such that $f$ satisfies the following assumptions.

(H1) $f \in C([0, b], [0, \frac{b}{a+\alpha+\beta}])$ is an odd function, $f(0) = 0$, $(a + \alpha + \beta)f'(0) > 1$ and there exists $K > 0$ with $K \leq b$ such that

$$(a + \alpha + \beta)f(K) = K \quad \text{and} \quad |f(u) - f(v)| \leq f'(0)|u - v| \quad \text{for} \ u, v \in [0, b].$$

(H2) $(a + \alpha + \beta)f(u) > 0$ for $u \in (0, K)$ and $(a + \alpha + \beta)f(u) < u$ for $u \in (K, b]$.

Define functions $f^-(u)$ and $f^+(u)$ by

$$f^-(u) = \min_{u \leq v \leq b} \{f(v)\} \quad \text{and} \quad f^+(u) = \min\{f'(0)u, \frac{b}{a+\alpha+\beta}\} \quad \text{for} \ u \in [0, b].$$

Then $f^-(u)$ and $f^+(u)$ satisfy the following properties (see [47]).

**Lemma 3.1.** Assume that (H1)-(H2) hold.

(i) $f^-(u)$ and $f^+(u)$ are nondecreasing and $|f^+(u) - f^-(v)| \leq f'(0)|u - v|$ for all $u, v \in [0, b]$.

(ii) $f^-(u) \leq f(u) \leq f^+(u)$ for all $u \in [0, b]$.

(iii) $f'(0)u \geq f^-(u) > 0$ and $f'(0)u \geq f^+(u) > 0$ for all $u \in (0, b]$.

(iv) $f^+(0) = (a + \alpha + \beta)f^+(b) - b = 0$ and there exists an $u^*_+ < 0 < u^*_- < K$ such that $f^-(0) = (a + \alpha + \beta)f^-(u^*_+) - u^*_+ = 0$. Moreover, we have

$$(a + \alpha + \beta)f^-(u) > u^*_+ for any u \in (0, u^*_+) \quad \text{and} \quad (a + \alpha + \beta)f^+(u) > u^*_- for any u \in (0, b).$$

**Lemma 3.2.** Assume that (H1)-(H2) hold. For any $w^0 = \{w^0_n\}_{n \in \mathbb{Z}} \in \mathcal{C}_b$, (4) has a unique global solution $w(t) = \{w_n(t)\}_{n \in \mathbb{Z}}$ through $w^0$ with $0 \leq w(t) \leq b$ for any $t \geq 0$. 

Proof. We omit the proof since it is essentially the same as that of the monotone case (see, Theorem 3.1 [43]). This completes the proof. \[\square\]

We also have the following proposition.

**Proposition 3.1.** Assume that (A) and (H1)-(H2) hold. For any \(w^0, w^0_+ \in C_0\) and \(w^0_- \in C_{a^+}\), with \(w^0_- \leq w^0 \leq w^0_+\), let \(w(t, w^0)\) be the solution of (4) through \(w^0\) and \(w_+(t, w^0_+)\) \((w_-(t, w^0_-))\) be the solution of (4) with \(f\) replaced by \(f^+\) \((f^-)\) through \(w^0_+\) \((w^0_-)\). Then \(w_-(t, w^0_-) \leq w(t, w^0) \leq w_+(t, w^0_+), \ t \in \mathbb{R}^+\).

**Proof.** Let \(v_n(t) = w_-(t) - w_n(t), \ t \geq 0\). Then \(v_n(t)\) and \(Z(t) := \sup_{n \in \mathbb{Z}} v_n(t)\) are continuous and bounded on \([0, +\infty)\). Suppose the assertion does not hold, then there exists \(t_0 > 0\) such that \(Z(t_0) > 0\) and

\[
Z(t_0)e^{-\delta t_0} = \max_{t \in [0, t_0]} [Z(t)e^{-\delta t} > Z(s)e^{-\delta s}, \ \forall s \in [0, t_0)],
\]

where \(\delta\) is chosen such that

\[
\delta > (\alpha + a + \beta)f'(0) - 1 > 0.
\]

It is easy to see that there exists a sequence \(\{n_k\}_{k=1}^\infty\) such that

\[
v_{n_k}(t_0) > 0, \ \forall k \in \mathbb{N}^+ \text{ and } \lim_{k \to \infty} v_{n_k}(t_0) = Z(t_0).
\]

Let \(\{t_k\}_{k=1}^\infty\) be a sequence in \([0, t_0]\) such that

\[
v_{n_k}(t_k)e^{-\delta t_k} = \max_{t \in [0, t_0]} v_{n_k}(t)e^{-\delta t}.
\]

For any \(\epsilon \in (0, t_0)\), take \(L_\epsilon := \max_{t \in [0, t_0-\epsilon]} [Z(t)e^{-\delta t}].\) By (25), we obtain

\[
\lim_{k \to \infty} v_{n_k}(t_0)e^{-\delta t_0} = Z(t_0)e^{-\delta t_0} > L_\epsilon.
\]

Thus, there is \(k_\epsilon\) such that for all \(k > k_\epsilon\), it holds

\[
v_{n_k}(t_0)e^{-\delta t_0} > L_\epsilon \geq Z(t)e^{-\delta t} \geq v_{n_k}(t)e^{-\delta t}, \ \forall t \in [0, t_0 - \epsilon].
\]

Thus, we obtain \(t_k \in [t_0 - \epsilon, t_0]\) for all \(k \geq k_\epsilon\). Thus, it holds \(\lim_{k \to \infty} t_k = t_0\). Since

\[
v_{n_k}(t_0)e^{-\delta t_0} \leq v_{n_k}(t_k)e^{-\delta t_k} \leq Z(t_k)e^{-\delta t_k} \leq Z(t_0)e^{-\delta t_0},
\]

we have

\[
v_{n_k}(t_0) \leq v_{n_k}(t_k)e^{-\delta (t_0-t_k)} \leq Z(t_0).
\]

Thus, we obtain that \(\lim_{k \to \infty} v_{n_k}(t_k) = Z(t_0)\).

From (25), we can obtain

\[
0 \leq e^{\delta t_k} \frac{d}{dt} v_{n_k}(t_k)e^{-\delta t} \bigg|_{t=t_k} = \frac{d}{dt} v_{n_k}(t_k) - \delta v_{n_k}(t_k)
\]

\[
= -(\delta + 1)v_{n_k}(t_k) + \sum_{i=1}^m \alpha_i \int_0^\tau J_i(s)[-f(w_{n_k-i}(t_k-s)) - f(w_{n_k-i}(t_k-s))]ds
\]

\[
+ a \int_0^\tau J_{m+1}(s)[-f(w_{n_k}(t_k-s)) - f(w_{n_k}(t_k-s))]ds
\]

\[
+ \sum_{j=1}^l \beta_j \int_0^\tau J_{m+1+j}(s)[-f(w_{n_k+j}(t_k-s)) - f(w_{n_k+j}(t_k-s))]ds.
\]

(27)

According to Lemma 3.2, we have

\[
f^-(u) \leq f(u) \text{ and } |f^-(u) - f^-(v)| \leq f'(0)|u - v|, \ \text{for all } u, v \in [0, b].
\]
Thus, it follows from (25) that
\begin{align*}
0 \leq - (\delta + 1) v_n(t_k) + \sum_{i=1}^{m} \alpha_i \int_{0}^{\tau} J_i(s) \left[ f^{-}(w_{-n_k-i}(t_k - s)) - f^{-}(w_{n_k-i}(t_k - s)) \right] ds \\
+ a \int_{0}^{\tau} J_{m+1}(s) \left[ f^{-}(w_{-n_k}(t_k - s)) - f^{-}(w_{n_k}(t_k - s)) \right] ds \\
+ \sum_{j=1}^{l} \beta_j \int_{0}^{\tau} J_{m+1+j}(s) \left[ f^{-}(w_{-n_k+j}(t_k - s)) - f^{-}(w_{n_k+j}(t_k - s)) \right] ds \\
\leq - (\delta + 1) v_n(t_k) + f'(0) \sum_{i=1}^{m} \alpha_i \int_{0}^{\tau} J_i(s) \max\{0, Z(t_k - s)\} ds \\
+ a f'(0) \int_{0}^{\tau} J_{m+1}(s) \max\{0, Z(t_k - s)\} ds \\
+ f'(0) \sum_{j=1}^{l} \beta_j \int_{0}^{\tau} J_{m+1+j}(s) \max\{0, Z(t_k - s)\} ds.
\end{align*}

Taking $k \to \infty$ in (28), we obtain
\begin{align*}
0 \leq - (\delta + 1) Z(t_0) + f'(0) \sum_{i=1}^{m} \alpha_i \int_{0}^{\tau} J_i(s) e^{\delta(t_0 - s)} \max\{0, Z(t_0 - s)\} e^{-\delta(t_0 - s)} ds \\
+ a f'(0) \int_{0}^{\tau} J_{m+1}(s) e^{\delta(t_0 - s)} \max\{0, Z(t_0 - s)\} e^{-\delta(t_0 - s)} ds \\
+ f'(0) \sum_{j=1}^{l} \beta_j \int_{0}^{\tau} J_{m+1+j}(s) e^{\delta(t_0 - s)} \max\{0, Z(t_0 - s)\} e^{-\delta(t_0 - s)} ds \\
\leq \left[ - \delta - 1 + f'(0) \int_{0}^{\tau} \left( \sum_{i=1}^{m} \alpha_i J_i(s) + a J_{m+1}(s) + \sum_{j=1}^{l} \beta_j J_{m+1+j}(s) \right) e^{-\delta s} ds \right] Z(t_0) \\
\leq \left[ - \delta - 1 + f'(0) \int_{0}^{\tau} \left( \sum_{i=1}^{m} \alpha_i J_i(s) + a J_{m+1}(s) + \sum_{j=1}^{l} \beta_j J_{m+1+j}(s) \right) e^{-\delta s} ds \right] Z(t_0) \\
\leq \left[ - \delta - 1 + f'(0) (\alpha + a + \beta) \right] Z(t_0).
\end{align*}

According to (24) and (29), it follows that $Z(t_0) \leq 0$, which is a contradiction. Therefore, we have $w_n(t) \geq w_{-n}(t)$ for all $t \geq 0$ and $n \in \mathbb{Z}$.

Similar to the above argument, we can also prove $w_n(t) \leq w_{+n}(t)$ for all $t \geq 0$ and $n \in \mathbb{Z}$. This completes the proof.

\[ \square \]

3.1. **Existence of spreading speeds.** In this subsection, we mainly study the existence of spreading speeds by using the results of spreading speeds for the monotone case and squeezing the given output function between two appropriate nondecreasing functions.

**Theorem 3.1.** Assume that (A) and (H1)-(H2) hold. Let $w(t)$ be a solution of (4) with the initial condition $w^0 \in C_b$. $c^*_-$ and $c^*_+$ are given in Propositions 2 and 3, respectively. Then the following statements are valid:
(i) For any $c > c_+^*$ and $c' > c_-^*$, if $w^0 \in C_b$ with $w^0_n(s) = 0$ for $s \in [-\tau, 0]$ and $n$ outside a bounded interval, then $\lim_{t \to \infty, n \geq c t} w_n(t) = 0$ and $\lim_{t \to \infty, n \leq -c t} w_n(t) = 0$.

(ii) For any $c < c_+^*$ and $c' < c_-^*$, if $w^0 \in C_b \setminus \{0\}$, then

$$u_*^+ \leq \liminf_{t \to \infty, -c' t \leq n \leq c t} w_n(t) \leq \limsup_{t \to \infty, -c' t \leq n \leq c t} w_n(t) \leq b.$$  

In addition, if $\frac{f(u)}{u}$ is strictly decreasing for $u \in [u_-, b]$ and the property (P) holds, then $\lim_{t \to \infty, -c' t \leq u \leq c t} w_n(t) = K$, where the property (P) means that for any $u, v \in [u_-, b]$ satisfying $u \leq K \leq v$, $u \geq (a + \alpha + \beta) f(v)$ and $v \leq (a + \alpha + \beta) f(u)$, then $u = v$.

**Proof.** Since $f'(0) > 0$, there exists a positive number $\delta \in (0, u^*_+)$ such that $f^\pm(u) = f(u)$ for all $u \in (0, \delta)$ and hence $(f^\pm)'(0) = f'(0)$. By Theorem 2.1, it follows that the spreading speeds are determined by the linearized systems at $0$ for the systems $w_n'(t)$

$$w_n'(t) = -w_n(t) + \sum_{i=1}^{m} a_i \int_{0}^{y} J_i(y) f^+(w_{n-i}(t-y)) dy + \alpha \int_{0}^{y} J_{m+1}(y) f^+(w_n(t-y)) dy 
+ \sum_{j=1}^{l} \beta_j \int_{0}^{y} J_{m+1+j}(y) f^+(w_{n+j}(t-y)) dy $$

(30)

and

$$w_n'(t) = -w_n(t) + \sum_{i=1}^{m} a_i \int_{0}^{y} J_i(y) f^-(w_{n-i}(t-y)) dy + \alpha \int_{0}^{y} J_{m+1}(y) f^-(w_n(t-y)) dy 
+ \sum_{j=1}^{l} \beta_j \int_{0}^{y} J_{m+1+j}(y) f^-(w_{n+j}(t-y)) dy.$$  

(31)

Therefore, $c_+^*$ and $c_-^*$ are the right and left spreading speed for (30) on $C_b$, respectively, and are also for (31) on $C_{u^-}^*$.

(i). For given $w^0 \in C_b$ with compact support, according to Lemma 2.2, it follows that

$$0 \leq w(t, w^0) \leq w_+(t, w^0), \forall t \geq 0.$$  

For any $c > c_+^*$ and $c' > c_-^*$, Theorem 2.1 (i) implies that $\lim_{t \to \infty, n \geq c t} w_n(t, w^0) = 0$ and $\lim_{t \to \infty, n \leq -c t} w_n(t, w^0) = 0$. Then $\lim_{t \to \infty, n \geq c t} w_n(t, w^0) = 0$ and $\lim_{t \to \infty, n \leq -c t} w_n(t, w^0) = 0$.

(ii). For any $w^0 \in C_b$, we have $\bar{w}^0 \in C_{u^-}^*$, where

$$\bar{w}^0(\theta) = \min\{w^0(\theta), u^*_+\} \text{ for any } \theta \in [-\tau, 0].$$

According to Lemma 2.2 and $\bar{w}^0(\theta) \leq w^0(\theta)$ for $\theta \in [-\tau, 0]$, it follows that

$$w_-(t, \bar{w}^0) \leq w(t, w^0) \leq w_+(t, w^0), \forall t \geq 0.$$  

For any $c < c_+^*$ and $c' < c_-^*$, Theorem 2.1 (ii) implies if $w^0 \in C_b \setminus \{0\}$, then $\lim_{t \to \infty, -c' t \leq n \leq c t} w_n(t, w^0) = b$, and if $\bar{w}^0 \in C_{u^-} \setminus \{0\}$, then $\lim_{t \to \infty, -c' t \leq n \leq c t} w_-(t, \bar{w}^0) = 0$. 


Therefore, for any \( c < c^*_+ \) and \( c' < c^*_+ \), if \( w^0 \in C_0 \setminus \{0\} \), then

\[
   u^* \leq \liminf_{t \to \infty, t \leq t \leq t} w_n(t) \leq \limsup_{t \to \infty, t \leq t \leq t} w_n(t) \leq b.
\]

Next, we prove the upward convergence. Setting

\[
   F(u, v) := \begin{cases} \min_{w \in [u, v]} f(w), & \text{if } u \leq v, \\ \max_{w \in [u, v]} f(w), & \text{if } u \geq v, \end{cases}
\]

we obtain that \( F \in C(\mathbb{R}^2_+, \mathbb{R}) \), which is nondecreasing in the first variable and nonincreasing in the second one. Furthermore, \( F \) satisfies

\[
   F(u, u) = f(u) \quad \text{for all } u \geq 0. \tag{32}
\]

Now we verify that \( \lim_{t \to \infty, n \leq t} w_n(t) = K \) for any \( c < c^*_+ \).

Letting \( \beta \leq c^*_+ \), we define

\[
   W_*(\beta) = \liminf_{t \to \infty, n \leq t} w_n(t) \quad \text{and} \quad W^*(\beta) = \limsup_{t \to \infty, n \leq t} w_n(t),
\]

where \( w_n(t) \) is the solution of (4). We choose sequences \((t_k)_{k \in \mathbb{N}}, (n_k)_{k \in \mathbb{N}}\) such that \( t_k \geq 0, n_k \leq \beta t_k \) for all \( k \in \mathbb{N} \) and \( t_k \to +\infty \) for \( k \to +\infty \), and

\[
   \liminf_{k \to +\infty} w_{n_k}(t_k) = W_*(\beta). \tag{33}
\]

We note that (4) has an equivalent form as follows:

\[
   w_n(t) = e^{-t} w_n^0(0) + \int_0^t e^{-t-s} \left[ \sum_{i=1}^m a_i \int_0^t J_i(y) f(w_{n-i}(s-y)) dy \right] ds \\
   + \int_0^t e^{-t-s} \left[ \frac{\alpha}{\beta} \sum_{i=1}^m J_m(y) f(w_{n}(s-y)) dy \right] ds \\
   + \int_0^t e^{-t-s} \left[ \frac{\alpha}{\beta} \sum_{i=1}^m J_{m+1+j}(y) f(w_{n+j}(s-y)) dy \right] ds
\]

\[
   = e^{-t} w_n^0(0) + \int_{-t}^0 e^s \left[ \sum_{i=1}^m a_i \int_0^t J_i(y) F(w_{n-i}(t+s-y), w_{n-i}(t+s-y)) dy \right] ds \\
   + \int_{-t}^0 e^s \left[ \frac{\alpha}{\beta} \sum_{i=1}^m J_m(y) F(w_{n}(t+s-y), w_{n}(t+s-y)) dy \right] ds \\
   + \int_{-t}^0 e^s \left[ \frac{\alpha}{\beta} \sum_{i=1}^m J_{m+1+j}(y) F(w_{n+j}(t+s-y), w_{n+j}(t+s-y)) dy \right] ds.
\]

Thus, we have

\[
   W_*(\beta) \geq \liminf_{k \to +\infty} \left\{ \int_{-t_k}^0 e^s \left[ \sum_{i=1}^m a_i \int_0^t J_i(y) F(w_{n_k-i}(t_k+s-y), w_{n_k-i}(t_k+s-y)) dy \right] ds \right\}
\]
According to (A) and the boundedness of $w_n$, we can apply Fatou’s lemma to (34) and obtain that

\[
W_{e}(\beta) \geq \int_{-\infty}^{0} e^{s} \left[ \sum_{i=1}^{m} a_i \int_{0}^{\tau} J_i(y) \liminf_{k \to +\infty} F(w_{n_k-i}(t_k + s - y), w_{n_k-i}(t_k + s - y))dy \right] ds \\
+ \int_{-\infty}^{0} e^{s} \left[ \sum_{j=1}^{l} \beta_j \int_{0}^{\tau} J_{m+1+j}(y) \liminf_{k \to +\infty} F(w_{n_k+j}(t_k + s - y), w_{n_k+j}(t_k + s - y))dy \right] ds.
\]

(35)

Let $\beta < \beta' < e^s$. For any given $s \in \mathbb{R}$ and $y \in [0, \tau]$, there exists a sufficiently large number $k \in \mathbb{N}$ such that $n_k + l \leq \beta'(t_k + s - y)$, then it follows that

\[
W^*(\beta') \geq \limsup_{k \to +\infty} w_{n_k-i}(t_k + s - y) \geq \liminf_{k \to +\infty} w_{n_k-i}(t_k + s - y) \geq W_{e}(\beta'),
\]

\[i = 0, 1, \cdots, m. \tag{36}\]

Similarly, we can obtain

\[
W^*(\beta') \geq \limsup_{k \to +\infty} w_{n_k}(t_k + s - y) \geq \liminf_{k \to +\infty} w_{n_k}(t_k + s - y) \geq W_{e}(\beta'),
\]

\[j = 1, \cdots, l. \tag{37}\]

and

\[
W^*(\beta') \geq \limsup_{k \to +\infty} w_{n_k+j}(t_k + s - y) \geq \liminf_{k \to +\infty} w_{n_k+j}(t_k + s - y) \geq W_{e}(\beta'),
\]

\[j = 1, \cdots, l. \tag{38}\]

Therefore, (36)-(38) and the monotone properties of $F$ imply that

\[
W_{e}(\beta) \geq \int_{-\infty}^{0} e^{s} \left[ \sum_{i=1}^{m} a_i \int_{0}^{\tau} J_i(y) F(W_{e}(\beta'), W^*(\beta'))dy \right] ds \\
+ \int_{-\infty}^{0} e^{s} \left[ \alpha \int_{0}^{\tau} J_{m+1}(y) F(W_{e}(\beta'), W^*(\beta'))dy \right] ds \\
+ \int_{-\infty}^{0} e^{s} \left[ \sum_{j=1}^{l} \beta_j \int_{0}^{\tau} J_{m+1+j}(y) F(W_{e}(\beta'), W^*(\beta'))dy \right] ds. \tag{39}\]

Since $\int_{-\infty}^{0} e^{s} ds = 1$ and $\int_{0}^{\tau} J_i(y)dy = 1$, $i = 1, 2, \cdots, m + l$, it follows from (39) that

\[
W_{e}(\beta) \geq \left( \sum_{i=1}^{m} a_i + \alpha + \sum_{j=1}^{l} \beta_j \right) F(W_{e}(\beta'), W^*(\beta')) \\
= (a + \alpha + \beta) F(W_{e}(\beta'), W^*(\beta')). \tag{40}\]
In a similar way, we can obtain
\[ W^\ast(\beta) \leq (a + \alpha + \beta)F(W^\ast(\beta'), W_\ast(\beta')). \] (41)

Let \( c < c' < c_\ast \). Setting
\[ V_\ast(c, c') := \inf_{c < \beta < c'} W_\ast(\beta) \text{ and } V^\ast(c, c') := \inf_{c < \beta < c'} W^\ast(\beta), \] (42)
and by the properties of \( F \), we have
\[ V_\ast(c, c') \geq (a + \alpha + \beta)F(V_\ast(c, c'), V^\ast(c, c')) \] (43)
and
\[ V^\ast(c, c') \leq (a + \alpha + \beta)F(V^\ast(c, c'), V_\ast(c, c')). \] (44)

According to the definition of \( F \), there exist \( u_0, v_0 \in [V_\ast(c, c'), V^\ast(c, c')] \subseteq [u_\ast, b] \) such that
\[ F(V_\ast(c, c'), V^\ast(c, c')) = f(v_0) \text{ and } F(V^\ast(c, c'), V_\ast(c, c')) = f(u_0). \] (45)

Hence, by (43)-(45), it follows that
\[ (a + \alpha + \beta)f(v_0) \leq V_\ast(c, c') \leq u_0, \ v_0 \leq V^\ast(c, c') \leq (a + \alpha + \beta)f(u_0), \] (46)
which implies that
\[ \frac{(a + \alpha + \beta)f(v_0)}{v_0} \leq 1 = \frac{(a + \alpha + \beta)f(K)}{K} \leq \frac{(a + \alpha + \beta)f(u_0)}{u_0}. \] (47)

Thus, we have
\[ \frac{f(v_0)}{v_0} \leq \frac{f(K)}{K} \leq \frac{f(u_0)}{u_0}. \] (48)

By (48) and the monotonicity of \( \frac{f(u_\ast)}{u} \) on \( [u_\ast, b] \), it is easily seen that \( u_0 \leq K \leq v_0 \).

On the other hand, by the property (P) and (48), we can obtain \( u_0 = v_0 \). Thus \( u_0 = K = v_0 = V^\ast(c, c') = V_\ast(c, c') \). It follows from the definitions of \( V^\ast(c, c') \) and \( V_\ast(c, c') \) that \( W_\ast(c) = K = W^\ast(c) \).

Similarly, we can prove \( \lim_{t \to -\infty, n \geq -c't} w_n(t) = K \). This completes the proof. \( \square \)

**Remark 3.1.** By the same argument as in [22], it follows that either of the following two conditions is sufficient for (P) to hold:

- (P1) \( uf(u) \) is strictly increasing for \( u \in [u^\ast, b] \), or
- (P2) \( f(u) \) is nonincreasing for \( u \in [u^\ast, b] \) and \( \frac{f(\bar{f}(u))}{u} \) is strictly decreasing for \((0, a^\ast)\), where \( \bar{f}(u) = (a + \alpha + \beta)f(u) \).

We would like to point out that the functions of logistic type \( f(u) = pu(1 - \frac{u}{K}) \) and Ricker type \( f(u) = pue^{-qu} \) both satisfy (H1)-(H2) and (P) under the suitable parameters (see [22]).

**Remark 3.2.**

(i) The spreading speeds of non-monotone CNNs with distributed delays (4) are also linearly determined.

(ii) For the non-monotone case, the corresponding results hold similar to Theorem 2.3.
3.2. Existence of travelling waves. In this subsection, we mainly study the existence of critical travelling waves for non-monotone CNNs (4), and non-existence of travelling waves with the help of the spreading theory. Moreover, under the condition that \( \frac{f(u)}{u} \) is strictly decreasing and the condition (P) holds, we can also obtain the travelling wave solution converges to the positive equilibrium \( K \) at negative infinity, which are different from the assumption (F4) in [47].

In order to obtain the existence of traveling waves, we need to impose the following assumption (see [46, 47]).

(H3) There exist \( \sigma > 0 \), \( \delta > 0 \) and \( M > 0 \) such that
\[
f(u) \geq f'(0)u - Mu^{1+\sigma} \quad \text{for } u \in [0, \delta].
\]

**Theorem 3.2.** Assume that (A) and (H1)-(H3) hold, in addition, \( \alpha f'(0) \geq 1 \). Letting \( c^*_+ \) be given in Propositions 2 and 3, the following assertions are valid.

(i) For any \( c \geq c^*_+ \), (4) has a travelling wave solution \( w_n(t) = \phi(n-ct) \) such that \( \phi \in C(\mathbb{R}, [0, b]) \setminus \{0\} \) and
\[
\lim_{\xi \to +\infty} \phi(\xi) = 0 \quad \text{and} \quad u^*_+ \leq \liminf_{\xi \to -\infty} \phi(\xi) \leq \limsup_{\xi \to -\infty} \phi(\xi) \leq b.
\]

Moreover, for any \( c < c^*_+ \), (4) has no travelling wave solution \( \phi(n-ct) \) with \( \phi \in C(\mathbb{R}, [0, b]) \setminus \{0\} \) and \( \lim_{\xi \to +\infty} \phi(\xi) = 0 \).

If, in addition, \( \frac{f(u)}{u} \) is strictly decreasing for \( u \in [u^*_+, b] \) and the condition (P) holds, then for any \( c \geq c^*_+ \), the wave profile function \( \phi \) satisfies \( \lim_{\xi \to -\infty} \phi(\xi) = K \).

(ii) For any \( c \geq c^*_+ \), (4) has a travelling wave solution \( w_n(t) = \psi(n+ct) \) such that \( \psi \in C(\mathbb{R}, [0, b]) \setminus \{0\} \) and
\[
\lim_{\xi \to -\infty} \psi(\xi) = 0 \quad \text{and} \quad u^*_+ \leq \liminf_{\xi \to +\infty} \psi(\xi) \leq \limsup_{\xi \to +\infty} \psi(\xi) \leq b.
\]

Moreover, for any \( c < c^*_+ \), (4) has no travelling wave solution \( \psi(n+ct) \) with \( \psi \in C(\mathbb{R}, [0, b]) \setminus \{0\} \) and \( \lim_{\xi \to -\infty} \psi(\xi) = 0 \).

If, in addition, \( \frac{f(u)}{u} \) is strictly decreasing for \( u \in [u^*_+, b] \) and the condition (P) holds, then for any \( c \geq c^*_+ \), the wave profile function \( \psi \) satisfies \( \lim_{\xi \to +\infty} \psi(\xi) = K \).

**Proof.** (i). According to \( \alpha f'(0) \geq 1 \) and Remark 2.4, it follows \( c^*_+ > 0 \). For \( c > c^*_+ \), this is a part of results in Theorem 1.2 [47]. Next, we only consider the case \( c = c^*_+ \). We first prove that, for any \( \rho \in (0, u^*_+) \), there exists a travelling wave solution \( \phi^\rho \) with the wave speed \( c^*_+ \) such that \( \phi^\rho(0) = \rho \) and \( \phi^\rho(\xi) < \rho \) for all \( \xi > 0 \) and \( u^*_+ \) holds, then for any \( c > c^*_+ \), the wave profile function \( \phi \) satisfies \( \lim_{\xi \to -\infty} \phi(\xi) = 0 \) and \( \lim_{\xi \to -\infty} \phi(\xi) = u^*_+ \). Thus we may assume that \( \phi_k(0) = \rho < u^*_+ \) and \( \phi_k(\xi) \leq \rho \).
for any $\xi > 0$, $k \in \mathbb{N}$. According to the process of the proof in Theorem 1.2 [47], $\phi_k$ is a fixed point of $\phi(\xi) = T(\phi)(\xi)$, where $\gamma > \frac{1}{k}$,

$$T(\phi)(\xi) = e^{\gamma \xi} \int_{\xi}^{+\infty} e^{-\gamma y} H(\phi)(y) dy$$

and

$$H(\phi)(\xi) = (\gamma - \frac{1}{c})\phi(\xi) + \frac{1}{c} \sum_{i=1}^{m} a_i \int_{0}^{\tau} J_i(y) f(\phi(\xi - i + cy)) dy$$

$$+ \frac{\alpha}{c} \int_{0}^{\tau} J_{m+1}(y) f(\phi(\xi + cy)) dy + \frac{1}{c} \sum_{j=1}^{l} \int_{0}^{\tau} J_{m+1+j}(y) f(\phi(\xi + j + cy)) dy.$$
\[
= [(a + \alpha + \beta)f'(0)(1 - \epsilon_0) - 1] \int_\xi^{\eta} \phi^\varepsilon(\theta) d\theta
\]

\[
+ f'(0)(1 - \epsilon_0) \sum_{i=1}^{m} a_i \int_0^\tau \left\{ J_i(y) \int_\xi^{\eta} [\phi^\varepsilon(\theta - i + c^*_+ y) - \phi^\varepsilon(\theta)] d\theta \right\} dy
\]

\[
+ f'(0)(1 - \epsilon_0) \alpha \int_0^\tau \left\{ J_{m+1}(y) \int_\xi^{\eta} [\phi^\varepsilon(\theta + c^*_+ y) - \phi^\varepsilon(\theta)] d\theta \right\} dy
\]

\[
+ f'(0)(1 - \epsilon_0) \sum_{j=1}^{m} \beta_j \int_0^\tau \left\{ J_{m+1+j}(y) \int_\xi^{\eta} [\phi^\varepsilon(\theta + j + c^*_+ y) - \phi^\varepsilon(\theta)] d\theta \right\} dy.
\]

(49)

Since \( \phi^\varepsilon(\xi) \) is differential and bounded on \( \mathbb{R} \), it follows that

\[
\int_\xi^{\eta} \phi^\varepsilon(\theta - i + c^*_+ y) - \phi^\varepsilon(\theta) d\theta = \int_\xi^{\eta} \int_0^{-i+c^*_+ y} (\phi^\varepsilon)'(\theta + \theta)d\theta d\theta
\]

\[
= \int_0^{-i+c^*_+ y} [\phi^\varepsilon(\eta + \theta) - \phi^\varepsilon(\xi + \theta)] d\theta
\]

is also bounded for \( i = 1, \ldots, m \) and \( y \in [0, \tau] \). Similarly, we can also obtain

\[
\int_\xi^{\eta} \phi^\varepsilon(\theta + c^*_+ y) - \phi^\varepsilon(\theta) d\theta = \int_0^{c^*_+ y} [\phi^\varepsilon(\eta + \theta) - \phi^\varepsilon(\xi + \theta)] d\theta
\]

and

\[
\int_\xi^{\eta} \phi^\varepsilon(\theta + j + c^*_+ y) - \phi^\varepsilon(\theta) d\theta = \int_0^{j+c^*_+ y} [\phi^\varepsilon(\eta + \theta) - \phi^\varepsilon(\xi + \theta)] d\theta
\]

are bounded for \( j = 1, \ldots, l \) and \( y \in [0, \tau] \). Therefore, it then follows from (49) that \( \int_\xi^{\eta} \phi^\varepsilon(\theta) d\theta \) is bounded. Since \( \int_0^{\eta} \phi^\varepsilon(\theta) d\theta \) is increasing and bounded on \( \eta \), we have

\[
\int_0^{\infty} \phi^\varepsilon(\theta) d\theta < \infty
\]

for small \( \theta \). Next, we only need to prove that \( \phi^\varepsilon(+\infty) \) exists. By (6), we obtain

\[
-c^*_+ [\phi^\varepsilon(\eta) - \phi^\varepsilon(0)]
\]

\[
= - \int_0^{\eta} \phi^\varepsilon(\theta) d\theta + \int_0^{\eta} \left[ \sum_{i=1}^{m} a_i \int_0^\tau J_i(y) f(\phi^\varepsilon(\theta - i + c^*_+ y)) dy \right] d\theta
\]

\[
+ \int_0^{\eta} \left[ \alpha \int_0^\tau J_{m+1}(y) f(\phi^\varepsilon(\theta + c^*_+ y)) dy \right] d\theta
\]

\[
+ \int_0^{\eta} \left[ \sum_{j=1}^{l} \beta_j \int_0^\tau J_{m+1+j}(y) f(\phi^\varepsilon(\theta + j + c^*_+ y)) dy \right] d\theta
\]

\[
= - \int_0^{\eta} \phi^\varepsilon(\theta) d\theta + f'(0) \sum_{i=1}^{m} a_i \int_0^\tau \left[ J_i(y) \int_0^{\eta} \phi^\varepsilon(\theta - i + c^*_+ y) d\theta \right] dy
\]

\[
+ \alpha f'(0) \int_0^\tau \left[ J_{m+1}(y) \int_0^{\eta} \phi^\varepsilon(\theta + c^*_+ y) d\theta \right] dy
\]

\[
+ f'(0) \sum_{j=1}^{l} \beta_j \int_0^\tau \left[ J_{m+1+j}(y) \int_0^{\eta} \phi^\varepsilon(\theta + j + c^*_+ y) d\theta \right] dy
\]
Obviously, \( \int_0^\eta \phi^\varepsilon(\theta)d\theta \) has the limit when \( \eta \to +\infty \). Let

\[
g_1(\eta) \triangleq \int_0^\eta \left[ J_i(y) \int_0^\eta \phi^\varepsilon(\theta - i + c_p^* y) d\theta \right] dy.
\]

It is obvious that \( g_1(\eta) \) is nondecreasing on \( \eta \). On the other hand,

\[
g_1(+\infty) = \int_0^\tau \left[ J_i(y) \int_0^+ \phi^\varepsilon(\theta - i + c_p^* y) d\theta \right] dy
\]

\[
= \int_0^\tau \left[ J_i(y) \int_{i-c_p^* y}^{+\infty} \phi^\varepsilon(\theta) d\theta \right] dy
\]

\[
= \int_0^\tau J_i(y) \left[ \int_0^{+\infty} \phi^\varepsilon(\theta) d\theta + \int_{i-c_p^* y}^{0} \phi^\varepsilon(\theta) d\theta \right] dy
\]

\[
\leq \int_0^\tau J_i(y) M_1 dy + \int_0^\tau J_i(y) b| - i + c_p^* y| dy
\]

\[
\leq M_1 + b i + c_p^* \int_0^\tau J_i(y) dy < \infty,
\]

where \( M_1 \) can be chosen such that \( \int_0^{+\infty} \phi^\varepsilon(\theta) d\theta \leq M_1 \) and \( \phi^\varepsilon(\xi) \leq b \) for all \( \xi \in \mathbb{R} \). Thus the limit of \( g_1(\eta) \) exists as \( \eta \to +\infty \). Similarly, we can prove the limits of both

\[
\int_0^\tau \left[ J_{m+1}(y) \int_0^{\eta} \phi^\varepsilon(\theta + c_p^* y) d\theta \right] dy
\]

and

\[
\int_0^\tau \left[ J_{m+1+j}(y) \int_0^{\eta} \phi^\varepsilon(\theta + j + c_p^* y) d\theta \right] dy
\]

exist as \( \eta \to +\infty \).

Let

\[
g_2(\eta) \triangleq -\int_0^\tau \left\{ J_i(y) \int_0^{\eta} [f(\phi^\varepsilon(\theta - i + c_p^* y)) - f'(0)\phi^\varepsilon(\theta - i + c_p^* y)] d\theta \right\} dy.
\]

Since \( f(u) \leq f'(0)u \) for all \( u \in [0,b] \), \( g_2(\eta) \) is nondecreasing on \( \eta \). On the other hand,

\[
g_2(+\infty) = -\int_0^\tau \left\{ J_i(y) \int_{0}^{+\infty} [f(\phi^\varepsilon(\theta - i + c_p^* y)) - f'(0)\phi^\varepsilon(\theta - i + c_p^* y)] d\theta \right\}
\]

\[
\leq \int_0^\tau \left\{ J_i(y) \int_0^{\eta} f'(0)\phi^\varepsilon(\theta - i + c_p^* y) d\theta \right\}
\]

\[
= f'(0) g_1(+\infty) < +\infty.
\]
Hence, the limit of \( g_2(\eta) \) exists as \( \eta \to +\infty \). According to the similar argument, we can obtain the limits of both

\[
\int_0^\tau \left\{ J_{m+1+j}(y) \int_0^n [f(\phi(\theta + j + c_+^*) y)) - f'(0)\phi(\theta + j + c_+^*) y)] d\theta \right\}
\]

and

\[
\int_0^\tau \left\{ J_{m+1}(y) \int_0^n [f(\phi(\theta + c_+^*) y)) - f'(0)\phi(\theta + c_+^*) y)] d\theta \right\}
\]

also exist as \( \eta \to +\infty \).

By the above argument, we can obtain \( \lim_{\eta \to +\infty} \phi(\eta) \) exists and \( \lim_{\eta \to +\infty} \phi(\eta) = 0 \) according to \( \int_0^\tau \phi(\eta)d\eta < \infty \).

For \( c < c_+^* \), let \( w_n(t) = \phi(n - ct) \). Choosing a \( \tilde{c} < c_+^* \) and by Theorem 2.1 (ii), it follows that

\[
u_\tilde{c}^* \leq \lim \inf_{t \to +\infty, c \leq \tilde{c}} w_n(t) \leq \lim \sup_{t \to +\infty, c \leq \tilde{c}} w_n(t) \leq b.
\]

Hence, letting \( \tilde{c} \in (c, c_+^*) \) and \( n = \tilde{c} t \)

\[
u_\tilde{c}^* \leq \lim \inf_{t \to +\infty} \phi((\tilde{c} - c)t) \leq \lim \sup_{t \to +\infty} \phi((\tilde{c} - c)t) \leq b.
\]

This implies that

\[
u_\tilde{c}^* \leq \lim \inf_{\xi \to +\infty} \phi(\xi) \leq \lim \sup_{\xi \to +\infty} \phi(\xi) \leq b
\]

which is a contradiction to \( \lim_{\xi \to +\infty} \phi(\xi) = 0 \).

For \( c \geq c_+^* \), let \( w_n(t) = \phi(n - ct) \) and \( \tilde{c} \in (c_+^* - 1, c_+^*) \). If, in addition, \( f(u)_u \) is strictly decreasing for \( u \in [u^*_\tilde{c}, b] \) and the condition (P) holds, it follows from Theorem 3.1 (ii) that

\[
limit_{t \to +\infty, n \leq \tilde{c} t} w_n(t) = K, \forall \tilde{c} < c_+^*.
\]

Thus, we have

\[
limit_{t \to +\infty, n \leq \tilde{c} t} w_n(t) = K
\]

which implies that \( \lim_{t \to +\infty} \omega_{c_+^*}(\omega) = K \) uniformly for \( \omega \in [c_+^* - 1, \tilde{c}] \) and \( \lim_{t \to +\infty} \phi(-(c - \omega)t) = K \) uniformly for \( \omega \in [c_+^* - 1, \tilde{c}] \). Hence, according to \( c \geq c_+^* \) and \( \omega \in [c_+^* - 1, \tilde{c}] \), it holds \( \lim_{\xi \to +\infty} \phi(\xi) = K \).

Similarly, we can prove that (ii) also holds. This completes the proof. \( \square \)

**Remark 3.3.**

(i) In [47], authors only obtained the existence of traveling wave solutions for \( c > c_+^* \). For the critical wave speed \( c = c_+^* \), the existence of traveling wave solutions was open. Theorem 3.2 answers this problem. Moreover, we can obtain the non-existence of traveling waves under the weaker conditions by using the theory of spreading speeds.

(ii) Theorems 3.1 and 3.2 imply that the spreading speed is also equal to the minimal wave speed for the non-monotone case.

(iii) For the non-monotone case, the corresponding results hold similar to Theorem 2.4.
4. Simulations. In this section, we do some numerical simulations to illustrate our analytic results. For illustration, we choose $\tau = 2, m = 2, l = 1, a_1 = 0.35, \alpha = 0.25, \beta_1 = 0.1, \beta_2 = 0.3, J(x) = \frac{e^{-x^2}}{1 + x^2}$ and output function $f(x) = xe^{1-0.5x}$. Using Propositions 2 and 3, we can numerically compute the left spreading $c_-^* = 1.9629$ and $c_+^* = 0.9356$. Take $n \in \mathbb{Z}$. Figure 1 shows the spatial spread the of the solution through the initial condition

$$w_n(\theta) = \begin{cases} 1.3, & \text{if } |n| \leq 25, \theta \in [-2, 0], \\ \frac{13}{25}(50 - |n|), & \text{if } 25 \leq |n| \leq 50, \theta \in [-2, 0], \\ 0, & \text{if } |n| \geq 50, \theta \in [-2, 0]. \end{cases}$$

The population in the drift $n_d$ and on the benthos $n_b$ spread in one direction towards downstream.

To get rightward traveling waves, we choose the initial condition as

$$w_n(\theta) = \begin{cases} 2, & \text{if } n \leq -25, \theta \in [-2, 0], \\ \frac{1}{25}(25 - n), & \text{if } |n| \leq 25, \theta \in [-2, 0], \\ 0, & \text{if } n \geq 25, \theta \in [-2, 0]. \end{cases}$$

The evolution of the solution is shown in Figure 2. Figure 3 gives the evolution of solution for the leftward case. From Figures 2 and 3, we may verify that the rightward wave speed is less than the leftward wave speed. A natural question then arises: under what A-templates constructions, the observation is true.

To observe the special case mentioned in the last section, we let $\alpha = 0.35, \beta_1 = 0, \beta_2 = 0, a = 0.4, f(x) = \frac{3}{4}xe^{1-0.5x}$. Figure 4 suggests the rightward spreading is positive, the leftward should be no more than zero, which we leave for the future investigation.

Appendix. In this Appendix, we first introduce some necessary notations and assumptions and then recall the abstract results developed in [26, 27] on spreading speeds and travelling wave solutions for abstract monotone evolution systems. As
Figure 2. The rightward traveling waves observed for $w_n(t)$ in different views.

Figure 3. The leftward traveling waves observed for $w_n(t)$ in different views.

mentioned in [27], this theory is a generalization of that in [26] to continuous-time semiflows under a weaker compactness assumption.

Let $C$ be the set of all bounded and continuous functions from $[-\tau, 0] \times \mathbb{Z}$ to $\mathbb{R}$. Clearly, any number in $\mathbb{R}$ and any element in the space $\bar{C} := C([-\tau, 0], \mathbb{R})$ can be regarded as a function in $C$. For any $r > 0$, we set $[0, r] := \{ u \in \mathbb{R} : 0 \leq u \leq r \}$ and $C_r := \{ u \in C : 0 \leq u \leq r \}$. For any $u(\theta) = \{ u(\theta, j) \}_{j \in \mathbb{Z}} \triangleq \{ u_j(\theta) \}_{j \in \mathbb{Z}}$, $v(\theta) = \{ v(\theta, j) \}_{j \in \mathbb{Z}} \triangleq \{ v_j(\theta) \}_{j \in \mathbb{Z}}$, we write $u(\theta) \geq v(\theta) (u(\theta) \gg v(\theta))$ provided $u_j(\theta) \geq v_j(\theta) (u_j(\theta) \gg v_j(\theta))$, $\forall j \in \mathbb{Z}$, $\theta \in [-\tau, 0]$, and $u > v$ provided $u \geq v$ but $u \neq v$. We equip $C$ with the compact open topology, that is, $u^m \to u$ in $C$ means that the sequence of $u_j^m$ converges to $u_j$, as $m \to \infty$, uniformly for $j$ in any compact set of $\mathbb{Z}$. Define

$$
||u|| = \sum_{k=1}^{\infty} \max_{|j| \leq k, \theta \in [-\tau, 0]} \frac{|u_j(\theta)|}{2^k}, \ \forall u \in C.
$$
Then \((C, \| \cdot \|)\) is a normed space. It follows that the topology in the metric space \((C_r, \| \cdot \|)\) is the same as the compact open topology in \(C_r\). Moreover, \(C_r\) is a complete metric space.

Recall that a family of operators \(\{Q_t\}_{t=0}^\infty\) is said to be a semiflow on a metric space \((C, d)\) provided \(Q_t\) has the following properties:

(i) \(Q_0(v) = v\), \(\forall v \in C_r\).

(ii) \(Q_{t_1}[Q_{t_2}[v]] = Q_{t_1 + t_2}[v]\), \(\forall t_1, t_2 \geq 0\), \(v \in C_r\).

(iii) \(Q(t, v) := Q_t(v)\) is continuous in \((t, v)\) on \([0, \infty) \times C_r\).

It is easy to see that the property (iii) holds if \(Q(\cdot, v)\) is continuous on \([0, \infty)\) for each \(v \in C_r\), and \(Q(t, \cdot)\) is uniformly continuous for \(t\) in bounded interval in the sense for any \(v_0 \in C_r\), bounded interval \(I\) and \(\epsilon > 0\), there exists \(\delta = \delta(v_0, I, \epsilon) > 0\) such that if \(d(v, v_0) < \delta\), then \(d(Q_t[v], Q_t[v_0]) < \epsilon\) for all \(t \in I\).

Define the translation operator \(T_y\) by \(T_y[u](j) = u(j - y)\) for any given \(y \in \mathbb{Z}\).

Let \(Q : C_K \to C_K\). In order to state the theory developed in [26, 27], we need the following assumptions on \(Q\):

(A1) \(T_y|Q[u]| = Q[T_y|u|],\) \(y \in \mathbb{Z}\).

(A2) \(Q : C_K \to C_K\) is continuous with respect to the compact open topology.

(A3) The set \(Q[C_K](0, \cdot)\) is precompact in the space \(C(\mathbb{Z}, \mathbb{R})\) equipped with compact open topology, and there is an equivalent norm \(\| \cdot \|^*\) in \(C_K\) such that for any number \(r \geq 0\), there exists \(k = k(r) \in [0, 1)\) such that for any interval \(I = [a, b]\) of the length \(r\) and any \(U \subset C_K\) with \(U(0, \cdot)\) precompact in \(C(\mathbb{Z}, \mathbb{R})\), we have \(\kappa(Q(U[I])) \leq \kappa(U[I])\), where \(\kappa\) is the Kuratowski measure of noncompactness on \(C_I\) with \((C_K, \| \cdot \|^*)\).

(A4) \(Q : C_K \to C_K\) is monotone(order preserving) in the sense that \(Q[u] \geq Q[w]\) whenever \(u \geq w\).

(A5) \(Q : [0, K] \to [0, K]\) admits exactly two fixed points 0 and \(K\), and \(\lim_{k \to \infty} Q^k[z] = K\) for any \(z \in [0, K] \setminus \{0\}\).

**Theorem A.** ([27, Theorem 3.4]) Let \(\{Q_t\}_{t=0}^\infty\) be a semiflow on \(C_K\) with \(Q_t[0] = 0, Q_t[K] = K\) for all \(t \geq 0\). Suppose that \(Q = Q_1\) satisfies all hypotheses (A1)-(A5), and let \(c^*_+\) and \(c^*_–\) be the rightward and leftward spreading speeds of \(Q_1\), respectively. Then the following statements are valid:
Theorem B. ([27, Theorems 4.3-4.4]) Suppose that for any $t > 0$, $Q_t$ satisfies hypotheses (A1)-(A5), and let $c^*_+$ and $c^*_-$ be the rightward and leftward spreading speed of $Q_t$, respectively. Then the following conclusions hold.

1. For any $c \geq c^*_+$, $\{Q_t\}_{t=0}^\infty$ has a rightward travelling wave solution $W(n-ct)$ such that $W(\xi)$ is continuous and nonincreasing on $\xi \in \mathbb{R}$, and $W(-\infty) = K$ and $W(\infty) = 0$. Moreover, for any $c < c^*_+$, $\{Q_t\}_{t=0}^\infty$ has no the rightward travelling wave $W$ connecting $K$ and 0.

2. For any $c \geq c^*_-$, $\{Q_t\}_{t=0}^\infty$ has a travelling wave solution $V(n+ct)$ such that $V(\xi)$ is continuous and nondecreasing on $\xi \in \mathbb{R}$, and $V(-\infty) = 0$ and $V(\infty) = K$. Moreover, for any $c < c^*_-$, $\{Q_t\}_{t=0}^\infty$ has no leftward travelling wave $V(n+ct)$ connecting $\theta$ and $K$.

In order to estimate the spreading speed, we introduce a linear operator $M$. Let $M : C \to C$ be a linear operator with the following properties:

(C1) $M$ is continuous with respect to the compact open topology.

(C2) $M$ is a positive operator; that is, $M[v] \geq 0$ whenever $v \geq 0$.

(C3) $M$ satisfies (A3) with $C_K$ replaced by any uniformly bounded subset of $C$.

(C4) $T_y[M[u]] = M[T_y[u]], \forall u \in C, y \in \mathbb{Z}$.

(C5) $M$ can be extended to a linear operator on the linear space $D$ of all functions $v \in C([-\tau, 0] \times \mathbb{Z}, \mathbb{R})$ have the form

$$v(\theta, j) = v_1(\theta, j) e^{\mu_1 j} + v_2(\theta, j) e^{\mu_2 j}, \ v_1, v_2 \in C, \mu_1, \mu_2 \in \mathbb{R},$$

such that $v_n, v \in D$ and $M[v](\theta, j) \to M[v](\theta, j)$ uniformly on any bounded set, then $v_n(\theta, j) \to v(\theta, j)$ uniformly on any bounded set.

(C6) For any $\chi \geq 0$, $B_\chi$ is a positive operator, and there is $n_0 \in \mathbb{N}$ such that $B_\chi^{n_0}$ is a compact and strongly positive linear operator on $\bar{C}$, where the linear map $B_\chi : \bar{C} \to \bar{C}$ by $B_\chi[\gamma](\theta) = M[\gamma e^{-\chi \cdot}](\theta, 0), \ \forall \theta \in [-\tau, 0], \gamma \in \bar{C}$.

(C7) The principal eigenvalue $\nu(0)$ of $B_0$ is large than 1.

Theorem C. ([26, Theorem 3.5]) Let $c^*_{+}$ be the rightward (leftward) spreading speed of $Q_t$. Assume that there is a sequence of linear operators $M_k$ satisfying (C1)-(C7) such that the the rightward and leftward spreading speed $c^*_{+}$ of $M_k$ converges to $c^*_{+}$ as $k \to \infty$ and that for each $k$ there is $\sigma_k > 0$ such that $M_k[w^0] \leq Q[w^0]$ for any $w^0 \in C_K$ with $w^0 \leq \sigma_k$. Then we can choose $r_\sigma$ in Theorem A to be independent of $\sigma \gg 0$.

Let $\Phi(\chi) = \frac{\ln \nu(\chi)}{\chi}, \ \chi > 0$, where $\nu(\chi)$ is the principal eigenvalue of $B_\chi$. We recall the following result on the estimate of spreading speeds in [26].
Theorem D. ([26, Theorem 3.10]) Let $Q$ be an operator on $C_K$ satisfying (A1)-(A5) and $c^\pm_1$ be the rightward (leftward) spreading speed of $Q$. Assume that the linear operator $M$ satisfies (C1)-(C7), and that the infimum of $\Phi(\chi)$ is attained at some finite value $\chi^*$ and $\Phi(\infty) > \Phi(\chi^*)$. Then the following statements are valid:

1. If $Q[w^0] \leq M[w^0]$ for all $w^0 \in C_K$, then $c^\pm_1 \leq \inf_{\chi > 0} \Phi(\chi)$.

2. If there is some real number $\eta > 0$ such that $Q[w^0] \geq M[w^0]$ for all $w^0 \in C_{\eta}$, then $c^\pm_1 \geq \inf_{\chi > 0} \Phi(\chi)$.

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