Nonnegative mean squared prediction error estimation in small area estimation

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Summary

Small area estimation has received enormous attention in recent years due to its wide range of application, particularly in policy making decisions. The variance based on direct sample size of small area estimator is unduly large and there is a need of constructing model based estimator with low mean squared prediction error (MSPE). Estimation of MSPE and in particular the bias correction of MSPE plays the central piece of small area estimation research. In this article, a new technique of bias correction for the estimated MSPE is proposed. It is shown that that the new MSPE estimator attains the same level of bias correction as the existing estimators based on straight Taylor expansion and jackknife methods. However, unlike the existing methods, the proposed estimate of MSPE is always nonnegative. Furthermore, the proposed method can be used for general two-level small area models where the variables at each level can be discrete or continuous and, in particular, be nonnormal.

Some key words: Best predictor; Bootstrap; Mean squared prediction error; small area.

1 INTRODUCTION

Small area estimation is an important statistical research area due to its growing demand from public and private agencies. The variance of a small area estimator is unduly large due to smallness of the area-level sample size. Use of models has proven to be unavoidable to control the mean squared prediction error (MSPE) of a small area predictor. The bias correction of the estimated MSPE is the central piece of small area estimation research. See Rao (2003) and
The standard small area models are usually two-level models, where one is a sampling model and the other one is a population model. Prasad and Rao (1990) assumed normality at both levels and used ANOVA estimates of the model parameters to derive second order correct MSPE estimates. Lahiri and Rao (1995) relaxed the normality assumption at the population level and re-establish the Prasad-Rao result on second order correct MSPE estimation. Datta and Lahiri (2000) investigated properties of Prasad-Rao (PR) type MSPE estimators for maximum likelihood and restricted maximum likelihood estimates of the model parameters, retaining the normal distribution assumption at both the levels. Recently, Jiang, Lahiri and Wan (2002) proposed a jackknife based MSPE estimators where normality is not a requirement. However, the Jiang-Lahiri-Wan method (JLW) requires a closed form expression for the posterior risk which is not often available (e.g., the binomial-normal model). Moreover, the JLW estimator has the undesirable property that it may produce negative MSPE estimates (Bell, 2002). Although the PR type MSPE estimates are nonnegative for the normal-normal case, the nonnegativity property is unknown for other situations. The PR type MSPE estimators correct the bias of the estimated MSPE using Taylor’s expansion. On the other hand, the JLW MSPE estimator corrects the bias of the posterior risk using the jackknifing method. In this article, we propose a new technique of MSPE bias correction which attains the same level of accuracy as that of the PR type or the JLW MSPE estimators. In addition, the new MSPE estimates are guaranteed to be nonnegative. Moreover, the new method is valid for any family of parametric distributions, discrete or continuous. Thus, unlike the traditional methods, neither the normality assumption nor the choice of a specific parameter estimation method are required for the validity of the proposed approach.

The organization of the paper is as follows: The next section introduces the two-level small area models and discusses the existing MSPE estimation methods in this framework. The Section 3 proposes the new MSPE estimator. Some technical properties of the proposed estimator are discussed and compared with the existing methods in Section 4. Section 5 reports finite sample properties of the new estimator using a simulation study. Some conclusions and
comments are made in Section 6. Proofs of the technical results are given in the Appendix.

2 Existing Methods of MSPE Estimation

Consider the two-level small area model

\begin{align}
  y_i &= \theta_i + e_i, \quad e_i \sim F(.; D_i) \\
  \theta_i &= x_i^T \beta + u_i, \quad u_i \sim G(.; \gamma)
\end{align}

\quad i = 1, \ldots, m, \text{ where } y_1, \ldots, y_m \text{ are direct estimators with sampling errors } e_1, \ldots, e_m, \text{ independently distributed with cumulative distribution functions } F(.; D_1), \ldots, F(.; D_m), \text{ respectively; } \\
\quad u_1, \ldots, u_m \text{ are independent and identically distributed (iid) random variables with common distribution function } G(.; \gamma), \text{ and } x_1, \ldots, x_m \text{ are } p\text{-dimensional nonrandom covariates. We suppose that the unknown parameters of the model are given by the regression parameter } \beta \text{ and the } p\text{-dimensional parameter } \gamma \text{ of the random effects distribution } G(.; \gamma) \text{ in (2.2), but the values of } D_1, \ldots, D_m \text{ are known, as typically assumed in two-level small area models. We also assume that the sampling errors } e_1, \ldots, e_m \text{ and random effects } u_1, \ldots, u_m \text{ are mutually independent with } E(e_i) = 0 = E(u_i), i = 1, \ldots, m. \text{ Note that neither the } e_i\text{'s nor the } u_i\text{'s are required to be normally distributed. In fact, under (2.1) and (2.2), the } e_i\text{'s and the } u_i\text{'s are allowed to have arbitrary parametric families of discrete or continuous distributions.}

Suppose that the quantity of interest for prediction is given by

\[ h(\theta_i), \quad i = 1, \ldots, m \]

for some smooth function \( h : \mathbb{R} \to \mathbb{R} \). For example, \( h(x) = x \) is the most commonly used function, which may correspond to area level means or totals. An important example of \( h(.) \) includes exponentiation in U.S. Census Bureau’s ongoing Small Area Income and Poverty Estimation (SAIPE) project. For county level poverty estimation in SAIPE, the model (2.1) and (2.2) applies after log transformation of the original data.
The best predictor (BP) of \( h(\theta_i) \) is given by

\[
H_i(\delta) \equiv E_{\delta}(h(\theta_i)|y_i), \quad i = 1, \ldots, m
\]  

(2.4)

where \( \delta = (\beta^T, \gamma^T)^T \) is the vector of model parameters. Since the true value of \( \delta \) is unknown, \( H_i(\delta) \) is not directly usable in practice. It is customary to substitute an estimator \( \hat{\delta} \), say, of \( \delta \) and predict \( h(\theta_i) \) by using the estimated best predictor (EBP) as

\[
H_i(\hat{\delta}), \quad i = 1, \ldots, m.
\]  

(2.5)

Performance of the EBP is measured by the mean squared prediction error (MSPE):

\[
M_i(\delta) = E_{\delta} \left( H_i(\hat{\delta}) - h(\theta_i) \right)^2, \quad i = 1, \ldots, m.
\]  

(2.6)

Further, like the EBP, an estimator of the MSPE is obtained by \( M_i(\hat{\delta}), \quad i = 1, \ldots, m \). However, as pointed out by Prasad and Rao (1990) in their seminal paper, this naive plug-in estimator is not very useful. To appreciate why, note that \( M_i(\delta) \) can be decomposed as

\[
M_i(\delta) = E_{\delta} \left( H_i(\hat{\delta}) - h(\theta_i) \right)^2 = E_{\delta} \left( H_i(\hat{\delta}) - h(\theta_i) \right)^2 + E_{\delta} \left( H_i(\hat{\delta}) - H_i(\delta) \right)^2
\]  

\[
\equiv M_{1i}(\delta) + M_{2i}(\delta), \quad i = 1, \ldots, m
\]  

(2.7)

where the cross-product term vanishes as a consequence of the fact that \( E_{\delta} (H_i(\delta) - h(\theta_i)) Z = 0 \) for any \( \sigma(y_1, \ldots, y_m) \)-measurable random variable \( Z \). In (2.7), the first term \( M_{1i}(\delta) \) is the optimal prediction error using the unknown ideal predictor \( H_i(\delta) \) and is of order \( O(1) \) as \( m \to \infty \). The second term \( M_{2i}(\delta) \) arises from the error in estimating the unknown model parameters \( \delta \) in the BP \( H_i(\delta) \), and, typically, it is of the order \( O(m^{-1}) \) as \( m \to \infty \). Prasad
and Rao (1990) showed that by substituting $\hat{\delta}$ for $\delta$ to define the naive plug-in estimator

$$M_i(\hat{\delta}) = M_{1i}(\hat{\delta}) + M_{2i}(\hat{\delta}),$$

one introduces an additional bias of the order $O(m^{-1})$, which is of the same order as the second term $M_{2i}(\delta)$ in (2.7). As a result, the naive estimator has a masking effect on the bias of the EBP and hence, it is not a good estimator of $M_i(\delta)$, particularly when $m$ is not too large.

Prasad and Rao (1990) suggested a bias corrected estimator of the MSPE $M_i(\delta)$ for a normal-normal model. The key idea there is to estimate the (leading term of the) bias of $M_i(\hat{\delta})$ using explicit analytical expressions. The bias corrected estimated MSPE, proposed by Prasad and Rao (1990), is of the form

$$\hat{M}_i^{PR} = M_{1i}(\hat{\delta}) - \hat{\text{Bias}}_{i}^{PR} + M_{2i}(\hat{\delta}),$$

(2.8)

where $\hat{\text{Bias}}_{i}^{PR}$ is obtained by estimating higher order terms in the Taylor’s expansion of the function $M_{1i}(\cdot)$ around $\delta$.

An alternative approach, put forward by Jiang, Lahiri and Wan (2002), involves using the jackknife method to correct the $O(m^{-1})$-order bias term in the naive estimator $M_i(\hat{\delta})$. More specifically, the bias-corrected estimator of $M_i(\delta)$ of JLW is given by

$$\hat{M}_i^{JLW} = M_{1i}(\hat{\delta}) - \hat{\text{Bias}}_{i}^{JLW} + M_{2i}(\hat{\delta}),$$

(2.9)

where $\hat{\text{Bias}}_{i}^{JLW}$ is the Jackknife estimator of the bias of $M_{1i}(\cdot)$.

Although, the estimators $\hat{M}_i^{PR}$ and $\hat{M}_i^{JLW}$ have superior bias properties, an undesirable feature of both of these estimators is that they may produce negative MSPE estimate with positive probabilities. This results from the sampling variability of the bias estimators, which may dominate the value of the unadjusted naive estimator $M_i(\hat{\delta})$ and thereby, may lead to a negative value of the bias corrected MSPE estimators.

In this article, we propose a different approach to bias correction that is guaranteed to
produce a nonnegative estimate of the MSPE. The key idea here is to tilt suitably the value of \( \hat{\delta} \), an initial estimator of \( \delta \), before evaluating the function \( M_i(\cdot) \), such that the difference between the true MSPE \( M_i(\delta) \) and the value of the function \( M_i(\cdot) \) at the new value of the argument, say \( \tilde{\delta} \), is smaller on the average. Since the MSPE function \( M_i(\cdot) \) is always nonnegative, the resulting estimator of the true MSPE is always nonnegative. The tilted value \( \tilde{\delta} \) is constructed from \( \hat{\delta} \) using the data-values only and hence it is itself an estimator of \( \delta \). In constructing \( \tilde{\delta} \), we implicitly correct the bias of \( M_i(\hat{\delta}) \), by making use of estimates of linear combination of the bias and the variance of the initial estimator \( \hat{\delta} \). Here, we employ the bootstrap method (Efron, 1979) to derive the bias and variance of the estimators of model parameters, although other methods such as the jackknife and the delta methods, are equally applicable. The details of the correct construction are given in the next section.

3 The Proposed Estimator of the MSPE

3.1 Motivation

To motivate the definition of the proposed MSPE estimator, consider a related deterministic approximation problem, where we wish to approximate the value of a smooth function \( f: \mathbb{R} \to \mathbb{R} \) at a point \( a \in \mathbb{R} \) using its values over an interval \( I \) containing \( a \). For a given \( c \neq 0 \), setting \( x_m = a + \frac{c}{\sqrt{m}}, \, m \geq 1 \) and using Taylor’s expansion, we get

\[
 f(x_m) = f(a) + (x_m - a)f'(a) + \frac{1}{2}(x_m - a)^2 f''(a) + O(m^{-\frac{3}{2}}). \tag{3.1}
\]

This suggests that starting with \( x_m \), we may now construct a new point \( \tilde{x}_m \in I_m \) of the form \( \tilde{x}_m = x_m + cm, \) such that

\[
 f(\tilde{x}_m) = f(a) + O(m^{-\frac{3}{2}}). \tag{3.2}
\]

Indeed, by Taylor’s expansion of \( f(\tilde{x}_m) \) around \( a \), we have

\[
 f(\tilde{x}_m) = f(a) + (x_m + cm - a)f'(a) + \frac{1}{2}(x_m + cm - a)^2 f''(a) + O(m^{-\frac{3}{2}}), \tag{3.3}
\]
which satisfies (3.2) if
\[(x_m + c_m - a)f'(a) + \frac{1}{2}(x_m + c_m - a)^2 f''(a) = 0.\] (3.4)

Now equation (3.4) can be solved for \(c_m\) (yielding the solution \(c_m = -\frac{2f'(a)}{f''(a)} - (x_m - a)\)) to find the desired point \(\tilde{x}_m\). In deriving the proposed MSPE estimator, we employ an extension of this simple idea to the function \(f(\cdot) = M_{1i}(\cdot)\) which is now a function (of several real variables) from \(\mathbb{R}^k \rightarrow \mathbb{R}\). The role of the point \(x_m\) is played by an initial estimator \(\hat{\delta}\). Some additional care is needed to ensure that the analog of the tilted point \(\tilde{x}_m\), now denoted by \(\tilde{\delta}\), is truly an estimator, i.e., a function of the data alone and does not involve any parameters (e.g., it may not involve the point “\(a\)” in \(\tilde{x}_m\), which represents the true parameter value \(\delta\) in our application).

3.2 Definition of the proposed estimator

Let \(\hat{\delta}\) be a given estimator of \(\delta\) and let \(b = b(\hat{\delta}) = E_{\hat{\delta}}(\hat{\delta} - \delta)\) denote the bias and \(V = V(\delta) = \text{Var}_{\hat{\delta}}(\hat{\delta})\) denote the variance matrix of \(\hat{\delta}\) at \(\delta\). We shall suppose that some consistent estimators \(\hat{b}\) and \(\hat{V}\) of the bias and the variance matrix of the initial estimator \(\hat{\delta}\) are available. For example, these may be generated by a suitable resampling method; see Section 5 where we use a parametric bootstrap method for this purpose. To define the tilted estimator of \(\delta\), we also suppose that for \(i = 1, \ldots, m\)
\[
\sum_{j=1}^{k} |M^{(j)}_{1i}(\delta)| \neq 0, \tag{3.5}
\]
where for a differentiable function \(f : \mathbb{R}^k \rightarrow \mathbb{R}\), \(f^{(j)}\) and \(f^{(jl)}\) denote the first and the second order partial derivatives with respect to the \(j\)-th co-ordinate and the \((j,l)\)-th co-ordinates, respectively, \(j,l = 1, \ldots, k\). Condition (3.5) says that at least one of the first order partial derivatives of the function \(M_{1i}(\cdot)\) is nonzero at the true value of the parameter \(\delta\) for each \(i\). For notational simplicity, without loss of generality, we suppose that \(M^{(1)}_{1i}(\delta) \neq 0\). Then, we define
the preliminary-tilted-estimator of $\delta$ for the $i$-th small area by

$$
\bar{\delta}_i = \delta - \left[ \sum_{j=1}^{k} M_{1i}^{(j)}(\delta) \hat{b}(j) + \frac{1}{2} \sum_{j=1}^{k} \sum_{l=1}^{k} M_{1i}^{(jl)}(\delta) \hat{V}(j,l) \right] \left\{ M_{1i}^{(1)}(\delta) \right\}^{-1} e_1
$$

(3.6)

where $e_1 = (1, 0, \cdots, 0)^T \in \mathbb{R}^k$, $\hat{b}(j)$ denote the $j$-th component of $\hat{b}$ and $\hat{V}(j,l)$ denote the $(j,l)$-th element of $\hat{V}$. Thus, the estimator $\bar{\delta}_i$ is obtained from the initial estimator $\hat{\delta}$ by adding a correction factor to the first component of $\hat{\delta}$ only. Note that if instead of $M_{1i}^{(1)}(\delta)$, a different partial derivative $M_{1i}^{(l)}(\delta)$ were nonzero, then we would define the preliminary tilted estimator $\bar{\delta}_i$ by replacing the factor $\left\{ M_{1i}^{(1)}(\delta) \right\}^{-1} e_1$ in (3.6) with $\left\{ M_{1i}^{(l)}(\delta) \right\}^{-1} e_l$, where the vector $e_l \in \mathbb{R}^k$ has 1 in the $l$-th position and zeros elsewhere, $1 \leq l \leq k$.

Next, let $\Delta$ denote the set of possible values of the parameter $\delta$ under the model (2.1) and (2.2). Then the tilted estimator of $\delta$ for the $i$-th small area is defined by

$$
\tilde{\delta}_i = \begin{cases} 
\bar{\delta}_i & \text{if } \bar{\delta}_i \in \Delta \text{ and } |M_{1i}^{(1)}(\delta)|^{-1} \leq (1 + \log m)^2 \\
\hat{\delta} & \text{otherwise}
\end{cases}
$$

(3.7)

$i = 1, \cdots, m$. Thus, if the preliminary estimator $\bar{\delta}_i$ takes values inside the parameter space $\Delta$ and the value of the partial derivative $M_{1i}^{(1)}(\delta)$ at $\hat{\delta}$ is not too small, the tilted estimator of $\delta$ is given by $\bar{\delta}_i$ itself. However, in the event that either $\bar{\delta}_i$ falls outside $\Delta$ or $M_{1i}^{(1)}(\delta)$ becomes too small, we replace it with the original estimator $\hat{\delta}$. Small values of $M_{1i}^{(1)}(\delta)$ make the estimator $\bar{\delta}_i$ unstable and hence, truncated below. It will be shown in Section 4 that under appropriate regularity conditions, the probability of getting a preliminary estimator $\bar{\delta}_i$ outside $\Delta$ or that of getting a value of $M_{1i}^{(1)}(\delta)$ below the threshold $(1 + \log m)^{-2}$ tends to zero rapidly as $m \to \infty$, uniformly in $i$. As a consequence, the tilted estimator $\tilde{\delta}_i$ coincides with the preliminary tilted estimator $\bar{\delta}_i$ with high probability. The proposed estimator of the MSPE is now defined as

$$
\widehat{M_i}(\delta) = M_{1i}(\delta) + M_{2i}(\delta), i = 1, \cdots, m.
$$

(3.8)

Note that by the construction, the MSPE estimator is always positive. In the next section, we
show that under some regularity conditions, it has a bias that is of the order $o(m^{-1})$. Therefore, the proposed estimator attains the same level of accuracy as the previously proposed estimators $\hat{M}_i^{PR}$ and $\hat{M}_i^{JLW}$, while at the same time, guarantees positivity.

4 Theoretical Properties of the Proposed Estimators

In this section, we describe some theoretical properties of the tilted estimator $\tilde{\delta}_i$ of (3.7) and of the bias corrected MSPE estimator $\hat{M}_i(\delta)$ of (3.8). For proving the result of this section, we shall assume the following regularity conditions on the model (2.1) and (2.2).

Condition S:

1. $\delta$, the true value of the parameter, is an interior point of $\Delta$.

2. $M_{1i}$ is twice continuously differentiable on $\Delta$ and there exists a constant $C_1 \in (0, \infty)$ such that

$$|M_{1i}^{(j)}(x)| + |M_{1i}^{(jl)}(x)| < C_1$$

for all $x \in \Delta, j, l = 1, \cdots, k$ and $i = 1, \cdots, m, m \geq 1$.

3. (i) $M_{2i}$ is differentiable on $\Delta$.
   (ii) There exist $C_2, \epsilon_0 \in (0, \infty)$ and $\gamma \in (0, 1]$ such that

$$|M_{2i}^{(j)}(x) - M_{2i}^{(jl)}(\delta)| + m|M_{2i}^{(jl)}(x) - M_{2i}^{(jl)}(\delta)| < C_2\|x - \delta\|^\gamma$$

for all $x \in \mathcal{N} \equiv \{\|x - \delta\| \leq \epsilon_0\}$ for $j, l = 1, \cdots, k; i = 1, \cdots, m, m \geq 1$.
   (iii) There exist a constant $C_3 \in (0, \infty)$ and a function $G : \mathbb{R}^k \rightarrow [0, \infty)$ with $EG(\hat{\delta}) < \infty$ such that

$$|M_{2i}(x)| \leq m^{-1}G(x) \quad \text{for all} \quad x \in \Delta,$$

and

$$|M_{2i}^{(jl)}(\delta)| \leq C_3m^{-1},$$
for all \( j = 1, \ldots, k; i = 1, \ldots, m, m \geq 1. \)

We now briefly comment on the regularity condition S. Condition S requires the functions \( M_{1i} \) and \( M_{2i} \) to be smooth, which typically holds under suitable smoothness conditions on the parametric model (2.1) and (2.2). As mentioned earlier, in most applications the function \( M_{1i} \) is of the order \( O(1) \) while \( M_{2i} \) is of the order \( O(m^{-1}) \) as \( m \to \infty \). Condition S requires that the partial derivatives of these functions also have the same orders, respectively. Condition S.3(iii) is a local Lipschitz condition of order \( \eta \in (0,1] \) on \( M_{1i} \) and \( M_{2i} \). This condition holds with \( \eta = 1 \) if \( M_{1i} \) is three-times continuously differentiable and \( M_{2i} \) two-times continuously differentiable on a neighborhood of the true parameter value \( \delta \).

Next, suppose that the bias and the variance matrix of the given estimator \( \hat{\delta} \) are of the form:

\[
\begin{align*}
b &\equiv E(\hat{\delta} - \delta) = \frac{a}{m} + o\left(\frac{1}{m}\right) \quad \text{as} \quad m \to \infty \quad (4.1) \\
V &\equiv Var(\hat{\delta}) = \frac{\Sigma}{m} + o\left(\frac{1}{m}\right) \quad \text{as} \quad m \to \infty \quad (4.2)
\end{align*}
\]

Let \( \hat{a} \) and \( \hat{\Sigma} \) be estimators of the parameters \( a \) and \( \Sigma \) in (4.1) and (4.2) respectively, such that for some \( \eta \in (0,1] \),

\[
\begin{align*}
E\|\hat{a} - a\|^{1+\eta} &= o(1) \quad \text{as} \quad m \to \infty \quad (4.3) \\
E\|\hat{\Sigma} - \Sigma\|^{1+\eta} &= o(1) \quad \text{as} \quad m \to \infty \quad (4.4)
\end{align*}
\]

Note that in the notation of Section 3, the quantities \( \hat{b}, \hat{a}, \hat{V} \) and \( \hat{\Sigma} \) are related as \( \hat{b} = m^{-1}\hat{a} \) and \( \hat{V} = m^{-1}\hat{\Sigma} \).

With this, we are now ready to state the main results of this section. The first result shows that the preliminary-titled-estimators \( \bar{\delta}_i \) converge to the true parameter \( \delta \) in probability uniformly in \( i = 1, \ldots, m \), and also that the first order partial derivative \( M_{1i}^{(1)}(\hat{\delta}) \) falls below the given threshold \( (1 + \log m)^{-2} \) with very small probability, uniformly in \( i = 1, \ldots, m \).

**Theorem 1:** Suppose that (4.1)-(4.4) and condition S hold. Then
(i) For any $\epsilon \in (0, \infty)$,
\[
\max_{1 \leq i \leq m} P\left( \|\bar{\delta}_i - \delta\| > \epsilon \right) = O(m^{-1}) \quad \text{as} \quad m \to \infty.
\]

(ii) As $m \to \infty$,
\[
\max_{1 \leq i \leq m} P\left( |M_i^{(1)}(\hat{\delta})| < (1 + \log m)^{-2} \right) = O(m^{-1}).
\]

**Proof:** A proof of the theorem is given in the Appendix.

As a direct consequence of the above result, we get the following.

**Theorem 2:** Under the conditions of Theorem 1,
\[
\max_{1 \leq i \leq m} P\left( \bar{\delta}_i = \bar{\delta}_i \right) = 1 - O(m^{-1}) \quad \text{as} \quad m \to \infty.
\]

**Proof:** A proof of the theorem is given in the Appendix.

Theorem 2 shows that uniformly in $i$, the titled estimator $\bar{\delta}_i$ coincides with the preliminary-titled-estimator $\bar{\delta}_i$ with high probability when $m$ is large. Thus, the typical value of the titled estimator has a correction term added to the first component of the given initial estimator $\hat{\delta}$ (cf. (3.6)). The next result shows that this correction factor indeed reduces the bias of the proposed MSPE estimator $\hat{M}_i(\delta)$ to order $o(m^{-1})$, as desired.

**Theorem 3:** Suppose that (4.1)-(4.4) and condition S hold. Further suppose that
\[
\left\{ \|\sqrt{m}(\hat{\delta} - \delta)\|^2 \right\}_{m \geq 1}
\]

is uniformly integrable. Then
\[
\max_{1 \leq i \leq m} |EM_i(\hat{\delta}) - M_i(\delta)| = o(m^{-1}) \quad \text{as} \quad m \to \infty.
\]
Proof: A proof of the theorem is given in the Appendix.

5 Simulation Study

We conduct a small simulation study to check small sample performance of our proposed MSPE estimator and compare it with its competitors. In order to mimic a real life study, we consider the example in Battese, Harter and Fuller (1988) to estimate the area under corn and soybeans for twelve counties of north-central Iowa. Originally, Battese et al. (1988) applied a nested error regression model. We consider here the area level version of their model for simplicity and we think that this is adequate for illustration purposes. Let $y_{ij}$ be the area under corn for $j$-th segment in $i$-th county and let $\bar{X}_i$ be the (population) average number of pixels classified as corn in the $i$-th county. We consider the area level model as

$$\bar{y}_i = \beta_0 + \beta_1 \bar{X}_i + u_i + e_i, \quad i = 1, \ldots, m \quad (5.1)$$

where $\bar{y}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} y_{ij}$ = the sample average area under corn in the $i$-th county. Here, $u_i$’s are independently distributed with each following the $N(0, \sigma^2_u)$ distribution and the $e_i$’s are independent with $e_i \sim N(0, D_i)$ for $i = 1, \ldots, m$ where $D_i = \frac{\sigma^2_e}{n_i}$. Further, the $u_i$’s and the $e_i$’s are independent. In our simulation, we take $\beta_0 = 43.00, \beta_1 = 0.25, \sigma^2_u = 140.00, \sigma^2_e = 147.00$. The $n_i$’s are as given in Battese et al. (1988) with $\min_{1 \leq i \leq m} n_i = 1, \max_{1 \leq i \leq m} n_i = 6$ and $m = 12$. For the simulation study, we generated $R = 20,000$ sets of samples using model (5.1) and computed $\hat{\delta} = (\hat{\beta}_0, \hat{\beta}_1, \hat{\sigma}_u)^T$ each time.

For estimating the bias and the variance of the estimator vector $\hat{\delta}$ used in the definition of the titled estimators $\tilde{\delta}_i$’s, we employed a parametric bootstrap method. For the sake of completeness, here we briefly point out the main steps of the bootstrap procedure.

- Step (I): Generate independent random variables $\{e^*_i\}_{i=1}^m$ and $\{u^*_i\}_{i=1}^m$ with $e^*_i \sim N(0, D_i)$ and $u^*_i \sim N(0, \hat{\sigma}^2_u)$. 

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• Step (II): Define the bootstrap variables, \( y^*_i = \theta^*_i + e^*_i; \quad \theta^*_i = x^T_i \hat{\beta} + u^*_i; \quad i = 1, \ldots, m. \)

• Step (III): Define the bootstrap version \( \delta^*_i \) of \( \hat{\delta} \) by replacing \( y_1, \ldots, y_m \) in \( \hat{\delta} \) with \( y^*_1, \ldots, y^*_m. \)

The bootstrap estimators of the bias and the variance matrix of \( \hat{\delta} \) are now given by

\[
\hat{b} = E_\ast \delta^* - \hat{\delta} \tag{5.2}
\]

\[
\hat{V} = E_\ast (\delta^* - E_\ast \delta^*)(\delta^* - E_\ast \delta^*)^T \tag{5.3}
\]

where \( E_\ast \) denote the conditional expectation given the data. In simulation, Steps (I)-(III) are repeated a large number of times and the average of the bootstrap versions \( \delta^*_i \)'s gives the Monte-Carlo approximation to \( E_\ast \delta^* \) while the sample covariance matrix of the \( \delta^*_i \)'s give the numerical value of the right side of (5.3).

Next for each of the three MSPE estimators (namely, the Prasad-Rao estimator \( \hat{M}^{PR}_i \), the Jiang, Lahiri and Wan estimator \( \hat{M}^{JLW}_i \), and the proposed estimator \( \hat{M}_i(\delta) \)) of the small area parameter \( \theta_i \), we calculate the following measures:

• Relative bias with respect to the empirical MSPE:

\[
RB_i = \frac{E\{\hat{MSPE}(\hat{\theta}_i)\} - \text{SMSE}(\hat{\theta}_i)}{\text{SMSE}(\hat{\theta}_i)}, \quad i = 1, \ldots, 12
\]

where, \( E\{\hat{MSPE}(\hat{\theta}_i)\} = \frac{1}{R} \sum_{r=1}^{R} \hat{MSPE}(\hat{\theta}_i)^{(r)} \).

• Empirical coefficient of variation:

\[
CV_i = \frac{E\{\hat{MSPE}(\hat{\theta}_i) - \text{SMSE}(\hat{\theta}_i)\}^2}{\text{SMSE}(\hat{\theta}_i)}, \quad i = 1, \ldots, 12
\]

where, \( E\{\hat{MSPE}(\hat{\theta}_i) - \text{SMSE}(\hat{\theta}_i)\}^2 = \frac{1}{R} \sum_{r=1}^{R} (\hat{MSPE}(\hat{\theta}_i)^{(r)} - \text{SMSE}(\hat{\theta}_i))^2 \).

Table 1 reports a summary result of the simulation study. The proposed estimator is denoted as ‘New’ in the table.
Table 1: Summary of simulation study

(a) Relative Bias

|     | min | Q₁   | median | mean  | Q₃   | max  |
|-----|-----|------|--------|-------|------|------|
| PR  | -.164 | -.114 | -.055  | -.061 | -.025 | .048 |
| JLW | -.210 | -.142 | -.067  | -.095 | -.040 | -.022|
| New | -.163 | -.113 | -.054  | -.060 | -.024 | .048 |

(b) Empirical CV

|     | min | Q₁   | median | mean  | Q₃   | max  |
|-----|-----|------|--------|-------|------|------|
| PR  | .010 | .033 | .055   | .074  | .114 | .164 |
| JLW | .082 | .094 | .120   | .132  | .151 | .212 |
| New | .009 | .034 | .054   | .074  | .113 | .163 |

From the above table, it is clear that the proposed estimator and the PR estimator performs at par and both perform better than the jackknife-based estimator, particularly in terms of the coefficient of variation. We should also mention that, in this simulation study, fortunately the jackknife method did not produce any negative MSPE estimates. This is perhaps due to the fact that the true parameter values are far away from the boundary of the parameter space.

6 Conclusions

In this paper, we described a new method of bias correction for the naive ‘plug-in’ estimator of the MSPE of a function of the small area means \( h(\theta_i), i = 1, \ldots, m \). Unlike the existing methods which may produce a negative estimate of the MSPE with positive probability, the estimates of the MSPE produced by the proposed method is always nonnegative. Theoretical properties of the method are investigated, which in particular show that the resulting estimator of the
MSPE attains the same level of accuracy as the existing methods in correcting the bias of the naive MSPE estimator. Further, the numerical results presented in the paper shows that the proposed method performs at per with the Prasad-Rao (1990) method, and has a slightly better performance compared to the estimator based on the jackknife method. A key difference of the new method with the existing methods is that while the existing methods apply various bias correction techniques to the MSPE function itself, the new method reduces the bias implicitly by suitably tilting the value of argument of the MSPE function.

**APPENDIX**

For a vector $x \in \mathbb{R}^k$, let $x(j)$ denote the $j$th component of $x$, $j = 1, \ldots, k$. Let $C, C(\cdot)$ denote generic positive constants that may depend on the argument(s) (if any) but not on $i = 1, \ldots, m$ or $m$. Also, unless explicitly specified, limits in order symbols are taken letting $m \to \infty$.

**Proof of Theorem 1:** Since $M_{1i}(\delta) \neq 0$, by condition S.3.(ii), there exists $\epsilon_1, \epsilon_2 \in (0, \epsilon_0)$ such that $|M_{1i}(x)| > \epsilon_2$ for all $x$ with $\|x - \delta\| \leq \epsilon_1$. Hence, again by condition S, there exists a constant $C = C(\epsilon_2) \in (0, \infty)$ such that on the set $\{\|\hat{\delta} - \delta\| \leq \epsilon_1,$

$$\|\hat{\delta}_i - \delta\| \leq C[\|\hat{b}\| + \|\hat{V}\|]$$

uniformly in $i = 1, \ldots, m$, $m \geq 1$. Hence, by Chebychev’s inequality, for any $\epsilon \in (0, \infty)$,

$$\max_{i=1,\ldots,m} P\left(\|\hat{\delta}_i - \delta\| > \epsilon\right) \leq P\left(\|\hat{\delta} - \delta\| > \epsilon_1\right) + P\left(C[\|\hat{b}\| + \|\hat{V}\|] > \epsilon\right) \leq \epsilon_1^{-2}E\|\hat{\delta} - \delta\|^2 + \epsilon^{-1}CE\left[\|\hat{b}\| + \|\hat{V}\|\right]$$
This proves part (i). For part (ii), note that

\[
\max_{i=1,\ldots,m} P\left( |M_{1i}^{(1)}(\hat{\delta})| \leq (1 + \log m)^{-2} \right)
\leq P\left( |M_{1i}^{(1)}(\hat{\delta}) - M_{1i}^{(1)}(\delta)| > \frac{\epsilon_2}{2} \right)
\leq P\left( \|\hat{\delta} - \delta\| > \epsilon_1 \right) + P\left( C\|\hat{\delta} - \delta\|^{\gamma} > \frac{\epsilon_2}{2} \right)
= O(m^{-1}).
\]

**Proof of Theorem 2:** Since \( \delta \) is an interior point of \( \Delta \), there exists a \( \epsilon_3 \in (0, \epsilon_0) \) such that \( \{x : \|\delta - x\| \leq \epsilon_3\} \subset \Delta \). Hence, by Theorem 1,

\[
\max_{i=1,\ldots,m} P\left( \hat{\delta}_i \neq \delta \right)
\leq P(\hat{\delta} \notin \Delta) + \max_{i=1,\ldots,m} P\left( |M_{1i}^{(1)}(\hat{\delta})| \leq (1 + \log m)^{-2} \right)
\leq P\left( \|\hat{\delta} - \delta\| > \epsilon_3 \right) + O(m^{-1})
= O(m^{-1}),
\]

where the last step follows by an application of Chebychev’s inequality as in the proof of Theorem 1 above. This proves Theorem 2.

**Proof of Theorem 3:** By Taylor’s expansion and condition S, on the set \( \{\hat{\delta} \in \mathcal{N}\} \),

\[
M_{2i}(\hat{\delta}) = M_{2i}(\delta) + (\hat{\delta} - \delta)^T \nabla M_{2i}(\delta) + R_{1i}
\]

where \( \nabla M_{2i}(\cdot) \) is the \( k \times 1 \) vector of first order partial derivatives of \( M_{2i} \) and \( R_{1i} \) is a remainder term. By condition S, \( R_{1i} \) admits the bound

\[
|R_{1i}| \leq \|\hat{\delta} - \delta\| \|\nabla M_{2i}(\delta) - \nabla M_{2i}(\delta^{0})\| \leq Cm^{-1}\|\hat{\delta} - \delta\|^{1+\gamma}
\]

(A.1)
uniformly in \( i = 1, \cdots, m, m \geq 1 \) where \( \delta^0 \) is a point on the line joining \( \hat{\delta} \) and \( \delta \), so that 
\[
\| \delta^0 - \delta \| \leq \| \hat{\delta} - \delta \|. 
\]
Hence, by (3.1), (3.2), (A.1), (A.2) and the dominated convergence theorem (DCT),

\[
\begin{align*}
\max_{1 \leq i \leq m} |EM_{2i}(\hat{\delta}) - M_{2i}(\delta)| & \leq \max_{1 \leq i \leq m} |E\{M_{2i}(\hat{\delta}) - M_{2i}(\delta)\}| + \max_{1 \leq i \leq m} E\{M_{2i}^{(\hat{\delta})} + M_{2i}(\delta)\} \mathbb{1}(\hat{\delta} \notin \mathcal{N}) \\
& \leq \max_{1 \leq i \leq m} \left[ \|E(\hat{\delta} - \delta)\|_{\mathcal{N}} \cdot \|\nabla M_{2i}(\delta)\| + E|R_{1i}| \mathbb{1}(\hat{\delta} \in \mathcal{N}) \right] \\
& \quad + m^{-1} E\{G(\hat{\delta}) + G(\delta)\} \mathbb{1}(\hat{\delta} \notin \mathcal{N}) \\
& \leq \max_{1 \leq i \leq m} \left[ \left\{ \|E(\hat{\delta} - \delta)\| + m^{-2} \|\nabla M_{2i}(\delta)\| + Cm^{-1} \|\hat{\delta} - \delta\|^{1+\gamma} \right\} + Cm^{-1} \left( P(\hat{\delta} \notin \mathcal{N}) + EG(\hat{\delta}) \right) \mathbb{1}(\hat{\delta} \notin \mathcal{N}) \right] \\
& \leq C \left[ m^{-2} + m^{-1} \left( P(\hat{\delta} \notin \mathcal{N}) \right) \right] + m^{-1} \left( E\|\hat{\delta} - \delta\|^2 \right)^{\frac{1+\gamma}{2}} + o(m^{-1}) \\
& = o(m^{-1}), 
\end{align*}
\]

as \( P(\hat{\delta} \notin \mathcal{N}) \leq \epsilon_0^{-2} E\|\hat{\delta} - \delta\|^2 = O(m^{-1}) \). Without loss of generality, suppose that \( \epsilon_0 \) (in the definition of \( \mathcal{N} \)) is small enough so that for some \( C \in (0, \infty) \), \( \sup\{|M_{1i}^{(j)}(x)|^{-1} : x \in \mathcal{N}, j, l = 1, \cdots, k; i = 1, \cdots, m\} < C \). Let

\[
\begin{align*}
\hat{L}_i &= \sum_{j=1}^{k} M_{1i}^{(j)}(\delta) \hat{b}(j) + \sum_{j=1}^{k} \sum_{l=1}^{k} c(j, l) M_{1i}^{(j,l)}(\delta) \hat{V}(j, l) \\
\tilde{L}_i &= \sum_{j=1}^{k} M_{1i}^{(j)}(\delta) \tilde{b}(j) + \sum_{j=1}^{k} \sum_{l=1}^{k} c(j, l) M_{1i}^{(j,l)}(\delta) \tilde{V}(j, l),
\end{align*}
\]
i = 1, \cdots, m, \text{ where } c(j, l) = 1/2 \text{ for } j \neq l \text{ and } c(j, l) = 1 \text{ for } j = l. \text{ Then by Taylor’s expansion, it follows that there exists a constants } C \in (0, \infty) \text{ (not depending on } i) \text{ such that on the set } \{\delta \in \mathcal{N}\},

\[
\hat{L}_i = \tilde{L}_i + R_{2i}, \text{ say}
\]

(A.4)
\[ \delta_i = \delta - \frac{L_i}{M^{(1)}_i(\delta)} e_1 + R_{3i}e_1 \quad (A.5) \]

for \( i = 1, \ldots, m \) where \( \max_{1 \leq i \leq m} |R_{2i}| \leq C \left\{ \| \hat{b} \|, \| \hat{\delta} - \delta \| + \| \hat{\delta} - \delta \|^\gamma \| \hat{V} \| \right\} \) and for all \( i = 1, \ldots, m, \)

\[ |R_{3i}| \leq C \left\{ |\hat{L}_i|, \| \hat{\delta} - \delta \| + |R_{2i}| \right\}. \quad (A.6) \]

By similar arguments, on the set \( A_{1i} = \{ \hat{\delta} \in N \} \cap \{ \delta_i \in \Delta \} \), we may write

\[
\begin{aligned}
&\sum_{j=1}^{k} \sum_{l=1}^{k} c(j, l) M^{(j)}_{1i}(u\delta_i + (1 - u)\delta)(\delta_i(j) - \delta(j))(\delta_i(l) - \delta(l)) \\
&\quad = \sum_{j=1}^{k} \sum_{l=1}^{k} c(j, l) M^{(j)}_{1i}(\delta)(\delta_i(j) - \delta(j))(\delta_i(l) - \delta(l)) + R_{4i}(u)
\end{aligned}
\]

where \( i = 1, \ldots, m; u \in [0, 1] \), where

\[ \sup_{u \in [0, 1]} \max_{1 \leq i \leq m} |R_{4i}(u)| \leq C \left[ \| \delta_i - \delta \|^2 + \| \delta_i - \hat{\delta}_i \| \cdot \| \hat{\delta}_i - \delta \| + \| \delta_i - \hat{\delta}_i \|^2 \right] \quad (A.7) \]

for some \( C \in (0, \infty) \).

On the set \( A_{2i} = A_{1i} \cap \{ \hat{\delta}_i \in N \} = \{ \hat{\delta} \in N \} \cap \{ \delta_i \in \Delta \} \), by Taylor’s expansion, there exists a point \( \delta^*_i \) on the line joining \( \hat{\delta}_i \) and \( \delta \) such that

\[
\begin{aligned}
&\frac{M_{1i}(\hat{\delta}_i) - M_{1i}(\delta)}{M^{(1)}_{1i}(\delta)} \\
&\quad = \sum_{j=1}^{k} M^{(j)}_{1i}(\delta)\delta_i(j) - M^{(1)}_{1i}(\delta) - \frac{L_i}{M^{(1)}_{1i}(\delta)} + R_{3i} \\
&\quad = \sum_{j=1}^{k} M^{(j)}_{1i}(\delta)\delta_j - \frac{L_i}{M^{(1)}_{1i}(\delta)} + R_{3i} \\
&\quad + \sum_{j=1}^{k} c(j, l) M^{(j)}_{1i}(\delta)\delta_i(j) - \delta_i(l) + R_{4i} \\
&\quad = \sum_{j=1}^{k} M^{(j)}_{1i}(\delta)\{ \delta(j) - \delta_i(j) \} - b(j)
\end{aligned}
\]

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+ \sum_{j=1}^{k} \sum_{l=1}^{k} c(j, l) M_{1i}^{(j,l)}(\delta) \left\{ (\delta(j) - \delta(j)) (\delta(l) - \delta(l)) - \bar{V}(j, l) \right\} \\
+ M_{1i}^{(1)}(\delta) R_{3i} + R_{4i}^* \\
\equiv Q_{1i} + M_{1i}^{(1)}(\delta) R_{3i} + R_{4i}^*, \text{ say} \tag{A.8}

where $R_{4i}^* = R_{4i}(u)$ with the $u$ corresponding to $\delta_i^*$.

Hence for $i = 1, \ldots, m$, with $A_{3i} = \{ |M_{1i}^{(1)}(\delta)|^{-1} \leq (1 + \log m)^2 \},$

$M_{1i}(\bar{\delta}) - M_{1i}(\delta)$

$= [M_{1i}(\bar{\delta}_i) - M_{1i}(\delta)] \mathbb{I} \left( \{ \bar{\delta}_i \in \Delta \} \cap A_{3i} \right) + [M_{1i}(\bar{\delta}) - M_{1i}(\delta)] \mathbb{I} \left( \{ \bar{\delta}_i \notin \Delta \} \cup A_{3i}^c \right)$

$= [M_{1i}(\bar{\delta}_i) - M_{1i}(\delta)] \mathbb{I} \left\{ \mathbb{I}(A_{2i}) + \mathbb{I}(\bar{\delta}_i \in \Delta) - \mathbb{I}(A_{2i}) \right\} \mathbb{I}(A_{3i})$

$\quad + [M_{1i}(\bar{\delta}) - M_{1i}(\delta)] \mathbb{I} \left( \{ \bar{\delta}_i \notin \Delta \} \cup A_{3i}^c \right)$

$\equiv \left[ Q_{1i} + M_{1i}^{(1)}(\bar{\delta}) R_{3i} + R_{4i}^* \right] \mathbb{I}(A_{2i} \cap A_{3i}) + R_{5i}, \text{ say}$

$\equiv Q_{1i} + R_{6i}, \text{ say,} \tag{A.9}$

where $|R_{6i}| \leq |R_{5i}| + |R_{3i} + R_{4i}^*| \mathbb{I}(A_{2i}) + |Q_{1i}| \mathbb{I}(A_{2i}^c \cap A_{3i}^c)$ and

$|R_{5i}| \leq |M_{1i}(\bar{\delta}_i) - M(\delta)| \cdot \mathbb{I}(\bar{\delta}_i \in \Delta) - \mathbb{I}(A_{2i}) \mathbb{I}(A_{3i})$

$\quad + |M_{1i}(\bar{\delta}) - M_{1i}(\delta)| \mathbb{I} \left( \{ \bar{\delta}_i \notin \mathcal{N} \} \cup A_{3i}^c \right)$

$\equiv R_{5ii}, \text{ say.}$

Note that by definition,

$| \mathbb{I}(\bar{\delta}_i \in \Delta) - \mathbb{I}(A_{2i}) |$

$\leq \mathbb{I}(\bar{\delta}_i \in \Delta) \mathbb{I}(A_{2i}^c) + \mathbb{I}(\bar{\delta}_i \notin \Delta) \mathbb{I}(A_{2i})$

$\leq \{ \mathbb{I}(\bar{\delta} \notin \mathcal{N}) + \mathbb{I}(\bar{\delta}_i \in \Delta \setminus \mathcal{N}) \} + \mathbb{I}(\emptyset). \tag{A.10}$
Hence, with \( A_{3i}^c = \{ \hat{\delta}_i \notin \mathcal{N} \} \cap A_{3i} \),

\[
R_{51i} \leq | M_{1i}(\delta) - M_{1i}(\hat{\delta}) | \{1(A_{3i}) \{1(\hat{\delta} \notin \mathcal{N}) + 1(\delta_i \in \Delta \setminus \mathcal{N})\} \\
+ 2 | M_{1i}(\hat{\delta}) - M_{1i}(\delta) | \left\{1(\hat{\delta} \notin \mathcal{N}) + 1(\{\delta_i \notin \mathcal{N} \} \cap A_{3i}) + 1(A_{3i}^c) \right\}
\]

\[
\leq C \| \hat{\delta}_i - \hat{\delta} \| 1(A_{3i}) \left\{1(\hat{\delta} \notin \mathcal{N}) + 1(\delta_i \in \Delta \setminus \mathcal{N})\right\} \\
+C \| \hat{\delta} - \delta \| \left\{1(\hat{\delta} \notin \mathcal{N}) + 1(A_{4i}^c) + 1(A_{3i}^c) \right\}
\]

\[
\leq C \cdot (\log m)^2 \{ \| \hat{b} \| + \| \hat{V} \| \} \{1(\hat{\delta} \notin \mathcal{N}) + 1(A_{4i}^c) \}
\]

\[
+C \cdot \| \hat{\delta} - \delta \| \left\{1(\hat{\delta} \notin \mathcal{N}) + 1(A_{4i}^c) + 1(A_{3i}^c) \right\}. \tag{A.11}
\]

By condition S, there exist \( C \in (0, \infty) \) and \( \epsilon_1 \in (0, \frac{c}{m}) \) such that

\[
A_{4i}^c \subset \{ \| \hat{\delta} - \delta \| > \frac{\epsilon_0}{2} \} \cup \{ \| \delta_i - \hat{\delta} \| > \frac{\epsilon_0}{2} \} \\
\subset \{ \| \hat{\delta} - \delta \| > \epsilon_1 \} \cup \{ (\log m)^2 (\| \hat{b} \| + \| \hat{V} \|) > C \} \tag{A.12}
\]

and \( A_{3i}^c \subset \{ \| \hat{\delta} - \delta \| > \epsilon_1 \} \) for all \( i = 1, \cdots, m, m \geq 1 \). Hence, it follows that

\[
R_{51i} \leq C \cdot (\log m)^2 \{ \| \hat{b} \| + \| \hat{V} \| \} \left[1(\| \hat{\delta} - \delta \| > \epsilon_1) + 1 \left( (\log m)^2 (\| \hat{b} \| + \| \hat{V} \|) > C \right) \right] \\
+ C \cdot \| \hat{\delta} - \delta \| \left[1(\| \hat{\delta} - \delta \| > \epsilon_1) + 1 \left( (\log m)^2 (\| \hat{b} \| + \| \hat{V} \|) > C \right) \right]. \tag{A.13}
\]

for all \( i = 1, \cdots, m, m \geq 1 \). Let \( W_1 = (\| \hat{a} \| + \| \hat{\Sigma} \|) \). Note that by uniform integrability of \( \{(\sqrt{m}\| \hat{\delta} - \delta \|)^2\}_{m \geq 1} \) and the fact that \( E \| W_1 \|^{1+\eta} = O(1) \),

\[
\max_{1 \leq i \leq m} E(R_{51i}) \\
\leq C m^{-1}(\log m)^2 \left( E \| W_1 \|^{1+\eta} \right)^{1+\eta} \left( P(\| \hat{\delta} - \delta \| > \epsilon_1) \right)^{1+\eta} + E \| W_1 \|^{1+\eta} \left\{ m^{-1}(\log m)^2 \right\}^{\eta} \\
+ C \left( \epsilon_1^{-1} E \| \hat{\delta} - \delta \| ^2 P(\| \hat{\delta} - \delta \| > \epsilon_1) + \left( E \| \hat{\delta} - \delta \| ^2 \right)^{1/2} \left\{ P(m^{-1}(\log m)^2 |W_1| > C) \right\} \right)^{1/2} \\
= o(m^{-1}) \quad \text{as} \quad m \to \infty. \tag{A.14}
\]

This completes the proof of Theorem 3.
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