THE APPROXIMATION PROPERTY FOR SPACES OF LIPSCHITZ FUNCTIONS

A. JIMÉNEZ-VARGAS

Abstract. Let Lip₀(X) be the space of all Lipschitz scalar-valued functions on a pointed metric space X. We characterize the approximation property for Lip₀(X) with the bounded weak* topology using as tools the tensor product, the ε-product and the linearization of Lipschitz mappings. A necessary and sufficient condition for a Banach space E to have the approximation property is that Lip₀(E) with the bounded weak* topology, or the Lipschitz-free space F(E), have the approximation property. We thus establish Lipschitz versions of classical results of Aron and Schottenloher [1] and Mujica [21] on the approximation property for spaces of holomorphic mappings and their preduals.

Introduction

Let (X, d) be a pointed metric space with a base point which we always will denote by 0 and let F be a Banach space. The space Lip₀(X, F) is the Banach space of all Lipschitz mappings f from X to F that vanish at 0, with the Lipschitz norm defined by

$$\text{Lip}(f) = \sup \{ \frac{\|f(x) - f(y)\|}{d(x, y)} : x, y \in X, x \neq y \}.$$ 

The elements of Lip₀(X, F) are frequently called Lipschitz operators. If K is the field of real or complex numbers, Lip₀(X, K) is denoted by Lip₀(X). The closed linear subspace of the dual of Lip₀(X) spanned by the functionals δₓ on Lip₀(X) with x ∈ X, given by δₓ(f) = f(x), is a predual of Lip₀(X). This predual is called the Lipschitz-free space over X and denoted by F(X) in [10]. We refer the reader to Weaver’s book [25] for the basic theory of Lip₀(X) and its predual F(X), which is called the Arens–Eells space of X and denoted by E(X) there.

The study of the approximation property is a topic of interest for many researchers. Let us recall that a Banach space E has the approximation property (in short, (AP)) if for each compact set K ⊂ E and each ε > 0, there exists a bounded finite-rank linear operator T : E → E such that supₓ∈K ∥T(x) − x∥ < ε. If ∥T∥ ≤ λ for some λ ≥ 1, it is said that E has the λ-bounded approximation property (in short, λ-(BAP)).

To our knowledge, little is known about the (AP) for Lip₀(X). Johnson [17] observed that if X is the closed unit ball of Enflo’s space [9], then Lip₀(X) fails the (AP). Godefroy and Ozawa [11] showed that there exists a compact pointed metric space X such that F(X) fails the (AP) and hence so does Lip₀(X). For positive results, Lip[0,1] is isomorphic to L∞[0,1] (see [14] p. 224) and thus Lip[0,1] has the (AP). If (X, d) is a doubling compact pointed metric space, in particular a compact subset of a finite dimensional Banach space, and X(α) with α ∈ (0,1) denotes the metric space (X, dα), then the space Lip₀(X(α)) is isomorphic to ℓ∞ by [15] Theorem 6.5, and therefore Lip₀(X(α)) has the (AP). In [10] (see also [15]), Johnson proved that Lip₀(X) has the (AP) if and only if, for each Banach space F, every Lipschitz compact operator from X to F can be approximated in the Lipschitz norm by Lipschitz finite-rank operators.

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The most recent research on the (AP) has been directed toward $F(X)$ rather than on $\text{Lip}_0(X)$. Godefroy and Kalton \cite{10} proved that a Banach space $E$ has the $\lambda$-(BAP) if and only if $F(E)$ has the same property. Lancien and Pernecká \cite{20} showed that $F(X)$ has the $\lambda$-(BAP) whenever $X$ is a doubling metric space. For stronger approximation properties as the existence of finite-dimensional Schauder decompositions or Schauder bases for certain spaces $F(E)$, one can see the papers of Borel-Mathurin \cite{3}, Lancien and Pernecká \cite{20} and Hájek and Pernecká \cite{13}. The results in those works provide apparently a limited information about the (AP) for $\text{Lip}_0(X)$ since the (AP) of a Banach space follows from the (AP) of its dual space but the converse does not always hold.

Our aim in this paper is to study the (AP) for the space $\text{Lip}_0(X)$, with the bounded weak* topology. In the seminal paper \cite{1}, Aron and Schottenloher initiated the investigation about the (AP) for spaces of holomorphic mappings on Banach spaces. Mujica \cite{21} extended this study to the preduals of such spaces. Their techniques, based on the tensor product, the $\epsilon$-product and the linearization of holomorphic mappings, work just as well for spaces of Lipschitz mappings.

We now describe the contents of this paper. In Section 1 we briefly recall the compact-open topology $\tau_0$, the approximation property, the $\epsilon$-product and the linearization of Lipschitz mappings.

We address the study of the topology of bounded compact convergence $\tau$ on $\text{Lip}_0(X)$ in Section 2. In the terminology of Cooper \cite{5}, we prove that $\tau$ is the mixed topology $\gamma[\text{Lip}, \tau_0]$ and $(\text{Lip}_0(X), \tau_0)$ is a Saks space. Furthermore, it is shown that $\tau$ agrees with the bounded weak* topology $\tau_{bw}$. We give a pair of descriptions of $\tau$ by means of seminorms in Section 3. Assuming $X$ is compact, we first identify $\tau$ with the classical strict topology $\beta$ introduced by Buck \cite{4}. A second, and perhaps more interesting, seminorm description for $\tau$ motivates the introduction of a new locally convex topology $\gamma\tau$ on $\text{Lip}_0(X, F)$.

Section 4 deals with the (AP) for $(\text{Lip}_0(X), \tau_0)$. We identify topologically the space $(\text{Lip}_0(X, F), \gamma\tau)$ with the $\epsilon$-product of $(\text{Lip}_0(X), \tau_0)$ and $F$, and this permits us to prove that the following properties are equivalent:

(i) $(\text{Lip}_0(X), \tau_0)$ has the (AP).

(ii) Every Lipschitz operator from $X$ into $F$ can be approximated by Lipschitz finite-rank operators within the topology $\gamma\tau$ for all Banach spaces $F$.

(iii) $F(X)$ has the (AP).

In Section 5 we establish a representation of the dual space of $(\text{Lip}_0(X, F), \gamma\tau)$ and, in Section 6 we apply this description to obtain, in analogy with the linear theory with $\text{Lip}_0(E)$ being regarded as a Lipschitz dual of $E$, that a necessary and sufficient condition for a Banach space $E$ to have the (AP) is that so does $(\text{Lip}_0(E), \tau_0)$.

1. Preliminaries

**Topologies on spaces of Lipschitz functions.** Let $X$ be a pointed metric space and let $E$ be a Banach space. The compact-open topology or topology of compact convergence on $\text{Lip}_0(X, E)$ is the locally convex topology generated by the seminorms of the form

$$|f|_K = \sup_{x \in K} \|f(x)\|, \quad f \in \text{Lip}_0(X, E),$$

where $K$ varies over the family of all compact subsets of $X$. We denote by $\tau_0$ the compact-open topology on $\text{Lip}_0(X, E)$, or on any vector subspace of $\text{Lip}_0(X, E)$.

The topology of pointwise convergence on $\text{Lip}_0(X, E)$ is the locally convex topology $\tau_p$ generated by the seminorms of the form

$$|f|_F = \sup_{x \in F} \|f(x)\|, \quad f \in \text{Lip}_0(X, E),$$

where $F$ ranges over the family of all finite subsets of $X$. 


Finally, we denote by $\tau_{\text{Lip}}$ the topology on $\text{Lip}_0(X,E)$ generated by the Lipschitz norm $\text{Lip}$. It is clear that $\tau_{\text{Lip}} \subseteq \tau_0$, and the inclusion $\tau_0 \subseteq \tau_{\text{Lip}}$ follows easily since $|f|_K \leq \text{Lip}(f)\text{diam}(K \cup \{0\})$ for all $f \in \text{Lip}_0(X)$ and each compact set $K \subseteq X$.

**Approximation property and $\epsilon$-product.** Let $E$ and $F$ be locally convex Hausdorff spaces. Let $\mathcal{L}(E;F)$ denote the vector space of all continuous linear mappings from $E$ into $F$, let $\mathcal{L}_b(E;F)$ denote the vector space $\mathcal{L}(E;F)$ with the topology of uniform convergence on the bounded subsets of $E$ and let $\mathcal{L}_c(E;F)$ denote the vector space $\mathcal{L}(E;F)$ with the topology of uniform convergence on the convex balanced compact subsets of $E$. That last topology coincides with the compact-open topology if the closed convex hull of each compact subset of $E$ is compact (for example, if $E$ is quasi-complete). When $F = \mathbb{K}$, we write $E'$ instead of $\mathcal{L}(E;\mathbb{K})$, $E'_b$ in place of $\mathcal{L}_b(E;\mathbb{K})$, and $E'_c$ instead of $\mathcal{L}_c(E;\mathbb{K})$. Unless stated otherwise, if $E$ and $F$ are normed spaces, $\mathcal{L}(E;F)$ is endowed with its natural norm topology.

Let $E \otimes F$ denote the tensor product of $E$ and $F$, and $E' \otimes F$ can be identified with the subspace of all finite-rank mappings in $\mathcal{L}(E;F)$.

A locally convex space $E$ is said to have the approximation property (in short, (AP)) if the identity mapping on $E$ lies in the closure of $E' \otimes E$ in $\mathcal{L}(E;E)$. This is Schwartz’s definition of the (AP) in [23], which is slightly different from Grothendieck’s definition in [12], though both definitions coincide for quasi-complete locally convex spaces.

The $\epsilon$-product of $E$ and $F$, denoted by $E\epsilon F$ and introduced by Schwartz [23][24], is the space $\mathcal{L}_c(F'_c;E)$, that is the vector space $\mathcal{L}_c(F'_c;E)$, with the topology of uniform convergence on the equicontinuous subsets of $F'$. Notice that if $E$ is a normed space, then equicontinuous sets and norm bounded sets in $F'$ coincide. The topology on $\mathcal{L}_c(F'_c;E)$ is generated by the seminorms

$$\alpha \beta(T) = \sup \{|(T(\mu),\nu)\colon \mu \in F', |\mu| \leq \alpha, \nu \in E', |\nu| \leq \beta\}, \quad T \in \mathcal{L}_c(F'_c;E),$$

where $\alpha$ ranges over the continuous seminorms on $F'$ and $\beta$ over the continuous seminorms on $E$.

We will use the subsequent results which follow from results of Grothendieck [12], Schwartz [23] and Bierstedt and Meise [2].

**Proposition 1.1.** [23] Let $E$ and $F$ be locally convex spaces. Then the transpose mapping $T \mapsto T^t$ from $E\epsilon F$ to $F\epsilon E$ is a topological isomorphism.

**Theorem 1.2.** [2][12][23] A locally convex space $E$ has the (AP) if and only if $E \otimes F$ is dense in $E\epsilon F$ for every Banach space $F$.

**Proposition 1.3.** [2][12][23] A locally convex space $E$ has the (AP) if $E'_c$ has the (AP).

Detailed proofs of the preceding results can be found in the paper [6] by Dineen and Mujica.

**Linearization of Lipschitz mappings.** The study of the preduals of $\text{Lip}_0(X)$ was approached by Weaver [25] by using a procedure to linearize Lipschitz mappings. A similar process of linearization was presented by Mujica for bounded holomorphic mappings on Banach spaces in [21].

**Theorem 1.4.** [24] Let $X$ be a pointed metric space. Then there exist a unique, up to an isometric isomorphism, Banach space $F(X)$ and an isometric embedding $\delta_X\colon X \to F(X)$ such that

(i) $F(X)$ is the closed linear hull in $\text{Lip}_0(X)'$ of the evaluation functionals $\delta_x\colon \text{Lip}_0(X) \to \mathbb{K}$ with $x \in X$, where $\delta_x(g) = g(x)$ for all $g \in \text{Lip}_0(X)$.

(ii) The Dirac map $\delta_X\colon X \to F(X)$ is the map given by $\delta_X(x) = \delta_x$.

(iii) For each Banach space $E$ and each $f \in \text{Lip}_0(X,E)$, there is a unique operator $T_f \in \mathcal{L}(F(X);E)$ such that $T_f \circ \delta_X = f$. Furthermore, $\|T_f\| = \text{Lip}(f)$.

(iv) The evaluation map $f \mapsto T_f$ from $\text{Lip}_0(X,E)$ to $\mathcal{L}(F(X);E)$, defined by $T_f(\varphi) = \varphi(f)$, is an isometric isomorphism.

(v) $\text{Lip}_0(X)$ is isometrically isomorphic to $F(X)'$ via the evaluation map.
(vi) $F(X)$ coincides with the space of all linear functionals $\varphi$ on $\text{Lip}_0(X)$ such that the restriction of $\varphi$ to the closed unit ball $B_{\text{Lip}_0}(X)$ of $\text{Lip}_0(X)$ is continuous when $B_{\text{Lip}_0}(X)$ is equipped with the topology of pointwise convergence $\tau_p$, and hence with the compact-open topology $\tau_0$.

The statements (i)–(v) of Theorem 1.4 were proved by Weaver (see [25, Theorem 2.2.4]). The statement (vi) was stated in [15, Lemma 1.1] for $\tau_p$ and recall that, by [19, p. 232], the topology $\tau_p$ agrees with $\tau_0$ on the equicontinuous subsets of $\text{Lip}_0(X)$, and in particular on $B_{\text{Lip}_0}(X)$.

Viewing $\text{Lip}_0(X)$ as the dual of $F(X)$, we can consider its weak* topology. We recall that the weak* topology on $\text{Lip}_0(X)$ is the locally convex topology $\tau_{w^*}$ generated by the seminorms of the form

$$p_G(f) = \sup_{\varphi \in G} |\varphi(f)|, \quad f \in \text{Lip}_0(X),$$

where $G$ ranges over the family of all finite subsets of $F(X)$. Let us recall that $\tau_{w^*}$ is the smallest topology for $\text{Lip}_0(X)$ such that, for each $\varphi \in F(X)$, the linear functional $f \mapsto \varphi(f)$ on $\text{Lip}_0(X)$ is continuous with respect to $\tau_{w^*}$.

It is easy to check that $\tau_p \subset \tau_{w^*} \subset \tau_{\text{lip}}$. Indeed, on a hand, if $F$ is a finite subset of $X$, then $G = \delta_X(F)$ is a finite subset of $F(X)$ and

$$|f|_F = \sup_{x \in F} |f(x)| = \sup_{x \in F} |\delta_x(f)| = \sup_{\varphi \in G} |\varphi(f)| = p_G(f)$$

for all $f \in \text{Lip}_0(X)$, and this proves that $\tau_p \subset \tau_{w^*}$. On the other hand, if $G$ is a finite subset of $F(X)$, then $G$ is a norm bounded subset of $\text{Lip}_0(X)'$ and $p_G(f) \leq \sup_{\varphi \in G} \|\varphi\| \text{Lip}(f)$ for all $f \in \text{Lip}_0(X)$, and this shows that $\tau_{w^*} \subset \tau_{\text{lip}}$.

The ensuing result was proved by Godefroy and Kalton in [10].

**Theorem 1.5.** [10] Let $E$ and $F$ be Banach spaces.

(i) For every mapping $f \in \text{Lip}_0(E, F)$, there exists a unique operator $\hat{f} \in \mathcal{L}(F(E); F(F))$ such that $\hat{f} \circ \delta_E = \delta_F \circ f$. Furthermore, $\|\hat{f}\| = \text{Lip}(f)$.

(ii) If $E$ is a subspace of $F$ and $i: E \to F$ is the canonical embedding, then $i: F(E) \to F(F)$ is an isometric embedding.

2. The Topology of Bounded Compact Convergence for $\text{Lip}_0(X)$

We recall (see [5, Definition 3.2]) that a Saks space is a triple $(E, \|\cdot\|, \tau)$, where $E$ is a vector space, $\tau$ is a locally convex topology on $E$ and $\|\cdot\|$ is a norm on $E$ so that the closed unit ball $B_E$ of $(E, \|\cdot\|)$ is $\tau$-bounded and $\tau$-closed.

Given a pointed metric space $X$, we consider on $\text{Lip}_0(X)$ the following topologies:

- $\tau_p$: the topology of pointwise convergence.
- $\tau_0$: the topology of compact convergence.
- $\tau_{w^*}$: the weak* topology $\sigma(\text{Lip}_0(X), F(X))$.
- $\tau_{\text{lip}}$: the topology of the norm $\text{Lip}$.

The triple $(\text{Lip}_0(X), \text{Lip}, \tau_0)$ is a Saks space since $B_{\text{Lip}_0}(X)$ is $\tau_0$-compact by the Ascoli theorem (see [19, p. 234]). Then, by [5, 3.4], we can form the mixed topology $\gamma[\text{Lip}, \tau_0]$ on $\text{Lip}_0(X)$. Following [5, Definition 1.4], $\gamma[\text{Lip}, \tau_0]$ is the locally convex topology on $\text{Lip}_0(X)$ generated by the base of neighborhoods of zero $\{\gamma(U)\}$, where $U = \{U_n\}$ is a sequence of convex balanced $\tau_0$-neighborhoods of zero and

$$\gamma(U) := \bigcup_{n=1}^{\infty} \left( U_1 \cap B_{\text{Lip}_0}(X) + U_2 \cap 2B_{\text{Lip}_0}(X) + \cdots + U_n \cap 2^{n-1}B_{\text{Lip}_0}(X) \right).$$

Since $\tau_p \subset \tau_{w^*} \subset \tau_0$ on $B_{\text{Lip}_0}(X)$ (the second inclusion follows from Theorem 1.4 (vi)) and $B_{\text{Lip}_0}(X)$ is $\tau_0$-compact, then $\tau_p = \tau_{w^*} = \tau_0$ on $B_{\text{Lip}_0}(X)$. Then [5, Corollary 1.6] yields

$$\gamma[\text{Lip}, \tau_p] = \gamma[\text{Lip}, \tau_{w^*}] = \gamma[\text{Lip}, \tau_0],$$

where $\gamma[\text{Lip}, \tau_p]$ is the topology of compact convergence, $\gamma[\text{Lip}, \tau_{w^*}]$ is the weak* topology, and $\gamma[\text{Lip}, \tau_0]$ is the topology of pointwise convergence.
and we denote this topology by $\tau_\gamma$. We gather next some properties of $\tau_\gamma$.

**Theorem 2.1.** Let $X$ be a pointed metric space.

(i) $\tau_0$ is smaller than $\tau_\gamma$, and $\tau_\gamma$ is smaller than $\tau_{\text{Lip}}$.

(ii) $\tau_\gamma$ is the largest locally convex topology on $\text{Lip}_0(X)$ which coincides with $\tau_0$ on each norm bounded subset of $\text{Lip}_0(X)$.

(iii) If $F$ is a locally convex space and $T: \text{Lip}_0(X) \to F$ is linear, then $T$ is $\tau_\gamma$-continuous if and only if $T|_{B}$ is $\tau_0$-continuous for each norm bounded subset $B$ of $\text{Lip}_0(X)$.

(iv) A sequence in $\text{Lip}_0(X)$ is $\tau_\gamma$-convergent to zero if and only if it is norm bounded and $\tau_0$-convergent to zero.

(v) A subset of $\text{Lip}_0(X)$ is $\tau_\gamma$-bounded if and only if it is norm bounded.

(vi) A subset of $\text{Lip}_0(X)$ is $\tau_\gamma$-compact (precompact, relatively compact) if and only if it is norm bounded and $\tau_0$-compact (precompact, relatively compact).

(vii) $(\text{Lip}_0(X), \tau_\gamma)$ is a complete semi-Montel space.

**Proof.** The statements (i)–(vii) follow immediately from the theory of [5, Chapter I]. More namely, (i) and (ii) follow from Proposition 1.5; (iii) from Corollary 1.7; (iv) from Proposition 1.10; (v) from Proposition 1.13 and 1.26 and the Ascoli theorem.

The property (ii) above justifies the name of topology of bounded compact convergence for $\tau_\gamma$. We next improve this property.

**Theorem 2.2.** Let $X$ be a pointed metric space.

(i) $\tau_\gamma$ is the largest topology on $\text{Lip}_0(X)$ which agrees with $\tau_0$ on each norm bounded subset of $\text{Lip}_0(X)$.

(ii) A subset $U$ of $\text{Lip}_0(X)$ is open (closed) in $(\text{Lip}_0(X), \tau_\gamma)$ if and only if $U \cap B$ is open (closed) in $(B, \tau_0)$ for each norm bounded subset $B$ of $\text{Lip}_0(X)$.

(iii) $F(X) = (\text{Lip}_0(X), \tau_\gamma, \gamma)' = (\text{Lip}_0(X), \tau_\gamma)'_c$.

(iv) The evaluation map $f \mapsto T_f$ from $(\text{Lip}_0(X), \tau_\gamma)$ to $F(X)'_c$ is a topological isomorphism.

**Proof.** The statements (i) and (ii) follow from [5, Corollary 4.2]. We now prove (iii). From Theorem 1.1 (vi) and Theorem 2.1 (ii)-(iii), we deduce that $F(X) = (\text{Lip}_0(X), \tau_\gamma)'$ algebraically. Since $F(X)$ is a linear subspace of $(\text{Lip}_0(X), \tau_{\text{Lip}})'$ by Theorem 1.1 (i) and both spaces $(\text{Lip}_0(X), \tau_{\text{Lip}})$ and $(\text{Lip}_0(X), \tau_\gamma)$ have the same bounded sets by Theorem 2.1 (v), we infer that $F(X) = (\text{Lip}_0(X), \tau_\gamma)'_c$. The identification $(\text{Lip}_0(X), \tau_\gamma)'_c = (\text{Lip}_0(X), \tau_\gamma)'_c$ follows from the fact that $(\text{Lip}_0(X), \tau_\gamma)$ is a semi-Montel space, and the proof of (iii) is finished.

To prove (iv), notice that \{n$\text{B}_{\text{Lip}_0(X)}(X)$\} is an increasing sequence of convex, balanced and $\tau_\gamma$-compact subsets of $\text{Lip}_0(X)$ (by Theorem 2.1 (vi) and the Ascoli theorem) with the property that a set $U \subset \text{Lip}_0(X)$ is $\tau_\gamma$-open whenever $U \cap n\text{B}_{\text{Lip}_0(X)}$ is open in $(n\text{B}_{\text{Lip}_0(X)} \cap \text{Lip}_0(X), \tau_\gamma)$ for every $n \in \mathbb{N}$. This property can be proved easily using the statements (i) and (ii). Then, by applying [4, Theorem 4.1], the evaluation map from $(\text{Lip}_0(X), \tau_\gamma)$ to $((\text{Lip}_0(X), \tau_\gamma)'_c$ is a topological isomorphism. Since $F(X) = (\text{Lip}_0(X), \tau_\gamma)'_c$ by (iii), the statement (iv) holds.

**Remark 2.1.** All assertions of Theorems 2.1 and 2.2 are valid if the topology $\tau_0$ is replaced by $\tau_p$ or $\tau_{bw}$.

We now recall that if $E$ is a Banach space, then the bounded weak* topology on its dual $E'$, denoted by $\tau_{bw}$, is the largest topology on $E'$ agreeing with the topology $\tau_{w^*}$ on norm bounded sets [3, V.5.3]. According to the Banach-Dieudonné theorem [3, V.5.4], $\tau_{bw}$ is just the topology of uniform convergence on sequences in $E$ which tend in norm to zero.

Since $\tau_{w^*} = \tau_0$ on $B_{\text{Lip}_0(X)}$, the assertion (i) of Theorem 2.2 gives the following.

**Corollary 2.3.** Let $X$ be a pointed metric space. On the space $\text{Lip}_0(X)$, the bounded weak* topology $\tau_{bw}$ is the topology $\tau_\gamma$. 

3. Seminorm descriptions of \( \tau_\gamma \) on \( \text{Lip}_0(X) \)

Our aim in this section is to give a pair of descriptions for \( \tau_\gamma \) by means of seminorms. By Theorem 2.2 (ii) and Remark 2.1, the convex balanced sets \( U \subset \text{Lip}_0(X) \) such that \( U \cap nB_{\text{Lip}_0(X)} \) is a neighborhood of zero in \( (nB_{\text{Lip}_0(X)}, \tau_{w^*}) \) for every \( n \in \mathbb{N} \), form a base of neighborhoods of zero for \( \tau_\gamma \). For our purposes, we will need the next lemma. For \( f \in \text{Lip}_0(X) \) and \( A \subset X \), define

\[
\text{Lip}_A(f) = \sup \left\{ \frac{|f(x) - f(y)|}{d(x,y)} : x, y \in A, \ x \neq y \right\}.
\]

Notice that if \( F \subset X \) is finite, then \( \text{Lip}_F(f) = p_G(f) \) (see Section 1) where \( G \) is the finite subset of \( \mathcal{F}(X) \) given by

\[
G = \left\{ \frac{\delta_x - \delta_y}{d(x,y)} : x, y \in F, \ x \neq y \right\},
\]

and hence, for each \( \varepsilon > 0 \), the set \( \{ f \in \text{Lip}_0(X) : \text{Lip}_F(f) \leq \varepsilon \} \) is a neighborhood of 0 in \( (\text{Lip}_0(X), \tau_{w^*}) \).

**Lemma 3.1.** Let \( X \) be a pointed metric space. Then the sets of the form

\[
U = \bigcap_{n=1}^{\infty} \{ f \in \text{Lip}_0(X) : \text{Lip}_{F_n}(f) \leq \lambda_n \}
\]

where \( \{ F_n \} \) is a sequence of finite subsets of \( X \) and \( \{ \lambda_n \} \) is a sequence of positive numbers tending to \( \infty \), form a base of neighborhoods of zero in \( (\text{Lip}_0(X), \tau_\gamma) \).

**Proof.** We first claim that if \( \{ F_k \} \) and \( \{ \lambda_k \} \) are sequences as above, then the set

\[
\bigcap_{k=1}^{\infty} \{ f \in \text{Lip}_0(X) : \text{Lip}_{F_k}(f) \leq \lambda_k \}
\]

is a neighborhood of 0 in \( (\text{Lip}_0(X), \tau_\gamma) \). Indeed, given \( n \in \mathbb{N} \), if \( m \in \mathbb{N} \) is chosen so that \( \lambda_k \geq n \) for \( k > m \), then

\[
\bigcap_{k=1}^{\infty} \{ f \in \text{Lip}_0(X) : \text{Lip}_{F_k}(f) \leq \lambda_k \} \cap nB_{\text{Lip}_0(X)} = \bigcap_{n=1}^{m} \{ f \in \text{Lip}_0(X) : \text{Lip}_{F_k}(f) \leq \lambda_k \} \cap nB_{\text{Lip}_0(X)}.
\]

The latter is a neighborhood of 0 in \( (nB_{\text{Lip}_0(X)}, \tau_{w^*}) \), and this proves our claim.

We now must prove that if \( U \) is a neighborhood of 0 in \( (\text{Lip}_0(X), \tau_\gamma) \), then there are sequences \( \{ F_k \} \) and \( \{ \lambda_k \} \) as above for which

\[
\bigcap_{k=1}^{\infty} \{ f \in \text{Lip}_0(X) : \text{Lip}_{F_k}(f) \leq \lambda_k \} \subset U.
\]

Indeed, we can take a set \( U \subset \text{Lip}_0(X) \) such that \( U \cap nB_{\text{Lip}_0(X)} \) is an open neighborhood of 0 in \( (nB_{\text{Lip}_0(X)}, \tau_{w^*}) \) for every \( n \in \mathbb{N} \). In particular, \( U \cap B_{\text{Lip}_0(X)} \) is a neighborhood of 0 in \( (B_{\text{Lip}_0(X)}, \tau_{lip}) \) and then there exists \( \varepsilon > 0 \) such that \( \varepsilon B_{\text{Lip}_0(X)} \subset U \). In order to prove that there exists a finite set \( F_1 \subset X \) such that

\[
\{ f \in \text{Lip}_0(X) : \text{Lip}_{F_1}(f) \leq \varepsilon \} \cap B_{\text{Lip}_0(X)} \subset U,
\]

assume on the contrary that the set

\[
\{ f \in \text{Lip}_0(X) : \text{Lip}_F(f) \leq \varepsilon \} \cap (B_{\text{Lip}_0(X)} \setminus U)
\]

is nonempty for every finite set \( F \subset X \). These sets are closed in \( (B_{\text{Lip}_0(X)} \setminus U, \tau_{w^*}) \) and have the finite intersection property. Since the set \( B_{\text{Lip}_0(X)} \setminus U \) is a closed, and therefore compact, subset of \( (B_{\text{Lip}_0(X)}, \tau_{w^*}) \), we infer that there exists some \( f \in B_{\text{Lip}_0(X)} \setminus U \) such that \( \text{Lip}_F(f) \leq \varepsilon \) for each finite set \( F \subset X \). This implies that \( f \in \varepsilon B_{\text{Lip}_0(X)} \setminus U \) which is impossible, and thus proving our assertion.
Proceeding by induction, suppose that we can find finite subsets \( F_2, \ldots, F_n \) of \( X \) such that
\[
\bigcap_{k=1}^{n} \{ f \in \text{Lip}_0(X): \text{Lip}_{F_k}(f) \leq \lambda_k \} \cap nB_{\text{Lip}_0(X)}(x) \subset U \cap nB_{\text{Lip}_0(X)}.
\]
where \( \lambda_1 = \varepsilon \) and \( \lambda_k = k - 1 \) for \( k > 1 \). We will prove that there exists a finite set \( F_{n+1} \subset X \) such that
\[
\bigcap_{k=1}^{n+1} \{ f \in \text{Lip}_0(X): \text{Lip}_{F_k}(f) \leq \lambda_k \} \cap (n+1)B_{\text{Lip}_0(X)}(x) \subset U \cap (n+1)B_{\text{Lip}_0(X)}.
\]
We argue by contradiction. If no such finite set \( F_{n+1} \) exists, then the set
\[
C_F := \bigcap_{k=1}^{n} \{ f \in \text{Lip}_0(X): \text{Lip}_{F_k}(f) \leq \lambda_k \} \cap \{ f \in \text{Lip}_0(X): \text{Lip}_F(f) \leq n \}
\]
has nonempty intersection with the \( \tau_{v^*} \)-compact set \((n+1)B_{\text{Lip}_0(X)} \setminus U \) for each finite set \( F \subset X \). So, by the finite intersection property, there is an \( f_0 \in \{(n+1)B_{\text{Lip}_0(X)} \setminus U \} \cap \bigcap C_F \). Therefore \( \text{Lip}_{F}(f_0) \leq n \) for each \( F \) and so \( \text{Lip}(f_0) \leq n \). Then \( f_0 \in U \cap nB_{\text{Lip}_0(X)} \subset U \cap (n+1)B_{\text{Lip}_0(X)} \) which is a contradiction.

Then we can construct, by induction, a sequence \( \{F_k\} \) of finite subsets of \( X \) so that
\[
\bigcap_{k=1}^{n} \{ f \in \text{Lip}_0(X): \text{Lip}_{F_k}(f) \leq \lambda_k \} \cap nB_{\text{Lip}_0(X)}(x) \subset U
\]
for every \( n \in \mathbb{N} \). Since \( \text{Lip}_0(X) = \bigcup_{n=1}^{\infty} nB_{\text{Lip}_0(X)} \), we conclude that
\[
\bigcap_{k=1}^{\infty} \{ f \in \text{Lip}_0(X): \text{Lip}_{F_k}(f) \leq \lambda_k \} \subset U.
\]

\[ \square \]

A first characterization of \( \tau_\gamma \) by means of seminorms lies over the concept of strict topology, introduced by Buck in [4], for spaces of continuous functions on locally compact spaces.

Let \( X \) be a pointed metric space. We denote
\[
\tilde{X} = \{(x, y) \in X^2: x \neq y \}.
\]
Let \( C_b(\tilde{X}) \) be the space of bounded continuous scalar-valued functions on \( \tilde{X} \) with the supremum norm, and let \( \Phi \) be De Leeuw’s map from \( \text{Lip}_0(X) \) into \( C_b(\tilde{X}) \) defined by
\[
\Phi(f)(x, y) = \frac{f(x) - f(y)}{d(x, y)}. \]
Clearly, \( \Phi \) is an isometric isomorphism from \( \text{Lip}_0(X) \) onto the closed subspace \( \Phi(\text{Lip}_0(X)) \) of \( C_b(\tilde{X}) \).

**Definition 3.1.** Let \( X \) be a compact pointed metric space. The strict topology \( \beta \) on \( \text{Lip}_0(X) \) is the strict topology on \( \Phi(\text{Lip}_0(X)) \), that is the locally convex topology generated by the seminorms of the form
\[
\|f\|_\phi = \sup_{(x, y) \in \tilde{X}} |\phi(x, y)| \frac{|f(x) - f(y)|}{d(x, y)}, \quad f \in \text{Lip}_0(X),
\]
where \( \phi \) runs through the space \( C_0(\tilde{X}) \) of continuous functions from \( \tilde{X} \) into \( \mathbb{K} \) which vanish at infinity.

**Theorem 3.2.** Let \( X \) be a compact pointed metric. On the space \( \text{Lip}_0(X) \), the strict topology \( \beta \) is the topology \( \tau_\gamma \).
Proof. We first show that the identity is a continuous mapping from \((\text{Lip}_0(X), \tau_\gamma)\) to \((\text{Lip}_0(X), \beta)\). By Theorem 2.1 (iii), it is enough to show that the identity on \(nB_{\text{Lip}_0(X)}(x)\) is continuous on \((nB_{\text{Lip}_0(X)}, \tau_0)\) for every \(n \in \mathbb{N}\). Let \(n \in \mathbb{N}\) and fix \(\phi \in C_0(\bar{X})\) and \(\varepsilon > 0\). Then there is a compact set \(K \subset \bar{X}\) such that \(|\phi(x, y)| < \varepsilon/2n\) if \((x, y) \in \bar{X} \setminus K\). Take

\[
U = \left\{ f \in \text{Lip}_0(X) : \sup_{(x,y) \in K} \frac{|f(x) - f(y)|}{d(x, y)} \leq \frac{\varepsilon}{2(1 + \|\phi\|_\infty)} \right\}.
\]

We now prove that \(U\) is a neighborhood of 0 in \((\text{Lip}_0(X), \tau_\gamma)\). Indeed, define \(\sigma : \bar{X} \to \mathcal{F}(X)\) by

\[
\sigma(x, y) = \frac{\delta_x - \delta_y}{d(x, y)}.
\]

Since the maps \(x \mapsto \delta_x\) and \((x, y) \mapsto d(x, y)\) are continuous, so is also \(\sigma\). Then \(\sigma(K)\) is a compact subset of \(\mathcal{F}(X)\) and therefore the polar

\[
\sigma(K)^\circ := \left\{ F \in \mathcal{F}(X)' : \sup_{(x,y) \in K} |F(\sigma(x,y))| \leq 1 \right\}
\]

is a neighborhood of 0 in \(\mathcal{F}(X)\). Then, by Theorem 2.2 (iv), the set \(\{ f \in \text{Lip}_0(X) : T_f \in \sigma(K)^\circ \}\), that is

\[
\left\{ f \in \text{Lip}_0(X) : \sup_{(x,y) \in K} \frac{|f(x) - f(y)|}{d(x, y)} \leq 1 \right\},
\]

is a neighborhood of 0 in \((\text{Lip}_0(X), \tau_\gamma)\), and hence so is \(U\) as required. It follows that \(U \cap nB_{\text{Lip}_0(X)}\) is a neighborhood of 0 in \((nB_{\text{Lip}_0(X)}, \tau_0)\) by Theorem 2.2 (ii). If \(f \in U \cap nB_{\text{Lip}_0(X)}\), we have

\[
\|f\|_\phi \leq \sup_{(x,y) \in K} |\phi(x,y)| \frac{\|f(x) - f(y)\|}{d(x, y)} + \sup_{(x,y) \in \bar{X} \setminus K} |\phi(x,y)| \frac{\|f(x) - f(y)\|}{d(x, y)}
\]

\[
\leq \|\phi\|_\infty \frac{\varepsilon}{2(1 + \|\phi\|_\infty)} + \frac{\varepsilon}{2n} < \varepsilon.
\]

Conversely, let \(U\) be a neighborhood of 0 in \((\text{Lip}_0(X), \tau_\gamma)\). By Lemma 3.1 we can suppose that

\[
U = \bigcap_{n=1}^{\infty} \{ f \in \text{Lip}_0(X) : \text{Lip}_{\lambda_n}(f) \leq \lambda_n \}
\]

where \(\{F_n\}\) is a sequence of finite subsets of \(X\) and \(\{\lambda_n\}\) is a sequence of positive numbers tending to \(\infty\). We can further suppose that \(F_n \subset F_{n+1}\) and \(\lambda_n < \lambda_{n+1}\) for all \(n \in \mathbb{N}\). We can construct a function \(\phi \in C_0(\bar{X})\) with \(\{(x,y) \in \bar{X} : \phi(x,y) \neq 0\} \subset \cup_{n=1}^{\infty} F_n\) so that \(\phi(x,y) = 1/\lambda_1\) if \((x, y) \in F_1\) and \(1/\lambda_{n+1} \leq \phi(x,y) \leq 1/\lambda_n\) for all \((x, y) \in F_{n+1} \setminus F_n\). Then \(\{ f \in \text{Lip}_0(X) : \|f\|_\phi \leq 1 \} \subset U\) and this proves the theorem.

The second description of \(\tau_\gamma\) in terms of seminorms is the ensuing.

Theorem 3.3. Let \(X\) be a pointed metric space. The topology \(\tau_\gamma\) is generated by the seminorms of the form

\[
p(f) = \sup_{n \in \mathbb{N}} \alpha_n \frac{|f(x_n) - f(y_n)|}{d(x_n, y_n)}, \quad f \in \text{Lip}_0(X),
\]

where \(\{\alpha_n\}\) varies over all sequences in \(\mathbb{R}^+\) tending to zero and \(\{(x_n, y_n)\}\) runs over all sequences in \(\bar{X}\).
Proof. Let \( V \) be the base of neighborhoods of 0 in \((\operatorname{Lip}_0(X), \tau_\gamma)\) formed by the sets of the form

\[
U = \bigcap_{n=1}^{\infty} \{ f \in \operatorname{Lip}_0(X) : \operatorname{Lip}_{F_n}(f) \leq \lambda_n \}
\]

where \( \{F_n\} \) and \( \{\lambda_n\} \) are sequences as in Lemma 3.3. If, for each \( U \in V \), \( p_U \) is the Minkowski functional of \( U \), then the family of seminorms \( \{p_U : U \in V\} \) generates the topology \( \tau_\gamma \) on \( \operatorname{Lip}_0(X) \), but justly we have

\[
p_U(f) = \sup_{n \in \mathbb{N}} \lambda_n^{-1} \operatorname{Lip}_{F_n}(f)
\]

for all \( f \in \operatorname{Lip}_0(X) \), and the result follows. \( \square \)

Let \( E \) be a Banach space. The polar of \( M \subset E \) and the prepolar of \( N \subset E' \) are respectively

\[
M^\circ = \left\{ f \in E' : \sup_{x \in M} |f(x)| \leq 1 \right\},
\]

\[
N_\circ = \left\{ x \in E : \sup_{f \in N} |f(x)| \leq 1 \right\}.
\]

\( \Gamma M \) stands for the closed, convex, balanced hull of \( M \) in \( E \). The next lemma will be needed later.

**Lemma 3.4.** Let \( X \) be a pointed metric space. For each compact set \( L \subset \mathcal{F}(X) \), there exist sequences \( \{\alpha_n\} \in c_0(\mathbb{R}^+) \) and \( \{(x_n, y_n)\} \in \bar{X}^\mathbb{N} \) such that

\[
L \subset \Gamma \left\{ \alpha_n \frac{\delta_{x_n} - \delta_{y_n}}{d(x_n, y_n)} : n \in \mathbb{N} \right\}.
\]

**Proof.** If \( L \) is a compact subset of \( \mathcal{F}(X) \), then the polar \( L^0 := \{ F \in \mathcal{F}(X)' : \sup_{\varphi \in L} |F(\varphi)| \leq 1 \} \) is a neighborhood of 0 in \( \mathcal{F}(X)' \). Then, by Theorem 2.2 (iv), the set \( \{ f \in \operatorname{Lip}_0(X) : T_f \in L^0 \} \) is a neighborhood of 0 in \( (\operatorname{Lip}_0(X), \tau_\gamma) \). Hence, by Theorem 3.3, there exist sequences \( \{\alpha_n\} \in c_0(\mathbb{R}^+) \) and \( \{(x_n, y_n)\} \in \bar{X}^\mathbb{N} \) such that

\[
\left\{ f \in \operatorname{Lip}_0(X) : \sup_{n \in \mathbb{N}} \frac{|f(x_n) - f(y_n)|}{d(x_n, y_n)} \leq 1 \right\} \subset \left\{ f \in \operatorname{Lip}_0(X) : \sup_{\varphi \in L} |T_f(\varphi)| \leq 1 \right\}.
\]

We have

\[
\left\{ \alpha_n \frac{\delta_{x_n} - \delta_{y_n}}{d(x_n, y_n)} : n \in \mathbb{N} \right\}^0 = \left\{ F \in \mathcal{F}(X)' : \sup_{n \in \mathbb{N}} \left| F \left( \alpha_n \frac{\delta_{x_n} - \delta_{y_n}}{d(x_n, y_n)} \right) \right| \leq 1 \right\}
\]

\[
= \left\{ T_f : f \in \operatorname{Lip}_0(X), \sup_{n \in \mathbb{N}} \left| \frac{f(x_n) - f(y_n)}{d(x_n, y_n)} \right| \leq 1 \right\}
\]

\[
\subset \left\{ T_f : f \in \operatorname{Lip}_0(X), \sup_{\varphi \in L} |T_f(\varphi)| \leq 1 \right\}
\]

\[
= \left\{ F \in \mathcal{F}(X)' : \sup_{\varphi \in L} |F(\varphi)| \leq 1 \right\}
\]

\[
= L^0,
\]

and then the bipolar theorem yields

\[
L \subset (L^0)_0 \subset \left\{ \alpha_n \frac{\delta_{x_n} - \delta_{y_n}}{d(x_n, y_n)} : n \in \mathbb{N} \right\}^0 = \Gamma \left\{ \alpha_n \frac{\delta_{x_n} - \delta_{y_n}}{d(x_n, y_n)} : n \in \mathbb{N} \right\}.
\]

\( \square \)
4. THE APPROXIMATION PROPERTY FOR $(\text{Lip}_0(X), \tau_\gamma)$

We devote this section to the study of the (AP) for the space $\text{Lip}_0(X)$ with the topology of bounded compact convergence. For it, we introduce the subsequent topology on $\text{Lip}_0(X, F)$.

**Definition 4.1.** Let $X$ be a pointed metric and let $F$ be a Banach space. The topology $\gamma \tau_\gamma$ on $\text{Lip}_0(X, F)$ is the locally convex topology generated by the seminorms of the form

$$q(f) = \sup_{n \in \mathbb{N}} \alpha_n \frac{\|f(x_n) - f(y_n)\|}{d(x_n, y_n)}, \quad f \in \text{Lip}_0(X, F),$$

where $\{\alpha_n\}$ ranges over the sequences in $\mathbb{R}^+$ tending to zero and $\{(x_n, y_n)\}$ over the sequences in $\bar{X}$.

We study the relation between the topologies $\gamma \tau_\gamma$, $\gamma_\gamma$ and $\tau_0$.

**Proposition 4.1.** Let $X$ be a pointed metric space and let $F$ be a Banach space.

(i) $\tau_0$ is smaller than $\gamma \tau_\gamma$ on $\text{Lip}_0(X, F)$.

(ii) $\gamma_\gamma$ agrees with $\gamma \tau_\gamma$ on $\text{Lip}_0(X)$.

**Proof.** To prove (i), let $K$ be a compact subset of $X$. Then $\delta_X(K)$ is a compact subset of $\mathcal{F}(X)$ and, by Lemma 3.4, there are sequences $\{\alpha_n\} \subset c_0(\mathbb{R}^+)$ and $\{(x_n, y_n)\} \subset \bar{X}^\mathbb{N}$ such that

$$\delta_X(K) \subset \left\{ \alpha_n \frac{\delta_{x_n} - \delta_{y_n}}{d(x_n, y_n)} : n \in \mathbb{N} \right\}.$$

It follows that

$$|f|_{K} = \sup_{x \in K} \|f(x)\| \leq \sup_{n \in \mathbb{N}} \alpha_n \frac{\|f(x_n) - f(y_n)\|}{d(x_n, y_n)} = q(f),$$

for all $f \in \text{Lip}_0(X, F)$, as desired. (ii) is deduced from Theorem 3.3 and Definition 4.1. \qed

If $f \in \text{Lip}_0(X, F)$, we define the Lipschitz transpose of $f$ to the linear mapping $f^\dagger : F' \to \text{Lip}_0(X)$ given by $f^\dagger(\psi) = \psi \circ f$ for all $\psi \in F'$. Our next result shows that the Lipschitz transpose can be used to identify the space $(\text{Lip}_0(X, F), \gamma \tau_\gamma)$ with $(\text{Lip}_0(X), \tau_\gamma)\epsilon F$.

By Section 3, notice that the seminorms

$$\sup \left\{ \alpha_n \frac{|T(\psi)(x_n) - T(\psi)(y_n)|}{d(x_n, y_n)} : n \in \mathbb{N}, \psi \in F', \|\psi\| \leq 1 \right\}, \quad T \in (\text{Lip}_0(X), \tau_\gamma)\epsilon F,$

where $\{\alpha_n\}$ and $\{(x_n, y_n)\}$ are sequences as above, determine the topology of $\mathcal{L}_e(F'; (\text{Lip}_0(X), \tau_\gamma)) = (\text{Lip}_0(X), \tau_\gamma)\epsilon F$.

**Theorem 4.2.** Let $X$ be a pointed metric space and let $F$ be a Banach space. The mapping $f \mapsto f^\dagger$ is a topological isomorphism from $(\text{Lip}_0(X, F), \gamma \tau_\gamma)$ onto $(\text{Lip}_0(X), \tau_\gamma)\epsilon F$.

**Proof.** If $f \in \text{Lip}_0(X, F)$, the mapping $f^\dagger : F' \to \text{Lip}_0(X)$ is continuous from $F'$ into $(\text{Lip}_0(X), \tau_\gamma)$. To prove this, let $p$ be a continuous seminorm on $(\text{Lip}_0(X), \tau_\gamma)$. By Theorem 3.3, we can suppose that

$$p(g) = \sup_{n \in \mathbb{N}} \alpha_n \frac{|g(x_n) - g(y_n)|}{d(x_n, y_n)}, \quad g \in \text{Lip}_0(X),$$

where $\{\alpha_n\}$ is a sequence in $\mathbb{R}^+$ tending to zero and $\{(x_n, y_n)\}$ is a sequence in $\bar{X}$. Since

$$\left\| \alpha_n \frac{f(x_n) - f(y_n)}{d(x_n, y_n)} \right\| \leq \alpha_n \text{Lip}(f)$$

for all $n \in \mathbb{N}$, the set

$$K = \left\{ \alpha_n \frac{f(x_n) - f(y_n)}{d(x_n, y_n)} \right\} \cup \{0\}$$

is a sequence in $\mathbb{R}^+$ tending to zero and $\{(x_n, y_n)\}$ is a sequence in $\bar{X}$. Since

$$\left\| \alpha_n \frac{f(x_n) - f(y_n)}{d(x_n, y_n)} \right\| \leq \alpha_n \text{Lip}(f)$$

for all $n \in \mathbb{N}$, the set

$$K = \left\{ \alpha_n \frac{f(x_n) - f(y_n)}{d(x_n, y_n)} \right\} \cup \{0\}$$
is compact in \( F \). For each \( \psi \in F' \), we have

\[
\alpha_n \frac{|f^n(\psi)(x_n) - f^n(\psi)(y_n)|}{d(x_n, y_n)} = \psi \left( \alpha_n \frac{f(x_n) - f(y_n)}{d(x_n, y_n)} \right) \leq |\psi|_K
\]

for all \( n \in \mathbb{N} \), and consequently \( p(f^n(\psi)) \leq |\psi|_K \) as required.

Clearly, the mapping \( f \mapsto f^t \) from \( \text{Lip}_0(X, F) \) to \( \mathcal{L}_t(F'_c; (\text{Lip}_0(X), \tau_\gamma)) \) is linear and, since \( F' \) separates the points of \( F \), is injective. To prove that it is surjective, let \( T \in \mathcal{L}_t(F'_c; (\text{Lip}_0(X), \tau_\gamma)) \). Then its transpose \( T^t \) is in \( \mathcal{L}_t((\text{Lip}_0(X), \tau_\gamma); F) = \mathcal{L}_t(F(X); F) \) by Proposition 1.1 and Theorem 2.2 (iii). Notice that \( T \in \mathcal{L}(F'; \text{Lip}_0(X)) \) since the closed unit ball \( B_{F'} \) of \( F' \) is a compact subset of \( (F', \tau_0) \), then \( T(B_{F'}) \) is a bounded subset of \( (\text{Lip}_0(X), \tau_\gamma) \) and hence norm bounded by Theorem 2.2 (v). Consider now the Dirac map \( \delta_X : X \to F(X) \). Then the mapping \( f = T^t \circ \delta_X \) maps \( X \) into \( F \), vanishes at 0 and is Lipschitz since

\[
\|f(x) - f(y)\| \leq \|T^t\| \|\delta_X(x) - \delta_X(y)\| = \|T\| d(x, y)
\]

for all \( x, y \in X \). For every \( \psi \in F' \) and \( x \in X \), we have

\[
f^n(\psi)(x) = \langle \psi, f(x) \rangle = \langle \psi, T^t \delta_X(x) \rangle = \langle T(\psi), \delta_X(x) \rangle = T(\psi)(x),
\]

and thus \( f^t = T \). Hence the mapping \( f \mapsto f^t \) is a linear bijection from \( \text{Lip}_0(X, F) \) onto \( (\text{Lip}_0(X), \tau_\gamma) \) with inverse given by \( T \mapsto T^t \circ \delta_X \).

It remains to show that it is continuous with continuous inverse. For it, let \( \{\alpha_n\} \) and \( \{(x_n, y_n)\} \) be sequences as above. By Definition 4.1,

\[
q(f) = \sup_{n \in \mathbb{N}} \alpha_n \frac{|f(x_n) - f(y_n)|}{d(x_n, y_n)}, \quad f \in \text{Lip}_0(X, F),
\]

is a continuous seminorm on \((\text{Lip}_0(X, F), \gamma_{\tau_\gamma})\). If \( n \in \mathbb{N} \) and \( \psi \in F' \) with \( \|\psi\| \leq 1 \), we have

\[
\alpha_n \frac{|f^n(\psi)(x_n) - f^n(\psi)(y_n)|}{d(x_n, y_n)} = \alpha_n \frac{\|f(x_n) - f(y_n)\|}{d(x_n, y_n)} \leq \alpha_n \frac{|f(x_n) - f(y_n)|}{d(x_n, y_n)},
\]

therefore

\[
\sup \left\{ \alpha_n \frac{|f^n(\psi)(x_n) - f^n(\psi)(y_n)|}{d(x_n, y_n)} : n \in \mathbb{N}, \psi \in F', \|\psi\| \leq 1 \right\} \leq q(f)
\]

and this proves that the mapping \( f \mapsto f^t \) is continuous. To see that its inverse \( T \mapsto T^t \circ \delta_X \) is continuous, let \( q \) be a continuous seminorm on \((\text{Lip}_0(X, F), \gamma_{\tau_\gamma})\). By definition 4.1, we can suppose that

\[
q(f) = \sup_{n \in \mathbb{N}} \alpha_n \frac{|f(x_n) - f(y_n)|}{d(x_n, y_n)}, \quad f \in \text{Lip}_0(X, F),
\]

where \( \{\alpha_n\} \) and \( \{(x_n, y_n)\} \) are sequences as above. For each \( n \in \mathbb{N} \), take \( \psi_n \in B_{F'} \) such that

\[
\|T^t \delta_X(x_n) - T^t \delta_X(y_n)\| = \|\langle \psi_n, T^t \delta_X(x_n) - T^t \delta_X(y_n) \rangle\|
\]

and then we have

\[
\alpha_n \frac{|T^t \delta_X(x_n) - T^t \delta_X(y_n)|}{d(x_n, y_n)} = \alpha_n \frac{|\langle T\psi_n, \delta_X(x_n) - \delta_X(y_n) \rangle\|}{d(x_n, y_n)} = \alpha_n \frac{|T(\psi_n)(x_n) - T(\psi_n)(y_n)|}{d(x_n, y_n)}.
\]

It follows that

\[
q(T^t \circ \delta_X) \leq \sup \left\{ \alpha_n \frac{|T(\psi)(x_n) - T(\psi)(y_n)|}{d(x_n, y_n)} : n \in \mathbb{N}, \psi \in F', \|\psi\| \leq 1 \right\},
\]

and the proof is finished. \( \Box \)
Our next aim is to identify linearly the tensor product \( \text{Lip}_0(X) \otimes F \) with the space of all Lipschitz finite-rank operators from \( X \) to \( F \). Let us recall that a mapping \( f \in \text{Lip}_0(X,F) \) is called a Lipschitz finite-rank operator if the linear hull of \( f(X) \) in \( F \) has finite dimension in whose case this dimension is called the rank of \( f \) and denoted by \( \text{rank}(f) \). We represent by \( \text{Lip}_{0F}(X,F) \) the vector space of all Lipschitz finite-rank operators from \( X \) to \( F \). This space can be generated linearly as follows.

**Lemma 4.3.** Let \( X \) be a pointed metric space and \( F \) a Banach space.

(i) If \( g \in \text{Lip}_0(X) \) and \( u \in F \), then the map \( g \cdot u : X \to F \), given by \( (g \cdot u)(x) = g(x)u \), belongs to \( \text{Lip}_{0F}(X,F) \) and \( \text{Lip}(g \cdot u) = \text{Lip}(g)\|u\| \). Moreover, \( \text{rank}(g \cdot u) = 1 \) if \( g \neq 0 \) and \( u \neq 0 \).

(ii) Every element \( f \in \text{Lip}_{0F}(X,F) \) has a representation in the form \( f = \sum_{j=1}^{m} g_j \cdot u_j \), where \( m = \text{rank}(f) \), \( g_1, \ldots, g_m \in \text{Lip}_0(X) \) and \( u_1, \ldots, u_m \in F \).

**Proof.** (i) Clearly, \( g \cdot u \) is well-defined. Let \( x, y \in X \). For any \( u \in F \), we obtain
\[
\|g(x)u - g(y)u\| = \|g(x)u - g(y)u\| = \|g(x) - g(y)\|u\| \leq \text{Lip}(g)d(x,y)\|u\|,
\]
and so \( g \cdot u \in \text{Lip}_0(X,F) \) and \( \text{Lip}(g \cdot u) \leq \text{Lip}(g)\|u\| \). For the converse inequality, note that
\[
\|g(x) - g(y)\|u\| = \|g \cdot u(x) - g \cdot u(y)\| \leq \text{Lip}(g \cdot u)d(x,y)
\]
for all \( x, y \in X \), and therefore \( \text{Lip}(g \cdot u) \leq \text{Lip}(g \cdot u) \).

(ii) Suppose that the linear hull \( \{f(X)\} \) of \( f(X) \) in \( F \) is \( m \)-dimensional and let \( \{u_1, \ldots, u_m\} \) be a base of \( \text{lin}(f(X)) \). Then, for each \( x \in X \), the element \( f(x) \in f(X) \) is expressible in a unique form as \( f(x) = \sum_{j=1}^{m} \lambda_j^{(x)} u_j \), where \( \lambda_1^{(x)}, \ldots, \lambda_m^{(x)} \in K \). For each \( j \in \{1, \ldots, m\} \), define the linear map \( y^j : \text{lin}(f(X)) \to K \) by \( y^j(f(x)) = \lambda_j^{(x)} \) for all \( x \in X \). Let \( g_j = y^j \circ f \). Clearly, \( g_j \in \text{Lip}_0(X) \) and \( f(x) = \sum_{j=1}^{m} \lambda_j^{(x)} u_j = \sum_{j=1}^{m} g_j(x) u_j \) for all \( x \in X \). Hence \( f = \sum_{j=1}^{m} g_j \cdot u_j \).

**Proposition 4.4.** Let \( X \) be a pointed metric space and \( F \) a Banach space. Then \( \text{Lip}_0(X) \otimes F \) is linearly isomorphic to \( \text{Lip}_{0F}(X,F) \) via the linear bijection \( K : \text{Lip}_0(X) \otimes F \to \text{Lip}_{0F}(X,F) \) given by
\[
K \left( \sum_{j=1}^{m} g_j \otimes u_j \right) = \sum_{j=1}^{m} g_j \cdot u_j.
\]

**Proof.** Let \( \sum_{j=1}^{m} g_j \otimes u_j \in \text{Lip}_0(X) \otimes F \). The mapping \( K \) is well defined. Indeed, if \( \sum_{j=1}^{m} g_j \otimes u_j = 0 \), then \( \sum_{j=1}^{m} \varphi(g_j)u_j = 0 \) for every \( \varphi \in \text{Lip}_0(X)' \) by [22, Proposition 1.2]. In particular, \( \sum_{j=1}^{m} \delta_x(g_j)u_j = 0 \) for every \( x \in X \) and thus \( \sum_{j=1}^{m} g_j \cdot u_j = 0 \) as required. Clearly, \( K \) is linear and, by Lemma 4.3, is onto. To see that it is injective, assume that \( K(\sum_{j=1}^{m} g_j \otimes u_j) = 0 \). Then \( \sum_{j=1}^{m} \delta_x(g_j)u_j = 0 \) for every \( x \in X \), and since \( \{\delta_x : x \in X\} \) is a separating subset of \( \text{Lip}_0(X)' \), we infer that \( \sum_{j=1}^{m} g_j \otimes u_j = 0 \) (see [22, p. 3-4]).

From the preceding results we deduce the next result that characterizes the \( \text{AP} \) for the space \( \text{Lip}_0(X) \) with the topology of bounded compact convergence.

**Corollary 4.5.** Let \( X \) be a pointed metric space. The following are equivalent.

(i) \( \text{Lip}_0(X, \tau_r) \) has the \( \text{AP} \).

(ii) \( \mathcal{F}(X) \) has the \( \text{AP} \).

(iii) \( \text{Lip}_0(X) \otimes F \) is dense in \( (\text{Lip}_0(X, \tau_r) \otimes F) \) for every Banach space \( F \).

(iv) \( \text{Lip}_{0F}(X,F) \) is dense in \( (\text{Lip}_0(X,F), \tau_r \otimes \tau_r) \) for every Banach space \( F \).

**Proof.** (i) \( \Leftrightarrow \) (ii): Assume that \( \text{Lip}_0(X, \tau_r) \) has the \( \text{AP} \). Since \( \text{Lip}_0(X, \tau_r) = \mathcal{F}(X)' \) by Theorem 2.2 (iv), then \( \mathcal{F}(X) \) has the \( \text{AP} \) by Proposition 1.3. Conversely, if \( \mathcal{F}(X) \) has the \( \text{AP} \), we use that \( \mathcal{F}(X) = (\text{Lip}_0(X, \tau_r)' \) by Theorem 2.2 (iii) to obtain that \( \text{Lip}_0(X, \tau_r) \) has the \( \text{AP} \) by Proposition 1.3.
5. The dual space of \((\text{Lip}_0(X,F),\gamma\tau)\)

The following theorem describes the dual of the space \((\text{Lip}_0(X,F),\gamma\tau)\). Recall that a linear functional \(T\) on a topological vector space \(Y\) is continuous if and only if there is a neighborhood \(U\) of \(0\) in \(Y\) such that \(T(U)\) is a bounded subset of \(\mathbb{K}\). Hence \(T \in (\text{Lip}_0(X,F),\gamma\tau)'\) if and only if there exist a constant \(c > 0\) and sequences \(\{\alpha_n\} \in c_0(\mathbb{R}^+)\) and \(\{(x_n,y_n)\} \in \bar{X}^\mathbb{N}\) such that

\[
|T(f)| \leq c \sup_{n \in \mathbb{N}} \alpha_n \frac{\|f(x_n) - f(y_n)\|}{d(x_n,y_n)}
\]

for every \(f \in \text{Lip}_0(X,F)\).

**Theorem 5.1.** Let \(X\) be a pointed metric and let \(F\) be a Banach space. Then a linear functional \(T\) on \(\text{Lip}_0(X,F)\) is in the dual of \((\text{Lip}_0(X,F),\gamma\tau)\) if and only if there exist sequences \(\{\phi_n\} \in F'\) and \(\{(x_n,y_n)\} \in \bar{X}\) such that \(\sum_{n=1}^{\infty} \|\phi_n\| < \infty\) and

\[
T(f) = \sum_{n=1}^{\infty} \phi_n \left( \frac{f(x_n) - f(y_n)}{d(x_n,y_n)} \right)
\]

for all \(f \in \text{Lip}_0(X,F)\).

**Proof.** Assume that \(T\) is a linear functional on \(\text{Lip}_0(X,F)\) of the preceding form. Since \(\sum_{n=1}^{\infty} \|\phi_n\| < \infty\), we can take a sequence \(\{\lambda_n\} \in \mathbb{R}^+\) tending to \(\infty\) so that \(\sum_{n=1}^{\infty} \lambda_n \|\phi_n\| = c < \infty\). Then we have

\[
|T(f)| \leq \sum_{n=1}^{\infty} \|\phi_n\| \frac{\|f(x_n) - f(y_n)\|}{d(x_n,y_n)} \leq c \sup_{n \in \mathbb{N}} \lambda_n \frac{\|f(x_n) - f(y_n)\|}{d(x_n,y_n)}
\]

for all \(f \in \text{Lip}_0(X,F)\). This proves that \(T\) is continuous on \((\text{Lip}_0(X,F),\gamma\tau)\).

Conversely, if \(T \in (\text{Lip}_0(X,F),\gamma\tau)'\), then there are sequences \(\{\alpha_n\} \in c_0(\mathbb{R}^+)\) and \(\{(x_n,y_n)\} \in \bar{X}\) such that

\[
|T(f)| \leq \sup_{n \in \mathbb{N}} \alpha_n \frac{\|f(x_n) - f(y_n)\|}{d(x_n,y_n)}
\]

for every \(f \in \text{Lip}_0(X,F)\). Consider the linear subspace

\[
Z = \left\{ \left\{ \alpha_n \frac{f(x_n) - f(y_n)}{d(x_n,y_n)} \right\} : f \in \text{Lip}_0(X,F) \right\}
\]

of \(c_0(F)\), and the functional \(S\) on \(Z\) given by

\[
S\left( \left\{ \alpha_n \frac{f(x_n) - f(y_n)}{d(x_n,y_n)} \right\} \right) = T(f)
\]

for every \(f \in \text{Lip}_0(X,F)\). It follows easily that \(S\) is well defined and linear. Since

\[
\left| S\left( \left\{ \alpha_n \frac{f(x_n) - f(y_n)}{d(x_n,y_n)} \right\} \right) \right| = |T(f)| \leq \sup_{n \in \mathbb{N}} \alpha_n \frac{\|f(x_n) - f(y_n)\|}{d(x_n,y_n)} = \left\| \left\{ \alpha_n \frac{f(x_n) - f(y_n)}{d(x_n,y_n)} \right\} \right\|_\infty
\]

for all \(f \in \text{Lip}_0(X,F)\), \(S\) is continuous on \(Z\). By the Hahn-Banach theorem, \(S\) has a norm-preserving continuous linear extension \(\hat{S}\) to all of \(c_0(F)\). Since \(c_0(F)'\) is just \(\ell_1(F')\), there exists a sequence \(\{\psi_n\} \in F'\) such that \(\sum_{n=1}^{\infty} \|\psi_n\| = \|\hat{S}\|\) and \(\hat{S}(u_n) = \sum_{n=1}^{\infty} \psi_n(u_n)\) for any \(\{(u_n)\} \in c_0(F)\). Taking \(\phi_n = \alpha_n \psi_n\) for each \(n \in \mathbb{N}\), we conclude that \(\sum_{n=1}^{\infty} \|\phi_n\| \leq \left\| \left\{ \alpha_n \right\}_\infty \right\| \|\hat{S}\| < \infty\) and

\[
T(f) = \hat{S}\left( \left\{ \alpha_n \frac{f(x_n) - f(y_n)}{d(x_n,y_n)} \right\} \right) = \sum_{n=1}^{\infty} \phi_n \left( \frac{f(x_n) - f(y_n)}{d(x_n,y_n)} \right)
\]
for all \( f \in \text{Lip}_0(X,F) \).

Since \( \mathcal{F}(X) = (\text{Lip}_0(X), \tau_\gamma)_0 \) by Theorem 2.2 (iii) and \( \tau_\gamma = \gamma \tau_\gamma \) on \( \text{Lip}_0(X) \) by Proposition 1.1 (ii), we next apply Theorem 5.1 to describe the members of \( \mathcal{F}(X) \).

**Corollary 5.2.** Let \( X \) be a pointed metric. Then \( \mathcal{F}(X) \) consists of all functionals \( T \in \text{Lip}_0(X)' \) of the form

\[
T(f) = \sum_{n=1}^{\infty} \alpha_n \frac{f(x_n) - f(y_n)}{d(x_n, y_n)}, \quad f \in \text{Lip}_0(X),
\]

where \( \{\alpha_n\} \in \ell_1 \) and \( \{(x_n, y_n)\} \in \hat{X}^2 \).

### 6. The Approximation Property for \((\text{Lip}_0(E), \tau_\gamma)\)

We now study the relation between the (AP) for a Banach space \( E \) and for the space \((\text{Lip}_0(E), \tau_\gamma)\).

**Definition 6.1.** Let \( E \) be a Banach space. It is said that \( E \) has the Lipschitz approximation property (in short, \((\text{LAP})\)) if for each compact set \( K \subset E \) and each \( \varepsilon > 0 \), there exists \( f \in \text{Lip}_0(E,E) \) such that \( \|f(x) - x\| < \varepsilon \) for every \( x \in K \).

A bounded version of the preceding property was introduced by Godefroy and Kalton in [10].

**Theorem 6.1.** Let \( E \) be a Banach space. The following are equivalent:

(i) \( E \) has the (AP).
(ii) \( E \) has the (LAP).
(iii) \( \mathcal{F}(E) \) has the (AP).
(iv) \( (\text{Lip}_0(E), \tau_\gamma) \) has the (AP).

**Proof.** (i) \( \Rightarrow \) (ii) is obvious. To prove (ii) \( \Rightarrow \) (iii), we appeal to the proof of [10] Theorem 5.3. Let \( \varphi \in \mathcal{F}(E) \) and \( \varepsilon > 0 \). Then there exist sequences \( \{\lambda_n\} \in \mathbb{K} \) and \( \{x_n\} \in E \) such that \( \varphi = \sum_{n=1}^{\infty} \lambda_n \delta_E(x_n) \), where \( \delta_E : E \to \mathcal{F}(E) \) is the Dirac map. Let \( m \) be any positive integer and consider \( u_m = \sum_{n=1}^{m} \lambda_n \delta_E(x_n) \). Write \( c = 1 + \sum_{n=1}^{m} |\lambda_n| \). Then, by (ii), there exists \( f \in \text{Lip}_0(E,E) \) such that \( \|f(x_n) - x_n\| < \varepsilon/4c \) for all \( n \in \{1, \ldots, m\} \). Let \( E_0 = \text{lin}(f(E)) \) and let \( \iota : E_0 \to E \) be the canonical embedding. By Theorem 1.5 there exist an operator \( \hat{f} \in \mathcal{L}(\mathcal{F}(E); \mathcal{F}(E_0)) \) and an isometric embedding \( \hat{i} \in \mathcal{L}(\mathcal{F}(E_0); \mathcal{F}(E)) \) such that \( \hat{f} \circ \delta_E = \delta_{E_0} \circ f \) and \( \hat{i} \circ \delta_{E_0} = \delta_E \circ \iota \). Since \( E_0 \) is finite dimensional, then \( \mathcal{F}(E_0) \) has the (AP) by [10] Proposition 5.1 and therefore there exists a finite-rank operator \( S \in \mathcal{L}(\mathcal{F}(E_0); \mathcal{F}(E_0)) \) such that \( \|S(F) - F\| < \varepsilon/4c \) for all \( F \in \mathcal{F}(E_0) \). Clearly, \( \hat{i}S\hat{f} \) is a finite-rank operator in \( \mathcal{L}(\mathcal{F}(E); \mathcal{F}(E)) \) and

\[
\left\| \hat{i}S\hat{f}(u_m) - u_m \right\| \leq \sum_{n=1}^{m} |\lambda_n| \left\| \hat{i}S\delta_E(x_n) - \delta_E(x_n) \right\|
\leq \sum_{n=1}^{m} |\lambda_n| \left( \left\| \hat{i}S\delta_E(x_n) - \hat{i}\delta_E(x_n) \right\| + \left\| \hat{i}\delta_E(x_n) - \delta_E(x_n) \right\| \right)
= \sum_{n=1}^{m} |\lambda_n| \left( \left\| \hat{f}\delta_E(x_n) - \hat{i}\delta_E(x_n) \right\| + \left\| f(x_n) - x_n \right\| \right)
< \frac{\varepsilon}{2}.
\]

Therefore

\[
\left\| \hat{i}S\hat{f}(\varphi) - \varphi \right\| = \lim_{m \to \infty} \left\| \hat{i}S\hat{f}(u_m) - u_m \right\| < \varepsilon,
\]

and this proves (iii).
To see that (iii) implies (i), assume that \( \mathcal{F}(E) \) has the (AP) but \( E \) fails the (AP). Then the identity operator \( I_E \) on \( E \) is not in the closure of \( \text{Lip}_0(E,E) \) in \( (\text{Lip}_0(E,E), \gamma_0) \) and thus it is not in the closure of \( \text{Lip}_0(E,E) \) in \( (\text{Lip}_0(E,E), \gamma_\tau) \) by Proposition 6.1 (i). Then, by the Hahn–Banach theorem, there exists \( T \in (\text{Lip}_0(E,E), \gamma_\tau)' \) such that \( T(f) = 0 \) for every \( f \in \text{Lip}_0(E,E) \) and \( T(I_E) = 1 \). By Theorem 4.3 there exist sequences \( \{\phi_n\} \) in \( E' \) and \( \{(x_n, y_n)\} \) in \( E \) such that \( \sum_{n=1}^\infty \|\phi_n\| < \infty \) and

\[
T(f) = \sum_{n=1}^\infty \phi_n \left( \frac{f(x_n) - f(y_n)}{\|x_n - y_n\|} \right)
\]

for all \( f \in \text{Lip}_0(E,E) \). Fix \( g \in \text{Lip}_0(E,E) \) and \( e \in E \). Then \( g \cdot e \in \text{Lip}_0(E,E) \) by Lemma 4.3 and therefore

\[
0 = T(g \cdot e) = \sum_{n=1}^\infty \phi_n \left( \frac{g(x_n)e - g(y_n)e}{\|x_n - y_n\|} \right) = \sum_{n=1}^\infty \phi_n(e) \frac{g(x_n) - g(y_n)}{\|x_n - y_n\|} = \sum_{n=1}^\infty \phi_n(e) \frac{\delta_{x_n} - \delta_{y_n}}{\|x_n - y_n\|}.
\]

Since \( g \) is arbitrary, we have

\[
0 = \sum_{n=1}^\infty \phi_n(e) \frac{\delta_{x_n} - \delta_{y_n}}{\|x_n - y_n\|} = \sum_{n=1}^\infty T_{\phi_n} \left( \frac{\delta_{x_n} - \delta_{y_n}}{\|x_n - y_n\|} \right)
\]

for every \( e \in E \). Since the linear hull of the set \( \{\delta_e : e \in E\} \) is dense in \( \mathcal{F}(E) \), we obtain that

\[
0 = \sum_{n=1}^\infty T_{\phi_n} \left( \frac{\delta_{x_n} - \delta_{y_n}}{\|x_n - y_n\|} \right)
\]

for all \( \varphi \in \mathcal{F}(E) \). Moreover, \( \{T_{\phi_n}\} \) is a sequence in \( \mathcal{F}(E)' \) and \( \{(\delta_{x_n} - \delta_{y_n})/\|x_n - y_n\|\} \) is a sequence in \( \mathcal{F}(E) \) such that

\[
\sum_{n=1}^\infty \|T_{\phi_n}\| \left( \frac{\|\delta_{x_n} - \delta_{y_n}\|}{\|x_n - y_n\|} \right) = \sum_{n=1}^\infty \|\phi_n\| < \infty.
\]

Since \( \mathcal{F}(E) \) has the (AP), it follows that

\[
\sum_{n=1}^\infty T_{\phi_n} \left( \frac{\delta_{x_n} - \delta_{y_n}}{\|x_n - y_n\|} \right) = 0.
\]

by applying [7] Theorem 4, p. 239], but

\[
1 = T(I_E) = \sum_{n=1}^\infty \phi_n \left( \frac{x_n - y_n}{\|x_n - y_n\|} \right) = \sum_{n=1}^\infty T_{\phi_n} \left( \frac{\delta_{x_n} - \delta_{y_n}}{\|x_n - y_n\|} \right).
\]

This contradiction shows that (iii) implies (i).

Finally, (iii) \( \iff \) (iv) has been proved in Corollary 4.3 and the proof is finished. \( \square \)

**Corollary 6.2.** Let \( E \) and \( F \) be Lipschitz isomorphic Banach spaces. If \( E \) has the (AP), then \( F \) also has the (AP).

**Proof.** If \( h : E \to F \) is a bi-Lipschitz map such that \( h(0) = 0 \), then it is clear that \( h^* : \text{Lip}_0(F) \to \text{Lip}_0(E) \) defined by \( h^*(f) = f \circ h \) is a topological isomorphism from \( (\text{Lip}_0(F), \tau_\gamma) \) onto \( (\text{Lip}_0(E), \tau_\gamma) \). If \( E \) is a Banach space with the (AP), then \( (\text{Lip}_0(E), \tau_\gamma) \) also has the (AP) by Theorem 6.1. If \( E \) and \( F \) are Lipschitz isomorphic, then \( (\text{Lip}_0(E), \tau_\gamma) \) and \( (\text{Lip}_0(F), \tau_\gamma) \) are topological isomorphic so that \( (\text{Lip}_0(F), \tau_\gamma) \) has the (AP) and it follows that \( F \) has the (AP) by Theorem 6.1. \( \square \)
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