A Description of Dark Energy in Terms of Two-Component Massive Spin-One Uncharged Fields on Spacetimes with Torsionful Affinities

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Abstract
The two-component spinor formalisms for curved spacetimes that are endowed with torsionful affine connections presumably supply a geometric description of the physical nature of dark energy in terms of classical massive spin-one uncharged fields. It appears that the relevant wave functions are related to torsional affine potentials which are invariant under the generalized Weyl gauge group. Such potentials may thus be taken to carry an observable character and arise directly from contracted spin affinities whose patterns are chosen in a suitable way. New covariant calculational techniques are then developed towards deriving explicitly the wave equations that control the propagation in spacetime of the dark energy background.

1 Introduction
Since the discovery of the cosmic dark energy [1, 2], many cosmological attempts have been made at designing elementary approaches and technical procedures that might provide a macroscopic explanation of the presently observable acceleration of the universe (see, for instance, Refs. [3-8]). Some of these works turned out to bring forth the need for implementing a torsional version of the standard cosmological model [9-12], in conjunction with a theoretical possibility of particularly explaining the cosmic acceleration. Noticeably enough, among the works we have just referred to, there is one [6] involving a non-minimally coupled interaction between gravity and a real massive spin-one field, which

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supplies a late-time accelerated expansion of a De Sitter type. In this context, the cosmological constant is naively related to the rest mass of the field allowed for, but no intrinsically geometric character at all was ascribed to it thereabout. Another notable approach [8] takes the cosmological constant to be generated by a small vacuum-energy contribution in a way which apparently avoids the occurrence of the cosmological constant problem while exhibiting consistency with some astronomical observations. Nevertheless, it may surely be said [11] that no definite microscopic characterization of dark energy fields has been achieved hitherto.

The present work amounts to the second part of the programme we had brought up earlier in Ref. [13] where the essentially unique torsionful extension of the classical two-component spinor formalisms for general relativity [14-16] was exhibited as the first part. At this stage, we take account of such torsional formalisms to bring forward a supposedly realistic description of dark energy on a purely geometric basis. As we believe, the true origin of dark energy relies geometrically upon the feasibility of choosing suitable asymmetric torsional spin-affine connections that supply gauge-invariant potentials for massive spin-one uncharged fields. Our view is that the universe could have been expected beforehand to host two physical backgrounds, namely, a torsionless electromagnetic one which would accordingly be predictable by the old two-component spinor formalisms such as emphasized in Refs. [16, 17], and a torsionful one which would presumably be describable in terms of geometric Proca fields within a suitably extended spinor framework. The former would thus be identified with the cosmic microwave background (CMB) whilst the latter would be thought of as constituting the cosmic dark energy.

We shall account for the well-established observational fact that the CMB and dark energy permeate together the whole of the universe. However, we realize that the propagation of the CMB in regions of the universe where the values of torsional affinities are negligible, may be described alone within the framework of Ref. [17]. On the other hand, as was pointed out in Ref. [13], any torsional affine potential must be accompanied by proper torsionless contributions. In actuality, the implementation of this situation gives rise to the striking physical feature that the spacetime description of dark energy has to be united together with that of the CMB. The completion of our main procedures will be accomplished without making it necessary to call upon any ordinary cosmological presuppositions like those concerning homogeneities, inflation and shape of physical densities. The only assumptions underlying these procedures are the same as the geometric ones made in Ref. [13], according to which the structure of manifold mapping groups and the matrices that constitute the generalized Weyl gauge group [18] remain all unaltered when any classical spacetime consideration is shifted to the torsional framework. We stress that the defining prescriptions for any of the geometric world and spin densities tied in with the old formalisms [16], may be applicable equally well herein. Loosely speaking, the information on the wave functions for both physical backgrounds is carried by adequately contracted spin curvatures which directly emerge as sums of typical bivector contributions from the action on arbitrary spin vectors of
a characteristic torsionful second-order covariant derivative operator [13]. The additivity property borne by such contracted curvatures is really passed on to the wave functions.

We will allow for the notation adhered to in Ref. [16]. Unless otherwise indicated in an explicit manner, the usual designation of the traditional framework as \( \gamma \)-formalisms [14, 16] will henceforward be attributed to the torsionful spinor formalisms mentioned previously. Upon writing down the world form of the pertinent field equations, we shall take into consideration classical electromagnetic and uncharged Proca fields settled in a curved spacetime \( M \) that carries a world metric tensor \( g_{\mu\nu} \) having the local signature \((++--)\) along with a torsionful, metric compatible, covariant derivative operator \( \nabla_\mu \). The spinor form of the field equations will be obtained by putting into effect a straightforward transcription of the respective world statements. We will see that the resulting spinor field equations involve pairs of complex conjugate current densities for each physical background, which absorb outer products carrying appropriate torsion spinors and the wave functions themselves. In either formalism, it appears that some pieces of such new geometric sources must be subject to certain gauge invariant subsidiary conditions which are brought about by the inherent symmetry of the corresponding wave functions. In order to carry out systematically the derivation of the wave equations that control the propagation of the fields in \( M \), we shall have to adapt to the torsional framework the differential calculational techniques employed for the first time in the work of Ref. [16].

Without any risk of confusion, we will use the same indexed symbol \( \nabla_\mu \) to write covariant derivatives in both formalisms. The symbol \( g \) will sometimes be used for denoting the determinant of \( g_{\mu\nu} \). For the world affine connection associated with \( \nabla_\mu \), we have the splitting

\[
\Gamma_{\mu\nu\lambda} = \tilde{\Gamma}_{\mu\nu\lambda} + T_{\mu\nu\lambda},
\]

where \( \tilde{\Gamma}_{\mu\nu\lambda} \) may eventually be identified with a Christoffel connection, and \( T_{\mu\nu\lambda} \) is by definition the torsion tensor of \( \nabla_\mu \). We take the elements of the Weyl gauge group as non-singular complex \((2 \times 2)\)-matrices whose entries are defined by

\[
\Lambda_A^B = \exp(i\theta)\delta_A^B,
\]

where \( \delta_A^B \) denotes the Kronecker symbol and \( \theta \) is the gauge parameter of the group which shows up as an arbitrary differentiable real-valued function on \( M \). The determinant \( \exp(2i\theta) \) of \( (\Lambda_A^B) \) will be denoted as \( \Delta_\Lambda \). We will also adopt the natural system of units in which \( c = \hbar = 1 \). A horizontal bar lying over some kernel letter will denote the operation of complex conjugation. Some minor conventions shall be explained in due course.

Our outline has been set as follows. In Section 2, we bring out the contracted spin curvatures as built up in Ref. [13] together with the world field equations. The definition of all wave functions and the spinor field equations, are shown in Section 3. The torsional calculational techniques are developed in Section 4. There, we will have to consider spin curvatures somewhat further. Yet the derivation of several of the curvature formulae displayed in Ref. [13] shall be
taken for granted at the outset. In Section 5, the wave equations are deduced. We make some remarks on our work in Section 6.

2 World Field Equations

The key curvature object for either formalism is a world-spin quantity $C_{\mu\nu AB}$ that occurs in the configuration

$$D_{\mu\nu}\zeta^B = C_{\mu\nu A} B^A \zeta^A,$$

where $\zeta^A$ is an arbitrary spin vector and $D_{\mu\nu}$ amounts to the characteristic second-order covariant derivative operator of the torsional framework, namely,

$$D_{\mu\nu} \equiv 2(\nabla_{[\mu} \nabla_{\nu]} + T_{\mu\nu} \Lambda^\lambda \nabla_{\lambda}).$$

In the $\gamma$-formalism, we have the tensor law

$$C'_{\mu\nu AB} = \Lambda_A C_{\mu C D} = \Delta A C_{\mu\nu AB},$$

whereas the object $C_{\mu\nu AB}$ for the $\epsilon$-formalism is taken as an invariant spin-tensor density of weight $-1$, that is to say,

$$C'_{\mu\nu AB} = (\Delta A)^{-1} \Lambda_A C_{\mu C D} = C_{\mu\nu AB}.$$

The contracted curvature $C_{\mu\nu A}^A$ possesses the additivity property

$$C_{\mu\nu A}^A = \tilde{C}_{\mu\nu A}^A + C_{(T)\mu A} A.$$

In particular, $C_{(T)\mu A}^A$ accounts for the torsionfulness of $\nabla_{\mu}$ while the whole $\tilde{C}_{\mu\nu AB}$ is taken up by the torsionless commutator

$$2\nabla_{[\mu} \tilde{\nabla}_{\nu]} \zeta^B = \tilde{C}_{\mu\nu A} B^A \zeta^A,$$

where $\tilde{\nabla}_{\mu}$ is indeed the covariant derivative operator for $\tilde{\Gamma}_{\mu\nu\lambda}$. It turns out [13] that we can write down the simultaneous relations

$$\tilde{C}_{\mu\nu A} = 2\partial_{[\mu} \psi_{\nu]}^A, \quad C_{(T)\mu A}^A = 2\partial_{[\mu} \psi_{(T)\nu]}^A,$$

with the involved $\psi$-pieces thus coming from the skew contributions that make up in each formalism a suitably chosen asymmetric spin affinity for $\nabla_{\mu}$, in agreement with Eq. [4]. Hence, making use of the standard patterns [13]

$$\tilde{\psi}_{\mu A} = \partial_{\mu} \log E - 2i\Phi_\mu, \quad \psi_{(T)\mu A}^A = -2iA_\mu,$$

1We should emphasize that the uncontracted object $C_{\mu\nu A}^B$ for either formalism does not hold the additivity property.
yields the purely imaginary expression

\[ C_{\mu\nu}A^\lambda = -2i(\tilde{F}_{\mu\nu} + F^{(T)}_{\mu\nu}), \]  

(9)

along with the bivectors

\[ \tilde{F}_{\mu\nu} \doteq 2\partial\{\Phi\mu\nu\}, \quad F^{(T)}_{\mu\nu} \doteq 2\partial\{A\mu\nu\}, \]  

(10)

with \( \Phi\mu \) and \( A\mu \) amounting to affine potentials subject to the gauge behaviours

\[ \Phi'\mu = \Phi\mu - \partial\mu\theta, \quad A'\mu = A\mu. \]  

(11)

It is worthwhile to recast each of the derivatives of Eq. (10) as a piece that looks formally like

\[ \partial\{\Omega\mu\nu\} = \nabla\{\Omega\mu\nu\} + T^{\lambda}_{\mu\nu}\lambda\Omega, \]  

(12)

We mention, in passing, that the quantity \( E \) carried by the prescriptions (8) is a real positive-definite world-invariant spin-scalar density of absolute weight +1. In the \( \gamma \)-formalism, it carries a manifestly spin-metric character, but this ceases holding for the \( \varepsilon \)-formalism. The potentials \( \Phi\mu \) and \( A\mu \) are the same in both formalisms. They arise naturally from a reality property of the covariant derivative expansions for the Hermitian connecting objects of the formalisms (for further details, see Ref. [13]).

It can be seen from Eq. (11) that \( \Phi\mu \) is a Maxwell potential, which we take to be physically associated to the CMB. In turn, \( A\mu \) bears gauge invariance and is likewise looked upon as a potential of mass \( m \) for the dark energy background. The world form of the first half of the relevant field equations emerges from the usual least-action principles for Maxwell and real Proca fields in curved spacetimes [19]. It follows that, allowing for the relation

\[ \frac{1}{\sqrt{-g}} \partial\mu(\sqrt{-g}F^{\mu\lambda}) = \nabla\mu F^{\mu\lambda} + 2T^{\mu}_{\mu\tau}\lambda F^{\mu\tau}, \]  

(13)

with \( T^{\mu}_{\mu\tau} \) and the kernel letter \( F \) standing for either \( \tilde{F} \) or \( F^{(T)} \), we get the first half of Maxwell’s equations

\[ \nabla^{\mu}\tilde{F}_{\mu\lambda} + 2T^{\mu}_{\mu\tau}\tilde{F}_{\mu\lambda} - T^{\mu\nu}_{\lambda}\tilde{F}_{\mu\nu} = 0, \]  

(14)

along with the first half of Proca’s equations

\[ \nabla^{\mu}F^{(T)}_{\mu\lambda} + 2T^{\mu}_{T\nu}\tilde{F}^{(T)}_{\mu\nu} - T^{\mu\nu}_{\lambda}F^{(T)}_{\mu\nu} + m^2A_{\lambda} = 0. \]  

(15)

Obviously, in accordance with our picture, the statements (14) and (15) are the dynamical world field equations in \( \mathcal{M} \) for CMB photons and dark energy fields. Both of the second halves come about as the corresponding Bianchi identities, which may be expressed by

\[ \nabla^{\mu}sF_{\lambda\mu} = -2sT_{\lambda\mu\tau}F_{\mu\tau}, \]  

(16)

with the kernel-letter notation of (13), as well as some of the dualization prescriptions given in Ref. [20], having been utilized for writing Eq. (16).
3 Spinor Field Equations

The wave functions for both backgrounds are supplied by the spinor decomposition of the bivectors carried by Eq. (10). We have, in effect,

\[ S_{AA'}^\mu S_{BB'}^\nu \bar{F}_{\mu\nu} = M_{AB'}\phi_{AB} + M_{AB}\phi_{A'B'} \quad (17) \]

and

\[ S_{AA'}^\mu S_{BB'}^\nu F_{\mu\nu}^{(T)} = M_{AB'}\psi_{AB} + M_{AB}\psi_{A'B'} \quad (18) \]

where the S-symbols are some of the connecting objects for the formalism occasionally allowed for, and the entries of the pair \((M_{AB}, M_{AB'})\) just denote the respective covariant metric spinors.\(^2\)

Thus, the wave functions carried by \((\phi_{AB}, \phi_{A'B'})\) and \((\psi_{AB}, \psi_{A'B'})\) come into play as massless and massive spin-one uncharged fields of opposite handednesses, with their gauge characterizations incidentally coinciding with those exhibited by Eqs. (3) and (4). By invoking Eq. (12) together with the torsion decomposition

\[ T_{AA'BB'}^\mu = M_{AB'}\tau_{AB}\mu + M_{AB}\tau_{A'B'}\mu \quad (19) \]

we obtain the field-potential relationships

\[ \phi_{AB} = -\nabla_{(A}^{C'}\Phi_{B)C'} + 2\tau_{AB}\mu \Phi_\mu \quad (20) \]

and

\[ \psi_{AB} = -\nabla_{(A}^{C'}\tau_{B)C'} + 2\tau_{AB}\mu \tau_\mu \quad (21) \]

The contravariant form of (20) and (21) is written in both formalisms as

\[ \phi^{AB} = \nabla^{(A}^{(C'}\Phi_{B)C')} + 2\tau_{AB}\mu \Phi_\mu \quad (22) \]

and

\[ \psi^{AB} = \nabla^{(A}^{(C'}\tau_{B)C')} + 2\tau_{AB}\mu \tau_\mu \quad (23) \]

where we have particularly implemented the eigenvalue equations [13]

\[ \nabla_\mu \gamma_{AB} = i\alpha_\mu \gamma_{AB}, \quad \nabla_\mu \gamma_{AB} = -i\alpha_\mu \gamma_{AB} \]

(24)

together with their conjugates and the definition

\[ \alpha_\mu \doteq \partial_\mu \Phi + 2(\Phi_\mu + A_\mu) \]

(25)

with the quantity \(\Phi\) being nothing else but the polar argument of the independent component of \(\gamma_{AB}\) (see Eq. (10) below).

We next carry out the spinor translation of the individual pieces of Eqs. (14)-(16), with the purpose of paving the way for deriving the field equations at issue. Evidently, it will suffice to carry through the apposite procedures for either of the \(F\)-bivectors of Eq. (13). For \(\nabla_\mu F_{\mu\lambda}^{(T)}\), say, we have

\[ \nabla^{AA'}F_{AA'BB'}^{(T)} = \nabla^{AA'}(M_{AB'}\psi_{AB}) + c.c., \quad (26) \]

\(^2\)The kernel letter \(M\) will henceforth denote either \(\gamma\) or \(\epsilon\).
with the symbol "c.c." denoting an overall complex conjugate piece. In the \( \gamma \)-formalism, the right-hand side of Eq. (26) reads
\[
\nabla A A' (\gamma A'B \psi_{AB}) + \text{c.c.} = (\nabla A B \psi_{AB} - i \alpha A'B \psi_{AB}) + \text{c.c.}.
\] (27)
As \( \nabla \mu \varepsilon_{AB} = 0 \) in both formalisms, the \( \varepsilon \)-formalism counterpart of (27) may be obtained by dropping the \( \alpha \)-term from it. By combining Eqs. (18) and (19), we readily find the patterns
\[
T^{AA'} F^{(T)}_{AA'BB'} = (\tau^{AM} M B' - \tau B'M' AM') \psi_{AB} + \text{c.c.}.
\] (28)
and
\[
T^{AA'M'M'} F^{(T)}_{AA'M'M'} = 2 \tau^{AM} M B' \psi_{AM} + \text{c.c.},
\] (29)
which just represent \( T^{\mu} F^{(T)}_{\mu} \) and \( T^{\mu \nu} F^{(T)}_{\mu \nu} \) in either formalism. The \( \gamma \)-formalism version of the left-hand side of Eq. (16) is given by
\[
\nabla A A' (\gamma B B' \psi_{AB'}) + \frac{1}{2} m^2 A B B' = s_{BB'},
\] (33)
with the complex dark energy source
\[
s_{BB'} = 2 (\tau^{AM} M B' \psi_{AM} - T^A_{BB'} \psi_{AB}).
\] (34)
It should be pointed out that the term \( \tau B^{AM} B' \psi_{AM} \), which is borne by Eq. (31), cancels out at an intermediate step of the manipulations that yield the statement \( \text{KK} \), and thence also so does its complex conjugate. In the \( \gamma \)-formalism, we then have
\[
\nabla A B' \psi_{AB} - i \alpha A'B \psi_{AB} + \frac{1}{2} m^2 A B B' = s_{BB'},
\] (35)
with the corresponding \( \varepsilon \)-formalism statement being spelt out as
\[
\nabla A B' \psi_{AB} + \frac{1}{2} m^2 A B B' = s_{BB'}.
\] (36)
For the CMB, we get the $\gamma$-formalism massless field equation
\[ \nabla^A \phi_{AB} - i\alpha^{AB} \phi_{AB} = s_{BB'}, \tag{37} \]
along with its $\varepsilon$-formalism counterpart
\[ \nabla^A \phi_{AB} = s_{BB'}, \tag{38} \]
and the geometric source
\[ s_{BB'} = 2(\tau^{AM} - T_{AB})\phi_{AM}. \tag{39} \]

It was demonstrated in Ref. [16] that the wave function $\phi_{AB}$ for the torsionless framework bears a commonness feature in that it is the same in both the classical formalisms. Inasmuch as the traditional set of algebraic definitions for metric spinors and connecting objects is formally appropriate for the torsionful framework as well, we can right away write the $\gamma\varepsilon$-relationships
\[ C^{(\gamma)}_{\mu\nu} D = C^{(\varepsilon)}_{\mu\nu} D \iff C^{(\gamma)}_{\mu\nu C} D = \gamma C^{(\varepsilon)}_{\mu\nu C} D, \tag{40} \]
where $\gamma$ is the independent component of $\gamma_{AB}$. Consequently, we can say that each of the pairs $(\phi_A^B, \phi_A^{B'})$ and $(\psi_A^B, \psi_A^{B'})$ possesses a commonness property which is seemingly similar to the classical one, in addition to holding in both formalisms a gauge invariant spin-tensor character. In each formalism, we thus have the field equations
\[ \nabla^{AB'} \psi_A^B + \frac{1}{2} m^2 \psi^{AB'} = s_{BB'}, \tag{41} \]
and
\[ \nabla^{AB'} \phi_A^B = s_{BB'}, \tag{42} \]
where the $\phi$-field relation, for instance,
\[ \gamma_{CB} \nabla^{AB'} \phi_A^C = \nabla^{AB'} \phi_{AB} - i\alpha^{AB'} \phi_{AB}, \tag{43} \]
has been used in the $\gamma$-formalism case.

## 4 Calcutational Techniques

By this point, we shall build up the techniques that make it feasible to derive in both formalisms the wave equations for the fields being considered. In fact, these techniques constitute a torsional version of the differential ones which had been developed originally within the classical $\gamma\varepsilon$-framework [16, 21]. Let us begin with the operator decomposition
\[ S_{\mu AB} S_{\nu BB'} D_{\mu\nu} = M_{A'B'} D_{AB} + M_{AB} D_{A'B'}. \tag{44} \]

\(^{3}\)We will henceforth stop staggering the indices of any symmetric two-index configuration.
Whence, implementing Eqs. (2) and (19), yields

\[ \tilde{D}_{AB} = \Delta_{AB} + 2\tau_{AB\mu}\nabla_\mu, \quad \Delta_{AB} \doteq -\nabla^C(A\nabla_B)C', \]  

(45)

together with the complex conjugate of (45). The operators \( \tilde{D}_{AB} \) and \( \Delta_{AB} \) both are linear and possess the Leibniz rule property.

It may be useful to utilize Eq. (24) for reexpressing the \( \gamma \)-formalism operator \( \Delta_{AB} \) as

\[ \Delta_{AB} = \nabla^C(A\nabla_B)C' - i\alpha_C(A\nabla_B). \]  

(46)

In the \( \epsilon \)-formalism, one has

\[ \Delta_{AB} = -\nabla^C(A\nabla_B)C' = \nabla^C(A\nabla_B). \]  

(47)

It is worth noticing that the \( \gamma \)-formalism contravariant form of \( \Delta_{AB} \) appears as

\[ \Delta^A_B = -(\nabla^C(A\nabla_B) + i\alpha_C(A\nabla_B)), \]  

(48)

or, equivalently, as

\[ \Delta^A_B = \nabla^C(A\nabla_B)C'. \]  

(49)

Because \( \alpha_\mu \) bears gauge invariance [13], the conjugate \( \tilde{D} \)-operators for the \( \gamma \)-formalism behave under gauge transformations as covariant spin tensors. In the \( \epsilon \)-formalism, they correspondingly behave as invariant spin-tensor densities of weight \(-1\) and antiweight \(-1\).

Equations (1) and (44) suggest that some of the most elementary \( \tilde{D} \)-derivatives should be prescribed in either formalism by

\[ \tilde{D}_{AB\xi}^C = \varpi_{ABM}C^M\xi^C, \quad \tilde{D}_{A'B'\xi}^C = \varpi_{A'B'M}C^M\xi^C, \]  

(50)

with the spin-curvature expansion

\[ C_{AA'BB'C'D} = M_{A'B'}\varpi_{ABCD} + M_{AB}\varpi_{A'B'CD}, \]  

(51)

and the relationships

\[ \varpi^{(\gamma)}_{ABCD} = \gamma^2\varpi^{(\epsilon)}_{ABCD}, \quad \varpi^{(\gamma)}_{A'B'CD} = |\gamma|^2 \varpi^{(\epsilon)}_{A'B'CD}, \]  

(52)

which clearly agree with Eq. (40). It can be shown [13] that the spinor pair \( (\varpi_{AB(CD)}, \varpi_{A'B'(CD)}) \) constitutes the irreducible decomposition of the Riemann tensor for \( \nabla_\mu \), with its unprimed entry being expandable as

\[ X_{ABCD} = \Psi_{ABCD} - M_{(A|(C\xi_D)|B)} - \frac{1}{3} \kappa M_{(C|D)B}, \]  

(53)

where

\[ \Psi_{ABCD} = X_{(ABCD)} \quad \xi_{AB} = X^M_{(AB)M}, \quad \kappa = X^{LM}_{LM}. \]  

\[ ^4\text{From now on, we will for simplicity employ the usual definitions } X_{ABCD} \doteq \varpi_{AB(CD)} \text{ and } \Xi_{A'B'CD} \doteq \varpi_{A'B'(CD)}. \]
Likewise, the contracted pieces $(\varpi_{ABM} M, \varpi_{A'B'M} M)$ fulfill the additivity relations (5) and (9), thereby being proportional to the wave functions of (17) and (18) according to the schemes

\begin{equation}
\varpi_{ABM} M = -2i\phi_{AB}, \quad \varpi_{A'B'M} M = -2i\phi_{A'B'}.
\end{equation}

and

\begin{equation}
\varpi_{(T)}_{ABM} M = -2i\psi_{AB}, \quad \varpi_{(T)}_{A'B'M} M = -2i\psi_{A'B'}.
\end{equation}

Hence, we can cast the prescriptions (50) into the form

\begin{equation}
\hat{D}_{AB} \zeta^C = X_{ABM} C \zeta^M - i(\phi_{AB} + \psi_{AB}) \zeta^C
\end{equation}

and

\begin{equation}
\hat{D}_{A'B'} \zeta^C = \Xi_{A'B'M} C \zeta^M - i(\phi_{A'B'} + \psi_{A'B'}) \zeta^C.
\end{equation}

The prescriptions for computing $\hat{D}$-derivatives of a covariant spin vector $\eta_A$ can be obtained out of Eq. (50) by assuming that

\begin{equation}
\hat{D}_{AB} (\zeta^C \eta_C) = 0, \quad \hat{D}_{A'B'} (\zeta^C \eta_C) = 0,
\end{equation}

and carrying out Leibniz expansions thereof. We thus have

\begin{equation}
\hat{D}_{AB} \eta_C = -[X_{ABC} M \eta_M - i(\phi_{AB} + \psi_{AB}) \eta_C]
\end{equation}

and

\begin{equation}
\hat{D}_{A'B'} \eta_C = -[\Xi_{A'B'M} C \eta_M - i(\phi_{A'B'} + \psi_{A'B'}) \eta_C],
\end{equation}

along with the complex conjugates of Eqs. (57)-(61).

The $\hat{D}$-derivatives of a differentiable complex spin-scalar density $\alpha$ of weight $w$ are written out explicitly as

\begin{equation}
\hat{D}_{AB} \alpha = 2i w \alpha (\phi_{AB} + \psi_{AB}), \quad \hat{D}_{A'B'} \alpha = 2i w \alpha (\phi_{A'B'} + \psi_{A'B'}).
\end{equation}

These configurations may in both formalisms be regarded as immediate consequences of the integrability condition

\begin{equation}
D_{\mu\nu} \alpha = 2i w \alpha (\bar{F}_{\mu\nu} + F_{\mu\nu}^{(T)}).
\end{equation}

When acting on a world-spin scalar $h$ on $\mathfrak{M}$, the $\hat{D}$-operators then recover the defining relation $D_{\mu\nu} h = 0$ as

\begin{equation}
\hat{D}_{AB} h = 0, \quad \hat{D}_{A'B'} h = 0,
\end{equation}

whence

\begin{equation}
\Delta_{AB} h = -2\tau_{AB}^\mu \nabla_\mu h.
\end{equation}

The patterns of $\hat{D}$-derivatives of some spin-tensor density can of course be specified from Leibniz expansions like

\begin{equation}
\hat{D}_{AB} (\alpha B_{C...D}) = (\hat{D}_{AB} \alpha) B_{C...D} + \alpha \hat{D}_{AB} B_{C...D},
\end{equation}

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with $B_{C...D}$ being a spin tensor.

As for the old $\gamma\varepsilon$-framework, whenever $D$-derivatives of Hermitian quantities are actually computed in either formalism, the wave function contributions carried by the expansions (57)-(61) will be cancelled. Such a cancellation likewise happens when we let the $D$-operators act freely upon spin tensors having the same number of covariant and contravariant indices of the same kind. For $w < 0$, it still occurs in the expansion (66) when $B_{C...D}$ is taken to carry $-2w$ indices and $\text{Im}\alpha \neq 0$ everywhere. A similar property also holds for situations that involve outer products between contravariant spin tensors and complex spin-scalar densities having suitable positive weights. The gauge behaviours specified in the foregoing Section tell us that such weight-valence properties neatly fit in with the case of the $\varepsilon$-formalism wave functions.

In carrying out the procedures for deriving our wave equations, it may become necessary to take account of the algebraic rules

$$2\nabla^{A'}_{[B}\nabla^{A}_{A']A'} = M_{AB}\square = \nabla^{A'}_{C}(M_{BA}\nabla^{C}_{A}) \quad (67)$$

and

$$2\nabla^{[A'}_{A'}\nabla^{B]A'} = M^{AB}\square = \nabla^{C}_{A'}(M^{BA}\nabla^{A'}_{C}) \quad (68)$$

along with the operator splittings

$$\nabla^{C'}_{A}\nabla_{BC'} = \frac{1}{2}M_{BA}\square - \Delta_{AB}, \quad \nabla^{A'}_{A'}\nabla^{B} = \Delta^{AB} + \frac{1}{2}M^{AB}\square \quad (69)$$

and the gauge-invariant definition

$$\square \equiv \nabla_{AA'}\nabla^{AA'}. \quad (70)$$

Owing to the applicability in both formalisms of the metric compatibility condition

$$\nabla_{\mu}(M_{AB}M_{A'B'}) = 0, \quad (71)$$

we can reset (70) as

$$\square = \nabla^{AA'}\nabla_{AA'}. \quad (72)$$

In addition, with the help of the equations

$$\square \gamma_{AB} = \Theta \gamma_{AB}, \quad \square \gamma_{AB} = \overline{\Theta} \gamma_{AB}, \quad (73)$$

whose derivation involves using the eigenvalue carried by (24) together with

$$\Theta \equiv -\alpha^{\mu}\alpha_{\mu} + i\nabla_{\mu}\alpha^{\mu}, \quad (74)$$

we also get the symbolic $\gamma$-formalism devices

$$(\square \epsilon^{A}_{C})\gamma_{CB} = (\square - 2i\alpha^{\mu}\nabla_{\mu} + \overline{\Theta})\epsilon_{AB} \quad (75)$$

and

$$\gamma^{AC} (\square \epsilon^{B}_{C}) = (\square + 2i\alpha^{\mu}\nabla_{\mu} + \Theta)\epsilon^{AB}, \quad (76)$$
which obey the valence-interchange rule \(^5\)

\[
i_\alpha^\mu \nabla_\mu \leftrightarrow -i_\alpha^\mu \nabla_\mu, \quad \Theta \leftrightarrow \overline{\Theta}.
\]  

(77)

The \(\Box\)-correlations for \(i_{AB}\) and \(i^{AB}\) can be achieved from

\[
M_{AC}M_{BD}\Box i^{CD} = (\Box - 4i_\alpha^\mu \nabla_\mu - \Upsilon)i_{AB}
\]  

(78)

and

\[
M^{AC}M^{BD}\Box i_{CD} = (\Box + 4i_\alpha^\mu \nabla_\mu - \Upsilon)i^{AB},
\]  

(79)

which conform to Eq. (77) with \(\Upsilon = 2(\alpha^\mu \alpha_\mu - \Theta)\).

5 Wave Equations

To obtain the entire set of wave equations that govern the propagation of both physical backgrounds in \(\mathfrak{M}\), we initially follow up the simpler procedure which consists in implementing the calculational techniques exhibited anteriorly to work out the field equation of either formalism for the common dark energy wave function \(\psi_A^B\). It will become manifest that a gauge invariant condition for each entry of the conjugate pairs \((\psi_A^B, \psi_A^{B'})\) and \((\phi_A^B, \phi_A^{B'})\) may be established as a geometric consequence of the symmetry of the underlying fields. Instead of elaborating upon Eq. (35), which could unnecessarily produce some complicated manipulations, we will deduce the \(\gamma\)-formalism wave equations for the unprimed pair \((\psi_A^B, \psi_A^{B'})\) by appealing to the differential devices \((75)\) and \((76)\). We will certainly get the wave equations for any primed \(\psi\)-fields by taking complex conjugates. The wave equations for all \(\phi\)-fields shall then arise in a trivial way provided that the field equations for both backgrounds formally the same couplings between the wave functions and torsion spinors.

We start by operating with \(\nabla_C^B\) on the configuration of Eq. (41) and recalling the contravariant splitting of (69). This leads us to the statement

\[
\Delta^{AC}\psi^B_A - \frac{1}{2}M^{AC}\Box \psi^B_A + \frac{1}{2}m^2\nabla^C_B A^{BB'} = \nabla^C_B s^{BB'}.
\]  

(80)

It is obvious that both first-order derivative kernels of (80) are of the type

\[
\nabla^C_B u^{BB'} = \nabla^C_B (u^{(B} A^{C)})^{B'} - \frac{1}{2}M^{BC}\nabla_\mu u^\mu,
\]  

(81)

with the \(\gamma\)-formalism symmetric piece for the potential being given as

\[
\nabla^C_B (u^{B} A^{C})^{B'} = \psi^{BC} - 2\tau^{BC} A^\mu_\mu,
\]  

(82)

in accordance with (23). In view of the relation (43), the \(\Delta\)-piece of (80) may be rewritten in either formalism as

\[
\Delta^{AC}\psi^B_A = \mathcal{D}^{AC}\psi^B_A - 2\tau^{AC} A^\mu_\mu \nabla_\mu \psi^B_A.
\]  

(83)

\(^5\)The rule (77) had also arisen in Ref. [16] in connection with the obtainment of the wave equations for the CMB and gravitons in torsionless environments.
Furthermore, calling for Eqs. (57) and (60) along with the expansion (53), after some computations, we get the contributions

\[ \hat{D}^{A(B}_A \psi^{C)}_A = \Psi^{ABC}_M \psi^M_A + \frac{2}{3} \kappa \psi^{BC} - \psi^{(B}_M \xi^{C)}_M \]  

(84)

and

\[ \hat{D}^{A[}_A \psi^{B]}_A = M^{BC} \psi^{AM}_A \xi^{AM}. \]  

(85)

We can see that the symmetry property of the wave functions entails imparting symmetry in the indices \( B \) and \( C \) to the \( \Box \)-block of (80), which means that

\[ M^{A[}_A \Box^{B]}_A = \frac{1}{2} M^{BC} M^A D \Box^{B}_A = 0. \]  

(86)

In both formalisms, Eq. (86) thus implies that

\[ 2 \Delta^{A[}_A \psi^{B]}_A = M^{BC} \left( \frac{1}{2} m^2 \nabla^\mu A^\mu - \nabla^\mu s^\mu \right), \]  

(87)

while the relations (83) and (85) yield the expression

\[ \Delta^{A[}_A \psi^{B]}_A = M^{BC} \left( \tau^A_M \nabla^\mu \psi^M_A - \psi^{AM}_A \xi^{AM} \right). \]  

(88)

So, utilizing Eq. (34) and working out the \( \tau \nabla \psi^\text{-term of (88)} to the extent that \[
\tau^A_M \nabla^\mu \psi^M_A = -\frac{1}{2} \nabla^\mu s^\mu + \nabla_{CB'} (T^{AB'} \psi^C_A) + \psi^M_A \nabla^\mu \tau^A_M, \]  

(89)

we arrive at the condition\footnote{When Eqs. (81)–(84) are combined together, the terms that involve \( \nabla^\mu s^\mu \) explicitly get cancelled.}

\[ \frac{1}{4} m^2 \nabla^\mu A^\mu + \nabla_{CB'} (T^{AB'} \psi^C_A) + \psi^M_A \nabla^\mu \tau^A_M - \psi^{AM}_A \xi^{AM}_M = 0. \]  

(90)

For \( \phi^B_A \), we similarly obtain the massless condition

\[ \nabla_{CB'} (T^{AB'} \phi^C_A) + \phi^M_A \nabla^\mu \tau^A_M - \phi^{AM}_A \xi^{AM}_M = 0, \]  

(91)

along with the complex conjugates of (90) and (91).

The property (86) stipulates in either formalism that the only contributions to the wave equation for \( \psi^B_A \) are those produced by the symmetric pieces in \( B \) and \( C \) of the corresponding configuration (80). Hence, carrying out a symmetrization over the indices \( B \) and \( C \) of (80), likewise fitting together the pieces of Eqs. (82)–(84) and rearranging indices adequately thereafter, we end up with the dark energy equation

\[ (\Box + \frac{4}{3} \kappa + m^2) \psi^B_A + 2 \Psi^{LB}_M \psi^M_L = 2 \beta^B_A, \]  

(92)
with
\[ \beta^{AB} = \nabla_{B^*}B' + \psi^{(A)}_M^{*B} + 2(\nabla_\mu\psi^{(A)}_M^{*B}M) + m^2\tau^{AB}A_\mu. \] (93)

We should emphasize that the statements (90)-(92) are formally the same in both formalisms, and additionally bear gauge invariance because of the behaviour of \( A_\mu \) as specified by Eq. (11). Indeed, it is the masslessness of the CMB fields that ensures the absence from (91) of a term proportional to \( \nabla_\mu\Phi^\mu \).

It now becomes clear that the application to Eq. (92) of the correlations supplied by Eqs. (75) and (76), allows us to attain quite easily the \( \gamma \)-formalism version of the wave equations for \( \psi^{AB} \) and \( \psi^{AB} \). In effect, we have
\[ (\Box - 2i\alpha^\mu\nabla_\mu + \Theta + 4/3\kappa + m^2)\psi^{AB} - 2\Psi^{LM}_{AB}\psi^{LM} = 2\beta^{AB}. \] (94)

and
\[ (\Box + 2i\alpha^\mu\nabla_\mu + \Theta + 4/3\kappa + m^2)\psi^{AB} - 2\Psi^{LM}_{AB}\psi^{LM} = 2\beta^{AB}, \] (95)
which satisfy the rule (77). For the \( \varepsilon \)-formalism, we obtain
\[ (\Box + 4/3\kappa + m^2)\psi^{AB} - 2\Psi^{LM}_{AB}\psi^{LM} = 2\beta^{AB} \] (96)
and
\[ (\Box + 4/3\kappa + m^2)\psi^{AB} - 2\Psi^{LM}_{AB}\phi^{LM} = 2\beta^{AB}. \] (97)

We notice that the \( \varepsilon \)-formalism lower-index version of \( \beta^{AB} \) is expressed simply as
\[ \beta^{AB} = \nabla_{B^*}^{(A)}B' - \psi^{M}_M^{(A)B} + 2(\nabla_\mu\psi^{(A)}_M^{*B}M + m^2\tau^{AB}A_\mu - 0. \] (98)

Due to the occurrence of the same formal geometric patterns on the right-hand sides of the field equations of Section 3, we can promptly deduce the CMB wave equations from the statements (92)-(97) by first setting \( m = 0 \) and then replacing wave functions appropriately. In either formalism, we thus have
\[ (\Box - 2i\alpha^\mu\nabla_\mu + \Theta + 4/3\kappa)\phi^{A}_A + 2\Psi^{LB}_{MA}\phi^{M} = 2\eta^{B}_A \] (99)
with
\[ \eta^{AB} = \nabla_{B^*}^{(A)B'} + \phi^{(A)}_M^{*B} + 2(\nabla_\mu\phi^{(A)}_M^{*B}M + m^2\tau^{AB}A_\mu. \] (100)
and \( A_\mu \) being given by Eq. (39). The \( \gamma \)-formalism equations for \( \phi^{AB} \) appear as
\[ (\Box - 2i\alpha^\mu\nabla_\mu + \Theta + 4/3\kappa)\phi^{AB} - 2\Psi^{LM}_{AB}\phi^{LM} = 2\eta^{AB} \] (101)
and
\[ (\Box + 2i\alpha^\mu\nabla_\mu + \Theta + 4/3\kappa)\phi^{AB} - 2\Psi^{LM}_{AB}\phi^{LM} = 2\eta^{AB}. \] (102)
whereas the $\varepsilon$-formalism counterparts of (101) and (102) are stated as

$$ (\Box + \frac{4}{3}\varepsilon)\phi_{AB} - 2\Psi_{AB}^{LM}\phi_{LM} = 2\eta_{AB} \quad (103) $$

and

$$ (\Box + \frac{4}{3}\varepsilon)\phi^{AB} - 2\Psi^{AB}_{LM}\phi^{LM} = 2\eta^{AB}. \quad (104) $$

## 6 Concluding Remarks and Outlook

The description of the CMB and dark energy backgrounds presented here has been based upon the torsional extension of the traditional two-component spinor formalisms for curved spacetimes as given in Ref. [13]. Because of the fact that any torsional affine potentials should always enter geometric prescriptions together with adequate torsionless companions, we could definitely establish that any torsional spinor description of the dark energy background must be accompanied by a description of the CMB. We saw that all wave functions for both backgrounds couple naturally to the spinors which occur in the decomposition of the torsion tensor of $\mathfrak{M}$. However, they do not interact with one another whence we can say that one background propagates in $\mathfrak{M}$ as if the other were absent.

One of the most remarkable features of the procedures implemented in Section 5, is related to the gauge invariance of the condition (91), which takes place because the masslessness of the CMB fields annihilates either $\gamma\varepsilon$-contribution that carries the non-invariant piece $\nabla_{\mu}\Phi^{\mu}$. It should be stressed that the occurrence of the massive condition (90) rests essentially upon the torsionfulness intrinsically borne by Eq. (88). In the limiting case of the torsionless framework, the derivative $\Delta_{A}^{[C}\phi_{A}^{B]}$ becomes a vanishing contribution in both formalisms, and the dark energy wave equations (92)-(97) all "evaporate" together with the source $\sigma^{\mu}$ and the curvature spinor $\xi_{AB}$. Under this circumstance, the worldspin scalar $\varepsilon$ bears reality and satisfies the equality

$$ 4\varepsilon = R, $$

with $R$ being the Ricci scalar of $\nabla_\mu$. Hence, the electromagnetic wave equations of Ref. [16] are effectively recovered. We expect that the subsidiary configurations involved in our derivation of the wave equations for the backgrounds, could shed some light on the physical meaning of the right-hand side of Einstein-Cartan’s field equations. Moreover, it is considerably interesting to point out that the calculational techniques developed in Section 4 may supply geometric means for describing the propagation of gravitons in $\mathfrak{M}$.

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