Flux corrections to anomaly cancellation in M-theory on a manifold with boundary

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We show how the coupling of gravitinos and gauginos to fluxes modifies anomaly cancellation in M-theory on a manifold with boundary. Anomaly cancellation continues to hold, after a shift of the definition of the gauge currents by a local gauge invariant expression in the curvatures and $E_8$ fieldstrengths. We compute the first nontrivial such correction.

Warning: Ian Moss has called into question several of the numerical coefficients in the extended Dirac operators in this paper. We have not confirmed this but the reader is warned not to trust the precise coefficients in the formulae for the Dirac operators and heat kernel expansions. We believe these possible errors do not change our qualitative conclusions. One of us intends to return to the issue and recheck the formulae. We thank Ian Moss for pointing out these problems.

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1. Introduction and Conclusion

M-theory on a manifold with boundary exhibits some extraordinary features, first noted by Horava and Witten \cite{Horava:2009uw}. First among these features is a subtle anomaly cancellation, requiring the presence of an independent $E_8$ super-Yang-Mills multiplet (of either chirality) on each boundary component. In general, anomaly cancellation is best addressed in the geometric framework of determinant line bundles with connection \cite{Witten:1985bt, Witten:1985zz, Witten:1986qs, Witten:1988ze, Witten:1988hc, Witten:1988ze, Gross:1986mw, Seiberg:1994rs}. For recent discussions see, for example \cite{Gross:1985hu, Witten:1988ze, Seiberg:1994rs}. This framework is conceptually clear, is the best approach to cancellation of global anomalies, and is in any case the basis for the descent formalism. In a word, it states that the effective action after integrating out fermions must be a section of a geometrically trivialized line bundle, that is, a topologically trivial line bundle with a trivial connection.

Anomaly cancellation in M-theory was discussed in the geometric framework in \cite{Gross:1985hu}. The present paper begins to fill a gap left open in \cite{Gross:1985hu} and indeed left open in the entire literature on anomaly cancellation in 10- and 11-dimensional supergravities. Namely, in \cite{Gross:1985hu} the coupling of gravitinos and gauginos to fluxes was omitted. In this paper it will be retained. The natural connection on a determinant line bundle for an operator $D$ is a regularized version of $\text{Tr}D^{-1}\delta D$. Therefore, including couplings to the flux results in a change in the connection on the determinant line bundle and hence in the curvature. In \cite{Gross:1985hu} it was shown that if we omit these couplings then there is a canonical geometrical trivialization (termed there a canonical “setting of the integrand”) of the line with connection $\mathcal{L}_{\text{Fermi}} \otimes \mathcal{L}_{\text{CS}}$. Here the fermion effective action is a section of $\mathcal{L}_{\text{Fermi}}$ while $\mathcal{L}_{\text{CS}}$ accommodates the Chern-Simons term. (See \cite{Seiberg:1994rs} for an in depth discussion of this line bundle and its connection). Including the couplings of the fermions to the fluxes spoils the geometrical trivialization. Nevertheless, as we show here, the curvature of $\mathcal{L}_{\text{Fermi}} \otimes \mathcal{L}_{\text{CS}}$ is of the form $\mathcal{F} = dA$ where $A$ is a globally well-defined 1-form on the space of (gauge-equivalence classes of) bosonic fields. Moreover, $A$ is of the form $\int_X I_{11}$ where $I_{11}$ is local in the fields, and $X$ is the 10-dimensional boundary. Physically this means that although there is a change in the anomaly polynomial $I_{12}$, it changes by $dI_{11}$ where $I_{11}$ is gauge invariant. There is still a physical consequence of this change - the change of connection needed to restore geometrical trivialization corresponds to a change of the definition of the gauge current. We give an explicit formula for this change, to lowest order in fluxes and in flat space, in equation (4.58) below.

This research could be continued in several possibly fruitful directions. The functionals that describe the flux corrections to the curvature of the line bundle are of the same type
as the ones which were introduced by N. Nekrasov \cite{13} to define actions for fluxes on manifolds of special holonomy. There are also analogous corrections to the gauge current in the $SO(32)$ heterotic string. These corrections might be relevant to the open string sector recently proposed by J. Polchinski \cite{14}. Finally, it would be interesting to carry out a similar investigation in the formulation by I. Moss of M-theory on a boundary \cite{15} \cite{16}. His formulation has several advantages over that of \cite{1} \cite{2}. There are no $\delta$-functions, and his approach is local at each boundary. His formulation uses different boundary conditions for the gravitinos and does not have the $\chi \gamma (\nu G^9) \chi$ term for the gauginos, which plays an important role in our analysis.

The organization of this paper is as follows: section 2 contains a definition of the one-loop effective action in M-theory, taking into account their couplings with the flux. We derive explicitly the contributions from the bulk and the boundary, and thus determine the line bundle $L_{\text{Fermi}}$, where the exponentiated effective action is defined. In section 3, we analyze the geometry of this line bundle. The contribution from the boundaries yields a non-vanishing local curvature $F_{\text{Fermi}} \in \Omega^2 (T)$ for $L_{\text{Fermi}}$. Here $T$ is the space of (gauge inequivalent) bosonic field configurations. After including the contribution of $L_{\text{CS}}$, the total curvature is a globally exact form $F = dA$. Thus, it is possible to obtain a geometrical trivialization by changing the connection. Similarly, the contribution from the bulk gives rise to possible $\mathbb{Z}_2$-holonomies for loops in $\pi_1 (T)$, due to an ambiguity in the definition of the sign of the Rarita-Schwinger determinant \cite{17} \cite{10}. We show how the flux corrections do not alter the usual $\mathbb{Z}_2$ (or parity) anomaly cancellation mechanism. Section 4 provides explicit formulas for the curvature of the line bundle when the boundaries of $Y$ are flat Euclidean space. We show how our calculations, based on heat kernel expansions and the descent formalism, confirm the general arguments given in section 3. For completeness, we also study this local anomaly using Fujikawa’s method, determining the flux correction to the gauge current as a gauge invariant 9-form in $\Omega^9 (\mathbb{R}^{10})$. Appendix A states our Clifford algebra conventions. Appendix B briefly indicates the connection to supersymmetric quantum mechanics.

2. The one-loop effective action

In this section we sketch the gravitino partition function in the case of M-theory on a spin 11-dimensional manifold $Y$, which might have a nonempty boundary.
The supergravity multiplet consists of the metric $g$, a gravitino $\psi$, and a 3-form gauge potential with corresponding field strength $G$. The low energy limit of M-theory is described by 11-dimensional supergravity [18]. Here we focus on the quadratic part of the action for the gravitino

$$-\frac{1}{2} \int_Y \text{vol}(g) \left[ \bar{\psi} I^{JJK} D_J \psi_K + \frac{\ell^3}{96} (\bar{\psi} I^{JJKLMN} \psi_N + 12 \bar{\psi} J^{KLM} \psi) G_{JKLM} \right]$$

(2.1)

with $I, J, \ldots$ worldindices, $D_I$ the spin connection and $\ell$ the eleven dimensional Planck length. We are neglecting higher order terms in $\psi_I$. The local supersymmetry transformation for the gravitino up to 3-fermi terms, is

$$\delta \psi_I = D_I \epsilon + \frac{\ell^3}{288} (\gamma_I^{JKLM} - 8 \delta_I^{JKLM}) G_{JKLM} \epsilon := \hat{D}_I \epsilon.$$  

(2.2)

We will write (2.2) as $\delta \psi_I = \hat{D}_I \epsilon$, and will refer to $\hat{D}_I$ as the supercovariant derivative. We will abbreviate the action as

$$\int_Y \bar{\psi} R \psi.$$  

(2.3)

Denote by $S$ the spin bundle on $Y$. The generalized Rarita-Schwinger operator $R : \Gamma(S \otimes T^*Y) \to \Gamma(S \otimes T^*Y)$, fits into the complex

$$0 \to \Omega^0(S) \xrightarrow{\hat{D}} \Omega^1(S) \xrightarrow{R} \Omega^1(S) \xrightarrow{\hat{D}^*} \Omega^0(S) \to 0,$$

(2.4)

if we require the vanishing of $R \circ \hat{D}$. Furthermore, at the level of principal symbols the complex is exact so (2.4) defines an elliptic complex. To check the exactness of (2.4) at the level of symbols it is enough to work in flat space, thus if $\sigma_{\hat{D}}(k) = k \in T^*Y$ is the principal symbol associated to $\hat{D}$ and the symbol for $R$ is $\sigma_R(k) = \gamma^{MNP} k_N$, then $\text{Ker}(\sigma_R(k))$ consists of the elements $\sigma_{\hat{D}}(k)$ for a spinor $s$.

The consistency condition $R \circ \hat{D} = 0$ requires that the equations of motion for the bosonic fields must be satisfied as we show below. Hence, if we write the equations of motion for the gravitino field as [18]\footnote{Note that we are expressing the Rarita-Schwinger operator $R$ in two equivalent ways.}

$$R \psi = \gamma^{MNP} \hat{D}_N \psi_P = 0,$$

(2.5)

we can write the condition $R \circ \hat{D} = 0$ as

$$R \circ \hat{D} = \gamma^{MNP} [\hat{D}_N, \hat{D}_P] = 0.$$  

(2.6)
We can describe the bosonic configurations satisfying (2.6) by considering the seemingly simpler relation
\[ \gamma^P[\hat{D}_N, \hat{D}_P] = 0. \] (2.7)

We claim that (2.6) and (2.7) are equivalent. That (2.7) implies (2.5) follows from
\[ \gamma^{MNP} = \gamma^M \gamma^N \gamma^P + g^{NP} \gamma^M - g^{MP} \gamma^N + g^{MN} \gamma^P. \] To prove the converse observe that
\[ 0 = (g_{QM} + \frac{1}{n-2} \gamma_Q \gamma_M) \times (\gamma^{MNP} [D_N, D_P]) = 2 \gamma^P [D_N, D_P]. \] (2.8)

By a straightforward computation can express the condition (2.6) on the bosonic fields, using the relation (2.7) as follows
\[ \gamma^N[\hat{D}_M, \hat{D}_N] = -\frac{\ell^3}{288} (D_{[N} G_{PQRS]}) \gamma^{MNPQRS} + \frac{5 \ell^3}{144} (D_{[M} G_{NMPQR]}) \gamma^{NPQR} \]
\[ - \frac{\ell^3}{72} \left( D^N G_{NPQR} + \frac{\ell^3}{4 \times 288} G^{I_1 \ldots I_4} G^{J_1 \ldots J_4} \varepsilon_{I_1 \ldots I_4 J_1 \ldots J_4 PQR} \right) g_{MT} \gamma^{TPQR} \]
\[ + \frac{\ell^3}{12} \left( D^N G_{NPQR} + \frac{\ell^3}{4 \times 288} G^{I_1 \ldots I_4} G^{J_1 \ldots J_4} \varepsilon_{I_1 \ldots I_4 J_1 \ldots J_4 PQR} \right) \gamma^{PQ} \]
\[ - \frac{1}{2} \left( R_{MN} - \frac{\ell^6}{6} (G_{MPQR} G^P_{NQR} - \frac{1}{12} g_{MN} G_{PQRS} G^{PQRS}) \right) \gamma^N = 0. \] (2.9)

Here, we expand (2.7) in terms of completely antisymmetrized products of gamma matrices (see Appendix A), hence (2.9) implies the following constraints for the bosonic fields
\[ dG = 0 \] (2.10)
\[ d \ast G = -\frac{\ell^3}{2} G \wedge G \] (2.11)
\[ R_{MN} = \frac{\ell^6}{6} (G_{MPQR} G^P_{NQR} - \frac{1}{12} g_{MN} \ast (G \wedge \ast G)) \] (2.12)

where \( R_{MN} \) is the Ricci tensor. These are just the classical equations of motion of 11-dimensional supergravity.

2.1. The gravitino partition function.

Since the local fermionic gauge symmetries of \( n = 11 \) supergravity do not close into a super Lie algebra for off-shell bosonic backgrounds, we should in principle use the BV quantization procedure to get a correct gauge fixed action. In this paper we determine the gauge fixed action for backgrounds that satisfy (2.11), and (2.12). This allows us to use standard BRST procedures \[19\] [20] and simplifies the discussion considerably. Of course, it...
leaves an important gap in our treatment. Accordingly, we consider the gravitino partition function

$$Z = \int_{\Omega(S)/\text{Im}D}[d\psi]e^{-\int_Y \bar{\psi} R \psi}, \quad (2.13)$$

It is useful to introduce the notation:

$$G_{\gamma} = \gamma^{PQRN}G_{PQRN} \quad (2.14)$$
$$G_{\gamma}^N = \gamma^{PQR}G_{PQRN} = -\gamma^{PQR}G_{NPQR} \quad (2.15)$$
$$G_{\gamma}^{RN} = \gamma^{PQ}G_{PQRN} \quad (2.16)$$

A direct calculation shows that

$$\gamma_M^{PQRN}G_{PQRN} = \gamma_M G_{\gamma} - 4G_{\gamma}^M, \quad (2.17)$$
and therefore we can write (2.2) as

$$\hat{D}_M \epsilon = D_M \epsilon + \frac{\ell^3}{288}\gamma_M G_{\gamma} \epsilon + \frac{\ell^3}{72}G_{\gamma} \epsilon. \quad (2.18)$$

Since \(\gamma_M \gamma_M = -11\) and \(\gamma_M G_{\gamma} = -G_{\gamma}\), the associated supercovariant Dirac operator will be

$$\hat{\mathcal{D}} = \gamma^M \hat{D}_M = \mathcal{D} - \frac{5\ell^3}{96} G_{\gamma}. \quad (2.19)$$

Thus we can write the action (2.1) as

$$-\frac{1}{2} \int_Y \text{vol}(g) \left[ \bar{\psi}_I \gamma^{IJK} D_J \psi_K + \frac{\ell^3}{96} \bar{\psi}_I (\gamma^{IK} G_{\gamma} - 8\gamma^I G_{\gamma^{JK}} - 24G_{\gamma^{IK}} \psi_K) \right]. \quad (2.20)$$

We now use the formal BRST procedure to determine the gravitino gauge fixed action, and choose the gauge \(s = \gamma \cdot \psi\) for an arbitrary spinor \(s \in \Omega^0(S)\). This leaves unfixed zeromodes of the Dirac equation, constituting a finite dimensional space which we will deal with presently. Following standard procedure we write

$$1 = \int_{\Omega^0(S)^\perp} [d\epsilon] \delta(s - \gamma^M(\psi_M + \hat{D}_M \epsilon)) (\det \hat{\mathcal{D}})^{-1} \quad (2.21)$$

with \(\Omega^0(S)^\perp = (\text{Ker} \hat{\mathcal{D}})^\perp\) and where \(\hat{D}_M\) and \(\hat{\mathcal{D}}\) were defined in (2.18) and (2.19). We now insert (2.21) into

$$\int [d\psi] e^{-\int_Y \bar{\psi} R \psi} \quad (2.22)$$
and divide by the volume of the supergauge group to obtain the gauge-fixed expression

\[ \int [d\psi] \delta(s - \gamma \cdot \psi)(\text{det}' \hat{D})^{-1} e^{-\int_Y \bar{\psi} R \psi}. \] (2.23)

Ghost fields are introduced by writing the determinant (2.21) in terms of commuting ghost \( \epsilon \) and antighost \( \beta \) fields as

\[ (\text{det}' \hat{D})^{-1} = \int [d\beta][d\epsilon] e^{-\int \beta \hat{D} / \epsilon}, \] (2.24)

the prime in the determinant denotes the omission of the null eigenvalues.

Furthermore we invoke the following algebraic identity for \( \phi_M = \psi_M + \frac{1}{2} \gamma_M (\gamma \cdot \psi) \), which allows us to split the gauge fixed action as a sum of functionally independent quadratic terms, i.e. we have the relation

\[ -\bar{\phi} \hat{D}_{T^*Y} \phi = \bar{\psi} R \psi - \frac{1}{4} (n - 2) (\gamma \cdot \psi) \hat{D} (\gamma \cdot \psi), \] (2.25)

where \( R \) was defined in (2.20) to be

\[ R^{IK} = \gamma^{JK} D_J + \frac{\ell^3}{96} (\gamma^{IK} G^I - 8 \gamma^I G^K - 24 G^{IK}), \] (2.26)

while \( \hat{D} \) and \( \hat{D}_{T^*Y} \) are uniquely fixed to be the generalized Dirac operators

\[ \hat{D} = D + \frac{\ell^3}{288} G \] (2.27)

and

\[ \hat{D}_{T^*Y} = D_{T^*Y} - \frac{\ell^3}{96} G. \] (2.28)

Here, the subscript \( T^*Y \) denotes the coupling with the cotangent bundle of \( Y \). The identity (2.25) is easy to check when we substitute the \( \phi \)-field and the operators \( \hat{D}_{T^*Y} \) and \( \hat{D} \) in it and use the following relations for \( G \) and the gamma matrices

\[ \gamma^M G^I - G^M \gamma^I = 8 G^M, \]
\[ \gamma^M G^P + G^P \gamma^M = -6 G^{MP}, \]
\[ \gamma^{IK} = \gamma^I \gamma^K + g^{IK}. \] (2.29)

At this point, rather than setting \( s = 0 \) we average over \( s = (\gamma \cdot \psi) \) using the expression

\[ 1 = \frac{1}{(\text{det}' \hat{D})^{1/2}} \int_{(\Omega^0(\mathcal{S}))^\perp} [ds] e^{-\int \bar{\psi} \hat{D} s}. \] (2.30)
Formally, using (2.25) the gauge fixed partition function for the gravitino can be written as

\[ Z' = \frac{1}{(\det \hat{\mathcal{D}})^{1/2}} \int [d\psi] [d\beta] [d\epsilon] \exp \left( -2\pi \int_Y \text{vol}(g)(\overline{\psi} \hat{\mathcal{D}} \gamma Y \psi - \beta \hat{D} \gamma \epsilon) \right). \] (2.31)

We still must fix the remaining global fermionic symmetries given by supercovariantly constant spinors. We will assume the procedure described in [10], eq. (A.11) continues to hold. The net result is the following key statement.

Let \( T \) denote the space of bosonic M-theory data on \( Y \), i.e., the Riemannian metrics and \( G \)-fluxes, and introduce the fibration \( Y \to T \) whose fiber is the spacetime manifold \( Y \). This yields a family of operators \((\hat{\mathcal{D}} \gamma T, \tilde{\mathcal{D}} \gamma, \hat{\mathcal{D}} \gamma T \gamma Y)\) built up fiberwise in \( Y \) through the geometric data parametrized by \( T \):

\[ \hat{\mathcal{D}} = \mathcal{D} \gamma - \frac{5\ell^3}{96} \mathcal{G}, \] (2.32)

\[ \tilde{\mathcal{D}} = \mathcal{D} \gamma + \frac{\ell^3}{288} \mathcal{G}, \] (2.33)

\[ \hat{\mathcal{D}} \gamma T \gamma Y = \mathcal{D} \gamma T \gamma Y - \frac{\ell^3}{96} \mathcal{G}. \] (2.34)

Then, the gravitino partition function \( \exp(-\Gamma_{\text{gravitino}}) \) is a section of the line bundle

\[ \mathcal{L}_{\text{gravitino}} := \text{Pfaff} \hat{\mathcal{D}} \gamma T \gamma Y \otimes (\text{Pfaff} \tilde{\mathcal{D}} \gamma)^{-1} \otimes (\text{Det} \hat{\mathcal{D}})^{-1} \to T, \] (2.35)

In fact, this is a line bundle with connection, as we will discuss below. In addition, the Chern-Simons term of M-theory is also a section of a line bundle with connection \( \mathcal{L}_{\text{CS}} \to T \), and hence the M-theory measure is a section of

\[ \mathcal{L}_{\text{gravitino}} \otimes \mathcal{L}_{\text{CS}} \to T \] (2.36)

2.2. Boundary contribution to the effective action.

Let us now turn to the case where \( Y \) has a boundary. We denote by \( \partial Y \) the different connected components and by \( \omega^0 \) Clifford multiplication by the volume 10-form on the

\[ \text{2 We are unaware of an adequate treatment of the ghost zeromodes in the gravitino partition function in the literature. In this paper we sidestep that issue and assume that the gravitino effective action is a section of eq. (2.35).} \]
boundary. We follow closely the discussion of boundary conditions in [10]. We fix a spatial boundary condition for the spinor field $\Psi$, by imposing

$$\epsilon_i \Psi^\partial = \Psi^\partial \quad \text{with} \quad \epsilon_i = i \omega^\partial \text{ or } \epsilon_i = -i \omega^\partial$$

at each connected component $\partial Y_i$. The presence of boundaries produces local anomalies in the theory.

The fermionic content at the boundary in the low energy description of M-theory comes from the restriction of the gravitino and the presence of gauginos. We generalize the discussion of Horava and Witten and attach an independent $N = 1$ super Yang-Mills multiplet with gauge group $E_8$ and chirality $\epsilon_i$ to each connected component of the boundary. According to [1], we should write the quadratic part of the action for the gauginos, as

$$S_i = -\frac{1}{4\pi \ell^6} \int_{\partial Y_i} \text{vol}(g^\partial) \text{tr} \left[ \nabla \slashed D_{E_8} \chi - \frac{\ell^3}{24} \tau^\nu \gamma(t_{\nu} G^\partial) \chi^a \right].$$

The superscript $^\partial$ denotes restriction of the field on the boundary and $t_{\nu}$ is the contraction with the unit outward normal vector field to $\partial Y_i$. As shorthand, we can write the action using the generalized Dirac operator

$$\hat{D}_{E_8} = \slashed D_{E_8} - \frac{\ell^3}{24} \gamma(t_{\nu} G^\partial),$$

so the exponentiated effective action for $\chi$ is section of the line bundle

$$L_{\text{gaugino}} = \bigotimes_{\epsilon_i = -i \omega^\partial} \left( \text{Pfaff } \hat{D}_{\partial Y_i} \right) \bigotimes_{\epsilon_i = i \omega^\partial} \left( \text{Pfaff } \hat{D}_{\partial Y_i} \right)^{-1} \rightarrow \mathcal{T},$$

It is useful to decompose the boundary 4-form $G$, in its tangential and normal components

$$G^\partial = \nu^b \wedge t_{\nu} G^\partial + (1 - \nu^b \wedge t_{\nu}) G^\partial = G^\partial_N + G^\partial_T$$

with $\nu^b$ the 1-form dual to the unit normal vector field $\nu$. Also, we introduce the local “torsion”

$$\mathfrak{h} = -\frac{1}{24} \ell^3 t_{\nu} G^\partial.$$

For the gravitino sector, we have to generalize the gauge fixed action $\exp(-\Gamma_{\text{gravitino}})$ to the case $\partial Y \neq \emptyset$. To do this recall the relation between a Dirac-like operator on the boundary $\partial Y_i$ and that in the bulk close to the boundary $\partial Y_i \times [0, \epsilon)$

$$\hat{D} = \gamma^\nu (\partial_{\nu} - \hat{D}^\partial)$$

(2.43)
where \( \nu \) is the normal unit vector field to the boundary. Then, as \( (\gamma^\nu)^2 = -1 \), the generalized Dirac operators that we have to study on the boundary are

\[
\hat{\nabla}_{T^*Y} = \nabla_{T^*Y} - \frac{\ell^3}{96} g, \quad \Rightarrow \quad \hat{\nabla}^\partial_{T^*Y} = \nabla^\partial_{T^*Y} - \frac{\ell^3}{24} \gamma (\nu^\nu G_N + \nu^5 G_T)
\]

\[
\hat{\nabla} = \nabla + \frac{\ell^3}{288} g, \quad \Rightarrow \quad \hat{\nabla} = \nabla + \frac{\ell^3}{72} \gamma (\nu^\nu G_N + \nu^5 G_T) \tag{2.44}
\]

\[
\hat{\nabla} = \nabla - \frac{5\ell^3}{96} g, \quad \Rightarrow \quad \hat{\nabla} = \nabla - \frac{5\ell^3}{24} \gamma (\nu^\nu G_N + \nu^5 G_T)
\]

as \( \nu^\nu G_N \) is a 3-form and \( \nu^5 \wedge G_T \) a 5-form, the operators \( \hat{\nabla}_{T^*Y} \), \( \hat{\nabla}^\partial \) and \( \hat{\nabla} \) anticommute with \( \omega^\partial \) and hence have a well defined index.

The restriction \( \psi^\partial \) of the Rarita-Schwinger field \( \psi \in \Omega^1(S) \) to \( \partial Y_i \) decomposes into tangential and normal components:

\[
\psi^\partial = \psi^\partial_T + \psi^\partial_\nu \tag{2.45}
\]

and their boundary conditions are given by the following definite choice of sign

\[
\omega^\partial \psi^\partial_T = +i \psi^\partial_T \tag{2.46}
\]
\[
\omega^\partial \psi^\partial_\nu = -i \psi^\partial_\nu.
\]

These boundary conditions imply that the gauge group must be restricted by

\[
\omega^\partial \hat{\nabla}_T e^\partial = +i \hat{\nabla}_T e^\partial \tag{2.47}
\]
\[
\omega^\partial \hat{\nabla}_\nu e^\partial = -i \hat{\nabla}_\nu e^\partial
\]

where

\[
\hat{\nabla}_\nu = \nu^M \hat{D}_M
\]
\[
\hat{\nabla}_T = \hat{D}_M - \nu_M \hat{\nabla}_\nu \tag{2.48}
\]

and \( \hat{D}_M \) is the supersymmetric variation of the gravitino

\[
\hat{D}_M = D_M + \frac{\ell^3}{288} \gamma_M g + \frac{\ell^3}{72} g_M.
\]

We then choose boundary conditions on the other ghost \( \beta \), so that \( \hat{\nabla}^\partial \) is skew-adjoint:

\[
\omega^\partial \hat{\nabla}_T e^\beta = -i \hat{\nabla}_T e^\beta
\]
\[
\omega^\partial \hat{\nabla}_\nu e^\beta = +i \hat{\nabla}_\nu e^\beta \tag{2.50}
\]
The third ghost that comes from integrating over $s$ \((2.30)\), has the same boundary conditions as $\gamma \cdot \psi$:

$$\omega^\partial (s^\partial) = -is^\partial. \quad (2.51)$$

As the chiralities of $\epsilon$ and $\beta^\partial$ are opposite, \((2.47)\), and \((2.50)\), lead to pfaffian line bundles which cancel, therefore $\tilde{\mathcal{D}}^\partial$ does not appear in our analysis. On the other hand, as $\psi_\nu$ comes from a component of the Rarita-Schwinger field in $\Omega^1(S)$, it couples to

$$\tilde{\mathcal{D}}^\partial_\nu = \mathcal{D}^\partial - \ell^3 96 \gamma (\nu G_N + \nu^b \wedge G_T) \quad (2.52)$$

and $s$ couples to $\tilde{\mathcal{D}}^\partial$, as defined in \((2.44)\). Therefore, according to the theorems stated in \([10]\), based on theorems proved by M. Scholl \([21]\), the boundary contribution to the exponentiated effective action $\exp(-\Gamma_{\text{gravitino}})$ is section of

$$\mathcal{L}_{\text{gravitino}} = \bigotimes_{-i\omega^\partial} \left[ (\text{Pfaff } \tilde{\mathcal{D}}^\partial_{T,Y})^{1/2} \otimes (\text{Pfaff } \tilde{\mathcal{D}}^\partial_{\nu})^{-1/2} \otimes (\text{Pfaff } \tilde{\mathcal{D}}^\partial_{Y_i})^{-1/2} \right] \bigotimes_{+i\omega^\partial} \left[ (\text{Pfaff } \tilde{\mathcal{D}}^\partial_{T,Y})^{-1/2} \otimes (\text{Pfaff } \tilde{\mathcal{D}}^\partial_{\nu})^{1/2} \otimes (\text{Pfaff } \tilde{\mathcal{D}}^\partial_{Y_i})^{1/2} \right] \rightarrow \mathcal{T} \quad (2.53)$$

where we are taking into account the contribution from every connected component of the boundary. Finally we have

$$\mathcal{L}_{\text{Fermi}} = \mathcal{L}_{\text{gaugino}} \otimes \mathcal{L}_{\text{gravitino}}. \quad (2.54)$$

In the following sections, we study the curvatures of the determinant line bundles associated to generalized Dirac operators. The $G$-dependent contributions to the curvature of \((2.54)\), are given by terms constructed with the exterior derivatives $d(\nu^b \wedge G_T)$ and $d\nu G_N$. Now

$$d(\nu^b \wedge G_T) = d\nu^b \wedge G_T - \nu^b \wedge dG_T = 0. \quad (2.55)$$

To see this, we work in the neighborhood of the boundary $\partial Y_i \times [0, \epsilon)$ such that $d\nu^b = 0$. Also, as $G_T$ is closed on the boundary we have $dG_T = 0$. Thus we can neglect the contributions from $\nu^b \wedge G_T$, and just work with the local torsion $h$ of \((2.42)\).
2.3. Hořava-Witten reduction

It is useful to connect our formalism to the standard Hořava-Witten setup $Y = X \times [0, 1]$, used to describe the strongly coupled heterotic string with gauge group $E_8 \times E_8$, in its low energy limit.

The H flux of heterotic string theory is recovered from the M-theory data according to

$$H = \int_{[0, 1]} dt G_{11} \wedge dx^M \wedge dx^N \wedge dx^P$$

with $t = x^{11}$ and $1 \leq M, N, P \leq 10$. On the other hand, using the decomposition of the G-flux in terms of tangential and normal components to the 11th-coordinate

$$G = G_{11} \wedge dx^M \wedge dx^N \wedge dx^P + G_{QRST} \wedge dx^Q \wedge dx^R \wedge dx^S \wedge dx^T = G_N + G_T$$

with the indices $M, N, \ldots$ running between 1 and 10. On the boundaries $\iota^*(G_T)$ at $t = 0, 1$ we have $\iota^* G_T = \text{tr} F_t^2 - \frac{1}{2} \text{tr} \mathcal{R}_t^2 \in \Omega^4(X)$ where $F_t$, $t = 0, 1$ is the curvature of the $E_8$ bundle on the boundary $X_t$. If we extend $G_T$ as a family of closed forms on $X$ then

$$0 = d_{11} G = \left(dt \wedge \frac{\partial}{\partial t} + d\right)(G_N + G_T) = dG_N + dt \wedge \frac{\partial}{\partial t} G_T$$

$(d$ and $d_{11}$ are exterior derivatives on $X$ and $Y$, respectively). Therefore, from (2.58)

$$dG_N = - dt \wedge \frac{\partial}{\partial t} G_T$$

Using (2.56) and (2.59) we recover the usual formula

$$d H = \text{tr} F_1^2 + \text{tr} F_2^2 - \text{tr} \mathcal{R}^2$$

Finally we would like to see how the interaction term

$$\Delta S = \frac{1}{96 \pi \ell^3} \int_X \text{vol}(g_X) \text{Tr}_{496} \left[ \overline{\chi} \gamma(H) \chi \right],$$

in heterotic string theory, is recovered from the boundary interactions of M-theory (2.38)

$$\Delta S_i = \frac{1}{96 \pi \ell^3} \int_{\partial Y_i} \text{vol}(g_{\partial}) \text{Tr}_{248} \left[ \overline{\chi} \gamma(h) \chi \right],$$

with $i = 1, 2$ labeling the boundaries of the cylinder $X \times [0, 1]$. In the zeromode limit we have

$$L_t G_N = 0, \quad \iota_t G_N = \iota_t G_N^{0_1} = \iota_t G_N^{0_2} = H = h_1 = h_2$$

i.e. $G_N$ is $t$-independent and the non-trivial $t$-dependence of $G$ comes from $G_T$. Therefore $\Delta S = \Delta S_1 + \Delta S_2$. 
3. Setting the bosonic measure in the presence of fluxes

In this section we will describe a connection on the gravitino and gaugino line bundles and compute its curvature. Without loss of generality, we can fix attention on one boundary component, and fix a chirality. We choose to study
\[(\text{Pfaff} \hat{\mathcal{D}}_{E_8}) \otimes (\text{Pfaff} \hat{\mathcal{D}}_{T \star Y})^{1/2} \otimes (\text{Pfaff} \hat{\mathcal{D}}_{\nu})^{-1/2} \otimes (\text{Pfaff} \hat{\mathcal{D}}_{Y})^{-1/2} \rightarrow \mathcal{T}^\partial,\] (3.1)

where \(\mathcal{T}^\partial\) is the space of bosonic fields on the boundary and the generalized Dirac operators in (3.1) are
\[
\begin{align*}
\hat{\mathcal{D}}_{E_8}^\partial &= \mathcal{D}_{E_8}^\partial + \gamma(\mathbb{H}), \\
\hat{\mathcal{D}}_{T \star Y}^\partial &= \mathcal{D}_{T \star Y}^\partial + \gamma(\mathbb{H}), \\
\hat{\mathcal{D}}_{\nu}^\partial &= \mathcal{D}_{\nu}^\partial + \gamma(\mathbb{H}), \\
\hat{\mathcal{D}}^\partial &= \mathcal{D}^\partial - \frac{1}{3} \gamma(\mathbb{H}).
\end{align*}
\]

where \(\gamma(\cdot)\) denotes Clifford multilication by elements in \(\Omega^\star(X)\), with \(X := \partial Y\).

A natural choice of connection on the determinant and Pfaffian line bundles follows the discussion of Bismut and Freed \[22\]\[23\]. Working fiberwise in \(\mathcal{X} \rightarrow \mathcal{T}^\partial\), we can define generalized Dirac operators \(\hat{\mathcal{D}}\) on \(X\), as the ones which appear in the definition of the effective action, i.e., the operators (3.2), (3.3), (3.4) and (3.5). We now drop the superscript \(\partial\) in the remainder of this section. The generalized Dirac operator
\[
\hat{\mathcal{D}} = \mathcal{D} + \alpha_0 \gamma(\mathbb{H}), \hspace{1cm} \mathbb{H} \in \Omega^2(X),
\]

(\(\alpha_0\) is \(\alpha_0 = 1, -1/3\) in the case of interest here) can be viewed as an odd endomorphism acting on the Hilbert bundle of spinors
\[
\Omega^0(S_+) \oplus \Omega^0(S_-) \rightarrow \mathcal{T}^\partial,
\]

where the subindices + and − denote the chirality of the spinor. In the Weyl basis \(\hat{\mathcal{D}}\) decomposes as
\[
\hat{\mathcal{D}} = \begin{pmatrix} 0 & \hat{\mathcal{D}}_- \\ \hat{\mathcal{D}}_+ & 0 \end{pmatrix},
\]

Next, using a Riemannian structure on \(\mathcal{T}^\partial\) we can then introduce a connection \(\tilde{\nabla}\) on the Hilbert bundle \(\Omega(S) \otimes \Lambda^\star(\mathcal{T}^\partial) \rightarrow \mathcal{T}^\partial\). This connection allows us to study the geometry.
of the determinant line bundle where the effective action lives, i.e. given the Hilbert bundle (3.7) it is possible to define its associated determinant line bundle

$$\text{Det} \hat{\mathcal{D}} / \rightarrow \mathcal{T}^\partial,$$

which can be also written as

$$\text{det}\Omega^0(S_+) \otimes \text{det}(\Omega^0(S_-)^\vee) \rightarrow \mathcal{T}^\partial.$$

This line bundle has a natural connection on it which can be determined using heat kernel expansions [23]. More concretely, when restricted to a 2 dimensional submanifold \(\Sigma \hookrightarrow \mathcal{T}^\partial\), one can compute its curvature as [22, 24]

$$\int_\Sigma \mathcal{F}(\text{Det} \hat{\mathcal{D}}_+ \rightarrow \mathcal{T}^\partial) = 2\pi i \int_{\pi^{-1}(\Sigma)} \left[ \text{Tr}_s a_6(\hat{\mathcal{P}}) \right]_{(12)},$$

with \(\mathcal{F} \in \Omega^2(\Sigma)\) and \(\pi: \mathcal{X} \rightarrow \mathcal{T}^\partial\) the defining fibration of the family, with fiber \(\mathcal{X}\). \footnote{See [1] and [22], for a rigorous definition of such infinite dimensional bundles.} In (3.11) we are using the heat kernel expansion

$$\text{Tr}_s(\exp(-t\hat{\mathcal{P}}^2)) = \frac{\text{Tr}_s a_0}{t^6} + \frac{\text{Tr}_s a_1}{t^5} + \ldots + \text{Tr}_s a_6 + \mathcal{O}(t),$$

where \(\text{Tr}_s(\cdot) = \text{Tr}(\Gamma^{13}\cdot)\), and \(\hat{\mathcal{P}}\) the generalized Dirac operator on the spin bundle of the 12 manifold \(\pi^{-1}(\Sigma)\), defined as

$$\hat{\mathcal{P}} = \mathcal{P} + \alpha_0 \Gamma(\mathcal{H}),$$

with \(\mathcal{P}\) the usual Dirac operator on \(\pi^{-1}(\Sigma)\), \(\mathcal{H} \in \Omega^3(\mathcal{X})\) and \(\Gamma(\cdot)\) denotes the Clifford multiplication in the Clifford algebra Cliff(12).

This approach allows us to compute the curvature of the line bundle form the integral over two-dimensional submanifolds \(\Sigma \hookrightarrow \mathcal{T}^\partial\).

\footnote{The theorems of [23] and [24] that we use here were stated for families of ordinary Dirac operators and not generalized Dirac operators. However the argument using Eq.(1.56) of [23], as well as the identity Eq.(5.4) of [24] can be shown to extend to the case of generalized Dirac operators. One need only require some mild conditions on the generalized Dirac operators, which turn out to be compatible with the physics of our problem.}
3.1. Flux corrections to the line bundle’s curvature

If $\text{Tr}_s a_6(\hat{\mathcal{D}})$ is the heat kernel coefficient associated to the generalized Dirac operator $\hat{\mathcal{D}}$, the curvature of the physical line bundle which appears in M-theory (3.1), can be expressed as

$$F(\mathcal{L}_{\text{gaugino}} \otimes \mathcal{L}_{\text{gravitino}} \otimes \mathcal{L}_{CS} \to \mathcal{T}^\partial) = F(\mathcal{L}_{CS} \to \mathcal{T}^\partial) + \frac{2\pi i}{4} \left[ \int_X 2\text{Tr}_s a_6(\hat{\mathcal{D}}_{E_8}) + \text{Tr}_s a_6(\hat{\mathcal{D}}_{\mathcal{T}^\partial Y}) - \text{Tr}_s a_6(\hat{\mathcal{D}}_{\mathcal{T}^\partial \nu}) - \text{Tr}_s a_6(\hat{\mathcal{D}}_{\mathcal{T}^\partial \nu}) \right]_{(2)},$$

(3.14)

where $[ \cdot ]_{(2)}$ extracts the two-form part. Thus, evaluating the curvature of (3.1) is equivalent to computing certain heat kernel coefficients.

Without evaluating the heat kernel coefficients we can make the following observation just based on index theory. From [10] we know that the curvature (3.14), is zero for $\mathbb{H} = 0$. Since the flux can be turned on by a compact perturbation the curvature will be an exact 2-form on $\mathcal{T}^\partial$

$$F(\mathcal{L}_{\text{gaugino}} \otimes \mathcal{L}_{\text{gravitino}} \otimes \mathcal{L}_{CS} \to \mathcal{T}^\partial) = dA,$$

(3.15)

for some globally well-defined 1-form $A \in \Omega^1(\mathcal{T}^\partial)$. As we have said, $\mathcal{T}^\partial$ is the space of gauge inequivalent field configurations, that is, the base of the $\mathcal{G} := \text{Diff}(Y) \times \text{Aut}(\mathcal{E})$ bundle

$$0 \to \mathcal{G} \to \text{Met}(Y) \times \mathcal{A} \xrightarrow{\pi^\gamma} \mathcal{T}^\partial \to 0,$$

(3.16)

with $\text{Met}(Y)$ the space of Riemannian metrics on $Y$ and $\mathcal{A}$ the affine space of $E_8$-gauge connections on the $E_8$ gauge bundle $\mathcal{E} \to X$. We can write the 12-form $I_{12}$, used to define the curvature of the line bundle $F = \int_X I_{12}$, as the exterior differential of a $\mathcal{G}$-equivariant 11-form $I_{11}(\mathcal{R}, F, G)$. Therefore, the descent formalism suggests that such flux corrections do not contribute to the anomaly.

In order to justify the above claim we proceed as follows. As we showed above, we can construct a generalized Dirac operator acting on the Hilbert bundle (3.7). If we now restrict to an arbitrary 2-dimensional family $\Sigma \subset \mathcal{T}^\partial$ then the index of this operator, which we will denote by $\text{Index} \hat{\mathcal{D}}$ is given by

$$\text{Index} \hat{\mathcal{D}} = \int_{\pi^{-1}(\Sigma)} \text{Tr}_s a_6(\hat{\mathcal{D}}).$$

(3.17)

One the other hand, since $\hat{\mathcal{D}} = \mathcal{D} + \gamma(\mathbb{H})$ differ by a compact perturbation

$$\text{Index} \hat{\mathcal{D}} = \text{Index} \mathcal{D}.$$

(3.18)
Since this applies to arbitrary families $\Sigma$ we learn that
\[
\int_X \text{Tr}_s a_6(\hat{\mathcal{D}}) = \int_X \text{Tr}_s a_6(\mathcal{D}) + d\alpha,
\]
for some globally well-defined 1-form $A$ on $\mathcal{T}^\partial$. However, since the heat kernel expression is a local expression in the fields we must have
\[
\text{Tr}_s a_6(\hat{\mathcal{D}}) = \text{Tr}_s a_6(\mathcal{D}) + d\alpha,
\]
for some 11-form $\alpha$, that becomes zero when $\mathbb{h} = 0$. In the next section we will verify this explicitly for the case of flat space to lowest order in $\mathbb{h}$.

3.2. The $\mathbb{Z}_2$-anomaly.

As noted in [10] there is a natural real structure on the gravitino line bundle, respected by the Bismut-Freed connection, and hence the holonomy group is at most $\mathbb{Z}_2$. In fact, it can very well be equal to $\mathbb{Z}_2$. The coupling of the gravitino to the $G$-flux respects this real structure, and hence coupling to the $G$-flux cannot modify the $\mathbb{Z}_2$ anomaly cancellation. It will, however change the one-loop measure. Here we give an expression for that change.

We need to compute
\[
\xi(\mathcal{P}_{RS} + \ell^3 \Xi \cdot G) := \xi\left(\mathcal{P}_{T-Y} - \frac{\ell^3}{96} \mathcal{G}\right) - \xi\left(\mathcal{P} + \frac{\ell^3}{288} \mathcal{G}\right) - 2\xi\left(\mathcal{P} - \frac{5\ell^3}{96} \mathcal{G}\right).
\]
(3.21)

where $\xi$ is the invariant appearing in the APS index theorem. We introduce a 1-parameter family of such operators by scaling $G \to tG$ and constructing the 12-dimensional operator:
\[
\hat{\mathcal{P}} = \sigma^2 \otimes \frac{\partial}{\partial t} + \sigma^1 \otimes \mathcal{P} + \ell^3 t \sigma^1 \otimes \mathcal{G},
\]
acting on spinors in the twelve-manifold $Z = Y \times \mathbb{R}$. In order to apply index theory we should think of the Dirac operator as
\[
\hat{\mathcal{P}} = \mathcal{P} + t\ell^3 \Gamma(\star G),
\]
(3.23)

where $\Gamma(\star G)$ is the Clifford multiplication by $\star G \in \Omega^7(Z)$ in Cliff(12), $\star$ is the 11-dimensional Hodge operator defined on $\Omega^*(Y)$, and
\[
\mathcal{P} = \sigma^2 \otimes \frac{\partial}{\partial t} + \sigma^1 \otimes \mathcal{P}
\]
(3.24)
is the Dirac operator in 12-dimensions. Then we have
\[
\frac{\partial \xi (\mathcal{D} + \ell^3 G)}{\partial t} dt = \int_Y \text{Tr}_s (a_6 (\mathcal{D}))_{(12)},
\] (3.25)
with \(a_6\) being the \(t\)-independent part of the heat kernel expansion for \(\exp(-t \mathcal{D}^2)\). We can write the tensor products by the Pauli matrices in (3.22) as gamma matrices in 12 dimensions. The 12-form that we integrate on \(Y\) in (3.25) can be interpreted as the index density of \(\mathcal{D}\), hence in order to extract information on the \(G\)-dependence of (3.21), we can use results from geometric index theory for families of operators \(\mathcal{D}\), as we did in the case of the local anomaly.

The index of (3.23), is not modified by the presence of the \(G\)-flux, hence the flux-correction to the 12-form \(\text{Tr}_s (a_6 (\mathcal{D}))_{(12)}\) will be
\[
\text{Tr}_s (a_6 (\mathcal{D}))_{(12)} = \text{Tr}_s (a_6 (\mathcal{P}))_{(12)} + \ell^3 d \varphi (\ast G, \mathcal{R}),
\] (3.26)
with \(\int_Y \varphi (\ast G, \mathcal{R}) : \mathcal{T} \mapsto \mathbb{R}\) a well defined diffeomorphism-invariant function defined on the functional space of bosonic configurations. Adding the contributions of the various terms we obtain an expression of the form:
\[
\xi ((\mathcal{D}^{\ast} Y - \frac{\ell^3}{96} G^2) - \xi (\mathcal{D} + \frac{\ell^3}{288} G^2) - 2 \xi (\mathcal{D} - \frac{5\ell^3}{96} G^2) = \xi (\mathcal{D}_{RS}) + \ell^3 \int_Y \varphi (\ast G, \mathcal{R}).
\] (3.27)
Since \(\varphi (\ast G, \mathcal{R})\) is local and gauge invariant we see explicitly that the \(\mathbb{Z}_2\) anomaly cancellation is unchanged.

4. Example: Eleven manifold with flat boundaries

Let \(X := \partial Y = \mathbb{R}^{10}\) be flat 10-dimensional Euclidean space. Let \(E \rightarrow X\) be the adjoint \(E_8\)-vector bundle, and \(D_M = \partial_M + A_M\) the gauge connection on \(E\), i.e. \(D_M : \Omega^0 (S \otimes E) \rightarrow \Omega^1 (S \otimes E)\). Thus the quadratic action for the gaugino is constructed through the generalized Dirac operator
\[
\mathcal{D}_{E_8} = \gamma^M D_M + \gamma (\mathcal{H})
\] (4.1)
where \(\mathcal{H} = -\frac{\ell^3}{24} i_{\nu} G^{\nu}\) is the 3-form that comes from contracting the M-theory G-flux in the bulk \(Y\), with the normal unit vector field to the boundary \(\partial Y = X\) and
\[
\gamma (\mathcal{H}) = \gamma^{M_1 M_2 M_3} \mathbb{H}_{M_1 M_2 M_3}.
\] (4.2)
We consider the fibration $\mathcal{X} \to \mathcal{T}^\partial$ encoding the family of geometric data on the fiber $X$, i.e. gauge connections and fluxes, and calculate the curvature of the Pfaffian line bundle $\text{Pfaff} \, \hat{\mathcal{D}}_{E_8} \to \Sigma \hookrightarrow \mathcal{T}^\partial$ using (3.11) as follows

$$F(\text{Pfaff} \, \hat{\mathcal{D}}_{E_8} \to \Sigma \hookrightarrow \mathcal{T}^\partial) = \pi i \int_X \text{Tr}_a \hat{a}_6(\hat{\mathcal{D}}_{E_8})$$

where $\text{Tr}_a \hat{a}_6(\hat{\mathcal{D}}_{E_8})$ is the $t$-independent finite part of the heat kernel expansion for $\text{Tr}_a \exp(-t \hat{\mathcal{D}}_{E_8}^2)$ when $t \to 0$ while $t > 0$. $\text{Tr}_a(\cdot) \equiv \text{Tr}(\gamma^1 \cdot)$ means supertrace. In contrast to the case with zero flux, there are nonzero divergent terms in the $t \to 0$ expansion. However, these may be easily cancelled by gauge invariant counterterms, so we focus on the $t$-independent terms.

4.1. Determining $a_6$ up to $O(h^2)$

Formally, we can expand $\text{Tr}_a(a_6)$ as a series in $h$

$$\text{Tr}_a(a_6) = \alpha_0(h) + \alpha_1(h) + \alpha_2(h) + \alpha_3(h) + \ldots$$

with $\alpha_i(h)$ a 2-form in $\mathcal{T}$ which scales homogeneously under scalings of the torsion, i.e. $\alpha_i(\lambda h) = \lambda^i \alpha_i(h)$. For simplicity, we determine only the lowest correction $\alpha_1(h)$ to $\text{Tr}_a(a_6)$.

In order to evaluate $\text{Tr}_a a_6$, we are going to use known results on heat kernel expansions for generalized Laplacians of the type

$$\Delta = -(\nabla_N \nabla^N + V)$$

with $\nabla_N = \partial_N + Q_N$ a first order partial differential operator, $Q_N dx^N$ a matrix of one-forms and $V$ a scalar matrix. For such operators, the $t$-independent finite part of the heat kernel expansion for

$$\exp(-t(\nabla_N \nabla^N + V))$$

has been calculated in flat space using different methods, see [23] and [26]. Thus we want to write $\hat{\mathcal{D}}_{E_8}^2$ as an operator of the type (4.6). If we introduce the connection

$$\nabla_M = \partial_M + A_M + 3h_{NM_1M_2} \gamma^{M_1} \gamma^{M_2},$$
we find
\[
\tilde{D}^2_{E_8} = -\nabla_N \nabla^N + F_{MN} \gamma^{MN} + \partial_{M_1} \hbar_{M_2 M_3 M_4} \gamma^{M_1 M_2 M_3 M_4} + 4 \hbar_{M_1 M_2 M_3} \hbar_{M_1 M_2 M_3}.
\]
(4.8)
with \( F = dA + A \wedge A \), the curvature of the vector bundle \( E \rightarrow X \). Hence, as \( \nabla_N = \partial_N + A_N + 3 \hbar_{NM_1 M_2} \gamma^{M_1 M_2} \), \( V \) in (4.6) is fixed to be
\[
V = -F_{MN} \gamma^{MN} - \partial_{M_1} \hbar_{M_2 M_3 M_4} \gamma^{M_1 M_2 M_3 M_4} - 4 \hbar_{M_1 M_2 M_3} \hbar_{M_1 M_2 M_3}.
\]
(4.9)
Now, having written \( \tilde{D}^2_{E_8} \) as a generalized Laplacian, we can use the coefficient calculated in [25][26],
\[
a_6 = \frac{1}{6!} \left[ V^6 + 6V^2 \nabla^N(V) V \nabla_N(V) + 4V^3 \nabla^N(V) \nabla_N(V) + O(V^4) \right],
\]
(4.10)
to evaluate the lowest order flux correction in \( \text{Tr}_s a_6(\hat{\tilde{D}}_{E_8}) \), neglecting the \( O(\hbar^2) \) terms in \( V \).

We now compute the contribution of every term in (4.10) as follows:
- \( \text{Tr}_s [V^6] \). The most obvious contribution is the leading term \( \text{Tr}(F^6) \). The first order contribution in \( \hbar \) is
\[
6 \text{Tr}_s \left[ \partial_{M_1} \hbar_{M_2 M_3 M_4} \gamma^{M_1 M_2 M_3 M_4} (F)^5 \right].
\]
(4.11)
As we are working with a 12 dimensional Clifford algebra, only the term proportional to \( \gamma^{M_1 M_2 \ldots M_{12}} \) contributes to the supertrace in (4.11). Thus, we determine the contribution from (4.11) by studying the irreps of the rank 14 tensor
\[
\partial_{M_1} \hbar_{M_2 M_3 M_4} F_{M_5 M_6} \ldots F_{M_{13} M_{14}},
\]
(4.12)
defined in dimension 12 under the group \( SO(12) \). Some of the symmetries of (4.12) under the permutation of indices are already known, for instance each curvature tensor \( F_{M_i M_j} \) contributes antisymmetric couples \( M_i, M_j \), also we know that \( \hbar \) is a completely antisymmetric rank 3 tensor, etc. A detailed analysis along these lines, shows how just the symmetric part of \( M_1 \) with the triad \( M_2, M_3 \) and \( M_4 \) in (4.12), gives a non zero contribution to the supertrace (4.11). Therefore, we find
\[
6 \text{Tr}_s \left[ \partial_{M_1} \hbar_{M_2 M_3 M_4} \gamma^{M_1 M_2 M_3 M_4} (F)^5 \right] = 6(2 - 12) \partial^M \hbar_{M_1 M_2} dx^{M_1} dx^{M_2} \text{Tr}(F^5).
\]
(4.13)
\( \textbf{Tr}_s [V^2 \nabla^N (V) V \nabla_N (V)] \). Here, the \( \mathbb{h} \) term can come from the \( \nabla_N \)-derivative or from the matrix \( V \). When it comes from the \( (\nabla_N = D_N + 3\mathbb{h}_{NM_1M_2} \gamma^{M_1} \gamma^{M_2}) \)-derivative, with \( D_N = \partial_N + A_N \) the usual gauge differential, we find

\[
-6 \text{Tr}_s [\gamma(F)^2 \cdot \mathbb{h}_{NM_1M_2} \gamma^{M_1} \gamma^{M_2} \cdot \gamma(F)^2 \mathcal{D}^N (\gamma(F))],
\]

which is the same as

\[
-6 \mathbb{h}_{M_1M_2M_3} dx^M dx^M \text{Tr}(D^{M_1}(F) F^4).
\]

If the \( \mathbb{h} \)-term comes from \( V \), it cannot come from \( \partial^N \mathbb{h}_{NM_1M_2} \gamma^{M_1} \gamma^{M_2} \) because we would combine less than 12 gamma matrices. Thus, up to global numerical factors we find

\[
\text{Tr}_s \left[ (d \mathbb{h})_{M_1M_2M_3M_4} \gamma^{M_1} \gamma^{M_2} \gamma^{M_3} \gamma^{M_4} \gamma(F) \mathcal{D}^N (\gamma(F)) \gamma(F) D_N (\gamma(F)) \right], \tag{4.16}
\]

and

\[
\text{Tr}_s \left[ \gamma(F)^2 \mathcal{D}^N ((d \mathbb{h})_{M_1M_2M_3M_4} \gamma^{M_1} \gamma^{M_2} \gamma^{M_3} \gamma^{M_4} \gamma(F) D_N (\gamma(F)) \right]. \tag{4.17}
\]

\( \textbf{Tr}_s [V^3 \nabla^N (V) \nabla_N (V)] \). These terms are of the same type as in the previous case, just differing in the order of terms. For example, we find

\[
\text{Tr}_s \left[ (d \mathbb{h})_{M_1M_2M_3M_4} \gamma^{M_1} \gamma^{M_2} \gamma^{M_3} \gamma^{M_4} \gamma(F)^2 \mathcal{D}^N (\gamma(F)) D_N (\gamma(F)) \right], \tag{4.18}
\]

\[
-6 \text{Tr}_s [\gamma(F)^3 \cdot \mathbb{h}_{NM_1M_2} \gamma^{M_1} \gamma^{M_2} \cdot \gamma(F) \mathcal{D}^N (\gamma(F))], \tag{4.19}
\]

etc.

\( \textbf{Tr}_s [\mathcal{O}(V^4)] \). It is easy to check that these terms only contribute to \( \mathcal{O}(\mathbb{h}^2) \).

Thus, the terms above determined are the only ones that contribute to \( \alpha_1(\mathbb{h}) \) in the expansion of the line bundle curvature in “powers” of \( \mathbb{h} \). Furthermore, we can group the terms in \( \alpha_1(\mathbb{h}) \) that scale as \( \lambda^5 \alpha_1(\mathbb{h}) \) under scalings \( F \mapsto \lambda \cdot F \), of the gauge connection curvature \( F \) by \( \lambda \in \mathbb{R} \). This set of terms coming from (4.11), (4.15) and (4.19), can be written as

\[
\text{Tr}_s (V^6 + 6V^2 \nabla^N (V) V \nabla_N (V) + 4V^3 \nabla^N (V) \nabla_N (V)) =
\]

\[
-60 \partial^M \mathbb{h}_{MPQ} dx^P dx^Q \text{Tr}(F^5) - 60 \mathbb{h}_{MPQ} dx^P dx^Q \text{Tr}(D^M (F) F^4) + \ldots \tag{4.20}
\]

where the \( \ldots \) refer to terms with different scaling properties.
After evaluating the other supertraces, we find the forms $D_N(d\mathbb{h}) \wedge \text{Tr}(D^N(F)F^3)$ and $d\mathbb{h} \wedge \text{Tr}(F D_N(F) D^N(F))$ or $d\mathbb{h} \wedge \text{Tr}(F^2 D_N(F) D^N(F))$, depending if it comes from $\text{Tr}_s[V^2 \nabla^N(V) V \nabla_N(V)]$ or $\text{Tr}_s[V^3 \nabla^N(V) \nabla_N(V)]$.

Taking into account the numerical factors and recalling $F(Pfaff \hat{D}/E) = \pi i \int_X \text{Tr}_a 6 \hat{D}/E$, we obtain the curvature for the Pfaffian line bundle $F(Pfaff \hat{D}/E) = -\pi i \int_X \text{Tr}_a 6 \hat{D}/E$.

The formula (4.22) allows us to compute the curvature of the M-theory line bundle $L_G \to T\partial$. The curvature of the Chern-Simons bundle $L_{CS} \to T\partial$ exactly cancels the $\mathbb{h}$-independent part of the curvature. Furthermore $L_{\text{gravitino}} \to T\partial$ does not contribute terms to $\alpha_0(\mathbb{h})$ nor $\alpha_1(\mathbb{h})$ in flat space. Therefore only the terms in (4.22) contribute to $\mathcal{F}(L_G \to T\partial)$, up to $O(\mathbb{h}^2)$.

For the first correction, we work with the set of terms

$$-60 \partial^M \mathbb{h}_M \text{Tr}(F^5) - 60 \mathbb{h}_M \text{Tr}(D^M(F)F^4),$$

which have identical behavior under scalings of $\mathbb{h}$ and $F$. An obvious candidate to write (4.23) as a total divergence seems to be

$$-60 d(\mathbb{h}_N \text{Tr}(F_M P d x^P F^4)) \delta^{NM}$$

with $\delta^{NM}$ the Kronecker delta, which is the metric for the twelve dimensional space that we are dealing with. Expanding (4.24) we find

$$d(\mathbb{h}_N \text{Tr}(F_P M d x^P F^4)) \delta^{NM} = d\mathbb{h}_N \text{Tr}(F_P M d x^P F^4) \delta^{NM} + \mathbb{h}_N d \text{Tr}(F_P M d x^P F^4) \delta^{NM}.$$
Furthermore, we can write the exterior differential of a trace over the color indices, as the trace of the covariant exterior differential, i.e.,

\[ d\text{Tr}(F_{PM}dx^PF^4) = \text{Tr}(D(F_{PM}dx^PF^4)) \]  

(4.26)

with \( D = dx^N D_N \cdot = dx^N (\partial_N \cdot + [A_N, \cdot]) \). The expression (4.26) holds because the trace of a commutator is zero. Therefore, recalling the Bianchi identity \( DF = 0 \), we get

\[ \text{Tr}(D(F_{PM}dx^PF^4)) = \text{Tr}(D(F_{PM}dx^PF^4)) = \text{Tr}(F_{M}(F^4)), \]  

(4.27)

where we have used the identity \( D(F_{PM}dx^PF^4) = D_{M}F^4 \), which follows from the antisymmetry under permutation of couples of indices in the rank 3 tensor \( D_{M}F_{PQ} \). Using (4.27)

we can write (4.24) as

\[ d h l_{NM} \text{Tr}(F_{PM}dx^PF^4) + h l_{NM} \text{Tr}(D_{M}F^4) \]  

(4.28)

which is not yet clearly equal to (4.23), because the first term. To show how

\[ \partial_{M_1} \ll_{M_2 M_3 M_4} \text{Tr}(F_{M_5 M_6}F_{M_7 M_8} \ldots F_{M_{13} M_{14}}) \]  

(4.30)

we study the irreps of the rank 14 tensor

that after contracting with \( \delta^{M_2 M_6} \) becomes identical to (4.29) as we wanted to prove.

Hence

\[ -60\partial^M \ll_{M} \text{Tr}(F^5) - 60 \ll_{M} \text{Tr}(D^M(F)F^4) = -60d(\ll_{N} \text{Tr}(F_{PM}dx^PF^4)\delta^{NM}). \]  

(4.32)

21
The second correction in (4.22), can be written as,

$$-20D_N(dh) \wedge \text{Tr}(D^N(F)F^3) = -20\partial_N(dh) \wedge \text{Tr}(\partial^N(F)F^3),$$  \hspace{1cm} (4.33)

because $D_N dh = \partial_N dh + [A_N, dh]$ and $[A_N, dh] = 0$. On the other hand, $\text{Tr}([A^N, F]F^3) = 0$, thus

$$-20\partial_N(dh) \wedge \text{Tr}(\partial^N(F)F^3) = -5\partial_N(dh) \wedge \partial^N\text{Tr}(F^4) = -\frac{5}{2}\left[\partial_N\partial^N(dh \wedge \text{Tr}(F^4))\right],$$  \hspace{1cm} (4.34)

or using the Hodge Laplacian $\partial_N\partial^N = \star \star d + d \star \star$ in Cartesian coordinates for the Euclidean space $X$, we write (4.34), as

$$-20\partial_N(dh) \wedge \text{Tr}(\partial^N(F)F^3) = -5\partial_N(dh) \wedge \text{Tr}(F^4) - dh \wedge \partial_N\partial^N(\text{Tr}(F^4)).$$  \hspace{1cm} (4.35)

The operator $\star \star d$ never appears, because it always acts on closed forms.

Finally, the third and fourth correction in (4.22), can be written using the covariant Laplacian $D_ND^N$, as

$$-dh \wedge \text{Tr}(12F \partial^N(F)DD^N(F) + 18F^2 \partial^N(F)D^N(F)) =$$

$$\text{Tr}(6D_ND^N(F)F^3 - 3D_ND^N(F)D^N(F) + 3D_ND^N(F^2)F^2).$$  \hspace{1cm} (4.36)

Also, we can use a more transparent notation, using the covariant exterior derivative

$$D = dx^N \wedge D_N = d + [A, \cdot]$$  \hspace{1cm} (4.37)

we can write the curvature $F$ as $F = D^2$, and the covariant Laplacian as

$$D_ND^N = \star D \star D + D \star \star$$  \hspace{1cm} (4.38)

with $\star$ being the Hodge operator. Using the Bianchi identity $DF = 0$, we rewrite (4.36), as

$$dh \wedge \text{Tr}(6D_ND^N(F)F^3 - 3D_ND^N(F^2)D^N(F) - 3D_ND^N(F^2)F^2 =$$

$$\text{Tr}(6D \star D \star (F^3 - 3D \star D \star (F^4) + 3D \star D \star (F^2)F^2).$$  \hspace{1cm} (4.39)

Now note that

$$\text{Tr}(D \star D \star (F^4)) = d\text{Tr}(\star D \star (F^4)) + \text{Tr}([A, \star D \star (F^4)]) = d\text{Tr}(\star D \star (F^4))$$  \hspace{1cm} (4.40)
is an exact form. On the other hand, consider the 6-forms $\text{Tr}(D \star D \star (F) F^2)$ and $\text{Tr}(D \star D \star (F^2) F)$, and differentiate them twice

$$d^2 \text{Tr}(D \star D \star (F) F^2) = d \text{Tr}(D[D \star D \star (F) F^2]) = d \text{Tr}(D \star D \star (F) F^3) = \text{Tr}(D \star D \star (F) F^3)$$

and

$$d^2 \text{Tr}(D \star D \star (F^2) F) = d \text{Tr}(D[D \star D \star (F^2) F]) = d \text{Tr}(D \star D \star (F^2) F^2) = \text{Tr}(D \star D \star (F^2) F^2)$$

therefore, by construction (4.41) and (4.42) are zero. This means that we can write (4.39), as

$$d \llh \wedge \text{Tr}(6D \star D \star (F) F^3 - 3D \star D \star (F^4) + 3D \star D \star (F^2) F^2) =$$

$$-3d \llh \wedge d \text{Tr}(\star D \star (F^4)).$$

(4.43)

Using the identities (4.32), (4.35) and (4.43), we can write the curvature of the M-theory line bundle as

$$\mathcal{F}(\mathcal{L}_G \rightarrow \mathcal{T}^\partial) = -\frac{\pi i}{6!} \int_X \left( 60d(\llh N \text{Tr}(F_{PM} dx^P F^4) \delta^{NM}) + \frac{5}{2} d \star d \star (d \llh \wedge \text{Tr}(F^4)) - \frac{5}{2} d \star d \star (d \llh \wedge \text{Tr}(F^4)) - \frac{1}{2} d \llh \wedge d \star d \star (\text{Tr}(F^4)) \right) + \mathcal{O}(\llh^2).$$

(4.44)

This formula agrees with the results explained in section 3, where we claimed that the curvature of $\mathcal{L}_G \rightarrow \mathcal{T}^\partial$ is an exact form $dA$, with $A$ being a $G$-equivariant one form on $\text{Met}(Y) \times A$. From (4.44), we can write $A$ as:

$$A = -\frac{\pi i}{6!} \int_X \left( 60(\llh N \text{Tr}(F_{PM} dx^P F^4) \delta^{NM}) + \frac{5}{2} d \star d \star (d \llh \wedge \text{Tr}(F^4)) - \frac{5}{2} d \star d \star (d \llh \wedge \text{Tr}(F^4)) - \frac{1}{2} d \llh \wedge d \star d \star (\text{Tr}(F^4)) \right) + \ldots$$

(4.45)

up to a globally exact form. The Hodge $\star$ depends on a metric on $\Sigma \hookrightarrow \mathcal{T}^\partial$. Note that there is a natural metric on $\mathcal{T}^\partial$, induced by the Riemannian metric itself.

4.3. Covariant form of the Anomaly

To get a better understanding of these flux corrections to the anomaly, it is instructive to calculate the contribution from the fluxes to the divergence of the gauge current using the gaussian cutoff proposed by Fujikawa. This approach to anomaly cancellation leads to the so-called covariant form of the anomaly. See [27][28]. Fujikawa proposed to account...
for the local chiral anomaly from the variation of the measure \( [d\chi][d\bar{\chi}] \) under the action of the gauge group in the path integral
\[
\int [d\chi][d\bar{\chi}] \exp \left( \int_X \bar{\chi} \hat{D} \chi \right)
\]  
(4.46)

If \( \{ T_a \} \) is a basis for the Lie algebra of the gauge group \( G = \mathcal{E}_8 \), then an infinitesimal gauge transformation can be expressed as \( g = I + \Lambda^a T_a + \mathcal{O}(\Lambda^2) \). We can compute
\[
\frac{|d_{T_a} \det \hat{D}_{\mathcal{E}_8}|}{|\det \hat{D}_{\mathcal{E}_8}|} := dj_a = 2i \text{Tr} \left[ T_a \gamma^{11} \exp \left( -t \hat{D}_{\mathcal{E}_8}^2 \right) \right],
\]  
(4.47)

where \( j_a \in \Omega^0(X) \) is the gauge current.

Of course, \( \text{Tr}(T_a \gamma^{11} \exp( -t \hat{D}_{\mathcal{E}_8}^2 )) \) must be regulated, and we do so by taking
\[
\text{Tr} \left[ T_a \gamma^{11} \exp( -t \hat{D}_{\mathcal{E}_8}^2 ) \right],
\]

where \( t = 1/\Lambda \) should tend to zero. In stark contrast to the case without fluxes, the expression for \( dj_a \) has divergent terms for \( t \to 0 \). These divergent terms can be shown to be total covariant divergences of local gauge invariant expressions in the fields by a method explained below for the \( t \)-independent part of the heat kernel. Thus the current must be renormalized by adding these terms.

In order to evaluate the regulator independent part of the supertrace (4.47) we have to determine the heat kernel coefficient \( a_5 \). We can use again the results of [25], to calculate (4.47) up to second order in \( \mathbb{H} \), i.e.
\[
dj_a = 2i \text{Tr} \left( T_a \gamma^{11} a_5 (\hat{D}_{\mathcal{E}_8}^2) \right) = \beta_0(\mathbb{H}) + \beta_1(\mathbb{H}) + \ldots,
\]  
(4.48)

where \( \beta_k(\mathbb{H}) \) are terms that scale homogeneously under dilations of \( \mathbb{H} \), i.e. if \( \lambda \) is a real parameter then \( \beta_k(\lambda \mathbb{H}) = \lambda^k \beta_k(\mathbb{H}) \).

Therefore, the only terms in \( a_5 \) which contribute up to first order are
\[
a_5 = \frac{1}{5!} \left[ V^5 + 2V \nabla_N(V) V \nabla_N(V) + 3V^2 \nabla_N(V) \nabla_N(V) + \mathcal{O}(V^3) \right].
\]  
(4.49)

Doing a similar calculation as we did above for the heat kernel coefficient \( a_6 \), we find
\[
dj_a = \frac{2i}{3!} \text{Tr} \left( T_a F^5 \right) + \frac{4i}{3!} \left[ 20 \partial^N \mathbb{H}_N \wedge \text{Tr}(F^4) + 4 \mathbb{H}_N \wedge \text{Tr}(FD^N(F)F^2) + 
4 \mathbb{H}_N \wedge \text{Tr}(F^3 D^N(F)) + d\mathbb{H} \wedge \text{Tr} \left( T_a D_N(F) D^N(F) + 4T_a F D_N(F) D^N(F) + 4T_a F D_N(F) F D^N(F) \right) +
\partial_N(d\mathbb{H}) \wedge \text{Tr} \left( 4T_a F^2 D^N(F) + T_a F D^N(F) F \right) \right] + \beta_2(\mathbb{H}) + \ldots
\]  
(4.50)

24
where \( D_N = \partial_N + A_N \) is the gauge covariant derivative. The “Chern-Simons” terms, exactly cancel the expression \( \frac{2i}{3} \text{Tr} \left( T_a F^5 \right) \) in (4.50). We can then write the flux corrections to the anomalous divergence of the gauge current as the covariant exterior derivative of a gauge invariant 9-form \( \Delta_j(h) \):

\[
D \Delta j = \frac{4i}{5} \left[ 20 \partial^N h_L \wedge F^4 + 4 h_L \wedge F D^N (F) F^2 + 16 h_L \wedge F^3 \wedge D^N (F) + d h_L \wedge \right. \\
\left. \left( D_N (F) F D^N (F) + 4 F D_N (F) D^N (F) \right) + \partial_N (d h_L) \wedge \left( 4 F^2 D^N (F) + F D^N (F) F \right) \right] + \ldots
\]

(4.51)

where we have written the expression as a Lie-algebra valued form.

We now show explicitly how this can be written as a total divergence of a gauge invariant quantity. For the three first terms in (4.51), one can show how

\[
20 \partial^N h_L F^4 + 4 h_L F D^N (F) F^2 + 16 h_L F^3 D^N (F) = D \left( 4 h_L F F_{PM} dx^P F^2 + 16 h_L F^3 F_{PM} dx^P \right) \delta^{NM}.
\]

(4.52)

Expanding (4.52), using the identity \( D_N (F) = D (F_{MN} dx^M) \) gives us

\[
D \left( 4 h_L F F_{PM} dx^P F^2 + 16 h_L F^3 F_{PM} dx^P \right) \delta^{NM} = \left( 4 d h_L F F_{PM} dx^P F^2 + 16 d h_L F^3 F_{PM} dx^P \right) \delta^{NM} + 4 h_L F D^N (F) F^2 + 16 h_L F^3 D^N (F),
\]

(4.53)

thus, we have to prove the identity

\[
20 \partial^N h_L F^4 = \left( 4 d h_L F F_{PM} dx^P F^2 + 16 d h_L F^3 F_{PM} dx^P \right) \delta^{NM}.
\]

(4.54)

This can be achieved by analyzing the irreps of the rank 12 tensor

\[
\partial_{M_1} h_{M_2 M_3 M_4} F_{M_5 M_6} F_{M_7 M_8} \ldots F_{M_{11} M_{12}},
\]

(4.55)

with antisymmetry under permutations of the sets of indices \( \{ M_2, M_3, M_4 \} \) and \( \{ M_1, M_3, M_4, M_5, M_6, M_7, \ldots M_{12} \} \) and symmetry under permutations of the couple \( \{ M_2, M_8 \} \). The only irreducible representation of \( SO(10) \) in \( (\mathbb{R}^{10}) \otimes^{12} \) which satisfies such properties under permutations of indices, also verifies the complete symmetry of the set \( \{ M_1, M_2 \) and \( M_8 \} \), therefore we can prove the identity (4.54) by using the symmetry under permutations of \( M_1 \) and \( M_8 \), contracting \( M_2 \) and \( M_8 \) with the Kronecker delta \( \delta^{M_2 M_8} \) and contracting the other indices with their corresponding grassmann differentials. One should also consider the same argument with \( M_{12} \) playing the role of \( M_8 \) in order to achieve the full proof.
For the second set of terms, we realize that $D_N(F)FD^N(F) + 4FD_N(F)D^N(F)$ using
the Laplacian $D_ND^N = \star D \star D + D \star D \star$. A short calculation yields

\[
D_N(F)FD^N(F) + 4FD_N(F)D^N(F) = \frac{1}{2} D \star D \star (F^3) + \frac{3}{2} FD \star D \star (F^2)
- \frac{1}{2} D \star D \star (F^2)F - \frac{3}{2} FD \star D \star (F)F - 2F^2 D \star D \star (F).
\]

(4.56)

If again we use the identity $D_N(F) = D(F_{MN}dx^M) := D(F_N)$, then we can write (4.51) as

\[
D\Delta^j = \frac{4i}{5!} \left[ D\left(4 \text{ln}_N FF_M F^2 + 16 \text{ln}_N F^3 F_M\right) \delta^N^M + d\text{ln} \wedge \left(\frac{1}{2} D \star D \star (F^3) + \frac{3}{2} FD \star D \star (F^2) - \frac{1}{2} D \star D \star (F^2)F - \frac{3}{2} FD \star D \star (F)F - 2F^2 D \star D \star (F)\right) + \partial^N (d\text{ln}) \wedge \left(4F^2 D(F_N) + FD(F_N)F\right) \right] + \ldots
\]

(4.57)

Finally, using the Bianchi identity $DF = 0$, it is easy to prove that

\[
\Delta^j = \frac{4i}{5!} \left[ \left(4 \text{ln}_N FF_M F^2 + 16 \text{ln}_N F^3 F_M\right) \delta^N^M + d\text{ln} \wedge \left(\frac{1}{2} \star D \star (F^3) + \frac{3}{2} F \star D \star (F^2) - \frac{1}{2} \star D \star (F^2)F - \frac{3}{2} F \star D \star (F)F - 2F^2 \star D \star (F)\right) + \partial^N (d\text{ln}) \wedge \left(4F^2 F_N + FF_N F\right) \right] + \ldots
\]

(4.58)

This gives a non-trivial redefinition of the gauge current, by gauge invariant flux-dependent 9-forms $\Delta^j(\text{ln})$.

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Appendix A. Clifford algebras in dimensions 10, 11 and 12

In this appendix we summarize our conventions for the Clifford algebras Cliff($n$). (We follow [29].) We follow the Clifford algebra multiplication convention

\[
\{\gamma^M, \gamma^N\} = -2g^{MN}.
\]

(A.1)
A natural basis for Cliff\((n)\), is given by the set of matrices
\[
\gamma^{M_1M_2...M_p} = \gamma^{[M_1\gamma^{M_2}...\gamma^{M_p}]} \quad p = 0,1,...,n.
\] (A.2)

For \(n\) even, Cliff\((n)\) is isomorphic to End\((S) = S^\vee \otimes S\), the vector space of endomorphisms of the spinor bundle.

For \(n\) odd, there is a two to one correspondence between elements in Cliff\((n)\) and elements in \(S^\vee \otimes S\). This map between vector spaces is understood through the action of the volume element \(\omega\) in Cliff\((n)\), i.e. in local coordinates
\[
\omega = \gamma^1\gamma^2...\gamma^n = \frac{1}{n!}\epsilon_{M_1M_2...M_n} \gamma^{M_1}\gamma^{M_2}...\gamma^{M_n}
\] (A.3)

verifies
\[
\omega^2 = (-1)^{(n+1)/2}1
\] (A.4)

where \(1\) is the identity matrix in \(S^\vee \otimes S\). As the volume element \(\omega\) commutes with every element in Cliff\((n)\) and the Clifford algebra is irreducible, Schur’s lemma implies that \(\omega\) must be represented by \(\pm 1\). For \(n = 11\), we choose \(\omega = 1\), by convention. In local coordinates, Clifford multiplication by the volume form \(\omega\) acts as a Hodge dual, that is, if \(H\) is a \(p\)-form and \(\gamma\) its associated Clifford multiplication
\[
\gamma(H) = H_{M_1M_2...M_p} \gamma^{M_1M_2...M_p},
\] (A.5)

then
\[
\omega \gamma(H) = \gamma(\star H)
\] (A.6)

with \(\star\) the Hodge star operator. Thus in odd dimensions, Clifford multiplication by a form and by its Hodge dual are represented by the same element in \(S^\vee \otimes S\).

We will also use the relation between irreducible representations of Cliff\((2n)\) and Cliff\((2n - 1)\), i.e. if \(\gamma^M\) is an irrep of Cliff\((2n - 1)\), an irrep \(\Gamma^M\) for Cliff\((2n)\) is given by
\[
\Gamma^M = \sigma^1 \otimes \gamma^M \quad M = 1,...,2n - 1
\] (A.7)
\[
\Gamma^{2n} = \sigma^2 \otimes 1
\] (A.8)
\[
\Gamma^{2n+1} = \sigma^3 \otimes 1
\] (A.9)

where \(\sigma^i\) are the \(2 \times 2\) Pauli matrices.
Appendix B. Heat kernel expansions and Quantum Mechanics

There are several algorithms to evaluate the trace $\text{Tr}(\exp(-t\hat{D}^2))$. As we show in the main text of this paper, the coefficients associated with the expansion of such a trace in powers of $t = 1/M^2$ determine the curvature of determinant line bundles and hence the anomalous divergence of the gauge current. Although there are explicit calculations of such expansions for flat space, see [25], we review here some of the techniques used to determine such coefficients, and explain qualitatively the one based on path integrals in supersymmetric quantum mechanics.

The main idea is to separate the interacting heat kernel

$$\langle x|K(t)|y \rangle = \langle x|\exp(-t\hat{D}^2)|y \rangle,$$  \hspace{1cm} (B.1)

as the product of the free heat kernel

$$\langle x|K_0(t)|y \rangle = \frac{1}{(4\pi t)^{n/2}}\exp\left(-\frac{(x-y)^2}{4t}\right)$$  \hspace{1cm} (B.2)

with $n$ the dimension of the $x$-space, and an interacting part $H$

$$H(x, y; t) = \sum_{k=0}^{\infty} a_k(x, y)t^k,$$  \hspace{1cm} (B.3)

i.e., we compute (B.1) through the ansatz

$$\langle x|K(t)|y \rangle = \frac{1}{(4\pi t)^{n/2}}\exp\left(-\frac{(x-y)^2}{4t}\right)H(x, y; t).$$  \hspace{1cm} (B.4)

There is a large variety of algorithms to calculate the coefficients $a_k$; they roughly fall in three categories:

- **Recursive $x$-space algorithms based on recursive relations among different heat kernel coefficients** [30] [31].
- **Nonrecursive algorithms based on the insertion of a momentum basis** [32] [33].
- **The method of Zuk, based on graphical representations of the heat kernel coefficients** [34].

If the supertrace of (B.4) is taken, we can evaluate the expansion using path integrals in quantum mechanics. We can follow the ideas of [35], [36] and [37], to determine the coefficients of the supertrace of the heat kernel expansion associated to a generalized Dirac
operator $\hat{\mathcal{D}}$ in 12-dimensions, as the ones which appear in the definition of the curvature of the M-theory line bundle (3.1). Thus, given the operator $\hat{\mathcal{D}}$, the expansion

$$\text{Tr}_s \left( \exp \left( -t\hat{\mathcal{D}}^2 \right) \right) = \frac{\text{Tr}_s a_0}{t^6} + \frac{\text{Tr}_s a_1}{t^5} + \ldots + \text{Tr}_s a_6 + O(t),$$  \hspace{1cm} (B.5)

can be determined through the partition function of a supersymmetric quantum mechanical model. The idea is to interpret (B.5) as the time evolution operator of a quantum mechanical system with Hamiltonian $H = \hat{\mathcal{D}}^2$, and calculate explicitly the expansion (B.5), through the path integral approach to quantum mechanics. A novelty introduced by the generalized Dirac operators is that there are coefficients $\text{Tr}_s a_k$ with $k < 6$ which are not zero. This differs sharply from the super heat kernel expansions for standard Dirac operators, where the coefficients with inverse powers of $t$ are known to be zero.

In the standard case the vanishing of the coefficients $\text{Tr}_s a_k$ with $k < 6$ allows us to determine $\text{Tr}_s a_6$ by evaluating the path integral in the limit $t \to 0$. In the case of generalized Dirac operators we find non-zero terms with inverse powers of $t$. Thus we have to be more careful and evaluate the path integral for a finite time interval instead of taking the limit $t \to 0$. Path integrals in supersymmetric quantum mechanics for a finite time interval were analyzed in detail by [39] [40], and used in [41], to determine index densities of generalized Dirac operators in 4-dimensions, which agree with the older results of [42] [43].

The type of quantum mechanical theory that we consider, is a supersymmetric non-linear sigma model with target the 12-dimensional manifold $Z = \pi^{-1}(\Sigma)$ where $\pi : X \to T^\partial$ is the projection to the space of M-theory and $\Sigma \hookrightarrow T^\partial$ is any surface where the curvature of $\text{Det} \hat{\mathcal{D}} \to T^\partial$ is to be evaluated. Here, $\hat{\mathcal{D}} = \mathcal{D} + \gamma(\mathfrak{h})$ stands for any of the chiral generalized Dirac operators that couple to the M-theory fermions at the boundary.

Therefore, let $\mathbb{R}^{1|1}$ denote the super Euclidean space with one even variable and one odd variable; i.e., $C^\infty(\mathbb{R}^{1|1}) = C^\infty(\mathbb{R}) \otimes \wedge^*(\mathbb{R})$. And let $\tau$ and $\theta$ be the natural even and odd variables, respectively. We consider a quantum theory of maps

$$X : \mathbb{R}^{1|1} \to Z \hspace{1cm} (B.6)$$

\[5\] In [38] E. Getzler calculates index densities for generalized Dirac operators. In his approach he introduces further scalings of the fluxes by the regulating parameter $t = 1/M^2$. The first non-vanishing term in his alternative expansion to (B.5) is $t$-independent. However such scalings of the field variables are not appropriate for our application.
where the action we take is

\[ S_{SQM} = -\frac{1}{2} \int_{\mathbb{R}^{3+1}} d\tau d\theta \left\{ g_{MN}(X) \frac{dX^M}{d\tau} DX^N + DX^M DX^M_1 DX^M_2 DX^M_3 \mathbb{H}_{M_1 M_2 M_3}(X) \right\}, \]  

(B.7)

with \( g_{MN} \) the metric tensor on \( Z \) and \( D \) is the superdifferential

\[ D = \frac{\partial}{\partial \theta} - \theta \frac{\partial}{\partial \tau}. \]  

(B.8)

The superfield that appears in (B.7), can be written in a local coordinate chart as

\[ X^M = x^M + \theta \psi^M \]  

(B.9)

where \( x \) is a local chart for \( Z \). The supersymmetry transformations are generated by the supercharge operator \( Q \)

\[ Q = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial \tau}, \]  

(B.10)

with \( \delta X^M = Q X^M \). In quantizing (B.7), we construct the Hilbert space of the theory as the space \( L^2(S(Z)) \), i.e. the space of \( L^2 \)-sections of the spin bundle \( S \rightarrow Z \) tensored by the half-densities on \( Z \). This space has a natural \( \mathbb{Z}/2\mathbb{Z} \)-grading induced by the chiral decomposition \( S = S_+ \oplus S_- \rightarrow Z \). Also, the quantum supercharge operator \( Q \) is (see [14] for a derivation):

\[ Q = \hat{P} = \mathcal{P} + \gamma(\mathcal{H}), \]  

(B.11)

which acts naturally on the quantum Hilbert space \( L^2(S(Z)) \).

Thus, the super heat kernel expansion for \( Q_+ \) can be expressed as the quantum mechanical partition function

\[ Z = \text{Tr}_s \left[ \exp \left\{ -t(Q_- Q_+ + Q_+ Q_-)/2 \right\} \right] = \int [dX] \exp(-S_{SQM}), \]  

(B.12)

where we take the action \( S_{SQM} \) defined in (B.7). More concretely, writing (B.7) in the field variables and recalling that the path integral matches with the left-hand-side of (B.12) iff the supercircle \( X : S^{11} \rightarrow Z \) is chosen to be a supercircle of length \( t \), we find

\[ S_{SQM} = \frac{1}{t} \int_0^1 d\tau \left[ \frac{1}{2} g_{MN} \frac{dx^M}{d\tau} \frac{dx^N}{d\tau} + \frac{1}{2} g_{MN} \psi^M \frac{D\psi^N}{D\tau} - \frac{1}{2} (d\mathcal{H})_{MNP\psi^N \psi^P \psi^O \psi^P} \right], \]  

(B.13)

See [14] for more details. There are terms of order \( O(t) \) which have to be included in the action, in order to make the path integral well defined for a finite time interval due to Weyl ordering ambiguities. Here, we just write out the classical expression derived by expanding (B.7) in the field variables.

6 See [14] for more details. There are terms of order \( O(t) \) which have to be included in the action, in order to make the path integral well defined for a finite time interval due to Weyl ordering ambiguities. Here, we just write out the classical expression derived by expanding (B.7) in the field variables.
where
\[
\frac{D\psi^N}{D\tau} = \frac{d\psi^N}{d\tau} + \Gamma^N_{MQ} x^M \psi^Q - 3\hbar^N_{MQ} \dot{x}^M \psi^Q.
\] (B.14)

One can use the background field approximation and expand the fields as classical fields plus quantum fluctuations
\[
x^M = x^M_0 + \delta x^M
\]
\[
\psi^Q = \psi^Q_0 + \delta \psi^Q.
\] (B.15)

We are now ready to compute the path integral (B.13) via a loop expansion in the parameter \( t \), with \( t \) playing the role of \( \hbar \). We only need compute graphs of 12\textsuperscript{th} order in the background fermions \( \psi_0 \) in order to saturate the Grassmann integration. Due to the four-fermi interaction in (B.13), the tree level contribution after integrating \([d^{12}\psi_0]\) yields terms of order \( O(t^{-3}) \). For instance,
\[
-\frac{1}{2^3 \cdot 3! \hbar^3} (d\hbar)^3.
\] (B.16)

Thus, in order to extract the full \( O(t^0) \) contribution we should take into account up to four-loop diagrams which are of order \( O(t^{-3}) \times O(t^3) \), since each loop order \( L \) contributes \( O(t^{L-1}) \). In this formalism, it becomes clear how inverse powers of \( t \) appear in the expansion due to the presence of a non-vanishing \( \hbar \)-flux.

In other words, we have shown how the super heat kernel expansion will be of the type
\[
\frac{1}{t^3} \text{Tr}_s(a_3) + \frac{1}{t^2} \text{Tr}_s(a_4) + \frac{1}{t} \text{Tr}_s(a_5) + \text{Tr}_s(a_6) + \ldots,
\] (B.17)

and we will have to evaluate up to four-loop Feynman diagrams, in order to determine the heat kernel coefficient \( \text{Tr}_s a_6 \). Note that five-loop Feynman diagrams are at least of order \( O(t) \) and hence do not contribute to \( \text{Tr}_s a_6 \).

Finally, we remark that if we had put in an appropriate extra scaling in \( \hbar \), as is done in [38] we would have had no divergent terms for \( t \to 0 \) and would have obtained the index density
\[
\int A e^{d\hbar}.
\] (B.18)
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