Strong connectivity and directed triangles in oriented graphs. Partial results on a particular case of the Caccetta-Häggkvist conjecture

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Abstract

A particular case of Caccetta-Häggkvist conjecture, says that a digraph of order $n$ with minimum out-degree at least $\frac{1}{3}n$ contains a directed cycle of length at most 3. In a recent paper, Kral, Hladky and Norine (see [7]) proved that a digraph of order $n$ with minimum out-degree at least $0.3465n$ contains a directed cycle of length at most 3 (which currently is the best result). A weaker particular case says that a digraph of order $n$ with minimum semi-degree at least $\frac{1}{3}n$ contains a directed triangle. In a recent paper (see [8]), by using the result of [7], the author proved that for $\beta \geq 0.343545$, any digraph $D$ of order $n$ with minimum semi-degree at least $\beta n$ contains a directed cycle of length at most 3 (which currently is the best result). This means that for a given integer $d \geq 1$, every digraph with minimum semi-degree $d$ and of order $md$ with $m \leq 2.91082$, contains a directed cycle of length at most 3. In particular, every oriented graph with minimum semi-degree $d$ and of order $md$ with $m \leq 2.91082$, contains a directed triangle. In this paper, by using again the result of [7], we prove that every oriented
graph with minimum semi-degree \( d \), of order \( md \) with \( 2.91082 < m \leq 3 \) and of strong connectivity at most 0.679\( d \). contains a directed triangle. This will be implied by a more general and more precise result, valid not only for \( 2.91082 < m \leq 3 \) but also for larger values of \( m \). As application, we improve two existing results. The first result (Authors Broersma and Li in \([2]\)) concerns the number of the directed cycles of length 4 of a triangle free oriented graph of order \( n \) and of minimum semi-degree at least \( n \). The second result (Authors Kelly, Kühn and Osthus in \([10]\)) concerns the diameter of a triangle free oriented graph of order \( n \) and of minimum semi-degree at least \( \frac{5}{4} \).

**Keywords**: Oriented graph, strong connectivity, girth, triangle

1 **Introduction and definitions**

The definitions which follow are those of \([1]\).

We consider digraphs without loops and without parallel arcs. \( V(D) \) is the *vertex set* of \( D \) and the *order* of \( D \) is the cardinality of \( V(D) \). \( A(D) \) is the set of the arcs of \( D \).

We denote by \( a(D) \) the number of the arcs of \( D \) (size of \( D \)). Two arcs \((x,y)\) and \((x',y')\) are *independent* if the pairs \( \{x,y\} \) and \( \{x',y'\} \) are disjoint.

We say that a vertex \( y \) is an *out-neighbor* of a vertex \( x \) (*in-neighbour* of \( x \)) if \((x,y)\) (resp. \((y,x)\)) is an arc of \( D \). \( N^+_D(x) \) is the set of the out-neighbors of \( x \) and \( N^-_D(x) \) is the set of the in-neighbors of \( x \). The cardinality of \( N^+_D(x) \) is the *out-degree* \( d^+_D(x) \) of \( x \) and the cardinality of \( N^-_D(x) \) is the *in-degree* \( d^-_D(x) \) of \( x \). We also put \( N_D(x) = N^+_D(x) \cup N^-_D(x) \) and \( N'_D(x) = N^+_D(x) \cup N^-_D(x) \cup \{x\} \). When no confusion is possible, we omit the subscript \( D \).

We denote by \( \delta^+(D) \) the minimum out-degree of \( D \) and by \( \delta^-(D) \) the minimum in-degree of \( D \). The *minimum semi-degree* of \( D \) is \( \delta^0(D) = \min\{\delta^+(D), \delta^-(D)\} \).

For a vertex \( x \) of \( D \) and for a subset \( S \) of \( V(D) \), \( N^+_S(x) \) is the set of the out-neighbors of \( x \) which are in \( S \), and \( d^+_S(x) \) is the cardinality of \( N^+_S(x) \). Similarly, \( N^-_S(x) \) is the set of the
in-neighbors of \( x \) which are in \( S \), and \( d^-_S(x) \) is the cardinality of \( N^-_S(x) \).

A directed path of length \( p \) of \( D \) is a list \( x_0, \ldots, x_p \) of distinct vertices such that \( (x_{i-1}, x_i) \in \mathcal{A}(D) \) for \( 1 \leq i \leq p \). A directed cycle of length \( p \geq 2 \) is a list \( (x_0, \ldots, x_p, x_0) \) of vertices with \( x_0, \ldots, x_p \) distinct, \( (x_{i-1}, x_i) \in \mathcal{A}(D) \) for \( 1 \leq i \leq p - 1 \) and \( (x_{p-1}, x_0) \in \mathcal{A}(D) \). From now on, we omit the adjective ” directed”. A \( p \)-cycle of \( D \) is a directed cycle of length \( p \).

A digon is a 2-cycle, and a triangle is a 3-cycle of \( D \) of length 3. The girth \( g(D) \) of \( D \) is the minimum length of the cycles of \( D \). The digraph \( D \) is said to be strongly connected (for briefly strong) if for every distinct vertices \( x \) and \( y \) of \( D \), there exists a path from \( x \) to \( y \). It is known that in a non-strong digraph \( D \), there exists a partition \( (A, B) \) of \( V(D) \) with \( A \neq \emptyset \) and \( B \neq \emptyset \) such that there are no arcs from a vertex of \( B \) to a vertex of \( A \). (one say that \( A \) dominates \( B \)). We say that a subset \( S \) of \( V(D) \) disconnects \( D \), if the digraph \( D - S \) is non-strong. The strong connectivity \( k(D) \) of \( D \) is the smallest of the positive integers \( m \) such that there exists a subset of \( V(D) \) of cardinality \( m \) disconnecting \( D \). \( D \) is said to be \( p \)-strong connected if \( k(D) \geq p \). It is well known that in a \( p \)-strong connected digraph, if \( S \) is a subset of \( V(D) \) such that \( |S| \geq p \) and \( |V(D) \setminus S| \geq p \), then there exist \( p \) independent arcs with starting vertices in \( S \) and with ending vertices in \( V(D) \setminus S \).

In a strong digraph \( D \), for vertices \( x \) and \( y \) of \( D \), the distance \( d(x, y) \) from \( x \) to \( y \) is the length of a shortest path from \( x \) to \( y \). The diameter \( \text{diam}(D) \) is the maximum of the distances \( d(x, y) \). The eccentricity \( \text{ecc}(x) \) of a vertex \( x \) is the maximum of the distances \( d(x, y), y \in V(D) \). It is clear that \( \text{ecc}(x) \leq \text{diam}(D) \) for every vertex \( x \) of \( D \).

An oriented graph, is a digraph \( D \) such that for any two distinct vertices \( x \) and \( y \) of \( D \), at most one of the ordered pairs \( (x, y) \) and \( (y, x) \) is an arc of \( D \). The author proved in \([9]\) that the strong connectivity \( k \) of an oriented graph \( D \) of order \( n \), satisfy \( k \geq \frac{2(\delta^+(D) + \delta^-(D) + 1) - n}{3} \), and this shows that an oriented graph of order \( n \) and of
minimum semi-degree at least $\frac{n}{3}$, is strongly connected.

Caccetta and Häggkvist (see [3]) conjectured in 1978 that the girth of any digraph of order $n$ and of minimum out-degree at least $d$ is at most $\lceil n/d \rceil$.

The conjecture is still open when $d \geq n/3$, in other words it is not known if any digraph of order $n$ and minimum out-degree at least $n/3$ contains a cycle of length at most 3.

In fact it is also unknown if any digraph of order $n$ with both minimum out-degree and minimum in-degree at least $n/3$ contains a cycle of length at most 3 and then a special case of the Caccetta-Häggkvist conjecture is :

**Conjecture 1.1** Every digraph of order $n$ and of minimum semi-degree at least $\frac{n}{3}$, contains a cycle of length at most 3.

Two questions were naturally raised :

**Question Q$_1$** What is the minimum constant $c$ such that any digraph of order $n$ with minimum out-degree at least $cn$ contains a cycle of length at most 3.

**Question Q$_2$** What is the minimum constant $c'$ such that any digraph of order $n$ with both minimum out-degree and minimum in-degree at least $c'n$ contains a cycle of length at most 3.

It is known that $c \geq c' \geq 1/3$ and the conjecture is that $c = c' = 1/3$. In a very recent paper (See [7]), Hladký, Král' and Norine proved that $c \leq 0.3465$, which currently is the best result.

By using this result, the author proved in [8] that $c' \leq 0.343545$, which currently is the best result. In other terms, this means :

**Theorem 1.2** For $d \geq 1$, any digraph with minimum semi-degree $d$ and of order at most $2.91082d$ contains a cycle of length at most 3.
In our paper, we will see that in an oriented graph $D$ of minimum semi-degree $d$ and of order $md$ with $2.91082 < m < \frac{2}{c}$, an adequate upper bound on the connectivity of $D$ forces the existence of a triangle. More precisely, we prove:

**Theorem 1.3** Let $D$ be an oriented graph of minimum semi-degree $d$, of order $n = md$ with $2.91082 < m < \frac{2}{c}$. If the connectivity $k$ of $D$ verifies $k \leq \max \left\{ \frac{5 - m - 4c + c^2}{(1 - c)(2 - c)} d, \frac{2 - cm}{2 - c} d \right\}$, then $D$ contains at least a triangle.

Since $c \leq 0.3465$, an easy consequence will be:

**Theorem 1.4** Let $D$ be an oriented graph of minimum semi-degree $d$, of order $n = md$ with $2.91082 < m \leq 3$. If the connectivity $k$ of $D$ verifies $k \leq 0.679d$, then $D$ contains at least a triangle.

Broersma and Li proved in [2] that in a triangle-free oriented graph of order $n$ and of minimum semi-degree at least $\frac{n}{3}$, every vertex is in more than $1 + \frac{28}{15}(11 - 4\sqrt{6})$ 4-cycles. We improve this result by proving:

**Theorem 1.5** Let $D$ be a triangle-free oriented graph of minimum semi-degree $d$, of order $n = md$ with $m \leq 3$. Then every vertex $x$ of $D$ is contained in more than $\frac{2(5 - m - 4c + c^2)d}{(1 - c)(2 - c)} + (2 - m)d + 1$ cycles such that two of these cycles have only the vertex $x$ in common.

If we allow distinct 4-cycles with others vertices than $x$ in common, we give an even more spectacular improvement, by proving:

**Theorem 1.6** Let $D$ be a triangle-free oriented graph of minimum semi-degree $d$, of order $n = md$ with $m \leq 3$. Then every vertex $x$ of $D$ is contained in more than $\frac{11 - 15c + 7c^2 - c^3 - (c^2 - 3c + 3)m}{(1 - c)^2(2 - c)} d$ 4-cycles.
Kelly, Kühn and Osthus proved in [10] that if $D$ is an oriented graph of order $n$ and of minimum semi-degree greater than $\frac{n}{3}$, then either the diameter of $D$ is at most 50 or $D$ contains a triangle. We will considerably improve this result by proving:

**Theorem 1.7** If $D$ is a triangle-free oriented graph of minimum semi-degree $d$ and of order $n = md$ with $m \leq 5$, then the diameter of $D$ is at most 9.

A result of Chudnovsky, Seymour and Sullivan (see [5]) asserts that one can delete $k$ edges from a triangle-free digraph $D$ with at most $k$ non-edges to make it acyclic. Hamburger, Haxell, and Kostochka used this to prove in [6] that in a triangle-free digraph $D$ with at most $k$ non-edges, $\delta^+(D) < \sqrt{2k}$ (and $\delta^-(D) < \sqrt{2k}$ also).

Chen, Karson, and Shen improved in [4] the initial result of [5] by asserting that one can delete $0.8616k$ edges from a triangle-free digraph $D$ with at most $k$ non-edges to make it acyclic. From this result, by using the reasoning of Hamburger, Haxell and Kostochka in [6], it is easy to prove that in a triangle-free digraph $D$ with at most $k$ non-edges, $\delta^+(D) < \sqrt{1.7232k}$ and $\delta^-(D) < \sqrt{1.7232k}$. As the maximum size of an oriented graph of order $n$ is $\frac{n(n-1)}{2}$, an immediate consequence is:

**Lemma 1.8** If $D$ is a triangle-free oriented graph of order $n$, then $a(D) < \frac{n^2}{2} - \frac{(\delta^+(D))^2}{1.7232}$ and $a(D) < \frac{n^2}{2} - \frac{(\delta^-(D))^2}{1.7232}$.

## 2 Proofs of Theorems 1.3 and 1.4

By hypothesis, $D$ is an oriented graph of minimum semi-degree $d$, of order $n = md$ with $2.91082 < m < \frac{2}{c}$ and of strong connectivity $k$. We put $k' = \frac{k}{d}$. Let $K$ be a set of $k$ vertices disconnecting $D$. Then there exists a partition of $V(D) \setminus K$ into two subsets $A$ and $B$, such that there are no arcs from a vertex of $B$ to a vertex of $A$. Without loss of generality, we
may suppose that \(|B| \leq |A|\). We put \(a = \frac{|A|}{d}\) and \(b = \frac{|B|}{d}\). Since \(b \leq a\), it holds \(b \leq \frac{m - k'}{2}\).

First we claim that:

**Lemma 2.1** If \(D\) is triangle-free, then for every arc \((y, x)\) of \(D\) with \(y \in A\) and \(x \in B\), it holds \(d_B^+(x) + d_A^-(y) \geq 2d - k'd\).

**Proof.** Since \(x\) has no out-neighbors in \(A\), \(x\) has \(d^+(x) - d_B^+(x)\) out-neighbors in \(K\), which means \(|N_K^+(x)| = d^+(x) - d_B^+(x)\). Since \(y\) has no in-neighbors in \(B\), \(y\) has \(d^-(y) - d_A^-(y)\) in-neighbors in \(K\), which means \(|N_K^-(y)| = d^-(y) - d_A^-(y)\). Since \(N_K^+(x)\) and \(N_K^-(y)\) are vertex-disjoint (for otherwise, we would have a triangle), we have \(d^+(x) - d_B^+(x) + d^-(y) - d_A^-(y) \leq k'd\), hence \(d_B^+(x) + d_A^-(y) \geq d^+(x) + d^-(y) - k'd\) and since \(d^+(x) \geq d\) and \(d^-(y) \geq d\), the result follows \(\blacksquare\)

Now, we claim:

**Lemma 2.2** Suppose that \(2.91082 < m < 5 - 4c + c^2\). If the connectivity \(k\) of \(D\) verifies \(k \leq \frac{5 - m - 4c + c^2}{(1-c)(2-c)}d\), then \(D\) contains at least a triangle.

**Proof.** We put \(k' = \frac{k}{d}\). Suppose, for the sake of a contradiction, that \(D\) does not contain triangles. Let \(sd\) be the minimum out-degree of \(D|B|\), and let \(x\) be a vertex of \(B\) with \(d_B^+(x) = sd\). It is easy to verify that \(\frac{5 - m - 4c + c^2}{(1-c)(2-c)} < 1\) and since all the out-neighbors of \(x\) are in \(B \cup K\), it follows that \(N_B^+(x) \neq \emptyset\), and so \(s > 0\). There exists a vertex \(x'\) of \(N_B^+(x)\), such that \(d_{N_B^+(x)}^+(x') < csd\). It follows that \(x'\) has more than \((s - cs)d = (1 - c)sd\) out-neighbors in \(B\) but not in \(N_B^+(x)\), and these out-neighbors cannot be in-neighbors of \(x\) (for otherwise, we would have a triangle). We get then \(d_{B\cup K}^+(x) < [b + k' - 1 - (1-c)s]d\).

Suppose that \(b + k' - 1 \geq 1\). Then \(k' \geq 2 - b\), and since \(b \leq \frac{m - k'}{2}\), we get \(k' \geq 2 - \frac{m - k'}{2}\), hence \(k' \geq 4 - m\). Then, since \(k' \leq \frac{5 - m - 4c + c^2}{(1-c)(2-c)}\), we get \(4 - m \leq \frac{5 - m - 4c + c^2}{(1-c)(2-c)}\), hence \((4 - m)(c^2 - 3c + 2) \leq 5 - m - 4c + c^2\). This yields \(m(c^2 - 3c + 1) \geq 3c^2 - 8c + 3\), hence \(m(c^2 - 3c + 1) \geq 3(c^2 - 3c + 1) + c\). Since \(c^2 - 3c + 1 > 0\), we get \(m \geq 3 + \frac{c}{c^2 - 3c + 1}\). It
is easy to verify that for \( \frac{1}{3} \leq c \leq 0.3465 \), it holds \( \frac{c}{c^2 - 3c + 1} > 1 \). We get then \( m > 4 \), and it is easy to verify that this is contradictory with \( m < 5 - 4c + c^2 \). Consequently, we have \( b + k' - 1 < 1 \). We deduce \( d_{B \cup K}^c(x) < d \), which means that \( N^-_A(x) \neq \emptyset \) (in fact, by the above reasoning, this is true for every vertex of \( B \)). More precisely, we have

\[
d^-_A(x) > [2 - k' - b + (1 - c)s]d
\]

There exists a vertex \( y \) of \( N^-_A(x) \) with fewer than \( cd^-_A(x) \) in-neighbors in \( N^-_A(x) \) (for otherwise \( D[N^-_A(x)] \) would contain a triangle). It follows \( d^-_A(y) < cd^-_A(x) + ad - d^-_A(x) \), hence \( d^-_A(y) < ad - (1 - c)d^-_A(x) \). From Lemma 2.1, we get \( d^-_A(y) \geq (2 - k')d - d^-_B(x) \), that is \( d^-_A(y) \geq (2 - k' - s)d \). We deduce \((2 - k' - s)d < ad - (1 - c)d^-_A(x)\), hence

\[
sd > (2 - k' - a)d + (1 - c)d^-_A(x)
\]

From (1) and (2), we deduce \( sd > (2 - k' - a)d + (1 - c)[2 - k' - b + (1 - c)s]d \), hence \( s > 2 - k' - a + 2 - 2c - k' + ck' - b + bc + (1 - c)^2 s \). It follows \( (2c - c^2)s > 4 - 2k' - a - b - 2c + ck' + bc \), and since \( a + b = m - k' \), we get \( (2c - c^2)s > 4 - m - k' - 2c + ck' + bc \). Since \( s < bc \) (for otherwise \( D[B] \) would contain a triangle), we get \( (2c - c^2)bc > 4 - m - k' - 2c + ck' + bc \), hence \( (1 - c)^2 bc < m + 2c - 4 + (1 - c)k' \). Since all the out-neighbors of \( x \) are in \( B \cup K \), we have \( 1 - s \leq k' \), hence \( s \geq 1 - k' \), and since \( s < bc \), we get \( bc > 1 - k' \). It follows \( (1 - k')(1 - c)^2 < m + 2c - 4 + (1 - c)k' \), hence \( k'(1 - c)(2 - c) > 1 - 2c + c^2 - m - 2c + 4 \). This implies \( k' > \frac{5 - m - 4c + c^2}{(1 - c)(2 - c)} \), which is contradictory with the hypothesis on \( k \). Consequently \( D \) contains at least a triangle, and so, the result is proved. \[\blacksquare\]

We claim also:

**Lemma 2.3** Suppose that \( 2.91082 < m < \frac{2}{c} \). If the connectivity \( k \) of \( D \) verifies \( k \leq \frac{2 - cm}{2 - c}d \), then \( D \) contains at least a triangle.

**Proof.** Suppose, for the sake of a contradiction, that \( D \) does not contain triangles. Let \( sd \) be the minimum out-degree of \( D[B] \), and let \( x \) be a vertex of \( B \) with \( d_B^c(x) = sd \). We have
then \( k' \geq 1 - s \), hence \( s \geq 1 - k' \). Since \( s < bc \) (for otherwise we would have a triangle), we get \( bc > 1 - k' \). Since \( b \leq \frac{m - k'}{2} \), it follows \( \frac{(m - k')c}{2} > 1 - k' \), hence \( mc - k'c > 2 - 2k' \). It follows \( k' > \frac{2 - cm}{2 - c} \), which is contradictory with the hypothesis on \( k = k'd \). So, the result is proved.

It is easy to prove that \( 5 - 4c + c^2 < \frac{2}{c} \). By using these two lemmas, we get Theorem 1.3.

It is easy to see that we have \( \frac{5 - m - 4c + c^2}{(1 - c)(2 - c)} \geq \frac{2 - cm}{2 - c} \) if and only if \( m \leq \frac{3 - 2c + c^2}{1 - c + c^2} \). Then Theorem 1.3 means that when \( 2.91082 < m \leq \frac{3 - 2c + c^2}{1 - c + c^2} \), a strong connectivity not greater than \( \frac{5 - m - 4c + c^2}{(1 - c)(2 - c)}d \) forces a triangle in \( D \), and when \( \frac{3 - 2c + c^2}{1 - c + c^2} < m < \frac{2}{c} \), a strong connectivity not greater than \( \frac{2 - cm}{2 - c}d \) forces a triangle in \( D \).

It is easy to see that for \( 2.91082 < m \leq 3 \), we have \( m < \frac{3 - 2c + c^2}{1 - c + c^2} \). Since \( c \leq 0.3465 \), it is easy to see that we have \( 0.679d < \frac{5 - m - 4c + c^2}{(1 - c)(2 - c)}d \). Then by Lemma 2.2, a strong connectivity no greater than \( 0.679d \) forces a triangle, and so Theorem 1.4 is proved. Since a digraph which is not oriented contains a digon, it is easy to see that proving Conjecture 1.1, amounts to proving that every oriented graph, of minimum semi-degree at least \( d \), of order \( md \) with \( 2.91082 < m \leq 3 \) and of connectivity \( k > 0.679d \), contains at least a triangle.

### 3 Proofs of Theorems 1.5, 1.6 and 1.7

a) Proof of Theorem 1.5

By hypothesis \( D \) is a triangle-free oriented graph of minimum semi-degree \( d \), of order \( n = md \) with \( m \leq 3 \), and \( x \) is a vertex of \( D \). Let \( k \) be the strong connectivity of \( D \) (and \( k' = k/d \)).

We have \( k > 0 \) (for otherwise, by Theorem 1.3 we would have triangles). Clearly, we have \( d^+(x) + d^-(x) < md \), and since \( k \leq d^-(x) \), it follows \( d^+(x) + k < md \), hence \( md - d^+(x) > k \). As we have also \( d^+(x) \geq k \), there exist \( k \) independent arcs \((y_1, z_1), \ldots, (y_k, z_k)\) with
Let \( y_i \in N^+(x) \), \( z_i \notin N^+(x) \) and \( z_i \neq x \) for \( 1 \leq i \leq k \). Since \( D \) is triangle-free, we have also \( z_i \notin N^-(x) \) for \( 1 \leq i \leq k \). It follows that the set \( S_i = \{z_1, \ldots, z_k\} \) is contained in \( V(D) \setminus N'(x) \). Similarly, there exist \( k \) independent arcs \((v_1, u_1), \ldots, (v_k, u_k)\) with \( u_i \in N^-(x) \), \( v_i \notin N^-(x) \) and \( v_i \neq x \) for \( 1 \leq i \leq k \). Since \( D \) is triangle-free, we have also \( v_i \notin N^+(x) \) for \( 1 \leq i \leq k \). It follows that the set \( S_2 = \{v_1, \ldots, v_k\} \) is contained in \( V(D) \setminus N'(x) \).

We have \(|S_1 \cap S_2| = |S_1| + |S_2| - |S_1 \cup S_2|\). Since \(|S_1| = |S_2| = k'd\) and \(|S_1 \cup S_2|\) is contained in \( V(D) \setminus N'(x) \), it follows \(|S_1 \cap S_2| \geq 2k'd - (md - d^+(x) - d^-(x) - 1)\), hence \(|S_1 \cap S_2| \geq 2k'd - md + d^+(x) + d^-(x) + 1\). Since \( d^+(x) \geq d \) and \( d^-(x) \geq d \), it follows \(|S_1 \cap S_2| \geq (2k' + 2 - m)d + 1\). This implies the existence of at least \((2k' + 2 - m)d + 1\) 4-cycles containing \( x \) and such that any two of these cycles have only \( x \) in common. Now since \( D \) is triangle-free, we deduce from Theorem 1.3 that \( k' > \frac{5 - m - 4c + c^2}{(1 - c)(2 - c)}\), and then Theorem 1.5 is proved.

Since \( c \leq 0.3465 \) and \( m \leq 3 \), it is easy to see that the number \( n_D(x, 4) \) of 4-cycles of \( D \) containing \( x \), and such that any two of these cycles have only \( x \) in common, is at least \[ \frac{2 \times (5 - 3 - 4 \times 0.3465 + 0.3465^2)d}{0.6535 \times 1.6535} - d + 1, \] hence \( n_D(x, 4) > 0.358d + 1 \), and since \( d \geq \frac{n}{3} \) (\( n \) being the order of \( D \)), we get \( n_D(x, 4) > 0.119n + 1 \). Since \( 1 + \frac{n}{10}(11 - 4\sqrt{6}) \approx 1 + 0.08014n \) (exceeding value), it is clear that our result improve that of Broersma and Li.

b) Proof of Theorem 1.6

Let \( k = k'd \) be the strong connectivity of \( D \). By Theorem 1.4, we have \( k > 0.679d \).

Clearly the eccentricity \( \text{ecc}(x) \) of \( x \) is at least 3 (for otherwise, we would have a triangle). The author proved in [9] that the diameter of an oriented graph of order \( n \) and of minimum semi-degree at least \( \frac{n}{2} \) is at most 4. By this result, we have \( \text{ecc}(x) \leq 4 \), and consequently \( 3 \leq \text{ecc}(x) \leq 4 \). For \( 1 \leq i \leq \text{ecc}(x) \) let \( R_i \) be the set of the vertices \( z \) of \( D \) such that \( d(x, z) = i \). Since \( D \) is triangle-free, all the in-neighbors of \( x \) are in \( R_3 \cup \cdots \cup R_{\text{ecc}(x)} \).
We claim that $d^-_{R_3}(x) > d - \frac{m - 2 - k'}{1 - c}d$ (Assertion (Ass)).

We observe first that $m - 2 - k' > 0$. Indeed, for an arbitrary vertex $u$ of $D$, there exists $k'd$ independent arcs with starting vertices in $N^+(u)$ and ending vertices in $V(D) \setminus N^+(u)$. Since $D$ is triangle-free these ending vertices are not in $N^-(u)$. It follows $2d + k'd < md$, hence $m - 2 - k' > 0$.

Suppose first that $\text{ecc}(x) = 3$. Then all the in-neighbors of $x$ are in $R_3$. This implies $d^-_{R_3}(x) \geq d$, and since $d > d - \frac{m - 2 - k'}{1 - c}d$, the assertion (Ass) is proved.

Suppose now that $\text{ecc}(x) = 4$. Since $R_2$ disconnects $D$, we have $r_2 \geq k'd$. Suppose first that $r_3 \geq d$. We have $r_4 = md - r_1 - r_2 - r_3 - 1$, hence $r_4 < md - d - k'd - d$, that is $r_4 < (m - 2 - k')d$. It follows $d^-_{R_3}(x) > d - (m - 2 - k')d$, and since $d - (m - 2 - k')d > d - \frac{m - 2 - k'}{1 - c}d$, the Assertion (Ass) is proved. Suppose now that $r_3 < d$. Clearly, all the in-neighbors of a vertex of $R_4$ are in $R_3 \cup R_4$. It follows that every vertex of $R_4$ has at least $d - r_3$ in-neighbors in $R_4$. Since $D[R_3]$ is triangle-free, it holds $d - r_3 < cr_4$, hence $r_4 > \frac{d - r_3}{c}$, hence $r_4 > \frac{(1 - m)d + r_1 + r_2 + r_4}{c}$. Since $r_1 \geq d$ and $r_2 \geq k'd$, we get $r_4 > \frac{(2 - m + k')d + r_4}{c}$, hence $(1 - c)r_4 < (m - 2 - k')d$, and then $r_4 < \frac{m - 2 - k'}{1 - c}d$. It follows $d^-_{R_3}(x) > d - \frac{m - 2 - k'}{1 - c}d$, which is the assertion (Ass). It is easy to see that an in-neighbor $z$ of $x$ which is in $R_3$ has an in-neighbor $z_2$ in $R_2$ and that $z_2$ has an in-neighbor $z_1$ in $R_1$. Then $C_z = (x, z_1, z_2, z, x)$ is a 4-cycle of $D$, containing $x$. It is clear that the cycles $C_z$, $z \in N^-_{R_3}(x)$ are distinct. Consequently the vertex $x$ is contained in more than $d - \frac{m - 2 - k'}{1 - c}d$ 4-cycles. Since $k > \frac{5 - m - 4c + c^2}{(1 - c)(2 - c)}d$ (By Theorem 1.3), the result follows.

Since $c \leq 0.3465$, $m \leq 3$ and $k' > 0.679$, it holds $d^-_{R_3}(x) > d - \frac{3 - 2 - 0.679}{1 - 0.3465}d$, hence $d^-_{R_3}(x) > 0.5087d$, hence $d^-_{R_3}(x) > 0.169n$. So $D$ possess more than $0.169n$ 4-cycles containing $x$, which is much better that the result of Broersma and Li.
c) Proof of Theorem 1.7

By hypothesis $D$ is a triangle-free oriented graph of minimum semi-degree $d$, of order $n = md$ with $m \leq 5$. Suppose, for the sake of a contradiction, that the diameter of $D$ is at least 10. Then let $x$ and $y$ be two vertices of $D$ such that $d(x, y) \geq 10$. For $1 \leq i \leq 6$, let $R_i$ be the set of the vertices $z$ of $D$ such that $d(x, z) = i$, and for $1 \leq i \leq 3$, let $R_{-i}$ be the set of the vertices $z$ of $D$ such that $d(y, z) = i$. For $1 \leq i \leq 6$, $r_i$ is the cardinality of $R_i$ and for $1 \leq i \leq 3$, $r_{-i}$ is the cardinality of $R_{-i}$. The sets $R_i$, $1 \leq i \leq 6$ are mutually vertex-disjoint, the sets $R_{-i}$, $1 \leq i \leq 3$ are also mutually vertex-disjoint, and a set $R_j$, $1 \leq i \leq 6$ is a vertex-disjoint with a set $R_{-j}$, $1 \leq j \leq 3$ (for otherwise the diameter of $D$ would be at most 9). For $2 \leq i \leq 6$ we put $R'_i = R_1 \cup \cdots \cup R_i$, for $2 \leq i \leq 3$ we put $R'_{-i} = R_{-1} \cup \cdots \cup R_{-i}$, and $r'_i$, $r'_{-i}$ are the respective cardinalities.

We claim that $r'_3 \geq 2.239d$. Indeed, since $D[R_1]$ is triangle-free, there exists a vertex $u$ of $R_1$ with fewer than $0.346d$ out-neighbors in $R_1$, and then we have $r_2 > 0.6535d$, hence $r_1 + r_2 > 1.6535d$. Now, if $r_3 \geq d$, it follows $r'_3 \geq 2.6535d$, and the assertion is proved.

Suppose now that $r_3 < d$. It is easy to see that a vertex of $R_2$ has all its out-neighbors in $R'_3$. It follows that a vertex of $R_2$ has at least $d - r_3$ out-neighbors in $R'_3$. Since every vertex of $R_1$ has all its out-neighbors in $R'_3$, it follows $a(D[R'_2]) \geq r_1d + r_2(d - r_3)$, hence:

$$a(D[R'_2]) \geq r_1d + r_2d - r_2r_3 \quad (3)$$

On the other hand by Theorem 1.7, we have

$$a(D[R'_2]) \leq \frac{(r'_3)^2}{2} - \frac{(d - r_3)^2}{1.7232} \quad (4)$$

From (3) and (4), we deduce $r_1d + r_2d - r_2r_3 \leq \frac{r'_1r'_2 + 2r_1r_2}{2} - \frac{d^2 - 2dr_3 + r_3^2}{1.7232}$, hence $3.4464r_1d + 3.4464r_2d - 3.4464r_3r_3 \leq 1.7232r_1^2 + 3.4464r_1r_2 + 1.7232r_2^2 - 2d^2 + 4r_3d - 2r_3^2$. An easy calculation yields: $1.7232(r_2 + r_3 + r_1 - d)^2 \geq 3.7232r_3^2 - (7.4464d - 3.4464r_1)r_3 + 3.7232d^2$.

Since $r_1 \geq d$, we get $1.7232(r_2 + r_3 + r_1 - d)^2 \geq 3.7232r_3^2 - 4r_3d + 3.7232d^2$, that is $1.7232(r_2 +
$r_3 + r_1 - d)^2 \geq f(r_3)$, $f$ being the function defined by $f(t) = 3.7232t^2 - 4dt + 3.7232d^2$.

By a classical result on the functions of second degree, we have $f(r_3) \geq f\left(\frac{2d}{3.7232}\right)$, hence $f(r_3) > 2.648d^2$. We deduce then $1.7232(r_2 + r_3 + r_1 - d)^2 > 2.648d^2$, hence $r_2 + r_3 + r_1 - d > 1.239d$ which yields $r'_3 > 2.239d$, and the assertion is still proved. Similarly, we have $r'_1 > 2.239$. Since $D$ is triangle-free, by Theorem 1.3, the strong connectivity $k$ of $D$ verifies $k > \frac{2 - 5c}{2 - c}d$, and since $c \leq 0.3465$, we get $k > 0.161d$. It is clear that each of the sets $R_4$, $R_5$ and $R_6$ disconnects $D$, and then $r_i > 0.161d$ for $4 \leq i \leq 6$. Suppose that $r_4 < 0.205d$.

Then $D[R'_3]$, which is triangle-free, is of minimum out degree at least $0.795d$. It follows $0.795 < 0.3465r'_3$, hence $r'_3 > 2.2943d$. We have then $v(D) > 2.2943d + 2.239d + 3 \times 0.161d$,

that is $v(D) > 5.0163d$, which is not possible. It follows $r_4 \geq 0.205d$. We deduce then $v(D) > 2.239d + 2.239d + 0.205d + 2 \times 0.161d$, that is $v(D) > 5.005d$, which is still impossible. Consequently, the diameter of $D$ is at most 9, and the result is proved. □

4 An open problem

Theorem 1.3 gives rise to the following question:

Open Problem . For $r$ with $2 < r < \frac{2}{c}$, what is the maximum number $\psi(r) \in [0, 1]$ such that every oriented graph $D$ of minimum semi-degree $d$ of order $n \leq rd$ and of connectivity $k(D) \leq \psi(r)d$, contains a triangle?

By the result of [8], we have $\psi(r) = 1$ for $2 < r \leq 2.91082$. By Theorem 1.3, for $2.91082 < r < \frac{2}{c}$ we have $\psi(r) \geq \max\left\{\frac{5 - r - 4c + c^2}{(1 - c)(2 - c)}d, \frac{2 - cr}{2 - c}d\right\}$. Thus, since $c \leq 0.3465$, we get $\psi(3) > 0.679$, $\psi(3.5) > 0.476$, $\psi(4) > 0.371$ $\psi(4.5) > 0.266$, $\psi(5) > 0.161$ and $\psi(5.5) > 0.057$. Observe that Conjecture 1.1 is true, if and only if $\psi(3) = 1$. 
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