$D$-dimensional spin projection operators for arbitrary type of symmetry via Brauer algebra idempotents

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Abstract

A new class of representations of Brauer algebra which centralizes the action of orthogonal and symplectic groups in tensor spaces was found. These representations makes it possible to apply the technique of constructing the primitive orthogonal idempotents of the Brauer algebra to the description of Behrends-Fronsdal type projectors of arbitrary type of symmetries.
1 Introduction

In this paper we construct projectors onto the spaces of all finite-dimensional irreducible tensor representations of the Poincaré group $ISO(1, D - 1)$ for the case $D > 4$. We stress that tensor representations are analogs of integer spin representations for $D = 4$ case. The projectors onto the spaces of these irreducible representations are realized as operators acting in the spaces $\mathcal{F}(j)$ of rank $j$ tensors $\phi(k) \in (\mathbb{R}^{1,D-1})^{\otimes j}$ with components $\phi^{\ell_1, ..., \ell_j}(k)$ which are functions of $D$-dimensional momentum vector $k \in \mathbb{R}^{1,D-1}$; Minkowski space $\mathbb{R}^{1,D-1}$ has the metric $\eta = \text{diag}(1, -1, ..., -1)$. The components $\phi^{\ell_1, ..., \ell_j}(k)$ are symmetric under special permutations of indices $\ell_1, ..., \ell_j$ and satisfy conditions

$$k_{n_1}^{\ell_1} \phi^{n_2, ..., n_j}_{n_1, ..., n_j} = 0, \quad \eta_{n_i, n_j} \phi^{n_i, ..., n_j} = 0.$$  

(1.1)

These conditions are called respectively as transversality and tracelessness conditions. The well-known first examples of this type projectors are the Behrends-Fronsdal (BF) spin operators $\Theta^{(j)}$ constructed in papers [9], [10] for the case $D = 4$ and devoted to the fundamentals of the theory of higher spin fields (see, e.g., [11], [12], [13] and references therein). The operators $\Theta^{(j)}$ project the space $\mathcal{F}(j)$ onto invariant (under the action of Poincaré group $ISO(1,3)$) subspace of completely symmetric rank $j$ tensors with components (1.1).

A $D$-dimensional ($D > 4$) generalization of the Behrends-Fronsdal projectors $^3$ for the case of the integer spins was found for the first time in [3] (see also [2], [4]). The $D$-dimensional BF spin projection operator for the half-integer spins was constructed in [4] and [5]. Operators of these type are used in different fields of modern theoretical physics $^4$ therefore new interesting forms of representation of these projectors appear in literature (see, e.g., [2], [6], [7]).

Our aim is to find the explicit projectors onto the spaces of rank $j$ tensors (1.1) with different types of symmetries which form the spaces of irreducible representations of the group $ISO(1, D - 1)$ for arbitrary $D > 4$. As it is known (see, e.g., [8] and references therein), such symmetries can be associated with Young diagrams. In particular the completely symmetric rank $j$ tensors are associated to the one row Young diagram with $j$ boxes.

In the case of orthogonal groups $SO$, the finite dimensional irreducible tensor representations can be constructed by means of the Schur–Weyl–Brauer duality. The key ingredient here is Brauer algebra and its primitive orthogonal idempotents (see [15] and [8]). The projectors onto the spaces of rank $j$ tensors with special type of symmetries (in which irreducible representations of the group $SO(1, D - 1)$ act) are realized as images of the primitive orthogonal idempotents of the Brauer algebra $Br_j(D)$ acting in the tensor spaces $(\mathbb{R}^{1,D-1})^{\otimes j}$. These images are analogs of the Young symmetrizers of group algebra of the permutation group that solve a similar problem for the linear groups (of $SL$ and $SU$ types).

As we mentioned above the irreducibility of tensor representations of the Poincaré group $ISO(1, D - 1)$ requires transversality and tracelessness properties (1.1) of tensors of representation spaces. In this paper, we propose a fairly general algebraic method of constructing $D$-dimensional projectors of the Behrends-Fronsdal type with an arbitrary type of symmetry. Our method is based on making use of new type representations of Brauer algebra. The

$^3$Projector onto the spaces of completely symmetric rank $j$ tensors (1.1).

$^4$See for example the work [23] where the Behrends-Fronsdal projector is used to construct higher-spin Cotton tensor controlling a conformal geometry in three dimensional spacetime (see also [21], [24] and [25]).
images of idempotents in these representations are obviously project any tensors to transverse and traceless tensors \([1,1]\) in all indices.

The work is organized as follows. In second section we recall the definition of the Behrends-Fronsdal spin projection operators in \(D\)-dimensional case. In section three we present the definition of the Brauer algebra. We introduce a system of its generators and present defining relations for these generators. Then we briefly expose the procedure of constructing primitive orthogonal idempotents of the Brauer algebra. In section four we construct new class of representations of the Brauer algebra \(\mathcal{B}_{rj}(\omega)\) acting in the space of rank \(j\) tensors. Then, by using these representations, we present the images of the primitive idempotents corresponding to the rank \(j\) symmetrizers \(\Theta_{(\{j\}^r)}\) that project on the spaces of complete symmetric \(j\)-rank tensors which are the spaces of irreducible representations (of the group \(ISO(1, D - 1)\)) associated to 1-row Young diagrams with \(j\) boxes. We prove that these symmetrizers are equal to \(D\)-dimensional Behrends-Fronsdal spin \(j\) projectors \(\Theta^{(j)}(k)\) constructed in [2] and [4]. In section five, we describe another approach to construct the completely symmetric projectors \(\Theta_{(\{j\}^r)}\). This approach is based on using of Zamolodchikov solution [19] of Yang-Baxter equations and gives the different construction [16] of complete symmetrizers \((3.34)\) in Brauer algebra \(\mathcal{B}_{rj}\). In this Section we also obtain new recurrence relations for \(D\)-dimensional Behrends-Fronsdal spin \(j\) projectors \(\Theta^{(j)}(k)\). In last Section 6, by using the new Brauer algebra representations constructed in Sect.4, we present few examples of images of the primitive idempotents that project on the tensor spaces (with conditions \([1,1]\)) of irreducible representations of the group \(ISO(1, D - 1)\) associated to Young diagrams with \(m > 1\) rows (note that \(m\) can not exceed \(r\), where \((D - 1) = 2r, 2r + 1\).

## 2 Behrends-Fronsdal spin projection operator in general case.

Here we recall the definition [2, 4] of the generalized \(D\)-dimensional Behrends-Fronsdal projector onto the spaces of irreducible completely symmetric tensor representations of the group \(ISO(p, q)\), where \(p + q = D\).

**Definition 1.** The operator \(\Theta^{(j)}(k)\) in the space \((\mathbb{R}^{p,q})^{\otimes j}\), where \((p + q) = D\) and \(k \in \mathbb{R}^{p,q}\), with matrix \((\Theta^{(j)}_{(\{j\}^r)})_{r_1\ldots r_j}^{n_1\ldots n_j}(k)\) is called \(D\)-dimensional Behrends-Fronsdal projector if \(\Theta^{(j)}\) has the following properties:

1. **Projective property and hermiticity:** \((\Theta^{(j)})^2 = \Theta^{(j)}, \quad (\Theta^{(j)})^\dagger = \Theta^{(j)}\).
2. **Complete symmetry:** \((\Theta^{(j)}_{r_1\ldots r_j})_{n_1\ldots n_j}^{n_1\ldots n_j} = (\Theta^{(j)}_{-r_1\ldots -r_j})_{n_1\ldots n_j}^{n_1\ldots n_j}, \quad (\Theta^{(j)}_{r_1\ldots r_j})_{n_2\ldots n_j}^{n_2\ldots n_j} = (\Theta^{(j)}_{-r_1\ldots -r_j})_{n_2\ldots n_j}^{n_2\ldots n_j}\).
3. **Transversality:** \(k_{r_1}(\Theta^{(j)}_{r_1\ldots r_j})^{n_1\ldots n_j}_{r_1\ldots r_j} = 0, \quad k_{n_1}(\Theta^{(j)}_{r_1\ldots r_j})^{n_1\ldots n_j}_{r_1\ldots r_j} = 0\).
4. **Tracelessness:** \(\eta_{r_1r_2}(\Theta^{(j)}_{r_1r_2\ldots r_j})^{n_1\ldots n_j}_{r_1r_2\ldots r_j} = 0, \quad \eta_{n_1n_2}(\Theta^{(j)}_{r_1\ldots r_j})^{n_1n_2\ldots n_j}_{r_1\ldots r_j} = 0\).

Here
\[||\eta_{kl}|| = ||\eta^{kl}|| = \text{diag}(+1, \ldots, +1, -1, \ldots, -1)\],

is the metric in the space \(\mathbb{R}^{p,q}\). We note that, for real matrices \(\Theta^{(j)}\), the hermiticity condition
Proposition 1. The components of $\Theta^{(j)}_{r_1\ldots r_j}$ defined uniquely by properties 1)-4) in Definition 1 and their generating function (2.2) has the form

$$\Theta^{(j)}(x, u) = u_{n_1} \cdots u_{n_j} (\Theta^{(j)})^{n_1\ldots n_j}_{r_1\ldots r_j} x^{r_1} \cdots x^{r_j} .$$

Remark. Instead of the matrix components $(\Theta^{(j)})^{n_1\ldots n_j}_{r_1\ldots r_j}$ symmetrized in the upper and lower indices it is convenient to consider the generating function

$$\Theta^{(j)}(x, u) = \sum_{A=0}^{[j/2]} a^{(j)}_A (\Theta^{(u)}_{(u)} \Theta^{(x)}_{(x)})^A (\Theta^{(u)}_{(x)})^{j-2A} ,$$

where $x^r$ and $u^n$ are components of vectors $x, u \in \mathbb{R}^{p,q}$ and $u_r = \eta_{rn} u^n$.

Proposition 2. The components of $D$-dimensional spin projection operator $\Theta^{(j)}$ are defined uniquely by properties 1)-4) in Definition 1 and their generating function (2.2) has the form

$$\Theta^{(j)}(x, u) = \sum_{A=0}^{[j/2]} a^{(j)}_A (\Theta^{(u)}_{(u)} \Theta^{(x)}_{(x)})^A (\Theta^{(u)}_{(x)})^{j-2A} ,$$

where $[j/2]$ – integer part of $j/2$, the coefficients $a^{(j)}_A$ (for $A = 0$ and $A \geq 1$) are

$$a^{(j)}_0 = 1, \quad a^{(j)}_A = \frac{(-1/2)^A j!}{(j-2A)! A! (2j + D - 5)(2j + D - 7) \cdots (2j + D - 2A - 3)} ,$$

and $\Theta^{(u)}_{(x)} = \Theta^{(x)}_{(u)}$ denotes the function

$$\Theta^{(u)}_{(x)} (x, u) = x^r u_n \Theta^{(u)}_{(u)} (k) , \quad \Theta^{(u)}_{(x)} (k) = \Theta^{(u)}_{(u)} (k) \delta_n - \frac{k_r k^n}{k^2} .$$

Remark. Constants (2.4) satisfy the recurrence relations

$$a^{(j)}_A = \frac{1}{2} \frac{(j - 2A + 2)(j - 2A + 1)}{A (2j + D - 3)} a^{(j)}_{A-1} = \frac{1}{2} \frac{(j - 2A + 1)j}{A (2j + D - 5)} a^{(j-1)}_{A-1} ,$$

which are used below.

Proposition 2. For the generation function (2.2) the following recurrence relation holds

$$\Theta^{(j)}(x, u) = \frac{1}{(j-1)!} \left( \Theta^{(x)}_{(u)} - \frac{1}{\omega + 2(j-2)} \Theta^{(x)}_{(x)} (u_k \partial_{x_k}) \right) (\Theta^{(x)}_{(x)})^{j-1} \Theta^{(j-1)}(z, u) ,$$

where $\partial_{x_k} = \frac{\partial}{\partial x_k}$; function $\Theta^{(x)}_{(u)}$ is defined in (2.7) and $\omega = (D-1)$.

Proof. In the right-hand side of (2.7) the differential operator $(\Theta^{(x)}_{(x)})^{j-1}$ acts to the generating function $\Theta^{(j-1)}(z, u)$ which is given in (2.3) where coefficients $a^{(j-1)}_A$ are defined in (2.4). The result of this action is

$$(\Theta^{(x)}_{(x)})^{j-1} \Theta^{(j-1)}(z, u) = \sum_{A=0}^{[j-1]} a^{(j-1)}_A (\Theta^{(u)}_{(u)} \Theta^{(x)}_{(x)})^A (\Theta^{(u)}_{(x)})^{j-2A} = (j-1)! \sum_{A=0}^{[j-1]} a^{(j-1)}_A (\Theta^{(u)}_{(u)} \Theta^{(x)}_{(x)})^A (\Theta^{(u)}_{(x)})^{j-1-2A} .$$
where the second equality in (2.8) follows from the formula
\[
(\Theta^{(x)}_{(\partial_x)})^{j-1} \left( (\Theta^{(z)}_{(z)})^A (\Theta^{(u)}_{(z)})^{j-1-2A} \right) = (j-1)! (\Theta^{(x)}_{(x)})^A (\Theta^{(u)}_{(x)})^{j-1-2A}.
\] (2.9)
Now, making use of (2.8), we write the right hand side of (2.7) in the form
\[
ad^{(j-1)}_0 (\Theta^{(u)}_{(x)})^j + \sum_{A=1}^{[j-1]} B_A^{(j)} (\Theta^{(u)}_{(u)} \Theta^{(x)}_{(x)})^A (\Theta^{(u)}_{(x)})^{j-2A} - \frac{(j+1-2A)}{(\omega + 2(j-2))} a^{(j-1)}_{A-1} (\Theta^{(u)}_{(u)} \Theta^{(x)}_{(x)})^A (\Theta^{(u)}_{(x)})^{j-2A} \bigg|_{A=\left\lfloor \frac{j-1}{2} \right\rfloor + 1},
\] (2.10)
where the coefficients \(B_A^{(j)}\) are determined as follows
\[
B_A^{(j)} = a^{(j-1)}_A - \frac{2A}{(\omega + 2(j-2))} a^{(j-1)}_A - \frac{(j-2A+1)}{(\omega + 2(j-2))} a^{(j-1)}_{A-1} = a^{(j)}_A.
\] (2.11)
Here, in the last equality, we have used the recurrence relations (2.6) for the coefficients \(a^{(j)}_A\) and condition \(\omega = (D - 1)\). Note that for odd and even \(j\) we have respectively the relations \([\frac{j}{2}] = \left\lfloor \frac{j}{2} \right\rfloor\) and \([\frac{j-1}{2}] = \left\lfloor \frac{j}{2} \right\rfloor - 1\). Besides, for odd \(j\), the last term in (2.10) is zero, while for even \(j\) this term is equal to
\[
a^{(j)}_A (\Theta^{(u)}_{(u)} \Theta^{(x)}_{(x)})^A (\Theta^{(u)}_{(x)})^{j-2A} \bigg|_{A=\left\lfloor \frac{j}{2} \right\rfloor}.
\]
As a result, the whole expression in (2.10) can be written (for both even and odd \(j\)) as
\[
\sum_{A=0}^{\left\lfloor \frac{j}{2} \right\rfloor} a^{(j)}_A (\Theta^{(u)}_{(u)} \Theta^{(x)}_{(x)})^A (\Theta^{(u)}_{(x)})^{j-2A} \equiv \Theta^{(j)}(x, u),
\] (2.12)
which proves the identity (2.7).

3 Brauer algebra and its idempotents

In this Section we follow the exposition of [8] (see also [15, 16, 17] and references therein).

**Definition 2.** Unital associative algebra \(Br_j(\omega)\) over the field of complex numbers with generators \(\sigma_i\) and \(\kappa_i\) \((i = 1, \ldots, j-1)\) and defining relations (see [17] and, e.g., [15, 16])
\[
\sigma_i^2 = e, \quad \kappa_i^2 = \omega \kappa_i, \quad \sigma_i \kappa_i = \kappa_i \sigma_i = \kappa_i, \quad i = 1, \ldots, j-1,
\]
\[
\sigma_i \sigma_\ell = \sigma_\ell \sigma_i, \quad \kappa_i \kappa_\ell = \kappa_\ell \kappa_i, \quad \sigma_i \kappa_\ell = \kappa_\ell \sigma_i, \quad |i - \ell| > 1,
\]
\[
\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad \kappa_i \kappa_{i+1} \kappa_i = \kappa_i \kappa_{i+1} \kappa_i \kappa_{i+1} = \kappa_{i+1},
\]
\[
\sigma_i \kappa_{i+1} \kappa_i = \sigma_{i+1} \kappa_i, \quad \kappa_{i+1} \kappa_i \sigma_{i+1} = \kappa_{i+1} \sigma_i, \quad i = 1, \ldots, j-2,
\]
is called **Brauer algebra**. Here \(e\) – unit element and \(\omega\) is a real parameter characterizing the algebra.
All basis elements of the algebra $\mathcal{B}r_j(\omega)$ are constructed as products of the generators $\sigma_i$ и $\kappa_i$. The dimension of the Brauer algebra $\mathcal{B}r_j(\omega)$ is

$$\dim(\mathcal{B}r_j) = (2n - 1)!! = (2n - 1)(2n - 3) \cdots 3 \cdot 1.$$  \hspace{1cm} (3.14)

One can consider the algebra $\mathcal{B}r_j(\omega)$ as an extension of the group algebra $\mathbb{C}[S_j]$ of the permutation group $S_j$. The algebra $\mathcal{B}r_j(\omega)$ plays the same role in the theory of representations of the orthogonal groups $SO(N, \mathbb{C})$ (and their real forms $SO(p, q)$) as the group algebra $\mathbb{C}[S_j]$ in the theory of representations of linear groups $SL(N, \mathbb{C})$ (and their real forms $SU(N)$).

Define \cite{15} a set of special elements $y_m \in \mathcal{B}r_j(\omega)$ ($m = 1, \ldots j$)

$$y_1 = 0, \quad y_m = \sum_{k=1}^{m-1} (\sigma_{k,m} - \kappa_{k,m}) = \sum_{k=1}^{m-1} \sigma_{m-1} \cdots \sigma_{k+1} (\sigma_k - \kappa_k) \sigma_{k+1} \cdots \sigma_{m-1}, \quad m = 2, 3, \ldots, j,$$ \hspace{1cm} (3.15)

where

$$\sigma_{k,m} = \sigma_{m-1} \cdots \sigma_{k+1} \sigma_k \sigma_{k+1} \cdots \sigma_{m-1}, \quad \kappa_{k,m} = \sigma_{m-1} \cdots \sigma_{k+1} \kappa_k \sigma_{k+1} \cdots \sigma_{m-1}.$$ \hspace{1cm} (3.16)

Operators $y_m \in \mathcal{B}r_j(\omega)$ are called Jucys-Murphy elements and play an important role in constructing primitive orthogonal idempotents of $\mathcal{B}r_j(\omega)$. Note, that the Jucys-Murphy elements can be expressed via the recurrence relation

$$y_1 = 0, \quad y_{n+1} = \sigma_n - \kappa_n + \sigma_n y_n \sigma_n,$$ \hspace{1cm} (3.17)

and for illustration we present first two nontrivial elements

$$y_2 = \sigma_1 - \kappa_1, \quad y_3 = \sigma_2 - \kappa_2 + \sigma_2 (\sigma_1 - \kappa_1) \sigma_2.$$ \hspace{1cm} (3.18)

We define a subalgebra $Y_j \in \mathcal{B}r_j(\omega)$ generated by all Jucys-Murphy elements $\{y_1, y_2, \ldots, y_j\}$. For brevity below we omit $\omega$ in the notation $\mathcal{B}r_j(\omega)$ and write $\mathcal{B}r_j$. The following statement holds (see \cite{8,15}).

**Proposition 3** Jucys-Murphy elements, defined in (3.15), form a complete set of commuting generators in $\mathcal{B}_j$

$$[y_i, y_\ell] = 0, \quad \forall i, \ell.$$ \hspace{1cm} (3.19)

The algebra $Y_j$ is a maximal commutative subalgebra in $\mathcal{B}_j$.

Now we briefly discuss the procedure for constructing the complete system of primitive orthogonal idempotents $e_\alpha \in \mathcal{B}r_j$ which satisfy relations

$$e_\alpha e_\beta = \delta_{\alpha\beta} e_\alpha, \quad \sum_\alpha e_\alpha = 1.$$ \hspace{1cm} (3.20)

In addition, we require that, in the left regular representation of $\mathcal{B}r_j$, elements $e_\alpha \in \mathcal{B}r_j$ are eigenvectors of the Jucys-Murphy generators:

$$y_m e_\alpha = a_m^{(\alpha)} e_\alpha, \quad a_m^{(\alpha)} \in \mathbb{R}, \quad \forall m = 1, 2, \ldots, j.$$ \hspace{1cm} (3.21)
Such a choice of idempotents $e_a \in Br_j$ is always possible, since Jucys-Murphy elements \{y_1, \ldots, y_j\} commute with each other. Moreover, it can be shown (see [8]) that idempotents $e_a$, satisfying (3.20) and (3.21), commute with all Jucys-Murphy elements $y_m$ (it means that $e_a$ are functions of $y_m$) and all eigenvalues $a_m^{(a)}$ in (3.21) are real numbers. According to (3.21) each primitive idempotent $e_a \in Br_j$ is characterized by the set of eigenvalues $a_m^{(a)}$ which form a spectral vector
\[
\Lambda_a = (a_1^{(a)}, a_2^{(a)}, \ldots, a_j^{(a)}) \in \mathbb{R}^j,
\]
where in view of (3.15) we have $a_1^{(a)} = 0$. We denote the set of all spectral vectors (3.22) as $\text{Spec}(y_1, \ldots, y_n)$. It is possible to prove (the proof is given in [8]) that eigenvalues $a_i^{(a)}$ of $y_i$ satisfy condition
\[
a_i^{(a)} \in \{[1 - i, i - 1], [2 - i, i - 2] + (1 - \omega)\},
\]
where the bracket $[-z, z]$ denotes a set of integers
\[
[-z, z] = \{-z, \ldots, -1, 0, 1, \ldots, z\}, \quad z \in \mathbb{Z}_{\geq 0},
\]
and $[-z, z] + a$ denotes a set of integers shifted by $a$
\[
[-z, z] + a = \{a - z, \ldots, a - 1, a, a + 1, \ldots, a + z\}.
\]
The remaining conditions that completely determine the elements of the set $\text{Spec}(y_1, \ldots, y_j)$ can be found in [8, 15].

**Proposition 4** Elements of the set $\text{Spec}(y_1, \ldots, y_j)$ correspond one-to-one to the elements of a set of oscillating Young tableaux.

In formulation of this Proposition we use the notion of oscillating Young tableau. Now we recall (see, e.g., [8, 16, 17]) the notions of Young diagrams and oscillating Young tableaux. Young diagram $\lambda$ with $r$ boxes and $k$ rows is a set of integers $\{m_1, m_2, \ldots, m_k\}$ such that $m_1 \geq m_2 \geq \ldots \geq m_k > 0$ and $\sum_{i=1}^{k} m_k = r$. The standard notation is $\lambda = [m_1, m_2, \ldots, m_k] \vdash r$. Consider a sequence of Young diagrams $\Lambda = \{\lambda_0, \lambda_1, \ldots, \lambda_j\}$, which starts with a trivial diagram $\lambda_0 = \emptyset$, and the diagram $\lambda_{k+1}$, standing in the sequence $\Lambda$ after $\lambda_k$, is obtained either by adding one box to the outer angle of diagram $\lambda_k$, or by deleting one box in the inner angle of diagram $\lambda_k$. Such a sequence $\Lambda$ of Young diagrams is called oscillating Young tableau (or updown Young tableau) of length $j$. In the figure below, we give an example of possible transitions from $\lambda_k$ to $\lambda_{k+1}$ in the oscillating Young tableau. In the first line we depict the Young diagram $\lambda_k^{(a)} = [3, 2, 2]$. The second line contains five diagrams that can be obtained from $\lambda_k^{(a)}$ by the operations which were described above (adding or removing one box). Here, in the notation of Young diagrams $\lambda_k^{(a)}$ in addition to the lower index $k$, we introduce the upper multi-index $(a) = (a_1, a_2, \ldots, a_k) \in \mathbb{R}^k$ which is content vector of the oscillating Young tableau, where the last Young diagram is $\lambda_k$. Now using the tables from Fig.1 as an example, we will explain according to what rules we select the value of the element $a_{k+1}$ when moving from the table $\lambda_k^{(a)}$ to the table $\lambda_{k+1}^{(a_k+1)}$, that is, when passing from the vector $(a) \in \mathbb{R}^k$ to the vector $\lambda_{k+1} = (a_1, a_2, \ldots, a_k, a_{k+1}) \in \mathbb{R}^{k+1}$.

Define the coordinates $(s, t) \in \mathbb{Z}_+^2$ of boxes of an arbitrary Young diagram $\lambda$, where $s$ – the row number and $t$ – the column number of the diagram $\lambda$. For example, the upper left angle of the diagram has the coordinates $s = 1$ and $t = 1$. The value $(t - s)$ will be called the content of a box with the coordinates $(s, t)$. The diagram boxes in Fig.1 are filled with
the content, according to this rule. In the general case, when the box with the coordinates
\((s, t)\) is added to the diagram \(\lambda_k^{(a)}\), then the coordinate \(a_{k+1} = (t - s)\) is added to the vector
\((a) \in \mathbb{R}^k\) of the new diagram \(\lambda_{k+1}^{(a,a_{k+1})}\) thus obtained, and if the box with the coordinates \((s, t)\)
is deleted, then the coordinate \(a_{k+1} = (t - s)' = (1 - \omega) - (t - s)\) is added to the vector \((a)\) in
the notation of the new diagram. For example, consider the diagram \(\lambda_{k+1}^{(a',x')}\) from the second
line of Figure 1. It is obtained from \(\lambda_k^{(a)}\) by deleting the box with the content \(2\), therefore
the value \((a_1, \ldots, a_k)\) is added to the vector \(a_{k+1} = 2' = (1 - \omega) - 2\).

Thus, each oscillating Young tableau corresponds to a sequence of diagrams with transitions
between them:

\[
\Lambda = \{ \emptyset \overset{a_1=0}{\rightarrow} \lambda_1 \overset{a_2}{\rightarrow} \lambda_2 \overset{a_3}{\rightarrow} \lambda_3 \overset{a_4}{\rightarrow} \cdots \overset{a_j}{\rightarrow} \lambda_j \},
\]

(3.26)

where each transition \(\lambda_k \rightarrow \lambda_{k+1}\) is assigned a value \(a_{k+1}\) determined from the content value
of the added or deleted box. It follows that to each oscillating Young tableau \(\Lambda\) of length \(j\)
there corresponds a vector called contents vector of length \(j\)

\[
A = (a_1, a_2, \ldots, a_j),
\]

(3.27)

where, according to the rules for determining the numbers \(a_i\) for added or removed boxes, we have (cm. (3.28))

\[
a_i \in \{ [1 - i, i - 1], [2 - i, i - 2] + 1 - \omega \}
\]

(3.28)

The sequence set \((a_1, a_2, \ldots, a_j)\) for all oscillating Young tableaux of length \(j\) is called the
set of content vectors of length \(j\). The elements of this set uniquely correspond to oscillating
Young tables, as indicated in Proposition 4. On the other hand (see 8) the set of content
vectors coincides with the set \(\text{Spec}(y_1, \ldots, y_j)\).

We introduce the concept of colored oscillating Young graph \(\Gamma\) of the algebra \(B_{r_j}\), which
is a convenient visual representation of all possible oscillating Young tableaux of \(3.26\) of
fixed length \(j\). On the one hand, the vertices of such a graph at the level \(j\) correspond to
irreducible representations of the algebra \(B_{r_j}\), on the other hand, such a graph indicates
the branching rules of these representations (that is, indicates possible transitions from
the diagram at the level of \(k\) to the diagrams at the level of \((k + 1)\)). Note that the dimension
of the representation \(B_{r_k}\), corresponding to the vertex \(\lambda\) at the level \(k\) of the Young graph is
equal to the number of paths starting at the very top vertex \(\emptyset\) and ending at this vertex \(\lambda\).
Thus, the oscillating Young graph encodes information on irreducible representations of the
Brauer algebras, including the branching rules of these representations. For instance, Fig. 2

\[\text{Figure 1: Examples of possible transitions between the diagrams } \lambda_k \text{ and } \lambda_{k+1} \text{ in an oscillating Young tableau.}\]
(taken from the book [8]) shows the oscillating Young graph for $B_4$, which gives information on irreducible representations of the Brauer algebras $B_k$, with $k = 1, 2, 3, 4$.

$$y_1 =$$

$$y_2 =$$

$$y_3 =$$

$$y_4 =$$

Figure 2: The colored oscillating Young graph for the Brauer algebra $B_{r_4}$. The indices assigned to the edges (arrows) of the graph are the eigenvalues of the Jucys-Murphy operators $y_k \in B_{r_4}$, which are indicated on the left.

We now proceed to the derivation (see, for example, [8]) of general formulas for primitive orthogonal idempotents of the Brauer algebra. The following figure shows a schematic view of the Young diagram $\lambda$ located at the $n$-th level of the oscillating Young graph.

$$\lambda = \begin{array}{c}
\phantom{0} \\
\phantom{0} \\
\phantom{0} \\
\phantom{0} \\
\phantom{0} \\
\phantom{0} \\
\end{array}$$

Here $(n_i, \lambda_{(i)})$ - coordinates of the boxes located in the inner angles of the $\lambda$. Note that the number of boxes $|\lambda|$ of all diagrams $\lambda$ at the level $n$ obeys the inequality $|\lambda| \leq n$.

Consider in the oscillating Young graph of the algebra $Br_{n+1}(\omega)$ any path $T_{(\lambda;n)}$, going down from the vertex $\{\emptyset;0\}$ to the vertex of $\{\lambda;n\}$, i.e., consider the path corresponding to the oscillating tableau $\Lambda = \{\lambda_1 = \emptyset, \lambda_2, \ldots, \lambda_n = \lambda\}$. Let $E_{T_{(\lambda;n)}} \in Br_n(\omega)$ be a primitive orthogonal idempotent corresponding to $T_{(\lambda;n)}$. Using the branching rule, which is dictated by the oscillating Young graph for the Brauer algebra $Br_{n+1}(\omega)$ (see the example in Figure 2), we can conclude that in order to move from the diagram $\lambda$ at the $n$ level to the diagram $\lambda_{n+1}$ at the $(n+1)$ level along the path $T_{(\lambda_{n+1};n+1)}$, we need to add one box to the $\lambda$ diagram (with the content $(\lambda_{(r)} - n_{r-1})$) to the outer angle of $\lambda$, or remove one box (with the content

---

$^6$Here, in the notation for the vertex $\{\lambda;n\}$ the second character $n$ in braces indicates the level at which the diagram $\lambda$ appears. The level is indicated since in the oscillating Young graph (see Figure 2) the same diagrams can appear at different levels.
of \((\lambda_{(r)} - n_r)\), from the inner angle of the diagram \(\lambda\). Knowing the contents of added or deleted boxes, we know all possible eigenvalues \((\lambda_{(r)} - n_{r-1})\) or \((1 - \omega + n_r - \lambda_{(r)})\) of the element \(y_{n+1}\) in the representation defined by \(E_{T(\lambda, n)}\) and therefore we have the identity

\[
E_{T(\lambda, n)} \prod_{r=1}^{k+1} (y_{n+1} - (\lambda_{(r)} - n_{r-1})) \prod_{r=1}^{k} (y_{n+1} - (1 - \omega + n_r - \lambda_{(r)})) = 0, \tag{3.29}
\]

where we put \(\lambda_{(k+1)} = n_0 = 0\). Thus, for the new diagram \(\lambda_{n+1} = \lambda'\), obtained by adding the box to the diagram \(\lambda\) with the coordinates \((n_{i-1} + 1, \lambda_{(i)} + 1)\), the corresponding primitive idempotent (after suitable normalization) has the form [8]

\[
E_{T(\lambda', n+1)} = E_{T(\lambda, n)} \prod_{r=1}^{k+1} \frac{(y_{n+1} - (\lambda_{(r)} - n_{r-1}))}{(\lambda_{(i)} - n_{i-1} - (\lambda_{(r)} - n_{r-1}))} \prod_{r=1}^{k} \frac{(y_{n+1} - (1 - \omega + n_r - \lambda_{(r)}))}{(\lambda_{(i)} - n_{i-1} - (1 - \omega + n_r - \lambda_{(r)}))}. \tag{3.30}
\]

For the new diagram \(\lambda_{n+1} = \lambda''\) obtained from \(\lambda\) by deleting the box with the coordinates \((n_i, \lambda_{(i)})\) we get a primitive idempotent [8]

\[
E_{T(\lambda'', n+1)} = E_{T(\lambda, n)} \prod_{r=1}^{k+1} \frac{(y_{n+1} - (\lambda_{(r)} - n_{r-1}))}{(1 - \omega + n_i - \lambda_{(i)} - (\lambda_{(r)} - n_{r-1}))} \prod_{r=1}^{k} \frac{(y_{n+1} - (1 - \omega + n_r - \lambda_{(r)}))}{(n_i - (\lambda_{(i)} - (n_r - \lambda_{(r)}))}. \tag{3.31}
\]

Using these formulas, as well as the initial data: \(E_{T(1; 1)} = 1\), step by step we get explicit expressions for all primitive idempotents corresponding to paths in the oscillating Young graph for the Brauer algebra \(Br_j(\omega)\).

**Example.** For the algebra \(Br_j\) we construct an explicit expression for the complete symmetrizer \(E_{T(1; j)}\). In the space of the left regular representation of the algebra \(Br_j\) the idempotent \(E_{T(1; 1; j-1)}\) extracts the subspace in which the element \(y_j\) can be equal to three eigenvalues. Indeed, according to the branching rule discussed above, there are two possibilities to add a box to the diagram \([j - 1]\) as indicated by the asterisks in the figure below, or delete the last box in the row with the content \((j - 2)\):

\[
\begin{array}{cccc}
0 & 1 & \ldots & j - 3 & j - 2 \\
* & & & & \\
\end{array}
\]

Following the procedure for constructing idempotents described above, we deduce identity for the element \(y_j\)

\[
E_{T(1; 1; j-1)} \cdot (y_j - (j - 1)) \cdot (y_j + 1) \cdot (y_j + \omega + j - 3) = 0 \tag{3.32}
\]

Since the symmetrizer \(E_{T(1; j)}\) corresponds to the Young diagram obtained by adding the box with the content \((j - 1)\), the expression \(3.32\) gives us the recurrence relation for symmetrizer \((j \geq 2)\)

\[
E_{T(1; j)} = E_{T(1; 1; j-1)} \cdot \frac{(y_j + 1) \cdot (y_j + \omega + n - 3)}{j \cdot (2j - 4 + \omega)}. \tag{3.33}
\]

Solving this recurrence relation, we obtain an explicit formula for the complete symmetrizer

\[
E_{T(1; j)} = \frac{(y_2 + 1) \cdots (y_j + 1)}{j!} \cdot \frac{(y_2 + \omega - 1) \cdot (y_3 + \omega) \cdots (y_j + \omega + j - 3)}{\omega \cdot (2 + \omega) \cdots (2j - 4 + \omega)}. \tag{3.34}
\]

One can check that \(3.34\) obeys identities (\(\forall r = 1, \ldots, j - 1\))

\[
\sigma_r \cdot E_{T(1; j)} = E_{T(1; j)} \cdot \sigma_r = E_{T(1; j)} \quad \kappa_r \cdot E_{T(1; j)} = 0 = E_{T(1; j)} \cdot \kappa_r. \tag{3.35}
\]

The proof of these identities for general \(q\)-deformed case is given in [18].
4 Realization of the Brauer algebra in tensor spaces.

Introduce the triple $(\theta, \hat{\theta}, \check{\theta})$ of $D \times D$ real matrices $\hat{\theta} = ||\hat{\theta}_{nn}||$, $\check{\theta} = ||\check{\theta}_{nm}||$ and $\theta = ||\theta^m_n|| = ||\theta^m_n||$ such that

$$\check{\theta}_{m\ell} \hat{\theta}_{n\ell} = \theta^m_n, \quad \hat{\theta}_{m\ell} \check{\theta}_{n\ell} = \check{\theta}_{m\ell} = \check{\theta}_{n\ell} = \check{\theta}_{m\ell} = \hat{\theta}_{mn}.$$  \hfill (4.1)

**Proposition 5** The triple $(\theta, \hat{\theta}, \check{\theta})$ with relations (4.1) satisfy conditions

$$\theta^m_n \hat{\theta}_{mn} = \hat{\theta}_{nn}, \quad \theta^m_n \check{\theta}_{mn} = \check{\theta}_{nn}, \quad \hat{\theta}_{m\ell} \theta^\ell_n = \check{\theta}_{m\ell} = \check{\theta}_{n\ell} = \theta^m_n.$$  \hfill (4.2)

$$\text{Tr}(\theta) = \theta^\ell_\ell = \hat{\theta}_{nn} \check{\theta}_{nn} = \omega,$$  \hfill (4.3)

where $\omega$ is an integer number: $0 \leq \omega \leq D$.

**Proof.** The relations (4.2) follow directly from the equalities (4.1). The formula (4.3) follows from the last relation in (4.2) which means that $\theta$ is a projector and eigenvalues of $\theta$ are equal to 0 or +1. Thus, $\omega$ is a rank of $\theta$. \hfill \blacksquare

Consider operators $P_r^{(\theta)}$ and $K_r^{(\theta)}$ $(r = 1, \ldots, j - 1)$ acting in the space $(\mathbb{R}^D)^{\otimes j}$ according to the following formulas

$$P_r^{(\theta)} \cdot (e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e_{i_{r+1}} \otimes \cdots \otimes e_{i_j}) =$$

$$(e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e_{i_{r+1}} \otimes \cdots \otimes e_{i_j}) \theta^1_{i_1} \cdots \theta^r_{i_r} \theta^{r+1}_{i_{r+1}} \theta^{r+2}_{i_{r+2}} \cdots \theta^j_{i_j},$$

$$K_r^{(\theta)} \cdot (e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e_{i_{r+1}} \otimes \cdots \otimes e_{i_j}) =$$

$$(e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e_{i_{r+1}} \otimes \cdots \otimes e_{i_j}) \theta^1_{i_1} \cdots \theta^{r-1}_{i_{r-1}} \theta^r_{i_r} \theta^{r+1}_{i_{r+1}} \theta^{r+2}_{i_{r+2}} \cdots \theta^j_{i_j},$$  \hfill (4.4)

where $i_\ell = 1, \ldots, D$ and $e_i$ are basis vectors in the space $\mathbb{R}^D$.

**Proposition 6** If the triple $(\theta, \hat{\theta}, \check{\theta})$ satisfies (4.1), (4.2) and (4.3), then the map $S_\theta : Br_j(\omega) \rightarrow \text{End}(\mathbb{R}^D)^{\otimes j}$ defined on the generators $\sigma_r, \kappa_r \in Br_j(\omega)$:

$$S_\theta(\sigma_r) = P_r^{(\theta)}, \quad S_\theta(\kappa_r) = K_r^{(\theta)},$$  \hfill (4.5)

is extended to the whole algebra $Br_j(\omega)$ as a homomorphism (i.e. $S_\theta$ is a representation of $Br_j(\omega)$).

**Proof.** Making use of formulas (4.1), (4.2) and (4.3), one can check directly that the operators (4.5) satisfy defining relations (3.13). It means that the map (4.5) can be extended to the whole algebra $Br_j(\omega)$ as a homomorphism and $S_\theta$ is a representation of $Br_j(\omega)$. \hfill \blacksquare

**Remark 1.** The triple of $D \times D$ matrices $\hat{\theta}_n = \delta^r_n, \check{\theta}_n = \eta_{rn}$, $\check{\theta}_{rn} = \eta_{rn}$, where $\delta^r_n$ is the Kronecker delta and $\eta_{rn}$ is a metric in $\mathbb{R}^{p,q}$ $(p + q = D)$, satisfy (4.1), (4.2) and (4.3) with
\( \omega = D \). Thus, according to Proposition 6 this triple defines the representation \( S \) of the Brauer algebra \( \mathcal{B}_r(D) \), which acts in the space \( (\mathbb{R}^{p,q})^\otimes j \) (see (4.4), (4.5))

\[
S(\sigma_r) \cdot (e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e_{i_{r+1}} \otimes \cdots \otimes e_{i_j}) = (e_{\ell_1} \otimes \cdots \otimes e_{\ell_r} \otimes e_{\ell_{r+1}} \otimes \cdots \otimes e_{\ell_j}) \delta_{i_1}^{\ell_1} \cdots \delta_{i_{r-1}}^{\ell_{r-1}} \delta_{i_r}^{\ell_r} \delta_{i_{r+1}}^{\ell_{r+1}} \cdots \delta_{i_j}^{\ell_j}
\]

\[
S(\kappa_r) \cdot (e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e_{i_{r+1}} \otimes \cdots \otimes e_{i_j}) = (e_{\ell_1} \otimes \cdots \otimes e_{\ell_r} \otimes e_{\ell_{r+1}} \otimes \cdots \otimes e_{\ell_j}) \delta_{i_1}^{\ell_1} \cdots \delta_{i_{r-1}}^{\ell_{r-1}} \eta_{i_r i_{r+1}}^{\ell_r} \delta_{i_{r+2}}^{\ell_{r+2}} \cdots \delta_{i_j}^{\ell_j},
\]

where \( e_\ell \) are basis vectors in \( \mathbb{R}^{p,q} \). It is known (see e.g. [8]) that the action of the algebra \( \mathcal{B}_r(D) \) in the space of representation (4.4) centralizes the action of the group \( SO(p,q) \) in the same space \( (\mathbb{R}^{p,q})^\otimes j \) of the representation \( T^{(j)} \equiv T^\otimes j \), where \( T \) – defining representations of \( SO(p,q) \). It means that any operator \( X \) which is a linear combination

\[
X = \sum_{i=1}^{(2j-1)!!} x_i S(a_i),
\]

where summation runs over all basis elements \( a_i \in \mathcal{B}_r(D) \) and \( x_i \) are complex coefficients, commutes with any element \( g \in SO(p,q) \) in the representation \( T^{(j)} \):

\[
X \cdot (T^{(j)}(g)) = (T^{(j)}(g)) \cdot X.
\]

In view of Schur’s Lemma, it means that the tensor product \( T^\otimes j \) of the defining representations \( T \) of \( SO(p,q) \) is reducible and it can be decomposed into the direct sum of its irreducible components. Subspaces of these irreducible components are extracted from the space of the representation \( T^\otimes j \) by acting of special projectors which are images (in the representation \( S \)) of the primitive orthogonal idempotents of the Brauer algebra \( \mathcal{B}_r(D) \) (for details see [8]).

**Remark 2.** Let the triple of matrices \((\theta, \bar{\theta}, \bar{\theta})\) be

\[
\theta^m_n = \Theta^m_n(k), \quad \bar{\theta}^{nm} = \Theta^m_n(k) \eta^{ln}, \quad \bar{\theta}^{mn} = \Theta^m_n(k) \eta^{lm},
\]

where the matrix \( \Theta^m_n(k) \) depends on the momentum \( k \equiv \vec{k} \in \mathbb{R}^{p,q} \) and is defined in (2.5). The triple (4.9) satisfies (1.1), (1.2) and (4.3) with \( \omega = (D-1) \). Thus, according to Proposition 6 the operators (4.4) with the choice (4.9) define the representation \( S_{(\vec{k})} \equiv S_{(k)} \) of \( \mathcal{B}_r(D-1) \).

Consider the image of the complete symmetrizer \((3.34)\) in the representation \( S_{(\vec{k})} \)

\[
\Theta^{(j)}_{(\vec{k})} \equiv S_{(\vec{k})}(E_{T^{(j)}(\vec{k})}).
\]

Operator \( \Theta^{(j)}_{(\vec{k})} \) acts in the space \( (\mathbb{R}^{p,q})^\otimes j \) and, in view of (3.35), satisfies conditions

\[
\Theta^{(j)}_{(\vec{k})}(\sigma_{i,\ell}) \cdot S_{(\vec{k})}(\sigma_{i,\ell}) = \Theta^{(j)}_{(\vec{k})}, \quad \Theta^{(j)}_{(\vec{k})}(\kappa_{i,\ell}) \cdot S_{(\vec{k})}(\kappa_{i,\ell}) = \Theta^{(j)}_{(\vec{k})},
\]

where the elements \( \sigma_{i,\ell}, \kappa_{i,\ell} \in \mathcal{B}_r(D) \) are defined in (3.16).
Proposition 7. Operator $\Theta_{\{j;j\}}$ given in (4,10) is equal to the D-dimensional spin projection operator $\Theta^{(j)}(k)$ (see Definition 1 and Proposition 7)

$$\Theta_{\{j;j\}} = \Theta^{(j)}(k).$$ (4.12)

Proof. The component form of conditions (4.11) is

$$\theta_{ij}^{n1\ldots n_j} \theta_{i'j'}^{n1'\ldots n_j'} = \theta_{ii'}^{n1\ldots n_j} \theta_{ii'}^{n1'\ldots n_j'} \Theta_{\{j;j\}}^{n1\ldots n_j}, \quad (\Theta_{\{j;j\}}){\{j;j\}}^{n1\ldots n_j} = \Theta_{\{j;j\}}^{n1\ldots n_j}, \quad (\Theta_{\{j;j\}}){\{j;j\}}^{n1\ldots n_j} = \Theta_{\{j;j\}}^{n1\ldots n_j}, \quad (\Theta_{\{j;j\}}){\{j;j\}}^{n1\ldots n_j} = \Theta_{\{j;j\}}^{n1\ldots n_j}, \quad (\Theta_{\{j;j\}}){\{j;j\}}^{n1\ldots n_j} = \Theta_{\{j;j\}}^{n1\ldots n_j}, \quad (\Theta_{\{j;j\}}){\{j;j\}}^{n1\ldots n_j} = \Theta_{\{j;j\}}^{n1\ldots n_j}. \quad (4.13)

We contract relations (4.13) with momentum $k^{ri}$ and $k_{ni}$ and use properties $\theta^{a} k^{r} = k^{a} \theta_{r} = 0$. As a result we obtain from relations (4.13) that operator $\Theta_{\{j;j\}}$ obeys properties 2.) and 3.) of Definition 1 in Section 2. Then, we substitute explicit forms (2.5) of matrices $\theta^{e}$ in equations (4.14) and use the transversality property 3.) for operator $\Theta_{\{j;j\}}$. In this way we deduce that $\Theta_{\{j;j\}}$ obeys property 4.) of Definition 1 in Section 2. Since the symmetrizer (3.34) is idempotent and satisfies projection identity $E_{j}^{2} = E_{j}$, the property 1.) of Definition 1 is fulfilled automatically for matrix $\Theta_{\{j;j\}}$. Thus, operator $\Theta_{\{j;j\}}$ obeys all four conditions of Definition 1 in Section 2 which determine spin projection operator uniquely (see Proposition 7). It leads to the identity (4.12).

5. New factorization formula for Behrends-Fronsdal symmetrizer. Recurrence relation.

Now we describe another approach to construct of the completely symmetric projectors $\Theta_{\{j;j\}}$. This approach is based on the different construction (see [16]) of complete symmetrizer (3.34) in Brauer algebra $Br_{j}$.

Consider the rational function $\hat{R}_{i}(w)$ with values in algebra $Br_{j}$

$$\hat{R}_{i}(w) = \sigma_{i}(1 - \frac{\sigma_{i}}{w} + \frac{\kappa_{i}}{w - \kappa}), \quad \kappa = \frac{\omega}{2} - 1,$$ (5.15)

where argument $w$ is usually called the spectral parameter. This function is a solution of Yang-Baxter equation in braid group form

$$\hat{R}_{i}(w)\hat{R}_{i+1}(w + v)\hat{R}_{i}(v) = \hat{R}_{i+1}(v)\hat{R}_{i}(w + v)\hat{R}_{i+1}(w).$$ (5.16)

Note, that for $w = -1$ the function $\hat{R}_{i}(-1)$ has the following properties:

$$\sigma_{i}\hat{R}_{i}(-1) = \hat{R}_{i}(-1)\sigma_{i} = \hat{R}_{i}(-1), \quad \kappa_{i}\hat{R}_{i}(-1) = \hat{R}_{i}(-1)\kappa_{i} = 0.$$(5.17)

Define the element $\Xi_{j} \in Br_{j}$ by means of the recurrence relations

$$\Xi_{j} = \Xi_{j-1} \left( \prod_{i=j-1}^{j} \hat{R}_{i}(-i) \right) = \left( \prod_{i=1}^{j-1} \hat{R}_{i}(-i) \right) \Xi_{j-1}, \quad (5.18)$$

where $\Xi_{1} = 1$. Here and below we use the following convention in the ordering of products of noncommutative operators:

$$\prod_{i=j-1}^{1} \hat{R}_{i} \equiv \hat{R}_{j-1} \cdots \hat{R}_{2} \hat{R}_{1}, \quad \prod_{i=1}^{j-1} \hat{R}_{i} \equiv \hat{R}_{1} \hat{R}_{2} \cdots \hat{R}_{j-1}.$$
Let us define the following elements:

\[ R_{ik}(w) = \sigma_{i,k} \cdot \hat{R}_{ik}(w), \quad R_{ik}(w) = \left(1 - \frac{\sigma_{i,k}}{w} + \frac{\kappa_{i,k}}{w - \kappa}\right) \in \mathcal{B}r_j(w), \tag{5.19} \]

where \( \sigma_{i,k} \) and \( \kappa_{i,k} \) are defined in (3.16); parameter \( \kappa \) is given in (5.15). Then the following relation holds

\[ \Xi_j = \prod_{k=j-1}^{1} \left( \prod_{i=1}^{k} \sigma_i \right) \cdot \prod_{i=j}^{2} \left( \prod_{k=i-1}^{1} R_{ki}(k - i) \right), \tag{5.20} \]

which is directly deduced from (5.18) and (5.19). We note that the elements \( R_{ik} \) from (5.19) are solutions of the standard Yang-Baxter equation (cf. 5.16)

\[ R_{ik}(w)R_{i\ell}(w + v)R_{k\ell}(v) = R_{k\ell}(v)R_{i\ell}(w + v)R_{ik}(w). \tag{5.21} \]

The image of the solution (5.19) in the representation (4.6) is known as Zamolodchikov’s \( R \)-matrix [19].

**Remark 3.** Note that we can omit the dependence on spectral parameters \( (k - i) \) in \( R \)-matrices \( R_{ki}(k-i) \) in the equation (5.20), since all spectral parameters are restored from lower indices of the elements \( R_{ki} \). Moreover, if we fix spectral parameters in (5.21) as \( w = (i - k) \) and \( v = (k - \ell) \) then Yang-Baxter equation (5.21) can be written in concise form

\[ R_{ik} R_{i\ell} R_{k\ell} = R_{k\ell} R_{i\ell} R_{ik}. \tag{5.22} \]

**Proposition 8.** For the element \( \Xi_j \) the following representation holds [19]:

\[ \Xi_j = \prod_{k=1}^{j-1} \left( \prod_{l=j-1}^{k} \hat{R}_{l}(\ell - j) \right) \equiv \prod_{\ell=j-1}^{1} \hat{R}_{\ell}(\ell - j) \prod_{\ell=j-1}^{2} \hat{R}_{\ell}(\ell - j) \cdots \prod_{\ell=j-1}^{j-1} \hat{R}_{\ell}(\ell - j). \tag{5.23} \]

The element \( \Xi_j \) defined in (5.18) and (5.23) satisfies the conditions (cf. (3.35))

\[ \sigma_r \cdot \Xi_j = \Xi_j \cdot \sigma_r = \Xi_j, \quad \kappa_r \cdot \Xi_j = 0 = \Xi_j \cdot \kappa_r \quad (r = 1, \ldots, j - 1), \tag{5.24} \]

and we have the identity

\[ E_{T_{(j-1)!}} = \frac{1}{j!} \Xi_j, \tag{5.25} \]

where idempotent \( E_{T_{(j-1)!}} \) is given in (3.34).

**Proof.** We prove the identity (5.23) for the element \( \Xi_j \) defined in (5.18) by induction. First, we rewrite the right-hand side of eq. (5.23) in terms of the elements (5.19):

\[ \prod_{k=1}^{j-1} \left( \prod_{l=j-1}^{k} \hat{R}_{l}(\ell - j) \right) = \prod_{k=1}^{j-1} \left( \prod_{l=j-1}^{k} \sigma_i \right) \cdot \prod_{1 \leq i < k \leq j} R_{ik}, \tag{5.26} \]

where we use the concise notation \( R_{ik} \equiv R_{ik}(i - k) \) and define the double product as

\[ \prod_{1 \leq i < k \leq j} R_{ik} = \left( \prod_{k=2}^{j} R_{1k} \right) \cdot \left( \prod_{k=3}^{j} R_{2k} \right) \cdots \left( \prod_{k=j-1}^{j} R_{j-2,k} \right) \cdot \left( \prod_{k=j}^{j} R_{j-1,k} \right). \tag{5.27} \]
Then, we note that the prefactors containing only elements \( \sigma_i \) in right hand sides of (5.20) and (5.26) are equal
\[
\prod_{k=j-1}^1 \left( \prod_{i=1}^k \sigma_i \right) = \prod_{k=1}^{j-1} \left( \prod_{i=1}^k \sigma_i \right),
\]
(5.28)
(one can prove this identity by means of induction over \( j \)). So, to prove (5.23) it is sufficient to show that the following equality holds
\[
\prod_{i=j}^2 \left( \prod_{k=1}^1 R_{ki} \right) = \prod_{1 \leq i<k \leq j} R_{ik}(i-k).
\]
(5.29)
For \( j = 3 \) we have the base of induction \( R_{23}R_{13}R_{12} = R_{12}R_{13}R_{23} \) which is valid in view of the Yang-Baxter equation (5.22). Below we need the identity
\[
\left[ \prod_{i=j-1}^k R_{ij} \right] \left( \prod_{\ell=k+1}^j R_{k\ell} \right) = \left( \prod_{\ell=k+1}^j R_{k\ell} \right) \left[ \prod_{i=j-1}^k R_{ij} \right],
\]
(5.30)
which can be deduced by applying the Yang-Baxter equation (5.22) many times. Suppose that (5.29) is valid for \( j \to (j-1) \). Then we consider the left-hand side of (5.29)
\[
\prod_{i=j}^2 \left( \prod_{k=1}^1 R_{ki} \right) = \prod_{i=j-1}^1 R_{ij} \cdot \prod_{1 \leq i<k \leq j} R_{i\ell} =
\]
\[
= \left[ \prod_{i=j-1}^1 R_{ij} \right] \cdot \left( \prod_{\ell=2}^{j-1} R_{1\ell} \right) \cdot \left( \prod_{\ell=3}^{j-2} R_{2\ell} \right) \cdots \left( \prod_{\ell=j-3}^{j-2} R_{j-3,\ell} \right) \cdot \left( \prod_{\ell=j-2}^{j-1} R_{j-2,\ell} \right) =
\]
\[
= \left( \prod_{\ell=2}^j R_{1,\ell} \right) \cdot \left[ \prod_{i=j-1}^1 R_{ij} \right] \cdot \left( \prod_{\ell=3}^{j-2} R_{2,\ell} \right) \cdots \left( \prod_{\ell=j-3}^{j-2} R_{j-3,\ell} \right) \cdots \left( \prod_{\ell=j-1}^{j-2} R_{j-2,\ell} \right) = \cdots =
\]
\[
\cdots = \left( \prod_{\ell=2}^j R_{1,\ell} \right) \cdot \left[ \prod_{\ell=3}^{j-2} R_{2,\ell} \right] \cdots \left( \prod_{\ell=j-3}^{j-2} R_{j-3,\ell} \right) \cdots \left( \prod_{\ell=j-1}^{j-2} R_{j-2,\ell} \right) \cdot \left[ R_{j-1,j} \right] = \prod_{1 \leq i<k \leq j} R_{i\ell},
\]
where in the first equality we use the induction hypothesis and in the following equalities we apply (5.30) many times (everywhere we use the concise notation from the Remark 3).

Now the conditions (5.24) are deduced from the representation (5.23), recurrence relations (5.18) and conditions (5.17). Finally we prove formula (5.25). Since the properties (3.35) for idempotent \( E_{T(y,j)} \) are the same as properties (5.24) for element \( \Xi_j \), then \( \Xi_j \) have to be proportional to idempotent \( E_{T(y,j)} \). Therefore to prove (5.29), it is enough to show that the element in right hand side of (5.25) satisfies the projection property, i.e. we need to verify the identity \( \Xi_j \cdot \Xi_j = j! \cdot \Xi_j \). Indeed, it follows from the chain of equalities
\[
\Xi_j \cdot \Xi_j = \Xi_{j-1} \cdot \hat{R}_{j-1} \cdots \hat{R}_2 \cdot \hat{R}_1 \cdot \Xi_j = j \cdot \Xi_{j-1} \cdot \Xi_j =
\]
\[
= j \cdot \Xi_{j-2} \cdot \hat{R}_{j-2} \cdots \hat{R}_1 \cdot \Xi_j = j(j-1) \cdot \Xi_{j-3} \cdot \hat{R}_{j-3} \cdots \hat{R}_1 \cdot \Xi_j = \cdots = j! \cdot \Xi_j,
\]
where \( \hat{R}_k \equiv \hat{R}_k(-k) \). Here we use the recurrence formula (5.18) for the elements \( \Xi_\ell \) and then apply relations
\[
\hat{R}_k(-k) \Xi_j = \frac{k+1}{k} \Xi_j, \quad \forall k = 1, \ldots, (j-1),
\]
where we take into account the explicit form (5.15) of the element \( \hat{R}_k(-k) \) and conditions (5.24).

\textbf{Remark 4.} Relations (5.32) lead to the formula \((j - 1)! \Xi_j = \Xi_{j-1} \cdot \Xi_j\). Making use the second part of (5.18) in the right hand side of this formula we obtain new recurrence identity for symmetrizers

\[
\Xi_j = \frac{1}{(j-2)!} \Xi_{j-1} \hat{R}_{j-1} \Xi_{j-1},
\]

which is useful in many applications.

In view of the identity (5.25) the element \((j!)^{-1} \Xi_j\) in the representation \(S_{(\bar{k})}\) (see Remark 2) is equal to the \(D\)-dimensional Behrends-Fronsdal symmetrizer (4.10)

\[
\frac{1}{j!} S_{(\bar{k})}(\Xi_j) = \Theta^{(j)} \equiv \Theta_{(j|j)}
\]

where the generating function for matrix \(\Theta^{(j)}\) is given in (2.3).

Now we prove identities (5.33) directly without using properties of Definition 1 in Sect. 2. For this purpose we show that matrices \((j!)^{-1} S_{(\bar{k})}(\Xi_j)\) and \(\Theta^{(j)}\) satisfy the same recurrence relations.

These recurrence relations have a unique solution and this means that the equality (5.33) holds.

Introduce the generating function for the matrix \((j!)^{-1} S_{(\bar{k})}(\Xi_j)\) (cf. (2.2))

\[
\Xi^{(j)}(x, u) = \frac{1}{j!} u_{n_1} \cdots u_{n_j} (S_{(\bar{k})}(\Xi_j))^{n_1 \cdots n_j}_{r_1 \cdots r_j} x^{r_1} \cdots x^{r_j},
\]

where representation \(S_{(\bar{k})}\) is defined in (4.4), (4.9).

\textbf{Proposition 9} For the generation function (5.34) the following recurrence relation holds (cf. (2.7))

\[
\Xi^{(j)}(x, u) = \frac{1}{(j-1)!} \left( \Theta^{(j)}_{(u)} - \frac{1}{(\omega + 2(j-2))} \Theta^{(x)}_{(x)}(u_k \partial_{x_k}) \right) (\Theta^{(x)}_{(\partial_x)})^{j-1} \Xi^{(j-1)}(z, u),
\]

where \(\partial_{x_k} = \frac{\partial}{\partial x_k}\), function \(\Theta^{(x)}_{(u)}\) is defined in (2.2) and \(\omega = (D - 1)\).

\textbf{Proof.} Formula (5.35) is a consequence of the last equality in (5.18). Introduce notation

\[
\tau_i := \sigma_i - \frac{2}{(\omega + 2(i-1))} \kappa_i \quad \Rightarrow \quad \hat{R}_i(-i) = \left( \frac{1}{i} + \tau_i \right).
\]

Then one can write the last equality in (5.18) in the form

\[
\Xi_j = \Pi j-1 \Pi j-1,
\]

where we have taken into account conditions (5.24) and introduce 1-shuffle element (about shuffle elements see [20] and [18] and references therein)

\[
\Pi j-1 = \left(1 + \tau_{j-1} + \tau_{j-2} \tau_{j-1} + \cdots + \tau_{j-k} \cdots \tau_{j-2} \tau_{j-1} + \cdots + \tau_{1} \tau_{2} \cdots \tau_{j-1} \right).
\]
Now, in the representation $S(\tilde{\tau})$, identity (5.36) is written for the generating function (5.34) as
\[
\Xi(x) = \frac{1}{x!} x^{\alpha_1} \cdots x^{\alpha_j} \left( \prod_{j=1}^{n} \Theta_{\alpha_{j-1}} \cdots \Theta_{\alpha_1} \right) \frac{\partial}{\partial x_{j-1}} \cdots \frac{\partial}{\partial x_1} \Xi(1) \Theta(t, u). \tag{5.38}
\]
Here we defined element
\[
\prod_{j=1}^{n} \left( \tilde{\tau}_{j-1} + \tilde{\tau}_{j-1} + \cdots + \tilde{\tau}_{j-k} + \cdots \tilde{\tau}_{j-2} + \cdots + \tilde{\tau}_{j-1} \right). \tag{5.39}
\]
We now show that the formula (5.43) is equivalent to (5.42). We carry out the proof by two simple terms
\[
\left( \tilde{\tau}_{j-1} \right)_{n_1 \cdots n_j} = \left( \Theta_{n_1} \cdots \Theta_{n_{j-1}} \Theta_{n_j} \right)_{\ldots} - \left( \frac{2}{\omega + 2(j-2)} \right) \Theta_{n_1} \cdots \Theta_{n_{j-1}} \Theta_{n_j} \tag{5.40}
\]
where $\omega = D - 1$. Let us simplify the formula (5.38). Note, we will need it later, that the following formula is hold
\[
\prod_{j=1}^{n} \left( \tilde{\tau}_{j-1} \right) = \left( \tilde{\tau}_{j-1} \right). \tag{5.41}
\]
Consider the differential operator
\[
x^{\alpha_1} \cdots x^{\alpha_j} \left( \prod_{j=1}^{n} \Theta_{\alpha_{j-1}} \cdots \Theta_{\alpha_1} \right) \frac{\partial}{\partial x_{j-1}} \cdots \frac{\partial}{\partial x_1} \tag{5.42}
\]
from the right hand-side (5.38). Next we show, that (5.42) reduces to the following sum of two simple terms
\[
j \cdot \Theta^{(x)}(\partial_{\alpha_1}) (\Theta^{(x)}(\alpha_{j-1}))^{j-1} \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_{j-1}} \Theta^{(x)}(\partial_{\alpha_1}) (\Theta^{(x)}(\alpha_{j-1}))^{j-2} \tag{5.43}
\]
We now show that the formula (5.43) is equivalent to (5.42). We carry out the proof by induction on $j$. For $j = 2$ the formula (5.39) is represented as
\[
\prod_{j=1}^{n} \Theta^2_{n_1} \Theta^2_{n_2} = \Theta^2_{n_1} \Theta^2_{n_2} \Theta^2_{n_3} - \frac{2}{\omega} \Theta_{n_1 n_2} \Theta^2_{n_3} \tag{5.44}
\]
We make a contraction similar to (5.42) for $j = 2$, using the formula (5.44) and the definition of the generating function $\Theta^{(x)}(\partial_{\alpha_1})$. As a result we have
\[
x^{\alpha_1} x^{\alpha_2} \left( \prod_{j=1}^{n} \Theta^2_{n_1} \Theta^2_{n_2} \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_{j-1}} \right) = 2 \cdot \Theta^{(x)}(\partial_{\alpha_1}) (\Theta^{(x)}(\partial_{\alpha_1}))^{j-1} \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_{j-1}} \Theta^{(x)}(\partial_{\alpha_1}) \tag{5.45}
\]
it follows that the formulas (5.43) and (5.42) is equivalent for $j = 2$. Now we consider the sequence of the transformations for the (5.42) in case any $j$
\[
x^{\alpha_1} \cdots x^{\alpha_j} \left( \prod_{j=1}^{n} \Theta^j_{n_1} \cdots \Theta^j_{n_{j-1}} \frac{\partial}{\partial x_{j-1}} \cdots \frac{\partial}{\partial x_1} \right) = x^{\alpha_1} \cdots x^{\alpha_j} \left( \Theta^j_{n_1} \cdots \Theta^j_{n_{j-1}} + \left( \prod_{j=1}^{n} \Theta^j_{n_1} \cdots \Theta^j_{n_{j-1}} \frac{\partial}{\partial x_{j-1}} \cdots \frac{\partial}{\partial x_1} \right) \right) \tag{5.46}
\]
where the operator $\tilde{\tau}_{j-1}$ really plays the role of a unit, since it trivially acts on all covariant combinations constructed from $\Theta = |\Theta_m^n|$ and space-time metric $\eta = |\eta_{nm}|$.

\[\text{Here the operator $\tilde{\tau}_{j-1}$ really plays the role of a unit, since it trivially acts on all covariant combinations constructed from $\Theta = |\Theta_m^n|$ and space-time metric $\eta = |\eta_{nm}|$.}\]
here in the first equality we used the recurrence relation (5.41), in the second equality we used the induction hypothesis (equivalence of formulas (5.43) and (5.42) for \( \Pi_{j-2} \)), in the third equation we applied differentiation with respect to the auxiliary \( D \)-vector \( c \). Based on (5.46), we can conclude that the formulas (5.42) and (5.43) are equivalent.

Next, substituting (5.43) in (5.38), and then taking the derivative with respect to the variable \( t \), we get

\[
\Xi^{(j)}(x, u) = \frac{1}{(j-1)!} \left( \left( \Theta^{(x)} (\Theta^{(x)} / \partial_{x_i}) \right)^{j-1} - \frac{(j-1)}{(\omega+2(j-2))} \Theta^{(x)} (\Theta^{(x)} / \partial_{x_i}) \left( \Theta^{(x)} / \partial_{x_i} \right)^{j-2} \right) \Xi^{(j-1)}(z, u). \tag{5.47}
\]

Now, taking the expression \( (\Theta^{(x)} / \partial_{x_i})^{j-1} \) out of the bracket of the right side (5.47), we reach the formula (5.33).

6 Explicit examples of spin projectors related to Young diagrams with \( m > 1 \) rows.

In this section, as an example, we give explicit construction of two idempotents of the Brauer algebra \( Br_j \). In representation \( S_{\bar{c}} \) (see Section 4) these idempotens correspond to the projectors of the Behrends-Fronsdal type with symmetries, which related of the \( m \)-row Young diagrams.

Example 1. At the end of Section 3 we gave explicit formula (3.34) for symmetrizer \( E_{T_{(\bar{1}, \bar{1})}} \). Here we construct idempotent \( E_{T_{(\bar{1}, \bar{1})}} \), which corresponds to antisymmetrizer, i.e. it corresponds to Young diagram \( \lambda = [1^j] \) consisting of one column of height \( j \). According to the general formula (3.29), we have the identity for the antisymmetrizer \( E_{T_{(\bar{1}, \bar{1})}} \) and element \( y_j \)

\[
E_{T_{(\bar{1}, \bar{1})}} \cdot (y_j - 1)(y_j + j - 1)(y_j + \omega - j + 1) = 0. \tag{6.48}
\]

Using this identity and formula (3.30), we obtain the recurrence relation for the antisymmetrizer \( E_{T_{(\bar{1}, \bar{1})}} \)

\[
E_{T_{(\bar{1}, \bar{1})}} = \frac{E_{T_{(\bar{1}, \bar{1})}} \cdot (1 - y_j)(y_j + \omega - j + 1)}{j(-2j + \omega + 2)}. \tag{6.49}
\]

Solution of the recurrence relation (6.49) with the initial condition \( E_{T_{(\bar{1}, \bar{1})}} = 1 \) is

\[
E_{T_{(\bar{1}, \bar{1})}} = \frac{(1 - y_2)(1 - y_3) \cdots (1 - y_j)(y_j + \omega - 1)(y_3 + \omega - 2) \cdots (y_j + \omega - j + 1)}{j!(\omega - 2)(\omega - 4) \cdots (\omega - 2(j - 1))}. \tag{6.50}
\]

Note, that for the antisymmetrizer \( E_{T_{(\bar{1}, \bar{1})}} \in Br_j \), one can write simpler formula than (6.50). To derive this simple formula we write conditions for \( E_{T_{(\bar{1}, \bar{1})}} \), which are analogs of (3.35):

\[
\sigma_r \cdot E_{T_{(\bar{1}, \bar{1})}} = -E_{T_{(\bar{1}, \bar{1})}} = E_{T_{(\bar{1}, \bar{1})}} \cdot \sigma_r, \tag{6.51}
\]

\[
\kappa_r \cdot E_{T_{(\bar{1}, \bar{1})}} = 0 = E_{T_{(\bar{1}, \bar{1})}} \cdot \kappa_r, \quad \forall r = 1, \ldots, j - 1. \tag{6.52}
\]

It is clear, that relations (6.52) follows from relations (6.51). It is seen from the chain of equalities

\[
\kappa_r \cdot E_{T_{(\bar{1}, \bar{1})}} = \kappa_r \cdot \sigma_r \cdot E_{T_{(\bar{1}, \bar{1})}} = -\kappa_r \cdot E_{T_{(\bar{1}, \bar{1})}} \Rightarrow \kappa_r \cdot E_{T_{(\bar{1}, \bar{1})}} = 0, \tag{6.53}
\]

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where we use relation $\kappa_r = \kappa_r \sigma_r$ and conditions (6.51). The second equality have the analogous proof. Therefore conditions (6.51) completely determine the antisymmetrizer $E_{T_{(1[j]j)}}$. Conditions (6.51) coincide to the conditions for the antisymmetrizer in the group algebra of the permutation group $C[S_j]$. It means that the expression for $E_{T_{(1[j]j)}}$ does not include the elements $\kappa_i \in Br_j$ and have the form which is specific for $C[S_j]$ (see for example [3])

$$E_{T_{(1[j]j)}} = \frac{1}{j!} \left( 1 - \sigma_{1,2} \right) \left( 1 - \sigma_{1,3} - \sigma_{2,3} \right) \cdots \left( 1 - \sigma_{1,j} - \cdots - \sigma_{j-1,j} \right). \quad (6.54)$$

We expand the brackets in the right hand side of eq. (6.54), and take it in the representation \(S_{(k)}\). As a result we obtain the explicit expression for completely antisymmetric $D$-dimensional spin projector (antisymmetric analog of the Behrends-Fronsdal projector)

$$\left( S_{(k)}(E_{T_{(1[j]j)}}) \right)_{d_1 \ldots d_j}^{n_1 \ldots n_j} = \frac{1}{j!} \sum_{\sigma \in S_j} (-1)^{p(\sigma)} \Theta_{n_{\sigma(1)}}^{d_1} \Theta_{n_{\sigma(2)}}^{d_2} \cdots \Theta_{n_{\sigma(j)}}^{d_j}, \quad (6.55)$$

where the sum runs over all elements $\sigma$ of the permutation group $S_j$ and $p(\sigma)$ is the parity of $\sigma$. Note that antisymmetric projector (6.55) is (by construction) orthogonal to the symmetric Behrends-Fronsdal projector $2.3$.

**Example 2.** Consider a second nontrivial example of the Brauer algebra idempotent (and its image in the representation $S_{(k)}$), which corresponds of the Young diagram $\lambda = [\lambda_1, \ldots, \lambda_m]$ with number of lines $m > 1$. For this we consider two oscillating Young tableaux $\Lambda_1$ и $\Lambda_2$ (see Section 3)

$$\Lambda_1 = \{ 0 \overset{a_1=0}{\rightarrow} [1] \overset{a_2=1}{\rightarrow} [2] \overset{a_3=1'}{\rightarrow} [2,1] \}, \quad \Lambda_2 = \{ 0 \overset{a_1=0}{\rightarrow} [1] \overset{a_2=1'}{\rightarrow} [1,1] \overset{a_3=1}{\rightarrow} [2,1] \}. \quad (6.56)$$

Both tableaux have the length 3 and in both cases the final Young diagram is a hook

$$[2,1] = \begin{array}{c} 0 \ \ 1 \\ -1 \end{array}. \quad (6.56)$$

Let $e_s$ and $e_a$ are primitive idempotents, corresponding by the diagrams $[2]$ and $[1^2]$ respectively, and having the following explicit expressions

$$e_s = \frac{(1 + y_2)(y_2 + \omega - 1)}{2\omega}, \quad e_a = \frac{(1 - y_2)(y_2 + \omega - 1)}{2(\omega - 2)}, \quad (6.57)$$

which can be easily obtained from the general formula (3.30). By using the branching rules (see Fig. 2) and oscillating Young tableaux $\Lambda_1$ и $\Lambda_2$, given in (6.56), we derive two identities (3.29) for the elements $e_s$, $e_a$ и $y_3$

$$e_s \cdot (y_3 - 2)(y_3 + 1)(y_3 + \omega) = 0, \quad e_a \cdot (y_3 + 2)(y_3 - 1)(y_3 + \omega - 2) = 0. \quad (6.58)$$

By means of these identities we construct (see formulas (3.30) and (3.31)) two primitive idempotents

$$e_{\Lambda_1} = \frac{e_s \cdot (2 - y_3)(y_3 + \omega)}{3(\omega - 1)}, \quad e_{\Lambda_2} = \frac{e_a \cdot (2 + y_3)(y_3 + \omega - 2)}{3(\omega - 1)}, \quad (6.59)$$

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for which (by construction) we have \( e^2_{\Lambda_\alpha} = e_{\Lambda_\alpha} \) and \( e_{\Lambda_1} \cdot e_{\Lambda_2} = 0 \). Opening the brackets in formulas (6.60), we obtain

\[
e_{\Lambda_1} = \frac{1}{6(\omega-1)} \left( 2\omega(\omega-1) + 2\omega^2y_2 + (\omega(3-\omega) - 2)y_3 + \omega(2-\omega)y_2y_3 + 
+ 2\omega y_2^2 + (1-\omega)y_2^2 - \omega y_2y_3 - y_2^2y_3^2 \right),
\]

\[
e_{\Lambda_2} = \frac{1}{6(\omega-1)} \left( 2(\omega(\omega-3) + 2) + 2(\omega(1-\omega) - 4)y_2 + \omega(\omega-1)y_3 + \omega(2-\omega)y_2y_3 + 
+ 2(2-\omega)y_2^2 + (\omega-1)y_2y_3 + (2-\omega)y_2y_3^2 - y_2^2y_3^2 \right).
\]

(6.61)

Taking into account identity \( \kappa_1y_3 = 0 \) and definition (3.13) for \( y_2 \), one can rewrite relations (6.60), (6.61) in concise form

\[
e_{\Lambda_1} = \frac{1}{6(\omega-1)} \left( (1 + \sigma_1)(2\omega - (\omega-2)y_3 - y_2^2) - 4\kappa_1 \right),
\]

(6.62)

\[
e_{\Lambda_2} = \frac{1}{6(\omega-1)} \left( (1 - \sigma_1)(2(\omega - 2) + \omega y_3 + y_2^2) \right).
\]

(6.63)

Now we use definition (3.13) for \( y_3 \) and express idempotents (6.62) and (6.63) in terms of generators \( \sigma_i, \kappa_i \in B_{r_3} \): (6.64)

\[
e_{\Lambda_1} = \frac{1}{6} \left( 2 - (\sigma_1\sigma_2 + \sigma_2\sigma_1) - \sigma_1\sigma_2\sigma_1 + 2\sigma_1 - \sigma_2 + \frac{1}{(\omega-1)}(2(\kappa_1\kappa_2 + \kappa_2\kappa_1)) + 
+ 2(\kappa_1\sigma_2 + \sigma_2\kappa_1) - (\kappa_2\sigma_1 + \sigma_1\kappa_2) - 4\kappa_1 - \kappa_2 - \sigma_1\kappa_2\sigma_1 \right),
\]

\[
e_{\Lambda_2} = \frac{1}{6} \left( 2 - (\sigma_1\sigma_2 + \sigma_2\sigma_1) + \sigma_1\sigma_2\sigma_1 - 2\sigma_1 + \sigma_2 + \frac{1}{(\omega-1)}(3(\kappa_2\sigma_1 + \sigma_1\kappa_2) - 
- 3\kappa_2 - 3\sigma_1\kappa_2\sigma_1) \right).
\]

(6.65)

The images of idempotents \( e_{\Lambda_1} \) in representation \( S(\vec{\kappa}) \) have the form

\[
(S(\vec{\kappa})e_{\Lambda_1})_{n_1n_2n_3} = \frac{1}{6} \left( 2\Theta_{n_1}^{d_1} \Theta_{n_2}^{d_2} \Theta_{n_3}^{d_3} - (\Theta_{n_2}^{d_1} \Theta_{n_3}^{d_2} \Theta_{n_1}^{d_3} + \Theta_{n_3}^{d_1} \Theta_{n_1}^{d_2} \Theta_{n_2}^{d_3}) - 
- \Theta_{n_2}^{d_1} \Theta_{n_3}^{d_2} \Theta_{n_1}^{d_3} + 2\Theta_{n_2}^{d_1} \Theta_{n_1}^{d_2} \Theta_{n_3}^{d_3} - \Theta_{n_1}^{d_2} \Theta_{n_2}^{d_3} \Theta_{n_3}^{d_1} - 
- \frac{1}{(\omega-1)} \cdot \left( 4\Theta_{n_3}^{d_1} \Theta_{n_1}^{d_2} \Theta_{n_2}^{d_3} + \Theta_{n_1}^{d_1} \Theta_{n_2}^{d_2} \Theta_{n_2}^{d_3} + \Theta_{n_2}^{d_2} \Theta_{n_1}^{d_1} \Theta_{n_3}^{d_3} - 
- 2(\Theta_{n_3}^{d_1} \Theta_{n_1}^{d_2} \Theta_{n_2}^{d_3} + \Theta_{n_1}^{d_1} \Theta_{n_2}^{d_2} \Theta_{n_2}^{d_3}) - 2(\Theta_{n_3}^{d_1} \Theta_{n_1}^{d_2} \Theta_{n_1}^{d_3} + \Theta_{n_2}^{d_1} \Theta_{n_2}^{d_1} \Theta_{n_2}^{d_3} \Theta_{n_1}^{d_3}) + 
+ \Theta_{n_1}^{d_2} \Theta_{n_2}^{d_1} \Theta_{n_2}^{d_3} + \Theta_{n_2}^{d_1} \Theta_{n_2}^{d_2} \Theta_{n_2}^{d_3} \Theta_{n_1}^{d_3} \right) \right),
\]

\[
(S(\vec{\kappa})e_{\Lambda_2})_{n_1n_2n_3} = \frac{1}{6} \left( 2\Theta_{n_1}^{d_1} \Theta_{n_2}^{d_2} \Theta_{n_3}^{d_3} - (\Theta_{n_2}^{d_1} \Theta_{n_3}^{d_2} \Theta_{n_1}^{d_3} + \Theta_{n_3}^{d_1} \Theta_{n_1}^{d_2} \Theta_{n_2}^{d_3}) + 
+ \Theta_{n_2}^{d_1} \Theta_{n_3}^{d_2} \Theta_{n_1}^{d_3} - 2\Theta_{n_2}^{d_1} \Theta_{n_1}^{d_2} \Theta_{n_3}^{d_3} + \Theta_{n_1}^{d_2} \Theta_{n_2}^{d_3} \Theta_{n_3}^{d_1} - 
- \frac{3}{(\omega-1)} \cdot \left( \Theta_{n_1}^{d_1} \Theta_{n_2}^{d_3} \Theta_{n_2}^{d_3} + \Theta_{n_2}^{d_2} \Theta_{n_1}^{d_3} \Theta_{n_2}^{d_3} - \Theta_{n_1}^{d_1} \Theta_{n_2}^{d_3} \Theta_{n_2}^{d_3} - \Theta_{n_2}^{d_2} \Theta_{n_1}^{d_3} \Theta_{n_2}^{d_3} \Theta_{n_1}^{d_3} \right) \right).
\]

(6.66)

(6.67)
It is clear that operators $S_{\{k\}}(e_{A_1})$ and $S_{\{k\}}(e_{A_2})$ given in (6.63), (6.67) are traceless and transverse (see conditions 3 and 4 in Definition 1, Section 2). It means that these operators are projectors onto two irreducible representations of the group $ISO(1, D - 1)$ acting in the space of 3-rank tensors. These two representations are equivalent to each other.

Define in the Brauer algebra $Br_3$ the central idempotent

$$e_{[2,1]} = \left( e_{A_1} + e_{A_2} \right), \quad (6.68)$$

associated to the Young diagram [2, 1]. The element $e_{[2,1]}$ satisfies relation $e_{[2,1]}^2 = e_{[2,1]}$ and belongs to the center because it is a symmetric function of elements $y_2, y_3$ (it means that $e_{[2,1]}$ commutes with all elements of the Brauer algebra). To prove the last statement we substitute expressions (6.60), (6.61) for idempotents $e_{A_1}$ and $e_{A_2}$ into the right hand side of the formula (6.68). Then we use the definition (3.18) of the elements $y_2, y_3$ and apply relation $\kappa_1 y_3 = 0$. As the result we obtain explicit form of $e_{[2,1]}$ in terms of Jucys-Murphy elements $y_2$ and $y_3$ as following

$$e_{[2,1]} = \frac{1}{3} \left( 2 - y_2 y_3 + \frac{1}{(w-1)} \left( 2(y_2 + y_3) - (y_2 y_3 + y_3 y_2) \right) \right). \quad (6.69)$$

Thus, the element $e_{[2,1]}$ is indeed a symmetric function of the elements $y_2$ and $y_3$. In terms of the generators $\sigma_i, \kappa_i \in Br_3$ ($i = 1, 2$), expression (6.69) is written in the form

$$e_{[2,1]} = \frac{1}{3} \left( 2 - (\sigma_1 \sigma_2 + \sigma_2 \sigma_1) + \frac{1}{(w-1)} \left( (\kappa_1 \kappa_2 + \kappa_2 \kappa_1) + (\kappa_1 \sigma_2 + \sigma_2 \kappa_1) + (\kappa_2 \sigma_1 + \sigma_1 \kappa_2) - 2 (\kappa_1 + \kappa_2 + \sigma_1 \sigma_2) \right) \right). \quad (6.70)$$

Note, that formula (6.70) is mirror-symmetric, or in another words it is invariant under simultaneous substitution: $\sigma_1 \leftrightarrow \sigma_2$ and $\kappa_1 \leftrightarrow \kappa_2$. Finally, the image of the central idempotent $e_{[2,1]}$ in the representation $S_{\{k\}}$ has the form

$$\left( S_{\{k\}}(e_{[2,1]}) \right)^{d_1 d_2 d_3}_{n_1 n_2 n_3} = \frac{1}{3} \left( 2 \Theta_{n_1 n_2 n_3}^d \Theta_{n_1 n_2 n_3}^d - \Theta_{n_2 n_3 n_1}^d \Theta_{n_2 n_3 n_1}^d + \Theta_{n_3 n_1 n_2}^d \Theta_{n_3 n_1 n_2}^d - \Theta_{n_1 n_2 n_3}^d \Theta_{n_1 n_2 n_3}^d - \Theta_{n_1 n_2 n_3}^d \Theta_{n_1 n_2 n_3}^3 - \Theta_{n_1 n_2 n_3}^d \Theta_{n_1 n_2 n_3}^d \right) -$$

$$- \frac{1}{(w-1)} \cdot \left( 2 \cdot \left( \Theta_{n_1 n_2 n_3}^d \Theta_{n_1 n_2 n_3}^d + \Theta_{n_1 n_2 n_3}^d \Theta_{n_2 n_3 n_1}^d + \Theta_{n_2 n_3 n_1}^d \Theta_{n_2 n_3 n_1}^d \right) -$$

$$- \Theta_{n_1 n_2 n_3}^d \Theta_{n_1 n_2 n_3}^d \Theta_{n_1 n_2 n_3}^d \Theta_{n_1 n_2 n_3}^d - \Theta_{n_2 n_3 n_1}^d \Theta_{n_2 n_3 n_1}^d \Theta_{n_2 n_3 n_1}^d \Theta_{n_2 n_3 n_1}^d \right). \quad (6.71)$$

From the right hand side of (6.71) we see that $S_{\{k\}}(e_{[2,1]})$ is transversal and traceless with respect to all pairs of lower and upper tensor indices (Properties 3 and 4 in Definition 1, Section 2), i.e. the following equalities hold

$$k_{\sigma_1}^{n_1} \left( S_{\{k\}}(e_{[2,1]}) \right)^{d_1 d_2 d_3}_{n_1 n_2 n_3} = 0 = k_{\sigma_2}^{n_2} \left( S_{\{k\}}(e_{[2,1]}) \right)^{d_1 d_2 d_3}_{n_1 n_2 n_3} \quad (6.72)$$

$$\eta^{n_1 n_2} \left( S_{\{k\}}(e_{[2,1]}) \right)^{d_1 d_2 d_3}_{n_1 n_2 n_3} = 0 = \eta^{n_1 n_2} \left( S_{\{k\}}(e_{[2,1]}) \right)^{d_1 d_2 d_3}_{n_1 n_2 n_3} ,$$

where $\sigma$ is an arbitrary permutation from the group $S_3$. 

21
7 Conclusion.

In this paper the new class of representations of the Brauer algebra is found. This allows us to apply the method of constructing irreducible finite dimensional representations of orthogonal and symplectic Lie groups (based on using of the idempotents of the Brauer algebra), to the construction of irreducible representations of $D$-dimensional Poincaré group. Using new representations of the Brauer algebra we derive a new recurrence formula for the $D$-dimensional completely symmetric BF projector. In particular, we derive new explicit formulae for $D$-dimensional BF type projectors related to any symmetries which correspond to the Young diagrams with two and more rows (in contrast to fully symmetric BF projectors which correspond to the single-row Young diagram). To illustrate obtained results we find images of some special idempotents of Brauer algebra in new representations.

We hope, that the generalizations of the BF projectors obtained in this paper, will have useful applications. For example, generalized BF projectors could be useful for constructing and investigating of different higher spin field theories. In particularly, we know (see e.g. [22]), that the squares of BF spin projectors are used as building blocks of invariant constructions included in Lagrangians of higher spin field theories.

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