Open Topological String Amplitudes on Calabi-Yau Threefolds by Extended Holomorphic Anomaly Equation

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Abstract

In this paper, we study the open topological string amplitudes on Calabi-Yau threefolds by the extended holomorphic anomaly equation. The disk two-point function determined by the domainwall tension, together with the Yukawa couplings, solves the amplitudes with high genus and boundaries recursively. The BPS invariants encoded in the amplitudes are extracted by mirror symmetry.

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1 Introduction

In open topological string theory, the BPS domainwall tensions, as the super-potential difference on two vacua\cite{46}, are generating functions of disk amplitudes at tree-level, which are given by the dimensional reduction of the holomorphic Chern-Simons functional\cite{47} and satisfy the inhomogeneous Picad-Fuchs equations\cite{6,21,45}. Based on the assumption of the absence of open string moduli and disk one-point function, the extended holomorphic anomaly equation is conjectured in\cite{44}, as analogy of the BCOV holomorphic anomaly equation\cite{12}, to calculate open topological string amplitudes recursively. The partition functions $F^{g,h}$ with $g$ genus and $h$ boundaries can be expressed in terms of the lower genus and boundaries partition functions. The open topological string amplitudes in terms of the Feymann rule are formulated and proved by shifting closed string variables\cite{15}, which is interpreted as boundary-condition dependent states in the Hilbert space\cite{38}.

The open mirror symmetry relates A-model topological string on a Calabi-Yau threefold with as A-branes to the B-model topological string on the mirror Calabi-Yau threefold with holomorphic submanifolds as B-branes\cite{30,40}. It leads to a powerful technique to understand A-model amplitudes, which have important implications on physics and mathematics. One the one hand, the open topological string partition functions underlay the so-called open Gromov-Witten invariants\cite{17,20}, which count the holomorphic maps from Riemann surfaces with boundaries to the Calabi-Yau threefolds, with boundaries mapped to the Lagrangian submanifolds\cite{26,34}. A mathematical definition of the open Gromov-Witten invariants is provided in\cite{35} as formal relative Gromov-Witten invariants of a relative formal toric Calabi-Yau threefold, relating to topological vertex\cite{3}.

On the other hand, the open topological string partition functions also counts the numbers of BPS state in M-theory compactified to five dimension when the un-oriented worldsheet contributions are included to obtain a consistent decoupling of A- and B-model at all genera\cite{29,31,32,42,43}. In addition, the topological string partition function on a Calabi-Yau threefold is related to the partition function of a four-dimensional BPS black hole constructed by compactifying type II superstrings on the same Calabi-Yau by the Ooguri, Strominger, and Vafa (OSV) conjecture\cite{39,41}, i.e. $Z_{BH} = |\psi_{top}|^2$, which has been tested on non-compact toric Calabi-Yau in\cite{1,16}, and given several general proof in\cite{10,13,19}. The open version conjecture relates the open topological string partition function with the black hole partition function that sums over the black hole bound states to BPS excition on D-branes wrapping cycles of the Calabi-Yau, i.e. $Z_{BPS}^{open} = |\psi_{top}^{open}|^2$\cite{2}.
In this paper, we study the open string partition functions on several Calabi-Yau threefolds. With domainwall tensions found by extremal transition [4], the disk amplitudes with two insertion are obtained as the covariant differentiation of the domainwall tension. Other amplitudes of low genus and boundary numbers are obtained by solving extended holomorphic anomaly equations. The organization of this paper is as follows: In Section 2, we review some background knowledge about open topological string amplitudes and the extended holomorphic anomaly equation. In Section 3, we review the extremal transition on the moduli space of Calabi-Yau threefolds and BPS domainwall tension on two threefolds related by extremal transition. In Section 4, we study the open topological string amplitudes on three compact complete intersection Calabi-Yau threefolds (CICY) ($X_{3,6}$, $X_{2,4}$, and $X_{2,6}$) and two non-compact threefolds ($X_4$ and $X_6$). In Section 5, there is a short summary and further discussion about this paper. Then, in Appendix A, we summarize the genus one BPS invariants on models of this work. In Section B and C, we list the amplitudes and invariants on another two models and omit certain details.

\section{The Extended Holomorphic Anomaly Equation}

The B-model on a Calabi–Yau threefold $X$ depends on the complex structures moduli space $M_{CS}(X)$ with local coordinates $z_i$, $i=1,...,h^{1,2}(X)$. The Weil-Petersson metric on $M_{CS}(X)$ is a Kahler metric,

$$G_{ij} = \partial_i \partial_j K,$$

with $K = - \log i \int_X \bar{\Omega} \wedge \Omega$ the Kahler potential. Taking the holomorphic limits, one obtains,

$$e^{-K} = \omega_0, \quad G_{zz} = 2\pi i \frac{dt}{dz},$$

Here $t = \frac{\omega_1}{\omega_0}$ is the special coordinate on $M_{CS}(X)$ that is defined as the ratio between the logarithmic period $\omega_1$ and the fundamental period $\omega_0$, and related to the mirror map that connects the complex structure moduli space $M_{CS}(X)$ of $X$ with the Kahler moduli $M_K(X^*)$ of the manifold $X^*$.

The topological string partition function $F^{g,h}$ of genus $g$ and boundaries $h$ are section of a line bundle $\mathcal{L}^{2g-h}$ over $M_{CS}(X)$. It is defined as the integration over worldsheet moduli $M^{g,h}$,

$$F^{g,h} = \int_{M^{g,h}} [dm][dl] \langle \prod_{a=1}^{3g-3+h} \mu_a G^- \int \mu_a \bar{G}^- \prod_{b=1}^{h} \int (\lambda_b G^- + \bar{\lambda} \bar{G}^-) \rangle,$$
where $\mu_a, a = 1, \ldots, 3g - 3$ are the Beltrami differentials associated with the moduli of the bulk, and $\mu_a, a = 3g - 3 + 1, \ldots, 3g - 3 + h$ and $\lambda_b, b = 1, \ldots, h$ are the Beltrami differentials associated to the positions and the length of the boundaries.

It is argue in [44] that the partition function is recursively related to the partition functions of lower genus and and less boundaries by the extended holomorphic anomaly equations. The torus amplitude and the annulus amplitude satisfies the equations,

\[
\partial F^{1,0}_{j} = \frac{1}{2}C_{jkl}C^{kl}_{i} + (1 - \frac{\chi}{24})G_{ji},
\]

\[
\partial F^{0,2}_{j} = -\Delta_{j} \Delta_{i} + \frac{N}{2}G_{ji},
\]

where $\chi$ is the Euler character of the manifold, and $N$ is the rank of a bundle over $M_{CS}(X)$. And for $2g - 2 + h > 0$, partition functions satisfies the equation,

\[
\partial F^{g,h}_{0} = \frac{1}{2}C^{jk}_{i} \sum_{g_1 + g_2 = g, h_1 + h_2 = h} D_{j}F^{g_1,h_1}_{g_2} + \frac{1}{2}C^{jk}_{i}D_{j}D_{k}F^{g-1,h} - \Delta_{i} \Delta_{j} + f^{0,2}_{j}.
\]

The low genus and boundaries partition functions are solved by direct integral,

\[
F^{1,0}_{i} = \frac{1}{2}C_{ijk}S^{jk} + (1 - \frac{\chi}{24})K_{i} + f^{1,0}_{i},
\]

\[
F^{1,1} = \frac{1}{2}S^{jk} \Delta_{j} - F^{1,0}_{j} \Delta_{i} + \frac{1}{2}C_{jkl}S^{kl} \Delta_{i} - (\frac{\chi}{24} - 1) \Delta + f^{1,1},
\]

\[
F^{0,2}_{i} = \frac{1}{2}C_{ijk} \Delta_{i} \Delta_{k} + \frac{N}{2}K_{i} + f^{0,2}_{i},
\]

\[
F^{0,3} = -F^{0,2}_{j} \Delta_{i} + \frac{N}{2} \Delta - \frac{1}{2} \Delta_{j} \Delta_{i} \Delta_{k} - \frac{1}{6}C_{jkl} \Delta_{i} \Delta_{k} \Delta_{l} + f^{0,3},
\]

where $f^{1,0}, f^{1,1}, f^{0,2}$ and $f^{0,3}$ are holomorphic ambiguities. And on one-parameter models, they can be written as,

\[
F^{1,0} = \frac{1}{2} \log \left[ \left( \frac{q}{z} \frac{dz}{dq} \right) (\omega_{0})^{\frac{q}{z} - 4\pi - \frac{\omega}{z} \text{diss} - \frac{q}{z}} \right],
\]

\[
F^{1,1} = -F^{1,0}_{j} \Delta_{i} - (\frac{\chi}{24} - 1) \Delta + f^{1,1},
\]

5
\[
F^0_{z} = \frac{1}{2} C_{ijk} A^{2}_{zz} + f^0_{z},
\]

\[
F^0_{z} = -F^0_{z} A^{2} - \frac{1}{3} A_{zz} A^{2} A^{z} + f^0_{z},
\]

where \( c_2 \) is the second Chern class and diss is the discriminant of the models.

These formulas can be interpreted by Feymann rules. The two fundamental vertices are the Yukawa couplings \( C_{ijk} \) given by the covariant derivative of the prepotentials \( F_0 \),

\[
C_{ijk} = D_i D_j D_k F_0,
\]

and the disk amplitudes with two insertions \( \Delta_{ij} \) given by the covariant derivative of the domainwall tension,

\[
\Delta_{ij} = D_i D_j \mathcal{W} - C_{ijk} \epsilon^{K} G^{k \bar{k}} D_{\bar{k}} \mathcal{W},
\]

and in the holomorphic limits,

\[
\lim_{\bar{z} \to 0} \Delta_{ij} = \partial_i \partial_j \mathcal{W}. \quad (2.8)
\]

Furthermore, the propagators contain \( S, S^i, S^{ij} \) for closed string and \( \Delta^i, \Delta \) for open string, which are related to \( C_{ijk} \) and \( \Delta_{ij} \),

\[
\partial_i S^{ij} = C^{ij}_i, \quad \partial_i S^i = G_{ii} S_{ij}, \quad \partial_i S = G_{ii} S^i, \quad \partial_i \Delta^i = \Delta^i, \quad \partial_i \Delta = G_{ii} \Delta_i,
\]

In particular, for one-modulus models, the propagators are given by,

\[
S^{zz} = C_{zz}^{-1} \partial_z \log (G_z G^{zz} (ze^K)^2),
\]

\[
S^{z} = C_{zz}^{-1} \left[ (\partial_z \log (ze^K))^2 - D_z \partial_z \log (ze^K) \right],
\]

\[
S = \left[ S^{z} - \frac{1}{2} D_z S^{zz} - \frac{1}{2} (S^{zz})^2 C_{zz} \right] \partial_z \log (ze^K) + \frac{1}{2} D_z S^{z} + \frac{1}{2} S^{zz} S^z C_{zz}, \quad (2.9)
\]

\[
\Delta^{z} = -A_{zz} C_{zz}^{-1},
\]

\[
\Delta = D_z \Delta^{z}.
\]
In this paper, we also consider the unoriented worldsheet contribution. The Klein bottle partition function satisfies the holomorphic anomaly equation,

$$\partial_i \partial_j B = \frac{1}{2} C_{jk} C_{i}^{kl} - G_{ij},$$

It has a general solution with the form,

$$B = \frac{1}{2} \log(\det G_{ij}^{-1} e^{K(n-1)} |g|^2),$$

and it can be written as

$$B = -\frac{1}{2} \log \left[ \left( \frac{q}{z} d\bar{z} \frac{dz}{q} \right) \text{diss}^{-\frac{1}{4}} \right].$$

(2.10)
on a one-parameter model under the holomorphic limits. In addition, the partition function at $\chi = 1$ satisfies the equation,

$$\partial_i B^{1,1} = \frac{1}{2} C_{ij}^{P P k} \Delta_{jk} - \mathcal{K}_{ij} \Delta^i,$$

and can be expressed by

$$B^{1,1} = -\mathcal{K}_{\bar{z}} \Delta^z - \Delta + h_{1,1}^{1,1}$$

(2.11)

with $h$ the holomorphic ambiguity.

### 3 Extremal Transition and Domainwall Tensions

Given a Calabi-Yau threefold $X$ that admits a birational contraction to a singular threefold $Y$, $X \rightarrow Y$, if $Y$ can be deformed into a smooth Calabi-Yau threefold $Y'$, then the transition from $X$ to $Y'$ is called an extremal transition.

In the IIA description, the transition is realized by contracting exceptional divisors to the curve of $A_{N-1}$ singularities, and the singular threefold deforms to a smooth Calabi-Yau threefold, with Hodge number changes,

$$h^{1,1} \mapsto h^{1,1} - (N - 1), \quad h^{2,1} \mapsto h^{2,1} + (2g - 2) \left( \frac{N}{2} \right) - (N - 1).$$

In the frame of Batyrev-Borisov mirror construction [8, 36], given a reflexive polyhedron $P$, the associated family of Calabi-Yau hypersurfaces is defined by the equation,

$$f_{P} = \sum_{a \in P \cap M} c_{a} \mathcal{X}_{a},$$
embedded in a ambient toric variety \(V_P\). If the polyhedra \(Q\) is a subpolyhedron of \(Q\), then all monomials appearing in \(f_Q\) also appear in \(f_P\). The manifold \(X_Q \subset V_Q\) associated to \(Q\) can be regarded as being limits of the hypersurfaces \(X_P \subset V_P\) associated to \(P\), with singularities fully resolved by further triangulation.

The singularities on the complex structure moduli space of \(X_P\) come from certain coefficients in \(f_P = 0\) being zero, which can be desingularized by extremal contractions. An open set \(S\) in the complex structure moduli space of \(X_P\) is given by,

\[ S := \mathbb{C}^{P^rM} / \text{Aut}(V_P) \times \mathbb{C}^*. \]

There is a subsets of \(\hat{S} \subset S\),

\[ \hat{S} := \mathbb{C}^{Q^rM} / \text{Aut}(V_Q) \times \mathbb{C}^*. \]

where the Calabi-Yau degenerates as a singular space \(\bar{X}\). All coefficients \(c_a\) in \(f_P = 0\) with \(a \notin Q\) have been set to be zero. It is where the extremal transition occurs. Some explicit examples are studied in \([11, 14, 27, 28]\).

The domainwall tensions on Calabi-Yau threefolds are solutions of the inhomogeneous Picard-Fuchs equations obtained

\[ \mathcal{L}_{PF} W(z) = f(z), \]

from the Griffiths-Dwork reduction method \([18, 37, 42]\), and are also computed by subsystem integration \([4]\). It is observed that the domainwall tensions on some multiple-parameter Calabi-Yau hypersurfaces \(\bar{X}\) are related to the domainwall tensions on certain one-parameter Calabi-Yau complete intersections \(X\) when restricted to special locus in the complex structure moduli spaces by extremal transition, and the BPS invariants on the two threefolds are also related by

\[ n_i(X) = \sum_{j,k} n_{i,j,k}(\bar{X}). \]

### 4 Amplitudes and BPS Invariants on Calabi-Yau Threefolds

In this section, we study open topological string amplitudes and BPS invariants on Calabi-Yau threefolds. Some useful geometry information is listed in the following table \([9, 5, 33]\).
Table 1: Geometric Information of Calabi-Yau Threefolds

| $X$                          | $\chi$ | $c_2$ | diss                  |
|------------------------------|--------|-------|-----------------------|
| $X_{3,6} \subset \mathbb{P}^5_{(1,1,1,2,3)}$ | $-204$ | 52    | $1 - 2^4 3^6 z$       |
| $X_{2,4} \subset \mathbb{P}^5$              | $-176$ | 56    | $1 - 2^{10} z$        |
| $X_{2,6} \subset \mathbb{P}^5_{(1,1,1,1,3)}$ | $-256$ | 52    | $1 - 2^8 3^3 z$       |
| $X_{4} \subset \mathbb{P}^4_{(2,1,1,1,1)}$   | $-36$  | $-16$ | $1 + 2^6 z$           |
| $X_{6} \subset \mathbb{P}^5_{(3,2,1,1,1,-1)}$| $-60$  | $-14$ | $1 + 2^4 3^3 z$       |

4.1 $X_{3,6}$

The A-incarnation $X_{3,6}^*$ in the weighted projective space $\mathbb{P}^5_{(1,1,1,2,3)}$ is described by the five-dimensional polyhedron $\Delta^*$. There is only one internal integer point $v_0^* = (0,0,0,0,0)$ and six integer vertices, following six vertices,

$$ v_1^* = (-1, -1, -1, -2, -3), \quad v_2^* = (1, 0, 0, 0, 0), \quad v_3^* = (0, 1, 0, 0, 0), $$

$$ v_4^* = (0, 0, 1, 0, 0), \quad v_5^* = (0, 0, 0, 1, 0), \quad v_6^* = (0, 0, 0, 0, 1), $$

The linear relation $l$ among vertices corresponds to the maximal triangulation of $\Delta^*$,

$$ l = (-3, -6; 1, 1, 1, 2, 3), $$

such that $\sum_i l^i v_i^* = 0$.

The nef-partition, $E_1 = \{v_1^*, v_2^*, v_3^*\}, E_2 = \{v_4^*, v_5^*, v_6^*\}$ determines the mirror geometry $X_{(2,6)}$ that can be written as the complete intersection of the vanishing locus of the following Laurent polynomials with torus coordinates $X_i, i = 2, \ldots, 6$ [24, 25],

$$ P_1 = a_{1,0} - a_1 (X_2 X_3 X_4 X_5^2 X_6^3)^{-1} - a_2 X_2, $$

$$ P_2 = a_{2,0} - a_3 X_3 - a_4 X_4 - a_5 X_5 - a_6 X_6, $$

The period integrals of $X_{3,6}$ defined as

$$ \sigma(a) = \int \frac{a_{1,0} a_{2,0}}{P_1 P_2} \prod_{i=2}^6 \frac{dX_i}{X_i}, $$
are annihilated by the GKZ operator associated to \( I \),
\[
\mathcal{L} = \frac{4}{\prod_{i=1}^{4} \frac{\partial}{\partial a_i \partial a_5}} \frac{2}{\prod_{i=1}^{4} \frac{\partial}{\partial a_6}} \left( \frac{\partial}{\partial a_{1,0}} \right)^3 \left( \frac{\partial}{\partial a_{2,0}} \right)^6,
\]
or
\[
\mathcal{L} = \theta^4 \prod_{i=0}^{1} (2\theta - i) \prod_{j=0}^{2} (3\theta - j) - z \prod_{m=1}^{3} \prod_{n=1}^{6} (3\theta + m)(6\theta + n),
\]
in terms of the logarithmic derivatives, \( \theta = \frac{z \bar{z}}{dz} \), and the coordinate on \( M_{CS}(X_{3,6}) \),
\[
z = \frac{a_1 a_2 a_3 a_4 a_5 a_6^2}{a_1 a_2 a_3 a_4 a_5 a_6}.
\]
Above equation can be reduced to the Picard-Fuchs equation,
\[
\mathcal{L}_{PF} = \theta^4 - 3^2 z (6\theta + 1)(6\theta + 2)(6\theta + 4)(6\theta + 5)
\]
and solves the mirror map from \( M_{CS}(X_{3,6}) \) to \( M_{K}(X_{3,6}^*) \),
\[
z = q - 2772q^2 + 1980126q^3 - 4010268048q^4 - 8360302475q^5 + \ldots.
\]

The domainwall tension on \( X \) satisfies the inhomogeneous Picard-Fuchs equation,
\[
\mathcal{L}_{PF} W = \frac{3}{(2\pi i)^2} z^{1/2},
\]
which is found by extremal transition from the hypersurface \( X_{18} \subset \mathbb{P}^4_{(1,2,3,3,9)} \) at the point \( t_2 = t_3 = 0 \) of SU(3) gauge enhancement [4]. After inserting the mirror map, \( \mathcal{F}(q) \) is obtained,
\[
W(q) = 96q^{1/2} + \frac{70592}{3} q^{3/2} + \frac{1432848096}{25} q^{5/2} + \frac{959191722592}{49} q^{7/2} + \ldots,
\]

The disk amplitude \( \Delta_{zz} \) is related to the domainwall tension by the equation
\[
- i\Delta_{zz} = 24q^{1/2} + 52944q^{3/2} + 358212024q^{5/2} + 2397979305648q^{7/2} + \ldots.
\]
and the Yukawa coupling is given by,
\[
C_{zzz} = 1 + 2628q + 16078500q^2 + 107103757608q^3 + \ldots.
\]

Then, inserting \( \Delta_{zz} \) and \( C_{zzz} \) into equation 2.5, the amplitude \( F_{\bar{z}}^{(0,2)} \) with one insertion is obtained,
\[
F_{\bar{z}}^{(0,2)} = -288q - 513792q^2 - 4017768768q^3 - 26851097548800q^4 + \ldots,
\]
and the partition function with zero genus and two boundaries is solved by direct integration,

\[ F^{0,2} = -288q - 256896q^2 - 1339256256q^3 - 6712774387200q^4 + \ldots \quad (4.3) \]

In addition, there are Klein bottle contribution \( B \) in the one loop level by equation 2.10,

\[ B = 72q + 678618q^2 + 4722711552q^3 + 31235476080258q^4 + \ldots \quad (4.4) \]

The genus one BPS invariants are extracted from the summation of \( F^{0,2} \) and \( B \) as in table 2.

At the two-loop level, the partition functions \( F^{0,3}, F^{1,1}, \) and \( B^{1,1} \) can be computed by solving the corresponding holomorphic anomaly equations respectively.

To begin with, \( F^{0,3} \) is solved by the equation 2.6,

\[ F^{0,3} = -2304q^{3/2} - 3138048q^{5/2} - 30306251520q^{7/2} \\
- 200402209737216q^{9/2} - 1394079763155261696q^{11/2} \\
- 9848995118139591641088q^{13/2} - 70386802081464082901031936q^{15/2} + \ldots \]

Here \( \Delta^2 \) is from equation 2.9.

Secondly, \( F^{1,1} \) can be obtained as follow

\[ F^{1,1} = -166q^{1/2} + 209756q^{3/2} + 70750818q^{5/2} + 466675366116q^{7/2} \\
+ 2062060525554428q^{9/2} + 11406521758300319916q^{11/2} \\
+ 7155253525041475172882q^{13/2} + 481939015584770078062938336q^{15/2} + \ldots, \]

Here we use the formula of \( F^{(1,0)} \) under the holomorphic limit equation 2.4. The discriminant, Euler characteristic, and second Chern class can be read from table 1.

Furthermore, the unoriented contribution \( B^{(1,1)} \) has to be considered, solving by equation 2.11,

\[ B^{1,1} = 12q^{1/2} - 13464q^{3/2} + 29205468q^{5/2} + 302085076824q^{7/2} \\
+ 2763351204278184q^{9/2} + 23231692148609680776q^{11/2} \\
+ 188521376343057222140124q^{13/2} + 1500381806456846910099106944q^{15/2} + \ldots \]
4.2 \( X_{2,4} \)

The polyhedron \( \Delta^* \) of \( X^\ast_{(2,4)} \subset \mathbb{P}^5 \) consists of the following vertices,

\[
\begin{align*}
\nu_1^* &= (-1, -1, -1, -1, -1), \\
\nu_2^* &= (1, 0, 0, 0, 0), \\
\nu_3^* &= (0, 1, 0, 0, 0), \\
\nu_4^* &= (0, 0, 1, 0, 0), \\
\nu_5^* &= (0, 0, 0, 1, 0), \\
\nu_6^* &= (0, 0, 0, 0, 1),
\end{align*}
\]

and has the maximally triangulation,

\[
l = (-2, -4; 1, 1, 1, 1, 1, 1).
\]

The mirror threefold \( X_{2,4} \) can be defined as the following equations,

\[
\begin{align*}
P_1 &= a_{1,0} - a_1(X_2X_3X_4X_5X_6) - a_2X_2, \\
P_2 &= a_{2,0} - a_3X_3 - a_4X_4 - a_5X_5 - a_6X_6.
\end{align*}
\]

The period integral of \( X_{2,4} \) satisfies the Picard-Fuchs equation,

\[
\mathcal{L}_{PF} = \theta^4 - 2^4z(4\theta + 1)(2\theta + 1)^2(4\theta + 3)
\]

with \( z = \frac{a_{1,0}a_{2,0}a_3a_4a_5a_6}{a_1a_2a_3a_4a_5a_6} \), and determines the mirror map \( z(q) \),

\[
z = q - 256q^2 + 19296q^3 - 2836480q^4 - 378262992q^5 + \ldots
\]

The domainwall tension on \( X_{2,4} \) is obtained from the domainwall tension on \( X_8 \subset \mathbb{P}^4_{1,1,2,2,2} \)[4]. At special locus \( y \rightarrow 8z, z_2 \rightarrow \frac{1}{4} \) on the complex moduli space,

\[
\mathcal{W}_{X_{2,4}}(z) = \mathcal{W}_{X_8}(8z, \frac{1}{4}).
\]

It satisfies the inhomogeneous Picard-Fuchs equation[4],

\[
\mathcal{L}_{PF} \mathcal{W}(z) = \frac{224z}{(2\pi i)^2}(1 + 272z + \frac{285120}{7}z^2 + 4925440z^3 + \ldots)
\]

and given by,

\[
\mathcal{W}_{X_{2,4}}(z(q)) = 384q + 29384q^2 + \frac{22954496}{3}q^3 + 2592661938q^4 + \ldots
\]
By the disk amplitude $\Delta_{zz}$ and Yukawa coupling,

$$-i\Delta_{zz} = 384q + 117536q^2 + 68863488q^3 + 41482591008q^4 + \ldots,$$

$$C_{zzz} = 8 + 1280q + 739584q^2 + 422690816q^3 + \ldots$$

the annulus partition function $F^{0,2}$ and Klein bottle partition function $B$ can be solved, respectively,

$$F^{0,2} = -4608q^2 - 1389056q^3 - 662529808q^4 - \frac{1706701489664}{5}q^5 - \frac{534044833761344}{3}q^6 - \frac{66828499880930304}{7}q^7 - 52332365789557579912q^8 + \ldots,$$

$$B = 2912q^2 + 2176000q^3 + 1320520800q^4 + 777151744000q^5 + \frac{1367654285858816}{3}q^6 + 268401212960489472q^7 + 158865301270593238112q^8 + \ldots.$$

It seems that $F^{0,2} + B$ can not give integer BPS invariant unless the holomorphic ambiguity is further explored.

Furthermore, $F^{0,3}, F^{1,1},$ and $B^{1,1}$ are solved by equation 2.6, equation 2.4, and equation 2.11 as before,

$$F^{0,3} = -147456q^3 - 88215552q^4 - 61506997248q^5 - \frac{127153359452416}{3}q^6 - \frac{84116510137784320}{3}q^7 - 18295779654814501120q^8 + \ldots,$$

$$F^{1,1} = -512q - 94316q^2 - 74532736q^3 - \frac{149541540172}{3}q^4 - \frac{86792064346880}{3}q^5 - \frac{50874394964035840}{3}q^6 - 10051275752143892480q^7 + \ldots,$$

$$B^{1,1} = 48q + 14024q^2 + 9425088q^3 + 6406868624q^4 + 3841512413248q^5 + 2315965178004736q^6 + 1410688247617024q^7 + \ldots.$$
4.3 $X_{2,6}$

Similar to the last case, the domainwall tension on $X_{2,6}$ is related to the domainwall tension on $X_{12} \subset \mathbb{P}^4_{1,1,2,2,6}$ at degeberate locus $y \to 2z^4, z_2 \to \frac{1}{4}$ [42].

$$\mathcal{W}_{X_{2,6}}(z) = \mathcal{W}_{X_{12}}(2z^4, \frac{1}{4}),$$

which is the solution of the inhomogeneous Picard-Fuchs equation,

$$\mathcal{L}_{PF} \mathcal{W}(z) = \frac{4z^{1/3} + 112z^{2/3}}{27(1 - 8z^{1/3})^{5/2}},$$

and given by,

$$\mathcal{W}_{X_{2,6}}(z(q)) = 12q^{1/3} + 99q^{2/3} + \frac{2368}{3}q + \frac{9867}{4}q^{4/3} + \frac{525312}{25}q^{5/3} + \ldots$$

in $q$-coordinate.

By the disk amplitude $\Delta_{zz}$ and Yukawa coupling ,

$$-i\Delta_{zz} = 4z^{1/3} + 44z^{2/3} + \frac{2368}{3}q + \frac{13156}{3}q^{4/3} + 58368q^{5/3} + 1926240q^2 + \ldots,$$

$$C_{zz} = 4 + 4992q + 19115136q^2 + 73765625856q^3 + 294375479225472q^4 + \ldots$$

$F^{0,2}$ and $B$ are obtained,

$$F^{0,2} = -\frac{1}{3}q^{2/3} - \frac{44}{3}q + \frac{2273}{6}q^{4/3} - \frac{88804}{15}q^{5/3} - \frac{572722}{9}q^2 - \frac{13672096}{21}q^{7/3} - \frac{108971905}{12}q^{8/3} - \frac{1079462156}{9}q^3 \ldots$$

$$B = 187488q^2 + 811219968q^3 + 3196262986848q^4 + 12484041857390592q^5 + 49037805709065086976q^6 + 194195672782104702468096q^7 + \ldots$$
In addition, $F^{0,3}, F^{1,1}$, and $B^{1,1}$ are also obtained,

\[
F^{0,3} = -\frac{2}{81} q - \frac{22}{9} q^{4/3} - \frac{3362}{27} q^{5/3} - \frac{35672}{9} q^2 - \left(\frac{2347268}{27} q^{7/3} - \frac{3835658}{27} q^{8/3}\right),
\]

\[
-1629947366 q^3 - \frac{8010604964}{27} q^{10/3} - \frac{120472884100}{27} q^{11/3} - \frac{5267892155720}{81} q^4
\]

\[
+ \frac{6354568170748}{135} q^{13/3} + \frac{10545415896272828}{135} q^{14/3} + \frac{392405345133455632}{945} q^5 + \ldots,
\]

\[
F^{1,1} = -\frac{109}{54} q^{1/3} - \frac{1969}{18} q^{2/3} - \frac{24568}{9} q^{4/3} - \frac{551431}{54} q^{5/3} + \frac{42224}{3} q^{5/3} - \frac{14722220}{9} q^2
\]

\[
+ \frac{195803224}{9} q^{7/3} + \frac{1525413785}{6} q^{8/3} - \frac{201709714880}{9} q^3 + \frac{347304066304}{27} q^{10/3}
\]

\[
+ \frac{330930764006212}{15} q^{11/3} + \frac{15719806195687040}{3} q^4 + \frac{2690798900534401408}{315} q^{13/3}
\]

\[
- \frac{6315571723672854104}{5} q^{14/3} - \frac{32340926084578462080}{q^5} + \ldots,
\]

\[
B^{1,1} = \frac{1}{9} q^{1/3} + \frac{22}{3} q^{2/3} + \frac{592}{3} q + \frac{8164}{9} q^{4/3} + 1440 q^{5/3} + 470576 q^2 - \frac{3690512}{3} q^{7/3}
\]

\[
- 13429304 q^{8/3} + 2032199296 q^3 + \frac{1224982720}{9} q^{10/3} - \frac{89835051090776}{5} q^{11/3}
\]

\[
- 428159082396160 q^4 + \frac{736519297946563792}{105} q^{13/3} + \frac{510219884409434976}{5} q^{14/3}
\]

\[
+ 2627791095598842624 q^5 + \ldots.
\]

### 4.4 Non-compact Threefold $X_4$

Non-compact hypersurface $X_4 \subset \mathbb{P}^4(2,1,1,1,1)$ is obtained from the degree-12 hypersurface $X_{12} \subset \mathbb{P}^4_{(1,2,3,3,3)}$ at the degenerate locus $z_2 = z_3 = 0$ on $M_{CS}(X_{12})$.

\[
P = x_1^2 + x_2^4 + x_3^4 + x_4^4 + x_5^{-4} + \psi x_1 x_2 x_3 x_4 x_5
\]

The closed string periods on $X_4$ are solutions of the Picard-Fuchs operator,

\[
\mathcal{L}_{PF} = (\theta^2 - 4z(4\theta + 3)(4\theta + 1))(-z) \cdot \theta
\]

and the domainwall tension $\mathcal{W}$ satisfies the inhomogeneous equation[4],

\[
\mathcal{L}_{PF} \mathcal{W} = -\frac{1}{2\pi^2} z^{1/2}
\]
In terms of the mirror map,
\[ z(q) = q + 12q^2 + 6q^3 + 688q^4 - 15375q^5 + \ldots, \]
domain wall tension can be written as,
\[ \mathcal{W}(z(q)) = 16q^{1/2} - \frac{416}{9} q^{3/2} + \frac{16016}{25} q^{5/2} - \frac{664032}{49} q^{7/2} + \frac{28436416}{81} q^{9/2} + \ldots. \]

The disk amplitude \( \Delta_{zz} \) is the covariant derivative of \( \mathcal{W} \) in the holomorphic limits, and Yukawa coupling is solved in [33],
\[ -i\Delta_{zz} = 4q^{1/2} - 104q^{3/2} + 4004q^{5/2} - 166008q^{7/2} + 7109104q^{9/2} + \ldots, \]
\[ C_{zzz} = -2 + 56q - 2120q^2 + 87536q^3 - 3741768q^4 + 162980056q^5 + \ldots. \]

\( F^{0,2} \) and \( B \) are obtained,
\[ F^{0,2} = 4q - 48q^2 + \frac{3784}{3} q^3 - 39360q^4 + \frac{6754004}{5} q^5 + \ldots, \]
\[ B = -2q + 58q^2 - \frac{5876}{3} q^3 + 71658q^4 - \frac{13747882}{5} q^5 + \ldots. \]

from which the BPS invariants are extracted as in table 3.

Moreover, \( F^{0,3}, F^{1,1}, \) and \( B^{1,1} \) are listed here,
\[ F^{0,3} = -\frac{8}{3} q^{3/2} + \frac{176}{3} q^{5/2} - \frac{7160}{3} q^{7/2} + \frac{298672}{3} q^{9/2} - \frac{12834392}{3} q^{11/2} + 186896160q^{13/2} - \frac{24758224912}{3} q^{15/2} + \frac{1101434756720}{3} q^{17/2} - \frac{4926957604136}{3} q^{19/2} + \ldots, \]
\[ F^{1,1} = \frac{7}{6} q^{1/2} - 5q^{3/2} + \frac{535}{6} q^{5/2} - \frac{3901}{3} q^{7/2} - 14106q^{9/2} + \frac{8609479}{3} q^{11/2} - \frac{410665069}{2} q^{13/2} + \frac{36071531510}{3} q^{15/2} - \frac{3888132212435}{6} q^{17/2} + 33426120758501q^{19/2} + \ldots, \]
\[ B^{1,1} = -q^{1/2} - 2q^{3/2} - 209q^{5/2} + 10842q^{7/2} - 538412q^{9/2} + 26001074q^{11/2} - 1239688911q^{13/2} + 58676641972q^{15/2} - 2764397103467q^{17/2} + 129822762435106q^{19/2} + \ldots. \]
4.5 Non-compact Threefold $X_6$

Non-compact hypersurface $X_4 \subset \mathbb{P}^4(3, 2, 1, 1, -1)$ is obtained from the degree-12 hypersurface $X_{12} \subset \mathbb{P}^4_{(1, 2, 3, 3, 3)}$ at the degenerate locus $z_2 = z_3 = 0$ on $M_{CS}(X_{12})$,

$$P = x_1^2 + x_2^3 + x_3^6 + x_4^6 + x_5^{-6} + \psi x_1 x_2 x_3 x_4 x_5$$

with the period integral annihilated by the following Picard-Fuchs operator,

$$\mathcal{L}_{PF} = (\theta^2 - 12 z (6 \theta + 5) (6 \theta + 1)) (-z) \cdot \theta.$$ 

It is find that the domainwall tension $\mathcal{W}$ satisfies the inhomogeneous equation[4],

$$\mathcal{L}_{PF} \mathcal{W} = -\frac{1}{\pi^2} z^{1/2}$$

and is written in $q$-coordinate,

$$\mathcal{W}(q) = 16 q^{1/2} - \frac{416}{9} q^{3/2} + \frac{16016}{25} q^{5/2} - \frac{664032}{49} q^{7/2} + \frac{28436416}{81} q^{9/2} + \ldots.$$ 

The disk amplitude $\Delta_{zz}$ is the covariant derivative of $\mathcal{W}$ in the holomorphic limits, and Yukawa coupling is solved in [33],

$$-i \Delta_{zz} = 4 q^{1/2} - 104 q^{3/2} + 4004 q^{5/2} - 166008 q^{7/2} + 7109104 q^{9/2} + \ldots,$$

$$C_{zzz} = -2 + 56 q - 2120 q^2 + 87536 q^3 - 3741768 q^4 + 162980056 q^5 + \ldots.$$ 

$F^{0,2}$ and $B$ are obtained,

$$F^{0,2} = 4 q - 48 q^2 + \frac{3784}{3} q^3 - 39360 q^4 + \frac{6754004}{5} q^5 + \ldots,$$

$$B = -2 q + 58 q^2 - \frac{5876}{3} q^3 + 71658 q^4 - \frac{13747882}{5} q^5 + \ldots$$

underling the BPS invariants as in table[3].
Moreover, $F^{0,3}, F^{1,1},$ and $B^{1,1}$ are listed here,

\[
F^{0,3} = \frac{8}{3} q^{3/2} + \frac{176}{3} q^{5/2} - \frac{7160}{3} q^{7/2} + \frac{298672}{3} q^{9/2} - \frac{12834392}{3} q^{11/2} + 186896160 q^{13/2} - \frac{24758224912}{3} q^{15/2} + \frac{1101434756720}{3} q^{17/2} - \frac{49269576041336}{3} q^{19/2} + \ldots,
\]

\[
F^{1,1} = \frac{7}{6} q^{1/2} - 5 q^{3/2} + \frac{535}{6} q^{5/2} - \frac{3901}{3} q^{7/2} - 14106 q^{9/2} + \frac{8609479}{3} q^{11/2} - \frac{410665069}{2} q^{13/2} + \frac{36071531510}{3} q^{15/2} - \frac{3888132212435}{6} q^{17/2} + 33426120758501 q^{19/2} + \ldots,
\]

\[
B^{1,1} = -q^{1/2} - 2 q^{3/2} - 209 q^{5/2} + 10842 q^{7/2} - 538412 q^{9/2} + 26001074 q^{11/2} - 1239688911 q^{13/2} + 58676641972 q^{15/2} - 2764397103467 q^{17/2} + 129822762435106 q^{19/2} + \ldots.
\]
5 Summary and Conclusion

In this paper, the holomorphic anomaly equation in presence of D-branes and extremal transition are reviewed. Then, Yukawa couplings and disk amplitudes are used as initial data to solve open topological string amplitudes recursively. They are generated by the holomorphic prepotential and BPS domainwall tension respectively. The amplitudes with first several genus and boundaries are computed for several one-parameter models by solving the extended holomorphic anomaly equations. The genus one BPS invariants are extracted from the annulus partition function amended by the Klein bottle contributions.

In the future, we hope to study the open topological string amplitudes further. The amplitudes and invariants on multiple-parameter Calabi-Yau hypersurfaces can be computed if the form of unoriented worldsheet contribution is identified. Furthermore, the relation of high genus BPS invariant between two Calabi-Yau threefolds under extremal transition is worth to explore. Also, it is phenomenologically interesting to the domainwall tension and invariants from inhomogeneous Picard-Fuchs equation with more complicated form. In addition, we’re interested in the polynomial structure and algebraic structure of the open string amplitudes in this work and wish to reproduce the result by algebraic methods.

Acknowledgement

This work is dedicated to our dear supervisor Prof. Fu-Zhong Yang who sadly passed away while the paper was being prepared.
A Genus One BPS Invariants

| $d$ | $n_d^{(1,\text{real})}$ |
|-----|------------------------|
| 2   | $-108$                 |
| 4   | 210861                 |
| 6   | 1691727684             |
| 8   | 12261350846529         |
| 10  | 85281547794525216      |
| 12  | 589741364496798435519  |
| 14  | 4088168398606663732226004 |
| 16  | 28473212562534359781492702609 |
| 18  | 199323405502548694553853261163032 |
| 20  | 1402176885853036915691702511225343488 |
| 22  | 99082473457336233632273338976995370571108 |
| 24  | 70299505376206587341692199773751852403628923 |
| 26  | 500599345846694349151535150084728041871513503840 |
| 28  | 3576426516203804180327813075957186288625200126974263 |
| 30  | 25626335416567506505515281172221112652927148559385617024 |
| 32  | 184109711917919362541496902594727432879224527932144282767313 |
| 34  | 1325899145047854560008073848155458009321006047727139826816296540 |
| 36  | 9569566064471974175729715027817667767095503179190445437747223728244 |

Table 2: Real BPS Invariants $n_d^{(1,\text{real})}$ on $X_{3,6}$. 


Table 3: Real BPS Invariants $\rho_{d, \text{real}}^{(1)}$ on Non-compact Threefolds $X_4$ and $X_6$

| $d$ | $X_4$         | $X_6$         |
|-----|--------------|--------------|
| 2   | 1            | -8           |
| 4   | 5            | 1633         |
| 6   | -349         | -400976      |
| 8   | 16149        | 107371973    |
| 10  | -699388      | -30230378688 |
| 12  | 29875727     | 8794612573059 |
| 14  | -1275403373  | -2618260738724480 |
| 16  | 54624885845  | 792974210880311061 |
| 18  | -2349706860286 | -243367747015245246824 |
| 20  | 101523052724116 | 75483488157699279826308 |
| 22  | -4404975038898593 | -23614530611208021420992640 |
| 24  | 191868347966729663 | 7440700914891506376095639375 |
| 26  | -8386687184785991814 | -2358697261688376821462850303280 |
| 28  | 367755330860124252031 | 751581357963857262692571222870515 |
| 30  | -16172473752326376335406 | -240560009891829964002898566936525608 |
| 32  | 713057845437257413599573 | 77298289878094678379649029834584140885 |
| 34  | -31513704812660854146557542 | -24923803319003434549062941934478799991216 |
| 36  | 1395752726831387569678298474 | 8061010159313660492971159894519500378800514 |
The B-model on $X_{2,12}$ is determined by the Picard-Fuchs equation,

$$\mathcal{L}_{PF} = \theta^4 - 2^4 3^2 z (12\theta + 1) (12\theta + 5) (12\theta + 7) (12\theta + 11)$$

with $z = \frac{a_1 a_2 a_3 a_4 a_6}{a_1^4 a_2^2 a_3^2 a_4^2}$ the coordinate on $M_{CS}(X_{2,12})$.

The domainwall tension is,

$$\mathcal{W}(z(q)) = 960 q^{1/2} + \frac{180147200}{3} q^{3/2} + \frac{196676435515392}{5} q^{5/2} + \ldots$$

The disk amplitude $\Delta_{zz}$ and Yukawa coupling are,

$$-i\Delta_{zz} = 240 q^{1/2} + 135110400 q^{3/2} + 245845544394240 q^{5/2} + \ldots,$$

$$C_{zzz} = 1 + 678816q + 1101481164576q^2 + 1865163478016858112q^3 + \ldots$$

which are used to derive partition function of high genus and boundaries.

$$F^{0,2} = -28800q - 6438297600q^2 - 922281453875200q^3 + \ldots,$$

$$B = 7200q + 52534580832q^2 + 86588737272520704q^3 + \ldots,$$

$$F^{0,3} = -2304000q^{3/2} - 763195392000q^{5/2} - 2097503571419136000q^{7/2}$$

$$- 3445339410339074408448000q^{9/2} - 6124894420630434562124716032000q^{11/2}$$

$$- 11014922284544639624390498823929856000q^{13/2} + \ldots,$$

$$F^{1,1} = -3000q^{1/2} + 949808000q^{3/2} - 16656103503360q^{5/2}$$

$$+ 94398257477211770880q^{7/2} + 84976266200715615951736640q^{9/2}$$

$$+ 118049404247105886752601970897920q^{11/2} + \ldots,$$

$$B^{1,1} = 120q^{1/2} - 39980160q^{3/2} + 25928505768960q^{5/2}$$

$$+ 55289630967273123840q^{7/2} + 126457589032857480954260160q^{9/2}$$

$$+ 266264979410195659051004092216320q^{11/2} + \ldots.$$
Table 4: Real BPS Invariants $n_{d}^{(0,\text{real})}$ on $X_{2,12}^{*}$

| $d$ | $n_{d}^{(0,\text{real})}$ |
|-----|-------------------------|
| 1   | 960                     |
| 3   | 60048960                |
| 5   | 39335287103040          |
| 7   | 33965566243528503360    |
| 9   | 36197061864551407599321600 |
| 11  | 4346798496165937083680841939840 |
| 13  | 5652213245339999803268480700206137920 |
| 15  | 7777089717782482021179338585251524621584320 |
| 17  | 111624419056025313789781039227134845629190053821760 |
| 19  | 165548092685033680177182361810397736122582109535366939200 |

Table 5: Real BPS Invariants $n_{d}^{(1,\text{real})}$ on $X_{2,12}^{*}$

| $d$ | $n_{d}^{(1,\text{real})}$ |
|-----|-------------------------|
| 2   | -10800                  |
| 4   | 23048141616             |
| 6   | 38683227909326352       |
| 8   | 65584802398584428929584 |
| 10  | 111079935597958581154390188912 |
| 12  | 190181454375481134199906176153316080 |
| 14  | 329138224418155481457451326812852126931024 |
| 16  | 5750886955979291688927479275984201338454448688 |
| 18  | 1013084664647843312101016011973479395822662477629518720 |
| 20  | 179717265319870698029746817565454857096808527614959148885904 |

The BPS invariants $n_{d}^{0,\text{real}}$ and $n_{d}^{1,\text{real}}$ are extracted as in table 4 and 5.

C Threefold in $\mathbb{P}^{1}\times\mathbb{P}^{1}\times\mathbb{P}^{1}\times\mathbb{P}^{1}$

The Calabi-Yau hypersurface in $(\mathbb{P}^{1})^{4}$ is described by the Picard-Fuchs equation [8].

$$L_{PF} = \theta^{4} - 4z(5\theta^{2} + 5\theta + 2)(2\theta + 1)^{3} + 64z^{2}(2\theta + 3)(2\theta + 2)^{2}(2\theta + 1).$$
The domainwall tension is,
\[ \mathcal{W}(z(q)) = 16q^{1/2} + \frac{160}{9}q^{3/2} + \frac{3216}{25}q^{5/2} + \frac{85472}{49}q^{7/2} + \frac{364512}{81}q^{9/2} + \ldots, \]

The disk amplitude \( \Delta_{zz} \) and Yukawa coupling are,
\[ -i\Delta_{zz} = 4q^{1/2} + 16q^{3/2} + 80q^{5/2} + 21368q^{7/2} + 911128q^{9/2} + \ldots, \]
\[ C_{zzz} = 1 + 4q + 164q^2 + 5800q^3 + 196772q^4 + 6564004q^5 + \ldots. \]

Then, other partition functions can be obtained,
\[ F_{0,2}^0 = -8q - 64q^2 - \frac{2192}{3}q^3 - 10368q^4 - \frac{1980488}{5}q^5 + \ldots, \]
\[ B = -6q^2 + 144q^3 + 6402q^4 + 253248q^5 + 9033564q^6 + \ldots \]
\[ F_{0,3}^0 = -\frac{32}{3}q^{3/2} - \frac{704}{3}q^{5/2} - \frac{12256}{3}q^{7/2} - \frac{177344}{3}q^{9/2} - \frac{8268064}{3}q^{11/2} - \frac{31870272}{3}q^{13/2} \]
\[ - 3672400000q^{15/2} - 122306249664q^{17/2} - \frac{12539594603776}{3}q^{19/2} + \ldots, \]
\[ F_{1,1}^1 = -\frac{86}{3}q^{1/2} - 252q^{3/2} - \frac{406}{3}q^{5/2} + 171884q^{7/2} + 147124q^{9/2} - 1020652q^{11/2} \]
\[ + 162369550q^{13/2} + 6472815456q^{15/2} + \frac{436660840264}{3}q^{17/2} + \frac{11668556346176}{3}q^{19/2} + \ldots, \]
\[ B_{1,1}^1 = 2q^{1/2} + 36q^{3/2} + 82q^{5/2} - 19476q^{7/2} + 113148q^{9/2} + 6571668q^{11/2} + 237579138q^{13/2} \]
\[ + 9447170112q^{15/2} + 381805635992q^{17/2} + 14830760334208q^{19/2} + \ldots. \]

The BPS invariants \( n(0, \text{real}) \) and \( n(1, \text{real}) \) are extracted as in
| $d$ | $n_{d}^{(1,\text{real})}$ |
|-----|------------------|
| 1   | 16               |
| 3   | 16               |
| 5   | 128              |
| 7   | 1744             |
| 9   | 44992            |
| 11  | 1006480          |
| 13  | 24154752         |
| 15  | 617583584        |
| 17  | 16508007216      |
| 19  | 455415438960     |

Table 6: Real BPS Invariants $n_{d}^{(0,\text{real})}$ on threefold in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

| $d$ | $n_{d}^{(1,\text{real})}$ |
|-----|------------------|
| 2   | $-4$             |
| 4   | $-35$            |
| 6   | $-292$           |
| 8   | $-1983$          |
| 10  | $-71424$         |
| 12  | $-1339313$       |
| 14  | $-12136660$      |
| 16  | 401290385        |
| 18  | 31690274392      |
| 20  | 1540632062720    |
| 22  | 64928687564668   |
| 24  | 2557463371902331 |
| 26  | 97003475592104320|
| 28  | 359583796606911047|
| 30  | 131322242315758797456 |
| 32  | 47475266166793404369 |
| 34  | 170409919553761528120468 |
| 36  | 6085384390825832081907124 |

Table 7: Real BPS Invariants $n_{d}^{(1,\text{real})}$ on threefold in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$
References

[1] Mina Aganagic, Daniel Jafferis, and Natalia Saulina. “Branes, black holes and topological strings on toric Calabi-Yau manifolds”. In: *JHEP* 12 (2006), p. 018. arXiv: [hep-th/0512245](https://arxiv.org/abs/hep-th/0512245).

[2] Mina Aganagic, Andrew Neitzke, and Cumrun Vafa. “BPS microstates and the open topological string wave function”. In: *Adv. Theor. Math. Phys.* 10.5 (2006), pp. 603–656. arXiv: [hep-th/0504054](https://arxiv.org/abs/hep-th/0504054).

[3] Mina Aganagic et al. “The Topological Vertex”. In: *Commun.Math.Phys.* 254 (2005) 425-478 (May 2003). arXiv: [hep-th/0305132](https://arxiv.org/abs/hep-th/0305132).

[4] Murad Alim et al. “Type II/F-theory Superpotentials with Several Deformations and N=1 Mirror Symmetry”. In: *JHEP* 06 (2011), p. 103. arXiv: [1010.0977](https://arxiv.org/abs/1010.0977).

[5] Gert Almkvist et al. “Tables of Calabi–Yau equations”. In: *arXiv preprint math/0507430* (2005).

[6] Pedro Luis del Angel and Stefan Müller-Stach. “Differential equations associated to families of algebraic cycles”. In: *Annales de l’institut Fourier*. Vol. 58. 6. 2008, pp. 2075–2085.

[7] Paul S Aspinwall, Brian R Greene, and David R Morrison. “Multiple mirror manifolds and topology change in string theory”. In: *Physics Letters B* 303.3-4 (1993), pp. 249–259.

[8] Victor V Batyrev and Lev A Borisov. “Dual cones and mirror symmetry for generalized Calabi-Yau manifolds”. In: *arXiv preprint alg-geom/9402002* (1994).

[9] Victor V Batyrev and Duco Van Straten. “Generalized hypergeometric functions and rational curves on Calabi-Yau complete intersections in toric varieties”. In: *Communications in mathematical physics* 168.3 (1995), pp. 493–533.

[10] Chris Beasley et al. “Why Z(BH) = |Z(top)|**2**”. In: (Aug. 2006). arXiv: [hep-th/0608021](https://arxiv.org/abs/hep-th/0608021).

[11] Per Berglund, Sheldon H. Katz, and Albrecht Klemm. “Mirror symmetry and the moduli space for generic hypersurfaces in toric varieties”. In: *Nucl. Phys. B* 456 (1995), pp. 153–204. arXiv: [hep-th/9506091](https://arxiv.org/abs/hep-th/9506091).
[12] M. Bershadsky et al. “Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes”. In: *Communications in Mathematical Physics* 165.2 (Oct. 1994), pp. 311–427.

[13] Jan de Boer et al. “A Farey Tail for Attractor Black Holes”. In: *JHEP* 11 (2006), p. 024. arXiv: [hep-th/0608059](https://arxiv.org/abs/hep-th/0608059).

[14] Philip Candelas et al. “Mirror symmetry for two parameter models. 1.” In: *Nucl. Phys. B* 416 (1994). Ed. by B. Greene and Shing-Tung Yau, pp. 481–538. arXiv: [hep-th/9308083](https://arxiv.org/abs/hep-th/9308083).

[15] Paul L. H. Cook, Hirosi Ooguri, and Jie Yang. “Comments on the Holomorphic Anomaly in Open Topological String Theory”. In: *Phys. Lett. B* 653 (2007), pp. 335–337. arXiv: [0706.0511 [hep-th]](https://arxiv.org/abs/0706.0511).

[16] Atish Dabholkar et al. “Precision counting of small black holes”. In: *JHEP* 10 (2005), p. 096. arXiv: [hep-th/0507014](https://arxiv.org/abs/hep-th/0507014).

[17] Bohan Fang and Chiu-Chu Melissa Liu. “Open Gromov-Witten Invariants of Toric Calabi-Yau 3-Folds”. In: *Commun. Math. Phys.* 323 (2013), pp. 285–328. arXiv: [1103.0693 [math.SG]](https://arxiv.org/abs/1103.0693).

[18] Xu Feng-Jun and Yang Fu-Zhong. “Another representation of the $β$ form of the inhomogeneous Picard-Fuchs equation”. In: *Chin. Phys. C* 37.10 (2013), p. 101001.

[19] Davide Gaiotto, Andrew Strominger, and Xi Yin. “From AdS(3)/CFT(2) to black holes/topological strings”. In: *JHEP* 09 (2007), p. 050. arXiv: [hep-th/0602046](https://arxiv.org/abs/hep-th/0602046).

[20] Tom Graber and Eric Zaslow. “Open string Gromov-Witten invariants: Calculations and a mirror ‘theorem’”. In: (Sept. 2001). arXiv: [hep-th/0109075](https://arxiv.org/abs/hep-th/0109075).

[21] Phillip A. Griffiths. “A Theorem Concerning the Differential Equations Satisfied by Normal Functions Associated to Algebraic Cycles”. In: *American Journal of Mathematics* 101.1 (1979), pp. 94–131. ISSN: 00029327, 10806377. (Visited on 07/12/2022).

[22] Mark Gross. “Deforming Calabi-Yau threefolds”. In: *arXiv preprint alg-geom/9506022* (1995).

[23] Mark Gross. “Primitive Calabi-Yau threefolds”. In: *Journal of Differential Geometry* 45.2 (1997), pp. 288–318.
[24] S. Hosono, B. H. Lian, and Shing-Tung Yau. “GKZ generalized hypergeometric systems in mirror symmetry of Calabi-Yau hypersurfaces”. In: Commun. Math. Phys. 182 (1996), pp. 535–578. arXiv:alg-geom/9511001.

[25] S. Hosono et al. “Mirror symmetry, mirror map and applications to complete intersection Calabi-Yau spaces”. In: Nucl. Phys. B 433 (1995). Ed. by B. Greene and Shing-Tung Yau, pp. 501–554. arXiv:hep-th/9406055.

[26] Sheldon H. Katz and Chiu-Chu Melissa Liu. “Enumerative geometry of stable maps with Lagrangian boundary conditions and multiple covers of the disc”. In: Adv. Theor. Math. Phys. 5 (2001). Ed. by David Auckly and Jim Bryan, pp. 1–49. arXiv:math/0103074.

[27] Sheldon H. Katz, David R. Morrison, and M. Ronen Plesser. “Enhanced gauge symmetry in type II string theory”. In: Nucl. Phys. B 477 (1996), pp. 105–140. arXiv:hep-th/9601108.

[28] Albrecht Klemm and Peter Mayr. “Strong coupling singularities and non-Abelian gauge symmetries in N=2 string theory”. In: Nucl. Phys. B 469 (1996), pp. 37–50. arXiv:hep-th/9601014.

[29] Johanna Knapp and Emanuel Scheidegger. “Towards Open String Mirror Symmetry for One-parameter Calabi-Yau Hypersurfaces”. In: Adv. Theor. Math. Phys. 13.4 (2009), pp. 991–1075. arXiv:0805.1013 [hep-th].

[30] Maxim Kontsevich. “Homological Algebra of Mirror Symmetry”. In: (Nov. 1994). arXiv:alg-geom/9411018.

[31] Daniel Krefl and Johannes Walcher. “Real Mirror Symmetry for One-parameter Hypersurfaces”. In: JHEP 09 (2008), p. 031. arXiv:0805.0792 [hep-th].

[32] Daniel Krefl and Johannes Walcher. “The Real Topological String on a local Calabi-Yau”. In: (Feb. 2009). arXiv:0902.0616 [hep-th].

[33] W. Lerche, P. Mayr, and N. P. Warner. “Noncritical strings, Del Pezzo singularities and Seiberg-Witten curves”. In: Nucl. Phys. B 499 (1997), pp. 125–148. arXiv:hep-th/9612085.

[34] Jun Li and Yun S. Song. “Open string instantons and relative stable morphisms”. In: Adv. Theor. Math. Phys. 5 (2001). Ed. by David Auckly and Jim Bryan, pp. 67–91. arXiv:hep-th/0103100.

[35] Jun Li et al. “A Mathematical theory of the topological vertex”. In: Geom. Topol. 13.1 (2009), pp. 527–621. DOI: 10.2140/gt.2009.13.527 arXiv:math/0408426.
[36] David R. Morrison. “Through the looking glass”. In: arXiv preprint alg-geom/9705028 (1997).

[37] David R. Morrison and Johannes Walcher. “D-branes and Normal Functions”. In: Adv. Theor. Math. Phys. 13.2 (2009), pp. 553–598. arXiv:0709.4028 [hep-th].

[38] Andrew Neitzke and Johannes Walcher. “Background independence and the open topological string wavefunction”. In: Proc. Symp. Pure Math. 78 (2008), p. 285. arXiv:0709.2390 [hep-th].

[39] Hirosi Ooguri, Andrew Strominger, and Cumrun Vafa. “Black hole attractors and the topological string”. In: Phys. Rev. D 70 (2004), p. 106007. arXiv:hep-th/0405146.

[40] Andrew Strominger, Shing-Tung Yau, and Eric Zaslow. “Mirror symmetry is T duality”. In: Nucl. Phys. B 479 (1996), pp. 243–259. arXiv:hep-th/9606040.

[41] Cumrun Vafa. “Two dimensional Yang-Mills, black holes and topological strings”. In: (June 2004). arXiv:hep-th/0406058.

[42] Johannes Walcher. “Calculations for Mirror Symmetry with D-branes”. In: JHEP 09 (2009), p. 129. arXiv:0904.4905 [hep-th].

[43] Johannes Walcher. “Evidence for Tadpole Cancellation in the Topological String”. In: Commun. Num. Theor. Phys. 3 (2009), pp. 111–172. arXiv:0712.2775 [hep-th].

[44] Johannes Walcher. “Extended holomorphic anomaly and loop amplitudes in open topological string”. In: Nucl. Phys. B 817 (2009), pp. 167–207. arXiv:0705.4098 [hep-th].

[45] Johannes Walcher. “Opening mirror symmetry on the quintic”. In: Commun. Math. Phys. 276 (2007), pp. 671–689. arXiv:hep-th/0605162.

[46] Edward Witten. “Branes and the dynamics of QCD”. In: Nucl. Phys. B 507 (1997), pp. 658–690. arXiv:hep-th/9706109.

[47] Edward Witten. “Chern-Simons gauge theory as a string theory”. In: Prog. Math. 133 (1995), pp. 637–678. arXiv:hep-th/9207094.