New Designs of Universal Reversible Gate Library

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Abstract. We present new algorithms to synthesize exact universal reversible gate library for various types of gates and costs. We use the powerful algebraic software GAP for implementation and examination of our algorithms and the reversible logic synthesis problems have been reduced to group theory problems. It is shown that minimization of arbitrary cost functions of gates and orders of magnitude are faster than its previously counterparts for reversible logic synthesis. Experimental results show that a significant improvement over the previously proposed synthesis algorithm is obtained compared with the existing approaches to reversible logic synthesis.

Keywords: Reversible gates; Reversible circuits; Quantum circuits; Quantum cost; Universal library; Universal gate.

1. Introduction

A logic circuit is synthesized using a set of elementary components. This set is called the library of synthesis, where the members of this set are logic gates \cite{1}. A gate is said to be reversible if it is used to synthesize a reversible function or a reversible circuit \cite{7}. A function is reversible if the number of inputs is equal to the number of outputs, and each input pattern maps to a uniquely output pattern (bijection) \cite{8}. Synthesizing circuits with pure quantum gates improves the cost and the time efficiency of these circuits since no heat dissipation and no information destroyed from the system, in addition, we gain the advantages of quantum computing \cite{7}. The studies to improve the efficiency of the reversible circuits have focused on either developing algorithms that improves the quantum cost of the circuit by optimizing the gates used to synthesize the circuits, as in \cite{2,3,4}, or by defining new universal libraries with less number of gates that leads
to circuits with better size on average, as in \[5, 6\].

Methods developed to improve the quantum cost of reversible circuits using the \(NCT\) library are proposed in many studies such as \[3, 11, 12, 13\]. An algorithm that reduces the reversible logic synthesis problem into permutation group using group theory is presented in \[2\]. An algorithm that uses a method that finds the tight bounds on the synthesized 3-bits reversible circuits using \(NCT\) library to reduce the quantum cost of the circuits is presented in \[4\]. A library based synthesis methodology for reversible circuits is presented in \[7, 14\] to introduce optimization methods by decomposing the functions into smaller building blocks. A new gate type that is universal for \(n\)-in/out reversible circuits is proposed in \[5\]. A minimal universal library which includes only two gates, such that all the \(n \times n\) reversible circuits can be synthesized by these two gates is proposed in \[6\].

The aim of this paper is to propose a new universal reversible gate library to be used in the synthesis of reversible circuits. The results presented in this paper are implemented and tested using the group theory algebraic software GAP \[10\]. The results shown in this paper are compared with the results shown in \[2, 5, 14\] and \[15\]. This paper is organized as follows: Sect.2 gives a short introduction to the elementary quantum gates used to build quantum circuits and defines the terminologies used in this paper. Sect.3 describes the proposed gate library. Sect.4 shows the experimental results and compares the proposed universal gate library with the libraries proposed by others. Finally the conclusion is shown in Sect.5.

2. Synthesizing Reversible Circuit

Here we introduce some basic definitions and terminologies used in the synthesizing of the reversible circuit.

**Definition 2.1** A Boolean function \(f\) is reversible such that, \(f : x \rightarrow y\) if and only if each input vector \(x \in X^n\) maps to a unique output vector \(y \in X^n\). For \(X^n\) there are \(n\) input vector \((x_1, x_2, x_3, \ldots, x_n)\) and \(n\) output vector \((y_1, y_2, y_3, \ldots, y_n)\). For \(n\)-inputs and \(n\)-outputs, there are \((2^n)!\) reversible functions, \(\forall X = \{0, 1\}\) and \(n \in Z\) \[4\].

**Definition 2.2** \(C^n\text{NOT}\) is the main reversible gate that is commonly used to synthesize any reversible circuit. It is defined as follows,

\[
C^n\text{NOT}(x_1, x_2, \ldots, x_{n-1}; f_{in}) = C^n\text{NOT}(y_1, y_2, \ldots, y_{n-1}; f_{out}),
\]

where \(y_i = x_i\) for \(1 \leq i \leq n - 1\) and \(f_{out} = f_{in} \oplus x_1x_2\ldots x_{n-1} \forall n \in Z\), \(x \in X^n\) and \(X = \{0, 1\}\). \(x_1x_2\ldots x_{n-1}\) are called the control bits and \(f_{in}\) is called the target bit \[4\].

**Definition 2.3** The minimum cost \(Minc(g)\) is the realization of reversible gate \(g\) with a minimum cost \(Minc(g)\), such that; there is no realization with cost less than the \(Minc(g)\) \[2\].
**Definition 2.4** The cost of the circuit is the summation of the total costs of all the gates used to synthesize that circuit \[2\].

**Definition 2.5** A gate \(g\) is said to be reversible if it computes a reversible function \(f\) and is bijective \[4, 9\].

**Definition 2.6** A logic gate \(g\) is said to be universal if it is sufficient to synthesize an arbitrary logical operation on these \(n\) inputs \[1\].

**Definition 2.7** The set of reversible gates that can be used to build a reversible circuit is called gate library \(L\) \[4\].

**Definition 2.8** The universal library of synthesis is the smallest set of building blocks used in the synthesis process \[1\].

**Definition 2.9** A universal reversible gate library \(L\) is a set of reversible gates such that any reversible \(n\)-bit circuit can be synthesized by cascading gates in \(L\). \[5\].

**Definition 2.10** A universal reversible gate sub library \(SL \subseteq L\) is a set of reversible gates such that any \(n\)-bit reversible circuit can be synthesized using \(SL\). \[5\].

**Definition 2.11** Utilization is the percentage of universal sub libraries from all sub libraries \(SL\) of a certain library. \[5\].

**Definition 2.12** A permutation \(\sigma\) is a bijection such that, \(\sigma:A \rightarrow A\) which maps an input to an output from a finite set \(= \{1, 2, \ldots, N\}\), which can be written as follows,

\[
\sigma = \begin{pmatrix}
1 & 2 & 3 & \ldots & N \\
\sigma(1) & \sigma(2) & \sigma(3) & \ldots & \sigma(N)
\end{pmatrix}
\]  

(2)

The top row can be eliminated and written as follows,

\[
\sigma = \begin{pmatrix}
\sigma(1) & \sigma(2) & \sigma(3) & \ldots & \sigma(N)
\end{pmatrix}
\]  

(3)

Another notation having a permutation in the form of \(\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
8 & 2 & 1 & 4 & 7 & 5 & 6 & 3\end{pmatrix}\), it can be written as \((1,8,3)(5,7,6)\), this notation is called the product of disjoint cycles \[4\].

2.1. Reversible Gates

The reversible gates that can be used to synthesize any 3-bit reversible circuits are: NOT \((N)\) gate, Feynman \((C)\) gate, Toffoli \((T^3)\) gate, Fredkin \((F)\) gate, Peres \((P)\) gate, the \(G^3\) gate and the square-root NOT gate which are the controlled-\(v\) (\(v\)) and the controlled-\(v^\dagger\) (\(u\)), such that \(v.u = u.v = N, u.v = v.u = I, v.N = N.v = u, u.N = N.u = v\) and \(I\) is the identity gate \[2, 5\]. For a 3-bit reversible circuit, there are three different \(N\) gates, six different \(C\) gates as shown in Fig[1], three different \(T^3\) gates, three different \(F\) gates as shown in Fig[2], six different \(P\) gates as shown in Fig[3] six different \(G^3\) gates
as shown in Fig. 4, six different $v$ gates and six different $u$ gates as shown in Fig. 5.

The $N$ gate is used to flip the input bits unconditionally with quantum cost equal to zero. Eqn. 4 shows the functionality of the $N$ gate. There are 8 circuits that can
be realized using the three possible \( N \) gates [5].

\[
N^3_j : y_j = x_j \oplus 1, y_k = x_k, y_l = x_l,
\]

\[
N^3_1 : (x_1, x_2, x_3) \rightarrow (1,5)(2,6)(3,7)(4,8),
\]

\[
N^3_2 : (x_1, x_2, x_3) \rightarrow (1,3)(2,4)(5,7)(6,8),
\]

\[
N^3_3 : (x_1, x_2, x_3) \rightarrow (1,2)(3,4)(5,6)(7,8),
\]

where \( j, k \) and \( l \in \{1, 2, 3\} \) in any order.

The \( C \) gate is used to flip the target bit if the controlled bit is set to 1 with quantum cost equals to 1 [14]. Eqn.5 shows the functionality of the \( C \) gate. There are 168 circuits that can be realized by the six possible \( C \) gates [5].

\[
C^3_{j,k} : y_j = x_j, y_k = x_k \oplus x_j, y_l = x_l,
\]

\[
C^3_{1,2} : (x_1, x_2, x_3) \rightarrow (5,7)(6,8),
\]

\[
C^3_{1,3} : (x_1, x_2, x_3) \rightarrow (5,6)(7,8),
\]

\[
C^3_{2,3} : (x_1, x_2, x_3) \rightarrow (3,4)(7,8),
\]

\[
C^3_{2,1} : (x_1, x_2, x_3) \rightarrow (3,7)(4,8),
\]

\[
C^3_{3,2} : (x_1, x_2, x_3) \rightarrow (2,4)(6,8),
\]

\[
C^3_{3,1} : (x_1, x_2, x_3) \rightarrow (2,6)(4,8).
\]

The \( T^3 \) gate is used to flip the target bit if the control bits are set to 1 with quantum cost equals to 5 [14]. Eqn.6 shows the functionality of the \( T^3 \) gate. There are 24 circuits that can be realized by the three possible \( T^3 \) gates [5].

\[
T^3_{j,k,l} : y_j = x_j, y_k = x_k, y_l = x_l \oplus x_j x_k,
\]

\[
T^3_{1,2,3} : (x_1, x_2, x_3) \rightarrow (7,8),
\]

\[
T^3_{1,3,2} : (x_1, x_2, x_3) \rightarrow (6,8),
\]

\[
T^3_{2,3,1} : (x_1, x_2, x_3) \rightarrow (4,8).
\]

The \( F \) gate is used to perform conditional swap on two of its inputs if the third input is set to 1 with quantum cost equals to 5 [13]. Eqn.7 shows the functionality of the \( F \) gate. There are 6 circuits that can be realized by the three possible \( F \) gates [5].

\[
F^3_{j,k,l} : y_j = x_j,
\]

\[
x_j = \begin{cases} 1 & y_k = x_l, y_l = x_k \\ 0 & y_k = x_k, y_l = x_l \end{cases}
\]

\[
F^3_{1,2,3} : (x_1, x_2, x_3) \rightarrow (6,7),
\]

\[
F^3_{2,1,3} : (x_1, x_2, x_3) \rightarrow (4,7),
\]

\[
F^3_{3,2,1} : (x_1, x_2, x_3) \rightarrow (4,6).
\]

The \( P \) gate combines the functions of \( T^3 \) gate and \( C \) gate in a single gate; it acts on an arbitrary 3-bits \( x_j, x_k \) and \( x_l \), \( C \) gate is applied on \( x_j \) and \( x_k \) using \( x_j \) as a controller bit and \( x_k \) as a target bit, then \( T^3 \) gate is applied on \( x_j, x_k \) and \( x_l \) using \( x_j \) and \( x_k \) as
controller bits and \( x_l \) as a target bit. The quantum cost of \( P \) is 4 [13]. Eqn.8 shows the functionality of the \( P \) gate. There are 5040 circuits that can be realized by the six possible \( P \) gates [5].

\[
P_{j,k,l}^3 : y_j = x_j, y_k = x_j \oplus x_k, y_l = x_l \oplus x_j x_k,
\]

\[
P_{123} : (x_1, x_2, x_3) \rightarrow (5, 7, 6, 8),
P_{132} : (x_1, x_2, x_3) \rightarrow (5, 6, 7, 8),
P_{213} : (x_1, x_2, x_3) \rightarrow (3, 7, 4, 8),
P_{231} : (x_1, x_2, x_3) \rightarrow (3, 4, 7, 8),
P_{312} : (x_1, x_2, x_3) \rightarrow (2, 6, 4, 8),
P_{321} : (x_1, x_2, x_3) \rightarrow (2, 4, 6, 8).
\]

The \( G^3 \) gate combines the function of \( N \), \( C \) and \( T^3 \) gates in a single gate; it acts on an arbitrary 3-bits \( x_j, x_k \) and \( x_l \). \( x_l \) is flipped if \( x_j \) and \( x_k \) are set to 1, then \( x_k \) is flipped if \( x_j \) is set to 1, finally the bit \( x_j \) is flipped unconditionally. The quantum cost of \( G^3 \) is 5 [5]. Eqn.9 shows the functionality of the \( G^3 \) gate. There are 40320 circuits that can be realized by the six possible \( G^3 \) gates, thus \( G \) gate is universal [5].

\[
G_{j,k,l}^3 : y_j = x_j \oplus 1, y_k = x_j \oplus x_k, y_l = x_l \oplus x_j x_k,
\]

\[
G_{123} : (x_1, x_2, x_3) \rightarrow (1, 5, 3, 7, 2, 6, 4, 8),
G_{132} : (x_1, x_2, x_3) \rightarrow (1, 5, 2, 6, 3, 7, 4, 8),
G_{213} : (x_1, x_2, x_3) \rightarrow (1, 3, 5, 7, 2, 4, 6, 8),
G_{231} : (x_1, x_2, x_3) \rightarrow (1, 3, 2, 4, 5, 7, 6, 8),
G_{312} : (x_1, x_2, x_3) \rightarrow (1, 2, 5, 6, 3, 4, 7, 8),
G_{321} : (x_1, x_2, x_3) \rightarrow (1, 2, 3, 4, 5, 6, 7, 8).
\]

2.2. Universal Libraries

Many universal libraries have been defined from the combination of the reversible gates, such as \( NCT \) (NOT- Feynman- Toffoli), \( NCTF \) (NOT- Feynman- Toffoli- Fredkin), \( NCP \) (NOT- Feynman- Peres), \( NCPF \) (NOT- Feynman- Peres- Fredkin), \( NCPT \) (NOT- Feynman- Peres- Toffoli), \( NT \) (NOT- Toffoli), \( NP \) (NOT- Peres), \( NCF \) (NOT- Feynman- Fredkin), \( NFT \) (NOT- Feynman- Toffoli), \( NCTPF \) (NOT- Feynman- Toffoli- Peres- Fredkin) and \( G \)-gate Library [2] [5] [15].

3. The Proposed Universal Reversible Library

This section proposes a new universal \( n \)-bit reversible gate \( R^n \) for \( n \)-bits input/output reversible circuits, which is a universal gate on its own. To reduce the quantum cost of the circuits synthesized with \( R^n \) gate, \( N \) gate might be added to form another library called \( NR^n \) which is also universal.
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3.1. 1-bit Gate ($R^1$ gate)

$R^1$ is a 1-bit gate. It acts as $N$ gate which inverses the input bit unconditionally. There are 2 possible 1-bit input/output reversible circuits. The $R^1$ gate is sufficient to realize these two circuits. For 1-bit reversible circuits built using $R$-gate library, there is one $R^1$ gate as shown in Fig. 6, it function as shown in Eqn. (10) and its quantum cost is 0.

\[
R^1_1: (x_1) = x_1 \oplus 1 = (1, 2). \tag{10}
\]

3.2. 2-bits Gate ($R^2$ gate)

$R^2$ is a 2-bits gate. It acts as a combination between $N$ gate and $C$ gate. It uses one bit as a controller to flip the other bit and then it flips the controller bits unconditionally. For 2 input/output reversible circuits there are 24 possible circuits. $R^2$ gate is sufficient to realize these 24 circuits. For 2-bits reversible circuits built using $R$-gate library, there are two possible $R^2$ gates as shown in Fig. 7, they function as shown in Eqn. (11) and their quantum cost is 1.

\[
R^2_{j,k}: y_j = x_j \oplus 1, \\
y_k = x_k \oplus x_j, \tag{11}
\]

\[
R^2_{1,2}: (x_1, x_2) \rightarrow (1, 3, 2, 4), \\
R^2_{2,1}: (x_1, x_2) \rightarrow (1, 2, 3, 4),
\]

where $j$ and $k \in \{1, 2\}$ in any order.

3.3. 3-bits Gate ($R^3$ gate)

$R^3$ is a 3-bits gate. It combines the action of the three gates $N$, $C$ and $T^3$. It acts on an arbitrary 3-bits $x_j$, $x_k$ and $x_l$ in any order. First $x_j$ and $x_k$ are used as controller bits to flip $x_l$, then $x_j$ is used as a controller bit to flip $x_k$, after that $x_l$ is flipped unconditionally and finally $x_l$ is used as a controller bit to flip $x_j$. For 3-bits reversible circuits built using $R$-gate library, there are three possible $R^3$ gates.
circuits built using $R$-gate library, there are six possible $R^3$ gates as shown in Fig.8 and they function as shown in Eqn.12

$$R^3_{j,k,l} : y_j = x_j \oplus x_k \oplus x_l, x_1 \oplus 1, $$
$$y_k = x_k \oplus x_j, x_1 \oplus 1,$$
$$y_l = x_l \oplus x_j.$$  \hspace{1cm} (12)$$

$R^3_{1,2,3} : (x_1, x_2, x_3) \rightarrow (1, 7, 6, 5, 4, 2, 8, 3),$  
$R^3_{3,2,1} : (x_1, x_2, x_3) \rightarrow (1, 4, 6, 2, 7, 5, 8, 3),$  
$R^3_{3,1,2} : (x_1, x_2, x_3) \rightarrow (1, 4, 7, 3, 6, 5, 8, 2),$  
$R^3_{1,3,2} : (x_1, x_2, x_3) \rightarrow (1, 6, 7, 5, 4, 3, 8, 2),$  
$R^3_{2,3,1} : (x_1, x_2, x_3) \rightarrow (1, 6, 4, 2, 7, 3, 8, 5),$  
$R^3_{2,1,3} : (x_1, x_2, x_3) \rightarrow (1, 7, 4, 3, 6, 2, 8, 5),$ 

where $j, k$ and $l \in \{1, 2, 3\}$ in any order.

The quantum cost for the $R^3$ gate is 4, Fig.10 shows the decomposition of the gate. Fig.10(a) shows the gate representation of $R^3_{1,2,3}$ gate, Fig.10(b) shows the four component gates of the $R^3_{1,2,3}$ gate, and Fig.10(c) shows the representation of the $R^3_{1,2,3}$ gate into its five elementary gates after applying circuit optimization over it as defined in [14]. The first gate is $v_{3,2}$, the second gate is $C_{1,3}$, the third gate is $u_{3,2}$, the fourth gate is $N_2$ and the last gate is $vC_{2,1}$, which is a merging gate between $v_{1,2}$ and $C_{2,1}$ in order, as shown in Fig.9 the process of merging gates is defined in [11].
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Figure 10. The circuit representation for the decomposition of the $R_{1,2,3}^3$ gate where: (a) the representation of the gate, (b) the decomposition of the gate into its four components two $C$ gates, one $N$ gate and one $T^3$ gate, and (c) the optimized decomposition of $R_{1,2,3}^3$ into its five elementary quantum gates.

$$R_{j,k,l,m}^4 : y_j = x_j \oplus x_k \oplus x_j x_l \oplus 1,$$
$$y_k = x_k \oplus x_j x_l \oplus 1,$$
$$y_l = x_l \oplus x_j,$$
$$y_m = x_m \oplus x_j x_k x_l,$$

\begin{align*}
R_{1,2,3,4}^4 &: (x_1, x_2, x_3, x_4) \rightarrow (1, 13, 11, 9, 7, 3, 15, 6, 2, 14, 2, 10, 8, 4, 16, 5), \\
R_{3,2,1,4}^4 &: (x_1, x_2, x_3, x_4) \rightarrow (1, 7, 11, 3, 13, 9, 15, 6, 2, 8, 12, 4, 14, 10, 16, 5), \\
R_{3,1,2,4}^4 &: (x_1, x_2, x_3, x_4) \rightarrow (1, 7, 13, 5, 11, 9, 15, 4, 2, 8, 14, 6, 12, 10, 16, 3), \\
R_{1,3,2,4}^4 &: (x_1, x_2, x_3, x_4) \rightarrow (1, 11, 13, 9, 7, 5, 15, 4, 2, 12, 14, 10, 8, 6, 16, 3), \\
R_{2,3,1,4}^4 &: (x_1, x_2, x_3, x_4) \rightarrow (1, 11, 7, 3, 13, 5, 15, 10, 2, 12, 8, 4, 14, 6, 16, 9), \\
R_{2,1,3,4}^4 &: (x_1, x_2, x_3, x_4) \rightarrow (1, 13, 7, 5, 11, 3, 15, 10, 2, 14, 8, 6, 12, 4, 16, 9), \\
R_{1,2,4,3}^4 &: (x_1, x_2, x_3, x_4) \rightarrow (1, 13, 10, 9, 6, 2, 14, 7, 3, 15, 12, 11, 8, 4, 16, 5), \\
R_{2,3,2,4}^4 &: (x_1, x_2, x_3, x_4) \rightarrow (1, 6, 10, 2, 13, 9, 14, 7, 3, 8, 12, 4, 15, 11, 6, 5), \\
R_{2,1,4,2}^4 &: (x_1, x_2, x_3, x_4) \rightarrow (1, 6, 13, 5, 10, 9, 14, 4, 3, 8, 15, 7, 12, 11, 6, 2), \\
R_{1,3,2,4}^4 &: (x_1, x_2, x_3, x_4) \rightarrow (1, 11, 13, 4, 2, 12, 14, 9, 7, 5, 15, 10, 8, 6, 16, 3), \\
R_{2,3,4,1}^4 &: (x_1, x_2, x_3, x_4) \rightarrow (1, 10, 6, 2, 13, 5, 14, 11, 3, 12, 8, 4, 15, 7, 16, 9), \\
R_{2,1,4,3}^4 &: (x_1, x_2, x_3, x_4) \rightarrow (1, 13, 6, 5, 10, 2, 14, 11, 3, 15, 8, 7, 12, 4, 16, 9), \\
R_{1,4,2,3}^4 &: (x_1, x_2, x_3, x_4) \rightarrow (1, 11, 10, 9, 4, 2, 12, 7, 5, 15, 14, 13, 8, 6, 16, 3), \\
R_{1,3,2,1}^4 &: (x_1, x_2, x_3, x_4) \rightarrow (1, 4, 10, 2, 11, 9, 12, 7, 5, 8, 14, 6, 15, 13, 16, 3), \\
R_{2,4,1,2}^4 &: (x_1, x_2, x_3, x_4) \rightarrow (1, 4, 11, 3, 10, 9, 12, 6, 5, 8, 15, 7, 14, 13, 16, 2), \\
R_{1,4,3,2}^4 &: (x_1, x_2, x_3, x_4) \rightarrow (1, 10, 11, 9, 4, 3, 12, 6, 5, 14, 15, 13, 8, 7, 16, 2), \\
R_{2,4,3,1}^4 &: (x_1, x_2, x_3, x_4) \rightarrow (1, 10, 4, 2, 11, 3, 12, 13, 5, 14, 8, 6, 15, 7, 16, 9), \\
R_{2,4,1,3}^4 &: (x_1, x_2, x_3, x_4) \rightarrow (1, 11, 4, 3, 10, 2, 12, 13, 5, 15, 8, 7, 14, 6, 16, 9), \\
R_{1,2,3}^4 &: (x_1, x_2, x_3, x_4) \rightarrow (1, 7, 6, 5, 4, 2, 8, 11, 9, 15, 14, 13, 12, 10, 16, 3), \\
R_{4,3,2,1}^4 &: (x_1, x_2, x_3, x_4) \rightarrow (1, 4, 6, 2, 7, 5, 8, 11, 9, 12, 14, 10, 15, 13, 16, 3), \\
R_{4,3,1,2}^4 &: (x_1, x_2, x_3, x_4) \rightarrow (1, 4, 7, 3, 6, 5, 8, 10, 9, 12, 15, 11, 14, 13, 16, 2), \\
R_{4,1,3,2}^4 &: (x_1, x_2, x_3, x_4) \rightarrow (1, 6, 7, 5, 4, 3, 8, 10, 9, 14, 15, 13, 12, 11, 16, 2), \\
R_{4,2,3,1}^4 &: (x_1, x_2, x_3, x_4) \rightarrow (1, 6, 4, 2, 7, 3, 8, 13, 9, 14, 12, 10, 15, 11, 16, 5), \\
R_{4,2,1,3}^4 &: (x_1, x_2, x_3, x_4) \rightarrow (1, 7, 4, 3, 6, 2, 8, 13, 9, 15, 12, 11, 14, 10, 16, 5),
3.4. 4-bits Gate ($R^4$ gate)

$R^4$ is 4-bits gate, which combines the action of the four gates $N$, $C$, $T^3$ and $T^4$. The four gates are applied on an arbitrary 4-bits $x_j, x_k, x_l, x_m$ in any order as follows, First the 3 bits $x_j, x_k$ and $x_l$ are used as controller bits to flip the bit $x_m$, then the 2 bits $x_j$ and $x_k$ are used as controller bits to flip $x_l$. After that $x_j$ is used as a controller bit to flip $x_k$, then $x_l$ is flipped unconditionally and finally $x_l$ is used as a controller bit to flip $x_j$. Fig.11 shows the decomposition of the gate, Fig.11(a) shows the representation of the $R^4_{1,2,3,4}$ gate, and Fig.11(b) shows the decomposition of the $R^4_{1,2,3,4}$ gate into its five components. There are 24 possible $R^4$ gates. These 24 gates are sufficient to realize the $(2^4)!$ circuits. For 4-bits reversible circuits built using $R$-gate library, there are 24 possible $R^4$ gates, their function is shown in Eqn.13.

![Figure 11](image)

**Figure 11.** The circuit representation for the decomposition of the $R^4$ gate where: (a) the representation of the gate, (b) the decomposition of the gate into its 5 components one $T^4$ gate, one $T^3$ gate, two $C$ gates and one $N$ gate.

3.5. $n$-bits Gate ($R^n$ gate)

The $R^n$ gate is universal of $n$-bit gate, where it can be extended according to the value of $n$. For $n ≥ 3$, $R^n$ combines the action of the $n$ gates $N, C, T^3, T^4, ..., T^{n-1}, T^n$ as shown in Fig.12. There are $n!$ possible $R^n$ gates which are sufficient to realize any $n$-bits circuit. The function of the $n!$ $R^n$ gates are shown in Eqn.14.
New Designs of Universal Reversible Gate Library

\[ R_{a_1,a_2,a_3,a_4,a_5,a_6,...,a_{n-1},a_n}^n: \]
\[ y_{a_1} = x_{a_1} \oplus x_{a_3} \oplus x_{a_1} \cdot x_{a_2} \oplus 1, \]
\[ y_{a_2} = x_{a_2} \oplus x_{a_1} \cdot x_{a_3} \oplus 1, \]
\[ y_{a_3} = x_{a_3} \oplus x_{a_1}, \]
\[ y_{a_4} = x_{a_4} \oplus x_{a_1} \cdot x_{a_2} \cdot x_{a_3}, \]
\[ y_{a_5} = x_{a_5} \oplus x_{a_1} \cdot x_{a_2} \cdot x_{a_3} \cdot x_{a_4}, \]
\[ y_{a_6} = x_{a_6} \oplus x_{a_1} \cdot x_{a_2} \cdot x_{a_3} \cdot x_{a_4} \cdot x_{a_5}, \]
\[ \vdots \]
\[ y_{a_{n-1}} = x_{a_{n-1}} \oplus x_{a_1} \cdot x_{a_2} \cdot x_{a_3} \cdot x_{a_4} \cdot x_{a_5} \cdots x_{a_{n-2}}, \]
\[ y_{a_n} = x_{a_n} \oplus x_{a_1} \cdot x_{a_2} \cdot x_{a_3} \cdot x_{a_4} \cdot x_{a_5} \cdots x_{a_{n-1}}, \]

where \( a_1, a_2, a_3, \ldots \) and \( a_n \in \{1, 2, 3, \ldots, n\} \) in any order. While the total number of possible gates for the general \( T_n \) is shown in Eqn. [15] [3].

\[ n \sum_{r=0}^{n-1} \left( \begin{array}{c} n-1 \\ r \end{array} \right), \]

where \( n \) is the number of bits and \( r \geq 0 \) is the number of controls per gate.

![Circuit Diagram](image-url)

**Figure 12.** The circuit representation for the decomposition of the \( R_{1,2,3,4,...,n}^n \) gate where: (a) the representation of the gate, (b) the decomposition of the gate into its main components gates.
Table 1. Utilization of gates in universal sub Libs.

| Lib  | Lib Size | Num of sub Lib | Num of universal sub Lib | Utilization  |
|------|----------|----------------|--------------------------|--------------|
| NT   | 6        | 64             | 4                        | 6.250%       |
| NP   | 9        | 512            | 333                      | 65.039%      |
| NCT  | 12       | 4096           | 1960                     | 47.852%      |
| NCF  | 12       | 4096           | 2460                     | 60.059%      |
| NCP  | 15       | 32768          | 26064                    | 79.541%      |
| NCTF | 15       | 32768          | 23132                    | 70.593%      |
| NCPF | 18       | 262144         | 217384                   | 82.925%      |
| G^3  | 6        | 64             | 51                       | 79.688%      |
| R^3  | 6        | 64             | 55                       | 85.938%      |
| NR^3 | 9        | 512            | 340                      | 66.406%      |

4. Experimental Results

This section compares the performance of the proposed two gate libraries $R^3$ and $NR^3$, with the known libraries in [2], [4], [5], [14] and [15]. It is found that the permutation group of the $R^3$ generators is of size 40320, thus the six $R^3$ generators are universal. There are 64 possible sub libraries from the main $R^3$ gate library, 55 of them are universal for the 3-bits reversible circuits. It is also found that the permutation group of the $NR^3$ generators is of size 40320, thus the nine generators in $NR^3$ library are universal. There are 512 possible sub libraries from the main library $NR^3$, 340 of them are universal. Table 1 compares the utilization of the different universal libraries. It can be seen that the gate library $R^3$ gives the best utilization of 85.938% and $NR^3$ gives a utilization of 66.406%, which is better than the utilization of the libraries $NT$, $NP$, $NCT$ and $NCF$.

The size of the minimum universal sub libraries from the main $R^3$ library and $NR^3$ library is two. For $R^3$, there are 13 universal sub libraries of size two, such as $\{R_{1,3,2}, R_{2,1,3}\}$, $\{R_{1,3,2}, R_{2,3,1}\}$ and $\{R_{1,3,2}, R_{3,1,2}\}$, while for $NR^3$ there are 8 universal sub libraries of size two, such as $\{R_{1,3,2}, R_{3,1,2}\}$, $\{R_{1,3,2}, R_{3,2,1}\}$ and $\{R_{3,1,2}, R_{3,2,1}\}$. Table 2 compares the utilization of gates in the smallest universal sub libraries. The utilization of the universal sub libraries with minimum size for $R^3$ is 86.667%, while for $NR^3$ is 22.222%. It shows that $R^3$ gives the best utilization, while $NR^3$ is better than $NP$, $NCT$, $NCF$, $NCP$, $NCTF$, $NCPT$ and $NCPF$.

Table 3 compares the minimum length for the 3-bits circuits using different libraries. It shows that the average minimum length for $R^3$ is 6.425, while for $NR^3$ it is 5.325 which is less than $NT$, $NP$, $NCT$, $NCF$, $NCTF$, $G^3$ and $R^3$. 
Table 2. Utilization of gates in the smallest universal sub libraries.

| Lib   | Size of min universal sub Lib | Num of sub Lib with min size | Num of universal sub Lib with min size | utilization |
|-------|-------------------------------|-------------------------------|----------------------------------------|-------------|
| NT    | 5                             | 6                             | 3                                      | 50%         |
| NP    | 3                             | 84                            | 18                                     | 21.429%     |
| NCT   | 4                             | 495                           | 21                                     | 4.242%      |
| NCF   | 4                             | 495                           | 60                                     | 12.121%     |
| NCP   | 3                             | 455                           | 30                                     | 6.593%      |
| NCTF  | 4                             | 1365                          | 105                                    | 7.692%      |
| NCPT  | 3                             | 816                           | 36                                     | 4.412%      |
| NCPF  | 3                             | 816                           | 42                                     | 5.147%      |
| $G^3$ | 2                             | 15                            | 9                                      | 60%         |
| $R^3$ | 2                             | 15                            | 13                                     | 86.667%     |
| NR$^3$| 2                             | 36                            | 8                                      | 22.222%     |

Table 3. Minimum length of 3-bits reversible circuits using $NT$, $NP$, $NCT$, $NCF$, $NCP$, $NCTF$, $NCPT$, $NCPF$, $G^3$, $R^3$ and $NR^3$ libraries.

| Min Len | $NT$ | $NP$ | $NCT$ | $NCF$ | $NCP$ | $NCTF$ | $NCPT$ | $NCPF$ | $G^3$ | $R^3$ | $NR^3$ |
|---------|------|------|-------|-------|-------|--------|--------|--------|-------|-------|--------|
| 0       | 1    | 1    | 1     | 1     | 1     | 1      | 1      | 1      | 1     | 1     | 1      |
| 1       | 6    | 9    | 12    | 12    | 15    | 15     | 18     | 18     | 6     | 6     | 9      |
| 2       | 24   | 69   | 102   | 101   | 174   | 143    | 228    | 248    | 36    | 33    | 72     |
| 3       | 88   | 502  | 625   | 676   | 1528  | 1006   | 1993   | 2356   | 207   | 180   | 541    |
| 4       | 296  | 3060 | 2780  | 3413  | 8968  | 5021   | 10503  | 12797  | 1097  | 960   | 3774   |
| 5       | 870  | 13432| 8921  | 11378 | 23534 | 15083  | 23204  | 22794  | 4946  | 4686  | 18027  |
| 6       | 2262 | 21360| 17049 | 17970 | 6100  | 17261  | 4373   | 2106   | 13819 | 14611 | 17556  |
| 7       | 5097 | 1887 | 10253 | 6739  | 0     | 1790   | 0      | 0      | 14824 | 15257 | 340    |
| 8       | 9339 | 0    | 577   | 0     | 0     | 0      | 0      | 0      | 5208  | 4555  | 0      |
| 9       | 12237| 0    | 0     | 0     | 0     | 0      | 0      | 0      | 31    | 0     | 0      |
| 10      | 8363 | 0    | 0     | 0     | 0     | 0      | 0      | 0      | 0     | 0     | 0      |
| 11      | 1690 | 0    | 0     | 0     | 0     | 0      | 0      | 0      | 0     | 0     | 0      |
| 12      | 47   | 0    | 0     | 0     | 0     | 0      | 0      | 0      | 0     | 0     | 0      |
| Avg     | 8.500| 5.516| 5.866 | 5.649 | 4.839 | 5.330  | 4.730  | 4.598  | 6.403 | 6.425 | 5.325  |

The optimization rules defined in [14] has been applied on the circuits built using $R$, $NR$ and $NT$ Libraries, which can be summarized as follows: first the $R$ gate is decomposed into its five main components, then the $T^3$ gate is decomposed into its five main components as defined in [14]. The adjacent gates are compared with each other, if the two adjacent gates work on the same input/output vectors, then they are merged to form one new gate, such that: if gate $C_{i,j}^3$ is adjacent to gate $C_{i,j}^3$, then the two gates are merged to give identity, otherwise if it is adjacent to $C_{j,i}^3$ or $v_{j,i}$ or $u_{j,i}$, then these gates are merged to form one gate. If $C_{i,j}^3$ is adjacent to $v_{i,j}$, then they are merged to form $u_{i,j}$, otherwise if it is adjacent to $u_{i,j}$, then they are merged to form $v_{i,j}$. If $v_{i,j}$ is adjacent to $v_{i,j}$, then they are merged to form $C_{i,j}$, or if it is adjacent to $u_{i,j}$, then these
two gates are merged to give identity, otherwise if it is adjacent to $v_{j,i}$ or $u_{j,i}$, then they are merged to form one gate. If $u_{i,j}$ is adjacent to $u_{i,j}$, then they are merged to $C_{i,j}$, otherwise if it is adjacent to $u_{j,i}$ or $v_{j,i}$, then these two gates are merged to form one gate, where $i, j, k \in \{1, 2, 3\}$ in any order. Some adjacent gates can be swapped with each other, if the target vector of one gate is not affecting the controller vector of the other gate, as defined in [14].

The costs of all the 40320 specifications synthesized by the $R^3$, $NR^3$ and $NT$ gate libraries are calculated and compared before and after optimization in Table 4. For circuits built using $R^3$-gate library, the maximum cost is 36 and the average cost is 25.701 before optimization, while after optimization, the maximum cost is 34 and the average cost is 23.954, giving 6.8% of improvement. After adding the $N$ gate to the gate library, the maximum cost has been reduced to 24, having an average cost of 14.062 before optimization and 13.388 after optimization, giving 44.1% of improvement. For circuits built using $NT$-gate library, the maximum cost is 40 and the average cost is 25.766 before optimization, while after optimization the maximum cost has been reduced to 36 and the average cost became 22.321, giving 13.4% of improvement. As shown in Table 4 the cost of the circuits built using $NR^3$-gate library have improved by 40% over those built using $NT$-gate library.

Table 4 compares the minimum cost for 3-bits circuits using different libraries. The average minimum cost for the circuits built using $R^3$ is 23.954, while the average minimum cost for those built using $NR^3$ is 13.388, which is less than those built using $NT$, $NCF$, $NCTF$ and $R^3$.

5. Conclusion

We have presented new algorithms to synthesize exact universal reversible gate library. Experimental results have been used to compares the proposed universal gate library with the existing approaches to reversible logic synthesis. We compares the minimum cost for 3-bits circuits using different libraries. It is shown that the cost of 40320 specifications synthesized by the $R^3$, $NR^3$ and $NT$ gate library can be calculated and optimized. Also, the average minimum cost for $R^3$ is 23.954, while the average minimum cost for $NR^3$ is 13.388, which is less than $NT$, $NCF$, $NCTF$ and $R^3$. For $n \geq 3$, $R^n$ combines the action of the $n$ gates $N, C, T^3, T^4, ..., T^{n-1}, T^n$. There are $n!$ possible $R^n$ gates which are sufficient to realize any $n$-bits circuit. We have found that for $n$-bit circuits built using $NR$-gate library, there are $n!$ $R^n$ gates and $n$ $N$ gates. Finally, we have applied our optimal synthesis tool to obtain much smaller circuits than previous methods.
Table 4. Comparing the cost of the 3-bits circuits built using $R^3$, $NR^3$ and $NT$-gate libraries before and after optimization.

| Min Cost | Num spc in $R^3$ bfr optm. | Num spc in $R^3$ aft optm. | Num spc in $NR^3$ bfr optm. | Num spc in $NR^3$ aft optm. | Num spc in $NT$ bfr optm. | Num spc in $NT$ aft optm. | Num spc in $NT$ aft optm. |
|----------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
| 0        | 1                           | 1                           | 7                           | 7                           | 7                           | 7                           | 7                           |
| 1        | 0                           | 0                           | 0                           | 0                           | 0                           | 0                           | 0                           |
| 2        | 0                           | 0                           | 0                           | 0                           | 0                           | 0                           | 0                           |
| 3        | 0                           | 0                           | 0                           | 0                           | 0                           | 0                           | 0                           |
| 4        | 6                           | 6                           | 192                         | 192                         | 0                           | 0                           | 0                           |
| 5        | 0                           | 0                           | 0                           | 0                           | 96                          | 94                          |
| 6        | 0                           | 0                           | 0                           | 0                           | 0                           | 0                           | 0                           |
| 7        | 0                           | 9                           | 851                         | 0                           | 0                           | 0                           | 0                           |
| 8        | 33                          | 24                          | 3442                        | 2591                        | 0                           | 16                          |
| 9        | 0                           | 0                           | 0                           | 0                           | 0                           | 0                           | 340                         |
| 10       | 0                           | 16                          | 636                         | 648                         | 288                         |
| 11       | 0                           | 68                          | 6050                        | 0                           | 32                          |
| 12       | 180                         | 96                          | 16040                       | 9353                        | 0                           | 179                         |
| 13       | 0                           | 28                          | 0                           | 39                          | 0                           | 790                         |
| 14       | 0                           | 162                         | 0                           | 2829                        | 0                           | 1487                        |
| 15       | 0                           | 394                         | 0                           | 7175                        | 2694                        | 324                         |
| 16       | 960                         | 422                         | 16676                       | 6331                        | 0                           | 574                         |
| 17       | 0                           | 341                          | 344                         | 0                           | 2052                        |
| 18       | 0                           | 1121                         | 0                           | 1200                        | 0                           | 3616                        |
| 19       | 0                           | 1919                         | 0                           | 1278                        | 0                           | 1462                        |
| 20       | 4686                        | 1798                         | 3928                        | 1053                        | 7640                        | 1041                        |
| 21       | 0                           | 1798                         | 0                           | 9                           | 0                           | 3405                        |
| 22       | 0                           | 4218                         | 0                           | 14                           | 0                           | 5357                        |
| 23       | 0                           | 5553                         | 0                           | 2                            | 0                           | 2894                        |
| 24       | 4059                        | 495                          | 34                          | 7                            | 0                           | 1435                        |
| 25       | 0                           | 3097                         | 0                           | 0                            | 12881                        | 3191                        |
| 26       | 0                           | 4934                         | 0                           | 0                            | 0                            | 4369                        |
| 27       | 0                           | 4578                         | 0                           | 0                            | 0                            | 2436                        |
| 28       | 15257                       | 2410                         | 0                           | 0                            | 806                          |
| 29       | 0                           | 1407                         | 0                           | 0                            | 0                            | 1444                        |
| 30       | 0                           | 1273                         | 0                           | 0                            | 11502                        | 1482                        |
| 31       | 0                           | 524                          | 0                           | 0                            | 761                          |
| 32       | 4555                        | 56                           | 0                           | 0                            | 125                          |
| 33       | 0                           | 4                            | 0                           | 0                            | 126                          |
| 34       | 0                           | 4                            | 0                           | 0                            | 109                          |
| 35       | 0                           | 0                            | 0                           | 0                            | 4489                         | 60                          |
| 36       | 31                          | 0                            | 0                           | 0                            | 0                            | 6                           |
| 37       | 0                           | 0                            | 0                           | 0                            | 0                            |
| 38       | 0                           | 0                            | 0                           | 0                            | 0                            |
| 39       | 0                           | 0                            | 0                           | 0                            | 0                            |
| 40       | 0                           | 0                            | 0                           | 0                            | 362                          | 0                            |
| Avg      | 25.701                      | 23.954                       | 14.062                      | 13.388                      | 25.766                       | 22.321                      |
Table 5. Optimal quantum cost of 3-bits reversible circuits using $NT$, $NCT$, $NCF$, $NFT$, $NCTF$, $NCTPF$, $NR^3$ and $R^3$ libraries.

| Min Cost | Num spc in $NT$ | Num spc in $NCT$ | Num spc in $NCF$ | Num spc in $NFT$ | Num spc in $NCTF$ | Num spc in $NCTPF$ | Num spc in $NR^3$ | Num spc in $R^3$ |
|----------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0        | 7               | 7               | 1               | 7               | 1               | 1               | 5               | 1               |
| 1        | 0               | 48              | 9               | 48              | 9               | 9               | 0               | 0               |
| 2        | 0               | 324             | 51              | 192             | 51              | 51              | 0               | 0               |
| 3        | 0               | 607             | 187             | 408             | 187             | 187             | 0               | 0               |
| 4        | 0               | 601             | 393             | 480             | 393             | 405             | 195             | 6               |
| 5        | 94              | 1148            | 474             | 288             | 477             | 609             | 0               | 0               |
| 6        | 0               | 2462            | 215             | 592             | 260             | 998             | 0               | 0               |
| 7        | 0               | 3576            | 17              | 1962            | 338             | 2648            | 851             | 9               |
| 8        | 16              | 2710            | 48              | 3887            | 1335            | 4397            | 2591            | 24              |
| 9        | 340             | 2855            | 408             | 2916            | 3224            | 2712            | 0               | 0               |
| 10       | 288             | 5601            | 1919            | 1299            | 3686            | 5994            | 636             | 16              |
| 11       | 32              | 6567            | 3931            | 3683            | 902             | 10249           | 6050            | 68              |
| 12       | 179             | 3183            | 2634            | 7221            | 933             | 1750            | 9354            | 96              |
| 13       | 790             | 2043            | 462             | 6059            | 4053            | 3488            | 396             | 28              |
| 14       | 1487            | 3771            | 5               | 1465            | 8690            | 6640            | 2829            | 162             |
| 15       | 324             | 3496            | 78              | 3562            | 4903            | 182             | 7175            | 394             |
| 16       | 574             | 1284            | 1038            | 4201            | 244             | 0               | 6331            | 422             |
| 17       | 2052            | 36              | 6079            | 2049            | 1094            | 0               | 344             | 341             |
| 18       | 3616            | 0               | 9571            | 0               | 4346            | 0               | 1200            | 1121            |
| 19       | 1462            | 0               | 2036            | 0               | 4724            | 0               | 1278            | 1919            |
| 20       | 1041            | 0               | 12              | 0               | 470             | 0               | 1053            | 1798            |
| 21       | 3405            | 0               | 0               | 0               | 0               | 0               | 9               | 1798            |
| 22       | 5357            | 0               | 24              | 0               | 0               | 0               | 14              | 4218            |
| 23       | 2894            | 0               | 732             | 0               | 0               | 0               | 2               | 5553            |
| 24       | 1435            | 0               | 5496            | 0               | 0               | 0               | 7               | 4059            |
| 25       | 3191            | 0               | 4482            | 0               | 0               | 0               | 0               | 3097            |
| 26       | 4369            | 0               | 18              | 0               | 0               | 0               | 0               | 4934            |
| 27       | 2436            | 0               | 0               | 0               | 0               | 0               | 0               | 4578            |
| 28       | 806             | 0               | 0               | 0               | 0               | 0               | 0               | 2410            |
| 29       | 1444            | 0               | 0               | 0               | 0               | 0               | 0               | 1407            |
| 30       | 1482            | 0               | 0               | 0               | 0               | 0               | 0               | 1273            |
| 31       | 761             | 0               | 0               | 0               | 0               | 0               | 0               | 524             |
| 32       | 125             | 0               | 0               | 0               | 0               | 0               | 0               | 56              |
| 33       | 126             | 0               | 0               | 0               | 0               | 0               | 0               | 4               |
| 34       | 109             | 0               | 0               | 0               | 0               | 0               | 0               | 4               |
| 35       | 60              | 0               | 0               | 0               | 0               | 0               | 0               | 0               |
| Avg      | 22.321          | 10.348          | 17.468          | 11.770          | 13.740          | 10.520          | 13.388          | 23.954          |
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