Sub-Manifolds of a Riemannian Manifold

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Abstract

In this chapter, we introduce the theory of sub-manifolds of a Riemannian manifold. The fundamental notations are given. The theory of sub-manifolds of an almost Riemannian product manifold is one of the most interesting topics in differential geometry. According to the behaviour of the tangent bundle of a sub-manifold, with respect to the action of almost Riemannian product structure of the ambient manifolds, we have three typical classes of sub-manifolds such as invariant sub-manifolds, anti-invariant sub-manifolds and semi-invariant sub-manifolds. In addition, slant, semi-slant and pseudo-slant sub-manifolds are introduced by many geometers.

Keywords: Riemannian product manifold, Riemannian product structure, integral manifold, a distribution on a manifold, real product space forms, a slant distribution

1. Introduction

Let \( i : M \rightarrow \tilde{M} \) be an immersion of an \( n \)-dimensional manifold \( M \) into an \( m \)-dimensional Riemannian manifold \((\tilde{M}, \tilde{g})\). Denote by \( g = i^* \tilde{g} \) the induced Riemannian metric on \( M \). Thus, \( i \) becomes an isometric immersion and \( M \) is also a Riemannian manifold with the Riemannian metric \( g(X, Y) = \tilde{g}(X, Y) \) for any vector fields \( X, Y \) in \( M \). The Riemannian metric \( g \) on \( M \) is called the induced metric on \( M \). In local components, \( \tilde{g}_{ij} = g_{AB} B_i^B B_j^A \) with \( g = g_{\mu} dx^\mu dx^\nu \) and \( \tilde{g} = \tilde{g}_{BA} dU^B dU^A \).

If a vector field \( \xi_p \) of \( \tilde{M} \) at a point \( p \in M \) satisfies
\[
\tilde{g}(X_p, \xi_p) = 0
\]
for any vector \( X_p \) of \( M \) at \( p \), then \( \xi_p \) is called a normal vector of \( M \) in \( \tilde{M} \) at \( p \). A unit normal vector field of \( M \) in \( \tilde{M} \) is called a normal section on \( M \) [3].
By $T^\perp M$, we denote the vector bundle of all normal vectors of $M$ in $\tilde{M}$. Then, the tangent bundle of $\tilde{M}$ is the direct sum of the tangent bundle $TM$ of $M$ and the normal bundle $T^\perp M$ of $M$ in $\tilde{M}$, i.e.,

$$T\tilde{M} = TM \oplus T^\perp M.$$  \hfill (2)

We note that if the sub-manifold $M$ is of codimension one in $\tilde{M}$ and they are both orientable, we can always choose a normal section $\xi$ on $M$, i.e.,

$$g(X, \xi) = 0, \quad g(\xi, \xi) = 1,$$  \hfill (3)

where $X$ is any arbitrary vector field on $M$.

By $\tilde{\nabla}$, denote the Riemannian connection on $\tilde{M}$ and we put

$$\tilde{\nabla}_XY = \nabla_XY + h(X, Y)$$  \hfill (4)

for any vector fields $X, Y$ tangent to $M$, where $\nabla_XY$ and $h(X, Y)$ are tangential and the normal components of $\tilde{\nabla}_XY$, respectively. Formula (4) is called the Gauss formula for the sub-manifold $M$ of a Riemannian manifold $(\tilde{M}, \tilde{g})$.

**Proposition 1.1.** $\nabla$ is the Riemannian connection of the induced metric $g = i^*\tilde{g}$ on $M$ and $h(X, Y)$ is a normal vector field over $M$, which is symmetric and bilinear in $X$ and $Y$.

**Proof:** Let $\alpha$ and $\beta$ be differentiable functions on $M$. Then, we have

$$\tilde{\nabla}_{aX}(\beta Y) = \nabla_{aX}(\beta Y) + \beta \nabla_{aX}Y$$

$$= a\{X(\beta)Y + \beta \nabla_XY + \beta h(X, Y)\}$$

$$\nabla_{aX}\beta Y + h(aX, \beta Y) = a\beta \nabla_XY + aX(\beta)Y + a\beta h(X, Y)$$  \hfill (5)

This implies that

$$\nabla_{aX}(\beta Y) = aX(\beta)Y + a\beta \nabla_XY$$  \hfill (6)

and

$$h(aX, \beta Y) = a\beta h(X, Y).$$  \hfill (7)

Eq. (6) shows that $\nabla$ defines an affine connection on $M$ and Eq. (4) shows that $h$ is bilinear in $X$ and $Y$ since additivity is trivial [1].

Since the Riemannian connection $\tilde{\nabla}$ has no torsion, we have

$$0 = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y] = \nabla_X Y + h(X, Y) - \nabla_Y X - h(Y, X) - [X, Y].$$  \hfill (8)

By comparing the tangential and normal parts of the last equality, we obtain
\[ \nabla_X Y - \nabla_Y X = [X, Y] \]  
(9)

and

\[ h(X, Y) = h(Y, X). \]  
(10)

These equations show that \( \nabla \) has no torsion and \( h \) is a symmetric bilinear map. Since the metric \( \tilde{g} \) is parallel, we can easily see that

\[
(\nabla_X g)(Y, Z) = (\tilde{\nabla}_X \tilde{g})(Y, Z) \\
= \tilde{g}(\tilde{\nabla}_X Y + h(X, Y), Z) + \tilde{g}(Y, \tilde{\nabla}_X Z + h(X, Z)) \\
= \tilde{g}(\nabla_X Y, Z) + \tilde{g}(Y, \nabla_X Z) \\
= g(\nabla_X Y, Z) + g(Y, \nabla_X Z)
\]  
(11)

for any vector fields \( X, Y, Z \) tangent to \( M \), that is, \( \nabla \) is also the Riemannian connection of the induced metric \( g \) on \( M \).

We recall \( h \) the second fundamental form of the sub-manifold \( M \) (or immersion \( i \)), which is defined by

\[ h : \Gamma(TM) \times \Gamma(TM) \to \Gamma(T^1M). \]  
(12)

If \( h = 0 \) identically, then sub-manifold \( M \) is said to be totally geodesic, where \( \Gamma(T^1M) \) is the set of the differentiable vector fields on normal bundle of \( M \).

Totally geodesic sub-manifolds are simplest sub-manifolds.

**Definition 1.1.** Let \( M \) be an \( n \)-dimensional sub-manifold of an \( m \)-dimensional Riemannian manifold \( (\tilde{M}, \tilde{g}) \). By \( h \), we denote the second fundamental form of \( M \) in \( \tilde{M} \).

\[ H = \frac{1}{n} \text{trace}(h) \] is called the mean curvature vector of \( M \) in \( \tilde{M} \). If \( H = 0 \), the sub-manifold is called minimal.

On the other hand, \( M \) is called pseudo-umbilical if there exists a function \( \lambda \) on \( M \), such that

\[ \tilde{g}\left(h(X, Y), H\right) = \lambda g(X, Y) \]  
(13)

for any vector fields \( X, Y \) on \( M \) and \( M \) is called totally umbilical sub-manifold if

\[ h(X, Y) = g(X, Y)H. \]  
(14)

It is clear that every minimal sub-manifold is pseudo-umbilical with \( \lambda = 0 \). On the other hand, by a direct calculation, we can find \( \lambda = \tilde{g}(H, H) \) for a pseudo-umbilical sub-manifold. So, every
totally umbilical sub-manifold is a pseudo-umbilical and a totally umbilical sub-manifold is totally geodesic if and only if it is minimal [2].

Now, let \( M \) be a sub-manifold of a Riemannian manifold \( (\tilde{M}, \tilde{g}) \) and \( V \) be a normal vector field on \( M \), \( X \) be a vector field on \( M \). Then, we decompose

\[
\tilde{\nabla}_X V = -A_V X + \nabla^\perp_X V, \tag{15}
\]

where \( A_V X \) and \( \nabla^\perp_X V \) denote the tangential and the normal components of \( \nabla^\perp_X V \), respectively.

We can easily see that \( A_V X \) and \( \nabla^\perp_X V \) are both differentiable vector fields on \( M \) and normal bundle of \( M \), respectively. Moreover, Eq. (15) is also called Weingarten formula.

**Proposition 1.2.** Let \( M \) be a sub-manifold of a Riemannian manifold \( (\tilde{M}, \tilde{g}) \). Then

(a) \( A_V X \) is bilinear in vector fields \( V \) and \( X \). Hence, \( A_V X \) at point \( p \in M \) depends only on vector fields \( V_p \) and \( X_p \).

(b) For any normal vector field \( V \) on \( M \), we have

\[
g(A_V X, Y) = g\left(h(X, Y), V\right). \tag{16}
\]

**Proof:** Let \( \alpha \) and \( \beta \) be any two functions on \( M \). Then, we have

\[
\tilde{\nabla}_{aX} (\beta V) = a \tilde{\nabla}_X (\beta V)
\]

\[
= a \{ X(\beta) V + \beta \tilde{\nabla}_X V \}
\]

\[
- A_{\beta V} aX + \nabla^\perp_{aX} \beta V = aX(\beta) V - a\beta A_V X + a\beta \nabla^\perp_X V. \tag{17}
\]

This implies that

\[
A_{\beta V} aX = a\beta A_V X \tag{18}
\]

and

\[
\nabla^\perp_{aX} \beta V = aX(\beta) V + a\beta \nabla^\perp_X V. \tag{19}
\]

Thus, \( A_V X \) is bilinear in \( V \) and \( X \). Additivity is trivial. On the other hand, since \( g \) is a Riemannian metric,

\[
X_{\tilde{g}}(Y, V) = 0, \tag{20}
\]

for any \( X, Y \in \Gamma(TM) \) and \( V \in \Gamma(T^\perp M) \).

Eq. (12) implies that

\[
\tilde{g}(\tilde{\nabla}_X Y, V) + \tilde{g}(Y, \tilde{\nabla}_X V) = 0. \tag{21}
\]

By means of Eqs. (4) and (15), we obtain
\[ \tilde{g}\left(h(X, Y), V\right) - g(A_V X, Y) = 0. \] (22)

The proof is completed [3].

Let \( M \) be a sub-manifold of a Riemannian manifold \( (\tilde{M}, \tilde{g}) \), and \( h \) and \( A_V \) denote the second fundamental form and shape operator of \( M \), respectively.

The covariant derivative of \( h \) and \( A_V \) is, respectively, defined by

\[ (\tilde{\nabla}_X h)(Y, Z) = \nabla^\perp_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) \] (23)

and

\[ (\nabla_X A)_V Y = \nabla_X (A_V Y) - A_{\nabla^\perp_X V} Y - A_V \nabla_X Y \] (24)

for any vector fields \( X, Y \) tangent to \( M \) and any vector field \( V \) normal to \( M \). If \( \nabla_X h = 0 \) for all \( X \), then the second fundamental form of \( M \) is said to be parallel, which is equivalent to \( \nabla_X A = 0 \). By direct calculations, we get the relation

\[ g\left((\nabla_X h)(Y, Z), V\right) = g\left((\nabla_X A)_V Y, Z\right). \] (25)

**Example 1.1.** We consider the isometric immersion

\[ \phi : \mathbb{R}^2 \to \mathbb{R}^4, \]

\[ \phi(x_1, x_2) = (x_1, \sqrt{x_1^2-1}, x_2, \sqrt{x_2^2-1}) \] (27)

we note that \( M = \phi(\mathbb{R}^2) \subset \mathbb{R}^4 \) is a two-dimensional sub-manifold of \( \mathbb{R}^4 \) and the tangent bundle is spanned by the vectors

\[ TM = sp\left\{ e_1 = \left(\sqrt{x_1^2-1}, x_1, 0, 0\right), e_2 = \left(0, 0, \sqrt{x_2^2-1}, x_2\right) \right\} \]

and the normal vector fields

\[ T^1M = sp\left\{ w_1 = \left(-x_1, \sqrt{x_1^2-1}, 0, 0\right), w_2 = (0, 0, -x_1, \sqrt{x_2^2-1}) \right\}. \] (28)

By \( \tilde{\nabla} \), we denote the Levi-Civita connection of \( \mathbb{R}^4 \), the coefficients of connection, are given by

\[ \tilde{\nabla}_{e_1} e_1 = \frac{2x_1 \sqrt{x_1^2-1}}{2x_1^2-1} e_1 - \frac{1}{2x_1^2-1} w_1, \]

\[ \tilde{\nabla}_{e_2} e_2 = \frac{2x_2 \sqrt{x_2^2-1}}{2x_2^2-1} e_2 - \frac{1}{2x_2^2-1} w_2 \] (29)
and

$$\nabla_{e_2} e_1 = 0.$$  \hfill (31)

Thus, we have $h(e_1, e_1) = -\frac{1}{2x_1^2 - 1} w_1$, $h(e_2, e_2) = -\frac{1}{2x_2^2 - 1} w_2$ and $h(e_2, e_1) = 0$. The mean curvature vector of $M = \phi(\mathbb{R}^2)$ is given by

$$H = -\frac{1}{2} (w_1 + w_2).$$  \hfill (32)

Furthermore, by using Eq. (16), we obtain

$$g(A_{w_1} e_1, e_1) = g(h(e_1, e_1), w_1) = -\frac{1}{2x_1^2 - 1} (x_1^2 + x_1^2 - 1) = -1,$$

$$g(A_{w_2} e_2, e_2) = g(h(e_2, e_2), w_1) = -\frac{1}{2x_2^2 - 1} g(w_1, w_2) = 0,$$

$$g(A_{w_1} e_1, e_2) = 0,$$

and

$$g(A_{w_2} e_1, e_1) = g(h(e_1, e_1), w_2) = 0,$$

$$g(A_{w_2} e_1, e_2) = 0, g(A_{w_2} e_2, e_2) = 1.$$  \hfill (34)

Thus, we have

$$A_{w_1} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad A_{w_2} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}. \hfill (35)$$

Now, let $M$ be a sub-manifold of a Riemannian manifold $(\tilde{M}, \tilde{g})$, $\tilde{R}$ and $R$ be the Riemannian curvature tensors of $\tilde{M}$ and $M$, respectively. From then the Gauss and Weingarten formulas, we have

$$\tilde{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$$

$$\begin{aligned}
&= \nabla_X \left( \nabla_Y Z + h(Y, Z) \right) - \nabla_Y \left( \nabla_X Z + h(X, Z) \right) - \nabla_{[X, Y]} Z - h([X, Y], Z) \\
&= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_X h(Y, Z) - \nabla_Y h(X, Z) - \nabla_{[X, Y]} Z - h(X, Z) + h(Y, Z) \\
&\quad - A_{h(Y, Z)} X - A_{h(X, Z)} Y - \nabla_{h(X, Z)} Y - h(Y, Z) - h(X, Z) \\
&= R(X, Y) Z + (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) + A_{h(X, Z)} Y - A_{h(Y, Z)} X
\end{aligned} \hfill (36)$$
Next, we will define the curvature tensor \( \tilde{R}(X, Y)Z = R(X, Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X + (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) \) \( (37) \)
for any vector fields \( X, Y \) and \( Z \) tangent to \( M \). For any vector field \( W \) tangent to \( M \), Eq. (37) gives the Gauss equation
\[
g(\tilde{R}(X, Y)Z, W) = g(R(X, Y)Z, W) + g(h(Y, W), h(X, Z)) - g(h(Y, Z), h(X, W)). \quad (38)
\]
On the other hand, the normal component of Eq. (37) is called equation of Codazzi, which is given by
\[
\left( \tilde{R}(X, Y) \right)^\bot = (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z). \quad (39)
\]
If the Codazzi equation vanishes identically, then sub-manifold \( M \) is said to be curvature-invariant sub-manifold [4].

In particular, if \( \tilde{M} \) is of constant curvature, \( \tilde{R}(X, Y)Z \) is tangent to \( M \), that is, sub-manifold is curvature-invariant. Whereas, in Kenmotsu space forms, and Sasakian space forms, this is not true.

Next, we will define the curvature tensor \( R^\bot \) of the normal bundle of the sub-manifold \( M \) by
\[
R^\bot(X, Y)V = \nabla^\bot_X \nabla^\bot_Y V - \nabla^\bot_Y \nabla^\bot_X V - \nabla^\bot_{[X,Y]} V \quad (40)
\]
for any vector fields \( X, Y \) tangent to sub-manifold \( M \), and any vector field \( V \) normal to \( M \). From the Gauss and Weingarten formulas, we have
\[
\tilde{R}(X, Y)V = \tilde{\nabla}_X \tilde{\nabla}_Y V - \tilde{\nabla}_Y \tilde{\nabla}_X V - \tilde{\nabla}_{[X,Y]} V
\]
\[
= \tilde{\nabla}_X (-A_V Y + \nabla^\bot_X V) - \tilde{\nabla}_Y (-A_V X + \nabla^\bot_Y V) + A_V [X, Y] - \nabla^\bot_{[X,Y]} V
\]
\[
= -\nabla_X A_V Y + \tilde{\nabla}_Y A_V X + \tilde{\nabla}_X \nabla^\bot_Y V - \tilde{\nabla}_Y \nabla^\bot_X V + A_V [X, Y] - \nabla^\bot_{[X,Y]} V
\]
\[
= -\nabla_X A_V Y + \nabla_Y A_V X + h(Y, A_V X)
\]
\[
+ \nabla^\bot_X \nabla^\bot_Y V - \nabla^\bot_Y \nabla^\bot_X V - A_{V^\bot} V X + A_{V^\bot} Y + A_V [X, Y] - \nabla^\bot_{[X,Y]} V
\]
\[
= \nabla^\bot_X \nabla^\bot_Y V - \nabla^\bot_Y \nabla^\bot_X V - \nabla^\bot_{[X,Y]} V - A_{V^\bot} V X + A_{V^\bot} Y + A_V [X, Y]
\]
\[
- \nabla_X A_V Y + \nabla_Y A_V X + h(Y, A_V X)
\]
\[
= R^\bot(X, Y)V + h(A_V Y, X) - h(X, A_V Y) - (\nabla_X A)_V Y + (\nabla_Y A)_V X. \quad (41)
\]
For any normal vector \( U \) to \( M \), we obtain
\[
g\left(\tilde{R}(X, Y)V, U\right) = g\left(R^\perp(X, Y)V, U\right) + g\left(h(A_U X, Y), U\right) - g\left(h(X, A_V Y), U\right)
\]
\[
= g\left(R^\perp(X, Y)V, U\right) + g(A_U Y, A_V X) - g(A_V Y, A_U X)
\]
\[
= g\left(R^\perp(X, Y)V, U\right) + g(A_V A_U Y, X) - g(A_U A_V Y, X)
\]
(42)

Since \([A_U, A_V] = A_U A_V - A_V A_U\), Eq. (42) implies
\[
g\left(\tilde{R}(X, Y)V, U\right) = g\left(R^\perp(X, Y)V, U\right) + g([A_U, A_V]Y, X).
\]
(43)

Eq. (43) is also called the Ricci equation.

If \(R^\perp = 0\), then the normal connection of \(M\) is said to be flat [2].

When \(\left(\tilde{R}(X, Y)V\right)^\perp = 0\), the normal connection of the sub-manifold \(M\) is flat if and only if the second fundamental form \(M\) is commutative, i.e. \([A_U, A_V] = 0\) for all \(U, V\). If the ambient space \(\tilde{M}\) is real space form, then \(\left(\tilde{R}(X, Y)V\right)^\perp = 0\) and hence the normal connection of \(M\) is flat if and only if the second fundamental form is commutative. If \(\tilde{R}(X, Y)Z\) tangent to \(M\), then equation of codazzi Eq. (37) reduces to
\[
(\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z)
\]
(44)

which is equivalent to
\[
(\nabla_X A)Y = (\nabla_Y A)X.
\]
(45)

On the other hand, if the ambient space \(\tilde{M}\) is a space of constant curvature \(c\), then we have
\[
\tilde{R}(X, Y)Z = c\{g(Y, Z)X - g(X, Z)Y\}
\]
(46)

for any vector fields \(X, Y\) and \(Z\) on \(\tilde{M}\).

Since \(\tilde{R}(X, Y)Z\) is tangent to \(M\), the equation of Gauss and the equation of Ricci reduce to
\[
g\left(R(X, Y)Z, W\right) = c\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\}
\]
\[
+ g\left(h(Y, Z), h(X, W)\right) - g\left(h(Y, W), h(X, Z)\right)
\]
(47)

and
\[
g\left(R^\perp(X, Y)V, U\right) = g([A_U, A_V]X, Y),
\]
(48)

respectively.
Proposition 1.3. A totally umbilical sub-manifold $M$ in a real space form $\tilde{M}$ of constant curvature $c$ is also of constant curvature.

Proof: Since $M$ is a totally umbilical sub-manifold of $\tilde{M}$ of constant curvature $c$, by using Eqs. (14) and (46), we have

$$g(R(X, Y)Z, W) = c\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\}$$

$$+ g(H, H)\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\}$$

$$= \{c + g(H, H)\} \{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\}. \quad (49)$$

This shows that the sub-manifold $M$ is of constant curvature $c + \|H\|^2$ for $n > 2$. If $n = 2$, $\|H\| = \text{constant}$ follows from the equation of Codazzi [3].

This proves the proposition.

On the other hand, for any orthonormal basis $\{e_a\}$ of normal space, we have

$$g(Y, Z)g(X, W) - g(X, Z)g(Y, W) = \sum_a \left[ g(h(Y, Z), e_a) g(h(X, W), e_a) \right]$$

$$- g(h(X, Z), e_a) g(h(Y, W), e_a)$$

$$= \sum_a [g(A_{e_a} Y, Z)g(A_{e_a} X, W) - g(A_{e_a} X, Z)g(A_{e_a} Y, W)] \quad (50)$$

Thus, Eq. (45) can be rewritten as

$$g(R(X, Y)Z, W) = c\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\}$$

$$+ \sum_a [g(A_{e_a} Y, Z)g(A_{e_a} X, W) - g(A_{e_a} X, Z)g(A_{e_a} Y, W)] \quad (51)$$

By using $A_{e_a}$, we can construct a similar equation to Eq. (47) for Eq. (23).

Now, let $S$ be the Ricci tensor of $M$. Then, Eq. (47) gives us

$$S(X, Y) = c\{ng(X, Y) - g(e_i, X)g(e_i, Y)\}$$

$$+ \sum_{e_a} [g(A_{e_a} e_a, X)g(A_{e_a} Y) - g(A_{e_a} Y, X)g(A_{e_a} e_a, X)]$$

$$= c(n-1)g(X, Y) + \sum_{e_a} [Tr(A_{e_a})g(A_{e_a} X, Y) - g(A_{e_a} X, A_{e_a} Y)], \quad (52)$$

where $\{e_1, e_2, \ldots, e_n\}$ are orthonormal basis of $M$.

Therefore, the scalar curvature $r$ of sub-manifold $M$ is given by
\[ r = cn(n-1) \sum_{n} Tr^2(A_{e_n}) - \sum_{n} Tr(A_{e_n})^2 \quad (54) \]

\[ \sum_{n} Tr(A_{e_n})^2 \] is the square of the length of the second fundamental form of \( M \), which is denoted by \( |A_{e_n}|^2 \). Thus, we also have

\[ \| h^2 \| = \sum_{i,j=1}^{n} g \left( h(e_i, e_j), h(e_i, e_j) \right) = \| A^2 \|. \quad (55) \]

2. Distribution on a manifold

An \( m \)-dimensional distribution on a manifold \( \tilde{M} \) is a mapping \( D \) defined on \( \tilde{M} \), which assigns to each point \( p \) of \( \tilde{M} \) an \( m \)-dimensional linear subspace \( D_p \) of \( T_{\tilde{M}}(p) \). A vector field \( X \) on \( \tilde{M} \) belongs to \( D \) if we have \( X_p \in D_p \) for each \( p \in \tilde{M} \). When this happens, we write \( X \in \Gamma(D) \). The distribution \( D \) is said to be differentiable if for any \( p \in \tilde{M} \), there exist \( m \)-differentiable linearly independent vector fields \( X_j \in \Gamma(D) \) in a neighborhood of \( p \).

The distribution \( D \) is said to be involutive if for all vector fields \( X, Y \in \Gamma(D) \) we have \([X, Y] \in \Gamma(D)\). A sub-manifold \( M \) of \( \tilde{M} \) is said to be an integral manifold of \( D \) if for every point \( p \in M \), \( D_p \) coincides with the tangent space to \( M \) at \( p \). If there exists no integral manifold of \( D \) which contains \( M \), then \( M \) is called a maximal integral manifold or a leaf of \( D \). The distribution \( D \) is said to be integrable if for every \( p \in \tilde{M} \), there exists an integral manifold of \( D \) containing \( p \) [2].

Let \( \tilde{\nabla} \) and distribution be a linear connection on \( \tilde{M} \), respectively. The distribution \( D \) is said to be parallel with respect to \( \tilde{M} \), if we have

\[ \tilde{\nabla}_X Y \in \Gamma(D) \] for all \( X \in \Gamma(T\tilde{M}) \) and \( Y \in \Gamma(D) \) \( (56) \)

Now, let \((\tilde{M}, \tilde{g})\) be Riemannian manifold and \( D \) be a distribution on \( \tilde{M} \). We suppose \( \tilde{M} \) is endowed with two complementary distribution \( D \) and \( D^\perp \), i.e., we have \( T\tilde{M} = D \oplus D^\perp \). Denoted by \( P \) and \( Q \) the projections of \( T\tilde{M} \) to \( D \) and \( D^\perp \), respectively.

**Theorem 2.1.** All the linear connections with respect to which both distributions \( D \) and \( D^\perp \) are parallel, are given by

\[ \nabla_X Y = PV_XPY + QV_XQY + PS(X, PY) + QS(X, QY) \quad (57) \]

for any \( X, Y \in \Gamma(T\tilde{M}) \), where \( V \) and \( S \) are, respectively, an arbitrary linear connection and arbitrary tensor field of type \( (1, 2) \) on \( \tilde{M} \).

**Proof:** Suppose \( \tilde{V} \) is an arbitrary linear connection on \( \tilde{M} \). Then, any linear connection \( V \) on \( \tilde{M} \) is given by
\[ \nabla_X Y = \nabla_X' Y + S(X, Y) \quad (58) \]

for any \( X, Y \in \Gamma(T \tilde{M}) \). We can put
\[ X = PX + QX \quad (59) \]

for any \( X \in \Gamma(T \tilde{M}) \). Then, we have
\[
\nabla_X Y = \nabla_X (PY + QY) = \nabla_X PY + \nabla_X QY = \nabla'_X PY + S(X, PY) \\
+ \nabla'_X QY + S(X, QY) = PV'_X PY + QV'_X PY + PS(X, PY) + QS(X, PY) \\
+ PV'_X QY + QV'_X QY + PD(X, QY) + QS(X, QY) \quad (60)
\]

for any \( X, Y \in \Gamma(T \tilde{M}) \).

The distributions \( D \) and \( D^\perp \) are both parallel with respect to \( \nabla \) if and only if we have
\[ \phi(\nabla_X PY) = 0 \text{ and } P(\nabla_X QY) = 0. \quad (61) \]

From Eqs. (58) and (61), it follows that \( D \) and \( D^\perp \) are parallel with respect to \( \nabla \) if and only if
\[ QV'_X PY + QS(X, PY) = 0 \text{ and } PV'_X QY + PD(X, QY) = 0. \quad (62) \]

Thus, Eqs. (58) and (62) give us Eq. (57).

Next, by means of the projections \( P \) and \( Q \), we define a tensor field \( F \) of type \((1,1)\) on \( \tilde{M} \) by
\[ FX = PX - QX \quad (63) \]

for any \( X \in \Gamma(T \tilde{M}) \). By a direct calculation, it follows that \( F^2 = I \). Thus, we say that \( F \) defines an almost product structure on \( \tilde{M} \). The covariant derivative of \( F \) is defined by
\[ (\nabla_X F)Y = \nabla_X FY - F\nabla_X Y \quad (64) \]

for all \( X, Y \in \Gamma(T \tilde{M}) \). We say that the almost product structure \( F \) is parallel with respect to the connection \( \nabla \), if we have \( \nabla_X F = 0 \). In this case, \( F \) is called the Riemannian product structure [2].

**Theorem 2.2.** Let \((\tilde{M}, \tilde{g})\) be a Riemannian manifold and \( D, D^\perp \) be orthogonal distributions on \( \tilde{M} \) such that \( T \tilde{M} = D \oplus D^\perp \). Both distributions \( D \) and \( D^\perp \) are parallel with respect to \( \nabla \) if and only if \( F \) is a Riemannian product structure.

**Proof:** For any \( X, Y \in \Gamma(T \tilde{M}) \), we can write
\[
\tilde{\nabla}_Y PX = \tilde{\nabla}_{PY} PX + \tilde{\nabla}_{QY} PX \quad (65)
\]

and

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http://dx.doi.org/10.5772/65948
\[ \nabla Y X = \nabla_{PY} PX + \nabla_{PY} QX + \nabla_{QY} PX + \nabla_{QY} QX, \]  
(66)

from which

\[ g(\nabla_{QY} PX, QZ) = QY g(PX, QZ) - g(\nabla_{QY} QZ, PX) = 0 - g(\nabla_{QY} QZ, PX) = 0, \]  
(67)

that is, \( V_{QY} PX \in \Gamma(\mathcal{D}) \) and so \( P\nabla_{QY} PX = \nabla_{QY} PX, \)

\[ Q\nabla_{QY} PX = 0. \]  
(68)

In the same way, we obtain

\[ g(\nabla_{PY} QX, PZ) = PY g(QX, PZ) - g(QX, \nabla_{PY} PZ) = 0, \]  
(69)

which implies that

\[ P\nabla_{PY} QX = 0 \quad \text{and} \quad Q\nabla_{PY} QX = \nabla_{PY} QX. \]  
(70)

From Eqs. (66), (68) and (70), it follows that

\[ P\nabla_{Y} X = \nabla_{PY} PX + \nabla_{QY} PX. \]  
(71)

By using Eqs. (64) and (71), we obtain

\[ (\nabla_{Y} P)X = \nabla_{Y} PX - P\nabla_{Y} X = \nabla_{PY} PX + \nabla_{QY} PX - \nabla_{PY} PX - \nabla_{QY} PX = 0. \]  
(72)

In the same way, we can find \( \nabla Q = 0. \) Thus, we obtain

\[ \nabla F = \nabla (P - Q) = 0. \]  
(73)

This proves our assertion [2].

**Theorem 2.3.** Both distributions \( \mathcal{D} \) and \( \mathcal{D}^\perp \) are parallel with respect to Levi-Civita connection \( \nabla \) if and only if they are integrable and their leaves are totally geodesic in \( \tilde{M} \).

**Proof:** Let us assume both distributions \( \mathcal{D} \) and \( \mathcal{D}^\perp \) are parallel. Since \( \nabla \) is a torsion free linear connection, we have

\[ [X, Y] = \nabla_X Y - \nabla_Y X \in \Gamma(\mathcal{D}), \text{ for any } X, Y \in \Gamma(\mathcal{D}) \]  
(74)

and

\[ [U, V] = \nabla_U V - \nabla_V U \in \Gamma(\mathcal{D}^\perp), \text{ for any } U, V \in \Gamma(\mathcal{D}^\perp) \]  
(75)

Thus, \( \mathcal{D} \) and \( \mathcal{D}^\perp \) are integrable distributions. Now, let \( M \) be a leaf of \( \mathcal{D} \) and denote by \( h \) the second fundamental form of the immersion of \( M \) in \( \tilde{M} \). Then by the Gauss formula, we have
for any \( X, Y \in \Gamma(D) \), where \( \nabla' \) denote the Levi-Civita connection on \( M \). Since \( D \) is parallel from Eq. (76) we conclude \( h = 0 \), that is, \( M \) is totally in \( \tilde{M} \). In the same way, it follows that each leaf of \( D^\perp \) is totally geodesic in \( \tilde{M} \).

Conversely, suppose \( D \) and \( D^\perp \) be integrable and their leaves are totally geodesic in \( \tilde{M} \). Then by using Eq. (4), we have

\[
\nabla_X Y \in \Gamma(D) \quad \text{for any} \quad X, Y \in \Gamma(D) \tag{77}
\]

and

\[
\nabla_U V \in \Gamma(D^\perp) \quad \text{for any} \quad U, V \in \Gamma(D^\perp). \tag{78}
\]

Since \( g \) is a Riemannian metric tensor, we obtain

\[
g(\nabla_Y Y, V) = -g(Y, \nabla_V V) = 0 \tag{79}
\]

and

\[
g(\nabla_Y V, Y) = -g(V, \nabla_Y Y) = 0 \tag{80}
\]

for any \( X, Y \in \Gamma(D) \) and \( U, V \in \Gamma(D^\perp) \). Thus, both distributions \( D \) and \( D^\perp \) are parallel on \( \tilde{M} \).

3. Locally decomposable Riemannian manifolds

Let \((\tilde{M}, \tilde{g})\) be \( n \)-dimensional Riemannian manifold and \( F \) be a tensor \((1,1)\)-type on \( \tilde{M} \) such that \( F^2 = I, F \neq \pm I \).

If the Riemannian metric tensor \( \tilde{g} \) satisfying

\[
\tilde{g}(X, Y) = \tilde{g}(FX, FY) \tag{81}
\]

for any \( X, Y \in \Gamma(T\tilde{M}) \) then \( \tilde{M} \) is called almost Riemannian product manifold and \( F \) is said to be almost Riemannian product structure. If \( F \) is parallel, that is, \( (\nabla_X F)Y = 0 \), then \( \tilde{M} \) is said to be locally decomposable Riemannian manifold.

Now, let \( \tilde{M} \) be an almost Riemannian product manifold. We put

\[
P = \frac{1}{2} (I + F), \quad Q = \frac{1}{2} (I-F). \tag{82}
\]

Then, we have
\[ P + Q = I, \quad P^2 = P, \quad Q^2 = Q, \quad PQ = QP = 0 \quad \text{and} \quad F = P - Q. \quad (83) \]

Thus, \( P \) and \( Q \) define two complementary distributions \( P \) and \( Q \) globally. Since \( F^2 = I \), we easily see that the eigenvalues of \( F \) are 1 and \(-1\). An eigenvector corresponding to the eigenvalue 1 is in \( P \) and an eigenvector corresponding to \(-1\) is in \( Q \). If \( F \) has eigenvalue 1 of multiplicity \( p \) and eigenvalue \(-1\) of multiplicity \( q \), then the dimension of \( P \) is \( p \) and that of \( Q \) is \( q \). Conversely, if there exist in \( \tilde{M} \) two globally complementary distributions \( P \) and \( Q \) of dimension \( p \) and \( q \), respectively. Then, we can define an almost Riemannian product structure \( F \) on \( \tilde{M} \) by \( \tilde{M} \) by \( F = P - Q \) [7].

Let \((\tilde{M}, \tilde{g}, F)\) be a locally decomposable Riemannian manifold and we denote the integral manifolds of the distributions \( P \) and \( Q \) by \( M^p \) and \( M^q \), respectively. Then we can write \( \tilde{M} = M^p \times M^q \), \((p, q > 2)\). Also, we denote the components of the Riemannian curvature \( R \) of \( \tilde{M} \) by \( R_{dcb} \) if \( 1 \leq a, b, c, d \leq n = p + q \).

Now, we suppose that the two components are both of constant curvature \( \lambda \) and \( \mu \). Then, we have

\[ R_{dcb} = \lambda \{ g_{da} S_{cb} - g_{ca} S_{db} \} \quad (84) \]

and

\[ R_{zxyw} = \mu \{ g_{zv} S_{yx} - g_{yz} S_{vx} \}. \quad (85) \]

Then, the above equations may also be written in the form

\[ R_{kij} = \frac{1}{4} (\lambda + \mu) \{ (g_{kh} S_{ji} - g_{jh} S_{ki}) + (F_{kh} F_{ji} - F_{jh} F_{ki}) \} \]

\[ + \frac{1}{4} (\lambda - \mu) \{ (F_{kh} S_{ji} - F_{jh} S_{ki}) + (g_{kh} F_{ji} - g_{jh} F_{ki}) \}. \quad (86) \]

Conversely, suppose that the curvature tensor of a locally decomposable Riemannian manifold has the form

\[ R_{kij} = a \{ (g_{kh} S_{ji} - g_{jh} S_{ki}) + (F_{kh} F_{ji} - F_{jh} F_{ki}) \} \]

\[ + b \{ (F_{kh} S_{ji} - F_{jh} S_{ki}) + (g_{kh} F_{ji} - g_{jh} F_{ki}) \}. \quad (87) \]

Then, we have

\[ R_{dcb} = 2(a + b) \{ g_{da} S_{cb} - g_{ca} S_{db} \} \quad (88) \]

and

\[ R_{zxyw} = 2(a - b) \{ g_{zv} S_{yx} - g_{yz} S_{vx} \}. \quad (89) \]

Let \( \tilde{M} \) be an \( m \)-dimensional almost Riemannian product manifold with the Riemannian structure \((F, \tilde{g})\) and \( M \) be an \( n \)-dimensional sub-manifold of \( \tilde{M} \). For any vector field \( X \) tangent to \( M \), we put
where $fX$ and $wX$ denote the tangential and normal components of $FX$, with respect to $M$, respectively. In the same way, for $V \in \Gamma(T^\perp M)$, we also put

$$FV = BV + CV,$$

where $BV$ and $CV$ denote the tangential and normal components of $FV$, respectively.

Then, we have

$$f^2 + Bw = I, Cw + wf = 0 \quad (92)$$

and

$$fB + BC = 0, wB + C^2 = I. \quad (93)$$

On the other hand, we can easily see that

$$g(X, fY) = g(fX, Y) \quad (94)$$

and

$$g(X, Y) = g(fX, fY) + g(wX, wY) \quad (95)$$

for any $X, Y \in \Gamma(TM)$ [6].

If $wX = 0$ for all $X \in \Gamma(TM)$, then $M$ is said to be invariant sub-manifold in $\tilde{M}$, i.e., $F(T_M(p)) \subset T_M(p)$ for each $p \in M$. In this case, $f^2 = I$ and $g(fX, fY) = g(X, Y)$. Thus, $(f, g)$ defines an almost product Riemannian on $M$.

Conversely, $(f, g)$ is an almost product Riemannian structure on $M$, the $w = 0$ and hence $M$ is an invariant sub-manifold in $\tilde{M}$.

Consequently, we can give the following theorem [7].

**Theorem 3.1.** Let $M$ be a sub-manifold of an almost Riemannian product manifold $\tilde{M}$ with almost Riemannian product structure $(F, \tilde{g})$. The induced structure $(f, g)$ on $M$ is an almost Riemannian product structure if and only if $M$ is an invariant sub-manifold of $\tilde{M}$.

**Definition 3.1.** Let $M$ be a sub-manifold of an almost Riemannian product $\tilde{M}$ with almost product Riemannian structure $(F, \tilde{g})$. For each non-zero vector $X_p \in T_M(p)$ at $p \in M$, we denote the slant angle between $FX_p$ and $T_M(p)$ by $\theta(p)$. Then $M$ said to be slant sub-manifold if the angle $\theta(p)$ is constant, i.e., it is independent of the choice of $p \in M$ and $X_p \in T_M(p)$ [5].

Thus, invariant and anti-invariant immersions are slant immersions with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. A proper slant immersion is neither invariant nor anti-invariant.
Theorem 3.2. Let $M$ be a sub-manifold of an almost Riemannian product manifold $\tilde{M}$ with almost product Riemannian structure $(F, \tilde{g})$. $M$ is a slant sub-manifold if and only if there exists a constant $\lambda \in (0, 1)$, such that

$$f^2 = \lambda I.$$ \hspace{1cm} (96)

Furthermore, if the slant angle is $\theta$, then it satisfies $\lambda = \cos^2 \theta$ [9].

Definition 3.2. Let $M$ be a sub-manifold of an almost Riemannian product manifold $\tilde{M}$ with almost Riemannian product structure $(F, \tilde{g})$. $M$ is said to be semi-slant sub-manifold if there exist distributions $D^\theta$ and $D^T$ on $M$ such that

(i) $TM$ has the orthogonal direct decomposition $TM = D \oplus D^T$.

(ii) The distribution $D^\theta$ is a slant distribution with slant angle $\theta$.

(iii) The distribution $D^T$ is an invariant distribution, i.e., $F(D^T) \subseteq D^T$.

In a semi-slant sub-manifold, if $\theta = \frac{\pi}{2}$, then semi-slant sub-manifold is called semi-invariant sub-manifold [8].

Example 3.1. Now, let us consider an immersed sub-manifold $M$ in $\mathbb{R}^7$ given by the equations

$$x_1^2 + x_2^2 = x_5^2 + x_6^2, x_3 + x_4 = 0.$$ \hspace{1cm} (97)

By direct calculations, it is easy to check that the tangent bundle of $M$ is spanned by the vectors

$$z_1 = \cos \theta \frac{\partial}{\partial x_1} + \sin \theta \frac{\partial}{\partial x_2} + \cos \beta \frac{\partial}{\partial x_3} + \sin \beta \frac{\partial}{\partial x_6},$$

$$z_2 = -u \sin \theta \frac{\partial}{\partial x_1} + u \cos \theta \frac{\partial}{\partial x_2}, z_3 = \frac{\partial}{\partial x_3},$$

$$z_4 = -u \sin \beta \frac{\partial}{\partial x_5} + u \cos \beta \frac{\partial}{\partial x_6}, z_5 = \frac{\partial}{\partial x_7},$$ \hspace{1cm} (98)

where $\theta, \beta$ and $u$ denote arbitrary parameters.

For the coordinate system of $\mathbb{R}^7 = \{(x_1, x_2, x_3, x_4, x_5, x_6, x_7) | x_i \in \mathbb{R}, 1 \leq i \leq 7\}$, we define the almost product Riemannian structure $F$ as follows:

$$F\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_i}, F\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial x_j}, 1 \leq i \leq 3 \text{ and } 4 \leq j \leq 7.$$ \hspace{1cm} (99)

Since $Fz_1$ and $Fz_3$ are orthogonal to $M$ and $Fz_2, Fz_4, Fz_5$ are tangent to $M$, we can choose a $\mathcal{D} = S_p\{z_2, z_4, z_5\}$ and $\mathcal{D}^\perp = S_p\{z_1, z_3\}$. Thus, $M$ is a 5-dimensional semi-invariant sub-manifold of $\mathbb{R}^7$ with usual almost Riemannian product structure $(F, < , >)$.

Example 3.2. Let $M$ be sub-manifold of $\mathbb{R}^8$ by given

$$(u + v, u-v, u \cos \alpha, u \sin \alpha, u + v, u-v, u \cos \beta, u \sin \beta)$$ \hspace{1cm} (100)
where \(u, v\) and \(\beta\) are the arbitrary parameters. By direct calculations, we can easily see that the tangent bundle of \(M\) is spanned by
\[
e_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \cos \alpha \frac{\partial}{\partial x_3} + \sin \alpha \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_6} + \cos \beta \frac{\partial}{\partial x_7} + \sin \beta \frac{\partial}{\partial x_8}
\]
\[
e_2 = -\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6} - \frac{\partial}{\partial x_7} + \frac{\partial}{\partial x_8},
\]
\[
e_3 = -u \sin \beta \frac{\partial}{\partial x_7} + u \cos \beta \frac{\partial}{\partial x_8},
\]
\[
e_4 = -u \sin \beta \frac{\partial}{\partial x_7} + u \cos \beta \frac{\partial}{\partial x_8}.
\]

For the almost Riemannian product structure \(F\) of \(\mathbb{R}^4 \times \mathbb{R}^4\), \(F(TM)\) is spanned by vectors
\[
Fe_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \cos \alpha \frac{\partial}{\partial x_3} + \sin \alpha \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_6} - \cos \beta \frac{\partial}{\partial x_7} - \sin \beta \frac{\partial}{\partial x_8},
\]
\[
Fe_2 = -\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_6} + \frac{\partial}{\partial x_7} + \frac{\partial}{\partial x_8},
\]
\[
Fe_3 = e_3 \quad \text{and} \quad Fe_4 = -e_4.
\]

Since \(Fe_1\) and \(Fe_2\) are orthogonal to \(M\) and \(Fe_3\) and \(Fe_4\) are tangent to \(M\), we can choose \(D^T = \text{Sp}\{e_3, e_4\}\) and \(D^\perp = \text{Sp}\{e_1, e_2\}\). Thus, \(M\) is a four-dimensional semi-invariant sub-manifold of \(\mathbb{R}^8 = \mathbb{R}^4 \times \mathbb{R}^4\) with usual Riemannian product structure \(F\).

**Definition 3.3.** Let \(M\) be a sub-manifold of an almost Riemannian product manifold \(\tilde{M}\) with almost Riemannian product structure \((F, \tilde{g})\). \(M\) is said to be pseudo-slant sub-manifold if there exist distributions \(D_0\) and \(D^\perp\) on \(M\) such that

i. The tangent bundle \(TM = D_0 \oplus D^\perp\).

ii. The distribution \(D_0\) is a slant distribution with slant angle \(\theta\).

iii. The distribution \(D^\perp\) is an anti-invariant distribution, i.e., \(F(D^\perp) \subseteq T^\perp M\).

As a special case, if \(\theta = 0\) and \(\theta = \frac{\pi}{2}\), then pseudo-slant sub-manifold becomes semi-invariant and anti-invariant sub-manifolds, respectively.

**Example 3.3.** Let \(M\) be a sub-manifold of \(\mathbb{R}^6\) by the given equation
\[
(\sqrt{3}u, v \sin \theta, v \cos \theta, s \cos t, -s \cos t)
\]
where \(u, v, s\) and \(t\) arbitrary parameters and \(\theta\) is a constant.

We can check that the tangent bundle of \(M\) is spanned by the tangent vectors
\[
e_1 = \sqrt{3} \frac{\partial}{\partial x_1}, \quad e_2 = \frac{\partial}{\partial y_1} + \sin \theta \frac{\partial}{\partial x_2} + \cos \theta \frac{\partial}{\partial y_2},
\]
\[
e_3 = \cos t \frac{\partial}{\partial x_3} - \cos t \frac{\partial}{\partial y_3}, \quad e_4 = -s \sin t \frac{\partial}{\partial x_3} + s \sin t \frac{\partial}{\partial y_3}.
\]

For the almost product Riemannian structure \(F\) of \(\mathbb{R}^6\) whose coordinate systems \((x_1, y_1, x_2, y_2, x_3, y_3)\) choosing
\[ F\left( \frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i}, \quad 1 \leq i \leq 3, \]  
\[ F\left( \frac{\partial}{\partial y_j}\right) = \frac{\partial}{\partial x_j}, \quad 1 \leq j \leq 3, \]  
(105)

Then, we have
\[ F e_1 = \sqrt{3} \frac{\partial}{\partial y_1}, \quad F e_2 = -\frac{\partial}{\partial x_1} + \sin \theta \frac{\partial}{\partial y_2} - \cos \theta \frac{\partial}{\partial x_2}, \]  
\[ F e_3 = \cos \frac{\partial}{\partial y_3} + \cos \frac{\partial}{\partial x_3}, \quad F e_4 = -\sin t \frac{\partial}{\partial y_3} - \sin t \frac{\partial}{\partial x_3}. \]  
(106)

Thus, \( D_\theta = S_p\{e_1, e_2\} \) is a slant distribution with slant angle \( \alpha = \frac{\pi}{4} \). Since \( F e_3 \) and \( F e_4 \) are orthogonal to \( M \), \( D^\perp = S_p\{e_3, e_4\} \) is an anti-invariant distribution, that is, \( M \) is a 4-dimensional proper pseudo-slant sub-manifold of \( \mathbb{R}^6 \) with its almost Riemannian product structure \( (F, <, >) \).

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**References**

[1] Katsuei Kenmotsu, editor. Differential Geometry of Submaifolds. Berlin: Springer-Verlag; 1984. 134 p.

[2] Aurel Bejancu. Geometry of CR-Submanifolds. Dordrecht: D. Reidel Publishing Company; 1986. 172 p. DOI: QA649.B44

[3] Bang-Yen Chen. Geometry of Submanifolds. New York: Marcel Dekker, Inc.; 1973. 298 p.

[4] Kentaro Yano and Masahiro Kon. Structures on Manifolds. Singapore: World Scientific Publishing Co. Pte. Ltd.; 1984. 508 p. DOI: QA649.Y327

[5] Meraj Ali Khan. Geometry of Bi-slant submanifolds "Some geometric aspects on submanifolds Theory". Saarbrücken, Germany: Lambert Academic Publishing; 2006. 112 p.
[6] Mehmet Atçeken. Warped product semi-invariant submanifolds in almost paracontact Riemannian manifolds. Mathematical Problems in Engineering. 2009;2009:621625. DOI: doi:10.1155/2009/621625

[7] Tyuzi Adati. Submanifolds of an almost product Riemannian manifold. Kodai Mathematical Journal. 1981;4(2):327–343.

[8] Mehmet Atçeken. A condition for warped product semi-invariant submanifolds to be Riemannian product semi-invariant Sub-manifolds in locally Riemannian product manifolds. Turkish Journal of Mathematics. 2008;33:349–362.

[9] Mehmet Atçeken. Slant submanifolds of a Riemannian product manifold. Acta Mathematica Scientia. 2010;30(1):215–224. DOI: doi:10.1016/S0252-9602(10)60039-2
