Nonlinear Connections and Description of Photon-like Objects

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Abstract

This paper aims to present a general idea for description of spatially finite physical objects with a consistent nontrivial translational-rotational dynamical structure and evolution as a whole, making use of the mathematical concepts and structures connected with the Frobenius integrability/nonintegrability theorems given in terms of distributions on manifolds with corresponding curvature defined by the Nijenhuis operator. The idea is based on consideration of nonintegrable subdistributions of some appropriate completely integrable distribution (differential system) on a manifold and then to make use of the corresponding curvatures as generators of measures of interaction, i.e. of energy-momentum exchange among the physical subsystems mathematically represented by the nonintegrable subdistributions. The concept of photon-like object is introduced and description of such objects in these terms is given.

1 Introduction

At the very dawn of the 20th century Planck (Planck 1901) proposed and a little bit later Einstein (Einstein 1905) appropriately used the well known and widely used through the whole last century simple formula \( E = h\nu, \) \( h = \text{const} > 0. \) This formula marked the beginning of a new era and became a real symbol of the physical science during the following years. According to the Einstein’s interpretation it gives the full energy \( E \) of really existing light quanta of frequency \( \nu = \text{const}, \) and in this way a new understanding of the nature of the electromagnetic field was introduced: the field has structure which contradicts the description given by Maxwell vacuum equations. After De Broglie’s (De Broglie 1923) suggestion for the particle-wave nature of the electron obeying the same energy-frequency relation, one could read Planck’s formula in the following way: there are physical objects in Nature the very existence of which is strongly connected to some periodic (with time period \( T = 1/\nu) \) process of intrinsic for the object nature and such that the Lorentz invariant product \( ET \) is equal to \( h. \) Such a reading should suggest that these objects do NOT admit point-like approximation since the relativity principle for free point particles requires straight-line uniform motion, hence, no periodicity should be allowed.

Although the great (from pragmatic point of view) achievements of the developed theoretical approach, known as quantum theory, the great challenge to build an adequate description of individual representatives of these objects, especially of light quanta called by Lewis photons (Lewis 1926) is still to be appropriately met since the efforts made in this direction, we have to admit, still have not brought satisfactory results. Recall that Einstein in his late years recognizes (Speziali 1972) that ”the whole fifty years of conscious brooding have not brought me nearer to the answer

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to the question "what are light quanta", and now, half a century later, theoretical physics still needs progress to present a satisfactory answer to the question "what is a photon". We consider the corresponding theoretically directed efforts as necessary and even urgent in view of the growing amount of definite experimental skills in manipulation with individual photons, in particular, in connection with the experimental advancement in the "quantum computer" project. The dominating modern theoretical view on microobjects is based on the notions and concepts of quantum field theory (QFT) where the structure of the photon (as well as of any other microobject) is accounted for mainly through the so called structural function, and highly expensive and delicate collision experiments are planned and carried out namely in the frame of these concepts and methods (see the 'PHOTON' Conferences Proceedings, some recent review papers: Dainton 2000; Stumpf, Borne 2001; Godbole 2003; Nisius 2001). Going not in details we just note a special feature of this QFT approach: if the study of a microobject leads to conclusion that it has structure, i.e. it is not point-like, then the corresponding constituents of this structure are considered as point-like, so the point-likeness stays in the theory just in a lower level.

In this paper we follow another approach based on the assumption that the description of the available (most probably NOT arbitrary) spatial structure of photon-like objects can be made by continuous finite/localized functions of the three space variables. The difficulties met in this approach consist mainly, in our view, in finding adequate enough mathematical objects and solving appropriate PDE. The lack of sufficiently reliable corresponding information made us look into the problem from as general as possible point of view on the basis of those properties of photon-like objects which may be considered as most undoubtedly trustful, and in some sense, identifying. The analysis made suggested that such a property seems to be the the available translational-rotational dynamical structure, so we shall focus on this property in order to see what useful for our purpose suggestions could be deduced and what appropriate structures could be constructed. All these suggestions and structures should be the building material for a step-by-step creation of a self-consistent system. From physical point of view this should mean that the corresponding properties may combine to express a dynamical harmony in the inter-existence of appropriately defined subsystems of a finite and time stable larger physical system. (for another approach based on slight modification of Maxwell equations see Funaro,D., arXiv:physics/0505068)

The plan of this paper is the following. In Sec.2 we introduce and comment the concept of photon-like object. In Sec.3 we recall some basic facts from Frobenius integrability theory, then we consider its possibilities to describe interaction between/among subsystems, mathematically represented by non-integrable subdistributions of an integrable distribution. In Sec.4 we introduce and consider the concept of nonlinear connection and deduce some important from our point of view relations. In Secs.5,6 we consider electromagnetic photon-like objects. In Sec.7 we interpret geometrically the translational-rotational consistency and obtain another look at the equations of motion obtained in Sec.5. In the concluding Sec.8 we give a short overview of the results obtained.

2 The notion of photon-like object

We begin with the notice that any notion of a physical object must unify two kinds of properties of the object considered: identifying and kinematical. The identifying properties being represented by quantities and relations, stay unchanged throughout the existence, i.e. throughout the time-evolution, of the object, they represent all the intrinsic structure and relations. The kinematical properties describe those changes, called admissible, which do NOT lead to destruction of the object, i.e. to the destruction of any of the identifying properties. Correspondingly, physics introduces two kinds of quantities and relations, identifying and kinematical. From theoretical point of view the more important quantities used turn out to be the dynamical quantities which, as a rule, are
functions of the identifying and kinematical ones, and the joint relations they satisfy represent
the necessary interelations between them in order this object to survive under external influence.
This view suggests to introduce the following notion of Photon-like object(s) (we shall use the
abbreviation "PhLO" for "Photon-like object(s)"):

**PhLO are real massless time-stable physical objects with a consistent
translational-rotational dynamical structure.**

We give now some explanatory comments, beginning with the term **real.** **First** we emphasize
that this term means that we consider PhLO as *really* existing physical objects, not as appropriate
and helpful but imaginary (theoretical) entities. Accordingly, PhLO necessarily carry energy-
momentum, otherwise, they could hardly be detected. **Second,** PhLO can undoubtedly be
created and destroyed, so, no point-like and infinite models are reasonable: point-like objects are
assumed to have no structure, so they can not be destroyed since there is no available structure
to be destroyed; creation of infinite physical objects (e.g. plane waves) requires infinite quantity
of energy to be transformed from one kind to another for finite time-period, which seems also
unreasonable. Accordingly, PhLO are spatially finite and have to be modeled like such ones, which
is the only possibility to be consistent with their "created-destroyed" nature. It seems hardly
reasonable to believe that PhLO can not be created and destroyed, and that spatially infinite and
indestructible physical objects may exist at all. **Third,** "spatially finite" implies that PhLO may
carry only finite values of physical (conservative or non-conservative) quantities. In particular,
the most universal physical quantity seems to be the energy-momentum, so the model must allow
finite integral values of energy-momentum to be carried by the corresponding solutions. **Fourth,**
"spatially finite" means also that PhLO propagate, i.e. they do not "move" like classical particles
along trajectories, therefore, partial differential equations should be used to describe their evolution
in time.

The term **"massless"** characterizes physically the way of propagation in terms of appropriate
dynamical quantities: the integral energy $E$ and integral momentum $p$ of a PhLO should satisfy
the relation $E = cp$, where $c$ is the speed of light in vacuum, and in relativistic terms this means
that their integral energy-momentum vector must be isotropic, i.e. it must have zero module with
respect to Lorentz-Minkowski (pseudo)metric in $\mathbb{R}^4$. If the object considered has spatial and time-
stable structure, so that the translational velocity of every point where the corresponding field
functions are different from zero must be equal to $c$, we have in fact null direction in the space-
time intrinsically determined by a PhLO. Such a direction is formally defined by a null vector field
$X, X^2 = 0$. The integral trajectories of this vector field are isotropic (or null) straight lines as is
traditionally assumed in physics. It follows that with every PhLO a null direction is necessarily
associated, so, canonical coordinates $(x^1, x^2, x^3, x^4) = (x, y, z, \xi = ct)$ on $\mathbb{R}^4$ may be chosen such
that in the corresponding coordinate frame $X$ to have only two non-zero components of magnitude 1:
$X^\mu = (0, 0, -\varepsilon, 1)$, where $\varepsilon = \pm 1$ accounts for the two directions along the coordinate $z$ (further such
a coordinate system will be called $X$-adapted and will be of main usage). Our PhLO propagates
as a whole along the $X$-direction, so the corresponding energy-momentum tensor $T_{\mu\nu}$ of the model
must satisfy the corresponding local isotropy (null) condition, namely, $T_{\mu\nu}T^{\mu\nu} = 0$ (summation
over the repeated indices is throughout used).

The term **"translational-rotational"** means that besides translational component along $X$, the
propagation necessarily demonstrates some rotational (in the general sense of this concept)
component in such a way that both components exist simultaneously and consistently. It seems
reasonable to expect that such kind of behavior should be consistent only with some distinguished
spatial shapes. Moreover, if the Planck relation $E = h\nu$ must be respected throughout the evolution,
the rotational component of propagation should have time-periodical nature with time period $T =$
ν⁻¹ = h/E = \text{const}, and one of the two possible, left or right, orientations. It seems reasonable also to expect periodicity in the spatial shape of PhLO, which somehow to be related to the time periodicity.

The term "dynamical structure" means that the propagation is supposed to be necessarily accompanied by an internal energy-momentum redistribution, which may be considered in the model as energy-momentum exchange between (or among) some appropriately defined subsystems. It could also mean that PhLO live in a dynamical harmony with the outside world, i.e. any outside directed energy-momentum flow should be accompanied by a parallel inside directed energy-momentum flow.

Finally, note that if the time periodicity and the spatial periodicity should be consistently related somehow, the simplest integral feature of such consistency would seem like this: the spatial size along the translational component of propagation λ is equal to cT: λ = cT, where λ is some finite positive characteristic constant of the corresponding solution. This would mean that every individual PhLO determines its own length/time scale.

We are going now to formulate shortly the basic idea, i.e. the basic mathematical identification, inside which this study will be carried out.

3 Curvature of Distributions and Physical Interaction

Any physical system with a dynamical structure is characterized by some internal energy-momentum redistributions, i.e. energy-momentum fluxes, during evolution. Any system of energy-momentum fluxes (as well as fluxes of other interesting for the case physical quantities subject to change during evolution, but we limit ourselves just to energy-momentum fluxes here) can be considered mathematically as generated by some system of vector fields. A physically isolated, consistent and interrelated time-stable system of energy-momentum fluxes can be considered to correspond directly or indirectly to a completely integrable distribution ∆ of vector fields (or differential system (Godbillon 1969)) according to the principle some local objects can generate integral object. Every distribution on a manifold defines its own curvature form (given further in the section). Let ∆₁ and ∆₂ be two distributions on the same manifold with corresponding curvature forms Ω₁ and Ω₂, each of them carries couples of vector fields inside their distributions outside ∆₁ and ∆₂ correspondingly, i.e. Ω₁(Y₁,Y₂) is out of ∆₁ and Ω₂(Z₁,Z₂) is out ∆₂, where (Y₁,Y₂) live in ∆₁ and (Z₁,Z₂) live in ∆₂. Let now ∆₁ and ∆₂ characterize two interacting physical systems, or two interacting subsystems of a larger physical system. It seems reasonable to assume as a working tool the following geometrization of the concept of local physical interaction: two distributions ∆₁ and ∆₂ on a manifold will be said to interact infinitesimally (or locally) if at least one of the corresponding two curvature forms Ω₁/Ω₂ takes values, or generates objects taking values, respectively in ∆₂/∆₁.

The above geometric concept of infinitesimal interaction is motivated by the fact that, in general, an integrable distribution ∆ may contain various nonintegrable subdistributions ∆₁, ∆₂,... which subdistributions may be associated physically with interacting subsystems of a larger time stable physical system. Any physical interaction between 2 subsystems is necessarily accompanied with available energy-momentum exchange between them, this could be understood mathematically as nonintegrability of each of the two subdistributions of ∆ and could be naturally measured directly or indirectly by the corresponding curvatures. For example, if ∆ is an integrable 3-dimensional distribution spent by the vector fields (X₁, X₂, X₃) then we may have, in general, three non-integrable, i.e. geometrically interacting, 2-dimensional subdistributions (X₁, X₂), (X₁, X₃), (X₂, X₃). Finally, some interaction with the outside world can be described by curvatures of distributions (and their subdistributions) in which elements from ∆ and vector fields outside ∆ are involved (such processes
will not be considered in this paper).

There are two basic ways to formalize the above statements. The first one is known as the **Frobenius integrability approach**, and the second one (been developed recently) is known as **nonlinear connections** (Vacaru, S. et al. 2005). We consider briefly the first one here and then go in a more detail to the nonlinear connections approach.

According to the Frobenius integrability theorem on a $n$-dimensional manifold $M^n$ (further all manifolds are assumed smooth and finite dimensional and all objects defined on $M^n$ are also assumed smooth) if the system of vector fields $\Delta = \{X_1(x), X_2(x), \ldots, X_p(x)\}$, $x \in M$, $1 < p < n$, satisfies $X_1(x) \land X_2(x) \land \ldots \land X_p(x) \neq 0$, $x \in M$, then $\Delta$ is completely integrable iff all Lie brackets $[X_i, X_j], \ i, j = 1, 2, \ldots, p$ are representable linearly through the very $X_i, i = 1, 2, \ldots, p : [X_i, X_j] = C_{ij}^k X_k$, where $C_{ij}^k$ are functions. Clearly, an easy way to find out if a distribution is completely integrable is to check if the exterior products

$$[X_i, X_j] \land X_1(x) \land X_2(x) \land \ldots \land X_p(x), \quad i, j = 1, 2, \ldots, p$$

are identically zero. If this is not the case (which means that at least one such Lie bracket "sticks out" of the distribution $\Delta$) then the corresponding coefficients, which are multilinear combinations of the components of the vector fields and their derivatives, represent the corresponding curvatures.

We note finally that if two subdistributions contain at least one common vector field it seems naturally to expect interaction.

In the dual formulation of Frobenius theorem in terms of differential 1-forms (i.e. Pfaff forms), having the distribution $\Delta$, we look for $(n-p)$-Pfaff forms $(\alpha^1, \alpha^2, \ldots, \alpha^{n-p})$, i.e. a $(n-p)$-codistribution $\Delta^*$, such that $\left\langle \alpha^m, X_j \right\rangle = 0$, and $\alpha^1 \land \alpha^2 \land \cdots \land \alpha^{n-p} \neq 0$, $m = 1, 2, \ldots, n-p, \ j = 1, 2, \ldots, p$. Then the integrability of the distribution $\Delta$ is equivalent to the requirements

$$d\alpha^m \land \alpha^1 \land \alpha^2 \land \cdots \land \alpha^{n-p} = 0, \quad m = 1, 2, \ldots, (n-p),$$

where $d$ is the exterior derivative.

Since the idea of curvature associated with, for example, an arbitrary 2-dimensional distribution $(X, Y)$ is to find out if the Lie bracket $[X, Y]$ has components along vector fields outside the 2-plane defined by $(X, Y)$, in our case we have to evaluate the quantities $\left\langle \alpha^m, [X, Y] \right\rangle$, where all linearly independent 1-forms $\alpha^m$ annihilate $(X, Y): \left\langle \alpha^m, X \right\rangle = \left\langle \alpha^m, Y \right\rangle = 0$. In view of the formula

$$d\alpha^m(X, Y) = X(\langle \alpha^m, Y \rangle) - Y(\langle \alpha^m, X \rangle) - \langle \alpha^m, [X, Y] \rangle = -\langle \alpha^m, [X, Y] \rangle$$

we may introduce explicitly the curvature 2-form for the distribution $\Delta(X) = \{X_1, \ldots, X_p\}$. In fact, if $\Delta(Y) = \{Y_1, \ldots, Y_{n-p}\}$ define a distribution which is complimentary (in the sense of direct sum) to $\Delta(X)$ and $\left\langle \alpha^m, X_i \right\rangle = 0$, $\left\langle \alpha^m, Y_1 \right\rangle = \delta^m_n$, i.e. $(Y_1, \ldots, Y_{n-p})$ and $(\alpha^1, \ldots, \alpha^{n-p})$ are dual bases, then the corresponding curvature 2-form $\Omega_{\Delta(X)}$ should be defined by

$$\Omega_{\Delta(X)} = -d\alpha^m \otimes Y_m, \quad \text{since} \quad \Omega_{\Delta(X)}(X_i, X_j) = -d\alpha^m(X_i, X_j)Y_m = \langle \alpha^m, [X_i, X_j] \rangle Y_m,$$
\( m = 1, 2, \ldots, q < (n - p) \) in general, the important moment is that the two distributions (or subdistributions) can "communicate" differentially through their curvature 2-forms.

Hence, from physical point of view, if the quantities \( \Omega_{\Delta(X)}(X_i, X_j) \) participate somehow in building the components of the energy-momentum locally transferred from the system \( \Delta(X) \) to the system \( \Delta(Y) \), then, naturally, we have to make use of the quantities \( \Omega_{\Delta(Y)}(Y_m, Y_n) \) to build the components of the energy-momentum transferred from \( \Delta(Y) \) to \( \Delta(X) \). It deserves to note that it is possible a dynamical equilibrium between the two systems \( \Delta(Y) \) and \( \Delta(X) \) to exist: each system to gain as much energy-momentum as it loses, and this to take place at every space-time point. On the other hand, the restriction of \( \Omega_{\Delta(X)} = -d\alpha^m \otimes Y_m, m = 1, \ldots, q \) to the system \( \Delta(Y) \), i.e. the quantities \( \Omega_{\Delta(X)}(Y_m, Y_n) \), and the restriction of \( \Omega_{\Delta(Y)} = -d\beta^i \otimes X_i, i = 1, \ldots, p \), acquire the sense of objects in terms of which the local change of the corresponding energy-momentum, i.e. differences between energy-momentum gains and losses, should be expressed. Therefore, if \( W_{(X,Y)} \) denotes the energy-momentum transferred locally from \( \Delta(X) \) to \( \Delta(Y) \), \( W_{(Y,X)} \) denotes the energy-momentum transferred locally from \( \Delta(Y) \) to \( \Delta(X) \), and \( \delta W_{(X)} \) and \( \delta W_{(Y)} \) denote respectively the local energy-momentum changes of the two systems \( \Delta(X) \) and \( \Delta(Y) \), then according to the local energy-momentum conservation law we can write

\[
\delta W_{(X)} = W_{(Y,X)} + W_{(X,Y)}, \quad \delta W_{(Y)} = -(W_{(X,Y)} + W_{(Y,X)}) = -\delta W_{(X)}.
\]

For the case of dynamical equilibrium we have \( W_{(X,Y)} = -W_{(Y,X)} = 0 \), so in such a case we obtain

\[
\delta W_{(X)} = 0, \quad \delta W_{(Y)} = 0, \quad W_{(Y,X)} + W_{(X,Y)} = 0. \tag{4}
\]

As for how to build explicitly the corresponding representatives of the energy-momentum fluxes, probably, universal procedure can not be offered. If, for example, the mathematical representative of the entire system containing \( \Delta(X) \) and \( \Delta(Y) \) as subsystems, is a differential form \( G \), then the most simple procedure seems to be to "project" the curvature components \( \Omega_{\Delta(X)}(X_i, X_j) \) and \( \Omega_{\Delta(Y)}(Y_m, Y_n) \), as well as the components \( \Omega_{\Delta(X)}(Y_i, Y_j) \) and \( \Omega_{\Delta(Y)}(X_m, X_n) \) on \( G \), i.e. to consider the corresponding interior products. For every special case, however, appropriate quantities constructed out of the members of the introduced distributions and co-distributions must be worked out.

### 4 Non-linear connections

#### 2.1 Projections: These are linear maps \( P \) in a linear space \( W^n \) sending all elements of \( W^n \) to some subspace \( P(W^n) \subset W^n \), so that \( P \circ P = P \). Let \( (e_1, \ldots, e_n) \) and \( (\varepsilon^1, \ldots, \varepsilon^n) \) be two dual bases in \( W^n \) and \( (W^n)^* \), such that \( P(W^n) \) is spent by \( (e_{p+1}, \ldots, e_n) \) and the dual to \( P(W^n) \) is spent by \( (\varepsilon^{p+1}, \ldots, \varepsilon^n) \). The map \( P \) is reduced to the identity map in \( P(W^n) \), so, it is given there by the tensor \( \varepsilon^a \otimes e_a \), where \( a = p + 1, \ldots, n \). The linear map \( P \), restricted to some other subspace \( \mathbb{H} \) of \( W^n \), such that \( \mathbb{H} \oplus Im(P) = W^n \) should be represented by some appropriate matrix \( N^a_i \) in the corresponding bases, so the map \( P \) looks like in these bases as follows:

\[
P = \varepsilon^a \otimes e_a + (N_i)^a \varepsilon^i \otimes e_a, \quad i = 1, \ldots, p; \quad a = p + 1, \ldots, n. \tag{5}
\]

Let now \( \phi \) and \( \psi \) be two arbitrary linear maps, \( \mathfrak{B} \) be a bilinear map in \( W^n \), and \( (x, y) \) be two arbitrary vectors in \( W^n \). We consider the expression

\[
A(\mathfrak{B}; \phi, \psi)(x, y) \equiv \frac{1}{2} \left( \mathfrak{B}(\phi(x), \psi(y)) + \mathfrak{B}(\psi(x), \phi(y)) + \phi \circ \psi(\mathfrak{B}(x, y)) + \psi \circ \phi(\mathfrak{B}(x, y)) \right)
\]
\[ -\phi(\mathfrak{B}(x, \psi(y))) - \phi(\mathfrak{B}(\psi(x), y)) - \psi(\mathfrak{B}(x, \phi(y))) - \psi(\mathfrak{B}(\phi(x), y)) \] .

Assuming \( \phi = \psi \) are projections denoted by \( P \) this expression becomes

\[ A(\mathfrak{B}; P)(x, y) = P(\mathfrak{B}(x, y)) + \mathfrak{B}(P(x), P(y)) - P(\mathfrak{B}(x, P(y))) - P(\mathfrak{B}(P(x), y)) . \]

Denoting the identity map of \( W^n \) by \( id \) and adding and subtracting \( P[\mathfrak{B}(P(x), P(y))] \), after some elementary transformations we obtain

\[ A(\mathfrak{B}; P)(x, y) = P[\mathfrak{B}((id - P)(x), (id - P)(y))] + (id - P)[\mathfrak{B}(P(x), P(y)] . \]

Recalling that \( P \) and \( (id - P) \) project on two subspaces of \( W^n \), the direct sum of which generates \( W^n \), and naming \( P \) as vertical projection denoted by \( V \), then \( (id - P) \), denoted by \( H \), gets naturally the name horizontal projection. So the above expression gets the final form of

\[ A(\mathfrak{B}; P)(x, y) = V[\mathfrak{B}(H(x), H(y))] + H[\mathfrak{B}(V(x), V(y))]. \quad (6) \]

Hence, the first term on the right measures the vertical component of the \( \mathfrak{B} \)-image of the horizontal projections of \( (x, y) \), and the second term measures the horizontal component of the \( \mathfrak{B} \)-image of the vertical projections of \( (x, y) \), which is in correspondence with the well known fact that if \( Ker(P) \) is the kernal space of \( P \) and \( Im(P) \) is the image space of \( P \) then the vector space \( W^n \) is a direct sum of \( Ker(P) \) and \( Im(P) \): \( W^n = Ker(P) \oplus Im(P) \).

We carry now this pure algebraic construction to the tangent bundle of a smooth manifold \( M^n \), where the above bilinear map \( \mathfrak{B} \) will be interpreted as the Lie bracket of vector fields, and the linear maps will be just linear endomorphisms of the tangent/cotangent bundles of \( M^n \). Under these assumptions the image of the above initial expression is called Nijenhuis bracket of the two linear endomorphisms \( \Phi \) and \( \Psi \), and is usually denoted by \( [\Phi, \Psi] \). It has two important for us properties: the first one is that it is linear with respect to the smooth functions on the manifold, so, the Nijenhuis bracket allows, starting with two \((1,1)\)-tensors on \( M^n \), to construct through differentiations one \((2,1)\)-tensor field being antisymmetric with respect to the covariant indices, i.e. a 2-form that is valued in the tangent bundle of \( M^n \); the second property is that if \( \Phi = \Psi \) then \([\Phi, \Phi]\) is not necessarily zero.

2.2 Nonlinear connections

Let now \( (x^1, \ldots, x^n) \) be any local coordinate system on our real manifold \( M^n \). We have the corresponding local frames \( \{dx^1, \ldots, dx^n\} \) and \( \{\partial_{x^1}, \ldots, \partial_{x^n}\} \). Let for each \( x \in M \) we are given a projection \( P_x \) of the same constant rank \( p \), i.e. \( p \) does not depend on \( x \), in every tangent space \( T_x(M) \). The space \( Ker(P_x) \subset T_x(M) \) is usually called \( P \)-horizontal, and the space \( Im(P_x) \subset T_x(M) \) then is called \( P \)-vertical. Thus, we have two distributions on \( M \) the direct sum of which gives the tangent bundle: \( T(M) = Ker(P) \oplus Im(P) \). The above result shows that each of these two distributions can be endowed with corresponding 2-form, valued in the other distribution, and depending on some binar operation in \( TM^n \). As we mentioned the combination ”Nijenhuis bracket plus Lie bracket” leads to tensor field. Therefore, assuming that the corresponding curvatures are defined by means of the combination ”Nijenhuis bracket of \( P \) plus Lie bracket of vector fields” we say that \( P \) defines a nonlinear connection on \( M \). Denoting by \( \mathcal{R} \) the so defined curvature 2-form of \( Ker(P) \) and by \( \mathcal{R} \) the analogically defined curvature 2-form of \( Im(P) \), by \( V_P \) the restriction of \( P \) to \( Ker(P) \) and by \( H_P \) the restriction of \( P \) to \( Im(P) \), we can write

\[ [P, P](X, Y) = \mathcal{R}(X, Y) + \mathcal{R}(X, Y), \quad (7) \]
where
\[ \mathcal{R}(X,Y) = V_P ([H_P X, H_P Y]), \quad \tilde{\mathcal{R}}(X,Y) = H_P ([V_P X, V_P Y]), \]

\((X,Y)\) are any two vector fields and the Lie bracket is denoted by \([\cdot,\cdot]\). Recalling the contents of the preceding section, we can say that \(\mathcal{R}(X,Y) \neq 0\) measures the nonintegrability of the corresponding horizontal distribution, and \(\tilde{\mathcal{R}}(X,Y) \neq 0\) measures the nonintegrability of the vertical distribution.

If the vertical distribution is given beforehand and is completely integrable, i.e. \(\mathcal{R} = 0\), then \(\mathcal{R}(X,Y)\) is called \textit{curvature} of the nonlinear connection \(P\) if there exist at least one couple of vector fields \((X,Y)\) such that \(\mathcal{R}(X,Y) \neq 0\).

5 Photon-like nonlinear connections

We assume now that our manifold is \(\mathbb{R}^4\) endowed with standard coordinates \((x^1, x^2, x^3, x^4 = x, y, z, \xi = ct)\), and make some preliminary considerations in order to make the choice of our projection \(V\) consistent with the introduced concept of PhLO. The intrinsically defined straight-line translational component of propagation of the PhLO will be assumed to be parallel to the horizontal plane \((z,\xi)\). Also, \(\frac{\partial}{\partial x}\) and \(\frac{\partial}{\partial y}\) will be vertical coordinate fields, so every vertical vector field \(Y\) can be represented by \(Y = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}\). It is easy to check that any two such linearly independent vertical vector fields \(Y_1\) and \(Y_2\) define an integrable distribution, hence, the corresponding curvature will be zero. It seems very natural to choose \(Y_1\) and \(Y_2\) to coincide correspondingly with the vertical projections of \(X_1\) and \(X_2\). Moreover, let’s restrict ourselves to PhLO of electromagnetic nature and denote further the vertical projection by \(\mathcal{R}\). Then, since this vertical structure is meant to be smoothly straight-line translated along the plane \((z,\xi)\) with the velocity of light, a natural suggestion comes to mind these two projections \(\mathcal{R}_1 = V)\) and \(\mathcal{R}_2 = V)\) to be physically interpreted as representatives of the electric and magnetic components. Now we know from classical electrodynamics that the situation described corresponds to zero invariants of the electromagnetic field, therefore, we may assume that \(Y_1\) and \(Y_2\) are orthogonal to each other and with the same modules with respect to the euclidean metric in the 2-dimensional space spent by \(\frac{\partial}{\partial x}\) and \(\frac{\partial}{\partial y}\). It follows that the essential components of \(Y_1\) and \(Y_2\) should be expressible only with two independent functions \((u, p)\). The conclusion is that our projection should depend only on \((u, p)\). Finally, we note that these assumptions lead to the horizontal nature of \(dz\) and \(d\xi\).

Note that if the translational component of propagation is along the vector field \(X\) then we can define two new distributions: \((Y_1, X)\) and \((Y_2, X)\), which do not seem to be integrable in general even if \(X\) has constant components as it should be. Since these two distributions are nontrivially intersected (they have a common member \(X\)), it is natural to consider them as geometrical images of two consistently interacting physical subsystems of our PhLO. Hence, we must introduce two projections with the same image space but with different kernel spaces, and the components of both projections must depend only on the two functions \((u, p)\).

Let now \((u, p)\) be two smooth functions on \(\mathbb{R}^4\) and \(\varepsilon = \pm 1\). We introduce two projections \(V\) and \(\tilde{V}\) in \(T^* \mathbb{R}^4\) as follows:

\[
V = dx \otimes \frac{\partial}{\partial x} + dy \otimes \frac{\partial}{\partial y} - \varepsilon u dz \otimes \frac{\partial}{\partial x} - u d\xi \otimes \frac{\partial}{\partial y} - \varepsilon p dz \otimes \frac{\partial}{\partial y} - p d\xi \otimes \frac{\partial}{\partial y},
\]

\[
\tilde{V} = dx \otimes \frac{\partial}{\partial x} + dy \otimes \frac{\partial}{\partial y} + p dz \otimes \frac{\partial}{\partial x} + \varepsilon p d\xi \otimes \frac{\partial}{\partial x} - u dz \otimes \frac{\partial}{\partial y} - \varepsilon u d\xi \otimes \frac{\partial}{\partial y}.
\]

So, in both cases we consider \((\frac{\partial}{\partial x}, \frac{\partial}{\partial y})\) as vertical vector fields, and \((dz, d\xi)\) as horizontal 1-forms. By corresponding transpositions we can determine projections \(V^*\) and \(\tilde{V}^*\) in the cotangent bundle.
The projections of the coordinate bases are:

\[ T^* \mathbb{R}^4. \]

\[ V^* = dx \otimes \frac{\partial}{\partial x} + dy \otimes \frac{\partial}{\partial y} - \varepsilon u \, dx \otimes \frac{\partial}{\partial z} - u \, dx \otimes \frac{\partial}{\partial \xi} - \varepsilon \, p \, dy \otimes \frac{\partial}{\partial z} - p \, dy \otimes \frac{\partial}{\partial \xi}. \]

\[ \tilde{V}^* = dx \otimes \frac{\partial}{\partial x} + dy \otimes \frac{\partial}{\partial y} + p \, dx \otimes \frac{\partial}{\partial z} + \varepsilon p \, dx \otimes \frac{\partial}{\partial \xi} - u \, dy \otimes \frac{\partial}{\partial z} - \varepsilon u \, dy \otimes \frac{\partial}{\partial \xi}. \]

The corresponding horizontal projections, denoted by \((H, H^*; \tilde{H}, \tilde{H}^*)\) look as follows:

\[ H = dz \otimes \frac{\partial}{\partial z} + d\xi \otimes \frac{\partial}{\partial \xi} + \varepsilon u \, dz \otimes \frac{\partial}{\partial x} + u \, dz \otimes \frac{\partial}{\partial y} + \varepsilon p \, dz \otimes \frac{\partial}{\partial y} + p \, d\xi \otimes \frac{\partial}{\partial y}. \]

\[ \tilde{H} = dz \otimes \frac{\partial}{\partial z} + d\xi \otimes \frac{\partial}{\partial \xi} - p \, dz \otimes \frac{\partial}{\partial x} - \varepsilon p \, dz \otimes \frac{\partial}{\partial y} + u \, dz \otimes \frac{\partial}{\partial y} + \varepsilon u \, dz \otimes \frac{\partial}{\partial y}. \]

\[ H^* = dz \otimes \frac{\partial}{\partial z} + d\xi \otimes \frac{\partial}{\partial \xi} + \varepsilon u \, dx \otimes \frac{\partial}{\partial x} + u \, dx \otimes \frac{\partial}{\partial y} + \varepsilon p \, dx \otimes \frac{\partial}{\partial y} + p \, dy \otimes \frac{\partial}{\partial \xi}. \]

\[ \tilde{H}^* = dz \otimes \frac{\partial}{\partial z} + d\xi \otimes \frac{\partial}{\partial \xi} - p \, dx \otimes \frac{\partial}{\partial z} - \varepsilon p \, dx \otimes \frac{\partial}{\partial \xi} + u \, dy \otimes \frac{\partial}{\partial \xi} + \varepsilon u \, dy \otimes \frac{\partial}{\partial \xi}. \]

The corresponding matrices look like:

\[
V = \begin{pmatrix}
1 & 0 & -\varepsilon u & -u \\
0 & 1 & -\varepsilon p & -p \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad H = \begin{pmatrix}
0 & 0 & \varepsilon u & u \\
0 & 0 & \varepsilon p & p \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

\[
V^* = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-\varepsilon u & -\varepsilon p & 0 & 0 \\
-u & -p & 0 & 0
\end{pmatrix}, \quad H^* = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\varepsilon u & \varepsilon p & 1 & 0 \\
u & p & 0 & 1
\end{pmatrix},
\]

\[
\tilde{V} = \begin{pmatrix}
1 & 0 & p & \varepsilon p \\
0 & 1 & -u & -\varepsilon u \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad \tilde{H} = \begin{pmatrix}
0 & 0 & -p & -\varepsilon p \\
0 & 0 & u & \varepsilon u \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

\[
\tilde{V}^* = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
p & -u & 0 & 0 \\
\varepsilon p & -\varepsilon u & 0 & 0
\end{pmatrix}, \quad \tilde{H}^* = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-p & u & 1 & 0 \\
-\varepsilon p & \varepsilon u & 0 & 1
\end{pmatrix}.
\]

The projections of the coordinate bases are:

\[
\left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial \xi} \right) . V = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, -\varepsilon u \frac{\partial}{\partial x} - \varepsilon p \frac{\partial}{\partial y}, -u \frac{\partial}{\partial x} - p \frac{\partial}{\partial y} \right); \\
\left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial \xi} \right) . H = \left( 0, 0, \varepsilon u \frac{\partial}{\partial x} + \varepsilon p \frac{\partial}{\partial y} + \frac{\partial}{\partial z}, u \frac{\partial}{\partial x} + p \frac{\partial}{\partial y} + \frac{\partial}{\partial \xi} \right); \\
(dx, dy, dz, d\xi) . V^* = (dx - \varepsilon u dz - ud\xi, dy - \varepsilon pdz - pd\xi, 0, 0)
We obtain (in our coordinate system)

\[
(\partial_x, \partial_y, \partial_z, \partial_{\xi}) \cdot H^* = (\varepsilon udz + ud\xi, \varepsilon pdz + pd\xi, dz, d\xi)
\]

\[
\left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial \xi} \right) \cdot \tilde{V} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial \xi} \right) - u \frac{\partial}{\partial y} \varepsilon p \frac{\partial}{\partial x} - \varepsilon u \frac{\partial}{\partial y} \frac{\partial}{\partial \xi};
\]

\[
\left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial \xi} \right) \cdot \tilde{H} = \left( 0, 0, -p \frac{\partial}{\partial x} + u \frac{\partial}{\partial y} + \frac{\partial}{\partial z}, -\varepsilon p \frac{\partial}{\partial x} + \varepsilon u \frac{\partial}{\partial y} + \frac{\partial}{\partial \xi} \right);
\]

\[
(\partial_x, \partial_y, \partial_z, \partial_{\xi}) \cdot \tilde{V}^* = (\partial_x + p dz + \varepsilon pd\xi, dy - u dz - \varepsilon ud\xi, 0, 0)
\]

\[
(\partial_x, \partial_{\xi}, \partial_y, \partial_{\xi}) \cdot \tilde{H}^* = (-p dz - \varepsilon p d\xi, u dz + \varepsilon u d\xi, \partial_x, \partial_{\xi}).
\]

We compute now the two curvature 2-forms $R$ and $\tilde{R}$. The components $R_{\mu\nu}^\sigma$ of $R$ in coordinate basis are given by $V_{\rho}^\sigma \left( [H_{\frac{\partial}{\partial x}}, H_{\frac{\partial}{\partial y}}]^\rho \right)$, and the only nonzero components are just

\[
R_{\xi\xi}^x = R_{34}^{x} = -\varepsilon (u_{\xi} - \varepsilon u_{z}), \quad R_{\xi\xi}^y = R_{34}^{y} = -\varepsilon (p_{\xi} - \varepsilon p_{z}).
\]

For the nonzero components of $\tilde{R}$ we obtain

\[
\tilde{R}_{\xi\xi}^x = \tilde{R}_{34}^{x} = (p_{\xi} - \varepsilon p_{z}), \quad \tilde{R}_{\xi\xi}^y = \tilde{R}_{34}^{y} = -(u_{\xi} - \varepsilon u_{z}).
\]

The corresponding two curvature forms are:

\[
R = -\varepsilon (u_{\xi} - \varepsilon u_{z}) dz \wedge d\xi \otimes \frac{\partial}{\partial x} - \varepsilon (p_{\xi} - \varepsilon p_{z}) dz \wedge d\xi \otimes \frac{\partial}{\partial y} \quad (10)
\]

\[
\tilde{R} = (p_{\xi} - \varepsilon p_{z}) dz \wedge d\xi \otimes \frac{\partial}{\partial x} - (u_{\xi} - \varepsilon u_{z}) dz \wedge d\xi \otimes \frac{\partial}{\partial y}, \quad (11)
\]

We obtain (in our coordinate system) $-\frac{1}{2} tr (V \circ H^*) = -\frac{1}{2} tr (\tilde{V} \circ \tilde{H}^*) = u^2 + p^2$, and

\[
V \left( \left[ H \left( \frac{\partial}{\partial z} \right), H \left( \frac{\partial}{\partial \xi} \right) \right] \right) = \left[ H \left( \frac{\partial}{\partial z} \right), H \left( \frac{\partial}{\partial \xi} \right) \right] = -\varepsilon (u_{\xi} - \varepsilon u_{z}) \frac{\partial}{\partial x} - \varepsilon (p_{\xi} - \varepsilon p_{z}) \frac{\partial}{\partial y} \equiv Z_1,
\]

\[
\tilde{V} \left( \left[ \tilde{H} \left( \frac{\partial}{\partial z} \right), \tilde{H} \left( \frac{\partial}{\partial \xi} \right) \right] \right) = \left[ \tilde{H} \left( \frac{\partial}{\partial z} \right), \tilde{H} \left( \frac{\partial}{\partial \xi} \right) \right] = (p_{\xi} - \varepsilon p_{z}) \frac{\partial}{\partial x} - (u_{\xi} - \varepsilon u_{z}) \frac{\partial}{\partial y} \equiv Z_2,
\]

where $Z_1$ and $Z_2$ coincide with the values of the two curvature forms $R$ and $\tilde{R}$ on the coordinate vector fields $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \xi}$ respectively:

\[
Z_1 = R \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial \xi} \right), \quad Z_2 = \tilde{R} \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial \xi} \right).
\]

We evaluate now the vertical 2-form $V^* (dx) \wedge V^* (dy)$ on the bivector $Z_1 \wedge Z_2$ and obtain $\varepsilon K^2$, where

\[
K^2 = (u_{\xi} - \varepsilon u_{z})^2 + (p_{\xi} - \varepsilon p_{z})^2.
\]

An important parameter, having dimension of length (the coordinates are assumed to have dimension of length) and denoted by $l_o$, turns out to be the square root of the quantity

\[
-\frac{1}{2} tr (V \circ H^*) \frac{K^2}{K^2} = \frac{u^2 + p^2}{(u_{\xi} - \varepsilon u_{z})^2 + (p_{\xi} - \varepsilon p_{z})^2}.
\]
Clearly, if $l_o$ is finite constant it could be interpreted as some parameter of extension of the PhLO described, so it could be used as identification parameter in the dynamical equations and in lagrangians, but only if $(u\xi - \varepsilon u_z) \neq 0$ and $(p\xi - \varepsilon p_z) \neq 0$. This goes along with our concept of PhLO which does not admit spatially infinite extensions. Finally we’d like to note that the right-hand side of the above relation does not depend on which projection $V$ or $\tilde{V}$ is used, i.e. $[\tilde{V}^*(dx) \wedge \tilde{V}^*(dy)](Z_1 \wedge Z_2) = \varepsilon K^2$ too, so

$$l_o^2 = \frac{-\frac{1}{2}tr\left(\tilde{V} \circ \tilde{H}^*\right)}{K^2} = \frac{-\frac{1}{2}tr\left(V \circ H^*\right)}{K^2} = \frac{u^2 + p^2}{(u\xi - \varepsilon u_z)^2 + (p\xi - \varepsilon p_z)^2}. \quad (12)$$

The parameter $l_o$ has the following symmetry. Denote by $V_o = dx \otimes \frac{\partial}{\partial x} + dy \otimes \frac{\partial}{\partial y}$, then $V = V_o + V_1$ and $\tilde{V} = V_o + \tilde{V}_1$, where, in our coordinates, $V_1$ and $\tilde{V}_1$ can be seen above how they look like. We form now $W = aV_o - b\tilde{V}_1$ and $\tilde{W} = bV_o + a\tilde{V}_1$, where $(a, b)$ are two arbitrary real numbers. The components of the corresponding projections $P_W = V_o + W$ and $P_{\tilde{W}} = V_o + \tilde{W}$ can be obtained through the substitutions: $u \rightarrow (au + b\varepsilon p)$; $p \rightarrow (\varepsilon b p - ap)$. Now $-\frac{1}{2}tr(V \circ H^*)$ transforms to $(a^2 + b^2)u^2 + p^2)$ and $K^2$ transforms to $(a^2 + b^2)[(u\xi - \varepsilon u_z)^2 + (p\xi - \varepsilon p_z)^2]$, so, $l_o(V) = l_o(W)$.

This corresponds in some sense to the dual summertime of classical vacuum electrodynamics. We note finally that the squared modules of the two curvature forms $|\mathcal{R}|^2$ and $|\mathcal{\tilde{R}}|^2$ are equal to $(u\xi - \varepsilon u_z)^2 + (p\xi - \varepsilon p_z)^2$ in our coordinates, therefore, the nonzero values of $|\mathcal{R}|^2$ and $|\mathcal{\tilde{R}}|^2$, as well as the finite value of $l_o$ guarantee that the two functions $u$ and $p$ are NOT plane running waves.

6 Connection to classical electrodynamics

From formal point of view the relativistic formulation of classical electrodynamics in vacuum ($\rho = 0$) is based on the following assumptions. The configuration space is the Minkowski space-time $M = (\mathbb{R}, \eta)$ where $\eta$ is the pseudometric with sign($\eta$) = $(-, -, -, +)$ with the corresponding volume 4-form $\omega_0 = dx \wedge dy \wedge dz \wedge d\xi$ and Hodge star $*$ defined by $\alpha \wedge \beta = \eta(\alpha, \beta)\omega_0$. The electromagnetic filed is describe by two closed 2-forms $(F, *F) : dF = 0, d*F = 0$. The physical characteristics of the field are deduced from the following stress-energy-momentum tensor field

$$T_{\mu}^{\nu}(F, *F) = -\frac{1}{2}[F_{\mu\sigma}F^{\nu\sigma} + (*F)_{\mu\sigma}(*F^{\nu\sigma})]. \quad (13)$$

In the non-vacuum case the allowed energy-momentum exchange with other physical systems is given in general by the divergence

$$\nabla_\nu T_{\mu}^{\nu} = \frac{1}{2}\left[F^{\alpha\beta}(dF)_{\alpha\beta\mu} + (*F)^{\alpha\beta}(d*F)_{\alpha\beta\mu}\right] = F_{\mu\nu}(\delta F)_{\nu} + (*F)_{\mu\nu}(\delta *F)_{\nu}, \quad (14)$$

where $\delta = *d*$ is the coderivative. If the field is free: $dF = 0, d*F = 0$, this divergence is obviously equal to zero on the vacuum solutions since its both terms are zero. Therefore, energy-momentum exchange between the two component-fields $F$ and $*F$, which should be expressed by the terms $(*F)_{\alpha\beta}(dF)_{\alpha\beta\mu}$ and $(F^{\alpha\beta}(d*F)_{\alpha\beta\mu}$ is NOT allowed on the solutions of $dF = 0, d*F = 0$. This shows that the widely used 4-potential approach (even if two 4-potentials $A, A^*$ are introduced so that $dA = F, dA^* = *F$ locally) to these equations excludes any possibility to individualize two energy-momentum exchanging time-stable subsystems of the field that are mathematically represented by $F$ and $*F$.

On the contrary, our concept of PhLO does NOT exclude such two physically interacting subsystems of the field to really exist, and therefore, to be mathematically individualized. The intrinsically
From these last relations we see that we can represent respect to \( \tilde{\mathcal{V}} \) connected two projections \( V \) and \( \tilde{V} \) and the corresponding two curvature forms give the mathematical realization of this idea: \( V \) and \( \tilde{V} \) individualize the two subsystems, and the corresponding two curvature 2-forms \( \mathcal{R} \) and \( \tilde{\mathcal{R}} \) represent the instruments by means of which mutual energy-momentum exchange between these two subsystems could be locally performed. Moreover, as we already mentioned, the energy-momentum tensor for a PhLO must satisfy the additional local isotropy (null) condition \( T_{\mu\nu}(F, *F)\mathcal{T}^{\mu\nu}(F, *F) = 0 \).

So, we have to construct appropriate quantities and relations having direct physical sense in terms of the introduced and considered two projections \( V \) and \( \tilde{V} \). The above well established in electrodynamics relations say that we need two 2-forms to begin with.

Recall that our coordinate 1-forms \( dx \) nd \( dy \) have the following vertical and horizontal projections:

\[
V^*(dx) = dx - \varepsilon u\, dz - u\, d\xi, \quad H^*(dx) = \varepsilon u\, dz + u\, d\xi,
\]

\[
V^*(dy) = dy - \varepsilon p\, dz - p\, d\xi, \quad H^*(dy) = \varepsilon p\, dz + p\, d\xi.
\]

We form now the 2-forms \( V^*(dx) \land H^*(dx) \) and \( V^*(dy) \land H^*(dy) \):

\[
V^*(dx) \land H^*(dx) = \varepsilon u\, dx \land dz + u\, dx \land d\xi,
\]

\[
V^*(dy) \land H^*(dy) = \varepsilon p\, dy \land dz + p\, dy \land d\xi.
\]

Summing up these last two relations and denoting the sum by \( F \) we obtain

\[
F = \varepsilon u\, dx \land dz + u\, dx \land d\xi + \varepsilon p\, dy \land dz + p\, dy \land d\xi.
\] (15)

Doing the same steps with \( \tilde{V}^* \) and \( \tilde{H}^* \) we obtain

\[
\tilde{F} = -p\, dx \land dz - \varepsilon p\, dx \land d\xi + u\, dy \land dz + \varepsilon u\, dy \land d\xi.
\] (16)

Noting that our definition of the Hodge star requires \( (*F)_{\mu\nu} = -\frac{1}{2} \varepsilon_{\mu\nu}^{\quad \sigma\rho} F_{\sigma\rho} \), it is now easy to verify that \( \tilde{F} = *F \). Moreover, introducing the notations

\[
A = u\, dx + p\, dy, \quad A^* = -\varepsilon p\, dx + \varepsilon u\, dy, \quad \zeta = \varepsilon\, dz + d\xi,
\]

we can represent \( F \) and \( \tilde{F} \) in the form

\[
F = A \land \zeta, \quad \tilde{F} = *F = A^* \land \zeta.
\]

From these last relations we see that \( F \) and \( *F \) are isotropic: \( F \land F = 0, F \land *F = 0 \), i.e. the field \( (F, *F) \) has zero invariants: \( F_{\mu\nu} F^{\mu\nu} = F_{\mu\nu} (*F)^{\mu\nu} = 0 \). The following relations are now easy to verify:

\[
V^*(F) = H^*(F) = V^*(*F) = H^*(*F) = \tilde{V}^*(F) = \tilde{H}^*(F) = \tilde{V}^*(*F) = \tilde{H}^*(*F) = 0,
\] (17)

d.e \( F \) and \( *F \) have zero vertical and horizontal projections with respect to \( V \) and \( \tilde{V} \). Since, obviously, \( \zeta \) is horizontal with respect to \( V \) and \( \tilde{V} \) it is interesting to note that \( A \) is vertical with respect to \( \tilde{V} \) and \( A^* \) is vertical with respect to \( V \): \( \tilde{V}^*(A) = A, V(A^*) = A^* \). In fact, for example,

\[
\tilde{V}^*(A) = \tilde{V}^*(u\, dx + p\, dy) = u\tilde{V}^*(dx) + p\tilde{V}^*(dy) = u[dx + p\, dz + \varepsilon p\, d\xi] + p[dy - u\, dz - \varepsilon u\, d\xi] = u\, dx + p\, dy.
\]
We are going to establish now that there is real energy-momentum exchange between the $F$-component and the $\ast F$-component of the field. To come to this we compute the quantities $i(Z_1)F$, $i(Z_2)\ast F$, $i(Z_1)\ast F$, $i(Z_2)F$. We obtain:

$$i(Z_1)F = i(Z_2)\ast F = \langle A, Z_1 \rangle \zeta = \langle A^\ast, Z_2 \rangle \zeta = \frac{1}{2} [(u^2 + p^2)x - \varepsilon (u^2 + p^2)z] \zeta =$$

$$= \frac{1}{2} F^\sigma\rho(dF)_{\sigma\rho\mu}dx^\mu = \frac{1}{2} (\ast F)^\sigma\rho(dF)_{\sigma\rho\mu}dx^\mu = \frac{1}{2} \nabla_\nu T_{\mu}^\nu(F, \ast F),$$

$$i(Z_1)\ast F = -i(Z_2)F = \langle A^\ast, Z_1 \rangle \zeta = -\langle A, Z_2 \rangle \zeta = [u(p_x - \varepsilon p_z) - p(u_x - \varepsilon u_z)] \zeta =$$

$$= -\frac{1}{2} F^\sigma\rho(d \ast F)_{\sigma\rho\mu}dx^\mu = \frac{1}{2} (\ast F)^\sigma\rho(dF)_{\sigma\rho\mu}dx^\mu.$$  \hspace{1cm} (18)

If our field is free then $\nabla_\nu T_{\mu}^\nu(F, \tilde{F}) = 0$. Moreover, in view of the divergence of the stress-energy-momentum tensor given above, these last relations show that some real energy-momentum exchange between $F$ and $\ast F$ takes place: the magnitude of the energy-momentum, transferred from $F$ to $\ast F$ and given by $i(Z_2)\ast F = \frac{1}{2} (\ast F)^\sigma\rho(dF)_{\sigma\rho\mu}dx^\mu$, is equal to that, transferred from $\ast F$ to $F$, which is given by $-i(Z_2)F = -\frac{1}{2} F^\sigma\rho(d \ast F)_{\sigma\rho\mu}dx^\mu$. On the other hand, as it is well known, the $\ast$-invariance of the stress-energy-momentum tensor in case of zero invariants leads to $F_{\mu\nu} F^{\nu\sigma} = (\ast F)_{\mu\sigma} (\ast F)^{\nu\sigma}$, so, $F$ and $\ast F$ carry equal quantities of stress-energy-momentum. Physically this could mean that the electromagnetic PhLO exist through a special internal dynamical equilibrium between the two subsystems of the field, represented by $F$ and $\ast F$, as mentioned in Section 1, namely, both subsystems carry the same stress-energy-momentum and the mutual energy-momentum exchange between them is always in equal quantities. This individualization does NOT mean that any of the two subsystems can exist separately, independently on each other. Moreover, NO spatial "part" of PhLO is considered to represent a physical object and to be energy-momentum carrier, as it is assumed, for example, when mass and charge distributions are defined in classical electrodynamics.

7 Equations of motion for electromagnetic PhLO

Every system of equations describing the time-evolution of some physical system should be consistent with the very system in the sense that all identification characteristics of the system described must not change. In the case of electromagnetic PhLO we assume the couple $(F, \tilde{F})$ to represent the field, and in accordance with our notion for PhLO one of the identification characteristics is straight-line translational propagation of the energy-density with constant velocity "c", therefore, with every PhLO we may associate appropriate direction, i.e. a geodesic null vector field $X, X^2 = 0$ on the Minkowski space-time. We choose further $X = -\varepsilon \frac{\partial}{\partial x} + \frac{\partial}{\partial z}$, which means that we have chosen the coordinate system in such a way that the translational propagation is parallel to the plane $(z, \xi)$. For another such parameter we assume that the finite longitudinal extension of any PhLO is fixed and is given by an appropriate positive number $\lambda$. In accordance with the "consistent translational-rotational dynamical structure" of PhLO we shall assume that no translation is possible without rotation, and no rotation is possible without translation, and in view of the constancy of the translational component of propagation we shall assume that the rotational component of propagation is periodic, i.e. it is characterized by a constant frequency. The natural period $T$ suggested is obviously $T = \frac{\pi}{\omega}$. An obvious candidate for "rotational operator" is the linear map $J$ transforming $F$ to $\tilde{F}$, which map coincides with the reduced to 2-forms Hodge-$\ast$, it rotates the 2-frame $(A, A^\ast)$ to $\frac{\pi}{2}$, so if such a rotation is associated with a translational advancement of $l_o$, then a full rotation should correspond to translational advancement of $4l_o = \lambda$. The simplest and
most natural translational change of the field $(F, \tilde{F})$ along $X$ should be given by the Lie derivative
of the field along $X$. Hence, the simplest and most natural equations should read

$$ \kappa l_o L_X(F) = \varepsilon \tilde{F}, \quad (20) $$

where $F$ and $\tilde{F}$ are given in the preceding section, $\kappa = \pm 1$ is responsible for left/right orientation
of the rotational component of propagation, and $l_o = \text{const}$. Vice versa, since $J \circ J = -id$ and $J^{-1} = -J$ the above equation is equivalent to

$$ \kappa l_o L_X(\tilde{F}) = -\varepsilon F. $$

It is easy to show that these equations are equivalent to

$$ \kappa l_o L_X(V - V_o) = \varepsilon (\tilde{V} - V_o), \quad (21) $$

where $V_o = dx \otimes \frac{\partial}{\partial x} + dy \otimes \frac{\partial}{\partial y}$ in our coordinates is the identity map in $Im(V) = Im(\tilde{V})$. Another equivalent form is given by

$$ \kappa l_o Z_1 = \tilde{A}^*, \quad \text{or} \quad \kappa l_o Z_2 = -\tilde{A}, $$

where $\tilde{A}^*$ and $\tilde{A}$ are $\eta$-corresponding vector fields to the 1-forms $A^*$ and $A$.

Appropriate lagrangian for these equations ($l_o=\text{const}$.) is

$$ \mathbb{L} = -\frac{1}{2} \left( \kappa l_o X^\sigma \frac{\partial F_{\alpha \beta}}{\partial x^\sigma} - \varepsilon \tilde{F}_{\alpha \beta} \right) \tilde{F}^{\alpha \beta} + \frac{1}{2} \left( \kappa l_o X^\sigma \frac{\partial \tilde{F}_{\alpha \beta}}{\partial x^\sigma} + \varepsilon F_{\alpha \beta} \right) F^{\alpha \beta}, \quad (22) $$

where $F$ and $\tilde{F}$, are considered as independent. The corresponding Lagrange equations read

$$ \kappa l_o X^\sigma \frac{\partial \tilde{F}_{\alpha \beta}}{\partial x^\sigma} + \varepsilon F_{\alpha \beta} = 0, \quad \kappa l_o X^\sigma \frac{\partial F_{\alpha \beta}}{\partial x^\sigma} - \varepsilon \tilde{F}_{\alpha \beta} = 0. \quad (23) $$

The stress-energy-momentum tensor is given by (13) under the additional condition $T_{\mu \nu} T^{\mu \nu} = 0$. It deserves noting that this isotropy condition leads to zero invariants:

$$ I_1 = F_{\mu \nu} F^{\mu \nu} = 0, \quad I_2 = F_{\mu \nu} (\ast F)^{\mu \nu} = 0, \quad \text{and to} \quad F_{\mu \sigma} F^{\nu \sigma} = (\ast F)_{\mu \sigma} (\ast F)^{\nu \sigma}. $$

Hence, the two subsystems represented by $F$ and $\ast F$ carry the same stress-energy-momentum, therefore, $F \equiv \ast F$ energy-momentum exchange is possible only in equal quantities.

In our coordinates the above equations reduce to

$$ \kappa l_o (u_\xi - \varepsilon u_z) = -p, \quad \kappa l_o (p_\xi - \varepsilon p_z) = u, $$

it is seen that the constant $l_o$ satisfies the above given relation (12).

From these last equations we readily obtain the relations

$$ (u^2 + p^2)\xi - \varepsilon (u^2 + p^2)_z = 0, \quad u(p_\xi - \varepsilon p_z) - p(u_\xi - \varepsilon u_z) = \frac{\kappa}{l_o} (u^2 + p^2). $$

Now, the substitution $u = \Phi \cos \psi, \quad p = \Phi \sin \psi$, leads to the relations

$$ L_X \Phi = 0, \quad L_X \psi = \frac{\kappa}{l_o}. $$

Recalling now that $\Phi^2 = -\frac{1}{2} tr(V \circ H^*)$ and computing $\frac{1}{2} tr(V \circ L_X \tilde{H}^*) = \varepsilon [u(p_\xi - \varepsilon p_z) - p(u_\xi - \varepsilon u_z)] = \Phi^2 \varepsilon L_X \psi$ the last two relations can be equivalently written as

$$ L_X [tr(V \circ H^*)] = 0, \quad tr(V \circ L_X \tilde{H}^*) = -\frac{\varepsilon \kappa}{l_o} tr(V \circ H^*). $$

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It seems important to note the following. Another natural equations appear to be the vacuum equations of Extended Electrodynamics (Donev, Tashkova 1995) describing the internal energy-momentum redistribution during evolution, namely,

\[ i(F) dF = 0, \quad i(\tilde{F}) d\tilde{F} = 0, \quad i(F) d\tilde{F} = -i(F) dF \]

The class of nonlinear solutions to these equations, i.e. those satisfying \( d\tilde{F} \neq 0, \quad dF \neq 0 \), incorporates the solutions to (23), however, at these conditions we obtain only one equation, namely,

\[ L_X \Phi^2 = (u^2 + p^2) \xi - \varepsilon (u^2 + p^2) z = 0, \]

which gives the energy conservation.

### 8 Another look at the translational-rotational consistency

In order to look at the translational-rotational consistency from a point of view mentioned in the previous section we recall the concept of local symmetry of a distribution: a vector field \( Y \) is a local (or infinitesimal) symmetry of a p-dimensional distribution \( \Delta \) defined by the vector fields \( (Y_1, \ldots, Y_p) \) if every Lie bracket \( \{Y_i, Y\} \) is in \( \Delta \): \( \{Y_i, Y\} \in \Delta \). Clearly, if \( \Delta \) is completely integrable, then every \( Y_i \) is a symmetry of \( \Delta \), and the flows of these vector fields move the points of each completely integral manifold of \( \Delta \) inside this completely integral manifold, that’s why they are called sometimes internal symmetries. If \( Y \) is outside \( \Delta \) then it is called shuffling symmetry, and in such a case the flow of \( Y \) transforms a given completely integral manifold to another one. We are going to show that our vector field \( X = -\varepsilon \frac{\partial}{\partial z} + \frac{\partial}{\partial \xi} \) is a shuffling symmetry for the distribution \( \Delta_o \) defined by the vector fields \( (\bar{A}, \bar{A}^*) \). But \( \Delta_o \) coincides with our vertical distribution generated by \( (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) \), so it is completely integrable and its integral manifold coincides with the \( (x, y) \)-plane. From physical point of view this is expectable because the allowed translational propagation of our PhLO along null straight lines should not destroy it: this propagation just transforms the 2-plane \( (x, y) \) passing through the point \( (z_1, \xi_1) \) to a parallel to it 2-plane passing through the point \( (z_2, \xi_2) \), and these two points lay on the same trajectory of our field \( X \).

The corresponding Lie brackets are

\[ [\tilde{A}, X] = (u_\xi - \varepsilon u_z) \frac{\partial}{\partial x} + (p_\xi - \varepsilon p_z) \frac{\partial}{\partial y}, \quad [\bar{A}^*, X] = -\varepsilon (p_\xi - \varepsilon p_z) \frac{\partial}{\partial x} + \varepsilon (u_\xi - \varepsilon u_z) \frac{\partial}{\partial y}. \]

We see that \([\tilde{A}, X]\) and \([\bar{A}^*, X]\) are generated by \((\frac{\partial}{\partial x}, \frac{\partial}{\partial y})\), but \( X \) is outside \( \Delta_o \), so our field \( X \) is a shuffling local symmetry of \( \Delta_o \).

We notice now that at each point our projections \( V \) and \( \tilde{V} \) generate two frames: \( (\tilde{A}, \bar{A}^*, \partial_z, \partial_\xi) \) and \( ([A, X], [\bar{A}^*, X], \partial_z, \partial_\xi) \). Physically this would mean that the internal energy-momentum redistribution during propagation transforms the first frame into the second one and vice versa, since both are defined by the dynamical nature of our PhLO. Taking into account that only the first two vectors of these two frames change during propagation we write down the corresponding linear transformation as follows:

\[ ([A, X], [\bar{A}^*, X]) = (\tilde{A}, \bar{A}^*) \left[ \begin{array}{ccc} \alpha & 0 & \beta \\ 0 & \gamma & 0 \\ \delta & 0 & \xi \end{array} \right]. \]
Solving this system with respect to the real numbers \((\alpha, \beta, \gamma, \delta)\) we obtain

\[
\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \frac{1}{\varphi^2} \begin{pmatrix} -\frac{2}{3} L_X \Phi^2 \\ -\varepsilon \mathbf{R} \\ -\frac{2}{3} L_X \Phi^2 \end{pmatrix} = -\frac{1}{2} \frac{L_X \Phi^2}{\varphi^2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \varepsilon L_X \psi \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]

where \(\mathbf{R} = u (p_\xi - \varepsilon p_z) - p (u_\xi - \varepsilon u_z)\). If the translational propagation is governed by the conservation law \(L_X \Phi^2 = 0\), then we obtain that the rotational component of propagation is governed by the matrix \(\varepsilon L_X \psi\), where \(J\) denotes the canonical complex structure in \(\mathbb{R}^2\), and since \(\Phi^2 L_X \psi = u (p_\xi - \varepsilon p_z) - p (u_\xi - \varepsilon u_z) \neq 0\) we conclude that the rotational component of propagation would be available if and only if \(\mathbf{R} \neq 0\). We may also say that a consistent translational-rotational dynamical structure is available if the amplitude \(\Phi^2 = u^2 + p^2\) is a running wave along \(X\) and the phase \(\psi = \arctg \frac{p}{u}\) is NOT a running wave along \(X : L_X \psi \neq 0\). Physically this means that the rotational component of propagation is entirely determined by the available internal energy-momentum exchange: \(i(\tilde{F}) d\tilde{F} = -i(F) d\tilde{F}\).

Now if we have to guarantee the conservative and constant character of the rotational aspect of the PhLO nature, we can assume \(L_X \psi = \text{const} = \kappa l_o^{-1}, \kappa = \pm 1\). Thus, the frame rotation \((A, A^*, \partial_z, \partial_\xi) \rightarrow ([A, X], [A^*, X], \partial_z, \partial_\xi), \text{i.e.} [A, X] = -\varepsilon A^* L_X \psi\) and \([A^*, X] = \varepsilon A L_X \psi\), gives the following equations for the two functions \((u, p)\):

\[
u_\xi - \varepsilon u_z = -\frac{\kappa}{l_o} p, \quad p_\xi - \varepsilon p_z = \frac{\kappa}{l_o} u.
\]

If we now quite independently from the projections considered introduce the complex valued function \(\Psi = u I + p J\), where \(I\) is the identity map in \(\mathbb{R}^2\), the above two equations are formally equivalent to

\[L_X \Psi = \frac{\kappa}{l_o} J(\Psi),\]

which apparently demonstrates the translational-rotational consistency in the above declared form that no translation is possible without rotation, and no rotation is possible without translation, where the rotation is represented by the complex structure \(J\).

The quantity \(\mathbf{R} = u (p_\xi - \varepsilon p_z) - p (u_\xi - \varepsilon u_z) = \Phi^2 L_X \psi = \kappa l_o^{-1} \Phi^2\) suggests to find an integral characteristic of the PhLO rotational nature. In fact, the two co-distributions \((A, \zeta)\) and \((A^*, \zeta)\) define the two (equal in our case) Frobenius 4-forms \(dA \wedge A \wedge \zeta = dA^* \wedge A^* \wedge \zeta\). Each of these two 4-forms is equal to \(\varepsilon \mathbf{R} \omega_o\). Now, multiplying by \(l_o/c\) any of them we obtain:

\[
l_o/c \ dA \wedge A \wedge \zeta = l_o/c \ dA^* \wedge A^* \wedge \zeta = l_o/c \varepsilon \mathbf{R} \omega_o = \varepsilon \kappa \frac{\Phi^2}{c} \omega_o.\tag{24}
\]

Integrating over the 4-volume \(\mathbb{R}^3 \times (\lambda = 4l_o)\) (and having in view the spatially finite nature of PhLO) we obtain the finite quantity \(\mathcal{H} = \varepsilon \kappa E T\), where \(E\) is the integral energy of the PhLO, \(T = \frac{\lambda}{c}\), which clearly is the analog of the Planck formula \(E = h \nu\), i.e. \(h = E T\). The combination \(\varepsilon \kappa\) means that the two orientations of the rotation, defined by \(\kappa = \pm 1\), may be observed in each of the two spatial directions of translational propagation of the PhLO along the \(z\)-axis: from \(-\infty\) to \(+\infty\), or from \(+\infty\) to \(-\infty\).

9 Solutions

We consider the equations obtained in terms of the two functions \(\Phi = \sqrt{u^2 + p^2}\) and \(\psi = \arctg \frac{\mathcal{L}}{u}\). The equation for \(\Phi\) in our coordinates is \(\Phi_\xi - \varepsilon \Phi_z = 0\), therefore, \(\Phi = \Phi(x, y, \xi + \varepsilon z)\). The equation
for \( \psi \) is \( \psi\xi - \varepsilon\psi_z = \frac{\kappa}{l_0} \). Two families of solutions for \( \psi \), depending on an arbitrary function \( \varphi \) can be given by

\[ \psi_1 = -\varepsilon\kappa \frac{\kappa}{l_0} z + \varphi(x, y, \xi + \varepsilon z), \quad \text{and} \quad \psi_2 = \frac{\kappa}{l_0} \xi + \varphi(x, y, \xi + \varepsilon z). \]

Since \( \Phi^2 \) is a spatially finite function representing the energy density we see that the translational propagation of our PhLO is represented by a spatially finite running wave along the \( z \)-coordinate. Let’s assume that the phase is given by \( \psi_1 \) and \( \varphi = \text{const} \). The form of this solution suggests to choose the initial condition

\[ u_t(x, y, \varepsilon z) = 0, \quad p_t(x, y, \varepsilon z) = 0. \]

Let for \( z = 0 \) the initial condition be located on a disk \( D = D(x, y; a, b; r_0) \) of small radius \( r_0 \), the center of the disk to have coordinates \((a, b)\), and the value of \( \Phi_t(x, y, 0) = \sqrt{u_t^2 + p_t^2} \) to be proportional to some appropriate for the case bump function \( f \) on \( D \) of the distance \( \sqrt{(x - a)^2 + (y - b)^2} \) between the origin of the coordinate system and the point \((x, y, 0)\), such that it is centered at the point \((a, b)\), so

\[ f(x, y) = f(\sqrt{(x - a)^2 + (y - b)^2}), \quad D \text{ defined by } D = \{(x, y) | \sqrt{(x - a)^2 + (y - b)^2} \leq r_0\}, \quad \text{and } f(x, y) \text{ is zero outside } D. \]

Let also the dependence of \( \Phi_t(x, y, 0) \) on \( z \) be given by be the corresponding bump function \( \theta(z; \lambda) \) of an interval \((z, z + \lambda)\) of length \( \lambda = 4l_0 \) on the \( z \)-axis. If \( \gamma \) is the proportionality coefficient we obtain

\[ u = \gamma \Phi(x, y, z, ct + \varepsilon z; a, b, \lambda) \theta(ct + \varepsilon z; \lambda) \cos(\psi_1), \]
\[ p = \gamma \Phi(x, y, z, ct + \varepsilon z; a, b, \lambda) \theta(ct + \varepsilon z; \lambda) \sin(\psi_1). \]

We see that because of the available sine and cosine factors in the solution, the initial condition for the solution will occupy a 3d-spatial region of shape that is close to a helical cylinder of height \( \lambda \), having internal radius of \( r_0 \) and wrapped up around the \( z \)-axis. Also, its center will always be \( \sqrt{a^2 + b^2} \)-distant from the \( z \)-axis. Hence, the solution will propagate translationally along the coordinate \( z \) with the velocity \( c \), and, rotationally, inside the corresponding infinitely long helical cylinder because of the \( z \)-dependence of the available periodical multiples.

On the two figures below are given two theoretical examples with \( \kappa = -1 \) and \( \kappa = 1 \) respectively, amplitude function \( \Phi \) located inside a one-step helical cylinder with height of \( \lambda \), and phase \( \psi = \kappa \frac{2\pi z}{\lambda} \). The solutions propagate left-to-right, i.e. \( \varepsilon = -1 \), along the coordinate \( z \).

Figure 1: Theoretical example with \( \kappa = -1 \). The translational propagation is directed left-to-right.

Figure 2: Theoretical example with \( \kappa = 1 \). The translational propagation is directed left-to-right.
The curvature $K$ and the torsion $T$ of the screwline through the point $(x, y, 0) \in D$ will be

$$K = \frac{R(x, y, 0)}{R^2(x, y, 0) + b^2}, \quad T = \frac{\kappa b}{R^2(x, y, 0) + b^2},$$

where $b = \lambda/2\pi$. The rotational frequency $\nu$ will be $\nu = c/2\pi b$, so we can introduce period $T = 1/\nu$ and elementary action $h = E.T$, where $E$ is the (obviously finite) integral energy of the solution defined as 3d-integral of the energy density $\Phi^2$.

## 10 Conclusion

We introduced a notion of PhLO as a spatially finite physical object with a consistent translational-rotational dynamical structure, and built a corresponding mathematical model making use of the geometry of nonintegrable distributions, i.e. nontrivial nonlinear connections, on a manifold. This approach to PhLO we consider as an illustration of the general idea that physical objects with dynamical structure seem to be in a good, local as well as integral, correspondence with an completely integrable distribution $\Delta$ plus an appropriate set $\Sigma$ of nonintegrable subdistributions $\Delta_k, k = 1, \ldots, p; p < n$ on a manifold $M^n$, such that their curvature 2-forms $\Omega_k$ send couples of vector fields from $\Delta_k \in \Sigma$ into $\Delta_m \in \Sigma$, where $k \neq m$, so that, $\Omega_k(X, Y) \in \Sigma_m, (X, Y) \in \Delta_k$.

The two basic features of our approach to describe the dynamical structure and behaviour of electromagnetic PhLO are: first, from physical viewpoint, two dynamically interacting subsystems of a PhLO can be individualized, these subsystems carry the same stress-energy-momentum, and they exchange energy-momentum locally always in equal quantities, so they exist in a dynamical equilibrium; second, from mathematical viewpoint, to every PhLO a couple of two nonlinear connections $V$ and $\tilde{V}$ is associated, such that they have a common image space, and their intercommunication is carried out and guaranteed by the two nonzero curvature forms $\Omega$ and $\tilde{\Omega}$. The values of $\Omega$ and $\tilde{\Omega}$ define two linearly independent 1-dimensional spaces, so, the corresponding two exterior products with the direction of translational propagation gives the mathematical images $F$ and $\tilde{F}$ of the two subsystems. This approach allows to get some information concerning the dynamical nature of the PhLO structure not only algebraically, i.e. only through $V$ and $\tilde{V}$, but also infinitesimally, i.e. through the curvature forms. While the energy density $\Phi^2$ of a PhLO propagates only translationally along straight isotropic lines, the available interaction of the two subsystems of a PhLO demonstrates itself through a rotational component of the entire propagational behaviour. The mutual energy-momentum exchanges are given by the interior products of the images of $\Omega$ and $\tilde{\Omega}$ with $F$ and $\tilde{F}$, in particular, the dynamical equilibrium between $F$ and $\tilde{F}$ is given by $i_{\tilde{F}}(dF) = -i_F(d\tilde{F})$.

Besides the spatially finite nature of PhLO that is allowed by our model and illustrated with the invariant parameter $l_o$, two basic identifying properties of PhLO were substantially used: straight-line translational propagation with constant speed, and constant character of the rotational component of propagation. The physical characteristics of a PhLO are represented by an analog of the Maxwell-Minkowski stress-energy-momentum tensor. An interesting moment is that $F$ and $\tilde{F}$ have zero horizontal and vertical components with respect to the two nonlinear connections. The equations of motion can be viewed from different viewpoints: as consistency conditions between the rotational and translational components of propagation, as Lagrange equations for an action principle, as part of the solutions of the vacuum equations of extended electrodynamics, and also as naturally defined transformation of 2-dimensional frames. In all these aspects of the equations of motion the curvature forms play essential role through controlling the inter-communication between $F$ and $\tilde{F}$. Moreover, the Frobenius curvature turns out to be proportional to the energy
density, which allows an analog of the famous Planck formula to be introduced. Moreover, this "energy-density - curvature" proportionality goes along with the main idea of General Relativity.

The solutions considered illustrate quite well the positive aspects of our approach. It is interesting to note that the phase terms depend substantially only on spatial variables, so, the spatial structure of the solutions considered participates directly in the rotational component of the PhLO dynamical structure.

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