STATIONARY COALESCING WALKS ON THE LATTICE

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ABSTRACT. We consider translation invariant measures on families of nearest-neighbor semi-infinite walks on the integer lattice. We assume that once walks meet, they coalesce. In 2d, we classify the collective behavior of these walks under mild assumptions: they either coalesce almost surely or form bi-infinite trajectories. Bi-infinite trajectories form measure-preserving dynamical systems, have a common asymptotic direction in 2d, and possess other nice properties. We use our theory to classify the behavior of non-crossing semi-infinite geodesics in stationary first- and last-passage percolation. We also partially answer a question raised by C. Hoffman about the limiting empirical measure of weights seen by geodesics. We construct several examples: our main example is a standard first-passage percolation model where geodesics coalesce almost surely, but have no asymptotic direction.

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1. INTRODUCTION

Let \((\Omega, \mathcal{F}, \mathbb{P}, \{T^z\}_{z \in \mathbb{Z}^d})\) be a \(\mathbb{Z}^d\) measure-preserving dynamical system. Let \(\mathcal{W}(\omega)\) be a stationary or translation-covariant subset —that is, \(\mathcal{W}(T^z \omega) = \mathcal{W}(\omega) - z\)— of the lattice \(\mathbb{Z}^d\), and suppose that it has at least one point with positive probability; i.e., \(\mathbb{P}(0 \in \mathcal{W}(\omega)) > 0\). Consider a family of measurable walks on the lattice \(\{X_z\}_{z \in \mathcal{W}}\), where each \(X_z : \Omega \times \mathbb{Z}^+ \to \mathbb{Z}^d\) is a nearest-neighbor path that starts at \(z\). We assume that these walks have been created in a stationary way: almost surely for all \(k \in \mathbb{Z}^+\),

\[X_z(T^x \omega, k) = X_{x+z}(\omega, k),\]

and that they are compatible:

\[X_z(\omega, k) = z + X_0(T^z \omega, k), \text{ and } X_0(\omega, k + 1) = X_{X_0(\omega, 1)}(\omega, k).\]

The compatibility condition implies that if two walks meet at a point at some time, then they remain together in the future; i.e., the two walks must coalesce and cannot cross each other.
Because walks coalesce when they meet, we may assume that there is a stationary vector-field $\alpha$ that is the discrete time-derivative of the walks:

$$\alpha(\omega, z) = X_z(\omega, 1) - X_z(\omega, 0).$$

(1)

The $\alpha$ function takes values in $\mathbb{A} \subset \{\pm e_1, \ldots, \pm e_d\}$, and we call a particular $\alpha$ value an arrow. We generally study two cases: the first where the walks satisfy a mild line-crossing assumption, and there is no restriction on $\mathbb{A}$, and the second with directed walks, where $\mathbb{A} = \{e_1, \ldots, e_d\}$. One might think of the walks as the flow generated by the stationary vector field of arrows. We call an arrow configuration non-trivial if $\alpha$ is not constant almost surely. The canonical walk $X(\omega)$ starts at the origin and $\alpha(\omega)$ is its (discrete) derivative at time 0. We will omit the $\omega$ from the notation when it is clear from context.

We frequently speak of configurations on the lattice: for any $\omega$, this refers to the collection of walks $\{X_z(\omega)\}_{z \in \mathcal{W}}$ or equivalently, the collection of arrows $\{\alpha(T^z \omega)\}_{z \in \mathcal{W}}$.

As a first example, consider independent and identically distributed (iid) arrows taking values in $\mathbb{A} = \{e_1, e_2\}$ (with probabilities $p$ and $1 - p$) on each point of the lattice $\mathbb{Z}^2$. Each walk is a classical simple random walk, where $e_1$ corresponds to stepping up and $e_2$ corresponds to stepping down. Consider any two random walks starting at $x \neq y$ on an anti-diagonal of the form $\{z: z_1 + z_2 = c\}$ for some fixed constant $c$. As long as $X_x(\omega, k) \neq X_y(\omega, k)$, the projection of the difference $(X_x(\omega, k) - X_y(\omega, k)) \cdot (-1, 1)$ is a one-dimensional simple random walk where time proceeds along the main diagonal in $\mathbb{Z}^2$. The $k$th step of the walk involves arrows on the antidiagonal line $\{z: z_1 + z_2 = c + k - 1\}$. Unless the two walks have coalesced previously, the $k$th step is independent of the previous steps, takes the values $\pm 2$ with equal probability $p(1 - p)$, and is 0 otherwise. Hence $(X_x - X_y) \cdot (-1, 1)$ is almost surely recurrent to 0, and the walks must coalesce. Thus, every pair of walks from points $x, y \in \mathbb{Z}^2$ must coalesce almost surely. In contrast, the periodic system in Fig. 1 (also a measure-preserving ergodic $\mathbb{Z}^2$ system) has bi-infinite trajectories.

![Figure 1](image-url)  

**Figure 1.** The space is $\Omega = \{\omega_1, \omega_2\}$ with uniform measure. The arrows are given by $\alpha(\omega_i) = e_i$ for $i = 1, 2$. The translation operators $T^{e_i}$ simply swap between $\omega_1$ and $\omega_2$.

In this paper, we completely classify the collective behavior of the trajectories in $d = 2$ under mild assumptions. There is a behavioural dichotomy (Theorem 2.4): with probability 1,

(1) the walks from all $x, y \in \mathbb{Z}^2$ coalesce, and no bi-infinite trajectories exist,
(2) a positive fraction of the walks in a configuration form bi-infinite trajectories that are themselves measure-preserving dynamical systems. Thus, all the walks have the same asymptotic direction (Theorem 2.12), and no two bi-infinite trajectories coalesce (Theorem 2.8). In fact, all other other trajectories must eventually coalesce with bi-infinite trajectories (Corollary 2.6).

There is a sequel to this paper that explores various entropic properties of bi-infinite trajectories. For example, we show that in (factors of) iid systems, bi-infinite trajectories must carry entropy. We also construct a discrete symmetric simple exclusion process that has bi-infinite trajectories carrying entropy.

All the nice properties that the bi-infinite trajectories possess are not shared by almost surely coalescing walks. For example, asymptotic velocity is no longer guaranteed. We demonstrate this by constructing an explicit example (Theorem 2.15).

In dimensions higher than 2, the bi-infinite trajectories/almost-sure coalescence dichotomy is not true. We construct an example (Corollary 2.17) in $d = 3$ where almost surely,

(i) every trajectory does not have an asymptotic direction,
(ii) every configuration does not have bi-infinite trajectories, and
(iii) we do not have almost sure coalescence.

1.1. First- and last-passage percolation. Our model is motivated by questions about the behavior of infinite geodesics in first- and last-passage percolation. Let $\Omega = \{\omega_z \in \mathbb{R}\}_{z \in \mathbb{Z}^d}$ with product $\sigma$-algebra and a translation invariant measure $\mathbb{P}$. The $\omega_z$ are called weights and they are typically nonnegative random variables. Let $X_{x,y}$ be a path from $x$ to $y$ and let the total weight of the path be the sum $W(X_{x,y}) := \sum_{z \in X_{x,y}} \omega_z$. Define the first-passage time from $x$ to $y$ to be

$$T(x, y) = \inf_{X_{x,y}} W(X_{x,y}).$$

The models may have weights on the edges of $\mathbb{Z}^d$ instead of the vertices. The first-passage time $T(x, y)$ satisfies a triangle inequality; if the weights are strictly positive, it defines a random metric on the lattice $\mathbb{Z}^d$. A geodesic for this random metric is a nearest-neighbor path that minimizes the passage time between every vertex that lies on it.

By considering the geodesic from 0 to $ne_1$, and then looking at a subsequence as $n \to \infty$, it is clear that there is at least one semi-infinite geodesic from the origin. Furstenberg (communicated in Kesten [1] page 134) asked if there exist bi-infinite geodesics (bigeodesics) in first-passage percolation with iid weights. This question has not been answered completely. However, there are several partial answers under different assumptions on the so-called time-constant. We survey a few important results below.

For $u \in S^1$, define the time-constant $g(u)$ of first-passage percolation as

$$g(u) = \lim_{n \to \infty} \frac{T(0, [nu])}{n}.$$  

Assuming that the weights are $L^1$, the limit exists almost surely and in $L^1$ by the subadditive ergodic theorem [2].

In a seminal paper [3], Licea and Newman prove theorems about the non-existence of bigeodesics under strong assumptions on the time-constant. They
assume that 1) the level sets of \( g(u) \) are “uniformly curved”, 2) the weights satisfy a property called finite-energy (see Burton and Keane [4]) and 3) the weights have continuous distribution. Then, with probability 1, except for a deterministic Lebesgue measure 0 set of directions \( u \in S^1 \),

(1) there exists exactly one geodesic from each point \( x \in \mathbb{Z}^2 \) in direction \( u \),
(2) geodesics from different points coalesce almost surely, and
(3) there are no bigeodesics in direction \((−u, u)\).

Busemann functions are a useful tool in the analysis of infinite geodesics. It is defined as the limit (if it exists)

\[
B_u(x, y) = \lim_{n \to \infty} T(x, z_n) - T(y, z_n)
\]

such that \( z_n/n \to u \in S^1 \). Newman [5] showed the existence of the above limit under certain strong hypotheses. Notably, Hoffman [6] realized their importance, and used Busemann functions to prove (concurrently with [7]) that there are at least two semi-infinite geodesics under no assumptions on the time-constant \( g(u) \).

Busemann functions have many useful properties, but from our perspective, the most useful property is that they encode geodesic behavior in a stationary manner. Moreover, Busemann geodesics coalesce when they meet. These two properties motivate our assumptions about the stationary coalescing walks we consider.

In first- and last-passage percolation, dual variational descriptions of the time-constant \( g(u) \) have recently been proved by Krishnan [8] and Georgiou, Rassoul-Agha, and Seppäläinen [9]. Here, the time constant is expressed as a minimization problem over functions instead of paths. It turns out that certain special minimizers of the formula are the Busemann functions. In stochastic homogenization, Busemann functions are known as correctors.

Damron and Hanson built a theory of generalized Busemann functions in first-passage percolation [10]. Associated with each direction \( u \in S^1 \), they construct a stationary function \( \tilde{B}_u(x, y) : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{R} \) called a reconstructed Busemann function. By building geodesics associated with these reconstructed Busemann functions, they obtain a directed geodesic graph \( G_u \) with vertices in \( \mathbb{Z}^2 \) such that

(1) each directed path on \( G_u \) is a geodesic,
(2) if there is a path from \( x \) to \( y \in \mathbb{Z}^2 \) in \( G_u \), then \( B_u(x, y) = T(x, y) \),
(3) \( G_u \) has no loops even as an undirected graph,
(4) each vertex has out degree 1.

The edges of this directed graph are the analogs of our arrow configurations \( \{\alpha(T^z \omega)\} \). They then show that under the upward finite-energy assumption [10], the geodesic graph coalesces almost surely. Ahlberg and Hoffman [11] prove similar results using different methods under weaker assumptions. Our dichotomy theorem classifies the behavior of such geodesic walks even when finite-energy does not hold (see Corollary 2.10).

Georgiou, Rassoul-Agha, and Seppäläinen [12, Theorem 2.3] prove the last-passage version of the results of [10] in \( d = 2 \): If the last-passage time-constant \( g \) is differentiable at \( u \in S^1 \), there exists a stationary function \( \tilde{B}_u(\omega, x, y) \) that can be used to create a family of compatible semi-infinite geodesics going in direction \( u \) by following the arrows defined by

\[
\alpha(x, \omega) = \arg\min_{z = e_1, e_2} B_u(\omega, x, x + z).
\]
Again, under the assumption of finite-energy, they prove almost sure coalescence of geodesics using the Licea-Newman argument. When the underlying graph is changed, finite-energy is no longer enough to prove almost sure coalescence. Benjamini and Tessera [13] show that first-passage percolation on hyperbolic graphs with iid weights have bigeodesics.

The property that one can create geodesics using (3) is rather special. In the general setting with stationary-ergodic weights, minimizers of the variational formulas with this property do not necessarily exist. Nevertheless, one can still create paths (that are not necessarily geodesics) from any minimizer using the recipe in (3). Our paper is an attempt to classify the behavior of these paths to better understand the behavior of the variational formulas of first- and last-passage percolation.

A different set of results that follow from our theorems comes from a question asked by C. Hoffman at the American Institute of Mathematics workshop in 2016: “On a semi-infinite geodesic, does the empirical measure of weights seen on the geodesic converge?” We interpret this as follows. Given a family of compatible geodesics \( \{X_x\}_{x \in \mathbb{Z}^d} \) and vertex weights \( \{\omega_z\}_{z \in \mathbb{Z}^d} \), does the measure defined by \( n^{-1} \sum_{i=1}^n \delta_{\omega(X_x(i))} \) converge as \( n \to \infty \)? Under an integrability condition, one can also ask if there is a limiting asymptotic weight on a geodesic: i.e., does \( n^{-1} \sum_{i=1}^n \omega(X(i)) \) converge? When bigeodesics exist (Theorem 2.12), these limits exist (the latter exists when \( \omega(0) \) in integrable), though the limit may not be deterministic.

However, these limits do not exist on geodesics in general. We use the example constructed in Theorem 2.15 to build a standard first-passage percolation model with edge-weights such that with probability 1, there exists a positive density of points on the lattice with compatible geodesics that do not have an asymptotic direction and do not carry an empirical measure of weights.

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2. Main results

Notation We will generally use Greek or calligraphic letters to denote events \( E \in \mathcal{F} \). We write subsets of \( \mathbb{Z}^d \), random or deterministic, using the Latin alphabet. We will frequently speak of “configurations having a density of points with a certain property”. We explain what we mean by this in the following.

Definition 2.1 (Rectangular subsets of \( \mathbb{Z}^d \)). Let the rectangle centered at \( x \in \mathbb{Z}^d \) with side lengths \((N_1, \ldots, N_d)\) be

\[
\text{Rect}_x(N_1, \ldots, N_d) = \prod_{i=1}^d [x_i - N_i, x_i + N_i].
\]

If the side-lengths are equal, then we write \( \text{Rect}_x(N) \). The boundary of any \( R \subset \mathbb{Z}^d \) is written as \( \partial R \) and consists of the set of points in \( R \) that have at least one point in \( \mathbb{Z}^d \setminus R \) as a nearest neighbor.

\(^1\)see Lions and Souganidis [14] in the context of continuum stochastic homogenization
A subset $A \subset \mathbb{Z}^d$ has density if there is a number $c \in [0, 1]$ such that $\text{Rect}_0(N)^{-1}|A \cap \text{Rect}_0(N)| \to c$. Given an event $\mathcal{E}$, the ergodic theorem ensures that almost surely in every configuration, the random subset $A(\omega) = \{x \in \mathbb{Z}^d : T^x \omega \in \mathcal{E}\}$ occurs with density $\mathbb{P}(\mathcal{E})$. Motivated by this interpretation, we will sometimes abuse notation and write for a set $A \subset \mathbb{Z}^d$ and an event $\mathcal{E}$,

$$\mathcal{E}(\omega) := \{x \in A : T^x \omega \in \mathcal{E}\}.$$  \hspace{1cm} (4)

**Definition 2.2** (Coalescence of points). Given a configuration, we say that the points $x$ and $y$ coalesce if the walks $X_x$ and $X_y$ coalesce in the future. That is, for some $k_0, k_1 \in \mathbb{Z}^+$, $X_x(\omega, k_0) = X_y(\omega, k_1)$. We say we have almost sure coalescence if almost surely for all $x, y \in \mathcal{W}$, the walks through $x$ and $y$ coalesce.

**Definition 2.3** (Bi-infinite walks and points). Fix a point $z \in \mathbb{Z}^d$. We say that the point $z$ is bi-infinite if for every $n \in \mathbb{Z}^+$, there is a sequence of points $\{a_n\}_{n=0}^\infty \in \mathbb{Z}^d$ such that for each $n$, $X_{a_n}(\omega, i) = a_{n-i}$ for $i = 1, \ldots, n-1$ and $X_{a_n}(\omega, n) = z$. We call this union of (one-sided) walks $\bigcup_{n \in \mathbb{Z}^+} \cup_{i=0}^n X_{a_n}(\omega, i)$ a bi-infinite trajectory.

**Theorem 2.4.** In $\mathbb{Z}^2$, suppose we have a positive density of walks $\mathcal{W}$ that each

1. have no loops, and
2. cross every vertical line a last time, and stays strictly to the right of it thereafter. Precisely, for each $z \in \mathcal{W}$ and $a \in \mathbb{Z}$ such that $a > z \cdot e_1$, there is a $k_0$ (which depends on $a$ and $z$) such that $X_z(\omega, k_0) \cdot e_1 = a$ and $X_z(\omega, k) \cdot e_1 > a$ for every $k > k_0$.

Then, if we do not have almost sure coalescence, there is a positive density of bi-infinite trajectories in every configuration with probability 1.

**Remark 2.5.** Vertical lines may clearly be replaced with horizontal lines or diagonal lines ($x = \pm y$). The crossing can be changed from right to left. We also believe that the proofs ought to generalize to any set of parallel lines, but do not pursue this here. Note that if our set of arrows excludes one of $\{e_1, -e_1, e_2, -e_2\}$ then assumption (2) is automatically satisfied with either vertical or horizontal lines.

Since not all walks in $\mathcal{W}$ have to be contained in some bi-infinite trajectory, we can ask about the behavior of these other walks.

**Corollary 2.6** (of Theorem 2.4). In $d = 2$ walks in $\mathcal{W}$, as defined in Theorem 2.4, that are not part of bi-infinite trajectories must eventually coalesce with bi-infinite trajectories.

**Remark 2.7.** Corollary 2.6 is used to show that ergodic averages converge on all trajectories in $\mathcal{W}$ when bi-infinite trajectories exist in Corollary 2.13.

Next, we investigate the behavior of bi-infinite trajectories in some detail. The following theorem is the converse of Theorem 2.4.

**Theorem 2.8.** Almost surely in each configuration, no two bi-infinite trajectories in $\mathcal{W}$ may coalesce, and thus we cannot have almost sure coalescence.

**Question 2.9.** Is there a natural measure of randomness that is weaker than finite-energy (like strong-mixing or total-ergodicity) that distinguishes between the bi-infinite trajectories and almost sure coalescence situations?
In first- and last-passage percolation, Theorem 2.8 is well-known in a slightly different setting. We provide Theorem 2.8, whose proof is analogous to the proof in that setting because it is a technical step in some of our results. It also motivates the term bi-infinite trajectory in Definition 2.3 (as opposed to a graph or tree of bi-infinite trajectories). We briefly recall the argument in first/last passage percolation. First, one constructs families of non-crossing semi-infinite geodesics. This requires unproven but reasonable assumptions on the differentiability of the time-constant. Then, assuming the weights satisfy the finite-energy condition, one shows that these geodesics coalesce almost surely. Second, one assumes that bi-infinite trajectories exist in the presence of the almost sure coalescence of these geodesics, and this results in a contradiction. Thus, for example, Theorem 4.6 in [12] and Theorem 6.9 in [10] prove that bi-infinite geodesics cannot exist in these families of almost surely coalescing geodesics. However, the heart of the matter in the second part is that bi-infinite trajectories cannot coalesce, and this is the content of Theorem 2.8.

Next, we state a simple corollary of Theorem 2.4 in first-passage percolation. It completely classifies the behavior of all compatible geodesic families that cross-lines in the sense of Theorem 2.4. Damron and Hanson [10] constructed such families under two sets of assumptions, A1 and A2. A1 assumes iid weights and hence they automatically satisfy the finite-energy condition. So the argument of Licea and Newman shows that these geodesics coalesce almost surely. Assumption A2 does not imply finite-energy in general, and hence we do not always have almost sure coalescence.

A2.

1. \((\Omega, \mathcal{F}, \mathbb{P})\) is an ergodic \(\mathbb{Z}^2\) system of weights.
2. \(\mathbb{P}\) has all the symmetries of \(\mathbb{Z}^2\).
3. \(\mathbb{E}[\omega^2 + \epsilon] < \infty\) for some \(\epsilon > 0\).
4. the limit shape for \(\mathbb{P}\) is bounded.
5. \(\mathbb{P}\) has unique passage times; i.e., if \(X_1\) and \(X_2\) are two finite paths, \(W(X_1) \neq W(X_2)\) almost surely.

The next corollary of the dichotomy completes the picture when finite-energy does not hold under assumption A2.

Corollary 2.10. Let \(G_u\) be the directed geodesic graph in direction \(u \in \mathbb{R}^2\) constructed in [10, Proposition 5.1 and Proposition 5.2] under assumption A2. Then infinite paths on \(G_u\) either coalesce almost surely or form bi-infinite trajectories.

Proof. Proposition 5.1 of [10] shows that there exists a semi-infinite path from each \(x \in \mathbb{Z}^2\). [10, Proposition 5.2] shows that these paths must be non-crossing. Together, [10, Proposition 5.2, Theorem 5.3 and Lemma 6.2] verify assumptions 1 and 2 of Theorem 2.4 in particular, they show that the family of geodesics is asymptotically directed in a sector of angular size at most \(\pi / 2\). Hence, Corollary 2.10 follows.

The arrows induce a map \(T_\alpha\) along walks defined by

\[
T_\alpha \omega = T^{(\omega)} \omega.
\]

The \(T_\alpha\) map is neither measure preserving nor invertible in general. Along bi-infinite trajectories, however, it is both invertible and measure preserving. This observation and Theorem 2.8 are used in the next theorem. Let \(S\) be the event
that the origin is in a bi-infinite trajectory. For any \( A \in \mathcal{F} \), let \( \mathbb{P}_\alpha(A) = \mathbb{P}(A \cap \mathcal{S}) \) to obtain the measure space \((\mathcal{S}, \mathcal{F}_\alpha, \mathbb{P}_\alpha)\).

**Definition 2.11 (Asymptotic velocity).** We say that a walk \( X : \mathbb{Z}^+ \to \mathbb{Z}^d \) has asymptotic velocity if

\[
\lim_{k \to \infty} \frac{X(k) \cdot e_i}{k}
\]

exists for each \( i = 1, \cdots, d \).

**Theorem 2.12.** The bi-infinite trajectories form a measure-preserving \( \mathbb{Z} \)-system \((\mathcal{S}, \mathcal{F}_\alpha, \mathbb{P}_\alpha, T_\alpha)\).

Hence, almost surely, ergodic averages converge on all bi-infinite trajectories and every bi-infinite trajectory has an asymptotic velocity.

**Corollary 2.13.** Suppose the walks in \( \mathcal{W} \) satisfy the assumptions in Theorem 2.4. In \( d = 2 \), when bi-infinite trajectories exist, almost surely, ergodic averages converge on all walks in \( \mathcal{W} \) in that configuration, and moreover all walks have the same asymptotic velocity. When the \( \mathbb{Z}^2 \) system \((\Omega, \mathcal{F}, \mathbb{P}, T)\) is ergodic, this direction is deterministic.

Corollary 2.13 is a simple consequence of Theorem 2.12, Theorem 2.8, and Corollary 2.6. Note that in general, ergodic averages do not have to have the same limit on all walks.

**Question 2.14.** Under what conditions is the invariant measure in Theorem 2.12 ergodic? In the context of first- and last-passage percolation, if the walks are Busemann geodesics that form bi-infinite trajectories, is there some natural assumption on the weights that ensures that the invariant measure on the bi-infinite trajectories is ergodic?

After establishing these nice properties of the bi-infinite trajectories, we show that in the case of almost sure coalescence, none of these properties need to hold. We do this by building an example in \( \mathbb{Z}^2 \) using a cutting and stacking construction (cutting and stacking was initiated by Chacon [15]).

**Theorem 2.15.** There exists an ergodic \( \mathbb{Z}^2 \) dynamical system, \((\Omega, \mathcal{F}, \mathbb{P}, \{T_z\}_{z \in \mathbb{Z}^2})\), and a stationary arrow map \( \alpha : \Omega \times \mathbb{Z}^2 \to \{e_1, e_2\} \) that defines walks from every point on \( \mathbb{Z}^2 \) such that almost surely, all walks coalesce but no walk has an asymptotic direction.

**Corollary 2.16.** In the setting of Theorem 2.15, the weight function defined in (11) gives a standard first-passage percolation model with edge-weights on \( \mathbb{Z}^2 \) and a family of compatible geodesics that coalesce almost surely. Here, with probability 1, a positive density of points have geodesics with no asymptotic direction.

See Section 5 for more details about Corollary 2.16. Remark 5.13 modifies the weights in (11) to give a counter-example to C. Hoffman’s question in the almost surely coalescent setting. Here, the average weight on the geodesic does not converge for a positive fraction of geodesics.

We then use Theorem 2.15 to construct an example in \( \mathbb{Z}^3 \) where we have neither almost sure coalescence nor bi-infinite trajectories. In other words, the dichotomy theorem in \( \mathbb{Z}^2 \) no longer holds.
Corollary 2.17. There exists an ergodic $\mathbb{Z}^3$ system defining walks where almost surely in every configuration,

(i) every walk does not have an asymptotic direction,
(ii) there are no bi-infinite trajectories, and
(iii) there are infinitely many walks that do not coalesce with each other.

3. Noncoalescence implies bi-infinite trajectories

Before proceeding with the proof of Theorem 2.4, we prove an elementary lemma that we will use repeatedly.

Lemma 3.1. Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{Z}^d)$ be an ergodic $\mathbb{Z}^d$ system, and let $M(\omega) \subset \mathbb{Z}^d$ be a random, translation covariant $(x \in M(\omega) \Leftrightarrow x - z \in M(T^z \omega) \forall \omega)$ set of points such that $|M(\omega)| \geq 1$ occurs with positive probability. Then, almost surely, $M(\omega)$ must have density $\rho > 0$.

Proof. Let $U_N = \{ \omega : |M(\omega) \cap \text{Rect}_0(N, N)| \geq 1 \}$. Then, there is an $N$ such that $\mathbb{P}(U_N) \geq c > 0$. By the ergodicity of $(\Omega, \mathbb{P}, \mathbb{Z}^d)$ for almost every $\omega$, the density of $U_N \cap \mathbb{Z}^d$ is at least $c$. Since $M(T^z \omega) \cap \text{Rect}_0(N, N) = M(\omega) \cap \text{Rect}_z(N, N)$, it follows that each point in $M(\omega)$ is counted in at most $(2N + 1)^2$ translates of the form $U_N(T^z \omega)$. Thus, the density of $M(\omega)$ is at least $\frac{c}{(2N + 1)^2}$.

In a given configuration $\omega$, we say that a point $x \in \mathbb{Z}^d$ has a past of length $n$ if there is a $z$ such that $X_z(\omega, n) = x$. Define the random set of points

$$P_n(\omega) := \{ x \in \mathbb{Z}^d : x \text{ has a past of length } n \}.$$ 

A point is in a bi-infinite trajectory iff it is in $\cap_{n \geq 0} P_n(\omega)$. When $d = 2$, the following proposition shows that the probability that the origin is in a trajectory of length $n$ is bounded by a constant $\rho > 0$ that is independent of $n$. Therefore $\cap_{n \geq 0} P_n(\omega)$ has positive density, and Theorem 2.4 follows.

Proposition 3.2. Under assumptions 1 and 2 of Theorem 2.4 if walks do not coalesce almost surely, there exists a constant $\rho > 0$ so that for all $n$, $P_n(\omega)$ has density at least $\rho$ almost surely.

Prop. 3.2 will be proved over several steps in this section.

Let $\text{Vert}_a = \{ a \} \times [-k, k]$ and write $\text{Vert}_a$ for the entire vertical line with $e_1$ coordinate $a$. Since there is no almost sure coalescence, there must be a pair of points $x, y \in \mathbb{Z}^2$ such that $x$ and $y$ do not coalesce. By assumption 2 in Theorem 2.4 we may assume without loss of generality that $y$ is of the form $x + ke_2$ for some integer $k \in \mathbb{Z}$. Since the walks cross lines, the walks from $x$ and $x + ke_2$ must cross $\text{Vert}_{x-e_1}$ a last time. The last-crossing points must be of the form $z, z + re_2$ such that $z \cdot e_1 = x \cdot e_1$. So let

$$L_r(\omega) := \{ x \in \mathbb{Z}^2 : x \text{ and } x + re_2 \text{ do not coalesce}$$

and $\forall k > 0, X_y(\omega, k) \cdot e_1 > y$, for $y = x, x + re_2$}

be set the of last-crossing points in $\mathbb{Z}^d$.

Lemma 3.3. There exists $r$, such that $L_r(\omega)$ has density $\xi > 0$ almost surely.

Proof. This follows directly from Lemma 3.1 since $L_r(\omega)$ is translation covariant.
The next lemma says that if one walk is below another it does not coalesce with at their last crossing points on a line, then they must continue to maintain this ordering at future last-crossing points. We say that \( x < y \) for two points \( x, y \in \text{Vert}_a \) if \( x \cdot e_2 < y \cdot e_2 \).

**Lemma 3.4.** Let \( x, y \in \mathbb{Z}^2 \) such that \( x < y \in \text{Vert}_p \cap L_r(\omega) \) for \( p \in \mathbb{Z} \). If \( p < q \), such that \((q, i) \in X_x \) and \((q, j) \in X_y \) and \((q, i), (q, j) \in L_{r'} \) for some \( r' > 0 \), then we must have \( i > j \).

**Proof.** Consider \( X_y \), the semi-infinite infinite walk from \( y \). Connect the points in the walk to obtain a semi-infinite trajectory \( M_y \subset \mathbb{R}^2 \). Consider the union of this semi-infinite trajectory and \( \text{Vert}_p \). It divides the plane into 3 regions: one on the left of \( \text{Vert}_p(w) \) and two to the right which are divided by the line \( M_y \). Now consider \( M_x \), the continuous line obtained from the semi-infinite walk \( X_x \). After its first step, it steps into the interior of the region below the infinite line \( M_y \) and to the right of \( \text{Vert}_p \). To exit this region, it must cross \( M_y \) and hence coalesce with it. Therefore, \( X_x \) must stay below \( X_y \).

Consider the set of points \( L_r(\omega) \cap \text{Vert}_a \), and vertically order these points as \( \cdots < x_{-1} < x_0 < x_1 < \cdots \) such that \( x_0 \) is the point with the smallest \( e_2 \) coordinate in absolute value (with some tie-breaking rule). Let \( S_a \subset L_r(\omega) \cap \text{Vert}_a \) be the subset of points \( \{ \cdots < x_{i_{-1}} < x_{i_0} < x_{i_1} \cdots \} \) such that \( i_0 = 0 \) and \( |x_{i_j} - x_{i_k}| > r \). We call \( S_a(\omega) \) a separating set.

**Corollary 3.5** (of Lemma 3.4). The walks from two distinct points \( x, y \in S_a \) do not coalesce to the right of \( \text{Vert}_a \).

**Proof.** Let \( x < y \in S_a \). By definition, \( x \) cannot coalesce with \( x + re_2 \leq y \). Since \( |y - x|_1 \geq r \), and all three of them (if \( x + re_2 \) and \( y \) are distinct) are last crossing points, \( x \) and \( y \) cannot coalesce. □

**Lemma 3.6.** Suppose \( \text{Rect}_0(N) \) has \( c(2N + 1)^2 \) points in \( L_r(\omega) \). Then, at least \( (c/2)(2N + 1) \) lines must have at least \( (c/2r)(2N + 1) \) separating points.

**Proof.** Let \( M = 2N + 1 \). Suppose at most \( cM/2 \) lines have at least \( cM/2 \) points in \( L_r(\omega) \), then the number of points in \( \text{Rect}_0(N) \cap L_r(\omega) \) is at most

\[
\frac{cM}{2}M + \left(1 - \frac{c}{2}\right)M\frac{cM}{2} < cM^2.
\]

This contradicts the assumption in the lemma. Therefore, there must be at least \( cM/2 \) lines with at least \( (c/2)M \) points in \( L_r(\omega) \). On each of these lines, at least \( (c/2r)M \) must be in the separating set.

The following lemma states that each separating point in \( \text{Vert}_k^m \) corresponds to a unique point on the boundary of \( \text{Rect}_{(k,0)}(n, m + n) \) with past of length at least \( n \). This follows from Corollary 3.5 which says that walks from the separating set must stay apart in the future.

**Lemma 3.7.** Let \( S_k(\omega) \) be a separating set. For all \( m, n > 0 \) we have

\[
|P_n(\omega) \cap \partial \text{Rect}_{(k,0)}(n, m + n)| \geq |S_k(\omega) \cap \text{Vert}_k^m(\omega)|.
\]

**Proof.** By the line-crossing assumption in Theorem 2.4 if \( (k, c) \in S_k \) and \( c \in [-m, m] \), then the walk \( X_{(k,c)} \) must cross \( \partial \text{Rect}_{(k,0)}(n, m + n) \). Clearly, \( X_{(k,c)} \) must take at least \( n \) steps before it crosses \( \partial \text{Rect}_{(k,0)}(n, m + n) \), and hence the
crossing point must be in $P_n(\omega)$. By Corollary [3.3] distinct points in $S_k$ cross at distinct points in $\partial \text{Rect}_{(k,0)}(n, m + n)$.

**Corollary 3.8.** Let $\xi$ and $r$ be as in Lemma [3.3] For each $n$ and $\epsilon > 0$, there exists $N_0(\epsilon, n)$ so that for all $N > N_0$,

$$
\mathbb{P} \left( \left\{ \omega : |P_n(\omega) \cap \partial \text{Rect}_0(N)| > \frac{\xi}{4r} N \right\} \right) > (1 - \epsilon).
$$

**Proof.** By the previous lemma it suffices to show that for each $\epsilon > 0$ and $n$ there exist $N > n$ and $|k| \leq N - n$ so that $|S_k(\omega) \cap \text{Vert}_{k}^{N-n}| > \frac{\xi}{2r} N$ with probability at least $1 - \epsilon$. For $N$ large enough, the ergodic theorem guarantees that there will be $(\xi/2)(2(N-n) + 1)^2$ points in $L_r(\omega) \cap \text{Rect}_0(N-n)$ with probability greater than $1 - \epsilon$. Lemma [3.6] shows that at least one of the $2(2N-n)+1$ vertical lines in $\text{Rect}_0(N-n)$ must have $(\xi/2r)(N-n)$ separating points. Finally we choose $N_0$ so large such that $2(N-n) > N$ for all $N > N_0$.

The next lemma shows that with positive probability, there is a positive density of points ($\geq \xi/16r$) in each configuration that have past of length at least $n$.

**Lemma 3.9.** Let $\xi$ be as in Lemma [3.3] There exists $M_0$ so that

$$
\mathbb{P} \left( \left\{ \omega : |P_n(\omega) \cap \text{Rect}_0(M)| > \frac{\xi}{256r}(2M+1)^2 \right\} \right) > \frac{1}{4}
$$

for all $M > M_0$.

**Proof.** Let $\mathcal{G}_N := \{ \omega : |P_n(\omega) \cap \partial \text{Rect}_0(N)| > \frac{\xi}{2r} N \}$ be the event that there is a rectangle of side-length $2N+1$ centered at the origin whose boundary has a significant number of points in $P_n(\omega)$. By Corollary [3.8] there exists $N$ so that $\mathbb{P}(\mathcal{G}_N) > \frac{1}{2}$. Therefore by the ergodic theorem for $M > N$, we must have

$$
\mathbb{P} \left( \omega : |\text{Rect}_0(M) \cap \mathcal{G}_N| \geq \frac{1}{4}(2M+1)^2 \right) \geq \frac{1}{4}.
$$

Hence, for each point $(p, q) \in \text{Rect}_0(M-N) \cap \mathcal{G}_N$ we have at least $\frac{\xi}{2r} N$ points in $\partial \text{Rect}_{(p,q)}(N)$ that are in $P_n(\omega)$. Each such point in $P_n(\omega)$ can appear in at most $8N$ different rectangular boundaries of the form $\partial \text{Rect}_{(p,q)}(N)$ for different points $(p, q) \in \text{Rect}_0(M-N) \cap \mathcal{G}_N$. Thus, we obtain the following lower bound on points with past of length $n$ inside a rectangle of size $M$. With probability at least $1/4$,

$$
|P_n(\omega) \cap \text{Rect}_0(M)|
\geq \left( |\mathcal{G}_N \cap \text{Rect}_0(M)| - |\text{Rect}_0(M) \setminus \text{Rect}_0(M-N)| \right) \frac{\xi}{4r} N
\geq (2M+1)^2 \frac{1}{4} - 4MN)(\xi/32r),
$$

where the subtraction in the second inequality accounts for boundary effects. By choosing $M$ sufficiently large (given $N$) the lemma follows.

**Proof of Prop. 3.2** Let $\mathcal{P}_n$ be the event that the origin has a past of length $n$. From the ergodic theorem, it follows that if $\mathbb{P}(\mathcal{P}_n) \leq \xi/256r - \epsilon$ for small enough $\epsilon > 0$, then

$$
\lim_{M \to \infty} \mathbb{P} \left( \omega : \frac{1}{(2M+1)^2} |\mathcal{P}_n \cap \text{Rect}_0(M, M)| > \frac{\xi}{256r} \right) = 0.
$$
This contradicts Lemma 3.9 and shows that \( \mathbb{P}(P_n) \geq \xi/256r \). \( \square \)

Finally, we prove Corollary 2.6, which says that walks that are not on bi-infinite trajectories must coalesce with bi-infinite trajectories. Recall \( S := \{ \omega \in \Omega: \mathbf{0} \in W, \text{ and } X_0(\omega) \text{ is bi-infinite} \} \).

**Proof of Corollary 2.6.** Let \( W' := W \cap (\Omega \setminus S) \) be the random set of points with walks that do not coalesce with bi-infinite trajectories. These walks inherit stationarity and compatibility from the original walks, and are disjoint from the bi-infinite points. If such points exist with positive probability, by Lemma 3.1, \( W' \) is a positive density subset of \( \mathbb{Z}^d \).

By the ergodic decomposition for the \( T^e_1 \) map, there must be at least one vertical line that has a positive density of points in \( W' \). Consider a bi-infinite trajectory such that there is at least one point in \( W' \) above and below it. This shows that they cannot coalesce, and Lemma 3.1 shows that there is a density of such points in \( W' \).

Therefore, Theorem 2.4 says that these walks in \( W' \) must contain a positive density of bi-infinite points. This is a contradiction. \( \square \)

## 4. Bi-infinite trajectories

In this section we assume that we have bi-infinite trajectories with positive probability. The results in this section apply to all \( \mathbb{Z}^d \) systems except for Corollary 2.13, which is only proved for \( d = 2 \).

**Definition 4.1.** We say that a point \( x \in \mathbb{Z}^d \) is cataclysmic if it is a point of coalescence of two distinct bi-infinite trajectories in a configuration.

The following Lemma is a standard application of the Burton-Keane argument [16].

**Lemma 4.2.** Given any bounded rectangle \( R \in \mathbb{Z}^d \), the number of points in \( \partial R \) crossed by a bi-infinite trajectory is at least the number of cataclysmic points in \( R \).

**Proof.** Consider two bi-infinite trajectories to be equivalent if they coalesce together at cataclysmic points in \( R \). Each equivalence class of trajectories forms a tree. It suffices to prove the result on each equivalence class of bi-infinite trajectories. Fix an equivalence class and put an order on it. After two bi-infinite trajectories coalesce, consider the coalesced trajectory to be the largest trajectory that has coalesced. At each cataclysmic point visited by the equivalence class, select the smallest of the trajectories that coalesce with it. By construction, each trajectory in the equivalence class is selected at most once. Repeating this selection procedure for all equivalence classes, we see that the number of bi-infinite trajectories is an upper bound for the number of cataclysmic points. Each selected trajectory gives at least one distinct crossing of the boundary because the region is bounded and the trajectories have an infinite past. \( \square \)

**Proof of Theorem 2.8.** Assume that at least one cataclysmic point occurs with positive probability. By Lemma 3.1, cataclysmic points occur with density \( \rho > 0 \). Choose \( N \) so that \( \rho N^d > 2|\partial \text{Rect}(N)| \). Then with positive probability, there are at least \( \frac{\xi}{2} N^d \) cataclysmic points in \( \text{Rect}(N) \). By the previous lemma, each cataclysmic point can be associated with a distinct point of \( \partial \text{Rect}(N) \). However,
there aren’t enough points on \( \partial \text{Rect}(N) \) to accommodate them all and this is a contradiction. \( \square \)

**Remark 4.3.** Using the ergodic decomposition we may relax our assumption to \( \mathbb{Z}^d \)-preserved measures.

Recall the measure space \((S, \mathcal{F}_\alpha, \mathbb{P}_\alpha)\) and the map \(T_\alpha\) map defined in [5]. Theorem \[\text{2.12}\] states that \((S, \mathcal{F}_\alpha, \mathbb{P}_\alpha, T_\alpha)\) is a measure-preserving, invertible \(\mathbb{Z}^d\) dynamical system.

**Proof of Theorem 2.12.** Since from Theorem 2.8 the bi-infinite trajectories cannot coalesce, \(T_\alpha\) must be almost surely invertible on the bi-infinite points \(S\). For each \(y \in A\), let \(V_y = \{\omega \in S : \alpha(T_\alpha^{-1}\omega) = y\}\).

Clearly, \(\{V_y\}_{y \in A}\) is a partition of \(S\). Thus, for any \(E \subset S\),

\[
\mathbb{P}_\alpha(T_\alpha^{-1}(E)) = \mathbb{P}(T_\alpha^{-1}(E) \cap S) = \mathbb{P}(T_\alpha^{-1}(E \cap S)) = \sum_{y \in A} \mathbb{P}(T_\alpha^{-1}(E \cap V_y)) = \sum_{y \in A} \mathbb{P}(T^{-y}(E \cap V_y)) = \sum_{y \in A} \mathbb{P}(E \cap V_y) = \mathbb{P}(E).
\]

The first equality is by the definition of \(\mathbb{P}_\alpha\). The second is because \(S\) is \(T_\alpha\) invariant. The third follows from the definition of the \(V_y\). The fourth is because on \(V_y\) we have \(T_\alpha = T^y\), and the fifth is because each \(T^{-y}\) is \(\mathbb{P}\) measure preserving. \(\square\)

Let \(F = \{a_1, \ldots, a_m\}\) be any finite set of symbols in \(A\). Then, we say \(F\) appears on a walk if \(\{\alpha(T_\alpha^{-i}\omega)\}_{i=1}^m = F\) for some \(n \in \mathbb{Z}^+\).

**Proposition 4.4.** On each bi-infinite trajectory, each finite block appears with some density \(\rho \in [0, 1]\).

**Proof.** There is an ergodic decomposition of \((S, \mathcal{F}_\alpha, \mathbb{P}_\alpha, T_\alpha)\). This means that almost every bi-infinite trajectory is ergodic for some measure. Therefore, almost surely, each finite block appears with (a possibly 0) density on all bi-infinite trajectories. \(\square\)

Applying the previous proposition to each singleton \(F_i^\pm = \{\pm e_i\}, i = 1, \ldots, d\) shows that all bi-infinite trajectories have asymptotic velocity.

Next, we prove Corollary 2.13 that shows that in dimension 2, almost surely, ergodic averages converge on all walks in \(\mathcal{W}\), where \(\mathcal{W}\) satisfies the assumptions in Theorem 2.4. Moreover, all walks in a configuration have the same asymptotic direction.

**Proof of Corollary 2.13.** Theorem 2.12 shows that \((S, \mathcal{F}_\alpha, \mathbb{P}_\alpha, T_\alpha)\) is a measure-preserving dynamical system. In \(d = 2\), Corollary 2.6 shows that all walks in \(\mathcal{W}\) must coalesce with bi-infinite trajectories, and hence almost surely, ergodic averages converge on all walks.

Next, we show that all bi-infinite walks in a configuration have the same velocity. We may assume without loss of generality that \(\mathbb{P}\) is ergodic. If it is not ergodic, we may restrict our attention to an ergodic component of \(\mathbb{P}\) that has bi-infinite trajectories.
Figure 2. Three steps of the cutting and stacking construction of the intervals $X$ and $Y$. The figure shows the case $n_1 = n_2 = n_3 = 2$. The dotted lines show where the intervals are cut. At each step, the intervals are cut, stacked horizontally, and a new “spacer” $K_i$ is appended. The arrows show the mapping $S_i$, $i = 1, 2$.

Assume for the sake of contradiction that there is no common asymptotic direction for configurations drawn from $\mathbb{P}$. Then, there exists $c$ so that a $\mathbb{P}$-positive measure set of points are on bi-infinite trajectories with slope at least $c$ and a $\mathbb{P}$-positive measure set of points are on bi-infinite trajectories with slope strictly less than $c$. Call these sets $S_1$ and $S_2$ respectively. By the ergodicity of $\mathbb{P}$, almost surely, there exist $i,j,k,\ell \in \mathbb{Z}$ with $i < k$ and $j > \ell$ so that $T_{(i,j)} \omega \in S_2$ and $T_{(k,\ell)} \omega \in S_1$. Thus, the bi-infinite trajectory through $(i,j)$ must coalesce with the bi-infinite trajectory through $(k,\ell)$. By Theorem 2.8, this must have zero probability. Therefore almost surely, all bi-infinite trajectories and consequently all walks in $W$ have the same asymptotic direction. If $\mathbb{P}$ is ergodic, this direction is deterministic. □

5. Examples

We first prove Theorem 2.15. We build this example as a product space $X \times Y$, with maps $S_1 : X \to X$ and $S_2 : Y \to Y$ such that for $(a,b) \in \mathbb{Z}$, we have $T_{(a,b)}(x,y) = (S_1^ax, S_2^by)$ for all $(x,y) \in X \times Y$. $X$ and $Y$ will be intervals in $\mathbb{R}$ with Lebesgue measure.

We build $(S_1, X)$ and $(S_2, Y)$ by cutting and stacking construction. In fact they are Rank 1 and therefore ergodic [17, Lemma 3].

**Building** $(S_1, X)$: Let $\{n_i\}_{i=1}^{\infty}$ be a sequence of integers that are at least 2. Let $X = [0, 1 + \sum_{i=2}^{\infty} \frac{1}{n_1 \cdots n_i})$. $X$ is a disjoint union of intervals $(\cup_{i=1}^{n_1} J_i) \cup (\cup_{i=2}^{\infty} K_i)$ that we define below. We begin with $n_1$ disjoint intervals of size $\frac{1}{n_1}$: $J_1, \ldots, J_{n_1}$ such that $\cup_{i=1}^{n_1} J_i = [0, 1]$. Let $S_1(J_i) = J_{i+1}$ for $i < n_1$. Now subdivide each $J_i$ into $n_2$ intervals of size $\frac{1}{n_1 n_2}$ called $J_i^{(j)}$ and add an interval $K_i$ of size $\frac{1}{n_1 n_2}$. Let $S_1(J_i^{(j)}) = J_i^{(j+1)}$ for all $i < n_1$, $S_1(J_{n_1}^{(j)}) = J_1^{(j+1)}$ if $j < n_2$ and lastly $S_2(J_{n_2}^{(n_2)}) = \ldots$.
$K_2$. Inductively at step $k$ we have $J_i^{(j_2,\ldots j_k)}$, $\ldots$, $K_k^{(j_r,\ldots j_k)}$, $\ldots$, $K_k$ of size $\frac{1}{n_1\cdots n_k}$. An index $i_s$ or $j_s$ in the superscript runs from 1 to $n_s$. We add another interval $K_{k+1}$ of size $\frac{1}{n_1\cdots n_{k+1}}$ and subdivide the other intervals into $n_k+1$ intervals of size $\frac{1}{n_1\cdots n_{k+1}}$. Call these $J_i^{(j_2,\ldots j_k+1)}$, $\ldots$, $K_r^{(j_r,\ldots j_k+1)}$, $\ldots$, $K_k^{(r,\ldots j_k+1)}$, where again, $\ell$ runs from 1 through $n_k+1$.

Let
\begin{align*}
S_1 J_i^{(j_2,\ldots j_k+1)} &= J_i^{(j_2,\ldots j_k+1)} \quad \text{for } i < n_1, \\
S_1 J_i^{(j_2,\ldots j_k+1)} &= J_i^{(j_2+1,\ldots j_k+1)} \quad \text{if } i_2 < n_2, \\
S_1 J_i^{(n_2,i_3,\ldots j_k+1)} &= K_2^{(j_3,\ldots j_k+1)} \\
S_1 K_i^{(j_r+1,\ldots j_k+1)} &= J_i^{(1,\ldots j_r+1,\ldots j_k+1)} \quad \text{if } i_r+1 < n_r+1, \\
S_1 K_i^{(n_r+1,i_r+2,\ldots j_k+1)} &= K_r^{(i_r+2,\ldots j_k+1)} \quad \text{otherwise}
\end{align*}

Lemma 5.1. For all $x \in X$ and $r \geq 2$, $S_1^i(x) \in \cup_{i=1}^{\infty} K_i$ for some $0 \leq \ell \leq n_1\cdots n_r$.

Proof. We prove this by induction on $r$. First we establish the base case of $r = 2$. Observe that if $x \in \cup_{i=1}^{\infty} K_i$, then it is obviously true. Otherwise, $x \in J_i^{(j_2)}$ and $S_1^{n_2-1} x \in J_i^{(j_1)}$. Now if $j < n_2$, we have that $S_1^{n_2-1}(J_i^{(j)}) = J_i^{(j+1)}$ and if $j = n_2$ then $S_1(J_i^{(j)}) = K_2$. Applying this $n_2 - j$ times we see $S_1^{(n_2-j)}(J_i^{(j)}) = K_2$. Combining these, there exists $r \leq n_1 - 1 + (n_2 - 1)n_1 + 1 = n_1n_2$ so that $S_1^i(J_i^{(j)}) \subset \cup_{i=1}^{\infty} K_i$.

The inductive step is similar. Assuming the result for $K_\ell$ we prove it for $K_{\ell+1}$. If $x \notin \cup_{i=1}^{\infty} K_i$, there exists $a \leq n_1 \cdots n_{\ell+1} - 1$ so that $S_1^i(x) \in K_i$ for some $i \geq \ell$. Since $i > \ell$ we are done and so we assume that $S_1^i(x) \in K_i^{(j)}$ for some $1 \leq \ell \leq n_\ell+1$. Similar to before $S_1^{(n_\ell+1-j)}(j_i) \subset K_{\ell+1}$ where $r \leq n_1 \cdots n_\ell$. Since $n_1 \cdots n_\ell(n_\ell+1 - 1) + 1 + a \leq n_1 \cdots n_\ell+1$ we have the lemma.

Lemma 5.2. If $j < n_{\ell+1}$ then $S_1^iK_i^{(j)} \cap \cup_{i=1}^{\infty} K_\ell = \emptyset$ for all $0 \leq r < n_1\cdots n_r$.

This is similar to the proof of the previous lemma.

Building $(S_2,Y)$: This is similar. Let $\{m_i\}_{i=1}^{\infty}$ be a sequence of integers that are at least 2. Let $Y = [0,1 + \sum_{i=2}^{\infty} \frac{1}{m_1\cdots m_i}]$. As before we define intervals $J_\ell^{(i_2,\ldots)}$, $K_\ell^{(i_2,\ldots)}$ and the map $S_2$. Analogously to before we have the following:

Lemma 5.3. For all $y \in Y$ and $r \geq 2$ we have $S_2^i(y) \in \cup_{i=1}^{\infty} K_i$ for some $0 \leq \ell \leq m_1\cdots m_r$.

Lemma 5.4. If $j < m_{\ell+1}$ then $S_2^iK_i^{(j)} \cap \cup_{i=1}^{\infty} K_\ell = \emptyset$ for all $0 \leq r < m_1\cdots m_i$.

For clarity we denote Lebesgue measure on $X$ by $\mu$ and Lebesgue measure on $Y$ by $\tilde{\mu}$. Let $B$ and $\mathcal{B}$ be the usual Borel $\sigma$-algebras on the intervals $X$ and $Y$.

Proposition 5.5. $(X \times Y, \mathcal{B} \times \mathcal{B}, \mu \times \tilde{\mu}, \{S_1^i \times S_2^i\}_{i,j} \in \mathbb{Z}^2)$ is an ergodic $\mathbb{Z}^2$ dynamical system.

Proof. This is straightforward and included for the reader’s convenience. In short, let $A \in \mathcal{B} \times \mathcal{B}$ be an invariant set under $S_1^i \times S_2^i$ for all $(i,j) \in \mathbb{Z}^2$. Each section $A_x = \{y : (x,y) \in A\}$ is invariant under $S_1^i$ and thus $\tilde{\mu}(A_x)$ has full or zero measure. But $\tilde{\mu}(A_x)$ is measurable function of $x$ that is invariant under $S_1$ and is therefore a constant $\mu$-almost surely.  

**Definition 5.6.** For any \( r > 0 \), define \( \alpha : X \times Y \to \{ e_1, e_2 \} \) by
\[
\alpha(x, y) = \begin{cases} 
  e_1 & \text{if } (x, y) \in \bigcup_{i=1}^{n_r} J_i \times Y, \text{ or } (x, y) \in K_r \times \left( \bigcup_{j=r}^{\infty} K_j \right) \\
  e_2 & \text{otherwise}
\end{cases}
\]  
(7)

This defines an arrow map on \( X \times Y \) from which we can construct walks [1].

**Proposition 5.7.** If
\[
\lim_{i \to \infty} \frac{m_1 \cdots m_i}{m_1 \cdots m_{i-1} i^2} = \infty = \lim_{i \to \infty} \frac{m_1 \cdots m_i}{m_1 \cdots m_{i-1} i^2}
\]
then almost every point of \( X \times Y \) defines a trajectory without an asymptotic direction.

**Remark 5.8.** The sequences defined by \( n_r = 2^{2r-1} \) and \( m_r = 2^r \) for \( r \geq 1 \) satisfy the condition in Prop. 5.7.

Since we cannot have a \( \mathbb{Z}^2 \) system with bi-infinite trajectories that don’t have asymptotic direction, Prop. 5.7 proves almost-sure coalescence and Theorem 2.15.

We need a few lemmas to prove Prop. 5.7. Let
\[
\tilde{G}_r = Y \setminus \bigcup_{j=0}^{m_1 \cdots m_r} \mathcal{S}_2^{-1}(\bigcup_{i=r}^{\infty} K_i).
\]
(9)
The next lemma says that when \( y \in \tilde{G}_r \), the walk goes vertically for a period of time that is much longer than its previous horizontal excursion, and hence has a very large vertical fluctuation.

**Lemma 5.9.** If \( r \geq 2 \) and \( y \in \tilde{G}_r \) then \( X_0(\omega, m_1 \cdots (\frac{m_r}{x}) - 1) = (a, b) \) where \( \omega = (x, y) \), and \( \frac{b}{a} \geq \frac{m_1 \cdots (m_j - 1) - m_1 \cdots m_r}{m_1 \cdots m_r} \).

**Proof.** We first show that the lemma will follow from the fact that if \( y \in \tilde{G}_r \) then
\[
\mathcal{S}_2 \in \bigcup_{i=r}^{\infty} K_i \text{ for } 0 \leq i \leq m_1 \cdots (\frac{m_r}{x} - 1).
\]
(10)
By Lemma 5.1, \( \mathcal{S}_2^i(x) \in \bigcup_{i=r}^{\infty} K_i \) for some \( 0 \leq q \leq n_1 \cdots n_r \). By Definition 5.6, once \( \mathcal{S}_2^i x \in K_r \), subsequent arrows will be in the \( e_2 \) direction. That is, if \( T_{a}^i(x, y) \) has its first coordinate in \( \bigcup_{i=r}^{\infty} K_i \), then for \( j > r \), \( T_{a}^j(x, y) = T_{a}^2 T_{a}^{j-2}(x, y) \) until the second coordinate of \( T_{a}^{j-1}(x, y) \) is in \( \bigcup_{i=r}^{\infty} K_i \). So if (10) holds then \( T_{a}^j(x, y) \) moves \( e_1 \) at most \( n_1 \cdots n_r \) in it first \( m_1 \cdots (\frac{m_r}{x} - 1) \) steps.

Similarly, let
\[
\tilde{G}_r = X \setminus \bigcup_{j=0}^{n_r-1} \mathcal{S}_1^{-1}(\bigcup_{i=r}^{\infty} K_i)
\]

**Lemma 5.10.** If \( r \geq 2 \) and \( x \in \tilde{G}_r \) then \( X_0(\omega, n_1 \cdots (\frac{m_r}{x}) - 1) = (a, b) \) where \( \frac{b}{a} \geq \frac{n_1 \cdots (m_r - 1) - m_1 \cdots m_{r-1}}{m_1 \cdots m_{r-1}} \).

The proof of this Lemma is identical to the proof of Lemma 5.9.

**Lemma 5.11.** If \( r \geq 2 \), we have \( \hat{\mu}(\tilde{G}_r) \leq \frac{1}{2^{2r}} \).

**Proof.** From (9) and since we are assuming \( m_i \geq 2 \) for all \( i \), it follows that
\[
\hat{\mu}(\tilde{G}_r) = \frac{1}{m_1 \cdots m_{r-1} i^2} + \sum_{j=r}^{\infty} \hat{\mu}(K_j) \leq \frac{1}{2^{2r}}.
\]
The next lemma has an analogous proof.

**Lemma 5.12.** If \( r \geq 2 \), we have \( \mu(G_r) \leq \frac{1}{2^{r-1}} \).

**Proof of Prop. 5.7.** First observe that by Lemmas \([5.11][5.12]\) and the Borel-Cantelli lemma, 
\[ \bigcup_{i=1}^{\infty} \bigcap_{r=i}^{\infty} X \times \hat{G}_r \text{ and } \bigcup_{i=1}^{\infty} \bigcap_{r=i}^{\infty} G_r \times Y \]
have full measure. Next by \([ \square ]\) and Lemmas \([5.9][5.10]\) we have that any such trajectory approximates both the vertical and the horizontal on infinite sequences of times. \( \square \)

Proposition \([5.7]\) proves that the trajectories have no asymptotic direction. We turn this into a geodesic walk by assigning weights to edges on the lattice, and considering the first-passage percolation model on these edge-weights. Define 
\[ w_z: X \times Y \to \mathbb{R} \]
by
\[
\begin{cases} 
\frac{1}{2} & \text{if } \alpha((x,y)) = z = e_1, e_2 \\
1 & \text{otherwise}
\end{cases}
\]
(11)

**Proof of Corollary 2.16.** Note that (11) is an everywhere well-defined map. Our walks in the previous proposition are geodesics for this set of weights because they only cross weights of \( \frac{1}{2} \). \( \square \)

**Remark 5.13** (No asymptotic weight distribution on geodesics). The previous example can be modified to give an example where there is no asymptotic weight distribution on the geodesic (c.f. Corollary \([2.13]\)). Let \( \tilde{w}_{e_1}(x,y) = w_{e_1}(x,y) \), but 
\[ \tilde{w}_{e_2} = \frac{3}{4} \text{ if } w_{e_2} = \frac{1}{2} \]. Walks using these arrows form geodesics since they only cross edges with weight \( 1/2 \) when going horizontally, and edges with weight \( 3/4 \) when going vertically. But horizontal edges weigh \( 1/2 \) or \( 1 \), and vertical edges weigh either \( 3/4 \) or \( 1 \). A calculation similar to the one in Lemma \([5.9]\) and Lemma \([5.10]\) shows that the weight on \( X_0(\omega,k) \) oscillates between two values \( c_1k \) and \( c_2k \) infinitely often as \( k \to \infty \), where \( 0 < c_1 < c_2 \).

**Proof of Corollary 2.17.** Consider the space \( X \times Y \times Z \) with product measure \( (\mu \times \tilde{\mu}) \otimes \pi \) and translation map \( S_1 \times S_2 \times T \) where \( T \) is simply the shift on the third coordinate. Let the arrow map be as in Definition \([5.6]\) where it simply ignores the third coordinate. Observe that almost every trajectory remains in a 2-plane where it behaves as a trajectory from our previous model. Therefore we do not have an asymptotic direction. However, almost every trajectory stays in a 2-plane and so we do not have almost sure coalescence. \( \square \)

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