Investigating climate tipping points under various emission reduction and carbon capture scenarios with a stochastic climate model

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We study the mitigation of climate tipping point transitions using an energy balance model. The evolution of the global mean surface temperature is coupled with the CO₂ concentration through the green-house effect. We model the CO₂ concentration with a stochastic delay differential equation (SDDE), accounting for various carbon emission and capture scenarios. The resulting coupled system of SDDEs exhibits a tipping point phenomena: if CO₂ concentration exceeds a critical threshold (around 478 ppm), the temperature experiences an abrupt increase of about six degrees Celsius. We show that the CO₂ concentration exhibits a transient growth which may cause a climate tipping point, even if the concentration decays asymptotically. We derive a rigorous upper bound for the CO₂ evolution which quantifies its transient and asymptotic growths, and provides sufficient conditions for evading the climate tipping point. Combining this upper bound with Monte Carlo simulations of the stochastic climate model, we investigate the emission reduction and carbon capture scenarios that would avert the tipping point.

1. Introduction

It is highly likely that anthropogenic greenhouse gases are responsible for more than half of the increase in the global mean surface temperature between 1951 and 2010 [1]. Therefore, reducing the atmospheric concentration of greenhouse gases must be a central component of any climate change mitigation strategy.
This reduction can be achieved in two ways. One involves reducing CO$_2$ emissions through alternative energy sources, increased fuel efficiency and reduced consumption [2]. A second approach is expanding carbon sinks, e.g. by employing carbon capture and storage technologies, which capture CO$_2$ from large emitting sources (such as factories) or directly from the atmosphere and prepares it for long-term storage [3,4].

What complicates these matters is the possible existence of climate *tipping points*, i.e. climate regimes where small changes significantly alter the future state of the system [5–7]. This tipping behaviour can involve the sudden disruption of climatological and ecological processes such as melting of ice sheets or large-scale death of rainforests [8]. Even if emission reduction and carbon capture strategies lead to long-term CO$_2$ reduction, its transient growth can trigger a climate tipping point with adversely irreversible impact. Therefore, mitigation strategies not only have to ensure long-term reduction of greenhouse gases but also ensure that their transient response remains below the critical tipping point levels.

The purpose of the present study is to quantify emission reduction and carbon capture scenarios that mitigate climate tipping points. To this end, we use a Budyko–Sellers-type model of the climate [9,10]. This energy balance model assumes that the Earth’s radiation output is balanced by radiation input, and represents key aspects of glacial–interglacial climate transitions [11]. It relies on the hypothesis that glacial cycles are triggered by variations in the Earth’s orbit which alter the incoming solar radiation to the earth. These orbital cycles are amplified by limited feedback between greenhouse gases and temperature [12]. The resulting governing equations are a one-way coupling between the global mean surface temperature and CO$_2$ concentration, where the temperature is affected by CO$_2$ concentration through the greenhouse effect.

We model the evolution of the CO$_2$ concentration by a stochastic delay differential equation (SDDE), which takes into account the carbon emission and capture rates. This model allows us to parameterize the possible reduction in CO$_2$ emission rates as well as the capability of carbon sinks to absorb atmospheric CO$_2$. In particular, we show that, even when the emission rates are reduced, the CO$_2$ concentration exhibits a transient growth which may instigate a climate tipping point. We investigate the response of global mean surface temperature to various emission reduction and carbon capture scenarios, identifying the scenarios which will mitigate the climate tipping point transition.

(a) Related work

The study of climate tipping points can be divided into three broad categories: modelling, prediction and mitigation. From the modelling perspective, it is important to accurately parameterize various contributing factors such as greenhouse gases, water vapour, clouds and ice sheets [13–16]. Although here we use a very simple energy balance model, it takes into account the main culprit, i.e. CO$_2$ emissions. Furthermore, we choose the model parameters such that the resulting climate sensitivity agrees with the available estimates from more elaborate models.

Prediction is concerned with early warning signs embedded in observational data that may indicate an upcoming climate tipping point [17]. Several such precursors have been proposed, including critical slowing down [18] and increased variability [19]. Moreover, it has been observed that the stochastic component of the system changes its characteristics close to a tipping point. For instance, Held & Kleinen [20] and Livina & Lenton [21] model the North Atlantic thermohaline circulation (THC) and note that the stochastic component of the data changes from white noise to red noise. The same transition is observed by Prettyman et al. [22] who studied tropical cyclones.

From the mitigation standpoint, it is widely accepted that a significant reduction in CO$_2$ emissions is a necessity [7,23,24]. A complementary solution is to remove CO$_2$ from the atmosphere using carbon capture technologies [25]. For instance, studying the Atlantic THC, Bahn et al. [26] find that a drastic and fast CO$_2$ reduction is required to avoid disrupting the THC. However, it is argued by Ritchie et al. [27] that crossing a threshold will not necessarily result in a simultaneous extreme change in the climate system. They find that the determining factor is how far the tipping point threshold is exceeded and for how long.
Previous studies mainly focus on the asymptotic climate state as a result of increasing CO$_2$ concentration. By contrast, a main focus of our work is the transient climate dynamics in response to various emission reduction and carbon capture scenarios. In particular, we show that, under certain emission reduction and carbon capture scenarios, the CO$_2$ concentration exhibits a transient growth large enough to trigger a climate tipping point, even though the concentration decays asymptotically. We note that, beyond emission reduction and carbon capture, geoengineering ideas have been proposed [28]. These methods seek to increase the reflected energy of the Sun by releasing aerosol particles into the stratosphere. These geoengineering ideas are beyond the scope of the present work and are not considered here.

This paper is organized as follows. We discuss the climate model and its parameters in §§2 and 3. In §4, we derive a rigorous upper bound for the CO$_2$ levels under various emission reduction and carbon capture scenarios. We discuss the temperature variations under each scenario in §5 using direct numerical simulations. Section 6 contains our concluding remarks.

2. Stochastic climate model

We use two stochastic differential equations for the climate system, modelling the global mean surface temperature, $T$, and average concentration of CO$_2$, $C$, in the atmosphere. The equation for temperature is a Budyko–Sellers-type model, derived from the balance between incoming and outgoing radiations [9,10]. The resulting temperature model reads

$$\frac{dT}{dt} = F(T, C) := Q_0(1 - \alpha(T)) + S + A \ln \left( \frac{C}{C_p} \right) - \varepsilon \sigma T^4,$$

where the temperature $T$ is in units of kelvin, $C$ is the concentration of CO$_2$ in parts per million (ppm) and $c_T$ is the thermal inertia in units of J m$^{-2}$ K$^{-1}$. The term $Q_0(1 - \alpha(T))$ represents short-wave radiation from the Sun, where $Q_0$ is the solar input in units of W m$^{-2}$. The multiplier $\alpha(T)$ denotes temperature dependent albedo that accounts for the light reflecting off the Earth surface. The term $S + A \ln(C/C_p)$ models the effect of greenhouse gases, where $A$ (in units of W m$^{-2}$) is the direct forcing of CO$_2$ and determines the sensitivity of the climate equilibrium [29]. The parameter $S$ represents the trapping of outgoing radiations by greenhouse gases [11]. The constant $C_p$ denotes the preindustrial concentration of CO$_2$ in units of ppm. Finally, the last term $-\varepsilon \sigma T^4$ represents long-wave radiation from the Sun, where $\sigma T^4$ represents the outgoing long-wave radiation that is modified by the emissivity $\varepsilon$.

Our choice of the temperature dependent albedo function $\alpha(T)$ shown in figure 1 is similar to [30], although we use a different set of parameters as reported in table 1. In particular, we define

$$\alpha(T) = \alpha_1(1 - \Sigma(T)) + \alpha_2 \Sigma(T),$$

which transitions smoothly between albedo parameters $\alpha_1$ and $\alpha_2$, where $\alpha_1$ represents the current global albedo, while $\alpha_2$ represents the global albedo of the Earth. The case $\alpha_1 > \alpha_2$ corresponds to the melting of ice on the Earth’s surface, whereas the case $\alpha_1 < \alpha_2$ represents the formation of ice. Here, we only consider the case $\alpha_1 > \alpha_2$ which conforms to the present trends.

The melting of ice depends on the temperature levels. The function $\Sigma(T)$ is chosen to reflect this temperature dependence, so that $\alpha(T)$ transitions smoothly between the albedo $\alpha_1$ at temperature $T_1$ and the threshold of $\alpha_2$ at temperature $T_2$. More precisely, we define

$$\Sigma(T) = \frac{T - T_1}{T_2 - T_1} H(T - T_1) H(T_2 - T) + H(T - T_2),$$

where

$$H(T) = \frac{1 + \tanh(T/T_\alpha)}{2}$$

is a smooth approximation of the unit Heaviside function. The parameter $T_\alpha$ controls the transition rate between temperatures $T_1$ and $T_2$. Following [30], we model the unresolved subgrid processes by a stochastic term $\eta_T dW_T$, where $\eta_T > 0$ is a constant amplitude and $W_T(t)$ denotes the
standard Wiener process. The subscript $T$ is used to distinguish the stochastic processes affecting the temperature from those affecting CO$_2$, to be described shortly. Adding this stochastic term, the temperature model reads

$$c_T dT = F(T, C) dt + \eta_T dW_T,$$  \hspace{1cm} \text{(2.5)}

where $F(T, C)$ is defined in equation (2.1). Note that, although the driving stochastic force is a Gaussian white noise, the response $T$ itself is non-Gaussian and non-white owing to the nonlinear nature of $F(T, C)$.

To close the equations, we also need to model the evolution of CO$_2$. Dijkstra & Viebahn [29] prescribe $C(t)$ as an explicit function of time. Ashwin & von der Heydt [30] model the CO$_2$ evolution as a random walk confined to an interval $[C_1, C_2]$. To mimic the current trends and
to allow for possible emission reduction and carbon capture, we model the CO$_2$ evolution as an SDDE:

$$\frac{dC}{dt} = \left[\beta(t)C(t) - GC(t - \tau)\right]dt + \eta_C dW_C,$$

(2.6)

where $\beta(t)$ is a time-dependent emission rate, $G$ is a time-independent carbon capture rate and $\tau$ is a time delay. The uncertainties in CO$_2$ concentration are modelled with the stochastic term $\eta_C dW_C$, where $\eta_C$ is a constant noise intensity and $W_C(t)$ is a standard Wiener process.

The source term in equation (2.6) models the CO$_2$ emission into the atmosphere. We allow for a time-dependent rate $\beta(t)$ to model reduction or enhancement of the CO$_2$ emissions. The sink term, on the other hand, models the CO$_2$ captured from the atmosphere either through natural phenomena (e.g. photosynthesis [31]) or through artificial technologies (e.g. chemical looping [32]). For the CO$_2$ sinks, we allow for a time delay $\tau$ to model the time between carbon capture and its effect being felt in the atmospheric concentration. This type of delay is typical in control theory, where there is often a lag between a modifying action and its actualization [33,34]. If this delay is non-existent or negligible, one can set $\tau = 0$. Here, we set $\tau = 1$ year, representing the time it takes for CO$_2$ data to be updated and the carbon capture strategies adjusted correspondingly. Nonetheless, we have varied the time delay in the interval 1–20 years (not shown here) and only observed variations of approximately 10 ppm in the CO$_2$ evolution, which do not significantly alter the results.

Equations (2.5) and (2.6) form a closed set of equations modelling the global mean temperature $T$ and CO$_2$ concentration $C$. Our main goal here is to investigate whether a combination of emission reduction and carbon capture may avert a possible upcoming climate tipping point. To this end, we consider various scenarios in terms of emission reduction and carbon capture as discussed in §3.

3. Tipping points and model parameters

The parameter values used in the model are listed in table 1 and their choice is discussed in appendix B. For a prescribed CO$_2$ concentration, the temperature model (2.5) exhibits three distinct regimes as shown in figure 2. Regime 1: If $C < C_1 = 378$ ppm, the temperature has a single stable equilibrium satisfying $F(T, C) = 0$. Regime 2: For CO$_2$ concentrations $C \in (C_1, C_2)$, a bifurcation takes place, whereby two additional equilibria are born. One of the new equilibria is unstable (dashed line in figure 2) whereas the other one is stable. As a result, in this regime the temperature model is bistable. Regime 3: If $C > C_2 = 478.6$, the model switches back to a single stable equilibrium. However, in this regime, the global mean temperature is significantly higher than the equilibrium temperature in regime 1.

In our model, the CO$_2$ concentration is not constant; instead it evolves according to the SDDE (2.6). We choose the initial conditions $T(0) = 15$°C and $C(0) = 410$ ppm which reflect the current global mean temperature and CO$_2$ concentration, respectively [35–37]. The model parameters, listed in table 1, are chosen such that the current climate lies in bistable regime 2 near the lower branch of equilibria. If the current trends continue, i.e. the CO$_2$ concentration keeps increasing, the temperature $T$ also continues to increase gradually. Eventually, one reaches the tipping point $C = C_2$ where a slight increase in CO$_2$ concentration leads to a dramatic increase in the temperature by about six degrees Celsius. Our goal is to determine the emission reduction and carbon capture scenarios that would avert this catastrophic climate tipping point.

To this end, we allow for a time-dependent emission rate $\beta(t)$ as shown in figure 3. It contains three adjustable periods. First is a period of inaction, up to time $t = t_1$, where the emission rate remains constant at its current level $a$. It is followed by a reduction period, $t_1 < t < t_2$ where the emission rate decreases linearly towards its terminal value $b = a/n$. This reduction continues for a period of $\Delta t = t_2 - t_1$ years until it plateaus at time $t = t_2$. The period $t > t_2$ constitutes the terminal stage where the emission rate remains at the constant level $b$. 

\[C_1 = \frac{378}{10}, C_2 = \frac{478.6}{10} \]

\[\Delta t = \frac{t_2 - t_1}{10} \]

\[b = \frac{a}{n} \]

\[\eta_C = \frac{\text{oC}}{t} \]

\[W_C(t) \text{ is a standard Wiener process.} \]
Figure 2. Bifurcation diagram of the temperature $T$ versus the CO$_2$ concentration $C$. The curve marks the equilibrium states of equation (2.1), obtained by setting $F(T, C) = 0$, with parameters in Table 1. Solid parts of the curve denote stable equilibria and the dashed part denotes unstable equilibria. (Online version in colour.)

Figure 3. The carbon emission rate $\beta(t)$. We assume the rate is initially equal to $a$ and remains so until time $t_1$. It is then followed with a linear reduction over the time window $\Delta t = t_2 - t_1$. Eventually, the emission rate reaches the plateau $b = a/n$ for $t > t_2$. (Online version in colour.)
Therefore, the emission rate has the general form

\[
\beta(t) = \begin{cases} 
  a, & 0 \leq t < t_1, \\
  a + \frac{b - a}{t_2 - t_1} (t - t_1), & t_1 \leq t \leq t_2, \\
  b, & t_2 < t \leq t_f,
\end{cases}
\]  

(3.1)

where \( b = a/n \) for an integer \( n \geq 2 \). The current emission rate \( a \) is estimated from the data in [38]. They estimate that there are \( 11.8 \pm 0.9 \) gigatons of carbon emissions each year from industrial processes and land usage. Converting this to concentration yields \( 5.556 \pm 0.4237 \) ppm added to the atmosphere each year. The current emission rate \( a \) is then estimated from \( aC(0) = 5.556/(365 \times 24 \times 60 \times 60) \) ppm s\(^{-1}\), where \( C(0) \approx 410 \) ppm is the current \( \text{CO}_2 \) concentration. This yields \( a = 4.2971 \times 10^{-10} \) s\(^{-1}\).

To conclude the choice of parameters, we need to specify the parameters \( G \) and \( \tau \) in the \( \text{CO}_2 \) model (2.6). Recall that the parameter \( G \) denotes the capacity of carbon sinks to remove \( \text{CO}_2 \) from the atmosphere. Assuming that carbon capture takes place only due to natural phenomena, the value of \( G \) can be estimated from the data in [38]. They estimate that approximately \( 6.5 \) gigatonnes of carbon are taken from the atmosphere each year from natural processes such as photosynthesis or absorption by the oceans. Converting this to concentration yields \( 3.0603 \) ppm removed from the atmosphere each year. The parameter \( G \) then can be estimated by \( GC(0) = 3.0603/(365 \times 24 \times 60 \times 60) \) ppm s\(^{-1}\), which yields \( G = 2.3669 \times 10^{-10} \) s\(^{-1}\).

Finally, we allow for a delay \( \tau \) in the \( \text{CO}_2 \) sinks, which can be set to zero if no such delay exists. We have not been able to determine this parameter from the available literature. The results in §5 are reported for the delay time \( \tau \) equal to 1 year. However, we examined delay times up to 3 years and did not observe a significant effect on the results.

### 4. Transient and asymptotic growth of \( \text{CO}_2 \)

The system of equations (2.5) and (2.6) is a one-way coupling between the temperature \( T \) and the \( \text{CO}_2 \) concentration \( C \), whereby the \( \text{CO}_2 \) concentration evolves independently of the temperature. As discussed in §3, if the \( \text{CO}_2 \) concentration increases beyond \( C_2 \approx 478.6 \) ppm, the climate system undergoes a tipping point transition leading to an abrupt increase of the global mean surface temperature (figure 2).

The main objective of this paper is to determine emission reduction and carbon capture scenarios that ensure the mitigation of this tipping point. Extreme events, such as tipping point transitions, have been the subject of much research, with an emphasis on their causal mechanisms [39–41], probabilistic quantification [42–45], and data-driven prediction [18,46–48]. Only recently, control strategies for mitigating extreme events have been proposed [34,46,49]. In particular, Farazmand [34] proposes a time-delay feedback control for mitigating noise-induced transitions in multistable systems. This control strategy relies on the stationary equilibrium density of the system. Given the time-dependent emission rate \( \beta(t) \) and the linearity of the \( \text{CO}_2 \) model (2.6), it does not possess such an equilibrium density. Consequently, the framework of [34] is not applicable here.

Therefore, we take a different approach here and derive a quantitative upper bound for the \( \text{CO}_2 \) concentration which determines whether the climate tipping point transitions can be averted. To this end, we consider the \( \text{CO}_2 \) model (2.6) without the stochastic term:

\[
\frac{dC}{dt} := \beta(t)C(t) - GC(t - \tau), \quad C(s) = C_0, \quad \forall s \in [-\tau, 0].
\]  

(4.1)

Note that (4.1) is a delay differential equation which requires the initial condition \( C(s) \) to be specified as a function over the interval \([-\tau, 0]\). Since we consider a relatively short delay \( \tau \) of one year, the \( \text{CO}_2 \) concentration does not change significantly over this time interval. As a result, we assume the constant initial condition \( C(s) = C_0 \) for all \( s \in [-\tau, 0] \), where \( C_0 = 410 \) ppm is the...
current level of CO₂ concentration. The following theorem provides an upper bound for the solutions $C(t)$. Adding the stochastic term $\eta_C dW_C$ only leads to small fluctuations around this upper bound, without fundamentally altering the results.

**Theorem 4.1.** Let $0 < T \leq \infty$, and $C_0 \geq 0$. Assume that $\beta : [0, T) \to \mathbb{R}^+$ is locally integrable and that $C : [-\tau, T) \to \mathbb{R}^+$ is a measurable, locally integrable function solving the delay differential equation (4.1). Then

$$C(t) \leq C_0 \exp \left( \int_0^t (\beta(s) + \gamma(s)) \, ds \right),$$

(4.2)

where

$$\gamma(t) = -G \exp \left( - \int_{t-\tau}^t \beta(s) \, ds \right).$$

(4.3)

**Proof.** See appendix A. ■

Recall that the climate tipping point occurs if the CO₂ levels exceed $C_2 = 478.6$ ppm. Therefore, ensuring that the upper bound (4.2) is uniformly below $C_2$ is a sufficient condition for avoiding the tipping point. This upper bound encapsulates the competition between CO₂ emission rate $\beta(t)$ and the carbon capture rate $G$. The negative-valued function $\gamma(t)$, appearing in the exponent of the upper bound, is proportional to the carbon capture rate $G$. But it also depends on the emission rate $\beta(t)$ due to the delay $\tau$. This leads to a non-trivial dependence of the upper bound on the carbon emission and capture rates. In §5, we investigate the shape of this upper bound for various parameter values.

The upper bound (4.2) also provides a sufficient condition for the asymptotic decay of CO₂ concentration.

**Corollary 4.2.** Assume the conditions of theorem 4.1. If the carbon capture rate satisfies $G > be^{bt}$, then we have $\lim_{t \to \infty} C(t) = 0$.

**Proof.** Note that for $t > t_2$, we have $\beta(t) = b$ (see equation (3.1)). Therefore, for all $t > t_2 + \tau$, upper bound (4.2) implies

$$C(t) \leq C_0 \exp \left[ \int_0^{t_2 + \tau} (\beta(s) + \gamma(s)) \, ds \right] \exp \left( \left( b - Ge^{-bt} \right) (t - t_2 - \tau) \right).$$

As a result, if $G > be^{bt}$, the CO₂ concentration $C(t)$ tends to zero as $t \to \infty$. ■

We emphasize that, to avert the climate tipping point, it is not sufficient for the CO₂ concentration to decay asymptotically. As we show in §5, even a transient growth of CO₂ that exceeds the critical threshold $C_2$ will lead to a tipping point transition.

### 5. Results and discussion

In this section, we present the numerical results obtained from the model (2.5) and (2.6). First, we focus on the CO₂ model (2.6) and investigate the behaviour of its upper bound (4.2). Recall that to avoid the climate tipping point it is sufficient for this upper bound to remain below the critical level $C_2$.

**Figure 4** shows the upper bound (4.2) with the delay time $\tau = 1$ year, emission reduction time span $\Delta t = 50$ years, and the carbon capture rate $G = G_0 = 2.37 \times 10^{-10}$ s⁻¹. The initial CO₂ level is set at $C(0) = C_0 = 410$ ppm, which is the estimated CO₂ concentration in the year 2019 [37]. **Figure 4a** represents the optimistic scenario where emission reductions begin immediately ($t_1 = 0$) and continue to decrease linearly for $\Delta t = 50$ years. The figure shows three terminal emission rates $b = a/n$ with $n = 2, 5, 10$. In every case, we observe a transient growth of CO₂ levels. But none of them reach the tipping point $C_2$ and they decay asymptotically.

On the other hand, **figure 4b** shows the case where the transient growth surpasses the tipping point $C = C_2$. In this figure, we fix the terminal CO₂ emission rate at $b = a/3$ with the time window for reaching this rate being $\Delta t = 50$ years. We vary the number of years $t_1$ before the linear
the equations by defining the temperature $T$ of the stochastic climate model and compute the ensemble average of the global mean surface temperature through the greenhouse effect. By contrast, the CO$_2$ concentration evolves independently of the scenarios, we simulate the model over a 200 year time span for a range of parameters $n$. Recall that in dimensional time.

However, the more germane factor is whether the transient growth of CO$_2$ would exceed the developed by Cao. In figure 5, the tipping point may be reached and consequently the global mean surface temperature may sharply increase by about six degrees. These observations accentuate the need for immediate action on carbon emission reduction.

We emphasize that in every case shown in figure 4 the CO$_2$ levels decay asymptotically. However, the more germane factor is whether the transient growth of CO$_2$ would exceed the climate tipping point $C = C_2$.

Next, we consider the full model, i.e. the coupled system of equations (2.5) and (2.6). Recall that this is a one-way coupling, where the temperature $T$ is affected by the CO$_2$ concentration through the greenhouse effect. By contrast, the CO$_2$ concentration evolves independently of the temperature.

This system of SDDEs is integrated numerically using the predictor-corrector scheme developed by Cao et al. The numerical integration is carried out after non-dimensionalizing the equations by defining $\hat{C} = C/C_p$, $\hat{T} = T/T_0$, and $\hat{t} = t/t_0$, where $C_p = 280$ ppm denotes the preindustrial CO$_2$ level, $T_0 = 288$ K denotes the emissivity threshold given by Ashwin & von der Heydt, and $t_0 = 10^7$ s is an arbitrary time scale (approx. one-third of a year). The time step of the numerical integrator in the non-dimensional time $\hat{t}$ is 0.0086, which is equivalent to one day in dimensional time.

To investigate the behaviour of the climate system with regards to various emission reduction scenarios, we simulate the model over a 200 year time span for a range of parameters $n$ and $\Delta t$. Recall that $n$ determines the terminal emission rate $b = a/n$ and $\Delta t$ determines the time it takes to reach this plateau (see figure 3). For each set of parameters $(n, \Delta t)$, we simulate $10^3$ realizations of the stochastic climate model and compute the ensemble average of the global mean surface temperature $T$ and the CO$_2$ concentration. In all simulations, the initial conditions are set to $T(0) = 15^\circ$C and $C(0) = 410$ ppm, which reflect the estimated values in the year 2019 [35,36].

Figure 5 shows the ensemble means of CO$_2$ concentration $C$ and temperature $T$ as a function of time and the parameters $(n, \Delta t)$. This figure has three panels as described below:

(i) Figure 5a: $t_1 = 0$ and $G = G_0 = 2.37 \times 10^{-10}$ s$^{-1}$.
(ii) Figure 5b: $t_1 = 0$ and $G = G_0/2 = 1.18 \times 10^{-10}$ s$^{-1}$.
(iii) Figure 5c: $t_1 = 25$ years and $G = G_0 = 2.37 \times 10^{-10}$ s$^{-1}$.

Figure 5a,b correspond to the scenario where emission reductions begin immediately ($t_1 = 0$). In figure 5a, the carbon capture rate is equal to the empirical value $G_0 = 2.37 \times 10^{-10}$ s$^{-1}$. In this

Figure 4. The CO$_2$ upper bound (4.2) for parameters $\tau = 1$ year, $\Delta t = 50$ years, $G = 2.3669 \times 10^{-10}$ s$^{-1}$. The horizontal dashed line marks the tipping point $C = C_2 = 478.6$. Time $t = 0$ corresponds to the year 2019. (a) $b = a/n$ and $t_1 = 0$, (b) $b = a/3$ and $t_1 > 0$. (Online version in colour.)
Figure 5. Ensemble averages of the CO$_2$ concentration $C$ in the first column and the temperature $T$ in the second column, given the terminal emission parameter $n$ and reduction time $\Delta t$. The time $t = 0$ corresponds to the year 2019. (a) $t_1 = 0$ and $G = G_0$. (b) $t_1 = 0$ and $G = G_0/2$. (c) $t_1 = 25$ years and $G = G_0$. Note the transient decrease in CO$_2$ concentration after 20 years for appropriate choices of parameters $n$ and $\Delta t$. (Online version in colour.)

case, the CO$_2$ concentration increases transiently, but does not reach its critical value $C_2$. As a result, no tipping point phenomenon is observed. In fact, the temperature does not exceed 16 degrees Celsius and decays towards 12 degrees asymptotically.

Figure 5b is identical to figure 5a except that we use a lower carbon capture rate $G = G_0/2 = 1.18 \times 10^{-10}$ s$^{-1}$. The carbon capture rate may decrease over time due to various factors, but most prominently due to deforestation [51,52]. In this case, the CO$_2$ concentration still undergoes a transient growth before eventually decreasing. Because of the lower carbon capture rate, however, the transient growth surpasses the critical threshold $C_2$, and leads to a drastic increase in the global mean surface temperature. This tipping point does not occur for all carbon emission scenarios. If the transition period $\Delta t$ is short enough and the terminal emission rate $b = a/n$ is small enough, the tipping point can still be mitigated.
In fact, as shown in figure 6a, there is a nonlinear boundary in the parameter space \((n, \Delta t)\) which separates the tipping point transition from its successful mitigation. For \(n = 2\), the tipping point occurs regardless of the transition time \(\Delta t\). Therefore, the terminal carbon emission rate has to reduce to at least one-third of its current value to mitigate the climate tipping point. Larger \(n\) allows for longer transition period \(\Delta t\). In figure 6b, we show the ensemble average of the temperature \(T\) for three parameter values \((n, \Delta t)\). The shaded areas mark one standard deviation from the mean. The point on the boundary \((n = 4, \Delta t = 29)\) has a large standard deviation because the stochastic process \(\eta_c \, dW_c\) can kick the trajectory towards the tipping regime or away from it. The parameters \(n = 3\) and \(\Delta t = 29\) lead to the tipping behaviour with high probability, where the temperature rises to about 24 °C. By contrast, \(n = 6\) successfully averts the climate tipping point, after a slight transient increase in the global mean surface temperature.

Finally, figure 5c shows the case where emission reduction is delayed by 25 years \(t_1 = 25\), and the carbon capture rate stays at its current level \((G = G_0)\). In this case, CO₂ concentration again exhibits a transient growth past the critical value \(C_2\), and as a result, a drastic increase in global temperature ensues. Although the CO₂ concentration reduces asymptotically to as low as 150 ppm, the transient tipping point will have dire consequences, such as sea level rise and droughts. This observation reaffirms the need for immediate action towards significant emission reduction. Otherwise, the only alternative would be to significantly increase the CO₂ sinks \(G\), through artificial carbon capture technologies.

6. Conclusion

We considered a stochastic climate model as a one-way coupling between CO₂ concentration \(C\) and the global mean surface temperature \(T\). The CO₂ concentration is governed by a SDDE, allowing for the modelling of various emission reduction and carbon capture scenarios. The temperature \(T\) satisfies a stochastic differential equation derived from energy balance (Budyko–Sellers model). The temperature is coupled with the CO₂ equation to model the effect of greenhouse gases.

The model has a tipping point behaviour: if the CO₂ concentration exceeds the critical value of \(C = 478.6\) ppm, the global mean surface temperature will increase abruptly by about six degrees Celsius. The CO₂ model exhibits transient growth which allows for reaching this tipping point in finite time, even when CO₂ decreases asymptotically. We derived a tight upper bound for the CO₂ concentration, which provides sufficient conditions for mitigating the climate tipping point (see
theorem 4.1). This upper bound depends on several parameters such as the CO$_2$ emission rate $\beta(t)$, the carbon capture rate $G$, and the delay time $\tau$. However, since the upper bound is explicitly known, it can be analysed numerically with low computational cost in order to determine the emission reduction and carbon capture scenarios which would mitigate the climate tipping point.

We examined various emission reduction scenarios by combining our analytic results with Monte Carlo simulations of the climate model. In particular, we find that the climate tipping point in our model can be averted if CO$_2$ emission reductions begin immediately and the emission rate decreases to one-third of its current level within 50 years. However, if these reductions are delayed by as much as ten years, the transient growth of the CO$_2$ concentration will exceed the tipping point value, leading to a drastic and abrupt increase in the global mean surface temperature. In the latter case, increasing the carbon capture rate could still avert the tipping point.

It remains to be seen whether our conclusions carry over to more sophisticated climate models, such as box models or general circulation models. We will investigate such models in future work. Although the temperature model would be more complex, our analysis of the CO$_2$ concentration can be used to derive sufficient conditions that guarantee tipping point evasion. Furthermore, deriving an invariant probability distribution for the CO$_2$ model is of great interest as it would quantify the probability of transitions which in turn leads to sufficient conditions for mitigating the climate tipping point. Finally, the CO$_2$ model itself can be improved, e.g. by using data-driven discovery methods to obtain simple models that agree with observational data [53].

Data accessibility. The data and code for reproducing the results of this paper are publicly available at https://github.com/mfarazmand/ClimateMitigation.

Authors’ contributions. M.F. conceptualized and supervised the research. A.M. performed the research. A.M and M.F. contributed equally to writing the manuscript. All authors gave final approval for publication and agree to be held accountable for the work performed therein.

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Appendix A. Proof of theorem 4.1

We begin by writing equation (4.1) as the integral equation

$$ C(t) = C_0 + \int_0^t \beta(s) C(s) \, ds - \int_0^t G C(s - \tau) \, ds. \tag{A 1} $$

We derive a tight upper bound for $C(t)$ using a modified version of a Grönwall-type inequality for delay differential equations given in [54]. We first prove a result with $\beta \equiv 0$, which we later use to address the general case where the emission rate $\beta$ is non-zero.

Proposition A.1. Let $t_0 \in \mathbb{R}$, $t_0 < T \leq \infty$, $C_0 \geq 0$, and $a : [t_0, T) \to \mathbb{R}^+$ be locally integrable. Let $C : [t_0, T) \to \mathbb{R}^+$ be Borel measurable and locally bounded such that

$$ C(t) \leq C_0 - \int_{t_0}^t a(u) C(u - \tau) \, du, \tag{A 2} $$

and $\gamma : [t_0 - \tau, T) \to \mathbb{R}$ be any locally integrable function that satisfies the inequality

$$ -a(t) \exp \left( -\int_{t-\tau}^t \gamma(s) \, ds \right) \leq \gamma(t). \tag{A 3} $$

Then

$$ C(t) \leq K \exp \left( \int_{t_0}^t \gamma(s) \, ds \right), $$
where

\[ K := \max \left\{ C_0 \exp \left( \int_{t_0}^{t_1} y(u) \, du \right), \sup_{t_0 - \tau \leq s \leq t_0} \exp \left( \int_{t_0}^{t_1} \gamma(s) \, ds \right) \right\}. \]

**Proof.** Note that in the context of our climate model, we have \( a(t) = G \). Györi & Horváth [54] proved a similar result with a plus sign in front of the integral in equation (A.2). We generalize their result to the case with a minus sign in front of the integral as required by our climate model. We first define

\[ y(t) = C(t) \exp \left( -\int_{t_0 - \tau}^{t} \gamma(s) \, ds \right). \]

Inequality (A.2) implies

\[ y(t) \leq C_0 \exp \left( -\int_{t_0 - \tau}^{t} \gamma(s) \, ds \right) - \exp \left( -\int_{t_0 - \tau}^{t} \gamma(s) \, ds \right) \times \int_{t_0}^{t} a(u)y(u - \tau) \exp \left( -\int_{u - \tau}^{u} \gamma(s) \, ds \right) \exp \left( \int_{t_0 - \tau}^{u} \gamma(s) \, ds \right) \, du. \]

The inequality (A.3) for \( \gamma \) then yields

\[ y(t) \leq C_0 \exp \left( -\int_{t_0 - \tau}^{t} \gamma(s) \, ds \right) + \exp \left( -\int_{t_0 - \tau}^{t} \gamma(s) \, ds \right) \int_{t_0}^{t} \gamma(u) \exp \left( \int_{t_0 - \tau}^{u} \gamma(s) \, ds \right) y(u - \tau) \, du. \]

Defining

\[ L := \max \left\{ C_0, \sup_{t_0 - \tau \leq s \leq t_0} C(s) \exp \left( -\int_{t_0 - \tau}^{s} \gamma(u) \, du \right) \right\}, \]

we have \( y(t) \leq L \) for all \( t \in [t_0 - \tau, t_0] \). We now prove that, in fact, \( y(t) \leq L \) for all \( t \in [t_0 - \tau, T] \).

Let \( L_1 > L \). By the definition of \( L \), we have

\[ y(t) \leq L < L_1, \quad t_0 - \tau \leq t \leq t_0. \]

Since \( y \) is continuous on \([t_0, T]\), there exists \( q > 0 \) such that \( t_0 + q < T \) and

\[ y(t) < L_1, \quad t_0 \leq t \leq t_0 + q. \]

Assume there exists \( t_1 \in (t_0 + q, T) \) such that \( y(t_1) = L_1 \). Since \( y \) is continuous on \([t_0, T]\), we can assume that \( y(t) < L_1 \) for all \( t \in [t_0, t_1) \).

It then follows from \( L_1 > L \geq C_0 \) that

\[ y(t_1) < C_0 \exp \left( -\int_{t_0 - \tau}^{t_1} \gamma(s) \, ds \right) + \exp \left( -\int_{t_0 - \tau}^{t_1} \gamma(s) \, ds \right) \int_{t_0}^{t_1} \gamma(u) \exp \left( \int_{t_0 - \tau}^{u} \gamma(s) \, ds \right) L_1 \, du \]

\[ = C_0 \exp \left( -\int_{t_0 - \tau}^{t_1} \gamma(s) \, ds \right) + L_1 \left( 1 - \exp \left( -\int_{t_0}^{t_1} \gamma(s) \, ds \right) \right) \]

\[ = L_1 + \exp \left( -\int_{t_0}^{t_1} \gamma(s) \, ds \right) \left( C_0 - L_1 \exp \left( \int_{t_0 - \tau}^{t_0} \gamma(s) \, ds \right) \right) \]

\[ < L_1. \]

This contradicts \( y(t_1) = L_1 \). Therefore, we must have \( y(t) < L_1 \) for all \( t \in [t_0 - \tau, T] \). Since \( L_1 > L \) is arbitrarily close to \( L \) and \( y \) is continuous, we have \( y(t) \leq L \) for all \( t \in [t_0 - \tau, T] \).
Finally, $y(t) \leq L$ implies
\[
C(t) = y(t) \exp \left( \int_{t_0 - \tau}^{t} \gamma(s) \, ds \right)
\leq L \exp \left( \int_{t_0 - \tau}^{t} \gamma(s) \, ds \right)
= K \exp \left( \int_{t_0}^{t} \gamma(s) \, ds \right).
\]

Now we return to equation (A 1) and derive an upper bound for $C(t)$ for the general case with $\beta(t) \geq 0$.

\textbf{Theorem A.2.} Let $t_0 \in \mathbb{R}$, $t_0 < T \leq \infty$, and $C_0 \geq 0$, $\beta : [t_0, T) \to \mathbb{R}^+$ be locally integrable. Let $C : [t_0 - \tau, T) \to \mathbb{R}^+$ be Borel measurable and locally bounded such that
\[
C(t) \leq C_0 + \int_{t_0}^{t} \beta(u) C(u) \, du - \int_{t_0}^{t} \mathcal{G} C(u - \tau) \, du,
\]
and $\gamma : [t_0 - \tau, T) \to \mathbb{R}$ be any locally integrable function satisfying
\[
- \mathcal{G} \exp \left( - \int_{t_0}^{t} (\beta(s) + \gamma(s)) \, ds \right) \leq \gamma(t).
\]
Then
\[
C(t) \leq K \exp \left( \int_{t_0}^{t} (\beta(s) + \gamma(s)) \, ds \right),
\]
where
\[
K := \max \left\{ C_0 \exp \left( \int_{t_0 - \tau}^{t_0} \gamma(u) \, du \right), \sup_{t_0 - \tau \leq s \leq t_0} C(s) \exp \left( \int_{s}^{t_0} \gamma(u) \, du \right) \right\}.
\]

\textit{Proof.} This result was first proved in [54] with a plus sign before the last integral in (A 4). We generalize their result to the case with a minus sign in front of the integral as required by our climate model. We also point out that our CO$_2$ model satisfies (A 4) with equality.

Applying variation of constants to (A 4), we obtain
\[
C(t) \leq C_0 \exp \left( \int_{t_0}^{t} \beta(u) \, du \right) - \int_{t_0}^{t} \mathcal{G} \left( \int_{u}^{t} \beta(v) \, dv \right) \exp \left( \int_{u}^{t} \beta(v) \, dv \right) \, du.
\]

Defining
\[
y(t) = C(t) \exp \left( \int_{t_0}^{t} -\beta(v) \, dv \right),
\]
we have
\[
y(t) \leq C_0 - \int_{t_0}^{t} \mathcal{G} \exp \left( - \int_{u - \tau}^{u} \beta(v) \, dv \right) y(u - \tau) \, du.
\]

The function
\[
t \to \mathcal{G} \exp \left( - \int_{t_0 - \tau}^{t} \beta(v) \, dv \right)
\]
is locally measurable. Thus, applying proposition A.1, we obtain
\[
y(t) \leq K \exp \left( \int_{t_0}^{t} \gamma(s) \, ds \right),
\]
where
\[
K := \max \left\{ C_0 \exp \left( \int_{t_0 - \tau}^{t_0} \gamma(u) \, du \right), \sup_{t_0 - \tau \leq s \leq t_0} C(s) \exp \left( \int_{s}^{t_0} \gamma(u) \, du \right) \right\}.
\]

The conclusion then follows immediately.
Although this theorem provides an upper bound for equation (4.1), the issue of finding a suitable choice of $\gamma(t)$ remains. To obtain a tight upper bound, $\gamma(t)$ must be strictly negative. To this end, we choose

$$\gamma(t) = -G \exp \left( -\int_{t-r}^{t} \beta(s) \, ds \right),$$

which is negative definite and satisfies the inequality (A.5). Recall that we assumed the constant initial condition $C(s) = C_0$ for all $s \in [-r, 0]$ (cf. equation (4.1)). This observation, together with the fact that $\gamma$ is negative, yields $K = C_0$. Therefore, the upper bound in theorem A.2 simplifies to

$$C(t) \leq C_0 \exp \left( \int_{0}^{t} (\beta(s) + \gamma(s)) \, ds \right),$$

which is the desired result of theorem 4.1.

**Appendix B. Model parameters**

Our model parameters mostly agree with those chosen by Dijkstra & Viebahn [29]. However, there are a few differences that we justify in this section. These changes are mostly made so that our model parameters more closely approximate the current climate state.

The main differences appear in the albedo function (2.2). Dijkstra & Viebahn [29] use the initial albedo value $\alpha_1 = 0.7$ which does not agree with the estimates of the current albedo of the Earth. We instead use $\alpha_1 = 0.31$ which agrees with the value inferred from empirical satellite observations [55–57]. The temperature-dependent albedo (2.2) requires albedo values to be paired with a corresponding temperature threshold. A reasonable choice is to pair the current albedo of Earth with the temperature $T_1 = 289$ K, which roughly corresponds to the current global mean surface temperature.

The terminal albedo value $\alpha_2 = 0.2$, with the corresponding threshold temperature $T_2 = 295$ K, is chosen to reflect the state of the Earth after significant amounts of warming. We point out that the terminal albedo $\alpha_2 = 0.2$ is most likely unrealistic since it will require a significant reduction in cloud coverage in addition to sea level rise. However, the model does not allow for much larger values of $\alpha_2$ without losing the climate tipping point. As seen in figure 7, for any $\alpha_2 > 0.22$, the fold in the bifurcation curve ceases to exist and, as a result, the irreversible climate tipping point vanishes.
Dijkstra & Viebahn [29] use the smoothness parameter $T_\alpha = 0.273$ which leads to a rapid change between the initial albedo $\alpha_1$ and the terminal albedo $\alpha_2$. Following Ashwin & von der Heydt [30], we use $T_\alpha = 3$ which corresponds to a more gradually decreasing albedo as shown in figure 1.

We use the CO2 forcing parameter $A = 20.5 \text{ W m}^{-2}$. The direct effect of CO2 forcing yields the lower value $A \approx 5 \text{ W m}^{-2}$. However, this value neglects other effects such as ice albedo and water vapour [11,30], and results in a climate sensitivity that is much lower than the range given in AR5 [1]. We note that Ashwin & von der Heydt [30] use $A = 5.35 \text{ W m}^{-2}$, but they allow for a temperature-dependent emissivity function $\epsilon(T)$ to account for the water vapour feedback. With CO2 forcing parameter $A = 20.5 \text{ W m}^{-2}$, the transient climate response (TCR) of our model is about 2.6°C and its equilibrium climate sensitivity (ECS) is about 6°C. Note that both the TCR and ECS of our model are only slightly larger than the current likely ranges given in AR5 [1], which are 1–2.5°C and 1.5–4.5°C, respectively.

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