ON LINEAR ELLIPTIC AND PARABOLIC EQUATIONS WITH GROWING DRIFT IN SOBOLEV SPACES WITHOUT WEIGHTS

N.V. KRYLOV

ABSTRACT. We consider uniformly elliptic and parabolic second-order equations with bounded zeroth-order and bounded VMO leading coefficients and possibly growing first-order coefficients. We look for solutions which are summable to the \( p \)-th power with respect to the usual Lebesgue measure along with their first and second-order derivatives with respect to the spatial variable.

1. Introduction

In this paper we concentrate on problems in the whole space for uniformly elliptic and parabolic second-order equations with bounded leading and zeroth-order coefficients and possibly growing first-order coefficients. We look for solutions which are summable to the \( p \)-th power with respect to the usual Lebesgue measure along with their first- and second-order derivatives with respect to the spatial variables.

There exists a quite extensive literature related to equations with growing coefficients in Sobolev-Hilbert spaces with weights. Since here no weights are used we only refer the reader to [1], [2], [4], [5], [6] where one can find further references as well.

It is generally believed that introducing weights is the most natural setting for equations with growing coefficients. The present paper seems to be the first one treating the unique solvability of these equations in Sobolev spaces \( W^2_p \) for \( p \in (1, \infty) \) without weights and without imposing any special conditions on the relations between the coefficients or on their smoothness. In the elliptic case, in rough terms, it is sufficient for us that the drift term \( b^i D_i u \) be, say, such that

\[
\lim_{\alpha \to 0} \sup_{x,y : |x-y| \leq \alpha} |x - y| \varepsilon |b(x) - b(y)| = 0,
\]

where (possibly negative) \( \varepsilon < (d-1)/(d \vee p) \). This condition has nothing to do with any continuity property of \( b \) since \( \varepsilon \) is allowed to be positive.

2000 Mathematics Subject Classification. 35K10, 35J15.

Key words and phrases. Linear elliptic and parabolic equations, growing coefficients, usual Sobolev spaces.

The work was partially supported by NSF grant DMS-0653121.
It is worth noting that many issues for divergence-type equations with time independent growing coefficients in $L_p$ spaces without weights were treated previously in the literature. This was done mostly by using the semigroup approach. We briefly mention only a few recent papers sending the reader to them for additional references.

In [9] a strongly continuous in $L_p$ semigroup is constructed corresponding to elliptic operators with measurable leading coefficients and Lipschitz continuous drift coefficients. This did not lead to the solvability of elliptic equations in $W^{1}_p$ for $p > 2$ because of low regularity of the leading coefficients. In [11] it is assumed that if, for $|x| \to \infty$, the drift coefficient grows, then the zeroth-order coefficient should grow, basically, as the square of the drift. There is also a condition on the divergence of the drift coefficient. In [12] there is no zeroth-order term and the semigroup is constructed under some assumptions one of which translates into the monotonicity of $\pm b(x) - Kx$, for a constant $K$, if the leading term is the Laplacian. In [3] the drift coefficient is assumed to be globally Lipschitz continuous if the zeroth-order coefficient is constant.

Some conclusions in the above cited papers are quite similar to ours but the corresponding assumptions are not as general in what concerns the regularity of the coefficients. However, these papers contain a lot of additional important information not touched upon in the present paper (in particular, it is shown in [9] that the corresponding semigroup is not analytic).

The technique, we apply, originated from [8] and uses special cut-off functions whose support evolves in time in a manner adapted to the drift. Another less important feature is that the leading coefficients of the equations are assumed to be only measurable in time and VMO in $x$. In fact, the reader will see from our proofs that nothing special is needed from the leading terms and one can add the drift term satisfying our conditions to any equation for which the Sobolev space theory is available. In particular, this can be done for divergence form equations with measurable coefficients if $p = 2$. However, for the sake of brevity and clarity we concentrate only on nondivergence type equations. The main emphasis here is that we allow $b(t, x)$ to grow as $|x| \to \infty$ and still measure the size of the second-order derivatives with respect to Lebesgue measure thus avoiding using weights.

Let $\mathbb{R}^d$ be a Euclidean space of points $x = (x^1, \ldots, x^d)$. We consider the following second-order operator $L$:

$$Lu(t, x) = a^{ij}(t, x)D_{ij}u(t, x) + b^i(t, x)D_iu(t, x) - c(t, x)u(t, x),$$

acting on functions defined on $\mathbb{R}^{d+1}_T$, which is $[T, \infty) \times \mathbb{R}^d$ if $T \in (-\infty, \infty)$ and on $\mathbb{R}^{d+1}$ if $T = -\infty$ (the summation convention is enforced throughout the article). Here

$$D_i = \frac{\partial}{\partial x^i}, \quad D_{ij} = D_iD_j.$$

We are dealing with the parabolic equation

$$\partial_t u(t, x) + Lu(t, x) = f(t, x), \quad (t, x) \in (T, \infty) \times \mathbb{R}^d,$$

(1.3)
where \( \partial_t = \partial / \partial t \), and, in case the coefficients are independent of \( t \), with the elliptic equation

\[
Lu(x) = f(x), \quad x \in \mathbb{R}^d.
\]

The solutions of (1.4) are sought in \( W^2_p(\mathbb{R}^d) \), usual Sobolev space, and the space of solutions of (1.3) will be \( W^2_p(T) \) which is defined as follows.

We write \( u \in W^2_p(T) \) if \( u = u(t,x) \) is a (measurable) function defined on \( \mathbb{R}^{d+1}_T \) such that

\[
\|u\|_{L^p(\mathbb{R}^{d+1}_T)} + \|Du\|_{L^p(\mathbb{R}^{d+1}_T)} + \|D^2u\|_{L^p(\mathbb{R}^{d+1}_T)} < \infty
\]

and \( \partial_t u := \partial u / \partial t \) is locally summable on \( \mathbb{R}^{d+1}_T \). Of course, \( Du \) and \( D^2u \) are the gradient and the Hessian matrix of \( u \), respectively. Observe that we do not include \( \partial_t u \) into the left-hand side of (1.5) because we believe that, generally, in our situation \( \partial_t u \notin L^p(\mathbb{R}^{d+1}_T) \) (see Remark 1.2).

Our main results are presented in Sections 2 (elliptic case) and 3 (parabolic case). Theorem 2.1 saying that under appropriate conditions the elliptic equation \( Lu - \lambda u = f \) is uniquely solvable in \( W^2_p(\mathbb{R}^d) \) if \( \lambda \) is large enough is proved in Section 3. Interestingly enough, even if \( b \) is constant we do not know any other proof of Theorem 2.1 not using the parabolic theory.

We prove Theorems 3.1 and 3.2 in Section 5 and 6, respectively, after we prepare necessary tools in Section 4. In Section 7 we give an example showing that for elliptic equations one cannot take \( \lambda_0 > 0 \) arbitrary small in contrast with the case of bounded coefficients as described in Section 11.6 of [7]. This fact is known from [10], where the spectrum of the Ornstein-Uhlenbeck operator is found in the multidimensional case in \( L^p \) spaces and it is shown that the spectrum depends on \( p \).

As usual when we speak of “a constant” we always mean “a finite constant”.

The author is sincerely grateful to A. Lunardi for the fruitful discussion of the results.

2. MAIN RESULT FOR ELLIPTIC CASE

For \( p \in (1, \infty) \), \( p \neq d \), define

\[
q = d \lor p,
\]

and if \( p = d \) let \( q \) be a fixed number such that \( q > d \).

Assumption 2.1. (i) The functions \( a^{ij}, b^i, c \) are measurable, \( a^{ij} = a^{ji} \), \( c \geq 0 \).

(ii) There exist constants \( K, \delta > 0 \) such that for all values of arguments and \( \xi \in \mathbb{R}^d \)

\[
\delta |\xi|^2 \leq a^{ij} \xi^i \xi^j \leq K |\xi|^2, \quad c \leq K.
\]

(iii) The function \( |b|^q \) is locally integrable on \( \mathbb{R}^d \).

The following assumptions contain parameters \( \gamma_0, \gamma_b \in (0, 1] \) whose value will be specified later. For \( \alpha > 0 \) we denote \( B_\alpha = \{ x \in \mathbb{R}^d : |x| < \alpha \} \).
Assumption 2.2 ($\gamma_b$). There exists an $\alpha \in (0, 1]$ such that on $\mathbb{R}^d$

$$\alpha^{-d} \int_{B_\alpha} \int_{B_\alpha} |b(x + y) - b(x + z)|^q \, dy \, dz \leq \gamma_b. \tag{2.1}$$

It is easy to check that Assumption 2.2 is satisfied with any $\gamma_b > 0$ if (1.1) holds. For instance, we allow $b$ such that $|b(x) - b(y)| \leq K$ if $|x - y| \leq 1$. We see that $|b(x)|$ can grow to infinity as $|x| \to \infty$.

Assumption 2.3 ($\gamma_a$). There exists an $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, $x \in \mathbb{R}^d$, and $i,j = 1, \ldots, d$ we have

$$\varepsilon^{-2d} \int_{B_\varepsilon} \int_{B_\varepsilon} |a^{ij}(x + y) - a^{ij}(x + z)| \, dy \, dz \leq \gamma_a. \tag{2.2}$$

Obviously, the left-hand side of (2.2) is less than

$$N(d) \sup_{|x - y| \leq 2\varepsilon} |a^{ij}(x) - a^{ij}(y)|,$$

which implies that Assumption 2.3 is satisfied with any $\gamma_a > 0$ if, for instance, $a$ is a uniformly continuous function. Recall that if Assumption 2.3 is satisfied with any $\gamma_a > 0$, then one says that $a$ is in VMO.

Here is one of the main results of the paper.

Theorem 2.1. There exist constants

$$\gamma_a = \gamma_a(d, \delta, K, p) > 0, \quad \gamma_b = \gamma_b(d, \delta, K, p, \varepsilon_0) > 0,$$

$$N = N(d, \delta, K, p, \varepsilon_0), \quad \lambda_0 = \lambda_0(d, \delta, K, p, \varepsilon_0, \alpha) \geq 0$$

such that, if the above assumptions are satisfied, then for any $u \in W^2_p(\mathbb{R}^d)$ and $\lambda \geq \lambda_0$ we have

$$\lambda \|u\|_{L^p(\mathbb{R}^d)} + \|D^2u\|_{L^p(\mathbb{R}^d)} \leq N \|Lu - \lambda u\|_{L^p(\mathbb{R}^d)}. \tag{2.3}$$

Furthermore, for any $f \in L^p(\mathbb{R}^d)$ and $\lambda \geq \lambda_0$ there is a unique $u \in W^2_p(\mathbb{R}^d)$ such that $Lu - \lambda u = f$.

We prove this theorem in Section 3. One of surprising features of (2.3) is that $N$ is independent of $b$ if $b$ is constant. Another one is that the set $(L - \lambda)W^2_p(\mathbb{R}^d)$ may not coincide with $L^p(\mathbb{R}^d)$ if $|b|$ grows and yet it always contains $L^p(\mathbb{R}^d)$. Some consequences of this peculiarity are discussed in [8].

3. MAIN RESULTS FOR PARABOLIC CASE

Assumption 3.1. (i) Assumptions 2.1 (i) and (ii) are satisfied.

(ii) For any $x \in \mathbb{R}^d$ and $\alpha \in (0, 1]$ the function

$$\int_{B_\alpha} |b(t, x + y)| \, dy$$

is locally integrable to the power $p/(p - 1)$ with respect to $t$. 
Notice that a simple covering argument shows that for any \( \alpha \in (0, \infty) \) the function
\[
\sup_{|x| \leq \alpha} \int_{B_\alpha} |b(t, x + y)| \, dy
\]
is also locally integrable to the power \( p/(p-1) \) with respect to \( t \).

**Assumption 3.2** \((\gamma_b)\). There exists an \( \alpha \in (0, 1] \) such that on \( \mathbb{R}^{d+1} \) (a.e.)
\[
\alpha^{-d} \int_{B_\alpha} \int_{B_\alpha} |b(t, x + y) - b(t, x + z)|^q \, dy \, dz \leq \gamma_b.
\]

**Assumption 3.3** \((\gamma_a)\). There exists an \( \varepsilon_0 > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_0] \), \( s \in \mathbb{R} \), and \( i, j = 1, \ldots, d \), we have
\[
\varepsilon^{-2d+2} \int_s^t \left( \sup_{x \in \mathbb{R}^d} \int_{B \varepsilon} \int_{B \varepsilon} |a^{ij}(t, x+y) - a^{ij}(t, x+z)| \, dy \, dz \right) \, dt \leq \gamma_a. \quad (3.1)
\]

The following is a parabolic analog of the estimate in Theorem 2.1.

**Theorem 3.1.** There exist constants
\[
\gamma_a = \gamma(d, \delta, K, p) > 0, \quad \gamma_b = \gamma(d, \delta, K, p, \varepsilon_0) > 0,
\]
\[
N = N(d, \delta, K, p, \varepsilon_0), \quad \lambda_0 = \lambda_0(d, \delta, K, p, \varepsilon_0, \alpha) \geq 0
\]
such that, if the above assumptions are satisfied, then for any \( T \in [-\infty, \infty) \), \( u \in W^{2,p}_T \), and \( \lambda \geq \lambda_0 \) we have
\[
\lambda \|u\|_{L^p_\mu(\mathbb{R}^{d+1}_T)} + \|D^2u\|_{L^p_\mu(\mathbb{R}^{d+1}_T)} \leq N \|Lu + \partial_t u - \lambda u\|_{L^p_\mu(\mathbb{R}^{d+1}_T)}. \quad (3.2)
\]

Observe that, if the right-hand side of (3.2) is finite, then \( \partial_t u + b^i D_i u \) is in \( L^p_\mu(\mathbb{R}^{d+1}_T) \) and, since \( \partial_t u \) is locally summable, the same is true for \( b^i D_i u \).

Therefore, not surprisingly, to prove the existence of solutions of parabolic equations we impose one more assumption on \( b \), which would guarantee that \( b^i D_i u \) is locally summable if \( u \in W^{2,p}_T \). For \( p < d \) set
\[
q_1 = \frac{pd}{(p-1)d + p}, \quad r_1 = \frac{(p-1)d + p}{(p-1)d},
\]
and for \( p \geq d \) let \( q_1 \in (1, p) \) be any fixed number and
\[
r_1 = \frac{p}{(p-1)q_1}.
\]

Observe that \( 1 < q_1 < q \).

**Assumption 3.4.** For any \( s, t \in \mathbb{R} \), such that \( s < t \), and \( R \in (0, \infty) \) we have
\[
\int_s^t \left( \int_{B_R} |b(\tau, x)|^{q_1} \, dx \right)^{r_1} \, d\tau < \infty. \quad (3.3)
\]

Notice that this assumption coincides with Assumption 3.1 (ii) if \( b \) is independent of \( x \).
Theorem 3.2. Take the constants $\gamma_a, \gamma_b,$ and $\lambda_0$ from Theorem 3.1 and suppose that Assumptions 3.1-3.4 are satisfied. Then for any $\lambda \geq \lambda_0$, $f \in L_p(\mathbb{R}^{d+1})$, and $T \in [-\infty, \infty)$, there is a unique $u \in W^2_p(T)$ such that $\partial_t u + Lu - \lambda u = f$ in $\mathbb{R}^{d+1}$.

We also have a result for the Cauchy problem. Fix $T, S \in \mathbb{R}$ such that $T < S$ and write $u \in W^2_p(T, S)$ if $u \in W^2_p(T)$ and $u(t, x) = 0$ for $t \geq S$.

Theorem 3.3. Take the constants $\gamma_a$, and $\gamma_b$ from Theorem 3.1 and suppose that Assumptions 3.1-3.4 are satisfied. Then for any

$$ f \in L_p((T, S) \times \mathbb{R}^d), \quad v \in W^{1,2}_p((T, \infty) \times \mathbb{R}^d) $$

there exists a unique $u \in W^2_p(T)$ such that $\partial_t u + Lu = f$ in $(T, S) \times \mathbb{R}^d$ and $u - v \in W^2_p(T, S)$.

Proof. It suffices to prove the theorem for the equation $\partial_t u + Lu - \lambda u = f$ with $\lambda$ as large as we like. We take it so large that we can apply Theorem 3.2. Next, we change the coefficients of $L$ for $t \geq S$ if needed in such a way that $L = \Delta$ for $t \geq S$. Finally, we change $f$ for $t \geq S$ if necessary and set it to be $(\partial_t + \Delta - \lambda)(\zeta v)$ for $t \geq S$, where $\zeta(t)$ is any $C^\infty_0(\mathbb{R})$ function such that $\zeta(t) = 1$ for $t \in (T, S)$. With this new objects according to Theorem 3.2, we can find a $\tilde{u} \in W^2_p(T)$ such that $\partial_t \tilde{u} + L\tilde{u} - \lambda \tilde{u} = f$ in $\mathbb{R}^{d+1}_T$. After applying Theorem 3.1 with $S$ and $\tilde{u} - \zeta v$ in place of $T$ and $u$, respectively, we see that $\tilde{u}(t, x) = \zeta(t, x)v(t, x)$ for $t \geq S$. Then $u := \tilde{u} + (1 - \zeta)v$ is obviously a solution we are after. Its uniqueness follows immediately from Theorem 3.1. The theorem is proved.

The above proof allows one to get corresponding estimates for the solution. We leave this to the interested reader.

Proof of Theorem 2.1. Take $\gamma_b, \gamma_a$, and $\lambda_0$ from Theorem 3.1, $f \in L_p(\mathbb{R}^d)$, $\lambda \geq \lambda_0$, and consider the equation

$$ \partial_t v + Lv - \lambda v = e^{-t}f $$

in $\mathbb{R}^{d+1}_0$. As we have pointed out above, we have $q_1 < q$. Therefore, Assumption 2.1 (iii) implies that Assumption 3.4 is satisfied for equation (3.4). Other assumptions stated before Theorem 3.1 are obviously satisfied too. Hence, by Theorem 3.2, equation (3.4) admits a unique solution $v \in W^2_p(0)$. One easily checks that for any $s \geq 0$ the function $v(s + t, x)e^s$ as a function of $(t, x) \in \mathbb{R}^{d+1}_0$ also satisfies (3.4). By uniqueness $v(s + t, x)e^s = v(t, x)$, which implies that $v(s, x) = e^{-s}u(x)$, where $u \in W^2_p(\mathbb{R}^d)$. After that (3.4) is written as $Lu - (\lambda + 1)u = f$. This proves the existence in Theorem 2.1. To prove uniqueness and estimate (2.3) (with $\lambda + 1$ in place of $\lambda$) it suffices to introduce $v(t, x) = e^{-t}u(x)$, observe that $v$ satisfies (3.4), and use Theorem 3.1. The theorem is proved.
4. Auxiliary results

To emphasize which $b$ is used in the definition of the operator $L$, write $L = L_b$.

**Lemma 4.1.** There exist constants
\[ \gamma_\alpha = \gamma(d, \delta, K, p) > 0, \quad N = N(d, \delta, K, p, \varepsilon_0), \]
\[ \lambda_0 = \lambda_0(d, \delta, K, p, \varepsilon_0) \geq 0 \]
such that, if Assumption 3.1 (i) and Assumption 3.3 $(\gamma_\alpha)$ are satisfied, then for any $T \in [-\infty, \infty)$, $u \in W^2_p(T)$, $\lambda \geq \lambda_0$, and any $\mathbb{R}^d$-valued locally integrable to the power $p/(p - 1)$ function $\bar{b} = \bar{b}(t)$ on $\mathbb{R}$ we have
\[ \lambda \|u\|_{L^p(\mathbb{R}^{d+1}_T)} + \|D^2 u\|_{L^p(\mathbb{R}^{d+1}_T)} \leq N \|L \bar{b} u + \partial_t u - \lambda u\|_{L^p(\mathbb{R}^{d+1}_T)}. \] (4.1)

Proof. First assume that $\bar{b} \equiv 0$. Since the coefficients of $L_0$ are bounded and $u \in W^2_p(T)$, the right-hand side of (4.1) is infinite unless $\partial_t u \in L^p(\mathbb{R}^{d+1}_T)$, that is unless $u \in W^{1,2}_{\bar{b}}(\mathbb{R}^{d+1}_T)$. In that case our assertion is true by Theorem 6.4.1 and Remark 6.3.1 of [7].

In the case of general $\bar{b}$ take $u \in W^2_p(T)$ and introduce
\[ B(t) = \int_0^t \bar{b}(s) \, ds, \quad v(t, x) = u(t, x + B(t)), \quad f = L \bar{b} u + \partial_t u - \lambda u. \]
As is easy to see, the function $|\bar{b}(t)| |Du(t, x + B(t))|$ is locally summable in $\mathbb{R}^{d+1}_T$ so that $v \in W^2_p(T)$ and
\[ \partial_t v(t, x) + [a^{ij}(t, x + B(t))D_{ij} - (\lambda + c(t, x + B(t))]v(t, x) = f(t, x + B(t)) =: g(t, x). \]
By the above
\[ \lambda \|v\|_{L^p(\mathbb{R}^{d+1}_T)} + \|D^2 v\|_{L^p(\mathbb{R}^{d+1}_T)} \leq N \|g\|_{L^p(\mathbb{R}^{d+1}_T)}, \]
which immediately yields (4.1). The lemma is proved.

**Remark 4.1.** In [7] the assumption corresponding to Assumption 3.3 is much weaker since in the corresponding counterpart of (3.1) there is no supremum over $x \in \mathbb{R}^d$. We need our stronger assumption because we need $a^{ij}(t, x + B(t))$ to satisfy the assumption in [7] for any function $\bar{b}$.

**Remark 4.2.** The above proof and the results in [7] also show that for any $f \in L^p(\mathbb{R}^{d+1}_T)$ there exists a solution $u \in W^2_p(T)$ of the equation $L \bar{b} u + \partial_t u - \lambda u = f$. Since the solution has the form $v(t, x - B(t))$ with $v \in W^{1,2}_{\bar{b}}(\mathbb{R}^{d+1}_T)$, generally, $\partial_t u$ is only locally summable in $t$.

**Lemma 4.2.** Suppose that Assumptions 3.1 and 3.3 $(\gamma_\alpha)$ are satisfied and let $n \in \{1, 2, \ldots\}$. Then one can find a nonnegative function $\xi \in C^0_c(B_\alpha)$ which integrates to one and a constant $\beta_n = \beta(n, \gamma_\alpha, d, \alpha)$ such that, for
almost any \( t \), we have \( |D^n\overline{b}_\alpha(t,x)| \leq \beta_n \) on \( \mathbb{R}^d \), where \( D^n\overline{b} \) is any derivative of \( \overline{b}_\alpha \) of order \( n \) with respect to \( x \)

\[
\overline{b}_\alpha(t,x) = \int_{B_\alpha} b(t,x-y)\xi(y) \, dy = \int_{\mathbb{R}^d} b(t,y)\xi(x-y) \, dy.
\]

Proof. Take and fix any nonnegative function \( \eta \in C^\infty_0(B_1) \) which integrates to one and set \( \xi(x) = \alpha^{-d} \eta(x/\alpha) \). Due to Assumption 3.1 for almost any \( t \) the function \( b(\cdot, \cdot) \) is locally integrable on \( \mathbb{R}^d \) and hence (for almost any \( t \)) the function \( \overline{b}_\alpha \) is well defined and infinitely differentiable with respect to \( x \). Observe that

\[
D^n\overline{b}_\alpha(t,x) = \int_{B_\alpha} b(t,x-y)D^n\xi(y) \, dy = \int_{B_\alpha} (b(t,x-y) - \overline{b}_\alpha(t,x))D^n\xi(y) \, dy.
\]

It follows that

\[
|D^n\overline{b}_\alpha(t,x)| \leq N \int_{B_\alpha} |b(t,x-y) - \overline{b}_\alpha(t,x)| \, dy
\]

\[
= N \int_{B_\alpha} |b(t,x-y) - \int_{B_\alpha} b(t,x-z)\xi(z) \, dz| \, dy
\]

\[
= N \int_{B_\alpha} \int_{B_\alpha} |b(t,x-y) - b(t,x-z)|\xi(z) \, dz \, dy
\]

\[
\leq N \int_{B_\alpha} \int_{B_\alpha} |b(t,x-y) - b(t,x-z)| \, dz \, dy
\]

and to get our assertion it only remains to use Hölder’s inequality. The lemma is proved.

**Corollary 4.3.** Under Assumptions [3.1] and [3.2] \( \gamma_b \) there exists a locally integrable to the power \( p/(p-1) \) function \( K(t) \) on \( \mathbb{R} \) such that, for almost any \( t \), we have on \( \mathbb{R}^d \) that

\[
g(t,x) := |\overline{b}_\alpha(t,x)| \leq K(t)(1 + |x|) \quad (4.2)
\]

Indeed by Lemma [4.2] we have \( g(t,x) \leq g(t,0) + \beta |x| \) and from Assumption [3.1] (ii) we know that \( g(t,0) \) is locally integrable to the power \( p/(p-1) \).

**5. Proof of Theorem 3.1**

We split the proof into several steps.

**Step 1. Introducing cut-off functions with time-dependent support.** First we take some \( \gamma_b > 0 \) to be specified later, suppose that Assumption [3.2] \( \gamma_b \) is satisfied with some \( \alpha > 0 \), take \( \beta = \beta_1 \) from Lemma [1.2] assume without loss of generality that \( \beta \geq 1 \), and take \( u \in W^2_p(0) \) such that \( u(t,x) = 0 \) for \( t \geq \beta^{-1} \). Next, fix a nonnegative \( \zeta \in C^\infty_0(\mathbb{R}^d) \) with support in \( B_\alpha \) and such that

\[
\int_{B_\alpha} \zeta^p(x) \, dx = 1. \quad (5.1)
\]
Also take a point $x_0 \in \mathbb{R}^d$ and introduce $x(t) = x_{x_0}(t)$ as a solution of the problem

$$x(t) = x_0 + \int_0^t b_\alpha(s, x(s)) \, ds, \quad t \in \mathbb{R},$$

(5.2)

where $b_\alpha$ is introduced in Lemma 4.2. Owing to Lemma 4.2 and Corollary 4.3, equation (5.2) admits a unique solution which is infinitely differentiable with respect to $x_0$ because $b_\alpha(t, x)$ is infinitely differentiable in $x$.

Set

$$\bar{b}_{x_0}(t) = b_\alpha(t, x_{x_0}(t)).$$

$$L_{x_0} = a^{ij}(t, x) D_{ij} + \bar{b}_{x_0}(t) D_i - c(t, x),$$

$$\eta_{x_0}(t, x) = \zeta(x - x_{x_0}(t)), \quad v_{x_0}(t, x) = u(t, x) \eta_{x_0}(t, x),$$

$$f := Lu + \partial_t u - \lambda u.$$

Observe that

$$\partial_t \eta_{x_0}(t, x) + \bar{b}_{x_0}(t) D_i \eta_{x_0}(t, x) = 0,$$

which implies that

$$\partial_t v_{x_0} + L_{x_0} v_{x_0} - \lambda v_{x_0} = \eta_{x_0} (\partial_t u + L_{x_0} u - \lambda u) + u(\partial_t \eta_{x_0} + L_{x_0} \eta_{x_0} + c \eta_{x_0}) + 2a^{ij}(D_i \eta_{x_0}) D_j u$$

$$= \eta_{x_0} f - f_{x_0}^1 + f_{x_0}^2 + f_{x_0}^3,$$

(5.3)

where

$$f_{x_0}^1 = \eta_{x_0} (b_i - \bar{b}_{x_0}^i) D_i u, \quad f_{x_0}^2 = u a^{ij} D_i \eta_{x_0},$$

$$f_{x_0}^3 = 2a^{ij}(D_i \eta_{x_0}) D_j u.$$

Step 2. Estimating the right-hand side of (5.3). Observe that if $\eta_{x_0}(t, x) \neq 0$, then $|x - x_{x_0}(t)| \leq \alpha$ and we may certainly assume that $\zeta^p \leq N(d) \alpha^{-d}$, so that

$$\|f_{x_0}^1\|_{L^p(B_0^d)} = \int_0^\infty \int_{B_0 + x_{x_0}(t)} \| (b_i - \bar{b}_{x_0}^i) \eta_{x_0} D_i u \|^p \, dx \, dt$$

$$\leq N(d) \alpha^{-d} \int_0^\infty I(t) \, dt,$$

where

$$I(t) = \int_{B_0 + x_{x_0}(t)} |(b_i - \bar{b}_{x_0}^i) D_i u|^p(t, x) \, dx.$$

If $p \leq d$ we use Hölder’s inequality, and embedding theorems to obtain

$$I(t) \leq \left( \int_{B_0 + x_{x_0}(t)} |b - \bar{b}_{x_0}|^q \, dx \right)^\frac{p}{q} \left( \int_{B_0 + x_{x_0}(t)} |Du|^\frac{q}{q-p} \, dx \right)^\frac{q-p}{q}$$

$$\leq N \mu^\frac{p}{q}(t, x_{x_0}(t); x_0) [v(t, x_{x_0}(t)) + w(t, x_{x_0}(t))],$$

where $N = N(d, p)$,

$$\mu(t, y, x_0) := \int_{B_0 + y} |b(t, x) - \bar{b}_{x_0}(t)|^q \, dx,$$
\[ v(t, y) = \alpha^{p-p'} \int_{B_{\alpha+y}} |D^2u(t, x)|^p \, dx, \quad w(t, y) = \alpha^{-p-p'} \int_{B_{\alpha+y}} |u(t, x)|^p \, dx, \]

and \( p' = pd/q \ (\leq p) \).

We also note that \( \xi \leq N(d)\alpha^{-d} \) (see Lemma 4.2) and by Assumption 2.2 \((\gamma_b)\) we find that

\[
\mu(t, y, x_0) = \int_{B_{\alpha+x_0}(t)} \int_{B_{\alpha+x_0}(t)} [b(t, x) - b(t, y)]\xi(x_{x_0}(t) - y) \, dy \, dx
\]

\[
\leq N(d)\alpha^{-d} \int_{B_{\alpha+x_0}(t)} \int_{B_{\alpha+x_0}(t)} |b(t, x) - b(t, y)|^\eta \, dx \, dy \leq N(d)\gamma_b.
\]

Hence

\[
I(t) \leq N^{-p/q}_b [v(t, x_{x_0}(t)) + w(t, x_{x_0}(t))],
\]

where \( N = N(d, p) \). This estimate also holds if \( p > d \), which is seen if we start like

\[
I(t) \leq \int_{B_{\alpha+x_0}(t)} |b - \bar{b}_{x_0}|^p \, dx \sup_{B_{\alpha+x_0}(t)} |Du|^p.
\]

Thus,

\[
\| f_{x_0} \|^p_{L^p(\mathbb{R}^{d+1})} \leq N(d, p)\bar{b}_b^{p/q} \alpha^{-d} \int_0^\infty [v(t, x_{x_0}(t)) + w(t, x_{x_0}(t))] \, dt.
\]

The following estimates of \( f^2 \) and \( f^3 \) are straightforward:

\[
\| f_{x_0}^2 \|^p_{L^p(\mathbb{R}^{d+1})} \leq N \int_{\mathbb{R}^d} \int_0^\infty I_{B_{\alpha}}(x_{x_0}(t) - x)|u(t, x)|^p \, dx \, dt,
\]

\[
\| f_{x_0}^3 \|^p_{L^p(\mathbb{R}^{d+1})} \leq N \int_{\mathbb{R}^d} \int_0^\infty I_{B_{\alpha}}(x_{x_0}(t) - x)|Du(t, x)|^p \, dx \, dt,
\]

where \( N = N(d, p, K, \alpha) \).

Now, provided that \( \lambda \geq \lambda_0 \) with \( \lambda_0 \) from Lemma 4.1, equation (5.3) and Lemma 4.4 lead to

\[
\lambda^p \| u_{x_0} \|^p_{L^p(\mathbb{R}^{d+1})} + \| D^2(u_{x_0}) \|^p_{L^p(\mathbb{R}^{d+1})} \leq N_1 \| \eta_{x_0} \|^p_{L^p(\mathbb{R}^{d+1})} + N_1 \bar{b}_b^{p/q} \alpha^{-d} \int_0^\infty [v(t, x_{x_0}(t)) + w(t, x_{x_0}(t))] \, dt
\]

\[
+ N_2 \int_{\mathbb{R}^d} \int_0^\infty I_{B_{\alpha}}(x_{x_0}(t) - x)(|u(t, x)|^p + |Du(t, x)|^p) \, dx \, dt,
\]

where and below by \( N_1 \) we denote generic constants depending only on \( d, \delta, K, p, \varepsilon_0 \) and by \( N_2 \) constants depending only on the same parameters and \( \alpha \). By writing what \( D^2(u_{x_0}) \) is, we conclude

\[
\lambda^p \| u_{x_0} \|^p_{L^p(\mathbb{R}^{d+1})} + \| \eta_{x_0} D^2u \|^p_{L^p(\mathbb{R}^{d+1})} \leq N_1 \| \eta_{x_0} \|^p_{L^p(\mathbb{R}^{d+1})} + N_1 \bar{b}_b^{p/q} \alpha^{-d} \int_0^\infty [v(t, x_{x_0}(t)) + w(t, x_{x_0}(t))] \, dt
\]

\[
+ N_2 \int_{\mathbb{R}^d} \int_0^\infty I_{B_{\alpha}}(x_{x_0}(t) - x)(|u(t, x)|^p + |Du(t, x)|^p) \, dx \, dt.
\]

(5.4)
Step 3. Integrating through (5.4) with respect to \( x_0 \). One knows that for each \( t \), the mapping \( x_0 \rightarrow x_{x_0}(t) \) is a diffeomorphism with Jacobian determinant given by
\[
\left| \frac{\partial x_{x_0}(t)}{\partial x_0} \right| (x_0) = \exp \int_0^t \sum_{i=1}^d [D_i b^i_{x_0}] (s, x_{x_0}(s)) \, ds.
\]
By Lemma 4.2
\[
e^{-N\beta t} \leq \left| \frac{\partial x_{x_0}(t)}{\partial x_0} \right| (x_0) \leq e^{N\beta t},
\]
where \( N \) depends only on \( d \). Therefore, for any nonnegative Lebesgue measurable function \( w(x) \) we have
\[
e^{-N\beta t} \int_{\mathbb{R}^d} w(y) \, dy \leq \int_{\mathbb{R}^d} w(x_{x_0}(t)) \, dx_0 \leq e^{N\beta t} \int_{\mathbb{R}^d} w(y) \, dy.
\]
In particular, since
\[
\int_{\mathbb{R}^d} |\eta_{x_0}(t,x)|^p \, dx_0 = \int_{\mathbb{R}^d} |\zeta(x - x_{x_0}(t))|^p \, dx_0,
\]
we have
\[
e^{-N\beta t} = e^{-N\beta t} \int_{\mathbb{R}^d} |\zeta(x - y)|^p \, dy \leq \int_{\mathbb{R}^d} |\eta_{x_0}(t,x)|^p \, dx_0 \leq e^{N\beta t},
\]
so that
\[
\int_0^\infty \int_{\mathbb{R}^d} [v(t, x_{x_0}(t)) + w(t, x_{x_0}(t))] \, dx_0 \, dt
\]
\[
\leq \int_0^\infty e^{N\beta t} \int_{\mathbb{R}^d} [v(t, y) + w(t, y)] \, dy \, dt
\]
\[
= N(d)\alpha^d \int_0^\infty e^{N\beta t} \int_{\mathbb{R}^d} [\alpha^{p-p'}|D^2 u|^p + \alpha^{-p-p'}|u|^p(t,x)] \, dx \, dt.
\]
Similarly one treats other terms in (5.4). For instance,
\[
\int_{\mathbb{R}^d} \|f\eta_{x_0}\|_{\mathcal{L}_p(\mathbb{R}^{d+1})}^p \, dx_0 \leq \int_0^\infty e^{N\beta t} \int_{\mathbb{R}^d} |f(t,x)|^p \, dx \, dt,
\]
\[
\int_{\mathbb{R}^d} \|u\eta_{x_0}\|_{\mathcal{L}_p(\mathbb{R}^{d+1})}^p \, dx_0 \geq \int_0^\infty e^{-N\beta t} \int_{\mathbb{R}^d} |u(t,x)|^p \, dx \, dt.
\]
We also observe that we need not integrate with respect to \( t \) beyond \( \beta^{-1} \) which allows us to conclude from (5.4) that
\[
\lambda^p \|u\|_{\mathcal{L}_p(\mathbb{R}^{d+1})}^p + \|D^2 u\|_{\mathcal{L}_p(\mathbb{R}^{d+1})}^p \leq N_1 \|f\|_{\mathcal{L}_p(\mathbb{R}^{d+1})}^p
\]
\[
+ N_1 \gamma_b^{p/q} \alpha^{p-p'} \|D^2 u\|_{\mathcal{L}_p(\mathbb{R}^{d+1})}^p + N_2 \|D u\|_{\mathcal{L}_p(\mathbb{R}^{d+1})}^p + N_2 (1 + \gamma_b^{p/q}) \|u\|_{\mathcal{L}_p(\mathbb{R}^{d+1})}^p,
\]
where still \( N_1 = N_1(d, \delta, K, p, \varepsilon_0) \) and \( N_2 \) depends only on the same parameters and \( \alpha \).
We now can specify $\gamma_0(d, \delta, K, p, \varepsilon_0)$. We take it in such a way that $N_1^{p/q} \leq 1/2$. Then (recall that $\alpha \in (0, 1]$ and $p \geq p'$) we obtain

$$
\lambda^p \|u\|_{L^p(\mathbb{R}^{d+1}_0)}^p + \|D^2 u\|_{L^p(\mathbb{R}^{d+1}_0)}^p \leq N_1 \|f\|_{L^p(\mathbb{R}^{d+1}_0)}^p + N_2 \|Du\|_{L^p(\mathbb{R}^{d+1}_0)}^p + N_2 \|u\|_{L^p(\mathbb{R}^{d+1}_0)}^p.
$$

(5.5)

Now we show how to choose $\lambda_0(d, \delta, K, p, \varepsilon_0, \alpha)$. We take it larger than the one from Lemma 4.1 and such that for $\lambda \geq \lambda_0$ interpolation inequalities would allow us to absorb the last two terms in (5.5) into its left-hand side. Then for $\lambda \geq \lambda_0$ we get

$$
\lambda^p \|u\|_{L^p(\mathbb{R}^{d+1})}^p + \|D^2 u\|_{L^p(\mathbb{R}^{d+1})}^p \leq N \|f\|_{L^p(\mathbb{R}^{d+1})}^p
$$

with $N = N(d, \delta, K, p, \varepsilon_0)$, provided that $u \in W^2_p(0)$ is such that $u(t, x) = 0$ for $t \geq \beta^{-1}$.

Step 4. Case of $u$ not compactly supported in $t$. To pass to the general case we take a nonnegative function $\chi \in C^\infty_0(0, 1)$ such that

$$
\int_0^1 \chi^p(t) dt = 1
$$

and set $\kappa_{t_0}(t) = \beta^{1/p} \chi(\beta t - \beta t_0)$. Recall that $\gamma_0$ is fixed above, so that in Lemma 4.2 we have $\beta_1 = \beta = \beta(d, \delta, K, p, \varepsilon_0, \alpha)$. Next, for each $t_0$ we have that $u_{t_0}(t, x) = u(t, x)\kappa_{t_0}(t)$ belongs to $W^2_p(T \vee t_0)$ and $u_{t_0}(t, x) = 0$ for $t \geq T \vee t_0 + \beta^{-1}$. The result of the above particular case implies that

$$
\int_T^\infty \int_{\mathbb{R}^d} |\lambda^p |u|^{p'} + |D^2 u|^{p'}|t, x)\kappa_{t_0}^p(t) dx dt \\
\leq N_1 \int_T^\infty \int_{\mathbb{R}^d} |f|^{p'} \kappa_{t_0}^p + |u|^{p'}(\kappa_{t_0})^{p'}|t, x) dx dt,
$$

provided that $\lambda \geq \lambda_0(d, \delta, K, p, \varepsilon_0, \alpha)$. By integrating through this with respect to $t_0$ over $\mathbb{R}$ we obtain

$$
\lambda^p \|u\|_{L^p(\mathbb{R}^{d+1}_T)}^p + \|D^2 u\|_{L^p(\mathbb{R}^{d+1}_T)}^p \leq N_1 \|f\|_{L^p(\mathbb{R}^{d+1}_T)}^p + N_2 \|u\|_{L^p(\mathbb{R}^{d+1}_T)}^p.
$$

Now it only remains to increase $\lambda_0$ if needed to absorb the last term on the right into the left-hand side.

The theorem is proved.

6. Proof of Theorem 3.2

We will only concentrate on the existence since uniqueness follows from (3.2). We start with the following.

Lemma 6.1. Let $b$ be a measurable function and let Assumption 3.4 be satisfied. Let $b_n(t, x)$, $n = 1, 2, \ldots$, be $\mathbb{R}^d$-valued measurable functions on $\mathbb{R}^{d+1}$ such that

$$
\int_s^t \left( \int_{\mathbb{B}_R} |b_n(\tau, x) - b(\tau, x)|^{p_1} \, dx \right)^{\gamma_1} d\tau \to 0
$$

(6.1)
as long as \( s, t \in \mathbb{R} \), \( s < t \), and \( R \in (0, \infty) \). Finally, let \( u, u_n, Du, Du_n, D^2 u, D^2 u_n \in L_p(\mathbb{R}^{d+1}) \), \( n = 1, 2, \ldots \), and assume that
\[
(u_n, Du_n, D^2 u_n) \to (u, Du, D^2 u)
\]
weakly in \( L_p(\mathbb{R}^{d+1}) \).

Then \( b^i_n D_i u_n \to b^i D_i u \) in the sense of distributions on \( \mathbb{R}^{d+1} \) and \( b^i D_i u \) is locally summable on \( \mathbb{R}^{d+1} \).

Proof. Take \( \kappa > 0 \), \( T < s < t \), \( R > 0 \), and \( \phi, \psi \in C_0^\infty(\mathbb{R}^{d+1}) \) with support in \( Q := (s, t) \times B_R \) such that
\[
\int_s^t \int_{B_R} |b - \psi|^q(\tau, x) \, dx \, d\tau \leq \kappa.
\]
Use the notation \( (g, h) \) for the integral of \( gh \) over \( \mathbb{R}^{d+1} \) and write
\[
I_n := |(\phi, b^i_n D_i u_n) - (\phi, b^i D_i u)| \leq (|\phi|, |b_n - b| |Du_n|)
\]
\[
+ (|\phi|, |b - \psi| |Du_n|) + (|\phi|, |b - \psi| |Du|) + (\phi \psi^i, D_i u_n - D_i u).
\]
Here the last term goes to zero since \( D_i u_n \to D_i u \) weakly. To estimate the remaining terms on the right we use embedding theorems. For instance,
\[
(|\phi|, |b - \psi| |Du_n|) \leq N \int_s^t \int_{B_R} |b - \psi| |Du_n(\tau, x)| \, dx \, d\tau
\]
\[
\leq N \int_s^t J_1(\tau) J_{2,n}(\tau) \, d\tau \leq \left( \int_s^t J_1^{p/(p-1)}(\tau) \, d\tau \right)^{(p-1)/p} \left( \int_s^t J_{2,n}^p(\tau) \, d\tau \right)^1/p,
\]
where
\[
J_1(\tau) = \left( \int_{B_R} |b - \psi|^q_1(\tau, x) \, dx \right)^{1/q_1}
\]
\[
J_{2,n}(\tau) = \left( \int_{B_R} |Du_n(\tau, x)|^{q_1/(q_1 - 1)} \, dx \right)^{(q_1 - 1)/q_1}.
\]
In light of (6.3) and the fact that \( p/[q_1(p-1)] = r_1 \) we have
\[
\int_s^t J_1^{p/(p-1)}(\tau) \, d\tau \leq \kappa.
\]
Furthermore, by embedding theorems and the fact that \( q_1/(q_1 - 1) = pd/(d-p) \) for \( p < d \) and \( q_1/(q_1 - 1) < \infty \) in any case, we obtain
\[
J_{2,n}(\tau) \leq N(\|u_n(\tau, \cdot)\|_{L_p(\mathbb{R}^d)} + \|D^2 u_n(\tau, \cdot)\|_{L_p(\mathbb{R}^d)}),
\]
so that
\[
\lim_{n \to \infty} \int_s^t J_{2,n}^p(\tau) \, d\tau \leq N \lim_{n \to \infty} (\|u_n\|^p_{L_p(\mathbb{R}^{d+1})} + \|D^2 u_n\|^p_{L_p(\mathbb{R}^{d+1})})
\]
which is finite due to the assumed weak convergence of \( u_n \) and \( D^2 u_n \).

Similarly one estimates \(|\phi|, |b - \psi| |Du|\) and \(|\phi|, |b_n - b| |Du_n|\) invoking (6.1) in the case of the latter. Hence
\[
\lim_{n \to \infty} I_n \leq N \kappa^{(p-1)/p}
\]
with $N$ independent of $\kappa$ and, since $\kappa > 0$ is arbitrary, $I_n \to 0$ as $n \to \infty$. The arbitrariness of $\phi$ finishes proving our claim.

The reader might have noticed that the above computations are only valid if we knew that $\phi b^i D_i u$ and $\phi b^i_n D_i u_n$ are summable at least for large $n$. To close this gap it suffices to fix $n$, $u_n = 0$. This would prove that $\phi b^i D_i u$ is summable. Due to (6.1) the functions $b_n$ satisfy (5.3) for all large $n$ and, as for $\phi b^i D_i u$, this implies that $\phi b^i_n D_i u_n$ are summable for large $n$. The lemma is proved.

**Proof of Theorem 3.2.** Recall that $\gamma_a$, $\gamma_b$, and $\lambda_0$ are taken from Theorem 3.1 for $n = 1, 2, \ldots$ define $\kappa_n(t) = (-n) \vee t \wedge n$ and $b^i_n = \kappa_n(b^i)$, and observe that, since the $b_n$ are bounded for each $n$, by Theorem 6.4.1 of [7], for any $\lambda \geq \lambda_0$ and $f \in L_p(\mathbb{R}^{d+1}_T)$, there exist $u_n \in W^{1,2}_p(\mathbb{R}^{d+1}_T)$ satisfying

$$L_{b_n} u_n + \partial_t u_n - \lambda u_n = f$$

(6.4)

in $\mathbb{R}^{d+1}_T$.

Notice that $\kappa_n$ are Lipschitz continuous functions with Lipschitz constant 1. It follows that $b_n$ satisfies Assumption 3.2 ($\gamma_b$) (with the same $\gamma_b$). Hence by Theorem 3.1,

$$\|u_n\|_{L_p(\mathbb{R}^{d+1}_T)} + \|D^2 u_n\|_{L_p(\mathbb{R}^{d+1}_T)} \leq N < \infty,$$

(6.5)

where $N$ is independent of $n$.

Owing to (6.5) there is a subsequence $n' \to \infty$ and a function $u$ such that

$$u_{n'} \to u, \quad Du_{n'} \to Du, \quad D^2 u_{n'} \to D^2 u$$

weakly in $L_p(\mathbb{R}^{d}_T)$. Consequently, also weakly in $L_p(\mathbb{R}^{d}_T)$, we have

$$L_0 u_{n'} \to L_0 u.$$

(6.6)

Next, $|b_n| \leq |b|$ and $b_n \to b$ as $n \to \infty$ (a.e.). By the dominated convergence theorem we have that condition (6.1) is satisfied. Then, according to Lemma 6.1 we have that $b^i_n D_i u_n \to b^i D_i u$ in the sense of distributions on $\mathbb{R}^{d+1}_T$ and $b^i D_i u$ is locally summable on $\mathbb{R}^{d}_T$. Now by passing to the limit in the sense of distributions in (6.4) we see that $Lu + \partial_t u - \lambda u = f$ in $\mathbb{R}^{d+1}_T$ and $\partial_t u$ is locally summable. It follows that $u \in W^{3,2}_p(T)$ and the theorem is proved.

7. An example

Let $d = 1$ and consider the elliptic equation

$$Lu := u'' - 2bxu' - 2u = -2f.$$

(7.1)

We claim that if $b \geq p$, then there exist functions $f \in L_p(\mathbb{R})$ such that equation (7.1) does not have solutions $u \in W^{2,2}_p(\mathbb{R})$. This fact is, actually, known from a very interesting article [10], where the spectrum of the Ornstein-Uhlenbeck operator is found in the multidimensional case in $L_p$ spaces. In the case of (7.1) the result of [10] says that this equation is uniquely solvable in $W^{2,2}_p(\mathbb{R})$ iff $p > b$. We give an independent and short proof of our claim.
for completeness of presentation. A very rough idea why this happens in our situation is that if the resolvent operator of this equation were bounded in $L^p(\mathbb{R})$, then its adjoint would also be bounded, but this adjoint is the resolvent operator of the adjoint equation that has $c$ with a wrong sign.

We know (see, for instance, [8]) that for any $f \in C_0^{\infty}(\mathbb{R})$ there exists a unique smooth and bounded solution $u$ of (7.1). It is well known that $u$ is given by

$$u(x) = \int_{\mathbb{R}^d} f(y) g(x, y) \, dy =: Rf(x),$$

where

$$g(x, y) = \int_0^\infty e^{-t} p(t, x, y) \, dt,$$

$$\sigma^2(t) = \int_0^t e^{-2bs} \, ds, \quad p(t, x, y) = \frac{1}{\sqrt{2\pi\sigma^2(t)}} \exp\left[-\frac{(y-xe^{-bt})^2}{2\sigma^2(t)}\right].$$

By the maximum principle any solution of class $W^2_p(\mathbb{R})$ coincides with $Rf$. Now to prove our claim, it suffices to show that, if $f \geq 0$ on $\mathbb{R}$, then we have $u = Rf \notin L^2(\mathbb{R})$. Observe that

$$u(x) \geq \int_0^\infty e^{-t} \left( \int_0^1 p(t, x, y) \, dy \right) \, dt$$

and since for $t \geq 1$ the function $\sigma^2(t)$ is bounded away from zero and infinity, we have

$$u(x) \geq \varepsilon \int_1^\infty e^{-t} \left( \int_0^1 \exp\left[-N(y-xe^{-bt})^2\right] \, dy \right) \, dt,$$

where $\varepsilon > 0$ and $N$ are some constants. Obviously, the interior integral is bigger than a constant $\varepsilon_1 > 0$ if $0 \leq xe^{-bt} \leq 1$, that is if $x \geq 0$ and $t \geq b^{-1} \log x$. Thus, for $x \geq e^b$

$$u(x) \geq \varepsilon \varepsilon_1 \int_{b^{-1} \log x}^\infty e^{-t} \, dt = \frac{\varepsilon \varepsilon_1}{x^{1/b}},$$

which is not in $L_p(e^b, \infty)$ for $b \geq p$.

**References**

[1] A. Bensoussan and J.-L. Lions, “Applications of variational inequalities in stochastic control”, North-Holland, Amsterdam, 1982.

[2] P. Cannarsa and V. Vespri, *Existence and uniqueness results for a nonlinear stochastic partial differential equation*, in Stochastic Partial Differential Equations and Applications Proceedings, G. Da Prato and L. Tubaro (eds.), Lecture Notes in Math., Vol. 1236, pp. 1-24, Springer Verlag, 1987.

[3] G. Cupini and S. Fornaro, *Maximal regularity in $L^p(\mathbb{R}^N)$ for a class of elliptic operators with unbounded coefficients*, Differential Integral Equations, Vol. 17 (2004), No. 3-4, 259-296.

[4] M. Geissert and A. Lunardi, *Invariant measures and maximal $L^2$ regularity for nonautonomous Ornstein-Uhlenbeck equations*, J. Lond. Math. Soc. (2), Vol. 77 (2008), No. 3, 719-740.

[5] I. Gyöngy, *Stochastic partial differential equations on Manifolds, I*, Potential Analysis, Vol. 2 (1993), 101-113.
[6] I. Gyöngy and N.V. Krylov, *On stochastic partial differential equations with unbounded coefficients*, Potential Analysis, Vol. 1 (1992), No. 3, 233-256.

[7] N.V. Krylov, “Lectures on elliptic and parabolic equations in Sobolev spaces”, Amer. Math. Soc., Providence, RI, 2008.

[8] N.V. Krylov and E. Priola, *Elliptic and parabolic second-order PDEs with growing coefficients*, submitted to Comm in PDEs, [http://arXiv.org/abs/0806.3100](http://arXiv.org/abs/0806.3100)

[9] A. Lunardi and V. Vespri, *Generation of strongly continuous semigroups by elliptic operators with unbounded coefficients in* $L^p(\mathbb{R}^n)$, Rend. Istit. Mat. Univ. Trieste 28 (1996), suppl., 251-279 (1997).

[10] G. Metafune, *$L^p$-spectrum of Ornstein-Uhlenbeck operators*, Annali della Scuola Normale Superiore di Pisa - Classe di Scienze, Sér. 4, Vol. 30, No. 1 (2001), 97-124.

[11] G. Metafune, J. Prüss, R. Schnaubelt, and A. Rhandi, *$L^p$-regularity for elliptic operators with unbounded coefficients*, Adv. Differential Equations, Vol. 10 (2005), No. 10, 1131-1164.

[12] J. Prüss, A. Rhandi, and R. Schnaubelt, *The domain of elliptic operators on* $L^p(\mathbb{R}^d)$ *with unbounded drift coefficients*, Houston J. Math., Vol. 32 (2006), No. 2, 563-576.

127 Vincent Hall, University of Minnesota, Minneapolis, MN, 55455, USA

*E-mail address: krylov@math.umn.edu*