Universal statistics of non-linear energy transfer in turbulent models

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A class of shell models for turbulent energy transfer at varying the inter-shell separation, \( \lambda \), is investigated. Intermittent corrections in the continuous limit of infinitely close shells (\( \lambda \to 1 \)) have been measured. Although the model becomes, in this limit, non-intermittent, we found universal aspects of the velocity statistics which can be interpreted in the framework of log-poisson distributions, as proposed by She and Waymire (1995, Phys. Rev. Lett. 74, 262). We suggest that non-universal aspects of intermittency can be adsorbed in the parameters describing statistics and properties of the most singular structure. On the other hand, universal aspects can be found by looking at corrections to the monofractal scaling of the most singular structure. Connections with similar results reported in other shell models investigations and in real turbulent flows are discussed.

One of the basic questions in understanding the physics of fully developed three dimensional turbulence is the dynamical mechanism characterizing the energy transfer from large to small scales. According to the Kolmogorov theory (K41) of fully developed turbulence, the energy should be transferred downwards from large scale to small scales following a self-similar and homogeneous process entirely dependent on the energy transfer rate, \( \epsilon \), and the scale, \( l \). By assuming local homogeneity and isotropy, the K41 theory allows us to predict the scaling properties of the structure functions \( S_p(l) \equiv \langle |v(x + l) - v(x)|^p \rangle \), which for \( \epsilon \) and \( l \) fixed follow a power-law scaling properties:

\[
S_p(l) \sim l^{-5/3} \epsilon^{p/3} \quad (l \ll \eta) \quad \text{and} \quad S_p(l) \sim \eta^{5/3} \epsilon^{p/3} \quad (l \gg \eta).
\]

The K41 theory has been questioned by several authors because of strong scale dependent fluctuations of the energy dissipation (intermittency). Because of intermittency, the scaling properties of the velocity structure functions acquire anomalous scaling, i.e.,

\[
S_p(l) \sim \eta^5 \epsilon^{p/3} \quad (l \ll \eta) \quad \text{and} \quad S_p(l) \sim l^{-5/3} \epsilon^{p/3} \quad (l \gg l_\eta).
\]

The multifractal character of energy transfer statistics has been also questioned. Different \( \zeta(p) \) exponents have been measured in anisotropic flows (shear flows and boundary layers) and strong dependencies on active quantities as the temperature in Rayleigh-Benard or the magnetic field in MHD have been quoted. The question of finding unifying and universal aspects common to all turbulent flows naturally arises. Recently, cascade descriptions based on random multiplicative processes and infinitely divisible distributions for random multipliers have been applied in order to clarify the energy transfer physics. In particular it has been recently shown that a log-Poisson distribution is able to provide an extremely good fit to experimental data. In authors proposed that the random multipliers \( W_{L_2,L_1} \) connecting velocity fluctuations at different scales \( L_2 \) and \( L_1 \)

\[
\delta v(L_2) = W_{L_2,L_1} \delta v(L_1)
\]

should follow a log-Poisson statistics. Log-poissonity should naturally arise in turbulent flows as the limit of a Bernoulli fragmentation process between infinitely close scales. In authors argued that the energy transfer between two adjacent scales \( l_1 \) and \( l_2 \) with \( \log(l_1) - \log(l_2) << 1 \) can be described in terms of only two bricks. The first one corresponds to the most singular structure characterized by a local scaling exponents \( h_0 \): \( \delta v(l_2) = (l_2/l_1)^{h_0} \delta v(l_1) \). The second brick is a defect-like energy transfer which modulates the most singular events by a factor \( \beta < 1 \); in this second case we would have a typical scaling:

\[
\delta v(l_2) = \beta(l_2/l_1)^{h_0} \delta v(l_1).
\]

Let us also assume that, in the limit \( \log(l_1/l_2) \to 0 \), defects along the cascade happen with a probability that goes to zero proportionally to the logarithm of the scale separation \( \sim d_0 \log(l_1/l_2) \), where \( d_0 \) is the parameter which controls how probable a defect is. It is now simple to show, following authors, that the finite-scale-separation transfer must have a log-Poisson statistics:

\[
< (W_{L_2,L_1})^p >= (L_2/L_1)^{h_0} d_0 \delta P - d_0 (1 - \beta^p),
\]

which corresponds to the She-Leveque proposal for intermittent exponents:

\[
\zeta(p) = h_0 p - d_0 (1 - \beta^p).
\]

The multifractal interpretation of is that \( h_0 \) is the most singular scaling exponent and \( d_0 \) corresponds to the codimension of the fractal set where the most singular scaling is observed. One may argue that the structure and the statistics of the most singular event could be strongly non-universal.

As a consequence, a possible scenario can be proposed where \( h_0 \) and \( d_0 \) could be system-dependent, while \( \beta \) maybe constant in a wider universality class. Some evidences of this universal character of \( \beta \) have already been reported in where it has been shown that the differences of intermittent exponents measured in Rayleigh-Benard, MHD, shear flows and in boundary
layers can all be re-adsorbed by properly changing \( h_0 \) and \( d_0 \) at constant \( \beta \). In [13] it has been shown that also viscous effects can be included in a suitable dependency on the scale of \( h_0 \) and \( d_0 \), showing for the first time that non-trivial intermittent corrections, due to the \( \beta \) dependent part of \( \zeta(p) \) curve, can be detected at viscous scales.

In the following we will show how log-poisson description of intermittency and its underlying interpretation can be applied for describing some important aspects of energy transfer in a class of dynamical models of turbulence [12-14]. In particular, we will be able to explain the continuous transition towards the K41 non-intermittent statistics in terms of the statistical properties of the most singular structure of the model. This trend towards K41 statistics is observed in a class of shell models at varying (diminishing) the characteristic shell separation, i.e. in the so-called continuous limit of infinitely close shells.

Universal aspects of the shell statistics are recovered by measuring the \( \beta \)-dependent part of the probability distribution. As proposed by She and Waymire, we find that the \( \beta \)-dependent part of the intermittent statistics is remarkably constant for all values of the shell separation \( \lambda \) explored.

Shell models have demonstrated to be very useful for the understanding of many properties connected to the non-linear turbulent energy transfer (13-20). The most popular shell model, the Gledzer-Ohkitani-Yamada (GOY) model (13-20), has been shown to predict scaling properties for \( \zeta(p) \) similar to what is found experimentally (for a suitable choice of the free parameters).

The GOY model can be seen as a severe truncation of the Navier-Stokes equations: it retains only one complex mode \( u_n \) as a representative of all Fourier modes in the shell of wave numbers \( k \) between \( k_n = k_0 \lambda^n \) and \( k_{n+1} \), \( \lambda \) being an arbitrary scale parameter (\( \lambda > 1 \)), usually taken equal to 2.

In two recent works [13,14] the GOY model has been generalized in terms of shell variables, \( u_n^+ \) and \( u_n^- \), transporting positive and negative helicity, respectively. These models have at least one inviscid invariant non-positive defined which is very similar to the 3d Navier-Stokes helicity. In the following we will focus on the intermittent properties of one of such models at varying the separation between shells, \( \lambda \). The time evolution for positive-helicity shells is [14]:

\[
\frac{d}{dt} u_n^+ = ik_n(u_{n+1}^-u_{n+1}^- + bu_{n+1}^-u_{n+1}^- + cu_{n-1}^-u_{n-2}^-)^* - \nu k_n^2 u_n^+ + \delta_{n,n_0} f^+,
\]

and the same, but with all helical signs reversed, holds for \( u_n^- \).

The coefficients \( b, c \) are determined imposing inviscid conservation of energy, \( E = \sum_n (|u_n^+|^2 + |u_n^-|^2) \), and helicity, \( H = \sum_n k_n (|u_n^+|^2 - |u_n^-|^2) \). In particular we have:

\[
b = -\frac{(1 + \lambda^2)}{\lambda(\lambda^2 + \lambda^2)} \quad \text{and} \quad c = \frac{(1 - \lambda)}{\lambda(\lambda^2 + \lambda^2)}.
\]

Structure functions for these models are naturally defined as \( S_p(n) = \langle \sqrt{|u_n^p|^2 + |u_n^-|^2} \rangle \sim k_n^{-\zeta(p)} \). Many investigations have been done on the intermittent properties of this model and of the original GOY model [13],[18],[20] at varying the coefficients such as to have different conserved quantities, in order to investigate the importance of inviscid invariants in determining the energy transfer properties. Some evidences supporting non-trivial effects introduced by non-positive invariants have been quoted. In this letter we discuss the dependency on the other important free parameter entering shell-modelization: the separation between neighboring shells \( \lambda \). By decreasing \( \lambda \) one describes turbulent energy transfer in terms of interactions which become more and more local in Fourier space. In the limit of \( \lambda \rightarrow 1 \) one recovers a 1-dimensional partial differential equation [21].

Locality of interactions has always been a long-debated issue of the K41 picture. In fig. 1 we show the 6th order structure function of model [3] for different inter-shell separations: \( \lambda = 2 \) and \( \lambda = 1.05 \). In both cases the total number of shells is chosen such that the physical length of the inertial range stays constant. Clearly, there is a net trend toward a less intermittent state by decreasing the shell-ratio. Nevertheless, the exceptionally good scaling properties allow us to estimate, using the Extended Self Similarity method [2], small deviations from K41 scaling also for the less intermittent value \( \lambda = 1.05 \).

In view of the discussion about log-poisson statistics this analysis could not be sufficient. Non-universal aspects can be masked by trivial properties of the most singular structure. We have therefore tried to highlight possible universal aspects of the chaotic energy transfer by focalizing our attention on the non-linear part of intermittent corrections and in particular on the parameter \( \beta \) which can be extracted by assuming log-poisson intermittency. The best way of extracting \( \beta \) from numerical data is to look if structure functions verify the log-poisson hierarchy [6]:

\[
\frac{S_{p+1}(n)}{S_p(n)} = A_n \left( \frac{S_p(n)}{S_{p-1}(n)} \right)^{\beta'},
\]

where \( \beta' = \beta^{1/3} \) and \( A_n \) has a scaling dependency on \( k_n \) but it is \( p \)-independent. If structure functions follow a log-poisson statistics one can extract \( \beta' \) from a linear fit of a log-log plot of (4), at varying \( p \) and fixed \( n \).

In fig. 2 we show relation (4) for the two inter-shell separations \( \lambda = 2, 1.05 \) and for \( p = 1, \ldots, 7 \). As it is possible to see, the straight line behaviour in the log-log plot is nicely verified (supporting the log-poisson assumption) and the slope of the two lines is the same, indicating that although the full intermittent corrections are different, the part directly affected by the \( \beta \) parameter is the same. In table 1 we collect all our estimates of \( \beta' \) as a function of \( \lambda \). From these data one can safely conclude
that $\beta$ stays constant by approaching the continuous limit $\lambda \to 1$. The remarkable change in the statistics reflected by changes in the $\zeta(p)$ exponents of fig. 1 may be only due to changes of the most singular energy-burst statistics, i.e., to changes of the parameters $h_0$ and $d_0$ in the log-poison jargon. Let us notice that already other authors have focused the attention on burst-like solutions in GOY models as possible explanation of its intermittent properties \[22\]. Moreover, it is very simple to interpret the measured tendency to K41 scaling as the results of a smoothing in the energy transfer due to the increasing local character of dynamical interactions, in the limit $\lambda \to 1$. In other words, we interpret the corrections to K41 for $\lambda \gg 1$ as the consequence of burst-structures travelling along the inertial range. Very singular bursts (with $h_0$ less than the Kolmogorov value 1/3) appear only when there is a relevant mismatch between eddy-turnover times of neighboring shells, i.e. only when shell-ratios are much larger than one. Otherwise, energy tends to flow smoothly toward small scales following K41 statistics. These bursts are non-universal in the sense that their statistics and their degree of singularity is a function of the inter-shell ratio, $\lambda$.

On the other hand, the mechanism underlying burst transfer seems only determined by the non-linear structure of shell-model dynamics and it seems to fix a universal value of $\beta$ in the log-poison description.

To conclude, we have analysed and discussed the intermittent properties of a class of shell models for turbulent energy transfer at varying the inter-shell ratio parameter. We have shown that there exists a suitable limit, $\lambda \to 1$, for which intermittent fluctuations disappear and the scaling properties of the system become close to the K41 theory. By employing a log-poison description of intermittent corrections, we have disentangled the limit $\lambda \to 1$ in two main components: from one hand the most singular structure seems to tend continuously toward a K41 scaling ($h_0 = 1/3$); from the other hand the probability distribution of intermittent fluctuations is still log-Poisson and described by the same defect parameter $\beta$. One possible way of describing this result, in terms of the scaling exponents $\zeta(p)$, consists in saying that the quantity
\[
\rho_{pq} = \frac{\zeta(p) - p/3 \zeta(3)}{\zeta(q) - q/3 \zeta(3)}
\]
(5)
is constant for $\lambda \to 1$. Similar universal properties have already been reported in shell models at varying the dissipation mechanism, i.e. using hyperviscosities \[23\]. Our results are qualitatively similar to what observed in real turbulent flows going from the inertial subrange to the viscous subrange. Indeed in this case anomalous scaling disappears (i.e. $\zeta(p)/\zeta(3) \to p/3$) while $\rho_{pq}$ stays constant, as recently observed in \[1\].

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FIGURE CAPTIONS

- FIGURE 1: Log-log plot of the 6th order structure function vs $k$, for the two cases $\lambda = 2$ (circles) and $\lambda = 1.05$ (diamonds). A linear fit in the inertial range, using Extended Self Similarity, gives: $\zeta(6)/\zeta(3) = 1.983 \pm 0.005$ for $\lambda = 1.05$ and $\zeta(6)/\zeta(3) = 1.76 \pm 0.01$ for $\lambda = 2$.

- FIGURE 2: Log-poison hierarchy (eq. 4) in a log-log plot for $\lambda = 2$ (circles) and $\lambda = 1.05$ (diamonds). For each $\lambda$, we took $p = 1, ..., 7$ for three
different scales $n$ in the inertial range and we shifted the sets along y-axis in order to perform a single linear fit (solid lines).

**TABLE CAPTIONS**

- **TABLE 1**: $\beta'$ at varying $\lambda$. Each value is the average of slopes evaluated as in fig. 2 for all scales $n$ in the inertial range. The theoretical prediction \[ \beta' = (2/3)^{1/3} = 0.87. \]

| $\lambda$ | $\beta'$ |
|-----------|---------|
| 1.05      | 0.86 ± 0.01 |
| 1.2       | 0.88 ± 0.01  |
| 1.5       | 0.88 ± 0.01  |
| 2         | 0.86 ± 0.01  |