Elastodynamic response of an orthotropic layered elastic half-space to time-harmonic loading

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Abstract
The elastodynamics of an orthotropic half-space coated by a thin orthotropic layer is theoretically investigated in this article. We newly propose explicit expressions of free Rayleigh waves in a layered half-space that are dependent on only one unknown constant representing amplitude. The main contribution is on deriving, in a simple manner, the theoretical predictions of far-field Rayleigh wave motion arising from time-harmonic loads using elastodynamic reciprocity theorems. These are the very first closed-form exact solutions found for the forced motion of Rayleigh waves in a layered half-space of orthotropic materials. To demonstrate the theoretical results, computation of Rayleigh wave motion in a jointed rock, including a layer of quartz-schist and a half-space of soil, is considered. We present the phase and group dispersion curves superimposed with the amplitude spectra that provide useful information on wave modes, frequencies, and displacement amplitudes. The inclusion of the amplitude spectra in the dispersion curves is a significant improvement over other dispersion curves currently available in the literature. The analytical predictions are compared with numerical results found by finite element analysis, and they show excellent agreement for the cases of a uniform distributed load and a varying distributed load both applied over a strip on the layer surface. The calculations obtained in the current study could generally be very useful for applications in seismology and materials characterization of coated structures.

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Keywords
Rayleigh waves, layered half-space, orthotropic materials, reciprocity theorem, time-harmonic loads

Introduction

The study of elastic waves in layered half-space structures is of great importance in seismology and geophysics. Seismologists and geophysicists are interested in understanding and predicting the behavior of seismic waves, including P, S, and Rayleigh surface waves, under different earthquake loadings. A numerical analysis of stress wave propagation through a joint rock mass, including three orthogonal sets, was explored. It used orthotropic continua to replace the discontinuous media and transferred the problem into an equivalent continuous model. The dynamic response agreed with the in-situ records and the response obtained by discrete modeling. Rayleigh waves in a self-reinforced layer over an inhomogeneous half-space were studied for geophysical and civil engineering applications. In particular, they discussed the influence of inhomogeneity parameters and layer thickness on the Rayleigh wave phase velocity. In another study, Gupta and Ahmed obtained analytical solutions of Rayleigh waves in a layered structure consisting of an anisotropic layer and a sandy medium. Moreover, Rayleigh waves are commonly employed to inspect and evaluate mechanical and civil engineering structures such as fiber-reinforced polymer-retrofitted concrete structures and composite coating structures.

Fundamentals of propagation of Rayleigh surface waves in a layered half-space are widely available in the literature. The topic is also discussed in detail in numerous research papers. In particular, Achenbach and Keshava found the dispersion curves of Rayleigh waves in a layered half-space and Tiersten explored the effect of thin films on the propagation of guided waves. The approximate characteristic equations for Rayleigh waves in an orthotropic layered half-space with a sliding contact at the interface were reported by Vinh et al. Elastodynamics of a layered half-space including a thin, soft layer with a clamped upper face overlying a half-space was recently considered. The authors derived the non-traditional boundary conditions along the substrate surface combined with the effect of the layer using a long-wave high-frequency procedure.

The problem of Rayleigh wave motions in layered half-spaces induced by forces is rather complicated and thus there are few published works. The conventional approach is based on integral transform techniques, commonly used to solve wave motions in single solids such as a thin plate and a half-space, becomes extremely difficult to use in solving the dynamic responses of Rayleigh waves in an orthotropic layered half-space. Xu and Ma used the direct stiffness method and Fourier transform to investigate the responses of a multilayered half-space to a spatially periodic moving load. They derived the displacement and traction components along layer interfaces for both the internal soil layers and the soil half-space. However, the soil materials were all considered homogeneous and isotropic. Based on the dual vector form and the Hankel transform, elastodynamic responses of a multi-layered transversely isotropic piezoelectric medium
under time-harmonic loadings were studied. To obtain the piezoelectric field, however, a numerical scheme is required to evaluate the complex integrals that involve products of polynomial and Bessel functions. Recently, Bratov et al. proposed an asymptotic hyperbolic-elliptic expression for Rayleigh waves in a layered half-space subjected to a vertical load.

In recent years, another method has been introduced based on the reciprocity theorem, i.e. a relation between displacements, traction components and forces for two different loading states of a body, to obtain the closed-form solutions of guided wave motions. The approach is simpler than the conventional approach using integral transform techniques and applicable to inhomogeneous solids, multilayered structures and anisotropic materials. The validity of the reciprocity-based approach was verified for Rayleigh waves in a half-space; the results found by the reciprocity application are identical to the ones by the integral transform methods.

One of the main contributions of this article is to derive the closed-form solutions of Rayleigh wave motions in an orthotropic layered half-space under time-harmonic loads using reciprocity theorems. The reciprocity approach was successfully employed to calculate Rayleigh wave fields in a layered half-space where both the layer and the half-space were homogeneous and isotropic solids. Preliminary results on the progress involved in the current work have also been reported by the authors. The rest of the paper is organized into four sections. The main purpose of Section 2 is to propose explicit expressions of Rayleigh waves propagating in an orthotropic layered half-space. Section 3 discusses the procedure to obtain the closed-form solutions of Rayleigh wave motion due to various time-harmonic loads by application of reciprocity. Section 4 presents several calculations of Rayleigh wave motions in a jointed rock including a quartz-schist layer and a soil half-space.

Moreover, we present the phase and group dispersion curves superimposed with the amplitude spectra that provide useful information on wave modes, frequencies, and displacement amplitudes. The analytical predictions are compared with numerical results found by finite element analysis, and they show excellent agreement for several loading conditions. The calculations obtained in the current study could generally be beneficial for applications in seismology and materials characterization of coated structures. Major conclusions drawn from the present work are given in Section 5.

**Propagation of free Rayleigh waves in an orthotropic layered half-space**

We propose here the explicit solutions of Rayleigh surface waves traveling in an orthotropic half-space coated by an orthotropic thin layer. These solutions are also important for the computation of Rayleigh wave fields in the layered half-space due to the application of time-harmonic sources using the reciprocity theorem in the following section. Let us begin by considering a layered structure including a layer \( \Omega \) of thickness \( h \) perfectly bonded with a half-space \( \bar{\Omega} \) along \( z = 0 \) of the coordinate system \( (x, z) \) as shown in Figure 1. Both the layer and the half-space are of homogeneous orthotropic elastic solids with the properties indicated by the stiffness matrices \( C \) and \( \bar{C} \), respectively.
The governing equation for elastic waves traveling in an orthotropic medium can be written as
\[ \sigma_{ij,j} = \rho \ddot{u}_i, \quad i, j = x, z, \tag{1} \]
where \( \sigma_{ij} \) represent stresses, \( u_j \) indicate displacements, and \( \rho \) denotes mass density. Equation (1) describes the relationship between stresses and displacements without the presence of force terms. This equation is therefore used to illustrate the propagation of free Rayleigh waves.

In general, guided waves that propagate in a thin layer can be separated into six partial waves, including three incident waves \( P^+, SV^+, SH^+ \) and three reflected waves \( P^-, SV^-, SH^- \). Meanwhile, there are only three partial waves in a half-space since there is no reflection back to the surface of the half-space. In this study, we are interested in a two-dimensional plane strain problem so that the \( SH \) waves are ignored. The displacement fields of the layer may then be expressed as a superposition of four partial waves
\[ u_x = (A_1 e^{ik_1z} + A_2 e^{ik_2z} + A_3 e^{ik_3z} + A_4 e^{ik_4z}) e^{ik(x-ct)} \tag{2} \]
\[ u_z = (W_1 A_1 e^{ik_1z} + W_2 A_2 e^{ik_2z} + W_3 A_3 e^{ik_3z} + W_4 A_4 e^{ik_4z}) e^{ik(x-ct)} \tag{3} \]
where \( k \) is wavenumber in the \( x \)-direction, \( i \) is the imaginary unit, \( c \) denotes phase velocity while \( A_j \) \( (j = 1, 2, 3, 4) \) and \( \hat{A}_j \) \( (j = 1, 2) \) are unknown amplitudes of the partial waves in the layer and the half-space, respectively. The expressions of dimensionless quantities \( W_j, \hat{W}_j \) and parameters \( \alpha_j, \hat{\alpha}_j \) which represent the ratios of the wavenumbers between \( z \)-direction and \( x \)-direction are given in the Appendix.

**Figure 1.** Free Rayleigh waves in a layered half-space.
Note that the corresponding stress components \( \sigma_{xz}, \sigma_{xx}, \sigma_{zz} \) can be easily calculated using Hooke's law. By assuming a perfectly bonded boundary condition at the interface of the layer and the half-space, we may write

\[
\sigma_{zz} = 0; \quad \sigma_{xz} = 0 \quad \text{at} \quad z = -h
\]

Thus

\[
u_x = \hat{u}_x; \quad u_z = \hat{u}_z; \quad \sigma_{zz} = \hat{\sigma}_{zz}; \quad \sigma_{xz} = \hat{\sigma}_{xz} \quad \text{at} \quad z = 0
\]

(6)

Substituting the displacements and stresses into the boundary conditions yields,

\[
TA = 0
\]

(7)

where \( A = [A_1 \ A_2 \ A_3 \ A_4 \ \hat{A}_1 \ \hat{A}_2]^T \) and \( T \) is a six-by-six matrix whose expression is provided in the Appendix. For nontrivial solutions of coefficients in \( A \), the determinant of matrix \( T \) must vanish, i.e. \( \det(T) = 0 \), leading to a relation called the characteristic equation of Rayleigh waves in an orthotropic layered half-space. Solving this equation, which is generally complicated and requires a numerical procedure, results in dispersion curves shown in Figure 3.

Note that, in Eq. (7), there are only five independent equations with six unknowns \( A_1, A_2, A_3, A_4, \hat{A}_1 \) and \( \hat{A}_2 \). We find a general solution in the form of

\[
A_1 = Ad_1; \quad A_2 = Ad_2; \quad A_3 = Ad_3; \quad A_4 = Ad_4; \quad \hat{A}_1 = \hat{A}d_1; \quad \hat{A}_2 = \hat{A}d_2
\]

(8)

where \( A \) represents the only relative amplitude while \( d_1, d_2, d_3, d_4, \hat{d}_1 \) and \( \hat{d}_2 \) are dimensionless coefficients depending on material properties of the layered half-space (see the Appendix).

For the layer, the displacement and stress fields can now be rewritten as

\[
u_x = AU_x(z)e^{ik(x-ct)}, \quad u_z = AU_z(z)e^{ik(x-ct)}
\]

(9)

\[
\sigma_{xx} = ikC_{55}AT_{xx}(z)e^{ik(x-ct)}, \quad \sigma_{xz} = ikC_{55}AT_{xz}(z)e^{ik(x-ct)}
\]

(10)

where

\[
U_x(z) = d_1 e^{ik_1z} + d_2 e^{ik_2z} + d_3 e^{-ik_1z} + d_4 e^{-ik_2z}, \quad U_z(z) = W_1 d_1 e^{ik_1z} + W_2 d_2 e^{ik_2z} - W_1 d_3 e^{-ik_1z} - W_2 d_4 e^{-ik_2z}
\]

(11)

\[
T_{xx}(z) = \frac{D_{31}}{C_{55}} d_1 e^{ik_1z} + \frac{D_{32}}{C_{55}} d_2 e^{ik_2z} + \frac{D_{31}}{C_{55}} d_3 e^{-ik_1z} + \frac{D_{32}}{C_{55}} d_4 e^{-ik_2z},
\]

\[
T_{xz}(z) = (a_1 + W_1) d_1 e^{ik_1z} + (a_2 + W_2) d_2 e^{ik_2z} - (a_1 + W_1) d_3 e^{-ik_1z} - (a_2 + W_2) d_4 e^{-ik_2z}.
\]

(12)

For the half-space,

\[
\hat{u}_x = A\hat{U}_x(z)e^{ik(x-ct)}, \quad \hat{u}_z = iA\hat{U}_z(z)e^{ik(x-ct)}
\]

(13)

\[
\hat{\sigma}_{xx} = i\hat{C}_{55}A\hat{T}_{xx}(z)e^{ik(x-ct)}, \quad \hat{\sigma}_{xz} = -k\hat{C}_{55}A\hat{T}_{xz}(z)e^{ik(x-ct)}
\]

(14)
where
\[
\begin{align*}
\hat{U}_x(z) &= \hat{d}_1 e^{-k_1 z} + \hat{d}_2 e^{-k_2 z}, \\
\hat{U}_z(z) &= \hat{W}_1 \hat{d}_1 e^{-k_1 z} + \hat{W}_2 \hat{d}_2 e^{-k_2 z},
\end{align*}
\]

\[
\begin{align*}
\hat{T}_{xx}(z) &= \frac{\hat{D}_{31}}{\hat{C}_{55}} \hat{d}_1 e^{-k_1 z} + \frac{\hat{D}_{32}}{\hat{C}_{55}} \hat{d}_2 e^{-k_2 z}, \\
\hat{T}_{xz}(z) &= (\hat{\alpha}_1 + \hat{W}_1) \hat{d}_1 e^{-k_1 z} + (\hat{\alpha}_2 + \hat{W}_2) \hat{d}_2 e^{-k_2 z}.
\end{align*}
\]

Here, the terms $C_{ij}$ denote components of material stiffness matrix and the quantities $D_{ij}$ can be found in the Appendix. The Rayleigh wave fields are now expressed via only one unknown amplitude $A$ instead of six, as in Eqs. (2)–(5). This amplitude depends on the loading and will be computed using the reciprocity theorem in the following section.

**Reciprocity application for the calculation of Rayleigh wave motions**

This section aims to derive closed-form solutions of Rayleigh waves in an orthotropic layered half-space subjected to a time-harmonic source. It will be shown that the amplitudes of generated Rayleigh waves are obtained, in a simple manner, using reciprocity relations between two loading states of an elastic body expressed as
\[
\int_{\Omega} (f_j^A u_j^A - f_j^B u_j^B) d\Omega + \int_{\hat{\Omega}} (\hat{f}_j^A \hat{u}_j^A - \hat{f}_j^B \hat{u}_j^B) d\hat{\Omega} = \int_S (\sigma_{ij}^B u_j^A - \sigma_{ij}^A u_j^B) n_i dS + \int_{\hat{S}} (\hat{\sigma}_{ij}^B \hat{u}_j^A - \hat{\sigma}_{ij}^A \hat{u}_j^B) \hat{n}_i d\hat{S}
\]

Here, $S$ and $\hat{S}$ are external boundaries of the layer and the half-space without the interface, respectively, $n$ and $\hat{n}$ are normal vectors, and $f$ represents body force, see Figure 2. The

![Figure 2. Layered half-space under a time-harmonic load.](image-url)
superscripts \( A \) and \( B \) stand for state \( A \) (actual state) and state \( B \) (virtual state), respectively. For this problem, State \( A \) is the Rayleigh wave motion due to the loading, while state \( B \) is generally chosen as a free Rayleigh wave propagating in the layered structure.

We first consider a vertical load applied at \((x_0, z_0)\) in the form of Delta function as

\[
f^A_z = P \delta(z - z_0) \delta(x - x_0) e^{-ikct}
\]  

(18)

Under the loading, both body waves and Rayleigh waves are generated. The body waves, however, rapidly attenuate after several wavelengths making the Rayleigh waves more dominant. In the far-field, the motion of Rayleigh waves propagating in the positive \( x \)-direction can be expressed as a summation of wave modes as

\[
u^m_x = \sum_{m=0}^{\infty} A_m^P U^m_x(z) e^{ikm(x - c_m t)} , \quad u^m_z = \sum_{m=0}^{\infty} A_m^P U^m_z(z) e^{ikm(x - c_m t)}
\]  

(19)

\[
sigma^m_{xx} = \sum_{m=0}^{\infty} ik_m C_{55} A_m^P T^m_{xx}(z) e^{ikm(x - c_m t)} , \quad \sigma^m_{xz} = \sum_{m=0}^{\infty} ik_m C_{55} A_m^P T^m_{xz}(z) e^{ikm(x - c_m t)}
\]  

(20)

and

\[
\hat{v}^m_x = \sum_{m=0}^{\infty} A_m^P \hat{U}^m_x(z) e^{ikm(x - c_m t)} , \quad \hat{v}^m_z = \sum_{m=0}^{\infty} iA_m^P \hat{U}^m_z(z) e^{ikm(x - c_m t)}
\]  

(21)

\[
\hat{\sigma}^m_{xx} = \sum_{m=0}^{\infty} ik_m \hat{C}_{55} A_m^P \hat{T}^m_{xx}(z) e^{ikm(x - c_m t)} , \quad \hat{\sigma}^m_{xz} = \sum_{m=0}^{\infty} -k_m \hat{C}_{55} A_m^P \hat{T}^m_{xz}(z) e^{ikm(x - c_m t)}
\]  

(22)
The integrals property. After some manipulations, Eq. (27) can be simplified where and here the integral along previously in Eqs. (11)–(12) and Eqs. (15)–(16).

We choose virtual state \( B \) as a free Rayleigh wave of mode \( n \) traveling in negative \( x \)-direction written as

\[
\hat{u}_x^n = -B_n U_x^n(z) e^{-ik_n(x+a,t)} \quad \hat{u}_z^n = B_n U_z^n(z) e^{-ik_n(x+a,t)}
\]  

(23)

\[
\hat{\sigma}_{xx}^n = ik_n C_{55} B_n T_{xx}^n(z) e^{-ik_n(x+c_n t)} \quad \hat{\sigma}_{xz}^n = -ik_n C_{55} B_n T_{xz}^n(z) e^{-ik_n(x+c_n t)}
\]  

(24)

and

\[
\hat{u}_x^n = -B_n \hat{U}_x^n(z) e^{-ik_n(x+c_n t)} \quad \hat{u}_z^n = iB_n \hat{U}_z^n(z) e^{-ik_n(x+c_n t)}
\]  

(25)

\[
\hat{\sigma}_{xx}^n = ik_n \hat{C}_{55} B_n \hat{T}_{xx}^n(z) e^{-ik_n(x+c_n t)} \quad \hat{\sigma}_{xz}^n = k_n C_{55} B_n T_{xz}^n(z) e^{-ik_n(x+c_n t)}
\]  

(26)

where \( B_n \) imply the amplitudes of free Rayleigh mode \( n \).

Substituting these expressions into Eq. (17) yields

\[
P B_n U_z^n(z_0) e^{-ik_n(x_0+c_n t)} e^{-ik_n(x_0+a,t)} = \sum_{m=0}^{\infty} \left( \int_{-h}^{0} F_{AB}^{nm} |_{x=b} dz + \int_{0}^{\infty} \hat{F}_{AB}^{nm} |_{x=b} dz \right)
\]  

(27)

where

\[
F_{AB}^{nm} = \sigma_{xx}^n u_{x}^m + \sigma_{xz}^n u_{z}^m - \sigma_{xx}^m u_{x}^n - \sigma_{xz}^m u_{z}^n
\]  

(28)

\[
\hat{F}_{AB}^{nm} = \hat{\sigma}_{xx}^n \hat{u}_{x}^m + \hat{\sigma}_{xz}^n \hat{u}_{z}^m - \hat{\sigma}_{xx}^m \hat{u}_{x}^n - \hat{\sigma}_{xz}^m \hat{u}_{z}^n
\]  

(29)

Here the integral along \( x = a \) is neglected due to the counter-propagating wave property. After some manipulations, Eq. (27) can be simplified as

\[
PU_z^n(z_0) e^{-ik_n(x_0)} = \sum_{m=0}^{\infty} A_{m}^{P+} e^{i(k_m - k_n) b} (C_{55} I_{mn} + \hat{C}_{55} \hat{I}_{mn})
\]  

(30)

where

\[
I_{mn} = \int_{-h}^{0} i[k_n(T_{xx}^n(z) U_x^n(z) - T_{xz}^n(z) U_z^n(z)) + k_m(T_{xx}^n(z) U_x^n(z) - T_{xz}^n(z) U_z^n(z))] dz
\]  

(31)

\[
\hat{I}_{mn} = \int_{0}^{\infty} i[k_n(T_{xx}^n(z) \hat{U}_x^n(z) - \hat{T}_{xz}^n(z) \hat{U}_z^n(z)) + k_m(T_{xx}^n(z) \hat{U}_x^n(z) - \hat{T}_{xz}^n(z) \hat{U}_z^n(z))] dz
\]  

(32)

The integrals \( I_{mn} \) and \( \hat{I}_{mn} \) only yield non-zero values in the case of \( m = n \) because of the orthogonality relation, which has been meticulously discussed by Phan et al. The amplitude of Rayleigh waves is then obtained as

\[
A_{m}^{P+} = \frac{PU_z^n(z_0) e^{-ik_n x_0}}{2(C_{55} I_{mn} + \hat{C}_{55} \hat{I}_{mn})}
\]  

(33)
where

\[
I_{mn} = ik_n \int_{-h}^{0} (T^n_{xx}(z)U^{n}(z) - T^n_{xz}(z)U^{n}_{z}(z))dz
\]

(34)

\[
\hat{I}_{mn} = ik_n \int_{0}^{\infty} (\hat{T}^{n}_{xx}(z)\hat{U}^{n}_{x}(z) - \hat{T}^{n}_{xz}(z)\hat{U}^{n}_{z}(z))dz.
\]

(35)

If we choose state \( B \) in the positive \( x \)-direction, the amplitude is calculated as

\[
A_{m}^{p-} = \frac{PU_{z}^{p}(z_{0})e^{ik_{n}x_{0}}}{2(C_{55}I_{mn} + \hat{C}_{55}\hat{I}_{mn})}.
\]

(36)

Consider now Rayleigh wave motion due to a horizontal load of the form

\[
f^{A}_{z} = Q\delta(z - z_{0})\delta(x - x_{0})e^{-ik_{ct}}
\]

(37)

Following a similar procedure as illustrated using the vertical load, we find the amplitudes of Rayleigh waves as

\[
A_{m}^{Q+} = -\frac{QU_{x}^{p}(z_{0})e^{-ik_{n}x_{0}}}{2(C_{55}I_{mn} + \hat{C}_{55}\hat{I}_{mn})}, \quad A_{m}^{Q-} = \frac{QU_{x}^{p}(z_{0})e^{ik_{n}x_{0}}}{2(C_{55}I_{mn} + \hat{C}_{55}\hat{I}_{mn})}.
\]

(38)

Based on the calculations of Rayleigh wave motion due to a time-harmonic load, we may derive solutions of Rayleigh waves generated by a distribution of loadings. In the case of a uniformly distributed vertical load applied on a strip from \( x = -a \) to \( x = a \), amplitudes of Rayleigh waves in the positive and negative \( x \)-directions are computed as

\[
A_{m}^{U+} = \frac{2A_{m}^{p+} \sin (ka)}{k}, \quad A_{m}^{U-} = \frac{2A_{m}^{p-} \sin (ka)}{k}.
\]

(39)

In case of varying vertical loadings of the form \( \bar{p}(x_{0}) = Pe^{ikx_{0}} \) on a strip from \( x = -a \) to \( x = a \), the amplitudes in the positive and negative \( x \)-directions are

\[
A_{m}^{V+} = 2aA_{m}^{p+}, \quad A_{m}^{V-} = \frac{A_{m}^{p-} \sin (2ka)}{k}.
\]

(40)

Results and discussions

The analytical solutions derived in the earlier sections are illustrated here by computing the Rayleigh wave motion in a rock mass modeled as a quartz-schist layer overlaid a soil half-space. Material properties of the layer and the half-space are given in Table 1. The layer thickness is chosen as \( h = 1000m \). In order to obtain the phase and group velocity dispersion curves of Rayleigh waves in the joint structure via Eq. (7), a computer code is built based on the numerical root-searching algorithms and extrapolation schemes. The displacement amplitudes of the Rayleigh waves generated by vertical and horizontal sources of unit magnitude applied on the free surface are calculated using our proposed method. These results are then integrated into the dispersion curves to highlight the amplitude spectra of different wave modes.
The phase and group velocity dispersion curves superimposed by the displacement amplitudes are plotted in Figure 3. Here, Figure 3(a) and Figure 3(c) show the horizontal-amplitude spectra while Figure 3(b) and Figure 3(d) show the vertical-amplitude spectra. There are upper and lower bounds for the phase velocities in the layered half-space shown in Figure 3(a) and Figure 3(b). In particular, as the thickness of the quartz-schist layer becomes small, the phase velocity of the first Rayleigh mode approaches the value of the Rayleigh wave velocity of the soil half-space (\( \hat{c}_R \)). The transverse wave velocity of the soil half-space (\( \hat{c}_T \)) defines the upper bound. This threshold velocity is a consequence of the existing condition of Rayleigh waves in the half-space that satisfies \( 0 < \rho c^2 < \min(\hat{C}_{11}, \hat{C}_{55}) \). The Rayleigh wave phase velocity of the quartz-schist layer (\( c_R \)) serves as the lower bound. When the dimensionless quantity \( f h \) increases to infinity, all dispersion curves essentially become asymptotic to the lower limit.

With the presence of the amplitude spectra, the curves produced in this study are considerably superior to other dispersion diagrams available in the literature. They facilitate the analysis of seismic waves, such as source localization or earthquake prediction, based on the observed wave amplitudes. In ultrasonic nondestructive evaluation, the superimposed dispersion curves could offer instant optimal selections of wave modes and frequencies, increasing the signal-to-noise ratio and simplifying signal-processing stages.

Analytical predictions of Rayleigh wave fields presented above are now verified using a finite element simulation carried out by COMSOL Multiphysics software. In this benchmark test, we compute the amplitudes of the Rayleigh waves generated by both a uniform distributed load and a varying distributed load applied over a strip of width 2\( a \) on the surface of the 1000m quartz-schist layer. By the use of a 0.1 Hz vertical distributed load, the lowest mode of the Rayleigh waves is triggered.

The horizontal displacement amplitudes (\( u_1 \)) and the vertical displacement amplitudes (\( u_2 \)) of Rayleigh waves are shown in Figure 4 for the uniform distributed loading and Figure 5 for the varying distributed loading. It can be seen in Figure 4 that both displacement amplitudes \( u_1, u_2 \) first increase and then decrease with increase of the width 2\( a \) because of the presence of the term \( \sin ka \) in Eq. (39). For this type of loading, the displacement amplitudes in the positive and negative directions are the same because this is a symmetric problem. On the other hand, the amplitudes of the Rayleigh surface waves due to the varying load in the positive and negative directions are different in Figure 5. Here, the displacement amplitudes in the positive direction are linearly proportional to the strip width where the loads are applied. This can be explained by Eq. 40. The amplitudes in the negative direction, which depend on the

### Table 1. Material properties of the quartz-schist layer and the soil half-space.

| Material       | \( C_{11} \) (GPa) | \( C_{13} \) (GPa) | \( C_{33} \) (GPa) | \( C_{55} \) (GPa) | \( \rho \) (kg / m\(^3\)) |
|----------------|--------------------|--------------------|--------------------|--------------------|--------------------------|
| Quartz-schist  | 13.55              | 3.00               | 12.36              | 3.75               | 2660                     |
| Soil           | 201.92             | 86.54              | 201.92             | 57.69              | 2000                     |
term $\sin 2ka$, increase at low values of $ka$ and then decrease with the increase of the loading width $2a$.

In both cases of loading conditions, comparisons between the analytical prediction and numerical results are in excellent agreement. Small differences appearing in the
comparisons are accounted for by the limitation of the numerical approach in dealing with the problems of multi-mode and dispersive guided waves. Due to the closed-form nature of solutions, analytical calculations are performed swiftly in these testing cases, while numerical simulation takes significant time to reproduce these results. Thus, the proposed method could be appropriate for solving inverse problems such as material characterization or defect quantification, which typically necessitate a large amount of data or a fast forward-problem solver. Furthermore, the most intriguing aspect of our approach is that reciprocity calculations can be feasible for individual modes at high frequencies. Thus, the method can perform effectively in highly dispersive regions with numerous overlapping modes, which could be challenging for numerical simulations.

Conclusions

The motion of Rayleigh waves in an orthotropic half-space coated by a thin orthotropic layer due to the application of time-harmonic loadings is discussed in this article. The explicit expressions of Rayleigh surface waves propagating in a layered half-space and that depend on only one unknown amplitude have been introduced. The main contribution is the derivation of the analytical solutions of far-field Rayleigh waves using elastodynamic reciprocity theorems. The proposed approach has shown an advantage for obtaining the Rayleigh wave fields in a simple manner and in closed-form exact solutions. Computation of Rayleigh wave motion in a jointed rock mass, including a quartz-schist layer and a soil half-space, has been demonstrated for the cases of a uniform distributed load and a varying distributed load both applied over a strip on the surface of the quartz-schist layer. In particular, we have presented the dispersion curves superimposed with the amplitude spectra which could offer useful information on Rayleigh wave modes, frequencies, and displacement amplitudes. The comparisons between the analytical predictions and numerical results by finite element analysis have shown excellent agreement. With the closed-form property and applicability to complex structures, the expressions and results reported in this research might be beneficial for application in seismology and materials characterization of coated half-space. Further studies on the propagation of Rayleigh waves in multilayered half-space can be conducted using the reciprocity theorem. These results can serve as fundamental solutions for scattering problems of Rayleigh waves due to a cavity in structures.

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Le Nguyen et al.

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**Appendix**

*Expressions of \( W_j \) and \( \hat{W}_j \):*

\[
W_j = \frac{\rho c^2 - C_{11} - C_{55}a_j^2}{(C_{13} + C_{55})a_j}, \quad \hat{W}_j = \frac{\hat{C}_{11} - \hat{C}_{55}\alpha_j^2 - \rho \hat{c}^2}{(\hat{C}_{13} + \hat{C}_{55})\hat{\alpha}_j}, \quad j = 1, 2
\]

where \( C_{ij} \) and \( \hat{C}_{ij} \) are components of the stiffness matrices \( C \) and \( \hat{C} \), respectively.

*Expressions of \( \alpha_j \) and \( \hat{\alpha}_j \):*

\[
\alpha_1 = -\alpha_3 = \sqrt{\frac{S + \sqrt{S^2 - 4\mathcal{P}}}{2}}, \quad \alpha_2 = -\alpha_4 = \sqrt{\frac{S - \sqrt{S^2 - 4\mathcal{P}}}{2}}
\]

\[
\hat{\alpha}_1^2 + \hat{\alpha}_2^2 = \hat{S}, \quad \hat{\alpha}_1^2 \hat{\alpha}_2^2 = \hat{\mathcal{P}}
\]

where

\[
S = \frac{(C_{13} + C_{55})^2 - (C_{11} - \rho c^2)C_{33} - (C_{55} - \rho c^2)C_{55}}{C_{33}C_{55}}, \quad \mathcal{P} = \frac{(C_{11} - \rho c^2)(C_{55} - \rho c^2)}{C_{33}C_{55}}
\]

\[
\hat{S} = \frac{(\hat{C}_{11} - \hat{\rho} c^2)\hat{C}_{33} + (\hat{C}_{55} - \hat{\rho} c^2)\hat{C}_{55} - (\hat{C}_{13} + \hat{C}_{55})^2}{\hat{C}_{33}\hat{C}_{55}}, \quad \hat{\mathcal{P}} = \frac{(\hat{C}_{11} - \hat{\rho} c^2)(\hat{C}_{55} - \hat{\rho} c^2)}{\hat{C}_{33}\hat{C}_{55}}
\]

*Matrix \( T \):*

\[
T = \begin{bmatrix}
D_{11}e^{-ik_{1}h} & D_{12}e^{-ik_{2}h} & D_{11}e^{ik_{1}h} & D_{12}e^{ik_{2}h} & 0 & 0 \\
D_{21}e^{-ik_{1}h} & D_{22}e^{-ik_{2}h} & -D_{12}e^{ik_{1}h} & -D_{22}e^{ik_{2}h} & 0 & 0 \\
1 & 1 & 1 & 1 & -1 & -1 \\
W_1 & W_2 & -W & -W_2 & -i\hat{W}_1 & -i\hat{W}_2 \\
D_{11} & D_{12} & D_{11} & D_{12} & -\hat{D}_{11} & -\hat{D}_{12} \\
D_{21} & D_{22} & -D_{21} & -D_{22} & -i\hat{D}_{21} & -i\hat{D}_{22}
\end{bmatrix}
\]
where

\[ D_{ij} = C_{i3} + C_{j3} \alpha_j W_j, \quad D_{2j} = C_{55}(\alpha_j + W_j), \quad D_{3j} = C_{11} + C_{13} \alpha_j W_j, \quad j = 1, 2 \]

\[ \hat{D}_{ij} = \hat{C}_{i3} - \hat{C}_{j3} \hat{\alpha}_j \hat{W}_j, \quad \hat{D}_{2j} = \hat{C}_{55}(\hat{\alpha}_j + \hat{W}_j), \quad \hat{D}_{3j} = \hat{C}_{11} - \hat{C}_{13} \hat{\alpha}_j \hat{W}_j, \quad j = 1, 2 \]

**Expressions of \( d_1, d_2, d_3, d_4, \hat{d}_1, \hat{d}_2 \):**

\[ d_1 = H_{11} \hat{d}_1 + H_{12} \hat{d}_2, \quad d_2 = H_{21} \hat{d}_1 + H_{22} \hat{d}_2 \]

\[ d_3 = H_{31} \hat{d}_1 + H_{32} \hat{d}_2, \quad d_4 = H_{41} \hat{d}_1 + H_{42} \hat{d}_2 \]

\[ \hat{d}_1 = \frac{D_{11}}{s_1} (H_{12} + s_1^2 H_{32}) + \frac{D_{12}}{s_2} (H_{22} + s_2^2 H_{42}) \]

\[ \hat{d}_2 = -\frac{D_{11}}{s_1} (H_{11} + s_1^2 H_{31}) - \frac{D_{12}}{s_2} (H_{21} + s_2^2 H_{41}) \]

where

\[ s_1 = e^{i \kappa_1 h}, \quad s_2 = e^{i \kappa_2 h} \]

\[ H_{11} = \frac{\hat{D}_{11} - D_{12}}{2(D_{11} - D_{12})} + i \frac{\hat{D}_{21} W_2 - D_{22} \hat{W}_1}{2(D_{21} W_2 - D_{22} W_1)} \]

\[ H_{12} = \frac{\hat{D}_{12} - D_{12}}{2(D_{11} - D_{12})} + i \frac{\hat{D}_{22} W_2 - D_{22} \hat{W}_2}{2(D_{21} W_2 - D_{22} W_1)} \]

\[ H_{21} = \frac{D_{11} - \hat{D}_{11}}{2(D_{11} - D_{12})} + i \frac{D_{21} \hat{W}_1 - \hat{D}_{21} W_1}{2(D_{21} \hat{W}_2 - \hat{D}_{21} W_1)} \]

\[ H_{22} = \frac{D_{11} - \hat{D}_{12}}{2(D_{11} - D_{12})} + i \frac{D_{21} \hat{W}_2 - \hat{D}_{22} W_1}{2(D_{21} \hat{W}_2 - \hat{D}_{22} W_1)} \]

\[ H_{31} = \frac{\hat{D}_{11} - D_{12}}{2(D_{11} - D_{12})} - i \frac{\hat{D}_{21} W_2 - D_{22} \hat{W}_1}{2(D_{21} W_2 - D_{22} W_1)} \]

\[ H_{32} = \frac{\hat{D}_{12} - D_{12}}{2(D_{11} - D_{12})} - i \frac{\hat{D}_{22} W_2 - D_{22} \hat{W}_2}{2(D_{21} W_2 - D_{22} W_1)} \]

\[ H_{41} = \frac{D_{11} - \hat{D}_{11}}{2(D_{11} - D_{12})} - i \frac{D_{21} \hat{W}_1 - \hat{D}_{21} W_1}{2(D_{21} \hat{W}_2 - \hat{D}_{21} W_1)} \]

\[ H_{42} = \frac{D_{11} - \hat{D}_{12}}{2(D_{11} - D_{12})} - i \frac{D_{21} \hat{W}_2 - \hat{D}_{22} W_1}{2(D_{21} \hat{W}_2 - \hat{D}_{22} W_1)} \]