THE REVERSE OPERATOR ORBITS ON $\Delta(1)$ AND A CONJECTURE OF PANYUSHEV

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Abstract. We verify conjecture 5.11 of Panyushev [Antichains in weight posets associated with gradings of simple Lie algebras, Math Z 281(3):1191–1214, 2015].

1. Introduction

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. The associated root system is $\Delta = \Delta(\mathfrak{g}, \mathfrak{h}) \subseteq \mathfrak{h}^*$. Recall that a decomposition

$$g = \bigoplus_{i \in \mathbb{Z}} g(i)$$

is a $\mathbb{Z}$-grading of $\mathfrak{g}$ if $[g(i), g(j)] \subseteq g(i + j)$ for any $i, j \in \mathbb{Z}$. In particular, in such a case, $g(0)$ is a Lie subalgebra of $\mathfrak{g}$. Since each derivation of $\mathfrak{g}$ is inner, there exists $h_0 \in g(0)$ such that $g(i) = \{x \in g \mid [h_0, x] = ix\}$. The element $h_0$ is said to be defining for the grading (1).

Without loss of generality, one may assume that $h_0 \in \mathfrak{h}$. Then $\mathfrak{h} \subseteq g(0)$. Let $\Delta(i)$ be the set of roots in $g(i)$. Then we can choose a set of positive roots $\Delta(0)^{+}$ for $\Delta(0)$ such that

$$\Delta^{+} := \Delta(0)^{+} \sqcup \Delta(1) \sqcup \Delta(2) \sqcup \cdots$$

is a set of positive roots of $\Delta(\mathfrak{g}, \mathfrak{h})$. Let $\Pi$ be the corresponding simple roots, and put $\Pi(i) = \Delta(i) \cap \Pi$. Note that the grading (1) is fully determined by $\Pi = \bigsqcup_{i \geq 0} \Pi(i)$. We refer the reader to Ch. 3, §3 of [2] for generalities on gradings of Lie algebras. Each $\Delta(i)$, $i \geq 1$, inherits a poset structure from the usual one of $\Delta^{+}$. That is, let $\alpha$ and $\beta$ be two roots of $\Delta(i)$, then $\beta \geq \alpha$ if and only if $\beta - \alpha$ is a nonnegative integer combination of simple roots.

Recently, Panyushev initiated the study of the rich structure of $\Delta(1)$ in [3]. In particular, he raised five conjectures concerning the $M$-polynomial, $N$-polynomial and the reverse operator of $\Delta(1)$. Note that Conjectures 5.1, 5.2 and 5.12 have been solved by Weng and the author [1]. The current paper aims to handle conjecture 5.11 of [3]. Let us prepare more notation.

Recall that a subset $I$ of a finite poset $(P, \leq)$ is a lower (resp., upper) ideal if $x \leq y$ in $P$ and $y \in I$ (resp. $x \in I$) implies that $x \in I$ (resp. $y \in I$). We collect the lower ideals of $P$ as $\text{J}(P)$. Since antichains of $P$ are non-comparable under $\leq$. We collect the antichains of $P$ as $\text{An}(P)$. For any $x \in P$, let $I_{\leq x} = \{y \in P \mid y \leq x\}$. Given an antichain $A$ of $P$, let $I(A) = \bigcup_{a \in A} I_{\leq a}$. The reverse operator $X$ is defined by $X(A) = \min(P \setminus I(A))$. Since antichains of $P$ are

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in bijection with lower (resp. upper) ideals of $P$, the reverse operator acts on lower (resp. upper) ideals of $P$ as well. Note that the current $\mathcal{X}$ is inverse to the reverse operator $\mathcal{X}'$ in Definition 1 of [3], see Lemma [2.1]. Thus replacing $\mathcal{X}'$ by $\mathcal{X}$ does not affect our forthcoming discussion on orbits.

We say the $\mathbb{Z}$-grading (1) is extra-special if

\begin{equation}
\mathfrak{g} = \mathfrak{g}(-2) \oplus \mathfrak{g}(-1) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2) \quad \text{and} \quad \dim \mathfrak{g}(2) = 1,
\end{equation}

Up to conjugation, any simple Lie algebra $\mathfrak{g}$ has a unique extra-special $\mathbb{Z}$-grading. Without loss of generality, we assume that $\Delta(2) = \{ \theta \}$, where $\theta$ is the highest root of $\Delta^+$. Namely, we may assume that the grading (2) is defined by the element $\theta^\vee$, the dual root of $\theta$. In such a case, we have

\begin{equation}
\Delta(1) = \{ \alpha \in \Delta^+ \mid (\alpha, \theta^\vee) = 1 \}.
\end{equation}

Let $ht$ be the height function. Recall that $h := ht(\theta) + 1$ is the Coxeter number of $\Delta$. Let $h^*$ be the dual Coxeter number of $\Delta$. That is, $h^*$ is the height of $\theta^\vee$ in $\Delta^\vee$. As noted on p. 1203 of [3], we have $|\Delta(1)| = 2h^* - 4$. We call a lower (resp. upper) ideal $I$ of $\Delta(1)$ Lagrangian if $|I| = h^* - 2$. Write $\Delta_l$ (resp. $\Pi_l$) for the set of all (resp. simple) long roots. In the simply-laced cases, all roots are assumed to be both long and short. Note that $\theta$ is always long, while $\theta^\vee$ is always short.

Now Conjecture 5.11 of [3] is stated as follows.

**Panyushev conjecture.** In any extra-special $\mathbb{Z}$-grading of $\mathfrak{g}$, the number of $\mathcal{X}_{\Delta(1)}$-orbits equals $|\Pi_l|$, and each orbit is of size $h - 1$. Furthermore, if $h$ is even (which only excludes the case $A_{2k}$ where $h = 2k + 1$), then each $\mathcal{X}_{\Delta(1)}$-orbit contains a unique Lagrangian lower ideal.

Originally, the conjecture is stated in terms of upper ideals and the reverse operator $\mathcal{X}'$. One agrees that we can equivalently phrase it using lower ideals and $\mathcal{X}$. The main result of the current paper is the following.

**Theorem 1.1.** *Panyushev conjecture is true.*

After collecting necessary preliminaries in Section 2, the above theorem will be proven in Section 3. Moreover, we note that by our calculations in Section 3, one checks easily that for any extra-special 1-standard $\mathbb{Z}$-grading of $\mathfrak{g}$, all the statements of Conjecture 5.3 in [3] hold.

**Notation.** Let $\mathbb{N} = \{0, 1, 2, \ldots \}$, and let $\mathbb{P} = \{1, 2, \ldots \}$. For each $n \in \mathbb{P}$, $[n]$ denotes the poset $\{1, 2, \ldots , n\}, \leq$.

2. **Preliminaries**

Let us collect some preliminary results in this section. Firstly, let us compare the two reverse operators. Let $(P, \leq)$ be any finite poset. For any $x \in P$, let $I_{\geq x} = \{ y \in P \mid y \geq x \}$. For any antichain $A$ of $P$, put $I_+(A) = \bigcup_{\alpha \in A} I_{\geq \alpha}$. Recall that in Definition 1 of [3], the reverse operator $\mathcal{X}'$ is given by $\mathcal{X}'(A) = \max(P \setminus I_+(A))$.

**Lemma 2.1.** The operators $\mathcal{X}$ and $\mathcal{X}'$ are inverse to each other.
Proof. Take any antichain \( A \) of \( P \), note that
\[
I_+(\min(P \setminus I(A))) = P \setminus I(A) \quad \text{and} \quad I(\max(P \setminus I_+(A))) = P \setminus I_+(A).
\]
Then the lemma follows. \( \square \)

Let \( (P_1, \leq) \), \( i = 1, 2 \) be two finite posets. One can define a poset structure on \( P_1 \times P_2 \) by setting \( (u_1, v_1) \leq (u_2, v_2) \) if and only if \( u_1 \leq u_2 \) in \( P_1 \) and \( v_1 \leq v_2 \) in \( P_2 \). We simply denote the resulting poset by \( P_1 \times P_2 \). The following well-known lemma describes the lower ideals of \([m] \times P\).

**Lemma 2.2.** Let \( P \) be a finite poset. Let \( I \) be a subset of \([m] \times P\). For \( 1 \leq i \leq m \), denote \( I_i = \{ a \in P \mid (i, a) \in I \} \). Then \( I \) is a lower ideal of \([m] \times P\) if and only if each \( I_i \) is a lower ideal of \( P \), and \( I_m \subseteq I_{m-1} \subseteq \cdots \subseteq I_1 \).

In this section, by a *finite graded poset* we always mean a finite poset \( P \) with a rank function \( r \) from \( P \) to the positive integers \( \mathbb{P} \) such that all the minimal elements have rank 1, and \( r(x) = r(y) + 1 \) if \( x \) covers \( y \). In such a case, let \( P_i \) be the set of elements in \( P \) with rank \( i \). The sets \( P_i \) are said to be the *rank levels of \( P \)*. Suppose that \( P = \bigsqcup_{j=1}^d P_j \). Let \( P_0 \) be the empty set \( \emptyset \). Put \( L_i = \bigsqcup_{j=1}^i P_j \) for \( 1 \leq j \leq d \), and let \( L_0 \) be the empty set. We call those \( L_i \) rank level lower ideals.

Let \( X \) be the reverse operator on \([m] \times P\). In view of Lemma 2.2, we denote by \((I_1, \cdots, I_m)\) a general lower ideal of \([m] \times P\), where each \( I_i \in J(P) \) and \( I_m \subseteq \cdots \subseteq I_1 \). We say that the lower ideal \((I_1, \cdots, I_m)\) is full rank if each \( I_i \) is a rank level lower ideal of \( P \). Let \( O(I_1, \cdots, I_m) \) be the \( X_{[m] \times P} \)-orbit of \((I_1, \cdots, I_m)\). The following lemma will be helpful in determining \( X_{[m] \times P} \)-orbits consisting of rank level lower ideals.

**Lemma 2.3.** Keep the notation as above. Then for any \( n_0 \in \mathbb{N} \), \( n_i \in \mathbb{P} \) (\( 1 \leq i \leq s \)) such that \( \sum_{i=0}^s n_i = m \), we have
\[
X_{[m] \times P}(L_{d_0}^{n_0}, L_{i_1}^{n_1}, \cdots, L_{i_s}^{n_s}) = (L_{i_1+1}^{n_1}, L_{i_2+1}^{n_2}, \cdots, L_{i_s+1}^{n_s-1}, L_0^{n_0-1}),
\]
where \( 0 \leq i_s < \cdots < i_1 < d \), \( L_{d_0}^{n_0} \) denotes \( n_0 \) copies of \( L_0 \) and so on.

**Proof.** Note that under the above assumptions, \((L_{d_0}^{n_0}, L_{i_1}^{n_1}, \cdots, L_{i_s}^{n_s})\) is a lower ideal of \([m] \times P\) in view of Lemma 2.2. Then analyzing the minimal elements of \(([m] \times P) \setminus (L_{d_0}^{n_0}, L_{i_1}^{n_1}, \cdots, L_{i_s}^{n_s})\) leads one to (4). \( \square \)

**Lemma 2.4.** Let \((I_1, \cdots, I_m)\) be an arbitrary lower ideal of \([m] \times P\). Then \((I_1, \cdots, I_m)\) is full rank if and only if each lower ideal in the orbit \( O(I_1, \cdots, I_m) \) is full rank.

**Proof.** Use Lemma 2.3. \( \square \)

The above lemma tells us that there are two types of \( X \)-orbits: in the first type each lower ideal is full rank, while in the second type each lower ideal is not. We call them *type I* and *type II*, respectively.

For any \( n \geq 2 \), let \( K_{n-1} = [n-1] \oplus ([1] \sqcup [1]) \oplus [n-1] \) (the ordinal sum, see p. 246 of [4]). We label the elements of \( K_{n-1} \) by \( 1, 2, \cdots, n-1, n, n', n+1, \cdots, 2n-2, 2n-1 \). Figure 1 illustrates the labeling for the Hasse diagram of \( K_3 \). Note that \( L_i \) (\( 0 \leq i \leq 2n-1 \)) are all
the full rank lower ideals. For instance, we have $L_n = \{1, 2, \cdots, n, n'\}$. Moreover, we put $I_n = \{1, \cdots, n-1, n\}$ and $I_{n'} = \{1, \cdots, n-1, n'\}$. The following lemma will be helpful in analyzing the $X_{[m]} \times K_{n-1}$-orbits of type II.

**Lemma 2.5.** Fix $n_0 \in \mathbb{N}$, $n_i \in \mathbb{P}$ ($1 \leq i \leq s$), $m_j \in \mathbb{P}$ ($0 \leq j \leq t$) such that $\sum_{i=0}^{s} n_i + \sum_{j=0}^{t} m_j = m$. Take any $0 \leq j_t < \cdots < j_1 < n \leq i_s < \cdots < i_1 < 2n - 1$, we have

$$X_{[m]} \times K_{n-1}(L_0^{n_0}, L_1^{n_1}, \cdots, L_s^{n_s}, I_0^{m_0}, L_1^{m_1}, \cdots, L_t^{m_t}) =$$

$$\begin{cases}
(L_{i_1+1}^{n_0+1}, L_{i_2+1}^{n_1}, \cdots, L_{i_s+1}^{n_s}, I_{n'}, L_{j_1+1}^{m_0}, L_{j_2+1}^{m_1}, \cdots, L_{j_t+1}^{m_t-1}, L_0^{m_t-1}) & \text{if } j_1 < n - 1; \\
(L_{i_1+1}^{n_0+1}, L_{i_2+1}^{n_1}, \cdots, L_{i_s+1}^{n_s}, I_{n'}, L_{j_1+1}^{m_0}, L_{j_2+1}^{m_1}, \cdots, L_{j_t+1}^{m_t-1}, L_0^{m_t-1}) & \text{if } j_1 = n - 1.
\end{cases}$$

**Proof.** Analyzing the minimal elements of

$$( [m] \times K_{n-1}) \setminus (L_0^{n_0}, L_1^{n_1}, \cdots, L_s^{n_s}, I_0^{m_0}, L_1^{m_1}, \cdots, L_t^{m_t})$$

leads one to the desired expression. \qed

3. **Panyushev conjecture**

This section is devoted to proving Theorem 1.1.
Thus the type I orbit $O(L_i, L_i)$ consists of $2n + 1$ elements. Moreover, in this orbit, $(L_{2n-1}, L_{2n-2})$ is the unique ideal with size $2n$ when $i$ is even, $(L_{n+1}, L_n)$ is the unique ideal with size $2n$ when $i$ is odd. Similarly, the orbit $O(L_0, L_0)$ consists of $2n - 1$ elements and contains a unique ideal with size $2n$: $(L_{n-1}, L_{n-2})$. Since there are $(n-1)(2n-1)$ lower ideals in $[2] 	imes [2n-3]$ by Lemma 2.2, one sees that all the $X$-orbits have been exhausted, and Theorem 1.1 holds for $B_n$.

Let us consider $D_{n+2}$, where the extra-special $\Delta(1) \cong [2] \times K_{n-1}$. We adopt the notation as in Section 2. For simplicity, we write $X_{[2] \times K_{n-1}}$ by $X$. We propose the following.

**Claim.** $O(L_i, L_i)$, $0 \leq i \leq n - 1$, $O(L_n, L_n)$, and $O(L_{n'}, L_{n'})$ exhausts the orbits of $X$ on $[2] \times K_{n-1}$. Moreover, each orbit has size $2n + 1$ and contains a unique lower ideal with size $2n$.

Indeed, firstly, for any $0 \leq i \leq n - 1$, observe that by Lemma 2.3 we have

$$
\begin{align*}
X(L_i, L_i) &= (L_{i+1}, L_0), \\
X^{2n-i-2}(L_{i+1}, L_0) &= (L_{2n-1}, L_{2n-i-2}), \\
X(L_{2n-1}, L_{2n-i-2}) &= (L_{2n-i-1}, L_{2n-i-1}), \\
X(L_{2n-i-1}, L_{2n-i-1}) &= (L_{2n-i}, L_0), \\
X^{i-1}(L_{2n-i}, L_0) &= (L_{2n-i-1}, L_{i-1}), \\
X(L_{2n-1}, L_{i-1}) &= (L_i, L_i).
\end{align*}
$$

Thus the type I orbit $O(L_i, L_i)$ consists of $2n + 1$ elements. Moreover, in this orbit, $(L_{2n-i+1}, L_{2n-i-1})$ is the unique ideal with size $2n$ when $i$ is odd, $(L_{n+1}, L_n)$ is the unique ideal with size $2n$ when $i > 0$ is even, while $(L_n, L_{n-1})$ is the unique ideal with size $2n$ when $i = 0$. 

**Proof of Theorem 1.1.** Note that when $g$ is $A_n$, the extra-special $\Delta(1) \cong [n-1] \sqcup [n-1]$; when $g$ is $C_n$, the extra-special $\Delta(1) \cong [2n-2]$. One can verify Theorem 1.1 for these two cases without much effort. We omit the details.

For $g = B_n$, the extra-special $\Delta(1) = [2] \times [2n-3]$. Now $|\Pi| = n - 1$, $h = 2n - 1$, and $h^* - 2 = 2n - 3$. As in Section 2, let $L_i$ ($0 \leq i \leq 2n - 3$) be the rank level lower ideals. For simplicity, we simply denote $X_{[2] \times [2n-3]}$ by $X$. For any $1 \leq i \leq n - 2$, let us analyze the type I $X$-orbit $O(L_i, L_i)$ via the aid of Lemma 2.3.

$$
\begin{align*}
X(L_i, L_i) &= (L_{i+1}, L_0), \\
X^{2n-1-i}(L_{i+1}, L_0) &= (L_{2n-3}, L_{2n-i}), \\
X(L_{2n-3}, L_{2n-i}) &= (L_{2n-3-i}, L_{2n-i-1}), \\
X(L_{2n-3-i}, L_{2n-3-i}) &= (L_{2n-2-i}, L_0), \\
X^{i-1}(L_{2n-2-i}, L_0) &= (L_{2n-3}, L_{i-1}), \\
X(L_{2n-3}, L_{i-1}) &= (L_i, L_i).
\end{align*}
$$

Thus $O(L_i, L_i)$ consists of $2n - 1$ elements. Moreover, in this orbit, $(L_{2n-2}, L_{2n-4})$ is the unique ideal with size $2n$ when $i$ is odd, $(L_{n+1}, L_n)$ is the unique ideal with size $2n$ when $i$ is even. Similarly, the orbit $O(L_0, L_0)$ consists of $2n - 1$ elements and contains a unique ideal with size $2n$: $(L_{n-1}, L_{n-2})$. Since there are $(n-1)(2n-1)$ lower ideals in $[2] \times [2n-3]$ by Lemma 2.2, one sees that all the $X$-orbits have been exhausted, and Theorem 1.1 holds for $B_n$. 

Let us consider $D_{n+2}$, where the extra-special $\Delta(1) \cong [2] \times K_{n-1}$. We adopt the notation as in Section 2. For simplicity, we write $X_{[2] \times K_{n-1}}$ by $X$. We propose the following.

**Claim.** $O(L_i, L_i)$, $0 \leq i \leq n - 1$, $O(L_n, L_n)$, and $O(L_{n'}, L_{n'})$ exhausts the orbits of $X$ on $[2] \times K_{n-1}$. Moreover, each orbit has size $2n + 1$ and contains a unique lower ideal with size $2n$.

Indeed, firstly, for any $0 \leq i \leq n - 1$, observe that by Lemma 2.3 we have

$$
\begin{align*}
X(L_i, L_i) &= (L_{i+1}, L_0), \\
X^{2n-i-2}(L_{i+1}, L_0) &= (L_{2n-1}, L_{2n-i-2}), \\
X(L_{2n-1}, L_{2n-i-2}) &= (L_{2n-i-1}, L_{2n-i-1}), \\
X(L_{2n-i-1}, L_{2n-i-1}) &= (L_{2n-i}, L_0), \\
X^{i-1}(L_{2n-i}, L_0) &= (L_{2n-i-1}, L_{i-1}), \\
X(L_{2n-1}, L_{i-1}) &= (L_i, L_i).
\end{align*}
$$

Thus the type I orbit $O(L_i, L_i)$ consists of $2n + 1$ elements. Moreover, in this orbit, $(L_{2n-i+1}, L_{2n-i-1})$ is the unique ideal with size $2n$ when $i$ is odd, $(L_{n+1}, L_n)$ is the unique ideal with size $2n$ when $i > 0$ is even, while $(L_n, L_{n-1})$ is the unique ideal with size $2n$ when $i = 0$. 


Secondly, assume that \( n \) is even and let us analyze the orbit \( \mathcal{O}(I_n, I_n) \). Indeed, by Lemma \ref{lem:2.5} we have

\[
\begin{align*}
\mathcal{X}(I_n, I_n) &= (I_n', L_0), \\
\mathcal{X}^{-1}(I_n', L_0) &= (I_n, L_{n-1}), \\
\mathcal{X}(I_{n'}, I_{n-1}) &= (L_n, I_n), \\
\mathcal{X}^{-1}(L_n, I_n) &= (I_{2n-1}, I_n'), \\
\mathcal{X}(L_{2n-1}, I_n') &= (I_n, I_n).
\end{align*}
\]

Thus the type II orbit \( \mathcal{O}(I_n, I_n) \) consists of \( 2n + 1 \) elements. Moreover, in this orbit, \( (I_n, I_n) \) is the unique ideal with size \( 2n \). The analysis of the orbit \( \mathcal{O}(I_{n'}, I_{n'}) \) is entirely similar.

Finally, assume that \( n \) is odd and let us analyze the orbit \( \mathcal{O}(I_n, I_n) \). Indeed, by Lemma \ref{lem:2.5} we have

\[
\begin{align*}
\mathcal{X}(I_n, I_n) &= (I_n', L_0), \\
\mathcal{X}^{-1}(I_n', L_0) &= (I_n, L_{n-1}), \\
\mathcal{X}(I_{n'}, I_{n-1}) &= (L_n, I_n'), \\
\mathcal{X}^{-1}(L_n, I_n') &= (I_{2n-1}, I_n'), \\
\mathcal{X}(L_{2n-1}, I_n') &= (I_n, I_n).
\end{align*}
\]

Thus the type II orbit \( \mathcal{O}(I_n, I_n) \) consists of \( 2n + 1 \) elements. Moreover, in this orbit, \( (I_n, I_n) \) is the unique ideal with size \( 2n \). The analysis of the orbit \( \mathcal{O}(I_{n'}, I_{n'}) \) is entirely similar.

To sum up, we have verified the claim since there are \((n + 2)(2n + 1)\) lower ideals in \( \mathbb{Z} \times K_{n−1} \) by Lemma \ref{lem:2.2}. Note that \(|\Pi_l| = n + 2\), \( h = h^* = 2n + 2 \) for \( g = D_{n+2} \), one sees that Theorem \ref{thm:1.1} holds for \( D_{n+2} \).

Theorem \ref{thm:1.1} has been verified for all exceptional Lie algebras using \texttt{Mathematica}. We only present the details for \( E_6 \), where \( \Delta(1) = [\alpha_2] \), and the Dynkin diagram is as follows.

\[
\begin{align*}
\circ \alpha_6 & \circ \alpha_5 & \circ \alpha_4 & \circ \alpha_3 & \circ \alpha_2 \\
\end{align*}
\]

Note that \(|\Pi_l| = 6\), \( h - 1 = 11\), \( h^* - 2 = 10\). On the other hand, \( \mathcal{X} \) has six orbits on \( \Delta(1) \), each has 11 elements. Moreover, the size of the lower ideals in each orbit is distributed as follows:

\begin{itemize}
  \item 0, 1, 2, 4, 7, \textbf{10}, 13, 16, 18, 19, 20;
  \item 3, 4, 5, 6, 9, \textbf{10}, 11, 14, 15, 16, 17;
  \item 3, 4, 5, 6, 9, \textbf{10}, 11, 14, 15, 16, 17;
  \item 7, 7, 8, 9, \textbf{10}, 11, 12, 13, 13;
  \item 5, 6, 6, 9, \textbf{10}, 11, 12, 14, 14, 15;
  \item 7, 7, 8, 9, \textbf{10}, 11, 12, 12, 13, 13.
\end{itemize}

One sees that each orbit has a unique Lagrangian lower ideal.

This finishes the proof of Theorem \ref{thm:1.1}. \( \square \)
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