THEORY OF INCOMPRESSIBLE MAGNETOHYDRODYNAMIC TURBULENCE WITH SCALE-DEPENDENT ALIGNMENT AND CROSS-HelicITY

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ABSTRACT

A phenomenological anisotropic theory of MHD turbulence with nonvanishing cross-helicity is constructed based on Boldyrev’s phenomenology and probabilities p and q for fluctuations δu⊥ and δb⊥ to be positively or negatively aligned. The positively aligned fluctuations occupy a fractional volume p and the negatively aligned fluctuations occupy a fractional volume q. Guided by observations suggesting that the normalized cross-helicity σc and the probabilities p and q are approximately scale invariant in the inertial range, a generalization of Boldyrev’s theory is derived that depends on the three ratios w+/w−, ε+ε−, and p/q. It is assumed that the cascade processes for positively and negatively aligned fluctuations are both in a state of critical balance and that the eddy geometries are scale invariant. The theory reduces to Boldyrev’s original theory when σc = 0, ε+ = ε−, and p = q and extends the theory of Perez and Boldyrev when σc ≠ 0. The theory is also an anisotropic generalization of the theory of Dobrowolny, Mangeney, and Veltri.

Key words: solar wind – turbulence – waves

Online-only material: color figures

1. INTRODUCTION

Phenomenological theories of incompressible MHD turbulence that take into account the anisotropy of the fluctuations with respect to the direction of the mean magnetic field B0 were pioneered in the 1980s by Montgomery & Turner (1981, 1982), Higdon (1984), and others. An important idea to emerge from this early work, an idea further developed by Goldreich & Sridhar (1995, 1997), is the idea that the timescales (coherence times) for motions of a turbulent eddy parallel and perpendicular to B0 are equal to each other and that this unique timescale defines the energy cascade time of the turbulence. This concept, called critical balance by Goldreich & Sridhar (1995, 1997), leads to the perpendicular energy spectrum E(k⊥) ∝ k⊥−5/3 and to the scaling relation k⊥ ∝ k2/3⊥ describing the anisotropy of the turbulent eddies in their theory (see also Sridhar 2010).

The decade following the publication of the paper by Goldreich & Sridhar (1995) was a time when numerical simulations of three-dimensional (3D) MHD turbulence led to some interesting new results. Simulations of incompressible 3D MHD turbulence during this time showed that when the mean magnetic field is sufficiently strong, B2 0 >> (δB)2, the perpendicular energy spectrum exhibits a power-law scaling closer to k−3/2 than k−5/3 (Maron & Goldreich 2001; Müller et al. 2003; Müller & Grappin 2005). Motivated by this result, Boldyrev modified the Goldreich & Sridhar theory to explain the k−3/2 power law seen in simulations (Boldyrev 2005, 2006). A new concept that emerged in Boldyrev’s theory is the scale-dependent alignment of velocity and magnetic field fluctuations whereby the angle θ formed by δu⊥ and δb⊥ scales like λ−1/4 in the inertial range. This alignment effect weakens the nonlinear interactions and yields the perpendicular energy spectrum E(k⊥) ∝ k−3/2⊥. Evidence for this alignment effect has been found in numerical simulations of incompressible MHD turbulence (Mason et al. 2006, 2008) and in studies of solar wind data (Podesta et al. 2008, 2009).

The phenomenological theory of Galtier et al. (2005) can also be used to explain the observed k−3/2 energy spectrum. Using a slightly modified critical balance relation that retains the k∥ ∝ k⊥2/3 scaling of the Goldreich & Sridhar theory, their model admits the k−3/2⊥ energy spectrum (as well as other solutions). However, the theory of Galtier et al. (2005) does not include the scale-dependent alignment that arises in Boldyrev’s theory and, more importantly, is seen in the solar wind (Podesta et al. 2008, 2009).

The theories discussed so far (Goldreich & Sridhar 1995, 1997; Boldyrev 2005, 2006; Galtier et al. 2005) all assume that the cross-helicity vanishes and, therefore, these theories cannot be applied to solar wind turbulence. When the cross-helicity of the turbulence is nonzero, it is necessary to take into account the cascades of both energy and cross-helicity. A generalization of the Goldreich & Sridhar theory to turbulence with nonvanishing cross-helicity, also called imbalanced turbulence, has been developed by Lithwick et al. (2007). Other theories of imbalanced turbulence have been derived by Beresnyak & Lazarian (2008), Chandran (2008), and Chandran et al. (2009). However, none of these theories contains the scale-dependent alignment of velocity and magnetic field fluctuations seen in the solar wind. Therefore, to develop a theory that may be applicable to solar wind turbulence it is of interest to generalize Boldyrev’s theory to incompressible MHD turbulence with nonvanishing cross-helicity.

An extension of Boldyrev’s theory to imbalanced turbulence has been discussed by Perez & Boldyrev (2009). The purpose of the present paper is to develop a theoretical framework which generalizes the results of Perez & Boldyrev and is consistent with solar wind observations.

Our theory is founded, in part, on two new solar wind observations presented in this paper. The first is the observation that the normalized cross-helicity σc, the ratio of cross-helicity to energy, is approximately scale invariant in the inertial range,
that is, $\sigma_v$ is approximately constant in the inertial range. The second is the observation that the probabilities $p$ and $q$, that fluctuations are positively or negatively aligned, respectively, are also scale invariant. Experimental evidence for the scale invariance of $p$ and $q$ is obtained from solar wind observations by Marsch & Tu (1990), Podesta (2010), see Figure 2 below, and from numerical simulations (Verma et al. 1996; Perez & Boldyrev 2009, 2010). Evidence for the scale invariance of $p$ and $q$ is shown in Figure 3. Assuming that these quantities are all scale invariant, we deduce expressions for the energy cascade rates and the rms fluctuations that generalize the results in Boldyrev (2006) and Perez & Boldyrev (2009) and are consistent with the concept of scale-dependent alignment of velocity and magnetic field fluctuations. The resulting theory, which is founded on the concept of scale invariance and grounded in solar wind observations, contains the theories of Boldyrev (2006) and Perez & Boldyrev (2009) as special cases, but opens up a broader range of physical possibilities.

Consistent with numerical simulations and solar wind observations, in our approach the fluctuations at a given point may assume one of two possible states referred to as positively aligned $\delta v_+ \cdot \delta b_+ > 0$ and negatively aligned $\delta v_- \cdot \delta b_- < 0$. Each state is characterized by its own rms energy $v^\perp$, alignment angle $\theta$, and nonlinear timescale $\tau$. Positively aligned fluctuations have a characteristic spatial gradient which determines their nonlinear timescale and negatively aligned fluctuations have a different spatial gradient which determines their nonlinear timescale. These timescales are estimated from the nonlinear terms in the MHD equations as described in Sections 2 and 3. Section 2 describes the geometries of velocity and magnetic field fluctuations that are either aligned “↑” or anti-aligned “↓” and these are used to form estimates of the nonlinear terms in the MHD equations. From this foundation, estimates of the energy cascade times are constructed in Section 3 and the theory of the energy cascade process is developed in Section 4. The summary and conclusions are presented in Section 5.

2. FLUCTUATIONS IN IMBALANCED TURBULENCE

Consider velocity and magnetic field fluctuations measured between two points separated by a distance $\lambda_\perp$ in the field perpendicular plane. Let $\mathbf{v}$ and $\mathbf{b}$ denote the fluctuations in the plane perpendicular to the local mean magnetic field, where $\mathbf{v}$ and $\mathbf{b}$ are both measured in velocity units. Suppose that $\mathbf{v}$ and $\mathbf{b}$ are aligned with some small angle $\theta > 0$ and assume, as for Alfven waves, that $|v| = |b|$. Then $\mathbf{w}^+ = \mathbf{v} + \mathbf{b}$ is nearly aligned with $\mathbf{v}$ and $\mathbf{w}^- = \mathbf{v} - \mathbf{b}$ is nearly perpendicular to $\mathbf{v}$ as sketched in Figure 1. It follows from the identity $\mathbf{w}^\pm \times \mathbf{v} = \mp \mathbf{v} \times \mathbf{b}$ that

$$w^\pm \sin \theta^\pm = v \sin \theta = w^- \sin \theta^-,$$

where $\theta^\pm$ is the angle formed by $\mathbf{w}^\pm$ and $\mathbf{v}$, $\theta^-$ is the angle formed by $\mathbf{w}^-$ and $\mathbf{v}$, and $\theta$ is the angle formed by $\mathbf{v}$ and $\mathbf{b}$. In addition, $\theta^+ + \theta^- = \pi/2$. Following Boldyrev (2006), suppose that the gradient of the fluctuations is in the direction perpendicular to $\mathbf{v}$. In this case,

$$\mathbf{v} \cdot \nabla \mathbf{w}^- \sin \theta^- = v \frac{\partial}{\partial t} \mathbf{w}^-, \mathbf{v} \cdot \nabla \mathbf{w}^+ \sin \theta^+ = v \frac{\partial}{\partial t} \mathbf{w}^+.$$

The time rate of change caused by nonlinear interactions is estimated from the relations

$$\frac{\partial}{\partial t} \frac{|\mathbf{v}|^2}{2} = \mathbf{v} \cdot (\mathbf{w}^- \nabla) \mathbf{w}^-,$$

and

$$\frac{\partial}{\partial t} \frac{|\mathbf{v}|^2}{2} = \mathbf{v} \cdot (\mathbf{w}^+ \nabla) \mathbf{w}^+.$$

If $\mathbf{v}$ and $\mathbf{b}$ are aligned with some small angle $\theta$, then the fluctuations are called “positively aligned” and denoted by “↑” (Figure 1(a)). Similarly, if $\mathbf{v}$ and $-\mathbf{b}$ are aligned with some small angle $\theta$, then the fluctuations are called “negatively aligned” or “anti-aligned” and denoted by “↓” (Figure 1(b)). For positively aligned fluctuations, Equations (2)–(5) imply

$$\frac{\partial}{\partial t} \left( \frac{v^2}{2} \right) = \frac{v^2}{2} \frac{\partial}{\partial t} \frac{\sin \theta^\pm}{\lambda_\perp} = \frac{w^\pm}{\lambda_\perp} \sin \theta_\perp,$$

and

$$\frac{\partial}{\partial t} \left( \frac{v^2}{2} \right) = \frac{v^2}{2} \frac{\partial}{\partial t} \frac{\sin \theta^\pm}{\lambda_\perp} = \frac{w^\pm}{\lambda_\perp} \sin \theta_\perp,$$

where $\theta_\perp$ is the angle formed by $\mathbf{v}$ and $\mathbf{b}$ and quantities with the subscript $\downarrow$ describe positively aligned fluctuations. It is clear from the middle term in Equation (6) that the time rate of change of $w^\perp_\perp$ depends on $w^-_\perp$, consistent with the nonlinear terms in the MHD equations, although this dependence is not immediately apparent in the last term in Equation (6). For negatively aligned fluctuations, Equations (2)–(5) imply

$$\frac{\partial}{\partial t} \left( \frac{v^2}{2} \right) = \frac{v^2}{2} \frac{\partial}{\partial t} \frac{\sin \theta^-}{\lambda_\perp} = \frac{w^-}{\lambda_\perp} \sin \theta^-,$$

and

$$\frac{\partial}{\partial t} \left( \frac{v^2}{2} \right) = \frac{v^2}{2} \frac{\partial}{\partial t} \frac{\sin \theta^+}{\lambda_\perp} = \frac{w^+}{\lambda_\perp} \sin \theta^+,$$

where $\theta_\perp$ is the angle formed by $\mathbf{v}$ and $\mathbf{b}$ and quantities with the subscript $\uparrow$ describe negatively aligned fluctuations. Here, $0 < \theta_\perp < \pi/2$ and $0 < \theta_\perp < \pi/2$.

In general, the fluctuations $\mathbf{v}$ and $\mathbf{b}$ observed at any point $(x, t)$ are either positively or negatively aligned. For a point
(\(x, t\)) picked at random, let \(p\) and \(q\) be the probabilities that the alignment is positive or negative, respectively (\(p + q = 1\)). Then, on average,
\[
\frac{\partial}{\partial t} \left( \bar{w}^+ \right)^2 \simeq \frac{1}{\lambda_\perp} \left[ p(w_i^+)^2 v_i \sin \theta_i + q(w_i^-)^2 v_i \sin \theta_i \right]
\]
and
\[
\frac{\partial}{\partial t} \left( \bar{w}^- \right)^2 \simeq \frac{1}{\lambda_\perp} \left[ p(w_i^-)^2 v_i \sin \theta_i + q(w_i^+)^2 v_i \sin \theta_i \right],
\]
where the rms values \(\bar{w}^\pm\) are defined by
\[
(\bar{w}^+)^2 = p(w_i^+)^2 + q(w_i^-)^2, \tag{12}
\]
\[
(\bar{w}^-)^2 = p(w_i^-)^2 + q(w_i^+)^2. \tag{13}
\]
The following relations also hold. For a positively aligned fluctuation, assuming \(|v| = |b|\),
\[
w_i^+ \cdot w_i^+ = 2v_i^2(1 + \cos \theta_i), \tag{14}
\]
\[
w_i^- \cdot w_i^- = 2v_i^2(1 - \cos \theta_i), \tag{15}
\]
and \(w_i^+ w_i^- = 2v_i^2 \sin \theta_i\). The energy of a positively aligned fluctuation is \(v_i^2\). For a negatively aligned fluctuation
\[
w_i^+ \cdot w_i^- = 2v_i^2(1 - \cos \theta_i), \tag{16}
\]
\[
w_i^- \cdot w_i^+ = 2v_i^2(1 + \cos \theta_i), \tag{17}
\]
and \(w_i^+ w_i^- = 2v_i^2 \sin \theta_i\). The energy of a negatively aligned fluctuation is \(v_i^2\). Thus, the rms values (12) and (13) are
\[
(\bar{w}^+)^2 = 2\left[ pv_i^2(1 + \cos \theta_i) + qv_i^2(1 - \cos \theta_i) \right], \tag{18}
\]
\[
(\bar{w}^-)^2 = 2\left[ pv_i^2(1 - \cos \theta_i) + qv_i^2(1 + \cos \theta_i) \right]. \tag{19}
\]
If the angles are small, \(\theta_i \ll 1\) and \(\theta_i \ll 1\), then the small parameter \(\theta\) can be used to order the terms in Equations (18) and (19) so that to leading order
\[
(\bar{w}^+)^2 \simeq 4v_i^2 p \quad \text{and} \quad (\bar{w}^-)^2 \simeq 4v_i^2 q, \tag{20}
\]
where \(p + q = 1\). This may be derived as follows. In Equations (18) and (19) assume that the angles are both small and then substitute \(1 + \cos \theta \simeq 2\) and \(1 - \cos \theta = 2 \sin^2(\theta/2)\) to obtain
\[
(\bar{w}^+)^2 \simeq 4\left[ pv_i^2 + qv_i^2 \sin^2(\theta_i/2) \right] \tag{21}
\]
and
\[
(\bar{w}^-)^2 \simeq 4\left[ pv_i^2 \sin^2(\theta_i/2) + qv_i^2 \right]. \tag{22}
\]
As \(\lambda_\perp \to 0\), both \(\theta_i \to 0\) and \(\theta_i \to 0\) and, therefore, to first order, the terms proportional to \(\sin^2(\theta)\) may be neglected. Alternatively, note that
\[
\left( \frac{\bar{w}^+}{\bar{w}^-} \right)^2 \simeq \left( \frac{pv_i^2/qv_i^2 \sin^2(\theta_i/2)}{pv_i^2/qv_i^2 \sin^2(\theta_i/2)} \right) + 1. \tag{23}
\]
As discussed below, solar wind observations show that this quantity is approximately constant in the inertial range. Now, as \(\theta_i \to 0\) and \(\theta_i \to 0\) the only way that this can remain constant is if \(pv_i^2/qv_i^2\) is bounded away from zero and
\[
\left( \frac{\bar{w}^+}{\bar{w}^-} \right)^2 \simeq \frac{pv_i^2}{qv_i^2}. \tag{24}
\]
This justifies the approximation in Equation (20).

Equation (20) shows that at a given scale \(\lambda_\perp\) the total energy \([\bar{w}^+)^2 + (\bar{w}^-)^2]/4\) is partitioned into two parts, the energy \((\bar{w}^+)^2/4\) associated with positive alignment and the energy \((\bar{w}^-)^2/4\) associated with negative alignment. The normalized cross-helicity \(\sigma_c\) is defined as the ratio of the cross-helicity to the energy at a given scale and can be written
\[
\sigma_c = \frac{(\bar{w}^+)^2 - (\bar{w}^-)^2}{(\bar{w}^+)^2 + (\bar{w}^-)^2}. \tag{25}
\]
For small angles, Equations (10) and (11) become, to leading order,
\[
\frac{\partial}{\partial t} \left( \bar{w}^+ \right)^2 \simeq \frac{4pv_i^2 \theta_i}{\lambda_\perp}, \tag{26}
\]
\[
\frac{\partial}{\partial t} \left( \bar{w}^- \right)^2 \simeq \frac{4pv_i^2 \theta_i}{\lambda_\perp}. \tag{27}
\]
To express these in terms of the rms values \(\bar{w}^\pm\), eliminate \(v_i\) and \(v_i\) using Equation (20). This yields
\[
\frac{\partial}{\partial t} \left( \bar{w}^+ \right)^2 \simeq \frac{\left( \bar{w}^+ \right)^3 \theta_i}{2\lambda_\perp p^{1/2}}, \tag{28}
\]
\[
\frac{\partial}{\partial t} \left( \bar{w}^- \right)^2 \simeq \frac{\left( \bar{w}^- \right)^3 \theta_i}{2\lambda_\perp q^{1/2}}. \tag{29}
\]
These estimates shall be used to derive the cascade times.

3. ENERGY CASCADE TIME

When nonlinear interactions are strong and a large number of Fourier modes are excited, fluctuations occur continuously in time and space. During a time \(\tau\), the fractional change in the quantity \((w^+)^2\) is, from Equation (28),
\[
\chi^+(\tau) \simeq \frac{(w^+)^3 \theta_i}{2\lambda_\perp p^{1/2}} \cdot \frac{2\tau}{(w^+)^2} = \frac{w_i^+ \theta_i \tau}{\lambda_\perp p^{1/2}}, \quad \tau \ll \tau^+, \tag{30}
\]
where \(\tau^+\) is the cascade time at the lengthscales \(\lambda_\perp\) and the tides have been dropped. Similarly, the fractional change in the quantity \((\bar{w}^-)^2\) is, from Equation (29),
\[
\chi^-\left( \frac{(w^-)^3 \theta_i}{2\lambda_\perp q^{1/2}} \cdot \frac{2\tau}{(w^-)^2} = \frac{w^- \theta_i \tau}{\lambda_\perp q^{1/2}}, \quad \tau \ll \tau^-, \tag{31}
\]
where \(\tau^-\) is the cascade time of \(\bar{w}^-\) and the tides have been dropped for brevity. Hereafter, the tides will be omitted and \(w^+\) and \(w^-\) will always represent the rms values.

According to the definition of the energy cascade time, the fractional change \(\chi^+\) is of order unity when the interaction time \(\tau\) is equal to the cascade time \(\tau^+\). Therefore, the relations (30) and (31) imply
\[
\tau^+ \simeq \frac{\lambda_\perp p^{1/2}}{w^+ \theta_i}, \quad \tau^- \simeq \frac{\lambda_\perp q^{1/2}}{w^- \theta_i}. \tag{32}
\]
By similar reasoning, Equations (6) and (9) imply
\[ \tau_+ \simeq \frac{\lambda_\perp}{2v_\perp \theta_\perp}, \quad \tau_- \simeq \frac{\lambda_\perp}{2v_\perp \theta_\perp}. \] (33)

Moreover, Equations (32), (33), and (20) imply \( \tau^+ = \tau_+ \) and \( \tau^- = \tau_- \). Thus, the energy cascade times for the rms Elsasser amplitudes are equal to the energy cascade times for the positively and negatively aligned fluctuations.

For balanced turbulence, \( \sigma_c \rightarrow 0 \), \( w^+/w^- \rightarrow 1 \), \( p = q \), \( \theta_+ = \theta_- \), and the energy cascade times (32) reduce to the cascade time in Boldyrev’s original theory (Boldyrev 2006).

For imbalanced turbulence, \( \sigma_c \neq 0 \), the cascade times (32) are different from the cascade times \( \tau^\pm \sim \xi/\theta^\pm \theta^\mp \) in the theory of Perez & Boldyrev (2009). The theory presented here is different from the theory of Perez & Boldyrev (2009) because the latter theory does not take into account the existence of two separate types of fluctuations, positively and negatively aligned, with separate probabilities of occurrence \( p \) and \( q \). Taking this into account and also the definitions of the rms amplitudes (12) and (13), it follows from the preceding analysis that the timescales for the rms amplitudes take the form (32).

As pointed out by Kraichnan (1965), Dobrowolny et al. (1980), and others, the energy cascade in MHD turbulence occurs through collisions between Alfvén wavepackets propagating in opposite directions along the mean magnetic field. In other words, it is the interaction between \( w^\perp \) and \( w^\perp \) waves that causes the energy to cascade to smaller scales in MHD turbulence. Consequently, the cascade time for \( w^\perp \), say, should depend on \( w^\perp \). While it may appear from Equations (32) and (33) that the timescale for \( w^\perp \) fluctuations depends only on \( w^\perp \) and, therefore, the interaction with the \( w^\perp \) waves is absent, this is not true. The interactions are still present in the expressions (32) and (33) through the dependence on the angles and other parameters as will be shown in the next section.

4. THEORY OF THE ENERGY CASCADE PROCESS

Assuming that there is no direct injection of energy or cross-helicity within the inertial range and there is no dissipation of energy or cross-helicity within the inertial range, the energy cascade rate \( \varepsilon \) and the cross-helicity cascade rate \( \varepsilon_c \) are scale invariant in the inertial range. It follows that the energy cascade rates for the two Elsasser variables \( \varepsilon^\pm = \varepsilon \pm \varepsilon_c \) are also scale invariant. The theory of the energy cascade process is based on Kolmogorov’s relations
\[ \frac{(w^+)^2}{2\tau^+} = \varepsilon^+ \quad \text{and} \quad \frac{(w^-)^2}{2\tau^-} = \varepsilon^-, \] (34)
where the nonzero constants \( \varepsilon^+ \) and \( \varepsilon^- \) are the energy cascade rates per unit mass for the two Elsasser variables \( w^\perp \) and \( w^\perp \), respectively. These equations describe the conservation of energy flux in \( k \)-space (Fourier space). In addition to Kolmogorov’s relations, there are two observational constraints that must be taken into account.

Solar wind observations show that the energy and cross-helicity spectra of the turbulence follow approximately the same power law in the inertial range (Figure 2) which implies that the normalized cross-helicity \( \sigma_c \) is approximately constant. In other words, the quantity \( \sigma_c \) is approximately scale invariant. Similar results have been found in simulations of incompressible MHD turbulence (Verma et al. 1996; Perez & Boldyrev 2009; Beresnyak & Lazarian 2008). In particular, the 3D simulations of Perez & Boldyrev (2009) indicate that the perpendicular Elsasser spectra are proportional to each other in Fourier space. Solar wind observations also suggest that the probabilities \( p \) and \( q \) are approximately scale invariant as shown in Figure 3. These observations will now be taken into account in the theory.

Assuming that \( \sigma_c \) and \( \varepsilon \) are both scale-invariant quantities, then \( w^+/w^- \), \( v_\perp/v_\parallel \), \( \tau^+/\tau^- \), and \( \theta_\perp/\theta_\parallel \) are scale invariant by Equations (25), (24), (34), and (32), respectively. In all, there
are six different scale-invariant ratios in the theory

$$\frac{w^+}{w^-}, \frac{ε^+}{ε^-}, \frac{p}{q}, \frac{v_t}{v_i}, \frac{τ^+}{τ^-}, \frac{θ_t}{θ_i}. \quad (35)$$

At most, only three of these are independent, say, the first three. Equations (24), (34), and (32) imply

$$\frac{v_t}{v_i} = \sqrt{\frac{q}{p}} \cdot \frac{w^+}{w^-}, \quad (36)$$

$$\frac{τ^+}{τ^-} = \left( \frac{w^+}{w^-} \right)^2 \frac{ε^-}{ε^+}, \quad (37)$$

$$\frac{θ_t}{θ_i} = \sqrt{\frac{q}{p}} \left( \frac{w^+}{w^-} \right)^3 \frac{ε^-}{ε^+}. \quad (38)$$

Therefore, the six scale-invariant ratios (35) can all be expressed in terms of the first three $w^+/w^-, ε^+/ε^-, \text{ and } p/q$.

To be able to solve Kolmogorov’s relations (34) for $w^\pm$, it is necessary to express the alignment angle $θ_t$ in terms of $w^\pm$. In general, $θ_t$ can depend on $w^+, w^-$, the Alfvén speed $v_A$, the lengthscale $λ_\perp$, the cascade rates $ε^+$ and $ε^-$, and the probabilities $p$ and $q$. By dimensional analysis, $θ_t$ must be a function of the following six dimensionless quantities

$$\frac{w^+}{v_A}, \frac{w^-}{v_A}, \frac{ε^+λ_\perp}{v_A^2}, \frac{ε^-λ_\perp}{v_A^2}, p, q. \quad (39)$$

Moreover, $θ_t$ must change into $θ_i$ when $w^+, ε^+$, and $p$ are interchanged with $w^-, ε^-$, and $q$, respectively, to be consistent with the nonlinear terms (28) and (29). For a theory composed of power-law functions, the only forms that satisfy all these requirements are

$$θ_t \propto \left( \frac{w^+}{v_A} \right)^α \left( \frac{w^-}{v_A} \right)^β \left( \frac{ε^+λ_\perp}{v_A^3} \right)^γ \left( \frac{ε^-λ_\perp}{v_A^3} \right)^δ \epsilon^{μ} q^{ν}, \quad (40)$$

$$θ_i \propto \left( \frac{w^-}{v_A} \right)^α \left( \frac{w^+}{v_A} \right)^β \left( \frac{ε^-λ_\perp}{v_A^3} \right)^γ \left( \frac{ε^+λ_\perp}{v_A^3} \right)^δ \epsilon^{μ} p^{ν}, \quad (41)$$

where $α, β, γ, δ, μ, \text{ and } ν$ are constants that must be determined by the theory. In addition, there is a leading coefficient which is omitted.

The substitution of Equations (40) and (41) into Equation (38) yields $β = α + 3, δ = γ − 1, \text{ and } ν = μ − 1/2$. The parameters are further constrained by considering the geometry of the “turbulent eddies” associated with the fluctuations $v_t$ and $v_i$.

The parallel correlation length is defined by $λ_\parallel = v_A τ_t$ and the correlation length in the direction of the velocity fluctuation is $ξ_t = v_t τ_t$. Similarly, the correlation lengths for negatively aligned fluctuations are $λ_\perp = v_A τ_i$ and $ξ_⊥ = v_i τ_i$. In the plane perpendicular to the local mean magnetic field $ξ$ is parallel to $v$, the gradient direction is perpendicular to $v$ with lengthscale $λ_\perp$, and the eddy dimensions are $ξ × λ_\perp$. The dimension parallel to the mean magnetic field is $λ_\parallel$. Hence, in physical space the turbulent eddies can be visualized as 3D structures with dimensions $λ_\parallel × ξ × λ_\perp$.

Therefore, the eddy geometry.

The condition of Goldreich & Sridhar which is also implicit in the work of Higdon (1984). Equation (33) and the definitions of the correlation lengths in the last paragraph immediately yield the critical balance condition

$$τ_t = \frac{ξ_⊥}{v_i} \approx \frac{λ_⊥}{2v_i θ_i}. \quad (42)$$

with a similar condition for the negatively aligned fluctuations

$$τ_i = \frac{ξ_∥}{v_A} \approx \frac{λ_∥}{2v_A θ_i}. \quad (43)$$

Now consider the eddy geometry. When the mean magnetic field is strong enough that $w^±/v_A < 1$, then $λ_⊥ > ξ > λ_∥$ and the eddies are elongated in the parallel direction. The condition $w^±/v_A < 1$ is assumed hereafter. Equation (42) shows that the aspect ratio in the field perpendicular plane is $φ_t = λ_∥/ξ_∥ = 2θ_t$ and the aspect ratio in the parallel direction is, from Equations (42) and (20),

$$ψ_t = \frac{ξ_∥}{λ_∥} = \frac{v_t}{v_A} = \frac{w^+}{2v_A p^{1/2}}. \quad (44)$$

The two aspect ratios will scale in the same way if $φ_t/ψ_t$ is scale invariant. This implies that $α = 1$ and $γ = 1/2$. The assumption that the ratio $φ_t/ψ_t$ is scale invariant is different from Boldyrev’s original approach in which he assumed that the alignment angles in and out of the field perpendicular plane are simultaneously minimized. Nevertheless, our assumption retains the spirit of Boldyrev’s original theory which implies the geometry of turbulent fluctuations is scale invariant.

Solving Kolmogorov’s relation (34) using Equations (32), (40), (41), and the parameter values obtained so far, one finds

$$\frac{w^±}{v_A} \simeq \left( \frac{w^+}{w^-} \right)^{\pm 1/2} \left( \frac{ε^+}{ε^-} \right)^{\pm 1/8} \left( \frac{ε^+ λ_⊥}{v_A^3} \right)^{1/4} (pq)^{ν/4}. \quad (45)$$

and the total energy cascade rate $ε = (ε^+ + ε^-)/2$ is

$$ε = \frac{(w^+ − w^-)^2}{4v_A λ_⊥} \sqrt{\frac{ε^+}{ε^-} + \sqrt{\frac{ε^-}{ε^+}} (pq)^ν}. \quad (46)$$
The total energy at scale \( \lambda_\perp \) is
\[
\frac{(w^+)^2 + (w^-)^2}{4} = \frac{w^+w^-}{4} \left( \frac{w^+}{w^+} + \frac{w^-}{w^+} \right) = v^2. \tag{47}
\]

Therefore, the energy cascade rate can be written
\[
\varepsilon = \frac{4v^4}{v_A\lambda_\perp} \left( \frac{v_A^+}{v_A} \right)^2 \left( \frac{v_A^-}{v_A} \right)^{1/2} \left( \frac{v_A^-}{v_A} \right)^{-1/2} \left( \frac{v_A^+}{v_A} \right)^{-1/2} \left( \frac{w^+}{w^-} \right)^2 \left( \frac{w^-}{w^+} \right)^{-2} (pq)^\nu. \tag{48}
\]

Assuming the rms energy \( v^2 \) at scale \( \lambda_\perp \) is held constant, the terms on the right-hand side describe the dependence of the energy cascade rate on the ratios \( \varepsilon^+/\varepsilon^- \) and \( w^+/w^- \). The value of \( \nu \) may be determined by comparison with experiment or possibly by further physical considerations. This parameter does not affect the inertial range scaling laws and is left undetermined for the moment.

At this point, it is of interest to return to the expressions (32) for the cascade times and ask: how do the cascade times depend \( \varepsilon \)? Using the parameter values obtained previously, Equation (40) becomes
\[
\tau_\perp \sim \left( \frac{w^+}{v_A} \right)^{-1} \left( \frac{w^-}{v_A} \right)^2 \left( \frac{v_A^+}{v_A} \right)^{1/2} \left( \frac{v_A^-}{v_A} \right)^{-1/2} \left( \frac{w^+}{w^-} \right)^2 \left( \frac{w^-}{w^+} \right)^{-2} \left( \frac{v_A^+}{v_A} \right)^{-1/2} \left( \frac{v_A^-}{v_A} \right)^{-1/2} \left( \frac{w^+}{w^-} \right)^2 \left( \frac{w^-}{w^+} \right)^{-2} (pq)^\nu. \tag{49}
\]

and the substitution of this result into Equation (32) yields
\[
\tau^+ \simeq \lambda_\perp \left( \frac{v_A}{w^+} \right)^2 \left( \frac{w^-}{w^+} \right)^{1/2} (pq)^\nu. \tag{50}
\]

A similar expression holds for \( \tau^- \) so that the ratio \( \tau^+/\tau^- \) satisfies Equation (37). Ignoring scale-invariant factors, the preceding equation shows that
\[
\tau^+ \propto \lambda_\perp \left( \frac{v_A}{w^+} \right)^{-2} \quad \text{and} \quad \tau^- \propto \lambda_\perp \left( \frac{v_A}{w^+} \right)^{-2}. \tag{51}
\]

In this form, the angle dependence has been eliminated. Note that the simple estimate \( \tau^+ \sim \lambda_\perp/w^- \), suggested by the nonlinear term in the MHD equations is modified by the factor \( v_A/w^- \) which accounts for the weakening of nonlinear interactions caused by scale-dependent alignment. The presence of this algebraic factor is one of the hallmarks of Boldyrev’s original (2006) theory which is generalized here to imbalanced turbulence. Remarkably, the relations (51) are identical to those in the isotropic theory of imbalanced turbulence developed by Dobrowolny, Mangeney, and Veltri; see Equation (10) in Dobrowolny et al. (1980). Recall that Dobrowolny, Mangeney, and Veltri concluded from their expressions for the cascade times that steady-state turbulence with nonvanishing cross-helicity is impossible. On the contrary, the theory presented here allows such a steady state because the additional coefficients shown in Equation (50) but not (51) maintain the relation (37) even when \( \varepsilon^+ \neq \varepsilon^- \). Thus, the theory presented here is also a generalization of the theory of Dobrowolny et al. (1980).

A remark about the timescales in the theory should be made. If \( w^+ > w^- \), then Equation (37) implies that it is possible that \( \tau^+ < \tau^- \) since there is nothing in the theory that prevents this. That is, the energy of the more energetic Elsasser species may be transferred to smaller scales in less time than the energy of the less energetic Elsasser species. This is not inconsistent with dynamic alignment, a well-known effect seen in simulations of decaying incompressible MHD turbulence where the minority species usually decays more rapidly than the dominant species causing the magnitude of the normalized cross-helicity to increase with time (Dobrowolny et al. 1980; Mattheus et al. 1983; Mattheus & Montgomery 1984; Pouquet et al. 1986).

In freely decaying turbulence, dynamic alignment occurs whenever the total energy decays more rapidly than the cross-helicity, that is, \( \varepsilon > |\varepsilon_c| \), where the cascade rate of cross-helicity \( \varepsilon_c \) may be positive or negative. From the relations \( \varepsilon > 0 \) and \( \varepsilon^+ = \varepsilon \pm |\varepsilon_c| \), it follows that dynamic alignment occurs if and only if \( \varepsilon^+ > 0 \) and \( \varepsilon^- > 0 \). If \( w^+ > w^- \), it is not necessary that \( \tau^+ > \tau^- \), only that
\[
\frac{\tau^+}{\tau^-} > \frac{\varepsilon^-}{\varepsilon^+}. \tag{52}
\]

as can be seen from Equation (37). Therefore, even though the relation \( \tau^+ < \tau^- \) may seem counter intuitive, it is not inconsistent with dynamic alignment.

5. SUMMARY AND CONCLUSIONS

Observations of scale-dependent alignment of velocity and magnetic field fluctuations \( \delta v_\perp \) and \( \delta b_\perp \) in the solar wind suggest that this effect must be included in any theory of solar wind turbulence (Podesta et al. 2008, 2009). Perez & Boldyrev (2009) have recently discussed a theory of imbalanced turbulence that includes scale-dependent alignment of the fluctuations \( \delta v_\perp \) and \( \delta b_\perp \) in the inertial range. We have extended the Perez–Boldyrev theory by including the probabilities \( p \) and \( q \) which solar wind observations indicate are not necessarily equal. Operationally, the probabilities \( p \) and \( q \) may be defined as follows. Suppose space is covered by a uniform cartesian grid or 3D mesh. At each grid point, one may compute the fluctuations \( \delta v_\perp \) and \( \delta b_\perp \) and the angle between them \( \theta \). If the angle lies in the range \( 0 < \theta < \pi/2 \), then the fluctuation is positively aligned and if \( \pi/2 < \theta < \pi \), then the fluctuation is negatively aligned. By counting the number of positively and negatively aligned fluctuations in a large volume \( V \), much larger than the length scales of the turbulent eddies, the probabilities \( p \) and \( q \) may be defined as the fractional numbers of positively and negatively aligned fluctuations in the volume \( V \).

The phenomenological theory developed in this paper was guided primarily by two new solar wind observations. It should be noted that both of these solar wind observations are necessary for the development of the theory. At first glance, it may seem that the condition \( \sigma_c = \text{const} \). implies that \( p \) and \( q \) are both constants. Or that these two conditions are somehow equivalent. However, the relation \( (w^+/w^-)^2 \simeq \nu v_x^2/\nu v_y^2 \), Equation (24), shows that \( p/q \) can vary with the lengthscale even if \( w^+/w^- \) is constant. Therefore, it is essential to have separate observations of the scale invariance of \( \sigma_c \) and the scale invariance of \( p \) and \( q \) to support the theoretical framework developed here.

In summary, using estimates of the cascade times derived from the nonlinear terms in the incompressible MHD equations and two new observational constraints derived from studies of solar wind data, we have constructed a generalization of Boldyrev’s theory (Boldyrev 2006) that depends on the three parameters \( w^+/w^- \), \( \varepsilon^+/\varepsilon^- \), and \( p/q \). The theory reduces to the original theory of Boldyrev (2006) when \( w^+ = w^- \), \( \varepsilon^+ = \varepsilon^- \), and \( p = q \) since in this limit \( \theta_\parallel = \theta_\perp \) and the cascade times (32) become equal to those of Boldyrev (2006). For imbalanced turbulence \( w^+ \neq w^- \), \( p \neq q \), and the theory predicts the scaling laws \( w^+ \propto \lambda_\perp^{1/4}, \theta_\parallel \propto \lambda_\perp^{1/4}, \) and \( \lambda_\parallel \propto \lambda_\perp^{1/2} \). Interestingly, the scaling laws for balanced and imbalanced turbulence are the
same. The perpendicular energy spectrum defined by $k_{\perp}E^\pm \sim |w^{\pm}|^2$ has the inertial range scaling $E^\pm \propto k_{\perp}^{-3/2}$ with

$$E^+ = \left(\frac{w^+}{w^-}\right)^2 = \left(\frac{1 + \sigma}{1 - \sigma}\right) = \text{const.} \quad (53)$$

The theory assumes that the cascades for positively and negatively aligned fluctuations are both in a state of critical balance (42), although they are governed by different timescales, and that the eddy geometry is scale invariant. The positively aligned fluctuations occupy a fractional volume $p$ and the negatively aligned fluctuations occupy a fractional volume $q$ so that the energy cascade rate is

$$\varepsilon = \frac{p}{\tau_i} + \frac{q}{\tau_i} \quad (54)$$

or, equivalently,

$$\varepsilon = \frac{(w^+)^2}{4\tau^+} + \frac{(w^-)^2}{4\tau^-}. \quad (55)$$

It should be emphasized that the theory developed here applies for any arbitrary value of the normalized cross helicity, $-1 < \sigma_c < 1$, and not just for high values $|\sigma_c| \gtrsim 0.9$. In connection with this, it is of interest to note that the fields are “highly aligned” whether the normalized cross-helicity is small or large. For Alfvén waves, the fluctuations are precisely aligned so that $\sin \theta = 0$. More precisely, either $\theta = 0$ for propagation anti-parallel to $B_0$ or $\theta = \pi$ for propagation parallel to $B_0$. In MHD turbulence, the fluctuations are predominantly Alfvénic, meaning they behave like Alfvén waves, and the sine of the alignment angle is usually small although not zero (see Mason et al. 2008; Biskamp 2003, and references therein). This holds for both parallel and anti-parallel propagating fluctuations for which the angles are observed to cluster near $\pi$ and 0, respectively. In general, MHD turbulence consists of a mixture of fluctuations of both types, parallel and anti-parallel propagating, with a possible imbalance in the fraction of parallel versus anti-parallel propagating waves. However, whether the turbulence is balanced or imbalanced, in either case the alignment angles of the fluctuations are still clustered in the neighborhoods of 0 and $\pi$ so that $\sin \theta$ is small throughout the inertial range. In other words, even balanced turbulence ($\sigma_c = 0$) is inherently “highly aligned.”

In the discussion following Equation (35), it was shown that at most three of the ratios $w^+/w^-$, $\varepsilon^+/\varepsilon^-$, and $p/q$ can be independent. However, the two ratios $w^+/w^-$ and $\varepsilon^+/\varepsilon^-$ cannot be independent since in the case of homogeneous steady-state turbulence $w^+ = w^-$ implies $\varepsilon^+ = \varepsilon^-$ and vice versa. This is because the injection of cross-helicity into the system, $\varepsilon_c \neq 0$ or $\varepsilon^+ \neq \varepsilon^-$, will create a nonzero cross-helicity spectrum and a cascade of cross-helicity from large to small scales which implies a net accumulation of cross-helicity within the volume ($\sigma_c \neq 0$). Hence, at most two of the ratios $w^+/w^-$ and $p/q$ are independent. Whether $p/q$ can be expressed in terms of $w^+/w^-$ and $\varepsilon^+/\varepsilon^-$ is an open question.

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