The Tremblay-Turbiner-Winternitz system as extended Hamiltonian

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Abstract
We generalize the idea of “extension of Hamiltonian systems” – developed in a series of previous articles – which allows the explicit construction of Hamiltonian systems with additional non-trivial polynomial first integrals of arbitrarily high degree, as well as the determination of new superintegrable systems from old ones. The present generalization, that we call “modified extension of Hamiltonian systems”, produces the third independent first integral for the (complete) Tremblay-Turbiner-Winternitz (TTW) system, as well as for the caged anisotropic oscillator in dimension two.

1 Introduction
We further improve a research started in [1] about a class of Hamiltonian systems admitting recursively computed, polynomial first integrals of high degree. In [3] we introduced a procedure that we called extensions of Hamiltonian systems, lately generalized to \((m,n)\)-extensions in [6]. Until now, our methods were not able to include two of the most important examples of two-dimensional superintegrable Hamiltonians with high-degree first integrals in whole generality (i.e., without setting some of the parameters appearing in the potential to be zero): the Tremblay-Turbiner-Winternitz (TTW) system [8]

\[
H = \frac{1}{2} p_x^2 + \frac{1}{2} p_\theta^2 + \left( \frac{1}{2} p_\theta^2 + \frac{\alpha_1}{\cos^2 \lambda \theta} + \frac{\alpha_2}{\sin^2 \lambda \theta} \right) + \omega r^2, \quad \lambda \in \mathbb{Q}, \tag{1}
\]

and the two-dimensional caged anisotropic harmonic oscillator [7]

\[
H = \frac{1}{2} p_x^2 + \frac{1}{2} p_y^2 - \omega^2 (\lambda^2 x^2 + y^2) + \frac{b}{x^2} + \frac{c}{y^2}, \quad \lambda \in \mathbb{Q}. \tag{2}
\]

In the present paper, we slightly modify the procedure for building the first integral in the \((m,n)\)-extensions to include both the above-mentioned systems.
The "modified \((m,n)\)-extensions" that we introduce here still keep the relevant features of the extension method, namely: the high-degree first integrals are explicitly determined, the extension procedure can be applied to \(n\)-degrees of freedom systems and it is linked to their geometry, the modified procedure remains a powerful tool for the creation of new superintegrable systems from old ones, as we plan to show in a future paper that will generalise [5].

2 \((m,n)\)-extensions and the TTW Hamiltonian

We adopt the notation introduced in [6] (dropping the tildes for simplicity). Let \(L\) be a Hamiltonian on a \(d\)-dimensional Poisson manifold \(Q\). For any pair of positive integers \(m, n\), we denote by \(H_{m,n}\) its \((m,n)\)-extension, that is the function

\[
H_{m,n} = \frac{1}{2}p_u^2 + \frac{m^2}{n^2} \alpha(u)L + \frac{m^2}{n^2} \beta(u),
\]

(3)
defined on the \((d+2)\)-dimensional Poisson manifold \(T \times Q\) where \(T\) has canonical symplectic form \(dp_u \wedge du\), and the functions \(\alpha\) and \(\beta\) are listed in Table 1 below. In [4] (for \(n = 1\)) and in [6] (for any \(n \in \mathbb{N} - \{0\}\)) we proved that, if there exist constants \(c\) and \(L_0\) (not both zero) such that the equation

\[
X_L^2(G) = -2(cL + L_0)G,
\]

(4)

admits a solution \(G\) on \(Q\) (\(X_L\) denotes the Hamiltonian vector field of \(L\)), then a first integral of \(H_{m,n}\) is given by

\[
K_{m,n} = U_{m,n}^m(G_n) = \left(p_u + \frac{m}{n^2}\gamma(u)X_L\right)^m(G_n)
\]

(5)

where \(G_n\) is the \(n\)-th term of the recursion

\[
G_1 = G, \quad G_{n+1} = X_L(G_n)G_n + \frac{1}{n}G X_L(G_n),
\]

(6)

and \(\gamma(u)\) is given in Table 1.

| \(c = 0\) | \(c \neq 0\) |
|-----------------|-----------------|
| \(\alpha = -\gamma' = A\) | \(\frac{c}{S_2^2(cu)}\) |
| \(\beta = L_0\gamma^2 = L_0A^2u^2\) | \(0\) |
| \(\gamma = -Au\) | \(\frac{1}{T_0(cu)}\) |

Table 1: Functions involved in the \((m,n)\)-extension of \(L\)

We remark that

• if [4] has a solution \(G\) for \(c \neq 0\), then we may assume without loss of generality \(L_0 = 0\);
in Table 1, $A$ and $\kappa$ are arbitrary constants and the functions $S_\kappa$ and $T_\kappa$ are the trigonometric tagged functions

$$S_\kappa(x) = \begin{cases} 
\sin\sqrt{\kappa}x & \kappa > 0 \\
\frac{x}{\sqrt{|\kappa|x}} & \kappa = 0 \\
\sinh\sqrt{|\kappa|x} & \kappa < 0
\end{cases}$$

and

$$C_\kappa(x) = \begin{cases} 
\cos\sqrt{\kappa}x & \kappa > 0 \\
1 & \kappa = 0 \\
cosh\sqrt{|\kappa|x} & \kappa < 0
\end{cases},$$

$$T_\kappa(x) = \frac{S_\kappa(x)}{C_\kappa(x)},$$

(see [6] for a summary of their properties).

• in [4] (where a slightly different notation was adopted) it is shown that the functions (3) and (5) Poisson-commute only for the $\alpha$, $\beta$, $\gamma$ appearing in Table 1.

The method of extensions can be effectively applied in two ways: it allows to construct new superintegrable systems from known superintegrable ones with one less degree of freedom (see [5]), or it can be a constructive proof that a given Hamiltonian $H_\lambda$, depending on a parameter $\lambda = m/n$, admits a first integral of degree depending on $\lambda$, provided we are able to write $H_\lambda$ as the $(m,n)$ extension of a Hamiltonian $L$ admitting a solution for (4). In [4] we give a geometric characterization of the natural Hamiltonians $H$ that can be written as the extension of a Hamiltonian $L$ with one less degree of freedom.

In both types of applications the key point is equation (4). For a natural Hamiltonian $L$ on the cotangent bundle of a Riemannian $N$-dimensional manifold $M$, this equation has been studied in the case of $G$ independent of the momenta (see [2, 3]) and linear in the momenta ([4]). For natural Hamiltonians, the recursion (6) provides solutions $G_n$ of (4) polynomial in the momenta with degree depending on $n$, starting from any known solution $G$ independent from the momenta.

When $N = 1$ and $G$ is a polynomial in $p$ of degree $r$ i.e.,

$$L = \frac{1}{2}p^2 + V(q), \quad G = \sum_{i=0}^r \eta_i(q)p^i,$$

the equation (4) is equivalent to

$$\begin{cases} 
\eta''_i + c\eta_i = 0, & i = r - 1, r, \\
\eta''_{i-2} + c\eta_{i-2} = (2i + 1)V'\eta'_i - (2cV + 2L_0 - iV'')\eta_i, & i = r - 1, r, \\
\eta''_{i-2} + c\eta_{i-2} = (2i + 1)V'\eta'_i - (2cV + 2L_0 - iV'')\eta_i - (i + 1)(i + 2)(V'')^2\eta_{i+2}, & 1 < i \leq r - 2, \\
(2i + 1)V'\eta'_i - (2cV + 2L_0 - iV'')\eta_i = (i + 1)(i + 2)(V'')^2\eta_{i+2}, & i = 0, 1,
\end{cases}$$

(7)

where $\eta_j = 0$ for $j < 0$ or $r < j$. We remark that the $\eta_i$ with indices even and odd are not appearing together into (7). Since we are interested in a particular solution $G$, it is not restrictive to assume that the solutions $G$ of
(7) are polynomials containing even (resp. odd) powers of \( p \) only when \( r \) is even (resp. odd). Moreover, when \( V \) is given, the system can be solved iteratively, starting from \( \eta_r \) to \( \eta_0 \) (resp. \( \eta_1 \)). The (7.4) gives a further condition on \( \eta_0 \) (resp. \( \eta_1 \)) for \( r \) even (resp. odd) that makes the system generally unsolvable.

For \( G \) linear and homogeneous in \( p \), the equations (7) become

\[
\begin{align*}
\eta'' + c\eta &= 0, \\
3V'\eta' + \eta V'' - 2\eta(eV + L_0) &= 0.
\end{align*}
\]

The solution of the first equation is

\[
\eta_r = a_1 S_c(q) + a_2 C_c(q).
\]

then, the second equation can be solved for \( V(q) \) to determine all the possible potentials in \( L \) admitting an extension for \( G = \eta p \). The possible solutions are summarized in Table 2.

| \( c \neq 0 \) | \( c = 0, a_1 \neq 0 \) | \( c = 0, a_1 = 0 \) |
|---|---|---|
| \( \frac{c_1 + c_2 \eta'}{\eta} + \frac{L_0}{c} \) | \( \frac{L_0}{4a_1} \eta^2 + \frac{c_1}{\eta} + c_2 \) | \( L_0 q^2 + c_1 q + c_2 \) |
| \( (a_1 S_c(q) + a_2 C_c(q))p \) | \( (a_1 q + a_2)p \) | \( a_2 p \) |

Table 2: Solutions of system (8).

The solution of the first equation is (9), then, the second equation can be solved for \( V(q) \) to determine all the possible potentials in \( L \) admitting an extension for \( G = \eta p \). The solution of the last equation is, since (8) holds,

\[
V = \frac{c_1 + c_2 \eta'}{\eta} + \frac{L_0}{2} \frac{F(q)}{\eta'}, \tag{10}
\]

where \( F' = \eta \) and \( c(F'' + cF) = 0 \) for \( \eta' \) not identically zero. If \( \eta' = 0 \), then we have necessarily \( c = 0 \) and

\[
V = L_0 q^2 + c_1 q + c_2.
\]

In the case \( c = 1 \), we have \( \beta = 0 \) and the \((m, n)\) extension (3) of \( L \) is

\[
H = \frac{1}{2} p_r^2 + m^2 \cos^2 \lambda \theta \left( \frac{1}{2} p_r^2 + V \right).
\]

Moreover, up to ineffective translations in \( q \), we get \( G = (\sin q)p \) and (10) becomes, up to additive constants,

\[
V = \frac{c_1 + c_2 \cos q}{\sin^2 q}. \tag{11}
\]

**Proposition 1.** The TTW Hamiltonian without harmonic term

\[
H = \frac{1}{2} p_r^2 + \frac{1}{r^2} \left( \frac{1}{2} p_\theta^2 + \frac{\alpha_1}{\cos^2 \lambda \theta} + \frac{\alpha_2}{\sin^2 \lambda \theta} \right), \quad \lambda = \frac{m}{n}, \tag{12}
\]
is the \((2m, n)\)-extension, with \(c = 1\), \(L_0 = 0\) and \(\kappa = 0\), of
\[
L = \frac{1}{2}p_u^2 + \frac{c_1 + c_2 \cos q}{\sin^2 q},
\]
(13)
with
\[
q = 2\lambda \theta, \quad c_1 = \frac{\alpha_1 + \alpha_2}{2\lambda^2}, \quad c_2 = \frac{\alpha_2 - \alpha_1}{2\lambda^2}.
\]
(14)
\[
\text{Proof.} \quad \text{Since the potential of (13) is of the form (11), for } c = 1 \text{ and } L_0 = 0, \text{ the Hamiltonian } L \text{ admits the } (2m, n) \text{-extension}
\]
\[
H = \frac{1}{2}p_u^2 + \frac{4m^2}{n^2} \frac{S^2_k(u)}{L}, \quad m, n \in \mathbb{N} - \{0\},
\]
(15)
Through the coordinate transformation \(q = 2\lambda \theta, u = r\) and with \(c_1\) and \(c_2\) given by (14), we get (12). Vice versa, to any \((m, n)\)-extension of (13) corresponds (12) with \(\lambda = \frac{n}{2m}\). \(\square\)

3 Modified extensions

For any \(m, n, s, k \in \mathbb{N} - \{0\}\), let us consider the functions \(\alpha, \beta, \gamma\), the recursion \(G_n\), and the operator
\[
U_{m, n} = p_u + \frac{m}{n^2} \gamma X L
\]
introduced in (4), (6), and in Table 1 for the \((m, n)\)-extension \(H_{m, n}\) of \(L\). We define the functions
\[
\tilde{H}_{m, n} = \frac{1}{2}p_u^2 + \frac{m^2}{n^2} \alpha L + \frac{m^2}{n^2} \beta + \omega \gamma - 2, \quad \omega \in \mathbb{R},
\]
(15)
\[
\tilde{K}_{2s, k} = (U_{2s, k}^2 + 2\omega \gamma - 2)^2 G_k.
\]
(16)
The main result of this paper is
\[
\textbf{Theorem 2.} \quad \text{For } m = 2s, \text{ we have}
\]
\[
\{\tilde{H}_{2s, n}, \tilde{K}_{2s, n}\} = 0.
\]
For \(m = 2s + 1\),
\[
\{\tilde{H}_{2s+1, n}, \tilde{K}_{4s+2, 2n}\} = 0.
\]
The proof of Theorem 2 depends on the following Lemma.
\[
\textbf{Lemma 3.} \quad \text{Let us consider the Hamiltonian}
\]
\[
H = \frac{1}{2}p_u^2 + f(u) + \left(\frac{2m}{n}\right)^2 \alpha(u) L, \quad m, n \in \mathbb{N} - \{0\},
\]
(17)
where the Hamiltonian $L$ does not depend on $(u, p_u)$, and the operator $W$ defined by

$$W(G) = \left( p_u + \frac{2m}{n^2} \gamma(u) X_L \right)^2 G + (2f(u) + h(u)) G.$$  \hspace{1cm} (18)

We have $\{ H, W^m(G) \} = 0$ iff

$$X_L^2(G) = -2n^2(cL + L_0)G,$$  \hspace{1cm} (19)

$$\gamma'' + 2c\gamma'\gamma = 0,$$  \hspace{1cm} (20)

$$\alpha(u) = -\gamma',$$  \hspace{1cm} (21)

$$f(u) = \frac{4m^2}{n^2} L_0 \gamma^2 + \frac{f_0}{\gamma^2} + \frac{h_0}{2},$$  \hspace{1cm} (22)

$$h(u) = -\frac{8m^2}{n^2} L_0 \gamma^2 - h_0,$$  \hspace{1cm} (23)

for some constants $c, L_0$ not both zero.

**Proof.** We follow the same path of the proof of Proposition 1 of [3]. It is well known that, if two operators $A, B$ satisfy

$$[A, B] = 0,$$  \hspace{1cm} (19)

then

$$AB^m(G) = B^{m-1} (m[A, B] + BA) (G),$$

for any function $G$. If $B$ is injective, then $AB^m = 0$ iff $m[A, B] + BA = 0$. When $A$ and $B$ are the Hamiltonian vector field $X_H$ of $H_{2m,n}$ and $U_{2m,n}^2 + 2\omega\gamma^{-2}$ respectively, the equation $AB^m = 0$ is equivalent to the statement of the Theorem. First, we check that $[[X_H, W], W] = 0$: we have

$$[X_H, W] = 4 \frac{m}{n^2} \gamma' p_u^2 + \left( h' - \frac{8m^2}{n^2} \alpha' L \right) p_u - 4 \left( \frac{m}{n^2} f' \gamma + \frac{4m^3}{n^4} \alpha' \gamma L \right) X_L + \frac{8m^2}{n^4} p_u \gamma' X_L^2,$$

and therefore the condition $[[X_H, W], W] = 0$ is evidently satisfied. For $G(q^i, p_i)$ such that $X_L(G) \neq 0 W$ is injective and

$$m[X_H, W]G + WX_H G = p_u^2 \frac{4m^2}{n^2} (\gamma' + \alpha) X_L G +$$

$$+ p_u \left( \left( mh' - \frac{8m^3}{n^2} \alpha' L \right) G + \frac{8m^3}{n^4} \gamma (\gamma' + 2\alpha) X_L^2 G \right) +$$

$$+ \frac{4m^2}{n^2} \left( \alpha (h + 2f) - f' \gamma - \frac{4m^2}{n^2} \alpha' \gamma L \right) X_L G + \frac{16m^4}{n^6} \alpha \gamma^2 X_L^3 G,$$

which is a polynomial in $p_u$. By requiring that the coefficient of $p_u^2$ vanishes, under the non-restrictive assumption $\gamma' \neq 0$, we obtain (21). The requirement that the coefficient of $p_u$ vanishes, by separating terms in $u$ from those in $q^i, p_i$, is equivalent to (19), (20) and

$$h' + 16 \frac{m^2}{n^2} L_0 \gamma' = 0,$$
where $c$, $L_0$ are separation constants, whose integration gives (up to an additive constant) (23). By imposing (21), the coefficient of degree 0 in $p_u$ becomes

$$16\frac{m^4}{n^6} \gamma^2 X_L G \left( -\frac{n^4}{4\gamma^2 m^2} (\gamma' (h + 2f) + f') + \frac{n^2\gamma''}{\gamma'\gamma'} L - \frac{X_L^2 G}{X_L G} \right).$$

(24)

By (19) and (20), we have

$$X_3^3 X_L G X_L G = -2n^2 (cL + L_0), \quad \frac{\gamma''}{\gamma\gamma'} = -2c, \quad \frac{n^2}{4\gamma^2 m^2} (h + h_0) = -2L_0,$$

and the bracket (24) reduces to

$$-\frac{n^4}{4\gamma^2 m^2} (2\gamma' f - \gamma' h_0 + f' \gamma) + 4n^2 L_0,$$

which vanishes if and only if

$$f' + 2\frac{\gamma'}{\gamma} f - h_0 \frac{\gamma'}{\gamma} - 16L_0 \frac{m^2}{n^2} \gamma' = 0,$$

i.e., iff (22) holds. We remark that $2f + h = f_0 \gamma^{-2}$. Thus, we have $W(G) = U^{2m,n}_{2m,n}(G)$ only for $f_0 = 0$ i.e., when $f = (2m/n)^2 \beta$, up to an additive constant.

Proof. of Theorem 2. We apply Lemma 3 to $H = \bar{H}_{2s,n}$ and $W = U^{2m,n}_{2m,n} + 2\omega \gamma^{-2}$. Then, we get that the functions are in involution iff $\alpha$, $\beta$ and $\gamma$ are the functions appearing in Table 1. Moreover, we have, up to inessential additive constants, $f(u) = \frac{2\sqrt{\omega}}{n} \beta + \omega \gamma^{-2}$ and $2f(u) + h(u) = \omega \gamma^{-2}$. Since $H_{m,n} = H_{2m,2n}$ the second part follows directly.

Hence, we can give the following definition:

Definition 1. We call the Hamiltonian $\bar{H}_{m,n}$ admitting the first integral $\bar{K}_{2s,n}$, if $m = 2s$, or $K_{2m,2m}$, if $m = 2s + 1$, the modified extended Hamiltonian of $L$.

For $\omega = 0$, we drop back to $(m, n)$-extensions.

We determine now an explicit expression for the first integrals.

Lemma 4. For $r \leq m$, $\Lambda = -2(cL + L_0)$, we have

$$U_{m,n}^r (G_n) = P_{m,n,r} G_n + D_{m,n,r} X_L (G_n),$$

with

$$P_{m,n,r} = \sum_{k=0}^{[r/2]} \binom{r}{2k} \left( \frac{m}{n} \right)^{2k} p_u^{r-2k} \lambda^k,$$

$$D_{m,n,r} = \frac{1}{n} \sum_{k=0}^{[r-1/2]} \binom{r}{2k+1} \left( \frac{m}{n} \right)^{2k+1} p_u^{r-2k-1} \lambda^k, \quad m > 1,$$

where $[\cdot]$ denotes the integer part and $D_{1,n,1} = \frac{1}{n} \gamma$.

Proof. It follows directly from the analogous expressions in [6].
Proposition 5. The expansion of the first integral (10) is

\[ \dot{K}_{2m,n} = \sum_{j=0}^{m} \binom{m}{j} \left( \frac{2\omega}{\gamma^2} \right)^j U^{2(m-j)}_{2m,n}(G_n). \]  

(26)

Proof. Since \( U_{2m,n}^2 \) and \( 2\omega\gamma^{-2} \) commute as operators, the \( s \)-th power in (10) coincides with the power of a binomial. \( \Box \)

Theorem 6. Let \( \{L_1 = L, \ldots, L_k\} \) be a set of functionally independent first integrals of the Hamiltonian \( L \) on \( Q \). If \( H_{2s,n}, \dot{K}_{2s,n} \) determine a non-trivial modified \((m,n)\)-extension of \( L \), then \( \{H_{2s,n}, \dot{K}_{2s,n}, L_1, \ldots, L_k\} \) are all functionally independent.

Proof. i) The rank of the Jacobian matrix of the \((\dot{H}_{2s,n}, \dot{K}_{2s,n}, L_1, \ldots, L_k)\) w.r. to the coordinates \((u, p, q)\), where \( i = 1, \ldots, k \) and \( q^i \) denote here both momenta and configuration coordinates, is equal to the rank of the square \((k+2) \times (k+2)\) matrix \( J \) given by

\[
J = \begin{pmatrix}
\frac{\partial H_{2s,n}}{\partial u} & \frac{\partial H_{2s,n}}{\partial p} & \frac{\partial H_{2s,n}}{\partial q} & \cdots & \frac{\partial H_{2s,n}}{\partial q^{k+1}} \\
\frac{\partial K_{2s,n}}{\partial u} & \frac{\partial K_{2s,n}}{\partial p} & \frac{\partial K_{2s,n}}{\partial q} & \cdots & \frac{\partial K_{2s,n}}{\partial q^{k+1}} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \frac{\partial L_1}{\partial q} & \cdots & \frac{\partial L_{k+1}}{\partial q^{k+1}} \\
\end{pmatrix}
\]  

(27)

where the order of the \( q^j \) is chosen so that the rank of the \( k \times k \) submatrix in the bottom-right corner, which we denote by \( J_k \), is \( k \). The determinant of \( J \) is given by

\[
\det(J) = \left( \frac{\partial H_{2s,n}}{\partial u} \frac{\partial K_{2s,n}}{\partial p} - \frac{\partial H_{2s,n}}{\partial p} \frac{\partial K_{2s,n}}{\partial u} \right) \det(J_k).
\]

Therefore, because \( \det(J_k) \neq 0 \) by assumption, \( \det(J) = 0 \) iff

\[
\frac{\partial H_{2s,n}}{\partial u} \frac{\partial K_{2s,n}}{\partial p} - \frac{\partial H_{2s,n}}{\partial p} \frac{\partial K_{2s,n}}{\partial u} = 0.
\]

Since \( \{H_{2s,n}, K_{2s,n}\} = 0 \) and \( \alpha \neq 0 \), it follows \( \{L, K_{2s,n}\}_{(q^i)} = 0 \). In the last equation the highest-degree term in \( p_u \) is \( p_u^2 \{L, G_n\} = p_u^2 X_{L}(G_n) \) and as we assumed \( X_{L}(G_n) \neq 0 \).

\( \Box \)

4 Examples

4.1 The TTW family

The complete TTW system in the Euclidean plane with \( \lambda = \frac{m}{r_n} \) is

\[
H = \frac{1}{2}p_u^2 + \frac{m^2}{r_n u^2} \left( \frac{1}{2}p_\psi^2 + \frac{\alpha_1}{\cos^2 \psi} + \frac{\alpha_2}{\sin^2 \psi} \right) + \omega u^2.
\]
From Proposition 1 we have that $H$ coincides with

$$\hat{H}_{2m,n} = \frac{1}{2}p_a^2 + \frac{4m^2}{n^2}a^2 + \frac{1}{2}p_q^2 + \frac{c_1 + c_2 \cos q}{\sin^2 q} + \omega u^2,$$

where $q = 2\psi$ and $c_1 = \frac{a + \omega}{2\lambda}$, $c_2 = \frac{a - \omega}{2\lambda}$. For $\lambda = 1$ the corresponding first integral of $\hat{H}_{2,1}$ is

$$\bar{K}_{2,1} = p_q p_u \sin q + \frac{4}{u} p_a p_q^2 \cos q - \frac{4}{u} p_q^3 \sin q + \frac{4}{u} p_q \frac{c_2 (\cos^2 q + 1) + 2c_1 \cos q}{\sin^2 q} + \frac{2}{u^2} \omega u^4 \sin^2 q - 4(c_1 + c_2 \cos q) \sin q.$$

4.2 The two-dimensional anisotropic caged oscillator

If we put $c = 0$, $V = aq^2 + b/q^2$ into (8) we get $\eta = q$ and $a = L_0/4$. The modified $(m, n)$-extension of $L = \frac{1}{2}p_a^2 + V$ is therefore

$$\hat{L}_{m,n} = \frac{1}{2}p_a^2 + \frac{m^2}{n^2} A \left( \frac{1}{2}p_q^2 + \frac{L_0}{4} q^2 + \frac{b}{q^2} \right) + \frac{m^2}{n^2} L_0 A^2 u^2 + \frac{\omega}{A^2 u^2}.$$

By putting $A = 1$, $x = \frac{a}{m^2} q$, we have

$$\hat{H}_{m,n} = \frac{1}{2}p_a^2 + \frac{1}{2}p_q^2 + L_0 \frac{m^2}{n^2} \left( \frac{m^2}{4n^2} x^2 + u^2 \right) + \frac{b}{x^2} + \frac{\omega}{u^2},$$

that is the generic two-dimensional superintegrable anisotropic caged oscillator.

Remark 1. It is easy to see that $\hat{H}_{1,1}$ is the same for examples 4.1 and 4.2 (it is enough to pass from Cartesian to polar coordinates), but the rational parameter appears in two different ways, leading to different types of extensions.

5 Conclusions

With the inclusion of the TTW system and the caged anisotropic oscillator into the scheme of extensions, we somehow complete a work started in [1] about systems admitting polynomial first integrals of high degree. Through several papers, our “extension” approach showed several unexpected possibilities, such as the creation of new superintegrable systems from old ones [5], the connection with warped manifolds theory [6]. With the present article, we gave an essay of the flexibility of our approach, whose possible developments, also for quantum systems, will be analysed in future.

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