GLOBAL CLASSICAL SOLUTION AND STABILITY TO A COUPLED CHEMOTAXIS-FLUID MODEL WITH LOGISTIC SOURCE

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Abstract. In this paper, we deal with a coupled chemotaxis-fluid model with logistic source $\gamma n - \mu n^2$. We prove the existence of global classical solution for the chemotaxis-Stokes system in a bounded domain $\Omega \subset \mathbb{R}^3$ for any large initial data. On the basis of this, we further prove that if $\gamma > 0$, the zero solution is not stable; if $\gamma = 0$, the zero solution is globally asymptotically stable; and if $0 < \gamma < 16\mu^2$, the nontrivial steady state $(\frac{\gamma}{\mu}, \frac{\gamma}{\mu}, 0)$ is globally asymptotically stable.

1. Introduction. In this paper, we consider a chemotaxis-Stokes model of characterizing a signal production mechanism,

$$
\begin{array}{l}
\frac{\partial n}{\partial t} + u \cdot \nabla n = \Delta n - \nabla \cdot (n(1 + n)^{-\alpha} \nabla c) + \gamma n - \mu n^2, \quad \text{in } Q, \\
\frac{\partial c}{\partial t} + u \cdot \nabla c = \Delta c - c + n, \quad \text{in } Q, \\
u = \Delta u - \nabla \pi + n \nabla \varphi, \quad \text{in } Q, \\
\n\cdot u = 0, \quad \text{in } Q, \\
\frac{\partial n}{\partial \nu} \big|_{\partial \Omega} = \frac{\partial c}{\partial \nu} \big|_{\partial \Omega} = 0, \quad u \big|_{\partial \Omega} = 0, \\
n(x, 0) = n_0(x), \quad c(x, 0) = c_0(x), \quad u(x, 0) = u_0(x), \quad x \in \Omega,
\end{array}
$$

where $Q = \Omega \times \mathbb{R}^+$, $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary, $n, c$ denote the bacterial or cell density, the chemical concentration respectively, $J = n(1 + n)^{-\alpha} \cdot \nabla c$ with $\alpha > 0$ is the chemotactic flux, $\gamma \geq 0, \mu > 0$ are parameters reflecting proliferation and death of cells in a logistic law, $-c + n$ is the kinetics/source term, $n$ represents the spontaneous production of the attractant and is proportional to the number of amoebae $n$, while $-c$ represents decay of attractant activity, $u = (u_1, u_2, u_3)$, $\pi$ are the fluid velocity and the associated pressure.

We see that this model is a coupled system of the chemotaxis model and the incompressible fluid model. The derivation of the chemotaxis model begins with a

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cell equation. Applying conservation law to the cell density $n$ leads to the following equation

$$\frac{\partial n}{\partial t} + \text{div}(J_{\text{flux}}) = R_n,$$

where $R_n$ is the proliferation or death of cells, which obeys a logistic growth law. As for the flux $J_{\text{flux}}$, in addition to random motion, cells migrate toward higher concentrations of diffusible chemoattractant, so we take $J_{\text{flux}} = J_{\text{random}} + J_{\text{chemotaxis}}$. Taking into account the above factors, one obtain the following equation for the cell motion,

$$\frac{\partial n}{\partial t} = \nabla \cdot \left( D_n \nabla n \right) - \alpha \nabla \cdot \left( g(n) \tau'(c) \nabla c \right) + \mu n \left( 1 - \frac{n}{\gamma} \right), \quad (1.2)$$

where $D_n$ is the intrinsic dispersion coefficient, $\alpha$ is the signal detection coefficient, $\mu$ is the proliferation rate of the cells, $\gamma$ is the maximum sustainable cell, $c$ is the signal concentration, and $\tau$ is the mechanism of tactic responses in cell populations, such as chemotaxis. $\chi(c) = \alpha \tau'(c)$ is the chemotactic sensitivity. Thus the taxis is positive or negative according to whether $\alpha \tau'(c)$ is positive or negative. (Here the word ‘taxis’ stems from the Greek ‘taxis’, meaning to arrange. In a living system, the individuals will always sense the environment where they reside and respond to the external stimulus, the response will lead to opposite behavior: movement toward or away from the external stimulus, and such a response is called taxis behavior [18].)

For the positive taxis case, the equation includes two opposite phenomena, one is diffusion of cells due to their random walks, another is their aggregations toward higher concentrations of the chemical, which may result in blow up. In the second equation, the coupling between the cell equation and the chemical equation is a positive feedback, in which, the term $+n$ implies that the more cells are aggregated, the more signals are produced to attract other cells, and $-c$ represents the decay of attractant activity. While if one considers the chemotaxis phenomena in the fluid, then the velocity of the fluid must be considered, and the chemotaxis model and the fluid coupled together through transports and gravitational force $\nabla \varphi$. For the derivation of a chemotaxis-fluid model, we also refer to a recent work [1] by Bellomo et al.

If we omit the fluid velocity $u$, then the system is reduced to the classical chemotaxis model, which was first introduced by Keller and Segel in 1970 [10]. In the past several decades, the study on the Keller-Segel models have attracted a great deal of attention, and the study on the chemotaxis models mainly concentrated into two types, one is the parabolic-elliptic Keller-Segel model, and another is the parabolic-parabolic model. In particular, a large amount of work has been devoted to determining whether the solutions are global in time or blow up in finite time. For example, for the Keller-Segel model of the form

$$\begin{align*}
\frac{\partial n}{\partial t} &= \Delta n - \nabla \cdot (n \nabla c), x \in \Omega, t > 0, \\
\tau c_t - \Delta c + c &= n \quad (\tau = 0, \text{ or } 1), x \in \Omega, t > 0, \\
\left. \frac{\partial n}{\partial \nu} \right|_{\partial \Omega} &= \left. \frac{\partial c}{\partial \nu} \right|_{\partial \Omega}, \\
n(x, 0) &= n_0(x), c(x, 0) = c_0(x), x \in \Omega.
\end{align*}$$

It is easy to see that this is a mass conservation model. Denote $M := \int n_0(x) dx = \int n(x, t) dx$. For the parabolic-elliptic Keller-Segel model ($\tau = 0$) [2, 3, 7, 16], it was
shown that in dimension 2, there exists a critical mass $M^*$, such that if $M < M^*$, the solution exists globally; while if $M > M^*$, the solution may blow up in finite time. However, for the three dimensional space, finite-time blow-up may occur for arbitrarily small values of $M$, meaning that no mass threshold exists [2, 7, 16]. Similar conclusions are true for the parabolic-parabolic Keller-Segel model ($\tau = 1$), but the available results were derived much latter and are less complete. While, the logistic-type growth restriction has been detected to prevent chemotactic collapse.

In fact, it has been shown that for the following system,

$$\begin{align*}
n_t &= \Delta n - \nabla \cdot (n \nabla c) + \gamma n(1 - n), \quad x \in \Omega, \quad t > 0, \\
c_t - \Delta c + \gamma n &= n, \quad x \in \Omega, \quad t > 0, \\
\frac{\partial n}{\partial \nu} \bigg|_{\partial \Omega} &= \frac{\partial c}{\partial \nu} \bigg|_{\partial \Omega} = 0, \\
n(x, 0) &= n_0(x), \quad c(x, 0) = c_0(x), \quad x \in \Omega,
\end{align*}$$

all solutions exist globally for the two dimensional space [17], and all solutions exist globally for the three dimensional space if $\gamma$ is large [27].

The coupled chemotaxis-fluid model was first introduced by Tuval et. al [23] in 2005, which describes the dynamics of bacterial swimming and oxygen transport near contact lines, and the model is a coupled system of the chemotaxis model and the viscous incompressible Navier-Stokes equations. While if the fluid motion is slow, the Navier-Stokes equations can be replaced by the Stokes equations [15], namely, the nonlinear convective term $u \cdot \nabla u$ can be abandoned. Recently, there has been an increasing interest in the study of this subject. Specially, the global existence of solutions of chemotaxis-fluid models have attracted considerable attention. Firstly, for the following problem with the chemical consumption

$$\begin{align*}
n_t + u \cdot \nabla n &= \Delta n - \nabla \cdot (\chi(c)n \nabla c), \\
c_t + u \cdot \nabla c &= \Delta c - k(c)n, \\
u_t + \tau u \cdot \nabla u &= \Delta u - \nabla \pi + n \nabla \varphi, \\
\nabla \cdot u &= 0,
\end{align*}$$

(1.3)

In 2011, Liu and Lorz [13] considered the global solvability of weak solutions for chemotaxis(-Navier)-Stokes system, obtained global existence of weak solutions for the chemotaxis-Navier-Stokes system in two space dimensions and global existence of weak solutions of the chemotaxis-Stokes system with nonlinear diffusion in three space dimensions under some conditions. In 2012, Winkler [26] considered this problem in a bounded domain with zero-flux boundary for $n, c$ and no-slip boundary for $u$, and obtained global existence of classical solution (with any $\tau \in \mathbb{R}$) in two space dimensions and global existence of weak solution (with $\tau = 0$) in three space dimensions. Recently, Lankeit [12] studied the weak solution (with $\tau = 1$) in three space dimensions for the system (1.3) with a logistic growth term, and the author also prove that after some waiting time the weak solution becomes smooth and finally converges to a steady state. When a signal production mechanism and a logistic reaction term appear in the chemotaxis-fluid model, it leads to the system (1.1). For the case $\alpha = 0$, Espejo and Suzuki [4], Tao and Winkler [22] showed the global existence of weak solution for the two-dimensional case, Tao and Winkler [21] obtained the global classical solution under the condition $\mu > 23$ for the three-dimensional case, however, the global existence or blow-up property for small $\mu > 0$ remains open. While it is known that the cells will stop aggregating when
a certain size of the aggregation is reached, so a modified chemotaxis model with
volume-filling effect is proposed, that is the chemical term $-\nabla \cdot (n V c)$ can be re-
placed by $-\nabla \cdot (n (1 + n)^{-\alpha} V c)$. For the problem (1.1) without the logistic growth
term, that is the case $\gamma = \mu = 0$, Wang and Xiang [25] studied the global existence
of solutions in two dimensional space for any $\alpha > 0$. When the logistic growth
term is considered, Liu, Wang [14] and Zheng [29] studied the system (1.1) with
the diffusion term $\Delta n$ being replaced by $\Delta n^\alpha$, and proved the global existence
of weak solutions for the case $\alpha > \frac{6}{5} - m$. Clearly, for the equations (1.1), the global
existence is only proved when $\alpha > \frac{1}{5}$, and for the case $\alpha < \frac{1}{5}$, it is still open.

In this paper, we study the problem (1.1), we aim to prove the existence of global
bounded solution for any $\alpha > 0$. The main difficulty lies in improving regularity of
the cell density $n$. For this purpose, we use an iterative technique to improve the
regularity of $\Delta c$ and $n$ alternately. On the basis of this, we use the Neumann heat
semigroup to further raise the regularity of the solution, and the global bounded
solution finally is obtained. On the other hand, we are also concerned with the
stability of steady states. More precisely, we prove that if $\gamma = 0$, the zero solution is
globally asymptotically stable; if $\gamma > 0$, the zero solution is not stable; and specially,
if $0 < \gamma < 16\mu^2$, the nontrivial steady state $(\frac{\gamma}{\mu}, \frac{\gamma}{\mu}, 0)$ is globally asymptotically
stable.

Throughout this paper, we assume that the boundary conditions, the initial data
and $\varphi$ satisfy
\[
\partial \Omega \subset C^{2+\sigma} \text{ with } \sigma \in (0, 1),
n_0 \in C(\Omega) \cap H^2(\Omega), n_0 \geq 0,
c_0 \in W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega), c_0 \geq 0,
u_0 \in W^{2,2}(\Omega) \cap W^{1,2}(\Omega), \text{ and } \text{div} \nu_0 = 0,
\]}

We give the main theorems of this paper as follows.

**Theorem 1.1.** Assume $\alpha > 0$ and (1.4) holds. Then the problem (1.1) admits a
global classical solution $(n, c, u, \pi)$ with $n, c \geq 0$, and
\[
n \in C^0(\Omega \times [0, \infty)) \cap C^{2,1}(\Omega \times (0, \infty)),
c \in C^0(\Omega \times [0, \infty)) \cap C^{2,1}(\Omega \times (0, \infty)),
u \in C^0(\Omega \times [0, \infty)) \cap C^{2,1}(\Omega \times (0, \infty)).
\]

**Theorem 1.2.** Assume $\alpha > 0$ and (1.4) holds, and $(n, c, u, \pi)$ is a global classical
solution. Then
(i) if $\gamma = 0$, then for any initial data satisfying (1.4), we have
\[
\|n(\cdot, t)\|_{L^\infty} + \|c(\cdot, t)\|_{L^\infty} + \|u(\cdot, t)\|_{L^\infty} \to 0, \text{ as } t \to \infty;
\]
(ii) if $\gamma > 0$, then the zero solution is unstable;
(iii) if $0 < \gamma < 16\mu^2$, then for any initial data satisfying (1.4), we have
\[
\left\| \frac{n - \frac{\gamma}{\mu}}{\mu} \right\|_{L^\infty} + \left\| \frac{c - \frac{\gamma}{\mu}}{\mu} \right\|_{L^\infty} + \|u\|_{L^\infty} \to 0, \text{ as } t \to \infty.
\]
2. Preliminaries. We first give some notations, which will be used throughout this paper.

**Notations.** $C_0^\infty(\Omega)$ denotes the set of all $C^\infty(\Omega)$-real functions $\phi = (\phi_1, \cdots, \phi_n)$ with compact support in $\Omega$, such that $\text{div}\phi = 0$. The closure of $C_0^\infty(\Omega)$ with respect to norm $L^r$, is denoted by $L^r_0(\Omega)$. By [5], each $u \in L^r$ has a unique decomposition

$$u = v + \nabla p, \quad v \in L^r_0, \nabla p \in G^r,$$

with $G^r = \{\nabla p; \nabla p \in L^r; p \in L^r_{loc}\}$, and the projection $P : L^r(\Omega) \to L^r_0(\Omega)$ is called Helmholtz projection. Let $A\omega := -P\Delta \omega$, then $A$ generates a bounded analytic semigroup $\{e^{-tA}\}_{t \geq 0}$ on $L^r_0$, and the solution $u$ of (1.1) can be expressed as

$$u = e^{-tA}u_0 + \int_0^t e^{-(t-s)A}P(n(s)\nabla \varphi(s))ds.$$  

(2.1)

For more details, please refer to [11].

In this paper, let $C, C_i, \bar{C}, M$ denote some different constants, which depend at most on $\Omega, \alpha, \mu, \gamma, \nabla \varphi, n_0, c_0, u_0$ unless otherwise specified.

Before going further, we list some inequalities, which will be used throughout this paper.

By Gagliardo-Nirenberg interpolation inequality, we have

**Lemma 2.1.** For functions $u : \Omega \to \mathbb{R}$ defined on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$, we have

$$\|u\|_{L^p} \leq C_1\|Du\|^\alpha_{L^q} \|u\|^1 - \alpha_{L^r} + C_2\|u\|_{L^r},$$

where $p \geq q, r > 1, s > 0, 0 \leq \alpha \leq 1, \left(\frac{1}{q} + \frac{1}{s} - \frac{1}{r}\right)\alpha = \left(\frac{1}{q} - \frac{1}{p}\right)$. Especially, if $u \big|_{\partial \Omega} = 0$, then

$$\|u\|_{L^p} \leq C\|Du\|^\alpha_{L^q} \|u\|^{1 - \alpha}_{L^r}.$$

By [20], we have the following lemma.

**Lemma 2.2.** Let $T \geq 0, \tau \in (0, T), a > 0, b > 0$, and suppose that $y : [0, T) \to [0, \infty)$ is absolutely continuous such that

$$y'(t) + ay(t) \leq h(t), \text{ for } t \in [0, T),$$

where $h \geq 0, h(t) \in L^1_{loc}([0, T))$ and

$$\sup_{t \in (\tau, T_{\max})}\int_{t-\tau}^t h(s)ds \leq b.$$

Then

$$\sup_{t \in (0, T)} y(t) \leq \max\{y(0) + b, \frac{b}{a\tau} + 2b\}.$$

**Lemma 2.3.** Let $T \geq 0, \tau \in (0, T), \alpha > 0, \beta > 0$, and suppose that $f : [0, T) \to [0, \infty)$ is absolutely continuous, and satisfies

$$f'(t) - g(t)f(t) + f^{1+\sigma}(t) \leq h(t), t \in \mathbb{R}^+,$$

(2.2)

where $\sigma > 0$ is a constant, $g(t), h(t) \geq 0$ with $g(t), h(t) \in L^1_{loc}([0, T))$ and

$$\sup_{t \in [\tau, T]} \int_{t-\tau}^t g(s)ds \leq \alpha, \quad \sup_{t \in [\tau, T]} \int_{t-\tau}^t h(s)ds \leq \beta.$$
Then for any $t > t_0$, we have
\[ f(t) \leq f(t_0) e^\int_{t_0}^t g(s) ds + \int_{t_0}^t h(\tau) e^\int_\tau^t g(s) ds d\tau, \] (2.3)
and
\[ \sup_{t \in (0, T)} f(t) \leq \sigma \left( \frac{2A}{1 + \sigma} \right)^{1+\alpha} + 2B, \quad \sup_{t \in [\tau, T]} \int_{t-\tau}^t f^{1+\sigma}(s) ds \leq (1 + \alpha) \sup_{t \in (0, T)} \{ f(t) \} + \beta, \] (2.4)
where
\[ A = \tau^{-\frac{1+\sigma}{\sigma}} (1 + \alpha) e^{2\alpha}, \quad B = \tau^{-\frac{1+\sigma}{\sigma}} \beta e^{2\alpha} + 2\beta e^{2\alpha} + f(0)e^\alpha. \]

Proof. Firstly, omitting the term $f^{1+\sigma}$, and by a direct calculation, we see that
\[ f(t) \leq f(0) e^\int_{t_0}^t g(s) ds + \int_0^t h(\tau) e^\int_\tau^t g(s) ds d\tau. \] (2.5)
Then we have
\[ \sup_{t \in (0, T)} f(t) \leq f(0) e^\alpha + \beta e^\alpha. \] (2.6)
Next, for any $t \in (\tau, T)$, integrating the inequality (2.2) from $t - \tau$ to $t$, and using mean value theorem of integrals, we see that there exists $t_0 \in (t - \tau, t)$, such that
\[ f^{1+\sigma}(t_0) = \frac{1}{\tau} \int_{t-\tau}^t f^{1+\sigma}(s) ds \leq \frac{1}{\tau} f(t - \tau) + \frac{1}{\tau} \int_{t-\tau}^t g(s) f(s) ds + \frac{1}{\tau} \int_{t-\tau}^t h(s) ds \]
\[ \leq \frac{1}{\tau} \sup_{t \in (0, T)} f(t) + \frac{\alpha}{\tau} \sup_{t \in (0, T)} f(t) + \frac{\beta}{\tau}, \]
going back to (2.2), we see that
\[ f(t) \leq f(t_0) e^\int_{t_0}^t g(s) ds + \int_{t_0}^t h(\tau) e^\int_\tau^t g(s) ds d\tau. \]
Take $t_1 = \min\{t_0 + 2\tau, T\}$, and we obtain
\[ \sup_{t \in (\tau, T)} f(t) \leq f(t_0) e^\int_{t_0}^{t_1} g(s) ds + e^\int_{t_0}^{t_1} g(s) ds \int_{t_0}^{t_1} h(t) dt \]
\[ \leq \tau^{-\frac{1+\sigma}{\sigma}} ((1 + \alpha) \sup_{t \in (0, T)} f(t) + \beta) e^{2\alpha} + 2\beta e^{2\alpha} \]
\[ \leq \tau^{-\frac{1+\sigma}{\sigma}} ((1 + \alpha) e^{2\alpha} + \beta e^{2\alpha}) \]
combining (2.6) and (2.7), we obtain
\[ \sup_{t \in (0, T)} f(t) \leq A \left( \sup_{t \in (0, T)} f(t) \right)^{1+\sigma} + B \leq \frac{1}{2} \sup_{t \in (0, T)} f(t) + \frac{\sigma}{2} \left( \frac{2A}{1 + \sigma} \right)^{1+\sigma} + B, \]
which implies the first inequality in (2.4). Integrating (2.2) from $t - \tau$ to $t$ gives
\[ \int_{t-\tau}^t f(s) ds \leq f(t - \tau) + \alpha \sup_{t \in (0, T)} \{ f(t) \} + \beta \leq (1 + \alpha) \sup_{t \in (0, T)} \{ f(t) \} + \beta, \]
and the proof is complete. \qed
By using a fixed point method, the local solvability of system (1.1) can be proved by using a similar process of Lemma 2.1 in [26].

**Lemma 2.4.** Assume \( \alpha > 0 \) and (1.4) holds. Then there exists \( T_{\text{max}} \in (0, \infty] \) such that the problem (1.1) admits a classical solution \((n, c, u, \pi)\) with \( n, c \geq 0 \), and

\[
\begin{align*}
    n & \in C^0(\overline{\Omega} \times [0, T_{\text{max}}]) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\text{max}})), \\
    c & \in C^0(\overline{\Omega} \times [0, T_{\text{max}}]) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\text{max}})), \\
    u & \in C^0(\overline{\Omega} \times [0, T_{\text{max}}]) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\text{max}})).
\end{align*}
\]

There is exact one of the following alternatives: either \( T_{\text{max}} = \infty \), or

\[
\limsup_{t \to T_{\text{max}}} (\|n(\cdot, t)\|_{L^\infty(\Omega)} + \|c(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|A^3 u(\cdot, t)\|_{L^2(\Omega)}) = \infty,
\]

for all \( \beta \in \left(\frac{3}{4}, 1\right) \).

3. **Energy estimates.** To prove the global existence of classical solution, we give some energy estimates for the solution \((n, c, u, \pi)\) of (1.1) under the assumptions (1.4). Throughout this section, we fix \( \tau = \min\{1, \frac{T_{\text{max}}}{2}\} \leq 1 \). So, in what follows, all these constants \( C, M, C_i \) are independent of \( \tau \). For reader’s convenience, here, we give an explanation of why these constants are independent of \( \tau \). In fact, if \( \tau = 1 \), by Lemma 2.2 and Lemma 2.3, we see that the constants can be fixed. While if \( \tau < 1 \), it implies that \( T_{\text{max}} < 2 \), so the following estimates of \( \int_{t-\tau}^t \int_{\Omega} n^2 \, dx \, ds \) can be replaced by \( \int_0^{T_{\text{max}}} \| \cdot \| \, ds \).

We first have the following lemma.

**Lemma 3.1.** Assume (1.4) holds, and let \((n, c, u, \pi)\) be the solution of the problem (1.1). Then there exists a constant \( M > 0 \), such that

\[
\sup_{t \in (0, \tau]} \int_{\Omega} n(x, t) \, dx + \sup_{t \in (\tau, T_{\text{max}}]} \int_{t-\tau}^t \int_{\Omega} n^2 \, dx \, ds \leq M,
\]

where \( M \) is independent of \( T_{\text{max}} \) and \( \tau \).

**Proof.** In fact, by integrating the first equation of (1.1) over \( \Omega \), it is easy to see that

\[
\frac{d}{dt} \int_{\Omega} n \, dx + \mu \int_{\Omega} n^2 \, dx = \gamma \int_{\Omega} n \, dx.
\]

By Hölder’s inequality and Cauchy’s inequality with \( \varepsilon \), we get

\[
\mu \int_{\Omega} n^2 \, dx \geq \frac{\mu}{|\Omega|} \left( \int_{\Omega} n \, dx \right)^2 \geq 2\gamma \int_{\Omega} n \, dx - \frac{\gamma^2 |\Omega|}{\mu}.
\]

Then we have

\[
\frac{d}{dt} \int_{\Omega} n \, dx + \gamma \int_{\Omega} n \, dx \leq \frac{\gamma^2 |\Omega|}{\mu}.
\]

By a direct calculation, we obtain

\[
\sup_{t \in (0, T_{\text{max}}]} \int_{\Omega} n \, dx \leq \|n_0\|_{L^1} + \frac{\gamma |\Omega|}{\mu}.
\]

Recalling that \( \tau \leq 1 \), integrating the equality (3.2) from \( t - \tau \) to \( t \), we obtain

\[
\int_{t-\tau}^t \int_{\Omega} n^2 \, dx \, ds \leq C \sup_{t \in (0, T_{\text{max}}]} \int_{\Omega} n \, dx,
\]

and (3.1) is obtained. \( \square \)
Lemma 3.2. Assume (1.4) holds, and let \((n, c, u, \pi)\) be the solution of the problem (1.1), then
\[
\sup_{t \in (0, T_{\text{max}})} \|u\|_{H^1}^2 + \sup_{t \in (\tau, T_{\text{max}})} \int_{t-\tau}^t (\|u\|_{H^2}^2 + \|u_t\|_{L^2}^2) \, ds \leq C,
\] (3.3)
where \(C\) is a constant, which is independent of \(T_{\text{max}}\) and \(\tau\).

Proof. Multiplying the third equation of (1.1) by \(u\), and integrating it over \(\Omega\), we see that
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 \, dx + \int_{\Omega} |\nabla u|^2 \, dx = \int_{\Omega} n \nabla \varphi \cdot u \, dx \leq \|\nabla \varphi\|_{L^\infty} \|n\|_{L^2} \|u\|_{L^2},
\]
by Poincaré inequality and Young’s inequality, we get
\[
\frac{d}{dt} \int_{\Omega} u^2 \, dx + C \|u\|_{H^1}^2 \leq C \|n\|_{L^2}^2.
\] (3.4)
Applying the Helmholtz projection operator \(P\) to both sides of the third equation of (1.1), we have
\[
u_t - \Delta u = P(n \nabla \varphi).
\] (3.5)
Multiplying this equation by \(-\Delta u\), then integrating it over \(\Omega\), we get
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla u|^2 \, dx + \int_{\Omega} |\Delta u|^2 \, dx = - \int_{\Omega} P(n \nabla \varphi) \Delta u \, dx \leq \frac{1}{2} \|\Delta u\|_{L^2}^2 + C \|n\|_{L^2}^2.
\]
If \(T_{\text{max}} \geq 2\), then \(\tau = 1\), and by virtue of Lemma 2.2 and Lemma 3.1, we obtain
\[
\sup_{t \in (0, T_{\text{max}})} \|u\|_{H^1}^2 + \sup_{t \in (\tau, T_{\text{max}})} \int_{t-\tau}^t \|u\|_{H^2}^2 \, ds \leq C.
\]
If \(T_{\text{max}} < 2\), then \(\tau < 1\), and by a direct calculation, the above inequality still holds.

Next, we estimate the term \(u_t\). Multiplying the third equation of (1.1) by \(u_t\), and integrating it over \(\Omega\), we obtain
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla u|^2 \, dx + \int_{\Omega} |u_t|^2 \, dx = \int_{\Omega} n \nabla \varphi u_t \, dx \leq \frac{1}{2} \|u_t\|_{L^2}^2 + C \|n\|_{L^2}^2,
\]
for any \(t \in (\tau, T_{\text{max}})\), integrating it from \(t - \tau\) to \(t\) yields
\[
\int_{t-\tau}^t \|u_t\|_{L^2}^2 \, ds \leq C,
\]
and the proof is completed. \(\Box\)

Lemma 3.3. Assume (1.4) holds, and \((n, c, u, \pi)\) be the solution of the problem (1.1), then
\[
\sup_{t \in (0, T_{\text{max}})} \|c\|_{H^1}^2 + \sup_{t \in (\tau, T_{\text{max}})} \int_{t-\tau}^t (\|c\|_{H^2}^2 + \|c_t\|_{L^2}^2) \, ds \leq C,
\]
where \(C\) is a constant independent of \(T_{\text{max}}\) and \(\tau\).
Proof. Multiplying the second equation of (1.1) by $c$, and integrating it over $\Omega$, we get
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega c^2 \, dx + \int_\Omega |\nabla c|^2 \, dx + \frac{1}{2} \int_\Omega c^2 \, dx \leq \|n\|_{L^2}^2,
\]
and thus by Lemma 2.2, we obtain
\[
\sup_{t \in (0,T_{\text{max}})} \|c\|_{L^2}^2 + \sup_{t \in (\tau,T_{\text{max}})} \int_{t-\tau}^t \|c\|_{H^1}^2 \, ds \leq C. \tag{3.6}
\]
Multiplying the second equation of (1.1) by $\Delta c$, integrating it over $\Omega$, and using Lemma 2.1 and Young's inequality, we get
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla c|^2 \, dx + \int_\Omega |\Delta c|^2 \, dx + \int_\Omega |\nabla c|^2 \, dx = \int_\Omega u \nabla c \Delta c \, dx = \int_\Omega n \Delta c \, dx \\
\leq \|u\|_{L^1}[\|\nabla c\|_{L^2}[\|\Delta c\|_{L^2} + \|n\|_{L^2}[\|\Delta c\|_{L^2}] \\
\leq C_1 \|u\|_{H^1} \|\nabla c\|_{L^2} \|\Delta c\|_{L^2} + C_2 \|u\|_{H^1} \|\nabla c\|_{L^2} \|\Delta c\|_{L^2} + \|n\|_{L^2} \|\Delta c\|_{L^2} \\
\leq \frac{1}{2} \|\Delta c\|_{H^1}^2 + C_3 \|u\|_{L^1} \|\nabla c\|_{H^1}^2 + C_4 \|u\|_{H^1} \|\nabla c\|_{L^2}^2 + \|n\|_{L^2}^2.
\]
Combining (3.1), (3.3) and (3.6), then using Lemma 2.2, we obtain
\[
\sup_{t \in (0,T_{\text{max}})} \|\nabla c\|_{L^2}^2 + \sup_{t \in (\tau,T_{\text{max}})} \int_{t-\tau}^t \|\nabla c\|_{H^1}^2 \, ds \leq C. \tag{3.7}
\]
Multiplying the second equation of (1.1) by $c_1$, and integrating it over $\Omega$, we conclude
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega (|\nabla c|^2 + c_1^2) \, dx + \int_\Omega |c_1|^2 \, dx = -\int_\Omega u \nabla c_1 \, dx + \int_\Omega n c_1 \, dx \\
\leq \frac{1}{2} \|c_1\|_{L^2}^2 + \|u\|_{L^1} \|\nabla c_1\|_{H^1}^2 + \|n\|_{L^2}^2,
\]
which implies that
\[
\sup_{t \in (\tau,T_{\text{max}})} \int_{t-\tau}^t \|c_1\|_{L^2}^2 \, ds \leq C.
\]
The proof is complete. \qed

Lemma 3.4. Assume (1.4) holds, and let $(n,c,u,\pi)$ be the solution of the problem (1.1). For any given $r > 0$, we have
\[
\sup_{t \in (\tau,T_{\text{max}})} \int_{t-\tau}^t \|\Delta c\|_{L^2}^2 \, ds \leq C,
\]
\[
\sup_{t \in (0,T_{\text{max}})} \|n(t)\|_{L^{\frac{r+1}{2}}}^{r+1} + \sup_{t \in (\tau,T_{\text{max}})} \int_{t-\tau}^t \int_\Omega \left( |\nabla n_{\frac{r+1}{2}}|^2 + \int_\Omega n^{r+2} \right) \, dx \, ds \leq C,
\]
\[
\sup_{t \in (0,T_{\text{max}})} \|\Delta c(t)\|_{L^2}^2 + \sup_{t \in (\tau,T_{\text{max}})} \int_{t-\tau}^t \|\nabla \Delta c\|_{L^2}^2 \, ds \leq C,
\]
where $C$ depends on $r$, $\alpha$, $\Omega$, $c_0$, $\n_0$, and it is independent of $T_{\text{max}}$ and $\tau$. 

Proof. Multiplying the first equation of (1.1) by \(n^r\) for any \(r > 0\), and integrating it over \(\Omega\), we get
\[
\frac{1}{r+1} \frac{d}{dt} \int_{\Omega} n^{r+1} dx + \frac{4r}{(r+1)^2} \int_{\Omega} |\nabla n^{r+1}|^2 dx + \mu \int_{\Omega} n^{r+2} dx
\]
\[
= - \int_{\Omega} g(n) \Delta c dx + \gamma \int_{\Omega} n^{r+1} dx
\]
\[
\leq - \int_{\Omega} g(n) \Delta c dx + \frac{\mu}{4} \int_{\Omega} n^{r+2} dx + C,
\]
where \(g(n) = r \int_0^n s^r(1+s)^{-\alpha}\). Take \(r = 2\alpha\), use Young’s inequality, and thus we see that
\[
- \int_{\Omega} g(n) \Delta c dx \leq \|g(n)\|_{L^2} \|\Delta c\|_{L^2} \leq \frac{\mu}{4} \|n\|^2_{L^{2(\alpha+1)}} + C \|\Delta c\|^2_{L^2}.
\]
Substituting this inequality into (3.8), we obtain
\[
\frac{1}{2\alpha+1} \frac{d}{dt} \int_{\Omega} n^{2\alpha+1} dx + \frac{8\alpha}{(2\alpha+1)^2} \int_{\Omega} |\nabla n^{2\alpha+1}|^2 dx + \frac{\mu}{2} \int_{\Omega} n^{2\alpha+2} dx
\]
\[
\leq C(1 + \|\Delta c\|^2_{L^2}).
\]
If \(\tau = 1\), recalling Lemma 2.3, and using Lemma 3.3, we obtain
\[
\sup_{t \in (0, T_{\text{max}})} \int_{\Omega} n^{2\alpha+1} dx + \sup_{t \in (\tau, T_{\text{max}})} \int_{t-\tau}^t \int_{\Omega} \left(|\nabla n^{2\alpha+1}|^2 + n^{2\alpha+2}\right) dx ds \leq C_1; \quad (3.9)
\]
if \(\tau < 1\), it implies \(T_{\text{max}} < 2\), so by a direct calculation, (3.9) still holds, and \(C_1\) is independent of \(\tau\) and \(T_{\text{max}}\). Using Lemma 2.1, we see that
\[
\|u \nabla c\|_{L^{\frac{5}{2}}} \leq \|u\|_{L^5} \|\nabla c\|_{L^{\infty}} \leq C \|\nabla u\|_{L^2} \|\nabla c\|_{L^2} \frac{1}{5} \|\Delta c\|_{L^2}^{4/5} + C \|\nabla u\|_{L^2} \|\nabla c\|_{L^2},
\]
and recalling Lemma 3.2 and Lemma 3.3, we have
\[
\sup_{t \in (\tau, T_{\text{max}})} \int_{t-\tau}^t \|u \nabla c\|_{L^{\frac{5}{2}}}^2 ds \leq C_1 \sup_{t \in (\tau, T_{\text{max}})} \int_{t-\tau}^t \|\Delta c\|_{L^2}^2 ds + C_2 \leq C_3. \quad (3.10)
\]
If \(2(1+\alpha) \geq \frac{5}{2}\), recalling the second equation of (1.1), then by the maximal regularity results of linear parabolic problems [8], and combining with (3.9), (3.10), we have
\[
\sup_{t \in (\tau, T_{\text{max}})} \int_{t-\tau}^t \|\Delta c\|_{L^{\frac{5}{2}}}^2 ds \leq C. \quad (3.11)
\]
While if \(2(1+\alpha) < \frac{5}{2}\), then by (3.9), (3.10), we have
\[
\sup_{t \in (\tau, T_{\text{max}})} \int_{t-\tau}^t \|\Delta c\|_{L^{2(1+\alpha)}}^{2(1+\alpha)} ds \leq C.
\]
The inequality (3.11) is important to improve the regularity of \(n\). In what follows, we use an iterative method to show (3.11) for the case \(2(1+\alpha) < \frac{5}{2}\). In fact, if
\[
\sup_{t \in (\tau, T_{\text{max}})} \int_{t-\tau}^t \|\Delta c\|_{L^{2(1+\alpha)}}^{2(1+\alpha)} ds \leq C \text{ with } 2(1+\alpha) < \frac{5}{2},
\]
we take $r = 2(1 + \alpha)^{k+1} - 2$ in (3.8) and obtain
\[
\frac{1}{2(1 + \alpha)^{k+1} - 1} \frac{d}{dt} \int_{\Omega} n^{2(1+\alpha)^{k+1}-1} dx + \frac{8((1 + \alpha)^{k+1} - 1)}{(2(1 + \alpha)^{k+1} - 1)^2} \int_{\Omega} \|
abla n\|_{L^2(\Omega)} 2L^{(2(1+\alpha)^{k+1}-1)} dx \\
+ \mu \int_{\Omega} n^{2(1+\alpha)^{k+1}} dx \leq \|g(n)\|_{L^{2(1+\alpha)^{k+1}}} \|\Delta c\|_{L^{2(1+\alpha)^{k}}} + \frac{\mu}{4} \int_{\Omega} n^{2(1+\alpha)^{k+1}} dx + C \\
\leq \|n\|_{L^{2(1+\alpha)^{k+1}}} \|\Delta c\|_{L^{2(1+\alpha)^{k}}} + \frac{\mu}{4} \int_{\Omega} n^{2(1+\alpha)^{k+1}} dx + C \\
\leq C \|\Delta c\|_{L^{2(1+\alpha)^{k}}}^2 + \frac{\mu}{2} \int_{\Omega} n^{2(1+\alpha)^{k+1}} dx + C,
\]
by Lemma 2.3, we arrive at
\[
\sup_{t \in (0,T_{\max})} \int_{\Omega} n^{2(1+\alpha)^{k+1}-1} dx + \sup_{t \in (\tau,T_{\max})} \int_{t-\tau}^{t} \int_{\Omega} \|
abla n\|_{L^2(\Omega)} 2L^{(2(1+\alpha)^{k+1}-1)} dx ds \\
+ \sup_{t \in (\tau,T_{\max})} \int_{t-\tau}^{t} \int_{\Omega} n^{2(1+\alpha)^{k+1}} dx ds \leq C_{k+1}.
\]
Then, repeating the above process, we obtain
\[
\int_{t-\tau}^{t} \|\Delta c\|_{L^{q^*}}^2 dx ds \leq C,
\]
where $q^* = \min\{2(1 + \alpha)^{k+1}, \frac{5}{2}\}$. Note that there must exist an integer $k^* > 0$ such that $2(1 + \alpha)^{k^*-1} < \frac{5}{2} \leq 2(1 + \alpha)^{k^*}$. Thus (3.11) holds. Next, let’s go back to (3.8). For any $r > \max\{\frac{5\alpha}{2} - 1, 0\}$, clearly, we have $2 < \frac{10(r - \alpha + 1)}{3(r+1)} < 6$, then using Lemma 2.1 and Lemma 3.1, we see that
\[
\int_{\Omega} g(n) \Delta c dx \leq C \|n^{r-1}\|_{L^{10(r - \alpha + 1)/(r+1)}} \|\Delta c\|_{L^2}^2 \\
\leq C_1 \|
abla n\|_{L^2(\Omega)}^{\frac{6(r-1)+15\alpha}{5(r+1)}} \|n^{r-1}\|_{L^2(\Omega)}^{\frac{5(r+1)}{6(r+1)+15\alpha}} \|\Delta c\|_{L^2}^2 + C_2 \|n^{r-1}\|_{L^{10(r - \alpha + 1)/(r+1)}} \|\Delta c\|_{L^2}^2 \\
\leq \frac{2r}{(r+1)^2} \|
abla n\|_{L^2(\Omega)}^{\frac{5(r-1)}{2}} \|n^{r-1}\|_{L^2(\Omega)}^2 + C_3 \|n^{r-1}\|_{L^2(\Omega)}^{\frac{5(r+1)}{2(r+1)+10\alpha}} \|\Delta c\|_{L^2}^2 + C_4 \|\Delta c\|_{L^2}^2 \\
\leq \frac{2r}{(r+1)^2} \|
abla n\|_{L^2(\Omega)}^{\frac{5(r^+)}{2}} \|n^{r^+}\|_{L^2(\Omega)}^2 + C_5 \left( \|n^{r^+}\|_{L^2(\Omega)}^2 \|\Delta c\|_{L^2(\Omega)}^{\frac{5}{2}} + \|n^{r^+}\|_{L^2(\Omega)}^2 + \|\Delta c\|_{L^2(\Omega)}^{\frac{5}{2}} + 1 \right) \\
\leq \frac{2r}{(r+1)^2} \|
abla n\|_{L^2(\Omega)}^{\frac{5(r^+)}{2}} \|n^{r^+}\|_{L^2(\Omega)}^2 + \frac{\mu}{4} \int_{\Omega} n^{r^+} dx + C_6 \left( \|n^{r^+}\|_{L^2(\Omega)}^2 \|\Delta c\|_{L^2(\Omega)}^{\frac{5}{2}} + \|\Delta c\|_{L^2(\Omega)}^{\frac{5}{2}} + 1 \right);
\]
while if $0 < r \leq \frac{5\alpha}{2} - 1$, that is $\frac{5(r - \alpha + 1)}{3} \leq r + 1$ we see that
\[
\int_{\Omega} g(n) \Delta c dx \leq \|g(n)\|_{L^{\frac{5}{2}}} + \|\Delta c\|_{L^2}^{\frac{5}{2}} \leq \frac{\mu}{4} \int_{\Omega} n^{r^+} dx + \|\Delta c\|_{L^2}^{\frac{5}{2}} + C.
\]
Substituting the estimate for $\int_{\Omega} g(n) \Delta c dx$ into (3.8), we obtain
\[
\frac{d}{dt} \int_{\Omega} n^{r^+} dx + \frac{2r}{r+1} \int_{\Omega} \|
abla n\|_{L^{r+1}}^{\frac{r+1}{2}} dx + \frac{\mu(r + 1)}{2} \int_{\Omega} n^{r^+} dx \\
\leq C \left( \|n\|_{L^{r+1}}^{\frac{r+1}{2}} \|\Delta c\|_{L^2}^{\frac{5}{2}} + \|\Delta c\|_{L^2}^{\frac{5}{2}} + 1 \right) (3.12)
\]
for any \( r > 0 \). By Lemma 2.3, and using (3.11), we see that
\[
\sup_{t \in (0,T_{\max})} |n(t)|_{L^{r+1}}^{r+1} \leq C,
\]
where \( C \) depends on \( r, \|n_0\|_{L^{r+1}} \). By (3.12), we further obtain
\[
\int_{t-\tau}^{t} \int_{\Omega} |\nabla n(s)|^2 dx ds + \int_{t-\tau}^{t} \int_{\Omega} n^{r+2} dx ds \leq C \quad \text{for any} \quad t \in [\tau, T_{\max}).
\]
Recalling Lemma 2.1, applying \( \nabla \) to the second equation of (1.1), and multiplying it by \( \nabla \Delta c \), then integration over \( \Omega \) yields
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\Delta c|^2 dx + \int_{\Omega} |\nabla \Delta c|^2 dx + \int_{\Omega} |\Delta c|^2 dx
\]
\[
= \int_{\Omega} \nabla (u \nabla c \nabla \Delta c dx - \int_{\Omega} \nabla n \nabla \Delta c dx
\]
\[
\leq \|\nabla \Delta c\|_{L^p}(\|u\|_{L^q}\|\Delta c\|_{L^q} + \|\nabla u\|_{L^q}\|\nabla c\|_{L^q}) + \|\nabla n\|_{L^q}\|\nabla \Delta c\|_{L^q}
\]
\[
\leq \|\nabla \Delta c\|_{L^p}(\|u\|_{H^1}\|\Delta c\|_{L^2} + \|u\|_{H^1}\|\nabla \Delta c\|_{L^2} + \|u\|_{H^1}\|\nabla c\|_{H^1})
\]
\[
+ \|\nabla n\|_{L^q}\|\nabla \Delta c\|_{L^q}
\]
\[
\leq \frac{1}{2} \|\nabla \Delta c\|_{L^2}^2 + C(1 + \|\Delta c\|_{L^2}^2 + \|\nabla u\|_{H^1}^2 + \|\nabla u\|_{H^1}^2 \|\Delta c\|_{L^2}^2 + \|\nabla n\|_{L^2}^2),
\]
which implies that
\[
\frac{d}{dt} \int_{\Omega} |\Delta c|^2 dx + \int_{\Omega} |\nabla \Delta c|^2 dx
\]
\[
\leq C(1 + \|\Delta c\|_{L^2}^2 + \|\nabla u\|_{H^1}^2 + \|\nabla u\|_{H^1}^2 \|\Delta c\|_{L^2}^2 + \|\nabla n\|_{L^2}^2). \quad (3.15)
\]
If \( T_{\max} \leq 2 \), then by a direct calculation, we obtain
\[
\sup_{t \in (0,T_{\max})} \|\Delta c(t)\|_{L^2}^2 \leq \|\Delta c_0\|_{L^2}^2 e^{C \int_{T_{\max}}^T \|\nabla u(s)\|_{H^1}^2 ds}
\]
\[
+ C \int_{T_{\max}}^T \left( 1 + \|\Delta c\|_{L^2}^2 + \|\nabla u\|_{H^1}^2 + \|\nabla n\|_{L^2}^2 \right) e^{C \int_{T_{\max}}^s \|\nabla u(s)\|_{H^1}^2 ds} dt. \quad (3.16)
\]
If \( T_{\max} > 2 \), it implies that \( \tau = \min\{1, \frac{T_{\max}}{2}\} = 1 \). By a direct calculation, we obtain
\[
\sup_{t \in (0,\tau)} \|\Delta c(t)\|_{L^2}^2 \leq \|\Delta c_0\|_{L^2}^2 e^{\int_0^\tau \|\nabla u(s)\|_{H^1}^2 ds}
\]
\[
+ C \int_0^\tau \left( 1 + \|\Delta c\|_{L^2}^2 + \|\nabla u\|_{H^1}^2 + \|\nabla n\|_{L^2}^2 \right) e^{\int_0^s \|\nabla u(s)\|_{H^1}^2 ds} ds. \quad (3.17)
\]
For any \( t \in (\tau, T_{\max}) \), using mean value theorem of integrals, there exists \( t_0 \in (t-\tau, t) \), such that
\[
\|\Delta c(t_0)\|_{L^2}^2 = \frac{1}{\tau} \int_{t-\tau}^t \|\Delta c(s)\|_{L^2}^2 ds \leq C.
\]
Recall (3.15), omit the second term \( \|\nabla \Delta c\|_{L^2}^2 \), and it follows
\[
\|\Delta c(t)\|_{L^2}^2 \leq \|\Delta c(t_0)\|_{L^2}^2 e^{\int_0^t \|\nabla u(s)\|_{H^1}^2 ds}
\]
\[
+ C \int_{t_0}^t \left( 1 + \|\Delta c\|_{L^2}^2 + \|\nabla u\|_{H^1}^2 + \|\nabla n\|_{L^2}^2 \right) e^{\int_{t_0}^s \|\nabla u(s)\|_{H^1}^2 ds} ds. \quad (3.18)
\]
Combining Lemma 3.2, Lemma 3.3, inequalities (3.14) with \( r = 1 \), and (3.17), we obtain

\[
\sup_{t \in (0, T_{\text{max}})} \| \Delta c(t) \|_{L^2}^2 \leq C. \tag{3.19}
\]

Then integrating (3.15) from \( t - \tau \) to \( t \) gives

\[
\int_{t-\tau}^{t} \| \nabla \Delta c \|_{L^2}^2 ds \leq C. \tag{3.20}
\]

The proof is complete. \( \square \)

Using Lemma 3.4, we can prove the following lemma

**Lemma 3.5.** Assume (1.4) and Lemma 3.4 hold. Then

\[
\sup_{t \in (0, T_{\text{max}})} \| \nabla n \|_{L^2}^2 + \sup_{t \in (\tau, T_{\text{max}})} \left( \int_{t-\tau}^{t} \int_{\Omega} (|n_t|^2 + |\Delta n|^2 + n|\nabla n|^2) dx ds \right) \leq C,
\]

where \( C \) is independent of \( T_{\text{max}} \) and \( \tau \).

**Proof.** Multiplying the first equation of (1.1) by \( \Delta n \), and integrating it over \( \Omega \), we get

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla n|^2 dx + \int_{\Omega} |\Delta n|^2 dx + 2\mu \int_{\Omega} n|\nabla n|^2 dx = \int_{\Omega} u \nabla n \Delta n dx + \int_{\Omega} \nabla \cdot (n(1 + n)^{-\alpha} \nabla c) \Delta n dx + \gamma \int_{\Omega} |\nabla n|^2 dx = I_1 + I_2 + I_3.
\]

For \( I_1 \), by virtue of Lemma 2.1 and Young’s inequality, we see that

\[
I_1 \leq \| u \|_{L^6} \| \nabla n \|_{L^6} \| \Delta n \|_{L^2} \\
\leq C \| u \|_{H^1} \| \nabla n \|_{L^2}^{1/2} \| \Delta n \|_{L^2}^{3/2} + C \| u \|_{H^1} \| \nabla n \|_{L^2} \| \Delta n \|_{L^2} \\
\leq \frac{1}{4} \| \Delta n \|_{L^2}^2 + C \| \nabla n \|_{L^2}^2.
\]

Moreover for \( I_2 \), by Lemma 3.4, we have

\[
I_2 = \int_{\Omega} n(1 + n)^{-\alpha} \Delta c \Delta n dx + \int_{\Omega} \left( (1 + n)^{-\alpha} - \frac{\alpha n}{(1 + n)^{1+\alpha}} \right) \nabla n \nabla c \Delta n dx \\
\leq \| n \|_{L^6} \| \Delta c \|_{L^6} \| \Delta n \|_{L^2} + \| \nabla n \|_{L^6} \| \nabla c \|_{L^6} \| \Delta n \|_{L^2} \\
\leq \| n \|_{L^6} \| \Delta c \|_{H^1} \| \Delta n \|_{L^2} + \| \nabla n \|_{L^6} \| \Delta n \|_{H^1}^2 \| \nabla c \|_{H^1} + \| \nabla n \|_{L^6} \| \nabla c \|_{H^1} \| \Delta n \|_{L^2} \\
\leq \frac{1}{4} \| \Delta n \|_{L^2}^2 + C(\| \nabla n \|_{L^2}^2 + \| \Delta c \|_{H^1}^2).
\]

Summing up, we obtain

\[
\frac{d}{dt} \int_{\Omega} |\nabla n|^2 dx + \int_{\Omega} |\Delta n|^2 dx + 4\mu \int_{\Omega} n|\nabla n|^2 dx \\
\leq C(\| \nabla n \|_{L^2}^2 + \| \Delta c \|_{H^1}^2).
\]

By Lemma 3.4, and similarly to the proof of (3.19) and (3.20), we obtain

\[
\sup_{t \in (0, T_{\text{max}})} \| \nabla n \|_{L^2}^2 + \sup_{t \in (\tau, T_{\text{max}})} \left( \int_{t-\tau}^{t} (|\Delta n|^2 + n|\nabla n|^2) dx ds \right) \leq C.
\]
Multiplying the first equation of (1.1) by $n_t$, integrating it over $\Omega$, and using a similar method as above, we also obtain

$$\sup_{t \in (\tau, T_{max})} \int_{t}^{T} \int_{\Omega} |n_t|^2 dx ds \leq C.$$  

Summing up, and we finally obtain the desired estimate. \hfill \Box

4. Global existence and stability. By Lemma 2.4, we see that to prove the global existence of classical solutions, we only need to prove the boundedness of the following norm

$$\|n(\cdot, t)\|_{L^\infty(\Omega)} + \|c(\cdot, t)\|_{W^{1, \infty}(\Omega)} + \|A^\beta u(\cdot, t)\|_{L^2(\Omega)}$$

with $\beta \in (\frac{1}{2}, 1)$.

\textbf{Proof of Theorem 1.1.} We see that the first equation of (1.1) is equivalent to

$$n_t - \Delta n + n = F(u, n, c),$$

where

$$F(x, t) = F(u(x, t), n(x, t), c(x, t)) = -u \cdot \nabla n - \nabla \cdot (n(1 + n)^{-\alpha} \nabla c) + (\gamma + 1)n - \mu n^2.$$  

By Duhamel's principle, we see that the solution of (4.1) can be expressed as follows

$$n(x, t) = e^{-t} e^{\Delta} n_0 + \int_{0}^{t} e^{-(t-s)} e^{(t-s)\Delta} F(x, s) ds,$$

where $\{e^{\Delta}\}_{t \geq 0}$ is the heat semigroup on the domain $\Omega$ under Neumann boundary condition, for more properties of Neumann heat semigroup, please refer to [28].

Then for any $3 < p < 6$, by using the regularity estimates in Lemma 3.2–Lemma 3.5, we have

$$\|\nabla n\|_{L^p} \leq e^{-t}\|\nabla n_0\|_{L^p} + \int_{0}^{t} e^{-(t-s)} \left\| \nabla \left( e^{(t-s)\Delta} \left( (\gamma + 1)n - \mu n^2 \right) \right) \right\|_{L^p} ds$$

$$+ \int_{0}^{t} e^{-(t-s)} \left\| \nabla \left( e^{(t-s)\Delta} \left( u \cdot \nabla n + \nabla \cdot (n(1 + n)^{-\alpha} \cdot \nabla c) \right) \right) \right\|_{L^p} ds$$

$$\leq e^{-t}\|\nabla n_0\|_{L^p} + \int_{0}^{t} e^{-(t-s)} (t-s)^{-\frac{1}{2} - \frac{3}{2}(\frac{1}{2} - \frac{1}{p})} \|u \cdot \nabla n + \nabla \cdot (n(1 + n)^{-\alpha} \cdot \nabla c)\|_{L^p} ds$$

$$+ \int_{0}^{t} e^{-(t-s)} (t-s)^{-\frac{1}{2} - \frac{3}{2}(\frac{1}{2} - \frac{1}{p})} (\gamma + 1)n - \mu n^2\|_{L^2} ds$$

$$\leq e^{-t}\|\nabla n_0\|_{L^p} + \int_{0}^{t} e^{-(t-s)} (t-s)^{-\frac{1}{2} - \frac{3}{2}(\frac{1}{2} - \frac{1}{p})} \|n\|_{L^2} + \mu \|n\|_{L^3}^2 ds$$

$$+ \|n\|_{L^p} \|\nabla n\|_{L^3} + \|n\|_{L^\infty} \|\Delta c\|_{L^2} + \|\nabla n\|_{L^3} \||\nabla c\|_{L^6} ds$$

$$\leq e^{-t}\|\nabla n_0\|_{L^p} + C \int_{0}^{t} e^{-(t-s)} (t-s)^{-\frac{1}{2} - \frac{3}{2}(\frac{1}{2} - \frac{1}{p})} \left( \|\nabla n\|_{L^2}^{\frac{2(p-3)}{p-2}} \|\nabla n\|_{L^p}^{\frac{3}{p-2}} + 1 \right) ds$$

$$\leq e^{-t}\|\nabla n_0\|_{L^p} + C \left( 1 + \sup_{t \in (0, T_{max})} \{ \|\nabla n\|_{L^p}^{\frac{3}{p-2}} \} \right) \int_{0}^{\infty} e^{-\tau} \tau^{-\frac{1}{2} - \frac{3}{2}(\frac{1}{2} - \frac{1}{p})} d\tau.$$
Noticing that \( \frac{1}{2} + \frac{3}{2} \left( \frac{1}{p} - \frac{1}{p} \right) < 1 \) since \( p < 6 \), then we further obtain
\[
\sup_{t \in (0, T_{\text{max}})} \| \nabla n \|_{L^p} \leq C_1 + C_2 \left( \sup_{t \in (0, T_{\text{max}})} \| \nabla n \|_{L^p} \right)^{\frac{p}{4(p-2)}},
\]
which implies that there exists a constant \( M > 0 \) independent of \( T_{\text{max}} \), such that
\[
\sup_{t \in (0, T_{\text{max}})} \| \nabla n \|_{L^p} \leq M, \quad \forall 3 < p < 6
\]
(4.2) since \( \frac{p}{3(p-2)} < 1 \). By Sobolev embedding theorem, \( n \in L^\infty(\Omega \times (0, T_{\text{max}})) \).

For \( u \), we see that
\[
\| A^\beta u \|_{L^2} \leq e^{-t} \| A^\beta u_0 \|_{L^2} + \int_0^t e^{-s} \| A^\beta e^{-\beta(t-s)} A^\beta A^\beta e^{-\beta(t-s)} A^\beta \| \| n \|_{L^p} \| \nabla \phi \|_{L^p} ds
\]
\[
\leq e^{-t} \| A^\beta u_0 \|_{L^2} + \int_0^t e^{-\lambda(t-s)} (t-s)^{-\beta} \| n \|_{L^p} \| \nabla \phi \|_{L^p} ds
\]
\[
\leq e^{-t} \| A^\beta u_0 \|_{L^2} + \int_0^t e^{-\lambda(t-s)} (t-s)^{-\beta} \| n \|_{L^\infty} \| \nabla \phi \|_{L^p} ds
\]
\[
\leq C.
\]
(4.3)

By embedding theorem, \( u \in L^\infty(\Omega \times (0, T_{\text{max}})) \) since \( \beta > \frac{3}{4} \).

For \( c \), clearly, we have \( c \in L^\infty(\Omega \times (0, T_{\text{max}})) \) since \( c \in L^\infty((0, T_{\text{max}}); H^2(\Omega)) \).

Similarly, for \( \nabla c \), we have
\[
\| \nabla c \|_{L^\infty} \leq e^{-t} \| \nabla c_0 \|_{L^\infty} + \int_0^t e^{-s} \| \nabla \left( e^{(t-s)} \Delta \left( -u \cdot \nabla n + n \right) \right) \|_{L^\infty} ds
\]
\[
\leq e^{-t} \| \nabla c_0 \|_{L^\infty} + \int_0^t e^{-\lambda(t-s)} (t-s)^{-\frac{1}{2}} \| -u \cdot \nabla c + n \|_{L^6} ds
\]
\[
\leq e^{-t} \| \nabla c_0 \|_{L^\infty} + \int_0^t e^{-\lambda(t-s)} (t-s)^{-\frac{1}{2}} \left( \| u \|_{L^\infty} \| \nabla c \|_{L^6} + \| n \|_{L^6} \right) ds
\]
\[
\leq C.
\]
(4.4)

Summing up, we obtain
\[
\sup_{t \in (0, T_{\text{max}})} \left\{ \| n(t) \|_{L^\infty(\Omega)} + \| c(t) \|_{W^{1, \infty}(\Omega)} + \| A^\beta u(t) \|_{L^2(\Omega)} \right\} \leq M,
\]
where \( M \) is independent of \( T_{\text{max}} \). Recalling Lemma 2.4, we see that the solution of problem (1.1) exists globally. \( \square \)

Next, we show that if \( \gamma > 0 \), the zero solution is unstable; and if \( \gamma = 0 \), the zero solution is globally asymptotically stable.

**Lemma 4.1.** If \( \gamma > 0 \), then \( \| n(t) \|_{L^\infty} \not\to 0 \) as \( t \to \infty \).

**Proof.** Suppose to the contrary, then there exists \( t_0 > 0 \) such that \( \| n(t) \|_{L^\infty} < \frac{\gamma}{2\mu} \) for \( t > t_0 \), and \( \| n(t, t_0) \|_{L^\infty} > 0 \). Then integrating the first equation of (1.1) over \( \Omega \) yields
\[
\frac{d}{dt} \int_\Omega ndx = \gamma \int_\Omega ndx - \mu \int_\Omega n^2dx \geq \gamma \int_\Omega ndx, \quad \text{for} \ t \geq t_0,
\]
which implies that
\[
\int_\Omega n(x, t)dx \geq e^{\frac{\gamma}{2}(t-t_0)} \int_\Omega n(x, t_0)dx \to \infty, \text{ as } t \to \infty.
\]
It is a contradiction. \( \square \)
Lemma 4.2. If $\gamma = 0$, then
\[
\int_0^t \int_\Omega n^2 dx dt \leq \frac{1}{\mu} \int_\Omega n_0 dx, \tag{4.5}
\]
\[
\|n(\cdot, t)\|_{L^1} \leq \left(\|n_0\|_{L^1}^{-1} + \frac{\mu t}{|\Omega|}\right)^{-1}, \tag{4.6}
\]
\[
\|c(t)\|_{L^1} \leq \frac{M_2}{1 + t}, \tag{4.7}
\]
\[
\|u\|_{L^1}^2 \leq \frac{M_2}{1 + t}, \tag{4.8}
\]
where $M_1, M_2$ are constants. It implies that
\[
\|n(\cdot, t)\|_{L^\infty} + \|c(\cdot, t)\|_{L^\infty} + \|u(\cdot, t)\|_{L^\infty} \to 0,
\]
as $t \to \infty$.

Proof. Integrating the first equation of (1.1) over $\Omega$ yields
\[
\frac{d}{dt} \int_\Omega ndx = -\mu \int_\Omega n^2 dx \leq -\frac{\mu}{|\Omega|} \left(\int_\Omega ndx\right)^2.
\]
Integrating the above equality from 0 to $t$, then (4.5) is proved. Note that
\[
\mu \int_\Omega n^2 dx \geq \frac{\mu}{|\Omega|} \left(\int_\Omega ndx\right)^2,
\]
which implies (4.6). Integrating the second equation of (1.1) over $\Omega$ yields
\[
\frac{d}{dt} \int_\Omega cdx + \int_\Omega cdx = \int_\Omega ndx \leq \frac{C}{1 + t},
\]
then we further have
\[
\|c(t)\|_{L^1} \leq \|c_0\|_{L^1} e^{-t} + \int_0^t \frac{C}{1 + s} e^{s-t} ds
\]
\[
\leq \|c_0\|_{L^1} e^{-t} + \int_0^t \frac{2}{1 + s} e^{s-t} ds + \int_t^{t+1} \frac{C}{1 + s} e^{s-t} ds
\]
\[
\leq \|c_0\|_{L^1} e^{-t} + C e^{-\frac{t}{2}} \ln(1 + \frac{t}{2}) + \frac{C}{1 + \frac{t}{2}}
\]
\[
\leq \frac{M}{1 + t}.
\]
Recalling (3.4), we see that
\[
\frac{d}{dt} \|u\|_{H^1}^2, dx + C_1 \|u\|_{H^1}^2 \leq \hat{C} \|n\|_{L^2}^2 \leq C_2 \|n\|_{L^1}.
\]
Similarly to the proof above, we obtain (4.8).

Next, we show that $\|n(\cdot, t)\|_{L^\infty} + \|c(\cdot, t)\|_{L^\infty} + \|u(\cdot, t)\|_{L^\infty} \to 0$ as $t \to \infty$. Recalling Lemma 2.1, and using (4.2) and (4.6), we obtain
\[
\|n(\cdot, t)\|_{L^\infty} \leq C \|n(\cdot, t)\|_{L^2} \|n(\cdot, t)\|_{L^1} \leq C_1 \|
\]
\[
\|c(\cdot, t)\|_{L^\infty} \leq C \|c(\cdot, t)\|_{L^2} \|c(\cdot, t)\|_{L^1} \leq C_2 \|c(\cdot, t)\|_{L^1} \leq C(1 + t)^{-\frac{1}{4}}.
\]
By fractional Gagliardo-Nirenberg interpolation inequality [6], recalling (4.3) and noticing that \( \beta > \frac{3}{4} \), we arrive at
\[
\|u(\cdot, t)\|_{L^\infty} \leq C_1 \|u(\cdot, t)\|_{L^6}^{\frac{48-3}{24-2}} \|A^\beta u\|_{L^2}^{\frac{1}{24-2}} + C_2 \|u(\cdot, t)\|_{H^1},
\]
\[
\leq C(1 + t)^{-\frac{4\beta - 2}{2\beta - 1}}.
\]

Summing up, we obtain
\[
\|n(\cdot, t)\|_{L^\infty} + \|c(\cdot, t)\|_{L^\infty} + \|u(\cdot, t)\|_{L^\infty} \to 0, \quad \text{as } t \to \infty.
\]

Next, we use an idea of [9] to show the stability of the nontrivial steady state \((\frac{\gamma}{\mu}, \frac{\gamma}{mu}, \omega(x))\), here \(\omega(x) = 0\) is the solution of the elliptic problem.

\[
\begin{aligned}
-\Delta \omega + \nabla \pi_1 &= \frac{\gamma}{\mu} \nabla \varphi, \quad \text{in } \Omega, \\
\nabla \cdot \omega &= 0, \quad \text{in } \Omega, \\
\omega|_{\partial \Omega} &= 0.
\end{aligned}
\]  

(4.9)

In fact, multiplying the first equation of (4.9) by \(\omega\), and noting that \(\nabla \cdot \omega = 0\) then we have
\[
\int_{\Omega} |\nabla \omega|^2 dx = 0.
\]

Using Poincaré inequality, it follows that \(\omega = 0\). Let \(v = u - \omega\) and \(\tilde{\pi} = \pi - \pi_1\). Then
\[
\begin{aligned}
v_t - \Delta v + \nabla \tilde{\pi} &= (n - \frac{\gamma}{\mu}) \nabla \varphi, \quad \text{in } \Omega, \\
\nabla \cdot v &= 0, \quad \text{in } \Omega, \\
v|_{\partial \Omega} &= 0.
\end{aligned}
\]  

(4.10)

**Lemma 4.3.** Assume \(0 < \gamma < 16\mu^2\). Then
\[
\|n - \frac{\gamma}{\mu}\|_{L^\infty} + \|c - \frac{\gamma}{\mu}\|_{L^\infty} + \|u\|_{L^\infty} \to 0, \quad \text{as } t \to \infty.
\]

**Proof.** Let
\[
A(t) = \int_{\Omega} (n - \frac{\gamma}{\mu}) \frac{\gamma}{\mu} \left( \ln n - \ln \frac{\gamma}{\mu} \right), \quad B(t) = \frac{\gamma}{8\mu} \int_{\Omega} \left( c - \frac{\gamma}{\mu} \right)^2.
\]

By the mean value theorem, we see that
\[
(n - \frac{\gamma}{\mu}) \frac{\gamma}{\mu} \left( \ln n - \ln \frac{\gamma}{\mu} \right) = \frac{1}{\xi} \left( \xi - \frac{\gamma}{\mu} \right) \left( n - \frac{\gamma}{\mu} \right) \geq 0,
\]
where \(\xi\) is between \(n\) and \(\frac{\gamma}{\mu}\). By a direct calculation, we see that
\[
A'(t) + B'(t) = \int_{\Omega} \frac{1}{n} \left( n - \frac{\gamma}{\mu} \right) n_t + \frac{\gamma}{4\mu} \left( c - \frac{\gamma}{\mu} \right) c_t dx
\]
\[
= \int_{\Omega} \frac{1}{n} \left( n - \frac{\gamma}{\mu} \right) \left( \Delta n - \nabla \cdot (n(1 + n)^{-\alpha} \cdot \nabla c) - u \cdot \nabla n + \gamma n - \mu n^2 dx \right)
\]
\[ + \frac{\gamma}{4\mu} \int_{\Omega} \left( c - \frac{\gamma}{\mu} \right) (\Delta c - u \cdot \nabla c - c + n) \, dx \]
\[ = \int_{\Omega} \left( -\frac{\gamma}{\mu} \frac{\nabla n}{n^2} + \frac{\gamma}{\mu n} (1 + n)^{-\alpha} \nabla c \nabla n - \mu \left( n - \frac{\gamma}{\mu} \right)^2 \right) \, dx \]
\[ + \int_{\Omega} \left( -\frac{\gamma}{4\mu} |\nabla c|^2 - \frac{\gamma}{4\mu} \left( c - \frac{\gamma}{\mu} \right)^2 + \frac{\gamma}{4\mu} \left( c - \frac{\gamma}{\mu} \right) \left( n - \frac{\gamma}{\mu} \right) \right) \, dx. \]

By Young’s inequality, we see that
\[ \frac{\gamma}{\mu n} (1 + n)^{-\alpha} \nabla c \nabla n \leq \frac{\gamma}{\mu} \frac{\nabla n^2}{n^2} + \frac{\gamma}{4\mu} |\nabla c|^2, \]
and
\[ \frac{\gamma}{4\mu} \left( c - \frac{\gamma}{\mu} \right) \left( n - \frac{\gamma}{\mu} \right) \leq \frac{1}{4\sigma} \frac{\gamma}{4\mu} \left( c - \frac{\gamma}{\mu} \right)^2 + \frac{\gamma}{4\mu} \left( n - \frac{\gamma}{\mu} \right)^2. \]

Take \( \frac{1}{4} < \sigma < \frac{4\mu^2}{\gamma} \), then there exists \( \varepsilon > 0 \) such that
\[ A'(t) + B'(t) \leq -\varepsilon \int_{\Omega} \left( n - \frac{\gamma}{\mu} \right)^2 + \left( c - \frac{\gamma}{\mu} \right)^2 \, dx. \]

Integrating this inequality from \( t_0 \) to \( \infty \) yields
\[ \varepsilon \int_{t_0}^{\infty} \int_{\Omega} \left( n - \frac{\gamma}{\mu} \right)^2 + \left( c - \frac{\gamma}{\mu} \right)^2 \, dx \leq A(t_0) + B(t_0). \]

Recalling Lemma 3.3, Lemma 3.5, we see that \( c_t, n_t \in L^2(\Omega \times (s, s + 1)) \) for any \( s \in \mathbb{R}^+ \). Then
\[ \sup_{s \in (0, +\infty)} \int_s^{s+1} \left| \frac{d}{dt} \int_{\Omega} \left( n - \frac{\gamma}{\mu} \right)^2 + \left( c - \frac{\gamma}{\mu} \right)^2 \, dx \right| \, dt \leq C, \]
which implies that \( \int_{\Omega} \left( n - \frac{\gamma}{\mu} \right)^2 + \left( c - \frac{\gamma}{\mu} \right)^2 \, dx \) is uniformly continuous. Thus, we have
\[ \lim_{t \to \infty} \left\| n - \frac{\gamma}{\mu} \right\|_{L^2}^2 + \left\| c - \frac{\gamma}{\mu} \right\|_{L^2}^2 = 0. \quad (4.11) \]

By Lemma 2.1, we further have
\[ \left\| n - \frac{\gamma}{\mu} \right\|_{L^\infty} \leq C \| \nabla n \|_{L^4}^\frac{2}{3} \left\| n - \frac{\gamma}{\mu} \right\|_{L^2}^{\frac{1}{3}} + \left\| n - \frac{\gamma}{\mu} \right\|_{L^2} \to 0, \text{ as } t \to \infty, \quad (4.12) \]
\[ \left\| c - \frac{\gamma}{\mu} \right\|_{L^\infty} \leq C \| \nabla c \|_{L^6}^\frac{3}{2} \left\| c - \frac{\gamma}{\mu} \right\|_{L^2}^\frac{1}{2} + \left\| c - \frac{\gamma}{\mu} \right\|_{L^2} \to 0, \text{ as } t \to \infty. \quad (4.13) \]

Multiplying equation (4.10) by \( v \) and \( \Delta v \) respectively, and integrating them over \( \Omega \), we obtain
\[ \frac{d}{dt} \int_{\Omega} v^2 \, dx + \int_{\Omega} |\nabla v|^2 \, dx \leq C \left\| n - \frac{\gamma}{\mu} \right\|_{L^2}^2 \| \nabla \varphi \|_{L^\infty}^2, \]
\[ \frac{d}{dt} \int_{\Omega} |\nabla \varphi|^2 \, dx + \int_{\Omega} |\Delta v|^2 \, dx \leq C \left\| n - \frac{\gamma}{\mu} \right\|_{L^2}^2 \| \nabla \varphi \|_{L^\infty}^2, \]
which implies that
\[ \int_0^\infty \| v \|_{H^2}^2 \leq C \| \nabla \varphi \|_{L^\infty}^2 \int_0^\infty \left( n - \frac{\gamma}{\mu} \right)^2 \| v(\cdot, t_0) \|_{H^1}^2 \, dt \]
\[ + \int_0^\infty \| n - \gamma \mu \|_{L^2}^2 \, dt + \| v(\cdot, t_0) \|_{H^1}^2. \]
Note that \( \| v \|_{H^2}^2 \) is uniformly continuous, then we have
\[ \lim_{t \to \infty} \| v(\cdot, t) \|_{H^2}^2 = 0, \]
and we further have
\[ \| v \|_{L^\infty} \leq \| v(\cdot, t) \|_{H^2} \to 0, \quad \text{as} \quad t \to \infty. \]
The proof is complete. □

Theorem 1.2 is a direct result of Lemma 4.1, Lemma 4.2 and Lemma 4.3.

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