ON CONTINUOUS BUT DIFFERENTIABLE NOWHERE SOLUTIONS IN THE MOMENT PROBLEM

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Since the times of Chebyshev and, especially, after Stieltjes seminal memoir [9], the moment problem has always been a subject of remarkable interest. Its notorious influence and appearance in many areas is well-known [1, 7, 8].

Let us briefly present the fundamentals of the moment problem. Let \( I \subseteq \mathbb{R} \) be an interval. For a positive measure \( \mu \) on \( I \), the \( n \)th moment is defined as \( \int_I x^n \, d\mu(x) \), provided the integral exists. Consider now the set \( \{ \gamma_n \}_{n=0}^{\infty} \), an infinite but countable sequence of real numbers. Then, the moment problem on \( I \) consists in the solution of the following questions:

0.0.1. Does there exist a positive measure on \( I \) with moments \( \{ \gamma_n \}_{n=0}^{\infty} \)? If so:

0.0.2. Is this positive measure uniquely determined by the moments \( \{ \gamma_n \}_{n=0}^{\infty} \)?

One may also want to consider a third question, in case the previous one is answered in the negative:

0.0.3. How can all the positive measures on \( I \), with moments \( \{ \gamma_n \}_{n=0}^{\infty} \), be described?

This last question is the main focus here. The simplest way to show the non-uniqueness feature is to recall Stieltjes famous computation:

\[
\int_0^\infty x^n e^{-k^2 \log^2 x} \sin \left( 2\pi \frac{\log x}{\log q} \right) \, dx = 0, \quad n \in \mathbb{Z},
\]

where \( q = e^{-1/2k^2} \). Thus, independently of \( \lambda \), we have:

\[
\int_0^\infty x^n e^{-k^2 \log^2 x} \left( 1 + \lambda \sin \left( 2\pi \frac{\log x}{\log q} \right) \right) \, dx = \frac{1}{\sqrt{\pi}} \int_0^\infty x^n e^{-k^2 \log^2 x} \, dx = q^{(n+1)^2/2}.
\]

Our main point is almost trivial but apparently absent in the literature: instead of (0.1), one can more generally write:

\[
\int_0^\infty x^n e^{-k^2 \log^2 x} \sin \left( 2\pi \frac{\log x}{\log q} \right) \, dx = 0, \quad n, j \in \mathbb{Z},
\]

which means that one can construct the Fourier series:

\[
f(x) = \frac{1}{\sqrt{\pi}} e^{-k^2 \log^2 x} \left( 1 + \lambda \sum_{n=1}^{m} a_n \sin \left( 2\pi b_n \frac{\log x}{\log q} \right) \right),
\]

\[1\text{Stieltjes considered } k = 1.\]
and the integer moments keep their (log-normal) value $k^2$. Now, it is manifest that a suitable choice of $a_n$ and $b_n$ and the case $m \to \infty$ lead to functions that are continuous but differentiable nowhere. The sum-integral exchange in the moment computation of (1.4) (term by term integration) is, by virtue of Lebesgue dominated convergence theorem, not a problem in the $m \to \infty$ case, as soon as the series is convergent.

One can get a more compact and elegant description by employing a more general characterization of the non-uniqueness, first found by Chihara in the seventies [3] and more recently exploited in [4]. It says that the moments remain unchanged if

$$f(x) = \frac{1}{\sqrt{\pi}} e^{-k^2 \log^2 x}$$

is modified in the following way:

$$f(x) \to f(x) g(x), \quad \text{with } g(x) = g(qx).$$

This $q$-periodic function $g(x)$ is well known in the $q$-calculus literature and is the most general solution of $D g(x) = 0$ where $D g(x) = \frac{g(x) - g(qx)}{(q-1)x}$ denotes the usual $q$-derivative [5]. Interestingly enough, this $q$-periodic function has been studied in the context of fractal geometry [5]. It turns out that nowhere differentiable functions are perfectly well-behaved under the $q$-derivative so, while in $q$-calculus a $q$-periodic function essentially plays the role of a constant, it may easily be differentiable nowhere under the ordinary derivative.

The property (0.5) was found employing the functional equation satisfied by the log-normal function, $f(xq) = \sqrt{q} x f(x)$. This functional equation actually corresponds to the $q$-Pearson equation associated to the Stieltjes-Wigert orthogonal polynomials. It can be easily shown that the same argument in [4] can be identically applied to any orthogonal polynomials system that is an indeterminate moment problem (all within the $q$-deformed Askey scheme [6]). Thus, the property (0.5), also holds for all of them and hence, the existence of differentiable nowhere solutions appears in the same way. A detailed account of this and other related results will appear elsewhere [10].

**Remark 0.1.** To conclude, we would like to make a remark of historical nature. In addition to Stieltjes seminal memoir, another relevant mathematical document is the correspondence between Hermite and Stieltjes [2]. There are two well-known letters that are still often quoted.

One is a letter from Hermite to Stieltjes (number 351, dated 22 October 1892) which shows the admiration and respect for the results of Stieltjes. This was a reply to previous correspondence from Stieltjes, where he described his discoveries to him. Among these results, the one we have discussed here. In his own words:

"L’existence de ces fonctions $\varphi(u)$ qui, sans être nulles, son telles que

$$\int_0^\infty u^k \varphi(u) \, du = 0 \quad (k = 0, 1, 2, 3, \ldots)$$

me paraît très remarquable."

The second one, dated 20 May (1893), is even more well-known, and is the one where Hermite emphatically expresses his dislike for the then recently introduced continuous but differentiable nowhere functions.

Therefore, it seems interesting that the same results that impressed Hermite, actually contained, as we have shown, mathematics that he despised so much. Namely, continuous but differentiable nowhere functions.
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