Metric and ultrametric inequalities for resistances in directed graphs.

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Abstract

Consider an electrical circuit $G$ each directed edge $e$ of which is a semiconductor with a monomial conductance function $y^*_e = f_e(y_e) = y_e^2/\mu_e$ if $y_e \geq 0$ and $y^*_e = 0$ if $y_e \leq 0$. Here $y_e$ is the potential difference (voltage), $y^*_e$ is the current in $e$, and $\mu_e$ is the resistance of $e$; furthermore, $r$ and $s$ are two strictly positive real parameters common for all edges. In particular, case $r = s = 1$ corresponds to the Ohm law, while $r = \frac{1}{2}, s = 1$ may be interpreted as the square law of resistance typical for hydraulics and gas dynamics.

We will show that for every ordered pair of nodes $a, b$ of the circuit, the effective resistance $\mu_{a,b}$ is well-defined. In other words, any two-pole network with poles $a$ and $b$ can be effectively replaced by two oppositely directed edges, from $a$ to $b$ of resistance $\mu_{a,b}$ and from $b$ to $a$ of resistance $\mu_{b,a}$.

Furthermore, for every three nodes $a, b, c$ the inequality $\mu^{s/r}_{a,c} + \mu^{s/r}_{c,b} \geq \mu^{s/r}_{a,b}$ holds, in which the equality is achieved if and only if every directed path from $a$ to $b$ contains $c$.

Some limit values of parameters $s$ and $r$ correspond to classic triangle inequalities. Namely, (i) the length/time of a shortest directed path, (ii) the inverse width of a bottleneck path, and (iii) the inverse capacity (maximum flow per unit time) between any ordered pair of terminals $a$ and $b$ are assigned to: (i) $r = s \to \infty$, (ii) $r = 1, s \to \infty$, (iii) $r \to 0, s = 1$, respectively.

These results generalize ones obtained in 1987 for the isotropic monomial circuits, modelled by undirected graphs. In this special case resistance distances form a metric space, while in general only a quasi-metric one: symmetry, $\mu_{a,b} = \mu_{b,a}$ is lost.

In linear symmetric case these results are known from 1960-s and were generalized to linear the non-symmetric case in 2016-th.

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1 Introduction
Two-Pole Circuits

We consider a circuit modeled by a directed graph (digraph) \( G = (V, E) \) in which each directed edge \( e \in E \) is a semiconductor with the monomial conductivity law

\[
y_e^* = f_e(y_e) = y_e^r/\mu_e^s
\]

if \( y_e \geq 0 \) and \( y_e^* = 0 \) if \( y_e \leq 0 \). Here \( y_e \) is the voltage, or potential difference, \( y_e^* \geq 0 \) current, and \( \mu_e \) is the resistance of \( e \), while \( r \) and \( s \) are two strictly positive real parameters, the same for all \( e \in E \).

In particular, the case \( r = 1 \) corresponds to Ohm’s law, while \( r = \frac{1}{4} \) is the so-called square law of resistance typical for hydraulics and gas dynamics. In the first case \( y_e \) is the drop of potential (voltage) and \( y_e^* \) is the current; in the second case \( y_e \) is the drop of pressure and \( y_e^* \) is the flow. Parameter \( s \), in contrast to \( r \), looks redundant, yet, it plays an important role helping to interpret some limit cases.

Given a circuit \( G = (V, E) \), let us fix an ordered pair of nodes \( a, b \in V \). We will show that the obtained two-pole circuit \( (G, a, b) \) satisfies the same monomial conductivity law. Let \( y_{a,b}^* \) denote the total current that comes from \( a \) into \( b \) and \( y_{a,b} \) the drop of potential (voltage) between \( a \) and \( b \). It will be shown that

\[
y_{a,b}^* = f_{a,b}(y_{a,b}) = y_{a,b}^r/\mu_{a,b}^s
\]

when \( y_{a,b} \geq 0 \), and there exists a directed path from \( a \) to \( b \) in \( G \). If there is no such path then \( y_{a,b}^* = 0 \) for any \( y_{a,b} \geq 0 \); in this case we set \( \mu_{a,b} = +\infty \). Also, by convention, we set \( y_{a,b}^* = 0 \) when \( y_{a,b} < 0 \) or \( a = b \). In the latter case \( y_{a,b} = 0 \) always holds and we set \( \mu_{a,b} = 0 \), by convention.

In other words, each two-pole circuit \( (G, a, b) \) can be effectively replaced by two oppositely directed edges: from \( a \) to \( b \) of resistance \( \mu_{a,b} \) and from \( b \) to \( a \) of resistance \( \mu_{b,a} \). Both numbers are 0 when \( a = b \).

Main inequality

For arbitrary three nodes \( a, b, c \in G \), we will prove inequality

\[
\mu_{a,b}^{s/r} \leq \mu_{a,c}^{s/r} + \mu_{c,b}^{s/r} \tag{1}
\]

Furthermore, the inequality in (1) is strict if and only if there exists a directed path from \( a \) to \( b \) that does not contain \( c \).

Obviously, equality holds if at least one of the three considered resistances equals 0 or \(+\infty\), that is, if at least two of the considered three nodes coincide, or at least one of three directed paths, from \( a \) to \( c \), from \( c \) to \( b \), or from \( a \) to \( b \) fails to exist. In the latter case, \(+\infty = +\infty\) by convention.

Clearly, if \( s \geq r \) then (1) implies the standard metric inequality:

\[
\mu_{a,b} \leq \mu_{a,c} + \mu_{c,b} \tag{2}
\]

Thus, a circuit can be viewed as a quasi-metric space in which the distance from \( a \) to \( b \) is the effective resistance \( \mu_{a,b} \). Note that equality \( \mu_{a,b} = \mu_{b,a} \) holds only in symmetric case, but may fail in general.
Quasi-metric and quasi-ultrametric spaces corresponding to asymptotics of parameters $r$ and $s$

Playing with parameters $r$ and $s$, one can get several interesting (but well-known) examples. Let $r = r(t)$ and $s = s(t)$ depend on a real parameter $t$. Then, these two functions define a curve in the positive quadrant $r \geq 0, s \geq 0$. For the next four limit transitions, as $t \to \infty$, for all pairs of poles $a, b \in V$, the limits $\mu_{a,b} = \lim_{t \to \infty} \mu_{a,b}(t)$ exist and can be interpreted as follows:

- (i) Effective resistance of an Ohm semiconductor circuit from pole $a$ to pole $b$; $s(t) = r(t) \equiv 1$, or more generally, $s(t) \to 1$ and $r(t) \to 1$.
- (ii) Standard length (travel time or cost) of a shortest route from terminal $a$ to terminal $b$ in a circuit of one-way roads; $s(t) = r(t) \to \infty$, or more generally, $s(t) \to \infty$ and $s(t)/r(t) \to 1$.
- (iii) The inverse width of a widest bottleneck path from terminal $a$ to terminal $b$ in a circuit of one way-roads; $s(t) \to \infty$ and $r(t) \equiv 1$, or more generally, $r(t) \leq \text{const}$, or even more generally $s(t)/r(t) \to \infty$.
- (iv) The inverse capacity (maximum flow per unit time) from terminal $a$ to terminal $b$ in a one-way pipeline; $s(t) \equiv 1$ and $r(t) \to 0$; or more generally, $s(t) \to 1$, while $r(t) \to 0$.

![Figure 1: Three types of limit transitions for $s$ and $r$.](image)

All four examples define quasi-metric spaces, since in all cases $s(t) \geq r(t)$ for any sufficiently large $t$ and we assume that $t \to \infty$. Moreover, for the last two examples the ultrametric inequality

$$\mu_{a,b} \leq \max(\mu_{a,c}, \mu_{c,b})$$

(3)
holds for any three nodes $a, b, c$, because $s(t)/r(t) \to \infty$, as $t \to \infty$, in cases (iii) and (iv).

These four examples allow us to interpret $s$ and $r$ as some important parameters of transportation problems.

Parameter $s$ can be viewed as a measure of divisibility of a transported material; $s(t) \to 1$ in examples (i) and (iv), because liquid, gas, or electrical charge are fully divisible; in contrast, $s(t) \to \infty$ for (ii) and (iii), because a car, a ship, or an individual travelling from $a$ to $b$ is indivisible.

Ratio $s/r$ can be viewed as a measure of subadditivity of the transportation cost; so $s(t)/r(t) \to 1$ in examples (i) and (ii), because in these cases the cost of transportation along a directed path is additive, i.e., is the sum of the costs or resistances of the directed edges that form this path; in contrast, $s(t)/r(t) \to \infty$ for (iii) and (iv), because in these cases only edges of the maximum cost (the width of a bottleneck) or capacity of a critical cut matter.

Other values of parameters $s$ and $s/r$, between 1 and $\infty$, correspond to an intermediate divisibility of the transported material and subadditivity of the transportation cost, respectively.

The metric inequality in case (ii) is obvious. Let $\mu_{a,c}$ and $\mu_{c,b}$ be the lengths (or the travel times) of the shortest directed paths $p_{a,c}$ from $a$ to $c$ and $p_{c,b}$ from $c$ to $b$, respectively. Combining these two paths we obtain a walk $p_{a,b}$ from $a$ to $b$. Thus, the triangle inequality follows.

The ultrametric inequality of case (iii) can be proven in a similar way. Let $\lambda_{a,c}$ and $\lambda_{c,b}$ be the largest width (or weight) of an object that can be transported from $a$ to $c$ and from $c$ to $b$, respectively, and let $p_{a,c}$ and $p_{c,b}$ be the corresponding transportation paths. Combining them we obtain a walk $p_{a,b}$ from $a$ to $b$. Obviously, an object of width $w$ can be transported along this walk, i.e., $\lambda_{a,b} \geq w$, whenever it can be transported from $a$ to $c$ and from $c$ to $b$, i.e., $\lambda_{a,c} \geq w$ and $\lambda_{c,b} \geq w$. This implies the ultrametric inequality.

Note that in both above (“indivisible”) cases the inequalities are strict whenever directed paths $p_{a,c}$ and $p_{c,b}$ intersect not only in $c$. Yet, we will show that they may be strict in some other cases too.

“Divisible” case (iv) requires a different approach. Given a circuit with three fixed nodes $a, c$, and $b$, let $\lambda_{a,c}$ and $\lambda_{c,b}$ denote the capacities (that is, the maximum feasible flows) from $a$ to $c$ and from $c$ to $b$, respectively. Furthermore, let $w$ be the value of a flow feasible in both cases, in other words, let inequalities $\lambda_{a,c} \geq w$ and $\lambda_{c,b} \geq w$ hold. Then, $\lambda_{a,b} \geq w$ holds too, implying. However, to show this, one cannot just combine (sum up) two flows realizing $\lambda_{a,c}$ and $\lambda_{c,b}$, because the resulting flow may exceed capacities of some edges, thus becoming not feasible.

Instead, inequality $\lambda_{a,b} \geq w$ can be easily derived from the classic “Max Flow - Min Cut Theorem”. According to it, each minimum $(a - b)$-cut $C : V = V_a \cup V_b$ (such that $a \in V_a$, $b \in V_b$, and $V_a \cap V_b = \emptyset$) is of capacity $\lambda_{a,b}$.

If $c \in V_a$ then $C$ is a $(c - b)$-cut too and, hence, $\lambda_{c,b} \leq \lambda_{a,b}$;

if $c \in V_b$ then $C$ is a $(a - c)$-cut too and, hence, $\lambda_{a,c} \leq \lambda_{a,b}$.
Thus, \( \min(\lambda_{a,c}, \lambda_{c,b}) \leq \lambda_{a,b} \), which is equivalent with (3).

When (1) holds with equality

In this paper we will prove that (1) holds with equality if and only if each directed path from \( a \) to \( b \) contains \( c \). (For the symmetric case this was shown in [15, 9].) The statement holds for any strictly positive real \( r \) and \( s \); in particular, in case (i), when \( r = s = 1 \). Yet, for the asymptotic cases (ii, iii, iv) only "if part" holds, while "only if" one may fail. Three examples are as follows:

- cases (ii) and (iii). Let \( G = (V, E) \) be the directed triangle in which \( V = \{a, b, c\} \) and \( E = \{(a, b), (a, c), (c, b)\} \). Obviously, \( G \) contains a directed \((a - b)\)-path avoiding \( c \) (just the edge \((a, b)\)). Set \( \mu_{a,c} = \mu_{c,b} = 1 \) and \( \mu_{a,b} = 2 \). Thus, (2) holds with equality: \( 1 + 1 = 2 \). In case (iii) we have \( \mu_{a,c} = \mu_{c,b} = \mu_{a,b} = 1 \). Indeed, edge \((a, b)\) of the width \( \lambda_{a,b} = 1/3 \) is useless. Thus, (3) holds with equality: \( \max(1, 1) = 1 \).

- cases (iv). Define digraph \( G = (V, E) \) by \( V = \{a, b, c, k, \ell\} \) and \( E = \{(a, k), (k, c), (c, \ell), (\ell, b), (k, \ell)\} \). Again \( G \) contains a directed \((a - b)\)-path avoiding \( c \); it is given by vertex-sequence \( a, k, c, \ell, b \). Set \( \mu_e = 1 \) for all \( e \in E \). Then again \( \mu_{a,c} = \mu_{c,b} = \mu_{a,b} = 1 \), since edge \((k, \ell)\) of capacity \( \lambda_{k,\ell} = 1 \) is not needed for transportation, and (3) holds with equality: \( \max(1, 1) = 1 \).

Known special cases of the main inequality

Our main result (1) generalizes some well (or maybe, not so well) known inequalities obtained earlier for the following special cases.

**Symmetric case.** A digraph \( G \) is called *symmetric* if its edges are split into pairs of oppositely directed edges \( e' = (v', v''), e'' = (v'', v') \). Respectively, a circuit is called *symmetric* if its graph is symmetric and \( \mu_{e'} = \mu_{e''} \) for each pair \( e', e'' \) introduced above. In this case one can replace each such pair \( e', e'' \) by a non-directed edge \( e \), thus, replacing the digraph of the circuit by a non-directed graph. For this case, the main inequality (1) was shown in [13]; see also [9, 10, 11] for more details.

The equality holds in (1) if and only if every path from \( a \) to \( b \) contains \( c \). First it was shown in Section 16.9 of [15]; see also [9]. It was also shown in Section 16.9 that the monomial conductance law is the only when the effective resistance \( \mu_{a,b} \) of the two-pole circuit \((G, a, b)\) is a real number. In general, it is a monotone non-decreasing function for an arbitrary monotone circuit [17].

Clearly, equality \( \mu_{a,b} = \mu_{b,a} \) holds in the isotropic (symmetric) case. Thus, resistance distances of symmetric circuits form a metric spaces. Yet, in anisotropic (non-symmetric) case the above equality may fail and we obtain only quasi-metric spaces, in general.

**Linear case**, \( r = s = 1 \). In the symmetric linear case, the metric resistance inequality was discovered by Gerald E. Subak-Sharpe [32, 33];
see also \[16, 27, 31, 34, 35, 36, 37, 14, 6, 14\] and preceding works \[ 39, 28, 26, 29, 30 \]. This result was rediscovered several times later.

Let us notice that the proof of (1) given in \[13, 9\] for the isotropic monomial conductance differs a lot from the proof of \[32, 33\] for the symmetric linear case. In this paper we extend the first proof to the anisotropic case or, in other words, to digraphs. To make the presentation self-contained we copy here some parts of \[9\].

For the linear non-symmetric case the quasi-metric inequality, along with many related results, was recently obtained in \[38\].

Continuum.

It would be natural to conjecture that the above approach can be developed not only for the discrete circuits but for continuum as well: inequality (1) and its corollaries should hold in this case too. Sooner or later, this will become the subject of a separate research. The same four subcases appear: linear and monomial, isotropic and anisotropic.

2 Resistances of two-pole circuits

Conductance law

Let \( e \) be a semi-conductor with the monomial conductivity law

\[
y_e^* = f_e(y_e) = \lambda_e y_e^e = \frac{y_e^e}{\mu_e^e} \quad \text{if } y_e \geq 0 \quad \text{and} \quad 0 \quad \text{if } y_e \leq 0.
\]  

(4)

Here \( y_e \) is the voltage or potential difference, \( y_e^e \) current, \( \lambda_e \) conductance, and \( \mu_e = \lambda_e^{-1} \) resistance of \( e \); furthermore, \( r \) and \( s \) are two strictly positive real parameters independent of \( e \). Obviously, the monomial function \( f_e \) is continuous, strictly monotone increasing when \( y_e \geq 0 \), and taking all non-negative real values.

Main variables and related equations

A semi-conductor circuit is modeled by a weighted digraph \( G = (V, E, \mu) \) in which weights of the edges are their positive resistances \( \mu_e, e \in E \).

Let us introduce the following four groups of real variables; two for each \( v \in V \) and \( e \in E \): potential \( x_v \); difference of potentials, or voltage \( y_e \); current \( y_e^* \); sum of currents, or flux \( x_v^* \).

The above variables are not independent. By (4), current \( y_e^* \) depends on voltage \( y_e \). Furthermore, the voltage (respectively, flux) is a linear function of the potentials (respectively, of the currents). These functions are defined by the node-edge incidence function of the digraph \( G \):

\[
\text{inc}(v, e) = \begin{cases} +1, & \text{if node } v \text{ is the beginning of } e; \\ -1, & \text{if node } v \text{ is the end of } e; \\ 0, & \text{if } v \text{ and } e \text{ are not incident.} \end{cases}
\]  

(5)
We will assume that the next two systems of linear equations always hold:

\[
y_e = \sum_{v \in V} \text{inc}(v, e)x_v; \quad (6)
\]
\[
x_v^* = \sum_{e \in E} \text{inc}(v, e)y_e^*. \quad (7)
\]

Let us notice that equation (6) for a directed edge \(e = (v', v'')\) can be reduced to \(y_e = x_{v'} - x_{v''}\).

We say that the first Kirchhoff law holds for a node \(v\) if \(x_v^* = 0\).

Let us introduce four vectors, one for each group of variables:

\[x = (x_v \mid v \in V), \quad x^* = (x_v^* \mid v \in V), \quad y = (y_e \mid e \in E), \quad y^* = (y_e^* \mid e \in E),\]

where \(n = |V|\) and \(m = |E|\) are the numbers of nodes and edges of the digraph \(G = (V, E)\). Let \(A = A_G\) be the edge-node \(m \times n\) incidence matrix of graph \(G\), that is, \(A(v, e) = \text{inc}(v, e)\) for all \(v \in V\) and \(e \in E\). Equations (6) and (7) can be rewritten in this matrix notation as \(y = Ax\) and \(x^* = A^T y^*\), respectively.

It is both obvious and well known that these two equations imply the following chain of identities:

\[(x, x^*) = \sum_{v \in V} x_v x_v^* = \sum_{e \in E} y_e y_e^* = (y, y^*).\]

Recall that \(y^*\) is uniquely defined by \(y\) according to the conductance law (4). Thus, given vector \(x\), the remaining three vectors \(y, y^*, \) and \(x^*\) are uniquely defined by \(x\) (6)(7). This triple will not change if we add an arbitrary real constant \(c\) to all coordinates of \(x\), while multiplying \(x\) by \(c\) will result in multiplying \(y\) by \(c\) and \(y^*, x^*\) by \(c^r\). More precisely, the following scaling property clearly holds.

**Lemma 1.** For any positive constant \(c\), two quadruples \((x, y, y^*, x^*)\) and \((cx, cy, c^r y^*, c^r x^*)\) can satisfy all equations of (6, 4, 7) only simultaneously.

**Two-pole boundary conditions**

In general theory of monotone circuits, one can consider arbitrary monotone functions: a non-decreasing one \(y_e^* = f_e(y_e)\) for each \(e \in E\) and a non-increasing one \(x_v^* = g_v(x_v)\) for each \(v \in V\); see [8, 17, 24, 25] and also [14, 15, 20, 21, 23].

In case of the two-pole circuits we restrict ourselves by the monomial conductance law (4). Fix an ordered pair of poles \(a, b\), potentials

\[x_a = x_a^0, \quad x_b = x_b^0, \quad (8)\]

in them, and require the first Kirchhoff law for any other node:
\[ x_v^* = 0, \ v \in V \setminus \{a, b\}. \quad (9) \]

By convention, \( y_a^* = 0 \) if \( a = b \) or \( y_{a,b} = x_a - x_b \leq 0 \). So, w.l.o.g. we can assume that \( a \neq b \) and \( x_a^0 \geq x_b^0 \).

Remark 1. By Lemma 1, it would be sufficient to replace (8) by \( x_a^0 = 1 \) and \( x_b^0 = 0 \). It would be also possible to replace it by \( x_a^* = x_a^u \) (or \( x_b^* = 1 \)). Then, \( x_b^* = -x_a^0 \) (resp., \( x_b^0 = -1 \)) will automatically hold, by (7).

We call a vector \( x = x(a, b) \) a solution of the two-pole circuit \((G, a, b)\) if the corresponding quadruple \((x, y, y^*, x^*)\) satisfies all equations (4 - 9).

In [9] the monomial symmetric case was considered and it was shown that there exists a unique solution \( x = x(a, b) \) whenever \( a \) and \( b \) belong to the same connected component of \( G \).

Yet, this claim cannot be extended directly to digraphs. For example, let \((G, a, b)\) be a directed \((a, b)\)-k-path \( a = v_0, v_1, \ldots, v_k = b \) from \( a \) to \( b \). Then potential \( x \) is unique if \( x_a^0 \geq x_b^0 \), but otherwise, when \( x_a^0 < x_b^0 \), any non-decreasing \( x_v^0 = x_0 \leq x_{v_1} \leq \ldots \leq x_{v_k} = x_b^0 \) will be a solution, with no current, that is, \( y_v^* = 0 \) for all \( v = (v_j-1, v_j) \) for \( j = 1, \ldots, k \).

In the directed case we will prove that \( y^* \) (rather than \( x^* \)) is the same in all solutions. Furthermore, let \( G^* \) be the subgraph of \( G \) defined by all directed edges \( e \in E \) such that \( y_e^* > 0 \). Then in all solutions potentials \( x \) are uniquely defined on vertices of \( G^* \).

Existence of a solution

We will apply Method of Successive Approximation (MSA) increasing potentials of some nodes, one by one in a certain order.

Obviously, when we increase \( x_v \) (keeping all remaining potentials \( x_u \) unchanged) the corresponding flux \( x_v^* \) is non-decreasing; furthermore, it is strictly increasing if and only if \( G \) contains an edge \((v, u)\) with \( x_v \geq x_u \) or an edge \((u, v)\) with \( x_v \leq x_u \). Respectively, \( x_v^* \) in any other node \( u \in V \setminus \{v\} \) is non-increasing; furthermore, it is strictly decreasing if and only if \((v, u)\) is an edge and \( x_v \geq x_u \) or \((u, v)\) is an edge and \( x_v \leq x_u \).

Let us set \( x_a = x_a^0 \) and \( x_b = x_b^0 \) for all nodes \( v \in V \setminus \{a\} \), including \( b \). In the course of iterations, potentials \( x_a = x_a^0 \) and \( x_b = x_b^0 \) will remain unchanged, while the all other potentials \( x_v \), on the nodes from \( W = V \setminus \{a, b\} \), will be recomputed by MSA as follows.

Order arbitrarily the nodes of \( W \) and consider them one by one in this order repeating cyclically. If \( x_v^* = 0 \), skip this node and go to the next one. If \( x_v^* < 0 \), increase \( x_v \) until \( x_v^* \) becomes 0. The latter is possible, since, by (4), \( f_e \) is continuous and \( y_v^* \to +\infty \) as \( y_e \to +\infty \). Let us notice that \( x_v^* \) may remain 0 for some time, but we stop increasing \( x_v \) the first moment when \( x_v^* \) becomes 0, and proceed to the next node.

The following claims can be easily proven together by induction on the number of iterations.

- (i) For any node \( v \in W \) its potential \( x_v \) is monotone non-decreasing and it remains bounded by \( x_v^0 \) from above. Hence, it tends to a limit \( x_v^0 \) between \( x_a^0 \) and \( x_b^0 \).
• (ii) These limit potentials solve the two-pole circuit \((G, a, b)\).

• (iii) Fluxes \(x^v_a\) remain non-positive for all \(v \in V \setminus \{a\}\). In contrast, \(x^a_v\) remains non-negative. Furthermore, \(x^a_a\) and \(x^b_b\) are monotone non-increasing.

• (iv) For the limit values of potentials and fluxes we have:
  If \(G\) contains no directed path from \(a\) to \(b\) then \(x^0_v = 0\) for all \(v \in V\), including \(a\) and \(b\). In this case \(x^0_v = x^0_a\) if \(G\) contains a directed path from \(a\) to \(v\), otherwise \(x^0_v = 0\). If \(G\) contains a directed path from \(a\) to \(b\) then \(x^0_a > 0\). (Respectively, \(x^0_v = -x^0_a < 0\) and \(x^0_v = 0\) for all \(v \in W\), in accordance with (9.).

• (v) The limit potentials \(x^v_a\) take only values \(x^0_a\) and \(x^0_b\) if and only if \(G\) contains no directed path from \(a\) to \(b\) or every such path consists of only one edge.

Existence of a solution is implied by (ii).

Remark 2. A very similar monotone potential reduction (pumping) algorithm for stochastic games with perfect information was suggested in [1, 2].

Uniqueness of the solution

A solution \(x\) of a two-pole circuit \((G, a, b)\) may be not unique. Suppose that \(G\) contains an induced directed path \(P\) from \(u\) to \(v\) of length greater than 1 and that \((G, a, b)\) has a solution \(x\) with \(x_u < x_v\). Then every monotone non-decreasing sequence of potentials on \(P\) is feasible. Notice, however, that the current along \(P\) will be zero for any such sequence.

We will demonstrate that, in general, vector of currents \(y^*\) (and, hence, \(x^*\) too) is unique for all solutions of \((G, a, b)\). It follows directly from an old classical result relating solutions of an arbitrary monotone circuit with a pair of dual problems of convex programming [3, 17, 21, 25]; see also [20, 21, 22, 23, 13, 14, 15].

First, note that, by Lemma 4 we can replace the boundary conditions (8) by

\[ x^a_a = x^0_a, \quad x^b_b = -x^0_a, \]  \(\text{(10)}\)

and recall that \(x^v_a = 0\) for all \(v \in W = V \setminus \{a, b\}\) by (9).

The Joule-Lenz heat on \(e\) is defined by the current \(y^e_\ast \geq 0\) as the integral

\[ F^e_\ast(y^e_\ast) = \int_{f_e}^{-1}(y^e_\ast) \ dy^e_\ast = \frac{\mu^e/r}{1 + 1/r} y^e_\ast^{1+1/r}, \]  \(\text{(11)}\)

which is a strictly convex function of \(y^e_\ast \geq 0\). Furthermore, the total heat dissipated in the circuit is additive:

\[ F^*(y^*) = \sum_{e \in E} F^e_\ast(y^e_\ast). \]  \(\text{(12)}\)

It is a strictly convex function of \(y^*\) defined on the positive ortant, \(y^* \geq 0\).

By the classical results (see, for example, Rockafellar [24]) solving \((G, a, b)\) is equivalent with minimizing dissipation \(F^*(y^*)\) subject to the following constraints
\[ x_a^* = x_a^{*0}, \quad (x_b^* = -x_a^{*0}), \quad x_v^* = 0 \forall v \in V \setminus \{a, b\}, \quad \text{and} \quad y^* \geq 0. \quad (13) \]

Since \( x^* = Ay^* \), we obtain the minimization problem for a strictly convex function of \( y^* \) subject to linear constraints on \( y^* \). It is known from calculus that solution \( y^{*0} \) is unique in this case.

**Remark 3.** In the non-directed (isotropic) case the above equivalence is well-known in physics as the minimum dissipation principle. It is applicable for arbitrary "boundary conditions" not only to the two-pole circuits.

Thus, all solutions of \((G, a, b)\) have the same current vector \( y^{*0} \). This implies the uniqueness of the flux vector \( x^{*0} = A^*y^{*0} \) as well.

Let us denote by \( G^+ = (V^+, E^+) \) the subgraph of \( G \) formed by the edges \( e \in E \) with positive currents, \( y^{*0} > 0 \). The following properties of \( G^+ \) are obvious:

- (j) Digraph \( G \) contains a directed path from \( a \) to \( b \) if and only if \( G^+ \) is not empty.
- (jj) In the latter case it contains the poles, \( a, b \in V^+ \), and at least one directed path from \( a \) to \( b \), but not necessarily all such paths. Yet, any vertex or edge of \( G^+ \) belongs to such a path.
- (jjj) Potentials are strictly decreasing on each edge of \( G^+ \) and, hence, it has no directed cycles.

**Conductance functions of two-pole networks**

Given a two-pole network \((G, a, b)\), let us define the potential drop and current from \( a \) to \( b \) as

\[ y_{a,b} = x_a - x_b, \quad y_{a,b}^* = x_a^* = -x_b^*. \]

If there is no directed path from \( a \) to \( b \) in \( G \), let us set \( \mu_{a,b} = +\infty \), since in this case \( y_{a,b}^* = 0 \) for any \( y_{a,b} \). Otherwise, by Lemma 1, \( y_{a,b}^* \) depends on \( y_{a,b} \) as in (13):

\[ y_{a,b}^* = f_{a,b}(y_{a,b}) = \lambda_{a,b} y_{a,b} r_{a,b} + \frac{y_{a,b}}{\mu_{a,b}} \quad \text{if} \quad y_{a,b} \geq 0 \quad \text{and} \quad 0 \quad \text{if} \quad y_{a,b} \leq 0. \quad (14) \]

Two strictly positive real values \( \lambda_{a,b} \) and \( \mu_{a,b} = \lambda_{a,b}^{-1} \) are called conductance and, respectively, resistance of \((G, a, b)\).

**Remark 4.** It is shown in Section 6.9 of [15] that among all monotone conductance laws the monomial one is the only case when resistance of a two-pole network is a real number; in other words, up to a real factor, the same function \( f \) describes the conductances \( f_e(y_e) \) and \( f_{a,b}(y_{a,b}) \).
Monotonicity of effective resistances and Braess’ Paradox

Given a two-pole circuit $(G, a, b)$, where $G = (V, E, \mu)$, let us fix an edge $e_0 \in E$, replace the resistance $\mu_{e_0}$ by a smaller one, $\mu'_{e_0} \leq \mu_{e_0}$, and denote by $G' = (V, E, \mu')$ the obtained circuit.

Of course, the total resistance will not increase either, that is, $\mu'_{a,b} \leq \mu_{a,b}$ will hold. Yet, how to prove this “intuitively obvious” statement? Somewhat surprisingly, the simplest way is to apply the minimum dissipation principle again; see, for example, [19].

Let $y^*$ and $y'^*$ be the (unique) current vectors that solve $(G, a, b)$ and $(G', a, b)$, respectively. Since $\mu'_{e_0} \leq \mu_{e_0}$, inequality $F_{e_0}^*(y'_{e_0}) \leq F_{e_0}^*(y_{e_0})$ is implied by (11). Furthermore, $F_e^*(y_e) = F_e^*(y_e)$ for all other $e \in E$, distinct from $e_0$. Hence, $F^e(y^*) \leq F^e(y'^*)$ holds by (12). As we know, all solutions of $(G', a, b)$ have the same vector of currents $y'^*$, which may differ from $(y^*)$ and, by the minimum dissipation principle, we have $F^e(y'^*) \leq F^e(y^*)$. From this, by transitivity, we conclude that $F^e(y^*) \leq F^e(y'^*)$ and, by (11,12), conclude that $\mu'_{a,b} \leq \mu_{a,b}$ holds.

In particular, when an edge $e$ is eliminated from $G$, its finite resistance $\mu_e$ is replaced by $\mu'_e = +\infty$. It was just shown that, by this operation, the effective resistance is not reduced, that is, $\mu_{a,b} \leq \mu'_{a,b}$ holds.

In general, for monotone circuits [8, 17] the conductance function of its edge may be an arbitrary, not necessarily monomial, monotone non-decreasing function:

\[ y^*_e = f_e(y_e) \quad \text{if} \quad y_e \geq 0 \quad \text{and} \quad 0 \quad \text{if} \quad y_e \leq 0, \]

Then, by results of [17], the conductance law of a two-pole network $(G, a, b)$ is represented by a similar formula:

\[ x^*_a = -x^*_b = y^*_{a,b} = f_{a,b}(y_{a,b}) \quad \text{if} \quad y_{a,b} \geq 0 \quad \text{and} \quad 0 \quad \text{if} \quad y_{a,b} \leq 0, \]

where $f_{a,b}$ is a monotone non-decreasing function too.

When we reduce conductance function $f_e(y_e)$ of an edge $e \in E$, the effective conductance function $f_{a,b}$ may increase for some (certainly, not for all) values of its argument $y_{a,b} = x_a - x_b$. This phenomenon is known as Braess paradox [3].

The above monotonicity principle implies that this paradox is not possible for circuits with the monomial conductance law provided parameters $r$ and $s$ are the same for all edges $e \in E$. Indeed, in this case resistance $\mu_{a,b}$ between the poles is a real number; moreover, it is a monotone function of resistances $\mu_e$ of edges $e \in E$, as it was shown above. Yet, the paradox can appear for monomial circuits in which parameter $r = r_e$ depends on $e$, or when some edges have non-monomial monotone conductance functions.

3 Proof of the main inequality and some related claims

Here we prove our main result generalizing the triangle inequality of [13, 9] from graphs to digraphs as follows.
Theorem 1. Given a weighted digraph $G = (V, E, \mu)$ with strictly positive weights-resistances ($\mu_e | e \in E$), three arbitrary nodes $a, b, c \in V$, and strictly positive real parameters $r$ and $s$, inequality (1) holds: $\mu_{a,b}^{x/r} \leq \mu_{a,c}^{x/r} + \mu_{b,c}^{x/r}$. Moreover, it holds with equality if and only if node $c$ belongs to every directed path from $a$ to $b$ in $G$.

Proof. W.l.o.g. we can assume that $G$ contains directed paths from $a$ to $c$ and from $c$ to $b$. Indeed, otherwise $\mu_{a,c}$ or $\mu_{c,b}$ is $+\infty$ and there is nothing to prove. By this assumption, $G$ contains a directed walk from $a$ to $b$ passing through $c$. Hence, $G$ also contains a directed path from $a$ to $b$, but the latter may avoid $c$.

Anyway, there exists a (not necessarily unique) solution of $(G, a, b)$ for any fixed potentials $x_a^0, x_b^0$ in the poles $a, b$. If $x_a^0 \leq x_b^0$ then $y_{a,b} = 0$, by definition, and again there is nothing to prove. Thus, w.l.o.g. we assume that $x_a^0 > x_b^0$ (Moreover, we could assume w.l.o.g. that $x_a^0 = 1$ and $x_b^0 = 0$, but will not do this.)

We make use of the same arguments as in subsection "Existence of a solution". Consider a solution $x_0 = x(a, b)$ constructed there and denote by $x_c^0$ the obtained potential in $c$. By construction, $x_a^0 \geq x_c^0 \geq x_b^0$ and at least one of these two inequalities is strict. (Actually, such inequalities hold for any solution $x = x(a, b)$, but one chosen $x_c^0$ will be enough for our purposes.)

Now let us consider the two-pole circuit $(G, a, c)$ and fix in it $x_a = x_a^0$ and $x_c = x_c^0$, standardly requiring the first Kirchhoff law, $x_b^0 = 0$ for all other vertices $v \in W = V \setminus a, c$, including $v = b$.

Lemma 2. The obtained currents in the circuits $(G, a, b)$ and $(G, a, c)$ satisfy inequality $y_{a,b}^{x_0} \geq y_{a,c}^{x_0}$. Moreover, the equality holds if and only if $c$ belongs to every directed path from $a$ to $b$.

Proof. First, recall that the current vectors (and, hence, the values of $x^*(a, b) = y_{a,b}^{x_0}$ and $x^*(a, c) = y_{a,c}^{x_0}$ too) are well-defined, that is, remain the same for any solutions $x(a, b)$ and $x(a, c)$ of $(G, a, b)$ and $(G, a, c)$ with boundary conditions $x_a^0, x_b^0$ and $x_a^0, x_c^0$, respectively.

Again we apply MSA to compute $x(a, c)$, yet, this time we take $x(a, b)$ as the initialization. Thus, in the beginning we have

$$x_0^*(a, b) = -x_b^0(a, b), \quad x_0^*(a, b) = 0$$

and at the end we will have

$$x_0^*(a, c) = -x_0^*(a, c), \quad x_0^*(a, c) = 0.$$

Potentials $x_a = x_a^0$ and $x_c = x_c^0$ satisfy the boundary conditions and will stay unchanged in the course of iterations, while the remaining potentials $x_b$ on the nodes from $W = V \setminus \{a, c\}$ will be determined by MSA as follows. Order arbitrarily the nodes of $W$ and consider them one by one in this order repeating cyclically. If $x_c^0 = 0$, skip this node $v$ and go to the next one. If $x_c^0 < 0$, increase $x_c$ until (the very first moment when) $x_c^0$ becomes 0. Then proceed with the next node. The following claims can be easily proven together, by induction on the number of iterations.
• (i) Fluxes $x^*_a$ and $x^*_c$ are both monotone non-decreasing and remain non-negative and non-positive, respectively. Moreover, $x^*_v$ remain non-positive for all $v \in V \setminus \{a\}$.

• (ii) All potentials $x_v$ are monotone non-decreasing and remain bounded by $x^0(a, c)$ from above. Hence, $x_v$ tends to a limit $x^v(a, c)$ between $x^0(a, a)$ and $x^0(a, b)$.

• (iii) These limit potentials solve the two-pole circuit $(G, a, c)$.

• (iv) Since $G$ contains directed pathes from $a$ to $c$ and from $c$ to $b$, for the limit values of the fluxes we have: $x^a(a, c) > 0$, $x^0(a, c) = -x^a(a, c)$, and $x^0(a, c) = 0$ for all $v \in V \setminus \{a, c\}$, in accordance with (9).

• (v) $x^a(a, b) = y^a_{a,b} \geq y^a_{a,c} = x^a(a, c)$

Existence of a solution is implied by (iii). The last inequality holds, because potential $x_a$ is constant, while all other potentials are not decreasing. Hence, the flux from $a$ cannot increase.

If all directed pathes from $a$ to $b$ contain $c$ then equality holds in (v). Indeed, in this case $x_v$ will not be changed by MSA for any vertex $v$ that belongs to a path from $a$ to $c$. Hence, the flux $x^*_a$ remains constant, resulting in $y^a_{a,b} = y^a_{a,c}$.

Suppose conversely that $G$ contains a directed path $P$ from $a$ to $b$ avoiding $c$. Then, order the nodes of $W = V \setminus \{a, c\}$ so that the nodes of $P$ go first ordered from $b$ to $a$. Obviously, in $|P| - 1$ steps the flux $x^*_a$ will be strictly reduced. Furthermore, $x^*_a$ is non-increasing, and its initial and limit values are $y^a_{a,b}$ and $y^a_{a,c}$, respectively. Thus, $y^a_{a,b} > y^a_{a,c}$. Recall that these two numbers are well-defined, although solutions of $(G, a, b)$ and $(G, a, c)$ are not necessarily unique. This proves the lemma.

Remark 5. The same arguments prove that inequality

$$f_{a,b}(x_a - x_b) = y^a_{a,b} \geq y^a_{a,c} = f_{a,c}(x_a - x_c)$$

holds not only for monomial but for arbitrary monotone non-decreasing conductivity functions.

In the exactly same way we can apply MSA to $(G, c, b)$ again taking a solution of $(G, a, b)$ as an initial approximation. Clearly this will result in inequality $y^b_{a,b} \geq y^b_{a,c}$, in which the equality holds if and only if $c$ belongs to every path between $a$ and $b$. Summarizing we obtain the following statement:

**Proposition 1.** For an arbitrary weighted digraph $G$ and nodes $a, b, c$ in it, the inequality $y^*_{a,b} \geq \max(y^*_{a,c}, y^*_{c,b})$ holds and the following five statements are equivalent:

• (ac) $y^*_{a,b} = y^*_{a,c}$;

• (bc) $y^*_{a,b} = y^*_{c,b}$;

• (ab) $y^*_{a,c} = y^*_{c,b}$;

• (acb) every directed path from $a$ to $b$ contains $c$;

• (e) $\mu^s_{a,b} = \mu^s_{a,c} + \mu^s_{c,b}$.
For the rest of the proof of Theorem 1 we will need only elementary "high-school" transformations:

\[ y^*_a,b = \frac{(x^0_a - x^0_b)^r}{\mu_{a,b}} \geq \frac{(x^0_c - x^0_b)^r}{\mu_{a,c}} = y^*_a,c; \quad y^*_a,b = \frac{(x^0_a - x^0_b)^r}{\mu_{a,b}} \geq \frac{(x^0_c - x^0_b)^r}{\mu_{a,c}} = y^*_a,c, \]

(15)

which can be obviously rewritten as follows

\[ \left( \frac{\mu_{a,c}}{\mu_{a,b}} \right)^{s/r} \geq \frac{x^0_c - x^0_a}{x^0_a - x^0_b}; \quad \left( \frac{\mu_{c,b}}{\mu_{a,b}} \right)^{s/r} \geq \frac{x^0_c - x^0_b}{x^0_a - x^0_b} \]

(16)

Summing up these two inequalities we obtain (1).

The above computations show that (1) holds with equality if and only if \( y^*_a,b = y^*_a,c \) and \( y^*_a,b = y^*_c,b \). By Propositions 1 these two equations are equivalent and hold if and only if \( c \) belongs to each directed path from \( a \) to \( b \).

\[
\]

4 Three limit cases

Parallel and series connection of edges

Let us consider two simplest two-pole circuits given in Figure 2.

\[
\]

Figure 2: Parallel and series connection. All edges are directed from left to right.

**Proposition 2.** The resistances of these two circuits can be determined, respectively, from formulas

\[ \mu^{\tilde{s}}_{a,b} = (\mu^{\tilde{s}}_{e'e} + \mu^{\tilde{s}}_{e''e}) \quad \text{and} \quad \mu^{s/f}_{a,b} = (\mu^{s/f}_{e'e} + \mu^{s/f}_{e''e}). \]

(17)

Proof. If \( r = s = 1 \) then (17) turns into familiar high-school formulas. The general case is just a little more difficult. Without loss of generality let us assume that \( y_{a,b} = x_a - x_b \geq 0 \).

In case of the parallel connection we obtain the following chain of equalities.

\[ y_{a,b} = f_{a,b}(y_{a,b}) = \frac{y_{e'a}^{r}}{\mu_{a,b}} + f_{e'e}(y_{a,b}) = \frac{y_{e'a}^{r}}{\mu_{a,b}} + \frac{y_{e''a}^{r}}{\mu_{a,b}} = \frac{y_{e'a}^{r}}{\mu_{a,b}} + \frac{y_{e''a}^{r}}{\mu_{a,b}}. \]

Let us compare the third and the last terms; dividing both by the numerator \( y_{a,b}^{r} \) we arrive at (17).
In case of the series connection, let us start with determining $x_c$ from the first Kirchhoff law:

$$y_{a,b}^r = \frac{y_{a,b}^r}{\mu_{a,b}^r} = \frac{(x_a - x_b)^r}{\mu_{a,b}^r} = y_{e'}^r = \frac{y_{e'}^r}{\mu_{e'}^r} = \frac{(x_a - x_c)^r}{\mu_{e'}^r} = y_{e''}^r = \frac{y_{e''}^r}{\mu_{e''}^r} = \frac{(x_c - x_b)^r}{\mu_{e''}^r}.$$ 

It is sufficient to compare the last and eighth terms to get

$$x_c = x_b \frac{\mu_{e'}^{s/r} + x_b \mu_{e''}^{s/r}}{\mu_{e'}^{s/r} + \mu_{e''}^{s/r}}.$$

Then, let compare the last and forth terms, substitute the obtained $x_c$, and get \ref{eq:17}.

Now, let us consider the convolution $\mu(t) = (\mu_{e'}^t + \mu_{e''}^t)^{1/t}$; it is well known and easy to see that

$$\mu(t) \to \max(\mu_{e'}, \mu_{e''}), \text{ as } t \to +\infty, \text{ and } \mu(t) \to \min(\mu_{e'}, \mu_{e''}), \text{ as } t \to -\infty.$$

### Main Four Examples of Resistance Distances

Let us fix a weighted digraph $G = (V, E, \mu)$ and two strictly positive real parameters $r$ and $s$. As we proved, the obtained circuit can be viewed as a quasi-metric space in which the distance from $a$ to $b$ is defined as the effective resistance $\mu_{a,b}$. As announced in the introduction, this model results in several interesting examples of quasi-metric and quasi-ultrametric spaces. Yet, to arrive to them we should allow for $r$ and $s/r$ to take values $0$ and $+\infty$. More accurately, let $r = r(t)$ and $s = s(t)$ depend on a real positive parameter $t$, or in other words, these two functions define a curve in the positive quadrant $s \geq 0, r \geq 0$.

We proved that resistances $\mu_{a,b}(t)$ are well-defined for every two nodes $a, b \in V$ and each $t$. We will show that, for the four limit transitions listed below, limits $\mu_{a,b}(t) = \lim_{t \to \infty} \mu_{a,b}(t)$, exist for all $a, b \in V$ and can be interpreted as follows:

#### Example 1: the effective Ohm resistance of an electrical circuit.

Let a weighted digraph $G = (V, E, \mu)$ model an electrical circuit in which $\mu_e$ is the resistance of a directed edge (semiconductor) $e$ and $r(t) = s(t) \equiv 1$, or more generally, $r(t) \to 1$ and $s(t) \to 1$, as $t \to +\infty$. Then, $\mu_{a,b}$ is the effective Ohm resistance from $a$ to $b$. For parallel and series connection of two directed edges $e'$ and $e''$, as in Figure 2 we obtain, respectively, $\mu_{a,b}^3 = \mu_{e'}^{-1} + \mu_{e''}^{-1}$ and $\mu_{a,b} = \mu_{e'} + \mu_{e''}$, which is known from the high school.

#### Example 2: the length of a shortest route.

Let a weighted digraph $G = (V, E, \mu)$ model a road network in which $\mu_e$ is the length (milage, traveling time, or gas consumption) of a one-way
road $e$. Then, $\mu_{a,b}$ can be viewed as the distance from $a$ to $b$, that is, the length of a shortest directed path between them. In this case, for parallel and series connection of $e$ and $e''$, we obtain, respectively, $\mu_{a,b} = \min(\mu_{e'}, \mu_{e''})$ and $\mu_{a,b} = \mu_{e'} + \mu_{e''}$. Hence, by (18), $-s(t) \to -\infty$ and $s(t) \equiv r(t)$ for all $t$, as in Figure 1 or more generally, $s(t) \to \infty$ and $s(t)/r(t) \to 1$, as $t \to +\infty$.

Example 3: the inverse width of a bottleneck route.

Now, let digraph $G = (V, E, \mu)$ model a system of one-way passages (rivers, canals, bridges, etc.), where the conductance $\lambda_e = \mu_e^{-1}$ is the "width" of a passage $e$, that is, the maximum size (or tonnage) of a ship or a car that can pass $e$, yet. Then, the effective conductance $\lambda_{a,b} = \mu_{a,b}^{-1}$ is interpreted as the maximum width of a (bottleneck) path between $a$ and $b$, that is, the maximum size (or tonnage) of a ship or a car that can still pass between terminals $a$ and $b$. In this case, $\lambda_{a,b} = \max(\lambda_{e'}, \lambda_{e''})$ for the parallel connection and $\lambda_{a,b} = \min(\lambda_{e'}, \lambda_{e''})$ for the series connection. Hence, $s(t) \to \infty$ and $s(t)/r(t) \to \infty$, as $t \to \infty$; in particular, $r$ might be bounded by a constant, $r(t) \leq \text{const}$, or just $r(t) \equiv 1$ for all $t$, as in Figure 1.

Example 4: the inverse value of a maximal flow.

Finally, let digraph $G = (V, E, \mu)$ model a pipeline or transportation network in which the conductance $\lambda_e = \mu_e^{-1}$ is the capacity of a one-way pipe or road $e$. Then, $\lambda_{a,b} = \mu_{a,b}^{-1}$ is the capacity of the whole two-pole network $(G, a, b)$ from terminal $a$ to $b$. (Standardly, the capacity is defined as the amount of material that can be transported through $e$, or from $a$ to $b$ in the whole circuit, per unit time.) In this case, $\lambda_{a,b} = \lambda_{e'} + \lambda_{e''}$ for the parallel connection and $\lambda_{a,b} = \min(\lambda_{e'}, \lambda_{e''})$ for the series connection. Hence, $-s(t) \equiv -1$ and $s(t)/r(t) \to \infty$, that is, $s(t) \equiv 1$ and $r(t) \to 0$, as in Figure 1 or more generally, $s(t) \to 1$, while $r(t) \to 0$, as $t \to \infty$.

Theorem 2. In all four examples, the limits $\mu_{a,b} = \lim_{t \to +\infty} \mu_{a,b}(t)$ exist and equal the corresponding distances from $a$ to $b$ for all $a, b \in V$. In all four cases these distances define quasi-metric spaces and in the last two - quasi-ultrametric spaces.

Proof. (sketch) For Example 1 there is nothing to prove. Also, for the series-parallel circuits the statement is obvious in all cases. It remains to consider Examples 2, 3, and 4 for general circuits. In each case our analysis will be based on the minimum dissipation principle (??).

For simiplicity, we will omit argument $t$ in $r, s, \mu_e, \mu_a, b, \lambda_e, \lambda_{a,b}, y_e^*, y_{a,b}$, remembering, however, that all these variables depend on $t$, as indicated in the definitions of Examples 2, 3 and 4.

Example 2. In this case $F_e(y_e^*) \sim \mu_e y_e^*$, as $t \to +\infty$.

Hence, "moving some current to a shorter directed path" reduces the total dissipation $F^*(y^*)$ when $t$ is large enough. More precisely, let $p' = p'(s, t)$ and $p''(s, t)$ be two directed paths from $s$ to $t$ in $G$ such that the first one is shorter, that is, $\mu(p') = \sum_{e \in p'} \mu_e < \sum_{e \in p''} \mu_e = \mu(p'')$. 
Suppose that \( \min(y_e^* \mid e \in p' \cup p'') = \gamma_0 > 0 \). Obviously, dissipation \( F''(y^*) \) will be reduced by about \( (\mu(p'') - \mu(p'))\gamma_0 \) if we move the current \( \gamma_0 \) from \( p'' \) to \( p' \), that is, we subtract \( \gamma_0 \) from \( y_e^* \) for each \( e \in p'' \) and add it to \( y_e^* \) for each \( e \in p' \).

Recall that for every fixed \( t > 0 \) each solution of the circuit \( (G, a, b) \) has a unique distribution of currents \( y^* \), which minimizes the total dissipation \( F''(y^*) \). Thus, the above observation implies that all currents will tend to the shortest directed paths from \( a \) to \( b \), as \( t \to +\infty \). Moreover, \( y_e^* \) well become just 0 for any \( e \) that does not belong to some shortest \((a-b)\)-path if \( t \) is large enough.

Clearly, by an arbitrary small perturbation of \( \mu_e \), one can make the lengths of all directed paths from \( a \) to \( b \) distinct. After such perturbation, the shortest \((a-b)\)-path \( p_0 \) in \( G \) becomes unique and all currents outside of it become 0, that is, \( y_e^* = 0 \) whenever \( e \notin p_0 \), in particular, \( \mu_{a,b} \) becomes the length of \( p_0 \) when \( t \) is large enough.

\textbf{Example 3.} In this case \( F_0(y_e^*) \sim \frac{1}{2} \mu_e y_e^2 \), where \( s = s(t) \to +\infty \), as \( t \to +\infty \). Thus, all currents will tend to the widest bottleneck directed paths from \( a \) to \( b \), as \( t \to +\infty \). Moreover, \( y_e^* \) well become just 0 for any \( e \) that does not belong to such a path if \( t \) is large enough.

Recall that the widest bottleneck directed \((a-b)\)-path \( p \) in \( G \) is defined as one maximizing \( \min(\mu_e \mid e \in p) \).

Unlike the shortest \((a-b)\)-path the widest bottleneck one is "typically" not unique in \( G \). Let us refine slightly this concept and introduce the lexicographically widest bottleneck directed \((a,b)\)-path in \( G \). To do so, consider all widest bottleneck directed \((a-b)\)-paths in \( G \). Among them choose those that maximize the second smallest width, etc. In several steps (at most \(|V| \)) we will obtain the required path.

Clearly, by an arbitrary small perturbation of \( \mu_e \), one can make the widths of all directed paths from \( a \) to \( b \) distinct. Under this condition, the lexicographically widest bottleneck directed \((a,b)\)-path \( p_0 \) in \( G \) becomes unique, and all currents outside of it become 0, that is, \( y_e^* = 0 \) whenever \( e \notin p_0 \), in particular, \( \mu_{a,b} \) becomes the width of \( p_0 \) when \( t \) is large enough.

\textbf{Example 4.} In this case, setting \( s = 1 \), we obtain
\[
F_0(y_e^*) \sim r y_e^* (\mu_e y_e^*)^{\frac{1}{2}} = r y_e^* (\frac{\gamma_0}{\lambda_e})^{\frac{1}{2}},
\] where \( r = r(t) \to 0 \) as \( t \to +\infty \).

Let us recall the concept of the so-called \textit{balanced flow} introduced in [12] for the multi-pole circuits. Given a weighted digraph \( G = (V, E) \) consider the following boundary conditions: \( x_v^* = x_v^0 \) for all \( v \in V \).

A flow \( y^* \) is called \textit{satisfactory} if it satisfies these conditions. We will assume that such a flow exists. In particular, \( \sum_{v \in V} x_v^0 = 0 \) must hold.

Two-pole boundary conditions [13], considered in the present paper, form a special case of the multi-pole conditions. In this case a satisfactory flow exists if and only if digraph \( G \) contains a directed path from \( a \) to \( b \).

Introduce resistances \( \mu_e \) (and conductances \( \lambda_e = \mu_e^{-1} \)) for all directed edges \( e \in E \), thus getting a weighted multi-pole circuit \( G = (V, E, \mu) \).

Among all satisfactory flows choose all that minimize \( \max(\frac{\lambda_e^2}{\mu_e^2} \mid e \in E) \); then among them choose all that minimize the second largest value of \( \frac{\lambda_e^2}{\mu_e^2} \).
etc. For all $e \in E$ order the ratios $\frac{y^*_e}{\Lambda(e)}$ non-decreasingly. A satisfactory flow $y^*$ realizing the lex-min over all such vectors is called balanced.

Let us briefly recall the algorithm from [12] constructing a balanced flow in a multi-pole circuit. A cut in a digraph $G = (V, E)$, is defined as an ordered pair $C = (V', V'')$ that partitions $V$ properly, that is,

$$V' \neq \emptyset, V'' \neq \emptyset, V' \cap V'' = \emptyset \text{ and } V' \cup V'' = V.$$  

We say that a directed edge $e = (u, w)$ is in $C$ if $u \in V'$ and $w \in V''$. Consider a multi-pole circuit defined by a weighted digraph $G = (V, E, \mu)$ and boundary conditions $x^o_v = x^v$, $v \in V$; fix a cut $C$ in $G$. Its deficiency and capacity are defined by formulas:

$$D(C) = \sum_{v \in V'} x^o_v = -\sum_{v \in V''} x^v; \quad \Lambda(C) = \sum_{e \in C} \lambda_e.$$  

Note that $\Lambda(C) \geq 0$, by this definition, while $D(C)$ may be negative.

A multi-pole problem has no satisfactory vector if and only if there exists a cut $C$ such that $R(C) = +\infty$, or in other words, such that $D(C) > 0$ and $\Lambda(C) = 0$. In a two-pole circuit this happens if and only if there is no directed path from $a$ to $b$. Note also that in the two-pole case we have $D(C) = 0$ whenever $a, b \in V'$ or $a, b \in V''$.

Cut $C$ is called critical if it realizes the maximum of the ratio $\frac{D(C)}{\Lambda(C)}$. Choose such $C$ and set $y^*_e = R(C)\lambda_e$ for each $e \in C$. Then, $\frac{y^*_e}{\lambda_e}$ take the same value $R(C)$ for all $e \in C$ and we have $\sum_{e \in C} y^*_e = \lambda(C)$.

Reduce digraph $G$ eliminating all edges of $C$ from it. Recompute new multi-pole boundary conditions in the obtained reduced digraph $G'$ taking into account flows $y^*_e$ on the deleted edges $e \in C$. Then, find a critical cut $C'$ in $G'$, and repeat. It is shown in [12] that the values $R, R', \ldots$ are monotone non-increasing. Hence, the ratio $\frac{y^*_e}{\lambda_e}$ takes the largest values on $e \in C$ at the first stage. It is also shown in [12] that the balanced flow is unique.

Recall that $F_t(y^*_e) \sim (ry^*_e)(\frac{y^*_e}{\lambda_e})^t$, where $r = r(t) \to 0$ as $t \to +\infty$. From this we conclude that a satisfactory flow $y^*$, minimizing the total dissipation $F^r(y^*)$, becomes a balanced flow when $t$ is large enough.

A satisfactory flow $y^*$ is called feasible if $y^*_e \leq \lambda_e$. Obviously, the (unique) balanced flow $y^{*0}$ is feasible whenever a latter exists. We saw that in this case $y^{*0}$ is a solution of the multi-pole network problem, since $y^{*0}$ minimizes $F^r(y^*)$ when $t$ is large enough.

In particular, this is true for the two-pole problems. In this case let us set $x^o_a = -x^0_b = \Lambda(a, b)$, where $\Lambda(a, b)$ is the capacity from $a$ to $b$ of the circuit $G = (V, E, \mu)$ with poles $a$ and $b$. Obviously, on the first stage of the algorithm we obtain a cut $C$ of the unit ratio $R(C) = 1$, that is, $y^*_e = \lambda e$ for all $e \in C$. Thus, $\lambda(a, b) \to \Lambda(a, b)$, as $t \to +\infty$.

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