Graphop Mean-Field Limits for Kuramoto-Type Models

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Abstract

Originally arising in the context of interacting particle systems in statistical physics, dynamical systems and differential equations on networks/graphs have permeated into a broad number of mathematical areas as well as into many applications. One central problem in the field is to find suitable approximations of the dynamics as the number of nodes/vertices tends to infinity, i.e., in the large graph limit. A cornerstone in this context are Vlasov-Fokker-Planck equations (VFPEs) describing a particle density on a mean-field level. For all-to-all coupled systems, it is quite classical to prove the rigorous approximation by VFPEs for many classes of particle systems. For dense graphs converging to graphon limits, one also knows that mean-field approximation holds for certain classes of models, e.g., for the Kuramoto model on graphs. Yet, the space of intermediate density and sparse graphs is clearly extremely relevant. Here we prove that the Kuramoto model can be approximated in the mean-field limit by far more general graph limits than graphons. In particular, our contributions are as follows. (I) We show, how to introduce operator theory more abstractly into VFPEs by considering graphops. Graphops have recently been proposed as a unifying approach to graph limit theory, and here we show that they can be used for differential equations on graphs. (II) For the Kuramoto model on graphs we rigorously prove that there is a VFPE equation approximating it in the mean-field sense. (III) This mean-field VFPE involves a graphop, and we prove the existence, uniqueness, and continuous graphop-dependence of weak solutions. (IV) On a technical level, our results rely on designing a new suitable metric of graphop convergence and on employing Fourier analysis on compact abelian groups to approximate graphops using summability kernels.

Keywords: Kuramoto model on graphs, mean field limit, Vlasov Fokker-Planck equation, graphops, o-convergence, summability kernel.

1 Introduction

Synchronization, or in other words the effect under which a system of coupled oscillators with different individual initial frequencies pulses, after a while, under the same single global frequency, is a phenomenon which can be found in various biological, ecological, social and technological processes [18]. An important first model for synchronization was developed by Kuramoto [11]. This model considers a finite number of $N$ different oscillators. Each oscillator has an intrinsic frequency $\omega_i \in \mathbb{R}$ for $i = 1, \ldots, N$. The frequencies are distributed according to a symmetric probability density function $g : \Omega \to [0, \infty)$. The phase of each oscillator $u_i(t) \in [0, 2\pi) =: \mathbb{T}$ are the unknowns satisfying the following system of ordinary differential equations (ODEs)

$$
\dot{u}_i = \omega_i + \frac{C}{N} \sum_{j=1}^{N} \sin(u_j - u_i), \quad i \in \{1, 2, \ldots, N\},
$$

(1.1)
where the parameter \( C > 0 \) is the coupling strength. Further, let \( \rho(u, \omega, t) \, du \) denote the fraction of oscillators with frequency \( \omega \) and phase between \( u \) and \( du \) for time \( t \). Sakaguchi [17] proposed that the Kuramoto model (1.1) can be approximated, as \( N \to \infty \), by the single mean field Vlasov-Fokker-Planck equation (VFPE)

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial u}(\rho V(\rho)) = 0,
\]

with the characteristic field

\[
V(\rho)(u, \omega, t) = \omega + C \int_{0}^{2\pi} \int_{\mathbb{R}} \sin(\tilde{u} - u) \rho(\tilde{u}, \tilde{\omega}, t) g(\tilde{\omega}) \, d\tilde{\omega} \, d\tilde{u}.
\]

Although the formal derivation of (1.2) from (1.1) has been extensively studied in the literature (see for example [2, 18] and references therein), it was only proved rigorously around fifteen years ago by Lancelloti [13]. His approach was to view (1.2) as an abstract continuity equation of measures and then apply Neunzert’s fixed point argument [15, 16]. Of course, the classical Kuramoto model (1.1) makes the unrealistic assumption that every oscillator equally affects everyone else and that coupling takes place exactly via the first Fourier mode represented by the sine nonlinearity. For more precise models we should take into account the network coupling structure of the system and more general Fourier modes. This generalized Kuramoto-type model on an arbitrary network/graph takes the form

\[
\dot{u}_i = \omega_i + C \sum_{j=1}^{N} A_{i,j}^N D(u_j - u_i), \quad i \in [N],
\]

where \( A = (A_{i,j}^N)_{i,j=1,...,N} \in \mathbb{R}^{N \times N} \) is the adjacency matrix of the network of oscillators and the coupling function \( D : \mathbb{T} \to \mathbb{R} \) satisfies the Lipschitz condition

\[
|D(u) - D(v)| \leq |u - v|, \quad \forall u, v \in \mathbb{T}.
\]

Further, without loss of generality we may assume that

\[
\max_{u \in \mathbb{T}} |D(u)| \leq 1.
\]

The recent development of graph limit theory [12] enabled the rigorous treatment of approximating limits as \( N \to \infty \) for several classes of graphs converging, in a suitable sense, towards a graph limit [14, 3, 4, 6]. For example, Kaliuzhnyi-Verbovetskyi and Medvedev [4] treat the case that there exists a graphon limit \( W : [0, 1] \times [0, 1] \to [0, 1] \), i.e., \( W \) is a measurable and symmetric function such that the weights \( A_{i,j}^N \) are given by

\[
A_{i,j}^N := \int_{I_j^N} \int_{I_i^N} W(x, y) \, dx \, dy.
\]

Here, \( \{I_i^N\}_{i=1,...,N} \) is the partition of \( I := [0, 1] \) given (up to measure 0) by the intervals \( I_i^N := \left[ \frac{i-1}{N}, \frac{i}{N} \right] \).
Consider the family of empirical measures

\[
\nu_{n,M}^x(S) = M^{-1} \sum_{j=1}^{M} \chi_S(u_{(i-1)M+j}^N(t)), \quad S \in \mathcal{B}(\mathbb{T}), \quad x \in I_i^N,
\]

where \( \mathcal{B}(\mathbb{T}) \) denotes the Lebesgue \( \sigma \)-algebra on \( \mathbb{T} \), \( \chi_S \) is the indicator function of \( S \), and \( u_i^N(t) \) is the solution of (1.4). Then one may prove [4] that the empirical measure (1.8) approximates as \( N \to \infty \),
in a suitable distance and under certain initial conditions of the Kuramoto model (1.4), the family of continuous measures

$$\nu^x_t(S) = \int_S \int \rho(t, \tilde{u}, \tilde{\omega}, x) \, d\tilde{\omega} \, d\tilde{u}, \quad S \in B(\mathbb{T}), \quad x \in I,$$

where $\rho$ solves the mean field equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial u} (\rho V[W](\rho)) = 0. \quad (1.10)$$

As we might expect, this VFPE is similar to (1.2), with the only difference that the characteristic field $V = V[W]$ now depends on the graphon and is explicitly given by

$$V[W](\rho)(u, \omega, x, t) = \omega + C \int_I \int_0^{2\pi} \int_\mathbb{R} D(\tilde{u} - u)W(x, y)\rho(\tilde{u}, \tilde{\omega}, y, t) g(\tilde{\omega}) \, d\tilde{\omega} \, d\tilde{u} \, dy. \quad (1.11)$$

All recent approaches for the mean-field limit relied on the fact that the limiting graph is given by a graphon (i.e., it is a dense graph) and a natural question arises is, how to treat the case of limiting graphs with intermediate densities or sparse graphs? A major obstacle was that up until recently, infinite sparse and dense graphs had been extensively studied in the graph limit theory but using very different convergence notions, which relied frequently on combinatorial ideas difficult to incorporate into analysis-based methods used for differential equations. Even beyond this challenge, no unified theory existed for the treatment of graphs of intermediate density. Only recently Backhausz and Szegedy [1] provided a far more general framework unifying dense and sparse graph limit theory. The novel viewpoint is that graphs can be represented via suitable operators, the so-called graphops.

The main goal of the present paper is to prove that mean field approximation of the Kuramoto-type models of the form (1.4) is possible in many cases if the limiting graph is given by a graphop. Since graphops cover a very large and general class of graph limits [1], we believe that our results can also provide the basis for a very broad use of graphops in differential equations arising from dynamics on networks/graphs. Next, we are going to introduce some basic facts about graphops. After that, we come back to discuss the central question in this paper in more detail.

### 1.1 Representation of graphs via graphops

Graphops were introduced in [1] as a new way for representing graphs to unify the language provided for dense graph limits and Benjamini-Schramm limits, and to include also graphs of intermediate density. We quickly summarize some basic notions and results given in [1]. Let $(\Omega, \Sigma, m)$ be a Borel probability space and $\Omega$ a compact set. A $P$-operator is a linear operator $A : L^\infty(\Omega, m) \to L^1(\Omega, m)$ which is bounded, i.e., it has a finite operator norm

$$\| A \|_{\infty \to 1} := \sup_{v \in L^\infty(\Omega)} \frac{\| Av \|_1}{\| v \|_\infty} < \infty.$$  

More generally, for a $P$-operator $A$, the operator norm $\| A \|_{p \to q}$, for any real numbers $p, q \in [1, \infty]$, is given by

$$\| A \|_{p \to q} := \sup_{v \in L^\infty(\Omega)} \frac{\| Av \|_q}{\| v \|_p}.$$  

$\mathcal{PB}(\Omega, \Sigma, m)$ denotes the space of all P-operators on $\Omega$; if the underlying measure $m$ is clear we simply write $\mathcal{PB}(\Omega)$. In the space of P-operators, objects which represent graphs are the so-called graphops.
A P-operator $A$ is called a graphop if it is positivity preserving and self-adjoint. To be more precise, positivity preserving means that

$$v(x) \geq 0 \text{ for m-a.e. } x \in \Omega \Rightarrow Av(x) \geq 0 \text{ for m-a.e. } x \in \Omega,$$

and self-adjoint here means that for any $v, w \in L^\infty(\Omega, m)$ we have

$$\langle Av, w \rangle = \langle v, Aw \rangle,$$

with the bilinear from $\langle v, w \rangle := \int_\Omega v(x)w(x) \, dm(x)$. We also write $\langle v, w \rangle_A := \langle Av, w \rangle$.

Intuitively, the space $\Omega$ represents the node set of the graph. Its edge set is represented by a symmetric fiber measure $\nu$ on the product set $\Omega \times \Omega$, which exists for any graphop $A$ according to the following theorem:

**Theorem 1.1.** (Measure representation of graphops)

Assume that $A : L^\infty(\Omega, m) \to L^1(\Omega, m)$ is a graphop. Then following statements are true:

1. There is a unique finite measure $\nu$ on $(\Omega \times \Omega, \Sigma \times \Sigma)$ with the following properties:
   (i) $\nu$ is symmetric.
   (ii) The marginal distribution $\pi_\nu$ of $\nu$ on $\Omega$ is absolutely continuous with respect to $m$. Here $\pi : \Omega \times \Omega \to \Omega$ denotes the canonical projection and $\pi_*$ is the associated pushforward.
   (iii) For every $f, g \in L^\infty(\Omega, m)$ holds:
   $$\langle f, g \rangle_A = \int_{\Omega^2} f(x)g(y) \, d\nu(x, y) = \int_{\Omega^2} g(x)f(y) \, d\nu(x, y) = \langle g, f \rangle_A.
   $$

2. There is a family $\{\nu_x\}_{x \in \Omega}$ of finite measures (called fiber measures), such that for all $f \in L^\infty(\Omega, m)$ we have
   $$(Af)(x) = \int_{\Omega} f(y) \, d\nu_x(y) \quad \text{m-a.e. } x \in \Omega.$$

For this family we have additionally for any $h \in L^\infty(\Omega^2, \nu)$

$$\int_{\Omega^2} h(x, y) \, d\nu_x(y) \, dm(x) = \int_{\Omega^2} h(x, y) \, d\nu(x, y).$$

For a given graphop $A$, we call the family $\{\nu_x\}_{x \in \Omega}$ the fiber measures associated to the graphop $A$. We sometimes also write $\nu^A_x$ to make the graphop dependence clear. Notice that the second statement in Theorem 1.1 follows immediately from the first one, using the disintegration theorem. For a node $x \in \Omega$, the measure $\nu_x$ represents the neighborhood of $x$. Moreover, the number $A\chi_{\Omega}(x) = \nu_x(\Omega)$ is the degree of $x$. A particular interesting case occurs, when all edges have the same degree, or in other words, when there exists a constant $c > 0$ such that $A\chi_{\Omega}(x) = c\chi_{\Omega}(x)$. In this case the graphop $A$ is called $c$-regular. A Markov graphop is a 1-regular graphop. Another important case occurs when all fiber measures $\{\nu_x\}_{x \in \Omega}$ are absolutely continuous with respect to $m$. In this case, by the Radon-Nikodym theorem, we have $\nu_x(dy) = W(x, y) \, dm(y)$ for a symmetric function $W : \Omega \times \Omega \to [0, \infty)$.

**Definition 1.2.** A graphon is a measurable, symmetric, bounded and positive function $W : \Omega \times \Omega \to [0, \infty)$.

The other way around, given a graphon $W$ we may easily obtain a graphop $A_W : L^\infty(\Omega, m) \to L^1(\Omega, m)$, via setting

$$A_W f(x) := \int_{\Omega} W(x, y) f(y) \, dm(y). \quad (1.12)$$

Thus we may view the graphon space as a true subspace of the space of graphops. In the following, we will often call $A_W$ given in (1.12) itself a graphon and $W$ the corresponding (graphon) kernel.

\[\text{In the classical literature, a graphon is considered to be a function } W : I \times I \to I, \text{ but it is also known that we may identify } \Omega = I \text{ with the unit interval.}\]
1.2 Main problem

Our starting point is an observation made by the second author in [9]: If we compare the original VFPE (1.2) with the VFPE on a graphon (1.10), we observe that in equation (1.2) we formally replace

\[ \rho(\tilde{u}, \tilde{\omega}, t) \quad \text{by} \quad \int_I W(x,y) \rho(\tilde{u}, \tilde{\omega}, y, t) \, dy. \]

Therefore, in the case that the sequence of adjacency matrices \( A^N \) for the discrete Kuramoto model (1.4) is converging in the sense of dense graph convergence towards a graphon, the effect on the mean field VFPE is best viewed as an operator action. Having in mind this observation and linking it to the new operator framework for representing graphs via graphops, one may conjecture [9] formally that if the limiting object of the sequence of graphs, in the sense of P-operator convergence, is a general graphop \( A \), then in equation (1.2) we should replace

\[ \rho(\tilde{u}, \tilde{\omega}, t) \quad \text{by} \quad A \rho(x; \tilde{u}, \tilde{\omega}, t). \]

Our main goal in this paper is to prove this conjecture rigorously. Furthermore, we want to provide a suitable solution theory for VFPEs involving graphops. Let us quickly discuss the main idea, how we are going to prove the approximation properties of the mean-field VFPE based upon the results for graphons.

From now on in this paper, for notational simplicity, we shall restrict to the case that all frequencies \( \omega_i = 0 \), for all \( i \in [N] \) are identical zero, but all results can be extended in a straightforward way to the general case of unequal frequencies, see the discussion in Section [6]. Let \( A : L^\infty(\Omega, m) \to L^1(\Omega, m) \) be a fixed graphop with node set given by a compact abelian group \( \Omega \) equipped with the Haar measure \( \mu_{\Omega} \) on the Lebesgue sets. Assume that for the graphop \( A \) we have found a sequence of graphons \( A^K \) with corresponding kernels \( W^K : \Omega \times \Omega \to [0, \infty) \), such that

\[ A^K \to A \quad \text{as} \quad K \to \infty, \]

where the convergence takes place in a carefully chosen topology. Further let \((\Omega^n_i)_{i=1,...,n}\) be a sequence partitions of \( \Omega \) satisfying \( m(\Omega^n_i) = \frac{1}{n} \) for all \( i \in [n] \). For any fixed \( M \in \mathbb{N} \) we set \( N = nM \) and assume additionally that the partition satisfies \( \Omega^n_i \subset \Omega^n_i \) for all \( n, M \in \mathbb{N} \). We can then define, for any \( N, K \in \mathbb{N} \), the weights

\[ A_{i,j}^{N,K} := N^2 \int_{\Omega^n_i \times \Omega^n_j} W^K(x,y) \, dx \, dy \]  

(1.13)

and consider the (generalized) Kuramoto model

\[ u_i^{N,K} = CN^{-1} \sum_{j=1}^N A_{i,j}^{N,K} D(u_j^{N,K} - u_i^{N,K}), \]  

(1.14a)

\[ u_i^{N,K}(0) = u_i^{N,0}, i \in [N]. \]  

(1.14b)

We recall that the coupling function \( D : \mathbb{T} \to \mathbb{R} \) satisfies conditions (1.5) and (1.6). Since \( A^K \) is for any fixed \( K \in \mathbb{N} \) a graphon, we can conclude by [4], that the empirical measure

\[ \nu_{\alpha,M,K,t}^n(x) = M^{-1} \sum_{j=1}^M \chi_D(u_{(i-1)M+j}^{N,K}(t)), \quad x \in \Omega^n_i, \quad S \in \mathcal{B}(\Omega). \]  

(1.15)
approximates, as $N \to \infty$, the $K$-th continuous measure

$$\nu^{x,K}_t(S) = \int_S \rho^K(t,u,x) \, du, \quad x \in \Omega, \quad S \in \mathcal{B}(\Omega). \quad (1.16)$$

where $\rho^K$ is the unique solution of the following mean field initial value problem (IVP), denoted by $\text{VFPE}^K$,

$$\begin{align*}
\partial_t \rho^K(t,u,x) &= -\partial_u (\rho^K V[A^K] \rho^K)(t,u,x), \quad (t,u,x) \in [0,T] \times \mathbb{T} \times \Omega, \\
\rho^K(0,u,x) &= \rho^0(u,x),
\end{align*} \quad (1.17a)$$

$$\begin{align*}
\rho^0(u,x) &= \rho_0(u,x), \quad (1.17b)
\end{align*}$$
corresponding to the graphon $A^K$ with initial condition $\rho^0$. We further define the limiting measure for $K \to \infty$ by

$$\nu^x_t(S) = \int_S \rho(t,u,x) \, du, \quad x \in \Omega, \quad S \in \mathcal{B}(\Omega), \quad (1.18)$$

where $\rho$ is the unique solution (cf. Theorem 2.10 below) of the limiting IVP, denoted by $\text{VFPE}^\infty$,

$$\begin{align*}
\partial_t \rho(t,u,x) &= -\partial_u (\rho V[A] \rho)(t,u,x), \quad (t,u,x) \in [0,T] \times \mathbb{T} \times \Omega, \\
\rho(0,u,x) &= \rho^0(u,x),
\end{align*} \quad (1.19a)$$

$$\begin{align*}
\rho^0(u,x) &= \rho_0(u,x), \quad (1.19b)
\end{align*}$$
corresponding to the graphop $A$, with the characteristic field $V[A]$ be given by

$$V[A] \rho(t,u,x) := C \int_0^{2\pi} (A \rho)(t,\tilde{u},x) D(\tilde{u} - u) \, d\tilde{u}. \quad (1.20)$$

If the convergence of the graphon approximation $A^K$ towards $A$ is strong enough, we can then hope that, under suitable assumptions (which we will discuss in more detail later), the $K$-th continuous measure (1.16) will be close to the measure (1.18). In particular, the following diagram summarizes the proof technique:

$$\text{Kuramoto's model} \xrightarrow{N \to \infty} \text{VFPE}^K \xrightarrow{N,K \to \infty} \text{VFPE}^\infty \xrightarrow{K \to \infty}$$

The basic advantage of this approach is that, once we passed to the first limit $N \to \infty$, we can forget the discrete Kuramoto model (1.14) and we only have to work with a VFPE. A central point for this approach to succeed is that the convergence of the approximating sequence $A^K$ towards $A$ should be

- **weak enough**, to allow approximation via graphons of a big enough class of graphops.
- **strong enough**, to guarantee that solutions of the VFPE for different graphops, which are “close enough” with respect to this topology, are themselves arbitrary close.

In particular, this is the analytic translation of the key problem in graph limit theory on the level of VFPEs. In our case, the following new convergence notion will actually work:

**Definition 1.3. (o-graphop convergence)**

For the graphops $A^n, A$ on the same probability space $(\Omega, \Sigma, m)$ with associated fiber measures $\nu^n_x$ and $\nu_x$ we write

$$A^n \to_o A \iff \nu^n_x \to_w \nu_x \quad m \text{-a.e.} \ x \in \Omega.$$  

**Remark 1.4.** By Portmanteau’s Theorem [8, Theorem 13.16], it follows immediately that $o$-convergence is equivalent to $A^n f(x) \to A f(x)$ $m$-a.e. $x \in \Omega$ for all $f \in C_0(\Omega)$.
1.3 Outline of the paper

In Section 2 we build up necessary results so that our main question concerning mean field approximation for the Kuramoto model (1.14) is well posed. In particular, we start by studying the general VFPE (1.19) with a graphop $A$, defined on an arbitrary compact Borel probability space $(\Omega, \Sigma, m)$. For this equation, we prove existence and uniqueness of solutions, cf. Theorem 2.9 and Theorem 2.10. Then, in Section 3 we prove that the solutions of the VFPE depend continuously on the graphop $A$, i.e., for all $\nu_{x,i} \in \Omega$ are fixed representatives defined on the whole space $\Omega$. In this section we are interested in proving existence and uniqueness of weak solutions for the general VFPE (1.19) via the VFPE (1.19), cf. Theorem 3.1. We prove the mean field approximation for a big class of graphops on compact abelian groups, which covers the case that $A$ is any $c$-regular or Markov graphop, cf. Corollary 3.2. Finally, in Section 4 we summarize our results and discuss further generalizations and open problems. We also discuss the straightforward adaptations needed for the treatment of general initial frequencies $\{\omega_i\}_{i \in \mathbb{N}}$.

2 Existence and uniqueness of solutions for the VFPE

Assume $(\Omega, \Sigma, m)$ is a compact Borel probability space and that $A \in \mathcal{PB}(\Omega, \Sigma, m)$ is a fixed graphop with corresponding measure $\nu = \nu^A$ on $\Omega^2$, cf. Theorem 1.1. Further, let $\{\nu_{x,i}\}_{x \in \Omega}$ be the family of fiber measures associated to the graphop $A$, i.e., for all $f \in L^\infty(\Omega, m)$ we have:

$$ (Af)(x) = \int_\Omega f(y) \, d\nu^A_x(y), \quad m\text{-a.e. } x \in \Omega, $$ (2.1)

cf. Theorem 1.1. Since we work with the fixed graphop $A$, we can (and will) always assume in this section that $\{\nu_{x,i}\}_{x \in \Omega}$ are fixed representatives defined on the whole space $\Omega$. In this section we are interested in proving existence and uniqueness of weak solutions for the general VFPE (1.19) with the graphop $A$. We consider following initial value problem (IVP) for the VFPE

$$ \partial_t \rho(t, u, x) = - \partial_u (\rho V(\rho))(t, u, x) \quad (t, u, x) \in [0, T] \times \mathbb{T} \times \Omega, \quad (2.2a) $$
$$ \rho(0, u, x) = \rho^0(u, x), \quad (2.2b) $$

where the characteristic field $V(t, u, x) = V[A, \rho, x](t, u)$ is given by

$$ V[A, \rho, x](t, u) := C \int_0^{2\pi} (Ap)(t, \tilde{u}, x)D(\tilde{u} - u) \, d\tilde{u}. \quad (2.3) $$

For any positive time $T > 0$ we set $T := [0, T]$. Following [16, 4], we state following definition:

Definition 2.1. (Weak solutions for the VFPE)

A measurable function $\rho : T \times \mathbb{T} \times \Omega \to \mathbb{R}$ is called a weak solution of the IVP (2.2) for the VFPE, if following conditions hold for $\nu_x$-a.e. $y \in \Omega$, for all $x \in \Omega$:

1. $\rho(t, u, x)$ is weakly continuous in $t \in \mathbb{T}$, i.e., the map $t \mapsto \int_\mathbb{T} \rho(t, u, x)f(u) \, du$ is continuous for every $f \in C(\mathbb{T})$;
2. for every $w \in C^1(\mathbb{T} \times \mathbb{T})$ with support in $[0, T) \times \mathbb{T}$ it holds that

$$ \int_0^T \int_\mathbb{T} \rho(t, u, x) \left( \partial_t w(t, u) + V(t, u, x) \partial_u w(t, u) \right) \, du \, dt + \int_0^T w(0, u) \rho_0(u, x) \, du = 0. $$
It can be shown \[16\] (remarks after eq. (10)) that if \(\rho\) and \(V\) are both sufficiently smooth, then \(\rho\) is also a classical solution of the IVP of the VFPE (2.2). As we are going to see later in Section 2.4 Neunzert’s fixed point argument \[15\] \[16\] translates the VFPE (2.2) to a fixed point equation for measures. Hence, let us now define the measure spaces we are going to work with.

2.1 The measure spaces

Let \(M_f = M_f(T)\) denote the space of finite Borel measures equipped with the bounded Lipschitz metric

\[
d_{BL}(\mu, \nu) := \sup_{f \in \mathcal{L}} \left| \int_T f(v) \, d(\mu - \nu)(v) \right|
\]

where

\[
\mathcal{L} := \{ f : T \to [0, 1], f \text{ is Lipschitz with Lipschitz constant } \leq 1 \}.
\]

It is known that \((M_f, d_{BL})\) is a complete metric space. Further, for any \(b > 0\), we define the space \(\bar{M}_b := \{ \bar{\mu} : \Omega \to M_f(T) : \bar{\mu} \text{ is measurable, and } \sup_{x \in \Omega} \mu^x(T) \leq b \}\), where \(\mu^x \in M_f(T)\) denotes the evaluation of the family of measures \(\bar{\mu}\) at \(x\). In the following we assume additionally that for the graphop \(A\) the following condition is satisfied

\[
\gamma_A := \sup_{x \in \Omega} \left( \nu^A_x(\Omega) \right) \leq 1. \tag{2.4}
\]

Then, on \(\bar{M}^b \times \bar{M}^b\) and with the graphop \(A\), we set

\[
\bar{d}^b(A, \bar{\kappa}) := \sup_{x \in \Omega} \left( \int_{\Omega} d_{BL}(\mu^y, \kappa^y) \, d\nu^A_x(y) \right).
\]

We further define the sets

\[
\mathcal{G} := \{ B \in \mathcal{PB}(\Omega, \Sigma, l) : l \text{ is a probability measure, } \gamma_B \leq 1 \},
\]

\[
\mathcal{G}^m := \{ B \in \mathcal{PB}(\Omega, \Sigma, m) : B \text{ is a graphop, } \gamma_B \leq 1 \},
\]

and the following metric (cf. Lemma A.1) on \(\bar{M}^b\)

\[
\bar{d}^b(\bar{\mu}, \bar{\kappa}) := \sup_{B \in \mathcal{G}} \bar{d}^b(B, \bar{\mu}, \bar{\kappa}) = \sup_{B \in \mathcal{G}} \left( \int_{\Omega} d_{BL}(\mu^y, \kappa^y) \, d\nu^B_x(y) \right) \tag{2.5}
\]

It can be shown that \((\bar{M}^b, \bar{d}^b)\) is a complete metric space, cf. Lemma A.2. We also define the space

\[
\bar{M}_T^b := C(T, (\bar{M}^b, \bar{d}^b))
\]

and equip it with following metric for a fixed \(\alpha > 0\)

\[
d_{\alpha}^b(\bar{\mu}, \bar{\nu}) := \sup_{t \in T} e^{-\alpha t} d^b(\bar{\mu}_t, \bar{\nu}_t). \tag{2.6}
\]

We note that

\[
e^{-\alpha T} d^b(\bar{\mu}, \bar{\nu}) \leq d_{\alpha}^b(\bar{\mu}, \bar{\nu}) \leq d^b(\bar{\mu}, \bar{\nu}),
\]

where the metric

\[
d^b(\bar{\mu}, \bar{\nu}) := \sup_{t \in T} d^b(\bar{\mu}_t, \bar{\nu}_t)
\]

generates the usual uniform topology on \(\bar{M}_T^b\). Hence, from Lemma A.2 it follows that the space \((\bar{M}_T^b, d_{\alpha}^b)\) is complete as well.
2.2 The extended graphop

Associated with the graphop $A$ we can define an operator $A$ on a family $\{\mu^y\}_{y \in \Omega} \in \mathcal{M}^b$ via

\[
(A\mu)^x := \int_{\Omega} \mu^y \, d\nu^A_x(y). \tag{2.7}
\]

Here, the integral in the right side is to be understood in the following sense

\[
(A\mu)^x(S) = \int_{\Omega} \mu^y(S) \, d\nu^A_x(y) \quad \text{for any Borel set } S \subset T \text{ and } x \in \Omega.
\]

Note especially that for the given fixed family $\{\nu^A_x\}_{x \in \Omega}$, $\{(A\mu)^x\}_{x \in \Omega}$, is a family of finite measures with

\[
(A\mu)^x(\Omega) \leq \nu^A_x(\Omega) \sup_{y \in \Omega} \mu^y(\Omega) \leq b \gamma_A \leq b. \tag{2.8}
\]

Note that the operator $A$ depends directly on fiber measures $\{\nu^A_x\}_{x \in \Omega}$. We will very often make use of the following lemma:

**Lemma 2.2. (A and integration)**

For any nonnegative Borel measurable function $f : T \to \mathbb{R}_{\geq 0}$ and any family $\{\mu^y\}_{y \in \Omega} \in \mathcal{M}^b$ we have

\[
\int_T f(v) \, d(A\mu^x)(v) = \int_{\Omega} \int_T f(v) \, d\mu^y(v) \, d\nu^A_x(y) = A(\int_T f(v) \, d\mu^y(v))(x).
\]

We also write in short notation

\[
d(A\mu^x)(v) = d\mu^y(v) \, d\nu^A_x(y).
\]

**Proof.** For the special case that $f = \chi_S$, where $S \subset T$ is Borel set, is a characteristic function we calculate immediately

\[
\int_T f(v) d(A\mu^x)(v) = \int_S d(A\mu^x)(v) = \left(\int_{\Omega} \mu^y \, d\nu^A_x(y)\right)(S) = \int_{\Omega} \mu^y(S) \, d\nu^A_x(y) = \int_{\Omega} \int_T f(v) \, d\mu^y(v) \, d\nu^A_x(y) = A(\int_T f(v) \, d\mu^y(v))(x).
\]

In the same way, using linearity, we can verify the claim in the case that $f = \sum_{k=1}^n \alpha_k \chi_{S_k}$ is a simple function. For a general $f$ we approximate it by simple functions. \qed

**Remark 2.3. (A is the canonical extention of the graphop A)**

Using the previous lemma and Fubini’s theorem, it is easy to check that if $\tilde{\mu} \in \mathcal{M}^b$ is absolutely continuous with respect to the Lebesgue measure with Radon-Nikodym derivative $\rho$, i.e., $d\mu^x(u) = \rho(u,x) \, du$, then $A\mu^x$ is also absolutely continuous with $d(A\mu)^x(u) = (A\rho(u, \cdot))(x) \, du$. This implies that the definition of $A$ we have provided is the correct extension of the graphop $A$ to measures to work with.

**Lemma 2.4.** For any $\tilde{\mu}, \tilde{\kappa} \in \mathcal{M}^b$ and $x \in \Omega$ we have

\[
d_{BL}(A\mu^x, A\kappa^x) \leq d_b(A, \mu^x, \kappa^x) \leq d^b(A, \mu^x, \kappa^x) \leq d^b(\mu, \kappa)
\]
Proof. This follows immediately from Lemma 2.2 and the definition of $d_{BL}$, since
\[
d_{BL}(\mathcal{A}^{x^2}, \mathcal{K}^{x^2}) = \sup_{f \in \mathcal{C}} \left| \int_{\Omega} \int_{\mathbb{T}} f(v) \, d(\mu^y - \kappa^y)(v) \, d\nu_x^A(y) \right|
\leq \int_{\Omega} \sup_{f \in \mathcal{C}} \left| \int_{\mathbb{T}} f(v) \, d(\mu^y - \kappa^y)(v) \right| \, d\nu_x^A(y)
\leq \int_{\Omega} d_{BL}(\mu^y, \kappa^y) \, d\nu_x^A(y) = d^{b,A,x}(\bar{\mu}, \bar{\kappa})
\leq \sup_{x \in \Omega} \left( \int_{\Omega} d_{BL}(\mu^y, \kappa^y) \, d\nu_x^A(y) \right) = d^{b}(\bar{\mu}, \bar{\kappa})
\leq d^{b}(\bar{\mu}, \bar{\kappa}).
\]
This finishes the proof. □

Lemma 2.5. (Lipschitz continuity of $A$)
The map $A : \mathcal{M}^b \to \mathcal{M}^b$ is well-defined. Further, for every $\bar{\mu}, \bar{\kappa} \in \mathcal{M}^b$ holds
\[
d^{b}(A\bar{\mu}, A\bar{\kappa}) \leq d^{b}(\bar{\mu}, \bar{\kappa}).
\]
Proof. Due to equation (2.8) it is easy to see that $A$ maps $\mathcal{M}^b$ to $\mathcal{M}^b$. For the second statement, note that by Lemma 2.4 we know that
\[
d_{BL}(\mathcal{A}^{x^2}, \mathcal{K}^{x^2}) \leq d^{b}(\bar{\mu}, \bar{\kappa}).
\]
Hence the claim follows by integrating over $\nu^B_y$ and taking the supremum over all $y \in \Omega$ and $B \in \mathcal{G}$. □

2.3 The extended characteristic field

Via the map $A$, we can now extend the mean field vector field $V$, defined in (2.3), from densities to measures, by defining
\[
V[A, \mu, x](t, u) := C \int_{\mathbb{T}} D(\bar{u} - u) \, d(A\mu)^x(\bar{u})
\]
for any $x \in \Omega, t \in \mathbb{T}, u \in \mathbb{T}$ and $\mu \in \mathcal{M}^b_T$. 

Lemma 2.6. (Regularity of the characteristic field $V$)
The following statements are true:
(I) $V$ satisfies a Lipschitz condition in $\mu$ in the sense that for all $x \in \Omega, u \in \mathbb{T}, t \in \mathbb{T}$ and any $\mu, \kappa \in \mathcal{M}^b_T$ we have
\[
|V[A, \mu, x](t, u) - V[A, \kappa, x](t, u)| \leq 2C d^{b,A,x}(\bar{\mu}_t, \bar{\kappa}_t) \leq 2C d^{b}(\bar{\mu}_t, \bar{\kappa}_t)
\]
(II) For any $\mu \in \mathcal{M}^b_T$ and $x \in \Omega$, the map $V[A, \mu, x](\cdot)$ is continuous in $(t, u)$ and Lipschitz continuous in $u$ uniformly in $t$ with Lipschitz constant bounded by $b \gamma_A$.

Proof. (I) We compute that
\[
|V[A, \mu, x](t, u) - V[A, \kappa, x](t, u)| = C \left| \int_{0}^{2\pi} D(\bar{u} - u) \, d(A(\mu)^x - A(\kappa)^x)(\bar{u}) \right|
\leq C \left( \int_{\mathbb{T}} \chi_{I \geq 0}(u) D(\bar{u} - u) \, d(A(\mu)^x - A(\kappa)^x)(\bar{u}) \right)
+ \left| \int_{\mathbb{T}} \chi_{I \leq 0}(u) D(\bar{u} - u) \, d(A(\mu)^x - A(\kappa)^x)(\bar{u}) \right|
\]
where \( I_{\geq 0}(u) \subset \mathbb{T} \) is the interval where \( D(\bar{u} - u) \), as a function of \( \bar{u} \), is positive and \( I_{\leq 0}(u) \) is the complementary set. Note that the functions \( f^{u}_{\lambda}(\bar{u}) := \chi_{I_{\geq 0}(u)} D(\bar{u} - u) \) and \( f^{u}_{\lambda}(\bar{u}) := -\chi_{I_{\leq 0}(u)} D(\bar{u} - u) \) both lie in the set \( \mathcal{L} \), so that we obtain, continuing the previous calculation and using Lemma 2.4

\[
|V[A, \mu, x](t, u) - V[A, \kappa, x](t, u)| \leq 2Cd_{BL}(A \mu^x, A \kappa x) \leq 2Cd^{b,A,x}(\bar{\mu}, \bar{\kappa}) \leq 2Cd^{b}(\bar{\mu}, \bar{\kappa}).
\]

(II) The proof proceeds in three steps. First, we are going to show Lipschitz continuity in \( u \), then continuity in \( t \) and finally continuity in \( (t, u) \). We start with Lipschitz continuity in \( u \). For any \( u \) and \( u_0 \in \mathbb{T} \) we have

\[
|V[A, \mu, x](t, u) - V[A, \mu, x](t, u_0)| = C \int_0^{2\pi} \left| D(\bar{u} - u) - D(\bar{u} - u_0) \right| d(A\mu_t)^x(\bar{u}) \leq \nu^2(\Omega) \sup_{t \in \mathbb{T}} |\mu(y)| \leq b\gamma_A u - u_0|.
\]

Next, we consider continuity in \( t \). For any \( u \in \mathbb{T} \) and \( t_0 \in \mathcal{T} \) we have for \( t \to t_0 \), using Lemma 2.4 and the fact that \( \mu \in \mathcal{M}^b_{\mathbb{T}} \) and with a similar calculation as in (I):

\[
|V[A, \mu, x](t, u) - V[A, \mu, x](t_0, u)| \leq 2Cd_{BL}(A(\mu_t)^x, A(\mu_{t_0})^x) \leq 2Cd^{b,A,x}(\bar{\mu}_t, \bar{\mu}_{t_0}) \to 0 \quad \text{as } t \to t_0.
\]

It remains to show continuity in \( (t, u) \). For \((u, t), (u_0, t_0) \in \mathbb{T} \times \mathcal{T} \) we have

\[
V[A, \mu, x](t, u) - V[A, \mu, x](t_0, u_0) \leq V[A, \mu, x](t, u) - V[A, \mu, x](t, u_0) + V[A, \mu, x](t, u_0) - V[A, \mu, x](t_0, u_0).
\]

The second difference goes to 0 as \( t \to t_0 \) due to continuity in \( t \). For the first difference we have

\[
V[A, \mu, x](t, u) - V[A, \mu, x](t_0, u_0) = C \int_0^{2\pi} \left( D(\bar{u} - u) - D(\bar{u} - u_0) \right) d(A\mu_t)^x(\bar{u}) \leq C|u - u_0|t_x^2(\Omega) \sup_{y \in \mathbb{T}} \mu(y) \to 0 \quad \text{as } (t, u) \to (t_0, u_0).
\]

The claim follows.

\[\square\]

2.4 The equation of characteristics and the fixed point equation

For an arbitrary \( \mu \in \mathcal{M}^b_{\mathbb{T}} \) and for any \( x \in \Omega \) we define following equation of characteristics for a point \( P = u \in \hat{G} := \mathbb{T} \):

\[
\frac{dP}{dt} = V[A, \mu, x](t, P), \quad (2.10a)
\]

\[
P(t_0) = P_0. \quad (2.10b)
\]

Note that, due to Lemma 2.6 we have that equation (2.10) generates the flow

\[
T_{t, t_0}[A, \mu, x] : \hat{G} \to \hat{G}, \quad P_0 \mapsto P(t), P(t) \text{ solves } (2.10). \quad (2.11)
\]

Note further that, if \( D \) is smooth, then the regularity of \( V[A, \mu, x](\cdot) \) in \( u \), implies that \( T_{t, t_0} \) is a \( C^\infty \) diffeomorphism, satisfying \( T_{t, t_0}^{-1} = T_{t_0, t} \).
Definition 2.7. We say that a measure $\kappa \in \mathcal{M}^b_T$ satisfies the fixed point equation associated with the VFPE \((2.2)\) with initial condition $\bar{\mu}_0 \in \bar{\mathcal{M}}^b$, if $\kappa$ satisfies
\[
\kappa^y_t = \mu^y_0 \circ T_{0,t} [A, \kappa, y], \quad \text{for all } y \in \Omega.
\] (2.12)

Lemma 2.8. (Properties of the characteristic flow)
The following statements are true:
(i) $V[A, \cdot](\cdot)$ is uniformly bounded (in $x \in \Omega$, $t \in \mathcal{T}$, $u \in \mathcal{T}$ and $\mu \in \mathcal{M}^b_T$).
(ii) The corresponding flow $T^x_{t,t_0} [\mu]$ is uniformly bounded (in $x \in \Omega$, $t, t_0 \in \mathcal{T}$, $u \in \mathcal{T}$ and $\mu \in \mathcal{M}^b_T$).
(iii) $T^x_{t,t_0} [\mu]$ is Lipschitz continuous with Lipschitz constant $e^{Tb\gamma_A}$.

Proof. (i) We calculate using equation \((2.8)\)
\[
\left| V[A, \mu, x](t, u) \right| \leq C \int_0^{2\pi} \left| D(\tilde{u} - u) \right| \, d\mu^x_T(\tilde{u}) \\
\leq \| D \|_{\infty} \, A \mu^x_T(\mathcal{T}) \\
\leq C \| D \|_{\infty} b\gamma_A.
\]

(ii) This follows from (i), the fact that $T^x_{t,t_0} u = u + \int_{t_0}^t V[A, \mu, x](s, T^x_{s,t} u) \, ds$ and the compactness of $\Omega \times \mathcal{T}$.

(iii) For any fixed $t_0 \in \mathcal{T}$, we define
\[
\lambda(t) := \left| T^x_{t,t_0} u - T^x_{t,t_0} w \right|.
\]
Using Lemma 2.6 and the calculation in (ii) we get
\[
\lambda(t) \leq |u - w| + \mathcal{T}b\gamma_A \int_0^t \lambda(s) \, ds.
\]
Applying Gronwall's Lemma (cf. Lemma A.3) the claim follows.

Theorem 2.9. (Existence and Uniqueness of solutions for the fixed point equation) Assume $(\Omega, \Sigma, m)$ is a compact Borel probability space and that $A \in \mathcal{G}^m$ is a fixed graphop with corresponding measure $\nu^A$ on $\Omega^2$ in the sense of Theorem 1.1 and family of fiber measures $\{\nu^A_x\}_{x \in \Omega}$. Then following statements are true:
(I) For any initial condition $\bar{\mu}_0 \in \bar{\mathcal{M}}^b$, the map $\mathcal{F} : \mathcal{M}^b_T \rightarrow \mathcal{M}^b_T$ given by
\[
\mathcal{F}_{\kappa}^y : = \mu^y_0 \circ T_{0,t} [A, \kappa, y], \quad \forall y \in \Omega,
\]
is well-defined and a contraction on $(\mathcal{M}^b_T, d^b_\alpha)$ for any $\alpha > 2Cb + b\gamma_A$.
(II) For any initial condition $\bar{\mu}_0 \in \bar{\mathcal{M}}^b$ there is a unique fixed point $\kappa \in \mathcal{M}^b_T$ of the map $\mathcal{F}$, i.e., there is a unique $\kappa \in \mathcal{M}^b_T$ satisfying
\[
\mathcal{F}\kappa = \kappa.
\]
Furthermore, for any startpoint $\kappa^0 \in \mathcal{M}^b_T$ the fixed point iteration given by
\[
\kappa^{n+1} := \mathcal{F}\kappa^n
\]
converges to $\kappa$.\[\Box\]
The proof of Theorem 2.9 uses exactly the same argument used in \cite[Theorem 2.4]{4} and is included in Appendix A for convenience. The previous lemmas and discussed extensions provide the main ingredients for the proof to succeed. In particular, the key steps were to design a suitable metric space setting to work with graphops, which we accomplished above. Likewise, a second result we immediately obtain is the following theorem:

**Theorem 2.10. (Existence and uniqueness of a weak solution for the VFPE with graphops)**

Assume \((\Omega, \Sigma, m)\) is a compact Borel probability space and that \(A \in \mathcal{G}^m\) is a fixed graphop with corresponding measure \(\nu^A\) on \(\Omega^2\) in the sense of Theorem 1.1 and family of fiber measures \(\{\nu^A_x\}_{x \in \Omega}\). Moreover, assume that the initial condition \(\bar{\mu}_0 \in \mathcal{M}^b\) is absolutely continuous with density \(\rho \in L^{\infty}(\Omega, m)\), i.e. it holds

\[
\mu^y_0 = \rho^0(u, y) \, du \quad \text{for all } x \in \Omega.
\]

Then there is a unique weak solution \(\rho\) of the IVP (2.2).

Having provided a suitable new metric setting above, the proof of Theorem 2.10 can be deduced from \cite[Theorem 3.2]{4}.

### 3 Continuous dependence of the solution of the fixed point equation on the graphop

After having established existence and uniqueness of solutions for the VFPE (2.2) we now want to ensure that small perturbations of the graphop will have small effect on the solution. Recall that to the VFPE (2.2) corresponds the fixed point equation (2.12). To compare solutions of this fixed point equation, we introduce following pseudometric on \(\bar{\mathcal{M}}^b\):

\[
\bar{d}^{b,m} = \int_{\Omega} d_{BL}(\mu^y, \kappa^y) \, dm(y) = \int_{\Omega} \sup_{f \in \mathcal{L}} \left| f(v) d(\mu^y - \kappa^y)(v) \right| \, dm(y).
\]  

We note that proofs could possible work in various different topologies, but this is beyond the scope of the current work, since we are only interested in the existence of a suitable topology, where continuous dependence holds.

**Lemma 3.1. (Estimation for varying measure)**

Let \(\mu, \kappa \in \mathcal{M}^b\) and \(A \in \mathcal{G}^m\) with \(\| A \|_{p \to q} < \infty\) for \(p, q \in [1, \infty]\). Then, for any \(t \in \mathcal{T}, u \in \mathbb{T}\) we have

\[
\int_{\Omega} \left| V[A, \mu, x](t, u) - V[A, \kappa, x](t, u) \right| \, dm(x) \leq 2C \| A \|_{p \to q} \left( \int_{\Omega} d_{BL}(\mu^y, \kappa^y)^p \, dm(y) \right)^{\frac{1}{p}}.
\]

Especially, for \(\| A \|_{1 \to q} < \infty\) we have

\[
\int_{\Omega} \left| V[A, \mu, x](t, u) - V[A, \kappa, x](t, u) \right| \, dm(x) \leq 2C \| A \|_{1 \to q} \bar{d}^{b,m}(\bar{\mu}_t, \bar{\kappa}_t).
\]
Proof. By Hölder’s inequality we have \( \| f \|_1 \leq \| f \|_q \) for any \( f \in L^q(\Omega, m) \). Thus,

\[
\begin{align*}
\int_{\Omega} \left| V[C, \mu, x](t, u) - V[C, \kappa, x](t, u) \right| \ dm(x) \\
&= C \int_{\Omega} \left| A \left( \int_{\bar{T}} D(\bar{u} - u)( \ d\mu_t(\bar{u}) - d\kappa_t(\bar{u})) \right) \right| \ dm(x) \\
&\leq \left( \int_{\Omega} \left| A \left( \int_{\bar{T}} D(\bar{u} - u)( \ d\mu_t(\bar{u}) - d\kappa_t(\bar{u})) \right) \right|^q \ dm(x) \right)^{\frac{1}{q}} \\
&\leq C \| A \|_{p \rightarrow q} \left( \int_{\Omega} \left| \int_{\bar{T}} D(\bar{u} - u)( \ d\mu_t(\bar{u}) - d\kappa_t(\bar{u})) \right|^p \ dm(x) \right)^{\frac{1}{p}} \\
&\leq 2C \| A \|_{p \rightarrow q} \left( \int_{\Omega} d_{BL}(\mu_t^y, \kappa_t^y)^p \ dm(y) \right)^{\frac{1}{p}}.
\end{align*}
\]

This finishes the proof. \( \square \)

Now, for any \( n \in \mathbb{N} \), let \( A^n, A \in \mathbb{G}^n \) be graphops with corresponding canonical extensions \( \mathcal{A}^n, \mathcal{A} \) on the space \( M^b \), as in (2.7). For an initial condition \( \tilde{\mu}_0 \in \mathcal{M}^b \), let \( \mu^n, \mu \in \mathcal{M}^b_T \) be the solutions of following fixed point equations

\[
\begin{align*}
\mu^n_t = \mu^n_0 \circ T_{t,0}[A^n, \mu^n, y], \quad & \text{for all } y \in \Omega, \quad (3.2a) \\
\mu^n_t = \mu^n_0 \circ T_{t,0}[A, \mu, y], \quad & \text{for all } y \in \Omega, \quad (3.2b)
\end{align*}
\]

cf. Theorem 2.9

**Proposition 3.2. (Continuous dependence of fixed point solutions on graphops)**

Assume that \( A^n \to A \) and \( \| A \|_{1 \rightarrow q} < \infty \), for a \( q \in [1, \infty) \). Further assume that for any Borel set \( D \subset T \) with \( \lambda(\partial D) = 0 \) and \( t \in [0, T] \) the function \( x \mapsto \mu^n_t(D) \) is continuous. Then,

\[
\sup_{t \in T} d_{b,m}(\tilde{x}_n, \tilde{x}) \to 0 \quad \text{as } n \to \infty. \tag{3.3}
\]

**Proof.** The proof starts as in [4] Lemma 2.7 and then uses a new argument. Following the same steps as in the proof of Theorem 2.9 (equation (A.2)) we can show that

\[
\begin{align*}
\hat{d}_{b,m}(\tilde{x}_t, \tilde{x}) &= \int_{\Omega} \int_{\overline{T}} [T_{s,0}^y[A, \mu]^y v - T_{s,0}^y[A^n, \mu^n]^y v] \ dm(y) \ dm(v) =: \lambda(t) \\
&\leq \int_{0}^{T} \int_{\Omega} \int_{\overline{T}} \left| V[A, \mu, y](T_{s,0}^y[A, \mu]^y v, s) - V[A^n, \mu^n, y](T_{s,0}^y[A^n, \mu^n]^y v, s) \right| \ dm(y) \ dm(v) \ ds.
\end{align*}
\]

Using the triangle inequality we have

\[
\lambda(t) \leq \lambda_1(t) + \lambda_2(t) + \lambda_3(t),
\]

with

\[
\begin{align*}
\lambda_1(t) &= \int_{0}^{T} \int_{\Omega} \int_{\overline{T}} \left| V[A, \mu, y](T_{s,0}^y[A, \mu]^y v, s) - V[A^n, \mu^n, y](T_{s,0}^y[A^n, \mu^n]^y v, s) \right| \ dm(v) \ dm(y) \ ds, \\
\lambda_2(t) &= \int_{0}^{T} \int_{\Omega} \int_{\overline{T}} \left| V[A^n, \mu^n, y](T_{s,0}^y[A^n, \mu^n]^y v, s) - V[A^n, \mu^n, y](T_{s,0}^y[A^n, \mu^n]^y v, s) \right| \ dm(v) \ dm(y) \ ds, \\
\lambda_3(t) &= \int_{0}^{T} \int_{\Omega} \int_{\overline{T}} \left| V[A^n, \mu^n, y](T_{s,0}^y[A^n, \mu^n]^y v, s) - V[A^n, \mu^n, y](T_{s,0}^y[A^n, \mu^n]^y v, s) \right| \ dm(v) \ dm(y) \ ds.
\end{align*}
\]
For the first term we obtain, using Lemma 3.1
\[ \lambda_1(t) \leq 2C b \| A \|_{1 \to q} \int_0^t \int_\Omega d_{BL}(\mu^n_s, \mu^n_\alpha) \, dm(y) \, ds = 2C b \| A \|_{1 \to q} \int_0^t d_{BL}(\tilde{\mu}_s, \tilde{\mu}_\alpha^n) \, ds. \]
For the third we get by Lemma 2.6 that
\[ \lambda_3(t) \leq b \int_0^t \lambda(s) \, ds. \]
All in all, we find
\[ \lambda(t) \leq 2C b \| A \|_{1 \to q} \int_0^t d_{BL}(\tilde{\mu}_s, \tilde{\mu}_\alpha^n) \, ds + \lambda_2(t) + b \int_0^t \lambda(s) \, ds. \]
Hence, by Gronwall’s inequality (cf. Lemma A.3) we have
\[ d_{BL}(\tilde{\mu}_t, \tilde{\mu}_\alpha^n) \leq e^{bt} \left( C_1 \int_0^t d_{BL}(\tilde{\mu}_s, \tilde{\mu}_\alpha^n) e^{-bs} \, ds + \lambda_2(t) \right), \]
with \( C_1 := 2C b \| A \|_{1 \to q}. \) Defining \( \phi(t) := e^{-bt} d_{BL}(\tilde{\mu}_t, \tilde{\mu}_\alpha^n) \) and applying Gronwall’s inequality for a second time we see that
\[ \phi(t) \leq e^{C_1 t} \lambda_2(t), \]
which implies that
\[ \sup_{t \in T} d_{BL}(\tilde{\mu}_t^n, \tilde{\mu}_\alpha) \leq \lambda_2(T) e^{\lambda_2(T) + bT}. \tag{3.4} \]
Thus, we have to deal with the term \( \lambda_2(T). \) From \( A^n \to A \) and the definition of weak convergence follows immediately that for any Borel set \( S \subset \Omega \) with \( \lambda(\partial S) = 0 \) we have
\[ A^n \mu_t^n(S) \to A \mu_t(S) \quad \text{for all } t \in [0, T] \text{ and } m\text{-a.e. } x \in \Omega. \]
By Portmanteau’s theorem (see for instance [8, Theorem 13.16]) this implies that
\[ A^n \mu_t^n \to_w A \mu_t \quad \text{for all } t \in [0, T] \text{ and } m\text{-a.e. } x \in \Omega. \]
Hence, due to the equivalence of the Lévy-Prokhorov metric with the bounded Lipschitz distance \([19]\) we have
\[ d_{BL}(A^n \mu_t^n, A \mu_t^n) \to 0 \quad \text{as } n \to \infty \text{ for all } t \in [0, T] \text{ and } m\text{-a.e. } x \in \Omega, \]
which in turn implies by the dominated convergence theorem (applicable due to the compactness of \( \Omega \times [0, T] \)) that
\[ \int_0^T \int_\Omega \int_T d_{BL}(A^n \mu_t^n, A \mu_t^n) \, d\mu_\alpha^n(v) \, dm(y) \, ds \to 0 \quad \text{as } n \to \infty. \]
Thus, we see that
\[ \lambda_2(t) = \int_0^t \int_\Omega \int_T |V[A^n, \tilde{\mu}, y](T^n_{s,0}[A^n, \tilde{\mu}_\alpha^n]v, s) - V[A, \mu, y](T^n_{s,0}[A^n, \tilde{\mu}_\alpha^n]v, s)| \, d\mu_\alpha^n(v) \, dm(y) \, ds \]
\[ \leq \int_0^T \int_\Omega \int_T \left| D(\tilde{u} - T^n_{s,0}[A^n, \tilde{\mu}_\alpha^n]v) d(A^n \mu_s^n - A \mu_t^n)(\tilde{u}) \right| \, d\mu_\alpha^n(v) \, dm(y) \, ds \]
\[ \leq 2d_{BL}(A^n \mu_t^n, A \mu_t^n) \]
we see that
\[ \lim_{n \to \infty} \lambda_2(T) = 0. \tag{3.5} \]
Combining (3.4) with (3.5) finishes the proof. □
Having completed all the necessary existence, uniqueness, and continuous dependence results, we now know that VFPEs involving graphops are suitably well-posed. The next step is to show that they are indeed mean-field limits for the generalized Kuramoto model on graphs.

4 Graphon approximation of graphops on compact groups

Our first step is to find, for a given graphop $A$, a suitable graphon regularization $A^n$, which approximates $A$ in the sense of Definition 1.3 where we defined our main new topology adapted to graphops. To find the right approximation via graphops, a key idea is to employ Fourier methods. We recall the notions we need briefly.

A **locally compact abelian (LCA) group** is an abelian group $G$ which is a locally compact Hausdorff space and such that the group operations are continuous (in other words, $G$ is topological group which is abelian, locally compact and Hausdorff). To be more precise, the maps

$$G \to G, \quad x \mapsto -x,$$

$$G \times G \to G, \quad (x, y) \to x + y,$$

are both continuous. Standard examples for LCA groups are $\mathbb{R}^d$ and $\mathbb{T}^d$ with the usual topologies and $(\mathbb{Z}, +)$ with the discrete topology.

**Definition 4.1.** A **Haar measure** $\mu_G$ on a locally compact group $G$ is a positive, regular, Borel measure having the following two properties:

(i) $\mu_G$ is finite on compact sets, i.e., we have

$$\mu_G(E) < \infty \quad \text{if } E \text{ is compact};$$

(ii) $\mu_G$ is invariant under translation, i.e., we have

$$\mu_G(x + E) = \mu_G(E) \quad \text{for all measurable } E \subset G \text{ and all } x \in G.$$

One can prove that the Haar measure always exists and is unique up to multiplication by a positive constant. One can also prove that the Haar measure is finite if and only if $G$ is compact, see [7]. In the following, in the case that $G$ is a **Compact Abelian (CA) group** we always assume that $\mu_G$ is the normalized probability measure.

**Definition 4.2.** (**summability kernel on LCA group**)

A **summability kernel** on the LCA group $G$ is a sequence $\{k_n\}_{n \in \mathbb{N}}$ satisfying the following conditions:

(1) \[
\int_G k_n(x) \, d\mu_G(x) = 1.
\]

(2) \[
\int_{\mathbb{T}} |k_n(x)| \, d\mu_G(x) \leq \text{const}.
\]

(3) For any neighborhood $V$ of 0 in $G$ we have

$$\lim_{n \to \infty} \int_{G \setminus V} |k_n(x)| \, d\mu_G(x) = 0.$$

Furthermore, we say that the summability kernel $\{k_n\}_{n \in \mathbb{N}}$ is:

(i) **positive**, provided that $k_n(x) \geq 0$ for all $x$ and $n$. 

16
(ii) symmetric, provided that $k_n(x) = k_n(-x)$ for all $x \in G$.

Standard examples for symmetric and positive summability kernels are the Poisson and the Gauss kernels. We note that we can view any given summability kernel $k_n$ as a P-operator $K_n : L^\infty(G, \mu_G) \to L^1(G, \mu_G)$, with action given by the convolution

$$K_nf(x) := k_n \ast f(x) = \int_G k_n(x-y)f(y) \, d\mu_G(y).$$

(4.2)

If moreover $\{k_n\}_{n \in \mathbb{N}}$ is a positive and symmetric summability kernel, then the function $W^n : G \times G \to \mathbb{R}$ given by $W^n(x,y) := k_n(x-y)$ is a graphon and the associated graphop $A_{W^n} : L^\infty(G, \mu_G) \to L^1(G, \mu_G)$ is given by the convolution

$$A_{W^n}f(x) = \int_T k_n(x-y)f(y) \, d\mu_G(y) = K_nf(x).$$

(4.3)

**Theorem 4.3. (Approximation by summability kernels)**

Let $\{k_n\}_{n \in \mathbb{N}}$ be a summability kernel on the CA group $G$. Then, for every $f \in L^p(G, \mu_G)$, $1 \leq p < \infty$, we have

$$\|K_n f - f\|_{L^p} = 0.$$ (4.4)

Moreover, if $f \in C(G)$ then the convergence is uniform, i.e.,

$$\|K_n f - f\|_{\infty} = 0.$$ (4.5)

**Proof.** This is a classical result in harmonic analysis. See for example [7, Chapter 7.2, Theorem 2.11] and references therein.

**Lemma 4.4. (graphops preserve uniform convergence of continuous functions)**

Let $A : L^\infty(G, \mu_G) \to L^1(G, \mu_G)$ be a graphop on the CA group $G$ with Haar measure $\mu_G$. Assume that $\gamma_A < \infty$ and $\{k_n\}_{n \in \mathbb{N}}$ is a positive and symmetric summability kernel. Then, the regularization $K_nAK_n$ defines a sequence of graphons such that $K_nAK_n \to_o A$.

**Proposition 4.5. (o-graphon approximability for graphops on $L^\infty(\Omega, \nu)$)**

Assume that $A : L^\infty(G, \mu_G) \to L^1(G, \mu_G)$ is a graphop on the CA group $G$ with Haar measure $\mu_G$. Assume that $\gamma_A < \infty$ and $\{k_n\}_{n \in \mathbb{N}}$ is a positive and symmetric summability kernel. Then, the regularization $K_nAK_n$ defines a sequence of graphons such that $K_nAK_n \to_o A$. 

17
The claim follows now by Remark 1.4. By Theorem 5.1 we have that \( (VFPE \text{ approximates the discrete Kuramoto's model}) \) uniformly, since \( K_n f \rightarrow f \) uniformly which implies by Lemma 4.4 that \( AK_n f \rightarrow Af \) uniformly, since \( K_n f \) is continuous. Thus, we have for any \( x \in G \)

\[
|K_n AK_n f(x) - Af(x)| \leq \| A^n f - Af \|_\infty \int_T |k_n(x-y)|dy = \| A^n f - Af \|_\infty \to 0.
\]

The claim follows now by Remark 1.4.

\[\square\]

### 5 Mean field Approximation

We now have everything we need to solve the main problem, which is to show that the VFPE (1.19) with the graphop \( A \) approximates the discrete Kuramoto problem (1.14).

We come back to the setup of Section 1.2. Recall that for the compact Borel probability space \( (\Omega, \Sigma, m) \) we have additionaly that \( \Omega = G \) is a CA group and \( m = \mu_G \) is the Haar probability measure. \( A \in G^m \) is assumed to be a graphop \( A : L^\infty(G, \mu_G) \to L^1(G, \mu_G) \) with \( \| A \|_{1\to q} < \infty \) for a \( q \in [1, \infty] \). Recall that by Proposition 4.5 there exists a sequence of graphons \( AK : L^\infty(G, \mu_G) \to L^1(G, \mu_G) \) with graphon kernels \( W^K \), such that \( AK \). We assume that the weights \( A^{N,K} \) in (1.13) are given by these kernels. Further let \( \xi_i^{N} \) be a sequence of points such that for an \( i \in [n], \xi_i^{N} \) are independent, identical distributed according to \( m \) restricted to \( \Omega_i \) for all \( k \in [M] \) and the initial values \( \nu_j^{N,0} \in T, j \in [N] \) are independent random variables, whose distribution have densities \( \rho^0_j(\cdot, \xi_j^{N}), j \in [N] \) (w.r.t. to the Lebesque measure on \( T \)). Assume additionally that for the initial condition \( \rho^0 \) we have that the function \( x \to \int_T \rho^0(u,x) \) is Riemann integrable for every \( f \in C(T) \).

**Theorem 5.1. (VFPE approximates the discrete Kuramoto’s model)**

Under the previous assumptions, for any given \( \epsilon > 0 \) there exists a \( K_1 \in \mathbb{N} \) such that for any \( K \geq K_1 \) there exist \( M_1(K), N_1(K) \in \mathbb{N} \) such that for all \( M \geq M_1(K), n \geq N_1(K) \) we have

\[
\sup_{t \in T} d^\nu_{m,K,t}(\tilde{\nu}_n, \tilde{\nu}_t) < \epsilon \quad \text{a.s.}
\]

**Proof.** Let \( \epsilon > 0 \). Since, by Proposition 4.5 \( AK \). By Proposition 3.2 we can find a \( K_1 \in \mathbb{N} \) such that for all \( K \geq K_1 \) we have

\[
\sup_{t \in T} d^\nu(\tilde{\nu}_t^K, \tilde{\nu}_t) < \frac{\epsilon}{2} \quad \text{a.s.}
\]

Furthermore, by Theorem 3.9 we can find for any \( K \in \mathbb{N} \) an \( M_1(K), N_1(K) \), such that for all \( M \geq M_1(K), n \geq N_1(K) \)

\[
\sup_{t \in T} d^\nu_{m,K,t}(\tilde{\nu}_n, \tilde{\nu}_t^K) < \frac{\epsilon}{2} \quad \text{a.s.}
\]
Thus, by the triangle inequality we have for all $M \geq M_1(K), n \geq N_1(K)$:

$$\sup_{t \in T} d^h_{m}(\bar{\nu}_{n,M,K,t}, \bar{\nu}_t) \leq \sup_{t \in T} d^h_{m}(\bar{\nu}_{n,M,K,t}, \bar{\nu}_t^K) + \sup_{t \in T} d^h_{m}(\bar{\nu}_t^K, \bar{\nu}_t) < \epsilon \ \text{a.s.}, \quad (5.4)$$

which finishes the proof.

A big class of graphops, which satisfy all conditions of the previous theorem are the $c$-regular graphops:

**Corollary 5.2. (Mean field approximation for $c$-regular graphops on $L^\infty(G, \mu_G)$)**

If $A : L^\infty(G, \mu_G) \to L^1(G, \mu_G)$ is a $c$-regular graphop with $c \leq 1$, then the statement of Theorem 5.1 is satisfied. Especially, the claim holds for any Markov graphop on $L^\infty(G, \mu_G)$.

**Proof.** Since the graphop $A$ is $c$-regular, we check that for all $x \in \Omega$ we have

$$\nu_x(G) = A\chi_{G}(x) = c,$$

which implies that

$$\gamma_A = \sup_{x \in G} \nu_x^A(G) = c \leq 1.$$ 

Thus, $A \in \mathcal{G}^m$. Further, see $A$ is self-adjoint, we have for any $f \in L^1(G, \mu_G)$:

$$\int_{\Omega} \left| Af(x) \right| \, dx = \int_{G} \left| f(x) A\chi_{G}(x) \right| \, dx = c \int_{G} \left| f(x) \right| \, dx, \quad (5.5)$$

which implies that

$$\| A \|_1 = c < \infty.$$

Thus, all assumptions in Theorem 5.1 are satisfied.

A concrete example for a graphop satisfying the assumptions is the so called spherical graphop:

**Example 5.3. (Kuramoto’s model on the spherical graphop)**

We consider the Borel probability space $\Omega := S^2 := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ with the uniform measure $\mu$. The spherical graphop, discussed in [11], is the graphop $S : L^\infty(\Omega, \mu) \to L^1(\Omega, \mu)$ given by

$$\left( Sf \right)(a) := \int_{S^a} f(b) \, d\nu_a(b), \quad (5.6)$$

where $S^a$ denotes the set of normalized vectors in $\mathbb{R}^3$, which are orthogonal to $a$ (which is a circle on the sphere) and $\nu_a$ is the uniform measure on $S^a$; note that we can also view $a$ as a normal vector corresponding to the circle $S^a$. In other words, the spherical graphop defines the graph for which any point on the sphere is connected only to its orthogonal vectors. Hence, every neighborhood is one-dimensional. The spherical graphop is Markov graphop, which is neither a graphon (dense graph) nor a graphing (sparse graph); it is a prototypical example for an infinite graph of intermediate density. It is not difficult to see that the sphere $S^2$ with the usual topology and vector addition is a CA group, and the Haar measure $\mu_{S^2}$ is simply the uniform measure on $S^2$. Hence, the spherical graphop satisfies all assumptions of Corollary 5.2 (and Theorem 5.1).
6 Conclusion and Outlook

Previous results on mean-field approximation for the discrete Kuramoto model on finite networks were restricted to the case that the limiting graph is a graphon, i.e., a dense structure. In this paper we have significantly generalized and extended them to the case that the finite graphs converge towards graphs of intermediate density, or even sparse graphs. Our approach was based upon the operator-functional representation of graphs via graphons provided recently in [1]. Introducing tools from harmonic analysis we were able to approximate any graphop, defined on a CA group with the Haar measure, by graphons. Since for any graphon mean field approximation is guaranteed, we could then bypass working directly with the generalized Kuramoto model. With this idea we managed to prove mean field approximation for a big class of graphops defined on a CA group with the Haar measure, which contains any -regular graphop and any Markov graphop. Furthermore, we showed existence, uniqueness, and continuous dependence on the graphop for the limiting PDE.

As already mentioned in the beginning, to simplify the notation and calculations, we have always assumed that in the discrete Kuramoto model all initial frequencies are the same: \( \omega_i = 0, \forall i \in [N] \), but all of our results extend to the general case of a frequency distribution. In this case, the Kuramoto model reads as

\[
\dot{u}_i = \omega_i + C \sum_{j=1}^{N} A_{i,j} N D(u_j - u_i), \quad i \in [N] := \{1, 2, ..., N\}. \quad (6.1)
\]

The simplification \( \omega_i = 0, \forall i \in [N] \) had as a result that the equation of characteristics was defined on the one-dimensional compact space \( \hat{G} = \mathbb{T} \). In the general case of of distributed frequencies \( \{\omega_i\}_{i=1}^{N} \), we set \( \hat{G} = \mathbb{T} \times \mathbb{R} \) and the characteristic field is given by

\[
V[A, \mu, x](t, P) := \left( \omega + C \int_{\hat{G}} g(\tilde{\omega}) D(\tilde{u} - u) \, d(\mu_\omega) \right).
\]

Since the added component is simply 0, the characteristic field \( V \) again satisfies the regularity properties of Lemma 2.6 which implies that we obtain a regular flow \( T_{t,t_0}[A, \mu, x] : \hat{G} \to \hat{G} \), and we can thus repeat the proof of existence and uniqueness of solutions for the fixed point equation from Theorem 2.9 see also [6] and [4, Section 4], where the additional frequencies effectively lead to one more integral but do not take any effect on the main argument of the proof for mean-field limits.

With our new view using graphops for large-scale dynamics on graphs, there are evidently now a lot of interesting open questions and challenges. We mention just a few open problem directly connected to our setting here:

- Generalize our analysis, with the underlying assumption that the limiting graphop \( A \) has bounded \((1, q)\) norm, to cover general \((p, q)\) bounded graphops.

- Another limitation of our analysis is the assumption that the node space \( \Omega \) is a CA group with the Haar measure, which means that the edges are distributed somewhat uniformly on the nodes. Hence, graphs with very inhomogeneous distribution of edges, like for instance star graphs with a “giant” node connected to every other node, are excluded from our analysis. One main goal for future research will be to allow such inhomogeneous structures by considering the general case of any compact Borel probability space \((\Omega, \Sigma, m)\).

- One may also expect that further variations of the Kuramoto model on complex networks, e.g., the Kuramoto model involving second-order time derivatives [10], also have mean-field limits leading again to VFPEs on graphops.
• Even more generally, instead of Kuramoto-type models one could consider a completely abstract kinetic model of the form

$$\dot{u}_i = \sum_{j=1}^{N} A_{N}^{N} f(u_j, u_i), \quad i \in [N] := \{1, 2, \ldots, N\},$$

(6.2)

defined on finite networks which converge, as $N \to \infty$, towards a limiting graphop $A$. It is known that for the corresponding kinetic problem

$$\dot{u}_i = \sum_{j=1}^{N} f(u_j, u_i), \quad i \in [N] := \{1, 2, \ldots, N\},$$

(6.3)

the mean field VFPE reads as

$$\partial_t \rho = -\partial_u \left( \rho V[f](\rho) \right),$$

(6.4)

where the characteristic field $V[f]$ can be entirely computed from $f$ [5]. As conjectured in [9], the mean field VFPE for the kinetic model (6.2) should be given by

$$\partial_t \rho = -\partial_u \left( \rho V[f](A\rho) \right).$$

(6.5)

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A Appendix: Some technical results

Lemma A.1. $d^b$ is a metric on $\mathcal{M}^b$.

Proof. We first note that $\bar{d}^b$ takes positive finite values since for any probability measure on $(\Omega, \Sigma)$, any $B \in \mathcal{G}^l$ and for $l$-a.e. $x \in \Omega$ we have ($d_{TV}$ denotes the total variation of measures):

$$0 \leq \int_{\Omega} d_{BL}(\mu^y, \kappa^y) \, d\nu_x^B(y) \leq \text{diam}(\Omega) \int_{\Omega} d_{TV}(\mu^y, \kappa^y) \, d\nu_x^B(y) \leq \text{diam}(\Omega) \cdot b \cdot \nu_x^B(\Omega) \leq b \cdot \text{diam}(\Omega).$$

Note that for the graphop $A$ and any Borel measurable functions $f, g \in L^\infty(\Omega, m)$ we have (due to the definition of the fiber measures and Theorem 1.1)

$$\int_{\Omega^2} f(x)g(y) \, d\nu_x^A(y) \, dm(x) = \int_{\Omega^2} f(x)g(y) \, d\nu(x, y) = \int_{\Omega^2} g(x)f(y) \, d\nu_x^A(y) \, dm(x)$$

Thus, for two elements $\bar{\mu}, \bar{\kappa} \in \mathcal{M}^b$ we have following implications:

$$\sup_{B \in \mathcal{G}^l} \sup_{x \in \bar{\Omega}} \int_{\bar{\Omega}} d_{BL}(\mu^y, \kappa^y) \, d\nu_x^B(y) = 0 \Rightarrow \int_{\bar{\Omega}} d_{BL}(\mu^y, \kappa^y) \, d\nu_x^B(y) = 0 \text{ for } l - a.e. x \in \bar{\Omega} \quad \forall B \in \mathcal{G}^l \quad \forall \text{ probability measures } l.$$

$$\Rightarrow \int_{\bar{\Omega}} \int_{\bar{\Omega}} d_{BL}(\mu^y, \kappa^y) \, d\nu_x^B(y) \, dl(x) = 0 \forall B \in \mathcal{G}^l \quad \forall \text{ probability measures } l.$$

$$\Rightarrow \nu_x^B(\Omega) d_{BL}(\mu^y, \kappa^y) \, dl(x) = 0 \forall B \in \mathcal{G}^l$$

$$\Rightarrow \nu_x^B(\Omega) d_{BL}(\mu^x, \kappa^x) \, dl(x) = 0 \, l-a.e. \, x \in \bar{\Omega} \quad \forall B \in \mathcal{G}^l \quad \forall \text{ probability measures } l.$$
Now, by the observation that the set $\mathcal{G}$ surely contains graphops for which $\nu^B_x(\Omega) \neq 0$ for l-a.e $x \in \Omega$ holds (we can consider for example the special case of graphons with $d\nu^B_x(y) := W(x, y) \, dl(y)$ and $W$ can be an arbitrary kernel) and that the measure $l$ can be concentrated at any point $x \in \Omega$ (we can consider for example the case that $l = \delta_x$ is as Dirac measure) we conclude that
\[
\mu^y = \kappa^y \text{ for all } y \in \Omega.
\]
The other properties of the metric are easy to check. \hfill \Box

Lemma A.2. (Completeness of $\tilde{\mathcal{M}}^b$)
The metric space $(\mathcal{M}^b, \tilde{d}^b)$ is complete.

Proof. The proof is adapted from [4, Lemma 2.1]. Let $\{\tilde{\mu}_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in $\mathcal{M}^b$. We have to show that this sequence converges in $\mathcal{M}^b$. Since $\{\tilde{\mu}_n\}$ is a Cauchy sequence, there is an increasing sequence of indices $n_k$ such that
\[
\tilde{d}^b(\tilde{\mu}_{n_k}, \tilde{\mu}_{n_{k+1}}) = \sup_{B \in \mathcal{G}} \sup_{x \in \Omega} \int_B \nu^B_x(y) \, dl(y) < \frac{1}{2^{k+1}}, \quad k = 1, 2, \ldots.
\]
Using the monotone convergence theorem, this implies that for any probability measure $l$ on $(\Omega, \Sigma)$, any $B \in \mathcal{G}$, and for l-a.e. $x \in \Omega$ we have that
\[
\int_\Omega \sum_{k=1}^\infty d_{BL}(\mu^y_{n_k}, \mu^y_{n_{k+1}}) \, dl(y) = \sum_{k=1}^\infty \int_\Omega d_{BL}(\mu^y_{n_k}, \mu^y_{n_{k+1}}) \, dl(y) < \infty
\]
or in other words, the function $f : \Omega \to \mathbb{R}$ given $f(y) = \sum_{k=1}^\infty d_{BL}(\mu^y_{n_k}, \mu^y_{n_{k+1}})$ is $\nu^B_x$-integrable for all $x \in \Omega$. This implies especially that
\[
\sum_{k=1}^\infty d_{BL}(\mu^y_{n_k}, \mu^y_{n_{k+1}}) < \infty \quad \text{for } \nu^B_x \text{-a.a. } y \in \Omega, \text{ for every } x \in \Omega.
\]
Since for every indices $i, j$ with $j > i$ we have
\[
d_{BL}(\mu^y_{n_i}, \mu^y_{n_j}) \leq \sum_{k=i}^{j-1} d_{BL}(\mu^y_{n_k}, \mu^y_{n_{k+1}}) \to 0 \quad \text{as } i, j \to 0,
\]
the sequence $\{\mu^y_{n_k}\}$ is Cauchy for $\nu^B_x$-a.a. $y \in \Omega$, for every $x \in \Omega$. Hence, since the metric space $(\mathcal{M}^b, d_{BL})$ is complete, there exists the limit
\[
\mu^y = \lim_{k \to \infty} \mu^y_{n_k}, \quad \text{for } \nu^B_x \text{-a.a. } y \in \Omega, \text{ for every } x \in \Omega
\]
which is a measurable function as a limit of measurable functions. Note further that for $\nu^B_x$-a.a. $y \in \Omega$, for every $x \in \Omega$ we have that
\[
|\mu^y_{n_k}(T) - \mu^y(T)| \leq d_{BL}(\mu^y_{n_k}, \mu^y) \to 0 \quad \text{as } k \to \infty
\]
which implies, due to the fact that $\tilde{\mu}_{n_k} \in \mathcal{M}^b$ that
\[
\mu^y(T) \leq b \quad \text{for } \nu^B_x \text{-a.a. } y \in \Omega, \text{ for every } x \in \Omega.
Since \( B \in \mathcal{G} \) can be any graphop, w.r.t. an arbitrary probability measure \( l \), this implies (similarly to the proof of Lemma A.1) that for the limit \( \mu, \bar{\mu} \in \mathcal{M} \) holds. To show that \( \bar{\mu} \) is also the limit of the whole sequence, we note that for every indices \( j > i \) we have

\[
\bar{d}(\bar{\mu}_{n_i}, \bar{\mu}_{n_j}) = \sup_{B \in \mathcal{G}} \sup_{x \in \Omega} \int_{\Omega} d_B(\mu_{n_i}^y, \mu_{n_j}^y) \, d\nu_x^B(y)
\]

\[
\leq \sup_{B \in \mathcal{G}} \sup_{x \in \Omega} \int_{\Omega} \sum_{k=i}^{j-1} d_B(\mu_{n_k}^y, \mu_{n_{k+1}}^y) \, d\nu_x^B(y)
\]

\[
\leq \sum_{k=i}^{j-1} \sup_{B \in \mathcal{G}} \sup_{x \in \Omega} \int_{\Omega} d_B(\mu_{n_k}^y, \mu_{n_{k+1}}^y) \, d\nu_x^B(y)
\]

\[
\leq \sum_{k=i}^{j-1} \sup_{B \in \mathcal{G}} \sup_{x \in \Omega} \int_{\Omega} d_B(\mu_{n_k}^y, \mu_{n_{k+1}}^y) \, d\nu_x^B(y)
\]

\[
\leq \frac{1}{j+1}
\]

\[
\leq \frac{1}{2^i} \to 0 \quad \text{as } i \to \infty.
\]

Now using the continuity of the metric and the dominated convergence theorem we obtain

\[
\bar{d}(\bar{\mu}_{n_i}, \bar{\mu}) = \sup_{B \in \mathcal{G}} \sup_{x \in \Omega} \int_{\Omega} d_B(\mu_{n_i}^y, \mu^y) \, d\nu_x^B(y)
\]

\[
= \sup_{B \in \mathcal{G}} \sup_{x \in \Omega} \lim_{j \to \infty} \int_{\Omega} d_B(\mu_{n_i}^y, \mu_{n_j}^y) \, d\nu_x^B(y)
\]

\[
= \sup_{B \in \mathcal{G}} \lim_{j \to \infty} \sup_{x \in \Omega} \int_{\Omega} d_B(\mu_{n_i}^y, \mu_{n_j}^y) \, d\nu_x^B(y)
\]

\[
\leq \lim_{j \to \infty} \sup_{B \in \mathcal{G}} \sup_{x \in \Omega} \int_{\Omega} d_B(\mu_{n_i}^y, \mu_{n_j}^y) \, d\nu_x^B(y)
\]

\[
\leq \frac{1}{2^i}
\]

\[
\to 0 \quad \text{as } i \to \infty.
\]

Hence, the subsequence \( \bar{\mu}_{n_i} \) converges to \( \bar{\mu} \). Since it is a subsequence of the Cauchy sequence \( \bar{\mu}_{n_i} \), this implies already the convergence of the whole sequence towards \( \bar{\mu} \), i.e.

\[
\bar{d}(\bar{\mu}_{n_i}, \bar{\mu}) \to 0 \quad \text{as } n \to \infty.
\]

Hence, the space \((\bar{\mathcal{M}}^b, \bar{d})\) is complete.

\[\square\]

**Lemma A.3.** [4, Lemma 2.5] (Gronwall’s lemma)

Let \( \phi(t) \) and \( \alpha(t) \) be continuous functions on \([0, T]\) and

\[
\phi(t) \leq A \int_0^t \phi(s) \, ds + B \int_0^t \alpha(s) \, ds + C, \quad t \in [0, T],
\]

where \( A \geq 0 \). Then

\[
\phi(t) \leq e^{At} \left( B \int_0^t \alpha(s)e^{-As} \, ds + C \right).
\]
Proof of Theorem 2.7. We follow the lines of the proof of [3 Theorem 2.4]. (I) First of all, it is easy to see that for any $\mu \in \mathcal{M}_T^b$, $\overline{\mu \circ T_{t_0}} \in \mathcal{M}_T^b$ holds. Further, for any times $t_0, t \in T$, w.l.o.g. $t \geq t_0$, we calculate, using a change of variables (in the third equation),

$$
\overline{d^b}(\mathcal{F}[\mathcal{F}][t, \cdot], \mathcal{F}[\mathcal{F}][t_0, \cdot]) = \overline{d^b}(\mu_0 \circ T_{t_0}, [\mu, \cdot], \overline{\mu_0 \circ T_{t_0}}[\mu, \cdot])
$$

$$
= \sup_{B \in G} \sup_{x \in \Omega} \int \int_{\Omega} d_{BL}(\mu_0^y \circ T_{t_0}^y[\mu], \mu_0^y \circ T_{t_0}^y[\mu]) \ d\nu^B_x(y)
$$

$$
= \sup_{B \in G} \sup_{x \in \Omega} \int \int_{\Omega} \left( f(T_{t_0}^y[\mu]v) - f(T_{t_0}^y[\kappa]v) \right) \ d\mu^y_0(v) \ d\nu^B_x(y)
$$

$$
\leq \sup_{B \in G} \int \int_{T} \left| T_{t_0}^y[\mu]v - T_{t_0}^y[\kappa]v \right| \ d\mu^y_0(v) \ d\nu^B_x(y)
$$

$$
\leq \int \int_{t_0}^{t} \sup_{B \in G} \sup_{x \in \Omega} \int \int_{\Omega} \left| V[A, \mu, x](s, T_{s_0}^y[\mu]v) - V[\mu, x](s, T_{s_0}^y[\kappa]v) \right| \ d\mu^y_0(v) \ d\nu^B_x(y) \ ds
$$

$$
\leq C \left\| D \right\|_{\infty} l^2(t - t_0) \to 0 \text{ as } t \to t_0. \tag{A.1}
$$

This shows that $\mathcal{F} \mu \in \mathcal{M}_T^b$. Thus, $\mathcal{F}$ is well-defined. Now let $\mu, \kappa \in \mathcal{M}_T^b$. As before we calculate that

$$
\overline{d^b}(\mathcal{F}[\mathcal{F}][t, \cdot], \mathcal{F}[\mathcal{F}][t_0, \cdot]) = \overline{d^b}(\mu_0 \circ T_{t_0}, [\mu, \cdot], \overline{\mu_0 \circ T_{t_0}}[\kappa, \cdot])
$$

$$
= \sup_{B \in G} \int \int_{\Omega} d_{BL}(\mu_0^y \circ T_{t_0}^y[\mu], \mu_0^y \circ T_{t_0}^y[\kappa]) \ d\nu^B_x(y)
$$

$$
= \sup_{B \in G} \int \int_{\Omega} \left( f(T_{t_0}^y[\mu]v) - f(T_{t_0}^y[\kappa]v) \right) \ d\mu^y_0(v) \ d\nu^B_x(y)
$$

$$
\leq \sup_{B \in G} \int \int_{T} \left| T_{t_0}^y[\mu]v - T_{t_0}^y[\kappa]v \right| \ d\mu^y_0(v) \ d\nu^B_x(y) \ dy
$$

$$
\leq C \left\| D \right\|_{\infty} l^2(t - t_0) \to 0 \text{ as } t \to t_0. \tag{A.2}
$$

Here we used again a change of variables. Using the triangular inequality we calculate

$$
\lambda(t) = \sup_{B \in G} \int \int_{\Omega} \left| T_{t_0}^y[\mu]v - T_{t_0}^y[\kappa]v \right| \ d\mu^y_0(v) \ d\nu^B_x(y)
$$

$$
\leq \sup_{B \in G} \int \int_{0}^{t} \int \int_{\Omega} \left| V[\mu, y](s, T_{s_0}^y[\mu]v) - V[\kappa, y](s, T_{s_0}^y[\kappa]v) \right| \ d\mu^y_0(v) \ d\nu^B_x(y) \ ds
$$

$$
\leq \sup_{B \in G} \int \int_{0}^{t} \int \int_{\Omega} \left| V[\mu, y](s, T_{s_0}^y[\mu]v) - V[\kappa, y](s, T_{s_0}^y[\kappa]v) \right| \ d\mu^y_0(v) \ d\nu^B_x(y) \ ds
$$

$$
+ \sup_{B \in G} \int \int_{0}^{t} \int \int_{\Omega} \left| V[\kappa, y](s, T_{s_0}^y[\mu]v) - V[\kappa, y](s, T_{s_0}^y[\kappa]v) \right| \ d\mu^y_0(v) \ d\nu^B_x(y) \ ds. \tag{A.3}
$$

With this notation we calculate for the first difference, using Lemma 2.6:

$$
\sup_{B \in G} \int \int_{0}^{t} \int \int_{\Omega} \left| V[\mu, y](s, T_{s_0}^y[\mu]v) - V[\kappa, y](s, T_{s_0}^y[\mu]v) \right| \ d\mu^y_0(v) \ d\nu^B_x(y) \ ds
$$

$$
\leq 2C \left\| D \right\|_{\infty} l^2(t) \int \sup_{B \in G} \int \int_{\Omega} \left( \nu^B_y(\Omega) \right) \ d\mu^y_0(T) \ d\overline{d^b}(\bar{\mu}_s, \bar{\kappa}_s) \ ds
$$

$$
\leq 2Cb \int \overline{d^b}(\bar{\mu}_s, \bar{\kappa}_s) \ ds
$$

25
For the second difference we calculate, again using Lemma 2.6

\[
\sup_{B \in \mathcal{G}} \sup_{x \in \Omega} \int_0^t \int_\Omega \int_T |V[\kappa, y](s, T_{s,0}^y[\mu]v) - V[\kappa, y](s, T_{s,0}^y[\kappa]v)| \, d\mu_0^y(v) \, d\nu_x^B(y) \, ds
\]

\[
\leq b\gamma_A \int_0^t \sup_{B \in \mathcal{G}} \sup_{x \in \Omega} \int_\Omega \int_T |T_{s,0}^y[\mu]v - T_{s,0}^y[\kappa]v| \, d\mu_0^y(v) \, d\nu_x^B(y) \, ds
\]

\[
= b\gamma_A \int_0^t \lambda(s) \, ds.
\]

We set \( C_1 := 2Cb \) and \( C_2 := b\gamma_A \). Substituting both expressions in (A.3) we get

\[
\lambda(t) \leq C_1 \int_0^t \bar{d}^b(\bar{\mu}_s, \bar{\nu}_s) \, ds + C_2 \int_0^t \lambda(s) \, ds.
\]

Using Gronwall’s inequality, (cf. Lemma A.3) we obtain

\[
\lambda(t) \leq C_1 e^{C_2 t} \int_0^t \bar{d}^b(\bar{\mu}_s, \bar{\nu}_s) e^{-C_2 s} \, ds. \tag{A.4}
\]

Using (A.2) this implies that

\[
\bar{d}^b(F[\mu](t, \cdot), F[\kappa](t, \cdot)) \leq C_1 e^{C_2 t} \int_0^t \bar{d}^b(\bar{\mu}_s, \bar{\nu}_s) e^{-C_2 s} \, ds.
\]

Hence,

\[
d^b_\alpha(F[\mu](t, \cdot), F[\kappa](t, \cdot)) = \sup_{t \in T} \left\{ e^{-\alpha t} \bar{d}^b(F[\mu](t, \cdot), F[\kappa](t, \cdot)) \right\}
\]

\[
\leq \sup_{t \in T} C_1 e^{-(\alpha-C_2)t} \int_0^t \bar{d}^b(\bar{\mu}_s, \bar{\nu}_s) e^{-C_2 s} \, ds
\]

\[
\leq C_1 \bar{d}^b_\alpha(\bar{\mu}, \bar{\kappa}) \sup_{t \in T} e^{-(\alpha-C_2)t} \int_0^t e^{(\alpha-C_2)s} \, ds
\]

\[
\leq C_1 (\alpha - C_2)^{-1} \bar{d}^b_\alpha(\bar{\mu}, \bar{\kappa}).
\]

This proves the claim. (II) This follows immediately from (I) and the Banach contraction principle in the complete metric space \( \mathcal{M}_T^b \). \( \square \)