Entire solutions of quasilinear elliptic systems on Carnot Groups

Lorenzo D’Ambrosio
Dipartimento di Matematica, Università degli Studi di Bari
via E. Orabona, 4, I-70125 Bari, Italy, dambros@dm.uniba.it

Enzo Mitidieri∗
Dipartimento di Matematica e Geoscienze, Università degli Studi di Trieste
via A.Valerio, 12/1, I-34127 Trieste, Italy, mitidier@units.it

To the memory of Professor Lev Dmitrievich Kudryavtsev
September 15, 2012

Abstract
We prove general a priori estimates of solutions of a class of quasilinear elliptic system on Carnot groups. As a consequence, we obtain several non–existence theorems. The results are new even in the Euclidean setting.

Keywords: Quasilinear elliptic systems; A priori estimates; Non–existence theorems; Positive solutions; Carnot groups.

1 Introduction
As it is well known, one of the main problems in the theory of nonlinear partial differential equations is to find a priori bounds on the possible solutions of the problem under consideration. This information is crucial from several point of view.

On one hand the bounds that one can prove may be used for improved regularity properties of the solutions and on the other hand these results are crucial for establishing special qualitative properties of them. For a recent contribution in this direction see D’Ambrosio, Farina, Mitidieri and Serrin [11] and D’Ambrosio and Mitidieri [15].

In this paper we consider a class of quasilinear elliptic systems on Carnot groups and prove general a priori estimates of positive solutions in an open set of $\mathbb{R}^N$.

There are several recent studies dealing with this problem in the Euclidean framework. See for instance [18 21 8 22 2 24 9 12].

∗Corresponding author, e-mail mitidier@units.it.
To our knowledge, this is the first attempt to prove general estimates of solutions of quasilinear systems on structures which are not necessarily Euclidean.

Among other possibilities, this allows to extend known existence results related to the classical Dirichlet problem in Euclidean setting to the Carnot framework by using topological methods via blow-up procedure in the same spirit of [19, 8, 1, 27].

In this paper we prove *a priori* estimates for the solutions of elliptic systems in an open set $\Omega \subseteq \mathbb{R}^N$ involving quasilinear operators in divergence form. As a consequence, we obtain some nonexistence results for these problems in all of $\mathbb{R}^N$.

Earlier contributions on the nonexistence question for semilinear scalar subelliptic problems with power nonlinearities were obtained by Capuzzo-Dolcetta and Cutrì [6], Birindelli, Capuzzo-Dolcetta and Cutrì [3]. The quasilinear case was studied by D’Ambrosio [9]. More recently, for general nonlinearities, the quasilinear scalar case has been studied in D’Ambrosio and Mitidieri [12], [13] and [15].

The results proved in this paper are new even in the Euclidean setting.

To be more precise our aim is to study problems of the type,

$$
\begin{align*}
-\text{div}(\mathcal{A}_p(x, u, \nabla u)) &\geq f(x, u, v) \quad \text{on } \Omega, \\
-\text{div}(\mathcal{A}_q(x, v, \nabla v)) &\geq g(x, u, v) \quad \text{on } \Omega, \\
\quad u \geq 0, \ v \geq 0 \quad \text{on } \Omega,
\end{align*}
(P)
$$

where,

$$
\mathcal{A}_p, \mathcal{A}_q : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N
$$

are strongly-$p$-coercive and strongly-$q$-coercive ($p, q > 1$) respectively, and

$$
f, g : \Omega \times [0, \infty) \times [0, \infty) \rightarrow [0, \infty)
$$

are Carathéodory functions. On the possible solution $(u, v)$ of (P), we do not require any kind of behavior near the boundary of $\Omega$ or at infinity.

Throughout this work, we shall essentially use the same ideas as in [12], where we deal with general estimates of solutions of scalar differential inequalities.

One of the typical result proved in this paper in the Euclidean setting is the following. Let $\Omega = \mathbb{R}^N$, $f(x, u, v) = f(v)$ and $g(x, u, v) = g(u)$. Suppose that the following local assumptions on the behavior near zero of $f$ and $g$ hold:

$$
\liminf_{t \to 0} \frac{f(t)}{t^a} > 0 \ (\text{possibly } + \infty), \quad \text{with } \ a > 0, \quad (f_0)
$$

$$
\liminf_{t \to 0} \frac{g(t)}{t^b} > 0 \ (\text{possibly } + \infty), \quad \text{with } \ b > 0. \quad (g_0)
$$

Let $(u, v)$ be a weak solution of (P) such that $\text{ess inf}_{\mathbb{R}^N} u = \text{ess inf}_{\mathbb{R}^N} v = 0$. If

$$
\min \left\{ N - p - (p - 1) \frac{q}{b}, \ N - q - (q - 1) \frac{p}{a} \right\} \leq N \frac{(p - 1)(q - 1)}{ab}, \quad (1.1)
$$
then \( u = v = 0 \ a.e \ in \ \mathbb{R}^N \). See Theorem 5.6 below.

We point out that the results proved in this paper are sharp. To see this, one needs only to slightly modify the examples contained in [12, 14]. We shall omit the tedious details.

As a concrete illustration of other results proved in this paper (see Section 4) we have.

Let \( a \in \mathbb{R} \) and let \( h : \mathbb{R} \to [0, +\infty[ \) be a continuous function, then the problem
\[
\begin{cases}
-\Delta u \geq v^a, & \text{on } \mathbb{R}^3, \\
-\Delta v \geq h(u)(1 - \cos u), & \text{on } \mathbb{R}^3, \\
u > 0, \ v > 0
\end{cases}
\] (1.2)

has no non constant weak solutions. For details see Example 4.10.

Another example of application is the following.

Let \( \gamma, \delta \in \mathbb{R} \) and let \( h : \mathbb{R}^+ \times \mathbb{R}^+ \to [0, +\infty[ \) be a continuous nonnegative function.

Then the system
\[
\begin{cases}
-\text{div}(|\nabla u|^{p-2} \nabla u) \geq |v - 1|^{\gamma}, & \text{on } \mathbb{R}^N, \\
-\text{div}(|\nabla v|^{q-2} \nabla v) \geq v^\delta + h(u, v), & \text{on } \mathbb{R}^N
\end{cases}
\] (1.3)

has no weak solutions. See Example 4.11 for a generalized version and details.

The paper is organized as follows. In Section 2 we give some definitions and present few preliminary results focusing on the weak Harnack inequality and some of its consequences. Section 3 is devoted to the general a priori estimates for weak solutions of problem (P), while in Section 4 we prove our main results concerning the non–existence of non–trivial solutions of (P). Section 5 contains some indications on some extensions of the results obtained in the preceding sections to general classes of quasilinear differential operators. Finally in Appendix 6 we quote some well known facts on Carnot groups.

2 Preliminaries

Throughout this paper we shall use some concepts briefly described in the Appendix 6. For further details related to Carnot groups the interested reader may refer to [4].

To begin with let us fix a homogeneous norm \( S \). For \( R > 0 \), we consider \( B_R \) the ball of radius \( R > 0 \) generated by \( S \), i.e. \( B_R := \{x : S(x) < R\} \). We shall also denote by \( A_R \)
the annulus $B_{2R} \setminus \overline{B_R}$. By using the dilation $\delta_R$ and the fact that the Jacobian of $\delta_R$ is $R^Q$, we have
\[
|B_R| = \int_{B_R} dx = R^Q \int_{B_1} dx = w_S R^Q \quad \text{and} \quad |A_R| = w_S (2^Q - 1) R^Q,
\]
where $w_S$ is the Lebesgue measure of the unit ball $B_1$ in $\mathbb{R}^N$.

Let $p > 1$ and denote by $W^{1,p}_{L,\text{loc}}$ the space
\[
W^{1,p}_{L,\text{loc}}(\Omega) := \{ u \in L^p_{\text{loc}}(\Omega) : |\nabla u| \in L^p_{\text{loc}}(\Omega) \}.
\]

Consider the system of inequalities
\[
\begin{cases}
-\text{div}_L(|\nabla u|^{p-2} \nabla u) \geq f(x, u, v) \quad \text{on } \Omega, \\
-\text{div}_L(|\nabla v|^{q-2} \nabla v) \geq g(x, u, v) \quad \text{on } \Omega,
\end{cases}
\]
where $\Omega \subseteq \mathbb{R}^N$ is an open set, and $f, g : \Omega \times [0, \infty) \times [0, \infty) \to [0, \infty)$ are Carathéodory functions.

**Definition 2.1** A pair of functions $(u, v) \in W^{1,p}_{L,\text{loc}}(\Omega) \times W^{1,q}_{L,\text{loc}}(\Omega)$ is a weak solution of (2.1) if $f(\cdot, u, v), g(\cdot, u, v) \in L^1_{\text{loc}}(\Omega)$, and the following inequalities hold
\[
\begin{align*}
\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \phi_1 &\geq \int_{\Omega} f(x, u, v) \phi_1 \quad (2.2) \\
\int_{\Omega} |\nabla v|^{q-2} \nabla v \cdot \nabla \phi_2 &\geq \int_{\Omega} g(x, u, v) \phi_2 \quad (2.3)
\end{align*}
\]
for all non-negative functions $\phi_1, \phi_2 \in C^1_0(\Omega)$.

**Lemma 2.2 (Weak Harnack inequality [5, 23, 25, 26])** Let $Q > p > 1$. If $u \in W^{1,p}_{L,\text{loc}}(\mathbb{R}^N)$ is a weak solution of
\[
\begin{cases}
-\text{div}_L(|\nabla u|^{p-2} \nabla u) \geq 0 \quad \text{on } \Omega, \\
u \geq 0 \quad \text{on } \Omega,
\end{cases}
\]
then for any $\sigma \in \left(0, \frac{Q(p-1)}{Q-p}\right)$ there exists a constant $c_H > 0$ independent of $u$ such that for all $R > 0$
\[
\left(\frac{1}{|B_R|} \int_{B_R} u^\sigma\right)^{1/\sigma} \leq c_H \text{ ess inf}_{B_{R/2}} u.
\]

Motivated by the above results, as in [12], we introduce the following definition.
Remark 2.3 Inequality (WH) implies immediately that if \( u \in W^{1,p}_{L,loc}(\mathbb{R}^N) \) is a weak solution of
\[
\begin{cases}
-\text{div}_L(|\nabla_L u|^{p-2} \nabla_L u) \geq 0 & \text{on } \mathbb{R}^N, \\
u \geq 0 & \text{on } \mathbb{R}^N,
\end{cases}
\]
then either \( u \equiv 0 \) or \( u > 0 \) a.e. on \( \mathbb{R}^N \). Therefore, without loss of generality we shall limit to study positive solutions.

Lemma 2.4 (Lemma 3.1 of [12]) Let \( u : \mathbb{R}^N \rightarrow [0, \infty) \) be a measurable function such that \( \text{ess inf}_{\mathbb{R}^N} u = 0 \). Assume that (WH) holds with exponent \( \sigma > 0 \), then for all \( \varepsilon > 0 \)
\[
\lim_{R \to \infty} \frac{|A_{R/2} \cap T^u_{\varepsilon}|}{|A_{R/2}|} = 1, \quad \lim_{R \to \infty} \frac{|B_{R} \cap T^u_{\varepsilon}|}{|B_{R}|} = 1,
\]
where \( T^u_{\varepsilon} = \{ x \in \mathbb{R}^N : u(x) < \varepsilon \} \).

Finally, few words on the hypothesis \( Q > p \). This assumption is quite natural since the following holds.

Theorem 2.5 Let \( u \in W^{1,p}_{L,loc}(\mathbb{R}^N) \) is a weak solution of
\[
\begin{cases}
-\text{div}_L(|\nabla_L u|^{p-2} \nabla_L u) \geq f(x, u, v) & \text{on } \Omega, \\
u \geq 0 & \text{on } \mathbb{R}^N,
\end{cases}
\]
\[
\begin{cases}
-\text{div}_L(|\nabla_L v|^{q-2} \nabla_L v) \geq g(x, u, v) & \text{on } \Omega, \\
u \geq 0, \ v \geq 0 & \text{on } \mathbb{R}^N.
\end{cases}
\]
If \( p \geq Q \), then \( u \) is constant.

See [22] for earlier results of this nature, and [9] for the proof and related theorems in the Carnot group setting.

3 General a priori estimates

In this section we give some a priori bounds for the solutions of the system of inequalities
\[
\begin{cases}
-\text{div}_L(|\nabla_L u|^{p-2} \nabla_L u) \geq f(x, u, v) & \text{on } \Omega, \\
u \geq 0, \ v \geq 0 & \text{on } \mathbb{R}^N.
\end{cases}
\]
where, we remind, \( \Omega \subseteq \mathbb{R}^N \) is an open set and \( f, g : \Omega \times [0, \infty) \times [0, \infty) \to [0, \infty) \) are nonnegative Carathéodory functions.
Theorem 3.1 Let \((u, v)\) be a weak solution of \(3.1\). Then for all test functions \(\phi_1, \phi_2\), every \(\ell \geq 0\) and every \(\alpha, \beta < 0\), we get

\[
\int_{\Omega} f(x, u, v)\phi_1 + c_1 \int_{\Omega} |\nabla_{L} u|^p u_{\ell}^\alpha \phi_1 \leq c_2 \int_{\Omega} u_{\ell}^{\alpha-1+p} \frac{|\nabla_{L} \phi_1|^p}{\phi_1^{p-1}},
\]

\[
\int_{\Omega} g(x, u, v)\phi_2 + c_1 \int_{\Omega} |\nabla_{L} v|^q v_{\ell}^\beta \phi_2 \leq \hat{c}_2 \int_{\Omega} v_{\ell}^{\beta-1+q} \frac{|\nabla_{L} \phi_2|^q}{\phi_2^{q-1}},
\]

(3.2)

where \(c_1 := |\alpha| - \eta p'/p', \ c_2 := \eta p/p, \ \eta > 0, \ \hat{c}_1 := |\beta| - \mu q'/q', \ \hat{c}_2 := \mu ^{-q}/q, \ \mu > 0, \ u_\ell := u + \ell\) and \(v := v + \ell\).

If \(\eta, \mu\) are so small that \(c_1, \hat{c}_1 > 0\), then for all \(\alpha, \beta < 0\) and \(\ell \geq 0\)

\[
\int_{\Omega} f(x, u, v)\phi_1 \leq c_3 \left( \int_{\Omega} u_{\ell}^{\alpha-1+p} \frac{|\nabla_{L} \phi_1|^p}{\phi_1^{p-1}} \right)^{1/p'} \left( \int_{\Omega} u_{\ell}^{(1-\alpha)(p-1)} \frac{|\nabla_{L} \phi_1|^p}{\phi_1^{p-1}} \right)^{1/p},
\]

\[
\int_{\Omega} g(x, u, v)\phi_2 \leq \hat{c}_3 \left( \int_{\Omega} v_{\ell}^{\beta-1+q} \frac{|\nabla_{L} \phi_2|^q}{\phi_2^{q-1}} \right)^{1/q'} \left( \int_{\Omega} v_{\ell}^{(1-\beta)(q-1)} \frac{|\nabla_{L} \phi_2|^q}{\phi_2^{q-1}} \right)^{1/q},
\]

(3.3)

where \(c_3 := (c_2/c_1)^{1/p'}\) and \(\hat{c}_3 := (\hat{c}_2/\hat{c}_1)^{1/q'}\).

If \(u_{\ell}^{\alpha-1+p}, u_{\ell}^{(1-\alpha)(p-1)} \in L_{loc}^{1}(A_{R}), \ v_{\ell}^{\beta-1+q}, v_{\ell}^{(1-\beta)(q-1)} \in L_{loc}^{1}(A_{R}), \) with \(R > 0\) such that \(B_{2R} \subseteq \Omega\), then for all \(\alpha, \beta < 0\) there exist \(c_4, \hat{c}_4 > 0\) for which

\[
\frac{1}{|B_{R}|} \int_{B_{R}} f(x, u, v) \leq c_4 R^{-p} \left( \frac{1}{|A_{R}|} \int_{A_{R}} u_{\ell}^{\alpha-1+p} \right)^{1/p'} \left( \frac{1}{|A_{R}|} \int_{A_{R}} u_{\ell}^{(1-\alpha)(p-1)} \right)^{1/p},
\]

\[
\frac{1}{|B_{R}|} \int_{B_{R}} g(x, u, v) \leq \hat{c}_4 R^{-q} \left( \frac{1}{|A_{R}|} \int_{A_{R}} v_{\ell}^{\beta-1+q} \right)^{1/q'} \left( \frac{1}{|A_{R}|} \int_{A_{R}} v_{\ell}^{(1-\beta)(q-1)} \right)^{1/q}.
\]

(3.4)

If there exist \(\sigma > p - 1, \ \delta > q - 1\) such that \(u^\sigma, v^\delta \in L_{loc}^{1}(\Omega), \) then

\[
\frac{1}{|B_{R}|} \int_{B_{R}} f(x, u, v) \leq c_4 R^{-p} \left( \frac{1}{|A_{R}|} \int_{A_{R}} u^\sigma \right)^{(p-1)/\sigma},
\]

\[
\frac{1}{|B_{R}|} \int_{B_{R}} g(x, u, v) \leq \hat{c}_4 R^{-q} \left( \frac{1}{|A_{R}|} \int_{A_{R}} v^\delta \right)^{(q-1)/\delta}.
\]

(3.5)

In particular, if (WII) holds with exponent \(\sigma\) for \(u\) and with exponent \(\delta\) for \(v\), then the following inequalities hold for some appropriate constants \(c_5, \hat{c}_5 > 0\)

\[
\frac{1}{|B_{R}|} \int_{B_{R}} f(x, u, v) \leq c_5 R^{-p} \left( \text{ess inf}_{B_{R}} u \right)^{(p-1)},
\]

\[
\frac{1}{|B_{R}|} \int_{B_{R}} g(x, u, v) \leq \hat{c}_5 R^{-q} \left( \text{ess inf}_{B_{R}} v \right)^{(q-1)}.
\]

(3.6)
Proof. We shall prove the first inequality of (3.2), (3.3), (3.4), (3.5) and (3.6). The remaining inequalities will follow similarly.

Let \( \phi_1 \in C_0^1(\Omega) \) be a nonnegative test function and set \( r := \text{dist}(\text{supp}(\phi_1), \partial \Omega), \Omega_r := \{ y \in \Omega \mid \text{dist}(y, \partial \Omega) > r \} \). For \( \varepsilon \in (0, r) \) and \( \ell > 0 \) we define

\[
w_\varepsilon(x) := \begin{cases}
\ell + \int_{\Omega_r} D_\varepsilon(x - y) u(y) dy, & \text{if } x \in \Omega_r, \\
0, & \text{if } x \in \Omega \setminus \Omega_r,
\end{cases}
\]

where \((D_\varepsilon)_\varepsilon\) is a family of mollifiers. See [4, 14]. Thus, choosing \( w_\varepsilon^0 \phi_1 \) as test function in (2.2) we have

\[
\int_\Omega f(x, u, v) w_\varepsilon^0 \phi_1 + |\alpha| \int_\Omega |\nabla L u|^{p-2} \nabla L u \cdot \nabla L w_\varepsilon \phi_1^{\alpha-1} \leq \int_\Omega |\nabla L u|^{p-1} |\nabla L \phi_1| w_\varepsilon^\alpha.
\]

Since \( w_\varepsilon \to u \), \( \nabla L w_\varepsilon \to \nabla L u \) in \( L^p_{\text{loc}}(\Omega_r) \) as \( \varepsilon \to 0 \), by Lebesgue’s dominated convergence theorem and by duality, we get

\[
\int_\Omega f(x, u, v) u_\varepsilon^0 \phi_1 + |\alpha| \int_\Omega |\nabla L u|^p u_\varepsilon^{\alpha-1} \phi_1 \leq \int_\Omega |\nabla L u|^{p-1} |\nabla L \phi_1| u_\varepsilon^\alpha
\]

\[
= \int_\Omega |\nabla L u|^{p-1} u_\varepsilon^{(\alpha-1)/p'} \phi_1^{1/p'} \cdot u_\varepsilon^{(\alpha-1+p)/p} |\nabla L \phi_1| \phi_1^{-1/p'}
\]

\[
\leq \frac{p'}{p} \int_\Omega |\nabla L u|^p u_\varepsilon^{\alpha-1} \phi_1 + \frac{1}{p^{1/p}} \int_\Omega u_\varepsilon^{\alpha-1+p} |\nabla L \phi_1|^p \phi_1^{-1-p},
\]

where in the last step we have used the Young’s inequalities. This completes the proof of the first inequality in (3.2) when \( \ell > 0 \). The case \( \ell = 0 \) follows immediately from the case \( \ell > 0 \), by an application of Beppo-Levi’s theorem letting \( \ell \to 0 \).

In order to prove the first inequality in (3.3), we use \( \phi_1 \) as test function in (2.2). Let \( \ell > 0 \), by Hölder’s inequality with exponent \( p \) and (3.2) we obtain

\[
\int_\Omega f(x, u, v) \phi_1 \leq \int_\Omega |\nabla L u|^{p-1} |\nabla L \phi_1|
\]

\[
\leq \left( \int_\Omega |\nabla L u|^p u_\varepsilon^{(\alpha-1)/p} \phi_1 \right)^{1/p'} \cdot \left( \int_\Omega u_\varepsilon^{(1-\alpha)/(p-1)} |\nabla L \phi_1|^p \phi_1^{-1-p} \right)^{1/p}
\]

\[
\leq c_3 \left( \int_\Omega u_\varepsilon^{\alpha-1+p} |\nabla L \phi_1|^p \phi_1^{-1-p} \right)^{1/p'} \cdot \left( \int_\Omega u_\varepsilon^{(1-\alpha)/(p-1)} |\nabla L \phi_1|^p \phi_1^{-1-p} \right)^{1/p},
\]

which is the claim for \( \ell > 0 \). An application of Beppo-Levi’s monotone convergence theorem implies the validity also for \( \ell = 0 \).

Let \( \phi_0 \in C_0^1(\mathbb{R}) \) be such that \( 0 \leq \phi_0 \leq 1 \), \( c_{\phi_0} := \| \phi_0^p / \phi_0^{p-1} \|_{\infty} < \infty \) and

\[
\phi_0(t) = \begin{cases}
1, & \text{if } |t| < 1,
0, & \text{if } |t| > 2.
\end{cases}
\]
Define $\phi_1(x) := \phi_0(S(\delta_1/Rx))$, so that
\[
\frac{||\nabla L \phi_1(x)||^p}{\phi_1(x)^{p-1}} = |\phi_0(S(\delta_1/Rx))|^p |\nabla L S|(\delta_1/Rx)^{R-p} \leq c_{\phi_0} ||\nabla L S||_\infty R^{-p} \leq cR^{-p}.
\]

Hence, using $\phi_1$ as test function in (3.3) with $\ell = 0$, we get
\[
\int_{\Omega} f(x, u, v) \phi_1 \leq c_3 \left( \int_{A_R} u^{\alpha-1+p} cR^{-p} \right)^{1/p'} \left( \int_{A_R} u^{(1-\alpha)(p-1)} cR^{-p} \right)^{1/p},
\]
and so, being $|A_R| = w_S(2^Q - 1) R^Q = (2^Q - 1)|B_R|$, we have
\[
\frac{1}{|B_R|} \int_{B_R} f(x, u, v) \leq c_4 (2^Q - 1) cR^{-p} \left( \frac{1}{|A_R|} \int_{A_R} u^{\alpha-1+p} \right)^{1/p'} \left( \frac{1}{|A_R|} \int_{A_R} u^{(1-\alpha)(p-1)} \right)^{1/p},
\]
which gives (3.4), with $c_4 := c_3 (2^Q - 1) c_{\phi_0} ||\nabla L S||_\infty$.

Estimate (3.5) easily follows from (3.4), by applying Hölder’s inequality.

Finally, since (WH) holds, by (3.5) we obtain
\[
\frac{1}{|B_R|} \int_{B_R} f(x, u, v) \leq c_5 R^{-p} \left( \text{ess inf}_{B_R} u \right)^{p-1},
\]
with $c_5 := c_4 \left( 1 - \frac{1}{2^Q} \right) \frac{(1-p)/\sigma}{(1/p-1)/\sigma}$. Which is the first inequality in (3.6) and the proof is complete. 

4 Some Liouville Theorems

In what follows $f, g : \mathbb{R}^N \times [0, +\infty[ \times [0, +\infty[ \rightarrow [0, +\infty[$ are supposed to be Charatheodory functions. Let $Q > p, q > 1$ and consider the problem,
\[
\begin{cases}
\begin{aligned}
- \text{div}_L (|\nabla u|^{p-2} \nabla u) &\geq f(x, u, v), & \text{on } \mathbb{R}^N \\
- \text{div}_L (|\nabla v|^{q-2} \nabla v) &\geq g(x, u, v), & \text{on } \mathbb{R}^N \\
u &> 0, & v > 0.
\end{aligned}
\end{cases}
\]

Our first result is the following.

**Theorem 4.1** Let $f(x, u, v) = f(v)$ with $f : [0, +\infty[ \rightarrow [0, +\infty[$ be continuous. No extra assumption on $g$. Then (4.1) has no weak solutions.
Proof. If \( \inf_{t \geq 0} f(t) > 0 \), then the non-existence of solutions is a consequence of Theorem 4.5 with \( q = 0 \) in [9].

Assume that \( \inf_{t \geq 0} f(t) = 0 \). From [3.6] we have that for any \( R > 0 \)

\[
R^{-p} \left( \essinf_{B_R} u \right)^{p-1} \geq c \frac{1}{|B_R|} \int_{B_R} f(v).
\]

Now let \( 0 < \sigma < \frac{Q(q-1)}{Q-q} \), \( R_0 > 0 \) sufficiently large and set \( h(t) := f(t^{1/\sigma}) \) and \( M := (\essinf_{B_{R_0}} u)^{p-1} \). For \( R > R_0 \), we have

\[
R^{-p} M \geq c \frac{1}{|B_R|} \int_{B_R} h(v^\sigma) .
\] (4.2)

Since \( h \) is continuous and positive on \([0, +\infty[\), then there exists a positive convex non-increasing function \( h^* \) such that \( h(t) \geq h^*(t) > 0 \) and such that \( h(t) \to 0 \) as \( t \to +\infty \) (for an explicit construction of \( h^* \), see [7]).

Therefore we obtain

\[
CR^{-p} \geq \int_{B_R} h(u^\sigma) \geq \int_{B_R} h^*(v^\sigma) \geq h^* \left( \int_{B_R} v^\sigma \right).
\]

Letting \( R \to +\infty \) in the last inequality we get

\[
h^* \left( \int_{B_R} v^\sigma \right) \to 0 \quad \text{as} \; R \to +\infty.
\]

Therefore, by construction of \( h^* \), we have

\[
\int_{B_R} v^\sigma \to +\infty \quad \text{as} \; R \to +\infty.
\]

Since \( 0 < \sigma < \frac{Q(q-1)}{Q-q} \) an application of Harnack inequality, implies

\[
\essinf_{B_R} v \to +\infty \quad \text{as} \; R \to +\infty.
\]

This contradiction completes the proof. \( \Box \)

**Corollary 4.2** Let \( f(x, v, u) = f(v), \; g(x, v, u) = g(u) \), with \( f, g : ]0, +\infty[ \to ]0, +\infty[ \) be continuous. If \((u, v)\) is a weak solution of (4.1), then necessarily

\[
\liminf_{t \to 0} g(t) = 0 = \liminf_{t \to 0} f(t) \quad (4.3)
\]

and \( \essinf_{\mathbb{R}^N} u = \essinf_{\mathbb{R}^N} v = 0 \).
Proof. From Theorem 4.1 we easily deduce that (4.3) holds.

Set $m := \text{ess inf}_{\mathbb{R}^N} v$. We shall argue by contradiction. Let us assume that $m > 0$. Set

$$v_1 := v - m/2$$

and

$$f_1(t) := f(t + m/2).$$

Clearly $(u, v_1)$ is a weak solution of

$$\begin{cases}
-\text{div}_L(|\nabla_L u|^{p-2} \nabla_L u) \geq f_1(v_1), & \text{on } \mathbb{R}^N \\
-\text{div}_L(|\nabla_L v_1|^{q-2} \nabla_L v_1) \geq g(u), & \text{on } \mathbb{R}^N \\
u > 0, & v_1 > 0.
\end{cases}$$

Since $f_1$ is positive in $[0, +\infty[$, from Theorem 4.1 we reach a contradiction. Similarly, we deduce that $\text{ess inf}_{\mathbb{R}^N} u = 0$. The proof is complete. \(\square\)

Theorem 4.3 Let $f(x, v, u) = f(v)$ with $f : [0, +\infty[ \to [0, +\infty[$ be continuous. No extra assumption on $g$.

Let $(u, v)$ be a weak solution of (4.1) and let $\alpha := \text{ess inf}_{\mathbb{R}^N} v$. Then, $f(\alpha) = 0$.

Proof. Since the differential operator appearing in (4.1) is translation invariant, by replacing $f$ with $f(\cdot + \alpha)$ we shall assume that $\alpha = 0$.

We proceed by contradiction assuming $m := f(0) > 0$. By using the same argument and notations of the proof of Theorem 4.1 we deduce that (4.2) holds and $h(0) = m > 0$. Now, by a standard continuity argument it follows that there exists $\alpha_1 > 0$ such that $h(t) > m/2$ for $t \in [0, \alpha_1]$.

Let $h^*$ be the continuous function defined as follows

$$h^*(t) := \begin{cases} 
\frac{m}{\alpha_1}(\alpha_1 - t) & \text{if } 0 \leq t \leq \alpha_1, \\
0 & \text{if } t > \alpha_1.
\end{cases}$$

(4.5)

By the convexity of $h^*$, from (4.2) we deduce

$$CR^{-p} \geq \int_{B_R} h(u^\sigma) \geq \int_{B_R} h^*(v^\sigma) \geq h^* \left( \int_{B_R} v^\sigma \right).$$

Letting $R \to +\infty$ in the last inequality we obtain that

$$\lim_{R \to +\infty} h^* \left( \int_{B_R} v^\sigma \right) = 0.$$
Therefore, taking into account the construction of $h^*$, we obtain

$$\liminf_{R \to +\infty} \int_{B_R} v^\sigma \geq \alpha_1,$$

which, in turn, by Harnack inequality, implies that $0 = \text{ess inf}_{\mathbb{R}^N} \geq \alpha_1$. This contradiction completes the proof. \qed

**Corollary 4.4** Let $f(v, u) = f(v)$, $g(x, u, v) = g(u)$ with $f, g : [0, +\infty] \to [0, +\infty]$ be continuous functions. Let $(u, v)$ be a solution of (4.1). Then $\alpha := \text{ess inf}_{\mathbb{R}^N} v$ is a zero of the function $f$ and $\beta := \text{ess inf}_{\mathbb{R}^N} u$ is a zero of the function $g$.

**Theorem 4.5** Let $f(x, v, u) = f(v)$, with $f : [0, +\infty] \to [0, +\infty]$ be a continuous function satisfying

$$\liminf_{t \to 0} \frac{f(t)}{t^a} > 0 \quad (\text{possibly } +\infty), \quad \text{with } a > 0. \quad (f_0)$$

Let $(u, v)$ be a weak solution of (4.1) such that $\text{ess inf}_{\mathbb{R}^N} v = 0$. Then there exists $c > 0$ such that for $R$ sufficiently large, the following estimates hold

$$\text{ess inf}_{B_R} v \leq c R^{-p/a} (\text{ess inf}_{B_R} u)^{\frac{\rho-1}{\rho}}, \quad (4.6)$$

$$\int_{B_R} g(x, u, v) \leq c R^{Q-q-(q-1)p/a} (\text{ess inf}_{B_R} u)^{\frac{(p-1)(q-1)}{a}}, \quad (4.7)$$

$$\int_{B_R} g(x, u, v) \leq c R^{Q-q-(q-1)/a} \left( \int_{A_{R/2}} f(v) \right)^{\frac{q-1}{a}}. \quad (4.8)$$

Moreover if $\text{ess inf}_{\mathbb{R}^N} u = 0$, and $g(x, u, v) = g(u)$ with $g : [0, +\infty] \to [0, +\infty]$ a continuous function satisfying

$$\liminf_{t \to 0} \frac{g(t)}{t^b} > 0 \quad (\text{possibly } +\infty), \quad \text{with } b > 0, \quad (g_0)$$

then we have

$$\left( \text{ess inf}_{B_R} \right)^{ab-(p-1)(q-1)} v \leq c R^{-aq-p(q-1)}, \quad \left( \text{ess inf}_{B_R} \right)^{ab-(p-1)(q-1)} u \leq c R^{-bp-q(p-1)}, \quad (4.9)$$

$$\int_{B_R} f(v)dx \leq c R^{Q-p-(p-1)\frac{q}{a}-(q-1)\frac{p}{ab}} \left( \int_{A_{R/2}} f(v)dx \right)^{\frac{(p-1)(q-1)}{ab}}. \quad (4.10)$$

**Proof.** From $(f_0)$, we deduce that there exist $c_f > 0$ and $\epsilon > 0$ such that

$$f(t) \geq c_f t^a \text{ for } 0 < t < \epsilon.$$
Set \( T^v_\varepsilon := \{ x \in \mathbb{R}^N : v(x) < \varepsilon \} \). From the first inequality of (3.6), we have

\[
c_5 R^{-p} \left( \operatorname{ess inf}_{B_R} u \right)^{p-1} \geq \frac{1}{|B_R|} \int_{B_R} f(v) \geq \frac{1}{|B_R|} \int_{B_R \cap T^v_\varepsilon} c_f v^a \geq \frac{|B_R \cap T^v_\varepsilon|}{|B_R|} c_f (\operatorname{ess inf}_{B_R \cap T^v_\varepsilon} v)^a.
\]

Next, since \( \operatorname{ess inf}_{B_R \cap T^v_\varepsilon} v \geq \operatorname{ess inf}_{B_R} v \), from Lemma 2.4 we obtain (4.6).

Combining the second inequality in (3.6) and (4.6) we deduce (4.7). Now, in order to show that (4.8) holds, we shall argue as follows. From (3.6) we have

\[
\frac{1}{|B_R|} \int_{B_R} g(x, u, v) \leq c_5 R^{-q} \left( \operatorname{ess inf}_{B_R} v \right)^{q-1} \leq c_5 R^{-q} \left( \frac{1}{|A_{R/2} \cap T^v_\varepsilon|} \int_{A_{R/2} \cap T^v_\varepsilon} \left( \operatorname{ess inf}_{A_{R/2} \cap T^v_\varepsilon} v \right)^a \right)^{\frac{q-1}{a}} \leq c_5 R^{-q} \left( \frac{1}{|A_{R/2} \cap T^v_\varepsilon|} \int_{A_{R/2}} f(v) \right)^{\frac{q-1}{a}},
\]

which, by Lemma 2.4, implies the claim.

Assume that \((g_0)\) holds. Then, from the first part of the theorem, for \( R \) large, it follows that

\[
\operatorname{ess inf}_{B_R} u \leq c R^{-q/b} \left( \operatorname{ess inf}_{B_R} v \right)^{\frac{q-1}{a}},
\]

which, together with (4.6), implies the estimates in (4.9).

Similarly, we have that for \( R \) large there holds

\[
\int_{B_R} f(v) \leq c R^{q-p-Q(p-1)/b} \left( \int_{A_{R/2}} g(u) \right)^{\frac{p-1}{b}},
\]

which, combined with (4.8), implies inequality (4.10), thereby concluding the proof.

\[
\square
\]

\textbf{Theorem 4.6} Let \( f(x, v, u) = f(v), \ g(x, u, v) = g(u) \) with \( f, g : [0, +\infty[ \rightarrow [0, +\infty[ \) be continuous functions satisfying \((f_0)\) and \((g_0)\) respectively. If

\[
\min \left\{ Q - p - (p-1) \frac{q}{b}, \ Q - q - (q-1) \frac{p}{a} \right\} \leq Q \frac{(p-1)(q-1)}{ab}
\]

then (4.1) has no weak solution \((u, v)\) such that \( \operatorname{ess inf}_{\mathbb{R}^N} u = \operatorname{ess inf}_{\mathbb{R}^N} v = 0 \).
Remark 4.7 Notice that condition \((4.11)\) can be also written as
\[ \max \{abp + aq(p - 1), abq + bp(q - 1)\} \geq Q (ab - (p - 1)(q - 1)) \] (4.12)
and in the particular case \(p = q\), it reads as
\[ \max \{a + p - 1, b + p - 1\} \geq \frac{Q - p}{p(p - 1)} (ab - (p - 1)(q - 1)) . \] (4.13)
Finally, in the special case \(p = q = 2\) all the above conditions become
\[ \max \{a + 1, b + 1\} \geq \frac{Q - 2}{2} (ab - 1) , \] (4.14)
which, in the Euclidean case is the inequality discovered in [20].

Remark 4.8 Notice that form \((4.12)\), it is evident that if \(ab \leq (p - 1)(q - 1)\), then the hypothesis \((4.11)\) is satisfied.

Proof. Assume, by contradiction, that \((u, v)\) is a non trivial weak solution of \((4.1)\) with \(\text{ess inf}_{\mathbb{R}^N} u = \text{ess inf}_{\mathbb{R}^N} v = 0\).

First consider the case \(ab \leq (p - 1)(q - 1)\). Clearly condition \((4.11)\) holds. By letting \(R \to +\infty\) in \((4.9)\) of Theorem 4.5 we reach a contradiction.

Let \(ab > (p - 1)(q - 1)\). Assume that
\[ Q - p - (p - 1)\frac{q}{b} \leq Q \frac{(p - 1)(q - 1)}{ab} . \]

From \((4.10)\) of Theorem 4.5 for any \(R\) large we have
\[ \int_{B_R} f(v)dx \leq c \left( \int_{A_{R/2}} f(v)dx \right)^{(p-1)(q-1)}_{ab} , \] (4.15)
which implies that
\[ \left( \int_{B_R} f(v)dx \right)^{ab-(p-1)(q-1)}_{ab} \leq c . \]

Therefore, we obtain that \(f(v) \in L^1(\mathbb{R}^N)\). Hence, from \((4.15)\) it follows that,
\[ f(v(x)) = 0 \quad \text{a.e. on } \mathbb{R}^N . \]

Using this information in \((4.15)\) and the condition \((f_0)\), for \(\epsilon > 0\) sufficently small, (small enough such that \(f(t) \geq c f t^a\) for \(t \in ]0, \epsilon]\)), we obtain
\[ cf \int_{T_\epsilon^a} v^a \leq \int_{T_\epsilon^a} f(v)dx = 0 . \]

where \( T_\epsilon^v = \{ x \in \mathbb{R}^N : v(x) < \epsilon \} \).

Now since \( v \neq 0 \), by Harnack’s inequality \( v^a > 0 \) a.e. on \( \mathbb{R}^N \), therefore, necessarily \( |T_\epsilon^v| = 0 \). This implies that \( v \geq \epsilon \) a.e. contradicting the fact that \( \text{ess inf}_{\mathbb{R}^N} u = 0 \).

If

\[
Q - q - (q - 1) \frac{p}{a} \leq \frac{Q(p-1)(q-1)}{ab},
\]

the proof is similar.

\[ \square \]

**Example 4.9** Consider,

\[
\begin{cases}
-\text{div}_L(|\nabla_L u|^{p-2} \nabla_L u) \geq f(v), & \text{on } \mathbb{R}^N, \\
-\text{div}_L(|\nabla_L v|^{q-2} \nabla_L v) \geq g(x,u,v), & \text{on } \mathbb{R}^N, \\
u > 0, v > 0.
\end{cases}
\]

i) If \( f(v) = v^{-\gamma} \) with \( \gamma > 0 \) and \( g : \mathbb{R}^N \times \mathbb{R}^+ \times \mathbb{R}^+ \to ]0, +\infty[ \) is Carathéodory, then the problem (4.16) has no weak solution.

ii) If \( f(v) = \frac{1}{1 + v \gamma} \) with \( \gamma > 0 \) and \( g : \mathbb{R}^N \times \mathbb{R}^+ \times \mathbb{R}^+ \to ]0, +\infty[ \) is Carathéodory, then the problem (4.16) has no weak solution.

Indeed, in both cases the claim follows from Theorem 4.4.

Notice that we do not assume any growth assumption on \( g \).

**Example 4.10** Let \( a \in \mathbb{R} \) and let \( h : \mathbb{R} \to ]0, +\infty[ \) be a continuous function.

Consider,

\[
\begin{cases}
-\Delta u \geq v^a, & \text{on } \mathbb{R}^3, \\
-\Delta v \geq h(u)(1 - \cos u), & \text{on } \mathbb{R}^3, \\
u > 0, v > 0.
\end{cases}
\]

The problem has no non constant weak solutions. Indeed, in the case \( a \leq 0 \) the claim follows from the previous example. Let \( a > 0 \). From Corollary 4.4 it follows that \( \text{ess inf}_{\mathbb{R}^N} v = 0 \) and \( \text{ess inf}_{\mathbb{R}^N} u = 2k\pi \) where \( k \) is an integer. By translation invariance we can assume that \( k = 0 \). Now we are in the position to apply Theorem 4.6. In this case \( b = 2 \) and the hypothesis (4.11), or equivalently (4.14), is satisfied provided

\[
\max\{a + 1, 3\} \geq \frac{1}{2}(2a - 1) = a - \frac{1}{2}.
\]

Notice that the above inequality holds for any \( a > 0 \).
Example 4.11 Let $\gamma, \delta \in \mathbb{R}$ and let $h : \mathbb{R}^+ \times \mathbb{R}^+ \to [0, +\infty]$ be a continuous nonnegative function. Consider the system

\[
\begin{align*}
-\text{div}_L(|\nabla_L u|^{p-2} \nabla_L u) & \geq |v - 1|^{\gamma}, \quad \text{on} \quad \mathbb{R}^N, \\
-\text{div}_L(|\nabla_L v|^{q-2} \nabla_L v) & \geq v^\delta + h(u,v), \quad \text{on} \quad \mathbb{R}^N, \\
& u > 0, \quad v > 0.
\end{align*}
\]  

Our claim is that this problem has no weak solutions.
Indeed, in the case $\gamma \leq 0$ or $\delta \leq 0$, this follows by applying Theorem 4.1 or Corollary 2.4 of [12] respectively.

Consider now the case $\gamma, \delta > 0$. From Theorem 4.3 it follows that $\text{ess inf}_{\mathbb{R}^N} v = 1$.

Using this information, and setting $v_1 := v - 1$, we see that $(u, v_1)$ is a weak solution of

\[
\begin{align*}
-\text{div}_L(|\nabla_L u|^{p-2} \nabla_L u) & \geq v_1^\gamma, \quad \text{on} \quad \mathbb{R}^N, \\
-\text{div}_L(|\nabla_L v_1|^{q-2} \nabla_L v_1) & \geq 1, \quad \text{on} \quad \mathbb{R}^N, \\
& u > 0, \quad v_1 > 0.
\end{align*}
\]  

In other words $(u, v_1)$ is a weak solution of (4.1) with $g = g(u) = 1 > 0$. An application of Theorem 4.1 implies the claim.

Example 4.12 Let $\gamma, \delta > 0$ and let $h : \mathbb{R}^+ \to [0, +\infty]$ be a continuous positive function. By an argument similar to the one used in the above example, we can show that the system

\[
\begin{align*}
-\text{div}_L(|\nabla_L u|^{p-2} \nabla_L u) & \geq |v - 1|^{\gamma}, \quad \text{on} \quad \mathbb{R}^N, \\
-\text{div}_L(|\nabla_L v|^{q-2} \nabla_L v) & \geq v^\delta h(u), \quad \text{on} \quad \mathbb{R}^N, \\
& u > 0, \quad v > 0,
\end{align*}
\]  

has no weak solutions. We omit the details.

5 Some Extensions

In what follows we suppose that $f, g : \mathbb{R}^N \times [0, +\infty] \to [0, +\infty]$ are two nonnegative Carathéodory functions.

Let $p_1, p_2 > 1$ and for $i = 1, 2$, $\mathcal{A}_{p_i} : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}^l$ denotes a Carathéodory function. We assume that the function $\mathcal{A}_{p_i}$ is $W$-$p_i$-C, weakly-$p_i$-coercive, namely there exists a constant $k > 0$ such that

$$
(\mathcal{A}_{p_i}(x,t,\xi) \cdot \xi) \geq k|\mathcal{A}_{p_i}(x,t,\xi)|^{p_i'} \quad \text{for all} \quad (x,t,\xi) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^l.
$$  

(W-$p_i$-C).
See [2] [22] [23] for details.

Consider the following,

\[
\begin{cases}
- \text{div}_L(\varphi_1(x, u, \nabla L u)) \geq f(x, u, v), & \text{on } \mathbb{R}^N \\
- \text{div}_L(\varphi_2(x, v, \nabla L v)) \geq g(x, u, v), & \text{on } \mathbb{R}^N,
\end{cases}
\]

(5.1)

As usual, a pair of functions \((u, v) \in W_{L,loc}^{1,p_1}(\mathbb{R}^N) \times W_{L,loc}^{1,p_2}(\mathbb{R}^N)\) is a weak solution of (5.1) if

\[
\int_{\mathbb{R}^N} (\varphi_1(x, u, \nabla L u) \cdot \nabla L \phi_1) \geq \int_{\mathbb{R}^N} f(x, u, v) \phi_1
\]

(5.2)

\[
\int_{\mathbb{R}^N} (\varphi_2(x, v, \nabla L v) \cdot \nabla L \phi_2) \geq \int_{\mathbb{R}^N} g(x, u, v) \phi_2
\]

(5.3)

for all non-negative functions \(\phi_1, \phi_2 \in C^1_0(\mathbb{R}^N)\).

We shall assume that \(Q > p_i > 1\). This restriction is justified by the fact that an analogue of Theorem 2.5 holds. Namely, if \(w\) is a weak solution of the problem (5.4) below and \(p_i > Q\), then \(w\) is a constant. See [9].

Furthermore we shall suppose that a weak Harnack inequality (WH) holds for solutions of

\[
\begin{cases}
- \text{div}(\varphi_i(x, w, \nabla L w)) \geq 0 & \text{on } \mathbb{R}^N, \\
w \geq 0 & \text{on } \mathbb{R}^N,
\end{cases}
\]

(5.4)

Namely, for \(i = 1, 2\) there exists \(\sigma_i > p_i - 1\) and \(c_H\) such that if \(w\) is a weak solution of (5.4), then for any \(R > 0\) we have

\[
\left( \frac{1}{|B_R|} \int_{B_R} w^\sigma_i \right)^{1/\sigma_i} \leq c_H \text{ ess inf}_{B_{R/2}} w. \quad (\text{WH})
\]

Examples of operators for which the weak Harnack inequality holds are given by the following.

**Lemma 5.1** (Weak Harnack Inequality, see [5]) Let \(Q > p_i > 1\). Let \(\varphi_i\) be \(S-p_i-C\) (strongly-\(p_i\)-coercive), that is there exist two constants \(k, h > 0\) such that

\[
(\varphi_i(x, t, \xi) \cdot \xi) \geq h|\xi|^{p_i} \geq k|\varphi_i(x, t, \xi)|^{p_i'} \quad \text{for all } (x, t, \xi) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^l. \quad (S-p_i-C)
\]

Then for any \(\sigma_i \in (0, \frac{Q(p_i-1)}{Q-p_i})\) there exists \(c_H > 0\) such that for any \(u \in W_{L,loc}^{1,p_i}(\mathbb{R}^N)\) weak solution of (5.4), and \(R > 0\), (WH) holds.
The results of the previous Section can be reformulated for the problem (5.1) as follows.

**Theorem 5.2** Let \( f(x,v,u) = f(v) \) with \( f : [0, +\infty[ \to [0, +\infty[ \) be continuous. No extra assumption on \( g \). Then (5.1) has no solutions.

**Corollary 5.3** Let \( f(x,v,u) = f(v), \ g(x,v,u) = g(u) \), with \( f, g : [0, +\infty[ \to [0, +\infty[ \) be continuous. If \((u,v)\) is a solution of (5.1), then (4.3) holds and \( \text{ess inf}_{R^N} u = \text{ess inf}_{R^N} v = 0 \).

**Theorem 5.4** Let \( \mathcal{A}_{p_i} = \mathcal{A}_{p_i}(x, \xi) \) for \( i = 1, 2 \). Let \( f(x,v,u) = f(v) \) with \( f : [0, +\infty[ \to [0, +\infty[ \) be continuous. No extra assumption on \( g \). Let \((u,v)\) be a weak solution of (5.1) and let \( \alpha := \text{ess inf}_{R^N} v \). Then, \( f(\alpha) = 0 \).

**Theorem 5.5** Let \( f(x,v,u) = f(v) \), with \( f : [0, +\infty[ \to [0, +\infty[ \) be a continuous function satisfying \((f_0)\). Let \((u,v)\) be a weak solution of (5.1) such that \( \text{ess inf}_{R^N} v = 0 \). Then there exists \( c > 0 \) such that for \( R \) sufficiently large, the following estimates hold

\[
\text{ess inf}_{B_R} v \leq R^{-p_1/a} \left( \text{ess inf}_{B_R} u \right)^{\frac{p_1-1}{a}}, \quad (5.5)
\]

\[
\int_{B_R} g(x,u,v) \leq c R^{Q-p_2-(p_2-1)p_1/a} \left( \text{ess inf}_{B_R} u \right)^{\frac{p_1-1}{a}}, \quad (5.6)
\]

\[
\int_{B_R} g(x,u,v) \leq c R^{Q-p_2-Q(p_2-1)/a} \left( \int_{A_R/2} f(v) \right)^{\frac{p_2-1}{a}}, \quad (5.7)
\]

Moreover if \( \text{ess inf}_{R^N} u = 0 \) and \( g(x,u,v) = g(u) \) with \( g : [0, +\infty[ \to [0, +\infty[ \) a continuous function satisfying \((g_0)\), then we have

\[
\left( \text{ess inf}_{B_R} u \right)^{(ab-(p_1-1)(p_2-1))/a} \leq c R^{-ap_2-p_1(p_2-1)}, \quad (5.8)
\]

\[
\left( \text{ess inf}_{B_R} v \right)^{(ab-(p_1-1)(p_2-1))/a} \leq c R^{-bp_1-p_2(p_1-1)}, \quad (5.9)
\]

\[
\int_{B_R} f(v)dx \leq c R^{Q-p_1-(p_1-1)(p_2-1)} \left( \int_{A_R/2} f(v)dx \right)^{\frac{(p_1-1)(p_2-1)}{ab}}, \quad (5.10)
\]

**Theorem 5.6** Let \( f(x,v,u) = f(v), \ g(x,u,v) = g(u) \) with \( f, g : [0, +\infty[ \to [0, +\infty[ \) be continuous satisfying \((f_0)\) and \((g_0)\) respectively and assume that (4.11) holds. Then (5.1) has no weak solution \((u,v)\) such that \( \text{ess inf}_{R^N} u = \text{ess inf}_{R^N} v = 0 \).

**Remark 5.7** As a final observation, we point out that most of the results proved in this section hold for systems associated to \((W,p,C)\) operators and power nonlinearities. We refer the interested reader to [2] and [22] for the Euclidean setting, and to [17] for precise formulation and interesting open problems in the Carnot group framework.
6 Appendix

We quote some facts on Carnot groups and refer the interested reader to [4, 16, 17] for more detailed information on this subject.

A Carnot group is a connected, simply connected, nilpotent Lie group $G$ of dimension $N$ with graded Lie algebra $G = V_1 \oplus \cdots \oplus V_r$ such that $[V_i,V_j] = V_{i+j}$ for $i = 1 \ldots r - 1$ and $[V_i,V_r] = 0$. Such an integer $r$ is called the step of the group. We set $l = n_1 = \dim V_1$, $n_2 = \dim V_2, \ldots, n_r = \dim V_r$. A Carnot group $G$ of dimension $N$ can be identified, up to an isomorphism, with the structure of a homogeneous Carnot Group $(\mathbb{R}^N, \circ, \delta_R)$ defined as follows; we identify $G$ with $\mathbb{R}^N$ endowed with a Lie group law $\circ$. We consider $\mathbb{R}^N$ split in $r$ subspaces $\mathbb{R}^N = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_r}$ with $n_1 + n_2 + \cdots + n_r = N$ and $\xi = (\xi^{(1)}, \ldots, \xi^{(r)})$ with $\xi^{(i)} \in \mathbb{R}^{n_i}$. We shall assume that for any $R > 0$ the dilation $\delta_R(\xi) = (R\xi^{(1)}, R^2\xi^{(2)}, \ldots, R^r\xi^{(r)})$ is a Lie group automorphism. The Lie algebra of left-invariant vector fields on $(\mathbb{R}^N, \circ)$ is $G$. For $i = 1, \ldots, n_1 = l$ let $X_i$ be the unique vector field in $G$ that coincides with $\partial/\partial \xi^{(i)}$ at the origin. We require that the Lie algebra generated by $X_1, \ldots, X_l$ is the whole $G$.

We denote with $\nabla_L$ the vector field $\nabla_L := (X_1, \ldots, X_l)^T$ and we call it horizontal vector field and by $\text{div}_L$ the formal adjont on $\nabla_L$, that is (6.2). Moreover, the vector fields $X_1, \ldots, X_l$ are homogeneous of degree 1 with respect to $\nabla_L$ and we call it homogeneous dimension of $G$. The canonical sub-Laplacian on $G$ is the second order differential operator defined by

$$\Delta_G = \sum_{i=1}^l X_i^2 = \text{div}_L(\nabla_L \cdot)$$

and for $p > 1$ the $p$-sub-Laplacian operator is

$$\Delta_{G,p} u := \sum_{i=1}^l X_i (|\nabla_L u|^{p-2} X_i u) = \text{div}_L(|\nabla_L u|^{p-2} \nabla_L u).$$

Since $X_1, \ldots, X_l$ generate the whole $G$, the sub-Laplacian $\Delta_G$ satisfies the Hörmander hypoellipticity condition.

In this paper $\nabla$ and $| \cdot |$ stand respectively for the usual gradient in $\mathbb{R}^N$ and the Euclidean norm.

Let $\mu \in \mathcal{C}(\mathbb{R}^N; \mathbb{R}^l)$ be a matrix $\mu := (\mu_{ij})$, $i = 1, \ldots, l$, $j = 1, \ldots, N$. For $i = 1, \ldots, l$, let $X_i$ and its formal adjoint $X_i^*$ be defined as

$$X_i := \sum_{j=1}^N \mu_{ij}(\xi) \frac{\partial}{\partial \xi_j}, \quad X_i^* := -\sum_{j=1}^N \frac{\partial}{\partial \xi_j}(\mu_{ij}(\xi)\cdot),$$

and let $\nabla_L$ be the vector field defined by $\nabla_L := (X_1, \ldots, X_l)^T = \mu \nabla$ and $\nabla_L^* := (X_1^*, \ldots, X_l^*)^T$. 

18
For any vector field $h = (h_1, \ldots, h_l)^T \in \mathcal{C}^1(\Omega, \mathbb{R}^l)$, we shall use the following notation
\[ \text{div}_L(h) := \text{div}(\mu^T h), \]
that is
\[ \text{div}_L(h) = -\sum_{i=1}^l X^*_i h_i = -\nabla^*_L \cdot h. \tag{6.2} \]

An assumption that we shall made (which actually is an assumption on the matrix $\mu$) is that the operator
\[ \Delta_G u = \text{div}_L(\nabla_L u) \]
is a canonical sub-Laplacian on a Carnot group (see below for a more precise meaning).

The reader, which is not acquainted with these structures, can think to the special case of $\mu = I$, the identity matrix in $\mathbb{R}^N$, that is the usual Laplace operator in Euclidean setting.

A nonnegative continuous function $S : \mathbb{R}^N \to \mathbb{R}_+$ is called a homogeneous norm on $\mathbb{G}$, if $S(\xi^{-1}) = S(\xi)$, $S(\xi) = 0$ if and only if $\xi = 0$, and it is homogeneous of degree 1 with respect to $\delta_R$ (i.e. $S(\delta_R(\xi)) = RS(\xi)$). A homogeneous norm $S$ defines on $\mathbb{G}$ a pseudo-distance defined as $d(\xi, \eta) := S(\xi^{-1} \eta)$, which in general is not a distance. If $S$ and $\tilde{S}$ are two homogeneous norms, then they are equivalent, that is, there exists a constant $C > 0$ such that $C^{-1} S(\xi) \leq \tilde{S}(\xi) \leq CS(\xi)$. Let $S$ be a homogeneous norm, then there exists a constant $C > 0$ such that $C^{-1} |\xi| \leq S(\xi) \leq C |\xi|^{1/r}$, for $S(\xi) \leq 1$. An example of homogeneous norm is $S(\xi) := \left( \sum_{i=1}^r |\xi_i|^{2r_i/\rho_i} \right)^{1/2r_i}$.

Notice that if $S$ is a homogeneous norm differentiable a.e., then $|\nabla_L S|$ is homogeneous of degree 0 with respect to $\delta_R$; hence $|\nabla_L S|$ is bounded.

We notice that in a Carnot group, the Haar measure coincides with the Lebesgue measure.

Special examples of Carnot groups are the Euclidean spaces $\mathbb{R}^Q$. Moreover, if $Q \leq 3$ then any Carnot group is the ordinary Euclidean space $\mathbb{R}^Q$.

The simplest nontrivial example of a Carnot group is the Heisenberg group $\mathbb{H}^1 = \mathbb{R}^3$. For an integer $n \geq 1$, the Heisenberg group $\mathbb{H}^n$ is defined as follows: let $\xi = (\xi^{(1)}, \xi^{(2)})$ with $\xi^{(1)} := (x_1, \ldots, x_n, y_1, \ldots, y_n)$ and $\xi^{(2)} := t$. We endow $\mathbb{R}^{2n+1}$ with the group law $\xi \circ \tilde{\xi} := (\tilde{x} + \tilde{x}, \tilde{y} + \tilde{y}, \tilde{t} + t + 2 \sum_{i=1}^n (\tilde{x}_i \tilde{y}_i - \tilde{x}_i \tilde{y}_i))$. We consider the vector fields
\[ X_i := \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \quad Y_i := \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}, \]
for $i = 1, \ldots, n$, and the associated Heisenberg gradient $\nabla_H := (X_1, \ldots, X_n, Y_1, \ldots, Y_n)^T$. The Kohn Laplacian $\Delta_H$ is then the operator defined by $\Delta_H := \sum_{i=1}^n X_i^2 + Y_i^2$. The family of dilations is given by $\delta_R(\xi) := (Rx, Ry, R^2t)$ with homogeneous dimension $Q = 2n + 2$. In $\mathbb{H}^n$ a canonical homogeneous norm is defined as $|\xi|_H := \left( \left( \sum_{i=1}^n x_i^2 + y_i^2 \right)^2 + t^2 \right)^{1/4}$.

**Acknowledgements**

This work is supported by the Italian MIUR National Research Project: Quasilinear Elliptic Problems and Related Questions.
References

[1] C. Azizieh, Ph. Clément and E. Mitidieri, Existence and a Priori Estimates for Positive Solutions of p-Laplace Systems, *J. Differential Equations* 184 (2002) 422–442.

[2] M.F. Bidaut–Véron and S.I. Pohozaev, Nonexistence results and estimates for some nonlinear elliptic problems, *J. Anal. Math.* 84 (2001) 1–49.

[3] I. Birindelli, I. Capuzzo–Dolcetta and A. Cutrì, A Liouville theorems for semilinear equations on the Heisenberg group, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 14 (1997) 295–308.

[4] A. Bonfiglioli, E. Lanconelli and F. Uguzzoni, Stratified Lie Groups and Potential Theory for their sub-Laplacians, Springer Monogr. Math., 26 New York, Springer-Verlag, 2007.

[5] L. Capogna, D. Danielli and N. Garofalo, Embedding theorem and the Harnack inequality for solutions of nonlinear subelliptic equations, *Comm. Partial Differential Equations* 18 (1993) 1765–1794.

[6] I. Capuzzo–Dolcetta and A. Cutrì, On the Liouville property for sub-Laplacians, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* 25 (1997) 239–256.

[7] G. Caristi, L. D’Ambrosio and E. Mitidieri, Liouville Theorems for some Nonlinear Inequalities, *Proc. Steklov Inst. Math.* 260 (2008) 90–111.

[8] Ph. Clément, J. Fleckinger, E. Mitidieri and F. de Thélin, Existence of Positive Solutions for a Nonvariational Quasilinear Elliptic System, *J. Differential Equations* 166 (2000) 455–477.

[9] L. D’Ambrosio, Liouville Theorems for Anisotropic Quasilinear Inequalities, *Nonlinear Anal.* 70 (2009) 2855–2869.

[10] L. D’Ambrosio, A new critical curve for a class of quasilinear elliptic systems, *Nonlinear Anal.* 78 (2013) 62–78.

[11] L. D’Ambrosio, A. Farina, E. Mitidieri and J. Serrin, Comparison principle, uniqueness and symmetry of entire solutions of quasilinear elliptic equations and inequalities, preprint (2012).

[12] L. D’Ambrosio and E. Mitidieri, A priori estimates, positivity results, and nonexistence theorems for quasilinear degenerate elliptic inequalities, *Adv. Math.* 224 (2010) 967–1020.

[13] L. D’Ambrosio and E. Mitidieri, Nonnegative solutions of some quasilinear elliptic inequalities and applications, *Sb. Math.* 201 (2010) 856–871.
[14] L. D’Ambrosio and E. Mitidieri, A Priori Estimates and Reduction Principles for Quasilinear Elliptic Problems and Applications, Adv. Differential Equations 17 (2012) 935–1000.

[15] L. D’Ambrosio and E. Mitidieri, Uniform bounds of solutions of some quasilinear reaction-diffusion systems, prerint (2012).

[16] G.B. Folland, Subelliptic estimates and function spaces on nilpotent Lie groups, Ark. Mat. 13 (1975), 161–207.

[17] G.B. Folland and E.M. Stein, Hardy spaces on homogeneous groups, Math. Notes, vol. 28, Princeton University Press, Princeton, NJ, 1982.

[18] B. Gidas and J. Spruck, Global and local behavior of positive solutions of nonlinear elliptic equations, Comm. Pure Appl. Math. 34 (1981) 525–598.

[19] B. Gidas and J. Spruck, A priori bounds for positive solutions of nonlinear elliptic equations, Comm. Partial Differential Equations 6 (1981) 883–901.

[20] E. Mitidieri, Non existence of positive solutions of semilinear elliptic systems in \( \mathbb{R}^N \), Differential Integral Equations 9 (1996) 465–479.

[21] E. Mitidieri and S.I. Pohozaev, Absence of positive solutions for systems of quasilinear elliptic equations and inequalities in \( \mathbb{R}^N \), Dokl. Akad. Nauk. 366 (1999) 13–17.

[22] E. Mitidieri and S.I. Pohozaev, A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities, Proc. Steklov Inst. Math. 234 (2001) 1–362.

[23] J. Serrin, Local behavior of solutions of quasi-linear equations, Acta Math. 111 (1964) 247–302.

[24] J. Serrin and H. Zou, Cauchy-Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities, Acta Math. 189 (2002) 79–142.

[25] N.S. Trudinger, On Harnack type inequalities and their application to quasilinear elliptic equations, Comm. Pure Appl. Math. 20 (1967) 721–747.

[26] N.S. Trudinger and X.J. Wang, On the weak continuity of elliptic operators and applications to potential theory, Amer. J. Math. 124 (2002) 369–410.

[27] H. Zou, A priori estimates and existence for quasilinear elliptic equations, Calc. Var. Partial Differential Equations 33 (2008) 417–437.