HODGE-RIEMANN RELATIONS FOR POTTS MODEL PARTITION FUNCTIONS

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Abstract. We prove that the Hessians of nonzero partial derivatives of the (homogenous) multivariate Tutte polynomial of any matroid have exactly one positive eigenvalue on the positive orthant when $0 < q \leq 1$. Consequences are proofs of the strongest conjecture of Mason and negative dependence properties for $q$-state Potts model partition functions.

1. Introduction

Several conjectures have been made regarding unimodality and log-concavity of sequences arising in matroid theory. Only recently have some of these been solved using combinatorial Hodge theory [AHK18, HSW18]. A conjecture that has resisted the approach of [AHK18] is the strongest conjecture of Mason regarding independent sets in a matroid [Mas72]. The purpose of this paper is to give a self-contained proof of the strongest conjecture avoiding, but inspired by, Hodge theory. We prove that the Hessian of the homogenous multivariate Tutte polynomial (or the $q$-state Potts model partition function) of a matroid has exactly one positive eigenvalue on the positive orthant when $0 < q \leq 1$. In a forthcoming paper we will take a more general approach and see that the results proved in this paper fit into a wider context.

Let $n$ be an integer larger than 1, and let $M$ be a matroid on $[n] = \{1, \ldots, n\}$. Mason [Mas72] offered the following three conjectures of increasing strength. Several authors studied correlations in matroid theory partly in pursuit of these conjectures [SW75, Wag08, BBL09, KN10, KN11].

Conjecture. For any $n$-element matroid $M$ and any positive integer $k$,

1. $I_k(M)^2 \geq I_{k-1}(M)I_{k+1}(M),$

2. $I_k(M)^2 \geq \frac{k+1}{k}I_{k-1}(M)I_{k+1}(M),$

3. $I_k(M)^2 \geq \frac{k+1}{k}\frac{n-k+1}{n-k}I_{k-1}(M)I_{k+1}(M),$

where $I_k(M)$ is the number of $k$-element independent sets of $M$.

1In related forthcoming papers, Anari, Liu, Gharan and Vinzant have independently developed methods that overlap with our work. In particular, they also prove Mason’s conjecture (3).
Conjecture (1) was proved in [AHK18], and Conjecture (2) was proved in [HSW18]. Note that Conjecture (3) may be written
\[
\frac{I_k(M)^2}{\binom{n}{k}^2} \geq \frac{I_{k+1}(M)}{\binom{n}{k+1}} \cdot \frac{I_{k-1}(M)}{\binom{n}{k-1}},
\]
and the equality holds when all \((k + 1)\)-subsets of \([n]\) are independent in \(M\). Conjecture (3) is known to hold when \(n\) is at most 11 or \(k\) is at most 5 [KN11]. We refer to [Sey75, Dow80, Mah85, Zha85, HK12, HS89, Len13] for other partial results. We prove Conjecture (3) in Corollary 7 by uncovering concavity properties of the multivariate Tutte polynomial of \(M\).

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2. The Hessian of the Multivariate Tutte Polynomial

Let \(\text{rk}_M : 2^{[n]} \to \mathbb{Z}_{\geq 0}\) be the rank function of \(M\). For a nonnegative integer \(k\) and a positive real parameter \(q\), consider the degree \(k\) homogeneous polynomial in \(n\) variables
\[
Z_k^M(q, w_1, \ldots, w_n) = \sum_A q^{-\text{rk}_M(A)} \prod_{i \in A} w_i,
\]
where the sum is over all \(k\)-element subsets \(A\) of \([n]\). We define the homogeneous multivariate Tutte polynomial of \(M\) by
\[
Z^M(q, w) = \sum_{k=0}^n Z_{n-k}^M w_0^k w_1^{n-k},
\]
which is a homogeneous polynomial of degree \(n\) in \(w = (w_0, w_1, \ldots, w_n)\). When \(w_0 = 1\), the function \(Z^M\) agrees with the partition function of the \(q\)-state Potts model, or the random cluster model [Pem00, Sok05, Gri06]. The Hessian of \(Z^M\) is the matrix
\[
\mathcal{H}_{Z^M}(w) = \left( \frac{\partial^2 Z^M}{\partial w_i \partial w_j} \right)_{i,j=0}^n.
\]
When \(w \in \mathbb{R}_{>0}^{n+1}\), the largest eigenvalue of \(\mathcal{H}_{Z^M}\) is simple and positive by the Perron-Frobenius theorem. We prove the following analogue of the Hodge-Riemann relations for \(Z^M\).

Theorem 1. The Hessian of \(Z^M\) has exactly one positive eigenvalue for all \(w \in \mathbb{R}_{>0}^{n+1}\) and \(0 < q \leq 1\).

It follows that the Hessian of \(\log Z^M\) is negative semidefinite on \(\mathbb{R}_{>0}^{n+1}\), and hence \(\log Z^M\) is concave on \(\mathbb{R}_{>0}^{n+1}\) when \(0 < q \leq 1\) [AGV, Lemma 2.7]. We deduce Theorem 1 from the following more precise statement. Let \(c = (c_0, c_1, \ldots, c_n)\) be a sequence of \(n + 1\) positive real numbers. We say that \(c\) is strictly log-concave if
\[
c_m^2 > c_{m-1} c_{m+1} \text{ for } 0 < m < n.
\]
For any strictly log-concave sequence \( c \) as above, set
\[
Z_{M,c} = \sum_{k=0}^{n} c_{n-k} Z_{M}^{n-k} w_0^k.
\]

For \( \alpha \in \mathbb{Z}_{\geq 0}^{n+1} \), we write \( \partial_i = \frac{\partial}{\partial w_i} \) and \( \partial^\alpha = \partial_0^{\alpha_0} \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} \).

**Theorem 2.** If \( \partial^\alpha Z_{M,c} \) is not identically zero, then

(i) the Hessian of \( \partial^\alpha Z_{M,c} \) is nonsingular for all \( w \in \mathbb{R}_{>0}^{n+1} \) and \( 0 < q \leq 1 \), and

(ii) the Hessian of \( \partial^\alpha Z_{M,c} \) has exactly one positive eigenvalue for all \( w \in \mathbb{R}_{>0}^{n+1} \) and \( 0 < q \leq 1 \).

Theorem 1 can be deduced from Theorem 2 for \( \alpha = 0 \) by approximating the constant sequence 1 by strictly log-concave sequences. Theorem 2 will be proved by induction on the degree of \( \partial^\alpha Z_{M,c} \). For undefined matroid terminologies, see [Oxl11].

**Lemma 3.** Let \( A = (a_{ij})_{i,j=1}^{n} \) be a symmetric matrix with at least one positive eigenvalue. The following statements are equivalent.

1. \( A \) has exactly one positive eigenvalue.
2. For any \( u, v \in \mathbb{R}^n \) with \( u^T A u > 0 \), \( (u^T A v)^2 \geq (u^T A u)(v^T A v) \).
3. There is a vector \( u \in \mathbb{R}^n \) with \( u^T A u > 0 \), such that \( (u^T A v)^2 \geq (u^T A u)(v^T A v) \) for all \( v \in \mathbb{R}^n \).

**Proof.** Since \( A \) has a positive eigenvalue, (2) implies (3).

If (3) holds, then \( A \) is negative semidefinite on the hyperplane \( \{ v \in \mathbb{R}^n \mid u^T A v = 0 \} \). Since \( A \) has a positive eigenvalue, Cauchy’s interlacing theorem implies (1).

Assume (1), \( u^T A u > 0 \), and that \( u \) and \( v \) are linearly independent. Let \( Q(w) = w^T A w \). The discriminant \( \Delta \) of the polynomial \( t \mapsto Q(u + tv) = (u^T A v)^2 - (u^T A u)(v^T A v) \). If \( \Delta < 0 \), then \( Q \) is positive on the plane spanned by \( u \) and \( v \). This contradicts the fact that \( A \) has exactly one positive eigenvalue, by Cauchy’s interlacing theorem. Hence \( \Delta \geq 0 \), and (2) follows. \( \square \)

**Lemma 4.** Theorem 2 holds when the degree of \( \partial^\alpha Z_{M,c} \) is two.

**Proof.** It is enough to consider the case \( \partial^\alpha = \partial_{0}^{\alpha_{0} - 2 - k} \prod_{i \in S} \partial_i \), where \( S \) is a \( k \)-element subset of \( E = [n] \). Note that \( \partial_i Z_{M}^{i} = q^{-r(i)}Z_{M/i}^{i-1} \), where \( M/i \) is the contraction of \( M \) by \( i \). We need to prove that the Hessian of the quadratic form
\[
Q = \frac{\sum_{i \in S} q^{-r(i)}}{(n-k-2)!} \partial^\alpha Z_{M,c} = c_k \binom{n-k}{2} w_0^2 + (n-k-1)c_{k+1}Z_{M/S}^1(w)w_0 + c_{k+2}Z_{M/S}^2(w)
\]
is nonsingular and has exactly one positive eigenvalue. By contraction, we may assume that \( S = \emptyset \) and \( k = 0 \). Write \( Q(w) = w^T A w \), where \( 2A = \mathcal{K}_Q \). We prove that the inequality in the third statement of Lemma 3 is satisfied with strict inequality whenever \( w = (1,0,\ldots,0)^T \), and
Again the proof reduces to the case when \( e \) proved that \( Z \) of variables \( w \) of positive eigenvalue. In other words, we will prove, \[ Z_M^1(w)^2 > 2t \frac{n}{n-1} Z_M^2(w) \] for all \( w \in \mathbb{R}^n \setminus \{0\} \), where \( t = \frac{c_0 c_2}{c_1} \). (a)

Let \( E_0 \) be the set of loops in \( E \), and let \( E_1, E_2, \ldots, E_\ell \) be the parallel classes of \( M \). By the change of variables \( w_j \rightarrow qw_j \) for all non-loops \( j \), we get \( Z_M^1 = e_1(E) \) and

\[ Z_M^2 = e_2(E) - (1-q)(e_2(E_1) + \cdots + e_2(E_\ell)), \] where \( e_k(U) \) denotes the degree \( k \) elementary symmetric polynomial in the variables indexed by \( U \subseteq E \).

We prove (a) for \( t = 1 \) with \( > \) replaced by \( \geq \). Moreover, we prove that if \( Z_M^1(w) = 0 \) for \( w \neq 0 \), then \( Z_M^2(w) < 0 \). The inequality (a) for \( t = \frac{c_0 c_2}{c_1} \) then follows since \( 0 < \frac{c_0 c_2}{c_1} < 1 \). Note that for \( q = 1 \) the desired inequality is an instance of the Cauchy-Schwarz inequality:

\[ (w_1 + \cdots + w_n)^2 \leq n \left( w_1^2 + \cdots + w_n^2 \right), \quad w \in \mathbb{R}^n. \] (c)

By (b), the inequality therefore reduces to the case when \( e_2(E_1) + \cdots + e_2(E_\ell) < 0 \). By monotonicity in \( q \) it suffices to consider the case \( q = 0 \). Then the inequality reduces to

\[ e_1(E)^2 \leq n \sum_{i=1}^\ell e_1(E_i)^2 + n \sum_{j \in E_0} w_j^2, \]

which follows from (c). Suppose \( Z_M^1(w) = 0 \) for \( w \neq 0 \). It remains to prove \( Z_M^2(w) < 0 \). Since \( e_1(E) = 0 \) and \( w \neq 0 \), it follows from the identity \( e_1(E)^2 = 2e_2(E) + \sum_{i=1}^n w_i^2 \) that \( e_2(E) < 0 \). Again the proof reduces to the case when \( e_2(E_1) + \cdots + e_2(E_\ell) < 0 \), by (b). We have already proved that \( Z_M^2(w) \leq 0 \) when \( q = 0 \). But then \( Z_M^2(w) < 0 \) when \( 0 < q \leq 1 \), by (b). This completes the proof of the lemma.

We prepare the proof of Theorem 2 with a lemma.

**Lemma 5.** Let \( F \) be a degree \( d \) homogeneous polynomial in \( \mathbb{R}[w_0, w_1, \ldots, w_n] \). If \( w \in \mathbb{R}_{\geq 0}^{n+1} \) and \( \mathcal{H}_{\partial_i F}(w) \) has exactly one positive eigenvalue for each \( i = 0, 1, \ldots, n \), then

\[ \ker \mathcal{H}_F(w) = \bigcap_{i=0}^n \ker \mathcal{H}_{\partial_i F}(w). \]

**Proof.** We fix \( w \in \mathbb{R}_{\geq 0}^{n+1} \) and write \( \mathcal{H}_F \) for \( \mathcal{H}_F(w) \). We may suppose \( d \geq 3 \). By Euler’s formula for homogeneous functions,

\[ (d-2) \mathcal{H}_F = \sum_{i=0}^n w_i \mathcal{H}_{\partial_i F}, \]

and hence the kernel of \( \mathcal{H}_F \) contains the intersection of the kernels of \( \mathcal{H}_{\partial_i F} \).

For the other inclusion, let \( z \) be a vector in the kernel of \( \mathcal{H}_F \). By Euler’s formula again,

\[ (d-2) e_i^T \mathcal{H}_F = w^T \mathcal{H}_{\partial_i F}, \]
where $e_i$ is the $i$-th standard basis vector in $\mathbb{R}^{n+1}$, and hence $w^T \mathcal{H}_{\partial_i F} z = 0$. We have $w^T \mathcal{H}_{\partial_i F} w > 0$ because $w \in \mathbb{R}_{>0}^{n+1}$ and $\partial_i F$ has nonnegative coefficients. It follows that $\mathcal{H}_{\partial_i F}$ is negative semidefinite on the kernel of $w^T \mathcal{H}_{\partial_i F}$, by e.g. Lemma 3. In particular,

$$z^T \mathcal{H}_{\partial_i F} z \leq 0,$$

with equality if and only if $\mathcal{H}_{\partial_i F} z = 0$.

To conclude, we write zero as the positive linear combination

$$0 = (d - 2) \left( z^T \mathcal{H}_F z \right) = \sum_{i=0}^{n} y_i \left( z^T \mathcal{H}_{\partial_i F} z \right).$$

Since every summand in the right-hand side is non-positive by the previous analysis, we must have $z^T \mathcal{H}_{\partial_i F} z = 0$ for every $i$, and hence $\mathcal{H}_{\partial_i F} z = 0$ for each $i$. \hfill \Box

Proof of Theorem 2. The proof is by induction on the degree $m$ of $F = \partial^m Z_{M,c}$. The case when $m = 2$ is Lemma 4. By relabeling the variables we may assume that $w_0, w_1, \ldots, w_n$ are the active variables in $F$. Suppose the theorem is true when the degree of $F$ is at most $m$, where $m \geq 2$.

Suppose $F$ has degree $m + 1$. We first prove (i). By induction, the Hessian of any derivative of $F$ is non-singular and has exactly one positive eigenvalue. Hence (i) for $F$ follows from Lemma 5.

When $q = 1$, $F$ has the form

$$F = (\ell - 1)! c_{\ell - 1} e_{m+1}([n]) + \ell! c_\ell e_m([n]) w_0 + \frac{1}{2} (\ell + 1)! c_{\ell + 1} e_{m-1}([n]) w_0^2 + \cdots.$$

If we choose $c$ so that $c_i = 0$ unless $i \in \{\ell - 1, \ell\}$, $c_{\ell - 1} = 1/(\ell - 1)!$ and $c_\ell = 1/\ell!$, then $F$ is equal to the degree $m + 1$ elementary symmetric polynomial in $w_0, w_1, \ldots, w_n$. The Hessian of $F$ evaluated at the all ones vector is equal to a constant multiple of the matrix $J_{n+1}$, which has all diagonal entries equal to zero and all off-diagonal entries equal to 1. Clearly $J_{n+1}$ is nonsingular and has exactly one positive eigenvalue. We may approximate $c$ with a strictly log-concave positive sequence. This implies that that there is a strictly log-concave sequence $c$ for which the Hessian of $F$ is nonsingular and has exactly one positive eigenvalue when $w = (1, \ldots, 1)^T$ and $q = 1$. Since (i) holds for all $0 < q \leq 1$ and $w \in \mathbb{R}_{>0}^{n+1}$, and (ii) holds for at least one choice of the parameters, by continuity of the eigenvalues, (ii) holds for all $0 < q \leq 1$ and $w \in \mathbb{R}_{>0}^{n+1}$. \hfill \Box

Theorems 1 and 2 suggest that there is an algebraic structure satisfying the Poincaré duality and the hard Lefschetz theorem whose degree 1 Hodge-Riemann form is given by the Hessian of $Z_M$. We refer to [Huh18] for a discussion of the one positive eigenvalue condition and the Hodge-Riemann relations.

3. Consequences

We collect some corollaries of Theorem 2. It has been conjectured that the $q$-state Potts model should exhibit negative dependence properties when $0 < q \leq 1$, see [Pem00, Sok05, Gri06,
Proof.\ Let \( f \) denote the homogeneous polynomial in \( n \) variables whose lattice of flats is isomorphic to that of \( M \). By Euler’s formula for homogeneous functions,

\[
\left( \frac{\partial f}{\partial w} \right)_0 = m \left( \frac{\partial f}{\partial w} \right)_0 = \left( \frac{\partial f}{\partial w} \right)_0 = m \left( \frac{\partial f}{\partial w} \right)_0 .
\]

The proof follows by continuity, letting \( w_0 = 0 \).

Let \( \mathcal{I}_M^m \) be the collection of independent sets of \( M \) of size \( m \). The \( m \)-th generating function of \( M \) is the homogeneous polynomial in \( n \) variables

\[
f_m^M(w) = \sum_{I \in \mathcal{I}_M^m} \prod_{i \in I} w_i , \quad w = (w_1, \ldots, w_n) .
\]

Note that \( f_m^M(1, \ldots, 1) \) is the number of independent sets of \( M \) of size \( m \).

**Corollary 6.** For any \( 0 < m < n \) and any \( 0 < q \leq 1 \), we have

\[
\frac{Z_M^m(q, w)^2}{\binom{n}{m}} \geq \frac{Z_M^{m+1}(q, w)}{\binom{n}{m+1}} \frac{Z_M^{m-1}(q, w)}{\binom{n}{m-1}} , \quad \text{for all} \ w \in \mathbb{R}_{\geq 0}^n .
\]

**Proof.** The proof follows by Corollary 6.

Let \( \ell \) be the number of rank one flats of \( M \). The simplification \( \overline{M} \) of \( M \) is a matroid on \( [\ell] \) whose lattice of flats is isomorphic to that of \( M \) [Oxl11, Section 1.7]. Applying Corollary 7 to the simplification \( \overline{M} \), we get the stronger inequality

\[
\frac{f_M^m(w)^2}{f_M^{m+1}(w)f_M^{m-1}(w)} \geq \frac{\binom{\ell}{m}}{\binom{\ell}{m+1}} \frac{\binom{n}{m-1}}{\binom{\ell}{m}} \quad \text{for all} \ w \in \mathbb{R}_{\geq 0}^n .
\]

As a closing remark, we note that Corollary 7 disproves a conjecture of Pemantle [Pem00, Conjecture 6]. In the language of [Pem00], Corollary 7 proves that the uniform measure concentrated on the independent sets of a matroid is \( \text{ULC}_+ \). Pemantle [Pem00] conjectured that \( \text{ULC}_+ \) implies \( \text{CNA}_+ \). It is known however that there are matroids that are not even negatively correlated (see [HSW18]), and hence Pemantle’s conjecture does not hold.
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