On the roots of a hyperbolic polynomial pencil

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Abstract. Let $\nu_0(t), \nu_1(t), \ldots, \nu_n(t)$ be the roots of the equation $R(z) = t$, where $R(z)$ be a rational function of the form

$$R(z) = z - \sum_{k=1}^{n} \frac{\alpha_k}{z - \mu_k},$$

$\mu_k$ are pairwise different real numbers, $\alpha_k > 0$, $1 \leq k \leq n$. Then for each real $\xi$, the function $e^{\xi \nu_0(t)} + e^{\xi \nu_1(t)} + \cdots + e^{\xi \nu_n(t)}$ is exponentially convex on the interval $-\infty < t < \infty$.

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1. Roots of the equation $R(z) = t$ as functions of $t$.

In the present paper we discuss questions related to properties of roots of the equation

$$R(z) = t$$

as functions of the parameter $t \in \mathbb{C}$, where $R$ is a rational function of the form

$$R(z) = z - \sum_{1 \leq k \leq n} \frac{\alpha_k}{z - \mu_k},$$

$\mu_k$ are pairwise different real numbers, $\alpha_k > 0$, $1 \leq k \leq n$. We adhere to the enumeration agreement

$$\mu_1 > \mu_2 > \cdots > \mu_n.$$  

The function $R$ is representable in the form

$$R(z) = \frac{P(z)}{Q(z)},$$

\footnote{We assume that $n \geq 1$.}
where
\[ Q(z) = (z - \mu_1) \cdot (z - \mu_2) \cdot \cdots \cdot (z - \mu_n), \quad (1.5) \]
\[ P(z) \overset{\text{def}}{=} R(z) \cdot Q(z) \quad (1.6) \]
are monic polynomials of degrees
\[ \deg P = n + 1, \quad \deg Q = n. \quad (1.7) \]
Since \( P(\mu_k) = -\alpha_k Q'(\mu_k) \neq 0 \), the polynomials \( P \) and \( Q \) have no common roots. Thus the ratio in the right hand side of (1.4) is irreducible. The equation (1.4) is equivalent to the equation
\[ P(z) - tQ(z) = 0. \quad (1.8) \]
Since the polynomial \( P(z) - tQ(z) \) is of degree \( n + 1 \), the latter equation has \( n + 1 \) roots for each \( t \in \mathbb{C} \).

The function \( R \) possess the property
\[ \Im R(z)/\Im z > 0 \quad \text{if} \quad \Im z \neq 0. \quad (1.9) \]
Therefore if \( \Im t > 0 \), all roots of the equation (1.1), which is equivalent to the equation (1.8), are located in the half-plane \( \Im z > 0 \). Some of these roots may be multiple.

However if \( t \) is real, all roots of the equation (1.1) are real and simple, i.e. of multiplicity one. Thus for real \( t \), the equation (1.1) has \( n + 1 \) pairwise different real roots \( \nu_k(t) : \nu_0(t) > \nu_1(t) > \cdots > \nu_{n-1}(t) > \nu_n(t) \). Moreover for each real \( t \), the poles \( \mu_k \) of the function \( R \) and the roots \( \nu_k(t) \) of the equation (1.1) are interspersed:
\[ \nu_0(t) > \mu_1 > \nu_1(t) > \mu_2 > \nu_2(t) > \cdots > \nu_{n-1}(t) > \mu_n > \nu_n(t), \quad \forall t \in \mathbb{R}. \quad (1.10) \]
In particular for \( t = 0 \), the roots \( \nu_k(0) = \lambda_k \) of the equation (1.1) are the roots of the polynomial \( P \):
\[ P(z) = (z - \lambda_0) \cdot (z - \lambda_1) \cdot \cdots \cdot (z - \lambda_n), \quad (1.11) \]
\[ \lambda_0 > \mu_1 > \lambda_1 > \mu_2 > \lambda_2 > \cdots > \lambda_{n-1} > \mu_n > \lambda_n. \quad (1.12) \]
Since \( R'(x) > 0 \) for \( x \in \mathbb{R}, x \neq \mu_1, \ldots, \mu_n \), each of the functions \( \nu_k(t), k = 0, 1, \ldots, n \), can be continued as a single valued holomorphic function to some neighborhood of \( \mathbb{R} \). However the functions \( \nu_k(t) \) can not be continued as single-valued analytic functions to the whole complex \( t \)-plane. According to (1.4),
\[ R'(z) = \frac{P'(z)Q(z) - Q'(z)P(z)}{Q^2(z)}. \quad (1.13) \]
The polynomial \( P'Q - Q'P \) is of degree \( 2n \) and is strictly positive on the real axis. Therefore this polynomial has \( n \) roots \( \zeta_1, \ldots, \zeta_n \) in the upper half-plane \( \Im(z) > 0 \) and \( n \) roots \( \overline{\zeta}_1, \ldots, \overline{\zeta}_n \) in the lower half-plane \( \Im(z) < 0 \). (Not all roots \( \zeta_1, \ldots, \zeta_n \) must be different.) The points \( \zeta_1, \ldots, \zeta_n \) and \( \overline{\zeta}_1, \ldots, \overline{\zeta}_n \) are the critical points of the function \( R \): \( R'(\zeta_k) = 0, R'((\zeta_k) = 0, 1 \leq k \leq n \). The
critical values $t_k = R(\zeta_k)$, $\overline{t_k} = R(\overline{\zeta_k})$, $1 \leq k \leq n$, of the function $R$ are the ramification points of the function $\nu(t)$:

$$R(\nu(t)) = t \quad \text{(1.14)}$$

(Even if the critical points $\zeta'$ and $\zeta''$ of $R$ are different, the critical values $R(\zeta')$ and $R(\zeta'')$ may coincide.) We denote the set of critical values of the function $R$ by $\mathcal{V}$:

$$\mathcal{V} = \mathcal{V}^+ \cup \mathcal{V}^-,$$

$$\mathcal{V}^+ = \{t_1, \ldots, t_n\}, \quad \mathcal{V}^- = \{\overline{t_1}, \ldots, \overline{t_n}\}. \quad \text{(1.15)}$$

Not all values $t_1, \ldots, t_n$ must be different. However $\mathcal{V} \neq \emptyset$. In view of (1.9), $\text{Im} t_k > 0$, $1 \leq k \leq n$. So

$$\mathcal{V}^+ \subset \{t \in \mathbb{C} : \text{Im} t > 0\}, \quad \mathcal{V}^- \subset \{t \in \mathbb{C} : \text{Im} t < 0\}. \quad \text{(1.16)}$$

Ler $G$ be an arbitrary simply connected domain in the $t$-plane which does not intersect the set $\mathcal{V}$. Then the roots of equation (1.1) are pairwise different for each $t \in G$. We can enumerate these roots, say $\nu_0(t), \nu_1(t), \ldots \nu_n(t)$, such that all functions $\nu_k(t)$ are holomorphic in $G$.

The strip $S_h$,

$$S_h = \{t \in \mathbb{C} : |\text{Im} t < h|\}, \quad \text{where} \quad h = \min_{1 \leq k \leq n} \text{Im} t_k, \quad \text{(1.17)}$$

does not intersect the set $\mathcal{V}$. So $n+1$ single valued holomorphic branches of the function $\nu(t)$, (1.14), are defined in the strip $S_h$. We choose such enumeration of these branches which agrees with the enumeration (1.10) on $\mathbb{R}$. The set $\{}$.

Let $L$ be a Jordan curve in $\mathbb{C}$ which possess the properties:

1. $L \subset \{t \in \mathbb{C} : \text{Im} t > -h\}$;
2. The set $\mathcal{V}^+$ is contained in the interior of the curve $L$.
3. $L \cap \mathbb{R} \neq \emptyset$.

Let us choose and fix a point $t_0 \in L \cap \mathbb{R}$. We consider the curve $L$ as a loop with base point $t_0$ oriented counterclockwise. Each branch $\nu_k$ of the function $\nu(t)$, (1.14), can be continued analytically along $L$ from a small neighborhood of the point $t_0$ considered as an initial point of the loop $L$ to the same neighborhood of the point $t_0$ but considered as a final point of this loop. Continuing analytically the branch indexed as $\nu_k$, we come to the branch indexed as $\nu_{k-1}$, $k = 0, 1, \ldots, n$. (We put $\nu_{-1} \overset{\text{def}}{=} \nu_n$.)

From (1.6) and (1.2) it follows that the polynomial $P$ is representable in the form

$$P(z) = zQ(z) - \sum_{k=1}^{n} \alpha_k Q_k(z), \quad \text{(1.18a)}$$

where

$$Q_k(z) = Q(z)/(z - \mu_k), \quad k = 1, 2, \ldots, n. \quad \text{(1.18b)}$$
2. Determinant representation of the polynomial pencil
\( P(z) - tQ(z) \).

The polynomial pencil \( P \) is hyperbolic: for each real \( t \), all roots of the equation (1.8) are real.

Using (1.18), we represent the polynomial \( P(z) - tQ(z) \) as the characteristic polynomial \( \det(zI - (A + tB)) \) of some matrix pencil, where \( A \) and \( B \) are self-adjoint \((n + 1) \times (n + 1)\) matrices, \( \text{rank} B = 1 \). We present these matrices explicitly.

**Lemma 2.1.** Let \( A = \|a_{p,q}\| \) and \( B = \|b_{p,q}\| \), \( 0 \leq p, q \leq n \), be \((n + 1) \times (n + 1)\) matrices with the entries
\[
a_{0,0} = 0, \quad a_{p,p} = \mu_p \text{ for } p = 1, 2, \ldots, n,
\]
\[
a_{p,q} = 0 \text{ for } p = 1, 2, \ldots, n, \quad q = 1, 2, \ldots, n, \quad p \neq q,
\]
and
\[
b_{0,0} = 1, \quad \text{all other } b_{p,q} \text{ vanish.}
\]

Then the equality
\[
\det(zI - A - tB) = (z - t) \cdot Q(z) - \sum_{k=1}^{n} |a_{k,n+1}|^2 Q_k(z).
\]
holds.

**Proof.** The matrix \( zI - (A + tB) \) is of the form
\[
zI - (A + tB) = \begin{bmatrix}
z - t & -a_{0,1} & -a_{0,2} & \cdots & -a_{0,n-1} & -a_{0,n} \\
-a_{0,1} & z - \mu_1 & 0 & \cdots & 0 & 0 \\
-a_{0,2} & 0 & z - \mu_2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-a_{0,n-1} & 0 & 0 & \cdots & z - \mu_{n-1} & 0 \\
-a_{0,n} & 0 & 0 & \cdots & 0 & z - \mu_n
\end{bmatrix}
\]

We compute the determinant of this matrix using the cofactor formula. \( \square \)

Comparing (1.18) and (2.3), we see that if the conditions
\[
|a_{0,p}|^2 = \alpha_p, \quad p = 1, 2, \ldots, n
\]
are satisfied, then the equality
\[
P(z) - tQ(z) = \det(zI - A - tB)
\]
holds for every \( z \in \mathbb{C}, t \in \mathbb{C} \).

The following result is an immediate consequence of Lemma 2.1.
Theorem 2.2. Let \( R \) be a function of the form (1.2), where \( \mu_1, \mu_2, \ldots, \mu_n \) are pairwise different real numbers and \( \alpha_1, \alpha_2, \ldots, \alpha_n \) are positive numbers. Let \( Q \) and \( P \) be the polynomials related to the function \( R \) by the equalities (1.5) and (1.18).

Then the pencil of polynomials \( P(z) - tQ(z) \) is representable as the characteristic polynomial of the matrix pencil \( A + tB \), i.e. the equality (2.5) holds for every \( z \in \mathbb{C}, t \in \mathbb{C} \), where \( B \) is the matrix with the entries (2.2), and the entries of the matrix \( A \) are defined by by (2.1) with

\[
a_{0,p} = \sqrt{\alpha_p} \omega_p, \quad p = 1, 2, \ldots, n,
\]

\( \omega_p \) are arbitrary \(^{2}\) complex numbers which absolute value equals one:

\[
|\omega_p| = 1, \quad p = 1, 2, \ldots, n.
\]

Corollary 2.3. Let \( R, A, B \) be the same that in Theorem 2.2. For each \( t \in \mathbb{C} \), the roots \( \nu_0(t), \nu_0(t), \ldots, \nu_n(t) \) of the equation (1.2) are the eigenvalues of the matrix \( A + tB \).

Lemma 2.4. Let \( R, A, B \) be the same that in Theorem 2.2, \( \nu_0(t), \nu_0(t), \ldots, \nu_n(t) \) be the roots of the equation (1.2) and \( h(z) \) be an entire function. Then the equality

\[
\sum_{k=0}^{n} h(\nu_k(t)) = \text{trace} \{h(A + tB)\}
\]

holds for every \( t \in \mathbb{C} \).

Proof. We refer to Corollary 2.3. If \( \nu \) is an eigenvalue of some square matrix \( M \), then \( h(\nu) \) is an eigenvalue of the matrix \( h(M) \). In (2.8), we interpret the trace of the matrix \( h(A + tB) \) as its spectral trace, that is as the sum of all its eigenvalues. \( \square \)

3. Exponentially convex functions.

Definition 3.1. A function \( f(t) \) on the interval \( a < t < b \) is said to be belongs to the class \( W_{a,b} \) if \( f \) is continuous on \((a, b)\) and if all forms

\[
\sum_{r,s=1}^{N} f(t_r + t_s) \zeta_r \zeta_s \quad (N = 1, 2, 3, \ldots)
\]

are non-negative for every choice of complex numbers \( \zeta_1, \zeta_2, \ldots, \zeta_N \) and for every choice of real numbers \( t_1, t_2, \ldots, t_N \) assuming that all sums \( t_r + t_s \) are within the interval \((a, b)\).

The class \( W_{a,b} \) was introduced by S.N.Bernstein, \([\text{Be}]\), see §15 there. Somewhat later, D.V.Widder also introduced the class \( W_{a,b} \) and studied it. S.N.Bernstein call functions \( f(x) \in W_{a,b} \) exponentially convex.

Properties of the class of exponentially convex functions.

P 1. If \( f(t) \in W_{a,b} \) and \( c \geq 0 \) is a nonnegative constant, then \( cf(t) \in W_{a,b} \).

\(^{2}\)We will use the freedom in choosing \( \omega_p \) to prescribe signs \( \pm \) to the entries \( a_{0,p} \).
P 2. If \( f_1(t) \in W_{a,b} \) and \( f_2(t) \in W_{a,b} \), then \( f_1(t) + f_2(t) \in W_{a,b} \).

P 3. If \( f_1(t) \in W_{a,b} \) and \( f_2(t) \in W_{a,b} \), then \( f_1(t) \cdot f_2(t) \in W_{a,b} \).

P 4. Let \( \{f_n(t)\}_{1 \leq n < \infty} \) be a sequence of functions from the class \( W_{a,b} \). We assume that for each \( t \in (a, b) \) there exists the limit \( f(t) = \lim_{n \to \infty} f_n(t) \), and that \( f(t) < \infty \) \( \forall t \in (a, b) \). Then \( f(t) \in W_{a,b} \).

From the functional equation for the exponential function it follows that for each real number \( u \), for every choice of real numbers \( t_1, t_2, \ldots, t_N \) and complex numbers \( \zeta_1, \zeta_2, \ldots, \zeta_N \), the equality holds

\[
\sum_{r,s=1}^N e^{(t_r+t_s)\zeta_r\zeta_s} = \left| \sum_{p=1}^N e^{t_p\zeta_p} \right|^2 \geq 0. \tag{3.2}
\]

The relation (3.2) can be formulated as

**Lemma 3.2.** For each real number \( \xi \), the function \( e^{t\xi} \) of the variable \( t \) belongs to the class \( W_{-\infty, \infty} \).

The term exponentially convex function is justified by an integral representation for any function \( f(t) \in W_{a,b} \).

**Theorem 3.3 (The representation theorem).** In order the representation

\[
f(x) = \int_{\xi \in (-\infty, \infty)} e^{\xi x} \sigma(d\xi) \quad (a < x < b) \tag{3.3}
\]

be valid, where \( \sigma(d\xi) \) is a non-negative measure, it is necessary and sufficient that \( f(x) \in W_{a,b} \).

The proof of the Representation Theorem can be found in [A], Theorem 5.5.4, and in [W], Theorem 21.

**Corollary 3.4.** The representation (3.3) shows that \( f(x) \) is the value of a function \( f(z) \) holomorphic in the strip \( a < \text{Re} \, z < b \).

4. Herbert Stahl’s Theorem.

In the paper [BMV] a conjecture was formulated which now is commonly known as the BMV conjecture:

**The BMV Conjecture.** Let \( U \) and \( V \) be Hermitian matrices of size \( l \times l \). Then the function

\[
\varphi(t) = \text{trace} \left\{ e^{U+tV} \right\} \tag{4.1}
\]

of the variable \( t \) belongs to the class \( W_{-\infty, \infty} \).

If the matrices \( U \) and \( V \) commute, the exponential convexity of the function \( \varphi(t) \), (4.1), is evident. In this case, the sum

\[
\sum_{r,s=1}^N \varphi(t_r + t_s)\xi_r\xi_s = \text{trace} \left\{ e^{U/2} \left( \sum_{r=1}^N e^{t_r V} \xi_r \right) \left( \sum_{s=1}^N e^{t_s V} \xi_s \right)^* \left( e^{U/2} \right)^* \right\}
\]
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is non-negative because this sum is the trace of a non-negative matrix. The measure $\sigma$ in the integral representation (3.3) of the function $\varphi(t)$, (4.1), is an atomic measure supported on the spectrum of the matrix $V$.

In general case, if the matrices $U$ and $V$ do not commute, the BMV conjecture remained an open question for longer than 40 years. In 2011, Herbert Stahl gave an affirmative answer to the BMV conjecture.

**Theorem 4.1.** (H.Stahl) Let $U$ and $V$ be Hermitian matrices of size $l \times l$.

Then the function $\varphi(t)$ defined by (4.1) belongs to the class $W_{-\infty,\infty}$ of functions exponentially convex on $-\infty, \infty$.

The first arXiv version of H.Stahl’s Theorem appeared in [S1], the latest arXiv version - in [S2], the journal publication - in [S3].

The proof of Herbert Stahl is based on ingenious considerations related to Riemann surfaces of algebraic functions. In [E], a simplified version of the Herbert Stahl proof is presented.

We present a toy version of Theorem 4.1 which is enough for our goal.

**Theorem 4.2.** Let $U$ and $V$ be Hermitian matrices of size $l \times l$. We assume moreover that

1. All off-diagonal entries of the matrix $U$ are non-negative.
2. The matrix $V$ is diagonal.

Then the function $\varphi(t)$ defined by (4.1) belongs to the class $W_{-\infty,\infty}$.

**Proof.** For $\rho \geq 0$, let $U_{\rho} = U + \rho I$, where $I$ is the identity matrix. If $\rho$ is large enough, then all entries of the matrix $U_{\rho}$ are non-negative. Let us choose and fix such $\rho$. It is clear that

$$e^{U+\rho V} = e^{-\rho} e^{U_{\rho}+\rho V}.$$ (4.2)

We use the Lie product formula

$$e^{U_{\rho}+\rho V} = \lim_{m \to \infty} \left(e^{U_{\rho}/m} e^{\rho V/m}\right)^m.$$ (4.3)

All entries of the matrix $e^{U_{\rho}/m}$ are non-negative numbers. Since matrix $V$ is Hermitian, its diagonal entries are real numbers. Thus

$$e^{\rho V/m} = \text{diag}(e^{tv_1/m}, e^{tv_2/m}, \ldots, e^{tv_m/m}),$$

where $v_1, v_2, \ldots, v_m$ are real numbers. The exponentials $e^{tv_j/m}$ are functions of $t$ from the class $W_{-\infty,\infty}$. Each entry of the matrix $e^{U_{\rho}/m} e^{\rho V/m}$ is a linear combination of these exponentials with non-negative coefficients. According to the properties P1 and P2 of the class $W_{-\infty,\infty}$, the entries of the matrix $e^{U_{\rho}/m} e^{\rho V/m}$ are functions of the class $W_{-\infty,\infty}$. Each entry of the matrix $\left(e^{U_{\rho}/m} e^{\rho V/m}\right)^m$ is a sum of products of some entries of the matrix $e^{U_{\rho}/m} e^{\rho V/m}$. According to the properties P2 and P3 of the class $W_{-\infty,\infty}$, the entries of the matrix $\left(e^{U_{\rho}/m} e^{\rho V/m}\right)^m$ are functions of $t$ belonging to the class $W_{-\infty,\infty}$. From the limiting relation (4.3) and from the property P4 of the class $W_{-\infty,\infty}$ it follows that all entries of the matrix $e^{U_{\rho}+\rho V}$ are function
of \( t \) belonging to the class \( W_{-\infty,\infty} \). From (4.2) it follows that all entries of the matrix \( e^{U+tV} \) belong to the class \( W_{-\infty,\infty} \). All the more, the function \( \varphi(t) = \text{trace}\{e^{U+tV}\} \), which is the sum of diagonal entries of the matrix \( e^{U+tV} \), belongs to the class \( W_{-\infty,\infty} \). □

5. Exponential convexity of the sum \( e^{\xi \nu_0(t)} + \ldots + e^{\xi \nu_n(t)} \).

Let \( \xi \) be a real number. Taking \( h(z) = e^{\xi z} \) in Lemma 2.4, we obtain

**Lemma 5.1.** Let \( R \) be the rational function of the form (1.2), \( \nu_0(t), \nu_1(t), \ldots, \nu_n(t) \) be the roots of the equation (1.1). Let \( A \) and \( B \) be the matrices (2.1), (2.6), (2.2) which appear in the determinant representation (2.5) of the matrix pencil \( P(z) - tQ(z) \).

Then the equality

\[
\sum_{k=0}^{n} e^{\xi \nu_k(t)} = \text{trace}\{e^{\xi A+t(\xi B)}\}
\]

holds.

Now we choose \( \omega_p \) in (2.6) so that all off-diagonal entries of the matrix \( U = \xi A \) are non-negative: if \( \xi > 0 \), then \( \omega_p = +1 \), if \( \xi < 0 \), then \( \omega_p = -1 \), \( 1 \leq p \leq n \).

Applying Theorem 4.2 to the matrices \( U = \xi A, V = \xi B \), we obtain the following result

**Theorem 5.2.** Let \( R \) be the rational function of the form (1.2), \( \nu_0(t), \nu_1(t), \ldots, \nu_n(t) \) be the roots of the equation (1.1). Then for each \( \xi \in \mathbb{R} \), the function

\[
g(t, \xi) \overset{\text{def}}{=} \sum_{k=0}^{n} e^{\xi \nu_k(t)}
\]

of the variable \( t \) belongs to the class \( W_{-\infty,\infty} \).

**Theorem 5.3.** Let \( f \in W_{u,v} \), where \( -\infty \leq u < v \leq +\infty \). Let \( R \) be the rational function of the form (1.2), \( \nu_0(t), \nu_1(t), \ldots, \nu_n(t) \) be the roots of the equation (1.1). Assume that for some \( a, b \), \( -\infty \leq a < b \leq +\infty \), the inequalities

\[
u_k(t) < v, \quad a < t < b, \quad k = 0, 1, \ldots, n
\]

hold.

Then the function

\[
F(t) \overset{\text{def}}{=} \sum_{k=0}^{n} f(\nu_k(t))
\]

belongs to the class \( W_{a,b} \).

**Proof.** According to Theorem 5.3, the representation

\[
f(x) = \int_{\xi \in (-\infty, \infty)} e^{\xi x} \sigma(d\xi), \quad \forall x \in (u, v)
\]
holds, where $\sigma$ is a non-negative measure. Substituting $x = \nu_k(t)$ to the above formula, we obtain the equality

$$f(\nu_k(t)) = \int_{\xi \in (-\infty, \infty)} e^{\xi \nu_k(t)} \sigma(d\xi), \quad \forall \ t \in (a, b), \ k = 0, 1, \ldots, n.$$ 

Hence

$$F(t) = \int_{\xi \in (-\infty, \infty)} g(t, \xi) \sigma(d\xi), \quad \forall \ t \in (a, b). \quad (5.5)$$

Theorem 5.4 is a consequence of Theorem 5.2 and of the properties P1,P2,P4 of the class of exponentially convex functions. □

**Example** For $\gamma > 0$, the function $f(x) = e^{\gamma x^2}$ is exponentially convex on $(-\infty, \infty)$: $e^{\gamma x^2} = \int_{\xi \in (-\infty, \infty)} e^{\xi x} \sigma(d\xi)$, where $\sigma(d\xi) = \frac{1}{2\sqrt{\pi} \gamma} e^{-\xi^2/4\gamma} d\xi$.

Thus the function $F(t) = \sum_{k=0}^{n} e^{\gamma (\nu_k(t))^2}$ is exponentially convex on $(-\infty, \infty)$.

**Remark 5.4.** Familiarizing himself with our proof of Theorem 5.2, Alexey Kuznetsov ( www.math.yorku.ca/~akuznets/ ) gave a new proof of a somewhat weakened version of this theorem. His proof is based on the theory of stochastic processes Lévy.

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