Rainbow Monochromatic $k$-Edge-Connection Colorings of Graphs

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Abstract

A path in an edge-colored graph is called a monochromatic path if all edges of the path have a same color. We call $k$ paths $P_1, \ldots, P_k$ rainbow monochromatic paths if every $P_i$ is monochromatic and for any two $i \neq j$, $P_i$ and $P_j$ have different colors. An edge-coloring of a graph $G$ is said to be a rainbow monochromatic $k$-edge-connection coloring (or $RMC_k$-coloring for short) if every two distinct vertices of $G$ are connected by at least $k$ rainbow monochromatic paths. We use $rmc_k(G)$ to denote the maximum number of colors that ensures $G$ has an $RMC_k$-coloring, and this number is called the rainbow monochromatic $k$-edge-connection number. We prove the existence of $RMC_k$-colorings of graphs, and then give some bounds of $rmc_k(G)$ and present some graphs whose $rmc_k(G)$ reaches the lower bound. We also obtain the threshold function for $rmc_k(G(n,p)) \geq f(n)$, where $\left\lceil \frac{n}{2} \right\rceil > k \geq 1$.

Keywords  Monochromatic path · Rainbow monochromatic path · Rainbow monochromatic $k$-edge-connection coloring (number) · Threshold function

1 Introduction

The monochromatic connection coloring of a graph, introduced in [4], allows that any two vertices are connected by a monochromatic path. In order to generalize this concept, we consider an edge-coloring of a given graph $G$ with any two vertices are connected by at least $k$ (a fixed integer) edge-disjoint monochromatic paths. If we

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allow some of those $k$ monochromatic paths to have different colors, then the edge-coloring is called $MC_k$-coloring of $G$. If we require that those $k$ monochromatic paths have the same color, then the edge-coloring is called $UMC_k$-coloring of $G$. The two generalized concepts are introduced in [12]. In this paper, we discuss the third generalized concept, $RMC_k$-coloring, which requires that the colors of those $k$ monochromatic paths are pairwise differently. We will introduce the above four concepts systematically, and also introduce some notations and previous work below.

For a graph $G$, let $C : E(G) \rightarrow [k]$ be an edge-coloring of $G$ that allows a same color to be assigned to adjacent edges, here and in what follows $[k]$ denotes the set $\{1, 2, \ldots, k\}$ of integers for a positive integer $k$. For an edge $e$ of $G$, we use $C(e)$ to denote the color of $e$. If $H$ is a subgraph of $G$, we also use $C(H)$ to denote the set of colors on the edges of $H$ and use $|C(H)|$ to denote the number of colors in $C(H)$. For all other terminology and notation not defined here we follow Bondy and Murty [2].

A monochromatic $uv$-path is a $uv$-path of $G$ whose edges are colored with a same color, and $G$ is monochromatically connected if for any two vertices of $G$, $G$ has a monochromatic path connecting them. An edge-coloring $C$ of $G$ is a monochromatic connection coloring (or MC-coloring for short) if it makes $G$ monochromatically connected. The monochromatic connection number of a connected graph $G$, denoted by $mc(G)$, is the maximum number of colors that are allowed in order to make $G$ monochromatically connected. An extremal MC-coloring of $G$ is an MC-coloring that uses $mc(G)$ colors.

The notion monochromatic connection coloring was introduced by Caro and Yuster [4]. Huang and Li [10] recently showed that it is NP-hard to compute the monochromatic number for a given graph. Some results were obtained in [3, 9, 11, 13, 14]. Later, González-Moreno et al. in [8] generalized the above concept to digraphs.

We list the main results in [4] below.

**Theorem 1** ([4]) Let $G$ be a connected graph with $n \geq 3$. If $G$ satisfies any of the following properties, then $mc(G) = m - n + 2$.

1. $\overline{G}$ (the complement of $G$) is a 4-connected graph;
2. $G$ is triangle-free;
3. $\Delta(G) < n - \frac{2m - 3(n - 1)}{n - 3}$;
4. $diam(G) \geq 3$;
5. $G$ has a cut vertex.

The Erdős–Rényi random graph model $G(n, p)$ will be studied in this paper. The graph $G(n, p)$ is defined on $n$ labeled vertices (informally, we use $[n]$ to denote the $n$ labeled vertices) in which each edge is chosen independently and randomly with probability $p$. A property of graphs is a subset of the set of all graphs on $[n]$ (such as connectivity, minimum degree, et al). If a property $Q$ has $Pr[G \sim G(n, p) \text{satisfies } Q] \rightarrow 1$ when $n \rightarrow +\infty$, then we call the property $Q$ almost surely. A property $Q$ is monotone increasing if whenever $H$ is a graph
obtained from $H'$ by adding some edges and $H'$ has property $Q$, then $H$ also has the property $Q$.

A function $h(n)$ is a threshold function for an increasing property $Q$, if for any two functions $h_1(n) = o(h(n))$ and $h(n) = o(h_2(n))$, $G(n, h_1(n))$ does not have property $Q$ almost surely and $G(n, h_2(n))$ has property $Q$ almost surely. Moreover, $h(n)$ is called a sharp threshold function of $Q$ if there exist two positive constants $c_1$ and $c_2$ such that $G(n, p(n))$ does not have property $Q$ almost surely when $p(n) \leq c_1 h(n)$ and $G(n, p(n))$ has property $Q$ almost surely when $p(n) \geq c_2 h(n)$. It was proved in [6] that every monotone increasing graph property has a sharp threshold function. The property monochromatic connection coloring of a graph (and also the properties monochromatic $k$-edge-connection coloring, uniformly monochromatic $k$-edge-connection coloring and rainbow monochromatic $k$-edge-connection coloring of graphs which are defined later) is monotone increasing, and therefore it has a sharp threshold function.

**Theorem 2** ([9]) Let $f(n)$ be a function satisfying $1 \leq f(n) < (\frac{n}{2})$. Then

$$p = \begin{cases} \frac{f(n) + n \log \log n}{n^2}, & \text{if } f(n) = \Omega(n \log n) \text{ and } f(n) < \left(\frac{n}{2}\right); \\ \log n, & \text{if } f(n) = o(n \log n). \end{cases}$$

is a sharp threshold function for the property $mc(G(n,p)) \geq f(n)$.

Now we generalize the concept monochromatic connection coloring of graphs. There are three ways to generalize this concept.

The first generalized concept is called the monochromatic $k$-edge-connection coloring (or MC$_k$-coloring for short) of $G$, which requires that every two distinct vertices of $G$ are connected by at least $k$ edge-disjoint monochromatic paths (allow some of the paths to have different colors). The monochromatically $k$-edge-connection number of a connected $G$, denoted by $mc_k(G)$, is the maximum number of colors that are allowed in order to make $G$ monochromatically $k$-edge-connected.

The second generalized concept is called the uniformly monochromatic $k$-edge-connection coloring (or UMC$_k$-coloring for short) of $G$, which requires that every two distinct vertices of $G$ are connected by at least $k$ edge-disjoint monochromatic paths such that all these $k$ paths have the same color (note that for different pairs of vertices the paths may have different colors). The uniformly monochromatically $k$-edge-connection number of a connected $G$, denoted by $umc_k(G)$, is the maximum number of colors that are allowed in order to make $G$ uniformly monochromatically $k$-edge-connected. These two concepts were studied in [12].

It is obvious that a graph has an MC$_k$-coloring (or UMC$_k$-coloring) if and only if $G$ is $k$-edge-connected. We mainly study the third generalized concept in this paper, which is called the rainbow monochromatic $k$-edge-connection coloring (or RMC$_k$-coloring for short) of a connected graph. One can see later, compare the results for MC-colorings, MC$_k$-colorings, UMC$_k$-colorings and RMC$_k$-colorings of graphs, the concept RMC$_k$-coloring has the best form among all the generalized concepts of the MC-coloring.
The definition of the third generalized concept goes as follows. For an edge-colored simple graph $G$ (if $G$ has parallel edges but no loops, the following notions are also reasonable), if for any two distinct vertices $u$ and $v$ of $G$, $G$ has $k$ edge-disjoint monochromatic paths connecting them, and the colors of these $k$ paths are pairwise differently, then we call such $k$ monochromatic paths $k$ rainbow monochromatic $uv$-paths. An edge-colored graph is rainbow monochromatically $k$-edge-connected if every two vertices of the graph are connected by at least $k$ rainbow monochromatic paths in the graph. An edge-coloring $\Gamma$ of a connected graph $G$ is a rainbow monochromatic $k$-edge-connection coloring (or $RMC_k$-coloring for short) if it makes $G$ rainbow monochromatically $k$-edge-connected. The rainbow monochromatically $k$-edge-connection number of a connected graph $G$, denoted by $rmc_k(G)$, is the maximum number of colors that are allowed in order to make $G$ rainbow monochromatically $k$-edge-connected. An extremal $RMC_k$-coloring of $G$ is an $RMC_k$-coloring that uses $rmc_k(G)$ colors.

If $k = 1$, then an $RMC_k$-coloring (also $MC_k$-coloring and $UMC_k$-coloring) is reduced to a monochromatic connection coloring for any connected graph.

In an edge-colored graph $G$, if a color $i$ only colors one edge of $E(G)$, then we call the color $i$ a trivial color, and call the edge (tree) a trivial edge (trivial tree). Otherwise we call the edges (colors, trees) nontrivial. A subgraph $H$ of $G$ is called an $i$-induced subgraph if $H$ is induced by all the edges of $G$ with the same color $i$. Sometimes, we also call $H$ a color-induced subgraph.

If $\Gamma$ is an extremal $RMC_k$-coloring of $G$, then each color-induced subgraph is connected. Otherwise we can recolor the edges in one of its components by a fresh color, then the new edge-coloring is also an $RMC_k$-coloring of $G$, but the number of colors is increased by one, which contradicts that $\Gamma$ is extremal. Furthermore, each color-induced subgraph does not have cycles; otherwise we can recolor one edge in a cycle by a fresh color. Then the new edge-coloring is also an $RMC_k$-coloring of $G$, but the number of colors is increased, a contradiction. Therefore, we have the following result.

**Proposition 1.** If $\Gamma$ is an extremal $RMC_k$-coloring of $G$, then each color-induced subgraph is a tree.

If $\Gamma$ is an extremal $RMC_k$-coloring of $G$ for $i \in \Gamma(G)$, we call an $i$-induced subgraph of $G$ an $i$-induced tree or a color-induced tree. We also call it a tree sometimes if there is no confusion.

The paper is organized as follows. Section 2 will give some preliminary results. In Sect. 3, we study the existence of $RMC_k$-colorings of graphs. In Sect. 4, we give some bounds of $rmc_k(G)$, and present some graphs whose $rmc_k(G)$ reaches the lower bound. In Sect. 5, we obtain the threshold function for $rmc_k(G) \geq f(n)$, where $\lfloor \frac{n}{2} \rfloor > k \geq 1$. 

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2 Preliminaries

Suppose that \( a = (a_1, \ldots, a_q) \) and \( b = (b_1, \ldots, b_p) \) are two positive integer sequences whose lengths \( p \) and \( q \) may be different. Let \( \prec \) be the lexicographic order for integer sequences, i.e., \( a \prec b \) if for some \( h \geq 1 \), \( a_j = b_j \) for \( j < h \) and \( a_h < b_h \), or \( p > q \) and \( a_j = b_j \) for \( j \leq q \).

Let \( D, n, s \) be integers with \( n \geq 5 \) and \( 1 \leq s \leq n - 4 \). Let \( r \) be an integer satisfying \( D < r^{(n-s)/2} \). For an integer \( t \geq r \), suppose \( f(x_i) = f(x_1, \ldots, x_t) = \sum_{i \in [t]} (x_i - 1) \) and \( g(x_i) = g(x_1, \ldots, x_t) = \sum_{i \in [t]} (x_i - 2) \), where \( x_i \in \{3, 4, \ldots, n-s\} \). We use \( S_t \) to denote the set of optimal solutions of the following problem:

\[
\min \quad s.t.
\]

**Lemma 1** There are integers \( r, x \) with \( r \leq t \) and \( 3 \leq x < n - s \), such that the above problem has a solution \( x_1, \ldots, x_t \) in \( S_t \) satisfying that \( x_i = n - s \) for \( i \in [r-1] \), \( x_r = x \) and \( x_j = 3 \) for \( j \in \{r+1, \ldots, t\} \).

**Proof** Let \( c_i = (c_1, \ldots, c_t) \) be a maximum integer sequence of \( S_t \). Then \( c_i = c_{i+1} \) for \( i \in [r-1] \). Since \( D < r^{(n-s)/2} \), there is an integer \( r \leq t \) such that \( c_i = n-s \) for \( i \leq r-1 \) and \( 3 \leq c_i < n-s \) for \( i \in [r, \ldots, t] \). Let \( x = c_r \). Then \( 3 \leq x < n-s \). We need to show \( c_i = 3 \) for each \( i \in \{r+1, \ldots, t\} \). Otherwise, suppose \( j \) is the maximum integer of \( \{r+1, \ldots, t\} \) with \( n-s > c_j > 3 \). Let \( d_i = (d_1, \ldots, d_t) \), where \( d_i = c_i \) when \( i \notin \{r, j\} \), \( d_r = c_r + 1 \) and \( d_j = c_j - 1 \). Then \( f(d_i) \geq f(c_i) \geq D \), \( 3 \leq d_i < n-s \) for each \( i \in [t] \), and \( g(c_i) = g(d_i) \). i.e., \( d_i \in S_t \). However, \( c_i \prec d_i \), which contradicts that \( c_i \) is a maximum integer sequence of \( S_t \). \( \square \)

**Lemma 2** Suppose \( t \geq r, a_i \in S_t \) and \( b_i \in S_r \). Then \( g(b_r) \leq g(a_t) \).

**Proof** The result holds for \( t = r, \) so let \( t > r \). W.l.o.g., suppose \( a_i = (a_1, \ldots, a_t) \), where \( a_1 = \cdots = a_{t-1} = n-s, 3 \leq a_t < n-s \) and \( a_{t+1} = \cdots = x_t = 3 \). Since \( t > r \) and \( D < r^{(n-s)/2} \), \( t \leq t \) and \( a_t = 3 \). Let \( c_{t-1} = (c_1, \ldots, c_{t-1}) \), where \( c_1 = \cdots = c_{t-1} = n-s, c_t = a_t + 1 \) and \( c_{t+1} = \cdots = x_{t-1} = 3 \). Then \( f(c_{t-1}) \geq D \) and \( g(c_{t-1}) = g(a_t) \). Let \( d_{t-1} \in S_{t-1} \). Then \( g(c_{t-1}) \geq g(d_{t-1}) \). By induction on \( t-r \), \( g(b_r) \leq g(d_{t-1}) \). Thus \( g(b_r) \leq g(a_t) \). \( \square \)

The following result is easily seen.

**Lemma 3** If \( a, b, c \) are positive integers with \( c + a - 1 \geq 2 \) and \( a + b = c \), then \( \binom{c}{2} - \binom{a}{2} \geq b \).

Suppose \( X \) is a proper vertex set of \( G \). We use \( E(X) \) to denote the set of edges whose ends are in \( X \). For a graph \( G \) and \( X \subseteq V(G) \), to shrink \( X \) is to delete \( E(X) \) and then merge the vertices of \( X \) into a single vertex. A partition of the vertex set \( V \) is to divide \( V \) into some mutual disjoint nonempty sets. Suppose \( \mathcal{P} = \{V_1, \ldots, V_s\} \) is a partition of \( V(G) \). Then \( G/\mathcal{P} \) is a graph obtained from \( G \) by shrinking every \( V_i \) into a single vertex.
The spanning tree packing number (STP number) of a graph is the maximum number of edge-disjoint spanning trees contained in the graph. We use \( T(G) \) to denote the number of edge-disjoint spanning trees of \( G \). The following theorem was proved by Nash-Williams and Tutte independently.

**Theorem 3** ([15, 16]) A graph \( G \) has at least \( k \) edge-disjoint spanning trees if and only if 
\[
e(G/\mathcal{P}) \geq k(|G/\mathcal{P}| - 1)
\]
for any vertex-partition \( \mathcal{P} \) of \( V(G) \).

We denote \( s(G) = \min_{|\mathcal{P}| \geq 2} \left( \frac{e(G/\mathcal{P})}{|G/\mathcal{P}| - 1} \right) \). Then Nash–Williams–Tutte Theorem can be restated as follows.

**Theorem 4** \( T(G) = k \) if and only if \( s(G) = k \).

If \( \Gamma \) is an extremal \( RMC_k \)-coloring of \( G \), then we say that \( \Gamma \) wastes \( \omega = \sum_{i \in [r]} (|T_i| - 2) \) colors, where \( T_1, \ldots, T_r \) are all the nontrivial color-induced trees of \( G \). Thus \( rmc_k(G) = m - \omega \).

Suppose that \( \Gamma \) is an edge-coloring of \( G \) and \( v \) is a vertex of \( G \). The nontrivial color degree of \( v \) under \( \Gamma \) is denoted by \( d^n(v) \), that is, the number of nontrivial colors appearing on the edges incident with \( v \).

**Lemma 4** Suppose that \( \Gamma \) is an \( RMC_k \)-coloring of \( G \) with \( k \geq 2 \). Then \( d^n(v) \geq k \) for every vertex \( v \) of \( G \).

**Proof** Since every two vertices have \( k \geq 2 \) rainbow monochromatic paths connecting them and \( G \) is simple, every two vertices have at least one nontrivial monochromatic path connecting them, i.e., \( d^n(v) \geq 1 \) for each \( v \in V(G) \). Let \( e = vu \) be a nontrivial edge. Then there are \( k - 1 \) rainbow monochromatic paths of order at least three connecting \( u \) and \( v \). Since these \( k - 1 \) rainbow monochromatic paths are nontrivial, \( d^n(v) \geq k \) for each \( v \in V(G) \). \( \square \)

### 3 Existence of \( RMC_k \)-Colorings

We knew that there exists an \( MC_k \)-coloring or a \( UMC_k \)-coloring of \( G \) if and only if \( G \) is \( k \)-edge-connected. It is natural to ask how about \( RMC_k \)-colorings? It is obvious that any cycle of order at least 3 is 2-edge-connected, but it does not have an \( RMC_2 \)-coloring.

We mainly think about simple graphs in this paper, but in the following result, all graphs may have parallel edges but no loops.

**Theorem 5** A graph \( G \) has an \( RMC_k \)-coloring if and only if \( \tau(G) \geq k \).

**Proof** If \( G \) has \( k \) edge-disjoint spanning trees \( T_1, \ldots, T_k \), then we can color the edges of each \( T_i \) by \( i \) and color the other edges of \( G \) by colors in \( [k] \) arbitrarily. Then the coloring is an \( RMC_k \)-coloring of \( G \). Therefore, \( G \) has an \( RMC_k \)-coloring when \( \tau(G) \geq k \).

We will prove that if there exists an \( RMC_k \)-coloring of \( G \), then \( G \) has \( k \) edge-disjoint spanning trees, i.e., \( \tau(G) \geq k \). Before proceeding to the proof, we need a critical claim as follows.
Claim If $G$ has an $RMC_k$-coloring, then $e(G) \geq k(n - 1)$.

**Proof** Suppose that $\Gamma$ is an extremal $RMC_k$-coloring of $G$ and $G_1, \ldots, G_\ell$ are all the color-induced trees of $G$ (say $G_i$ is the $i$-induced tree). If there are two color-induced trees $G_i$ and $G_j$ satisfying that all the three sets $V(G_i) - V(G_j)$, $V(G_j) - V(G_i)$ and $V(G_i) \cap V(G_j)$ are nonempty, then we use $P(G, \Gamma, i, j)$ to denote the graph $(G - E(G_i \cup G_j)) \cup T_1 \cup T_2$, where $T_1$ and $T_2$ are two new trees with $V(T_1) = V(G_i) \cup V(G_j)$ and $V(T_2) = V(G_i) \cap V(G_j)$ (note that $T_1, T_2$ and $G - E(G_i \cup G_j)$ are mutually edge disjoint, then $P(G, \Gamma, i, j)$ may have parallel edges); we also use $\Upsilon(G, \Gamma, i, j)$ to denote the edge-coloring of $P(G, \Gamma, i, j)$, which is obtained from $\Gamma$ by coloring $E(T_1)$ with $i$ and coloring $E(T_2)$ with $j$, respectively. Then $|G| = |P(G, \Gamma, i, j)|$ and $e(G) = e(P(G, \Gamma, i, j)).$

We claim that $\Upsilon(G, \Gamma, i, j)$ is an $RMC_k$-coloring of $P(G, \Gamma, i, j)$, and we prove it below. For any two vertices $u, v$ of $G$, if at least one of them is in $V(G) - V(G_i \cup G_j)$, or one is in $V(G_i) - V(G_j)$ and the other is in $v \in V(G_i) - V(G_j)$, then none of rainbow monochromatic $uv$-paths of $G$ are colored by $i$ or $j$, these rainbow monochromatic $uv$-paths of $G$ are kept unchanged.

Thus there are at least $k$ rainbow monochromatic $uv$-paths in $P(G, \Gamma, i, j)$ under $\Upsilon(G, \Gamma, i, j)$; if both of $u, v$ are in $V(G_i) \cap V(G_j)$, then there are at least $k - 2$ rainbow monochromatic $uv$-paths of $G$ with colors different from $i$ and $j$, and these rainbow monochromatic $uv$-paths are kept unchanged. Since $T_1$ and $T_2$ provide two rainbow monochromatic $uv$-paths, one is colored by $i$ and the other is colored by $j$, there are at least $k$ rainbow monochromatic $uv$-paths in $P(G, \Gamma, i, j)$ under $\Upsilon(G, \Gamma, i, j)$; if, by symmetry, $u$ and $v$ are in $G_i$ and at most one of them is in $V(G_i) \cap V(G_j)$, then there are at least $k - 1$ rainbow monochromatic $uv$-paths with colors different from $i$ and $j$, and these rainbow monochromatic $uv$-paths are kept unchanged. Since $T_1$ provides a monochromatic $uv$-path with color $i$, there are at least $k$ rainbow monochromatic $uv$-paths in $P(G, \Gamma, i, j)$ under $\Upsilon(G, \Gamma, i, j)$.

We now introduce a simple algorithm on $G$. Setting $H := G$ and $\Gamma^* := \Gamma$. If there are two color-induced subgraphs $H_i$ and $H_j$ of $H$ satisfying that all the three sets $V(H_i) - V(H_j)$, $V(H_j) - V(H_i)$ and $V(H_i) \cap V(H_j)$ are nonempty, then replace $H$ by $P(H, \Gamma^*, i, j)$ and replace $\Gamma^*$ by $\Upsilon(H, \Gamma^*, i, j)$.

We now show that the algorithm will terminate in a finite steps. In the $i$th step, let $H = H_i$ and $\Gamma^* = \Gamma_i$, and let $G_1, \cdots, G_\ell_i$ be all the color-induced subgraphs of $H_i$ such that $|G_1| \geq |G_2| \geq \cdots \geq |G_\ell_i|$ (in fact, in each step, each color-induced subgraph is a tree), and let $l_i = (|G_1|, |G_2|, \cdots, |G_\ell_i|)$ be an integer sequence. Suppose $H_{i+1} = P(H_i, \Gamma_i, s, t)$, i.e., $H_{i+1} = H_i - E(G_s \cup G_t) \cup T_1 \cup T_2$, where $V(T_1) = V(G_s) \cup V(G_t)$ and $V(T_2) = V(G_s) \cap V(G_t)$. Then $|T_1| > \max\{|G_s|, |G_t|\}$. Therefore, $l_i < l_{i+1}$. Since $G$ is a finite graph and $e(H_i) = e(G)$ in each step, the algorithm will terminate in a finite step.

Let $H'$ be the resulting graph and $\Gamma'$ be the resulting $RMC_k$-coloring of $H'$, and $T_1', \ldots, T_r'$ be the color-induced trees of $H'$ with $|T_1'| \geq \cdots \geq |T_r'|$. Then $T_k'$ is a spanning tree of $H'$; otherwise, there is at least one vertex $w$ in $V(G) - V(T_k)$. Suppose $u \in V(T_k)$. Since $T_1', \ldots, T_{k-1}'$ provide at most $k - 1$ rainbow monochromatic $uw$-paths, there is a tree of $\{T_{k+1}', \ldots, T_r'\}$, say $T_a'$, containing $u$ and $w$. Then
$V(T_k') - V(T_a') \neq \emptyset$; otherwise $|T_k'| < |T_a'|$, a contradiction. Thus $V(T_k') - V(T_a')$, $V(T_a') \cap V(T_k')$ and $V(T_a') - V(T_k')$ are nonempty sets, which contradicts that $H'$ is the resulting graph of the algorithm. Therefore, there are at least $k$ spanning trees of $H'$, i.e., $e(G) = e(H') \geq k(n - 1)$. □

Now, we are ready to prove $\tau(G) \geq k$ by contradiction. Suppose that $I'$ is an RMC$_k$-coloring of $G$ but $\tau(G) < k$. By Theorem 3, there exists a partition $P = \{V_1, \ldots, V_t\}$ of $V(G)$ ($|P| = t \geq 2$), such that $e(G/P) < k(|P| - 1)$. Let $G^* = G/P$ be the graph obtained from $G$ by shrinking each $V_i$ into a single vertex $v_i$, $1 \leq i \leq t$.

Suppose that $I^*$ is an edge-coloring of $G^*$ obtained from $I'$ by keeping the color of every edge of $G$ not being deleted (we only delete edges contained in each $V_i$). It is obvious that $I^*$ is an RMC$_k$-coloring of $G^*$. However, $e(G^*) < k(|G^*| - 1)$, a contradiction to Claim 3. So, $\tau(G) \geq k$. □

We will turn to discuss simple graphs below. Because a simple graph is also a loopless graph, Theorem 5 holds for simple graphs. For a connected simple graph $G$, since $1 \leq \tau(G) \leq \tau(K_n) = \left\lceil \frac{e(K_n)}{n - 1} \right\rceil = \lceil \frac{n}{2} \rceil$, we have the following result.

**Corollary 1** If $G$ is a simple graph of order $n$ and $G$ has an RMC$_k$-coloring, then $1 \leq k \leq \lceil \frac{n}{2} \rceil$.

By Theorem 5, if $\tau(G) \geq k$, a trivial RMC$_k$-coloring of a graph $G$ is a coloring that colors the edges of the $k$ edge-disjoint spanning trees of $G$ by colors in $[k]$, respectively, and then colors the other edges trivial. Since the edge-coloring wastes $k(n - 2)$ colors, rmck$_k(G) \geq m - k(n - 2)$. Thus, $m - k(n - 2)$ is a lower bound of rmck$_k(G)$ if $G$ has an RMC$_k$-coloring.

**Corollary 2** If $G$ is a graph with $\tau(G) \geq k$, then rmck$_k(G) \geq m - k(n - 2)$.

## 4 Some Graphs with Rainbow Monochromatic $k$-Edge-Connection Number $m - k(n - 2)$

In this section, we mainly study the graphs with rainbow monochromatic $k$-edge-connection number $m - k(n - 2)$ (graphs in the following theorem).

**Theorem 6** Let $G$ be a graph with $\tau(G) \geq k$. If $G$ satisfies any of the following properties, then rmck$_k(G) = m - k(n - 2)$.

1. $G$ is triangle-free;
2. $\text{diam}(G) \geq 3$;
3. $G$ has a cut vertex;
4. $G$ is not $k + 1$-edge-connected.

We will prove this theorem separately by four propositions below (the second result is a corollary of Proposition 3).

**Proposition 2** If $G$ is a triangle-free graph with $\tau(G) \geq k$, then rmck$_k(G) = m - k(n - 2)$.
Proof By Theorem 1, the result holds for \( k = 1 \). Therefore, let \( k \geq 2 \) (this requires \( n \geq 4 \)). Since \( G \) is a triangle-free graph, by Turán’s Theorem, \( e(G) \leq \frac{n^2}{4} \). Then
\[
k \leq \tau(G) \leq \frac{e(G)}{|G| - 1} \leq \frac{n + 1}{4} + \frac{1}{4(n - 1)}.
\]
So, \( n \geq 4k - 1 - \frac{1}{n - 1} \), i.e., \( n \geq 4k - 1 \).

Suppose \( \Gamma \) is an extremal \( RMC_k \)-coloring of \( G \). If there is a color-induced tree, say \( T \), that forms a spanning tree of \( G \), then \( \Gamma \) is an extremal \( RMC_{k-1} \)-coloring restricted on \( G - E(T) \). Otherwise, suppose \( \Gamma \) is not an extremal \( RMC_{k-1} \)-coloring restricted on \( G - E(T) \). Since \( \Gamma \) is obviously an \( RMC_{k-1} \)-coloring restricted on \( G - E(T) \), there is an \( RMC_{k-1} \)-coloring \( \Gamma' \) of \( G - E(T) \) such that \( |\Gamma'(G - E(T))| < |\Gamma(G - E(T))| \). Let \( \Gamma'' \) be an edge-coloring of \( G \) obtained from \( \Gamma' \) by assigning \( E(T) \) with a new color. Then \( \Gamma'' \) is an \( RMC_k \)-coloring of \( G \). However, \( |\Gamma'(G)| < |\Gamma''(G)| \), a contradiction. Since \( G - E(T) \) is triangle-free, by induction on \( k \),
\[
rmc_{k-1}(G - E(T)) = e(G - E(T)) - (k - 1)(n - 2) = m - k(n - 2) - 1.
\]

Therefore,
\[
rmc_k(G) = 1 + |\Gamma'(G - E(T))| = 1 + rmc_{k-1}(G - E(T)) = m - k(n - 2).
\]

Now, suppose that each color-induced tree is not a spanning tree. We use \( \mathcal{F} \) to denote the set of nontrivial color-induced trees of \( G \). We will prove that \( \Gamma \) wastes at least \( k(n - 2) \) colors below.

Case 1. There is a vertex \( v \) of \( G \) such that \( d^v(v) = k \).

Suppose that \( \mathcal{F} = \{T_1, \ldots, T_k\} \) is the set of the \( k \) nontrivial color-induced trees containing \( v \). Since each vertex connects \( v \) by at least \( k - 1 \geq 1 \) nontrivial rainbow monochromatic paths, \( V(G) = \bigcup_{i \in [k]} V(T_i) \). Let \( S = \bigcap_{i \in [k]} V(T_i) \) and \( S_i = V(T_i) - S \).

For any \( i, j \in [k] \), both \( S_i - S_j \) and \( S_j - S_i \) are nonempty. Otherwise, suppose \( S_i \subseteq S_j \). Since \( T_j \) is not a spanning tree, there is a vertex \( u_j \in V(G) - V(T_j) \). Then there are at most \( k - 2 \) nontrivial rainbow monochromatic \( u_jv \)-paths, a contradiction.

According to the above discussion, \( S, S_1, \ldots, S_k \) are all nonempty sets. Moreover, since \( k \geq 2 \), \( |V(G) - S| \geq 2 \).

For each \( i \in [k] \) and a vertex \( u \) in \( S_i \), there is an \( i_u \in [k] \) such that \( u \notin V(T_{i_u}) \). Furthermore, \( u \in V(T_j) \) for each \( j \in [k] - \{i_u\} \); for otherwise, there are at most \( k - 2 \) nontrivial rainbow monochromatic \( uv \)-paths, which contradicts that \( \Gamma \) is an \( RMC_k \)-coloring of \( G \). Therefore, there are exactly \( k - 1 \) nontrivial rainbow monochromatic \( uv \)-paths. This implies that \( uv \) is a trivial edge of \( G \). Thus, \( v \) connects each vertex of \( V(G) - S \) by a trivial edge. Since \( G \) is triangle-free, \( V(G) - S \) is an independent set. It is easy to verify that \( \mathcal{F} \) wastes
\[ \sum_{i \in [k]} (|T_i| - 2) = \sum_{i \in [k]} |T_i| - 2k = k|S| + (k - 1)(n - |S|) - 2k = k(n - 2) + |S| - n \]

colors.

Let \( T = T - T \) (recall that \( T \) is the set of nontrivial trees of \( G \)). Since each two vertices of \( V(G) - S \) are in at most \( k - 1 \) trees of \( T \) and \( V(G) - S \) is an independent set, there is at least one tree of \( T \) containing them. Moreover, such a tree contains at least one vertex of \( S \). Suppose that \( F_1, \ldots, F_t \) are trees of \( T \) with \( |V(F_i) \cap (V(G) - S)| = x_i \geq 2 \) and \( x_1 \geq x_2 \geq \cdots \geq x_t \). Let \( w_i \in V(F_i) \cap S \) and \( W_i = V(F_i) \cap (V(G) - S) \cup \{w_i\} \). Then \( 3 \leq |W_i| \leq n - |S| + 1 \) for each \( i \in [t] \), and

\[ \sum_{i \in [t]} \left( \frac{|W_i| - 1}{2} \right) \geq \left( \frac{n - |S|}{2} \right). \quad (1) \]

Let \( T \) wastes at least \( \sum_{i \in [t]} (|F_i| - 2) \geq \sum_{i \in [t]} (|W_i| - 2) \) colors.

For any \( i, j \in [k] \), since both \( S_i - S_j \) and \( S_j - S_i \) are nonempty, there are at most \( k - 2 \) rainbow monochromatic paths connecting every vertex of \( S_i - S_j \) and every vertex of \( S_j - S_i \) in \( T \). Thus there are at least two trees of \( T \) containing the two vertices, i.e., \( t \geq 2 \).

If \( k = 2 \) and \( |S| - 1 = 3 \), then \( T \) wastes at least two colors, and thus \( T \) wastes at least \( k(2 - 2) \) colors. Otherwise, \( |S| - 1 \geq 4 \). Then by Lemma 1, the expression \( \sum_{i \in [t]} (|W_i| - 2) \), subjects to (1), \( n - |S| + 1 \geq |W_i| \geq 3 \) and \( t \geq 2 \), is minimum when \( |W_i| = n - |S| + 1 \), and \( |W_i| = 3 \) for \( i = 2, 3, \ldots, t \). Then \( T \) wastes at least \( n - |S| \) colors, and thus \( T \) wastes at least \( k(n - 2) \) colors.

**Case 2.** each vertex \( v \) of \( G \) has \( d^*(v) \geq k + 1 \).

Suppose \( T = \{T_1, \ldots, T_r\} \) and \( |T_i| \geq |T_{i+1}| \) for \( i \in [r - 1] \). Since \( d^*(v) \geq k + 1 \) for each vertex \( v \) of \( G \), \( \sum_{i \in [r]} |T_i| \geq (k + 1)n \).

If \( r \leq \frac{n}{2} + k \), then \( \sum_{i \in [r]} (|T_i| - 2) \geq k(n - 2) \). This implies that \( T \) wastes at least \( k(n - 2) \) colors. Thus, we consider \( r > \frac{n}{2} + k \).

Since each pair of non-adjacent vertices is connected by at least \( k \) rainbow monochromatic paths of order at least three, and each pair of adjacent vertices are connected by at least \( k - 1 \) rainbow monochromatic paths of order at least three, there are at least \( k \left( \frac{n}{2} - e(G) \right) + (k - 1)e(G) = k \left( \frac{n}{2} \right) - e(G) \) such paths. Since each \( T_i \) of \( T \) provides \( \left( \frac{|T_i| - 1}{2} \right) \) paths of order at least three, we have

\[ \sum_{i \in [r]} \left( \frac{|T_i| - 1}{2} \right) \geq k \left( \frac{n}{2} \right) - e(G). \]

Since \( e(G) \leq \frac{n^2}{4} \),

\[ \sum_{i \in [r]} \left( \frac{|T_i| - 1}{2} \right) \geq k \left( \frac{n}{2} \right) - \frac{n^2}{4}. \quad (2) \]

If \( |T_i| = n - 1 \) for each \( i \in [r] \), since \( r > \frac{n}{2} + k \), \( T \) wastes \( r(n - 3) > k(n - 2) \)
By Lemma 1, there are integers $t, x$ with $t < r$ and $3 \leq x \leq n - 2$, such that the expression $\sum_{i \in [r]}(|T_i| - 2)$, subject to (2) and $3 \leq |T_i| \leq n - 1$, is minimum when $|T_i| = n - 1$ for $i \in [r]$, $|T_{t+1}| = x$ and $|T_j| = 3$ for $j \in \{t + 1, \cdots, r\}$. By (2),

$$t\left(\frac{n-2}{2}\right) + \left(x - 1\right) + r - t - 1 \geq k\left(\frac{n}{2}\right) - \frac{n^2}{4}. \quad (3)$$

This implies that $\Gamma$ wastes at least

$$w(\Gamma) = t(n - 3) + x - 2 + r - t - 1 \quad (4)$$

colors.

If $t \geq k$, or $t = k - 1$ and $x \geq \frac{n}{2} + k - 1$, then $\Gamma$ wastes at least

$$(k - 1)(n - 3) + x - 2 + r - k = k(n - 2) + (r + x + 1 - 2k - n) \geq k(n - 2)$$

colors.

If $t = k - 1$ and $x < \frac{n}{2} + k - 1$, then suppose $y$ is a positive integer such that $x + y = \left[\frac{n}{2} + k - 1\right]$. Let $z = \left[\frac{n}{2} + k - 1\right]$. Recall that $n \geq 4k - 1$ and $x \geq 3$, and then $x + z - 3 \geq 7$. By Lemma 3, $(z - 1)_2 - (x - 1)_2 \geq y - 1$. We have

$$\sum_{i \in [r]}\left(\frac{|T_i| - 1}{2}\right) = (k - 1)\left(\frac{n-2}{2}\right) + \left(x - 1\right) + r - k$$

$$\leq (k - 1)\left(\frac{n-2}{2}\right) + \left(z - 1\right) + y + 1 + r - k$$

$$\leq (k - 1)\left(\frac{n-2}{2}\right) + \left(\frac{n}{2} + k - 1\right) - y + 1 + r - k$$

$$= \frac{4k - 3}{8}n^2 - \frac{8k - 7}{4}n + \frac{(k - 1)(k + 2)}{2} + r - y$$

$$= k\left(\frac{n}{2}\right) - \frac{n^2}{4} - \left(\frac{n^2}{8} + \frac{6k - 7}{4}n - \frac{(k + 2)(k - 1)}{2}\right) + r - y.$$
Then \( h(n) \geq 0 \) when \( n \geq \frac{1}{2} (\sqrt{160k^2 - 384k + 292} - 12k + 18) \). Thus \( h(n) \geq 0 \) when \( n \geq \frac{k}{2} + 9 \). Recall that \( n \geq 4k - 1 \), and then \( n \geq \frac{k}{2} + 9 \) holds for \( k \geq 3 \). So \( \Gamma \) wastes at least \( k(n - 2) \) colors if \( k \geq 3 \). If \( k = 2 \), then \( h(n) = \frac{1}{8} (n^2 + 6n - 32) \). Since \( n \geq 4k - 1 = 7 \), \( h(n) \geq 0 \). Therefore, \( \Gamma \) wastes at least \( k(n - 2) \) colors when \( k = 2 \).

If \( t \leq k - 2 \), then the number of trees of order 3 is at least \( r - t - 1 \). Recall that \( n \geq 4k - 1 \geq 7 \) and \( k \geq 2 \). By (3),

\[
  r - t - 1 \geq k \left( \frac{n}{2} - \frac{n^2}{4} - t \left( \frac{n - 2}{2} \right) - \left( \frac{x - 1}{2} \right) \right)
  \geq k \left( \frac{n}{2} - \frac{n^2}{4} - (k - 1) \left( \frac{n - 2}{2} \right) \right)
  \geq k(2n - 3) + \frac{1}{4} (n^2 - 10n + 12)
  \geq k(2n - 3) - \frac{9}{4} \geq k(n - 2).
\]

Thus, \( \Gamma \) wastes at least \( k(n - 2) \) colors. \( \square \)

For a graph \( G \), we use \( N_{uv} \) to denote the set of common neighbors of \( u \) and \( v \), and let \( n_{uv} = |N_{uv}| \), \( n_G = \min\{n_{uv} : u, v \in V(G) \text{ and } u \neq v\} \).

**Proposition 3** If \( G \) is a graph with \( \tau(G) \geq k \), then \( rmc_k(G) \leq m - k(n - 2) + n_G \).

**Proof** Suppose \( \Gamma \) is an extremal \( RM C_k \) coloring of \( G \). Let \( u, v \) be two vertices of \( G \) with \( n_{uv} = n_G \). Let \( V(G) = N[v] - \{u\} = A, N_{uv} = C \) and \( N(v) - \{u\} = B \). Then \( C \subseteq B \). Suppose that \( \mathcal{F} \) is the set of nontrivial trees containing \( u \) and \( v \), \( \mathcal{F} \) is the set of nontrivial trees containing \( u \) and at least one vertex of \( B \) but not \( v \), and \( \mathcal{H} \) is the set of nontrivial trees containing \( v \) and at least one vertex of \( A \) but not \( u \). Thus, \( \mathcal{F}, \mathcal{F} \) and \( \mathcal{H} \) are pairwise disjoint.

The vertex set \( A \) is partitioned into \( k + 1 \) pairwise disjoint subsets \( A_0, \ldots, A_k \) (some sets may be empty) such that every vertex of \( A_i \) is in exactly \( i \) nontrivial trees of \( \mathcal{F} \) for \( i \in \{0, \ldots, k - 1\} \) and every vertex of \( A_k \) is in at least \( k \) nontrivial trees of \( \mathcal{F} \). The vertex set \( B \) can also be partitioned into \( k + 1 \) pairwise disjoint subsets \( B_0, \ldots, B_k \) (some sets may be empty) such that every vertex of \( B_i \) is in exactly \( i \) nontrivial trees of \( \mathcal{F} \) for \( i \in \{0, \ldots, k - 1\} \) and every vertex of \( B_k \) is in at least \( k \) nontrivial trees of \( \mathcal{F} \). Then \( \mathcal{F} \) wastes

\[
  w_1 = \Sigma_{T \in \mathcal{T}}(|T| - 2) \geq \sum_{i=0}^{k} (\sum_{j=0}^{i} |A_j| + |B_j|)
\]

For every vertex \( w \) of \( A_i \), since \( N(v) \cap A = \emptyset \), there are at least \( k \) nontrivial trees containing \( v \) and \( w \). Since there are \( i \) such trees in \( \mathcal{F} \) for \( i \neq k \), there are at least \( k - i \)
nontrivial trees connecting \( v \) and \( w \) in \( \mathcal{H} \). Since every nontrivial tree of \( \mathcal{H} \) must contain \( v \) and a vertex of \( B \), \( \mathcal{H} \) wastes

\[
w_2 = \sum_{H \in \mathcal{H}} (|H| - 2) \geq \sum_{i=0}^{k} |A_i|
\]
colors.

Let \( C_i = \{ w : w \in B_i \cap C \text{ and } uw \text{ is a trivial edge } \} \). For each vertex \( w \) of \( B \), if \( w \in B_i - C_i \), then there are at least \( k \) nontrivial trees containing \( u \) and \( w \); if \( w \in C_i \), there are at least \( k-1 \) nontrivial trees containing \( u \) and \( w \). This implies that each vertex of \( B_i - C_i \), \( i \in \{0, \ldots, k-1\} \), is in at least \( k-i \) nontrivial trees of \( \mathcal{F} \), and each vertex of \( C_i \) is in at least \( k-i-1 \) nontrivial trees of \( \mathcal{F} \). Now we partition \( \mathcal{F} \) into two parts, \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \), such that

\[
\mathcal{F}_1 = \{ F \in \mathcal{F} : V(F) \subseteq B \cup \{u\} \}
\]

and

\[
\mathcal{F}_2 = \mathcal{F} - \mathcal{F}_1.
\]

Then for every \( F \) of \( \mathcal{F}_1 \), \( u \) connects a vertex of \( C \) in \( F \). Thus, there are at most \( |C| - \sum_{i=0}^{k} |C_i| \) trees in \( \mathcal{F}_1 \). Therefore, \( \mathcal{F} \) wastes

\[
w_3 = \sum_{F \in \mathcal{F}} (|F| - 2) \\ \geq \sum_{i=0}^{k} (k-i)|B_i - C_i| + \sum_{i=0}^{k-1} (k-i-1)|C_i| - \left( |C| - \sum_{i=0}^{k-1} |C_i| \right) \\ = -|C| + \sum_{i=0}^{k} (k-i)|B_i|
\]
colors.

According to the above discussion, \( \Gamma \) wastes at least

\[
w_1 + w_2 + w_3 \geq -|C| + \sum_{i=0}^{k} |A_i| + \sum_{i=0}^{k} |B_i| = k(n-2) - n_G
\]
colors. Therefore, \( rmc_k(G) \leq m - k(n-2) + n_G \). \( \Box \)

If \( G \) is not an \( s+1 \)-connected graph, then \( n_G \leq s \). Thus, we have the following result.

**Corollary 3** If \( G \) is a graph with \( \tau(G) \geq k \) and \( G \) is not \( s+1 \)-connected, then \( rmc_k(G) \leq m - k(n-2) + s \).

The next theorem decreases this upper bound by one when \( s = 1 \).

**Proposition 4** If \( G \) has a cut vertex and \( \tau(G) \geq k \geq 2 \), then \( rmc_k(G) = m - k(n-2) \).

**Proof** Let \( \Gamma \) be an extremal \( RMC_k \)-coloring of \( G \). Suppose that \( a \) is a vertex cut of \( G \) and \( A_1, \ldots, A_t \) are components of \( G - \{a\} \). Let \( w \) be a vertex of \( A_1 \), and let \( \mathcal{F} = \{ T_1, \ldots, T_r \} \) be the set of nontrivial trees connecting \( w \) and some vertices of
\[ \text{Proposition 6} \quad (\[4\]) \quad \text{If } G \text{ is a cycle of order } n, \text{ then } mc(G) \geq e(G) - \left\lfloor \frac{2n}{3} \right\rfloor. \]

By Proposition 6, if \( P \) is a Hamiltonian path of \( K_n \) with \( n \geq 4 \), then \( mc(G\setminus P) \geq e(G\setminus P) - \left\lfloor \frac{2n}{3} \right\rfloor \). The following result is obvious.

**Corollary 4** \( \text{rmc}_2(K_n) \geq \left\lfloor \frac{3n^2 - 13n}{6} \right\rfloor + 2, \quad n \geq 4. \)

**Remark 1** The above corollary implies that there are indeed some graphs with rainbow monochromatic \( k \)-edge-connection number greater than the lower bound. In fact, for any \( k \geq 2 \) and \( s \geq 2 \), there exist graphs with rainbow monochromatic \( k \)-
edge-connection number greater than or equal to \(m - k(n - 2) + s - 1\). We construct the \((k, s)\)-perfectly-connected graphs below. A graph \(G\) is called a \((k, s)\)-perfectly-connected graph if \(V(G)\) can be partitioned into \(s + 1\) parts \(\{v\}, V_1, \ldots, V_s\), such that \(\tau(G|V_i|) \geq k\), \(V_i, \ldots, V_s\) induces a corresponding complete \(s\)-partite graph (call it \(K^s\)), and \(v\) has precisely \(k\) neighbors in each \(V_i\). Since \(\tau(G|V_i|) \geq k\), each \(G|V_i|\) has \(k\) edge-disjoint spanning trees (say \(T^*_i, \ldots, T^*_{i}^k\)). Let the \(k\) neighbors of \(v\) in \(V_i\) be \(u_i^1, \ldots, u_i^k\) and let \(e_i^1 = vu_i^1, \ldots, e_i^k = vu_i^k\). Let \(T_j = \bigcup_{i \in [s]} e_j^i \cup \bigcup_{i \in [s]} T^*_{ij}\) for \(j \in \{2, \ldots, k\}\). Let \(\Gamma\) be an edge-coloring of \(G\) such that \(\Gamma(T^*_i \cup e_i^j) = i\) for \(i \in [s]\), \(\Gamma(T^*_j) = s + j - 1\) for \(j \in \{2, \ldots, k\}\), and the other edges are trivial. Then \(\Gamma\) is an \(RMC_k\)-coloring of \(G\) and \(|\Gamma(G)| = m - k(n - 2) + s - 1\), and thus \(rmc_k(G) \geq m - k(n - 2) + s - 1\). □

We propose an open problem below. If the answer for the problem is true, then it will cover our main Theorem 6.

**Problem 1** For an integer \(k \geq 2\) and a graph \(G\) with \(\tau(G) \geq k\), does \(rmc_k(G) \leq mc(G) - (k - 1)(n - 2)\) hold? More generally, does \(rmc_k(G) \leq rmc_t(G) - (k - t)(n - 2)\) hold for any integer \(1 \leq t < k\)?

## 5 Random Results

The following result can be found in text books.

**Lemma 5** ([1], Chernoff Bound) If \(X\) is a binomial random variable with expectation \(\mu\), and \(0 < \delta < 1\), then

\[
Pr[X < (1 - \delta)\mu] \leq \exp\left(-\frac{\delta^2 \mu}{2}\right)
\]

and if \(\delta > 0\),

\[
Pr[X > (1 + \delta)\mu] \leq \exp\left(-\frac{\delta^2 \mu}{2 + \delta}\right).
\]

Let \(p = \frac{\log n + a}{n}\). The authors in [5] proved that

\[
Pr[G(n, p) \text{ is connected}] \rightarrow \begin{cases} 
1, & a \rightarrow +\infty; \\
\exp^{-e^{-a}}, & |a| = O(1); \\
0, & a \rightarrow -\infty.
\end{cases}
\]

Thus, \(p = \frac{\log n}{n}\) is the threshold function for \(G(n, p)\) being connected.

A sufficient condition for \(G(n, p)\) to have an \(RMC_k\)-coloring almost surely is that \(T(G(n, p)) \geq k\) almost surely. For the STP number problem of \(G(n, p)\), Gao et al. proved the following results.

**Lemma 6** ([7]) For every \(p \in [0, 1]\), we have
\[ T(G(n,p)) = \min \left\{ \delta(G(n,p)), \left[ \frac{e(G(n,p))}{n-1} \right] \right\} \]

almost surely.

In this section, we denote \( \beta = \frac{2}{\log e - \log 2} \approx 6.51778 \).

**Lemma 7** ([7]) If

\[ p \geq \frac{\beta(\log n - \log \log n/2) + o(1)}{n-1}, \]

then \( T(G(n,p)) = \left[ \frac{e(G(n,p))}{n-1} \right] \) almost surely; if

\[ p \leq \frac{\beta(\log n - \log \log n/2) - o(1)}{n-1}, \]

then \( T(G(n,p)) = \delta(G(n,p)) \) almost surely.

We knew that \( m - k(n - 2) \) is a lower bound of \( rmck(G) \). Next is an upper bound of \( rmck(G) \). Although the upper bound is rough, it is useful for the subsequent proof.

**Proposition 7** If \( G \) is a graph with \( \tau(G) \geq k \), then \( rmck(G) \leq m - (k - 1)(n - 2) \).

**Proof** Since the result holds for \( k = 1 \), we only consider \( k \geq 2 \). Suppose \( \Gamma \) is an extremal \( RMCk \)-coloring of \( G \) and \( \mathcal{F} = \{T_1, \ldots, T_r\} \) is the set of nontrivial color-induced trees with \( |T_1| \geq \cdots \geq |T_r| \). Then

\[ k \binom{n}{2} - e(G) \leq \sum_{i \in [r]} \left( \frac{|T_i| - 1}{2} \right). \tag{5} \]

**Case 1.** \( T_1 \) is a spanning tree of \( G \).

Then \( \Gamma' \) is an extremal \( RMCk - 1 \)-coloring restricted on \( G' = G - E(T_1) \) (this result has been proved in Theorem 2). By induction on \( k \),

\[ |\Gamma(G')| = rmck_{k-1}(G') \leq e(G') - (k - 2)(n - 2). \]

Then

\[ rmck(G) = 1 + |\Gamma(G')| = 1 + rmck_{k-1}(G') \leq 1 + e(G') - (k - 2)(n - 2) \leq m - (k - 1)(n - 2). \]

**Case 2.** \( |T_i| \leq n - 1 \) for each \( i \in [r] \).

By Lemmas 1 and 2, the expression \( \sum_{i \in [r]} (|T_i| - 2) \), subjects to (5) and \( 3 \leq |T_i| \leq n - 1 \), is minimum when \( |T_1| = \cdots = |T_{r-1}| = n - 1 \) and \( |T_r| = x + 1 \), where \( x \) is an integer with \( 3 \leq x + 1 \leq n - 2 \).

If \( r \leq k - 1 \), then \( \sum_{i \in [r]} \left( \frac{|T_i| - 1}{2} \right) < (k - 1) \binom{n-2}{2} < k \binom{n}{2} - e(G) \), a contradiction to (5).

If \( r > k \), then \( \Gamma \) wastes at least \( k(n - 3) \geq (k - 1)(n - 2) \) colors. Thus \( rmck(G) \leq m - (k - 1)(n - 2) \).

If \( r = k \), then

\[ \text{...} \]
So, \(x^2 - x - \alpha \geq 0\), where
\[
\alpha = 2\binom{n}{2} + (2n - 3)(k - 1) - e(G) = 2\left[(2n - 3)(k - 1) + e(G)\right].
\]
The inequality holds when \(x \geq \frac{1 + \sqrt{1 + 4\alpha}}{2} \geq \sqrt{x}\). Thus, \(I\) wastes at least
\[
\Sigma_{i \in [k]}(|T_i| - 2) = (k - 1)(n - 2) + x - 1 \geq (k - 1)(n - 2) + \sqrt{x} - 1.
\]
Since \(k \geq 2\), \(\sqrt{x} \geq 1\). Thus \(\text{rmck}(G) \leq m - (k - 1)(n - 2)\).

**Theorem 7** Let \(k = k(n)\) be an integer such that \(\left\lceil \frac{n}{k}\right\rceil > k \geq 1\) and let \(\text{rmck}(K_n) > f(n) \geq k(n - 1)\). Then
\[
p = \begin{cases} 
\frac{f(n) + kn}{n^2}, & f(n) \geq O(n \log n) \text{ and } k = o(n); \\
\min \left\{ \frac{k \cdot \log n}{n}, \frac{n}{f(n)} \right\}, & f(n) = o(n \log n) \text{ and } k = o(n); \\
1, & k = O(n) \text{ and } f(n) < \text{rmck}(K_n).
\end{cases}
\]
is a sharp threshold function for the property \(\text{rmck}(G, p) \geq f(n)\).

**Proof** Let \(c\) be a positive constant and let \(E(||G(n, cp)||)\) be the expectation of the number of edges in \(G(n, cp)\). Then
\[
E(||G(n, cp)||) = \begin{cases} 
\frac{c(n - 1)}{2n} f(n) + \frac{c \cdot k(n - 1)}{2}, & f(n) \geq O(n \log n) \text{ and } k = o(n); \\
\frac{c \cdot k(n - 1)}{2}, & f(n) = o(n \log n), k = o(n) \text{ and } k > \log n; \\
\frac{c \log n(n - 1)}{2}, & f(n) = o(n \log n), k = o(n) \text{ and } k \leq \log n; \\
\frac{c^n}{2}, & k = O(n) \text{ and } f(n) < \text{rmck}(K_n).
\end{cases}
\]
By Lemma 5, both inequalities
\[
\Pr[||G(n, cp)|| < \frac{1}{2} E(||G(n, cp)||)] \leq \exp \left( -\frac{1}{8} E(||G(n, cp)||) \right) = o(1)
\]
and
\[
\Pr[||G(n, cp)|| > \frac{3}{2} E(||G(n, cp)||)] \leq \exp \left( -\frac{1}{10} E(||G(n, cp)||) \right) = o(1)
\]
hold for each \(p\).

**Case 1.** \(k = O(n)\), i.e., there is an \(l \in \mathbb{R}^+\) such that \(l \cdot n \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor\).

Since \(G(n, p) = K_n\), \(\text{rmck}(G(n, p)) \geq f(n)\) always holds. On the other hand, we have
\[ ||G(n, l \cdot p)|| \leq \frac{3}{2} E(||G(n, l \cdot p)||) = \frac{3l}{2} \cdot \binom{n}{2} < k(n - 2) \]

almost surely. By Claim 3, \( G(n, l \cdot p) \) does not have \( RMC_k \)-colorings almost surely.

**Case 2.** \( k = o(n) \).

**Case 2.1.** \( f(n) \geq O(n \log n) \).

Then, there is an \( s \in \mathbb{R}^+ \) and \( f(n) \geq s \cdot n \log n \). Let

\[
c_1 = \begin{cases} 
\beta + 1, & s \geq 1; \\
\beta + 1, & 0 < s < 1.
\end{cases}
\]

Since \( f(n) \geq s \cdot n \log n \), we have

\[
c_1 p \geq \frac{(\beta + 1)(\log n + kn)}{n} \geq \frac{\beta (\log n - \log \log n/2) + o(1)}{n - 1}.
\]

Since

\[
||G(n, c_1 p)|| \geq \frac{1}{2} E(||G(n, c_1 p)||) = \frac{\beta + 1}{2} \cdot \frac{n - 1}{2n} f(n) + \frac{k(n - 1)(\beta + 1)}{4}
\]

almost surely, by Lemma 7, \( T(G(n, c_1 p)) = \left[ \frac{||G(n, c_1 p)||}{n - 1} \right] > k \) almost surely, i.e., \( G(n, c_1 p) \) has \( RMC_k \)-colorings almost surely. Therefore,

\[
rmc_k(G(n, c_1 p)) \geq ||G(n, c_1 p)|| - k(n - 2)
\]

\[
\geq \frac{\beta + 1}{2} \cdot \frac{n - 1}{2n} f(n) + \frac{k(n - 1)(\beta + 1)}{4} - k(n - 2)
\]

\[
> \frac{(\beta + 1)(n - 1)}{4n} f(n)
\]

\[
> f(n)
\]

almost surely.

Let \( c_2 = \frac{2}{3} \). Then

\[
||G(n, c_2 p)|| \leq \frac{3}{2} E(||G(n, c_2 p)||)
\]

\[
\leq \frac{3c_2}{2} \cdot \frac{n - 1}{2n} f(n) + \frac{3c_2}{2} \cdot \frac{k(n - 1)}{2}
\]

\[
< \frac{1}{2} [f(n) + k(n - 1)]
\]

almost surely. Thus, either \( G(n, c_2 p) \) does not have \( RMC_k \)-colorings almost surely, or

\[
rmc_k(G(n, c_2 p)) < ||G(n, c_2 p)|| - (k - 1)(n - 2) < \frac{1}{2} f(n)
\]

almost surely [(recall that \( rmc_k(G) \leq m - (k - 1)(n - 2) \) by Proposition 7)].
Case 2.2. \( f(n) = o(n \log n) \).
If \( k \leq \log n \), then \( p = \frac{\log n}{n} \). Let \( c_1 = \beta + 1 \) and \( c_2 = \frac{1}{2} \) be two constants. Since
\[
c_1p > \frac{(\beta + 1) \log n}{n} \geq \frac{\beta(\log n - \log \log n/2 + \omega(1))}{n - 1},
\]
by Lemma 7, \( T(G(n, c_1p)) = \left\lfloor \frac{||G(n, c_1p)||}{n-1} \right\rfloor \) almost surely. Since
\[
||G(n, c_1p)|| \geq \frac{1}{2} E(||G(n, c_1p)||) = \frac{\log n(n-1)(\beta + 1)}{4}
\]
almost surely, \( T(G(n, c_1p)) \geq \log n \geq k \) almost surely, i.e., \( G(n, c_1p) \) has \( RMC_k \)-coloring almost surely. Therefore,
\[
rmc_k(G(n, c_1p)) \geq ||G(n, c_1p)|| - k(n - 2)
\]
\[
\geq \frac{\log n(n-1)(\beta + 1)}{4} - k(n - 2)
\]
\[
\geq \frac{3\log n(n-1)}{4} > f(n)
\]
almost surely. For \( G(n, c_2p) \), since \( c_2p = \frac{\log n}{2n} \), \( G(n, c_2p) \) is not connected almost surely, i.e., \( G(n, c_2p) \) does not have \( RMC_k \)-colorings almost surely.
If \( k > \log n \) and \( k = o(n) \), then \( p = \frac{k}{n} \). Let \( c_1 = \beta + 1 \) and \( c_2 = 1 \). Then
\[
c_1p = \frac{(\beta + 1) k}{n} > \frac{(\beta + 1) \log n}{n} \geq \frac{\beta(\log n - \log \log n/2 + \omega(1))}{n - 1},
\]
i.e., \( T(G(n, c_1p)) = \left\lfloor \frac{||G(n, c_1p)||}{n-1} \right\rfloor \) almost surely. Since
\[
||G(n, c_1p)|| \geq \frac{1}{2} E(||G(n, c_1p)||) = \frac{k(n-1)(\beta + 1)}{4}
\]
almost surely, \( T(G(n, c_1p)) \geq k \) almost surely, i.e., \( G(n, c_1p) \) has \( RMC_k \)-colorings almost surely. Thus
\[
rmc_k(G(n, c_1p)) \geq ||G(n, c_1p)|| - k(n - 2) \geq \frac{3}{4} k(n - 1) > \frac{3}{4} (n - 1) \log n > f(n)
\]
almost surely. For \( G(n, c_2p) \), since
\[
||G(n, c_2p)|| \leq \frac{3}{2} E(||G(n, c_2p)||) = \frac{3}{4} k(n-1) < k(n - 2)
\]
almost surely. By Claim 3, \( G(n, c_2p) \) does not have \( RMC_k \)-colorings almost surely.

Remark 2 Since \( rmc_k(G) = rmc_k(K_n) \) if and only if \( G = K_n \), we only concentrate on the case \( 1 \leq f(n) < rmc_k(K_n) \). If \( n \) is odd, then \( G \) has \( RMC_{[\frac{4}{2}]} \)-colorings if and only if \( G = K_n \). So, we are not going to consider the case \( k = [\frac{4}{2}] \). \( \square \)
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