Schneider-Teitelbaum duality for locally profinite groups

Tomoki Mihara

Abstract. We define monoidal structures on several categories of linear topological modules over the valuation ring of a local field, and study module theory with respect to the monoidal structures. We extend the notion of the Iwasawa algebra to a locally profinite group as a monoid with respect to one of the monoidal structure, which does not necessarily form a topological algebra. This is one of the main reasons why we need monoidal structures. We extend Schneider–Teitelbaum duality to duality applicable to a locally profinite group through the module theory over the generalised Iwasawa algebra, and give a criterion of the irreducibility of a unitary Banach representation.

1 Introduction

Let \( k \) denote a non-Archimedean local field, and \( O_k \subset k \) the valuation ring of \( k \). The paper is devoted to two topics. One topic is to give monoidal structures on several categories of linear topological \( O_k \)-modules. We are interested mainly in the closed symmetric monoidal category \( \mathcal{C}_\ell^{\text{CG}} \) of CG.
linear topological $O_k$-modules. A CG linear topological $O_k$-module is a linear topological $O_k$-module given as the colimit of totally bounded $O_k$-submodules. By the definition, it is a module theoretic analogue of a compactly generated topological space. We show that every Banach $k$-vector space and every compact linear topological $O_k$-module are CG. Therefore $\mathcal{C}_\ell^{cg}$ contains both of the categories of Banach $k$-vector spaces and compact Hausdorff flat linear topological $O_k$-modules, which play the roles of the foundation in Schneider–Teitelbaum duality (cf. [13] Theorem 2.3).

The other topic is to define a generalised Iwasawa algebra $O_k[[G]]$ associated to a locally profinite group $G$, and to extend Schneider–Teitelbaum duality, which is applicable to a profinite group, to duality applicable to $G$ by using module theory over $O_k[[G]]$. We note that $O_k[[G]]$ is defined as a monoid in $\mathcal{C}_\ell^{cg}$, and does not necessarily form a topological $O_k$-algebra. This is one of the main reasons why we need monoidal structures. As the classical Iwasawa algebra associated to a profinite group is naturally identified with the $O_k$-algebra of $O_k$-valued measures, $O_k[[G]]$ is naturally identified with the $O_k$-algebra of $O_k$-valued measures on $G$ satisfying a certain property called the normality. As the original Schneider–Teitelbaum duality is given by a module theoretic interpretation of a Banach $k$-linear representations through the integration of the action along measures (cf. [13] Corollary 2.2), the generalised Schneider–Teitelbaum duality is given by a module theoretic interpretation through the integration of the action of $G$ by normal measures.

As applications, we establish a criterion of the irreducibility of a unitary Banach $k$-linear representation of $G$, and give a description of the continuous induction of a unitary Banach $k$-linear representation of a closed subgroup $P \subset G$ such that the homogeneous space $P \backslash G$ is compact. In particular, we give an explicit description of the continuous parabolic induction for the case $G$ is an algebraic group over a local field so that the representation space of the continuous parabolic induction is independent of the choice of the action of $P$.

We explain the contents of this paper. In §2.1, we study several categories of linear topological $O_k$-modules. In §2.2, we introduce a notion of the normality of an $O_k$-valued measure on a topological space. In §3.1, we define monoidal structures on several categories of linear topological $O_k$-modules. In §3.2, we define a notion of a CGLT $O_k$-algebra as a monoid in
\( \mathcal{C}_c^g \), which is a counterpart of a topological \( O_k \)-algebra, and define \( O_k[[G]] \) as a CGLT \( O_k \)-algebra. In §3.3, we define a notion of a CGLT module over a CGLT \( O_k \)-algebra, which is a counterpart of a topological left module over a topological \( O_k \)-algebra. In §4.1, we recall a unitary Banach \( k \)-linear representation of \( G \) and interpret it in terms of a CGLT \( O_k[[G]] \)-module. In §4.2, we interpret a continuous action of \( G \) on a compact Hausdorff flat linear topological \( O_k \)-module in terms of a CGLT \( O_k[[G]] \)-module. In §4.3, we define a notion of the dual of a unitary Banach \( k \)-linear representation of \( G \), and extend Schneider–Teitelbaum duality to duality applicable to \( G \). In §5.1, we study the dual of several operations on Banach \( k \)-linear representations such as the continuous induction. In §5.2, we give an explicit description of the continuous parabolic induction in the case where \( G \) is an algebraic group.

2 Preliminaries

Let \( k \) denote a local field, that is, a complete discrete valuation field with finite residue field, \( O_k \subset k \) the valuation ring of \( k \), and \( G \) a locally profinite group. We denote by \( \omega \) the set of natural numbers. For a set \( X \), we denote by \( \mathcal{P}_{<\omega}(X) \) the set of finite subsets of \( X \). Since we deal with many pairs, we abbreviate \((\bullet_i)_{i=0}^1 \) to \((\bullet_i)\), \( \sum_{i=0}^1 \bullet_i \) to \( \sum \bullet_i \), and \( \prod_{i=0}^1 \bullet_i \) to \( \prod \bullet_i \).

Let \( \Theta \) be a category. We say that \( \Theta \) is \( \omega \)-cocomplete (respectively, cocomplete, complete) if it admits all small filtered colimits (respectively, colimits, limits), and is bicomplete if it is cocomplete and complete. Let \( F \) be a functor. We say that \( F \) is \( \omega \)-cocontinuous (respectively, cocontinuous, continuous) if it commutes with all small filtered colimits (respectively, colimits, limits), and is bicontinuous if it is cocontinuous and continuous. We denote by \( \text{Set} \) the bicomplete category of sets and maps, and by \( \text{Top} \) the bicomplete category of topological spaces and continuous maps. We abbreviate \( \text{Hom}_{\text{Top}} \) to \( C \).

2.1 Linear topological modules Let \( M \) be a topological \( O_k \)-module, and \( C \subset M \) a subset. We say that \( C \) is pre-compact (respectively, complete) if \( C \) is totally bounded (respectively, complete) with respect to the restriction of the uniform structure on \( M \) associated to the structure as a topological Abelian group to \( C \). By the definition of the uniformity on \( M \), \( C \) is
totally bounded if and only if for any open neighbourhood $U \subset M$ of $0 \in M$, there exists a finite subset $C_0 \subset C$ such that $C \subset \{m_0 + m_1 \mid (m_0, m_1) \in U \times C_0\}$. The following are well-known facts (cf. [3] 8.3.2 Theorem, [4], and [3] 8.3.16 Theorem, respectively) on the pre-compactness:

**Proposition 2.1.** (i) A $C \subset M$ is pre-compact if and only if every subset of the closure of $C$ in $M$ is pre-compact.

(ii) A $C \subset M$ is compact, that is, every open covering admits a finite subcovering, if and only if $C$ is pre-compact and every Cauchy net in $C$ is a convergent net in $C$.

(iii) A $C \subset M$ is compact and Hausdorff if and only if $C$ is pre-compact and complete.

We denote by $\mathcal{O}(M)$ the set of open $O_k$-submodules of $M$, and by $\mathcal{K}(M)$ the set of pre-compact $O_k$-submodules of $M$. We say that $M$ is linear if $\mathcal{O}(M)$ forms a fundamental system of neighbourhoods of $0 \in M$. We have two examples of linear topological $O_k$-modules.

**Example 2.2.** (i) We denote by $\overline{M}$ the underlying $O_k$-module of $M$ equipped with the topology generated by $\{m + L \mid (m, L) \in M \times \mathcal{O}(M), \#(M/L) < \infty\}$. Then $\overline{M}$ forms a pre-compact linear topological $O_k$-module, and the identity map $\pi_M : M \to \overline{M}$ is continuous.

(ii) Let $S$ be a set. A map $f : S \to M$ is said to vanish at infinity if for any $L \in \mathcal{O}(M)$, there is an $S_0 \in \mathcal{P}_{<\omega}(S)$ such that $f(s) \in L$ for any $s \in S \setminus S_0$. We denote by $C_0(S, M)$ the $O_k$-module of maps $f : S \to M$ vanishing at infinity equipped with the topology generated by $\{f + C_0(S, L) \mid (f, L) \in C_0(S, M) \times \mathcal{O}(M)\}$. Then $C_0(S, M)$ forms a linear topological $O_k$-module.

We denote by $\mathcal{C}_\ell$ the $O_k$-linear category of linear topological $O_k$-modules and continuous $O_k$-linear homomorphisms. We abbreviate $\text{Hom}_{\mathcal{C}_\ell}$ to $\mathcal{L}$. Since the pre-image of an open $O_k$-submodule by a continuous $O_k$-linear homomorphism is an open $O_k$-submodule, the correspondence $M \rightsquigarrow \mathcal{O}(M)$ gives a functor $\mathcal{O} : \mathcal{C}_\ell^{\text{op}} \to \text{Set}$. On the other hand, the correspondence $M \rightsquigarrow \mathcal{K}(M)$ gives a functor $\mathcal{K} : \mathcal{C}_\ell \to \text{Set}$ by the following:

**Proposition 2.3.** Let $(M_i) \in \text{ob}(\mathcal{C}_\ell^2)$ and $f \in \mathcal{L}((M_i))$. For any pre-compact subset $C_0 \subset M_0$, $f(C_0) \subset M_1$ is pre-compact.

**Proof.** The assertion follows from [3] p. 445 by the uniform continuity of $f$.
We will use $\mathcal{O}(M)$ and $\mathcal{K}(M)$ as index sets of limits and colimits. They are filtered and cofiltered with respect to inclusions by Proposition 2.1 (i) and the following:

**Proposition 2.4.** The sets $\mathcal{O}(M)$ and $\mathcal{K}(M)$ are closed under finite sum.

**Proof.** The assertion for $\mathcal{O}(M)$ immediately follows from [3] p. 433. The assertion for $\mathcal{K}(M)$ immediately follows from Proposition 2.3 and [3] 8.3.3 Theorem, because $\sum M_j$ is the image of the addition $\prod M_i \to M$ for any $(M_i) \in \mathcal{K}(M)^2$.

As a consequence, we obtain the following variant of [13] Lemma 1.5 i:

**Corollary 2.5.** For any pre-compact subset $C \subset M$, $\sum_{m \in C} O_k m$ is pre-compact.

**Proof.** Let $L \in O(M)$. Take a $C_0 \in \mathcal{P}_{<\omega}(C)$ satisfying $C \subset \bigcup_{m \in C_0} (m + L)$. We have $O_k m \in \mathcal{K}(M)$ for any $m \in M$ by Proposition 2.3, and hence $\sum_{m \in C_0} O_k m \in \mathcal{K}(M)$ by Proposition 2.4. Take a $K_0 \in \mathcal{P}_{<\omega}(\sum_{m \in C_0} O_k m)$ satisfying $\sum_{m \in C_0} O_k m \subset \bigcup_{m \in K_0} (m + L)$. We obtain

$$\sum_{m \in C} O_k m \subset \bigcup_{m \in C_0} O_k(m + L) = \bigcup_{m \in C_0} (O_k m + L) \subset \bigcup_{m \in K_0} (m + L).$$

It implies $\sum_{m \in C} O_k m \in \mathcal{K}(M)$.

We denote by $\mathcal{C}$ the category of $O_k$-modules and $O_k$-linear homomorphisms. We denote by $U : \mathcal{C}_\ell \to \text{Top}$ and $F : \mathcal{C}_\ell \to \mathcal{C}$ the forgetful functors.

**Proposition 2.6.** The category $\mathcal{C}_\ell$ is bicomplete, and $U$ (respectively, $F$) is $\omega$-cocontinuous and continuous (respectively, bicontinuous).

**Proof.** The completeness of $\mathcal{C}_\ell$ and the continuity of $U$ and $F$ follow from the definition of the limits in Top and $\mathcal{C}$. The $\omega$-cocontinuity of $\mathcal{C}_\ell$ and the $\omega$-cocontinuity of $U$ and $F$ follow from [6] Proposition 1.3. For any small family $(M_s)_{s \in S}$ in $\mathcal{C}_\ell$, $\bigoplus_{s \in S} F(M_s)$ forms a linear topological $O_k$-module with respect to the topology generated by $\{ m + \bigoplus_{s \in S} F(L_s) | (m, (L_s)_{s \in S}) \in (\bigoplus_{s \in S} F(M_s)) \times \prod_{s \in S} O(M_s) \}$, and satisfies the universality of the direct sum of $(M_s)_{s \in S}$ in $\mathcal{C}_\ell$. Thus $\mathcal{C}_\ell$ is cocomplete, and $F$ is cocontinuous.
Since we will introduce several full subcategories of $C_\ell$, we prepare a convention for colimits (respectively, limits) in order to avoid the ambiguity of categories in which we consider the universality. Let $(M_s)_{s \in S}$ be a small diagram in a full subcategory $\Theta \subset C_\ell$. We always denote by $\lim_{\leftarrow s \in S} M_s$ (respectively, $\lim_{\rightarrow s \in S} M_s$) the colimit (respectively, limit) of $(M_s)_{s \in S}$ in $C_\ell$ but not in $\Theta$. As an immediate consequence of Proposition 2.6, we obtain the following:

**Corollary 2.7.** Let $(M_s)_{s \in S}$ be a small diagram in $C_\ell$. For any subset $U \subset \lim_{\leftarrow s \in S} M_s$ (respectively, $U \subset \lim_{\rightarrow s \in S} M_s$), $U$ is open if and only if the preimage of $U$ in $M_s$ is open for any $s \in S$ (respectively, if and only if for any $m \in U$, there is an $(L_s)_{s \in S} \in \prod_{s \in S} \mathcal{O}(M_s)$ satisfying $\{s \in S \mid L_s \neq M_s\} \in \mathcal{P}_{(<\omega}S)$ and $m + \prod_{s \in S} \mathcal{F}(L_s) \subset U$.

We denote by $C^{c}_\ell \subset C_\ell$ the full subcategory of pre-compact linear topological $O_k$-modules and by $\mathcal{F}^c$ the inclusion $C^{c}_\ell \hookrightarrow C_\ell$. We put $\mathcal{U}^c := \mathcal{U} \circ \mathcal{F}^c$ and $\mathcal{F}^c := \mathcal{F} \circ \mathcal{F}^c$.

**Proposition 2.8.** (i) The correspondence $M \rightsquigarrow M$ gives a functor $\overline{\bullet}: C_\ell \rightarrow C^{c}_\ell$ left adjoint to $\mathcal{F}^c$ such that the counit is given as a natural equivalence.

(ii) The topological $O_k$-module $M$ is linear and pre-compact if and only if $\pi^c_M$ is an open map.

(iii) The category $C^{c}_\ell$ is bicomplete, and the colimit of a small diagram $(M_s)_{s \in S}$ in $C^{c}_\ell$ is given by $\lim_{\leftarrow S} \mathcal{F}^c(M_s)$.

**Proof.** The functoriality of $\overline{\bullet}$ and the assertion (ii) immediately follow from the definition. The assertion (iii) immediately follows from the assertion (i) and Proposition 2.6. We show the assertion (i). We consider two functors $F, G: C^{c}_\ell \times C^{c}_\ell \rightarrow \text{Set}$ given as $F := \mathcal{L}(\overline{\bullet}, \bullet_1)$ and $G := \mathcal{L}(\bullet_0, \mathcal{F}^c(\bullet_1))$. The correspondence $M \rightsquigarrow \pi^c_M$ gives a unit $\pi^c: \text{id}_{\mathcal{C}^c} \Rightarrow \mathcal{F}^c \circ \overline{\bullet}$. We have a counit $(\pi^c_{\mathcal{F}^c})^{-1}: \overline{\bullet} \circ \mathcal{F}^c \Rightarrow \text{id}_{\mathcal{C}^c}$, which is a natural equivalence by the assertion (ii). For a $K \in \text{ob}(\mathcal{C}^c)$, we consider maps $T_{M,K}: F(M, K) \rightarrow G(M, K), f \mapsto f \circ \pi^c_M$ and $T'_{M,K}: G(M, K) \rightarrow F(M, K), f \mapsto (\pi^c_{\mathcal{F}^c(K)})^{-1} \circ \overline{f}$. The correspondences $(M, K) \rightsquigarrow T_{M,K}, T'_{M,K}$ give natural transformations $T: F \Rightarrow G$ and $T': G \Rightarrow F$ satisfying $T \circ T' = \text{id}_G$ and $T' \circ T = \text{id}_F$ by the bijectivity of of values of $\pi^c$. We obtain adjunction data $(\overline{\bullet}, \mathcal{F}^c, T, \pi^c, (\pi^c_{\mathcal{F}^c})^{-1})$ between $\mathcal{C}^{c}_\ell$ and $\mathcal{C}_\ell$. It implies that $(\overline{\bullet})$ is left adjoint to $\mathcal{F}^c$. \qed
Suppose that $M$ is linear in the following in this subsection. Then $\mathcal{X}(M)$ forms a small filtered diagram in $\mathcal{C}_\ell$ by Proposition 2.4. We put $M_\mathcal{X} := \lim_{\longrightarrow} K \in \mathcal{X}(M)$. By the universality of the colimit, the system of inclusions induces a continuous injective $O_k$-linear homomorphism $\iota_M^{cg} : M_\mathcal{X} \to M$. By Corollary 2.5, $\iota_M^{cg}$ is bijective. We show that $\iota_M^{cg}$ preserves the pre-compactness of $O_k$-submodules.

**Proposition 2.9.** Let $K \subset M$ be an $O_k$-submodule of $M$. Put $K' := (\iota_M^{cg})^{-1}(K)$.

(i) If $K$ is pre-compact, then $\iota_M^{cg}|_{K'}$ is a homeomorphism onto $K$.

(ii) The pre-compactness of $K$ is equivalent to that of $K'$.

**Proof.** The assertion (ii) follows from Proposition 2.3 and the assertion (i). We show the assertion (i). By $K \in \mathcal{X}(M)$, we have $\iota_M^{cg}(K') = K$. Let $L \in \mathcal{O}(M_\mathcal{X})$. By $\iota_M^{cg}(K') = K$ and the injectivity of $\iota_M^{cg}$, we have $\iota_M^{cg}(L \cap K') = \iota_M^{cg}(L) \cap K$, and hence $\iota_M^{cg}(L \cap K') \in \mathcal{O}(K)$. It implies that $\iota_M^{cg}|_{K'}$ is an open map onto $K$.

We say that $M$ is $CG$ if $\iota_M^{cg}$ is an isomorphism in $\mathcal{C}_\ell$. We denote by $\mathcal{C}^{cg}_\ell \subset \mathcal{C}_\ell$ the full subcategory of CG linear topological $O_k$-modules and by $\mathcal{I}^{cg}$ the inclusion $\mathcal{C}^{cg}_\ell \hookrightarrow \mathcal{C}_\ell$. We put $\mathcal{U}^{cg} := \mathcal{U} \circ \mathcal{I}^{cg}$ and $\mathcal{F}^{cg} := \mathcal{F} \circ \mathcal{I}^{cg}$. We study properties of $\mathcal{C}^{cg}_\ell$ analogous to those of the category of compactly generated topological spaces.

**Corollary 2.10.** (i) The correspondence $M \rightsquigarrow M_\mathcal{X}$ gives a functor $(\bullet)_\mathcal{X} : \mathcal{C}_\ell \to \mathcal{C}^{cg}_\ell$ right adjoint to $\mathcal{I}^{cg}$ such that the counit is given as a natural equivalence.

(ii) The category $\mathcal{C}^{cg}_\ell$ is bicomplete, and the colimit of a small diagram $(M_s)_{s \in S}$ in $\mathcal{C}^{cg}_\ell$ is given by $(\lim_{\longrightarrow} s \in S (\mathcal{I}^{cg}(M_s)))_\mathcal{X}$.

**Proof.** To begin with, we show that $\mathcal{C}^{cg}_\ell$ is closed under small colimits in $\mathcal{C}_\ell$. Let $(M_s)_{s \in S}$ be a small diagram in $\mathcal{C}^{cg}_\ell$. Put $M := \lim_{\longrightarrow} s \in S \mathcal{I}^{cg}(M_s)$. In order to verify that $M$ is pre-compactly generated, it suffices to show $\iota_M^{cg}(L) \in \mathcal{O}(M)$ for any $L \in \mathcal{O}(M_\mathcal{X})$. Let $s \in S$. We denote by $L_s$ the preimage of $\iota_M^{cg}(L)$ in $M_s$. Let $K_0 \in \mathcal{X}(M_s)$. We denote by $K \subset M$ the image of $K_0$. By Proposition 2.3 and Proposition 2.9 (ii), we have $\iota_M^{cg}((\iota_M^{cg})^{-1}(K)) \in \mathcal{X}(M_\mathcal{X})$. It ensures $L \cap (\iota_M^{cg})^{-1}(K) \in \mathcal{O}((\iota_M^{cg})^{-1}(K))$. By Proposition 2.9 (i), we obtain
\[ \iota_M^\text{cg}(L) \cap K \in \mathcal{O}(K) \] and hence \( L_s \cap K_0 \in \mathcal{O}(K_0) \). It ensures \( L_s \in \mathcal{O}(M_s) \) because \( M_s \) is CG. It implies \( \iota_M^\text{cg}(L) \in \mathcal{O}(M) \) by Corollary 2.7.

We show the assertion (i). Since \( \mathcal{C}_\ell^\text{cg} \) is closed under small colimits in \( \mathcal{C}_\ell \), the correspondence \( M \rightsquigarrow M_\mathcal{X} \) gives a functor \((\bullet)_\mathcal{X} : \mathcal{C}_\ell \to \mathcal{C}_\ell^\text{cg} \) by Proposition 2.3 and Proposition 2.9 (i). We consider two functors \( F, G : (\mathcal{C}_\ell^\text{cg})^\text{op} \times \mathcal{C}_\ell \to \text{Set} \) given as \( F := \mathcal{L}(\mathcal{I}_\mathcal{X}(\bullet_0), \bullet_1) \) and \( G := \mathcal{L}(\bullet_0, (\bullet_1)_\mathcal{X}) \). The correspondence \( M \rightsquigarrow \iota_M^\text{cg} \) gives a unit \( \iota^\text{cg} : \mathcal{I}_\mathcal{X}(\bullet)_\mathcal{X} \to \text{id}_{\mathcal{C}_\ell} \), and we also have a counit \((\iota^\text{cg})^{-1} : \text{id}_{\mathcal{C}_\ell^\text{cg}} \Rightarrow (\bullet)_\mathcal{X} \circ \mathcal{J}^\text{cg} \), which is a natural equivalence by definition. For an \((M_i) \in \text{ob}(\mathcal{C}_\ell^\text{cg} \times \mathcal{C}_\ell)\), we consider maps \( T_{(M_i)} : F((M_i)) \to G((M_i)) \), \( f \mapsto f \circ \iota^\text{cg} \circ (\iota^\text{cg})^{-1} \) and \( T'_{(M_i)} : G((M_i)) \to F((M_i)) \), \( f \mapsto \iota^\text{cg} \circ f \). The correspondences \((M_i) \rightsquigarrow T_{(M_i)}\), \( T'_{(M_i)} \) give natural transformations \( T : F \Rightarrow G \) and \( T' : G \Rightarrow F \) satisfying \( T \circ T' = \text{id}_G \) and \( T' \circ T = \text{id}_F \) by the bijectivity of values of \( \iota \). We obtain adjunction data \((\mathcal{J}^\text{cg}, (\bullet)_\mathcal{X}, T, \iota^\text{cg}, (\iota^\text{cg})^{-1}) \) between \( \mathcal{C}_\ell \) and \( \mathcal{C}_\ell^\text{cg} \). It implies that \((\bullet)_\mathcal{X} \) is right adjoint to \( \mathcal{J}^\text{cg} \).

We show the assertion (ii). By the assertion (i), \((\bullet)_\mathcal{X} \) is continuous and \( \mathcal{J}^\text{cg} \) is cocontinuous. Since the counit \((\iota^\text{cg})^{-1} \) is a natural equivalence, \( \mathcal{C}_\ell^\text{cg} \) is complete by Proposition 2.6. Since we have already verified that \( \mathcal{C}_\ell^\text{cg} \) is closed under small colimits in \( \mathcal{C}_\ell \), it implies the assertion (ii) by Proposition 2.6.

We have three criteria of CG linear topological \( O_k \)-modules.

**Proposition 2.11.** (i) If \( M \) is CG, then so is every closed \( O_k \)-submodule of \( M \).

(ii) If \( M \) is locally compact, then \( M \) is CG.

(iii) If \( M \) is first countable, then \( M \) is CG.

*Proof.* The assertion (ii) follows from Proposition 2.1 (ii) and Proposition 2.9 (i), because \( M \) is locally compact if and only if \( M \) admits a compact clopen \( O_k \)-submodule. We verify the assertion (i). Let \( M_0 \subset M \) be a closed \( O_k \)-submodule. Since \( \iota_M^\text{cg} \) is an isomorphism in \( \mathcal{C}_\ell \), \((\iota_M^\text{cg})^{-1}(M_0) \) is closed in \( M_\mathcal{X} \). Therefore \( \iota_M^\text{cg} \) induces a homeomorphism \( \lim_{\to K \in \mathcal{X}(M_0)} (\mathcal{U}^\text{cg}(K) \cap \mathcal{U}(M_0)) \to \mathcal{U}(M_0) \) by [6] Lemma 2.23. By Corollary 2.7, we obtain an isomorphism \( \lim_{\to K \in \mathcal{X}(M)} (K \cap M_0) \to M_0 \). By Proposition 2.1 (i), \( K \cap M_0 \) lies in \( \mathcal{X}(M_0) \) for any \( K \in \mathcal{X}(M) \). It implies that \( M_0 \) is CG by Corollary 2.10 (i).
We verify the assertion (iii). Let $L \in \mathcal{O}(M_{\mathcal{K}})$. We show $i^c_{\mathcal{M}}(L) \notin \mathcal{O}(M)$. Assume $i^c_{\mathcal{M}}(L) \notin \mathcal{O}(M)$. Take an decreasing sequence $(L_r)_{r \in \omega} \in \mathcal{O}(M)^{\omega}$ such that $\{L_r \mid r \in \omega\}$ forms a fundamental system of neighbourhoods of $0 \in M$. By the assumption, we have $L_r \setminus i^c_{\mathcal{M}}(L) \neq \emptyset$ for any $r \in \omega$. Take an $(m_r)_{r \in \omega} \in \prod_{r \in \omega}(L_r \setminus i^c_{\mathcal{M}}(L))$. Put $C := \{m_r \mid r \in \omega\}$. We have $C = \bigcup_{h=0}^r m_h + L_r$ for any $r \in \omega$, and hence $C$ is pre-compact. Put $K := \sum_{m \in C} O_k m \subset M$. By Corollary 2.5, we have $K \in \mathcal{K}(M)$. It ensures $i^c_{\mathcal{M}}(L) \cap K \in \mathcal{O}(K)$. By $0 \in i^c_{\mathcal{M}}(L) \cap K$, there is an $r \in \omega$ such that $L_r \cap K \subset i^c_{\mathcal{M}}(L) \cap K$. We obtain $m_r \in L_r \cap K \subset i^c_{\mathcal{M}}(L) \cap K$, which contradicts $m_r \notin i^c_{\mathcal{M}}(L)$. It implies $i^c_{\mathcal{M}}(L) \notin \mathcal{O}(M)$. Thus $M$ is CG. 

We survey Schikhof duality (cf. [10] Theorem 4.6, [13] Theorem 1.2, and [7] Theorem 2.2). We follow the convention of Banach $k$-vector space in [7] §1.2. We denote by $\mathcal{C}^c_{\ell} \subset \mathcal{C}_{\ell}$ the full subcategory of compact Hausdorff flat linear topological $O_k$-modules, by $\text{Ban}(k)$ the $k$-linear category of Banach $k$-vector spaces and bounded $k$-linear homomorphisms, by $\text{Ban}_{\leq}(k) \subset \text{Ban}(k)$ the $O_k$-linear subcategory of submetric $k$-linear homomorphisms, and by $\text{Ban}^{ur}_{\leq}(k) \subset \text{Ban}_{\leq}(k)$ the full subcategory of unramified Banach $k$-vector spaces. By Proposition 2.1 (ii), $\mathcal{C}^c_{\ell}$ is a full subcategory of $\mathcal{C}^c_{\ell}$. For a $(V_i) \in \text{ob}(\text{Ban}^{ur}_{\leq}(k))^2$, we denote by $\mathcal{J}((V_i))$ the $O_k$-module $\text{Hom}_{\text{Ban}^{ur}_{\leq}(k)}((V_i))$ equipped with the topology of pointwise convergence. For a $V \in \text{ob}(\text{Ban}^{ur}_{\leq}(k))$, we put $V^{D_a} := \mathcal{J}(V, k)$. For a $K \in \text{ob}(\mathcal{C}^c_{\ell})$, we denote by $K^{D_c}$ the $k$-vector space $\mathcal{L}(K, k)$ equipped with the supremum norm. The correspondence $V \mapsto V^{D_a}$ gives a functor $D_a : \text{Ban}^{ur}_{\leq}(k)^{\text{op}} \rightarrow \mathcal{C}^c_{\ell}$, and the correspondence $K \mapsto K^{D_c}$ gives a functor $D_c : \mathcal{C}^c_{\ell} \rightarrow \text{Ban}^{ur}_{\leq}(k)^{\text{op}}$.

**Theorem 2.12** (Schikhof duality). The pair $(D_a, D_c)$ is an $O_k$-linear equivalence between $\text{Ban}^{ur}_{\leq}(k)^{\text{op}}$ and $\mathcal{C}^c_{\ell}$.

### 2.2 Normal Measures

We study a non-Archimedean analogue of the normality of a measure. For this purpose, we introduce a convention of infinite sums. Let $S$ be a set. For an $f \in k^{S}$, we denote by $\sum_{s \in S} f(s)$ the limit of the net $(\sum_{s \in S_0} f(s))_{S_0 \in S_{<\omega}(S)}$, where $S_{<\omega}(S)$ is directed by inclusions. It is elementary to show the following:

**Proposition 2.13.** Let $S$ be a set. For any $f \in k^{S}$ (respectively, $O^S_k$), $\sum_{s \in S} f(s)$ converges in $k$ (respectively, $O_k$) if and only if $f \in C_0(S, k)$ (respectively, $C_0(S, O_k)$).
Let $X$ be a topological space. We denote by $\text{CO}(X)$ the set of clopen subsets of $X$, and by $\mathcal{P}(X)$ the set of subsets $P \subset \text{CO}(X)$ satisfying $X = \bigsqcup_{U \in P} U$. An \textit{$O_k$-valued measure on $X$} is a map $\mu : \text{CO}(X) \to O_k$ such that $\mu(U_0 \cup U_1) = \sum \mu(U_i)$ for any $(U_i) \in \text{CO}(X)^2$ satisfying $U_0 \cap U_1 = \emptyset$. An $O_k$-valued measure $\mu$ on $X$ is said to be \textit{normal} if $\sum_{U' \in P} \mu(U')$ converges to $\mu(U)$ for any $U \in \text{CO}(X)$ and $P \in \mathcal{P}(U)$.

Let $P \in \mathcal{P}(X)$. For a subset $U \subset X$, we put $P|_U := \{ U' \in P \mid U' \subset U \}$. We define a partial order $P_0 \leq P_1$ on $(P_i) \in \mathcal{P}(X)^2$ as $(P_0|_U)_{U \in P_1} \in \prod_{U \in P_1} \mathcal{P}(U)$. Let $(P_i) \in \mathcal{P}(X)^2$. Then $\{ U_0 \cup U_1 \mid (U_i) \in \prod_{i \in P} P_i \} \in \mathcal{P}(X)$ forms the least upper bound of $\{ P_0, P_1 \}$ with respect to \leq. In particular, $\mathcal{P}(X)$ is directed with respect to \leq. Suppose $P_0 \leq P_1$. Let $f \in C_0(P_0, O_k)$ and $U \in P_1$. By $P_0|_U \subset P_0$ and Proposition 2.13, $\tilde{f}(U) := \sum_{U' \in P_0|_U} f(U')$ is a converging sum. For any $\epsilon \in (0, \infty)$, there is a $P'_0 \in \mathcal{P}_{<\omega}(P_0)$ such that $|\tilde{f}(U')| < \epsilon$ for any $U' \in P_0 \setminus P'_0$, and hence $P'_1 := \{ U \in P_1 \mid P'_0 \cap (P_0|_U) \neq \emptyset \}$ is a finite set satisfying $|\tilde{f}(U)| < \epsilon$ for any $U \in P_1 \setminus P'_1$. It implies that the map $\tilde{f} : P_1 \to X, U \mapsto \tilde{f}(U)$ lies in $C_0(P_1, O_k)$. We obtain a continuous $O_k$-linear homomorphism $C_0(P_0, O_k) \to C_0(P_1, O_k)$, $f \mapsto \tilde{f}$ for each $(P_i) \in \mathcal{P}(X)^2$ satisfying $P_0 \leq P_1$, for which $(C_0(P, O_k))_{P \in \mathcal{P}(X)}$ forms a cofiltered diagram in $\mathcal{C}_k$.

We put $\mathcal{M}(X) := \varprojlim_{P \in \mathcal{P}(X)} C_0(P, O_k)$ and $O_k[[X]] := \mathcal{M}(X)_{X'}$. The abuse of the notation with the classical Iwasawa algebra is harmless, because we will show in Proposition 2.21 that $O_k[[X]]$ is its generalisation. For a $(\mu, U) \in \mathcal{M}(X) \times \text{CO}(X)$, we denote by $\mu(U)$ the image of $\mu$ by the composite of the $\{ U, X \setminus U \}$-th projection $\mathcal{M}(X) \to C_0(\{ U, X \setminus U \}, O_k)$ and the evaluation $C_0(\{ U, X \setminus U \}, O_k) \to O_k$ at $U$. For a $(P, \epsilon) \in \mathcal{P}(X) \times (0, 1]$, we set $\mathcal{M}(X; P, \epsilon) := \{ \mu \in \mathcal{M}(X) \mid \forall U \in P, |\mu(U)| < \epsilon \}$. By Corollary 2.7 and the continuity of $i_{\mathcal{M}(X)}^\text{co}$, we obtain the following:

\textbf{Proposition 2.14.} The linear topological $O_k$-modules $\mathcal{M}(X)$ and $O_k[[X]]$ are Hausdorff, and the set $\{ \mathcal{M}(X; P, \epsilon) \mid (P, \epsilon) \in \mathcal{P}(X) \times (0, 1] \}$ forms a fundamental system of neighbourhoods of $0 \in \mathcal{M}(X)$.

The evaluation map $\mathcal{M}(X) \to O_k^{\text{CO}(X)}$, $\mu \mapsto (\mu(U))_{U \in \text{CO}(X)}$ is injective. We identify $\mathcal{S}(\mathcal{M}(X))$ with the $O_k$-module of normal $O_k$-valued measures on $X$ through the evaluation map. For a $U \in \text{CO}(X)$, we denote by $1_U : X \to k$ the characteristic function of $U$. 
Proposition 2.15. If $X$ is compact, then $\mathcal{M}(X)$ is a compact Hausdorff flat linear topological $O_k$-module, and the map $C(X,k)^{D_A} \to O_k^{CO(X)}$, $\mu \mapsto (\mu(1_U))_{U \in CO(X)}$ (cf. [7] Example 1.4) induces an isomorphism $C(X,k)^{D_A} \to \mathcal{M}(X)$ in $\mathcal{C}^\text{ch}_{it}$.

Proof. By the compactness of $X$, every $O_k$-valued measure on $X$ is normal, and hence the map in the assertion gives an $O_k$-linear homomorphism $C(X,k)^{D_A} \to \mathcal{M}(X)$, which is continuous by the finiteness of pairwise disjoint clopen coverings of $X$. On the other hand, again by the compactness of $X$, every continuous $k$-valued function is uniformly approximated by a finite $k$-linear combination of characteristic functions of clopen subsets. Therefore we obtain the inverse $\mathcal{M}(X) \to C(X,k)^{D_A}$, which is continuous because $C(X,k)^{D_A}$ is compact and $\mathcal{M}(X)$ is Hausdorff.

We denote by $\delta_{X,x} \in \mathcal{M}(X)$ the normal $O_k$-valued measure which assigns 1 if $x \in U$ and 0 otherwise to each $U \in CO(X)$ for an $x \in X$, by $\delta_{X} : X \to \mathcal{M}(X)$ the map given by setting $\delta_{X}(x) := \delta_{X,x}$ for an $x \in X$, and by $O_k^{\oplus \delta_{X}} : O_k^{\oplus X} \to \mathcal{M}(X)$ the $O_k$-linear extension of $\delta_{X}$.

Proposition 2.16. (i) The map $\delta_{X}$ is continuous.

(ii) If $X$ is zero-dimensional, that is, $CO(X)$ generates the topology of $X$, and Hausdorff, then $O_k^{\oplus \delta_{X}}$ is injective.

(iii) The image of $O_k^{\oplus \delta_{X}}$ is dense.

Proof. We show the assertion (i). Let $U_1 \subseteq \mathcal{M}(X)$ be an open subset. For any $x \in X$ satisfying $\delta_{X,x} \in U_1$, there is a $(P, \epsilon) \in \mathcal{P}(X) \times (0, 1]$ such that $\delta_{X,x} + \mathcal{M}(X; P, \epsilon) \subseteq U_1$, and hence for any $U_0 \in P$, $x \in U_0$ implies $U_0 \subseteq \delta_{X}^{-1}(U_1)$. Therefore $\delta_{X}$ is continuous. We show the assertion (ii). Suppose that $X$ is zero-dimensional and Hausdorff. Let $m \in O_k^{\oplus X} \setminus \{0\}$. Let $X_0 \subseteq X$ denote a unique non-empty finite subset for which $m$ is presented as $\sum_{x \in X_0} c_x x$ for a $(c_x)_{x \in X_0} \in (O_k \setminus \{0\})^{X_0}$. By the assumption, there is a $P \in \mathcal{P}(X)$ such that $\#(U \cap X_0) \leq 1$ for any $U \in P$. Then $O_k^{\oplus \delta_{X}}(m)(U) = c_x \neq 0$ for any $(U, x) \in P \times X$ satisfying $x \in U$. It implies $\ker(O_k^{\oplus \delta_{X}}) = \{0\}$.

We show the assertion (iii). Let $U \subseteq \mathcal{M}(X)$ be an open neighbourhood of a $\mu \in U$. By Corollary 2.7, there is a $(P, \epsilon) \in \mathcal{P}(X) \times (0, 1]$ such that $\mu + \mathcal{M}(X; P, \epsilon) \subseteq U$. Put $P_0 := \{U' \in P \mid |\mu(U')| \geq \epsilon \} \in \mathcal{P}_{<\omega}(P) \setminus \{\emptyset\}$.

For each $U' \in P_0$, take an $x_{U'} \in U'$. Then $\mu' := O_k^{\oplus \delta_{X}}(\sum_{U' \in P_0} \mu(U') x_{U'})$.
satisfies $|\mu'(U') - \mu(U')| < \epsilon$ for any $U' \in P$. It ensures $\mu' \in U$. Therefore
the image of $O_0^{\oplus \delta_X}$ is dense.

We put $d_X := (\iota^G_{M(X)})^{-1} \delta_X$ and $O_0^{\oplus d_X} := (\iota^G_{M(X)})^{-1} O_0^{\oplus \delta_X}$. We
consider $d_G$ and $O_0^{\oplus d_G}$.

**Proposition 2.17.** (i) The map $\delta_G$ is a homeomorphism onto the image.

(ii) The map $d_G$ is a homeomorphism onto the image.

(iii) The image of $O_0^{\oplus d_G}$ is dense.

In order to verify Proposition 2.17, we study pre-compact subsets of $M(G)$.

**Lemma 2.18.** Let $C \subset M(G)$ be a pre-compact subset. For any $\epsilon \in (0, 1]$, there is a compact clopen subset $G_0 \subset G$ such that $|\mu(U)| < \epsilon$ for any $(\mu, U) \in C \times CO(G \setminus G_0)$.

**Proof.** Take an open profinite subgroup $K \subset G$. Assume that there is an $\epsilon \in (0, 1]$ such that for any compact clopen subset $G_0 \subset G$, some $(\mu, U) \in C \times CO(G \setminus G_0)$ satisfies $|\mu(U)| \geq \epsilon$. In particular, $G$ is not compact, because $G_0 = G$ satisfies $CO(G \setminus G_0) = 1$ and $\mu(\emptyset) = 0$ for any $\mu \in C$. Therefore $G/K$ is an infinite set. We construct $(\mu_r, U_r, C_r) \in C \times CO(G) \times G/K$ inductively on $r \in \omega$ so that $C_r \neq K$ for any $r \in \omega$, $|\mu_r(U_r)| \geq \epsilon$ for any $r \in \omega$, $U_r \subset C_r$ for any $r \in \omega$, and $C_{r_0} \neq C_{r_1}$ for any $(r_i) \in \omega^2$ satisfying $r_0 \neq r_1$.

By the assumption, there is a $(\mu_0, U_0) \in C \times CO(G \setminus K)$ such that $|\mu_0(U_0)| \geq \epsilon$. By the normality of $\mu_0$, we have $\mu_0(U_0) = \sum_{C \in G/K} \mu_0(U_0 \cap C)$, and hence $|\mu_0(U_0 \cap C)| \geq \epsilon$ for some $C_0 \in G/K$ satisfying $C_0 \neq K$. Replacing $U_0$ by $U_0 \cap C_0$, we may assume $U_0 \subset C_0$. Let $r \in \omega \setminus \{0\}$. Suppose that we have constructed $(\mu_h, U_h, C_h)_{h=0}^{n-1} \in (C \times CO(G) \times G/K)^n$ such that $C_h \neq K$, $|\mu_h(U_h)| \geq \epsilon$, and $U_h \subset C_h$ for any $h \in \omega$ satisfying $h < n$, and $C_{h_0} \neq C_{h_1}$ for any $(h_i) \in \omega^2$ satisfying $h_0 \neq h_1$, $h_0 < r$, and $h_1 < r$. By the assumption, there is a $(\mu_r, U_r) \in C \times CO(G \setminus (K \sqcup \bigsqcup_{h=0}^{n-1} C_h))$ such that $|\mu_r(U_r)| \geq \epsilon$. By the normality of $\mu_r$, we may assume that $U_r$ is contained in a $C_r \in G/K$ satisfying $C_r \neq K$. By induction on $r \in \omega$, we obtain a desired family $(\mu_r, U_r, C_r)_{r \in \omega}$.

Since $(C_r)_{r \in \omega}$ is a system of pairwise disjoint subsets of $G$, $U_\omega := G \setminus \bigsqcup_{r \in \omega} U_r$ is a clopen subset of $G$. Put $P := \{U_r \mid r \in \omega \sqcup \{\omega\}\} \in \mathcal{P}(G_0)$. 

Since \( C \) is pre-compact, so is its image \( C_P \) in \( C_0(P, O_k) \) by Proposition 2.3. Therefore there is a \( C_{P, 0} \in \mathcal{P}_{<\omega}(C_P) \) satisfying \( C_P \subset \{ \mu \in C_0(P, O_k) \mid \exists \mu' \in C_{P, 0}, \forall U \in P, |\mu(U) - \mu'(U)| < \epsilon \} \). By \( C_{P, 0} \in \mathcal{P}_{<\omega}(C_P) \), there is a \( P_0 \in \mathcal{P}_{<\omega}(P) \) satisfying \( \mu(U) < \epsilon \) for any \( (\mu, U) \in C_{P, 0} \times (P \setminus P_0) \). It ensures \( \mu(U) < \epsilon \) for any \( (\mu, U) \in C_P \times (P \setminus P_0) \) by the choice of \( C_{P, 0} \). It contradicts that the inequality \( |\mu_r(U_r)| \geq \epsilon \) holds for any \( r \in \omega \). This completes the proof of the assertion. \( \square \)

For an increasing sequence \( (X_r)_{r \in \omega} \) of compact clopen subsets of \( X \) and a decreasing sequence \( (\epsilon_r)_{r \in \omega} \in (0, 1)^\omega \), we put \( M(X; (X_r)_{r \in \omega}, (\epsilon_r)_{r \in \omega}) := \{ \mu \in \mathcal{M}(X) \mid \forall r \in \omega, \forall U \in CO(X \setminus X_r), |\mu(U)| < \epsilon_r \} \).

**Lemma 2.19.** Let \( \epsilon \in (0, 1) \). A subset of \( \mathcal{M}(G) \) is pre-compact if and only if it is contained in \( \mathcal{M}(G; (G_r)_{r \in \omega}, (\epsilon^r)_{r \in \omega}) \) for an increasing sequence \( (G_r)_{r \in \omega} \) of compact clopen subsets of \( G \).

**Proof.** Let \( C \subset \mathcal{M}(G) \) be a subset. Suppose that \( C \) is pre-compact. For each \( r \in \omega \), there is a compact clopen subset \( G_{r, 0} \subset G \) such that \( C \subset \{ \mu \in \mathcal{M}(G) \mid \forall U \in CO(G \setminus G_{r, 0}), |\mu(U)| < \epsilon^r \} \) by Lemma 2.18. For an \( r \in \omega \), put \( G_r := \bigcup_{s=0}^r G_{s, 0} \in CO(G) \). Then \( (G_r)_{r \in \omega} \) forms an increasing sequence of compact clopen subsets of \( G \) satisfying \( C \subset \mathcal{M}(G; (G_r)_{r \in \omega}, (\epsilon^r)_{r \in \omega}) \).

On the other hand, suppose that \( C \) is contained in \( \mathcal{M}(G; (G_r)_{r \in \omega}, (\epsilon^r)_{r \in \omega}) \) for an increasing sequence \( (G_r)_{r \in \omega} \) of compact clopen subsets of \( G \). Let \( L \in \mathcal{O}(\mathcal{M}(G)) \). By Corollary 2.7, there is a \( (P, \epsilon') \in \mathcal{P}(X) \times (0, 1] \) such that \( \mathcal{M}(G; P, \epsilon') \subset L \). By \( \epsilon \in (0, 1) \), there is a \( r \in \omega \) such that \( \epsilon^r \leq \epsilon' \). By the compactness of \( G_r \), there is a \( P_0 \in \mathcal{P}_{<\omega}(P) \) such that \( G_r \subset \bigcup_{U \in P_0} U \). Since \( O_k \) is compact, there is a \( S \in \mathcal{P}_{<\omega}(O_k) \) such that \( O_k = \bigcup_{c' \in S} c' \in O_k \mid |c' - c| < \epsilon^r \). By \#S\#P_0 = (\#S)\#P_0 < \infty \), there is a \( C_0 \in \mathcal{P}_{<\omega}(C) \) such that \( C = \bigcup_{\mu \in C_0} \{ \mu \in C \mid \forall U \in P_0, |\mu'(U) - \mu(U)| < \epsilon^r \} \). It implies \( C \subset \bigcup_{\mu \in C_0} \mu + L \). Thus \( C \) is pre-compact. \( \square \)

**Lemma 2.20.** Let \( M \in \text{ob}(C^0) \). Then a map \( f : G \to M \) is continuous if and only if \( (\iota^c_M)^{-1} \circ f \) is continuous.

**Proof.** Take an open profinite subgroup \( K \subset G \). The direct implication follows from the continuity of \( \iota^c_M \). Suppose that \( f \) is continuous. Let \( U \subset M^c \) be an open subset. Let \( g \in G \). Suppose \( (\iota^c_M)^{-1}(f(g)) \in U \). Since \( f(gK) \subset M \) is compact, \( \iota^c_M(U) \cap f(gK) \) is an open subset of \( f(gK) \) by
Proposition 2.1 (ii) and Corollary 2.5. By the continuity of \( f \), \( f^{-1}(\iota_M^c(U) \cap f(gK)) \) is an open subset of \( f^{-1}(f(gK)) \). It ensures that \( f^{-1}(\iota_M^c(U)) \cap gK \) is an open subset of \( gK \). Since \( gK \) is an open subset of \( G \), \( f^{-1}(\iota_M^c(U)) = ((\iota_M^c)^{-1} \circ f)^{-1}(U) \) is an open neighbourhood of \( g \) in \( G \). It implies that \( (\iota_M^c)^{-1} \circ f \) is continuous. \( \square \)

**Proof of Proposition 2.17.** Take an open profinite subgroup \( K \subset G \). Then \( G/K \) gives an element \( \{gK \mid g \in G\} \) of \( \mathbb{P}(G) \). For any \( g \in G \), \( \delta_G|_{gK} \) is a closed continuous map by Proposition 2.16 (i) because \( gK \) is compact and \( \mathcal{M}(G) \) is Hausdorff, and its image is contained in \( \delta_{G,g} + \mathcal{M}(G;G/K,1) \). Therefore \( \delta_G \) is an injective local homeomorphism onto the image by Proposition 2.16 (ii), because \( \{\delta_{G,g} + \mathcal{M}(G;G/K,1) \mid g \in G\} \) forms a covering of the image of \( \delta_G \) consisting of pairwise disjoint clopen subsets of \( \mathcal{M}(G) \). It implies that \( \delta_G \) is a homeomorphism onto the image, and so is \( d_G \) by Lemma 2.20.

Let \( U \subset O_k[[G]] \) be a non-empty open subset. Take a \( \mu \in U \). By Lemma 2.19, the pre-compact subset \( \{\iota_M^c(\mu)\} \subset \mathcal{M}(G) \) is contained in \( K := \mathcal{M}(G; (G_r)_{r \in \omega}, (\epsilon^r)_{r \in \omega}) \) for an increasing sequence \( (G_r)_{r \in \omega} \subset \text{CO}(G)^\omega \) and an \( \epsilon \in (0,1) \), and \( K \) itself is a pre-compact \( O_k \)-submodule of \( \mathcal{M}(G) \). By Corollary 2.7, there is a \( (P, \epsilon') \in \mathbb{P}(G) \times (0,1] \) such that \( \{\mu' \in K \mid \forall U' \in P, |\mu'(U') - \iota_M^c(\mu)(U')| < \epsilon'\} \subset \iota_M^c(\mu)(U) \). By \( \epsilon \in (0,1) \), there is an \( r \in \omega \) such that \( \epsilon^r < \epsilon' \). By the compactness of \( G_r \), there is a \( P_0 \in \mathcal{P}_{\omega} \setminus \{\emptyset\} \) such that \( G_r \subset \bigcup_{U' \in P_0} U' \). For each \( U' \in P_0 \), take an \( x_{U'} \in U' \). Then \( \mu' := O_k^{\delta_G} \sum_{U' \in P_0} \iota_M^c(\mu)(U')x_{U'} \) satisfies \( |\mu'(U') - \iota_M^c(\mu)(U')| < \epsilon^r \) for any \( U' \in P \). It ensures \( (\iota_M^c)^{-1}(\mu') \subset U \). Therefore the image of \( O_k^{\delta_G} \) is dense. \( \square \)

We show the relation between \( O_k[[G]] \) and the classical Iwasawa algebra. We denote by \( \mathcal{O}(G) \) the set of open normal subgroups of \( G \), which is filtered and cofiltered by inclusions. For a \( (\varphi, K) \in \mathcal{O}(O_k) \times \mathcal{O}(G) \), we equip \( (O_k/\varphi)[G/K] \) with the discrete topology so that it forms a linear topological \( O_k \)-module.

**Proposition 2.21.** Suppose that \( G \) is a profinite group. Then the system of the canonical projections \( O_k[G] \to (O_k/\varphi)[G/K] \) indexed by \( (\varphi, K) \in \mathcal{O}(O_k) \times \mathcal{O}(G) \)
$\mathcal{O}(O_k) \times \mathcal{O}(G)$ induces a unique isomorphism

$$O_k[[G]] \rightarrow \lim_{(\varphi, K) \in \mathcal{O}(O_k) \times \mathcal{O}(G)} (O/\varphi)[G/K]$$

in $\mathcal{C}_\ell$. In particular, $O_k[[G]]$ forms a compact Hausdorff flat linear topological $O_k$-module.

Proof. The assertion follows from Proposition 2.1 (ii), Proposition 2.8 (ii), Proposition 2.15, and the fact that the classical Iwasawa algebra over $O_k$ associated to $G$ has an interpretation as an $O_k$-module of $O_k$-valued measures on $G$. \hfill \Box

3 Monoidal structures

We define symmetric monoidal structures on the categories introduced in §2.1, and an $O_k$-algebra structure on $O_k[[G]]$ in terms of a monoid in one of them. We note that $O_k[[G]]$ does not necessarily form a topological $O_k$-algebra, that is, a monoid object in the Cartesian monoidal category of topological $O_k$-modules and continuous $O_k$-linear homomorphisms. This is one of the main reasons why we need monoidal structures.

3.1 Topological tensor products We define symmetric monoidal structures on $\mathcal{C}_\ell$, $\mathcal{C}_\ell^c$, and $\mathcal{C}_\ell^{cg}$. First, we study $\mathcal{C}_\ell$. Let $(M_i) \in \text{ob}(\mathcal{C}_\ell^2)$. We denote by $(L_i)_{(M_i)} \subset \mathcal{F}(M_0) \otimes_{O_k} \mathcal{F}(M_1)$ the kernel of the natural projection $\mathcal{F}(M_0) \otimes_{O_k} \mathcal{F}(M_1) \rightarrow \mathcal{F}(M_0/L_0) \otimes_{O_k} \mathcal{F}(M_1/L_1)$ for $O_k$-submodules $L_0 \subset M_0$ and $L_1 \subset M_1$, and by $M_0 \otimes^\ell M_1$ the $O_k$-module $\mathcal{F}(M_0) \otimes_{O_k} \mathcal{F}(M_1)$ equipped with the topology generated by the set $\{m + (L_i)_{(M_i)} \mid (m, (L_i)) \in (\mathcal{F}(M_0) \otimes_{O_k} \mathcal{F}(M_1)) \times \prod \mathcal{O}(M_i)\}$. Then $M_0 \otimes^\ell M_1$ forms a linear topological $O_k$-module. By the definition of the topology of $M_0 \otimes^\ell M_1$, the $O_k$-bilinear homomorphism $\nabla_{(M_i)} : \prod \mathcal{U}(M_i) \rightarrow \mathcal{U}(M_0 \otimes^\ell M_1)$, $(m_i) \mapsto m_0 \otimes m_1$ is continuous. The correspondence $(M_i) \mapsto M_0 \otimes^\ell M_1$ gives a functor $\otimes^\ell : \mathcal{C}_\ell^2 \rightarrow \mathcal{C}_\ell$, and the correspondence $(M_i) \mapsto \nabla_{(M_i)}$ gives a natural transformation $\nabla : \prod \mathcal{U}(\bullet_i) \Rightarrow \mathcal{U}(\bullet_0 \otimes^\ell \bullet_1)$. Let $(M_s)_{s \in S}$ be a small diagram in $\mathcal{C}_\ell$. By the functoriality of $\otimes^\ell$ and the universality of the colimit, the
system of canonical morphisms $M_{s_0} \to \lim_{s \in S} M_s$ indexed by $s_0 \in S$ induces a morphism $S(M_{s})_{s \in S,M} : \lim_{s \in S}(M_s \otimes^\ell M) \to (\lim_{s \in S} M_s) \otimes^\ell M$ for an $M \in \text{ob}({\mathcal{C}}^c)$. We note that $\otimes^\ell$ seems not to be cocontinuous.

**Proposition 3.1.** The triad $({\mathcal{C}}^c, \otimes, O_k)$ forms a symmetric monoidal category.

**Proof.** We denote by $(A, L, R, B)$ the data of the associator, the left unitor, the right unitor, and the braiding of $({\mathcal{C}}^c, \otimes, O_k)$. We have

$$\mathcal{F}(\bullet_0 \otimes^\ell \bullet_1) = \mathcal{F}(\bullet_0) \otimes_{O_k} \mathcal{F}(\bullet_1)$$

by definition. Since $\mathcal{F} : {\mathcal{C}}^c \to \mathcal{C}$ is faithful, it suffices to verify that every value of $\Phi \circ \mathcal{F}$ lies in the image of $\mathcal{F}$ for any $\Phi \in \{A, L, R, B\}$. By $O_k \in \mathcal{O}(O_k)$, every value of $L \circ \mathcal{F}$ (respectively, $R \circ \mathcal{F}$) lies in the image of $\mathcal{F}$. By the symmetry of the sub-base of the topology of every value of $\otimes^\ell$, every value of $B \circ \mathcal{F}$ lies in the image of $\mathcal{F}$. Let $(M_i)_{i=0}^2 \in \text{ob}({\mathcal{C}}^c)$. We show that the $O_k$-linear homomorphism $A_{M_0, M_1, M_2} : (M_0 \otimes^\ell M_1) \otimes^\ell M_2 \to M_0 \otimes^\ell (M_1 \otimes^\ell M_2), m \mapsto A_{\mathcal{F}(M_0), \mathcal{F}(M_1), \mathcal{F}(M_2)}(\mathcal{F}(m))$ is continuous. Let $(L_0, L_{1,2}) \in \mathcal{O}(M_0) \times \mathcal{O}(M_1 \otimes^\ell M_2)$. Take an $(L_{i+1}) \in \prod \mathcal{O}(M_{i+1})$ satisfying $(L_{i+1})_{(M_{i+1})} \subset L_{1,2}$. We have

$$(L_i)_{(M_i)(L_i)}(L_2)_{M_0 \otimes^\ell M_1, M_2} = (L_0, (L_{i+1})_{(M_{i+1})})_{M_0, M_1 \otimes^\ell M_2} \subset A_{M_0, M_1, M_2}^{-1}((L_0, L_{1,2})_{M_0, M_1 \otimes^\ell M_2})$$

by the right exactness of $\otimes_O$. Therefore $A_{M_0, M_1, M_2}$ is a continuous map satisfying $\mathcal{F}(A_{M_0, M_1, M_2}) = A_{\mathcal{F}(M_0), \mathcal{F}(M_1), \mathcal{F}(M_2)}$. \hfill $\square$

Next, we study $\mathcal{C}_k^c$. Let $(K_i) \in \text{ob}((\mathcal{C}_k^c)^2)$. Then $K_0 \otimes^\ell K_1$ is pre-compact by $\#((K_0 \otimes^\ell K_1)/(L_i)) = \#(K_0/L_0 \otimes^\ell K_1/L_1) \leq \prod \#(K_i/L_i) < \infty$ for any $(L_i) \in \prod \mathcal{O}(K_i)$. Therefore the correspondence $(K_i) \mapsto K_0 \otimes^\ell K_1$ gives a functor $\otimes^c : (\mathcal{C}_k^c)^2 \to \mathcal{C}_k^c$, and the correspondence $(K_i) \mapsto \nabla(K_i)$ gives a natural transformation $\nabla^c : \prod \mathcal{C}_k^c(\bullet_i) \Rightarrow \mathcal{C}_k^c(\bullet_0 \otimes^c \bullet_1)$. Since $\mathcal{C}_k^c$ is a full subcategory of $\mathcal{C}_k$, we obtain the following by Proposition 3.1:

**Proposition 3.2.** The triad $({\mathcal{C}}_k^c, \otimes^c, O_k)$ forms a symmetric monoidal category.
We put $\mathcal{L}((K_0, M_1), L) := \{f \in \mathcal{L}(K_0, M_1) \mid f(K_0) \subset L\}$ for an $L \in \mathcal{O}(M_1)$, and denote by $\mathcal{H}\text{om}^c(K_0, M_1)$ the $O_K$-module $\mathcal{L}(K_0, M_1)$ equipped with the topology generated by the set $\{f + \mathcal{L}((K_0, M_1), L) \mid (f, L) \in K_0 \times \mathcal{O}(M_1)\}$. Then $\mathcal{H}\text{om}^c(K_0, M_1)$ forms a linear topological $O_K$-module.

By Proposition 2.3 and Corollary 2.10 (i), the correspondence $(K_0, M_1) \mapsto \mathcal{H}\text{om}^c(K_0, M_1)$ gives a functor $\mathcal{H}\text{om}^c : (\mathcal{C}_K)^{op} \times \mathcal{C}_\ell \to \mathcal{C}_\ell$. By Theorem 2.12, the transpose map $T(\bullet)_{(K_i)} : \mathcal{H}\text{om}^c((K_i)) \to \mathcal{S}((K_1^{D_{\mathcal{C}}}))$ is bijective. We have a comparison of the endomorphism algebras, which corresponds to [13] Lemma 1.6 in the case $\text{ch}(k) = 0$.

**Proposition 3.3.** The map $T(\bullet)_{(K_i)}$ is an isomorphism in $\mathcal{C}_\ell$.

**Proof.** Let $(v, \epsilon) \in K_1^{D_{\mathcal{C}}} \times (0, \infty)$. Put $L := \{f \in \mathcal{S}((K_1^{D_{\mathcal{C}}})) \mid |f(v)| < \epsilon\}$. We show $T(\bullet)_{(K_i)}^{-1}(L) \in \mathcal{O}(\mathcal{H}\text{om}^c((K_i)))$. Put $L_1 := \{m \in K_1 \mid |v(m)| < 2^{-1}\epsilon\} \in \mathcal{O}(K_1)$. Let $f \in \mathcal{L}((K_1), L_1)$. We have $\|Tf_{(K_i)}(v)\| = \sup_{m \in K_1} |f(v(m))| \leq \sup_{m \in L_1} |v(m)| \leq 2^{-1}\epsilon < \epsilon$. It ensures $Tf_{(K_i)} \in L$. It implies $\mathcal{L}((K_1), L_1) \subset T(\bullet)_{(K_i)}^{-1}(L)$. We obtain $T(\bullet)_{(K_i)}^{-1}(L) \in \mathcal{O}(\mathcal{H}\text{om}^c((K_i)))$. Therefore $T(\bullet)_{(K_i)}$ is continuous.

Let $L_1 \in \mathcal{O}(K_1)$. We show $T(\bullet)_{(K_i)}(\mathcal{L}((K_i), L_1)) \in \mathcal{O}(\mathcal{H}\text{om}^c((K_i)))$. By Theorem 2.12, there is an $(S, \epsilon) \in \mathcal{P}_{<\omega}(K_0^{D_{\mathcal{C}}}) \times (0, \infty)$ such that $\{m \in K_1 \mid \forall v \in S, |v(m)| < \epsilon\} \subset L_1$. Put $L := \{f \in \mathcal{S}((K_1^{D_{\mathcal{C}}})) \mid \forall v \in S, \|f(v)\| < \epsilon\} \in \mathcal{O}(\mathcal{S}((K_1^{D_{\mathcal{C}}}))$. Let $f \in L$. We show $T(\bullet)_{(K_i)}^{-1}(f) \in \mathcal{L}((K_i), L_1)$. Let $m \in K_0$. We have $|v(T(\bullet)_{(K_i)}^{-1}(f)(m))| = |f(v(m))| \leq \|f(v)\| < \epsilon$ for any $v \in S$, and hence $T(\bullet)_{(K_i)}^{-1}(f)(m) \in L_1$. It ensures $T(\bullet)_{(K_i)}^{-1}(f) \in \mathcal{L}((K_i), L_1)$. It implies $L \subset T(\bullet)_{(K_i)}(\mathcal{L}((K_i), L_1))$. Therefore $T(\bullet)_{(K_i)}$ is an open map. □

We denote by $C_{\mathcal{C}, \mathcal{R}} : ((\mathcal{C}_K)^{op})^2 \times \mathcal{C}_\ell \to \text{Set}$ the functors given as $C_{\mathcal{L}, \mathcal{R}} := \mathcal{L}((\mathcal{C}_\ell \circ \mathcal{C}_K(\bullet, \mathcal{C}_\ell), \bullet, \bullet, \bullet)$ and $C_{\mathcal{R}, \mathcal{R}} := \mathcal{L}((\mathcal{C}_\ell(\bullet), \mathcal{H}\text{om}^c((\bullet + 1)))$. We construct an adjunction $T^c : C_{\mathcal{L}, \mathcal{R}} \Rightarrow C_{\mathcal{R}, \mathcal{R}}$. Let $f$ be an $O_K$-linear homomorphism $K_0 \otimes^c K_1 \to M_2$ for a $((K_i), M_2) \in \text{ob}((\mathcal{C}_\ell)^2 \times \mathcal{C}_\ell)$. We characterise the continuity of $f$.

**Proposition 3.4.** The map $f$ is continuous if and only if $f \circ \nabla^c_{(K_i)}$ is continuous.

**Proof.** The inverse implication follows from the continuity of $\nabla^c_{(K_i)}$. Suppose that $f \circ \nabla^c_{(K_i)}$ is continuous. Let $L_2 \in \mathcal{O}(M_2)$. We show $f^{-1}(L_2) \in \mathcal{O}(M_1)$. On the other hand, if $f \circ \nabla^c_{(K_i)} \in \mathcal{O}(M_1)$, then $f^{-1}(L_2) \in \mathcal{O}(M_1)$.
\( C(K_0 \otimes^c K_1) \). By the continuity of \( f \circ \nabla_{(K_i)} \), there is an \((L_i) \in \prod C(K_i)\) such that \( \prod L_i \subset (f \circ \nabla_{(K_i)})^{-1}(L_2) \). Put \( i_0 := 0 \) (respectively, \( i_0 := 1 \)). Take a \( K_{i_0,0} \in \mathcal{P}_{<\omega}(K_0) \) satisfying \( K_{i_0} \subset \bigcup_{m \in K_{i_0,0}} (m + L_{i_0}) \). For each \( m \in K_{i_0,0} \), there is an \( L_{i_0,0,m} \in C(K_i) \) such that \( L_{i_0,0,m} \times \{m\} \) (respectively, \( \{m\} \times L_{i_0,0,m} \)) is contained in \( (f \circ \nabla_{(K_i)})^{-1}(L_2) \) by the continuity of \( f \circ \nabla_{(K_i)} \). Therefore \( \tilde{f} \) is continuous. We have

Suppose that \( f \) is continuous. Let \( m_0 \in K_0 \). We denote by \( f(m_0 \otimes^c \bullet) \) the \( O_k \)-linear homomorphism \( K_1 \to M_2 \), \( m_1 \mapsto f(m_0 \otimes m_1) \). Then \( f(m_0 \otimes^c \bullet) \) is the composite of \( f \), \( \nabla_{(K_i)}^c \), and the map \( \bar{\mathcal{U}}^c(K_1) \hookrightarrow \prod \mathcal{U}^c(K_i) \), \( m_1 \mapsto (m_i) \), and hence is continuous. We obtain an \( O_k \)-linear homomorphism \( T_{(K_i),M_2}^c(f) : K_0 \to \mathcal{H}om^c(K_1, M_2) \), \( m_0 \mapsto f(m_0 \otimes^c \bullet) \).

**Proposition 3.5.** The \( O_k \)-linear homomorphism \( T_{(K_i),M_2}^c(f) \) is continuous.

**Proof.** Let \( L_2 \in C(M_2) \). By the continuity of \( f \), there is an \((L_i) \in \prod C(K_i)\) such that \( (L_i)_{(K_i)} \subset f^{-1}(L_2) \). Take a \( K_{i,0} \in \mathcal{P}_{<\omega}(K_1) \) satisfying \( K_i \subset \bigcup_{m_1 \in K_{i,0}} (m_1 + L_1) \). For each \( m_1 \in K_{i,0} \), there is an \( L_{0,0,m_1} \in C(K_0) \) such that \( f(m_0 \otimes m_1) \in L_2 \) for any \( m_0 \in L_{0,0,m_1} \) by the continuity of \( f \), \( \nabla_{(K_i)}^c \), and the map \( K_0 \hookrightarrow \prod K_i \), \( m_0 \mapsto (m_i) \). By \( 0 \in K_1 \), we have \( K_{i,0} \neq \emptyset \). Put \( L_{0,0} := L_0 \cap \bigcap_{m_1 \in K_{i,0}} L_{0,0,m_1} \in C(K_0) \). By \( L_2 + L_2 = L_2 \), we obtain \( f(m_0 \otimes m_1) \in L_2 \) for any \( (m_i) \in L_{0,0} \times K_1 \). It ensures \( L_{0,0} \subset T_{(K_i),M_2}^c(f)^{-1}(\mathcal{L}((K_1, M_2), L_2)) \). Thus \( T_{(K_i),M_2}^c(f) \) is continuous.

By Proposition 3.5, the correspondence \( ((K_i), M_2) \leadsto T_{(K_i),M_2}^c(f) \) gives a natural transformation \( T^c : C_L \Rightarrow C_R^c \).

**Proposition 3.6.** The natural transformation \( T^c \) is a natural equivalence.

**Proof.** We have \( \mathcal{F}^c(\bullet_0 \otimes^c \bullet_1) = \mathcal{F}^c(\bullet_0) \otimes_{O_k} \mathcal{F}^c(\bullet_1) \) and \( \mathcal{F}^c \circ T^c \) coincides with the restriction of the adjunction between \( \otimes_{O_k} \) and the internal-hom functor on \( \mathcal{C} \). Since \( \mathcal{F}^c \) is faithful, \( T_{(K_i),M_2}^c \) is injective. Let \( f \in C_R^c((K_i), M_2) \). We show that the \( O_k \)-linear homomorphism \( \tilde{f} : K_0 \otimes^c K_1 \to M_2 \), \( (m_i) \mapsto f(m_0)(m_1) \) is continuous. Let \( L_2 \in C(M_2) \). By the continuity of \( f \), there is an \( L_0 \in C(K_0) \) such that \( L_0 \subset f^{-1}(\mathcal{L}((K_1, M_2), L_2)) \). It ensures \( (L_0, K_1)_{(K_i)} \subset \tilde{f}^{-1}(L_2) \). Therefore \( \tilde{f} \) is continuous. We have
\( T^c_{(K_i),M_2}(f) = f \). It implies that \( T^c_{(K_i),M_2} \) is surjective. Thus \( T^c \) is a natural equivalence.

By Proposition 3.6, we obtain an adjoint property between \( \otimes^c \) and \( \mathcal{H} \text{om}^c \). It does not ensure that \( \otimes^c \) is cocontinuous, because we used \( \mathcal{H} \text{om} \) in the description of the adjoint property. On the other hand, we have a commutativity between \( \otimes^c \) and colimits in \( \mathcal{C}^c_{\ell} \) in a special case. Let \((K_s)_{s \in S} \) be a small diagram in \( \mathcal{C}^c_{\ell} \). We put \( M := \lim_{\longrightarrow \, s \in S} \mathcal{H} \text{om}^c(K_s) \). We recall that the colimit of \((K_s)_{s \in S} \) in \( \mathcal{C}^c_{\ell} \) is given as \( \overline{M} \) by Proposition 2.8 (i). Therefore if \( M \) is pre-compact, then \( S(\mathcal{H} \text{om}^c(K_s))_{s \in S}, \mathcal{H} \text{om}^c(K) \) gives a morphism \( S^c_{(K_s)_{s \in S}, K} : \lim_{\longrightarrow \, s \in S} \mathcal{H} \text{om}^c(K_s \otimes^c K) \to \mathcal{H} \text{om}^c(M \otimes^c K) \) in \( \mathcal{C}^c_{\ell} \) for any \( K \in \text{ob}(\mathcal{C}^c_{\ell}) \).

**Proposition 3.7.** If \( M \) is pre-compact, then \( S(\mathcal{H} \text{om}^c(K_s))_{s \in S}, K \) is an isomorphism in \( \mathcal{C}^c_{\ell} \) for any \( K \in \text{ob}(\mathcal{C}^c_{\ell}) \).

**Proof.** For any \( M' \in \text{ob}(\mathcal{C}^c_{\ell}) \), \( \mathcal{L}(S(\mathcal{H} \text{om}^c(K_s))_{s \in S}, K, M') \) is given as the composite of \( T^c_{M,K,M'} \), the natural map

\[
C^c_R(M,K,K') \to \lim_{\longrightarrow \, s \in S} C^c_R(K_s,K,K') , \quad \lim_{\longrightarrow \, s \in S} (T^c_{K_s,K,M'})^{-1},
\]

and the natural map \( \lim_{\longrightarrow \, s \in S} C^c_L(K_s,K,M') \to \mathcal{L}(\lim_{\longrightarrow \, s \in S} \mathcal{H} \text{om}^c(K_s \otimes^c K), M') \), which are bijective by Proposition 3.6 and the universality of colimits. Therefore \( S(\mathcal{H} \text{om}^c(K_s))_{s \in S}, K \) is an isomorphism in \( \mathcal{C}^c_{\ell} \).

Finally, we study \( \mathcal{C}^c_{\ell} \). We put \( M_0 \otimes^c M_1 := \lim_{\longrightarrow \,(K_i) \in \prod \mathcal{K}(M_i)} K_0 \otimes^\ell K_1 \) and \( M_0 \times^c M_1 := \lim_{\longrightarrow \,(K_i) \in \prod \mathcal{K}(M_i)} \prod \mathcal{K}(K_i) \). By Proposition 2.9 (i) and Corollary 2.10, \( M_0 \otimes^c M_1 \) forms a CG linear topological \( O_k \)-module. By Corollary 2.7 and the naturality of \( \nabla^c \), the system \( (\nabla^c_{(K_i)}(K_i)_{(K_i) \in \prod \mathcal{X}(M_i)}) \) induces a continuous \( O_k \)-bilinear homomorphism

\[
\nabla^c_{(M_i)} : M_0 \times^c M_1 \to \mathcal{V}^c(M_0 \otimes^c M_1).
\]

Suppose \( (M_i) \in \text{ob}(\mathcal{C}^c_{\ell})^2 \) in the following in this subsection. By the universality of the colimit, the system of the inclusions \( \prod \mathcal{V}^c(K_i) \hookrightarrow \prod \mathcal{V}^c(M_i) \) indexed by \( (K_i) \in \prod \mathcal{X}(M_i) \) induces a bijective continuous map \( \nabla^c_{(M_i)} : M_0 \times^c M_1 \to \prod \mathcal{V}^c(M_i) \). By Proposition 2.3, the correspondences \( (M_i) \leadsto M_0 \otimes^c M_1, M_0 \times^c M_1 \) give functors \( \otimes^c : (\mathcal{C}^c_{\ell})^2 \to \mathcal{C}^c_{\ell} \)

\[\text{Proof.}\]
and $\bullet_0 \times_{\text{cg}} \bullet_1 : (\mathcal{C}_\ell^{\text{cg}})^2 \to \text{Top}$, respectively, and the correspondences $(M_i) \rightsquigarrow \nabla_{(M_i)}^{\text{cg} \otimes}, \nabla_{(M_i)}^{\text{cg} \times}$ give natural transformations $\nabla_{(M_i)}^{\text{cg} \otimes} : \bullet_0 \times_{\text{cg}} \bullet_1 \Rightarrow \mathcal{H}_{\text{cg}}(\bullet_0 \otimes_{\text{cg}} \bullet_1)$ and $\nabla_{(M_i)}^{\text{cg} \times} : \bullet_0 \times_{\text{cg}} \bullet_1 \Rightarrow \prod \mathcal{H}_{\text{cg}}(\bullet_i)$, respectively.

**Theorem 3.8.** The triad $(\mathcal{C}_\ell^{\text{cg}}, \otimes_{\text{cg}}, O_k)$ forms a closed symmetric monoidal category.

We construct an exponential functor on $\mathcal{C}_\ell^{\text{cg}}$. We put $\mathcal{L}((M_i), K, L) := \{f \in \mathcal{L}((M_i)) \mid f(K) \subset L\}$ for a $(K, L) \in \mathcal{H}(M_0) \times \mathcal{O}(M_1)$, and denote by $\mathcal{H}_{\text{om}}^{\text{cg}}((M_i))$ the $O_k$-module $\mathcal{L}((M_i))$ equipped with the topology generated by the set \( \{f + \mathcal{L}((M_i), K, L) \mid (f, K, L) \in M_0 \times \mathcal{H}(M_0) \times \mathcal{O}(M_1)\} \). Then $\mathcal{H}_{\text{om}}^{\text{cg}}((M_i))$ forms a linear topological $O_k$-module. We put $M_{1_i}^{M_0} := \mathcal{H}_{\text{om}}^{\text{cg}}((M_i))_{\mathcal{X}}$. By Proposition 2.3 and Corollary 2.10 (i), the correspondence $((M_i)) \rightsquigarrow M_{1_i}^{M_0}$ gives a functor $(\bullet_1)^{\bullet_0} : (\mathcal{C}_\ell^{\text{cg}})^{\text{op}} \times \mathcal{C}_\ell^{\text{cg}} \to \mathcal{C}_\ell^{\text{cg}}$. We denote by $C_L^{\text{cg}}, C_R^{\text{cg}} : ((\mathcal{C}_\ell^{\text{cg}})^{\text{op}})^2 \times \mathcal{C}_\ell^{\text{cg}} \to \text{Set}$ the functors given as $C_L^{\text{cg}} := \mathcal{L}(\bullet_0 \otimes_{\text{cg}} \bullet_1, \bullet_2)$ and $C_R^{\text{cg}} := \mathcal{L}(\bullet_0, \bullet_2^1)$. We construct an adjunction $T_{\text{cg}} : C_L^{\text{cg}} \Rightarrow C_R^{\text{cg}}$. Let $m_0 \in M_0$.

**Lemma 3.9.** The map $(m_0, \bullet) : \mathcal{H}_{\text{cg}}(M_1) \hookrightarrow M_0 \times_{\text{cg}} M_1, m_1 \mapsto (m_i)$ is continuous.

**Proof.** Let $U \subset M_0 \times_{\text{cg}} M_1$ be an open subset. By Corollary 2.5, we have $O_k m_0 \subset \mathcal{H}(M_0)$. For any $K \in \mathcal{H}(M_1)$, the map $(m_0, \bullet)_K : K \hookrightarrow O_k m_0 \times K, m_1 \mapsto (m_0, M_1)$ is continuous, and hence $(m, \bullet)^{-1}(U) \subset K = (m, \bullet)^{-1}(U \cap (O_k m_0 \times K))$ is open in $K$. It implies $(m, \bullet)^{-1}(U)$ is open in $M_1$ by Corollary 2.7. Thus $(m_0, \bullet)$ is continuous.

Let $f$ be an $O_k$-linear homomorphism $M_0 \otimes_{\text{cg}} M_1 \to M_2$ for a $M_2 \in \text{ob}(\mathcal{C}_\ell^{\text{cg}})$. By Corollary 2.7 and Proposition 3.4, we have the following characterisation of the continuity of $f$:

**Proposition 3.10.** The map $f$ is continuous if and only if $f \circ \nabla_{(M_i)}^{\text{cg} \otimes}$ is continuous.

Suppose that $f$ is continuous. By Lemma 3.9, $f \circ \nabla_{(M_i)}^{\text{cg} \otimes} \circ (m_0, \bullet)$ is continuous. We obtain an $O_k$-linear homomorphism

\[ f_k : M_0 \to \mathcal{H}_{\text{om}}^{\text{cg}}((M_{i+1})), m_0 \mapsto f \circ \nabla_{(M_i)}^{\text{cg} \otimes} \circ (m_0 \otimes \bullet). \]

**Lemma 3.11.** The $O_k$-linear homomorphism $f_k$ is continuous.
Proof. Let \((K_1, L_2) \in \mathcal{K}(M_1) \times O(M_2)\). Put \(L := f_{R}^{-1}(\mathcal{L}((M_1), K_1, L_2))\). We show \(L \in O(M_0)\). Let \(K_0 \in \mathcal{K}(M_0)\). We denote by \(f_{(K_1)}: K_0 \otimes^c K_1 \to M_2\) the composite of \(f\) and the canonical morphism \(K_0 \otimes^c K_1 \to M_0 \otimes^c M_1\). By the continuity of \(f\), \(f_{(K_1)}\) is continuous. By Proposition 3.5, we have \(L \cap K_0 \subset O(K_0)\). By Corollary 2.7, we obtain \(L \in O(M_0)\). Thus \(f_R\) is continuous.

The \(O_k\)-linear homomorphism \(T_{M_0,M_1,M_2}^{cg}(f): M_0 \to M_2^{M_1}\) given as the composite \((t^{cg}_{\mathcal{O}(M_0),M_1,M_2})^{-1} \circ f_R\) is continuous by Corollary 2.10 (i) and Lemma 3.11. We obtain a map

\[
T_{M_0,M_1,M_2}^{cg}: C_{L_0}^{cg}(M_0, M_1, M_2) \to C_{R}^{cg}(M_0, M_1, M_2), \quad f \mapsto T_{M_0,M_1,M_2}^{cg}(f).
\]

The correspondence \((M_i)^2_{i=0} \rightsquigarrow T_{M_0,M_1,M_2}^{cg}\) gives a natural transformation \(T^{cg}: C_{L}^{cg} \Rightarrow C_{R}^{cg}\).

Proof of Theorem 3.8. We denote by \((A, L, R, B)\) the data of the associator, the left unitor, the right unitor, and the braiding of \((\mathcal{C}_\ell, \otimes^c, O_k)\). Let \(M \in \text{ob}(\mathcal{C}_\ell^{cg})\). The system \((L_K)^{(O_k, (K_0, K_1))} \in \mathcal{K}(O_k)\times \mathcal{K}(M)\) induces a morphism \(\tilde{L}_M: O_k \otimes^c M \to M\) in \(\mathcal{C}_\ell^{cg}\) by the functoriality of \(L\) and the universality of the colimit. We show that \(\tilde{L}_M\) is an isomorphism in \(\mathcal{C}_\ell^{cg}\). Let \(L \in O(O_k \otimes^c M)\). Since the preimage of \(L\) in \(O_k \otimes^c K\) is open and \(L_K\) is a homeomorphism, we have \(\tilde{L}_M(L) \cap K \in O(K)\) for any \(K \in \mathcal{K}(M)\). It ensures \(\tilde{L}_M(L) \in O(M)\) by Corollary 2.7. Therefore \(\tilde{L}_M\) is an isomorphism in \(\mathcal{C}_\ell^{cg}\). The correspondence \(M \rightsquigarrow \tilde{L}_M\) gives a natural equivalence \(\tilde{L}: O_k \otimes^c \bullet \Rightarrow \text{id}_{\mathcal{C}_\ell^{cg}}\). Similarly, we also have a natural equivalence \(\tilde{R}: \bullet \otimes^c O_k \Rightarrow \text{id}_{\mathcal{C}_\ell^{cg}}\). Let \((M_i) \in \text{ob}(\mathcal{C}_\ell^{cg})^2\). The system \((B_{(K_i)})^{(K_0, K_1)}_{(K_0, K_1)} \in \prod \mathcal{K}(M_i)\) induces a morphism \(\tilde{B}_{(M_i)}: M_0 \otimes^c M_1 \to M_1 \otimes^c M_0\) in \(\mathcal{C}_\ell^{cg}\) by the functoriality of \(B\) and the filtered colimit. The correspondence \((M_i) \rightsquigarrow \tilde{B}_{(M_i)}\) gives a natural transformation \(\tilde{B}: \bullet_0 \otimes^c \bullet_1 \Rightarrow \bullet_1 \otimes^c \bullet_0\). By \(B^2 = \text{id}_{\mathcal{C}_\ell^{cg} \times \mathcal{C}_\ell^{cg}}\), we obtain \(\tilde{B}^2 = \text{id}_{\mathcal{C}_\ell^{cg} \times \mathcal{C}_\ell^{cg}}\).

Let \((M_i)^2_{i=0} \in \text{ob}(\mathcal{C}_\ell^{cg})^3\). We define a morphism \(\tilde{A}_{M_0,M_1,M_2}: (M_0 \otimes^c M_1) \otimes^c M_2 \to M_0 \otimes^c (M_1 \otimes^c M_2)\) in \(\mathcal{C}_\ell^{cg}\). Let \((K_{0,1,0}, K_2) \in \mathcal{K}(M_0 \otimes^c M_1) \times \mathcal{K}(M_2)\). We denote by \(K_{0,1} \subset M_0 \otimes^c M_1\) the closure of \(K_{0,1,0}\), which is pre-compact by Proposition 2.1 (i). Let \((K_i) \in \prod \mathcal{K}(M_i)\). We denote by
The composite of the inclusion \(((K_i)_{K_0,1} \hookrightarrow K_0 \otimes^c (K_1)_{K_0,1,1})\) and the natural morphism \(K_0 \otimes^c (K_1)_{K_0,1,1} \rightarrow M_0 \otimes^{cg} (M_0, M_1, M_2)\) in \(\mathcal{C}_{\ell}\). By Proposition 2.6 and [6] Lemma 2.23, the system of inclusions \((K_i)_{K_0,1} \hookrightarrow K_0,1 \rightarrow K_0,1,1\) induces an isomorphism \(\text{lim} \rightarrow \mathcal{M}(K_i)_{K_0,1} \rightarrow K_0,1,1\) in \(\mathcal{C}_{\ell}\). Therefore the system \((\tilde{A}_{K_0,1,0,K_1,2}(K_i)_{K_0,1} \rightarrow K_0,1,1)\) indexed by \(K_i \in \mathcal{M}(M_i)\) induces a morphism \(A_{K_0,1,0,K_1,2} : K_0,1 \otimes^c K_2 \rightarrow M_0 \otimes^{cg} (M_0, M_1, M_2)\) in \(\mathcal{C}_{\ell}\) by Proposition 3.7. We denote by \(\tilde{A}_{M_0,1,2} : (M_0 \otimes^{cg} M_1) \otimes^{cg} M_2 \rightarrow M_0 \otimes^{cg} (M_0, M_1, M_2)\) in \(\mathcal{C}_{\ell}^{cg}\). The correspondence \((M_i)_{i=0}^2 \rightarrow \tilde{A}_{M_0,1,2}\) induces a natural transformation \(\tilde{A} : (\bullet \otimes^{cg} \bullet_1) \otimes^{cg} \bullet_2 \rightarrow \bullet \otimes^{cg} (\bullet_1 \otimes^{cg} \bullet_2)\). Similarly, we obtain a natural formation of the opposite direction, which is the inverse of \(\tilde{A}\).

By the construction, the data \(\tilde{A}, \tilde{L}, \tilde{R}, \tilde{B}, T^{cg}\) is sent to the data of the associator, the left unitor, the right unitor, the braiding, and the Currying of \((\mathcal{C}, \otimes_{O_k}, O_k)\) through \(\mathcal{F}_{cg}\) and \(\nu^{cg}\). Since \(\mathcal{F}_{cg}\) is faithful, it ensures the coherence so that \(\tilde{A}, \tilde{L}, \tilde{R}, \tilde{B}\) forms data of an associator, a left unitor, a right unitor, a braiding, and an injective Currying of \((\mathcal{C}_{\ell}^{cg}, \otimes^{cg}, O_k)\). We have only to verify that \(T^{cg}_{M_0,1,2} : (M_0 \otimes^{cg} M_1) \otimes^{cg} M_2 \rightarrow M_0 \otimes^{cg} (M_0, M_1, M_2)\) is surjective. Let \(f \in C^{cg}_{\mathcal{R}}(M_0, M_1, M_2)\). Put

\[ f' := \nu^{cg}_{\mathcal{K}_{om}^{cg}((M_{i+1}))} \circ f : M_0 \rightarrow \mathcal{K}_{om}^{cg}((M_{i+1})). \]

Let \((K_i) \in \prod \mathcal{M}(M_i)\). We show that the \(O_k\)-linear homomorphism \(f'_{(K_i)} : K_0 \otimes^c (K_1)_{K_0,1} \rightarrow M_2, (m_i) \mapsto f'(m_0)(m_1)\) is continuous. Let \(L_2 \in \mathcal{O}(M_2)\). Put \(L_0 := (f')^{-1}(\mathcal{L}(K_1, M_2), L_2) \cap K_0\). By the continuity of \(f'\), we have \(L_0 \in \mathcal{O}(K_0)\). It implies \((f'_{(K_i)})^{-1}(L_2) \in \mathcal{O}(K_0)\) by \((L_0, K_1, K_2) \subseteq (f'_{(K_i)})^{-1}(L_2)\). Therefore \(f'_{(K_i)}\) is continuous. By the universality of the colimit, the system \((f'_{(K_i)})_{K_i} \in \prod \mathcal{M}(M_i)\) gives a morphism \(f_{\tilde{L}} : M_0 \otimes^{cg} M_1 \rightarrow M_2\) in \(\mathcal{C}_{\ell}\). By the construction, we have \(T^{cg}_{M_0,1,2}(f_{\tilde{L}}) = f\). Thus \(T^{cg}_{M_0,1,2}\) is surjective.

As a consequence of Theorem 3.8, we obtain the following:
Corollary 3.12. The functor $\otimes^{cg}$ is cocontinuous.

3.2 CGLT algebras A CGLT $O_k$-algebra is a monoid in $(\mathcal{C}^{cg}_k, \otimes^{cg}, O_k)$. We will verify that $O_k[[G]]$ forms a CGLT $O_k$-algebra. Before that, we give examples of CGLT $O_k$-algebras. For this purpose, we compare $\otimes$, the tensor product $\hat{\otimes}_k$ of Banach $k$-vector spaces (cf. [1] p. 12), and the tensor product of compact Hausdorff flat linear topological $O_k$-modules given as the inverse limit of the algebraic tensor product of finite quotients. For this purpose, we recall an elementary property of $\hat{\otimes}_k$.

Proposition 3.13. For any $(X,V) \in \text{ob}(\text{Top} \times \text{Ban}^{ur}_k)$, the multiplication $C(X,k) \times V \rightarrow C(X,V)$ extends to a unique isomorphism $C(X,k) \hat{\otimes}_k V \rightarrow C(X,V)$ in $\text{Ban}^{ur}_k$.

Proof. The assertion immediately follows from the orthonormalisability of an unramified Banach $k$-vector space (cf. [8] IV 3 Corollaire 1, [2] 2.5.2 Lemma 2, and the proof of [11] Proposition 10.1).

The underlying linear topological $O_k$-module of any Banach $k$-vector space is CG by Proposition 2.11 (iii). We denote by $\mathcal{I}_k: \text{Ban}(k) \rightarrow \mathcal{C}^{cg}_k$ the forgetful functor. Let $(V_i) \in \text{ob}(\text{Ban}(k)^2)$. By the definition of $\otimes^\ell$, $\mathcal{I}^{cg}(\mathcal{I}_k(V_0)) \otimes^\ell \mathcal{I}^{cg}(\mathcal{I}_k(V_1))$ is first countable. The natural embedding $\mathcal{I}^{cg}(\mathcal{I}_k(V_0)) \otimes^\ell \mathcal{I}^{cg}(\mathcal{I}_k(V_1)) \hookrightarrow \mathcal{I}^{cg}(\mathcal{I}_k(V_0 \hat{\otimes}_k V_1))$ is a homeomorphism onto the dense image by the definition of $\otimes^\ell$ and $\hat{\otimes}_k$, and hence induces a homeomorphism $T^{\hat{\otimes}_k, \otimes^{cg}}_{(V_i)}: \mathcal{I}_k(V_0) \otimes^{cg} \mathcal{I}_k(V_1) \hookrightarrow \mathcal{I}_k(V_0 \hat{\otimes}_k V_1)$ onto the dense image by Proposition 2.11 (iii). The correspondence $(V_i) \sim T^{\hat{\otimes}_k, \otimes^{cg}}_{(V_i)}$ gives a natural transformation $T^{\hat{\otimes}_k, \otimes^{cg}}: \mathcal{I}_k(\bullet_0) \otimes^{cg} \mathcal{I}_k(\bullet_1) \rightarrow \mathcal{I}_k(\bullet_0 \hat{\otimes}_k \bullet_1)$. As a consequence, we obtain the following:

Proposition 3.14. Every Banach $k$-algebra, that is, monoid in $(\text{Ban}(k), \hat{\otimes}_k, k)$, forms a CGLT $O_k$-algebra through $\mathcal{I}_k$ and $T^{\hat{\otimes}_k, \otimes^{cg}}$.

By [7] Corollary 2.8 (i), if $G$ is a profinite group, then $C(G,k)$ admits a unique Hopf monoid structure in $(\text{Ban}(O_k), \hat{\otimes}_k, k)$ extending the pointwise $k$-algebra structure. Therefore by Proposition 3.14, we obtain the following:

Corollary 3.15. If $G$ is a profinite group, then $C(G,k)$ admits a unique structure of a commutative CGLT $O_k$-algebra such that the multiplication is a continuous $O_k$-linear extension of the pointwise multiplication.
Every compact topological $O_k$-module is CG by Proposition 2.1 (ii) and Proposition 2.11 (ii). We denote by $\mathcal{O}_k: C^{ch} \hookrightarrow C^{cg}$ the inclusion. The natural $O_k$-linear homomorphism $\mathcal{I}(K_0) \otimes \mathcal{I}(K_1) \rightarrow \mathcal{O}_k(K_0 \hat{\otimes} O_k K_1)$ is a homeomorphism onto the dense image by the definition of $\otimes$ and $\hat{\otimes} O_k$, and it induces a homeomorphism $T_{(K_i)} : \mathcal{O}_k(K_0) \otimes^{cg} \mathcal{O}_k(K_1) \hookrightarrow \mathcal{O}_k(K_0 \hat{\otimes} O_k K_1)$ onto the dense image. The correspondence $(K_i) \rightsquigarrow T_{(K_i)}$ gives a natural transformation $T_{\hat{\otimes} O_k, \otimes^{cg}}$.

As a consequence, we obtain the following:

**Proposition 3.16.** Every monoid in $(C^{ch}, \hat{\otimes} O_k, O_k)$ forms a CGLT $O_k$-algebra through $\mathcal{O}_k$ and $T_{\hat{\otimes} O_k, \otimes^{cg}}$.

By Proposition 2.21 and [7] Proposition 2.7, if $G$ is a profinite group, then $O_k[[G]]$ admits a unique Hopf monoid structure in $(C^{ch}, \hat{\otimes} O_k, O_k)$ extending the Hopf $O_k$-algebra structure of $O_k[G]$. Therefore by Proposition 3.16, we obtain the following:

**Corollary 3.17.** If $G$ is a profinite group, then $O_k[[G]]$ admits a unique structure of a CGLT $O_k$-algebra extending the Hopf $O_k$-algebra structure of $O_k[G]$.

We note that Corollary 3.17 will be extended to the case where $G$ is not necessarily a profinite group, as we mentioned in the beginning of this subsection. Another simple example of a CGLT $O_k$-algebra is given by a topological $O_k$-algebra.

**Proposition 3.18.** Let $A$ be a topological $O_k$-algebra. If the underlying topological $O_k$-module $M$ of $A$ is linear and CG, then $M$ admits a unique structure of a CGLT $O_k$-algebra whose multiplication is an $O_k$-linear extension of the multiplication of $A$ through $\nabla^{cg}_{M,M} \circ (\nabla^{cr}_{M,M})^{-1}$.

**Proof.** We denote by $f_{(K_i)} : K_0 \otimes \mathcal{I}(K_1) \rightarrow M$ the $O_k$-linear extension of the multiplication of $A$ restricted to $\prod \mathcal{I}(K_i) \subset \mathcal{I}(M)^2$, which is continuous for any $(K_i) \in \mathcal{K}(M)^2$ by Proposition 3.4. The system $(f_{K_0,K_1}(K_0,K_1))_{(K_0,K_1) \in \mathcal{K}(M)^2}$ induces a continuous $O_k$-linear homomorphism $f : M \otimes^{cg} M \rightarrow M$ by the
universality of the colimit. We denote by \( \epsilon \) the map \( O_k \to \mathcal{M} \), \( c \mapsto cl \). Since the identity map \( A \to (M, f, \epsilon) \) preserves the multiplication and the unit, \( (M, f, \epsilon) \) satisfies the axiom of a monoid in \( \mathcal{C}_\ell \).

The rest of this subsection is devoted to the following extension of Corollary 3.17:

**Theorem 3.19.** The CG linear topological \( O_k \)-module \( O_k[[G]] \) admits a unique structure of a CGLT \( O_k \)-algebra such that \( O_k^{cg} \) is an \( O_k \)-algebra homomorphism.

In order to verify Theorem 3.19, we define a convolution product on \( \mathcal{M}(G) \). Let \( (\mu_i) \in \mathcal{M}(G)^2 \). We define elements \( \prod \mu_i \in \mathcal{M}(G^2) \) and \( \mu_0 * \mu_1 \in \mathcal{M}(G) \). Let \( U' \in CO(G^2) \). To begin with, suppose that \( U' \) is compact. Take an \( S \in \mathcal{P}_{<\omega}(CO(G)^2) \) satisfying \( U' = \bigsqcup \{U_i \in S \prod U_i \}. We put \( (\prod \mu_i)(U') := \sum_{(U_i) \in S} \prod \mu_i(U_i) \). By the finite additivity of \( \mu_0 \) and \( \mu_1 \), \( (\prod \mu_i)(U') \) depends only on \( U' \). In particular, the equality \( (\prod \mu_i)(\prod U_i) = \prod \mu_i(U_i) \) holds for any compact clopen subsets \( U_0 \) and \( U_1 \) of \( G \).

Next, we consider the case where \( U' \) is not necessarily compact. Take a compact clopen subgroup \( K \subset G \). Then \( (G/K)^2 = \{(g, K) \mid (g_i) \in G^2 \} \) gives an element of \( \mathcal{P}(G^2) \) consisting of compact clopen subsets. Put \( (\prod \mu_i)(U') := \sum_{(C_i) \in (G/K)^2} (\prod \mu_i)(U' \cap \prod C_i) \). By the normality of \( \mu_0 \) and \( \mu_1 \), the infinite sum in the right hand side actually converges, and \( (\prod \mu_i)(U') \) is independent of the choice of \( K \). We obtain a normal \( O_k \)-valued measure \( \prod \mu_i \) on \( G^2 \).

For a \( U \in CO(G) \), we denote by \( \tilde{U} \subset G^2 \) the preimage of \( U \) by the multiplication \( G^2 \to G \). Set \( (\mu_0 * \mu_1)(U) := (\prod \mu_i)(\tilde{U}) \). Since \( \prod \mu_i \) is a normal \( O_k \)-valued measure on \( G^2 \), so is \( \mu_0 * \mu_1 \) on \( G \). We have constructed an element \( \mu_0 * \mu_1 \in \mathcal{M}(G) \). By the construction, the convolution product \( *_G : \mathcal{F}(\mathcal{M}(G))^2 \to \mathcal{F}(\mathcal{M}(G)) : (\mu_i) \mapsto \mu_0 * \mu_1 \) is compatible with \( O_k^{cg} \) and the multiplication \( O_k[G]^2 \to O_k[G] \). We note that \( *_G \) is not necessarily continuous.

**Lemma 3.20.** For any \( (K_i) \in \mathcal{H}(\mathcal{M}(G))^2 \), \( \{\mu_0 * \mu_1 \mid (\mu_i) \in \prod K_i \} \subset \mathcal{M}(G) \) is pre-compact.

**Proof.** Put \( K := \{\mu_0 * \mu_1 \mid (\mu_i) \in \prod K_i \} \). For each \( i \in \{0, 1\} \), there is an increasing sequence \( (G_i, r)_{r \in \omega} \) of compact clopen subsets such that
By the bijectivity of $i_{\CGG}^{cg}$ and $\nabla_{\CGG,\CGG}^{cg}$, $*_{G}$ induces an $O_{k}$-bilinear homomorphism $*_{G}^{cg}: K[[G]] \times_{cg} K[[G]] \to \CGG^{cg}(K[[G]])$, $(\mu_{i}) \mapsto \mu_{0} * \mu_{1}$ compatible with $O_{k}^{\otimes d_{G}}$ and the multiplication $O_{k}[G]^{2} \to O_{k}[G]$.

**Lemma 3.21.** The convolution product $*_{G}^{cg}$ is continuous.

**Proof.** Let $U \subset O_{k}[[G]]$ be an open neighbourhood of $\mu_{0} * \mu_{1}$ for a $(\mu_{i}) \in \CGG(G)^{2}$. It suffices to show that for any $(K_{i}) \in \mathscr{X}(\CGG(G))^{2}$ satisfying $(\mu_{i}) \in \prod K_{i}$, the preimage of $(*_{G}^{cg})^{-1}(U)$ satisfying $(\mu_{i}) \in \prod K_{i}$ is open. Put $K := \{\mu_{0}' * \mu_{1}' \mid (\mu_{i}') \in \prod K_{i}\}$. By Lemma 3.20, $K$ lies in $\mathscr{X}(\CGG(G))$. By Corollary 2.7, $i_{\CGG(G)}^{cg}(U) \cap K$ is an open subset of $K$, and there is a $(P, \epsilon) \in P(G) \times (0,1]$ such that $(\mu_{0} + \CGG(G; P, \epsilon)) \cap K \subset i_{\CGG(G)}^{cg}(U) \cap K$. Let $i \in \{0,1\}$. By Lemma 2.18, there is a compact clopen subset $G_{i} \subset G$ such that $|\mu(U)| < \epsilon$ for any $(\mu, U) \in K_{i} \times \CGG(G \setminus G_{i})$. We obtain $\mu_{0}' * \mu_{1}' \in i_{\CGG(G)}^{cg}(U) \cap K$ for any $(\mu_{i}') \in \prod((\mu_{i} \in \CGG(G; \{G_{i}, G \setminus G_{i}\}, \epsilon)) \cap K_{i})$ by the definition of $*_{G}$. It implies that the preimage of $(*_{G}^{cg})^{-1}(U)$ in $\prod K_{i}$ is open.

**Proof of Theorem 3.19.** The uniqueness follows from Proposition 2.14 and Proposition 2.17 (iii). By Corollary 2.7 and the cocontinuity of the forgetful functor $\text{Top} \to \text{Set}$, $*_{G}$ induces an $O_{k}$-linear homomorphism $\otimes_{G}^{cg}: O_{k}[[G]] \otimes_{cg} O_{k}[[G]] \to O_{k}[[G]]$. The composite $\otimes_{G} \circ \nabla_{\CGG,\CGG}^{cg}$ coincides with $*_{G}^{cg}$ by the construction, and hence is continuous by Lemma 3.21. The embedding $O_{k}^{\otimes_{d_{G}}}$ sends the multiplication of $O_{k}[G]$ to $\otimes_{G}^{cg}$ and the identity to $d_{G,1}$ by the construction. Since $O_{k}[G]$ satisfies the axiom of a monoid in $\mathcal{C}$, $O_{k}[[G]]$ forms a CGLT $O_{k}$-algebra with respect to the convolution product $\otimes_{G}^{cg}$ and the unit $O_{k} \to O_{k}[[G]]$, $c \mapsto cd_{G,1}$ by Proposition 2.14, Proposition 2.17 (iii), and the continuity of $\otimes_{G}^{cg}$.

We have examples of the computation of the generalised Iwasawa algebra $O_{k}[[G]]$.

**Example 3.22.** (i) If $G$ is discrete, then $O_{k}[[G]]$ is identified with $C_{0}(G, O_{k})$ equipped with the unique continuous extension of the $O_{k}$-algebra structure of the group algebra $O_{k}[G]$ through the correspondence in Proposition 3.14.
(ii) If $G$ is a profinite group, then the algebra structure of $O_k[[G]]$ coincides with the one induced by the homeomorphic $O_k$-linear isomorphism in Proposition 2.21, and $O_k[[G]]$ is identified with the classical Iwasawa algebra associated to $G$ through the correspondence in Proposition 3.16.

(iii) If $G$ admits a closed subgroup $H \subset G$ and a compact subset $C \subset G$ such that the multiplication $H \times C \to G$ is bijective, then the multiplication is actually a homeomorphism by [6] Lemma 2.13, and hence $O_k[[G]]$ admits a natural homeomorphic $O_k$-linear isomorphism to $O_k[[H \times C]]$.

(iv) For any open subgroup $H \subset G$ and a discrete subset $D \subset G$ such that the multiplication $H \times D \to G$ is bijective, the multiplication is a homeomorphism by [3] p. 433, and hence $O_k[[G]]$ admits a natural homeomorphic $O_k$-linear isomorphism to $O_k[[H \times D]]$. In particular, $\mathcal{F}^{cg}(O_k[[G]])$ admits a natural $O_k$-linear isomorphism to the ideal-adic completion of $\mathcal{F}^{cg}(O_k[[H]]) \oplus D$ by Lemma 2.19.

(v) If $G$ admits an increasing sequence $(G_r)_{r \in \omega}$ of open subgroups satisfying $\bigcup_{r \in \omega} G_r = G$, then $\mathcal{F}^{cg}(O_k[[G]])$ admits a natural $O_k$-algebra isomorphism to the ideal-adic completion of $\mathcal{F}^{cg}(O_k[[G]])$ by Lemma 2.19.

As an application of Example 3.22 (ii) and (vi), we immediately obtain the following:

**Proposition 3.23.** Let $p$ denote the residual characteristic of $k$, and $\varpi \in O_k$ a uniformiser. Then $\mathcal{F}^{cg}(O_k[[\mathbb{Q}_p]])$ admits a natural $O_k$-algebra isomorphism to the $\varpi$-adic completion of the filtered colimit of $\mathcal{F}^{cg}(O_k[[T]])$ with respect to the continuous $O_k$-algebra homomorphism $O_k[[T]] \to O_k[[T]], T \mapsto (T + 1)^p - 1$.

### 3.3 CGLT modules

Let $A$ be a CGLT $O_k$-algebra. A **CGLT $A$-module** is a left $A$-module in $(\mathcal{C}^{cg}_{\ell}, \otimes^{cg}, O_k)$. We give three examples of CGLT modules as immediate consequences of Proposition 3.14, Proposition 3.16, and Proposition 3.18, respectively:

**Proposition 3.24.** Let $\mathcal{A}$ be a Banach $k$-algebra. Then every Banach left $\mathcal{A}$-module, that is, left $\mathcal{A}$-module in $(\text{Ban}(k), \hat{\otimes}_k, k)$, forms a CGLT $\mathcal{A}_k(\mathcal{A})$-module through $\mathcal{A}_k$ and $T \mapsto (T + 1)^p - 1$. 

Proposition 3.25. Let $\mathcal{A}$ be a monoid in $(\mathcal{C}_\ell^{\text{ch}}, \hat{\otimes}_{\text{Ok}}, \text{Ok})$. Then every left $\mathcal{A}$-module in $(\mathcal{C}_\ell^{\text{ch}}, \hat{\otimes}_{\text{Ok}}, \text{Ok})$ forms a CGLT $\mathcal{J}_{\text{Ok}}(\mathcal{A})$-module through $\mathcal{J}_{\text{Ok}}$ and $T^{\hat{\otimes}_{\text{Ok}}, \text{cg}}$.

Proposition 3.26. Let $\mathcal{A}$ be a topological $\text{Ok}$-algebra whose underlying topological $\text{Ok}$-module is linear and CG. Then every topological left $\mathcal{A}$-module whose underlying topological $\text{Ok}$-module is linear and CG forms a CGLT $\mathcal{A}$-module through $\nabla_{\text{cg}} \circ (\nabla_{\text{cg}})^{-1}$.

A $\text{BT} A$-module is a CGLT $A$-module $V$ whose underlying $\text{Ok}$-module structure extends to a $k$-vector space structure equipped with a complete non-Archimedean norm on the underlying $k$-vector space of $V$ giving its original topology. Let $V$ be a $\text{BT} A$-module. Then $V$ forms a topological $k$-vector space because it forms a Banach $k$-vector space. We say that $V$ is bounded if there is an $R \in (0, \infty)$ such that $\|fv\| \leq R\|v\|$ for any $(f, v) \in A \times V$, is submetric if $\|fv\| \leq \|v\|$ for any $(f, v) \in A \times V$, and is unitary if it is submetric and the underlying Banach $k$-vector space $V$ is unramified. We denote by $\text{BT}(A)$ the $k$-linear category of bounded $\text{BT} A$-modules and bounded $A$-linear homomorphisms, by $\text{BT}_{\leq}(A) \subset \text{BT}(A)$ the $\text{Ok}$-linear subcategory of submetric $\text{BT} A$-modules and submetric $A$-linear homomorphisms, and by $\text{BT}_{\text{ur}}(A) \subset \text{BT}_{\leq}(A)$ the full subcategory of unitary $\text{BT} A$-modules.

Let $M$ be a CGLT $A$-module. We say that $M$ is a CHFLT $A$-module if the underlying linear topological $\text{Ok}$-module of $M$ is a compact Hausdorff flat linear topological $\text{Ok}$-module. We give a characterisation of a CHFLT $A$-module.

Proposition 3.27. Let $K \in \text{ob}(\mathcal{C}_c^c)$. For any a map $\rho: A \times K \to K$, $K$ forms a CGLT $A$-module with respect to the $\text{Ok}$-linear extension of $\rho$ if and only if $\mathcal{F}_{\text{cg}}(K)$ forms a left $\mathcal{F}_{\text{cg}}(A)$-module and $\rho$ is continuous.

Proof. We denote by $\hat{\rho}: A \otimes_{\text{cg}} K \to K$ the $\text{Ok}$-linear extension of $\rho$. The direct implication follows from Proposition 3.10 and the continuity of $\nabla_{\text{cg}}^{\otimes}$. Suppose that $K$ forms a CGLT $A$-module with respect to $\hat{\rho}$. By the naturality of $\nabla_{\text{cg}}$, $\mathcal{F}_{\text{cg}}(K)$ forms a left $\mathcal{F}_{\text{cg}}(A)$-module with respect to $\rho$. Let $U \subset K$ be an open subset. Let $(f, m) \in A \times K$. We show that if $\rho(f, m) \in U$, then $\rho^{-1}(U)$ is an open neighbourhood of $(f, m)$. Put $L := \{ f' \in A \mid \forall m' \in K, \rho(f + f', m') \in U \}$. Let $K_0 \in \mathcal{K}(A)$. By Proposition 2.1 (ii) and the continuity of the $\hat{\rho}$, there is an $(L_i) \in \mathcal{O}(K_0) \times \mathcal{O}(K)$.
such that \( \rho(f', m') \in U \) for any \( (f', m') \in \left( (f + L_0) \times K \right) \cup (K_0 \times (m + L_1)) \).
In particular, we have \( L_0 \subset L \cap K_0 \) and hence \( L \cap K_0 \in \mathcal{O}(K_0) \). It implies \( L \in \mathcal{O}(A) \) by Corollary 2.7. By \( L \times K \subset \rho^{-1}(U) \), \( \rho^{-1}(U) \) is an open neighbourhood of \( (f, m) \). Thus \( \rho \) is continuous. \qed

A left \( A \)-submodule \( K \subset M \) is said to be a core of \( M \) if \( K \) is compact, the inclusion \( K \hookrightarrow M \) induces an isomorphism \( k \otimes_{O_k} \mathcal{F}^c(K) \to \mathcal{F}^c(M) \)
in \( \mathcal{C} \), and every \( O_k \)-submodule \( L \subset M \) satisfying \( cL \cap K \in \mathcal{O}(K) \) for any \( c \in O_k \setminus \{0\} \) is open. We say that \( M \) is a CGHLT \( A \)-module if \( M \) is Hausdorff and admits a core. If \( M \) is a CGHLT \( A \)-module, then \( M \) forms a topological \( k \)-vector space because \( \mathcal{O}(M) \) is stable under the action of \( k^X \). We denote by \( \text{Mod}^\text{ch}_k(A) \) the \( O_k \)-linear category of CHFLT \( A \)-modules and continuous \( A \)-linear homomorphisms, and by \( \text{Mod}^\text{ch}_{cg}(A) \) the \( k \)-linear category of CGHLT \( A \)-modules and continuous \( A \)-linear homomorphisms.

We give an example of a CGHLT \( A \)-module. Let \( K \in \text{ob} (\text{Mod}^\text{ch}_k(A)) \). We denote by \( K_k \) the left \( \mathcal{F}^c(A) \)-module \( k \otimes_{O_k} \mathcal{F}^c(K) \) equipped with the strongest topology for which \( K_k \) forms a topological \( k \)-vector space and the natural embedding \( \iota_K^c: K \hookrightarrow K_k \) is continuous. We identify \( \mathcal{F}^c(K) \) with its image in \( k \otimes_{O_k} \mathcal{F}^c(K) \). The following is an analogue of [13] Lemma 1.4:

**Proposition 3.28.** The linear topological \( O_k \)-module \( K_k \) forms a CGHLT \( A \)-module, and \( \iota_K^c \) is a homeomorphism onto a core.

In order to verify Proposition 3.28, we characterise the topology of \( K_k \).

**Lemma 3.29.** A subset \( U \subset K_k \) is open if and only if \( (\iota_K^{-1}(cU))^{-1} \subset K \) is open for any \( c \in O_k \setminus \{0\} \).

**Proof.** We denote by \( \mathcal{O} \) the set of subsets \( U \subset K_k \) such that \( (\iota_K^{-1}(cU))^{-1} \subset K \) is open for any \( c \in O_k \setminus \{0\} \). Then \( \mathcal{O} \) satisfies the open set axiom of the underlying set of \( K_k \), for which \( \iota_K^c \) is continuous and \( K_k \) forms a topological \( k \)-vector space because \( \mathcal{O} \) is stable under the action of \( k^X \). Therefore by the universality of the strongest topology, \( \mathcal{O} \) coincides with the set of open subsets of \( K_k \). \qed

**Proof of Proposition 3.28.** Take a uniformiser \( \varpi \in O_k \). Put \( K_r := K \) for an \( r \in \omega \), and denote by \( K_\omega \) the colimit in \( \mathcal{C} \) of \( (K_r)_{r \in \omega} \) with respect to the transition maps \( K_r \to K_{r+1}, m \mapsto cm \) indexed by \( r \in \omega \). Then \( K_\omega \) forms a CG linear topological \( O_k \)-module by Proposition 2.1 (ii), Proposition 2.9 (i),
and Corollary 2.10. It is Hausdorff by the same computation as that in the proof of [6] Proposition 1.27 using Corollary 2.7 and a well-known property of T$_1$ normal topological spaces. By Corollary 3.12 and the functoriality of the colimit, the scalar multiplication $A \otimes^{cg} K \to K$ induces a continuous $O_k$-linear homomorphism $A \otimes^{cg} K_\omega \to K_\omega$, for which $K_\omega$ forms a CGLT $A$-module.

By the universality of the colimit and the flatness of $K$, $\iota^c_K$ induces a continuous bijective $O_k$-linear homomorphism $k\iota^c_K: K_\omega \to K_k$. By Corollary 2.7, the map $K_\omega \to K_\omega$, $m \mapsto \varpi m$ is an isomorphism in $c\ell$, and hence $K_\omega$ forms a topological $k$-vector space. We show that $k\iota^c_K$ is an open map. Let $L \in O_k(K_\omega)$. For any $c \in O_k \setminus \{0\}$, $(\iota^c_K)^{-1}(c(k\iota^c_K)(L))$ coincides with the preimage of $cL$ in $K_0$, and hence is open by the continuity of the canonical embedding $K_0 \hookrightarrow K_\omega$. It ensures $(k\iota^c_K)(L) \in \mathcal{O}(K_k)$ by Lemma 3.29. Therefore $k\iota^c_K$ is an isomorphism in $c\ell$, and $K_k$ forms a Hausdorff CGLT $A$-module. Since $K$ is compact and $K_k$ is Hausdorff, $\iota^c_K$ is a homeomorphism onto the image, which is a core of $K_k$. \hfill \Box

We obtain a characterisation of a CGHLT $A$-module.

**Proposition 3.30.** If $M$ is a CGHLT $A$-module with a core $K \subset M$, then the bijective $O_k$-linear homomorphism $K_k \to M$ induced by the inclusion $K \hookrightarrow M$ is an isomorphism in $\text{Mod}^{\text{ch}}_{c\ell}(A)$.

**Proof.** We denote by $\varphi: K_k \to M$ the map in the assertion, and by $i: K \hookrightarrow K_k$ the canonical embedding. By the universality of the strongest topology, $\varphi$ is continuous. Let $L \in K_k$. For any $c \in O_k \setminus \{0\}$, we have $c\varphi(L) \cap K = \varphi(cL) \cap K = i^{-1}(cL) \in \mathcal{O}(K)$. It implies $\varphi(L) \in \mathcal{O}(M)$. Therefore $\varphi$ is an open map. \hfill \Box

The correspondence $K \mapsto K_k$ gives an $O_k$-linear functor $\Phi_A: \text{Mod}^{\text{ch}}_{c\ell}(A) \to \text{Mod}^{\text{cg}}_{c\ell}(A)$ by Proposition 3.28. We denote by $\Phi_{A,k}: \text{kMod}^{\text{ch}}_{c\ell}(A) \to \text{Mod}^{\text{cg}}_{c\ell}(A)$ its $k$-linear extension.

**Proposition 3.31.** The $k$-linear functor $\Phi_{A,k}$ is fully faithful and essentially surjective.

**Proof.** The faithfulness of $\Phi_{A,k}$ follows from the faithfulness of $\Phi_A$ and the flatness of hom objects. The fullness follows from the same computation as that in the proof of [13] Lemma 1.5 ii and iii using Baire category theorem. The essential surjectivity follows from Proposition 3.30. \hfill \Box
By Proposition 2.1 (ii), Proposition 3.28, and Proposition 3.29, $O_k$ forms a commutative CGLT $O_k$-algebra. By Proposition 3.27, $\mathcal{O}_{O_k}$ induces an equivalence $\mathcal{C}^{\text{ch}}_{\ell} \to \text{Mod}^{\text{gh}}_{\ell}(O_k)$ of categories. By Proposition 3.28 and Proposition 3.29, we obtain an $O_k$-linear functor $\mathcal{C}^{\text{ch}}_{\ell} \to \text{Mod}^{\text{gh}}_{\ell}(O_k)$, which extends to a fully faithful essentially surjective $k$-linear functor $k\mathcal{C}^{\text{ch}}_{\ell} \to \text{Mod}^{\text{gh}}_{\ell}(O_k)$.

4 Modules over Iwasawa algebras

We study relation between module theory over $O_k[[G]]$ and representation theory of $G$. As a main result, we generalise Schneider–Teitelbaum duality to duality applicable to $G$, and give a criterion of the irreducibility of unitary Banach $k$-linear representations of $G$.

4.1 Unitary Banach representations A Banach $k$-linear representation of $G$ is a pair $(V, \rho)$ of a $V \in \text{ob}(\text{Ban}(k))$ and a continuous map $\rho: G \times V \to V$ giving a $k$-linear action of $G$ on $V$. Let $(V, \rho)$ be a Banach $k$-linear representation of $G$. We say that $(V, \rho)$ is unitarisable if there is an $R \in (0, \infty)$ such that $\|\rho(g, v)\| \leq R\|v\|$ for any $(g, v) \in G \times V$, is isometric if $\|\rho(g, v)\| = \|v\|$ for any $(g, v) \in G \times V$, and is said to be unitary if $V$ is unramified and $(V, \rho)$ is isometric. A map between Banach $k$-linear representations is said to be a $k[G]$-linear homomorphism if it is a $G$-equivariant $k$-linear homomorphism. We denote by $\text{Rep}_G(\text{Ban}(k))$ the $k$-linear category of unitarisable Banach $k$-linear representations of $G$ and bounded $k[G]$-linear homomorphisms, by $\text{Rep}_G(\text{Ban}_{\leq}(k)) \subset \text{Rep}_G(\text{Ban}(k))$ the $O_k$-linear subcategory of isometric Banach $k$-linear representations of $G$ and submetric $k[G]$-linear homomorphisms, and by $\text{Rep}_G(\text{Ban}^{\text{ur}}_{\leq}(k)) \subset \text{Rep}_G(\text{Ban}_{\leq}(k))$ the full subcategory of unitary Banach $k$-linear representations of $G$.

We compare the notion of a BT $O_k[[G]]$-module and the notion of a Banach $k$-linear representation of $G$. For this purpose, we consider a partial generalisation of Banach–Steinhaus theorem (cf. [11] Corollary 6.16). Let $(X_0, (V_i)) \in \text{ob}(\text{Top} \times \text{Ban}(k)^2)$.

Proposition 4.1. A map $\varphi: X_0 \to \mathcal{S}((V_i))$ is continuous if and only if the map $X_0 \times V_1 \to V_2: (x, v) \mapsto \varphi(x)(v)$ is continuous.
Proof. We denote by $\rho: X_0 \times V_1 \to V_2$ the induced map. The direct implication follows from the continuity of the map $X_0 \to X_0 \times V_1$, $x \mapsto (x, v)$ for any $v \in V_1$. Suppose that $\varphi$ is continuous. Let $U_2 \subset V_2$ be an open subset. Let $(x, v) \in X_0 \times V_1$. Suppose $\rho(x, v) \in U_2$. Take an $\epsilon \in (0, \infty)$ satisfying $\{v' \in V_2 \mid \|v' - \rho(x, v)\| < \epsilon\} \subset U_2$. Put $U_1 := \{v' \in V_1 \mid \|v' - v\| < \epsilon\}$. By the continuity of $\varphi$, there is an open neighbourhood $U_0 \subset X_0$ of $x$ such that $\|\rho(x', v) - \rho(x, v)\| < \epsilon$ for any $x' \in U_0$. We obtain $\|\rho(x', v') - \rho(x, v)\| \leq \max\{\|\rho(x', v' - v)\|, \|\rho(x', v) - \rho(x, v)\|\} < \epsilon$ for any $(x', v') \in \prod U_i$, and hence $\prod U_i \subset \rho^{-1}(U_2)$. It implies that $\rho$ is continuous.

Let $(X, (V, \rho)) \in \text{ob}(\text{Top} \times \text{Rep}_G(\text{Ban}_<(k)))$. By Proposition 4.1, the monoid homomorphism $\varphi_\rho: G \to \mathcal{S}(V)^\times$ induced by $\rho$ is continuous. In order to obtain a submetric $\text{BT} O_k[[G]]$-module structure on $V$ associated to $\rho$, we prepare a partial generalisation of [13] Lemma 2.1 for the Banach space side.

**Proposition 4.2.** The map $\mathcal{L}(\mathbb{M}(X), \mathcal{S}(V)) \to \text{Hom}_{\text{Top}}(X, \mathcal{U}(\mathcal{S}(V))), F \mapsto F \circ \delta_X$ is bijective.

**Proof.** Denote by $\delta_X^*$ the map in the assertion. By Proposition 2.16 (iii), $\delta_X^*$ is injective. Let $\varphi \in \text{Hom}_{\text{Top}}(X, \mathcal{U}(\mathcal{S}(V)))$. Denote by $O_k^{\oplus \varphi}: O_k^{\oplus X} \to \mathcal{S}(V)$ the $O_k$-linear extension of $\varphi$. Let $(v, \epsilon) \in V \times (0, 1]$. Put $L := \{f \in \mathcal{S}(V) \mid |f(v)| < \epsilon\}$. By the continuity of $\varphi$, the set of the preimages of open balls in $V$ of radius $\epsilon$ by the map $X \to V$, $x \mapsto \varphi(x)(v)$ gives a $P \in \mathbb{F}(X)$. We have $(O_k^{\oplus \varphi})^{-1}(\mathbb{M}(X; P, \epsilon)) \subset (O_k^{\oplus \varphi})^{-1}(L)$. Therefore $O_k^{\oplus \varphi}$ extends to a unique continuous $O_k$-linear homomorphism $\tilde{\varphi}: \mathbb{M}(X) \to \mathcal{S}(V)$ by Proposition 2.16 (iii). We have $\delta_X^*(\tilde{\varphi}) = \varphi$. Thus $\delta_X^*$ is surjective.

As a consequence of Theorem 3.19 and Proposition 4.2, we obtain the following:

**Corollary 4.3.** For any continuous monoid homomorphism $\varphi: G \to \mathcal{S}(V)^\times$ there is a unique continuous $O_k$-linear homomorphism $F: \mathbb{M}(G) \to \mathcal{S}(V)$ satisfying $F \circ \delta_L = \varphi$, and $F \circ \iota_{\mathbb{M}(G)}^{cg}$ preserves the multiplication and the unit.

By Corollary 4.3, $\varphi_\rho$ induces a continuous $O_k$-linear homomorphism $\tilde{\Pi}_\rho: \mathbb{M}(G) \to \mathcal{S}(V)$ such that $\tilde{\Pi} \circ \iota_{\mathbb{M}(G)}^{cg}$ preserves the multiplication and the unit. We have a comparison between the closed unit balls of $\mathcal{H} \text{om}^{cg}$ and $\mathcal{B}$. 
Proposition 4.4. The identity map
\[ \mathcal{H}\text{om}^{\mathfrak{Ck}}(\mathcal{J}_k(V), \mathcal{J}_k(V)) \cap \text{End}_{\text{Ban}_{\leq}(k)}(V) \to \mathcal{I}(V) \]
is an isomorphism in \( \mathcal{C}_\ell \).

Proof. Denote by \( i \) the map in the assertion. By Corollary 2.5, \( i \) is continuous. Let \( L \in \mathcal{C}(\mathcal{H}\text{om}^{\mathfrak{Ck}}(\mathcal{J}_k(V), \mathcal{J}_k(V)) \cap \text{End}_{\text{Ban}_{\leq}(k)}(V)) \). Take a \((K, \epsilon) \in \mathcal{K}(\mathcal{J}_k(V)) \times (0, \infty) \) satisfying \( \{ f \in \text{End}_{\text{Ban}_{\leq}(k)}(V) \mid \forall v \in K, |f(v)| < \epsilon \} \subset L \) and a \( K_0 \in \mathcal{P}_{<\omega}(K) \) satisfying \( K \subset \bigcup_{v \in K_0} \{ v' \in V \mid |v' - v| < \epsilon \} \). We have \( \{ f \in \mathcal{I}(V) \mid \forall v \in K_0, |f(v)| < \epsilon \} \subset i(L) \), and hence \( i(L) \in \mathcal{C}(\mathcal{I}(V)) \). Therefore \( i \) is an isomorphism in \( \mathcal{C}_\ell \).

By Corollary 2.10 (i) and Proposition 4.4, \( \tilde{\Pi}_\rho \) induces a continuous \( O_k \)-linear homomorphism \( \Pi_\rho : O_k[[G]] \to \mathcal{J}_k(V) \mathcal{J}_k(V) \) preserving the multiplication and the unit. By Theorem 3.8, \( \Pi_\rho \) gives a CGLT \( O_k[[G]] \)-module structure on \( \mathcal{J}_k(V) \), for which \( \mathcal{J}_k(V) \) forms a submetric BT \( O_k[[G]] \)-module \( \int_G(V, \rho) \). By the construction, the correspondence \((V, \rho) \sim \int_G(V, \rho) \) gives \( O_k \)-linear functors \( \int_{G, \leq}^d : \text{Rep}_G(\text{Ban}_{\leq}(k)) \to \text{BT}_{\leq}(O_k[[G]]) \) and

\[ \int_{G, \text{ur}}^d : \text{Rep}_G(\text{Ban}_{\leq}^\text{ur}(k)) \to \text{BT}_{\leq}^\text{ur}(O_k[[G]]). \]

Each step of the construction of \( \int_G(V, \rho) \) is obviously invertible, and hence we obtain a comparison between the notion of a submetric BT \( O_k[[G]] \)-module and the notion of an isometric Banach \( k \)-linear representation of \( G \).

Theorem 4.5. The functors \( \int_{G, \leq}^d \) and \( \int_{G, \text{ur}}^d \) are equivalences of \( O_k \)-linear categories.

We also consider a similar comparison without the assumption of the submetric condition. Let \((V, \rho) \in \text{ob}(\text{Rep}_G(\text{Ban}(k))) \). Take a \( c \in k^\times \) satisfying \( \|\rho(g, v)\| \leq |c| \|v\| \) for any \((g, v) \in G \times V \). By Proposition 4.1, the map \( G \to \mathcal{J}(V) \) induced by the continuous map \( G \times V \to V, (g, v) \mapsto c^{-1}\rho(g, v) \) is continuous. By Proposition 4.2, it induces a continuous \( O_k \)-linear homomorphism \( \mathbb{M}(G) \to \mathcal{J}(V) \), which does not necessarily preserve the multiplication. By Corollary 2.10 (i) and Proposition 4.4, it induces a continuous \( O_k \)-linear homomorphism \( O_k[[G]] \to \mathcal{J}_k(V) \mathcal{J}_k(V) \). Multiplying \( c \), we
obtain a continuous $O_k$-linear homomorphism $\Pi : O_k[[G]] \rightarrow \mathcal{I}_k(V)$ independent of the choice of $c$ preserving the multiplication and the unit. By Theorem 3.8, $\Pi$ gives a CGLT $O_k[[G]]$-module structure on $\mathcal{I}_k(V)$, for which $\mathcal{I}_k(V)$ forms a bounded BT $O_k[[G]]$-module $\mathcal{I}_G(V,\rho)$. By the construction, the correspondence $(V,\rho) \leadsto \mathcal{I}_G(V,\rho)$ gives a $k$-linear functor $\mathcal{I}_G^d : \text{Rep}_G(\text{Ban}(k)) \rightarrow \text{BT}(O_k[[G]])$. Each step of the construction of $\mathcal{I}_G(V,\rho)$ is obviously invertible, and hence we obtain a comparison between the notion of a bounded BT $O_k[[G]]$-module and the notion of a unitarisable Banach $k$-linear representation of $G$.

**Theorem 4.6.** The functor $\mathcal{I}_G^d$ is a $k$-linear equivalence of categories.

### 4.2 CHFLT modules

A CGLT $O_k$-linear representation of $G$ is a pair $(M,\rho)$ of an $M \in \text{ob}(\mathcal{E}_{\ell}^{cg})$ and a continuous map $\rho : G \times M \rightarrow M$ giving an $O_k$-linear action of $G$ on $M$. A map between CGLT $O_k$-linear representations is said to be an $O_k[G]$-linear homomorphism if it is a $G$-equivariant $O_k$-linear homomorphism. Let $(M,\rho)$ be a CGLT $O_k$-linear representation of $G$. We say that $(M,\rho)$ is a *CHFLT $O_k$-linear representation of $G$* if $M \in \text{ob}(\mathcal{E}_{\ell}^{ch})$. We denote by $\text{Rep}_G(\mathcal{E}_{\ell}^{cg})$ the $O_k$-linear category of CGLT $O_k$-linear representations of $G$ and continuous $O_k[G]$-linear homomorphisms, and by $\text{Rep}_G(\mathcal{E}_{\ell}^{ch}) \subset \text{Rep}_G(\mathcal{E}_{\ell}^{cg})$ the full subcategory of CHFLT $O_k$-linear representations of $G$.

We compare the notion of a CHFLT $O_k[[G]]$-module and the notion of a CHFLT $O_k$-linear representation of $G$. For this purpose, we consider a compact analogue of Banach–Steinhaus theorem (cf. [11] Corollary 6.16). We denote by $\text{Unf}$ the category of compact uniform spaces and uniformly continuous maps. Let $(X_0, (C_{i+1})) \in \text{ob}(\text{Top} \times \text{Unf}^2)$. We equip $\text{Hom}_{\text{Unf}}((C_{i+1}))$ the topology of uniform convergence.

**Proposition 4.7.** A map $\varphi : X_0 \rightarrow \text{Hom}_{\text{Unf}}((C_{i+1}))$ is continuous if and only if the induced map $X_0 \times C_1 \rightarrow C_2 : (x,m) \mapsto \varphi(x)(m)$ is continuous.

**Proof.** If $C_1 = \emptyset$, then the assertion is obvious. We assume $C_1 \neq \emptyset$. We denote by $\rho : X_0 \times C_1 \rightarrow C_2$ the induced map. Suppose that $\varphi$ is continuous. Let $U_2 \subset C_2$ be an open neighbourhood of $\rho(x_0,m_0)$ for $(x_0,m_0) \in X_0 \times C_1$. Take entourages $E_0, E_1 \subset C_2^2$ satisfying $\{m_1 \in C_2 \mid (\rho(x_0,m_0),m_1) \in E_0\} \subset U_2$ and that for any $(m_i)_{i=0}^2 \in C_2^3$, $((m_i),(m_{i+1})) \in E_1^2$ implies $(m_{2i}) \in E_0$. 

By the uniform continuity of \( \varphi(x_0) \), there is an entourage \( E_2 \subset C_1^2 \) such that every \( (m_i) \in E_2 \) satisfies \( (\varphi(x_0)(m_i)) \in E_1 \). By the continuity of \( \varphi \), there exists an open neighbourhood \( U_0 \subset X \) of \( x_0 \) such that \( (\varphi(x_{1-i})(m_1)) \in E_1 \) for any \( (x_1, m_1) \in U_0 \times C_1 \). Put \( U_1 := \{m_1 \in C_1 \mid (m_i) \in E_2\} \). Then for any \( (x_1, m_1) \in \prod U_i \), we have \( ((\rho(x_0, m_i)), (\rho((x_i, m_1))) \in E_1^2 \), and hence \( (\rho(x_i, m_i)) \in E_0 \). It implies that \( \prod U_i \subset \rho^{-1}(U_2) \). Therefore \( \rho \) is continuous.

Suppose that \( \rho \) is continuous. Let \( U \subset \text{Hom}_{\text{Unf}}((C_{i+1})) \) be an open neighbourhood of \( \varphi(x_0) \) for a \( x_0 \in X_0 \). For an entourage \( E \subset C_2^2 \), set \( U_E := \{f \in \text{Hom}_{\text{Unf}}((C_{i+1})) \mid \forall m \in C_1, (\varphi(x_0)(m), f(m)) \in E\} \). Then the collection of subsets of the form \( U_E \) forms a fundamental system of neighbourhoods of \( \varphi(x_0) \). Take entourages \( E_0, E_1 \subset C_2 \) satisfying \( U_{E_0} \subset U \) and that for any \( (m_i)_{i=0}^2 \subset C_2 \), \( (m_i), (m_{2i}) \in E_1^2 \) implies \( (m_{i+1}) \in E_0 \). For each \( m_0 \in C_1 \), there are open neighbourhoods \( U_0 \subset X \) and \( U_1 \subset C_1 \) of \( x_0 \) and \( m_0 \), respectively, such that \( (\rho((x_i, m_i))) \in E_1 \) for any \( (x_1, m_1) \in \prod U_i \) by the continuity of \( \rho \). We denote by \( S \) the set of such an \( (m_0, (U_i)) \) satisfying \( m_0 \in C_1 \). Since \( C_1 \) is compact and non-empty, there is an \( S_0 \in \mathcal{P}_{<\omega}(S) \) such that \( C_1 = \bigcup_{(m_0, (U_i)) \in S_0} U_1 \). Put \( V_0 := \bigcap_{(m_0, (U_i)) \in S_0} U_0 \). Let \( x_1 \in V_0 \). We show \( \varphi(x_1) \in U_{E_0} \). Let \( m_0 \in C_1 \). Take an \( (m_1, (U_i)) \in S_0 \) satisfying \( m_0 \in U_1 \). We have \( ((\rho(x_0, m_i)), (\rho((x_i, m_i))) \in E_1^2 \) by the choice of \( m_1 \) and \( U_1 \). Therefore we obtain \( (\rho((x_i)(m_1))) = (\rho((x_i, m_1))) \in E_0 \) by the choice of \( E_1 \). It ensures \( \varphi(x_1) \in U_{E_0} \). It implies that \( V_0 \subset \varphi^{-1}(U_{E_0}) \). Thus \( \varphi \) is continuous.

Let \( (K, \rho) \in \text{ob}(\text{Rep}_G(C_{\text{f\ell}}^c(K,K))^X) \). The monoid homomorphism \( \varphi_\rho: G \to \mathcal{H}\text{om}^c(K,K)^X \) induced by \( \rho \) is continuous by Proposition 4.7. In order to obtain a CHFLT \( O_k[[G]] \)-module structure on \( K \) associated to \( \rho \), we prepare a partial generalisation of [13] Lemma 2.1 for the compact side. By Proposition 3.3 and Proposition 4.2, we obtain the following:

**Proposition 4.8.** The map

\[ \mathcal{L}([\mathbb{M}(X), \mathcal{H}\text{om}^c(K,K)]) \to C(X, \mathcal{H}\text{om}^c(K,K)), \ F \mapsto F \circ \delta_X \]

is bijective.

By Theorem 3.19 and Proposition 4.8, we obtain a locally profinite counterpart of [13] Corollary 2.2 for the compact side.
Corollary 4.9. For any continuous monoid homomorphism 
\[ \varphi : G \to \mathcal{H}\text{om}^c(K,K)^\times, \]
there is a unique continuous \( O_k \)-linear homomorphism 
\[ F : \mathcal{M}(G) \to \mathcal{H}\text{om}^c(K,K) \]
such that \( F \circ \delta_G = \varphi \), and \( F \circ \iota_{\mathcal{M}(G)}^c \) preserves the multiplication and the unit.

By Corollary 2.10 (i) and Corollary 4.9, \( \varphi_\rho \) induces a continuous \( O_k \)-linear homomorphism \( \Pi_\rho : O_k[[G]] \to \mathcal{J}_{O_k}(K)\mathcal{I}_{O_k}(K) \) preserving the multiplication and the unit. By Theorem 3.8, \( \Pi_\rho \) gives a CGLT \( O_k[[G]] \)-module structure on \( \mathcal{I}_{O_k}(K)\mathcal{I}_{O_k}(K) \). By the construction, the correspondence \((K,\rho) \mapsto \int_G^c(K,\rho)\) gives an \( O_k \)-linear functor \( \int_G^c : \text{Rep}_{G}(\mathcal{C}\text{ch}\ell) \to \text{Mod}_{\mathcal{C}\text{ch}\ell}(O_k[[G]]) \). Each step of the construction of \( \int_G^c(K,\rho) \) is obviously invertible, and hence we obtain a comparison between the notion of a CHFLT \( O_k[[G]] \)-module and the notion of a CHFLT \( O_k \)-linear representation of \( G \).

Theorem 4.10. The functor \( \int_G^c \) is an \( O_k \)-linear equivalence of categories.

Let \((M,\rho) \in \text{ob}(\text{Rep}_{G}(\mathcal{C}\text{ch}\ell_k))\). A \( G \)-stable \( O_k \)-submodule \( K \subset M \) is said to be a core of \((M,\rho)\) if \( K \) is compact, the inclusion \( K \hookrightarrow M \) induces an isomorphism \( k \otimes_{O_k} \mathcal{F}^c(K) \to \mathcal{F}^c(M) \) in \( \mathcal{C} \), and every \( O_k \)-submodule \( L \subset M \) satisfying \( cL \cap K \in \mathcal{O}(K) \) for any \( c \in O_k \setminus \{0\} \) is open. We say that \((M,\rho)\) is a CGHLT \( k \)-linear representation of \( G \) if \( M \) is Hausdorff and \((M,\rho)\) admits a core. If \((M,\rho)\) is a CGHLT \( k \)-linear representation of \( G \), then \( M \) forms a topological \( k \)-vector space because \( \mathcal{O}(M) \) is closed under the action of \( k^\times \). We denote by \( \text{Rep}_G(k\mathcal{C}^\text{ch}\ell_k) \subset \text{Rep}_G(\mathcal{C}^\text{ch}\ell_k) \) the full subcategory of CGHLT \( k \)-linear representations of \( G \). We give an example of a CGHLT \( k \)-linear representation of \( G \). We denote by \((K,\rho)_k\) the pair of \( K_k \in \text{ob}(\text{Mod}_{\mathcal{C}^\text{ch}\ell_k}(O_k)) \) and the \( k \)-linear extension of \( \rho \).

Proposition 4.11. The pair \((K,\rho)_k\) forms a CGHLT \( k \)-linear representation of \( G \), and \( \iota_{K_k}^c \) is a homeomorphic \( O_k[G] \)-linear isomorphism onto a core.
Proof. By Proposition 3.28 applied to $A = O_k$, $K_k$ is a CGHLT $O_k$-module, and $\nu_K^c$ is a homeomorphism onto a core of $K_k$. By Corollary 2.7, the $k$-linear extension of $\rho$ gives a continuous map $G \times K_k \to K_k$. Therefore $(K, \rho)_k$ forms a CGLT $O_k$-linear representation of $G$. Since $\nu_K^c$ is $O_k[[G]]$-linear, $\nu_K^c(K)$ forms a core of $(K, \rho)_k$. 

By Proposition 4.11, the correspondence $(K, \rho) \leadsto (K, \rho)_k$ gives an $O_k$-linear functor $\Psi : \text{Rep}_G(k'_{\text{ch}}) \to \text{Rep}_G(k'_{\text{ch}})$. We denote by $\Psi_k : k\text{Rep}_G(k'_{\text{ch}}) \to \text{Rep}_G(k'_{\text{ch}})$ its $k$-linear extension. By a similar argument to that in the proof of Proposition 3.31, we obtain a characterisation of a CGHLT $k$-linear representation of $G$.

**Proposition 4.12.** The $k$-linear functor $\Psi_k$ is fully faithful and essentially surjective.

We compare the notion of a CGHLT $O_k[[G]]$-module and the notion of a CGHLT $k$-linear representation of $G$. Let $(M, \rho) \in \text{ob}(\text{Rep}_G(k'_{\text{ch}}))$. Take a core $K \subset (M, \rho)$. We abbreviate the pair of $K$ and the restriction $G \times K \to K$ of $\rho$ to $(K, \rho)$. The scalar multiplication $O_k[[G]] \otimes^g \int_G(K, \rho) \to \int_G(K, \rho)$ induces a continuous $O_k$-linear homomorphism $O_k[[G]] \otimes^g K_k \to K_k$ by Corollary 3.12 and the functoriality of the colimit. Through the isomorphism $(K, \rho)_k \to (M, \rho)$ in $\text{Rep}_G(k'_{\text{ch}})$ induced by the inclusion $K \hookrightarrow M$, we obtain a continuous $O_k$-linear homomorphism $O_k[[G]] \otimes^g M \to M$, for which $M$ forms a CGHLT $O_k$-module $\int_G(M, \rho)$ with a core $K$. By the construction, the correspondence $(M, \rho) \leadsto \int_G(M, \rho)$ gives a $k$-linear functor $\int_{G,k}^c : \text{Rep}_G(k'_{\text{ch}}) \to \text{Mod}^{\text{cgh}}_k(O_k[[G]])$. We obtain a comparison between the notion of a CGHLT $O_k[[G]]$-module and the notion of a CGHLT $k$-linear representation of $G$.

**Theorem 4.13.** The functor $\int_{G,k}^c$ is a $k$-linear equivalence of categories.

Proof. We construct an inverse. Let $M \in \text{ob}(\text{Mod}^{\text{cgh}}_k(O_k[[G]]))$. We denote by $M_0$ the underlying CGHLT $O_k$-module of $M$. We show that the map $\rho_M : G \times M_0 \to M_0$, $(g, m) \mapsto d_{G,g}m$ is continuous. Take a core $K_1 \subset M$. We consider the composite $O_k[[G]] \otimes^g K_1 \to M$ of the $O_k$-linear
homomorphism $O_k[[G]] \otimes^c K_1 \to O_k[[G]] \otimes^c M$ induced by the inclusion $K_1 \hookrightarrow M$, which is continuous by the functoriality of $\otimes^c$, and the scalar multiplication $O_k[[G]] \otimes^c M \to M$. Since $K_1$ is a left $O_k[[G]]$-submodule, it factors through $K_1 \subset M$. We obtain a continuous $O_k$-linear homomorphism $O_k[[G]] \otimes^c K_1 \to K_1$, for which $K_1$ forms a CHFLT $O_k[[G]]$-module.

Let $U_2 \subset M_0$ be an open subset. Take an open profinite subgroup $H \subset G$. For a $g \in G$, put $U_{g,1} := gH$ and $U_{g,2} := \{m \in M_0 \mid \forall g' \in U_{g,1}, \rho_M(g', m) \in U_2\}$. Then we have $\rho_M^{-1}(U_2) = \bigcup_{g \in G} \prod U_{g,i}$. Therefore in order to show that $\rho_M^{-1}(U_2)$ is open, it suffices to show $U_{g,2}$ is open for any $g \in G$. Let $g \in G$. We show that $cU_{g,2} \cap K_1$ is open in $K_1$ for any $c \in O_k \setminus \{0\}$. Let $c \in O_k \setminus \{0\}$. Let $m \in cU_{g,2} \cap K_1$. By Proposition 2.1 (ii), Corollary 2.5, and Proposition 2.17 (ii), we have $K_0 := \sum_{g' \in H} O_k d_{G,g,g'} \in \mathcal{K}(O_k[[G]])$. By Proposition 2.1 (ii) and the continuity of the scalar multiplication $O_k[[G]] \otimes^c K_1 \to K_1$, there is an $(L_i) \in \prod \mathcal{O}(K_i)$ such that $(d_{G,g} \otimes m) + (L_i)(K_i)$ (cf. §3.1) is contained in the preimage of $\rho_M^{-1}(cU_2)$ in $K_0 \otimes^c K_1$. In particular, we have $m + L_1 \in cU_{g,2} \cap K_1$. It ensures that $cU_{g,2} \cap K_1$ is open in $K_1$. By Lemma 3.29 and Proposition 3.30, $U_{g,2}$ is open. It implies that $\rho_M$ is continuous. We obtain a CGHT $k$-linear representation $(M_0, \rho_M)$ with a core $K_1$. The correspondence $M \rightsquigarrow (M_0, \rho_M)$ gives a functor $\mathrm{Mod}^{c\mathrm{ch}}(O_k[[G]]) \to \mathrm{Rep}_G(k^{\mathrm{c\mathrm{ch}}}_W)$ which is a strict inverse of $\int_G^c$.

4.3 Generalised Schneider-Teitelbaum duality

Imitating the method of [13] Theorem 2.3, we extend $(D_d, D_c)$ to an $O_k$-linear equivalence $(\mathcal{D}_d, \mathcal{D}_c)$ of $\mathrm{Rep}_G(\mathrm{Ban}^\mathrm{ur}_k)^{\mathrm{op}}$ and $\mathrm{Mod}^{\mathrm{ch}}_W(O_k[[G]])$. Let $(V, \rho) \in \mathrm{ob}(\mathrm{Rep}_G(\mathrm{Ban}^\mathrm{ur}_k))$. For a $(g, m) \in G \times V^{D_d}$, we denote by $\rho^{D_d}(g, m)$ the submtric $k$-linear homomorphism $V \to k$, $v \mapsto m(\rho(g^{-1}, v))$. We obtain a map $\rho^{D_d}: G \times V^{D_d} \to V^{D_d}: (g, m) \mapsto \rho^{D_d}(g, m)$.

**Proposition 4.14.** The map $\rho^{D_d}$ is continuous.

**Proof.** By Proposition 4.1, $\rho$ induces a continuous monoid homomorphism $\varphi: G \to \mathcal{H}(V)^\times$. The map $G \to \mathcal{H}\mathrm{om}_c(V^{D_d}, V^{D_d})^\times$, $g \mapsto T(\bullet)^1_{V^{D_d}, V^{D_d}}(\varphi(g^{-1}))$ is a continuous by Proposition 3.3. Therefore $\rho^{D_d}$ is continuous by Proposition 4.7. 

By Proposition 4.14, the correspondence $(V, \rho) \rightsquigarrow (V^{D_d}, \rho^{D_d})$ gives an
$O_k$-linear functor $d\mathcal{D}_d: \text{Rep}_G(\text{Ban}_\text{ur}(k))^{\text{op}} \to \text{Rep}_G(\mathcal{C}^\text{ch}_H)$. We denote by $\mathcal{D}_d: \text{Rep}_G(\text{Ban}_\text{ur}(k))^{\text{op}} \to \text{Mod}_H^\text{ch}(O_k[[G]])$ the composite of $\int_G^c$ and $d\mathcal{D}_d$.

Let $K \in \text{ob}(\text{Mod}_H^\text{ch}(O_k[[G]]))$. We denote by $K_0 \in \text{ob}(\mathcal{C}^\text{ch}_H)$ the underlying topological $O_k$-module of $K$. For a $(g,v) \in G \times K_0^{\text{Dc}}$, we denote by $\rho_K(g,v)$ the continuous $O_k$-linear homomorphism $K_0 \to k$, $m \mapsto v(d_G,g^{-1}m)$. We obtain a map $\rho_K: G \times K_0^{\text{Dc}} \to K_0^{\text{Dc}}: (g,v) \mapsto \rho_K(g,v)$.

**Proposition 4.15.** The map $\rho_K$ is continuous.

**Proof.** By Proposition 2.17 (ii) and Proposition 3.27, the map $G \times K \to K$, $(g,m) \mapsto d_G,g,m$ is continuous. By Proposition 4.7, it induces a continuous monoid homomorphism $\varphi: G \to \mathcal{H}\text{om}^c(K_0,K_0)^\times$. The map $G \to \mathcal{H}\text{om}^c(K_0^{\text{Dc}},K_0^{\text{Dc}})^\times$, $g \mapsto T\varphi(g^{-1})_{K_0,K_0}$ is continuous by Proposition 3.3. Therefore $\rho_K$ is continuous by Proposition 4.1.

We put $K^{\mathcal{D}_c} := (K_0^{\text{Dc}},\rho_K)$. By Proposition 4.15, the correspondence $K \mapsto K^{\mathcal{D}_c}$ gives an $O_k$-linear functor $\mathcal{D}_c: \text{Mod}_H^\text{ch}(O_k[[G]]) \to \text{Rep}_G(\text{Ban}_\text{ur}(k))^{\text{op}}$. By Proposition 2.12, we obtain the following:

**Theorem 4.16.** The pair $(\mathcal{D}_d, \mathcal{D}_c)$ is an $O_k$-linear equivalence between $\text{Rep}_G(\text{Ban}_\text{ur}(k))^{\text{op}}$ and $\text{Mod}_H^\text{ch}(O_k[[G]])$.

We obtain a generalised Schneider–Teitelbaum duality (cf. [13] Theorem 2.3).

**Theorem 4.17.** The composite $k\text{Mod}_H^\text{ch}(O_k[[G]]) \to \text{Rep}_G(\text{Ban}(k))$ of $k\mathcal{D}_c$ and the $k$-linear extension $k\text{Rep}_G(\text{Ban}_\text{ur}(k)) \to \text{Rep}_G(\text{Ban}(k))$ of the inclusion $\text{Rep}_G(\text{Ban}_\text{ur}(k)) \hookrightarrow \text{Rep}_G(\text{Ban}(k))$ is fully faithful and essentially surjective.

**Proof.** The assertion follows from Theorem 4.5, Theorem 4.6, and Theorem 4.16 because the composite of the $k$-linear functor $k\text{Rep}_G(\text{Ban}_\text{ur}(k)) \to \text{Rep}_G(\text{Ban}(k))$ and $\int_G^d$ coincides with the composite of $k\int_{G,\text{ur}}$ and the $k$-linear extension $k\text{BT}_\text{ur}(O_k[[G]]) \to \text{BT}(O_k[[G]])$ of the inclusion $\text{BT}_\text{ur}(O_k[[G]]) \hookrightarrow \text{BT}(O_k[[G]])$. 

\[\square\]
Let \((V, \rho) \in \text{ob}(\text{Rep}_G(\text{Ban}(k)))\) (respectively, \(M \in \text{ob}(\text{Mod}^{\text{cgh}}(O_k[[G]])\)). We say that \((V, \rho)\) (respectively, \(M\)) is irreducible (respectively, simple) if it admits exactly two closed \(G\)-stable \(k\)-vector subspaces (respectively, closed left \(O_k[[G]]\)-submodules which are \(k\)-vector spaces). As an analogue of [13] Corollary 3.6, we obtain a criterion for the irreducibility.

**Theorem 4.18.** Suppose that \((V, \rho)\) is unitary. Then \((V, \rho)\) is irreducible if and only if \(((V, \rho)^{D_d})_k\) is simple.

**Proof.** Suppose that \(((V, \rho)^{D_d})_k\) is simple. We show that \((V, \rho)\) is irreducible. We have \(((V, \rho)^{D_d})_k \neq \{0\}\) by \(((V, \rho)^{D_d})_k \neq \{0\}\), and hence \(V \neq \{0\}\). Let \(V_0 \subset V\) be a proper closed \(G\)-stable \(k\)-vector subspace. Then \((V_0, \rho)\) forms a unitary Banach \(k\)-linear representation of \(G\). By Hahn–Banach theorem (cf. [5] Theorem 3 and [11] Proposition 9.2), the restriction map \(\pi: ((V, \rho)^{D_d})_k \to (V_0^{D_d})_k\) is surjective and \(\ker \pi\) is a non-zero closed \(O_k[[G]]\)-submodule of \(((V, \rho)^{D_d})_k\) which is a \(k\)-vector space. Since \(((V, \rho)^{D_d})_k\) is simple, we obtain \(\ker \pi = ((V, \rho)^{D_d})_k\). It ensures \(V_0^{D_d} = \{0\}\), and hence \(V_0 = \{0\}\) again by Hahn–Banach theorem. It implies that \((V, \rho)\) is irreducible.

Suppose that \((V, \rho)\) is irreducible. We show that \(((V, \rho)^{D_d})_k\) is simple. Let \(M_0 \subset ((V, \rho)^{D_d})_k\) be a proper closed \(O_k[[G]]\)-submodule which is a \(k\)-vector space. The identity map \(\mathcal{F}_c((V, \rho)^{D_d}) \to \text{Hom}_{\text{Ban}^w(k)}(V, k)\) induces a bijective \(k\)-linear homomorphism \(\mathcal{F}_c((V, \rho)^{D_d})_k \to \text{Hom}_{\text{Ban}(k)}(V, k)\), through which we regard \(\mathcal{F}(M_0)\) as a \(k\)-vector subspace of \(\text{Hom}_{\text{Ban}(k)}(V, k)\).

We have \(((V, \rho)^{D_d})_k \neq \{0\}\) by \(V \neq \{0\}\) and Hahn–Banach theorem. Put \(V_0 := \bigcap_{m \in M_0} \ker(m) \subset V\). We show \(V_0 \neq \{0\}\). Let \(M\) denote the quotient \((V^{D_d})_k/M_0\). Since \(M_0\) is a proper closed \(O_k[[G]]\)-submodule of \((V^{D_d})_k\) which is a \(k\)-vector space, \(M\) is a non-zero Hausdorff linear topological \(O_k\)-module which is a topological \(k\)-vector space. Therefore there is a non-zero continuous \(O_k\)-linear homomorphism \(\overline{\nu}: M \to k\) by [6] Theorem 2.1. By the compactness of \(V^{D_d}\) and the continuity of \(\iota_{V^{D_d}}\) and the canonical projection \(((V, \rho)^{D_d})_k \to M\), we have \(\sup_{m \in V^{D_d}} \overline{\nu}(\iota_{V^{D_d}}(m) + M_0) < \infty\). Therefore there is a \(v \in V \setminus \{0\}\) such that \(\overline{\nu}(\iota_{V^{D_d}}(m) + M_0) = m(v)\) for any \(m \in V^{D_d}\) by Theorem 2.12. It ensures \(m(v) = \overline{\nu}(0) = 0\) for any \(m \in M_0\). We obtain \(v \in V_0\) and hence \(V_0 \neq \{0\}\). Since \(V_0\) is a closed \(G\)-stable \(k\)-vector subspace of \((V, \rho)\) and \((V, \rho)\) is irreducible, we obtain \(V_0 = V\). It ensures \(M_0 = \{0\}\). It implies that \(((V, \rho)^{D_d})_k\) is simple. 

\(\square\)
5 Applications

As applications of the module theory in the monoidal structure, we give an explicit description of a continuous parabolic induction of unitary Banach $k$-linear representations.

5.1 Duality of operations Let $P \subset G$ be a closed subgroup. Suppose that $P \setminus G$ is compact. We study relations between the dual functors in §4.3 and operations on representations. Let $(V, \rho) \in \text{ob}(\text{Rep}_G(\text{Ban}_{ur}^\mathbb{R}(k)))$. We put $\text{Res}^G_P(V, \rho) := (V, \rho|_{P \times V})$. The correspondence $(V, \rho) \mapsto \text{Res}^G_P(V, \rho)$ gives an $O_k$-linear functor $\text{Res}^G_P : \text{Rep}_G(\text{Ban}_{ur}^\mathbb{R}(k)) \to \text{Rep}_P(\text{Ban}_{ur}^\mathbb{R}(k))$.

Let $K \in \text{ob}(\text{Mod}_{\ell}^{\text{ch}}(O_k[[G]]))$. We denote by $\text{Res}^{O_k[[G]]}_{O_k[[P]]}(K)$ the scalar restriction of $K$ by the natural embedding $O_k[[P]] \hookrightarrow O_k[[G]]$. The correspondence $K \leadsto \text{Res}^{O_k[[G]]}_{O_k[[P]]}(K)$ gives an $O_k$-linear functor

$$
\text{Res}^{O_k[[G]]}_{O_k[[P]]} : \text{Mod}_{\ell}^{\text{ch}}(O_k[[G]]) \to \text{Mod}_{\ell}^{\text{ch}}(O_k[[P]]).
$$

We have $\mathcal{D}_d \circ \text{Res}^G = \text{Res}^{O_k[[G]]}_{O_k[[P]]} \circ \mathcal{D}_d : \text{Rep}_G(\text{Ban}_{ur}^\mathbb{R}(k)) \to \text{Mod}_{\ell}^{\text{ch}}(O_k[[P]])$ by the construction.

Let $(V_0, \rho_0) \in \text{ob}(\text{Rep}_P(\text{Ban}_{ur}^\mathbb{R}(k)))$. We denote by $\rho : G \times C_{bd}(G, V_0) \to C_{bd}(G, V_0)$ the map given by setting $\rho(g, f)(g') := f(g'g)$ for an $(f, g, g') \in C_{bd}(G, V_0) \times G^2$, which is not necessarily continuous. We set $\text{Ind}^G_P(V_0) := \{f \in C_{bd}(G, V_0) \mid \forall (h, v) \in P \times G, f(hg) = \rho_0(h, f(g))\}$. Then $\text{Ind}^G_P(V_0) \subset C_{bd}(G, V_0)$ is a closed $G$-equivariant $k$-vector subspace. We denote by $\text{Ind}^G_P(\rho_0) : G \times \text{Ind}^G_P(V_0) \to \text{Ind}^G_P(V_0)$ the restriction of $\rho$. It can be easily verified that $\text{Ind}^G_P(\rho_0)$ is continuous by Banach–Steinhaus theorem (cf. [11] Corollary 6.16), and $\text{Ind}^G_P(V_0, \rho_0) := (\text{Ind}^G_P(V_0), \text{Ind}^G_P(\rho_0))$ forms a unitary Banach $k$-linear representation of $G$. The correspondence $(V_0, \rho_0) \leadsto \text{Ind}^G_P(V_0, \rho_0)$ gives an $O_k$-linear functor

$$
\text{Ind}^G_P : \text{Rep}_P(\text{Ban}_{ur}^\mathbb{R}(k)) \to \text{Rep}_G(\text{Ban}_{ur}^\mathbb{R}(k)).
$$

Let $K_0 \in \text{ob}(\text{Mod}_{\ell}^{\text{ch}}(O_k[[P]]))$. We describe $\text{Ind}^G_P(K_0 \mathcal{D}_d)$ explicitly by $G$ and $K_0$. Since the underlying topological space of $G$ is a disjoint union of compact clopen subspaces, a map $\varphi : G \to K_0^{\mathcal{D}_d}$ is continuous if and
only if the induced map \( G \times K_0 \to k: (g, m) \mapsto \varphi(g)(m) \) is continuous by Proposition 4.7. Therefore we obtain an isometric \( k \)-linear homomorphism \( C_{bd}(G,K_0^2) \hookrightarrow C_{bd}(G \times K_0,k) \) onto the closed image. We consider the map 
\[
\rho: G \times C_{bd}(G \times K_0,k) \to C_{bd}(G \times K_0,k) 
\]
given by setting \( \rho(g,f)(g',m) := f(g'g,m) \) for \((g,g',m) \in G \times C_{bd}(G \times K_0,k) \times G \times K_0\), which is not necessarily continuous. The inclusion \( \text{Ind}^G_P(K_0^2) \hookrightarrow C_{bd}(G,K_0^2) \subset C_{bd}(G \times K_0,k) \) is an isometric \( G \)-equivariant \( k \)-linear homomorphism, and its image is the closed \( G \)-stable \( k \)-vector subspace consisting of functions 
\[
f: G \times K_0 \to k
\]
satisfying the following:

(I) The equality \( f(g,cm) = cf(g,m) \) holds for any \((g,c,m) \in G \times O_k \times K_0\).

(II) The equality \( f(g,\sum m_i) = \sum f(g,m_i) \) holds for any \((g,(m_i)) \in G \times K^2_0\).

(III) The equality \( f(hg,m) = f(g,\delta^{-1}_g h m) \) holds for any \((h,g,m) \in P \times G \times K_0\).

The inclusion \( \text{Ind}^G_P(K_0^2) \hookrightarrow C_{bd}(G \times K_0,k) \) induces a continuous surjective \( G \)-equivariant \( O_k \)-linear homomorphism \( \varphi_{G,P}: C_{bd}(G \times K_0,k)^{Da} \to \text{Ind}^G_P(K_0^2)^{Da} \) by Hahn–Banach theorem (cf. [5] Theorem 3 and [11] Proposition 9.2). Since the target and the source of \( \varphi_{G,P} \) are compact and Hausdorff, the target is homeomorphic to the coimage. We determine \( \ker(\varphi_{G,P}) \) in order to describe the target. We denote by \( e_{g,m} \) the \( k \)-linear homomorphism \( C_{bd}(G \times K_0,k) \to k; f \mapsto f(g,m) \) for a \((g,m) \in G \times K_0\). We put 
\[
\mu^1_{g,c,m} := e_{g,m} - e_{g,cm}, \quad \mu^I_{g,c,m} := e_{g,\sum m_i} - \sum e_{g,m_i}, \quad \mu^II_{g,h,m} := e_{hg,m} - e_{g,d_{G,h}^{-1}m}, \quad \mu^III_{g,h,m} := e_{hg,m} - e_{g,d_{G,h}^{-1}m}
\]
for \((g,m) \in G \times O_k \times K_0\), and \( \mu^I_{g,c,m}, \mu^II_{g,h,m}, \mu^III_{g,h,m} \) the closed \( O_k \)-submodule generated by the union of \( \{\mu^I_{g,c,m} | (g,c,m) \in G \times O_k \times K_0\} \), \( \{\mu^II_{g,(m_i)} | (g,(m_i)) \in G \times K^2_0\} \), and \( \{\mu^III_{g,h,m} | (g,h,m) \in G \times P \times K_0\} \).

**Proposition 5.1.** The equality \( \ker(\varphi_{G,P}) = \mu^I + \mu^II + \mu^III \) holds.

**Proof.** We have \( \mu^I + \mu^II + \mu^III \subset \ker(\varphi_{G,P}) \) by the characterisation of the image of \( \text{Ind}^G_P(K_0^2) \) in \( C_{bd}(G \times K_0,k) \). Let \( \mu \in \ker(\varphi_{G,P}) \). We show \( \mu \in \mu^I + \mu^II + \mu^III \). Let \( f \in C_{bd}(G \times K_0,k) \) and \( \epsilon \in (0,\infty) \). We verify that there is a \( \mu' \in \mu^I + \mu^II + \mu^III \) such that \( |\mu(f) - \mu'(f)| < \epsilon \). In the case...
$f \in \text{Ind}_P^G(K_0^{\mathcal{D}_c})$, we have $\mu(f) = \varphi_{G,P}(\mu)(f) = 0$, and hence $\mu' := 0$ satisfies the desired inequality. Suppose $f \notin \text{Ind}_P^G(V_0)$. Then $f$ does not satisfy at least one of the conditions (I)–(III) in the characterisation of the image of $\text{Ind}_P^G(K_0^{\mathcal{D}_c})$ in $\text{C}^{\text{bd}}(G \times K_0, k)$. First, suppose that $f$ does not satisfy (I). Take a $(g, c, m) \in G \times O_k \times K_0$ satisfying $f(g, cm) - cf(g, m) \neq 0$. Set $\mu' := (f(g, cm) - cf(g, m))^{-1}\mu(f)\mu_{g,c,m}^1$. Then we have $\mu'(f) = \mu(f)$ by the construction, and hence $|\mu(f) - \mu'(f)| = 0 < \epsilon$. Next, suppose that $f$ does not satisfy (II). Take a $(g, c, m) \in G \times O_k \times K_0$ satisfying $f(g, \sum m_i) - \sum f(g, m_i) \neq 0$. Set $\mu' := (f(g, \sum m_i) - \sum f(g, m_i))^{-1}\mu(f)\mu_{g,m,m'}^\text{II}$. Then we have $\mu'(f) = \mu(f)$ by the construction, and hence $|\mu(f) - \mu'(f)| = 0 < \epsilon$. Finally, suppose that $f$ does not satisfy (III). Take a $(g, h, m) \in G \times P \times K_0$ satisfying $f(hg, m) - f(g, \delta_{G,h}^{-1}m) \neq 0$. Set $\mu' := (f(hg, m) - f(g, \delta_{G,h}^{-1}m))^{-1}\mu(f)\mu_{g,h,m}^\text{III}$. Then we have $\mu'(f) = \mu(f)$, and hence $|\mu(f) - \mu'(f)| = 0 < \epsilon$. It ensures $\mu \in \mu^I + \mu^\text{II} + \mu^\text{III}$. We obtain $\ker(\varphi_{G,P}) = \mu^I + \mu^\text{II} + \mu^\text{III}$.

We set $\text{Ind}_{O_k[[G]]}^{O_k[[P]]}(K_0) := \text{C}^{\text{bd}}(G \times K_0, k)^{\mathcal{D}_d}/(\mu^I + \mu^\text{II} + \mu^\text{III})$. By Proposition 5.1, we obtain the following:

**Theorem 5.2.** The continuous surjective $O_k$-linear homomorphism $\varphi_{G,P}$ induces a homeomorphic $O_k$-linear isomorphism

$$\text{Ind}_{O_k[[P]]}^{O_k[[G]]}(K_0) \rightarrow \text{Ind}_P^G(K_0^{\mathcal{D}_c})^{\mathcal{D}_d}.$$ 

We equip $\text{Ind}_{O_k[[P]]}^{O_k[[G]]}(K_0)$ with a CHFLT $O_k[[G]]$-module structure by pulling back that of $\text{Ind}_P^G(K_0^{\mathcal{D}_c})^{\mathcal{D}_d}$ by the isomorphism in Theorem 5.2. The correspondence $K_0 \rightsquigarrow \text{Ind}_{O_k[[P]]}^{O_k[[G]]}(K_0)$ gives an $O_k$-linear functor

$$\text{Ind}_{O_k[[P]]}^{O_k[[G]]} : \text{Mod}_{\text{ch}}^{\text{ch}}(O_k[[P]]) \rightarrow \text{Mod}_{\text{ch}}^{\text{ch}}(O_k[[G]]).$$

By Theorem 4.16 and Theorem 5.2, we obtain the following:

**Corollary 5.3.** There is a natural equivalence $\text{Ind}_P^G \Rightarrow \mathcal{D}_c \circ \text{Ind}_{O_k[[P]]}^{O_k[[G]]} \circ \mathcal{D}_d$.

### 5.2 Continuous parabolic inductions

As an application of Corollary 5.3, we compute the continuous parabolic induction. For this purpose,
we give a more practical description of $\text{Ind}_{O_{k}[[G]]}^{O_{k}[[P]]}$. To begin with, we prepare a compact complete representative $C \subset G$ of $P \setminus G$. We denote by $\Sigma$ the set of open subsets $U \subset P \setminus G$ admitting a continuous section $U \hookrightarrow G$ of the canonical projection $G \twoheadrightarrow P \setminus G$. Take an open profinite subgroup $G_{0} \subset G$. Since $G$ is a topological group, the canonical projection $G \twoheadrightarrow P \setminus G$ is an open map. Therefore the image $\overline{G_{0}g} \subset P \setminus G$ of $G_{0}g$ is an open subset, and the map $G_{0} \hookrightarrow G$, $h \mapsto hg$ induces a homeomorphism $(P \cap G_{0}) \setminus G_{0} \rightarrow \overline{G_{0}g}$ for any $g \in G$. It implies that $\Sigma$ forms an open covering of $P \setminus G$ by [9] Theorem 2. Take a $\Sigma_{0} \in \mathcal{S}_{\omega}(\Sigma)$ satisfying $P \setminus G = \bigsqcup_{U \in \Sigma_{0}} U$. Gluing continuous sections on each $U \in \Sigma_{0}$, we obtain a continuous section $P \setminus G \hookrightarrow G$, whose image forms a compact subset $C \subset G$ such that the multiplication $P \times C \rightarrow G$ is a continuous bijective map. Conversely, let $C \subset G$ be an arbitrary compact subset such that the multiplication $P \times C \rightarrow G$ is a continuous bijective map. As is mentioned in Example 3.22 (iii), the multiplication $P \times C \rightarrow G$ is a homeomorphism, and induces a $O_{k}$-linear isomorphism $O_{k}[[P \times C]] \rightarrow O_{k}[[G]]$. We denote by $\pi_{0}: G \rightarrow P$ (respectively, $\pi_{1}: G \rightarrow C$) the composite of the inverse $G \rightarrow P \times C$ of the multiplication and the canonical projection $P \times C \rightarrow P$ (respectively, $P \times C \rightarrow P$). As a result, $C$ is obtained as the image of the continuous section $P \setminus G \hookrightarrow G$ induced by $\pi_{1}$.

Let $F$ be a local field, $G$ an algebraic group over $\text{Spec}(F)$, and $P \subset G$ a parabolic subgroup. Then $G(F)$ forms a locally profinite group with respect to the topology induced by the valuation of $F$, and $P(F)$ is naturally identified with a closed subgroup of $G(F)$. Since $P \setminus G$ forms a proper algebraic variety over $\text{Spec}(F)$, $P(F) \setminus G(F)$ forms a totally disconnected compact Hausdorff topological space. Henceforth, we consider the case $G = G(F)$ and $P = P(F)$.

Let $(V_{0}, \rho_{0}) \in \text{ob}(\text{Rep}_{P}(\text{Ban}_{\leq}(k)))$. We consider the composite $r_{C,V_{0}}: \text{Ind}_{P}^{G}(V_{0}) \rightarrow C(C,V_{0})$ of the inclusion $\text{Ind}_{P}^{G}(V_{0}) \hookrightarrow C_{\text{bd}}(G,V_{0})$ and the restriction map $C_{\text{bd}}(C,V_{0}) \rightarrow C(C,V_{0})$. Then $r_{C,V_{0}}$ is injective by the conditions (III) in §5.1 and $PC = G$. The quotient norm on the source of $r_{C,V_{0}}$ coincides with the norm restricted to the image of $r_{C,V_{0}}$ because $P$ acts isometrically on $V_{0}$. Therefore $r_{C,V_{0}}$ is isometric. For any $f \in C(C,V_{0})$, the map $\tilde{f}: G \rightarrow V_{0}$, $g \mapsto \rho_{0}(\pi_{0}(g), (f \circ \pi_{1}(g)))$ lies in $\text{Ind}_{P}^{G}(K_{0})$. We obtain an isometric section $C(C,V_{0}) \rightarrow \text{Ind}_{P}^{G}(V_{0})$, $\tilde{f} \mapsto \tilde{f}$, and hence $r_{C,V_{0}}$ is an isomorphism in $\text{Ban}_{\leq}(k)$. Pulling back $\text{Ind}_{P}^{G}(\rho_{0})$ by $r_{C,V_{0}}$ and the isomorphism
Schneider-Teitelbaum duality for locally profinite groups

C(C, k) \hat{\otimes}_k V_0 \to C(C, V_0) in \text{Ban}_{ur}(k) introduced in Proposition 3.13, we equip C(C, k) \hat{\otimes}_k V_0 with a continuous action C \hat{\otimes}_k \rho_0 of G. By Theorem 5.2, we obtain an isomorphism \text{Ind}^{G}_{O_k[[G]]}((V_0, \rho_0) \oplus d) \to (C(C, k) \hat{\otimes}_k V_0, C \hat{\otimes}_k \rho_0) \oplus d in \text{Mod}_{ch}((O_k[[G]])). By Proposition 2.15 and [7] Theorem 2.2, we have a natural isomorphism O_k[[C]] \hat{\otimes}_k V_0^{D_c} \to (C(C, k) \hat{\otimes}_k V_0) \oplus d in \text{CHFLT}_{ch}. Pulling back the scalar multiplication of O_k[[G]] on (C(C, k) \hat{\otimes}_k V_0, C \hat{\otimes}_k \rho_0) \oplus d, we regard O_k[[C]] \hat{\otimes}_k V_0^{D_c} as a CHFLT O_k[[G]]-module. By Theorem 4.16, we obtain the following:

**Theorem 5.4.** The continuous parabolic induction \text{Ind}^{G}_{P}(V_0, \rho_0) admits a natural isomorphism to \((O_k[[C]] \hat{\otimes}_k V_0^{D_c}) \oplus \iota_c in \text{Rep}_G(\text{Ban}_{ur}(k)).

The induced action of O_k[[G]] on \((O_k[[C]] \hat{\otimes}_k V_0^{D_c}) \oplus \iota_c is a little complicated, but this presentation enable us to describe the deformation of \text{Ind}^{G}_{P}(V_0, \rho_0) associated to a deformation of \rho_0 as a deformation of actions of G on a single Banach k-vector space \((O_k[[C]] \hat{\otimes}_k V_0^{D_c})^{D_c}.

**Example 5.5.** Let \(n \in \omega\). We denote by B_n^{+}(k) \subset GL_n(k) the Borel subgroup consisting of upper triangular invertible matrices, by C_n^{-} \subset GL_n(k) the compact subset consisting of lower triangular invertible matrix whose entries are contained in O_k and whose diagonals are 1, and by S_n \subset GL_n(k) the finite subgroup consisting of permutations of the canonical basis. By the LUP-decomposition, GL_n(k) is expressed as the product B_n^{+}(k)C_n^{-} S_n, and the multiplication B_n^{+}(k) \times C_n^{-} S_n \to GL_n(k) is bijective. Therefore for a \((V_0, \rho) \in \text{ob(Rep}_{B_n^{+}(k)}(\text{Ban}_{ur}(k)))\), we have a natural isomorphism \text{Ind}^{GL_n(k)}_{rB_n^{+}(k)}(V_0, \rho_0) \to \((O_k[[C_n^{-} S_n]] \hat{\otimes}_k V_0^{D_c}) \oplus \iota_c in \text{Rep}_{GL_n(k)}(\text{Ban}_{ur}(k)) by the argument above, and also a natural isomorphism \((O_k[[C_n^{-} S_n]] \hat{\otimes}_k V_0^{D_c}) \oplus \iota_c \to \((O_k[[C_n^{-} S_n]] \hat{\otimes}_k K_0) V_0^{D_c} \oplus \iota_c in \text{Ban}_{ur}(k).

**Acknowledgement**

I am extremely grateful to Takeshi Tsuji for constructive advices in seminars. I express my deep gratitude to Atsushi Yamashita for instructing me on topological groups. I am profoundly thankful to Takuma Hayashi and Frédéric Paugum for instructing me on the elementary categorical convention. I thank my colleague for daily discussions. I greatly appreciate my
family’s deep affection. I was a research fellow of Japan Society for the Promotion of Science. I am also thankful to the referee for the careful reading and the helpful comments. This work was supported by the Program for Leading Graduate Schools, MEXT, Japan.

References

[1] Berkovich, V.G., “Spectral Theory and Analytic Geometry over non-Archimedean Fields”, Mathematical Surveys and Monographs 33, Amer. Math. Soc. 1990.

[2] Bosch, S., G"untzer, U., and Remmert, R., “Non-Archimedean Analysis: A Systematic Approach to Rigid Analytic Geometry”, Springer, 1984.

[3] Engelking, R., “General Topology”, Polish Scientific Publishers, 1977.

[4] Frank, D.L., *A totally bounded, complete uniform space is compact*, Proc. Amer. Math. Soc. 16 (1965), p. 514.

[5] Ingleton, A.W., *The Hahn-Banach theorem for non-Archimedean valued fields*, Math. Proc. Cambridge Philos. Soc. 48(1) (1952), 41-45.

[6] Mihara, T., *Hahn-Banach theorem and duality theory on non-Archimedean locally convex spaces*, J. Convex Anal. 24(2) (2017), 587-619.

[7] Mihara, T., *Duality theory of p-adic Hopf algebras*, Categ. General Algebraic Struct. Appl. 14(1) (2020), 81-117.

[8] Monna, A.F., “Analyse Non-Archimedienne”, Springer, 1970.

[9] Mostert, P.S., *Local cross sections in locally compact groups*, Proc. Amer. Math. Soc. 4(4) (1953), 645-649.

[10] Schikhof, W.H., *A perfect duality between p-adic Banach spaces and compactoids*, Indag. Math. (N.S.) 6(3) (1995), 325-339.

[11] Schneider, P., “Non-Archimedean Functional Analysis”, Springer, 2002.

[12] Demazure, M. and Grothendieck, A., “Seminaire de Geometrie Algebrique du Bois Marie - 1962-64 - Schemas en groupes - SGA3 - Tome 1”, Springer, 1970.

[13] Schneider, P. and Teitelbaum, J., *Banach space representations and Iwasawa theory*, Israel J. Math. 127(1) (2002), 359-380.
Tomoki Mihara, Division of Mathematics, University of Tsukuba, 1-1-1 Tennodai, Tsukuba, Ibaraki 305-8577, Japan
Email: mihara@math.tsukuba.ac.jp
