An odd feature of the ‘most classical’ states of $SU(2)$ invariant quantum mechanical systems

László B. Szabados
Wigner Research Centre for Physics,
H-1525 Budapest 114, P. O. Box 49, EU
e-mail: lbszab@rmki.kfki.hu

March 9, 2023

Abstract

Complex and spinorial techniques of general relativity are used to determine all the states of the $SU(2)$ invariant quantum mechanical systems in which the equality holds in the uncertainty relations for the components of the angular momentum vector operator in two given directions. The expectation values depend on a discrete quantum number and two parameters, one of them is the angle between the two angular momentum components and the other is the quotient of the two standard deviations. Allowing the angle between the two angular momentum components to be arbitrary, a new genuine quantum mechanical phenomenon emerges: it is shown that although the standard deviations change continuously, one of the expectation values changes discontinuously on this parameter space. Since physically neither of the angular momentum components is distinguished over the other, this discontinuity suggests that the genuine parameter space must be a double cover of this classical one: it must be diffeomorphic to a Riemann surface known in connection with the complex function $\sqrt{z}$. Moreover, the angle between the angular momentum components plays the role of the parameter of an interpolation between the continuous range of the expectation values in the special case of the orthogonal angular momentum components and the discrete point spectrum of one angular momentum component. The consequences in the simultaneous measurements of these angular momentum components are also discussed briefly.

Keywords: coherent spin states, spin weighted spherical harmonics, edth operators

1 Introduction

If the algebra of basic observables of the quantum system is the Heisenberg algebra, then the so-called canonical coherent states, defined to be the eigenstates of the annihilation operator, can be characterized in various equivalent ways (see e.g. [1, 2]). E.g. these are the smallest uncertainty states for the basic canonically conjugate observables, and, with the Hamiltonian of the harmonic oscillator, they do not spread out in space during their time evolution. Thus, these states are usually interpreted as the ‘most classical’ states of the quantum system. These states have particular significance e.g. in the characterization of the resulting state after the most accurate simultaneous measurement of two non-commuting observables [3, 4]; or in the characterization of the system in the classical limit.
The notion of coherent states has already been introduced for systems whose algebra of basic observables is more complicated than the Heisenberg algebra, e.g. when it is $su(2)$ (see e.g. [2]-[8]). In addition to applications e.g. in quantum industry (in particular, in quantum optics see e.g. [1],[2], and in quantum computation see e.g. [9]), these states could have particular significance in basic science, e.g. in the characterization of the classical limit of the spin systems [6], or in the determination of the large spin limit of the spin network model of the quantum spacetime suggested by Penrose in [10].

Our aim in the present paper is to determine all the ‘most classical’ states of the general $SU(2)$-invariant quantum mechanical systems (even with given intrinsic spin) in the sense that, in these states, the equality holds in the uncertainty relation for the components of the angular momentum vector operator in two arbitrary, given directions. The analogous problem was solved by Aragone et al in [7] in the framework of the spin coherent states of Radcliffe [5] in the special case when the two directions were orthogonal to one another (and the resulting states were called the ‘intelligent states’). As far as we know, the general case has not been considered hitherto. As we will see, allowing the angle between the two angular momentum components to be arbitrary, a new quantum mechanical phenomenon emerges: the parameter space parameterizing the expectation values must be diffeomorphic to the non-trivial Riemann surface appearing in connection with the complex function $\sqrt{z}$ in complex analysis, rather than the classical parameter space which is a simply connected subset of the complex plane.

In deriving our results, complex and spinorial techniques of general relativity are used. It is these techniques that made it possible to determine the spectrum and the eigenfunctions of the relevant non-self-adjoint operator in explicit, closed form in a relatively straightforward (but technically a bit involved) way. It is these techniques that can be (and, in fact, have been) used to carry out the analogous investigations in the considerably more complicated Euclidean invariant systems [11]. To make the present paper self-contained, these ideas and techniques are summarized in Appendix A.1. For a more detailed discussion of them, see e.g. [12]-[17].

In section 2, we determine the most classical states of the $SU(2)$-invariant systems and discuss its properties, namely we calculate the expectation values, the standard deviations and find the wave functions. There we raise the possibility that the proper parameter space for the expectation value is the Riemann surface above. Section 3 is devoted to a brief discussion of how could it be possible to decide experimentally, at least in principle, that the genuine parameter space is this Riemann surface. Section 4 is a more detailed summary of the results and our conclusions. Several technical details of the analyses in section 2 are presented in Appendix A.2.

Our conventions are mostly those of [12, 13], except that the signature of the metric of the Euclidean 3-space is positive definite. Here we do not use abstract indices: every index is a concrete (name, or component) index referring to some basis. Round brackets around indices denote symmetrization.

2 The most classical states

As is well known, using the Cauchy–Schwarz inequality, for the standard deviation (or uncertainty) $\Delta_\phi A$ and $\Delta_\phi B$ of any two observables $A$ and $B$ in any normalized state $\phi$, respectively, Schrödinger derived the inequality

$$(\Delta_\phi A)^2(\Delta_\phi B)^2 \geq \frac{1}{4} |\langle [A,B] \rangle_\phi|^2 + \left( \frac{1}{2} \langle A B + B A \rangle_\phi - \langle A \rangle_\phi \langle B \rangle_\phi \right)^2.$$  \hspace{1cm} (2.1)

Here, e.g. $\langle A \rangle_\phi$ is the expectation value of $A$, and the standard deviation in the state $\phi$ is given by $\Delta_\phi A = \sqrt{\langle A^2 \rangle_\phi - \langle A \rangle_\phi^2}$. The necessary and sufficient condition of the equality in (2.1) is well known to be the vanishing of the second term on the right, which is the first order correlation between the two expectation values, and the equality in the Cauchy–Schwarz inequality. These
two conditions together yield that the state $\phi$ must be a solution of the eigenvalue equation

$$(A - i\lambda B)\phi = (\langle A \rangle_\phi - i\lambda \langle B \rangle_\phi)\phi, \quad (2.2)$$

where, for non-commuting observables, $\lambda$ is a nonzero real constant. Then, as immediately implies, $|\lambda| = \Delta_\phi A / \Delta_\phi B$ holds; and, for the sake of simplicity, $\lambda$ will be chosen to be positive.

In the present paper, we use these general ideas to find the most classical states with a given intrinsic spin when the basic observables form the $su(2)$ Lie algebra. In the companion paper [11], we consider systems for which this is the Lie algebra $e(3)$ of the Euclidean group.

### 2.1 $SU(2)$ invariant elementary systems

The generators of the Lie algebra $su(2)$ are denoted by $J_i$, $i = 1, 2, 3$, satisfying $[J_i, J_j] = i\hbar \varepsilon_{ijk} J^k$, where the lowering and raising of the small Latin indices are defined by $\delta_{ij}$ and its inverse, and $\varepsilon_{ijk}$ is the alternating Levi-Civita symbol. Its only Casimir operator is $J_i J^i$.

We search for the general most classical states in the form of superpositions of such states belonging to the irreducible representations of $SU(2)$ with given, but arbitrary intrinsic spin. The finite dimensional irreducible representations of $SU(2)$ and $su(2)$ in their traditional, purely algebraic bra-ket formalism by unitary and self-adjoint operators, respectively, are well known. Nevertheless, the representations that we use here are less well known outside the general relativity community, and they are based on the complex line bundles $O(-2s)$ over the unit 2-sphere $S \approx S^2$, where $2s \in \mathbb{Z}$ is fixed. The integral of the first Chern class of $O(-2s)$ is $2s$. This is a topological invariant, characterizing the global non-triviality (or ‘twist’) of the bundle, and it represents the intrinsic spin of the system. The carrier space of such a representation is an appropriate finite dimensional subspace of the Hilbert space $H_s = L_2(S, dS)$ of the square-integrable cross sections $\phi$ of the line bundle, where $dS$ is the natural metric area element on $S$. The reason of using this less well known form of the unitary, irreducible representations of $su(2)$ is twofold: first, the intrinsic spin of the system appears naturally in this formalism (see Appendix A.1), although it is not linked to any Casimir operator of $su(2)$; and, second, it is this form that fits smoothly to the most convenient representations of the $E(3)$-invariant elementary quantum mechanical systems that we will consider in [11], where the results of the present paper will also be used.

The form of the angular momentum operators that we use are (densely defined) self-adjoint differential operators acting on the smooth cross sections of $O(-2s)$. Explicitly, they are

$$J^i \phi = s \hbar n^i \phi + \hbar (m^i \delta^i \phi - \bar{m}^i \delta \phi), \quad (2.3)$$

where $n^i$ are the Cartesian components of the unit normal of $S$ in $\mathbb{R}^3$, $m^i$ and $\bar{m}^i$ are those of the complex null tangents of $S$, and $\delta$ and $\delta^i$ are the so-called edth operators, are the covariant directional derivative operators in the corresponding complex null directions. The spin weight $s$ of $\phi$, i.e. the intrinsic spin of the system, will be fixed in what follows, and it may take any integer or half-odd-integer value.

In this representation, the $su(2)$ Casimir operator is $J_i J^i = s^2 \hbar^2 \phi - \hbar^2 (\delta \delta^i + \delta^i \delta)\phi$, where $(\delta \delta^i + \delta^i \delta)$ is just the familiar Laplacian on the unit 2-sphere. In Appendix A.1.4, we summarize the construction and the key properties of the null tangents, the line bundles $O(-2s)$, the operators $\delta$ and $\delta^i$ and the related concepts, as well as the derivation of (2.3) and the form of the Casimir operator (see, in particular, equations (A.17)-(A.18)). A more detailed discussion of the line bundles $O(-2s)$ and the related concepts are given e.g. in [14] (see also [12] [13]). Appendix A.1.1 and A.1.3 are also intended to provide some informal ‘dictionary’ to translate the geometrical notion used here to the more familiar ones in quantum mechanics/quantum technology.
2.2 The spectrum

For any $\alpha^i, \beta^j \in \mathbb{R}^3$ satisfying $\alpha^i \alpha^j \delta_{ij} = \beta^i \beta^j \delta_{ij} = 1$, we form the operators $J(\alpha) := \alpha^i J_i$ and $J(\beta) := \beta^j J_j$, i.e. the components of the angular momentum vector operator determined by the directions $\alpha^i$ and $\beta^j$. However, without loss of generality, we may assume that e.g. $\beta^1 = \beta^2 = 0$ and $\beta^3 = 1$, because by an appropriate rotation of the Cartesian coordinate system this can always be achieved. Moreover, we assume that $\alpha_3 \neq \pm 1$, because otherwise $\alpha^i = \pm \beta^i$ would be allowed, and for these $J(\alpha) = \pm J(\beta)$ would hold. Then by (2.1) and the commutation relations we have $\Delta_{\phi} J(\alpha) \Delta_{\phi} J(\beta) \geq (\hbar/2)\alpha^i \beta^j \varepsilon_{ijk} (\mathcal{J}_k)_{\phi}$, in which, by (2.2), the equality holds precisely when $\phi$ solves the eigenvalue equation

$$
(J(\alpha) - i\lambda J(\beta))\phi = \left(\langle J(\alpha) \rangle_{\phi} - i\lambda \langle J(\beta) \rangle_{\phi}\right)\phi =: \hbar C \phi
$$

for some $\lambda > 0$. By (2.3) this eigenvalue equation is

$$
(\alpha^i - i\lambda \beta^i)\left(m_i \delta^i \phi - m_i \delta^i \phi + s n_i \phi\right) = C \phi.
$$

We solve this eigenvalue problem in two steps: first, using spinorial techniques of general relativity, we determine the spectrum of the operator $J(\alpha) - i\lambda J(\beta)$, and then, in the next subsection, we determine the eigenfunctions both as series and in a closed form, too.

Since $J, J'$ and $J(\alpha) - i\lambda J(\beta)$ are commuting, they have a system of common eigenfunctions. Hence, it might appear to be natural to search for the eigenfunctions of the latter in the form of a linear combination of the spin weighted spherical harmonics $\gamma_{j,m}$, where $j = |s|, |s|+1, |s|+2, \ldots$ and $m = -j, -j+1, \ldots, j$ (see Appendix A.1.1-A.1.3). However, the direct use of these harmonics in (2.5) would yield a coupled system of algebraic equations even for given $j$, and it would be difficult to get its solutions in a simple, closed form. Our strategy, suggested by Paul Tod [18], is based on the trick that, instead of the Cartesian spinor basis (see Appendix A.1.1), we use the principal spinors of the (unitary spinor form $\gamma_{AB}$ of the) purely spatial complex vector $\alpha_i - i\lambda \beta_i$. This makes the calculation of the spectrum of $J(\alpha) - i\lambda J(\beta)$ easier, giving it in a simple closed form. The next two paragraphs are mostly based on Paul Tod’s ideas and calculations.

Thus, we form $\gamma_{AA'} := (\alpha_i - i\lambda \beta_i) \sigma_{AA'}^{i}$, and, since $\alpha_i - i\lambda \beta_i$ is a spatial vector, its unitary spinor form, $\gamma_{AB} := \gamma_{AA'} \mathbb{V}^{A'B}$, is symmetric. (Here, $\sigma_{AA'}^{i} := (\sigma^{0}_{AA'}, \sigma^{i}_{AA'})$ are the standard $SL(2, \mathbb{C})$ Pauli matrices, including the factor $1/\sqrt{2}$, and raising and lowering of the capital Latin indices are defined by the symplectic metric $\varepsilon^{AB}$ and its inverse. For our conventions, see Appendix A.1.1.) Then (2.5) takes the form

$$
C \phi = \gamma_{AA'} \sigma^A \iota^A \delta^i \phi - \gamma_{AA'} \iota A^\prime \sigma^A \iota^A \delta^i \phi + \sqrt{2} s \gamma_{AA'} \sigma^A \iota^A \delta^i \phi
$$

$$
= -\gamma_{AB} \sigma^A \sigma^B \delta^i \phi - \gamma_{AB} \iota A \iota^B \delta^i \phi + \sqrt{2} s \gamma_{AB} \sigma^A \iota^B \phi,
$$

(2.6)

where $\sigma^A$ and $\iota^A$ are the vectors of the Newman–Penrose spinor basis (see Appendix A.1.1-A.1.2). Next, for any given $j$, let us consider the function

$$
\phi = \phi_{A_1 \ldots A_2} \sigma^A \ldots \sigma^{A_{j+1}} \iota^{A_{j+1}} \ldots \iota^{A_2j}
$$

defined on $S$, where the coefficients $\phi_{A_1 \ldots A_2}$ are constant and completely symmetric in its indices. Recalling how the operators $\delta$ and $\delta'$ act on the spinors $\sigma^A$ and $\iota^A$ (see Appendix A.1.1-A.1.3) and how the operator $J, J'$ is built from $\delta$ and $\delta'$ (see Appendix A.1.4), it is easy to check that these functions are, in fact, eigenfunctions of $J, J'$ with eigenvalue $j(j+1)\hbar^2$. Substituting this $\phi$ into (2.6) and using that the symplectic metric $\varepsilon^{AB}$ on the space $S_A$ of spinors can be written as $\varepsilon^{AB} = \sigma^A \sigma^B - \iota^A \iota^B$, after some algebra we obtain

$$
C \phi_{A_1 \ldots A_2} \sigma^A \ldots \sigma^{A_{j+1}} \iota^{A_{j+1}} \ldots \iota^{A_2j}
$$

$$
= -\frac{1}{\sqrt{2}} (j + s)\gamma_{BA_1} \iota^B \phi_{A_2 \ldots A_2} + (j - s)\gamma_{BA_1} \iota^B \phi_{A_2 \ldots A_2},
$$

(2.7)
Since $\gamma_{AB}$ is symmetric, there are spinors $\mu_A$ and $\nu_A$, the so-called principal spinors of $\gamma_{AB}$, such that $\gamma_{AB} = \sqrt{2}\mu_A\nu_B$. Substituting this into (2.7) we find

$$C\phi_{A_1...A_{2j}} = -j\left(\mu_A\phi^A_{(A_1...A_{2j-1})A_{2j}} + \nu_A\phi^A_{(A_1...A_{2j-1})A_{2j}}\right).$$  \hspace{1cm} (2.8)$$

We solve this equation when $\gamma_{AB}$ is not null (`generic case'), i.e. if $\mu_A$ and $\nu_A$ are not proportional to each other, and when $\gamma_{AB}$ is null (`exceptional case') separately.

If $\gamma_{AB}$ is not null, then $0 \neq |\nu_A\mu^A|^2 = -\gamma_{AB}\gamma^{AB} = -\gamma_{AA'}\gamma^{AA'} = (\alpha_i - i\beta_i)(\alpha^i - i\beta^i) = 1 - \lambda^2 - 2i\lambda\alpha_3$, and $\mu_A$ and $\nu_A$ span the space $S_A$. Hence the spinors of the form

$$\phi_{A_1...A_{2j}} = \mu_{(A_1...A_{2j-1}以外)},$$  \hspace{1cm} (2.9)$$
m = -j, -j + 1, ..., j, form a basis in the space $S(A_1...A_{2j})$ of the totally symmetric spinors of rank $2j$ (see also Appendix A.1.1). Substituting this into (2.8), we obtain

$$C\mu_{(A_1...A_{2j-1}以外)}\nu_{A_{2j}} = m(\nu_A\mu^A)\mu_{(A_1...A_{2j-1}以外)}\nu_{A_{2j}};$$

i.e. the eigenvalues are

$$C = m(\nu_A\mu^A) = \pm j\sqrt{1 - \lambda^2 - 2i\lambda\alpha_3},$$  \hspace{1cm} (2.10)$$

and the corresponding (not normalized) eigenfunctions are

$$\phi_{s,j,m} := \mu_{(A_1...A_{2j-1}以外)}\nu_{A_{2j}}\alpha^{A_{2j}}...\alpha^{A_{2j-1}}...\alpha^{A_{2j}},$$  \hspace{1cm} (2.11)$$

For the sake of concreteness, in the rest of the paper, we will choose the upper sign in (2.10).

The principal spinors of $\gamma_{AB}$ are determined only up to the scale ambiguity, $(\mu_A, \nu_A) \mapsto (\chi\mu_A, \chi^{-1}\nu_A)$, where $\chi$ is any non-zero complex constant; and up to their order. However, $\nu_A\mu^A$ is invariant under the rescalings by $\chi$, while under the interchange of the principal spinors both $\nu_A\mu^A$ and $m$ change sign (see (2.9)). Therefore, the eigenvalue $C$ is independent of these ambiguities (as it should be). Under such transformations the eigenfunctions change according to $\phi_{s,j,m} \mapsto \chi^{2m}\phi_{s,j,m}$ and $\phi_{s,j,m} \mapsto \phi_{s,j,-m}$, respectively. The first of these ambiguities is reduced to a phase ambiguity by the normalization of the eigenfunctions.

If $\gamma_{AB} = \sqrt{2}\mu_A\mu_B$ (i.e. when $\gamma_{AB}$ is null), then let us consider the spinor of the form

$$\phi_{A_1...A_{2j}} = \mu_{A_1...A_{2j}},$$

where $\chi_{A_1...A_{2j-1}}$ is some totally symmetric spinor for which $\mu_{A_1}\chi_{A_1...A_{2j-1}} \neq 0$, $n = 0, 1, ..., 2j$. Substituting this into (2.8) we obtain

$$C\mu_{A_1...A_{2j}} = (2j + 1)\mu_{A_1...A_{2j}}\mu_{A_{n+1}}\chi_{A_{n+2}...A_{2j}}\mu^A = (2j - n)\mu_{A_1...A_{2j}}\chi_{A_{n+2}...A_{2j}}\mu^A.$$  \hspace{1cm} (2.12)$$

Since, however, the number of the spinors $\mu_A$ with free index on the left hand side is $n$ while on the right hand side it is $n + 1$, this equation can hold true precisely when $n = 2j$; i.e. the eigenvalue is zero, $C = 0$, the constant spinor $\phi_{A_1...A_{2j}}$ is null with $\mu_A$ as its $2j$-fold principal spinor, and the corresponding (not normalized) eigenfunction is

$$\phi_{s,j} := \mu_{A_1...A_{2j}}\alpha^{A_1}...\alpha^{A_{2j-1}}...\alpha^{A_{2j}}.$$  \hspace{1cm} (2.13)$$

Thus, the solution of (2.8) in the exceptional case is not the special case of that in the generic case: while the eigenvalue is the $m = 0$ special case of (2.10), the eigenfunction is the $m = -j$ special case of (2.11).

Remarkably enough, the eigenvalue (2.10) does not directly depend on $j$ or on $s$; it depends on them only indirectly through $m$ since $m = -j, -j + 1, ..., j$ and $j = |s|$, $|s| + 1, ....$ Moreover,
we got no restriction on the parameters $\alpha_3$ and $\lambda$. Therefore, even with fixed $s$, there are infinitely many eigenfunctions, labelled by $j$, with the given eigenvalue (2.11). Let $\phi$ be any normalized solution of the eigenvalue equation (2.4) with the given eigenvalue. Then by (2.10) $(J(\alpha))_{\phi} - i\lambda (J(\beta))_{\phi} = m \hbar \sqrt{1 - \lambda^2 - 2i\lambda\alpha_3}$ holds. Since both $J(\alpha)$ and $J(\beta)$ are self-adjoint, their expectation value must be real, and hence an elementary algebraic calculation yields that

$$\langle J(\alpha) \rangle_{\phi} = m \frac{\hbar}{\sqrt{2}} \sqrt{1 - \lambda^2 + \sqrt{(1 - \lambda^2)^2 + 4\lambda^2 \alpha_3^2}}, \quad (2.13)$$

$$\lambda \langle J(\beta) \rangle_{\phi} = m \text{sign}(\alpha_3) \frac{\hbar}{\sqrt{2}} \sqrt{\lambda^2 - 1 + \sqrt{(1 - \lambda^2)^2 + 4\lambda^2 \alpha_3^2}} \quad (2.14)$$

if $\alpha_3 \neq 0$; these are $m \hbar \sqrt{1 - \lambda^2}$ and 0, respectively, if $\alpha_3 = 0$ and $\lambda < 1$; while these are 0 and $m \hbar \sqrt{\lambda^2 - 1}$, respectively, if $\alpha_3 = 0$ and $\lambda > 1$. In the exceptional case both these expectation values are vanishing.

Therefore, apart from the discrete quantum number $m$, the expectation values are functions of two parameters: $\alpha_3 \in (-1, 1)$, as we expected, and $\lambda \in (0, \infty)$. The appearance of $\lambda$ in the expectation values is a bit surprising, because, by its very meaning, it characterizes the ratio of the two standard deviations. Now we discuss the behaviour of the expectation values as functions of these two parameters.

First, for fixed non-zero $m$, $\langle J(\alpha) \rangle_{\phi}$ and $\langle J(\beta) \rangle_{\phi}$, as functions of $\lambda$, behave in complementary ways: $|\langle J(\alpha) \rangle_{\phi}| = |\langle J(\beta) \rangle_{\phi}|$ precisely when $\lambda = 1$; but in the $\lambda \to 0$ limit $\langle J(\alpha) \rangle_{\phi} \to m\hbar$ and $\langle J(\beta) \rangle_{\phi} \to m\hbar\alpha_3$; while in the $\lambda \to \infty$ limit $\langle J(\alpha) \rangle_{\phi} \to m\hbar|\alpha_3|$ and $\langle J(\beta) \rangle_{\phi} \to \text{sign}(\alpha_3)m\hbar$. The expectation values can be zero only if $\alpha_3 = 0$ (see above); and, for $\alpha_3 \neq 0$ and $m \neq 0$, $\langle J(\alpha) \rangle_{\phi}/m$, as a function of $\lambda$, is strictly monotonically decreasing from $\hbar$ to $|\alpha_3|$, while $\text{sign}(\alpha_3)\langle J(\beta) \rangle_{\phi}/m$ is strictly monotonically increasing from $\hbar|\alpha_3|$ to $\hbar$. Both expectation values are smooth functions of $\lambda$.

On the other hand, also for fixed non-zero $m$, although $\langle J(\alpha) \rangle_{\phi}$ is a continuous function of $\alpha_3$, but it is not differentiable on the $\lambda > 1$ portion of the $\alpha_3 = 0$ line (on which it is vanishing). $\langle J(\beta) \rangle_{\phi}$ behaves just in the opposite way: it is not continuous at $\alpha_3 = 0$ for $\lambda > 1$, its left/right limit is $\mp m\hbar \sqrt{\lambda^2 - 1}/\lambda$; but its derivative with respect to $\alpha_3$ is zero there both from the $\alpha_3 > 0$ and the $\alpha_3 < 0$ directions. Thus, $\langle J(\beta) \rangle_{\phi}$ has a jump, while its square, $(\langle J(\beta) \rangle_{\phi})^2$, is differentiable there. Therefore, as functions on the parameter space $\mathcal{P} := \{(\alpha_3, \lambda) | \alpha_3 \in (-1, 1), \lambda \in (0, \infty)\}$, the two expectation values behave in asymmetric ways.

It follows from these properties that, for given $j$, the range of the expectation values $\langle J(\alpha) \rangle_{\phi}$ and $\langle J(\beta) \rangle_{\phi}$ is a connected interval of $\mathbb{R}$, parameterized by $\lambda$, precisely when the directions $\alpha^i$ and $\beta^j$ are orthogonal to each other (see above). This case was considered by Aragone et al in [7]. If $j$ may take any allowed value, then the range of the expectation values is the whole $\mathbb{R}$. If the two directions are not orthogonal to each other, then the range of them is a union of intervals of length $|m|\hbar(1 - |\alpha_3|)$. In the limit $|\alpha_3| \to 1$ (i.e. when $J(\alpha) \to \pm J(\beta)$), these intervals are getting to be disjoint and, ultimately, shrink to points. Thus, in this limit, the range of the expectation values reduces to the discrete (point) spectrum of the operator $J(\beta)$. Therefore, $\alpha_3$ provides the parameter in an interpolation between the purely continuous, connected range of the expectation values and the completely discrete spectrum of $J(\beta)$.

We return to the discussion of the expectation values and the structure of the proper parameter space in subsection 2.5 and section 3.

### 2.3 The eigenfunctions

The eigenfunctions $\phi_{s,j,m}$ and $\phi_{s,j}$ given by (2.11) and (2.12) in the generic and exceptional cases, respectively, are not normalized. The corresponding normalized eigenfunctions will be denoted by $W_{s,j,m} := N_{s,j,m} \phi_{s,j,m}$ and $W_{s,j} := N_{s,j} \phi_{s,j}$, respectively, where the factors of normalization,
\[ N_{s,j,m} \text{ and } N_{s,j}, \text{ (as well as the explicit coordinate form of the eigenfunctions) will be determined in Appendix A.2.1 and A.2.2. However, as we will see there, although the functions } W_{s,j,m} \text{ and } W'_{s',j',m'} \text{ are L}_2\text{-orthogonal to one another if } s \text{ and } s' \text{ or } j \text{ and } j' \text{ are different, } W_{s,j,m} \text{ and } W'_{s,j,m'} \text{ are, in general, not. In fact, since (in contrast to } J_3 \text{ or } J_\alpha J_\beta) \text{ the operator } J(\alpha) - i\lambda J(\beta) \text{ is not self-adjoint (and, in fact, not even normal), the orthogonality of its eigenfunctions with different eigenvalues does not follow.}\]

Since the eigenfunctions \( W_{s,j,m} \) and \( W_{s,j} \) belong to the eigenvalue labeled by \( m \) and zero, respectively, the general solution \( \phi \) of the eigenvalue problem (2.5) with given \( s \) can be written as a combination of these functions:

\[
\phi = \sum_{j = \max(|s|,|m|)}^\infty c^j W_{s,j,m}, \quad \text{or} \quad \sum_{j = |s|}^\infty c^j W_{s,j}, \quad (2.15)
\]

respectively, where \( \sum_{j} |c^j|^2 = 1 \). Next we determine the eigenfunctions of the eigenvalue equation (2.5) in closed form.

Since, as we will see, all the functions in (2.5) are polynomial in the complex stereographic coordinates, and polynomial equations are easier to solve than trigonometrical ones, we search for its solutions in these coordinates. In addition, since \( \alpha_1 \alpha_2 = 1 \), we can (and will) write \( (\alpha_1, \alpha_2) = \sqrt{1 - \alpha^2_3}(\cos \alpha, \sin \alpha) \), where \( \alpha \in [0, 2\pi) \).

Using \( \beta^i = (0, 0, 1) \) and the explicit form of \( m^i \) and the edth operators (see Appendix A.1.2 and A.1.3), equation (2.5) in the coordinate system \( (\zeta, \xi) \) takes the form:

\[
\left( \frac{1}{2} \sqrt{1 - \alpha^2_3} (1 - \exp[-2i\alpha|\xi|^2]) + (\alpha_3 - i\lambda) \exp[-i\alpha|\zeta|] \right) \exp[i\alpha] \frac{\partial \ln \phi}{\partial \zeta} - \left( \frac{1}{2} \sqrt{1 - \alpha^2_3} (1 - \exp[2i\alpha|\xi|^2]) + (\alpha_3 - i\lambda) \exp[i\alpha|\zeta|] \right) \exp[-i\alpha\frac{\partial \ln \phi}{\partial \zeta} + \frac{1}{2}s \left( \sqrt{1 - \alpha^2_3} (\exp[i\alpha|\zeta| + \exp[-i\alpha|\zeta|]) - 2(\alpha_3 - i\lambda) \right) = C.
\]

Thus, it seems useful to rotate the complex coordinates according to \( \zeta \mapsto \xi := \exp[-i\alpha|\zeta|], \) and in the new coordinates \( (\xi, \bar{\xi}) \) this equation simplifies to

\[
\left( \frac{1}{2} \sqrt{1 - \alpha^2_3} (1 - \xi^2) + (\alpha_3 - i\lambda)\xi \right) \frac{\partial \ln \phi}{\partial \xi} - \left( \frac{1}{2} \sqrt{1 - \alpha^2_3} (1 - \bar{\xi}^2) + (\alpha_3 - i\lambda)\bar{\xi} \right) \frac{\partial \ln \phi}{\partial \bar{\xi}} = C + s(\alpha_3 - i\lambda) - \frac{1}{2}s \sqrt{1 - \alpha^2_3}(\xi + \bar{\xi}).
\]

Denoting the complex vector field on the left of this equation by \( X \), this equation can be written in the compact form:

\[
X(\ln \phi) = C - \frac{1}{2}s \left( \sqrt{1 - \alpha^2_3} \xi - (\alpha_3 + i\lambda) \right) - \frac{1}{2}s \left( \sqrt{1 - \alpha^2_3} \bar{\xi} - (\alpha_3 + i\lambda) \right),
\]

i.e. the directional derivative of \( \ln \phi \) in the direction \( X \) is a given function on \( S \). Rewriting \( X \) in the polar coordinate system, it becomes clear that its real part is \( \lambda \)-times the rotation generator about \( \beta^1 \), while its imaginary part is the rotation generator about \( \alpha^1 \). Since \( \alpha^1 \neq \pm \beta^1 \), \( X \) does not have any zero on \( S \).

To find its solution, let us observe that

\[
-(\sqrt{1 - \alpha^2_3} \xi - (\alpha_3 + i\lambda)) = \frac{\partial}{\partial \xi} \left( \frac{1}{2} \sqrt{1 - \alpha^2_3} (1 - \xi^2) + (\alpha_3 - i\lambda)\xi \right)
\]

\[
= X \left( \ln \left( \frac{1}{2} \sqrt{1 - \alpha^2_3} (1 - \xi^2) + (\alpha_3 - i\lambda)\xi \right) \right),
\]

\]

\[ 7 \]
and hence the last two terms on the right of (2.16) can be written in the form

\[ X \left( \ln \left( \frac{\sqrt{1 - \alpha^2(1 - \xi^2) + 2(\alpha_3 - i\lambda)\xi}}{\sqrt{1 - \alpha^2(1 - \bar{\xi}^2) + 2(\alpha_3 - i\lambda)\bar{\xi}}} \right) \right)^{s/2} \].

Introducing the notation

\[ \xi_{\pm} := \frac{\alpha_3 - i\lambda \pm \sqrt{1 - \lambda^2 - 2i\lambda\alpha_3}}{\sqrt{1 - \alpha_3^2}}, \]  

(2.17)

this can be rewritten into the remarkably simple form

\[ X \left( \ln \left( \frac{\sqrt{1 - \alpha_3^2(1 - \xi^2) + 2(\alpha_3 - i\lambda)\xi}}{\sqrt{1 - \alpha_3^2(1 - \bar{\xi}^2) + 2(\alpha_3 - i\lambda)\bar{\xi}}} \right) \right)^{s/2} = X \left( \ln \left( \frac{(\xi - \xi_+)(\xi - \xi_-)}{(\xi - \xi_-)(\xi - \xi_+)} \right) \right)^{s/2}; \]  

(2.18)

while the vector field \( X \) is

\[ X = -\frac{1}{2} \sqrt{1 - \alpha_3^2} \left( (\xi - \xi_+)(\xi - \xi_-) \frac{\partial}{\partial \xi} - (\xi - \xi_-)(\xi - \xi_+) \frac{\partial}{\partial \xi} \right). \]  

(2.19)

It might be worth noting that \( \xi_{\pm} \) is in a one-to-one correspondence with the expectation value given by (2.14):

\[ |\xi_{\pm}|^2 = \frac{m\hbar \pm \langle J_3 \rangle_\phi}{m\hbar \mp \langle J_3 \rangle_\phi}, \]

(2.20)

which can be inverted to express \( \langle J_3 \rangle_\phi \) by \( |\xi_{\pm}|^2 \).

If \( u \) were a function on \( S \) for which \( X(u) = 1 \) held, then the first term on the right of (2.16) would have the form \( X(Cu) \), and hence we already would have found a particular solution of the inhomogeneous equation (2.16). We also need the general solution \( \phi \) of the corresponding homogeneous equation \( X(\ln \phi) = 0 \), i.e., \( X(\phi_0) = 0 \). To find this \( u \), let us observe that if the functions \( u_1 = u_1(\xi) \) and \( u_2 = u_2(\xi) \) solve

\[
\frac{du_1}{d\xi} = -\frac{1}{\sqrt{1 - \alpha_3^2}} \frac{1}{(\xi - \xi_+)(\xi - \xi_-)}, \quad \frac{du_2}{d\xi} = -\frac{1}{\sqrt{1 - \alpha_3^2}} \frac{1}{(\xi - \xi_-)(\xi - \xi_+)},
\]

(2.21)

respectively, then \( X(u_1 - u_2) = 1 \) (and \( X(u_1 + u_2) = 0 \), too). However, since

\[
\frac{d}{d\xi} \ln \left( \frac{\xi - \xi_+}{\xi - \xi_-} \right) = \frac{\xi_+ - \xi_-}{(\xi - \xi_+)(\xi - \xi_-)},
\]

we find that

\[
u_1 = -\frac{1}{2\sqrt{1 - \lambda^2 - 2i\lambda\alpha_3}} \ln \frac{\xi - \xi_+}{\xi - \xi_-}, \quad u_2 = -\frac{1}{2\sqrt{1 - \lambda^2 - 2i\lambda\alpha_3}} \ln \frac{\xi - \xi_-}{\xi - \xi_+},
\]

(2.22)

provided that \( 1 - \lambda^2 - 2i\lambda\alpha_3 \neq 0 \), i.e., \( (1 - \lambda)^2 + \alpha_3^2 > 0 \) (‘generic case’). Therefore, in the generic case, \( u = u_1 - u_2 \) holds and the general local solution of (2.16) is

\[
\phi = \phi_0 \left( \frac{(\xi - \xi_-)(\xi - \xi_+)}{(\xi - \xi_+)(\xi - \xi_-)} \right)^{m/2} \left( \frac{(\xi - \xi_-)(\xi - \xi_-)}{(\xi - \xi_-)(\xi - \xi_-)} \right)^{s/2},
\]

(2.23)

where we used the expression (2.10) (with the upper sign) for the eigenvalue \( C \), and \( \phi_0 \) is the general solution of the homogeneous equation. Clearly, an arbitrary complex function \( \phi_0 \) of \( v := u_1 + u_2 \) solves \( X(\phi_0) = 0 \); and it is easy to see that this is, in fact, the general solution of \( X(\phi_0) = 0 \). Indeed, let us define \( \psi_0(u_1, u_2) := \phi_0(\xi, \bar{\xi}) \) and observe that by (2.19) \( X(\phi_0) = 0 \) is equivalent to

\[
0 = (\xi - \xi_+)(\xi - \xi_-) \frac{\partial \phi_0}{\partial \xi} - (\xi - \xi_-)(\xi - \xi_+) \frac{\partial \phi_0}{\partial \bar{\xi}} = -\frac{1}{\sqrt{1 - \alpha_3^2}} \left( \frac{\partial \psi_0}{\partial u_1} - \frac{\partial \psi_0}{\partial u_2} \right).
\]
However, its general solution is an arbitrary complex valued smooth function of \( v := u_1 + u_2 \), i.e. \( \phi_0 \) is an arbitrary complex valued smooth function of

\[
w := \frac{(\xi - \xi_+)(\xi - \xi_+)}{(\xi - \xi_-)(\xi - \xi_-)}.
\]

In particular,

\[
\omega := \frac{(\xi - \xi_+)^a(\xi - \xi_-)^a(\xi - \xi_+)^b(\xi - \xi_-)^b}{(1 + \xi \xi)^{a+b}} = w^a \left( \frac{\xi_+ - \xi_-}{\xi_+ - \xi_- + \xi} \right)^{a+b}
\]

is a solution of \( X(\phi_0) = 0 \) even with arbitrary real \( a \) and \( b \). However, with such a general \( \phi_0 \) is only a local solution of \( (2.22) \): we still have to ensure that \( \phi \) be well defined even on small circles surrounding the poles, and be square integrable, too. These requirements restrict the structure of \( \phi_0 \), and, in particular, \( a \) and \( b \) in \( \omega \) are restricted to be only non-negative integers or half-odd-integers. In particular, by \( (A.28) \) and \( (A.32)-(A.33) \) the eigenfunctions \( W_{s,j,m} \) are, in fact, combinations of functions of the form \( (2.22) \) with \( \phi_0 = \omega \) in which \( a \) and \( b \) are non-negative integer or half-odd-integer such that \( a + b = j \).

If \( \lambda = 1 \) and \( \alpha_3 = 0 \) (‘exceptional case’), then \( \xi_\pm = -i \) and the solution of equation \( (2.21) \) is given by

\[
u_1 = \frac{1}{i + \xi}, \quad u_2 = \frac{1}{i + \xi}.
\]

Thus \( u = u_1 - u_2 \), and the general solution of \( X(\phi_0) = 0 \) is an arbitrary smooth complex function of \( v := u_1 + u_2 \). Hence, in the exceptional case, the local solution of \( (2.16) \) is

\[
\phi = \phi_0 \left( \frac{i + \xi}{i + \xi} \right)^{s,j},
\]

where we have used that in the exceptional case the eigenvalue \( C \) in \( (2.10) \) is zero. In particular,

\[
\omega := \frac{(\xi + i)(\xi + i)}{(1 + \xi \xi)^a} = \frac{1}{(1 - i \xi)^a}
\]

is a solution of \( X(\phi_0) = 0 \) even with arbitrary real \( a \). This \( a \) is restricted to be a non-negative integer or half-odd-integer by the requirement that the corresponding \( \phi \) be well defined and square integrable. In particular, by \( (A.29) \) and \( (A.31) \) the eigenfunctions \( W_{s,j} \) are combinations of functions of the form \( (2.25) \) with \( \phi_0 = \omega \) in which \( a + b = j \).

Therefore, the eigenfunctions even with fixed eigenvalue depend on one (almost completely) free function; or, in their series expansion, on infinitely many freely specifiable complex constants \( c^j \) (see \( (2.15) \)).

### 2.4 The standard deviations

Taking the \( L_2 \)-scalar product of the eigenvalue equation \( J(\alpha)\phi = hC\phi + iJ(\beta)\phi \) with itself and using the consequence \( \langle \phi, J(\alpha)\phi \rangle - i\lambda \langle \phi, J(\beta)\phi \rangle = hC \) of the eigenvalue equation, we obtain

\[
\langle \phi, (J(\alpha))^2 \rangle \phi = \langle J(\alpha)\phi, J(\alpha)\phi \rangle = \langle \phi, J(\alpha)\phi \rangle^2 - \lambda^2 \langle \phi, J(\beta)\phi \rangle^2 + \lambda^2 \langle \phi, (J(\beta))^2 \phi \rangle.
\]

This does, in fact, imply that \( \Delta_\phi J(\alpha) = \lambda \Delta_\phi J(\beta) \), yielding \( \Delta_\phi J(\alpha) \Delta_\phi J(\beta) = \lambda (\Delta_\phi J(\beta))^2 \), too. Hence, to calculate the standard deviations, we need to compute only \( \langle (J(\beta))^2 \rangle_\phi \). Thus, let \( \phi \) be given by \( (2.15) \). Then

\[
\langle (J(\beta))^2 \rangle_\phi = \langle J(\beta)\phi, J(\beta)\phi \rangle = \sum_{j,j' = \max\{s,j,m\}} \infty c^{j,j'} N_{s,j',m} N_{s,j,m} \langle J_3 \phi_{s,j',m}, J_3 \phi_{s,j,m} \rangle.
\]
Therefore, we should calculate \( \langle J_3 \phi_{s,j,m}, J_3 \phi_{s,j,m} \rangle \). This, with the formal substitution \( m = -j \), will give the general form of \( \langle J_3 \phi_{s,j'}, J_3 \phi_{s,j,m} \rangle \) in the exceptional case, too.

To calculate \( \langle J_3 \phi_{s,j'}, J_3 \phi_{s,j,m} \rangle \), we need to know the action of \( J_3 \) on the eigenfunctions \( \phi_{s,j,m} \). Repeating the analysis behind the equations \((2.27)-(2.28)\), we obtain that

\[
J_i \phi_{s,j,m} = \frac{\hbar}{\sqrt{2}} \left\{ (j + s)\sigma^B_i A \phi_{BA_2 \ldots A_2} o^A o^{A_2} \ldots o^{A_j} \phi_{A^{j+1} \ldots A_2} + (j - s)\sigma^B_i A \phi_{BA_2 \ldots A_2} l^A o_{A_2} \ldots o_{A^{j+1}} \phi_{A^{j+2} \ldots A_2} \right\}.
\]

(2.26)

As a by-product, this formula makes it possible to calculate all the expectation values \( \langle J_i \phi \rangle \), too. In fact, using the techniques of Appendix A.2.2, a straightforward calculation yields

\[
\langle \phi_{s,j',m}, J_i \phi_{s,j,m} \rangle = -(-)^{j'j} \sqrt{2} h_{j'2}^{4\pi} \frac{(j-s)!(j+s)!}{(2j+1)!} C_i B C \phi_{CA_2 \ldots A_2},
\]

(2.27)

where the adjoint \( C_i \) of the spinor \( \phi_{A_1 \ldots A_2} \) is defined according to the standard convention

\[
\phi_{B}^\dagger := \phi_A \sqrt{2} \omega_{0A}.
\]

Since this is non-zero only if \( j' = j \), the expectation value of \( J_i \) in the state \( \phi \) given by (2.15) is simply \( \langle J_i \phi \rangle = \sum_{s,j,m}^{\infty} |c|^2 N_{s,j,m}^2 \langle \phi_{s,j,m}, J_i \phi_{s,j,m} \rangle \). Using (2.26) and the techniques of Appendix A.2.2, the expectation values \( \langle \phi_{s,j,m}, J_i \phi_{s,j,m} \rangle \) can be expressed as

\[
\langle \phi_{s,j,m}, (J_1 + iJ_2) \phi_{s,j,m} \rangle = h_{2}^{4\pi} \frac{(j-s)!(j+s)!}{(2j+1)!} \sum_{k=0}^{2j} \binom{2j}{k} \Phi_{k,j,m} \Phi_{k+1,j,m},
\]

\[
\langle \phi_{s,j,m}, J_3 \phi_{s,j,m} \rangle = -h_{2}^{4\pi} \frac{(j-s)!(j+s)!}{(2j+1)!} \sum_{k=0}^{2j} \binom{2j}{k} \Phi_{k,j,m}^2,
\]

(2.28)

where, analogously to (2.11) (see also Appendix A.1.1), we introduced the notation

\[
\Phi_{k,j,m} := \phi_{A_1 \ldots A_2} o^{A_1} \ldots o^{A_k} l^{A_{k+1} \ldots A_2}.
\]

(2.29)

These, together with the expression (A.38) for the factor of normalization and the explicit form (A.39) for \( \Phi_{k,j,m} \) given in Appendix A.2.2, yield the general form for the expectation value of the operators \( J_i \) in the state \( W_{s,j,m} \). In particular, the latter for \( J_3 \) yields (2.14) in that particular state. We will use this form of the expectation value of \( J_3 \) in Appendix A.2.4.

Using (2.26), a bit longer but straightforward spinorial calculation, similar to that behind (2.27), yields that \( \langle J_3 \phi_{s,j',m}, J_3 \phi_{s,j,m} \rangle \) is zero unless \( j' = j \), and that

\[
\langle J_3 \phi_{s,j,m}, J_3 \phi_{s,j,m} \rangle = (-)^{2j} h_{2}^{4\pi} \frac{(j-s)!(j+s)!}{(2j+1)!} \left( \beta^B \beta^C \beta^D \phi_{BA_2 \ldots A_2} o^{A_2} \phi_{A^{j} \ldots A_2} \right)
\]

\[
+ (2j-1) \beta^B \beta^C \phi_{BA_2 \ldots A_2} o^{A_2} \phi_{A^{j} \ldots A_2} \right)
\]

\[
= (-)^{2j} h_{2}^{4\pi} \frac{(j-s)!(j+s)!}{(2j+1)!} \left( j \phi_{A_1 \ldots A_2} o^{A_2} \phi_{A^{j} \ldots A_2} \right)
\]

\[
+ (2j-1) O^A o^{A} I_B \phi_{BA_2 \ldots A_2} o^{A_2} \phi_{A^{j} \ldots A_2} \right).
\]

(2.30)

Here, in the second step, we used \( I^A O_B - O^A I_B = \delta^A_B \) and \( 2 \beta^A \beta^B - \delta^A_B \), where \( \beta_{AB} \) is the
unitary spinor form of $\beta_i$. Then, using the definition of the adjoint $\phi_{A1\ldots A_{2j}}^\dagger$, we obtain
\[
\langle \mathbf{J}_3 \phi_{s,j,m}, \mathbf{J}_3 \phi_{s,j,m} \rangle = i\hbar 2 \pi \frac{(j-s)!(j+s)!}{(2j+1)!} \prod_{k=0}^{2j} \frac{1}{k} \left| \phi_{A1\ldots A_{2j}} O^A_1 \ldots O^A_k I^{A_{k+1}} \ldots I^{A_{2j}} \right|^2
\]
\[
-\hbar^2 4\pi \frac{(j-s)!(j+s)!}{(2j+1)!} 2j(2j-1) \sum_{k=0}^{2j-2} \binom{2j-2}{k} \left| O^A_i I^B \phi_{ABA1\ldots A_{2j-2}} O^A_1 \ldots O^A_k I^{A_{k+1}} \ldots I^{A_{2j-2}} \right|^2
\]
\[
= \hbar^2 4\pi \frac{(j-s)!(j+s)!}{(2j+1)!} \sum_{k=0}^{2j} (j-k)^2 \binom{2j}{k} \left| \Phi_{k,j,m} \right|^2.
\] (2.30)

Since for any normalized state $\Delta_\phi \mathbf{J}(\alpha) \Delta_\phi \mathbf{J}(\beta) = \lambda(\Delta_\phi \mathbf{J}(\beta))^2$ holds, the normalization condition $\sum_{j=\max\{|s|,|m|\}}^\infty |c|^2 = 1$ and equation (2.14) give that the ‘product uncertainty’ in the state $\phi$ is
\[
\Delta_\phi \mathbf{J}(\alpha) \Delta_\phi \mathbf{J}(\beta) = \sum_{j=\max\{|s|,|m|\}}^\infty |c|^2 \lambda \left( \langle \mathbf{J}_3 W_{s,j,m}, \mathbf{J}_3 W_{s,j,m} \rangle - \langle W_{s,j,m}, \mathbf{J}_3 W_{s,j,m} \rangle \right). (2.31)
\]

Hence, it is enough to evaluate the product uncertainty only in the normalized eigenstates $W_{s,j,m}$; i.e., by (2.30), to clarify the properties of $\Phi_{k,j,m}$ only. This will be done in Appendix A.2.2. In particular, in Appendix A.2.3 we show that $\langle \mathbf{J}_3 W_{s,j,m}, \mathbf{J}_3 W_{s,j,m} \rangle$ does not depend on the sign of $m$, while $\langle W_{s,j,m}, \mathbf{J}_3 W_{s,j,m} \rangle$ changes sign if $m$ is replaced by $-m$. Moreover, $\langle \mathbf{J}_3 W_{s,j,m}, \mathbf{J}_3 W_{s,j,m} \rangle$ is continuous even at $\alpha = 0$ for $\lambda > 1$, while $\langle W_{s,j,m}, \mathbf{J}_3 W_{s,j,m} \rangle$ has a jump there. In addition, in Appendix A.2.4 we determine the asymptotic behaviour of the standard deviations in the limits $\lambda \to 0$ and $\lambda \to \infty$. We will see that the standard deviation for $\mathbf{J}(\beta)$ is finite and that for $\mathbf{J}(\alpha)$ tends to zero as $\lambda \to 0$; while the standard deviation for $\mathbf{J}(\beta)$ tends to zero as $1/\lambda$ and that for $\mathbf{J}(\alpha)$ is finite if $\lambda \to \infty$. Hence, the product uncertainty tends to zero in both limits. Therefore, the uncertainties do not indicate any asymmetry between the $\mathbf{J}(\alpha)$ and $\mathbf{J}(\beta)$ angular momentum components, just as we expect it on physical grounds.

Finally, in the exceptional case, (2.30) together with the expression of $\mu_A O^A$ and $\mu_A I^A$ given in Appendix A.2.2 yield $\langle \mathbf{J}_3 W_{s,j}, \mathbf{J}_3 W_{s,j} \rangle = j\hbar^2 / 2$. Since $\langle W_{s,j}, \mathbf{J}_3 W_{s,j} \rangle = 0$, we find that the product uncertainty in the state $W_{s,j}$ is $j\hbar^2 / 2$. Hence, in the general eigenstate $\phi = \sum_{j=|s|}^\infty c_j W_{s,j}$, the product uncertainty is $(\hbar^2 / 2) \sum_{|s|=j}^\infty |c_j|^2$, which can be zero only for $j = |s| = 0$.

### 2.5 The extension of the classical parameter space

Since physically $\mathbf{J}(\alpha)$ and $\mathbf{J}(\beta)$ are on equal footing, both should have the same qualitative properties. Indeed, the behaviour of the standard deviations does not break this symmetry between the two observables. However, the behaviour of the expectation values $\langle \mathbf{J}(\alpha) \rangle_\phi$ and $\langle \mathbf{J}(\beta) \rangle_\phi$ as functions on the classical parameter space $\mathcal{P} := \{(\alpha_3, \lambda) | \alpha_3 \in (-1, 1), \lambda \in (0, \infty)\}$ apparently does break this symmetry. Hence, this odd behaviour indicates that either the expectation values should be extended to be double valued on $\mathcal{P}$, or they should in fact be functions on a non-trivial Riemann surface $\mathcal{R}$ larger than $\mathcal{P}$, which must be homeomorphic to the one known in elementary complex analysis in connection with the function $\sqrt{z}$.

In fact, this second possibility seems more natural, because, for fixed $m$ and apart form the factor $\hbar$, the complex eigenvalue of $\mathbf{J}(\alpha) - i\mathbf{J}(\beta)$ is the square root of the complex function $1 - \lambda^2 - 2i\lambda \alpha_3$ on $\mathcal{P}$ (see equation (2.10)). The corresponding Riemann surface $\mathcal{R}$ is obtained from $\mathcal{P}$ by cutting it along the $\lambda \geq 1$ part of the $\alpha_3 = 0$ axis, and identifying the resulting edges for $\lambda > 1$ with the opposite edges of another copy of $\mathcal{P}$ that has been cut in the similar way. The
branch point of the resulting Riemann surface is at \( \lambda = 1, \alpha_3 = 0 \), which corresponds just to the exceptional case of subsections 2.2 and 2.3. Then the two expectation values are extended from the first to the second copy of \( \mathcal{P} \) in \( \mathcal{R} \) just to be \(-1\) times the ones on the original ‘classical’ copy \( \mathcal{P} \). With this extension the resulting expectation values behave in the same way, and, in particular, will be differentiable on the whole of \( \mathcal{R} \).

3 On the simultaneous measurement of \( J(\alpha) \) and \( J(\beta) \)

In the pioneering work [3], Arthurs and Kelly raised the possibility of the simultaneous measurement of the conjugate, and hence not commuting, observables in a Heisenberg system. Soon later, it was argued [4] that every measurement of the momentum or of the position is, in fact, such a simultaneous measurement, although while we measure one quantity precisely, we measure the conjugate one rather imprecisely. For example, when we do a precise measurement of the momentum of a particle, then the particle must be in the measuring apparatus (or at least in the laboratory), so we do a rude position measurement as well; and when we measure the position of the particle accurately and the measuring apparatus is not destroyed in the measuring procedure, then we can be sure that the particle’s momentum could not have arbitrarily large momentum, so we did a rude momentum measurement, too. This idea of the simultaneous measurement of the conjugate observables was discussed further e.g. in [4, 19, 20], clarifying, in particular, the precise relationship between the errors of the measurements and the standard deviations, deriving inequalities for the former, etc.

The basis of the possibility of such simultaneous measurements is that the non-commuting observables are different kinds of quantities: to measure them we need different devices, and during the simultaneous running of them one does not make impossible to measure the other. If, however, the two non-commuting observables are of the same kind, e.g. the \( J(\alpha) \) and \( J(\beta) \) components of angular momentum, then it does not seem to be possible to carry out such a measurement in a direct way. For example, in the Stern–Gerlach apparatus we cannot have two different magnetic fields at the same time to measure the spin of the particle in the corresponding two different directions.

Nevertheless, the phenomenon of the quantum entanglement makes it possible, at least in principle, to measure \( J(\alpha) \) and \( J(\beta) \) simultaneously in an indirect way. In fact, it is known that even the coherent spin states can be entangled [5], and one particle in its most classical state can be maximally entangled with a second one. Thus, preparing two such particles to be maximally entangled and then spatially separating them, we can measure \( J(\alpha) \) for one, and \( J(\beta) \) for the second in two independent Stern–Gerlach apparatus.

In this way, in principle, one can verify experimentally whether or not the proper parameter space preferred by Nature is the classical \( \mathcal{P} \) or the Riemann surface \( \mathcal{R} \). (The parameter \( \alpha_3 \) has the obvious meaning in the experimental apparatus, and \( \lambda \) can also be controlled in an indirect way since it is the quotient of the two standard deviations.) Indeed, let us consider the closed path in the classical parameter space \( \mathcal{P} \) consisting of the following four straight line segments:

1. the initial point is given by \( \alpha_3 = 1/\sqrt{2} \) and \( \lambda = 2 \) (i.e. the angle between the directions \( \alpha^i \) and \( \beta^i \) is \( \pi/4 \), and \( \Delta_\phi J(\alpha) \) is just twice of \( \Delta_\phi J(\beta) \)), and the end point is defined by \( \alpha_3 = 1/\sqrt{2} \) and \( \lambda = 1/2 \);
2. the end point of the second segment is at \( \alpha_3 = -1/\sqrt{2} \) and \( \lambda = 1/2 \) (i.e. we enlarge the angle between \( \alpha^i \) and \( \beta^i \) from \( \pi/4 \) to \( 3\pi/4 \) while keeping \( \lambda \) to be \( 1/2 \);
3. the end point of the third segment is at \( \alpha_3 = -1/\sqrt{2} \) and \( \lambda = 2 \); and
4. the fourth segment closes the path, returning to the initial point of the first segment.
Next consider a 1-parameter family of the most classical states with fixed non-zero $m$ and parameterized by the parameter of this closed path, and measure the two expectation values. If we find that both expectation values change continuously along the closed path from their initial value at the initial point of the path to their own negative at the end point of the path, then the genuine parameter space is the Riemann surface $R$; but if only one changes continuously along the path but the other had a discontinuity (or, equivalently, one does not change sign along the closed path but the other does), then the Nature’s parameter space is the classical $P$.

4 Results, discussion and conclusions

In [7], Aragone et al determined the most classical states with respect to two Cartesian components of the angular momentum vector operator in quantum systems with the $su(2)$ as the algebra of its basic observables. These states with given total angular momentum number $j$ and spin $s$ depend on a discrete quantum number $m$ and a single parameter $\lambda$. However, as the functions of $\lambda$, the expectation values of the two components change in an asymmetric way: for $\lambda < 1$ one is zero and the other is decreasing with $\lambda$, while for $\lambda > 1$ the former is increasing with $\lambda$ while the latter is zero (see the discussion following equation (2.14)).

In the present paper, we extended (and, in fact, completed) the investigations above: we determined all the most classical states for the components of the angular momentum vector operator in any two given directions. Allowing the angle between the two angular momentum components to be arbitrary, we found that the expectation values and the standard deviations depend on one discrete quantum number, $m$, and two continuous parameters. In addition, the angle between the two directions provided an interpolation between the continuous range of the expectation values found by Aragone et al and the discrete point spectrum of one angular momentum component.

Since the two expectation values are the real and imaginary parts of the square root of a complex function on this two-dimensional parameter space, one of the expectation values changes continuously but the other discontinuously on this (classical) parameter space. As far as we know, this is a new quantum mechanical phenomenon. The asymmetric behaviour of the expectation values in the special case considered by Aragone et al and mentioned above is a special manifestation of this phenomenon. However, physically, neither of the angular momentum components is distinguished over the other; and the standard deviations do, in fact, show this symmetry. As a possible resolution of the contradiction between the symmetry of the angular momentum components and the apparently asymmetric behaviour of the expectation values we suggest that the genuine parameter space should be diffeomorphic to the Riemann surface appearing in connection with the function $\sqrt{z}$ in complex analysis. With this extension of the classical parameter space the asymmetric behaviour of the expectation values disappears.

The odd behaviour of the expectation values on the classical parameter space seems to be analogous to that of the scissors in Dirac’s scissors problem (see e.g. [12], page 43); but, being independent of the value of the intrinsic spin, this is independent of the fermionic nature of the system. Based on the use of the entangled coherent states, we raise the possibility of a potential experimental verification of this phenomenon.

5 Acknowledgments

Thanks are due to Paul Tod for the idea how the spectrum of the operator $J(\alpha) - \lambda J(\beta)$ of subsection 2.2 can be calculated in the most economical way, as well as for his helpful remarks and comments; to Péter Vecsernyés for his remarks on the classical limit of non-relativistic quantum systems; to Zoltán Zimborás for the discussion of the quantum information theoretic
interpretation of the various spinorial notions, as well as for suggesting reference [9]; and to the Referee for suggesting to add the ‘informal dictionary’ in Appendix A.1.5.

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors. The author has no conflicts to disclose.

A Appendices

A.1 The irreducible representations of \(su(2)\) by spin weighted functions

In Appendices [A.1.1–A.1.4] mostly to fix the notations and the technical background, we summarize the spinorial and complex techniques that we used in the calculations in the main part (as well as in Appendix A.2) of the paper. To make these spinorial notions more understandable for a wider readership, we also add Appendix A.1.5 in which we show how some of these basic notions correspond to those e.g. in quantum optics and quantum information theory. (Thanks are due to one of the Referees for suggesting to provide such an informal ‘dictionary’.)

A.1.1 An algebraic introduction of the spin weighted spherical harmonics

As is well known, any unitary irreducible (and hence finite dimensional) representation of \(SU(2)\) can be labeled by a non-negative integer or half-odd-integer \(j\), i.e. for which \(2j = 0, 1, 2, \ldots\). The carrier space \(H_j\) of such a representation is a \(2j + 1\) dimensional Hilbert space, in which the vectors of an orthonormal basis are usually denoted by \(|j, m\rangle\), where \(m = -j, -j + 1, \ldots, j\).

One concrete realization of this abstract Hilbert space can be based on the use of completely symmetric spinors. The space \(S^A\) of (two-component, unitary) spinors is a two-complex dimensional vector space endowed by a symplectic and a positive definite Hermitean metric that are compatible with each other. These metrics can be given in terms of the vectors of an orthonormal basis are usually denoted by \(|\ldots\rangle\).

As is well known, any unitary irreducible (and hence finite dimensional) representation of \(SU(2)\) can be labeled by a non-negative integer or half-odd-integer \(j\), i.e. for which \(2j = 0, 1, 2, \ldots\). The carrier space \(H_j\) of such a representation is a \(2j + 1\) dimensional Hilbert space, in which the vectors of an orthonormal basis are usually denoted by \(|j, m\rangle\), where \(m = -j, -j + 1, \ldots, j\).

One concrete realization of this abstract Hilbert space can be based on the use of completely symmetric spinors. The space \(S^A\) of (two-component, unitary) spinors is a two-complex dimensional vector space endowed by a symplectic and a positive definite Hermitean metric that are compatible with each other. These metrics can be given in terms of the vectors \(O_A\) and \(I_A\) of a basis in \(S^A\); these, respectively, are \(\varepsilon_{AB} := O_A I_B - O_B I_A\) and \(\sqrt{2} T_{AA'} := O_A O_{A'} + I_A I_{A'}\), where over-bar denotes complex conjugation. (For a general background, see e.g. [12] [13]). Then, lowering and raising the spinor indices by \(\varepsilon_{AB}\) and its inverse \(\varepsilon^{AB}\), respectively, it is easy to check that this basis \(\{O^A, I^A\}\) is normalized with respect to the symplectic and orthonormal with respect to the Hermitean metric: \(O_A O^A = \varepsilon_{AB} O_B O_A = 1\) and \(\sqrt{2} T_{AA'} O^A O^A' = \sqrt{2} T_{AA'} I^A I^A' = 1\), \(\sqrt{2} T_{AA'} I^A I^A' = 0\). We call such a basis a Cartesian spin frame; which is unique only up to \(SU(2)\) transformations. These imply that \(T_{AA'} T^{AA'} = 1\), \(T_{AA'} O^{A'} = -I_A / \sqrt{2}\) and \(T_{AA'} I^{A'} = O_A / \sqrt{2}\) also hold. With the choice \(O^A = \delta^A_0, I^A = \delta^A_1\) this Hermitean metric is just the unit matrix \(\sqrt{2} \sigma^A_{AA'}\), where \(\delta^A_{AA'}\) denotes the zeroth of the four \(SL(2, \mathbb{C})\) Pauli matrices \(\sigma^A_{AA'}\) (including the factor \(1 / \sqrt{2}\), according to the conventions of [12]). With the identification \(O^A = [\frac{1}{2}, -\frac{1}{2}])\) the space \(S^A\) can be identified with the Hilbert space \(H_j\) above with \(j = \frac{1}{2}\), and the Hermitean scalar product of the two states \(\phi^A = |\phi\rangle\) and \(\psi^A = |\psi\rangle\) is just \(\langle \phi | \psi \rangle = \sqrt{2} T_{AA'} \phi^{A'} \psi^A\).

Note, however, that while in quantum mechanics the relative phase of the vectors of an orthonormal basis is usually not fixed, the relative phase of \(O^A\) and \(I^A\) in \(S^A\) is fixed by the condition \(O_A I^A = 1\).

The space \(S_{(A_1 A_2)}\) of the completely symmetric spinors \(\phi_{A_1 \ldots A_2}\) of rank \(2j\), i.e. for which \(\phi_{A_1 \ldots A_2} = \phi_{(A_1 \ldots A_2)}\), is the symmetrized \(2j\)-fold tensor product of the spin space \(S_A\) with itself. If \(\{O_A, I_A\}\) is any basis in \(S_A\), then the spinors of the form \(Z(j, m)_{A_1 \ldots A_2} := O_{(A_1} \cdots O_{A_{2j}} I_{A_{2j+1}} \cdots I_{A_2)}\) form a basis in \(S_{(A_1 \ldots A_2)}\). If the basis \(\{O_A, I_A\}\) is chosen as above, then the Hermitean scalar product makes it possible to normalize the basis vectors \(Z(j, m)_{A_1 \ldots A_2}\). In fact, the vectors

\[
\sqrt{(2j)! / (j + m)!(j - m)!} Z(j, m)_{A_1 \ldots A_2}
\]
have not only unit norm with respect to $\sqrt{2}T^{AA'}$, but they are orthogonal to one another as well. Thus they can be identified with the abstract basis vectors $[j,m]$ in $H_j$ above. Hence, a completely symmetric spinor $\phi_{A_1...A_{2j}} \in S(A_1...A_{2j})$ can equivalently be represented by its components in this basis, or by the contractions $\phi_{A_1...A_{2j}}Z(j,m)^{A_1...A_{2j}}$, which are $2j+1$ complex numbers.

Another basis in the space $S_A$ is provided by the so-called Newman–Penrose spin frame $\{o_A, \iota_A\}$. Such a frame can be chosen to be

$$o_A = \frac{-i}{\sqrt{1+\zeta}}(\zeta O_A + I_A), \quad \iota_A = \frac{-i}{\sqrt{1+\zeta}}(O_A - \zeta I_A), \quad (A.1)$$

where $\zeta \in \mathbb{C}$ (see e.g. [12]). The matrix of the basis transformation $\{O_A, I_A\} \mapsto \{o_A, \iota_A\}$ is an $SU(2)$ matrix. Hence, in particular, $\sqrt{2}t_{AA'} := o_A\bar{\sigma}_{A'} + \iota_A\bar{\iota}_{A'} = O_A\bar{O}_{A'} + I_AI_{A'} = \sqrt{2}T_{AA'}$ holds. The frame $\{o_A, \iota_A\}$ is a two-real-parameter family of bases, which can be considered to be defined on the (e.g. unit) 2-sphere $S$, where $\zeta$ is the complex stereographic coordinate on the 2-plane (see the next subsection). Hence, the spinors of the form $\phi_{(A_1...A_{2j})} := o_{(A_1...A_{2j})} \iota_{(A_{j+1}...A_{2j})}$, $s = -j, -j+1, ..., j$, form another basis in $S(A_1...A_{2j})$. These basis vectors can also be normalized, and

$$\sqrt{(2j)!}{(j-s)!(j+s)!}Z(j,s)^{A_1...A_{2j}}$$

form another orthonormal basis with respect to $\sqrt{2}T^{AA'} = \sqrt{2}T^{AA'}$. Hence, the contractions

$$U(j)_{m,s} := \frac{(2j)!}{(j+m)!(j-m)!(j-s)!(j+s)!}Z(j,m)^{A_1...A_{2j}}$$

form a $(2j+1) \times (2j+1)$ unitary matrix, taking one orthonormal basis of $S(A_1...A_{2j})$ to another one. Hence, the spinor $\phi_{A_1...A_{2j}} \in S(A_1...A_{2j})$ can also be represented by the contractions $\phi_{A_1...A_{2j}}z(j,s)^{A_1...A_{2j}}$, which are $2j+1$ special complex valued functions on $S$.

Apart from a numerical coefficient, the familiar spin weighted spherical harmonics are just the components of the matrix $U(j)_{m,s}$:

$$sY_{j,m} := (-)^{j+m}\sqrt{\frac{2j+1}{4\pi}}U(j)_{m,s}, \quad (A.2)$$

where the numerical coefficient makes these to be normalized with respect to the $L_2$-scalar product on the unit sphere; and, for $s = 0$, these are just the familiar ordinary spherical harmonics $Y_{j,m}$. For further properties of these harmonics, see [15, 16, 12, 13].

### A.1.2 Spinorial coordinates on $S$

Let $p^i, i = 1, 2, 3$, denote Cartesian coordinates in $\mathbb{R}^3$, in which the components of the independent rotation Killing vector fields are $k_{im}^m = p^i(\delta_{jm}\delta_{im} - \delta_{jm}\delta_{in}), m, n = 1, 2, 3$. Thus here $i$ is the vector index, while $m$ and $n$ are the name indices of the Killing fields. These are tangent to the 2-spheres of radius $P$, $S := \{p^i \in \mathbb{R}^3|P^2 := \delta_{ij}p^ip^j = \text{const}\}$, and vanish at the point $p^i = 0$. These Killing fields with the Lie bracket generate the Lie algebra $su(2)$.

The complex stereographic coordinates, projected from the north pole, are defined on $U_n := S - \{(0,0,P)\}$, the sphere minus its north pole, by $\zeta := \exp(i\varphi) \cot(\theta/2)$, where $(\theta, \varphi)$ are the standard spherical polar coordinates. In terms of $(\zeta, \bar{\zeta})$, the Cartesian coordinates of the point $p^i \in U_n$ are

$$p^i = P\left(\frac{\zeta + \bar{\zeta}}{1 + \zeta\bar{\zeta}}, i\frac{\zeta - \bar{\zeta}}{1 + \zeta\bar{\zeta}}, \frac{\zeta\bar{\zeta} - 1}{1 + \zeta\bar{\zeta}}\right). \quad (A.3)$$
The outward pointing unit normal to \( S \) at the point \( p^i = n^i := p^i/P \). This normal is completed to be a basis by the complex vector field

\[
m^i := \frac{1}{\sqrt{2}} \left( \frac{1-\zeta^2}{1+\zeta^2}, \frac{1+\zeta^2}{1+\zeta^2}, \frac{2\zeta}{1+\zeta^2} \right)
\]

(A.4)

and its complex conjugate \( \bar{m}^i \). These are orthogonal to \( n^i \), null (i.e. \( m^i n_i = 0 \)), normalized with respect to each other (i.e. \( m^i \bar{m}_i = 1 \)), and \( p^i m^j \bar{m}_k \epsilon_{ijk} = iP \) holds. (Recall that, in the present paper, the metric on \( \mathbb{R}^3 \) is chosen to be the positive definite \( \delta_{ij} \), rather than the negative definite spatial part of the Minkowski metric \( \eta_{ab} := \text{diag}(1,-1,-1,-1) \), where \( a, b = 0, i \).) In fact, apart from a phase factor, the complex null vectors \( m^i \) and \( \bar{m}^i \) are uniquely determined by the intrinsic complex structure of \( S \) (see e.g. \([13]\)). The vector field \( m^i \), as a differential operator, is given by

\[
m^i(\partial/\partial p^j) = \frac{1}{\sqrt{2P}}(1+\zeta \bar{\zeta}) \left( \partial/\partial \zeta \right).
\]

(A.5)

Hence, \( \zeta \) is a local anti-holomorphic coordinate on \( U_n \). Also in these coordinates, the line element of the metric and the corresponding area element on \( S \) of radius \( P \), respectively, are

\[
d\zeta^2 = \frac{4P^2}{(1+\zeta \bar{\zeta})^2}d\zeta d\bar{\zeta}, \quad dS = \frac{-2iP^2}{(1+\zeta \bar{\zeta})^2}d\zeta \wedge d\bar{\zeta}.
\]

(A.6)

These are just the metric and area element inherited from the metric and volume element of \( \mathbb{R}^3 \), respectively. There are analogous constructions on \( U_s := S - \{(0,0,-P)\} \), on the 2-sphere minus the south pole, too; and the structures defined on \( U_n \) are related to those introduced on \( U_s \) smoothly on the overlap \( U_n \cap U_s \).

Considering \( \mathbb{R}^3 \) to be the \( p^0 = P \) hyperplane of the Minkowski space \( \mathbb{R}^{1,3} \) with the Cartesian coordinates \( p^a = (p^0, p^i) \) and the flat metric \( \eta_{ab} \), the 2-sphere \( S \) is just the intersection of the \( p^0 = P \) hyperplane with the null cone of the origin \( p^a = 0 \) in \( \mathbb{R}^{1,3} \). Hence, for any \( p^i \in S \), there is a spinor \( \pi^A \), the ‘spinor constituent’ of \( p^i \), such that \( p^i = \sigma^i_A \pi^A \bar{\pi}^A \), and \( \pi^A \) is unique only up to the phase ambiguity \( \pi^A \mapsto \exp(i\gamma)\pi^A \), \( \gamma \in \{0, 2\pi\} \). (Note that we lower and raise the small Latin indices by the positive definite \( \delta_{ij} \) and its inverse.) Since \( P^2 = \delta_{ij}p^i p^j = -\eta_{ij} \sigma^i_A \sigma^j_B \pi^A \pi^B \bar{\pi}^A \bar{\pi}^B = (-\eta_{ij} \sigma^i_A \sigma^j_B - \sigma^0_A \sigma^0_B) \pi^A \pi^A \bar{\pi}^B \bar{\pi}^B = (\sigma_{AA}^i \pi^A \bar{\pi}^A)^2 \), the norm of \( \pi^A \) with respect to \( \sqrt{\eta} \pi_A^i \) is the norm of \( \pi^A \) on \( S \), i.e. \( \sqrt{\frac{d\zeta^2}{2P}} \), which is \( \sqrt{\frac{\sqrt{2P}}{2}} \). However, \( \pi^A \) as a spinor field is well defined only on \( S \) minus one point (see e.g. \([13, 14]\)). In particular, on \( U_n \), this spinor field, up to a phase, is \( \pi^A = (\sqrt{2P})^{1/2} \sigma^A \), where \( \sigma^A \) is given by \((A.1)\). Thus \( \pi^A \) on \( U_n \) and the analogous one on \( U_s \) are only locally defined ‘spinorial coordinates’ on \( S \).

Using the Newman–Penrose spinor basis \{\( \sigma^A, i \lambda^A \)\}, given explicitly on \( U_n \) by \((A.1)\) with the choice \( \lambda^A = \delta^A_0 \) and \( I^A = \delta^A_i \), it is easy to verify that \( m^i \sigma^A_i \pi^A = -\sigma^A \pi^i \) and \( p^i \sigma^A_i \pi^A = P(\pi^2 \pi^A - \sigma^A \sigma^A) / \sqrt{2} \) also hold.

**A.1.3 The line bundles \( \mathcal{O}(-2s) \) over \( S \)**

One way of defining the complex line bundles \( \mathcal{O}(-2s) \) over \( S \) can be based on the concept of the bundle of totally symmetric N-type spinor fields of rank \( 2|s| \) on \( S \): if \( s = -|s| \leq 0 \) then these spinor fields are unprimed and their principal spinor at the point \( p^i = \sigma^i_A \pi^A \bar{\pi}^A \) is \( \pi^A \), and if \( s = |s| > 0 \) then the spinor fields are primed and their principal spinor at \( p^i = \bar{\pi}^A \). (Recall that e.g. \( \lambda^A \) is a \( 2|s| \)-fold principal spinor of the totally symmetric spinor \( \phi^{A_1 \cdots A_{2|s|}} \) if \( \phi^{A_1 \cdots A_{2|s|}} \lambda_{A_{2|s|}} = 0 \) holds, in which case \( \phi^{A_1 \cdots A_{2|s|}} \lambda_{A_{2|s|}} \) necessarily has the form \( \phi^{A_1 \cdots A_{2|s|}} \), and the anti-linear type of the spinor is called null or of type \( N \), see e.g. \([12, 13]\).) Hence, e.g. on the domain \( U_n \), these spinor fields have the form \( \phi^{A_1 \cdots A_{2|s|}} = \phi^{A_1 \cdots A_{2|s|}} \) and \( \chi^{A_1 \cdots A_{2|s|}} = \chi^{A_1 \cdots A_{2|s|}} \), where \( \phi \) and \( \chi \) are complex functions on \( U_n \). Thus, the fibers of these
bundles are one complex dimensional, and the line bundle \(O(-2s)\) is just the abstract bundle of these fibers over \(S\). \(U_n\) and \(U_s\) are local trivialization domains of \(O(-2s)\), and the functions \(\phi\) for \(s = -|s|\) (and \(\chi\) for \(s = |s|\)) are local cross sections of \(O(-2s)\) on \(U_n\). \(O(-2s)\) is globally trivializable precisely when \(s = 0\).

The phase ambiguity \(\pi^A \mapsto \exp(i\gamma)\pi^A\) in the principal spinor yields the ambiguity \(\phi \mapsto \exp(-2i|s|\gamma)\phi\), where \(\gamma\) is an arbitrary \([0, 2\pi]\)-valued locally defined function on \(S\). The analogous ambiguity in the function \(\chi\) is \(\chi \mapsto \exp(2is\gamma)\chi\). Therefore, despite this ambiguity, the Hermitian scalar product of any two cross sections, representing e.g. the spinor fields \(\phi^{A_1 \ldots A_2|s|}\) and \(\psi^{A_1 \ldots A_2|s|}\) on \(S\) and given by

\[
\langle \phi^{A_1 \ldots A_2|s|}, \psi^{A_1 \ldots A_2|s|} \rangle_s := \int_S \bar{\phi}\psi dS,
\]

is well defined. The space of the square-integrable cross sections of \(O(-2s)\) is a Hilbert space, denoted by \(\mathcal{H}_s\). An alternative, and perhaps more familiar form of this scalar product can be given in terms of the spinor fields themselves. To rewrite \(\langle A.7 \rangle\), let us recall from subsection \(A.1.2\) that \(\sigma^0_{AB} = (OAO_A' + IA'I_A)/\sqrt{2} = (\lambda A\bar{\sigma}_A' + \lambda^{-1}A'I_A)/\sqrt{2}\). Using this, the integrand of \(\langle A.7 \rangle\) can be rewritten as

\[
\bar{\phi} \psi = (\sqrt{2}P)^{-2|s|}\bar{\epsilon}_{A_1} \ldots \bar{\epsilon}_{A_2|s|} \bar{\sigma}^{A_1 \ldots A_2|s|}(\lambda A_1 \ldots \lambda A_2|s|) \psi^{A_1 \ldots A_2|s|} = P^{-2|s|}\sigma^0_{A_1 A_1} \ldots \sigma^0_{A_2 A_2} \bar{\sigma}^{A_1 \ldots A_2|s|}(\lambda A_1 \ldots \lambda A_2|s|) \psi^{A_1 \ldots A_2|s|}.
\]

If we think of \(\sigma^0_{AB}\) as the components of the spinor form of the timelike vector of the orthonormal vector basis in a Lorentzian vector space whose spatial vectors span \(\mathbb{R}^3\), then by \(\langle A.8 \rangle\) the scalar product \(\langle A.7 \rangle\) is just \((\sqrt{2}P)^{-2|s|}\) times the scalar product that is analogous to the familiar, standard \(L_2\)-scalar product of two spinor fields on the mass shell in Poincaré-invariant quantum theory (see [21]).

Recalling that a scalar \(\phi\) is said to have the spin weight \(\frac{1}{2}(p - q)\) if under the rescaling \(\{\sigma^A, \nu^A\} \mapsto \{\lambda \sigma^A, \lambda^{-1}\nu^A\}\), where \(\lambda\) is any nowhere vanishing complex function on the domain of the spin frame, the scalar \(\phi\) transforms as \(\phi \mapsto \lambda^p\bar{\phi}\phi\) (see e.g. [12] [13]), we can see that \(O(-2s)\) is just the bundle of spin weighted scalars of weight \(s\) on \(S\). In particular, the components of the vectors \(\hat{m}\) and \(\tilde{m}\) are of types \((1, -1)\) and \((-1, 1)\), respectively, while those of \(p\) are sums of \((1, 1)\) and \((-1, -1)\) type scalar. Thus, the spin weight of \(\hat{m}\), \(\tilde{m}\) and \(p\) is \(1\), \(-1\) and \(-1\), respectively; and the spin weight of the spherical harmonics \(\psi^s_{j,m}\), defined by \(\langle A.2 \rangle\), is \(s\).

If \(\delta_i\) denotes the (Cartesian components of the) covariant derivative operator of the induced Levi-Civita connection acting on the spinor fields on \(S\), then the edth and edth-prime operators of Newman and Penrose [15] are defined as the covariant directional derivative operators \(m^i\delta_i\) and \(m^i\delta_i\), respectively, acting on the cross sections \(\phi\) of the line bundles \(O(-2s)\). Explicitly, if \(\phi_{A_1 \ldots A_2|s|} = \phi \sigma_{A_1} \ldots \pi_{A_2|s|}\), then \(\delta_i \phi := m^i \delta_i((\sqrt{2}P)^{-|s|}\phi_{A_1 \ldots A_2|s|})\nu^{A_1} \ldots \nu^{A_2|s|}\) and \(\delta_i \phi := m^i \delta_i((\sqrt{2}P)^{-|s|}\phi_{A_1 \ldots A_2|s|})\nu^{A_1} \ldots \nu^{A_2|s|}\) (see also [12] [13] [14] [16]). \(\delta_s\) and \(\delta_s\) acting on cross sections of \(O(-2s)\) with \(s = |s|\) are defined analogously. \(\delta_s\) increases, and \(\delta_s\) decreases the spin weight by one. In the complex stereographic coordinates on \(U_n\), the explicit form of these operators, acting on a function \(\phi\) of spin weight \(s\), is

\[
\delta_s \phi = \frac{1}{\sqrt{2}P}(1 + \zeta \frac{\partial \phi}{\partial \zeta} + s \zeta \phi), \quad \delta_s^* \phi = \frac{1}{\sqrt{2}P}(1 + \zeta \frac{\partial \phi}{\partial \zeta} - s \zeta \phi).
\]

If no confusion arises, simply we write \(\delta\) and \(\delta^*\) instead of \(\delta_s\) and \(\delta_s^*\). These operators link the spinors \(\sigma^A\) and \(\nu^A\): \(\partial \sigma^A = 0\), \(\partial \nu^A = \nu^A/(\sqrt{2}P)\), \(\partial \nu^A = -\sigma^A/(\sqrt{2}P)\) and \(\partial \nu^A = 0\); which imply \(\partial \sigma = m^i\), \(\partial \sigma = m^i\) and \(\partial \sigma = 0\). Using these formulæ, it is easy to see that

\[
\delta z(j, s)_{A_1 \ldots A_2} = -\frac{1}{\sqrt{2}P}(j - s)z(j, s + 1)_{A_1 \ldots A_2},
\]

\[
\delta^* z(j, s)_{A_1 \ldots A_2} = \frac{1}{\sqrt{2}P}(j + s)z(j, s - 1)_{A_1 \ldots A_2}.
\]
where the spinor \( z(j, s)_{A_1...A_2j} \) was introduced in Appendix A.1.1. Thus, while the ladder operators \( J_{\pm} := J_1 \pm iJ_2 \) change the quantum number \( m \) of the canonical angular momentum basis vectors \( |j, m \rangle \) by one, \( \delta \) increases and \( \delta' \) decreases the ‘quantum number’ \( s \) of \( z(j, s)_{A_1...A_2j} \) by one. Hence, \( \delta \) and \( \delta' \) play the role analogous to the ladder operators.

The spin weighted spherical harmonics \( sY_{j,m} \) can also be defined (up to phase and normalization) by the pair of equations \( \delta sY_{j,m} = -\sqrt{2P} \sqrt{(j+s+1)(j-s)}s + 1Y_{j,m} \) and \( \delta' sY_{j,m} = \frac{1}{\sqrt{2P}} \sqrt{(j-s+1)(j+s)}s - 1Y_{j,m} \). These imply that the harmonics are eigenfunctions of \( \delta \delta' \) and \( \delta' \delta \), and also of the Laplacian: \( (\delta \delta' + \delta' \delta)sY_{j,m} = -P^2(j(j+1) - s^2)sY_{j,m} \). (For more details, see [15, 16, 12, 13].)

For the general, abstract definition of the bundles \( \mathcal{O}(-2s) \) over complex projective spaces, see e.g. [12, 13, 14]. For their introduction and a discussion of some of the global topological properties of the operators \( \delta \) and \( \delta' \) on closed metric 2-surfaces even with any genus, see [17].

### A.1.4 The angular momentum operators on \( \mathcal{O}(-2s) \)

In this subsection, we determine the specific form of the angular momentum operators acting on spin weighted functions. This geometrical form is more natural than the usual one when the algebra of the basic quantum observables, \( su(2) \), is considered to be a subalgebra of the Euclidean algebra \( e(3) \), which will be considered in [11]. The form of this representation is similar to that of the generators of the Poincaré algebra of relativistic quantum systems [21], too.

The action of the group \( SU(2) \) on \( \mathbb{R}^3 \) is defined by \( \hat{p}^i \mapsto \Lambda_{ij}(A)p^j \), where \( \Lambda_{ij}(A) := -\sigma^{AB}_{\mu \nu} A^\mu_B \tilde{A}^\nu_A \sigma_{ij}^{BB'} \), in which \( A^\mu_B \in SU(2) \) and over-bar denotes complex conjugation. (The \((-)\) sign in the expression of \( \Lambda_{ij}(A) \) is due to our convention that we lower and raise the small Latin indices by the positive definite \( \delta_{ij} \) and its inverse.) Thus \( SU(2) \) is the (universal covering group of the) group of those isometries of the flat Riemannian 3-manifold \( (\mathbb{R}^3, \delta_{ij}) \) that leave the origin \( p^i = 0 \) fixed. The surfaces of transitivity of \( SU(2) \) are the 2-spheres \( S \) with radius \( P > 0 \) and the origin \( p^i = 0 \). The latter case is uninteresting for us in the present paper, because that yields the trivial representation.

One way of determining the irreducible representations of \( SU(2) \) is by means of the method of induced representations. In this way, first we should find the representations of the stabilizer subgroup for a point \( \hat{p}^i \in S \) in \( SU(2) \). This is \( U(1) \subset SU(2) \), and, by Schur’s lemma, all of its irreducible representations are one-dimensional, and these are labeled by \( s = 0, \pm \frac{1}{2}, \pm 1, \ldots \). If \( \hat{\pi}^A \) is the spinor constituent of \( \hat{p}^i \) (see Appendix A.1.2), then this one-dimensional representation space is chosen to be spanned by the spinor of the form \( \hat{\pi}^{A_1} \cdots \hat{\pi}^{A_{2|s|}} \) if \( s = -|s| \leq 0 \), and \( \hat{\pi}^{A_1} \cdots \hat{\pi}^{A_{2|s|}} \) if \( s = |s| > 0 \). The next step is the generation of the representation space for the whole group \( SU(2) \) from this one dimensional space by the elements of \( SU(2) \) that do not leave \( \hat{p}^i \) fixed.

Geometrically, the above method (by using the group action from \( \hat{p}^i \)) is the construction of the bundle of totally symmetric unprimed N-type spinors \( \phi^{A_{1}\cdots A_{2|s|}} \) on \( S \) if \( s = -|s| \), and of the totally symmetric primed N-type spinors \( \chi^{A_1\cdots A_{2|s|}} \) on \( S \) if \( s = |s| \). (Clearly, these are equivalent to the line bundles \( \mathcal{O}(-2s) \) with the corresponding \( s \).) The 2\(|s|\)-fold principal spinor of them at the point \( \hat{p}^i = \sigma^{AB}_{\mu \nu} \pi^A \pi^{A'} \) is \( \pi^A \) and \( \bar{\pi}^{A'} \), respectively.

To determine the explicit form of the representation of \( SU(2) \) by operators \( U(A) \) acting on the spinor fields, let us recall that the rotation Killing vectors \( \kappa_{mn}^{i} \) are tangent to \( S \) and also generate its isometries. Then the action of \( SU(2) \) e.g. on any \( \phi^{A_{1}\cdots A_{2|s|}} \) is defined by \( (U(A)\phi)^{A_{1}\cdots A_{2|s|}}(\hat{p}^i) := A_{1}^{B_1} \cdots A_{2|s|}^{B_{2|s|}} \sigma^{B_{1}\cdots B_{2|s|}}\bar{\phi}^{B_{1}\cdots B_{2|s|}}(A^{-1})^\mu_{ij}p^\mu) \). In particular, apart from a phase factor, \( (U(A)\pi)^A \) is just \( \pi^A \), i.e. the spinor constituent of the position vector field is \( SU(2) \)-invariant up to a phase. Hence, any 2\(|s|\)-rank spinor field that belongs to the carrier space of an irreducible representation of \( SU(2) \) is necessarily \( N \)-type with \( \pi^A \) and \( \bar{\pi}^{A'} \) as its 2\(|s|\)-fold principal spinor for \( s = -|s| \) and \( s = |s| \), respectively. Note that the function \( \phi \) appearing in
\( \phi^{A_1...A_{2|s|}} = \phi \pi^{A_1} \cdots \pi^{A_{2|s|}} \) has spin weight \( s \). Using (A.3), it is straightforward to check that the operator \( U(A) \) is unitary with respect to the scalar product \( \langle A, A \rangle \). As we will see in (11), the \( L_2 \) space \( H_s \) of the \( N \)-type spinor fields with given \( s \) provides the carrier space of an irreducible representation for the \( E(3) \) group, but it is not irreducible for the \( SU(2) \).

In this representation, still for \( s = -|s| \leq 0 \), the operators \( J_{ij} := \varepsilon_{ijk} J^k \) are defined to be the densely defined self-adjoint generators of these transformations: let \( A^{A_B} \in SU(2) \) be a 1-parameter subgroup in \( SU(2) \) generated by \( \lambda^A B \in su(2) \). Then its trajectories on \( S \) are necessarily integral curves of some rotation Killing vector field, say of subgroup in \( \text{densely defined self-adjoint generators of these transformations}: \) let \( \sigma^1 B \) be the rotation Killing fields. Repeating the analogous analysis for \( \sigma^2 B \), still for \( s = -|s| \leq 0 \), this is just minus the Lie derivative of the spinor field along the Killing vector \( M^{mn} k_m^J \) (see e.g. [12] [13]). Here the limit in the definition of the derivative is meant in the strong topology of \( H_s \). To evaluate this, let us calculate the tangent of the trajectories \( \Lambda_J^t (A(u)) p^j \) at \( p^j \). This is \( k^i = -\sigma^i_A A (\lambda^A B \delta^t_{B'}) \sigma_{B'} B p^j \), which must coincide with \( M^{mn} k_m^J \). Since \( \lambda^A B \in su(2) \), its complex conjugate is not independent of \( \lambda^A B \in su(2) \), because \( \lambda_{AB} = 2 \lambda^A B \sigma_{A'B'} B B B \) holds. Thus, introducing the standard \( SU(2) \) Pauli matrices \( \sigma^A B \) (including the factor \( 1/\sqrt{2} \)) as the unitary spinor form of the three non-trivial \( SL(2, C) \) Pauli matrices (i.e. which are given explicitly by \( \sigma_{A'B'} B = \delta_{ij} \varepsilon_{AC} A C B' \sqrt{2} \delta_{0B'} B' ) \), we can express \( \lambda^A B \) in terms of \( M^{ij} \):

\[
\lambda^A B = -\frac{i}{\sqrt{2}} M^{ij} \varepsilon_{ij} k^k \sigma^A B. \tag{A.12}
\]

In deriving this formula we also used the identity

\[
\sigma^A B \sigma^B C = -\frac{i}{\sqrt{2}} \varepsilon_{ij} k^j \sigma^A C + \frac{1}{2} \delta_{ij} \delta_{AB}. \]

Now, using (A.12), we are able to evaluate the equation defining the operator \( J_{ij} \). We find that, still for \( s = -|s| \),

\[
J_{ij} \phi^{A_1...A_{2|s|}} = i \hbar \left( p^i \frac{\partial}{\partial p^i} - p_i \frac{\partial}{\partial p_i} \right) \phi^{A_1...A_{2|s|}} + \sqrt{2} \hbar s \varepsilon_{ij} k^k \sigma^A_k (B_1 \delta_{A_0 A_1} \cdots \delta_{A_{2|s|}} A_{B_2} \cdots B_{2|s|}) \phi B_1...B_{2|s|}. \tag{A.13}
\]

Thus, \( J_{ij} \) is \( i \hbar \)-times the Lie derivative operator along the Killing vector \( k_{ij} \), and hence is well defined on the dense subspace of the smooth spinor fields in \( H_s \). Then it is a straightforward calculation to check that these operators do, indeed, satisfy the defining commutation relations of \( su(2) \) on the appropriate dense subspaces, i.e. provide a representation of the Lie algebra of the rotation Killing fields. Repeating the analogous analysis for \( s = |s| \), we obtain the same expression (A.13) for \( J_{ij} \).

Next, using the form \( J_i = \varepsilon_{ijk} J_{jk} / 2 \) of the angular momentum operator, we determine its contraction with the basis vectors \( p^i \), \( m^i \) and \( \tilde{m}^i \). First, recalling from Appendix A.12 that \( p^i \sigma^A A' = P (\tau^A A' - o^A A') / \sqrt{2} \), for its unitary spinor form we obtain

\[
p^i \sigma^A B := p^i \sigma^A A' \sqrt{2} \sigma_0 B' = -\frac{P}{\sqrt{2}} (\tau^A A' - o^A A' - \tilde{A}' \tilde{A}) (\tilde{A}' o_A + \tilde{A} A') = -\frac{P}{\sqrt{2}} (\tau^A o_B + o^A A').
\]

Since on the domain \( U_n \) the spinor field has the form \( \phi^{A_1...A_{2|s|}} = \phi \pi^{A_1} \cdots \pi^{A_{2|s|}} \), we find that

\[
p^i J_i \phi^{A_1...A_{2|s|}} = s P \hbar \phi^{A_1...A_{2|s|}}. \tag{A.14}
\]

It might be worth noting that in the unitary, irreducible representation of the Euclidean group \( E(3) \) exactly the same expression emerges as the analog of the Pauli–Lubanski spin operator [11], which is one of the two Casimir operators of \( e(3) \).

Similarly, one can show that \( m^i p^j \varepsilon_{ijk} = p^i m^j \varepsilon_{ijk} m_k = -i P m_k \) and \( \tilde{m}^i p^j \varepsilon_{ijk} = i P \tilde{m}_k \) (see Appendix A.12), as well
as the definition of the $\delta$ and $\delta'$ operators (Appendix [A.1.3]), we find

\[ m^i J_i \phi_{A_1 \ldots A_{2|s|}} = -\hbar \phi (\delta \phi) \pi^{A_1} \ldots \pi^{A_{2|s|}}, \tag{A.15} \]

\[ \bar{m}^i J_i \phi_{A_1 \ldots A_{2|s|}} = \hbar \phi (\delta' \phi) \pi^{A_1} \ldots \pi^{A_{2|s|}}. \tag{A.16} \]

Hence, by (A.14)-(A.16),

\[ J_i \phi_{A_1 \ldots A_{2|s|}} = \left( m_i \bar{m}^j + \bar{m}_i m^j + \frac{1}{P^2} p_i p^j \right) J_i \phi_{A_1 \ldots A_{2|s|}} = \hbar \left( m_i \delta' \phi - \bar{m}_i \delta \phi + s \frac{p_i}{P} \phi \right) \pi^{A_1} \ldots \pi^{A_{2|s|}}. \]

Thus, $J_i$ preserves the algebraic, viz. the N-type of the spinor spinor fields. Therefore, defining the action of $J_i$ on the spin weighted function $\phi$ with spin weight $s$ simply by $J_i \phi := (\sqrt{2}P)^{-|s|} (J_i \phi_{A_1 \ldots A_{2|s|}}) \pi^{A_1} \ldots \pi^{A_{2|s|}}$, we obtain

\[ J_i \phi = \hbar \left( m_i \delta' \phi - \bar{m}_i \delta \phi + s \frac{p_i}{P} \phi \right). \tag{A.17} \]

Hence, the operators $J_i$ map cross sections of $\mathcal{O}(-2s)$ into cross sections of $\mathcal{O}(-2s)$. In $E(3)$ invariant elementary quantum mechanical systems the same expression will be interpreted in $[11]$ as the decomposition of the total angular momentum into its orbital and spin parts, where $P$ and $s$ are fixed by the two Casimir operators of $e(3)$.

By (A.17) it is easy to compute the only Casimir operator $J_i J^i$ of the $su(2)$ algebra. It is

\[ J_i J^i = \hbar^2 \left( -P^2 (\delta' + \delta') \phi + s^2 \phi \right). \tag{A.18} \]

where, as we noted at the end of Appendix [A.1.3], $\delta' + \delta$ is just the Laplace operator on $S$, and there we determined its spectrum and eigenfunctions. Using these, we find that the eigenvalues of $J_i J^i$ are $j(j+1)\hbar^2$, $j = |s|, |s| + 1, \ldots$, and the corresponding eigenfunctions are the spin weighted spherical harmonics $sY_{j,m}$, $m = -j, -j + 1, \ldots, j$. To find the $J_i$-invariant subspaces of $\mathcal{H}_s$, it is enough to recall that $J_i$ and $J_i J^j$ commute, and hence we find that, for fixed $s$ and $j$, the $SU(2)$-irreducible subspace $\mathcal{H}_{s,j} \subset \mathcal{H}_s$ is spanned just by the spin weighted spherical harmonics $sY_{j,m}$, $m = -j, -j + 1, \ldots, j$. Thus all these irreducible subspaces are $2j + 1$ dimensional, and the edth and edth-prime operators, which are adjoint of each other, are maps between them: $\delta : \mathcal{H}_{s,j} \rightarrow \mathcal{H}_{s+1,j}$ and $\delta' : \mathcal{H}_{s,j} \rightarrow \mathcal{H}_{s-1,j}$. This structure of the $SU(2)$-irreducible subspaces together with the structure (A.17) of the operators $J_i$ and its interpretation in $E(3)$-invariant systems justify the interpretation of $s$ as the intrinsic spin of the system, even though it is not the value of any Casimir operator of $su(2)$.

A.1.5 A quantum information theoretical interpretation of the spinorial notions

The spin-space $S_A$ could be interpreted as the Hilbert space of the states of a two-states quantum system (‘qubit’); while $S_{(A_1 \ldots A_{2j})}$, the symmetrized $2j$-fold tensor product of the spin-space with itself, as the Hilbert space of the states of $2j$ identical and indistinguishable (‘bosonic’) qubits. The basis vectors $O_A$ and $I_A$ in $S_A$ are usually denoted in quantum information theory by $|0\rangle$ and $|1\rangle$, respectively. The basis vector $Z(j, m)_{A_1 \ldots A_{2j}}$, $m = -j, -j + 1, \ldots, j$, in $S_{(A_1 \ldots A_{2j})}$ corresponds to the one obtained by complete symmetrization of the tensor product basis vectors in $S_{A_1} \otimes \cdots \otimes S_{A_{2j}}$ according to

\[ |(0, \ldots, 0, 1, \cdots, 1)\rangle := \frac{1}{(2j)!} \sum_{\pi} |\pi(0, \cdots, 0, 1, \cdots, 1)\rangle. \]
Here \( \pi(0, \ldots, 0, 1, \ldots, 1) \) denotes a permutation of the array \((0, \ldots, 0, 1, \ldots, 1)\), the summation is on all the permutations, and the number of 0’s in each term is \((j - m)\) and that of 1’s is \((j + m)\). This vector is usually denoted by \([j, m]\). In particular, \([j, j] = [0, \ldots, 0]\) and \([j, j] = [1, \ldots, 1]\).

As is well known (see e.g. [9]), the set of the pure quantum states of a qubit is homeomorphic to the unit sphere \(S^2\), the so-called Bloch sphere. Parameterizing this sphere by the familiar angle coordinates \((\theta, \varphi)\), the coordinates \((\theta', \varphi')\) of its anti-podal point on \(S^2\) are \(\theta' = \pi - \theta, \varphi' = \varphi + \pi\). Then, in the basis \(\{|0\}, |1\}\) = \(\{O^A, I^A\}\), the quantum states corresponding to these points of the Bloch sphere, \(|\psi\rangle = \psi^A\) and \(|\psi'\rangle = \psi'^A\), are

\[
\psi^A := \cos \frac{\theta}{2} O^A + \exp(i \varphi) \sin \frac{\theta}{2} I^A,
\]

\[
\psi'^A := -\cos \frac{\theta}{2} O^A + \exp(i \varphi') \sin \frac{\theta}{2} I^A = -\sin \frac{\theta}{2} O^A + \exp(i \varphi) \cos \frac{\theta}{2} I^A.
\]

Clearly, \(\{\psi^A, \psi'^A\}\) is a basis in \(S^A\), this is orthonormal with respect to the Hermitian scalar product \(\sqrt{T}_{AA'}\), but it is not a normalized spin frame: \(\varepsilon_{AB} \psi^A \psi'^B = \psi^A \psi'^A = \exp(i \varphi) \neq 1\). In fact, the determinant of the matrix of the basis transformation \(\{O^A, I^A\} \rightarrow \{\psi^A, \psi'^A\}\) given by (A.19) - (A.20) is \(\exp(i \varphi)\), rather than 1. Hence this matrix is not an \(SU(2)\) spin transformation.

The vectors \(\omega^A\) and \(\iota^A\) of the Newman–Penrose spin frame on \(S\), given by (A.1), can also be interpreted as quantum states of the qubit. In fact, identifying the point \((\theta, \varphi)\) of the Bloch sphere with the point with the complex conjugate stereographic coordinate \(\eta := \bar{\zeta}\) on \(S\) according to \(\eta = \exp(i \varphi) \cot(\theta/2)\), the vectors of the Newman–Penrose spin frame at the point \((\zeta, \bar{\zeta})\in S\) take the form

\[
\omega^A = -\frac{i}{\sqrt{1 + \zeta}} \left(\zeta O^A + I^A\right) = -i \exp(-i \varphi) \left(\cos \frac{\theta}{2} O^A + \exp(i \varphi) \sin \frac{\theta}{2} I^A\right),
\]

\[
\iota^A = -\frac{i}{\sqrt{1 + \zeta}} \left(O^A - \zeta I^A\right) = i \left(-\sin \frac{\theta}{2} O^A + \exp(i \varphi) \cos \frac{\theta}{2} I^A\right).
\]

Thus, \(\omega^A\) is proportional to \(\psi^A\) and \(\iota^A\) to \(\psi'^A\) above, the factors of proportionality are only pure phases, but these factors are different. Since \(\{\omega^A, \iota^A\}\) is connected to the basis \(\{O^A, I^A\}\) by an \(SU(2)\) basis transformation, it is slightly more natural to use the Newman–Penrose basis than \(\{\psi^A, \psi'^A\}\).

It is well known [4] that, using the angular momentum ladder operators \(J_\pm := (J_1 \pm i J_2)\), for any fixed \(j\) and \(w \in \mathbb{C}\) the states

\[
|w\rangle \pm := \frac{1}{\sqrt{1 + |w|^2}} \exp\left(\frac{w J_\pm}{\hbar}\right) |j, \mp j\rangle = \frac{1}{\sqrt{1 + |w|^2}} \sum_{k=0}^{2j} w^k \sqrt{\binom{2j}{k}} |j, \mp (j - k)\rangle,
\]

the so-called \(SU(2)\) coherent states, are analogous to the canonical coherent states of Heisenberg systems; and the latter are usually considered to be the ‘most classical states’ of these systems.

States analogous to \(|w\rangle \pm\) were introduced in [3], which are based on states of the form \(\psi_{A_1} \cdots \psi_{A_2j}\) or \(\psi'_{A_1} \cdots \psi'_{A_2j}\) above, instead of \([j, \pm j]\). (In quantum information theory, these states are usually denoted by \(|\psi\rangle \otimes 2^j\) and \(|\psi'\rangle \otimes 2^j\), respectively.) Now we show how these analogous states can be introduced in a quite simple way, using the \(\delta\) and \(\delta'\) operators of Appendix A.1.3. Recalling from Appendix A.1.3 how \(\delta\) and \(\delta'\) act on the spinor field \(z(j, s)_{A_1 \cdots A_{2j}}\) (see equations (A.10) - (A.11)), it is easy to see that, for any \(k = 0, 1, \ldots, 2j\),

\[
\delta^k z(j, -j)_{A_1 \cdots A_{2j}} = \left(\frac{-1}{\sqrt{2^P}}\right)^k \binom{2j}{k} z(j, -j + k)_{A_1 \cdots A_{2j}},
\]

\[
\delta'^k z(j, j)_{A_1 \cdots A_{2j}} = \left(\frac{1}{\sqrt{2^P}}\right)^k \binom{2j}{k} z(j, j - k)_{A_1 \cdots A_{2j}}.
\]
hold. Hence, for any \( w \in \mathbb{C} \), it is straightforward to show that

\[
|w\rangle_{+} := \frac{1}{1 + |w|^2} \exp\left(-w\sqrt{2}P\phi\right)z(j,-j)_{A_1...A_{2j}} = \frac{1}{1 + |w|^2} \sum_{k=0}^{2j} w^k \binom{2j}{k} z(j,-j+k)_{A_1...A_{2j}}.
\]

\begin{equation}
|w\rangle_{-} := \frac{1}{1 + |w|^2} \exp\left(w\sqrt{2}P\phi'\right)z(j,j)_{A_1...A_{2j}} = \frac{1}{1 + |w|^2} \sum_{k=0}^{2j} w^k \binom{2j}{k} z(j,j-k)_{A_1...A_{2j}}.
\end{equation}

(A.23)

(A.24)

It is a simple calculation to demonstrate that the states \(|w\rangle_{+}\) and \(|w\rangle_{-}\) have unit norm. \(|w\rangle_{+}\) is orthogonal to \(|w\rangle_{+}\), and \(|w\rangle_{-}\) to \(|w\rangle_{-}\), precisely when \(w' = 1/w\) (i.e. when \(w\) and \(w'\) are the complex stereographic coordinates of points of \(S^2\) that are anti-podal of each other); and \(|w\rangle_{+}\) is orthogonal to \(|w\rangle_{-}\) precisely when \(w' = -w\). A key property of the coherent states is well known to be that they provide the partition of unity. Since the spinors \(z(j,s) = z(j,s)_{A_1...A_{2j}}\), \(s = -j,\ -j + 1, ..., j\), form a basis in \(\mathbb{S}(A_1...A_{2j})\), this key property is equivalent to

\[
z(j,s)_{A_1...A_{2j}} = \frac{1}{\pi} \int_{\mathbb{C}} |w\rangle\langle w| z(j,s) \rangle d^2w
\]

for some weight function \(m = m(w)\), which may depend on \(j\). Here \(|w\rangle\) is any of \(|w\rangle\)\(\pm\), and the integral measure on \(\mathbb{C}\) is the natural one: \(d^2w = \rho d\rho d\chi\), where \(w\) is represented by its Euler form \(w = \rho \exp(i\chi)\). A direct calculation does, in fact, show that, with the weight function \(m = (2j + 1)(1 + |w|^2)^{-2}\) as given in [3], (A.23) holds true.

A.2 The explicit form of the eigenfunctions \(W_{s,j,m}\) and the asymptotics of the standard deviations

A.2.1 The explicit form of the eigenfunctions \(\phi_{s,j,m}\)

The strategy of the determination of the explicit coordinate form of \(W_{s,j,m}\) and \(W_{s,j}\) on \(S\) (and in particular the factors of normalization, \(N_{s,j,m}\) and \(N_{s,j}\)) is just that for the spin weighted spherical harmonics \(sY_{j,m}\) given in [12]: the eigenfunction

\[
\phi_{s,j,m} = \mu(A_1 \ldots \mu_{A_{2j}} \nu_{A_{j-m+1}} \ldots \nu_{A_{2j}}) \sigma^{A_1} \ldots \sigma^{A_{j+m}} \tau^{A_{j+m}} \ldots \tau^{A_{2j}} \quad (A.26)
\]

is the combination of terms of the form

\[
\frac{1}{(2j)!} (\mu_A^A \sigma^A)^r (\mu_B^B \tau^B)^k (\nu_C^C \tau^C)^l (\nu_D^D \tau^D)^{2j-r-k-l}
\]

with non-negative integer coefficients. However, in such a term the number of the \(\sigma^A\) spinors must be \(j+s\), and the number of the \(\nu^A\) spinors must be \(j-s\), and hence \(l = j+s-r\). In a similar way, the number of the \(\mu_A\) spinors is \(j-m\) and that of \(\nu_A\) is \(j+m\), and hence \(k = j-m\). Therefore, for given \(r\), the structure of the terms in (A.26) is

\[
\frac{1}{(2j)!} (\mu_A^A \sigma^A)^r (\mu_B^B \tau^B)^{j-m-r} (\nu_C^C \tau^C)^{j+s-r} (\nu_D^D \tau^D)^{r+m-s} \quad (A.27)
\]

\(r\) takes its minimum value when all the \(\nu_A\) spinors are contracted with \(\sigma^A\) (but no \(\nu^A\) spinors; and in this case, if \(s-m \geq 0\), the number of the \(\sigma^A\) spinors that remain to contract with the \(\mu_A\) spinors is \(s-m\). Hence, \(r \geq \max\{0, s-m\}\). The maximum value of \(r\) is the maximal number
of contractions $\mu_A o^A$, which is $j + s$ if $j - m \geq j + s$, and it is $j - m$ if $j + s \geq j - m$. Therefore, $r \leq \min\{j - m, j + s\}$. Finally, to determine the number of the terms of the form (A.27) in (A.26), let us rewrite (A.27) in the form
\[
\frac{1}{(2j)!} \left( \mu_{A_1} \cdots \mu_{A_j} \nu_{A_{j+1}} \cdots \nu_{A_{2j+s}} \right) o^{A_1} \cdots o^{A_{j+s}} \times \left( \mu_{B_1} \cdots \mu_{B_{j-m-r}} \nu_{B_{j-m-r+1}} \cdots \nu_{B_{2j-s}} \right) t^{B_1} \cdots t^{B_{j-s}}.
\]

The number of ways in which the $\mu_A$ and the $\nu_A$ spinors can be chosen in this manner is $(j^m_r)$ and $(j^m_{r-s})$, respectively; and each of these choices can be contracted with the $o^A$ spinors in $(j + s)!$ ways and with the $t^B$ spinors in $(j - s)!$ ways. Thus, their total number is $(j^m_r) \cdot (j^m_{r-s}) \cdot (j + s)! \cdot (j - s)!$. Hence
\[
\phi_{s,j,m} = \frac{(j + m)!((j - m)!(j + s)!(j - s)!}{(2j)!} \times \sum_r \frac{(\mu_A o^A) r^{(j^m_r)(j^m_{r-s})\nu_{A_{j+s}}(\nu_{B_{j+m-s}}) t^{B_{j-s}}}}{r!(j - m - r)!(j + s - r)!(r + m - s)!},
\]

where $\max\{0, s - m\} \leq r \leq \min\{j - m, j + s\}$. In the exceptional case $r = j + s$, and hence (A.28) reduces to
\[
\phi_{s,j} = (\mu_A o^A)^j s(j^m_r)^j^{-s};
\]
which can be derived directly even from (A.26), too.

Next, we determine the explicit form of the contractions $\mu_A o^A$, $\nu_A o^A$, $\mu_A t^A$, and $\nu_A t^A$. (These are not only the factors in (A.28), but also these are the eigenfunctions $\phi_{s,j,m}$ with $(s, m) = (1/2, -1/2), (1/2, 1/2), (-1/2, -1/2)$ and $(-1/2, 1/2)$, respectively.) First, suppose that $\gamma_{AB}$ is not null. Then
\[
\gamma_{AB} = \frac{1}{\sqrt{2}} \begin{pmatrix} -\alpha_1 + i\alpha_2 & \alpha_3 - i\lambda \\ \alpha_3 - i\lambda & \alpha_1 + i\alpha_2 \end{pmatrix},
\]
and hence the solution of $\sqrt{2} \gamma_{AB} = \mu_A \nu_B + \mu_B \nu_A$ is given by
\[
\mu_A = \mu_o O_A - \mu_o I_A = -\left(\xi_+ \exp[i\alpha]O_A + I_A\right) \mu_o,
\]
\[
\nu_A = \nu_o O_A - \nu_o I_A = \frac{1}{2} \sqrt{1 - \alpha_3^2} \left(\xi_+ O_A + \exp[-i\alpha] I_A\right) \mu_o^{-1};
\]
where $\mu_o := \mu_A o^A$ is an arbitrary nonzero constant and $\xi_\pm$ has been defined by (2.17). However, this solution gives $\nu_A t^A = \mp(1/2) \exp[-i\alpha] \mu_o^{-1}$, and hence, to be compatible with our sign convention in (2.10), we must choose $\xi_+$ in $\mu_1$ and $\xi_-$ in $\nu_1$. Therefore,
\[
\mu_A o^A = -i \frac{\xi - \xi_+}{\sqrt{1 + \xi}} \mu_o \exp[i\alpha], \quad \nu_A o^A = i \frac{\xi - \xi_+}{\sqrt{1 + \xi}} \mu_o^{-1},
\]
\[
\mu_A t^A = -i\xi_+ \frac{\xi - \xi_+}{\sqrt{1 + \xi}} \mu_o, \quad \nu_A t^A = \frac{1}{2} \xi_+ \sqrt{1 - \alpha_3^2} \frac{\xi - \xi_+}{\sqrt{1 + \xi}} \mu_o \exp[i\alpha] \mu_o^{-1}.
\]
In the exceptional case, i.e. when $\lambda = 1$ and $\alpha_3 = 0$, the components of the principal spinor $\mu_A$ are $\mu_0 := \mu_A o^A = \pm(i/\sqrt{2}) \exp[-i\alpha] \mu_1 := \mu_A t^A = \mp(1/2) \exp[i\alpha] \mu_1$. Hence, the corresponding solutions $\phi_{1/2,1/2}$ and $\phi_{-1/2,1/2}$, respectively, are
\[
\mu_A o^A = \pm \frac{1}{\sqrt{2}} \frac{\xi + i}{\sqrt{1 + \xi}} \exp[i\alpha], \quad \mu_A t^A = \mp \frac{1}{\sqrt{2}} \frac{\xi + i}{\sqrt{1 + \xi}} \exp[-i\alpha],
\]
Therefore, the eigenfunctions $\phi_{s,j,m}$ and $\phi_{s,j}$ are polynomials in $\xi$ and $\bar{\xi}$ (with the overall order $2j$) and divided by $(1 + \xi \bar{\xi})^j$. Thus, in particular, all these are bounded on $S$ and smooth on $S$ minus the ‘north pole’ of the coordinate system $(\xi, \bar{\xi})$. 23
A.2.2 Orthogonality properties and the factors of normalization

To find the factor of normalization, and also to clarify whether or not the functions $\phi_{s,j,m}$ and $\phi_{s,j}$ form orthogonal systems, we need Lemma 4.15.86 of [12], viz. that

$$\int o_{A_1} \ldots o_{A_k} B_1 \ldots t^{B_k} dS = \frac{4\pi}{k+1} \delta^{(B_1 \ldots B_k)}_{(A_1 \ldots A_k)};$$  \hspace{1cm} (A.35)

and that the integral is zero if the number of $o_A$ and $t^B$ spinors under the integral is different. Here, and in the rest of this Appendix, $S$ is assumed to be a unit sphere.

Let $\phi_{s,j,m}$ and $\phi'_{s',j',m'}$ be two eigenfunctions, where the latter is built from $\phi'_{A_1 \ldots A_{2j'}}$ according to (2.9) but in which $m$ is replaced by $m'$. Recalling that $2\sigma_0^{A'B} \sigma_0^{BB'} = \delta_{B'}^{A'}$ and that the Hermitean adjoint e.g. of a spinor $\phi_A$ is defined by $\phi_A^\dagger := \bar{\phi}_A \sqrt{2} \sigma_0^{A'}$, we have that

$$\langle \phi_{s',j',m'}, \phi_{s,j,m} \rangle = \int \phi'_{D_{j'} \ldots D_{j'+1}} \phi_A^{A_1 \ldots A_{j+1} A_{j+1} \ldots A_{2j}} \bar{\phi}_A^{A_1 \ldots A_{j+1} A_{j+1} \ldots A_{2j}} \frac{4\pi}{j+j'+1} \delta^{(D_{j'} \ldots D_{j'+1})}_{A_1 \ldots A_{j+j'}} dS.$$  \hspace{1cm} (A.36)

However, as we noted above, this integral can be non-zero only if the number of the $o_A$ and $t^D$ spinors is the same, i.e. if $s' = s$. Hence, by (A.35), this integral is

$$\langle \phi_{s',j',m'}, \phi_{s,j,m} \rangle = (-)^{j+s} \delta_{s's'} \frac{4\pi}{(j+s)!((j+s)!)!} \phi_A^{A_1 \ldots A_{2j}} \phi_A^{A_1 \ldots A_{2j}}.$$  \hspace{1cm} (A.37)

Finally, rewriting $\phi_A^{A_1 \ldots A_{2j}}$ by the complex conjugate spinor and recalling that $\sqrt{2} \sigma_0^{AA'} = O^A O^{A'} + I^A I^{A'}$, we obtain

$$\langle \phi_{s',j',m'}, \phi_{s,j,m} \rangle = \frac{(j-s)!((j+s)!}{(2j+1)!} \sum_{k=0}^{2j} \binom{2j}{k} \Phi_{k,j,m} \bar{\Phi}_{k,j,m'},$$  \hspace{1cm} (A.38)

where $\Phi_{k,j,m}$ was defined by (2.20). If in $\phi_{A_1 \ldots A_{2j}}$ the spinors $\mu_A$ and $\nu_A$ were $O_A$ and $I_A$, respectively, then $\langle \phi_{s',j',m'}, \phi_{s,j,m} \rangle$ would be proportional to $\delta_{m'm}$, too, and the whole expression would reduce to the expression of the norm of the not normalized spin weighted spherical harmonics $^{s} Z(j, m)$ [12]. However, with our $\mu_A$ and $\nu_A$, the eigenfunctions $\phi_{s,j,m}$ and $\phi_{s,j,m'}$, $m \neq m'$, are not orthogonal to each other. Nevertheless, (A.37) provides the factor of normalization $N_{s,j,m}$:

$$N_{s,j,m}^{-2} := \langle \phi_{s,j,m}, \phi_{s,j,m} \rangle = \frac{(j-s)!((j+s)!}{(2j+1)!} \sum_{k=0}^{2j} \binom{2j}{k} |\Phi_{k,j,m}|^2.$$  \hspace{1cm} (A.38)
To see its dependence on the parameters \(\alpha_3\) and \(\lambda\), we should find the explicit form of \(\Phi_{k,j,m}\). We determine this in the next subsection.

The previous analysis is valid even in the exceptional case. The only difference between the generic and exceptional cases is that now \(m = m' = -j\), and hence, according to (A.37), the eigenfunctions \(\phi_{s,j}\) form an orthogonal system. Since, in the exceptional case, the modulus of the spinor components, \(\mu_0 := \mu_A O^A\) and \(\mu_1 := \mu_A I^A\), is \(1/\sqrt{2}\), the factor of normalization \(N_{s,j} = N_{s,j,-j}\), given by (A.38), can be written as

\[
N_{s,j}^{-2} = 4\pi \frac{(j-s)!(j+s)!}{(2j+1)!} \sum_{k=0}^{2j} \binom{2j}{k} |\mu_0|^{2k} |\mu_1|^{2(2j-k)} = 4\pi \frac{(j-s)!(j+s)!}{(2j+1)!},
\]

(A.39)

which is a pure number, independently of the (only) free parameter \(\alpha\) in \(\mu_0\) and \(\mu_1\).

### A.2.3 The continuity of the standard deviations

(A.28) gives the structure of \(\Phi_{k,j,m}\), too, if \(O^A\) is replaced by \(O^A, I^A\) by \(I^A\), and \(s = k - j\). Thus, using (A.30)-(A.31) and taking into account (A.28), we find that

\[
\Phi_{k,j,m} = (-)^k \mu_0^{-2m} \frac{\sqrt{2}}{2} \left(1 - \alpha^2\right)^{j-m} \exp[i\alpha]^{j-m-k} (\xi_-)^{2m-k}
\]

\[
\times \frac{k!(2j-k)!}{(2j)!} \sum_r (-)^r \binom{j-m}{m} \binom{j+m}{k-r} (\xi_-)^{2r},
\]

(A.40)

where \(\max\{0, k - j - m\} \leq r \leq \min\{j - m, k\}\). However, by equations (A.38), (2.28) and (2.30), in the expression of \(\langle J(\beta)^2\rangle\) and \(\langle J(\beta)^2\rangle_{\phi}\) it is only the absolute value of \(S_{k,j,m} := (\xi_-)^{-k} \sum_r (-)^r \binom{j-m}{m} \binom{j+m}{k-r} (\xi_-)^{2r}\)

(A.41)

that appears. In fact, in terms of this, \(\langle W_{s,j,m}, J^3 W_{s,j,m} \rangle\) and \(\langle J^3 W_{s,j,m}, J^3 W_{s,j,m} \rangle\) take the form

\[
\langle W_{s,j,m}, J^3 W_{s,j,m} \rangle = -\hbar \sum_{k=0}^{2j} \frac{(j-k)k!(2j-k)!|S_{k,j,m}|^2}{k!(2j-k)!|S_{k,j,m}|^2},
\]

(A.42)

\[
\langle J^3 W_{s,j,m}, J^3 W_{s,j,m} \rangle = \hbar^2 \frac{k!(2j-k)!|S_{k,j,m}|^2}{k!(2j-k)!|S_{k,j,m}|^2}.
\]

(A.43)

These depend on \(\alpha_3\) and \(\lambda\) only through \(\xi_-\) according to (2.17) via \(S_{k,j,m}\), but they do not depend on \(\alpha\) or \(\lambda\) (as they should not).

To see the structure of \(S_{k,j,m}\) and its dependence on \(\alpha_3\) and \(\lambda\), let us recall that the range of the summation in (A.41) is \(\max\{0, k - j - m\} \leq r \leq \min\{j - m, k\}\). Hence, we should split the range of \(k\) in (A.42) into three disjoint domains: i. when \(0 \leq k < j - m\), then \(r = 0,\ldots,k\); ii.a. when \(j - m \leq j + |m|\) and \(m \geq 0\), then \(r = 0,\ldots,j - m, m \geq 0\), then \(r = k - j - m,\ldots,k\); iii. when \(j + |m| < k \leq 2j\), then \(r = k - j - m,\ldots,j - m\). Hence, redefining \(r\) and its range if needed, in the respective cases

\[
\begin{align*}
\text{i.} & \quad S_{k,j,m} = (\xi_-)^{-k} \sum_{r=0}^{k} (-)^r \binom{j-m}{m} \binom{j+m}{k-r} (\xi_-)^{2r}, \\
\text{ii.a.} & \quad S_{k,j,m} = (\xi_-)^{-k} \sum_{r=0}^{j-m} (-)^r \binom{j-m}{m} \binom{j+m}{k-r} (\xi_-)^{2r}, \\
\text{ii.b.} & \quad S_{k,j,m} = (\xi_-)^{-k-j-m} \sum_{r=0}^{j+m} (-)^r \binom{j-m}{2j-k-r} (\xi_-)^{2r},
\end{align*}
\]

(A.44)

(A.45)

(A.46)

\[
\begin{align*}
\text{iii.} & \quad S_{k,j,m} = (\xi_-)^{-k-j-m} \sum_{r=0}^{2j-k} (-)^r \binom{j-m}{2j-k-r} (\xi_-)^{2r}.
\end{align*}
\]

(A.47)
Next we show that these expressions admit a rather non-trivial, hidden symmetry: for given \( k, j, m \) let us define \( \tilde{k} := 2j - k \), and clearly \( k = 0, \ldots, j - |m| - 1 \) precisely when \( \tilde{k} = j + |m| + 1, \ldots, 2j \); and \( k = j - |m|, \ldots, j + |m| \) precisely when \( \tilde{k} = j - |m|, \ldots, j + |m| \). Then e.g. for \( k = 0, \ldots, j - |m| - 1 \) and \( m \geq 0 \) by (A.44) and (A.47) with the notation \( \tilde{r} := r - k + j - m \) we have that

\[
S_{k,j,-m} = (\xi_\cdot)^{-k} \sum_{r=0}^{k} (-1)^r \binom{j + m}{r} \binom{j - m}{k - r} (\xi_\cdot)^{2r} = (\xi_\cdot)^{-\tilde{k} - 2j} \sum_{r=k-j-m}^{j-m} (-1)^r + m - \tilde{k} \binom{j + m}{\tilde{k} - \tilde{r} + j + m} \binom{j - m}{\tilde{r} - j - m} (\xi_\cdot)^{2(\tilde{r} - \tilde{k} - j + m)}
\]

\[
= (-1)^{j + m - \tilde{k}} (\xi_\cdot)^{2m - \tilde{k}} S_{\tilde{k},j,m}.
\]

Analogous calculations in the other cases show that \( S_{k,j,-m} = (-1)^{j + m - \tilde{k}} (\xi_\cdot)^{2m} S_{\tilde{k},j,m} \) holds in general.

This symmetry of \( S_{k,j,m} \) yields that the common denominator in (A.42) and (A.43) is

\[
\sum_{k=0}^{2j} k!(2j-k)!(|S_{k,j,-m}|^2) = |\xi_\cdot|^{4m} \left( \sum_{k=0}^{2j} k!(2j-k)!(|S_{k,j,m}|^2) \right),
\]

while their numerators, respectively, are

\[
\sum_{k=0}^{2j} (j-k)k!(2j-k)!(|S_{k,j,-m}|^2) = -|\xi_\cdot|^{4m} \left( \sum_{k=0}^{2j} (j-k)k!(2j-k)!(|S_{k,j,m}|^2) \right),
\]

\[
\sum_{k=0}^{2j} (j-k)^2k!(2j-k)!(|S_{k,j,-m}|^2) = |\xi_\cdot|^{4m} \left( \sum_{k=0}^{2j} (j-k)^2k!(2j-k)!(|S_{k,j,m}|^2) \right).
\]

These immediately imply that \( \langle W_{s,j,-m}, J_3 W_{s,j,-m} \rangle = -\langle W_{s,j,m}, J_3 W_{s,j,m} \rangle \), as we expected, and

\( \langle J_3 W_{s,j,-m}, J_3 W_{s,j,-m} \rangle = \langle J_3 W_{s,j,m}, J_3 W_{s,j,m} \rangle \).

Next we show that \( \langle J_3 W_{s,j,m}, J_3 W_{s,j,m} \rangle \) is a continuous function of \( \alpha_3 \), although the expectation value \( \langle W_{s,j,m}, W_{s,j,m} \rangle \) has a jump at \( \alpha_3 = 0 \) for \( \lambda > 1 \).

\( \xi_\cdot \), as a function of \( \alpha_3 \) (see (2.17)), is not continuous at \( \alpha_3 = 0 \) for \( \lambda > 1 \): its \( \alpha_3 \to 0 \) limit from the left is \(-i(\lambda + \sqrt{\lambda^2 - 1})\), while from the right it is \(-i(\lambda - \sqrt{\lambda^2 - 1})\). Hence, \( S_{k,j,m} \) is not continuous there. Now we calculate the left limit of \( S_{k,j,m} \), denoted by \( S_{k,j,m}^- \), and relate it to the right limit \( S_{k,j,m}^+ \) of \( S_{k,j,m} \). The key observation is that \( \lambda + \sqrt{\lambda^2 - 1} = (\lambda - \sqrt{\lambda^2 - 1})^{-1} \), but the strategy of the calculation is the same that we followed in the derivation of (A.43). In particular, for \( k = 0, \ldots, j - m - 1 \), \( m \geq 0 \), with the notation \( \tilde{r} := j - m - r \) the main steps are

\[
S_{k,j,m}^- = (-i)^{-k} (\lambda + \sqrt{\lambda^2 - 1})^{-k} \sum_{r=0}^{k} (-1)^r \binom{j + m}{r} \binom{j - m}{k - r} (-i(\lambda + \sqrt{\lambda^2 - 1}))^{2r}
\]

\[
= (-i)^{k-2m} (\lambda - \sqrt{\lambda^2 - 1})^{-k+2m} \sum_{r=j-m}^{k-j-m} (-1)^r \binom{j - m}{\tilde{r}} \binom{j + m}{k - \tilde{r}} (-i(\lambda - \sqrt{\lambda^2 - 1}))^{2\tilde{r}}
\]

\[
= (-1)^{-k} (\lambda - \sqrt{\lambda^2 - 1})^{2m} S_{k,j,m}^+.
\]
\[ \sqrt{\lambda^2 - 1} \] times their \( \alpha_3 \to 0 \) limit from the right, and hence the left and right limits of \( \langle J_3 W_{s,j,m}, J_3 W_{s,j,m} \rangle \) coincide. Similar argumentation confirms that \( \langle W_{s,j,m}, J_3 W_{s,j,m} \rangle \) changes sign at \( \alpha_3 = 0 \) for \( \lambda > 1 \), i.e. it jumps there.

Since \( \langle J_3 W_{s,j,m}, J_3 W_{s,j,m} \rangle \) and \( (\langle W_{s,j,m}, J_3 W_{s,j,m} \rangle)^2 \) are continuous and \( \Delta_\phi J(\alpha) = \lambda \Delta_\phi J(\beta) \) holds, \( \text{(2.31)} \) implies that the standard deviations, both for \( J(\alpha) \) and \( J(\beta) \), are continuous functions of the parameters \( \alpha_3 \) and \( \lambda \).

### A.2.4 The \( \lambda \to 0 \) and \( \lambda \to \infty \) limits of the standard deviations

First consider the \( \lambda \to 0 \) limit. By \( \text{(2.17)} \), in this limit,

\[ \xi_\pm = -\sqrt{1 - \alpha_3^2} + O(\lambda^2) + iO(\lambda). \]

Hence, by \( \text{(A.41)} \), all \( S_{k,j,m} \) are bounded in this limit, implying that \( \langle J_3 W_{s,j,m}, J_3 W_{s,j,m} \rangle = \langle J_3 W_{s,j,m}, J_3 W_{s,j,m} \rangle |_{\lambda=0} + O(\lambda) \). Since, as we saw in subsection \( \text{(2.2)} \) the expectation value is also finite in this limit, the standard deviation for \( J(\beta) \) is finite. Then \( \Delta_\phi J(\alpha) = \lambda \Delta_\phi J(\beta) \) implies that both the standard deviation for \( J(\alpha) \) and the product uncertainty, \( \Delta_\phi J(\alpha) \Delta_\phi J(\beta) \), tend to zero as \( \lambda \to 0 \) limit.

In the \( \lambda \to \infty \) limit, \( \text{(2.17)} \) gives that

\[
\begin{align*}
\xi_- &= -\frac{1}{2} \sqrt{1 - \alpha_3^2} (i \lambda^{-1} + \alpha_3 \lambda^{-2} + O(\lambda^{-3})) \quad \text{if } \alpha_3 > 0, \\
\xi_- &= -\frac{2}{\alpha_3} (i \lambda + |\alpha_3| + O(\lambda^{-1})) \quad \text{if } \alpha_3 \leq 0.
\end{align*}
\]

To calculate the standard deviations in this limit, let us recall from the previous subsection that \( (W_{s,j,-m}, J_3 W_{s,j,m}) = -\langle W_{s,j,m}, J_3 W_{s,j,m} \rangle \) and \( \langle J_3 W_{s,j,-m}, J_3 W_{s,j,-m} \rangle = \langle J_3 W_{s,j,m}, J_3 W_{s,j,m} \rangle \), and hence the standard deviation of \( J_3 \) in the states \( W_{s,j,m} \) and \( W_{s,j,-m} \) is the same. Therefore, it is enough to consider the \( m = 0 \) and \( m > 0 \) disjoint cases.

First suppose that \( m = 0 \). Then, with the notations of the previous subsection, the hidden symmetry \( \text{(A.43)} \) yields that \( S_{k,j,0} = (-)^{k-j} S_{k,j,0} \). Hence the denominator and the numerator, respectively, in \( \text{(A.43)} \) are

\[ (j!)^2 |S_{j,j,0}|^2 + \sum_{k=0}^{j-1} k!(2j - k)! |S_{k,j,0}|^2, \quad 2 \sum_{k=0}^{j-1} (j - k)!^2 |S_{k,j,0}|^2; \]

while the numerator in \( \text{(A.43)} \) is vanishing. Therefore, the square of the standard deviation of \( J_3 \) in the states \( W_{s,j,0} \) is simply the quotient of these two. Thus the only thing to do is to determine the leading order terms of these two expressions in the cases \( \text{(A.50)} \) and \( \text{(A.51)} \). However, in both cases

\[ S_{k,j,0} = (\xi_-)^{-1} \sum_{r=0}^{k} (-)^r \binom{j}{r} \binom{j}{k-r} (\xi_-)^{2r} = \left( \frac{2i}{\sqrt{1 - \alpha_3^2}} \right)^k \binom{j}{k} \lambda^k + O(\lambda^{k-1}), \]

yielding that the order of the leading term of the numerator is \( k = j - 1 \), and in the denominator it is \( k = j \). Therefore,

\[ \langle J_3 W_{s,j,0}, J_3 W_{s,j,0} \rangle = \frac{1}{2} i (j + 1) (1 - \alpha_3^2) \lambda^{-2} + O(\lambda^{-3}); \]

and hence, by \( m = 0 \), the standard deviation for \( J(\beta) \) tends to zero as \( 1/\lambda \), and that for \( J(\alpha) \) remains finite. Thus, by \( \text{(2.31)} \) in the states \( \phi \) with eigenvalue \( m = 0 \), the product uncertainty \( \Delta_\phi J(\alpha) \Delta_\phi J(\beta) = \lambda (\Delta_\phi J(\beta))^2 \) tends to zero as \( 1/\lambda \) in the \( \lambda \to \infty \) limit.
Next suppose that $m > 0$. If $j$ is an integer, then the common denominator in (A.42) and (A.43) is

$$D := \sum_{k=0}^{j-m-1} k!(2j - k)!|S_{k,j,m}|^2 + \sum_{k=j-m}^{j-1} k!(2j - k)!|S_{k,j,m}|^2$$

$$+(j!)^2|S_{j,j,m}|^2 + \sum_{k=j-m+1}^{j+m} k!(2j - k)!|S_{k,j,m}|^2 + \sum_{k=j+m+1}^{2j} k!(2j - k)!|S_{k,j,m}|^2.$$ 

Then, using (A.44), (A.45), (A.47) and (A.50), it is a tedious but routine calculation to determine the leading order terms in $D$ when $\alpha_3 > 0$. It is

$$D = (j - m)(j + m)\left(\frac{2}{1 - \alpha_3^2}\right)^{j+m} \lambda^{2(j+m)} + O(\lambda^{2(j+m-1)}).$$

We obtain the same expression when $j$ is a half-odd-integer. Similar calculations show that the numerators $N_1$ and $N_2$ in (A.42) and (A.43), respectively, are

$$N_1 = -m(j + m)(j - m)\left(\frac{2}{1 - \alpha_3^2}\right)^{j+m} \lambda^{2(j+m)} + O(\lambda^{2(j+m-1)}),$$

$$N_2 = m^2(j + m)(j - m)\left(\frac{2}{1 - \alpha_3^2}\right)^{j+m} \lambda^{2(j+m)} + O(\lambda^{2(j+m-1)}).$$

If $\alpha_3 \leq 0$, then, using (A.51), analogous calculations yield the same expressions for $D$, $N_1$ and $N_2$ except that $m$ should be replaced by $-m$. Hence,

$$\langle W_{s,j,m}, J_3 W_{s,j,m} \rangle = \text{sign}(\alpha_3) m\hbar + O(\lambda^{-2}), \quad \langle J_3 W_{s,j,m}, J_3 W_{s,j,m} \rangle = m^2\hbar^2 + O(\lambda^{-2});$$

which, by (2.31), imply that, in the states $\phi$ with eigenvalue $m > 0$, the standard deviation for $J(\beta)$ tends to zero as $1/\lambda$, and that for $J(\alpha)$ remains finite. These imply that the product uncertainty $\Delta_\phi J(\alpha) \Delta_\phi J(\beta) = \lambda (\Delta_\phi J(\beta))^2$ tends to zero as $1/\lambda$ in the $\lambda \to \infty$ limit.

References

[1] J. R. Klauder, B.-S. Skagerstam, Coherent states, Applications in physics and mathematical physics, in *Coherent States*, World Scientific, Singapore 1985

[2] W.-M. Zhang, D. H. Feng, R. Gilmore, Coherent states: Theory and some applications, Rev. Mod. Phys. 62, 867 (1990)

[3] E. Arthurs, J. L. Kelly, On the simultaneous measurements of a pair of conjugate observables, Bell System Technical J. 44, 725 (1965)

[4] C. Y. She, H. Heffner, Simultaneous measurement of noncommuting observables, Phys. Rev. 152, 1103 (1966)

[5] J. M. Radcliffe, Some properties of coherent spin states, J. Phys. A: Gen. Phys. 4, 313 (1971)

[6] E. H. Lieb, The classical limit of quantum spin systems, Commun. Math. Phys. 31, 327 (1973)

[7] C. Aragone, G. Guerri, S. Salamo, J. L. Tani, Intelligent spin states, J. Phys. A: Math., Nucl. Gen., 7, L149 (1974)

28
[8] X. Wang, B. C. Sanders, S.-h. Pan, Entangled coherent states for systems with $SU(2)$ and $SU(1,1)$ symmetries, J. Phys. A: Math. Gen. 33, 7451 (2000)

[9] M. A. Nielsen, I. L. Chuang, Quantum Computation and Quantum Information, Cambridge Univ. Press, Cambridge 2010

[10] R. Penrose, Combinatorial quantum theory and quantized directions, in Advances in Twistor Theory, Eds. L. P. Houghston, R. S. Ward, Pitman Publ. Ltd, London 1979,
R. Penrose, Angular momentum: An approach to combinatorial spacetime, in Quantum Theory and Beyond, Ed. T. Bastin, Cambridge Univ. Press, Cambridge 1971
R. Penrose, On the nature of quantum geometry, in Magic without Magic, Ed. J. Klauder, Freeman, San Francisco 1972

[11] L. B. Szabados, The ‘most classical’ states of Euclidean invariant elementary quantum mechanical systems, J. Math. Phys. 64 (2023), doi: 10.1063/5.0109613, arXiv: 2111.11876 [quant-ph]
L. B. Szabados, Three-space from quantum mechanics, Foundations of Physics (2022) 52 102, arXiv: 2203.04827 [quant-ph]

[12] R. Penrose, W. Rindler, Spinors and Spacetime, vol 1, Cambridge University Press, Cambridge 1984

[13] S. A. Hugget, K. P. Tod, An Introduction to Twistor Theory, London Mathematical Society Student Texts 4, 2nd edition, Cambridge University Press, Cambridge 1994

[14] M. Eastwood, P. Tod, Edth – a differential operator on the sphere, Math. Proc. Camb. Phil. Soc. 92, 317 (1982)

[15] E. T. Newman, R. Penrose, Note on the Bondi–Metzner–Sachs group, J. Math. Phys. 7, 863 (1966)

[16] J. N. Goldberg, A. J. Macfarlane, E. T. Newman, F. Rochlich, E. C. G. Sudarshan, Spin-spherical harmonics and $\delta$, J. Math. Phys. 8, 2155 (1967)

[17] J. Frauendiener, L. B. Szabados, The kernel of the edth operators on higher-genus spacelike 2-surfaces, Class. Quantum Grav. 18, 1003 (2001), [gr-qc/0010089]

[18] K. P. Tod, (private communication, 2021 January)

[19] S. Stenholm, Simultaneous measurement of conjugate variables, Ann. Phys. (N.Y.) 218, 233 (1992)

[20] M. G. Raymer, Uncertainty principle for joint measurement of noncommuting variables, Am. J. Phys. 62, 986 (1994)

[21] R. F. Streater, A. S. Wightman, PCT, Spin and Statistics, and All That, W. A. Benjamin, INC, New York 1964