Gravitational waves from binary systems in circular orbits: Does the post-Newtonian expansion converge?

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Abstract. Gravitational radiation can be expressed in terms of an infinite series of radiative, symmetric trace-free (STF) multipole moments which can be connected to the behavior of the source. We consider a truncated model for gravitational radiation from binary systems in which each STF mass and current moment of order \( l \) is given by the lowest-order, Newtonian-like \( l \)-pole moment of the orbiting masses; we neglect post-Newtonian corrections to each STF moment. Specializing to orbits which are circular (apart from the radiation-induced inspiral), we find an explicit infinite series for the energy flux in powers of \( v/c \), where \( v \) is the orbital velocity. We show that the series converges for all values \( v/c < 2/e \) when one mass is much smaller than the other, and \( v/c < 4/e \) for equal masses, where \( e \) is the base of natural logarithms. These values include all physically relevant values for compact binary inspiral. This convergence cannot indicate whether or not the full series (obtained from the exact moments) will converge. But if the full series does not converge, our analysis shows that this failure to converge does not originate from summing over the Newtonian part of the multipole moments.

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1. Introduction

The possibility of detection of gravitational waves from inspiralling compact binaries using laser interferometric gravitational-wave observatories such as the U.S. LIGO and the French-Italian VIRGO projects has brought into sharp focus the accuracy of calculations of gravitational waves using approximation methods. The ability to measure the source parameters using matched filtering of theoretical templates against the tens of thousands of cycles observed in, say, a double neutron-star inspiral, assumes that the templates are sufficiently accurate, especially in the evolution of the phase of the waves, that the errors are smaller than the errors arising from noise in the detectors [1, 2]. This may require knowledge of the damping of the orbit via gravitational-radiation reaction (which determines the non-linear evolution of the phase) that incorporates corrections to the lowest-order quadrupole approximation as high as order \((v/c)^6\) [3, 4]. Corrections at \(O((v/c)^4)\) and \(O((v/c)^5)\) have recently been calculated [5, 6, 7, 8, 9].

On the other hand, there is evidence that such post-Newtonian expansions of gravitational radiation (weak-field, slow-motion expansions in powers of \(\frac{\epsilon}{2} = \frac{v}{c} \sim \frac{Gm/rc^2}{c^2} \sim \left(\frac{Gm}{c^2r}\right)^{1/2}\)) do not converge rapidly, if at all. For the case of a test body in circular orbit around a black hole, perturbation calculations carried to very high order in \(v/c\) show slow convergence — the coefficients of successive powers of \(v/c\) in the expansions grow alarmingly quickly [10, 11]. This could call into question the accuracy of any post-Newtonian approximation truncated at a finite order, and by implication, the use of such approximations in templates used in data analysis for LIGO and VIRGO.

To date, no method has been identified to study the convergence properties of the post-Newtonian expansion rigorously and in generality, because of the difficulty in generating higher-order corrections explicitly. In the case of post-Newtonian calculations for systems of arbitrary masses [6, 7], the complexity of the computations grows rapidly with each succeeding order. Although the “third” post-Newtonian order (3PN), corresponding to corrections at \(O((v/c)^6)\) may be achievable, progress beyond that is unlikely. Besides, 3PN order may be adequate from the data-analysis point of view, given the expected noise characteristics of the advanced LIGO/VIRGO detectors, provided one had some understanding of the convergence properties, or could bound the errors neglected.

There is one situation in which the convergence properties can be studied at all orders in \(v/c\). We call this the “bare-multipole truncation”, and although it is unphysical, it may provide clues concerning the convergence of the true (and physical) post-Newtonian expansion. This truncation is best discussed with reference to figure 1.

Gravitational radiation can be expressed in terms of an infinite set of radiative mass and current “symmetric, trace-free (STF)” multipole moments, which can be related to multipole moments of the source [12]. For example, the rate of energy loss \(\dot{E}\) and the gravitational waveform far from the source \(h_{TT}^{ij}\) can be written

\[
\dot{E} = \frac{G}{c^2} \sum_{l=2}^{\infty} \left( \alpha_{l} (l+1) I_{a_{1}...a_{l}(l+1)} I_{a_{1}...a_{l}} + \beta_{l} (l+1) J_{a_{1}...a_{l}(l+1)} J_{a_{1}...a_{l}} \right), \tag{1a}
\]

\[
h_{TT}^{ij} = \frac{G}{Rc^4} \sum_{l=2}^{\infty} \left( \frac{4}{l!} (l)^{i} I_{a_{1}...a_{l-2}} N_{a_{1}...a_{l-2}} + \frac{8l}{(l+1)!} \epsilon_{pq(i} (l)^{j} p_{a_{1}...a_{l-2}} N_{qa_{1}...a_{l-2}} \right)_{TT}. \tag{1b}
\]
with
\[ \alpha_l = \frac{(l+1)(l+2)}{(l-1)!(2l+1)!!}, \quad \beta_l = \frac{4l(l+2)}{(l-1)(l+1)!(2l+1)!!}, \]  
where \( I^{a_1\ldots a_l} \) and \( J^{a_1\ldots a_l} \) are respectively the STF mass and current multipole moments of order \( l \), \( N^{a_1\ldots a_l} \) denotes an \( l \)-dimensional product of unit vectors pointing from the source’s center of mass to the observer at a distance \( R \), \( \varepsilon_{pqj} \) is the completely antisymmetric Levi-Civita symbol, and the subscript “TT” denotes the transverse-traceless part. The indices on the left of the moments denote a number of derivatives with respect to retarded time. Summation over repeated indices is implied.

Each STF multipole moment is then expressed as a post-Newtonian (PN) expansion, effectively in powers of \( \epsilon^{1/2} \), starting with a lowest-order multipole moment given in general by

\[ I^{< l >}_{(0)} = \left( \sum_A m_A x_A^i x_A^j \cdots x_A^l \right)_{\text{STF}}, \] (3a)
\[ J^{< l >}_{(0)} = \left( \varepsilon_{i_1 a b} \sum_A m_A x_A^a x_A^b x_A^i \cdots x_A^l \right)_{\text{STF}}, \] (3b)

where \( m_A, x_A^i, \) and \( v_A^i \) are the suitably defined mass, position, and velocity of each body, and the superscript notation \( < l > \) is short-hand for the \( l \) indices.

Figure 1 represents these STF moments in an array with multipole order \( l \) increasing horizontally beginning with \( l = 2 \) (for each \( l \), \( I^{< l >} \) and \( J^{< l-1 >} \) are grouped together), and with PN order for each multipole moment increasing downward. (Note that \( I^{< 0 >}, I^{< 1 >}, \) and \( J^{< 1 >} \) are related to the non-radiative total mass, center of mass, and total angular momentum of the system.) It is straightforward to show that the first-order PN correction to each multipole moment is \( O(\epsilon) \), followed by a “tail” correction at \( O(\epsilon^3/2) \), then \( O(\epsilon^2) \), and so on. We ignore finite-size effects, such as the effects of spin, in this discussion. There is evidence from black-hole perturbation studies that the PN expansions become non-analytic at high enough order, with the appearance of \( \ln\epsilon \) terms [10, 11].

Because each multipole moment is differentiated with respect to retarded time a number of times corresponding to its multipole order, and because \( d/dt \sim \epsilon^{1/2}/r_A \), a specific moment contributes to \( h_{TT}^{(2)} \) and to \( \dot{E} \) at a PN order related to its multipole order. For example, a determination of \( \dot{E} \) to 2PN order requires knowledge only of the first three columns of figure 1, \( l = 2, 3, 4 \), together with corrections through 2PN order of the quadrupole STF mass moment \( (I^{<2>}) \), the 1PN corrections of the octopole mass and quadrupole current moments \( (I^{<3>}, J^{<2>}) \), and the lowest-order hexadecapole mass moment \( (I^{<4>}) \) and octopole current moment \( (J^{<3>}) \). These correspond to the shaded region in figure 1. For the waveform at 2PN order, on the other hand, the moments and their corrections indicated by the dark border are required. The difference derives from the fact that the waveform is linear in the moments, while the energy flux is quadratic.

The “bare-multipole truncation” consists in keeping only the first row of figure 1, but keeping \( all \) orders in \( l \). This is not a consistent PN approximation to either the waveform or the energy loss. Moreover, this truncation does not correspond to any sort of physical model for the system, such as one in which non-gravitational forces are responsible for keeping the two bodies in orbital motion. This can be seen as follows:
Consider the $O(\epsilon)$ corrections to a particular multipole moment. These contain terms of the form $Gm/rc^2$, which can be thought of as the gravitational contribution to the effective stress-energy tensor, and terms of the form $(v/c)^2$, which can be thought of as special relativistic corrections in the matter contribution to the effective stress-energy tensor. Now, it is clear that the first set of correction terms would be discarded in a non-gravitational model for the source motion, although it would presumably be replaced by something else (related to the stresses responsible for maintaining the orbit). However, it is also clear that the second set of terms cannot be discarded in any sort of physical model for the source. The fact that both sets are discarded in our truncation clearly makes such a procedure unphysical.

Although it is unphysical, the bare-multipole truncation may illustrate some of the convergence properties of the full formulae. For two-body systems, we show that the bare moments have the general form

\[ I^{(l)}_{(0)} = \mu r f_l(\eta) S_l^0|_{TF}, \]
\[ J^{(l)}_{(0)} = \mu v f_{l+1}(\eta) [\hat{L} \circ S_l^{l-1}]_{TF}, \]

(4)

where $\mu = m_1 m_2/(m_1 + m_2)$ is the reduced mass, $\eta = \mu/(m_1 + m_2)$, $S_l^0$ is a symmetrized product of $l$ unit radial vectors directed from body 2 to body 1, $\hat{L}$ is a unit vector in the direction of the orbital angular momentum, $r$ and $v$ are the magnitudes of the relative separation and orbital velocity, respectively, and $\circ$ denotes a symmetrized product. The function $f_l(\eta)$ is given by

\[ f_l(\eta) = \rho^{l-1} + (-1)^l / (1 + \rho)^{l-1}, \]
\[ \rho \equiv m_2/m_1 = \frac{1}{2\eta} \left[ 1 - 2\eta - \sqrt{1 - 4\eta} \right], \]

(5)

(6)

where we choose the convention $m_2 \leq m_1$, so that $0 < \rho \leq 1$.

Specializing to quasi-circular orbits with angular velocity $\Omega \equiv v/r$, we find, after considerable manipulations of STF tensors (Section 3), the closed-form result

\[ \dot{E} = \dot{E}_Q \left\{ 1 + \sum_{l=3}^{\infty} B_l f_l^2(\eta) \left( \frac{v}{c} \right)^{2l-4} \right\}, \]

(7)

where $\dot{E}_Q$ represents the quadrupole approximation,

\[ \dot{E}_Q = \frac{32G}{5c^2} \mu^2 v^4 \Omega^2. \]

(8)

The coefficients $B_l$ are given explicitly by (38). It is interesting to note here that (7) does not depend on any assumption of an equation of motion for the bodies; only the existence of a quasi-circular orbit with angular velocity $\Omega$ is assumed.

A completely equivalent series, applicable in the test mass limit ($\eta \to 0$, $|f_l(\eta)| \to 1$) may be derived using black-hole perturbation theory (see Section 4 for details). The coefficients $B_l$ resulting from that method have a different, albeit numerically equivalent representation, given by (44, 45). Most importantly, they can easily be shown to have the form, for large $l$,

\[ B_l \sim \frac{5}{64} \left( \frac{l}{\pi} \right)^{1/2} \left( \frac{c}{2} \right)^{2l} \left[ 1 + O(l^{-1}) \right], \]

(9)
where $e$ is the base of natural logarithms.

By applying the standard Cauchy ratio test, we find that the radius of convergence of the sequence is given by

$$\left(\frac{v}{c}\right)_{\text{converge}} = \frac{4}{e(1 + \sqrt{1 - 4\eta})}. \quad (10)$$

The series thus converges for values of $v/c$ less than a critical value ranging from 0.74 ($\eta = 0$) to 1.47 ($\eta = 1/4$). These values encompass all physically relevant values in binary inspiral, until the final coalescence phase.

The infinite series (7) can be used to study the error made in truncating the method at a given PN order. (By this we mean a truncation of the sum appearing in Eq. (7) at the value of $l$ corresponding to the specified PN order. For example, a truncation at 2PN order involves keeping only the $l = 3$ and $l = 4$ terms.) Figure 2 shows, as a function of $v$, the fractional difference between a given PN truncation and the full series. Notice that, at $v \approx 0.4$, corresponding to the innermost stable orbit for a test-body orbiting a black hole, the errors at 2PN and 3PN order are 2.1 and 0.6 percent, respectively. For equal masses, odd-numbered mass moments and even-number current moments vanish, thus there are no terms in $\dot{E}$ at odd-PN order in the bare-multipole truncation. This is partly responsible for the improved convergence in the equal-mass case: at $v \approx 0.4$, the fractional difference between the 2PN/3PN approximation and the full series is $2 \times 10^{-5}$.

The remainder of this paper provides details of the calculations. In Section 2, we describe the bare-multipole truncation and set up our definitions and conventions. Section 3 derives the energy flux, and discusses the convergence of the series. In Section 4, we describe the analogous approach using black hole perturbation theory, and find an analytic expression for the radius of convergence. Section 5 gives concluding remarks. A number of technical details are relegated to Appendices.

2. Bare-multipole truncation for gravitational radiation

We describe the motion of a system of two bodies whose size is negligible compared to their relative separation. We work in a coordinate system whose origin is the center of mass, and define the relative position vector $\mathbf{x} \equiv \mathbf{x}_1 - \mathbf{x}_2$, where $\mathbf{x}_1$ and $\mathbf{x}_2$ are the positions of each body. If the masses of the bodies are $m_1$ and $m_2$, we define $m \equiv m_1 + m_2$, $\mu \equiv m_1 m_2/m$, $\eta \equiv \mu/m$, and $\delta m \equiv m_1 - m_2$. Also, $r \equiv |\mathbf{x}|$ and $v \equiv \dot{r}$.

Henceforth, we use units in which $G = c = 1$.

The radiative multipole moments in (1a) and (1b) are expanded in powers of $\epsilon^{1/2} \sim v \sim (m/r)^{1/2} \sim \dot{r}$, in the generic form

$$I^{<l>} \sim I^{<l>}_{(0)}(1 + \epsilon + \epsilon^{3/2} + \epsilon^2 + \ldots), \quad (11)$$

with a similar expansion for $\mathbf{J}^{<l>}$, where the $\epsilon^{3/2}$ contribution signals the first appearance of the effects of radiative “tails”, caused by backscatter of the waves off the background spacetime curvature. In calculating $\dot{E}$, one takes each multipole moment of order $l$, differentiates it $l + 1$ times with respect to retarded time, and squares the result (contracting on all indices). Because the $l$-pole moment is differentiated once more than the $(l - 1)$-pole moment, its leading contribution to $\dot{E}$ is of order $v^2$ higher than that of the $(l - 1)$-pole moment. Additionally, the current multipole moments are proportional to the angular momentum of the system, through a factor $\varepsilon_{iab} x^i v^b$. 


which adds an extra velocity $v^b$ to their expressions; consequently the contribution of a current multipole moment is $O(v^2)$ higher than that of the mass multipole moment of the same rank. As a result, (1a) is a post-Newtonian expansion of the luminosity in powers of $e$. Similar considerations apply to the expressions for the gravitational waveform.

Explicit expressions for the radiative mass and current multipole moments for general two-body systems are now known sufficiently accurately to calculate the energy flux to 5/2PN order beyond the quadrupole approximation [10], and the waveform to 2PN order [11]. By way of illustration, we quote here the moments sufficient to calculate the 2PN energy flux [11,12]:

$$I^{ij} = \mu \left(1 + \frac{29}{42} (1 - 3\eta) v^2 - \frac{1}{7} (5 - 8\eta) \left(\frac{m}{r}\right) x^i x^j \right.$$  
$$- \frac{4}{7} (1 - 3\eta) r \dot{r} x^i v^j + \frac{11}{21} (1 - 3\eta) r^2 v^i v^j$$  
$$+ x^i x^j \left[\frac{1}{504} (253 - 1835\eta + 3545\eta^2) v^4 + \frac{1}{756} (2021 - 5947\eta - 4883\eta^2) v^2 \left(\frac{m}{r}\right) \right.$$  
$$- \frac{1}{252} (355 + 1906\eta - 337\eta^2) \left(\frac{m}{r}\right)^2 - \frac{1}{756} (131 - 907\eta + 1273\eta^2) v^2 \left(\frac{m}{r}\right) \right]$$  
$$+ r^2 v^i v^j \left[\frac{1}{189} (742 - 335\eta - 985\eta^2) \left(\frac{m}{r}\right) + \frac{1}{126} (41 - 337\eta + 733\eta^2) v^2 \right.$$  
$$+ \frac{5}{63} (1 - 5\eta + 5\eta^2) v^2$$  
$$- r \dot{r} x^j \left[\frac{1}{378} (1085 - 4057\eta - 1463\eta^2) \left(\frac{m}{r}\right) + \frac{1}{63} (26 - 202\eta + 418\eta^2) v^2 \right]\right\}_{STF}$$  

$$+ I^{ij}_{tail},$$  

$$I^{ijk} = -\mu \frac{\delta m}{m} \left[1 + \frac{1}{6} (5 - 19\eta) v^2 - \frac{1}{6} (5 - 13\eta) \left(\frac{m}{r}\right) \right] x^i x^j x^k$$  
$$+ (1 - 2\eta) \left(r^2 v^i v^j x^k - r \dot{r} v^i x^j x^k \right) \right\}_{STF},$$  

$$I^{ijkl} = \mu \left(1 - 3\eta\right) \left(x^i x^j x^k x^l\right)_{STF},$$  

$$J^{ij} = -\mu \frac{\delta m}{m} \left[\varepsilon_{iab} \left[1 + \frac{1}{2} (1 - 5\eta) v^2 + 2 (1 + \eta) \left(\frac{m}{r}\right) \right] x^i x^a v^b \right.$$  
$$+ \frac{1}{28} d \left[ (1 - 2\eta)(3r^2 v^i - r \dot{r} x^i) x^a v^b \right]\right\}_{STF}$$  

$$J^{ijk} = \mu \left(1 - 3\eta\right) \left(\varepsilon_{iab} x^a v^b x^j x^k\right)_{STF}. $$  

(12)

Explicit expressions for the “Tail” term may be found in [11,12], for example.

We now specialize to quasi-circular orbits, i.e. orbits which are circular, apart from the slow inspiral caused by radiation damping (at sufficiently high PN order the non-circularity of the orbits must be taken into account). We thus approximate: \( \dot{r} = 0, \) \( x = r \hat{n}, \) \( v = \nu \hat{\lambda}, \) where \( \hat{n} \) and \( \hat{\lambda} \) are unit vectors in the radial and tangential directions, respectively, and \( \hat{n} \times \hat{\lambda} = \hat{L} \). We define the following symmetric tensors (see Appendix A):
where the sum is over all possible distinct permutations \( \mathcal{N} = \binom{I}{k} \) of the set of indices \( \{i_j\} \). The STF tensors \( I^{<l>} \) and \( J^{<l>} \) can then be expressed in terms of these symmetric tensors, and the post-Newtonian terms can be converted, via the appropriate equations of motion, into terms involving \( v \) alone. The result is, schematically,

\[
I^{<2>} = \mu^2 \left[ (1 + \alpha_1 v^2 + \alpha_2 v^4) S_0^2 + v^2 (1 + \alpha_3 v^2) S_2^{[STF]} \right] + I^{<2>}_{\text{Tail}},
\]

\[
I^{<3>} = \mu^3 \left[ \left( -\frac{\delta m}{m} \right) \left[ (1 + \beta_1 v^2) S_0^3 + \beta_2 v^2 S_2^{[STF]} \right] \right],
\]

\[
I^{<4>} = \mu^4 \left[ \gamma_1 S_0^4 \right]^{[STF]},
\]

\[
J^{<2>} = \mu^2 v \left[ \left( -\frac{\delta m}{m} \right) (1 + \delta_1 v^2) \left[ L \circ S_0^2 \right]^{[STF]} \right],
\]

\[
J^{<3>} = \mu^3 v \varepsilon_1 \left[ L \circ S_0^2 \right]^{[STF]},
\]

where the coefficients \( \alpha_i, \beta_i, \) etc., result from combining expressions in (12), and depend only on \( \eta \); \( \circ \) denotes a symmetrized product, defined by

\[
\hat{L} \circ S_0^{T-1} \equiv \hat{L}^{(i_1 \cdot \ldots \cdot i_T)} \equiv \frac{1}{T} \sum_{\Pi_T} \hat{L}^{i_1} \hat{n}^{i_2} \ldots \hat{n}^{i_T},
\]

where \( \Pi_T \) are all the \( T \) distinct permutations of the indices \( \{i_j\} \). We now restrict these expressions to the leading order in \( v \), and denote them the “bare” multipole moments \( I^{<l>}_{(0)} \) and \( J^{<l>}_{(0)} \), as follows

\[
I^{<2>}_{(0)} = \mu^2 S_0^2|^{[STF]} \equiv \mu^2 f_2(\eta) S_0^2|^{[STF]},
\]

\[
I^{<3>}_{(0)} = \mu^3 \left[ \left( -\frac{\delta m}{m} \right) S_0^3 \right]^{[STF]} \equiv \mu^3 f_3(\eta) S_0^3|^{[STF]},
\]

\[
I^{<4>}_{(0)} = \mu^4 (1 - 3\eta) S_0^4|^{[STF]} \equiv \mu^4 f_4(\eta) S_0^4|^{[STF]},
\]

\[
J^{<2>}_{(0)} = \mu^2 v \left[ \left( -\frac{\delta m}{m} \right) \left[ L \circ S_0^2 \right]^{[STF]} \right] \equiv \mu^2 v f_3(\eta) \left[ L \circ S_0^2 \right]^{[STF]},
\]

\[
J^{<3>}_{(0)} = \mu^3 v (1 - 3\eta) \left[ L \circ S_0^2 \right]^{[STF]} \equiv \mu^3 v f_4(\eta) \left[ L \circ S_0^2 \right]^{[STF]},
\]

where the function \( f_l(\eta) \) is given by (6) and is plotted in figure 3 for different values of \( \eta \) within the possible range \( (0, 0.25) \). Note that \( |f_l(\eta)| < 1 \), for all \( \eta \neq 0 \); \( |f_l(0)| = 1 \), for all \( l \) and \( f_l(0.25) = 0 \) for odd \( l \) (see Appendix B). Equations (16) suggest, and Appendix B confirms, that the general forms for \( I^{<l>}_{(0)} \) and \( J^{<l>}_{(0)} \) at leading order are given by (4). Notice that the original expressions for \( I^{<l>}_{(0)} \) and \( J^{<l>}_{(0)} \) were expressed in harmonic coordinates. However, these expressions are also valid in Schwarzschild coordinates, as the transformation between coordinate systems, both for \( r \) and \( v \), introduces terms that contribute only higher-order corrections.
Finally, we explicitly extract the traces in the previous expressions, and get $I^{<l>}_{(0)}$ and $J^{<l>}_{(0)}$ in terms of the symmetric tensors $S_k^{l-k}$ only,

$$I^{<l>}_{(0)} = \mu r f_1(\eta) \sum_{j=0}^{[\frac{l}{2}]} \frac{(-1)^j(j)! (l-j)!}{(2j)!} \left(D^{(j)} \circ S_0^{l-2j}\right),$$  

(17a)

$$J^{<l>}_{(0)} = \mu r v f_{l+1}(\eta) \sum_{j=0}^{[\frac{l}{2}]} \frac{(-1)^j(j)! (l-j)! (l-2j)}{l} \left(D^{(j)} \circ (\hat{L} \circ S_0^{l-2j-1})\right).$$  

(17b)

Here $[a/b]$ denotes the integer part of $a/b$, and $D^{(j)}$ is a symmetrized product of $j$ Kronecker deltas, given by

$$D^{(j)} = \delta^{i_1 i_2} \delta^{i_3 i_4} \ldots \delta^{i_{2j-1} i_{2j}} = \frac{1}{\mathcal{N}(j,2)} \sum_{\Pi_{\mathcal{N}(j,2)}} \delta^{i_1 i_2} \delta^{i_3 i_4} \ldots \delta^{i_{2j-1} i_{2j}}$$  

(18)

where

$$\mathcal{N}(j,2) = \frac{1}{j!} \left(\begin{array}{c} 2j \\ 2 \end{array}\right) \left(\begin{array}{c} 2j-2 \\ 2 \end{array}\right) \ldots \left(\begin{array}{c} 2 \\ 2 \end{array}\right) = (2j-1)!!$$  

(19)

is the number of distinct products of deltas. The factor $(l-2j)/l$ in (17b) comes from taking into account the fact that the unit vector $\hat{L}$ is perpendicular to $\hat{n}$ when applying traces to the tensor $\hat{L} \circ S_0^{l-2j-1}$.

Notice that each term in (17a) and (17b) has rank $l-2j$, varying for each $j$ in the sum. This fact complicates considerably the process of multiplying two of these objects, as prescribed in (1a). In the next section we will describe how to simplify this calculation.

3. Energy flux in the bare-multipole truncation

We now express the energy flux approximately in terms of the bare multipole moments

$$\dot{E} \approx \sum_{l=2}^{\infty} \alpha_l (l+1) I^{<l>}_{(0)} \cdot (l+1) I^{<l>}_{(0)} + \beta_l (l+1) J^{<l>}_{(0)} \cdot (l+1) J^{<l>}_{(0)},$$  

(20)

where the notation $(\cdot)$ stands for the inner product that saturates all the indices $\{i_l\} \equiv \{i_1, i_2, \ldots, i_l\}$ of the two tensors. As we are dealing henceforth with bare multipole moments, we drop the subscript $(0)$ notation.

In order to simplify the product of derivatives in each of the sums of (20), we first write:

$$(l+1) I^{<l>} \cdot (l+1) I^{<l>} = \frac{d}{dt} \left[I^{<l>} \cdot (l+1) I^{<l>}\right] - (l) I^{<l>} \cdot (l+2) I^{<l>}$$

$$= \frac{d}{dt} \left[\begin{array}{c} (l+1) I^{<l>} \cdot (l) I^{<l>} \\ (l+2) I^{<l>} \cdot (l) I^{<l>} \end{array}\right] - (l+1) I^{<l>} \cdot (l+3) I^{<l>}$$

$$= \frac{d}{dt} \left[\begin{array}{c} (l+1) I^{<l>} \cdot (l+2) I^{<l>} \\ (l+3) I^{<l>} \cdot (l+1) I^{<l>} \end{array}\right]$$

$$= \frac{d}{dt} \left[\begin{array}{c} (l+1) I^{<l>} \cdot (l+3) I^{<l>} \\ (l+3) I^{<l>} \cdot (l+1) I^{<l>} \end{array}\right]$$

$$\ldots$$

$$= \frac{d}{dt} \left[\begin{array}{c} (l+1) I^{<l>} \cdot (l+3) I^{<l>} \\ (l+3) I^{<l>} \cdot (l+1) I^{<l>} \end{array}\right].$$  

(21)

The first term is the total derivative of several terms, each of which is the product of an even and an odd number of derivatives of \( I^{(l)} \), respectively. Since each time derivative converts \( \mathbf{n} \) into \( \mathbf{\lambda} \), each term is thus a product of a tensor with an even number of \( \mathbf{\lambda} \) unit vectors by a tensor with an odd number of \( \mathbf{\lambda} \)’s; consequently, every term in the bracket is null, because \( \mathbf{n} \cdot \mathbf{\lambda} = 0 \). The same can be done with the magnetic terms, so that we may write

\[
(l+1) I^{(l)} , (l+1) J^{(l)} = (−1)^{l+1} I^{(l+2)} , (l+1) J^{(l)} = (−1)^{l+1} J^{(l+2)}.
\]

\[
(I+1), (l+1) I^{(l)} , (l+1) J^{(l)} = (−1)^{l+1} I^{(l+2)} , (l+1) J^{(l)} = (−1)^{l+1} J^{(l+2)}.
\]

Now, \( I^{(l)} \) and \( J^{(l)} \) are trace-free tensors, as are their derivatives. But the terms in \( I^{(l)} \) that contain Kronecker deltas, such as \( δ^{i_1i_2}(Rest)^{i_3\cdots i_l} \), are just trace operators on the subspace \( \{ i_1, i_2 \} \); so their product with the time-differentiated STF tensor \( (2l+2) I^{(l)} \) will vanish. Thus, the only non-vanishing term is the symmetric product of \( l \) unit vectors \( [\mathbf{n}^{i_1} \mathbf{n}^{i_2} \cdots \mathbf{n}^{i_l}] \) multiplied by \( (2l+2) I^{(l)} \). A similar argument applies to the current moments. The result is

\[
(l+1), (l+1) I^{(l)} , (l+1) J^{(l)} = (−1)^{l+1} \mu r^l \left[ \mathbf{n}^{i_1} \mathbf{n}^{i_2} \cdots \mathbf{n}^{i_l} \right] , (2l+2) I^{(l)} = (−1)^{l+1} \mu r^l S_0^{(l)} , (2l+2) I^{(l)} = (−1)^{l+1} \mu r^l \left[ \mathbf{\hat{L}}^{(i_1i_2i_3\cdots i_l)} \right] , (2l+2) J^{(l)} = (−1)^{l+1} \mu r^l \left[ \mathbf{\hat{L}} \circ S_0^{(l)} \right] .
\]

We now need an expression for the \((2l+2)\)-time-derivative of \( I^{(l)} \) and \( J^{(l)} \). These tensors \((17a, 17b)\) are sums of products of constants with respect to time (like \( D^{(j)} \) and \( \mathbf{\hat{L}} \)) with a symmetric tensor \( S_0^{(l)} \), which has only radial components. We will concentrate first on calculating the \( 2p = 2l + 2 \) derivatives of a symmetric tensor \( S_0^T \) where \( T = l - 2j = l, l - 2, l - 4, \cdots \).

The derivative of any symmetric product of unit vectors, radial and/or tangential is (see Appendix A.2)

\[
\frac{d}{dt} \left( S_{k}^{(l-k)} \right) = \Omega \left[ (l-k) S_{k+1}^{(l-k-1)} - k S_{k-1}^{(l-k+1)} \right].
\]

Thus, even if we have only radial components in \( S_{k}^T \), tangential components will appear after differentiation. Notice that \((24)\) is the same rule as the derivative of a product of \( k \) cosine and \( l-k \) sine functions

\[
\frac{d}{dx} \left( \sin^{l-k} \Omega x \cos^{k} \Omega x \right) = \Omega \left[ (l-k) \left( \sin^{l-k-1} \Omega x \cos^{k+1} \Omega x \right) - k \left( \sin^{l-k+1} \Omega x \cos^{k-1} \Omega x \right) \right].
\]

We may thus simplify the calculation considerably by associating each symmetric tensor with a product of sines and cosines in the form

\[
S_{k}^{(l-k)} \leftrightarrow \sin^{l-k} \Omega x \cos^{k} \Omega x.
\]

Any higher-order derivative of \( S_{k}^{(l-k)} \), expanded in terms of others \( S_{k'}^{(l-k')} \), will have the same coefficients as the corresponding derivative of the analogous product of sines.
and cosines, expanded in products of sines and cosines, provided that we can express such a derivative in terms of the equivalent products of sines and cosines (in Appendix A.3 we show that there is a subtlety in this procedure).

We are interested in the $(2p)$-derivative of $S_T^0$, which we associate with $\sin^T \Omega x$, with $T = l, l - 2, l - 4, \ldots$. To get a closed expression, we first expand $\sin^T \Omega x$ as a sum of sines of multiple angles [13]:

\[
\sin^T \Omega x = \frac{1}{2^{T-1}} \sum_{k=0}^{\left\lfloor \frac{T-1}{2} \right\rfloor} (-1)^{\frac{q}{2}} \binom{T}{k} \cos \Omega(T - 2k)x + \frac{1}{2} \left( \frac{T}{T} \right), \quad \text{if } T = 2n,
\]

\[
\sin^T \Omega x = \frac{1}{2^{T-1}} \sum_{k=0}^{\left\lfloor \frac{T-1}{2} \right\rfloor} (-1)^{\frac{q}{2}} \binom{T}{k} \sin \Omega(T - 2k)x, \quad \text{if } T = 2n - 1. \quad (27)
\]

We then take the $2p$ derivative of (27):

\[
(2p)(\sin^T \Omega x) = \frac{(-1)^p \Omega^{2p}}{2^{T-1}} \sum_{k=0}^{\left\lfloor \frac{T-1}{2} \right\rfloor} (-1)^{\frac{q}{2}} \binom{T}{k} (T - 2k)^{2p} \cos \Omega(T - 2k)x, \quad \text{if } T = 2n,
\]

\[
(2p)(\sin^T \Omega x) = \frac{(-1)^p \Omega^{2p}}{2^{T-1}} \sum_{k=0}^{\left\lfloor \frac{T-1}{2} \right\rfloor} (-1)^{\frac{q}{2}} \binom{T}{k} (T - 2k)^{2p} \sin \Omega(T - 2k)x, \quad \text{if } T = 2n - (28)
\]

Now we can express $\cos \Omega(T - 2k)x$ and $\sin \Omega(T - 2k)x$ back in terms of powers of sines and cosines, to recover terms analogous to the symmetric products $S_k^{(l)}$:

\[
\cos \Omega(T - 2k)x = \sum_{q=0}^{\left\lfloor \frac{T-1}{2} \right\rfloor} (-1)^{q} \binom{T-2k}{2q} \sin^{2q} \Omega x \cos^{(T-2k-2q)} \Omega x, \quad \text{if } T = 2n,
\]

\[
\sin \Omega(T - 2k)x = \sin \Omega x \sum_{q=0}^{\left\lfloor \frac{T-1}{2} \right\rfloor} (-1)^{q} \binom{T-2k}{2q+1} \sin^{2q} \Omega x \cos^{(T-2k-2q-1)} \Omega x, \quad \text{if } T = 2n \quad (29)
\]

Combining (28) and (29), one can get the $2p$-derivative of a product of $T$ sines. The $2p$-derivative of $S_T^0$ will then have a similar expansion, with identical coefficients:

\[
(3p)(S_T^0) = \frac{(-1)^p \Omega^{2p}}{2^{T-1}} \sum_{k=0}^{\left\lfloor \frac{T-1}{2} \right\rfloor} (-1)^{\frac{q}{2}} \binom{T}{k} (T - 2k)^{2p} \]

\[
\times \sum_{q=0}^{\left\lfloor \frac{T-1}{2} \right\rfloor} (-1)^{q} \binom{T-2k}{2q} S_{T-2k-2q}^{2q}, \quad \text{if } T = 2n, \quad (30a)
\]

\[
(2p)(S_T^0) = \frac{(-1)^p \Omega^{2p}}{2^{T-1}} \sum_{k=0}^{\left\lfloor \frac{T-1}{2} \right\rfloor} (-1)^{\frac{q}{2}} \binom{T}{k} (T - 2k)^{2p} \]

\[
\times \sum_{q=0}^{\left\lfloor \frac{T-1}{2} \right\rfloor} (-1)^{q} \binom{T-2k}{2q+1} S_{T-2k-2q-1}^{2q+1}, \quad \text{if } T = 2n - 1. \quad (30b)
\]
However, this expansion of \((2p)\langle S_0^T \rangle\) is not yet what we need. Notice that the derivative rule (24) preserves the rank of the tensors involved: \(\text{rank}(S_{k-1}^{l-k}) = \text{rank}(S_{k+1}^{l-k+1})\). This is no longer true for (30a, 30b), where each term is of rank \(T = 2k\), different for each \(k\). This is an artifact created by expanding the \(\sin^T \Omega x\) in terms of sines and cosines of multiple angles, which we resolve in Appendix A.3. The product of these expressions with \(S_0^{l-k}\) or with \(\hat{\mathbf{L}} \circ S_0^{l-k-1}\) in (23), selects only the terms of the form \(S_0^l\) in equations (30a, 30b). The expressions simplify considerably (see Appendix A.3), giving the same result for even or odd \(T\). Using (A23), we get the following product, valid for any \(T, T'\) (even or odd) and for any \(p\):

\[
S_0^{l-k} \cdot \langle 2p \rangle S_0^T = A(T, p) \left( S_0^{T'} \cdot S_0^T \right),
\]

where

\[
A(T, p) = \frac{(-1)^p \Omega^{2p}}{2^{T-1}} \sum_{k=0}^{\frac{T-1}{2}} \binom{T}{k} (T - 2k)^{2p}.
\]

As the angular momentum \(\hat{\mathbf{L}}\) or the Kronecker delta operators \(D^{(j)}\) are constants with respect to time derivatives, we also find that

\[
S_0^{T'} \cdot \left[ D^{(j)} \circ \langle 2p \rangle S_0^T \right] = A(T, p) \left( S_0^{T'} \cdot \left[ D^{(j)} \circ S_0^T \right] \right),
\]

\[
\left[ \hat{\mathbf{L}} \circ S_0^T \right] \cdot \left[ D^{(j)} \circ \left[ \hat{\mathbf{L}} \circ \langle 2p \rangle S_0^T \right] \right] = A(T, p) \left( \left[ \hat{\mathbf{L}} \circ S_0^{l-k} \right] \cdot \left[ D^{(j)} \circ \left[ \hat{\mathbf{L}} \circ S_0^T \right] \right] \right).
\]

With these simplifications we can now express the luminosity in terms of products of symmetric tensors containing only radial components. This makes it possible to calculate these products in closed form, for any multipole order, \(l\). Using the previous expressions and the following rules (see Appendix A.2):

\[
S_0^l \cdot \left[ D^{(j)} \circ S_0^{l-k} \right] = \frac{1}{7},
\]

the products (23), with \(T = l - 2j, 2p = 2l + 2\) and \(v = \Omega r\) are

\[
\begin{align*}
\langle l+1 \rangle \langle j+1 \rangle & = 2^{-(l-1)} \mu^2 v^2 \Omega^2 f_l^2(\eta) C_l, \\
\langle l+1 \rangle \langle j \rangle & = 2^{-(l-2)} \mu^2 v^2 \Omega^2 f_l^2(\eta) D_l,
\end{align*}
\]

where \(C_l\) and \(D_l\) are given by

\[
C_l = \sum_{j=0}^{l-2j} \frac{(-1)^j 2^{2j} \binom{l}{j} \binom{l}{2j}}{2^{2j}} \sum_{k=0}^{\frac{l-1}{2}} \frac{(l - 2j)(l - 2k)^2}{k},
\]

\[
D_l = \sum_{j=0}^{l-2j} \frac{(-1)^j 2^{2j} \binom{l}{j} \binom{l}{2j}}{2^{2j}} \sum_{k=0}^{\frac{l-1}{2}} \frac{(l - 2j)(l - 1 - 2j)(l - 1 - 2k)^2}{k}.
\]

Finally, the luminosity in terms of \(C_l\), \(D_l\), and \(f_l(\eta)\) is

\[
\dot{E} = \dot{E}_0 \left( 1 + \sum_{l=3}^{\infty} B_l f_l^2(\eta) v^{2l-4} \right),
\]
where \( \dot{E}_Q = \frac{32 \pi^2}{3} \mu^2 v^4 \Omega^2 \) is the luminosity due to the quadrupole term and \( B_l \) is given by

\[
B_l = \frac{5}{16} \frac{(l + 1)(l + 2)}{l(l - 1)(2l + 1)!} \left[ C_l + \frac{16(2l + 1)l}{(l - 2)(l + 2)} D_l \right].
\]

(38)

To study the convergence of this series, we apply the standard Cauchy ratio test, requiring that

\[
\lim_{l \to \infty} \frac{B_{l+1} f_{l+1}^2(\eta)}{B_l f_l^2(\eta)} v^2 < 1.
\]

(39)

Evaluating this numerically up to \( l = 250 \), we find the constraint \(|f_{l+1}(\eta)/f_l(\eta)|v \to (1 + \sqrt{1 - 4\eta})v/2 < 0.74\). The case \( \eta = 0.25 \) must be treated separately, since \( f_l(0.25) = 0 \) for odd \( l \). For this case, the Cauchy ratio test takes the form \(|f_{l+2}/f_l|^{1/2} v \to v/2 < 0.74\), which is the continuous limit of the previous statement. Note that, for a given \( v \), the convergence is worst for \( \eta = 0 \) (test mass limit) and best for \( \eta = 1/4 \) (equal masses).

4. Black-hole perturbation theory and the bare-multipole truncation

In this section we consider the calculation of the bare multipole moments in the restricted context of a binary system with small mass ratio. We shall therefore demand \( \eta \ll 1 \) throughout this section. The method of calculation used here is completely different from the one used in the previous two sections.

For the problem considered in this paper, the internal structure of the orbiting masses is of no consequence. For simplicity, in this section we take the larger mass \( m_1 \) to be a nonrotating black hole. The smaller mass \( m_2 \) then creates a small perturbation in the gravitational field of the black hole. This perturbation propagates away from the source as a gravitational wave. By virtue of the small mass approximation, the gravitational perturbations are accurately described by solving a linear wave equation on the Schwarzschild spacetime. This equation is called the Teukolsky equation [14], and it can be solved exactly, for example, using numerical methods [10].

We assume that the mass \( m_2 \) moves on a circular orbit with radius \( r \), such that \( v \equiv (m/r)^{1/2} \) is much smaller than unity. (Here, \( m = m_1 + m_2 \approx m_1 \) is the total mass.) In this limit the Teukolsky equation can be solved analytically, and the bare multipole moments of the radiative field can be evaluated. Such a calculation was first carried out by Poisson [13], and then completed by Poisson and Sasaki [14], hereafter referred to as PS.

The Teukolsky equation is analyzed by first separating the variables. This means that the radiative field is expressed not in terms of symmetric trace-free tensors, but in terms of (spin-weighted) spherical harmonics [17]. The two representations are entirely equivalent [12], and in both cases the gravitational-wave luminosity takes the form of (7). Here, of course, \( \eta \to 0 \) so that \(|f_l(\eta)| = 1\).

The bare multipole moments were calculated explicitly by PS, who express the gravitational-wave luminosity as

\[
\dot{E} = \dot{E}_Q \left( 1 + \sum_{l=3}^{\infty} \sum_{m=1}^{l} \eta_l m \right).
\]

(40)
apart from a slight change of notation; cf. their equation (5.14). The contributions \( \eta_{lm} \) take different forms according to whether \( l + m \) is odd or even:

\[
\eta_{lm} = \begin{cases} 
p_{lm} v^{2l-4} & \text{for } l + m \text{ even} \\
q_{lm} v^{2l-2} & \text{for } l + m \text{ odd}.
\end{cases}
\] (41)

The coefficients \( p_{lm} \) and \( q_{lm} \) are explicitly given in equations (5.15) and (5.16) of PS. Simple manipulations bring (40) into the form of (7), which we write as

\[
\dot{E} = \dot{E}_Q \left( 1 + \sum_{l=3}^{\infty} B_l v^{2l-4} \right). \tag{42}
\]

We find

\[
B_l = p_{ll} + \sum_{n=0}^{N} (p_{lm} + q_{l-1,m}). \tag{43}
\]

Here, \( m \) is to be considered to be a function of \( n \): for odd \( l \), \( m = 2n + 1 \) and \( N = (l - 3)/2 \); for even \( l \), \( m = 2n + 2 \) and \( N = (l - 4)/2 \). This ensures that the sum is properly restricted to values of \( m \) such that \( l + m \) is even. The largest value of \( m \) contributing to the sum is therefore \( l - 2 \).

Some algebra, using the explicit expressions for \( p_{lm} \) and \( q_{lm} \) provided by PS, converts (43) into

\[
B_l = \frac{5(l + 1)(l + 2)}{16(2l + 1)!((l - 1)!)} \left( 1 + b_l \right)^{2l+2},
\] (44)

where

\[
b_l = \frac{l^2}{(2l)!^{2l+2}} \sum_{n=0}^{N} \frac{m^{2l+2}(l - m)! (l + m)!}{[(l - m)/2]^2[(l + m)/2]^2} \left[ 1 + \frac{4(2l - 1)(2l + 1)(l - m)(l + m)}{(l - 2)(l + 2)m^2} \right]. \tag{45}
\]

This expression for \( B_l \) is entirely equivalent to that given in (38). We have indeed verified that these \( B_l \)'s are numerically equal to the ones derived in section 3. However, we have not been able to establish their equality algebraically.

We now wish to examine the convergence of the sequence \( B_l v^{2l} \). This analysis will greatly benefit from the simplicity of our current expression for \( B_l \). We point out first that the sequence \( b_l \) converges. This was established by numerical experiment, which also reveals that \( b_{\infty} \simeq 0.01 \). This implies that the behavior of \( B_l \) as \( l \to \infty \) is determined entirely by the factor \( (2l+2)/(2l+1)! \) in (44). Writing \( (2l+1)! \) as \((2l)!^2 \Gamma(2l)(1 + O(1/l)) \) and using the Stirling approximation \[18\]

\[
\Gamma(z) = (2\pi)^{1/2}e^{-z}z^{-1/2} \left[ 1 + O(1/z) \right], \tag{46}
\]

we quickly arrive at the asymptotic form

\[
B_l = \frac{5(1 + b_{\infty})}{64} \left( \frac{l}{\pi} \right)^{1/2} \left( \frac{e}{2} \right)^{2l} \left[ 1 + O(l^{-1}) \right]. \tag{47}
\]

We therefore find that the sequence \( B_l v^{2l} \) behaves asymptotically as \((ev/2)^{2l}\). A direct application of the Cauchy ratio test then reveals that the sequence converges provided

\[
v < 2/e \simeq 0.7358. \tag{48}
\]

The generalization of this result, appropriate for the case of nonvanishing mass ratios, was given in section 1.
5. Concluding remarks

We have shown that a truncated model for gravitational radiation from binary systems in circular orbits converges for values of the orbital velocity that encompass all inspirals of physical interest. However, our model is admittedly non-physical, and may only be revealing that, whatever poor convergence properties have been seen to date in the PN expansion, they do not arise from summing over the Newtonian part of the multipole moments, but arise instead from the PN corrections to the moments.

Figure 4 illustrates the limitations of our truncation and reinforces the notion that it is unphysical. Shown is a comparison, in the test body limit, between our bare-multipole series and the physical results of black-hole perturbation theory, including both the numerical “exact” results [4] and the true PN series, accurate to 3.5PN order [11]. It is apparent that the truncated model compares rather poorly; this is mostly due to the fact that while our series is a sum of positive terms, the true PN series is alternating. Nevertheless, our analysis can be improved somewhat. The lowest-order tail corrections to each STF moment can be written down explicitly, and could thus be added in a straightforward way. It is possible that the \( O(\epsilon) \) corrections to each moment could also be calculated without too much difficulty. Whether such a “dressed-multipole” truncation shows better agreement with the exact results — and still converges — will be a subject for future work.

Appendix A. Symmetric products of unit vectors \( S^{l-k}_{k} \)

The multipole moments in (12) can be expressed in terms of products of components of unit vectors,

\[
(P^{l-k}_{k})^{i_1 \cdots i_l} \equiv \hat{\lambda}^{i_1} \hat{\lambda}^{i_2} \cdots \hat{\alpha}^{i_k} \hat{n}^{i_{k+1}} \hat{n}^{i_{k+2}} \cdots \hat{n}^{i_l}.
\]  

Here \( k \) components are in the tangential direction to the circular orbit, \( \hat{\lambda}^i \), and \( (l-k) \) are in the radial direction, \( \hat{n}^i \), with \( \hat{n} \cdot \hat{n} = \hat{n}^i \hat{n}^i = 1 \), \( \hat{\lambda} \cdot \hat{\lambda} = \hat{\lambda}^i \hat{\lambda}^i = 1 \), \( \hat{n} \cdot \hat{\lambda} = \hat{n}^i \hat{\lambda}^i = 0 \), and \( \hat{n} \times \hat{\lambda} = \hat{\Lambda} \) (or \( \hat{\Lambda}^i = \varepsilon_{ijk} \hat{n}^j \hat{\lambda}^k \)).

A symmetric tensor can be constructed from \( (P^{l-k}_{k})^{i_1 \cdots i_l} \),

\[
(S^{l-k}_{k})^{i_1 \cdots i_l} \equiv \hat{\lambda}^{(i_1} \hat{\lambda}^{i_2} \cdots \hat{\alpha}^{i_k} \hat{n}^{i_{k+1}} \hat{n}^{i_{k+2}} \cdots \hat{n}^{i_l)}
\]

\[
= \frac{1}{\mathcal{N}} \sum_{\Pi} \left[ \hat{\lambda}^{i_1} \hat{\lambda}^{i_2} \cdots \hat{\alpha}^{i_k} \hat{n}^{i_{k+1}} \hat{n}^{i_{k+2}} \cdots \hat{n}^{i_l} \right]
\]

\[
= \frac{1}{\mathcal{N}} \sum_{\Pi} (P^{l-k}_{k})^{i_1 \cdots i_l},
\]  

(A2)

where the sum is over all the \( \mathcal{N} \equiv \binom{l}{k} \) permutations of the indices \( \{i_l\} \).

For simplicity, from now on we will drop the set of indices \( \{i_1 \cdots i_l\} \) in the notation for \( P^{l-k}_{k} \) and \( S^{l-k}_{k} \).

Appendix A.1. Differentiation rule for \( S^{l-k}_{k} \)

In a system of two bodies following a circular orbit, the time derivative of the components of the normal and tangential unit vectors are, respectively,

\[
\frac{d\hat{n}^i}{dt} = \Omega \hat{\lambda}^i, \quad \frac{d\hat{\lambda}^i}{dt} = -\Omega \hat{n}^i,
\]  

(A3)
where $\Omega$ is the angular velocity, such that $v = \Omega r$.

We need to calculate the time derivative of a symmetrized product of $l$ components of unit vectors, where $k$ of them are tangential ($\xi^i$'s) and $(l - k)$ are normal ($\hat{n}^i$'s), as in (A2). The time derivative of one of the $P_{l-k}^i$'s that belongs to $S_{k+1}^l$:

$$
\frac{d}{dt} (P_{k}^{l-k}) = \Omega \left[ - \left( \hat{n}^{i_1} \xi^{i_2} \ldots \xi^{i_k} \hat{n}^{i_{k+1}} \xi^{i_{k+2}} \ldots \hat{n}^{i_l} 
+ \lambda^{i_1} \lambda^{i_2} \ldots \lambda^{i_k} \hat{n}^{i_{k+1}} \xi^{i_{k+2}} \ldots \hat{n}^{i_l} 
\ldots \ldots 
+ \lambda^{i_1} \lambda^{i_2} \ldots \lambda^{i_k} \hat{n}^{i_{k+1}} \lambda^{i_{k+2}} \ldots \hat{n}^{i_l} 
\ldots \ldots 
+ \lambda^{i_1} \lambda^{i_2} \ldots \lambda^{i_k} \hat{n}^{i_{k+1}} \lambda^{i_{k+2}} \ldots \lambda^{i_l} \right) \right], \quad (A4)
$$

is the sum of two parts: the first parenthesis with $k$ terms of the type $P_{k-1}^{l-k+1}$, and the second parenthesis with $(l - k)$ terms of the type $P_{k+1}^{l-k-1}$.

Taking the derivative of the whole $S_k^{l-k}$ creates $\binom{l}{k}$ sets of terms like (A4). In the expression for $d (S_k^{l-k}) / dt$, all permutations of indices will be present because they were present in the original $S_k^{l-k}$. We just need to collect all the terms of the type $P_{k-1}^{l-k+1}$, that belong to $S_{k+1}^{l-k+1}$, and all the terms of the type $P_{k+1}^{l-k-1}$, that belong to $S_{k+1}^{l-k+1}$. There are $\binom{l}{k-1} P_{k-1}^{l-k+1}$-terms and $\binom{l}{k+1} P_{k+1}^{l-k-1}$-terms. So the expression for the time derivative of $S_k^{l-k}$ is

$$
\frac{d}{dt} (S_k^{l-k}) = \frac{1}{\binom{l}{k}} \Omega \left[ \binom{l}{k-1} \sum_{\Pi_j (t_j)} P_{k-1}^{l-k+1} - \binom{l}{k+1} \sum_{\Pi_j (t_j)} P_{k+1}^{l-k-1} \right] 
= \Omega \left[ (l-k) S_{k+1}^{l-k+1} - k S_{k-1}^{l-k+1} \right]. \quad (A5)
$$

For this procedure to be valid, at least one of each of the new symmetrized products $S_{k+1}^{l-k+1}$ and $S_{k-1}^{l-k+1}$ has to be obtained. In other words,

$$
\frac{\binom{l}{k} (l-k)}{\binom{l}{k+1}} \geq 1, \quad \text{and} \quad \frac{\binom{l}{k} k}{\binom{l}{k-1}} \geq 1, \quad (A6)
$$

which are trivially true, with the equality corresponding to $k = l$ (for $S_l^0$), and $k = 0$ (for $S_0^l$), the two possible limit cases. Then, (A5) is the rule of differentiation for the symmetric tensors $S_k^{l-k}$.

**Appendix A.2. Products of $S_0^T$'s**

In this section we will limit the discussion to symmetric products that contain only radial components. In this case, (with the same conventions as in (15), (18), and (19)), we have

$$
S_0^T = \hat{n}^{i_1} \hat{n}^{i_2} \ldots \hat{n}^{i_T} = \hat{n}^{i_1} \hat{n}^{i_2} \ldots \hat{n}^{i_T}, \quad (A7)
$$
\[ \mathbf{L} \circ \mathbf{S}_0^{T-1} = \mathbf{L}^{i_1 i_2 \ldots i_T} = \frac{1}{T} \sum_{i_T} \mathbf{L}^{i_1 i_2 \ldots i_T}, \quad \text{(A8)} \]

\[ \mathbf{D}^{(j)} \circ \mathbf{S}_0^{T-2j} = \delta^{i_1 i_2 \delta^{i_3 i_4} \ldots \delta^{i_{2j-1} i_{2j} i_{2j+1} \ldots i_T}} \]

\[ = \frac{1}{N(T; j, 2)} \sum_{\Pi} \delta^{i_1 i_2 \delta^{i_3 i_4} \ldots \delta^{i_{2j-1} i_{2j} i_{2j+1} \ldots i_T}}, \quad \text{(A9)} \]

\[ \mathbf{D}^{(j)} \circ [\mathbf{L} \circ \mathbf{S}_0^{T-1-2j}] = \delta^{i_1 i_2 \delta^{i_3 i_4} \ldots \delta^{i_{2j-1} i_{2j} i_{2j+1} \ldots i_T} \hat{L}^{i_{2j+1} i_{2j+2} i_{2j+2} \ldots i_T}}, \quad \text{(A10)} \]

where \( \Pi \) in the latter two equations simply denotes the distinct permutations, and

\[ N(T; j, 2) = \frac{1}{j!} \left( \frac{T}{2} \right)^2 \left( \frac{T-2}{2} \right) \ldots \left( \frac{T-2j}{2} \right), \quad \text{(A11)} \]

Using the orthonormality of \( \hat{n} \) and \( \hat{L} \), and the definition \( \mathbf{D}^{(1)} \cdot \hat{n} \hat{n} = \delta^{ij} \hat{n}^{i} \hat{n}^{j} = 1 \), we calculate the products: \( [\mathbf{D}^{(j)} \circ \mathbf{S}_0^{T-2j}] \cdot \mathbf{S}_0^T \) and \( [\mathbf{D}^{(j)} \circ [\mathbf{L} \circ \mathbf{S}_0^{T-1-2j}]] \cdot [\mathbf{L} \circ \mathbf{S}_0^T], \)

which occur in (23). Notice that we use as “target” (or second factor in the product), the simpler of the two factors. This simplifies the counting of resulting terms. For both cases, the expression is, schematically,

\[ \left[ \frac{1}{N_1} \sum \text{Term}_1 \right] \cdot \left[ \frac{1}{N_2} \sum \text{Term}_2 \right], \quad \text{(A12)} \]

where \( N_i \) is the number of distinct terms in each sum. Next, we calculate the effect of multiplying one term only from the symmetric first factor by the whole of the target, say

\[ [\text{Term}_1] \cdot \left[ \frac{1}{N_2} \sum \text{Term}_2 \right]. \quad \text{(A13)} \]

There are \( N_1 \) identical products like this in the final result, so

\[ \left[ \frac{1}{N_1} \sum \text{Term}_1 \right] \cdot \left[ \frac{1}{N_2} \sum \text{Term}_2 \right] = [\text{Term}_1] \cdot \left[ \frac{1}{N_2} \sum \text{Term}_2 \right]. \quad \text{(A14)} \]

For the first product, \( \text{(A13)} \) gives

\[ \left[ \delta^{i_1 i_2 \delta^{i_3 i_4} \ldots \delta^{i_{2j-1} i_{2j} i_{2j+1} \ldots i_T}} \right] \cdot [\hat{n}^{i_1 i_2 \ldots i_T}] = 1 \], \quad \text{(A15)} \]

trivially, as the target has only one term. The result is

\[ [\mathbf{D}^{(j)} \circ \mathbf{S}_0^{T-2j}] \cdot \mathbf{S}_0^T = 1. \quad \text{(A16)} \]

For the second product, \( \text{(A13)} \) gives

\[ \left[ \delta^{i_1 i_2 \delta^{i_3 i_4} \ldots \delta^{i_{2j-1} i_{2j} i_{2j+1} \ldots i_T}} \hat{L}^{i_{2j+1} i_{2j+2} \ldots i_T} \right] \cdot \left[ \frac{1}{T} \sum_{i_T} \mathbf{L}^{i_1 i_2 \ldots i_T} \right] = \frac{1}{T}, \quad \text{(A17)} \]

because only one of the \( T \) terms in the target has the angular momentum component in position \( i_{2j+1} \), as in the first factor. The result is

\[ [\mathbf{D}^{(j)} \circ [\mathbf{L} \circ \mathbf{S}_0^{T-1-2j}]] \cdot [\mathbf{L} \circ \mathbf{S}_0^{T-1}] = \frac{1}{T}. \quad \text{(A18)} \]
Appendix A.3. Canonical form of an even number of derivatives of $S_0^T$

Here we discuss the derivation of an expression for the $(2p)$-derivative of $S_0^T$ in which all terms are of rank $T$. We will carry out the derivation explicitly only for the terms proportional to $S_0^T$, since only these survive the contraction in (23). We can illustrate the problem appearing in the discussion of Section 2 following (25), with this example: take, say, $T = 4$ and $p = 2$ in (30a). In the sine–cosine representation, this is

$$
\frac{d^4}{dx^4}(\sin^4 \Omega x) = \Omega^4 \left[ (32 \sin^4 \Omega x - 192 \sin^2 \Omega x \cos^2 \Omega x + 32 \cos^4 \Omega x) \\
+ (8 \sin^2 \Omega x - 8 \cos^2 \Omega x) \right]. \tag{A19}
$$

The first parenthesis contains products of “rank” 4 while the second contains products of “rank” 2. Direct differentiation, however, gives

$$
\frac{d^4}{dx^4}(\sin^4 \Omega x) = \Omega^4 \left[ 40 \sin^4 \Omega x - 192 \sin^2 \Omega x \cos^2 \Omega x + 24 \cos^4 \Omega x \right], \tag{A20}
$$

where all the terms are of “rank” 4. The expansion of $\sin^T \Omega x$ in terms of sines and cosines of multiples angles before differentiation produced this effect. However, equations (A19) and (A20) are identical: in (A19), multiplying the second parenthesis by $\cos^2 \Omega x + \sin^2 \Omega x = 1$ yields $8 \sin^4 \Omega x - 8 \cos^4 \Omega x$.

We need to perform the same regrouping of terms on (30a, 30b). This is very difficult to do in generic form. However, we should remember that these derivatives are to be multiplied by a factor proportional to $S_0^T$, which has components only in the radial direction. Consequently, all terms in the derivatives possessing tangential components (terms with cosines in the sine–cosine representation) will have no effect. The only surviving term in (30a, 30b) is the one that has only radial components (all-sines term). Explicitly, this condition is $T - 2k - 2q = 0$ in (30a) and $T - 2k - 2q - 1 = 0$ in (30b).

One can demonstrate that in the regrouping process, the coefficient of the final all-sines term is obtained by simply adding the coefficients of all the all-sines terms of different rank $T - 2k$ in the original sum (the same conclusion holds for the all-cosines term). The demonstration generalizes the example above, and involves multiplying the lowest ranked terms by $\cos^2 \Omega x + \sin^2 \Omega x$, combining with the next higher ranked terms, and repeating the process until all terms are the same rank.

In (30a, 30b), we thus need to add the coefficients of all $S_{T-2k-2q}^{2q}$, for $T - 2k - 2q = 0$ and all $k$, in (30a), and the coefficients of all $S_{T-2k-2q-1}^{2q+1}$, for $T - 2k - 2q - 1 = 0$ and all $k$, in (30b), to get the coefficient of $S_0^T$.

For (30a), the condition $2q = T - 2k$, eliminates all the terms in the sum over $q$ except the last one, so the coefficient of $S_0^{T-2k}$ is

$$
(-1)^{\lfloor \frac{T}{2} \rfloor - k} \binom{T}{k} (T - 2k)^{2p} \left( -1 \right)^{\lfloor \frac{T}{2} \rfloor - k} \binom{T - 2k}{T - 2k} = \binom{T}{k} (T - 2k)^{2p}. \tag{A21}
$$

For (30b), the condition $2q + 1 = T - 2k$, also eliminates all the terms in the sum over $q$ except the last one, so the coefficient of $S_0^{T-2k}$ is

$$
(-1)^{\lfloor \frac{T+1}{2} \rfloor + k} \binom{T}{k} (T - 2k)^{2p} \left( -1 \right)^{\lfloor \frac{T+1}{2} \rfloor - k} \binom{T - 2k}{T - 2k} = \binom{T}{k} (T - 2k)^{2p}. \tag{A22}
$$
The result is the same for $T$ even or odd. Thus, the term in the $(2p)$ derivative of $S^T_0$ that will survive the product by $S^T_0$ is:

$$
(2p)S^T_0 \bigg|_{\text{all-sines}} = \left( \frac{(-1)^p \Omega^{2p}}{2^{T-1}} \sum_{k=0}^{T-1} \binom{T}{k} (T-2k)^{2p} \right) S^T_0 = A(T, p) S^T_0 .
$$ (A23)

Appendix B. Mass functions $f_l(\eta)$

In a general coordinate system, the expressions for the leading order of the radiative multipole moments are simply

$$
I^{<l>} = \left\{ m_1 x_1^i x_2^i \cdots x_1^i + m_2 x_2^i x_2^i \cdots x_2^i \right\}_{\text{STF}} ,
$$

$$
J^{<l>} = \left\{ \varepsilon_{1ab} [m_1 x_1^a b x_2^i \cdots x_1^i + m_2 x_2^a b x_2^i \cdots x_2^i] \right\}_{\text{STF}} .
$$ (B24)

Changing to the CM-coordinate system defined by the Newtonian relations (consistent with our bare-multipole approach) $m_1 \mathbf{x}_1 + m_2 \mathbf{x}_2 = 0$, and with $\mathbf{x} \equiv \mathbf{x}_1 - \mathbf{x}_2$, and

$$
x_1 = \frac{m_2}{m} \mathbf{x} , \quad x_2 = -\frac{m_1}{m} \mathbf{x} ,
$$ (B25)

$$
v_1 = \frac{m_2}{m} \mathbf{v} , \quad v_2 = -\frac{m_1}{m} \mathbf{v} ,
$$ (B26)

the moments become

$$
I^{<l>} = \left[ m_1 \left( \frac{m_2}{m} \right)^l + (-1)^l m_2 \left( \frac{m_1}{m} \right)^l \right] x^{<l>}_{\text{STF}},
$$

$$
J^{<l>} = \left[ m_1 \left( \frac{m_2}{m} \right)^{l+1} + (-1)^{l+1} m_2 \left( \frac{m_1}{m} \right)^{l+1} \right] \mu^{-1} \left[ \mathbf{L} \circ \mathbf{x}^{<l-1>}_{\text{STF}} \right] .
$$ (B27)

where $\mathbf{L} = \mu \mathbf{x} \times \mathbf{v}$ is the angular momentum vector, and $\mathbf{x}^{<l>} \equiv x^i_1 x^i_2 \cdots x^i_l$. We define $\rho$ such that

$$
m_2 = \rho m_1 , \quad m = (1 + \rho) m_1 , \quad \mu = \frac{\rho}{(1 + \rho)^2} m , \quad \eta = \frac{\rho}{(1 + \rho)^2} ,
$$ (B28)

with the convention $m_2 \leq m_1$. Inverting the last expression, we have

$$
\rho(\eta) = \frac{1}{2\eta} \left[ 1 - 2\eta - \sqrt{1 - 4\eta} \right] .
$$ (B29)

In terms of $\mu$ and $\rho$, (B27) becomes

$$
I^{<l>} = \mu \frac{\rho^{l-1} + (-1)^l}{(1 + \rho)^{l-1}} x^{<l>}_{\text{STF}} \equiv \mu f_l(\eta) x^{<l>}_{\text{STF}},
$$

$$
J^{<l>} = \frac{\rho^l + (-1)^{l+1}}{(1 + \rho)^l} \left[ \mathbf{L} \circ \mathbf{x}^{<l-1>}_{\text{STF}} \right] \equiv f_{l+1}(\eta) \left[ \mathbf{L} \circ \mathbf{x}^{<l-1>}_{\text{STF}} \right] .
$$ (B30)

Notice that we have not used any information about the trajectory of the bodies in this section. This expression for $f_l(\eta)$ is then valid for any system of two bodies.

Figure 3 shows the dependence of $f_l$ on the reduced mass $\eta$, for odd and even values of $l$. Limiting values for $f_l$ in the range $\eta = (0, 0.25]$, are $|f_l(0)| = 1$, $f_l(0.25) = 0$, if $l$ is odd, and $f_l(0.25) = 1/2^{l-2}$, if $l$ is even.
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Figure captions

Figure A1. STF multipole moments displayed in an array with multipole order increasing to the right, and the order of post-Newtonian corrections of each multipole moment increasing downward. Tail terms first appear at $O(\epsilon^{3/2})$ for each moment. For each $l$, $I^{<l>}$ and $J^{<l-1>}$ are grouped together. The polygons below indicate the moments and PN accuracies required to calculate the indicated quantity through 2PN order.

Figure A2. Fractional difference between series truncated at the labelled PN order and the exact series, as a function of $v$, for $\eta = 0$ (solid lines) and $\eta = 0.25$ (dashed lines). Innermost stable orbit for test body motion is at $v \approx 0.4$. In the equal-mass case, vanishing of odd-parity moments leads to degeneracy between adjacent PN approximations.

Figure A3. The function $|f_l(\eta)|$, for odd (dashed lines) and even (solid lines) values of $l$. Note that $f_2(\eta) \equiv 1$.

Figure A4. A comparison between the bare-multipole series (upper dashed curve), the "exact" numerical results from black-hole perturbation theory (solid curve), and the true PN series truncated to 3.5PN order (lower dashed curve). The three curves are plots of $\dot{E}/\dot{E}_Q$ as a function of orbital velocity $v$, in the physically relevant interval $0 < v < 0.4$. The comparison is valid in the test-body limit $\eta \ll 1$. 

Multipole moments ($I^L, J^{L-1}$)

Relative PN Order

$\varepsilon$

$\varepsilon^{3/2}$

$\varepsilon^2$

$\varepsilon^{5/2}$

$\varepsilon^3$

$\varepsilon^4$

$\varepsilon^5$

$\varepsilon^6$

$\varepsilon^7$

$\varepsilon^8$

$T A I L S$

$E$

$h^{ij}$
