THE VIRTUALLY GENERATING GRAPH
OF A PROFINITE GROUP

ANDREA LUCCHINI

Abstract. We consider the graph $\Gamma_{\text{virt}}(G)$ whose vertices are the elements of a finitely generated profinite group $G$ and where two vertices $x$ and $y$ are adjacent if and only if they topologically generate an open subgroup of $G$. We investigate the connectivity of the graph $\Delta_{\text{virt}}(G)$ obtained from $\Gamma_{\text{virt}}(G)$ by removing its isolated vertices. In particular we prove that for every positive integer $t$, there exists a finitely generated prosoluble group $G$ with the property that $\Delta_{\text{virt}}(G)$ has precisely $t$ connected components. Moreover we study the graph $\tilde{\Gamma}_{\text{virt}}(G)$, whose vertices are again the elements of $G$ and where two vertices are adjacent if and only if there exists a minimal generating set of $G$ containing them. In this case we prove that the subgraph $\tilde{\Delta}_{\text{virt}}(G)$ obtained removing the isolated vertices is connected and has diameter at most 3.

1. Introduction

The generating graph of a finite group is an interesting notion and has been much studied in the literature. In [7] the investigation on this topic has been extended to the context of profinite groups. Let $G$ be a 2-generated profinite group. The generating graph $\Gamma(G)$ is defined with vertex set $G$ where two elements $x$ and $y$ are connected if and only if $x$ and $y$ (topologically) generate $G$. An open conjecture is that if $G$ is a finite group, then the subgraph $\Delta(G)$ obtained from $\Gamma(G)$ by removing its isolated vertices is connected. In [7] it is proved that the conjecture is true if $G$ is prosoluble but an example is given showing that it fails for arbitrary finitely generated profinite groups. Recently, during the workshop ‘Groups with Geometrical and Topological Flavours’ Yiftach Barnea proposed to investigate a related graph. Following his suggestion we define the virtually generating graph $\Gamma_{\text{inv}}(G)$ associated to a profinite group $G$ as the graph whose vertices are the elements of $G$ and in which two elements $x$ and $y$ are connected if and only if $x$ and $y$ generate (topologically) an open subgroup of $G$. In particular we address the question whether the graph $\Delta_{\text{inv}}(G)$ obtained from $\Gamma_{\text{inv}}(G)$ by removing its isolated vertices is connected. The answer is negative. Let $G_p = (\text{SL}(2, 2^p))^\delta_p$, where $p$ is a prime and $\delta_p$ is the largest positive integer with the property that the direct power $(\text{SL}(2, 2^p))^\delta_p$ can be generated by 2-elements and consider $G = \prod_p G_p$. In [7] Section 3, it is proved that the graph $\Delta(G)$ has $2^{\aleph_0}$ connected components. This is a consequence of the fact that the diameter of $\Delta(G_p)$ tends to infinity with $p$. In Section 2 we will see that also $\Delta_{\text{virt}}(G)$ has $2^{\aleph_0}$ connected components. In Section 3 we will prove a more surprising results, considering the graph $\Delta_{\text{virt}}(G)$, when $G$ is prosoluble. In [8] it was proved that if $G$ is a finite soluble group, then $\Delta(G)$ is connected, with diameter at most 3, and this statement has been

1991 Mathematics Subject Classification. 20D60, 05C25.
Theorem 2. Let sizes of the minimal generating set of $G$ be extended to the profinite case. In this case we define $k$ generating set of cardinality $d$ with $d(G)$ and $m(G)$, respectively, the smallest and the largest cardinality of a minimal generating set of a finite group $G$. A nice result in universal algebra, the Tarski irredundant basis theorem (see for example [1, Theorem 4.4]), implies that if $G$ is a finite group, then, for every positive integer $k$ with $d(G) \leq k \leq m(G)$, $G$ contains an independent generating set of cardinality $k$. In section III we will prove that this result can be extended to the profinite case. In this case we define $m(G)$ as the supremum of the sizes of the minimal generating set of $G$.

**Theorem 1.** For every positive integer $t$, there exists a finitely generated prosoluble group $G$ with the property that $\Delta_{\text{virt}}(G)$ has precisely $t$ connected components.

In the second part of the paper we extend to the profinite groups the notion of independence graph, given in [8] for finite groups, and we introduce the new definition of virtually independence graph of a profinite group. Let $G$ be a finitely generated profinite group. A generating set $X$ of $G$ is said to be minimal if no proper subset of $X$ (topologically) generates $G$. We denote by $d(G)$ and $m(G)$, respectively, the smallest and the largest cardinality of a minimal generating set of a finite group $G$. A nice result in universal algebra, the Tarski irredundant basis theorem (see for example [1, Theorem 4.4]), implies that if $G$ is a finite group, then, for every positive integer $k$ with $d(G) \leq k \leq m(G)$, $G$ contains an independent generating set of cardinality $k$. In section III we will prove that this result can be extended to the profinite case. In this case we define $m(G)$ as the supremum of the sizes of the minimal generating set of $G$.

**Theorem 2.** Let $G$ be a finitely generated profinite group. For every positive integer $k$ with $d(G) \leq k \leq m(G)$, $G$ contains an independent generating set of cardinality $k$. Moreover $m(G)$ is finite if and only if the Frattini subgroup of $G$ has finite index in $G$.

We denote by $\hat{\Gamma}(G)$ the graph whose vertices are the elements $G$ and in which two vertices $x$ and $y$ are joined by an edge if and only if $x \neq y$ and there exists a finite minimal generating set of $G$ containing $x$ and $y$. Roughly speaking, $x$ and $y$ are adjacent vertices of $\hat{\Gamma}(G)$ if they are ‘independent’, so we call $\hat{\Gamma}(G)$ the independence graph of $G$. In Section III extending results proved in [8] in the case of finite groups, we show that if $G$ is a finitely generated profinite group and $g$ is an isolated vertex of $\hat{\Gamma}(G)$, then either $g$ belongs to the Frattini subgroup of $G$ or it is a topological generator of $G$. Moreover the graph obtained from $\hat{\Gamma}(G)$ by removing the isolated vertices is always connect. Also in this case instead of generating subsets, we may consider virtually generating subsets, i.e. subsets generating an open subgroup of $G$. In particular we say that two elements $x$ and $y$ are virtually independent if there exists a minimal generating set of an open subgroup of $G$ containing them. In Section III we define the virtually independent graph $\hat{\Gamma}_{\text{virt}}(G)$ of $G$ as the graph whose vertices are the elements of $G$ and in which two vertices are joined by an edge if and only if they are virtually independent. Moreover we denote by $\hat{\Delta}_{\text{virt}}(G)$ the subgraph induced by the non-isolated vertices of the virtually independent graph of $G$. We prove in particular the following results:

**Theorem 3.** Let $G$ be a finitely generated profinite group. If $\hat{\Gamma}_{\text{virt}}(G)$ contains a non-trivial isolated vertex $g$, then either $g$ is a topological generator of $G$ or one of the following occurs:

1. $G \cong C_{p^n}$ is a cyclic group of $p$-power order and all the vertices of $\hat{\Gamma}_{\text{virt}}(G)$ are isolated.
2. $G \cong \mathbb{Z}_p$, the group of the $p$-adic integers, and all the vertices of $\hat{\Gamma}_{\text{virt}}(G)$ are isolated.
3. $G \cong Q_{2^n} = \langle x, y \mid x^{2^n-1}, y^2 = x^{2^{n-2}}, x^{-1}yx = y^{-1} \rangle$ is a generalized quaternion group and $g = y^2$ is the unique non-trivial isolated vertex of $\hat{\Gamma}_{\text{virt}}(G)$. 


Lemma 7. Virt by the previous theorem is best possible.

Corollary 6. If $x$ then $\Delta_{\text{virt}}(G)$ is connected and $\text{diam}(\Delta_{\text{virt}}(G)) \leq 3$.

We will exhibit an example, showing that the bound $\text{diam}(\Delta_{\text{virt}}(G)) \leq 3$ given by the previous theorem is best possible.

2. AN EXAMPLE OF A PROFINITE GROUP $G$ SUCH THAT $\Delta_{\text{inv}}(G)$ HAS $2^{\aleph_0}$ CONNECTED COMPONENTS.

Let $G_p = (\text{SL}(2, 2^p))^{\delta_p}$, where $p$ is a prime and $\delta_p$ is the largest positive integer with the property that the direct power $(\text{SL}(2, 2^p))^{\delta_p}$ can be generated by $2$-elements. The graph $\Delta(G_p)$ is connected for every prime $p$, and, by [2 Theorem 1.3], there exists an increasing sequence $(p_n)_{n \in \mathbb{N}}$ of odd primes, such that $\text{diam}(\Delta(G_{p_n})) \geq 2^n$ for every $n \in \mathbb{N}$. Consider the cartesian product

$$G = \prod_{n \in \mathbb{N}} G_{p_n}$$

with the product topology, and for any $n \in \mathbb{N}$, let $\pi_n : G \to G_{p_n}$ be the projection homomorphism.

Lemma 5. Let $x = (x_n)_{n \in \mathbb{N}}$, $y = (y_n)_{n \in \mathbb{N}} \in G$. The subgroup $\langle x, y \rangle$ is open in $G$ if and only if there exists $m \in \mathbb{N}$ such that $(x_n, y_n) = G_{p_n}$ for any $n \geq m$.

Proof. Let $H = \langle x, y \rangle$ and, for $n \in \mathbb{N}$, set $H_n = \pi_n(G)$. Clearly $H \leq \prod_{n \in \mathbb{N}} H_n$ and $|G : H| \geq |G : \prod_{n \in \mathbb{N}} H_n| = \prod_{n \in \mathbb{N}} |G_{p_n} : H_{p_n}|$. If $H$ is open in $G$, then $|G : H|$ is finite and therefore $H_n = G_{p_n}$ for all but finitely many $n$. Conversely, assume that $H_n = G_{p_n}$ for any $n \leq m$ and set $K = \prod_{n < m} G_{p_n}$. If follows from the Goursat’s Lemma (see for example [4 Theorem 5.5.1]) that $HK = G$, and so $H$ is open in $G$.

Let $V_{\text{virt}}(G)$ be the set of the non-isolated vertices of $\Gamma_{\text{virt}}(G)$. Moreover, given $x = (x_n)_{n \in \mathbb{N}} \in G$, define $\Lambda(x) = \{ n \in \mathbb{N} \mid x_n \in V(G_{p_n}) \}$.

Corollary 6. $x = (x_n)_{n \in \mathbb{N}}$ is a non-isolated vertex of $\Gamma_{\text{virt}}(G)$ if and only if $n \in \Lambda(x)$ for all but finitely many $n \in \mathbb{N}$.

Lemma 7. Let $x = (x_n)_{n \in \mathbb{N}} \in V_{\text{virt}}(G)$ and let $\Omega_x$ be the connected component of $\Delta_{\text{virt}}(G)$ containing $x$. Then $y = (y_n)_{n \in \mathbb{N}}$ belongs to $\Omega_x$ if and only if

$$\sup_{n \in \Lambda(x) \cap \Lambda(y)} \text{dist}_{\Delta(G_{p_n})}(x_n, y_n) < \infty.$$ 

Proof. Assume that $y = (y_n)_{n \in \mathbb{N}} \in \Omega_x$ and let $m = \text{dist}(G, \Delta(x, y))$. It follows from Lemma 5 that $\text{dist}_{\Delta(G_{p_n})}(x_n, y_n) \leq m$ for all but finitely many $n \in \Lambda(x) \cap \Lambda(y)$. Conversely assume $y = (y_n)_{n \in \mathbb{N}} \in V_{\text{virt}}(G)$ and $\text{dist}_{\Delta(G_{p_n})}(x_n, y_n) \leq m$ for every $n$ in a cofinite subset $\Lambda$ of $\Lambda(x) \cap \Lambda(y)$. If $n \in \Lambda$, then there is a path

$$x_n = x_{n,0}, x_{n,1}, \ldots, x_{n,\mu_n} = y_n,$$

with $\mu_n \leq m$, joining $x_n$ and $y_n$ in the graph $\Delta(G_{p_n})$. For $0 \leq i \leq m$, set

$$\tilde{x}_{n,i} = \begin{cases} x_n & \text{if } n \notin \Lambda, \\ x_{n,i} & \text{if } n \in \Lambda, i < \mu_n, \\ x_{n,\mu_n} & \text{if } n \in \Lambda, i \geq \mu_n \text{ and } m - \mu_n \text{ is even,} \\ x_{n,\mu_n-1} & \text{if } n \in \Lambda, i \geq \mu_n \text{ and } m - \mu_n \text{ is odd.} \end{cases}$$
Then
\[ x = \tilde{x}_0 = (\tilde{x}_{n,0})_{n \in \mathbb{N}}, \quad \tilde{x}_1 = (\tilde{x}_{n,1})_{n \in \mathbb{N}}, \ldots, y = \tilde{x}_m = (\tilde{x}_{n,m})_{n \in \mathbb{N}} \]
is a path joining \( x \) and \( y \) in the graph \( \Delta_{\text{virt}}(G) \), so \( y \in \Omega_x \).

**Proposition 8.** \( \Delta_{\text{virt}}(G) \) has \( 2^n \) different connected components.

**Proof.** Fix \( x = (x_n)_{n \in \mathbb{N}} \in \Delta_{\text{virt}}(G) \). Let \( \tau \) be a real number with \( \tau > 1 \). Since
\[
\text{diam}(\Delta(G_{p_n})) \geq 2^n \geq 1 + \lfloor n/\tau \rfloor,
\]
for every \( n \in \mathbb{N} \) there exists \( y_{\tau,n} \in G_{p_n} \) such that
\[
\text{dist}(G_{p_n})(x_n,y_{\tau,n}) = 1 + \lfloor n/\tau \rfloor.
\]
If \( \tau_2 > \tau_1 \), then
\[
\text{dist}(y_{\tau_2,n},y_{\tau_1,n}) \geq \text{dist}(x_n,y_{\tau_1,n}) - \text{dist}(x_n,y_{\tau_2,n}) = \lfloor n/\tau_1 \rfloor - \lfloor n/\tau_2 \rfloor
\]
tends to infinity with \( n \), so, by Lemma 7, \( \Omega_{y_{\tau_1}} \neq \Omega_{y_{\tau_2}} \).

**3. Proof of Theorem 4**

We consider the set \( \Omega \) of the elements \((x_1,x_2,\ldots,x_{2t-1},x_{2t}) \in \mathbb{F}_2^{2t}\) such that \((x_{2i-1},x_{2i}) \neq (0,0)\) for \( 1 \leq i \leq t \). To any \( \omega \in \Omega \) we associate a different odd prime number \( p_{\omega} \). Let \( H = \langle y_1,y_2,\ldots,y_{2t-1},y_{2t} \rangle \) be an elementary abelian 2-group of rank \( 2t \). For any \( \omega = (x_1,x_2,\ldots,x_{2t-1},x_{2t}) \in \Omega \), we define an action of \( H \) on the direct product \( N_{\omega} = (\mathbb{Z}_{p_{\omega}})^3 \) of three copies of the group \( \mathbb{Z}_{p_{\omega}} \) of the \( p_{\omega} \)-adic integers as follows:
\[
(z_1,z_2,z_3)^{\nu} = (z_1,z_2,(-1)^{\nu_1} \cdot z_3).
\]
We consider the semidirect product
\[
G = \left( \prod_{\omega \in \Omega} N_{\omega} \right) \rtimes H.
\]
Notice that \( G \) is a prosoluble group that can be generated by \( 2t \) elements, as it follows easily from the following lemma.

**Lemma 9.** Let \( \omega = (x_1,x_2,\ldots,x_{2t-1},x_{2t}) \in \Omega \) and \( v_\omega = (1,0,1) \), \( w_\omega = (1,0,-1) \in N_\omega \). Then \( N_\omega \leq \bigcap_{1 \leq i \leq t} \langle v_\omega y_{2i-1}, w_\omega y_{2i} \rangle \).

**Proof.** We prove \( N_\omega \leq \langle v_\omega y_1,w_\omega y_2 \rangle \) (the other inclusions \( N_\omega \leq \langle v_\omega y_{2i-1},w_\omega y_{2i} \rangle \) can be proved with a similar argument). Let \( \rho_1 = v_\omega y_1 \) and \( \rho_2 = w_\omega y_2 \). Notice that
\[
\rho_1^2 = (2,0,0^{z_1} \cdot 2), \quad \rho_2^2 = (2,0,0^{z_2} \cdot 2), \quad [\rho_1,\rho_2] = \begin{cases} (0,0,4) & \text{if } (x_1,x_2) = (1,1) \\ (0,0,-2) & \text{otherwise.} \end{cases}
\]
Since \( p_{\omega} \neq 2 \), it follows \( N_\omega = \langle (1,0,0),(0,1,0),(0,0,1) \rangle \leq \langle \rho_1,\rho_2 \rangle \).

Now, for \( 1 \leq i \leq t \), set
\[
\sigma_{2i-1} = ((v_\omega)_{\omega \in \Omega}) y_{2i-1}, \quad \sigma_{2i} = ((w_\omega)_{\omega \in \Omega}) y_{2i}.
\]
It follows immediately from Lemma 7 that \( \langle \sigma_1,\sigma_2,\ldots,\sigma_{2t-1},\sigma_{2t-2} \rangle \) are open subgroups of \( G \) (with index \( 2^{t-2} \)) and therefore \( \langle \sigma_1,\sigma_2,\ldots,\sigma_{2t-1},\sigma_{2t-2} \rangle \) are edges of \( \Gamma_{\text{virt}}(G) \). Now let
\[
N = \prod_{\omega \in \Omega} N_{\omega} \quad \text{and} \quad \Sigma_i = \{ nh \in G \mid n \in N \text{ and } h \in \langle y_{2i-1},y_{2i} \rangle \} \text{ for } 1 \leq i \leq t.
\]
Proposition 10. Suppose that $g_1$ and $g_2$ are two adjacent vertices of $\Gamma_{\text{virt}}(G)$. Then $\{g_1, g_2\} \subseteq \Sigma_i$ for some $1 \leq i \leq t$.

Proof. Suppose that $g_1 = n_1 h_1$, $g_2 = n_2 h_2$ with $n_1, n_2 \in N$, $h_1 = \prod_{1 \leq j \leq 2t} y_j^a$ and $h_2 = \prod_{1 \leq j \leq 2t} y_j^b$. We claim that the system

$$
\begin{align*}
  a_1 x_1 + \cdots + a_{2t} x_{2t} &= 0 \\
  b_1 x_1 + \cdots + b_{2t} x_{2t} &= 0
\end{align*}
$$

(3.1)

where the coefficients are viewed as elements of $\mathbb{F}_2$, has no solution in $\Omega$. Indeed if $\omega = (x_1, \ldots, x_{2t})$ is a solution of the previous system, then $h_1$ and $h_2$ centralize $N_\omega$ and consequently $\langle g_1, g_2 \rangle \cap N_\omega = \langle n_1, n_2 \rangle$ is a 2-generated subgroup of $N_\omega = (\mathbb{Z}_{p_\omega})^3$. But this would imply

$$|G : \langle g_1, g_2 \rangle| \geq |N_\omega : \langle g_1, g_2 \rangle| = |N_\omega : N_\omega \cap \langle g_1, g_2 \rangle| = \infty$$

and therefore $(g_1, g_2)$ is not an open subgroup of $G$.

For $1 \leq i \leq t$, consider the following matrices, with coefficients in $\mathbb{F}_2$:

$$
C_i = \begin{pmatrix}
  a_{2i-1} & a_{2i} \\
  b_{2i-1} & b_{2i}
\end{pmatrix},

D_i := \begin{pmatrix}
  a_1 & \cdots & a_{2i-2} & a_{2i+1} & \cdots & a_{2t} \\
  b_1 & \cdots & b_{2i-2} & b_{2i+1} & \cdots & b_{2t}
\end{pmatrix}.
$$

Our statement is equivalent to stating that if $C_i \neq 0$, then $D_i = 0$. First we prove that if $C_i$ is invertible, then $D_i = 0$. Let $\Omega^*$ be the set of the elements $(r_1, r_2, \ldots, r_{t-3}, r_{t-2}) \in \mathbb{F}_2^{2(t-1)}$ such that $(r_{2i-1}, r_{2i}) \neq (0,0)$ for $1 \leq i \leq t-1$. Assume, by contradiction, $D_i \neq 0$. Since $\langle \Omega^* \rangle = \mathbb{F}_2^{2(t-1)}$, there exists an element $(x_1, x_{2i-2}, x_{2i+1}, \ldots, x_{2t}) \in \Omega^*$ such that

$$
D_i \begin{pmatrix}
  x_1 \\
  \vdots \\
  x_{2i-2} \\
  x_{2i+1} \\
  \vdots \\
  x_{2t}
\end{pmatrix} = \begin{pmatrix}
  z_1 \\
  z_2 \\
  \vdots \\
  z_t
\end{pmatrix} \neq \begin{pmatrix}
  0 \\
  0 \\
  \vdots \\
  0
\end{pmatrix}.
$$

If $C_i$ is invertible, then there exists $(x_{2i-1}, x_{2i}) \neq (0,0)$ such that

$$
C \begin{pmatrix}
  x_{2i-1} \\
  x_{2i}
\end{pmatrix} = \begin{pmatrix}
  z_1 \\
  z_2
\end{pmatrix}
$$

and $(x_1, \ldots, x_{2t}) \in \Omega$ is a solution of (3.1). We remain with the case where none of the matrices $C_1, \ldots, C_t$ is invertible. However in this case, for any $1 \leq i \leq t$, there exists $(x_{2i-1}, x_{2i}) \neq (0,0)$ such that

$$
C_i \begin{pmatrix}
  x_{2i-1} \\
  x_{2i}
\end{pmatrix} = \begin{pmatrix}
  0 \\
  0
\end{pmatrix},
$$

and again $(x_1, \ldots, x_{2t}) \in \Omega$ is a solution of (3.1). \qed

For $1 \leq i \leq t$, let $\Gamma_i$ be the set of the non-isolated vertices of $\Gamma_{\text{virt}}(G)$ contained in $\Sigma_i$. By the previous lemma, $\Gamma_i$ is a disjoint union of connected components of $\Gamma_{\text{virt}}(G)$. We are going to prove that indeed $\Gamma_i$ is a connected components.

Lemma 11. Let $g = ((n_\omega)_{\omega \in \Omega}) h \in \Sigma_i$, with $h \in H$ and $n_\omega = (z_1, z_2, z_3, \omega) \in N_\omega$. If $g$ is a non-isolated vertex of $\Gamma_\text{inv}(G)$, then following conditions hold:
has finite index in $N_\omega$.

(2) \((z_{1,\omega}, z_{2,\omega}) \neq (0, 0)\) for any $\omega \in \Omega$;
(3) if $h$ centralized $N_\omega$, then $z_{3,\omega} \neq 0$.

**Proof.** We may assume $i = 1$. Suppose that there exists $g^* = ((n^*_\omega)_{\omega \in \Omega}) h^*$ such that $X := (g, g^*)$ is an open subgroup of $G$. By the previous lemma and its proof, $K := (h, h^*) = \langle a_1, a_2 \rangle$. Let

$$A_\omega := \{(z_1, z_2, 0) \mid z_1, z_2 \in \mathbb{Z}_p \} \leq N_\omega, \quad B_\omega := \{(0, 0, z_3) \mid z_3 \in \mathbb{Z}_p \} \leq N_\omega.$$ 

Since $X$ is open in $G$, \((X \cap N_\omega)B_\omega / B_\omega = \langle (z_{1,\omega}, z_{2,\omega}, 0)B_\omega, (z_{1,\omega}^*, z_{2,\omega}^*, 0)B_\omega \rangle\) has finite index in $N_\omega / B_\omega \cong (\mathbb{Z}_p)^2$ and this implies $(z_{1,\omega}, z_{2,\omega}) \neq (0, 0)$. Finally assume $[h, N_\omega] = 0$ and set $M_\omega = \left( \prod_{\omega \neq \omega} N_\omega \right) \times A_\omega$. Consider $G_\omega := G / M_\omega$. We have that $X M_\omega / M_\omega$ is an open subgroup of $G_\omega$ and this implies that $\langle (0, 0, z_{3,\omega})h, (0, 0, z_{3,\omega})h^* \rangle$ has finite index in $B_\omega K \cong \mathbb{Z}_p \times K$. By definition, $[K, B_\omega] \neq 0$. Since $[h, B_\omega] = 0$ and $K = (h, h^*)$, we deduce $[h^*, B_\omega] \neq 0$. This implies in particular $((0, 0, z_{3,\omega})h)^2 = 1$. If $z_{3,\omega} = 0$, then $(0, 0, z_{3,\omega})h, (0, 0, z_{3,\omega})h^*$ are two commuting involutions and do not generated a subgroup of finite index of $B_\omega K$. \hfill \square

**Lemma 12.** If $g \in \Sigma_i$ satisfies condition (1), (2), (3) of Lemma 13, then $g$ is a non-isolated vertex of $\Gamma_{\text{inv}}(G)$. In particular if $g_1$ and $g_2$ are two elements of $\Sigma_i$ satisfying (1), (2), (3), then there exists $\tilde{g} \in \Sigma_i$, adjacent to both $g_1$ and $g_2$ in $\Gamma_{\text{inv}}(G)$.

**Proof.** We may assume $i = 1$. Let $g_1 = ((n_{1,\omega})_{\omega \in \Omega}) h_1$ and $g_2 = ((n_{2,\omega})_{\omega \in \Omega}) h_2$, with $h_1, h_2 \in K = \langle a_1, a_2 \rangle$. Choose $\tilde{h} \in K \setminus \{1, h_1, h_2\}$. We want to construct $\tilde{n}_\omega$ in $N_\omega$ such that $\langle (\tilde{n}_\omega)_{\omega \in \Omega} \rangle \tilde{h}$ is adjacent to both $g_1$ and $g_2$ in $\Gamma_{\text{inv}}(G)$. It suffices to choose, for any $\omega \in \Omega$, $\tilde{n}_\omega$ in $N_\omega$ with the property that $\langle n_{1,\omega} h_1, \tilde{n}_\omega \rangle \cap N_\omega$ and $\langle n_{2,\omega} h_2, \tilde{n}_\omega \rangle \cap N_\omega$ have finite index in $N_\omega$. Suppose $n_{1,\omega} = (z_{11}, z_{12}, z_{13})$ and $n_{2,\omega} = (z_{21}, z_{22}, z_{23})$. We may choose $(\tilde{z}_1, \tilde{z}_2) \in (\mathbb{Z}_p)^2$ with the property that $\langle (\tilde{z}_1, \tilde{z}_2), (z_{11}, z_{12}) \rangle$ and $\langle (\tilde{z}_1, \tilde{z}_2), (z_{21}, z_{22}) \rangle$ have finite index in $(\mathbb{Z}_p)^2$. For $j \in \{1, 2\}$, set $\eta_j = 0$ if $[h, N_\omega] = 0$ \(\eta_j = 1\) otherwise. Similarly set $\tilde{\eta} = 0$ if $[\tilde{h}, N_\omega] = 0$, $\tilde{\eta} = 0$ otherwise. Choose $\tilde{z} \in \mathbb{Z}_p / Z_p$ such that $\tilde{z}_3 \notin \langle \tilde{\eta}(-1)^{j+1} z_{13}, \tilde{\eta}(-1)^{j+1} z_{23} \rangle$. We claim that $\tilde{n}_\omega = (\tilde{z}_1, \tilde{z}_2, \tilde{z}_3)$ satisfies the requested property. More precisely, for $j \in \{1, 2\}$, the subgroup $\langle (n_{j,\omega} h_j)^2, (\tilde{n}_\omega \tilde{h})^2, [n_{1,\omega} h_j, \tilde{n}_\omega \tilde{h}] \rangle$ has finite index in $N_\omega$. Indeed we have the following two possibilities:

1) $\eta_j = 0$. In this case, by hypothesis, $z_{j3} \neq 0$. Moreover, since $K = \langle h_j, \tilde{h} \rangle$, we must have $[N_\omega, \tilde{h}] \neq 0$. So

$$\langle (n_{j,\omega} h_j)^2, (\tilde{n}_\omega \tilde{h})^2, [n_{j,\omega} h_j, \tilde{n}_\omega \tilde{h}] \rangle = \langle (2z_{j1}, 2z_{j2}, 2z_{j3}), (2\tilde{z}_1, 2\tilde{z}_2, 0), (0, 0, -2z_{j3}) \rangle$$

has finite index in $N_\omega$.

2) $\eta_j \neq 0$. In this case

$$\langle (n_{j,\omega} h_j)^2, (\tilde{n}_\omega \tilde{h})^2, [n_{j,\omega} h_j, \tilde{n}_\omega \tilde{h}] \rangle$$

$$= \langle (2z_{j1}, 2z_{j2}, 0), (2\tilde{z}_1, 2\tilde{z}_2, 2(1 - \tilde{\eta})\tilde{z}_3), (0, 0, 2\tilde{\eta}z_{j3} + (-1)^{\tilde{\eta}}z_{j3}) \rangle$$

$$= \langle (z_{j1}, z_{j2}, 0), (\tilde{z}_1, \tilde{z}_2, (1 - \tilde{\eta})\tilde{z}_3), (0, 0, \tilde{\eta}z_{j3} + (-1)^{\tilde{\eta}}\tilde{z}_3) \rangle$$

has finite index in $N_\omega$ since $\tilde{\eta} z_{j3} + (-1)^{\tilde{\eta}}\tilde{z}_3 \neq 0$. \hfill \square
Combining Proposition 10, Lemma 11 and Lemma 12 we reach the following conclusion.

**Proposition 13.** \( \Lambda_1, \ldots, \Lambda_t \) are the connected components of the graph \( \Delta_{\text{virt}}(G) \).

4. **Minimal generating sets**

Let \( G = \langle x_1, \ldots, x_d \rangle \) be a finitely generated profinite group. If \( Y \) is a minimal generating set of \( G \), then \( Y \) is finite. Indeed, for any \( 1 \leq i \leq d \), there exists a finite subset \( Y_i \) of \( Y \) such that \( x_i \in (Y_i) \). But then \( G = \langle x_1, \ldots, x_d \rangle \leq \langle Y_1, \ldots, Y_d \rangle \) so \( Y = Y_1 \cup \cdots \cup Y_d \) is finite. Let \( m(G) = \sup \{|Y| \mid Y \text{ is a minimal generating set of } G \} \). By [5] Theorem 5, \( m(G) \) is finite if and only if \( G \) contains only finitely many maximal subgroups, i.e. if and only if the Frattini subgroup of \( G \) has finite index in \( G \).

**Theorem 14.** Let \( G \) be a finitely generated profinite group. For every positive integer \( k \) with \( d(G) \leq k \leq m(G) \), there exists a minimal generating set of \( G \) of size \( k \).

**Proof.** Let \( k \in \mathbb{N} \) with \( d(G) \leq k \leq m(G) \). By the way in which \( m(G) \) is defined, there exists a minimal generating set \( X = \{x_1, \ldots, x_t\} \) of \( G \) with \( k \leq t \). For \( 1 \leq i \leq t \), let \( X_i := X \setminus \{x_i\} \). Since \( \langle X_i \rangle \neq G \), there exists an open normal subgroup \( N_i \) of \( G \) such that \( \langle X_i \rangle N_i \neq G \). Set \( N := N_1 \cap \cdots \cap N_t \). We have \( \langle X_i \rangle N \leq \langle X_i \rangle N_i < G \) for any \( 1 \leq i \leq t \), so \( \{x_1N, \ldots, x_tN\} \) is a minimal generating set for \( G/N \). By the Tarski irredundant basis theorem [1] Theorem 4.4, since \( d(G/N) \leq d(G) \leq k \leq t \leq m(G/N) \), there exists a minimal generating set \( \{y_1N, \ldots, y_kN\} \) of \( G/N \). By the profinite version of the Gaschütz Lemma (see for example [3] 15.30), there exists \( n_1, \ldots, n_k \in N \) such that \( \{y_1n_1, \ldots, y_kn_k\} \) is a (minimal) generating set of \( G \). \( \square \)

5. **The independence graph**

We will denote by \( \tilde{V}(G) \) the set of the non-isolated vertices of the independence graph \( \tilde{\Gamma}(G) \) and by \( \tilde{\Delta}(G) \) the subgraph of \( \tilde{\Gamma}(G) \) induced by \( \tilde{V}(G) \).

**Lemma 15.** Let \( G \) be a finitely generated profinite group and let \( g \in G \). Then \( g \) is isolated in \( \tilde{\Gamma}(G) \) if and only if either \( G = \langle g \rangle \) or \( g \in \text{Frat}(G) \).

**Proof.** Suppose \( g \notin \text{Frat}(G) \). There exists an open maximal subgroup \( M \) of \( G \) with \( g \notin M \). Since \( M \) is an open subgroup of a finitely generated profinite group, there exists a finite subset \( Y \) of \( M \) with \( M = \langle Y \rangle \). The set \( X = \{g\} \cup Y \) contains a minimal generating \( X \) of \( G \) and \( g \in X \) (otherwise \( G = \langle X \rangle \leq M \)). If \( X \neq \{g\} \), then \( g \) is not isolated in \( \tilde{\Gamma}(G) \), otherwise \( \langle g \rangle = G \). \( \square \)

**Lemma 16.** Let \( N \) be an open normal subgroup of a finitely generated profinite group \( G \). If \( Y = \{y_1N, \ldots, y_tN\} \) is a minimal generating set of \( G/N \), then there exist \( n_1, \ldots, n_u \in N \) such that \( \{y_1, \ldots, y_t, n_1, \ldots, n_u\} \) is a minimal generating set of \( G \).

**Proof.** Let \( Z \) be a finite generating set of \( N \). Since \( G = \langle Y, Z \rangle \), \( Y \cup Z \) contains a minimal generating set \( X \) of \( G \), and the minimality property of \( Y \) implies \( Y \subseteq X \). \( \square \)

We will write \( x_1 \sim_G x_2 \) if \( x_1 \) and \( x_2 \) belong to the same connected component of \( \tilde{\Delta}(G) \). The following lemma is an immediate consequence of Lemma 16.
Lemma 17. Let \( N \) be an open normal subgroup of a finite group \( G \) and let \( x, y \in G \). If \( xN, yN \in \tilde{V}(G/N) \) and \( xN \sim_{G/N} yN \), then \( x \sim_G y \).

Theorem 18. If \( G \) is a finitely generated profinite group, then the graph \( \tilde{\Delta}(G) \) is connected.

**Proof.** Let \( x_1, x_2 \in \tilde{V}(G) \). If follows from Lemma 14 that there exists two open normal subgroups, \( N_1 \) and \( N_2 \) of \( G \) with \( x_1N_1 \in V(G/N_1) \) and \( x_2N_2 \in \tilde{V}(G/N_2) \). Moreover there exists an open normal subgroup \( N_3 \) with \( x_1x^{-1}_2 \notin N_3 \). Let \( N = N_1 \cap N_2 \cap N_3 \). Since \( x_1N, x_2N \) are distinct elements of \( \tilde{V}(G/N) \), it follows from Theorem 1 that \( x_1N_1 \sim_{G/N} x_2N_2 \). By Lemma 17, \( x_1 \sim_G x_2 \). \( \square \)

6. THE VIRTUALLY INDEPENDENCE GRAPH

Let \( G \) be a (finitely generated) profinite group. We say that two elements \( x \) and \( y \) are virtually independent if there exists a minimal generating set on an open subgroup of \( G \) containing them. We define the virtually independent graph \( \tilde{\Gamma}_{\text{virt}}(G) \) of \( G \) as the graph whose vertices are the elements of \( G \) and in which two vertices are joined by an edge if and only if they are virtually independent. Moreover we denote by \( \tilde{\Delta}_{\text{virt}}(G) \) the subgraph induced by the non-isolated vertices of the virtually independent graph of \( G \).

Theorem 19. Let \( G \) be a finite group. If \( \tilde{\Gamma}_{\text{virt}}(G) \) contains a non-trivial isolated vertex \( x \), then either \( G = \langle x \rangle \) or one of the following occurs:

1. \( G \cong C_{p^n} \) is a cyclic group of \( p \)-power order and all the vertices of \( \tilde{\Gamma}_{\text{virt}}(G) \) are isolated.
2. \( G \cong Q_{2^n} = \langle a, b \mid a^{2^{n-1}}, b^2 = a^{2^{n-2}}, a^{-1}ba = b^{-1} \rangle \) is a generalized quaternion group and \( x = b^2 \) is the unique non-trivial isolated vertex of \( \tilde{\Gamma}_{\text{virt}}(G) \).

**Proof.** Let \( x \) be a non-trivial isolated vertex of \( \tilde{\Gamma}_{\text{virt}}(G) \). If \( y \in G \), then \( \{x, y\} \) is not a virtually independent subset, so either \( x \notin \langle y \rangle \) or \( y \notin \langle x \rangle \). In any case, \(|x, y| = 1\). So \( x \in Z(G) \). Assume \( G \neq \langle x \rangle \) and let \( p \) be a prime divisor of \(|x|\). Then \( \langle x \rangle / \langle x^p \rangle \) is the unique minimal subgroup of \( G / \langle x^p \rangle \) (if \( \langle y \rangle / \langle x^p \rangle \) were another minimal subgroup of \( G \), then \( x \) and \( y \) would be virtually independent). In particular \( G / \langle x^p \rangle \) is a non-trivial \( p \)-group and \( p \) is the unique prime divisor of \(|x|\). All the elements of order \( p \) in \( G \) belong to \( \langle x \rangle \), so \( G \) contains a unique minimal subgroup and therefore \( G \) is either cyclic of \( p \)-power order or generalized quaternion (see [3, 5.3.6]). \( \square \)

Lemma 20. Let \( G \) be a finitely generated profinite group and let \( N \) be an open normal subgroup of \( G \). If \( xN \) and \( yN \) are adjacent vertices of \( \tilde{\Gamma}_{\text{virt}}(G/N) \), then \( x \) and \( y \) are adjacent vertices of \( \tilde{\Gamma}_{\text{virt}}(G) \).

**Proof.** Let \( \{n_1, \ldots, n_t\} \) be a finite generating set of \( N \) and set \( A := \{x, y, n_1, \ldots, n_t\} \). Notice that \( H := \langle A \rangle \) is an open subgroup of \( G \) and that \( A \) contains a minimal generating set \( B \) of \( H \). Since \( xN \) and \( yN \) are adjacent vertices of \( \tilde{\Gamma}_{\text{virt}}(G/N) \), \( \langle x, y \rangle N \neq \langle x \rangle N \neq \langle y \rangle N \) and therefore \( x, y \in B \) and \( x \) and \( y \) are adjacent in \( \Gamma_{\text{virt}}(G) \). \( \square \)

Theorem 21. Let \( G \) be a finitely generated profinite group. If \( G \) is infinite and \( \tilde{\Gamma}_{\text{virt}}(G) \) contains a non-trivial isolated vertex, then \( G \) is procyclic.
Proof. Assume that \( x \) is a non-trivial isolated vertex in \( \tilde{\Gamma}_{\text{virt}}(G) \) and that \( G \) is not topologically generated by \( x \). Let \( \mathcal{N} \) be the set of the open normal subgroups \( N \) of \( G \) with the property that \( x \notin N \) and \( G \neq \langle x \rangle N \). Let \( N \in \mathcal{N} \). By Lemma 20, \( xN \) is an isolated vertex of \( \tilde{\Gamma}_{\text{virt}}(G/N) \), so, by Theorem 19, \( G/N \) is either a cyclic \( p \)-group or a generalized quaternion group. Assume that \( G/M \) is a generalized quaternion group for some \( M \in \mathcal{N} \). Since \( \cap_{N \in \mathcal{N}} N = 1 \) and no proper epimorphic image of a generalized quaternion group is generalized quaternion, we would have \( M = 1 \) and \( G \) would be finite, against our assumption. So \( G/N \) is cyclic for every \( N \in \mathcal{N} \), and therefore \( G \) is procyclic.

Lemma 22. If \( G \) is a finite group, then the graph \( \tilde{\Delta}_{\text{virt}}(G) \) is connected and \( \text{diam}(\tilde{\Delta}_{\text{virt}}(G)) \leq 3 \).

Proof. We prove the statement by induction. For this purpose, notice that it follows from Lemma 20 that if \( N \) is a normal subgroup of \( G \) and \( x_1N \) and \( x_2N \) are adjacent vertices of \( \tilde{\Delta}_{\text{virt}}(G/N) \), then \( x_1 \) and \( x_2 \) are adjacent in \( \tilde{\Delta}_{\text{virt}}(G) \) and \( \text{dist}_{\tilde{\Delta}_{\text{virt}}(G)}(x_1, x_2) \leq \text{dist}_{\tilde{\Delta}_{\text{virt}}(G/N)}(x_1N, x_2N) \). Assume that \( x_1 \) and \( x_2 \) are distinct vertices of \( \tilde{\Delta}_{\text{virt}}(G) \). If neither \( \langle x \rangle \leq \langle y \rangle \) nor \( \langle y \rangle \leq \langle x \rangle \), then \( \{x, y\} \) is a minimal generating set of \( \langle x, y \rangle \) and \( \text{dist}_{\tilde{\Delta}_{\text{virt}}(G)}(x, y) = 1 \). So we may assume \( \langle x \rangle \leq \langle y \rangle \). Let 
\[
|x| = p_1^{a_1} \cdots p_i^{a_i}, \quad |y| = p_1^{b_1} \cdots p_i^{b_i}, \quad [G] = p_1^{c_1} \cdots p_i^{c_i},
\]
with \( a_i \leq b_i \leq c_i \) for \( 1 \leq i \leq t \). First assume \( t \neq 1 \). If \( b_j = 0 \) for some \( 1 \leq j \leq t \), let \( g \) be an element of \( G \) of order \( p_j \): then \( g \) is adjacent to \( x \) and \( y \) and \( \text{dist}_{\tilde{\Delta}_{\text{virt}}(G)}(x, y) \leq 2 \). So we may assume \( b_j \neq 0 \) for every \( 1 \leq j \leq t \). Since \( y \) is not isolated in \( \tilde{\Delta}_{\text{virt}}(G) \), we have \( G \neq \langle y \rangle \). In particular there exists a prime \( p \) and a \( p \)-element \( g \in G \) with \( g \notin \langle y \rangle \). It is not restrictive to assume \( p = p_1 \). If \( a_i \neq 0 \) for some \( 1 \leq i \leq t - 1 \), then \( g \) is adjacent to \( x \) and \( y \) and \( \text{dist}_{\tilde{\Delta}_{\text{virt}}(G)}(x, y) \leq 2 \). Otherwise \( x - y^{p_i} - g - y \) is a path in \( \tilde{\Delta}_{\text{virt}}(G) \) and \( \text{dist}_{\tilde{\Delta}_{\text{virt}}(G)}(x, y) \leq 3 \). We remain with the case when \( G \) is a \( p \)-group. If there exists a minimal subgroup \( \langle y \rangle \) of \( G \) not contained in \( \langle x \rangle \), then \( x \) and \( y \) are adjacent to \( g \). So we may assume that \( G \) has a unique minimal subgroup, and consequently \( G \) is either a cyclic \( p \)-group or a generalized quaternion group.

We may exclude the first possibility, since in that case all the vertices of \( \tilde{\Gamma}_{\text{virt}}(G) \) are isolated. In the second case, \( G \) has a unique minimal subgroup, say \( N \), and \( xN, yN \) are distinct non-isolated vertices of \( G/N \) (which is a dihedral group), so \( \text{dist}_{\tilde{\Delta}_{\text{virt}}(G)}(x, y) \leq \text{dist}_{\tilde{\Delta}_{\text{virt}}(G/N)}(x, y) \leq 3 \) by induction. \( \square \)

The bound \( \text{diam}(\tilde{\Delta}_{\text{virt}}(G)) \leq 3 \) is best possible. Consider
\[
G = \langle a, b \mid a^4 = 1, b^3 = 1, b^a = b^{-1} \rangle
\]
and let \( x = a^2, y = a^2b \). The elements of \( G \) that are adjacent to \( y \) in \( \tilde{\Gamma}_{\text{virt}}(G) \) are precisely those of the form \( a^ib^j \) with \( i \) and odd integer. If \( g \) is one of these elements, then \( g^2 = a^2 = x \), so \( g \) is not adjacent to \( x \) and therefore \( \text{diam}(\tilde{\Delta}_{\text{virt}}(G)) \geq 3 \).

Theorem 23. If \( G \) is a finitely generated profinite group, then \( \tilde{\Delta}_{\text{virt}}(G) \) is connected and \( \text{diam}(\tilde{\Delta}_{\text{virt}}(G)) \leq 3 \).

Proof. By Proposition 22 we may assume that \( G \) is infinite and that \( G \) is not a procyclic pro-\( p \)-groups (otherwise all the vertices of \( \tilde{\Gamma}_{\text{virt}}(G) \) are isolated). Assume that \( x \) and \( y \) are non-isolated distinct vertices of \( \tilde{\Delta}_{\text{virt}}(G) \). There exists three open normal subgroups \( N_1, N_2 \) and \( N_3 \) with \( x \notin N_1, y \notin N_2 \) and \( xy^{-1} \notin N_3 \). Set \( M = N_1 \cap N_2 \cap N_3 \). Then \( xM \) and \( yM \) are two different vertices non-trivial...
vertices of $G/M$. We may choose $M$ in such a way that $G/M$ is neither a cyclic $p$-group nor a generalized quaternion group (this is because a proper epimorphic image of a generalized quaternion group is not generalized quaternion). But then $xM$ and $yM$ are non-isolated vertices of $\tilde{\Gamma}_{\text{virt}}(G/M)$, so by Lemmata 20 and 22 $\text{dist}_{\Delta_{\text{virt}}(G)}(x, y) \leq \text{dist}_{\Delta_{\text{virt}}(G/N)}(xN, yN) \leq 3$. □

References

1. S. Burris and H. P. Sankappanavar, A course in universal algebra. Graduate Texts in Mathematics, 78. Springer-Verlag, New York-Berlin, 1981. xvi+276 pp. ISBN: 0-387-90578-2.
2. E. Crestani and A. Lucchini, The non-isolated vertices in the generating graph of a direct power of simple groups, J. Algebraic Combin. 37 (2013), no. 2, 249–263.
3. M. Fried and M. Jarden, Field Arithmetic, Springer-Verlag, Berlin, 1986.
4. M. Hall, The theory of groups, The Macmillan Co., New York, N.Y. 1959 xiii+434 pp.
5. A. Lucchini, The largest size of a minimal generating set of a finite group, Arch. Math. (Basel) 101 (2013), no. 1, 1–8.
6. A. Lucchini, The diameter of the generating graph of a finite soluble group, J. Algebra 492 (2017), 28–43.
7. A. Lucchini, The generating graph of a profinite group, Arch. Math. (2020) https://doi.org/10.1007/s00013-020-01502-y.
8. A. Lucchini, The independence graph of a finite group, Monatsh. Math. (2020) https://doi.org/10.1007/s00605-020-01445-0.
9. D. Robinson, A course in the theory of groups. Second edition. Graduate Texts in Mathematics, 80, Springer-Verlag, New York, 1996.