COMPUTATION OF CENTRAL VALUE OF QUADRATIC
TWISTS OF MODULAR L–FUNCTIONS

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1. Introduction

Let \( f \in S_2(p) \) be a newform of weight two, prime level \( p \). If \( f(z) = \sum_{m=1}^{\infty} a(m)q^m \), where \( q = e^{2\pi iz} \), and \( D \) is a fundamental discriminant, we define the twisted \( \mathcal{L} \)-function
\[
L(f, D, s) = \sum_{m=1}^{\infty} a(m) \frac{m^s}{D^m}.
\]
It will be convenient to also allow \( D = 1 \) as a fundamental discriminant, in which case we write simply \( L(f, s) \) for \( L(f, 1, s) \).

In this paper we consider the question of computing the twisted central values \( \{ L(f, D, 1) : |D| \leq x \} \) for some \( x \).

It is well known that the fact that \( f \) is an eigenform for the Fricke involution yields a rapidly convergent series for \( L(f, D, 1) \). Computing \( L(f, D, 1) \) by means of this series, which we call the standard method, takes time very roughly proportional to \( |D| \) and therefore time very roughly proportional to \( x^2 \) to compute \( L(f, D, 1) \) for \( |D| \leq x \). We will see that this can be improved to \( x^{3/2} \) by using an explicit version of Waldspurger’s theorem \( \{W\} \); this theorem relates the central values \( L(f, D, 1) \) to the \( |D|-\)th Fourier coefficient of weight \( 3/2 \) modular forms in Shimura correspondence with \( f \).

Concretely, the formulas we use have the basic form
\[
(1.1) \quad L(f, D, 1) = \star \kappa_\pm \frac{|c_\pm(|D|)|^2}{\sqrt{|D|}}, \quad \text{sign}(D) = \mp,
\]
where \( \star = 1 \) if \( p \nmid D \), \( \star = 2 \) if \( p \mid D \), \( \kappa_- \) and \( \kappa_+ \) are positive constants independent of \( D \), and \( c_-(|D|) \) (resp. \( c_+(|D|) \)) is the \( |D|-\)th Fourier coefficient of a certain modular form \( g_- \) (resp. \( g_+ \)) of weight \( 3/2 \).

Gross \( [G] \) proves such a formula, and gives an explicit construction of the corresponding form \( g_- \), in the case that \( L(f, 1) \neq 0 \). The purpose of this paper is to extend Gross’s work to all cases. Specifically, we give an explicit construction of both \( g_- \) and \( g_+ \), regardless of the value of \( L(f, 1) \), together with the corresponding values of \( \kappa_- \) and \( \kappa_+ \) in \( [11] \). The proof of the validity of this construction will be given in a later publication and relies partly in the results of \( [BM] \).

The construction gives \( g_- \) and \( g_+ \) as linear combinations of (generalized) theta series associated to positive definite ternary quadratic forms. Computing the Fourier coefficients of these theta series up to \( x \) is tantamount to running over all lattice...
points in ellipsoid of volume proportional to $x^{3/2}$. Doing this takes time roughly proportional to $x^{3/2}$ which yields our claim above.

This approach to computing $L(f,D,1)$ has several other advantages over the standard method. First, the numbers $c(|D|)$ are algebraic integers and are computed with exact arithmetic. Once $c(|D|)$ is known it is trivial to compute $L(f,D,1)$ to any desired precision. Second, the $c(|D|)$’s have extra information; if $f$ has coefficients in $\mathbb{Z}$, for example, (1.1) gives a specific square root of $L(f,D,1)$ (if non-zero), whose sign remains a mystery.

Moreover, the actual running time of our method vs. the standard method is, in practice, significantly better even for small $x$.

2. Quaternion algebras and Brandt matrices

A quaternion algebra $B$ over a field $K$ is a central simple algebra of dimension 4 over $K$. When $2 \neq 0$ in $K$ we can give $B$ concretely by specifying a $K$-basis \{1, $i$, $j$, $k$\} such that

\[ i^2 = \alpha, \quad j^2 = \beta, \quad \text{and} \quad k = ij = -ji, \]

for some non-zero $\alpha, \beta \in K$. If $K = \mathbb{Q}$ we typically rescale and assume that $\alpha, \beta \in \mathbb{Z}$. A general element of $B$ then has the form $b = b_0 + b_1 i + b_2 j + b_3 k$, with $b_i \in K$ and multiplication in $B$ is determined by the above defining relations and $K$-linearity.

The conjugate of $b$ is defined as

\[ \overline{b} = b_0 - b_1 i - b_2 j - b_3 k. \]

We define the (reduced) norm and trace of $b$ by

\[ N(b) := b\overline{b} = b_0^2 - \alpha b_1^2 - \beta b_2^2 + \alpha \beta b_3^2, \quad Tr(b) := b + \overline{b} = 2b_0. \]

Let $B$ be a quaternion algebra over $K = \mathbb{Q}$. For $\nu$ a rational prime we let $\mathbb{Q}_\nu$ be the field of $\nu$-adic numbers and for $\nu = \infty$ we let $\mathbb{Q}_\nu = \mathbb{R}$. We call $\nu$, a rational prime or $\infty$, a place of $\mathbb{Q}$.

The localization $B_\nu := B \otimes \mathbb{Q}_\nu$ is a quaternion algebra over $\mathbb{Q}_\nu$. It is a fundamental fact of Number Theory that $B_\nu$ is either isomorphic to the algebra $M_2(\mathbb{Q}_\nu)$ of $2 \times 2$ matrices, or a division algebra, which is unique up to isomorphism. (A division algebra is an algebra in which every non-zero element has a multiplicative inverse.) The two options are encoded in the Hilbert symbol $(\alpha, \beta)_\nu$, defined as $+1$ if $B_\nu$ is a matrix algebra, $-1$ if it is a division algebra. In the first case we say that $B$ is split at $\nu$, in the second, that $B$ is ramified at $\nu$.

For example, if $\nu = \infty$ so $\mathbb{Q}_\nu = \mathbb{R}$ then $(\alpha, \beta)_\infty = -1$ if and only if $\alpha < 0, \beta < 0$ in which case $B_\infty$ is isomorphic to the usual Hamilton quaternions. A quaternion algebra $B$ is definite if it ramifies at $\infty$ otherwise it is indefinite (this notation is consistent with the nature of the quadratic form on $B_\infty$ determined by the norm $N$).

A quaternion algebra $B$ is ramified at a finite number of places and the total number of ramified places must be even (e.g. the Hilbert reciprocity law says that $\prod \nu (\alpha, \beta)_\nu = 1$). The set of ramified places determines $B$ up to isomorphism (the local-global principle). For any finite set $S$ with an even number of places there is a (unique up to isomorphism) $B$ which ramifies exactly at places in $S$.

Let $B$ be a quaternion algebra over $\mathbb{Q}$. An order in $B$ is a (full rank) lattice $R \subseteq B$ which is also a ring with $1 \in R$. As for number fields, an element of an
order must be integral over \( \mathbb{Z} \), i.e., must satisfy a monic equation with coefficients in \( \mathbb{Z} \) (or even more concretely must have integral trace and norm). Unlike in the commutative case, however, the set of all integral elements of \( B \) is not a ring. The best next thing is to consider maximal orders (which always exist), i.e., orders not properly contained in another order. But maximal orders are not unique. In fact, if \( B \) is definite, a maximal order is in general not even unique up to isomorphism though there always is only a finite number of isomorphism classes of maximal orders in \( B \).

As an illustration consider the classical case \( \alpha = \beta = -1 \) of the Hamilton quaternions. The algebra is definite and hence ramifies at \( \nu = \infty \). It must ramify at a least one other prime, which turns out to be only \( \nu = 2 \). To see this note that

\[
N(b_0 + b_1i + b_2j + b_3k) = b_0^2 + b_1^2 + b_2^2 + b_3^2.
\]

There always is a non-trivial solution to the congruence \( b_0^2 + b_1^2 + b_2^2 + b_3^2 \equiv 0 \mod p \) for \( p \) prime. If \( p \) is odd we can lift this solution to a solution in \( \mathbb{Z}_p \) by Hensel’s lemma obtaining a non-zero quaternion in \( B_p \) of zero norm. This implies that \( B_p \) cannot be a division algebra and hence \( (-1, -1)_p = 1 \) for \( p \) odd. We must necessarily have then that \( (-1, -1)_2 = -1 \).

If we want to study the representation of numbers as sum of four squares it is natural to consider, as Lipschitz did, the arithmetic of the quaternions with \( b_i \in \mathbb{Z} \). These quaternions form an order \( R' \), but, as it turns out, it is not maximal. Indeed, as Hurwitz noted, \( \rho := \frac{1}{2}(1 + i + j + k) \) is integral (\( N\rho = 1 \) and \( \text{Tr}\rho = 1 \)) and \( R := R' + \mathbb{Z}\rho \) is also an order of \( B \) strictly containing \( R' \).

Moreover, \( R \) is maximal and hence its arithmetic is significantly simpler than that of \( R' \). Hurwitz showed, for example, that there is a left and right division algorithm in \( R \), from which it follows that every positive integer is a sum of four squares.

Fix a prime \( p \) and let \( B \) be the quaternion algebra over \( \mathbb{Q} \) ramified precisely at \( \infty \) and \( p \). Let \( R \) be a fixed maximal order in \( B \). A right ideal \( I \) of \( R \) is a lattice in \( B \) that is stable under right multiplication by \( R \). Two right ideals \( I \) and \( J \) are in the same class if \( J = bI \) with \( b \in B^\times \). The set of right ideal classes is finite; let \( n \) be its number. Chose a set of representatives \( \{I_1, \ldots, I_n\} \) of the classes. (We should emphasize here that contrary to the commutative setting there is \textit{no} natural group structure on the set of classes.)

Consider the vector space \( V \) of formal linear combinations

\[
\sum_{i=1}^{n} a_i [I_i], \quad a_i \in \mathbb{C}
\]

(here \([I]\) denotes the class of \( I \)). For each integer \( m \) there is an \( n \times n \) matrix \( B_m \) acting on \( V \). Pizer \cite{P} gives an efficient algorithm for computing these \textit{Brandt matrices}: its coefficients are given by the representation numbers of the norm form for certain quaternary lattices in \( B \).

The Brandt matrices commute with each other and are self-adjoint with respect to the \textit{height pairing} on \( V \) (see \S 1 and \S 2 of \cite{G} for an account of this.) From this it follows that there is basis of \( V \) consisting of simultaneous eigenvectors of all \( B_m \).
It follows from Eichler’s trace formula that there is a one to one correspondence between Hecke eigenforms of weight 2 and level \( p \) (cf. [G, §5]) and eigenvectors in \( V \) of all Brandt matrices (up to a constant multiple).

If \( f \) is the Hecke eigenform we let \( e_f \) be the corresponding eigenvector (well defined up to a constant). Then \( B_m e_f = a_m e_f \) where \( T_m f = a_m f \) and \( T_m \) is the \( m \)-th Hecke operator.

3. Construction of \( g_- \) and \( g_+ \)

Let \( e_f \) be the eigenvector for the Brandt matrices for \( R \) corresponding to \( f \) as in the last section. One can use linear algebra to find its coefficients

\[
e_f = \sum_{i=1}^{n} a_i [I_i],
\]

by computing the Brandt matrices, and from the knowledge of a few eigenvalues (i.e. Fourier coefficients) of \( f \).

We will describe below the construction of certain generalized theta series \( \Theta_{l^*}([I_i]) \) corresponding to each ideal class \([I_i]\), and then define

\[
\Theta_{l^*}(e_f) := \sum_{i=1}^{n} a_i \Theta_{l^*}([I_i]) = \sum_{n=1}^{\infty} c_{l^*}(n) q^n.
\]

Here \( l^* \) is a fundamental discriminant for which we will consider three cases: \( l^* = 1 \), which is Gross’s construction of \( g_- \); \( l^* = l \) for an odd prime \( l \neq p \) such that \( l \equiv 1 \) (mod 4), which will generalize Gross’s construction of \( g_- \); and \( l^* = -l \) for an odd prime \( l \neq p \) such that \( l \equiv 3 \) (mod 4), which will give a construction of \( g_+ \).

Furthermore, for any fundamental discriminant \( D \) such that \( Dl^* < 0 \), the following formula holds

\[
\left( L(f, l^*, 1) L(f, D, 1) = \star \kappa_f \frac{|c_{l^*}([D])|^2}{\sqrt{|Dl^*|}}, \right.
\]

where \( \star = 1 \) if \( p \nmid D \), \( \star = 2 \) if \( p \mid D \), and \( \kappa_f := \frac{(L(f,e_f,e_f)}{(e_f,e_f)} \) is a positive constant independent of \( D \) or \( l^* \). Here \( (e_f,e_f) \) is the height of \( e_f \), and \( (f,f) \) is the Petersson norm of \( f \) (cf. §4 and §7 of [G].) For \( l = 1 \), it was proved by Gross in [G Proposition 13.5]. The proof of this formula for the case \( l \neq 1 \) will be given in a later publication.

Note that, as a corollary, we have \( \Theta_{l^*}(e_f) \neq 0 \) if and only if \( L(f,l^*,1) \neq 0 \), and this happens for infinitely many \( l^* > 0 \) and for infinitely many \( l^* < 0 \), as follows from [BFH].

3.1. Gross’s construction of \( \Theta_1 \). Let \( R_i := \{ b \in B : bI_i \subset I_i \} \) be the left order of \( I_i \). The \( R_i \) are maximal orders in \( B \), and each conjugacy class of maximal orders has a representative \( R_i \) for some \( i \).

We let \( S_i^0 := \{ b \in \mathbb{Z} + 2R_i : \text{Tr} b = 0 \} \), a ternary lattice, and define

\[
\Theta_1([I_i]) := \frac{1}{2} \sum_{b \in S_i^0} q^{N b}.
\]

Then \( \Theta_1([I_i]) \) is a weight \( 3/2 \) modular form of level \( 4p \) and trivial character.
3.2. **Weight functions and $\Theta_l$.** Fix an odd prime $l \neq p$. In order to generalize Gross’s method, we need to construct certain weight functions $\omega_l(I, \cdot)$ on $S_0^0$ with values in $\{0, \pm 1\}$. There is a choice of sign in the construction, and some care is needed to ensure that the choice is consistent from one ideal to another. It will be the case that $\omega_l(I, b) = 0$ unless $l \mid N b$, and thus we define a *generalized theta series*

$$\Theta_l([I_i]) := \frac{1}{2} \sum_{b \in S_0^0} \omega_l(I, b) q^{N b/l},$$

a modular form of weight $3/2$ and level $4p$ with trivial character. In addition, $\Theta_l([I_i])$ is already a cusp form whenever $l \neq 1$, although it might be zero.

**Definition 3.1.** Given a pair $(L, v)$, where $L$ is an integral $\mathbb{Z}_l$-lattice of rank 3 with $l \nmid \det L$, and $v \in L$ is such that $l \mid \mathcal{N} v$ but $v \notin lL$, we define its weight function $\omega_{l,v} : L \to \{0, \pm 1\}$ to be

$$\omega_{l,v}(v') := \begin{cases} 0 & \text{if } l \nmid \mathcal{N} v', \\ \chi_l((v, v')) & \text{if } l \mid (v, v'), \\ \chi_l(k) & \text{if } v' - k v \in lL. \end{cases}$$

Here $\chi_l$ is the quadratic character of conductor $l$, and $\mathcal{N} v := \frac{1}{2} \langle v, v \rangle$.

This is well defined, because if $v, v' \in L$ are such that $\mathcal{N} v \equiv \mathcal{N} v' \equiv \langle v, v' \rangle \equiv 0$ (mod $l$), then $v$ and $v'$ must be collinear modulo $l$, since $L$ is unimodular. This means that, assuming $v \notin lL$, there is indeed a well defined $k \in \mathbb{Z}/l\mathbb{Z}$ such that $v' - k v \in lL$.

Note that there are, for different choices of $v$, two different weight functions for each $L$, opposite to each other; the definition above singles out the one for which $\omega_{l,v}(v) = +1$.

We will apply the above definition to the ternary lattices $S_0^0(\mathbb{Z}_l) := S_0^0 \otimes \mathbb{Z}_l$. Fix a quaternion $b_0 \in S_0^0 := \{ b \in \mathbb{Z} + 2R : \Tr b = 0 \}$, and such that $l \mid \mathcal{N} b_0$ but $b_0 \notin lS_0^0$. For each $I_i$, find $x_i \in I_i$ such that $l \mid n_i := \mathcal{N} x_i / \mathcal{N} I_i$. Then $x_i$ is a local generator of $I_i$, and $b_i := x_i b_0 x_i^{-1} \in S_0^0(\mathbb{Z}_l)$. We finally set

$$\omega_l(I, b) := \chi_l(n_i) \omega_{l,b_0}(b),$$

where $\omega_{l,b_0}$ is the weight function of the pair $(S_0^0(\mathbb{Z}_l), b_i)$.

3.3. **Odd weight functions and $\Theta_{-l}$.** When $l \equiv 3 \pmod{4}$ the weight functions $\omega_l(I, \cdot)$ are odd, since $\chi_l$ is odd. Therefore, we will have $\Theta_l = 0$. To address this problem, we will construct a different kind of weight functions $\omega_p(I, \cdot)$, and then define

$$\Theta_{-l}([I_i]) := \frac{1}{2} \sum_{b \in S_0^0} \omega_p(I, b) \omega_l(I, b) q^{N b/l},$$

which will be a modular form of weight $3/2$, this time of level $4p^2$. Again, $\Theta_{-l}([I_i])$ is a cusp form, which might be zero. Note that we could have used the product of two odd weight functions $\omega_l$ and $\omega_p$, but this construction would only lead us to the same $g_\cdot$. By using the weight functions $\omega_p$ we get a construction of $g_\cdot$ instead.

**Definition 3.2.** Given a triple $(L, v, \psi)$ where $L$ is an integral $\mathbb{Z}_p$-lattice of rank 3 with level $p$ and determinant $p^2$ (i.e. $L$ is $\mathbb{Z}_p$-equivalent to $S_0^0(\mathbb{Z}_p)$), $v \in L$ is
such that $p \nmid N v$, and $\psi : \mathbb{Z}/p\mathbb{Z} \to \mathbb{C}$ is a periodic function modulo $p$, the weight function $\omega_{\psi,v} : L \to \mathbb{C}$ is defined by

$$\omega_{\psi,v}(v') := \psi((v,v')).$$

Clearly, the weight function $\omega_{\psi,v}$ will be odd if and only if $\psi$ itself is odd. From now on we assume that $\psi$ is a fixed odd periodic function such that

$$|\psi(t)| = 1 \quad \text{for } t \not\equiv 0 \pmod{p}. \quad (3.2)$$

Now fix $b_0 \in S^0$ such that $p \nmid N b_0$. Find $x_i \in I_i$ such that $p \nmid N x_i/N I_i$; since $B$ is ramified at $p$, the maximal order at $p$ is unique, and so $b_i := x_i b_0 x_i^{-1} \in S^0(\mathbb{Z}_p)$. We define

$$\omega_p(I_i,b) := \omega_{\psi,b_i}(b),$$

to be the weight function for the triple $(S^0(\mathbb{Z}_p), b_i, \psi)$. In practice, one can use the same $b_0$ and $b_i$ for the definitions of both $\omega(I_i,\cdot)$ and $\omega_p(I_i,\cdot)$.

Note that different choices of $\psi$ will, in general, yield different forms $\Theta_{-i}([I_i])$, but as long as (3.2) holds their coefficients will be the same up to a constant of absolute value 1; thus formula (3.1) will not be affected. Moreover, given two such odd periodic functions is not difficult to produce another periodic function $\chi$ with the property that the ratio of the $m$-th coefficients of the respective theta series will be $\chi(m)$.

The case when $\psi$ is actually a character of conductor $p$ is of particular interest, since the generalized theta series $\Theta_{-i}([I_i])$ will be a modular form of level $4p^2$ and character $\psi_1$, where $\psi_1(m) = \left(\frac{1}{p}\right) \psi(m)$. From a computational point of view, however, it will always be preferable to choose a real $\psi$, whose values will be either 0 or ±1, and so that the coefficients of $\Theta_{-i}([I_i])$ will be rational integers. Only in case $p \equiv 3 \pmod{4}$ it is possible to satisfy both requirements at the same time, by taking for $\psi$ the quadratic character of conductor $p$.

4. Examples

4.1. 11A. Let $f = f_{11A}$, the modular form of level 11, and consider $B = B(-1,-11)$, the quaternion algebra ramified precisely at $\infty$ and 11. A maximal order, and representatives for its right ideals classes, are given by

$$R = I_1 = \left\langle 1, i, -\frac{1+j}{2}, -\frac{i+k}{2} \right\rangle \quad \text{with } N I_1 = 1,$$

$$I_2 = \left\langle 2, 2i, -\frac{1+2i+j}{2}, -\frac{2+3i+k}{2} \right\rangle \quad \text{with } N I_2 = 2.$$

By computing the Brandt matrices (see §6 of [12] for this example), we find a vector

$$e_f = [I_2] - [I_1]$$

of height $\langle e_f, e_f \rangle = 5$ corresponding to $f$. Since $L(f,1) \approx 0.25384186$, Gross’s method works, and it’s easy to compute

$$\Theta_1(e_f) = \Theta_1([I_2]) - \Theta_1([I_1]) = q^3 - q^4 - q^{11} - q^{12} + q^{15} + 2q^{16} + O(q^{20}),$$

as the difference of two regular theta series corresponding to the ternary quadratic forms $[I_1]$ and $[I_2]$. 


The norm form in the given bases will be
\[
\begin{aligned}
S_0^1 &= \langle 2i, j, i+k \rangle \\
S_0^2 &= \left\langle 4i, 2i+j, \frac{7i+k}{2} \right\rangle
\end{aligned}
\]

with \(b_1 = (0, 0, 1)\),
\[
\begin{aligned}
S_0^1 &= \langle 2i, j, i+k \rangle \\
S_0^2 &= \left\langle 4i, 2i+j, \frac{7i+k}{2} \right\rangle
\end{aligned}
\]

with \(b_2 = \left(-\frac{3}{2}, 0, 2\right)\).

The norm form in the given bases will be
\[
\begin{aligned}
N_1(x_1, x_2, x_3) &= 4x_1^2 + 11x_2^2 + 12x_3^2 + 4x_1x_3,
N_2(x_1, x_2, x_3) &= 16x_1^2 + 15x_2^2 + 15x_3^2 + 14x_2x_3 + 28x_1x_3 + 16x_1x_2.
\end{aligned}
\]

This information is all that we need to compute \(\Theta_{-3} \). As an example, we show how to compute \(\Theta_{-3}(I_1)\). A simple calculation shows that

\[
\omega_3(I_1, \cdot) = \chi_3(x_1 + 24x_3 \equiv x_1 \pmod{3},
\]

and thus \(\omega_3(I_1, \cdot)\) can be computed by

\[
\omega_3(I_1, (x_1, x_2, x_3)) = \begin{cases} 
0 & \text{if } 3 \nmid N_1(x_1, x_2, x_3), \\
\chi_3(x_1) & \text{if } x_1 \equiv 0 \pmod{3}, \\
\chi_3(x_3) & \text{otherwise.}
\end{cases}
\]

Similarly, \(\omega_{11}(I_1, \cdot)\) will be given by

\[
\omega_{11}(I_1, (x_1, x_2, x_3)) = \chi_{11}(4x_1 + 2x_3).
\]

Hence we compute
\[
\Theta_{-3}(I_1) = -2q^4 + 2q^5 + 2q^9 + 2q^{12} + 2q^{20} + 2q^{25} - 2q^{37} + O(q^{48}).
\]

In a similar way one can easily get
\[
\Theta_{-3}(I_2) = q + q^4 - 3q^5 - 3q^{12} + 4q^{16} - 3q^{20} + 2q^{25} - 6q^{36} + 3q^{37} + O(q^{48}).
\]

Table 1 shows the values of \(c_{-3}(d)\) and \(L(f, d, 1)\), where \(0 < d < 200\) is a fundamental discriminant such that \(\left(\frac{d}{11}\right) = 1\). The formula
\[
L(f, d, 1) = k_{-3} \frac{c_{-3}(d)^2}{\sqrt{d}}
\]
is satisfied, where
\[ k_{-3} = \frac{1}{5} \cdot \frac{(f, f)}{L(f, -3, 1)^{\sqrt{3}}} = L(f, 1) \approx 0.2538418608559106843377589233509... \]

Note that when \( \left( \frac{d}{37} \right) \neq 1 \) it is trivial that \( c_{-3}(d) = L(f, d, 1) = 0 \).

4.2. \textbf{37A}. Let \( f = f_{37, A} \), the modular form of level 37 and rank 1, and consider \( B = B(-2, -37) \), the quaternion algebra ramified precisely at \( \infty \) and 37. A maximal order, and representatives for its right ideal classes, are given by

\[
R = I_1 = \left< 1, i, \frac{1+i+j}{2}, \frac{2+3i+k}{4} \right> \quad \text{with } N I_1 = 1,
\]
\[
I_2 = \left< 2, 2i, \frac{1+3i+j}{2}, \frac{6+3i+k}{4} \right> \quad \text{with } N I_2 = 2,
\]
\[
I_3 = \left< 4, 2i, \frac{3+3i+j}{2}, \frac{6+i+k}{2} \right> \quad \text{with } N I_3 = 4.
\]

By computing the Brandt matrices, we find a vector
\[ e_f = \frac{[I_3] - [I_2]}{2} \]
of height \( \langle e_f, e_f \rangle = \frac{1}{2} \) corresponding to \( f \). Since \( L(f, 1) = 0 \) we know that \( 2 \Theta_1(e_f) = \Theta_1([I_3]) - \Theta_1([I_2]) = 0 \). Indeed, one checks that \( R_2 \) and \( R_3 \) are conjugate, which explains the identity \( \Theta_1([I_2]) = \Theta_1([I_3]) \).

4.2.1. \textit{Imaginary twists in a case of rank 1}. Let \( l = 5 \). One can compute \( L(f, 5, 1) \approx 5.3548616 \), and thus we expect \( \Theta_5(e_f) \) to be nonzero. We note that, by the same reason that the orders are conjugate, we have \( \Theta_5([I_1]) = -\Theta_5([I_2]) \), except now there’s an extra sign, ultimately coming from the fact that \( \left( \frac{37}{5} \right) = -1 \). Thus, \( \Theta_5(e_f) = \Theta_5([I_3]) \). A basis for \( S_3^0 \) is given by
\[ S_3^0 = \left< 4i, 3i + j, \frac{3i + 2j + k}{4} \right>, \]
with the norm in this basis
\[ N_3(x_1, x_2, x_3) = 32x_1^2 + 55x_2^2 + 15x_3^2 + 46x_2x_3 + 12x_1x_3 + 48x_1x_2. \]

Choose \( b_3 = (0, 0, 1) \), with norm 15. Then
\[ \langle (x_1, x_2, x_3), b_3 \rangle = 12x_1 + 46x_2 + 30x_3 \equiv 2x_1 + x_2 \quad \text{(mod 5)}, \]
so that
\[ \omega_5(I_3, (x_1, x_2, x_3)) = \begin{cases} 
0 & \text{if } 5 \nmid N_3(x_1, x_2, x_3), \\
\chi_5(2x_1 + x_2) & \text{if } 2x_1 + x_2 \equiv 0 \pmod{5}, \\
\chi_5(x_3) & \text{otherwise}.
\end{cases} \]

Table 2 shows the values of \( c_5(d) \) and \( L(f, -d, 1) \), where \(-200 < -d < 0\) is a fundamental discriminant such that \( \left( \frac{-d}{37} \right) \neq -1 \). The formula
\[ L(f, -d, 1) = k_5 \frac{c_5(d)^2}{\sqrt{d}} \cdot \begin{cases} 
1 & \text{if } \left( \frac{-d}{37} \right) = +1, \\
2 & \text{if } \left( \frac{-d}{37} \right) = 0, \\
0 & \text{if } \left( \frac{-d}{37} \right) = -1,
\end{cases} \]
\[
\text{Table 2. Coefficients of } \Theta_5(e_f) \text{ and central values for } f = f_{37A} \\
\begin{array}{|c|c|c|c|c|c|c|}
\hline
-d & c_5(d) & L(f, -d, 1) & -d & c_5(d) & L(f, -d, 1) & -d & c_5(d) & L(f, -d, 1) \\
\hline
-3 & 1 & 2.830621 & -95 & 0 & 0.000000 & -139 & 0 & 0.000000 \\
-4 & 1 & 2.451389 & -104 & 0 & 0.000000 & -148 & -3 & 7.254107 \\
-7 & -1 & 1.853076 & -107 & 0 & 0.000000 & -151 & -2 & 1.595930 \\
-11 & 1 & 1.478243 & -111 & 1 & 0.930702 & -152 & -2 & 1.590671 \\
-40 & 2 & 3.100790 & -115 & -6 & 16.458713 & -155 & 2 & 1.575203 \\
-47 & -1 & 0.715144 & -120 & -2 & 1.790242 & -159 & 1 & 0.388816 \\
-67 & 6 & 21.562911 & -123 & 3 & 3.978618 & -164 & -1 & 0.382843 \\
-71 & 1 & 0.538150 & -127 & 1 & 0.435051 & -165 & 0 & 0.000000 \\
-83 & -1 & 0.538150 & -132 & 3 & 3.840589 & -195 & 2 & 1.404381 \\
-84 & -1 & 0.534937 & -136 & 4 & 6.726557 & & & \\
\hline
\end{array}
\]

\[
\text{Table 3. Coefficients of } \Theta_{-3}(e_f) \text{ and central values for } f = f_{37A} \\
\begin{array}{|c|c|c|c|c|c|c|}
\hline
-d & c_{-3}(d) & L(f, d, 1) & -d & c_{-3}(d) & L(f, d, 1) & -d & c_{-3}(d) & L(f, d, 1) \\
\hline
5 & 1 & 5.354862 & 76 & 1 & 1.373493 & 133 & 1 & 1.038263 \\
8 & 1 & 4.233390 & 88 & 1 & 1.276415 & 140 & -3 & 9.107764 \\
13 & -1 & 3.320944 & 89 & -1 & 1.269224 & 156 & -1 & 0.958674 \\
17 & 1 & 2.904081 & 92 & 2 & 4.993434 & 161 & -2 & 3.774681 \\
24 & -1 & 2.444149 & 93 & 2 & 4.966515 & 165 & 1 & 0.932162 \\
29 & 2 & 8.893941 & 97 & 0 & 0.000000 & 168 & -1 & 0.923801 \\
56 & -1 & 1.600071 & 105 & 1 & 1.168527 & 172 & 1 & 0.912996 \\
57 & 1 & 1.585973 & 109 & -1 & 1.146885 & 177 & 0 & 0.000000 \\
60 & -1 & 1.558158 & 113 & 0 & 0.000000 & 193 & -1 & 0.861895 \\
61 & 0 & 0.000000 & 124 & 0 & 0.000000 & & & \\
69 & 0 & 0.000000 & 129 & 1 & 1.054237 & & & \\
\hline
\end{array}
\]

is satisfied, where

\[
k_5 = 2 \cdot \frac{(f, f)}{L(f, 5, 1) \sqrt{5}} \approx 4.902778763973580121708449663733... \]

Note that in the case \((-d^3) = -1\) it is trivial that \(c_5(d) = L(f, -d, 1) = 0\).

4.2.2. Real twists in a case of rank 1. Let \(l = 3\), since \(L(f, -3, 1) \approx 2.9934586\). Keep \(b_3\) as above, and let \(\psi\) be the odd periodic function modulo 37 such that

\[
\psi(x) = \begin{cases} 
+1 & \text{if } 1 \leq x \leq 18, \\
-1 & \text{if } 19 \leq x \leq 36.
\end{cases}
\]

Using again \(\Psi\), we have that

\[
\omega_3(I_1, (x_1, x_2, x_3)) = \begin{cases} 
0 & \text{if } 3 \nmid N_1(x_1, x_2, x_3), \\
\chi_3(x_2) & \text{if } x_2 \not\equiv 0 \pmod{3}, \\
\chi_3(3) & \text{otherwise}.
\end{cases}
\]

and \(\omega_{11}(I_1, \cdot)\) will be given by

\[
\omega_{37}(I_1, (x_1, x_2, x_3)) = \psi(12x_1 + 9x_2 + 30x_3).
\]
Table 3 shows the values of $c_{-3}(d)$ and $L(f,d,1)$, where $0 < d < 200$ is a fundamental discriminant such that $(\frac{d}{37}) = -1$. The formula

$$L(f,d,1) = k_{-3} \frac{c_{-3}(d)^2}{\sqrt{d}}$$

is satisfied, where

$$k_{-3} = 2 \cdot \frac{(f,f)}{L(f,-3,1)^{3/2}} \approx 11.97383458492783851932803991781...$$

Note that in the case $(\frac{d}{37}) \neq -1$ it is trivial that $c_{-3}(d) = L(f,d,1) = 0$.

4.3. 43A. Let $f = f_{43A}$, the modular form of level 43 and rank 1. Let $B = B(-1, -43)$, the quaternion algebra ramified precisely at $\infty$ and 43. A maximal order, and representatives for its right ideals classes, are given by

$$R = I_1 = \langle 1, i, \frac{1+j}{2}, \frac{i+k}{2} \rangle$$

with $\mathcal{N} I_1 = 1$,

$$I_2 = \langle 2, 2i, \frac{1+2i+j}{2}, \frac{2+3i+k}{2} \rangle$$

with $\mathcal{N} I_2 = 2$,

$$I_3 = \langle 3, 3i, \frac{1+2i+j}{2}, \frac{2+5i+k}{2} \rangle$$

with $\mathcal{N} I_3 = 3$,

$$I_4 = \langle 3, 3i, \frac{1+4i+j}{2}, \frac{4+5i+k}{2} \rangle$$

with $\mathcal{N} I_4 = 3$.

By computing the Brandt matrices, we find a vector

$$e_f = \frac{[I_4] - [I_3]}{2}$$

of height $\langle e_f, e_f \rangle = \frac{1}{2}$ corresponding to $f$. 

| $-d$ | $c_5(d)$ | $L(f,-d,1)$ | $-d$ | $c_5(d)$ | $L(f,-d,1)$ | $-d$ | $c_5(d)$ | $L(f,-d,1)$ |
|------|----------|-------------|------|----------|-------------|------|----------|-------------|
| -3   | 1        | 3.148135    | -91  | 1        | 0.571601    | -151 | 1        | 0.443737    |
| -7   | 1        | 2.060938    | -104 | 1        | 0.534684    | -155 | 1        | 0.437974    |
| -8   | -1       | 1.927831    | -115 | -3       | 4.576227    | -159 | 1        | 0.432430    |
| -19  | 2        | 5.003768    | -116 | -1       | 0.506273    | -163 | 7        | 20.927447   |
| -20  | -1       | 1.219267    | -119 | -1       | 0.499851    | -168 | -2       | 1.682749    |
| -39  | -1       | 0.873136    | -120 | 0        | 0.000000    | -179 | -1       | 0.467556    |
| -43  | 2        | 6.552268    | -123 | -5       | 12.291402   | -184 | -3       | 3.617825    |
| -51  | 1        | 0.763535    | -131 | 0        | 0.000000    | -191 | 0        | 0.000000    |
| -55  | 1        | 0.735246    | -132 | 3        | 4.271393    | -199 | 0        | 0.000000    |
| -71  | 0        | 0.000000    | -136 | -1       | 0.467568    |       |          |             |
| -88  | 3        | 5.231366    | -148 | -4       | 7.171386    |       |          |             |

Table 4. Coefficients of $\Theta_5(e_f)$ and central values for $f = f_{43A}$
4.3.1. **Imaginary twists in a case of rank 1.** We can use \( l = 5 \), since \( L(f, 5, 1) \approx 4.8913446 \) is nonzero; again, we find \( \Theta_5(e_f) = \Theta_5([I_4]) \). Table 4 shows the values of \( c_5(d) \) and \( L(f, -d, 1) \), where \(-200 < -d < 0\) is a fundamental discriminant such that \( (\frac{-d}{43}) \neq -1 \). The formula

\[
L(f, -d, 1) = k_5 \frac{c_5(d)^2}{\sqrt{d}} \cdot \begin{cases} 
1 & \text{if } (\frac{-d}{43}) = +1, \\
2 & \text{if } (\frac{-d}{43}) = 0, \\
0 & \text{if } (\frac{-d}{43}) = -1, 
\end{cases}
\]

is satisfied, where

\[
k_5 = 2 \cdot \frac{(f, f)}{L(f, 5, 1)\sqrt{5}} \approx 5.452729672681734385570722785283...
\]

Note that in the case \( (\frac{-d}{43}) = -1 \) it is trivial that \( c_5(d) = L(f, -d, 1) = 0 \).

4.3.2. **Real twists in a case of rank 1.** We can use \( l = 3 \), since \( L(f, -3, 1) \approx 3.1481349 \), and let \( \psi = \chi_{43} \) be the quadratic character of conductor 43. Table 5 shows the values of \( c_{-3}(d) \) and \( L(f, d, 1) \), where \( 0 < d < 200 \) is a fundamental discriminant such that \( (\frac{d}{43}) = -1 \). The formula

\[
L(f, d, 1) = k_{-3} \frac{c_{-3}(d)^2}{\sqrt{d}}
\]

is satisfied, where

\[
k_{-3} = 2 \cdot \frac{(f, f)}{L(f, -3, 1)\sqrt{3}} \approx 10.937379059935167648758735438779...
\]

Note that in the case \( (\frac{d}{43}) \neq -1 \) it is trivial that \( c_{-3}(d) = L(f, d, 1) = 0 \).

4.4. **389A.** Let \( f = f_{389A} \), the modular form of level 389 and rank 2. Let \( B = B(-2, -389) \), the quaternion algebra ramified precisely at \( \infty \) and 389. A maximal order, with 33 ideal classes, is given by

\[
R = \left\{ 1, i, \frac{1+i+j}{2}, \frac{2+3i+k}{4} \right\}.
\]

There is a vector \( e_f \) of height \( \langle e_f, e_f \rangle = \frac{5}{2} \) corresponding to \( f \).
4.4.1. Imaginary twists in a case of rank 2. We can use $l = 5$, since $L(f, 5, 1) \approx 8.9092552$. We have omitted the 33 ideal classes; however, the computation of $\Theta_l(e_f)$ involves only 14 distinct theta series. In Table 6 we give the value of $e_f$ and the coefficients of the norm form $N_i$ and of $b_i$ on chosen bases of $S_0^i$.

Each row in the table allows one to compute an individual theta series $h_i(z) := \frac{1}{2} \sum_{b \in \mathbb{Z}^3} w_5(I_i, b) q^{N_i(b)/5}$. The ternary form corresponding to a sextuple $(A_1, A_2, A_3, A_{23}, A_{13}, A_{12})$ is

$$N_i(x_1, x_2, x_3) = A_1 x_1^2 + A_2 x_2^2 + A_3 x_3^2 + A_{23} x_1 x_2 + A_{13} x_1 x_3 + A_{12} x_2 x_3,$$

and $\omega_5(I_i, \cdot)$ is the weight function of the pair $(\mathbb{Z}^3, b_i)$. As an example, we show how to compute $h_1(z)$. First, we have

$$N_1(x_1, x_2, x_3) = 15x_1^2 + 107x_2^2 + 416x_3^2 - 100x_2 x_3 - 8x_1 x_3 - 14x_1 x_2.$$

A simple calculation shows that

$$\langle (x_1, x_2, x_3), (2, 4, 0) \rangle \equiv 4x_1 + 3x_2 + 4x_3 \pmod{5}.$$

Thus, $\omega_5$ can be computed as

$$\omega_5(I_1, (x_1, x_2, x_3)) = \begin{cases} 0 & \text{if } 5 \nmid N_1(x_1, x_2, x_3), \\ \chi_5(4x_1 + 3x_2 + 4x_3) & \text{if } \not\equiv 0 \pmod{5}, \\ \chi_5(x_2) & \text{otherwise,} \end{cases}$$

and we have

$$h_1(z) = q^3 - q^{12} - q^{27} + q^{39} + q^{40} + q^{48} - q^{83} - 2q^{92} + O(q^{100}).$$

Finally, we combine all of the theta series in

$$\Theta_5(e_f) = \sum_{i=1}^{14} a_i h_i(z)$$

| $i$ | $a_i$ | $N_i$ | $b_i$ |
|-----|-------|-------|-------|
| 1   | 1/2   | 15, 107, 416, -100, -8, -14 | 2, 4, 0 |
| 2   | -1/2  | 15, 104, 415, 104, 2, 4 | 0, 4, 1 |
| 3   | -1/2  | 23, 136, 203, 68, 2, 8 | 2, 1, 4 |
| 4   | 1/2   | 23, 72, 407, 72, 10, 20 | 1, 1, 0 |
| 5   | -1/2  | 31, 51, 407, -46, -26, -10 | 1, 2, 0 |
| 6   | 1/2   | 31, 103, 204, 56, 20, 18 | 2, 0, 3 |
| 7   | 1/2   | 39, 128, 160, -116, -8, -36 | 1, 1, 4 |
| 8   | -1/2  | 39, 40, 399, 40, 2, 4 | 1, 1, 4 |
| 9   | 1/2   | 47, 101, 135, 42, 22, 38 | 2, 4, 3 |
| 10  | -1/2  | 56, 84, 139, 56, 4, 12 | 3, 1, 4 |
| 11  | 1/2   | 56, 92, 151, 76, 52, 44 | 4, 2, 3 |
| 12  | 1/2   | 71, 83, 132, -16, -12, -70 | 2, 3, 4 |
| 13  | -1/2  | 71, 103, 124, -36, -64, -66 | 4, 0, 2 |
| 14  | -1/2  | 71, 103, 124, -36, -64, -66 | 4, 0, 2 |

Table 6. Coefficients of the ternary forms and of $b_i$.
COMPUTATION OF CENTRAL VALUE

\[ -d \quad c_5(d) \quad L(f, -d, 1) \quad -d \quad c_5(d) \quad L(f, -d, 1) \quad -d \quad c_5(d) \quad L(f, -d, 1) \]

| -d |  1 |  4.553533 | -83 | -1 | 0.865705 | -139 | -1 | 0.668962 |
| -8 | -1 |  2.788458 | -84 |  1 | 0.860537 | -148 |  6 | 23.338921 |
| -15 | -1 |  2.036402 | -88 | -4 | 13.452028 | -151 |  2 | 2.567324 |
| -23 |  1 |  1.644543 | -103 |  0 | 0.000000 | -152 | -1 | 0.639716 |
| -31 |  1 |  1.416538 | -104 | -1 | 0.773379 | -155 |  3 | 5.701456 |
| -39 |  1 |  1.262923 | -107 |  0 | 0.000000 | -163 |  8 | 39.536232 |
| -40 |  1 |  1.247036 | -115 | -1 | 0.735462 | -167 | -1 | 0.610311 |
| -43 | -3 | 10.824738 | -116 | -2 | 2.929140 | -191 |  1 | 0.570680 |
| -47 |  0 | 0.000000 | -123 |  3 | 6.400282 | -195 |  1 | 0.564796 |
| -51 | -2 |  4.417576 | -131 |  1 | 0.689086 | -199 | -1 | 0.559091 |
| -56 |  1 |  1.053938 | -132 | -2 | 2.745884 | -202 | -2 | 7.652032 |
| -71 |  1 |  0.936009 | -136 | -2 | 2.705202 |

Table 7. Coefficients of \( \Theta_5(\epsilon_f) \) and central values for \( f = f_{389A} \)

Table 7 shows the values of \( c_5(d) \) and \( L(f, -d, 1) \), where \( 0 < -d < 200 \) is a fundamental discriminant such that \( \left( \frac{-d}{389} \right) \neq +1 \). The formula

\[
L(f, -d, 1) = k_5 \frac{c_5(d)^2}{\sqrt{d}} \cdot \begin{cases} 
1 & \text{if } \left( \frac{-d}{389} \right) = -1, \\
2 & \text{if } \left( \frac{-d}{389} \right) = 0, \\
0 & \text{if } \left( \frac{-d}{389} \right) = +1,
\end{cases}
\]

is satisfied, where

\[
k_5 = \frac{2}{5} \cdot \frac{(f, f)}{L(f, 5, 1)\sqrt{5}} \approx 7.886950806206592817689630792605...
\]

Note that when \( \left( \frac{-d}{389} \right) = +1 \) it is trivial that \( c_5(d) = L(f, -d, 1) = 0 \).

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