The Polytrope Index Revealed: Implications for Planet, Solar and Material Models

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ABSTRACT
Techniques to model the interior of planets are varied. We introduce a new approach to a century old assumption which enhances not only planetary interior calculations but also solar models and high pressure material physics. Our methodology uses the polytrope assumption which was used to model main sequence and white dwarf stars by Eddington (1916) and Chandrasekhar (1931). A polytrope is a simple structural assumption between a material’s pressure and volume, \( PV^n = C \), where \( C \) is a constant and \( n \) is the polytrope index. We derive that the polytropic index is the derivative of the bulk modulus with respect to pressure. We then augment the theory by including a variable polytrope index which produces a high quality universal equation of state, within the confines of the Lane-Emden differential equation (Chandrasekhar 1939), making it a robust tool with the potential for excellent predictive power. Unlike most previous equations of state, which have pressure as the dependent variable, the theoretical foundation of our equation of state is the same elastic observable which we found equivalent to the polytrope index. We calculate the density-pressure of six common materials up to \( 10^{18} \) Pa, mass-radius relationships for the same materials, and produce plausible density-radius models for Mars, Jupiter, and Uranus. An examination of the diversity exhibited by universal equations of state follows, specifically how they functionally negotiate the pressure derivative of the bulk modulus. We analyze the potential of our model using planet Earth, our best static laboratory, ascertaining the ability of our model to include temperature. We end by constraining the material surface observables for the inner core, outer core, and mantle of planet Earth.

Key words: planets and satellites:formation - stars:formation - methods:analytical - methods: numerical

1 INTRODUCTION
As the number of exoplanets discovered increases (ExoSolar Planets Encyclopedia 2014), the need for a systematic method to develop planetary models becomes more pronounced (Zapolsky & Salpeter 1969; Stacey & Davis 2004; Sotin, Grasset & Mocquet 2007; Seager et al. 2007; Grasset, Schneider & Sotin 2009; Lecante & Chabrier 2012; Swift et al. 2012; Wagner et al. 2012; Zeng & Sasselov 2013; Alibert 2014). This work begins with the well known polytrope assumption at its foundation. A polytrope is a simple structural assumption between a material’s pressure and volume, \( PV^n = C \), or similarly it’s pressure and density, \( P = C \rho^n \). The \( C \) is a constant and \( n \) is the polytrope index which we will find to be equivalent to the derivative of the bulk modulus with respect to pressure.

Polytropes first made their appearance in the 19th century in the context of perfect (ideal) gases. The two most important being the isothermal, \( PV = C \), and adiabatic conditions, \( PV^n = C \), where the polytrope indices are 1 and \( \gamma \) (the traditional heat capacity ratio) respectively. From this humble beginning the use of polytropes soon expanded to astrophysics (Eddington 1916).
2 THE POLYTROPE INDEX IS A DERIVATIVE OF THE BULK MODULUS

We begin our exposition of the polytrope index with an example from introductory chemistry. Starting with the ideal gas law, $PV^n = C$ (assuming $C$ is constant), we take the derivative with respect to volume, $\frac{d}{dV}(PV^n) = 0$. After applying the product rule we have

$$-V \frac{dP}{dV} \equiv B = nP.$$  \hspace{1cm} (1)

The isotropic bulk modulus, $B$, is a standard stress definition (the inverse of the bulk modulus is the compressibility, $K$). This ideal equation of state (EOS), which is written in polytrope form, has a bulk modulus, $B$, proportional to the product of the polytrope index and pressure, $nP$. If the polytrope index is assumed constant, the derivative of the bulk modulus with respect to pressure, $\frac{dB}{dP}$, is plainly the polytrope index $n$. Although this example was limited, the validity of this statement extends well beyond the confines of the ideal gas law as we will show in the next subsection.

This ideal gas example began with a polytrope equation of state as an assumption which confines its applicability (an interaction like van der Waals, $(P + a/v^2)(v - b) = RT$, cannot be written in that form). However the result, that the exponent $n$ is the pressure derivative of the bulk modulus, is congruent with Chandrasekhar’s ideal gas result

$$n = \left(\frac{dQ}{dP} - \frac{dQ}{dT}\right)/\left(\frac{dQ}{dV} - \frac{dQ}{dT}\right),$$  \hspace{1cm} (2)

for the special cases of isobaric ($n = 0$), isothermal ($n = 1$), adiabatic ($n = \gamma = c_p/c_v$), and isochoric($n = \infty$) (Chandrasekhar 1939). We will now show that the polytropic index is equivalent to the derivative of the bulk modulus with respect to pressure ($n \equiv \frac{dB}{dP}$) in all equations of state where this elastic observable can be treated as a constant.

2.1 Using Bulk Modulus as Starting Point

Starting with the definition of the bulk modulus we manipulate:

$$\text{from } B = K^{-1} = \rho \frac{dP}{d\rho},$$

then $K^{-1} \frac{d\rho}{dP} = \rho$. 

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then \( \frac{dK}{dP} K^{-1} \frac{d\rho}{dP} = \frac{dK}{dP} \rho \),

(3)

and by using the relationship, \( \frac{dB}{dP} K^{-1} = -\frac{dK}{dP} K = -\frac{d\rho}{dP} K \) we are able to progress to

\[ \frac{dK}{dP} = -\frac{dB}{dP} K \]

(4)

and then multiplying both sides by the awkward \( \rho \frac{dK}{dP} K^{-1} \)

\[ \rho \frac{dK}{dP} K^{-1} \frac{d\rho}{dP} = -\frac{dB}{dP} K \]

(5)

Though this looks complicated we are able to notice a derivative relationship of the form \( \frac{d}{dP}(X^n Y) = X^{n-1} n Y \frac{dX}{dP} + \frac{dY}{dP} X^n \)

and so we have the intermediate result

\[ \frac{d}{dP}(\rho K) = 0 \quad \text{or} \quad \rho \frac{dK}{dP} K = C', \]

(6)

where \( C' \) is a constant. From the \( P = 0 \) boundary condition we have

\[ C' = \rho_0^n K_0 = \rho_0^n K_0, \]

(7)

where the naught subscript represents the values of these variables at zero pressure. This derivation only requires that the exponent, the pressure derivative of the bulk modulus, remains constant because a derivative power law was used to derive it. This relationship, between density and the bulk modulus, has been shown previously by Stacey & Davis (2004, 2008a) in the context of the Murnaghan equation of state (EOS) (Murnaghan 1944). This EOS also assumes that the bulk modulus pressure derivative is constant and thus it reproduces Eq. (7). The derivation using the Murnaghan EOS is reproduced in Appendix A.

This result, \( \rho^n K = C' = \rho_0^n K_0 \), can be demonstrated as equivalent to a polytrope when \( \frac{dB}{dP} \) is a constant. Because this derivative is unchanging, it is equivalent to the value at zero pressure, \( \frac{dB}{dP} \equiv B_0' \). The first step in producing the polytrope is replacing the compressibility with its definition

\[ \rho^n K = \frac{1}{\rho} \frac{d\rho}{dP} C'. \]

(8)

This equation is separable with \( \rho \) and \( P \) so multiplying both sides by \( dP \) and integrating

\[ \int \rho^{n-1} d\rho = \int C' dP, \]

(9)

we find

\[ \frac{\rho^n}{n} = C' P + D, \]

(10)

which is in a modified polytrope form allowing for an additional intercept \( D = \rho_0^n / n \) which can be non-zero (solids and liquids described by the Murnaghan EOS or self-bound neutron stars as in Lattimer & Prakash (2001)). In the special case when \( D = 0 \) (if \( \rho \approx 0 \) when \( P = 0 \)) we recover the normal form Eddington used:

\[ P = \frac{\rho^n}{n C'} C' \rho^n, \]

(11)

thus using Eq. (8)

\[ \frac{1}{C'} = n C' = n \rho_0^n K_0 = \frac{n \rho_0^n}{B_0}. \]

(12)

We have successfully recovered a mathematically equivalent polytrope, Eq. (8), using the bulk modulus and compressibility instead of pressure. More importantly we have shown that the polytrope index is the derivative with respect to pressure of the bulk modulus when this polytrope index is constant.

The variables that make up our modified polytrope, \( \rho^n K = C' \), are not independent but are intertwined by derivative relationships, \( n = \frac{dB}{dP} = \frac{dK}{dP} B \), which when this equation is multiplied by \( B = \rho \frac{dB}{dP} \) produces

\[ n B = \rho \frac{dB}{dP} B \quad \text{or} \quad n = \frac{dB}{dP} = \frac{dK}{dP} B. \]

(13)

These interdependencies provide benefits which will be discussed later.

Our modified polytrope form of Eq. (10) is also equivalent to the Murnaghan EOS (Murnaghan 1944), which is often expressed in this form

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\[ P = C \rho^n - nCD = \frac{B_i}{n} \left( \rho \frac{\rho_i}{\rho_0^n} \right)^n - 1, \]  

(14)

where again \( n \equiv \frac{dB}{dP} \) (see Appendix A for a derivation). So in summary we have derived that the polytrope index is mathematically equivalent to the definition of the pressure derivative of the bulk modulus assuming it is constant. Since these same assumptions make up the Murnaghan EOS than the modified polytrope assumption is also equivalent to the Murnaghan EOS. Although the Murnaghan is well established, its connection to the polytrope has been under appreciated. In essence the Murnaghan equation is an explicit derivation of the polytrope in a more general form (Murnaghan noted that his equation resembled the Laplace Law used to fit the interior of the Earth).

2.2 Ramifications of revealing the index as an elastic observable

The insight gained is significant when recognizing that the polytrope index is equivalent to the derivative of the bulk modulus with respect to pressure. When the traditional polytrope is successful \( (P = C \rho^n) \), most notably in the interior of stars, it can be said that the polytrope exponent, \( n \), is the derivative with respect to pressure of the bulk modulus. It can further be said that this derivative in that star interior is close to constant. So the interior of a main sequence star, which fits closely to a 4/3 index, implies that the derivative with respect to pressure of the bulk modulus is also 4/3.

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Likewise the interior of a cold white dwarf star, which fits closely to a 5/3 index, implies that the derivative with respect to pressure of the bulk modulus is 5/3. A hot white dwarf star, which follows a relativistic derivation, implies that the derivative with respect to pressure of the bulk modulus is 4/3 and its polytropic index is also 4/3. Likewise the interior of neutron stars, which fits closely to a polytropic index of 2, implies that the derivative with respect to pressure of the bulk modulus is also 2. These statements work in reverse also. If the pressure derivative of the bulk modulus is fixed then \( n = \frac{dB}{dP} \) and Eqs. (10,14) hold as well as

\[ \rho^n K = \rho_i^n K_i, \]  

(15)

where \( i \) is any spot where the derivative with respect to pressure of the bulk modulus is near constant.

Another accurate statement is that \( C = \frac{\rho}{n \rho^n} \) is a constant throughout the interior of the star which puts powerful constraints on the material that makes up the interior of the star. It requires that \( \frac{B_i}{\rho_i^n} = nC = \text{constant} \) where \( B_i \) and \( \rho_i \) are the bulk modulus and density at any spot in the interior of the star respectively when \( n \equiv \frac{dB}{dP} \) is constant. Also from

\[ P = \frac{B_i}{\rho_i^n} \rho^n, \]  

(16)

we find, using Eq. (12), that

\[ n = \frac{B_i}{P_i}. \]  

(17)

This result, that at high pressures \( \frac{dB}{dP} \) is a ratio of the bulk modulus over pressure is obvious but often unexploited, it is also found in Stacey (2000); Stacey & Davis (2004). It is true of any material which can be fit to a traditional constant polytrope including the interior of stars and isothermal and adiabatic ideal gases.

For specific classes of stars the polytropic constant,

\[ C = \frac{B_i}{n \rho_i^n}, \]  

(18)

can be equated with the thermodynamic results used to derive the polytrope EOS. For example the "Eddington solution" for main sequence stars (Eddington 1916) would equate the ratio of the bulk modulus over the pressure derivative at the core of the star with radiative and gas pressure variables. Likewise the white dwarf stars produce a polytrope index of 5/3 (classically) or 4/3 (relativistically) based on the standard statistical mechanical derivation treating the interior as a degenerate electron gas thus connects the same ratio, \( \frac{dB}{dP} \), to the mass of the electron, proton, and quantum mechanical ion concentration. The polytrope constant can also be used to put constraints on the mass-radius relationship.

Another wonderful example involves neutron stars. Lattimer and Prakash (2001) compared fourteen neutron-neutron microscopic interactions as they were extrapolated to model the interior of a neutron star. The fourteen models produced dramatically different density-pressure ranges yet they all fit polytropes with an index of very close to 2. So in the extreme conditions of the interior of a neutron-neutron star the models differ on the calculation of the core bulk modulus and pressure but all ratios are similar \( \frac{dB}{dP} \approx 2 \). As in the white dwarf stars this polytropic constant also puts constraints on the mass-radius relationship for the star. Note the inference that the polytrope index, with its physical insight \( (n = \frac{dB}{dP}) \), now becomes a revitalized tool for modern research.
3 DEVELOPING THE FIXED POLYTROPE TECHNIQUE

Now on to deriving the gravitational differential equation for large spheres. This derivation is similar to those traditional derivations found in Chandrasekhar (1939) and Horedt (2004) except now we use the modified form of our polytrope, $\rho^n K = \rho_0^n K_0$, from Eq. (1), instead of the traditional Eddington form, $P = C \rho^n$. One advantage being that Eq. (1) is more general and holds even when the density does not approach zero in the limit of zero pressure, another is that we replace the pressure as our independent variable with elastic constants.

If $P$ (pressure) and $g$ (gravity) are functions of the radial distance $r$, as measured from the core, one may start by considering the equilibrium of radial force components on a concentric spherical shell element of interior radius $r$ and exterior radius $r + dr$, within which differential changes $dr$, $dP$, $dp$ and $dg$ alter their respective variables. Writing the forces as pressure times area and mass times gravitational acceleration and equating radial force components to zero, with the neglect of second order differentials, one obtains a differential equation for the pressure of the form

$$\frac{dP}{dr} = -\rho(r) g(r)$$  \hspace{1cm} (19)

This equation expresses the Newtonian equilibrium of hydrostatic forces. The density is affected by pressure through a compressibility relation of the form

$$K(\rho) = -\frac{1}{V} \frac{dV}{dP} = \frac{1}{\rho} \frac{d\rho}{dP}$$  \hspace{1cm} (20)

In writing these expressions it is to be noted that in a volume element whose mass is conserved, $dm = 0$, and thus $0 = d(\rho V) = \rho dV + d\rho$ and thus $\frac{d\rho}{\rho} = -\frac{dV}{V}$. From Eq. (20) it is easily seen that $\rho K(\rho) = dp/dP$, while $d\rho/dr = (dp/dP)(dP/dr)$. Therefore, Eq. (19) can be written as

$$\frac{dP}{dr} = -\rho^2(r) K(\rho) g(r).$$  \hspace{1cm} (21)

This expression depends upon an understanding of how the compressibility $K$ varies as a function of density. First one must observe from Eq. (21) that

$$g(r) = -\frac{1}{\rho K(\rho)} \frac{dp}{dr}$$  \hspace{1cm} (22)

Also, the mass $m(r)$ within a sphere of radius $r$ is given by $m(r) = \int_0^r 4\pi r^2 \rho(r)dr$. It is now clear that $g(r)$ can also be written as

$$g(r) = \frac{Gm(r)}{r^2} = \frac{G}{r^2} \int_0^r 4\pi r^2 \rho(r)dr,$$  \hspace{1cm} (23)

where $G$ is the Newtonian universal gravitation constant. Differentiating this expression, one now obtains

$$\frac{dg}{dr} = G \left[ 4\pi \rho(r) - \frac{2}{r^2} \int_0^r 4\pi r^2 \rho(r)dr \right].$$  \hspace{1cm} (24)

Using this result along with Eq. (22), we may now write

$$\frac{d}{dr} \left[ \frac{1}{\rho^2 K(\rho)} \frac{dp}{dr} \right] = G \left[ 4\pi \rho(r) - \frac{2}{r^2} \int_0^r 4\pi r^2 \rho(r)dr \right].$$  \hspace{1cm} (25)

In this expression, the second term on the right side can be identified with the help of Eq. (23), as $2G/r^3 \int_0^r 4\pi r^2 \rho(r)dr = 2g(r)/r$, which with the further aid of Eq. (22), leads finally to

$$\frac{d}{dr} \left[ \frac{1}{\rho^2 K(\rho)} \frac{dp}{dr} \right] - \frac{2}{r^2 \rho^2 K(\rho)} \frac{dp}{dr} - 4\pi G \rho(r) = 0.$$  \hspace{1cm} (26)

This can be put in a more explicit form by performing the indicated differentiation with respect to $r$ in the leading term. In so doing it is convenient to make a substitution of the form $\nu(\rho) = \rho^2 K(\rho)$, and using this relationship: $\frac{d\nu}{d\rho} = \frac{d\rho}{d\nu} \frac{d\rho}{d\nu}$. The differentiation with respect to $r$ will clearly yield a nonlinear term containing a factor of the squared derivative $(dp/dr)^2$. The algebra is straightforward though tedious, but finally leads to a nonlinear differential equation of the form

$$\frac{d^2 \rho}{dr^2} = \left[ \frac{2}{\rho} + \frac{K'}{K} \right] \left( \frac{dp}{dr} \right)^2 + \frac{2 dp}{r \rho dr} + 4\pi G \rho^3 K(\rho) = 0,$$  \hspace{1cm} (27)

where $K'$ is the derivative of the compressibility with respect to density. One can also alternatively use Eqs. (26) (28) to remove compressibility for pressure and get a version equivalent to the original Lane-Emden form used by Eddington,

$$\frac{d^2 P}{dr^2} + \left[ \frac{2}{r} - \frac{1 dp}{\rho dr} \right] \frac{dP}{dr} + 4\pi G \rho^2 = 0.$$  \hspace{1cm} (28)
This traditional version is mathematically simpler than the version derived above but it cannot use our modified polytrope assumption, $\rho_0 K = \rho^n K$, so we keep the more complicated Eq. (27).

Now assuming the density-compressibility relationship of our modified polytrope

$$\rho_0 K = \rho^n K \quad \text{or} \quad K = \frac{\rho_0^n K_0}{\rho^n},$$

Where $\rho_0$ and $K_0$ are the density and compressibility of a material at vacuum pressure, and then differentiating to get

$$K' = \frac{-n \rho_0^n K_0}{\rho^{n+1}}.$$

We can reconstruct Eq. (27) as

$$\frac{d^2 \rho}{dr^2} - \left( \frac{2 - n}{\rho} \right) \left( \frac{d \rho}{dr} \right)^2 + \frac{2 d \rho}{r} + 4 \pi G \rho_0^n K_0 \rho^{3-n} = 0.$$

A difference between our version of the Lane-Emden equation, Eq. (31), and the traditional equation, Eq. (28), is that ours already has assumed the polytrope assumption, Eq. (29), while Eq. (28) has not. If the polytrope assumption, $P = \frac{\rho_0^n K_0}{\rho^n} - \frac{D}{\rho}$, is used to remove pressure dependence in Eq. (28) than it will be mathematically congruent with our Eq. (31). This equivalence will be true for both zero and non-zero $D$.

In Fig. 1 we show a density profile of planet Earth from the seismic analysis of the Preliminary Reference Earth Model (Dziewonski & Anderson 1981) (PREM) and the newer Reference Earth Model (Kustowski, Ekström & Dziewoński 2008) (REF) giving a profile of Earth’s inner core, outer core, and three stages of the mantle. Most of the seismic data is obscured by our calculation of the solution to Eq. (31) using our fixed polytrope assumption. In all cases our calculation returned excellent results for the density profile and the gravitational and pressure profiles as well (not shown) using a variety of different vacuum densities, $\rho_0$, bulk moduli, $B_0$, and polytrope indices which are equal to the bulk modulus pressure derivative $(\frac{dB_0}{dP})$. The inner most boundary had its density set at the value from the experimental data set, the first derivative initial value was constrained using Eq. (21) and Eq. (29). The poorest fit was for the liquid outer core but results were still well under 1% error. However the predictive power of this fixed index analysis is weak. With an ideal theory, one would assume that by choosing vacuum values which best fit the experimental seismic data one could elicit more detail about the composite materials comprising the interior Earth. Unfortunately a plethora of values for $\rho_0$, $B_0$, and $B'_0$ give excellent simultaneous fits to all the density-radius and the density-pressure profile. The inset of Fig. 1 details this lack of clarity. Each theoretical result which provided a good agreement to the subset seismic data of the interior earth is provided by a point on a line of the form $\log_{10}(\rho_0^n K_0) = m n + b$. The polytropic index $n$, between 2.7 and 4.3 worked well as long as $\rho_0$ and $K_0$ were constrained to the displayed line. However it is that as the pressure significantly increases the pressure derivative of the bulk modulus
should decrease. By using an EOS which fixes this polytrope observable means that the best fits will reflect only the best 
average values of \( \frac{d B}{d P} \) for that span of the interior of the Earth. For example the core is best reflected with \( 3.35 < n < 3.55 \) 
but we know that at the surface the same materials, rich in iron, will have \( n > 4 \). Having a fixed polytropic index for planets 
does not represent the physical reality adequately, despite the fact that the pressure-density and density-radius profiles were 
described satisfactorily. Having limited physical insight we must add more information to the model.

## 4 MAKING THE POLYTROPIC INDEX A VARIABLE

The ansatz that the polytropic index is constant is constraining and unrealistic. Eddington inherently recognized this and 
attempted a variable index analysis \( 1938 \). It is well known that for all materials the derivative of the bulk modulus with 
respect to pressure does change \( \left( \frac{d B}{d P} \neq 0 \right) \) \cite{Zhang2007,Garai2007,Singh2010}. We show later that the bulk modulus 
pressure derivative can best be approximated as a constant only at pressures below \( 10^{10} \) Pa (small rocky planets or asteroids) 
or above \( 10^{14} \) Pa (interiors of stars and very massive planets). Most planets have a pressure at their core which is between 
these extremes, including our Earth and all the planets in our solar system. Likewise the Murnaghan EOS was quickly 
deemed unsatisfactorily because of its constant derivative of the bulk modulus with respect to pressure so Birch proposed 
an extension \( \{\text{Birch, 1947} \} \). A more realistic approach would allow the index to vary while at the same time retaining the 
definition for the polytrope index, \( n = \frac{4B}{P} \). 

### 4.1 Modifying the Polytrope

So first assuming a modified polytrope with a variable index we must allow for the contingent that the polytrope expression, 
derived in Sect. 2 is no longer constant

\[
\rho^{n(\rho)} K = \rho_0^{n(\rho)} K_0 h(\rho),
\]

(32)

where we have introduced a new function, \( h(\rho) \) which is a measure of the non-conservation of the fixed expression or a 
weighting function. In the limit of the fixed case \( h(\rho) = 1 \).

By taking the natural log of this equation and then taking the pressure derivative

\[
\frac{d}{dP} \left( n(\rho) \ln \frac{\rho}{\rho_0} + \ln \frac{K}{K_0} \right) = \frac{d}{dP} \ln h(\rho),
\]

(33)

and progressing by realizing that \( B = K^{-1}, \frac{dB}{dP} \ln p = K, \) and thus \( \frac{d}{dP} \ln K = -\frac{\frac{dB}{dP}}{B} \)

\[
\frac{dn(\rho)}{dP} \ln \frac{\rho}{\rho_0} + \frac{n(\rho)}{B} = \frac{dB}{dP} \frac{B}{B} = \frac{d}{dP} \ln h(\rho),
\]

(34)

We chose to maintain \( n = \frac{4B}{P} \), thus two terms cancel and we have

\[
\frac{dn(\rho)}{dP} \ln \frac{\rho}{\rho_0} = \frac{d}{dP} \ln h(\rho).
\]

(35)

This is a simple prescription on how use a modified polytrope with a variable index equivalent to the pressure derivative of 
the bulk modulus. We now have a new replacement for compressibility, using Eq. 32 we have

\[
K = \frac{\rho_0^{n(\rho)} K_0 h(\rho)}{\rho^{n(\rho)}},
\]

(36)

which is analogous to the fixed \( n \) version, Eq. 29, with the addition of a weighting function, \( h(\rho) \).

### 4.2 Changes to the Lane-Emden Equation

The new modified Lane-Emden equation, which now includes a variable \( n \). Starting with

\[
\frac{d^2 \rho}{dr^2} - \left[ \frac{2}{\rho} + \frac{K'}{K} \right] \left( \frac{d \rho}{dr} \right)^2 + \frac{2}{r} \frac{d \rho}{dr} + 4\pi G \rho \rho^3 K(\rho) = 0,
\]

(37)

it is trivial to remove \( K \) in terms of \( \rho \) using Eq. 35. We also need to find \( K' = \frac{dK}{dr} \), which is a larger undertaking since \( n \) is 
no longer constant and needs to be treated with care. Starting again with Eq. 32 we take the logarithm of both sides

\[
\frac{d}{d\rho} \log K = \frac{d}{d\rho} \left( \log h + \log \rho_0^n K_0 - n \log \rho \right)
\]

\[
= \frac{d}{d\rho} \log h - \frac{dn}{d\rho} \log \rho - n \frac{d}{d\rho} \log \rho + \frac{dn}{d\rho} \log \rho_0
\]

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recognizing that we can use earlier relationships if we assume a natural log
\[
\frac{dK}{d\rho} = K \left( \frac{d\ln h}{d\rho} + \frac{dn}{d\rho} \ln \frac{\rho_0}{\rho} - \frac{n}{\rho} \right)
\]  
and then using Eq. (33) and the definition \( K = \frac{1}{\rho} \frac{d\rho}{dP} \) a cancellation occurs and we are left with the same result as with a fixed \( n \)
\[
\frac{dK}{d\rho} = K' = K\left(-\frac{n}{\rho}\right),
\]  
except now \( n \) is no longer a constant but a density dependent function. Thus the only symbolic change to the Lane-Emden equation is in the final term
\[
\frac{d^2\rho}{dr^2} - \left[\frac{2 - n(\rho)}{\rho}\right] \left(\frac{d\rho}{dr}\right)^2 + \frac{2d\rho}{r dr} + 4\pi G h(\rho) \rho_0 n(\rho) K_0^{3-n(\rho)} = 0,
\] where the function \( h(\rho) \) is introduced and \( n \), the polytrope index and equivalent to the derivative of the bulk modulus with respect to pressure, is also assumed a function of density as expected. This derivation is independent of the EOS function chosen for \( n(\rho) \).

A comment on the stability of this result is in order. The polytrope index, \( n \), is rather limited in range. The density may change by three orders of magnitude, the pressure by ten orders of magnitude, and at the same time the polytrope index, \( n = \frac{d\rho}{dP} \), changes by only a factor of five (from perhaps 6 to 1.7). This is an incredibly stable physical variable, a good place to develop a dynamic model of materials under extreme conditions in the center of planets and stars. This stability is a large reason for the success of the fixed polytrope index method and also a large reason for the stability of the variable method we propose. Because of the small derivatives of the polytrope index the weighting function of the variable method, \( h(\rho) \), is also well behaved varying between zero and one.

### 4.3 Our Equation of State

To use the modified version with the changing polytrope index we must inform our theory with an equation of state (EOS) that dictates how the index changes with respect to density. There is a cottage industry of universal EOSs (see Roy & Roy (2005, 2006); Poirier (2000) for surveys), started in the 1940’s, with an updated Murnaghan EOS, the Birch-Murnaghan (Birch 1947) (still commonly used) which to third order has the functional form (Poirier 2000)

\[
P(V) = \frac{3B_0}{2}[(V_0/V)^{7/3} - (V_0/V)^{10/3}][1 + \frac{3}{4}(B'_0 - 4)[(V_0/V)^{7/3} - 1]] + \frac{9B_0}{16}[(V_0/V)^{10/3} - (V_0/V)^{7/3}][1 + B_0 B'_0 + B'_0(B'_0 - 7)][(V_0/V)^{7/3} - 1]^2.
\]  
We show an example EOS to impress upon the reader that it contains many of the same variables discussed thus far: the density at vacuum pressure \((V_0/V = \rho/\rho_0)\), the bulk modulus at zero pressure \((B_0)\), and the first \((B'_0 = n_0)\) and 2nd \((B''_0)\) pressure derivatives of the bulk modulus also at zero pressure. For over seventy years the bulk modulus and its pressure derivatives have been used to determine equations of state, to now identify the pressure derivative of the bulk modulus as the polytrope index adds validity to this search for universal equations of state.

To have a working hypothesis for a dynamic polytrope index EOS we examined previous equations of state (Roy & Roy 2005, 2006) and pressure-density relationships (Seager et al. 2007; Swift et al. 2012) for an empirical function form for the changing polytrope index. To be consistent we developed our own EOS which uses the informed polytrope index as the function of reference in place of the traditional pressure equation. We found that

\[
n(\rho) = \frac{dB}{dP} = A_0 \left(\frac{\rho_0}{\rho}\right)^{A_1} + A_2
\]  
works well (more detail on how this equation developed is proffered in Sect. 5). It is a respectable EOS for pressures up to the \(10^{13} \) Pa range. The resulting functional form for the polytrope index is also exceedingly simple. All other major relations to this polytrope index can be found analytically by integration or differentiation, using fundamental relations found in Appendix B. Specifically applying these relations to our EOS as expressed in Eq. (42) we find for pressure

\[
P = \frac{B_0 e^{\Delta n}}{A_1} \left[\left(\frac{\rho}{\rho_0}\right)^{A_2} E_n \left(\frac{A_1 + A_2}{A_1}, \frac{A_0}{A_1} \left(\frac{\rho_0}{\rho}\right)^{A_1}\right) - E_n \left(\frac{A_1 + A_2}{A_1}, \frac{A_0}{A_1}\right)\right].
\]  
This pressure equation contains a special function, the generalized exponential integral \((E_n)\) which is defined as

\[
E_n(n, x) = \int_1^\infty \frac{e^{-xt}}{t^n} dt.
\]
The parameters $A_0, A_1, A_2$ are dimensionless, connected to experiment, and well behaved, always greater than zero and less than ten. They are connected to the extremes of the polytrope index: the infinite pressure derivative of the bulk modulus with respect to pressure, $B'_0 = n_\infty$ and the zero pressure asymptote of the same observable, $B'_0 = n_0$. They were fit as follows: $A_2 = B'_0, A_0 = B'_0 - A_2$, and $A_1 = \frac{B'_0 B'_0}{B'_0 - A_2}$. In this work we used the parameter settings in a predictive method, setting $B_0$, and $B_0$ close to experimental results, choosing a near fixed, but yet to be determined, infinite pressure derivative $(1.75 < B'_0 < 2.1)$, and assuming a universal ratio for $B'_0$ by setting
\[
A_1 = \frac{B'_0}{B'_0 - A_2} \equiv \frac{B'_0}{A_0}
\]
which we borrowed from Roy & Roy (2006). Thus for any given material there are only three parameters $(\rho_0, B_0, B'_0)$. We will show our EOS compared to experiment in subsection 4.5 but first we require that it be continuous with a higher pressure theory to extend its validity.

4.4 Pressures above Ten Tera-Pascal

Inspired by Seager et al. (2007) we went higher in pressure, to $10^{18}$ Pa, by following their technique to match at some critical density our empirical EOS to the Thomas-Fermi-Dirac theory (Salpeter & Zapolsky 1967) which treats at extreme pressures the quantum mechanic Fermi-Dirac gas whose pressure we approximated as
\[
P(\rho) = P_0 + F_3 \rho^{4/3} + F_2 \rho^{5/3}
\]
(46)
where
\[
F_1 = 5.16 \times 10^{12} Pa (2690 \frac{p}{\rho})^{5/3}
\]
\[
F_2 = 5.16 \times 10^{12} Pa (0.40726Z^{2/3} + 0.20732)
\]
\[
F_3 = 5.16 \times 10^{12} Pa (0.01407).
\]
(47)
the variable $P_0$ is a small, relatively low pressure, addition to keep pressure continuity across the matching boundary and will be determined last. If the material is not monatomic we use a weighted average for $A$ and $Z$ (atomic and proton number respectively).

Following the common definition used between the pressure and the bulk modulus, $B = \rho \frac{\partial P}{\partial \rho}$, we have for this high pressure theory
\[
B = \frac{5}{3} F_1 \rho^{5/3} - \frac{4}{3} F_2 \rho^{4/3} - \frac{3}{3} F_3 \rho^{3/3}.
\]
Likewise $n$, our polytrope index equals $\frac{dB}{dP} = n = \frac{dF}{F} \frac{dB}{dP}$ (Eq. 13), thus
\[
n = \frac{5}{3} F_1 - \frac{4}{3} F_2 \rho^{-1/3} - \frac{3}{3} F_3 \rho^{-2/3}.
\]
(49)
A key aspect of this is to choose the critical density in which to switch from $n(\rho) = A_0 (\rho / \rho_0)^{A_1} + A_2$, to the $n$ of Eq. 49. We found that by requiring at the boundary that the bulk modulus, $B$, was continuous and using the Newton-Raphson method to find this critical density was all that was required. We also found that at this critical density $n$, the derivative of the bulk modulus, also was continuous. By adding a relatively small constant to pressure ($P_0$ of Eq. 46) we can also set the pressures equivalent at the critical density boundary. The reason we get a match between both $B$ and $n$ at the boundary is because these two variables are not independent. Recalling Eq. 13, $\frac{dF}{dP} = n = \frac{dF}{dP}$, we see that if $\rho$ and $B$ are attuned we get the two derivatives, $n = \frac{dB}{dP}$ and $\frac{dF}{dP}$ as bonus. With this extension to a high pressure EOS, we now feel comfortable in fitting up to $10^{18}$ Pa.

Pragmatically we assume informed values for a material’s $\rho_0, B_0$, and $B'_0$. We then make initial approximations for $A_2 \approx 2$. We then solve for $A_0$, and $A_1$ of Eq. 12. Then a Newton-Raphson method adjusts $A_2$ and then determines $A_0$, and $A_1$ at the critical density, $\rho_c$, the low pressure EOS becomes continuous with the higher pressure Thomas-Fermi-Dirac EOS (TFD) of Eq. 18 by finding at what critical density $B_{TFD} = B_{TFD} = 0$. We then ensure that $n_{TFD} = n_{TFD} = 0$ and then we lastly solve for $P_0$ of Eq. 16 to establish continuity for pressure, bulk modulus, and the pressure derivative of the bulk modulus across the critical boundary.
4.5 Comparison with experimental results

In Fig. 2 we plot density versus pressure for six common materials that make up planet interiors in our solar system ranging from molecular hydrogen to atomic iron. The materials are either solids or liquids throughout the calculation and are also low temperature isothermal calculations (cold curves, 0K - 300K). We choose to examine an intermediate pressure range, $10^9$ Pa to $10^{12}$ Pa, because it is challenging; it is at the extreme of our experimental capabilities and the interiors of the intra-solar...
planets are in this range. The solid black line is our EOS which was input into the Lane-Emden equation of Eq. (40) through the polytropic index (as a check of our work we used Eq. (43)). At these lower pressures we used the EOS described with Eq. (42), the values of the parameters can be also found in Appendix C. The filled triangles of various colors are experimental data, the dashed green line is a common empirical theory from the twentieth century (Birch-Murnaghan (Birch 1947) or Vinet (Vinet et al. 1987, 1989; Poirier 2000) which begins to fail in the $10^{11}$ Pascal range, the dotted red and orange lines are density functional theories and/or quantum Monte Carlo theories from the twenty-first century. Our theory has the correct general trends, it fits theories and experimental data well considering that it has no adjustable parameters. Our method was that we first chose average experimental values for our material vacuum values of $\rho_0, B_0$, and $n_0$ which in many cases the literature values varied widely. Our parameter $A_2$ was chosen as the critical boundary value for $n = dB/dP$ when the EOS transitioned from our own to the Thomas-Fermi-Dirac EOS (Salpeter & Zapolsky 1967) ($1.75 < A_2 < 2.1$) and we fixed out remaining parameters by constraining $A_0 + A_2 = n_0$ and $A_0 A_1 = n_0$. All the parameters used to make the plots can be found in Appendix C.

This success is proof of concept, that it is possible to develop an experimentally informed EOS which uses a polytrope function form, equivalent to $dB/dP$, which can be naturally and consistently input into the dynamic index Lane-Emden equation. It matches well with even the more sophisticated present day first-principle theories (French et al. 2009; Geng et al. 2012; Khairallah & Militzer 2008; Hermann, Ashcroft & Hoffmann 2011; Driver et al. 2010; Ping et al. 2013). The inset of Fig. 2 plots the polytrope index as a function of the same pressure range for the six materials. The polytrope index, $n = dB/dP$ does change significantly over this pressure range, thus to keep the polytrope index fixed for planets in our solar system would be approximate.

Figure 3 shows our calculation over a wider pressure range, with a similar inset as Fig. 2. Note the general trends: up to $10^7$ Pa the pressure density profile is a nearly incompressible horizontal line, the polytrope index, equivalent to the pressure derivative of the bulk modulus, wanders little from from its zero pressure value. The most dynamic range of change for the polytrope index is from $10^8$ Pa to $10^{14}$ Pa as was illustrated in Fig. 2. As the pressure increases, our theory must switch from the low pressure EOS of Eq. (42) to the technique of Thomas-Fermi-Dirac of Sect. 4.3. The polytrope index again becomes nearly constant after $10^{14}$ Pa as each material begins its slow asymptotic drive towards the Thomas-Fermi-Dirac value of $5/3$ (Salpeter & Zapolsky 1967). The parameters used for the materials in this figure can be found in Appendix C.

We believe our calculation is among the most consistent universal EOSs, its strength is its strong predictive powers over the widest range of pressures and materials. The technique also has the ability to grow in complexity as the sophistication of the polytrope index EOS advances as we will discuss in Sect. 7.
4.6 Comparisons to the Seager Modified Polytrope Form

Our Fig. 3 pressure-density profile is similar in description and scope to a figure in Seager et al. (2007). For easy comparison to their work we start with our Eq. (10) and first multiply by the polytrope index, \( n \), then take both sides to the power of the inverse of the polytrope index, \( 1/n \)

\[
\rho = \left( nC' P + nD \right)^{1/n}.
\]  

(50)

Compare this to Seager’s modified polytrope form which is

\[
\rho = \rho_0 + cP^k,
\]  

(51)

The equations concur at the asymptotes (when \( P < 10^7 \) Pa and \( P > 10^{14} \) Pa) but in the intermediate range they are not equivalent. In some detail, at low pressure (\( < 10^7 \) Pa), their fit can be approximated in MacLaurin series form as

\[
\log \rho \approx \log \rho_0 + cP^k/\rho_0,
\]  

(52)

which given their parameter values of \( c \) and \( k \) produce an insignificant second term on the right side and thus the horizontal line, \( \log \rho \approx \log \rho_0 \), insensitive to the values of \( c \) and \( k \). At high pressure (\( > 10^{14} \) Pa) another MacLaurin series approximation:

\[
\log \rho \approx k \log P + \log c + \rho_0/(cP^k),
\]  

(53)

again the last term on the right side is insignificant at extreme pressures and another formula for a line emerges, now with non-zero slope \( k \). Since \( k = d \log \rho/d\log P \) we can use the definition that \( d \log \rho / d\log P = K = B^{-1} \) to find that \( k = P/B \), the inverse of Eq. (17) and also found in Stacey (2000); Stacey & Davis (2004). The curve is nearly linear over multiple orders of magnitude so this is not a bad approximation to \( k \approx \frac{dP}{d\rho} = 1/n \). Therefore Seager’s values for \( n \) (2007) match ours at these high pressures, \( 1.8 < n < 2.0 \), as their lines head towards the \( n = 5/3 \) asymptote since they also connect to the Thomas-Fermi-Dirac theory.

5 APPLICATION TO PLANETS

We now apply our method to planets of our solar system and beyond. For planets the changing polytrope index is a must for all but the smallest planets. One looming approximation that needs to be addressed is the use of cold isothermal material curves to describe planets with hot interiors. We will address this in Sect. 7 but for now let us proceed assuming that this simplification is feasible.
Figure 5. (color online) Jupiter, Uranus, and Mars density-radius profile predictions in log-log form using our EOS. The details of construction are in the text.

In Fig. 4 we solve the modified Lane-Emden equation with a variable index for the same six materials of the two previous figures except now we plot mass versus radius. This is a popular figure in planet modeling papers dating back to at least Zapolsky & Salpeter (1960). For comparison we designate the positions on the figure for the planets of the solar system. We also add as the top symbol, "K12", the extreme giant exo-planet Kepler 12b (Fortney et al. 2011). This planet is located seven times closer to its sun than Mercury. To be running above our hydrogen line shows that our calculation has problems with this outlier; an extremely hot light gas giant that is actually mostly gas (Jupiter is not) and has a possible interior heat source. An example of one of the exciting discoveries that will challenge us for years as our exoplanet catalogue continues to grow. Moving to our more mundane solar system we describe from vertical top to bottom: Jupiter, Saturn, Uranus and Neptune (almost on top of each other), Earth and Venus (almost on top of each other), Mars and Mercury. They are all placed close to their line of significant composition. Jupiter is close to hydrogen, Saturn has slightly more helium than Jupiter. Uranus and Neptune are between water and helium, Earth, Venus, Mars are between iron and the silicates. Mercury, the most metallic is closest to iron. These logical results give confidence to the validity of the method.

Turning now to specific models for three planets in our solar system. We assume Mars has a core and mantle, Jupiter to have a core and two more layers made of mostly hydrogen and helium. Uranus is also a three layer system similar to Jupiter however its middle layer is predominately ices. We constrain the boundary interfaces by substituting for the compressibility, $K$, in Eq. (53) our variable index assumption

$$K = \frac{\rho_n(\rho)K_0(\rho)}{\rho_n(\rho)}.$$  \hfill (54)

We are then left with a constraint on the derivative at each boundary

$$\frac{d\rho}{dr} = -\frac{\rho_n^2 K_0 h(\rho)}{\rho(\rho)^{n-2}} g(r).$$  \hfill (55)

The method of construction is illustrative. We assume the material makeup of the layers and the size of each layer, we then also assume the core pressure. There are no further adjustable parameters. The materials dictate the parameters in the equations of state, the core pressure determines the starting core density (by using Eq. (33), and the interface constraint defines the density values and their first derivatives at the boundaries uniquely, the program runs from the core ($r = 0$) to the surface solving the differential equation of Eq. (40) and using the EOS as summarized in Eq. (42). Numerically this differential equation is very stable and was solvable with relative ease, there were no variables that gave us numerical difficulty. Table I contains our pertinent values for these multi-layer constructions. The materials at each layer were picked by consulting the best ‘general knowledge’ about each of the layers and estimating their average density, bulk modulus, etc. Examining the cores we hypothesized that Mars has a metallic core, the core of Jupiter is icy, and the core of Uranus contains ice and silicates. The method portrays a standardized way to put a multi-layered planet together. To see an alternative method to solve a similar differential equation, but working from the surface inward, see Zeng & Seager (2008).

In Fig. 5 we plot these hypothetical density-radius profiles for the planets Mars (rocky), Jupiter (gas giant), and Uranus.
Table 1. Mars in two steps, Jupiter and Uranus in three steps. The variables $\rho_0$, $B_0$, and $n_0$ would be the values of those observables at zero vacuum. Thickness is the radial size of the core or mantle part, $P_C$ is the assumed pressure at the largest depth for each layer.

|                | $\rho_0$ kg/m$^3$ | $B_0$ GPa | $n_0$ | $A_{mean}$ | $Z_{mean}$ | thickness m | $P_C$ GPa |
|----------------|-------------------|-----------|-------|-------------|-------------|-------------|-----------|
| Mars           |                   |           |       |             |             |             |           |
| core           | 6200              | 160       | 5.5   | 47          | 22          | 1.600E6     | 3.9E1     |
| mantle         | 3400              | 140       | 4.5   | 36          | 18          | 1.776E6     | 2.097E1   |
| Jupiter        |                   |           |       |             |             |             |           |
| core           | 1375              | 160       | 4.3   | 20          | 10          | 1.200E7     | 4.800E3   |
| mantle 1       | 98.0              | 0.172     | 6.88  | 1.75        | 1.25        | 2.300E7     | 3.931E3   |
| mantle 2       | 77.0              | 0.165     | 6.75  | 1.42        | 1.14        | 3.522E7     | 1.429E3   |
| Uranus         |                   |           |       |             |             |             |           |
| core           | 3885.0             | 153.0     | 4.4   | 38          | 18          | 6.600E6     | 8.250E2   |
| mantle 1       | 645.0              | 5.80      | 5.45  | 16.5        | 8           | 9.800E6     | 3.793E2   |
| mantle 2       | 324.0              | 2.16      | 6.10  | 12.5        | 5.5         | 8.933E6     | 7.409E1   |

Table 2. Jupiter, Uranus, and Mars observable predictions using our EOS. The experimental results are taken from the National Aeronautics and Space Administration website.

|                | mass kg | radius m | mean density kg/m$^3$ | surf. gravity m/s$^2$ |
|----------------|---------|----------|-----------------------|-----------------------|
| Mars           |         |          |                       |                       |
| calculation    | 6.36E23 | 3.38E6   | 3.94E3                | 3.72                  |
| experiment     | 6.42E23 | 3.39E6   | 3.93E3                | 3.71                  |
| Jupiter        |         |          |                       |                       |
| calculation    | 1.88E27 | 7.02E7   | 1.30E3                | 25.5                  |
| experiment     | 1.90E27 | 6.99E7   | 1.33E3                | 24.8                  |
| Uranus         |         |          |                       |                       |
| calculation    | 8.56E25 | 2.53E7   | 1.26E3                | 8.90                  |
| experiment     | 8.68E25 | 2.54E7   | 1.27E3                | 8.87                  |

6 USING THE PRESSURE DERIVATIVE OF THE BULK MODULUS AS THE EQUATION OF STATE FOUNDATION

The role of the pressure derivative of the bulk modulus ($\frac{dB}{dP} = B'$), which we have shown to be the polytrope index, $n$, has grown in importance in material science. As already discussed, one of the first high pressure universal equations of state by Murnaghan 1944 used the pressure derivative of the bulk modulus at vacuum pressure as a constant parameter independent of pressure ($\frac{dB}{dP} = 0$). The trend of using this elastic observable at vacuum pressure as a parameter continued in later universal EOSs (Birch 1947; Keane 1953; Vinet et al. 1987, 1989; Poirier 2000; Roy & Roy 2006, 2005), but now the derivative varied as a function of pressure (a moving polytrope index). As strain theory continued to progress this elastic observable assumed a more prominent role. It is, frankly, a wonderful observable. It is dimensionless, of order one (for all materials and pressures), and because it is proportional to the second derivative of pressure-density it has tremendous sensitivity. This work has developed a technique where the variable polytrope index is equivalent to this elastic observable, $\frac{dB}{dP}$.
Figure 6. (color online) A plot of $n \equiv dB/dP$ for cold-curve molecular hydrogen using a variety of EOSs. The solid black line is the present calculation, the green dotted line is the EOS of Vinet (Vinet et al. 1987, 1989), the pink dashed line is the third order EOS of Birch-Murnaghan (Birch 1947). We also present the EOS of Stacey 2000 (blue dash - dot fixing the infinite $dB/dP$ at 2) and the EOS of Roy-Roy (Roy & Roy 2001, 1999) is depicted by a purple long dash - dot. A recent density functional theory (DFT) developed for molecular hydrogen (Geng et al. 2012) in red. All EOS (except the DFT) have fixed $\rho_0 = 79.43 \text{kg/m}^3$, $B_0 = 0.162 \text{GPa}$. The $n_0$ values are within 3% of each other (ours, Birch-Murnaghan is $n = 6.7$, Vinet is $n = 6.813$ and we have set Roy-Roy and Stacey to $n = 6.9$). Note the dramatic differences in the function $n(P)$ and its effects on the density-pressure relationship at extreme pressures. The experimental analysis, in blue triangles, was done by taking 2nd derivatives of experiments done by Loubeyre et al. (1996).

6.1 The importance of $dB/dP$ in material research

Stacey (2000) developed an EOS that used $B' \equiv dB/dP = n$ as the starting point. This work gives a thorough analysis of the history of this technique and rightly recognizes Murnaghan (1944) and Keane (1954) as instrumental in the development of the Stacey EOS. Stacey and Davis strongly believe that $B'$ should be the center-piece of a universal EOS (Stacey & Davis 2004, 2008a; Lal, Singh & Chauhan 2009; Singh & Dwivedi 2012).

Calculating from first principles the electronic contribution to the bulk modulus is well established in the literature (Ravindran et al. 1998). The ability for modern material theorists to develop excellent $B'$ theories from density functional theory is not in doubt (Zhang, Wang & Zhao 2010, for example). Classical solid state theory may also be of use. The intermediate polytropic expression we derived in the context of the fixed polytrope index, $\rho_n K = h(\rho) \rho_n K_0$ (Eq. (32)), in Sect. 4 has some remarkable similarities to the structure of the repulsive coulomb potential informed by Born-Madelung theory (Kittel 2005). Are the Born exponents related to the polytrope index, $n$?

There have been some recent high pressure models which include temperature as a direct extension of the pressure equation (Sotin, Grasset & Mocquet 2007; Wagner et al. 2011, 2012). Our calculation technique would use alternative methods which show the thermodynamic connection between $B'$, the Grüneisen parameter (Stacey & Davis 2004, 2008a; Shanker, Singh & Jitendra 2009) and temperature (Stacey & Davis 2004, 2008a; Sotin, Grasset & Mocquet 2007) gives a good overview of both techniques. The elastic constants, such as $B'$, are a natural environment to develop temperature dependence as shown in a variety of research at low pressures (Zhang et al. 2007), first principle calculations of copper (Narasimhan & de Gironcoli 2002), magnesium oxide (Li, Xu & Liu 2005) or high pressure iron (Sha & Cohen 2010) and high pressure salt (Singh 2010). We will examine this potential by adding a primitive temperature dependence in Sect. 7.

Phase changes would also have a natural place in a foundational $B'$ EOS. The discontinuities created by phase changes would be more significant in the bulk modulus and its derivative than in a pressure-density curve. The importance of magnetic effects has also recently been studied (Pourovskii et al. 2013) and similarly the $B'$ equation has been shown to be a natural place to add this interaction (Zhang, Wang & Zhao 2010).
6.2 Comparisons of $\frac{dB}{dP}$

As stated in Sect. 4 we found that

$$n(\rho) = \frac{dB}{dP} = B' = A_0 \left( \frac{\rho_0}{\rho} \right)^{A_1} + A_2$$  \hspace{1cm} (56)

works well. It is a respectable EOS for pressures up to $10^{13}$ Pa and it can be extended to higher pressures by attaching it to quantum mechanical models as also discussed in Sect. 4. This EOS was validated by examining the functional form of $\frac{dB}{dP}$ for the Vinet and Birch-Murnaghan EOS (Birch 1947; Poirier 2000) and also analyzing pressure-density relationships found in previous work (Seager et al. 2005; Swift et al. 2012; Roy & Rox 2005; 2006) including first principal calculations (French et al. 2006; Hermann, Ashcroft & Hoffmann 2011; Geng et al. 2012) and acquired numerical second derivatives. This was instrumental in developing the intermediate pressure ranges ($10^{11} - 10^{13}$ Pa) which older EOS often do not fit well. Our result is entirely empirical but it meets the criteria that it is using $B'$ as its foundation, that it is analytically simple, and that it strives to go at least an order of magnitude higher than the earlier EOSs.

The analysis included taking analytical second derivatives of EOSs using

$$B = \rho \frac{dP}{d\rho}$$
$$n = \frac{dB}{dP} = B' = \frac{d}{dP} \left( \rho \frac{dP}{d\rho} \right)$$
so $\frac{dB}{B} = \frac{n}{\rho} \frac{dP}{d\rho}$ \hspace{1cm} (57)

The analytical results for the Vinet, Birch-Murnaghan (Birch 1947; Poirier 2000), Stacey, Roy-Roy, and an EOS for hydrogen derived from density functional theory (Geng et al. 2012) are shown for completeness in Appendix D; these results are non-trivial fractions. One of the goals of this work was to develop a analytical formula for $\frac{dB}{dP}$ which is simpler than the two most used universal equations, the Vinet and Birch-Murnaghan, yet at the same time mimic their behavior.

We portray $\frac{dB}{dP}$ in Fig. 6 for molecular hydrogen as a function of pressure for a variety of EOSs and in the inset we show the pressure-density result. This figure is similar to Figs. 2 and 3 but now the focus is on the behavior of the polytrope index so the plot proper and inset have been switched. We choose hydrogen because it offers challenges at relatively low pressures because of its high compressibility. It was seen in Fig. 2 that a standard Vinet EOS does not do well as one approaches pressures of $10^{13}$ Pa, and Fig. 6 shows the probable cause, the Vinet becomes soft as it heads towards a lower asymptote for $n$ than the other EOSs. In contrast, the complete failure of the Birch-Murnaghan EOS with these chosen parameters is because the asymptote for $n$ is higher than the rest of the EOSs. All the EOSs (except the DFT by Geng et al. 2012) which is valid in a limited range only) approach a constant at extreme pressures but they disagree at what the infinite pressure constant ($B'_\infty$) should be. Analysis of the importance of this asymptote is good in detail in the research of Stacey and Davis (Stacey 2000; Stacey & Davis 2004; 2008a). The Birch-Murnaghan, in Fig. 6, has the highest $B'_\infty$ of 3. The Vinet has the lowest, $B'_\infty = 2/3$, the rest are $5/3 \leq B'_\infty \leq 2$. Our theory, because it connects to the Thomas-Fermi-Dirac high pressure theory has as its asymptote $B'_\infty = 5/3$, we set Stacey’s EOS to $B'_\infty = 2$ (they treat it as an adjustable parameter).

One must be careful with the Roy-Roy equation of state, they have two different forms. The one they consider superior (Roy & Rox 1999; 2004; 2005), which has a logarithmic form, does not have a physical $B'_\infty$ asymptote (it is designed to go no higher than $10^{12}$ Pa). The one we depict in Fig. 6 and Appendix D has a physical asymptotic form and has $B'_\infty = 5/3$ (Roy & Rox 2001; 1999).

Likewise note the peculiarity of the density function theory (Geng et al. 2012). The authors choose a pressure EOS (see Appendix D) which fit only the pressures applicable to their theory ($1 \times 10^{10}$ Pa to $4 \times 10^{12}$ Pa). When derivatives are taken to produce $\frac{dB}{dP}$ the results are complicated and quickly unstable at the extremes (the ends of the plotted curve head towards infinity). Their fit is satisfactory on the pressure-density profile but the sensitivity exhibited on the $B'$ plot is surprising. On reflection this discovery illustrates that at very high pressures (thus at a high bulk modulus) a small range of erratic derivative values (which are relatively very small compared to the bulk modulus) will not propagate significantly to the bulk modulus or pressure. However if this trend manifests itself over a larger range the consequence will be measurable. This insensitivity provides an additional lesson on how it is easy to fit the density-pressure results and yet have faulty derivatives, an authoritative EOS will fit the pressure-density, bulk modulus-density and bulk modulus pressure derivative and density profiles.

The experimental analysis technique we used, as depicted in Fig. 6, is detailed in Appendix D. To summarize, we fit a density-pressure curve to a high order polynomial and then take two derivatives, using Eqs. 58 to achieve a good analysis independently, directly from the data, of $n \equiv \frac{dB}{dP}$. We urge that this technique be used by experimentalists to further analyze their high precision data. With a good experimental set of density-pressure data one can also determine the bulk modulus.
and the pressure derivative of the bulk modulus over the same range. If one does want to extrapolate to vacuum pressure using a universal EOS, this information will help produce an authoritative EOS and further constrain the parameters chosen.

It has been argued by Stacey & Davis (2004) that \( B'_{\infty} \geq 5/3 \) but not necessarily \( B'_{\infty} = 5/3 \). The argument is that phase changes and proton number dependence make it difficult to rectify that all materials will approach the same asymptote. We also believe this is an open question but still found it convenient to set it to the Thomas-Fermi-Dirac value of 5/3 (Salpeter & Zapolsky 1967). Yet we recognize that the present work and others (Seager et al. 2007) have found that this asymptote is not reached quickly. It takes hydrogen to \( \approx 10^{14} \text{Pa} \) and iron to \( \approx 10^{16} \text{Pa} \) to approach this asymptote within 5%, we are still not there yet experimentally. These asymptotes are perhaps better tested in the interior of stars than planets. In neutron star research (Lattimer & Prakash 2001) the authors compiled fourteen EOSs with different theoretical foundations for their modelling the interior of a neutron star. Most of the lines (quark models excluded) follow a polytrope for much of the neutron star research (Lattimer & Prakash 2001) the authors compiled fourteen EOSs with different theoretical foundations.

The polytrope exponent (we are still not there yet experimentally. These asymptotes are perhaps better tested in the interior of stars than planets. In neutron star research (Lattimer & Prakash 2001) the authors compiled fourteen EOSs with different theoretical foundations for their modelling the interior of a neutron star. Most of the lines (quark models excluded) follow a polytrope for much of the interior. The polytrope exponent (\( B'_{\infty} \)) at the lower pressures (\( > 10^{30} \text{Pa} \)) of the interior are all about 2, at higher pressures (\( > 10^{35} \text{Pa} \)) some of the EOS abruptly go to a lower polytrope exponent of \( 4/3 \leq B'_{\infty} \leq 2 \).

Figure 6 was not created as an argument that our universal EOS is the best choice. A good experimental mapping of \( n = \frac{dP}{d\rho} = B' \) needs to be done independent of a universal EOS theory. What can be said is that generally Stacey and Roy-Roy are soft EOSs, their \( \frac{dP}{d\rho} \) are generally lower than the average while the Birch-Murnaghan EOS is the hardest. It is not a coincidence that the best fitting EOS to the lower pressure experimental data is the Vinet which has been the EOS of choice for low atomic number materials. Too often the universal EOSs are used to put limits on the bulk modulus and its derivative at zero pressure (\( B_0, B'_0 \)). All reasonable EOSs will fit any pressure-density curve if the parameters are adjusted enough as discussed in Stacey & Davis (2004), what Fig. 6 shows is the disparity that exists between the EOSs on the functional form and the asymptote of \( \frac{dP}{d\rho} \). This observable needs to be measured or analyzed experimentally as well as theoretical approaches from first principles (Ziambaras & Schroder 2003), independent of a universal EOS, calculated so the best functional forms can be developed. Hydrogen, helium, and water, because of their higher compressibilities, are a good place to start. These informed results would catalyze the development of EOSs. With a trusted functional form, which mimics experiment for pressure-density, bulk modulus-density, and especially \( B' \)-density (which is sensitive even at low pressures), there will be more faith in the physical interpretation of the parameters being adjusted. To constrain the universal functional form beyond the pressure-density profile would have significant impact on calculations of larger planets.

To illustrate this point dramatically the study of hexagonally closed packed (hcp) iron, one of our six materials under study, has had little consensus (Sha & Cohen 2010) for its vacuum observables (\( \rho_0, B_0, B'_0 \)). This is a material which does not exist at low pressures so the extrapolation to vacuum pressure is already problematic. The experimental results, which usually fit only an experimental pressure-density curve to a universal EOS, have in recent literature a bulk modulus range from 138 GPa to 202 GPa and the bulk modulus pressure derivative varies from 4.5 to 6.2. Even the vacuum density has disagreement ranging from 8180 kg/m\(^3\) to 8650 kg/m\(^3\) (Seager et al. 2007; Sha & Cohen 2010; Garai, Chen & Telekes 2011; Yamazaki et al. 2012). These discrepancies occur because the fitting universal EOSs do not agree on the functional form of \( \frac{dP}{d\rho} \) as depicted clearly with molecular hydrogen in Fig. 6.

### 7 ANALYSIS OF PLANET EARTH WITH OUR EOS

The Earth is our best pseudo-static laboratory for studying materials under high pressures. In this section we analyze the Earth using different constraints on our EOS to better elucidate its strengths and weaknesses.

To solve for planet Earth we assumed five layers. The initial theoretical density (innermost boundary of each layer) was fixed at the same value as the experimental data (seismic model results), the core pressure was checked by using Eq. 43, the initial derivatives at each stage were constrained by our polytrope equation to be \( \frac{dP}{d\rho} = -h(\rho)\frac{\rho_0^6 K_0}{\rho r^{\gamma-2}} g(r) \), where the gravitational function, \( g(r) \), and the initial density, \( \rho(r) \), were fixed by the seismic model results. The function \( h(\rho) \) was found by using the equations in Appendix E.

By using the seismic data of planet Earth we now constrain, with our EOS, the material of the inner core, outer core and mantle using the pressure-density profile, and the density-radius profile. The variable polytrope index EOS, our most sophisticated model, is now used to augment Fig. 1 when only the fixed polytropic index model was used. The best results still achieve an excellent fit and reveal a linear analysis similar to the inset of Fig. 1. (\( \rho_0^6 K_0 \) vs. \( n_0 \)). We show results for the inner core, outer core, and lower mantle of the Earth in Table 3. A reminder that the linear analysis constrains \( \rho_0^6 K_0 \) but does not guarantee that every point on the line will be a good fit, but all good fits are very near that line. We offer no explanation for this relationship but that its success does speak to the validity of universal EOSs and that the three variables (\( \rho_0, K_0, n_0 \)) are not independent. For example the equation of the lines for the inner core, outer core, and mantle gives these relationships:

\[
\frac{n_0}{n_{\text{inner core}}} = -\log \frac{650 \text{GPa}}{B_0} / \log \frac{\rho_0}{9860 \text{kg/m}^3}
\]
temperature and other dynamic variables are included in the analysis the ability to constrain the physical properties of the point that to infer a precise physical reality from a fit of an untested EOS is not sound. As already discussed, until the higher than any acceptable value, the surface pressure derivatives are well below any acceptable value). This emphasizes the weak. The best fits have produced vacuum bulk moduli and derivatives which are implausible (the bulk moduli are much

7.1 Temperature Analysis

Earth materials will be diminished. In the next subsection we discuss the potential for including temperature.

A constraints on

\[
\begin{align*}
\rho_{\text{outer core}} &= -\log \frac{685 \text{ GPa}}{B_0} / \log \frac{\rho_0}{9480 \text{ kg/m}^3} \\
\rho_{\text{mantle}} &= -\log \frac{521 \text{ GPa}}{B_0} / \log \frac{\rho_0}{5040 \text{ kg/m}^3}. 
\end{align*}
\]

(59)

It is extraordinary that the linear equation for the inner and outer core parameters match above to better than 5% exuding their commonality, perhaps this type of linear analysis can be used to categorize aspects of non congruent universal EOSs.

The surface observables of the inner core of Earth, as predicted by this model, are not a bad approximation for the unknown constituent material. In contrast the predictive powers of this technique to constrain the outer core and mantle are weak. The best fits have produced vacuum bulk moduli and derivatives which are implausible (the bulk moduli are much higher than any acceptable value, the surface pressure derivatives are well below any acceptable value). This emphasizes the point that to infer a precise physical reality from a fit of an untested EOS is not sound. As already discussed, until the universal functional form of the pressure derivative is better known no universal EOS should be authoritative. Likewise until temperature and other dynamic variables are included in the analysis the ability to constrain the physical properties of the Earth materials will be diminished. In the next subsection we discuss the potential for including temperature.

Table 3. Values that best fit the seismic data of planet earth (density-radius and pressure-density profile) along with the linear equation that constrains these good fits using the variable polytrope index found in this work. All curves here are isothermal cold-curves. The method is outlined in Sects. L5 and 5.

|               | \( \rho \) kg/m\(^3\) | \( B_0 \) GPa | \( n_0 \) | \( \log_{10}(\rho_0^{n_0}K_0) \) |
|---------------|---------------------|--------------|----------|-------------------------------|
| inner core    | 7400-7700           | 175-210      | 4.45-4.70| 3.994n_0 - 11.813             |
| outer core    | 7700-7950           | 360-440      | 2.5-2.9  | 3.977n_0 - 11.836             |
| lower mantle  | 4100-4170           | 300-330      | 2.3-2.7  | 3.702n_0 - 11.717             |

It is extraordinary that the linear equation for the inner and outer core parameters match above to better than 5% exuding their commonality, perhaps this type of linear analysis can be used to categorize aspects of non congruent universal EOSs.

The surface observables of the inner core of Earth, as predicted by this model, are not a bad approximation for the unknown constituent material. In contrast the predictive powers of this technique to constrain the outer core and mantle are weak. The best fits have produced vacuum bulk moduli and derivatives which are implausible (the bulk moduli are much

\[
\frac{dn}{d\rho} = -\frac{A_0 A_1 \rho_0 A_1}{A_0 + 1} 
\]

(60)

(see appendix B for more detail). The cold curves \( A_2 \) was set by a numerical Newton-Raphson search that connected our cold curve EOS to the Thomas-Fermi-Dirac EOS, the values for \( A_2 \) was always between 1.75 and 2.1. So by adjusting \( A_2 \) outside of this realm, we can examine the potential of this EOS to include temperature. The procedure is simple: assume a reasonable material with a given vacuum density (\( \rho_0 \)), bulk modulus (\( B_0 \)), and first pressure derivative of the bulk modulus (\( n_0 \)). The constraints on \( A_0, A_1, \) and \( A_2 \) are that \( n = A_0 + A_2 \) and \( n = A_0 A_1 \). We treat as free parameters \( \rho_0, B_0, n_0 \) and \( A_2 \) and adjust them using a Powell method (constraining \( A_0 \) and \( A_1 \) by \( A_0 + A_2 = A_0 A_1 = n_0 \)). So we are assuming that at the surface of the Earth the density, bulk modulus, first derivative with respect to pressure, and all higher order derivatives have the same value irregardless of whether the isothermal or adiabatic path is chosen. But as one moves into the interior \( B_{\text{adiab}} > B_{\text{isothermal}} \) thus the elastic derivatives are also disparate. In Stacey & Davis (2004, 2008b) one finds a thermodynamic derivation that provide a mapping

\[
B_{\text{adiab}} = B_{\text{isothermal}}(1 + \gamma \alpha \Delta T),
\]

(61)
Figure 7. (color online) This figure is in three panels: the top referencing the inner core, the middle the outer core, and the bottom is the inner mantle. In black, significantly buried under the theoretical calculations, is the newer Reference Earth Model (Kustowski, Ekström & Dziewoński 2008) (REF). The colored lines are all calculations of ours with different sophistication, all parameters are given in Table 4. The three models are named from the section from which they were introduced. Model (VI - two light blue short dashed) includes an adjustable second derivative \( \frac{d^2B}{dp} \) to simulate the effect of temperature in the interior of the Earth. Model (IV - short dark blue dashed-dot) is the same material as (1) but the second derivative functional form has now been fixed to the cold curve value effectively removing the temperature dependence. Model (III – purple long dash-dot) is the least sophisticated fixed polytrope index \( (\frac{d^2B}{dp} = 0). \) For reference the cold curve for hcp iron is included for the cores (dashed red) and magnesium oxide (dashed pink) and silicon dioxide (dashed brown) is included for the mantle. The inset contains a depiction of the function \( n(P) \equiv \frac{dB}{dp}(P) \) for our three models along with experimental analysis direct from the seismic wave data of the newer Reference Earth Model [Kustowski, Ekström & Dziewoński 2008] (REF) depicted by the thick black line.

As \( B_{\text{adiab}} \) increases faster relative to \( B_{\text{isothermal}} \) as the pressure increases, this implies directly that \( n_{\text{adiab}} \equiv \frac{dB_{\text{adiab}}}{dp} > n_{\text{isothermal}} \). Pragmatically this is not too difficult, we must have an \( A_2 \) greater than the cold isothermal curves \((> 2.2)\) and so we vary \( \rho_0, B_0, A_0 \) and \( A_2 \) to fit the interior of the Earth. This fitting has only one more adjustable parameter, \( A_2 \), than the cold isothermal curves. We will label this primitive temperature dependent method 'model VII', after the section from which it was introduced.

In Fig. 7 we examine the inner core, outer core and mantle of the Earth in more detail than Fig. 1. We begin by plotting the seismic results, as we did in Fig. 1, except now we plot the pressure-density profile (sold black line). Since the data sets are large and precise they are informative. Not only do they contain the pressure-density but they also implicitly contain the derivatives. Here we calculate the bulk modulus directly from the \( s \) and \( p \) wave velocities (see Stacey & Davis (2004))

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The constant\( \rho \) inverse, the bulk modulus. A small change in\( \rho \) can be used instead and the error will be very manageable, less than 5% for a rocky planet like Earth as detailed in the analysis of the ratio of Eq. (61) which we always found to be under 1.05 using our primitive temperature dependence model.

The simple form of our EOS cannot describe this dynamic and so we hope only to stay close. Luckily the sensitivity to the pressure derivative is manageable.

Our simple temperature dependence does an adequate job describing the changing polytrope index for the mantle and the outer core, especially away from the inner boundary layer. The direction is correct, the magnitude is of the right order for the thermal pressure and the bulk modulus differential of Eq. (61)\( (B_{\text{adiab}}/B_{\text{isothermal}} \approx 1.01) \) for the mantle and \( 1.04 \) for the outer core. Most importantly we can describe the functional form of the polytrope index adequately. The inner core has the right direction and magnitude also \( (B_{\text{adiab}}/B_{\text{isothermal}} \approx 1.04) \) for the inner core but we fail to find an adequate function to match the low experimental analysis of the inner polytrope index\( (n \approx 2.3) \). The most obvious potential source of error is that the functional form of our temperature dependence does not have the complexity to adequately describe the adiabatic curvature relative to the cold temperature curve at inner core pressures. For example, Eq. (61) has a complex functional form because the elastic constants \( \gamma \) and \( \alpha \) are dependent on pressure. It is believed that at lower pressures \( B_{\text{adiab}} > B_{\text{isothermal}} \), which our simple temperature dependence mimics, but in the the extreme limit of a core of suns \( (P \to \infty \text{ and } \Delta T \to \infty) \), it is thought that \( B_{\text{adiab}} \equiv B_{\text{isothermal}} \). It would seem possible that the Earth core is high enough pressure that it exhibits the opposite behavior \( (n_{\text{adiab}} < n_{\text{isothermal}}) \) as that of the outer core and mantle \( (n_{\text{adiab}} > n_{\text{isothermal}}) \) thus explaining the extremely low polytrope index. The Earth inner core was the only layer where the cold curve produced satisfactory results, compared to the temperature dependent model, adding some credence to this hypothesis. We also could easily be missing important magnetic\( [\text{Poulovskii et al. 2013}] \) or rotational effects. We do have some faith in our estimation for \( n_0 \) of the core because we match the bulk modulus number at the core to a high precision.

The results of the three models are systematically logical. For the best fit with realistic values for the vacuum density, bulk modulus, and pressure derivative of the bulk modulus the adiabatic curve version with an unsophisticated temperature dependence, model VII, had good success as the listings show in Table 4 and depicted in Fig. 7. This result implies that this technique has the potential to add temperature and any other dynamic variables if the effect of these variables on \( \frac{dB}{dT} \) are known. Even with our simple adiabatic temperature model we were able to fit the density-radius profile to \(< 2\%\), the density-pressure profile to \(< 1\%\), the bulk modulus of the core to \(< 3\%\), and the pressure derivative of the bulk modulus for the outer core and mantle to \(< 10\%\), the inner core we fit to 30%. Our three best fits, with temperature dependence, also fall very near the linear equations that were stated in Table 3 and Eq. (59). Again this linear analysis is a remarkable relationship that we offer without theoretical underpinning.

Recognizing that choosing our EOS as authoritative (the best functional forms for the cold curves) is a source of possible systematic error (when our EOS err it is likely to error on the hard side) as is our simple model for adiabatic temperature dependence (the slope of the polytrope index-pressure curve is low for mantle and outer core), we choose to constrain the interior of the Earth materials with conservative, fair errors:

**inner core**: \( \rho_0 = 7600 \pm 400 \text{ kg/m}^3, B_0 = 190 \pm 40 \text{ GPa}, \text{ and } B_0' = 4.7 \pm 0.5 \)

**outer core**: \( \rho_0 = 6800 \pm 300 \text{ kg/m}^3, B_0 = 130 \pm 30 \text{ GPa}, \text{ and } B_0' = 4.8 \pm 0.3 \)

**mantle**: \( \rho_0 = 4100 \pm 200 \text{ kg/m}^3, B_0 = 200 \pm 30 \text{ GPa}, \text{ and } B_0' = 4.2 \pm 0.3 \).

The constant \( \rho_0' B_0 \), which plays a pivotal role in this research, explains why the bulk modulus is difficult to constrain. This constant, which is important in the linear analysis depicted in Table 3 is extremely sensitive to the compressibility and its inverse, the bulk modulus. A small change in \( \rho \) or \( n \) will cause a much larger change in the bulk modulus if this term is to stay near constant.

If the effects of temperature and other dynamic variables are not specified explicitly then a reasonable material cold curve can be used instead and the error will be very manageable, less than 5% for a rocky planet like Earth as detailed in the analysis of the ratio of Eq. (61) which we always found to be under 1.05 using our primitive temperature dependence model.

This trend is further bolstered by examining the difference between the density-pressure curves of Fig. 7. So we recommend
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
Model & \(\rho_0\) & \(B_0\) & \(n_0\) & \(A_0\) & \(A_1\) & \(A_2\) \\
\hline
\textbf{Earth Inner Core} & & & & & & \\
Sect. VII & 7563.0 & 196.39 & 4.486 & 1.736 & 2.584 & 2.750 \\
Sect. IV & 7563.0 & 196.39 & 4.486 & 2.369 & 1.893 & 2.117 \\
Sect. III & 7372.6 & 194.74 & 3.525 & 0.0 & - & 3.525 \\
\textbf{Earth Outer Core} & & & & & & \\
Sect. VII & 6654.2 & 131.32 & 4.748 & 2.096 & 2.652 \\
Sect. IV & 6654.2 & 131.32 & 4.748 & 3.814 & 1.791 & 2.096 \\
Sect. III & 6596.7 & 153.33 & 3.500 & 0.0 & - & 3.500 \\
\textbf{Earth Mantle} & & & & & & \\
Sect. VII & 3988.4 & 207.30 & 4.104 & 1.804 & 2.275 & 2.300 \\
Sect. IV & 3988.4 & 207.30 & 4.104 & 2.199 & 1.867 & 1.905 \\
Sect. III & 3942.9 & 216.85 & 3.200 & 0.0 & - & 3.200 \\
\end{tabular}

Table 4. \(\rho_0, B_0, n_0\) are chosen to be a good fit to the models of the Earth based on experiment. Model (VII - two dashed light blue on Fig. 7) is the most realistic allowing the parameters \(A_0\) and \(A_1\) to vary to simulate temperature dependence. Model (IV - dark blue short dash-dot on Fig. 7) is the same material with the temperature dependence removed and thus not as good fit. The simplest model (III - purple long dash-dot on Fig. 7) is a fixed polytrope index. The fixed polytrope does not require the sophistication of the earlier models but it can still be reduced to a specific case of the variable model by setting \(A_0 = 0\), \(A_2 = n\) and noting that it is independent of \(A_1\). For reference we depict the cold curve prediction for \(\epsilon\)-iron in the core panels, we add magnesium oxide and silicon dioxide for the mantle.

The fixed polytrope index (horizontal dashed purple line) becomes a better result as the pressure lowers but the values for planet Earth are not constrained enough to warrant it a valid approximation, it is evident that the polytrope index varies dramatically from the surface value to the interior value. The fixed polytrope index may have some justification for asteroids or small rocky planets, like Mercury or Mars, but it is of limited value otherwise. Interestingly fixed polytrope does not require the sophistication of the earlier models but it can still be reduced to a specific case of the variable model by setting \(A_0 = 0\), \(A_2 = n\) and noting that it is independent of \(A_1\), the validity of this reduction has been checked. Ironically the overwhelming success of the fixed polytrope method to fit simultaneously the pressure-density profile and the density-radius profile speaks volumes about how a flawed method, with an unchanging \(\frac{dB}{dP}\), can be used successfully to model a planet, yet there should be little import placed on the parameters extrapolated from that model.

If one tries to fit only the pressure-density experimental data with a cold curve to elucidate the the vacuum elastic variables the results will also be suspect and highly dependent on the EOS used. As described in Fig. 6, the results for high pressure molecular hydrogen varied depending on the functional form of \(n = \frac{dB}{dP}\). This same variance was shown on planet Earth (Fig. 1, Table 3 and Fig. 7) throughout this work. A plethora of densities, bulk moduli, and bulk moduli pressure derivatives can fit the earth pressure-density-radius well and it would be really hard to constrain these variables to anything approaching a few percent. By numerically analyzing the experimental seismic data we have constrained our EOS further than most (fitting to pressure-density, density-radius, bulk modulus-pressure and \(\frac{dB}{dP}\) vs. pressure) and can thus eliminate many previous parameter choices. We have also added a primitive temperature dependence which has the correct direction and the right order of magnitude. So we felt confident, with our model VII, to constrain the materials in the interior of the Earth somewhat. For better analysis we would need a reasonable temperature, magnetic, and convection profile, prepared from first principles, which would more tightly constrain the parameters. One could also imagine an array of coupled differential equations which represent the interplay between gravity and the elastic strain variables of \(B\) and \(\frac{dB}{dP}\) (our version of the Lane-Emden equations), heat and the elastic strain variables, rotational forces and the elastic strain variables, and electric and magnetic forces and the elastic strain variables. In principle our method could fully elevate to meet the challenges of these more sophisticated equations of state.

### 7.2 Summary of Technique

The method presented here is systematic, builds on history, and is able to grow in complexity as the EOS becomes more complex. To summarize this theoretical apparatus to solve the radius-density profile for a multi-layered planet:

\(\odot\) 0000 RAS, MNRAS 000, 000-000
• Develop a functional form or differential equation(s) for the polytrope index as a function of density (and other dynamic variables that are appropriate): \( n(\rho) \equiv \frac{dB(\rho)}{d\rho} \) for an appropriate core material with a given \( \rho_0, B_0, B_0' \equiv n_0 \)
• Solve analytically or numerically for the weighting function: \( h(\rho) = \int_{\rho_0}^\rho \frac{d\rho}{\rho^{n(\rho)} \ln \left( \frac{\rho}{\rho_0} \right)} \)
• Starting at the inner core \((r = 0)\) pick a reasonable core density and set \( \frac{d\rho}{dr} = 0 \).
• Solve the Lane-Emden differential equation (Eq. 40) to a chosen radius (or if the surface, when \( \rho(r) = \rho_0 \)).
• If another layer is desired choose an appropriate material with a given \( \rho_0, B_0, B_0' \equiv n_0, n(\rho) \) and \( h(\rho) \).
• Choose an initial density for the next layer and a starting radius \((r = r_{final} \) of the previous layer).
• Find the initial derivative, \( \frac{d\rho}{dr} \) by using Eq. 55 and Eq. 40
• Solve the differential equation to a chosen radius (or if the surface, when \( \rho(r) = \rho_0 \)). Repeat the last 4 steps if desired.

If one uses a series of differential equations which produce numerical functions only for the polytrope index, \( n(\rho) \), it would probably be preferred to start at the surface of the planet and work inward. This would involve some more computational challenges, but they are not insurmountable as Zeng & Seager (2008) have demonstrated.

8 SUMMARY AND CONCLUSION

This work first derived, in Sect. 2, that the historic polytrope index is for all cases and materials equivalent to the derivative with respect to pressure of the bulk modulus \( (n = \frac{dB}{dP}) \). This result is the foundation of the article, the further development that follows is inspired and motivated by recognizing the physical implications of the polytropic index as an elastic constant thus giving it a life beyond merely index status. We further show the strong relationship between the Murnaghan equation of state and the extended polytrope form thus rewriting these EOSs as equivalent to conserving a relationship between density, the polytropic index, and the compressibility \( (\rho^*K \) is conserved as long as \( n \) is constant). Section 3 uses this conserving relationship as an input to the Lane-Emden differential equation. This is mathematically equivalent to earlier pioneering work used to describe the interior of suns by Eddington and Chandrasekhar. The difference is that our equation uses compressibility and bulk modulus relationships to make it a differential equation with elastic variables as input. This form, stressing the elastic variables, is better suited for further development of the polytrope relation. We solve this differential equation for plausible solutions to the density profile of the five layer interior of the Earth. We find that there is a wide range of constant polytropic indices \( (n = \frac{dB}{dP}) \) that agree with the seismic data of the interior of the Earth.

In Sect. 4 we expand our technique by making the polytropic index a variable. Since the index has been shown to be equivalent to the derivative with respect to pressure of the bulk modulus \( (n = \frac{dB}{dP}) \) this is a natural progression since all materials have a decreasing first derivative when the pressures become very large. We re-derive the Lane-Emden differential equation for a variable polytropic index in a consistent manner. We then introduce an empirically developed functional form for our variable polytropic index, \( n(\rho) = A_0 \left( \frac{\rho}{\rho_0} \right)^{A_1} + A_2 \), which is our submission for a universal equation of state. With this creation we can create analytical formulas for pressure and bulk modulus as a function of density. We, more importantly, use our variable polytropic index as input in the Lane-Emden differential equation, we are able to make pressure-density profiles for six materials which are common in the interior of planets which compare favorably to experiment. By connecting our results to high pressure theory (Salpeter & Zapolsky 1967) we are able to make theoretical calculations up to \( 10^{18} \) Pa. We turn to the planets of our solar system in Sect. 5 where we show that their masses and radii compare favorably with mass-radius curves for the six common materials. We then show that this technique can be applied to a multi-layered body as we fit our technique to the planets Mars, Jupiter, and Uranus.

In Sect. 6 and Sect. 7 we discuss the importance of developing a universal EOS from \( \frac{dB}{dP} \), we then compare our functional form for \( \frac{dB}{dP} \) to previous work and to experimental analysis. The importance of constraining a EOS fit beyond the mundane pressure-density profile is demonstrated. We show that our EOS has the advantage of being simple and thus easy to modify. One addition that is needed is the adding of temperature dependence, we give a suggestion on how this could be done and show a variety of results using the interior of the Earth as our laboratory. We also verify earlier work that shows that temperature modification is a relatively small effect, under 5% or the Earth’s density-pressure profile. Finally using a variety of experimental analysis, we are able to narrow the constituents densities of the Earth to 5%, the bulk moduli to 20%, and the pressure derivative of the bulk moduli to 10%.

Having revealed the polytrope index as the pressure derivative of the bulk modulus is fortuitous for future research advancement. There are many first principle EOS theories (density functional, quantum Monte-Carlo) which calculate the pressure derivative of the bulk modulus (Ravindran et al 1998; Sha & Cohen 2010; Xun, Xian-Ming & Zhao-Yi 2010; Li, Xu & Liu 2005). The prominence of the bulk modulus in this procedure may also lends itself to solid state Born-Madelung calculations (Kittel 2005). By using the modified polytrope assumption of this work one has a straightforward procedure to solve the Lane-Emden equation for stars and planets using more sophisticated equations of state. These methods may include dynamic convective interactions, heat flow, rotational dynamics, magnetic kinetics, nuclear reactions, etc. which would complement a new generation of realistic planetary and solar models.

Finally why was Eddington’s choice of the polytrope index set to \( 4/3 \) for main sequence stars fortuitous? Assume, in a
main sequence star, the most significant interactions under extreme pressure and temperature are gravity, electron repulsion, and heat flow. Eddington assumed only heat flow and gravity but completely neglected the significant electron repulsion in a dense solar gas yet still calculated satisfactory results. If we examine the derivative of the bulk modulus with respect to pressure, our newly revealed polytrope index, we discover why. For large massive objects like suns their partial bulk moduli with respect to pressure have similar values, greater than one and less than two (≈ 4/3 for heat flow and ≈ 5/3 for electron repulsion if the Thomas-Fermi-Dirac scheme of Salpeter & Zapolsky [1967] is followed). The thermal and electric contribution to the total bulk modulus and thus the compressibility is nearly the same as one follows the pressure gradient. Over ninety percent of the suns interior (all but the near surface) can be described well by a constant polytrope index that averages somewhere between 4/3 and 5/3. As seen in this work the sensitivity of the polytrope index is small enough that this 1/3 difference has little effect on the final results. In contemporary solar research the polytrope assumption is described as a first order reference curve before other dynamic interactions are included. As with the interior of planets this technique of varying the polytrope index can expand, with a sophisticated EOS, the relevance of the polytrope assumptions for solar research and continue what Lane, Eddington, and Chandrasekhar began a century ago.

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APPENDIX A: RELATIONS BETWEEN DENSITY AND COMPRESSIBILITY FROM THE MURNAUGHAN EOS

It was Stacey and Davis 2004; 2008a who we have seen state that in the context of the Murnaghan EOS the relationship

\[ \frac{B}{B_0} = \left( \frac{\rho}{\rho_0} \right)^{B'_0}, \]  

(A1)

which is equivalent to our own intermediate results for the fixed polytrope index (Eqs. (6,7)). The derivation of Eq. (A1) is straightforward but we include it in some detail because the relation is used extensively in this work.

Assuming only a linear change in the bulk modulus

\[ B = B_0 + B'_0 P, \]  

(A2)

where \( B'_0 \) is the fixed derivative with respect to pressure. One can then insert the definition of the bulk modulus

\[ \rho \frac{dP}{d\rho} = B_0 + B'_0 P, \]  

(A3)

and integrating and describing the zero pressure boundary conditions by using a naught subscript, one gets

\[ \ln \left( \frac{B}{B'_0} \right) = \ln \left( \frac{\rho}{\rho_0} \right) + \ln \frac{B_0}{B'_0}, \]  

(A4)

which reduces to

\[ (1 + \frac{B'_0 P}{B_0}) = \left( \frac{\rho}{\rho_0} \right)^{B'_0}. \]  

(A5)

Finally using Eq. (A2) to substitute \( B'_0 P = B - B_0 \) we have the desired relation, Eq. (A1).

The Murnaghan EOS is found by solving Eq. (A5) for pressure

\[ P = \frac{B_0}{B'_0} \left( \frac{\rho}{\rho_0} \right)^{B'_0} - 1. \]  

(A6)

The similarity of this equation to the polytrope is undeniable

APPENDIX B: OUR EMPIRICAL EQUATION OF STATE

We developed an equation of state from analysis of earlier empirical equations of state. The major difference is ours uses the polytrope index, the pressure derivative of the bulk modulus, as its place of development. It was first stated in the text proper as Eq. (42), again

\[ n(\rho) = \frac{dB}{dP} = A_0 \left( \frac{\rho_0}{\rho} \right)^{A_1} + A_2. \]  

(B1)

The constants are chosen as follows: \( A_1 \to B'_\infty \) (approximately 1.95 or connecting it to the results of Salpeter & Zapolsky [1967]), \( A_0 = B'_0 - A_2 \), and \( A_1 = \frac{B_0 B'_\infty}{B'_0 - A_2} \) where \( B'_\infty \) is the asymptotic value at very high densities, see Appendix C for more details. In this work we set \( B_0 \), and \( B'_0 \) to experimental results and assume a universal ratio for \( B''_0 \) by setting

\[ A_1 = \frac{B'_0}{B'_0 - A_2} = \frac{B'_0}{A_0}, \]  

(B2)

which we borrowed from Roy-Roy (2006). Now using these fundamental relations

\[ B = \rho \frac{dP}{d\rho}, \]  

\[ n = \frac{dB}{dP} = \frac{d}{dP} \left( \rho \frac{dP}{d\rho} \right) \]

so

\[ \frac{dB}{B} = \frac{n}{\rho} d\rho, \]  

\[ \ln h = n \ln \frac{\rho}{\rho_0} + \ln \frac{B_0}{B} \]  

(B4)

in conjunction with Eq. (B5)

\[ \frac{dn(\rho)}{dP} \ln \frac{\rho}{\rho_0} = \frac{d}{dP} \ln h(\rho). \]  

(B5)

It can thus be shown
\[
\frac{dn}{dP} = \frac{\rho_0}{\rho} \frac{dn}{d\rho} dp.
\] (B6)

Specifically applying these relations to our EOS as expressed in Eq. (42) we find

\[
n = \frac{dB}{dP} = A_0 \left( \frac{\rho_0}{\rho} \right)^{A_1} + A_2
\] (B7)

\[
B = B_0 e^{M(\rho)} \left( \frac{\rho}{\rho_0} \right)^{A_2}
\] (B8)

\[
P = \frac{B_0 e^{\frac{\rho}{A_1}}}{A_1} \left[ \left( \frac{\rho}{\rho_0} \right)^{A_2} E_n \left( \frac{1}{A_1} + \frac{A_0}{A_1} \left( \frac{\rho_0}{\rho} \right)^{A_1} \right) - E_n \left( \frac{1}{A_1} + \frac{A_2}{A_1} \right) \right]
\] (B9)

\[
\frac{dn}{dP} = \frac{d^2B}{dP^2} = -A_1 \left( \frac{dB}{dP} - A_2 \right) B
\] (B10)

\[
\ln h = \frac{A_0}{A_1} \left( \frac{\rho_0}{\rho} \right)^{A_1} \left( 1 + A_1 \ln \frac{\rho}{\rho_0} \right) - \frac{A_0}{A_1}
\] (B11)

\[
h = e^{-M(\rho)} \left( \frac{\rho}{\rho_0} \right)^{A_0 \left( \frac{\rho}{\rho_0} \right)^{A_1}}
\] (B12)

noting that only \(n(\rho)\) and \(h(\rho)\) are needed to solve the Lane-Emden differential equation, Eq. (40). The functions in the exponents are given by

\[
M(\rho) = \frac{A_0}{A_1} \left( 1 - \left( \frac{\rho_0}{\rho} \right)^{A_1} \right)
\] (B13)

The only equation which is non-analytical is pressure which contains a special integral function, the generalized exponential integral \(E_n\) which is defined as

\[
E_n(n, x) = \int_1^{\infty} e^{-xt} t^n dt.
\] (B14)

By using the assumption of Ref. Roy & Roy (2006), we are equating the second derivative as

\[
B_0'' = -\frac{B_0'}{B_0}
\] (B15)

This is true at the surface but we can find the general relation also for our EOS. First taking the derivative of the polytrope index with respect to density,

\[
\frac{dn}{d\rho} = -A_0 A_1 \frac{\rho_0^{A_1} \rho^{A_1 + 1}}{\rho_0^{A_1 + 1}}
\] (B16)

then switching variables to a pressure derivative we have

\[
\frac{dn}{dP} = -A_0 A_1 \frac{\rho_0^{A_1}}{\rho_0^{A_1 + 1}} \frac{dP}{d\rho} = -A_0 A_1 \frac{\rho_0^{A_1}}{B_0^{A_1}} \frac{dP}{d\rho} = -A_1 \left( \frac{dP}{d\rho} - A_2 \right). \] (B17)

By doing a substitution for \(A_1\) in terms of \(A_2\) (using Eq. (B2)) in Eq. (B17) we have finally

\[
\frac{dn}{dP} = \frac{d^2B}{dP^2} = -\left( \frac{B_0'}{B_0} - A_2 \right) \left( \frac{dP}{d\rho} - A_2 \right),
\] (B18)

which has an interesting symmetry. This relationship for the second derivative has been studied in some detail recently in Ref. Singh & Dwivedi (2012).

**APPENDIX C: PARAMETERS CHOSEN FOR MATERIALS**

In the text proper we used six materials commonly found in the interior of planets in our solar system. For completeness and reproducibility we include our parameters for the lower pressures (< 10^{13} Pa) in Table C1.

The parameters, \(\rho_0, B_0,\) and \(n_0 = \frac{dB}{dP}\), were not adjusted and were set to values which best reflect the experimental results (Zha et al. 1993; Khairallah & Militzer 2008; Sha & Cohen 2010; Duffy, Hemley & Mao 1995; Panero, Benedetti & Jeanloz 2003b; Driver et al. 2010) the rest of these parameters are derived from these three experimental values. To first approximation these parameters are \(A_2 = 1.95, A_1 = \frac{n_0}{n_0 - 1.95}\) (or adjustable for best fit) and \(A_0 = n_0 - A_2\) if one needs pressures only up to the Tera-Pascal range.

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First order approximate values are

\[
A_2 = 1.95, A_1 = \frac{n_0}{n_0 - A_2} \text{ and } A_0 = n_0 - A_2 \text{ where } n_0 = \frac{dB}{dP} = B'_{0}.
\]

### Table C1.

| Material | \(\rho_0\) \([\text{kg/m}^3]\) | \(B_0\) \([\text{GPa}]\) | \(n_0\) | \(A_0\) | \(A_1\) | \(A_2\) |
|----------|----------------|----------------|------|------|------|------|
| \(H_2\)  | 79.43          | 0.162          | 6.70 | 4.837 | 1.385 | 1.863 |
| \(He\)   | 291.73         | 0.224          | 7.15 | 5.141 | 1.391 | 2.009 |
| \(H_2O\) | 998.0          | 2.20           | 7.13 | 5.248 | 1.359 | 1.882 |
| \(MgO\)  | 3580.0         | 157.0          | 4.37 | 2.465 | 1.772 | 1.904 |
| \(SiO_2\)| 4287.0         | 305.0          | 4.75 | 2.983 | 1.592 | 1.767 |
| \(Fe\)   | 8300.0         | 165.0          | 5.15 | 3.080 | 1.672 | 2.070 |

\(B'_{0}\) are chosen to be a rough consensus of the experimental values at vacuum pressure. The values of \(A_0, A_1, A_2\) follow the prescription of Eq. (C1) which is slightly adjusted on each iteration of the Newton-Raphson method searching for the critical density.

### Table C2.

| Material | \(\rho_c\) \([\text{kg/m}^3]\) | \(B_{0}\) \([\text{GPa}]\) | \(n_c\) | \(P_{c}\) \([\text{GPa}]\) | \(P_{0}\) \([\text{GPa}]\) | \(A\) | \(Z\) |
|----------|----------------|----------------|------|------|------|------|------|
| \(H_2\)  | 1.690E4        | 1.154E5        | 1.866| 6.165E4 | 2.782E3 | 2   | 2   |
| \(He\)   | 1.229E4        | 1.623E4        | 2.037| 7.801E3 | 7.806E2 | 4   | 2   |
| \(H_2O\) | 3.758E5        | 7.360E6        | 1.883| 3.900E6 | 2.027E5 | 18  | 10  |
| \(MgO\)  | 1.262E6        | 4.478E7        | 1.905| 2.351E7 | 1.434E6 | 40  | 20  |
| \(SiO_2\)| 1.577E7        | 3.967E9        | 1.767| 2.245E9 | 4.396E7 | 60  | 30  |
| \(Fe\)   | 9.736E5        | 1.998E7        | 2.071| 9.631E6 | 1.280E6 | 55.85| 26  |

The critical values are found by solving for the critical density, \(\rho_c\), using the Newton-Raphson Method when the bulk modulus of our EOS, Eq. (B11) matches the bulk modulus of the degenerate Fermi gas bulk modulus of Thomas-Fermi-Dirac, Eq. (B5). The parameters \(P_{c}\) and \(P_{0}\) are found by forcing the pressure formula of Eq. (48) to equal the pressure formula listed in Eq. (B15) by choosing an appropriate \(P_{0}\).

At higher pressures we switch to a Thomas-Fermi-Dirac scheme of Ref. [Salpeter & Zapolsky 1967]. To make this transformation we find a critical density where we match \(B\) and \(\frac{dB}{dP} = n\), we then create a match with pressure by adding a constant \(P_{0}\) to the Thomas-Fermi-Dirac scheme. We first assume an initial guess using a Newton Raphson method to find the critical density. The density is then iterated until it meets the requirement \(B_{T F D}(\rho_{c}) - B_{\text{Classical}}(\rho_{c}) = 0\). At this critical density the derivative of the bulk modulus with respect to pressure also matches thanks to the relationship in Eq. (13). At each iteration of the Newton-Raphson Method we adjust \(A_2\) and then we are able to set \(A_0\) and \(A_1\):

\[
\begin{align*}
A_2 &= n_{T F D} - (n_0 - n_{T F D}) \ast \left(\frac{\rho_0}{\rho_c}\right)^{\frac{n_0}{n_0 - n_{T F D}}} \\
A_1 &= \frac{n_0}{n_0 - A_2} \\
A_0 &= n_0 - A_2,
\end{align*}
\]

where \(n_{T F D}\) is the past iterative guess using the present iteration of \(\rho_c\) in Eq. (49) while \(n_0\) is set to the rough experimental consensus value. The results of the Newton-Raphson Method with additional useful parameters (atomic number, \(A\), and proton number, \(Z\)) are in Table C2.

Again no adjustable parameters are in this table, they were found using the Newton-Raphson Method by requiring that the bulk modulus matches at the boundary between the low pressure and high pressure theories.

### APPENDIX D: \(N = \frac{dB}{dP}\) FOR VARIOUS EQUATIONS OF STATE

Our Equation of state is developed from a functional form of \(\frac{dB}{dP}\) which is detailed in Appendix B. We repeat them here:

\[
\begin{align*}
B' &\equiv n = \frac{dB}{dP} = A_0 \left(\frac{\rho_0}{\rho}\right)^{A_1} + A_2 \tag{D1} \\
B &\equiv B_{0} e^{\frac{A_0}{A_1} \left(1 - \left(\frac{\rho}{\rho_0}\right)^{A_1}\right)} \left(\frac{\rho}{\rho_0}\right)^{A_2} \tag{D2} \\
P &\equiv B_{0} e^{\frac{A_0}{A_1}} \left[\left(\frac{\rho}{\rho_0}\right)^{A_2} E_n \left(\frac{A_1 + A_2}{A_1}, \frac{A_0}{A_1} \left(\frac{\rho}{\rho_0}\right)^{A_1}\right) - E_n \left(\frac{A_1 + A_2}{A_1}, \frac{A_0}{A_1}\right)\right] \tag{D3}
\end{align*}
\]

The equation of state of Stacey also starts with \(B' = \frac{dB}{dP}\).
\[ \frac{1}{B} = \frac{1}{B_0} + \left(1 - \frac{B_0^\prime}{B_0} \right) \frac{P}{B} \]

which then one can derive a formula for the bulk modulus and the density ratio

\[ B = B_0 \left(1 - B_0^\prime \frac{P}{B} \right) \frac{\eta}{\eta^\prime} \]

\[ \ln \frac{\rho}{\rho_0} = -\frac{B_0^\prime}{B_0^\prime} \ln \left(1 - B_0^\prime \frac{P}{B} \right) - \left( \frac{B_0^\prime}{B_0^\prime} - 1 \right) \frac{P}{B} \]

Most equations of state start from a definition of pressure, one can calculate \( n = \frac{dP}{d\rho} \) from \( P(\rho) \) analytically

\[ \frac{dP}{d\rho} = \frac{dP(\rho dP/d\rho)}{d\rho} = \rho \frac{dB/d\rho}{B} \]

Many of the next equations use \( \eta = \frac{\rho}{\rho_0} \). The Vinet EOS \cite{Vinat1977, 1980MNRAS.196..491P} is

\[ P = 3B_0 \eta^\star \left(1 - \eta^\star \right) \exp \left( \frac{3}{2} (B_0^\prime - 1) (1 - \eta^\star) \right) \]

\[ B = 3B_0 \left( \frac{2}{3} \eta^\star - \frac{5}{3} \eta^\star + \frac{3}{4} (B_0^\prime - 1) \eta^\star - 1 \right) \exp \left( \frac{3}{2} (B_0^\prime - 1) (1 - \eta^\star) \right) \]

\[ B' = \frac{(\frac{2}{3} \eta^2 \eta^\star - (\frac{1}{3})^2 \eta^2 \eta^\star + \frac{1}{2} (B_0^\prime - 1) (\eta^\star - \frac{1}{3}) \eta^\star + \frac{1}{2} (B_0^\prime - 1)^2 (1 - \eta^\star))}{\eta^\star - \frac{1}{3} \eta^\star + \frac{1}{2} (B_0^\prime - 1) (\eta^\star - 1)} \]

The Birch-Murnaghan third order EOS is \cite{Birch1947, 2000RvMP...72..789P}

\[ P = \frac{3}{2} B_0 (\eta^\star - \eta^\prime) \left(1 + \frac{3}{4} (B_0^\prime - 4) (\eta^\star - 1) \right) \]

\[ B = \frac{3}{2} B_0 \left( \frac{7}{3} \eta^\star - \frac{5}{3} \eta^\star + \frac{3}{4} (B_0^\prime - 4) \left( \frac{9}{3} \eta^\star - \frac{7}{3} \eta^\star + \frac{5}{3} \eta^\star \right) \right) \]

\[ B' = \frac{(\frac{2}{3} \eta^2 \eta^\star - (\frac{1}{3})^2 \eta^2 \eta^\star + \frac{1}{2} (B_0^\prime - 4) \left( \frac{9}{3} \eta^\star - \frac{7}{3} \eta^\star + \frac{5}{3} \eta^\star \right))}{\eta^\star - \frac{2}{3} \eta^\star + \frac{1}{2} (B_0^\prime - 4) \left( \frac{9}{3} \eta^\star - \frac{7}{3} \eta^\star + \frac{5}{3} \eta^\star \right)} \]

The Roy-Roy EOS \cite{Roy1970} is

\[ \eta = 1 + aP(1 + bP)^c \]

\[ B = \frac{aP + abP^2 + (1 + bP)^{1-c}}{a + ab(1 + c)P} \]

\[ B' = \frac{abc(1 + bP)^{c}(-b(1 + c)P - 2) + a^2(1 + bP(b(1 + c)P + 2))}{(a + ab(1 + c)P)^2} \]

where

\[ a = \frac{1}{B_0} \]

\[ b = \frac{1 + 2B_0^\prime - 5(B_0^\prime)^2 - 1}{6B_0^\prime (1 - B_0^\prime)} \]

\[ c = \frac{(3 - B_0^\prime)^2}{2B_0^\prime - 5(B_0^\prime)^2 - 1} \]

The DFT EOS equation of state is

\[ P = 10^{N(\rho)} \]

\[ B = -\frac{10^{N(\rho)}(a_1 \rho^{4/3} + 2a_2 \rho + 3a_3 \rho^{2/3} + 4a_4 \rho^{1/3} + 5a_5) \ln(10)}{3 \rho^{1/3}} \]

\[ B' = -\frac{25a_2 \rho^{2/3} \ln(10) + 25a_5 \rho^{5/3} + 40a_4 \rho^{4/3} \rho^{1/3} (\ln(10) + 30a_3 \rho^{2/3} \ln(10))}{15a_5 \rho^{7/3} + 12a_4 \rho^{6/3} + 9a_3 \rho^{5/3} + 6a_2 \rho^{8/3} + 3a_1 \rho^{9/3}} \]

\[ + \frac{20a_2 \rho^{2/3} \ln(10) + 5a_5 a_3 \rho^{2/3} \ln(100)}{15a_5 \rho^{5/3} + 12a_4 \rho^{4/3} + 9a_3 \rho^{7/3} + 6a_2 \rho^{8/3} + 3a_1 \rho^{9/3}} \]

\[ - \frac{16a_2 \rho^{2/3} \ln(10) + 9a_3 \rho^{5/3} + 4a_4 \rho^{4/3} + a_1 \rho^{3/3} + \rho^{4/3} (3a_4 + 2a_2 \rho^{1/3} + a_1 \rho^{2/3} + 3a_1 \rho^{3/3} \ln(10))}{15a_5 \rho^{7/3} + 12a_4 \rho^{6/3} + 9a_3 \rho^{7/3} + 6a_2 \rho^{8/3} + 3a_1 \rho^{9/3}} \]

\[ - \frac{16a_4 \rho^{6/3} + 24a_4 \rho^{4/3} \ln(10) + 8a_4 \rho^{5/3} \ln(10) + 8a_4 \rho^{4/3} \ln(100)}{15a_5 \rho^{7/3} + 12a_4 \rho^{6/3} + 9a_3 \rho^{7/3} + 6a_2 \rho^{8/3} + 3a_1 \rho^{9/3}} \]
with \( a_0 = 1.0683, a_1 = 19.1824(2696^{1/3}), a_2 = -36.3776(2696^{2/3}), a_3 = 28.5165(2696^{3/3}), a_4 = -10.6068(2696^{4/3}), a_5 = 1.5224(2696^{5/3}) \). The power functions are

\[
N(\rho) = 9 + a_0 + \frac{a_1}{\rho^{1/3}} + \frac{a_2}{\rho^{2/3}} + \frac{a_3}{\rho^{3/3}} + \frac{a_4}{\rho^{4/3}} + \frac{a_5}{\rho^{5/3}}
\]  

(D23)

In Fig. 6 we also attempted to analyze experimental data independent of a universal EOS. We take high precision pressure-density data that enters the \( 10^{9}\) Pa range for light elements and the \( 10^{10}\) Pa range for heavier elements. It is also advantageous to have at least two points of low pressure data (< \( 10^8\) Pa) that anchors the fit towards the correct origin. In the case of the hydrogen data we found that a 5th order polynomial gave a very low chi-squared. Once we have a polynomial for \( P \), calculating the bulk modulus and the polytrope index are trivial.

\[
P(GPa) = a_0 + a_1\rho + a_2\rho^2 + a_3\rho^3 + a_4\rho^4 + a_5\rho^5
\]  

(D24)

\[
B(GPa) = a_1\rho + 2a_2\rho^2 + 3a_3\rho^3 + 4a_4\rho^4 + 5a_5\rho^5
\]  

(D25)

\[
n = \frac{a_1\rho + 4a_2\rho^2 + 9a_3\rho^3 + 16a_4\rho^4 + 25a_5\rho^5}{a_1\rho + 2a_2\rho^2 + 3a_3\rho^3 + 4a_4\rho^4 + 5a_5\rho^5}
\]  

(D26)

with \( a_0 = 2.26603GPa, a_1 = -4.52911 \times 10^{-2}GPa/(kg/m^3)^1, a_2 = 2.16832 \times 10^{-4}GPa/(kg/m^3)^2, a_3 = 1.77254 \times 10^{-6}GPa/(kg/m^3)^3, a_4 = 1.64780 \times 10^{-10}GPa/(kg/m^3)^4, a_5 = -1.48451 \times 10^{-13}GPa/(kg/m^3)^5 \). It also helps to normalize the fitting routine, have as your dependent fitting variable \( \rho/(2\rho_{max}) \) so that your domain is much less than one, the range should also be small, here we fit it to Giga Pascals.