1. Introduction

1.1. Suppose $X$ is a compact $n$-dimensional complex manifold. Each partition $I = \{i_1, i_2, \ldots, i_r\}$ of $n$ corresponds to a Chern number $c^I(X) = \epsilon(c^{i_1}(X) \cup c^{i_2}(X) \cup \ldots \cup c^{i_r}(X) \cap [X]) \in \mathbb{Z}$ where $c^k(X) \in H^{2k}(X; \mathbb{Z})$ are the Chern classes of the tangent bundle, $[X] \in H_{2n}(X; \mathbb{Z})$ is the fundamental class, and $\epsilon : H_0(X; \mathbb{Z}) \to \mathbb{Z}$ is the augmentation. Many invariants of $X$ (such as its complex cobordism class) may be expressed in terms of its Chern numbers ([Mi], [St]). During the last 25 years, characteristic classes of singular spaces have been defined in a variety of contexts: Whitney classes of Euler spaces [Su], [H-T], [Ak], Todd classes of singular varieties [BFM], Chern classes of singular algebraic varieties [Mac], L-classes of stratified spaces with even codimension strata [GM1], Wu classes of singular spaces [Go2], [GP] (to name a few). However, these characteristic classes are invariably homology classes and as such, they cannot be multiplied with each other. In some cases it has been found possible to “lift” these classes from homology to intersection homology, where (some) characteristic numbers may be formed ([BBF], [BW], [Go2], [GP], [T]).

The case of locally symmetric spaces is particularly interesting. Suppose $\Gamma$ is a torsion-free arithmetic group acting on a complex $n$-dimensional Hermitian symmetric domain $D = G/K$, where $G$ is the group of real points of a semisimple algebraic group $\mathfrak{g}$ defined over $\mathbb{Q}$ with $\Gamma \subset G(\mathbb{Q})$, and where $K \subset G$ is a maximal compact subgroup. Then $X = \Gamma \setminus D$ is a Hermitian locally symmetric space. When $X$ is compact, Hirzebruch’s proportionality theorem [Hr1] says that there is a number $v(\Gamma) \in \mathbb{Q}$ so that for every partition $I = \{i_1, i_2, \ldots, i_r\}$ of $n$, the Chern number satisfies $c^I(X) = v(\Gamma)c^I(\tilde{D})$, where $\tilde{D} = G_u/K$ is the compact dual symmetric space (and $G_u$ is a compact real form of $\mathfrak{g}$ containing $K$).

If $X = \Gamma \setminus D$ is noncompact, it has a canonical Baily-Borel (Satake) compactification, $\overline{X}$. This is a (usually highly singular) complex projective algebraic variety. To formulate a proportionality theorem in the noncompact case, one might hope that the tangent bundle $TX$ extends as a complex vectorbundle over $\overline{X}$, but this is false. In [Mu1], D. Mumford showed that $TX$ has a particular extension $\overline{E}_\Sigma \to \overline{X}_\Sigma$ over any toroidal resolution $\tau : \overline{X}_\Sigma \to \overline{X}$ of the Baily-Borel compactification and that for any partition $I$ of $n$, the resulting Chern numbers

$$c^I(\overline{E}_\Sigma) = \epsilon(c^{i_1}(\overline{E}_\Sigma) \cup c^{i_2}(\overline{E}_\Sigma) \cup \ldots \cup c^{i_r}(\overline{E}_\Sigma) \cap [\overline{X}_\Sigma]) \quad (1.1.1)$$
satisfy the same equation, $c^i(E_\Sigma) = v(\Gamma)c^i(\hat{D})$. (The toroidal resolution $\overline{X}_\Sigma$ is constructed in [AMRT]; it depends on a choice $\Sigma$ of polyhedral cone decompositions of certain self-adjoint homogeneous cones.) Mumford also showed that if $\Sigma'$ is a refinement of $\Sigma$ then there is a natural morphism $f : \overline{X}_{\Sigma'} \to \overline{X}_\Sigma$ and that $f^*(E_{\Sigma}) \cong E_{\Sigma'}$ (hence $f^*c^i(E_{\Sigma}) = c^i(E_{\Sigma'})$). Moreover, it is proven in [Har] that the coherent sheaf $\tau_*\overline{E}_{\Sigma}$ is independent of the choice of $\Sigma$. One is therefore led to suspect the existence of a closer relationship between the characteristic classes of the vector bundles $\overline{E}_\Sigma$ and the topology of the Baily-Borel compactification $\overline{X}$. In Theorem 11.6 and Theorem 13.2 we show that, at least for the variety $\overline{X}$, the original goal of constructing Chern numbers can be completely realized:

**Theorem.** Every Chern class $c^i(X)$ has a canonical lift $\overline{c}^i \in H^{2i}(\overline{X}; \mathbb{C})$ to the cohomology of the Baily-Borel compactification. Moreover, if $\tau : \overline{X}_\Sigma \to \overline{X}$ is any toroidal resolution of singularities then

$$\tau^*(\overline{c}^i) = c^i(\overline{E}_\Sigma) \in H^{2i}(\overline{X}_\Sigma; \mathbb{C}).$$

It follows ([13.3]) that the lifts $\overline{c}^i$ satisfy ([1.1]). In §13 we show that the homology image $\overline{c}^i \cap [\overline{X}] \in H_*(\overline{X})$ lies in integral homology and coincides with (MacPherson’s) Chern class $\tau_*\overline{E}_\Sigma$ of the constructible function which is 1 on $X$ and is 0 on $\overline{X} - X$.

1.2. Moreover, a similar result holds for any automorphic vector bundle. Let $\lambda : K \to GL(V)$ be a representation of $K$ on some finite dimensional complex vectorspace $V$. By [Mul], the automorphic vector bundle $E' = (\Gamma \backslash G) \times_K V$ on $X$ has a particular extension $\overline{E}_\Sigma$ over any toroidal resolution $\overline{X}_\Sigma$. We show that each Chern class $c^i(E_\Gamma)$ has a canonical lift $\overline{c}^i(E') \in H^{2i}(\overline{X}; \mathbb{C})$ and that these lifts also satisfy the proportionality formula. Moreover, $\tau^*(\overline{c}(E')) = c^i(\overline{E}_\Sigma)$ and the image of $\overline{c}(E')$ in $\text{Gr}_W^{2i}(\overline{X}; \mathbb{C})$ (the top graded piece of the weight filtration) is uniquely determined by this formula.

1.3. In §14 we consider the subalgebra $H^*_{\text{Chern}}(\overline{X}; \mathbb{C})$ of the cohomology of the Baily-Borel compactification that is generated by the (above defined lifts of) Chern classes of certain “universal” automorphic vector bundles, and show that

**Theorem.** Suppose the Hermitian symmetric domain $D$ is a product of irreducible factors $G_i/K_i$ (where $K_i$ is a maximal compact subgroup of $G_i$), and that each $G_i$ is one of the following: $Sp_n(\mathbb{R})$, $U(p,q)$, $SO(2n)$, or $SO(2,2)$ with $p$ odd or $p = 2$. Then there is a (naturally defined) surjection $h : H^*_{\text{Chern}}(\overline{X}; \mathbb{C}) \to H^*(\overline{D}; \mathbb{C})$ from this subalgebra to the cohomology of the compact dual symmetric space.

This result is compatible with the few known facts about the cohomology of the Baily-Borel compactification. Charney and Lee [CL] have shown, when $D = Sp_{2n}(\mathbb{R})/U(n)$ is the Siegel upper half space, and when $\Gamma = Sp_{2n}(\mathbb{Z})$ that the “stable” cohomology of $\overline{X}$ contains a polynomial algebra which coincides with the “stable” cohomology of the compact dual
symmetric space $\bar{D}$ (which is the complex Lagrangian Grassmannian). It is a general fact (cf. §16) that the intersection cohomology $IH^*(\bar{X}; \mathbb{C})$ contains a copy of $H^*(\bar{D}; \mathbb{C})$.

1.4. Here are the main ideas behind the proof of Theorem 11.6. In [Hr2], Hirzebruch shows that the Chern classes of Hilbert modular varieties have lifts to the cohomology of the Baily-Borel compactification because the tangent bundle has a trivialization in a neighborhood of each of the finitely many cusp points, cf. [ADS]. If $X = \Gamma \backslash D$ is a $\mathbb{Q}$-rank 1 locally symmetric space such that $\bar{X}$ is obtained from $X$ by adding finitely many cusps, then the tangent bundle is not necessarily trivial near each cusp, but it admits a connection which is flat near each cusp, and so the Chern forms vanish near each cusp, hence the Chern classes lift to the cohomology of the Baily-Borel compactification. A similar argument applies to arbitrary automorphic vectorbundles (cf. [HZ] §3.3.9).

In the general $\mathbb{Q}$-rank 1 case, the singular set of the Baily-Borel compactification $\bar{X}$ consists of finitely many disjoint smooth compact manifolds (rather than finitely many cusp points). If $Y$ denotes such a singular stratum, then it admits a neighborhood $\pi_Y : N_Y \to Y$ such that every slice $\pi_Y^{-1}(y) \cap X$ is diffeomorphic to a neighborhood of a cusp similar to the kind described above. It is then possible to construct a connection $\nabla$ (on the tangent bundle) which is “flat along each fiber $\pi_Y^{-1}(y)$”. (We call this the “parabolic connection”; it is constructed in Section 10.) Moreover, within the neighborhood $N_Y$, each Chern form $\sigma^i(\nabla)$ is the pullback $\pi_Y^*(\sigma_Y^i)$ of a certain differential form $\sigma_Y^i$ on $Y$. Differential forms with this “$\pi$-fiber property” form a complex whose cohomology is the cohomology of $\bar{X}$, as discussed in Section 4. So the Chern form $\sigma^i(\nabla)$ determines a class $\bar{c}^i(\nabla) \in H^{2i}(\bar{X}; \mathbb{C})$. (In fact, even the curvature form satisfies the $\pi$-fiber condition.)

1.5. In higher rank cases, there are more problems. If $Y_1 \subset Y_2 \subset \bar{X}$ are singular strata of the Baily-Borel compactification, then it is possible to define a “parabolic” connection in a neighborhood $N(Y_1)$ of $Y_1$ whose curvature form has the $\pi$-fiber property relative to the tubular projection $\pi_1 : N(Y_1) \to Y_1$. It is also possible to construct a “parabolic” connection in a neighborhood $N(Y_2)$ of $Y_2$ whose curvature form has the $\pi$-fiber property relative to the tubular projection $\pi_2 : N(Y_2) \to Y_2$. However these two connections do not necessarily agree on the intersection $N(Y_1) \cap N(Y_2) \cap X$, nor do their curvature forms. When we patch these two connections together using a partition of unity, the curvature form of the resulting connection fails to have the $\pi$-fiber property. Nevertheless it is possible (as explained in Remark 11.3) to patch together connections of this type so as to obtain a connection whose curvature form $\Omega \in \text{End}(V)$ differs from a $\pi$-fiber differential form by a nilpotent element $n \in \text{End}(V)$ which commutes with $\Omega$ (cf. §12.1). (Here, $V$ is the representation of $K$ that gives rise to the automorphic vectorbundle $E_\Gamma = (\Gamma \backslash G) \times_K V$ on $X = \Gamma \backslash G/K$.) This is enough to guarantee that the Chern forms of this “patched” connection are $\pi$-fiber differential forms (cf. Lemma 1.4). A standard argument shows that the resulting cohomology class is independent of the choices that were involved in the construction.
1.6. A number of interesting questions remain. We do not know whether the results on Chern classes which are described in this paper for Hermitian symmetric spaces may be extended to the “equal rank” case (when the real rank of $G$ and of $K$ coincide). We do not know if the lifts $\bar{c}(E') \in H^{2i}(X; \mathbb{C})$ are integer or even rational cohomology classes. We do not know to what extent these lifts are uniquely determined by the properties (11.7), (13.2.1), (15.5). We do not know whether similar techniques can be applied to the Euler class of automorphic vectorbundles (when such a class exists: see §4.9). We do not know whether the surjection $h$ of Theorem 16.4 admits a natural splitting. We expect that $\bar{c}^*(E') = 0$ whenever the automorphic vectorbundle $E'$ arises from a representation $\lambda : K \to GL(V)$ which extends to a representation of $G$. If $E_{\Gamma}^{\text{RBS}}$ denotes the canonical extension ([GT] §9) of the automorphic vectorbundle $E$ over the reductive Borel-Serre ([Z1] §4.2 p. 190, [GHM] §8) compactification $\nu : X_{\text{RBS}} \to X$ then it is likely that $\nu^*(\bar{c}(E')) = c^*(E_{\Gamma}^{\text{RBS}})$. We expect these results to have interesting applications to the study of the signature defect ([Hr2] §3, [ADS], [Mü], [St1]) and to variations of weight 1 (and some weight 2) Hodge structures ([Gr1], [Gr2]).

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2. Control Data

2.1. A weakly stratified space $W$ is a compact Hausdorff space with a decomposition into finitely many smooth manifolds $W = Y_1 \cup Y_2 \cup \ldots \cup Y_r$ (called the strata of $W$) which satisfy the axiom of the frontier: If $Y$ and $Z$ are strata and if $Z \cap \overline{Y} \neq \emptyset$ then $Z \subset \overline{Y}$; we write $Z < Y$ and say that $Z$ is incident to $Y$. The boundary $\partial Y = Y - Y = \cup_{Z < Y} Z$ of the stratum $Y$ is the union of all strata incident to $Y$. If $W = \overline{X}$ is the closure of a single stratum $X$ then we say that $X$ is the nonsingular part of $W$ and the other strata $Y < X$ are boundary or singular strata of $W$. Fix a positive real number $\epsilon > 0$.

2.2. Definition. An $\epsilon$-system of control data on a weakly stratified space $W$ is a collection $\{T_Y(\epsilon), \pi_Y, \rho_Y\}$ indexed by the boundary strata $Y \subset W$, where

1. $T_Y(\epsilon) \subset W$ is an open subset of $W$ containing $Y$,
2. $T_Y(\epsilon) \cap T_Z(\epsilon) = \emptyset$ unless $Y < Z$ or $Z < Y$.
3. The tubular projection $\pi_Y : T_Y(\epsilon) \to Y$ is a retraction of $T_Y(\epsilon)$ to $Y$ which is smooth on each stratum,
4. $\pi_Z \pi_Y = \pi_Z$ whenever $Z < Y$ and both sides of the equation are defined,
5. \( \rho_Y : T_Y(\epsilon) \rightarrow [0, \epsilon) \) is a continuous “distance function”, with \( \rho_Y^{-1}(0) = Y \), such that the mapping \( (\rho_Y, \pi_Y) : T_Y(\epsilon) \rightarrow [0, \epsilon) \times Y \) is proper and its restriction to each stratum is a submersion.

6. \( \rho_Z \pi_Y = \rho_Z \) whenever \( Z < Y \) and both sides of the equation are defined.

For \( \tau \leq \epsilon \), write \( T_Y(\tau) = \rho_Y^{-1}([0, \tau]) \). By shrinking the neighborhood \( T_Y(\epsilon) \) and scaling \( \rho_Y \) if necessary, we may assume that each \( \rho_Y \) is defined on a slightly larger neighborhood \( T_Y'(\epsilon') \) (where \( \epsilon' > \epsilon \)) and that the “boundary” of \( T_Y(\epsilon) \) is

\[
\partial T_Y(\epsilon) = T_Y(\epsilon) - T_Y(\epsilon') = \rho_Y^{-1}(\epsilon) = \rho_Y^{-1}(\epsilon) \cup \partial Y.
\]

Such neighborhoods are illustrated in the following diagram.

![Tubular neighborhoods](image)

**Figure 1.** Tubular neighborhoods

If \( W \) is the closure of a single stratum \( X \) we extend this notation by setting \( T_X(\epsilon) = X \), \( \pi_X(x) = x \) and \( \rho_X(x) = 0 \) for all \( x \in X \).

2.3. Any compact real or complex algebraic or analytic variety admits a Whitney stratification ([Ha1], [Ha2]). Any compact Whitney stratified subset of a smooth manifold admits a system of control data (see [Mat] or [Gi] Thm. 2.6). If \( W \) is a compact Whitney stratified set and if the mappings \( \{\pi_Y\} \) are preassigned so as to satisfy Conditions (3) and (4) above, then distance functions \( \rho_Y \) may be found which are compatible with the mappings \( \pi_Y \).

3. **Partition of Unity**

3.1. Throughout this paper we will fix a choice of a smooth nondecreasing function \( s : \mathbb{R} \rightarrow [0, 1] \) so that \( s(x) = 0 \) for all \( x \leq 1/2 \) and \( s(x) = 1 \) for all \( x \geq 3/4 \). For any \( \epsilon > 0 \) define \( s_\epsilon(\rho) = s(\rho/\epsilon) \).

Fix \( 0 < \epsilon \leq \epsilon_0 \). Let \( W \) be a weakly stratified space with an \( \epsilon_0 \) system of control data \( \{T_Y(\epsilon_0), \pi_Y, \rho_Y\} \). For each stratum \( Z \subset W \) define the modified distance function \( s'_Z : T_Z(\epsilon) \rightarrow [0, 1] \) by \( s'_Z(x) = s_\epsilon(\rho_Z(x)) \). Then \( s'_Z = 0 \) on \( T_Z(\epsilon_0/2) \) and \( s'_Z = 1 \) near the edge \( \partial T_Z(\epsilon) \) of the tubular neighborhood \( T_Z(\epsilon) \).

For each stratum \( Y \subset W \) define a smooth function \( t^Y_\epsilon : Y \rightarrow \mathbb{R} \) as follows: If \( y \in Y \) is not contained in the tubular neighborhood \( T_Z(\epsilon) \) of any stratum \( Z < Y \) then set \( t^Y_\epsilon(y) = 1 \).
Otherwise, there is a unique maximal collection of boundary strata $Z_1, Z_2, \ldots, Z_r$ such that $y \in T_{Z_1}(\epsilon) \cap T_{Z_2}(\epsilon) \cap \ldots \cap T_{Z_r}(\epsilon)$ and in this case, by (2.2) (Condition 3), these boundary strata form a flag $Z_1 < Z_2 < \ldots < Z_r$ (after possibly relabeling the indices). Define

$$t^Y_\epsilon(y) = s^\epsilon_{Z_1}(y).s^\epsilon_{Z_2}(y)\ldots s^\epsilon_{Z_r}(y).$$

Then the function $t^Y_\epsilon : Y \to [0, 1]$ is smooth and vanishes on

$$\bigcup_{Z < Y} T_Z(\epsilon/2) \cap Y.$$ 

Pull this up to a function $\pi^*_Y t^Y_\epsilon : T_Y(\epsilon) \to [0, 1]$ by setting $\pi^*_Y t^Y_\epsilon(x) = t^Y_\epsilon(\pi_Y(x))$.

3.2. For each stratum $Y \subset W$, the product

$$B^Y_\epsilon = (\pi^*_Y t^Y_\epsilon).(1 - s^\epsilon_Y) : T_Y(\epsilon) \to [0, 1]$$

is smooth, vanishes near $\partial T_Y(\epsilon)$, and also near $\partial Y$:

$$x \in \bigcup_{Z < Y} T_Z(\epsilon/2) \implies B^Y_\epsilon(x) = 0.$$

Hence $B^Y_\epsilon$ admits an extension to $W$ which is defined by setting

$$B^Y_\epsilon(x) = 0 \text{ if } x \notin T_Y(\epsilon).$$

This extension is smooth on each stratum of $W$ and satisfies the following conditions whenever $Z < Y$:

$$B^Z_\epsilon \pi_Y(x) = B^Z_\epsilon(x) \text{ for all } x \in T_Y(\epsilon)$$

$$B^Y_\epsilon(\pi_Y(x)) \geq B^Y_\epsilon(x) \text{ for all } x \in T_Y(\epsilon)$$

$$B^Y_\epsilon \pi_Y(x) = B^Y_\epsilon(x) = t^Y_\epsilon(x) \text{ for all } x \in T_Y(\epsilon/2)$$

$$B^Y_\epsilon(x) = 1 \text{ for all } x \in T_Y(\epsilon/2) - \bigcup_{Z < Y} T_Z(\epsilon).$$
3.3. Lemma. For every stratum $Y \subset W$ and for every point $y \in Y$ we have

$$B^\epsilon_Y(y) + \sum_{Z < Y} B^\epsilon_Z(y) = 1 \quad (3.3.1)$$

3.4. Proof. It suffices to verify (3.3.1) for $y \in Y$. Then $B^\epsilon_Y(y) = t^\epsilon_Y(y)$. If $y$ is not in any tubular neighborhood $T^\epsilon_Z \cap Y$ (for $Z < Y$) then $t^\epsilon_Y(y) = 1$ and $1 - s^\epsilon_Z(y) = 0$. Otherwise, let $\{Z_1, Z_2, \ldots Z_r\}$ be the collection of strata for which $Z_i < Y$ and $y \in T^\epsilon_{Z_i}$. By relabeling the indices, we may assume that $Z_1 < Z_2 < \ldots Z_r < Y$ form a flag of strata. The nonzero terms in the sum (3.3.1) involve only the functions $s_1, s_2, \ldots, s_r$ (where $s_i = s^\epsilon_{Z_i}$) and can be written:

$$(1 - s_1) + s_1(1 - s_2 + s_2(\ldots + s_{r-1}(1 - s_r + s_r)\ldots)) = 1. \quad (3.4.1)$$

![Figure 3. Partition of Unity for fixed $\epsilon$](image)

3.5. In §11.2 we will construct a connection on a modular variety by induction, patching together connections which have been previously defined on neighborhoods of boundary strata. For each step of this induction we will need a different partition of unity, which is obtained from (3.3.1) by shrinking the parameter $\epsilon$. The purpose of this subsection is to construct the family of partitions of unity.

Let $W$ be a weakly stratified space with an $\epsilon_0 > 0$ system of control data, $\{T^\epsilon_Y, \pi^Y, \rho^Y\}$. Suppose each stratum $Y$ is a complex manifold and define

$$\epsilon Y = \epsilon_0 / 2^{\dim_{\mathbb{C}}(Y)}.$$
(The complex structure is irrelevant to this construction and is only introduced so as to agree with the later sections in this paper.) By Lemma 3.3 for every point $x \in Y$ we have
\[ \sum_{Z \leq Y} B^Y_Z(x) = 1. \]

By (3.2.4) and (3.2.6) the same equation holds, in fact for all $x \in T_Y(\frac{\pi_Y}{2})$.

Let $x \in W$. Then there is a maximal collection of strata $Y_1, Y_2, \ldots, Y_r$ such that
\[ x \in T_{Y_1}(\epsilon_0) \cap T_{Y_2}(\epsilon_0) \cap \ldots \cap T_{Y_r}(\epsilon_0). \]
These strata form a partial flag which (we may assume) is given by $Y_1 < Y_2 < \cdots < Y_r$. Set
\[ B^m_n = B^Y_{Y_n} \text{ and } \pi_m(x) = \pi_{Y_m}(x). \]
In the following lemma we assume $1 \leq m, n, m', n' \leq r$, that $m \geq n$ and that $m' \geq n'$.

3.6. Lemma. If $B^m_n(\pi_n(x)) \neq 0$ then $B^{m'}_{n'}(x) = 0$ for all $n' < n$ and for all $m' > m$. If $B^m_n(x) \neq 0$ then $B^{m'}_{n'}(\pi_m'(x)) = 0$ for all $n' > n$ and for all $m' < m$.

3.7. Proof. If $B^m_n(\pi_n(x)) \neq 0$ then $\pi_n(x) \notin T_n(\epsilon m/2) \supset T_n(\epsilon m')$ by (3.2.2). Therefore $B^{m'}_{n'}(\pi_n(x)) = 0$ by (3.2.3). Hence $B^{m'}_{n'}(x) = 0$ by (3.2.4). The second statement is the contrapositive of the first. \hfill \Box

4. $\pi$-Fiber Differential Forms

4.1. Suppose $W$ is a stratified space with a fixed $\epsilon$-system of control data $\{T_Y(\epsilon), \pi_Y, \rho_Y\}$. Define a $\pi$-fiber differential form $\omega$ to be a collection $\omega = \{\omega_Y \in A^*(Y; \mathbb{C})\}$ of smooth differential forms (with complex coefficients) on the strata $Y$ of $W$, which satisfy the following compatibility condition whenever $Z \subset Y$: There exists a neighborhood $T(\omega) \subset T_Z(\epsilon)$ of $Z$ such that
\[ \omega_Y|(Y \cap T(\omega)) = \pi_Y^Z(\omega_Z)|(Y \cap T(\omega)). \quad (4.1.1) \]
(Here, $\pi_Y^Z$ denotes the restriction $\pi_Z|Y \cap T_Z(\epsilon)$.) We refer to equation (4.1.1) as the $\pi$-fiber condition. If $\omega = \{\omega_Y\}$ is a $\pi$-fiber differential form, define its differential to be the $\pi$-fiber differential form $d\omega = \{d\omega_Y\}$. Let $A^\bullet(W)$ denote the complex of $\pi$-fiber differential forms and let $H^\pi(W)$ denote the resulting cohomology groups. These differential forms are analogous to the $\pi$-fiber cocycles in [Go1]. The following result is proven in [V].

4.2. Theorem. The inclusion $\mathbb{C} \to A^\pi(W)$ of the constant functions into the complex of $\pi$-fiber differential forms induces an isomorphism
\[ H^i(W; \mathbb{C}) \cong H^\pi_i(W) \quad (4.2.1) \]
for all $i$. The restriction $i^* [\omega] \in H^\pi(\overline{Y})$ of the cohomology class $[\omega]$ represented by a closed, $\pi$-fiber differential form $\omega = \{\omega_Y\}_{Y \subset W}$ to the closure $\overline{Y}$ of a single stratum is given by the $\pi$-fiber differential form $\{\omega_Z\}_{Y \subset \overline{Y}}$ (where $i : \overline{Y} \to W$ denotes the inclusion).
4.3. Suppose the stratified space $W$ is the closure of a single stratum $X$. Then a smooth differential form $\omega_X \in A^i(X)$ is the $X$-component of a $\pi$-fiber differential form $\omega \in A^i_w(W)$ if and only if for each stratum $Y$ there exists a neighborhood $U_Y$ of $Y$ such that for each point $p \in U_Y \cap X$ and for every tangent vector $v \in T_pX$ the following condition holds:

$$\text{If } d\pi(p)(v) = 0 \text{ then } i_v\omega = 0$$  \hspace{1cm} (4.3.1)

where $i_v$ denotes the contraction with $v$. If (4.3.1) holds, then the $\pi$-fiber form $\omega$ is uniquely determined by $\omega_X$.

Now suppose that $W = \overline{X}$ is a compact subanalytic Whitney stratified subset of some (real) analytic manifold, and that $\tau: \widetilde{W} \rightarrow W$ is a (subanalytic) resolution of singularities (cf. [Hi1], [Hi2]). This means that $\widetilde{W}$ is a smooth compact subanalytic manifold, the mapping $\tau$ is subanalytic, its restriction $\tau^{-1}(X) \rightarrow X$ to $\tau^{-1}(X)$ is a diffeomorphism, and $\tau^{-1}(X)$ is dense in $\widetilde{W}$. Let $\omega_X \in A^i(X)$ be a $\pi$-fiber differential form on $W$.

4.4. Lemma. Suppose the differential form $\tau^*(\omega_X)$ is the restriction of a smooth closed differential form $\widetilde{\omega} \in A^i(\widetilde{W})$. Let $[\widetilde{\omega}] \in H^i(\widetilde{W};\mathbb{R})$ and $[\omega_X] \in H^i(W;\mathbb{R})$ denote the corresponding cohomology classes. Then $[\widetilde{\omega}] = \tau^*([\omega_X])$.

4.5. Proof. The cohomology classes $[\widetilde{\omega}]$ and $[\omega_X]$ are determined by their integrals over subanalytic cycles by [Ha1], [Ha2]. Any subanalytic cycle $\xi \in C^i(\widetilde{W};\mathbb{R})$ may be made transverse (within its homology class) to the “exceptional divisor” $\tau^{-1}(W - X)$ ([GM3] §1.3.6). Then

$$\int_{\xi} \widetilde{\omega} = \int_{\xi \cap \tau^{-1}(X)} \widetilde{\omega} = \int_{\tau(\xi) \cap X} \omega_X = \int_{\tau(\xi)} \omega_X$$ \hspace{1cm} \square

5. Homogeneous Vectorbundles

5.1. If $M$ is a smooth manifold and $E \rightarrow M$ is a smooth vectorbundle, let $A^i(M,E)$ denote the space of smooth differential $i$-forms with values in $E$. Throughout this section, $K$ denotes a closed subgroup of a connected Lie group $G$ with Lie algebras $\mathfrak{k} \subset \mathfrak{g}$ and with quotient $D = G/K$. We fix a representation $\lambda : K \rightarrow GL(V)$ on some finite dimensional (real or complex) vectorspace $V$. Write $\lambda' : \mathfrak{k} \rightarrow \text{End}(V)$ for its derivative at the identity, and note that its derivative at a general point $g \in K$ is given by

$$d\lambda(h)((L_h)_*(\hat{k})) = \lambda(h)\lambda'(\hat{k}) \in \text{End}(V)$$ \hspace{1cm} (5.1.1)

for any $\hat{k} \in \mathfrak{k}$, where $L_h : K \rightarrow K$ is multiplication from the left by $h \in K$. The quotient mapping $q : G \rightarrow D = G/K$ is a principal $K$-bundle. The fundamental vertical vectorfields $Y_{\hat{k}}(g) = L_{q*}(\hat{k})$ (for $\hat{k} \in \mathfrak{k}$) determine a canonical trivialization, $\ker(dq) \cong G \times \mathfrak{k}$.
5.2. The representation $\lambda : K \to GL(V)$ determines an associated homogeneous vector-bundle $E = G \times_K V$ on $D = G/K$, which consists of equivalence classes $[g, v]$ of pairs $(g, v) \in G \times V$ under the equivalence relation $(g, v) \sim (g, \lambda(k)v)$ for all $g \in G$, $k \in K$, $v \in V$. It admits the homogeneous $G$ action given by $g' \cdot [g, v] = [g'g, v]$. Smooth sections $s$ of $E$ may be identified with smooth mappings $s : G \to V$ such that
\[
s(gk) = \lambda(k^{-1})s(g)
\] (5.2.1)
by $s(gk') = [g, s(g)] \in E$. Then (5.2.1) implies
\[
ds(g)(L_{g^*}(\dot{k})) = -\lambda'(\dot{k})s(g)
\] (5.2.2)
for all $g \in G$ and $\dot{k} \in \mathfrak{k}$.

Similarly we identify smooth differential forms $\tilde{\eta} \in \mathcal{A}^i(D, E)$ (with values in the vector-bundle $E$) with differential forms $\eta \in \mathcal{A}^i_{bas}(G, V)$ which are “basic,” meaning they are both $K$-equivariant ($R^i_k(\eta) = \lambda(k)^{-1}\eta$ for all $k \in K$) and horizontal ($i(Y_k)\eta = 0$ for all $\dot{k} \in \mathfrak{k}$). Here, $i(Y)$ denotes the interior product with the vectorfield $Y$ and $R_k(g) = gk$ for $g \in G$.

A connection $\nabla$ on $E$ is determined by a connection 1-form $\omega \in \mathcal{A}^1(G, \text{End}(V))$ which satisfies $\omega(L_{g^*}(\dot{k})) = \lambda'(\dot{k})$ and $R^i_k(\omega) = \text{Ad}(\lambda(k^{-1}))\omega$ for any $\dot{k} \in \mathfrak{k}$, $k \in K$, and $g \in G$. The covariant derivative $\nabla_X$ with respect to a vectorfield $X$ on $D$ acts on sections $s : G \to V$ satisfying (5.2.1) by
\[
\nabla_X s(g) = ds(g)(\tilde{X}(g)) + \omega_g(\tilde{X}(g))(s(g))
\] (5.2.3)
where $\tilde{X}$ is any lift of $X$ to a smooth vectorfield on $G$. We write $\nabla = d + \omega$. The curvature form $\Omega \in \mathcal{A}^2(D, \text{End}(E))$ takes values in the vectorbundle $\text{End}(E) = G \times_K \text{End}(V)$ and it will be identified with the “basic” 2-form $\Omega \in \mathcal{A}^2_{bas}(G, \text{End}(V))$ which assigns to tangent vectors $X, Y \in T_gG$ the endomorphism
\[
\Omega(X, Y) = d\omega(X, Y) + [\omega(X), \omega(Y)]
\] (5.2.4)
If the Lie bracket is extended in a natural way to Lie algebra-valued 1–forms, then it turns out ([BGV] §1.12) that $[\alpha, \alpha](X, Y) = 2[\alpha(X), \alpha(Y)]$ so we may express the curvature form as $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$.

The connection $\nabla = d + \omega$ is $G$-invariant iff $L^*_g(\omega) = \omega$, in which case it is determined (on the identity component $G^0$) by its value $\omega_0 : \mathfrak{g} \to \text{End}(V)$ at the identity. Using [Wa] or [KN] Chapt. II Thm. 11.5. it is easy to verify the following result.

5.3. Proposition. Suppose $G$ is a connected Lie group and $K$ is a closed subgroup. Then the $G$-invariant connections on the homogeneous vectorbundle $E = G \times_K V$ are given by linear mappings $\omega_0 : \mathfrak{g} \to \text{End}(V)$ such that

1. $\omega_0(\dot{k}) = \lambda'(\dot{k})$ for all $\dot{k} \in \mathfrak{k}$ and
2. $\omega_0([g, \dot{k}]) = [\omega_0(\dot{g}), \lambda'(\dot{k})]$ for all $\dot{g} \in \mathfrak{g}$ and all $\dot{k} \in \mathfrak{k}$. 

Moreover the curvature $\Omega \in \mathcal{A}^2(G, \text{End}(V))$ of such a connection is the left-invariant “basic” differential form whose value $\Omega_0$ at the identity is given by
\[
\Omega_0(\dot{g}, \dot{h}) = [\omega_0(\dot{g}), \omega_0(\dot{h})] - \omega_0([\dot{g}, \dot{h}])
\] (5.3.1)
for any $\dot{g}, \dot{h} \in \mathfrak{g}$. The connection is flat iff $\omega_0$ is a Lie algebra homomorphism. \hfill \Box

5.4. Example. Suppose the representation $\lambda : K \to GL(V)$ is the restriction of a representation $\hat{\lambda} : G \to GL(V)$. Then we obtain a flat connection with $\omega_0(\dot{g}) = \hat{\lambda}'(\dot{g})$ for $\dot{g} \in \mathfrak{g}$.

5.5. Example. A connection in the principal bundle $G \to D$ is given by a connection 1-form $\theta \in \mathcal{A}^1(G, \mathfrak{k})$ such that $R^*_k(\theta) = \text{Ad}(k^{-1})(\theta)$ and $\theta(Y_k) = \dot{k}$ (for any $k \in K$ and any fundamental vectorfield $Y_k$). It determines a connection $\nabla = d + \omega$ in the associated bundle $E = G \times_K V$ by $\omega(X) = \lambda'(\theta(X))$. The principal connection $\theta$ is $G$-invariant iff $L^*_g(\theta) = \theta$ in which case it is determined by its value $\theta_0 : \mathfrak{g} \to \mathfrak{k}$ at the identity. Conversely, by $[\mathfrak{kg}]$, any linear mapping $\theta_0 : \mathfrak{g} \to \mathfrak{k}$ determines a $G$-invariant principal connection iff $\ker(\theta_0)$ is preserved under the adjoint action of $K$. If $K$ is a maximal compact subgroup of $G$ then the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ determines a canonical $G$ invariant connection in the principal bundle $G \to G/K$, and hence a connection on $E$ which we refer to as the Nomizu connection. It is given by $\omega_0(\dot{k} + \dot{p}) = \lambda'(\dot{k})$ for $\dot{k} \in \mathfrak{k}$ and $\dot{p} \in \mathfrak{p}$. By (5.3.1), its curvature is given by
\[
\Omega(L_{g*}(\dot{g}_1), L_{g*}(\dot{g}_2)) = -\lambda'(\dot{p}_1, \dot{p}_2)
\] (5.5.1)
(where $\dot{g}_i = \dot{k}_i + \dot{p}_i \in \mathfrak{k} \oplus \mathfrak{p}$, since $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$.

5.6. There is a further description of homogeneous vectorbundles on $D$ which are topologically trivial. Let $E = G \times_K V$ be a homogeneous vectorbundle corresponding to a representation $\lambda : K \to GL(V)$ of $K$. A (smooth) automorphy factor $J : G \times D \to GL(V)$ for $E$ (or for $\lambda$) is a (smooth) mapping such that
1. $J(gg', x) = J(g, g'x)J(g', x)$ for all $g, g' \in G$ and for all $x \in D$
2. $J(k, x_0) = \lambda(k)$ for all $k \in K$.
It follows (by taking $g = 1$) that $J(1, x) = I$. The automorphy factor $J$ is determined by its values $J(g, x_0)$ at the basepoint: any smooth mapping $j : G \to GL(V)$ such that $j(gk) = j(g)\lambda(k)$ (for all $k \in K$ and all $g \in G$) extends in a unique way to an automorphy factor $J : G \times D \to GL(V)$ for $E$, namely
\[
J(g, hx_0) = j(gh)j(h)^{-1}.
\] (5.6.1)
An automorphy factor $J$, if it exists, determines a (smooth) trivialization
\[
\Phi_J : G \times_K V \to (G/K) \times V
\] (5.6.2)
by \([g, v] \mapsto (gK, J(g, x_0)v)\). This trivialization is \(G\)-equivariant with respect to the following \(J\)-automorphic action of \(G\) on \((G/K) \times V\):
\[
g.(x, v) = (gx, J(g, x)v).
\]  

(5.6.3)

Conversely, any smooth trivialization \(\Phi : E \cong (G/K) \times V\) of the homogeneous vectorbundle \(E\) determines a unique automorphy factor \(J\) such that \(\Phi = \Phi_J\). A trivialization of \(E\) (if one exists) allows one to identify smooth sections \(s\) of \(E\) with smooth mappings \(r : D \to V\). If the trivialization is given by an automorphy factor \(J\) and the smooth section \(s\) of \(E\) is given by a \(K\)-equivariant mapping \(s : G \to V\) as in (5.2.1) then the corresponding smooth mapping is
\[
r(gK) = J(g, x_0)s(g)
\]  

(5.6.4)

which is easily seen to be well defined. Sections \(s\) which are invariant under \(\gamma \in G\) correspond to functions \(r\) such that \(r(\gamma x) = J(\gamma, x)r(x)\) for all \(x \in D\).

If \(D = G/K\) is Hermitian symmetric of noncompact type then the canonical automorphy factor (§5.3) \(J_0 : G \times D \to K(C)\) determines an automorphy factor \(J = \lambda_C \circ J_0\) for every homogeneous vectorbundle \(E = G \times_K V\), where \(\lambda_C : K(C) \to \text{GL}(V)\) is the complexification of \(\lambda\).

5.7. If \(J_1, J_2\) are two automorphy factors for \(E\), then the mapping \(\phi : D \times V \to D \times V\) which is given by \(\phi(gx_0, v) = (gx_0, J_1(g, x_0)J_2(g, x_0)^{-1}v)\) is a well defined \(G\)-equivariant isomorphism of trivial bundles, where \(G\) acts on the domain via the \(J_1\)-automorphic action and \(G\) acts on the target via the \(J_2\)-automorphic action.

5.8. (The following fact will be used in §[10.4]) Suppose that \(J : G \times D \to \text{GL}(V)\) is an automorphy factor for \(E\). Let \(\nabla = d + \omega\) be a connection on \(E\). The trivialization \(\Phi_J\) of \(E\) (5.6.2) determines a connection \(\nabla^J = d + \eta\) on the trivial bundle \(D \times V\) (with \(\eta \in \mathcal{A}^1(D, \text{End}(V))\)) as follows: if \(r(gK) = J(g, x_0)s(g)\) as in (5.6.4) then \(\nabla^J_{q*}r(gK) = J(g, x_0)\nabla_Xs(g)\) (for any \(X \in T_gG\)). It follows from (5.1.1) that the connection 1-forms are related by
\[
\eta(q_*(X)) = J(g, x_0)\omega(X)J(g, x_0)^{-1} - (d_XJ(g, x_0))J(g, x_0)^{-1}.
\]  

(5.8.1)

6. Lemmas on curvature

6.1. Suppose \(E = G \times_K V\) is a homogeneous vectorbundle over \(D = G/K\) arising from a representation \(K \to \text{GL}(V)\) of a closed subgroup \(K\) of a Lie group \(G\) on a complex vector space \(V\). If \(\theta \in \mathcal{A}^1(G, \text{End}(V))\) is a Lie algebra-valued 1-form, denote by \([\theta, \theta]\) the Lie algebra valued 2-form \((X, Y) \mapsto 2[\theta(X), \theta(Y)]\) (cf [BGV] §1.12). The proof of the following lemma is a direct but surprisingly tedious computation.
6.2. Lemma. Suppose \( \{f_1, f_2, \ldots, f_n\} \) form a smooth partition of unity on \( D \), that is \( 0 \leq f_i(x) \leq 1 \) and \( \sum_{i=1}^{n} f_i(x) = 1 \) for all \( x \in D \). Let \( \nabla_i = d + \omega_i \) be connections on \( E \) with curvature forms \( \Omega_i \) for \( 1 \leq i \leq n \). Let \( \nabla = \sum_{i=1}^{n} f_i \nabla_i \) be the connection with connection form \( \omega = \sum_{i=1}^{n} f_i \omega_i \). Then the curvature \( \Omega \) of \( \nabla \) is given by

\[
\Omega = \sum_{i=1}^{n} f_i \Omega_i - \frac{1}{2} \sum_{i,j} f_i f_j [\omega_i - \omega_j, \omega_i - \omega_j] + \sum_{i=1}^{n-1} df_i \wedge (\omega_i - \omega_n). \]

(Even if the \( \nabla_i \) are flat and the \( f_i \) are constant, the connection \( \nabla \) is not necessarily flat.)

6.3. Let \( \mathcal{E} \) be a complex vectorspace and let \( f : \mathcal{E} \rightarrow \mathbb{C} \) be a homogeneous polynomial of degree \( k \). The **polarization** of \( f \) is the unique symmetric \( k \)-linear form \( P : \mathcal{E} \times \mathcal{E} \times \ldots \times \mathcal{E} \rightarrow \mathbb{C} \) such that \( f(x) = P(x, x, \ldots, x) \) for all \( x \in \mathcal{E} \). If \( N \subset \mathcal{E} \) is a vectorsubspace such that \( f(x + n) = f(x) \) for all \( x \in \mathcal{E} \) and all \( n \in N \) then the polarization \( P \) satisfies

\[
P(x_1 + n_1, x_2 + n_2, \ldots, x_k + n_k) = P(x_1, x_2, \ldots, x_k)
\]

for all \( x_1, \ldots, x_k \in \mathcal{E} \) and all \( n_1, \ldots, n_k \in N \).

Now let \( K \rightarrow GL(V) \) be a representation on a complex vectorspace \( V \) as above, and let \( f : \mathcal{E} = \text{End}(V) \rightarrow \mathbb{C} \) be a homogeneous polynomial of degree \( k \), which is invariant under the adjoint action \( K \rightarrow GL(\mathfrak{k}) \rightarrow GL(\text{End}(V)) \) of \( K \). Then the polarization \( P \) of \( f \) is also \( \text{Ad}_K \)-invariant. If \( \nabla \) is a connection on \( E = G \times_K V \) with curvature form \( \Omega \in \mathcal{A}^2(G, \text{End}(V)) \) then the characteristic form associated to \( f \) is

\[
\sigma^f_{\nabla}(X_1, Y_1, \ldots, X_k, Y_k) = P(\Omega(X_1, Y_1), \ldots, \Omega(X_k, Y_k)) \in \mathcal{A}^{2k}(G, \mathbb{C}).
\]

It is “basic” (cf. §6.2) and hence descends uniquely to a differential form on \( D \) which we also denote by \( \sigma^f_{\nabla} \in \mathcal{A}^{2k}(D, \mathbb{C}) \). Throughout this paper we shall be concerned only with homogeneous polynomials \( f : \text{End}(V) \rightarrow \mathbb{C} \) which are invariant under the full adjoint action of \( GL(V) \) and which we shall refer to as a **Ad-invariant polynomials**. When \( f(x) \) is the \( i \)-th elementary symmetric function in the eigenvalues of \( x \), the resulting characteristic form is the \( i \)-th **Chern form** and it will be denoted \( \sigma^i(\nabla) \).

6.4. Lemma. Let \( V \) be a complex vectorspace, \( H = GL(V) \) and \( \mathfrak{h} = \text{End}(V) \). Let \( x, n \in \mathfrak{h} \) and suppose that \( [x, n] = 0 \) and that \( n \) is nilpotent. Then for any \( \text{Ad} \)-invariant polynomial \( f : V \rightarrow \mathbb{C} \) we have:

\[
f(x + n) = f(x).
\]

6.5. Proof. Let \( \mathfrak{t} \subset \mathfrak{h} \) be the Lie algebra of a Cartan subgroup \( T \subset H \) and let \( W \) denote its Weyl group. The **adjoint quotient** mapping \( \chi : \mathfrak{h} \rightarrow \mathfrak{t}/W \) associates to any \( a \in \mathfrak{h} \) the \( W \)-orbit \( C(a_s) \cap t \) where \( a = a_s + a_n \) is the Jordan decomposition of \( a \) into its semisimple and nilpotent parts (with \( [a_s, a_n] = 0 \)), and where \( C(a_s) \) denotes the conjugacy class of \( a_s \) in \( \mathfrak{h} \), cf. [Sp]. (The value \( \chi(a) \) may be interpreted as the coefficients of the characteristic polynomial...
of \( a \in \text{End}(V) \). Since \( x \) and \( n \) commute, \((x+n)_s = x_s + n_s = x_s\), hence \( \chi(x+n) = \chi(x) \). But every Ad-invariant polynomial \( f : h \to \mathbb{C} \) factors through \( \chi \), hence \( f(x+n) = f(x) \). \( \square \)

7. Hermitian Symmetric Spaces

7.1. Throughout the remainder of this paper, algebraic groups will be denoted by boldface type and the associated group of real points will be denoted in Roman. Fix a reductive algebraic group \( G \) which is defined over \( \mathbb{Q} \) and which (for convenience only) is assumed to be connected and simple over \( \mathbb{Q} \). Let \( D \) acts as the identity component of the group of automorphisms of a Hermitian symmetric space \( D \). A choice of basepoint \( x_0 \in D \) determines a Cartan involution \( \theta : G \to G \), a maximal compact subgroup \( K = G^\theta = \text{Stab}_G(x_0) \) and a diffeomorphism \( G/K \cong D \). Let \( D^* \) denote the Satake partial compactification of \( D \), in other words, the union of \( D \) and all its rational boundary components, with the Satake topology (cf. \cite{BB}). The closure \( D_1^* \) in \( D^* \) of a rational boundary component \( D_1 \) is again the Satake partial compactification of \( D_1 \).

If \( D_1 \subset D^* \) is a rational boundary component of \( D \), its normalizer \( P \) is the group of real points of a rationally defined maximal proper parabolic subgroup \( P \subset G \). Let \( U_P \) denote the unipotent radical of \( P \) and let \( L(P) = P/U_P \) be the Levi quotient with projection \( \nu_P : P \to L(P) \). It is well known that \( L(P) \) is an almost direct product (commuting product with finite intersection) of two subgroups, \( L(P) = G_hG_\ell \) (we include possible compact factors in the \( G_\ell \) factor), where the “hermitian part” \( G_h \) is semisimple and defined over \( \mathbb{Q} \). (It may be trivial). The choice of basepoint \( x_0 \in D \) determines a unique \( \theta \)-stable lift \( L_P(x_0) \subset P \) of the Levi quotient \( \text{BoS} \) (which, from now on, we shall use without mention), as well as basepoints \( x_1 \in D_1 \) in the boundary component \( D_1 \) such that \( \text{Stab}_{G_h}(x_1) = K \cap G_h \). We obtain a decomposition

\[
P = U_PG_hG_\ell. \tag{7.1.1}
\]

The group \( P \) acts on the boundary component \( D_1 \) through the projection \( \nu_h : P \to L(P) \to G_h/(G_h \cap G_\ell) \) which also determines a diffeomorphism \( G_h/K_h \cong D_1 \) (where \( K_h = K \cap G_h \)). This projection \( \nu_h \) also gives rise to a \( P \)-equivariant canonical projection

\[
\pi : D \to D_1 \tag{7.1.2}
\]

by \( \pi(ug_hg_\ell K_P) = g_hK_h \).

The “linear part” \( G_\ell \) is reductive and contains the 1-dimensional \( \mathbb{Q} \)-split torus \( S_P(\mathbb{R}) \) in the center of the Levi quotient. If \( \mathfrak{j} \subset \mathfrak{n}_P \) denotes the center of the Lie algebra \( \mathfrak{n}_P \) of the unipotent radical of \( P \), then the adjoint action of \( G_\ell \) on \( \mathfrak{j} \) has a unique open orbit \( C(P) \) which is a self adjoint homogeneous cone.

7.2. Let \( P_0 \subset G \) be a fixed minimal rational parabolic subgroup and define the standard parabolic subgroups to be those which contain \( P_0 \). Let \( S_0 \subset L(P_0) \) be the greatest \( \mathbb{Q} \)-split torus in the center of \( L(P_0) \) and let \( \Phi = \Phi(S_0, G) \) be the (relative) roots of \( G \) in \( S_0 \) with positive roots \( \Phi^+ \) consisting of those roots which appear in the unipotent radical \( U_{P_0} \) and
resulting simple roots $\Delta$. Each simple root $\alpha$ corresponds to a vertex in the (rational) Dynkin diagram for $G$ and also to a maximal standard parabolic subgroup $P$ such that $S_P \subset \ker(\beta)$ for each $\beta \in \Delta - \{\alpha\}$.

7.3. **Two maximal parabolic subgroups.** For simplicity, let us assume that $G$ is (almost) simple over $\mathbb{Q}$. The (rational) Dynkin diagram for $G$ is linear, of type BC, and determines a canonical ordering among the maximal standard rational parabolic subgroups with $P_1 \prec P_2$ iff $D_1 \prec D_2$ (meaning that $D_1 \subset D_2^* \subset D^*$) where $D_i$ is the rational boundary component fixed by $P_i$. Write $P_1 = U_1G_{1h}G_{1\ell}$ and $P_2 = U_2G_{2h}G_{2\ell}$ as in (7.1.1). If $P_1 \prec P_2$ then $G_{1h} \subset G_{2h}$ and $G_{1\ell} \supset G_{2\ell}$. Let $P = P_1 \cap P_2$. In Figure 4, $\Delta - \{\alpha_1\}$ and $\Delta - \{\alpha_2\}$ denote the simple roots corresponding to $P_1$ and $P_2$ respectively.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{dynkin_diagram.png}
\caption{Dynkin diagrams for $G$ and $P$}
\end{figure}

Then we have a commutative diagram
\[
\begin{array}{ccc}
U_1 \subset U_P \subset P \subset P_1 \\
\downarrow \quad \downarrow \quad \downarrow ^{\nu_1} \\
U_P \subset \overline{P} \subset L(P_1) = G_{1h}G_{1\ell}
\end{array}
\] (7.3.1)
where $\overline{P} = \nu_1(P) \subset L(P_1)$ is the image of $P$. Then $\overline{P} = G_{1h}P_\ell$ where $P_\ell \subset G_{1\ell}$ is a parabolic subgroup of $G_{1\ell}$ whose Levi factor decomposes as a commuting, almost direct product $L(P_\ell) = G'_\ell G_{2\ell}$. Writing $\overline{U}$ for the lift of $U_P = \overline{U}$ we conclude that $P$ has a decomposition
\[
P = U_1G_{1h}(\overline{U}G'_\ell G_{2\ell}) = U_PG_{1h}G_{P_\ell}
\] (7.3.2)
with $U_P = U_1\overline{U}$, $G_{P_\ell} = G'_\ell G_{2\ell}$ and $P_\ell = \overline{U}G'_\ell G_{2\ell} \subset G_{1\ell}$. Similarly, we have a diagram
\[
\begin{array}{ccc}
U_2 \subset U_P \subset P \subset P_2 \\
\downarrow \quad \downarrow \quad \downarrow ^{\nu_2} \\
U_P \subset \overline{P} \subset L(P_2) = G_{2h}G_{2\ell}
\end{array}
\] (7.3.3)
where $\overline{P} = \nu_2(P) \subset L(P_2)$ is the image of $P$. Then $\overline{P} = P_h G_{2\ell}$ with $P_h \subset G_{2h}$ a parabolic subgroup of $G_{2h}$ whose Levi factor decomposes as a product $L(P_h) = G_{1h} G'_{\ell}$. Writing $\overline{U}$ for the canonical lift of $U_P = U_{P_h}$ we obtain another decomposition,

$$P = U_2(\overline{U} G_{1h} G'_{\ell}) G_{2\ell} = U_P G_{1h} G_P \ell \quad (7.3.4)$$

with $P_h = \overline{U} G_{1h} G'_{\ell}$.

Similarly, an arbitrary standard parabolic subgroup $Q$ may be expressed in a unique way as an intersection $Q = P_1 \cap P_2 \cap \ldots \cap P_m$ of maximal standard parabolic subgroups, with $P_1 \prec P_2 \prec \ldots \prec P_m$. In this case we write $Q^b = P_1$ and $Q^\ell = P_m$. If $P_1 = U_1 G_{1h} G_{1\ell}$ (with projection $\nu_1 : P_1 \to G_{1h} G_{1\ell}$) then the Levi factor $L(Q)$ decomposes as an almost direct product of $m + 1$ factors

$$L(Q) = G_{1h} (G_1 G'_{2} \ldots G'_{m-1} G_{m\ell}) = G_{1h} G_{Q\ell} \quad (7.3.5)$$

where $G_{1h}$ is the hermitian part of $L(P_1)$, and $G_{Q\ell}$ consists of the remaining factors, including $G_{m\ell}$, the linear part of $L(P_m)$. Each factor in $G_{Q\ell}$ acts as an automorphism group of a certain symmetric cone in the boundary of the cone $C(P_1)$. The projection

$$\overline{Q} = \nu_1(Q) = G_{1h} U_{P_1} Q G_{Q\ell}$$

is parabolic in $L(P_1)$ with unipotent radical $U_{P_1} Q \subset G_{1\ell}$ which also has a lift depending on the choice of basepoint. In summary we obtain a decomposition

$$Q = U_1 G_{1h} U_{P_1} Q G_{Q\ell} \quad (7.3.6)$$

8. CAYLEY TRANSFORM

8.1. As in §7, suppose that $G$ is defined over $\mathbb{Q}$ and simple over $\mathbb{Q}$, that $G = G(\mathbb{R})$, and that $K$ is a maximal compact subgroup of $G$ with $D = G/K$ Hermitian. The following proposition is the key technical tool behind our construction of a connection which is flat along the fibers of $\pi$. The proof follows from the existence of a “canonical automorphy factor for $P_1$" as defined by M. Harris [Har2, [HZ1] (1.8.7). See also the survey in [Z3]. In this section we will approximately follow [Har2] and derive these results from known facts about the Cayley transform [WK], [Z3] Chapter III.

8.2. Proposition. Let $P_1 = U_1 G_{1h} G_{1\ell}$ be a maximal rational parabolic subgroup of $G$. Set $K_1 = K \cap P_1 = K_{1h} K_{1\ell}$. Let $\lambda : K \to GL(V)$ be a representation of $K$. Then the restriction $\lambda|_{K_1}$ admits a natural extension $\lambda_1 : K_{1h} G_{1\ell} \to GL(V)$.

8.3. Proof. The group $K$ is the set of real points of an algebraic group $K$ defined over $\mathbb{R}$. As in [Hd] VII §7, [Sa] II §4, [AMRT] III §2 when the Cartan decomposition $g = \mathfrak{k} \oplus \mathfrak{p}$ is complexified it gives rise to two abelian unipotent subgroups $P^+$ and $P^-$ of $G(\mathbb{C})$ such that the complex structure on $g(\mathbb{C})$ acts with eigenvalue $\pm i$ on $\text{Lie}(P^\pm)$. The natural mapping $P^+ K(\mathbb{C}) P^- \to G(\mathbb{C})$ is injective and its image contains $G = G(\mathbb{R})$. Let $j : P^+ K(\mathbb{C}) P^- \to$
$\mathbf{K}(\mathbb{C})$ denote the projection to the middle factor. The group $P^+$ is the unipotent radical of the maximal parabolic subgroup $P^+\mathbf{K}(\mathbb{C})$ and hence is normal in $P^+\mathbf{K}(\mathbb{C})$; similarly $P^-$ is normal in $\mathbf{K}(\mathbb{C})P^-$. It follows that, for all $h \in P^+\mathbf{K}(\mathbb{C})$, for all $h' \in \mathbf{K}(\mathbb{C})P^-$ and for all $g \in P^+\mathbf{K}(\mathbb{C})P^-$ we have

$$j(ghh') = j(h)j(g)j(h'). \quad (8.3.1)$$

In particular, $j(gk) = j(g)k$ whenever $k \in \mathbf{K}(\mathbb{C})$, so by equation (6.6.1), $j$ determines a unique automorphy factor (the “usual” canonical automorphy factor) $J_0 : G \times D \to \mathbf{K}(\mathbb{C})$ such that $J_0(g, x_0) = j(g)$ (for all $g \in G$).

Let $c_1 \in G(\mathbb{C})$ be the Cayley element [Sa], [WK], [BE], [Har2], [Z3], which corresponds to $P_1$. That is, $c_1$ is a lift to $G(\mathbb{C})$ of the standard choice of Cayley element in $\text{Ad}(g)(\mathbb{C})$. We follow Satake’s convention, rather than that of [WK] (which would associate $c_1^{-1}$ to $P_1$). The element $c_1$ satisfies the following properties, [Sa] (Chapt. III (7.8), (7.9), (2.4)), [Har2], [Z3]:

1. $c_1, c_1^{-1} \in P^+\mathbf{K}(\mathbb{C})P^-$ and $c_1G \subset P^+\mathbf{K}(\mathbb{C})P^-$;
2. $c_1$ commutes with $G_{1h}$;
3. $c_1u_1K_{1h}G_{1\ell}c_1^{-1} \subset P^+\mathbf{K}(\mathbb{C})$;
4. $c_1K_{1h}G_{1\ell}c_1^{-1} \subset \mathbf{K}(\mathbb{C})$, and hence
5. $j(c_1g) = j(c_1gc_1^{-1})j(c_1) = c_1gc_1^{-1}j(c_1)$ for all $g \in K_{1h}G_{1\ell}$.

Harris then defines the canonical automorphy factor (for $P_1$) to be the automorphy factor $J_1 : G \times D \to \mathbf{K}(\mathbb{C})$ which is determined by its values at the basepoint, $J_1(g, x_0) = j(c_1)^{-1}j(c_1g)$. This is well defined because $j(c_1)^{-1}j(c_1gk) = j(c_1)^{-1}j(c_1g)k$ by (8.3.1), see (6.6.1). So we may define the canonical extension

$$\lambda_1(g) = \lambda_{c_1}J_1(g, x_0) = \lambda_{c_1}(j(c_1)^{-1}j(c_1g)) \quad (8.3.2)$$

for any $g \in K_{1h}G_{1\ell}$, where $\lambda_{c_1}$ is the complexification of $\lambda$. Then $\lambda_1(k) = \lambda(k)$ for any $k \in K_1$. Moreover $\lambda_1$ is a homomorphism: if $k_{1h}g_{1\ell}$ and $k'_{1h}g'_{1\ell}$ are elements of $K_{1h}G_{1\ell}$ then

$$J_1(k_{1h}g_{1\ell}k'_{1h}g'_{1\ell}, x_0) = j(c_1)^{-1}j(c_1k_{1h}g_{1\ell}k'_{1h}g'_{1\ell}c_1^{-1})j(c)$$

$$= j(c_1)^{-1}c_1k_{1h}g_{1\ell}c_1^{-1}j(c_1)j(c_1)^{-1}c_1k'_{1h}g'_{1\ell}c_1^{-1}j(c_1)$$

$$= J_1(k_{1h}g_{1\ell}, x_0)J_1(k'_{1h}g'_{1\ell}, x_0).$$

Verification that $J_1(k_{1h}^{-1}g_{1\ell}^{-1}, x_0) = J_1(k_{1h}g_{1\ell}, x_0)^{-1}$ is similar. We remark, following [Har2] that modifying $c_1$ by any element $d \in P^+\mathbf{K}(\mathbb{C})$ will not affect the values of $J_1(k_{1h}g_{1\ell}, x_0)$.

Now suppose $P_1 < P_2$ are rational maximal parabolic subgroups of $\mathbf{G}$ with $P_i = U_iG_{ih}G_{i\ell}$ ($i = 1, 2$) and, as in (7.3.4),

$$P_1 \cap P_2 = U_2\overline{U}G_{ih}G'_{i\ell}G_{2\ell} \text{ with } P_h = \overline{U}G_{ih}G'_{i\ell} \subset G_{2h}.$$
8.4. Proposition. Let $\lambda_i : K_{ih}G_{i\ell} \to GL(V)$ be the canonical extensions of $\lambda|K_i = K \cap P_i$ ($i = 1, 2$) and let $\lambda_{21} : K_{1h}G'_\ell \to GL(V)$ be the canonical extension of $\lambda|K_{2h}$ corresponding to the canonical automorphy factor $J_{21}$ for $P_h \subset G_{2h}$. Then
\[ \lambda_1(g_{2\ell}) = \lambda_2(g_{2\ell}) \] and
\[ \lambda_1(g'_\ell) = \lambda_{21}(g'_\ell) \]
for all $g_{2\ell} \in G_{2\ell}$ and all $g'_\ell \in G'_\ell$.

8.5. Proof. Let $c_1, c_2 \in G(\mathbb{C})$ and $c_{21} \in G_{2h}(\mathbb{C})$ be the Cayley elements for $P_1, P_2 \subset G$ and $P_h \subset G_{2h}$ respectively. By \cite{Sa} Chapter III (9.5),
\[ c_{21} = c_1c_2^{-1} = c_2^{-1}c_1. \]
The lift $G_{2h} \subset G$ is stable under the Cartan involution on $G$ so the corresponding decomposition $P_{2h}^+K_{2h}(\mathbb{C})P_{2h}$ coincides with $(G_{2h} \cap P^+)(G_{2h} \cap K(\mathbb{C}))(G_{2h} \cap P^-)$ and in particular $j_{2h} : G_{2h} \to K_{2h}(\mathbb{C})$ is the restriction of $j : G \to K(\mathbb{C})$. Let $c_{21} = c_{21}^+c_{21}^-c_{21}$ be the resulting decomposition. Then $c_{22}$ commutes with each of the factors $c_{21}^-$. It follows from (8.3.1) that
\[ j(c_1) = j(c_2c_2) = j(c_{21}^+c_{21}^-c_{21}) = j(c_{21}^-c_{21}) = j(c_{21})j(c_2) \]
and similarly $j(c_1) = j(c_2j(c_2))$. Since $g_{2\ell} \in G_{2\ell}$ also commutes with $c_{21} \in G_{2h}(\mathbb{C})$ and with $j(c_{21}) \in K_{2h}(\mathbb{C})$, we find
\[ \lambda_1(g_{2\ell})\lambda_2(g_{2\ell})^{-1} = \lambda_C\left(j(c_1)^{-1}c_{11}g_{2\ell}c^{-1}_{21}j(c_1)j(c_2)^{-1}c_{22}g_{2\ell}^{-1}c_2^{-1}j(c_2)\right) \]
\[ = \lambda_C\left(j(c_1)^{-1}c_{11}g_{2\ell}c^{-1}_{21}j(c_2)c_{22}g_{2\ell}^{-1}c_2^{-1}j(c_2)\right) \]
\[ = \lambda_C\left(j(c_1)^{-1}c_{11}g_{2\ell}c^{-1}_{21}c_{22}g_{2\ell}^{-1}c_2^{-1}j(c_2)j(c_2)\right) \]
\[ = \lambda_C\left(j(c_1)^{-1}c_{22}c_{21}^+g_{2\ell}^{-1}c_2^{-1}j(c_1)\right) \]
\[ = \lambda_C\left(j(c_1)^{-1}c_{22}c_{21}^-c_2^{-1}j(c_1)\right) = 1. \]
Similarly if $g'_\ell \in G'_\ell$ then using (4) above,
\[ \lambda_{21}(g'_\ell) = \lambda_C\left(j(c_{21})^{-1}j(c_{21}g'_\ell c_{21}^-)j(c_{21})\right) \]
\[ = \lambda_C\left(j(c_{21})^{-1}j(c_{21}^-c_1g'_\ell c_{21}^-c_2)j(c_{21})\right) \]
\[ = \lambda_C\left(j(c_{21})^{-1}j(c_{21})^{-1}j(c_{21}g'_\ell c_{21}^-c_2)j(c_{21})\right) \]
\[ = \lambda_C\left(j(c_1)^{-1}j(c_{21}g'_\ell c_{21}^-c_2)j(c_{21})\right) = \lambda_1(g'_\ell). \]

9. Baily-Borel Satake compactification

9.1. As in \S 6, suppose that $\mathbf{G}$ is defined over $\mathbb{Q}$ and simple over $\mathbb{Q}$, that $G = \mathbf{G}(\mathbb{R})$, and that $K$ is a maximal compact subgroup of $G$ with $D = G/K$ Hermitian. Let $D^*$ be the Satake partial compactification of $D$, consisting of $D$ together with its rational boundary components $D_P$, one for each (proper) maximal rational parabolic subgroup $P \subset G$; with the Satake topology \cite{B2}. The action of $\mathbf{G}(\mathbb{Q})$ on $D$ extends continuously to an action of $\mathbf{G}(\mathbb{Q})$ on $D^*$. Let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be a neat arithmetic subgroup and let $q : D^* \to \overline{X} = \Gamma\bs D^*$
denote the quotient mapping. Then $\overline{X}$ is the Baily-Borel compactification of $X$ and it admits the structure of a complex projective algebraic variety with a canonical stratification with a single stratum for every $\Gamma$-conjugacy class of rational boundary components as follows. Let $D_1 \subset D^*$ be a rational boundary component with normalizing maximal parabolic subgroup $P = U_P G_h G_\ell$. Let $\nu_h : P \to G'_h = G_h/(G_h \cap G_\ell)$ and let $\nu_\ell : P \to G'_\ell = G_\ell/(G_h \cap G_\ell)$. The closure $D'_1$ of $D_1$ in $D^*$ is the Satake partial compactification of $D_1$. The group $P$ acts on $D_1$ through its projection to $G'_h$ and the group $\Gamma_P = \Gamma \cap P$ acts on $D'_1$ through its projection $\Gamma_h = \nu_h(\Gamma_P)$ to $G'_h$. Then $X_1 = \Gamma_P \backslash D'_1 = \Gamma_h \backslash D'_1$ is a stratum of $\overline{X}$. Its closure $\overline{X}_1 = \Gamma_h \backslash D'_1$ in $\overline{X}$ is the Baily-Borel compactification of $X_1$. The stratum $X_1$ is also the image of the (infinitely many) rational boundary components $D'_1$ which are $\Gamma$-conjugate to $D_1$.

9.2. Let $D_1 \subset D^*$ be a rational boundary component which projects to $X_1$. We will say that a neighborhood $\tilde{U} \subset D^*$ is a $\Gamma$-parabolic neighborhood of $D_1$ if the following holds: if $x_1, x_2 \in \tilde{U}$ and $\gamma \in \Gamma$ satisfy $x_2 = \gamma x_1$ then $\gamma \in \Gamma \cap P$. If $X_1 \subset \overline{X}$ is a stratum in the Baily-Borel compactification of $X$, we say that a neighborhood $U \subset \overline{X}$ of $X_1$ is parabolic if for some (and hence for any) boundary component $D_1 \subset D^*$ with $q(D_1) = X_1$, there is a $\Gamma$-parabolic neighborhood $\tilde{U} \subset D^*$ of $D_1$ such that $U = q(\tilde{U})$. This means that the covering $\Gamma_P \backslash D^* \to \Gamma \backslash D^*$ is one to one on $\tilde{U}$, and we have a commutative diagram

$$
D^* \supset \tilde{U} \supset D_1
$$

9.3. Lemma. Each stratum $X_1 \subset \overline{X}$ has a fundamental system of neighborhoods, each of which is $\Gamma$-parabolic.

9.4. Proof. \cite{Sa2} Let $D^{BS}$ be the Borel-Serre partial compactification of $D$ together with its “Satake” topology \cite{BoS}. It is a manifold with corners, having one corner $e(P)$ for each rational parabolic subgroup $P$. According to \cite{Z2} the identity mapping $D \to D$ has a unique continuous extension $\nu : D^{BS} \to D^*$, and it is surjective. If $P$ is standard then $\nu(e(P)) = D_P$. Let $D^\dagger$ denote the quotient topology on the underlying set $|D^*|$ which is induced by $\nu$. Then $D^\dagger \to D^*$ is a continuous bijection and the quotient mapping $\Gamma \backslash D^\dagger \to \Gamma \backslash D^*$ is a homeomorphism. In \cite{Sa1} Theorem 8.1, Saper constructs a basis of parabolic neighborhoods $U_P$ of each corner $e(P)$ in $D^{BS}$. If $P$ is maximal then $\nu(U_P)$ is open in $D^*$ as may be shown by verifying the condition at the bottom of page 264 in \cite[AMRT]{AMRT}. We remark that the image of $U_P$ is $\Gamma$-parabolic and is open in $D^\dagger$ by construction, and that the topology $D^\dagger$ may be substituted for the Satake topology $D^*$ throughout this paper.

9.5. Fix a standard rational boundary component $D_1 \subset D^*$ normalized by a standard maximal rational parabolic subgroup $P_1$. In the Satake topology (or in the topology $D^\dagger$) there is a natural neighborhood $T(D_1) = \bigcup \{D_2 \mid D_1 \prec D_2 \prec D\}$ consisting of the union of
all rational boundary components (including $D$, the nonproper boundary component) whose closures contain $D_1$. The projection $\pi : D \to D_1$ has a unique continuous extension $T(D_1) \to D_1$ to this neighborhood. Its restriction to each intermediate boundary component $D_2$ coincides with the canonical projection $D_2 \to D_1$ which is obtained by considering $D_2$ to be the symmetric space corresponding to the Hermitian part $G_{2h}$ of the Levi factor of $P_2$ and by considering $D_1 \subset D_2$ to be the rational boundary component preserved by the parabolic subgroup $P_h \subset G_{2h}$ (notation as in \S 9.3). It follows that $\pi_1(x) = \pi_1\pi_2(x)$ for all $x \in T(D_1) \cap T(D_2)$. (The above union can be quite large: if $D_{20}$ is a standard boundary component normalized by a standard parabolic subgroup $P_2 > P_1$ then the boundary components $D_2 \subset T(D_1)$ which are conjugate to the standard one $D_{20}$ are in one to one correspondence with elements of $P_1(\mathbb{Q})/(P_1(\mathbb{Q}) \cap P_2(\mathbb{Q}))$ cf. (7.3.2).)

9.6. Lemma. For $\epsilon_0 > 0$ sufficiently small, the Baily-Borel compactification $\overline{X}$ admits an $\epsilon_0$-system of control data \{$T_Y(\epsilon_0)$, $\pi_Y, \rho_Y$\} such that for each stratum $Y \subset \partial X$ and for any choice of boundary component $D_1 \subset D^*$ with $q(D_1) = Y$ we have

1. The neighborhood $T_Y(\epsilon_0) \subset \overline{X}$ is a parabolic neighborhood of $Y$; it is the image, say, of some $\Gamma$-parabolic neighborhood $\tilde{U}(\epsilon_0) \subset D^*$ of $D_1$, and
2. $\pi_Y(\nu_X(x)) = \pi(\nu_X(x))$ for all $x \in \tilde{U}(\epsilon_0)$ (where $\pi : D \to D_1$ is the canonical projection).

9.7. Proof. For any $g \in P$ and $x \in D$ we have $\pi(gx) = \nu_x(g)\pi(x) \in D_1$. It follows that the projection function $\pi$ passes to the quotient $\Gamma_P \backslash D^*$, where it may be restricted to a parabolic neighborhood $U$ of $Y = q(D_1)$; write $\pi_Y : U \to Y$ for the result. If $P$ and $P'$ are $\Gamma$ conjugate maximal rational parabolic subgroups corresponding to conjugate boundary components $D_1$ and $D'_1$ then the projections $T(D_1) \to D_1$ and $T(D'_1) \to D'_1$ are compatible with conjugation, which shows that the resulting projection $\pi_Y : U \to Y$ is independent of the choice of lift $D_1 \subset D^*$ of the stratum $Y \subset X$. The tubular neighborhood $T_Y(\epsilon_0)$ may be chosen inside $U$. The compatibility between these projections follows from (9.3). As mentioned in \S 9.3, by further shrinking the tubular neighborhoods if necessary, control data may be found for which the tubular projections agree with these $\pi_Y$. 

10. Parabolically Induced Connection

10.1. As in \S 7.4 we suppose that $G$ is semisimple, defined over $\mathbb{Q}$ and simple over $\mathbb{Q}$; that $G = G(\mathbb{R})$, and $K \subset G$ is a maximal compact subgroup with $D = G/K$ Hermitian symmetric. Fix $\Gamma \subset G(\mathbb{Q})$ a neat arithmetic subgroup. Let $\lambda : K \to GL(V)$ be a representation of $K$ on some complex vector space $V$ and denote by $E = G \times_K V$ the associated homogeneous vector bundle on $D$.

Let $D_1$ be a rational boundary component of $D$ with canonical projection $\pi : D \to D_1$. Let $P$ be the maximal parabolic subgroup of $G$ which preserves $D_1$. Write $P = UG_hG_\ell$ as in \S 7.4 and let $K_h = K \cap G_h$ and $K_\ell = K \cap G_\ell$ be the corresponding maximal compact subgroups. Let $g_h = t_h \oplus p_h$ and $g_\ell = t_\ell \oplus p_\ell$ denote the corresponding Cartan decompositions.
The restriction of \(\lambda\) to \(K_h\) determines a homogeneous vectorbundle \(E_1 = G_h \times K_h V\) over \(D_1\). By Proposition 5.2, the representation \(\lambda|\ K_hK_\ell\) admits an extension to a representation \(\lambda_1 : K_hG_\ell \to GL(V)\). This extension determines an action of \(P\) on \(E_1\) which is given by

\[
ug_hg_\ell [g_h', v] = [g_h g_h', \lambda_1(g_\ell)v]. \quad (10.1.1)
\]

We obtain a vectorbundle mapping (which covers \(\pi\)),

\[
\tilde{\Phi} : E = P \times K_\ell V \to G_h \times K_h V = E_1
\]

by \(\tilde{\Phi}([ug_hg_\ell, v]) = [g_h, \lambda_1(g_\ell)v]\). Then \(\tilde{\Phi}\) induces an isomorphism,

\[
\Phi : E \cong \pi^*(E_1); \quad [g, v] \mapsto (gK_P, \tilde{\Phi}([g, v])) \in D \times E_1 \quad (10.1.3)
\]

of \(P\)-homogeneous vectorbundles (where \(g \in P\) and \(v \in K\)).

10.2. Definition. Let \(\nabla_1 = d + \omega_1\) be a connection on \(E_1\). The parabolically induced connection \(\nabla = d + \omega\) on \(E\) is defined to be the pullback \(\nabla = \Phi^*(\nabla_1)\) of \(\nabla_1\) under the isomorphism \(\Phi\). It is the unique connection whose covariant derivative (5.2.3) satisfies

\[
\nabla_v(\Phi^*(s)) = \Phi^*((\nabla_1)_{\pi^*s}) \quad (10.2.1)
\]

for any section \(s\) of \(E_1\) and for any tangent vector \(v \in T_x D\).

10.3. Proposition. Suppose \(\nabla_1 = d + \omega_1\) is a connection on \(E_1 = G_h \times K_h V\). Let \(\nabla = d + \omega\) denote the parabolically induced connection on \(E = P \times K_\ell V\). Then

\[
\omega(L_{g^*}(\dot{u} + \dot{g}_h + \dot{g}_\ell)) = \lambda_1(\dot{g}_\ell) + Ad(\lambda_1(g_\ell^{-1}))(\omega_1(L_{g_h^*}(\dot{g}_h))) \quad (10.3.1)
\]

for any \(g = ug_hg_\ell \in P\) and any \(\dot{u} + \dot{g}_h + \dot{g}_\ell \in \text{Lie}(U_P) \oplus g_h \oplus g_\ell\).

10.4. Proof. Let \(J_1 : G_h \times D_1 \to GL(V)\) be an automorphy factor for \(E_1\), corresponding to a trivialization \(E_1 \cong D_1 \times V\). Composing this with the isomorphism \(\Phi : E \to \pi^*(E_1)\) determines an automorphy factor \(J : P \times D \to GL(V)\) with

\[
J(ug_hg_\ell, x_0) = J_1(g_h, x_1) \lambda_1(g_\ell) \quad (10.4.1)
\]

where \(x_1 = \pi(x_0) \in D_1\) denotes the basepoint in \(D_1\). To simplify notation we will write \(j(g)\) and \(j_1(g_h)\) rather than \(J(g, x_0)\) and \(J_1(g_h, x_1)\).

By (5.8.1), the connection \(\nabla_1\) in \(E_1\) determines a connection \(\nabla^{J_1} = d + \eta_1\) in the \(J_1\)-trivialization \(E_1 \cong D_1 \times V\) with

\[
\eta_1(q_1*(X_h)) = j_1(g_h)w_1(X_h)j_1(g_h)^{-1} - d_{X_h}(j_1(g)) \circ j_1(g)^{-1} \quad (10.4.2)
\]

for any \(g_h \in G_h\) and any \(X_h \in T_{g_h}G_h\), where \(q_1 : G_h \to D_1\) denotes the projection.

The parabolically induced connection \(\nabla\) in \(E\) determines a connection \(\nabla^J = d + \eta\) in the \(J\)-trivialization \(E \cong D \times V\) with

\[
\eta(q_*X) = j(g)\omega(X)j(g)^{-1} - d_X(j(g)) \circ j(g)^{-1}. \quad (10.4.3)
\]
By (10.2.1) and (5.2.3) the connection forms $\eta$ and $\eta_1$ are related by $\eta(q_*(X)) = \eta_1(\pi_*(q_*(X)))$ for any $X \in T_g G$. Take $X = L_{g*}(\dot{u} + \dot{g} + \dot{g}_t) \in T_g G$ and let $X_h = L_{g_h*}(\dot{g}_h) \in T_{g_h} G_h$ denote its projection to $G_h$. Then we have

$$j_1(g_h)\omega_1(X_h)j_1(g_h)^{-1} = j_1(g_h)\lambda_1(g_t)\omega(X)\lambda_1(g_t)^{-1}j_1(g_h)^{-1} - j_1(g_h)d_X(\lambda_1(g_t))\lambda_1(g_t)^{-1}j_1(g_h)^{-1}$$

or, using (5.1.1)

$$\omega_1(L_{g_h*}(\dot{g}_h)) = \lambda_1(g_t)\omega(X)\lambda_1(g_t)^{-1} - \lambda_1(g_t)\lambda_1'(\dot{g}_t)\lambda_1(g_t)^{-1}$$

\[\square\]

10.5. Corollary. Suppose $\omega_1 \in \mathcal{A}^1(G, \text{End}(V))$ commutes with the adjoint action of $\lambda_1 : G_\ell \to \text{GL}(V)$. Then the curvature form $\Omega$ of the parabolically induced connection $\nabla = \Phi^*(\nabla_1) = d + \omega$ satisfies the $\pi$-fiber condition,

$$\Omega = \pi^*(\Omega_1)$$

where $\Omega_1 \in \mathcal{A}^2_{\text{bas}}(G_\ell, \text{End}(V))$ is the curvature form of $\nabla_1 = d + \omega_1$.

10.6. Proof. Let us compute $\Omega(X, Y)$ where $X = L_{g*}(\dot{x})$ and $Y = L_{g*}(\dot{y})$, where $\dot{x}, \dot{y} \in \text{Lie}(P)$ and where $g = ug_h g_\ell \in P$. Set $\dot{x} = \dot{u}_x + \dot{g}_X h + \dot{g}_t \in \text{Lie}(U_P) \oplus \mathfrak{g}_h \oplus \mathfrak{g}_\ell$ (and similarly for $\dot{y}$). By (10.3.1),

$$\omega(X) = \omega_1(L_{g_h*}(\dot{g}_X h)) + \lambda_1'(\dot{g}_X \ell)$$

and similarly for $\omega(Y)$. Set $X_h = L_{g_h*}(\dot{g}_X h)$ and $Y_h = L_{g_h*}(\dot{g}_Y h)$. Using the structure equation (5.2.4) and the fact that $\text{Lie}(U_P)$ is an ideal in $\text{Lie}(P)$ gives

$$\Omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]) + [\omega(X), \omega(Y)]$$

$$= X(\omega_1(L_{g_h*}(\dot{g}_Y h))) + X(\lambda_1'(\dot{g}_Y \ell)) - Y(\omega_1(L_{g_h*}(\dot{g}_X h))) - Y(\lambda_1'(\dot{g}_X \ell)) - \omega_1(L_{g_h*}([\dot{g}_X, \dot{g}_Y])) - \lambda_1'([\dot{g}_X, \dot{g}_Y])$$

$$+ [\omega_1(L_{g_h*}(\dot{g}_X h)), \omega_1(L_{g_h*}(\dot{g}_Y h))] + [\lambda_1'(\dot{g}_X \ell), \lambda_1'(\dot{g}_Y \ell)]$$

$$= X_h \omega_1(L_{g_h*}(\dot{g}_Y h)) - Y_h \omega_1(L_{g_h*}(\dot{g}_X h)) - \omega_1([X_h, Y_h]) + [\omega_1(X_h), \omega_1(Y_h)]$$

$$= \Omega_1(X_h, Y_h).$$

\[\square\]

10.7. If $\nabla_1 = d + \omega_1$ is a connection on $E_1$ which is invariant under a subgroup $\Gamma_h \subset G_h$, then by (10.3.1) the induced connection $\Phi^*(\nabla_1)$ is invariant under the group $\nu_h^*(\Gamma_h) \subset P$ which is obtained by first projecting $\Gamma_h$ to $G'_h = G_h/(G_h \cap G_\ell)$ then taking the pre-image under the projection $\nu_h : P \to G'_h$ (cf. §6.1, §7.1).

As in §3 let $\Gamma \subset G$ be a neat arithmetic subgroup with $X = \Gamma \setminus D$. Write $X_1 = \Gamma_h \setminus D_1$ for the stratum in $\overline{X} = \Gamma \setminus \text{D}^*_h$ corresponding to the boundary component $D_1$. The homogeneous vectorbundles $E \to D$ and $E_1 \to D_1$ pass to automorphic vectorbundles $E' \to X$ and $E'_1 \to X_1$ respectively. Parabolic induction then passes to an operation on these vectorbundles as follows. Suppose $\nabla'_1$ is a connection on $E'_1$. It pulls back to a $\Gamma_h$-invariant connection
\( \nabla_1 \) on \( E_1 \to D_1 \). The parabolically induced connection \( \nabla = \Phi^* (\nabla_1) \) is invariant under \( \Gamma_P = \Gamma \cap P \subset \nu_P(\Gamma_h) \) so it passes to a connection on \( \Gamma_P \setminus E \to \Gamma_P \setminus X \). Since \( T_{X_1}(\epsilon_0) \) is a \( \Gamma \)-parabolic neighborhood of \( X_1 \) in \( X \) this defines a connection \( \nabla' = \Phi_{X_1 X_2}^* (\nabla'_1) \) on the restriction \( E'_1(\nabla) \cap T_{X_1}(\epsilon_0) \).

This procedure may be applied to any pair of strata, say, \( X_2 < X_1 \) of \( X \). Thus, if \( \nabla'_2 \) is any connection on the automorphic vectorbundle \( E'_2 \to X_2 \) defined by \( \lambda \), then we obtain a parabolically induced connection

\[
\nabla'_1 = \Phi_{X_1 X_2}^* (\nabla'_2)
\]

on \( E'_1(\nabla) \cap T_{X_2}(\epsilon_0) \). However, if \( X_3 < X_2 < X_1 \) are strata of \( X \) and if \( \nabla'_3 \) is a connection on \( E'_3 \to X_3 \) then the parabolically induced connection \( \Phi_{X_1 X_2}^* (\nabla'_3) \) does not necessarily agree with the connection \( \Phi_{X_1 X_2}^* (\nabla'_2) \), even in the neighborhood \( X_1 \cap T_{X_2}(\epsilon_0) \cap T_{X_3}(\epsilon_0) \) where they are both defined, cf. Proposition 10.9.

10.8. We will also need the following more technical result concerning parabolic induction for the proof of the main theorem. Suppose \( P_1 < P_2 < \ldots < P_r \) are standard maximal parabolic subgroups with corresponding rational boundary components \( D_1 < D_2 < \ldots < D_r \). Let \( Q = P_1 \cap P_2 \cap \ldots \cap P_r \). Then \( Q' = \mathcal{U}_1 G_{1h} G_{1t} \) so by (7.3.6), \( Q \) decomposes as \( Q = \mathcal{U}_Q G_{1h} G_{Q\ell} \).

Let \( \lambda \) be a representation of \( K \) with resulting homogeneous vectorbundles \( E_i \to D_i \) \((1 \leq i \leq r)\). By Proposition 8.2, \( \lambda \) extends to a representation \( \lambda_1 \) of \( K_{1h} G_{1t} \). For \( 1 \leq i < j \leq r \) let \( \Phi_{ji} : E_j \to \pi^*_j E_i \) denote the vectorbundle isomorphism of (10.1.2) which covers the canonical projection \( \pi_{ji} : D_j \to D_i \). Let \( \Phi_r : E \to \pi^*_r (E_r) \) be the vectorbundle isomorphism (10.1.2) which covers the canonical projection \( \pi_r : D \to D_r \).

10.9. Proposition. Suppose \( \nabla_1 = d + \omega_1 \) is a connection on \( E_1 \) and suppose that the connection form \( \omega_1 \in \mathcal{A}^1 (G_{1h}, \text{End}(V)) \) commutes with the adjoint action of \( \lambda_1 (G_{1t}) \). Let \( \nabla = d + \omega \) denote the connection

\[
\nabla = \Phi_r^* \Phi_{r-1}^* \ldots \Phi_{21}^* (\nabla_1).
\]

Then

\[
\omega (L_{g^*}(\hat{u}_Q + \hat{g}_{1h} + \hat{g}_{Q\ell})) = \omega_1 (L_{g_{1h}} \hat{g}_{1h}) + \lambda_1' (\hat{g}_{Q\ell}) \quad (10.9.1)
\]

for any \( g = u_Q g_{1h} g_{Q\ell} \in Q \) and any \( \hat{u}_Q + \hat{g}_{1h} + \hat{g}_{Q\ell} \in \text{Lie}(Q) = \text{Lie}(\mathcal{U}_Q) + \mathfrak{g}_{1h} + \mathfrak{g}_{Q\ell} \).

10.10. Proof. First we determine the connection form of the connection

\[
\nabla_r = \Phi_{r-1}^* \Phi_{r-2}^* \ldots \Phi_{21}^* (\nabla_1) = d + \omega_r
\]
on the vectorbundle \( E_r \to D_r \). Set \( P_r = \mathcal{U}_r G_{rh} G_{Q\ell} \). The images under the projection \( \nu_{rh} : P \to G_{rh} \) of \( Q \subset P_1 \cap P_r \) are parabolic subgroups \( Q \subset P_1 \subset G_{rh} \). In fact, \( Q \) is the parabolic
subgroup corresponding to the flag of rational boundary components $D_1 < D_2 < \cdots < D_{r-1}$ of $D_r$. By (7.3.4) there are compatible decompositions

$$P_r = U_r G_{rh} G_{r\ell}$$

$$P_1 \cap P_r = U_r(\overline{U} G_{1h} G_{1\ell} G_{r\ell})$$

with $P_1 = \overline{U} G_{1h} G_{1\ell}$

$$Q = U_r(\overline{U} G_{1h} (U_{\overline{Q}} L_{Q_\ell}) G_{r\ell})$$

with $Q = \overline{U} U_{\overline{Q}} G_{1h} G_{1\ell}$.

Corresponding to the maximal parabolic subgroup $P_1 \subset G_{rh}$ the representation $\lambda|K_{1h} K_{1\ell}'$ has a canonical extension

$$\lambda_{r1} : K_{1h} G_{1\ell}' \to GL(V).$$

According to Proposition 8.4, $\lambda_{r1}|G'_{1\ell} = \lambda_1|G'_{1\ell}$ which, by assumption, commutes with $\omega_1$. So by induction, for any $\bar{q} = u_{\overline{Q}} g_{1h} g_{Q_\ell} \in Q$ and for any $\bar{q} = \bar{u}_{\overline{Q}} + g_{1h} + g_{Q_\ell} \in Lie(Q)$,

$$\omega_r(L_{g_{1h}}(\bar{u}_{\overline{Q}} + g_{1h} + g_{Q_\ell})) = \omega_1(L_{g_{1h}}(g_{1h})) + \lambda_1'(g_{Q_\ell}).$$

Moreover $\nabla = \Phi^*_r(\nabla_r) = d + \omega$ so by Proposition 10.3, for any $g = u_{\overline{Q}} g_{1h} g_{Q_\ell} \in P_r$,

$$\omega(L_{g_{1h}}(\bar{u}_{\overline{Q}} + g_{1h} + g_{Q_\ell})) = \omega_r(L_{g_{1h}}(g_{1h})) + \lambda_1'(g_{Q_\ell})$$

for any $\bar{u}_{\overline{Q}} + g_{1h} + g_{Q_\ell} \in Lie(P_r)$. Taking $g_{1h} = \bar{q} = u_{\overline{Q}} g_{1h} g_{Q_\ell} \in Q$ and

$$g_{1h} = \bar{q} = \bar{u}_{\overline{Q}} + g_{1h} + g_{Q_\ell} \in Lie(Q)$$

gives equation (10.9.1):

$$\omega(L_{g_{1h}}(\bar{u}_{\overline{Q}} + g_{1h} + g_{Q_\ell} + g_{1h})) = \omega_1(L_{g_{1h}}(g_{1h})) + \lambda_1'(g_{Q_\ell} + g_{1h}).$$

\[\square\]

11. The Patched Connection

11.1 As §9.3, suppose that $D = G/K$ is a Hermitian symmetric space which is irreducible over $\mathbb{Q}$, and that $\Gamma \subset G(\mathbb{Q})$ is a neat arithmetic group. Let $\overline{X} = \Gamma \backslash D^*$ denote the Baily Borel compactification of $X = \Gamma \backslash D$ with projection $q : D^* \to \overline{X}$. By lemma 9.6, for any sufficiently small $\epsilon_0 > 0$ there exists an $\epsilon_0$-system of control data $\{T_Y(\epsilon_0), \pi_Y, \rho_Y\}$ (which we now fix) on $\overline{X}$, so that $\pi_Y$ is obtained from the canonical projection $D \to D_1$ whenever $q(D_1) = Y$ and so that $T_Y(\epsilon)$ is a $\Gamma$-parabolic neighborhood of $Y$ in $\overline{X}$. Applying §8.5 to this system of control data yields a partition of unity on each stratum $Y$ of $\overline{X}$,

$$B_Y^Y(\epsilon) + \sum_{Z < Y} B_Y^Z(\epsilon) = 1 \tag{11.1.1}$$

for all $y \in Y$, where $\epsilon Y = \epsilon_0 / 2^{\text{dim} Y}$.

A choice of representation $\lambda : K \to GL(V)$ on some complex vector space $V$ determines homogeneous vector bundles $E = G \times_K V$ on $D$ and $E_1 = G_{1h} \times_{K_{1h}} V$ on $D_1$ which pass to automorphic vector bundles $E' = \overline{E}$ on $X$ and $E_Y' \to Y$ on $Y$. Here, $D_1$ is a rational boundary component (with $q(D_1) = Y$), normalized by some maximal parabolic subgroup
\( P = \mathcal{U}_P G_h G_\ell \); and \( K_h = K \cap G_h \); cf. §{[7.1], [9.1]}. The Nomizu connections \( \nabla^\text{Nom} \) (on \( E \)) and \( \nabla^\text{Nom}_1 \) (on \( E_1 \)) pass to connections \( \nabla^\text{Nom}_X \) on \( E'_X \rightarrow X \) and \( \nabla^\text{Nom}_Y \) on \( E'_Y \rightarrow Y \) respectively. We use an inductive procedure to define the \textit{patched connection} \( \nabla^p \) on the vectorbundle \( E'_Y \rightarrow Y \), for any stratum \( Y \leq X \) as follows. If \( Y \subset \overline{X} \) is a minimal stratum set \( \nabla^p_Y = \nabla^\text{Nom}_Y \). Now suppose that the patched connection \( \nabla^p_Z \) has been constructed on every stratum \( Z < Y \).

11.2. \textbf{Definition.} The patched connection \( \nabla^p_Y \) on \( E'_Y \rightarrow Y \) is the connection

\[
\nabla^p_Y = B_Y^Y \nabla^\text{Nom}_Y + \sum_{Z < Y} B_Z^Y \Phi^*_{YZ} (\nabla^p_Z) \tag{11.2.1}
\]

(where the sum is taken over all strata \( Z < Y \) in the Baily Borel compactification of \( X \)).

11.3. \textbf{Remarks.} The idea behind this construction may be explained when there are two singular strata \( Z < Y < X \). A simpler candidate for a connection on \( X \) whose Chern forms might satisfy the \( \pi \)-fiber condition is

\[
\nabla'_X = B^{X} Z \Phi^*_{XZ} \nabla^\text{Nom}_Z + B_Y^X \Phi^*_{XY} \nabla^\text{Nom}_Y + B_X^X \nabla^\text{Nom}_X. \tag{11.3.1}
\]

In the region \( T_Y (e X / 2) \) only the first two terms contribute to \( \nabla'_X \). Both connection \( \Phi^*_{XZ} \nabla^\text{Nom}_Z \) and \( \Phi^*_{XY} \nabla^\text{Nom}_Y \) have curvature forms which satisfy the \( \pi \)-fiber condition with respect to \( Y \). However the curvature form of (and even the Chern forms of) any affine combination of these fails to satisfy the \( \pi \)-fiber condition. (cf. Figure 3.4: this occurs in the region where \( B_Z + B_Y = 1 \).) The remedy is to create a connection on \( X \) for which no nontrivial affine combination of \( \Phi^*_{XZ} \nabla^\text{Nom}_Z \) and \( \Phi^*_{XY} \nabla^\text{Nom}_Y \) ever occurs. Replacing \( \nabla^\text{Nom}_Y \) by \( \nabla^p_Y \) in (11.3.1) gives

\[
\nabla^p_X = B^{X} Z \Phi^*_{XZ} \nabla^\text{Nom}_Z + B_Y^X B^Y Z \Phi^*_{XY} \nabla^\text{Nom}_Y + B_X^X \nabla^\text{Nom}_X.
\]

Within the region \( T_Y (e X / 2) \) only the first three terms appear: the first term alone appears in \( T_Z (e X / 2) \); the first and second terms appear in \( T_Z (e X) - T_Z (e X / 2) \); the second term alone appears in \( T_Z (e Y / 2) - T_Z (e X) \); the second and third terms appear in \( T_Z (e Y) - T_Z (e Y / 2) \) and the third term alone appears outside \( T_Z (e Y) \). In the region \( T_Z (e Y) - T_Z (e Y / 2) \),

\[
\nabla^p_X = \Phi^*_{XY} (B^Y Z \Phi^*_{YZ} \nabla^\text{Nom}_Z + B^Y_Y \nabla^\text{Nom}_Y).
\]

So \( \nabla^p_X \) is parabolically induced from a connection on \( Y \), and by Corollary [0.3] its curvature form satisfies the \( \pi \)-fiber condition relative to \( Y \). In the region \( T_Z (e X) - T_Z (e X / 2) \),

\[
\nabla^p_X = (B^X_Z \Phi^*_{XZ} + B_Y^X B^Y_Z \Phi^*_{XY} \Phi^*_{YZ} (\nabla^\text{Nom}_Z)).
\]

In this region, the curvature form still does not satisfy the \( \pi \)-fiber condition however we show in §[2.10] that the difference \( (\Phi^*_{XZ} - \Phi^*_{XY} \Phi^*_{YZ}) \omega^\text{Nom}_Z \) is nilpotent and commutes with the curvature form. This turns out to be enough (Lemma 6.4) to imply that the Chern forms of \( \nabla^p_X \) satisfy the \( \pi \)-fiber condition with respect to \( Y \).
11.4. Returning to the general case, suppose $Z = Z(1) < Z(2) < \cdots < Z(r)$ is a chain of strata in $\overline{X}$. Write $Z \leq Y$ if $Z(r) = Y$, that is, if the chain ends in $Y$. Write $Y \leq Z$ if $Z(1) = Y$, that is, if the chain begins at $Y$. Suppose $Z$ is such a chain of strata and suppose $x \in T_{Z(i)}(\epsilon_0)$ for $1 \leq i \leq r$. Denote by

$$
\epsilon_i = \epsilon(Z(i)) = \epsilon_0 / 2^{\dim Z(i)}
$$

$$
B_i^\epsilon = B_{Z(i)}^\epsilon
$$

$$
\Phi_{ji}^* = \Phi_{Z(j)Z(i)}^* \text{ for } j > i
$$

$$
\pi_i(x) = \pi_{Z(i)}(x).
$$

Define

$$
B_Z(x) = B_{r-1}^\epsilon(\pi_r(x)) \cdots B_3^\epsilon(\pi_3(x)) B_2^\epsilon(\pi_2(x))
$$

$$
\Phi_Z^* = \Phi_{r,r-1}^* \cdots \Phi_{32}^* \Phi_{21}^*.
$$

If $Z = \{Y\}$ consists of a single element, set $B_Z(x) = 1$ and $\Phi_Z^* = \text{Id}$. The following lemma is easily verified by induction.

11.5. Lemma. The patched connection may be expressed as follows,

$$
\nabla_Y^p(x) = \sum_Z B_Z(x) \Phi_Z^* \left( B_{Z(1)}^\epsilon(\pi_{Z(1)}(x)) \nabla_{Z(1)}^{\text{Nom}} \right)
$$

(11.5.1)

where the sum is over all chains of strata $Z \leq Y$ ending in $Y$. \hfill \square

(One checks that, although the projection functions $x \mapsto \pi_{Z(i)}(x)$ are not everywhere defined, they occur in (11.5.1) with coefficient 0 unless $x$ lies in the region of definition.)

Definition 11.2 constructs a patched connection $\nabla_Y^p$ on each of the automorphic vector-bundles $E'_Y \to Y$. The proof of the following theorem will appear in §12.

11.6. Theorem. The Chern forms $\{\sigma^j(\nabla_Y^p)\}_{Y \leq X}$ of the patched connection $\nabla_Y^p$ constitute a closed $\pi$-fiber differential form.

11.7. Corollary. For each $j$, the Chern form $\sigma^j(\nabla_Y^p) \in A^{2j}(\overline{X}; \mathbb{C})$ of the patched connection determines a lift

$$
\tilde{c}^j(E') = [\sigma^j(\nabla_Y^p)] \in H^{2j}(\overline{X}; \mathbb{C})
$$

(11.7.1)

of the Chern class $c^j(E') \in H^{2j}(X; \mathbb{C})$ which is independent of the choices that were made in its construction. For any stratum closure $i : \overline{Y} \hookrightarrow \overline{X}$ the restriction $i^* \tilde{c}^j(E')$ is equal to the Chern class $\tilde{c}^j(E'_Y) \in H^{2j}(\overline{Y}; \mathbb{C})$ of the automorphic vectorbundle $E'_Y \to Y$. 
11.8. Proof of Corollary 11.7. The restriction map \( i^* : H^{2j}(\overline{X}) \to H^{2j}(X) \) associates to any \( \pi \)-fiber differential form \( \omega \in A^{2j}(\overline{X}) \) the cohomology class \([\omega_X]\) of the differential form \( \omega_X \in A^{2j}(X) \) on the nonsingular part. Hence \( i^*(c^j(E')) = c^j(E') \in H^{2j}(X; \mathbb{C}) \) since the latter is independent of the connection.

The patched connection \( \nabla^p_X \) depends on the choice of a pair (partition of unity, control data which is subordinate to the canonical projections \( \{ \pi_Z \} \)) (see §3 and §2). It is tedious but standard to check that two such choices are connected by a smooth 1-parameter family of choices (partition of unity, control data subordinate to \( \{ \pi_Z \} \)). The resulting patched connections \( \nabla^p_0 \) and \( \nabla^p_1 \) are therefore connected by a smooth 1-parameter family of patched connections \( \nabla^p_j \), each of whose Chern forms is a \( \pi \)-fiber differential form. So the usual argument (e.g. [KN] Chapt. XII lemma 5; [MiS]) produces a differential \( 2j - 1 \) form \( \Psi \) such that \( \sigma^j(\nabla^p_0) - \sigma^j(\nabla^p_1) = d\Psi \). It is easy to see that \( \Psi \in A^{2j-1}(\overline{X}; \mathbb{C}) \) is a \( \pi \)-fiber differential form. Consequently the \( \pi \)-fiber cohomology classes coincide: \([\sigma^j(\nabla^p_i)] \in H^{2j}(\overline{X}; \mathbb{C})\).

The second statement follows from the analogous statement in Theorem 11.6.

11.9. Remarks. Theorem 11.6 and Corollary 11.7 extend to the case that \( \mathcal{G} \) is semisimple over \( \mathbb{Q} \). The restriction map \( H^{2j}(\overline{X}; \mathbb{C}) \to H^{2j}(X; \mathbb{C}) \) factors as follows,

\[
H^{2j}(\overline{X}; \mathbb{C}) \to IH^{2j}(\overline{X}; \mathbb{C}) \to H_{2n-2j}(\overline{X}; \mathbb{C}) = H_{2n-2j}(\overline{X}, \partial \overline{X}; \mathbb{C}) \cong H^{2j}(X; \mathbb{C})
\]

where \( 2n = \dim_{\mathbb{R}}(X) \), and where \( \partial \overline{X} \) denotes the singular set of the Baily-Borel compactification \( \overline{X} \). The Chern class \( c^j(E') \in H^{2j}(X; \mathbb{C}) \) lives in the last group. For any toroidal resolution of singularities \( \tau : \overline{X}_\Sigma \to \overline{X} \), the pushdown

\[
c_{n-j}(E') = \tau_*(c^j(E'_\Sigma) \cap [\overline{X}_\Sigma]) \in H_{2n-2j}(\overline{X}; \mathbb{Z})
\]

of the Chern class of Mumford’s canonical extension ([Mull]) \( \overline{E}'_\Sigma \) of \( E' \) gives a canonical lift of \( c^j(E') \) to the homology of the Baily-Borel compactification. In §15 (in the case of the tangent bundle) we identify this with the (homology) Chern class of the constructible function \( 1_X \). In [BBF] it is shown that every algebraic homology class (including \( c_{n-j}(E') \)) admits a (non-canonical) lift to middle intersection homology with rational coefficients.

12. Proof of Theorem 11.6

12.1. Preliminaries. As in §9.3, let \( q : D^* \to \overline{X} \) denote the projection. If \( \epsilon_0 > 0 \) is sufficiently small, then for each stratum \( Z \) of \( \overline{X} \) the preimage

\[
q^{-1}(T_Z(\epsilon_0)) = \bigcap_{q(D_j) = Z} U_{D_j}(\epsilon_0)
\]
is a disjoint union of $\Gamma$-parabolic neighborhoods $U_{D_1}(\epsilon_0)$ of those boundary components $D_1$ such that $q(D_1) = Z$. For such a boundary component define

$$\chi_{D_1}(x) = \begin{cases} 1 & \text{if } x \in U_{D_1}(\epsilon_0) \\ 0 & \text{otherwise} \end{cases}$$

to be the characteristic function of $U_{D_1}$.

Fix a stratum $Y$ and a choice $D_2$ of rational boundary component such that $q(D_2) = Y$. Denote by $P_2 = U_2G_{2h}G_{2\ell}$ the rational maximal parabolic subgroup of $G$ which normalizes $D_2$ and by $\nu_\ell : P_2 \to G_{2h}$ the projection as in §9. By Proposition 8.2 the representation $\lambda|K \cap P_2$ extends to a representation $\lambda_2$ of $K_{2h}G_{2\ell}$. Set $\Gamma_h = \nu_\ell(\Gamma \cap P_2)$.

The partition of unity $B_Y^Y + \sum_{Z < Y} B_Z^Y = 1$ on $Y$ pulls back to a $\Gamma_h$-invariant locally finite partition of unity on $D_2$,

$$B_{D_2}^Y + \sum_{D_1 < D_2} B_{D_1}^Y = 1$$

where the sum is over all rational boundary components $D_1 < D_2$, where $B_{D_1}^Y = q^*(B_Z^Y)\chi_{D_1}$ (and similarly for $B_{D_2}^Y$, however $\chi_{D_2} = 1$ on $D_2$). The patched connection $\nabla_Y^p$ on the vectorbundle $E_Y \to Y$ pulls back to a $\Gamma_h$-invariant connection $\nabla_2^p = q^*(\nabla_Y^p)$ on the homogeneous vectorbundle $E_2 = G_{2h} \times_{K_{2h}} V$. This connection may also be described as the affine locally finite combination

$$\nabla_2^p = B_{D_2}^Y \nabla_2^{\text{Nom}} + \sum_{D_1 < D_2} B_{D_1}^Y \Phi_2^* (\nabla_1^p)$$

where, for each rational boundary component $D_1 < D_2$ the obvious notation holds: $\nabla_1^p$ is the patched connection on $E_1 \to D_1$ and $\Phi_2 : E_2 \to \pi_2^1(E_1)$ is the vectorbundle isomorphism which is obtained from (10.1.3) upon replacing $G$ by $G_{2h}$.

Denote by $\omega_2^p \in \mathcal{A}^1(G_{2h}, \text{End}(V))$ the connection form of $\nabla_2^p$. The curvature form $\Omega_Y^p \in \mathcal{A}^2(Y, \text{End}(E_Y^p))$ of $\nabla_Y^p$ coincides with the curvature form $\Omega_2^p \in \mathcal{A}_{2\text{bas}}^2(G_{2h}, \text{End}(V))$ of $\nabla_2^p$ under the canonical isomorphism

$$\mathcal{A}^2(Y, \text{End}(E_Y^p)) \cong \mathcal{A}^2(D_2, \text{End}(E_2))^\Gamma_h \cong \mathcal{A}_{2\text{bas}}^2(G_{2h}, \text{End}(V))^\Gamma_h$$

where the superscript $\Gamma_h$ denotes the $\Gamma_h$-invariant differential forms.

**12.2. Proposition.** Let $D_2$ be a rational boundary component of $D = G/K$. Then the connection form $\omega_2^p \in \mathcal{A}^1(G_{2h}, \text{End}(V))$ and the curvature form $\Omega_2^p \in \mathcal{A}_{2\text{bas}}^2(G_{2h}, \text{End}(V))$ commute with the adjoint action of $\lambda_2(G_{2\ell}) \subset \text{GL}(V)$.

**12.3. Proof.** The proof uses a double induction over boundary components $D_2$ in $D^\ast$. However, so as to avoid the horribly complicated notation which would arise in the proof, we rephrase the double induction as follows:
1. By induction we assume the proposition has been proven for every rational boundary component $D'_i$ of any Hermitian symmetric domain $D' = G'/K$ for which $\text{dim}(D') < \text{dim}(D)$. (The case $\text{dim}(D') = 0$ is trivial.)

2. For our given domain $D$, we assume the proposition has been proven for every rational boundary component $D_1$ of $D$ such that $\text{dim}(D_1) < \text{dim}(D_2)$. (The case $D_1 = \phi$ is trivial.)

To prove Proposition 12.2 for $D_2 \subset D^*$, it suffices to verify that $\lambda_2(G_{2\ell})$ commutes with the connection form of each of the connections appearing in the linear combination (12.1.1). The connection form $\omega^\text{Nom}_2$ of the Nomizu connection $\nabla^\text{Nom}_2$ is given by

$$\omega^\text{Nom}_2(L_{g_{2h*}})(\dot{g}_{2h}) = \lambda'(\dot{k}_{2h}) = \lambda'_2(\dot{k}_{2h})$$

for any $g_{2h} \in G_{2h}$ and for any $\dot{g}_{2h} \in \mathfrak{g}_{2h}$, where $\dot{g}_{2h} = \dot{k}_{2h} + \dot{p}_{2h}$ is its Cartan decomposition. This commutes with $\lambda_2(G_{2\ell})$ since $G_{2h}$ and $G_{2\ell}$ commute.

Now consider any boundary component $D_1 \varsubsetneq D_2$ which appears in the sum (12.1.1). Let $P_1 = \mathcal{U}_1 G_{1h}G_{1\ell}$ be the rational parabolic subgroup which normalizes $D_1$. Decompose the intersection

$$P = P_1 \cap P_2 = \mathcal{U}_2(\overline{U} G_{1h}G_{1\ell}) G_{2\ell}$$

according to (7.3.4), setting $P_h = \overline{U} G_{1h}G_{1\ell} \subset G_{2h}$. Let $\nabla^p_1$ be the patched connection on $E_1 \rightarrow D_1$. According to Proposition 10.3, the connection form $\omega_{21}$ of the parabolically induced connection $\Phi^p_{21}(\nabla^p_1)$ is given by

$$\omega_{21}(L_{g_{2h}}(\dot{u}) + \dot{g}_{1h} + \dot{g}_1) = \lambda_{21}'(\dot{g}_1) + Ad(\lambda_{21}(g_1^{-1}))(\omega^p_1(L_{g_{1h*}}(\dot{g}_{1h}))).$$

Here, $g = \tilde{u}g_{1h}g'_{1\ell} \in P_h$ and $\dot{u} + \dot{g}_{1h} + \dot{g}_1 \in \text{Lie}(P_h)$ and $\lambda_{21} : K_{1h}G_{1\ell} \rightarrow \text{GL}(V)$ is as in Proposition 8.2. Since $\text{dim}Y < \text{dim}X$ we may apply the first induction hypothesis and conclude that the adjoint action of $\lambda_1(G_{1\ell})$ commutes with the connection form $\omega^p_1 \in \mathcal{A}^1(G_{1h}, \text{End}(V))$. Hence, using Proposition 8.4,

$$\omega_{21}(L_{g_{2h}}(\dot{u}) + \dot{g}_{1h} + \dot{g}_1) = \lambda_1'(\dot{g}_1) + \omega^p_1(L_{g_{1h*}}(\dot{g}_{1h})).$$

The group $G'_{1\ell}$ commutes with $G_{2\ell}$ so the first term $\lambda_1'(\dot{g}_1)$ commutes with $\lambda_1(G_{2\ell})$. By the second induction hypothesis, the connection form $\omega^p_1 \in \mathcal{A}^1(G_{1h}, \text{End}(V))$ also commutes with $\lambda_1(G_{1\ell})$. But $\lambda_1(G_{1\ell}) \supset \lambda_1(G_{2\ell}) = \lambda_2(G_{2\ell})$ by Proposition 8.4 again, which completes the proof that the connection form of the patched connection commutes with $\lambda_2(G_{2\ell})$. □

12.4. Let $x \in X$ be a point near the boundary of $\overline{X}$. Then there is a maximal collection of strata $Y_1, Y_2, \ldots, Y_\ell$ such that $x \in T_{c_0}(Y_i)$ for each $i$, and we may assume they form a partial flag,

$$Y_1 < Y_2 < \cdots < Y_\ell = X. \quad (12.4.1)$$

Let $W \leq X$ be the largest stratum in this collection such that $B^W_{i}(\pi_W(x)) \neq 0$. Such a stratum exists since $B^W_{i}(\pi_{Y_1}(x)) = 1$. (Choosing $W$ in this way guarantees that, if $W \neq X$,
then for every stratum $Z > W$ in this partial flag, at the point $\pi_Z(x)$ the connection $\nabla^p_Z$ is an affine combination of connections induced from smaller strata and contains no contribution from $\nabla^\text{Nom}_Z$, because $B^Z_t (\pi(x)) = 0$.

12.5. Proposition. At the point $x \in X$, $\sum_{W \leq S \leq X} B_S(x) = 1$ and

$$\nabla^p_X(x) = \left( \sum_{W \leq S \leq X} B_S(x) \Phi_S^* \right) \left( \nabla^p_W(\pi_W(x)) \right) \quad (12.5.1)$$

where the sum is over sub-chains $S$ in the partial flag (12.4.1) which begin at $W$ and end at $X$.

12.6. Proof. By Lemma 11.5 applied to $\nabla^p_W$, we need to show that

$$\nabla^p_X(x) = \left( \sum_{W \leq S \leq X} B_S(x) \Phi_S \right) \left( \sum_{R \leq W} B_{R}(\pi_W(x)) \Phi_R^* \left( B^{\epsilon R(1)}_{R(1)} \nabla^\text{Nom}_{R(1)}(\pi_{R(1)}(x)) \right) \right) \quad (12.6.1)$$

By Lemma 11.3, $\nabla^p_X(x)$ is a sum over chains $Z \leq X$ of terms

$$B_{Z(x)} \Phi_Z^* B^{\epsilon Z(1)}_{Z(1)} \nabla^\text{Nom}_{Z(1)}(\pi_{Z(1)}(x)).$$

For any $\epsilon \leq \epsilon_0$, $B_Z(x) = 0$ unless $Z$ occurs in the collection $\{Y_1, Y_2, \ldots, Y_t = X\}$. By assumption the term $B^{\epsilon Z(1)}_{Z(1)} \pi_{Z(1)}(x)$ also vanishes unless $Z(1) \leq W$. Therefore each chain $Z = Z(1) < \cdots < X$ appearing in the sum may be assumed to occur as a sub-chain of $Y_1 < Y_2 < \cdots < Y_t = X$, and we may also assume the chain begins at $Z(1) \leq W$. We claim that if such a chain $Z$ occurs with nonzero coefficient, then the stratum $W$ must appear in the chain. For if not, then $Z(k) < W < Z(k + 1)$ for some $k$. But the term $B_{Z(x)}$ contains a factor

$$B^{\epsilon Z(k+1)}_{Z(k)} (\pi_{Z(k+1)}(x)).$$

Since $B^{\epsilon W}_{W} (\pi_W(x)) \neq 0$, this factor vanishes by Lemma 3.6, which proves the claim. Summarizing, every chain $Z$ which occurs with nonzero coefficient in the sum may be described as

$$R(1) < R(2) < \cdots < R(r) = W = S(1) < S(2) < \cdots < S(s) = X.$$  

The contribution to $\nabla^p_X(x)$ in (11.5.1) from such a chain is the product of

$$B^{\epsilon X}_{S(s-1)(\pi_X(x))} \cdots B^{\epsilon S(2)}_{S(2)(\pi_S(x))} \Phi^*_{X S(s-1)} \cdots \Phi^*_{S(2) W}$$

with

$$B^{\epsilon W}_{R(r-1)(\pi_W(x))} \cdots B^{\epsilon R(2)}_{R(2)(\pi_R(x))} \Phi^*_{W R(r-1)} \cdots \Phi^*_{R(2) R(1)}$$

applied to

$$B^{\epsilon R(1)}_{R(1)} \nabla^\text{Nom}_{R(1)}(\pi_{R(1)}(x)).$$
However this product is exactly a single term in (12.6.1) and every such product occurs exactly once, which verifies (12.5.1). Since the coefficients in (11.5.1) sum to 1, so also does \( \sum_{W \leq S \leq X} B_S(x) \).

12.7. We must prove that the Chern forms of the patched connection satisfy the \( \pi \)-fiber condition near each stratum of \( X \). Let \( Y \) be such a stratum and let \( x \in X \cap T_Y(\epsilon X/2) \). We will verify the \( \pi \)-fiber condition relative to the stratum \( Y \) at the point \( x \). The point \( x \in X \) lies in an intersection of \( \epsilon_0 \)-tubular neighborhoods of a maximal collection of strata, which (we may assume) form a partial flag \( Z_1 < Z_2 < \cdots < X \). Let \( W \) be the largest stratum in this chain such that \( B_W^W(\pi_W(x)) \neq 0 \). Then \( W \leq Y \) by (3.2.2). Consider the subchain lying between \( W \) and \( Y \), which we shall denote by

\[
W = Y_1 < Y_2 < \cdots < Y_t = Y.
\]

Fix corresponding boundary components \( D_1 < D_2 < \cdots < D_t \) and let \( P_1 < P_2 < \cdots < P_t \) be their normalizing maximal parabolic subgroups. Set \( P = P_1 \cap P_2 \cap \cdots \cap P_t \). Let \( \nabla^p_W \) be the patched connection on \( E'_W \to W \). According to Proposition 12.5,

\[
\nabla^p_X(x) = \left( \sum_{W \leq Y \leq X} B_S(x)\Phi^*_S \right) \nabla^p_W(\pi_W(x)). \tag{12.7.1}
\]

However the only chains \( S = \{S(1) < \cdots < S(s)\} \) which occur with nonzero coefficient in this sum satisfy

\[
\{S(1), S(2), \ldots, S(s-1)\} \subset \{Y_1, Y_2, \ldots, Y_t\}, \quad S(1) = W = Y_1, \quad S(s) = X \tag{12.7.2}
\]

for the following reason. Suppose a chain \( S \) contains a stratum larger than \( Y \) (but not equal to \( X \)). Let \( Z \) be the largest such stratum occurring in \( S \). Then the first factor in \( B_S(x) \) is \( B^X_Z(x) \), cf. (11.4.1). By assumption, \( x \in T_Y(\epsilon X/2) \). So by (3.2.2) (with \( \epsilon = \epsilon X \) and where the roles of \( Y \) and \( Z \) are reversed), \( B^X_Z(x) = 0 \).

If \( W = Y \) then (12.7.1) becomes \( \nabla^p_X = \Phi^*_X(\nabla^p_Y) \) so by Corollary 10.5 and Proposition 12.2 the curvature form \( \Omega^p_X \) of \( \nabla^p_X \) satisfies the \( \pi \)-fiber condition with respect to \( Y \). So the same is true of every Chern form which proves Theorem 11.6 in this case.

Therefore we may assume that \( W < Y \). As in §12.1 the connection \( \nabla^p_W \) on \( E'_W \to W \) pulls back to a \( \Gamma \)-invariant connection \( \nabla^p_E \) on \( E \to E \) and the connection \( \nabla^p_Y \) on \( E' \to X \) pulls back to a \( \Gamma \)-invariant connection \( \nabla^p_E \) on \( E \to D \). As in (12.1.2) we identify the curvature form \( \Omega^p_X \) of \( \nabla^p_X \) with the curvature form \( \Omega^p \) of \( \nabla^p_Y \) under the canonical isomorphism

\[
\mathcal{A}^2(X, \text{End}(E')) \cong \mathcal{A}^2(D, \text{End}(E))^\Gamma \cong \mathcal{A}_{bas}^2(G, \text{End}(V))^\Gamma.
\]

Similarly identify the curvature \( \Omega^p_1 \) of \( \nabla^p_1 \) with the curvature \( \Omega^p_W \) of \( \nabla^p_W \). Choose a lift \( \tilde{x} \in D \) of \( x \), which lies in the intersection

\[
U_{D_1}(\epsilon_0) \cap U_{D_2}(\epsilon_0) \cap \cdots \cap U_{D_t}(\epsilon_0)
\]
of Γ-parabolic neighborhoods of the boundary components \( D_1 < D_2 < \ldots < D_t \). Let \( \pi : D \to D_1 \) denote the canonical projection.

**12.8. Lemma.** For any tangent vectors \( U, V \in T_x D \),

\[
\Omega^p(U, V) = \pi^* \Omega^p_1(U, V) + n \in \text{End}(V)
\]

for some nilpotent element \( n \in \lambda'_1(g_{1\ell}) \). (Here, \( P_1 = U_{P_1}G_{1h}G_{1\ell}; g_{1\ell} = \text{Lie}(G_{1\ell}) \); and \( \lambda_1 \) is the extension (Proposition 8.2) of the representation \( \lambda | K \cap P_1 \).

**12.9. Proof.** We will compute the curvature \( \Omega^p_X \) of \( \nabla^p_X \). Each chain \( S \) satisfying (12.7.2) corresponds also to a chain of rational boundary components \( D_1 = D_{S(1)} < D_{S(2)} < \ldots < D_{S(s)} = D \). Let \( \tilde{B}_S = q^*B_S \) denote the pullback to \( D \). It follows from Proposition 12.5 that \( \sum_S \tilde{B}_S(\tilde{x}) = 1 \) (where the sum is over all chains \( S \) which appear in (12.7.1)). Choose any ordered labeling of the chains \( S \) which appear in (12.7.1), say \( S_1, S_2, \ldots , S_M \). By §2.1 (4),

\[
\sum_{i=1}^M \sum_{j=1}^M B_{S_i}(x)\pi^* \Omega^p_W = \sum_{i=1}^M B_{S_i}(x)\pi^* \Omega^p_W = \pi^* \Omega^p_W.
\]

Let \( U, V \in T_x X \). Pulling back the equation (12.7.1) to \( D \) and using Lemma 5.2 gives:

\[
\Omega^p(U, V) = \pi^*(\Omega^p_1(U, V)) + \sum_{i=1}^{M-1} \tilde{B}_{S_i} \wedge (\Phi^* \omega^p_1 - \Phi^* \omega^p_M) + \sum_{i<j} \tilde{B}_{S_i}(x)\tilde{B}_{S_j}(x) \left[ \Phi^* \omega^p_1(U) - \Phi^* \omega^p_M, \Phi^* \omega^p(V) - \Phi^* \omega^p(V) \right]
\]

where \( \Phi^* \omega^p \) denotes the connection form of \( \Phi^* \nabla^p_1 \). Let us compute this connection form. Suppose that \( S \) is a chain satisfying (12.7.2). For \( 1 \leq j < s-1 \) let \( P_{S(j)} \) be the corresponding normalizing maximal parabolic subgroup and set \( Q = P_{S(1)} \cap P_{S(2)} \cap \ldots \cap P_{S(s-1)} \). Then \( P \subset Q \) and \( P^* = Q^* = P_1 \) which implies (as in §7.3) that \( Q \subset P \subset P_1 \) have compatible decompositions,

\[
P_1 = U_1 G_{1h} G_{1\ell}
\]

\[
Q = U_1 G_{1h} (U_{P_1}G_{Q}) \quad \text{with} \quad U_Q = U_1 U_{P_1} G_Q
\]

\[
P = U_1 G_{1h} U_{P_1} Q (U_{Q_P}G_{P_{1\ell}}) \quad \text{with} \quad U_P = U_1 U_{P_1} U_{Q_P}.
\]

Here, \( U_{Q_P}G_{P_{1\ell}} \) is the parabolic subgroup of \( G_{Q_{1\ell}} \) determined by \( P \subset Q \). We also note that

\[
U_{P_1} = U_{P_1} U_{Q_{P_1}} U_{Q_P} \quad \text{(12.9.1)}
\]

is the unipotent radical of the parabolic subgroup \( \nu_{1\ell}(P) \subset G_{1\ell} \) determined by \( P \) (where \( \nu_{1\ell} : P_1 \to G_{1\ell} \) is the projection). Let \( \mathfrak{m}_{P_1} \) denote its Lie algebra.
By Proposition 12.2, the connection form \( \omega^p_1(x) \in \mathcal{A}^1(G_{1h}, \text{End}(V)) \) commutes with the adjoint action of \( \lambda_1(G_{1h}) \subset GL(V) \). So we may apply Proposition 10.9 to determine \( \Phi^*_S(\omega^p_1) \). Let \( g = u_P g_{1h} g_P \ell \in P = U_P G_{1h} G_P \ell \) and let
\[
g = \dot{u}_1 + \dot{g}_{1h} + \dot{u}_{PQ} + \dot{u}_{Qf} + \dot{g}_{Pf} \in \text{Lie}(U_P G_{1h} U_{PQ} U_{Qf} G_P f).
\]
Apply Proposition 10.3 using \( \dot{u}_Q = \dot{u}_1 + \dot{u}_{PQ} \in \text{Lie}(U_Q) \) and \( \dot{g}_{Qf} = \dot{u}_{Qf} + \dot{g}_{Pf} \in \mathfrak{g}_{Qf} \) to find:
\[
\Phi^*_S(\omega^p_1)(L_g(\dot{g})) = \omega^p_1(L_{g_{1h}}(\dot{g}_{1h})) + \lambda_1(\dot{u}_{Qf}) + \lambda_1(\dot{g}_{Pf}).
\]
Moreover, \( \lambda_1(\dot{u}_{Qf}) \in \lambda_1(\mathfrak{N}_{Pf}) \subset \lambda_1(\mathfrak{g}_{1h}). \)

Now suppose that \( \mathbf{R} \) is another chain in the sum \([12.7.1]\) which makes a nonzero contribution \( \Phi^*_R \nabla^p_1 = d + \Phi^*_R(\omega^p_1) \) to the connection \( \nabla^p \), say, \( W = R(1) < R(2) < \cdots < R(r-1) < X \). Then \( Q' = P_{R(1)} \cap P_{R(2)} \cap \ldots \cap P_{R(r-1)} \) and \( P \) also have compatible decompositions:
\[
\begin{align*}
Q' &= U_1 G_{1h} (U_{PQ} G_{Qf}) \text{ with } U_{Q'} = U_{FQ} U_{PQ} \\
P &= U_1 G_{1h} (U_{PQ} G_{Qf} G_P f) \text{ with } U_P = U_{FQ} U_{PQ} U_{Qf}.
\end{align*}
\]
The same element \( \dot{g} \in \text{Lie}(P) \) decomposes as
\[
\dot{g} = \dot{u}_1 + \dot{g}_{1h} + \dot{u}_{PQ} + \dot{u}_{Qf} + \dot{g}_{Pf}.
\]
So the same argument gives \( \Phi^*_R(\omega^p_1)(L_g(\dot{g})) = \omega^p_1(L_{g_{1h}}(\dot{g}_{1h})) + \lambda_1(\dot{u}_{Qf}) + \lambda_1(\dot{g}_{Pf}). \) We conclude that:
\[
(\Phi^*_S \omega^p_1 - \Phi^*_R \omega^p_1)(L_g(\dot{g})) = \lambda_1(\dot{u}_{Qf} - \dot{u}_{Qf}) \in \lambda_1(\mathfrak{N}_{Pf}) \subset \lambda_1(\mathfrak{g}_{1h}).
\]
Consequently each term in the sum \([12.9]\) (except for the first) lies in \( \mathfrak{N}_{Pf} \).

**12.10. Completion of the proof.** Using Lemma 12.8, at the point \( \mathbf{x} \) we may write
\[
\Omega^p(U, V) = \pi^*(\Omega^p_1(U, V)) + n
\]
where \( n \in \lambda_1(G_{1h}) \) is nilpotent and in fact lies in \( \mathfrak{N}_{Pf} \). Moreover, by Proposition 12.2, the curvature \( \Omega^p_1 \) commutes with \( n \). If \( f : \text{End}(V) \to \mathbb{C} \) is any \( \text{Ad} \)-invariant polynomial, it follows from Lemma 6.4 that \( f(\Omega^p(U, V)) = f(\Omega^p(U, V)) \) hence also \( f(\Omega^p_1(U, V)) = f(\Omega^p_1(U, V)) \). So it follows from \([6.3.1]\) that the corresponding characteristic form satisfies the \( \pi \)-fiber condition relative to \( W \). This completes the proof of Theorem 11.6.

### 13. Toroidal compactification

**13.1. Throughout this section we assume that** \( G = G(\mathbb{R}) \) is the set of real points of a connected semisimple algebraic group \( G \) defined over \( \mathbb{Q} \), that \( D = G/K \) is a Hermitian symmetric space, \( \Gamma \subset G(\mathbb{Q}) \) is a neat arithmetic group, \( X = \Gamma \setminus G/K \) is the corresponding locally symmetric space with Baily-Borel-Satake compactification \( \overline{X} = \Gamma \setminus D^* \). Fix a representation \( \lambda : K \to GL(V) \) on some complex vectorspace \( V \) and let \( E = G \times_K V \) be the corresponding homogeneous vectorbundle on \( D \), and \( E' = \Gamma \setminus E \) the automorphic vectorbundle on \( X \). Choose a system of control data on the Baily-Borel compactification \( \overline{X} \) and a
partition of unity as in §3, and let $\nabla_X^p$ denote the resulting patched connection on $E' \to X$. For each $i$, the Chern form $\sigma^i(\nabla_X^p) \in \Omega^{2i}(X; \mathbb{C})$ (cf §3) is a $\pi$-fiber differential form on $X$ so it determines a cohomology class $\bar{c}^i(E') = [\sigma^i(\nabla_X^p)] \in H^{2i}(X; \mathbb{C})$.

We also fix a nonsingular toroidal compactification $\overline{X}_\Sigma$. This corresponds to a $\Gamma$-compatible collection of simplicial polyhedral cone decompositions $\Sigma_F$ of certain self adjoint homogeneous cones. These compactifications were constructed in [AMRT] and are reviewed in [Har3], [HZ1], [FC], [Na]. In [Mu1], D. Mumford shows that the automorphic vector-bundle $E' \to X$ admits a canonical extension $\overline{E}_\Sigma'$ over the toroidal compactification $\overline{X}_\Sigma$. In [Har3] Theorem 4.2, M. Harris shows that Mumford’s canonical extension coincides with Deligne’s canonical extension [D] (for an appropriately chosen flat connection with unipotent monodromy).

The identity mapping $X \to X$ has a unique continuous extension, $\tau : \overline{X}_\Sigma \to \overline{X}$ of $X$, and this is a resolution of singularities.

**13.2. Theorem.** The patched connection $\nabla_X^p$ on $E' \to X$ extends to a smooth connection $\nabla_{\Sigma}'$ on $\overline{E}_\Sigma' \to \overline{X}_\Sigma$. Moreover for each $i$,

$$\tau^*\bar{c}^i(E') = \tau^*([\sigma^i(\nabla_X^p)]) = [\sigma^i(\nabla_{\Sigma}'^p)] = c^i(E_{\Sigma}') \in H^{2i}(\overline{X}_\Sigma; \mathbb{C}). \quad (13.2.1)$$

The proof will appear in Section 14. S. Zucker has pointed out that it follows from mixed Hodge theory that the image of $\bar{c}^i(E')$ in $\text{Gr}_W^V H^{2i}(\overline{X}; \mathbb{C})$ is uniquely determined by (13.2.1).

**13.3. Proportionality theorem.** Fix representations $\lambda_j : K \to GL(V_j)$ for $j = 1, 2, \ldots, r$ and fix nonnegative integers $I = (i_1, i_2, \ldots, i_r)$ with $i_1 + i_2 + \ldots + i_r = n = \dim(D)$. For $j = 1, 2, \ldots, r$ let $E'_j = \Gamma \backslash G \times_K V_j \to X$ be the resulting automorphic vectorbundle on $X$ and let $\tilde{E}_j = G_u \times_K V_j$ be the corresponding vectorbundle on the compact dual symmetric space $\tilde{D} = G_u / K$ (where $G_u$ is a compact real form of $G$ containing $K$). Define “generalized” Chern numbers

$$\bar{c}^I(\lambda_1, \lambda_2, \ldots, \lambda_r) = (c^{i_1}(\tilde{E}_1) \cup c^{i_2}(\tilde{E}_2) \cup \ldots \cup c^{i_r}(\tilde{E}_r)) \cap [\overline{D}] \in \mathbb{Z} \quad (13.3.1)$$

and

$$(\bar{c}^I(\lambda_1, \lambda_2, \ldots, \lambda_r) = c^{i_1}(\nabla_{\Sigma}_1^p) \cup c^{i_2}(\nabla_{\Sigma}_2^p) \cup \ldots \cup c^{i_r}(\nabla_{\Sigma}_r^p) \cap [\overline{X}] \in \mathbb{C} \quad (13.3.2)$$

where $\nabla_{\Sigma}^p$ denotes the patched connection on $E'_j \to X$ and where $[\overline{X}] \in H^{2n}(\overline{X}; \mathbb{C})$ denotes the fundamental class of the Baily-Borel compactification. Let $v(\Gamma) \in \mathbb{Q}$ denote the constant which appears in the proportionality theorem of Hirzebruch [Hr1], [Mu1].

**13.4. Proposition.** For any choice $\lambda_1, \lambda_2, \ldots, \lambda_r$ of representations and for any partition $I = (i_1, i_2, \ldots, i_r)$ of $n = \dim_D(D)$ we have

$$\bar{c}^I(\lambda_1, \lambda_2, \ldots, \lambda_r) = v(\Gamma)\bar{c}^I(\lambda_1, \lambda_2, \ldots, \lambda_r). \quad (13.4.1)$$
13.5. **Proof.** Each of the vector bundles $E'_j$ has a canonical extension $E'_{j,\Sigma} \to \overline{X}_\Sigma$. The same proof as in [Mu1] (which is the same proof as in [Hr1]) (cf. [Hr3]) shows that the Chern classes of these extended bundles satisfy the proportionality formula
\[(c^1(E'_{1,\Sigma}) \cup c^2(E'_{2,\Sigma}) \cup \ldots \cup c^r(E'_{r,\Sigma})) \cap [\overline{X}_\Sigma] = v(\Gamma)c^I(\lambda_1, \lambda_2, \ldots, \lambda_r). \tag{13.5.1}\]

The result now follows immediately from Theorem 13.2. \qed

14. **Proof of Theorem 13.2.**

14.1. Fix $\epsilon \leq \epsilon_0$. We claim that the partition of unity (3.3.1) $\sum_{Z \subseteq X} B^Z_\epsilon = 1$ on $\overline{X}$ pulls back to a smooth partition of unity on $\overline{X}_\Sigma$. Fix a pair of strata $Y,Z$ of $\overline{X}$. We must verify that $\tau^*B^Z_\epsilon$ is smooth near $\tau^{-1}(Y)$. It can be shown that the mapping $\tau : \overline{X}_\Sigma \to \overline{X}$ is a complex analytic morphism between complex analytic varieties, as is the projection $\pi_Y : T_Y(\epsilon) \to Y$. It follows that the composition $\pi_Y \circ \tau : \tau^{-1}(T_Y(\epsilon)) \to Y$ is a complex analytic morphism between smooth complex varieties, so it is smooth. If the stratum $Z$ is not comparable to $Y$ or if $Z > Y$ then $B^Z_\epsilon$ vanishes on $T_Y(\epsilon/2)$ hence $\tau^*B^Z_\epsilon$ vanishes on $\tau^{-1}(T_Y(\epsilon/2))$ so it is smooth. If $Z \leq Y$ then by (3.2.4) and (5.2.7), $\tau^*B^Z_\epsilon$ is smooth in this open set.

14.2. We may assume that $G$ is simple over $\mathbb{Q}$. The “boundary” $\overline{X}_\Sigma - X$ of the toroidal compactification has a distinguished covering by open sets $U_Y$, one for each stratum $Y \subset X$ of the Baily-Borel compactification, such that $\tau(U_Y) \subset X$ is a neighborhood of $Y$, and for which the restriction $E'_{\Sigma} \mid U_Y$ arises from an automorphy factor ([HZ1] §3.3).

14.3. **Proposition.** For any smooth connection $\nabla_Y$ on $(E'_Y,Y)$ the parabolically induced connection $\Phi_{XY}^\epsilon(\nabla_Y)$ (which is defined only on $E'_Y \mid (U_Y \cap X) = E'_{\Sigma}(U_Y \cap X)$) extends canonically to a smooth connection (which we denote by $\overline{\Phi}_{XY}^\epsilon(\nabla_Y)$) on $E'_{\Sigma} \mid U_Y$.

14.4. **Proof of Theorem 13.2.** We may assume that $\epsilon_0 > 0$ was chosen so small that $T_Y(\epsilon_0) \subset U_Y$ for each stratum $Y < X$ of $\overline{X}$. By §4.1 the partition of unity which is used to construct the patched connection $\nabla^p_X$ extends to a smooth partition of unity on $\overline{X}_\Sigma$. Hence, using Proposition 14.3,
\[\nabla^\epsilon_{\Sigma} = \tau^*B^\epsilon_X + \sum_{Y < X} \tau^*B^\epsilon_Y \Phi_{XY}^\epsilon(\nabla_Y)\]
is a smooth connection on $E'_{\Sigma} \to \overline{X}_\Sigma$ which coincides with the patched connection $\nabla^p_X$ on $E' \to X$. Therefore its Chern forms are smooth and everywhere defined and they restrict to the Chern forms of $\nabla^p_X$. It follows from Lemma 14.4 that each Chern class of $\nabla^\epsilon_{\Sigma}$ is the pullback of the corresponding Chern class of $\nabla^p_X$. \qed

The remainder of §13 is devoted to the proof of Proposition 14.3 which is essentially proven in [HZ1] (3.3.9) (following [Har2]). We will now verify the details.
14.5. Let us recall the construction [AMRT] of the toroidal compactification. Fix a rational boundary component $F$ with normalizing parabolic subgroup $P = UG_bG_\ell$ and let $Y = \Gamma_h\backslash F \subset X$ denote the corresponding stratum in the Baily-Borel compactification of $X = \Gamma\backslash D$. Let $Z_F = \text{Center}(U)$ and $\mathfrak{z} = \text{Lie}(Z_F)$. The vectorspace $\mathfrak{z}$ is preserved under the adjoint action of $G_\ell$ and contains a unique open orbit $C_F \subset \mathfrak{z}$; it is a rationally defined self-adjoint homogeneous cone. Its Satake compactification $C_F'$ consists of $C_F$ together with all its rational boundary components (in the Satake topology). The toroidal compactification is associated to a collection $\Sigma = \{\Sigma_F\}$ of rational polyhedral cone decompositions of the various $C_F'$ which are compatible under $\Gamma$.

Let $\tilde{D}$ denote the compact dual symmetric space (so $\tilde{D} = \mathfrak{g}(\mathbb{C})/K(\mathbb{C})P^-$ in the notation of Proposition 8.2; cf. [AMRT], [Sa]) and let $\beta : D \to \tilde{D}$ denote the Borel embedding. Set $D_F = Z_F(\mathbb{C}).\beta(D)$. The domain $D_F$ is homogeneous under $P.Z_F(\mathbb{C})$ and it admits “Siegel coordinates” $D(F) \cong Z_F(\mathbb{C}) \times \mathbb{C}^n \times F$ in which the subset $\beta(D) \subset D_F$ is defined by a certain well known inequality ([AMRT] p. 239, [Sa] §III (7.4)). Now consider the commutative diagram [HZ1] (1.2.5), reproduced in figure 5.

![Figure 5. Toroidal compactification](attachment:image.png)

Here, $\Gamma_F' = \Gamma \cap (G_bU)$ and $M_F' = \Gamma_F' \backslash D_F$. The algebraic torus $T_F = Z_F(\mathbb{C})/(\Gamma \cap Z_F)$ acts on $M_F'$ with quotient $A_F$, which is in turn an abelian scheme over $Y$. The choice $\Sigma_F$ of polyhedral cone decomposition determines a torus embedding $T_F \hookrightarrow T_{F,\Sigma}$ and a partial compactification $M_{F,\Sigma}' = M_F' \times_{T_F} T_{F,\Sigma}$ of $M_F'$. Let $D_{F,\Sigma}$ denote the interior of the closure of $M_F' = \Gamma_F' \backslash D$ in $M_{F,\Sigma}'$. The quotient mapping $\Gamma_F' \backslash D \to X$ extends to a local isomorphism $\varphi_{F,\Sigma} : D_{F,\Sigma} \to \overline{X}_\Sigma$ (cf. [AMRT] p. 250). In other words, $\varphi_{F,\Sigma}$ is an open analytic mapping with discrete fibers which, near the boundary, induces an embedding $D_{F,\Sigma}/(\Gamma_F' \backslash \Gamma_F) \hookrightarrow \overline{X}_\Sigma$ whose image is the neighborhood $U_Y$ referred to in §14.2. The mappings $\varphi_{F,\Sigma}$ for the various strata $Y \subset \overline{X}$ cover the boundary of $\overline{X}_\Sigma$. The composition $\theta_1\theta_2|D : D \to F$ coincides with the canonical projection $\pi$. 
14.6. Each of the spaces in Figure 3 comes equipped with a vectorbundle and most of the mappings in this diagram are covered by vectorbundle isomorphisms. We will make the following notational convention: If $E_1 \to M_1$ and $E_2 \to M_2$ are (smooth) vectorbundles, $\alpha : M_1 \to M_2$ is a (smooth) mapping, and $\Phi : E_1 \to E_2$ is a vectorbundle mapping which induces an isomorphism $E_1 \cong \alpha^*(E_2)$, then we will write $\Phi : (E_1, M_1) \sim (E_2, M_2)$ and refer to $\Phi$ as being a vectorbundle isomorphism which covers $\alpha$.

As in [Mu1], complexify $\lambda : K \to GL(V)$ and extend it trivially over $P$ to obtain a representation $\lambda : K(\mathbb{C})P^{-} \to GL(V)$. The homogeneous vectorbundle $E = G \times_K V$ has a canonical extension

$$\tilde{E} = G(\mathbb{C}) \times_{K(\mathbb{C})P^{-}} V$$

over the compact dual symmetric space $\hat{D}$. Its restriction $E_F$ to $D_F$ is a $P.Z_F(\mathbb{C})$-homogeneous bundle, and it passes to a vectorbundle $E^\prime_F \to M_F^\prime$ upon dividing by $\Gamma_F^\prime$. The restriction $E^\prime_F|\Gamma_F^\prime \setminus D_F$ coincides with the vectorbundle obtained from the homogeneous vectorbundle $E = G \times_K V$ upon dividing by $\Gamma_F^\prime$. We will denote this restriction also by $(E^\prime_F, \Gamma_F^\prime \setminus D_F)$.

Define $\tilde{E} = P.Z_F(\mathbb{C}) \times_{K_P.Z_F(\mathbb{C})} V$. This vectorbundle on $D_F/Z_F(\mathbb{C})$ is homogeneous under $P.Z_F(\mathbb{C})$ and it passes to a vectorbundle $E^A_F \to A_F$ upon dividing by $\Gamma_F^\prime$. As in [HZ1] (3.2.1) and (3.3.5), there is a canonical vectorbundle isomorphism

$$\psi : (E^\prime_F, M^\prime_F) \sim (E^A_F, A_F) \quad (14.6.1)$$

which covers $\pi_2$. In fact, the isomorphism $\psi$ is obtained from the canonical isomorphism of $P.Z_F(\mathbb{C})$-homogeneous vectorbundles,

$$\Psi : (E_F, D_F) \sim (\tilde{E}, D_F/Z_F(\mathbb{C})) \quad (14.6.2)$$

which covers $\theta_2$ and which is given by the quotient mapping

$$E_F = P.Z_F(\mathbb{C}) \times_{K_P} V \to \tilde{E} = P.Z_F(\mathbb{C}) \times_{K_P.Z_F(\mathbb{C})} V.$$ 

Let $\tilde{E}^\prime_{\Sigma}$ denote Mumford’s canonical extension of the vectorbundle $E^\prime = \Gamma \setminus E \to X$ to the toroidal compactification $\overline{X}_{\Sigma}$, and let $\tilde{E}_{F,\Sigma}^\prime = \varphi_{F,\Sigma}^*(E^\prime_{\Sigma})$ be its pullback to $D_{F,\Sigma}$. Then we have a further canonical identification $E^\prime = \tilde{E}_{F,\Sigma}^\prime|\Gamma_F^\prime \setminus D_F = E^\prime_F|\Gamma_F^\prime \setminus D_F$. We also have vectorbundles $E_h = G_h \times_{K_h} V$ on $F$ and its quotient $E^\prime_h \to Y = \Gamma_h \setminus F$.

According to [HZ1] (3.3.9) (which in turn relies on [Har2]), the canonical isomorphism

$$\psi_{\Sigma} : (\tilde{E}_{F,\Sigma}^\prime, D_{F,\Sigma}) \sim (E^A_F, A_F) \quad (14.6.3)$$

which covers $\pi_{2,\Sigma}$. This is the key point in the argument: the isomorphism $\psi_{\Sigma}$ identifies Mumford’s canonical extension (which is defined using a growth condition on a singular connection) with a vectorbundle, $\pi_{2,\Sigma}^*(E^A_F)$ which is defined topologically, and which is trivial on
Each torus embedding $\pi_{2,\Sigma}^{-1}(a) \cong T_{F,\Sigma}$. We will use this isomorphism to extend the parabolically induced connection over the toroidal compactification, because such a parabolically induced connection is also pulled up from $E'_F$.

As in (10.1.1), define an action of $P.Z_F(\mathbb{C})$ on the vectorbundle $E_h \to F$ by

$$ug_hg\ell z, [g_h', v] = [g_hg_h', \lambda_1(g\ell)v]$$

(14.6.4)

(\text{where } u \in U_F, g_h, g_h' \in G_h, g\ell \in G_\ell, z \in Z_f(\mathbb{C}), \text{ and } v \in V). Define a mapping

$$\tilde{\Phi}_F : PZ_F(\mathbb{C}) \times_{K_{\Sigma}Z_F(\mathbb{C})} V \to G_h \times_{K_h} V$$

(14.6.5)

by $\tilde{\Phi}_F([ug_hg\ell z, v]) = [g_h, \lambda_1(g\ell)v]$. Then $\tilde{\Phi}_F$ is well defined, it is $P.Z_F(\mathbb{C})$-invariant, and it gives a $P.Z_F(\mathbb{C})$-equivariant isomorphism of vectorbundles,

$$\Phi : (\tilde{E}, D_F/Z_F(\mathbb{C})) \sim (E_h, F)$$

(14.6.6)

which covers $\theta_1$. Moreover the composition $\Phi_F \Psi|(E, D)$ is precisely the isomorphism $\Phi : (E, D) \cong \pi^*(E_h, F)$ of (10.1.2). In summary, this array of vectorbundles appears in figure 6.

\begin{align*}
(E, D) \mod \Gamma'_F & \rightarrow (E'_F, \Gamma'_F/\Gamma_F(\mathbb{C}) \mod \Gamma'_F) \rightarrow \longrightarrow (E'_F, D_{\bar{F}}) \rightarrow \longrightarrow (E'_F, \bar{x}) \\
(E_h, F) & \rightarrow \longrightarrow (E'_F, Y)
\end{align*}

\textbf{Figure 6.} Vectorbundles on the toroidal compactification

14.7. Each of the vectorbundles in Figure 6 comes equipped with a connection. Let $\nabla_Y$ be a given connection on $E'_F \rightarrow Y$ and let $\nabla_h$ be its pullback to $E_h \rightarrow F$. Define $\tilde{\nabla} = \Phi_F(\nabla_h)$ to be the pullback of $\nabla_h$ under the isomorphism (14.6.6). Then $\Psi^*(\tilde{\nabla})$ is an extension of the parabolically induced connection $\Phi^*(\nabla_h)$ on $(E, D)$. Both connections are invariant under $\Gamma'_F$. Let $\nabla'_F$ denote the resulting connection on the quotient $(E'_F, M'_F)$ (where again we use the same symbol to denote this connection as well as its restriction to $(E'_F, \Gamma'_F/\Gamma_F(\mathbb{C}) \mod \Gamma'_F)$). We need to show that this connection $\nabla'_F$ on $(E'_F, \Gamma'_F/\Gamma_F(\mathbb{C}) \mod \Gamma'_F)$ has a smooth extension to a connection $\nabla'_{F,\Sigma}$ on $(\tilde{E}_{F,\Sigma}, D_{F,\Sigma})$ which is invariant under $\Gamma'_F \mod \Gamma_F$. 

The connection $\tilde{\nabla}$ passes to a connection $\nabla^A_F$ on $(E^A_F, A_F)$ such that $\nabla'_F = \psi^*(\nabla^A_F)$. Therefore the connection

$$\nabla'_{F, \Sigma} = \psi^*(\nabla^A_F)$$

on $(\bar{E}'_F, \Sigma, \bar{M}'_F, \Sigma)$ is a smooth extension of $\nabla'_F$. The $\Gamma'_F \backslash \Gamma_F$-invariance of $\nabla'_{F, \Sigma}$ follows from the $P.Z_F(\mathbb{C})$-invariance of $\tilde{\nabla}$. This completes the proof of Proposition 14.3. 

15. Chern classes and constructible functions

15.1. A constructible function $F : W \to \mathbb{Z}$ on a complete (complex) algebraic variety $W$ is one which is constant on the strata of some algebraic (Whitney) stratification of $W$. The Euler characteristic of such a constructible function $F$ is the sum

$$\chi(W; F) = \sum \chi(W_\alpha) F(W_\alpha)$$

over strata $W_\alpha \subset W$ along which the function $F$ is constant. If $f : W \to W'$ is an (proper) algebraic mapping, then the pushforward of the constructible function $F$ is the constructible function

$$f_*(F)(w') = \chi(f^{-1}(w'); F)$$

(15.1.1)

(for any $w' \in W'$). According to [Mac], for each constructible function $F : W \to \mathbb{Z}$ it is possible to associate a unique Chern class $c_*(W; F) \in H_*(W; \mathbb{Z})$ which depends linearly on $F$, such that $f^* c_*(W; F) = c_*(W'; f_* F)$ (whenever $f : W \to W'$ is a proper morphism), and such that $c_*(W; 1_W) = c^*(W) \cap [W]$ if $W$ is nonsingular. (Here, $[W] \in H_{2 \dim(W)}(W; \mathbb{Z})$ denotes the fundamental class of $W$.) The MacPherson-Schwartz Chern class of $W$ is the Chern class of the constructible function $1_W$.

15.2. Now let $\bar{Z}$ be a nonsingular complete complex algebraic variety and let $D = D_1 \cup D_2 \cup \ldots \cup D_m$ be a union of smooth divisors with normal crossings in $\bar{Z}$. Set $Z = \bar{Z} - D$. The tangent bundle $T_Z$ of $Z$ has a “logarithmic” extension to $\bar{Z}$,

$$T_{\bar{Z}}(- \log D) = \text{Hom}(\Omega^1_{\bar{Z}}(\log D), \mathcal{O}_{\bar{Z}})$$

which is called the “log-tangent bundle” of $(\bar{Z}, D)$. It is the vectorbundle whose sheaf of sections near any k-fold multi-intersection $\{z_1 = z_2 = \ldots = z_k = 0\}$ of the divisors is generated by $z_1 \frac{\partial}{\partial z_1}, z_2 \frac{\partial}{\partial z_2}, \ldots, z_k \frac{\partial}{\partial z_k}, z_{k+1}, \ldots, z_n$ (where $n = \dim(Z)$). The following result was discovered independently by P. Aluffi [Al].

15.3. Proposition. The Chern class of the log tangent bundle is equal to the Chern class of the constructible function which is 1 on $Z = \bar{Z} - D$, that is,

$$c^*(T_{\bar{Z}}(- \log D)) \cap [\bar{Z}] = c_*(1_Z).$$
15.4. **Proof.** For any subset $I \subset \{1, 2, \ldots, m\}$ let $D_I = \cap_{i \in I} D_i$, let
\[
D^I = D_I \cap \bigcup_{j \notin I} D_j
\]
denote the “trace” of the divisor $D$ in $D_I$, and let $D_i^I = D_I - D^I$ denote its complement. The restriction of the log tangent bundle of $(\overline{Z}, D)$ to any intersection $D_I$ is (topologically) isomorphic to the direct sum of vector bundles
\[
T_{\overline{Z}}(-\log D)|_{D_I} \cong T_{D_I}(-\log (D^I)) \oplus |I| \mathbf{1}
\]  
(15.4.1)
(the last symbol denoting $|I|$ copies of the trivial bundle). (This follows from the short exact sequence of locally free sheaves on $D_j$,
\[
0 \longrightarrow \Omega^1_{D_j}(\log D^{(j)}) \longrightarrow \Omega^1_{\overline{Z}}(\log D)|_{D_j} \longrightarrow \mathcal{O}_{D_j} \longrightarrow 0
\]
by dualizing and induction.) We will prove Proposition 15.3 by induction on the number $m$ of divisors, with the case $m = 0$ being trivial. For any constructible function $F$ on $\overline{Z}$, denote by $c(F) \in H^*(\overline{Z})$ the Poincaré dual of the (homology) Chern class of $F$. Each divisor $D_j$ carries a fundamental homology class whose Poincaré dual we denote by $[D_j] \in H^2(\overline{Z})$. The Chern class of the line bundle $\mathcal{O}(D_j)$ is $1 + [D_j]$. Let $\tilde{c}$ denote the Chern class of the bundle $T_{\overline{Z}}(-\log D)$. If $I \subset \{1, 2, \ldots, m\}$ and if $i : D_I \to \overline{Z}$ denotes the inclusion then
\[
\tilde{c} \cdot [D_I] = i_* (c(T_{\overline{Z}}(-\log D))|_{D_I}) = i_* c(T_{D_I}(-\log (D^I))) = i_* c(1_{D^I})
\]
by (15.4.1) and induction. Using [Tu] Proposition 1.2 we see,
\[
c(\overline{Z}) = \tilde{c} \cdot \prod_i (1 + [D_i]) = \tilde{c} + \tilde{c} \cdot \sum_I [D_I]
= \tilde{c} + \sum_I c(1_{D^I}) = \tilde{c} + c(1_D)
\]
since each point in $D$ occurs in exactly one multi-intersection of divisors. □

15.5. **Theorem.** Let $X = \Gamma \backslash G/K$ be a Hermitian locally symmetric space as in §9 with Baily-Borel compactification $\overline{X}$. Let $\tilde{c}^*(\overline{X}) \in H^{2i}(\overline{X}; \mathbb{C})$ denote the cohomology Chern class of the tangent bundle, constructed in Theorem 11.4. Then its homology image
\[
\tilde{c}^*(\overline{X}) \cap [X] = c_*(1_X) \in H_*(\overline{X}; \mathbb{Z})
\]
lies in integral homology and coincides with the (MacPherson) Chern class of the constructible function which is 1 on $X$ and is 0 on $\overline{X} - X$. 
15.6. **Proof.** Let \( \tau : \overline{X}_\Sigma \to \overline{X} \) denote a smooth toroidal resolution of singularities, having chosen the system of polyhedral cone decompositions \( \Sigma \) so that the exceptional divisor is a union of smooth divisors with normal crossings. Let \( T_{\overline{X}_\Sigma}(-\log D) \) denote the log tangent bundle of \( (\overline{X}_\Sigma, D) \). As in [Mu1] Prop. 3.4, this bundle is isomorphic to Mumford’s canonical extension \( T_{\overline{X},\Sigma} \) of the tangent bundle. Therefore

\[
\bar{c}^*([\overline{X}]) \cap [\overline{X}] = \tau_* (r^*\bar{c}^*([\overline{X}]) \cap [\overline{X}_\Sigma])
\]

\[
= \tau_* (\bar{c}^*([T_{\overline{X},\Sigma}]) \cap [\overline{X}_\Sigma])
\]

\[
= \tau_* c_\lambda (1_\Sigma) = c_\lambda (\tau_* (1_\Sigma)) = c_\lambda (1_\Sigma)
\]

by Theorem 13.2, Proposition 15.3 and (15.1.1). \( \Box \)

15.7. **Corollary.** The MacPherson-Schwartz Chern class of the Baily-Borel compactification \( \overline{X} \) is given by the sum over strata \( Y \subset \overline{X} \),

\[
c_\lambda (1_{\overline{X}}) = c_\lambda \left( \sum_{Y \subset \overline{X}} 1_Y \right) = \sum_{Y \subset \overline{X}} i_* \bar{c}^*([Y]) \cap [\overline{X}]
\]

where \( i : \overline{Y} \hookrightarrow \overline{X} \) is the inclusion of the closure of \( Y \) (which is also the Baily-Borel compactification of \( Y \)) into \( \overline{X} \). \( \Box \)

16. **Cohomology of the Baily-Borel Compactification**

16.1. Let \( K \) be a compact Lie group and let \( EK \to BK \) be the universal principal \( K \)-bundle. For any representation \( \lambda : K \to GL(V) \) on a complex vector space \( V \), let \( E\lambda = EK \times_K V \) be the associated vector bundle. The Chern classes \( c^i(E) \in H^{2i}(BK; \mathbb{C}) \) of all such vector bundles generate a subalgebra which we denote \( H^*_{\text{chern}}(BK; \mathbb{C}) \). Two cases are of particular interest: if \( K = U(n) \) then \( BK = \lim_{k \to \infty} G_n(\mathbb{C}^{n+k}) \) is the infinite Grassmann manifold and \( H^*(BK; \mathbb{C}) = H^*_{\text{chern}}(BK; \mathbb{C}) \). In fact, the standard representation \( \lambda : U(n) \to GL_n(\mathbb{C}) \) gives rise to a single vector bundle \( E_\lambda \to BK \) such that the algebra \( H^*(BK; \mathbb{C}) \) is canonically isomorphic to the polynomial algebra in the Chern classes \( c^1(E_\lambda), c^2(E_\lambda), \ldots, c^n(E_\lambda) \). If \( K = SO(n) \) then \( BK = \lim_{k \to \infty} G_\infty^k(\mathbb{R}^{n+k}) \) is the infinite Grassmann manifold of real oriented \( n \)-planes. Let \( \bar{\lambda} : SO(n) \to GL_n(\mathbb{R}) \) be the standard representation with resulting vector bundle \( E_{\bar{\lambda}} \to BK \), and let \( \lambda : SO(n) \to GL_n(\mathbb{C}) \) denote the composition of \( \bar{\lambda} \) with the inclusion \( GL_n(\mathbb{R}) \subset GL_n(\mathbb{C}) \). The associated vector bundle \( E_\lambda = E_{\bar{\lambda}}(\mathbb{C}) \) is the complexification of \( E_{\bar{\lambda}} \). If \( n \) is odd, then \( H^*(BK; \mathbb{C}) \) is canonically isomorphic to the polynomial algebra generated by the Pontrjagin classes \( p^i(E_{\bar{\lambda}}) = c^{2i}(E_{\bar{\lambda}}) \in H^{4i}(BK; \mathbb{C}) \) for \( i = 1, 2, \ldots, n \). Hence \( H^*(BK; \mathbb{C}) = H^*_{\text{chern}}(BK; \mathbb{C}) \). If \( n \) is even then the algebra \( H^*(BK; \mathbb{C}) \) has an additional generator, the Euler class \( e = e(E_{\bar{\lambda}}) \in H^n(BK; \mathbb{C}) \). (It satisfies \( e^2 = p^{n/2} \).) If \( n = 2 \) then \( e \) is the first Chern class of the line bundle corresponding to the representation \( SO(2) \cong U(1) \subset GL_1(\mathbb{C}) \).
16.2. Now suppose that \( K = K_1 \times K_2 \times \ldots \times K_r \) is a product of unitary groups, odd orthogonal groups, and copies of \( \text{SO}(2) \). According to the preceding paragraph, there are representations \( \lambda_1, \ldots, \lambda_r \) of \( K \) on certain complex vector spaces \( V_1, V_2, \ldots, V_r \) so that the Chern classes of the resulting “universal” complex vector bundles \( E_i = EK \times_K V_i \to BK \) generate the polynomial algebra \( H^*(BK; \mathbb{C}) = H^*_\text{Chern}(BK; \mathbb{C}) \).

16.3. Suppose that \( G \) is a semisimple algebraic group defined over \( \mathbb{Q} \), and that \( G(\mathbb{R})^0 \) acts as the identity component of the group of automorphisms of a Hermitian symmetric space \( D = G/K \). Recall ([He], §6, [Bo2]) that the irreducible components of \( D \) come from the following list:

| Type | Symmetric Space | Compact Dual |
|------|----------------|-------------|
| AIII | \( U(\mathfrak{p}, \mathfrak{q})/U(\mathfrak{p}) \times U(\mathfrak{q}) \) | \( U(\mathfrak{p} + \mathfrak{q})/U(\mathfrak{p}) \times U(\mathfrak{q}) \) |
| DIII | \( \text{SO}(2n)/U(n) \) | \( \text{SO}(2n)/U(n) \) |
| BDI | \( \text{SO}(p, 2)/\text{SO}(p) \times \text{SO}(2) \) | \( \text{SO}(p + 2)/\text{SO}(p) \times \text{SO}(2) \) |
| CI   | \( \text{Sp}(n, \mathbb{R})/U(n) \) | \( \text{Sp}(n)/U(n) \) |
| EIII | \( \text{E}_6^1/\text{Spin}(10) \times \text{SO}(2) \) | \( \text{E}_6/\text{Spin}(10) \times \text{SO}(2) \) |
| EVII | \( \text{E}_7^1/\text{E}_6 \times \text{SO}(2) \) | \( \text{E}_7/\text{E}_6 \times \text{SO}(2) \) |

Let \( X = \Gamma \backslash G/K \), with \( \Gamma \subseteq G(\mathbb{Q}) \) a neat arithmetic group, and let \( X \) denote the Baily-Borel compactification of \( X \). Let \( \tilde{D} = G_u/K \) be the compact dual symmetric space, where \( G_u \subseteq G(\mathbb{C}) \) is a compact real form containing \( K \). The principal bundles \( \Gamma \backslash G \to X \) and \( G_u \to \tilde{D} \) are classified by mappings \( \Phi : X \to BK \) and \( \Psi : \tilde{D} \to BK \) (respectively) which are uniquely determined up to homotopy. A theorem of Borel ([Bo2]) states that (in this Hermitian case) the resulting homomorphism \( \Psi^* : H^*(BK; \mathbb{C}) \to H^*(\tilde{D}; \mathbb{C}) \) is surjective.

Suppose the irreducible factors of \( D = G/K \) are of type AIII, DIII, CI, or BDI for \( p \) odd or \( p = 2 \). The construction of \( \pi \)-fiber Chern forms in Section 11 determines a homomorphism \( \Phi^* : H^*(BK; \mathbb{C}) \to H^*(X; \mathbb{C}) \) by setting \( \Phi^*(\tilde{c}(E_j)) = \tilde{c}(E'_j) \) (where \( E_j \to BK \) is the universal vector bundle corresponding to the representation \( \lambda_j \) of \( G/K \) and \( E'_j \to X \) is the corresponding automorphic vector bundle). Let us denote the image of \( \Phi^* \) by \( H^*_{\text{Chern}}(X; \mathbb{C}) \).

16.4. **Theorem.** Suppose \( X = \Gamma \backslash G/K \) is a Hermitian locally symmetric space such that the irreducible factors of \( D = G/K \) are of type AIII, DIII, CI, or BDI for \( p \) odd or \( p = 2 \). Then the mappings \( \Phi^* \) and \( \Psi^* \) determine a surjection

\[
h : H^*_{\text{Chern}}(X; \mathbb{C}) \to H^*(\tilde{D}; \mathbb{C})
\]

from this subalgebra of the cohomology of the Baily-Borel compactification, to the cohomology of the compact dual symmetric space. Moreover, for each “universal” vector bundle \( E_j \to BK \) we have

\[
h(\tilde{c}(E'_j)) = \tilde{c}(E_j)
\]
where \( E_j' \to X \) and \( \bar{E}_j \to \bar{D} \) are the associated automorphic and homogeneous vectorbundles, respectively.

16.5. **Proof.** Define the mapping \( h : H_{\text{Chern}}(\overline{X}; \mathbb{C}) \to H^\ast(\bar{D}; \mathbb{C}) \) by \( h\Phi^\ast(c) = \Psi^\ast(c) \) for any \( c \in H^\ast(BK; \mathbb{C}) \). If this is well defined, it is surjective by Borel’s theorem. To show it is well defined, let us suppose that \( \tilde{\Phi}^\ast(c) = 0 \). We must show that \( \Psi^\ast(c) = 0 \), so we assume the contrary.

Let \( x = \Psi^\ast(c) \in H^i(\bar{D}; \mathbb{C}) \). By Poincaré duality, there exists a complementary class \( y \in H^{2n-i}(\bar{D}; \mathbb{C}) \) so that \( (x \cup y) \cap [\bar{D}] \neq 0 \) (where \( n = \dim_\mathbb{C}(\bar{D}) \)). Then \( y \) has a lift, \( \sigma \in H^{2n-i}(BK; \mathbb{C}) \) with \( \Psi^\ast(d) = y \). Let us write \( K = K_1 K_2 \ldots K_r \) for the decomposition of \( K \) into irreducible factors. By §16.2 the polynomial algebra \( H^\ast(BK; \mathbb{C}) \) is generated by the Chern classes of the universal vectorbundles \( E_1, E_2, \ldots, E_r \) corresponding to representations \( \lambda_i : K_i \to GL(V_i) \). Hence, both \( c \) and \( d \) are polynomials in the Chern classes of the vectorbundles \( E_1, E_2, \ldots, E_r \). Hence \( (\tilde{\Phi}^\ast(c) \cup \tilde{\Phi}^\ast(d)) \cap [\overline{X}] \in \mathbb{C} \) is a sum of “generalized” Chern numbers which, by the Proposition 13.4, coincides with the corresponding sum of “generalized” Chern numbers for the compact dual symmetric space, \( v(\Gamma)(x \cup y) \cap [\bar{D}] \neq 0 \). This implies that \( \tilde{\Phi}^\ast(c) \neq 0 \) which is a contradiction. \( \square \)

16.6. **Remarks.** We do not know whether the surjection (16.4) has a canonical splitting. However, the intersection cohomology \( IH^\ast(\overline{X}; \mathbb{C}) \) contains, in a canonical way, a copy of the cohomology \( H^\ast(\bar{D}; \mathbb{C}) \) of the compact dual symmetric space. By the Zucker conjecture ([Lo] and [SS]), the intersection cohomology may be identified with the \( L^2 \) cohomology of \( X \) which, in turn may be identified with the relative Lie algebra cohomology \( H^\ast(g, K; L^2(\Gamma \backslash G)) \). But \( L^2(\Gamma \backslash G) \) contains a copy of the trivial representation \( 1 \) (the constant functions), whose cohomology \( H^\ast(g, K; 1) \cong H^\ast(\bar{D}; \mathbb{C}) \) is the cohomology of the compact dual symmetric space. We sketch a proof that the following diagram commutes.

\[
\begin{array}{ccc}
H^\ast_{\text{Chern}}(\overline{X}; \mathbb{C}) & \longrightarrow & IH^\ast(\overline{X}; \mathbb{C}) \\
\downarrow h & & \downarrow i \\
H^\ast(\bar{D}; \mathbb{C}) & \longrightarrow & H^\ast(\bar{D}; \mathbb{C})
\end{array}
\]

If \( E' \to X \) and \( \bar{E} \to \bar{D} \) are vectorbundles arising from the same representation \( \lambda \) of \( K \) then the class \( j(c^k(E')) \) is represented by the differential form \( \sigma^k(\nabla^p_X) \) which is \( \pi \)-fiber, hence bounded, hence \( L^2 \). The class \( i(c^k(\bar{E})) \) is represented by the differential form \( \sigma^k(\nabla^{\text{Nom}}_X) \) which is “invariant” (meaning that its pullback to \( D \) is invariant), hence \( L^2 \). The intersection cohomology of \( \overline{X} \) embeds into the ordinary cohomology of any toroidal resolution \( \overline{X}_\Sigma \). But when these two differential forms are considered on \( \overline{X}_\Sigma \), they both represent the same cohomology class, \( c^k(\overline{E}_\Sigma) \) (using Theorem 13.2 and [Mu1]). Alternatively, one may deform the connection \( \nabla^p_X \) to \( \nabla^{\text{Nom}}_X \), obtaining a differential form \( \Psi \in \mathcal{A}^{2k-1}(X) \) such that \( d\Psi = \sigma^k(\nabla^p_X) - \sigma^k(\nabla^{\text{Nom}}_X) \), and check that \( \Psi \) is \( L^2 \).
In the case BDI, the compact dual is $\tilde{D} = SO(p + 2)/SO(p) \times SO(2)$. Suppose $p$ is even. Its cohomology $H^*(\tilde{D}; \mathbb{C})$ has a basis $\{1, c_1, c_1^2, \ldots, c_1^{p-1}, e\}$ where $c_1$ is the Chern class of the complexification of the line bundle arising from the standard representation of $SO(2)$ and where $e$ is the Euler class of the vector bundle arising from the standard representation of $SO(p)$, [BoH] §16.5. All these classes lift canonically to $IH^*(\tilde{X}; \mathbb{C})$ and $c_1^j$ lifts further to the (ordinary) cohomology of the Baily-Borel compactification. However (except in the case $p = 2$) we do not know whether $e$ also lifts further to $H^*(\tilde{X}; \mathbb{C})$.

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