THE $V$-MONOID OF A WEIGHTED LEAVITT PATH ALGEBRA

BY

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ABSTRACT

We compute the $V$-monoid of a weighted Leavitt path algebra of a row-finite weighted graph, correcting a wrong computation of the $V$-monoid that exists in the literature. Further we show that the description of $K_0$ of a weighted Leavitt path algebra that exists in the literature is correct (although the computation was based on a wrong $V$-monoid description).

1. Introduction

The weighted Leavitt path algebras were introduced by R. Hazrat in [4]. They generalise the Leavitt path algebras. While the Leavitt path algebras only embrace Leavitt’s algebras $L_K(1, 1 + k)$ where $K$ is a field and $k \geq 0$, the weighted Leavitt path algebras embrace all of Leavitt’s algebras $L_K(n, n + k)$ where $K$ is a field, $n \geq 1$ and $k \geq 0$. In [5] linear bases for weighted Leavitt path algebras were obtained. They were used to classify the simple and graded simple weighted Leavitt path algebras and the weighted Leavitt path algebras which are domains. In [6] the Gelfand–Kirillov dimension of a weighted Leavitt path algebra $L_K(E, w)$, where $K$ is a field and $(E, w)$ is a row-finite weighted graph, was determined. Further finite-dimensional weighted Leavitt path algebras were investigated. In [7] locally finite weighted Leavitt path algebras were investigated.

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The $V$-monoid $V(R)$ of an associative, unital ring $R$ is the set of all isomorphism classes of finitely generated projective right $R$-modules, which becomes an abelian monoid by defining

$$[P] + [Q] := [P \oplus Q]$$

for any $[P], [Q] \in V(R)$. It can also be defined using matrices and the definition can be extended to include all associative rings; see Section 4. For an associative ring $R$ with local units, the Grothendieck group $K_0(R)$ is the group completion $V(R)^+$ of $V(R)$ (see [1, p. 77]).

A presentation for $V(L_K(E, w))$, where $K$ is a field and $(E, w)$ is a row-finite weighted graph, was given in [4, Theorem 5.21]. In [5] this presentation was used in order to show that there is a huge class $C$ of weighted Leavitt path algebras that are domains but neither are isomorphic to a Leavitt path algebra nor to a Leavitt algebra. Unfortunately, [4, Theorem 5.21] is wrong as we will show in Section 4. In this paper we correct the false [4, Theorem 5.21]. It turns out that the statement of [4, Theorem 5.21] is true at least for row-finite weighted graphs $(E, w)$ that have the property that for any vertex $v \in E^0$ all the edges in $s^{-1}(v)$ have the same weight. Further, it turns out, surprisingly, that [4, Theorem 5.23], which gives a presentation for the Grothendieck group $K_0(L_K(E, w))$ of a weighted Leavitt path algebra, is correct.

To any row-finite weighted graph $(E, w)$ we associate an abelian monoid $M(E, w)$, which is defined as follows. For each vertex $v \in E^0$ we partition the edges emitted by $v$ by their weights, i.e.,

$$s^{-1}(v) = \bigsqcup_{i=1}^{k_v} A_{w_i(v)}$$

where $w_1(v) < w_2(v) < \cdots < w_{k_v}(v)$ are the weights in the corresponding partition. Then $M(E, w)$ is the abelian monoid presented by the generating set \{\(v, q_{v,1}^v, \ldots, q_{v,k_v-1}^v \mid v \in E^0\}\} and the relations

$$q_{i-1}^v + (w_i(v) - w_{i-1}(v))v = q_i^v + \sum_{e \in A_{w_i(v)}} r(e), \quad (v \in E^0, 1 \leq i \leq k_v)$$

where $q_0^v = q_{k_v}^v = w_0(v) = 0; \text{ cf. Definition 11.}$ The main result of this paper is Theorem 14 which implies that

$$V(L_K(E, w)) \cong M(E, w)$$

for any field $K$ and row-finite weighted graph $(E, w)$. 
The rest of this paper is organised as follows. In Section 2 we recall some standard notation which is used throughout the paper. In Section 3 we recall the definition of a weighted Leavitt path algebra. In Section 4 we prove our main result, Theorem 14, and show that the description of $K_0(L_K(E,w))$ obtained in [4] is correct. Moreover, we show that there is a class $D$, containing the class $C$ mentioned above, that consists of weighted Leavitt path algebras that are domains but are isomorphic neither to a Leavitt path algebra nor to a Leavitt algebra. In the last section we determine the $V$-monoids of some concrete examples of weighted Leavitt path algebras.

2. Notation

Throughout the paper $K$ denotes a field; $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0$ the set of nonnegative integers. If $m, n \in \mathbb{N}$ and $R$ is a ring, then $M_{m \times n}(R)$ denotes the set of all $m \times n$-matrices whose entries are elements of $R$. Instead of $M_{n \times n}(R)$ we might write $M_n(R)$.

3. Weighted Leavitt path algebras

Definition 1: A directed graph is a quadruple $E = (E^0, E^1, s, r)$ where $E^0$ and $E^1$ are sets and $s, r : E^1 \rightarrow E^0$ maps. The elements of $E^0$ are called vertices and the elements of $E^1$ edges. If $e$ is an edge, then $s(e)$ is called its source and $r(e)$ its range; $E$ is called row-finite if $s^{-1}(v)$ is a finite set for any vertex $v$ and finite if $E^0$ and $E^1$ are finite sets.

Definition 2: A weighted graph is a pair $(E, w)$ where $E$ is a directed graph and $w : E^1 \rightarrow \mathbb{N}$ is a map. If $e \in E^1$, then $w(e)$ is called the weight of $e$; $(E, w)$ is called row-finite (resp. finite) if $E$ is row-finite (resp. finite). In this article all weighted graphs are assumed to be row-finite. For a vertex $v \in E^0$ we set

$$w(v) := \max\{w(e) \mid e \in s^{-1}(v)\}$$

with the convention $\max\emptyset = 0$.

Remark 3: In [4] and [5], $E^1$ was denoted by $E^{st}$. The set

$$\{e_i \mid e \in E^1, 1 \leq i \leq w(e)\}$$

was denoted by $E^1$. 
Definition 4: Let \((E,w)\) be a weighted graph. The associative \(K\)-algebra presented by the generating set \(E^0 \cup \{e_i, e_i^* \mid e \in E^1, 1 \leq i \leq w(e)\}\) and the relations

1. \(uv = \delta_{uv}u\ (u,v \in E^0)\),
2. \(s(e)e_i = e_i = e_ir(e), \ r(e)e_i^* = e_i^* = e_i^*s(e)\ (e \in E^1, 1 \leq i \leq w(e))\),
3. \(\sum_{e \in s^{-1}(v)} e_ie_j^* = \delta_{ij}v\ (v \in E^0, 1 \leq i,j \leq w(v))\) and
4. \(\sum_{1 \leq i \leq w(v)} e_i^*f_i = \delta_{ef}s(e)\ (v \in E^0, e,f \in s^{-1}(v))\)

is called the **weighted Leavitt path algebra of** \((E,w)\) and is denoted by \(L_K(E,w)\). In relations (iii) and (iv), we set \(e_i\) and \(e_i^*\) zero whenever \(i > w(e)\).

Example 5: If \((E,w)\) is a weighted graph such that \(w(e) = 1\) for all \(e \in E^1\), then \(L_K(E,w)\) is isomorphic to the usual Leavitt path algebra \(L_K(E)\).

Example 6: Let \(n \geq 1\) and \(k \geq 0\). Let \((E,w)\) be the weighted graph

\[
\begin{array}{c}
(e^{(n+k)}, n) \\
\vdots \\
(e^{(2)}, n) \\
(e^{(1)}, n) \\
(v) \\
(e^{(3)}, n)
\end{array}
\]

with one vertex \(v\) and \(n+k\) edges \(e^{(1)}, \ldots, e^{(n+k)}\) each of which has weight \(n\). Then \(L_K(E,w)\) is isomorphic to the Leavitt algebra \(L_K(n,n+k)\); for details see [4, Example 5.5] or [5, Example 4].

Remark 7: Let \((E,w)\) be a weighted graph. Then \(L_K(E,w)\) has the properties (a)–(e) below. For details regarding (a)–(d) see [4, Proposition 5.7]; for details regarding (e) see [5, Section 2].

(a) If \((E,w)\) is finite, then \(L_K(E,w)\) is a unital ring (with \(\sum_{v \in E^0} v\) as multiplicative identity).

(b) \(L_K(E,w)\) has a set of local units, namely the set of all finite sums of distinct elements of \(E^0\). Recall that an associative ring \(R\) is said to have a set of local units \(X\) in case \(X\) is a set of idempotents in \(R\) having the property that for each finite subset \(S \subseteq R\) there exists an \(x \in X\) such that \(xsx = s\) for any \(s \in S\).

(c) There is an involution \(*\) on \(L_K(E,w)\) mapping \(k \mapsto k, v \mapsto v, e_i \mapsto e_i^*\) and \(e_i^* \mapsto e_i\) for any \(k \in K, v \in E^0, e \in E^1\) and \(1 \leq i \leq w(e)\). If \(m,n \in \mathbb{N}\), then \(*\) induces a map \(\mathbb{M}_{m,n}(L_K(E,w)) \rightarrow \mathbb{M}_{n,m}(L_K(E,w))\)
mapping a matrix $\sigma$ to the matrix $\sigma^*$ one gets by transposing $\sigma$ and then applying the involution $\ast$ to each entry.

(d) Set $n := \sup\{w(e) \mid e \in E^1\}$. One can define a $\mathbb{Z}^n$-grading on $L_K(E, w)$ by setting $\deg(v) := 0$, $\deg(e_i) := \epsilon_i$ and $\deg(e_i^*) := -\epsilon_i$ for any $v \in E^0$, $e \in E^1$ and $1 \leq i \leq w(e)$. Here $\epsilon_i$ denotes the element of $\mathbb{Z}^n$ whose $i$-th component is 1 and whose other components are 0.

(e) Let $x_1 \ldots x_n$ be a nonempty word over the alphabet $E^0 \cup \{e_i, e_i^* \mid e \in E^1, 1 \leq i \leq w(e)\}$.

Then $x_1 \ldots x_n$ is called a \textbf{generalised path} if either

$x_1, \ldots, x_n \in E^1 \cup (E^1)^*$ and $r(x_i) = s(x_{i+1})$ (1 $\leq i \leq n - 1$)

or

$x_1 \in E^0$ and $n = 1$.

Here we use the convention $s(v) := v$, $r(v) := v$, $s(e_i) := s(e)$, $r(e_i) := r(e)$, $s(e_i^*) := r(e)$ and $r(e_i^*) := s(e)$ for any $v \in E^0$, $e \in E^1$ and $1 \leq i \leq w(e)$.

Now fix, for any $v \in E^0$ such that $s^{-1}(v) \neq \emptyset$, an edge $e^v \in s^{-1}(v)$ such that $w(e^v) = w(v)$. The $e^v$'s are called \textbf{special edges}. A generalised path $x_1 \ldots x_n$ is called \textbf{normal} if it does not contain any of the words $e^v_i(e^v_j)^* \ (v \in E^0, s^{-1}(v) \neq \emptyset, 1 \leq i, j \leq w(v))$ and $e^s_1f_1 (e, f \in E^1)$ as a subword. The images of the normal generalised paths in $L_K(E, w)$ form a linear basis for $L_K(E, w)$.

4. The $V$-monoid of a weighted Leavitt path algebra

Consider the weighted graphs

$(E, w) : u \xleftarrow{(e,2)} v \xrightarrow{(f,2)} x$ and $(E, w') : u \xleftarrow{(e,1)} v \xrightarrow{(f,2)} x$.

Set $L := L_K(E, w)$ and $L' := L_K(E, w')$. According to [4, Theorem 5.21] we should have

$V(L) \cong V(L') \cong \mathbb{N}^{E^0}_0 / \langle 2\alpha_v = \alpha_u + \alpha_x \rangle$

where for any $y \in E^0$, $\alpha_y$ denotes the element of $\mathbb{N}^{E^0}_0$ whose $y$-component is one and whose other components are zero. Hence $V(L')$ is not a refinement monoid. But in [6, Example 46] it was shown that $L' \cong \mathbb{M}_3(K) \oplus \mathbb{M}_3(K)$. Hence $V(L') \cong \mathbb{N}^2_0$ and therefore $V(L')$ is a refinement monoid. In view of this contradiction, is [4, Theorem 5.21] wrong?
First we consider the algebra $L$. By the relations for the generators of a weighted Leavitt path algebra (see Definition 4), the matrix

$$
A := \begin{pmatrix} e_1 & f_1 \\ e_2 & f_2 \end{pmatrix} \in M_2(L)
$$

defines a “universal” (in the sense of “as general as possible”) isomorphism $uL \oplus xL \rightarrow vL \oplus vL$ (by left multiplication). Set

$$
B_0 := K^{E^0} \quad \text{and} \quad B_1 := B_0 \langle i, i^{-1} : \alpha_u B_0 \oplus \alpha_x B_0 \cong \alpha_v B_0 \oplus \alpha_v B_0 \rangle
$$

(see Subsection 4.2 or [3, p. 38]) where for any $y \in E^0$, $\alpha_y$ denotes the element of $B_0$ whose $y$-component is one and whose other components are zero. One checks easily that $L \cong B_1$. It follows from [3, Theorem 5.2] that

$$
V(L) \cong N_0^{E^0} / \langle \alpha_u + \alpha_x = 2 \alpha_v \rangle
$$

and hence [4, Theorem 5.21] yields a correct presentation for $V(L)$.

For the algebra $L'$ the situation is a bit different. By the relations for the generators of a weighted Leavitt path algebra, the matrix

$$
A := \begin{pmatrix} e_1 & f_1 \\ 0 & f_2 \end{pmatrix} \in M_2(L')
$$

defines an isomorphism $uL' \oplus xL' \rightarrow vL' \oplus vL'$, but this isomorphism is not universal since an entry of $A$ is zero. Let $P$ be the image of the map $uL' \rightarrow vL'$ defined by $e_1$ and $Q$ the image of the map $xL' \rightarrow vL'$ defined by $f_1$. One checks easily that $vL' = P \oplus Q$. Further $e_1$ defines an isomorphism $uL' \rightarrow P$ and $(f_1, f_2)$ defines an isomorphism $xL' \rightarrow Q \oplus vL'$. We can describe $L'$ as follows. Set

$$
B_0 := K^{E^0},
B_1 := B_0 \langle \epsilon : \alpha_u B_0 \rightarrow \alpha_v B_0 ; \epsilon^2 = \epsilon \rangle,
B_2 := \langle i, i^{-1} : \alpha_u B_1 \cong \ker \epsilon \rangle \quad \text{and}
B_3 := \langle j, j^{-1} : \alpha_x B_2 \cong \im \epsilon \oplus \alpha_v B_2 \rangle
$$

(see Subsection 4.2 or [3, pp. 38–39]). One checks easily that $L' \cong B_3$ (for details see Theorem 14, Part II). It follows from [3, Theorems 5.1, 5.2] that

$$
V(L') \cong N_0^{E^0 \cup \{p, q\}} / \langle \alpha_p + \alpha_q = \alpha_v, \alpha_u = \alpha_p, \alpha_x = \alpha_q + \alpha_v \rangle \cong N_0^2.
$$

Thus [4, Theorem 5.21] indeed is wrong.
In this section we repair [4, Theorem 5.21]. Further we show that [4, Theorem 5.23], which gives a presentation for $K_0(L_K(E, w))$ where $(E, w)$ is any weighted graph, is correct. In particular, $K_0(L) \cong K_0(L')$ where $L$ and $L'$ are the weighted Leavitt path algebras defined above, while $V(L) \not\cong V(L')$.

We denote by $G^w$ the category whose objects are all weighted graphs and whose morphisms are the complete weighted graph homomorphisms between weighted graphs (see [4, p. 884]). Further, we denote the category of associative $K$-algebras by $A_K$ and the category of abelian monoids by $M^{ab}$. In Subsection 4.1 we define three functors, $L_K : G^w \to A_K$, $V : A_K \to M^{ab}$ and $M : G^w \to M^{ab}$. In Subsection 4.2 we recall some universal ring constructions by G. Bergman which will be used in the proof of our main result, Theorem 14. In Subsection 4.3 we prove Theorem 14 which states that $V \circ L_K \cong M$ and that $L_K(E, w)$ is left and right hereditary provided $(E, w)$ is finite. In Subsection 4.4 we compute the Grothendieck group of a weighted Leavitt path algebra.

4.1. The functors $L_K$, $V$ and $M$.

Definition 8: In Definition 4 we associated to any weighted graph $(E, w)$ an associative $K$-algebra $L_K(E, w)$. If $\phi : (E, w) \to (E', w')$ is a morphism in $G^w$, then there is a unique $K$-algebra homomorphism $L_K(\phi) : L_K(E, w) \to L_K(E', w')$ such that $L_K(\phi)(v) = \phi^0(v)$, $L_K(\phi)(e_i) = (\phi^1(e))_i$ and $L_K(\phi)(e^*_i) = (\phi^1(e))^*_i$ for any $v \in E^0$, $e \in E^1$ and $1 \leq i \leq w(e)$. One checks easily that $L_K : G^w \to A_K$ is a functor that commutes with direct limits.

Definition 9: Let $A$ be an associative $K$-algebra. Let $M_\infty(A)$ be the directed union of the rings $M_n(A)$ ($n \in \mathbb{N}$), where the transition maps $M_n(A) \to M_{n+1}(A)$ are given by

$$x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}.$$ 

Let $I(M_\infty(A))$ denote the set of all idempotent elements of $M_\infty(A)$. If $e, f \in I(M_\infty(A))$, write $e \sim f$ iff there are $x, y \in M_\infty(A)$ such that $e = xy$ and $f = yx$. Then $\sim$ is an equivalence relation on $I(M_\infty(A))$. Let $V(A)$ be the set of all $\sim$-equivalence classes, which becomes an abelian monoid by defining

$$[e] + [f] = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}.$$
for any $[e], [f] \in V(A)$. If $\phi : A \to B$ is a morphism in $A_K$, let $V(\phi) : V(A) \to V(B)$ be the canonical monoid homomorphism induced by $\phi$. One checks easily that $V : A_K \to M^{ab}$ is a functor that commutes with direct limits.

**Remark 10:** Let $A$ be an associative, unital $K$-algebra. Let $V'(A)$ denote the set of isomorphism classes of finitely generated projective right $A$-modules, which becomes an abelian monoid by defining

$$[P] + [Q] := [P \oplus Q]$$

for any $[P], [Q] \in V'(A)$. Then $V'(A) \cong V(A)$ as abelian monoids, see [1, Definition 3.2.1].

**Definition 11:** Let $(E, w)$ be a weighted graph. For any $v \in E^0$ write

$$w(s^{-1}(v)) = \{w_1(v), \ldots, w_{k_v}(v)\}$$

where $k_v \geq 0$ and $w_1(v) < \cdots < w_{k_v}(v)$ (hence $k_v$ is the number of different weights of the edges in $s^{-1}(v)$). Further, set $w_0(v) := 0$ for any $v \in E^0$ (note that with this convention one has $w_{k_v}(v) = w(v)$ for any $v \in E^0$). Let $M(E, w)$ be the abelian monoid presented by the generating set

$$\{v, q^{v}_1, \ldots, q^{v}_{k_v-1} \mid v \in E^0\}$$

and the relations

$$q^{v}_{i-1} + (w_i(v) - w_{i-1}(v))v = q^{v}_i + \sum_{e \in s^{-1}(v), w(e) = w_i(v)} r(e) \quad (v \in E^0, 1 \leq i \leq k_v)$$

where $q^{v}_0 = q^{v}_{k_v} = 0$. If $\phi : (E, w) \to (E', w')$ is a morphism in $G^w$, then there is a unique monoid homomorphism $M(\phi) : M(E, w) \to M(E', w')$ such that

$$M(\phi)([v]) = [\phi^0(v)] \quad \text{and} \quad M(\phi)([q^v_i]) = [q^{\phi^0}_i(v)]$$

for any $v \in E^0$ and $1 \leq i \leq k_v - 1$. One checks easily that $M : G^w \to M^{ab}$ is a functor that commutes with direct limits.

**Remark 12:** If $k_v \leq 1$ for any $v \in E^0$, then $M(E, w)$ is the abelian monoid $M_E$ defined in [4, Theorem 5.21].
4.2. SOME UNIVERSAL RING CONSTRUCTIONS BY G. BERGMAN. In this subsection all rings are assumed to be associative and unital. Let $k$ be a commutative ring and $R$ a $k$-algebra (i.e., $R$ is a ring given with a homomorphism of $k$ into its center). A $R$-ring$_k$ is a $k$-algebra $S$ given with a $k$-algebra homomorphism $R \to S$. In [3], G. Bergman described the following two key constructions:

- **ADJOINING MAPS.** Let $M$ be any $R$-module and $P$ a finitely generated projective $R$-module. Then there exists $R$-ring$_k$ $S$, having a universal module homomorphism $f : M \otimes S \to P \otimes S$, see [3, Theorem 3.1]. $S$ can be obtained by adjoining to $R$ a family of generators subject to certain relations; see [3, Proof of Theorem 3.1].

- **IMPOSING RELATIONS.** Let $M$ be any $R$-module, $P$ a projective $R$-module and $f : M \to P$ any module homomorphism. Then there exists an $R$-ring$_k$ $S$ such that $f \otimes S = 0$ and universal for that property. $S$ can be chosen to be a quotient of $R$; see [3, Proof of Theorem 3.2].

Using the above key constructions Bergman described more complicated constructions. Two of them are used in this paper:

- **ADJOINING ISOMORPHISMS.** Given two finitely generated projective $R$-modules $P$ and $Q$, one can adjoin a universal isomorphism between $P \otimes$ and $Q \otimes$ by first freely adjoining maps

  \[ i : P \otimes \to Q \otimes \quad \text{and} \quad i^{-1} : Q \otimes \to P \otimes \]

  (via adjoining maps) and then setting $1_{Q \otimes} - ii^{-1}$ and $1_{P \otimes} - i^{-1}i$ equal to 0 (via imposing relations); see [3, p. 38]. Bergman denoted the resulting $R$-ring$_k$ by

  \[ R\langle i, i^{-1} : \mathcal{P} \cong \mathcal{Q} \rangle. \]

- **ADJOINING IDEMPOTENT ENDOMORPHISMS.** Given a finitely generated projective $R$-module $P$, one can adjoin a universal idempotent endomorphism of $P \otimes$ by first freely adjoining a map $e : P \otimes \to P \otimes$ (via adjoining maps) and then setting $e - e^2$ equal to 0 (via imposing relations); see [3, p. 39]. Bergman denoted the resulting $R$-ring$_k$ by

  \[ R\langle e : \mathcal{P} \to \mathcal{P}; e^2 = e \rangle. \]

Note that the adjoined idempotent endomorphism $e$ yields a universal direct sum decomposition $P \otimes = \ker(e) \oplus \operatorname{im}(e)$. 

Set
\[ S := R\langle i, i^{-1} : \overline{P} \cong \overline{Q} \rangle \quad \text{and} \quad T := R\langle e : \overline{P} \to \overline{P}; e^2 = e \rangle. \]

Bergman proved the following (for these results he required that \( k \) is a field and that \( P \) and \( Q \) are nonzero): The abelian monoid \( V'(S) \) (see Remark 10) may be obtained from \( V'(R) \) by imposing one relation \([P] = [Q]\). The abelian monoid \( V'(T) \) may be obtained from \( V'(R) \) by adjoining two new generators \([P_1] \) and \([P_2] \) and one relation \([P_1] + [P_2] = [P] \). Further, the right global dimension of \( S \) (resp. \( T \)) equals the right global dimension of \( R \), unless the right global dimension of \( R \) is 0, in which case the right global dimension of \( S \) (resp. \( T \)) is \( \leq 1 \). See [3, Theorems 5.1, 5.2 and the last paragraph of p. 48].

4.3. THE MAIN RESULT. The lemma below will be used in the proof of Theorem 14.

**Lemma 13:** Let \( G \) be an abelian group (resp. an abelian monoid) presented by a generating set \( X \) and relations
\[ l_i = r_i \ (i \in I) \quad \text{and} \quad y = \sum_{x \in X \setminus \{y\}} n_x x, \]
where for any \( i \in I \), \( l_i \) and \( r_i \) are elements of the free abelian group (resp. the free abelian monoid) \( G\langle X \rangle \) generated by \( X \), \( y \) is an element of \( X \), the \( n_x \) are integers (resp. nonnegative integers) and only a finite number of them are nonzero. Let \( G\langle X \setminus \{y\} \rangle \) be the free abelian group (resp. the free abelian monoid) generated by \( X \setminus \{y\} \) and \( f : G\langle X \rangle \to G\langle X \setminus \{y\} \rangle \) the homomorphism which maps each \( x \in X \setminus \{y\} \) to \( x \) and \( y \) to \( \sum_{x \in X \setminus \{y\}} n_x x \). Then \( G \) is also presented by the generating set \( X \setminus \{y\} \) and the relations
\[ f(l_i) = f(r_i) \quad (i \in I). \]

**Proof.** Straightforward. \( \blacksquare \)

**Theorem 14:** \( V \circ L_K \cong M. \) Moreover, if \((E, w)\) is finite, then \( L_K(E, w) \) is left and right hereditary.

**Proof.** We have divided the proof into two parts, Part I and Part II. In Part I we define a natural transformation \( \theta : M \to V \circ L_K \). In Part II we show that \( \theta \) is a natural isomorphism and further that \( L_K(E, w) \) is left and right hereditary provided that \((E, w)\) is finite.
PART I. Let \((E, w)\) be a weighted graph. Let \(v \in E^0\) be a vertex that emits edges (i.e., \(s^{-1}(v) \neq \emptyset\)). Write

\[
s^{-1}(v) = \{e^{1,v}, \ldots, e^{n(v),v}\}
\]

where \(w(e^{1,v}) \leq \cdots \leq w(e^{n(v),v})\). Let \(A = A(v) \in \mathbb{M}_{w(v) \times n(v)}(L_K(E, w))\) be the matrix whose entry at position \((i, j)\) is \(e^{j,v}_i\) (we set \(e^{j,v}_i := 0\) if \(i > w(e^{j,v})\)). By relations (iii) and (iv) in Definition 4 we have that

\[
(2) \quad AA^* = \begin{pmatrix} v & \cdots & v \end{pmatrix} \in \mathbb{M}_{w(v)}(L_K(E, w))
\]

and

\[
(3) \quad A^*A = \begin{pmatrix} r(e^{1,v}) & \cdots & r(e^{n(v),v}) \end{pmatrix} \in \mathbb{M}_{n(v)}(L_K(E, w)).
\]

As in Definition 11, set \(w_0(v) := 0\) and write \(w(s^{-1}(v)) = \{w_1(v), \ldots, w_{k_v}(v)\}\) where \(w_1(v) < \cdots < w_{k_v}(v)\). For any \(0 \leq l \leq k_v\) set

\[
n_l(v) := |s^{-1}(v) \cap w^{-1}(\{w_0(v), \ldots, w_l(v)\})|\]

(note that \(n_0(v) = 0\) and \(n_{k_v}(v) = n(v)\)). For \(0 \leq l < t \leq k_v\) and \(0 \leq l' < t' \leq k_v\) let \(A^{n_{l'}, n_{l'}}_{w_l, w_{l'}} = A^{n_{l'}, n_{l'}}_{w_l, w_{l'}}(v) \in \mathbb{M}_{(w_t(v)-w_l(v)) \times (n_{t'}(v)-n_{l'}(v))}(L_K(E, w))\) be the matrix whose entry at position \((i, j)\) is \(e^{n_{l'}(v)+j,v}_{w_l(v)+i}\) (for any \(1 \leq i \leq w_t(v)-w_l(v)\) and \(1 \leq j \leq n_{t'}(v) - n_{l'}(v)\)). Then \(A\) has the block form

\[
A = \begin{pmatrix}
A^{n_{0}, n_{1}}_{w_0, w_1} & A^{n_{1}, n_{2}}_{w_0, w_1} & \cdots & A^{n_{k_v-1}, n_{k_v}}_{w_0, w_1} \\
0 & A^{n_{1}, n_{2}}_{w_0, w_1} & \cdots & A^{n_{k_v-1}, n_{k_v}}_{w_0, w_1} \\
0 & 0 & \ddots & \vdots \\
0 & 0 & 0 & A^{n_{k_v-1}, n_{k_v}}_{w_0, w_1}
\end{pmatrix}
\]

and \(A^*\) has the block form

\[
A^* = \begin{pmatrix}
(A^{n_{0}, n_{1}}_{w_0, w_1})^* & 0 & 0 & 0 \\
(A^{n_{1}, n_{2}}_{w_0, w_1})^* & (A^{n_{1}, n_{2}}_{w_0, w_1})^* & 0 & 0 \\
\vdots & \vdots & \ddots & 0 \\
(A^{n_{k_v-1}, n_{k_v}}_{w_0, w_1})^* & (A^{n_{k_v-1}, n_{k_v}}_{w_0, w_1})^* & \cdots & (A^{n_{k_v-1}, n_{k_v}}_{w_0, w_1})^*
\end{pmatrix}.
\]
For any $1 \leq l \leq k_v - 1$ set

$$
\epsilon_l = \epsilon_l(v) := A_{w_0,w_l}^{n_0,n_l} (A_{w_0,w_l}^{n_0,n_l})^* \in \mathbb{M}_{w_l(v)}(L_K(E,w)).
$$

It follows from equation (2) that

$$
(4) \quad \epsilon_l = \begin{pmatrix} v & \cdots & \cdots & v \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & \vdots \\ v & \cdots & \cdots & v \end{pmatrix} - A_{w_0,w_l}^{n_0,n_l} (A_{w_0,w_l}^{n_0,n_l})^*.
$$

By equation (3) we have

$$
(5) \quad (A_{w_0,w_l}^{n_0,n_l})^* A_{w_0,w_l}^{n_0,n_l} = \begin{pmatrix} r(e^{1,v}) & \cdots & \cdots & r(e^{n_l(v),v}) \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & \vdots \\ r(e^{n_l(v),v}) & \cdots & \cdots & r(e^{n_l(v),v}) \end{pmatrix}.
$$

Equations (4) and (5) imply that $\epsilon_l$ is an idempotent matrix for any $1 \leq l \leq k_v - 1$.

Let $F$ be the free abelian monoid generated by the set 

$$\{v, q_0^v, \ldots, q_{k_v-1}^v \mid v \in E_0^0\}.$$ 

There is a unique monoid homomorphism $\psi : F \to V(L_K(E,w))$ such that

$$
\psi(v) = [(v)] \quad \text{and} \quad \psi(q_i^v) = [\epsilon_l(v)]
$$

for any $v \in E_0$ and $1 \leq l \leq k_v - 1$. In order to show that $\psi$ induces a monoid homomorphism $M(E,w) \to V(L_K(E,w))$ we have to check that $\psi$ preserves the relations (1), i.e.,

$$
\psi(q_{l-1}^v + (w_l(v) - w_{l-1}(v))) = \psi(q_l^v + \sum_{e \in s^{-1}(v), w(e) = w_l(v)} r(e))
$$

(6) \quad \begin{bmatrix} \epsilon_{l-1}(v) & \cdots & \cdots & \cdots & \epsilon_{1}(v) & \cdots & \cdots & \cdots & \epsilon_{l}(v) \\ v & \cdots & \cdots & \cdots & r(e^{n_l-1(v)+1,v}) & \cdots & \cdots & \cdots & r(e^{n_l(v),v}) \end{bmatrix}

for any $v \in E_0^0$ and $1 \leq l \leq k_v$ (where $\epsilon_0(v)$ and $\epsilon_{k_v}(v)$ are the empty matrix). Set

$$
X_l(v) := \begin{pmatrix} \epsilon_l(v) & A_{w_0,w_l}^{n_{l-1},n_l}(v) \end{pmatrix} \quad \text{and} \quad Y_l(v) := (X_l(v))^* = \begin{pmatrix} \epsilon_l(v) \\ (A_{w_0,w_l}^{n_{l-1},n_l}(v))^* \end{pmatrix}.
$$
Clearly

\[ X_l(v)Y_l(v) = \epsilon_l(v) + A_{w_0, w_l}^{n_l-1, n_l}(v)(A_{w_0, w_l}^{n_l-1, n_l}(v))^*. \]

Writing \( A_{w_0, w_l}^{n_0, n_l}(v) \) in block form

\[ A_{w_0, w_l}^{n_0, n_l}(v) = \begin{pmatrix} A_{w_0, w_l-1}^{n_0, n_l-1}(v) & A_{w_0, w_l-1}^{n_l-1, n_l}(v) \\ 0 & A_{w_l-1, w_l}^{n_l-1, n_l}(v) \end{pmatrix} \]

it is easy to deduce from equation (4) that

\[ \epsilon_l(v) = \begin{pmatrix} \epsilon_{l-1}(v) & v & \cdots & v \\ & v & \cdots & v \\ & & \ddots & \vdots \\ & & & v \end{pmatrix} = A_{w_0, w_l}^{n_l-1, n_l}(v)(A_{w_0, w_l}^{n_l-1, n_l}(v))^*. \]

By equations (7) and (8) we have

\[ X_l(v)Y_l(v) = \begin{pmatrix} \epsilon_{l-1}(v) & v & \cdots & v \\ & v & \cdots & v \\ & & \ddots & \vdots \\ & & & v \end{pmatrix}. \]

On the other hand, one checks easily that

\[ Y_l(v)X_l(v) = \begin{pmatrix} \epsilon_l(v) & r(e^{n_l-1(v)}, 1, v) & \cdots & r(e^{n_l(v)}, v) \\ & & & \cdots \\ & & & \end{pmatrix} \]

(note that \((A_{w_0, w_l}^{n_l, n_{k_v}}(v))^* A_{w_0, w_l}^{n_l-1, n_l}(v) = 0\) by equation (3), hence \(\epsilon_l(v)A_{w_0, w_l}^{n_l-1, n_l}(v) = 0\) and \((A_{w_0, w_l}^{n_l-1, n_l}(v))^* \epsilon_l(v) = 0\)). Thus equation (6) holds for any \(v \in E^0\) and \(1 \leq l \leq k_v\) and therefore \(\psi\) induces a monoid homomorphism

\[ \theta_{(E, w)} : M(E, w) \to V(L_K(E, w)). \]

It is an easy exercise to show that \(\theta : M \to V \circ L_K\) is a natural transformation (note that for \(v \in E^0\) and \(1 \leq l \leq k_v - 1\), the matrix \(\epsilon_l(v)\) does not depend on the weight-respecting order of the elements of \(s^{-1}(v)\) chosen in the second line of Part I).
Part II. We want to show that the natural transformation \( \theta : M \to V \circ L_K \) defined in Part I is a natural isomorphism, i.e., that \( \theta_{(E, w)} : M(E, w) \to V(L_K(E, w)) \) is an isomorphism for any weighted graph \((E, w)\). By [4, Lemma 5.19] any weighted graph is a direct limit of a direct system of finite weighted graphs. Hence it is sufficient to show that \( \theta_{(E, w)} \) is an isomorphism for any finite weighted graph \((E, w)\) (note that \(M, V\) and \(L_K\) commute with direct limits).

Let \((E, w)\) be a finite weighted graph. Set

\[
B_0 := K^{E^0}.
\]

We denote by \(\alpha_v\) the element of \(B_0\) whose \(v\)-component is 1 and whose other components are 0. Let \(\{v_1, \ldots, v_m\}\) be the elements of \(E^0\) which emit vertices. Let \(1 \leq t \leq m\) and assume that \(B_{t-1}\) has already been defined. We define an associative \(K\)-algebra \(B_t\) as follows. Set

\[
C_{t,0} := B_{t-1}
\]

and let \(\beta^{t,0} : C_{t,0} \to C_{t,0}\) be the map sending any element to 0. For \(1 \leq l \leq k_{v_t} - 1\) define inductively

\[
C_{t,l} := C_{t,l-1} \langle \beta^{t,l} : \overline{O_{t,l}} \to \overline{O_{t,l}}; (\beta^{t,l})^2 = \beta^{t,l} \rangle
\]

(see Subsection 4.2 or [3, p. 39]) where

\[O_{t,l} = \text{im}(\beta^{t,l-1}) \oplus \bigoplus_{h=w_{l-1}(v_t)+1} \alpha_{v_t} C_{t,l-1}.
\]

Set

\[
D_{t,0} := C_{t,k_{v_t} - 1}.
\]

For \(1 \leq l \leq k_{v_t} - 1\) define inductively

\[
D_{t,l} := D_{t,l-1} \langle \gamma^{t,l} : \overline{P_{t,l}} \cong \overline{Q_{t,l}} \rangle
\]

(see Subsection 4.2 or [3, p. 38]) where

\[P_{t,l} = \bigoplus_{h=n_{l-1}(v_t)+1} \alpha_{r(e^h, v_t)} D_{t,l-1} \quad \text{and} \quad Q_{t,l} = \ker(\beta^{t,l}).
\]

Finally define

\[
B_t := D_{t,k_{v_t} - 1} \langle \gamma^{t,k_{v_t}} : \overline{P_{t,k_{v_t}}} \cong \overline{Q_{t,k_{v_t}}} \rangle.
\]
where

\[ P_{t,l} = \bigoplus_{h=n_{k_v-1}(v_t)+1}^{n_{k_v}(v_t)} \alpha_{r(e^h,v_t)} D_{t,k_{v_t}-1} \]

and

\[ Q_{t,k_{v_t}} = \text{im}(\beta_{t,k_{v_t}-1}) \oplus \bigoplus_{h=n_{k_v-1}(v_t)+1}^{w_{k_v}(v_t)} \alpha_{v_{t}} D_{t,k_{v_t}-1}. \]

We will show that

\[ L_K(E, w) \cong B_m. \]

Investigating the proofs of [3, Theorems 3.1, 3.2] we see that \( B_m \) is presented by the generating set

\[ X := \{ \alpha_v \mid v \in E_0 \} \]

\[ \cup \{ \beta_{t,l}^{i,j} \mid 1 \leq t \leq m, 1 \leq l \leq k_{v_t} - 1, 1 \leq i, j \leq w_{l}(v_t) \} \]

\[ \cup \{ \gamma_{t,l}^{i,j}, (\gamma_{t,l}^{i,j})^* \mid 1 \leq t \leq m, 1 \leq l \leq k_{v_t}, \]

\[ 1 \leq i \leq w_{l}(v_t), 1 \leq j \leq n_{l}(v_t) - n_{l-1}(v_t) \} \]

and the relations

(i) \( \alpha_u \alpha_v = \delta_{uv} \alpha_u \) (\( u, v \in E^0 \)),

(ii) \( \text{id}_{O_{t,l}} \beta_{t,l}^{i,j} = \beta_{t,l}^{i,j} \text{id}_{O_{t,l}} \) (\( 1 \leq t \leq m, 1 \leq l \leq k_{v_t} - 1 \)),

(iii) \( (\beta_{t,l})^2 = \beta_{t,l} \) (\( 1 \leq t \leq m, 1 \leq l \leq k_{v_t} - 1 \)),

(iv) \( \gamma_{t,l}^{i,j} \text{id}_{P_{t,l}} = \gamma_{t,l}^{i,j} = \text{id}_{Q_{t,l}} \gamma_{t,l}^{i,j} \) (\( 1 \leq t \leq m, 1 \leq l \leq k_{v_t} \)),

(v) \( (\gamma_{t,l}^{i,j})^* \text{id}_{Q_{t,l}} = (\gamma_{t,l}^{i,j})^* = \text{id}_{P_{t,l}} (\gamma_{t,l}^{i,j})^* \) (\( 1 \leq t \leq m, 1 \leq l \leq k_{v_t} \)),

(vi) \( (\gamma_{t,l}^{i,j})^* \gamma_{t,l}^{i,j} = \text{id}_{Q_{t,l}} \) (\( 1 \leq t \leq m, 1 \leq l \leq k_{v_t} \)),

(vii) \( (\gamma_{t,l}^{i,j})^* \gamma_{t,l}^{i,j} = \text{id}_{P_{t,l}} \) (\( 1 \leq t \leq m, 1 \leq l \leq k_{v_t} \)),

where \( \beta_{t,l}^{i,j} \in M_{w_{l}(v_t)}(K\langle X \rangle) \) (we denote by \( K\langle X \rangle \) the free associative \( K \)-algebra generated by \( X \)) is the matrix whose entry at position \( (i,j) \) is \( \beta_{i,j}^{t,l} \),

\[ \gamma_{t,l}^{i,j} \in M_{w_{l}(v_t) \times (n_{l}(v_t) - n_{l-1}(v_t))}(K\langle X \rangle) \]

is the matrix whose entry at position \( (i,j) \) is \( \gamma_{i,j}^{t,l} \),

\[ (\gamma_{t,l}^{i,j})^* \in M_{(n_{l}(v_t) - n_{l-1}(v_t)) \times w_{l}(v_t)}(K\langle X \rangle) \]
is the matrix whose entry at position \((i, j)\) is \((\gamma_{j,i}^t)^*\), and further

\[
\begin{pmatrix}
\beta_{t,l-1} & \alpha_v \v_t \\
& \ddots \\
& & \alpha_v \\
\end{pmatrix}
\in M_{w_1(v_t)}(K\langle X \rangle),
\]

\[
\begin{pmatrix}
\alpha_{r(e^{n_{l-1}(v_t)+1,v_t})} & \alpha_{r(e^{n_{l}(v_t),v_t})} \\
& \ddots \\
& & \ddots \\
\end{pmatrix}
\in M_{n_1(v_t)-n_{l-1}(v_t)}(K\langle X \rangle),
\]

\[
id_{Q_t,l} = \id_{O_t,l} - \beta_{t,l} \in M_{w_1(v_t)}(K\langle X \rangle)
\text{ if } l < k_{v_t}
\]

(we let \(\beta_{t,0}\) be the empty matrix). Define a \(K\)-algebra homomorphism \(\zeta : L_K(E, w) \to B_m\) by

\[
\zeta(v) = \alpha_v \quad (v \in E^0), \quad \zeta(A^{n_{l-1},n_l}_{w_0,w_l}(v_t)) = \gamma_{l,l}^t,
\]

\[
\zeta((A^{n_{l-1},n_l}_{w_0,w_l}(v_t))^*) = ((\gamma_{l,l}^t)^*)^* \quad (1 \leq t \leq m, 1 \leq l \leq k_{v_t})
\]

(meaning that each entry of \(A^{n_{l-1},n_l}_{w_0,w_l}(v_t)\) (resp. \((A^{n_{l-1},n_l}_{w_0,w_l}(v_t))^*)\) is mapped to the corresponding entry of \(\gamma_{l,l}^t\) (resp. \((\gamma_{l,l}^t)^*)\)). Define an \(K\)-algebra homomorphism \(\xi : B_m \to L_K(E, w)\) by

\[
\xi(\alpha_v) = v \quad (v \in E^0), \quad \xi(\beta_{t,l}) = \epsilon_l(v_t) \quad (1 \leq t \leq m, 1 \leq l \leq k_{v_t} - 1)
\]

\[
\xi(\gamma_{l,l}^t) = A^{n_{l-1},n_l}_{w_0,w_l}(v_t), \quad \xi((\gamma_{l,l}^t)^*) = (A^{n_{l-1},n_l}_{w_0,w_l}(v_t))^* \quad (1 \leq t \leq m, 1 \leq l \leq k_{v_t}).
\]

We leave it to the reader to show that \(\zeta\) and \(\xi\) are well-defined and also

\[
\xi \circ \zeta = \id_{L_K(E, w)} \quad \text{and} \quad \zeta \circ \xi = \id_{B_m}
\]

(a hint: in order to show that \(\zeta(\xi(\beta_{t,l})) = \beta_{t,l}\), it is convenient to use equation (8) and relation (vi) above). Thus \(L_K(E, w) \cong B_m\).
By [3, Theorems 5.1, 5.2], the abelian monoid \( V'(B_m) \) (see Remark 10) is presented by the generating set
\[
\{ v, p^v_1, \ldots, p^v_{k_v-1}, q^v_1, \ldots, q^v_{k_v-1} \mid v \in E^0 \}
\]
and the relations
\[
\begin{align*}
(i) \quad & q^v_{i-1} + (w_i(v) - w_{i-1}(v))v = q^v_i + p^v_i \quad (v \in E^0, 1 \leq i \leq k_v - 1), \\
(ii) \quad & p^v_i = \sum_{e \in s^{-1}(v), \, w(e) = w_i(v)} r(e) \quad (v \in E^0, 1 \leq i \leq k_v - 1) \text{ and} \\
(iii) \quad & q^v_{k_v-1} + (w_{k_v}(v) - w_{k_v-1}(v))v = \sum_{e \in s^{-1}(v), \, w(e) = w_{k_v}(v)} r(e) \quad (v \in E^0)
\end{align*}
\]
where \( q^v_0 = 0 \). It follows from Lemma 13 that
\[
M(E, w) \cong V'(B_m) \cong V'(L_K(E, w)) \cong V(L_K(E, w)).
\]
One checks easily that the monoid isomorphism \( M(E, w) \rightarrow V(L_K(E, w)) \) one gets in this way is precisely \( \theta_{(E, w)} \).

Furthermore, the right global dimension of \( B_m \cong L_K(E, w) \) is \( \leq 1 \) by [3, Theorems 5.1, 5.2], i.e., \( L_K(E, w) \) is right hereditary. Since \( L_K(E, w) \) is a ring with involution, we have
\[
L_K(E, w) \cong L_K(E, w)^{op}.
\]
Thus \( L_K(E, w) \) is also left hereditary.

**COROLLARY 15:** Let \( (E, w) \) be a weighted graph. If there is a vertex \( v \in E^0 \) such that \( k_v > 1 \) (i.e., there are \( e, f \in s^{-1}(v) \) such that \( w(e) \neq w(f) \)), then
\[
|V(L_K(E, w))| = \infty.
\]

**Proof.** Let \( F \) denote the free abelian monoid generated by the set
\[
\{ v, q^v_1, \ldots, q^v_{k_v-1} \mid v \in E^0 \}
\]
and \( \sim \) the congruence on \( F \) defined by the relations (1) in Definition 11. Let \( v \in E^0 \) be a vertex such that \( k_v > 1 \). For any \( n \in \mathbb{N}_0 \) let \( [nq^v_1] \) denote the \( \sim \)-congruence class of \( nq^v_1 \). In \( F \), one cannot write \( nq^v_1 \) as \( x + y \) where \( x \in F \) and \( y \) is the LHS or RHS of one of the relations (1) (note that in the LHS as well as in the RHS of each of the relations (1) a nonempty sum of vertices appears). Hence \( [nq^v_1] = \{ nq^v_1 \} \) for any \( n \in \mathbb{N}_0 \) (i.e., each \( nq^v_1 \) is only congruent to itself). Therefore the elements \( [nq^v_1] \) \( (n \in \mathbb{N}_0) \) are pairwise distinct in \( M(E, w) \) (and thus we have an embedding \( \mathbb{N}_0 \hookrightarrow M(E, w) \) defined by \( n \mapsto [nq^v_1] \)). It follows from Theorem 14 that \( |V(L_K(E, w))| = \infty \). \( \blacksquare \)
In [5, Section 4] it was “proved” by using the false [4, Theorem 5.21] that if 
$(E,w)$ is an LV-rose (see [5, Definition 38]) such that the minimal weight is 2, 
the maximal weight is $l \geq 3$ and the number of edges is $l + m$ for some $m > 0$, 
then the domain $L_K(E,w)$ is not isomorphic to any of the Leavitt algebras $L_K(n,n+k)$ where $n,k \geq 1$. Using Theorem 14 we prove a stronger statement:

**Corollary 16:** Let $(E,w)$ be an LV-rose such that there are edges of different 
weights. Then $L_K(E,w)$ is a domain that is $K$-algebra isomorphic neither to a 
Leavitt path algebra $L_K(F)$ nor to a Leavitt algebra $L_K(n,n+k)$.

**Proof.** First we show that $L_K(E,w)$ is not isomorphic to a Leavitt path algebra. 
By [5, Theorem 41], $L_K(E,w)$ is a domain (i.e., a nonzero ring without zero 
divisors). It is well-known that if $F$ is a directed graph such that $L_K(F)$ is a 
domain, then $F$ is either the graph $\bullet$ and we have $L_K(F) \cong K$, or the graph 
$\bullet \rightarrow$ and we have $L_K(F) \cong K[x,x^{-1}]$. In both cases we have 

$$V(L_K(F)) \cong \mathbb{N}_0$$

by Example 5 and Theorem 14. Assume that there is an isomorphism 
$\phi : \mathbb{N}_0 \to M(E,w)$. One checks easily that if $q_1^n = a + b$ for some $a,b \in M(E,w)$, 
then $a = 0$ and $b = q_1^n$ or vice versa. Hence $\phi(1) = q_1^n$. But then $\phi$ cannot be 
surjective (see the proof of the previous corollary). Hence 

$$V(L_K(E,w)) \not\cong \mathbb{N}_0$$

and therefore $L_K(E,w)$ is not isomorphic to a Leavitt path algebra $L_K(F)$.

Next we show that $L_K(E,w)$ is not isomorphic to a Leavitt algebra $L_K(n,n+k)$ where $n \geq 1$ and $k \geq 0$. It follows from Example 6 and Theorem 14 that 

$$V(L_K(n,n+k)) \cong \mathbb{N}_0/\langle n = n + k \rangle.$$ 

If $k = 0$, then 

$$V(L_K(n,n+k)) \cong \mathbb{N}_0$$

and therefore $L_K(E,w)$ is not isomorphic to $L_K(n,n+k)$ by the previous paragraph. Suppose now that $k \geq 1$. Then $|V(L_K(n,n+k))| = n + k < \infty$. But 
by Corollary 15, $|V(L_K(E,w))| = \infty$. Hence $L_K(E,w)$ is not isomorphic to a 
Leavitt algebra $L_K(n,n+k)$. \[\square\]
4.4. The Grothendieck group. One can use the adjacency matrix and the weighted identity matrix of a weighted graph \((E, w)\) to describe the Grothendieck group \(K_0(L_K(E, w))\). We define those matrices below.

**Definition 17:** Let \((E, w)\) be a weighted graph. The **adjacency matrix** of \((E, w)\) is the matrix \(N \in \mathbb{N}_0^{E^0 \oplus E^0}\) whose entry at position \((u, v)\) is the number of edges from \(u\) to \(v\). The **weighted identity matrix** of \((E, w)\) is the matrix \(I_w \in \mathbb{N}_0^{E^0 \oplus E^0}\) whose entry at position \((u, v)\) is \(w(v)\) if \(u = v\) and 0 otherwise.

Let \((E, w)\) be a weighted graph. Denote the transpose of its adjacency matrix \(N\) by \(N^t\). Multiplying the matrix \(N^t - I_w\) from the left defines a group homomorphism \(\mathbb{Z}^{E_0} \rightarrow \mathbb{Z}^{E_0}\) where \(\mathbb{Z}^{E_0}\) is the direct sum of copies of \(\mathbb{Z}\) indexed by \(E_0\). The theorem below shows that the cokernel of this map is the Grothendieck group of \(L_K(E, w)\).

**Theorem 18:** Let \((E, w)\) be a weighted graph. Then

\[
K_0(L_K(E, w)) \cong \text{coker}(N^t - I_w : \mathbb{Z}^{E_0} \rightarrow \mathbb{Z}^{E_0}).
\]

**Proof.** Since \(L_K(E, w)\) is a ring with local units, \(K_0(L_K(E, w))\) is the group completion \((V(L_K(E, w)))^+\) of the abelian monoid \(V(L_K(E, w))\), see [1, p. 77]. By Theorem 14, \((V(L_K(E, w)))^+ \cong (M(E, w))^+\). It follows from [4, Equation (45)] that the abelian group \((M(E, w))^+\) is presented by the generating set \(\{v, q^v_1, \ldots, q^v_{k_v-1} \mid v \in E^0\}\) and the relations

\[
q^v_{i-1} + (w_i(v) - w_{i-1}(v))v = q^v_i + \sum_{e \in s^{-1}(v), \ w(e)=w_i(v)} r(e) \quad (v \in E^0, 1 \leq i \leq k_v),
\]

where \(q^v_0 = q^v_{k_v} = 0\). We can rewrite the relations above in the form

\[
q^v_i = q^v_{i-1} + (w_i(v) - w_{i-1}(v))v - \sum_{e \in s^{-1}(v), \ w(e)=w_i(v)} r(e) \quad (v \in E^0, 1 \leq i \leq k_v).
\]

By successively applying Lemma 13 we get that \((M(E, w))^+\) is presented by the generating set \(E^0\) and the relations

\[
w(v)v = \sum_{e \in s^{-1}(v)} r(e) \quad (v \in E^0).
\]
Hence \((M(E, w))^+ \cong \mathbb{Z}^{E_0}/H\) where \(H\) is the subgroup of \(\mathbb{Z}^{E_0}\) generated by the set
\[
\left\{ \sum_{e \in s^{-1}(v)} \alpha_v(e) - w(v)\alpha_v \mid v \in E^0 \right\}
\]
(where for a vertex \(v\), \(\alpha_v\) denotes the element of \(\mathbb{Z}^{E_0}\) whose \(v\)-component is 1 and whose other components are 0). One checks easily that \(H\) is the image of the homomorphism \(N^t - I_w : \mathbb{Z}^{E_0} \to \mathbb{Z}^{E_0}\). Thus
\[
K_0(L_K(E, w)) \cong (V(L_K(E, w)))^+ \cong (M(E, w))^+ \cong \mathbb{Z}^{E_0}/H = \text{coker}(N^t - I_w : \mathbb{Z}^{E_0} \to \mathbb{Z}^{E_0}).
\]

Remark 19: Let \(\text{Sink}\) denote the subset of \(E_0\) consisting of all the vertices that emit no edges. Let \(\hat{N}\) and \(\hat{I}_w\) be the matrices one gets by removing from \(N\) resp. \(I_w\) all rows corresponding to vertices in \(\text{Sink}\). Multiplying the matrix \(\hat{N}^t - \hat{I}_w\) from the left defines a homomorphism \(\mathbb{Z}^{E_0\setminus\text{Sink}} \to \mathbb{Z}^{E_0}\). Clearly the image of this map equals the image of the map \(N^t - I_w : \mathbb{Z}^{E_0} \to \mathbb{Z}^{E_0}\) and hence
\[
\text{coker}(\hat{N}^t - \hat{I}_w : \mathbb{Z}^{E_0\setminus\text{Sink}} \to \mathbb{Z}^{E_0}) = \text{coker}(N^t - I_w : \mathbb{Z}^{E_0} \to \mathbb{Z}^{E_0}).
\]
Thus [4, Theorem 5.23] is correct.

5. Examples

Example 20: Consider the weighted graph
\[
(E, w) : u \leftarrow (e,1) v \xrightarrow{(f,2)} x.
\]
As mentioned at the beginning of the previous section,
\[
L_K(E, w) \cong \mathbb{M}_3(K) \oplus \mathbb{M}_3(K).
\]
By Theorem 14 and Lemma 13,
\[
V(L_K(E, w)) \cong N_0^{u,v,q_1^u,x}/\langle \alpha_v = \alpha_{q_1^u} + \alpha_u, \alpha_{q_1^u} + \alpha_v = \alpha_x \rangle \cong N_0^2.
\]
Example 21: Consider the weighted graph

\[
(E, w) : \begin{array}{c}
(e, 1) \\
\circ \ \\
(f, 2) \\
\end{array}
\]

and set \( L := L_K(E, w) \). Set \( u_1 := f_1^* f_1 \) and \( u_2 := f_2^* f_2 \). Then \( u = u_1 + u_2 \) and furthermore \( u_1 \) and \( u_2 \) are orthogonal idempotents. Hence \( Lu = Lu_1 \oplus Lu_2 \).

One checks easily that the maps

\[
\begin{pmatrix} e_1 & f_1 \end{pmatrix} : Lv \to Lu \oplus Lu_1 \quad \text{and} \quad f_2 : Lv \to Lu_2
\]

(both defined by right multiplication) are isomorphisms. Now consider the directed graph

\[
F : \begin{array}{c}
u_1 \\
\circ \\
u \\
\circ \\
u_2, \end{array}
\]

Set \( L' := L_K(F) \). One checks easily that the maps

\[
\begin{pmatrix} a & b & c \end{pmatrix} : L'v \to L'u_1 \oplus L'u_2 \oplus L'u_1 \quad \text{and} \quad d^* : L'v \to L'u_2
\]

are isomorphisms. Since \((E, w)\) and \(F\) seem to encode the same information, the author conjectured that \( L_K(E, w) \cong L_K(F) \). Indeed, there is a \(*\)-isomorphism \( \phi : L_K(E, w) \to L_K(F) \) mapping

\[
u \mapsto u_1 + u_2, \quad v \mapsto v, \quad e_1 \mapsto a + b, \quad f_1 \mapsto c, \quad f_2 \mapsto d^*.
\]

Its inverse \( \phi^{-1} \) maps

\[
u_1 \mapsto f_1^* f_1, \quad u_2 \mapsto f_2^* f_2, \quad v \mapsto v, \quad a \mapsto e_1 f_1^* f_1, \quad b \mapsto e_1 f_2^* f_2, \quad c \mapsto f_1, \quad d \mapsto f_2^*.
\]

By Theorem 14 and Lemma 13,

\[
V(L_K(E, w)) \cong N_0^{\{u, v, q_1^v\}} / \langle \alpha_v = \alpha_{q_1^v}, \alpha_{q_1^u} + \alpha_v = \alpha_u \rangle \cong N_0^2 / \langle (1, 0) = (1, 2) \rangle.
\]

[1, Theorem 3.2.5] (or Theorem 14, which generalises [1, Theorem 3.2.5]) yields the same result for \( V(L_K(F)) \).
Example 22: Consider the weighted graph
\[(E, w) : (e, 1) \circlearrowright v \circlearrowleft (f, 2)\]
By Theorem 14,
\[V(L_K(E, w)) \cong \mathbb{N}_0^{\{v, q_1\}} / \langle \alpha_v = \alpha_{q_1} + \alpha_v, \alpha_{q_1} + \alpha_v = \alpha_v \rangle \cong \mathbb{N}_0^2 / \langle (1, 0) = (1, 1) \rangle.\]
Let \(F\) be the directed graph
\[F : e \circlearrowright u \xrightarrow{f} v.\]
Its Leavitt path algebra \(L_K(F)\) is called the algebraic Toeplitz \(K\)-algebra; see [1, Example 1.3.6]. By [1, Theorem 3.2.5] we have \(V(L_K(F)) \cong V(L_K(E, w))\). But
\[\text{GKdim } L_K(F) = 2\]
by [2, Theorem 5] while \(\text{GKdim } L_K(E, w) = \infty\) by [6, Theorem 23]. Hence
\[L_K(F) \not\cong L_K(E, w).\]

Example 23: Consider the LV-roses
\[(E, w) : (g, 3) \circlearrowleft v \circlearrowright (e, 3) \quad \text{and} \quad (E, w') : (g, 3) \circlearrowleft v \circlearrowright (e, 2).\]
By Theorem 14,
\[V(L_K(E, w)) \cong \mathbb{N}_0^{\{v\}} / \langle 3\alpha_v = 4\alpha_v \rangle\]
and
\[V(L_K(E, w')) \cong \mathbb{N}_0^{\{v, q_1\}} / \langle 2\alpha_v = \alpha_{q_1} + \alpha_v, \alpha_{q_1} + \alpha_v = 3\alpha_v \rangle.\]
Since the image of \(L_K(E, w)\) in \(V(L_K(E, w))\) (resp. the image of \(L_K(E, w')\) in \(V(L_K(E, w'))\)) is \(\alpha_v\) (see Remark 10), the module type of \(L_K(E, w)\) is \((3, 1)\) and the module type of \(L_K(E, w')\) is \((2, 1)\).
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