Lattice study on kaon pion scattering length in the $I = 3/2$ channel

Chuan Miao$^a$, Xining Du$^a$, Guangwei Meng$^a$ and Chuan Liu$^a$

$^a$School of Physics
Peking University
Beijing, 100871, P. R. China

Abstract

Using the tadpole improved Wilson quark action on small, coarse and anisotropic lattices, $K\pi$ scattering length in the $I = 3/2$ channel is calculated within quenched approximation. The results are extrapolated towards the chiral and physical kaon mass region. Finite volume and finite lattice spacing errors are also analyzed and a result in the infinite volume and continuum limit is obtained. Our result is compared with the results obtained using Roy equations, Chiral Perturbation Theory, dispersion relations and the experimental data.

Key words: $K\pi$ scattering length, lattice QCD, improved actions.

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1 Introduction

Low-energy $K\pi$ scattering experiment is an important experimental method in the study of interactions among mesons [1,2,3]. It is also a good testing ground for our understanding of the low-energy structure of Quantum Chromodynamics (QCD). Chiral Perturbation Theory [4,5], Roy equations [6], dispersion relations [7,8] and other theoretical methods [9] have been used in the study of low-energy $K\pi$ scattering. However, if the hadrons being scattered involve strange quarks, as in the case of $K\pi$ scattering, predictions within Chiral perturbation theory usually suffer from sizable corrections due to $SU(3)$ flavor symmetry breaking, as compared with the case of pion-pion scattering. Lattice QCD is a genuine non-perturbative method which can handle hadron-hadron scattering at low-energies. Therefore, a lattice QCD calculation will offer an important and independent check on these results. In fact, pion-pion [10,11,12,13,14,15,16,17,18], pion-nucleon [11] and kaon-nucleon [19] scattering have been studied using lattice QCD. In this letter, we report our
quenched lattice results on the $K\pi$ scattering length in the $I = 3/2$ channel. This quantity is of phenomenological importance for $K\pi$ scattering since the value of the scattering length in the $I = 3/2$ channel is correlated with that in the $I = 1/2$ channel and the value of the latter is crucial for the determination of the possible $\kappa$ resonance in the $I = 1/2$ channel [8].

The lattice gauge action that is used in our study is the tadpole improved clover action on anisotropic lattices [20,21]. Using this gauge action, glueball and hadron spectra have been studied within quenched approximation [20,21,22,23,24,25,26]. In our previous works, configurations generated from this improved action have also been utilized to calculate the $\pi\pi$ scattering lengths in the $I = 2$ channel [13] and the $KN$ scattering length in the $I = 1$ channel [19]. The fermion action used in this study is the tadpole improved clover Wilson action on anisotropic lattices [27,28,13]:

\[
M_{xy} = \delta_{xy}\sigma + A_{xy}
\]

\[
A_{xy} = \delta_{xy}\left[\frac{1}{2\kappa_{\text{max}}} + \rho_t \sum_{i=1}^{3}\sigma_{0i}F_{0i} + \rho_s(\sigma_{12}F_{12} + \sigma_{23}F_{23} + \sigma_{31}F_{31})\right]
- \sum_{\mu}\eta_{\mu}\left[(1 - \gamma_{\mu})U_{\mu}(x)\delta_{x+\mu,y} + (1 + \gamma_{\mu})U_{\mu}^\dagger(x - \mu)\delta_{x-\mu,y}\right],
\]

where various coefficients in the fermion matrix $M$ are given by:

\[
\eta_i = \nu/(2u_s), \quad \eta_0 = \xi/2, \quad \sigma = 1/(2\kappa) - 1/(1\kappa_{\text{max}}), \quad \rho_t = c_{SW}(1 + \xi)/(4u_s^2), \quad \rho_s = c_{SW}/(2u_s^4).
\]

Among the parameters which appear in the fermion matrix, $c_{SW} = 1$ is the tree-level clover coefficient and $\nu$ is the so-called bare velocity of light, which has to be tuned non-perturbatively using the single pion dispersion relations [28]. In the fermion matrix (1), the bare quark mass dependence is singled out into the parameter $\sigma$ and the matrix $A$ remains unchanged if the bare quark mass is varied. This shifted structure of the matrix $M$ can be utilized to solve for quark propagators at various values of valence quark mass $m_0$ (or equivalently $\kappa$) at the cost of solving only the lightest valence quark mass value at $\kappa = \kappa_{\text{max}}$, using the so-called Multi-mass Minimal Residual ($M^3R$ for short) algorithm [29,30,31]. This is particularly advantageous in a quenched calculation since one needs the results at various quark mass values to perform the chiral extrapolation.

The basic procedure for the calculation of $K\pi$ scattering length is similar to that adopted in the $\pi\pi$ scattering length calculation [13].
2 Numerical calculation of the scattering length

In order to calculate the elastic scattering lengths for hadron-hadron scattering on the lattice, or the scattering phase shifts in general, one uses Lüscher’s formula which relates the exact energy level of two hadron states in a finite box to the elastic scattering phase shift in the continuum. In the case of $K\pi$ scattering at zero relative three momentum, this formula amounts to a relation between the exact energy $E^{(I)}_{K\pi}$ of the $K\pi$ system with vanishing relative momentum in a finite box of size $L$ with isospin $I$, and the corresponding scattering length $a^{(I)}_{0}$ in the continuum. This formula reads [32]:

$$E^{(I)}_{K\pi} - (m_K + m_\pi) = \frac{2\pi a^{(I)}_{0}}{\mu_{K\pi}L^3} \left[ 1 + c_1 \frac{a^{(I)}_{0}}{L} + c_2 \left( \frac{a^{(I)}_{0}}{L} \right)^2 \right] + O(L^{-6}) \quad (3)$$

where $c_1 = -2.837297$, $c_2 = 6.375183$ are numerical constants and $\mu_{K\pi} = m_Km_\pi/(m_K + m_\pi)$ is the reduced mass of the $K\pi$ system. In this letter, the $K\pi$ scattering length $a^{(3/2)}_{0}$ in the isospin $I = 3/2$ channel will be calculated.

To measure the hadron mass values $m_\pi$, $m_K$, $m_\rho$ and to extract the energy shift $\delta E^{(3/2)}_{K\pi}$, one constructs the correlation functions from the corresponding one meson and two meson operators in the appropriate symmetry channel. For example, the $K\pi$ operator in the $I = 3/2$ channel is given by:

$$O^{I=3/2}_{K\pi}(t) = K_0^+(t)\pi_0^+(t+1) \quad (4)$$

where $K_0^+(t)$ and $\pi_0^+(t)$ are the zero momentum kaon and pion operators respectively. The $K\pi$ correlation function in the $I = 3/2$ channel then reads:

$$C^{I=1}_{K\pi}(t) = \langle O^{I=3/2\dagger}_{K\pi}(t)O^{I=3/2}_{K\pi}(0) \rangle \quad (5)$$

Numerically, it is more advantageous to construct the ratio of the correlation functions defined above:

$$\mathcal{R}^{I=1}(t) = C^{I=3/2}_{K\pi}(t)/(C_K(t)C_\pi(t)) \quad (6)$$

In this case, one uses the linear fitting function:

$$\mathcal{R}^{I=3/2}(t) \sim > L > 1 - \delta E^{(3/2)}_{K\pi} t \quad (7)$$

to determine the energy shift $\delta E^{(3/2)}_{K\pi}$. $K\pi$ correlation function, or equivalently, the ratio $\mathcal{R}^{I=3/2}(t)$ is constructed from quark propagators, which are obtained
using the Multi-mass Minimal Residue algorithm with wall sources. Periodic boundary condition is applied to all three spatial directions while in the temporal direction, Dirichlet boundary condition is utilized.

Configurations used in this calculation are generated using the pure gauge action for $6^3 40$, $8^3 40$ and $10^3 50$ lattices with the gauge coupling $\beta = 1.9, 2.2, 2.4, 2.6$ and $3.0$. The spatial lattice spacing $a_s$ is roughly between $0.1$fm and $0.4$fm while the spatial physical size of the lattice ranges from $0.7$fm to $4.0$fm. For each set of parameters, several hundred decorrelated gauge field configurations are used to measure the fermionic quantities. Statistical errors are all analyzed using the usual jack-knife method. Single pion, kaon and rho mass values are obtained from the plateau of their corresponding effective mass plots. The fitting interval is automatically chosen by minimal $\chi^2$ per degree of freedom. Due to the usage of finer lattice spacing in the temporal direction, good plateau behavior was observed in these effective mass plots. Therefore, contaminations from excited states should be negligible. These mass values will be utilized in the chiral extrapolation.

After obtaining the energy shifts $\delta E_{K\pi}^{(3/2)}$, these values are substituted into Lüscher’s formula to solve for the scattering length $a_0^{(3/2)}$ for all possible hopping parameter pairs: $(\kappa_u, \kappa_s)$, which corresponds to the up (down) and the strange bare quark mass values, respectively. This is done for lattices of all sizes being simulated and for all values of $\beta$. From these results, attempts are made to perform an extrapolation towards the chiral, infinite volume and continuum limit.

### 3 Extrapolations of the scattering length

The chiral extrapolations of physical quantities involving both the up (down) and the strange quarks consists of two steps. In the first step, the bare strange quark mass, or equivalently the corresponding hopping parameter $\kappa_s$, is kept fixed while the hopping parameter of the up/down quark, $\kappa_u$, is brought to their physical value $\kappa_u^{(phy)}$. The precise value of $\kappa_u^{(phy)}$ can be obtained by inspecting the chiral behavior of the pseudo-scalar (pion) and the vector meson (rho) mass values. In the second step, one fixes the up/down quark hopping parameter at its physical value obtained in the first step and extrapolate/interpolate in the strange quark mass. The physical strange quark hopping parameter $\kappa_s^{(phy)}$ is determined by demanding the mass of the kaon being exactly at its physical value (the so-called K-input).

We now come to the chiral extrapolation of the scattering length. In the chiral limit, the $K\pi$ scattering length in the $I = 3/2$ channel is given by the current
algebra result:
\[ a^{(3/2)}_0 = -\frac{1}{8\pi} \frac{\mu_{K\pi}}{f_\pi^2} \], \quad (8)

where \( a^{(3/2)}_0 \) is the \( K\pi \) scattering length in the \( I = 3/2 \) channel and \( f_\pi \sim 93\text{MeV} \) is the pion decay constant. To perform the chiral extrapolation of the scattering length, it is more convenient to use the quantity \( F = a^{(3/2)}_0 \frac{m_\rho^2}{\mu_{K\pi}} \), which in the chiral limit reads: [13]

\[ F \equiv a^{(3/2)}_0 \frac{m_\rho^2}{\mu_{K\pi}} = -\frac{1}{8\pi} \frac{m_\rho^2}{f_\pi^2} \sim -2.728 , \quad (9) \]

where the final numerical value is obtained by substituting in the experimental values for \( m_\rho \sim 770\text{MeV} \) and \( f_\pi \sim 93\text{MeV} \). One-loop corrections reduce the central value of the scattering length by about 28\%, with an estimated error of 40\% [4,5]. The factor \( F \) can be calculated on the lattice with good precision \textit{without} the lattice calculation of meson decay constants. Since we have calculated the factor \( F \) for several different values of valance quark mass, we could make a chiral extrapolation and extract the corresponding results in the chiral limit.

In chiral extrapolations, we have adopted a quadratic functional form and the fitting range of the extrapolation is self-adjusted by the program to yield a minimal \( \chi^2 \) per degree of freedom. First, we extrapolate the factor \( F \) in the up/down quark mass values. Second, the extrapolated values for the factor \( F \) are then extrapolate/interpolate in the strange quark mass. In Fig. 1, we have shown the extrapolation in the strange quark mass for one of our simulation points. The resulting factor \( F \) after the up and down quark mass extrapolation are plotted in Fig. 1 versus the strange quark mass parameter \( 1/(2\kappa_s) \). The line represents the quadratic extrapolation/interpolation and the final result is depicted as a solid square at the physical strange quark mass. The fitting quality \( Q \) for this fit is also displayed.

After the chiral extrapolation, we now turn to study the finite volume effects of the simulation. According to formula (3), the quantity \( F \) obtained from finite lattices differs from its infinite volume value by corrections of the form \( 1/L^3 \). In Fig. 2, we have shown the infinite volume extrapolation for our simulation points at \( \beta = 3.0, 2.6, 2.4, 2.2 \) and 1.9. The extrapolated results for the factor \( F \) are then used for the continuum limit extrapolation.

Finally, we can make an extrapolation towards the continuum limit by eliminating the finite lattice spacing errors. Since we have used the tadpole improved clover Wilson action, all physical quantities differ from their continuum
Fig. 1. Chiral extrapolation for the quantity \( \frac{a_0 m_{\rho}^2}{\mu_KN} \) for our simulation results at \( \beta = 2.4 \) on \( 8^340 \) lattices. In this plot, quantity \( F \), with the up and down quark mass already extrapolated to the chiral limit, is plotted (open squares) as a function of strange quark mass parameter \( \frac{1}{2\kappa_s} \). The line represents the quadratic interpolation/extrapolation to the physical strange quark mass, where the result is depicted as a solid square.

Fig. 2. Infinite volume extrapolation for the quantity \( \frac{a_0 m_{\rho}^2}{\mu_K} \) obtained from our simulation results at \( \beta = 3.0, 2.6, 2.4, 2.2, \) and \( 1.9 \). The straight line represents the linear extrapolation in \( a_s/r_0 \). The extrapolated results are shown with solid squares at \( L^{-3} = 0 \).

counterparts by terms that are proportional to \( a_s \). The physical value of \( a_s \) for each value of \( \beta \) can be found from Ref. [21,26].

The result of the continuum extrapolation is shown in Fig. 3 where the results from the chiral and infinite volume extrapolation discussed above are
Fig. 3. Continuum extrapolation for the quantity $F = a_0^{(3/2)} m_{\rho}^2 / \mu_{K\pi}$ obtained from our simulation results at $\beta = 3.0, 2.6, 2.4, 2.2$ and $1.9$. The straight line represents the linear extrapolation in $a_s/r_0$. The extrapolated result is also shown as solid square with the corresponding error. Also shown are the result from current algebra and dispersion relations [7]. They coincide numerically and the value is shown as an open circle. The chiral perturbation theory result is shown as a solid circle with error bar.

indicated as data points in the plot for all 5 values of $\beta$ that have been simulated. The straight line shows the extrapolation towards the $a_s = 0$ limit and the extrapolated result is also shown as a solid square together with the chiral result from Ref.[4,5] which is shown as the open (tree-level, or current algebra result) and filled (one-loop) circles. Result from dispersion relations from Ref. [7] happens to coincide with the current algebra result (open circle) numerically. It is seen that these results are compatible within error bars.

To summarize, we obtain from the linear extrapolation the following result for the quantity $F = a_0^{(3/2)} m_{\rho}^2 / \mu_{K\pi} = (-1.87 \pm 0.55)$. If we substitute in the physical values, we obtain the $K\pi$ scattering length in the $I = 3/2$ channel: $a_0^{(3/2)} m_{\pi} = (-0.048 \pm 0.014)$, which is to be compared with the one-loop chiral result of $(-0.05 \pm 0.02)$ and the tree-level result (and also the early dispersion relation result) $-0.07$. Our lattice study also indicates a value consistent with the chiral result that is smaller in magnitude than the experimental result. The experimental results for the $K\pi$ scattering length $a_0^{(3/2)} m_{\pi}$ show sizable variation [1,2,3]. It lies between $-0.13$ and $-0.05$, significantly larger in magnitude compared with chiral and early dispersion relation results. In a recent analysis [8], a value of $a_0^{(3/2)} m_{\pi} = -0.129 \pm 0.006$ was obtained using dispersion relations incorporated with chiral perturbation theory to fit the experimental data.

4 Conclusions

In this paper, we have calculated kaon-pion scattering lengths in isospin $I = 3/2$ channel using quenched lattice QCD. It is shown that such a calculation is
feasible using relatively small, coarse and anisotropic lattices. The calculation is done using the tadpole improved clover Wilson action on anisotropic lattices. Simulations are performed on lattices with various sizes, ranging from 0.7fm to about 4fm and with five different values of lattice spacing. Quark propagators are measured with different valence quark mass values. These enable us to explore the finite volume and the finite lattice spacing errors in a systematic fashion. The infinite volume extrapolation is made. The lattice result for the scattering length is extrapolated towards the chiral and continuum limit where a result consistent with Chiral Perturbation Theory and early dispersion relation calculation is found. Our result for the scattering length is smaller in magnitude when compared with the experimental data results.

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