Three-point functions of twist-two operators in $\mathcal{N}=4$ SYM at one loop

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Abstract: We calculate three-point functions of two protected operators and one twist-two operator with arbitrary even spin $j$ in $\mathcal{N}=4$ super Yang-Mills theory to one-loop order. In order to carry out the calculations we project the indices of the spin $j$ operator to the light-cone and evaluate the correlator in a soft-limit where the momentum coming in at the spin $j$ operator becomes zero. This limit largely simplifies the perturbative calculation, since all three-point diagrams effectively reduce to two-point diagrams and the dependence on the one-loop mixing matrix drops out completely. The results of our direct calculation are in agreement with the structure constants obtained by F.A. Dolan and H. Osborn from the operator product expansion of four-point functions of half-BPS operators.

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1 Introduction and conclusions

The full solution of $\mathcal{N} = 4$ SYM theory is equivalent to the ability of calculating any observable of interest in this quantum field theory exactly or perturbatively to arbitrary order in the coupling constant. In recent years, such impressive achievements have indeed been made for certain observables in planar $\mathcal{N} = 4$ SYM, largely due to the existence of integrability [1–4] in the AdS/CFT correspondence [5–7], see [8] for a review.

Using the methods of integrability, all-order conjectures, e.g. for the cusp anomalous dimension of twist two operators could be obtained [9, 10]. Using this, also for scattering amplitudes of four and five external particles in planar $\mathcal{N} = 4$ SYM, all-order results became available [11, 12]. These all-order results were achieved using the duality between MHV amplitudes and light-like Wilson loops [13–15], see [16, 17] for reviews.

Given these results, it is natural to ask, whether similar advances can be made for three-point functions of gauge invariant operators. Not only are three-point functions the next natural correlators to address after two-point functions, but they possess a particular
importance due to the operator product expansion (OPE). Given the knowledge of all structure constants that appear in the theory, one can in principle construct any higher-point correlation function using the OPE. Therefore, all-order results for structure constants, together with all-order results for anomalous dimensions yield important information on any higher-point function.

The form of three-point functions is fixed by conformal symmetry and the only dependence on the coupling constant is contained in the anomalous dimensions of the operators and the structure constants which receive radiative corrections in the coupling

\[ C_{\alpha\beta\gamma}(g^2) = C_{\alpha\beta\gamma}^{(0)} + g^2 C_{\alpha\beta\gamma}^{(1)} + \ldots \]  

(1.1)

Similarly as for two-point functions, there are non-renormalisation theorems for three-point correlators of half-BPS operators, which guarantee that they do not get quantum corrections [18–23].

Direct computations of three-point functions in \( \mathcal{N} = 4 \) SYM theory have been performed in [24–32]. In [33] a large number of three-point functions involving scalar primary operators of up to and including length five is considered.

The role of integrability for three-point function calculations was first addressed in [27–29]. Applications of integrability methods can be found in [34–37] at tree-level and in [37, 38] for loop-level three-point functions of scalar single trace operators. It does indeed turn out, that three-point functions can be studied efficiently using integrability.

The question of a weak-strong coupling matching was addressed in [35, 39–44]. The SL(2) sector was adressed in [44] and agreement between the structure constants at weak and strong coupling was found, where a BPS state and two operators with large spin were considered at tree-level. It would be interesting to promote this study to the loop-level, which requires the knowledge of the correlator of one BPS operator and two operators with spin at one-loop level. Furthermore, it would be interesting to see, how one can make use of integrability methods in order to calculate this correlator at loop-level.

Here, we make a first step towards this direction by providing a direct field theory computation of three-point correlators with two protected operators and one twist-two operator. This calculation serves as a starting point for the case of two operators with spin, where several distinct space-time structures enter with a priori different structure constants, see e.g. [45, 46]. A direct calculation is possibly more suitable for the development of integrability methods than the extraction of the same coefficient from the OPE.

More explicitly, here we calculate three-point functions in \( \mathcal{N} = 4 \) SYM theory involving two protected scalar operators

\[ \mathcal{O}(x) = \text{Tr} \left( \tilde{\phi}_{12}(x) \tilde{\phi}_{13}(x) \right), \quad \tilde{\mathcal{O}}(x) = \text{Tr} \left( \tilde{\phi}_{12}(x) \phi^{13}(x) \right) \]  

(1.2)

and one twist-two operator with arbitrary even spin \( j \), which is totally symmetric and traceless in all indices and schematically of the form

\[ \mathcal{O}_{\mu_1 \ldots \mu_j}(x) = \text{Tr} \left( D_{\mu_1} \ldots D_{\mu_k} \phi^{12}(x) D_{\mu_{k+1}} \ldots D_{\mu_j} \phi^{12}(x) \right) + \ldots, \]  

(1.3)

where the ellipses stand for a certain distribution of the derivatives, which is given explicitly in section 2. Conformal symmetry fixes the three-point functions up to the structure
constants $C_{\mathcal{O}\hat{O}_j}$: We use the light-cone projection,\(^1\) where all indices are contracted with a light-like vector $z^\mu$ and the contraction is denoted by a hat, e.g.

$$\hat{x} = x_\mu z^\mu, \quad \hat{O}_j = \mathcal{O}_{\mu_1...\mu_j} z^{\mu_1} \ldots z^{\mu_j}, \quad z^2 = 0.$$  \hfill (1.4)

Then the correlator reads

$$\langle \mathcal{O}(x_1)\hat{O}(x_2)\hat{O}_j(x_3) \rangle = C_{\mathcal{O}\hat{O}_j} \frac{(\hat{Y}_{12,3})^j}{|x_{12}|^{\Delta_1+\Delta_2-\theta}|x_{13}|^{\Delta_1+\theta-\Delta_2}|x_{23}|^{\Delta_2+\theta-\Delta_1}},$$  \hfill (1.5)

where $\theta = \Delta_j - j$ is the twist (dimension minus spin) of the operator $\hat{O}_j$,

$$\hat{Y}_{12,3} = Y_{12,3}^{\mu} z^\mu, \quad Y_{12,3}^{\mu} = \frac{x_{13}^{\mu}}{x_{13}^2} - \frac{x_{23}^{\mu}}{x_{23}^2} = \frac{1}{2} \partial_{x_3} \ln \left( \frac{x_{23}^2}{x_{13}^2} \right)$$  \hfill (1.6)

and where $x_{ij}^{\mu} = x_i^{\mu} - x_j^{\mu}$. Note, that the scaling dimension of the twist operator and the structure constants both acquire corrections in perturbation theory

$$\theta = \Delta_j - j = 2 + \gamma_j (g^2), \quad C_{\mathcal{O}\hat{O}_j}(g^2) = C_{\mathcal{O}\hat{O}_j}^{(0)} + g^2 C_{\mathcal{O}\hat{O}_j}^{(1)} + \mathcal{O}(g^4).$$  \hfill (1.7)

In order to carry out the calculations we evaluate the correlator in a soft-limit, where the momentum coming in at the spin $j$ operator becomes zero. In position space this corresponds to an integration over the corresponding point $x_3$

$$\int \frac{d^4p}{(2\pi)^4} e^{ip\cdot x_{12}} \langle \mathcal{O}(p)\hat{O}(-p)\hat{O}_j(0) \rangle = \int d^4 x_3 \langle \mathcal{O}(x_1)\hat{O}(x_2)\hat{O}_j(x_3) \rangle.$$  \hfill (1.8)

This limit largely simplifies the perturbative calculation, since all three-point diagrams effectively reduce to two-point diagrams and the dependence on the one-loop mixing matrix drops out completely, as we will see in section 2. We find that the correction to the normalisation invariant structure constant takes the simple form

$$C'_{\mathcal{O}\hat{O}_j}(g) = C_{\mathcal{O}\hat{O}_j}^{(0)} \left( 1 + \frac{g^2 N}{8\pi^2} \left( 2H_j(H_j - H_{2j}) - H_{j,2} \right) + \mathcal{O}(g^4) \right),$$  \hfill (1.9)

where $H_{j,m}$ are generalised harmonic sums $H_j = \sum_n 1/n, \quad H_{j,r} = \sum_n 1/n^r$. The result is in agreement with the extraction of the structure constants from the analysis of the operator product expansion (OPE) of four-point functions of half-BPS operators [49]. Recently, the OPE and the three-loop expression for the four-point correlator of half-BPS operators [50] were used to extract the structure constants up to three loops [51]. Our calculation thus directly confirms the results obtained from the OPE.

In the following sections, we will only summarise the main steps of our calculation in order to keep the presentation short, more details will be made available in [52].

\(^1\)The indices can be recovered by repeated application of a second order differential operator in $z^\mu$ given by $\Delta_{\mu} = ((d/2 - 1) + z \cdot \partial)\partial_{z,\mu} - \frac{1}{2} z_{\nu} \partial_{z,\nu} \cdot \partial_z$ in the presence of the constraint $z^2 = 0$, see e.g. [47, 48] for more information.
2 Three-point functions of twist operators

We use the mostly minus metric and the conventions as given in appendix A and adapt the notation from [48]. The renormalised twist-two operators are given by

$$\hat{\hat{O}}_j(x) = \sum_k Z_{jk} \hat{D}^j \hat{\hat{\hat{O}}}_k(x),$$

(2.1)

where

$$\hat{\hat{O}}_j(x) = \sum_{k=0}^j a^{1/2}_{jk} \text{Tr} \left( \hat{D}^k \phi^{12}(x) \hat{D}^j \phi^{12}(x) \right),$$

(2.2)

is the tree-level operator, $\hat{D} = D^\mu z_\mu$ is the light-cone projected covariant derivative and the coefficients $a^{1/2}_{jk}$ are related to the Gegenbauer polynomials as in (A.2). The one-loop mixing matrix in the conformal scheme has the form

$$Z_{jk} = \delta_{jk} + \gamma_{jk} C_2(g^2) + O(g^4) = \delta_{jk} + g^2 \left( -B^{(1)}_{jk} + \frac{1}{\epsilon} \delta_{jk} Z^{(1)}_{jk} \right) + O(g^4).$$

(2.3)

At one-loop level the divergent part of the renormalisation matrix is diagonal with $Z^{(1)}_{jk} = H_j / (4 \pi d/2) \exp(\gamma_E)$ and determines the anomalous dimension via $\gamma_j = -\mu d/d\mu \ln Z_j(\mu)$. The finite matrix $B^{(1)}_{jk}(g^2)$ diagonalizes the anomalous dimension matrix at higher-loop level $\gamma_{jk}(g^2)\delta_{jk} = (B^{-1} \gamma B)_{jk}$ and accounts for non-diagonal contributions to the two-point functions at loop-level. We do not specify $B_{jk}$, since it drops out in the limit that we consider. The operators given by (2.1) then have diagonal, conformal two-point functions also at loop-level and are fixed up to the normalisation $C_j(g^2)$

$$\langle \hat{\hat{\hat{O}}}_j(x_1) \hat{\hat{O}}_k(x_2) \rangle = \delta_{jk} C_j(g^2) \left( \frac{I_{12}}{-x_{12}^2} \right)^{j+\theta} (j \text{ even}),$$

(2.4)

where $\theta = \Delta_j - j = 2 + \gamma_j(g^2)$ and $I_{12}^{\mu\nu} = \eta^{\mu\nu} - 2x_1^\mu x_2^\nu / x_{12}^2$. In order to read-off the structure constants in the limit (1.8) we have to integrate (1.5) over $x_3$, which yields

$$\int d^4 x_3 \langle \hat{O} \hat{\hat{O}} \hat{O} \rangle = N(g^2) \left( C_j^{(0)} C_k C_j + g^2 C_j^{(1)} C_k C_j \right) \left( \frac{x_{12}^2}{-x_{12}^2} \right)^{j+\theta}$$

(2.5)

where $N(g^2)$ is a normalisation factor explicitly given in (B.5). By calculating the left-hand side in the limit $p_1 + p_2 \to 0$ one can thus easily read-off the structure constants, after Fourier transforming the momentum space expression.\footnote{An analogue of this procedure appears in [53], where a deformation of the N = 4 SYM Lagrangian by integrated local operators is considered and two-point functions of the deformed theory are studied at leading order in the deformation parameter.}

3 Tree-level calculation

In momentum space, the tree-level three-point function with the twist-operator in the representation (A.3) reads

$$\langle \hat{O}(p_1) \hat{\hat{O}}(p_2) \hat{O}_j \rangle = \frac{1}{4} \gamma^{3+j} g^4 \delta^{aa} \int \frac{d^d k}{(2\pi)^d} \left( \hat{\hat{p}}_1 + \hat{\hat{p}}_2 \right)^j \frac{C_j^{1/2} \left( \frac{2k \cdot \hat{\hat{p}}_1 - \hat{\hat{p}}_2}{p_1 + p_2} \right)}{(2\pi)^d k^2 (p_1 - k)^2 (p_1 + p_2 - k)^2},$$

(3.1)
Now we take the limit $p_1 + p_2 \to 0$ in momentum space. Then, due to the factor $(\hat{p}_1 + \hat{p}_2)^j$ only the term with the highest power, i.e. $j$, in the Gegenbauer polynomial can survive. The corresponding coefficient reads
\[ c_{jj}^{1/2} = 2^{1-j} \frac{\Gamma(2j)}{\Gamma(j) \Gamma(j+1)} \]
where \[ C_j^{1/2}(x) = \sum_{k=0}^j c_{jk}^{1/2} x^k. \] (3.2)

Thus, in this limit the three-point integral in (3.1) becomes a two-point integral with a doubled propagator
\[ \int \frac{d^dk}{(2\pi)^d} \frac{(\hat{p}_1 + \hat{p}_2)^j C_j^{1/2} \left( \frac{2k-\hat{p}_1 - \hat{p}_2}{p_1 + p_2} \right)}{k^2 (p_1 - k)^2 (p_1 + p_2 - k)^2} \to 2^j c_{jj}^{1/2} \int \frac{d^dk}{(2\pi)^d} \frac{(k)^j}{k^4 (p_1 - k)^2}, \] (3.3)
which can easily be solved using (B.2) in the appendix. Fourier transformation to $x$-space using (B.1) and insertion of the coefficients yields
\[ \int d^dx \langle \mathcal{O} \hat{\mathcal{O}} \hat{\mathcal{O}}_j \rangle = -i g^6 \delta^{aa} \frac{2^{d-7+j} \Gamma(2j) \Gamma(d/2 - 1) \Gamma(j - 2 + d/2)}{\pi^d} \frac{\Gamma(j+1)}{(x_{12}^2)^{d-3-j}}. \] (3.4)

Comparing with (2.5) we can directly read off the tree-level structure constant for $d = 4$ and arbitrary even spin $j$
\[ C_j^{(0)} = g^6 \delta^{aa} \frac{2^{j-8}}{\pi^6} \Gamma(j+1). \] (3.5)

4 One-loop calculation

In order to read-off the one-loop structure constant we have to compute the renormalised three-point function using the renormalised operator given by (2.1)–(2.3). From
\[ \int d^dx \langle \mathcal{O} \hat{\mathcal{O}} \hat{\mathcal{O}}_j \rangle = \sum_k Z_{jk} \int d^dx \hat{\mathcal{O}}_k \hat{\mathcal{O}}_k = \sum_k Z_{jk} \delta_{jk} \int d^dx \langle \mathcal{O} \hat{\mathcal{O}} \hat{\mathcal{O}}_k \rangle \]

it is however immediately clear, that the finite matrix $B_{jk}$ drops out in the limit that we consider, which vastly simplifies the calculation. As can be seen from (2.3), we then only need to calculate the one-loop diagrams and subtract the tree-level contribution\footnote{It needs to be evaluated including $\mathcal{O}(\epsilon)$ terms, because it is multiplied with $1/\epsilon$.} multiplied with the renormalisation constant $Z_j$. The one-loop diagrams that enter the calculation are shown in figure 1.

It is instructive to write down all diagrams before employing the limit $p_1 + p_2 \to 0$ and to convince oneself that diagrams which could lead to ambiguities, such as the diagram in figure 1g, which vanishes in dimensional regularisation when taking the limit, cancel with contributions from other diagrams.

Indeed, all diagrams with a scalar four-point interaction vertex, i.e. figure 1c, 1g, 1k cancel against contributions from figure 1b, 1f, 1j, which becomes clear when considering the decomposition of diagrams with two gluon vertices into scalar integrals, see figure 2.
\[ \begin{array}{ccc}
(a) & (b) & (c) \\
\hline
(d) & (e) & (f)
\end{array} \]

\( (g) \quad (h) \quad (i) \quad (j) \quad (k) \quad (l) \)

**Figure 1:** Feynman diagrams contributing to the three-point function at one loop. For \( j \) even the diagrams in the second and third row are identical.

\[ \Delta = -2p_1^2 \Delta - \Delta + \Delta + \Delta + \Delta + \Delta \]

**Figure 2:** Due to the numerator momenta from the gluon vertices, the integrand decomposes into simpler ones, which partially cancel with the self-energy and four-scalar interaction terms.

For \( j = 0 \), the diagrams in the last column do not appear, since there is no covariant derivative that the gauge field at \( x_3 \) could arise from and half of the diagrams 1e, 1a cancels against a contribution from 1f, half of the diagrams 1a, 1i cancels against a contribution from 1j and half of the diagrams 1e, 1i cancels against a contribution from 1b. The remaining finite contributions from 1b, 1f, 1j cancel due to a relation between scalar three-point integrals given in [54]. Thus for \( j = 0 \) all contributions exactly cancel, which is due to the fact that in this case all of the operators are protected.

For \( j \neq 0 \) the above cancellations between divergent diagrams where the divergence is located at the points \( x_1 \) resp. \( x_2 \), i.e. where the BPS operators sit, remain true as one would expect.\(^4\) The cancellations between half of the diagrams 1e, 1i with a contribution from 1b does however not take place anymore, and the spin-\( j \) operator at \( x_3 \) acquires an anomalous dimension \( \gamma_j \). For the remaining diagrams in the limit \( p_1 + p_2 \), for the same reason as in the tree-level calculation, we only need the coefficient \( c_{jj}^{1/2} \) with the highest power of the Gegenbauer polynomial that was given in (3.2). Since \( p_1 = -p_2 = p \) and \( j \) is even we find the same result for all diagrams which are equal under \( p_1 \leftrightarrow p_2 \). The integrals

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\(^4\)Would divergences at these points remain, the operators acquired an anomalous dimension, we do however have protected operators with vanishing anomalous dimension at \( x_1, x_2 \).
turn into simple two-point integrals that can be easily solved using (B.2). Denoting from now on \( p_1 = p \) we find

\[
\langle \mathcal{O}\tilde{\mathcal{O}}\tilde{\mathcal{O}}\rangle^{1b} + \langle \mathcal{O}\tilde{\mathcal{O}}\mathcal{O}\rangle^{1c} = -c_{ij}^{1/2}2^{-1}i^{2+j}g^8\delta^{aa}b_j \left( 4 - \frac{d}{2}, 1 \right) (b_j(2, 1) + b_j(1, 1)) \frac{\hat{p}^j}{(-p^2)^{5-d}}.
\]

(4.1)

where \( b_n(\alpha_1, \alpha_2) \) is defined in (B.3). As mentioned before, half of the self-energy diagrams in figure 1e, 1i do not cancel for \( j \neq 0 \) and the expression is

\[
\frac{1}{2}\langle \mathcal{O}\tilde{\mathcal{O}}\mathcal{O}\tilde{\mathcal{O}}\rangle^{1e} + \frac{1}{2}\langle \mathcal{O}\tilde{\mathcal{O}}\tilde{\mathcal{O}}\rangle^{1i} = c_{ij}^{1/2}i^{2+j}g^8\delta^{aa}2^{-1}b_0(1, 1)b_j(4 - \frac{d}{2}, 1) \frac{\hat{p}^j}{(-p^2)^{5-d}}.
\]

(4.2)

Applying the limit to the diagrams where the divergence is located at \( x_1 \) we find

\[
\langle \mathcal{O}\tilde{\mathcal{O}}\mathcal{O}\tilde{\mathcal{O}}\rangle^{1f} + \langle \mathcal{O}\tilde{\mathcal{O}}\mathcal{O}\rangle^{1g} + \frac{1}{2}\langle \mathcal{O}\tilde{\mathcal{O}}\tilde{\mathcal{O}}\rangle^{1e} + \frac{1}{2}\langle \mathcal{O}\tilde{\mathcal{O}}\tilde{\mathcal{O}}\rangle^{1a}
\]

\[
= i^{2+j}g^8\delta^{aa}2^{-c_{ij}^{1/2}} \frac{\hat{p}^j}{(-p^2)^{d-5}} (2c_{0j}(1, 1, 1, 2, 1) + c_{0j}(1, 1, 1, 1, 1) + c_{0j}(1, 1, 1, 2, 0)).
\]

(4.3)

The integrals \( c_{nm}(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \) are defined in (C.2) and solved\(^5\) in appendix C using the IBP technique. All these integrals are finite. The sum of the diagrams in figures 1k, 1j, 1a, 1i yield the same result as those in (4.3), since they are related to these by \( p_1 \leftrightarrow p_2 \).

For \( j \neq 0 \) we have additional diagrams with exactly one gauge field from the covariant derivative in \( \tilde{\mathcal{O}}_j \). The diagrams are shown in figure 1d, 1h, 1l. Diagrams 1h and 1l are equal in our limit. The appearing integrals are simple two-point integrals and we find

\[
\langle \mathcal{O}(p)\tilde{\mathcal{O}}(-p)\tilde{\mathcal{O}}(0)\rangle^{1h} = -6^jg^8\delta^{aa}4b_j(4 - d/2, 1) \sum_{k=1}^j a_j^{1/2} \sum_{m=1}^{k-1} \sum_{n=0}^{m-1} \binom{k}{m} \binom{m-1}{n} (-1)^m \frac{\hat{p}^j}{(-p^2)^{5-d}} \left[ b_{k-n}(1, 1) - b_{k-n-1}(1, 1) + b_{j-n+k+1}(1, 1) - b_{j-k+n+1}(1, 1) \right].
\]

(4.4)

and one can proceed to solve the sums. Diagram 1d is calculated in a very similar way and reads

\[
\langle \mathcal{O}(p)\tilde{\mathcal{O}}(-p)\tilde{\mathcal{O}}(0)\rangle^{1d} = 6^jg^8\delta^{aa}2 \sum_{k=1}^j a_j^{1/2} \sum_{m=1}^{k} \sum_{n=0}^{m-1} \binom{k}{m} \binom{m-1}{n} (-1)^{k-m+n} \frac{\hat{p}^j}{(-p^2)^{5-d}} \left[ 2c_{k-1-n,j-k+n} + c_{k-n,j-k+n} + c_{k-1-n,j-k+n+1} \right].
\]

(4.5)

where all integrals \( c_{nm} = c_{nm}(1, 1, 1, 1, 1) \) are finite and the solution can be found in appendix C.

\(^5\)These integrals were considered before in [55].
4.1 Full bare three-point function

Taking into account the exact cancellations between diagrams and adding up the remaining contributions the three-point function is given by

\[
\langle O \tilde{O} \tilde{O} \rangle^{(1)} = \sum_{\alpha=a,...,l} \langle O \tilde{O} \tilde{O} \rangle^{1\alpha} = (4.1) + (4.2) + 2 \times (4.3) + 2 \times (4.4) + (4.5) .
\]

In order to read off the structure constant we Fourier transform the expression to position space using (B.1), since we have calculated all diagrams in momentum space. As a check of the calculation one can then extract the divergent part of (4.6) and read off the anomalous dimension from the three-point function. As mentioned before, only divergences located at \( x_3 \) remain and we get the following contributions to the anomalous dimension

\[
\gamma_j^{(1e+1l)/2} = \frac{g^2 N}{4\pi^2}, \quad \gamma_j^{1b+1c} = \frac{g^2 N}{4\pi^2} \left( -\frac{1}{j+1} \right), \quad \gamma_j^{1b+1l} = \frac{g^2 N}{4\pi^2} \left( 2H_j + \frac{1}{j+1} - 1 \right),
\]

such that we recover the well-known one-loop contribution to the anomalous dimension of twist-two operators

\[
\gamma_j = \left( \frac{g^2 N}{4\pi^2} \right) 2H_j + \mathcal{O}(g^4).
\]

Using the tree-level three-point function and (2.1), (2.3), (2.5) and (B.5) we find that the one-loop correction to the structure constant is

\[
C_{O\tilde{O}\tilde{O}}^{(1)} / C_{O\tilde{O}\tilde{O}}^{(0)} = \frac{g^2 N}{8\pi^2} \left( 5H(j)^2 - 4H(j)H(2j) - \sum_{r=1}^{j} \frac{1}{r^2} \right),
\]

4.2 Normalisation invariant structure constants

The coefficients that operators appear with in the operator product expansion are equal to the structure constants if the normalisation of the two-point functions of the operators is equal to one. The operators (A.3) in terms of Gegenbauer polynomials are not normalised to one, but they have the perturbative expansion

\[
\langle \hat{O}_j(x_1) \hat{O}_j(x_2) \rangle = \delta_{jk} \left( C_j^{(0)} + g^2 C_j^{(1)} \right) 2^{2j} \frac{(\hat{x}_{12})^{2j}}{(-x_{12}^2)^{2j+\bar{\sigma}}} + \mathcal{O}(g^4),
\]

and one can explicitly calculate \( C_j^{(0)}, C_j^{(1)} \). Using the Schwinger parametrisation of the propagator and some properties of the Gegenbauer polynomials as e.g. in [48] one finds that the operators (A.3) have the tree-level normalisation

\[
C_j^{(0)} = g^4 \delta^{aa} \Gamma(2j + 1) \frac{2^8 \pi^4}{2^8 \pi^4} .
\]
At one-loop level the diagrams shown in figure 3 appear. The calculation is well-known, technically similar to the three-point calculation and all appearing integrals can be found in the appendix C. We shall not reproduce the complete calculation here, but refer the reader to [52] for more details on this and all other calculations. For the one-loop result of the normalisation we find

\[
C_j^{(1)}/C_j^{(0)} = \frac{g^2 N}{4\pi^2} \left( 3H(j)^2 - 2H(j)H(2j) \right) .
\]

(4.11)

The BPS operators (1.2) have the normalisation \( C = 2^{-6}/\pi^4 \) and do not get quantum corrections. We can define a normalisation invariant structure constant \( C'_\tilde{O}_j \tilde{O}_j \) which is related to \( C_{\tilde{O}_j \tilde{O}_j} \) through

\[
C'_{\tilde{O}_j \tilde{O}_j} = \frac{C_{\tilde{O}_j \tilde{O}_j}(g^2)}{\sqrt{C_j^{(0)} C}} = \frac{C_{\tilde{O}_j \tilde{O}_j}^{(0)}}{\sqrt{C_j^{(0)} C}} \left( 1 + g^2 \left( \frac{C_{\tilde{O}_j \tilde{O}_j}^{(1)}}{C_{\tilde{O}_j \tilde{O}_j}^{(0)}} - \frac{1}{2} \frac{C_{\tilde{O}_j}^{(1)}}{C_{\tilde{O}_j}^{(0)}} \right) \right) = C_{\tilde{O}_j \tilde{O}_j}^{(0)} + g^2 C_{\tilde{O}_j \tilde{O}_j}^{(1)} .
\]

(4.12)

It corresponds to the structure constants calculated for operators, which are all normalised to one. Inserting (4.11) and (4.8) we thus find

\[
C'_{\tilde{O}_j \tilde{O}_j}(g^2) = C_{\tilde{O}_j \tilde{O}_j}^{(0)} \left( 1 + \frac{g^2 N}{8\pi^2} (2H_j(H_j - H_{2j}) - H_{j,2}) + \mathcal{O}(g^4) \right) ,
\]

(4.13)

which is the result quoted in the introduction.

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A Notation and conventions

Throughout this paper we use the Minkowski-space metric with signature \((+\ldots-\ldots)\).

A.1 \(\mathcal{N} = 4\) SYM Lagrangian

We use the following action of \(\mathcal{N} = 4\) SYM theory

\[
S = \frac{1}{g^2} \int d^d x \left( -\frac{1}{4} F_{\mu\nu}^a F^{a,\mu\nu} + \frac{1}{4} D_{\mu} \phi^{a,AB} D^\mu \phi_{AB} \right. 
\]

\[
- \frac{1}{16} f^{abc} f^{ade} \phi_{b,AB} \phi_{c,CD} \bar{\phi}_{d,\alpha\beta} \bar{\phi}_{e,\alpha\beta} + i \frac{\bar{\lambda}_{a,\alpha A}}{\sqrt{2}} (\sigma_\mu)_{\alpha\beta} \lambda^{a\beta}_A 
\]

\[
- i \frac{\sqrt{2}}{2} f^{abc} \lambda_{a\alpha A} \phi_{b,AB} \phi_{c} \bar{\lambda}_{d,\alpha B} 
\]

where we have taken the trace of the matrix valued fields, e.g. \(\bar{\phi}_{AB} = \bar{\phi}_{a}^{AB} \), using the normalisation of the generators \(T_a\) in the fundamental representation according to \(\text{Tr} (T_a T_b) = \delta_{ab} / 2\).

A.2 Representation in terms of Gegenbauer polynomials

The numerical coefficients \(a_{\nu,jk}\) that appear in (2.2) are related to the so-called Gegenbauer polynomials \(C^\nu_j(x)\) such that

\[
\sum_{k=0}^j a_{\nu,jk} x^k y^{j-k} = (x + y)^j C^\nu_j \left( \frac{x-y}{x+y} \right) 
\]

and \(\nu = d/2 - 3/2\). Therefore, we can rewrite the operators in the bi-local form \([56, 57]\)

\[
\hat{O}_{\text{tree}}^j = \left( \hat{\partial}_a + \hat{\partial}_b \right)^j C^1_{\nu} \left( \frac{\hat{\partial}_z - \hat{\partial}_\ell}{\hat{\partial}_a + \hat{\partial}_b} \right) \text{Tr} \left( \phi_{12}(x_a) \phi_{12}(x_b) \right) \bigg|_{x_a=x_b}. 
\]

Explicitly, the coefficients are given by \(a_{\nu,jk}^{1/2} = (-1)^k \left( \frac{j}{k} \right) \left( \frac{j}{k} \right)\). These operators have diagonal conformal two-point functions.

B Details of the calculation

B.1 Fourier transformation and bubble integrals

In Minkowski-space with signature \((+\ldots-\ldots)\) we have

\[
\int \frac{d^d p}{(2\pi)^d} e^{-ip\cdot x} \frac{\Gamma(\frac{d}{2} - k)}{\Gamma(k)} \frac{1}{4^k \pi^{\frac{d}{2}}} \frac{i}{(-x^2 + i\epsilon)^{\frac{d}{2} - k}}. 
\]

The two-point integral with momenta in the numerator is

\[
B_n(\alpha_1, \alpha_2) = \int \frac{d^d k}{(2\pi)^d} \frac{k^n}{(-k^2 - i\epsilon)^{\alpha_1} (-p^2 - i\epsilon)^{\alpha_2}} 
\]

\[
= b_n(\alpha_1, \alpha_2) \frac{p^n}{(-p^2 - i\epsilon)^{\alpha_1 + \alpha_2 - \frac{d}{2}}}, 
\]
where

\[ b_n(\alpha_1, \alpha_2) = \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(\frac{d}{2} + n - \alpha_1) \Gamma(\frac{d}{2} - \alpha_2) \Gamma(\alpha_1 + \alpha_2 - \frac{d}{2})}{\Gamma(d + n - \alpha_1 - \alpha_2) \Gamma(\alpha_1) \Gamma(\alpha_2)} \cdot \]

(B.3)

Note, that analogous expressions for the bubble integral and the Fourier transformation in \( x \)-space differ by minus signs, due to the different sign of \( i\epsilon \) in the propagators.

B.2 Normalisation factor of the \( x_3 \) integration

Integration of (1.5) over \( x_3 \) yields

\[ \int d^d x_3 \langle \hat{O} \hat{O} \rangle_j = N(g^2) \left( C^{(0)}_{\hat{O} \hat{O} j} + g^2 C^{(1)}_{\hat{O} \hat{O} j} \right) \]

where

\[ N(g^2, d) = -\frac{i}{\pi^{\frac{d}{2} - 3 + \delta}} \frac{\Gamma(\theta - d/2 + j) \Gamma((d - \theta)/2)^2 \Gamma(j + (\theta - 1)/2)}{\Gamma(d - \theta) \Gamma(j + \frac{d}{2}) \Gamma(j + \theta - 1)} \frac{2^{\theta + 2j - 2}}{\pi^{\frac{d}{2} - \frac{d}{2}}} \cdot \]

(B.5)

C Integrals with the IBP method

We define the following set of integrals

\[ f_{mn}(a_1, a_2, a_3, a_4, a_5) = \int \frac{d^d k \cdot l}{(2\pi)^{2d} k^{2a_1} (p + k)^{2a_2} (l - k)^{2a_3} (p + l)^{2a_5}} \]

where \( \hat{k} = z^\mu k_\mu \) is the contraction with a light-like vector \( z^2 = 0 \). The integral is symmetric under the simultaneous exchange \( (m, a_1, a_2) \leftrightarrow (n, a_4, a_5) \). In particular, in this work we need \( f_{mn}(1, 1, 1, 1, 1) \) and \( f_{j0}(2, 1, 1, 1, 1) \). Since the momentum dependence is the same everywhere we define \( c_{mn} \) by stripping off the equal coefficients

\[ f_{mn}(a_1, a_2, a_3, a_4, a_5) = c_{mn}(a_1, a_2, a_3, a_4, a_5) \frac{(\hat{k})^{m+n}}{(-p^2)^{\sum a_i - d}} \cdot \]

(C.2)

A detailed introduction to the IBP method can e.g. be found in [58]. For the integrals that appear here it just consists of inserting the identity in the form

\[ 1 = \frac{1}{d} \frac{\partial}{\partial k^\mu} (k^\mu - l^\mu) \]

(C.3)

into the integrals (C.1) and using integration by parts to obtain a relation between the original integral and simpler integrals, which are given in the following section.

C.1 Bubble integrals

The integrals (C.1) with one argument set to zero are easy to solve and are needed as the building blocks for solving the integrals with the IBP method. We have

\[ c_{mn}(a_1, a_2, 0, a_4, a_5) = (-1)^{\sum a_i} b_m(a_1, a_2) b_n(a_4, a_5) \]

(C.4)

These integrals were considered before in [55].
and
\[ c_{mn}(a_1, a_2, a_3, a_4, 0) = (-1)^n (-1)^{m-i} n! b_n(a_1, a_3) b_{m+n}(a_1 + a_3 + a_4 - \frac{d}{2}, a_2). \]

By the symmetry of the integral (C.1) \((m, a_1, a_2) \leftrightarrow (n, a_4, a_5)\) this also yields
\[ c_{mn}(a_1, 0, a_3, a_4, a_5) = c_{nm}(a_4, a_5, a_3, a_1, 0). \] (C.5)

Furthermore, we have
\[ c_{mn}(a_1, a_2, a_3, 0, a_5) = (-1)^n \sum_{j=0}^{n} \sum_{i=0}^{n-j} \binom{n}{j} \binom{n-j}{i} (-1)^n b_{n-j}(a_5, a_3) b_{m+n-i-j} \left( a_1, a_2 + a_3 + a_5 - \frac{d}{2}, a_1 - m \right). \] (C.6)

and \(c_{mn}(0, a_2, a_3, a_4, a_5)\) is related to this integral by the symmetry \((m, a_1, a_2) \leftrightarrow (n, a_4, a_5)\).

We can perform the sum over \(i\) and find
\[ c_{mn}(a_1, a_2, a_3, 0, a_5) = \frac{\Gamma(a_1 - m) \Gamma(a_1 + a_2 + a_3 + a_5 - d)}{\Gamma(a_1) \Gamma(a_1 + a_2 + a_3 + a_5 - d - m)} \sum_{j=0}^{n} \binom{n}{j} (-1)^{m+n-j} b_j(a_5, a_3) b_j \left( a_2 + a_3 + a_5 - \frac{d}{2}, a_1 - m \right). \] (C.7)

The sum can be performed, e.g. using Mathematica, in terms of generalised hypergeometric functions.

### C.2 Integrals \(f_{mn}(1, 1, 1, 1, 1)\)

The IBP method using (C.3) yields the recursion relation
\[ (d + m - 4) f_{mn}(1, 1, 1, 1, 1) = g(m, n) + m f_{m-1, n+1}(1, 1, 1, 1, 1), \] (C.8)

where \(g(m, n)\) is an abbreviation for the known bubble-integrals
\[ g(m, n) = f_{mn}(2, 1, 0, 1, 1) - f_{mn}(2, 1, 1, 0, 1) + f_{mn}(1, 2, 0, 1, 1) - f_{mn}(1, 2, 1, 1, 0). \] (C.9)

We can solve the recursion by writing down (C.8) for \(m = n + 1\) and \(n = m - 1\), i.e.
\[ (d + (n + 1) - 4) f_{n+1, m-1}(1, 1, 1, 1, 1) = g(n + 1, m - 1) + (n + 1) f_{nm}(1, 1, 1, 1, 1) \] (C.10)

and using \(f_{mn}(1, 1, 1, 1, 1) = f_{nm}(1, 1, 1, 1, 1).\) Solving (C.10) for \(f_{n+1, m-1}\) and inserting it into (C.8) we get \(f_{mn}\) in terms of known integrals:
\[ f_{mn}(1, 1, 1, 1, 1) = \frac{(n + d - 3) g(m, n) + m g(n + 1, m - 1)}{(d - 4)(d - 3 + m + n)}. \] (C.11)

All integrals that appear in \(g(m, n)\) were solved in section C.1.
C.3 Integrals $f_{mn}(2,1,1,1,1)$

For the three-point function calculation we need the integral $f_{j,0}(2,1,1,1,1)$. Using the IBP method, just as in section C.2 for $f_{mn}(1,1,1,1,1)$, we find

$$(d - 5 + m)f_{m,n}(2,1,1,1,1) = m f_{m-1,n+1}(2,1,1,1,1) + h(m,n),$$

(C.12)

where we have abbreviated the bubble integrals $h(m,n)$

$$h(m,n) = 2f_{mn}(3,1,0,1,1) - 2f_{mn}(3,1,1,0,1)$$

(C.13)

and all appearing integrals were solved in C.1. Thus $f_{mn}$ can be obtained recursively from (C.12).

References

[1] J. Minahan and K. Zarembo, The Bethe ansatz for $N=4$ super Yang-Mills, JHEP 03 (2003) 013 [hep-th/0212208] [INSPIRE].

[2] N. Beisert, C. Kristjansen and M. Staudacher, The dilatation operator of conformal $N=4$ super Yang-Mills theory, Nucl. Phys. B 664 (2003) 131 [hep-th/0303060] [INSPIRE].

[3] I. Bena, J. Polchinski and R. Roiban, Hidden symmetries of the $AdS_5 \times S^5$ superstring, Phys. Rev. D 69 (2004) 046002 [hep-th/0305116] [INSPIRE].

[4] N. Beisert and M. Staudacher, The $N=4$ SYM integrable super spin chain, Nucl. Phys. B 670 (2003) 439 [hep-th/0307042] [INSPIRE].

[5] J.M. Maldacena, The large-$N$ limit of superconformal field theories and supergravity, Adv. Theor. Math. Phys. 2 (1998) 231 [Int. J. Theor. Phys. 38 (1999) 1113] [hep-th/9711200] [INSPIRE].

[6] E. Witten, Anti-de Sitter space and holography, Adv. Theor. Math. Phys. 2 (1998) 253 [hep-th/9802150] [INSPIRE].

[7] S. Gubser, I.R. Klebanov and A.M. Polyakov, Gauge theory correlators from noncritical string theory, Phys. Lett. B 428 (1998) 105 [hep-th/9802109] [INSPIRE].

[8] N. Beisert, C. Ahn, L.F. Alday, Z. Bajnok, J.M. Drummond, et al., Review of AdS/CFT integrability: an overview, Lett. Math. Phys. 99 (2012) 3 [arXiv:1012.3982] [INSPIRE].

[9] M. Staudacher, The factorized S-matrix of CFT/AdS, JHEP 05 (2005) 054 [hep-th/0412188] [INSPIRE].

[10] N. Beisert, B. Eden and M. Staudacher, Transcendentality and crossing, J. Stat. Mech. 0701 (2007) P01021 [hep-th/0610251] [INSPIRE].

[11] Z. Bern, L.J. Dixon and V.A. Smirnov, Iteration of planar amplitudes in maximally supersymmetric Yang-Mills theory at three loops and beyond, Phys. Rev. D 72 (2005) 085001 [hep-th/0505025] [INSPIRE].

[12] J. Drummond, J. Henn, G. Korchemsky and E. Sokatchev, Dual superconformal symmetry of scattering amplitudes in $N=4$ super-Yang-Mills theory, Nucl. Phys. B 828 (2010) 317 [arXiv:0807.1098] [INSPIRE].
[13] L.F. Alday and J.M. Maldacena, *Gluon scattering amplitudes at strong coupling*, JHEP 06 (2007) 064 [arXiv:0705.0303] [nSPIRE].

[14] G. Korchemsky, J. Drummond and E. Sokatchev, *Conformal properties of four-gluon planar amplitudes and Wilson loops*, Nucl. Phys. B 795 (2008) 385 [arXiv:0707.0243] [nSPIRE].

[15] A. Brandhuber, P. Heslop and G. Travaglini, *MHV amplitudes in N = 4 super Yang-Mills and Wilson loops*, Nucl. Phys. B 794 (2008) 231 [arXiv:0707.1153] [nSPIRE].

[16] L.F. Alday and R. Roiban, *Scattering amplitudes, Wilson loops and the string/gauge theory correspondence*, Phys. Rept. 468 (2008) 153 [arXiv:0807.1889] [nSPIRE].

[17] J. Henn, *Duality between Wilson loops and gluon amplitudes*, Fortsch. Phys. 57 (2009) 729 [arXiv:0903.0522] [nSPIRE].

[18] S. Lee, S. Minwalla, M. Rangamani and N. Seiberg, *Three point functions of chiral operators in D = 4 N = 4 SYM at large-N*, Adv. Theor. Math. Phys. 2 (1998) 697 [hep-th/9806074] [nSPIRE].

[19] B. Eden, P.S. Howe and P.C. West, *Nilpotent invariants in N = 4 SYM*, Phys. Lett. B 463 (1999) 19 [hep-th/9905085] [nSPIRE].

[20] G. Arutyunov, B. Eden and E. Sokatchev, *On nonrenormalization and OPE in superconformal field theories*, Nucl. Phys. B 619 (2001) 359 [hep-th/0105254] [nSPIRE].

[21] P. Heslop and P.S. Howe, *OPEs and three-point correlators of protected operators in N = 4 SYM*, Nucl. Phys. B 626 (2002) 265 [hep-th/0107212] [nSPIRE].

[22] A. Basu, M.B. Green and S. Sethi, *Some systematics of the coupling constant dependence of N = 4 Yang-Mills*, JHEP 09 (2004) 045 [hep-th/0406231] [nSPIRE].

[23] M. Baggio, J. de Boer and K. Papadodimas, *A non-renormalization theorem for chiral primary 3-point functions*, JHEP 07 (2012) 137 [arXiv:1203.1036] [nSPIRE].

[24] M. Bianchi, S. Kovacs, G. Rossi and Y.S. Stanev, *Properties of the Konishi multiplet in N = 4 SYM theory*, JHEP 05 (2001) 042 [hep-th/0104016] [nSPIRE].

[25] N. Beisert, C. Kristjansen, J. Plefka, G. Semenoff and M. Staudacher, *BMN correlators and operator mixing in N = 4 super Yang-Mills theory*, Nucl. Phys. B 650 (2003) 125 [hep-th/0208178] [nSPIRE].

[26] C.-S. Chu, V.V. Khoze and G. Travaglini, *Three point functions in N = 4 Yang-Mills theory and pp waves*, JHEP 06 (2002) 011 [hep-th/0206005] [nSPIRE].

[27] R. Roiban and A. Volovich, *Yang-Mills correlation functions from integrable spin chains*, JHEP 09 (2004) 032 [hep-th/0407140] [nSPIRE].

[28] K. Okuyama and L.-S. Tseng, *Three-point functions in N = 4 SYM theory at one-loop*, JHEP 08 (2004) 055 [hep-th/0404190] [nSPIRE].

[29] L.F. Alday, J.R. David, E. Gava and K. Narain, *Structure constants of planar N = 4 Yang-Mills at one loop*, JHEP 09 (2005) 070 [hep-th/0502186] [nSPIRE].

[30] L.F. Alday, J.R. David, E. Gava and K. Narain, *Towards a string bit formulation of N = 4 super Yang-Mills*, JHEP 04 (2006) 014 [hep-th/0510264] [nSPIRE].

[31] G. Georgiou, V.L. Gili and R. Russo, *Operator mixing and the AdS/CFT correspondence*, JHEP 01 (2009) 082 [arXiv:0810.0499] [nSPIRE].
G. Georgiou, V.L. Gili and R. Russo, Operator mixing and three-point functions in $N=4$ SYM, *JHEP* **10** (2009) 009 [arXiv:0907.1567] [insPIRE].

G. Georgiou, V. Gili, A. Grossardt and J. Plefka, Three-point functions in planar $N=4$ super Yang-Mills theory for scalar operators up to length five at the one-loop order, *JHEP* **04** (2012) 038 [arXiv:1201.0992] [insPIRE].

J. Escobedo, N. Gromov, A. Sever and P. Vieira, Tailoring three-point functions and integrability, *JHEP* **09** (2011) 028 [arXiv:1012.2475] [insPIRE].

J. Escobedo, N. Gromov, A. Sever and P. Vieira, Tailoring three-point functions and integrability II. Weak/strong coupling match, *JHEP* **09** (2011) 029 [arXiv:1104.5501] [insPIRE].

N. Gromov, A. Sever and P. Vieira, Tailoring three-point functions and integrability III. Classical tunneling, *JHEP* **07** (2012) 044 [arXiv:1111.2349] [insPIRE].

I. Kostov, Classical limit of the three-point function of $N=4$ supersymmetric Yang-Mills theory from integrability, *Phys. Rev. Lett.* **108** (2012) 261604 [arXiv:1203.6180] [insPIRE].

N. Gromov and P. Vieira, Quantum integrability for three-point functions, arXiv:1202.4103 [insPIRE].

J. Russo and A. Tseytlin, Large spin expansion of semiclassical 3-point correlators in $AdS_5 \times S^5$, *JHEP* **02** (2011) 029 [arXiv:1012.2760] [insPIRE].

A. Bissi, C. Kristjansen, D. Young and K. Zoubos, Holographic three-point functions of giant gravitons, *JHEP* **06** (2011) 085 [arXiv:1103.4079] [insPIRE].

A. Bissi, T. Harmark and M. Orselli, Holographic 3-point function at one loop, *JHEP* **02** (2012) 133 [arXiv:1112.5075] [insPIRE].

G. Grignani and A. Zayakin, Matching three-point functions of BMN operators at weak and strong coupling, *JHEP* **06** (2012) 142 [arXiv:1204.3096] [insPIRE].

G. Grignani, A. Zayakin and A. Zayakin, Three-point functions of BMN operators at weak and strong coupling II. One loop matching, *JHEP* **09** (2012) 087 [arXiv:1205.5279] [insPIRE].

G. Georgiou, SL(2) sector: weak/strong coupling agreement of three-point correlators, *JHEP* **09** (2011) 132 [arXiv:1107.1850] [insPIRE].

E. Fradkin and M. Palchik, Conformal quantum field theory in $D$-dimensions, Springer, Berlin Germany (1994).

G. Sotkov and R. Zaikov, Conformal invariant two point and three point functions for fields with arbitrary spin, *Rept. Math. Phys.* **12** (1977) 375.

V. Dobrev, V. Petkova, S. Petrova and I. Todorov, Dynamical derivation of vacuum operator product expansion in Euclidean conformal quantum field theory, *Phys. Rev. D* **13** (1976) 887 [insPIRE].

A. Belitsky, J. Henn, C. Jarzak, D. Mueller and E. Sokatchev, Anomalous dimensions of leading twist conformal operators, *Phys. Rev. D* **77** (2008) 045029 [arXiv:0707.2936] [insPIRE].

F. Dolan and H. Osborn, Conformal partial wave expansions for $N=4$ chiral four point functions, *Annals Phys.* **321** (2006) 581 [hep-th/0412335] [insPIRE].
[50] B. Eden, P. Heslop, G.P. Korchemsky and E. Sokatchev, *Hidden symmetry of four-point correlation functions and amplitudes in N = 4 SYM*, *Nucl. Phys. B* 862 (2012) 193 [arXiv:1108.3557] [inSPIRE].

[51] B. Eden, *Three-loop universal structure constants in N = 4 SUSY Yang-Mills theory*, arXiv:1207.3112 [inSPIRE].

[52] K. Wiegandt, *Superconformal quantum field theories in string-gauge theory dualities*, Ph.D. Thesis, Humboldt-Universität zu Berlin, Berlin Germany (2012).

[53] M.S. Costa, R. Monteiro, J.E. Santos and D. Zoakos, *On three-point correlation functions in the gauge/gravity duality*, JHEP 11 (2010) 141 [arXiv:1008.1070] [inSPIRE].

[54] N.I. Usyukina and A.I. Davydychev, *New results for two loop off-shell three point diagrams*, Phys. Lett. B 332 (1994) 159 [hep-ph/9402223] [inSPIRE].

[55] D. Kazakov and A. Kotikov, *The method of uniqueness: multiloop calculations in QCD*, Theor. Math. Phys. 73 (1988) 1264 [inSPIRE].

[56] T. Ohrndorf, *Constraints from conformal covariance on the mixing of operators of lowest twist*, Nucl. Phys. B 198 (1982) 26 [inSPIRE].

[57] Y. Makeenko, *Conformal operators in quantum chromodynamics*, Sov. J. Nucl. Phys. 33 (1981) 440 [inSPIRE].

[58] V.A. Smirnov, *Evaluating Feynman integrals*, Springer Tracts Mod. Phys. 211 (2004) 1.