A generalized action for \((2 + 1)\)-dimensional Chern–Simons gravity

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Abstract

We show that the so-called semi-simple extended Poincaré (SSEP) algebra in \(D\) dimensions can be obtained from the anti-de Sitter algebra \(so(D − 1, 2)\) by means of the \(S\)-expansion procedure with an appropriate semigroup \(S\). A general prescription is given for computing Casimir operators for \(S\)-expanded algebras, and the method is exemplified for the SSEP algebra. The \(S\)-expansion method also allows us to extract the corresponding invariant tensor for the SSEP algebra, which is a key ingredient in the construction of a generalized action for Chern–Simons gravity in \((2 + 1)\) dimensions.

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1. Introduction

In [1–4], the Poincaré algebra of rotations \(J_{ab}\) and translations \(P_a\) in \(D\)-dimensional spacetime have been extended by the inclusion of the second-rank tensor generator \(Z_{ab}\) in the following way:

\[
[J_{ab}, J_{cd}] = \eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac},
\]

\[
[J_{ab}, P_c] = \eta_{bc}P_a - \eta_{ac}P_b,
\]

\[
[P_a, P_b] = cZ_{ab},
\]

\[
[J_{ab}, Z_{cd}] = \eta_{ad}Z_{bc} + \eta_{bc}Z_{ad} - \eta_{ac}Z_{bd} - \eta_{bd}Z_{ac},
\]

\[
[Z_{ab}, P_c] = \frac{4a^2}{c}(\eta_{bc}P_a - \eta_{ac}P_b),
\]
\[ [Z_{ab}, Z_{cd}] = \frac{4a^2}{c} [\eta_{bd}Z_{bc} + \eta_{bc}Z_{bd} - \eta_{ac}Z_{ad} - \eta_{ad}Z_{ac}], \]  

(6)

where \(a\) and \(c\) are constants. It is remarkable that the Lie algebra (1)–(6) is semi-simple, in contrast to the Poincaré and extended Poincaré algebras (cf equations (1.1) and (1.2) of [3]). Note that in the \(a \to 0\) limit, the algebra (1)–(6) reduces to the algebra in equation (1.2) of [3]. The semi-simple extended Poincaré (SSEP) algebra (1)–(6) can be rewritten in the form

\[ [N_{ab}, N_{cd}] = \eta_{bd} N_{bc} + \eta_{bc} N_{bd} - \eta_{ac} N_{ad} - \eta_{ad} N_{ac}, \]  

(7)

\[ [L_{AB}, L_{CD}] = \eta_{AD} N_{BC} + \eta_{BC} N_{AD} - \eta_{AC} N_{BD} - \eta_{BD} N_{AC}, \]  

(8)

\[ [N_{ab}, L_{cd}] = 0, \]  

(9)

where the metric tensor \(\eta_{AB}\) is given by

\[ \eta_{AB} = \begin{bmatrix} \eta_{ab} & 0 \\ 0 & -1 \end{bmatrix} \]  

(10)

and the \(N_{ab}\) generators read

\[ N_{ab} = J_{ab} - \frac{c}{4a^2} Z_{ab}. \]  

(11)

The \(N_{ab}\) generators form a basis for the Lorentz algebra \(\mathfrak{so} (D - 1, 1)\). The \(L_{AB}\) generators, on the other hand, are given by

\[ L_{AB} = \begin{bmatrix} L_{ab} & L_{a,D} \\ L_{D,a} & L_{D,D} \end{bmatrix} = \begin{bmatrix} \frac{c}{4a^2} Z_{ab} & \frac{1}{2a} P_a \\ -\frac{1}{2a} P_a & 0 \end{bmatrix} \]  

(12)

and form a basis for the anti-de Sitter (AdS) \(\mathfrak{so} (D - 1, 2)\) algebra. The SSEP algebra (7)–(9) is thus seen to be the direct sum \(\mathfrak{so} (D - 1, 1) \oplus \mathfrak{so} (D - 1, 2)\) of the \(D\)-dimensional Lorentz algebra and the \(D\)-dimensional AdS algebra.

Using (11) and (12) in (7)–(9), we find that the SSEP algebra (1)–(6) can be written in the form

\[ [N_{ab}, N_{cd}] = \eta_{bd} N_{bc} + \eta_{bc} N_{bd} - \eta_{ac} N_{ad} - \eta_{ad} N_{ac}, \]  

(13)

\[ [L_{ab}, L_{cd}] = \eta_{bd} L_{bc} + \eta_{bc} L_{bd} - \eta_{ac} L_{ad} - \eta_{ad} L_{ac}, \]  

(14)

\[ [L_{ab}, L_{c,D}] = \eta_{bc} L_{a,D} - \eta_{ac} L_{b,D}, \]  

(15)

\[ [L_{a,D}, L_{c,D}] = L_{ac}, \]  

(16)

\[ [N_{ab}, L_{cd}] = 0, \]  

(17)

\[ [N_{ab}, L_{c,D}] = 0. \]  

(18)

It is the purpose of this paper to show that the SSEP algebra \(\mathfrak{so} (D - 1, 1) \oplus \mathfrak{so} (D - 1, 2)\) can be obtained from the AdS algebra \(\mathfrak{so} (D - 1, 2)\) via the \(S\)-expansion procedure with an appropriate semigroup \(S\) [6, 7]. The \(S\)-expansion method also allows us to compute an invariant tensor for SSEP algebra, which is a key ingredient in the construction of the more general action for Chern–Simons gravity in \((2 + 1)\) dimensions.
2. The \( S \)-expansion procedure

In this section we briefly review the general Abelian semigroup expansion procedure (\( S \)-expansion for short; for details, see [6]). Consider a Lie algebra \( \mathfrak{g} \) and a finite Abelian semigroup \( S = \{\lambda_\alpha\} \). According to theorem 3.1 from [6], the direct product \( S \times \mathfrak{g} \) is also a Lie algebra. Interestingly, there are cases when it is possible to systematically extract subalgebras from \( S \times \mathfrak{g} \). Start by decomposing \( \mathfrak{g} \) in a direct sum of subspaces, as in \( \mathfrak{g} = \bigoplus_{I \in \mathbb{I}} \mathfrak{g}_I \), where \( \mathbb{I} \) is a set of indices. The internal structure of \( \mathfrak{g} \) can be codified through the mapping \( i : I \times I \rightarrow 2^I \) according to \( [\mathfrak{g}_I, \mathfrak{g}_J] \subseteq \bigoplus_{(p,q) \in \mathbb{I} \times \mathbb{I}} \mathfrak{g}_{Ipq} \). When the semigroup \( S \) can be decomposed in subsets \( S_p \), \( S = \bigcup_{p \in \mathbb{I}} S_p \), such that they satisfy the ‘resonant condition’ \( S_p \cdot S_q \subseteq \bigcap_{(p,q) \in \mathbb{I} \times \mathbb{I}} S_r \), then we have that \( \mathfrak{g}_R = \bigoplus_{p \in \mathbb{I}} S_p \times \mathfrak{g}_p \) is a ‘resonant subalgebra’ of \( S \times \mathfrak{g} \) (see theorem 4.2 from [6]).

An even smaller algebra can be obtained when there is a zero element in the semigroup, i.e. an element \( 0_\alpha \in S \) such that, for all \( \lambda_\alpha \in S, 0_\alpha \lambda_\alpha = 0_\alpha \). When this is the case, the whole \( 0_\alpha \times \mathfrak{g} \) sector can be removed from the resonant subalgebra by imposing \( 0_\alpha \times \mathfrak{g} = 0 \). The remaining piece, to which we refer as the \( 0_\alpha \)-reduced algebra, continues to be a Lie algebra (see \( 0_\alpha \)-reduction and theorem 6.1 from [6]).

3. \( S \)-expansion of the AdS algebra

In this section, we sketch the steps to be undertaken in order to obtain the SSEP algebra, \( \mathfrak{so}(D-1,1) \oplus \mathfrak{so}(D-1,2) \), as an \( S \)-expansion of the AdS algebra \( \mathfrak{so}(D-1,2) \).

The first step consists of splitting the AdS algebra in subspaces, i.e. \( \mathfrak{g} = \mathfrak{so}(D-1,2) = \mathfrak{V}_0 \oplus \mathfrak{V}_1 \), where \( \mathfrak{V}_0 \) corresponds to the Lorentz subalgebra \( \mathfrak{so}(D-1,1) \), which is generated by \( J_{ab} \), and \( \mathfrak{V}_1 \) corresponds to the AdS ‘boosts’, generated by \( P_a \). The generators \( J_{ab} \) and \( P_a \) satisfy the following commutation relations:

\[
\begin{align*}
[\mathfrak{P}_a, \mathfrak{P}_b] &= \mathfrak{J}_{ab}, \\
[\mathfrak{J}_{ab}, \mathfrak{P}_c] &= \eta_{bc} \mathfrak{P}_a - \eta_{ca} \mathfrak{P}_b, \\
[\mathfrak{J}_{ab}, \mathfrak{J}_{cd}] &= \eta_{ad} \mathfrak{J}_{bc} + \eta_{bd} \mathfrak{J}_{ac} - \eta_{ac} \mathfrak{J}_{bd} - \eta_{bc} \mathfrak{J}_{ad}.
\end{align*}
\]

The subspace structure can be written as

\[
\begin{align*}
[\mathfrak{V}_1, \mathfrak{V}_1] &\subseteq \mathfrak{V}_0, \\
[\mathfrak{V}_0, \mathfrak{V}_1] &\subseteq \mathfrak{V}_1, \\
[\mathfrak{V}_0, \mathfrak{V}_0] &\subseteq \mathfrak{V}_0.
\end{align*}
\]

The second step consists of finding an Abelian semigroup \( S \) which can be partitioned in a ‘resonant’ way with respect to equations (22)–(24). We shall consider the expansion using two different semigroups.

3.1. Semigroup \( S_{33} \)

Let us consider first the semigroup \( S_{33} = \{\tilde{\lambda}_0, \tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3\} \) defined by the following multiplication table:

\[
\begin{array}{cccc}
\tilde{\lambda}_0 & \tilde{\lambda}_1 & \tilde{\lambda}_2 & \tilde{\lambda}_3 \\
\tilde{\lambda}_0 & \tilde{\lambda}_2 & \tilde{\lambda}_3 & \tilde{\lambda}_0 \\
\tilde{\lambda}_1 & \tilde{\lambda}_3 & \tilde{\lambda}_1 & \tilde{\lambda}_3 \\
\tilde{\lambda}_2 & \tilde{\lambda}_0 & \tilde{\lambda}_3 & \tilde{\lambda}_2 \\
\tilde{\lambda}_3 & \tilde{\lambda}_3 & \tilde{\lambda}_3 & \tilde{\lambda}_3.
\end{array}
\]
A straightforward but important observation is that for each \( \lambda_\alpha \in S \), we have that \( \tilde{\lambda}_3 \lambda_\alpha = \tilde{\lambda}_3 \), so that \( \tilde{\lambda}_3 \) is seen to play the role of a zero element inside \( S \).

Consider now the partition \( S = S_0 \cup S_1 \), with

\[
S_0 = \{ \tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3 \}.
\]

\[
S_1 = \{ \tilde{\lambda}_0, \tilde{\lambda}_3 \}.
\]

This partition is said to be resonant, since it satisfies (cf equations (22)–(24))

\[
S_0 \cdot S_0 \subset S_0,
\]

\[
S_0 \cdot S_1 \subset S_1,
\]

\[
S_1 \cdot S_1 \subset S_0.
\]

Theorem 4.2 from [6] now ensures that

\[
\mathfrak{g}_R = W_0 \oplus W_1
\]

is a resonant subalgebra of \( S_{\mathfrak{g}_3} \times g \), where

\[
W_0 = (S_0 \times V_0) = \{ \tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3 \} \otimes \{ J_{ab} \} = \{ \tilde{\lambda}_1 J_{ab}, \tilde{\lambda}_2 J_{ab}, \tilde{\lambda}_3 J_{ab} \},
\]

\[
W_1 = (S_1 \times V_1) = \{ \tilde{\lambda}_0, \tilde{\lambda}_3 \} \otimes \{ P_a \} = \{ \tilde{\lambda}_0 P_a, \tilde{\lambda}_3 P_a \}.
\]

As a last step, impose the condition \( \lambda_3 \times g = 0 \) on \( \mathfrak{g}_R \) and relabel its generators as \( J_{ab,1} = \tilde{\lambda}_1 J_{ab}; J_{ab,2} = \tilde{\lambda}_2 J_{ab}; \) and \( P_{a,0} = \lambda_0 P_a \). This procedure leads us to the following commutation relations:

\[
[J_{ab,1}, J_{cd,1}] = \tilde{\lambda}_1 \lambda_1 [J_{ab}, J_{cd}] = \tilde{\lambda}_1 [J_{ab}, J_{cd}]
= \eta_{bc} J_{cd,1} + \eta_{ac} J_{bd,1} - \eta_{ad} J_{bc,1}
\]

\[
[J_{ab,2}, J_{cd,2}] = \tilde{\lambda}_2 \lambda_2 [J_{ab}, J_{cd}] = \tilde{\lambda}_2 [J_{ab}, J_{cd}]
= \eta_{bc} J_{cd,2} + \eta_{ac} J_{bd,2} - \eta_{ad} J_{bc,2}
\]

\[
[J_{ab,1}, J_{cd,2}] = \tilde{\lambda}_1 \lambda_2 [J_{ab}, J_{cd}] = \tilde{\lambda}_3 [J_{ab}, J_{cd}] = 0
\]

\[
[J_{ab,1}, P_{a,0}] = \tilde{\lambda}_1 \lambda_0 [J_{ab}, P_a] = \tilde{\lambda}_3 [J_{ab}, P_a] = 0
\]

\[
[P_{a,0}, P_{b,0}] = \tilde{\lambda}_0 \lambda_0 [P_a, P_b] = \tilde{\lambda}_2 [P_a, P_b] = \tilde{\lambda}_3 J_{ab} = J_{ab,2},
\]

where we have used the commutation relations of the AdS algebra and the multiplication law (25) of the semigroup.

The identification \( N_{ab} = J_{ab,1}; L_{ab} = J_{ab,2}; L_{abD+1} = P_{a,0} \) shows that the algebra (34)–(39), obtained by \( S_{\mathfrak{g}_3} \)-expansion and \( O_2 \)-reduction of the AdS algebra \( SO(D - 1, 2) \), coincides with the SSEP algebra (13)–(18) obtained by semi-simple extension of the Poincaré algebra in [1–3].
3.2. Semigroup $S_{22}$

Let us now consider the semigroup $S_{22} = \{\lambda_0, \lambda_1, \lambda_2\}$ defined by the multiplication law

$$\lambda_\alpha \lambda_\beta = \begin{cases} 
\lambda_{\alpha + \beta} & \text{if } \alpha + \beta \leq 2 \\
\lambda_{\alpha + \beta - 2} & \text{if } \alpha + \beta > 2
\end{cases} \quad (40)$$

or, equivalently, by the multiplication table

\[
\begin{array}{c|ccc}
\lambda_0 & \lambda_0 & \lambda_1 & \lambda_2 \\
\lambda_0 & \lambda_0 & \lambda_1 & \lambda_2 \\
\lambda_1 & \lambda_1 & \lambda_2 & \lambda_1 \\
\lambda_2 & \lambda_2 & \lambda_1 & \lambda_2
\end{array}
\]

(41)

Take now the partition $S = S_0 \cup S_1$, with

$$S_0 = \{\lambda_0, \lambda_2\}, \quad (42)$$

$$S_1 = \{\lambda_1\}. \quad (43)$$

This partition is said to be resonant, since it satisfies (cf equations (22)–(24))

$$S_0 \cdot S_0 \subseteq S_0, \quad (44)$$

$$S_0 \cdot S_1 \subseteq S_1, \quad (45)$$

$$S_1 \cdot S_1 \subseteq S_0. \quad (46)$$

Theorem 4.2 from [6] now ensures that

$$\mathfrak{g}_R = W_0 \oplus W_1 \quad (47)$$

is a resonant subalgebra of $S_{22} \times \mathfrak{g}$, where

$$W_0 = (S_0 \times V_0) = \{\lambda_0, \lambda_2\} \otimes \{\tilde{I}_{ab}\} = \{\lambda_0 \tilde{I}_{ab}, \lambda_2 \tilde{I}_{ab}\}, \quad (48)$$

$$W_1 = (S_1 \times V_1) = \{\lambda_1\} \otimes \{\tilde{P}_a\} = \{\lambda_1 \tilde{P}_a\}. \quad (49)$$

Relabeling the generators of the resonant subalgebra as $\tilde{J}_{ab,0} = \lambda_0 \tilde{I}_{ab}$; $\tilde{J}_{ab,2} = \lambda_2 \tilde{I}_{ab}$; and $\tilde{P}_{a,1} = \lambda_1 \tilde{P}_a$, we are left with the following commutation relations:

$$[\tilde{J}_{ab,0}, \tilde{J}_{cd,0}] = \lambda_0 [\tilde{J}_{ab,0}, \tilde{J}_{cd,0}] = \lambda_0 [\tilde{I}_{ab,0}, \tilde{I}_{cd,0}] = \eta_{ab} \tilde{I}_{bc,0} + \eta_{bc} \tilde{I}_{ab,0} - \eta_{ac} \tilde{I}_{bd,0} - \eta_{bd} \tilde{I}_{ac,0} \quad (50)$$

$$[\tilde{J}_{ab,2}, \tilde{J}_{cd,2}] = \lambda_2 [\tilde{J}_{ab,2}, \tilde{J}_{cd,2}] = \lambda_2 [\tilde{I}_{ab,2}, \tilde{I}_{cd,2}] = \eta_{ad} \tilde{I}_{bc,2} + \eta_{bc} \tilde{I}_{ad,2} - \eta_{ac} \tilde{I}_{bd,2} - \eta_{bd} \tilde{I}_{ac,2} \quad (51)$$

$$[\tilde{J}_{ab,0}, \tilde{J}_{cd,2}] = \lambda_0 \lambda_2 [\tilde{J}_{ab,0}, \tilde{J}_{cd,2}] = \lambda_2 [\tilde{I}_{ab,0}, \tilde{I}_{cd,2}] = \eta_{ad} \tilde{I}_{bc,2} + \eta_{bc} \tilde{I}_{ad,2} - \eta_{ac} \tilde{I}_{bd,2} - \eta_{bd} \tilde{I}_{ac,2} \quad (52)$$

$$[\tilde{J}_{ab,0}, \tilde{P}_{c,1}] = \lambda_0 \lambda_1 [\tilde{J}_{ab,0}, \tilde{P}_{c,1}] = \lambda_1 [\tilde{J}_{ab,0}, \tilde{P}_{c,1}] = \eta_{bc} \tilde{P}_{a,1} - \eta_{ac} \tilde{P}_{b,1} \quad (53)$$

$$[\tilde{J}_{ab,2}, \tilde{P}_{c,1}] = \lambda_2 \lambda_1 [\tilde{J}_{ab,2}, \tilde{P}_{c,1}] = \lambda_1 [\tilde{J}_{ab,2}, \tilde{P}_{c,1}] = \eta_{bc} \tilde{P}_{a,1} - \eta_{ac} \tilde{P}_{b,1} \quad (54)$$

$$[\tilde{P}_{a,1}, \tilde{P}_{b,1}] = \lambda_1 \lambda_1 [\tilde{P}_{a,1}, \tilde{P}_{b,1}] = \lambda_2 [\tilde{P}_{a,1}, \tilde{P}_{b,1}] = \lambda_2 \tilde{I}_{ab} = \tilde{J}_{ab,2}. \quad (55)$$
where we have used the commutation relations of the AdS algebra and the multiplication law (40) of the semigroup $S_{S^2}$.

The identification $\tilde{J}_{ab} = \tilde{J}_{ab,0}$; $\tilde{Z}_{ab} = \tilde{J}_{ab,2}$; and $\tilde{P}_a = \tilde{P}_{a,1}$ lead to the following algebra:

$$\{ \tilde{J}_{ab}, \tilde{J}_{cd} \} = \eta_{ad} \tilde{J}_{bc} + \eta_{bc} \tilde{J}_{ad} - \eta_{ac} \tilde{J}_{bd} - \eta_{bd} \tilde{J}_{ac},$$  
(56)

$$\{ \tilde{J}_{ab}, \tilde{P}_c \} = \eta_{bc} \tilde{P}_a - \eta_{ac} \tilde{P}_b,$$  
(57)

$$\{ \tilde{P}_a, \tilde{P}_b \} = \tilde{Z}_{ab},$$  
(58)

$$\{ \tilde{J}_{ab}, \tilde{Z}_{cd} \} = \eta_{ad} \tilde{Z}_{bc} + \eta_{bc} \tilde{Z}_{ad} - \eta_{ac} \tilde{Z}_{bd} - \eta_{bd} \tilde{Z}_{ac},$$  
(59)

$$\{ \tilde{Z}_{ab}, \tilde{P}_c \} = \eta_{bc} \tilde{P}_a - \eta_{ac} \tilde{P}_b,$$  
(60)

$$\{ \tilde{Z}_{ab}, \tilde{Z}_{cd} \} = \eta_{ad} \tilde{Z}_{bc} + \eta_{bc} \tilde{Z}_{ad} - \eta_{ac} \tilde{Z}_{bd} - \eta_{bd} \tilde{Z}_{ac},$$  
(61)

which matches the SSEP algebra (1)–(6) obtained in [1–4] up to (inessential) numerical factors.

### 3.3. Relationship between multiplication tables of semigroups $S_{S^3}$ and $S_{S^2}$

In section 3.1, the SSEP algebra (13)–(18) was obtained through an $S$-expansion using the semigroup $S_{S^3}$, whose multiplication table given in equation (25) was constructed by following the procedure given in [5]. The process also involves imposing the condition known as $0_S$-reduction [6].

In section 3.2, the SSEP algebra (1)–(6) was obtained (up to inessential numerical factors) through an $S$-expansion using the semigroup $S_{S^2}$, whose multiplication table is given in (41). In stark contrast with the previous case, the procedure does not involve imposing 0-reduction.

This curious state of affairs can be clarified by promoting the semigroup $S_{S^2}$ to a ring (here we do not require that the elements of the ring form a group under multiplication, but rather only a semigroup) and setting

$$\lambda_1 = \lambda_0 - \lambda_2$$

$$\lambda_2 = \lambda_2$$

$$\lambda_0 = \lambda_1.$$  
(62)

This amounts to a change of basis in $S_{S^2}$ and leads to the following multiplication table:

| $\tilde{\lambda}_0$ | $\tilde{\lambda}_1$ | $\tilde{\lambda}_2$ | 
|---------------------|---------------------|---------------------|
| $\lambda_0$        | $\tilde{\lambda}_2$ | $\lambda_0$         |
| $\lambda_1$        | $\tilde{\lambda}_1$ | $\lambda_1$         |
| $\lambda_2$        | $\tilde{\lambda}_0$ | $\lambda_2$         |

This multiplication table exactly matches the multiplication table of the $S_{S^3}$ semigroup (see equation (25)) except for the rows and columns involving $\lambda_3$. In place of $\lambda_3$, the symbol ‘0’ in (63) now stands for the additive zero of the $S_{S^2}$ ring.

The generators $N_{ab}$ and $L_{AB}$ can be recovered by setting

$$N_{ab} = \tilde{\lambda}_1 \tilde{J}_{ab} = (\lambda_0 - \lambda_2) \tilde{J}_{ab}$$  
(64)

$$L_{ab} = \tilde{\lambda}_2 \tilde{J}_{ab} = \lambda_2 \tilde{J}_{ab}$$  
(65)

$$L_{aD} = \tilde{\lambda}_0 \tilde{P}_a = \lambda_1 \tilde{P}_a$$  
(66)

without invoking $0_S$-reduction. The advantage of not using $0_S$-reduction is that it facilitates the construction of Casimir operators, as discussed in section 4.
4. Casimir operators for S-expanded Lie algebras

In this section we consider the construction of Casimir operators for S-expanded Lie algebras. We then compute the Casimir operators for the SSEP algebra obtained by Soroka et al in [1–4].

4.1. Construction of Casimir operators for S-expanded Lie algebras

Let $g$ be a Lie algebra and let $\{T_A, A = 1, \ldots, \dim g\}$ be the generators of $g$. A Casimir operator $C_m$ of degree $m$ can be written as

$$ C_m = C^{A_1 \ldots A_m} T_{A_1} \ldots T_{A_m} \tag{67} $$

which, by definition, satisfies the condition that $\forall T_{A_0} \in g$,

$$ [T_{A_0}, C_m] = 0. \tag{68} $$

where the coefficient $C^{A_1 \ldots A_m}$ forms a symmetric invariant tensor for the corresponding Lie group. This means that the operators $C_m$ ($m = 2, 3, \ldots$) are invariants of the enveloping algebra. From (67) and (68), we have

$$ [T_{A_0}, C_m] = \left( \sum_{p=1}^{m} s_{A_B}^{(A_1 \ldots A_p-1)B} A_p A_0 C_B^{A_1 \ldots A_m} \right) T_{A_1} \ldots T_{A_m} \tag{69} $$

where $f_{AB}^C$ are the structure constants of $g$. Therefore, the ‘Casimir condition’ (68) is seen to be equivalent to

$$ \sum_{p=1}^{m} s_{A_B}^{(A_1 \ldots A_p-1)B} A_p A_0 C_B^{A_1 \ldots A_m} = 0. \tag{70} $$

For the standard, quadratic (i.e. $m = 2$) Casimir operator, equation (70) reads

$$ f_{AB} A_1 C_B^{A_1 A_2} + f_{AB} A_2 C_B^{A_2 A_1} = 0. \tag{71} $$

The structure constants of an S-expanded Lie algebra are given by

$$ f_{(A, \alpha)(B, \beta)}^{(C, \gamma)} = K_{\alpha \gamma}^\beta f_{AB}^C, \tag{72} $$

where $K_{\alpha \gamma}^\beta$ stands for the ‘2-selector’ of the semigroup $S$ [6]. The (quadratic) Casimir condition for an S-expanded Lie algebra thus reads

$$ K_{\alpha \gamma}^\beta f_{AB}^{(A_1 \ldots A_2)(B_1 \ldots B_2)} + K_{\alpha \gamma}^\beta f_{AB}^{(A_2 \ldots A_1)(B_1 \ldots B_2)} = 0. \tag{73} $$

Consider now the following ansatz for the components of the (quadratic) Casimir operator of an S-expanded algebra

$$ C^{(A_1, \alpha_1)(B, \beta)} = m^{AB} C^{AB}, \tag{74} $$

where $C^{AB}$ are the components of the (quadratic) Casimir operator for the original algebra $g$ and $m^{AB}$ are the components of a symmetric tensor, associated with the semigroup $S$, which must be determined.

Introducing (72) in (73), we obtain

$$ K_{\alpha \gamma}^\beta f_{AB}^{(A_1 \ldots A_2)B_2} + K_{\alpha \gamma}^\beta f_{AB}^{(A_2 \ldots A_1)B_2} = 0. \tag{75} $$

Equation (75) is satisfied if the following condition holds:

$$ K_{\alpha \gamma}^\beta f_{AB}^{(A_1 \ldots A_2)} = K_{\alpha \gamma}^\beta f_{AB}^{(A_2 \ldots A_1)}. \tag{76} $$
To check this, let us plug equation (76) into equation (75) to find
\[ K_{\alpha_1\alpha_0\beta} \alpha_2 f_{\alpha_2\beta} A_1 C^{\alpha_1} + K_{\alpha_0\beta} \alpha_2 m^{\alpha_1\beta} f_{\alpha_2\beta} A_2 C^{\alpha_0} + f_{\alpha_2\beta} A_2 C^{\alpha_0} = 0, \]
where the expression in parentheses vanishes because \( C^{\alpha_1\beta} \) are the components of the (quadratic) Casimir operator for the original algebra \( g \) (cf equation (72)).

The following theorem provides us with a way of finding a tensor \( m^{\alpha\beta} \) with the required properties.

**Theorem.** Let \( K_{\alpha\beta} \) be the 2-selector for an Abelian semigroup \( S \), and define
\[ m_{\alpha\beta} = \alpha_y K_{\alpha\beta}^y, \]
where \( \alpha_y \) are numerical coefficients. If \( \alpha_y \) are chosen in such a way that \( m_{\alpha\beta} \) is an invertible ‘metric’, then its inverse \( m_{\alpha\beta} \), (which, by definition, satisfies \( m^{\alpha\lambda} m_{\lambda\beta} = \delta_{\alpha\beta} \)) fulfils equation (76).

**Proof.** From the associativity and commutativity of the inner binary operation (multiplication) of the semigroup \( S \), we have
\[ (\lambda_{\alpha\lambda\beta} \lambda_{\mu})_{\nu} = (\lambda_{\alpha\lambda\beta} \lambda_{\mu})_{\nu}. \]
In terms of the 2-selectors \( K_{\alpha\beta} \), equation (79) may be cast as
\[ K_{\alpha_1\lambda} \alpha_1 \lambda = m_{\alpha_1\lambda} K_{\lambda\mu} \lambda. \]
Multiplying (80) by \( \alpha_\lambda \), we find
\[ K_{\alpha_1\lambda} \alpha_1 \lambda = m_{\alpha_1\lambda} K_{\lambda\mu} \lambda, \]
so that
\[ m_{\alpha_1\lambda} = K_{\alpha_1\lambda} K_{\lambda\mu} \lambda. \]

This means that if \( C = C^{\alpha\beta} T_{\alpha} T_{\beta} \) is the (quadratic) Casimir operator for the original algebra \( g \), then
\[ C = m^{\alpha\beta} C^{\alpha\beta} T_{\alpha} T_{\beta}, \]
is the (quadratic) Casimir operator for the \( S \)-expanded Lie algebra.

### 4.2. Casimir operators for AdS algebra

In this section we apply the general procedure developed in section 5 to the case of the SSEP algebra.

Using the representation given by the Dirac matrices for the AdS algebra,
\[ P_a = \frac{1}{2} \Gamma_a, \]
\[ J_{ab} = \frac{1}{2} \Gamma_{ab}, \]
we have that the Killing metric $k_{AB}$ for the AdS algebra can be written as
\[ k_{AB} = \frac{1}{\text{Tr}T} \left( T_A T_B \right) \]
which for $d \geq 4$ is given by
\[ k_{a,b} = \frac{1}{4} \eta_{ab} \]
\[ k_{ab,cd} = -\frac{1}{2} \eta_{[ab][cd]} \]
\[ k_{ab,c} = 0. \]
where
\[ \eta_{[ab][cd]} = \delta_{mn} \eta_{mc} \eta_{nd}. \]
For an arbitrary algebra, the quadratic Casimir operator is given by
\[ C = k^{AB} T_A T_B, \]
where $k^{AB}$ stands for the inverse of the Killing metric $k_{AB}$.

For the AdS algebra, we have
\[ k_{a,b} = 4 \eta^{ab} \]
\[ k_{ab,c} = 0 \]
\[ k_{ab,cd} = -\eta^{[ab][cd]} \]
so that
\[ C_{\text{AdS}} = 4 \left[ P^a P_a - \frac{1}{2} J_{ab} J^{ab} \right]. \]
This result is valid for any dimension $d \geq 4$.

There is another Killing ‘metric’ that can be constructed only in $d = 4$. This is given by
\[ \tilde{k}_{(A,B)} = \frac{1}{\text{Tr}T} \text{Tr} \left( \Gamma^a T_A T_B \right), \]
where $\Gamma^a$ is the usual $\gamma_5$ matrix. A direct calculation shows that
\[ \tilde{k}_{a,b} = 0 \]
\[ \tilde{k}_{ab,cd} = -\frac{1}{2} \epsilon_{abcd} \]
\[ \tilde{k}_{ab,c} = 0. \]
This ‘metric’, however, is not invertible, so that we cannot construct a Casimir operator for the AdS algebra from it. On second thought, it is possible to use this ‘metric’ to construct a Casimir operator for the Lorentz subalgebra, because, when so restricted, the metric turns out to be invertible. We find
\[ \tilde{k}^{(ab,cd)} = -\epsilon^{abcd}. \]
This means that a (quadratic) Casimir operator for the Lorentz group is given by
\[ \tilde{C}_L = -\epsilon^{abcd} J_{ab} J_{cd}. \]
5. Casimir operators for the SSEP algebra

We consider the construction of the metric $m_{ab}$ corresponding to the semigroup $S_{S2}$, whose multiplication law is given in equations (40) and (41). The semigroup $S_{S2}$ is interesting because although it is not a group, it is cyclic (i.e. similar to $\mathbb{Z}_3$). The elements of the semigroup can be represented by the following set of matrices:

$$
\begin{align*}
\lambda_0 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \lambda_1 &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, & \lambda_2 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.
\end{align*}
$$

(103)

It is straightforward to verify that the representation (103) faithfully satisfies equations (40) and (41). The 2-selectors $K_{ab}^{\rho}$ of $S_{S2}$ can be represented as (cf equations (1) and (2) from [5])

$$
K_{ab}^{0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K_{ab}^{1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad K_{ab}^{2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.
$$

(104)

This, in turn, implies that a generic metric $m_{ab}$ for $S_{S2}$ reads

$$
m_{ab} = \alpha_2 K_{ab}^\lambda
$$

(105)

where $\alpha_2$ are numerical coefficients. The inverse metric is given by

$$
m^{\rho\sigma} = \frac{1}{\det(m_{ab})} \begin{pmatrix}
\alpha_2^2 - \alpha_1^2 & 0 & - (\alpha_2^2 - \alpha_1^2) \\
0 & \alpha_2 (\alpha_0 - \alpha_2) - \alpha_1 (\alpha_0 - \alpha_2) \\
- (\alpha_2^2 - \alpha_1^2) & -\alpha_1 (\alpha_0 - \alpha_2) & \alpha_0 \alpha_2 - \alpha_1^2
\end{pmatrix},
$$

(106)

where $\alpha_0, \alpha_1$ and $\alpha_2$ must be chosen so that

$$
\det(m_{ab}) = (\alpha_0 - \alpha_2) (\alpha_2 + \alpha_1) (\alpha_2 - \alpha_1) \neq 0.
$$

(107)

The quadratic Casimir operators for the SSEP algebra have the form

$$
C = C^{(a,A)(b,B)} T_{(a,A)} T_{(b,B)} - m^{ab} C^{ab} T_{(a,A)} T_{(b,B)}
$$

$$
= m^{ab} C^{ab} T_{(a,A)} T_{(b,B)}
$$

$$
= m^{00} C^{ab,cd} J_{ab} J_{cd} + m^{11} C^{ab} P_a P_b + 2 m^{02} C^{ab,cd} J_{ab} Z_{cd} + m^{22} C^{ab,cd} Z_{ab} Z_{cd},
$$

$$
= \frac{1}{\det(m_{ab})} \left[ (\alpha_2^2 - \alpha_1^2) C^{ab,cd} J_{ab} J_{cd} + \alpha_2 (\alpha_0 - \alpha_2) C^{ab} P_a P_b
$$

$$
- 2 (\alpha_2^2 - \alpha_1^2) C^{ab,cd} J_{ab} Z_{cd} + (\alpha_0 \alpha_2 - \alpha_1^2) C^{ab,cd} Z_{ab} Z_{cd} \right],
$$

(108)

where $C^{ab}$ are the components of the Casimir operator for the AdS algebra. The $m^{12}$ term is absent from the sum because the corresponding components of the Casimir operator for the AdS algebra in $d \geq 4$ vanish, $C^{ab,c} = C^{a,bc} = 0$ (see equation (94)).

Plugging equations (93)–(95) into equation (108), we find

$$
C = \frac{4}{\det(m_{ab})} \left[ \frac{1}{2} (\alpha_2^2 - \alpha_1^2) J_{ab} J^{ab} + \alpha_2 (\alpha_0 - \alpha_2) P_a P^a
$$

$$
- (\alpha_2^2 - \alpha_1^2) J_{ab} Z^{ab} + \frac{1}{2} (\alpha_0 \alpha_2 - \alpha_1^2) Z_{ab} Z^{ab} \right].
$$

(109)

Defining

$$
\alpha = \alpha_2 \alpha_0 - \alpha_2^2; \quad \beta = \alpha_2 \alpha_0 - \alpha_1^2,
$$

(110)
equation (109) can be cast in the form
\[
C = \frac{4}{\det(m_{\alpha \beta})} \left[ \frac{1}{2} (\beta - \alpha) J_{ab} J^{ab} + \alpha P_a P^a - (\beta - \alpha) J_{ab} Z^{ab} + \frac{1}{2} \beta Z_{ab} Z^{ab} \right].
\]
(111)

Since \( \alpha \) and \( \beta \) are arbitrary, subject only to the condition \( \det(m_{\alpha \beta}) \neq 0 \), we can conclude that equation (111) shows that the SSEP algebra possesses two independent Casimir operators, namely
\[
C_1 = \frac{4\alpha}{\det(m_{\alpha \beta})} \left( P_a P^a + J_{ab} Z^{ab} - \frac{1}{2} J_{ab} J^{ab} \right),
\]
(112)
\[
C_2 = \frac{2\beta}{\det(m_{\alpha \beta})} (Z_{ab} Z^{ab} - 2J_{ab} Z^{ab} + J_{ab} J^{ab}).
\]
(113)

There exists a third Casimir operator, but it is valid only for the subspace spanned by \( J_{ab} \) and \( Z_{ab} \), and not for the full SSEP algebra. This Casimir operator is constructed from \( \bar{C}_{IJZ} \) (cf equation (101)), and is given by
\[
\bar{C}_{IJZ} = -\frac{1}{\det(m_{\alpha \beta})} \left[ (\alpha_2 - \alpha_1) e^{abcd} J_{ab} J_{cd} - 2(\alpha_2 - \alpha_1) e^{abcd} J_{ab} Z_{cd} \right.
\]
\[
+ \left( \alpha_0 \alpha_2 - \alpha_1^2 \right) e^{abcd} Z_{ab} Z_{cd} \right] = -\frac{1}{\det(m_{\alpha \beta})} \left[ (\alpha_2 - \alpha_1) e^{abcd} Z_{ab} Z_{cd} - 2(\beta - \alpha) e^{abcd} J_{ab} Z_{cd} + (\beta - \alpha) e^{abcd} J_{ab} J_{cd} \right].
\]
(114)

The Casimir operators of the SSEP algebra obtained in [1–4] are apparently different from the ones shown in equations (112) and (113). The mismatch, however, is only superficial. Indeed, if we make \( c = 1 \) and \( a = i/2 \) in equations (2.2) and (2.3) of [3], we readily obtain the operators \( C_1 \) and \( \bar{C}_{IJZ} \) shown in equations (112) and (113).

Performing the same rescaling and choosing \( \alpha = 1 \) y \( \beta = 2 \) in \( \bar{C}_{IJZ} \), we can verify that the Casimir operator \( C_1 \) of [3] exactly matches our \( \bar{C}_{IJZ} \) Casimir operator.

6. A generalized action for Chern–Simons gravity in (2 + 1) dimensions

In this section we find a rank-2, symmetric invariant tensor for the SSEP algebra and use it to build the more general action for Chern–Simons gravity in (2 + 1) dimensions.

6.1. The invariant tensor

It is easy to see that the most general symmetric invariant tensor of rank 2 for the AdS algebra in three-dimensional spacetime is given by (see, e.g., [6])
\[
\langle J_{ab} J_{cd} \rangle = \tilde{\mu}_0 (\eta_{ad} \eta_{bc} - \eta_{ac} \eta_{bd})
\]
(115)
\[
\langle J_{ab} P_c \rangle = \tilde{\mu}_1 \epsilon_{abc}
\]
(116)
\[
\langle P_a P_b \rangle = \tilde{\mu}_0 \eta_{ab},
\]
(117)
where \( \tilde{\mu}_0 \) and \( \tilde{\mu}_1 \) are arbitrary constants. Theorem 7.2 from [6] ensures us that the only nonzero components of the corresponding symmetric invariant tensor for the SSEP algebra are:
\[
\langle N_{ab} N_{cd} \rangle = \alpha_0 (\eta_{ad} \eta_{bc} - \eta_{ac} \eta_{bd})
\]
(118)
\[
\langle L_{ab} L_{cd} \rangle = \alpha_2 (\eta_{ad} \eta_{bc} - \eta_{ac} \eta_{bd}) \\
\langle L_{ab} L_{c3} \rangle = \alpha_1 \epsilon_{abc} \\
\langle L_{a3} L_{b3} \rangle = \alpha_2 \eta_{ab},
\]
where \( \alpha_0, \alpha_1 \) and \( \alpha_2 \) are arbitrary constants.

### 6.2. Chern–Simons action for the SSEP algebra in \((2 + 1)\) dimensions

A generic Chern–Simons Lagrangian in \((2 + 1)\)-dimensional spacetime reads [8, 9]

\[
L_{\text{CS}}^{2+1} = 2k \int_0^1 dt \langle A(tdA + r^2 A^2) \rangle = k \left( A \left( dA + \frac{2}{3} A^3 \right) \right),
\]

where \( A \) is a Lie algebra-valued 1-form gauge connection and \( k \) is an arbitrary coupling constant. For the SSEP algebra, we may write

\[
A = \frac{1}{2} \sigma^{ab} N_{ab} + \frac{1}{2} \omega^{ab} L_{ab} + \omega^{a3} L_{a3}.
\]

For the sake of convenience, let us define the SSEP-valued 1-form gauge fields \( \sigma = \frac{1}{2} \sigma^{ab} N_{ab}, \omega = \frac{1}{2} \omega^{ab} L_{ab}, \varphi = \omega^{a3} L_{a3} \). In terms of these, \( A \) takes on the simple form

\[
A = \sigma + \omega + \varphi.
\]

A straightforward calculation shows that the Chern–Simons Lagrangian for the SSEP algebra in three-dimensional spacetime may be written as

\[
L_{\text{SSEP}}^{(2+1)} = k \left[ A(dA + \frac{2}{3} A^2) \right] = k \left[ \sigma d\sigma + \sigma d\omega + \sigma d\varphi + \frac{1}{2} \sigma [\sigma, \sigma] \right]
+ k \left[ \sigma d\omega + \sigma d\varphi + \sigma d\varphi + \frac{1}{2} \omega [\omega, \varphi] + \frac{1}{3} \omega [\varphi, \varphi] \right]
+ k \left[ \omega d\sigma + \varphi d\omega + \varphi d\varphi + \frac{1}{2} \varphi [\omega, \varphi] + \frac{1}{3} \varphi [\varphi, \varphi] \right].
\]

The SSEP 2-form curvature reads

\[
F = dA + AA
= d\sigma + d\omega + d\varphi + \sigma \sigma + \omega \omega + \varphi \varphi + [\omega, \varphi]
\]

so that it proves convenient to define the following partial curvatures:

\[
\tilde{K} = d\sigma + \sigma \sigma = d\sigma + \frac{1}{4} [\sigma, \sigma]
\]

\[
R = d\omega + \omega \omega = d\omega + \frac{1}{4} [\omega, \omega]
\]

\[
\tilde{T} = d\varphi + \varphi \varphi + [\omega, \varphi] = T + \frac{1}{3} \varphi, \varphi.
\]

where \( T = d\varphi + [\omega, \varphi] \).

From the definition of the covariant derivative

\[
D\phi = d\phi + [A, \phi] = d\phi + [\sigma, \phi] + [\omega, \phi] + [\varphi, \phi],
\]

we can write

\[
D\sigma = d\sigma + [\sigma, \sigma]
\]

\[
D\omega = d\omega + [\omega, \omega] + [\omega, \varphi]
\]

\[
D\varphi = d\varphi + [\omega, \varphi] + [\varphi, \varphi] = T + \varphi, \varphi.
\]
Inserting equations (127) and (129) in (125), we have

\begin{equation}
L_{\text{SSEP}}^{(2+1)} = k \left[ A \left( dA + \frac{2}{3} A^2 \right) \right] \quad (130)
\end{equation}

\begin{align*}
&= \frac{k}{4} \sigma^{ab} \left( d\sigma^{cd} + \frac{2}{3} \sigma^{c} \sigma^{ed} \right) \langle N_{ab} N_{cd} \rangle + \frac{k}{4} \omega^{ab} \left( d\omega^{cd} + \frac{2}{3} \omega^{c} \omega^{ed} \right) \langle L_{ab} L_{cd} \rangle \\
&+ k \left( R^{ab} \omega^c - \frac{2}{3} \omega^a \omega^b \omega^c \right) \langle L_{ab} L_{c3} \rangle + kD\omega^{a} \omega^{c} \langle L_{a3} L_{c3} \rangle \\
&- d \left( \frac{k}{2} \omega^{ab} \omega^c \langle L_{ab} L_{c3} \rangle \right). \\
&\quad (131)
\end{align*}

Introducing the invariant tensor (115)–(121) in equation (130), we find that the Chern–Simons action for the SSEP algebra, in the \{N_{ab}, L_{CD}\} basis, is given by

\begin{align*}
S_{\text{SSEP}}^{(2+1)} &= \int \frac{1}{M} \alpha_0 \sigma^{a} \left( d\sigma^{c} + \frac{2}{3} \sigma^{c} \sigma^{d} \right) + \alpha_1 \epsilon_{abc} \left( R^{ab} \omega^c + \frac{1}{3} \omega^a \omega^b \omega^c \right) \\
&+ \alpha_2 D\omega^a \omega^c + \frac{1}{2} \alpha_2 \omega^a \left( d\omega^c + \frac{2}{3} \omega^c \omega^d \right) - d \left( \alpha_1 \frac{1}{2} \epsilon_{abc} \omega^{ab} \omega^c \right),
\end{align*}

where we have absorbed \( k \) in \( \alpha \) constants.

Relabeling \( \omega^a \) \( = e^a/l \), where \( l \) is a length, we may write

\begin{align*}
S_{\text{SSEP}}^{(2+1)} &= \frac{\alpha_0}{2} \int \sigma^a \left( d\sigma^c + \frac{2}{3} \sigma^c \sigma^d \right) \\
&+ \frac{\alpha_1}{l} \left[ \int \epsilon_{abc} \left( R^{ab} \omega^c + \frac{1}{3} \omega^a \omega^b \omega^c \right) - \frac{1}{2} \int \epsilon_{abc} \omega^{ab} \omega^c \right] \\
&+ \frac{\alpha_2}{2} \int \epsilon^a \left( d\omega^c + \frac{2}{3} \omega^c \omega^d \right) + \frac{2}{l^2} \epsilon^a T^a, \\
&\quad (132)
\end{align*}

where we have used \( D\omega^a = (D e^a)/l = T^a/l \). The action (132) is probably the most general action for Chern–Simons gravity in \((2 + 1)\) dimensions.

7. Comments

We have shown that: (i) the SSEP algebra \( so(D - 1, 1) \oplus so(D - 1, 2) \) of [1–4] can be obtained from the AdS algebra \( so(D - 1, 2) \) via the S-expansion procedure [6, 6] with an appropriate semigroup \( S \); (ii) there exists a prescribed method for computing Casimir operator for \( S \)-expanded algebras, which is exemplified through the SSEP algebra; and (iii) the above-mentioned \( S \)-expansion methods allowed us to obtain an invariant tensor for the SSEP algebra, which in turn permits the construction of the more general action for Chern–Simons gravity in \((2 + 1)\) dimensions.

The interesting facts here are that the resultant theory goes beyond the sum of the corresponding Chern–Simons forms associated with the direct sum of \( so(D - 1, 1) \oplus so(D - 1, 2) \) of the Lorentz and the AdS Lie algebras.

The action (132) includes among its terms: (i) a term corresponding to the so-called exotic Lagrangian for the connection \( \sigma \), which is invariant under the Lorentz algebra [10]; and (ii) the topologically Mielke–Baekler action for three-dimensional gravity (for details, see [11]).
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