A fractional-order difference Cournot duopoly game
with long memory

Baogui Xin∗  Wei Peng†  Yekyung Kwon‡

Abstract

We reconsider the Cournot duopoly problem in light of the theory for long memory. We introduce the Caputo fractional-order difference calculus to classical duopoly theory to propose a fractional-order discrete Cournot duopoly game model, which allows participants to make decisions while making full use of their historical information. Then we discuss Nash equilibria and local stability by using linear approximation. Finally, we detect the chaos of the model by employing a 0-1 test algorithm.

Keywords: Fractional-order difference; Fractional-order discrete dynamical systems; Cournot duopoly game; Bifurcation and chaos; 0-1 test.

Contents

1 Introduction 2
2 Preliminaries 3
3 Model 6
4 Nash equilibrium and local stability 7
5 Numerical simulation 9
6 Conclusion 10

∗B. Xin, College of Economics and Management, Shandong University of Science and Technology, Qingdao 266590, China, e-mail: xin@tju.edu.cn, corresponding author
†W. Peng, College of Economics and Management, Shandong University of Science and Technology, Qingdao 266590, China, e-mail: pengweisd@foxmail.com
‡Y. Kwon, Division of Global Business Administration, Dongseo University, Busan 47011, Korea, e-mail: yiqing@hanmail.net
1 Introduction

The purpose of this paper are twofold. One purpose is to bring together two independent lines of research in applied mathematics and industrial economics: fractional-order difference equations and Cournot equilibria. The other is to introduce long memory to classical Cournot duopoly games by replacing integer-order difference equations with fractional-order forms.

1.1 Links between difference equation and Cournot equilibria in the literature

Cournot game theory [1] was proposed in 1838. Many researchers, such as Nash [2], Von Neumann, and Morgenstern [3], have since made landmark contributions to the development of game theory, which has become a powerful analytic tool in fields outside economics, such as cyberspace security [4–8], power systems [9, 10], cytobiology [11], image processing [12], human-machine systems [13], artificial intelligence [14, 15], safety engineering [16, 17], nuclear security [18], oncology [19], system control [20], and information science [21]. Considering the powerful dynamic characterization ability of difference equations, the discrete dynamical game with integer-order difference equations has become an important research direction in industrial economics field. Using integer-order difference equations, economics researchers have developed various interesting discrete Cournot duopoly game models, as shown in Table 1.

1.2 Links between fractional-order difference equations and scientific models in the literature

Díaz and Osler [49] first presented the theory of fractional-order differences in 1974. Miller and Ross [50] and Gray and Zhang [51] developed fractional-order difference calculus in 1988. Fractional-order difference calculus has attracted the interest of many scholars and practitioners, but it is still a young research field compared to fractional differential calculus, which has developed for more than 300 years. Researchers of the former include Atici, Eloe, and Şengül [52–56]; Bastos, Ferreira and Torres [57]; Wu, Baleanu et al. [58, 59]; Goodrich [60, 61]; Abdeljawad et al. [62–65]; Wei and Wang et al. [67, 68]; Čermák, Győri, and Nechvátal [69]; Mozyrska and Wyrwas [70, 71]; and Abu-Saris and Al-Mdallal [72]. Fractional-order difference calculus is applied in many scientific fields, as shown in Table 2.

1.3 Links between fractional-order difference equations and Cournot duopoly game

The theory of integer-order difference equation is not suitable to analyze the nonlinear dynamic characteristics of the fractional discrete Cournot duopoly game with long memory. We must employ new theories to analyze the stability, bifurcation, and chaos of the fractional discrete game. Fortunately, researchers have
produced some useful results for analyzing the stability of fractional-order difference equations, such as explicit stability conditions [69], explicit criteria for stability [70], stability by linear approximation [71, 94], asymptotic stability criteria [72], Lyapunov functions [95], finite-time stability [96], and chaos analysis [97].

The remainder of this paper is organized as follows. The relevant literature is reviewed in section 2. In section 3, a long-memory discrete Cournot duopoly game with Caputo fractional-order difference equations is presented. The Nash equilibrium points and their local stabilities are studied in section 4. In section 5, bifurcation diagrams, phase portraits, and 0-1 test algorithms are employed to validate the main results. This paper concludes with a summary in section 6.

2   Preliminaries

The following definitions of fractional-order difference calculus are introduced.
| Study                  | Research field     | Scientific problem                          | Type of fractional-order difference calculus                  |
|-----------------------|--------------------|---------------------------------------------|-------------------------------------------------------------|
| Tarasov [73]          | physics            | physical lattices                           | fractional-order difference equations                       |
| Wu et al. [74]        | physics            | anomalous diffusion                         | Riesz-Caputo difference equations                           |
| Huang et al. [75]; Ismail et al. [76]; Liu, Xia, & Wang [77, 78]; Kassim et al. [79] | cryptography | image encryption technique                  | fractional logistic difference equations                     |
| Xin et al. [80]; Wu et al. [81]; Shukla & Sharma [82]; Ouannas et al. [83, 84]; Liu [85] | system engineering | chaos synchronization, control               | fractional nonlinear difference equations                    |
| Mozyrska & Pawluszewicz [86–90] | system engineering | controllability, observability              | fractional-order difference equations                       |
| Sierociuk & Twardy [91] | system engineering | parameter identification                    | variable fractional-order difference equations              |
| Yin & Zhou [92]       | image processing   | image denoising                             | difference curvature driven fractional nonlinear diffusion equations |
| Liu et al. [93]       | signal processing  | Kalman filter                               | nonlinear difference fractional system with stochastic perturbation |

**Definition 1** (See [52].) For any real numbers \( v, t \in \mathbb{R} \), the \( v \) rising fractional factorial of \( t \) is defined as

\[
 t^{(v)} := \frac{\Gamma(t + 1)}{\Gamma(t + 1 - v)}, \quad t^{(0)} = 1.
\]

**Definition 2** (See [51, 52].) Let \( x : \mathbb{N}_a \to \mathbb{R}, a \in \mathbb{R}, t \in \mathbb{N}_{a+v}, \) and \( v > 0 \). Then the fractional sum of order \( v \) is defined as

\[
 \Delta_a^{-v} x(t) := \frac{1}{\Gamma(v)} \sum_{s=a}^{t-v} (t - \sigma(s))^{(v-1)} x(s),
\]

where \( a \) is the start point, \( \mathbb{N}_a = \{a, a+1, a+2, \ldots \} \) denotes the isolated time scale, and \( \sigma(s) = s + 1 \).

**Definition 3** (See [63].) Let \( v > 0, v \not\in \mathbb{N}, t \in \mathbb{N}_{a+n-v}, \) and \( n = [v] + 1 \). Then the \( v \)-order Caputo-like left delta difference is defined by

\[
 C\Delta_a^v x(t) := \Delta_a^{-(n-v)} \Delta^n x(t) = \frac{1}{\Gamma(n-v)} \sum_{s=a}^{t-n-v} (t - \sigma(s))^{(n-v-1)} \Delta^n x(s).
\]

**Theorem 1** (See [99].) For the Caputo fractional-order difference system

\[
 \begin{cases}
    C\Delta_a^v x(t) = f (t^+, x(t^+)), & t^+ = t + v - 1, \\
    \Delta^k x(a) = x_k, & k = 0, \ldots, m - 1, \quad m = [v] + 1,
\end{cases}
\]
the equivalent discrete integral system is written as

\[
x(t) = x_0(t) + \frac{1}{\Gamma(\nu)} \sum_{s=t+\nu-1}^{t-1} (t - \sigma(s))^{(\nu-1)} f(s + \nu - 1, x(s + \nu - 1)), \quad t \in \mathbb{N}_{\nu + m},
\]

where the initial iteration is

\[
x_0(t) = \sum_{k=0}^{m-1} \frac{(t - a)^k}{\Gamma(k + 1)} \Delta^k x(a).
\]

Using Definition 1, we can get

\[
(t - \sigma(s))^{(\nu-1)} = (t - s - 1)^{(\nu-1)} = t^{(\nu)} := \frac{\Gamma(t - s)}{\Gamma(t - s - \nu - 1)}.
\]

So, we can obtain the following proposition.

**Remark 1** If \(a = 0\), we rewrite system (2.2) in the following numerical form

\[
x(n) = x(0) + \frac{1}{\Gamma(\nu)} \sum_{i=1}^{n} \frac{\Gamma(n - i + \nu)}{\Gamma(n - i + 1)} f(x(i - 1)), \quad n \in \mathbb{N}.
\]

According to the linearization theorem [94], we obtain the following theorem, which is a special case of refs. [69, 71].

**Theorem 2** Assume that system (2.4) is nonlinear with \(\nu \in (0, 1)\), \(x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T\), and \(f(t) = (f_1(t), f_2(t), \cdots, f_n(t))^T\) is continuously differentiable at a fixed point \(x^o\). Then \(\forall t \in \mathbb{N}_{\nu + 1 - \nu}\), system (2.4) is locally asymptotically stable when all eigenvalues \(\lambda\) of the following Jacobian matrix

\[
J(x^o) = \left. \frac{\partial f(x)}{\partial x} \right|_{x = x^o} = \begin{pmatrix}
\frac{\partial f_1(x^o)}{\partial x_1} & \frac{\partial f_1(x^o)}{\partial x_2} & \cdots & \frac{\partial f_1(x^o)}{\partial x_n} \\
\frac{\partial f_2(x^o)}{\partial x_1} & \frac{\partial f_2(x^o)}{\partial x_2} & \cdots & \frac{\partial f_2(x^o)}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_n(x^o)}{\partial x_1} & \frac{\partial f_n(x^o)}{\partial x_2} & \cdots & \frac{\partial f_n(x^o)}{\partial x_n}
\end{pmatrix}
\]

are such that

\[
\lambda \in \left\{ z \in \mathbb{C} : |z| < \left( 2\cos \frac{|\arg z| - \pi}{2 - \nu} \right)^\nu \text{ and } |\arg z| > \frac{\nu\pi}{2} \right\}.
\]

Considering a two-dimensional case of system (2.4), we obtain the following theorem, which is a special case of Theorem 2.

\[\text{---}\]

5
Theorem 3 (See [69].) Two-dimensional system (2.4) is locally asymptotically stable if $\det J > 0$ and either

$$\frac{\text{tr} J}{2} > \sqrt{\det J}, \quad \nu > \log_2 \frac{\sqrt{\varpi} - \text{tr} J}{2},$$

(2.5)

or

$$\frac{|\text{tr} J|}{2} < \sqrt{\det J} < \left(2 \cos \frac{\kappa - \pi}{2 - \nu}\right)^\nu, \quad \nu < \frac{2\kappa}{\pi},$$

(2.6)

where $\varpi = |(\text{tr} J)^2 - 4 \det J|$, and $\kappa = \frac{\text{tr} J}{\sqrt{\varpi}}$.

In the following discussion, the time-dependence, subscripts, and superscripts will be omitted from notation if no confusion is caused.

3 Model

Let us consider a classical Cournot competition between firms 1 and 2, who produce homogeneous products which are perfect substitutes. Let $q_i(t), i = 1, 2$ denote the $i$–th firm’s output during discrete time periods $t = 0, 1, 2, \ldots$. For simplicity, as mentioned by Bischi & Naimzada [25] and Agiza & Elsadany [28, 29], we assume the market-clearing price $p(t)$ at period $t$, an inverse demand function, is linear and decreasing, as follows:

$$p(t) = b - d(q_i(t) + q_j(t)), \quad i, j = 1, 2, \quad i \neq j.$$  

(3.7)

where $b$ and $d$ are positive constants.

The cost function takes the following form:

$$C_i(t) = \frac{1}{2} c_i q_i^2(t), \quad i = 1, 2,$$

(3.8)

where $c_i, i = 1, 2$, is a positive constant.

The $i$–th firm’s profit can be written as

$$\Pi_i(q_i(t), q_j(t)) = p(t)q_i(t) - C_i(t), \quad i, j = 1, 2, \quad i \neq j.$$  

(3.9)

Then the $i$–th firm’s marginal profit can be obtained by differentiating with respect to $q_i$:

$$\Phi_i(t) = \frac{\partial \Pi_i(q_i(t), q_j(t))}{\partial q_i(t)} = b - (c_i + 2d)q_i(t) - d q_j(t), \quad i, j = 1, 2, \quad i \neq j.$$  

(3.10)
By simple algebraic computation, we obtain four Nash equilibria:

\[ q_i \text{ function} \]

where \( \alpha \) is the speed of adjustment, and \( \Phi(t) \) is the \( \nu \)-order left Caputo-like delta difference. For the function \( q(t) \), the delta difference operator \( \Delta \) is defined by \( \Delta q(t) = q(t+1) - q(t) \).

**Remark 2** When \( \nu = 1 \) and \( \Delta q(t) = q(t+1) - q(t) \), then equation (3.11) degenerates to

\[ q_i(t+1) = q_i(t) + \alpha_i q_i(t) \Phi(t), \quad i = 1, 2, \]

which agrees with the game model without long-memory effects proposed by Bischi & Naimzada [25] and Agiza & Elsadany [28, 29].

From the above assumptions, we obtain the following discrete fractional Cournot duopoly game:

\[
\begin{align*}
\Delta_\nu^\alpha q_1(t) &= \alpha_1 q_1(t^+) (b - (c_1 + 2d)q_1(t^+) - dq_2(t^+)), \\
\Delta_\nu^\alpha q_2(t) &= \alpha_2 q_2(t^+) (b - (c_2 + 2d)q_2(t^+) - dq_1(t^+)).
\end{align*}
\]

(3.13)

**4 Nash equilibrium and local stability**

The equilibrium points of system (3.13) satisfy

\[
\begin{align*}
\alpha_1 q_1 (b - (c_1 + 2d)q_1 - dq_2) &= 0, \\
\alpha_2 q_2 (b - (c_2 + 2d)q_2 - dq_1) &= 0.
\end{align*}
\]

(4.14)

By simple algebraic computation, we obtain four Nash equilibrium points: \( E_1 = (0, 0), \ E_2 = \left(0, \frac{b}{c_1+2d}\right), \ E_3 = \left(\frac{b}{c_1+2d}, 0\right), \) and \( E_4 = (m(c_2 + d), m(c_1 + d)) \), where \( m = \frac{b}{c_1+c_2+2d+2c_1d+3d} \). They contain the following economic information:

(i) At the equilibrium point \( E_1 = (0, 0) \), no single firm has anything to gain by producing its products if its opponent keeps producing nothing.

(ii) At the equilibrium point \( E_2 = \left(0, \frac{b}{c_1+2d}\right) \), the best strategy of firm 1 is to produce nothing if firm 2 adopts its equilibrium production strategy \( q_2^* = \frac{b}{c_1+2d} \). Similarly, the best output of firm 2 is \( q_2^* = \frac{b}{c_2+2d} \) if firm 1 keeps producing nothing as its equilibrium strategy.
(iii) At the equilibrium point $E_3 = \left( \frac{b}{c_1+2d}, 0 \right)$, firm 2 cannot obtain any payoff by producing its products if firm 1 adopts its equilibrium output strategy $q_1^* = \frac{b}{c_1+2d}$. In the same way, firm 1 will maintain its equilibrium output strategy $q_1^* = \frac{b}{c_2+2d}$ if firm 2 keeps zero output as its equilibrium strategy.

(iv) At the equilibrium point $E_4 = \left( m(c_2+d), m(c_1+d) \right)$, firm 1 can receive no incremental benefit from deviating unilaterally from its chosen strategy $q_1 = m(c_2+d)$, assuming firm 2 maintains its production strategy $q_2 = m(c_1+d)$, and vice versa.

Obviously, equilibrium points $E_1$, $E_2$, and $E_3$ are bounded equilibria [25]. So, we only analyze the stability of non-bounded equilibrium point $E_4$, as follows.

**Theorem 4** System (3.13) is locally asymptotically stable at Nash equilibrium point $E_4$ if

$$\nu > \log_2 \frac{\sqrt{(tr J)^2 - 4 \det J - tr J}}{2},$$

where

$$tr J = -\frac{b(2d^2(\alpha_1 + \alpha_2) + d(c_1(\alpha_1 + 2\alpha_2) + c_2(2\alpha_1 + \alpha_2)) + c_1c_2(\alpha_1 + \alpha_2))}{3d^2 + 2d(c_1 + c_2) + c_1c_2},$$

$$det J = \frac{\alpha_1\alpha_2b^2(c_1+d)(c_2+d)}{c_1c_2 + 2d(c_1 + c_2) + 3d^2}.$$  \hspace{1cm} (4.16) \hspace{1cm} (4.17)

**Proof.** The Jacobian matrix $J$ of system (3.13) evaluated at Nash equilibrium points $E_4$ is

$$J(E_4) = \begin{pmatrix} \alpha_1(b - m(2c_1c_2 + 5d^2 + d(3c_1 + 4c_2))) & -\alpha_1md(c_2 + d) \\ -\alpha_2md(c_1 + d) & \alpha_2(b - m(2c_1c_2 + 5d^2 + d(4c_1 + 3c_2))) \end{pmatrix},$$

whose characteristic equation is

$$p(\lambda) = \lambda^2 - tr J \lambda + det J,$$

where $tr J$ and $det J$ are the same as (4.16) and (4.17), respectively.

We algebraically obtain $det J > 0$, $tr J < 0$, $(tr J)^2 - 4 det J > 0$, and $\frac{-tr J}{2} > \sqrt{det J}$. So, we can use Theorem 2 to find that system (3.13) is locally asymptotically stable at Nash equilibrium point $E_4$ if eq. (4.15) holds.

This completes the proof. \Box
According to Theorem 4, we can get (3.13)

\[ \nu = 5 \]

Numerical simulation

We will use eqs. (5.18) to demonstrate stability, bifurcation, and chaos of system (3.13).

\[
\begin{align*}
q_1(n) &= q_1(0) + \frac{1}{\Gamma(\nu)} \sum_{i=1}^{n} \frac{\Gamma(n-i+\nu)}{\Gamma(n-i+1)} a_1 q_1(i-1) (b - (c_1 + 2d)q_1(i-1) - dq_2(i-1)), \\
q_2(n) &= q_2(0) + \frac{1}{\Gamma(\nu)} \sum_{i=1}^{n} \frac{\Gamma(n-i+\nu)}{\Gamma(n-i+1)} a_2 q_2(i-1) (b - (c_2 + 2d)q_2(i-1) - dq_1(i-1)).
\end{align*}
\]

(5.18)

We will use eqs. (5.18) to demonstrate stability, bifurcation, and chaos of system (3.13).

According to Remark 1, system (5.18) can be rewritten as

\[
\begin{align*}
q_1(n) &= q_1(0) + \frac{1}{\Gamma(\nu)} \sum_{i=1}^{n} \frac{\Gamma(n-i+\nu)}{\Gamma(n-i+1)} a_1 q_1(i-1) (b - (c_1 + 2d)q_1(i-1) - dq_2(i-1)), \\
q_2(n) &= q_2(0) + \frac{1}{\Gamma(\nu)} \sum_{i=1}^{n} \frac{\Gamma(n-i+\nu)}{\Gamma(n-i+1)} a_2 q_2(i-1) (b - (c_2 + 2d)q_2(i-1) - dq_1(i-1)).
\end{align*}
\]

(5.18)

We will use eqs. (5.18) to demonstrate stability, bifurcation, and chaos of system (3.13).

Let parameters \( \nu = 0.99, a_1 = 0.45, a_2 = 0.12, b = 6, d = 4.1, c_1 = 0.2, \) and \( c_2 = 0.3. \) Then non-bounded Nash equilibrium point \( E_4 = (0.4836, q_2 = 0.4726), trJ = -2.3101 < 0, \) and \( \det J = 0.67378 > 0. \)

According to Theorem 4, we can get \( (trJ)^2 - 4 \det J = 2.6415 > 0, \) \( \sqrt{\frac{trJ}{2}} - \sqrt{\det J} = 2.0079 > 0, \) and 
\( \nu = 0.99 > \log_2 \frac{\sqrt{(trJ)^2 - 4 \det J} - trJ}{2} = 0.9765, \) which is to say system (3.13) is local asymptotically stable at non-bounded Nash equilibrium point \( E_4, \) as shown in Figure 1.

To analyze bifurcation and chaos of the long-memory system (3.13) with fractional-order \( \nu, \) we vary the long-memory parameter, fractional-order \( \nu \in (0, 1), \) and fix the remaining parameters as \( a_1 = 0.45, a_2 = 0.12, b = 6, d = 4.1, c_1 = 0.2, \) and \( c_2 = 0.3, \) with the starting point \( (q_1(0), q_2(0)) = (0.3, 0.3), \) as shown in Figure 2. This figure includes the bifurcation diagram of firm 1’s output and the scatter diagram of \( K, \) which is the median value of the correlation coefficient of \( q_1. \) To obtain Figure 2, the long-memory parameter \( \nu \) varies from 0 to 1 with an increment of 0.002. The bifurcation diagram of \( q_1 \) is drawn by using 100 data points after dropping 500 transient data points. \( K \) is calculated by taking 3,000 data points after dropping 500 transient data points. The bifurcation diagram demonstrates the possible long-term values of firm 1’s output with long-memory parameter \( \nu \) varying from 0 to 1. The scatter diagram of \( K \) illustrates the possibility of chaos occurrence in system (3.13) corresponding to different values of \( \nu \in (0, 1). \)
Figure 2: Firm 1’s output bifurcation (blue) and K (red) of system (5.18) with varying $\nu \in (0, 1)$.

The criterion of the 0-1 chaos test [100–110] shows that bounded trajectories in the $(p, s)$–plane or $K \approx 0$ mean system (3.13) is regular, and Brownian-like trajectories in the $(p, s)$–plane or $K \approx 1$ indicate system (3.13) is chaotic. So, system (3.13) is chaotic when $\nu < 0.4$, i.e., when $K \approx 1$. Obviously, system (3.13) shows that the bifurcation diagram quite well coincides with the scatter diagram of $K$.

For Figure 2, let $\nu = 0.2$, which leads to $K = 0.9752$. We can know that system (3.13) is chaotic, as shown in Figure 3. To produce Figure 3, we iterate it 3,500 times and draw a duopoly output time series using the first 300 data points, as shown in Figure 3(a)-(b). We also create its phase portrait (as shown in Figure 3(c)) and calculate its median correlation coefficient using the last 3,000 data points after dropping 500 transient data points, and draw its trajectories in new coordinates $(p, s)$, as shown in Figure 3(d). From Figure 3, we can find that their time series are irregular and chaotic, their output phase portrait is a strange chaotic attractor, and their trajectories in the $(p, s)$–plane are Brownian-like.

6 Conclusions

We have proposed a nonlinear fractional-order discrete Cournot duopoly game model that shows the long-memory effect. We discussed its Nash equilibria and local stability by linear approximation, then numerically illustrated its phase portraits, bifurcation diagrams, and chaos attractors using the 0-1 test algorithm. The paradigm to analyze the Cournot duopoly game can be applied to other scientific fields. There are still some open issues regarding fractional-order difference calculus, such as qualitative analysis theories of bifurcation, and high efficient Lyapunov algorithms.
Figure 3: Chaos in system (5.18) with \((q_1(0), q_2(0)) = (0.1, 0.3)\).
Acknowledgments

This work is supported by Natural Science Foundation of Shandong Province (Grant No. ZR2016FM26) and National Social Science Foundation of China (Grant No.16FJY008).

References

[1] A. Cournot, Recherches sur les principes mathématiques de la théorie des richesses par Augustin Cournot. chez L. Hachette; 1838.

[2] J. Nash. Non-Cooperative Games, Ann. Math. 54(2)(1951)286-295.

[3] J. Von Neumann, O. Morgenstern, The Theory of Games and Economic Behavior, Princeton Univ. Press, 1944.

[4] L. Han, et al., Intrusion detection model of wireless sensor networks based on game theory and an autoregressive model. Inform. Sci. 476(2019)491-504.

[5] M. Shareh, H. Navidi, H. Javadi, M. HosseinZadeh, Preventing Sybil attacks in P2P file sharing networks based on the evolutionary game model. Inform. Sci. 470(2019)94-108.

[6] K. Lalropuia, V. Gupta, Modeling cyber-physical attacks based on stochastic game and Markov processes. Reliab. Eng. Sys. Safe. 181(2019)28-37.

[7] S. Khaliq, et al., Defence against PUE attacks in ad hoc cognitive radio networks: a mean field game approach, Telecom. Syst. 70(1) (2019)123-140.

[8] M. Ranjbar, M. Kheradmandi, A. Pirayesh, Game framework for optimal allocation of spinning reserve to confront intelligent physical attacks on power system. IET Gener. Transm. Dis. 13(1)(2019)92-98.

[9] H. Liu, SINR-based multi-channel power schedule under DoS attacks: A Stackelberg game approach with incomplete information. Automatica, 100 (2019)274-280.

[10] Q. Wang, et al. A two-layer game theoretical attack-defense model for a false data injection attack against power systems. Int. J. Elect. Power Energy Sys. 104(2019)169-177.

[11] M. Archetti, K. Pienta, Cooperation among cancer cells: applying game theory to cancer. Nat. Rev. Cancer, 19(2019)115.

[12] S. Bhowmik, R. Sarkar, B. Das, D. Doermann, GiB: A game theory inspired binarization technique for degraded document images. IEEE T. Image Process. 28(3) (2019)1443-1455.
[13] X. Liang, Z. Yan, A survey on game theoretical methods in HumanMachine Networks. Future Gener. Comp. Sys. 92(2019) 674-693.

[14] X. Ji, et al., Shared Steering Torque Control for Lane Change Assistance: A Stochastic Game-Theoretic Approach. IEEE T. Ind. Electron. 66(4) (2019)3093-3105.

[15] S. Farzi, S. Yousefi, J. Bagherzadeh, B. Eslamnour, Zone-based load balancing in two-tier heterogeneous cellular networks: a game theoretic approach. Telecom. Syst. 70(1) (2019)105-121.

[16] J. Guan, K. Wang, Towards pedestrian room evacuation with a spatial game. Appl. Math. Comput. 347(2019)492-501.

[17] Q. Liu, X. Li, X. Meng, Effectiveness research on the multi-player evolutionary game of coal-mine safety regulation in China based on system dynamics. Safety Sci. 111 (2019)224-233.

[18] T. Woo, Game theory based complex analysis for nuclear security using non-zero sum algorithm. Ann. Nuc. Energy, 125 (2019)12-17.

[19] K. Staňková, J. Brown, W. Dalton, R Gatenby, Optimizing cancer treatment using game theory: A review. JAMA oncology, 5(1)(2019)96-103.

[20] Y. Wu, et al., Cooperative Game Theory-Based Optimal Angular Momentum Management of Hybrid Attitude Control Actuator, IEEE Access 7(2019) 6853-6865.

[21] S. Meng, et al., Hierarchical evolutionary game based dynamic cloudlet selection and bandwidth allocation for mobile cloud computing environment. IET Commu. 13(1)(2019)16-25.

[22] T. Puu, Attractors, Bifurcations, and Chaos Nonlinear Phenomena in Economics, Springer-Verlag Berlin Heidelberg New York, 2000.

[23] M. Kopel, Simple and complex adjustment dynamics in Cournot duopoly models, Chaos Soliton. Fract. 12(1996)2031-48.

[24] W. Govaerts and G. Ghaziani, Stable cycles in a Cournot duopoly model of Kopel, J Comput. Appl. Math. 218(2008.)247-258

[25] G. Bischi, A. Naimzada, Global analysis of a dynamic duopoly game with bounded rationality, in: Advances in Dynamic Games and Applications, vol.5, Birkhaur, Boston, 1999.

[26] F. Cavalli, A. Naimzada, F. Tramontana, Nonlinear dynamics and global analysis of a heterogeneous Cournot duopoly with a local monopolistic approach versus a gradient rule with endogenous reactivity. Commun. Nonlinear Sci. Numer. Simul. 23(1-3)(2015)245-262.
[27] A. Agliari, A. Naimzada, N. Pecora, Nonlinear dynamics of a Cournot duopoly game with differentiated products. Appl. Math. Comput. 281(2016)1-15.

[28] H. Agiza, A. Elsadany, Nonlinear dynamics in the Cournot duopoly game with heterogeneous players, Physica A. 320(2003)512-524.

[29] H. Agiza, A. Elsadany, Chaotic dynamics in nonlinear duopoly game with heterogeneous players, Appl. Math. Comput. 149(2004)843-860.

[30] A. Elsadany, A. Awad, Dynamical analysis and chaos control in a heterogeneous Kopel duopoly game. Indian J. Pure Appl. Math. 47(4)(2016)617-639.

[31] Y. Fan, T. Xie, J. Du, Complex dynamics of duopoly game with heterogeneous players: a further analysis of the output model. Appl. Math. Comput. 218(15)(2012)7829-7838.

[32] T. Li, J. Xie, S. Lu, J. Tang, Duopoly game of callable products in airline revenue management. Eur. J. Oper. Res. 254(3)(2016)925-934.

[33] A. El-Sayed, A. Elsadany, A. Awad, Chaotic dynamics and synchronization of cournot duopoly game with a logarithmic demand function, Appl. Math. Inform. Sci. 9(6)(2015)3083.

[34] A. Awad, A. Elsadany, Nonlinear dynamics of cournot duopoly game with social welfare, Electron. J. Math. Anal. Appl. 4(2)(2016)173-191.

[35] A. Elsadany, Dynamics of a Cournot duopoly game with bounded rationality based on relative profit maximization. Appl. Math. Comput. 294(2017)253-263.

[36] A. Matsumoto, Controlling the cournot-nash chaos. J. optimiz. theor. app. 128(2)(2006)379-392.

[37] L. Fanti, L. Gori, The dynamics of a differentiated duopoly with quantity competition. Econ. Model. 29(2)(2012)421-427.

[38] B. Xin, J. Ma, Q. Gao, Complex dynamics of an adnascent-type game model. Discrete Dynamics in Nature and Society, 2008(2008)467972.

[39] J. Ma, Z. Guo, The parameter basin and complex of dynamic game with estimation and two-stage consideration, Appl. Math. Comput. 248(2014)131-142.

[40] S. Askar, A. Alshamrani, K. Alnowibet, The arising of cooperation in Cournot duopoly games, Appl. Math. Comput. 273(2016)535-542.

[41] Y. Yu, Complexity analysis of taxi duopoly game with heterogeneous business operation modes and differentiated products. J. Intell. Fuzzy Syst. 33(5)(2017)3059-3067.
[42] Y. Zhang, W. Zhou, T. Chu, Y. Chu, J Yu, Complex dynamics analysis for a two-stage Cournot duopoly game of semi-collusion in production. Nonlinear Dynam. 91(2)(2018)819-835.

[43] F. Tramontana, L. Gardini, T. Puu, Cournot duopoly when the competitors operate multiple production plants, J. Econ. Dynam. Control 33(2009)250-265.

[44] F. Tramontana, Heterogeneous duopoly with isoelastic demand function, Econ. Model. 27(2010)350-357.

[45] F. Tramontana, A. Elsadany, B. Xin, H. Agiza, Local stability of the Cournot solution with increasing heterogeneous competitors. Nonlinear Analysis: Real World Applications, 26(2015)150-160.

[46] L. Baiardi, A. Naimzada, Imitative and best response behaviors in a nonlinear Cournotian setting. Chaos 28(5)(2018)055913.

[47] L. Baiardi, A. Naimzada, An oligopoly model with best response and imitation rules, Appl. Math. Comput. 336(2018)193-205.

[48] L. Baiardi, A. Naimzada, An oligopoly model with rational and imitation rules. Math. Comput. Simul. 156(2019)254-278.

[49] J. Díaz, T. Osler. "Differences of fractional order." Math. Comput. 28.125 (1974): 185-202.

[50] K. Miller, B. Ross, Fractional difference calculus, in: Proceedings of the International Symposium on Univalent Functions, Fractional Calculus and Their Applications, Nihon University, Koriyama, Japan, May 1988, in: Ellis Horwood Ser. Math. Appl., Horwood, Chichester, 1989, pp.139152.

[51] H. Gray, N. Zhang, On a new definition of the fractional difference. Math. Comput. 50(182)(1988)513-529.

[52] F. Atici, P. Eloe, Initial value problems in discrete fractional calculus. Proc. Am. Math. Soc. 137(2007)9819

[53] F. Atici, P. Eloe, A transform method in discrete fractional calculus, Int. J. Differ. Equ. 2(2)(2007)165-176.

[54] F. Atici, P. Eloe, Discrete fractional calculus with the nabla operator, Electron. J. Qual. Theo. Differ. Equ. 3(2009)1-12.

[55] F. Atici, P. Eloe, Discrete fractional calculus with the nabla operator, Electron. J. Qual. Theo. Differ. Equ.(3)(2009)1-12.
[56] F. Atici, Şengül, Modeling with fractional difference equations, J. Math. Anal. Appl. 369(1)(2010)1-9.

[57] N. Bastos, R. Ferreira, D. Torres, Discrete-time fractional variational problems, Signal Process, 91(2011)51324.

[58] G.C. Wu, D. Baleanu, L. Huang, Novel Mittag-Leffler stability of linear fractional delay difference equations with impulse, Appl. Math. Lett. 82(2018)71-78.

[59] G.C. Wu, D. Baleanu, S. Zeng, Finite-time stability of discrete fractional delay systems: Gronwall inequality and stability criterion, Commun. Nonlinear Sci. Numer. Simul. 57(2018)299-308.

[60] C. Goodrich, A. Peterson, Discrete fractional calculus, Springer, 2016.

[61] C. Goodrich, Monotonicity and non-monotonicity results for sequential fractional delta differences of mixed order, Positivity 22(2)(2018)551-573.

[62] T. Abdeljawad, F. Atici, On the definitions of nabla fractional operators, Abstr, Appl, Anal, 2012(2012)406757.

[63] T. Abdeljawad, On Riemann and Caputo fractional differences. Comput. Math. Appl. 62(3)(2011)1602-1611.

[64] T. Abdeljawad, D. Baleanu, Discrete fractional differences with nonsingular discrete Mittag-Leffler kernels, Adv. Differ. Equ.(1)(2016)232.

[65] I. Suwan, T. Abdeljawad, F. Jarad, Monotonicity analysis for nabla h-discrete fractional Atangana-Baleanu differences. Chaos Solitons Fract. 117(2018)50-59.

[66] F. Chen, X. Luo, Y. Zhou, Existence results for nonlinear fractional difference equation. Adv. Differ. Equ.713201(2011)1-12.

[67] Y. Wei, Y. Chen, T. Liu, Y. Wang, Lyapunov functions for nabla discrete fractional order systems, ISA T. (2018)(In Press).

[68] Y. Wei, Q. Gao, D. Liu, Y. Wang, On the series representation of nabla discrete fractional calculus, Commun. Nonlinear Sci. Numer. Simul. 69(2019)198-218.

[69] J. Čermák, I. Győri, L. Nechvátal, On explicit stability conditions for a linear fractional difference system, Fract. Calc. Appl. Anal. 18(3)(2015)651-672.

[70] D. Mozyrska, M. Wyrwas, Explicit criteria for stability of fractional h-difference two-dimensional systems. Int. J. Dynam. Control 5(1)(2017)4-9.
[71] D. Mozyrska, M. Wyrwas, Stability by linear approximation and the relation between the stability of difference and differential fractional systems. Math. Method Appl. Sci. 40(11)(2017)4080-4091.

[72] R. Abu-Saris, Q. Al-Mdallal, On the asymptotic stability of linear system of fractional-order difference equations. Fract. Calc. Appl. Anal. 16(3)(2013)613-629.

[73] V. Tarasov, Fractional-order difference equations for physical lattices and some applications. J. Math. Phys. 56(10) (2015)103506.

[74] G.C. Wu, D. Baleanu, Z. Deng, S. Zeng, Lattice fractional diffusion equation in terms of a Riesz-Caputo difference. Phys. A 438(2015)335-339.

[75] Huang et al. A new application of the fractional logistic map, Rom. J. Phys. 61(7-8)(2016)1172-1179.

[76] S. Ismail et al., Generalized fractional logistic map encryption system based on FPGA, AEU-Int. J. Electron. Commun. 80(2017)114-126.

[77] Z. Liu, T. Xia, J. Wang, Image encryption technique based on new two-dimensional fractional-order discrete chaotic map and Menezes-Vanstone elliptic curve cryptosystem, Chinese Phys. B. 27(3)(2018)030502.

[78] Z. Liu, T. Xia, J. Wang, Fractional two-dimensional discrete chaotic map and its applications to the information security with elliptic-curve public key cryptography, J. Vib. Control, 24(20)(2018)4797-4824.

[79] S. Kassim, H. Hamiche, S. Djennoune, M. Bettayeb, A novel secure image transmission scheme based on synchronization of fractional-order discrete-time hyperchaotic systems. Nonlinear Dynamics, 88(4)(2017)2473-2489.

[80] B. Xin, L. Liu, G. Hou, Y. Ma, Chaos Synchronization of Nonlinear Fractional Discrete Dynamical Systems via Linear Control, Entropy 19(2017) 351.

[81] G.C. Wu et al. Chaos synchronization of fractional chaotic maps based on the stability condition, Physica A 460(2016)374-383.

[82] M. Shukla, B. Sharma, Investigation of chaos in fractional order generalized hyperchaotic Henon map, AEU-Int. J. Electron. Commun. 78(2017)265-273.

[83] A. Ouannas et al., Fractional Form of a Chaotic Map without Fixed Points: Chaos, Entropy and Control, Entropy 20(10)(2018)720.

[84] A. Ouannas, et al., The fractional form of the tinkerbell map is chaotic. Appl. Sci. 8(2018)2640.
[85] Y. Liu, Chaotic synchronization between linearly coupled discrete fractional Henon maps, Indian J. Phys. 90(3)(2016)313-317.

[86] D. Mozyrska, E. Pawluszewicz, Observability of linear q-difference fractional-order systems with finite initial memory, B. Pol. Acad. Sci. Tech. 58(4) (2010) 601-605.

[87] D. Mozyrska, Multiparameter fractional difference linear control systems. Discrete Dyn. Nat. Soc. 2014(2014)183782.

[88] D. Mozyrska, M. Wyrwas, The Z-Transform Method and Delta Type Fractional Difference Operators. Discrete Dyn. Nat. Soc. 2015(2015) 852734.

[89] D. Mozyrska, E. Pawluszewicz, Controllability of h-difference linear control systems with two fractional orders, Int. J. Sys. Sci. 46(4)(2015)662-669.

[90] E. Pawluszewicz, Constrained controllability of the h-difference fractional control systems with caputo type operator, Discrete Dyn. Nat. Soc. 2015(2015) 638420.

[91] D. Sierociuk, M. Twardy, Duality of variable fractional order difference operators and its application in identification. B. Pol. Acad. Sci. Tech. 62(4) (2014)809-815.

[92] X. Yin, S. Zhou, Image structure-preserving denoising based on difference curvature driven fractional nonlinear diffusion, Math. Probl. Eng. 2015(2015) 930984.

[93] T. Liu, et al. Fractional central difference Kalman filter with unknown prior information, Signal Process. 154(2019)294-303.

[94] C. Li, Y. Ma, Fractional dynamical system and its linearization theorem. Nonlinear Dyn. 71(4) (2013), 621633.

[95] G.C. Wu, D. Baleanu, W. Luo, Lyapunov functions for Riemann-Liouville-like fractional difference equations. Appl. Math.Comput. 314(2017)228-236.

[96] G.C. Wu, D. Baleanu, S. Zeng, Finite-time stability of discrete fractional delay systems: Gronwall inequality and stability criterion. Commun. Nonlinear Sci. Numer. Simul. 57(2018)299-308.

[97] A. Khennaoui, On fractionalorder discretetime systems: Chaos, stabilization and synchronization. Chaos Soliton Fract. 119(2019)150-162.

[98] K. Ahrendt, L. Castle, M. Holm, K. Yochman, Laplace transforms for the nabla-difference operator and a fractional variation of parameters formula, Commun. Appl. Anal. 16 (2012)317347.
[99] F. Chen, X. Luo, Y. Zhou, Existence Results for Nonlinear Fractional Difference Equation, Adv. Differ. Equ. 713201(2011)1-12.

[100] G.C. Gottwald and I. Melbourne, On the validity of the 0-1 test for chaos, Nonlinearity, 22(2009)1367-1382.

[101] G.C. Gottwald and I. Melbourne, Comment on "Reliability of the 0-1 test for chaos", Phys. Rev. E 77(2)(2008)028201.

[102] I. Falconer, G. Gottwald, I. Melbourne and K. Wormnes, Application of the 0-1 test for chaos to experimental data, SIAM J. Appl. Dyn. Syst. 6(2007)95-402.

[103] H. Bazine, M. Mabrouki, Chaotic dynamics applied in time prediction of photovoltaic production, Renew. Energ. 136(2019)1255-1265.

[104] H. Zhu, W. Qi, J. Ge, Y. Liu, Analyzing Devaney Chaos of a Sine-Cosine Compound Function System, Int. J. Bifurcat. Chaos 28(14)(2018)1850176.

[105] J. Ran, Y. Li, C. Wang, Chaos and Complexity Analysis of a Discrete Permanent-Magnet Synchronous Motor System, Complexity 2018(2018)7961214.

[106] L. Yuan, A Song, A. Zeeshan, Dynamics analysis and cryptographic application of fractional logistic map, Nonlinear Dynam. (2019)1-22.

[107] A. Tiwari, G. Rangan, Chaos in G7 stock markets using over one century of data: A note, Res. Int. Bus. Fin. 47 (2019): 304-310.

[108] T. Martinovic, Chaotic behaviour of noisy traffic data, Math, Meth, Appl, Sci, 41(6)(2018)2287-2293.

[109] R. Halfar, L. Marek, Dynamical properties of the improved FK3V heart cell model, Math. Meth. Appl. Sci. 41(17)(2018)7472-7480.

[110] S. Vaidyanathan, A. Akgul, S. Kaar, U. Cavusoglu, A new 4-D chaotic hyperjerk system, its synchronization, circuit design and applications in RNG, image encryption and chaos-based steganography, Eur. Phys. J. Plus 133(2)(2018)46.