STABLE INTERSECTIONS OF TROPICAL VARIETIES

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ABSTRACT. We give several characterizations of stable intersections of tropical cycles and establish their fundamental properties. We prove that the stable intersection of two tropical varieties is the tropicalization of the intersection of the classical varieties after a generic rescaling. Given $n$ polytopes in $\mathbb{R}^n$ the coefficients of their volume polynomial appear as multiplicities of the stable intersections of their tropical hypersurfaces. In particular we recover the volumes and mixed volumes of the polytopes. Using our characterization of stable intersections through generic scaling this gives a proof of Bernstein’s theorem. We prove that the tropical intersection ring is isomorphic to McMullen’s polytope algebra. It follows that every tropical cycle is a linear combination of pure powers of tropical hypersurfaces. Although in general the stable intersection does not preserve connectivity in codimension one, we prove that every stable intersection of constant coefficient tropical varieties defined by prime ideals is connected in codimension one.

1. Introduction

We study stable intersections of tropical cycles in $\mathbb{R}^n$. Tropical cycles are rational pure weighted balanced polyhedral complexes. Stable intersections first appeared under a different guise in [FS97] where Fulton and Sturmfels performed multiplication of Chow cohomology classes on a complete toric variety using the fan displacement rule. This rule was used in [RGST05, Mik06] to define stable intersection of tropical varieties and cycles. Allermann and Rau gave a different definition of the intersection product of tropical cycles using tropical Cartier divisors [AR10]. Katz and Rau then showed independently that these two notions of intersection products coincide [Kat12, Rau].

We give a different definition of stable intersection (Definition 2.4) and show that it is equivalent to both the fan displacement rule (Proposition 2.7) and the Allermann–Rau intersection (Proposition 2.13). Our definition is preferable for computations since it does not use limits or generic perturbations as in the fan displacement rule, and does not require performing intersections iteratively on the Cartesian product as the Allermann–Rau definition does. In Section 2 we give self-contained proofs of numerous fundamental results about stable intersections without relying on algebraic geometry results.

It has been known to tropical geometers that the stable intersection of tropical varieties is the tropicalization of the intersection of varieties after a generic

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change of coefficients. We give a rigorous statement and an elementary proof in Section 3 (Theorem 3.5), filling a gap in the literature.

Based on these results, we give a proof of Bernstein’s Theorem in Section 4. We show in Section 5 that the ring of tropical cycles with the stable intersection product is isomorphic to McMullen’s polytope algebra, based on a result by Fulton and Sturmfels. From this and McMullen’s work on polytope algebra, it follows that every tropical cycle is a linear combination of pure powers of tropical hypersurfaces.

Tropical varieties of prime ideals are known to be connected in codimension one. We prove that stable intersections of such tropical varieties are also connected in codimension one (Theorem 6.1), although for arbitrary tropical cycles the stable intersection does not preserve connectivity in codimension one (Example 6.2).

Our tropical cycles are polyhedral complexes in Section 2, and we assume that they are fans in Sections 3, 4, 5, and 6.

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2. Definitions and Basic Properties

Let $\mathbb{N}$ be a lattice, $\mathbb{N}_Q = \mathbb{N} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathbb{N}_R = \mathbb{N} \otimes_{\mathbb{Z}} \mathbb{R}$. We may also refer to $\mathbb{N}_R$ as $\mathbb{R}^n$ where $n$ is the dimension.

A tropical $k$-cycle in $\mathbb{N}_R$ is a pure $k$-dimensional weighted balanced rational polyhedral complex. A polyhedral complex is called \textit{weighted} if every facet $\sigma$ is assigned a number $\text{mult}_F(\sigma)$ which we call its multiplicity or weight. The multiplicity is usually an integer, but we also allow rational multiplicities in Section 4 and 5 and real multiplicities at the end of this section. If a point $x$ is in the relative interior of a facet $\sigma$, then let $\text{mult}_F(x) := \text{mult}_F(\sigma)$. For a face $\sigma \in \mathcal{F}$, let $N_\sigma$ denote the maximal sublattice of $\mathbb{N}$ parallel to the affine span of $\sigma$, and for any relative interior point $w \in \sigma$, let $N_w := N_\sigma$.

The \textit{support} $\text{supp}(\mathcal{F})$ of a tropical cycle $\mathcal{F}$ is the union of its closed facets with non-zero multiplicities. The \textit{link} of a polyhedron $\sigma \subseteq \mathbb{R}^m$ at a point $v \in \sigma$ is the polyhedral cone

$$\text{link}_v(\sigma) = \{u \in \mathbb{R}^m \mid \exists \delta > 0 : \forall \varepsilon \text{ between 0 and } \delta : v + \varepsilon u \in \sigma\}.$$  

The \textit{link} of a tropical cycle $\mathcal{F}$ at a point $v \in \text{supp}(\mathcal{F})$ is the tropical cycle

$$\text{link}_v(\mathcal{F}) = \{\text{link}_v(\sigma) \mid v \in \sigma \in \mathcal{F}\}$$  

with inherited multiplicities. This is always a fan. We also use notation $\text{link}_v(\sigma)$ and $\text{link}_v(\mathcal{F})$ to denote links with respect to relative interior points of $\sigma$. 

A weighted rational polyhedral complex $F$ is called \textit{balanced} if for any ridge (codim-1 face) $\tau$ of $F$,
\[
\sum_{\sigma \supset \tau} \text{mult}_F(\sigma) \cdot v_{\sigma/\tau} \in \text{span}_Q(N_\tau)
\]
where $\sigma$ runs over facets of $F$ containing $\tau$ and $v_{\sigma/\tau} \in N \cap \text{link}_F(\sigma)$ is a lattice element generating $N_\sigma$ together with $N_\tau$.

As we explain in the following the polyhedral complex structure is disregarded. For the purpose of disregarding the polyhedral structure we let the \textit{common refinement} $F \wedge G := \{ \sigma \cap \tau : \sigma \in F, \tau \in G \}$ of a tropical cycle $F$ and a complete complex $G$ inherit the multiplicities of $F$. Two cycles $F_1$ and $F_2$ are identified if there exists a complete complex $G$ such that $F_1 \wedge G = F_2 \wedge G$ with multiplicity.

Let $F$ be a tropical cycle in $N_\mathbb{R}$ and $A : N \to N'$ be a linear map, inducing a map $A : N_\mathbb{R} \to N'_\mathbb{R}$. We can define multiplicities on the image $AF$ as follows. A point $w$ in the support $\text{supp}(F)$ is called \textit{smooth} if $\text{supp}(\text{link}_w(F))$ is a linear space. For a smooth point $w \in AF$, where $AF$ is endowed with a polyhedral structure such that the image of every face of $F$ is a union of of faces of $AF$, let
\[
\text{mult}_{AF}(w) = \sum_v \text{mult}_F(v) \cdot [N'_w : AN_v],
\]
where the sum runs over one $v$ for each facet of $F$ meeting the preimage of $w$. If $F$ is the tropical variety of an ideal $I$ in the sense of Section 3 and $A$ is the tropicalization of a map $\alpha$ of tori, we have the relation $AF = \delta \mathcal{T}(\alpha(V(I)))$, where $\delta$ is the degree of $\alpha$ on $V(I)$. This can be seen from the Sturmfels–Tevelev projection formula \textsuperscript{ST08}. When $AF$ is the entire ambient space, $\mathcal{T}(\alpha(V(I)))$ has multiplicity one everywhere and $\delta$ is the multiplicity of $AF$. However, in general we cannot recover $\delta$ tropically from $A$ and $\mathcal{T}(I)$ as the following example shows.

\textbf{Example 2.1.} Let $I = \langle x_1 + x_2 + x_3 + 1, (x_3 - 2)(x_3 - 1) \rangle$ and $J = \langle x_1 + x_2 + 1, (x_3 - 2)(x_3 - 1) \rangle$ be ideals in $\mathbb{C}[x_1, x_2, x_3]$ and $A : \mathbb{Z}^3 \to \mathbb{Z}^2$ be the projection to the first two coordinates. Then $\mathcal{T}(I) = \mathcal{T}(J)$ consists of the three rays $-e_1, -e_2$ and $e_1 + e_2$, each with multiplicity 2. The ideal $J \cap \mathbb{C}[x_1, x_2]$ is a linear ideal. This makes the projection of $V(J)$ to the first two coordinates a degree-2 map. The ideal $I \cap \mathbb{C}[x_1, x_2]$, however, is generated in degree two, making the projection of $V(I)$ a degree-1 map.

Although a tropical cycle $F$ is pure dimensional, the image $AF$ need not be pure. For example let $F$ have support equal to the union of $\text{span}(e_1, e_2)$ and $\text{span}(e_3, e_4)$ in $\mathbb{R}^4$. With all multiplicities equal to one this is a tropical cycle. Let $A$ be the projection onto $\text{span}(e_1, e_2, e_3)$. Then $AF$ is a union of a plane and a line, hence not pure dimensional. However, the image $AF$ is balanced in the following sense.
Lemma 2.2. Let \( \tau \) be a ridge in \( \mathcal{F} \) such that \( A\tau \) also has codimension 1 in \( A(\text{link}_F(\tau)) \). Then \( A(\text{link}_F(\tau)) \) is balanced with multiplicity defined above.

**Proof.** Let \( \tau \) be a ridge in \( \mathcal{F} \) such that \( A\tau \) also has codimension 1 in \( A(\text{link}_F(\tau)) \). From the balancing condition on \( \mathcal{F} \) at \( \tau \), we have

\[
\sum_{\sigma \supset \tau} \text{mult}_\mathcal{F}(\sigma) \cdot v_{\sigma/\tau} \in \text{span}_Q(N_{\tau}).
\]

Applying the map \( A \) gives

\[
\sum_{\sigma \supset \tau} \text{mult}_\mathcal{F}(\sigma) \cdot Av_{\sigma/\tau} \in \text{span}_Q(AN_{\tau}).
\]

Observe that \([N'_{A\sigma} : N'_{A\tau} + \text{span}_Z(Av_{\sigma/\tau})]v_{A\sigma/A\tau} \equiv Av_{\sigma/\tau} \pmod{\text{span}_Q(AN_{\tau})}\) and

\[
[N'_{A\sigma} : AN_{\sigma}] = [N'_{A\sigma} : A(N_{\tau} + \text{span}_Z(v_{\sigma/\tau}))]
= [N'_{A\sigma} : AN_{\tau} + \text{span}_Z(Av_{\sigma/\tau})]
= [N'_{A\sigma} : N'_{A\tau} + \text{span}_Z(Av_{\sigma/\tau})][N'_{A\tau} : AN_{\tau}].
\]

Hence

\[
\sum_{\sigma \supset \tau} \text{mult}_\mathcal{F}(\sigma) \cdot [N'_{A\sigma} : AN_{\sigma}] \cdot v_{A\sigma/A\tau}
= [N'_{A\tau} : AN_{\tau}] \sum_{\sigma \supset \tau} \text{mult}_\mathcal{F}(\sigma) \cdot [N'_{A\sigma} : N'_{A\tau} + \text{span}_Z(Av_{\sigma/\tau})] \cdot v_{A\sigma/A\tau}
\equiv [N'_{A\tau} : AN_{\tau}] \sum_{\sigma \supset \tau} \text{mult}_\mathcal{F}(\sigma)Av_{\sigma/\tau} \equiv 0 \pmod{\text{span}_Q(AN_{\tau})}.
\]

This proves that the image of a neighborhood of \( \tau \) is balanced with the multiplicities given by formula (1).

**Corollary 2.3.** If \( A\mathcal{F} \) contains a full-dimensional polyhedron, then \( A\mathcal{F} = N'_{\mathcal{F}} \), and the multiplicity of a generic point is given by formula (1) above.

**Proof.** Suppose the image \( A\mathcal{F} \) is a union of finitely many polyhedra, but it is not all of \( N'_{\mathcal{F}} \). Then there is a face on the boundary of \( A\mathcal{F} \) with dimension \( \dim(N'_{\mathcal{F}}) - 1 \). However, the balancing condition cannot hold at this face, contradicting Lemma 2.2. □

For subsets \( A, B \subset N_{\mathcal{F}} \), we will use + for Minkowski sum \( A + B = \{x + y : x \in A, y \in B\} \). Also, let \( -A = \{-x : x \in A\} \) and \( A - B = A + (-B) \). For a tropical cycle \( \mathcal{F} \), \( -\mathcal{F} \) is also a tropical cycle with multiplicity \( \text{mult}_{\mathcal{F}}(-w) = \text{mult}_\mathcal{F}(w) \). If \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) are tropical cycles, then we can define multiplicities on \( \mathcal{F}_1 + \mathcal{F}_2 \) by applying the formula (1) to the projection of the Cartesian product \( \mathcal{F}_1 \times \mathcal{F}_2 \) onto \( \mathcal{F}_1 + \mathcal{F}_2 \) via \( (x, y) \mapsto x + y \). It can be verified straightforwardly that the product \( \mathcal{F}_1 \times \mathcal{F}_2 \) is a tropical cycle with multiplicity given by
mult_{\mathcal{F}_1 \times \mathcal{F}_2}((w_1, w_2)) := \text{mult}_{\mathcal{F}_1}(w_1) \text{ mult}_{\mathcal{F}_2}(w_2). \quad \text{More concretely, a generic point } v \in \mathcal{F}_1 + \mathcal{F}_2 \text{ has multiplicity:}

\begin{equation}
\text{mult}_{\mathcal{F}_1 + \mathcal{F}_2}(v) = \sum_{\sigma_1, \sigma_2} \text{mult}_{\mathcal{F}_1}(\sigma_1) \text{ mult}_{\mathcal{F}_2}(\sigma_2)[N_v : N_{\sigma_1} + N_{\sigma_2}]
\end{equation}

where the sum is over all pairs of facets \( \sigma_1 \in \mathcal{F}_1 \) and \( \sigma_2 \in \mathcal{F}_2 \) such that \( v \in \sigma_1 + \sigma_2 \).

**Definition 2.4.** Let \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) be tropical cycles in \( \mathbb{R}^n \) with positive multiplicities. We define the **stable intersection** as

\( \mathcal{F}_1 \cdot \mathcal{F}_2 := \{ \omega \in \mathbb{R}^n : \text{link}_\omega \mathcal{F}_1 - (\text{link}_\omega \mathcal{F}_2) = N_{\mathbb{R}} \} \).

For a facet \( \gamma \in \mathcal{F}_1 \cdot \mathcal{F}_2 \), the multiplicity \( \text{mult}_{\mathcal{F}_1 \cdot \mathcal{F}_2}(\gamma) \) is defined to be the multiplicity of \( \text{link}_\gamma \mathcal{F}_1 - (\text{link}_\gamma \mathcal{F}_2) \) given by the formula \( (2) \) above. That is, for a generic element \( v \in N_{\mathbb{R}} \),

\[ \text{mult}_{\mathcal{F}_1 \cdot \mathcal{F}_2}(\gamma) = \sum_{\sigma, \tau} \text{mult}_{\mathcal{F}_1}(\sigma) \text{ mult}_{\mathcal{F}_2}(\tau)[N : N_{\sigma} + N_{\tau}] \]

where the sum is over all pairs of facets \( \sigma \in \text{link}_\gamma(\mathcal{F}_1) \) and \( \tau \in \text{link}_\gamma(\mathcal{F}_2) \) such that \( v \in \gamma - \tau \) (or equivalently, \( \gamma \) meets \( \tau + v \)).

The formula on the right hand side coincides with the cup product formula for Chow cohomology of toric varieties given by the **fan displacement rule** of Fulton and Sturmfels \([FS97, \text{Theorem 4.2}]\).

The stable intersection is a subcomplex of \( \mathcal{F}_1 \cap \mathcal{F}_2 \), which has a natural polyhedral complex structure as the common refinement \( \{ \sigma_1 \cap \sigma_2 : \sigma_1 \in \mathcal{F}_1, \sigma_2 \in \mathcal{F}_2 \} \).

Let \( T^k \) be the set of tropical cycles in \( N_{\mathbb{R}} \) of codimension \( k \). Then \( T^k \) is an abelian group with respect to the following addition operation. For \( \mathcal{F}_1, \mathcal{F}_2 \in T^k \), their sum \( \mathcal{F}_1 \oplus \mathcal{F}_2 \) is obtained by taking the union \( \mathcal{F}_1 \cup \mathcal{F}_2 \) and adding multiplicities on the overlaps. For a generic point \( w \in \mathcal{F}_1 \cup \mathcal{F}_2 \), we have \( \text{mult}_{\mathcal{F}_1 \oplus \mathcal{F}_2}(w) = \text{mult}_{\mathcal{F}_1}(w) + \text{mult}_{\mathcal{F}_2}(w) \). If \( \text{mult}_{\mathcal{F}_1 \oplus \mathcal{F}_2}(w) = 0 \), then we remove the facet from \( \mathcal{F}_1 \oplus \mathcal{F}_2 \). If \( \text{mult}_{\mathcal{F}_1 \oplus \mathcal{F}_2}(w) = 0 \) for \( w \) in a dense set of \( \mathcal{F}_1 \cup \mathcal{F}_2 \), then \( \mathcal{F}_1 \oplus \mathcal{F}_2 \) is the zero cycle. Note that we only add tropical cycles of the same codimension and that the sum preserves codimension unless it is the zero cycle.

The stable intersection as defined is easily seen to be distributive over addition of tropical cycles, so it can be extended to tropical cycles with arbitrary (possibly negative) weights by first rewriting each cycle as a linear combination of cycles with all positive weights. This can be done, for example, by adding and subtracting tropical cycles that are affine spans of faces with negative weights.

**Remark 2.5.** A polyhedral complex in \( \mathbb{R}^n \) is called **locally balanced** if it is pure dimensional and the link of every codimension one face positively spans a linear subspace of \( \mathbb{R}^n \). We can define stable intersections of locally balanced complexes without multiplicities using the same definition, and the set-theoretic parts of the following results still hold.
The following result follows from the definition and is useful for computing stable intersections.

**Lemma 2.6.** For tropical cycles $F_1$ and $F_2$ in $N_\mathbb{R}$ with positive multiplicities,
\[
F_1 \cdot F_2 = \bigcup_{\sigma_1 \in F_1, \sigma_2 \in F_2, \dim(\sigma_1 + \sigma_2) = n} \sigma_1 \cap \sigma_2.
\]

**Proof.** For any $w \in F_1 \cap F_2$, link$_w F_1 - \text{link}_w F_2 = \mathbb{R}^n$ if and only if there are \( \sigma_1 \in F_1 \) and \( \sigma_2 \in F_2 \) containing $w$ such that \( \dim(\sigma_1 + \sigma_2) = \dim(\sigma_1 - \sigma_2) = n \). \( \square \)

In [RGST05] stable intersections were defined by taking limits of perturbed intersections. We will now show that this is equivalent to Definition 2.4.

**Proposition 2.7.** Let $F_1$ and $F_2$ be tropical cycles in $N_\mathbb{R}$ with positive multiplicities. Then for any sufficiently generic $v \in N_\mathbb{R}$, we have
\[
F_1 \cdot F_2 = \lim_{\varepsilon \to 0} F_1 \cap (F_2 + \varepsilon v).
\]

In particular, the limit set does not depend on the choice of generic $v$.

**Proof.** Let $w \in F_1 \cap F_2$, and $v \in N_\mathbb{R}$. Then $w \in \lim_{\varepsilon \to 0} F_1 \cap (F_2 + \varepsilon v)$ if and only if for every $\delta > 0$ there is an $\varepsilon > 0$ such that $F_1 \cap (F_2 + \varepsilon v)$ contains a point within distance $\delta$ from $w$. This holds if and only if \( \text{link}_w(F_1) \cap (\text{link}_w(F_2) + v) \neq \emptyset \), or equivalently \( v \in \text{link}_w F_1 - (\text{link}_w F_2) \).

When $F_1$ and $F_2$ are balanced with positive multiplicities, the support of $\text{link}_w F_1 - (\text{link}_w F_2)$ is either all of $N_\mathbb{R}$ or has positive codimension. There are only finitely many such positive-codimensional complexes as $w$ varies, so for generic $v$
\[
v \in \text{link}_w F_1 - (\text{link}_w F_2) \iff \text{link}_w F_1 - (\text{link}_w F_2) = N_\mathbb{R}.
\]
The result follows from the two equivalences above. \( \square \)

For any pure dimensional tropical cycles $F$ and $F'$, we say that the intersection $F \cap F'$ is *transverse* if for every generic point $w$ of $F \cap F'$, $w$ is also a smooth point of both $F$ and $F'$ and $\text{link}_w F$ and $\text{link}_w F'$ are linear spaces that together span $N_\mathbb{R}$. If $F$ and $F'$ intersect transversely, then it follows from Lemma 2.6 that $F \cdot F' = F \cap F'$. Any generic point $w$ in $F \cdot F'$ lies in the relative interiors of some facets $\sigma \in F$ and $\sigma' \in F'$ for some choice of polyhedral structure on $F$ and $F'$. By definition of the stable intersection,
\[
\text{mult}_{F, F'}(w) = \text{mult}_F(w) \text{mult}_{F'}(w)[N : N_\sigma + N_{\sigma'}].
\]

Let us fix generic $v \in N_\mathbb{R}$ and $\varepsilon > 0$. For any polyhedra $\sigma \in F_1$ and $\tau \in F_2$, the intersection $\sigma \cap (\tau + \varepsilon v)$ is either empty or contains a point in the relative interior of both $\sigma$ and $\tau + \varepsilon v$. In the later case, $\text{codim}(\sigma \cap (\tau + \varepsilon v)) = \text{codim}(\sigma) + \text{codim}(\tau)$. Hence the intersection $F_1 \cap (F_2 + \varepsilon v)$ is transverse and has codimension equal to $\text{codim}(F_1) + \text{codim}(F_2)$.
We can assign multiplicities to the limit in Proposition 2.7 as follows. We say that a facet $\gamma \in \mathcal{F}_1 \cdot \mathcal{F}_2$ is the limit of the facet $\sigma \cap (\tau + \varepsilon v)$ in $\mathcal{F}_1 \cap (\mathcal{F}_2 + \varepsilon v)$ for sufficiently small $\varepsilon > 0$ if $\sigma, \tau \supset \gamma$ and $\text{link}_\gamma(\sigma) \cap (\text{link}_\gamma(\tau) + v) \neq \emptyset$. Since for generic $v$ and $\varepsilon$ the intersection $\mathcal{F}_1 \cap (\mathcal{F}_2 + \varepsilon v)$ is transverse, hence stable, the facet $\sigma \cap (\tau + \varepsilon v)$ has multiplicity given by formula (3) above. Combining with the multiplicity formula in Definition 2.4, we see that
\[
\text{mult}_{\mathcal{F}_1, \mathcal{F}_2}(\gamma) = \sum_{\sigma, \tau} \text{mult}_{\mathcal{F}_1, (\mathcal{F}_2 + \varepsilon v)}(\sigma \cap (\tau + \varepsilon v))
\]
where $v$ is generic, $\varepsilon > 0$ is sufficiently small, and the sum is over facets $\sigma \in \mathcal{F}_1$ and $\tau \in \mathcal{F}_2$ such that the limit of $\sigma \cap (\tau + \varepsilon v)$ is $\gamma$.

We will now see that stable intersections and multiplicities behave well under taking links and quotienting out by lineality. In the following, for a rational linear space $L \subset N_{\mathbb{R}}$, $N_{\mathbb{R}}/L$ is equipped with the lattice $N/N_L$.

**Lemma 2.8.** Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be tropical cycles.

1. For $w \in \mathcal{F}_1 \cdot \mathcal{F}_2$, we have $\text{link}_w(\mathcal{F}_1 \cdot \mathcal{F}_2) = \text{link}_w(\mathcal{F}_1) \cdot \text{link}_w(\mathcal{F}_2)$.
2. For a rational linear space $L$ contained in the lineality spaces of both $\mathcal{F}_1$ and $\mathcal{F}_2$, we have $(\mathcal{F}_1 \cdot \mathcal{F}_2)/L = (\mathcal{F}_1/L) \cdot (\mathcal{F}_2/L)$.

**Proof.** The first statement follows easily from the definition and Lemma 2.6. For the second statement, first observe that for a unimodular coordinate change $U$ we have $U(\mathcal{F}_1 \cdot \mathcal{F}_2) = U(\mathcal{F}_1) \cdot U(\mathcal{F}_2)$. Pick a lattice basis of $N_L$ and extend it to a lattice basis of $N$. In this basis $L$ is a coordinate subspace and its presence in $\mathcal{F}_1$ and $\mathcal{F}_2$ does not affect the construction of the stable intersection other than having to take the product with $L$. \qed

**Lemma 2.9.** Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be polyhedral complexes in $\mathbb{R}^n$. If we identify $x \in \mathbb{R}^n$ with $(x, x) \in \mathbb{R}^n \times \mathbb{R}^n$ then $\mathcal{F}_1 \cdot \mathcal{F}_2 = (\mathcal{F}_1 \times \mathcal{F}_2) \cdot \{\Delta\}$, where $\Delta = \{(x, x) : x \in N_{\mathbb{R}}\}$ is the diagonal with multiplicity 1 in $N_{\mathbb{R}} \times N_{\mathbb{R}}$ and is identified with $N_{\mathbb{R}}$.

**Proof.** Let $\omega \in \mathcal{F}_1 \cap \mathcal{F}_2$, $\sigma_1 \in \text{link}_\omega(\mathcal{F}_1)$, and $\sigma_2 \in \text{link}_\omega(\mathcal{F}_2)$. Let $A_1$ and $A_2$ be matrices whose sets of columns are lattice bases for $N_{\sigma_1}$ and $N_{\sigma_2}$ respectively. Then the columns of the matrix $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ span $N_{\sigma_1 \times \sigma_2} + N_{\Delta}$ over $\mathbb{Z}$, while the columns of $\begin{pmatrix} A_1 & -A_2 \end{pmatrix}$ span $N_{\sigma_1-\sigma_2}$ over $\mathbb{Z}$. Hence $\dim((\sigma_1 \times \sigma_2) + \Delta) = 2n$ if and only if $\dim(\sigma_1 + \sigma_2) = n$. Moreover, these two matrices have the same index (gcd of maximal minors), so $[N \times N : N_{\sigma_1 \times \sigma_2} + N_{\Delta}] = [N : N_{\sigma_1} + N_{\sigma_2}]$. For (generic) $v_1, v_2 \in N_{\mathbb{R}}$, we have $(v_1, v_2) \in (\sigma_1 \times \sigma_2) - \Delta$ if and only if $v_1 - v_2 \in \sigma_1 - \sigma_2$. The assertion follows from Definition 2.4 \qed

We will need the following lemma to prove associativity of stable intersection.

**Lemma 2.10.** For any tropical cycle $\mathcal{F}$ and linear spaces $L_1$ and $L_2$ in $N_{\mathbb{R}}$, we have
\[
(\mathcal{F} \cdot L_1) \cdot L_2 = \mathcal{F} \cdot (L_1 \cdot L_2).
\]
Proof. By taking links, for equality of the supports it suffices to prove that $0 \in (\mathcal{F} \cdot L_1) \cdot L_2$ if and only if $0 \in \mathcal{F} \cdot (L_1 \cap L_2)$. Note that for any linear space $L$, $0 \in \mathcal{F} \cdot L$ if and only if the projection of $\mathcal{F}$ onto a complement of $L$ is surjective. Let $\pi_1$, $\pi_2$, and $\pi_{12}$ be projections from $N_{\mathbb{R}}$ onto the $L_1^\perp$, $L_2^\perp$, and $(L_1 \cap L_2)^\perp$ respectively. WLOG, we may assume that $L_1^\perp \perp L_2^\perp$.

Suppose $0 \in (\mathcal{F} \cdot L_1) \cdot L_2$. Then there is a cone $\sigma \in \mathcal{F}$ such that $\dim(\pi_1(\sigma)) = \dim(L_1^\perp)$ and $\dim(\pi_2(\sigma \cap L_1)) = \dim(L_2^\perp)$, so $\dim(\pi_{12}(\sigma)) = \dim(\pi_1(\sigma)) + \dim(\pi_2(\sigma)) \geq \dim(L_1^\perp) + \dim(L_2^\perp) = \dim((L_1 \cap L_2)^\perp)$. Thus $0 \in \mathcal{F} \cdot (L_1 \cap L_2)$.

Now suppose $0 \in \mathcal{F} \cdot (L_1 \cdot L_2)$. Then $\pi_{12}(\mathcal{F}) = L_1^\perp + L_2^\perp$, so there exists a $\sigma \in \mathcal{F}$ such that $\dim(\pi_{12}(\sigma)) = \dim(L_1^\perp) + \dim(L_2^\perp)$ and $\dim(\pi_{12}(\sigma \cap L_1)) = \dim(L_1^\perp)$. Then $\dim(\pi_1(\sigma)) = \dim(L_1^\perp)$, so $\sigma \cap L_1 \in \mathcal{F} \cdot L_1$. We will show that $\pi_2(\sigma \cap L_1) \subset \pi_{12}(\sigma) \cap L_2^\perp$. Let $v \in \pi_{12}(\sigma) \cap L_2^\perp$. Then there is a $v' \in \sigma$ such that $\pi_{12}(v') = v$. Then $\pi_1(v') \in L_1^\perp$, and also $\pi_1(v') = \pi_{12}(v') - \pi_2(v') \in L_2^\perp$. Thus $\pi_1(v') = 0$, so $v' \in L_1$, and $v = \pi_{12}(v') = \pi_2(v') \in \pi_2(\sigma \cap L_1)$. This proves that $\dim(\pi_2(\sigma \cap L_1)) \geq \dim(\pi_{12}(\sigma) \cap L_2^\perp) = \dim(L_2^\perp)$. Therefore $(\sigma \cap L_1) \cap L_2$ is a face of $(\mathcal{F} \cdot L_1) \cdot L_2$ proving that $0 \in (\mathcal{F} \cdot L_1) \cdot L_2$ as desired.

To compute multiplicities, after taking links and taking quotients, we may assume that the support of the tropical cycles on both sides consist only of the origin. For generic $v_2 \in \mathbb{R}^n$ we have

$$\text{mult}_{(\mathcal{F} \cdot L_1) \cdot L_2}(0) = \sum_{\tau \in \mathcal{F} \cdot L_1: (L_2 + v_2) \cap \tau \neq \emptyset} \text{mult}_{\mathcal{F} \cdot L_1}(\tau)[N : N_{L_2} + N_{\tau}]$$

where, for generic $v_1 \in \mathbb{R}^n$,

$$\text{mult}_{\mathcal{F} \cdot L_1}(\tau) = \sum_{\sigma \in \mathcal{F} \cdot (L_1 + v_1) \cap \sigma \neq \emptyset \text{ and } \tau = \sigma \cap L_1} \text{mult}_{\mathcal{F}}(\sigma)[N : N_{L_1} + N_{\sigma}].$$

Since $\text{mult}_{L_1, L_2}(L_1 \cap L_2) = [N : N_{L_1} + N_{L_2}]$, we on the other hand have

$$\text{mult}_{\mathcal{F} \cdot (L_1 \cdot L_2)}(0) = \sum_{\sigma \in \mathcal{F} \cdot (L_1 \cap L_2) + v_3 \cap \sigma \neq \emptyset} \text{mult}_{\mathcal{F}}(\sigma)[N : N_{L_1} + N_{L_2}][N : N_{L_1 \cap L_2} + N_{\sigma}]$$

for generic $v_3 \in \mathbb{R}^n$. We now argue that the double sum for the first computation of multiplicity runs over the same $\sigma$ as the last, with $\tau$ always chosen as $\sigma \cap L_1$. Furthermore, we are free to choose $v_1$ much smaller than $v_2$ and have $v_3 = v_1 + v_2$. For a fixed finite complex, the cones $\sigma$ containing $v_3$ are exactly those for which $v_2 \in \sigma \cap L_1$ and $(L_1 + v_1) \cap \sigma \neq \emptyset$, as desired.

Since $N_{\sigma} = N_{L_1} 

Thus, $N_{\sigma} = N_{L_1} n N_{\sigma}$, to prove that the two multiplicities are equal, it suffices to prove for saturated lattices $A, B, C$ (and $A \cap B$) with well defined indices:

$$[N : A + C][N : B + (A \cap C)] = [N : A + B][N : (A \cap B) + C].$$
All lattices of which we take indices contain $A \cap B$. Since $A \cap B$ is saturated, after quotienting out by $A \cap B$, we are in the case where $A \cap B = \{0\}$. Then

$$[N : (A \cap B) + C] = [N : C]$$

$$= [N : A + C][A + C : C]$$

$$= [N : A + C][A : A \cap C]$$

$$= [N : A + C][A + B : (A \cap C) + B]$$

$$= [N : A + C][N : B + (A \cap C)]/[N : A + B].$$

For the third equality, notice that $a+C = b+C$ if and only if $a+A\cap C = b+A\cap C$ for all $a, b \in A$. The fourth equality holds because $A \cap B = \{0\}$ and $A$ and $B$ are saturated.

Lemma 2.11. For tropical cycles $F_1, F_2, F_3$ in $N_R$ we have

$$(F_1 \cdot F_2) \times F_3 = (F_1 \times F_3) \cdot (F_2 \times N_R).$$

Proof. The result follows from Definition 2.4. The equation holds with multiplicities since the lattice indices of Definition 2.4 stay the same when going to the bigger lattices.

The following result was proved in [AR10, Theorem 9.10], but we include a self-contained proof here for completeness.

Proposition 2.12. Stable intersection is associative, i.e. for any three tropical cycles $F_1, F_2, F_3$, we have

$$(F_1 \cdot F_2) \cdot F_3 = F_1 \cdot (F_2 \cdot F_3).$$

Proof. Let $\pi : N_R \times N_R \to N_R$ and $\pi' : N_R \times N_R \times N_R \to N_R \times N_R$ be the projections to the first coordinates and the first and last coordinates, respectively. Let $\Delta = \{(x, x) : x \in N_R\}, \Delta' = \{(x, x, x) : x \in N_R\}, \Delta_{12} = \{(x, x, y) : x, y \in N_R\}$, and $\Delta_{13} = \{(x, y, x) : x, y \in N_R\}$. Using the diagonal trick from Lemma 2.9

$$(F_1 \cdot F_2) \cdot F_3 = \pi((\pi((F_1 \times F_2) \cdot \Delta) \times F_3) \cdot \Delta)$$

$$= \pi(\pi'(((F_1 \times F_2) \cdot \Delta) \times F_3) \cdot \Delta)$$

$$= \pi(\pi'(((F_1 \times F_2 \times F_3) \cdot \Delta_{12}) \cdot \Delta_{13}))$$

$$= (\pi \circ \pi')(((F_1 \times F_2 \times F_3) \cdot \Delta').$$

The third and fourth equality follow from Lemma 2.11 while the last follows from Lemma 2.10 and the fact that $\Delta_{12} \cdot \Delta_{13} = \Delta'$. The formula on the right hand side is clearly associative, and hence is the stable intersection.

Proposition 2.13. Our definition coincides with the Allermann–Rau intersection product of tropical cycles.
Proof. Using the diagonal trick from Lemma 2.9 and rewriting the diagonal as the stable intersection of hyperplanes, we can reduce to the case when one of the tropical cycles is a usual hyperplane $H$ with multiplicity 1 and the other is a tropical cycle $F$. Then both sets contain points in the support of $F$ whose link is not contained in $H$. To compute multiplicities, suppose $H$ is defined by $x_1 = 0$. By taking links if necessary, we may assume that $F \cdot H$ is a linear space $L \subseteq H$, and that $F$ consists of $k$ cones, each of which is spanned by one of the vectors $r^{(1)}, \ldots, r^{(k)}$ together with $L$. Let $m_1, \ldots, m_k$ be the multiplicities of those cones respectively. Using the formula (3) with perturbation $H + \varepsilon e_i$, we compute the multiplicity of $L$ in $F \cdot H$ to be

$$\sum_{i : r^{(i)}_1 > 0} m_i [N : N_H + N_{r^{(i)}_1 + L}] = \sum_{i : r^{(i)}_1 > 0} m_i r^{(1)}_1.$$

This is easily seen to coincide with the Allermann–Rau definition of multiplicities using the tropical polynomial $\max(0, x_1)$ that defines $H$. □

Next we will show that stable intersections of tropical cycles have expected dimensions and that they satisfy the balancing condition. Although these results follow from the equivalence of our definition of stable intersection with the Allermann–Rau intersection product, we include them for completeness.

Lemma 2.14. A two-dimensional tropical cycle $F$ with positive multiplicities cannot stably intersect a hyperplane $H$ in finitely many points, unless it is contained in a union of affine hyperplanes parallel to $H$.

Proof. Choose two-dimensional face of $F$ which is not parallel to $H$, and let $S$ be its affine span. Choose an $n - 2$ dimensional rational subspace $U$ of $H$ such that $U + S = N_R$. Since the multiplicities are positive, $F + U$ is a full-dimensional cycle, which must be complete and cover $H$. Thus there is a facet $\sigma$ of $F$ such that $\dim(\sigma + U) = n$ and $\dim((\sigma + U) \cap H) = n - 1$. Then $\sigma \cap H$ is in $F \cdot H$, and it has dimension at least one because $\dim((\sigma \cap H) + U) = n - 1$ and $\dim(U) = n - 2$. □

Theorem 2.15. For tropical cycles $F_1$ and $F_2$, the stable intersection $F_1 \cdot F_2$ is either the zero cycle or has $\text{codim}(F_1 \cdot F_2) = \text{codim}(F_1) + \text{codim}(F_2)$. Moreover, $F_1 \cdot F_2$ satisfies the balancing condition. In other words, the stable intersection of tropical cycles is a tropical cycle of expected dimension unless it is zero.

Proof. Every tropical cycle can be written as a linear combination of cycles with positive multiplicities, so we may assume that the cycles $F_1$ and $F_2$ have positive multiplicities.

Suppose $F_1 \cdot F_2 \neq 0$. Let us first show that $F_1 \cdot F_2$ is pure of expected codimension. First consider the case when $F_2$ is a hyperplane. Let $\sigma$ be a facet of $F_1 \cdot F_2$ and consider the link $\text{link}_\sigma(F_1 \cdot F_2) = \text{link}_\sigma(F_1) \cdot \text{link}_\sigma(F_2)$. Taking quotient by $\text{span}(\sigma)$ we may assume that this stable intersection is a point. By the previous
lemma \( \dim(\text{link}_\sigma(\mathcal{F}_1)) \leq \dim(\sigma) + 1 \). On the other hand, \( \dim(\sigma) < \text{link}_\sigma(\mathcal{F}_1) \). This shows that \( \dim(\sigma) = \dim(\mathcal{F}_1) - 1 \) as asserted.

By Lemma 2.9 the stable intersection equals \((\mathcal{F}_1 \times \mathcal{F}_2) \cdot \{\Delta\}\) where \(\Delta\) is the diagonal in \(N_\mathbb{R} \times N_\mathbb{R}\). The codimension of \((\mathcal{F}_1 \times \mathcal{F}_2)\) equals the sum of the codimensions. By the associativity of the stable intersection, we may apply the argument above repeatedly to get the stable intersection with the diagonal \(\Delta\). The codimension increases \(n\) times, but this is countered by the identification of \(N_\mathbb{R}\) with \(\Delta\).

We now verify the balancing condition. By taking the stable intersection of the Cartesian product with the diagonal, taking links at a ridge, and modding out by lineality, we may assume that \(\mathcal{F}_1\) is a two dimensional fan and that \(\mathcal{F}_2\) is a hyperplane. For a generic vector \(v\) and \(\varepsilon > 0\), the set-intersection \(\mathcal{F}_1 \cap (\mathcal{F}_2 + \varepsilon v)\) is transverse and of dimension one. It suffices to show that the transverse intersection is balanced because the stable intersection consists of the unbounded rays of the transverse intersection, counted with multiplicities.

We have now reduced to the case when \(\mathcal{F}_1 = \mathcal{F}\) is a two dimensional balanced fan with a one dimensional lineality space \(\tau\) and \(\mathcal{F}_2 = H\) is a hyperplane not containing \(\tau\). For any facet \(\sigma\) in \(\mathcal{F}\), let \(v_\sigma\) be the primitive lattice vector in \(\mathcal{F} \cap \mathcal{H}\). Then by the definition of multiplicities in stable intersections, we have

\[
\text{mult}_{\mathcal{F},H}(\sigma \cap H) = \frac{\text{mult}_{\mathcal{F}}(\sigma) \cdot [N : N_H + N_\tau]}{[N_\sigma : \text{span}_\mathbb{Q}(v_\sigma) + N_\tau]}.
\]

On the other hand, from balancedness of \(\mathcal{F}\), we have

\[
\sum_{\sigma \supseteq \tau} \text{mult}_{\mathcal{F}}(\sigma) \cdot v_{\sigma/\tau} \in \text{span}_\mathbb{Q}(N_\tau).
\]

Moreover we have \(v_\sigma - [N_\sigma : \text{span}_\mathbb{Q}(v_\sigma) + N_\tau] \cdot v_{\sigma/\tau} \in \text{span}_\mathbb{Q}(N_\tau)\). Combining the last two statements and multiplying through with \([N : N_H + N_\tau]\) gives

\[
\sum_{\sigma \supseteq \tau} \text{mult}_{\mathcal{F},H}(\sigma \cap H) \cdot v_\sigma \in \text{span}_\mathbb{Q}(N_\tau) \cap H = \{0\}.
\]

This completes the proof of the balancing condition for stable intersections. \(\square\)

Let \(n\) be a positive integer. For \(k = 0, 1, \ldots, n\), let \(T^k\) be the \(\mathbb{Q}\)-vector space of tropical codimension \(k\) cycles in \(N_\mathbb{R}\) with rational multiplicities, where addition is the union. Let \(T\) be the direct sum \(T = \bigoplus_{k=0}^n T^k\). The stable intersection gives multiplication on \(T\). We have shown that the stable intersection is commutative, associative, and distributive over cycle-addition.

**Theorem 2.16.** The set \(T\) of tropical cycles form a graded \(\mathbb{Q}\)-algebra where addition is union and multiplication is stable intersection.

We will see in Section 5 that the algebra \(T\) is isomorphic to the polytope algebra of McMullen.
3. Stable intersections as tropical varieties of ideals

In this section, we interpret stable intersections of tropical varieties as tropicalizations of intersections after generic perturbations by rescaling, as we will make precise below.

Until now we have not assumed that our tropical cycles are fans, but for the rest of this paper they will be. The definition of tropical varieties below can be extended to the case where a valuation of the coefficient field is taken into account. See [MS] for details. In that setting tropical varieties need not be fans. For simplicity we consider only the fan case here.

Let \( k \) be an algebraically closed field, and \( R \) be the Laurent polynomial ring of the torus \( N \otimes \mathbb{Z} k^* \), i.e. \( R = k[M] \) where \( M = \text{Hom}_\mathbb{Z}(N, \mathbb{Z}) \). The tropical variety of an ideal \( I \) in \( R \) is

\[
\mathcal{T}(I) = \{ \omega \in N_R : \text{in}_\omega(I) \neq R \}.
\]

It can be given a fan structure as a subfan of the Gröbner fan of \( I \) (after possibly a homogenization of \( I \)), and it is a tropical cycle with multiplicities given by

\[
\text{mult}_{\mathcal{T}(I)}(\omega) := \dim_k(k[\mathbb{Z}^n \cap C] / \langle \text{in}_\omega(I) \rangle)
\]

for generic points \( \omega \) in a facet \( C \). Equivalently the multiplicity can be defined as

\[
\text{mult}_{\mathcal{T}(I)}(\omega) := \sum_{P \in \text{Ass}(\text{in}_\omega(I))} \text{mult}(P, \text{in}_\omega(I)).
\]

See [BJS+07, MS] for more details. If \( L \) is a subspace of the lineality space of \( \mathcal{T}(I) \), then \( \mathcal{T}(I)/L \) is a tropical cycle in \( (N/N_L)_R \) with inherited multiplicities from \( \mathcal{T}(I) \), and

\[
(5) \quad \mathcal{T}(I)/L = \mathcal{T}(I \cap k[L^\perp \cap M])
\]

where the lattice \( L^\perp \cap M \) is naturally identified with \( \text{Hom}_\mathbb{Z}(N/N_L, \mathbb{Z}) \).

**Lemma 3.1.** Let \( I \subseteq k[x_1, \ldots, x_n] \) be an ideal with \( n \geq 1 \). Let \( \omega \in \mathbb{R}^n \) have \( \omega_1 = 0 \). Considered as ideals in \( k(\alpha)[x_1, \ldots, x_n] \) we have

\[
\text{in}_\omega((I) + \langle x_1 - \alpha \rangle) = \text{in}_\omega((I)) + \text{in}_\omega(\langle x_1 - \alpha \rangle).
\]

**Proof.** The inclusion \( \supseteq \) is clear and we will prove \( \subseteq \). The left hand side is generated by elements of the form \( \text{in}_\omega(\sum_{P \in \text{Ass}(\text{in}_\omega(I))} \frac{1}{p}(f + g \cdot (x_1 - \alpha))) \) with \( f \in \langle I \rangle \cap k[\alpha][x_1, \ldots, x_n], g \in k[\alpha][x_1, \ldots, x_n], p \in k[\alpha] \). Since \( p \) is a unit, we may ignore the \( 1/p \) factor in our argument.

We argue that without loss of generality \( f \in I \). That is, \( f \) does not involve \( \alpha \). If \( f \not\in I \), then consider the expression \( f = \sum c_i F_i \) with \( c_i \in k[\alpha][x_1, \ldots, x_n] \) and \( F_i \in I \) and perform polynomial division of \( c_i \) modulo \( \alpha - x_1 \) to obtain \( c_i = g_i'(x_1 - \alpha) + r_i \) for some \( g_i' \in k[\alpha][x_1, \ldots, x_n] \) and \( r_i \in k[x_1, \ldots, x_n] \). We now have \( f + g(x_1 - \alpha) = \sum c_i F_i + g(x_1 - \alpha) = \sum (g_i'(x_1 - \alpha) + r_i) F_i + g(x_1 - \alpha) = \sum r_i F_i + (\sum g_i F_i) + g(x_1 - \alpha) \). Here \( \sum r_i F_i \) indeed is in \( I \).
Lemma 3.2. Let $I$ be an ideal in $k[x_1, x_2, \ldots, x_n]$. Then

$$\mathcal{T}(I) \nsubseteq \{x : x_1 = 0\} \iff (I : (x_1x_2 \cdots x_n)\infty) \cap k[x_1] = \{0\}. $$

Proof. Consider the projection of $V(I)$ in $(k^*)^n$ onto the $x_1$ axis. By elimination theory, the defining ideal $J$ of the image is $(I : (x_1x_2 \cdots x_n)\infty) \cap k[x_1]$. By the fundamental theorem of tropical geometry [MS, JMM08], the tropical variety of the image is the image of $\mathcal{T}(I)$ under the projection onto the first axis. There are three possibilities for the image in $k^*$:

- Empty: LHS is false, and $J = k[x_1] \neq \{0\}$.
- Finitely many points: the tropicalization of the projection is $\{0\}$, so the LHS is true, and $J$ contains a nonzero polynomial in $x_1$ vanishing on those points.
- All of $k^*$: the LHS is true, and $J = \{0\}$.

Let $I$ be an ideal in $k[x_1, x_2, \ldots, x_n]$. For almost all $c \in k$, we have

$$\mathcal{T}(I + \langle x_1 - c \rangle) = \mathcal{T}((I + \langle x_1 - \alpha \rangle))$$

where the ideal in the right hand side is in $k(\alpha)[x_1, x_2, \ldots, x_n]$. Indeed for a homogeneous ideal $I$, to find the right hand side a finite number of Gröbner basis computations are required. Except for a finite number of choices of $c$, the Gröbner bases of $\langle I \rangle + \langle x_1 - \alpha \rangle$ are also Gröbner bases for $I + \langle x_1 - c \rangle$ after substituting $\alpha$ with $c$.

Lemma 3.3. Let $I$ be an ideal in $k[x_1, x_2, \ldots, x_n]$, and $H$ be the tropical cycle in $N_\mathbb{R}$ defined by $x_1 = 0$ with multiplicity one at all points. Then

$$\mathcal{T}(I) \cdot H = \mathcal{T}((I + \langle x_1 - \alpha \rangle)$$

as tropical cycles, where $\alpha$ is generic.

Proof. Let $\omega \in \mathbb{R}^n$ such that $\omega_1 = 0$. Then the following statements are equivalent, where $\langle I \rangle$ and $\langle x_1 - \alpha \rangle$ denote ideals in $k(\alpha)[x_1, \ldots, x_n]$:

1. $\omega \in \mathcal{T}(I) \cdot H$
2. $\mathcal{T}(\operatorname{in}_\omega(I)) \nsubseteq \{x_1 = 0\}$
3. $\langle \operatorname{in}_\omega(I) : (x_1x_2 \cdots x_n)\infty \rangle \cap k[x_1] = \{0\}$
4. $\operatorname{in}_\omega(\langle I \rangle) + \langle x_1 - \alpha \rangle$ is monomial free
5. $\omega \in \mathcal{T}(\langle I \rangle + \langle x_1 - \alpha \rangle)$. 

Consider the degree of $f, g \cdot (x_1 - \alpha)$ and $f + g \cdot (x_1 - \alpha)$ in the $\omega$ grading. Since $\operatorname{in}_\omega(g \cdot (x_1 - \alpha))$ contains $\alpha$, its terms of some $\omega$-degree cannot cancel completely with terms of $f$, if $g \neq 0$. Therefore, $\operatorname{in}_\omega(f + g \cdot (x_1 - \alpha))$ is either $\operatorname{in}_\omega(f)$, if the degree of $f$ is highest, or $\operatorname{in}_\omega(g \cdot (x_1 - \alpha))$ if the degree of $g \cdot (x_1 - \alpha)$ is highest, or $\operatorname{in}_\omega(f) + \operatorname{in}_\omega(g \cdot (x_1 - \alpha))$ if the degrees are equal. $\square$
The equivalence (1) ⇔ (2) follow from the definitions of tropical varieties and stable intersection and the statement linkω(T(I)) = T(ω). The equivalences (4) ⇔ (5) and (2) ⇔ (3) follow from Lemmas 3.1 and 3.2 respectively. To see (4) ⇒ (3), note that an ideal containing x, 1 − α and a non-zero polynomial in k[x1] must be the unit ideal.

To see (3) ⇒ (4), suppose (4) does not hold, then there is a monomial m such that m = ∑i pi fi + g · (x, 1 − α) where fi ∈ inω(I), pi ∈ k(α) and g ∈ k(α)[x1, x2, . . . , xn]. After clearing denominators, we may assume that all pi ∈ k[α], g ∈ k[α, x1, x2, . . . , xn] and m = q · xν with q ∈ k[α] \ {0} and ν ∈ Nn. By substituting α with x1 we get q · xν = ∑i pi fi with pi and q being in k[x1]. Hence q · xν is in inω(I) and q is a non-zero element in (inω(I) : x1 · · · xν) ∩ k[x1].

Now we need to show that multiplicities coincide on a dense open subset. By taking links and quotienting out linearity space as in Lemma 2.8 and equation (5) we can reduce to the case where I is one dimensional and the intersection is just {0}. In this case, using Definition 2.4 of stable intersection multiplicities, we have

\[ \text{mult}_{T(I),H}(0) = \sum_{\sigma} \text{mult}_{T(I)}(\sigma)[N : N_\sigma + N_H] \]

where σ runs over rays of T(I) such that σ + H contains a fixed generic element (a1, . . . , an) ∈ Nn. For a ray σ of T(I), let vs denote the generator of Nσ. Then

\[ \text{mult}_{T(I),H}(0) = \sum_{\sigma} \text{mult}_{T(I)}(\sigma) vs1 \]

where σ runs over all rays of T(I) such that the first coordinates vs1 and a1 have the same sign. The right hand side is equal to the degree of the projection of the curve V(I) onto the first coordinate, by the Sturmfels–Tevelev formula for push-forward of multiplicities [ST08]. See the paragraph above Example 2.4. This degree is the degree of I + ⟨x − α⟩ for a generic α, which is the multiplicity of the origin of T(I + ⟨x − α⟩).

Let I and J be ideals in k[x1, . . . , xn]. Let K = k(c1, c2, . . . , cn) be the field of rational functions in indeterminates c1, c2, . . . , cn. Define I′ to be the ideal in K[x1, . . . , xn] generated by I and J′ to be the ideal generated by the image of J in K[x1, . . . , xn] under the ring homomorphism k[x1, . . . , xn] → K[x1, . . . , xn] given by x, i → ci · x, i.

**Lemma 3.4.** The change of coordinates from J to J′ preserves the Gröbner fan and the tropical variety, i.e. Gfan(J) = Gfan(J′) and T(J) = T(J′).

The fans on the left hand sides are defined with respect to k while those on the right are with respect to K.

**Proof.** Let ⪯ be any term order. Buchberger’s S-pair algorithm for Gröbner bases commutes with the coordinate change x, i → ci · x, i, so for a Gröbner basis
$G$ of $J$ with respect to $\prec$, the image of $G$ under the map $x_i \mapsto c_i x_i$ forms a Gröbner basis of $J'$ with respect to $\prec$. □

**Theorem 3.5.** With the notation above, we have
\[ \mathcal{T}(I') \cdot \mathcal{T}(J') = \mathcal{T}(I' + J'). \]

*Proof.* Using Lemmas 3.4, 2.9, 3.3 and associativity of stable intersection we get
\[ \mathcal{T}(I') \cdot \mathcal{T}(J') = \mathcal{T}(I) \cdot \mathcal{T}(J) = (\mathcal{T}(I) \times \mathcal{T}(J)) \cdot \Delta = \mathcal{T}(I'' + J'') \]
where $I''$ is the ideal generated by $I$ in $k[x_1, x_2, \ldots, x_n, X_1, X_2, \ldots, X_n]$ and $J''$ the ideal generated by $J$ after substituting $X_i$ for $x_i$. The second and fourth equalities are after identification of $N_R$ with the diagonal in $N_R \times N_R$. □

For some problems such as computation of resultants studied in [JY13], we want to compute the Newton polytope of a polynomial after substituting one or more variables with generic constants. This amounts to projecting the Newton polytope onto the coordinate subspace of the remaining variables. By the following theorem, we can perform this operation on tropical hypersurfaces, as orthogonal projection of a polytope onto a linear space is equivalent to stably intersecting the tropical hypersurface of the polytope with the linear space.

**Theorem 3.6.** Let $P \subset \mathbb{R}^n$ be a polytope, $L \subset \mathbb{R}^n$ be a rational linear subspace, and $\pi : \mathbb{R}^n \to L$ be the orthogonal projection. Then
\[ \mathcal{T}(\pi(P)) = (\mathcal{T}(P) \cdot L) + L^\perp. \]

*Proof.* Since $L^\perp$ is contained in the lineality space of both sides of the equation, it suffices to show that
\[ \mathcal{T}(\pi(P)) \cap L = \mathcal{T}(P) \cdot L. \]
For $w \in \mathbb{R}^n$, let $P^w$ denote the face of $P$ supported by the hyperplane whose normal vector pointing toward $P$ is $w$. Then $\text{link}_w(\mathcal{T}(P)) = \mathcal{T}(P^w)$. For $w \in L$, we also have $(\pi(P))^w = \pi(P^w)$, and
\[ w \in \mathcal{T}(P) \cdot L \iff \text{link}_w \mathcal{T}(P) + L = \mathbb{R}^n \]
\[ \iff \mathcal{T}(P^w) + L = \mathbb{R}^n \]
\[ \iff \mathcal{T}(P^w) \text{ contains cone } C \text{ such that } \dim(L + C) = n \]
\[ \iff P^w \text{ contains an edge not in } L^\perp \]
\[ \iff (\pi(P))^w \text{ contains at least two distinct points} \]
\[ \iff w \in \mathcal{T}(\pi(P)). \]
□

The normal fan of the projection of a polytope onto a linear subspace is the restriction of the normal fan to the linear subspace [Zie95, Chapter 7].
4. Volume, mixed volume, and Bernstein’s Theorem

In this section we study stable intersections of hypersurfaces of polytopes. The tropical hypersurface \( \mathcal{T}(P) \) of a rational polytope \( P \) is the union of codimension-1 cones in the normal fan of \( P \), i.e. the normal cones to edges of \( P \). The multiplicity of a normal cone to an edge is the lattice length of the edge. For rational polytopes, this weight is a rational number. For polytopes \( P_1, P_2, \ldots, P_n \) in \( \mathbb{Q}^n \), the stable intersection of their hypersurfaces \( \mathcal{T}(P_0) \cdot \mathcal{T}(P_1) \cdot \cdots \cdot \mathcal{T}(P_n) \) is either empty or consists only of the origin.

Lemma 4.1. Let \( P \) and \( Q \) be polytopes such that \( P \cup Q \) is also a polytope. Then \( (P \cup Q) + (P \cap Q) = P + Q \), where + denotes the Minkowski sum.

Proof. For a convex body \( K \subset \mathbb{Q}^n \), let \( K(x) \) denote its support function, i.e. for any \( x \in \mathbb{Q}^n \), \( K(x) = \max\{x \cdot v : v \in K\} \). Our assumptions imply that \( (P \cap Q)(x) = \min(P(x), Q(x)) \). Therefore \( ((P \cup Q) + (P \cap Q))(x) = (P \cup Q)(x) + (P \cap Q)(x) = \max(P(x), Q(x)) + \min(P(x), Q(x)) = P(x) + Q(x) = (P + Q)(x) \). Since closed convex bodies are uniquely determined by their support functions, we get \( (P \cup Q) + (P \cap Q) = P + Q \). \( \square \)

For a polytope \( P \) and a non-negative integer \( r \), let \( \mathcal{T}^r(P) \) denote the stable intersection of \( \mathcal{T}(P) \) with itself \( r \) times. In particular, \( \mathcal{T}^0(P) = \mathbb{R}^n \) and \( \mathcal{T}^1(P) = \mathcal{T}(P) \).

Proposition 4.2. Let \( P \) and \( Q \) be polytopes in \( \mathbb{Q}^n \) such that \( P \cup Q \) is also a polytope. Then

\[
\begin{align*}
(1) \quad & \quad \mathcal{T}(P \cup Q) \cdot \mathcal{T}(P \cap Q) = \mathcal{T}(P) \cdot \mathcal{T}(Q) \\
(2) \quad & \quad \mathcal{T}^k(P \cup Q) + \mathcal{T}^k(P \cap Q) = \mathcal{T}^k(P) + \mathcal{T}^k(Q) \quad \text{for any } k \geq 0
\end{align*}
\]

In the language of polytope algebra [McM89], the second part says that \( \mathcal{T}^k \) is a valuation for every \( k \geq 0 \).

Proof. Let \( X = P \cup Q \) and \( Y = P \cap Q \). Since \( X + Y = P + Q \) by Lemma 4.1, both \( \mathcal{T}(X) \cdot \mathcal{T}(Y) \) and \( \mathcal{T}(P) \cdot \mathcal{T}(Q) \) are codimension-2 subfans of the normal fan of \( X + Y = P + Q \). For a codimension-2 cone \( \sigma \) in the normal fan, \( \dim(P^\sigma + Q^\sigma) = 2 \), and \( \sigma \) is in \( \mathcal{T}(P) \cdot \mathcal{T}(Q) \) if and only if \( \dim(P^\sigma) \geq 1 \) and \( \dim(Q^\sigma) \geq 1 \); similarly for \( \mathcal{T}(X) \cdot \mathcal{T}(Y) \). If \( P(\omega) \neq Q(\omega) \) for \( \omega \) in the relative interior of \( \sigma \), then \( \{P^\sigma, Q^\sigma\} = \{X^\sigma, Y^\sigma\} \). If \( P(\omega) = Q(\omega) \), then \( X^\sigma = P^\sigma \cup Q^\sigma \) and \( Y^\sigma = P^\sigma \cap Q^\sigma \). In any case, because \( \dim(X^\sigma) \leq \max(\dim(P^\sigma), \dim(Q^\sigma)) \), it follows that \( \sigma \in \mathcal{T}(P) \cdot \mathcal{T}(Q) \) if and only if \( \sigma \in \mathcal{T}(X) \cdot \mathcal{T}(Y) \).

To compare multiplicities, by taking the link at \( \sigma \) and quotienting out by the lineality space, by Lemma 2.8 we reduce the problem to the case when \( P, Q, X, Y \) lie in a two-dimensional plane. By equation 4, the multiplicity at 0 of two tropical hypersurfaces \( \mathcal{T}(P) \) and \( \mathcal{T}(Q) \) in the plane can be computed by translating one of them generically and adding up the intersection multiplicities.
By examining the dual (mixed) subdivision, we have
\[
\text{mult}_{\mathcal{T}(P)\cdot\mathcal{T}(Q)}(\sigma) = \text{vol}(P^\sigma + Q^\sigma) - \text{vol}(P^\sigma) - \text{vol}(Q^\sigma), \quad \text{and}
\text{mult}_{\mathcal{T}(X)\cdot\mathcal{T}(Y)}(\sigma) = \text{vol}(X^\sigma + Y^\sigma) - \text{vol}(X^\sigma) - \text{vol}(Y^\sigma).
\]
See [Stu02, Figures 3.1 and 9.7] for pictures of tropical plane curves and the dual subdivisions. To see that the two quantities on the right are equal we apply Lemma 4.1 and the observations of the previous paragraph to get \( \text{vol}(P^\sigma + Q^\sigma) = \text{vol}(X^\sigma + Y^\sigma) \). Furthermore, \( \text{vol}(P^\sigma) + \text{vol}(Q^\sigma) = \text{vol}(X^\sigma) + \text{vol}(Y^\sigma) \) either because \( \{P^\sigma, Q^\sigma\} = \{X^\sigma, Y^\sigma\} \) or because \( \text{vol}(P^\sigma) + \text{vol}(Q^\sigma) = \text{vol}(P^\sigma \cup Q^\sigma) + \text{vol}(P^\sigma \cap Q^\sigma) \). We conclude that multiplicities are equal.

To prove the statement (2), we proceed by induction on \( k \). Since \( \mathcal{T}^0(P) = \mathbb{R}^n \) for all \( P \), the assertion is true for \( k = 0 \). For \( k = 1 \), it follows from Lemma 4.1 and the fact that \( \mathcal{T}(P+Q) = \mathcal{T}(P)\oplus\mathcal{T}(Q) \) for any polytopes \( P \) and \( Q \). Suppose \( k \geq 2 \). Let \( p = \mathcal{T}(P), \ q = \mathcal{T}(Q), \ x = \mathcal{T}(P \cup Q), \) and \( y = \mathcal{T}(P \cap Q) \). We have \( x^{k-2} \oplus y^{k-2} = p^{k-2} \oplus q^{k-2} \) by the inductive hypothesis, and \( xy = pq \) from part (1). Multiplying them gives \( x^{k-1}y \oplus xy^{k-1} = p^{k-1}q \oplus pq^{k-1} \). Combining this with \( (x \oplus y)(x^{k-1} \oplus y^{k-1}) = (p \oplus q)(p^{k-1} \oplus q^{k-1}) \), which follows from the inductive hypothesis for 1 and \( k - 1 \), gives the desired identity \( x^k \oplus y^k = p^k \oplus q^k \).

\[ \blacksquare \]

**Theorem 4.3.** Let \( P \) be a rational polytope in \( \mathbb{Q}^n \) and \( \mathcal{T}(P) \) be its tropical hypersurface. Then
\[
\text{vol}(P) = \text{mult}_{\mathcal{T}(\dim P)}(0)
\]
where \( \text{vol} \) denotes the \( \dim(P) \)-dimensional volume normalized to the integer lattice parallel to the affine span of \( P \).

**Proof.** In [Tve74], Tverberg showed that every polytope can be decomposed into simplices by a finite number of hyperplane cuts. Combining this with Proposition 4.2(2) applied to the case when \( P \cap Q \) is lower dimensional, we reduce the problem to the case when \( P \) is a simplex.

Now we will show the result for the case where \( P \) is a simplex, by induction on the dimension of \( P \). When \( P \) is one-dimensional, it is a line segment, and the assertion is true, as the multiplicity equals the lattice length of the segment. Suppose \( P \) is a \( d \)-dimensional simplex. By quotienting out by the lineality space of \( \mathcal{T}(P) \) if necessary, we may assume that \( P \) is full-dimensional in its ambient space. By the inductive hypothesis, \( \mathcal{T}^{d-1}(P) \) is a one dimensional fan whose rays are facet normals of \( P \) with multiplicities equal to the respective \( d - 1 \) dimensional volumes of the facets.

The multiplicity of the origin in \( \mathcal{T}(P) \cdot \mathcal{T}^{d-1}(P) \) is, by definition, the multiplicity of the Minkowski sum \( \mathcal{T}(P) \cdot \mathcal{T}^{d-1}(P) \) which is equal to \( \mathbb{R}^n \) as a set. Each connected component \( C \) of the complement of \( \mathcal{T}(P) \) is a simplicial full-dimensional cone, containing exactly one ray \( R \) of \( -\mathcal{T}^{d-1}(P) \) in its interior because \( \mathcal{T}^{d-1}(P) \) has exactly \( d + 1 \) rays, \( d \) of which are rays of \( C \) and the remaining ray is the negative of a positive linear combination of the other \( d \) rays. Taking
the Minkowski sum of $R$ with each facet of $C$, we get a triangulation of $C$. Doing so for each complement component, we get a triangulation of $\mathbb{R}^n$ (which is the normal fan of $P - P$). These cones are precisely the full dimension cones of the form $\sigma + R$ where $\sigma$ is a facet of $T(P)$ and $R$ is a ray of the tropical curve $-T^{d-1}(P)$, and they have disjoint interiors. Hence, to compute the multiplicity of $T(P) - T^{d-1}(P)$, we need only to consider one such cone.

Suppose $0, v_1, \ldots, v_n$ are vertices of the simplex $P$. Let $r$ be the primitive vector perpendicular to the facet containing $0$ and $v_1, \ldots, v_n$. Let $\sigma$ be the maximal cone of $T(P)$ normal to the edge $\{0, v_n\}$ with multiplicity equal to the lattice length $l$ of the edge and $R$ be the cone spanned by $r$ with multiplicity $a$. Let $u_1, \ldots, u_{n-1}$ be a lattice basis of $v_n^\perp \cap \mathbb{Z}^n$. According to Definition 2.4, the multiplicity of the stable intersection of $T(P)$ and $T^{d-1}(P)$ is $l \cdot a \cdot |\det[r|u_1| \cdots |u_{n-1}]|$, which is equal to $a |r^T v_n|$. This is equal to the volume $|\det[v_1|v_2| \cdots |v_n]|$ of $P$.

**Corollary 4.4.** The function $\text{vol}_n(\lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_n P_n)$ is a degree $n$ homogeneous polynomial in $\lambda_1, \lambda_2, \ldots, \lambda_n$, and the coefficient of $\Pi_{i=1}^n \lambda_i^{n-i}$ is

$$\frac{n!}{a_1!a_2! \cdots a_n!} \text{mult}_{T^n(P_i)}(0).$$

**Proof.** We expand $\text{vol}_n(\lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_n P_n)$ using Theorem 4.3. The result now follows from the distributivity of stable intersection over cycle sums (unions) and the additivity of multiplicities in cycle sums.

**Corollary 4.5.** For rational polytopes $P_1, P_2, \ldots, P_n$ in $\mathbb{Q}^n$, we have

$$\text{mult}_{T(P_1) \cap T(P_2) \cdots T(P_n)}(0) = \text{MV}(P_1, P_2, \ldots, P_n)$$

where MV denotes the mixed volume.

**Proof.** The mixed volume $\text{MV}(P_1, \ldots, P_n)$ is the coefficient of $\lambda_1 \lambda_2 \cdots \lambda_n$ in the polynomial $\frac{1}{n!} \text{vol}_n(\lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_n P_n)$, which is equal to

$$\text{mult}_{T(P_1) \cap T(P_2) \cdots T(P_n)}(0)$$

by Corollary 4.4.

Alternatively, we can see that the function

$$(P_1, P_2, \ldots, P_n) \mapsto \text{mult}_{T(P_1) \cap T(P_2) \cdots T(P_n)}(0)$$

satisfies the axioms of the mixed volume, i.e. it is symmetric, multilinear, and

$$\text{mult}_{T^n(P)}(0) = \text{vol}_n(P).$$

We get a proof of Bernstein’s Theorem.

**Theorem 4.6** (Bernstein). Let $f_1, f_2, \ldots, f_n$ be generic Laurent polynomials in $\mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$, and let $I$ be the ideal generated by them. If $I$ is zero-dimensional, then it has length equal to the mixed volume of the Newton polytopes of $f_1, f_2, \ldots, f_n$. 


Proof. If the coefficients are generic, then by Theorem 3.5 the tropical variety of $I$ is the stable intersection of the tropical hypersurfaces of $f_1, f_2, \ldots, f_n$, which is either empty or consists only of the origin with multiplicity equal to the mixed volume. The length of the ideal is the multiplicity of the origin, by the definition of multiplicity. □

5. Relation to Polytope Algebra

For $r = 0, 1, \ldots, n$, let $T^r$ be the vector space over $\mathbb{Q}$ of rational tropical cycles of codimension $r$ in $\mathbb{R}^n$. Scalar multiplication acts on the multiplicities, and addition is taking union. Then $T = T^0 \oplus T^1 \oplus \ldots T^n$ is a graded algebra with stable intersection as multiplication.

Let $\Pi$ be the polytope algebra of $\mathbb{Q}^n$ [McM89] defined as follows. For a polytope $P \subset \mathbb{Q}^n$, let $[P]$ denote the equivalence class of $P$ under the equivalence relation $P \sim P + v$ for $v \in \mathbb{Q}^n$. Then $\Pi$ consists of formal $\mathbb{Q}$-linear combinations of $\{ [P] : P$ is a polytope in $\mathbb{Q}^n \}$, modulo relations

$$[P \cup Q] + [P \cap Q] = [P] + [Q]$$

whenever $P \cup Q$ is a polytope. The multiplication is Minkowski sum:

$$[P] \cdot [Q] = [P + Q].$$

The additive identity is $0 = [\emptyset]$, the class of the empty polytope, while the multiplicative identity is $1 = [\emptyset]$, the class of a point. In $\Pi$, $([P] - 1)^{n+1} = 0$ for every $n$ dimensional polytope $P$. Hence the logarithm $\log([P]) = \sum_{k \geq 1} (-1)^{k-1}([P] - 1)^k/k$ and the exponentiation $\exp(z) = \sum_{k \geq 0} z^k/k!$ are well-defined and are inverses of each other [McM89].

**Theorem 5.1.** There is an isomorphism of graded algebras

$$\phi : \Pi \rightarrow T$$

given by $\phi([P]) = 1 \oplus T(P) \oplus \frac{1}{2!} T^2(P) \oplus \cdots \oplus \frac{1}{n!} T^n(P)$ for polytopes $P$ and linearly extending to $\Pi$.

Under this map, $\log[P] \mapsto T(P)$, so tropicalization is the logarithm.

Proof. Any fan, after some subdivision, is a subfan of the normal fan of a polytope. This can be achieved, for example, by taking the arrangement of hyperplanes spanned by any collection of cones in the original fan. Any hyperplane arrangement gives rise to the normal fan of a zonotope. We can also make them simplicial fans (dual to simple polytopes) by further subdivisions. When restricted to this polytopal fan, the algebra of tropical cycles is isomorphic to the algebra of Minkowski weights defined by Fulton and Sturmfels, which was shown to be isomorphic to the polytope algebra [FS97]. □

**Corollary 5.2.** Every tropical cycle is a linear combination of pure powers of tropical hypersurfaces.
Proof. The \( k \)-th graded piece of \( \Pi \) is spanned by \( \{ \log([P])^k : P \text{ is a polytope} \} \) as shown in \cite{McM89}. For a polytope \( P \), \( \phi(\log([P])^k) = T^k(P) \), and the assertion follows because \( \phi \) is an isomorphism. \qed

This corollary can be made constructive. Given a tropical cycle, first make it a subfan of the normal fan of a simple polytope, for example, by extending it to a hyperplane arrangement and perturbing the facets of the dual zonotope. We can then find a linear basis \( F_1, \ldots, F_m \) of \( T^1 \) restricted on this fan, by linear programming, so that each \( F_i \) has non-negative weights and hence are tropical hypersurfaces of polytopes. Then the for every \( 1 \leq k \leq n \), the \( k \)-fold products \( F_{i_1} \cdot F_{i_2} \cdot \ldots \cdot F_{i_k} \) linearly span \( T^k \). We can then decompose the input fan as a linear combination of these stable intersections of tropical hypersurfaces.

6. Connectivity

A pure dimensional polyhedral complex is connected in codimension one if every pair of facets is connected by a path through ridges and facets of the complex. Let \( k \) be an algebraically closed field. Tropical varieties of prime ideals in \( k[x_1, \ldots, x_n] \) are connected in codimension one \cite{BJS07, CP13}.

Let \( T_1 \) and \( T_2 \) be tropical varieties connected in codimension one. Then the stable intersection \( T_1 \cdot T_2 \), or even transverse intersection, need not be connected in codimension one, as we will see in Example \ref{ex:transverse} below. However, for constant coefficient tropical varieties (i.e. tropical varieties with respect to the trivial valuation) of prime ideals, stable intersection preserves connectivity in codimension one. A version of the following result appeared in our earlier paper \cite{JY13}. This answers the last open question in \cite{CP13} affirmatively for constant coefficient tropical varieties of irreducible varieties.

**Theorem 6.1.** Let \( I_1, I_2, \ldots, I_k \subset k[x_1, \ldots, x_n] \) be prime ideals, where \( 2 \leq k \leq n \), and let \( T(I_1), T(I_2), \ldots, T(I_k) \) be their constant coefficient tropical varieties respectively. Then the stable intersection \( T(I_1) \cdot T(I_2) \cdot \ldots \cdot T(I_k) \) is connected in codimension one.

**Proof.** First consider the case when \( k = 2 \). Using the diagonal trick, we can reduce to the case when \( I_1 \) is a prime ideal and \( I_2 = \langle x_1 - \alpha \rangle \) for a generic constant \( \alpha \).

In general, for a prime ideal \( I \) and a generic \( \alpha \), specializing a variable \( x_1 \) to \( \alpha \) may not preserve primality, i.e. the ideal \( I_1 := I + \langle x_1 - \alpha \rangle \subseteq k(\alpha)[x_1, \ldots, x_n] \) need not be prime. However, all its irreducible components have the same tropical variety. To see this, note that \( T(I_1) = \{0\} \times T(I_2) \) where \( I_2 := I \subseteq k(x_1)[x_2, \ldots, x_n] \). The ideal \( I_3 := I \subseteq k(x_1)[x_2, \ldots, x_n] \) is prime because \( I \) remains prime under extension from \( k[x_1] \) to \( k(x_1) \), as primality is preserved under localization. Deciding whether a point is in a tropical variety can be done with reduced Gröbner bases which are independent of the field extension, so we have \( \{0\} \times T(I_3) = \{0\} \times T(I_2) = T(I_1) \). Furthermore, since \( I_1 \) is
prime, by [CP13, Proposition 4], all irreducible components of $I_2$ have the same tropicalization. Since the tropical varieties of the irreducible components of $I_3$ are the same as those of the irreducible components of $I_2$, the conclusion follows.

We have shown that $\text{supp}(\mathcal{T}(I_1) \cdot \mathcal{T}(I_2))$ is the support of the tropical variety of a prime ideal. The cases for $k > 2$ follow by induction. □

In general, for tropical cycles that do not come from prime ideals, the stable intersection need not be connected in codimension one, as the following example shows.

**Example 6.2.** Let $T_1 = \mathcal{T}^2(1 + x_1 x_2 + x_3 + x_4 + x_5)$ and $T_2 = \mathcal{T}^2(1 + x_1 x_4 + x_2 x_5 + x_3^2 x_4 + x_4^2 x_5 + x_5^2)$ be 3-dimensional fans in $\mathbb{R}^5$. They are both realizable as tropical varieties of ideals by Theorem 3.5 and connected in codimension 1 by Theorem 6.1. Both $T_1$ and $T_2$ contain the two dimensional cone spanned by $-e_1$ and $-e_2$, so the union $T_1 \cup T_2$ is connected in codimension 1. However for the hyperplane $H = \mathcal{T}(x_1 x_2 x_3 x_4 x_5 + 1)$ the stable intersection $(T_1 \cup T_2) \cdot H$ is not connected in codimension 1. To see this, we used gfan [Jen] to compute $T_1 \cdot H$ and $T_2 \cdot H$ using the command `gfan_tropicalintersection --stable` and verified that $(T_1 \cdot H) \cap (T_2 \cdot H) = \{0\}$ using `gfan_fancommonrefinement`. Thus the two dimensional fan $(T_1 \cdot H) \cup (T_2 \cdot H)$ is not connected in codimension 1.

### 7. Open Questions

Minkowski showed that the facet normals and facet volumes determine a polytope up to translation. Can we tropically recover the polytope from this data? That is, for an unknown $d$-dimensional polytope $P$, how can we reconstruct the tropical hypersurface $\mathcal{T}(P)$ from $\mathcal{T}^{d-1}(P)$?

Does analogous statements to Minkowski’s theorem hold for in-between dimensions: is a $d$-dimensional polytope $P$ determined by $\mathcal{T}^k(P)$ for $1 < k < d-1$?

Is every tropical cycle with positive multiplicities realizable as a tropical variety of an ideal after scaling the coefficients sufficiently? Another question is whether the set \{ $\mathcal{T}^k(P) : P \text{ is a polytope}$ \} positively span the positive orthant in $T^k$? The second statement is stronger and the answer is probably negative. In that case, find a counterexample.

We conjecture that the sum of the multiplicities of the stable intersection depend only on the “recession fans”. The collection of recession cones of a polyhedral complex may or may not form a fan. However, their union is a tropical cycle, balanced with well-defined multiplicities on a dense open subset.

**Conjecture 7.1.** Let $T_1$ and $T_2$ be tropical cycles (not necessarily fans) such that $T_1 \cdot T_2$ modulo lineality consists of finitely many points. Then the sum of multiplicities of the points is equal to the multiplicity of the lineality space of the stable intersection of the “recession fans” of $T_1$ and $T_2$. 
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