On the magnetic Hooke-Newton transmutation of electrons

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Abstract
The author investigates the Hooke-Newton transmutation of the electron in the uniform magnetic field. Two results are reported. First, the modulable quantum spectrum and spinor of the magnetic field are given. Spin exhibits its effect, even in the first order approximation of the power series expansion of the spectrum. Since the transmuted Coulomb interaction, attractive or repulsive, is determined by both the signatures of the charge and the total angular momentum, the pure spin states of different orientations experience the reverse Coulomb forces. This feature may be useful in grouping the pure spin states of the same orientation. Second, it is shown that the effect of the effective vector potential due to the geometry of conformal mapping can be generated by the vector potential of two fractional magnetic fluxes with the reverse directions which couple to the different components of the spinor. The equivalence shows the possibility of demonstrating the physics of the 2D electron in conformal space with an isotropic momentum modulation while interacting with the field of the magnetic fluxes.

1. Introduction

The Hooke-Newton (H-N) transmutation states a kind of duality relation between Hooke’s linear force law and the inverse square force law. Its discovery can be dated back to 1779 when it was presented in Newton’s book [1], in which he revealed the duality by demonstrating the transmutation between these two kinds of force laws (Cor. III, Prop. VII, [1]). It may be the first duality system discovered in history, and in the modern viewpoint it is generated by the quadratic conformal mapping (see, e.g., [2, 3], and section 6 in [4]). Recently, it was pointed out that transmutation can also appear in the system of a charge moving in the uniform magnetic field, and it can be physically realized by the reinterpretation of the conformal factor as an effective scalar potential [5, 6]. One of the interesting features is that the magnetic transmutation system itself possesses a novel kind of quantum spectrum with a threshold controlled by the strength of the magnetic field which may be used to select and trap extremely low energy charged particles in the uniform magnetic field. In the foregoing inspections, the system was investigated from the prospects of quantum and classical mechanics. The preparation of the paper was prompted by the understanding of the transmutation of the electron in the field. This purpose is completed through acquisition of the quantum spectrum and spinor, and a proposition of realizing the system. Our discussions are arranged as follows: in section 2, the electron moving in the uniform magnetic field is studied in 2D conformal space, which is generated by a general conformal mapping from the polar coordinates. The quantization rule of the angular momentum is determined by the single valuedness condition of the wave function and the physical requirement of the spin flip. The rule is then used to obtain the equations satisfied by the radial functions of the spinor in the conformal space. We devote section 3 to discussing the quadratic conformal mapping which leads to the electron’s H-N transmutation in the magnetic field. The quantum spectrum and the corresponding spinor of the electron are given. In section 4, a reinterpretation of the transformation structure is proposed for the physical construction of the duality system. It is shown that the effective vector potential due to conformal mapping can be substituted by the vector field of two fractional magnetic fluxes with reverse orientations. The transmutation can thus be achieved by offering the electron an environment of the uniform magnetic field and the magnetic flux of the corresponding spin state while
2. The dirac equation of electron moving in the uniform magnetic field in 2D conformal space

The equations for the radial wave functions of the electron moving in the uniform magnetic field in 2D conformal space are given in this section. The evolution of the electron in a general space-time with the invariant squared distance \( ds^2 = \sum_{j,k=1}^4 g_{jk} dx^j dx^k \) is described by the Dirac spinor \( \Psi \) which, pursuant to the invariant principle of Einstein, satisfies the covariant Dirac equation \[ (\mathcal{A}_i)_{\alpha\beta} = \gamma^\mu \partial_\mu - \gamma^5 M \Gamma_{\alpha\beta} \] (1)

where \( \gamma^5 \) satisfies the anticommutation relation \( \{ \gamma^i, \gamma^k \} = 2\delta^{ik} \), and \( \Gamma_{\alpha\beta} \) is the metric-associated spin connection. Throughout this paper we shall adopt the natural units \( h = c = 1 \), if not explicitly stated otherwise. For a space allowed to be described by an orthogonal frame \( ds^2 = dx^2 + dy^2 \).

Here the imaginary time interval \( dx^4 = d\tau = i dt \) was introduced for the unification of the signatures of the space-time metric. The Dirac equation can be simplified to become

\[ \left( \sum_{k=1}^4 \gamma^k (\partial_k - \Gamma_k) + M \right) \Psi = 0, \] (2)

where \( \gamma^k \) satisfies the anticommutation relation \( \{ \gamma^i, \gamma^k \} = 2\delta^{ik} \) of Minkowski space-time, and the components of the effective vector potential \( A_{k,\text{eff}} \) are given by (see appendix A)

\[ A_{k,\text{eff}} = -\frac{1}{2f_k} \partial_k \sum_{i=k} f_i. \] (3)

It is obvious that the vector potential is a geometric stretch factor, and is only associated with the geometric structure. In the coming discussions, we are interested in the dynamics of the electron in the uniform field while moving in a 2D surface generated from the mapping of the polar system \( (\rho, \varphi) \),

\[ \rho(\rho) = \frac{\rho^\nu}{\lambda^\nu - 1}, \quad \varphi(\varphi) = \nu \varphi, \] (4)

where \( \nu \) is a real number and the constant \( \lambda \) has the units of length used to maintain the units of \( \rho \). The mapping is conformal since the invariant distance \( ds^2 = d\rho^2 + \rho^2 d\varphi^2 = \left( \frac{d\rho}{d\rho} \right)^2 \rho^2 d\varphi^2 = \left( \frac{\rho^\nu - 1}{\lambda^\nu - 1} \right)^2 \rho^2 d\varphi^2 = \left( \frac{\nu \rho^\nu - 1}{\lambda^\nu - 1} \right)^2 (dx^2 + dy^2) \) (5)

is different from that of the Euclidean space by the conformal factor \( (\nu \rho^\nu - 1) / (\lambda^\nu - 1) \). Because the metric and the magnetic field under consideration are time independent, the Dirac spinor has the decomposition \( \Psi(x, \tau) = \Psi(x) \exp \{-E\tau\} \) and the spatial part \( \Psi(x) \) of the spinor for the electron satisfies the Dirac equation

\[ \left( \sum_{k=1}^4 \gamma^k (\partial_k - A_{k,\text{eff}} - iaA_{k,\text{em}}) + (M - \sigma^3 E) \right) \Psi(x) = 0. \] (6)

Here \( \sigma^k, k = 1, 2, 3 \), are the Pauli matrices, \( A_{k,\text{em}} \) are the components of the vector potential for the magnetic field, and the semi-metric \( f_k = \rho^k = \nu \rho^{\nu - 1} / \lambda^{\nu - 1} \). By the substitution of \( \Psi(x) = (\Psi_0, \Psi_1, \Psi_2) \) and the matrix representation of \( \sigma^k \) in equation (6) may decompose into the coupling system

\[ \frac{1}{f_1} \partial_1 \Psi_2 - A_{1,\text{eff}} \frac{f_1}{f_2} \Psi_2 - i f_1 \frac{A_{1,\text{em}}}{f_2} \Psi_2 = i \left[ \frac{1}{f_2} \partial_2 \Psi_1 - A_{2,\text{eff}} \frac{f_1}{f_2} \Psi_1 - \frac{i}{f_1} A_{1,\text{em}} \Psi_1 \right] + (M - E) \Psi_1 = 0, \] (7)

and

\[ \frac{1}{f_2} \partial_2 \Psi_1 - A_{2,\text{eff}} \frac{f_1}{f_2} \Psi_1 - i f_2 \frac{A_{1,\text{em}}}{f_1} \Psi_1 = -i \left[ \frac{1}{f_1} \partial_1 \Psi_2 - A_{1,\text{eff}} \frac{f_1}{f_2} \Psi_2 - \frac{i}{f_2} A_{2,\text{em}} \Psi_2 \right] + (M + E) \Psi_2 = 0. \] (8)

Let us discuss the system in the polar coordinates \( \rho = \sqrt{x^2 + y^2} \), and \( \varphi = \tan^{-1}(y/x) \). It follows that \( \partial_\rho = [\cos \varphi \partial_x - (\sin \varphi / \rho) \partial_y] \) and \( \partial_\varphi = [\sin \varphi \partial_x + (\cos \varphi / \rho) \partial_y] \), and one has the expressions of the effective vector potential
\[ \mathcal{K}^{\text{ef}} = - (\nu - 1) \frac{1}{2 \rho} (\cos \varphi \hat{e}_x + \sin \varphi \hat{e}_y) = - (\nu - 1) \frac{1}{2 \rho} \hat{e}_\rho, \]

(10)

and the vector potential of the uniform magnetic field

\[ \mathcal{A}^{\text{m}} = \frac{B \rho}{2} (- \sin \varphi \hat{e}_x + \cos \varphi \hat{e}_y) = \frac{B \rho}{2} \hat{e}_\varphi. \]

(11)

Both have axial symmetry. The spinor is then supposed to be

\[ \Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} R_1(\rho) e^{i \alpha_1 \varphi} \\ R_2(\rho) e^{i \alpha_2 \varphi} \end{pmatrix}. \]

(12)

The equations in (8) and (9) then turn into

\[ \left( \begin{array}{c} \cos \varphi \\ - \frac{i \sin \varphi}{f_1} \end{array} \right) \rho \frac{\partial R_2}{\partial \rho} e^{i \alpha_2 \varphi} + \left( - \frac{i \sin \varphi}{f_2} + \cos \varphi \right) \frac{\partial \alpha_2}{\partial \rho} R_2 e^{i \alpha_2 \varphi} + \left( - \frac{\cos \varphi}{f_1} b_1(\rho) + \frac{i \sin \varphi}{f_2} b_2(\rho) \right) R_2 e^{i \alpha_2 \varphi} = 0, \]

(13)

and

\[ \left( \begin{array}{c} \cos \varphi \\ + \frac{i \sin \varphi}{f_1} \end{array} \right) \rho \frac{\partial R_1}{\partial \rho} e^{i \alpha_1 \varphi} + \left( - \frac{i \sin \varphi}{f_2} - \cos \varphi \right) \frac{\partial \alpha_1}{\partial \rho} R_1 e^{i \alpha_1 \varphi} + \left( - \frac{\cos \varphi}{f_1} b_1(\rho) - \frac{i \sin \varphi}{f_2} b_2(\rho) \right) R_1 e^{i \alpha_1 \varphi} = 0, \]

(14)

where the brief notations

\[ b_1(\rho) = - \left( \frac{\partial}{\partial \rho} \right), \quad b_2(\rho) = \frac{1}{f_2} \left( \frac{\partial}{\partial \rho} f_1 \right) \]

(15)

are introduced. For the conformal metric, one has \( b_1 = b_2 \). Nevertheless, the notations \( b_k, k = 1, 2, \) are left in equations (13) and (14) for convenient operation later. To simplify the above coupling system, we note that the single valuedness of the wave function requires \( \Psi(\varphi) = \Psi(\varphi + 2 \pi) \), i.e.,

\[ \left( \begin{array}{c} R_1(\rho) e^{i \alpha_1 \varphi} \\ R_2(\rho) e^{i \alpha_2 \varphi} \end{array} \right) = \left( \begin{array}{c} R_1(\rho) e^{i \alpha_1 \varphi} \\ R_2(\rho) e^{i \alpha_2 \varphi} \end{array} \right), \]

(16)

which implies \( e^{i \alpha_1 \varphi + 2 \pi} = 1 \), and \( e^{i \alpha_2 \varphi + 2 \pi} = 1 \). So we can only have

\[ a_1 = \frac{n_1}{\nu}, \quad a_2 = \frac{n_2}{\nu}, \]

with \( n_1, n_2 = 0, \pm 1, \pm 2, \cdots \).

(17)

Moreover, since the first and second components of the spinor \( \Psi \) are different from a spin flip, the relation between \( n_1 \) and \( n_2 \) can only be

\[ n_2 = n_1 + 1. \]

(18)

Accordingly, the relation between \( a_1 \) and \( a_2 \) is given by

\[ a_2 = a_1 \pm \frac{1}{\nu}. \]

(19)

Without loss of generality, we shall take \( a_2 = a_1 + 1/\nu \) in the following discussions. Now multiply (13) by \( -i (a_1 \varphi + 2 \varphi) \), and then integrate the variable \( \varphi \) over the range \([0, 2\pi]\). One gets the equality

\[ \left( \frac{1}{f_1} + \frac{1}{f_2} \right) \rho \frac{\partial R_2}{\partial \rho} + \left( - \frac{1}{f_1} + \frac{1}{f_2} \right) \frac{\partial \alpha_2}{\partial \rho} R_2 + \left( \frac{1}{4 f_1} b_1(\rho) + \frac{1}{4 f_2} b_1(\rho) \right) R_2 = 0. \]

(20)

This is just a trivial identity since \( f_1 = f_2 \) and \( b_1 = b_2 \). Multiply (13) by \( -i a_1 \varphi \), and perform the integration of the variable \( \varphi \) over the range \([0, 2\pi]\). We obtain the first radial equation

\[ \frac{d R_2}{d \rho} + \left( \frac{\varphi}{f_1} \right) \frac{a_2}{\rho} R_2 - \frac{b_1(\rho)}{2 f_1} R_2 - \frac{q B \rho}{2 f_1} R_2 + (M - E) R_1 = 0, \]

(21)

where \( f_1 = f_2 = \bar{\rho} \) was used. A similar means of multiplying equation (14) by \( -i (a_2 \varphi - 2 \varphi) \) gives an alternative trivial identity.
The second equation can be evaluated by multiplying (14) by exp \{-ia_2\nu\phi\} yielding
\[
\frac{dR_1}{d\hat{\rho}} + \frac{1}{\hat{\rho}} \left( \frac{1}{2f_1} \right)_{\nu} \frac{\partial R_1}{\partial \nu} + \left( \frac{1}{2f_2} \right)_{\nu} \frac{\varphi' a_{\nu}}{\rho} R_1 + \left( -\frac{1}{4f_1} b_1(\rho) + \frac{1}{4f_2} b_2(\rho) \right) R_1 = 0.
\] (22)

It is easy to show that it has the following exact values for the factors
\[
\left( \frac{\varphi' \hat{\rho}}{f_2 \rho} \right) = 1, \text{ and } \frac{b_1(\rho)}{2f_1} = -\frac{(\nu - 1) 1}{2\nu} \hat{\rho}.
\] (24)

These make the conformal image of the radial wave functions of the electron satisfy the equations
\[
\frac{dR_2}{d\hat{\rho}} + \frac{[a_2 + (\nu - 1) / 2\nu]}{\hat{\rho}} R_2 - \left( \frac{qB\rho}{2f_1} \right) R_2 + (M - E)R_1 = 0,
\] (25)

and
\[
\frac{dR_2}{d\hat{\rho}} + \frac{[a_2 - (\nu - 1) / 2\nu]}{\hat{\rho}} R_2 + \left( \frac{qB\rho}{2f_1} \right) R_2 + (M + E)R_1 = 0.
\] (26)

With the substitution
\[
R_1 = F / \sqrt{g}, \text{ and } R_2 = G / \sqrt{g},
\] (27)

where \( \sqrt{g} = \sqrt{|g|} = g_1 = (\nu^2 \rho^{2\nu-2} / \lambda^{2\nu-2}) \). One finally gets
\[
\frac{dG}{d\hat{\rho}} + \frac{[a_2 - 3(\nu - 1) / 2\nu]}{\hat{\rho}} G - \left( \frac{qB\rho}{2f_1} \right) G + (M - E)F = 0,
\] (28)

and
\[
\frac{dF}{d\hat{\rho}} + \frac{[a_1 + 3(\nu - 1) / 2\nu]}{\hat{\rho}} F + \left( \frac{qB\rho}{2f_1} \right) F + (M + E)G = 0.
\] (29)

In the next section, we shall use equations (28) and (29) to discuss the case of the quadratic conformal mapping, i.e., \( \nu = 2 \), for the magnetic H-N transmutation of the electron.

### 3. The magnetic Hooke-Newton transmutation of the electron

For \( \nu = 2 \), the semi-metric \( f_1 = f_2 = 2\rho / \lambda \), and the factors in (28) and (29)
\[
a_2 - \frac{3(\nu - 1)}{2\nu} = \frac{n_1}{2} - \frac{1}{4}, \text{ and } a_1 + \frac{3(\nu - 1)}{2\nu} = \frac{n_1}{2} + \frac{3}{4}
\] (30)

for \( a_2 = a_1 + 1/2 \) and \( a_1 = n_1/2 \) with \( n_1 = 0, \pm 1, \pm 2, \cdots \). The radial equations become in this case
\[
\frac{dG}{d\hat{\rho}} + \frac{(n_1/2 - 1/4)}{\hat{\rho}} G - \left( \frac{qB\lambda}{4} \right) G + (M - E)F = 0,
\] (31)

and
\[
\frac{dF}{d\hat{\rho}} - \frac{(n_1/2 + 3/4)}{\hat{\rho}} F + \left( \frac{qB\lambda}{4} \right) F + (M + E)G = 0.
\] (32)

By taking the derivative with respect to \( \hat{\rho} \) to the equations, the radial functions \( G \) and \( F \) are shown to satisfy
\[
\frac{d^2G}{d\hat{\rho}^2} - \frac{1}{\hat{\rho}} \frac{dG}{d\hat{\rho}} + \left[ \frac{(n_1/2 - 1/4)(n_1/2 + 7/4)}{\hat{\rho}^2} + \left( \frac{qB\lambda}{4} \right) \frac{(n_1 + 1/2)}{\hat{\rho}} - \left( M^2 + \left( \frac{qB\lambda}{4} \right)^2 - E^2 \right) \right] G = 0,
\] (33)
and

\[
\frac{d^2 F}{dp^2} - \frac{1}{\rho} \frac{d F}{d \rho} + \left[ -\frac{(m/2 - 5/4)(n_1/2 + 3/4)}{\rho^2} + \left(\frac{qB\lambda}{4}\right)\left(\frac{n_1 + 1/2}{\rho} - M^2 + \left(\frac{qB\lambda}{4}\right)^2 - E^2\right) \right] F = 0. \tag{34}
\]

These exhibit that \(G\) and \(F\) in the conformal space \((\rho, \varphi)\) are like the radial wave functions of a charged particle with different angular momenta moving in the Coulomb field

\[
\left(\frac{qB\lambda}{4}\right)\left(\frac{n_1 + 1/2}{\rho}\right).
\tag{35}
\]

Therefore, the conformal mapping of \(\nu = 2\) indeed turns the interaction of the magnetic field with the electron into the Coulomb-like interaction. An interesting feature in this Coulomb field is that attractive interaction is not just dependent on the signature of the charge \(q\), but also on the signature of the electron’s angular momentum. Only when both signatures are the same, can the bound states be formed. This feature shows us that the signature of angular momentum is as important as that of the charge in the interaction of the system. This was pointed out in [5]. Additional information shown in the interaction equation (35) than to that shown in the former system without spin is that the pure spin states with the same orientation can be grouped by the types of interaction, i.e., attractive or repulsive force, when the orbital angular momentum quantum number is zero. This can be understood as follows: according to (17) and (19), we note that \(a_1 = n_1\) is an integer, and \(a_2 = n_1 + 1\) when \(\nu = 1\). This implies that \(n_1\) is actually the total angular momentum quantum number, and the second component of the spinor represents the spin up state since it is traditionally supposed that the quantum number of the angular momentum for spin up state leads the spin down state to a positive integer step. Thus, one can regard \(n_1 = 0\) \((n_1 = -1)\) corresponding to the purely spin up (spin down) state with no orbital angular momentum in the situation of the conformal index \(\nu > 0\). According to the inference, the Coulomb interaction in equation (35) for the electron, \(q < 0\), would exhibit attractive (repulsive) force for the pure spin down (up) state and forms a bound (scattering) state. The conclusion implies that the H-N system has the functionality of selecting the pure spin orientation states of Dirac particles. Both (33) and (34) should be able to determine the quantum spectrum of the electron. From (33), the condition to have a physical solution is given by (see appendix B)

\[
\tilde{n}(G) = -\left(\mu_1 \pm \frac{1}{2} + qB\lambda(n_1 + 1/2)/4 \right) = 0, 1, 2, \ldots.
\tag{36}
\]

where \(\mu_1 = \pm(n_1/2 + 3/4)\), and \(k^2 = M^2 + (qB\lambda/4)^2 - E^2\). We have used \(\tilde{n}(G)\) here to denote the natural number coming from the condition of the physical solution of \(G(\rho)\). The allowed energy levels of the transmutation system are then given by

\[
E = \pm M \left\{ 1 + \left(\frac{\omega \cdot \lambda}{4}\right)^2 \left[ 1 - \frac{\left(\frac{n_1 + 1/2}{\tilde{n}(G) + (n_1 + 1/2) + (1 \pm 1)^2} \right)^2}{\tilde{n}(G)} \right] \right\}^{1/2}.
\tag{37}
\]

The top (bottom) sign of ‘\(\pm\)’ in the middle bracket corresponding to quantum number \(n_1 \geq 0\) \((n_1 < 0)\) is for the positron (electron). For the second term being a small quantity, the energy formula has the expansion:

\[
E = \pm M \left\{ 1 + \frac{1}{2} \left(\frac{\omega \cdot \lambda}{4}\right)^2 \left[ 1 - \frac{\left(\frac{n_1 + 1/2}{\tilde{n}(G) + (n_1 + 1/2) + (1 \pm 1)^2} \right)^2}{\tilde{n}(G) + \left(\frac{n_1 + 1/2}{2}\right)^2 + (1 \pm 1)^2} \right] \right\}^{1/2} + \ldots
\tag{38}
\]

Compared with the non-relativistic energy spectrum, e.g., equation (37) in [6], the quantum number of the angular momentum was modified by the spin, and a new modification \(\pm 1/2\) appears in the final term of the denominator. It reveals a finer structure of the spectrum due to the dependence of the signature of the angular momentum. This is more accessible by letting \(n_1 = j - 1/2, j = \mp 1/2, \mp 3/2, \mp 5/2, \ldots\). The electron and positron spectra of the bound states can be expressed as

\[
E(\text{electron}) = M \left\{ 1 + \left(\frac{\omega \cdot \lambda}{4}\right)^2 \left[ 1 - \frac{(j/2)^2}{\tilde{n}(G) + \left[j/2\right]^2} \right] \right\}^{1/2}, \quad j < 0,
\tag{39}
\]
and

\[ E(\text{positron}) = -M \left\{ 1 + \left( \frac{\omega_c \lambda}{4} \right)^2 \left[ 1 - \frac{(j/2)^2}{\hat{\eta}(G) + j/2 + 1} \right] \right\}^{1/2}, \quad j > 0. \] (40)

An alternative remarkable point of (37) is that the distribution of quantization levels is between the interval \([M, M + M(\omega_c \lambda/4)^2/2]\) for the electron and \([-M, -M - M(\omega_c \lambda/4)^2/2]\) for the positron as shown in figure 1, where the physical units have been retrieved. The upper bound is subjected by the energy \(M(\omega_c/2)^2(\lambda/2)^2/2\) with the potential form of the simple harmonic oscillator. It is controlled by the strength of the magnetic field. To estimate the distribution range of the spectrum, one needs to determine the parameter \(\lambda\). It is noticed that the characteristic size of the usual Coulomb system with interaction \(V = e^2/r\) is the Bohr radius \(a_B = h^2/(Mq_e^2)\). Comparing this with the transmuted Coulomb interaction (35), a good choice of the parameter \(\lambda\) is the characteristic length of the transmutation system denoted by \(a_T\). It is given by

\[ a_T = \frac{8h^2}{\lambda \omega_c (n_1 + 1/2) h}, \] (41)

where the cyclotron frequency \(\omega_c = qB/M\) was introduced. The characteristic length \(\lambda\) can then be determined by the ratio

\[ \frac{\lambda}{a_T} = \frac{\rho^2/\lambda}{8h^2/(\lambda \omega_c (n_1 + 1/2) h)} = \frac{\rho^2/\sqrt{8h^2/(\lambda \omega_c (n_1 + 1/2) h)}}{\sqrt{8h^2/(\lambda \omega_c (n_1 + 1/2) h)}}, \] (42)

which implies

\[ \lambda = a_T = \frac{8h^2}{\lambda \omega_c (n_1 + 1/2) h}. \] (43)

With the quantity, one can estimate the energy range of the spectrum. In the situation of the strength of the magnetic field \(B = 1\) Tesa, \(\hbar \omega_c \approx 10^{-4}\) eV. Using (43), we have

\[ \frac{M}{2} \left( \frac{\omega_c \lambda}{4} \right)^2 \approx \frac{\hbar \omega_c}{4[n_1 + 1/2]} \approx \frac{10^{-4}}{4[n_1 + 1/2]} \text{ eV}. \] (44)

It shows that the quantum levels distribute only over a narrow range in the low energy region. Thus, the system is sensitive and suitable to be a selector for the trapping of the 2D spin-\(1/2\) charged fermions with extremely low energy. Another interesting feature from the spectrum is that the electron energy level approaches the upper bound \(M + M(\omega_c \lambda/4)^2/2\) when the radial quantum number \(\hat{n}(G) \to \infty\) and the angular momentum quantum number is given. On the other hand, given the radial quantum number, the energy levels approach the lower bound \(M\) when \(n_1 \to \infty\). The behavior is rather different from the electron spectrum in the normal Coulomb field.

An alternative way to obtain the electron spectrum is from equation (34). It can be shown that the spectrum is determined by the condition

![Figure 1. The electron spectrum of the magnetic Hooke-Newton transmutation system. The upper bound of the quantization levels is controllable by the strength of the magnetic field in the parameter \(M(\omega_c \lambda/4)^2/2\). The spectrum along the negative axis is for the positron.](image-url)
where $\mu_2 = \pm (n_1/2 - 1/4)$. The energy levels are then given by

$$E = \pm M \left\{ 1 + \left( \frac{\omega_n \lambda}{4} \right)^2 \left[ 1 - \frac{(n_1 + 1/2)^2}{[2\bar{n}(F) \pm (m + 1/2) + (1 \pm 1)^2]} \right] \right\}^{1/2},$$

where the top signs of ‘+’ and ‘–’ in the middle bracket correspond to $n_1 \geq 0$. The formulas in (37) and (46) should describe the same electron spectrum. It can be proved by the relation

$$\bar{n}(F) = \bar{n}(G) \pm 1$$

between $\bar{n}(F)$ and $\bar{n}(G)$. Substituting (47) into (46) results in (37). With the normalization condition of the spinor

$$\int \psi \psi^* dv = 1,$$

where $dv = \rho d\rho d\varphi$ is the volume element in the considered conformal space, one can determine the Dirac spinor of the transmutation system as follows:

3.1. $n_1 < 0$

The radial functions of the spinor corresponding to the electron spectrum are given by

$$G(\bar{\rho}) = C_1 e^{-k_\rho \bar{\rho}} \rho^{-(n_1/2 + 1/4)} L_{\bar{n}(G) - 1}^{(n_1 - 1/2)}(2k_\rho \bar{\rho}),$$

and

$$F(\bar{\rho}) = \frac{C_1}{E - M} \frac{2k_\rho(k_\rho + qB\lambda/4)}{\bar{n}(G)} e^{-k_\rho \bar{\rho}} \rho^{-(n_1/2 + 3/4)} L_{\bar{n}(G) - 1}^{(n_1 + 1/2)}(2k_\rho \bar{\rho}),$$

where

$$C_1 = \sqrt{\frac{(E - M)}{2E}} \frac{(2k_\rho)^{n_1 - 1/2} \bar{n}(G) !}{4\pi [2\bar{n}(G) + m - 1/2] \Gamma(\bar{n}(G) - n_1 - 1/2)},$$

and $L_{\alpha\beta}^{(n)}(z)$ is the generalized Laguerre polynomials. The condition shows that the electron can only be in the radial state of $G(\bar{\rho})$ when $\bar{n}(G) = 0$.

3.2. $n_1 \geq 0$

The case is for the positron. The normalized spinor has the radial functions

$$F(\bar{\rho}) = C_2 e^{-k_\rho \bar{\rho}} \rho^{-(n_1/2 + 1/4)} L_{\bar{n}(F) - 1}^{(n_1 - 1/2)}(2k_\rho \bar{\rho}),$$

and

$$G(\bar{\rho}) = \left( -\frac{C_2}{E + M} \right) \frac{2k_\rho(k_\rho - qB\lambda/4)}{\bar{n}(F)} e^{-k_\rho \bar{\rho}} \rho^{(n_1/2 + 3/4)} L_{\bar{n}(F) - 1}^{(n_1 + 3/2)}(2k_\rho \bar{\rho}),$$

where

$$C_2 = \sqrt{\frac{(M + E)}{2E}} \frac{(2k_\rho)^{n_1 + 3/2} \bar{n}(F) !}{4\pi [2\bar{n}(F) + n_1 + 1/2] \Gamma(\bar{n}(F) + n_1 + 1/2)}.$$

Before closing the section, let us present the radial wave function of the Schrödinger equation for a 2D charged particle moving in the Coulomb field for comparison

$$R_{n_0, m}(\rho) \sim \rho^{|m|} e^{-\rho / \bar{n}(n_0)} L_{|n_0|}^{|n_0|} \left( \frac{2\rho}{\bar{n}(n_0)} \right),$$

where $m = 0, \pm 1, \pm 2, \cdots$ is the angular momentum quantum number, $n_0 = 0, 1, 2, \cdots$ is the radial quantum number, and the principle quantum number $n = n_0 + |m| + 1/2 = 1/2, 3/2, 5/2, \cdots$. The wave function shows the size of the atom depending mainly on the factor $e^{-\rho / \bar{n}(n_0)}$ which is adjusted by the quantum number $n$. There is no other freedom to be adjusted in it. Compared with it, the components $F$ and $G$ depend on the similar factor

$$\exp \{-k_\rho \bar{\rho}\} = \exp \{-k_\rho \bar{\rho}^2 / \lambda\} = \exp \left\{ -k_\rho \rho^2 / \left( \frac{8\hbar^2}{M\omega_\rho (n_1 + 1/2) \hbar} \right) \right\}.$$

It is not only altered by the angular momentum $(n_1 + 1/2) \hbar$ but also the strength of the magnetic field which is easy to control in the laboratory. In the very weak magnetic field, $|\omega_\rho (n_1 + 1/2) | \hbar \to 0$, the radial distribution...
functions $F$ and $G$ of the electron may extend to an extensive range. Conversely, when $B \to \infty$, the spinor can only have a very finite distribution. The electron is likely to be frozen up around the origin.

4. Physical realization of the conformally geometric effect

In the notable works of Leonhardt and Pendry et al in optics [8, 9], the solution in a transformed space was shown to be possibly realized by reinterpreting the transformation structure as the material parameter which makes novel devices such as the invisibility cloak possibly coined technically. A similar means to the construction of the invisibility cloak for the matter wave can interpret the transformation structure as the anisotropic momentum parameter and an effective scalar potential [10]. Recently, it was reported that using conformal mapping to design a quantum device for splitting a matter wave only needs to offer a scalar potential in the controlling region without resorting to the modulation of mass [11]. It largely simplifies the construction of the transformation device. Nevertheless, the skill cannot be applied to construct the electron’s wave in conformal space since its geometric structure can only be reinterpreted as the anisotropic momentum modulation of the fermion [12] and the effective vector potential, a geometric stretch factor. A connection of the vector potential with a physical quantity has to be established for the realization of the solution. To achieve this purpose, let us reinspect the Dirac equation by multiplying equation (7) by $f_1$ from the left. It makes the Dirac equation become

$$\nonumber \sum_{k=1}^3 \sigma^k (\partial_k - A_k^\text{eff} - i\mathbf{q} A_k^\text{em}) + f_1 (M - \sigma^3 E) \Psi(x) = 0 \quad (57)$$

for $f_1 = f_{2}$. Obviously, the second term can be reinterpreted as

$$\nonumber E \longrightarrow \tilde{E} = f_1 E, \quad \text{and} \quad M \longrightarrow \tilde{M} = f_1 M \quad (58)$$

such that the momentum of the electron changes from $k$ to $\tilde{k}$, i.e.,

$$\nonumber k^2 \longrightarrow \tilde{k}^2 = \tilde{E}^2 - \tilde{M}^2. \quad (59)$$

The geometric effect on the second bracket can then be reinterpreted as the isotropic modulation of the momentum, $k^2 \rightarrow \tilde{k}^2 = f^2_1 k^2$. For the vector potential $A^\text{eff}$, there is no convenient reinterpretation. Nevertheless, it is noticed that the radial equations (21) and (23) under consideration can also be obtained by performing the replacement

$$\nonumber A^\text{eff} \longrightarrow iA^\text{em} \quad (60)$$

with the matrix-valued vector potential

$$\nonumber A^\text{em}(\sigma^3) = -\frac{(\nu - 1)}{2} \sigma^3 \hat{\epsilon}_z \rho, \quad (61)$$

where $\sigma^3$ is the Pauli matrix as usual. This is just the vector field for two fractional magnetic fluxes with the reverse direction along the $z$-axis. The magnitude of the corresponding fluxes is

$$\nonumber \left\{ \begin{array}{ll} \Omega_1 / \Omega_0 & 0 \\ 0 & \Omega_2 / \Omega_0 \end{array} \right\} = \left\{ \begin{array}{ll} -(\nu - 1)/2 & 0 \\ 0 & (\nu - 1)/2 \end{array} \right\} \quad (62)$$

where $\Omega_0 = hc/q$ is the fundamental magnetic flux quantum. The orientations of the fluxes depend on the conformal parameter $\nu$. It will be reversed when $\nu < 1$, and we note that no matter what kind of orientation there is, the top left vector field always couples to the first component $\Psi_1$ of the spinor, while the bottom right one always couples to the second component $\Psi_2$ of the spinor. For the H-N transmutation, $\nu = 2$, the top left one couples to the spin up state since the quantum number $a_2 = a_1 + 1/2$ leads to $a_1$, and traditionally the lead is supposed to be in the spin up state. Generally, the effect of the effective potential due to 2D conformal geometry can always be realized by a physical vector field. The proof is as follows:

$$\nonumber [\sigma^1(\partial_1 - A_1^\text{eff}) + \sigma^2(\partial_2 - A_2^\text{eff}) + f_1 (M - \sigma^3 E)]\Psi(x) = [\sigma^1(\partial_1 - (\sigma^3)^2 A_1^\text{eff}) + \sigma^2(\partial_2 - (\sigma^3)^2 A_2^\text{eff}) + f_1 (M - \sigma^3 E)]\Psi(x)$$

$$\nonumber = [\sigma^1(\partial_1 - iA_1^\text{em}(\sigma^3)) + \sigma^2(\partial_2 - iA_2^\text{em}(\sigma^3)) + f_1 (M - \sigma^3 E)]\Psi(x), \quad (63)$$

with $A_1^\text{em}(\sigma^3) = \sigma^3 A_1^\text{eff}$, and $A_2^\text{em}(\sigma^3) = -\sigma^3 A_1^\text{eff}$. We thus learn that the effect of $A^\text{eff}$ can be generated by the two-valued vector potential

$$\nonumber A^\text{em}(\sigma^3) = \sigma^3(A_1^\text{eff} \hat{\epsilon}_x - A_1^\text{eff} \hat{\epsilon}_y). \quad (64)$$
An alternative choice is

\[ \mathbf{A}^m(\sigma^3) = \sigma^3(-A_1^\text{eff} \hat{\epsilon}_x + A_2^\text{eff} \hat{\epsilon}_y). \]

It is possible to generalize the situation to a Dirac fermion moving in a \((2+1)\)D conformal space-time surface, so that the time-dependent spinor now satisfies the Dirac equation

\[ [\sigma^1(\partial_1 - A_1^\text{eff}) + \sigma^2(\partial_2 - A_2^\text{eff}) + \sigma^3(\partial_3 - A_3^\text{eff}) + f_0 M] \Psi(x, \tau) = 0, \]  

(65)

where the suffix 4 has been left for the dimension of the imaginary time. It is easy to show that the effect of the effective potential due to the \((2+1)\)D geometry can be produced by the physical fields

\[ \mathbf{A}^m(\sigma^3) = \sigma^3A_1^\text{eff} \hat{\epsilon}_x + (\sigma^3A_1^\text{eff} + \sigma^1A_4^\text{eff}) \hat{\epsilon}_y, \]  

(66)

or

\[ \mathbf{A}^m(\sigma^3) = -\sigma^3A_1^\text{eff} \hat{\epsilon}_x + (\sigma^3A_2^\text{eff} + \sigma^3A_4^\text{eff}) \hat{\epsilon}_y. \]  

(67)

A potential application of the fact is to investigate the quantum effect of the \((2+1)\)D conformal surface using the electron or hole with state near the Dirac point of the band structure for the graphene where the mass term in (65) should be ignored \cite{13}, and the physics of the fermions depends only on \(\mathbf{A}^\text{eff}\), whose effect can be produced by the physical fields in (66) or (67).

The reinterpretation of the conformal structure proposed is applicable to demonstrate the behavior of the electron in higher dimensional conformal space. Consider the electron moving in a 3D conformal surface with

\[ ds^2 = f^2((dx^1)^2 + (dx^2)^2 + (dx^3)^2), \]  

(68)

where the metric coefficient is \(f^2 = f(x)\), independent of the time variable. The evolution of the electron is described by the Dirac equation

\[ \sum_{k=1}^3 \gamma^k(\partial_k - A_k^\text{eff}) + f(M - \gamma^E) \Psi = 0, \]  

(69)

where the effective vector potential

\[ A_k^\text{eff} = -\frac{1}{2f_k} \sum_{i=k} f_i = -\partial_k \ln f \]

(70)

since \(f_1 = f_2 = f_3 \equiv f\). With the usual definitions of the \(\gamma\) matrices

\[ \gamma^k = \begin{pmatrix} 0 & -i\sigma^k \\ i\sigma^k & 0 \end{pmatrix}, \]

(71)

and spin matrices

\[ \Sigma^k = \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}, \]

(72)

it is easy to show that

\[ \gamma^1A_1^\text{eff} = \gamma^1(\Sigma^2)^2A_1^\text{eff} = \gamma^2(-i\Sigma^3)A_1^\text{eff}, \]

(73)

\[ \gamma^2A_2^\text{eff} = \gamma^2(\Sigma^3)^2A_2^\text{eff} = \gamma^1(i\Sigma^2)A_2^\text{eff}, \]

(74)

and

\[ \gamma^3A_3^\text{eff} = \gamma^3(\Sigma^1)^2A_3^\text{eff} = \gamma^3(i\Sigma^1)A_3^\text{eff}. \]

(75)

The effect of the effective potential

\[ \mathbf{A}^\text{eff} = A_1^\text{eff} \hat{\epsilon}_x + A_2^\text{eff} \hat{\epsilon}_y + A_3^\text{eff} \hat{\epsilon}_z, \]

can thus be generated by the matrix-valued magnetic vector potential

\[ \mathbf{A}^m = (\Sigma^1A_2^\text{eff}) \hat{\epsilon}_x + (\Sigma^2A_3^\text{eff} - \Sigma^3A_1^\text{eff}) \hat{\epsilon}_y, \]

(76)

which couples to different components of the spinor, and the Dirac equation turns into the form with the physical coupling

\[ \sum_{k=1}^3 \gamma^k(\partial_k - iA_k^\text{eff}) + f(M - \gamma^E) \Psi = 0, \]

(77)

where \(A_1^\text{eff} = (\Sigma^2A_2^\text{eff})\), and \(A_2^\text{eff} = (\Sigma^3A_3^\text{eff} - \Sigma^1A_1^\text{eff})\). Compared with the 2D case, the components of the vector potential are non-commutative. For \((3+1)\)D conformal space with the invariant distance...
\[ ds^2 = f^2[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2], \]  

The effective vector potential

\[ A_{\text{eff}}^{\text{eff}} = A_1^{\text{eff}} \hat{e}_x + A_2^{\text{eff}} \hat{e}_y + A_3^{\text{eff}} \hat{e}_z + A_4^{\text{eff}} \hat{e}_w, \]  

where \( A_4^{\text{eff}} = -3\partial_t \ln f / 2 \), and \( \hat{e}_u \) denotes the unit vector of the imaginary time axis. The effect of the effective potential can be created by the vector field

\[ A_{\text{em}} = (\Sigma^1 A_2^{\text{eff}}) \hat{e}_x + (\Sigma^1 A_3^{\text{eff}}) \hat{e}_y - (\Sigma^1 A_4^{\text{eff}}) \hat{e}_z, \]  

where

\[ \alpha^3 = \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}, \]  

and the evolution of the electron in the conformal space is equivalently described by the Dirac equation with physical coupling

\[ \left[ \sum_{k=1}^4 \gamma^{k} (\partial_{k} - i A_{k}^{\text{em}}) + fM \right] \Psi = 0. \]  

The interpretation is the reverse. Given the matrix-valued vector potential along with the corresponding isotropic modulation of the electron’s momentum, it can be reinterpreted as the dynamics of the electron in the corresponding conformal space. The means is easy to generalize to a 3D space with an orthogonal frame just by combining with anisotropic momentum modulation [12]. This is comprehended by interpreting the Dirac equation

\[ \left[ \sum_{k=1}^3 \gamma^{k} (\partial_{k} - A_{k}^{\text{eff}}) + (M - \gamma^4 E) \right] \Psi = 0 \]  

as

\[ \left[ \sum_{k=1}^3 \gamma^{k} (\partial_{k} - A_{k}^{\text{eff}}) + \left( 1 - \gamma^4 \frac{\text{tr}(E_{ij})}{\text{tr}(M_{ij})} \right) \right] \Psi = 0, \]  

where

\[ (E_{ij}) = \text{diag}(E_{11}, E_{22}, E_{33}) = \text{diag}(f_1, f_2, f_3) E, \]  

\[ \text{and } (M_{ij}) = \text{diag}(M_{11}, M_{22}, M_{33}) = \text{diag}(f_1, f_2, f_3) M, \]  

are the anisotropic energy and mass components. Since mass and energy are not independent, they are associated with the anisotropic momentum \( (k_{ij}) = \text{diag}(k_{11}, k_{22}, k_{33}) \) by

\[ k_{ij}^2 = (f_i E)^2 - (f_i M)^2 = f_i^2 (E^2 - M^2) = f_i^2 k^2. \]  

So the invariant distance in the momentum space becomes

\[ \vec{k}^2 = k_{11}^2 + k_{22}^2 + k_{33}^2 = (f_1^2 + f_2^2 + f_3^2) k^2. \]  

From this anisotropic perspective, the effects of geometry and non-commutative vector potential interacting with the momentum-modulated fermion are equivalent to each other within the framework of the Dirac theory.

5. Conclusion

The objectives of this paper are twofold. The first is to investigate the Hooke-Newton transmutation of the electron in the uniform magnetic field. The other is to study the realization of the system. One finishes the first one by studying the electron moving in the magnetic field in the 2D quadratic conformal space. The allowed quantum spectrum and the corresponding spinor of the electron are obtained. The second purpose is achieved by reinterpreting the effect of the effective vector potential caused by conformal geometry as the magnetic vector potential of two fractional magnetic fluxes with magnitude \( \Omega = (\mp 1/2) \Omega_0 \), where \( \Omega_0 = \hbar c / q \) is the fundamental flux quantum. Since the formulation is established in the general 2D conformal space, it is possible to establish the other transmutations of the electron in the uniform magnetic field in the presented scheme.

Several points associated with the presented results worth noticing are stated as follows:

(a) The power series expansion of the spectrum is meaningful only when the condition \( \omega, \lambda < 4c \) is true. Here \( c \) is the light speed in free space. This can be understood by retrieving the physical units in (37).
\[ E = \pm Me^2 \left\{ 1 + \left( \frac{\omega \lambda}{4c} \right)^2 \left[ 1 - \frac{(n_l + 1/2)^2}{2\hbar(G) \pm (n_l + 1/2) + (1 \pm 1)^2} \right] \right\}^{1/2}. \]  \hfill (89)

A meaningful expansion exists only if the second term is smaller than unity. For the current technique, the condition is always true since for \( B = 1 \) Testa, \( \omega_c \approx 1.6 \times 10^{11} \text{ Hz} \), the characteristic length of the system is

\[ \lambda = \frac{1}{\sqrt{|n_l + 1/2|}} \left( \frac{8\hbar}{M\omega_c} \approx \frac{8.12 \times 10^{-2} \mu m}{\sqrt{|n_l + 1/2|}}. \right) \hfill (90) \]

So we have the velocity parameter

\[ \omega_c \lambda \approx 10^4 \text{ m/s}. \]  \hfill (91)

Even when the strength of the magnetic field reaches \( B = 100 \) Testa, the parameter is just \( \omega_c \lambda \approx 10^5 \text{ m/s} \). Equation (38) is still a good approximation.

(b) The upper bound of the quantization levels is consistent with the classical critical energy of the transmutation system. It was shown that the critical energy for the closed trajectories is given by

\[ E_C = \frac{\hbar^2}{4} \left| \frac{\omega_c}{p_c} \right| \]  \hfill (92)

with \( p_c \) being the angular momentum of the charged particle [6]. In the presented note, the critical energy for the bound states of the electron is given by

\[ E_C = Me^2 + \frac{M}{2} \left( \frac{\omega_c \lambda}{4} \right)^2. \]  \hfill (93)

Ignoring the rest mass term and substituting the characteristic length (90) of the system into it gives

\[ E_C = \frac{\hbar^2}{4} \left| \frac{\omega_c}{(n_l + 1/2)\hbar} \right|. \]  \hfill (94)

This is the quantum version of equation (92) for the particle with spin.

(c) The action direction of the Coulomb interaction with respect to the pure spin states of the different orientations is the reverse. The interaction is given by \(- (qB\lambda/4)(1/2) / \hbar\) for the state with spin and orbital angular momenta \( m_s = 1/2 \) and \( m_l = 0 \), corresponding to \( n_l = 0 \) as discussed below equation (35). Since \( q < 0 \) for an electron, and \( \lambda \) is definitely positive, the interaction represents a repulsive force. Conversely, for the state with \( m_s = -1/2 \) and \( m_l = 0 \), corresponding to \( n_l = -1 \), the interaction is given by \((qB\lambda/4)(1/2) / \hbar\). The electron would experience an attractive interaction. This is similar to the energy splitting of the different spin orientations in the magnetic field due the the Hamiltonian \( \sigma \cdot B \). Here, the splitting is exhibited by the Coulomb-like energy.

Along with the presented discussions, two topics are worth discussing further. The first is the novel behavior of the quantum particle due to the invariant representation of the equations. It can be shown that the Schrödinger equation and the Dirac equation for the free particles keep form invariant in the surface described by the invariant metric which is generated by 2D conformal mapping. When we consider the interaction of the particles with the electromagnetic field by performing the minimal coupling, a requirement of the form invariant equations with the new coordinates means that new solutions to the particles caused by an electromagnetic interaction which keeps the form invariant can be obtained. This is meaningful not only in mathematics but also in physics since the transformation structure can be realized by the reinterpretation means. The second interesting topic associated with the presented H-N transmutation is the charged particle moving on a sphere while interacting with the field of the magnetic monopole. A quadratic conformal mapping of the sphere would result in the similar transmutation in the space. Moreover, the requirement of the single valuedness of the physical quantity on the conformal space of the sphere should generalize the Dirac quantization rule of the magnetic monopole. As usual, the geometric effect of the conformal structure can be realized by the reinterpretation of the transformation structure. Some of the discussions on these topics will be reported in the near future.

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Appendix A. The effective vector potential

As presented in equation (1), the evolution of the Dirac fermion in general space-time can be described by the solution of the covariant Dirac equation

\[
\left[ \sum_{k=1}^{4} \tilde{\gamma}^k (\partial_k - \Gamma_k) + M \right] \Psi = 0, \tag{A1}
\]

where \( \tilde{\gamma}^k \) satisfies the anticommutation relation

\[
[\tilde{\gamma}^k, \tilde{\gamma}^l] = 2g^{kl}, \tag{A2}
\]

and \( \Gamma_k \) is the metric-associated spin connection. For a space-time allowed to be described by an orthogonal frame

\[
ds^2 = \sum_{k=1}^{4} g_{kk} (dx^k)^2. \tag{A3}
\]

There exists the expression

\[
\Gamma_k = \frac{1}{4} \left( \partial_k g_{kk} \right) g^{kk} - \frac{1}{4} \sum_{j=1}^{4} \left( \partial_j g_{kk} \right) \tilde{\gamma}^k \tilde{\gamma}^j, \tag{A4}
\]

for the spin connection (see, e.g., [14], and appendix in [15]). For the metric coefficients defined by

\[
(g_{kk}) = \text{diag}(g_{11}, g_{22}, g_{33}, g_{44}) = \text{diag}(f_1^2, f_2^2, f_3^2, f_4^2), \tag{A5}
\]

the components of the spin connection are as follows:

\[
\Gamma_1 = -\frac{1}{4} \left[ (\partial_2 f_1^2) \tilde{\gamma}^{12} f_1 f_2 + (\partial_3 f_1^2) \tilde{\gamma}^{13} f_1 f_3 + (\partial_4 f_1^2) \tilde{\gamma}^{14} f_1 f_4 \right], \tag{A6}
\]

\[
\Gamma_2 = -\frac{1}{4} \left[ (\partial_1 f_2^2) \tilde{\gamma}^{21} f_2 f_1 + (\partial_3 f_2^2) \tilde{\gamma}^{23} f_2 f_3 + (\partial_4 f_2^2) \tilde{\gamma}^{24} f_2 f_4 \right], \tag{A7}
\]

\[
\Gamma_3 = -\frac{1}{4} \left[ (\partial_1 f_3^2) \tilde{\gamma}^{31} f_3 f_1 + (\partial_2 f_3^2) \tilde{\gamma}^{32} f_3 f_2 + (\partial_4 f_3^2) \tilde{\gamma}^{34} f_3 f_4 \right], \tag{A8}
\]

and

\[
\Gamma_4 = -\frac{1}{4} \left[ (\partial_1 f_4^2) \tilde{\gamma}^{41} f_4 f_1 + (\partial_2 f_4^2) \tilde{\gamma}^{42} f_4 f_2 + (\partial_3 f_4^2) \tilde{\gamma}^{43} f_4 f_3 \right], \tag{A9}
\]

where the \( \gamma \) matrices of the Minkowski space-time are associated with the \( \tilde{\gamma} \) matrices of the curved space-time by

\[
\tilde{\gamma}^k = \gamma^k \tag{A10}
\]

from (A2). Substituting these \( \Gamma_i \) into (A1), and collecting the connection terms associated with \( \gamma^1 \) give

\[
\gamma_1^{-1} \frac{1}{4f_1} \left\{ \frac{\partial_1 f_1^2}{f_1} f_2 + \frac{\partial_2 f_1^2}{f_2} f_1 + \frac{\partial_3 f_1^2}{f_3} f_1 + \frac{\partial_4 f_1^2}{f_4} f_1 \right\} \Psi = \gamma_1^{-1} \frac{1}{2f_1} \left( \partial_1 \sum_{i=1}^{4} f_i \right) \Psi. \tag{A11}
\]

The collection of the terms associated with \( \gamma^2 \) gives

\[
\gamma_2^{-1} \frac{1}{4f_1} \left\{ \frac{\partial_1 f_2^2}{f_1} f_1 + \frac{\partial_2 f_2^2}{f_2} f_2 + \frac{\partial_3 f_2^2}{f_3} f_2 + \frac{\partial_4 f_2^2}{f_4} f_2 \right\} \Psi = \gamma_2^{-1} \frac{1}{2f_2} \left( \partial_2 \sum_{i=1}^{4} f_i \right) \Psi. \tag{A12}
\]

The coefficients of the rest \( \gamma^k, k = 3, 4, \) have the same structure. Therefore, one can identify the components of the effective vector potential caused by the spin connection

\[
A_k^{\text{eff}} = -\frac{1}{2f_k} \partial_k \sum_{i=1}^{4} f_i, \tag{A13}
\]

This is the formula given in (4).
Appendix B. The energy spectrum and radial functions of the spinor

Equation (33) exhibits that $G(\rho)$ satisfies the equation
\[
\frac{d^2 G}{d \rho^2} - \frac{1}{\rho} \frac{d G}{d \rho} + \left[ -\frac{(n_l/2 - 1/4)(n_l/2 + 7/4)}{\rho^2} + \left( \frac{qB\lambda}{4} \right)(\frac{1}{\rho} + \frac{k_b^2}{\rho^2} \right) G = 0. \tag{B1}
\]
In the region $\rho \to \infty$, it has the asymptotic form
\[
\frac{d^2 G}{d \rho^2} - k_b^2 G = 0, \tag{B2}
\]
which shows $G(\rho \to \infty) \sim e^{-k_b \rho}$. The asymptotic representation of (B1) around the origin $\rho \to 0$ is given by
\[
\frac{d^2 G}{d \rho^2} - \frac{1}{\rho} \frac{d G}{d \rho} - \left( \frac{1}{\rho} + \frac{2k_b}{\rho} \right)(\frac{1}{\rho} - 1)(\frac{1}{\rho} + \frac{7}{4}) G = 0. \tag{B3}
\]
It follows that
\[
G(\rho \to 0) \sim \rho^{1 \pm (n_l/2 + 3/4)}, \tag{B4}
\]
The solution is physical only if the top (bottom) sign of $\pm$ is for the quantum number $n_l \geq 0$ ($n_l < 0$). Consequently, $G(\rho)$ should have the solution of the whole range
\[
G(\rho) \sim e^{-k_b \rho} \rho^{1 \pm (n_l/2 + 3/4)}, \tag{B5}
\]
where $\alpha = [1 \pm (n_l/2 + 3/4)]$. Substituting the ansatz into (B1) shows that $R(\rho)$ satisfies the equation
\[
\frac{d^2 R}{d \rho^2} + (-2k_b + \frac{2\alpha - 1}{\rho}) \frac{d R}{d \rho} + \left[ \alpha^2 - 2\alpha - D_1 \frac{1}{\rho^2} + \left( \frac{-2\alpha + 1}{\rho} \right) k_b + D_2 \right] R = 0. \tag{B6}
\]
Here for short we have defined $D_1 = (n_l/2 - 1/4)(n_l/2 + 7/4)$, and $D_2 = qB\lambda(n_l + 1/2)/4$. By introducing the new variable $z = 2k_b \rho$, it turns into
\[
z \frac{d^2 R}{dz^2} + \left[ -z + (2\mu_1 + 1) \right] \frac{d R}{dz} + \left[ \frac{(2\mu_1 + 1)}{2} + \frac{D_2}{2k_b} \right] R = 0, \tag{B7}
\]
where $\mu_1 = \pm (n_l/2 + 3/4)$. This is the standard form of the associated Leguerre equation (p. 243, [16]). A necessary and sufficient condition for the differential equation to have a bounded solution is determined by the condition
\[
\frac{-(2\mu_1 + 1)}{2} + \frac{D_2}{2k_b} = \bar{n}(G) = 0, 1, 2, 3, \cdots. \tag{B8}
\]
The energy spectrum in (37) then follows. The solution to (B7) is the associated Leguerre function
\[
R = L_{\alpha+1/2}^\pm(n_l/2 + 3/4) / (2k_b \rho). \tag{B9}
\]
The radial function is then given by
\[
G(\rho) = C_l e^{-k_b \rho} \rho^{1 \pm (n_l/2 + 3/4)} / (2k_b \rho). \tag{B10}
\]
According to equation (31), the first radial component of the spinor satisfies
\[
F = \frac{1}{(M - E)} \left[ -\frac{d G}{d \rho} - \frac{(n_l/2 - 1/4)}{\rho} G + \left( \frac{qB\lambda}{4} \right) G \right]. \tag{B11}
\]
In the following calculation, we discuss the case of $n_l < 0$ which is for the solution of the electron. This can be understood from the Coulomb interaction in (B1). With the differential formula $dL_n^\alpha(x)/dx = -L_{n-1}^{\alpha+1}(x)$ (p.241, [16]), one has
\[
\frac{d G}{d \rho} = C_l e^{-k_b \rho} \rho^{(-n_l/2 + 1/4)} \left\{ -k_b + \left( \frac{-n_l/2 + 1/4}{\rho} \right) L_{n_l}^{(-n_l/2 - 3/4)}(\rho) = 2k_b L_{n_l}^{(-n_l/2 - 1/2)}(\rho), \right. \tag{B12}
\]
and
\[
F = \frac{C_l e^{-k_b \rho} \rho^{(-n_l/2 + 1/4)}}{(M - E)} \left\{ \left( k_b + \frac{qB\lambda}{4} \right) L_{n_l}^{(-n_l/2 - 3/4)}(\rho) + 2k_b L_{n_l}^{(-n_l/2 - 1/2)}(\rho) \right\}. \tag{B13}
\]
A further simplification needs the formula (p.241, [16])
\[
L_{n_l}^{(-n_l/2 - 3/4)} = L_{n_l}^{(-n_l/2 - 1/2)} - L_{n_l}^{(-n_l/2 - 1/2)}(\rho), \tag{B14}
\]
which makes (B13) turn into
\[
F = \frac{G e^{-\frac{k_B \rho}{E}}}{(M - E)} \left\{ k_B + \frac{qB\lambda}{4} \right\} L_n^{(-n_1 - 1/2)}(\frac{x}{\bar{\eta}(G)}) = \left\{ k_B - \frac{qB\lambda}{4} \right\} L_n^{(-n_1 - 1/2)}(\frac{x}{\bar{\eta}(G)}).
\]
(B15)

With alternative relation (p.241, [16]),
\[
L_n^{(-n_1 - 1/2)}(x) = \frac{1}{\bar{\eta}(G)} \left\{ xL_n^{(-n_1 + 1/2)}(x) + (\bar{\eta}(G) - n_1 - 1/2)L_n^{(-n_1 - 1/2)}(x) \right\},
\]
the equality (B15) becomes
\[
F = \frac{G e^{-\frac{k_B \rho}{E}}}{(M - E)} \left\{ k_B + \frac{qB\lambda}{4} \right\} - \frac{2k_B}{\bar{\eta}(G)} L_n^{(-n_1 + 1/2)}(\frac{x}{\bar{\eta}(G)} - 1/2)
\]
\[
+ \left\{ k_B + \frac{qB\lambda}{4} \right\} \frac{(\bar{\eta}(G) - n_1 - 1/2)}{\bar{\eta}(G)} + \left\{ k_B - \frac{qB\lambda}{4} \right\} L_n^{(-n_1 - 1/2)}(\frac{x}{\bar{\eta}(G)}).
\]
(B17)

From (B8), there exists the relation
\[
\bar{\eta}(G) = \frac{k_B + qB\lambda}{2k_B}.
\]
(B18)

This makes the middle bracket of (B17) vanish. We finally have
\[
F = \frac{G e^{-\frac{k_B \rho}{E}}}{(M - E)} \left[ 2k_B \left( k_B + \frac{qB\lambda}{4} \right) \right] \frac{e^{-\frac{k_B \rho}{E}}}{\bar{\eta}(G)} L_n^{(-n_1 + 1/2)}(2k_B \rho).
\]
(B19)

This is the result given in (50). The function can also be obtained from equation (34). Nevertheless, using (34) it is hard to determine the coefficient of (B19). The functions in (52), and (53) can be obtained with a similar calculation. To have the spinor of the scattering state, one can take the replacement
\[
k_B \rightarrow i\kappa_B,
\]
with
\[
\kappa_B = \sqrt{E^2 - (M^2 + (qB\lambda/4)^2)},
\]
and \(E^2 > [M^2 + (qB\lambda/4)^2]\). The operation changes the asymptotic behavior of the radial function at \(\rho \rightarrow \infty\), i.e., \(e^{-k_B \rho} \rightarrow e^{i\kappa_B \rho}\). The function \(G(\rho)\) should then have the proportionality
\[
G(\rho) \sim \rho^\alpha e^{i\kappa_B \rho} R(\rho).
\]
(B22)

A substitution of the ansatz into (B1) shows that \(R(\rho)\) satisfies
\[
y \frac{d^2 R}{dy^2} + \left\{ y + (2\mu_1 + 1) \right\} \frac{dR}{dy} - \frac{(2\mu_1 + 1)}{2} - \frac{D_2}{2i\kappa_B} R = 0,
\]
(B23)

where \(y = 2i\kappa_B \rho\). This is the standard confluent hypergeometric equation (p. 268, [16]). The analytic solution can be expressed in terms of confluent hypergeometric function
\[
R \sim {}_1F_1(\mu_1 + 1/2 - D_2/(2i\kappa_B), 2\mu_1 + 1; 2i\kappa_B \rho).
\]
(B24)

It is in general an infinite series, and has the asymptotic representation behavior like \(1F_1 \sim e^{2i\kappa_B \rho}\) when \(\rho \rightarrow \infty\). Obviously, it cannot satisfy the boundary condition of the bound states. A means to make the function satisfy the boundary condition is to intercept the infinite series making it become a polynomial. This is achieved by requiring
\[
\frac{(2\mu_1 + 1)}{2} - \frac{D_2}{2i\kappa_B} = -\bar{\eta}(G),
\]
(B25)
from which we get the energy spectrum (37) again.

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