Tail terms in gravitational radiation reaction via effective field theory

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Gravitational radiation reaction affects the dynamics of gravitationally bound binary systems via "tail" terms which, at the lowest level, modify the conservative dynamics at fourth post-Newtonian order, as it was first computed by Blanchet and Damour. Here we re-produce this result using effective field theory techniques in the framework of the closed-time-path formalism. This tail term is the lowest order example of a short-distance singularity showing up in the conservative dynamics, and it is correctly taken into account within the effective field theory formalism.

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I. INTRODUCTION

The forces induced on an isolated system by reaction to the emission of gravitational waves and their impact on the motion of gravitationally bound binary systems has been studied in great accuracy since the first derivation of the radiation reaction force in General Relativity by Burke and Thorne \cite{1,2,3}. Their phenomenological impact is linked to the forthcoming observation runs of the Laser Interferometer Gravitational Observer (LIGO) and Virgo, see \cite{4} for the result of a recent search of coalescing binaries, and have already been observed

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to be at play in binary pulsar systems \cite{5, 6}.

The motion of coalescing binaries is imprinted in the shape of the emitted gravitational waves and the output of gravitational detectors is particularly sensitive to the time varying phase of the radiated wave, which has to be determined with high accuracy in order to ensure high efficiency of the detection algorithms and faithful source parameter reconstruction. The standard approach to describe the motion of coalescing binaries lies within the post-Newtonian (PN) approximation to General Relativity, describing the binary system dynamics as a perturbative series in terms of the relative velocity of the binary constituents, see e.g. \cite{7} for a review.

The leading effect of radiation reaction modifies the binary dynamics giving rise to a term \textit{odd} under time reversal which changes the energy of the system at 2.5 PN order: this is the lowest order at which linear effects of the gravitational radiation enter. In \cite{8, 9} the leading non-linear radiation reaction effect has been derived, and it is shown to modify the binary dynamics at 4PN order (i.e. at 1.5PN order relatively to the leading effect): it belongs to the species of terms dubbed \textit{hereditary}, as it depends on the entire history of the source. In particular it originates from radiation emitted and then scattered back into the system by the background curvature generated by the mass $M$ of the binary system, hence the name of \textit{tail} term. Such non-linear 4PN tail term has been shown to include both a term non-invariant and a term \textit{invariant} under time reversal, the latter can be incorporated in the conservative dynamics of a binary system, in good agreement with computations performed within the framework of the gravitational self-force analysis of circular orbits in Schwarzschild background, as found in \cite{10}.

Here we present the re-computation of the 4PN tail contribution to the conservative dynamics of inspiraling binaries via the use of the Effective Field Theory (EFT) methods for gravity introduced in \cite{11}. EFT methods turns out to be useful in problems admitting a clear scale separation: in the binary system case we have the size of the compact objects $r_s$, the orbital separation $r$ at which the system consists of point particles interacting via instantaneous potential, and the gravitational wave-length $\lambda$ (with hierarchy $r_s < r \sim r_s/v^2 < \lambda \sim r/v$, being $v$ the relative velocity between the two bodies) at which the binary system behaves as a particle of negligible size endowed with multipoles.

Using this approach several different groups have re-produced results in the PN analysis which have been previously computed in the standard approach both in the conservative
Figure 1: Diagram describing the gravitational radiation reaction force at leading order (left) and leading non-linear order (right). The thick line represents the massive binary system, the curly line the the gravitons emitted and absorbed by the system, the dashed line the potential graviton responsible for the Newtonian potential.

[12, 13] and in the dissipative [14] sector. Moreover new results on the PN analysis have been made available by the use of the EFT method in both sectors [15 16], as well as in the extreme mass ratio limit approach to binary coalescence [17]. We have recently shown [13] how to recover the Hamiltonian dynamics at 3PN order using effective field theory methods, by implementing an algorithm using the Mathematica [18] software to integrate out short range Newtonian gravitons to determine the conservative binary dynamics, paving the way to higher order computation not involving tail terms. However at 4PN order, the radiation reaction comes into play and we show in the present paper how EFT methods are able to recover the 4PN conservative tail, which involves emission and absorption of radiation by the gravitational system.

The leading order radiation reaction effect via effective field theory methods have been computed in [19] adopting the closed-time-path formalism, which is the suitable framework in which computing amplitudes with time-asymmetric boundary conditions, as the physical set up represented by coalescing binaries requires no incoming radiation from the past-null infinity.

In the following we use $c = 1$ units and the mostly plus signature convention. Contraction of space indices are taken with the Kronecker delta.

II. RADIATION REACTION FROM TAIL TERM
Following the EFT framework for non relativistic General Relativity, after integrating out the potential gravitons we are left with an effective action at the orbital scale \( r \), describing radiation gravitons coupled via moments of the two-particle distribution to a composite static object of total mass \( M \) given by the sum of the binary constituents masses \( m_{1,2} \). We adopt the decomposition of the metric suggested in [22]

\[
g_{\mu\nu} = e^{2\phi/m_{Pl}} \begin{pmatrix} -1 & A_j/m_{Pl} \\ A_i/m_{Pl} e^{-4\phi/m_{Pl}} (\delta_{ij} + \sigma_{ij}/m_{Pl}) - A_i A_j/m_{Pl}^2 & \end{pmatrix}.
\]

(1)

The coupling of the radiation gravitons can be read from eq. (24) of [11], here we report the relevant terms for the radiation reaction computation, that are

\[
S_{\text{eff}} \supset -\frac{1}{m_{Pl}} \int d\tau \left( M\phi + \frac{1}{2} I_{ij} R^0_{ij} \right),
\]

(2)

with \( G_N = 1/(32\pi m_{Pl}^2) \), \( R^0_{ij} \) denotes the appropriate component of the Riemann tensor and the moment \( I_{ij} \) is defined as \( I_{ij} \equiv \sum_{a=1}^{2} m_a x_{ai} x_{aj} \). [26]

At lowest order, by integrating out the radiation graviton, i.e. by computing the diagram in the left of fig. [1] the Burke-Thorne potential is obtained, which has been derived in the EFT framework in [19]. Corrections to the leading effect appears at relative 1PN order due to the inclusion of higher multipoles and the 1PN modified dynamics of the quadrupole [20, 21]. The genuinely non-linear effect appear at relative 1.5PN order and it is due to the diagram in the right of fig. [1] whose calculation is the subject of the remaining of this paper. The interactions involved in the 3-particle vertex can be read from eq. (6) of [13]. The effective action obtained by computing the right diagram in fig. [1] is given by the sum of six contributions, one for each interaction vertex which can be built out of the metric fields \( \phi, A \) and \( \sigma \), given that the \( M \) insertion sources a \( \phi \).

In order to perform the computation of the diagram, boundary conditions asymmetric in time have to be imposed, as no incoming radiation at past null infinity is required. Technically this is implemented by adopting the closed-time-path (CTP), or in-in formalism (first proposed in [23], see [24] for a review) as described in [19]. The CTP construction requires to double the field variables, e.g. for a scalar field \( \Psi \), the generating functional \( W \) for connected correlation functions has the path integral representation

\[
e^{iW[J_1, J_2]} = \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 \exp \left\{ \int d^4x \left[ -\frac{i}{2} (\partial\Psi_1)^2 + \frac{i}{2} (\partial\Psi_2)^2 + iJ_1 \Psi_1 - iJ_2 \Psi_2 \right] \right\}.
\]

(3)
It turns out to be useful to perform a change of variables from the 1, 2 base to a +, − base defined by

\[ \Psi_- \equiv \Psi_1 - \Psi_2, \quad \Psi_+ \equiv \frac{\Psi_1 + \Psi_2}{2}, \tag{4} \]

so that for generic fields \( \Psi_A, \Psi_B \) and \( \Psi_C \) the doubling of the degrees of freedom gives

\[ \Psi \to \Psi_-, \quad \Psi_A \Psi_B \to \Psi_A \Psi_B^1 - \Psi_A \Psi_B^2 = \Psi_A^+ \Psi_B^- + \Psi_A^- \Psi_B^+, \tag{5} \]

\[ \Psi_A \Psi_B \Psi_C \to (\Psi_A^+ \Psi_B^- + \Psi_A^- \Psi_B^+) \Psi_C^+ + \Psi_A^+ \Psi_B^+ \Psi_C^- + \frac{\Psi_A^- \Psi_B^- \Psi_C^-}{4}. \]

For free scalar fields, the path integral in eq. (3) can be performed exactly, leading to the effective action \( W[J_+, J_-] = i/2 \int d^4x d^4y J_D(x) G^{DE}(x - y) J_E(y) \) which can be written in the so-called Keldysh representation \[25\] (indices \( D \) and \( E \) taking values + and −) where

\[ G^{CD}(t, x) = \begin{pmatrix} 0 & -iG_A(t, x) \\ -iG_R(t, x) & \frac{1}{2}G_H(t, x) \end{pmatrix}, \tag{6} \]

where \( G^{++} = 0 \) and \( G_{A(R)} \) is the usual advanced (retarded) propagator

\[ iG_A(t, x) = \theta(-t) (\Delta_+(t, x) - \Delta_-(t, x)) = -i \frac{\delta(t + r)}{4\pi r}, \]

\[ -iG_R(t, x) = \theta(t) (\Delta_+(t, x) - \Delta_-(t, x)) = i \frac{\delta(t - r)}{4\pi r}, \tag{7} \]

\[ G_H(t - t', x) = \Delta_+(t, x) + \Delta_-(t, x), \]

where

\[ \Delta_\pm(t, x) = \int \frac{d^3k}{(2\pi)^3} e^{\pm i k \cdot x} \frac{e^{-ik \cdot r}}{2k}. \]

The computation of the right diagram in fig. 1 using eqs. (4) and (5), boils down to the
following contribution \( S_{eff}^{(QQM)} \) to the effective action \( S_{eff} \)

\[
i S_{eff}^{(QQM)} = \frac{1}{2} \left( \frac{i}{2\Lambda} \right)^2 \left( \frac{-iM}{\Lambda} \right) \left( \frac{i4\pi}{\Lambda} \right) \int dt \, dt' \, dt'' \, dt_i \, dr \, r^2 \, d\Omega \frac{d\Omega}{4\pi}
\]

\[
\left\{ (I_{ij}(t)R^0_{i-0j}(t) + I_{+ij}(t)R^{-0i}_{j-0})(t') \phi_-(t'', 0) \times \\
2\sigma_{-ab} \phi_+ \delta_{-bp} (\delta_{ab} \delta_{rp} + \delta_{ap} \delta_{br} - \delta_{ar} \delta_{bp}) \\
+ 4 \left( \sigma_{+ar} \partial_p \phi_+ \hat{A}_{-b} + \sigma_{-ar} \partial_p \phi_+ \hat{A}_{+b} \right) (\delta_{ab} \delta_{rp} + \delta_{rb} \delta_{ap} - \delta_{ar} \delta_{bp}) \\
+ 2 \left( 2\sigma_{+ar} \partial_b \phi_+ \partial_p \phi_+ + \sigma_{-ar} \partial_b \phi_+ \partial_p \phi_+ \right) (\delta_{ab} \delta_{rp} + \delta_{ap} \delta_{rb} - \delta_{ar} \delta_{bp}) \\
+ 8 \partial_a \hat{A}_{+r} \phi_+ \partial_b \hat{A}_{-p} (\delta_{ab} \delta_{rp} - \delta_{ap} \delta_{rb} + \delta_{ar} \delta_{bp}) \\
- 8 \left( \hat{A}_{+a} \partial_x \phi_+ \dot{\phi}_- + \hat{A}_{-a} \partial_x \phi_+ \dot{\phi}_+ \right) \\
- 32 \dot{\phi}_+ \phi_+ \dot{\phi}_- \right) (t_i, x) \times \\
(I_{-kl}(t')R^0_{-k0l}(t') + I_{+kl}(t')R^0_{+k0l}(t')) \right\},
\]

where all possible contractions between fields at different space-time points should be taken (ensuring that one of the two fields involved in contraction is at \((t_i, x)\)) and are understood, an over-dot denotes a time derivative and \(\partial_a \equiv \partial/\partial x^a\), \(r \equiv |x|\), and where

\[
I_{-ij} = \sum_{a=1}^{2} m_a \left( x_{-ai} x_{+aj} + x_{+ai} x_{-aj} \right),
\]

\[
I_{+ij} = \sum_{a=1}^{2} m_a x_{+ai} x_{+aj} + O(x_i^2).
\]

After taking the Wick contractions as summarized in eq. (A1), we obtain mixed terms of the type \(I_-I_+\) and terms quadratic in \(I_-\). In order to get the equations of motion for the \(x_i\)'s we have to take the variation with respect to \(x_{-i}\) and then evaluate the result at \(x_{-i} = 0\), \(x_{+i} = x_i\), so we can discard the terms involving \(D_H\) as they are quadratic in \(I_-\), as well as the terms quadratic in \(x_-\) present in in \(I_+\). Using the Einstein Hilbert action, the metric parametrization in eq. (1) and the gauge fixing term as in [13], the retarded propagators for
the \( \phi, A \) and \( \sigma \) fields are

\[
P[\phi(t, x) \phi(t', x)] = -\frac{1}{8} \left( \frac{1}{2} \delta_{ij} \right) \times G_R(t-t', x-x'),
\]

\[
P[A_i(t, x) A_j(t', x')] = \frac{1}{2} \delta_{ij}
\]

\[
P[\sigma_{ij}(t, x) \sigma_{kl}(t', x')] = -\frac{1}{2} \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} - 2 \delta_{ij} \delta_{kl} \right)
\]

where all other mixed propagators vanish. Introducing the quadrupole moments

\[
Q_{\pm ij} = I_{\pm ij} - \frac{1}{3} \delta_{ij} I_{\pm kk},
\]

after performing the calculations reported in the appendix, the final result turns out to be

\[
i S_{eff}^{(QQM)} = -i \frac{4}{5} G_N M \int dt Q_{-ij}(t) \int_{-\infty}^{t} dt' \frac{d^6 Q_{+ij}(t')}{dt'^6} \frac{1}{(t-t')}. \tag{12}
\]

This results exhibits a short-distance singularity, which is inherited by the contribution to the equations of motion containing the effect of the tail term \( \Delta^{(QQM)} x_{ai} \):

\[
\Delta^{(QQM)} \ddot{x}_{ai}(t) = \frac{1}{m_a} \left. \frac{\partial S_{eff}^{(QQM)}}{\partial x_{-ai}} \right|_{x_{-ai}=0,x_{+ai}=x_{ai}} = -\frac{8}{5} x_{aj}(t) \int_{-\infty}^{t} dt' \frac{d^6 Q_{ij}(t')}{dt'^6} \frac{1}{(t-t')} . \tag{13}
\]

The divergence can be regularized by standard methods, i.e. by renormalizing the bare source moment \( Q_{ij} \) to absorb the divergence which then disappears from the physical results expressed in terms of the renormalized moment. After integrating by part and regularizing, expressing the results in terms of the regularized moment, which we still denote by \( Q_{ij} \), one obtains:

\[
\Delta^{(QQM)} \ddot{x}_{ai}(t) = -\frac{8}{5} x_{aj}(t) \int_{-\infty}^{t} dt' \frac{d^6 Q_{ij}(t')}{dt'^6} \log \left[ \frac{(t-t')\mu}{\lambda} \right]. \tag{14}
\]

where the renormalization scale \( \mu \) has been introduced.

The explicit \( \mu \)-dependence in eq. \( \text{[14]} \) is absorbed by a compensating dependence on \( \mu \) in the renormalized quadrupole moment. Consistently with renormalization group invariance, physical results are independent on the arbitrary renormalization scale \( \mu \). In practice one can take \( \mu \) to be of the order of the typical graviton frequency \( \mu = v/r \), or of the typical short distance scale in the problem \( \mu = 1/r \). Following \[10\] we adopt \( \mu = 1/r \) and using the trivial identity

\[
\log \left( \frac{t-t'}{r} \right) = \log \left( \frac{t-t'}{\lambda} \right) - \log \left( \frac{r}{\lambda} \right) . \tag{15}
\]
valid for any arbitrary \( \lambda \), we obtain a piece in the equations of motion which contains no explicit dependence on time and which can be interpreted as a quadrupole-sourced contribution affecting the conservative motion of the sources \( \Delta_{\text{cons}}^{(QQM)} \ddot{x}_{ai} \). Using eqs. (14) and (15) one obtains

\[
\Delta_{\text{cons}}^{(QQM)} \ddot{x}_{ai} = \frac{8}{5} x_{aj} G_N^2 M \frac{\partial^6 Q_{ij}}{\partial \theta^6} \log \left( \frac{r}{\lambda} \right),
\]

which coincides with the equations of motion derived in [10] following the result of [8].

III. CONCLUSION

The conservative dynamics of gravitationally bound binary systems is known in literature up to the third post-Newtonian order. Such results has been derived by both traditional methods and within the context of effective field theory. In both methods long distance singularities are met and consistently dealt with. At fourth post-Newtonian order a radiation reaction tail term affects the conservative dynamics, involving the regularization of a short distance singularity. We have shown that the effective field theory approach consistently reproduces the result obtained by traditional methods, adding yet another ingredient to support the equivalence of the two approaches and to cross-check old results.

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Appendix A: Useful formulae

We give an explicit derivation of the Wick contractions taking from eq. (8) to the result (14). Within the CTP formalism, in the Keldysh representation, the Wick contractions give

\[
(I_- R_+ + I_+ R_-) (t) (\Psi_A - \Psi_{B+} + \Psi_{B+} \Psi_{A-}) (t_i) (I_- R_+ + I_+ R_-) (t') \phi_+ (t_i) \phi_- (t'') \rightarrow \\
\left\{ \begin{array}{l}
I_- (t) [-iG_R (t - t_i) \frac{1}{2} G_H (t_i - t') + \frac{1}{2} D_H (t - t_i) (-iG_A (t_i - t')) \\
+ \frac{1}{2} G_H (t - t_i) (-iG_A (t_i - t')) - iG_R (t - t_i) \frac{1}{2} G_H (t_i - t') ] I_- (t') \\
+ I_- (t) [2 (-iG_R (t - t_i)) (-iG_R (t_i - t'))] I_+ (t') \\
+ I_+ (t) [2 (-iG_A (t - t_i)) (-iG_A (t_i - t'))] I_- (t') \end{array} \right\} (-iG_R (t_i - t'')) ,
\]

where we have suppressed tensor indices and we have assumed the field \( R \) to have non-vanishing \( A \)-independent contraction with any of the fields \( \Psi_A, \phi \). In our case we have to use the Riemann tensor expanded to first order as

\[
R^0_{\ abij} = \frac{1}{2} \delta_{ij} - \frac{1}{2} \dot{A}_{ij} - \frac{1}{2} A_{ij} - \phi, ij - \delta_{ij} \ddot{\phi} + O (h^2) ,
\]

where \( h \) denotes generically the field \( \phi, A \) or \( \sigma \). After taking all the Wick contractions in eq. (8) and keeping only the terms linear in \( I_- \) one obtains:

\[
iS_{\ eff}^{(QQM)} = \frac{1}{2} \left( \frac{i}{2\Lambda} \right)^2 \left( -\frac{iM}{\Lambda} \right) \left( \frac{i 4\pi}{\Lambda} \right) \left( \frac{(-i)^3}{4(8\pi)^3} \right) \int dt \int dt' \int dt_i \int dr \int r^2 \int d\Omega \left\{ \right.

I_- ij (t) I_+ kl (t') \times \\
\left[ \begin{array}{l}
8 (\delta_{ij} \delta_{kl} - \delta_{ik} \delta_{jl}) \frac{\partial_i^2 \partial_j \delta (t - t_i - r) \partial_k \partial_l (\delta (t_i - t' - r))}{r} \\
+ 32 \delta_{ij} \frac{\partial_i^2 \delta (t - t_i - r) \partial_j \partial_k \partial_l (\delta (t_i - t' - r))}{r} \\
- 8 \frac{\partial_i^2 \delta (t - t_i - r) \partial_j \partial_k \partial_l (\delta (t_i - t' - r))}{r} \\
+ 16 \frac{\partial_i^2 \delta (t - t_i - r) \partial_j \partial_k \partial_l (\delta (t_i - t' - r))}{r} \\
+ 8 \frac{\partial_i^2 \delta (t - t_i - r) \partial_j \partial_k \partial_l (\delta (t_i - t' - r))}{r} \\
- 4 \frac{\partial_i \partial_j (\delta (t - t_i - r))}{r} \partial_i (\partial_k \partial_l + \delta_{kl} \partial_j) (\delta (t_i - t' - r)) \\
+ 8 \frac{\partial_i \partial_j (\delta (t - t_i - r))}{r} \partial_i (\partial_k \partial_l + \delta_{kl} \partial_j) (\delta (t_i - t' - r)) \\
+ 16 \frac{\partial_i \partial_j (\delta (t - t_i - r))}{r} \partial_i (\partial_k \partial_l + \delta_{kl} \partial_j) (\delta (t_i - t' - r)) \\
+ 8 \frac{\partial_i \partial_j (\delta (t - t_i - r))}{r} \partial_i (\partial_k \partial_l + \delta_{kl} \partial_j) (\delta (t_i - t' - r)) \\
- 4 \frac{\partial_i \partial_j (\delta (t - t_i - r))}{r} \partial_i (\partial_k \partial_l + \delta_{kl} \partial_j) (\delta (t_i - t' - r)) \left\} .
\]

Using the following

\[
\begin{align*}
\int \frac{d\Omega}{4\pi} \frac{r_ir_j}{r^2} &= \frac{1}{3} \delta_{ij}, \\
\int d\Omega \frac{r_ir_jr_kr_l}{4\pi r^4} &= \frac{1}{15} (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) , \\
\partial_a \left( \frac{\delta(t - r)}{r} \right) &= -\frac{r_a}{r^2} \left( \frac{\delta(t - r)}{r} + \frac{\delta(t - r)}{r} \right) , \\
\partial_a \partial_b \left( \frac{\delta(t - r)}{r} \right) &= \frac{1}{r} \left[ \frac{\delta(t - r)}{r^2} - \frac{\delta(t - r)}{r} \left( \delta_{ab} - 3 \frac{r_ar_b}{r^2} \right) \\
&\quad - \frac{\delta(t - r)}{r^2} \left( \delta_{ab} - 3 \frac{r_ar_b}{r^2} \right) \right] , \\
\partial_a \partial_b \partial_c \left( \frac{\delta(t - r)}{r} \right) &= \frac{1}{r} \left[ \frac{\delta(t - r)}{r^2} - \frac{\delta(t - r)}{r} \left( \delta_{ab} - 3 \frac{r_ar_b}{r^2} \right) \\
&\quad + 3 \frac{\delta(t - r)}{r^2} \left( \delta_{ab} - 3 \frac{r_ar_b}{r^2} \right) \right] ,
\end{align*}
\]

we obtain the following contributions from respectively the $\sigma^2$A, $\sigma A\phi$, $\sigma \phi^2$, $A^2 \phi$, $A \phi^2$ and $\phi^3$ vertices (we substitute $u \equiv t - t' - 2r$):

\[
i S_{\text{eff}}^{QQM} = -\frac{iG^2}{4} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' i_{ij}(t)I_{kl}(t') \int_0^\infty dr \frac{1}{r^2} \times
\]

\[
\left\{ \begin{array}{l}
8 \left[ r \delta^{(6)}(u) \left( \delta_{ik} \delta_{jl} - \delta_{ij} \delta_{kl} \right) \\
- \frac{32}{3} \left[ \left( \delta^{(5)}(u) + \frac{\delta^{(4)}(u)}{r} \right) \delta_{ik} \delta_{jl} \right] \\
- \frac{16}{15} \left[ \left( \delta^{(5)}(u) + \frac{\delta^{(4)}(u)}{r} \right) \left( \delta_{ik} \delta_{jl} + 3 \delta_{ij} \delta_{kl} \right) \right] \\
- \frac{16}{3} \left[ r \delta^{(6)}(u) \delta_{ik} \delta_{jl} + \left( \delta_{ik} \delta_{jl} - \frac{1}{3} \delta_{ij} \delta_{kl} \right) \right] \times \\
\left( 6 \delta^{(5)}(u) + 15 \frac{\delta^{(4)}(u)}{r} + 18 \frac{\delta^{(3)}(u)}{r^2} + 9 \frac{\delta^{(2)}(u)}{r^3} \right) \right]
\end{array} \right.
\]

\[
+ \frac{16}{15} \left[ \left( \delta_{ik} \delta_{jl} - \frac{1}{3} \delta_{ij} \delta_{kl} \right) \left( \delta^{(5)}(u) + 4 \frac{\delta^{(4)}(u)}{r} + 6 \frac{\delta^{(3)}(u)}{r^2} + 3 \frac{\delta^{(2)}(u)}{r^3} \right) \\
+ 10 \delta_{ij} \delta_{kl} \left( \delta^{(5)}(u) + \frac{\delta^{(4)}(u)}{r} \right) \right] \\
+ \frac{8}{15} \left[ r \delta^{(6)}(u) \left( \delta_{ik} \delta_{jl} + 13 \delta_{ij} \delta_{kl} \right) \\
+ \left( \delta_{ik} \delta_{jl} - \frac{1}{3} \delta_{ij} \delta_{kl} \right) \left( 6 \frac{\delta^{(5)}(u)}{r} + 15 \frac{\delta^{(4)}(u)}{r^2} + 18 \frac{\delta^{(3)}(u)}{r^3} + 9 \frac{\delta^{(2)}(u)}{r^4} \right) \right] \right\}.
\]
Now we use
\[
\int_{-\infty}^{\infty} dt' I(t') \int_{0}^{\infty} dr \frac{1}{r^a} \delta^{(7-a)}(t - t' - 2r) = \frac{(-2)^{a-1}}{\Gamma(a)} \int_{-\infty}^{t} dt' \frac{d^6 I(t')}{dt'^6} \frac{1}{(t - t')}
\] (A6)
to find that the terms contained in the second, third and fifth square brackets of the above eq. (A5) (corresponding respectively to the \(\sigma \phi A\), \(\phi \phi \sigma\) and \(\phi \phi A\) bulk interactions) are identically zero, and in the other lines only the \(\delta^{(6)}(u)\) terms contribute, summing up to
\[
iS_{eff}^{(QQM)} = -iG_N^2 M \frac{2}{5} \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} - \frac{2}{3} \delta_{ij} \delta_{kl} \right) \int_{-\infty}^{\infty} dt I_{-ij}(t) \int_{-\infty}^{t} dt' \frac{d^6 I_{+kl}(t')}{dt'^6} \frac{1}{(t - t')}
\] (A7)
The amplitude index structure projects the symmetric moments onto their traceless part, allowing us to replace \(I_{\pm ij}\) with the traceless quadrupole moments \(Q_{\pm ij}\) as expected.

Note that when integrating by parts the derivatives of the delta’s we repeatedly use
\[
\int_{-\infty}^{t} dt' \frac{Q(t')}{(t - t')^a} = \frac{1}{a - 1} \frac{Q(t')}{(t - t')^{a-1}} \bigg|_{t'=-\infty}^{t'} - \frac{1}{a - 1} \int_{-\infty}^{t} dt' \frac{dQ(t')}{dt'} \frac{1}{(t - t')^{a-1}}
\] (A8)
and imposed
\[
\frac{1}{a - 1} \frac{Q(t')}{(t - t')^{a-1}} \bigg|_{t'=-\infty}^{t'} = 0
\] (A9)
for any \(a \neq 1\). In dimensional regularization, only the \(t' = t\) divergence for \(a = 1\) (i.e. the logarithmic divergence) is physical and it introduces the standard \(1/(d - 3)\) divergent term in the renormalization constant, being \(d\) the number of space dimensions.

[1] W. L. Burke and K. S. Thorne, in “Relativity”, edited by M. Carmeli, S. I. Fickler and L. Witten, (Plenum, New York, 1970) pp.209-228
[2] W. L. Burke, J. Math. Phys. 12, 401 (1971).
[3] K. S. Thorne, Astrophys. J. 158, 997 (1969).
[4] The LIGO Scientific Collaboration and the Virgo Collaboration, Phys. Rev. D 83 (2011) 122005 [arXiv:1102.3781 [gr-qc]].
[5] R. A. Hulse, J. H. Taylor, Astrophys. J. 195 (1975) L51-L53.
[6] J. H. Taylor, J. M. Weisberg, Astrophys. J. 253 (1982) 908-920.
[7] L. Blanchet, Living Rev. Relativity 5, (2002), URL: http://www.livingreviews.org/lrr-2002-3
[8] L. Blanchet, T. Damour, Phys. Rev. D37 (1988) 1410.
[9] L. Blanchet, Phys. Rev. D47 (1993) 4392-4420.

[10] L. Blanchet, S. L. Detweiler, A. Le Tiec, B. F. Whiting, Phys. Rev. D81 (2010) 084033. arXiv:1002.0726 [gr-qc]; L. Blanchet, S. Detweiler, A. Le Tiec, B. F. Whiting, arXiv:1007.2614 [gr-qc].

[11] W. D. Goldberger and I. Z. Rothstein, Phys. Rev. D73 (2006) 104029 arXiv:hep-th/0409156.

[12] R. A. Porto, Phys. Rev. D73 (2006) 104031 [gr-qc/0511061]; M. Levi, Phys. Rev. D82 (2010) 064029 arXiv:0802.1508 [gr-qc]; J. B. Gilmore and A. Ross, Phys. Rev. D78, 124021 (2008) arXiv:0810.1328 [gr-qc]; Y. -Z. Chu, Phys. Rev. D79 (2009) 044031 arXiv:0812.0012 [gr-qc]; D. L. Perrodin, arXiv:1005.0634 [gr-qc]; R. A. Porto, Class. Quant. Grav. 27, 205001 (2010) arXiv:1005.5730 [gr-qc]; M. Levi, Phys. Rev. D82 (2010) 104004 arXiv:1006.4139 [gr-qc].

[13] S. Foffa, R. Sturani, Phys. Rev. D84 (2011) 044031 arXiv:1104.1122 [gr-qc].

[14] W. D. Goldberger and A. Ross, Phys. Rev. D81 (2010) 124015 arXiv:0912.4254 [gr-qc].

[15] R. A. Porto, I. Z. Rothstein, Phys. Rev. Lett. 97 (2006) 021101 [gr-qc/0604099]. R. A. Porto, I. Z. Rothstein, Phys. Rev. D78 (2008) 044012 arXiv:0802.0720 [gr-qc] [Erratum-ibid. D 81 (2010) 029904]; R. A. Porto, I. Z. Rothstein, Phys. Rev. D78 (2008) 044013 arXiv:0804.0260 [gr-qc] [Erratum-ibid. D 81 (2010) 029905]; R. A. Porto, Phys. Rev. D73 (2006) 104031 arXiv:gr-qc/0511061.

[16] R. A. Porto, A. Ross, I. Z. Rothstein, JCAP 1103 (2011) 009 arXiv:1007.1312 [gr-qc].

[17] C. R. Galley, B. L. Hu, Phys. Rev. D79 (2009) 064002 arXiv:0801.0900 [gr-qc]; C. R. Galley, arXiv:1012.4488 [gr-qc]; C. R. Galley, arXiv:1107.0766 [gr-qc];

[18] Wolfram Research, Inc., Mathematica, Version 6.0, Champaign, IL (2007).

[19] C. R. Galley, M. Tiglio, Phys. Rev. D79 (2009) 124027. arXiv:0903.1122 [gr-qc].

[20] B. R. Iyer and C. M. Will, Phys. Rev. Lett. 70 (1993) 113.

[21] L. Blanchet, T. Damour, Annales Poincare Phys. Theor. 50 (1989) 377.

[22] B. Kol, M. Smolkin, Class. Quant. Grav. 25 (2008) 145011 arXiv:0712.4116 [hep-th]; B. Kol, M. Levi, M. Smolkin, Class. Quant. Grav. 28 (2011) 145021. arXiv:1011.6024 [gr-qc].

[23] J. S. Schwinger, J. Math. Phys. 2 (1961) 407-432.

[24] B. DeWitt, Effectice action for expectation values, in Quantum concepts in Space and Time, edited by R. Penrose and C. J. Isham, Clarendon Press, Oxford, Clarendon, 1986.

[25] L. V. Keldysh, Zh. Eksp. Teor. Fiz. 47 (1964) 1515-1527.
[26] We will show that substituting the moment $I_{ij}$ with the quadrupole $Q_{ij} \equiv \sum_{a=1}^{2} m_a (x_{ia} x_{aj} - \delta_{ij} x_a \cdot x_a / 3)$ does not make any difference.