Invariant solutions and bifurcation analysis of the nonlinear transmission line model

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Abstract In this paper, the nonlinear transmission line model with the power law nonlinearity and the constant capacitance and voltage relationship is studied using Lie symmetry analysis. Corresponding to the infinitesimals obtained, using commutation relations, abelian and non-abelian Lie subalgebras are obtained. Also, using the adjoint table, a one-dimensional optimal system of subalgebra is presented. Based on this optimal system, the corresponding Lie symmetry reductions are obtained. Moreover, variety of new similarity solutions in the form of trigonometric functions, hyperbolic functions, are obtained. Corresponding to one similarity reduction, by bifurcation analysis of dynamical system, the stable and unstable regions are determined, which show the existence of soliton solutions from the nonlinear dynamics point of view. Some of the obtained solutions represented graphically and observations are also discussed.

Keywords Nonlinear transmission lines · Power law nonlinearity · Lie symmetry method · Exact solutions · Bifurcation analysis

1 Introduction

The concept of nonlinearity is observed in many areas of physics [1–12]. To understand the nonlinear phenomena, which are described by nonlinear partial differential equations (NLPDEs), we try to obtain their exact analytical solutions if it exists; otherwise, we try to obtain the numerical solutions. Recently, a lot of work has been done to study the exact solutions of these NLPDEs. Some of them include the Lie symmetry method [13,14], Direct method for symmetries [15], non-classical symmetry method [16], Bäcklund transformation method [17,18], solitary wave ansatz method [19,20], Hirota’s bilinear method [21,22], the modified simple equation method [23], the \((G'/G)\) expansion method [24,25] and so on. Some authors also developed some numerical techniques to obtain the approximate solutions of NLPDEs [26–28].

Among all the methods in the literature, the Lie symmetry method [13,14,29] is one of the most effective and powerful methods for finding the travelling and nontravelling wave solutions of NLPDEs. Using Lie symmetries, it is possible to reduce the number of independent variables and further, to reduce the order of ordinary differential equations, thus making them easier to solve. Once the symmetry group is constructed, it can be used to determine new solutions of the NLPDEs from the known ones. Only limitation of this method is that calculations involved are cumbersome. For higher-order NLPDEs, calculations become
more tedious. But nowadays, software like Mathematica, Maple, REDUCE, Maxima and wxMaxima makes it easy to apply this method. Some of the recent work in this field can be seen in [30–33].

In electronics and communication engineering, a transmission line is a cable designed to conduct the currents with a frequency high enough that their wave nature must be taken into account. In this paper, we consider a nonlinear transmission line (NLTL) [34] of the following form:

$$L \frac{\partial}{\partial t} \left[ C(v) \frac{\partial v}{\partial t} \right] = \frac{\partial^2 v}{\partial x^2} + \frac{\delta^2}{12} \frac{\partial^4 v}{\partial x^4}$$

(1.1)

where $C$ and $L$ denote the capacitance (depends on voltage $v$) and inductance, respectively. Afshari et al. [35] proposed and obtained travelling wave and numerical solutions of one- and two-dimensional non-uniform NLTL equations. El-Borai et al. [34] used extended tanh method to obtain the soliton solutions of (1.1) for the relation $C(v) = C_0(1 - bv)$. Mostafa [6] obtained analytical solutions of NLTL equation using improved tanh and sech methods for capacitance-voltage (C-V) relationships $C(v) = C_0(1 - bv)$ and $C(v) = C_0(1 + b_1 v + b_2 v^2)$.

In this work, Lie point symmetry analysis [36,37] of a class of nonlinear transmission line model, with respect to capacitance and voltage relationship, is discussed. Let us approximate the capacitor’s voltage dependence in the form of power law nonlinearity, linear and constant relations as follows:

(i) $C(v) = C_0(1 - bv)^n$, \hspace{1cm} where $b \neq 0$, (1.2)

(ii) $C(v) = C_0$ (1.3)

which reduces Eq. (1.1) to equations

$$(1 - bv)^n \frac{\partial^2 v}{\partial t^2} - nb(1 - bv)^{n-1} \left( \frac{\partial v}{\partial t} \right)^2 = \frac{1}{LC_0} \frac{\partial^2 v}{\partial x^2} + \frac{\delta^2}{12LC_0} \frac{\partial^4 v}{\partial x^4};$$

(1.4)

and

$$\frac{\partial^2 v}{\partial t^2} = \frac{1}{LC_0} \frac{\partial^2 v}{\partial x^2} + \frac{\delta^2}{12LC_0} \frac{\partial^4 v}{\partial x^4};$$

(1.5)

respectively.

As mentioned above, some special cases of NLTL model without power law nonlinearity were already discussed by several authors. The main purpose of this paper is to obtain symmetry reductions and exact solutions to the nonlinear transmission lines for the capacitor’s voltage dependence in the forms (1.2)–(1.3). To the best of our knowledge, in this work, NLTL for power law nonlinearity is first time considered for Lie symmetry analysis and for obtaining the analytical exact solutions. So far, some authors [6,7,34,35] have considered the particular case of NLTL (1.4) for $n = 0, 1$ or $n = 2$ and obtained their travelling wave solutions. We have considered the general form of NLTL with power law nonlinearity and obtained travelling wave as well as non-travelling wave solutions. In the literature, even bifurcation analysis of NLTL with power law nonlinearity has not been performed by any author. So, this shows the novelty and originality of this work.

The paper is organized as follows. Firstly, symmetries of nonlinear transmission lines (1.4)–(1.5) are obtained using the Lie classical method. Then, corresponding to the optimal system, the symmetry reductions and exact solutions of the equations are obtained. Some graphical representations of obtained solutions are also given. Furthermore, phase plane analysis, corresponding to one of similarity reduction, is also performed.

2 Lie symmetry analysis

Let us first consider the Lie group of point transformations

$$x^* = x + \epsilon \xi(x, t, v) + O(\epsilon^2),$$

$$t^* = t + \epsilon \tau(x, t, v) + O(\epsilon^2),$$

$$v^* = v + \epsilon \eta(x, t, v) + O(\epsilon^2),$$

(2.6)

where $\xi$, $\tau$, and $\eta$ are infinitesimals and $\epsilon$ is very small parameter; thus, $\epsilon \ll 1$.

Let

$$X = \xi(x, t, v) \frac{\partial}{\partial x} + \eta(x, t, v) \frac{\partial}{\partial v}$$

(2.7)

be the infinitesimal generator of the Lie group of point transformations (2.6). Let

$$X^{(k)} = \xi(x, t, v) \frac{\partial}{\partial x} + \eta(x, t, v) \frac{\partial}{\partial v} + \eta^{(1)}(x, t, v, \partial v) \frac{\partial}{\partial v} + \cdots + \eta_{12\ldots k}(x, t, v, \partial v, \partial^2 v, \ldots, \partial^k v) \frac{\partial}{\partial v^{12\ldots k}},$$

(2.8)

be the $k$th extended infinitesimal generator of (2.7). Then, the one-parameter Lie group of point transformations (2.6) is admitted by PDE $F(x, t, v, \partial v, \partial^2 v, \ldots, \partial^k v) = 0$ if and only if
\(X^{(k)} F(x, t, v, \partial v, \partial^2 v, \ldots, \partial^k v) = 0\) when \(F(x, t, v, \partial v, \partial^2 v, \ldots, \partial^k v) = 0\).

**Definition 2.1** (Lie Algebra) [38] A Lie algebra, \(\mathcal{L}\), is a vector space over some field with one more operation, commutation \([,\), connecting elements of \(\mathcal{L}\) such that the following properties are satisfied:

(i) If \(X \in \mathcal{L}, Y \in \mathcal{L}\), then \([X, Y] \in \mathcal{L}\).
(ii) \([X, Y] = -[Y, X]\)
(iii) If \(X, Y, Z \in \mathcal{L}\), then \([X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0\)

**Definition 2.2** (Abelian subalgebra) A subalgebra of Lie algebra \(\mathcal{L}\) is called abelian if the commutation vanishes, i.e., \([X, Y] = 0\), for all \(X\) and \(Y\) in subalgebra; otherwise, it is called non-abelian.

### 2.1 Determination of Lie symmetries and optimal system for equation (1.4)

Assuming that Eq. (1.4) is invariant under the transformations (2.6), the following invariance condition is obtained:

\[
(1 - bv)^n \eta^{tt} - bn(1 - bv)^{n-1} v_t \eta - n(n - 1)b^2(1 - bv)^{n-2} v_t^2 \eta - 2nb(1 - bv)^{n-1} v_t \eta' = \frac{1}{LC_0} \eta^{xx} + \frac{b^2}{12LC_0} \eta^{xxxx}, \tag{2.9}
\]

where \(\eta^{tt}, \eta^{xx}, \eta^{xxxx}\) are prolonged infinitesimals acting on an extended space that contains all the derivatives of the dependent variable i.e., \(v_t, v_{xx}, v_{xxxx}\).

The infinitesimal generator of a point symmetry admitted by Eq. (1.4) is of the form:

\[
X = \xi(x, t, v) \frac{\partial}{\partial x} + \tau(x, t, v) \frac{\partial}{\partial t} + \eta(x, t, v) \frac{\partial}{\partial v}. \tag{2.10}
\]

The values of infinitesimals up to the fourth prolongation [38] are given by

\[
\eta^{tt} = D_t(\eta^t) - v_{tt} D_t \xi - v_{ttt} (D_t \tau),
\]

\[
\eta^{xx} = D_x(\eta^x) - v_{xx} D_x \xi - v_{xxt} (D_x \tau),
\]

\[
\eta^{xxxx} = D_{xx}(\eta^{xx}) - v_{xxxx} D_x \xi - v_{xxxxx} (D_x \tau). \tag{2.11}
\]

Therefore, computing and substituting the values of \(\eta^{tt}, \eta^{xx}\) and \(\eta^{xxxx}\) in Eq. (2.9), we get the symmetry determining equations. Solving the determining equations, the following expressions for the infinitesimals \(\xi, \tau, \eta\) are obtained:

\[
\xi(x, t, v) = c_2,
\]

\[
\tau(x, t, v) = c_1 + tc_3,
\]

\[
\eta(x, t, v) = \frac{2}{bn} (bv - 1) c_3, \tag{2.12}
\]

where \(c_1, c_2, c_3\) are arbitrary parameters.

The point symmetry generators admitted by Eq. (1.4) are given by

\[
X_1 = \frac{\partial}{\partial t},
\]

\[
X_2 = \frac{\partial}{\partial x},
\]

\[
X_3 = \frac{t \partial}{\partial t} + \frac{2}{bn} (bv - 1) \frac{\partial}{\partial v}. \tag{2.13}
\]

The commutator table of the generators is given in Table 1.

From the table, it can be seen that the generators form a Lie algebra.

From commutator table, one can see that \(X_1, X_2\) generate abelian Lie subalgebra and \(X_1, X_3\) forms non-abelian Lie subalgebra [39]. The adjoint representation [14] is given by:

\[
\text{Ad}(\exp(\epsilon X_i) X_j) = X_j - \epsilon [X_i, X_j] + \frac{\epsilon^2}{2!} [X_i, [X_i, X_j]] - \ldots
\]

The three-dimensional Lie algebra \(\mathcal{L}^3\) is solvable, and the adjoint table of the generators is given in Table 2.

**Theorem 2.3** An optimal system [14] of Lie algebras is provided by:

(i) \(X_3 + \lambda X_2\) (ii) \(X_2 + \mu X_1\) (iii) \(X_1\).

**Proof** Consider the symmetry algebra of Eq. (1.4) whose adjoint representation is determined in Table 2.

Let us consider the nonzero vector in the form

\[
X = a_1 X_1 + a_2 X_2 + a_3 X_3. \tag{2.14}
\]

We need to simplify this vector field by using suitable adjoint maps. First, assume that \(a_3 \neq 0\) and assume \(a_3 = 1\).

Now if we apply on \(X\) by \(\text{Ad}(\exp(a_3 X_3)) X\), using Table 2, the coefficient of \(X_1\) will vanish as follows:

\[
\text{Ad}(\exp(a_3 X_3)) X = a_2 X_2 + X_3. \tag{2.15}
\]

It cannot be reduced further by applying adjoint operations. So, \(X\) is equivalent to \(a_2 X_2 + X_3\) under the adjoint representation. The remaining subalgebras can be spanned by vectors of the form \(X\) with \(a_3 = 0\). If \(a_2 \neq 0\), we may take \(a_2 = 1\), and then, we have

\[
X' = a_1 X_1 + X_2. \tag{2.16}
\]
We can further act on $X'$ by the group generated by $X_3$, which has the effect of scaling the coefficient of $X_1$:

$$\text{Ad}(e^{X_3})X' = a_1 e^{X_1} + X_2.$$  \hspace{1cm} (2.17)

Thus, any sub-algebra spanned by $X$ with $a_3 = 0$, $a_2 \neq 0$ is equivalent to one spanned by either $a_1 e^{X_1} + X_2$, where $a_1$ is arbitrary and may take values $+1$, $-1$ or $0$. Further, by assuming $a_3 = a_2 = 0$ and $a_1 = 1$, we are left with subalgebra $X_1$.

### 2.2 Determination of Lie symmetries for equation (1.5)

As procedure mentioned above, for Eq. (1.5), we get rich class of symmetries and the infinitesimals $\xi$, $\tau$ and $\eta$ are obtained as follows:

$$\begin{align*}
\xi(x, t, u) &= C_2 \\
\tau(x, t, u) &= C_1 \\
\eta(x, t, u) &= (C_6 t + C_{11}) \cos \left( \frac{2\sqrt{3}}{\delta} x \right) \\
&\quad + (C_5 t + C_{10}) \sin \left( \frac{2\sqrt{3}}{\delta} x \right) \\
&\quad + (C_4 x + C_3) t + C_9 x + C_7 v + C_8 \end{align*}$$ \hspace{1cm} (2.18)

where $C_1, C_2, \ldots, C_{11}$ are arbitrary constants.

The eleven-dimensional symmetry algebra admitted by Eq. (1.5) is given by

$$\begin{align*}
X_1 &= \frac{\partial}{\partial t}, \\
X_2 &= \frac{\partial}{\partial x}, \\
X_3 &= t \frac{\partial}{\partial v}, \\
X_4 &= xt \frac{\partial}{\partial v} \\
X_5 &= t \sin \left( \frac{2\sqrt{3}}{\delta} x \right) \frac{\partial}{\partial v}, \\
X_6 &= t \cos \left( \frac{2\sqrt{3}}{\delta} x \right) \frac{\partial}{\partial v} \\
X_7 &= v \frac{\partial}{\partial v}, \\
X_8 &= \frac{\partial}{\partial v}, \\
X_9 &= x \frac{\partial}{\partial v} \\
X_{10} &= \sin \left( \frac{2\sqrt{3}}{\delta} x \right) \frac{\partial}{\partial v}, \\
X_{11} &= \cos \left( \frac{2\sqrt{3}}{\delta} x \right) \frac{\partial}{\partial v}. \end{align*}$$ \hspace{1cm} (2.19)

In this case, the nonzero commutator relations are obtained as follows:

$$\begin{align*}
[X_1, X_3] &= X_8, \\
[X_1, X_4] &= X_9, \\
[X_1, X_5] &= X_{10}, \\
[X_1, X_6] &= X_{11}, \\
[X_2, X_4] &= X_3, \\
[X_2, X_5] &= \frac{2\sqrt{3}}{\delta} X_6, \\
[X_2, X_6] &= \frac{2\sqrt{3}}{\delta} X_5. \\
[X_2, X_9] &= X_8, \\
[X_2, X_{10}] &= \frac{2\sqrt{3}}{\delta} X_{11}, \\
[X_2, X_{11}] &= \frac{2\sqrt{3}}{\delta} X_{10}. \\
[X_3, X_7] &= X_3, \\
[X_4, X_7] &= X_4, \\
[X_5, X_7] &= X_5, \\
[X_6, X_7] &= X_6, \\
[X_7, X_8] &= -X_8, \\
[X_7, X_9] &= -X_9, \\
[X_7, X_{10}] &= -X_{10}, \\
[X_7, X_{11}] &= -X_{11}. \end{align*}$$ \hspace{1cm} (2.20)

So, Lie algebra generated by vector fields $X_i, i = 1, 2, \ldots, 11$, is non-abelian.

For reduction, we will consider the following linear combinations of vector fields:

$$\begin{align*}
(i) X_1 + \mu X_2 + \lambda X_7, \hspace{1cm} (2.21) \\
(ii) X_1 + \mu X_2 + C_3 X_3 + C_9 X_9, \hspace{1cm} (2.22) \\
(iii) X_1 + \mu X_2 + C_5 X_5 + C_6 X_6 + C_8 X_{10} + C_3 X_{11}. \hspace{1cm} (2.23)
\end{align*}$$

where $\mu, \lambda, C_3, C_5, C_6, C_9$ are arbitrary constants.
3 Symmetry reductions

In this section, the symmetry reductions in nonlinear transmission line Eqs. (1.4)–(1.5) are obtained.

3.1 Similarity reductions for equation (1.4)

As explained in Sect. 2.1, optimal system of vector fields for Eq. (1.4) is:

(i) $X_3 + \lambda X_2$ (ii) $X_2 + \mu X_1$ (iii) $X_1$.

For vector field $X_3 + \lambda X_2$

Corresponding to this vector field, solving the characteristic equation, we have the following similarity variables

\[ s = te^{-\frac{x}{\delta}}, \]

\[ v(x, t) = F(s)e^{\frac{2t}{\delta}} + \frac{1}{B}, \]  

(3.24)

where $s$ is new independent variable and $F$ is new dependent variable.

Using (3.24) in Eq. (1.4), we obtain the following ODE

\[
\begin{aligned}
&\left( - \left( - \frac{1}{n} \delta^2 \lambda^4 F + s \left( \frac{2}{n} - \frac{4}{12} \delta^2 - \frac{2}{n^2} \delta^2 \right) \right) F'' \\
&+ \left( \frac{88^2}{3n^3} - \frac{2\delta^2}{n} + \frac{4\lambda^2}{n} \\
&+ \frac{24^2}{12} \delta^2 \right) F'' \\
&- \frac{\delta^2}{12} \delta^2 F''' + \frac{\delta^2}{2} \left( \frac{2}{3n} - \frac{1}{2} \right) s F''
\end{aligned}
\]

\[ = \frac{4}{n^2} \left( \lambda^2 + \frac{\delta^2}{3n^2} \right) F = 0, \]  

(3.25)

where $'$ denotes the derivative with respect to $s$.

For vector field $X_2 + \mu X_1$

For vector field $X_2 + \mu X_1$, the similarity variables are

\[ \xi = x - \mu t, \]

\[ v(x, t) = G(\xi), \]  

(3.26)

where $\xi$ and $G$ are corresponding new variables.

Using (3.26) in (1.4), we have

\[
\left( (1 - bG) G' \right) - \frac{1}{\mu L C_0} G'' - \frac{\delta^2}{12 \mu^2 L C_0} G''' = 0. \]  

(3.27)

where $'$ denotes the derivative with respect to $\xi$.

For vector field $X_1$

Corresponding to this vector field, the similarity variables are as follows:

\[ \tau = x, \]

\[ v = H(\tau), \]  

(3.28)

where $\tau$ is new independent variable and $H$ is new dependent variable.

Using (3.28) in (1.4), we have

\[ H'' + \frac{\delta^2}{12} H''' = 0, \]  

(3.29)

where $'$ denotes derivative with respect to $\tau$.

3.2 Similarity reduction for equation (1.5)

For vector field $X_1 + \mu X_2 + \lambda X_7$

Solving the characteristic equations for the vector field (2.21), we have the following similarity variables

\[ \xi = x - \mu t, \]

\[ v(x, t) = F(\xi)e^{\lambda t}, \]  

(3.30)

where $\xi$ and $F$ are new independent variable and dependent variable, respectively.

Using (3.30) in (1.5), we have the following ODE

\[
\begin{aligned}
&\left( - \delta^2 \lambda^4 - 12 \lambda^2 F + \left( - 4 \delta^2 \lambda^3 - 24 \lambda \right) \frac{d}{d\xi} F(\xi) \\
&+ (12 C_1 \mu^2 - 6 \delta^2 \lambda^2 - 12) \frac{d^2}{d\xi^2} F(\xi) \\
&- 4 \left( \frac{d^3}{d\xi^3} F(\xi) \right) \delta^2 \lambda - \left( \frac{d^4}{d\xi^4} F(\xi) \right) \delta^2 = 0.
\end{aligned}
\]

(3.31)

For vector field $X_1 + \mu X_2 + C_3 X_3 + C_9 X_9$

For vector field (2.22), we have the following similarity variables of Eq. (1.5)

\[ \xi = x - \mu t, \]

\[ v(x, t) = \frac{C_9 x^2}{2\mu} + \frac{C_2}{2} t^2 + G(\xi). \]  

(3.32)

Using (3.32) in (1.5), we have

\[
\begin{aligned}
&LC_0 \left( C_3 + \left( \frac{d^2}{d\xi^2} G(\xi) \right) \mu^2 \right) \\
&- \frac{C_9}{\mu} - \frac{d^2}{d\xi^2} G(\xi) - \frac{\delta^2}{12} \left( \frac{d^4}{d\xi^4} G(\xi) \right) = 0.
\end{aligned}
\]

(3.33)
For vector field $X_1 + \mu X_2 + C_5 X_5 + C_6 X_6 + C_8 X_{10} + C_7 X_{11}$

For vector field (2.23), we have the following similarity variables of Eq. (1.5)

\[ \xi = x - \mu t, \]
\[ v(x, t) = \frac{\sqrt{3} \delta t}{6} \left( -C_5 \cos \left( \frac{2 \sqrt{3} x}{\delta} \right) \right. \]
\[ + C_6 \sin \left( \frac{2 \sqrt{3} x}{\delta} \right) \left. \right) + G(\xi). \] (3.34)

Using (3.34) into main Eq. (1.5), we have the following ODE

\[ LC_0 \left( \frac{d^2}{d\xi^2} G(\xi) \right) \mu^2 - \frac{\delta^2}{12} \left( \frac{d^4}{d\xi^4} G(\xi) \right) \]
\[ - \frac{d^2}{d\xi^2} G(\xi) = 0. \] (3.35)

4 Exact solutions

In this section, we will obtain the exact solutions of Eqs. (1.4)–(1.5).

4.1 For equation (1.4)

For finding the solutions of (1.4), firstly we will solve Eq. (3.25).

Let us assume the solution of Eq. (3.25) in the following form

\[ F(\sigma) = k \lambda^p, \] (4.36)

where $k$ and $p$ have to be determined.

Using (4.36) in (3.25) and simplifying, we get the following values

\[ p = 1/(1 + n), \]
\[ \lambda = \pm \frac{c(n + 1)\delta}{2\sqrt{3}n(n + 1)}. \] (4.37)

Using (4.36) and (4.37) in (3.24), solution of Eq. (3.25) is given by

\[ v(x, t) = kt \frac{1}{1 + n} \left[ \cos \left( \frac{2 \sqrt{3} x}{\delta} \right) \right. \]
\[ \left. + \xi \sin \left( \frac{2 \sqrt{3} x}{\delta} \right) \right] + \frac{1}{b}. \] (4.38)

For solving Eq. (3.27), substitute

\[ 1 - bG(\sigma) = V(\sigma), \] (4.39)

we obtain the following ODE

\[ \frac{\delta^2}{12} V^{'''} - \mu^2 C_0 L(V^n V')' + V'' = 0. \] (4.40)

Now integrating (4.40) twice and taking constant of integration equal to zero, we have

\[ \frac{\delta^2}{12} V'' - \mu^2 C_0 L \frac{V^{n+1}}{n + 1} + V = 0. \] (4.41)

Balancing the highest derivative term and nonlinear term, we have $m + 2 = (n + 1)m$. So that

\[ m = \frac{2}{n}. \] (4.42)

Clearly $m$ is integer only if $n = 1$ or $n = 2$. Case $n = 1$ will be discussed separately. For all other values of $n$, let us apply the transformation

\[ V(\sigma) = W(\sigma) \frac{1}{\sqrt{n}}. \] (4.43)

Using (4.43), Eq. (4.41) reduces to

\[ n \delta^2 (n + 1) W W'' - 12 \mu^2 C_0 L n^2 W^3 \]
\[ + 12 n^2 (n + 1) W^2 - \delta^2 (n^2 - 1) W'' = 0. \] (4.44)

Now for obtaining solutions corresponding to ODE (4.44), we will apply extended ($\frac{G'}{G}$)-expansion method [24].

4.1.1 Method description

Let us consider a general nonlinear PDE with constant coefficients as follows:

\[ F(u, u_x, u_t, u_{xx}, u_{tt}, u_{xt}, \ldots ) = 0, \] (4.45)

where $u = u(x, t)$ is an unknown function; $F$ is polynomial in $u(x, t)$ and its partial derivatives, in which the highest derivative term and nonlinear terms are involved. The following are main steps involved.

**Step 1.** To reduce Eq. (4.45) into ordinary differential equation (ODE), we consider a variable $\xi$ such that

\[ u(x, t) = u(\xi), \quad \xi = x - \mu t, \] (4.46)

where $\mu$ is speed of travelling wave.

Using (4.46) into (4.45), we have

\[ F(u, u', u'', \ldots ) = 0, \] (4.47)

where $'$ denotes derivative with respect to $\xi$. 
Step 3. Let us consider the solution of Eq. (4.47) in the following form:

\[
\begin{aligned}
\frac{u(\xi)}{G} &= a_0 + \sum_{i=0}^{m} \left[ a_i \left( \frac{G'}{G} \right)^i + b_i \left( \frac{G'}{G} \right)^{i-1} \right. \\
&\quad \left. \sqrt{\rho \left( 1 + \frac{1}{\mu_1} \left( \frac{G'}{G} \right)^2 \right)} + c_i \left( \frac{G'}{G} \right)^{-i} \right]
\end{aligned}
\]

(4.48)

where \( a_i, b_i, c_i, d_i (i = 1, \ldots, m) \) are arbitrary constants to be determined later, \( \rho = \pm 1, m \) is positive integer, and \( G = G(\xi) \) satisfies the following ODE \( G'' + \mu_1 G = 0 \),

(4.49)

where \( \mu_1 \) is to be determined.

Step 4. Now determine the value of \( m \) by homogeneous balance between the highest derivative term and highest nonlinear term for Eq. (4.47).

Step 5. Substitute (4.48) with (4.49) into Eq. (4.47), we have a equation in powers of \( \left( \frac{G'}{G} \right) \) and \( \left( \frac{G'}{G} \right)^k \)

\[
\begin{aligned}
\sqrt{\sigma \left( 1 + \frac{1}{\mu_1} \left( \frac{G'}{G} \right)^2 \right)} \end{aligned}
\]

Equating the coefficients of different powers of \( \left( \frac{G'}{G} \right) \) and \( \left( \frac{G'}{G} \right)^k \) equal to zero, yield system of algebraic equations. Solving this system of equation, we obtain values of parameters \( \mu, \mu_1, a_1, b_1, c_1, d_1 \) \((i = 1, \ldots, m)\).

Step 6. Substituting the values of different parameters \( \mu, \mu_1, a_1, b_1, c_1, d_1 \) \((i = 1, \ldots, m)\) with solutions of Eq. (4.49) into (4.48), we obtain different travelling wave solutions of Eq. (4.45).

Using the general solution of Eq. (4.49) in (4.48), we have the following values of \( \frac{G'}{G} \):

Family 1. When \( \mu_1 < 0 \), then we have hyperbolic type solutions as follows:

\[
\frac{G'}{G} = \sqrt{-\mu_1} \left( C_1 \sinh (\sqrt{-\mu_1} \xi) + C_2 \cosh (\sqrt{-\mu_1} \xi) \right)
\]

(4.50)

where \( C_1 \) and \( C_2 \) are two arbitrary constants.

Family 2. When \( \mu_1 > 0 \), then we have trigonometric type solutions as follows:

\[
\frac{G'}{G} = \sqrt{\mu_1} \left( C_1 \cos (\sqrt{\mu_1} \xi) - C_2 \sin (\sqrt{\mu_1} \xi) \right)
\]

(4.51)

where \( C_1 \) and \( C_2 \) are two arbitrary constants.

Family 3. If \( \mu_1 = 0 \), then we have rational type solutions as follows:

\[
H(\xi) = \frac{C_1}{C_2 + C_1 \xi}
\]

(4.52)

where \( C_1 \) and \( C_2 \) are two arbitrary constants.

Similar type of solutions for the nonlinear Schrödinger equation in the form Kuznetsov breather, the Akhmediev breather, and the Peregrine solution has also been obtained by some of the authors [40–64].

Now we will apply the method to ODE (4.44). Balancing the highest order derivative term and nonlinear term in Eq. (4.44), we have \( m = 2 \).

So, as per algorithm for the extended \( \left( \frac{G'}{G} \right) \) expansion method, value of solution \( W(\sigma) \) of Eq. (4.44) is given by

\[
W(\sigma) = a_0 + \sum_{i=0}^{2} \left[ a_i \left( \frac{G'}{G} \right)^i \right.
\]

\[
\left. + b_i \left( \frac{G'}{G} \right)^{i-1} \sqrt{\rho \left( 1 + \frac{1}{\mu_1} \left( \frac{G'}{G} \right)^2 \right)} + c_i \left( \frac{G'}{G} \right)^{-i} \right]
\]

(4.53)

where \( G \) satisfy Eq. (4.49) with \( \xi \) replaced by \( \sigma \), \( \rho = \pm 1 \) and \( a_0, a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 \) are to be determined.

Substituting (4.53) in (4.44) with (4.49) and taking coefficients of \( \left( \frac{G'}{G} \right) \) and \( \left( \frac{G'}{G} \right)^k \) \sqrt{\rho \left( 1 + \frac{1}{\mu_1} \left( \frac{G'}{G} \right)^2 \right)}

equal to zero yields a set of algebraic equations in \( a_0, a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, \mu_1, \mu_2 \) and \( \mu_1 \). Solving the system of algebraic equations using the software Maple, we have the following results:

Set 1:

\[
\mu = \frac{3a^2}{\delta^2 \mu_1}, a_0 = \frac{(n^2+3n+2)\delta^4 \mu_1^2}{18\delta^2 + LC_0}, c_2 = \frac{\delta^4 \mu_1^3(n^2+3n+2)}{18\delta^2 + LC_0},
\]

\[
a_1 = a_2 = b_1 = b_2 = c_1 = c_2 = d_1 = d_2 = 0, \text{ and } \mu_1 \text{ is arbitrary,}
\]

Set 2:

\[
\mu = \frac{3a^2}{\delta^2 \mu_1}, a_0 = \frac{(n^2+3n+2)\delta^4 \mu_1^2}{18\delta^2 + LC_0}, a_2 = \frac{(n^2+3n+2)\delta^4 \mu_1}{18\delta^2 + LC_0},
\]

\[
a_1 = b_1 = b_2 = c_1 = c_2 = d_1 = d_2 = 0, \text{ and } \mu_1 \text{ is arbitrary,}
\]
Set 3:

\[ \mu = \frac{3n^2}{4d^2\mu_1}, \quad a_0 = \frac{4(n^2+3n+2)d^2\mu_1^2}{9n^4LC_0}, \quad d_2 = \frac{2(n^2+3n+2)d^2\mu_1}{9n^4LC_0}, \]
\[ c_2 = \frac{2d^2\mu_1}{9n^4LC_0}, \quad a_1 = b_1 = b_2 = c_1 = d_1 = d_2 = 0 \]
and \( \mu_1 \) is arbitrary.

Set 4:

\[ \mu = \frac{12n^2}{d^2\mu_1}, \quad a_0 = \frac{(n^2+3n+2)d^2\mu_1^2}{144n^4LC_0}, \quad d_2 = \frac{(n+1)d^2\mu_1(n+2)}{144n^4LC_0}, \]
\[ b_2 = \pm \frac{(n+1)d^2\mu_1(n+2)}{144n^4LC_0}, \quad a_1 = b_1 = c_1 = c_2 = d_1 = d_2 = 0 \]

One can see that \( \mu_1 \) is arbitrary in all Sets of values of \( a_0, a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 \) and \( \mu \). So, on depending upon the values of \( \mu_1 \), three cases of solutions can be derived as follows:

(1) Hyperbolic solutions (if \( \mu_1 < 0 \))

If \( \mu_1 < 0 \) in sets 1, 2, 3 and 4, then using Eq. (4.49) and transformation (4.43), we obtain the following solutions of ODE (4.41):

Set 1 and Set 2 give:

\[
V(\sigma) = \left\{ \frac{(n+1)(n+2)\mu_1^2}{18n^4LC_0} \left( 1 - \frac{C_1 \sinh(\sqrt{\mu_1} \sigma) + C_2 \cosh(\sqrt{\mu_1} \sigma)}{C_1 \cosh(\sqrt{\mu_1} \sigma) + C_2 \sinh(\sqrt{\mu_1} \sigma)} \right)^2 \right\}^{\frac{1}{2}}. \tag{4.54}
\]

Set 3 gives:

\[
V(\sigma) = \left\{ \frac{2d^2(n+2)(n+1)\mu_1^2}{9C_0Ln^2} \left( 1 - \frac{C_1 \cosh(\sqrt{\mu_1} \sigma) + C_2 \sinh(\sqrt{\mu_1} \sigma)}{C_1 \sinh(\sqrt{\mu_1} \sigma) + C_2 \cosh(\sqrt{\mu_1} \sigma)} \right)^2 \left( 2 - \frac{C_1 \cosh(\sqrt{\mu_1} \sigma) + C_2 \sinh(\sqrt{\mu_1} \sigma)}{C_1 \sinh(\sqrt{\mu_1} \sigma) + C_2 \cosh(\sqrt{\mu_1} \sigma)} \right)^2 \left( C_1 \cosh(\sqrt{\mu_1} \sigma) + C_2 \sinh(\sqrt{\mu_1} \sigma) \right)^2 \right\}^{\frac{1}{2}}. \tag{4.55}
\]

Set 4 gives:

\[
V(\sigma) = \left\{ a_0 \left( 1 - \frac{C_1 \sinh(\sqrt{\mu_1} \sigma) + C_2 \cosh(\sqrt{\mu_1} \sigma)}{C_1 \cosh(\sqrt{\mu_1} \sigma) + C_2 \sinh(\sqrt{\mu_1} \sigma)} \right)^2 \pm \frac{C_1 \cosh(\sqrt{\mu_1} \sigma) + C_2 \sinh(\sqrt{\mu_1} \sigma)}{C_1 \sinh(\sqrt{\mu_1} \sigma) + C_2 \cosh(\sqrt{\mu_1} \sigma)} \right\} \left\{ a_0 \left( -1 + \frac{C_1 \cosh(\sqrt{\mu_1} \sigma) + C_2 \sinh(\sqrt{\mu_1} \sigma)}{C_1 \sinh(\sqrt{\mu_1} \sigma) + C_2 \cosh(\sqrt{\mu_1} \sigma)} \right)^2 \right\}^{\frac{1}{2}}. \tag{4.56}
\]

where \( a_0 \) is given by set 4, \( \mu_1 \) is arbitrary negative constant and \( \sigma = x - \frac{12n^2}{d^2\mu_1}t \).

Using the transformations (3.26) and (4.39), with above obtained solutions, the corresponding solutions of main Eq. (1.4) can be obtained.

(II) Trigonometric solutions (if \( \mu_1 > 0 \))

If \( \mu_1 > 0 \) in sets 1, 2, 3 and 4, then using Eq. (4.49) and transformation (4.43), we obtain the following solutions of ODE (4.41):

Set 1 and Set 2 give:

\[
V(\sigma) = \left\{ \frac{(n+1)(n+2)\mu_1^2}{18n^4LC_0} \left( 1 + \frac{C_1 \sin(\sqrt{\mu_1} \sigma) + C_2 \cos(\sqrt{\mu_1} \sigma)}{C_1 \cos(\sqrt{\mu_1} \sigma) - C_2 \sin(\sqrt{\mu_1} \sigma)} \right)^2 \right\}^{\frac{1}{2}}. \tag{4.57}
\]

Set 3 gives:

\[
V(\sigma) = \left\{ \frac{2d^2(n+2)(n+1)\mu_1^2}{9C_0Ln^2} \left( 2 + \frac{C_1 \cos(\sqrt{\mu_1} \sigma) - C_2 \sin(\sqrt{\mu_1} \sigma)}{C_1 \sin(\sqrt{\mu_1} \sigma) + C_2 \cos(\sqrt{\mu_1} \sigma)} \right)^2 \right\}^{\frac{1}{2}}. \tag{4.58}
\]

Set 4 gives:

\[
V(\sigma) = \left\{ a_0 \left( 1 + \frac{C_1 \sin(\sqrt{\mu_1} \sigma) + C_2 \cos(\sqrt{\mu_1} \sigma)}{C_1 \cos(\sqrt{\mu_1} \sigma) - C_2 \sin(\sqrt{\mu_1} \sigma)} \right)^2 \right\}^{\frac{1}{2}}. \tag{4.59}
\]

where \( a_0 \) is given by set 4, \( \mu_1 \) is arbitrary negative constant, and \( \sigma = x - \frac{12n^2}{d^2\mu_1}t \).

Using the transformations (3.26) and (4.39), with the above obtained solutions, corresponding solutions of main Eq. (1.4) can be obtained.

(III) Rational solutions (if \( \mu_1 = 0 \))

For rational solutions, put \( \mu_1 = 0 \), and also taking \( \mu = 0 \), in sets 1, 2, 3 and 4, we obtain trivial solution of ODE (4.41).

Using the transformations (3.26) and (4.39), with the above obtained trivial solution, corresponding constant solution of main Eq. (1.4) can be obtained as follows:

\[
v(x, t) = \frac{1}{b}. \tag{4.60}
\]
**Particular cases**

For particular values of $C_1$ and $C_2$ in aforementioned solutions, soliton, periodic, and complex solutions can be obtained when parameters take special values as follows:

(i) Soliton solutions

Taking $C_2 = 0$ in (4.54), we obtain the following solitary wave solution

$$V(\sigma) = \left\{ \frac{(n + 1)(n + 2)\mu_1^2\delta^4}{18n^4LC_0} \text{sech}^2\left(\sqrt{\mu_1}\sigma\right) \right\}^{\frac{1}{2}}. \quad (4.61)$$

where $\sigma = x - \frac{12n^2}{\delta^2\mu_1} t$.

Solution is represented by Fig. 1. Taking $C_1 = 0$ in (4.54), the following solitary wave solution is obtained

$$V(\sigma) = \left\{ -\frac{(n + 1)(n + 2)\mu_1^2\delta^4}{18n^4LC_0} \text{csch}^2\left(\sqrt{\mu_1}\sigma\right) \right\}^{\frac{1}{2}}. \quad (4.62)$$

Taking either $C_1 = 0$ or $C_2 = 0$ in (4.55), we have the following solitary wave solution

$$V(\sigma) = \left\{ \frac{(n + 1)(n + 2)\mu_1^2\delta^4}{18n^4LC_0} \left(\text{sech}^2\left(\sqrt{\mu_1}\sigma\right) - \text{csch}^2\left(\sqrt{\mu_1}\sigma\right)\right) \right\}^{\frac{1}{2}}. \quad (4.63)$$

Now taking $C_1 = 0$ in (4.56), we have

$$V(\sigma) = \left\{ a_0 \text{csch}^2\left(\sqrt{\mu_1}\sigma\right) \left(1 \pm \cosh\left(\sqrt{\mu_1}\sigma\right)\right) \right\}^{\frac{1}{2}}. \quad (4.64)$$

(ii) Periodic solutions

Setting $C_1 = 0$ in (4.57), we have

$$V(\sigma) = \left\{ \frac{(n + 1)(n + 2)\mu_1^2\delta^4}{18n^4LC_0} \text{csc}^2\left(\sqrt{\mu_1}\sigma\right) \right\}^{\frac{1}{2}}. \quad (4.65)$$

Setting $C_2 = 0$ in (4.57), we have

$$V(\sigma) = \left\{ \frac{(n + 1)(n + 2)\mu_1^2\delta^4}{18n^4LC_0} \sec^2\left(\sqrt{\mu_1}\sigma\right) \right\}^{\frac{1}{2}}. \quad (4.66)$$

Taking either $C_1 = 0$ or $C_2 = 0$ in (4.58), we have the following solitary wave solution

$$V(\sigma) = \left\{ \frac{(n + 1)(n + 2)\mu_1^2\delta^4}{18n^4LC_0} \left(\sec^2\left(\sqrt{\mu_1}\sigma\right) + \text{csc}^2\left(\sqrt{\mu_1}\sigma\right)\right) \right\}^{\frac{1}{2}}. \quad (4.67)$$

Taking $C_1 = 0$ in (4.59), we have

$$V(\sigma) = \left\{ a_0 \text{csc}^2\left(\sqrt{\mu_1}\sigma\right) \left(1 \pm \cos\left(\sqrt{\mu_1}\sigma\right)\right) \right\}^{\frac{1}{2}}. \quad (4.68)$$

Taking $C_2 = 0$ in (4.59), we have

$$V(\sigma) = \left\{ a_0 \text{sec}^2\left(\sqrt{\mu_1}\sigma\right) \left(1 \pm \sin\left(\sqrt{\mu_1}\sigma\right)\right) \right\}^{\frac{1}{2}}. \quad (4.69)$$

(iii) Complex solutions

Now taking $C_2 = 0$ in (4.56), we have

$$V(\sigma) = \left\{ a_0 \text{sech}^2\left(\sqrt{-\mu_1}\sigma\right) \left(1 \pm \sinh\left(\sqrt{-\mu_1}\sigma\right)\right) \right\}^{\frac{1}{2}}. \quad (4.70)$$

Now solution of Eq. (3.29) is given by

$$H(\tau) = C_1 + C_2 \tau + C_3 \sin\left(\frac{2\sqrt{3}\tau}{\delta}\right) + C_4 \cos\left(\frac{2\sqrt{3}\tau}{\delta}\right). \quad (4.71)$$

Corresponding solution of main Eq. (1.4) is given by

$$v(x, t) = C_1 + C_2 x + C_3 \sin\left(\frac{2\sqrt{3}x}{\delta}\right). \quad (4.72)$$
where $r_i, i = 0, 1, 2, 3$ are roots of the following polynomial in $m$

\[-\delta^2 \lambda^4 - 12 \lambda^2 + \left(-4 \delta^2 \lambda^3 - 24 \lambda\right) m^2 - \delta 2 \lambda^2 m^4 + 4 m^3 \delta^2 \lambda - m^4 \delta^2 = 0.
\] (4.74)

Corresponding solution of Eq. (1.5) can be given as

\[v(x, t) = \sum_{i=0}^{3} e^{r_i(x-\mu t)+3\lambda t},\] (4.75)

where $r_i, i = 0, 1, 2, 3$ are roots of Eq. (4.74).

Solution of (3.33) is obtained as

\[G(\xi) = -1/4 \frac{A_2 \delta^2}{-3 \lambda C_0 \mu^2 + 3} \sin\left(2 \frac{\sqrt{-3 \lambda C_0 \mu^2 + 3} \xi}{\delta}\right)
-1/4 \frac{A_1 \delta^2}{-3 \lambda C_0 \mu^2 + 3} \cos\left(2 \frac{\sqrt{-3 \lambda C_0 \mu^2 + 3} \xi}{\delta}\right)
+1/2 \frac{-C_0 (\mu C_3 + C_9) \xi^2}{\mu (L \lambda C_0 \mu^2 - 1)} + A_3 \xi + A_4.
\] (4.76)

where $A_1, A_2, A_3$ and $A_4$ are arbitrary constants.

Using (3.32) and (4.76), solution of Eq. (1.5) is given as

\[v(x, t) = \frac{C_0 x^2}{2 \mu} + \frac{C_3}{2} t^2 - \frac{A_2 \delta^2}{-12 \lambda C_0 \mu^2 + 12} \sin\left(2 \frac{\sqrt{-3 \lambda C_0 \mu^2 + 3} \xi}{\delta}\right)
- \frac{A_1 \delta^2}{-12 \lambda C_0 \mu^2 + 12} \cos\left(2 \frac{\sqrt{-3 \lambda C_0 \mu^2 + 3} \xi}{\delta}\right)
+ \frac{(-C_0 L \mu C_3 + C_9) (x - \mu t)^2}{2 \mu (L \lambda C_0 \mu^2 - 1)}
+A_3 (x - \mu t) + A_4.
\] (4.77)

Observation

One can observe that:

(i) As $\xi \rightarrow \pm \infty$, graph of solution is parabolic as shown in Fig. 2(a).

(ii) For transient value of $\xi$, solution is represented by bell shaped solutions as shown in Fig. 2(b).

Now, let us obtain the solution of ODE (3.35). Using Maple, we obtain the following solution of the ODE (3.35)

\[G(\xi) = C_1 + C_2 \xi + C_3 \sin\left(2 \frac{\sqrt{-3 \lambda C_0 \mu^2 + 3} \xi}{\delta}\right)
+ C_4 \cos\left(2 \frac{\sqrt{-3 \lambda C_0 \mu^2 + 3} \xi}{\delta}\right).
\] (4.78)

Corresponding solution of (1.5) is given by

\[v(x, t) = \frac{\sqrt{3} \delta t}{6} \left(-C_5 \cos\left(2 \frac{\sqrt{3} \xi}{\delta}\right)
+ C_6 \sin\left(2 \frac{\sqrt{3} \xi}{\delta}\right)\right) + C_1
+ C_2 (x - \mu t) + C_3 \sin\left(2 \frac{\sqrt{-3 \lambda C_0 \mu^2 + 3} (x - \mu t)}{\delta}\right)
+ C_4 \cos\left(2 \frac{\sqrt{-3 \lambda C_0 \mu^2 + 3} (x - \mu t)}{\delta}\right).
\] (4.79)

Solution (4.78) is represented in Fig. 3.

Observation

One can observe that:

(i) With taking coefficient of $\xi$ nonzero, the graph of the solution is parabolic as shown in Fig. 3(a).

(ii) For taking coefficient of $\xi$ equal to zero, the graph of the solution is periodic as shown in Fig. 3(b).

(iii) Also, with the increasing the value of $\mu$, the amplitude of travelling wave remains same, but width increases as shown in Fig. 3(b).

5 Bifurcation analysis

In this section, the transmission line model with power law nonlinearity will be studied using the dynamical system approach. The phase portraits will be displayed.
Invariant solutions and bifurcation analysis

(a) As $\xi \to \pm \infty$

(b) For transient value of $\xi$

Fig. 2 For $A_1 = A_2 = C_0 = C_3 = \delta = C_0 = 1, L = 1.2, \mu = 0.8$, graphs of the solutions (4.76)

(a) Quasi periodic solution

(b) Periodic solution

Fig. 3 Graphs of the solution (4.78) a For $A_1 = A_2 = A_3 = A_4 = \delta = C_0 = C_1 = C_2 = C_3 = C_4 = 1, L = 1.2, \mu = 0.8$ b For $A_1 = A_3 = A_4 = \delta = C_0 = C_1 = C_3 = C_4 = 1, C_2 = 0, A_2 = 0, L = 1.2$ and $\mu = 0.5, \mu = 0.8$ are shown corresponding to red, blue colors, respectively
5.1 Bifurcation phase portraits and qualitative analysis

Introducing the notation \( X = V, Y = V' \), let us reduce Eq. (4.41) to the autonomous system:

\[
X' = Y \\
Y' = \frac{a_1}{a_0} X^{n+1} - \frac{1}{a_0} X, \tag{5.80}
\]

where \( a_0 = \frac{\delta^2}{\tau^2} \) and \( a_1 = \frac{\mu^2 C_0 L}{a_{n+1}} \).

The system (5.80) is a Hamiltonian system with the following Hamiltonian function:

\[
H(X, Y) = Y^2 - \frac{2a_1}{a_0(n+2)} X^{n+2} + \frac{1}{a_0} X^2. \tag{5.81}
\]

If the system is not Hamiltonian, then there is no closed orbit in the phase portrait. So, in that case, the system will not have solitary and periodic wave solutions. For obtaining the phase portrait of (5.80), set

\[
f(X) = \frac{a_1}{a_0} X^{n+1} - \frac{1}{a_0} X. \tag{5.82}
\]

When \( n \) is even number and \( a_1 > 0 \), \( f(X) \) has three zeros, \( X_-, X_0 \) and \( X_+ \) given as follows:

\[
X_- = -\left(\frac{1}{a_1}\right)^\frac{1}{n}, \quad X_0 = 0, \quad X_+ = \left(\frac{1}{a_1}\right)^\frac{1}{n}. \tag{5.83}
\]

When \( n \) is odd number, \( f(X) \) has two zeros, \( X_0 \) and \( X_* \) given as follows:

\[
X_0 = 0, \quad X_* = \left(\frac{1}{a_1}\right)^\frac{1}{n}. \tag{5.84}
\]

Let \( (X_i, 0) \) be the critical point of (5.80), then the corresponding characteristic values of linearized system (5.80) at the singular points \( (X_i, 0) \) are given by

\[
\lambda_{\pm} = \pm \sqrt{f'(X_i)}. \tag{5.85}
\]

Using the qualitative theory of dynamical system, we know that

1. If \( f'(X_i) > 0 \), \( (X_i, 0) \) is a saddle point.
2. If \( f'(X_i) < 0 \), \( (X_i, 0) \) is a center point.
3. If \( f'(X_i) = 0 \), \( (X_i, 0) \) is a degenerate saddle point.

So, the bifurcation phase portraits of the system (5.80) are obtained as shown in Figs. 4 and 5.

If \( n \) is even and \( na_0 > 0 \), \( (0, 0) \) is center point and \( -\left(\frac{1}{a_1}\right)^\frac{1}{n}, 0 \), \( \left(\frac{1}{a_1}\right)^\frac{1}{n}, 0 \) are unstable saddle points for system (5.80) as shown in Fig. 4a. For \( n \) odd and \( na_0 > 0 \), \( (0, 0) \) is center point and \( \left(\frac{1}{a_1}\right)^\frac{1}{n}, 0 \) is unstable saddle point for system (5.80) as shown in Fig. 4b.

For even \( n \) and \( na_0 < 0 \), \( (0, 0) \) is stable point and \( -\left(\frac{1}{a_1}\right)^\frac{1}{n}, 0 \), \( \left(\frac{1}{a_1}\right)^\frac{1}{n}, 0 \) are center points.

For odd \( n \) and \( na_0 < 0 \), \( (0, 0) \) is saddle point and \( \left(\frac{1}{a_1}\right)^\frac{1}{n}, 0 \) is center point.

Let us consider

\[
H(X, Y) = h, \tag{5.86}
\]

where \( h \) is Hamiltonian. Now, we will study the relation between the system (5.80) and the Hamiltonian \( h \).

Set

\[
h* = |H(X_+, 0)| = |H(X_-, 0)| = |H(X_*, 0)|. \tag{5.87}
\]

From Fig. 4, we have the following results.

**Proposition 5.1** When \( n \) is even number and \( na_0 > 0 \), we have

(I) When \( h = 0 \), system (5.80) has two periodic orbits \( P1 \) and \( P1^* \).

(II) When \( 0 < h < h^* \), system (5.80) has three periodic orbits \( P2, P2^* \) and \( P2^{**} \).

(III) When \( h = h^* \), system (5.80) has two heteroclinic orbits \( P3 \) and \( P3^* \).

(IV) When \( h < 0 \) or \( h > h^* \), system (5.80) does not have close orbit.

**Proposition 5.2** When \( n \) is odd number and \( na_0 > 0 \), we have

(I) When \( h = h^* \), system (5.80) has a homoclinic orbit \( P4 \).

(II) When \( 0 < h < h^* \), system (5.80) has two periodic orbits \( P5 \) and \( P5^* \).

(III) When \( h = 0 \), system (5.80) has a periodic orbit \( P6 \).

(IV) When \( h < 0 \) or \( h > h^* \), system (5.80) does not have any close orbit.

Now according to Fig. 5, we have the following propositions.

**Proposition 5.3** When \( n \) is even number and \( na_0 < 0 \), we have

(I) When \( h > 0 \), system (5.80) has a periodic orbit \( P7 \).

(II) When \( h = 0 \), system (5.80) has two homoclinic orbits \( P8 \) and \( P8^* \).
Fig. 4  a For even $n$ and $n a_0 > 0$, $(0, 0)$ is center point and $\left(-\left(\frac{1}{a_0}\right)^{\frac{1}{n}}, 0\right)$, $\left(\left(\frac{1}{n}\right)^{\frac{1}{n}}, 0\right)$ are unstable saddle points. b For odd $n$ and $n a_0 > 0$, $(0, 0)$ is center point and $\left(-\left(\frac{1}{a_0}\right)^{\frac{1}{n}}, 0\right)$ is unstable saddle point.

Fig. 5  a For even $n$ and $n a_0 < 0$, $(0, 0)$ is saddle point and $\left(-\left(\frac{1}{a_0}\right)^{\frac{1}{n}}, 0\right)$, $\left(\left(\frac{1}{n}\right)^{\frac{1}{n}}, 0\right)$ are center points. b For odd $n$ and $n a_0 < 0$, $(0, 0)$ is saddle point and $\left(\left(\frac{1}{n}\right)^{\frac{1}{n}}, 0\right)$ is center point.
(III) When \(-h^* < h < 0\), system (5.80) has two periodic orbits P9 and P9*.
(IV) When \(h \leq -h^*\), system (5.80) does not have any close orbit.

Proposition 5.5 When \(n\) is even number and \(na_0 > 0\), we have

(I) For \(h = 0\), (1.4) has four periodic blow up wave solutions corresponding to the orbits P1 and P1* in Fig. 4a.

(II) For \(0 < h < h^*\), (1.4) has four periodic blow up wave solutions and a periodic wave solution corresponding to the orbits P2, P2* and P2** in Fig. 4a.

(III) When \(h = h^*\), (1.4) has two kink solitary wave solutions and two unbounded wave solutions corresponding to orbits P3 and P3* in Fig. 4a.

Proposition 5.6 When \(n\) is odd number and \(na_0 > 0\), we have

(I) When \(h = h^*\), (1.4) has a solitary wave solution corresponding to the homoclinic orbit P4 in Fig. 4b.

(II) When \(0 < h < h^*\), (1.4) has a periodic wave solution and two periodic blow up wave solutions corresponding to the two periodic orbits P5 and P5* in Fig. 4b.

(III) When \(h = 0\), (1.4) has two periodic blow up wave solutions corresponding to a periodic orbit P6 in Fig. 4b.

Now according to Fig. 5, we have the following propositions.

Proposition 5.7 When \(n\) is even number and \(na_0 < 0\), we have

(I) When \(h > 0\), (1.4) has two periodic wave solutions corresponding to the periodic orbits P7 in Fig. 5a.

(II) When \(h = 0\), (1.4) has two solitary wave solutions corresponding to the homoclinic orbits P8 and P8* in Fig. 5a.

(III) When \(-h^* < h < 0\), (1.4) has periodic wave solutions corresponding to the periodic orbits P9 and P9* in Fig. 5a.

Proposition 5.8 When \(n\) is odd number and \(na_0 < 0\), we have

(I) When \(h = 0\), (1.4) has a solitary wave solution corresponding to the homoclinic orbit P10 in Fig. 5b.

(II) When \(-h^* < h < 0\), (1.4) has a periodic wave solution and two periodic blow up wave solutions corresponding to the periodic orbits P11 and P11* in Fig. 5b.

(III) When \(h = -h^*\), (1.4) has two periodic blow up wave solutions corresponding to the periodic orbit P12 in Fig. 5b.

5.2 Travelling wave solutions

Firstly, let us consider the case when \(n\) is even and \(na_0 > 0\), we obtain the explicit expressions for travelling wave solutions for (1.4). From phase portrait, we have two special orbits P1 and P1* which have the same value of Hamiltonian as that of center point (0, 0). In \((X, Y)\) plane, the expressions of the orbits are given by

\[
Y = \pm \sqrt[2]{\frac{2a_1}{a_0(n+2)}X^n+2 - \frac{1}{a_0}X^2}. \tag{5.88}
\]

Substituting (5.94) into \(\frac{dX}{d\sigma} = Y\) and integrating along P1 and P2, we have

\[
\pm \int_{X_1}^X \frac{1}{\sqrt[2]{\frac{2a_1}{a_0(n+2)}s^n+2 - \frac{1}{a_0}s^2}} ds = \int_0^\sigma d\sigma,
\]
\[
\pm \int_{X_i}^{\infty} \frac{1}{\sqrt{a_0(n+2)X^{n+2} - \frac{1}{a_0}X^2}} dX = \int_0^\sigma d\sigma, \quad (5.89)
\]

where \(i = 1, 2\).

Evaluating the integrals, we have

\[
X = \pm \left( \frac{n+2}{2a_1} \sec^2 \left( \frac{n}{2\sqrt{a_0}} \sigma \right) \right)^{\frac{1}{n}}, \quad (5.90)
\]

\[
X = \pm \left( \frac{n+2}{2a_1} \csc^2 \left( \frac{n}{2\sqrt{a_0}} \sigma \right) \right)^{\frac{1}{n}}. \quad (5.91)
\]

Using the above expressions with \(V = X\) and Eqs. (3.26), (4.39), the solutions of main Eq. (1.4) are given by

\[
v = \frac{1}{b} \left[ 1 \mp \left( \frac{n+2}{2a_1} \sec^2 \left( \frac{n}{2\sqrt{a_0}} (x - \mu t) \right) \right)^{\frac{1}{n}} \right]. \quad (5.92)
\]

\[
v = \frac{1}{b} \left[ 1 \mp \left( \frac{n+2}{2a_1} \csc^2 \left( \frac{n}{2\sqrt{a_0}} (x - \mu t) \right) \right)^{\frac{1}{n}} \right]. \quad (5.93)
\]

where \(a_0 = \frac{\delta^2}{12}\) and \(a_1 = \frac{\mu^2C_0L}{n+1}\).

Now, on considering the case when \(n\) is even and \(na_0 < 0\), we obtain the explicit expressions for traveling wave solutions for (1.4). Again from phase portrait, we have two homoclinic orbits \(P8\) and \(P8^*\) intersecting at the saddle point \((0, 0)\). In \((X, Y)\) plane, the expressions of the orbits are given by

\[
Y = \pm \sqrt{\frac{2a_1}{a_0(n+2)}} X^{n+2} - \frac{1}{a_0} X^2. \quad (5.94)
\]

Substituting (5.94) into \(\frac{dX}{d\sigma} = Y\) and integrating along orbits \(P8\) and \(P8^*\), we have

\[
\pm \int_{X_i}^{\infty} \frac{1}{\sqrt{a_0(n+2)X^{n+2} - \frac{1}{a_0}X^2}} dX = \int_0^\sigma d\sigma, \quad (5.95)
\]

where \(i = 3, 4\).

Evaluating the integrals, we have

\[
X = \pm \left( \frac{n+2}{a_1 (1 - \cosh(\frac{n}{\sqrt{a_0}} \sigma))) \right)^{\frac{1}{n}}, \quad (5.96)
\]

Using the above expressions with \(V = X\) and Eqs. (3.26), (4.39), the singular solitary wave solutions of main Eq. (1.4) are given by

\[
v = \pm \left( \frac{n+2}{a_1 (1 - \cosh(\frac{n}{\sqrt{a_0}} (x - \mu t))) \right)^{\frac{1}{n}}, \quad (5.97)
\]

where \(a_0 = \frac{\delta^2}{12}\) and \(a_1 = \frac{\mu^2C_0L}{n+1}\).

### 6 Conclusion

In this work, nonlinear transmission line model (NLTL) with arbitrary power law nonlinearity and the constant capacitor voltage nonlinearity has been studied through application of Lie symmetry analysis. For both the cases, i.e., \(C(v) = C_0(1 - bv)\) and \(C(v) = C_0\) of NLTL, infinitesimal symmetries are obtained and partial differential equations are reduced to ordinary differential equations (ODEs). Furthermore, variety of new exact solutions in the form of trigonometric functions, hyperbolic functions solutions, are also obtained.

Additionally, corresponding to one reduction in NLTL with power law nonlinearity, phase plane analysis is carried out. This results in various propositions where various solution existence structures are examined that depend upon the parameter values. Finally, some more solutions of NLTL model in the form of periodic singular waves and singular soliton waves are obtained. Graphical representation of some of solutions is also given and observed effect of parameters in graphs is discussed.

### Data Availability

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

### Declarations

**Conflicts of interest** The authors declare that they have no conflict of interest.

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