Free harmonic oscillators, Jack polynomials and Calogero-Sutherland systems

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Abstract

The algebraic structure and the relationships between the eigenspaces of the Calogero-Sutherland model (CSM) and the Sutherland model (SM) on a circle are investigated through the Cherednik operators. We find an exact connection between the simultaneous non-symmetric eigenfunctions of the $A_{N-1}$ Cherednik operators, from which the eigenfunctions of the CSM and SM are constructed, and the monomials. This construction, not only, allows one to write down a harmonic oscillator algebra involving the Cherednik operators, which yields the raising and lowering operators for both of these models, but also shows the connection of the CSM with free oscillators and the SM with free particles on a circle. We also point out the subtle differences between the excitations of the CSM and the SM.

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I. INTRODUCTION

Exactly solvable and quantum integrable many-body Hamiltonians with non-trivial potentials are of great interest, since they lead to a deeper understanding of the role of interaction in these models. Among such systems, the Calogero-Sutherland model (CSM) and its generalizations enjoy a special status and have been found to be of relevance in diverse branches of physics, such as the universal conductance fluctuations in mesoscopic systems, quantum Hall effect, fluid dynamics, random matrix theory, fractional statistics, anyons, gravity and gauge theories. They describe \( N \)-identical particles in one-dimension with pair-wise inverse distance-square long-range interactions. The CSM is defined on the entire real line with or without the harmonic confinement, while the Sutherland model (SM) lives on a circle. The underlying algebraic structure and the commuting conserved operators of these models have been analyzed by the exchange operator formalism (EOF) and the quantum Lax formulation (QLF). Both of these models share the same algebraic structure. The celebrated Jack polynomials arise as the polynomial part of the orthogonal basis for the SM and their one parameter deformation, known as the Hi-Jack polynomials, play the same role for the CSM. Rodrigues-type formulae have been discovered for the Jack and the Hi-Jack polynomials. The SM is better understood as compared to the CSM, since, its exact density-density dynamical correlation functions have been computed. The eigenspectrum of the SM can be interpreted as arising from a set of free quasi-particles satisfying the generalized exclusion principle.

Yet another approach, towards the understanding of these correlated systems, is to map them to the corresponding interaction-free systems. Motivated by Calogero’s conjecture, the present authors have shown that the CSM is equivalent to a set of free harmonic oscillators. Later, the same model without the harmonic confinement was found to be unitarily equivalent to free particles on the real line; the corresponding eigenfunctions...
can also be realized as coherent states [49]. Recently, the present authors have developed a general method, by which, a host of non-trivial interacting models can be mapped to non-interacting systems [50]. This method treats both the CSM and the SM on equal footing; the former one is equivalent to decoupled harmonic oscillators and the later, to free particles on a circle.

This paper is organized as follows. The following section deals with the algebraic structure and the relationships between the Hilbert spaces of the CSM and the SM, via the Cherednik operators [51]. The origin of two orthogonal basis sets of the CSM, with two different innerproducts, is clearly pointed out. In the subsequent section, we find the exact connection between the eigenfunctions of the Cherednik operators and the non-symmetric monomials. From this, the raising and lowering operators for the eigenfunctions of the CSM and the SM are constructed. Subtle properties of these operators, with correlations built into them, are pointed out and contrasted with the free oscillators raising and lowering operators.

We conclude in the final section by pointing out the advantages of the present approach as compared to the others in the literature and the future directions of work. The appendix deals with the explicit derivation of the map, which is used to obtain the connection between the eigenfunctions of the Cherednik operators and the non-symmetric monomials.

II. ALGEBRAIC STRUCTURE OF THE CSM & SM, AND THE RELATIONSHIP BETWEEN THEIR HILBERT SPACES

In the following, we first touch upon some common aspects of both the CSM and the SM, since these are important for concluding the major results of the present paper. In particular, we show the equivalence of their underlying algebraic structure through the Cherednik operators; this naturally brings out the relationships of their respective Hilbert spaces.

As mentioned earlier, the EOF and the QLF establish that, both the CSM and the SM share the same algebraic structure which becomes exactly the same in the limit $\omega \to \infty$.
Keeping this in mind and also the method developed to map the SM to free particles, we begin with the $N$-particle CSM Hamiltonian, given by ($\hbar = m = 1$),

$$H_{CSM} = -\frac{1}{2} \sum_{i=1}^{N} \partial_i^2 + \frac{1}{2} \omega^2 \sum_{i=1}^{N} x_i^2 + \frac{1}{2} \beta(\beta - 1) \sum_{i,j=1, i \neq j}^{N} \frac{1}{(x_i - x_j)^2}, \quad (1)$$

where, $\partial_i \equiv \partial / \partial x_i$, $\omega$ is the frequency of the oscillators and $\beta \geq \frac{1}{2}$ is the coupling parameter.

The correlated ground-state of the $H_{CSM}$ is known to be of the form $\psi_0 = ZG$, where $Z \equiv \prod_{i<j}^{N} [|x_i - x_j|^{\beta}(x_i - x_j)^{\delta}]$ and $G \equiv \exp\{-\frac{1}{2} \omega \sum_i x_i^2\}$; here, $\delta = 0$ or 1 corresponds to the quantization of the CSM, as bosons or fermions, respectively. Without any loss of generality, we choose $\delta = 0$.

Performing a division by $\omega$ and a similarity transformation by $\hat{T} \equiv ZG \exp\{-\frac{1}{2} \omega \hat{A}\}$, where $\hat{A} \equiv \frac{1}{2} \sum_i \partial_i^2 + \beta \sum_{i \neq j} \frac{1}{(x_i - x_j) \partial_i}$, Eq. (1) can be brought to the following form \[47\],

$$\tilde{H}_{CSM} \equiv \hat{T}^{-1}(H_{CSM}/\omega)\hat{T} = \sum_i x_i \partial_i + \frac{1}{2} N(N - 1) \beta + \frac{1}{2} N , \quad (2)$$

which is independent of the parameter $\omega$. Using the following identity,

$$\sum_{i,j, i \neq j}^{N} \frac{x_i}{(x_i - x_j)^2} = \frac{1}{2} N(N - 1) \quad ,$$

Eq. (2) can be rewritten as

$$\tilde{H}_{CSM} = \sum_i \hat{D}_i + \frac{1}{2} N , \quad (3)$$

where, $\hat{D}_i \equiv D_i + \beta(i - 1) - \beta \sum_{j<i} (1 - K_{ij})$ is the well-known Cherednik operator for the $A_{N-1}$ root system \[51\] \[38\] \[52\]; here, $D_i \equiv x_i \nabla_i$, $\nabla_i \equiv \partial_i + \beta \sum_{j \neq i} (1 - K_{ij})$ is nothing but the Dunkl derivative \[33\] and $K_{ij}$ is the transposition or exchange operator \[26\] \[27\] \[53\] \[54\] \[55\] \[56\], whose action on an arbitrary state can be written as

$$K_{ij}|x_1, \cdots, x_i, \cdots, x_j, \cdots, x_N > = |x_1, \cdots, x_j, \cdots, x_i, \cdots, x_N > .$$

It is easy to check that, the Cherednik operators are in involution, i.e.,

$$[\hat{D}_i , \hat{D}_j] = 0 = [\tilde{H}_{CSM} , \hat{D}_k] , \quad (4)$$
for any \(i, j, k = 1, 2, 3, \ldots, N\). Henceforth, we follow Ref. [38] for the notations and Ref. [36], for the definitions of symmetric functions, ordering of partitions, e.t.c., which are not discussed explicitly in the present paper.

From Eq. (4), it is clear that, by constructing the simultaneous eigenfunctions, \(\chi_\lambda\), of \(\hat{D}_i\):

\[
\hat{D}_i \chi_\lambda = \delta_\lambda^i \chi_\lambda
\]

and symmetrizing them, one recovers the eigenfunctions, \(J_\lambda\), of the \(\tilde{H}_{CSM}\):

\[
J_\lambda = \sum_P \chi_\lambda ; \quad (P \text{ denotes the permutations})
\]

A generic form of \(\chi_\lambda\), for a given partition of \(\lambda\), can be expressed as,

\[
\chi_\lambda = \hat{m}_\lambda + \sum_{\mu<\lambda} u_{\mu\lambda} \hat{m}_\mu
\]

where, the non-symmetric monomial function \(\hat{m}_\lambda = \prod_i x_i^{\lambda_i}\) and \(u_{\mu\lambda}\)'s are some coefficients. In Eq. (6), the eigenvalues \(\delta_\lambda^i = \lambda_i + \beta(N-i)\) and \(\lambda = \sum_i \lambda_i\); \(\lambda = 0, 1, 2, \ldots, \infty\) and \(\lambda_i\)'s are non-negative integers obeying the dominance ordering [36]. Note that, the monomial symmetric function is given by \(m_\lambda = \sum_P \hat{m}_\lambda\). The inverse similarity transformation by \(\hat{T}\) on the eigenstates of \(\tilde{H}_{CSM}\) yields the eigenfunctions of the original \(H_{CSM}\). At this moment, it is important to note that, one can construct the following conserved operator [57,38,52]:

\[
\tilde{H}_{SM} \equiv \sum_i \left( \hat{D}_i^2 - (N-1)\beta \hat{D}_i \right) + \frac{1}{6} N(N-1)(N-2)/L^2
\]

which, when restricted to act on the symmetric functions of the variables, \(x_i\)'s, yields the differential equation for the Jack polynomials [34], and is nothing but the Sutherland Hamiltonian [3], which is gauged away by the ground-state wavefunction and divided by \((2\pi/L)^2\). Note that, in the Sutherland model, \(x_j = \exp\{\frac{2\pi i j \theta_j}{L}\}\), where, \(\theta_j\) is the location of the \(j\)-th particle on the circle and \(L\) is the length of the circumference.

Since, both the \(\tilde{H}_{CSM}\) and the \(\tilde{H}_{SM}\) can be expressed in terms of the same Cherednik operators, \(\hat{D}_i\), we also conclude that, these two models share the same algebraic structure.
Now, the symmetrized simultaneous eigenfunctions of the Cherednik operators coincide with the Jack polynomials, because, \([\tilde{H}_{\text{CSM}} , \tilde{H}_{\text{SM}} ] = 0\). Therefore, the polynomial part of the eigenfunctions of the \(H_{\text{CSM}}\), i.e., the Hi-Jack polynomials, \(j_\lambda\), can be expressed as,

\[
j_\lambda(\{x_i\}, \beta, \omega) = e^{-\frac{1}{2} \lambda^\dagger A} J_\lambda(\{x_i\}, \beta) ;
\]

this is Lassalle’s famous exponential formula \([58, 52]\). An alternate derivation of the Lassalle’s formula has been earlier given by Sogo \([58]\).

It is interesting to note from Eq. (2) that, akin to the Cherednik operators, one can also choose \(x_i \partial_i\) as the \(N\) commuting operators, whose simultaneous eigenfunctions are \(\hat{m}_\lambda\) with eigenvalues \(\lambda_i\). By symmetrizing the \(\hat{m}_\lambda\)’s, one obtains the monomial symmetric functions as the eigenfunctions of the \(\tilde{H}_{\text{CSM}}\) with eigenvalues \(\sum_i \lambda_i + \frac{1}{2} N(N-1) \beta + \frac{1}{2} N\). Hence, similar to the Hi-Jack polynomials, another set of eigenstates of the \(H_{\text{CSM}}\) can be written as

\[
P_\lambda(\{x_i\}, \beta, \omega) = e^{-\frac{1}{2} \lambda^\dagger A} m_\lambda(\{x_i\}) .
\]

Performing one more similarity transformation by \(\exp\{\frac{1}{4 \omega} \sum_i \partial_i^2\} G^{-1}\), \(\tilde{H}_{\text{CSM}}\) can be mapped to a set of \(N\) free harmonic oscillators \([47]\). Further, by the inverse transformation, one can define \(a_i^+ = \hat{T} x_i \hat{T}^{-1}\) and \(a_i^- = \hat{T} \partial_i \hat{T}^{-1}\) as the creation and annihilation operators:

\[
H_i/\omega = a_i^+ a_i^- + \frac{1}{2} (N-1) \beta + \frac{1}{2}, \quad [a_i^+, a_j^-] = \delta_{ij} \quad \text{and} \quad [H_i/\omega , a_j^\pm] = \pm a_j^\pm \delta_{ij}.
\]

Now, the \(H_{\text{CSM}}\) can be written completely in terms of the decoupled oscillators:

\[
H_{\text{CSM}}/\omega = \sum_i a_i^+ a_i^- + \frac{1}{2} N(N-1) \beta + \frac{1}{2} N .
\]

It can be verified by a direct computation that, the Hi-Jack polynomials, \(j_\lambda\), form an orthogonal basis with respect to the conventional inner product, whereas the \(P_\lambda\)’s do not \([58]\). However, it is clear that, \(H_{\text{CSM}}\) truly becomes a set of decoupled harmonic oscillators in the \(P_\lambda\) basis, but not in the \(j_\lambda\) basis, since, \(P_\lambda\)’s can be obtained by the repeated applications of the commuting creation operators, \(a_i^+\), on the ground-state. \(P_\lambda\)’s can be made orthogonal by postulating a new inner product \(<< \mu | \lambda >= \delta_{\mu \lambda}\), where, \(<< \mu| = <0|m_\mu(\{a_i^-\})\).
| | > \ m_{\lambda}(\{a_i^+\})>0 \text{ and } a_i^-|0>=0 =<<0|a_i^+\text{, for } i = 1, 2, 3, \cdots, N \ [17]. \text{ Explicit construction of this new orthogonal basis was achieved in Ref. \ [59].}

III. FREE HARMONIC OSCILLATORS AND THE JACK POLYNOMIALS

In the following, using the properties of the Cherednik operators [51,38,52] and the method developed to map the Sutherland model to free particles [50], we obtain the Jack polynomials akin to the $P_{\lambda}$'s. The Cherednik operators, along with certain creation and annihilation operators, are found to obey the free harmonic oscillator algebra. However, as one naturally expects, these operators, with correlations built into them, drastically differ in their properties, when compared with the creation and the annihilation operators for the monomial symmetric functions.

Rewriting Eq. (3) as,

$$(x_i \partial_i - \lambda_i + \hat{B}_i) \chi_{\lambda} = 0 ,$$

where, $\hat{B}_i \equiv \beta \sum_{j \neq i} \frac{x_i(x_i - x_j)}{(x_i - x_j)}(1 - K_{ij}) - \beta \sum_{j<i} (1 - K_{ij}) - \beta(N + 1 - 2i)$, the solution is found to be (see the appendix for the proof),

$$\chi_{\lambda} = \sum_{n=0}^{\infty} (-1)^n \left[ \frac{1}{(x_i \partial_i - \lambda_i)} \hat{B}_i \right]^n \hat{m}_{\lambda}$$

$$\equiv \hat{g}_{\lambda} \hat{m}_{\lambda} .$$

(11)

It can be straightforwardly verified that, $\hat{g}_{\lambda}$ is, indeed, independent of the particle index $i$, for a given partition of $\lambda$. Now, the Jack polynomials can be obtained by simply symmetrizing $\chi_{\lambda}$:

$$J_{\lambda} = \sum_P \chi_{\lambda} = \sum_P \hat{g}_{\lambda} \hat{m}_{\lambda} .$$

(12)

Comparing the above with an earlier derived formula for the Jack polynomials [50],

$$J_{\lambda}(\{x_i\}) = \sum_{n=0}^{\infty} (-\beta)^n \left[ \frac{1}{\sum_{i} (x_i \partial_i)^2 - \lambda_i^2} \sum_{i<j} x_i + x_j (x_i \partial_i - x_j \partial_j) - \sum_i (N + 1 - 2i) \lambda_i \right] n \times m_{\lambda}(\{x_i\}) ,$$

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we get the following operator identity, satisfied by the monomial functions:

\[
\sum_{\mathcal{P}} \left( \frac{1}{(x_i \partial_i - \lambda_i)} \left( \sum' x_i \left( 1 - K_{ij} - \sum_{j<i} (1 - K_{ij}) - (N + 1 - 2i) \right) \right)^n \hat{m}_\lambda \right) = \left( \frac{1}{\sum_i [(x_i \partial_i)^2 - \lambda_i^2]} \left( \sum x_i + x_j (x_i \partial_i - x_j \partial_j) - \sum_i (N + 1 - 2i) \lambda_i \right) \right)^n \sum_{\mathcal{P}} \hat{m}_\lambda .
\]

From Eq. (10), it can be verified that

\[
\hat{g}_\lambda^{-1} \hat{D}_i \hat{g}_\lambda = x_i \partial_i + \beta(N - i)
\]  

(13)

Due to the above result, Eq. (13) becomes,

\[
\hat{g}_\lambda^{-1} \hat{H}_{SM} \hat{g}_\lambda = \sum_i \left( (x_i \partial_i)^2 + \beta(N + 1 - 2i)x_i \partial_i \right)
\]  

(14)

Eqs. (13) and (14) depict the equivalence of the CSM to decoupled oscillators and the SM, to free particles on a circle. This is yet another proof of their equivalences [47,50].

By the inverse transformation, the creation and annihilation operators for the Jack polynomials can be defined:

\[
b^+_{i,\lambda} = \hat{g}_\lambda x_i \hat{g}_\lambda^{-1} ; \quad b^-_{i,\lambda} = \hat{g}_\lambda \partial_i \hat{g}_\lambda^{-1}
\]  

(15)

which satisfy,

\[
\hat{D}_i = b^+_{i,\lambda} b^-_{i,\lambda} + \beta(N - i)
\]

and

\[
[\hat{D}_i , \ b^\pm_{j,\lambda}] = \pm b^\pm_{j,\lambda} ; \quad [b^-_{i,\lambda} , \ b^+_{j,\lambda}] = \delta_{ij}
\]  

(16)

Note that, \(b^\pm_{j,\lambda}\) crucially depend on a given partition of \(\lambda\); which, in turn, implies that, each Cherednık operator can be written in terms of an infinite set of decoupled oscillators. The ground and excited states can be obtained from,

\[
b^-_{i,\lambda} |0 >_\lambda = 0 \quad \text{for} \quad i = 1, 2, \cdots, N
\]

and

\[
\prod_{i}(b^+_{i,\lambda})^{\mu_i} |0 >_\lambda = |\mu >_\lambda \quad \text{with} \quad \sum \mu_i = \mu,
\]

(17)

respectively. However, all the states, \(|\mu >_\lambda\), are not normalizable except those, for which \(\mu = \lambda\), i.e., \(|\lambda >_\lambda\).
Recollecting the earlier mapping of the SM to free particles [50],
\[
\hat{G}_\lambda^{-1} \hat{H}_{SM} \hat{G}_\lambda = \sum_i (x_i \partial_i)^2 + \beta \sum_i (N + 1 - 2i) \lambda_i ,
\]
(18)
where,
\[
\hat{G}_\lambda \equiv \sum_{n=0}^{\infty} (-\beta)^n \left[ \frac{1}{\sum_i (x_i \partial_i)^2 - \lambda_i^2} \sum_{i<j} \frac{x_i + x_j}{x_i - x_j} (x_i \partial_i - x_j \partial_j) - \sum_i \lambda_i (N + 1 - 2i) \right]^n ,
\]
and
\[
\hat{H}_{SM} \equiv \sum_i (x_i \partial_i)^2 + \beta \sum_{i<j} \frac{x_i + x_j}{x_i - x_j} (x_i \partial_i - x_j \partial_j) ,
\]
one can also define another set of creation and annihilation operators for the Jack polynomials:
\[
c^+_i,\lambda = \hat{G}_\lambda x_i \hat{G}_\lambda^{-1} ; \quad c^-_i,\lambda = \hat{G}_\lambda \partial_i \hat{G}_\lambda^{-1} ,
\]
and
\[
[c^-_i,\lambda , c^+_j,\lambda] = \delta_{ij} .
\]
(19)
Note that, $c^\pm_i,\lambda$'s also depend on $\lambda$, but, unlike $b^\pm_i,\lambda$'s, they are insensitive to the permutations of the particle coordinates. In order to have the normalizable eigenstates, one has to symmetrize the states created by the repeated application of $c^+_i,\lambda$'s on the ground-state which depends on the $\lambda$ and is annihilated by $c^-_i,\lambda$'s. This situation is analogous to the case encountered earlier, when the CSM is mapped to free harmonic oscillators [17,59]. Keeping this in mind, a generic state can be written as,
\[
S_{\mu,\lambda} = m_\mu(\{c^+_i,\lambda\}) \phi_{0,\lambda}(\{x_i\}) ,
\]
(20)
where, $m_\mu$'s are the monomial symmetric functions [30], and $c^-_i,\lambda \phi_{0,\lambda}(\{x_i\}) = 0$. It can be verified that, unlike the previous situation [17,59], all these symmetrized states, $S_{\mu,\lambda}$, are still not normalizable, except those, for which $\mu = \lambda$. In this case, $S_{\lambda,\lambda}$ coincides with the Jack polynomial $J_\lambda$, i.e., $S_{\lambda,\lambda} = J_\lambda$. 

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IV. CONCLUSIONS

In conclusion, we have found a mapping between the eigenstates of the Cherednik operators and the non-symmetric monomials. This not only allows one to find the common algebraic structure of these two models, but also enables one to map the CSM to free harmonic oscillators and the SM to free particles on the circle. Hence, both these models are treated on equal footing. The earlier known method of generating Jack polynomials used the creation operators, which were not commuting ones, and have to be acted in a particular order on the vacuum \[^{38}\]. The present method does not suffer from these difficulties.

The excited states of the Calogero-Sutherland model (CSM) and the Sutherland model (SM) can be thought to be arising from the excitations of an infinite set of free harmonic oscillators, labeled by the partitions of \(\lambda\). In other words, from each set of harmonic oscillators labeled by the partitions of \(\lambda\), one can construct an infinite number of towers such that, each tower contains an infinite number of excited states bounded from below. However, from each tower of these excited states, only one state survives as the normalizable one, which belongs to the Hilbert space. This rich structure needs further analysis, which is currently under progress and will be reported elsewhere.

It worth pointing out again that the excited states of the CSM can be interpreted in two ways due to the presence of two different inner products \[^{47,59}\]. In one case, they arise out of the decoupled oscillators, whereas in the other scenario, they originate from a correlated system. For the SM, only the later interpretation seems to be valid.

Finally, we would like to remark that, the present procedure can also be carried out for the root systems, other than the \(A_{N-1}\) \[^{4}\]. Extension of these analyses to the higher dimensional models \[^{60,63}\] may provide new insights; particularly, in the context of the two-dimensional systems, this may lead to a better understanding of some intriguing aspects of the anyons \[^{18,20}\]. Furthermore, this technique may also throw new light on the structure of the supersymmetric versions of these models \[^{54,55}\].
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APPENDIX

In the following, we connect the solutions of differential equations, involving the Dunkl derivatives, to the monomials. For that purpose, we extend the proof given in Ref. \[50\] for ordinary differential equations, to the equations involving Dunkl derivatives. First, we illustrate the procedure for the single variable case, and then extend it to the multivariable scenario.

Consider the most general and arbitrary linear differential equation \[50\],

\[
\left( F(D) + \hat{P} \right) y(x) = 0 ,
\]

where, \( D \equiv x \frac{d}{dx} \) and \( F(D) = \sum_{n=-\infty}^{\infty} a_n D^n \), is a diagonal operator. \( \hat{P} \) can be an arbitrary operator, having a well-defined action in the space spanned by \( x^n \). Here, \( a_n \)'s are some parameters. The following ansatz,

\[
y(x) = C_{y} \{ \sum_{m=0}^{\infty} (-1)^m \left[ \frac{1}{F(D)} \hat{P} \right]^m \} x^\lambda \equiv C_{y} \hat{G}_\lambda x^\lambda ,
\]

is a solution of the above equation, provided, \( F(D)x^\lambda = 0 \) and the coefficient of \( x^\lambda \) in \( y(x) - C_{y} x^\lambda \) is zero (no summation over \( \lambda \)); here, \( C_{y} \) is a constant. The later condition not only guarantees that, the solutions, \( y(x) \)'s, are non-singular, but also yields the eigenvalues.

Substituting Eq. (22), modulo \( C_{y} \), in Eq. (21),

\[
\left( F(D) + \hat{P} \right) \left\{ \sum_{m=0}^{\infty} (-1)^m \left[ \frac{1}{F(D)} \hat{P} \right]^m \right\} x^\lambda =
\]

\[
= F(D) \left[ 1 + \frac{1}{F(D)} \hat{P} \right] \left\{ \sum_{m=0}^{\infty} (-1)^m \left[ \frac{1}{F(D)} \hat{P} \right]^m \right\} x^\lambda
\]

\[
= F(D) \sum_{m=0}^{\infty} (-1)^m \left[ \frac{1}{F(D)} \hat{P} \right]^m x^\lambda
\]
\[ +F(D) \sum_{m=0}^{\infty} (-1)^m \left[ \frac{1}{F(D)} \hat{P} \right]^{m+1} x^\lambda \]
\[ = F(D)x^\lambda - F(D) \sum_{m=0}^{\infty} (-1)^m \left[ \frac{1}{F(D)} \hat{P} \right]^{m+1} x^\lambda \]
\[ +F(D) \sum_{m=0}^{\infty} (-1)^m \left[ \frac{1}{F(D)} \hat{P} \right]^{m+1} x^\lambda \]
\[ = 0 \quad . \]  

(23)

Note that, the detailed properties of \( \hat{P} \) are not needed to prove Eq. (22) as a solution of Eq. (21). However, naturally, they are required while constructing the explicit solutions of any given linear differential equation.

Eq. (22), which connects the solutions of a differential equation to the monomials, can be generalized to many-variables as follows [50].

Consider,

\[ \left( \sum_{n=-\infty}^{\infty} b_n (\sum_i D^n_i) + \hat{A} \right) Q_\lambda(\{x_i\}) = B_\lambda(\{x_i\}) , \]  

(24)

where, \( b_n \)'s are some parameters, \( D_i \equiv x_i \partial_i \), \( \hat{A} \) can be a function of \( x_i, \partial_i \) and also some other well-defined operators like the transposition operator \( K_{ij} \) and \( B_\lambda(\{x_i\}) \) is a source term.

Case (i) When \( B_\lambda(\{x_i\}) = 0 \) and \( \hat{A}m_\lambda = \epsilon_\lambda m_\lambda + \sum_{\mu<\lambda} C_{\mu\lambda} m_\mu \); where, \( m_\lambda \)'s are the monomial symmetric functions [36] and \( \epsilon_\lambda \) and \( C_{\lambda\mu} \) are some coefficients.

Using Eq. (22), the solutions can be obtained as,

\[ Q_\lambda(\{x_i\}) = \sum_{r=0}^{\infty} (-1)^r \left[ \frac{1}{((\sum_{n=-\infty}^{\infty} b_n(\sum_i D^n_i) - (\sum_{n=-\infty}^{\infty} b_n(\sum_i \lambda^n_i)))(\hat{A} - \epsilon_\lambda)} \right]^r m_\lambda(\{x_i\}) \]  

(25)

with, \( \sum_{n=-\infty}^{\infty} b_n(\sum_i \lambda^n_i) + \epsilon_\lambda = 0 \).

Case (ii) When \( B_\lambda(\{x_i\}) \neq 0 \).

\[ Q_\lambda(\{x_i\}) = \sum_{r=0}^{\infty} (-1)^r \left[ \frac{1}{((\sum_{n=-\infty}^{\infty} b_n(\sum_i D^n_i) - (\sum_{n=-\infty}^{\infty} b_n(\sum_i \lambda^n_i)))(\hat{A} - \epsilon_\lambda)} \right]^r \times \left[ \frac{1}{((\sum_{n=-\infty}^{\infty} b_n(\sum_i D^n_i) - (\sum_{n=-\infty}^{\infty} b_n(\sum_i \lambda^n_i)))} \right] B_\lambda(\{x_i\}) , \]  

(26)
provided, the coefficient of the divergent part in the right hand side of the above equation is zero. As mentioned earlier, this requirement yields the eigenvalues.

As an example, consider the hypergeometric differential equation [36],

$$\left( x(1-x) \frac{d^2}{dx^2} + [\gamma - (\alpha + \beta + 1)x] \frac{d}{dx} - \alpha \beta \right) y(x) = 0 \quad . \tag{27}$$

Multiplying by \(x\), the above can be written as,

$$\left( D(D + \gamma - 1) - \hat{A} \right) y(x) = 0 \quad , \tag{28}$$

where, \( D \equiv x \frac{d}{dx} \) and \( \hat{A} \equiv x(D + \alpha)(D + \beta) \). Now, \( F(D)x^\lambda = D(D + \gamma - 1)x^\lambda = 0 \) yields \( \lambda = 0 \) or \( 1 - \gamma \). From Eq. (22), the two linearly independent solutions, \( y_0(x) \) and \( y_{1-\gamma}(x) \), can be written as,

$$y_0(x) = C_0 \left\{ \sum_{m=0}^{\infty} (-1)^m \left[ \frac{1}{D(D + \gamma - 1)} \right]^m \right\} x^0 = C_0 \ e^{\frac{1}{D} - \hat{A}} 1 \quad , \tag{29}$$

and

$$y_{1-\gamma}(x) = C_{1-\gamma} \ e^{(1/D)\hat{A}} \ x^{1-\gamma} \ . \tag{30}$$

Solutions for many other differential equations, constructed by this method, can be found in Ref. [50]. Further, this technique can be applied to the bound-state problems of the Schrödinger equations with complicated potentials. These details are also available in the same reference.
REFERENCES

[1] F. Calogero, *J. Math. Phys.* **12**, 419 (1971).

[2] B. Sutherland, *J. Math. Phys.*, **12**, 246 (1971); **12**, 251 (1971).

[3] B. Sutherland, *Phys. Rev. A* **4** 2019 (1971); *A 5*, 1372 (1972); *Phys. Rev. Lett.* **34**, 1083 (1975).

[4] M.A. Olshanetsky and A.M. Perelomov, *Phys. Rep.* **94**, 6 (1983).

[5] For various connections, see the chart in B.D. Simons, P.A. Lee and B.L. Altshuler, *Phys. Rev. Lett.* **72**, 64 (1994).

[6] S. Tewari, *Phys. Rev. B* **46**, 7782 (1992).

[7] N.F. Johnson and M.C. Payne, *Phys. Rev. Lett.* **70**, 1513 (1993); **70**, 3523 (1993).

[8] M. Caselle, *Phys. Rev. Lett.* **74**, 2776 (1995).

[9] N. Kawakami, *Phys. Rev. Lett.* **71**, 275 (1993).

[10] H. Azuma and S. Iso, *Phys. Lett. B* **331**, 107 (1994).

[11] P.K. Panigrahi and M. Sivakumar, *Phys. Rev. B* **52**, 13742 (15) (1995).

[12] H.H. Chen, Y.C. Lee and N.R. Pereira, *Phys. Fluids.* **22**, 187 (1979).

[13] K. Nakamura and M. Lakshamanan, *Phys. Rev. Lett.* **57**, 1661 (1986).

[14] See M.L. Mehta, *Random Matrices*, Revised Edition (Academic Press N.Y, 1990).

[15] J.M. Leinaas and J. Myrheim, *Phys. Rev. B* **37**, 9286 (1988).

[16] A.P. Polychronakos, *Nucl. Phys. B* **324**, 597 (1989).

[17] M.V.N. Murthy and R. Shankar, *Phys. Rev. Lett.* **73**, 3331 (1994).

[18] A. Lerda, *Anyons*, Lecture notes in physics, Eds. H. Araki *et al.*, Springer-Verlag, Berlin Heidelberg, 1992 and references therein.
[19] S.B. Fsakov, G. Lozano and S. Ouvry, *Non-Abelian Chern-Simons Particles in an External Magnetic Field*, pre-print hep-th/9902028 and references therein.

[20] S. Ouvry, *On the relation between anyon and Calogero models*, pre-print cond-mat/9907239 and references therein.

[21] I. Andric, A. Jevicki and H. Levine, *Nucl. Phys. B* 312, 307 (1983).

[22] A. Jevicki, *Nucl. Phys.*, B 376, 75 (1992).

[23] G.W. Gibbons and P.K. Townsend, *Phys. Lett. B* 454, 187 (1999).

[24] J.A. Minahan and A.P. Polychronakos, *Phys. Lett. B* 312, 155 (1993); 336, 288 (1994).

[25] E. D’Hoker and D.H. Phong, *Nucl. Phys. B* 513, 405 (1998).

[26] M. Vasiliev, *Int. J. Mod. Phys. A* 6, 1115 (1991).

[27] A.P. Polychronakos, *Phys. Rev. Lett.* 69, 703 (1992).

[28] H. Ujino, K. Hikami and M. Wadati, *J. Phys. Soc. Jpn.* 61, 3425 (1992); 62, 3035 (1993).

[29] K. Hikami and M. Wadati, *J. Phys. Soc. Jpn.* 62, 4203 (1993).

[30] M. Wadati, T. Nagao and K. Hikami, *Physica D* 68, 162 (1993).

[31] M. Wadati, K. Hikami and H. Ujino *Chaos, Solitons & Fractals* 3, 627 (1993).

[32] H. Ujino and M. Wadati, *J. Phys. Soc. Jpn.* 63, 3585 (1994); *Chaos, Solitons & Fractals* 5, 109 (1995).

[33] C.F. Dunkl, *Trans. Amer. Math. Soc.* 311, 167 (1989).

[34] H. Jack, *Proc. R. Soc. Edinburgh* (A) 69, 1 (1970); 347, (1972).

[35] R.P. Stanley, *Adv. Math.* 77, 76 (1988).

[36] I.G. Macdonald, *Symmetric Functions and Hall Polynomials*, 2nd edition, Oxford:
Clarendon press, 1995.

[37] S. Chaturvedi, *Special Functions and Differential Equations*, Eds. K.S. Rao, R. Jagannathan, G.V. Berghe and J.V. Jeugt, Allied publishers, New Delhi, 1997.

[38] L. Lapointe and L. Vinet, *Commun. Math. Phys.* **178**, 425 (1996).

[39] H. Ujino and M. Wadati, *J. Phys. Soc. Jpn.* **65**, 653 (1996); 2423 (1996).

[40] F.D.M. Haldane and M.R. Zirnbauer, *Phys. Rev. Lett.* **71**, 4055 (1993).

[41] P.J. Forrester, *Nucl. Phys. B* **388**, 671 (1992); *Phys. Lett. A* **179**, 127 (1993).

[42] F. Lesage, V. Pasquier and D. Serban, *Nucl. Phys. B* **435**, 585 (1995).

[43] Z.N.C. Ha *Phys. Rev. Lett.* **73**, 1574 (1994); *Nucl. Phys. B* **435**, 604 (1995).

[44] F.D.M. Haldane, *Phys. Rev. Lett.* **67**, 937 (1991).

[45] A.P. Polychronakos, *Generalized statistics in one dimension* pre-print hep-th/9902157 and references therein.

[46] F. Calogero, *J. Math. Phys.* **10**, 2191 (1970).

[47] N. Gurappa and P.K. Panigrahi, *Phys. Rev. B*, R2490 (1999).

[48] T. Brzezinski, C. Gonera and P. Maslanka, *Phys. Lett. A* **254**, 185 (1999).

[49] N. Gurappa, P.S. Mohanty and P.K. Panigrahi, *A novel realization of Calogero-Moser scattering states as coherent states*, pre-print quant-ph/9908047.

[50] N. Gurappa and P.K. Panigrahi, *Equivalence of the Sutherland model to free particles on a circle*, pre-print hep-th/9908127.

[51] I. Cherednik, *Inv. Math.* **106**, 411 (1991).

[52] T.H. Baker and P.J. Forrester, *Nucl. Phys. B* **492**, 682 (1997); *The Calogero-Sutherland model and polynomials with prescribed symmetry*, pre-print solv-int/9609010.
[53] L. Brink, T.H. Hansson and M. Vasiliev, *Phys. Lett.* B 286, 109 (1992).

[54] L. Brink, T.H. Hansson, S.E. Konstein and M. Vasiliev, *Nucl. Phys.* B 384, 591 (1993).

[55] N. Gurappa and P.K. Panigrahi, *Mod. Phys. Lett.* A 11, 891 (1996).

[56] N. Gurappa, P.K. Panigrahi and V. Srinivasan *Mod. Phys. Lett.* A 13, 339 (1996).

[57] D. Bernard, M. Gaudin, F.D.M. Haldane and V. Pasquier, *J. Phys.* A 26, 5219 (1993).

[58] K. Sogo, *J. Phys. Soc. Jpn.* 65, 3097 (1996).

[59] H. Ujino, A. Nishino and M. Wadati, *J. Phys. Soc. Jpn.* 67, 2658 (1998); *Phys. Lett.* A 249, 459 (1998).

[60] M.V.N. Murthy, R.K. Bhaduri and D. Sen, *Phys. Rev. Lett.* 76, 4103 (1996).

[61] R.K. Bhaduri, A. Khare, M.V.N. Murthy and D. Sen, *J. Phys.* A: Math. Gen. 30, 2557 (1997).

[62] R.K. Ghosh and S. Rao, *Phys. Lett.* A 238, 213 (1998).

[63] N. Gurappa, P.K. Panigrahi and T.S. Raju, *A Unified Algebraic Approach to Few and Many-Body Hamiltonians having Linear Spectra*, pre-print cond-mat/9901073, and references therein.

[64] B. Basu-Mallik, H. Ujino and M. Wadati, *Exact spectrum and partition function of SU(m|n) supersymmetric Polychronakos model*, pre-print cond-mat/9904167.

[65] A.J. Bordner, N.S. Manton and R. Sasaki, *Calogero-Moser Models V: Supersymmetry and Quantum Lax Pair*, pre-print hep-th/9910033, and references therein.

[66] I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals, Series and Products* (Academic Press Inc., 1965).