Generalized Chebyshev problem in nonholonomic mechanics and control theory

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Abstract. The paper is based on the talk with the same name given at the International scientific conference on mechanics “The Ninth Polyakhov’s Reading” dedicated to the 200th anniversary of the birth of the prominent Russian scientist Pafnuty Lvovich Chebyshev. The generalized Chebyshev problem is formulated, in which the motion of a system in the presence of given generalized forces should satisfy an additional system of linear differential equations in which the order of each equation exceeds three. These problems constitute a new class of control problems in which the motion program is given in the form of the above additional system of differential equations. These equations can be considered as linear nonholonomic constraints of high order, whose reactions are the desired control forces. To solve such problems, two theories were developed at the Department of Theoretical and Applied Mechanics of St. Petersburg University. In the first theory, we construct a consistent system of differential equations for the generalized coordinates and the Lagrange multipliers, which are considered as equitable unknown functions of time. The second theory is based on the generalized Gauss principle. The application of the theory is illustrated by the solution of a real space mechanics problem about the motion of an Earth satellite after fixing the value of its acceleration at some point in time. Especially efficient is the application of the second theory to the determination of the optimal control force for transferring a mechanical system with a finite number of degrees of freedom from an existing phase state to a new specified state within a specified period of time. The new method is used to solve the model problem of controlled horizontal motion of a cart bearing the axes of several mathematical pendulums. It is shown that the use of the generalized Gauss principle for solving this problem is undoubtedly superior to that of the classical Pontryagin maximum principle.

1. Statement of the generalized Chebyshev problem. Two theories of its solution

The outstanding works of Pafnuty Lvovich Chebyshev (1821–1894), whose 200th anniversary is celebrated this year, are widely known in various fields of mathematics and mechanics. In particular, he created the theory of synthesis of mechanisms, which can be used, for example, in finding sizes of the bars of a four-bar mechanism that will ensure the stop of a required element in a given position. A number of such mechanisms, made under either by the supervision of Chebyshev or by him personally, some of which have his own handwritten notes [1], are carefully preserved at the Department of Theoretical and Applied Mechanics of the Faculty of Mathematics and Mechanics of St. Petersburg University. So, by the Chebyshev problem we will mean such a problem from the field of mechanism synthesis.
By the generalized Chebyshev problem we will understand the problem in which it is required to find additional control forces \( R_\kappa, \kappa = \overline{1,k} \) that, with given generalized forces \( Q_\pi, \pi = \overline{1,p} \), acting on a mechanical system with finite number of degrees of freedom, simultaneously satisfy the following additional system of differential equations

\[
f^{n\kappa}_\pi \equiv a^{l+\kappa}_{n\pi}(t,q,\dot{q},...,q^{(n-1)})q^{(n)} + a^{l+\kappa}_{n0}(t,q,\dot{q},...,q^{(n-1)}) \equiv 0,
\]

\( \pi = \overline{1,p}, \kappa = \overline{1,\bar{k}}, k \leq p, \quad l = p - k, \quad n \geq 3 \)  

(1)

for higher derivatives than the second. Thus, a new class of control problems is introduced in control theory, in which the motion program is given in the form of an additional system of differential equations of order \( n \geq 3 \).

Simultaneously, the above control problem (the generalized Chebyshev problem) can be considered as a constraint motion when the motion of a mechanical system, whose position is described by the generalized coordinates \( q^\pi, \pi = \overline{1,p} \), is subject to \( k \) ideal linear nonholonomic constraints of high order (1).

Two theories of motion of nonholonomic systems with high-order constraints have been created at the Department of Theoretical and Applied Mechanics of the Faculty of Mathematics and Mechanics of St. Petersburg University. In the first theory, the generalized reactions (the Lagrange multipliers) \( \Lambda_\kappa, \kappa = \overline{1,\bar{k}} \) are considered as \( k \) unknown functions of time \( t \), which should be determined in addition to the required \( p \) generalized coordinates. We construct the consistent system of differential equations

\[
\ddot{q}^\pi = F^\pi(t,q,\dot{q},\Lambda), \quad \pi = \overline{1,p},
\]

\[
\Lambda_\kappa^{(n-2)} = C^n_\kappa(t,q,\dot{q},\Lambda,\dot{\Lambda},...,\Lambda), \quad \kappa = \overline{1,\bar{k}}, \quad n \geq 3,
\]

(2)

for these \( p + k \) unknown functions (see the paper [2] and the books [3, 4]), where \( F^\pi, C^n_\kappa \) are known functions of their arguments.

The second theory is based on the generalized Gauss principle [5], according to which

\[
\delta'' Z = 0, \quad Z = \frac{M}{2} \left( W - \frac{Y}{M} \right)^2.
\]

(3)

The two primes in formula (3) indicate that only the second-order derivatives of the generalized coordinates are varied. For more on this principle and the notation, see [3, 4].

2. Motion of an Earth satellite after fixing the value of its acceleration

The application of both theories is illustrated by consideration of the motion of an Earth satellite after fixing the value of its acceleration at some time. This problem is one of the first examples of the motion of a real mechanical system (in our case, this is a space dynamics problem) with a high-order constraint [6, 7]. For numerical experiments, see [8]. Figures 1 and 2 show the results of [8] for the Soviet Molniya-type communication spacecraft after its acceleration was fixed at perigee.

It turns out that under the first theory (i.e., formulas (2)), after the fixation of the acceleration the spacecraft begins to move between two concentric circles centered in the Earth and touch them alternately. Under the second theory (i.e., formulas (3)) the spacecraft becomes a space body and asymptotically tends to a uniformly accelerated motion along a straight line.
3. Solving one of the most important problems of control theory with the help of the second theory

It turned out that the second theory has the most interesting applications in finding the control force that converts a mechanical system from an existing phase state to a new specified phase state in a specified period of time, and in particular, in solving problems of oscillation damping. A large number of such examples can be found in the book [9]. It is worth pointing out that in the above problems the generalized Gauss principle was found to be more efficient than the classical Pontryagin maximum principle.

The main idea of the proposed new method is demonstrated when solving the model problem of determination of the control force $F$ applied to a horizontally moving trolley carrying $s$ axes of mathematical pendulums. It is required to find an optimal control force that, within a specified time $\tilde{T}$, will transfer the cart with the pendulums from the given initial phase state to the specified final phase state. If the latter state corresponds to the equilibrium position, then here one speaks about oscillation damping of a mechanical system. For the convenience of calculations, we assume that the system was at rest in the initial phase state. This allows us to describe the motion of the cart with pendulums via Duhamel integrals. Thus, a boundary-value problem is formulated in which the values of the phase variables are given at $t = 0$ and $t = \tilde{T}$.

Initially, the above boundary-value problem was attacked using the Pontryagin maximum principle to minimize the functional of the squared control force, which we need to find. When writing the equations of small oscillations of the system in normal coordinates, it was possible to construct a differential equation for the required control force $F$ with constant coefficients

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**Figure 1.**
Motion of a Molniya-type spacecraft with fixed acceleration under the first motion theory.

**Figure 2.**
Motion of a Molniya-type spacecraft with fixed acceleration under the second motion theory.
that depend only on the nonzero eigenfrequencies of the system \( \Omega \), \( \pi = 1, p \),

\[
\frac{d^2}{dt^2} \left( \frac{d^2}{dt^2} + \Omega_1^2 \right) \left( \frac{d^2}{dt^2} + \Omega_2^2 \right) \cdots \left( \frac{d^2}{dt^2} + \Omega_p^2 \right) F = 0 .
\] (4)

The following interesting property of equation (4) was found: if we substitute into this equation
the control force taken from the original system of differential equations, we get a differential
equation of order \( 2s + 4 \) with respect to the dimensional generalized coordinates of the system.
For \( s = 2 \), it has the form

\[
a_{8,x} \frac{d^8 x}{dt^8} + a_{8,\varphi_1} \frac{d^8 \varphi_1}{dt^8} + a_{8,\varphi_2} \frac{d^8 \varphi_2}{dt^8} + a_{6,x} \frac{d^6 x}{dt^6} + a_{6,\varphi_1} \frac{d^6 \varphi_1}{dt^6} +
\]
\[
+ a_{6,\varphi_2} \frac{d^6 \varphi_2}{dt^6} + a_{4,x} \frac{d^4 x}{dt^4} + a_{4,\varphi_1} \frac{d^4 \varphi_1}{dt^4} + a_{4,\varphi_2} \frac{d^4 \varphi_2}{dt^4} = 0 ,
\] (5)

where \( x \) is the cart coordinate, \( \varphi_1, \varphi_2 \) are the of angles of deviation of the pendulums, and
the coefficients of equation (5) are constants, which can be found from the parameters of the
mechanical system.

In other words, it turned out that when solving the control problem under consideration by
applying the Pontryagin maximum principle in which the functional of the squared control force
is minimized, a high-order linear nonholonomic constraint changes continuously (even with two
pendulums, the constraint was found to be of the eighth order). This suggests the possibility
of using methods of nonholonomic mechanics with high-order constraints \( (n \geq 3) \) to solve the
boundary-value problem. Hence, instead of the Pontryagin maximum principle we will use
the generalized Gauss principle \([5]\). The latter principal, which is natural for this theory, was
proposed already in 1983. This allows us to get the control in the form of a polynomial.

For \( s = 2 \), the calculations (see \([2, 9]\)) were carried out for linearized dimensionless differential
equations with respect to the principal coordinates of the mechanical system. The following
notation was used: \( u \) is the dimensionless control force, \( x_0 \) is the dimensionless coordinate of the
center of mass of the system, \( x_1, x_2 \) are, respectively, the principal angular coordinates systems,
\( T \) is the dimensionless motion time, \( T_1 \) and \( T_2 \) are, respectively, the dimensionless greatest and
smallest periods of oscillation of the pendulums. The graphs in figures 3 and 4 show that for
a short-time motion the solutions obtained by the above two methods (of which the first is the
classical method of control theory) practically coincide. This allows us to say that the new
method, which depends the generalized Gauss principle, is quite workable. At the same time,
the results are substantially different for a long-time motion of the mechanical system. This

Figure 3.
Short-time motion of a mechanical system, \( T = T_2, T_2 = 0.5 T_1 \).
can be explained by the fact that the control obtained using the Pontryagin maximum principle contains harmonics with the eigenfrequencies of the system — this tends to bring the system into resonance. In contrast, the application of the generalized Gauss principle, as mentioned above, gives a control in the form of a polynomial, which ensures a relatively smooth motion of the system.

Another fact is also worth pointing out — the application of the Pontryagin maximum principle always creates jumps of the control force at the beginning and end of the motion. However, if the generalized Gauss principle is used, then such jumps disappear for a long-time motion (cf. figures 3 and 4). Therefore, the question arises whether it is possible to remove the control jumps also for a short-time motion of the system. It turns out that to do this it is sufficient to additionally require the fulfillment of two more boundary conditions: the acceleration of the cart should be zero at the beginning and end of motion. Such a problem will be called the extended (generalized) boundary-value problem of first order. It is important to emphasize that it is impossible to solve the above generalized boundary-value problem via the Pontryagin maximum principle, since the control obtained with its help will contain a number of unknown arbitrary constants, which are insufficient for the satisfaction of all boundary conditions. In contrast, the generalized Gauss principle is capable of dealing with the above extended boundary-value problem if the order of this principle is increased by two. In figure 5, the dashed lines correspond to the solution obtained by the generalized Gauss principle for the ordinary boundary-value problem, the solid lines correspond to the solution of the extended boundary-value problem considered in this section. It can be seen that we have managed to eliminate the jumps of the control force at the beginning and end of the system motion.

Figure 5. Comparison of the Gauss-solutions for the extended and ordinary boundary-value problems for $T = T_2$, $T_2 = 0.5T_1$. 
4. Singular points of solutions of extended boundary-value problems

However, the formulation and solution of the extended boundary-value problem are not always useful. The fact is that the calculations show that the results of motion of a mechanical system under the action of a control obtained as a result of the solution of an extended boundary-value problem depend substantially on the dimensionless parameter $K = T/T_1$. It turns out that there is a countable set of such values of the parameter $K$ near which intense oscillations develop in the system. Such values of $K$ are called singular points of solutions of extended boundary-value problems.

It is proposed to construct an analytical solution of the problem $u(\tau)$, which is free from singular points, as a linear combination of $u_1(\tau)$ and $u_2(\tau)$ ($\tau$ is the dimensionless time), which are solutions of the extended boundary-value problems of first and second orders:

$$u(\tau) = u_1(\tau) + \mu(u_2(\tau) - u_1(\tau)).$$

Note that the extended second-order boundary-value problem also involves the requirement that at the beginning and end of motion the time derivatives of the acceleration of the cart should vanish. In this case, by defining the parameter $\mu$ from the condition of minimality of the integral of the squared function $u(\tau)$ during the transfer time $T$ we can avoid evaluation of singular values of the solutions $u_1$ and $u_2$ and hence to construct an analytical solution continuously depending on the parameter $K$.

![Figure 6. Comparison of the Gauss-solutions for various boundary-value problems, $T = 4.02\pi$, $\omega_2 = 1.734\omega_1$.](image)

For one of the mechanical systems, for the case $K = 2.01$, we consider motions with different choices of the control force. The smooth solid line in figure 6 shows the displacement $x$ of the cart (measured in fractions of its dimensional total displacement $S$), which was obtained via the generalized Gauss principle for the ordinary boundary-value problem. At the same time, by solving the extended boundary-value problem we get the motion of the cart shown in figure 6 by the dashed line. We see that in this case the above problem of damping of oscillation is solved, but at the same time, intense large-amplitude oscillations of the cart develop in the system. This unexpected behavior of the cart is explained by the fact that the used values of the system parameters correspond to the accepted value $K = 2.01$, which is close to the value of the first singular point, which is $K = 2.01265$. The solution obtained via formulas (6) with the given parameters is shown in figure 6 by a dotted line.

The above theory, which is based on the machinery of motion of nonholonomic systems with high-order constraints, was applied to the solution of a number of practical problems [9]. We also mention recent studies aimed, for example, at finding the control moment that smoothly translates a solid body from one angular state to another within a specified time [10].
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