The $\kappa$-core and the $\kappa$-balancedness of TU games

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Abstract

We consider transferable utility cooperative games with infinitely many players. In particular, we generalize the notions of core and balancedness, and also the Bondareva-Shapley Theorem for infinite TU-games with and without restricted cooperation, to the cases where the core consists of $\kappa$-additive set functions. Our generalized Bondareva-Shapley Theorem extends previous results by Bondareva (1963), Shapley (1967), Schmeidler (1967), Faigle (1989), Kannai (1969, 1992), Pintér (2011) and Bartl and Pintér (2022).

Keywords: TU games with infinitely many players, Bondareva-Shapley Theorem, $\kappa$-core, $\kappa$-balancedness, $\kappa$-additive set function, duality theorem for infinite LPs

1 Introduction

The core (Shapley, 1953; Gillies, 1959) is definitely one of the most important solution concepts of cooperative game theory. In the transferable utility setting (henceforth games) the Bondareva-Shapley Theorem (Bondareva, 1963)

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Shapley (1967; Faigle, 1989) provides a necessary and sufficient condition for the non-emptiness of the core; it states that the core of a game with or without restricted cooperation is not empty if and only if the game is balanced. The textbook proof of the Bondareva-Shapley Theorem goes by the strong duality theorem for linear programs (henceforth LPs), see e.g. Peleg and Sudhölter (2007). The primal problem corresponds to the concept of balancedness and so does the dual problem to the notion of core. However, this result is formalized for games with finitely many players. It is a question how one can generalize this result to the infinitely many player case.

The finitely many player case is special in (at least) two counts: (1) it can be handled by finite linear programs, (2) since the power set of the player set is also finite, it is natural to take the solution of a game from the set of additive set functions (additive games).

There are two main directions to generalize the notion of additive set function. The first, when we weaken the notion of additivity; this leads to the notion of $k$-additive core (Grabisch and Miranda, 2008), where $k$ is a finite cardinal (natural number). The second, when we use a notion stronger than additivity (e.g. $\sigma$-additivity). Naturally, a stronger notion does matter only if there are infinitely many players. This latter approach is considered here.

Schmeidler (1967), Kannai (1969, 1992), Pintér (2011), and Bartl and Pintér (2022) considered games with infinitely many players. All these papers studied the additive core; that is, the case when the core consists of bounded additive set functions. Schmeidler (1967) and Kannai (1969) showed that the additive core of a non-negative game without restricted cooperation with infinitely many players is not empty if and only if the game is Schmeidler balanced (Definition 12). Bartl and Pintér (2022) extended these results and showed that the additive core of a game bounded below with our without restricted cooperation with infinitely many players is not empty if and only if the game is (bounded-)Schmeidler balanced.

Kannai (1992) raised the following two research questions: (1) When does there exist a bounded $\sigma$-additive set function in the core? (2) When are all elements in the core bounded $\sigma$-additive?

Kannai (1969) gave a necessary and sufficient condition for that the $\sigma$-additive core of a non-negative game without restricted cooperation and with infinitely many players is not empty; that is, he answered Question (1). This result (the necessary and sufficient condition) is, however, only slightly similar to the classical balancedness condition. Moreover, it works only for non-negative games without restricted cooperation. Schmeidler (1972) and Einy et al (1997) answered Question (2) respectively for exact and for continuous convex games.
In this paper we raise the following question, where $\kappa$ is an infinite cardinal number, and generalize Kannai’s first question thus: When does there exist a bounded $\kappa$-additive set function in the core? Moreover, we consider this question in the case of games with restricted cooperation too.

Addressing this question, we introduce the notions of $\kappa$-core and $\kappa$-balancedness (Definitions 11 and 14). Then, we apply the strong duality theorem for infinite LPs by Anderson and Nash (1987) (Proposition 3) and prove that the $\kappa$-core of a game with or without restricted cooperation and with arbitrarily many players is not empty if and only if the game is $\kappa$-balanced (Theorem 17).

The set-up of the paper is as follows. In the next section we introduce the main mathematical notions and results, which are related to infinite LPs, and used in this paper. In Section 3 we introduce the notion of $\kappa$-additive set functions and discuss some related concepts and results. In Section 4 we present game theory notions and define various cores (such as $\kappa$-core) and balancedness conditions (such as $\kappa$-balancedness) we consider in this paper. Section 5 presents our main result. We give an answer to the question we have raised: there is a bounded $\kappa$-additive set function in the core if and only if the game is $\kappa$-balanced (Theorem 17). The last section briefly concludes.

2 Duality theorem

In this section we discuss the duality theorem for infinite linear programs that we will use later.

Let $X$ and $Y$ be real vector spaces; the algebraic dual of $X$, which is the space of all linear functionals on $X$, is denoted by $X'$; similarly $Y'$ denotes the algebraic dual of $Y$. Moreover, $Y^* \subseteq Y'$ denotes a linear subspace of $Y'$ such that $(Y, Y^*)$ is a dual pair of spaces; that is, if $f \in Y$ is non-zero, then there exists a $y \in Y^*$ such that $y(f) \neq 0$. For any linear mapping $A: X \to Y$ its adjoint mapping is $A': Y' \to X'$ with $(A'(y))(x) = y(A(x))$ for all $x \in X$ and $y \in Y'$. Moreover, a subset $P \subseteq X$ of the vector space $X$ is a convex cone if $\alpha x + \beta y \in P$ for all $x, y \in P$ and all non-negative $\alpha, \beta \in \mathbb{R}$. For any two functionals $f, g: X \to \mathbb{R}$ we write $f \geq_P g$ if $f(x) \geq g(x)$ for all $x \in P$.

Now, given a linear mapping $A: X \to Y$, a point $b \in Y$ and a linear functional $c: X \to \mathbb{R}$, let us consider the following infinite LP-pair (cf. Anderson and Nash, 1987, Section 3.3):

\[
\begin{align*}
\text{(P}\_\text{LP}) & \quad c(x) \to \sup \\
\text{s.t.} & \quad A(x) = b \\
& \quad x \in P \\
\text{(D}\_\text{LP}) & \quad y(b) \to \inf \\
\text{s.t.} & \quad A'(y) \geq_P c \\
& \quad y \in Y^* 
\end{align*}
\]
where \( P \subseteq X \) is a convex cone and \( Y^* \) is a subspace of \( Y' \) such that \((Y,Y^*)\) is a dual pair of spaces.

**Definition 1.** The program \((D_{LP})\) is **consistent** if there exists a linear functional \( y \in Y^* \) such that \((A'(y))(x) \geq c(x)\) for all \( x \in P \). The **value** of a consistent program \((D_{LP})\) is \( \inf \{ y(b) : A'(y) \geq_P c, y \in Y^* \} \).

In the next definition we assume the weak topology on the space \( Y \) with respect to \( Y^* \). To introduce that, we describe all the neighborhoods of a point. A set \( U \subseteq Y \) is a **weak neighborhood** of a point \( f_0 \in Y \) if there exist a natural number \( n \) and functionals \( y_1, \ldots, y_n \in Y^* \) such that

\[
\bigcap_{j=1}^n \{ f \in Y : \left| y_j(f) - y_j(f_0) \right| < 1 \} \subseteq U.
\]

**Definition 2.** Put \( D = \{ (A(x), c(x)) : x \in P \} \). The program \((P_{LP})\) is **superconsistent** if there exists a \( z \in \mathbb{R} \) such that \((b, z) \in D\), where \( D \) is the closure of \( D \). The **supervalue** of a superconsistent program \((P_{LP})\) is \( \sup \{ z : (b, z) \in D \} \).

We recall that a pair \((I, \leq)\) is **right-directed** if \( I \) is a preordered set and for any \( i, j \in I \) there exists a \( k \in I \) such that \( i \leq k \) and \( j \leq k \). A **net** (generalized sequence) of \( X \) is \((x_i)_{i \in I}\) where \((I, \leq)\) is a right-directed pair and \( x_i \in X \) for all \( i \in I \).

Notice that the program \((P_{LP})\) is superconsistent if there exists a net \((x_i)_{i \in I}\) from \( P \) such that \( A(x_i) \xrightarrow{w} b \), which means that \( A(x_i) \) converges to \( b \) in the weak topology, and \((c(x_i))_{i \in I}\) is bounded. Furthermore, a number \( z^* \) is the supervalue of a superconsistent program \((P_{LP})\) if it is the least upper bound of all numbers \( z \) such that there exists a net \((x_i)_{i \in I}\) from \( P \) such that \( A(x_i) \xrightarrow{w} b \) and \( c(x_i) \xrightarrow{w} z \).

**Proposition 3.** Consider the programs in \((I)\). Program \((P_{LP})\) is superconsistent and \( z^* \) is its finite supervalue if and only if program \((D_{LP})\) is consistent and \( z^* \) is its finite value.

Proposition 3 is a restatement of Theorem 3.3, p. 41, in Anderson and Nash (1987). Notice that we differ from Anderson and Nash (1987) in the point that Anderson and Nash use slightly different notions of superconsistency and supervalue. However, they also remark that their notions and the ones we use here are equivalent (p. 41 above Theorem 3.3). This is why we omit the proof of Proposition 3 here.

### 3 The \( \kappa \)-structures

Throughout this section \( \kappa \) is an infinite cardinal number. Let \( N \) be a non-empty set and let \( \mathcal{A} \subseteq \mathcal{P}(N) \) be a field of sets; that is, if \( S_1, \ldots, S_n \in \mathcal{A} \),
then $$\bigcup_{j=1}^{n} S_j \in \mathcal{A}$$, and $$N \in \mathcal{A}$$ with $$N \setminus S \in \mathcal{A}$$ for any $$S \in \mathcal{A}$$. The pair $$(N, \mathcal{A})$$ is called chargeable space.

Given a chargeable space $$(N, \mathcal{A})$$, let $$\text{ba}(\mathcal{A})$$ and $$\text{ca}(\mathcal{A})$$ denote, respectively, the set of bounded additive set functions and the set of bounded $$\sigma$$-additive set functions $$\mu: \mathcal{A} \to \mathbb{R}$$.

Let $$(S_i)_{i \in I}$$ be a net of sets of $$\mathcal{A}$$; a net $$(S_i)_{i \in I}$$ is a $$\kappa$$-net if $$|I| \leq \kappa$$, where $$|I|$$ is the cardinality of the set $$I$$. In addition, the net $$(S_i)_{i \in I}$$ is monotone decreasing or monotone increasing if $$i \leq j$$ implies $$S_i \supseteq S_j$$ or $$S_i \subseteq S_j$$, respectively, for any $$i, j \in I$$.

Let $$\mu: \mathcal{A} \to \mathbb{R}$$ be a set function. We say that $$\mu$$ is upper $$\kappa$$-continuous or lower $$\kappa$$-continuous at $$S \in \mathcal{A}$$ if for any monotone decreasing or increasing $$\kappa$$-net $$(S_i)_{i \in I}$$ from $$\mathcal{A}$$ with $$\bigcap_{i \in I} S_i = S$$ or $$\bigcup_{i \in I} S_i = S$$, respectively, it holds that $$\lim_{i \in I} \mu(S_i) = \mu(S)$$. The set function $$\mu$$ is $$\kappa$$-continuous if it is both upper and lower $$\kappa$$-continuous at every set $$S \in \mathcal{A}$$.

Next we define the notion of $$\kappa$$-additivity. Our definition is similar to the one by Schervish et al (2017).

**Definition 4.** A set function $$\mu: \mathcal{A} \to \mathbb{R}$$ is $$\kappa$$-additive if it is additive and $$\kappa$$-continuous. Let $$\text{ba}^\kappa(\mathcal{A})$$ denote the set of $$\kappa$$-additive set functions over $$\mathcal{A}$$.

Note that $$\text{ba}^\kappa(\mathcal{A})$$ is a linear subspace of $$\text{ba}(\mathcal{A})$$. Furthermore, the following proposition is easy to see.

**Proposition 5.** If the set function $$\mu: \mathcal{A} \to \mathbb{R}$$ is additive, then it is

- upper $$\kappa$$-continuous if and only if it is lower $$\kappa$$-continuous;
- $$\kappa$$-continuous if and only if it is lower $$\kappa$$-continuous at $$\emptyset$$;
- $$\aleph_0$$-continuous if and only if it is $$\sigma$$-additive.

**Example 6.** The Lebesgue measure on $$B([0, 1])$$, the Borel $$\sigma$$-field of $$[0, 1]$$, is not $$\kappa$$-additive for any $$\kappa \geq \mathfrak{c}$$, where $$\mathfrak{c}$$ denotes the cardinality of the real numbers; but it is $$\kappa$$-additive for $$\kappa = \aleph_0$$.

If $$\kappa$$ is not countable and the field of sets on which the $$\kappa$$-additive set function is defined is rich enough, then one may ask whether there are enough or just few $$\kappa$$-additive set functions. Without going into the details we remark that this problem is related to the notion of measurable cardinal (Ulam, 1930).

The next example shows that there are many $$\kappa$$-additive set functions in the space $$\text{ba}^\kappa(\mathcal{A})$$ even in the case when the field $$\mathcal{A}$$ is large; that is, the theory is not trivial nor vacuous.
Example 7. Let $X$ be an arbitrary set such that $\#X = \kappa \geq \aleph_0$. Consider $\mathcal{P}(X)$, the power set of $X$. It is clear that the Dirac measures on $\mathcal{P}(X)$ are $\kappa$-additive. Let

$$\Delta = \left\{ \sum_{n=1}^{\infty} \alpha_n \delta_n : (\alpha_n)_{n=1}^{\infty} \in \ell^1, \right.$$ \hspace{1cm}
$$\delta_n \text{ being Dirac measures on } \mathcal{P}(X) \text{ for } n = 1, 2, 3, \ldots \left\}.$$ 

It is clear that each $\mu \in \Delta$ is a $\kappa$-additive set function on $\mathcal{P}(X)$. Notice that $\#\Delta \geq \# \text{ca}(\mathbb{N}, \mathcal{P}(\mathbb{N}))$; that is, even in the “worst” case, when there does not exist a non-trivial $\{0,1\}$-valued $\kappa$-additive set function on $\mathcal{P}(X)$, which means the cardinal $\kappa$ is not measurable (Ulam, 1930), the collection $\Delta$ of the trivial $\kappa$-additive set functions on $\mathcal{P}(X)$ is at least as large as the collection of the $\sigma$-additive ones on $\mathcal{P}(\mathbb{N})$. In other words, even in the “worst” case, the problem of the non-emptiness of the $\kappa$-core is at least as complex as the non-emptiness of the $\sigma$-core with player set $\mathbb{N}$ and all coalitions feasible, the case considered by Kannai (1969, 1992).

Given a set system $\mathcal{A}$, the space $\mathbb{R}(\mathcal{A})$ consists of all functions $\lambda : \mathcal{A} \rightarrow \mathbb{R}$ with a finite support; that is,

$$\mathbb{R}(\mathcal{A}) = \{ \lambda \in \mathbb{R}^\mathcal{A} : \#\{ S \in \mathcal{A} : \lambda S \neq 0 \} < \infty \}.$$

Denoting $\lambda(S)$ and the characteristic function of a set $S \in \mathcal{A}$ by $\lambda_S$ and $\chi_S$, respectively, let $\Lambda(\mathcal{A}) = \{ \lambda S_1 \chi_{S_1} + \cdots + \lambda S_n \chi_{S_n} : n \in \mathbb{N}, \lambda S_1, \ldots, \lambda S_n \in \mathbb{R}, S_1, \ldots, S_n \in \mathcal{A} \}$ be the space of all simple functions on $(\mathbb{N}, \mathcal{A})$; that is,

$$\Lambda(\mathcal{A}) = \left\{ \sum_{S \in \mathcal{A}} \lambda_S \chi_S : \lambda \in \mathbb{R}(\mathcal{A}) \right\}.$$ 

We introduce a norm on $\Lambda(\mathcal{A})$ as follows. For a simple function $f = \lambda S_1 \chi_{S_1} + \cdots + \lambda S_n \chi_{S_n} \in \Lambda(\mathcal{A})$ let

$$\| f \| = \sup_{x \in \mathbb{N}} |f(x)|.$$ 

Then the topological dual $(\Lambda(\mathcal{A}))^*$ of the vector space $\Lambda(\mathcal{A})$, which is the space of all continuous linear functionals on $\Lambda(\mathcal{A})$, is isometrically isomorphic to $\text{ba}(\mathcal{A})$, the space of all bounded additive set functions on $\mathcal{A}$ (see e.g. Dunford and Schwartz (1958), Theorem IV.5.1, p. 258). For simplicity, we shall identify the space $(\Lambda(\mathcal{A}))^*$ with $\text{ba}(\mathcal{A})$. Indeed, a set function $\mu \in \text{ba}(\mathcal{A})$ induces a continuous linear functional $\mu^* \in (\Lambda(\mathcal{A}))^*$ on $\Lambda(\mathcal{A})$ as follows:

$$\mu^*(f) = \lambda S_1 \mu(S_1) + \cdots + \lambda S_n \mu(S_n)$$

for any $f = \lambda S_1 \chi_{S_1} + \cdots + \lambda S_n \chi_{S_n} \in \Lambda(\mathcal{A})$. 


Lemma 8. It holds that \((Λ(\mathcal{A}), ba^\kappa(\mathcal{A}))\) is a dual pair of spaces.

Proof. Let \(f \in Λ(\mathcal{A})\) be non-zero, whence there is an \(x \in N\) such that \(f(x) \neq 0\). Then \(δ_x\), the Dirac measure concentrated at point \(x\) on \(\mathcal{A}\), is a \(κ\)-additive set function, and \(δ_x'(f) = f(x) \neq 0\). □

4 The \(κ\)-core and the \(κ\)-balancedness of TU games

Let \(κ\) be an arbitrary infinite cardinal number as in the previous section. First, we introduce the notion of TU games. Let \(N\) be a non-empty set of players, let \(A' \subseteq P(N)\) be a collection of sets such that \(\emptyset, N \in A'\), and let \(\mathcal{A}\) denote the field hull of \(A'\); that is, the smallest field of sets that contains \(A'\). Then a TU game (henceforth a game) on \(A'\) is a set function \(v : A' \to \mathbb{R}\) such that \(v(\emptyset) = 0\). We denote the class of games on \(A'\) by \(G^{A'}\). If \(A' = A\), then \(v \in G^{A'}\) is a game without restricted cooperation. Otherwise, if \(A'\) is not a field, \(v \in G^{A'}\) is a game with restricted cooperation.

In the following subsections we introduce the three notions of core and the three notions of balancedness that we consider in this paper.

4.1 The core of a TU game

First, we introduce the notion of additive core of a game, which was considered by Schmeidler (1967), Kannai (1969, 1992), Pinter (2011), and Bartl and Pinter (2022).

Definition 9. For a game \(v \in G^{A'}\) its additive core (henceforth ba-core) is defined as follows:

\[
ba-core(v) = \{ \mu \in ba(\mathcal{A}) : \mu(N) = v(N) \text{ and } \mu(S) \geq v(S) \text{ for all } S \in A' \}.
\]

We shall also need the notion of \(σ\)-additive core of a game.

Definition 10. For a game \(v \in G^{A'}\) its \(σ\)-additive core (henceforth ca-core) is defined as follows:

\[
ca-core(v) = \{ \mu \in ca(\mathcal{A}) : \mu(N) = v(N) \text{ and } \mu(S) \geq v(S) \text{ for all } S \in A' \}.
\]

In general, for an infinite cardinal number \(κ\) we introduce the notion of \(κ\)-core of a game.

Definition 11. For a game \(v \in G^{A'}\) its \(κ\)-core is defined as follows:

\[
κ-core(v) = \{ \mu \in ba^κ(\mathcal{A}) : \mu(N) = v(N) \text{ and } \mu(S) \geq v(S) \text{ for all } S \in A' \}.
\]
In words, the ba-core, the ca-core, and the $\kappa$-core consists of bounded additive, bounded $\sigma$-additive, and bounded $\kappa$-additive, respectively, set functions defined on the field hull $A$ of the feasible coalitions $A'$ that meet the conditions of efficiency ($\mu(N) = v(N)$) and coalitional rationality ($\mu(S) \geq v(S)$ for all $S \in A'$). Observe that the ca-core is a special case of the $\kappa$-core when $\kappa = \aleph_0$.

Notice that in the finite case all the three notions of ba-core, ca-core, and $\kappa$-core are equivalent with the notion of (ordinary) core.

### 4.2 Balancedness of a TU game

In the case of infinite games without restricted cooperation with additive core Schmeidler (1967) defined the notion of balancedness. Here, we generalize his notion to the restricted cooperation case, and call it Schmeidler balancedness.

**Definition 12.** We say that a game $v \in G^{A'}$ is Schmeidler balanced if

$$
\sup \left\{ \sum_{S \in A'} \lambda_S v(S) : \sum_{S \in A'} \lambda_S \chi_S = \chi_N, \lambda \in \mathbb{R}_{+}^{(A')} \right\} \leq v(N). \tag{3}
$$

Notice that for finite games the notions of Schmeidler balancedness and (ordinary) balancedness (Bondareva, 1963; Shapley, 1967; Faigle, 1989) coincide, hence Schmeidler balancedness is an extension of (ordinary) balancedness.

Recall that $Y^* = ba^*(A)$ is a linear subspace of $Y^* = ba(A)$, which can be identified with the topological dual of the normed linear space $Y = \Lambda(A)$. In the next two definitions, where we introduce two new notions of balancedness, we consider the weak topology on $Y = \Lambda(A)$ with respect to $Y^* = ba^*(A)$ (see Lemma 8).

First, for a game $v \in G^{A'}$ consider the convex cone

$$
K^+_v = \left\{ \left( \sum_{S \in A'} \lambda_S \chi_S, \sum_{S \in A'} \lambda_S v(S) \right) : \lambda \in \mathbb{R}_{+}^{(A')} \right\}. \tag{4}
$$

**Definition 13.** We say that a game $v \in G^{A'}$ is Schmeidler $\kappa$-balanced if

$$
z \leq v(N)
$$

for all $z \in \mathbb{R}$ such that $(\chi_N, z) \in \overline{K^+_v}$, where $\overline{K^+_v}$ is the closure of $K^+_v$.

Observe that Schmeidler $\kappa$-balancedness implies Schmeidler balancedness, which implies (ordinary) balancedness.
Lastly, for a game \( v \in \mathcal{G}^{A'} \) let
\[
\mathbb{R}^{(A')}_* = \{ \lambda \in \mathbb{R}^{(A')} : \lambda_S \geq 0 \text{ for all } S \in A' \setminus \{N\} \}
\]
and consider the convex cone
\[
K_v = \left\{ \left( \sum_{S \in A'} \lambda_S \chi_S, \sum_{S \in A'} \lambda_S v(S) \right) : \lambda \in \mathbb{R}^{(A')}_* \right\}.
\] (5)

**Definition 14.** A game \( v \in \mathcal{G}^{A'} \) is \( \kappa \)-balanced if
\[
z \leq v(N)
\]
for all \( z \in \mathbb{R} \) such that \( (\chi_N, z) \in K_v \), where \( K_v \) is the closure of \( K_v \).

**Remark 15.** The notion of \( \kappa \)-balancedness and Schmeidler \( \kappa \)-balancedness is very closely related to the notion of supravalue introduced in Definition 2. The cone \( K_v \) or \( K_v^+ \) is precisely the set \( D \) if \( A(\lambda) = \sum_{S \in A'} \lambda_S \chi_S \) and \( c(\lambda) = \sum_{S \in A'} \lambda_S v(S) \) with \( P = \mathbb{R}_+^{(A')} \) or \( P = \mathbb{R}_*^{(A')} \), respectively, in Definition 2. Then the game is \( \kappa \)-balanced or Schmeidler \( \kappa \)-balanced, respectively, if and only if the supravalue of the related primal problem \( (P_{LP}) \) is not greater than \( v(N) \).

Notice that the notion of \( \kappa \)-balancedness is a “double” extension of Schmeidler balancedness. First, we do not take the balancing weight system alone, but we take nets of balancing weight systems. Second, we let the weight of the grand coalition be sign unrestricted. It is worth noticing that the notion of \( \kappa \)-balancedness applies its full strength when in a net of balancing weight systems the net of the weights of the grand coalition is not bounded below (see Lemma 16 below).

The insight why we need the “double” extension is the following: As we shall see, the proof of our generalized Bondareva-Shapley theorem is based on the strong duality theorem for infinite LPs (Proposition 3), which is based on separation of a closed convex set from a point (not in the set). Therefore, we need to take the weak closure of a convex set in our proof. This is why we use the nets of balancing set systems.

Regarding that the weight of the grand coalition is sign unrestricted, notice that the linear combinations of Dirac measures are \( \kappa \)-additive for any \( \kappa \), moreover, it is easy to see that the linear space spanned by the Dirac measures is weak* dense in the set of bounded additive set functions. Hence, by the results of Schmeidler (1967), Kannai (1969, 1992), and Bartl and Pintér (2022), we have a necessary and sufficient condition for the non-emptiness of
the “approximate” $\kappa$-core for any $\kappa$ for free: Schmeidler balancedness. However, we analyze the non-emptiness of the (exact) $\kappa$-core for any $\kappa$. Therefore, we set the appropriate variable (the weight of the grand coalition) in the primal problem be sign unrestricted, by which we get equality in the related constraint in the dual problem (the total mass of an allocation must exactly be the value of the grand coalition), hence we will have a necessary and sufficient condition for the non-emptiness of the $\kappa$-core for any $\kappa$: $\kappa$-balancedness.

Between Schmeidler balancedness and $\kappa$-balancedness, there lies the “double” extension of the former one, Schmeidler $\kappa$-balancedness, where only the first step is taken: we take nets of balancing weight systems. Even though we shall see later that Schmeidler $\kappa$-balancedness does not lead to new characterization results, it provides deeper understanding of the problem.

Since Schmeidler $\kappa$-balancedness is the same as $\kappa$-balancedness except that $K_\nu$ in Definition 13 is replaced by $K^+_{\nu}$ in Definition 13, by $K^+_{\nu} \subseteq K_\nu$, it is clear that $\kappa$-balancedness implies Schmeidler $\kappa$-balancedness. Furthermore, Schmeidler $\kappa$-balancedness and $\kappa$-balancedness are related by the following lemma.

**Lemma 16.** For a game $v \in \mathcal{G}^A$ it holds

\[
\sup_{(\lambda^i)_{i \in I} \subseteq \mathbb{R}^{A'}} A(\lambda^i) \xrightarrow{w} \chi_N \quad \text{if and only if} \quad \sup_{(\lambda^j)_{j \in J} \subseteq \mathbb{R}^{A'}} A(\lambda^j) \xrightarrow{w} \chi_N \quad \text{and for any } \lambda \in \mathbb{R}^{A'} \quad \lim \inf \lambda^j_N \geq -\infty
\]

where $A(\lambda) = \sum_{S \subseteq A'} \lambda_S \chi_S$ and $c(\lambda) = \sum_{S \subseteq A'} \lambda_S v(S)$ for any $\lambda \in \mathbb{R}^{A'}$.

**Proof.** The “if” part is obvious. Given a net $(\lambda^i)_{i \in I} \subseteq \mathbb{R}^{A'}$, consider the same net $(\lambda^j)_{j \in J} = (\lambda^i)_{i \in I} \subseteq \mathbb{R}^{A'}$. Notice that $\lim \inf \lambda^j_N \geq 0$.

We prove the “only if” part indirectly. Suppose the right-hand side does not hold. Then there exists a net $(\lambda^j)_{j \in J} \subseteq \mathbb{R}^{A'}$ such that $\lim \inf \lambda^j_N = L > -\infty$ and $A(\lambda^j) \xrightarrow{w} \chi_N$ with $c(\lambda^j) \xrightarrow{w} z > v(N)$.

If $L > 0$, then there exists a $j_0 \in J$ such that $j \geq j_0$ implies $\lambda^j_N \geq 0$. Consider the index set $I = \{j \in J : j \geq j_0\}$ and the net $(\lambda^i)_{i \in I} \subseteq \mathbb{R}^{A'}$, which satisfies $A(\lambda^i) \xrightarrow{w} \chi_N$ and $c(\lambda^i) \xrightarrow{w} z > v(N)$.

Assume $L \leq 0$. There exists a subnet $(\tilde{\lambda}^i)_{i \in I}$ of $(\lambda^j)_{j \in J}$ such that $\lambda^j_N \xrightarrow{w} L$. Define the net $(\bar{\lambda}^i)_{i \in I}$ as follows: for any $i \in I$ and for any $S \in A'$ let

\[
\bar{\lambda}^i_S = \begin{cases} 
0 & \text{if } S = N, \\
\lambda^j_S/(1 - L) & \text{otherwise}.
\end{cases}
\]

10
Then

\[ A(\tilde{\lambda}^i) = \sum_{\substack{S \in A' \\lambda^i \neq N}} \lambda^i_S \chi_S \frac{1}{1 - L} = \frac{A(\lambda^i) - \lambda^i_N \chi_N}{1 - L} \]

\[ \rightarrow z - L \chi(N) > \frac{v(N) - Lv(N)}{1 - L} = v(N). \]

and

\[ c(\tilde{\lambda}^i) = \sum_{\substack{S \in A' \\lambda^i \neq N}} \lambda^i_S v(S) \frac{1}{1 - L} = \frac{c(\lambda^i) - \lambda^i_N v(N)}{1 - L} \]

It follows that the left-hand side does not hold in either case, which concludes the proof.

\[ \square \]

5 The main result

The next result is our generalized Bondareva-Shapley Theorem.

**Theorem 17.** For any game \( v \in G^A \) it holds that \( \kappa \)-core \((v) \neq \emptyset \) if and only if the game is \( \kappa \)-balanced.

**Proof.** Put \( X = \mathbb{R}^{(A')} \), \( P = \mathbb{R}^{(A')}_{\geq 0} \), \( Y = \Lambda(A) \), and \( Y^* = ba^* \kappa(A) \), moreover define the mapping \( A: \mathbb{R}^{(A')} \to \Lambda(A) \) by \( A(\lambda) = \sum_{S \in A'} \lambda_S \chi_S \), let \( b = \chi_N \), and define the functional \( c: \mathbb{R}^{(A')} \to \mathbb{R} \) by \( c(\lambda) = \sum_{S \in A'} \lambda_S v(S) \). Now, consider the programs \((P_{LP})\) and \((D_{LP})\) of (1).

Notice that program \((P_{LP})\) is superconsistent and its supervalue is at least \( v(N) \). (Consider that \((A(\lambda), c(\lambda)) \in K_v \subseteq K_v \) for \( \lambda \in \mathbb{R}^{(A')} \) with \( \lambda_N = 1 \) and \( \lambda_S = 0 \) for \( S \neq N \).) Then the game is \( \kappa \)-balanced (Definition 11) if and only if the supervalue of \((P_{LP})\) is finite and not greater than \( v(N) \) (Remark 15).

Moreover, observe that a set function \( \mu \in ba^* \kappa(A) \) is feasible for \((D_{LP})\) if and only if \( \mu(S) \geq v(S) \) for all \( S \in A' \) and \( \mu(N) = v(N) \). Thus program \((D_{LP})\) is equivalent to finding an element of \( \kappa \)-core \((v)\), and its value is \( v(N) \) if it is consistent, and its value is \( +\infty \) otherwise.

Therefore by Proposition 3 the game has a non-empty \( \kappa \)-core (program \((D_{LP})\) is consistent) if and only if it is \( \kappa \)-balanced (the supervalue of program \((P_{LP})\) is not greater than \( v(N) \)).

If the player set \( N \) is finite, then so is \( A' \subseteq P(N) \), whence the cone \( K_v \) is closed. Then by Lemma \( 15 \) \( \kappa \)-balancedness reduces to Schmeidler
balancedness, which is (ordinary) balancedness \(\text{[Bondareva, 1963; Shapley, 1967; Faigle, 1989]}\), and the \(\kappa\)-core is the (ordinary) core in the finite case. We thus obtain the classical Bondareva-Shapley Theorem as a corollary of Theorem 17.

**Corollary 18** (Bondareva-Shapley Theorem). If \(N\) is finite, then the core of a game with or without restricted cooperation is non-empty if and only if the game is balanced.

Regarding Theorem 17, it is worth mentioning that while Bondareva (1963) applied the strong duality theorem to prove the Bondareva-Shapley Theorem, Shapley (1967) used a different approach. We do not go into the details, but we remark that the common point in both approaches is the application of a separating hyperplane theorem. In other words, both Bondareva’s and Shapley’s approaches are based on the same separating hyperplane theorem, practically their result is a direct corollary of that. Here we use the strong duality theorem for infinite LPs (Proposition 3 Anderson and Nash, 1987), which is also a direct corollary of the same separating hyperplane theorem.

5.1 The \(\sigma\)-additive case

In this subsection let \(\kappa = \aleph_0\). Then \(\text{ba}^{\kappa}(\mathcal{A}) = \text{ca}(\mathcal{A})\), the space of all bounded countably additive set functions on \(\mathcal{A}\). Given a game \(v \in \mathcal{G}^d\), its \(\kappa\)-core is the \(\sigma\)-additive core \(\text{ca-core}(v)\) introduced by Definition 10.

In the next example we demonstrate that there exists a Schmeidler \(\kappa\)-balanced (\(\aleph_0\)-balanced) non-negative game without restricted cooperation having its ca-core empty.

**Example 19.** Let the player set \(N = \mathbb{N}\), the system of coalitions \(\mathcal{A} = \mathcal{P}(\mathbb{N})\), and the game \(v\) be defined as follows: for any \(S \in \mathcal{A}\) let

\[
v(S) = \begin{cases} 1 & \text{if } \#(N \setminus S) \leq 1, \\ 0 & \text{otherwise.} \end{cases}
\]

We show that \(\text{ca-core}(v) = \emptyset\). If \(\mu \in \text{ca-core}(v)\), then \(\mu(N \setminus \{n\}) \geq v(N \setminus \{n\}) = 1\) and \(v(N) = 1\), whence \(\mu(\{n\}) \leq 0\). So \(0 \geq \sum_{n=1}^{\infty} \mu(\{n\}) = \mu(N) = v(N) = 1\), a contradiction.

We now show that, if \((\chi_N, z) \in K_{v}^+\), see (11), then \(z \leq 1 = v(N)\). We have \((\chi_N, z) \in K_{v}^+\) if and only if each neighborhood of the point \((\chi_N, z)\) intersects the cone \(K_{v}^+\). In particular, if \((\chi_N, z) \in K_{v}^+\), then for any natural number \(m\)
and for any \( \varepsilon > 0 \) there exists a point \((f, t) \in K^+_v\) such that \(f\) belongs to the weak neighborhood

\[
\{ f \in \Lambda(A) : \left| \delta'_i(f) - 1 \right| < \varepsilon \text{ for } i = 1, \ldots, m \},
\]

where \( \delta'_i \) is the continuous linear functional induced by the Dirac measure \( \delta_i \) concentrated at \(i\), see (2), and \(t\) belongs to the neighborhood \(\{ t \in \mathbb{R} : |t - z| < \varepsilon \}\). Hence, we have a natural number \(n\), some distinct sets \(S_0, S_1, \ldots, S_n \in A\), and some non-negative \(\lambda_{S_0}, \lambda_{S_1}, \ldots, \lambda_{S_n}\) such that \(f = \lambda_{S_0} \chi_{S_0} + \lambda_{S_1} \chi_{S_1} + \ldots + \lambda_{S_n} \chi_{S_n}\) and

\[
\sum_{j=0}^{n} \lambda_{S_j} - 1 < \varepsilon \quad \text{for } i = 1, \ldots, m \tag{6}
\]

with

\[
\sum_{j=0}^{n} \lambda_{S_j} v(S_j) - z = \left| \sum_{j=0}^{n} \lambda_{S_j} - z \right| < \varepsilon. \tag{7}
\]

We can assume w.l.o.g. that \(S_0 = N\), as well as \(\#(N \setminus S_j) = 1\) for \(j = 1, \ldots, n_1\) and \(\#(N \setminus S_j) > 1\) for \(j = n_1 + 1, \ldots, n\), where \(n_1 \leq n\).

Everything is clear if there exists an \(i \in \{1, \ldots, m\}\) such that \(i \in \bigcap_{j=1}^{n_1} S_j\). Then by (6)

\[
\sum_{j=0}^{n} \lambda_{S_j} = \sum_{j=0}^{n_1} \lambda_{S_j} \leq \sum_{j=0}^{n} \lambda_{S_j} < 1 + \varepsilon,
\]

whence \(z < 1 + 2 \varepsilon\) by (7).

In the other case we have \(m \leq n_1\) and, because the sets \(S_0, S_1, \ldots, S_n\) are pairwise distinct, for \(i = 1, \ldots, m\) we can assume w.l.o.g. that \(S_i = N \setminus \{i\}\).

By (6)

\[
\sum_{j=0}^{n} \lambda_{S_j} - \lambda_{S_i} = \sum_{j=0}^{n_1} \lambda_{S_j} - \sum_{j \neq i}^{n-n_1} \lambda_{S_j} \leq \sum_{j=0}^{n} \lambda_{S_j} < 1 + \varepsilon \quad \text{for } i = 1, \ldots, m.
\]

Summing up, we get \(m \sum_{j=0}^{n_1} \lambda_{S_i} - \sum_{i=1}^{m} \lambda_{S_i} < m + m\varepsilon\), whence \(m \sum_{j=0}^{n_1} \lambda_{S_j} - \sum_{j=0}^{n_1} \lambda_{S_i} < m + m\varepsilon\). It then follows

\[
\sum_{j=0}^{n} \lambda_{S_j} = \sum_{j=0}^{n} \lambda_{S_j} \leq \frac{m}{m-1} (1 + \varepsilon).
\]
Taking (7) into account, we obtain

\[ z < \frac{m}{m - 1}(1 + \varepsilon) + \varepsilon. \tag{8} \]

Since \( 1 + 2\varepsilon < (1 + \varepsilon)m/(m - 1) + \varepsilon \), inequality (5) holds in both cases. By that \( m \geq 2 \) and \( \varepsilon > 0 \) can be arbitrary, we conclude that \( z \leq 1 \).

**Remark 20.** Consider the game \( v \) from Example 19. Since \( \text{ca-core}(v) = \emptyset \), the game is not \( \kappa \)-balanced (\( R_0 \)-balanced). To see this, consider the sequence \( (\lambda^i)_{i=1}^\infty \), with \( \lambda^i \in \mathbb{R}_+^{(A)} \), defined as follows: for any \( i \in \mathbb{N} \) and for any \( S \in A \) let

\[ \lambda^i_S = \begin{cases} -(i - 2) & \text{if } S = N, \\ 1 & \text{if } S = N \setminus \{n\} \text{ for } n = 1, \ldots, i, \\ 0 & \text{otherwise}. \end{cases} \]

Then \( \sum_{S \in A} \lambda^i_S \chi_S = 2\chi_N - \chi_{\{1, \ldots, i\}} \xrightarrow{w} \chi_N \), where the weak convergence in the space \( \Lambda(A) \) is with respect to \( \text{ca}(A) \), and \( \sum_{S \in A} \lambda^i_S v(S) = 2 > 1 = v(N) \).

Notice again that the sequence \( (\lambda^i_N)_{i=1}^\infty = (2-i)_{i=1}^\infty \) is unbounded below. If \( (\lambda^i_N)_{i=1}^\infty \) were bounded below, then by Lemma 16 we would get a contradiction with Example 19.

Example 19 demonstrates that it is not sufficient to use \( \mathbb{R}_+^{(A)} \) and \( K_+^{A} \) in the definition of \( \kappa \)-balancedness; that is, Schmeidler \( \kappa \)-balancedness is unable to reveal that the ca-core is empty even for non-negative games without restricted cooperation.

**Remark 21.** Reconsidering Schmeidler balancedness for the additive case, it is somehow tempting to ask whether the following “\( \sigma \)-extension” of condition (3) could lead to a similar result in the \( \sigma \)-additive case too:

\[ \sup\left\{ \sum_{S \in A} \lambda_S v(S) : \sum_{S \in A} \lambda_S \chi_S = \chi_N, \lambda \in \mathbb{R}_+^{|A'|} \right\} \leq v(N), \tag{9} \]

where \( \mathbb{R}^{|A'|} \) = \( \{ \lambda \in \mathbb{R}^{A'} : \#\{ S \in A' : \lambda_S \neq 0 \} \leq 8 \} \) and \( \mathbb{R}_+^{|A'|} \) = \( \{ \lambda \in \mathbb{R}^{|A'|} : \lambda_S \geq 0 \text{ for all } S \in A' \} \). Moreover, the convergence of the sum \( \sum_{S \in A} \lambda_S v(S) \) is understood pointwise. In this case it is equivalent to say that the convergence is weak in the space \( \Lambda(A) \) with respect to \( \text{ca}(A) \). If the sum \( \sum_{S \in A} \lambda_S v(S) \) is convergent, but not absolutely convergent, then we put \( \sum_{S \in A} \lambda_S v(S) := +\infty \).

Denoting \( A(\lambda) = \sum_{S \in A'} \lambda_S \chi_S \) and \( c(\lambda) = \sum_{S \in A'} \lambda_S v(S) \), we can also consider the following generalization of (9). Let \( z \leq v(N) \) whenever there exists a net \( (\lambda_i)_{i \in I} \subseteq \mathbb{R}_+^{|A'|} \) such that \( A(\lambda_i) \xrightarrow{w} \chi_N \) and \( c(\lambda_i) \longrightarrow z \) where
z is finite. Then for each $i \in I$ there exists a sequence $(\lambda^m)_{m=1}^{\infty} \subseteq \mathbb{R}^{(A')}_{+}$ such that $A(\lambda^m) \xrightarrow{w} A(\lambda^i)$ and $c(\lambda^m) \rightarrow c(\lambda^i)$. Consequently, there exists a net $(\lambda_j)_{j \in J} \subseteq \mathbb{R}^{(A')}_{+}$ such that $A(\lambda^j) \xrightarrow{w} \chi_N$ and $c(\lambda^j) \rightarrow z$. In other words, Schmeidler $\kappa$-balancedness covers such extensions of Schmeidler balancedness (Definition 12) like (9).

Moreover, in Example 19 we presented a non-negative Schmeidler $\kappa$-balanced game. Therefore, the presented game is balanced according to (9) too, but the ca-core of the game is empty.

6 Conclusion

We have generalized the Bondareva-Shapley Theorem to TU games with and without restricted cooperation, with infinitely many players, and with at least $\sigma$-additive cores: we have proved for an arbitrary infinite cardinal $\kappa$ that the $\kappa$-core of a TU game with or without restricted cooperation is not empty if and only if the TU game is $\kappa$-balanced. The main conceptual messages of our results might be that in the proper notion of balancing weight system the weight of the grand coalition is sign unrestricted.

While $\kappa$-balancedness is universally a necessary and sufficient condition for that the $\kappa$-core of a game with or without restricted cooperation is not empty (Theorem 17), we have shown that Schmeidler $\kappa$-balancedness (which implies Schmeidler balancedness as well as its $\sigma$-extension, see Remark 21) is not suitable for this purpose even in the case of $\sigma$-additive core of a non-negative game without restricted cooperation (Example 19).

Notice that Kannai (1969, 1992) gave another necessary and sufficient condition for that the $\sigma$-additive core of a non-negative game without restricted cooperation is not empty. Kannai’s result is based on a very different approach and not related directly to our $\aleph_0$-balancedness condition.

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