A short essay on quantum black holes and underlying noncommutative quantized space-time

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Abstract
We emphasize the importance of noncommutative geometry or Lorenz-covariant quantized space-time towards the ultimate theory of quantum gravity and Planck scale physics. We focus our attention on the statistical and substantial understanding of the Bekenstein–Hawking area-entropy law of black holes in terms of the kinematical holographic relation (KHR). KHR manifestly holds in Yang’s quantized space-time as the result of kinematical reduction of spatial degrees of freedom caused by its own nature of noncommutative geometry, and plays an important role in our approach without any recourse to the familiar hypothesis, so-called holographic principle. In the present paper, we find a unified form of KHR applicable to the whole region ranging from macroscopic to microscopic scales in spatial dimension $d = 3$. We notice a possibility of nontrivial modification of area-entropy law of black holes which becomes most remarkable in the extremely microscopic system close to Planck scale.

Keywords: Kinematical reduction of spatial degrees of freedom, kinematical holographic relation (KHR), quantum black holes over arbitrary scales, noncommutative quantized space-time

1. Introduction

In our preceding paper, ‘Where does black-hole entropy lie?—Some remarks on area-entropy law, holographic principle and noncommutative space-time’ (2014) [1], hereafter referred to as I, we emphasized the importance of underlying noncommutative geometry or quantized space-time such as Snyder’s and Yang’s Lorenz-covariant quantized space-time [2–5] towards the ultimate theory of quantum gravity and Planck scale physics. We focused our attention there most importantly on the statistical and substantial understanding of the Bekenstein–Hawking...
area-entropy law of black holes [6–9] in terms of the kinematical holographic relation (KHR) [10]. Indeed, as will be simply reviewed in section 2, KHR manifestly holds in Yang’s quantized space-time as the result of kinematical reduction of spatial degrees of freedom caused by its own nature of noncommutative geometry, and plays an important role in our approach, without any recourse to the familiar hypothesis, the so-called holographic principle.

In the present paper, first of all, we find out in section 3 the following important fact that KHR given in I gets a simple and unified form equally applicable to the whole regions ranging from macroscopic to microscopic. As a result, the new form of KHR enables us to reconsider area-entropy law of black holes all over the regions from macroscopic to extremely microscopic (see section 4). And finally in section 5 we notice an important possibility of the nontrivial modification of area-entropy law of black holes, which becomes remarkable in the extremely microscopic black holes close to Planck scale.

The present paper is organized as follows. In section 2, we first review the derivation of the approximate form of KHR mentioned above for the subsequent arguments. Section 3 is devoted to our central concern in our present research, that is, in section 3.1 we notice first of all the unified form KHR: \(n_{\text{dof}}(V^L) = (L/\lambda)^d + 1\) in place of the approximate form of KHR and in section 3.2 we statistically derive the entropy and mass of \(D_0\) brane gas systems ranging from macroscopic- to microscopic- scales in \(d = 3\), under the full use of the novel KHR mentioned above. Section 4 is devoted to Schwarzschild black holes ranging from macroscopic to extremely microscopic scales in \(d = 3\), noticing there the existence and different behavior of two kinds of temperatures, \(T_{\text{HR}}\) and \(T_S\) of black holes in both regions. In the final section 5, ‘Concluding arguments and further outlook’, first we give the supplementary and summarizing arguments on two kinds of temperatures of black holes (section 5.1) and notice the possible limits of applicability of area-entropy law of black holes (section 5.2). Appendix A is devoted to the review of Yang’s quantized space-time and appendix B to the historical background of noncommutative quantized space-time, recollecting the related works by W K Heisenberg, P A M Dirac and H Yukawa.

2. Review of kinematical holographic relation (KHR) and area-entropy law of black holes

Let us briefly review here the derivation of KHR for the subsequent arguments. In I, we started with the kinematical holographic relation (KHR) mainly for the Macroscopic system given in the following form

\[
\text{KHR} \quad n_{\text{dof}}(V^L) = A(V^L)/f_d,
\]  

(2.1)

that is, the proportional relation between \(n_{\text{dof}}(V^L)\) and \(A(V^L)\) with proportional constant \(f_d^2\), where \(n_{\text{dof}}(V^L)\) and \(A(V^L)\), respectively, denote the number of degrees of freedom of any \(d\) dimensional bounded spatial region \(V^L_d\) with radius \(L\) in Yang’s quantized space-time, and the boundary area of \(V^L_d\) in units of Planck length.

First, the region \(V^L_d\) is defined on any \(d\)-dimensional quantized spatial coordinate operators,

\[
\hat{X}_0^2 + \hat{X}_1^2 + \cdots + \hat{X}_d^2 = L^2.
\]  

(2.2)

As was shown in detail in I (section 3), the most important concept, the number of degrees of freedom of \(V^L_d\), \(n_{\text{dof}}(V^L)\) in equation (2.1), is found in a certain irreducible representation of \(SO(d + 1)\), a minimum subalgebra of Yang quantized space-time, which includes the above

\(^2\)We use hereafter \(f_d\) instead of the misleading notation \(G_d\) used in I.
d spatial coordinate operators, $\hat{X}_1, \hat{X}_2, \cdots, \hat{X}_d$ in equation \((2.2)\) needed to properly describe $V_d^L$, and is really constructed by the generators $\hat{S}_{MN}$ with $M, N$ ranging over $1, 2, \cdots, d$ (See appendix A.). Let us denote the irreducible representation by $\rho(L\lambda)$ with the characteristic integer $l$ which indicates the maximal eigenvalue of any generators, $\hat{S}_{MN}$ of $SO(d+1)$. Then, $n_{\text{def}}(V_d^L)$ is reasonably identified with the dimension of $\rho(L\lambda)$ that is, $n_{\text{def}}(V_d^L) = \dim(\rho(L\lambda)(V_d^L))$.

According to the Weyl dimension formula applied to the irreducible representation of $SO(d+1)$, the dimension of $V_d^L$ is given by

$$\dim(\rho_l) = \frac{l + \nu}{\nu} \frac{(l + 2\nu - 1)!}{l!(2\nu - 1)!},$$

with $\nu = (d - 1)/2$ in the case $d \geq 2$. (see, more in detail \[10\])

One immediately finds that

$$n_{\text{def}}(V_d^L) = \dim(\rho(L\lambda)(V_d^L)) = \frac{2[L/\lambda] + d - 1}{2} \frac{([L/\lambda] - 1)!}{![L/\lambda] - 1]!} (d - 1)! \sim \frac{2}{(d - 1)!} [L/\lambda]^{d-1},$$

where $[L/\lambda]$ denotes the nearest integer of $L/\lambda$ and $\lambda$ the short scale parameter in Yang’s quantized space time (see appendix A) and identified with Planck length $l_G = \sqrt{\frac{\hbar c}{G}}$ in I and in what follows. The expression in the last line holds for a macroscopic system with $[L/\lambda] \gg d$, which was considered in I.

On the other hand, the boundary area of $V_d^L$ in the unit of $\lambda$, $A(V_d^L)$ is given by $S^{d-1}$ with radius $L/\lambda$, that is.

$$A(V_d^L) = \frac{(2\pi)^{d/2}}{d-2} \frac{(L/\lambda)^{d-1}}{1!!} \text{ for } d \text{ even}$$

$$= 2 \frac{(2\pi)^{(d-1)/2}}{(d-2)!!} \frac{(L/\lambda)^{d-1}}{1!!} \text{ for } d \text{ odd}.$$  \hspace{1cm} \text{(2.5)}

Comparing both equations \((2.4)\) and \((2.5)\), one finally arrives at KHR \((2.1)\) with $f_d$ given by

$$f_d \sim \frac{(2\pi)^{d/2}}{2} \frac{(d - 1)!!}{2(d - 2)!!} \text{ for } d \text{ even}$$

$$\sim \frac{(2\pi)^{(d-1)/2}}{2} \frac{(d - 1)!!}{2(d - 2)!!} \text{ for } d \text{ odd}.$$  \hspace{1cm} \text{(2.6)}

for the macroscopic system with $[L/\lambda] \gg d$, without any recourse to the familiar hypothesis, so-called holographic principle (see, for instance, \[20–22\]).

As was emphasized in I, the spatial structure of $V_d^L$ is described through some specific representation $\rho(L\lambda)(V_d^L)$. Let us denote its orthogonal basis-vector system in the representation space, which we called Hilbert space I, as follows

$$\rho(L\lambda)(V_d^L) : |m\rangle, \quad m = 1, 2, \cdots, n_{\text{def}}(V_d^L).$$

The labeling number $m$ of basis vectors in ‘Hilbert space I’, plays the role of classical spatial coordinates of the classical space inside $V_d^L$ and we called the point [site] or [site $m$].

It is easy to imagine that KHR equation \((2.1)\) strongly suggests that the entropy of any statistical system realized in the spatial region $V_d^L$ must be proportional not to the classical
volume of $V_d^L$, but to the degrees of freedoms $n_{\text{def}}(V_d^L) = (A(V_d^L)/\eta)$, namely, it yields a new area-entropy law. Indeed, in I, we derived the following form of a new area-entropy relation of a black hole in a purely statistical way, through a simple $D_0$ brane gas model constructed in Yang’s quantized space time

$$S_\text{site} = n_{\text{def}}(V_d^L)S_\text{site}[\text{site}].$$

(2.8)

Indeed, the relation was shown by equation (53) in I, and in the related argument we concluded that $S_\text{site}[\text{site}]$ represents a kind of universal unit of entropy of black holes, which appears as the entropy realized on each individual [site] in any black hole, by taking a proper specific value,

$$S_\text{site}[\text{site}] = 4\pi \eta$$

(2.9)

and thus

$$S_\text{site}[\text{site}] = \pi$$

(2.10)

under Bekenstein parameter $\eta = 1/4$.

In the present paper, the above argument given in I focusing our attention on the area-entropy law of macroscopic black holes will be reconsidered in the final section from the viewpoint of the limits of applicability of the Bekenstein–Hawking area-entropy law, on the basis of unified consideration of black holes ranging from the macroscopic to the extremely microscopic scales given in the next section.

3. Entropy $S(V_3^L)$ and mass $M(V_3^L)$ of $D_0$ Brane gas systems ranging from macroscopic to microscopic scales

3.1. KHR in Yang’s quantized space-time with $d = 3$

Now, let us examine more in detail—KHR equation (2.1) together with equation (2.4), specifically in $d = 3$. First of all, we notice that there holds the following simple and unified expression

$$n_{\text{def}}(V_3^L) = \dim (\rho_{(L/\lambda)}(V_3^L)) = ([L/\lambda] + 1)^2$$

(3.1)

corresponding to equation (2.4) specifically in the case of $d = 3$, without any approximation and thus it enables us safely to investigate the structure of the microscopic system, together with the macroscopic system which was reviewed in the preceding section.

Here, one should notice very importantly that the above relation (3.1) manifestly shows ‘Kinematical reduction of spatial degrees of freedom’ [10]. That is, $n_{\text{def}}(V_3^L) = ([L/\lambda] + 1)^2$ is not proportional to the order of $V_3^L$ or $(L/\lambda)^3$ but proportional to $([L/\lambda])^2$, remarkably for the macroscopic system, on account of its own nature of underlying noncommutative quantized space and time, and leads us automatically to the kinematical holographic relation KHR as shown below without any recourse to the so-called holographic principle [21, 22].

Indeed, in $d = 3$, the boundary area $A(V_3^L)$ is given by equation (2.5), that is, $A(V_3^L) = 4\pi (L/\lambda)^2$, so the relation (3.1) leads us to the following form of the kinematical holographic relation

$$\text{KHR} \quad n_{\text{def}}(V_3^L) \cong ((A(V_3^L)/4\pi)^{1/2} + 1)^2$$

(3.2)

taking into consideration the possible slight difference between $L/\lambda$ and $[L/\lambda]$. 
Meanwhile, as was remarked at the end of the preceding section and will be discussed in section 5.2 in the final section, one should notice that this form of KHR equation (3.2) has a possibility of causing significant change of area-entropy law of black holes ranging from macroscopic to microscopic scales, by rewriting equation (3.1) in the following form

$$KHR' \quad n_{\text{dof}}(V^L_3) \cong A(V^L_3)/4\pi + 2(A(V^L_3)/4\pi)^{3/2} + 1,$$  

(3.3)

which surely reproduces the relation equation (2.1) with \(f_2 \sim 4\pi\) given for the macroscopic system.

3.2. Statistical derivation of \(S(V^L_3)\) and \(M(V^L_3)\) based on KHR

According to I, let us consider the quantum system realized in Yang’s quantized space-time, which constitutes of \(D_0\)-branes or \(D\)-particles [11, 12]. As was done in I, \(D_0\) brane gas system formed inside \(V^L_3\) is most likely described in terms of the second-quantized field of \(D_0\) brane or \(D\)-particle defined in Yang quantized space-time, \(V^L_3\).

Corresponding to equation (2.7), the representation space of \(\rho_{[L/M]}(V^L_3)\), called ‘Hilbert space I’ in I (in distinction to ‘Hilbert space II’), one finds

$$\rho_{[L/M]}(V^L_3) : \quad |m\rangle, m = 1, 2, \ldots, ([L/\lambda] + 1)^2 (= n_{\text{dof}}(V^L_3)).$$  

(3.4)

Needless to say, under this representation, each spatial operator \(\hat{X}_i\)'s becomes expressed by \(((L/\lambda) + 1)^2 \times ((L/\lambda) + 1)^2\) matrix like \(\langle m | \hat{X}_i | n\rangle\).

Let us simply assume that the quantum states of \(D_0\) branes of microscopic system are constructed in ‘Hilbert space II’, in a similar way, as was done for the macroscopic system in I. Indeed, we assume that even in the present microscopic system, \(D_0\) brane gas model holds where all interactions among \(D_0\) branes are ignored so that the statistical operator at each [site \(m\)], \(W[m]\) is common to every [ site] and given in the following form,

$$W[m] = \sum_k w_k \langle m : k \rangle \langle k : m \rangle,$$  

(3.5)

with

$$|m : k\rangle = \frac{1}{\sqrt{k!}} (a^\dagger_m)^k |m : 0\rangle.$$  

(3.6)

In the above expression, \(a^\dagger_m\) and \(|m : k\rangle\) (\(k = 0, 1, \ldots\)), respectively, denote creation operator of \(D\)-particle at [ site \(m\)] (see I, section 4.1) and the normalized quantum-mechanical state in Hilbert space II with \(k D_0\) branes constructed by \(a^\dagger_m\) on \(|m : 0\rangle\), i.e. the vacuum state of [site \(m\)]. Namely, we assume that the \(D_0\) brane gas system is under a static and equilibrium state with temperature \(T\) and the statistical operator at each [site \(m\)] is common to every [ site] with the common values \(w_k\)'s:

$$w_k = e^{-\mu k T} Z(T),$$  

(3.7)

where

$$Z(T) \equiv \sum_{k=0}^{\infty} e^{-\mu k T} = 1/(1 - e^{-\mu/T})$$  

(3.8)

and \(\mu\) denotes the average energy or effective mass of the individual \(D_0\) brane in \(V^L_3\).
The statistical operator of total system in $V_L^3$, $W(V_L^3)$, is now given by

$$W(V_L^3) = W[1] \otimes W[2] \otimes \cdots \otimes W[(L/\lambda) + 1]^2].$$

(3.9)

Consequently, one finds that the entropy and the energy or effective mass of the total system, $S(V_L^3)$ and $M(V_L^3)$ are respectively given by

$$S(V_L^3) = -\text{Tr} \left[ W(V_L^3) \ln W(V_L^3) \right] = n_{\text{dof}}(V_L^3)S[\text{site}]$$

$$= (L/\lambda) + 1)^2S[\text{site}]$$

(3.10)

and

$$M(V_L^3) = n_{\text{dof}}(V_L^3)\mu \bar{N}[\text{site}] = (L/\lambda) + 1)^2 \mu \bar{N}[\text{site}]$$

(3.11)

corresponding to equations (32) and (33) in I, respectively. In the above expressions, $S[\text{site}]$ in equation (3.10) denotes the entropy of each site assumed to be common to every site and is given by

$$S[\text{site}] \equiv -\sum_k w_k \ln w_k = \frac{\mu \bar{N}[\text{site}]}{T} + \ln Z(T)$$

$$= -\ln(1 - e^{-\mu T}) + \frac{\mu}{T} (e^{\mu T} - 1)^{-1},$$

(3.12)

and $\bar{N}[\text{site}]$ in equation (3.11) the average occupation number of $D_0$ brane at each site

$$\bar{N}[\text{site}] \equiv \sum_k w_k = (e^{\mu T} - 1)^{-1}.$$  

(3.13)

At the end of this section, let us notice the following two relations

$$T = \mu \ln(1 + \bar{N}^{-1}[\text{site}])$$

(3.14)

and

$$S[\text{site}] = \ln(1 + \bar{N}[\text{site}]) + \bar{N}[\text{site}] \ln(1 + \bar{N}^{-1}[\text{site}]),$$

(3.15)

which are simply derived from equations (3.12) and (3.13), respectively.

4. Schwarzschild black holes ranging from macroscopic to extremely microscopic scales

Now, according to the consideration given in I, let us assume that the present $D_0$ brane gas system ranging from macroscopic- to microscopic-scales in $d = 3$, considered in the preceding section 3.2, transforms into a Schwarzschild black hole. Indeed, as was done in I, we assume that the relevant quantities acquire certain limiting values, such as $\mu_S$, $\bar{N}_S[\text{site}]$ and $S_\text{d}[\text{site}]$, while the size of the gas system, $L$, becomes $R_S$, that is, the so-called Schwarzschild radius given by

$$R_S = \frac{2GM_S(V_L^3)c^2}{c^2},$$

(4.1)

where $G$ and $c$ denote Newton’s constant and the light velocity, respectively, and $M_S(V_L^3)$ is given by equation (3.11) with $L = R_S$, $\mu = \mu_S$ and $\bar{N}[\text{site}] = \bar{N}_S[\text{site}]$. Indeed, inserting the above values into (3.11), we arrive at the important relation, called hereafter the black hole condition BHC, that is,
In the last expression, we assumed that \( \lambda \), i.e. the short scale parameter in Yang–quantized space-time (see appendix A) is identified with Planck length \( l_P = [\hbar c^3]/[\hbar c M_P] \), where \( M_P \) denotes Planck mass. In what follows, we will use Planck units in \( D = 4 \) or \( d = 3 \), with \( M_P = l_P = h = c = k = 1 \), where \( k \) is Boltzmann’s constant [22].

According to the above consideration, let us notice \( S_3(V_{S}^{R}) \) given through equation (3.10), that is,

\[
S_3(V_{S}^{R}) (= n_{dof}(V_{S}^{R})S_3[\text{site}] ) = ([R_3/\lambda] + 1)^2S_3[\text{site}].
\]

Furthermore, it is important here to notice that, by using equation (3.15), one finds the following relation

\[
S_3[\text{site}] = \ln(1 + \tilde{N}_S[\text{site}]) + \tilde{N}_S[\text{site}] \ln(1 + \tilde{N}_S^{-1}[\text{site}])
\]

which shows the fact that \( \tilde{N}_S[\text{site}] \) gets some universal and fixed value, that is,

\[
\tilde{N}_S[\text{site}] \sim 1/0.12,
\]

under \( S_3[\text{site}] = \pi \) equation (2.10), that is, our basic assumption on \( S_3[\text{site}] = 4\pi \eta \) equation (2.9) with \( \eta = 1/4 \).

4.1. Macroscopic black holes in \( d = 3 \)

Now, let us notice that the BHC (4.2) becomes for the macroscopic scales of black holes (\( M_3(V_{S}^{R}) \gg M_P \))

\[
\text{BHC} \quad M_3(V_{S}^{R}) = \frac{M_P^2}{4\mu_S \tilde{N}_S[\text{site}]} \quad (\text{for } M_3(V_{S}^{R}) \gg M_P),
\]

which reproduces equation (41) in I.

On the other hand, the above relation (4.6) leads us to the following universal relation for the macroscopic black holes

\[
\mu_S M_S \sim 0.03 \quad (\text{for } M_3(V_{S}^{R}) \gg M_P)
\]

in Planck units, on account of equation (4.5).

4.2. Two kinds of temperatures of macroscopic black holes, \( T_{\text{H.R.}} \) and \( T_S \) in \( d = 3 \)

As was once pointed out and argued in [12], one should notice here the fact that there exist two kinds of temperatures of black holes, \( T_{\text{H.R.}} \) and \( T_S \).

The first one \( T_{\text{H.R.}} \) is the familiar Hawking’s radiation temperature, which is given by using equation (4.3) in the following way:

\[
T_{\text{H.R.}}^{-1} = \frac{d}{dM_S} S_3(V_{S}^{R}) = \frac{d}{dM_S} ([R_3/\lambda] + 1)^2S_3[\text{site}]
\]

\[
= 4\pi(2M_S + 1)
\]
or

\[ T_{\text{HR}} = 1/(4\pi M_S + 1). \] (4.9)

In the above derivation of equations (4.8) and (4.9), we assumed implicitly that \( S_{\text{site}} \) is independent of \( M_S \) and \( S_{\text{site}} = \pi \) according to the preceding arguments on the idea of universality of \( S_{\text{site}} \), given in connection with equations (2.9) and (2.10). Further, one should notice that the relations equation (4.8) and thus equation (4.9) are based on the relation \( S(V^3_{\lambda}) = (R_S/\lambda + 1/2)S_{\text{site}} \) (equation (4.3)). In this connection, we notice that \( T_{\text{HR}} = 1/(4\pi M_S + 1) \) (equation (4.9)) reproduces nicely the familiar result \( T_{\text{HR}} = 1/(8\pi M_S) \) for the macroscopic black holes (see equation (48) in I). On the other hand, however, it implies a possibility of causing the nontrivial modification for the extremely microscopic black holes, as will be shown in the next sections 4.3 and 4.4.

The second one \( T_5 \) is derived through equation (3.14)

\[ T_5 = \mu_S/\ln(1 + \tilde{N}_S^{-1}[\text{site}]) \] (4.10)

or

\[ T_5 = \mu_S/\ln(1 + 4\mu_S M_S), \] (4.11)

on account of equation (4.6).

For the macroscopic black holes, equations (4.9) and (4.7) show some similarity between the order of magnitudes of \( T_{\text{HR}} \sim 1/(8\pi M_S) \sim 0.04/M_S \) and \( \mu_S \sim 0.03/M_S \), that is,

\[ T_{\text{HR}} \sim (0.04/0.03) \mu_S \sim 1.33\mu_S. \] (4.12)

In contrast, one finds in this case

\[ T_5 \sim (10.11) \mu_S \sim 9.09\mu_S \] (4.13)

from equation (4.10) or (4.11). The physical implication of equations (4.12) and (4.13) will be discussed at the end of the section 5.1 in comparison with the corresponding result of the extremely microscopic black hole given in the next sections 4.3 and 4.4.

### 4.3. Extremely microscopic black hole system in \( d = 3 \)

According to the argument given in the beginning of this section, now we consider the extremely microscopic black hole system with \( R_S = L = \lambda (= l_\text{p}) \) or \( [L/\lambda] = 1 \) in \( d = 3 \).

Let us denote hereafter the relevant quantities \( M_S, T_{\text{HR}}, T_5 \) and so on by attaching the tilde-mark for example \( \tilde{M}_S \), showing their specific values proper to the extremely microscopic system.

First of all, in the extremely microscopic system, one finds

\[ n_{\text{def}}(V^3_\lambda) = 4, \] (4.14)

from equation (3.1), and correspondingly

\[ \tilde{S}_S(V^3_\lambda) = n_{\text{def}}(V^3_\lambda)S_{\text{site}} = 4S_{\text{site}}, \] (4.15)

\[ \tilde{M}_S(V^3_\lambda) = n_{\text{def}}(V^3_\lambda)\tilde{M}_S[\text{site}] = 4\tilde{M}_S[\text{site}] \] (4.16)

to hold, from equations (3.10) and (3.11), respectively. The latter relation is further constrained from equation (4.1) as
\[ \mathcal{M}_S(V_3^{R(=\lambda)}) (= R_S/2 = \lambda/2) = 1/2 \]  

(4.17)

in Planck units.

Then, from equation (4.16) combined with equations (4.5) and (4.17), one finds

\[ \mu_S (= \mathcal{M}_S(V_3^{\lambda})/(4\tilde{N}[\text{site}]^\lambda)) = \frac{1}{8} \times 0.12 \sim 0.02 . \]  

(4.18)

Finally, let us notice that equation (4.15) tells us that the entropy of extremely microscopic black hole is given by

\[ S_S(V_3^{\lambda}) = n_{\text{det}}(V_3^{\lambda})S_S^\lambda[\text{site}] = (4\tilde{S}_S[\text{site}]) = 4\pi, \]  

(4.19)

under \( S_S^\lambda[\text{site}] = \pi \) equation (2.10).

4.4. Two kinds of temperatures of extremely microscopic black hole, \( \tilde{T}_{\text{H.R.}} \) and \( \tilde{T}_S \) in \( d = 3 \)

Corresponding to the arguments given in section 4.2, let us consider two kinds of temperatures, \( \tilde{T}_{\text{H.R.}} \) and \( \tilde{T}_S \) of an extremely microscopic black hole system.

First of all, let us notice from equation (4.17)

\[ \mathcal{M}_S = 1/2 . \]  

(4.20)

With respect to \( \tilde{T}_{\text{H.R.}} \), as was emphasized in section 4.2, the expression \( \tilde{T}_{\text{H.R.}} = 1/(4\pi(2\mathcal{M}_S + 1)) \) given in equation (4.9) holds ranging from a macroscopic to an extremely microscopic system, so one immediately gets the following result

\[ \tilde{T}_{\text{H.R.}} (= 1/(4\pi(2\mathcal{M}_S + 1))) = 1/(8\pi) \sim 0.04 \]  

(4.21)

on the basis of equation (4.20).

On the other hand, according to equation (3.14), \( \tilde{T}_S \) is simply given by

\[ \tilde{T}_S = \mu_S / \ln(1 + \tilde{N}_S^{-1}[\text{site}]). \]  

(4.22)

By using \( \tilde{N}_S[\text{site}] \sim 1/0.12 \) equation (4.5), one gets

\[ \tilde{T}_S = \mu_S / \ln(1 + 0.12) \sim (1/0.11)\mu_S \sim 9.09\mu_S , \]  

(4.23)

that is, the parallel result with equation (4.13). Further, by using the result equation (4.18), one finally arrives at the result

\[ \tilde{T}_S \sim 9.09 \times 0.02 \sim 0.18 . \]  

(4.24)

5. Concluding arguments and further outlook

5.1. Two kinds of temperatures of black holes

Let us reconsider the arguments about two kinds of temperatures of black holes given in sections 4.2 and 4.4. First, concerning section 4.2 devoted to macroscopic black holes, we note the following three relations

\[ T_S/T_{\text{H.R.}} (\sim 9.09/1.33) \sim 6.83 , \]  

\[ T_{\text{H.R.}} (\sim (0.04/0.03))\mu_S \sim 1.33\mu_S , \]  

\[ \tilde{T}_S \sim 9.09\mu_S . \]  

(5.1)
Next, concerning section 4.4 devoted to the extremely microscopic black hole, we note the corresponding three relations

\[ T_S / T_{\text{H.R.}} \sim 0.18 / 0.04 \sim 4.5, \]

\[ T_{\text{H.R.}} \sim (0.04 / 0.02) \mu_s \sim 2.00 \mu_s, \]

\[ T_S \sim 9.09 \mu_s. \]  

(5.2)

With respect to the marked difference between \( T_S \) and \( T_{\text{H.R.}} \) as seen in (5.1) and (5.2), one should notice that \( T_S \) means the statistical and equilibrium temperature of \( D_0 \) brane gas (see equation (3.7)), that is, the temperature inside of black hole, while \( T_{\text{H.R.}} \) is the thermodynamical temperature observed from outside of black hole.

We anticipate that the above arguments of possible existence and different behavior of two kinds of temperatures of black holes might be instructive for forthcoming researches on formation and evaporation of black holes which may be closely related to the whole scales of black holes ranging from macroscopic to extremely microscopic.

5.2. Possible modification of area-entropy law of black holes

Finally, we reconsider our central concern, the universality of area-entropy law of black holes. By applying KHR’ (3.3) to equation (3.10) which is derived through our simple D-particle gas model in section 3, we have

\[ S(V_4^2) \geq (A(V_4^2)/4\pi + 2(A(V_4^2)/4\pi)^{1/2} + 1)S[\text{site}]. \]  

(5.3)

For the macroscopic system, the first term on the right hand side becomes the dominant term and the relation finally leads us to the Bekenstein–Hawking area-entropy law of black holes under the assumption \( S[\text{site}] = \pi \) equation (2.10).

On the other hand, one finds that for the extremely microscopic system, the relation equation (5.3) just leads us to equations (4.15) and (4.19) on account of \( A(V_4^2) = 4\pi \). This fact, however, implies very importantly that for the black holes of intermediate scales between macroscopic and extremely microscopic, the second term on the right hand side of equation (5.3) has a possibility of causing the significant correction term proportional to \((A(V_4^2)/4\pi)^{1/2}\) to the familiar Bekenstein–Hawking area-entropy law.

We expect that such a possible modification of area-entropy law of black holes will shed a new light on the resolution of our serious question, Where does black hole entropy lie? [1] and related fundamental problems [6–9, 13, 14].

As was remarked at the end in I [1], Kinematical reduction of spatial degrees of freedom which underlies KHR may be expected to hold widely in the noncommutative space-time in general. Indeed, one can easily confirm that the original Snyder’s quantized space-time satisfies it entirely in the same way as in the case of Yang’s quantized space-time shown in sections 2 and 3.

Before closing this short essay, let us note another interesting possibility of Yang’s quantized space-time algebra (YSTA, see appendix A). Indeed, one should notice that YSTA is intrinsically equipped with the long scale parameter \( R \), together with the short scale parameter \( a \) which has been identified with Planck length in our present research so far. On the other hand, as was preliminarily pointed out in [25], \( R \) might be promisingly related to a fundamental cosmological constant in connection with the recent dark-energy problem, under the further idea that YSTA subject to the \( SO(D + 1, 1) \) algebra (see, appendix A) might be understood in terms of a some kind of local reference frame in the ultimate theory of quantum gravity, on the analogy of the familiar local Lorentz frame in Einstein’s general theory of relativity.
In this connection, we know that recently the issue of quantum space-time with nonvanishing cosmological constant has been addressed in the literature by several authors (See, for instance, [26, 27]). It is quite interesting to examine their possible relations with our present approach.

We emphasize again the importance and the necessity of noncommutative geometry or more specifically Yang’s quantized space-time towards the ultimate theory of quantum gravity and Planck scale physics. It is our urgent task to reconstruct M-theory [20] in terms of noncommutative quantized space-time along this line of thought.

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Appendix A. Yang’s Lorentz covariant quantized space-time

Let us here briefly review the Lorentz-covariant Yang’s quantized space-time [3, 4]. $D$-dimensional Yang’s quantized space-time algebra (YSTA) was introduced as the result of the so-called Inonu–Wigner contraction procedure with two contraction parameters, long $R$ and short $\lambda^3$, from $SO(D+1, 1)$ algebra with generators $\hat{\Sigma}_{MN}$ [9] (see, more in detail, [10])

\[ \hat{\Sigma}_{MN} \equiv i(q_M \partial \partial q_N - q_N \partial \partial q_M), \] (A.1)

which work on $(D + 2)$-dimensional parameter space $q_M (M = \mu, a, b)$ satisfying

\[ -q_0^2 + q_1^2 + \cdots + q_{D-1}^2 + q_a^2 + q_b^2 = R^2. \] (A.2)

Here, $q_0 = -i\eta_{0\mu}$ and $M = a, b$ denote two extra dimensions with space-like metric signature. $D$-dimensional space-time and momentum operators, $\hat{X}_\mu$ and $\hat{P}_\mu$, with $\mu = 1, 2, \cdots, D$, are defined in parallel by

\[ \hat{X}_\mu \equiv \lambda \hat{\Sigma}_{\mu a}, \] (A.3)
\[ \hat{P}_\mu \equiv \hbar/R \hat{\Sigma}_{a b}, \] (A.4)

together with $D$-dimensional angular momentum operator $\hat{M}_{\mu \nu}$

\[ \hat{M}_{\mu \nu} \equiv \hbar \hat{\Sigma}_{\mu \nu}, \] (A.5)

and the so-called reciprocity operator

\[ \hat{N} \equiv \lambda R \hat{\Sigma}_{a b}. \] (A.6)

Operators $\{\hat{X}_\mu, \hat{P}_\mu, \hat{M}_{\mu \nu}, \hat{N}\}$ defined above satisfy the so-called contracted algebra of the original $SO(D + 1, 1)$, or YSTA:

\[ [\hat{X}_\mu, \hat{X}_\nu] = -i\lambda^2/\hbar \hat{M}_{\mu \nu}. \] (A.7)

3 In the Yang’s article [4], the short scale parameter is denoted by $a$ after the original Snyder’s article [2].
\[ [\hat{P}_{\mu}, \hat{P}_{\nu}] = -i\hbar R^{2} \hat{M}_{\mu\nu} \]  
(A.8)

\[ [\hat{X}_{\mu}, \hat{P}_{\nu}] = -i\hbar \delta_{\mu\nu} \]  
(A.9)

\[ [\hat{\nabla}, \hat{X}_{\mu}] = -i\lambda^{2}/\hbar \hat{P}_{\mu} \]  
(A.10)

\[ [\hat{\nabla}, \hat{P}_{\mu}] = i\hbar R^{2} \hat{X}_{\mu} \]  
(A.11)

with other familiar relations concerning \( \hat{M}_{\mu\nu} \)'s omitted.

**Appendix B. Historical background of noncommutative quantized space and time**

In association with the argument about area-entropy law problem given in section 5.2, let us consider another key problem towards the ultimate theory of quantum gravity, that is, ‘Singularity problem’ in the local field theories, first disclosed by Heisenberg–Pauli (~1929). In this connection, we recollect H Yukawa’s ‘Theory of elementary domain’ (1966) whose preliminary version, ‘On probability amplitude in relativistic quantum mechanics’ (Talk in Japanese) started in the spring of 1934, stimulated by Dirac’s idea of ‘Generalized transformation function’ (g.t.f) (1933) presented in ‘The Lagrangian in quantum mechanics’ [15]. It means that the above Yukawa’s Talk was done just in the midst of his struggle with ‘Meson theory’ (1934). Indeed, one decade on from ‘Meson theory’, Dirac’s idea ‘g.t.f.’ was prominently referred in Yukawa’s elaborate work ‘On the foundation of the theory of fields’ (1942) [16]. Furthermore, after the subsequent ‘Quantum theories of non-local fields’ (~1947), Yukawa finally arrived at the thought of ‘Atomistics and the divisibility of space and time’ (1966) [17] under a novel concept of *Elementary domain* \( D \), in association with the microscopic limit of Dirac’s ‘Generalized transformation function’.

Yukawa’s ‘Theory of elementary domain’ remained unaccomplished. However, he left the following impressive statement (~1978, in Japanese) [18]: ‘When we will proceed in this direction, we shall be after all faced with the problem of quantization of space-time … The resolution, however, must be all left in future’.

Nearly in the midst of the 1990s, in accord with Yukawa’s anticipation, there appeared ‘tantalizingly’ [19] *Noncommutative position coordinates of D-particles* in front of M-theory, that is, in quantum mechanics of many-body system of D-particles [19, 20]. According to Yukawa’s viewpoint on ‘Second quantization of fields’ [16], this fact strongly suggests the real existence of *noncommutative quantized space-time* behind D-particles or M-theory itself. Motivated by this fact, our present research started in the form ‘Space-time quantization and matrix model’, [23] on the basis of the early works by H S Snyder and C N Yang (1942) [2–5], that is, ‘Lorentz-covariant quantized space-time’. The historical background of their pioneering works in relation with W K Heisenberg was referred to in I, according to R Jackiw’s comment [24].

Indeed, we emphasize the historical importance of their pioneering works, which will possibly play the ultimate role in clearing away *Twentieth-century clouds* over difficulties of ultra-violet divergence in quantum field theories and area-entropy law of black holes towards the ultimate theory of quantum gravity and Planck scale physics [28].

4Let us note that the central concept in our present approach, ‘Basis vector’s set in Hilbert space \( I \), that is, \( | m \rangle \)'s \( m = 1, 2, …, n_{D} \) in equation (2.7) plays the role of Yukawa’s ‘Complete set of \( D \)’ [17].
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