Wilsonian Black Hole Entropy in Quantum Gravity

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Abstract

Using Wilsonian procedure (renormalization group improvement) we discuss the finite quantum corrections to black hole entropy in renormalizable theories. In this way, the Wilsonian black hole entropy is found for GUTs (of asymptotically free form, in particularly) and for the effective theory of conformal factor aiming to describe quantum gravity in infrared region. The off-critical regime (where the coupling constants are running) for effective theory of conformal factor in quantum gravity (with or without torsion) is explicitly constructed. The correspondent renormalization group equations for the effective couplings are found using Schwinger-De Witt technique for the calculation of the divergences of fourth order operator.
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I. INTRODUCTION

Since the seminal papers by Bekenstein and Hawking [1], the exact formula for intrinsic entropy in black hole solutions of Einstein theory (so-called Bekenstein-Hawking formula) is quite well-known. Moreover, as it was found recently [2], higher derivative invariants which naturally appear, in particular, in the gravitational effective action [3] give the additional contributions to the entropy density. Hence, the Bekenstein-Hawking result becomes only the leading contribution to the entropy density in such circumstances.

Despite many efforts, we are still very far from the complete understanding of black hole thermodynamics and in particular from the understanding of black hole entropy. (For a different approaches to black hole entropy, see Refs. [4–10].)

Moreover, the recent studies of quantum corrections [4,5,7,11–17] (and references therein) to black hole entropy on different black hole backgrounds show that there appear explicit divergences in such calculations. One possibility to understand the origin of these divergences is connected with the equivalence principle [18,19] (see also Refs. [13,20]). From another side, as it has been proposed in Ref. [4], these divergent contributions to black hole entropy (which have been discovered long ago [3]) have the ultraviolet nature. Hence, it may be removed by the renormalization of the gravitational constants [4] - what is usual way to work with ultraviolet divergences in quantum field theory (for a review of renormalization of quantum field theory in curved background, see Ref. [21]).

Using cut-off regularization it has been checked that this indeed the case at least in the situation with non-extremal black holes (see Refs. [4,11–17]); That was also checked by Pauli-Villars regularization in Ref. [17]. For the extremal black holes, where entropy vanishes [22], it is not completely clear that it will be the same. However, it seems [17] that possibility to remove all divergences from black hole entropy via the renormalization of the gravitational coupling constants exists at least using Pauli-Villars regularization.

The purpose of this work will be to discuss the quantum corrections to black hole entropy using Wilsonian procedure [23] (or renormalization group (RG) improvement [24]). That
gives the way to find the Wilsonian black hole entropy beyond one-loop (making summation of the leading-logarithms of the perturbation theory). Such procedure is standard now in discussing of the effective potential in the Standard Model.

The paper is organized as following. In the next section, we discuss general renormalizable quantum field theory including scalars, spinors and vectors (GUT-like theory) on curved background with conical singularity. Using RG equations for effective couplings, the RG-improved black hole entropy is found. In section 3, we consider effective theory for conformal factor aiming to describe the quantum gravity in infrared limit. This effective theory for quantum gravity is considered off-critical regime. The Wilsonian black hole entropy is found in this theory as well. In section 4, the conformal sector of quantum gravity with torsion is considered near IR fixed stable point. The gravitational and torsion running couplings are found. Finally, some discussion is given in conclusion.

II. QUANTUM FIELD THEORY IN BLACK HOLE BACKGROUND AND RG IMPROVED BLACK HOLE ENTROPY

We will consider an arbitrary multiplicatively renormalizable field theory including the scalars $\varphi$, spinors $\psi$ and vectors $A_\mu$ in curved spacetime which will be chosen to be of black hole type form, i.e. including the conical singularity. General arguments tell that such an action to be multiplicatively renormalizable should have the form [21];

$$L = L_m + L_{ext},$$

$$L_m = L_{YM} + \frac{1}{2}(\nabla_\mu \varphi)^2 + \frac{1}{2}\xi R \varphi^2 - \frac{1}{4!}f \varphi^4 - \frac{1}{2}m^2 \varphi^2 + i\bar{\psi} (\gamma^\mu \nabla_\mu - h \varphi) \psi,$$

$$L_{ext} = a_1 R^2 + a_2 C_{\mu\nu\alpha\beta} + a_3 G + \Lambda - \frac{1}{16\pi G} R,$$  \hspace{1cm} (1)

where some gauge group is supposed to be chosen. Note that it is quite well-known (see Ref. [21] for a review) that the Lagrangian of the external fields $L_{ext}$ should be added to $L_m$ in order to have the theory to be multiplicatively renormalizable. As the example of black
hole type background we will consider the Euclidean space which topologically represents
the direct product of the singular surface (horizon surface) and two-dimensional cone with
angle deficit $2\pi(1-\alpha)$. The example of such type is given by the well-known Rindler space
which represents the infinite mass limit of the Schwarzschild black hole. The corresponding
Euclidean metric is given by

$$ds^2 = \rho^2 d\theta^2 + d\rho^2 + dx_2^2 + dx_3^2,$$

where Euclidean time $\theta$ is periodic with period $\beta$, space coordinates $x_2, x_3$ are restricted.
The subspace of constant $x_2, x_3$ forms a cone with angle deficit $2\pi(1-\alpha)$ (i.e. with conical
singularity at $\rho = 0$). However [4], in Lorentzian notations where $\rho = 0$ describes the black
hole horizon there is no singularity. Hence, for correct Euclidean continuation $\alpha = 1$ (i.e,
$\beta = \beta_H$ as $\alpha = \frac{\beta}{\beta_H}$), where $\beta_H = 2\pi$ for the metric (2).

Our purpose in this section will be to calculate the quantum matter corrections to the
black hole entropy for the theory of type (1) on the above described black hole background.
Note that since the work by t’Hooft [5] (see also Ref. [9]) where the attempt to calculate
the quantum corrections to black hole entropy has been first done, there appeared many
works where similar calculations have been done in various contexts and using the different
approaches to entropy [8,11,25] (for a review of discussions, for different definitions of black
hole entropy, see Ref. [6]). It has been checked by the direct calculations that quantum
corrections to black hole entropy contain divergences (see, for example, Refs. [4,11–17]). It
has been proposed in Ref. [9] that divergences which appear in the calculation of the quantum
corrections to black hole entropy (more precisely, the divergent part of these corrections)
have the ultraviolet nature and may be removed by the renormalization of the gravitational
constants. It has been shown by direct calculations, mainly using the example of free scalar
theory in the frames of cut-off (brick wall) regularization, that this is indeed the case at least
for canonical horizon. Moreover, the renormalization of gravitational coupling constants
removes the divergences from the quantum corrections to black hole entropy even in the
case of using Pauli-Villars regularization as it was shown in Ref. [17] recently. It is clear
that if this indeed the case then any other regularization (which may look less physical in above context) should work as well while one is working with renormalizable theories.

Here we suggest to use the dimensional regularization within scheme of minimal subtraction. Then, only logarithmic divergences appear in the effective action calculation. Moreover, the using of such technique may be useful presumably in the similar considerations for extremal black holes where extra cubic divergences in cut-off regularization have been found [13,15]. These extra divergences in cut-off regularization seems to be impossible to be removed by renormalization of gravitational couplings. This fact led the authors of Ref. [13] to argue that thermodynamics of such black holes is not well-defined.

We will start from the theory (1) on purely gravitational background of type (2). Then the classical (tree-level) entropy may be easily defined as following [2,26]:

\[
\sigma = \frac{A}{4G} - \int_\Sigma [8\pi (a_1 + \frac{1}{3}a_2 + a_3) R - 4\pi (2a_2 + 4a_3) R_{\mu\nu} n^\mu_i n^\nu_i + 8\pi (a_2 + a_3) R_{\mu\nu\lambda\rho} n^\mu_i n^\nu_i n^\lambda_j n^\rho_j], \tag{3}
\]

where the first term is well-known Bekenstein-Hawking entropy [1], \(n^k\) are two orthonormal vectors orthogonal to \(\Sigma\). Note that one can use different forms of higher order invariant corrections to Bekenstein-Hawking entropy (see Ref. [4] for details). Notice also that coupling constants which appear in Eq.(3) are classical tree-level coupling constants.

The entanglement entropy is defined by the standard relation:

\[
\sigma = (\beta \frac{\partial}{\partial \beta} - 1) \Gamma, \tag{4}
\]

where \(\Gamma\) is one-loop effective action (free energy) on the background (2), and after the calculation one has to put \(\beta = \beta_H\), i.e. \(\alpha = 1\) [4].

Now one can make the renormalization of the entanglement entropy via

\[
\sigma(G, a_1, a_2, a_3) + \sigma_{div} = \sigma_{ren}(G^R, a_1^R, a_2^R, a_3^R), \tag{5}
\]

where on the right-hand side the gravitational coupling constants \(G, a_1, a_2, a_3\) are renormalized ones (we will drop below the superscript \(R\) for simplicity). They are connected with the
bare couplings in the standard way \cite{21} via renormalization of the one-loop effective action on the regular spacetime. Schematically, these relations look like

\begin{align}
G_R^{-1} &= G_B^{-1} + \frac{\tilde{G}m^2}{(n-4)} , \\
\tilde{a}_i^R &= \tilde{a}_i^B + \frac{\tilde{a}_i}{(n-4)} , \quad (i = 1, 2, 3) ,
\end{align}

where coefficients $\tilde{G}, \tilde{a}_i$ originate from the well known $a_2$- coefficient of Schwinger-De Witt expansion \cite{3}. This coefficient is defined only by quadratic part of action \cite{1} for all quantum fields, and it was written explicitly for the fields of spin $0, \frac{1}{2}, 1, \frac{3}{2}, 2$ in the literature few hundred times, so it is not necessary to write it explicitly. It may be shown in a variety of ways \cite{1, 12, 13, 17, 26} that this renormalization \cite{3} is not influenced by the presence of the conical singularity. So, the ultraviolet divergences of the entropy are given by \cite{3} where instead of gravitational couplings one has to substitute their infinite parts (according to \cite{3} in dimensional regularization, or one can use cut-off regularization \cite{4, 5}). Now, after this discussion of the renormalization of the black hole entropy (which definitely works for the canonical horizon) one can try to get some universal information from the ultraviolet renormalized entropy.

For that purpose, one may apply the Wilsonian procedure \cite{16} (or renormalization group improvement) which gives the way to make the summation of leading logarithms of the whole perturbation series. This way is based on renormalization group and it was successfully applied in the study of the effective potential in Standard Model.

First of all, let us write explicitly the RG equations for gravitational coupling constants \cite{21} (they are induced by the one-loop renormalization \cite{3});

\begin{align}
\frac{da_1(t)}{dt} &= \frac{1}{(4\pi)^2} \left[ \xi(t) - \frac{1}{6} \right]^2 \frac{N_s}{2} , \\
\frac{da_2(t)}{dt} &= \frac{1}{120(4\pi)^2} (N_s + 6N_f + 12N_A) , \\
\frac{da_3(t)}{dt} &= -\frac{1}{360(4\pi)^2} (N_s + 11N_f + 62N_A) , \\
\frac{d\Lambda(t)}{dt} &= \frac{m^4(t)N_s}{2(4\pi)^2} ,
\end{align}
\[
\frac{d}{dt} \frac{1}{16\pi G(t)} = -\frac{m^2(t)}{(4\pi)^2} N_\ast [\xi(t) - \frac{1}{6}],
\]  
(7)

where \(N_s, N_f, N_A\) are the numbers of real scalars, spinors and vectors in the theory (1).

Thus, the universal form of RG improved quantum corrections to entropy looks now as:

\[
\sigma = \frac{A}{4G(t)} - \int [8\pi \{a_1(t) + \frac{1}{3}a_2(t) + a_3(t)\} R \\
- 4\pi \{2a_2(t) + 4a_3(t)\} R_{\mu\nu\iota\kappa} n^\mu_i n^\nu_j + 8\pi \{a_2(t) + a_3(t)\} R_{\mu\nu\lambda\rho} n^\mu_i n^\lambda_j n^\nu_i n^\rho_j].
\]  
(8)

Here, now we have running coupling constants in Eq.(8) instead of classical coupling constants.

For example, in GUT models which are asymptotically free in all interaction couplings (for a review, see Ref. [21]) one has

\[
g^2 = g^2(1 + \frac{B^2 g^2 t}{(4\pi)^2})^{-1},
\]

\[
h^2 = \kappa_1 g^2(t),\quad f(t) = \kappa_2 g^2(t),
\]

\[
\xi(t) = \frac{1}{6} + (\xi - \frac{1}{6})(1 + \frac{B^2 g^2 t}{(4\pi)^2})^b,
\]

\[
m^2(t) = m^2(1 + \frac{B^2 g^2 t}{(4\pi)^2})^b,
\]  
(9)

where \(\kappa_1, \kappa_2\) are numerical constants defined by the specific features of the theory (gauge group, number of scalars and spinor multiplets), the constant \(b\) may be positive or negative what depends on the model under consideration.

Then, solving Eq.(7) we will find (see also recent work Ref. [27], where application of the above running coupling constants for study of quantum corrections to Newtonian potential has been done):

\[
G(t) = G_0 \left\{1 - \frac{16\pi N_s G_0 m^2(\xi - \frac{1}{6})}{B^2 g^2(2b + 1)[(1 + \frac{B^2 g^2 t}{(4\pi)^2})^{2b+1} - 1]}\right\}^{-1},
\]
\[ a_1(t) = a_1 + \frac{N_s \left(\xi - \frac{1}{6}\right)^2}{2(2b + 1)B^2g^2}[1 + \frac{B^2g^2t}{(4\pi)^2}2^{b+1} - 1], \]

\[ a_2(t) = a_2 + \frac{t}{120(4\pi)^2}(N_s + 6N_f + 12N_A), \]

\[ a_3(t) = a_3 - \frac{t}{360(4\pi)^2}(N_s + 11N_f + 62N_A). \]  

Note that we found only universal quantum corrections to finite black hole entropy in GUT under consideration making summation of leading-logarithmic terms of whole perturbation series, and applying Wilsonian procedure \[23\]. Of course, there are also finite quantum corrections resulting from specific features of the black hole background under consideration. They were extensively studied recently in Refs. \[13-15\] in one-loop level. One may also hope that application of dimensional regularization may help to solve the problem which appears in extremal black hole (where cubic divergences in cut-off regularization have been found) just because powerlike divergences disappear in dimensional regularization.

Now the natural question appears: what is RG parameter \( t \) in our context? The choice of this parameter is normally connected with the presence of the mass parameters in the theory. In particular, in standard RG \( t = \ln \frac{\mu}{\mu_0} \) where \( \mu, \mu_0 \) are different mass scales, in the study of RG improved potential \[24\] \( t = \frac{1}{2} \ln \frac{\varphi^2}{\mu^2} \) (or \( \frac{1}{2} \ln \frac{\varphi^2}{\varphi_0^2} \)) where \( \varphi \) is scalar field and this choice is not changed in curved spacetime \[28\]. In the situation under discussion the natural choice is \( t = \ln \beta \mu \) where \( \beta \) is temperature and \( \mu \) is an arbitrary mass parameter. For extremal black hole, the correspondent parameter looks as \( t = \ln \frac{\beta}{\tau_H} \) where \( \tau_H \) is the horizon radius.

Thus, we found the (beyond one-loop) form of matter quantum corrections to black hole entropy which are universal and are caused by the ultraviolet structure of the theory.
III. QUANTUM CORRECTIONS TO BLACK HOLE ENTROPY IN THE EFFECTIVE THEORY FOR CONFORMAL FACTOR

Up to now almost all studies of quantum corrected black hole entropy have been done for free matter (mainly scalar). It is very interesting to understand what happens then in quantum gravity? Due to the fact that free energy is affected by ultraviolet divergences, the only consistent way to consider such questions is in the frames of consistent renormalizable quantum gravity. In the absence of such theory (for a general review of different quantum gravity models, see Ref. [21]), we suggest to consider the effective theory of quantum gravity, taking the so-called effective theory for conformal factor [29,30] (which is renormalizable one) as an example. We will consider the effective theory for conformal factor on curved background [30] which will be chosen of the black hole type as in previous section. Note that the effective theory for the conformal factor has been suggested [29] to describe the infrared sector of quantum gravity.

The action of such theory looks as following [30]:

\[
L = -\frac{Q^2}{(4\pi)^2} \sigma \Box^2 \sigma + \sigma [\xi_1 R_{\mu\nu} \nabla_\mu \nabla_\nu + \xi_2 R \Box + \xi_3 (\nabla_\mu R) \nabla_\mu] \sigma \\
- \zeta [2\tilde{\alpha}(\nabla_\mu \sigma)(\nabla^\mu \sigma) \Box \sigma + \tilde{\alpha}^2 ((\nabla_\mu \sigma)(\nabla^\mu \sigma))^2] + \eta_1 \tilde{\alpha}^2 e^{2\tilde{\alpha} \sigma} R \\
+ \eta_2 R(\nabla_\mu \sigma)(\nabla^\mu \sigma) + \gamma e^{2\tilde{\alpha} \sigma}(\nabla_\mu \sigma)(\nabla^\mu \sigma) - \frac{\lambda}{\tilde{\alpha}^2} e^{4\tilde{\alpha} \sigma} + \tilde{a}_1 R_{\mu\nu}^2 + \tilde{a}_2 G + \tilde{a}_3 R^2, 
\]

where \(\tilde{\alpha}\) is the scaling dimension of scalar field \(\sigma\). Note that action (11) represents the multiplicatively renormalizable generalization of the effective theory for the conformal factor [29] on curved background (we suppose the absence of \(\sigma\)- linear terms on classical and quantum level). Notice also that due to the presence of higher derivative terms, such a theory may have the problem with unitarity. As it is effective (not fundamental) theory, it is not such a big problem. Note also that a mechanism to ensure unitarity in such model may exist [31] in the original theory for conformal factor. One starts from the conformal anomaly for free conformally invariant matter:
\[ T = b(C_{\mu\nu\alpha\beta}^2 + \frac{2}{3} \Box R) + b' G + b'' \Box R. \] (12)

Working in the parameterization

\[ g_{\mu\nu} = e^{2\sigma(x)} \bar{g}_{\mu\nu}, \] (13)

where \( \sigma(x) \) is the conformal factor which is supposed to be mainly dominant on quantum level in infrared sector of quantum gravity [29] and \( \bar{g}_{\mu\nu} \) is a fixed fiducial metric, one can integrate Eq.(12) in the parameterization (13) (see Ref. [21] for example) and obtain the anomaly induced effective action. After addition to this anomaly induced effective action the standard Einstein action (with gravitational constant \( \kappa \) and cosmological constant \( \Lambda \)) in parameterization (13) one obtains the action of the from (11) with:

\[ \frac{Q^2}{(4\pi)^2} = 2b + 3b'', \quad \zeta = 2b + 2b' + 3b'', \quad \gamma = \frac{3}{\kappa}, \quad \lambda = \frac{\Lambda}{\kappa}, \]

\[ \xi_1 = 2(\zeta - \frac{Q^2}{(4\pi)^2}), \quad \xi_2 = -\zeta + \frac{2}{3} \frac{Q^2}{(4\pi)^2}, \quad \xi_3 = -\frac{Q^2}{3(4\pi)^2}. \] (14)

In order to have such a theory to be multiplicatively renormalizable, one has to add few more terms to such a action with arbitrary coupling constants \( (R^2 \text{- terms and terms with } \eta_1, \eta_2) \). That is done in the writing of the action (11).

Now, choosing as it was said before, the metric \( g_{\mu\nu} \) in the black hole type form (like (2)) and background \( \sigma \)- field to be zero, one can easily write the tree level black hole entropy as:

\[ \sigma = -4\pi \frac{A}{\alpha^2} \eta_1 - \int_\Sigma [8\pi(\alpha_3 + \alpha_2)R + 4\pi(\alpha_1 - 4\alpha_2)R_{\mu\nu}n^\mu_i n^\nu_i + 8\pi \alpha_2 R_{\mu\nu\lambda\rho} n^\mu_i n^\nu_i n^\lambda_j n^\rho_j]. \] (15)

We suppose that in the IR sector of quantum gravity (which may be relevant for black hole physics) the main contribution of quantum gravity to entropy comes from the conformal sector, i.e. rest gravitational modes are frozen as in the original paper [29].

Using the results of Refs. [30,32], we may find the running gravitational couplings which present in the expression for the entropy. The correspondent expressions are given by an asymptotically free solution for \( \zeta(t) \) in infrared [32].
\[ \zeta(t) = -\frac{4b'^2(4\pi)^2}{5\tilde{\alpha}^2 t}, \quad Q^2(t) = (4\pi)^2[\zeta(t) - 2b'], \quad \text{where} \ |t| \ \text{is big,} \ t < 0. \quad (16) \]

Using the running couplings (16) and the result of Ref. [30] one can easily estimate the gravitational couplings in (13) in IR sector \((t \to -\infty)\):

\[ \tilde{\alpha}_1(t) = \tilde{\alpha}_1 - \frac{2t}{15(4\pi)^2}, \quad \tilde{\alpha}_2(t) = \tilde{\alpha}_2 + \frac{t}{90(4\pi)^2}, \quad \tilde{\alpha}_3(t) = \tilde{\alpha}_3 + \frac{2t}{45(4\pi)^2}. \quad (17) \]

Note that only leading terms are kept in the expressions (17). In the similar way one can obtains the running coupling \(\eta_1(t)\) which plays the role of gravitational coupling in Eq.(15). It is given by (for simplicity \(\eta_1(0)\) is chosen to be zero);

\[ \eta_1(t) \approx (-t)^2 e^{(2-\frac{2\tilde{\alpha}}{\tilde{\alpha}(0)^2})t}. \quad (18) \]

One sees that even choosing \(\eta_1(0) = 0\) on classical level, we have this term induced by quantum corrections which leads also to its appearance in black hole entropy. Thus, substituting instead of tree level couplings \(\eta_1, \tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3\) in (13), the correspondent running couplings found before we immediately obtain the improved expression for the black hole entropy from quantum effects of infrared gravity. The choice of RG parameter \(t\) was already discussed at the end of the previous section.

In similar way one can find the universal form for the quantum corrections to black hole entropy in other renormalizable models of quantum gravity. One example may be \(R^2\)-gravity [21] (forgetting again about unitarity problem) where the gravitational running coupling constants are known [21] (for a recent discussion, see also Ref. [22]).

IV. CONFORMAL SECTOR OF QUANTUM GRAVITY WITH TORSION NEAR IR FIXED POINT

As we discussed in the previous sections, the running of gravitational couplings may be relevant for the study of quantum corrections to black hole entropy. Let us discuss now the running couplings in the effective theory of conformal factor in quantum gravity with torsion (see Ref. [33] where such a theory has been constructed). Due to technical problems which
result in the appearance of many new terms in conformal anomaly for theory with torsion, we will consider as in Ref. [33] the flat background (i.e. \( g_{\mu\nu} = e^{2\sigma} \eta_{\mu\nu} \)) and anti-symmetric part of torsion \( S_\mu \) to be non-zero only. Then the correspondent Lagrangian is given by:

\[
L = -\frac{Q^2}{(4\pi)^2} (\Box \sigma)^2 - \zeta [2\vec{\alpha} (\partial_\mu \sigma)^2 \Box \sigma + \vec{\alpha}^2 (\partial_\mu \sigma)^4] + \gamma e^{2\vec{\alpha} \sigma} (\partial_\mu \sigma)^2 \\
- \frac{\lambda}{\vec{\alpha}^2} e^{4\vec{\sigma}} + b_1 S^2 (\partial_\mu \sigma)^2 + b_2 (S^\nu \partial_\mu \sigma)^2 + \frac{h}{2k\vec{\alpha}^2} e^{2\vec{\alpha} \sigma} S^2 + \eta S^4.
\]

(19)

All the details of the construction of this Lagrangian, which consists basically of three pieces; first, anomaly-induced Lagrangian [21], second, the Einstein theory action with torsion and third, the last term in (19) introduced for the theory to be renormalizable, are given in Ref. [33].

Note that running of coupling constants \( Q^2, \zeta(t), \gamma(t), \lambda(t) \) near IR stable fixed point (\( \zeta = 0 \)) has been given in Ref. [32] (see also Eq. (16) of the previous section). Scaling dimensions of the theory in IR fixed point have been given in Ref. [29].

The calculation of the \( \beta \)-functions in the theory (19) at \( \zeta = 0 \) has been done in Ref. [29] and at \( \zeta \neq 0 \) (off-critical regime) in Ref. [32] for the case of zero torsion. Now, using the standard algorithm for the calculation of the divergences of 4-th derivative scalar operator (see Ref. [30]), we may get, in the sector connected with the torsion, the corresponding \( \beta \)-functions (we don’t give the details of this background field method calculation, for an introduction to this method, see for example Ref. [21]);

The \( \zeta \) and torsion dependent terms of the 2nd heat kernel coefficient \( a_2 \) (see Refs. [3,34] for definition) is given by

\[
a_2 = e^{2\vec{\alpha} \sigma} S^2 \left[ \frac{h}{kQ^2} + \gamma(4\pi)^2 Q^2 b_1 + \frac{1}{4} b_2 \right] + S^4 \left[ \frac{4(4\pi)^2}{2Q^4} [b_1^2 + \frac{1}{2} b_1 b_2 + \frac{1}{8} b_2^2] \\
+ S^2 (\partial \sigma)^2 \left[ \frac{\zeta \vec{\alpha}^2}{3Q^2} [9b_1 + 2b_2] - 2\zeta(6b_1 + b_2) \right] + (S^\nu \partial_\mu \sigma)^2 \frac{\zeta \vec{\alpha}^2}{3Q^2} b_2 (1 - 4\zeta) \right] (20)
\]

Using this \( a_2 \) coefficient, we have found the following RG equations;

\[
\frac{dh(t)}{dt} = -\frac{3(4\pi)^2 \vec{\alpha}^2 \zeta(t)}{Q^4(t)} h(t) + \frac{6(4\pi)^2 \vec{\alpha}^2 \gamma(t)}{Q^4(t)} (b_1(t) + \frac{1}{4} b_2(t)).
\]
\[
\frac{db_1(t)}{dt} = \frac{\zeta(t)\tilde{\alpha}^2}{3Q^2(t)}(9b_1(t) + 2b_2(t)) + O(\zeta^2(t)) ,
\]
\[
\frac{db_2(t)}{dt} = \frac{\zeta(t)\tilde{\alpha}^2}{3Q^2(t)}b_2(t) + O(\zeta^2(t)) ,
\]
\[
\frac{d\eta(t)}{dt} = \frac{(4\pi)^2}{2Q^4(t)}[b_1^2(t) + \frac{1}{2}b_1(t)b_2(t) + \frac{1}{8}b_2^2(t)] .
\]

(21)

A set of solution for the above RG equations in the limit \( t \to -\infty \) is
\[
b_1(t) = -\frac{1}{4}b_2(t) ,
\]
\[
b_2(t) \simeq c(-t)^{-\frac{Q^2}{15(4\pi)^2}} ,
\]
\[
\eta(t) \simeq -\frac{c^2(4\pi)^2}{15Q^4_0(1 - \frac{2Q^2_0}{15(4\pi)^2})}(-t)^{1-\frac{2Q^2}{5(4\pi)^2}} + c' ,
\]
\[
h(t) \simeq c^{''}(-t)^{\frac{7}{3}} .
\]

(22)

where \( c, c' \) and \( c'' \) are the integration constants. Note that in the solution for \( \eta(t) \) one has to suppose that the number of matter fields is large enough in order that the condition \( \frac{2Q^2}{15(4\pi)^2} > 1 \) would hold. Otherwise, \( \eta(t) \) would be the increasing function of \( t \). Notice also that, even though \( h(t) \) is a growing function, the coupling of \( e^{2\tilde{\alpha}\sigma}S^2 \) term in the Lagrangian (19) is \( h(t)\gamma(t) \simeq c^{''}(-t)^{\frac{7}{3}}e^{(2-2\tilde{\alpha} + \frac{2\tilde{\alpha}^2}{Q^2_0})} \), which is a decreasing function in far infrared provided \( 2-2\tilde{\alpha} + \frac{2\tilde{\alpha}^2}{Q^2_0} \) > 0. This condition is necessary also to ensure that \( \gamma(t) \) is a decreasing function in far infrared. Therefore, the decreasing of all running couplings in far infrared \( t \to -\infty \) for the Lagrangian (19) is guaranteed if the following conditions hold;
\[
\frac{2Q^2_0}{15(4\pi)^2} > 1 \quad \text{and} \quad (2 - 2\tilde{\alpha} + \frac{2\tilde{\alpha}^2}{Q^2_0}) > 0 .
\]

Thus, we constructed the off-critical regime in effective theory for conformal factor in quantum gravity with torsion. However, to apply these results to calculation of quantum corrections to black hole entropy one has to consider the original theory on curved background. That requires the addition of many more terms to the original Lagrangian. In addition, one has to construct the entropy for the effective gravitational Lagrangian which includes many torsion terms. These questions will be discussed in other place.
V. DISCUSSION

We discussed the RG improving procedure to obtain the universal quantum corrections to the entropy on the black hole background. The following renormalizable theories have been considered: an arbitrary renormalizable GUT and the effective model for quantum gravity. As usually, Wilsonian procedure gives the universal way to get some information about quantum properties of the system under consideration.

The (beyond one-loop) Wilsonian black hole entropy which has been discussed in this work has its origin in the logarithmic divergences of the whole perturbation series. It gives only universal part of necessary information. There are also quantum corrections which depend very much from the choice of black hole background under consideration. These finite corrections are not universal and should be calculated directly using specific black hole background.

We also constructed off-critical regime in the effective theory for conformal factor in quantum gravity (with torsion). In particularly, the running couplings have been defined. They may be useful not only for calculation of black hole entropy but in other cosmological applications, like quantum corrections to Newtonian potential or influence of the running gravitational couplings in galaxy formation processes.

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