Prefixes of the Fibonacci word that end with a cube

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Abstract

We study the prefixes of the Fibonacci word that end with a cube. Using Walnut we obtain an exact description of the positions of the Fibonacci word at which a cube ends.

1 Introduction

This paper is motivated by the following remarkable result, which was originally conjectured by Jeffrey Shallit and proved by Mignosi, Restivo, and Salemi [3]:

An infinite word \( w \) is ultimately periodic if and only if all sufficiently long prefixes of \( w \) end with a repetition of exponent at least \( \varphi^2 \), where \( \varphi \) is the golden ratio.

The exponent of a word is the ratio of its length to its minimal period. In particular, this result implies that no aperiodic infinite word can have all sufficiently long prefixes end with a cube (a word with exponent 3). Counting the number of prefixes of an infinite word that end with cubes can therefore provide a measure, in some sense, of how close the infinite word is to being ultimately periodic.

The first candidate that one would choose to investigate in regards to this measure is the Fibonacci word. Indeed, Mignosi et al. also proved that the Fibonacci word witnesses the optimality of their result in the following sense:

For any \( \epsilon > 0 \), all sufficiently long prefixes of the Fibonacci word

\[ f = 0100101001001010010100101001010010 \cdots \]

end with repetitions of exponent at least \( \varphi^2 - \epsilon \).

In this paper we examine the positions at which a cube ends in the Fibonacci word (the starting positions of cubes in the Fibonacci word have been characterized by Mousavi, Schaeffer, and Shallit [5]). Let \( \text{cubes}_f \) be the infinite word whose \( n \)-th term is

\[
\begin{cases} 
1 & \text{if a cube ends at position } n \text{ of } f, \\
0 & \text{otherwise.}
\end{cases}
\]

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For any \( n \geq 0 \), let \((n)_F\) denote the canonical representation of \( n \) in the Fibonacci (Zeckendorf) numeration system.

**Theorem 1.** There are arbitrarily long runs of 1’s in \( \text{cubes}_f \). More precisely, the runs of 1’s in \( \text{cubes}_f \) are characterized by the following: If \((i)_F\) has the form

\[
(i)_F \in (10)^*0(0 + 10)(00)^*0w,
\]

where \( w \in 0(10)^*(\epsilon + 1) \) then \( \text{cubes}_f \) contains a run of 1’s of length

- \( F_{2n+2} - 1 \), if \(|w| = 2n \) for some \( n \geq 0 \),
- \( F_{2n+3} - 1 \), if \(|w| = 2n + 1 \) for some \( n \geq 0 \),

beginning at position \( i \).

**Theorem 2.** The runs of 0’s in \( \text{cubes}_f \) have lengths 1, 2, 3, 7, 8, and 13. The only run of length 13 occurs at the beginning of \( \text{cubes}_f \). For each of the other lengths (1, 2, 3, 7, and 8), there are infinitely many runs of that length in \( \text{cubes}_f \).

The proofs of these theorems are given in the next section.

## 2 Walnut computations

Our main results are all obtained by computer using Walnut [4]. We begin with the command

\[
\text{eval fib_end_cubes } "?msd_fib \text{ Ei En n > 1 & j = i+3*n-1 & (Ak k < 2*n \Rightarrow F[i+k] = F[i+k+n])"}:
\]

which produces the automaton in Figure 1, which accepts the Zeckendorf representations of the positions at which a cube ends in \( f \).

(Proof of Theorem 1.) To determine the lengths of the runs of 1’s in \( \text{cubes}_f \), we use the command

\[
\text{eval fib_end_cubes_run } "?msd_fib n>1 & (At t<n => \$fib_end_cubes(i+t)) & \neg\$fib_end_cubes(i+n) & (i=0|\neg\$fib_end_cubes(i-1))":
\]
Figure 2: Automaton for runs of 1’s in \texttt{cubes}_f

which produces the automaton in Figure 2, which accepts the Zeckendorf representations of pairs \((i, \ell)\) such that there is a run of 1’s in \texttt{cubes}_f of length \(\ell\) starting at position \(i\).

By examining the structure of this automaton we see that for an accepted pair \((i, \ell)\), the representation \((i)_F\) has the form \((i)_F = (10)^{+}0(0 + 10)(00)^{*}0w\), where \(w \in 0(10)^{*}(\epsilon + 1)\). Furthermore, if \(|w| = 2n\), then \((\ell)_F = (10)^{n}\) and if \(|w| = 2n + 1\), then \((\ell)_F = (10)^{n}1\). Now, let \(F_m\) denote the \(m\)-th Fibonacci number and recall the identities:

\[
\sum_{j=0}^{n-1} F_{2j+1} = F_{2n} \quad \text{and} \quad \sum_{j=1}^{n} F_{2j} = F_{2n+1} - 1.
\]

Hence, if \(|w| = 2n\), we have

\[
\ell = \sum_{j=1}^{n} F_{2j+1} = F_{2n+1} + F_{2n} - F_1 = F_{2n+2} - 1
\]

and if \(|w| = 2n + 1\) we have

\[
\ell = \sum_{j=1}^{n+1} F_{2j} = F_{2n+2} + F_{2n+1} - 1 = F_{2n+3} - 1.
\]

\((Proof of Theorem 2.)\) To determine the lengths of the runs of 0’s in \texttt{cubes}_f, we use the command

\begin{verbatim}
 eval fib_no_cubes_run "?msd_fib n>=1 & (At t<n => ~$fib_end_cubes(i+t)) & $fib_end_cubes(i+n) & (i=0|$fib_end_cubes(i-1))":
\end{verbatim}

which produces the automaton in Figure 3, which accepts the Zeckendorf representations of pairs \((i, \ell)\) such that there is a run of 0’s in \texttt{cubes}_f of length \(\ell\) starting at position \(i\).

We can project this automaton onto the second component of its input with the command

\begin{verbatim}
 eval fib_no_cubes_run_length "?msd_fib Ei $fib_no_cubes_run(i,n)":
\end{verbatim}

which produces the automaton in Figure 4. We see that the only possible run lengths are \(\ell \in \{1, 2, 3, 7, 8, 13\}\).

The command
eval tmp "?msd_fib Ai Ej j>i & $fib_no_cubes_run(j,1)"

evaluates to TRUE, indicating that there are infinitely many runs of 0’s of length 1. This is also the case for run lengths 2, 3, 7, and 8. For length 13 however, we get a result of FALSE.

The positions of the runs of length 7 and 8 have a simple structure, so we describe these next.

**Theorem 3.**

- **The runs of 0’s in cubes** \( f \) **of length 8 begin at positions** \( i \) \( F \in (10)^{+0001} \).
- **The runs of 0’s in cubes** \( f \) **of length 7 begin at positions** \( i \) \( F \in (10)^{+01001} \).

**Proof.** These are obtained via the commands

```
eval tmp "?msd_fib Ai Ej j>i & fib_no_cubes_run(j,8)"
eval tmp "?msd_fib $fib_no_cubes_run(j,7)"
```

The descriptions of the starting positions for the other lengths of runs of 0’s in cubes \( f \) are a little more complicated, so we omit them here, but the reader can easily compute these with Walnut.

**Theorem 4.** **The density of 0’s in cubes** \( f \) **is zero.**

**Proof.** We examine the complement of the automaton in Figure 1. The Walnut command

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produces the automaton in Figure 5, which gives the positions in $f$ where no cube ends. To complete the proof, it suffices to show that there are only polynomially many strings of length $n$ that are accepted by this automaton. This can be seen directly from the structure of the automaton: since this automaton does not have two cycles that can both mutually reach each other, we can conclude that the number of strings of length $n$ accepted by this automaton is polynomially bounded (see, for example, [1]).

3 Other Sturmian words

Although the Fibonacci word is “optimal” with respect to the result of Mignosi et al. mentioned in the Introduction, some computer calculations suggest that there may be other Sturmian words that have even more prefixes that end with cubes than the Fibonacci word.

For any infinite word $w$, let us define cubes$_w$ to be the binary word whose $n$-th term is 1 if cubes$_w$ has a cube ending at position $n$, and 0 otherwise. Let max_no_cubes$(w)$ denote the largest $\ell$ such that cubes$_w$ contains infinitely many runs of 0’s of length $\ell$. Let us also define $S_w(n)$ to be the sum of the first $n$ terms of cubes$_w$. That is, $S_w(n)$ counts the number of positions $< n$ at which a cube ends in $w$.

Now Theorem 2 shows that max_no_cubes$(f) = 8$. Let $c_\alpha$ be the characteristic Sturmian word with slope $\alpha$. It is not hard to find a $\beta$ for which max_no_cubes$(c_\beta) = 3$. Let $\beta = (5 - \sqrt{13})/6 = [0; 4, 3]$. Then $c_\beta$ is a concatenation of the blocks 00001 and 0001, so for any given position, there is always an occurrence of 000 ending either at that position or within the next 3 positions. Hence, we have max_no_cubes$(c_\beta) = 3$.

Computationally, we can examine $S_{c_\beta}(n)$ and $S_f(n)$ and compare these two quantities. Table 1 gives some values of these two functions. Computer calculations show that $S_{c_\beta}(n) > S_f(n)$ for $2 \leq n \leq 3000$.

| $n$  | $S_f(n)$ | $S_{c_\beta}(n)$ |
|------|----------|------------------|
| 500  | 353      | 408              |
| 1000 | 779      | 860              |
| 2000 | 1722     | 1812             |
| 3000 | 2669     | 2716             |

Table 1: Comparing $S_{c_\beta}(n)$ and $S_f(n)$
We have the following open questions:

**Problem 1.** Is it possible to determine \( \max_{\text{no cubes}}(c_\alpha) \) from the continued fraction expansion of \( \alpha \)?

**Problem 2.** What is the least (resp. greatest) possible value of \( \max_{\text{no cubes}}(c_\alpha) \) over all \( \alpha \)? Is it 3 (resp. 8)?

**Problem 3.** Is there an \( \alpha \) such that for all other \( \alpha' \) the function \( S_{c_\alpha}(n) \) is eventually greater than \( S_{c_{\alpha'}}(n) \)?

**Problem 4.** Can one prove that the density of 0’s in \( \text{cubes}_{c_\alpha} \) is 0 for all \( \alpha \)?

One might also wish to investigate the relationship between the critical exponent of an infinite word \( w \) and the density of 0’s in \( \text{cubes}_w \). The critical exponent of \( w \) is the quantity

\[
\sup \{ r : w \text{ contains a factor with exponent } r \}.
\]

Note that it is easy to construct an aperiodic word with unbounded critical exponent for which “almost all” positions are the ending position of a cube: for example, the infinite word

\[
010^210^410^810^{16}10^{32}1\cdots
\]

has this property. So it is natural to restrict our attention to words with bounded critical exponent. The Fibonacci word has critical exponent \( 2 + \varphi \approx 3.618 \), and all Sturmian words have critical exponent at least this large. Are there words \( w \) with lower critical exponent for which the density of 0’s in \( \text{cubes}_w \) is still 0? The answer is “yes”. For instance, the fixed point \( x \) (starting with 0) of the morphism \( 0 \rightarrow 0001, 1 \rightarrow 1011 \) has critical exponent \( 10/3 \) [2, p. 99], and just as we did for the Fibonacci word, we can use Walnut to show that the density of 0’s in \( \text{cubes}_x \) is 0 (after computing the automaton for the 0’s in \( \text{cubes}_x \), one computes the eigenvalues of the adjacency matrix and finds that they are all strictly smaller than 4).

**Problem 5.** What is the infimum of the critical exponents among all infinite words \( w \) for which the density of 0’s in \( \text{cubes}_w \) is 0? Is it 3?

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**References**

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