One-loop Feynman Integral Reduction by Differential Operators

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ABSTRACT: For loop integrals, the standard method is reduction. A well-known reduction method for one-loop integrals is the Passarino-Veltman reduction. Inspired by the recent paper [1] where the tadpole reduction coefficients have been solved, in this paper we show the same technique can be used to give a complete integral reduction for any one-loop integrals. The differential operator method is an alternative version of the PV-reduction method. Using this method, analytic expressions of all reduction coefficients of the master integrals can be given by algebraic recurrence relation easily. We demonstrate our method explicitly with several examples.

KEYWORDS: One-loop Feynman integral, Integral Reduction, Differential Operator, Recursion Relation

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1 Introduction

The calculation of scattering amplitude at higher loop level is always like a chronic disease to block the evolution of High Energy Physics. Theoretical physicists have made many prescriptions to cure this problem started in the 1970s. The most significant receipt is to reduce a loop amplitude into a linear combination of some scalar master integrals (MIs) under dimensional regularization [2–18]

\[ I^{1-\text{loop}} = \sum_{i_{d_0+1}} C^{i_{d_0+1}} I^{i_{d_0+1}} + \sum_{i_{d_0}} C^{i_{d_0}} I^{i_{d_0}} + \cdots + \sum_{i_1} C^{i_1} I^{i_1}, \]  

(1.1)

where \( i_s \) is the set of propagators appearing in the master integrals. The coefficient \( C^{i_s} \) \((s = 1, \cdots, d_0+1)\) is simply a rational function of some Lorentz invariant such like the scalar product of external momenta while the terms \( I_s \) are the \( s \)-gon scalar integral. With the general expansion (1.1), the computation of general one-loop amplitudes has been switched to determining those coefficients of \( C^{i_s} \). Many tools have been invented to shovel the brambles, such as integration-by-parts (IBP) [19, 20], PV reduction [4], OPP reduction [17, 21–23], Unitarity cut [15, 18, 24–30] etc.

All these methods can be divided into two categories, i.e., the reduction at the integrand level or the integral level. For reduction at the integrand level, [17] shows how to extract
the coefficients of the 4-, 3-, 2- and 1-point one-loop scalar integrals from the full one-loop integrand of arbitrary scattering processes in an algebraical way. For the reduction at the integral level, an efficient way is the unitarity cut method. The main idea is to compare the imaginary part of two sides of (1.1). However, since the loop-integral is well-defined using the dimensional regularization, the unitarity cut method in pure 4D need to generalize to \((4 - 2\epsilon)\)-dimension, which has been done in \([29, 31]\). Based on this generalization, the analytic expressions for reduction coefficients (except the tadpole coefficients) have been derived in a series of papers \([32–36]\).

In our previous work \([1]\), we reconsider the problem by introducing differential operators \(D\) and \(T\). We first introduce an auxiliary vector \(R^\mu\) and reduce it to master integrals, then applying differential operators to the integrals with respect to \(R\). By comparing two sides of the expansion, We will achieve the recursion relations of those coefficients of master integrals in differential form. With the knowledge of the algebraical structure of the reduction coefficients, we transform those differential equation form relations into algebraical form. In \([1]\), we solve the remaining unsolved tadpole coefficients by this method. In this paper, we will provide a general algorithm for calculating all reduction coefficients for a general tensor one-loop integral and give the explicit analytic results.

In section 2, we will review the derivation of the differential equations of reduction coefficients and show how to get the recursion relations of expansion coefficients. In section 3, we will solve the recursion relations of reduction coefficients in the general case. In section 4, we provide some examples and summarize the algorithm for calculating the reduction of a general tensor one-loop Feynman integral. In appendix A, we list all reduction coefficients for tensor triangles, boxes and pentagons with rank 1 and 2.

### 2 Differential equations and recursion relations

We will review the differential operators in \([1]\) and show the way we obtain the recursion relations of every reduction coefficients. Starting with the following general one-loop \(m\)-rank tensor integral with \(n + 1\) propagators

\[
I_{n+1}^{\mu_1 \cdots \mu_m} = \int \frac{d^D \ell}{(2\pi)^D} \frac{\ell^{\mu_1} \ell^{\mu_2} \cdots \ell^{\mu_m}}{P_0 P_1 \cdots P_n},
\]

where the \(i\)-th propagator is \(P_i = (\ell - K_i)^2 - M_i^2\) with setting \(K_0 = 0\), we introduce an auxiliary vector \(R^\mu\) and contract \(\ell\) with \(R\) on (2.1) \(m\) times to get

\[
I_{n+1}^{(m)}[R] \equiv 2^m I_{n+1}^{\mu_1 \cdots \mu_m} R_{\mu_1} \cdots R_{\mu_m} = \int \frac{d^D \ell}{(2\pi)^D} \frac{(2\ell \cdot R)^m}{P_0 P_1 \cdots P_n}.
\]

The vector \(R\) lies in the same dimension as the \(\ell\) does. The (2.2) contains all information as in (2.1) but with much simpler organization of tensor structure. In this paper, we will focus on \(D = (4 - 2\epsilon)\)-dimensional space, although our method can obviously be applied to arbitrary dimension. With this assumption, the integral \(I_{n+1}^{(m)}[R]\) is reduced to the linear
combination of pentagon, box, triangle, bubble and tadpole master integrals

\[
I^{(m)}_{n+1}[R] = \sum_{a_1,a_2,\ldots,a_5} C^{(a_1,a_2,\ldots,a_5)}(m|n) I_5[a_1,a_2,\ldots,a_5] \\
+ \sum_{a_1,a_2,a_3,a_4} C^{(a_1,a_2,a_3,a_4)}(m|n) I_4[a_1,a_2,a_3,a_4] + \cdots + \sum_{a_1} C^{(a_1)}(m|n) I_1[a_1].
\]

(2.3)

The reduction coefficients \(C^{(a_1,\cdots,a_r)}(m|n), 1 \leq r \leq 5\) are the rational functions of external momenta \(K_i\), masses \(M_i\), and vector \(R\). The summation in (2.3) covers all possible combinations of \(r\) propagators \(\{P_{a_1},\ldots,P_{a_r}\} \subseteq \{P_0,P_1,\ldots,P_n\}\). We will use the abbreviation \(C^{ir}(m|n)\) instead of \(C^{(a_1,\ldots,a_r)}(m|n)\) with index set \(i_r = \{a_1,\ldots,a_r\} \subseteq \{0,1,2,\ldots,n\}\).

It is easy to see vector \(R\) only appears in the numerator of \(C^{ir}\) with the form \(R \cdot R\) or \(R \cdot K_i, i = 1,2,\ldots,n\). We introduce the following two operators

\[
\mathcal{D}_i \equiv K_i \cdot \frac{\partial}{\partial R}, \quad i = 1,\ldots,n; \quad \mathcal{T} \equiv \eta^\mu\nu \frac{\partial}{\partial R^\mu} \frac{\partial}{\partial R^\nu}. \tag{2.4}
\]

We take the derivative on the both sides of (2.3) by these two operators. The left-hand side will be

\[
\mathcal{D}_i I^{(m)}_{n+1}[R] = m I^{(m-1)}_{n+1;0} - m I^{(m-1)}_{n+1;\hat{i}} + m f_i I^{(m-1)}_{n+1}, \\
\mathcal{T} I^{(m)}_{n+1}[R] = 4m(m-1)M_0^2 I^{(m-2)}_{n+1} + 4m(m-1)I^{(m-2)}_{n+1;0}, \tag{2.5}
\]

where the constant \(f_i \equiv M_0^2 + K_i^2 - M_i^2\), and

\[
I^{(m-1)}_{n+1;\hat{i}}[R] = \int \frac{d^D \ell}{(2\pi)^D} \frac{(2\ell \cdot R)^{m-1}}{P_0P_1\cdots P_{i-1}P_{i+1}\cdots P_n} \tag{2.6}
\]

i.e., the \(i\)-th propagator has been removed. For the right-hand side of (2.3), since the master integrals contains no \(R\), the operators will directly act on coefficients \(C^{ir}(m|n)\). Therefore, we have the following equations

\[
\sum_{s=1}^{5} \sum_{i_s} (\mathcal{D}_i C^{is}(m|n)) I'^s_s = m \sum_{s=1}^{5} \sum_{i_s} \left(C^{is}(m - 1;n;\hat{0}) - C^{is}(m - 1;n;\hat{i}) + f_i C^{is}(m - 1;n)\right) I'^s_s, \tag{2.7}
\]

and

\[
\sum_{s=1}^{5} \sum_{i_s} (\mathcal{T} C^{is}(m|n)) I'^s_s = 4m(m-1) \sum_{s=1}^{5} \sum_{i_s} \left(C^{is}(m - 2;n;\hat{0}) + M_0^2 C^{is}(m - 2;n)\right) I'^s_s, \tag{2.8}
\]

where \(C^{is}(m - 1;\hat{i})\) is the coefficient of the master integrals \(I'^s_{s+1}\) in the reduced expansion of tensor integral \(I^{(m-1)}_{n+1;\hat{i}}[R]\). Assuming that all the reduction coefficient of tensor integral \(I^{(m')}_{n'+1}[R]\) with either \(m' < m\) or \(n' < n\) are known already, we can get a series of differential
equations of $C^i(r|m|n)$ by comparing the coefficients of each master integral of both side of (2.7) and (2.8). Without loss of generality, we choose $i_r = (0, 1, \ldots, r)$. Then we have

$$\mathcal{T}C^{(0,1,\ldots,r)}(m|n) = 4m(m-1)M_0^2C^{(0,1,\ldots,r)}(m-2|n),$$

(2.9)

and

$$\mathcal{D}_iC^{(0,1,\ldots,r)}(m|n) = -mC^{(0,1,\ldots,r)}(m-1|n;\hat{i}) + m f_iC^{(0,1,\ldots,r)}(m-1|n).$$

(2.10)

In the equation (2.10), $C^{(0,1,\ldots,r)}(m-1|n;\hat{i})$ is the reduction coefficient of the master integral $I_{r+1}^{(0,1,\ldots,r)}$. Since $\hat{i}$ means the propagator $P_\hat{i}$ has been removed, $C^{(0,1,\ldots,r)}(m-1|n;\hat{i})$ is zero when $i \leq r$.

Similar to the idea used in [1], we do not solve the differential equations directly, but expand the reduction coefficients according to its tensor structure

$$C^{(0,1,\ldots,r)}(m|n) = \sum_{2a_0 + \sum_{k=1}^{n} a_k = m} \left\{ c_{a_0,a_1,\ldots,a_n}^{(0,1,\ldots,r)}(m)(M_0^2)^{a_0+r-n} \prod_{k=0}^{n} s_{0k}^{a_k} \right\},$$

(2.11)

where the notation $s_{0k} \equiv (R \cdot R), s_{0i} \equiv (R \cdot K_i)$. The summation condition $2a_0 + \sum_{k=1}^{n} a_k = m$ guarantees vector $R$ appears $m$ times. The exponent of $M_0^2$ makes $c_{a_0,a_1,\ldots,a_n}^{(0,1,\ldots,r)}(m)$ dimensionless. The expansion coefficients $c_{a_0,a_1,\ldots,a_n}^{(0,1,\ldots,r)}(m)$ can only be a rational function of $(K_i \cdot K_j), i,j \neq 0$ and $M_i^2, (i = 0, 1, \ldots, n)$. Moreover, $c_{a_0,a_1,\ldots,a_n}^{(0,1,\ldots,r)}(m)$ vanish if any $a_k < 0, k = 0, 1, \ldots, n$.

For $C^{(0,1,\ldots,r)}(m-1|n;\hat{i})$ in (2.10), the expansion is

$$C^{(0,1,\ldots,r)}(m-1|n;\hat{i}) = \sum_{2a_0 + \sum_{k=1}^{n} a_k = m-1} \left\{ c_{a_0,a_1,\ldots,a_n}^{(0,1,\ldots,r)}(m-1;\hat{i})(M_0^2)^{a_0+r-n} \prod_{k=0}^{n} s_{0k}^{a_k} \right\}.$$

(2.12)

The absence of term $s_{00}$ is because propagator $P_{\hat{i}}$ has been removed. In the last line of (2.12), we add a factor $\delta_{0a_i}$ to make the expression simpler. The subscript $\hat{i}$ means index $a_i$ is absent.

To get the algebraic recursion relation for expansion coefficients, we need rewrite $\mathcal{D}_i$ and $\mathcal{T}$ in terms of

$$\mathcal{D}_i = K^\mu \frac{\partial}{\partial R^\mu} = 2s_{00} \frac{\partial}{\partial s_{00}} + \sum_{j=1}^{n} s_{ij} \frac{\partial}{\partial s_{0j}},$$

$$\mathcal{T} = 2D \frac{\partial}{\partial s_{00}} + 4s_{00} \frac{\partial^2}{\partial s_{00}^2} + 4 \sum_{i=1}^{n} s_{0i} \frac{\partial}{\partial s_{0i}} \frac{\partial}{\partial s_{00}} + \sum_{i=1}^{n} \sum_{j=1}^{n} s_{ij} \frac{\partial}{\partial s_{ij}} \frac{\partial}{\partial s_{00}}.$$
With above explanation, putting (2.11) and (2.12) to (2.9) and (2.10), comparing the expansion coefficients of \(\prod_{k=0}^{n} s_{0k}^{a_k}\), the two types of differential equations (2.9) and (2.10) become
\[
(m + 1 - \sum_{l=1}^{n} \beta_{il} c_{a_1, \ldots, a_{l-1}, \ldots, a_n}(m)) + \sum_{l=1}^{n} \beta_{il} c_{a_1, \ldots, a_{l-1}, \ldots, a_n}(m)
= m\alpha_j c_{a_1, \ldots, a_n}(m - 1) - m\delta_{0a_i} c_{a_1, \ldots, a_{i-1}, \ldots, a_n}(m - 1; \hat{i})
\]
for the \(D\)-type and
\[
4m(m - 1)c_{a_1, \ldots, a_n}(m - 2) = (m - \sum_{k=1}^{n} a_k)(D + m - \sum_{k=1}^{n} a_k - 2)c_{a_1, \ldots, a_n}(m)
+ \sum_{0 < i < j} 2(a_i + 1)(a_j + 1)\beta_{ij} c_{a_1, \ldots, a_{i+1}, \ldots, a_{j+1}, \ldots, a_n}(m)
+ \sum_{i=1}^{n} (a_i + 1)(a_i + 2)\beta_{ii} c_{a_1, \ldots, a_{i+2}, \ldots, a_n}(m)
\]
for the \(T\)-type, where \(\alpha_i \equiv f_i/M_0^2, \beta_{ij} \equiv s_{0i}/M_0^2\) for simplicity. Again we need to emphasize \(c_{\hat{a}_1, \ldots, \hat{a}_n}(m - 1; \hat{i}) = 0\) in the case \(i \leq r\) for the same reason as discussed before. In (2.14) and (2.15) we have ignored the subscript \(a_0\) because it has been uniquely determined by the restrictive condition \(2a_0 + \sum_{k=1}^{n} a_n = m\) in (2.11).

3 Algorithm for recursion relations

The recurrence relations (2.14) and (2.15) are the key relations throughout the whole paper. In this section we show how to solve expansion coefficients by these two relations systematically.

3.1 Reduction coefficient of \(I_{r+1}[0, 1, \cdots, r]\)

First, we choose the master basis contains propagator \(\{P_0, P_1, \cdots, P_r\}\). We start by rewriting \(D\)-type relations (2.14) in a compact form as
\[
\tilde{G} T \mathbf{c}^{(0,1,\cdots,r)}(a_1, \cdots, a_n; m) = \mathbf{O}^{(0,1,\cdots,r)}(a_1, \cdots, a_n; m),
\]
where \(\tilde{G} = [\beta_{ij}]\) is the \(n \times n\) rescaled Gram matrix and \(T\) is a diagonal matrix
\[
T = \text{diag}(a_1 + 1, a_2 + 1, \cdots, a_n + 1).
\]
The \(\mathbf{c}^{(0,1,\cdots,r)}(a_1, \cdots, a_n; m), \mathbf{O}^{(0,1,\cdots,r)}(a_1, \cdots, a_n; m)\) are two vectors defined as
\[
[\mathbf{c}^{(0,1,\cdots,r)}(a_1, \cdots, a_n; m)]_i = c_{a_1, a_2, \cdots, a_{i+1}, \ldots, a_n}(m),
\]
and
\[
[\mathbf{O}^{(0,1,\cdots,r)}(a_1, \cdots, a_n; m)]_i = m\alpha_j c_{a_1, \cdots, a_{i-1}, \ldots, a_n}(m - 1) - m\delta_{0a_i} c_{a_1, \cdots, a_{i-1}, \ldots, a_n}(m - 1; \hat{i})
- (m + 1 - \sum_{l=1}^{n} a_l) c_{a_1, \cdots, a_{i-1}, \ldots, a_n}(m). 
\]

\(\tilde{G}\) and \(T\) are two symmetric matrices. The characteristic polynomial of \(\tilde{G} T\) is given as
\[
\text{det}(\tilde{G} T - \lambda I) = \prod_{i=0}^{n} (\lambda + i) = \prod_{i=0}^{n} c_{a_1, \cdots, a_n}(\lambda + 1) = \prod_{i=0}^{n} O^{(0,1,\cdots,r)}(\lambda + 1).
\]
The definition of these two vectors are purposely for the recurrence construction. The vector \( \mathbf{c} \) contains coefficients with rank \( m \) and subscript with the summation \( 1 + \sum_i a_i \), while the vector \( \mathbf{O} \) contains coefficients of three different patterns: (1) the first term with coefficients of rank \( m - 1 \); (2) the second one with coefficients of master integrals with one less propagator and lower rank \( m - 1 \) rank; (3) the third one with coefficients of same rank \( m \), but the summation \( -1 + \sum_i a_i \) of subscript. By induction assumption, the first two terms are considered to be known already. Thus, by rewriting \( (3.1) \)

\[
\mathbf{c}^{(0,1,\cdots,r)}(a_1, \cdots, a_n; m) = \mathbf{T}^{-1}\tilde{\mathbf{G}}^{-1}\mathbf{O}^{(0,1,\cdots,r)}(a_1, \cdots, a_n; m),
\]

we have established the recurrence relations between expansion coefficients with higher summation of subscript and those of same rank but with the summation of subscript reduced by two.

Iteratively using \( (3.5) \), we have left two kinds of unknown expansion coefficients

\[
c^{(0,1,\cdots,r)}(m), \quad m = 2k,
\]

\[
c^{(0,1,\cdots,r)}_{0,0,\cdots,0}(m), \quad c^{(0,1,\cdots,r)}_{0,1,0,\cdots,0}(m), \cdots, c^{(0,1,\cdots,r)}_{0,0,0,\cdots,1}(m) = 2k + 1,
\]

depending on the parity of \( m \). For the odd case \( m = 2k + 1 \), we solve \( c^{(0,1,\cdots,r)}_{1,0,0,\cdots,0} \)'s by \( (3.5) \) again. To see it, setting \( a_1 = \cdots = a_n = 0 \) in \( (3.5) \), the left hand side becomes

\[
(c^{(0,1,\cdots,r)}_{1,0,0,\cdots,0}(2k + 1), c^{(0,1,\cdots,r)}_{0,0,0,\cdots,0}(2k + 1), \cdots, c^{(0,1,\cdots,r)}_{0,0,0,\cdots,1}(2k + 1))^T,
\]

while the right-hand side is

\[
\mathbf{T}^{-1}\tilde{\mathbf{G}}^{-1}\mathbf{O}^{(0,1,\cdots,r)}(0, \cdots, 0; 2k + 1),
\]

where

\[
[\mathbf{O}^{(0,1,\cdots,r)}(0, \cdots, 0; 2k + 1)]_i = m\alpha_i c^{(0,1,\cdots,r)}_{0,0,\cdots,0}(2k) - (2k + 1)c^{(0,1,\cdots,r)}_{0,0,\cdots,0}(2k; i)
\]

since the third term \( c^{(0,1,\cdots,r)}_{0,0,\cdots,0}(2k + 1) \) vanishes. Therefore, we reduced to the problem of solving \( c^{(0,1,\cdots,r)}_{0,0,\cdots,0}(2k) \).

Determining the value of \( c^{(0,1,\cdots,r)}_{0,0,\cdots,0}(2k) \) requires \( \mathcal{T} \)-type recursion relations. For \( m = 2k \) and \( a_1 = a_2 = \cdots = a_n = 0 \), \( \mathcal{T} \)-type recursion relation becomes

\[
8k(2k - 1)c^{(0,1,\cdots,r)}_{0,0,\cdots,0}(2k - 2) = 2k(D + 2k - 2)c^{(0,1,\cdots,r)}_{0,0,\cdots,0}(2k) + \sum_{0 < i < j < n} 2\beta_{ij}c^{(0,1,\cdots,r)}_{0,0,\cdots,0}(2k)
\]

\[
+ \sum_{i=1}^n 2\beta_{ii}c^{(0,1,\cdots,r)}_{0,0,\cdots,2,\cdots,0}(2k).
\]

In \( c^{(0,1,\cdots,r)}_{0,0,\cdots,0,0}(2k) \), indices 1 appear in the both \( i \)-th and \( j \)-th positions, while in \( c^{(0,1,\cdots,r)}_{0,0,\cdots,2,\cdots,0}(2k) \) index 2 appears in the \( i \)-th position. For \( c^{(0,1,\cdots,r)}_{0,0,\cdots,0}(2k) \) and \( c^{(0,1,\cdots,r)}_{0,0,\cdots,1,\cdots,0}(2k) \) in \( (3.10) \), we
use (3.5) again to reach \( c_{0,\cdots,0}^{(0,\cdots,r)}(2k) \). Then we establish the relation reduced rank \( 2k \) to \( 2k - 2 \)

\[
\begin{align*}
\frac{c_{0,\cdots,0}^{(0,\cdots,r)}(2k)}{c_{0,\cdots,0}^{(0,\cdots,r)}(2k)} &= \frac{2k - 1}{D + 2k - n - 2} \left[ (4 - \alpha^T \tilde{G}^{-1}) c_{0,\cdots,0}^{(0,\cdots,r)}(2k) - 2 + \alpha^T \tilde{G}^{-1} c_{0,\cdots,0}^{(0,\cdots,r)}(2k) \right],
\end{align*}
\]

(3.11)

where \( \alpha \) is a vector define as

\[
(\alpha)^T = (\alpha_1, \alpha_2, \cdots, \alpha_n)^T = \left( \frac{f_1}{M_{d_0}}, \frac{f_2}{M_{d_0}}, \cdots, \frac{f_n}{M_{d_0}} \right)^T.
\]

(3.12)

In the second term of right-hand side of (3.11), \( c_{0,\cdots,0}^{(0,\cdots,r)}(m) \) is a vector defined as

\[
\begin{align*}
\left( c_{0,\cdots,0}^{(0,\cdots,r)}(m) \right)^{n-1 \text{ times}} &= \left( c_{0,\cdots,0}^{(0,\cdots,r)}(m; 1), c_{0,\cdots,0}^{(0,\cdots,r)}(m; 2), \cdots, c_{0,\cdots,0}^{(0,\cdots,r)}(m; n) \right)^T
\end{align*}
\]

(3.13)

The zero of first \( r \) components has been explained under the Eq(2.10). Equation (3.11) reduced rank \( m \) by two. Furthermore, we see a propagator is removed in the second term of Right-Hand-Side. Therefore, if we utilize the \( T \)-type recursion relation repeatedly, we will end with one of the following two cases. (1) The rank \( m \) is reduced to zero, which related to the reduction coefficient of a master integral. So it is either 1 or 0. (2) One of the propagator \( P_i, i \leq r \) has been removed. In this case the coefficients must be zero because the master integral will not appear in the reduction.

To make a long story short, for \( d_0 = 4 \), we summarize the whole reduction process below.

- **Step 1:** For a tensor integral with more than 5 propagators, we reduce it to 5-,4-,3-,2-,1-gon tensor integral.
- **Step 2:** For an arbitrary rank \( m_0 \), we take each \( m \leq m_0 \) with arrangement from small to large. If \( m \) is even, we calculate the expansion coefficients \( c_{a_1,\cdots,a_n}^{(0,\cdots,r)}(m) \) in the order \( \sum_{i=1}^{n} a_i = 0, 2, 4, \cdots, m \) by using (3.5) and (3.10). If \( m \) is odd, we calculate in the order \( \sum_{i=1}^{n} a_i = 1, 3, 5, \cdots, m \) by (3.5).
- **Step 3:** We continue the Step 2 until \( m = m_0 \).
- **Final step:** Combining all expansion coefficients \( c_{a_1,\cdots,a_n}^{(0,\cdots,r)}(m_0) \), we obtain the reduction coefficient by (2.11).


3.2 Calculate general $C^{(j_0,j_1,\cdots,j_r)}(m|n)$ from $C^{(0,1,\cdots,r)}(m|n)$

In this subsection, we will show how to obtain the reduction coefficients of other MIs from the result of $C^{(0,1,\cdots,r)}(m|n)$. Let us begin with the case that the Master Integral contains propagators $P_0$. It is obvious that tensor integral $I^{(m)}_{n+1}[R]$ is invariant under a permutation of labels $\{1,2,\cdots,n\}$. Then the reduction coefficients $C^{(0,1,\cdots,r)}(m|n)$ is simply given by a proper replacement $\sigma : \{M_i,K_i\} \rightarrow \{M_{j_i},K_{j_i}\},(i = 1,2,\cdots,n)$

$$C^{(0,1,\cdots,r)}(m|n) = \sigma C^{(0,1,\cdots,r)}(m|n).$$

Now the remaining part is those MIs without $P_0$. Note that by a loop momenta shift $\ell \rightarrow \ell + K_{j_0}$ we have

$$I^{(m)}_{n+1}[R] \rightarrow \int \frac{d^D\ell}{(2\pi)^D} \frac{(2\ell \cdot R + 2K_{j_0} \cdot R)^m}{(\ell^2 - M_{j_0}^2) \prod_{i=1,i\neq j_0} (\ell - (K_i - K_{j_0})^2 - M_i^2)}$$

$$= \sum_{k=0}^{m} \frac{m}{k} (2R \cdot K_{j_0})^{m-k} \times \int \frac{d^D\ell}{(2\pi)^D} \frac{(2\ell \cdot R)^k}{(\ell^2 - M_{j_0}^2) \prod_{i=1,i\neq j_0} (\ell - (K_i - K_{j_0})^2 - M_i^2)}.$$

By variable substitution $K_{j_0} \rightarrow -K_{j_0}$, $K_i \rightarrow K_i - K_{j_0}$, $M_{j_0} \rightarrow M_0$ inside the integrand\(^3\), we arrive the same form as (2.1). Then we have

$$C^{(j_0,j_1,\cdots,j_r)}(m|n) = \sum_{k=0}^{m} \frac{m}{k} (2R \cdot K_{j_0})^{m-k} C^{(0,1,\cdots,r)}(k|n)$$

$$\left. \left[ C^{(0,1,\cdots,r)}(k|n) \right]_{K_{j_0} \rightarrow -K_{j_0}, K_i \rightarrow K_i - K_{j_0}, M_{j_0} \rightarrow M_0} \right)$$

$$= \left[ \sum_{k=0}^{m} \frac{m}{k} (-2R \cdot K_{j_0})^{m-k} C^{(0,1,\cdots,r)}(k|n) \right]_{K_{j_0} \rightarrow -K_{j_0}, K_i \rightarrow K_i - K_{j_0}, M_{j_0} \rightarrow M_0}.$$

(3.16)

4 Examples

Having presented the general algorithm, in this section we will use various examples to demonstrate the use of the algorithm. In the first subsection, we will show how to reduce any tensor bubble to the basis of scalar bubble and two scalar tadpoles. The reduction of tensor triangles, tensor boxes and tensor pentagons of rank 1 and 2 has been given in the Appendix. In the second subsection, we will show how to get the reduction coefficients of tensor box with rank 1 to scalar triangles without $P_0$ from the result of $C^{(0,1,2)}(1|3)$ by (3.16).

4.1 The reduction of tensor bubble

The reduction of tensor bubble $I^{(m)}_2$ will contain three MIs as below

$$Tadpoles : I_1[0], I_1[1],$$

$$Bubbles : I_2[0, 1],$$

(4.1)

\(^3\)Note that we don’t substitute $K_{j_0}$ in $(2R \cdot K_{j_0})^{m-k}$.
and we have the expansion

\[ I_2^{(m)} = C^{(0)}(m|1)I_1[0] + C^{(1)}(m|1)I_1[1] + C^{(0,1)}(m|1)I_2[0,1]. \]  \tag{4.2}

The way to achieve \( C^{(0)}(m|1) \) have been given in [1]. Here we only provide how to calculate \( C^{(1)}(m|1) \) and \( C^{(0,1)}(m|1) \). The coefficient of \( I_1[1] \) can be obtained by (3.16) from \( C^{(0)}(m|1) \). While for \( I_2[0,1] \), there is only one subscript in the expansion coefficients. So the expansion of \( C^{(0,1)}(m|1) \) is

\[ C^{(0,1)}(m|1) = \sum_i c_i^{(0,1)}(m)[M_0^2(R \cdot R)]^{m-i} (R \cdot K_1)^i. \]  \tag{4.3}

We have the corresponding \( \mathcal{D} \)-type recursion relation

\[
\begin{align*}
c_{i+2}^{(0,1)}(m) &= \frac{1}{(i+2)\beta_{11}} \left( m\alpha_1 c_{i+1}^{(0,1)}(m-1) - m\delta_{0,i+1} c_0^{(0,1)}(m-1) - (m-i)c_i^{(0,1)}(m) \right) \\
&= \frac{1}{(i+2)\beta_{11}} \left( m\alpha_1 c_{i+1}^{(0,1)}(m-1) - (m-i)c_i^{(0,1)}(m) \right), \tag{4.4}
\end{align*}
\]

and \( \mathcal{T} \)-type recursion relation

\[
\begin{align*}
c_0^{(0,1)}(2r) &= \frac{2r-1}{2r+D-3} \left[ \left( 4 - \frac{\alpha_1^2}{\beta_{11}} \right) c_0^{(0,1)}(2r-2) + \frac{\alpha_1}{\beta_{11}} c_0^{(0,1)}(2r-2) \right] \\
&= \frac{2r-1}{2r+D-3} \left( 4 - \frac{\alpha_1^2}{\beta_{11}} \right) c_0^{(0,1)}(2r-2), \tag{4.5}
\end{align*}
\]

where \( c_i^{(0,1)}(m) \) and \( c_i^{(0,1)}(2r-2) \) without subscript stands for \( c_{i_1}^{(0,1)}(m) \) and \( c_{i_1}^{(0,1)}(2r-2) \). These two terms vanish, because they come from the reduction coefficient of bubble \( I_2[0,1] \) for a tensor tadpole, i.e., the propagator \( P_1 \) has been removed.

Now we show the result for rank \( m \leq 4 \). The rank \( m = 0 \) is trivial. For other ranks:

- \( m = 1 \)

The reduction coefficients of tadpoles: \( I_1[0], I_1[1] \)

Using the result in [1], we have

\[ C^{(0)}(1|1) = \frac{R \cdot K_1}{K_1^2}. \]  \tag{4.6}

For \( I_1[1] \), by choosing \( j_0 = 1 \) in (3.16), we have

\[
\begin{align*}
C^{(1)}(1|1) &= \left. C^{(0)}(1|1) \right|_{K_1 \rightarrow -K_1, M_0 \leftrightarrow M_1} \\
&= \frac{R \cdot K_1}{K_1^2}. \tag{4.7}
\end{align*}
\]

The reduction coefficients of bubble: \( I_2[0,1] \)

The expansion of \( C^{(0,1)}(1|1) \) becomes

\[ C^{(0,1)}(1|1) = c_i^{(0,1)}(1)(R \cdot K_1). \]  \tag{4.8}
By (4.4), we have
\[
\begin{align*}
c_1^{(0,1)}(1) &= \frac{1}{\beta_{11}} \left( \alpha_1 c_0^{(0,1)}(0) - 2 c_{-1}^{(0,1)}(1) \right) \\
&= \frac{f_1}{s_{11}}, \quad (4.9)
\end{align*}
\]
where the boundary conditions are \( c_0^{(0,1)}(0) = 1, c_{-1}^{(0,1)}(1) = 0 \). Then
\[
C^{(0,1)}(1|1) = c_1^{(0,1)}(1) R \cdot K_1 = \left( \frac{K_1 \cdot K_1 + M_0^2 - M_1^2}{K_1 \cdot K_1} \right) R \cdot K_1. \quad (4.10)
\]

- **m = 2**

The reduction coefficients of tadpoles: \( I_1[0], I_1[1] \)

The reduction coefficient of tadpole \( I_1[0] \) is
\[
C^{(0)}(2|1) = \frac{\left( K_1 \cdot K_1 + M_0^2 - M_1^2 \right) \left( K_1 \cdot K_1 R \cdot R - D (R \cdot K_1)^2 \right)}{(D - 1) (K_1 \cdot K_1)^2}. \quad (4.11)
\]

For \( I_1[1] \), by choosing \( j_0 = 1 \) in (3.16), we have
\[
C^{(1)}(2|1) = 2(2R \cdot K_1) \left[ C^{(0)}(1|1) \bigg|_{K_1 \rightarrow K_1, M_0 \leftrightarrow M_1} + C^{(0)}(2|1) \bigg|_{K_1 \rightarrow K_1, M_0 \leftrightarrow M_1} \right] + \frac{4(R \cdot K_1)^2}{K_1^2} + \frac{\left( K_1 \cdot K_1 + M_0^2 - M_1^2 \right) \left( K_1 \cdot K_1 R \cdot R - D (R \cdot K_1)^2 \right)}{(D - 1) (K_1 \cdot K_1)^2}.
\]

The reduction coefficients of bubble: \( I_2[0,1] \)

The expansion of \( C^{(0,1)}(2|1) \) is
\[
C^{(0,1)}(2|1) = c_0^{(0,1)}(2) M_0^2 s_{00} + c_2^{(0,1)}(2) s_{01}^2. \quad (4.13)
\]

By setting \( r = 1 \) in (4.5), we have
\[
\begin{align*}
c_0^{(0,1)}(2) &= \frac{1}{D - 1} \left( 4 - \frac{\alpha_1^2}{\beta_{11}} \right) c_0^{(0,1)}(0) \\
&= \frac{1}{D - 1} \left( 4 - \frac{\alpha_1^2}{\beta_{11}} \right) \\
&= \frac{4}{D - 1} - \frac{f_1^2}{(D - 1) M_0^2 s_{11}}, \quad (4.14)
\end{align*}
\]
where the boundary conditions is \( c_0^{(0,1)}(0) = 1 \).

By setting \( i = 0 \) in (4.4), we have
\[
\begin{align*}
c_2^{(0,1)}(2) &= \frac{1}{2 \beta_{11}} \left( 2 \alpha_1 c_1^{(0,1)}(1) - 2 c_0^{(0,1)}(2) \right) \\
&= \frac{1}{\beta_{11}} \left( \alpha_1 f_1 \frac{1}{s_{11}} - \frac{1}{D - 1} \left( 4 - \frac{\alpha_1^2}{\beta_{11}} \right) \right) \\
&= \frac{D f_1^2}{(D - 1) s_{11}^2} - \frac{4 M_0^2}{(D - 1) s_{11}}. \quad (4.15)
\end{align*}
\]
where \( c_1^{(0,1)}(1) \) has been presented in the case \( m = 1 \). Then

\[
C^{(0,1)}(2|1) = \frac{s_{01}^2 (Df_1^2 - 4M_0^2 s_{11}) + s_{00}s_{11} (4M_0^2 s_{11} - f_1^2)}{(D - 1)s_{11}^2}. \tag{4.16}
\]

- \( m = 3 \)

The reduction coefficients of tadpoles: \( I_1[0], I_1[1] \)

The reduction coefficient of tadpole \( I_1[0] \) is

\[
C^{(0)}(3|1) = \frac{f_1^2}{D} \left( 3s_{00}s_{01}s_{11} - (D + 2)s_{01}^3 \right) + \frac{4M_0^2 s_{01} (2s_{01}^2 - 3s_{00}s_{11})}{Ds_{11}^2}. \tag{4.17}
\]

By choosing \( j_0 = 1 \) in (3.16), we have

\[
C^{(1)}(3|1) = 3(2R \cdot K_1)^2 \left[ C^{(0)}(1|1)\right]_{K_1 \rightarrow K_1, M_0 \leftrightarrow M_1} + 3(2R \cdot K_1) \left[ C^{(0)}(2|1)\right]_{K_1 \rightarrow K_1, M_0 \leftrightarrow M_1}
\]

\[
= \frac{s_{01} (7D^2 s_{01} + 12DM_0^2 s_{00} - 10D^2 s_{01}^2 - 12M_0^2 s_{00})}{(D - 1)s_{11}} + \frac{(D + 2) (M_0^2 - M_1^2)^2 s_{01}^3}{Ds_{11}^2}
\]

\[
+ \frac{4 (D^2 M_0^4 - DM_0^2 - 2M_0^2) s_{01}^3}{Ds_{11}^2} - \frac{3 (M_0^2 - M_1^2)^2 s_{00}s_{01}}{(D - 1)s_{11}^2} + \frac{3s_{00}s_{01}}{D - 1}. \tag{4.18}
\]

The reduction coefficients of bubble: \( I_2[0, 1] \)

The expansion of \( C^{(0,1)}(3|1) \) is

\[
C^{(0,1)}(3|1) = c_1^{(0,1)}(3)M_0^2 s_{00}s_{01} + c_3^{(0,1)}(3)s_{01}^3. \tag{4.19}
\]

By setting \( i = -1 \) in (4.4), we have

\[
c_1^{(0,1)}(3) = \frac{1}{\beta_{11}^2} \left( 3\alpha_1 c_0^{(0,1)}(2) - 4c_{-1}^{(0,1)}(3) \right)
\]

\[
= \frac{1}{\beta_{11}^2} \left( 3\alpha_1 \left( \frac{4}{D - 1} - \frac{f_1^2}{(D - 1)M_0^2 s_{11}} \right) \right)
\]

\[
= \frac{12f_1}{(D - 1)s_{11}} - \frac{3f_1^3}{(D - 1)M_0^2 s_{11}^2}. \tag{4.20}
\]

where we have used \( c_{-1}^{(0,1)}(3) = 0 \) and the result of expansion coefficient \( c_0^{(0,1)}(2) \) in the case \( m = 2 \).

By setting \( i = 1 \) in (4.4), we have

\[
c_3^{(0,1)}(3) = \frac{1}{3\beta_{11}^3} \left( 3\alpha_1 c_0^{(0,1)}(2) - 2c_{-1}^{(0,1)}(3) \right)
\]

\[
= \frac{1}{3\beta_{11}^3} \left[ 3\alpha_1 \left( \frac{D f_1^2}{(D - 1)s_{11}^2} - \frac{4M_0^2}{(D - 1)s_{11}} \right) - 2 \left( \frac{12f_1}{(D - 1)s_{11}} - \frac{3f_1^3}{(D - 1)M_0^2 s_{11}^2} \right) \right]
\]

\[
= \frac{(D + 2)f_1^3}{(D - 1)s_{11}^2} - \frac{12f_1 M_0^2}{(D - 1)s_{11}^2}. \tag{4.21}
\]
Then the reduction coefficient is

\[ C^{(0,1)}(3|1) = \frac{f_1 \left( s_{01}^3 \left( (D + 2) f_1^2 - 12 M_0^2 s_{11} \right) + 3 s_{00} s_{11} s_{01} \left( 4 M_0^2 s_{11} - f_1^2 + f_1 s_{00} \right) \right) + 3 \left( 4 s_{00} s_{11} s_{01} \right)}{(D - 1) s_{11}^3} \]  

(4.22)

- \quad m = 4

**The reduction coefficients of tadpoles: \( I_1[0], I_1[1] \)**

The reduction coefficient of tadpole \( I_1[0] \) is

\[ C^{(0)}(4|1) = -\frac{3 f_1 s_{00} \left( 8 D^2 M_0^2 s_{01}^2 + f_1^2 s_{00} + 16 D M_0^2 s_{01}^2 - 16 M_0^2 s_{01}^2 \right)}{(D - 1) D (D + 1) s_{11}^3} \]

\[ + \frac{2(D + 2) f_1 s_{01} \left( 3 D f_1^2 s_{00} + 10 D M_0^2 s_{01}^2 - 8 M_0^2 s_{01}^2 \right)}{(D - 1) D (D + 1) s_{11}^3} + \frac{12(2D - 1) f_1 M_0^2 s_{00}^2}{(D - 1) D (D + 1) s_{11}^3} \]

\[ - \frac{(D + 2)(D + 4) f_1^3 s_{01}^4}{(D - 1)(D + 1) s_{11}^3} \]  

(4.23)

By choosing \( j_0 = 1 \) in (3.16), we have

\[ C^{(1)}(4|1) = 4(2R \cdot K_1)^3 \left[ C^{(0)}(0|1) \big|_{K_1 \rightarrow -K_1, M_0 \leftrightarrow M_1} \right] + 6(2R \cdot K_1)^2 \left[ C^{(0)}(2|1) \big|_{K_1 \rightarrow -K_1, M_0 \leftrightarrow M_1} \right] + 4(2R \cdot K_1) \left[ C^{(0)}(3|1) \big|_{K_1 \rightarrow -K_1, M_0 \leftrightarrow M_1} \right] \]

\[ + \left[ C^{(0)}(4|1) \big|_{K_1 \rightarrow -K_1, M_0 \leftrightarrow M_1} \right] \]

\[ = \frac{5 D^2 + 6D - 8}{D (D^2 - 1) s_{11}^3} s_{00} + \frac{12 s_{00} \left( (2D - 1) M_1^2 s_{00} + 2D (D + 1) s_{01}^2 \right)}{D (D^2 - 1) s_{11}^3} \]

\[ - \frac{f_1}{D (D^2 - 1) s_{11}^3} \left[ \left( \frac{D^2 + 6D + 8}{(D^2 - 1) s_{11}^3} \right) + \frac{6(D + 2) s_{00} s_{01}^2}{(D^2 - 1) s_{11}^3} - \frac{3 s_{00}^2}{(D^2 - 1) s_{11}^3} \right] \]

\[ + \frac{f_1}{(D^2 - 1) s_{11}^3} \left[ \left( \frac{8(D + 2) s_{01}^4}{(D - 1) s_{11}^3} - \frac{24 s_{00} s_{01}^2}{(D - 1) s_{11}^3} \right) - \frac{64 M_1^2 s_{01}^4}{D s_{11}^3} + \frac{32 s_{01}^2}{D s_{11}^3} \right], \]  

(4.24)

where \( f_1 = K_1^2 + M_1^2 - M_0^2 \).

**The reduction coefficients of bubble: \( I_2[0,1] \)**

The expansion of \( C^{(0,1)}(4|1) \) is

\[ C^{(0,1)}(4|1) = c_0^{(0,1)}(4) M_0^2 s_{00}^2 + c_2^{(0,1)}(4) M_0^2 s_{00} s_{01}^2 + c_4^{(0,1)}(4) s_{01}^4. \]  

(4.25)

By setting \( r = 2 \) in (4.5), we have

\[ c_0^{(0,1)}(4) = \frac{4}{D + 1} \left[ 4 - \frac{\alpha_1^2}{\beta_{11}} \right] c_0^{(0,1)}(2) \]

\[ = \frac{4}{D + 1} \left[ \left( 4 - \frac{\alpha_1^2}{\beta_{11}} \right) \left( 4 - \frac{f_1^2}{(D - 1) M_0^2 s_{11}} \right) \right] \]

\[ = \frac{3 f_1^4}{(D^2 - 1) M_0^2 s_{11}^3} - \frac{24 f_1^2}{(D^2 - 1) M_0^2 s_{11}^3} + \frac{48}{D^2 - 1}, \]  

(4.26)
By setting \( i = 0, 2 \) in (4.4), we calculate \( c_2^{(0,1)}(4), c_4^{(0,1)}(4) \) iteratively

\[
c_2^{(0,1)}(4) = \frac{1}{2\beta_{11}} \left( 4\alpha_1 c_1^{(0,1)}(3) - 4c_0^{(0,1)}(4) \right)
= \frac{1}{2\beta_{11}} \left[ 4\alpha_1 \left( \frac{(D + 2)f_1^3}{(D - 1)s_{11}^3} - \frac{12f_1M_0^2}{(D - 1)s_{11}^2} \right) - 4 \left( \frac{3f_1^4}{(D^2 - 1)M_0^4s_{11}^2} - \frac{24f_1^2}{(D^2 - 1)M_0^2s_{11}} + \frac{48}{D^2 - 1} \right) \right]
= -\frac{6(D + 2)f_1^4}{(D^2 - 1)M_0^6s_{11}^2} + \frac{24(D + 3)f_1^2}{(D^2 - 1)s_{11}^2} - \frac{96M_0^2}{(D^2 - 1)s_{11}}.
\] (4.27)

\[
c_4^{(0,1)}(4) = \frac{1}{4\beta_{11}} \left( 4\alpha_1 c_3^{(0,1)}(3) - 2c_2^{(0,1)}(4) \right)
= \frac{1}{4\beta_{11}} \left[ 4\alpha_1 \left( \frac{(D + 2)f_1^3}{(D - 1)s_{11}^3} - \frac{12f_1M_0^2}{(D - 1)s_{11}^2} \right) - 2 \left( \frac{6(D + 2)f_1^4}{(D^2 - 1)M_0^6s_{11}^2} + \frac{24(D + 3)f_1^2}{(D^2 - 1)s_{11}^2} - \frac{96M_0^2}{(D^2 - 1)s_{11}} \right) \right]
= -\frac{24(D + 2)f_1^2M_0^2}{(D^2 - 1)s_{11}^4} + \frac{(D^2 + 6D + 8)f_1^4}{(D^2 - 1)s_{11}^2} + \frac{48M_0^4}{(D^2 - 1)s_{11}^2}.
\] (4.28)

Where we have used the results of the expansion coefficients with lower rank. Then the reduction coefficient is

\[
C^{(0,1)}(4|1) = -\frac{24f_1^2M_0^2(s_{01}^2 - s_{00}s_{11})(D + 2)s_{01}^3 - s_{00}s_{11})}{(D^2 - 1)s_{11}^3} + \frac{48M_0^4(s_{01}^2 - s_{00}s_{11})^2}{(D^2 - 1)s_{11}^2}
+ \frac{f_1^4((D^2 + 6D + 8)s_{01}^3 - 6(D + 2)s_{00}s_{11}s_{01}^2 + 3s_{00}^2s_{11}^2)}{(D^2 - 1)s_{11}^4}.
\] (4.29)

### 4.2 Reduce tensor box to scalar triangles

We will consider the reduction coefficients of triangle MIs of tensor integral \( I_4^{(1)} \) as another example to illustrate the algorithm in section 3.2. For simplicity, we denote \( G(i_1, i_2, \cdots, i_s; j_1, j_2, \cdots, j_r) \) as the determinant of the Gram matrix \( G \) with entries \( G_{ab} = K_a \cdot K_b = s_{ab} \). Specially, we denote \( G(i_1, i_2, \cdots, i_s) \equiv G(i_1, i_2, \cdots, i_s; i_1, i_2, \cdots, i_s) \).

The reduction coefficient of the scalar triangle \( I_3[0, 1, 2] \) is

\[
C^{(0,1,2)}(1|3) = -\frac{G(2, 3; 1, 2)s_{01} - G(1, 3; 1, 2)s_{02} + G(1, 2; 1, 2)s_{03}}{G(1, 2, 3)}
\] (4.30)

The reduction coefficients of \( I_3[0, 1, 3], I_3[0, 2, 3] \) are easy to obtained by simply changing
labels \{1, 2, 3\} \rightarrow \{1, 3, 2\} and \{1, 2, 3\} \rightarrow \{2, 1, 3\} respectively:

\[ C^{(0,1,3)}(1|3) = C^{(0,1,2)}(1|3) \bigg|_{K_2 \leftrightarrow K_3, M_2 \leftrightarrow M_3} \]
\[ = -\frac{G(3, 2; 1, 3)s_{01} - G(1, 2; 1, 3)s_{03} + G(1, 3; 1, 3)s_{02}}{G(1, 2, 3)}, \]
\[ C^{(0,2,3)}(1|3) = C^{(0,1,2)}(1|3) \bigg|_{K_1 \leftrightarrow K_2, M_1 \leftrightarrow M_2} \]
\[ = -\frac{G(2, 1; 3, 2)s_{03} - G(3, 1; 3, 2)s_{02} + G(3, 2; 3, 2)s_{01}}{G(1, 2, 3)}. \]  

(4.31)

Now we consider the reduction coefficient of the triangle without \(P_0\), i.e., \(I_3[1, 2, 3]\). In (3.16), choosing \(j_0 = 3, j_1 = 1, j_2 = 2\), we have

\[ C^{(1,2,3)}(1|3) = C^{(3,1,2)}(1|3) = (2R \cdot K_3) C^{(0,1,2)}(0|3) \bigg|_{K_1 \rightarrow K_1 - K_3, K_2 \rightarrow K_2 - K_3, K_3 \rightarrow -K_3, M_0 \leftrightarrow M_3} \]
\[ + C^{(0,1,2)}(1|3) \bigg|_{K_1 \rightarrow K_1 - K_3, K_2 \rightarrow K_2 - K_3, K_3 \rightarrow -K_3, M_0 \leftrightarrow M_3} \]
\[ = \frac{G(K_2 - K_3, K_3; K_1 - K_3, K_2 - K_3)(s_{01} - s_{03})}{G(K_1 - K_3, K_2 - K_3, K_3; K_1 - K_3, K_2 - K_3)} \]
\[ + \frac{-G(K_1 - K_3, K_3; K_1 - K_3, K_2 - K_3)(s_{02} - s_{03})}{G(K_1 - K_3, K_2 - K_3, K_3; K_1 - K_3, K_2 - K_3)} \]
\[ + \frac{G(K_1 - K_3, K_2 - K_3; K_1 - K_3, K_2 - K_3)s_{03}}{G(K_1 - K_3, K_2 - K_3, K_3; K_1 - K_3, K_2 - K_3, K_3)} \].  

(4.32)

where \(C^{(0,1,2)}(0|3) = 0\) is the reduction coefficient of a triangle MI from a box MI.

5 Discussion

In this paper, we show how to use the differential operators to get the analytical expressions for the reduction coefficients of all master basis. By these operators, one can establish the recurrence relations about reduction coefficients in differential equation form. Another crucial step in this method is that we use the information of tensor structure to avoid solving the intricate differential equations directly.

As we have reviewed, in [32–34], the analytical expressions for reduction coefficients can be solved by unitary method. However, there are some differences between these two approaches.

- The first difference is that the expression given by unitarity cut method is written using the spinor formalism, while results in this paper use the traditional Lorentz invariant contractions.

- The second difference is that in unitarity cut method, we have assumed the external momenta to be in pure 4D and only loop momentum is in general \((4 - 2\epsilon)\) dimension. For our new method, there is no such a constraint and the external momenta can be in 4D or in \((4 - 2\epsilon)\) dimension.
• The third difference is that results in the paper are defined in an iterated way, while expressions given by unitarity cut method are just one equation (although the differentiation has the spirit of iteration).

• The fourth difference is that expressions by unitarity cut method using the input of arbitrary forms, while the one in this paper using the standard input given in (2.2). The difference has a potential and huge impact on computation efficiency. The reason is that with the development of on-shell program, it is well known that tree-level amplitudes will be significantly simplified if we use spinor variables with spurious poles, such as these given by the recursion relation [37, 38]. Thus, it will be desirable to cooperate these advantages of unitarity cut method to our current new strategy.

Before ending this paper, let us emphasize that the purpose of the paper is to establish an independent and complete reduction framework for one-loop integrals using auxiliary vector $R$. In the previous work [1], we have discussed the reduction coefficients of tadpoles. In this paper we have complete the coefficients of other basis. However, for these two works, we have assumed the power of each propagator is just one. To be a complete reduction framework, we need to give the reduction of integrals with arbitrary tensor structures and propagators having arbitrary powers. We will show in the upcoming paper how to achieve this. After completing the framework of reduction with auxiliary vector $R$, we can discuss various limit cases, like the massless limit or the vanishing of Gram determinant, which will be presented in another paper soon. Another direction is to apply our new framework to higher loops. But unlike one-loop case, relations established by differentiation over $R$ are usually not enough. Besides, the master integrals are more complicated in higher-loop integrals. How to solve these difficulties will be another future project.

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**A  More examples**

In this appendix, we provide more examples to illustrate our method.\(^4\) There are three points we need to emphasize ahead.

- For the tensor integral $I_{n+1}^{(m)}$, we only list the reduction coefficient $C_i^r$ for $m \geq n - |i_r|$, because there are not enough $\ell \cdot R$ in the nominator to cancel $n - |i_r|$ propagators for $m < n - |i_r|$.

---

\(^4\)All results have been checked with Fire6. [39–43]
A.1 All reduction coefficients of tensor triangle with rank $m$.

The MI of a tensor triangle

- There is a permutation symmetry about the expansion coefficient.
- We merely list the results of $C^i(m|n)$.

\[
\begin{align*}
\text{Tadpoles} : & \quad C^{(i)}(m|n) = C^{(1)}(m|n) |_{1\leftrightarrow i} \\
\text{Bubbles} : & \quad C^{(i,j)}(m|n) = C^{(0,1)}(m|n) |_{1\leftrightarrow i} \\
\text{Triangles} : & \quad C^{(i,j,k)}(m|n) = C^{(1,2,3)}(m|n) |_{1\leftrightarrow i, 2\leftrightarrow j, 3\leftrightarrow k} \\
\text{Boxes} : & \quad C^{(i,j,k,l)}(m|n) = C^{(1,2,3,4)}(m|n) |_{1\leftrightarrow i, 2\leftrightarrow j, 3\leftrightarrow k, 4\leftrightarrow l}
\end{align*}
\]

where $0 < i < j < k < l$.

There is a permutation symmetry about the expansion coefficient $C_{a_1, \ldots, a_n}^{(j_0, \ldots, j_r)}(m)$. If the Master Integral $I_{r+1}[j_0, j_1, \ldots, j_r]$ is invariant under a label permutation $\sigma : \{1, 2, \ldots, n\} \rightarrow \{\sigma(1), \sigma(2), \ldots, \sigma(n)\}$, we have $C_{a_1, \ldots, a_n}^{(j_0, \ldots, j_r)}(m) = C_{a_{\sigma(1)}, \ldots, a_{\sigma(n)}}^{(\sigma(1), \ldots, \sigma(n))}(m)$.

For example, $I_3[0, 1, 2, 3]$ is invariant under the label permutation $\sigma : \{1, 2, 3, 4\} \rightarrow \{3, 1, 2, 4\}$, then we have

\[
C_{1,2,4,5}^{(0,1,2,3)}(14) = C_{2,4,1,5}^{(0,1,2,3)}(14) |_{\{1,2,3,4\} \rightarrow \{3,1,2,4\}}.
\]

A.1 All reduction coefficients of tensor triangle with rank $m = 1, 2$

The MI of a tensor triangle $I_3^{(m)}$ are

\[
\begin{align*}
\text{Tadpoles} : & \quad I_3[0], I_3[1], I_3[2], \\
\text{Bubbles} : & \quad I_3[0, 1], I_3[0, 2], I_3[1, 2], \\
\text{Triangles} : & \quad I_3[0, 1, 2].
\end{align*}
\]

Then the reduction of the tensor triangle is

\[
I_3^{(m)} = C^{(0)}(m|2)I_3[0] + C^{(1)}(m|2)I_3[1] + C^{(2)}(m|2)I_3[2] + C^{(0,1)}(m|2)I_3[0, 1] + C^{(0,2)}I_3[0, 2] + C^{(1,2)}I_3[1, 2] + C^{(0,1,2)}I_3[0, 1, 2].
\]

- $m = 1$

Reduction coefficients of tadpoles

All reduction coefficients vanish.
Reduction coefficient of bubbles

\[ C^{(0,1)}(1|2) = \frac{s_{01}s_{12} - s_{02}s_{11}}{G(1,2)}. \]  

(A.5)

Choosing \( j_0 = 1, j_1 = 2 \) in (3.16), we have

\[ C^{(1,2)}(1|2) = C^{(0,2)}(1|2) \bigg|_{M_0 \to M_1, K_1 \to -K_1, K_2 \to K_2 - K_1} = \frac{s_{02}s_{12} - s_{01}s_{22}}{G(1,2)} \bigg|_{M_0 \to M_1, K_1 \to -K_1, K_2 \to K_2 - K_1} = \frac{s_{02}(s_{11} - s_{12})}{G(1,2)} + (1 \leftrightarrow 2). \]  

(A.6)

Reduction coefficient of triangle

\[ C^{(0,1,2)}(1|2) = \frac{s_{01}(f_1s_{22} - f_2s_{12})}{G(1,2)} + (1 \leftrightarrow 2). \]  

(A.7)

\( m = 2 \)

Reduction coefficients of tadpoles

\[ C^{(0)}(2|2) = \frac{s_{11}s_{22}s_{01}s_{02} - s_{12}s_{22}s_{01}^2}{s_{11}s_{22}G(1,2)} + (1 \leftrightarrow 2). \]  

(A.8)

Choose \( j_0 = 1 \) in (3.16), we have

\[ C^{(1)}(2|2) = C^{(0)}(2|2) \bigg|_{M_0 \to M_1, K_1 \to -K_1, K_2 \to K_2 - K_1} = \frac{-s_{12}s_{22}s_{01}^2 + 2s_{02}s_{11}s_{22}s_{01} + s_{02}s_{11}s_{12}}{s_{11}s_{22}(s_{11}s_{22} - s_{12}^2)} \bigg|_{M_0 \to M_1, K_1 \to -K_1, K_2 \to K_2 - K_1} = \frac{-2s_{12}^2 + s_{22}s_{12} + s_{11}s_{22}}{s_{11}s_{22}^2} \frac{s_{01}^2 + 2s_{02}s_{11}(s_{12} - s_{22})s_{01} + s_{02}s_{11}(s_{12} - s_{11})}{s_{11}(s_{11} - 2s_{12} + s_{22})G(1,2)}. \]  

(A.9)

Reduction coefficients of bubbles

\[ C^{(0,1)}(2|2) = \frac{s_{01}c_{2,0}^{(0,1)}(2)}{M_0^2} + \frac{s_{02}s_{01}c_{1,1}^{(0,1)}(2)}{M_0^2} + \frac{s_{02}c_{0,2}^{(0,1)}(2)}{M_0^2} + s_{00}c_{0,0}^{(0,1)}(2), \]  

(A.10)

where

\[ c_{0,0}^{(0,1)}(2) = \frac{f_2s_{11} - f_1s_{12}}{(D - 2)G(1,2)}, \]  

(A.11)

\[ c_{1,1}^{(0,1)}(2) = \frac{2(D - 1)f_2M_0^2s_{11}s_{12}}{(D - 2)(s_{12}^2 - s_{11}s_{22})^2} - \frac{2f_1M_0^2((D - 2)s_{11}s_{22} + s_{12}^2)}{(D - 2)(s_{12}^2 - s_{11}s_{22})^2}. \]  

(A.12)
\[ c_{2,0}^{(0,1)}(2) = -\frac{M_0^2 (f_2 s_{11} ((D - 2)s_{12}^2 + s_{11}s_{22}) + f_1 s_{12} ((D - 2)s_{12}^2 + (3 - 2d)s_{11}s_{22}))}{(D - 2)s_{11} G(1, 2)^2}, \]
(A.13)

\[ c_{0,2}^{(0,1)}(2) = -\frac{(D - 1)M_0^2 s_{11} (f_2 s_{11} - f_1 s_{12})}{(D - 2)^2 G(1, 2)^2}, \]
(A.14)

\[ C^{(1,2)}(2|2) = 2(2R \cdot K_2) \left[ C^{(0,1)}(1, 2) \bigg|_\sigma \right] + C^{(0,1)}(2, 2) \left|_\sigma \right. \]
\[ = 4s_{02} \left[ \frac{s_{01} (f_1 s_{22} - f_2 s_{12})}{G(1, 2)} + (1 \leftrightarrow 2) \right] + C^{(0,1)}(2, 2) \left|_\sigma \right., \]
(A.15)

where

\[ \sigma = M_0 \leftrightarrow M_2, K_2 \rightarrow -K_2, K_1 \rightarrow K_1 - K_2. \]
(A.16)

Reduction coefficients of triangle

\[ C^{(0,1,2)}(2|2) = c_{0,0}^{(0,1,2)} M_0^2 s_{00} + c_{2,0}^{(0,1,2)} s_{01}^2 + c_{0,2}^{(0,1,2)} s_{02}^2 + c_{1,1}^{(0,1,2)} s_{01} s_{02}, \]
(A.17)

where

\[ c_{0,0}^{(0,1,2)}(2) = \frac{f_1^2 s_{22} - 2f_2 f_1 s_{12} + f_2^2 s_{11} + 4M_0^2 (s_{12}^2 - s_{11}s_{22})}{(D - 2)M_0^2 (s_{12}^2 - s_{11}s_{22})}, \]
(A.18)

\[ c_{2,0}^{(0,1,2)}(2) = \frac{f_2^2 ((D - 2)s_{12}^2 + s_{11}s_{22})}{(D - 2) (s_{12}^2 - s_{11}s_{22})^2} + \frac{2(D - 1)f_1 f_2 s_{12}s_{22}}{(D - 2) (s_{12}^2 - s_{11}s_{22})^2} + \frac{4M_0^2 s_{22}}{(D - 2) (s_{12}^2 - s_{11}s_{22})}, \]
(A.19)

\[ c_{0,2}^{(0,1,2)}(2) = \frac{f_1^2 ((D - 2)s_{12}^2 + s_{11}s_{22})}{(D - 2) (s_{12}^2 - s_{11}s_{22})^2} + \frac{2(D - 1)f_2 f_1 s_{11}s_{12}}{(D - 2) (s_{12}^2 - s_{11}s_{22})^2} + \frac{4M_0^2 s_{11}}{(D - 2) (s_{12}^2 - s_{11}s_{22})}, \]
(A.20)

\[ c_{1,1}^{(0,1,2)}(2) = -\frac{2(D - 1)f_2 f_1 s_{12}s_{22}}{(D - 2) (s_{12}^2 - s_{11}s_{22})^2} + \frac{2f_2 f_1 (D s_{12}^2 + (D - 2)s_{11}s_{22})}{(D - 2) (s_{12}^2 - s_{11}s_{22})^2} \]
\[ - \frac{2(D - 1)f_2^2 s_{11}s_{12}}{(D - 2) (s_{12}^2 - s_{11}s_{22})^2} - \frac{8M_0^2 s_{12}}{(D - 2) (s_{12}^2 - s_{11}s_{22})}. \]
(A.21)
A.2 All reduction coefficients of tensor box with rank \( m = 1, 2 \)

The MIs of a tensor box \( I_4^{(m)} \) are

\[
\text{Tadpoles: } I_1[0], I_1[1], I_1[2], I_1[3], \nonumber \\
\text{Bubbles: } I_2[0, 1], I_2[0, 2], I_2[0, 3], I_2[1, 2], I_2[1, 3], I_2[2, 3], \nonumber \\
\text{Triangles: } I_3[0, 1, 2], I_3[0, 1, 3], I_3[0, 2, 3], I_3[1, 2, 3], \nonumber \\
\text{Box: } I_4[0, 1, 2, 3]. \quad (A.22)
\]

Then the reduction of the tensor box is

\[
I_4^{(m)} = \sum_{i=0}^{3} C^{(i)}(m|3) I_1[i] + \sum_{0 \leq i_1 < i_2 \leq 3} C^{(i_1,i_2)}(m|3) I_2[i_1,i_2] 
+ \sum_{0 \leq i_1 < i_2 < i_3 \leq 3} C^{(i_1,i_2,i_3)}(m|3) I_3[i_1,i_2,i_3] 
+ C^{(0,1,2,3)}(m|3) I_4[0,1,2,3]. \quad (A.23)
\]

- \( m = 1 \)

**Reduction coefficients of tadpoles, bubbles**

All reduction coefficients vanish.

**Reduction coefficients of triangles**

\[
C^{(0,1,2)}(1|3) = -\frac{G(2,3;1,2) s_{01} - G(1,3;1,2) s_{02} + G(1,2;1,2) s_{03}}{G(1,2,3)}. \quad (A.24)
\]

In (3.16), choosing \( j_0 = 3, j_1 = 1, j_2 = 2 \), we have

\[
C^{(1,2,3)}(1|3) = C^{(3,1,2)}(1|3) = (2R \cdot K_3) C^{(0,1,2)}(1|3) \bigg|_{K_1 \rightarrow K_1 - K_3, K_2 ightarrow K_2 - K_3, K_3 ightarrow - K_3, M_0 \leftrightarrow M_3} 
+ C^{(0,1,2)}(1|3) \bigg|_{K_1 \rightarrow K_1 - K_3, K_2 ightarrow K_2 - K_3, K_3 ightarrow - K_3, M_0 \leftrightarrow M_3}
+ \frac{G(2 - 3, K_3; K_1 - K_3, K_2 - K_3, K_1 - K_3, K_2 - K_3)}{G(K_1 - K_3, 3; K_1 - K_3, K_2 - K_3)}
+ \frac{G(K_1 - K_3, K_2 - K_3, K_3; K_1 - K_3, K_2 - K_3)}{G(K_1 - K_3, K_2 - K_3, K_3)}
+ \frac{G(K_1 - K_3, K_2 - K_3, K_3; K_1 - K_3, K_2 - K_3) s_{03}}{G(K_1 - K_3, 3; K_1 - K_3, K_2 - K_3)}.
\quad (A.25)
\]

**Reduction coefficients of box**

\[
C^{(0,1,2,3)}(1|3) = \frac{f_3(s_{01}G(2,3;1,2) - s_{02}G(1,3;1,2) + s_{03}G(1,2;1,2))}{G(1,2,3)} 
- \frac{f_2(s_{01}G(2,3;1,3) - s_{02}G(1,3;1,3) + s_{03}G(1,3;1,2))}{G(1,2,3)} 
+ \frac{f_1(s_{01}G(2,3;2,3) - s_{02}G(2,3;1,3) + s_{03}G(2,3;1,2))}{G(1,2,3)}. \quad (A.26)
\]

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• $m = 2$

Reduction coefficients of tadpoles

All reduction coefficients vanish.

Reduction coefficients of bubbles

$$C^{(0,1)}(2|3) = (M_0^2)^{-2} \left[ c_{0,0,0}^{(0,1)}(2) M_0^2 s_{00} + c_{0,0,0}^{(0,1)}(2) s_{01}^2 + c_{0,0,0}^{(0,1)}(2) s_{02}^2 + c_{0,0,0}^{(0,1)}(2) s_{03}^2 
+ c_{1,1,0}^{(0,1)}(2) s_{01} s_{02} + c_{0,1,1}^{(0,1)}(2) s_{02} s_{03} + c_{1,0,1}^{(0,1)}(2) s_{01} s_{03} \right].$$  \hspace{1cm} (A.27)

where

$$c_{0,0,0}^{(0,1)}(2) = 0,$$  \hspace{1cm} (A.28)

$$c_{2,0,0}^{(0,1)}(2) = \frac{M_0^4 s_{13} G(2, 3; 1, 3)}{G(1, 3; 1, 3) G(1, 2, 3)} - \frac{M_0^4 s_{12} G(2, 3; 1, 2)}{G(1, 2; 1, 2) G(1, 2, 3)},$$  \hspace{1cm} (A.29)

$$c_{0,2,0}^{(0,1)}(2) = \frac{M_0^4 s_{11} G(1, 3; 1, 2)}{G(1, 2; 1, 2) G(1, 2, 3)},$$  \hspace{1cm} (A.30)

$$c_{0,0,2}^{(0,1)}(2) = c_{2,0,0}^{(0,1)}(2) \bigg|_{2=3},$$

$$c_{0,1,1}^{(0,1)}(2) = \frac{2 M_0^4 s_{13}}{G(1, 2, 3)},$$  \hspace{1cm} (A.31)

$$c_{1,1,0}^{(0,1)}(2) = \frac{2 M_0^4 s_{11} G(2, 3; 1, 2)}{G(1, 2; 1, 2) G(1, 2, 3)},$$

$$c_{1,0,1}^{(0,1)}(2) = c_{1,1,0}^{(0,1)}(2) \bigg|_{2=3}. \hspace{1cm} (A.32)$$

Choosing $j_0 = 2, j_1 = 1$ in (3.16) we have

$$C^{(1,2)}(2|3) = \left[ 2(-2R \cdot K_2) C^{(0,1)}(1|3) + C^{(0,1)}(2|3) \right] \bigg|_{K_1 \to K_1 - K_2, K_2 \to K_3, K_3 \to K_4 - K_2 M_0 + M_2}$$

$$\hspace{4cm} = C^{(0,1)}(2|3) \bigg|_{K_1 \to K_1 - K_2, K_2 \to K_3, K_3 \to K_3 - K_2 M_0 + M_2}. \hspace{1cm} (A.33)$$

Reduction coefficients of triangles

$$C^{(0,1,2)}(2|3) = M_0^{-2} \left[ c_{0,0,0}^{(0,1,2)}(2) M_0^2 s_{00} + c_{0,0,0}^{(0,1,2)}(2) s_{01}^2 + c_{0,0,0}^{(0,1,2)}(2) s_{02}^2 + c_{0,0,0}^{(0,1,2)}(2) s_{03}^2 
+ c_{1,1,0}^{(0,1,2)}(2) s_{01} s_{02} + c_{0,1,1}^{(0,1,2)}(2) s_{02} s_{03} + c_{1,0,1}^{(0,1,2)}(2) s_{01} s_{03} \right], \hspace{1cm} (A.34)$$

where

$$c_{0,0,0}^{(0,1,2)}(2) = \frac{f_1 G(2, 3; 1, 2)}{(D - 3) G(1, 2, 3)} - \frac{f_2 G(1, 3; 1, 2)}{(D - 3) G(1, 2, 3)} + \frac{f_3 G(1, 2; 1, 2)}{(D - 3) G(1, 2, 3)} \hspace{1cm} (A.35)$$
\[ c_{0,1,2}^{(1)}(2) = \frac{f_2 M_0^2 G(1, 1; 1, 2) G(2, 3; 2, 3)}{(D - 3) G(1, 2, 3)^2} - \frac{f_3 M_0^2 G(1, 2; 1, 2) G(2, 3; 2, 3)}{(D - 3) G(1, 2, 3)^2} \]

\[ + G(2, 3; 1, 2) \left( \frac{f_2 M_0^2 G(2, 3; 1, 3)}{G(1, 2, 3)^2} - \frac{(D - 2) f_1 M_0^2 G(2, 3; 2, 3)}{(D - 3) G(1, 2, 3)^2} \right) \]

\[ + \frac{M_0^2 (f_2 s_{12} - f_1 s_{22}) G(2, 3; 1, 2)}{G(1, 2; 1, 2) G(1, 2, 3)} - \frac{f_3 M_0^2 G(2, 3; 1, 2)^2}{G(1, 2, 3)^2}, \]  \hspace{1cm} (A.36)

\[ c_{0,0,2}^{(1)}(2) = \left. c_{0,2,0}^{(1)}(2) \right|_{1 \leftrightarrow 2}, \]  \hspace{1cm} (A.37)

\[ c_{0,0,2}^{(1)}(2) = G(1, 2; 1, 2) \left( \frac{(D - 2) f_2 M_0^2 G(1, 3; 1, 2)}{(D - 3) G(1, 2, 3)^2} - \frac{(D - 2) f_1 M_0^2 G(2, 3; 1, 2)}{(D - 3) G(1, 2, 3)^2} \right) \]

\[ - \frac{(D - 2) f_3 M_0^2 G(1, 2; 1, 2)^2}{(D - 3) G(1, 2, 3)^2}, \]  \hspace{1cm} (A.38)

\[ c_{1,0,1}^{(1)}(2) = -\frac{2(D - 2) f_2 M_0^2 G(1, 1; 3, 2) G(2, 3; 1, 3)}{(D - 3) G(1, 2, 3)^2} + \frac{2 f_3 M_0^2 G(1, 2; 1, 2) G(2, 3; 1, 3)}{(D - 3) G(1, 2, 3)^2} \]

\[ + G(2, 3; 1, 2) \left( \frac{2 f_2 M_0^2 G(2, 3; 1, 3)}{(D - 3) G(1, 2, 3)^2} + \frac{2 f_3 M_0^2 G(1, 1; 3, 2)}{(D - 3) G(1, 2, 3)^2} \right) \]

\[ + \frac{2 M_0^2 (f_2 s_{12} - f_1 s_{22}) G(2, 3; 1, 2)}{G(1, 2; 1, 2) G(1, 2, 3)} + \frac{2 f_1 M_0^2 G(1, 3; 1, 2) G(2, 3; 2, 3)}{G(1, 2, 3)^2}, \]  \hspace{1cm} (A.39)

\[ c_{0,1,1}^{(1)}(2) = \frac{2 f_1 M_0^2 G(1, 1; 3, 2) G(2, 3; 1, 3)}{(D - 3) G(1, 2, 3)^2} - \frac{2 f_2 M_0^2 G(1, 3; 1, 2)^2}{(D - 3) G(1, 2, 3)^2} \]

\[ + G(1, 2; 1, 2) \left( \frac{2(D - 2) f_3 M_0^2 G(1, 3; 1, 2)}{(D - 3) G(1, 2, 3)^2} + \frac{2 f_1 M_0^2 G(2, 3; 1, 3)}{G(1, 2, 3)^2} \right) \]

\[ - \frac{2 f_2 M_0^2 G(1, 2; 1, 2) G(1, 3; 1, 3)}{G(1, 2, 3)^2}, \]  \hspace{1cm} (A.40)

\[ c_{1,1,1}^{(1)}(2) = \left. c_{0,1,1}^{(1)}(2) \right|_{1 \leftrightarrow 2}. \]  \hspace{1cm} (A.41)

Choosing \( j_0 = 3, j_1 = 1, j_2 = 2 \) in (3.16), we have

\[ C^{(1,2,3)}(1) \mid 3 = \left. \left[ 2(-2 R \cdot K_3) C^{(0,1,2)}(1) \mid 3 + C^{(0,1,2)}(2) \mid 3 \right] \right|_{K_1 \to K_1 - K_3, K_2 \to K_2 - K_3, K_3 \to K_3, M_0 \to M_3} \]

\[ = 4 \delta_{03} \left\{ \begin{array}{l}
G(K_2 - K_3, K_3; K_1 - K_3, K_2 - K_3)(s_{01} - s_{03}) \\
G(K_1 - K_3, K_3; K_1 - K_3, K_2 - K_3)(s_{02} - s_{03}) \\
G(K_1 - K_3, K_2 - K_3; K_1 - K_3, K_2 - K_3) s_{03} \\
G(K_1 - K_3, K_2 - K_3; K_1 - K_3, K_2 - K_3) s_{02} \\
+ C^{(0,1,2)}(2) \mid 3 \right\} \right|_{K_1 \to K_1 - K_3, K_2 \to K_2 - K_3, K_3 \to K_3, M_0 \to M_3}, \]  \hspace{1cm} (A.42)
Reduction coefficient of box

\[ C^{(0,1,2,3)}(2|3) = c^{(0,1,2,3)}_{0,0,0}(2)M_0^2 s_{00} + c^{(0,1,2,3)}_{0,0,2}(2)s_{01}^2 + c^{(0,1,2,3)}_{0,2,0}(2)s_{02}^2 + c^{(0,1,2,3)}_{0,0,2}(2)s_{03}^2 + c^{(0,1,2,3)}_{1,1,0}(2)s_{01}s_{02} + c^{(0,1,2,3)}_{0,1,1}(2)s_{01}s_{03} + c^{(0,1,2,3)}_{1,0,1}(2)s_{02}s_{03} + c^{(0,1,2,3)}_{1,1,0}(2)s_{01}s_{03} , \]  

(A.43)

where

\[ c^{(0,1,2,3)}_{0,0,0}(2) = -\frac{f_2^2 G(2,3;2,3)}{(D-3)M_0^2 G(1,2,3)} + \frac{2f_2 f_1 G(2,3;1,3)}{(D-3)M_0^2 G(1,2,3)} - \frac{2f_3 f_1 G(2,3;1,2)}{(D-3)M_0^2 G(1,2,3)} - \frac{f_2^2 G(1,2;1,2)}{(D-3)M_0^2 G(1,2,3)} + \frac{f_2 \left[ f_2 G(1,3;1,3) G(2,3;2,3) - 2f_3 G(1,3;1,2) G(2,3;2,3) \right]}{(D-3)M_0^2 G(1,2,3)^2} + \frac{4}{D-3}, \]  

(A.44)

\[ c^{(0,1,2,3)}_{2,0,0}(2) = \frac{f_3^2 G(1,2;1,2) G(2,3;2,3)}{(D-3)G(1,2,3)^2} + \frac{2(D-2)f_1 f_3 G(2,3;1,2) G(2,3;2,3)}{(D-3)G(1,2,3)^2} - \frac{2(D-2)f_1 f_2 G(2,3;1,3) G(2,3;2,3)}{(D-3)G(1,2,3)^2} - \frac{f_2^2 G(2,3;1,3)}{G(1,2,3)^2} - \frac{4M_0^2 G(2,3;2,3)}{(D-3)G(1,2,3)^2} + \frac{(D-2)f_1^2 G(2,3;2,3)^2}{(D-3)G(1,2,3)^2} - \frac{2f_2 f_3 G(2,3;1,2) G(2,3;1,3)}{G(1,2,3)^2}, \]  

(A.45)

\[ c^{(0,1,2,3)}_{1,1,0}(2) = \frac{2f_2 \left[ (D-2)f_2 G(1,3;1,3) G(2,3;1,3) - (D-1)f_3 G(1,3;1,2) G(2,3;1,3) \right]}{(D-3)G(1,2,3)^2} - \frac{2(D-1)f_1 f_2 G(2,3;1,3)}{(D-3)G(1,2,3)^2} + \frac{2f_1 G(2,3;1,3) \left[ f_2 G(1,3;1,3) - f_3 G(1,3;1,2) \right]}{G(1,2,3)^2} + \frac{2(D-2)f_2^2 G(2,3;1,3) G(2,3;2,3)}{(D-3)G(1,2,3)^2} - \frac{2(D-1)f_1 f_3 G(2,3;1,3) G(2,3;1,2)}{(D-3)G(1,2,3)^2} + \frac{2f_3 G(2,3;1,2) \left( f_2 G(1,3;1,3) - f_3 G(1,3;1,2) \right)}{G(1,2,3)^2}. \]  

(A.46)

Other expansion coefficients can be got by using the permutation symmetry:

\[ c^{(0,1,2,3)}_{0,2,0}(2) = c^{(0,1,2,3)}_{2,0,0}(2) \bigg|_{1 \leftrightarrow 2}, \]

\[ c^{(0,1,2,3)}_{0,1,2}(2) = c^{(0,1,2,3)}_{2,0,0}(2) \bigg|_{1 \leftrightarrow 3}, \]

\[ c^{(0,1,2,3)}_{1,0,1}(2) = c^{(0,1,2,3)}_{2,0,0}(2) \bigg|_{2 \leftrightarrow 3}, \]

\[ c^{(0,1,2,3)}_{0,1,1}(2) = c^{(0,1,2,3)}_{2,0,0}(2) \bigg|_{1 \leftrightarrow 3}. \]  

(A.47)
A.3 All reduction coefficients of tensor pentagon with rank \( m = 1,2 \)

Consider the reduction of a tensor pentagon \( I_5^{(m)} \), the MIs are as below

**Tadpoles**: \( I_1[0], I_1[1], I_1[2], I_1[3], I_1[4], \)

**Bubbles**: \( I_2[0, 1], I_2[0, 2], I_2[0, 3], I_2[0, 4], I_2[1, 2], I_2[1, 3], I_2[1, 4], I_2[2, 3], I_2[2, 4], I_2[3, 4], \)

**Triangles**: \( I_3[0, 1, 2], I_3[0, 1, 3], I_3[0, 1, 4], I_3[0, 2, 3], I_3[0, 2, 4], I_3[0, 3, 4], I_3[1, 2, 3], \)

**Box**: \( I_4[0, 1, 2, 3], I_4[0, 2, 3, 4], I_4[0, 1, 2, 4], I_4[1, 2, 3, 4], \)

**Pentagon**: \( I_5[0, 1, 2, 3, 4]. \) \( \) (A.48)

Then the reduction of the tensor pentagon is given by

\[
I_5^{(m)} = \sum_{i=0}^{4} C_0^{(i)}(m|4) I_1[i] + \sum_{0 \leq i_1 < i_2 \leq 4} C^{(i_1,i_2)}(m|4) I_2[i_1,i_2] + \sum_{0 \leq i_1 < i_2 < i_3 \leq 4} C^{(i_1,i_2,i_3)}(m|4) I_3[i_1,i_2,i_3]
\]
\[
+ \sum_{0 \leq i_1 < i_2 < i_3 < i_4 \leq 4} C^{(i_1,i_2,i_3,i_4)}(m|4) I_4[i_1,i_2,i_3,i_4] + C^{(0,1,2,3,4)}(m|4) I_5[0,1,2,3,4]. \quad (A.49)
\]

\[ m = 1 \]

**Reduction coefficients of tadpoles, bubbles, triangles**

All reduction coefficients vanish.

**Reduction coefficients of boxes**

\[
C^{(0,1,2,3)}(1|4) = \frac{s_{01} G(2, 3, 4; 1, 2, 3) - s_{02} G(1, 3, 4; 1, 2, 3) + s_{03} G(1, 2, 4; 1, 2, 3)}{G(1, 2, 3, 4; 1, 2, 3, 4)}. \quad (A.50)
\]

Choosing \( j_0 = 4, j_1 = 1, j_2 = 2, j_3 = 3 \) in (3.16), we have

\[
C^{(1,2,3,4)}(1|4) = C^{(0,1,2,3)}(1|4) \bigg|_{K_1 \rightarrow K_1-K_4, K_2 \rightarrow K_2-K_4, K_3 \rightarrow K_3-K_4, K_4 \rightarrow K_4, M_0 \leftrightarrow M_4}
\]
\[
= \frac{(-s_{01} + s_{04}) G(K_2 - K_4, K_3 - K_4; K_1 - K_4, K_2 - K_4, K_3 - K_4)}{G(K_1 - K_4, K_2 - K_4, K_3 - K_4)}
\]
\[
+ \frac{(s_{02} - s_{04}) G(K_1 - K_4, K_3 - K_4, K_4; K_1 - K_4, K_2 - K_4, K_3 - K_4)}{G(K_1 - K_4, K_2 - K_4, K_3 - K_4, K_4; K_1 - K_4, K_2 - K_4, K_3 - K_4, K_4)}
\]
\[
+ \frac{(s_{03} - s_{04}) G(K_1 - K_4, K_2 - K_4, K_3 - K_4, K_4; K_1 - K_4, K_2 - K_4, K_3 - K_4, K_4)}{G(K_1 - K_4, K_2 - K_4, K_3 - K_4, K_4; K_1 - K_4, K_2 - K_4, K_3 - K_4, K_4)}
\]
\[
+ \frac{s_{04} G(K_1 - K_4, K_2 - K_4, K_3 - K_4, K_4; K_1 - K_4, K_2 - K_4, K_3 - K_4, K_4)}{G(K_1 - K_4, K_2 - K_4, K_3 - K_4, K_4; K_1 - K_4, K_2 - K_4, K_3 - K_4, K_4)}. \quad (A.51)
\]

**Reduction coefficients of pentagon**

\[
C^{(0,1,2,3,4)}(1|4) = c^{(0,1,2,3,4)}_{1,0,0,0,0}(1)s_{01} + c^{(0,1,2,3,4)}_{0,1,0,0}(1)s_{02} + c^{(0,1,2,3,4)}_{0,0,1,0}(1)s_{03} + c^{(0,1,2,3,4)}_{0,0,0,1}(1)s_{04}, \quad (A.52)
\]
where
\[
c^{(0,1,2,3,4)}_{0,1,0,0,0} (1) = \frac{f_1 G(2, 3, 4) - f_2 G(2, 3, 4; 1, 3, 4) + f_3 G(2, 3, 4; 1, 2, 4)}{G(1, 2, 3, 4; 1, 2, 3, 4)} - \frac{f_4 G(2, 3, 4; 1, 2, 3)}{G(1, 2, 3, 4; 1, 2, 3, 4)},
\]
\[
\begin{align*}
&c^{(0,1,2,3,4)}_{0,1,0,0,0} (1) = c^{(0,1,2,3,4)}_{0,1,0,0,0} (1) \bigg|_{1+2}, \\
&c^{(0,1,2,3,4)}_{0,0,1,0,0} (1) = c^{(0,1,2,3,4)}_{0,1,0,0,0} (1) \bigg|_{1+3}, \\
&c^{(0,1,2,3,4)}_{0,0,0,0,1} (1) = c^{(0,1,2,3,4)}_{0,1,0,0,0} (1) \bigg|_{1+4}.
\end{align*}
\] (A.53)

- \( m = 2 \)

**Reduction coefficients of tadpoles, bubbles**

All reduction coefficients vanish.

**Reduction coefficients of triangles**
\[
\begin{align*}
C^{(0,1,2)}(2|4) &= -\frac{s^2_{01} G(1, 2, 1, 2) G(1, 2, 4; 1, 2, 3)}{G(1, 2, 4; 1, 2, 3) G(1, 2, 3; 4; 1, 2, 3, 4)} - \frac{s^2_{02} G(1, 2, 1, 2) G(1, 4; 2, 3; 4; 1, 2, 3, 4)}{G(1, 2, 4; 1, 2, 3, 4; 1, 2, 3, 4)} \\
&\quad - \frac{s_{01} G(2, 3, 4; 1, 2, 3) (s_{01} G(2; 3, 1, 2) - 2 s_{02} G(1, 3; 1, 2) + 2 s_{03} G(1, 2; 1, 2))}{G(1, 2, 3) G(1, 2, 3; 4; 1, 2, 3, 4)} \\
&\quad + \frac{s_{02} G(1, 3, 4; 1, 2, 3) (2 s_{03} G(1, 2; 1, 2) - s_{02} G(1, 3; 1, 2))}{G(1, 2, 3) G(1, 2, 3; 4; 1, 2, 3, 4)} \\
&\quad + \frac{s^2_{01} G(2, 4; 1, 2) G(2, 3, 4; 1, 2, 4)}{G(1, 2, 2; 4; 1, 2, 4) G(1, 2, 3, 4; 1, 2, 3, 4)} \\
&\quad + \frac{s_{02} G(1, 4; 1, 2) (s_{02} G(1, 3, 4; 1, 2, 4) - 2 s_{01} G(2, 3, 4; 1, 2, 4))}{G(1, 2, 4; 1, 2, 4) G(1, 2, 3, 4; 1, 2, 3, 4)} \\
&\quad + \frac{2 s_{04} G(1, 2; 1, 2) (s_{01} G(2, 3, 4; 1, 2, 4) - s_{02} G(1, 3, 4; 1, 2, 4) + s_{03} G(1, 2, 4; 1, 2, 4))}{G(1, 2, 4; 1, 2, 4) G(1, 2, 3, 4; 1, 2, 3, 4)}.
\end{align*}
\] (A.55)

Choosing \( j_0 = 3, j_1 = 1, j_2 = 2 \) in (3.16), we have
\[
C^{(1,2,3)}(2|4) = \left( C^{(0,1,2)}(2|4) \right) \bigg|_{K_i \rightarrow K_i - K_3; j_3 \neq 3; K_3 \rightarrow - K_3, M_0 \leftrightarrow M_3}.
\] (A.56)

**Reduction coefficients of boxes**
\[
\begin{align*}
C^{(0,1,2,3)}(2|4) &= \frac{1}{M_0^2} \left[ c^{(0,1,2,3)}_{0,0,0,0} (2) M_0^2 s_{00} + c^{(0,1,2,3)}_{0,2,0,0} (2) s^2_{01} + c^{(0,1,2,3)}_{0,0,2,0} (2) s^2_{02} + c^{(0,1,2,3)}_{0,0,0,2} (2) s^2_{03}
\right.
\]
\[
+ c^{(0,1,2,3)}_{0,1,0,0} (2) s_{04} + c^{(0,1,2,3)}_{1,0,0,0} (2) s_{01} s_{02} + c^{(0,1,2,3)}_{0,1,1,0} (2) s_{02} s_{03} + c^{(0,1,2,3)}_{1,0,1,0} (2) s_{01} s_{03}
\]
\[
+ c^{(0,1,2,3)}_{0,0,0,1} (2) s_{01} s_{04} + c^{(0,1,2,3)}_{0,1,0,1} (2) s_{02} s_{04} + c^{(0,1,2,3)}_{0,0,1,1} (2) s_{03} s_{04}.\)
\] (A.57)
where

\[ c^{(0,1,2,3)}_{0,0,0,0}(2) = \frac{f_1 G(2,3,4;1,2,3)}{(4-D)G(1,2,3,4)} + \frac{f_2 G(1,3,4;1,2,3)}{(D-4)G(1,2,3,4)} + \frac{f_3 G(1,2,4;1,2,3)}{(4-D)G(1,2,3,4)} + \frac{f_4 G(1,2,3)}{(D-4)G(1,2,3,4)} \]  

(A.58)

\[ c^{(0,1,2,3)}_{2,0,0,0}(2) = \frac{(D-3)f_1 M_2^2 G(2,3,4;1,2,3)G(2,3,4)}{(D-4)G(1,2,3,4;1,2,3,4)^2} \]

\[ - \frac{M_2^2 G(2,3,4) (f_2 G(1,3,4;1,2,3) - f_3 G(1,2,4;1,2,3))}{(D-4)G(1,2,3,4;1,2,3,4)^2} \]

\[ + \frac{f_4 M_2^2 G(2,3,4;1,2,3)^2}{(D-4)G(1,2,3,4;1,2,3,4)^2} \]

\[ + G(2,3,4;1,2,3) \frac{M_2^2 (f_1 G(2,3;2,3) - f_2 G(2,3;1,3) + f_3 G(2,3;1,2))}{G(1,2,3)G(1,2,3,4;1,2,3,4)} \]

\[ - G(2,3,4;1,2,3) \frac{M_2^2 (f_2 G(2,3,4;1,3,4) - f_3 G(2,3,4;1,2,4))}{G(1,2,3,4;1,2,3,4)^2} \]  

(A.59)

\[ c^{(0,1,2,3)}_{0,0,0,2}(2) = \frac{(D-3)M_2^2 G(1,2,3)}{(D-4)G(1,2,3,4;1,2,3,4)^2} \left[ f_1 G(2,3,4;1,2,3) \right. \]

\[ - f_2 G(1,3,4;1,2,3) + f_3 G(1,2,4;1,2,3) - f_4 G(1,2,3) \right] \]  

(A.60)

\[ c^{(0,1,2,3)}_{1,1,0,0}(2) = \frac{2(D-3)f_2 M_2^2 G(1,3,4;1,2,3)G(2,3,4;1,3,4)}{(D-4)G(1,2,3,4;1,2,3,4)^2} \]

\[ + \frac{2M_2^2 G(2,3,4;1,2,3)}{(D-4)G(1,2,3,4;1,2,3,4)^2} \left[ f_4 G(1,2,3) - f_3 G(1,2,4;1,2,3) \right] \]

\[ + \frac{2M_2^2 G(2,3,4;1,2,3)}{G(1,2,3,4;1,2,3,4)^2} \left[ f_4 G(1,3,4;1,2,3) - f_1 G(2,3,4;1,3,4) \right. \]

\[ - \frac{2M_2^2 G(1,3,4;1,2,3)}{G(1,2,3,4;1,2,3,4)^2} \left[ f_1 G(2,3,4) - f_3 G(2,3,4;1,2,4) \right] \]

\[ - \frac{2M_2^2 G(2,3,4;1,2,3)}{G(1,2,3,4;1,2,3,4)} \left( f_1 G(2,3;1,3) - f_2 G(1,3;1,3) + f_3 G(1,3;1,2) \right) \]  

(A.61)

\[ c^{(0,1,2,3)}_{1,0,0,1}(2) = - \frac{2f_1 M_2^2 G(2,3,4;1,2,3)^2}{(D-4)G(1,2,3,4;1,2,3,4)^2} \]

\[ + \frac{2M_2^2 G(2,3,4;1,2,3)}{(D-4)G(1,2,3,4;1,2,3,4)^2} \left[ f_2 G(1,3,4;1,2,3) - f_3 G(1,2,4;1,2,3) \right] \]

\[ + \frac{2M_2^2 G(1,2,3)}{G(1,2,3,4;1,2,3,4)^2} \left[ \frac{(D-3)f_4 G(2,3,4;1,2,3)}{(D-4)} \right. \]

\[ - f_1 G(2,3,4) + f_2 G(2,3,4;1,3,4) - f_3 G(2,3,4;1,2,4) \]  

(A.62)
Other expansion coefficients can be got by using the permutation symmetry:

\[
c^{(0,1,2,3)}_{0,2,0,0} (2) = c^{(0,1,2,3)}_{2,0,0,0} (2) \\
c^{(0,1,2,3)}_{0,0,2,0} (2) = c^{(0,1,2,3)}_{2,0,0,0} (2) \\
c^{(0,1,2,3)}_{1,0,1,0} (2) = c^{(0,1,2,3)}_{1,1,0,0} (2) \\
c^{(0,1,2,3)}_{0,1,1,0} (2) = c^{(0,1,2,3)}_{1,1,0,0} (2) \\
c^{(0,1,2,3)}_{0,0,1,1} (2) = c^{(0,1,2,3)}_{1,0,0,1} (2) \quad (\text{A.63})
\]

Choosing \( j_0 = 4, j_1 = 1, j_2 = 2, j_3 = 3 \) in (3.16), we have

\[
C^{(1,2,3,4)}(2|4) = 4s_{04} \left[ C^{(0,1,2,3)}(1|4) \right]_{K_i \rightarrow K_i - K_4, i < 4; K_4 \rightarrow K_4, M_0 \leftrightarrow M_4} + \left( C^{(0,1,2,3)}(2|4) \right)_{K_i \rightarrow K_i - K_4, i < 4; K_4 \rightarrow K_4, M_0 \leftrightarrow M_4} \quad (\text{A.64})
\]

The reduction coefficient of pentagon

\[
C^{(0,1,2,3,4)}(2|4) = \left[ c^{(0,1,2,3,4)}_{0,0,0,0} (2) M_0^2 s_{00} + c^{(0,1,2,3,4)}_{2,0,0,0} (2) s_{01}^2 + c^{(0,1,2,3,4)}_{0,2,0,0} (2) s_{02}^2 + c^{(0,1,2,3,4)}_{0,0,2,0} (2) s_{03}^2 \\
+ c^{(0,1,2,3,4)}_{0,0,0,2} (2) s_{04}^2 + c^{(0,1,2,3,4)}_{1,1,0,0} (2) s_{01} s_{02} + c^{(0,1,2,3,4)}_{0,1,1,0} (2) s_{02} s_{03} + c^{(0,1,2,3,4)}_{1,0,0,1} (2) s_{01} s_{04} \\
+ c^{(0,1,2,3,4)}_{0,1,0,1} (2) s_{02} s_{04} + c^{(0,1,2,3,4)}_{0,0,1,1} (2) s_{03} s_{04} \right] \quad (\text{A.65})
\]

where

\[
c^{(0,1,2,3,4)}_{0,0,0,0} (2) = -\frac{f_2^2 G(2, 3, 4)}{(D - 4) M_0^2 G(1, 2, 3, 4; 1, 2, 3, 4)} + \frac{2 f_2 f_1 G(2, 3, 4; 1, 3, 4)}{(D - 4) M_0^2 G(1, 2, 3, 4; 1, 2, 3, 4)} \\
- \frac{1 f_2 f_1 G(2, 3, 4; 1, 2, 4)}{(D - 4) M_0^2 G(1, 2, 3, 4; 1, 2, 3, 4)} + \frac{2 f_2 f_1 G(2, 3, 4; 1, 2, 3)}{(D - 4) M_0^2 G(1, 2, 3, 4; 1, 2, 3, 4)} \\
- \frac{f_2^2 G(1, 2, 3)}{(D - 4) M_0^2 G(1, 2, 3, 4; 1, 2, 3, 4)} - \frac{2 f_2 f_1 G(1, 3, 4; 1, 2, 4)}{(D - 4) M_0^2 G(1, 2, 3, 4; 1, 2, 3, 4)} \\
- \frac{f_2^2 G(1, 2, 3)}{(D - 4) M_0^2 G(1, 2, 3, 4; 1, 2, 3, 4)} + \frac{2 f_2 f_1 G(1, 3, 4; 1, 2, 4)}{(D - 4) M_0^2 G(1, 2, 3, 4; 1, 2, 3, 4)} \\
- \frac{1 f_2 f_1 G(1, 3, 4; 1, 2, 3)}{(D - 4) M_0^2 G(1, 2, 3, 4; 1, 2, 3, 4)} + \frac{2 f_2 f_1 G(1, 3, 4; 1, 2, 3)}{(D - 4) M_0^2 G(1, 2, 3, 4; 1, 2, 3, 4)} \\
+ \frac{4}{D - 4} \quad (\text{A.66})
\]
\[
\begin{align*}
\mathcal{c}^{(0,1,2,3,4)}_{2,0,0,0} (2) &= \frac{(D - 3) f_2^2 G(2, 3, 4)^2}{(D - 4) G(1, 2, 3, 4; 1, 2, 3, 4)^2} - \frac{2(D - 3) f_2 f_1 G(2, 3, 4; 1, 3, 4) G(2, 3, 4)}{(D - 4) G(1, 2, 3, 4; 1, 2, 3, 4)^2} \\
&+ \frac{2(D - 3) f_3 f_1 G(2, 3, 4; 1, 2, 4) G(2, 3, 4)}{(D - 4) G(1, 2, 3, 4; 1, 2, 3, 4)^2} - \frac{2(D - 3) f_3 f_1 G(2, 3, 4; 1, 2, 3) G(2, 3, 4)}{(D - 4) G(1, 2, 3, 4; 1, 2, 3, 4)^2} \\
&+ f_2^2 \left( \frac{G(1, 3, 4) G(2, 3, 4)}{(D - 4) G(1, 2, 3, 4; 1, 2, 3, 4)^2} + \frac{G(1, 2, 3, 4; 1, 2, 3, 4)^2}{G(1, 2, 3, 4; 1, 2, 3, 4)^2} \right) \\
&+ f_3^2 \left( \frac{G(1, 2, 3) G(2, 3, 4)}{(D - 4) G(1, 2, 3, 4; 1, 2, 3, 4)^2} + \frac{G(1, 2, 3, 4; 1, 2, 3, 4)^2}{G(1, 2, 3, 4; 1, 2, 3, 4)^2} \right) \\
&+ f_2 f_3 \left( \frac{2G(1, 3, 4; 1, 2, 4) G(2, 3, 4)}{(D - 4) G(1, 2, 3, 4; 1, 2, 3, 4)^2} - \frac{2G(2, 3, 4; 1, 2, 4) G(2, 3, 4; 1, 3, 4)}{G(1, 2, 3, 4; 1, 2, 3, 4)^2} \right) \\
&+ f_2 f_4 \left( \frac{2G(1, 3, 4; 1, 2, 3) G(2, 3, 4)}{(D - 4) G(1, 2, 3, 4; 1, 2, 3, 4)^2} + \frac{2G(2, 3, 4; 1, 2, 3) G(2, 3, 4; 1, 3, 4)}{G(1, 2, 3, 4; 1, 2, 3, 4)^2} \right) \\
&+ f_3 f_4 \left( \frac{2G(1, 2, 4; 1, 2, 3) G(2, 3, 4)}{(D - 4) G(1, 2, 3, 4; 1, 2, 3, 4)^2} - \frac{2G(2, 3, 4; 1, 2, 3) G(2, 3, 4; 1, 3, 4)}{G(1, 2, 3, 4; 1, 2, 3, 4)^2} \right) \\
&- \frac{4M_0^2 G(2, 3, 4)}{(D - 4) G(1, 2, 3, 4; 1, 2, 3, 4)^2}, \\
\end{align*}
\]

\[
\begin{align*}
\mathcal{c}^{(0,1,2,3,4)}_{1,1,0,0} (2) &= -\frac{2(D - 3) G(2, 3, 4; 1, 3, 4) G(2, 3, 4) f_2^2}{(D - 4) G(1, 2, 3, 4; 1, 2, 3, 4)^2} + \frac{8G(2, 3, 4; 1, 3, 4) M_0^2}{(D - 4) G(1, 2, 3, 4; 1, 2, 3, 4)} \\
&+ \left( \frac{2(D - 2) G(2, 3, 4; 1, 3, 4)^2}{(D - 4) G(1, 2, 3, 4; 1, 2, 3, 4)^2} + \frac{2G(1, 3, 4) G(2, 3, 4)}{G(1, 2, 3, 4; 1, 2, 3, 4)^2} \right) f_2 f_1 \\
&+ \left( \frac{2(D - 2) G(2, 3, 4; 1, 2, 4) G(2, 3, 4; 1, 3, 4)}{(D - 4) G(1, 2, 3, 4; 1, 2, 3, 4)^2} - \frac{2G(1, 3, 4; 1, 2, 4) G(2, 3, 4; 2, 3, 4; 1, 3, 4)}{G(1, 2, 3, 4; 1, 2, 3, 4)^2} \right) f_3 f_1 \\
&+ \frac{2(D - 2) G(2, 3, 4; 1, 2, 3) G(2, 3, 4; 1, 3, 4)}{(D - 4) G(1, 2, 3, 4; 1, 2, 3, 4)^2} + \frac{2G(1, 3, 4; 1, 2, 3) G(2, 3, 4; 2, 3, 4; 1, 3, 4)}{G(1, 2, 3, 4; 1, 2, 3, 4)^2} f_3^2 \\
&+ \left( -\frac{2G(1, 3, 4; 1, 2, 4) G(2, 3, 4; 1, 2, 3)}{G(1, 2, 3, 4; 1, 2, 3, 4)^2} + \frac{2G(1, 2, 4; 1, 2, 3) G(2, 3, 4; 2, 3, 4; 1, 3, 4)}{G(1, 2, 3, 4; 1, 2, 3, 4)^2} \right) f_2 f_3 \\
&+ \frac{2G(1, 3, 4; 1, 2, 4) G(2, 3, 4; 1, 2, 3)}{G(1, 2, 3, 4; 1, 2, 3, 4)^2} + \frac{2(D - 2) G(1, 3, 4; 1, 2, 4) G(2, 3, 4; 2, 3, 4; 1, 3, 4)}{(D - 4) G(1, 2, 3, 4; 1, 2, 3, 4)^2} f_2 f_4 \\
&+ \frac{2G(1, 3, 4; 1, 2, 4) G(2, 3, 4; 1, 2, 3)}{G(1, 2, 3, 4; 1, 2, 3, 4)^2} + \frac{2(D - 2) G(1, 3, 4; 1, 2, 3) G(2, 3, 4; 2, 3, 4; 1, 3, 4)}{(D - 4) G(1, 2, 3, 4; 1, 2, 3, 4)^2} f_3 f_4 \\
&+ \frac{4G(1, 2, 4; 1, 2, 3) G(2, 3, 4; 1, 3, 4)}{(D - 4) G(1, 2, 3, 4; 1, 2, 3, 4)^2} f_3 f_4 - \frac{2(D - 3) G(1, 3, 4) G(2, 3, 4; 1, 3, 4) f_3^2}{(D - 4) G(1, 2, 3, 4; 1, 2, 3, 4)^2}. \\
\end{align*}
\]
Other expansion coefficients can be got by using permutation symmetry:

\[
\begin{align*}
\left(c_{0,2,0,0}^{(0,1,2,3,4)}(2)\right)_{1+2} &= \left(c_{2,0,0,0}^{(0,1,2,3,4)}(2)\right)_{1+2}, \\
\left(c_{0,0,0,2}^{(0,1,2,3,4)}(2)\right)_{1+3} &= \left(c_{2,0,0,0}^{(0,1,2,3,4)}(2)\right)_{1+3}, \\
\left(c_{0,0,1,0}^{(0,1,2,3,4)}(2)\right)_{1+4} &= \left(c_{1,0,0,0}^{(0,1,2,3,4)}(2)\right)_{1+4}, \\
\left(c_{0,0,1,1}^{(0,1,2,3,4)}(2)\right)_{1+4} &= \left(c_{1,0,0,0}^{(0,1,2,3,4)}(2)\right)_{1+4}.
\end{align*}
\]

(A.69)

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