Higher order cohomology of arithmetic groups

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Abstract: Higher order cohomology of arithmetic groups is expressed in terms of $(g,K)$-cohomology. Generalizing results of Borel, it is shown that the latter can be computed using functions of (uniform) moderate growth. A higher order versions of Borel’s conjecture is stated, asserting that the cohomology can be computed using automorphic forms.

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Introduction

In [2] we have defined higher order group cohomology in the following general context: Let $\Gamma$ be a group and $\Sigma$ a normal subgroup. For a ring $R$ we define a sequence of functors $H^0_q$ from the category of $R[\Gamma]$-modules to the category of $R$-modules. First, for an $R[\Gamma]$-module $V$, one defines $H^0_0(\Gamma, \Sigma, V) = H^0(\Gamma, V) = V^\Gamma$ as the fixed point module. Inductively, $H^0_{q+1}(\Gamma, \Sigma, V)$ is the module of all $v \in V$ such that $\sigma v = v$ for every $\sigma \in \Sigma$ and $\gamma v - v$ is in $H^1_q(\Gamma, \Sigma, V)$ for every $\gamma \in \Gamma$. For every $q \geq 1$ the functor $H^0_q(\Gamma, \Sigma, \cdot)$ is left-exact and we define the higher order group cohomology as the right derived functor $H^p_q = R^p H^0_q$.

In the case of a Fuchsian group the choice $\Sigma = \Gamma_{\text{par}} = \text{the subgroup generated by all parabolic elements}$, turned out to be the adequate choice for an Eichler-Shimura isomorphy result to hold, see [2]. For general arithmetic groups $\Gamma \subset G$, where $G$ is a reductive linear group over $\mathbb{Q}$, a replacement for the Eichler-Shimura isomorphism is the isomorphism to $(\mathfrak{g}, K)$-cohomology,

$$H^p(\Gamma, E) \cong H^p_{\mathfrak{g}, K}(C^\infty(\Gamma \backslash G) \otimes E),$$

where $E$ is a finite dimensional representation of $G$. In this paper we present a higher order analogue of this result, i.e., we will show isomorphy of higher order cohomology to $(\mathfrak{g}, K)$-cohomology,

$$H^p_q(\Gamma, \Sigma, E) \cong H^p_{\mathfrak{g}, K}(H^0_q(\Gamma, \Sigma, C^\infty(G)) \otimes E).$$

We will prove higher order versions of results of Borel by which one can compute the cohomology using spaces of functions with growth restrictions. We also state a higher order version of the Borel conjecture, proved by Franke [3], that the cohomology can be computed using automorphic forms.

Note that if $\text{Hom}(\Gamma, \mathbb{C}) = 0$, then $H^p_q = H^p_1 = H^p$ for every $q \geq 1$. Consequently, in the case of arithmetic groups, higher order cohomology is of interested only for rank-one groups.

1 General groups

Let $R$ be a commutative ring with unit. Let $\Gamma$ be a group and $\Sigma \subset \Gamma$ a normal subgroup. Let $I$ denote the augmentation ideal in the group algebra
$A = R[\Gamma]$. Let $I_\Sigma$ denote the augmentation ideal of $R[\Sigma]$. As $\Sigma$ is normal in $\Gamma$, the set $A I_\Sigma$ is a 2-sided ideal in $A$. For $q \geq 1$ consider the ideal

$$J_q \overset{def}{=} I^q + R[\Gamma]I_\Sigma.$$ 

So in particular, for $\Sigma = \{1\}$ one has $J_q = I^q$. On the other end, for $\Sigma = \Gamma$ one gets $J_q = I$ for every $q \geq 1$. For an $A$-module $V$ define

$$H^p_q(\Gamma, \Sigma, V) = \text{Ext}^p_A(A/J_q, V).$$ 

This is the higher order cohomology of the module $V$, see [2]. Note that in the case $q = 1$, we get back the ordinary group cohomology, so

$$H^p_1(\Gamma, \Sigma, V) = H^p(\Gamma, V).$$ 

For convenience, we will sometimes suppress the $\Sigma$ in the notation, so we simply write $H^p_q(\Gamma, V)$ or even $H^p_q(V)$ for $H^p_q(\Gamma, \Sigma, V)$.

For an $R$-module $M$ and a set $S$ we write $M^S$ for the $R$-module of all maps from $S$ to $M$. Then $M^\emptyset$ is the trivial module 0. Up to isomorphy, the module $M^S$ depends only on the cardinality of $S$. It therefore makes sense to define $M^c$ for any cardinal number $c$ in this way. Note that $J_q/J_{q+1}$ is a free $R$-module. Define

$$N_{\Gamma, \Sigma}(q) \overset{def}{=} \dim_R J_q/J_{q+1}.$$ 

Then $N_{\Gamma, \Sigma}(q)$ is a possibly infinite cardinal number.

**Lemma 1.1**  (a) For every $q \geq 1$ there is a natural exact sequence

$$0 \to H^0_q(\Gamma, V) \to H^0_{q+1}(\Gamma, V) \to H^0(\Gamma, V)^{N_{\Gamma, \Sigma}(q)} \to$$

$$\to H^1_q(\Gamma, V) \to H^1_{q+1}(\Gamma, V) \to H^1(\Gamma, V)^{N_{\Gamma, \Sigma}(q)} \to \ldots$$

$$\ldots \to H^p_q(\Gamma, V) \to H^p_{q+1}(\Gamma, V) \to H^p(\Gamma, V)^{N_{\Gamma, \Sigma}(q)} \to \ldots$$

(b) Suppose that for a given $p \geq 0$ one has $H^p(\Gamma, V) = 0$. Then it follows $H^p_q(\Gamma, V) = 0$ for every $q \geq 1$. In particular, if $V$ is acyclic as $\Gamma$-module, then $H^p_q(\Gamma, V) = 0$ for all $p, q \geq 1$. 
Proof: Consider the exact sequence
\[ 0 \to J_q/J_{q+1} \to A/J_{q+1} \to A/J_q \to 0. \]
As an \( A \)-module, \( J_q/J_{q+1} \) is isomorphic to a direct sum \( \bigoplus R_\alpha \) of copies of \( R = A/I \). So we conclude that for every \( p \geq 0 \),
\[ \Ext^p_A(J_q/J_{q+1}, V) \cong \prod_{\alpha} \Ext^p_A(R, V) \cong H^p(\Gamma, V)^{N_{\Gamma, \Sigma}(q)}. \]
The long exact Ext-sequence induced by the above short sequence is
\[ 0 \to \Hom_A(A/J_q, V) \to \Hom_A(A/J_{q+1}, V) \to \Hom_A(J_q/J_{q+1}, V) \to \]
\[ \to \Ext^1_A(A/J_q, V) \to \Ext^1_A(A/J_{q+1}, V) \to \Ext^1_A(J_q/J_{q+1}, V) \to \]
\[ \to \Ext^2_A(A/J_q, V) \to \Ext^2_A(A/J_{q+1}, V) \to \Ext^2_A(J_q/J_{q+1}, V) \to \ldots \]
This is the claim (a). For (b) we proceed by induction on \( q \). For \( q = 1 \) the claim follows from \( H^1_p(\Gamma, V) = H^p(\Gamma, V) \). Assume the claim proven for \( q \) and \( H^p(\Gamma, V) = 0 \). As part of the above exact sequence, we have the exactness of
\[ H^p_q(\Gamma, V) \to H^p_{q+1}(\Gamma, V) \to H^p(\Gamma, V)^{N_{\Gamma, \Sigma}(q)}. \]
By assumption, we have \( H^p(\Gamma, V)^{N_{\Gamma, \Sigma}(q)} = 0 \) and by induction hypothesis the module \( H^p_q(\Gamma, V) \) vanishes. This implies \( H^p_{q+1}(\Gamma, V) = 0 \) as well. \( \square \)

Lemma 1.2 (Cocycle representation) The module \( H^1_q(\Gamma, V) \) is naturally isomorphic to
\[ \Hom_A(J_q, V)/\alpha(V), \]
where \( \alpha : V \to \Hom_A(J_q, V) \) is given by \( \alpha(v)(m) = mv \).

Proof: This is Lemma 1.3 of [2]. \( \square \)

2 Higher order cohomology of sheaves

Let \( Y \) be a topological space which is path-connected and locally simply connected. Let \( C \to Y \) be a normal covering of \( Y \). Let \( \Gamma \) be the fundamental
group of $Y$ and let $X \xrightarrow{\pi} Y$ be the universal covering. The fundamental group $\Sigma$ of $C$ is a normal subgroup of $\Gamma$.

For a sheaf $\mathcal{F}$ on $Y$ define

$$H^0_q(Y, C, \mathcal{F}) \overset{\text{def}}{=} H^0_q(\Gamma, \Sigma, H^0(X, \pi^* \mathcal{F})).$$

Let $\text{Mod}(R)$ be the category of $R$-modules, let $\text{Mod}_R(Y)$ be the category of sheaves of $R$-modules on $Y$, and let $\text{Mod}_R(X)_\Gamma$ be the category of sheaves over $X$ with an equivariant $\Gamma$-action. Then $H^0_q(Y, C, \cdot)$ is a left exact functor from $\text{Mod}_R(Y)$ to $\text{Mod}(R)$. We denote its right derived functors by $H^p_q(Y, C, \cdot)$ for $p \geq 0$.

**Lemma 2.1** Assume that the universal cover $X$ is contractible.

(a) For each $p \geq 0$ one has a natural isomorphism $H^p_1(Y, C, \mathcal{F}) \cong H^p(Y, \mathcal{F})$.

(b) If a sheaf $\mathcal{F}$ is $H^0(Y, \cdot)$-acyclic, then it is $H^0_q(Y, C, \cdot)$-acyclic.

Note that part (b) allows one to use flabby or fine resolutions to compute higher order cohomology.

**Proof:** We decompose the functor $H^0(Y, C, \cdot)$ into the functors

$$\text{Mod}_R(Y) \xrightarrow{\pi^*} \text{Mod}_R(X)_\Gamma \xrightarrow{H^0(X, \cdot)} \text{Mod}(R[\Gamma]) \xrightarrow{H^0(\Gamma, \Sigma, \cdot)} \text{Mod}(R).$$

The functor $\pi^*$ is exact and maps injectives to injectives. We claim that $H^0(X, \cdot)$ has the same properties. For the exactness, consider the commutative diagram

$$\begin{array}{ccc}
\text{Mod}_R(X)_\Gamma & \xrightarrow{H^0} & \text{Mod}(R[\Gamma]) \\
\downarrow f & & \downarrow f \\
\text{Mod}_R(X) & \xrightarrow{H^0} & \text{Mod}(R),
\end{array}$$

where the vertical arrows are the forgetful functors. As $X$ is contractible, the functor $H^0$ below is exact. The forgetful functors have the property, that a sequence upstairs is exact if and only if its image downstairs is exact. This implies that the above $H^0$ is exact. It remains to show that $H^0$ maps
injective objects to injective objects. Let $\mathcal{J} \in \text{Mod}_R(X)_{\Gamma}$ be injective and consider a diagram with exact row in $\text{Mod}(R[\Gamma])$,

$$
0 \longrightarrow M \longrightarrow N \longrightarrow H^0(X, \mathcal{J}).
$$

The morphism $\varphi$ gives rise to a morphism $\phi : M \times X \to \mathcal{J}$, where $M \times X$ stands for the constant sheaf with stalk $M$. Note that $H^0(X, \phi) = \varphi$. As $\mathcal{J}$ is injective, there exists a morphism $\psi : N \times X \to \mathcal{J}$ making the diagram commutative. This diagram induces a corresponding diagram on the global sections, which implies that $H^0(X, \mathcal{J})$ is indeed injective.

For a sheaf $\mathcal{F}$ on $Y$ it follows that

$$
H^p(Y, \mathcal{F}) = R^p(H^0(Y, \mathcal{F})) = R^p H^0(\Gamma, \Sigma, \mathcal{F}) \circ H^0_\Gamma \circ \pi^* = H^p_\Gamma(Y, C, \mathcal{F}).
$$

Now let $\mathcal{F}$ be acyclic. Then we conclude $H^p_\Gamma(\mathcal{F}) = 0$ for every $p \geq 1$, so the $\Gamma$-module $V = H^0(X, \pi^* \mathcal{F})$ is $\Gamma$-acyclic. The claim follows from Lemma 1.1.

### 3 Arithmetic groups

Let $G$ be a semisimple Lie group with compact center and let $X = G/K$ be its symmetric space. Let $\Gamma \subset G$ be an arithmetic subgroup which is torsion-free, and let $\Sigma \subset \Gamma$ be a normal subgroup. Let $Y = \Gamma \backslash X$, then $\Gamma$ is the fundamental group of the manifold $Y$, and the universal covering $X$ of $Y$ is contractible. This means that we can apply the results of the last section.
Theorem 3.1 Let \((\sigma, E)\) be a finite dimensional representation of \(G\). There is a natural isomorphism

\[ H^p_q(\Gamma, \Sigma, E) \cong H^p_{q,K}(H^0_q(\Gamma, \Sigma, C^\infty(G)) \otimes E), \]

where the right hand side is the \((\mathfrak{g}, K)\)-cohomology.

Proof: Let \(\mathcal{F}_E\) be the locally constant sheaf on \(Y\) corresponding to \(E\). Let \(\Omega^p_Y\) be the sheaf of complex valued \(p\)-differential forms on \(Y\). Then \(\Omega^p_Y \otimes \mathcal{F}_E\) is the sheaf of \(\mathcal{F}_E\)-valued differential forms. These form a fine resolution of \(\mathcal{F}_E\):

\[ 0 \rightarrow \mathcal{F}_E \rightarrow \mathbb{C}^\infty \otimes \mathcal{F}_E \xrightarrow{d \otimes 1} \Omega^1_Y \otimes \mathcal{F}_E \rightarrow \ldots \]

Since \(\pi^*\Omega^\bullet_Y = \Omega^\bullet_X\), we conclude that \(H^p_q(\Gamma, \Sigma, E)\) is the cohomology of the complex \(H^0_q(\Gamma, \Sigma, H^0(X, \Omega^\bullet_X \otimes E))\). Let \(\mathfrak{g}\) and \(\mathfrak{k}\) be the Lie algebras of \(G\) and \(K\) respectively, and let \(\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}\) be the Cartan decomposition. Then \(H^0(X, \Omega^p \otimes \mathcal{F}_E) = (C^\infty(G) \otimes \Lambda^p \mathfrak{p})^K \otimes E\). Mapping a form \(\omega\) in this space to \((1 \otimes x^{-1})\omega(x)\) one gets an isomorphism to \((C^\infty(G) \otimes \Lambda^p \mathfrak{p} \otimes E)^K\), where \(K\) acts diagonally on all factors and \(\Gamma\) now acts on \(C^\infty(G)\) alone. The claim follows. \(\square\)

Let \(U(\mathfrak{g})\) act on \(C^\infty\) as algebra of left invariant differential operators. Let \(\|\cdot\|\) be a norm on \(G\), see [4], Section 2.A.2. Recall that a function \(f \in C^\infty(G)\) is said to be of moderate growth, if for every \(D \in U(\mathfrak{g})\) one has \(Df(x) = O(||x||^a)\) for some \(a > 0\). The function \(f\) is said to be of uniform moderate growth, if the exponent \(a\) above can be chosen independent of \(D\). Let \(C^\infty_{mg}(G)\) and \(C^\infty_{umg}(G)\) denote the spaces of functions of moderate growth and uniform moderate growth respectively.

Let \(\mathfrak{z}\) be the center of the algebra \(U(\mathfrak{g})\). Let \(\mathcal{A}(G)\) denote the space of functions \(f \in C^\infty(G)\) such that

- \(f\) is of moderate growth,
- \(f\) is right \(K\)-finite, and
- \(f\) is \(\mathfrak{z}\)-finite.
Proposition 3.2  (a) For $\Omega = C^\infty_{unr}(G), C^\infty_{mg}(G), C^\infty(G)$ one has

$$H^1_q(\Gamma, \Sigma, \Omega) = 0$$

for every $q \geq 1$.

(b) If $\text{Hom}(\Gamma, \mathbb{C}) \neq 0$, then one has

$$H^1(\Gamma, \mathcal{A}(G)) \neq 0.$$  

Proof: In order to prove (a), it suffices by Lemma 1.1 (b), to consider the case $q = 1$. A 1-cocycle is a map $\alpha : \Gamma \to \Omega$ such that $\alpha(\gamma\tau) = \gamma\alpha(\tau) \alpha(\gamma)$ holds for all $\gamma, \tau \in \Gamma$. We have to show that for any given such map $\alpha$ there exists $f \in \Omega$ such that $\alpha(\tau) = \tau f - f$. To this end consider the symmetric space $X = G/K$ of $G$. Let $d(xK,yK)$ for $x,y \in G$ denote the distance in $X$ induced by the $G$-invariant Riemannian metric. For $x \in G$ we also write $d(x) = d(xK,eK)$. Then the functions $\log \|x\|$ and $d(x)$ are equivalent in the sense that there exists a constant $C > 1$ such that

$$\frac{1}{C} d(x) \leq \log \|x\| \leq C d(x)$$

or

$$\|x\| \leq e^{Cd(x)} \leq \|x\|^C$$

holds for every $x \in G$. We define

$$\mathcal{F} = \{ y \in G : d(y) < d(\gamma y) \ \forall \gamma \in \Gamma \setminus \{e\} \}.$$ 

As $\Gamma$ is torsion-free, this is a fundamental domain for the $\Gamma$ left translation action on $G$. In other words, $\mathcal{F}$ is open, its boundary is of measure zero, and there exists a set of representatives $R \subset G$ for the $\Gamma$-action such that $\mathcal{F} \subset R \subset \overline{\mathcal{F}}$. Next let $\varphi \in C^\infty_c(G)$ with $\varphi \geq 0$ and $\int_G \varphi(x) \, dx = 1$. Then set $u = 1_x * \varphi$, where $1_A$ is the characteristic function of the set $A$ and $*$ is the convolution product $f * g(x) = \int_G f(y)g(y^{-1}x) \, dy$. Let $C$ be the support of $\varphi$, then the support of $u$ is a subset of $\mathcal{F}C$ and the sum $\sum_{\tau \in \Gamma} u(\tau^{-1}x)$ is locally finite in $x$. More sharply, for a given compact unit-neighborhood $V$ there exists $N \in \mathbb{N}$ such that for every $x \in G$ one has

$$\# \{ \tau \in \Gamma : u(\tau^{-1}xV) \subset \{0\} \} \leq N.$$
This is to say, the sum is uniformly locally finite. For a function \( h \) on \( G \) and \( x, y \in G \) we write \( L_y h(x) = h(y^{-1} x) \). Then for a convolution product one has \( L_y (f * g) = (L_y f) * g \), and so

\[
\sum_{\tau \in \Gamma} u(\tau^{-1} x) = \left( \sum_{\tau \in \Gamma} L_{\tau} 1_F \right) * \varphi.
\]

The sum in parenthesis is equal to one on the complement of a nullset. Therefore,

\[
\sum_{\tau \in \Gamma} u(\tau^{-1} x) \equiv 1.
\]

Set

\[
f(x) = -\sum_{\tau \in \Gamma} \alpha(\tau)(x) u(\tau^{-1} x).
\]

**Lemma 3.3** The function \( f \) lies in the space \( \Omega \).

**Proof:** Since the sum is uniformly locally finite, it suffices to show that for each \( \tau \in \Gamma \) we have \( \alpha(\tau)(x) u(\tau^{-1} x) \in \Omega \) where the \( O(\| \cdot \|_d) \) estimate is uniform in \( \tau \). By the Leibniz-rule it suffices to show this separately for the two factors \( \alpha(\tau) \) and \( L_{\tau} u \). For \( D \in U(g) \) we have

\[
D(L_{\tau} u) = (L_{\tau} 1_F) * (D \varphi).
\]

This function is bounded uniformly in \( \tau \), hence \( L_{\tau} u \in C_{\text{umg}}^\infty(G) \). Now \( \alpha(\tau) \in \Omega \) by definition, but we need uniformity of growth in \( \tau \). We will treat the case \( \Omega = C_{\text{umg}}^\infty(G) \) here, the case \( C_{\text{mg}}^\infty \) is similar and the case \( C^\infty(G) \) is trivial, as no growth bounds are required.

So let \( \Omega = C_{\text{umg}}^\infty(G) \) and set

\[
S = \{ \gamma \in \Gamma \setminus \{ e \} : \gamma F \cap \mathcal{F} \neq \emptyset \}.
\]

Then \( S \) is a finite symmetric generating set for \( \Gamma \). For \( \gamma \in \Gamma \), let \( \mathcal{F}_\gamma \) be the set of all \( x \in G \) with \( d(x) < d(\gamma x) \). Then

\[
\mathcal{F} = \bigcap_{\gamma \in \Gamma \setminus \{ e \}} \mathcal{F}_\gamma.
\]
Let $\tilde{F} = \bigcap_{s \in S} F_s$. We claim that $F = \tilde{F}$. As the intersection runs over fewer elements, one has $F \subset \tilde{F}$. For the converse note that for every $s \in S$ the set $s \tilde{F}/K$ lies in $X \setminus \tilde{F}/K$, therefore $F/K$ is a connected component of $\tilde{F}/K$. By the invariance of the metric, we conclude that $x \in F_\gamma$ if and only if $d(xK, eK) < d(xK, \gamma^{-1}K)$. This implies that $F_\gamma/K$ is a convex subset of $X$. Any intersection of convex sets remains convex, therefore $\tilde{F}/K$ is convex and hence connected, and so $\tilde{F}/K = F/K$, which means $\tilde{F} = F$.

Likewise we get $\mathcal{F} = \bigcap_{s \in S} F_s$. The latter implies that for each $x \in G \setminus \mathcal{F}$ there exists $s \in S$ such that $d(s^{-1}x) < d(x)$. Iterating this and using the fact that the set of all $d(\gamma x)$ for $\gamma \in \Gamma$ is discrete, we find for each $x \in G \setminus \mathcal{F}$ a chain of elements $s_1, \ldots, s_n \in S$ such that $d(x) > d(s_1^{-1}x) > \cdots > d(s_n^{-1} \cdots s_1^{-1}x)$ and $s_n^{-1} \cdots s_1^{-1}x \in \mathcal{F}$. The latter can be written as $x \in s_1 \ldots s_n \mathcal{F}$. Now let $\tau \in \Gamma$ and suppose $u(\tau^{-1}x) \neq 0$. Then $x \in \mathcal{F}C$, so, choosing $C$ small enough, we can assume $x \in s_\tau \mathcal{F}$ for some $s \in S \cap \{e\}$. As the other case is similar, we can assume $s = e$. It suffices to assume $x \in \tau \mathcal{F}$, as we only need the estimates on the dense open set $\Gamma \mathcal{F}$. So then it follows $\tau = s_1 \ldots s_n$.

Let $D \in U(g)$. As $\alpha$ maps to $\Omega = C^\infty_{\text{umg}}(G)$, for every $\gamma \in \Gamma$ there exist $C(D, \gamma), a(\gamma) > 0$ such that

$$|D\alpha(\gamma)(x)| \leq C(D, \gamma) \|x\|^{a(\gamma)}.$$ 

The cocycle relation of $\alpha$ implies

$$\alpha(\tau)(x) = \sum_{j=1}^n \alpha(\gamma_j)(s_j^{-1} \cdots s_1^{-1}x).$$
We get

\[ |D\alpha(\tau)(x)| \leq \sum_{j=1}^{n} C(D, s_j) ||s_{j-1}^{s-1} \ldots s_1^{s-1} x||^{a(s_j)} \]

\[ \leq \sum_{j=1}^{n} C(D, s_j) e^{Cd(s_{j-1}^{s-1} \ldots s_1^{s-1} x)a(s_j)} \]

\[ \leq \sum_{j=1}^{n} C(D, s_j) e^{Cd(x)a(s_j)} \]

\[ \leq \sum_{j=1}^{n} C(D, s_j) ||x||^{C^2a(s_j)} \]

\[ \leq nC_0(D)||x||^{a_0}, \]

where \( C(D) = \max_j C(D, s_j) \) and \( a_0 = C^2 \max_j d(s_j) \). It remains to show that \( n \) only grows like a power of \( ||x|| \). To this end let for \( r > 0 \) denote \( N(r) \) the number of \( \gamma \in \Gamma \) with \( d(\gamma) \leq r \). Then a simple geometric argument shows that

\[ N(r) = \frac{1}{\text{vol} \mathcal{F}} \text{vol} \left( \bigcup_{\gamma:d(\gamma) \leq r} \gamma \mathcal{F}/K \right) \leq C_1 \text{vol}(B_{2r}), \]

where \( B_{2r} \) is the ball of radius \( 2r \) around \( eK \). Note that for the homogeneous space \( X \) there exists a constant \( C_2 > 0 \) such that \( \text{vol} B_{2r} \leq e^{C_2r} \). Now \( n \leq N(d(x)) \) and therefore

\[ n \leq C_1 \text{vol} B_{2d(x)} \leq C_1 e^{C_2d(x)} \leq C_1 ||x||^{C_3} \]

for some \( C_3 > 0 \). Together it follows that there exists \( C(D) > 0 \) and \( a > 0 \) such that

\[ |D\alpha(\tau)(x)| \leq C(D) ||x||^a. \]

This is the desired estimate which shows that \( f \in \Omega \). The lemma is proven. \( \square \)
To finish the proof of part (a) of the proposition, we now compute for $\gamma \in \Gamma$,
\[
\gamma f(x) - f(x) = f(\gamma^{-1}x) - f(x)
\]
\[
= \sum_{\tau \in \Gamma} \alpha(\tau x)u(\tau^{-1}x) - \alpha(\gamma^{-1}x)u(\tau^{-1}\gamma^{-1}x)
\]
\[
= \sum_{\tau \in \Gamma} \alpha(\tau)(x)u(\tau^{-1}x) + \alpha(\gamma)(x)\sum_{\tau \in \Gamma} u((\gamma\tau)^{-1}x)
\]
\[
- \sum_{\tau \in \Gamma} \alpha(\gamma\tau)(x)u((\gamma\tau)^{-1}x)
\]

The first and the last sum cancel and the middle sum is $\alpha(\gamma)(x)$. Therefore, part (a) of the proposition is proven.

We now prove part (b). Let $Q = C^\infty(G)/A(G)$. We have an exact sequence of $\Gamma$-modules
\[
0 \to A(G) \to C^\infty(G) \to Q \to 0.
\]
This results in the exact sequence
\[
0 \to A(G)^\Gamma \to C^\infty(\Gamma\backslash G) \xrightarrow{\phi} Q^\Gamma \to H^1(\Gamma, A(G)) \to 0.
\]
The last zero comes by part (a) of the proposition. We have to show that the map $\phi$ is not surjective. So let $\chi : \Gamma \to \mathbb{C}$ be a non-zero group homomorphism and let $u \in C^\infty(G)$ as above with $\sum_{\tau \in \Gamma} u(\tau^{-1}x) = 1$, and $u$ is supported in $FC$ for a small unit-neighborhood $C$. Set
\[
h(x) = -\sum_{\tau \in \Gamma} \chi(\tau)u(\tau^{-1}x).
\]
Then for every $\gamma \in \Gamma$ the function
\[
h(\gamma^{-1}x) - h(x) = \chi(\gamma)
\]
is constant and hence lies in $A(G)^\Gamma$. This means that the class $[h]$ of $h$ in $Q$ lies in the $\Gamma$-invariants $Q^\Gamma$. As $\chi \neq 0$, the function $f$ is not in $C^\infty(\Gamma\backslash G)$, and therefore $\phi$ is indeed not surjective.

\begin{proposition}
For every $q \geq 1$ there is an exact sequence of continuous $G$-homomorphisms,
\[
0 \to H^0_q(\Gamma, \Sigma, C^\infty_*(G)) \xrightarrow{\phi} H^0_{q+1}(\Gamma, \Sigma, C^\infty_*(G)) \xrightarrow{\psi} C^\infty_*(\Gamma\backslash G)^{N_\Gamma, \Sigma(q)} \to 0,
\]
where $\phi$ is the inclusion map and $*$ can be $\emptyset, \text{umg}$, or $\text{mg}$.
\end{proposition}
Proof: This follows from Lemma 1.1 together with Proposition 3.2 (a).

The space $C^\infty(G)$ carries a natural topology which makes it a nuclear topological vector space. For every $q \geq 1$, the space $H^0_q(\Gamma, \Sigma, C^\infty(G))$ is a closed subspace. If $\Gamma$ is cocompact, then one has the isotypical decomposition

$$H^0_1(\Gamma, \Sigma, C^\infty(G)) = C^\infty(\Gamma \setminus G) = \bigoplus_{\pi \in \hat{G}} C^\infty(\Gamma \setminus G)(\pi),$$

and $C^\infty(\Gamma \setminus G)(\pi) \cong m_{\Gamma}(\pi)\pi^\infty$, where the sum runs over the unitary dual $\hat{G}$ of $G$, and for $\pi \in \hat{G}$ we write $\pi^\infty$ for the space of smooth vectors in $\pi$. The multiplicity $m_{\Gamma}(\pi) \in \mathbb{N}_0$ is the multiplicity of $\pi$ as a subrepresentation of $L^2(\Gamma \setminus G)$, i.e.,

$$m_{\Gamma}(\pi) = \dim \text{Hom}_G(\pi, L^2(\Gamma \setminus G)).$$

Finally, the direct sum $\bigoplus$ means the closure of the algebraic direct sum in $C^\infty(G)$. We write $\hat{G}_\Gamma$ for the set of all $\pi \in \hat{G}$ with $m_{\Gamma}(\pi) \neq 0$.

Let $\pi \in \hat{G}$. A smooth representation $(\beta, V_\beta)$ of $G$ is said to be of type $\pi$, if it is of finite length and every irreducible subquotient is isomorphic to $\pi^\infty$. For a smooth representation $(\eta, V_\eta)$ we define the $\pi$-isotype as

$$V_\eta(\pi) \overset{\text{def}}{=} \bigoplus_{\beta \in V_\eta \text{ of type } \pi} V_\beta,$$

where the sum runs over all subrepresentations $V_\beta$ of type $\pi$.

**Theorem 3.5** Suppose $\Gamma$ is cocompact and let $\ast \in \{\emptyset, mg, umg\}$. We write $V_q = V$. For every $q \geq 1$ there is an isotypical decomposition

$$V_q = \bigoplus_{\pi \in \hat{G}_\Gamma} V_q(\pi),$$

and each $V_q(\pi)$ is of type $\pi$ itself. The exact sequence of Proposition 3.4 induces an exact sequence

$$0 \to V_q(\pi) \to V_{q+1}(\pi) \to (\pi^\infty)^{m_{\Gamma}(\pi)\mathcal{N}_\Gamma, \Sigma(q)} \to 0$$

for every $\pi \in \hat{G}_\Gamma$. 

Proof: We will prove the theorem by reducing to a finite dimensional situation by means of considering infinitesimal characters and \(K\)-types. For this let \(\hat{\mathfrak{z}} = \text{Hom}(\mathfrak{z}, \mathbb{C})\) be the set of all algebra homomorphisms from \(\mathfrak{z}\) to \(\mathbb{C}\). For a \(\mathfrak{z}\)-module \(V\) and \(\chi \in \hat{\mathfrak{z}}\) let

\[ V(\chi) \overset{\text{def}}{=} \{ v \in V : \forall z \in \mathfrak{z} \exists n \in \mathbb{N} (z - \chi(z))^n v = 0 \} \]

be the generalized \(\chi\)-eigenspace. Since \(\mathfrak{z}\) is finitely generated, one has

\[ V(\chi) = \{ v \in V : \exists n \in \mathbb{N} \forall z \in \mathfrak{z} (z - \chi(z))^n v = 0 \} \]

For \(\chi \neq \chi'\) in \(\hat{\mathfrak{z}}\) one has \(V(\chi) \cap V(\chi') = 0\). Recall that the algebra \(\mathfrak{z}\) is free in \(r\) generators, where \(r\) is the absolute rank of \(G\). Fix a set of generators \(z_1, \ldots, z_r\). The map \(\chi \mapsto (\chi(z_1), \ldots, \chi(z_r))\) is a bijection \(\hat{\mathfrak{z}} \to \mathbb{C}^r\). We equip \(\hat{\mathfrak{z}}\) with the topology of \(\mathbb{C}^r\). This topology does not depend on the choice of the generators \(z_1, \ldots, z_r\).

Let \(\Gamma \subset G\) be a discrete cocompact subgroup. Let \(\hat{\mathfrak{z}}_\Gamma\) be the set of all \(\chi \in \hat{\mathfrak{z}}\) such that the generalized eigenspace \(C^\infty(\Gamma \backslash G)(\chi)\) is non-zero. The \(\hat{\mathfrak{z}}_\Gamma\) is discrete in \(\hat{\mathfrak{z}}\), more sharply there exists \(\varepsilon_\Gamma > 0\) such that for any two \(\chi \neq \chi'\) in \(\hat{\mathfrak{z}}_\Gamma\) there is \(j \in \{1, \ldots, r\}\) such that \(|\chi(z_j) - \chi'(z_j)| > \varepsilon_\Gamma\).

Proposition 3.6 Let \(* \in \{\emptyset, mg, umg\}\). For every \(q \geq 1\) and every \(\chi \in \hat{\mathfrak{z}}\) the space \(V_q(\chi) = H^0_q(\Gamma, \Sigma, C^\infty_*(G))(\chi)\) coincides with

\[ \bigcap_{z \in \mathfrak{z}} \ker(z - \chi(z))^{2q-1}, \]

and is therefore a closed subspace of \(V_q\). The representation of \(G\) on \(V_q(\chi)\) is of finite length.

The space \(V_q(\chi)\) is non-zero only if \(\chi \in \hat{\mathfrak{z}}_\Gamma\). One has a decomposition

\[ H^0_q(\Gamma, \Sigma, C^\infty_*(G)) = \bigoplus_{\chi \in \hat{\mathfrak{z}}_\Gamma} H^0_q(\Gamma, \Sigma, C^\infty_*(G))(\chi). \]

The exact sequence of Proposition 3.4 induces an exact sequence

\[ 0 \to V_q(\chi) \to V_{q+1}(\chi) \to \bigoplus_{\pi \in G_\chi} m_\pi N_{\Gamma, \Sigma}(q) \pi \to 0. \]
Proof: All assertions, except for the exactness of the sequence, are clear for \( q = 1 \). We proceed by induction. Fix \( \chi \in \hat{\mathfrak{z}}_\Gamma \). Since \( V_q(\chi) = V_q \cap V_{q+1}(\chi) \) one gets an exact sequence

\[
0 \to V_q(\chi) \to V_{q+1}(\chi) \xrightarrow{\psi_\chi} V_1(\chi)^{N_\Gamma \Sigma(q)}.
\]

Let \( v \in V_1(\chi)^{N_\Gamma \Sigma(q)} \). As \( \psi \) is surjective, one finds \( u \in V_{q+1} \) with \( \psi(u) = v \). We have to show that one can choose \( u \) to lie in \( V_q(\chi) \). Inductively we assume the decomposition to holds for \( V_q \), so we can write

\[
(z_j - \chi(z_j))u = \sum_{\chi' \in \hat{\mathfrak{z}}_\Gamma} u_{j,\chi'},
\]

for \( 1 \leq j \leq r \) and \( u_{j,\chi'} \in \ker(z - \chi(z))^{2q-1} \) for every \( z \in \mathfrak{z} \). For every \( \chi' \in \hat{\mathfrak{z}}_\Gamma \setminus \{ \chi \} \) we fix some index \( 1 \leq j(\chi') \leq r \) with \( |\chi(z_j(\chi')) - \chi'(z_j(\chi'))| > \varepsilon_\Gamma \). On the space

\[
\bigoplus_{\chi' : j(\chi') = j} V_q(\chi')
\]

the operator \( z_j - \chi(z_j) \) is invertible and the inverse \( (z_j - \chi(z_j))^{-1} \) is continuous. We can replace \( u \) with

\[
u - \sum_{\chi' \in \hat{\mathfrak{z}}_\Gamma \setminus \{ \chi \}} (z_j(\chi') - \chi(z_j(\chi')))^{-1} u_{j,\chi'},\]

We end up with \( u \) satisfying \( \psi(u) = v \) and

\[
(z_1 - \chi(z_1)) \cdots (z_r - \chi(z_r))u \in V_q(\chi) = \bigcap_{z \in \mathfrak{z}} \ker(z - \chi(z))^{2q-1}.
\]

So for every \( z \in \mathfrak{z} \) one has

\[
0 = (z_1 - \chi(z_1)) \cdots (z_r - \chi(z_r))(z - \chi(z))^{2q-1} u,
\]

which implies

\[
(z - \chi(z))^{2q-1} u \in \ker ((z_1 - \chi(z_1)) \cdots (z_r - \chi(z_r))).
\]

As the set \( \hat{\mathfrak{z}}_\Gamma \) is countable, one can, depending on \( \chi \), choose the generators \( z_1, \ldots, z_r \) in a way that \( \chi(z_j) \neq \chi'(z_j) \) holds for every \( j \) and every \( \chi' \in \hat{\mathfrak{z}}_\Gamma \).
\[ \hat{\mathfrak{g}} \setminus \{ \chi \}. \]

Therefore the operator \((z_1 - \chi(z_1)) \cdots (z_r - \chi(z_r))\) is invertible on \(V_q(\chi')\) for every \(\chi' \in \hat{\mathfrak{g}} \setminus \{ \chi \}\) and so it follows \((z - \chi(z))^{2r-1} u \in V_q(\chi) \subset \ker((z - \chi(z))^{2r-1})\). Since this holds for every \(z\) it follows \(u \in \ker((z - \chi(z))^{2r})\). Since this holds for every \(z\) it follows \(u \in V_{q+1}(\chi)\) and so \(\psi_\chi\) is indeed surjective. One has an exact sequence

\[
0 \rightarrow V_q(\chi) \rightarrow V_{q+1}(\chi) \rightarrow V_1(\chi)^{N_r, \Sigma(q)} \rightarrow 0.
\]

Taking the sum over all \(\chi \in \hat{\mathfrak{g}}\) we arrive at an exact sequence

\[
0 \rightarrow V_q \rightarrow \bigoplus_{\chi \in \hat{\mathfrak{g}}} V_{q+1}(\chi) \rightarrow V_1^{N_r, \Sigma(q)} \rightarrow 0.
\]

Hence we get a commutative diagram with exact rows

\[
\begin{array}{cccccc}
0 & \rightarrow & V_q & \rightarrow & \bigoplus_{\chi \in \hat{\mathfrak{g}}} V_{q+1}(\chi) & \rightarrow & V_1^{N_r, \Sigma(q)} & \rightarrow & 0 \\
& & \downarrow \cong & & \downarrow i & & \downarrow \cong & & \\
0 & \rightarrow & V_q & \rightarrow & V_{q+1} & \rightarrow & V_1^{N_r, \Sigma(q)} & \rightarrow & 0,
\end{array}
\]

where \(i\) is the inclusion. By the 5-Lemma, \(i\) must be a bijection. The proposition follows. \[\square\]

We now finish the proof of the theorem. We keep the notation \(V_q\) for the space \(\mathbb{H}_q^0(\Gamma, \Sigma, C_\infty^\infty(G))\). For a given \(\chi \in \hat{\mathfrak{g}}\) the \(G\)-representation \(V_q(\chi)\) is of finite length, so the \(K\)-isotypical decomposition

\[
V_q(\chi) = \bigoplus_{\tau \in K} V_q(\chi)(\tau)
\]

has finite dimensional isotypes, i.e., \(\dim V_q(\chi)(\tau) < \infty\). Let \(U(\mathfrak{g})^K\) be the algebra of all \(D \in U(\mathfrak{g})\) such that \(\text{Ad}(k)D = D\) for every \(k \in K\). Then the action of \(D \in U(\mathfrak{g})\) commutes with the action of each \(k \in K\), and so \(K \times U(\mathfrak{g})^K\) acts on every smooth \(G\)-module. For \(\pi \in \hat{G}\) the \(K \times U(\mathfrak{g})^K\)-module \(V_\pi(\tau)\) is irreducible and \(V_\pi(\tau) \cong V_{\pi^r}(\tau^r)\) as a \(K \times U(\mathfrak{g})^K\)-module implies \(\pi = \pi^r\) and \(\tau = \tau^r\), see [4], Proposition 3.5.4. As \(V_q(\chi)(\tau)\) is finite dimensional, one gets

\[
V_q(\chi)(\tau) = \bigoplus_{\pi \in \hat{G}} V_q(\chi)(\tau)(\pi)
\]

\[\pi \circ \chi = \chi\]
where $V_q(\chi)(\tau)(\pi)$ is the largest $K \times U(\mathfrak{g})^K$-submodule of $V_q(\chi)(\tau)$ with the property that every irreducible subquotient is isomorphic to $V_\pi(\tau)$. Let

$$V_q(\pi) = \bigoplus_{\tau \in K} V_q(\chi_\pi)(\tau)(\pi).$$

The claims of the theorem follow from the proposition. \hfill \Box

4 The higher order Borel conjecture

Let $(\sigma, E)$ be a finite dimensional representation of $G$. In [1], A. Borel has shown that the inclusions $C_{\text{umg}}(G) \hookrightarrow C_{\text{mg}}^\infty(G) \hookrightarrow C^\infty(G)$ induce isomorphisms in cohomology:

$$H^p_{\mathfrak{g},K}(H^0(\Gamma, C_{\text{umg}}^\infty(G)) \otimes E) \xrightarrow{\cong} H^p_{\mathfrak{g},K}(H^0(\Gamma, C_{\text{mg}}^\infty(G)) \otimes E) \xrightarrow{\cong} H^p_{\mathfrak{g},K}(H^0(\Gamma, C^\infty(G)) \otimes E).$$

In [3], J. Franke proved a conjecture of Borel stating that the inclusion $\mathcal{A}(G) \hookrightarrow C^\infty(G)$ induces an isomorphism

$$H^p_{\mathfrak{g},K}(H^0(\Gamma, \mathcal{A}(G)) \otimes E) \xrightarrow{\cong} H^p_{\mathfrak{g},K}(H^0(\Gamma, C^\infty(G)) \otimes E).$$

**Conjecture 4.1 (Higher order Borel conjecture)** For every $q \geq 1$, the inclusion $\mathcal{A}(G) \hookrightarrow C^\infty(G)$ induces an isomorphism

$$H^p_{\mathfrak{g},K}(H^0_q(\Gamma, \Sigma, \mathcal{A}(G)) \otimes E) \xrightarrow{\cong} H^p_{\mathfrak{g},K}(H^0_q(\Gamma, \Sigma, C^\infty(G)) \otimes E).$$

We can prove the higher order version of Borel’s result.

**Theorem 4.2** For each $q \geq 1$, the inclusions $C_{\text{umg}}^\infty(G) \hookrightarrow C_{\text{mg}}^\infty(G) \hookrightarrow C^\infty(G)$ induce isomorphisms in cohomology:

$$H^p_{\mathfrak{g},K}(H^0_q(\Gamma, \Sigma, C_{\text{umg}}^\infty(G)) \otimes E) \xrightarrow{\cong} H^p_{\mathfrak{g},K}(H^0_q(\Gamma, \Sigma, C_{\text{mg}}^\infty(G)) \otimes E) \xrightarrow{\cong} H^p_{\mathfrak{g},K}(H^0_q(\Gamma, \Sigma, C^\infty(G)) \otimes E).$$
Proof: Let $\Omega$ be one of the spaces $C^\infty_{\text{umg}}(G)$ or $C^\infty_{\text{mg}}(G)$.

We will now leave $\Sigma$ out of the notation. By Proposition 3.4 we get an exact sequence

$$0 \to H^0_q(\Gamma, \Omega) \to H^0_{q+1}(\Gamma, \Omega) \to H^0(\Gamma, \Omega)^{N_r, \Sigma(q)} \to 0,$$

and the corresponding long exact sequences in $(g, K)$-cohomology. For each $p \geq 0$ we get a commutative diagram with exact rows

$$
\begin{array}{ccc}
H^p_{\mathfrak{g}, K}(H^0_q(\Gamma, \Omega) \otimes E) & \to & H^p_{\mathfrak{g}, K}(H^0_{q+1}(\Gamma, \Omega) \otimes E) \\
\alpha \downarrow & & \beta \downarrow \\
H^p_{\mathfrak{g}, K}(H^0_q(\Gamma, C^\infty(G)) \otimes E) & \to & H^p_{\mathfrak{g}, K}(H^0_{q+1}(\Gamma, C^\infty(G)) \otimes E) \\
\gamma \downarrow & & \\
& & H^p_{\mathfrak{g}, K}(H^0(\Gamma, C^\infty(G)) \otimes E)^{N_r, \Sigma(q)}.
\end{array}
$$

Borel has shown that $\gamma$ is an isomorphism and that $\alpha$ is an isomorphism for $q = 1$. We prove that $\beta$ is an isomorphism by induction on $q$. For the induction step we can assume that $\alpha$ is an isomorphism. Since the diagram continues to the left and right with copies of itself where $p$ is replaced by $p - 1$ or $p + 1$, we can deduce that $\beta$ is an isomorphism by the 5-Lemma.

By Proposition 3.2 (b) this proof cannot be applied to $\mathcal{A}(G)$.

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