Design of Passive and Lossless Single Layer Metasurfaces for Far Field Beamforming

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Abstract—A general synthesis technique for beamforming metasurfaces is presented which utilizes accurate modeling techniques and rapid optimization methods. The metasurfaces considered consist of patterned metallic claddings supported by finite grounded dielectric substrates. The metasurfaces are modeled using integral equations which accurately account for all mutual coupling and finite dimensions. A beamforming metasurface is designed in three phases: an initial Direct Solve phase involving the solution of the integral equation via the method of moments to obtain a complex-valued initial design satisfying the desired far-field beam specifications, a subsequent Optimization phase to convert the complex-valued metasurface into a purely reactive metasurface, and a final Patterning phase to realize the metasurface as a patterned metallic cladding. The metasurface is optimized using gradient descent with a semi-analytic gradient obtained from the Adjoint field method. The optimization phase introduces surface waves which facilitate passivity. A novel surface impedance extraction approach for aperiodic unit cell environments is used to pattern the metallic cladding of the metasurfaces. Several examples are presented, including a scanned beam reflectarray design with low sidelobes, a low-profile directly-fed meta-antenna with 100% aperture efficiency, and a conformal metasurface reflectarray.

Index Terms—Metasurface, Beamforming, Gradient Descent, Adjoint field method

I. INTRODUCTION

COMMON approaches for the design of beamforming antennas include the Schelkunoff polynomial, Fourier Transform, Woodward Lawson, Dolph-Chebyshev, and Taylor Line Source methods [1]. These methods are general and can be used to shape far field radiation patterns for a given set of far field pattern specifications. However, they generally result in the need for complex-valued excitation coefficients and hence require local loss and gain making the fabrication difficult or expensive.

Metasurfaces can be used for far field beamshaping or beamforming in a completely passive and lossless manner requiring no local loss or gain, only a simple patterned RF substrate. Metasurfaces are thin subwavelength textured surfaces which are modeled using the Generalized Sheet Transition Condition or GSTC [2]. The texturing of the surface describes electric and magnetic admittances and impedances which form proportionality constants between induced surface currents and electric and magnetic fields tangential to the surface. By introducing surface waves to these textured surfaces to redistribute power transversally along the array from places where local loss is needed to places where local gain is needed, metasurfaces can perform far field beam shaping or beamforming in a completely passive and lossless manner so long as global power is conserved. For example, the metasurfaces in [3]–[17] use surface waves to control either the near or far fields in a passive and lossless way. However, the metasurfaces in [4], [15], [16], require multilayer constructions. Those in [15]–[17] are approaches for infinitely periodic metasurfaces only. In [6], [12], [15], [17], the true thickness of the metasurface is not modelled and instead is assumed to be infinitely thin. Not modelling the true thickness of the metasurface can lead to the need for perfect electric conducting (PEC) baffles separating the unit cells as in [4], [18]. The metasurfaces in [6], [12] are not designed with practicality in mind as their impedances are not limited in any way and may...
be unrealizable. This may also be why they do not offer any way to realize their optimized designs. In [11], the authors do design beamforming metasurfaces but their design process results in complex-valued sheet impedances. They mention that the attainment of complex-valued sheet impedances is a limitation of their design approach. They did not consider introducing surface waves to end up with purely reactive sheets.

In contrast, the approaches presented in [5], [8]–[10], [13], [14], [19], [20] and in this paper are for the design of metasurface beamformers which are not assumed infinite or periodic, model the true thickness of the metasurface, only require a single impedance sheet, and are designed with practicality in mind leading to realizable sheet impedances. Furthermore, most of the designs are either already realized as patterned metallic claddings or will be as part of the work presented in this paper proving the practicality of the presented metasurface beamformer design approach.

The required surface waves in all of [3]–[14], [16], [17], [20] were found through optimization. In [12], [16], [17], a derivative free simplex method (MATLAB fminsearch) is used. Although this algorithm can handle discontinuous cost functions, the algorithm in general requires many more iterations and is thus slow to converge. In [6], the Alternating Direction Method of Multipliers (ADMM) is used. ADMM is an algorithm that solves convex optimization problems by breaking them into smaller pieces, each of which are then easier to handle. This approach may not lend itself nicely to acceleration schemes using analytic or semi-analytic gradients. In [4], gradient descent is used, however, no acceleration schemes were discussed. Finally, in [11], the conjugate gradient method is used. This approach is similar to the gradient descent algorithm. The authors of [11] do not report the use of any acceleration techniques. All of these approaches may take many iterations to converge and/or the gradient may be very costly to calculate limiting the design space of practical metasurfaces.

In contrast, we have developed several accelerated gradient descent optimization strategies for beamforming metasurface design including acceleration strategies for gradient calculation based on the Woodbury matrix identity [5], [20] and the Adjoint field method [13]. In this paper, we present accurate modelling strategies, based on integral equations, coupled with very fast optimization methods, based on the Adjoint field method, to design completely passive and lossless metasurface beamformers in very fast and practical times. The beamforming metasurfaces considered consist of a single purely reactive electric impedance sheet supported by a grounded dielectric substrate (see Fig. 1). The impedance sheet is assumed transparent and hence both the dielectric spacer and ground plane are also modeled by the integral equations. Due to the modelling methods and design approach, arbitrary far field beams can be generated by simply patterning the upper cladding layer of a commonly available RF substrate as depicted in Fig. 2.

The design of the metasurface is broken down into three design phases (see Fig. 2). This paper begins with an overview of the proposed three phase design strategy in section II. The following three sections detail each of the design phases individually, phase 1 (Direct solve) is detailed in section III, phase 2 (Optimization) is detailed in section IV, and phase 3 (Patterning) is detailed in section V. Since phase 3 is covered in [19], this paper will concentrate on design phases 1 and 2 only. A subsequent paper will present further details on the results of phase 3. Three metasurface beamformer designs will be presented in section VI including a metasurface reflectarray designed for a scanned far field beam with a -20dB sidelobe limit, a directly-fed compact meta-antenna with 100% aperture
efficiency, and finally a conformal metasurface beamformer. A study on the loss tolerance and bandwidth of the metasurface beamformers is presented in section VII. Finally, the paper is concluded in section VIII. An appendix provides the derivations of the semi-analytic gradient used in the Adjoint field acceleration method.

An $e^{j\omega t}$ time convention is assumed and suppressed throughout.

II. BEAMFORMING METASURFACE DESIGN
Overview

The metasurface is designed in three phases as depicted in Fig. 2. In summary, in phase 1, the desired total aperture fields are defined. These may be defined directly from known incident fields and desired scattered aperture fields or can be obtained from a desired far field amplitude pattern and a guessed aperture field amplitude profile using the Gerchberg-Saxton algorithm [21]. In either case, the patterned metallic cladding is homogenized and represented as a spatially variant array of homogenized impedance sheet elements linking the desired aperture fields to the induced surface currents per the impedance boundary condition (IBC) derived from the GSTC [2]. From the IBC, a non-linear electromagnetic inverse problem is solved to obtain the complex-valued impedances required to transform the incident field into the desired scattered field (see section III). The complex-valued impedances arise because the power density is not locally conserved at all points on the metasurface [9]. Since the positive and negative resistances (real part) are difficult or costly to manufacture, a purely reactive sheet is desired. In phase 2, the complex-valued sheet is converted into a purely reactive sheet by way of gradient descent optimization accelerated by the Adjoint Field Method (see section IV). The reactances (imaginary part) of the complex-valued sheet obtained in phase 1 serve as the critical initial optimization point required for convergence in local optimization-based techniques. The optimization varies the reactances until the normalized far field amplitude scattered by the reactive sheet agrees with the normalized far field amplitude scattered by the complex-valued sheet from phase 1. The optimization introduces evanescent surface waves, which when added to the total field on the metasurface, satisfy a passive and lossless boundary condition. In phase 3, the optimized reactive sheet impedances are realized as patterned metallic claddings (see section V). A novel sheet impedance extraction technique for aperiodic metasurface environments is used to obtain the metallic cladding patterning mask. Full-wave simulation of the patterned cladding and grounded substrate validates the three-phase design approach. This paper concentrates on phases 1 and 2. Phase 3 is treated in [19] and will be detailed in a separate paper. Phase 1 is detailed next, where the metasurface beamformer model is constructed and solved to obtain the initial complex-valued design.

III. PHASE 1: DIRECT SOLVE

The 2-dimensional problem (with out-of-plane wavenumber $k_z=0$) that is considered is illustrated in Fig. 1. A metasurface of width $w$ is placed above a grounded substrate with $\epsilon = \epsilon_r \epsilon_0 (1 - j \tan \delta)$. The thickness of the substrate is $t$. The metasurface is divided into $N$ cells of width $w/N$ each. The metasurface can be described as consisting of three layers. The first layer, denoted layer 1, is the patterned metallic cladding (Metasurface), the second layer, denoted layer 2, is the dielectric spacer, and the third layer, denoted layer 3, is the PEC ground plane. The task of design phase 1 is, given the desired field transformation, to design the inhomogeneous impedance sheet (possibly complex-valued) of layer 1 such that the desired field transformation occurs.

To that end, the individual elements of the patterned metallic cladding are homogenized. The collection of homogenized elements is modeled as an array of homogeneous impedance sheets. Note, just as the elements of the patterned metallic cladding are electrically isolated by introducing a small gap between metallizations, each of the homogenized impedance sheets are separated by a small gap to electrically isolate them. Thus, the homogenized sheet impedances and the elements of the patterned metallic cladding each have width $\Delta = 0.9(w/N)$ (see Fig. 1).

An Electric Field Integral Equation (EFIE) is written to model the response of the metasurface to the exciting field, including the mutual coupling between the homogenized cells of the inhomogeneous impedance sheet, between the polarization currents in the dielectric substrate, and between the surface currents on the ground plane [8], [9]. By both the surface and volume form of the equivalence principle, all the currents radiate in free space. For Transverse Electric (TE) polarization, all quantities are $\hat{z}$-directed. The integral equation has the form

$$\bar{E}_i^{inc} = \bar{\eta}_i \odot \bar{J}_i + \sum_{j=1}^N j \omega \mu_0 \left[ G(\bar{\rho}, \bar{\rho}_j) \bar{J}_j(\bar{\rho}_j') d\bar{\rho}_j' \right]$$

(1)

where $\odot$ is the Hadamard (element-wise) multiplication operator, $\omega$ is the angular frequency of operation, $\mu_0$ is the magnetic permeability of free space, and $G$ is the free space Green’s function for two-dimensional problems

$$G(\bar{\rho}, \bar{\rho}') = \frac{1}{4\pi} H_0^{(2)}(k_0 |\bar{\rho} - \bar{\rho}'|)$$

(2)

$H_0^{(2)}(\cdot)$ is the Hankel function of the second kind or order 0, and the indices $i$ and $j$ denote the layer numbers (see Fig. 1). When $i$ or $j$ is 1 or 3, $\bar{I}_{or \ j}^i$ is an electric surface current density and when $i$ or $j$ is 2, $\bar{I}_{or \ j}^i$ is the volumetric polarization current density. $\bar{\eta}_i$ is the surface impedance of layer $i$, $\bar{\rho}_{or \ j}$ is a position vector to points in the $i$th or $j$th layer, and $\bar{E}_i^{inc}$ is the incident field on layer $i$.

The incident field is that radiated by an infinite $\hat{z}$-directed electric line source placed $F$ meters above (or below) the metasurface along the y-axis

$$\bar{E}_i^{inc} = \frac{-j I_o \eta_0 k_0}{4} H_0^{(2)} \left( k_0 \sqrt{x^2 + F^2} \right) \hat{z}$$

(3)

In (3), $I_o$ is the line current strength in Amps, $\eta_0$ is the free space wave impedance, and $k_0$ is the free space wavenumber.

In phase design 1, the integral equation is solved using the
method of moments (MoM) for the unknown sheet impedances given the desired aperture fields. The desired scattered aperture field amplitude distribution is defined based on the desired far field pattern. The amplitude level in V/m of the desired aperture field is found by conserving global power on the metasurface

\[
\int_{-\infty}^{\infty} \frac{1}{2} \left( E_{\text{inc}} \times H_{\text{inc}}^* \right) dx = \int_{-\infty}^{\infty} \frac{1}{2} \left( E_{\text{tot}} \times H_{\text{tot}}^* \right) dx = P_{\text{inc}}
\]

where \( P_{\text{inc}} \) is the incident power on the metasurface in Watts. From the desired scattered electric field, \( \vec{E}_{\text{ssc}} \), the associated magnetic field is found by first forming the plane wave spectrum of \( \vec{E}_{\text{ssc}}^\text{inc} \), dividing through by the TE wave impedance (\( \eta_{\text{TE}} = \eta_0 k_0 / k_y \)) to obtain the spectrum of the magnetic field, \( \vec{H}_{\text{ssc}}^\text{inc} \), and finally taking the inverse Fourier transform to obtain the magnetic field, \( \vec{H}_{\text{ssc}} \). With \( \vec{E}_{\text{ssc}}^\text{inc} \) and \( \vec{H}_{\text{ssc}}^\text{inc} \) defined, the scattered field power can be found. The scattered electric field amplitude, \( \vec{E}_{\text{ssc}} \), is scaled to satisfy (4). This ensures the scattered field has the same total power as the incident field which is a necessary requirement for passive operation. The normal power density associated with the incident field can be discarded and the retained reactances serve as the initial design for the optimization.

With both \( \vec{E}_{\text{ssc}}^\text{inc} \) and \( \vec{E}_{\text{ssc}} \) defined, (1) is solved for the unknown currents [8], [9] by making the replacement

\[
\vec{\eta} \circ \vec{J} = \vec{E}_{\text{inc}} + \vec{E}_{\text{ssc}} = \vec{E}_{\text{tot}}^\text{inc}
\]

and noting \( \eta = \text{diag}((j \omega \epsilon_0 (1 - \epsilon_r))^{-1}) \), following from volume equivalence, and \( \eta_\phi = 0 \), by definition, using the MoM. Note, \( \text{diag}() \) refers to the construction of a diagonal matrix with the enclosed elements appearing along the diagonal. The substitution in (5) linearizes the system allowing for its direct solution using MoM. Application of the MoM leads to the linear system representation of (1) [5], [8], [9], [20]

\[
\begin{bmatrix}
V_1 \\
V_2 \\
V_3
\end{bmatrix}
= \begin{bmatrix}
\eta \circ I_1 \\
\eta \circ I_2 \\
\eta \circ I_3
\end{bmatrix}
= \begin{bmatrix}
[Z_{11}] & [Z_{12}] & [Z_{13}] \\
[Z_{21}] & [Z_{22}] & [Z_{23}] \\
[Z_{31}] & [Z_{32}] & [Z_{33}]
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2 \\
I_3
\end{bmatrix}
\]

\[\vec{V} = \text{diag}(\vec{\eta}) \vec{I} + \vec{Z} \vec{I} = \left\{ \vec{Z} c + \text{diag}(\vec{\eta}) \right\} \vec{I} = \vec{Z} \vec{l}
\]

(6)

which can be solved by matrix inversion for the unknown induced surface currents. Note, \( \vec{\eta} = [\vec{\eta}_1, \vec{\eta}_2, \vec{\eta}_3] \). With the induced surface currents, (5) can be used again to obtain the sheet impedances \( \vec{\eta}_1 \)

\[
\vec{\eta}_1 = \vec{E}_{\text{tot}}^\text{inc} \circ \vec{J}
\]

where \( \circ \) is the Hadamard (element-wise) division operator.

The obtained sheet impedances will in general be complex-valued as the desired total field may not satisfy a reactive impedance boundary condition. Surface waves can be added to the total electric field of (7) allowing a passive and lossless result. These surface waves are determined through optimization next in phase 2.

IV. PHASE 2: OPTIMIZATION

The metasurface sheet impedance of layer 1 can be optimized such that the desired field transformation occurs and a number of engineered surface waves are excited which facilitate passivity. The added surface waves add locally to the desired total field, \( \vec{E}_{\text{tot}} = \vec{E}_{\text{inc}} + \vec{E}_{\text{ssc}} + \vec{E}_{\text{sw}} \), such that the boundary condition is satisfied in a passive and lossless manner. The surface waves are evanescent and hence carry no power away from the metasurface only redistribute it along the metasurface transversally in accordance with (4). Consequently, they do not contribute to the far field or the radiative near field. To setup the optimizer, the seed, cost function, domain boundaries, and convergence criteria must be defined. These parameters are defined in this section.

A. Optimization Cost Function and Seed

In local optimization methods, convergence strongly depends on obtaining a good initial solution. The real part of (7) can be discarded and the retained reactances serve as the initial solution. Thus, the optimization variables are purely reactive, and hence the final sheet will be lossless and passive. The field pattern scattered by the complex valued sheet obtained in the initial design can be used to construct an optimization cost function. The cost function is defined as

\[
g = \frac{1}{2} \Delta \vec{E}^T \Delta \vec{E}
\]

(8)

where \( T \) denotes the matrix transpose and \( \Delta \vec{E} \) is the difference between the normalized far field amplitude patterns at the \( M \) observation angles, respectively, calculated using (21). This cost function attempts to minimize the difference between the normalized far field amplitude pattern in dB scale radiated by the optimized purely reactive sheet and that of the complex-valued sheet of the initial design. The dB scale is used to give more weight to the sidewebs in the cost function and the amplitude pattern is used since the far field is generally not a smooth function of angle and is difficult to optimize over.

B. Optimization Domain Boundaries

The surface waves are necessary to obtain passive metasurfaces. However, the surface waves cannot have a tangential wavenumber which is too large such that the current across the elements can no longer be homogenized. For TE-polarization, each homogenized impedance sheet element can support a surface wave if the total input impedance of the parallel combination of the sheet and grounded dielectric substrate has a negative imaginary part [22]. Since the input impedance of the thin grounded substrate is inductive, this only occurs when the sheet has sufficient capacitive reactance.
Therefore, each capacitive reactive element supports a surface wave of the form
\[ E^\omega = A e^{-j\beta_x x} e^{j\beta_y y} z \] (10)
where \( A \) is an unknown complex coefficient, \( \beta_x \) is the tangential wavenumber of the surface wave and \( \alpha \) is the attenuation constant in the normal direction. The surface waves are excited due to the finite edges of the metasurface and due to small gaps between elements. These small gap fields contain wide evanescent spectrum which can excite the high tangential wave of the form
\[ E^\omega = A e^{-j\beta_x x} e^{j\beta_y y} z \] (10)
where \( A \) is an unknown complex coefficient, \( \beta_x \) is the tangential wavenumber of the surface wave and \( \alpha \) is the attenuation constant in the normal direction. The surface waves are excited due to the finite edges of the metasurface and due to small gaps between elements. These small gap fields contain wide evanescent spectrum which can excite the high tangential wavenumber (\( \beta_x \)) surface waves in (10).

For the assumed surface wave, the dispersion relation for the capacitive sheet over a grounded dielectric substrate is found by applying the Transverse Resonance Technique [23]
\[ -j\sqrt{\beta_x^2 - \beta_y^2} + Y_e - j\frac{\varepsilon_e k_0^2 - \beta_x^2}{\omega \mu} \cos\left(\sqrt{\varepsilon_e k_0^2 - \beta_x^2} \right) = 0 \] (11)
Equation (11) can be solved for \( \beta_x(Y_e) \) given \( \omega \). The surface admittance \( Y_e \) is equal to the reciprocal of the surface impedance \( X_s \). The result is plotted in Fig. 3 as a function of \( X_s \). As can be seen the surface wavenumber increases rapidly as \( -jX_s \rightarrow 0 \).

Thus, to ensure proper homogenizability, \( -jX_s \) should be limited. We will derive these limits next.

The attenuation constant can be obtained from the separation relation
\[ \alpha^2 = \beta_x^2 - k_0^2 \] (12)
Since the homogenized elements are electrically isolated form one another, the surface waves in (10) see an open circuit at both ends of the element, and hence a standing wave forms on each element. The standing wave can be described by
\[ E_s = \begin{cases} 2A e^{-\alpha y} \cos(\beta_x x), & y > 0, \frac{-\Delta}{2} < x < \frac{\Delta}{2} \\ 2A e^{-j\alpha y} \cos(\beta_x x), & y < 0, \frac{-\Delta}{2} < x < \frac{\Delta}{2} \end{cases} \] (13)
where \( k_{yd} = \sqrt{\varepsilon_e k_0^2 - \beta_x^2} \) and \( \Delta \) is the element width (see Fig. 1). The magnetic field can be obtained for the regions \( y > 0 \) and \( y < 0 \) following from Faraday’s Law. Then, the surface current density can be obtained following from the boundary condition \( \mathbf{J}_s = \mathbf{n} \times [\mathbf{H}_{y<0} - \mathbf{H}_{y>0}] \). The magnitude and phase of the resulting current density at the plane of the element \( (y = 0) \) is
\[ |J_s| = \frac{2|A|}{\omega \mu} \cos(\beta_x x) \sqrt{(k_{yd}^2)^2 \cos^2 \phi_s + (\alpha + k_{yd}^2)^2 \sin^2 \phi_s} \] (14)
\[ \angle J_s = \tan^{-1}\left(\frac{\alpha + k_{yd}^2}{k_{yd}^2} \tan \phi_s \right) \] (15)
where \( k_{yd} \) has been expressed as \( k_{yd} = k_y + jk_{yd}^0 \) and the complex coefficient \( A \) has been expressed in polar form as \( A = |A|e^{j\phi_A} \). Thus, the surface current density phase angle is constant over the element and the amplitude varies \( \propto \cos(\beta_x x) \).

Note, there is also a contribution to the surface current from the propagating scattered wave with tangential wavenumber within the light cone, however, since the scattered wave is a propagating wave, the tangential wavenumber will be much lower than the surface waves and hence can be ignored in this analysis.

For proper homogenization and efficient modelling of the surface current density using simple basis functions, both the amplitude and phase of the current density should be nearly constant. Fabrication tolerance and etching limits will determine the minimum element width \( \Delta \). Thus, given a \( \Delta \), one can solve (14) for a maximum allowable \( \beta_x \) which gives a nearly uniform amplitude variation. From (11) or Fig. 3, one can determine a range of negative reactances to exclude, \((-X_{s,exlude}, 0)\), in the optimization such that \( \beta_x < \beta_{max} \) always holds. For example, in the cases presented in this paper with the exception of the conformal metasurface in section VI.C, \( X_{s,exlude} \) is found by allowing the cosine of the amplitude of the surface current to vary by 0.43 \( \pi \) (77.2°) over the element dimension of \( \Delta = 0.9 \lambda_{0.7} \), leading to \( \beta_{max} = 997.7 \text{ rad/m} \), and hence a \( X_{s,exlude} = -j40 \Omega \) from Fig. 3.

To exclude the reactances from the optimization domain, a non-linear constraint can be used. The allowable reactances are formed by the disjoint set
\[ X_s = \left[ -X_{s,\min}, -X_{s,exlude} \right] \cup \left[ 0, X_{s,\max} \right] \] (16)
where \( X_{s,\min} \) and \( X_{s,\max} \) describe the minimum and maximum realizable reactances. The non-linear constraint function, \( c(X_s) \), is defined as the multidimensional parabola passing through the points \((-X_{s,exlude}, 0)\) and satisfies the inequality
\[ c(X_s) = -X_s^2 - X_{s,exlude} X_s \] (17)
\[ c(X_s) \leq 0, \quad \forall X_s \]
Only those \( X_s \) which lie in the disjoint set (16) satisfy the inequality in (17). Gradient descent optimization routines which accept non-linear constraints such as MATLAB’s fmincon can be used to optimize the metasurface reactances while avoiding the exclusion range.

Alternatively, if one excludes the inductive reactances as well, then the non-linear constraint functions can be avoided as the set is no longer disjoint. In all cases presented in this paper,
the inductive reactances are avoided and thus the optimization domain is limited to the convex set

$$X_s = \left[ -X_{s,\text{min}}, -X_{s,\text{exclude}} \right]$$  \hspace{1cm} (18)

An added benefit of the optimization domain of (18) is the avoidance of lossy thin traces carrying high current densities (see Fig. 20b). This will decrease the sensitivity of the final metasurface to conductor losses (see section VII).

From previous patterned geometry part libraries made for the same substrates [5], $X_{s,\text{min}} = -j4700\Omega$ and $X_{s,\text{max}} = +j135\Omega$. Thus, (16) becomes

$$X_i = \left[ -j4700, -j40 \right] \cup [0, j135] \hspace{1cm} (\Omega)$$  \hspace{1cm} (19)

and (18) becomes

$$X_i = \left[ -j4700, -j40 \right] \hspace{1cm} (\Omega)$$  \hspace{1cm} (20)

The limits expressed by (20) will be used to define the optimization domain boundaries.

C. Gradient Calculation: Adjoint Field Method

The gradient-based optimization method used in this work has been accelerated by the Adjoint Field method. This acceleration technique permits the calculation of the full $N$ component gradient in only two linear system solves. This is in contrast to finite difference based methods which require the $N+1$ total cost function evaluations to compute the full gradient [5], [20]. The derivations of the Adjoint Field method formulas are provided in the appendix and the key results are summarized here. The far fields in (9) are calculated from the vector potential

$$E^{\text{eff}}(\rho, \phi) = -\frac{k_i h_0}{4} \int \frac{2j}{\pi k_i \rho} e^{-j\rho_\theta} \sum_i \int J_i(\hat{n}_i) e^{j\kappa_\rho \hat{l}_\phi} dS,$$  \hspace{1cm} (21)

where the currents $J_i(\hat{n}_i)$ result from solving (6) for a given vector $\hat{n}_i$, and $\hat{\rho} = \cos \phi \hat{x} + \sin \phi \hat{y}$.

The adjoint field method calculates an analytic gradient of (8) by forming its components as (see appendix)

$$\nabla g(\vec{n}_i) = \text{Re} \left\{ j^T \text{diag} [\vec{l}] \right\}$$  \hspace{1cm} (22)

where $\vec{l}$ is the current density expansion coefficients obtained from the solution of the linear system in (6), and $\vec{l}_\phi$ is the solution to the adjoint problem

$$\vec{Z} \vec{l}_\phi = \vec{R} \Delta E_{\text{adj}}$$  \hspace{1cm} (23)

In (23), $\vec{R}$ is an $M$-by-$N$ matrix formed by evaluating (21) at the $M$ observation angles in the far field with the currents $J$ on all $N$ elements replaced by $1 \times 0^\circ$ when pulse basis functions are used (see (A.13)). Also, $\vec{Z}$ is the method of moments impedance matrix in (6) and $\Delta E_{\text{adj}}$ is the adjoint field defined in (A.17). The gradient in (22) is used in the classical gradient descent optimization method to optimize the impedance sheets.

D. Optimization Convergence Criteria

The optimization will run until (8) has reached its minimum value. This occurs when the gradient descent step size has reduced to below $10^{-10}$.

![Fig. 4. Metasurface geometry. The metasurface consists of a single layer of patterned metallic cladding over a grounded dielectric substrate. The metasurface is fed by an incident cylindrical wave. The patterned metallic cladding supports a surface wave and scatters the prescribed far field pattern of a scanned beam with Taylor aperture distribution sidelobe envelope.](image)

Fig. 4. Metasurface geometry. The metasurface consists of a single layer of patterned metallic cladding over a grounded dielectric substrate. The incident field is given by the fields radiated by the electric line source feed. The scattered field amplitude is a Taylor aperture field distribution, while the scattered field phase is designed to scan the far field beam to an angle of $30^\circ$ off broadside.

![Fig. 5. Amplitude of Incident and Scattered Fields](image)

![Fig. 5. Phase Angle of Incident and Scattered Fields](image)

Fig. 5. Incident and scattered field amplitude and phase at the metasurface plane. The incident field is given by the fields radiated by the electric line source feed. The scattered field amplitude is a Taylor aperture field distribution, while the scattered field phase is designed to scan the far field beam to an angle of $30^\circ$ off broadside.

V. PHASE 3: PATTERNING

A. Sheet Impedance Extraction for Aperiodic Metasurface Environments

The optimized impedance profiles will vary highly non-adiabatically (see Fig. 6b for example) and hence the metasurface cannot be accurately represented as a locally periodic environment. New extraction techniques must be invented to accurately assign a surface impedance to patterned geometries in the true aperiodic environment. That problem has been treated in [19] and since this paper focuses on Phases 1 and 2 only, won’t be discussed further here.

Now that the three phases have been discussed and detailed, several examples of metasurface beamformers designed using the three phase design approach will be presented in the next section.

VI. RESULTS

A. Scanned Beam with -20dB Maximum Sidelobe Envelope

The metasurface that is considered is illustrated in Fig. 4. A
A metasurface of width \( w = 16\lambda_\ell \) (\( f = 10\) GHz) is placed above a Rogers 6010 substrate with \( \varepsilon_r = 10.7 \) and \( \tan\delta = 0.0023 \). The thickness of the substrate is 0.00127m which is \( \lambda_\ell / 7.22 \), where \( \lambda_\ell \) is the wavelength in the dielectric. The metasurface is divided into 320 cells of width \( \lambda_\ell / 20 \) each. Each cell contains an element of width \( \Delta = 0.9\lambda / 20 \). The metasurface is designed to transform the cylindrical wave emanated from the line source at \( F = 4\lambda \) into collimated beam scanned to an angle of 30° off broadside and with a maximum sidelobe envelope of -20 dB. Hence, the aperture field amplitude is described by a Taylor aperture field distribution function and the aperture field phase should contain a linear phase gradient required to scan the beam to the prescribed angle. These desired aperture fields are shown in Fig. 5.

The results of phase 1 are shown in Fig. 6 through Fig. 9 as the curves labeled ‘Complex’. Fig. 6 shows the metasurface sheet impedances. As can be seen in Fig. 6a, the initial direct solve design contains complex-valued sheet impedances indicating the need for both loss and gain. The far field radiation patterns are shown in Fig. 7. The beam is scanned to the desired angle and contains the desired sidelobe envelope. The scattered field amplitude spectrum at the metasurface plane is shown in Fig. 8. The complex-valued sheet does not show significant evanescent spectrum. Also shown in the plot is the scattered field amplitude spectrum for the case of no metasurface present, only the feed grounded dielectric substrate. When the far fields of Fig. 7 are backprojected to the metasurface plane as seen in Fig. 9, it is observed that the desired aperture fields of Fig. 5 are obtained. These results, however, require a complex-valued sheet.

The optimization approach of phase 2 is applied to this complex-valued sheet design to arrive at the results shown in Fig. 6 through Fig. 9 in the curves labeled ‘OptReact’. Fig. 6b shows the optimized reactive metasurface sheet impedances. The reactances all fall within the range expressed by (20) and the resistances are all zero, and hence the metasurface is passive and lossless. This optimized reactive sheet scatters the same far fields as the complex-valued sheet from phase 1 as can be seen in Fig. 7. As discussed, the key to the similar operation is due to the introduced surface waves seen in the invisible spectrum portion of Fig. 8. The optimized reactive metasurface contains a much more prominent evanescent spectrum than the complex-valued sheet solution. However, the visible region of the scattered field amplitude spectrum is identical as per (8) and (9). The backprojected far fields in Fig. 9 again show the desired near fields are achieved by the optimized reactive metasurface. Finally, the optimized reactive metasurface was imported into COMSOL Multiphysics for full-wave verification. The results are shown superimposed in Fig. 7 and show excellent agreement validating design phases 1 and 2.

This example illustrates the power of the presented design approach for designing single layer passive and lossless metasurface beamforming antennas that are space fed. The next example will be of a metasurface which is directly fed from...
B. Metasurface Antenna with Perfect Aperture Efficiency

The next example is a directly-fed metasurface antenna with perfect aperture efficiency. The aperture efficiency is the product of taper efficiency (a measure of how uniform the electric field is across the radiating aperture) and the radiation efficiency (a measure of the losses associated with the antenna). Thus, the passive and lossless metasurface antenna will contain uniform aperture field amplitude and phase. The metasurface that is considered is illustrated in Fig. 10. The same substrate and discretization are used as in the previous example, however, in this case the infinite line source feed is placed inside the substrate at $F = -t/2$ as shown in Fig. 10. Both the incident field due to this substrate integrated line source feed and the uniform scattered aperture field amplitude and phase are shown in Fig. 11. The metasurface will transform the incident field (shown in the blue curves) to the scattered field (shown in the red curves). The results of phase 1 are shown in Fig. 12 through Fig. 15 as the curves labeled ‘Complex’, and the results of phase 2 in the same figures as the curves labeled ‘OptReact’. The far fields shown in Fig. 13 also include the COMSOL Multiphysics full-wave verification. The results show excellent agreement for the formed far field beams. The amplitude spectrum of the scattered field in Fig. 14 shows the developed surface waves in the invisible region of the spectrum. The farthest out sidelobes did not converge completely and thus the main beam peak amplitude of the optimized reactive metasurface is not at the same level as the complex-valued metasurface. The difference in the farthest out sidelobes can also be seen in at $k_x/k_0 = \pm 1$ in the amplitude spectrum in the lower half of Fig. 14. The backprojected far fields in Fig. 15 do show uniform aperture field amplitude and phase. This type of performance is not possible without the presented design approach.
C. Conformal Metasurface

The final example is a conformal metasurface as shown in Fig. 16. The conformal metasurface is described by the function

\[ y(x) = \frac{\lambda}{2} \sin \left( \frac{2\pi x}{w} \right) \]  

(24)

It contains 80 cells each \( \lambda/10 \) wide and hence \( w = 8\lambda \). Each cell contains a conformal element of width \( \Delta = 0.8\lambda/10 \). The metasurface is suspended in air \( \lambda/10 \) above a conformal PEC ground plane. Since the dielectric is now air, (20) becomes \( X_s = \{ -j2200, -j30 \} \). The metasurface is designed to generate a uniform amplitude and phase aperture field similar to the previous example only now from a conformal surface and fed from a line source placed \( F = 2\lambda \) above the center of the metasurface as in Fig. 16. Both the ground plane and the sheet impedance elements follow (24) (see inset of Fig. 16). The results of phases 1 and 2 are shown in Fig. 17 through Fig. 19. The impedances shown in Fig. 17 are plotted against the \( x \)-coordinate of each of the elements. In Fig. 16, the real part of the incident near field (a), the real part of the scattered near field when only the ground plane is present (b), the real part of the scattered near field for the complex-valued metasurface from phase 1 (c), and the real part of the scattered near field for the optimized reactive metasurface from phase 2 (d) is shown. The figure shows the development of planar wavefronts scattered from the conformal metasurface. In Fig. 19, the far fields for phases 1 and 2 are shown as well as the far fields of the case of
the ground plane only. The far fields show excellent agreement between the optimized reactive and complex metasurface designs. Thus, the design approach presented in this paper allows for complete control of the scattered near and far fields from conformal surfaces of any shape. This is critical for next-generation cloaks, electromagnetic illusions, and camouflage.

VII. LOSS AND BANDWIDTH

A. Loss

The dielectric loss is already included in the design implemented as a complex permittivity in the volumetric impedance $\eta_2$ of (6). To estimate the metal losses, locally periodic simulations of several patterned metallic cladding parts were made in order to extract the complex-valued $Z_{12}$ impedance matrix element. The parts (shown in Fig. 20a) were made from 18$\mu$m thick copper (Cu) and simulated in CST Microwave Studio using periodic boundaries. This thickness corresponds to $1/2$ ounce cladding commonly applied to RF substrates. The extracted reactances are shown in Fig. 20a and the extracted resistances as a function of the extracted reactances are shown in Fig. 20b. Figure 20 was used to estimate the loss (shown in Fig. 21b) associated with each homogenized sheet element of the optimized reactive metasurface of section III.B.

Fig. 18. (a) the real part of the incident field. (b) real part of the scattered field when only the ground plane is present. (c) real part of the scattered near field for the complex-valued sheet design. (d) real part of the scattered near field for the optimized reactive sheet design.

Fig. 19. Far field pattern comparison.

Fig. 20. Patterned metallic cladding part libraries in periodic environments for TE-polarization. Each element is $W=0.9(\lambda/20)$ wide by $H=2(\lambda/20)$ tall. (a) Extracted sheet reactance versus geometrical parameter $l$. (b) Loss resistance as a function of sheet reactance.

Fig. 21. (a) Far field comparison between optimized reactive design of section III.B with and without added Cu losses. (b) The added Cu loss implemented as a real part of the homogenized sheet impedances and calculated from Fig. 24 as a function of the (c) optimized reactances of the design in section III.B.

Fig. 19. Far Field Radiation Pattern

| $\phi$($^\circ$) |
|----------------|
| 0 | 30 | 60 | 90 | 120 |
| 0.05 | 0.1 | 0.15 | 0.2 |

| $|E_{ff}|(dB)$ |
|----------------|
| 0 | 30 | 60 | 90 | 120 |
| 0.05 | 0.1 | 0.15 | 0.2 |

| $R_{opt}$ (ohm) |
|----------------|
| -300 | -200 | -100 | 0 | 100 |
| 0.05 | 0.1 | 0.15 | 0.2 |

| $\eta_2$ (imag) |
|----------------|
| -8 | -6 | -4 | -2 | 0 |
| 0.05 | 0.1 | 0.15 | 0.2 |

| $\eta_2$ (real) |
|----------------|
| -0.2 | -0.1 | 0.05 | 0.1 |
| 0.05 | 0.1 | 0.15 | 0.2 |

| $\eta_2$ (total) |
|----------------|
| -0.2 | -0.1 | 0.05 | 0.1 |
| 0.05 | 0.1 | 0.15 | 0.2 |
which have exceeded an envelope of pattern directivity and the RMS value of the pattern values.

The directivity of the pattern for the sheet including boundaries imposed in (20) where the inductive elements were sensitive to metal losses. This is partly due to the optimization process according to (25) while the resistances were kept constant. The metasurface beamformers show a 1.3% 1dB directivity bandwidth and a 1.03% 3dB RMS pattern error bandwidth. The bandwidths are greater than those for the complex-valued sheet designs.

VIII. CONCLUSION

An approach for the design of beamforming metasurfaces was presented. The metasurfaces considered consists of a patterned metallic cladding supported by finite truncated grounded dielectric spacers. They are modeled using integral equations. The design approach consists of three phases, an initial direct solve phase, a subsequent optimization phase, and a final patterning phase. The three phase design approach results in passive and lossless metasurfaces which can be realized as patterned metallic claddings. The optimization phase introduces surface waves which transfer power transversely from places where loss is needed to places where gain was needed leading to the passive design. These optimizations were done using gradient descent accelerated by supplying a semi-analytic gradient obtained by the adjoint field method. The optimizations result in aperiodic metasurfaces.

New surface impedance extraction approaches for aperiodic unit cell environments that will be used to realize the metasurface beamformers was discussed. Examples of both planar and conformal beamforming metasurface reflectarrays were presented. In addition, a directly-fed meta-antenna with 100% aperture efficiency was also presented. Finally, the bandwidth and sensitivity to losses was investigated. It was found that due to how the optimization limited the final reactances of the metasurface, the beamformers are not sensitive to losses. It was also found that the metasurface beamformers exhibited over 1.3% bandwidth making them viable candidates for communications links or for electromagnetic illusions, camouflage, and cloaking. Future work includes extending the design concepts to 3D designs and extending the bandwidth of the beamformers.

APPENDIX: ADJOINT FIELD OPTIMIZATION EXPRESSIONS

The adjoint field optimization expressions are derived in this appendix. We will derive the expressions for the gradient of a cost function built around the un-normalized complex-valued far electric field first as it aids in understanding, then that for the cost function in (9) which is for the normalized far electric field amplitude only in dB scale. The optimization variables are the reactances of the metasurface elements

\[ \eta_i = [\eta_{i1}, \eta_{i2}, \ldots, \eta_{iN}] \]  

(A.1)

The cost function is defined in (8) and is repeated here as
where $\dagger$ denotes Hermitian transpose. If the cost function is to be built upon the un-normalized complex-valued far electric field, then

$$g(\eta_i) = \frac{1}{2} \Delta \bar{E}^* \Delta \bar{E}$$  \hspace{1cm} (A.2)$$

where $\dagger$ denotes Hermitian transpose. If the cost function is to be built upon the un-normalized complex-valued far electric field, then

$$\Delta \bar{E} = \begin{pmatrix} E_{ff \text{ca}(c)}^s(\eta_1) \\ \vdots \\ E_{ff \text{ca}(c)}^s(\eta_m) \\ E_{ff \text{ca}(a)}^s(\eta_M) \\ \vdots \\ E_{ff \text{ca}(a)}^s(\eta_m) \end{pmatrix} - \begin{pmatrix} E_{ff \text{ca}}^s(\eta_1) \\ \vdots \\ E_{ff \text{ca}}^s(\eta_m) \\ E_{ff \text{ca}}^s(\eta_M) \\ \vdots \\ E_{ff \text{ca}}^s(\eta_m) \end{pmatrix}$$

$$= E_{ff \text{ca}(a)}^s(\eta_M) - E_{ff \text{ca}}^s(\eta_m)$$

where $E_{ff \text{ca}}^s$ is found from (21) and $\eta_m = (\rho_m, \phi_m)$. The derivative of the cost function with respect to the $k$th optimization variable is

$$\frac{\partial g(\eta)}{\partial \eta_k} = \frac{\partial}{\partial \eta_k} \left( \frac{1}{2} \Delta \bar{E}^* \Delta \bar{E} \right) = \frac{1}{2} \left( \Delta \bar{E}^* \frac{\partial \Delta \bar{E}}{\partial \eta_k} + \frac{\partial \Delta \bar{E}^*}{\partial \eta_k} \Delta \bar{E} \right)$$

By the identity $\alpha^* \beta + \beta^* \alpha = 2 \text{Re} \{ \alpha^* \beta \}$, (A.4) becomes

$$\frac{\partial g(\eta)}{\partial \eta_k} = \text{Re} \left\{ \Delta \bar{E}^* \frac{\partial \Delta \bar{E}}{\partial \eta_k} \right\}$$

$$= \text{Re} \left\{ \Delta \bar{E}^* \frac{\partial}{\partial \eta_k} \left[ E_{ff \text{ca}(a)}^s(\eta_M) - E_{ff \text{ca}}^s(\eta_m) \right] \right\}$$

$$= \text{Re} \left\{ \Delta \bar{E}^* \frac{\partial}{\partial \eta_k} E_{ff \text{ca}(a)}^s(\eta_m) \right\}$$

(A.5)

Substituting in $E_{ff \text{ca}(a)}^s$ from (21)

$$\frac{\partial g(\eta)}{\partial \eta_k} = \text{Re} \left\{ \Delta \bar{E}^* \frac{\partial E_{ff \text{ca}(a)}^s(\eta_m)}{\partial \eta_k} \right\}$$

$$= \text{Re} \left\{ \Delta \bar{E} \frac{\partial}{\partial \eta_k} E_{ff \text{ca}(a)}^s(\eta_m) \right\}$$

$$= \text{Re} \left\{ \Delta \bar{E} \frac{\partial}{\partial \eta_k} \left[ \frac{k \rho_m}{4} \sqrt{\frac{2}{\pi k_p \rho_m}} e^{-\beta \rho_m} \int \bar{J}(r') e^{i 2 \rho_m \cos \theta_{jk} + i \phi_{jk}} d\nu \right] \right\}$$

$$= \text{Re} \left\{ \Delta \bar{E} \frac{k \rho_m}{4} \sqrt{\frac{2}{\pi k_p \rho_m}} e^{-\beta \rho_m} \int \bar{J}(r') e^{i 2 \rho_m \cos \theta_{jk} + i \phi_{jk}} d\nu \right\}$$

(A.6)

Next, we formulate an expression for $\frac{\partial \bar{J}(\eta)}{\partial \eta_k}$ using the solution to the linear system in (6)

$$\frac{\partial \bar{J}(\eta)}{\partial \eta_k} = \frac{\partial}{\partial \eta_k} \left\{ \bar{B} \bar{I} \right\}$$

(A.7)

where $\bar{B} = [\bar{f}_1, \bar{f}_2, ..., \bar{f}_n]$ and $\bar{I}$ are the basis functions (in our cases simple pulse basis functions) and are not a function of $\eta$ and hence (A.7) becomes

$$\frac{\partial \bar{J}(\eta)}{\partial \eta_k} = \frac{\partial}{\partial \eta_k} \left\{ \bar{B} \bar{I} \right\}$$

(A.8)

By the rule for differentiation of the inverse of a matrix [24] and since $\bar{V}$ is not a function of $\eta_k$

$$\frac{\partial \bar{J}(\eta)}{\partial \eta_k} = \bar{B} \left\{ \begin{array}{c} -\bar{V} \left[ \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} \right] \\ \vdots \\ -\bar{V} \left[ \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} \right] \end{array} \right\} \right\}$$

where we have explicitly factored out the $j$ from the reactances in $\eta_j$. Substituting back into (A.6),

$$\frac{\partial g(\eta)}{\partial \eta_k} = \text{Re} \left\{ \Delta \bar{E} \frac{k \rho_m}{4} \sqrt{\frac{2}{\pi k_p \rho_m}} e^{-\beta \rho_m} \int \bar{J}(r') e^{i 2 \rho_m \cos \theta_{jk} + i \phi_{jk}} d\nu \right\}$$

(A.9)

The result (A.11) is for the $k$th component of the gradient. For the full gradient, we form a matrix with (A.11) as the $k$th column

$$\nabla g(\eta) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{\partial \bar{J}(\eta)}{\partial \eta_k} \end{pmatrix}$$

(A.10)

Next, we formulate an expression for $\frac{\partial \bar{J}(\eta)}{\partial \eta_k}$ using the solution to the linear system in (6)

$$\frac{\partial \bar{J}(\eta)}{\partial \eta_k} = \frac{\partial}{\partial \eta_k} \left\{ \bar{B} \bar{I} \right\}$$

(A.7)

where $\bar{B} = [\bar{f}_1, \bar{f}_2, ..., \bar{f}_n]$ and $\bar{I}$ are the basis functions (in our cases simple pulse basis functions) and are not a function of $\eta$ and hence (A.7) becomes

$$\frac{\partial \bar{J}(\eta)}{\partial \eta_k} = \frac{\partial}{\partial \eta_k} \left\{ \bar{B} \bar{I} \right\}$$

(A.8)

By the rule for differentiation of the inverse of a matrix [24] and since $\bar{V}$ is not a function of $\eta_k$

$$\frac{\partial \bar{J}(\eta)}{\partial \eta_k} = \bar{B} \left\{ \begin{array}{c} -\bar{V} \left[ \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} \right] \\ \vdots \\ -\bar{V} \left[ \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} \right] \end{array} \right\} \right\}$$

where we have explicitly factored out the $j$ from the reactances in $\eta_j$. Substituting back into (A.6),

$$\frac{\partial g(\eta)}{\partial \eta_k} = \text{Re} \left\{ \Delta \bar{E} \frac{k \rho_m}{4} \sqrt{\frac{2}{\pi k_p \rho_m}} e^{-\beta \rho_m} \int \bar{J}(r') e^{i 2 \rho_m \cos \theta_{jk} + i \phi_{jk}} d\nu \right\}$$

(A.9)

The result (A.11) is for the $k$th component of the gradient. For the full gradient, we form a matrix with (A.11) as the $k$th column

$$\nabla g(\eta) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{\partial \bar{J}(\eta)}{\partial \eta_k} \end{pmatrix}$$

(A.10)

The result (A.12) is valid for the $m$th observation point in the far field. A matrix $\bar{R}$ can be formulated with the $m$th observation point appearing along the $m$th row and the $n$th basis function appearing on the $n$th column as (and hence $\bar{R}$ is of size $M \times N$)

$$\bar{R} = \frac{k \rho_m}{4} \sqrt{\frac{2}{\pi k_p \rho_m}} e^{-\beta \rho_m} \int \bar{J}(r') e^{i 2 \rho_m \cos \theta_{jk} + i \phi_{jk}} d\nu$$

(A.13)

which allows (A.12) to be written as

$$\nabla g(\eta) = \text{Re} \left\{ \Delta \bar{E} \frac{\bar{R} \bar{Z}^{-1} \text{diag}(\bar{I})}{\bar{V}} \right\}$$

(A.14)
The adjoint equation is therefore
\[ \Delta \vec{E} R \vec{Z}^{-1} = \vec{I}_{A} \]
\[ \vec{Z} \vec{I}_{A} = R \Delta \vec{E} \]  
(A.15)

The linear system of the adjoint problem in (A.15) can be solved leading to the final form for the gradient of the cost function
\[ \nabla g(\vec{\eta}) = \text{Re}\left\{ j \vec{f}_{A} \text{diag}(\vec{I}) \right\} \]  
(A.16)

Thus, to calculate the full gradient, the linear system in (6) is solved at each new optimization point yielding \( \vec{Z} \) and \( \vec{I} \) (the moment method impedance matrix and current vector of the forward problem solution), then \( \Delta \vec{E} \) is found from this result and (A.3), and finally, the adjoint equation is solved for \( \vec{I}_{A} \) yielding the full gradient at the optimization point following from (A.16). Note, \( \vec{R} \) is calculated from (A.13) and can be precalculated only once for the entire optimization. In this way, the entire gradient is found by solving only the forward linear system (6) and the adjoint linear system of (A.15), independent of the length of (A.1) (the number of unknowns in the optimization).

When the cost function is formulated around the normalized far field amplitude in dB scale as in (9) (the cost function in (A.3) is for the un-normalized far field complex field value), the derivation is the same except the \( \Delta \vec{E} \) in the right hand side of (A.15) becomes
\[ \Delta \vec{E}_{\text{ref}}^{\text{opt}} = \frac{20}{\ln 10} \left| E_{\text{ref,calc}}^{\text{opt}}(\rho, \phi) \right|_{\rho} \left( \Delta \vec{E} \bigcirc E_{\text{ref,calc}}^{\text{opt}} \right) - \left( \sum_{m} \Delta E_{m} \right)^{-1} \left( \sum_{m} E_{\text{ref,calc}}^{\text{opt}} \right)^{-1} \left( \left( E_{\text{ref,calc}}^{\text{opt}} \right)^{-1} \right) \left( E_{\text{ref,calc}}^{\text{opt}} \bigcirc E_{\text{ref,calc}}^{\text{opt}} \right) \]  
(A.17)

where a subscript \( n \) indicates a quantity normalized to its own maximum. \( E_{\text{ref,calc}}^{\text{opt}} \) is the Hadamard (element-wise) product of \( E_{\text{ref,calc}}^{\text{opt}} \) with itself \( p-1 \) times. In the work presented in this paper, a value of \( p=250 \) was chosen.

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