Inverse problems with a general transfer condition *

Sonja Currie †+ 
Marlena Nowaczyk ‡o 
Bruce A. Watson §+

+ School of Mathematics 
University of the Witwatersrand 
Private Bag 3, P O WITS 2050, South Africa

o AGH University of Science and Technology 
Faculty of Applied Mathematics 
al. A. Mickiewicza 30, 30-059 Krakow, Poland

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Abstract

We consider a Sturm-Liouville operator on a finite interval as well as a scattering problem 
on the real line both with transfer conditions at the origin. On a finite interval we show 
that the the Titchmarsh-Weyl $m$-function can be uniquely determined from two spectra 
for the same equation but with varied boundary conditions at one end of the interval. 
In addition, we prove that the $m$-function can also be uniquely reconstructed from one 
spectrum and the corresponding norming constants. For the scattering problem on 
the real line we assume that the potential has compact essential support. For a given 
symmetric finite intervals containing the essential-support of the potential and a pair of 
separated boundary conditions imposed at the ends of the interval, the spectrum and 
corresponding norming constants can be uniquely recoverable from the scattering data 
on $\mathbb{R}$. Consequently the potential and transfer matrix can be determined.

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1 Introduction

Inverse spectral problems for the Sturm-Liouville operator were studied as early as 1929 by Ambartsumyan. The applications to mathematical physics lead to a renewed interest in the problem in recent years. For a comprehensive discussion on the early development of the inverse Sturm-Liouville problem see the introduction of Levitan’s book [18]. More recently there has been interest in inverse Sturm-Liouville problems where the eigenparameter appears in the boundary conditions, for example see [4] and the references therein.

For an introduction to inverse scattering and inverse spectral problems including many direct applications we refer the reader to the book by Chadan et al. [7]. Inverse Sturm-Liouville problems with a discontinuity at an interior point were first studied by Hochstadt and Lieberman in [15]. These results were then generalized in the famous paper by Hald, [14], where it is shown that if the potential is known over half of the interval and if one of the boundary conditions is given, then the potential and the other boundary condition are uniquely determined by the eigenvalues. This in turn was extended by Willis in [22] to the case of two interior discontinuities. Similar techniques were then used by Kobayashi in [16] to give a uniqueness proof for the inverse Sturm-Liouville problem on a bounded interval with a symmetric potential having two interior jump discontinuities. A case with finitely many transmission conditions was discussed by Shahriari, Jodayree Akbarfam and Teschl in [21]. In particular, the authors prove that the specification of the Weyl function and the weight function uniquely determines the Sturm-Liouville operator on a finite interval for both Robin and separated eigenparameter dependent boundary conditions.

Ramm, [20], discusses inverse scattering and spectral problems on the half-line in detail. Some of the main topics included are, invertibility of the steps in the Gel’fand-Levitan and Marchenko inversion procedures, Krein inverse scattering theory and inverse problems. Aktosun and Weder, in [1], consider the Schrödinger equation on the half line with a real-valued, integrable potential having a finite first moment. They extend the well-known two-spectrum uniqueness theorem of Borg and Marchenko to the case where there is also a continuous spectrum. An elegant proof of the local Borg-Marchenko theorem is given by Bennewitz in [3].

In the last twenty years inverse problems on quantum graphs have received much attention. In particular Gerasimenko, in [13], develops a procedure for recovering the potential in the Schrödinger equation on a non-compact graph from the scattering data. Very recently, in [5], using the method of spectral mappings Buterin and Freiling prove that the specification of the spectral-scattering data uniquely determines the Sturm-Liouville operator on a non-compact graph. In addition, Delghani and Jodayree Akbarfam study an inverse spectral problem on a three-star graph and show that if four particular spectra do not intersect then it is possible to recover the potential uniquely, see [10]. It should be noted here that the problem we consider in Section 3 is a special case of a non-compact graph consisting of two infinite edges connected at one vertex.
In this paper we investigate the differential equation
\[ \ell y := -\frac{d^2 y}{dx^2} + q(x)y = \zeta^2 y, \quad \text{on } [-S,S], \] (1.1)
in \( L^2[-S,S] \) with point transfer condition
\[ \begin{bmatrix} y(0^+) \\ y'(0^+) \end{bmatrix} = M \begin{bmatrix} y(0^-) \\ y'(0^-) \end{bmatrix}. \] (1.2)

Here the entries of \( M \) are taken to be real, \( q \in L^2[-S,S] \) is assumed to be real valued. We assume that \( \det M > 0 \) and without loss of generality we will consider \( \det M = 1 \).

At the endpoints we impose the following boundary conditions
\[ y(-S)\cos \alpha - y'(-S)\sin \alpha = 0, \quad \alpha \in [0,\pi), \] (1.3)
\[ y(S)\cos \beta - y'(S)\sin \beta = 0, \quad \beta \in (0,\pi]. \] (1.4)

We will only consider point transfer matrices at the origin and henceforth will refer to them as transfer matrices. In the physical context the transfer matrix represents a change of medium which affects the incident wave as represented by components of the matrix. Our transfer matrices will be real constant transfer matrices i.e. all components will be constants.

We show that two distinct problems are spectrally related. Firstly there is the finite interval problem (1.1) and (1.2). For this problem we summarize, from [9], how the Titchmarsh-Weyl \( m \)-function, \( m(\lambda) \), of (1.1) on \([-S,S]\) obeying the transfer condition (1.2) with separated boundary conditions at the end points is defined. An asymptotic approximation for \( m(\lambda) \) is then obtained and consequently we show that \( m(\lambda) \) can be uniquely determined from two spectra. In addition, we show that from one spectrum and norming constants it is also possible to uniquely find \( m(\lambda) \). Secondly there is the scattering problem (1.1), (1.2) on \(( -\infty, 0) \cup (0, \infty) \) where the potential \( q \) has compact essential support in \([-S,S]\). In [9] we show that the scattering data uniquely determines the \( m \)-function, \( m(\lambda) \), and indeed the converse holds i.e. given \( m(\lambda) \) we can uniquely find the scattering data. In this paper we show that the scattering data also determines two spectra or one spectrum and norming constants. Thus the finite interval problem on \([-S,S]\) determines the scattering problem on the line with potential having compact essential support in \([-S,S]\) and vice-versa.

The paper is organised as follows. Section 2 is dedicated to asymptotic approximation for the \( m \)-function. Using these asymptotic estimates in Section 3 we prove that from two
spectra the $m$-function can be uniquely determined. Moreover using residue calculus and the Mittag-Leffler expansion it is shown that $m$-function can also be reconstructed from one spectrum and the corresponding norming constants. Inverse scattering problems on the line with potential having compact essential support form the topic of Section 4. We prove that given the scattering data on $\mathbb{R}$, i.e. the reflection coefficient and eigenvalues, two spectra or one spectrum and corresponding norming constants on the finite interval $[-S,S]$ with ess sup $q \subset [-S,S]$ can be reconstructed. Consequently the operator and transfer matrix can be uniquely found.

2 Titchmarsh-Weyl $m$-function

Let $v_\beta$ be the solution of (1.1) on $[-S,S]$ obeying the transfer condition (1.2) and satisfying the terminal conditions $v_\beta(S,\lambda) = \sin \beta$ and $v_\beta'(S,\lambda) = \cos \beta$. To define the Titchmarsh-Weyl $m$-function we use the approach given in [4], i.e. the $m$-function of (1.1) on $[-S,S]$ for boundary conditions (1.3), (1.4) and the transfer condition (1.2) is that value of $m_{\alpha,\beta}$ for which

$$\psi_{\alpha,\beta} := u_\alpha + m_{\alpha,\beta}w_\alpha$$ (2.1)

obeys the terminal condition (1.4). Here $u_\alpha, w_\alpha$ are solutions of (1.1) obeying the initial condition

$$W_\alpha(-S,\lambda) = H_\alpha = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}$$ (2.2)

where

$$W_\alpha(x,\lambda) := \begin{bmatrix} u_\alpha(x,\lambda) & w_\alpha(x,\lambda) \\ u'_\alpha(x,\lambda) & w'_\alpha(x,\lambda) \end{bmatrix}. \quad (2.3)$$

The entries of $W_\alpha(x,\lambda)$ are entire functions of $\lambda$ and the determinant of $W_\alpha(x,\lambda)$ is the Wronskian of $u_\alpha$ and $w_\alpha$, which is 1 for all $x$ and $\lambda$.

Let

$$\Delta_{\alpha,\beta}(\lambda) := \text{Wron}[w_\alpha, v_\beta] = w_\alpha v'_\beta - v_\beta w'_\alpha = v'_\beta(-S,\lambda) \sin \alpha - v_\beta(-S,\lambda) \cos \alpha. \quad (2.4)$$

The function $\Delta_{\alpha,\beta}(\lambda)$ is entire in $\lambda$ and the zeros of $\Delta_{\alpha,\beta}(\lambda)$ are the eigenvalues of (1.1) with boundary conditions (1.3) and (1.4) and the transfer condition (1.2). For $\lambda$ not an eigenvalue of (1.1) – (1.4), $v_\beta$ and $\psi_{\alpha,\beta}$ are linearly dependent, say $v_\beta = k\psi_{\alpha,\beta}$, giving

$$v_\beta(-S,\lambda) = -k \cos \alpha + km_{\alpha,\beta}(\lambda) \sin \alpha,$$

$$v'_\beta(-S,\lambda) = k \sin \alpha + km_{\alpha,\beta}(\lambda) \cos \alpha.$$ 

It thus follows that $k = v'_\beta(-S,\lambda) \sin \alpha - v_\beta(-S,\lambda) \cos \alpha = \Delta_{\alpha,\beta}(\lambda)$ and

$$\psi_{\alpha,\beta}(x,\lambda) = \frac{v_\beta(x,\lambda)}{\Delta_{\alpha,\beta}(\lambda)}.$$
Forming the linear combinations $v_\beta(-S, \lambda) \sin \alpha + v'_\beta(-S, \lambda) \cos \alpha = k m_{\alpha, \beta}(\lambda)$ we have

$$m_{\alpha, \beta}(\lambda) = \frac{v_\beta(-S, \lambda) \sin \alpha + v'_\beta(-S, \lambda) \cos \alpha}{\Delta_{\alpha, \beta}(\lambda)}. \tag{2.5}$$

**Lemma 2.1** The Titchmarsh-Weyl m-function for $\alpha = 0$ and $\beta = \pi$, $m_{0,\pi}$, has asymptotic approximation

$$m_{0,\pi}(\lambda) = -i \sqrt{\lambda} + O(1) \quad \text{as} \quad \lambda \to -\infty.$$  

**Proof:** From (2.4) and (2.5), for $\alpha = 0$,

$$m_{0, \beta}(\lambda) = -\frac{v'_\beta(-S, \lambda)}{v_\beta(-S, \lambda)}. \tag{2.6}$$

Denote the entries of the transfer matrix $M$ by $m_{ij}, i, j = 1, 2$. Note that, as $\det M = 1$ and $M$ has real entries not both $m_{12}$ and $m_{11} + m_{22}$ can simultaneously be zero. Let $\sqrt{\lambda} = ik, k > 0$, then from [9, Appendix], as $k \to \infty$,

$$v_\pi(-S, -k^2) = -\frac{km_{12} + m_{11} + m_{22}}{4k^2} e^{2kS} [k + O(1)], \tag{2.7}$$

$$v'_\pi(-S, -k^2) = \frac{km_{12} + m_{11} + m_{22}}{4k} e^{2kS} [k + O(1)]. \tag{2.8}$$

Therefore, as $k \to \infty$,

$$-\frac{v'_\pi(-S, -k^2)}{v_\pi(-S, -k^2)} = k + O(1) = -i \sqrt{\lambda} + O(1). \blacksquare$$

**Lemma 2.2** The Titchmarsh-Weyl m-function for $\alpha = 0$ and $\beta \in (0, \pi)$, $m_{0,\beta}$, has asymptotic approximation

$$m_{0,\beta}(\lambda) = -i \sqrt{\lambda} + O(1) \quad \text{as} \quad \lambda \to -\infty.$$  

**Proof:** We proceed from (2.6), again let $\sqrt{\lambda} = ik, k > 0$, then from [9, Appendix], as $k \to \infty$, for $\beta \in (0, \pi)$,

$$v_\beta(0^+, -k^2) = -\frac{\sin \beta}{2} e^{kS} \left(1 + O\left(\frac{1}{k}\right)\right), \tag{2.9}$$

$$v'_\beta(0^+, -k^2) = k \frac{\sin \beta}{2} e^{kS} \left(1 + O\left(\frac{1}{k}\right)\right). \tag{2.10}$$

Since $v_\beta(x, \lambda)$ satisfies transfer condition [12], we have

$$v_\beta(0^-, -k^2) = -(km_{12} + m_{22}) \frac{\sin \beta}{2} e^{kS} \left(1 + O\left(\frac{1}{k}\right)\right), \tag{2.11}$$

$$v'_\beta(0^-, -k^2) = (km_{11} + m_{21}) \frac{\sin \beta}{2} e^{kS} \left(1 + O\left(\frac{1}{k}\right)\right). \tag{2.12}$$
On \((-S, 0)\),
\[ v_\beta(x, -k^2) = v_\beta(0^-, -k^2) h_0(x) + v'_\beta(0^-, -k^2) h_1(x), \]
where \(h_0, h_1\) are solutions of (1.1) with \(h_0(0^-) = 1 = h'_0(0^-)\) and \(h'_0(0^-) = 0 = h_1(0^-)\). Here, as \(\lambda = -k^2 \to -\infty\), \(h_0(x) = \frac{e^{-kx}}{2} (1 + O\left(\frac{1}{k}\right))\) and \(h_1(x) = -\frac{e^{-kx}}{2k} (1 + O\left(\frac{1}{k}\right))\) with \(h'_0(x) = -k\frac{e^{-kx}}{2} (1 + O\left(\frac{1}{k}\right))\) and \(h'_1(x) = \frac{e^{-kx}}{2k} (1 + O\left(\frac{1}{k}\right))\). Thus
\[
v_\beta(-S, -k^2) = -(k^2 m_{12} + k(m_{11} + m_{22}) + m_{21}) \sin \frac{\beta}{4k} e^{2kS} \left(1 + O\left(\frac{1}{k}\right)\right), (2.13)
\]
\[
v'_\beta(-S, -k^2) = k(k^2 m_{12} + k(m_{11} + m_{22}) + m_{21}) \sin \frac{\beta}{4k} e^{2kS} \left(1 + O\left(\frac{1}{k}\right)\right). (2.14)
\]
Thus \(m_{0, \beta}(-k^2) = k + O(1)\).

**Lemma 2.3** The Titchmarsh-Weyl m-function given in (2.1) has the following asymptotic approximation for \(\alpha, \beta \in (0, \pi)\) as \(\lambda \to -\infty\)
\[
m_{\alpha, \beta}(\lambda) = \cot \alpha + O\left(\frac{1}{\sqrt{\lambda}}\right). \tag{2.15}
\]

**Proof:** Let \(\lambda = -k^2\) and \(k > 0\) for \(k \to \infty\) we have \(v_\beta(-S, -k^2)\) and \(v'_\beta(-S, -k^2)\) as given by (2.13)-(2.14). Direct computation give
\[
\Delta_{\alpha, \beta}(\lambda) = \frac{k^2 m_{12} + k(m_{11} + m_{22})}{4} \sin \alpha \sin \beta e^{2kS} \left(1 + O\left(\frac{1}{k}\right)\right),
\]
and
\[
v_\beta(-S, \lambda) \sin \alpha + v'_\beta(-S, \lambda) \cos \alpha = \frac{k^2 m_{12} + k(m_{11} + m_{22})}{4} \cos \alpha \sin \beta e^{2kS} \left(1 + O\left(\frac{1}{k}\right)\right).
\]
The result thus follows from (2.15).

**Lemma 2.4** The Titchmarsh-Weyl m-function given in (2.7) has the following asymptotic approximation for \(\alpha \in (0, \pi)\) and \(\beta = \pi\) as \(\lambda \to -\infty\)
\[
m_{\alpha, \pi}(\lambda) = \cot \alpha + O\left(\frac{1}{\sqrt{\lambda}}\right). \tag{2.16}
\]

**Proof:** Let \(\lambda = -k^2\) for \(k > 0\), then for \(k \to \infty\) from (2.3) and (2.8) we have
\[
\Delta_{\alpha, \pi}(\lambda) = \sin \alpha (km_{12} + m_{11} + m_{22})(k + \cot \alpha) \frac{e^{2sk}}{4k^2} [k + O(1)]
\]
and
\[
v_\beta(-S, \lambda) \sin \alpha + v'_\beta(-S, \lambda) \cos \alpha = \sin \alpha (km_{12} + m_{11} + m_{22})(k \cot \alpha - 1) \frac{e^{2sk}}{4k^2} [k + O(1)].
\]
Hence

\[ m_{\alpha,\pi}(\lambda) = \frac{k \cot \alpha - 1}{k + \cot \alpha} \left( 1 + O \left( \frac{1}{k} \right) \right). \]

Combining the four lemmas above we can write the following.

**Theorem 2.5** The Titchmarsh-Weyl \( m \)-function given in (2.1) has the following asymptotic behaviour as \( \lambda \to -\infty \),

\[ m_{\alpha,\beta}(\lambda) = \begin{cases} -i\sqrt{\lambda} + O(1), & \alpha = 0, \\ \cot \alpha + O \left( \frac{i}{\sqrt{\lambda}} \right), & \alpha \neq 0. \end{cases} \quad (2.17) \]

Note that this result is consistent with that obtained in [4].

### 3 Inverse problems on a compact interval

In this section we consider various inverse spectral problems for the differential equation (1.1) on the interval \([-S, S]\) with transmission condition (1.2) at 0 and boundary conditions (1.3) and (1.4).

**Theorem 3.1** Let \( \lambda_0 < \lambda_1 < \lambda_2 < \ldots \) be the eigenvalues of (1.1) and (1.2) with boundary conditions (1.3) and (1.4). Let \( \mu_0 < \mu_1 < \mu_2 < \ldots \) be the eigenvalues of (1.1) and (1.2) with boundary conditions (1.3) and (1.4), where \( \alpha \) in (1.3) has been replaced by \( \alpha' \neq \alpha, \alpha' \in [0, \pi) \). The Titchmarsh-Weyl \( m \)-function, \( m_{\alpha,\beta} \), can be uniquely reconstructed from \( (\lambda_n), (\mu_n), \alpha \) and \( \alpha' \).

**Proof:** The functions \( \Delta_{\alpha,\beta}(\lambda) \) and \( \Delta_{\alpha',\beta}(\lambda) \) are entire of order \( \frac{1}{2} \) in \( \lambda \). The zeros of the former are \( (\lambda_n) \) while the zeros of the latter are \( (\mu_n) \). Thus using the Hadamard product theorem we can write

\[ \Delta_{\alpha,\beta}(\lambda) = C_{\alpha,\beta} \prod_{n=0}^{\infty} \left( 1 - \frac{\lambda}{\lambda_n} \right) \]

and

\[ \Delta_{\alpha',\beta}(\lambda) = C_{\alpha',\beta} \prod_{n=0}^{\infty} \left( 1 - \frac{\lambda}{\mu_n} \right) . \]

Note that if for some \( n \), \( \lambda_n = 0 \) then we consider \( \frac{\Delta_{\alpha,\beta}(\lambda)}{\lambda} \) instead. Similarly if \( \mu_n = 0 \) for some \( n \) consider \( \frac{\Delta_{\alpha',\beta}(\lambda)}{\lambda} \). For convenience we denote

Now

\[ \cot(\alpha - \alpha') \begin{bmatrix} \sin \alpha \\ \cos \alpha \end{bmatrix} - \csc(\alpha - \alpha') \begin{bmatrix} \sin \alpha' \\ \cos \alpha' \end{bmatrix} = \begin{bmatrix} \cos \alpha \\ -\sin \alpha \end{bmatrix} = \begin{bmatrix} \sin(\frac{\pi}{2} + \alpha) \\ \cos(\frac{\pi}{2} + \alpha) \end{bmatrix} , \]
so
\[
\cot(\alpha - \alpha') \Delta_{\alpha, \beta} - \cosec(\alpha - \alpha') \Delta_{\alpha', \beta} = \Delta_{\alpha + \frac{\pi}{2}, \beta}.
\]
Hence
\[
m_{\alpha, \beta} = \frac{\Delta_{\alpha + \frac{\pi}{2}, \beta}}{\Delta_{\alpha, \beta}} = \cot(\alpha - \alpha') - \cosec(\alpha - \alpha') \frac{\Delta_{\alpha', \beta}}{\Delta_{\alpha, \beta}}
\]
giving
\[
m_{\alpha, \beta}(\lambda) = \cot(\alpha - \alpha') - C \cosec(\alpha - \alpha') \prod_{n=0}^{\infty} \left(1 - \frac{\lambda}{\mu_n}\right) \prod_{n=0}^{\infty} \left(1 - \frac{\lambda}{\lambda_n}\right)
\]
where \(C = C_{\alpha', \beta}/C_{\alpha, \beta}\). In particular if \(\alpha, \alpha' \neq 0\) then
\[
\lim_{\lambda \to -\infty} \frac{\sin \alpha' \prod_{n=0}^{\infty} \left(1 - \frac{\lambda}{\lambda_n}\right)}{\sin \alpha \prod_{n=0}^{\infty} \left(1 - \frac{\lambda}{\mu_n}\right)} = C
\]
while if \(\alpha = 0\) then \(\alpha' \in (0, \pi)\) and
\[
\lim_{\lambda \to -\infty} -i\sqrt{\lambda} \sin \alpha' \prod_{n=0}^{\infty} \left(1 - \frac{\lambda}{\lambda_n}\right) = C,
\]
from Corollary 2.5.

Finally, for \(\alpha' = 0\) and \(\alpha \in (0, \pi)\) to solve for \(C\) we need to consider the asymptotic behaviour of \(\Delta_{0, \beta}(\lambda)/\Delta_{\alpha, \beta}(\lambda)\) as \(\lambda \to -\infty\). From the previously computed asymptotic approximations, as \(k \to \infty\),
\[
\Delta_{0, \beta}(-k^2) = \left\{ \begin{array}{ll}
\frac{km_{12} + m_{11} + m_{22}}{4k^2} e^{2Sk} [k + O(1)], & \beta = \pi \\
(m_{12} + m_{11} + m_{22}) \sin \beta e^{2kS} [k + O(1)], & \beta \in (0, \pi)
\end{array} \right.
\]
and
\[
\Delta_{\alpha, \beta}(-k^2) = \left\{ \begin{array}{ll}
\sin \alpha \left(\frac{km_{12} + m_{11} + m_{22}}{4k^2} e^{2Sk} [k + O(1)]\right), & \beta = \pi \\
\frac{k}{m_{12} + m_{11} + m_{22}} \sin \alpha \sin \beta e^{2kS} [k + O(1)], & \beta \in (0, \pi)
\end{array} \right.
\]
giving
\[
\frac{\Delta_{0, \beta}(\lambda)}{\Delta_{\alpha, \beta}(\lambda)} = \frac{1}{-i\sqrt{\lambda} \sin \alpha} \left(1 + O \left(\frac{1}{\sqrt{\lambda}}\right)\right),
\]
as \(\lambda \to -\infty\) and
\[
\lim_{\lambda \to -\infty} \frac{1}{-i\sqrt{\lambda} \sin \alpha} \prod_{n=0}^{\infty} \left(1 - \frac{\lambda}{\lambda_n}\right) = C.
\]

We now show that the Titchmarsh-Weyl \(m\)-function can be determined from one spectrum together with the associated norming constants. Here we recall that if \(\lambda_n\) is an eigenvalue of (1.1)-(1.4) then the norming constant associated with the eigenvalue \(\lambda_n\) is
\[
a_n = \int_{-S}^{S} w_n^2(x, \lambda_n) \, dx.
\]
Theorem 3.2 For \((1.1)\) on \([-S,S]\) with transmission condition \((1.2)\) at \(x = 0\) and boundary conditions \((1.3)-(1.4)\) the eigenvalues \(\lambda_0 < \lambda_1 < \lambda_2 < \ldots\) together with the corresponding norming constants \((a_n)\), uniquely determine the Titchmarsh-Weyl m-function, \(m_{\alpha,\beta}\) for \(\alpha \neq 0\).

Proof: Since the eigenvalues are simple, for each \(n = 0, 1, 2, \ldots\), there is a non-zero \(k_n\) so that \(k_n w_\alpha(x, \lambda_n) = v_\beta(x, \lambda_n)\). Hence
\[
k_n = k_n \text{Wron}[u_\alpha, w_\alpha](x, \lambda_n) = \text{Wron}[u_\alpha, v_\beta](x, \lambda_n) = \Delta_{\alpha+\pi/2,\beta}(\lambda_n).
\]
Since \(w_\alpha(x, \lambda_n)\) and \(v_\beta(x, \lambda)\) both obey \((1.4)\) we have \(\text{Wron}[v_\beta(x, \lambda), w_\alpha(x, \lambda_n)](S) = 0\) and thus
\[
\int_{-S}^{S} (\lambda - \lambda_n) v_\beta(x, \lambda) w_\alpha(x, \lambda_n) \, dx = \int_{-S}^{S} [w_\alpha''(x, \lambda_n) v_\beta(x, \lambda) - w_\alpha(x, \lambda_n) v_\beta''(x, \lambda)] \, dx
\]
\[
= \text{Wron}[v_\beta(x, \lambda), w_\alpha(x, \lambda_n)]_{-S}^{S}
\]
\[
= \Delta_{\alpha,\beta}(\lambda_n).
\]
So, as \(\Delta_{\alpha,\beta}(\lambda_n) = 0\),
\[
\frac{\Delta_{\alpha,\beta}(\lambda) - \Delta_{\alpha,\beta}(\lambda_n)}{\lambda - \lambda_n} = \int_{-S}^{S} v(x, \lambda) w_2(x, \lambda_n) \, dx.
\]
Thus taking \(\lambda \to \lambda_n\), we have
\[
\Delta'_{\alpha,\beta}(\lambda_n) = \int_{-S}^{S} v_\beta(x, \lambda_n) w_\alpha(x, \lambda_n) \, dx = k_n a_n = \Delta_{\alpha+\pi/2,\beta}(\lambda_n) a_n.
\]
Combining the above with the relation
\[
m_{\alpha,\beta} = \frac{\Delta_{\alpha+\pi/2,\beta}}{\Delta_{\alpha,\beta}}
\]
(where \(\Delta_{\alpha+\pi/2,\beta} := \Delta_{(\alpha+\pi/2)\text{mod}\pi,\beta}\)) gives that
\[
\text{Res}_{\lambda=\lambda_n} m(\lambda) = \frac{1}{a_n}.
\]
Without loss of generality we may assume that 0 is not an eigenvalue (if it is shift the potential and the eigenparameter so that in the shifted eigenparameter 0 is not an eigenvalue). For \(\alpha \in (0, \pi)\) and \(\beta \in (0, \pi]\), using the asymptotic estimates from the Appendix, it follows that for large \(n \in \mathbb{N}\), \(|m_{\alpha,\beta}(\lambda)| \leq |\cot \alpha| + 1\) for on circle \(C_n\) with \(|\lambda| = (n + \frac{1}{2})\pi\). Hence the Mittag-Leffler expansion theorem can be applied to give
\[
m_{\alpha,\beta}(\lambda) = m_{\alpha,\beta}(0) + \sum_{n=0}^{\infty} \frac{1}{a_n} \left( \frac{1}{\lambda - \lambda_n} + \frac{1}{\lambda_n} \right).
\]
Applying Theorem 2.5 to (3.1) enables one to solve for the constant \( m_{\alpha,\beta}(0) \). □

**Remark** For \( \alpha = 0 \), let \( \mu < \min\{0, \lambda_0\} \). Using the asymptotic estimates given in the Appendix it can be seen that the function \( g(\lambda) := \frac{m_{0,\beta}(\lambda)}{\lambda - \mu} \) satisfies the conditions of the Mittag-Leffler expansion theorem. The poles of \( g(\lambda) \) are \( \mu < \lambda_0 < \lambda_1 < \ldots \) and a direct computation gives that the residues of \( g \) at these poles are \( m_{0,\beta}(\mu), \frac{1}{a_0(\lambda_0-\mu)}, \frac{1}{a_1(\lambda_1-\mu)}, \ldots \). Hence

\[
g(\lambda) = -\frac{m_{0,\beta}(0)}{\mu} + m_{0,\beta}(\mu) \left( \frac{1}{\lambda - \mu} + \frac{1}{\mu} \right) + \sum_{n=0}^{\infty} \frac{1}{a_n(\lambda_n - \mu)} \left( \frac{1}{\lambda - \lambda_n} + \frac{1}{\lambda_n} \right). \tag{3.2}
\]

Multiplying by \( \lambda - \mu \) we obtain

\[
m_{0,\beta}(\lambda) = m_{0,\beta}(0) + (m_{0,\beta}(\mu) - m_{0,\beta}(0)) \frac{\lambda}{\mu} + \sum_{n=0}^{\infty} \frac{\lambda}{a_n \lambda_n} \left( \frac{1}{\lambda_n - \mu} + \frac{1}{\lambda - \lambda_n} \right). \tag{3.3}
\]

As \( m_{0,\beta}(\mu) \) is analytic at \( \mu = 0 \) we can take \( \mu \to 0 \) in the above, giving

\[
m_{0,\beta}(\lambda) = m_{0,\beta}(0) + \lambda \left[ m_{0,\beta}'(0) + \sum_{n=0}^{\infty} \frac{1}{a_n \lambda_n} \left( \frac{1}{\lambda_n} + \frac{1}{\lambda - \lambda_n} \right) \right]. \tag{3.4}
\]

Dividing by \( \lambda \) and taking \( \lambda \to -\infty \) in (3.4), from Theorem 2.5 we have

\[
m_{0,\beta}'(0) = -\sum_{n=0}^{\infty} \frac{1}{a_n \lambda_n^2}. \tag{3.5}
\]

Using the asymptotics in the Appendix for \( v(x, \lambda_n) \) together with the fact that \( \lambda_n = O(n^2) \) one can show that \( \frac{1}{a_n \lambda_n^2} = O\left( n^{-2} \right) \). Therefore the series in (3.5) converges and thus

\[
m_{0,\beta}(\lambda) = m_{0,\beta}(0) + \lambda \sum_{n=0}^{\infty} \frac{1}{a_n \lambda_n (\lambda - \lambda_n)}. \tag{3.6}
\]

Note that if the constant term \( C \), for \( \alpha = 0 \), in (2.17) were known i.e.

\[
m_{0,\beta}(\lambda) = -i\sqrt{\lambda} + C + O\left( \frac{1}{\sqrt{\lambda}} \right),
\]

then \( m_{0,\beta}(0) \) in (3.6) is uniquely given by

\[
m_{0,\beta}(0) = \lim_{\lambda \to -\infty} \left( -i\sqrt{\lambda} - \lambda \sum_{n=0}^{\infty} \frac{1}{a_n \lambda_n (\lambda - \lambda_n)} \right) + C.
\]

Whether or not \( C \) and \( \sum_{n=0}^{\infty} \frac{1}{a_n \lambda_n (\lambda - \lambda_n)} \) are practically accessible remains an open question.
4 Scattering problem on the line

In this section we consider the inverse scattering problem on the real line i.e. for the
differential equation
\[ \ell y = \zeta^2 y, \quad \text{where} \quad \ell y := -\frac{d^2 y}{dx^2} + q(x)y, \quad \text{on} \ (-\infty, 0) \cup (0, \infty), \tag{4.1} \]
with point transfer condition
\[ \begin{bmatrix} y(0^+) \\ y'(0^+) \end{bmatrix} = M \begin{bmatrix} y(0^-) \\ y'(0^-) \end{bmatrix}. \tag{4.2} \]

Here the entries of $M$ are taken to be real, $q \in L^2(\mathbb{R})$ is assumed to be real valued and
have compact essential support, say $\text{ess supp}(q) \subset [-S, S]$ for some $S > 0$. The scattering
problem can now be treated as two half-line problems interacting via a matrix transfer
condition \[1.2\] at the origin.

**Definition 4.1** \[6, p.297\] The Jost solutions $f_{+,M}(x, \zeta)$ and $f_{-,M}(x, \zeta)$ are the solutions
of (4.1) and (4.2) with
\[ \lim_{x \to \infty} e^{-i\zeta x} f_{+,M}(x, \zeta) = 1 = \lim_{x \to -\infty} e^{i\zeta x} f_{-,M}(x, \zeta). \tag{4.3} \]

The Jost solutions $f_{+,M}(x, \zeta)$ and $f_{-,M}(x, \zeta)$ to (4.1), (4.2), can be expressed in terms
of the classical Jost solutions $f_{+}(x, \zeta)$ and $f_{-}(x, \zeta)$ (i.e. when $M = I$) by
\[ f_{+,M}(x, \zeta) := \begin{cases} f_{+}(x, \zeta), & x > 0 \\ h_{1}(x, \zeta), & x < 0 \end{cases} \tag{4.4} \]
\[ f_{-,M}(x, \zeta) := \begin{cases} f_{-}(x, \zeta), & x < 0 \\ h_{2}(x, \zeta), & x > 0 \end{cases} \tag{4.5} \]
where $h_{1}(x, \zeta)$ and $h_{2}(x, \zeta)$ are solutions of (4.1) on $(-\infty, 0)$ and $(0, \infty)$ respectively
obeying
\[ \begin{pmatrix} h_{1}(0^-, \zeta) \\ h'_{1}(0^-, \zeta) \end{pmatrix} = M^{-1} \begin{pmatrix} f_{+}(0^+, \zeta) \\ f'_{+}(0^+, \zeta) \end{pmatrix}, \]
\[ \begin{pmatrix} h_{2}(0^+, \zeta) \\ h'_{2}(0^+, \zeta) \end{pmatrix} = M \begin{pmatrix} f_{-}(0^-, \zeta) \\ f'_{-}(0^-, \zeta) \end{pmatrix}. \]

For $M = I$ the existence and asymptotic behaviour of the Jost solutions have been well
studied, see for example \[12\] \[19\]. In particular
\[ f_{+}(x, \zeta) = e^{i\zeta x} + O \left( \frac{C(x)\rho(x)e^{-\eta x}}{1 + |\zeta|} \right), \tag{4.6} \]
\[ f_{-}(x, \zeta) = e^{-i\zeta x} + O \left( \frac{C(-x)\bar{\rho}(x)e^{\eta x}}{1 + |\zeta|} \right), \tag{4.7} \]
as \(|x| + |\zeta| \to \infty\), where \(\eta = \Im(\zeta)\). Here \(C(x)\) is a non-negative, non-increasing function of \(x\) and
\[
\rho(x) = \int_{x}^{\infty} (1 + |\tau|)|q(\tau)| \, d\tau, \quad \tilde{\rho}(x) = \int_{-\infty}^{x} (1 + |\tau|)|q(\tau)| \, d\tau.
\] (4.8)

For \(\xi \in \mathbb{R}\), see [8, Sections 2 and 4], the conjugate Jost solutions take the form
\[
f_{+,M}(x, \xi) := \begin{cases} 
  f_{+}(x, -\xi), & x > 0 \\
  h_{1}(x, -\xi), & x < 0
\end{cases}
\] (4.9)
which obeys the transfer condition at \(x = 0\). The solutions \(f_{+M}(x, \xi)\) and \(\tilde{f}_{+M}(x, \xi)\) are independent for \(\xi \in \mathbb{R} \setminus \{0\}\) and span the solution space of (4.1), with (4.2). Hence there are unique coefficients \(A(\xi)\) and \(B(\xi)\) so that
\[
f_{-M}(x, \xi) = A(\xi) f_{+,M}(x, \xi) + B(\xi) f_{+,M}(x, \xi).
\] (4.10)
Here \(A(\xi)\) and \(B(\xi)\) are independent of whether \(x > 0\) or \(x < 0\), and they satisfy the equality \(|A(\xi)|^2 - |B(\xi)|^2 = 1\) for \(\xi \in \mathbb{R} \setminus \{0\}\). The reflection coefficient is defined as
\[
R(\xi) = \frac{B(\xi)}{A(\xi)} \quad \text{for} \quad \xi \in \mathbb{R} \setminus \{0\}.
\]

The next two results are a generalisation of [9, Lemma 4.1, Theorem 4.2]. The proofs follow in exactly the same manner, we will point out the main differences and provide the necessary asymptotics.

**Lemma 4.2** Let ess supp\(q\) \(\in [-S, S]\) for some \(S > 0\). Given the scattering data \(\{R(\xi), \eta_1, \ldots, \eta_N\}\), the matrix \(W(S, \zeta)\) is uniquely determined. Here, \(W(x, \zeta)\) is as given in (2.3) with \(W_{-S, \zeta}\) defined in (2.2).

**Proof:** Similarly to the proof of [9] Lemma 4.1] we now obtain
\[
[w_{1}(S, \xi) \ w_{2}(S, \xi)] = [e^{-i\xi S} e^{i\xi S}] \begin{pmatrix} A(\xi) & B(\xi) \\ B(\xi) & A(\xi) \end{pmatrix} \begin{pmatrix} e^{-i\xi S} \frac{2}{\xi} \\ e^{i\xi S} \frac{2}{\xi} \end{pmatrix} H_{\alpha}.
\] (4.11)
Therefore as in [9] we can find \(w_{1}(S, \xi)\) and \(w_{2}(S, \xi)\) and since \(w_{1}\) and \(w_{2}\) are entire, by analyticity we can extend them to \(w_{1}(S, \zeta)\) and \(w_{2}(S, \zeta)\). 

**Theorem 4.3** Given the Titchmarsh-Weyl m-function, \(m\), to (1.1) on \([-S, S]\) with boundary conditions (1.3) and (1.4) and the transfer condition (1.2) and \(\tilde{m}\), the Titchmarsh-Weyl m-function for the same problem but with the potential \(q\) replaced by \(\tilde{q}\). If \(m = \tilde{m}\) then \(q = \tilde{q}\).
Proof: The proof follows identically to that given in [9, Theorem 4.2]. It should be noted that only the asymptotics for \( v \) and \( w_2 \) used in Theorems 5.3 and 5.4 change. The new required asymptotics for \( v \) are given by (2.9), (2.10) and (5.9)-(5.12). The asymptotics for \( w_2 \) are as follows:

For \( -S \leq x < 0 \) we get

\[
\begin{align*}
  w_2(x, \lambda) &= \sin \alpha \cos \sqrt{\lambda}(x + S) + O\left(\frac{e^{\sqrt{\lambda}(x+S)}}{\sqrt{\lambda}}\right), \\
  w_2'(x, \lambda) &= -\sqrt{\lambda} \sin \alpha \sin \sqrt{\lambda}(x + S) + O(e^{\sqrt{\lambda}(x+S)}).
\end{align*}
\]

(4.12)

(4.13)

For \( S \geq x > 0 \) we have if \( m_{12} \neq 0 \) then

\[
\begin{align*}
  w_2(x, \lambda) &= -m_{12} \sqrt{\lambda} \sin \alpha \sin \sqrt{\lambda}S \cos \sqrt{\lambda}x + O(e^{\sqrt{\lambda}(x+S)}), \\
  w_2(x, \lambda) &= m_{12} \lambda \sin \alpha \sin \sqrt{\lambda}S \sin \sqrt{\lambda}x + O(\sqrt{\lambda}e^{\sqrt{\lambda}(x+S)}),
\end{align*}
\]

(4.14)

(4.15)

and if \( m_{12} = 0 \) then

\[
\begin{align*}
  w_2(x, \lambda) &= m_{11} \sin \alpha \cos \sqrt{\lambda}x \cos \sqrt{\lambda}S - m_{22} \sin \alpha \sin \sqrt{\lambda}x \sin \sqrt{\lambda}S \\
                    &+ O\left(\frac{e^{\sqrt{\lambda}(x+S)}}{\sqrt{\lambda}}\right), \\
  w_2'(x, \lambda) &= \sqrt{\lambda} \sin \alpha (-m_{11} \sin \sqrt{\lambda}x \cos \sqrt{\lambda}S - m_{22} \cos \sqrt{\lambda}x \sin \sqrt{\lambda}S) \\
                   &+ O\left(\frac{e^{\sqrt{\lambda}(x+S)}}{\sqrt{\lambda}}\right).
\end{align*}
\]

(4.16)

(4.17)

Combining Theorem 3.3 with Theorem 4.3 gives the following.

**Corollary 4.4** Given two spectra where one spectrum comes from the problem \( (1.1), (1.2) \) with boundary conditions \( (1.3) \) and \( (1.4) \) and the other spectrum comes from the same problem but with a different \( \beta \in (0, \pi] \) at the terminal end point, the potential \( q \) can be uniquely determined on \([-S, S]\).

Also combining Theorem 3.2 with Theorem 4.3 gives the result below.

**Corollary 4.5** Given the spectrum of the Neumann-Neumann problem and corresponding norming constants for equation \( (1.1) \) on \([-S, S]\) and transfer matrix, \( M \), at \( x = 0 \), the potential \( q \) may be uniquely determined on \([-S, S]\).

**Theorem 4.6** Given the scattering data \( \{R(\xi), \eta_1, \ldots, \eta_N\} \), i.e. the reflection coefficient and the eigenvalues of the problem \( (4.1), (4.2) \), the spectra of the Neumann-Neumann and the Neumann-Dirichlet problems on \([-S, S]\), with potential \( q_{[-S,S]} \) and transfer matrix, \( M \), as in the original scattering problem on the line, can be obtained. Moreover, the norming constants for the Neumann-Neumann problem on \([-S, S]\) can be determined.
Proof: Since \( w_1(x, \zeta) \) and \( w_2(x, \zeta) \) build a fundamental system and we know their values at \( x = -S \) and from Lemma 4.2 we can determined \( w_1(S, \zeta) \) and \( w_2(S, \zeta) \), we can obtain (from the scattering data) any spectrum, in particular the Neumann-Neumann spectrum, say \( \lambda_n \). Thus we can construct \( \Delta'(\lambda) \) for \( \alpha = \frac{\pi}{2} \) and therefore we know \( \Delta'(\lambda) \) for Neumann-Neumann boundary conditions. In addition from Lemma 4.2 we know \( w_2(S, \lambda) \) and

\[
\text{Wron}[w_1, v](S, \lambda) = -w'_1(S, \lambda) = \text{Wron}[w_1, v](-S, \lambda) = v(-S, \lambda).
\]

Thus since \( w'_1(S, \lambda) \) is known by Lemma 4.2, \( v(-S, \lambda) \) is also known. Hence

\[
\alpha_n = \frac{\Delta'(\lambda_n)}{v(-S, \lambda_n)},
\]

i.e. the norming constants for the Neumann-Neumann problem on \([-S, S]\), given in (??), can be uniquely determined.

To conclude we combine the above theorem together with the results from Section 2 as well as those given in [9, Sections 3 and 4] to obtain the following Corollary.

**Corollary 4.7** Given the scattering data i.e. the reflection coefficient and eigenvalues of (4.1), (4.2) it is possible to obtain the spectra of the Neumann-Neumann and Neumann-Dirichlet problems on \([-S, S]\) or the spectrum and norming constants of the Neumann-Neumann problem on \([-S, S]\). Consequently \( M \) can be uniquely reconstructed and \( q \) is unique on \( \mathbb{R} \).

**Remark** The fact that, in the above Corollary, we require only the reflection coefficient and eigenvalues (and not the norming constants) of (4.1), (4.2) in order to uniquely reconstruct \( q \) does not contradict the classical results since \( q \) has compact essential support, so is known on a very large portion of the line.

## 5 Appendix

The function \( v \) obeys the following asymptotics:

For \( \beta = 0 \) and any arbitrary \( \alpha \):

If \( 0 < x \leq S \)

\[
v(x, \lambda) = \frac{-\sin \sqrt{\lambda}(S - x)}{\sqrt{\lambda}} + O \left( \frac{e^{i3\sqrt{\lambda}(S-x)}}{\lambda} \right) \quad (5.1)
\]

\[
v'(x, \lambda) = \cos \sqrt{\lambda}(S - x) + O \left( \frac{e^{i3\sqrt{\lambda}(S-x)}}{\sqrt{\lambda}} \right). \quad (5.2)
\]
For \(-S \leq x < 0\) and \(m_{12} \neq 0\)

\[
v(x, \lambda) = -m_{12} \cos \sqrt{\lambda} S \cos \sqrt{\lambda} x + O\left(\frac{e^{\left|\sqrt{\lambda}(S-x)\right|}}{\sqrt{\lambda}}\right)
\]

\[
v'(x, \lambda) = m_{12} \sqrt{\lambda} \cos \sqrt{\lambda} S \sin \sqrt{\lambda} x + O\left(\frac{e^{\left|\sqrt{\lambda}(S-x)\right|}}{\sqrt{\lambda}}\right),
\]

if \(m_{12} = 0\) then

\[
v(x, \lambda) = -m_{22} \frac{\sin \sqrt{\lambda} S}{\sqrt{\lambda}} \cos \sqrt{\lambda} x + m_{11} \frac{\sin \sqrt{\lambda} x}{\sqrt{\lambda}} \cos \sqrt{\lambda} S + O\left(\frac{e^{\left|\sqrt{\lambda}(S-x)\right|}}{\lambda}\right)
\]

\[
v'(x, \lambda) = m_{22} \sin \sqrt{\lambda} S \sin \sqrt{\lambda} x + m_{11} \cos \sqrt{\lambda} S \cos \sqrt{\lambda} x + O\left(\frac{e^{\left|\sqrt{\lambda}(S-x)\right|}}{\sqrt{\lambda}}\right).
\]

For \(\beta \neq 0\) and any arbitrary \(\alpha\):

If \(0 < x \leq S\)

\[
v(x, \lambda) = -\sin \beta \cos \sqrt{\lambda}(S-x) + O\left(\frac{e^{\left|\sqrt{\lambda}(S-x)\right|}}{\sqrt{\lambda}}\right),
\]

\[
v'(x, \lambda) = -\sqrt{\lambda} \sin \beta \sin \sqrt{\lambda}(S-x) + O\left(\frac{e^{\left|\sqrt{\lambda}(S-x)\right|}}{\sqrt{\lambda}}\right).
\]

For \(-S \leq x < 0\), if \(m_{12} \neq 0\)

\[
v(x, \lambda) = \sqrt{\lambda} m_{12} \sin \beta \cos \sqrt{\lambda} x \sin \sqrt{\lambda} S + O\left(\frac{e^{\left|\sqrt{\lambda}(S-x)\right|}}{\lambda}\right),
\]

\[
v'(x, \lambda) = -\lambda m_{12} \sin \beta \sin \sqrt{\lambda} x \sin \sqrt{\lambda} S + O\left(\frac{e^{\left|\sqrt{\lambda}(S-x)\right|}}{\sqrt{\lambda}}\right),
\]

if \(m_{12} = 0\)

\[
v(x, \lambda) = -m_{22} \sin \beta \cos \sqrt{\lambda} x \cos \sqrt{\lambda} S - m_{11} \sin \beta \sin \sqrt{\lambda} x \sin \sqrt{\lambda} S + O\left(\frac{e^{\left|\sqrt{\lambda}(S-x)\right|}}{\lambda}\right),
\]

\[
v'(x, \lambda) = m_{22} \sqrt{\lambda} \sin \beta \sin \sqrt{\lambda} x \cos \sqrt{\lambda} S - m_{11} \sqrt{\lambda} \sin \beta \cos \sqrt{\lambda} x \sin \sqrt{\lambda} S + O\left(\frac{e^{\left|\sqrt{\lambda}(S-x)\right|}}{\lambda}\right).
\]
Working as in the Appendix of [9], setting $\alpha, \beta \in (0, \pi)$ we have for $m_{12} = 0$

$$\Delta_{\alpha, \beta}(\lambda) = -\sin \alpha \sin \beta \sqrt{\lambda} (m_{11} + m_{22}) \frac{\sin 2 \sqrt{\lambda} S}{2} + O \left( e^{3 \sqrt{\lambda} |S|^2} \right),$$

$$\Delta_{\alpha, \pi}(\lambda) = -\sin \alpha (m_{11} + m_{22}) \sin \sqrt{\lambda} S \sin \sqrt{\lambda} S + O \left( \frac{e^{3 \sqrt{\lambda} |S|^2}}{\sqrt{\lambda}} \right),$$

$$\Delta_{0, \pi}(\lambda) = (m_{22} + m_{11}) \frac{\sin \sqrt{\lambda} S}{\sqrt{\lambda}} \cos \sqrt{\lambda} S + O \left( \frac{e^{3 \sqrt{\lambda} |S|^2}}{\sqrt{\lambda}} \right),$$

$$\Delta_{0, \beta}(\lambda) = m_{22} \sin \beta \cos^2 \sqrt{\lambda} S - m_{11} \sin \beta \sin^2 \sqrt{\lambda} S + O \left( \frac{e^{3 \sqrt{\lambda} |S|^2}}{\sqrt{\lambda}} \right),$$

and for $m_{12} \neq 0$

$$\Delta_{\alpha, \beta}(\lambda) = \lambda \sin \alpha \sin \beta m_{12} \sin^2 \sqrt{\lambda} S + O \left( \sqrt{\lambda} e^{3 \sqrt{\lambda} |S|^2} \right),$$

$$\Delta_{\alpha, \pi}(\lambda) = -\sin \alpha m_{12} \sqrt{\lambda} \cos \sqrt{\lambda} S \sin \sqrt{\lambda} S + O \left( e^{3 \sqrt{\lambda} |S|^2} \right),$$

$$\Delta_{0, \pi}(\lambda) = m_{12} \cos^2 \sqrt{\lambda} S + O \left( \frac{e^{3 \sqrt{\lambda} |S|^2}}{\sqrt{\lambda}} \right),$$

$$\Delta_{0, \beta}(\lambda) = -m_{12} \sqrt{\lambda} \sin \beta \cos \sqrt{\lambda} S \sin \sqrt{\lambda} S + O \left( e^{3 \sqrt{\lambda} |S|^2} \right).$$

References

[1] T. Aktosun, R. Weder, Inverse spectral-scattering problem with two sets of discrete spectra for the radial Schrödinger equation, Inverse Problems, 22 (2006), 89–114.

[2] D. Alpay, I. Gohberg, Inverse problem for Sturm Liouville operators with rational reflection coefficient, Integral Equations and Operator Theory, 30 (1998), 317–325.

[3] C. Bennewitz, A Proof of the Local Borg-Marchenko Theorem, Commun. Math. Phys., 218 (2001), 131–132.

[4] P.A. Binding, P.J. Browne, B.A. Watson, Recovery of the $m$-function from spectral data for generalized Sturm-Liouville problems, J. Comput. Appl. Math., 171 (2004), 73–91.

[5] S. Buterin, G. Freiling, Inverse spectral-scattering problem for the Sturm-Liouville operator on a noncompact star-type graph, Tamkang J. Math., 44 no. 3 (2013), 327–349.

[6] K. Chadan, P.C. Sabatier, Inverse Problems in Quantum Scattering Theory, Springer-Verlag, (1977).

[7] K. Chadan, D. Colton, L. Paivarinta, W. Rundell An Introduction to Inverse Scattering and Inverse Spectral Problems, SIAM Philadelphia, (1997).
[8] S. Currie, M. Nowaczyk, B.A. Watson, Forward scattering on the line with a transfer condition, *Boundary value problem*, 2013 255 (2013), 14 pages.

[9] S. Currie, M. Nowaczyk, B.A. Watson, Inverse scattering on the line with a transfer condition, *Mediterranean Journal of Mathematics*, to appear.

[10] I. Dehghani, A. Jodayree Akbarfam Recovering Sturm-Liouville operators on a graph from pairwise disjoint spectra, *British Journal of Mathematics & Computer Science*, 3 issue 1 (2013), 52–72.

[11] P. Deift, E. Trubowitz Inverse scattering on the line, *Communications on Pure and Applied Mathematics*, 32 issue 2 (1979), 121–251.

[12] G. Freiling, V. Yurko, *Inverse Sturm-Liouville Problems and their Applications*, Nova Science, (2001).

[13] N.I. Gerasimenko Inverse scattering problem on a non-compact graph, *Theor. Math. Phys.*, 75 issue 2 (1988), 460–470.

[14] O.H. Hald, Discontinuous inverse eigenvalue problems, *Commun. Pure Appl. Math*, 37 (1984), 539–577.

[15] H. Hochstadt, B. Lieberman, An inverse Sturm-Liouville problem with mixed given data, *SIAM J. Appl. Math*, 34 (1978), 676–680.

[16] M. Kobayashi, A uniqueness proof for discontinuous inverse Sturm-Liouville problems with symmetric potentials, *Inverse Problems*, 5 (1989), 767–781.

[17] N. Levinson, The inverse Sturm-Liouville problem, *Matematisk Tidsskrifts*, B25 (1949a), 25–30.

[18] B.M. Levitan, *Inverse Sturm-Liouville Problems*, VNU Science Press BV, (1987).

[19] V.A. Marˇ cenko, *Sturm-Liouville Operators and Applications: Revised Edition* AMS, (2011).

[20] A.G.Ramm, One-dimensional inverse scattering and spectral problems, *CUBO a Math. Journal*, 6 N1 (2004), 313–426.

[21] M. Shahriari, A. Jodayree Akbarfam, G. Teschl, Uniqueness for inverse Sturm-Liouville problems with a finite number of transmission conditions, *J. Math. Anal. Appl.*, 395 N1 (2012), 19–29.

[22] C. Willis, Inverse Sturm-Liouville problems with two discontinuities, *Inverse Problems*, 1 (1985), 263–289.