Generalized Mittag Leffler distributions arising as limits in preferential attachment models

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Abstract: For $0 < \alpha < 1$, and $\theta > -\alpha$, let $(S_{\alpha,\theta+r}^{-\alpha \alpha})(r \geq 0)$ denote an increasing(decreasing) sequence of variables forming a time inhomogeneous Markov chain whose marginal distributions are equivalent to generalized Mittag Leffler distributions. We exploit the property that such a sequence may be connected with the two parameter ($\alpha, \theta$) family of Poisson Dirichlet distributions with law PD($\alpha, \theta$). We demonstrate that the sequences serve as limits in certain types of preferential attachment models. As one illustrative application, we describe the explicit joint limiting distribution of scaled degree sequences arising under a class of linear weighted preferential attachment models as treated in Mori[37], with weight $\beta > -1$. When $\beta = 0$ this corresponds to the Barabasi-Albert preferential attachment model. We are in fact primarily interested in distributional properties of $(S_{\alpha,\theta+r}^{-\alpha \alpha})(r \geq 0)$ and related quantities arising in more intricate exchangeable sampling mechanisms, with direct links to nested mass partitions governed by PD($\alpha, \theta$). We construct sequences of nested ($\alpha, \theta$) Chinese restaurant partitions of $[n]$. From this, we identify and analyze relevant quantities that may be thought of as mimics for vectors of degree sequences, or differences in tree lengths. We also describe connections to a wide class of continuous time coalescent processes that can be seen as a variation of stochastic flows of bridges related to generalized Fleming-Viot models. Under a change of measure our results suggest the possibilities for identification of limiting distributions related to consistent families of nested Gibbs partitions of $[n]$ that would otherwise be difficult by methods using moments or Laplace transforms. In this regard, we focus on special simplifications obtained in the case of $\alpha = 1/2$. That is to say, limits derived from a PD(1/2$|$t) distribution. Throughout we present some distributional results that are relevant to various settings. We close by describing nested schemes varying in ($\alpha, r$).

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1. Introduction

For each $0 < \alpha < 1$, let $S_\alpha$ denote a random variable whose law coincides with a positive stable random variable with index $\alpha$ specified by its Laplace transform $E[e^{-\omega S_\alpha}] = e^{-\omega^\alpha}$ and density denoted as $f_\alpha(t)$. Now define the variables $S_{\alpha,\theta}$ for each $\theta > -\alpha$, as having a density, denoted by $f_{\alpha,\theta}$, formed by polynomially tilting a stable density as follows

$$f_{\alpha,\theta}(t) = c_{\alpha,\theta} t^{-\theta} f_\alpha(t)$$

where $c_{\alpha,\theta} := \Gamma(\theta + 1)/\Gamma(\theta/\alpha + 1)$, and satisfies for $\delta + \theta > -\alpha$

$$E[S_{\alpha,\theta}^{-\delta}] = \frac{\Gamma(\theta + 1)}{\Gamma(\theta/\alpha + 1)} E[S_\alpha^{-\theta}] = \frac{\Gamma((\theta+\delta)/\alpha + 1)}{\Gamma(\theta + \delta + 1)} \Gamma(\theta/\alpha + 1).$$

The variable $S_{\alpha}^{-\alpha} \overset{d}{=} S_{\alpha,0}$ is often referred to as having a Mittag-Leffler distribution, hence it is natural to consider $S_{\alpha,\theta}$ as generalized Mittag-Leffler variables. In terms of combinatorial objects, versions of such variables arise as limits in the two parameter Poisson Dirichlet framework of [49] as follows: Let $(P_k)_{(k \geq 1)}$ be the collection of ranked probability masses summing to 1, whose law, denoted as $\text{PD}(\alpha, \theta)$, follows a Poisson-Dirichlet distribution with parameters $(\alpha, \theta)$ as described in Pitman and Yor [49] and Pitman [43, 44]. From those works, it follows that letting $K_n$ denote the number of blocks in a $\text{PD}(\alpha, \theta)$ partition of $\{n\} = \{1, 2, \ldots, n\}$, that is to say the well-known two parameter $(\alpha, \theta)$ Chinese restaurant process, then as $n \to \infty$, $n^{-\alpha} K_n \to S_{\alpha,\theta}^{-\alpha}$, almost surely. There are of course other known asymptotic results involving the number of blocks of a certain size etc. Following [41, 44, 49], a version of $S_{\alpha,\theta}^{-\alpha}$ may be interpreted in terms of the local time up to time 1 of a generalized Bessel process. The following relation shows that $S_{\alpha,\theta}$ is a measurable function of the $(P_k) \sim \text{PD}(\alpha, \theta)$ which makes sense of conditioning $(P_k) | S_{\alpha,\theta} = t$: $S_{\alpha,\theta} := \lim_{n \to \infty} (i\Gamma(1 - \alpha) P_n)^{-1/\alpha}$, almost surely. For general $\alpha$, they also arise in various Pólya urn and random tree growth models as described in for instance in [26, 32]. There are however instances where the limit is not recognized.

We also note that $S_{1/2,\theta}^{-1/2} \overset{d}{=} 2 G_{\theta+1/2}^{1/2}$, where $G_{\theta+1/2}$ denotes a Gamma$(\theta + 1/2, 1)$ variable. Random variables that are powers of gamma variables have played a key role in recent work by [16, 38, 39, 40, 41]. One can surmise that such results for $\alpha = 1/2$ can possibly be adapted for the general $\alpha$ case. Indeed, the works of [16, 18, 20, 30, 31, 41] have shown that $S_{\alpha,\theta}$ satisfies many interesting distributional identities demonstrating a notion of a beta-gamma-stable algebra. However, for example, one important problem considered in Peköz, Röllin and Ross [32] is results related to the scaled limiting distribution of the joint degree distribution of linearly weighted variations of the Barabasi-Albert [2] preferential attachment model. This requires quite specific information about the joint distributional behavior between variables in the limit. This is considerably more challenging in the general $\alpha$ setting. Although of interest, our aim in this present setting is not to mimic or adapt the methods in [28, 38, 40], but...
rather to provide more details and indicators on what types of limits might arise in various models, in $n$, having random limits when scaled by $n^\alpha$.

Let $B_{a,b}$ denote a Beta$(a, b)$ variable. James [31] notes there are versions of the generalized Mittag Leffler variables that satisfy the following exact equality for any $\theta > -\alpha$,

$$S_{\alpha,\theta}^{-\alpha} = S_{\alpha,\theta+\alpha}^{-\alpha} B_{\theta+1-\alpha}(\theta+\alpha, 1-\alpha) = S_{\alpha,\theta+1}^{-\alpha} B_{\theta+1-\alpha}(\theta+\alpha, 1-\alpha)$$  \tag{1.3}$$

where $S_{\alpha,\theta+\alpha}^{-\alpha}$ is independent of $B_{\theta+\alpha, 1-\alpha}$ and $B_{\theta+1-\alpha}(\theta+\alpha, 1-\alpha)$ is independent of $S_{\alpha,\theta+1}^{-\alpha}$. [See [31], eq. (2.11)] for an in distribution version of this result applied to a wider range of parameters]. By recursion, this leads to two sequences forming Markov chains. For an integer $r \geq 0$ an interpretation, in terms of size biased deletion of excursion intervals of certain generalized Bessel bridges, of the first such sequence $(S_{\alpha,\theta+r}^{-\alpha})_{r \geq 0}$, as is well known, may be read from Perman, Pitman and Yor [41, Corollary 3.15]. Perhaps more simply, the sequence represents a dual Markov chain corresponding to the operation of size biased deletion and insertion as described in Pitman and Yor [49, Proposition 34 and 35]. The sequence encodes such operations relative to a nested family of $(\text{PD}(\alpha, \theta+r))_{r \geq 0}$ distributions. Reading for $r$ increasing describes the states relative to a deletion operation.

Our primary interest is in the second sequence $(S_{\alpha,\theta+r}^{-\alpha})_{r \geq 0}$ which encodes Markov chains for the following family of distributions $(\text{PD}(\alpha, \theta+r))_{r \geq 0}$. Specifically this is encoded by the recursion formed from the equality,

$$S_{\alpha,\theta}^{-\alpha} = S_{\alpha,\theta+1}^{-\alpha} B_{\theta+1-\alpha}(\theta+\alpha, 1-\alpha):$$  \tag{1.4}$$

The family can be seen to coincide with discrete dual fragmentation coagulation operations described in [10, 21], although the particular role of the sequence $(S_{\alpha,\theta+r}^{-\alpha})_{r \geq 0}$ is not emphasized. These authors, as mentioned in [21, Remarks p.1712], do cite relations of their constructions to random recursive trees and other trees and graphs constructed under preferential attachment. One could say that $(S_{\alpha,\theta+r}^{-\alpha})_{r \geq 0}$ is a family of generalized Mittag Leffler distributions under a PD$(\alpha, \theta)$ discrete coagulation or fragmentation regime. We shall simply refer to $(S_{\alpha,\theta+r}^{-\alpha})_{r \geq 0}$ as a PD$(\alpha, \theta)$ sequence. The states $(\text{PD}(\alpha, \theta+r))_{r \geq 0}$, read for $r$ increasing correspond to fragmentation schemes.

The PD$(\alpha, 1-\alpha)$ case arises in Haas, Miermont, Pitman and Winkel [29]. There the sequence $(S_{\alpha,\theta+r+1-\alpha}^{-\alpha})_{r \geq 0}$ is interpreted as increasing lengths of nested families of trees. The general Markov chain associated with $(S_{\alpha,\theta+r+1-\alpha}^{-\alpha})_{r \geq 0}$ subject to a change of measure, is presented in James [31] which we shall reproduce here. One could also deduce this from [26, Proposition 18], by a change of measure, since their result involves any $\alpha$. Subsequent to these, the Markov chain based on constructions in Haas and Goldschmidt [23] involves a PD$(\alpha, \alpha)$ sequence for $\alpha \leq 1/2$. We note further that the M and L preferential attachment models in [39] correspond to the case of PD$(1/2, 0)$ and PD$(1/2, 1/2)$, respectively.

Both sets of Markov chains are well defined when conditioned on $S_{\alpha,\theta} = t$, leading to sequences governed by a PD$(\alpha|t)$, law as defined in Pitman [43, 44].
That is if \((P_{k,0})_{\{k \geq 1\}}\) is the mass partition having law \(PD(\alpha, \theta)\), then its conditional distribution is \(PD(\alpha|t)\), the distribution of the families \((P_{k,r})_{\{k \geq 1\}}\) in the Markov chain are then determined by transition rules of known form. The \(PD(1/2|t)\) case has special cancellation properties, which under the regime of the first sequence \((S_{1/2,\theta+1/2}^{-1/2})_{\{r \geq 0\}}\), translates into constructions for the standard additive coalescent and Brownian fragmentation processes in \(\cite{2, 5, 13, 14}\). Here we shall present some details for the \(PD(1/2|t)\) sequence under the second regime.

For general \(\alpha\), these conditioning arguments now allow one to mix \(t\) relative to any non-negative distribution, where the mixing distribution can be expressed as \(h(t)f_\alpha(t)\) for any non-negative function \(h(t)\), such that \(E[S_\alpha] = 1\). Thus we say \((P_k)_{\{k \geq 1\}}\) has the distribution \(PK_\alpha(h \cdot f_\alpha)\), if \(PK_\alpha(h \cdot f_\alpha) = \int_0^\infty PD(\alpha|t)h(t)f_\alpha(t)dt\). This distribution is a Poisson-Kingman distribution based on a stable subordinator with mixing distribution \(h(t)f_\alpha(t)\) as defined in Pitman \(\cite{13, 14}\). \(PD(\alpha, \theta)\) arises by choosing \(h(t) = t^{-\theta} c_\alpha,\theta\).

### 1.1. Outline

The paper will now progress as follows. In section 2 we will present a detailed description of the limiting joint degree distribution of preferential attachment models considered by Mori \(\cite{37}\) and others. In section 3, we will present a formal description of the pertinent Markov chain under general \(PK_\alpha(h \cdot f_\alpha)\) distributions models. This parallels Pitman, Perman and Yor \(\cite{41}\) Theorem 2.1] in the \(\alpha\)-stable setting. Section 4 describes nested families of random partitions of \([n]\) determined by a \(PD(\alpha, \theta)\) sequence of Chinese restaurant processes. We introduce and obtain some distributional results for an interesting class of variables, in \(n, (\xi_{n,0}, \ldots, \xi_{n,r})\). These can be seen as mimics for vectors of degree sequences. We further describe some joint limits where the idea is one can then map to various constructions of trees and graphs. By a change of measure these results can extended to any \(PK_\alpha(h \cdot f_\alpha)\). Hence, this allows one to describe limits for models based on nested sequences of general Gibbs partitions of \([n]\), \(\cite{2, 13, 14}\). We present relevant calculations for this general setting in section 4.2. We partially view these contributions as helping to provide a blueprint for construction of models having more flexible properties as demonstrated by their limiting distributions. We would add that it seems quite unlikely that one would be able to characterize(recognize) such limits by the usual methods. In Section 5, in terms of practical implementation, we can consider all possibilities in the \(\alpha = 1/2\) case by obtaining explicit results for \(PD(1/2|t)\). Section 6 describes how to further embed/nest \(PD(\alpha\delta, \theta)\) nested schemes into \(PD(\alpha, \theta)\) nested schemes for any \(0 < \delta < 1\), which in some sense offers a coalescent version of the results of \(\cite{17}\).

**Remark 1.1.** More details and related results in terms of basic properties of \((S_{\alpha,\theta+1/2}^{-\alpha})_{\{r \geq 0\}}\) are discussed in the unpublished manuscript of James \(\cite{31}\). See there for connections to models where \(h(t)f_\alpha(t) = [\zeta^{1-1/\alpha}/\alpha]e^{-t^{1/\alpha}}c\zeta f_\alpha(t)\). The entire range of \(PD(\alpha, \theta)\), for \(\theta > -\alpha\), is obtained by randomizing \(\zeta\) to have...
We now describe the limiting joint degree distribution of the linearly weighted of the scale free tree construction given in [13]. discussed for instance in Devroye [18, section 5], As such, we lift the description note that the model of [37] is equivalent to certain recursive tree models as and Sethuraman [5] for a more general extension. Bertoin and Uribe-Bravo [13], preferential attachment graph model obtained in Mori [37]. [See Athreya, Ghosh a single edge connecting 0 and 1. Then suppose that $T_n$ has been constructed for some $n \geq 1$, and for every $i \in \{0, 1, \ldots, n\}$, denote by $d_n(i)$ the degree of the vertex $i \in T_n$. Conditionally given $T_n$, the tree $T_{n+1}$ is derived from $T_n$ by incorporating the new vertex $n+1$ and creating an edge between $n+1$ and a vertex $v_n \in T_n$ chosen at random according to the law

$$
\mathbb{P}(v_n = i|T_n) = \frac{d_n(i) + \beta}{2n + \beta(n + 1)} \text{ for } i \in \{0, 1, \ldots, n\}
$$

For reference we shall call these models $\beta-$recursive trees.

Let $\xrightarrow{a.s.}$ denote convergence almost surely. From [37], one has for any $r \geq 0$, that as $n \to \infty$ the joint vector

$$ n^{-\frac{1}{\alpha+\beta}}(d_n(0), d_n(1), \ldots, d_n(r)) \xrightarrow{a.s.} (\xi_0, \xi_1, \ldots, \xi_r), \quad (2.1) $$

where $(\xi_0, \ldots, \xi_r)$ has joint moments specified in Mori [37]. Furthermore an important result for the scaled maximal degree is obtained

$$ n^{-\frac{1}{\alpha+\beta}} \max_{i \geq 0} d_n(i) \xrightarrow{a.s.} \max_{i \geq 0} \xi_i. $$

See also Durrett [22] and van der Hofstad [52, Section 8]. These results correspond to the model of Barabasi and Albert [6] when $\beta = 0$.

We will now show this model corresponds to components of a PD($\alpha, 1-2\alpha$) distribution. From (2.1) and (1.2), for each fixed $j \geq 1$, the density of $S_{\alpha,j-2\alpha}$ is given by $f_{\alpha,j-2\alpha}(t)$, and furthermore

$$ \mathbb{E}[S_{\alpha,j-2\alpha}^{-k\alpha}] = \frac{\Gamma\left(\frac{1}{\alpha} + k\right)\Gamma(j + 1 - 2\alpha)}{\Gamma\left(\frac{1}{\alpha}\right)\Gamma(j + 1 - 2\alpha + k\alpha)}. $$

**Proposition 2.1.** Set $\beta \alpha = 1 - 2\alpha > -\alpha$, and let $(S_{\alpha,r+1-2\alpha}^{-\alpha})_{r \geq 0}$ denote the sequence of PD($\alpha, 1-2\alpha$) $\alpha$-diversities satisfying the recursive identity

$$ S_{\alpha,j-2\alpha}^{-\alpha} = S_{\alpha,j+1-2\alpha}^{-\alpha} B_j \quad (2.2) $$

where $B_j = S_{\alpha,j-2\alpha}^{-\alpha}/S_{\alpha,j+1-2\alpha}^{-\alpha}$ are mutually independent Beta($\frac{1-\alpha}{\alpha}, \frac{1-\alpha}{\alpha}$) random variables. Furthermore $(B_1, \ldots, B_j)$ is independent of $S_{\alpha,\ell-2\alpha}^{-\alpha}$ for $\ell > j$. 


(i) Then for every integer $r \geq 0$, the joint distribution of the sequence
\[
(S_{\alpha,1-2\alpha}^\alpha, S_{\alpha,2-2\alpha}^\alpha - S_{\alpha,1-2\alpha}^\alpha, \ldots, S_{\alpha,r+1-2\alpha}^\alpha - S_{\alpha,r-2\alpha}^\alpha)
\] (2.3)
is equivalent in distribution component-wise and jointly to the vector
\[
(\xi_0, \xi_1, \ldots, \xi_r)
\] in (2.1).

(ii) It follows that, for $S_{\alpha,-2\alpha} = 0$,
\[
n^{-\alpha}\max_{i \geq 0} d_n(i) \xrightarrow{a.s} \max_{i \geq 0} (S_{\alpha,i+1-2\alpha} - S_{\alpha,i-2\alpha}^\alpha).
\]

(iii) One may set $\xi_0 = S_{\alpha,1-2\alpha}^\alpha$ and $\xi_1 = S_{\alpha,2-2\alpha}^\alpha - S_{\alpha,1-2\alpha}^\alpha$. Then noting that $B_1$ is a symmetric Beta($\frac{1-\alpha}{\alpha}, \frac{1-\alpha}{\alpha}$) random variable, there is the distributional identity
\[
\xi_1 = S_{\alpha,2-2\alpha}^\alpha [1 - B_1] \xrightarrow{d} S_{\alpha,2-2\alpha}^\alpha B_1 = \xi_0
\] (2.4)

(iv) Note for $(S_{\alpha,\theta}, S_{\alpha,1+\theta})$ in a PD($\alpha, \theta$) sequence, the correspondence in (2.4) only holds for the case $\theta = 1 - 2\alpha$.

Proof. One can verify [(i)] by checking that the joint moments of the vector in (2.3) correspond to joint moments of $(\xi_0, \xi_1, \ldots, \xi_r)$ provided in [37]. However, while true, this is rather tedious. Mori[37, Lemma 3] shows that for each $j$, $(\xi_0 + \ldots + \xi_j)B_j = (\xi_0 + \ldots + \xi_{j-1})$, where $B_j$ has the same beta distribution as in the PD($\alpha, 1-2\alpha$) sequence and $(\xi_0 + \ldots + \xi_j)$ is independent of $B_j$. Due to scaling, there are a myriad of potential solutions for the $(\xi_j)$. Nonetheless the recursion in (2.2) establishes the result if one can show that one can set $\xi_0 = S_{\alpha,1-2\alpha}^\alpha$. This is true since $E[S_{\alpha,1-2\alpha}^\alpha] = E[\xi_0^\alpha]$. \qed

Remark 2.1. The equivalence in distribution of $(\xi_0, \xi_1)$ in Proposition 2.1 was noted in [37], and is otherwise evident in the description of the $\beta$-recursive tree. The result in (2.4) shows the only PD($\alpha, \theta$) case we could have considered is $\theta = 1 - 2\alpha$.

Remark 2.2. The case $\beta = \infty$ corresponds to $\alpha \to 0$ which, by continuity,
\[
\lim_{\alpha \to 0} \text{PD}(\alpha, 1-2\alpha) = \text{PD}(0, 1),
\]
yields the Poisson Dirichlet model PD(0, 1). The rates become $\log(n)$ and $\xi_j = 1$ for all $j \geq 0$.

Remark 2.3. [39] first provide an explicit description of these limits when $\beta = 0$, that is $\alpha = 1/2$ and hence the PD(1/2, 0).
3. PK_{\alpha}(h \cdot f_{\alpha}) Markov chains

We now present the formal details of the Markov chain for \((S_{\alpha,\theta}^{-\alpha})_{r\geq 0}\) under PD(\(\alpha, \theta\)) and under a general change of measure to PK_{\alpha}(h \cdot f_{\alpha}) as described in [31]. As was noted earlier, one can also deduce this from [26]. It suffices to work with the basic case of a PD(\(\alpha, 0\)) sequence. Note from the recursion, there is the identity

\[ S_{\alpha,0}^{-\alpha} = S_{\alpha,r}^{-\alpha} \prod_{k=1}^{r} B_k \]  

(3.1)

where here \(B_k\) are independent Beta\((\frac{\alpha+k-1}{\alpha}, \frac{1-\alpha}{\alpha})\) variables independent of \(S_{\alpha,r}\).

**Proposition 3.1.** For each \(r\), let \((T_{\alpha,0}, T_{\alpha,1}, \ldots, T_{\alpha,r})\) denote a vector of random variables such that \(T_{\alpha,0} \overset{d}{=} S_{\alpha}\) and there is the relationship for each integer \(k\)

\[ T_{\alpha,(k-1)} = T_{\alpha,k} \times V_{k}^{-1/\alpha} \]  

(3.2)

where \(V_k\) has a Beta\((\frac{\alpha+k-1}{\alpha}, \frac{1-\alpha}{\alpha})\) distribution, independent of \(T_{\alpha,k}\) and marginally \(T_{\alpha,k} \overset{d}{=} S_{\alpha,k}\). Then, the conditional distribution of \(T_{\alpha,k}\) given \(T_{\alpha,k-1} = t\) is the same for all \(k\) and equates to the density,

\[ P(T_{\alpha,1} \in ds | T_{\alpha,0} = t) / ds = \frac{\alpha^2}{\Gamma(\frac{1-\alpha}{\alpha})} \frac{(s/t)^{\alpha-1}(1 - (s/t)^\alpha)^{(1-\alpha)/\alpha} f_{\alpha}(s)}{t^{2} f_{\alpha}(t)}, \]  

(3.3)

for \(s < t\). By a change of variable \(v = (s/t)^\alpha\) the density of \(V_1 | T_{\alpha,0} = t\) is given by

\[ P(V_1 \in dv | T_{\alpha,0} = t) / dv = \frac{\alpha}{\Gamma(\frac{1-\alpha}{\alpha})} \frac{(1 - v)^{(1-\alpha)/\alpha} f_{\alpha}(v^{1/\alpha} t)}{t f_{\alpha}(t)}. \]  

(3.4)

Furthermore \((V_1, \ldots, V_r)\) are independent variables, independent of \(T_{\alpha,r}\). The sequence is a Markov chain, governed by a PD(\(\alpha, 0\)) law.

**Proof.** Because of the independence between \(V_k\) and \(T_{\alpha,k}\) the proof just reduces to an elementary Bayes rule argument. Details are presented for clarity. The distribution of \(T_{\alpha,k-1} | \tilde{T}_{\alpha,k} = s\) is just \(V_k^{-1/\alpha}s\), where \(V_k \sim \text{Beta}(\frac{\alpha+k-1}{\alpha}, \frac{1-\alpha}{\alpha})\).

Use the fact that for each \(k\), \(T_{\alpha,k}\) has density

\[ f_{\alpha,k}(s) = \frac{\Gamma(k+1)}{\Gamma(\frac{k+\alpha}{\alpha})} s^{-k} f_{\alpha}(s), \]

to show that the joint density of \(T_{\alpha,k-1}, \tilde{T}_{\alpha,k}\) is,

\[ \frac{\alpha^2 \Gamma(k) t^{-(k+1)}}{\Gamma(\frac{k+\alpha}{\alpha}) \Gamma(\frac{1-\alpha}{\alpha})} (s/t)^{\alpha-1}(1 - (s/t)^\alpha)^{(1-\alpha)/\alpha} f_{\alpha}(s). \]  

(3.5)

Now divide (3.5) by the \(f_{\alpha,k-1}(t)\) density of \(T_{\alpha,k-1}\), to obtain (3.3). The Markov chain is otherwise evident from the exact equality statement.
Corollary 3.1. As consequences of Proposition 3.1, the distribution of the quantities above with respect to a $PK_\alpha(h \cdot f_\alpha)$ are given by (3.3) and specifying $T_{\alpha,0}$ to have density $h(t)f_\alpha(t)$.

(i) In particular, the joint law of $(V_1, \ldots, V_r, T_{\alpha,r})$ is given by,

\[
\left[ \prod_{k=1}^{r} f_{B_k}(v_k) \right] h(s) \prod_{i=1}^{r} \frac{1}{v_i^{1/\alpha}} f_{\alpha,r}(s) ds \tag{3.6}
\]

where $f_{B_k}$ denotes the density of a Beta($\frac{\alpha+k-1}{\alpha}, \frac{1-\alpha}{\alpha}$) variable. $f_{\alpha,r}(s) = c_{\alpha,r}s^{-\alpha}f_\alpha(s)$.

(ii) It follows that the conditional distribution of $T_{\alpha,r}|V_1, \ldots, V_r$ is proportional to $h(s)\prod_{i=1}^{r} v_i^{1/\alpha} f_{\alpha,r}(s)$.

(iii) Relative to $\tilde{T}_{\alpha,r}$, for each $j = 1, 2, \ldots, r$

\[
T_{\alpha,j-1} = T_{\alpha,r} \times \prod_{l=j}^{r} V_l
\]

Remark 3.1. The fact that the quantity in (3.6) integrates to 1, follows from the identity (3.3). Which reads as

\[
E_{\alpha,0}[h(S_\alpha)] = E_{\alpha,0}[h(S_{\alpha,r} \times \prod_{i=1}^{r} B_{\left(\frac{1}{\alpha}, \frac{1-\alpha}{\alpha}\right)})] = 1.
\]

Remark 3.2. Comparing (3.3) with Haas, Miermont, Pitman and Winkel [24], Proposition 18, (ii), (iii)] shows that under a PD($\alpha,1-\alpha$) model, where for each $k = 1, 2, \ldots; T_{\alpha,k-1} d S_{\alpha,k-1}, T_{\alpha,k-1}^{-\alpha}$ equates to the total length of $\bar{R}_{k}^{ord}$, say $D(\bar{R}_k^{ord}) = T_{\alpha,k-1}^{-\alpha}$. Where $\bar{R}_k^{ord}$ is a member of an increasing family ($\bar{R}_k^{ord}$) of leaf-labeled $\mathbb{R}$-trees with edge lengths, arising as limits in Ford’s sequential construction. It follows from (3.2) that, in this setting, $(V_k)$ can be interpreted as

\[
V_k = \frac{D(\bar{R}_{k+1}^{ord})}{D(\bar{R}_k^{ord})} d B_{\left(\frac{1-\alpha}{\alpha}, 1-\alpha\right)},
\]

which is independent of $D(\bar{R}_{k+1}^{ord}) = T_{\alpha,k}^{-\alpha} d S_{\alpha,k+1}^{-\alpha}$. In fact $(V_1, \ldots, V_k)$ are mutually independent and independent of $D(\bar{R}_{k+1}^{ord})$. See [24] for a more precise interpretation of ($\bar{R}_k^{ord}$). See also [24] for related discussions involving fragmentation by PD($\alpha,1-\alpha$) models.

Remark 3.3. Note there are other distributions besides PD($\alpha,\theta$) that may produce the same sequences $(S_{\alpha,\theta,t+1}^{\alpha})_{t \geq 0}$. From a point of view of wide applicability of our results, this is rather fortunate. A key word in our exposition is version. The explicit constructions via bridges in [37] or the analysis of [26], in the PD($\alpha,1-\alpha$) case, already verifies the existence of the appropriate versions of variables we identify via Corollary 3.1 with respect to a sequence of mass partitions $((P_{k,r})_{k \geq 1})_{t \geq 0}$, following a sequence of laws determined by $PK_\alpha(h \cdot f_\alpha)$.
In the next section we will work with characterizing features of such families. Namely nested random partitions of \([n]\) derived from the appropriate Chinese restaurant processes.

4. PD(\(\alpha, \theta\)) nested Chinese restaurant processes

For any fixed \(r \geq 0\), set \((P_{k,r})_{k \geq 1} \sim\) PD(\(\alpha, \theta + r\)), and independent of this let \((U_{k,r})_{k \geq 1}\) be a collection of iid Uniform[0,1] random variables. Then the random probability measure \(P_{\alpha,\theta+r}(y) := \sum_{k=1}^{\infty} P_{k,r}(U_{k,r} \leq y)\) is a PD(\(\alpha, \theta + r\))-bridge, also known as a Pitman-Yor process as coined in [29]. For \(j = 1, \ldots, r\) one may set \(B_j = S_{\alpha,\theta+j-1/\alpha}^{-\alpha} / S_{\alpha,\theta+j}^{-\alpha}\), which are independent Beta(\(\theta + r + 1, 1 - \alpha\)) variables independent of \(P_{\alpha,\theta+r}\). Let \(\mathcal{U}(y) = y \in [0,1]\) denote a Uniform[0,1] cdf. Now for each \(j\) define independent simple bridges

\[
\lambda_j(y) = B_j \mathcal{U}(y) + (1 - B_j) \mathcal{I}(\hat{U}_j \leq y)
\]  

Then the coagulation operation in [21] can be encoded by the compositional identity, for each \(r \geq 1\),

\[
P_{\alpha,\theta+r-1}(y) = P_{\alpha,\theta+r}(\lambda_r(y)) = P_{\alpha,\theta+r}(B_r y) + P_{\alpha,\theta+r}(1 - B_r) \mathcal{I}(\hat{U}_r \leq y)
\]

where \(P_{\alpha,\theta+r}(1 - B_r)\) has a Beta(\(1 - \alpha, \theta + \alpha + r - 1\)) distribution. Furthermore, one can show that \(P_{\alpha,\theta+r}(B_r y) / P_{\alpha,\theta+r}(B_r) = P_{\alpha,\theta+r}(y)\) independent of \(P_{\alpha,\theta+r}(B_r)\). More generally, for any \(r \geq 1\),

\[
P_{\alpha,\theta}(\cdot) = P_{\alpha,\theta+r} \circ \lambda_r \circ \cdots \circ \lambda_1(\cdot).
\]

It follows that if \(F^{-1}\) denotes a possibly random quantile function, then for every \(r \geq 1\)

\[
P_{\alpha,\theta+r}^{-1}(\cdot) = \lambda_1^{-1} \circ \cdots \circ \lambda_r^{-1} \circ P_{\alpha,\theta+r}^{-1}(\cdot).
\]

We shall use these properties to construct nested sequences of Chinese restaurant process partitions of \([n]\) = \{1, 2, \ldots, n\}. Note a dual fragmentation process can be deduced from [21] which will produce nested partitions with the same distributions in reverse order. For a Chinese restaurant process following a PD(\(\alpha, \theta + r\)) distribution, the sampling scheme proceeds as follows. The first customer with index \(\{1\}\) is seated to a new table \(A_{1,r}\). After \(n\) customers arrive in succession, a partition of \([n]\), \((A_{1,r}, \ldots, A_{K_{n,r},r})\) where \(K_{n,r} \leq n\) are the number of distinct blocks in the partition, and \(N_{i,r} = |A_{i,r}|\) are the sizes of each block, is produced. Given this configuration, customer \(\{n+1\}\) is seated to a new table with probability \((\theta + r + K_{n,r} \cdot \alpha)/(\theta + r + n)\) and sits at an existing table \(A_{i,r}\) with probability \((N_{i,r} - \alpha)/(\theta + r + n)\), for \(i = 1, \ldots, K_{n,r}\). We now describe the combinatorial scheme we have in mind.

**Nested PD(\(\alpha, \theta\)) partitions of \([n]\)**

(i) For any \(r \geq 1\), draw a random partition of \([n]\), \((A_{1,r}, \ldots, A_{K_{n,r},r})\) from a PD(\(\alpha, \theta + r\)) Chinese restaurant process scheme.
(ii) Draw \((U^*_1, \ldots, U^*_K)\) iid Uniform\([0,1]\) variables.

(iii) Recall that \(\tilde{U}_r\) is the atom of \(\lambda_r\), and has a Uniform\([0,1]\) distribution. A
PD\((\alpha, \theta + r - 1)\) partition of \([n]\) \((A_{1,r-1}, \ldots, A_{K_{n,r-1},r-1})\), is obtained as
follows. Blocks of \((A_{1,r}, \ldots, A_{K_{n,r},r})\) are merged into a set \(A'_{1,r-1}\) defined as
\[
A'_{1,r-1} = \{A_i : \lambda^{-1}_r(U^*_i) = \tilde{U}_r\},
\]
if \(A'_{1,r-1}\) is not empty, set \(A_{1,r-1} = A'_{1,r-1}\), the remaining \(K_{n,r} - |A'_{1,r-1}| = K_{n,r-1} - 1\) blocks of \((A_{1,r}, \ldots, A_{K_{n,r},r})\), are relabeled \(A_{2,r-1}, \ldots, A_{K_{n,r-1},r-1}\).
If \(A'_{1,r-1} = \emptyset\), \(K_{n,r-1} = K_{n,r}\) and one sets \(A_{k,r-1} = A_{k,r}\) for \(k = 1, \ldots, K_{n,r}\).

(iv) Repeat steps [(ii)] and [(iii)] for \(r-1, r-2, \ldots, 1\) to obtain nested partitions
of \([n]\) following PD\((\alpha, \theta + j)\) marginal distributions for \(j = 0, \ldots, r\).

**Remark 4.1.** Kuba and Panholzer [33, Proposition 3] point out that partitions
generated by the PD\((\alpha, \theta)\) Chinese restaurant can be equally generated by
the growth process of generalized plane-oriented recursive trees. As such, some vari-
ation of our scheme can be used to produce nested version of such trees.

**Remark 4.2.** Note in the general PK\(_\alpha(h \cdot f_\alpha)\) setting, one would use simple
bridges defined as
\[
\lambda_\alpha(y) = V_k U(y) + (1 - V_k) I_{\{U \leq y\}}
\]
for \(V_k = T^{-\alpha}_{\alpha,k-1} / T^{-\alpha}_{\alpha,k}\). These are the same entities subject to a change of mea-
Sure, where generally independence no longer holds.

**Remark 4.3.** It is a simple matter to show that PD\(_{\alpha, \theta + r}()\) converges almost
Surely to \(U()\) as \(r \to \infty\). Thus implying that \(\lambda_r \circ \cdots \circ \lambda_1()\) converges almost
Sure to \(P_{\alpha, \theta}\). See James [33], Section 6.4 and Proposition 6.6] for distributional
results related to the composition of bridges \(\lambda_r \circ \cdots \circ \lambda_1()\).

### 4.1. Mixed Binomial distributions, \(\bar{\eta}\)-mergers and \(\beta\)-splitting

Note the nested scheme described above provides nested versions of all the
statistics generally associated with random partitions, and appropriate limits.
For brevity we shall only concentrate on results for the sequence of the number of
blocks \((K_{n,r})_{r \geq 0}\). Let Bin\((m, p)\) denote a Binomial distribution based on
\(m\) Bernoulli trials with success probability \(p\). We now describe results for \(K_{n,r}\).
The first result is immediate from the description of the nested PD\((\alpha, \theta)\) scheme.

**Proposition 4.1.** For every \(r \geq 1\), consider the the blocks \((K_{n,0}, \ldots, K_{n,r})\)
produced by a nested PD\((\alpha, \theta)\) scheme. It follows that for each \(n\) \(K_{n,j-1} \leq K_{n,j}\)
for \(j = 1, \ldots, r\) with properties;

(i) For \(j = 0, \ldots, r\) the marginal distribution of each \(K_{n,j}\) is exactly that of
the number of blocks of a PD\((\alpha, \theta + j)\) partition of \([n]\). For \(j = 0, \ldots, r-1\)
\[
K_{n,j} = (K_{n,j+1} - |A'_{i,j}| + 1)I_{\{|A'_{i,j}| \geq 2\}} + K_{n,j+1}I_{\{|A'_{i,j}| \in \{0,1\}\}}
\]
(ii) For each \( j \) the conditional distribution of \( |A'_{1,j-1}| \) given \( (K_{n,j}, B_j) \) is 
\[ \text{Bin}(K_{n,j}, 1 - B_j) \]

Note that Proposition 4.1 shows that in step[(iii)] of the nested PD(\( \alpha, \theta \)) scheme one is repeatedly performing some sort of \( \tilde{\rho}_{\alpha, \theta + j} = 1 - B_j \) merger in the language of Berestycki [7, p. 69-70]. That is \( \ell \) of the \{\( A_{1,j}, \ldots, A_{K_n,j} \)\}, blocks are said to coalesce if \( |A'_{1,j-1}| = \ell \geq 2 \). The next results, which follow from elementary calculations, describes some more details about the distributions of \( |A'_{1,j-1}| \) and random variables \((\xi_{n,0}, \xi_{n,1}, \ldots, \xi_{n,r})\) we define as follows. Set

\[
\xi_{n,0} = K_{n,0}I_{|A'_{1,0}| \geq 2} = (K_{n,1} - |A'_{1,0}| + 1)I_{|A'_{1,0}| \geq 2},
\]

and for \( j = 1, \ldots, r \), define,

\[
\xi_{n,j} := K_{n,j} - K_{n,j-1} = (|A'_{1,j-1}| - 1)I_{|A'_{1,j-1}| \geq 2}
\]

As the notation suggests these are meant to be thought of as mimics for degree sequences. Write the Beta\((\frac{1-\alpha}{\alpha}, \frac{\theta + \alpha}{\alpha})\) density,

\[
\rho_{\alpha, \theta}(v) = \frac{\Gamma\left(\frac{1+\theta}{2}\right)}{\Gamma\left(\frac{2+\alpha}{2}\right)\Gamma\left(\frac{1-\alpha}{2}\right)}v^{1/2}(1 - v)^{\theta/\alpha}.
\]

**Proposition 4.2.** In the PD(\( \alpha, \theta \)) setting of Proposition 4.1, the general distribution of \( |A'_{1,0}| \), given \( K_{n,1} = b \) is a mixed Binomial distribution \( \text{Bin}(b, 1 - B_1) \), where \( 1 - B_1 \) is a Beta\((\frac{1-\alpha}{\alpha}, \frac{\theta + \alpha}{\alpha})\) random variable with density function \( \rho_{\alpha, \theta}(v) \). Hence the probability mass function of \( |A'_{1,0}| \), is

\[
p_{\alpha, \theta}(\ell | b) = \binom{b}{\ell} \frac{\Gamma\left(\frac{1+\theta}{2}\right)\Gamma\left(\frac{\theta + \alpha}{2} + b - \ell\right)\Gamma\left(\frac{1}{2} + \ell - 1\right)}{\Gamma\left(\frac{\theta + \alpha}{2}\right)\Gamma\left(\frac{1-\alpha}{2}\right)\Gamma\left(\frac{1+\theta}{2} + b\right)}
\]

Furthermore the distribution of \( |A'_{1,0}| \), given \( K_{n,1} = b, |A'_{1,0}| \geq 2 \), is for \( 2 \leq \ell \leq b \),

\[
\lambda_{\alpha, \theta}(\ell | b) = \frac{p_{\alpha, \theta}(\ell | b)}{1 - p_{\alpha, \theta}(0 | b) - p_{\alpha, \theta}(1 | b)}
\]

The corresponding conditional distributions of the random variable \( K_{n,1} - |A'_{1,0}| \) can be expressed as \( p_{\alpha, \theta}^+(\ell | b) := p_{\alpha, \theta}(b - \ell | b) \) and \( \lambda^+_{\alpha, \theta}(\ell | b) := \lambda_{\alpha, \theta}(b - \ell | b) \) respectively. Replace \( \theta \) with \( \theta + j - 1 \) to obtain corresponding results for \( |A'_{1,j-1}| \), given \( K_{n,j} \)

(i) In the Brownian cases, PD(1/2, \( \theta \)), \( \theta > -1/2 \),

\[
p_{1/2, \theta}(\ell | b) = \frac{(2\theta + 1)(2\theta + b - \ell)!b!}{(2\theta + b + 1)!b!(b - \ell)!}
\]

In particular \( p_{1/2, 0}(0 | b) = 1/(b+1) \) is the discrete uniform distribution on \( \{0, \ldots, b\} \), and \( p_{1/2, 1/2}(\ell | b) = 2(b + 1 - \ell)/[(b + 1)(b + 2)] \).
(ii) In the limiting Dirichlet case, PD(0, θ),

$$p_{0,\theta}(\ell|b) = \binom{b}{\ell} p_{\theta}^{\ell}(1 - p_{\theta})^{b-\ell}$$

is a proper Binomial distribution with success probability $p_{\theta} = 1/(\theta + 1)$, for $\theta > 0$.

Note starting from some PD($\alpha, \theta + r$) it follows that every layer of our PD($\alpha, \theta$) nested scheme produces proper consistent infinitely exchangeable partitions in $[n]$. As such, one may view our scheme, as discrete time coalescent process based on a sequence of merger rates determined by measures $(\Lambda_{\alpha,\theta+r})_{r \geq 0}$, where from $\lambda_{\alpha,\theta}(\ell|b)$, in $[12]$,

$$\Lambda_{\alpha,\theta}(dv) = \frac{\Gamma(\frac{1+\theta}{\alpha})}{\Gamma(\frac{2+\theta}{\alpha})\Gamma(\frac{\theta}{\alpha})} v^{1/\alpha}(1 - v)^{\theta/\alpha} dv.$$  

For a fixed time, these schemes are suggestive of relations to a class of $\Lambda_{\alpha,\theta}$-coalescents where $\Lambda_{\alpha,\theta}$ corresponds to a Beta($\frac{1+\theta}{\alpha}, \frac{\theta}{\alpha}$)-coalescent. However our models should not be confused with such processes. Rather, our models are interpreted in terms of renewal sequences in $\alpha,\theta$.

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(1 – B1) \sim \text{Beta}\left(\frac{1-\alpha}{\alpha}, \frac{1-\alpha}{\alpha}\right)\. Where for 2 \leq \ell \leq b,

\lambda_{\alpha,1-2\alpha}(\ell|b) \propto p_{\alpha,1-2\alpha}(\ell|b) = \binom{b}{\ell} \frac{\Gamma\left(\frac{2-2\alpha}{\alpha}\right)\Gamma\left(\frac{1-\alpha}{\alpha} + \frac{\ell}{b} + \ell - 1\right)}{\Gamma\left(\frac{1-\alpha}{\alpha}\right)\Gamma\left(\frac{2-2\alpha}{\alpha} + b\right)} .

(i) Setting \beta\alpha = 1 – 2\alpha > -\alpha it follows that

p_{\alpha,1-2\alpha}(\ell|b) \propto \tilde{q}_{\alpha,1-2\alpha}(\beta, \ell) for 1 \leq \ell \leq b - 1, equating to the splitting kernel in the \beta\text{-splitting model of Aldous}\cite{Aldous89b} for the range \beta > -1. Here we use the notation in \cite[p.1824]{Aldous89b},

(ii) Hence, the Yule model case of \beta = 0 corresponds to a PD\left(\frac{1}{2}, 0\right) model with p_{1/2,0}(\ell|b) = 1/(b + 1). The symmetric random trie case \beta \to \infty corresponds to a PD\left(0, 1\right) model with p_{0,1}(\ell|b) = \binom{b}{\ell} (1/2)^{b}. In particular,

\tilde{q}_{\alpha,1-2\alpha}(0) = \frac{1}{b - 1} and \tilde{q}_{\alpha,1-2\alpha}(\infty) = \binom{b}{\ell} \frac{1}{2^{b - 2}} .

Remark 4.4. For more on \Lambda\text{-coalescents see} \cite{Aldous99, MIM99, MIM01}.

Remark 4.5. See \cite{Aldous99, MIM99, MIM01} for more on the \beta\text{-splitting model and its connection to fragmentation trees. In addition See} \cite[Section 3.3, and Proposition 27]{Aldous99}, \cite[Proposition 18]{Aldous99} and \cite[p. 1738]{Aldous99} which describe relations to the \alpha\text{-model of Ford}\cite{Ford03} and the Brownian CRT of Aldous\cite{Aldous99, Aldous99a}.

4.2. Calculations for general PK_{\alpha}(h \cdot f_{\alpha}) nested Gibbs partition schemes

As we have discussed above, viewing (S_{\alpha,\theta+r})(t \geq 0) in reverse order, as times where mergers occur gives the formalism to recognize our nested scheme as continuous time coalescent processes based on a sequence of merger determining measures (\Lambda_{\alpha,\theta+r})(t \geq 0), that produce, at each merger time, consistent exchangeable partitions of \{n\}. Conditioning on \{n\}, gives in a distributional, rather than operational, sense the relevant quantities emerging under PD\left(\alpha(t|t)\right). schemes. Operationally, since consistent families of Chinese restaurant partitions are produced under this setting, one carries out the same nested scheme under a PD\left(\alpha(t|t)\right) distribution with time points \alpha,0 \sim t\sim \alpha and (T_{\alpha,r})(t \geq 0). PK_{\alpha}(h \cdot f_{\alpha}) nested schemes proceed in a similar fashion. Thus all producing coalescent schemes which are based on sequences of merger rate determining distributions (1 – V_k) whose distributions generally depend on the number of the blocks considered at the time where mergers are to occur. The exception to this are the PD\left(\alpha, \theta\right) distributions, which represents the only instances where the pairs (V_k, K_{n,k}) are independent. We now describe more explicit details of these distributions.
where, as in \cite{28, Theorem 4.1} suppressing dependence on $h$ pressed as Proposition 4.4. See also Gnedin and Pitman \cite{24} for other representations. The next result follows from the observations of Pitman \cite{43} for the case of $\alpha$ taking on fractional values. Thus extending \cite{43} for the case of $\alpha = 1/2$. Furthermore, they show that $\mathcal{G}_\alpha^{(n,b)}(t) = (\Gamma(n - ba) f_\alpha(t))^{-1} \alpha^b t^{-n} \int_0^t f_\alpha(t - v)(1 - v)^{n - ba - 1} dv$. While the expression for $\mathcal{G}_\alpha^{(n,b)}(t)$, seems to be generally intractable, the work of Ho, James and Lau \cite{28} shows that this quantity may be expressed in terms of special functions corresponding to Fox-H functions in the general $\alpha$ setting and Meijer-G functions in the case of $\alpha$ setting and $\alpha$-EPPF of a PD\((\alpha|t)\) distribution its EPPF, and distribution of $K_n$ under PD\((\alpha,0)\) given in \cite{44}. Hence

\begin{equation}
\Sigma_{\alpha,n}(ba) \overset{d} = S_{\alpha,ba} B_{\alpha,b \times -ba}^{-1} = S_{\alpha,n} B_{\alpha,b \times -ba}^{-1/\alpha},
\end{equation}

which interprets as the conditional density of $S_{\alpha,0}|K_n = b$ when the pair $(S_{\alpha,0}, K_n) := (S_{\alpha,0}, K_n, 0)$ are interpreted with respect to a PD\((\alpha,0)\) distribution. Note the identity on the right hand side of \cite{10} follows from \cite{40}, eq. (2.11)\cite{20}. This leads to the identity $f_\alpha(t) = \sum_{b=1}^n \mathbb{P}_{\alpha,0}(K_n = b) f_{\alpha,(n,b)}(t)$, where $\mathbb{P}_{\alpha,0}(K_n = b)$, denotes the distribution of $K_n$ under PD\((\alpha,0)\) given in \cite{44} or otherwise deduced from \cite{15}. Hence

\begin{equation}
p_\alpha(n_1, \ldots, n_b|s) = \frac{f_{\alpha,(n,b)}(s)}{f_\alpha(s)} p_{\alpha,0}(n_1, \ldots, n_b)
\end{equation}

where $p_{\alpha,0}(n_1, \ldots, n_b)$ is the EPPF under PD\((\alpha,0)\), and

\begin{equation}
\mathbb{P}_\alpha(K_{n,0} = b|S_{\alpha,0} = s) = \frac{f_{\alpha,(n,b)}(s)}{f_\alpha(s)} \mathbb{P}_{\alpha,0}(K_n = b)
\end{equation}

It follows that by integrating \cite{44} with respect to $h(s) f_\alpha(s)$, that for a general $\text{PK}_\alpha(h \cdot f_\alpha)$ distribution its EPPF, and distribution of $K_n$, can be represented as

\begin{equation}
p_\alpha^{(h)}(n_b) = W_{n,b} \times p_{\alpha,0}(n_b) \text{ and } \mathbb{P}_\alpha^{(h)}(K_n = b) = W_{n,b} \times \mathbb{P}_{\alpha,0}(K_n = b)
\end{equation}

where, as in \cite{28} Theorem 4.1 suppressing dependence on $h$, $W_{n,b}$ can be expressed as

\begin{equation}
W_{n,b} = \mathbb{E}_{\alpha,0}[h(S_{\alpha,0})|K_{n,0} = b] = \mathbb{E}[h(S_{\alpha,n}(ba))].
\end{equation}

See also Gnedin and Pitman \cite{24} for other representations. The next result follows from our discussion.

**Proposition 4.4.** Suppose that the law of $(K_{n,0}, T_{\alpha,0})$ is determined by $\text{PK}_\alpha(h \cdot f_\alpha)$. Then,
(i) the joint distribution of \((K_{n,0}, T_{\alpha,0})\) may be expressed as
\[
\mathbb{P}_{\alpha,0}(K_n = b) \frac{f_{\alpha,(n,b)}(s)}{f_{\alpha}(s)} \times h(s)f_{\alpha}(s).
\]

(ii) Hence by Bayes rule the conditional distribution of \(T_{\alpha,0} | K_{n,0} = b\) is given by
\[
f_{T_{\alpha,0} | K_{n,0}}^{(h)}(t|b) = W_{n,b}^{-1} h(s)f_{\alpha,(n,b)}(s).
\]

(iii) Statement [(ii)] implies that for any integrable function \(g\)
\[
W_{n,b}[g(T_{\alpha,0})|K_{n,0} = b] = \mathbb{E}_{\alpha,0}[g(S_{\alpha,0})h(S_{\alpha,0})|K_{n,0} = b]
\]
where \(\mathbb{E}_{\alpha,0}\) denotes expectation relative to the distribution of \((K_{n,0}, S_{\alpha,0})\) defined under a PD(\(\alpha, 0\)) distribution. This may be also expressed as
\[
\mathbb{E}_{\alpha}^{(h)}[g(T_{\alpha,0})|K_{n,0} = b] = \mathbb{E}[g(\Sigma_{\alpha,n}(ba))h(\Sigma_{\alpha,n}(ba))]W_{n,b}^{-1}
\]

Proof. As mentioned, the first two statements are just simple consequences of our description of the conditional EPPF. Statement [(iii)] follows by a simple manipulation of \(s^{-\theta}f_{\alpha,(n,b)}(s)\).

We now specialize to the important PD(\(\alpha, \theta\)) case. This result was originally deduced by [28].

**Proposition 4.5.** Suppose that the law of \((K_{n,0}, T_{\alpha,0})\) are determined by PD(\(\alpha, \theta\)) case where \(h(s) = s^{-\theta}c_{\alpha,\theta}\). Hence one may set \(T_{\alpha,0} = S_{\alpha,\theta}\). It follows that
\[
W_{n,b} = c_{\alpha,\theta}\mathbb{E}[\Sigma_{\alpha,n}^{-\theta}(ba)] = \frac{\Gamma(n)\Gamma(\theta + 1)\Gamma(\theta + ba)}{\Gamma(\theta + n)\Gamma(b)\Gamma(\theta + \alpha)}
\]
and \(W_{n,b}^{-1}s^{-\theta}c_{\alpha,\theta}f_{\alpha,(n,b)}(s)\) corresponds to the conditional density of the random variable \(S_{\alpha,\theta}|K_{n,0} = b\), which is equivalent in distribution to,
\[
\Sigma_{\alpha,n}(\theta + ba) \overset{d}{=} S_{\alpha,\theta+ba}B_{\theta+ba,n-ba}^{-1/\alpha} \overset{d}{=} S_{\alpha,n+\theta}B_{\alpha+b,\alpha-b}^{-1/\alpha}
\]

We now present the main result in this section which follows from these facts.

**Proposition 4.6.** Consider the variables described in Proposotion [4.1](T_{\alpha,r}^{-\alpha})\{r \geq 0\}, and consider the nested PD(\(\alpha, \theta\)) scheme where mergers and hence consistent partitions of \([n]\) occur at time points \(S_{\alpha,\theta+r}\), for \(r \) decreasing. Then conditioning on \(S_{\alpha,\theta} = t\), the law of the nested schemes are determined by a PD(\(\alpha|t\)) distribution, and in fact are equivalent in (conditional) distribution to a nested family of partitions that was initially constructed from any PK\(_{\alpha}(h \cdot f_{\alpha})\) distribution. Operationally, under PD(\(\alpha|t\)) the consistent family of nested exchangeable partitions can be generated for any \(r \geq 1\) by generating a Chinese restaurant process partition of \([n]\) according to the Poisson Kingman law defined by mixing PD(\(\alpha|s\)) relative to the density of \(T_{\alpha,\theta} | T_{\alpha,0} = t\), which can be determined from Proposition [4.7]. Denote this density as \(f_{\alpha,r}^{(c)}(s) = h_r(s|t)f_{\alpha}(s)\) and the
corresponding Poisson Kingman law as $PK_n(f_{α,r}^{(t)})$. This represents the configuration of the partition of $[n]$ at the time $T_n^α$. Mergers in the past occur at times $(T_{n-1}^α, \ldots, T_0^α)$ where $T_n^α = t$, producing partitions of $[n]$ with marginal laws determined by a $PK_n(f_{α,r}^{(t)})$ EPPF for $j = r - 1, \ldots, 0$. The distributional properties of the relevant quantities in Proposition 4.1 are now described.

(i) For each $r \geq 0$, the conditional distribution of $K_{n,r}$ given $(T_{α,r}, \ldots, T_{α,0})$ only depends on $T_{α,r}$ and is given by

$$\mathbb{P}_α(K_{n,r} = b|T_{α,r} = s) = \frac{f_{α,(n,b)}(s)}{f_α(s)}\mathbb{P}_{α,0}(K_n = b).$$

(ii) $|A_{j,j-1}^i|$ given $(K_{n,j}, V_j, T_{α,j-1}, \ldots, T_{α,0})$ has a Binomial$(K_{n,j}, 1 - V_j)$ distribution.

(iii) For any integer $j$, the distribution of $(K_{n,j}, V_j)|(T_{α,j-1} = s, \ldots, T_{α,0} = t_0)$ only depends on $T_{α,j-1} = s$ and is the same in form as the case where $j = 1$, with $T_{α,0} = s$.

(iv) From [(i)] and using [4.3], the joint distribution of $(K_{n,1} = b, V_1)|T_{α,0} = t$ is given by

$$\mathbb{P}_{α,0}(K_n = b)\frac{f_{α,(n,b)}(tv^{1/α})}{f_α(tv^{1/α})} \times \frac{α}{Γ(1-α)} \left(1 - v\right)^{(1-α)-1} f_α(v^{1/α}) t f_α(t).$$

which reduces to

$$\mathbb{P}_{α,0}(K_n = b)\frac{α}{Γ(1-α)} \left(1 - v\right)^{(1-α)-1} \frac{f_α,(n,b)(tv^{1/α})}{f_α(t)}.\frac{tf_α(t)}{}.$$  

(v) Noting, from Proposition 4.4, that $t^{-1} f_{α,(n,b)}(t)/\mathbb{E}[\Sigma_{α,n}^{-1}(ba)]$ is the density of $\Sigma_{α,n}(1 + ba)$ it follows that

$$\frac{αt^{-1}}{1-α} \int_0^1 \left(1 - v\right)^{(1-α)-1} f_{α,(n,b)}(tv^{1/α}) dv = \mathbb{E}[\Sigma_{α,n}^{-1}(ba)] f_{Y_{α,n}(b)}(t)$$

where $f_{Y_{α,n}(b)}(t)$ denotes the density of the random variable,

$$Y_{α,n}(b) \overset{d}{=} B_{1-\frac{1}{α-1}}^{-1/α}(1 + ba).$$

(vi) $Y_{α,n}(b)$ is the random variable corresponding to the conditional distribution of $S_{α,0}|K_{n,1} = b$, where $(S_{α,0}, K_{n,0}, K_{n,1})$ follow a PD$(α,0)$ distribution, and otherwise $S_{α,0} = S_{α,1} B_{1-\frac{1}{α-1}}^{-1/α}$.

(vii) It follows that the conditional distribution of $K_{n,1}|T_{α,0} = t$ is given by

$$\mathbb{P}(K_{n,1} = b|T_{α,0} = t) = \mathbb{P}_{α,0}(K_n = b) \frac{(1-α)\mathbb{E}[\Sigma_{α,n}^{-1}(ba)] f_{Y_{α,n}(b)}(t)}{Γ(\frac{1-α}{α}) f_α(t)}.$$
(viii) Hence, the conditional density of \((1 - V_1)|K_{n,1} = b, T_{\alpha,0} = t,\) is given by, for \(0 < p < 1,\)

\[
p^{-2} \Lambda_\alpha(dp|b,t) = \frac{t^{-1} \alpha^\left(\frac{k - 1}{\alpha}\right) - 1}{(1 - \alpha)} \frac{f_{\alpha,n}(b) f_{\alpha,n}(b)}{E[(\Sigma_{n,1}(b)) f_{\alpha,n}(b)(t)]} dp,
\]

which is the same as the distribution of \((1 - V_k)|K_{n,k} = b, T_{\alpha,k-1} = t,\) for each \(k \geq 1.\)

(ix) It follows that the distribution of \(|A_{1,0}'|\) given \(K_{n,1} = b, T_{\alpha,0} = t\) is specified by its probability mass function

\[
P_\alpha(|A_{1,0}'| = l|K_{n,1} = b, T_{\alpha,0} = t) = \left(\frac{b}{t}\right) \int_0^1 p^{t-2} (1 - p)^{b-l} \Lambda_\alpha(dp|b,t).
\]

We close this section with the following corollary.

**Corollary 4.1.** Suppose that \(T_{\alpha,0}\) has density \(h(t)f_\alpha(t)\) then,

(i) the conditional density of \(T_{\alpha,0}|K_{n,1} = b\) is

\[
f_{\alpha,K_{n,1}}(t|b) = \frac{h(t)f_{\alpha,n}(b)}{E[h(Y_{\alpha,n}(b))]}.
\]

(ii) It follows that the conditional density of \((1 - V_1)|K_{n,1} = b\) is given by, for \(0 < p < 1,\)

\[
p^{-2} \Lambda_\alpha(dp|b) = \frac{\alpha^{(1-\alpha)/\alpha} - 1}{(1 - \alpha)} \frac{E[h(\Sigma_{n,1}(1 + b\alpha)(1 - p)^{-1/\alpha})]}{E[h(Y_{\alpha,n}(b))]} dp.
\]

(iii) The conditional density of \((1 - V_1)|K_{n,1} = b\) can also be expressed as, for \(0 < p < 1,\)

\[
p^{-2} \Lambda_\alpha(dp|b) = \frac{\alpha^{(1-\alpha)/\alpha} E_{\alpha,0}[h(S_{\alpha,1}(1 - p)^{-1/\alpha})|K_{n,1} = b]}{E_{\alpha,0}[h(S_{\alpha,0})|K_{n,1} = b]} dp.
\]

(iv) Hence when \(h(t) = t^{-\theta} c_{\alpha,\theta},\) corresponding to the PD(\(\alpha, \theta\)) case, the conditional distribution of \((1 - V_1)|K_{n,1} = b\) has a Beta\(\left(\frac{1-\alpha}{\alpha}, \frac{\theta}{\alpha}\right)\) distribution independent of \(K_{n,1}.\)

### 4.3. Limits of joint vectors under the nested PD(\(\alpha, \theta\)) scheme

Our constructions allow one to utilize well-known results to easily establish limit theorems. We shall present one such result here. We again note that these results hold in the more general PK_\alpha(h \cdot f_\alpha) setting

**Proposition 4.7.** Let \(\{S_{\alpha,\theta+1}\}_{r \geq 0}\) denote the PD(\(\alpha, \theta\)) sequence associated with the nested scheme. For each \(r \geq 0\) let \((K_{n,r}), (\xi_{n,0}, \ldots, \xi_{n,r})\) be as previously defined. Then
(i) As \( n \to \infty \), jointly and component-wise, for any \( r \geq 0 \)
\[
n^{-\alpha}(K_{n,0}, K_{n,1}, \ldots, K_{n,r}) \overset{a.s.}{\to} (S^{\alpha}_{\theta+1}, \ldots, S^{\alpha}_{\theta+r})
\]

(ii) As \( n \to \infty \), jointly and component-wise, for any \( r \geq 0 \)
\[
n^{-\alpha}(\xi_{n,0}, \xi_{n,1}, \ldots, \xi_{n,r}) \overset{a.s.}{\to} (\xi_0, \xi_1, \ldots, \xi_r)
\]

where \( \xi_0 = S^{\alpha}_{\theta+1} \) and \( \xi_j = S^{\alpha}_{\theta+j} - S^{\alpha}_{\theta+j-1} \), for \( j = 1, \ldots, r \).

(iii) One may replace \( \xi_{n,0} \) with \( K_{n,0} \)

Proof. It suffices to show [(i)] as the arguments for [(iii)] are similar. We first note the fact that \( n^{-\alpha} K_{n,r} \overset{a.s.}{\to} S^{\alpha}_{\theta+r} \). We may also consider the \((B_j)\) to be fixed. Using the notation \( X \overset{a.s.}{\sim} Y \) to mean \( X / Y \to 1 \) a.s., it follows from Proposition 4.11 that \( K_{n,r-1} \overset{a.s.}{\sim} Y_r \), where \( Y_r = \sum_{k=1}^{K_{n,r}} b_{k,r} \), for \( b_{k,r} \) iid Bernoulli\((B_r)\) variables. Dividing by \( K_{n,r} \) it follows that \( Y_r = K_{n,r} Y_r / K_{n,r} \), hence \( n^{-\alpha}(K_{n,r-1}, K_{n,r}) \overset{a.s.}{\sim} n^{-\alpha} K_{n,r}(Y_r, 1) \overset{a.s.}{\sim} S^{\alpha}_{\theta+r}(B_r, 1) \), which is \((S^{\alpha}_{\theta+r-1}, S^{\alpha}_{\theta+r})\). Continuing in this way it follows that \( K_{n,j-1} \overset{a.s.}{\sim} K_{n,r} \prod_{j=1}^{r} Y_j \) for \( j = 1, \ldots, r \). The result is concluded by applying the law of large numbers to a vector of Bernoulli sample means and utilizing the definition of \((S^{\alpha}_{\theta+r})\).

By taking \( \alpha \to 0 \) and known results for PD\((0, \theta)\) distributions we obtain the following corollary

**Corollary 4.2.** Set \( \alpha = 0 \) in Proposition 4.7. Then this yields results for a PD\((0, \theta)\) nested scheme as follows:

(i) As \( n \to \infty \), jointly and component-wise, for any \( r \geq 0 \)
\[
(K_{n,0}, K_{n,1}, \ldots, K_{n,r}) \overset{a.s.}{\sim} \log(n)(\theta, \theta + 1, \ldots, \theta + r)
\]

(ii) As \( n \to \infty \), jointly and component-wise, for any \( r \geq 0 \)
\[
(\xi_{n,0}, \xi_{n,1}, \ldots, \xi_{n,r}) \overset{a.s.}{\sim} \log(n)(\theta, 1, \ldots, 1)
\]

5. The PD\((1/2|t)\) case

We now describe all possible limits under any scheme that is asymptotically equivalent to our nested schemes based on \( PK_{1/2}(h \cdot f_{1/2}) \) distributions. This is done by looking at their common generator under the PD\((1/2|t)\) model. Note again that if \( K_n \) is the number of distinct blocks in a partition of \([n]\) following a PD\((1/2|t)\) distribution then \( n^{-1/2} K_n \overset{a.s.}{\sim} 1/\sqrt{t} \). See Pitman [43, Section 8] for a nice treatment in regards to the Brownian excursion partition. The simple properties of the PD\((1/2|t)\) distribution related to the Markov chain in Perman, Pitman and Yor [11] are also exploited in [1, 8, 43]. What we are doing here
is expressing properties of this distribution relative to operations corresponding to Proposition 5.1. Recall that $S_{1/2,0} \sim S_{1/2} \sim 1/(4G_{1/2})$, where $G_{1/2}$ is a Gamma$(1/2,1)$ variable. This means that

$$f_{1/2}(t) = \frac{1}{2\Gamma(1/2)}t^{-3/2}e^{-1/4t}$$

Throughout, let $(e_k)$ denote an iid sequence of exponential(1) variables.

**Proposition 5.1.** Consider the setting in Proposition 5.1 then the joint distribution of $(T_{1/2,k})_{\{k \geq 1\}}|T_{1/2,0} = t$ is described as follows.

(i) Setting $\alpha = 1/2$ in (3.3), for each $k \geq 1$, the conditional density of $T_{1/2,k}|T_{1/2,k-1} = t_{k-1}$ is given by

$$f_{T_{1/2,k}|T_{1/2,k-1}}(t_k|t_{k-1}) = \frac{1}{4}t_k^{-2}e^{-1/4t_k}e^{-1/4t_{k-1}} \text{ for } t_k < t_{k-1}.$$

(ii) It follows that $(T_{1/2,k})_{\{k \geq 1\}}|T_{1/2,0} = t$ correspond to the points of an inhomogeneous Poisson point process with intensity

$$\tau(s|t) = \frac{1}{4}s^{-2}I_{\{s< t\}} \quad (5.1)$$

(iii) Hence, conditional on $T_{1/2,0} = t$, for each $k \geq 1$, one can set

$$T_{1/2,k} = \frac{1}{4\sum_{\ell=1}^{k} e_{\ell} + 1/t}$$

(iv) From (5.1) it follows that $2^{-1/2}(T_{1/2,k}^{-1})_{\{k \geq 1\}}|T_{1/2,0} = t$ are the points of an inhomogeneous Poisson point process with intensity

$$\rho(y|t) = y^2I_{\{y > 1/\sqrt{2t}\}} \quad (5.2)$$

**Proof.** The results follow by the simplifications unique to the $\alpha = 1/2$ case via the density $f_{1/2}(t)$ as applied to the transition density in (3.3). \qed

**Remark 5.1.** Setting $C_1 = 1/\sqrt{2t}$, $C_{k+1} = 1/\sqrt{2T_{1/2,k}}$, for $k \geq 1$, the spacings $C_{k+1} - C_k$ can be interpreted as the distribution of the edge lengths in Aldous’s construction of the Brownian Random Tree (CRT) when conditioned on $C_1 = 1/\sqrt{2t}$. As is well known Aldous’s construction corresponds to randomizing in the PD$(1/2,1/2)$ case. However, using our explicit descriptions in Propositions 5.1, such a construction makes sense for any choice of $t$, yielding different trees.

We next describe some implications of Proposition 5.1.

**Proposition 5.2.** For $\alpha = 1/2$, consider the setting in Proposition 5.1. The initial mass partition $(P_{k,0})|T_{1/2,0} = t$ follows a PD$(1/2|t)$ distribution. A description of the corresponding variables is provided as follows. Given, $T_{1/2,0} = t$. 

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**Lancelot F. James/Mittag Leffler limits**

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(i) \(T^{-1/2}_{1,2} = \sqrt{4e_1 + 1/t}\) and for \(k \geq 2\),

\[
T^{-1/2}_{1,2,k} = \sqrt{\sum_{\ell=1}^{k} e_\ell + 1/t}
\]

(ii) \(V_1 = 1/\sqrt{4te_1 + 1}\), with density, for \(0 < u < 1\),

\[
f_{V_1}(u|t) = \frac{1}{2t}u^{-3}e^{-\frac{(u-x)^2}{4}}, \quad (5.3)
\]

(iii) For \(k \geq 2\),

\[
V_k = \frac{\sqrt{\sum_{\ell=1}^{k} e_\ell + 1/t}}{\sqrt{\sum_{\ell=1}^{k} e_\ell + 1/t}}
\]

(iv) For every \(r \geq 0\), there exist random vectors \((\xi_{n,0}, \xi_{n,1}, \ldots, \xi_{n,r})\) defined under a PD(1/2) partition scheme, as in (4.3) and (4.2) such that,

\[
n^{-1/2}(\xi_{n,0}, \xi_{n,1}, \ldots, \xi_{n,r}) \sim (\xi_0, \xi_1, \ldots, \xi_r)
\]

where \(\xi_0 = 1/\sqrt{t}\), \(\xi_1 = \sqrt{4e_1 + 1/t} - 1/\sqrt{t}\), and for \(k \geq 2\),

\[
\xi_k = \sqrt{\sum_{\ell=1}^{k} e_\ell + 1/t} - \sqrt{\sum_{\ell=1}^{k-1} e_\ell + 1/t}
\]

(v) Results for any PK\(_{1/2}(h \cdot f_{1/2})\) distribution are obtained by randomizing \(t\) with respect to \(h(t)f_{1/2}(t)\) which can be any non-negative distribution.

Remark 5.2. Notice that \(\xi_{n,0}\) is not fixed, rather \(n^{-1/2}\xi_{n,0} \sim 1/\sqrt{t}\).

Remark 5.3. It is now easy to see that the limits for the M and L preferential attachment models in (4.3) correspond to the case of PD(1/2, 0) and PD(1/2, 1/2), respectively.

With respect to the calculations in section 4.2, it follows that

\[
\Sigma_{1/2,2}^{-1}(1 + b/2) \overset{d}{=} 4G_{n+3/2}(2+b, 2n-b) \overset{d}{=} 4G_{(b+3)/2}(2+b, (2n-b)/2)
\]

and

\[
Y_{1/2,2}^{-1}(b) \overset{d}{=} B_{(1,1)}(1+b/2) \overset{d}{=} B_{1/2,2, n}(1 + b/2).
\]

From [28, section 3.1] we have that \(G_{1/2,2}^{1}(t) = (\Gamma(n)f_{1/2}(t))^{-1}\Gamma(k)2^{-k+1}f_{1/2, (n,k)}(t)\), is expressible in terms of a ratio of Meijer G functions as

\[
G_{2,1}^{0,2}(\begin{array}{c}
- \frac{1+k}{2}, - \frac{k}{2} \\
- n
\end{array} | 4t)
\]

\[
G_{1,0}^{0,1}(\begin{array}{c}
- \frac{k}{2} \\
- \frac{n}{2}
\end{array} | 4t) = \frac{(4t)^{-\frac{1+k}{2}}e^{-\frac{1+k}{2}}U(-\frac{k}{2} + n, \frac{3}{2}, \frac{1}{4})}{(4t)^{-\frac{k}{2}}e^{-\frac{k}{4}}},
\]
where \( U(a, b, c) \) is the confluent hypergeometric function of the second kind (see \[34, p.263\]). The above ratio reduces to

\[
2^{-k+1} t^{-\frac{k}{2} + \frac{1}{2}} U\left(-\frac{k}{2} - \frac{1}{2} + n, \frac{1}{2}, \frac{1}{4} t\right)
\]

via an application of the recurrence relation \[51, p.505\]

\[
U(a, b, z) = z^{1-b} U(1+a-b, 2-b, z).
\]

A change of variable \( t = \frac{\lambda}{2} - 2y \) yields the expression

\[
2^{n-k} \frac{\lambda}{2}^{k-1} h_{k+1-2n}(\lambda)
\]

inside equation (110) in Pitman \[43\], where \( h_{\nu}(\lambda) \) is the Hermite function of index \( \nu \) \[34, Sec. 10.2\], based on the following relationship,

\[
h_{\nu}(\lambda) = 2^{\nu/2} U\left(-\frac{\nu}{2} - \frac{1}{2}, \frac{\lambda^2}{2}\right).
\]

Hence from Proposition \[4.6\] it follows that the joint distribution of \((K_{n,1} = k, V_{1/2,0} = t)\), can be expressed as,

\[
P_{1/2,0}(K_n = k) \frac{t^{-(k+1)}}{2\Gamma(k)} \Gamma(n) U\left(-\frac{k}{2} - \frac{1}{2} + n, \frac{1}{2}, \frac{1}{4} tv^2\right).
\]

where, as in Pitman \[43\],

\[
P_{1/2,0}(K_n = k) = \binom{2n-k-1}{n} 2^{k+1-2n}.
\]

6. Nesting across \( \alpha \)

Recently, within the context of \( \kappa \)-stable trees, for \( 1 < \kappa \leq 2 \), Curien and Haas \[17\], showed that one can construct all the stable trees simultaneously as a nested family. In particular for \( 1 < \kappa < \kappa' \leq 2 \), they relate a \( \kappa \)-stable tree with a \( \kappa' \)-stable tree rescaled by an independent Mittag-Leffler type distribution. We suspect that hidden in such a story are operations induced by a PD(\( \alpha, -\alpha\delta \))–Frag operator as described in Pitman \[42, Theorem 12\], for some choice of \( 0 \leq \delta < 1 \) and \( \theta > -\alpha\delta \). Rather than pursue that, we describe how one can produce nested PD(\( \alpha\delta, \theta \)) partitions from PD(\( \alpha, \theta \)) nested schemes via the corresponding PD(\( \delta, \theta \))–COAG operator. See also \[27, Corollary 10\].

**Proposition 6.1.** Suppose that a sequence of nested local times \( (S_{\alpha, \theta+r})_{(r \geq 0)} \) correspond to the states of a Markov chain \( (PD(\alpha, \theta + r))_{(r \geq 0)} \) produced by a PD(\( \alpha, \theta \)) nested CRP scheme as described in section 4. Let \( (Q_{k,r})_{(k \geq 1)} \) \( (r \geq 0) \) denote a nested sequence of mass partitions following precisely a nested sequence of laws \( (PD(\delta, -\alpha\delta))_{(r \geq 0)} \) whose relations will be described more formally below, and where for each fixed \( r \), \( (Q_{k,r})_{(k \geq 1)} \) is independent of the corresponding PD(\( \alpha, \theta + r \)) mass partition. Let \( P_{\delta, \theta+r} \) denote corresponding bridges.
PD(αδ, θ) nested sequence of partitions of [n] with \( (S^α_{n,\delta, \theta+r})_{r \geq 0} \), corresponding to the states of a Markov Chain \( \{PD(\alpha, \theta + r)\}_{r \geq 0} \), can be obtained in a distributional sense from the PD(α, θ) nested CRP scheme by using the coagulation operation of Pitman [43] encoded in the composition of independent bridges \( P_{\alpha, \delta, \theta} = P_{\alpha, \delta, \theta} \circ P_{\delta, \theta} \), where \( P_{\delta, \theta} \) corresponds to the PD(δ, θ) – COAG operator. More precisely this is encoded in the ordered operation of coagulations described by the relations

\[
P_{\alpha, \delta, \theta}^{-1} = P_{\delta, \theta}^{-1} \circ P_{\alpha, \delta, \theta}^{-1} = P_{\delta, \theta}^{-1} \circ \lambda_{\alpha, \delta, \theta}^{-1} \circ \cdots \circ \lambda_{\alpha, \delta, \theta}^{-1} \circ P_{\alpha, \delta, \theta}^{-1}
\]

where the simple bridge \( \lambda_{\alpha, \delta, \theta}(y) := \lambda_1(y) \), as in [4, 7]. A scheme that directly relates each pair \( (S_{\alpha, \delta, \theta+r}, S_{\alpha, \delta, \theta+r}) \), and hence \( PD(\alpha, \theta + r), PD(\alpha, \theta + r) \), for each step \( r \), can be deduced from the following results.

(i) There is the identity

\[
\lambda_{\alpha, \delta, \theta} \circ P_{\delta, \theta} \circ (\cdot) = P_{\delta, \theta} \circ \lambda_{\alpha, \delta, \theta}(\cdot)
\]

(ii) Statement [(i)] means that \( \lambda_{\alpha, \delta, \theta} \circ P_{\delta, \theta}(y) \) is equivalent to the bridge

\[
B(\alpha, \delta, \theta) P_{\delta, \theta} \circ (\cdot) = P_{\delta, \theta} \circ \lambda_{\alpha, \delta, \theta}(\cdot)
\]

where \( U' = P_{\delta, \theta}^{-1} (\tilde{U}_1) \) is the Uniform[0, 1] variable associated with the first size biased pick from a PD(δ, θ/α) distribution, and is further equivalent to the bridge representation

\[
P_{\delta, \theta} \circ B(\alpha, \delta, \theta) \circ (\cdot) = P_{\delta, \theta} \circ \lambda_{\alpha, \delta, \theta}(\cdot)
\]

(iii) The equivalences are then encoded by their local times or \( \delta \)-diverisities,

\[
S^{\alpha}_{\delta, \theta} + B^{\alpha}_{\delta, \theta} = S^{\alpha}_{\delta, \theta} + B^{\alpha}_{\delta, \theta}
\]

(iv) The identity \( P_{\alpha, \delta, \theta} \circ P_{\delta, \theta} \circ (\cdot) = P_{\alpha, \delta, \theta} \circ P_{\delta, \theta} \circ \lambda_{\alpha, \delta, \theta}(\cdot) \) shows that the \( \alpha \)-\( \delta \)-diversity of \( P_{\alpha, \delta, \theta} \) can be represented as,

\[
S^{\alpha}_{\alpha, \delta, \theta} = S^{\alpha}_{\alpha, \delta, \theta} + S^{\alpha}_{\alpha, \delta, \theta} + S^{\alpha}_{\alpha, \delta, \theta} = S^{\alpha}_{\alpha, \delta, \theta} + S^{\alpha}_{\alpha, \delta, \theta} + B^{\alpha}_{\alpha, \delta, \theta} = S^{\alpha}_{\alpha, \delta, \theta} + B^{\alpha}_{\alpha, \delta, \theta}
\]

(v) Replacing \( \theta \) with \( \theta + r \) establishes relationships between \( (S^{\alpha}_{\alpha, \delta, \theta+r})_{r \geq 0} \) and \( (S^{\alpha}_{\alpha, \delta, \theta+r})_{r \geq 0} := (S^{\alpha}_{\alpha, \delta, \theta+r} \times S^{\alpha}_{\alpha, \delta, \theta+r})_{r \geq 0} \) in the respective nested schemes. Which is encoded by,

\[
P_{\alpha, \delta, \theta}^{-1} = P_{\delta, \theta}^{-1} \circ P_{\alpha, \delta, \theta}^{-1} = \lambda_{\alpha, \delta, \theta}^{-1} \circ \cdots \circ \lambda_{\alpha, \delta, \theta}^{-1} \circ P_{\alpha, \delta, \theta}^{-1}
\]

where \( P_{\alpha, \delta, \theta}^{-1} = P_{\delta, \theta}^{-1} \circ P_{\alpha, \delta, \theta}^{-1} \), and for the simple bridge \( \lambda_{\alpha, \delta, \theta+r} \),

\[
B^{\alpha}_{\alpha, \delta, \theta+r} := \frac{S^{\alpha}_{\alpha, \delta, \theta+r} S^{\alpha}_{\alpha, \delta, \theta+r}}{S^{\alpha}_{\alpha, \delta, \theta} S^{\alpha}_{\alpha, \delta, \theta}}
\]
for \( r \geq 1 \). Note, \( P_{\theta,\delta}^{-1}(U_{\theta,\delta}^{-1}) = \lambda_{\delta+r-1}^{-1} \circ P_{\delta,\theta}^{-1} \), which follows from [(i)].

(vi) The relationship between the nested sequence of COAG-operators, following the laws \( \text{PD}(\delta, \theta) \{ r \geq 0 \} \) is encoded by the local time relations

\[
S_{\delta}^{-\delta} = S_{\delta}^{-\delta} B_{\theta}^{-\delta} \left( \frac{\delta}{\alpha}, \frac{1-\alpha}{\alpha} \right) B_{\theta}^{\delta} \left( \frac{\theta+\delta}{\alpha}, \frac{1-\alpha}{\alpha} \right),
\]

substituting generally \( \theta \) with \( \theta + r - 1 \). This follows from [(iv)] by using the identity \( S_{\alpha,1+\theta}^{-\alpha} = S_{\alpha}^{-\alpha} B_{\theta}^{-\delta} \left( \frac{\theta}{\alpha}, \frac{1-\alpha}{\alpha} \right) \), where the variables on the right are not independent.

**Proof.** Statement [(i)] is a just a consequence of Pitman’s coagulation operation combined with the simpler coagulation operator in [21] as follows. \( P_{\alpha,\delta} = P_{\alpha,\theta} \circ P_{\delta,\theta} = P_{\alpha,1+\theta} \circ \lambda_{\alpha,\theta} \circ P_{\delta,\theta} \). Also by [42] \( P_{\alpha,1+\theta} \circ P_{\delta,1+\theta} = P_{\alpha,\delta,1+\theta} \). So for each \( y \) there is the identity of distribution functions

\[
P_{\alpha,\delta,\theta}(y) = P_{\alpha,1+\theta}(\lambda_{\alpha,\theta}(P_{\delta,\theta}(y))) = P_{\alpha,1+\theta}(P_{\delta,1+\theta}(\lambda_{\alpha,\theta}(y))),
\]

which establishes [(i)]. Although equivalent, statement [(ii)] does not directly appeal to [(i)], and offers a result by direct construction. By definition, \( \lambda_{\alpha,\theta} \circ P_{\delta,\theta}(y) \) is given by

\[
B_{\theta}^{-\delta} \left( \frac{\delta}{\alpha}, \frac{1-\alpha}{\alpha} \right) P_{\delta,\theta}(y) + (1 - B_{\theta}^{-\delta} \left( \frac{\delta}{\alpha}, \frac{1-\alpha}{\alpha} \right)) E \{ P_{\delta,\theta}(U_1) \leq y \}
\]

where \( U_1 = P_{\delta,\theta}^{-1}(U_1) \) is, as noted, the point associated with the first size biased pick from \( \text{PD}(\delta, \theta) \), say \( 1 - W_1 \) following a Beta\((1 - \delta, \frac{\theta}{\alpha} + \delta)\) distribution. Use \( U_1 = P_{\delta,\theta}^{-1}(U_1) \), and otherwise replace \( P_{\delta,\theta} \) with its stick-breaking representation

\[
P_{\delta,\theta}(y) = W_1 P_{\delta,\theta}^1(y) + (1 - W_1) E \{ U_1 \}.
\]

Combining terms, one sees that for \( B = B_{\theta}^{-\delta} \left( \frac{\delta}{\alpha}, \frac{1-\alpha}{\alpha} \right) \), it remains to evaluate the distribution of \( BW_1 \) and \( 1 - W_1 \). It follows by the usual Beta Gamma algebra that \( BW_1 = B \left( \frac{\theta+\delta}{\alpha}, \frac{1-\alpha}{\alpha} \right) \). The remaining equivalence is apparent either by direct evaluation or an appeal to [49]. Proposition 21]. Statement [(iii)] follows from this and the remaining statements have been explained within the text of the Proposition.

We close by noting some connections to the Bolthausen-Sznitman coalescent [14].

**Corollary 6.1.** Consider the dynamics of the Bolthausen-Sznitman coalescent, or \( U \)-coalescent, as described in for instance [21, Theorem 14, Corollary 15] or [2, 11]. Then setting \( \alpha = e^{-s}, \delta = e^{-(t-s)} \) and \( \theta = 0 \) in Proposition 6.7 establishes various relations between the \( U \)-coalescent and corresponding nested CRP schemes.

**Remark 6.1.** This section is partially influenced by recent interactions with Anton Wakolbinger and Martin Möhle, whom I thank for their time. One sees...
that \((S_{(e^{-t},0)}, t \geq 0)\) is a version of M"ohle’s \cite{Mohle1994} Mittag Leffler process. Setting 
\((K_n(t) := N^{(n)}_t, t \geq 0)\) to match with the notation in \cite{Mohle1994}, denoting for fixed \(t\), the number of blocks of a \(PD(e^{-t},0)\) partition of \([n]\), one has for \(n \to \infty\),

\[
\frac{K_n(t)}{n e^{-t}} = \frac{N^{(n)}(t)}{n e^{-t}} \overset{a.s.}{\sim} S_{(e^{-t},0)},
\]

as was established in \cite{Mohle1994}, and otherwise corresponds to results for the \(e^{-t}\)-diversity.

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