A COMPARISON THEOREM FOR THE PRO-ÉTALE FUNDAMENTAL GROUP

JIU-KANG YU, LEI ZHANG

Abstract. Let $X$ be a connected scheme locally of finite type over $\mathbb{C}$, and let $X^{\text{an}}$ be its associated analytic space. In this paper, we provide a comparison map from the topological fundamental group of $X^{\text{an}}$ to the pro-étale fundamental group of $X$.

Introduction

Let $X$ be a connected, locally path-connected and semi-locally simply connected topological space, and let $x_0 \in X$ be a point in $X$. Then $\pi_1^{\text{top}}(X, x_0)$ classifies all the covering spaces of $X$, that is, there is an equivalence

| the category of covering spaces of $X$ | $\iff$ | the category of sets with a $\pi_1^{\text{top}}(X, x_0)$-action |

Inspired by the analogy between the above classification and the classical Galois theory, A. Grothendieck constructed in SGA1 the étale fundamental group $\pi_1^{\text{ét}}(X, x_0)$ for each connected scheme $X$ and a geometric point $x_0 \in X$. The fundamental group $\pi_1^{\text{ét}}(X, x_0)$ classifies all the finite étale covers of $X$, i.e. there is an equivalence

| the category of finite étale covers of $X$ | $\iff$ | the category of finite sets with a continuous $\pi_1^{\text{ét}}(X, x_0)$-action |

A. Grothendieck also showed that if $X$ is a scheme locally of finite type over $\mathbb{C}$, then $\pi_1^{\text{ét}}(X, x_0)$ is the profinite completion of $\pi_1^{\text{top}}(X^{\text{an}}, x_0)$, where $X^{\text{an}}$ is the complex analytic space associated with $X$. There are two main ingredients behind this comparison:

- If $p: Y \to X$ is a morphism of $\mathbb{C}$-schemes locally of finite type, then the analytification $p^{\text{an}}$ is a finite covering space if and only if $p$ is a finite étale cover (cf. [2, Exposé XII, Proposition 3.1 (iii), Proposition 3.2 (vi)]);
- The Riemann Existence Theorem ([2, Exposé XII, Théorème 5.1, p. 251]) which states that the analytification functor $(-)^{\text{an}}$ from the category of schemes locally of finite type over $X$ to the category of analytic spaces over $X^{\text{an}}$ induces an equivalence between the finite étale covers of $X$ and the finite covering spaces of $X^{\text{an}}$.

In [1], B. Bhatt and P. Scholze introduced the notion of the pro-étale fundamental group $\pi_1^{\text{proét}}(X, x_0)$, a topological group which classifies the geometric covers of $X$ (cf. [1, Definition 7.3.1]). The geometric covers are locally constant sheaves in the pro-étale topology, in particular, they include the finite étale covers. Therefore, $\pi_1^{\text{proét}}(X, x_0)$ refines Grothendieck’s étale fundamental group $\pi_1^{\text{ét}}(X, x_0)$, which classifies the finite étale covers. In fact, there is a

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natural morphism \( \pi_1^{\text{pro\-ét}}(X, x_0) \to \pi_1^\text{ét}(X, x_0) \) which makes \( \pi_1^\text{ét}(X, x_0) \) the profinite completion of \( \pi_1^{\text{pro\-ét}}(X, x_0) \). However, there has been no comparison between \( \pi_1^\text{top} \) and \( \pi_1^{\text{pro\-ét}} \). The main purpose of this paper is to achieve this comparison.

Let \( X \) be a scheme locally of finite type over \( \mathbb{C} \). Recall that if \( p': Y' \to X^\text{an} \) is a map of analytic spaces, then we say \( p' \) is algebraizable if there is a map \( p: Y \to X \) of \( \mathbb{C} \)-schemes locally of finite type whose analytification (cf. [2, Théorème et définition 1.1]) \( p^\text{an} \) is \( p' \).

**Theorem I.** (cf. 1.7, 2.1 and 3.1) Let \( X \) be a scheme locally of finite type over \( \mathbb{C} \).

1. If \( p: Y \to X \) is a morphism of \( \mathbb{C} \)-schemes locally of finite type, then its analytification \( p^\text{an} \) is a covering space if and only if \( p \) is a geometric cover;
2. The analytification functor \((-)\text{an}\) from the category of schemes locally of finite type over \( X \) to the category of analytic spaces over \( X^\text{an} \) induces an equivalence between the geometric covers of \( X \) and the algebraizable covering spaces of \( X^\text{an} \).

Moreover, the full subcategory of algebraizable covering spaces is stable under subobjects and quotients.

Therefore, we obtain a natural comparison map.

**Theorem II.** (cf. 4.1) Let \( X \) be a connected scheme locally of finite type over \( \mathbb{C} \), and let \( x_0 \in X \) be a geometric point. Then there is a natural comparison map

\[
\pi_1^\text{top}(X^\text{an}, x_0) \to \pi_1^{\text{pro\-ét}}(X, x_0)
\]

whose image is dense. Moreover, there is no nontrivial continuous homomorphism from \( \pi_1^{\text{pro\-ét}}(X, x_0) \) to a discrete group \( G \) whose restriction to \( \pi_1^\text{top}(X^\text{an}, x_0) \) is trivial.

Thus, the difference between \( \pi_1^\text{top} \) and \( \pi_1^{\text{pro\-ét}} \) for complex varieties lies exactly at those covering spaces, such as the exponential map \( \mathbb{G}_a^\text{an} \to \mathbb{G}_m^\text{an} \), which are not algebraizable.

It is remarkable that unlike the étale fundamental group, which is the profinite completion of the topological fundamental group, the pro-étale fundamental group is not determined by the topological fundamental group. For example, let \( X \) be the space \( \mathbb{P}^1_\mathbb{C} \) with 0 and \( \infty \) identified, then both \( \pi_1^\text{top}(X^\text{an}) \) and \( \pi_1^\text{top}(\mathbb{G}_m^\text{an}, \mathbb{C}) \) are \( \mathbb{Z} \). However, \( \pi_1^{\text{pro\-ét}}(X) = \mathbb{Z} \) (cf. [7, Theorem IV]) while \( \pi_1^{\text{pro\-ét}}(\mathbb{G}_m^\text{an}, \mathbb{C}) = \hat{\mathbb{Z}} \). Indeed, the comparison map \( \pi_1^\text{top}(X^\text{an}, x_0) \to \pi_1^{\text{pro\-ét}}(X, x_0) \) is an isomorphism by Theorem II, so all the covering spaces of \( X^\text{an} \) are algebraizable.

**Notation and conventions.** For analytic spaces (always over \( \mathbb{C} \)), we follow the convention of [3, Définition 2.1] and [2, Exposé XII], allowing non-separated analytic spaces. For topological spaces, we follow the convention of [10, 004C]. In particular, a proper map is assumed to be separated [10, 005M].

Let \( p: Y \to X \) be a map of topological spaces (resp. analytic spaces). The map \( p \) is called étale if it is a local isomorphism. We denote by \( \hat{\text{Ét}}(X) \) the category of étale maps with target \( X \). We denote by \( \text{Cov}(X) \) the category of covering spaces over \( X \) (resp. the full subcategory of \( \hat{\text{Ét}}(X) \) consisting of maps \( p: Y \to X \) which are covering maps for the underlying topological spaces). If \( X \) is a scheme, then \( \text{Cov}(X) \) denotes the category of geometric covers of \( X \).

A covering map \( p: Y \to X \) is called a trivial covering space if it is isomorphic (in \( \text{Cov}(X) \)) to \( \bigsqcup_J X \to X \) for some set \( J \), where \( \bigsqcup_J X \) denote the disjoint union of copies of \( X \) indexed by \( J \). Some authors would say that \( X \) is evenly covered by \( p \).
1. Geometric Covers are Covering Spaces

Lemma 1.1. (1) If $X$ is an analytic space, then the forgetful functor from the category of étale analytic spaces over $X$ to the category of topological spaces over $X$ is fully faithfule.

(2) If $u: X' \to X$ is a map of analytic spaces such that the induced map on the underlying topological spaces is a homoeomorphism, then the pullback along $u$ induces an equivalence $\text{Et}(X) \to \text{Et}(X')$.

Proof. Let $f_1: (Y_1, \mathcal{O}_{Y_1}) \to (X, \mathcal{O}_X)$ and $f_2: (Y_2, \mathcal{O}_{Y_2}) \to (X, \mathcal{O}_X)$ be two étale maps of analytic spaces. We want to show that if $g$ is a morphism of topological spaces $Y_1 \to Y_2$ over $X$, then there exists a unique map of sheaves $g^{-1}\mathcal{O}_{Y_2} \to \mathcal{O}_{Y_1}$ making the following diagram

$$
\begin{array}{c}
g^{-1}\mathcal{O}_{Y_2} \\
\downarrow g^{-1}(f_2)^* \\
\text{id}_{f_1^{-1}\mathcal{O}_X} \\
\downarrow f_1^{-1}(f_2)^* \\
\mathcal{O}_{Y_1}
\end{array}
$$

commutative i.e. making $g$ a map of $X$-analytic spaces $f_1 \to f_2$. The uniqueness is local on $Y_1$. Since $f_1, f_2$ are local isomorphisms, to prove the uniqueness we may assume that they are isomorphisms. Then the situation is clear. Having the uniqueness, the existence becomes local on $Y_1$, thus assuming that $f_1, f_2$ are isomorphisms again we finish the proof of (1).

To prove part (2), one first notices that (1) already provides the full faithfulness, so we only have to show the essential surjectivity. Suppose that $f': (Y', \mathcal{O}_{Y'}) \to (X', \mathcal{O}_{X'})$ is an étale map. We want to find an étale map $f: (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ whose pullback along $u$ is $f'$. It follows from the full faithfulness of the pullback that the existence of $f$ can be proved locally on $Y'$. Thus we may assume that $f'$ is an isomorphism, in which case the assertion is clear.

Remark 1.2. (1) An étale map of analytic spaces induces an étale map of the underlying topological spaces, but not vice versa.

(2) It follows from 1.1 (1) that if $X$ is an analytic spaces, then any map in Cov$(X)$ is locally on $X$ a trivial covering space – as an analytic space!

Lemma 1.3. Let $f: \bar{X} \to X$ be a closed map of topological spaces, and let $x \in X$ be a point. If $V \subseteq \bar{X}$ is an open subset containing $f^{-1}(x)$, then there exists an open neighborhood $U \subseteq X$ of $x$ such that $f^{-1}(U) \subseteq V$.

Proof. Set $U := X \setminus f(\bar{X} \setminus V)$.

Lemma 1.4. Let $X$ be a scheme locally of finite type over $\mathbb{C}$, and let $p: Y \to X$ be a geometric cover which is a disjoint union of finite étale covers. Then for each $x \in X^{\text{an}}$ there is an open neighborhood $U \subseteq X^{\text{an}}$ such that $p^{\text{an}}|_{U}$ is a trivial covering space.

Proof. An analytic space is locally simply connected because it is locally a CW-complex (cf. [9, Theorem 1]) and every CW-complex is locally contractible (cf. [4, A.4, p. 522]). Thus we can choose $U$ to be a simply connected open neighborhood of $x$. Then the claim follows from Remark 1.2 (2).

Lemma 1.5. Let $X$ be a topological space, and let $Z_1, Z_2$ be closed subspaces of $X$ such that $X = Z_1 \cup Z_2$. Then we have an equivalence given by the pullback functor:

$$
\text{Et}(X) \overset{\cong}{\longrightarrow} \text{Et}(Z_1) \times_{\text{Et}(Z)} \text{Et}(Z_2)
$$
where $Z = Z_1 \cap Z_2$.

**Proof.** We remark that since a proper map is separated, $\Delta_f$ is indeed a closed embedding. Set $Y := Z_1 \coprod Z_2$. By [5, 4.1, 4.7] or [11, Theorem 5.6] a Bourbaki-proper surjective map (simply called a proper map in loc. cit.) such as $\phi : Y \longrightarrow X$ is an effective étale descent map. Thus an object in $\text{Ét}(X)$ is equivalent to an object in $W \in \text{Ét}(Y) = \text{Ét}(Z_1) \times \text{Ét}(Z_2)$ together with an isomorphism $\phi : p_1^* W \cong p_2^* W$ in $\text{Ét}(Y \times_X Y)$ satisfying cocycle conditions, where $p_i$ ($i = 1, 2$) are the two projections of $Y \times_X Y$. Since

$$Y \times_X Y = Z_1 \coprod Z_2 \coprod (Z_1 \cap Z_2) \coprod (Z_2 \cap Z_1)$$

the cocycle condition is equivalent to saying that $\phi|_{Z_1} = \text{id}_{W|_{Z_1}}$, $\phi|_{Z_2} = \text{id}_{W|_{Z_2}}$ and

$$\phi|_{Z_1 \cap Z_2} \circ \phi|_{Z_2 \cap Z_1} = \text{id}_{W|_Z}.$$

In other words, the pair $(W, \phi)$ is nothing but an object in $\text{Ét}(Z_1) \times_{\text{Ét}(Z)} \text{Ét}(Z_2)$. \hfill $\square$

**Lemma 1.6.** Let $X$ be a topological space. Let $Z \subseteq X$ be a closed subspace, and suppose we have a proper surjective map

$$\tilde{X} \longrightarrow X$$

Denote $f^{-1}(Z)$ by $\tilde{Z}$. Suppose that the union of the images of the two closed subspaces

$$\Delta_f : \tilde{X} \longrightarrow \tilde{X} \times_X \tilde{X}, \quad \tilde{Z} \times \tilde{Z} \longrightarrow \tilde{X} \times_X \tilde{X}$$

is $\tilde{X} \times_X \tilde{X}$. Then the pullback functor induces an equivalence of categories:

$$\text{Ét}(X) \longrightarrow \tilde{\text{Ét}}(\tilde{X}) \times_{\tilde{\text{Ét}}(\tilde{Z})} \tilde{\text{Ét}}(Z).$$

**Proof.** Let us construct a quasi-inverse of the pullback functor. So we start with a triple $(Y, W, \phi)$, where $Y \in \text{Ét}(\tilde{X})$, $W \in \text{Ét}(Z)$, and $\phi : Y \times_X \tilde{Z} \cong W \times_X \tilde{Z}$ is an isomorphism. The idea is to construct a descent datum of $Y$ for the morphism $f$, then we get the desired object in $\tilde{\text{Ét}}(X)$ by applying [5, 4.1, 4.7] or [11, Theorem 5.6].

To construct this descent datum, we need to work on $\tilde{X} \times_X \tilde{X}$ using the full faithfulness part of the equivalence

$$\text{Ét}(\tilde{X} \times_X \tilde{X}) \cong \tilde{\text{Ét}}(\tilde{X}) \times_{\tilde{\text{Ét}}(\tilde{Z})} \tilde{\text{Ét}}(\tilde{Z} \times \tilde{Z})$$

which follows from Lemma 1.5, because the fiber product of the two closed immersions in the current lemma is the closed immersion $\tilde{Z} \xrightarrow{\Delta_f} \tilde{X} \times X \tilde{X}$. Under (1), $p_i^*(Y)$ corresponds to the triple $(Y, q_i^*(Y|_{\tilde{Z}}), \text{can}_i)$ for $i = 1, 2$. Here $p_i : \tilde{X} \times X \tilde{X} \longrightarrow \tilde{X}$, $q_i : \tilde{Z} \times X \tilde{Z} \longrightarrow X$ are projections, and $\text{can}_i$ is the obvious canonical isomorphism. The desired descent datum $\lambda : p_1^*(Y) \rightarrow p_2^*(Y)$ corresponds to $(Y, q_1^*(Y|_{\tilde{Z}}), \text{can}_1) \rightarrow (Y, q_2^*(Y|_{\tilde{Z}}), \text{can}_2)$ given by $\text{id}_{Y} : Y \rightarrow Y$, $\varphi : q_1^*(Y|_{\tilde{Z}}) \rightarrow q_2^*(Y|_{\tilde{Z}})$, where $\varphi$ is the canonical descent datum signifying the fact that $Y|_{\tilde{Z}}$ is isomorphic to $W \times_X \tilde{Z}$ via $\phi$.

We have to show that $\lambda$ is indeed a descent datum. That is, considering the triple fibred product, the fibred product and the projections:

$$\tilde{X} \times_X \tilde{X} \times_X \tilde{X} \xrightarrow{p_{12}} \tilde{X} \times_X \tilde{X} \xrightarrow{p_{1, 2}} \tilde{X}$$
we have to show the cocycle condition \( p_{23}^* \lambda \circ p_{12}^* \lambda = p_{13}^* \lambda \). Since the fibred product is covered by \( \Delta_f(\tilde{X}) \) and \( \tilde{Z} \times Z, \) the triple fibred product is covered by the triple diagonal \( \Delta_f^3(\tilde{X}) \) and the closed subset \( \tilde{Z} \times Z \tilde{Z} \times Z \tilde{Z} \). Applying 1.5 we get an equivalence

\[
(2) \quad \text{Et}(\tilde{X} \times X \times X \tilde{X}) \xrightarrow{\cong} \text{Et}(\tilde{X}) \times_{\text{Et}(Z)} \text{Et} \left( \tilde{Z} \times Z \tilde{Z} \times Z \tilde{Z} \right)
\]

By the faithfulness part of (2), it is enough to check the equality \( p_{23}^* \lambda \circ p_{12}^* \lambda = p_{13}^* \lambda \) on \( \Delta_f^3(\tilde{X}) \) and \( \tilde{Z} \times Z \tilde{Z} \times Z \tilde{Z} \) separately. On \( \Delta_f^3(\tilde{X}) \) all the pullbacks of \( \lambda \) are identities, so there is nothing to check. On \( \tilde{Z} \times Z \tilde{Z} \times Z \tilde{Z} \) the equality holds because \( \lambda|_{\tilde{Z} \times Z} = \varphi \) is a descent datum for \( \tilde{Z} \to Z \).

\[\square\]

**Theorem 1.7.** Let \( X \) be a scheme locally of finite type over \( \mathbb{C} \), and let \( p: Y \to X \) be a geometric cover. Then \( p^{an} \) is a covering space.

**Proof.** Let \( x \in X(\mathbb{C}) \) be a point. We want to find an open neighborhood of \( x \) in \( X^{an} \) over which \( p^{an} \) is a trivial covering space. For this we may and do assume that \( X = \text{Spec}(A) \), where \( A \) is reduced (cf. 1.1 (2)) and of finite type. Thus \( X \) is finite-dimensional.

We first remark that the theorem is true when \( X \) is normal. Indeed in this case [1, 7.4.10] implies that every geometric cover \( Y \) is a disjoint union of finite étale cover. So the theorem follows from 1.4.

We will apply induction on the dimension of \( X \). The theorem is trivial if \( \dim X = 0 \). By the induction hypothesis, if \( Z \subseteq X \) is the singular locus of \( X \), then \( p^{an}|_{Z^{an}} \) is a covering space.

Let \( f: \tilde{X} \to X \) be the normalization of \( X \) (cf. [10, 035E]). Then \( f \) is finite, and it is an isomorphism away from \( Z \). If \( x \notin Z \), then the desired result follows from the first remark and 1.4, so we may assume that \( x \in Z \).

Write \( f^{-1}(x) = \{ x_1, \ldots, x_n \} \subset \tilde{X}(\mathbb{C}) \). Since the topology of \( \tilde{X}^{an} \) is separated (\( \tilde{X} \) being affine), we can find open neighborhoods \( V_i \) of \( x_i \) in \( \tilde{X}^{an} \) such that \( V_i \cap V_j = \emptyset \) for \( i \neq j \). By 1.3, we can find an open neighborhood \( U \) of \( x \) such that \( f^{an-1}(U) \subset \coprod_{1 \leq i \leq n} V_i \).

We now choose \( \{ V_i \} \) and \( U \) satisfying the conditions in the preceding paragraph and in addition satifying that \( p^{an} \) is trivial on each \( V_i \) and on \( Z^{an} \cap U \). This is possible by our first remark, the induction hypothesis, and 1.4.

Write

\[
p^{an}|_{V_i} = \coprod_{J_i} V_i, \quad p^{an}|_{Z^{an} \cap U} = \coprod_{J} (Z^{an} \cap U)
\]

for some sets \( J_i \) and \( J \). Let \( W_i = V_i \cap f^{-1}(Z^{an} \cap U) \) and let \( \phi_i : (p^{an}|_{V_i})|_{W_i} \to (p^{an}|_{Z^{an} \cap U})|_{W_i} \) be the natural isomorphism. Choose a connected open neighborhood \( A_i \) of \( x_i \) in \( W_i \). Then \( \phi_i|_{A_i} \) determines a bijection \( J_i \simeq J \) (which may depend on the choice of \( A_i \)). Changing the sets \( J_i \) if necessary, we may and do assume that \( J_i = J \) for each \( i \) and the bijection \( J_i = J \to J \) induced by \( \phi|_{A_i} \) is the identity.

Finally, we write \( A_i = V_i' \cap f^{-1}(Z^{an} \cap U) \) with \( V_i' \) an open neighborhood of \( x_i \) in \( V_i \), and we find an open neighborhood \( U' \) of \( x \) in \( U \) such that \( f^{an-1}(U') \subset \coprod_{1 \leq i \leq n} V_i' \). We now claim that \( p^{an}|_{U'} \) is a trivial covering space.

Write \( \tilde{U}' = f^{an-1}(U') \) and \( Z_{U'} = Z^{an} \cap U' \). Applying 1.6 we have the following equivalence

\[
(3) \quad \text{Et}(U') \xrightarrow{\cong} \text{Et}(\tilde{U}') \times_{\text{Et}(Z_{U'})} \text{Et}(Z_{U'})
\]

where \( Z_{U'} = f^{-1}(Z_{U'}) \). We wish to show that \( p^{an}|_{U'} \), an object in \( \text{Et}(U') \), is a trivial cover. Consider the corresponding object on the right hand side of (3), which is the triple
(p^{an}|_U, p^{an}|_Z_U, \phi). We have $p^{an}|_U' = \bigsqcup_i U', \ p^{an}|_Z_{U'} = \bigsqcup_i Z_{U'}$, and $\phi$ is the identity since $\phi$ is simply $\phi_i$ on $f^{an-1}(U') \cap V'_i \subset A_i$. Clearly this triple corresponds to a trivial cover in $\text{Ét}(U')$. This proves the claim and hence the theorem. \hfill \Box

2. Algebraizable Covering Spaces are Geometric Covers

Theorem 2.1. Let $p: Y \to X$ be a map of schemes locally of finite type over $\mathbb{C}$. If $p^{an}$ is a covering space of $X^{an}$, then $p$ is a geometric cover of $X$.

Proof. It is easy to see that a covering map is separated. By [2, Exposé XII, Proposition 3.1 (iii), (viii)], $p$ is étale and separated. Since geometric covers are locally constant sheaves in the pro-étale topology, they satisfy effective pro-étale descent, so in particular effective Zariski descent. Thus we may assume that $X = \text{Spec}(A)$, where $A$ of finite type.

Now suppose that $R$ is a valuation ring with function field $K$, and suppose that we have the following commutative diagram

$$
\begin{array}{ccc}
\text{Spec}(K) & \xrightarrow{\eta} & Y \\
\downarrow & & \downarrow p \\
\text{Spec}(R) & \xrightarrow{(x, \xi)} & X
\end{array}
$$

We have to show that there exists a unique dashed arrow as above which makes all the triangles commutative.

If the morphism $\text{Spec} R \to X$ factors as $\text{Spec} R \to X' \to X$, we can put $Y' = Y \times_X X'$ and form the diagram

$$
\begin{array}{ccc}
\text{Spec}(K) & \xrightarrow{} & Y' \\
\downarrow & & \downarrow p' \\
\text{Spec}(R) & \xrightarrow{} & X'
\end{array}
$$

If the dashed arrow $\text{Spec}(R) \to Y'$ in (5) exists, clearly the composition $\text{Spec}(R) \to Y' \to Y$ can serve as the dashed arrow in (4). Thus we can reduce the problem for $p$ to the problem for $p'$, because [2, Exposé XII, 1.2] and $p^{an}$ is a covering space too.

Let $A'$ be the image of $A \to R$. Then we can apply the above reduction with $X' = \text{Spec}(A')$. Thus we may and do assume that $X$ is integral, and that $A \to R$ is injective.

Let $y \in Y'$ be the image of $\eta$. It lies above the generic point of $X$. Therefore the residue field $\kappa(y)$ of $y$ is a finite extension of $\text{Frac}(A)$. Let $A''$ be the integral closure of $A$ in $\kappa(y)$. Then we can apply the above reduction again with $X' = \text{Spec}(A'')$. Observe that the map $\text{Spec} K \to Y'$ factors through $\text{Spec} K \to \text{Spec} \kappa(y)$. Therefore, after replacing $X$ with $X'$, we may and do assume: $X$ is normal and $\kappa(y) = \text{Frac}(A)$.

Since $X$ is normal and $p$ is étale, $Y$ is normal. Thus if $V \subseteq Y$ denote the connected component of $Y$ containing $y$, then $V$ is irreducible. As $p$ is étale, $p(y) \in X$ is the generic point if and only if $y \in Y$ is a generic point, so $y$ is the generic point of $V$. By assumption $p^{an}$ is a covering space, since $V^{an} \subseteq Y^{an}$ is also a connected component, $(p|_V)^{an} = p|_{V^{an}}$ is also a covering space of $X^{an}$. Thus we may assume that $Y = V$, i.e. we may assume moreover that $Y$ is an integral normal scheme, and that $Y$ and $X$ has the same function field.

At this moment, we don’t know whether $Y$ is of finite type so we may not be able to apply Zariski’s Main Theorem (in the form of [8, §4.4, Corollary 4.6]) to $p$. But we can apply it to $p|_V$ for any quasi-compact open $V \subseteq Y$, and deduce that $p|_V$ is an open embedding for
any such $V$. We claim that $p$ itself is an open embedding. If $p(s) = p(t)$ for $s, t \in Y$, then we can find open affine neighborhoods $V_s, V_t$ of $s, t$ respectively so that $p|_{V_s}, p|_{V_t}$ are open embeddings. As $V := V_s \cup V_t$ is quasi-compact, $p|_V$ is also an open embedding. Thus $s = t$, so $p$ is injective. This proves the claim.

As $Y$ is non-empty, so is $Y^\text{an}$. Since $X^\text{an}$ is connected (cf. [2, Exposé XII, Proposition 2.4]), $p^\text{an}$ is surjective. Thus by [2, Exposé XII, Proposition 3.2 (i)], $p$ is surjective, or equivalently, $p$ is an isomorphism. Now we see clearly that (4) has a unique lift, as desired. \hfill $\square$

### 3. Full Faithfulness

**Theorem 3.1.** Let $X$ be a scheme locally of finite type over $\mathbb{C}$. Then the analytification functor $(-)^\text{an}$ restricting to the category of geometric covers $\text{Cov}(X)$ is fully faithful.

Moreover, the full subcategory $\text{Cov}(X) \subset \text{Cov}(X^\text{an})$ is stable under taking “subobjects” and “quotients”, i.e. if $p: Y \to X$ is a geometric cover and if $q': Y' \to X^\text{an} \in \text{Cov}(X^\text{an})$ admits either an injection $a: U' \to Y^\text{an}$ or a surjection $b: Y^\text{an} \twoheadrightarrow U'$ over $X^\text{an}$, then the covering space $q'$ is algebraizable.

**Proof.** For the first statement we assume, without loss of generality, that $X$ is connected. Then both $\text{Cov}(X)$ and $\text{Cov}(X^\text{an})$ are infinite Galois categories. By [2, Exp. XII Proposition 2.4] $(-)^\text{an}$ sends connected objects to connected objects, so the full faithfulness follows from [6, Proposition 2.64].

By the full faithfulness, the second statement is Zariski local on $X$, so we may assume that $X$ affine and connected.

Suppose that there is an injection $a: U' \hookrightarrow Y^\text{an}$. Then $a$ is étale, so it is an open embedding. Thus we can identify $U'$ as an open sub analytic space of $Y^\text{an}$ via $a$. Now the task is to show that $U'$ is a Zariski open. If $Y = \bigsqcap_{i \in I} Y_i$, where $Y_i$ is a finite étale cover over $X$, then $U'_i := U' \cap Y^\text{an}_i$ is a finite covering space over $X^\text{an}$, so it is algebraizable by the Riemann Existence Theorem. Therefore, $U' = \bigsqcap_{i \in I} U'_i$ is also algebraizable. In the general case, we consider the normalization map $f: \tilde{X} \to X$. Set $\tilde{Y} := \tilde{X} \times_X Y$ and $\tilde{U}' := \tilde{X}^\text{an} \times_{X^\text{an}} U'$. Since $\tilde{Y}$ is a disjoint union of finite étale covers of $\tilde{X}$, $\tilde{U}'$ is algebraizable, so it comes from a Zariski open $\tilde{U} \subseteq \tilde{Y}$ by 3.1. Then the complement

$$Y^\text{an} \setminus U' = f^\text{an}(\tilde{Y}^\text{an} \setminus \tilde{U}') = f(\tilde{Y} \setminus \tilde{U})^\text{an}$$

is Zariski-closed. Thus $U'$ is Zariski-open.

Suppose that there is a surjection $b: Y^\text{an} \twoheadrightarrow U'$. If $Y = \bigsqcap_{i \in I} Y_i$, where $Y_i$ is finite étale and connected, then $b(Y^\text{an}_i) \subseteq U'$ is both open and closed. Indeed, $b(Y^\text{an}_i)$ is open because $b$ is étale, and $b(Y^\text{an}_i)$ is closed because it is the image of the graph $Y^\text{an}_i \subseteq Y^\text{an}_i \times_{X^\text{an}} U'$ of $b|_{Y^\text{an}_i}$ under the proper map $Y^\text{an}_i \times_{X^\text{an}} U' \to U'$. Note that the graph of $b|_{Y^\text{an}_i}$ is closed since $q': U' \to X^\text{an}$ is separated. In this way, $U'$ becomes a disjoint union of connected components of the form $b(Y^\text{an}_i)$ which are finite covering spaces. Thus $U'$ is algebraizable by the Riemann Existence Theorem. In the general case, one considers the normalization map $f: \tilde{X} \to X$. Since the descent data of $q': U' \to X^\text{an}$ along $f^\text{an}$ are all algebraic, by the proper descent of $\text{Cov}(X)$, we find $q: U \to X$ in $\text{Cov}(X)$ whose analytification is $q'$, i.e. $q' = q^\text{an}$ is algebraizable. \hfill $\square$
4. The Comparison Map

Theorem 4.1. Let $X$ be a connected scheme locally of finite type over $\mathbb{C}$, and let $x_0 \in X$ be a geometric point. Then there is a group homomorphism with dense image

$$c_X^{\text{top} \rightarrow \text{proét}} : \pi_1^{\text{top}}(X, x_0) \longrightarrow \pi_1^{\text{proét}}(X, x_0)$$

making the following diagram commutative, where $c_X^{\text{top} \rightarrow \text{proét}}$ and $c_X^{\text{proét} \rightarrow \text{ét}}$ are the natural comparison maps.

Moreover, for each discrete quotient $u: \pi^{\text{proét}}(X, x_0) \twoheadrightarrow G$, i.e. a surjective homomorphism where $G$ is a discrete group, the composition $u \circ c_X^{\text{top} \rightarrow \text{proét}}$ is also surjective. In particular, there is no nontrivial continuous homomorphism from $\pi_1^{\text{proét}}(X, x_0)$ to a discrete group $G$ whose restriction to $\pi_1^{\text{top}}(X\text{an}, x_0)$ is trivial.

Proof. The existence of $c_X^{\text{top} \rightarrow \text{proét}}$ and the commutativity of the diagram follow readily from the infinite Galois theory à la Bhatt-Scholze (cf. [1, §7.2 and §7.4]).

The fact that $c_X^{\text{top} \rightarrow \text{proét}}$ has dense image follows from [6, Proposition 2.64] and 3.1.

For the last statement we can view $G$ as a set equipped with a transitive $\pi_1^{\text{proét}}(X, x_0)$-action. By 3.1 any $\pi_1^{\text{top}}(X\text{an}, x_0)$-equivariant subset of $G$ is also a $\pi_1^{\text{proét}}(X, x_0)$-equivariant subset. Thus $G$ is also a transitive $\pi_1^{\text{top}}(X\text{an}, x_0)$-set, i.e. $u \circ c_X^{\text{top} \rightarrow \text{proét}}$ is surjective as desired. \hfill $\square$

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Jiu-Kang Yu, The Chinese University of Hong Kong, Institute of Mathematical Science (IMS), Shatin, New Territories, Hong Kong

Email address: jkyu@ims.cuhk.edu.hk

Lei Zhang, Sun Yat-Sen University, School of Mathematics (Zhuhai), Zhuhai, Guangdong, P. R. China

Email address: cumt559@gmail.com