Abstract. Let $S$ be a standard graded polynomial ring over a field, and $I$ be a homogeneous ideal that contains a regular sequence of degrees $d_1, \ldots, d_n$. We prove the Eisenbud-Green-Harris conjecture when the forms of the regular sequence satisfy $d_i \geq \sum_{j=1}^{i-1} (d_j - 1)$, improving a result due to the first author and Maclagan.

1. Introduction

Let $S = \mathbb{k}[x_1, \ldots, x_n]$ be a standard graded polynomial ring over a field $\mathbb{k}$. Given a homogeneous ideal $I$, Macaulay’s Theorem states that there exists a lexicographic ideal with the same Hilbert function as $I$. A lexicographic ideal is a monomial ideal which, in each degree, is either the zero vector space or it contains exactly all the monomials which are greater or equal than a given one with respect to the lexicographic order. Another way to interpret Macaulay’s Theorem is in terms of growth of Hilbert functions: the knowledge of the Hilbert function of $I$ in a given degree $j$ allows to estimate the growth in degree $j+1$. In fact, Macaulay’s Theorem states that in degree $j+1$ the ideal $I$ grows at least as a lexicographic ideal which has the same Hilbert function as $I$ in degree $j$.

The Eisenbud-Green-Harris conjecture (henceforth EGH, see Conjecture 2.3) is an attempt to improve Macaulay’s Theorem by taking into account additional information about the ideal $I$. Namely, the information that $I$ contains a regular sequence of given degrees $d_1, \ldots, d_h$ should, in principle, give more restrictions on the growth of the Hilbert function. In this setup, the natural substitute for lexicographic ideals are the so-called lex-plus-power ideals (henceforth, LPP). As the name suggests, such objects are simply monomial ideals that can be written as a lexicographic ideal plus an ideal generated by pure powers of the variables, that is, $(x_1^{d_1}, \ldots, x_h^{d_h})$.

The EGH conjecture was also introduced and studied in relation to Cayley-Bacharach type theorems [EGH96]. For instance, see [GK13, CDS20] for results in this direction.

The EGH conjecture for ideals containing a regular sequence of degrees $d_1 \leq \ldots \leq d_h$ is known to hold in several specific cases, which include the following: when $h = 2$ [Ric04, Coo08], when $h = 3$ provided $d_1 = 2$ or $d_1 = 3$ and $d_2 = d_3$ [Coo12], if the regular sequence is monomial [CL69, MP06, CK13], if the ideal is generated by a quadratic regular sequence plus general quadratic forms [HP98, Gas99], if the regular sequence decomposes as a product of linear forms [Abe15], if the ideal is monomial [Abe16], or if $d_1 = \ldots = d_h = 2$ and $h \leq 5$ [GH19].

In 2008, the first author and Maclagan proved that the EGH conjecture holds if the degrees $d_1 \leq \ldots \leq d_h$ of the regular sequence contained in $I$ satisfy $d_i > \sum_{j=1}^{i-1} (d_j - 1)$ for all $i \geq 3$, see [CM08, Theorem 2]. We extend this result by improving each inequality by 1.

Theorem A (see Corollary 3.7). Let $I \subseteq S = \mathbb{k}[x_1, \ldots, x_n]$ be a homogeneous ideal containing a regular sequence of degrees $d_1 \leq \ldots \leq d_h$, such that $d_i \geq \sum_{j=1}^{i-1} (d_j - 1)$ for all $i \geq 3$. Then $I$...
satisfies the EGH conjecture, that is, there exists a lex-plus-power ideal containing \((x_1^{d_1}, \ldots, x_h^{d_h})\) with the same Hilbert function as \(I\).

We actually prove a stronger statement than Theorem A: if any homogeneous ideal \(I\) containing a given regular sequence \(f_1, \ldots, f_{h-1}\) of degrees \(d_1 \leq \ldots \leq d_{h-1}\) satisfies the EGH conjecture with respect to such a sequence, then \(I + (f_h)\) satisfies the EGH conjecture with respect to the sequence \(f_1, \ldots, f_{h-1}, f_h\) for any element \(f_h\) which is regular modulo \((f_1, \ldots, f_{h-1})\), and has degree at least \(\sum_{i=1}^{h-1}(d_i - 1)\) (see Theorem 3.6).

We point out that the strategy employed in [CM08] could not be directly used to prove Theorem A. In fact, [CM08, Theorem 2] relies on estimates on the Hilbert function of ideals containing shorter regular sequences, which are then glued together using linkage. In Theorem A, on the other hand, if \(d_i = \sum_{j=1}^{i-1}(d_j - 1)\) for some \(i\) there is one critical degree, namely \(d_i\), where the gluing process could in principle go wrong. We tackle this problem by carefully estimating the growth of the Hilbert function in that degree, and comparing it with the one of the corresponding LPP ideal. Using the same techniques, we can also prove some additional results which are not covered by the main theorem. For instance, with the notation used above, we recover the case \(h = 3, d_1 = 3\) and \(d_2 = d_3\) proved in [Coo12], and we prove the new case \(h = 4, d_1 = d_2 = 2\) and \(d_3 = d_4\) (see Proposition 3.8).

As a final remark, consider the following very concrete scenario. Suppose we are given a homogeneous ideal \(I \subseteq k[x_1, x_2, x_3, x_4]\), with the knowledge that \(I\) contains a regular sequence of degrees \((4, 5, 6)\), and that \(HF(I; 6) = 20\). While the EGH conjecture is not known to hold in this case, after possibly enlarging the field \(k\) (which does not affect our considerations) the ideal \(I\) will certainly contain a regular sequence of degrees \((4, 5, 7)\). Then we can apply Theorem A to obtain that \(HF(I; 7) \geq 41\). While this is not the sharpest estimate predicted by the EGH conjecture (that is, \(HF(I; 7) \geq 43\)), it still provides a significantly better estimate than the one coming from Macaulay’s Theorem (that is, \(HF(I; 7) \geq 35\)).

The strategy outlined in the example above illustrates that, while all the listed results on the EGH conjecture can only be applied when some specific conditions are met by the ideal or by the degrees of the regular sequence, the main result of [CM08] and our improvement, Theorem A, provide an estimate on the growth of the Hilbert function, which is more accurate than Macaulay’s Theorem, for any homogeneous ideal.

2. Preliminaries

Let \(k\) be a field, and \(S = k[x_1, \ldots, x_n]\) be a polynomial ring over \(k\), with the standard grading. Given a finitely generated graded \(S\)-module \(M = \bigoplus_{j \in \mathbb{Z}} M_j\), we will denote by \(HF(M; j) = \dim_k M_j\) its Hilbert function in degree \(j\).

On \(S\) we will consider the degree-lexicographic order, which we denote \(>_{\text{lex}}\), and the variables of \(S\) will be ordered by \(x_1 >_{\text{lex}} x_2 >_{\text{lex}} \ldots >_{\text{lex}} x_n\). A graded vector space \(V\) is a lex-segment if, whenever \(u, v \in S\) are monomials of the same degree with \(u >_{\text{lex}} v\), then \(u \in V\) whenever \(v \in V\). A monomial ideal \(L \subseteq S\) is called a lexicographic ideal if its graded components \(L_j\) are lex-segments for all \(j \in \mathbb{Z}\).

A degree sequence is a vector \(d = (d_1, \ldots, d_h) \in \mathbb{N}^h\), with \(1 \leq d_1 \leq \ldots \leq d_h\). Given a regular sequence \(f_1, \ldots, f_h \in S\) we say that it has degree \(d = (d_1, \ldots, d_h)\) if \(\deg(f_i) = d_i\) for \(i = 1, \ldots, h\).

**Definition 2.1.** Let \(S = k[x_1, \ldots, x_n]\), and \(d\) be a degree sequence. A monomial ideal \(L\) is called a \(d\)-LPP ideal if \(L = (x^d) + L\), where \(L\) is a lexicographic ideal.
Remark 2.2. If $\mathbb{L}$ is a $d$-LPP ideal then we may also write $\mathbb{L} = (x^d) \oplus V$, where $V = \bigoplus_j V_j$ is a graded $k$-vector space generated by monomials. Observe that, in general, $V$ is not an ideal.

We now recall the most current version of the Eisenbud-Green-Harris conjecture (see [CM08]).

Conjecture 2.3 (EGH$_{d,n}$). Let $I \subseteq S = k[x_1, \ldots, x_n]$ be a homogeneous ideal containing a regular sequence of degree $d$. We say that $I$ satisfies EGH$_{d,n}$ if there exists a $d$-LPP ideal with the same Hilbert function as $I$.

Observe that the case in which no regular sequence is taken into account is the well-known Macaulay’s Theorem on the existence of a lexicographic ideal with the same Hilbert function as $I$.

As in Macaulay’s Theorem, the $d$-LPP ideal of Conjecture 2.3 is unique, whenever it exists.

We make the following definition, based on [CM08, Definition 11].

Definition 2.4. Let $j$ be a non-negative integer, and $I \subseteq S = k[x_1, \ldots, x_n]$ be a homogeneous ideal containing a regular sequence of degree $d$. We say that $I$ satisfies EGH$_{d,n}(j)$ if there exists a $d$-LPP ideal $\mathbb{L}$ such that $HF(I; j) = HF(\mathbb{L}; j)$ and $HF(I; j + 1) \geq HF(\mathbb{L}; j + 1)$.

It is immediate to see that an ideal $I$ satisfies EGH$_{d,n}$ if and only if it satisfies EGH$_{d,n}(j)$ for all non-negative integers $j$.

3. Main result

We start by setting up some notation. Let $\mathfrak{f} = (f_1, \ldots, f_h)$ be an ideal of $S = k[x_1, \ldots, x_n]$ generated by a regular sequence of degrees $d = (d_1, \ldots, d_h)$, with $1 \leq d_1 \leq \ldots \leq d_h$. After possibly enlarging $k$, which does not affect any of our considerations, we may find $n - h$ linear forms $\ell_1, \ldots, \ell_{n-h}$ such that $f_1, \ldots, f_h, \ell_1, \ldots, \ell_{n-h}$ is a maximal regular sequence in $S$. After a change of coordinates, we may assume that $\ell_i = x_{h+i}$ for all $i = 1, \ldots, n - h$, and therefore we may view $\mathfrak{f} + (x_{h+1}, \ldots, x_n)$ as an ideal $\overline{\mathfrak{f}}$ inside $\overline{S}$, still generated by a regular sequence of degrees $d$. We will refer to $\overline{\mathfrak{f}}$ as an Artinian reduction of $\mathfrak{f}$. Clearly, the Artinian reduction depends on the choice of the original linear forms $\ell_1, \ldots, \ell_{n-h}$. However, by [CK14, Theorem 4.1] we have that if every homogeneous ideal of $\overline{S}$ that contains $\overline{\mathfrak{f}}$ satisfies EGH$_{d,h}$, then every homogeneous ideal of $S$ that contains $\mathfrak{f}$ satisfies EGH$_{d,n}$.

Remark 3.1. To the best of our knowledge, in all the cases where the EGH conjecture is known to hold the converse to the last statement is also true. For instance, most of the known cases only require numerical conditions on the degree sequence $d$ to be satisfied, independently of whether $\mathfrak{f}$ is Artinian or not.

Before proving the main theorem, we need some preparatory results.

Lemma 3.2. Let $d = (d_1, \ldots, d_h)$ be a degree sequence, and $\mathbb{L}$ be a $d$-LPP ideal such that $HF(\mathbb{L}; D) > HF((x^d); D)$ for some integer $D$. Given $D'$ such that $D \leq D' \leq \sum_{i=1}^h (d_i - 1)$, there exists a $d$-LPP ideal $\mathbb{L}'$ such that

$$HF(\mathbb{L}'; j) = \begin{cases} HF((x^d); j) & \text{if } j < D \\ HF(\mathbb{L}; j) - 1 & \text{if } D \leq j \leq D' \\ HF(\mathbb{L}; j) & \text{otherwise.} \end{cases}$$

Proof. As in Remark 2.2, we may write $\mathbb{L} = (x^d) \oplus V$, where $V = \bigoplus_j V_j$ is a graded vector space generated by monomials. By assumption $V_D \neq 0$, and this implies $V_j \neq 0$ for all $D \leq$
Lemma 3.3. Let $I \subseteq S = k[x_1, \ldots, x_n]$ be a homogeneous ideal generated in degree at most $D$. If $0 < \text{HF}(S/I; D) = \text{HF}(S/I; D + 1) \leq D$, then $\dim(S/I) = 1$.

Proof. The condition $\text{HF}(S/I; D) = \text{HF}(S/I; D + 1) \leq D$ implies, by Macaulay’s Theorem, that the lexicographic ideal with the same Hilbert function as $I$ has no minimal generator in degree $D + 1$. It follows from Gotzmann’s Persistence Theorem [Got78, Gre89] that $\text{HF}(S/I; D) = \text{HF}(S/I; j)$ for all $j \geq D$. As this value is positive by assumption, this forces $\dim(S/I) = 1$. □

The next proposition, even though not stated in this generality, is a direct consequence of the techniques used in [CM08].

Proposition 3.4. Let $I \subseteq S = k[x_1, \ldots, x_n]$ be a homogeneous ideal containing an ideal $\mathfrak{f}$ generated by a regular sequence $f_1, \ldots, f_h$ of degrees $d = (d_1, \ldots, d_h)$. Let $\mathfrak{f}' = (f_1, \ldots, f_{h-1})$, $d' = (d_1, \ldots, d_{h-1})$, and fix an Artinian reduction $\overline{\mathfrak{f}}$ of $\mathfrak{f}'$ inside $\overline{S} = k[x_1, \ldots, x_{n-1}]$. Assume that every homogeneous ideal of $\overline{S}$ containing $\overline{\mathfrak{f}}$ satisfies EGH$_{d', h-1}$. If $d_h > \sum_{i=1}^{h-1}(d_i - 1)$, then $I$ satisfies EGH$_{d, n}$.

Proof. By [CK14, Theorem 4.1], in order to prove the proposition we may assume that $h = n$, and that the image of $\mathfrak{f}' + (x_n)$ inside $S/(x_n)$ is identified with the fixed Artinian reduction $\overline{\mathfrak{f}}$ of $\mathfrak{f}'$. Again by [CK14, Theorem 4.1], we have that any ideal containing $\mathfrak{f}'$ satisfies EGH$_{d', n}$, since any ideal containing $\overline{\mathfrak{f}}$ satisfies EGH$_{d', n-1}$ by assumption. By [CM08, Lemma 12], it suffices to show that $I$ satisfies EGH$_{d, n}(j)$ for all $0 \leq j \leq d_n - 2$. Since $f_n$ has degree $d_n$ it is clear that, for $j$ in this range, $I$ satisfies EGH$_{d, n}(j)$ if and only if $I$ satisfies EGH$_{d', n}(j)$, and therefore the proof is complete. □

Notation 3.5. Let $\mathbb{L}$ be a $d$-LPP ideal, and $j \geq 0$ be an integer. If $\mathbb{L}_j \neq S_j$ we set $\text{out}(\mathbb{L}; j)$ to be the largest monomial with respect to the lexicographic order that does not belong to $\mathbb{L}_j$. If $\mathbb{L}_j = S_j$, then we set $\text{out}(\mathbb{L}; j) = 0$.

Theorem 3.6. Let $I \subseteq S = k[x_1, \ldots, x_n]$ be a homogeneous ideal containing an ideal $\mathfrak{f}$ generated by a regular sequence $f_1, \ldots, f_h$ of degrees $d = (d_1, \ldots, d_h)$. Let $\mathfrak{f}' = (f_1, \ldots, f_{h-1})$, $d' = (d_1, \ldots, d_{h-1})$, and fix an Artinian reduction $\overline{\mathfrak{f}}$ of $\mathfrak{f}'$ inside $\overline{S} = k[x_1, \ldots, x_{n-1}]$. Assume that every homogeneous ideal of $\overline{S}$ containing $\overline{\mathfrak{f}}$ satisfies EGH$_{d', h-1}$. If $d_h > \sum_{i=1}^{h-1}(d_i - 1)$, then $I$ satisfies EGH$_{d, n}$.

Proof. Without loss of generality, we may assume that $k$ is infinite. By [CK14, Theorem 4.1] we may assume that $h = n$. By Proposition 3.4, we may assume that $d_n = \sum_{i=1}^{n-1}(d_i - 1)$, so that $\sum_{i=1}^{n}(d_i - 1) = 2d_n - 1$. By [CM08, Lemma 12], in order to prove the theorem it suffices
to show that $I$ satisfies EGH$_{d,n}(j)$ for all $0 \leq j \leq d_n - 1$. As in the proof of Proposition 3.4 we have that $J$ satisfies EGH$_{d',n}$ by [CK14, Theorem 4.1], given that $I$ contains $f'$ and EGH$_{d',n-1}$ is assumed to be true. Since for $j \leq d_n - 2$ EGH$_{d',n}(j)$ is clearly equivalent to EGH$_{d,n}(j)$, it suffices to prove that $I$ satisfies EGH$_{d,n}(d_n - 1)$.

Let $\{v_1, \ldots, v_c\} \subseteq I_{d_n-1}$ be the pre-image in $S$ of a $k$-basis of $(I/\hat{f})_{d_n-1}$, and consider the ideal $Q = (f_1, \ldots, f_{n-1}, v_1, \ldots, v_c)$. First, assume that $f_n \not\in Q$. In this case, we have HF($I; d_n$) $\geq$ HF($Q; d_n$) + 1. By assumption, we have that $Q$ satisfies EGH$_{d,n}(d_n - 1)$, that is, there exists a $d'$-LPP ideal $L$ with the same Hilbert function as $Q$ (and thus as $I$) in degree $d_n - 1$, and such that HF($Q; d_n$) $\geq$ HF($L; d_n$). If $x_n^{d_n} \in L$, then $I_{d_n} = Q_{d_n} = S_{d_n}$, which is a contradiction since $f_n \not\in Q$. Therefore $x_n^{d_n} \notin L$, and we have that HF($L + (x_n^{d_n}); d_n$) $=$ HF($L; d_n$) + 1. In particular, we have that HF($I; d_n$) $\geq$ HF($L + (x_n^{d_n}); d_n$). Since $L + (x_n^{d_n})$ is a $d'$-LPP ideal with the same Hilbert function as $I$ in degree $d_n - 1$, it follows that $I$ satisfies EGH$_{d,n}(d_n - 1)$, as desired.

From now on, assume that $f_n \in Q$. Under this assumption, in order to prove the theorem we may replace $I$ by $Q$. Then, since $Q$ has height $n$ and is generated in degree at most $d_n - 1$, we may assume that $f_1, \ldots, f_{n-1}, v_c$ is a maximal regular sequence of degree $d_n - 1$, inside $Q$. Let $g = (f_1, \ldots, f_{n-1}, v_c)$ and $J = g : Q$. As the degree of the socle of $S/g$ is $2d_n - 2$, by linkage (for instance, see [Mig98, Corollary 5.2.19]), we have that

$$HF(S/Q; d_n) = HF(S/g; d_n - 2) - HF(S/J; d_n - 2),$$

and

$$HF(S/Q; d_n - 1) = HF(S/g; d_n - 1) - HF(S/J; d_n - 1).$$

Let $L_1$ be the $d'$-LPP ideal with the same Hilbert function as $J$, which exists by hypothesis. Observe that HF($L_1; d_n - 2$) $\geq$ HF($L_1; d_n - 2$) $= HF(g; d_n - 2)$. If equality holds, then by the linkage formula used above we conclude that HF($S/Q; d_n$) $= 0$, that is, $Q_{d_n} = S_{d_n}$. In this case, $Q$ trivially satisfies EGH$_{d,n}(d_n - 1)$. If HF($L_1; d_n - 2$) $>$ HF($L_1; d_n - 2$), then by Lemma 3.2 applied to $L_1$ with $D = d_n - 2$ and $D' = d_n - 1$, there exists a $d''$-LPP ideal $L_2$ such that

$$HF(L_2; j) = \begin{cases} 
HF((x^d); j) & \text{if } j < d_n - 2 \\
HF(L_1; j) - 1 & \text{if } j = d_n - 2, d_n - 1 \\
HF(L_1; j) & \text{if } j > d_n - 1
\end{cases}$$

In particular, $L_2 + (x_n^{d_n-1})$ is a $d''$-LPP ideal which has the same Hilbert function as $L_1$ in degree $d_n - 1$, and value one less than $L_1$ in degree $d_n - 2$. If we let $L_3 = (x^{d''}) : (L_2 + (x_n^{d_n-1}))$, again by linkage we have HF($Q; d_n - 1$) $= HF(L_3; d_n - 1)$ and HF($Q; d_n$) $= HF(L_3; d_n)$. Furthermore, by [MP06, Theorem 1.2] there exists a $d''$-LPP ideal $L_4$ with the same Hilbert function as $L_3$. As in Remark 2.2, we may write $L_4 = (x^{d''}) \oplus V$, where $V = \bigoplus_j V_j$ is a graded $k$-vector space. We now want to trade $x_n^{d_n-1} \in (x^d)$ for $x_n^{d_n} \in (x^d)$ in the ideal $L_4$ we have just defined. To do so, we need to adjust the vector space $V_{d_n-1}$ by adding one specific monomial of degree $d_n - 1$, and then estimate its growth.

More specifically, consider the $k$-vector space $W = V_{d_n-1} \oplus V_{d_n}$, and let $L_5$ be the ideal generated by $(x^d) \oplus W$. It can be shown that $L_5$ is a $d'$-LPP ideal. Let $u = \text{out}(L_5; d_n) - 1$, and observe that $u \neq 0$. In fact, if $u = 0$, then necessarily $x_n^{d_n-1} \in W$, and therefore $(L_5)_{d_n-1} = S_{d_n-1}$. This forces $(L_4)_{d_n-1} = S_{d_n-1}$, and thus $Q_{d_n-1} = S_{d_n-1}$, because $Q$ and $L_4$ have the same Hilbert function in degree $d_n - 1$. But this is a contradiction, since this would imply $Q_{d_n} = (L_4)_{d_n} = S_{d_n}$, while we are assuming that HF($Q; d_n$) $= HF(L_4; d_n) - 1$. 


Now we let $\mathbb{L}(d) = \mathbb{L}_5 + (u, x_{d_n}^n)$, which is a $d$-LPP ideal. Observe that $x_{d_n}^n$ is necessarily a minimal generator of $\mathbb{L}(d)$, by what we have observed above. Moreover, we have that

$$HF(\mathbb{L}(d); d_n - 1) = HF(\mathbb{L}_5 + (u); d_n - 1) = HF(\mathbb{L}_5 + (x_{d_n}^{d_n-1}); d_n - 1)$$

$$= HF(\mathbb{L}_4; d_n - 1) = HF(Q; d_n - 1),$$

and

$$HF(\mathbb{L}(d); d_n) = HF(\mathbb{L}_5; d_n) + HF((u, x_{d_n}^n); \mathbb{L}_5; d_n)$$

$$= HF(\mathbb{L}_4; d_n) - HF((u, x_{d_n}^n); \mathbb{L}_5; d_n) + HF((u, x_{d_n}^{d_n-1}); \mathbb{L}_5; d_n)$$

$$= HF(Q; d_n) + 1 - HF((u, x_{d_n}^{d_n-1}; \mathbb{L}_5; d_n) + HF((u, x_{d_n}^n); \mathbb{L}_5; d_n).$$

To conclude the proof, we want to show that $HF(\mathbb{L}(d); d_n) \leq HF(Q; d_n)$. Since the degree of $u$ is $d_n - 1 = \sum_{i=1}^{n-1} (d_i - 1) - 1$, it is easy to see from the fact that lex-segments are strongly stable that $HF((u, x_{d_n}^n); \mathbb{L}_5; d_n) = 3$ if and only if $u = x_{d_n}^{d_n-1} \ldots x_{d_n-1}^{d_n-2}$, and $HF((u, x_{d_n}^n); \mathbb{L}_5; d_n) \leq 2$ in all the other cases. However, if $u = x_{d_n}^{d_n-1} \ldots x_{d_n-2}$, then it follows by a direct computation that $HF(\mathbb{L}(d); d_n - 1) = HF((u); d_n - 1) + 1$. In our running assumptions, this forces $c = 1$, and $Q = (f_1, \ldots, f_{n-1}, v_1)$ to be generated by a regular sequence of degree $d''$. Since by [MP06, Theorem 1.2] there is a $d$-LPP ideal with the same Hilbert function as $(x^{d''})$, this concludes the proof in this case.

We will henceforth assume that $HF((u, x_{d_n}^n); \mathbb{L}_5; d_n) \leq 2$, so that by the above computation

$$HF(\mathbb{L}(d); d_n) \leq HF(Q; d_n) - HF((x_{d_n}^{d_n-1}; \mathbb{L}_5; d_n) + 3.$$

We now need to estimate the growth of $(x_{d_n}^{d_n-1}; \mathbb{L}_5$ in degree $d_n$. In order to do so, we let $\ell = \min\{i = 1, \ldots, n \mid x_i x_{d_n-1}^{d_n-1} \notin \mathbb{L}_5\}$. If $\ell \leq n - 2$, then $HF((x_{d_n}^{d_n-1}; \mathbb{L}_5; d_n) \geq 3$, and the proof is complete.

For the rest of the proof, assume that $\ell \geq n - 1$. Under this assumption, we necessarily have that $u \in \mathbb{k}[x_{n-1}, x_n]$ and, in fact, $u \leq_{\text{lex}} x_{d_n-1}^{d_n-1} x_n^{d_n-d_n-1}$. Moreover, a direct computation shows that $HF(S(\mathbb{L}(d); d_n) - 1) = HF(S(\mathbb{L}(d); d_n) - 1)$, Macaulay’s Theorem implies that $HF(S/Q; j) \leq HF(S/Q; d_n - 1)$ for all $j \geq d_n - 1$. If $HF(S(Q); d_n - 1) = HF(S(Q); d_n)$, then by Lemma 3.3 we would have $\dim(S(Q) = 1$. However, $f \subseteq Q$ by our running assumptions, and therefore $\dim(S/Q) = 0$, a contradiction. Thus, we have that $HF(S/Q; d_n) < HF(S/Q; d_n - 1)$. Therefore $HF(S/Q; d_n) \leq HF(S/Q; d_n - 1) - 1 = HF(S/\mathbb{L}(d); d_n - 1) - 1 = HF(S/\mathbb{L}(d); d_n)$, where the last equality follows from a direct computation, thanks to the knowledge of the monomial $u$.

As a consequence of Theorem 3.6, we obtain the main result of this article.

**Corollary 3.7.** Let $I \subseteq S = \mathbb{k}[x_1, \ldots, x_n]$ be a homogeneous ideal containing a regular sequence of degrees $d$, such that $d_i \geq \sum_{j=1}^{i-1} (d_j - 1)$ for all $i \geq 3$. Then $I$ satisfies EGH$_{d,n}$.

**Proof.** This follows immediately by induction on the length $h$ of the regular sequence, and Theorem 3.6. \hfill $\Box$

Consider the EGH conjecture in the case of the degree sequences $d = (2, d, d)$ and $d = (3, d, d)$, both covered in [Coo12]. While the first case is a consequence of Corollary 3.7, the same is not true for $d = (3, d, d)$. However, by means of the same techniques used in Theorem 3.6, we can recover the second case as well, and prove an additional one.

**Proposition 3.8.** Let $I \subseteq \mathbb{k}[x_1, \ldots, x_n]$ be a homogeneous ideal that contains a regular sequence of degree $d$, where $d$ is either $(3, d, d)$ or $(2, 2, d, d)$. Then $I$ satisfies EGH$_{d,n}$.
Proof. Let $\mathcal{f}$ be the ideal generated by a regular sequence of degree $d$ inside $I$, and $\mathcal{f} \subseteq \mathcal{f}$ be the ideal generated only by the forms of degree $d'$. If $d' = (3, d)$ if $d = (3, d, d)$, and $d' = (2, 2, d)$ if $d = (2, 2, d, d)$. Observe that Conjecture EGH$_{d', n}$ is known to hold for every ideal containing a regular sequence of degree $d'$ by [Ric04, Coo08] in the first case, and by Corollary 3.7 in the second. By [CM08, Lemma 12], it suffices to prove that $I$ satisfies EGH$_{d,n}(j)$ for $0 \leq j \leq d-1$. Moreover, as in the proof of Theorem 3.6, the critical case is $j = d-1$. Let $\{v_1, \ldots, v_c\}$ be the pre-image in $S$ of a $\mathbb{k}$-basis of $(\mathcal{f})_{d-1}$, and $Q = \mathcal{f} + (v_1, \ldots, v_c)$.

First assume $d = (3, d, d)$. If $f_3 \notin Q$, then since $Q$ satisfies EGH$_{d', n}$ there exists a $d'$-LPP ideal $\mathbb{L}$ with the same Hilbert function as $Q$ in degree $d-1$, and such that $HF(\mathbb{L}; d) \geq HF(\mathbb{L}; d)$. As in the proof of Theorem 3.6, one can check that $\mathbb{L} + (x_3^d)$ is a $d$-LPP ideal, and we have that $HF(I; d) \geq HF(Q; d) + 1 \geq HF(\mathbb{L}; d) + 1 = HF(\mathbb{L} + (x_3^d); d)$. On the other hand, if $f_3 \in Q$, then we may assume that $f_1, f_2, v_c$ is a regular sequence of degrees $3, d$ and $d-1$. If $d = 3$, then we let $d'' = (2, 3, 3)$, while if $d > 3$, then we let $d'' = (3, d - 1, d)$. Either way, $I$ satisfies EGH$_{d', n}$ by Corollary 3.7, and therefore there exists a $d''$-LPP ideal $\mathbb{L}$ with the same Hilbert function as $I$. In particular, since $L$ contains $(x_1^3, x_2^d, x_3^d)$, we have that $I$ satisfies EGH$_{d,n}$ by [MP06, Theorem 1.2].

The case $d = (2, 2, d, d)$ is similar. If $f_4 \notin Q$, then the proof goes as in the previous case, using that $I$ satisfies EGH$_{d', n}$. If $f_4 \in Q$, then we may assume that $f_1, f_2, f_3, v_c$ forms a regular sequence of degrees $2, 2, d$ and $d-1$. If $d = 2$ then we let $d'' = (1, 2, 2, 2)$, while if $d > 2$ we let $d'' = (2, 2, d - 1, d)$. Either way, $I$ satisfies EGH$_{d', n}$ by Corollary 3.7, and we conclude using [MP06, Theorem 1.2] as above that it satisfies EGH$_{d,n}$ as well.

□

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