INTERIOR AND EXTERIOR CONTACT PROBLEMS WITH FRICTION FOR HEMITROPIC SOLIDS: BOUNDARY VARIATIONAL INEQUALITY APPROACH

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Abstract We study the interior and exterior contact problems for hemitropic elastic solids. We treat the cases when the friction effects, described by Tresca friction (given friction model), are taken into consideration either on some part of the boundary of the body or on the whole boundary. We equivalently reduce these problems to a boundary variational inequality with the help of the Steklov-Poincaré type operator. Based on our boundary variational inequality approach we prove existence and uniqueness theorems for weak solutions. We prove that the solutions continuously depend on the data of the original problem and on the friction coefficient. For the interior problem necessary and sufficient conditions of solvability are established when friction is taken into consideration on the whole boundary.

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1 Introduction

The main goal of the present paper is the study of contact problems for hemitropic elastic solids with friction obeying the Tresca friction model, their mathematical modelling as nonsmooth boundary value problems and their analysis with the help of the boundary variational inequality technique.

Technological and industrial developments, and also essential success in biological and medical sciences require to use more generalized and refined models for elastic bodies. In recent years, theories of continuum mechanics with a complex microstructure have been the object of intensive research. Classical elasticity associates only the three translational degrees of freedom to material points of the body and all the mechanical characteristics are expressed by the corresponding
displacement vector. On the contrary, micropolar theory, by including intrinsic rotations of the particles, provides a rather complex model of an elastic body that can support body forces and body couple vectors as well as force stress vectors and couple stress vectors at the surface. Consequently, in micropolar theory all the mechanical quantities are written in terms of the displacement and microrotation vectors.

The origin of the rational theories of polar continua goes back to brothers E. and F. Cosserat [CC1], [CC2], who gave a development of the mechanics of continuous media in which each material point has the six degrees of freedom defined by 3 displacement components and 3 microrotation components (for the history of the theory of micropolar elasticity see [Dy1], [KGBB1], [Min1], [Now1], and the references therein).

A micropolar solid which is not isotropic with respect to inversion is called hemitropic, noncentrosymmetric, or chiral. Materials may exhibit chirality on the atomic scale, as in quartz and in biological molecules - DNA, as well as on a large scale, as in composites with helical or screw-shaped inclusions, certain types of nanotubes, bone, fabricated structures such as foams, chiral sculptured thin films and twisted fibers. Experiments have shown that micropolar materials possess quite different properties in comparison with the classical elastic materials. For example, the twisting behaviour under an axial stress is a purely hemitropic (chiral) phenomenon and has no counterpart in classical elasticity. For more details and applications see the references [AK1], [AK2], [CC1], [Dy1], [Er1], [HZ1], [La1], [LB1], [Mu1], [Mu2], [Now1], [Ro1], [Sh1], [YL1].

Refined mathematical models describing the hemitropic properties of elastic materials have been proposed by Aero and Kuvshinski [AK1], [AK2]. In the mathematical theory of hemitropic elasticity there are introduced the asymmetric force stress tensor and moment stress tensor, which are kinematically related with the asymmetric strain tensor and torsion (curvature) tensor. The governing equations in this model become very involved and generate $6 \times 6$ matrix partial differential operator of second order.

In [NGGS1], [NGS1], [NGZ1], [NS1] the fundamental matrices of the associated systems of partial differential equations of statics and steady state oscillations have been constructed explicitly in terms of elementary functions and the basic boundary value and transmission problems of hemitropic elasticity have been studied by the potential method for smooth and non-smooth Lipschitz domains. Particular problems of the elasticity theory of hemitropic continuum have been considered in [EL1], [La1], [LB1], [LVV1], [LVV2], [Now1], [Now2], [NN1], [We1]. The frictionless unilateral contact problems for hemitropic solids have been studied in [GGN1].

In classical elasticity similar contact problems have been considered in many monographs and papers (see, e.g., [DuLi1], [EJK], [Fi1], [Fi2], [GaNa1], [GS1], [HLNL1], [HH1], [Han1], [KiOd1], [Ro1], [SST], and the references therein).

The paper is organized as follows. First we give the general field equations of the linear theory of elasticity for hemitropic materials. Then we present a reasonable mathematical model for the boundary conditions that apply to hemitropic solids in contact with friction. We start with interior problems and consider the case when some portion of the boundary is mechanically fixed and the original problem is modelled as a coercive boundary variational inequality. Further, we treat a more complicated case when only traction-contact conditions are considered on the whole boundary. For this problem the corresponding bilinear form is not coercive any more and we need the more involved theory of semicoercive variational problems (see [EJ1], [Goe], [GG1] for related problems in classical linear elasticity and nonlinear elasticity). In this more involved case, we establish the necessary and sufficient conditions of solvability. Next we show that the similar exterior problems are uniquely solvable. On the basis of the results obtained we prove that solutions of the boundary variational inequality and, consequently, the corresponding solutions of the original contact problems continuously depend on the data of the problem and on the friction coefficient.
2 Field equations and Green’s formulas

2.1 Basic Equations

Let $\Omega^+ \subset \mathbb{R}^3$ be a bounded domain with a $C^\infty$ smooth (we can later relax this assumption), simply connected boundary $S := \partial \Omega^+ \cup \Omega^-$ and $\Omega := \mathbb{R}^3 \setminus \overline{\Omega^+}$. It is evident that $\partial \Omega^- = S$.

We assume that $\Omega \in \{\Omega^+, \Omega^-, \Omega^-\}$ is occupied by a homogeneous hemitropic elastic material. Denote by $u = (u_1, u_2, u_3)^T$ and $\omega = (\omega_1, \omega_2, \omega_3)^T$ the displacement vector and the micro-rotation vector, respectively. Here and in what follows the symbol $(\cdot)^T$ denotes transposition. Denote by $n(x) = (n_1(x), n_2(x), n_3(x))^T$ the outward normal vector to the surface $S$ at the point $x \in S$.

In hemitropic elasticity theory we have the following constitutive equations for the force stress tensor $\{\tau_{pq}\}$ and the couple stress tensor $\{\mu_{pq}\}$ for $p, q = 1, 2, 3$:

$$
\tau_{pq} = \tau_{pq}(U) := (\mu + \alpha) \partial_p u_q + (\mu - \alpha) \partial_q u_p + \lambda \delta_{pq} \text{div} u + \delta \delta_{pq} \text{div} \omega + (\chi + \nu) \partial_p \omega_q + (\chi - \nu) \partial_q \omega_p - 2\alpha \sum_{k=1}^3 \epsilon_{pqk} \omega_k, \quad (2.1)
$$

$$
\mu_{pq} = \mu_{pq}(U) := \delta \delta_{pq} \text{div} u + (\chi + \nu) \left[ \partial_p u_q - \sum_{k=1}^3 \epsilon_{pqk} \omega_k \right] + \beta \delta_{pq} \text{div} \omega + (\chi - \nu) \left[ \partial_q u_p - \sum_{k=1}^3 \epsilon_{pqk} \omega_k \right] + (\gamma + \varepsilon) \partial_p \omega_q + (\gamma - \varepsilon) \partial_q \omega_p, \quad (2.2)
$$

where $U = (u, \omega)^T$, $\partial = (\partial_1, \partial_2, \partial_3)$ with $\partial_j = \partial / \partial x_j$, $\delta_{pq}$ is the Kronecker delta, $\epsilon_{pqk}$ is the permutation (Levi-Civitá) symbol, and $\alpha, \beta, \gamma, \delta, \lambda, \mu, \nu, \chi$ and $\varepsilon$ are the material constants, see [AK1], [NGS1].

The components of the force stress vector $\tau^{(n)} = (\tau_1^{(n)}, \tau_2^{(n)}, \tau_3^{(n)})^T$ and the couple stress vector $\mu^{(n)} = (\mu_1^{(n)}, \mu_2^{(n)}, \mu_3^{(n)})^T$, acting on a surface element with the normal vector $n$ read as

$$
\tau_q^{(n)} = \sum_{p=1}^3 \tau_{pq} n_p, \quad \mu_q^{(n)} = \sum_{p=1}^3 \mu_{pq} n_p, \quad q = 1, 2, 3. \quad (2.3)
$$

Let us introduce the $6 \times 6$ matrix differential stress operator $T(\partial, n)$ [NGS1]

$$
T(\partial, n) = \begin{bmatrix}
T^{(1)}(\partial, n) & T^{(2)}(\partial, n) \\
T^{(3)}(\partial, n) & T^{(4)}(\partial, n)
\end{bmatrix}_{6 \times 6}, \quad T^{(j)} = [T^{(j)}_{pq}]_{3 \times 3}, \quad j = 1, 4, \quad (2.4)
$$

$$
T^{(1)}_{pq}(\partial, n) = (\mu + \alpha) \delta_{pq} \partial_n + (\mu - \alpha) n_q \partial_p + \lambda n_p \partial_q, \\
T^{(2)}_{pq}(\partial, n) = (\chi + \nu) \delta_{pq} \partial_n + (\chi - \nu) n_q \partial_p + \delta n_p \partial_q - 2\alpha \sum_{k=1}^3 \epsilon_{pqk} n_k, \\
T^{(3)}_{pq}(\partial, n) = (\chi + \nu) \delta_{pq} \partial_n + (\chi - \nu) n_q \partial_p + \delta n_p \partial_q, \\
T^{(4)}_{pq}(\partial, n) = (\gamma + \varepsilon) \delta_{pq} \partial_n + (\gamma - \varepsilon) n_q \partial_p + \beta n_p \partial_q - 2\nu \sum_{k=1}^3 \epsilon_{pqk} n_k, \quad (2.5)
$$

where $\partial_n = \partial / \partial n$ denotes the normal derivative.

From the formulas (2.1), (2.2) and (2.3) it can be easily checked that

$$
\left( \tau^{(n)}, \mu^{(n)} \right)^T = T(\partial, n) U. \quad (2.6)
$$
The equilibrium equations in the theory of hemitropic elasticity read as, see \[\text{[AK1, NGS1]}\]

\[\begin{align*}
\sum_{p=1}^{3} \partial_p \tau_{pq}(x) + \varrho \tilde{X}_q^{(1)}(x) &= 0, \\
\sum_{p=1}^{3} \partial_p \mu_{pq}(x) + \sum_{l,r=1}^{3} \varepsilon_{qlr} \tau_{lr}(x) + \varrho \tilde{X}_q^{(2)}(x) &= 0, \quad q = 1, 2, 3,
\end{align*}\]

where \(\varrho\) is the mass density of the elastic material, while \(\tilde{X}^{(1)} = (\tilde{X}_1^{(1)}, \tilde{X}_2^{(1)}, \tilde{X}_3^{(1)})^\top\), and \(\tilde{X}^{(2)} = (\tilde{X}_1^{(2)}, \tilde{X}_2^{(2)}, \tilde{X}_3^{(2)})^\top\) are the body force and body couple vectors, respectively.

Using the constitutive equations (2.1) and (2.2) we can rewrite the equilibrium equations in terms of the displacement and micro-rotation vectors, \(\Delta \omega(x) + (\varepsilon + \nu) \text{ grad div } u(x) + (\kappa + \mu) \text{ grad div } \omega(x) + 2\alpha \text{ curl } \omega(x) + \varrho \tilde{X}^{(1)}(x) = 0, \)

\(+(\gamma + \varepsilon) \Delta \omega(x) + (\beta + \gamma - \varepsilon) \text{ grad div } \omega(x) + 4\nu \text{ curl } \omega(x) - 4\alpha \omega(x) + \varrho \tilde{X}^{(2)}(x) = 0, \quad (2.7)\)

where \(\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2\) is the Laplace operator.

Let us introduce the matrix differential operator given by the left hand side of (2.7):

\[L(\partial) := \begin{bmatrix} L^{(1)}(\partial) & L^{(2)}(\partial) \\ L^{(3)}(\partial) & L^{(4)}(\partial) \end{bmatrix} \in \mathbb{R}^{6 \times 6}. \quad (2.8)\]

\[L^{(1)}(\partial) := (\mu + \alpha) \Delta I_3 + (\lambda + \mu - \alpha) Q(\partial), \]

\[L^{(2)}(\partial) := (\varepsilon + \nu) \Delta I_3 + (\delta + \varepsilon - \nu) Q(\partial) + 2\alpha R(\partial), \quad (2.7)\]

\[L^{(4)}(\partial) := ((\gamma + \varepsilon) \Delta - 4\alpha) I_3 + (\beta + \gamma - \varepsilon) Q(\partial) + 4\nu R(\partial), \]

where and in the sequel \(I_k\) stands for the \(k \times k\) unit matrix and

\[Q(\partial) := [\partial_k \partial_j]_{3 \times 3}, \quad R(\partial) := \begin{bmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{bmatrix} \in \mathbb{R}^{3 \times 3}. \]

It is easy to see that

\[R(\partial) u = \begin{bmatrix} \partial_2 u_3 - \partial_3 u_2 \\ \partial_3 u_1 - \partial_1 u_3 \\ \partial_1 u_2 - \partial_2 u_1 \end{bmatrix} = \text{curl } u, \quad Q(\partial) u = \text{grad div } u. \]

Thus (2.7) can be written in matrix form as

\[L(\partial) U(x) + X(x) = 0 \quad \text{with} \quad U = (u, \omega)^\top, \quad X = (X^{(1)}, X^{(2)})^\top := (\varrho \tilde{X}^{(1)}, \varrho \tilde{X}^{(2)})^\top. \]

Note that the operator \(L(\partial)\) is formally self-adjoint, i.e., \(L(\partial) = [L(-\partial)]^\top\).
2.2 Green’s formulas

For real-valued vector functions $U = (u, \omega)^\top$ and $U' = (u', \omega')^\top$ from the class $[C^2(\Omega^\pm)]^6$ the following Green formula holds \cite{NGS1}

$$
\int_{\Omega^+} \left[ L(\partial)U \cdot U' + E(U, U') \right] \, dx = \int_S \{T(\partial, n)U\}^+ \cdot \{U'\}^+ \, dS, \quad (2.9)
$$

where the symbols $\{ \cdot \}^\pm$ denote the one sided limits (trace operators) on $S$ from $\Omega^\pm$ respectively, while $E(\cdot, \cdot)$ is the bilinear form defined by

$$
E(U, U') = E(U', U) = \sum_{p, q=1}^3 \left\{ (\mu + \alpha)u_p' u_{pq} + (\mu - \alpha)u_{pq}' u_p + (\kappa + \varepsilon)u_p' u_{pq} + \delta(u_{pp}' u_{qq} + \omega_{pq}' u_{pp} + \lambda u_{pp}' u_{qq} + \beta u_{pp}' u_{qq}) \right\}, \quad (2.10)
$$

where $u_{pq}$ and $\omega_{pq}$ are the so called strain and torsion (curvature) tensors for hemitropic bodies,

$$
u_{pq} = u_{pq}(U) = \partial_p u_q - \sum_{k=1}^3 \varepsilon_{pqk} \omega_k, \quad \omega_{pq} = \omega_{pq}(U) = \partial_p \omega_q, \quad p, q = 1, 2, 3. \quad (2.11)
$$

Here and in what follows $a \cdot b := a^\top b$ is the usual scalar product of two vectors $a, b$. We can generalize Green’s formula \cite{2.10} to unbounded domains. We say that a vector $U = (u, \omega)^\top$ satisfies the decay condition $(Z)$ at infinity if for sufficiently large $|x|$.

$$
u_j(x) = O(|x|^{-1}), \quad \omega_j(x) = O(|x|^{-2}), \quad \frac{\partial \nu_j(x)}{\partial x_k} = O(|x|^{-2}), \quad \frac{\partial \omega_j(x)}{\partial x_k} = O(|x|^{-2}), \quad k, j = 1, 2, 3.
$$

Let $U = (u, \omega)^\top$ and $U' = (u', \omega')^\top$ belong to the class $[C^2(\Omega^-)]^6$ and satisfy the decay condition $(Z)$ at infinity. Then the following Green’s formula holds

$$
\int_{\Omega^-} \left[ L(\partial)U \cdot U' + E(U, U') \right] \, dx = - \int_S \{T(\partial, n)U\}^- \cdot \{U'\}^- \, dS.
$$

From formulas \cite{2.10} and \cite{2.11} we get

$$
E(U, U') = \frac{3\lambda + 2\mu}{3} \left( \text{div} \ u + \frac{3\delta + 2\kappa}{3\lambda + 2\mu} \text{div} \ \omega \right) \left( \text{div} \ u' + \frac{3\delta + 2\kappa}{3\lambda + 2\mu} \text{div} \ \omega' \right)
$$

$$
+ \frac{1}{3} \left( 3\beta + 2\gamma - \frac{(3\delta + 2\kappa)^2}{3\lambda + 2\mu} \right) \left( \text{div} \ \omega \right) \left( \text{div} \ \omega' \right)
$$

$$
+ \frac{\mu}{2} \sum_{k,j=1, k \neq j}^3 \left[ \frac{\partial u_k}{\partial x_j} + \frac{\partial u_j}{\partial x_k} + \frac{\kappa}{\mu} \left( \frac{\partial \omega_k}{\partial x_j} + \frac{\partial \omega_j}{\partial x_k} \right) \right]
$$

$$
\times \left[ \frac{\partial u_k'}{\partial x_j} + \frac{\partial u_j'}{\partial x_k} + \frac{\kappa}{\mu} \left( \frac{\partial \omega_k'}{\partial x_j} + \frac{\partial \omega_j'}{\partial x_k} \right) \right]$$

From formulas \cite{2.10} and \cite{2.11} we get

$$
E(U, U') = \frac{3\lambda + 2\mu}{3} \left( \text{div} \ u + \frac{3\delta + 2\kappa}{3\lambda + 2\mu} \text{div} \ \omega \right) \left( \text{div} \ u' + \frac{3\delta + 2\kappa}{3\lambda + 2\mu} \text{div} \ \omega' \right)
$$

$$
+ \frac{1}{3} \left( 3\beta + 2\gamma - \frac{(3\delta + 2\kappa)^2}{3\lambda + 2\mu} \right) \left( \text{div} \ \omega \right) \left( \text{div} \ \omega' \right)
$$

$$
+ \frac{\mu}{2} \sum_{k,j=1, k \neq j}^3 \left[ \frac{\partial u_k}{\partial x_j} + \frac{\partial u_j}{\partial x_k} + \frac{\kappa}{\mu} \left( \frac{\partial \omega_k}{\partial x_j} + \frac{\partial \omega_j}{\partial x_k} \right) \right]
$$

$$
\times \left[ \frac{\partial u_k'}{\partial x_j} + \frac{\partial u_j'}{\partial x_k} + \frac{\kappa}{\mu} \left( \frac{\partial \omega_k'}{\partial x_j} + \frac{\partial \omega_j'}{\partial x_k} \right) \right]$$
Let us note that, if the condition \(3\lambda\) imply positive definiteness of the energy density quadratic form are the following inequalities (see [AK2], [Dy1], [GGN1])

**Lemma 2.1** (see [NGS1]).

where

\[
\frac{\partial u_k}{\partial x_k} - \frac{\partial u_j}{\partial x_j} + \frac{\partial \omega_k}{\partial x_k} - \frac{\partial \omega_j}{\partial x_j}
\]

\[
\times \left[ \frac{\partial u_k'}{\partial x_k} - \frac{\partial u_j'}{\partial x_j} + \frac{\partial \omega_k'}{\partial x_k} - \frac{\partial \omega_j'}{\partial x_j} \right]
\]

\[
+ \left( \gamma - \frac{x^2}{\mu} \right) \sum_{k,j=1,k\neq j}^{3} \left[ \frac{1}{2} \left( \frac{\partial \omega_k}{\partial x_j} + \frac{\partial \omega_j}{\partial x_k} \right) \left( \frac{\partial \omega_k'}{\partial x_j} + \frac{\partial \omega_j'}{\partial x_k} \right) - \left( \frac{\partial \omega_k}{\partial x_k} - \frac{\partial \omega_j}{\partial x_j} \right) \left( \frac{\partial \omega_k'}{\partial x_k} - \frac{\partial \omega_j'}{\partial x_j} \right) \right]
\]

\[
+ \frac{1}{3} \left( \frac{\partial \omega_k}{\partial x_k} - \frac{\partial \omega_j}{\partial x_j} \right) \left( \frac{\partial \omega_k'}{\partial x_k} - \frac{\partial \omega_j'}{\partial x_j} \right)\]

\[
\alpha \left( \text{curl } u + \frac{\nu}{\alpha} \text{curl } \omega - 2\omega \right) \cdot \left( \text{curl } u' + \frac{\nu}{\alpha} \text{curl } \omega' - 2\omega' \right)
\]

\[
+ \left( \varepsilon - \frac{\nu^2}{\alpha} \right) \text{curl } \omega \cdot \text{curl } \omega'. \tag{2.12}
\]

The necessary and sufficient conditions for the potential energy density function \(E(U,U)\) to be a positive definite quadratic form are the following inequalities (see [AK2], [Dy1], [GGN1])

\[
\begin{align*}
\mu > 0, & \quad \alpha > 0, \quad \gamma > 0, \quad \varepsilon > 0, \quad \lambda + 2\mu > 0, \quad \mu \gamma - \varepsilon^2 > 0, \quad \alpha \varepsilon - \nu^2 > 0, \\
(\lambda + \mu)(\beta + \gamma) - (\delta + \varepsilon)^2 > 0, & \quad (3\lambda + 2\mu)(3\beta + 2\gamma) - (3\delta + 2\varepsilon)^2 > 0, \\
(\mu + \alpha)(\gamma + \varepsilon) - (\varepsilon + \nu)^2 > 0, & \quad (\lambda + 2\mu)(\beta + 2\gamma) - (\delta + 2\varepsilon)^2 > 0, \\
\mu(\lambda + \mu)(\beta + \gamma) - (\delta + \varepsilon)^2 > 0, & \quad (3\lambda + 2\mu)(3\beta + 3\gamma) - (3\delta + 2\varepsilon)^2 > 0, \\
\mu \left( (3\lambda + 2\mu)(3\beta + 3\gamma) - (3\delta + 2\varepsilon)^2 \right) + (3\lambda + 2\mu)(\mu \gamma - \varepsilon^2) > 0.
\end{align*}
\]

Let us note that, if the condition \(3\lambda + 2\mu > 0\) is fulfilled, which is very natural in classical elasticity [KGBB1], then the above conditions are equivalent to the following simultaneous inequalities

\[
\begin{align*}
\mu > 0, & \quad \alpha > 0, \quad \gamma > 0, \quad \varepsilon > 0, \quad 3\lambda + 2\mu > 0, \quad \mu \gamma - \varepsilon^2 > 0, \quad \alpha \varepsilon - \nu^2 > 0, \\
(\mu + \alpha)(\gamma + \varepsilon) - (\varepsilon + \nu)^2 > 0, & \quad (3\lambda + 2\mu)(3\beta + 2\gamma) - (3\delta + 2\varepsilon)^2 > 0.
\end{align*}
\]

For simplicity in what follows we assume that \(3\lambda + 2\mu > 0\) and therefore the above conditions imply positive definiteness of the energy density quadratic form \(E(U,U)\) with respect to the variables \(u_{pq}(U)\) and \(\omega_{pq}(U)\), i.e., there exists a positive constant \(c_0 > 0\) depending only on the material parameters, such that

\[
E(U,U) \geq c_0 \sum_{p,q=1}^{3} \left[ u_{pq}^2 + \omega_{pq}^2 \right]. \tag{2.13}
\]

The following assertion describes the null space of the energy density quadratic form \(E(U,U)\) (see [NGS1]).

**Lemma 2.1** Let \(U = (u,\omega) \in [C^1(\Omega^+)]^6\). Then \(E(U,U) = 0\) holds in \(\Omega^+\), if and only if

\[
u(x) = \chi^{(1)}(x) = [a \times x] + b, \quad \omega(x) = \chi^{(2)}(x) = a, \quad x \in \Omega^+,
\]

where \(a\) and \(b\) are arbitrary three-dimensional constant vectors and the symbol \(\cdot \times \cdot\) denotes the cross product of two vectors.
Vectors of type $\chi = (\chi^{(1)}, \chi^{(2)})^T = ([a \times x] + b, a)^T$ are called \textit{generalized rigid displacement vectors}. Observe that a generalized rigid displacement vector vanishes, i.e. $a = b = 0$, if it is zero at a single point.

Throughout the paper $L_p (1 \leq p \leq \infty)$, $H^s = H^s_2$, $s \in \mathbb{R}$, denote the Lebesgue and Bessel potential spaces (see, e.g., [Tr1]) with the norms $\| \cdot \|_{L_p}$ and $\| \cdot \|_{H^s}$, respectively. Moreover, $L_{p, \text{comp}}(\Omega^-)$ is the subspace of the $L_p$ functions with compact support in the unbounded domain $\Omega^-$. Denote by $\mathcal{D}(\Omega^\pm)$ the class of $C^\infty$ functions with support in the domains $\Omega^\pm$. If $M$ is an open proper part of the manifold $S$, i.e., $M \subset S$, $M \neq S$, then by $H^s(M)$ we denote the restriction of the space $H^s(S)$ on $M$, $H^s(M) := \{ r_M \varphi : \varphi \in H^s(S) \}$, where $r_M$ denotes the restriction operator on the set $M$. Further, let $\tilde{H}^s(M) := \{ \varphi \in H^s(S) : \text{supp} \varphi \subset M \}$.

From the positive definiteness (2.13) of the energy density form $E(\cdot, \cdot)$ it follows that the inequality

$$B(U,U) := \int_{\Omega^+} E(U,U) dx \geq c_1 \int_{\Omega^+} \left\{ \sum_{p,q=1}^3 \left[ (\partial_p u_q)^2 + (\partial_p \omega_q)^2 \right] + \sum_{q=1}^3 \left[ u_q^2 + \omega_q^2 \right] \right\} dx$$

$$- c_2 \int_{\Omega^+} \sum_{q=1}^3 \left[ u_q^2 + \omega_q^2 \right] dx \quad (2.14)$$

holds true for an arbitrary real-valued vector function $U \in [C^1(\Omega^+)]^6$ with some positive constants $c_1, c_2$ depending only on the material parameters. By standard limiting arguments we easily conclude that for any $U \in [H^1(\Omega^+)]^6$ there holds the following Korn’s type inequality (cf. [Fi1], Part I, §12, [Ci1], §6.3)

$$B(U,U) \geq c_1 \|U\|^2_{[H^1(\Omega^+)]^6} - c_2 \|U\|^2_{[L_2(\Omega^+)]^6}. \quad (2.15)$$

**REMARK 2.2** By standard limiting arguments Green’s formula (2.9) can be extended to Lipschitz domains and to vector functions $U, U' \in [H^1(\Omega^+)]^6$, hence $L(\partial) U \in [L_2(\Omega^+)]^6$ (see, [Ne1], [LiMall]),

$$\int_{\Omega^+} \left[ L(\partial) U \cdot U' + E(U,U') \right] \ dx = \langle \{ T(\partial, n) U^+ \}^+, \{ U'^+ \}^+ \rangle_{\partial\Omega^+}, \quad (2.16)$$

where $\langle \cdot, \cdot \rangle_{\partial\Omega^+}$ denotes the duality between the spaces $[H^{-1/2}(\partial\Omega^+)]^6$ and $[H^{1/2}(\partial\Omega^+)]^6$, which extends the inner product in the space $[L_2(\partial\Omega^+)]^6$. By this relation the generalized stress $\{ T(\partial, n) U^+ \}^+ \in [H^{-1/2}(\partial\Omega^+)]^6$ of the stress operator is well-defined.

Analogously, for the unbounded domain $\Omega^-$ and for vector functions $U, U' \in [H^1_{\text{loc}}(\Omega^-)]^6$, satisfying the decay condition (Z) along with the imbedding we have

$$\int_{\Omega^-} \left[ L(\partial) U \cdot U' + E(U,U') \right] \ dx = -\langle \{ T(\partial, n) U^- \}^-, \{ U'^- \}^- \rangle_{\partial\Omega^-}. \quad (2.17)$$
3 Contact problems with Tresca friction

3.1 Coulomb’s law and Tresca friction

Let the boundary $S$ of the domain $\Omega^+$ be divided into two open, connected and non-overlapping parts $S_1$ and $S_2$ of positive measure, $S = S_1 \cup S_2$, $S_1 \cap S_2 = \emptyset$. Assume that the hemitropic elastic body occupying the domain $\Omega^+$ is in bilateral contact with a foundation along the subsurface $S_2$, i.e., there is no gap between the body and the foundation. Denote by $F(x) = (F_1(x), F_2(x), F_3(x))^\top$ the reaction force stress vector which the foundation exerts on the hemitropic body at the point $x \in S_2$.

Throughout the paper $F_n$ and $F_s$ stand for the normal and tangential components of the vector $F$: $F_n = F \cdot n$ and $F_s = F - (F \cdot n)n$. Further, let $\mathcal{F}(x)$ be the friction coefficient at the point $x$. It is a nonnegative scalar function which depends on the geometry of the contacting surfaces and also on the physical properties of interacting materials.

Coulomb’s law describing friction for static bilateral contact reads as follows (see [DuLi1]): For the force stress vector $F$ there holds

$$|F_s(x)| \leq \mathcal{F}(x) |F_n(x)|.$$ 

Moreover, if

$$|F_s(x)| < \mathcal{F}(x) |F_n(x)|,$$

then the tangential component of the displacement vector vanishes, $u_s(x) = 0$, and if

$$|F_s(x)| = \mathcal{F}(x) |F_n(x)|,$$

then there exist nonnegative functions $\lambda_1$ and $\lambda_2$ which do not vanish simultaneously and such that

$$\lambda_1(x) u_s(x) = -\lambda_2(x) F_s(x).$$

In classical elasticity the fixed point approach to frictional contact as proposed by Panagiotopoulos [Pan], employed in the existence proofs (see [EJK]), and recently numerically realized in [HKD] leads to an approximating sequence of contact problems with given friction. In these approximations, also known as contact problems with Tresca friction (see [SST]), the unknown normal component is replaced by a given nonnegative slip stress.

Similarly we replace the unknown normal component $F_n$ by some given force $F_0$ and replace the above threshold by

$$g(x) := \mathcal{F}(x) |F_0(x)|.$$  

(3.1)

3.2 Pointwise and variational formulation of the bilateral contact problem

Consider the equation in the domain $\Omega^+$:

$$L(\partial) U + X = 0,$$  

(3.2)

where $L(\partial)$ is the matrix differential operator given by the formula (2.8), $U = (u, \omega)^\top$, $X := (\rho \mathcal{X}^{(1)}, \rho \mathcal{X}^{(2)})^\top \in [L_2(\Omega^+)]^6$.

A vector-function $U = (u, \omega)^\top \in [H^1(\Omega^+)]^6$ is a weak solution of equation (3.2) in $\Omega^+$ if

$$B(U, \Phi) = \int_{\Omega^+} X \cdot \Phi \, dx \quad \forall \Phi \in [D(\Omega^+)]^6,$$

where the bilinear form $B(\cdot, \cdot)$ is given by formula (2.14).
Due to the formulas \([2.1]-[2.6]\) for the force stress and couple stress vectors we have:
\[
\tau^{(n)}(U) = T U = T^{(1)}u + T^{(2)}\omega, \quad \mu^{(n)}(U) = M U = T^{(3)}u + T^{(4)}\omega.
\]
It is clear that
\[
\tau^{(n)}_n(U) := (TU)_n, \quad \tau^{(n)}_s(U) := (TU)_s = \tau^{(n)}_n(U) - \tau^{(n)}_s(U)n.
\]
Further, let \(X \in \mathbb{L}_2(\Omega^{+})^6, \varphi \in [H^{-1/2}(S_2)]^3, f \in [H^{1/2}(S_1)]^6, F_0 \in L_{\infty}(S_2),\) and \(F : S_2 \rightarrow [0, +\infty)\) be a bounded measurable function. Thus from formula \([3.1]\), the nonnegative function \(g \in L_{\infty}(S_2)\).

Consider the following bilateral mixed contact problem with friction.

Problem \((A)\) (coercive case). Find a weak solution \(U = (u, \omega)^T \in [H^1(\Omega^{+})]^6\) of equation \([3.2]\) satisfying the inclusion \(r_{s_2} \{r_s^{(n)}(U)\}^+ \in [L_{\infty}(S_2)]^3\) and the following conditions:

\[
\begin{align*}
(i) \quad & r_{s_1} \{U\}^+ = f \quad \text{on} \quad S_1; \\
(ii) \quad & r_{s_2} \{r_s^{(n)}(U)\}^+ = F_0 \quad \text{on} \quad S_2; \\
(iii) \quad & r_{s_2} \{MU\}^+ = \varphi \quad \text{on} \quad S_2; \\
(iv) \quad & \begin{aligned}
(a) \quad & |r_{s_2} \{r_s^{(n)}(U)\}^+| < g, \quad \text{then} \quad r_{s_2} \{u_s\}^+ = 0, \\
(b) \quad & |r_{s_2} \{r_s^{(n)}(U)\}^+| = g, \quad \text{then} \quad \text{there exist nonnegative functions} \ \lambda_1 \text{and} \ \lambda_2 \text{which do not vanish simultaneously and} \\
& \lambda_1 r_{s_2} \{u_s\}^+ = -\lambda_2 r_{s_2} \{r_s^{(n)}(U)\}^+.
\end{aligned}
\end{align*}
\]  

We emphasize that by the requirement \(r_{s_2} \{r_s^{(n)}(U)\}^+ \in [L_{\infty}(S_2)]^3\) the contact conditions in (iv) can be understood to hold almost everywhere on \(S_2\). Thus we are here more precise with the pointwise formulation than other expositions of contact problems in classical elasticity (see [KiiOd1], [SS1]).

To reduce Problem \((A)\) to a boundary variational inequality we need first to reduce the nonhomogeneous equation \([3.2]\) and nonhomogeneous condition \([3.3]\) to the homogeneous ones. To this purpose consider the following auxiliary linear mixed boundary value problem:

Find a vector function \(U_0 = (u_0, \omega_0)^T \in [H^1(\Omega^{+})]^6\), which is a weak solution of equation \([3.2]\) and satisfies the following conditions:
\[
\begin{align*}
& r_{s_1} \{U_0\}^+ = f \quad \text{on} \quad S_1; \\
& r_{s_2} \{T(\partial, n) U_0\}^+ = 0 \quad \text{on} \quad S_2.
\end{align*}
\]

This problem has a unique weak solution [NGS1]. Clearly, if \(W\) is a solution of Problem \((A)\) and \(U_0\) is a solution of the above mixed auxiliary problem, then the difference \(U := W - U_0\) will solve the following problem:

Problem \((A_0)\). Find a weak solution \(U = (u, \omega)^T \in [H^1(\Omega^{+})]^6\) of the equation
\[
L(\partial) U = 0 \quad \text{in} \quad \Omega^{+}, \tag{3.5}
\]
satisfying the following conditions \( r_{s_2} \{ \tau^{(n)}(U) \}^+ \in [L_\infty(S_2)]^3 \) and

\[
\begin{align*}
(\text{i}) & \quad r_{s_1} \{ U \}^+ = 0 \quad \text{on} \quad S_1; \\
(\text{ii}) & \quad r_{s_2} \{ \tau^{(n)}(U) \}^+ = F_0 \quad \text{on} \quad S_2; \\
(\text{iii}) & \quad r_{s_2} \{ MU \}^+ = \varphi \quad \text{on} \quad S_2; \\
(\text{iv}) & \quad (a) \quad \text{if} \quad |r_{s_2} \{ \tau^{(n)}(U) \}^+| < g, \quad \text{then} \quad r_{s_2} \{ u_s \}^+ = \varphi_0, \\
& \quad (b) \quad \text{if} \quad |r_{s_2} \{ \tau^{(n)}(U) \}^+| = g, \quad \text{then} \quad \text{there exist nonnegative} \\
& \quad \quad \quad \text{functions} \quad \lambda_1 \quad \text{and} \quad \lambda_2 \quad \text{which do not vanish simultaneously and} \\
& \quad \quad \quad \lambda_1 r_{s_2} \{ u_s \}^+ = -\lambda_2 r_{s_2} \{ \tau^{(n)}(U) \}^+ + \lambda_1 \varphi_0, \quad (3.10)
\end{align*}
\]

where \( g \) is defined by formula (3.1) and \( \varphi_0 := -r_{s_2} \{ u_0 \}^+ \in [H^{1/2}(S_2)]^3 \).

### 3.3 Reduction of Problem \((A_0)\) to a boundary variational inequality

To reduce equivalently Problem \((A_0)\) to a boundary variational inequality we recall that a solution vector \( U = (u, \omega) \in [H^1(\Omega^+)]^6 \) to equation (3.5) satisfying the Dirichlet boundary condition

\[ \{ U \}^+ = h \quad \text{on} \quad S \]

with \( h \in [H^{1/2}(S)]^6 \), can be uniquely represented as a single layer potential (see [NGS1])

\[ U(x) = V(\mathcal{H}^{-1} h)(x) := \int_S \Gamma(x-y) (\mathcal{H}^{-1} h)(y) dS_y, \quad x \in \Omega^+, \]

where \( \Gamma \) is a fundamental solution of the operator \( L(\partial) \) and \( \mathcal{H} \) is the boundary integral operator generated by the trace of the single layer potential on the boundary \( S \) (see the explicit expression for \( \Gamma \) in [GGN1], [NGS1]):

\[ (\mathcal{H}h)(x) = \lim_{\Omega^+ \ni x \to \partial S} \int_S \Gamma(z-y) h(y) dS_y = \{ V(h) \}^+ \equiv \{ V(h) \}^-. \quad (3.11) \]

Note that the single layer potential operator \( V \) and the integral operator \( \mathcal{H} \) have the following mapping properties (see [NGST])

\[
\begin{align*}
V : [H^r(S)]^6 & \to [H^{r+3/2}(\Omega^+)]^6 \quad \left[ [H^r(S)]^6 \to [H^{r+3/2}(\Omega^-)]^6 \right] \quad \forall r \in \mathbb{R}, \\
\mathcal{H} : [H^r(S)]^6 & \to [H^{r+1}(S)]^6 \quad \forall r \in \mathbb{R}.
\end{align*}
\]

Moreover, the operator \( \mathcal{H} \) is invertible and

\[ \mathcal{H}^{-1} : [H^r(S)]^6 \to [H^{r-1}(S)]^6 \quad \forall r \in \mathbb{R}. \quad (3.13) \]

Further, there hold the following limiting relations

\[ \{ T(\partial, n) V(h) \}^\pm = (\mp 2^{-1} I_6 + \mathcal{K}) h \quad \text{on} \quad S, \quad (3.14) \]

where

\[ \mathcal{K} h(x) = \int_S [T(\partial_x, n(x)) \Gamma(x-y)] h(y) dS_y, \quad (3.15) \]
It is shown in \[\text{NGS1}\] that

\[+2^{-1}I_6 + \mathcal{K} : [H^{-1/2}(S)]^6 \to [H^{-1/2}(S)]^6\]

is a singular integral operator of normal type with zero index.

Let \(G^+ : [H^{1/2}(S)]^6 \to [H^1(\Omega^+)]^6\) be the operator defined by the formula

\[G^+ h := V(\mathcal{H}^{-1} h).\] (3.16)

It is clear that \(L(\partial)G^+ h = 0\) in \(\Omega^+\) and \(\{G^+ h\}^+ = h\) on \(S\).

From the properties of the trace operator and mapping properties of the single layer potential operator it follows that there exist positive numbers \(c_1\) and \(c_2\), such that

\[c_1\|h\|_{[H^{1/2}(S)]^6} \leq \|G^+ h\|_{[H^1(\Omega^+)]^6} \leq c_2\|h\|_{[H^{1/2}(S)]^6} \quad \forall h \in [H^{1/2}(S)]^6.\] (3.17)

Define the Steklov-Poincaré type operator

\[A^+ h := \{T(\partial,n)(G^+ h)\}^+ = \{T(\partial,n)V(\mathcal{H}^{-1} h)\}^+.\]

The operator \(A^+\) is well-defined and due to (3.16) we have the following representation, see (3.15) and (3.16):

\[A^+ = (-2^{-1}I_6 + \mathcal{K}) \mathcal{H}^{-1}.\] (3.18)

Denote by \(\Lambda(S)\) the set of restrictions on \(S\) of rigid displacement vectors, i.e.,

\[\Lambda(S) := \{\chi(x) = ([a \times x] + b, a)^\top, \quad x \in S\},\] (3.19)

where \(a\) and \(b\) are arbitrary three-dimensional constant vectors.

With the help of Green’s formula (2.16) with \(U = U' = V(\mathcal{H}^{-1} h)\), the relations (3.14), (3.19) and the uniqueness theorem for the Dirichlet BVP, we infer that \(\ker A^+ = \Lambda(S)\).

Now we formulate the following technical lemma describing the properties of the Steklov-Poincaré operator.

**Lemma 3.1** The following relations are true:

(a) \(\langle A^+ h, \eta \rangle_S = \langle A^+ \eta, h \rangle_S \quad \forall h \in [H^{1/2}(S)]^6\) and \(\forall \eta \in [H^{1/2}(S)]^6;\)

(b) \(A^+ : [H^{1/2}(S)]^6 \to [H^{-1/2}(S)]^6\) is a continuous operator;

(c) \(\exists c' > 0, c'' > 0 : \langle A^+ h, h \rangle_S \geq c'\|h\|_{[H^{1/2}(S)]^6}^2 - c''\|h\|_{L_2(S)}^6 \quad \forall h \in [H^{1/2}(S)]^6;\)

(d) \(\exists c > 0 : \langle A^+ h, h \rangle_S \geq c\|h\|_{[H^{1/2}(S)]^6}^2 \quad \forall h \in [\tilde{H}^{1/2}(S^*)]^6, \quad S^* \subset S;\)

(e) \(\exists c > 0 : \langle A^+ h, h \rangle_S \geq c\|h - Ph\|_{[H^{1/2}(S)]^6}^2 \quad \forall h \in [H^{1/2}(S)]^6,\)

where \(S^*\) is a proper part of \(S\) and \(P\) is the operator of the orthogonal projection (in the sense of \(L_2(S)\)) of the space \([H^{1/2}(S)]^6\) onto the space \(\Lambda(S)\).

**Proof.** (a) Let \(h, \eta \in [H^{1/2}(S)]^6\). Taking into account the equality \(L(\partial)G^+ h = 0\), due to Green’s formula (2.16) we get the equality:

\[\langle A^+ h, \eta \rangle_S = \langle \{T(G^+ h)\}^+, \eta \rangle_S = B(G^+ h, G^+ \eta)\]

\[= B(G^+ \eta, G^+ h) = \langle \{T(G^+ \eta)\}^+, h \rangle_S = \langle A^+ \eta, h \rangle_S.\]

The item (b) is evident since by (3.18), \(A^+\) is the composition of the continuous operators \(\mathcal{H}^{-1}\) and \(-2^{-1}I_6 + \mathcal{K}\). The item (c) can be shown by the following arguments. For arbitrary \(h \in [H^{1/2}(S)]^6\) with the help of (2.15) we derive

\[\langle A^+ h, h \rangle_S = B(G^+ h, G^+ h) \geq c_1\|G^+ h\|_{[H^1(\Omega^+)]^6}^2 - c_2\|G^+ h\|_{L_2(\Omega^+)}^2.\]
Since \( \{G^+ h\}^+ = h \) on \( S \) by the trace theorem we have
\[
\|h\|_{[H^{1/2}(S)]^6} \leq c_3 \|G^+ h\|_{[H^1(\Omega^+)]^6},
\]
where \( c_3 \) is some positive constant independent of \( h \).

On the other hand, since \( [L_2(S)]^6 \) is continuously embedded into \( [H^{-1/2}(S)]^6 \), then by virtue of the properties (3.12) and (3.13) we have for \( G^+ = V \mathcal{H}^{-1} \):
\[
\|V(\mathcal{H}^{-1}h)\|_{L_2(\Omega^+)}^6 \leq c_4 \|\mathcal{H}^{-1}h\|_{[H^{-3/2}(S)]^6}^6 \leq c_5 \|h\|_{[H^{-1/2}(S)]^6} \leq c_6 \|h\|_{L_2(S)}^6
\]
with some positive constants \( c_4, c_5 \) and \( c_6 \) independent of \( h \). So, finally we obtain that
\[
\langle A^+ h, h \rangle_S \geq \frac{c_1}{c_3} \|h\|_{[H^{1/2}(S)]^6}^2 - c_2 c_6^2 \|h\|_{L_2(S)}^6.
\]
The item (e) follows from the item (c) (see also Lemma 5.1 in the reference [GGN1]). The item (d) follows from the item (e). The lemma is proved.

Our goal is to reduce equivalently Problem \((A_0)\) to a boundary variational inequality. To this end, we introduce the following convex continuous, but nondifferentiable functional on the space \([H^{1/2}(S_2)]^3\)
\[
j(\psi) := \int_{S_2} g |\psi_0 - \varphi_0| dS \quad \forall \psi \in [H^{1/2}(S_2)]^3.
\]
(3.20)

Further, let us define the closed subspace
\[
\mathbb{H} := \{h = (h^{(1)}, h^{(2)})^\top \in [H^{1/2}(S)]^6 : r_{S_1} h = 0\}
\]
and consider the variational inequality: Find a vector function \( h_0 = (h_0^{(1)}, h_0^{(2)})^\top \in \mathbb{H} \) such that the following inequality
\[
\langle A^+ h_0, h - h_0 \rangle_S + j(h^{(1)}) - j(h_0^{(1)}) \geq \int_{S_2} F_0 (h_0^{(1)} - h_0^{(1)}) dS + \langle \varphi, r_{S_2} (h^{(2)} - h_0^{(2)}) \rangle_{S_2}
\]
holds for all \( h = (h^{(1)}, h^{(2)})^\top \in \mathbb{H} \).

Now we show that the variational inequality (3.22) and Problem \((A_0)\) are equivalent.

THEOREM 3.2
The boundary variational inequality (3.22) and Problem \((A_0)\) are equivalent in the following sense: if \( U \in [H^1(\Omega^+)]^6 \) is a solution of Problem \((A_0)\), then \( \{U\}^+ \in [H^{1/2}(S)]^6 \) is a solution of the variational inequality (3.22) and vice versa, if \( h_0 \in \mathbb{H} \) is a solution of the variational inequality (3.22), then \( G^+ h_0 \in [H^1(\Omega^+)]^6 \) is a weak solution of Problem \((A_0)\).

Proof. Let \( U = (u, \omega)^\top \in [H^1(\Omega^+)]^6 \) be a solution of Problem \((A_0)\) and \( h_0 = (h_0^{(1)}, h_0^{(2)})^\top := \{U\}^+ \). In accordance with the definition of the operator \( G^+ \) we have \( U = G^+ h_0 \). We show that the conditions (3.9) and (3.10) yield the following inequality:
\[
\sum_{s_2} \{\tau^{(n)}_s (G^+ h_0)\}^+ \cdot r_{S_2} (h^{(1)}_s - h_0^{(1)}) + g |r_{S_2} (h^{(1)}_s - \varphi_0) - |r_{S_2} h^{(1)}_0 - \varphi_0| \geq 0 \quad \forall h = (h^{(1)}, h^{(2)})^\top \in \mathbb{H}.
\]
(3.23)

Indeed, let
\[
\tau^{(n)}_{0s} := \{\tau^{(n)}_s (G^+ h_0)\}^+ \quad \text{on} \ S.
\]
If \( |r_{S_2} r_{0s}^{(n)}| < g \), then \( r_{S_2} h_{10}^{(1)} = \varphi_0 \), and the left hand side of (3.23) becomes
\[
r_{S_2} r_{0s}^{(n)} \cdot (r_{S_2} h_{1}^{(1)} - \varphi_0) + g r_{S_2} h_{1}^{(1)} - \varphi_0 \geq ( - |r_{S_2} r_{0s}^{(n)}| + g ) r_{S_2} h_{1}^{(1)} - \varphi_0 \geq 0.
\]
Otherwise, \( |r_{S_2} r_{0s}^{(n)}| = g \). If in (3.10) \( \lambda_1 = 0 \), then \( \lambda_2 \neq 0 \), hence \( r_{S_2} r_{0s}^{(n)} = 0, g = 0 \) and (3.23) immediately holds. If \( \lambda_1 \neq 0 \), then \( \lambda := \lambda_2 / \lambda_1 \geq 0 \) and the left hand side of (3.23) becomes by (3.10)
\[
r_{S_2} r_{0s}^{(n)} \cdot r_{S_2} h_{1}^{(1)} - r_{S_2} r_{0s}^{(n)} \cdot (\varphi_0 - \lambda r_{S_2} r_{0s}^{(n)}) + g |r_{S_2} h_{1}^{(1)} - \varphi_0| - g \lambda |r_{S_2} r_{0s}^{(n)}| \geq - |r_{S_2} r_{0s}^{(n)}| |r_{S_2} h_{1}^{(1)} - \varphi_0| + g |r_{S_2} h_{1}^{(1)} - \varphi_0| = 0.
\]
Integrate the inequality (3.23) over \( S_2 \) to obtain
\[
\int_{S_2} \{ \tau_{n}^{(n)} (G^{+} h_{0}) \}^{+} \cdot (h_{1}^{(1)} - h_{0s}^{(1)}) dS + j(h_{1}^{(1)}) - j(h_{0}^{(1)}) \geq 0.
\]
Hence we have
\[
\int_{S_2} \{ \tau_{n}^{(n)} (G^{+} h_{0}) \}^{+} \cdot (h_{1}^{(1)} - h_{0s}^{(1)}) dS + j(h_{1}^{(1)}) - j(h_{0}^{(1)}) + \int_{S_2} \{ \tau_{n}^{(n)} (G^{+} h_{0}) \}^{+} (h_{n}^{(1)} - h_{0n}^{(1)}) dS
\]
\[
+ \langle r_{S_2} (\mathcal{M} (G^{+} h_{0}))^{+}, r_{S_2} (h_{1}^{(2)} - h_{0}^{(2)}) \rangle_{S_2} \geq \int_{S_2} F_{0} (h_{n}^{(1)} - h_{0n}^{(1)}) dS + \langle \varphi, r_{S_2} (h_{1}^{(2)} - h_{0}^{(2)}) \rangle_{S_2}.
\]
Since \( r_{S_1} (h - h_{0}) = 0 \) due to the inclusion \( h, h_{0} \in \mathbb{H} \), then finally we arrive at the inequality
\[
\langle A^{+} h_{0}, h - h_{0} \rangle_{S} + j(h_{1}^{(1)}) - j(h_{0}^{(1)}) \geq \int_{S_2} F_{0} (h_{n}^{(1)} - h_{0n}^{(1)}) dS + \langle \varphi, r_{S_2} (h_{1}^{(2)} - h_{0}^{(2)}) \rangle_{S_2}.
\]
\[
\forall h = (h_{1}^{(1)}, h_{0}^{(2)})^{\top} \in \mathbb{H}.
\]
Thus, \( \{ U \}^{+} \) solves the variational inequality (3.22).

Now, let \( h_{0} = (h_{0}^{(1)}, h_{0}^{(2)})^{\top} \in \mathbb{H} \) be a solution of the variational inequality (3.22). Denote \( U := G^{+} h_{0} \) in \( \Omega^{+} \). We have to show that \( U \) solves Problem (A0). Due to the definition of the operator \( G^{+} \) the vector \( G^{+} h_{0} \) is a weak solution of the equation (3.5) and \( r_{S_1} \{ U \}^{+} = r_{S_1} h_{0} = 0 \), since \( h_{0} \in \mathbb{H} \). Thus (3.5) and (3.6) hold.

Let \( h = (h_{1}^{(1)}, h_{0}^{(2)})^{\top} \in \mathbb{H} \) and \( h_{1}^{(1)} = h_{0s}^{(1)}, h_{2} = h_{0}^{(2)} \) and \( h_{n}^{(1)} = h_{0n}^{(1)} \pm \psi \), with arbitrary \( \psi \in \overline{H}^{1/2}(S_2) \). Then \( j(h_{1}^{(1)}) = j(h_{0}^{(1)}) \) and from (3.22) we obtain
\[
\langle r_{S_2} (\tau_{n}^{(n)} (G^{+} h_{0}))^{+}, r_{S_2} \psi \rangle_{S_2} = \int_{S_2} F_{0} \psi dS \quad \forall \psi \in \overline{H}^{1/2}(S_2),
\]
implying
\[
r_{S_2} (\tau_{n}^{(n)} (G^{+} h_{0}))^{+} = F_{0} \quad \text{on} \quad S_2.
\]
Thus (3.7) holds.
Select \( h = (h^{(1)}, h^{(2)})^\top \in \mathbb{H} \) such that \( h^{(1)} = h_0^{(1)} \) and \( h^{(2)} = h_0^{(2)} \pm \psi \), with arbitrary \( \psi \in [\widetilde{H}^{1/2}(S_2)]^3 \). From (3.22) we have

\[
\langle r_{S_2} \{\mathcal{M}(G^+ h_0)\}^+, r_{S_2} \psi \rangle_{S_2} = \langle \varphi, r_{S_2} \psi \rangle_{S_2} \quad \forall \psi \in [\widetilde{H}^{1/2}(S_2)]^3,
\]

hence

\[
r_{S_2} \{\mathcal{M}U\}^+ = r_{S_2} \{\mathcal{M}(G^+ h_0)\}^+ = \varphi \quad \text{on} \quad S_2,
\]

and (3.8) holds.

From (3.22), by virtue of (3.24) and (3.25) we derive

\[
\langle r_{S_2} \{\tau_s^{(n)}(G^+ h_0)\}^+, r_{S_2} (h_s^{(1)} - h_0^{(1)}) \rangle_{S_2} + \int_{S_2} g \left( |h_s^{(1)}| - \varphi_0 | - |h_0^{(1)}| - \varphi_0 | \right) dS \geq 0
\]

\[
\forall h = (h^{(1)}, h^{(2)})^\top \in \mathbb{H}.
\]

We rewrite this inequality as follows

\[
\langle r_{S_2} \{\tau_s^{(n)}(G^+ h_0)\}^+, r_{S_2} h_s^{(1)} - \varphi_0 \rangle_{S_2} - \langle r_{S_2} \{\tau_s^{(n)}(G^+ h_0)\}^+, r_{S_2} h_0^{(1)} - \varphi_0 \rangle_{S_2}
\]

\[
+ \int_{S_2} g |h_s^{(1)} - \varphi_0 | dS - \int_{S_2} g |h_0^{(1)} - \varphi_0 | dS \geq 0 \quad \forall h = (h^{(1)}, h^{(2)})^\top \in \mathbb{H}.
\]

We set

\[
\theta_s := r_{S_2} h_s^{(1)} - \varphi_0, \quad \theta_0 := r_{S_2} h_0^{(1)} - \varphi_0.
\]

Then (3.26) reads

\[
\langle r_{S_2} \{\tau_s^{(n)}(G^+ h_0)\}^+, \theta_s \rangle_{S_2} + \int_{S_2} g |\theta_s | dS - \langle r_{S_2} \{\tau_s^{(n)}(G^+ h_0)\}^+, \theta_0 \rangle_{S_2}
\]

\[
- \int_{S_2} g |\theta_0 | dS \geq 0.
\]

For arbitrary \( \psi \in [\widetilde{H}^{1/2}(S_2)]^3 \) we have

\[
\langle r_{S_2} \{\tau_s^{(n)}(G^+ h_0)\}^+, r_{S_2} \psi \rangle_{S_2} = \langle r_{S_2} \{\tau_s^{(n)}(G^+ h_0)\}^+, r_{S_2} \psi \rangle_{S_2}
\]

and \( |r_{S_2} \psi| \leq |r_{S_2} \psi| \). Therefore, if we take \( r_{S_2} \psi \) for \( \theta_s \) in (3.27), we obtain

\[
\langle r_{S_2} \{\tau_s^{(n)}(G^+ h_0)\}^+, r_{S_2} \psi \rangle_{S_2} + \int_{S_2} g |\psi | dS - \langle r_{S_2} \{\tau_s^{(n)}(G^+ h_0)\}^+, \theta_0 \rangle_{S_2}
\]

\[
+ \int_{S_2} g |\theta_0 | dS \geq 0 \quad \forall \psi \in [\widetilde{H}^{1/2}(S_2)]^3.
\]

Put \( \pm t \psi \) with \( t \geq 0 \) for \( \psi \) in (3.28):

\[
t \left\{ \pm \langle r_{S_2} \{\tau_s^{(n)}(G^+ h_0)\}^+, r_{S_2} \psi \rangle_{S_2} + \int_{S_2} g |\psi | dS \right\} - \langle r_{S_2} \{\tau_s^{(n)}(G^+ h_0)\}^+, \theta_0 \rangle_{S_2}
\]

\[
+ \int_{S_2} g |\theta_0 | dS \geq 0 \quad \forall t \geq 0, \forall \psi \in [\widetilde{H}^{1/2}(S_2)]^3.
\]

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First, sending \( t \) to \( +\infty \) and afterwards sending \( t \) to 0, we arrive at the inequalities:

\[
|r_{S_2}\{\tau_s^{(n)}(G^+h_0)\}^+,r_{S_2}\psi|_{S_2}| \leq \int g|\psi|dS \quad \forall \psi \in [\overline{H}^{1/2}(S_2)]^3; \tag{3.29}
\]

\[
\int_{S_2}\{\{\tau_s^{(n)}(G^+h_0)\}^+\cdot \theta_{0s} + g|\theta_{0s}|\}dS \leq 0. \tag{3.30}
\]

Consider the linear functional \( \Phi \) on the space \([\overline{H}^{1/2}(S_2)]^3\) given by

\[
\Phi(\psi) = \int_{S_2} \Phi^* \cdot \psi dS \quad \forall \psi \in [L_1(S_2)]^3.
\]

Hence

\[
r_{S_2}\{\tau_s^{(n)}(G^+h_0)\}^+ = \Phi^* \in [L_\infty(S_2)]^3.
\]

Using again the inequality (3.29) we derive

\[
\int_{S_2}\{\pm \{\tau_s^{(n)}(G^+h_0)\}^+\cdot \psi - g|\psi|\}dS \leq 0 \quad \forall \psi \in [\overline{H}^{1/2}(S_2)]^3 \tag{3.31}
\]

and the inequality

\[
|r_{S_2}\{\tau_s^{(n)}(G^+h_0)\}^+| \leq g \quad \text{almost everywhere on } S_2 \tag{3.32}
\]

follows.

Indeed, it is well known that for an arbitrary essentially bounded function \( \tilde{\psi} \in L_\infty(S_2) \) there is a sequence \( \tilde{\varphi}_l \in C^\infty(S_2) \) with \( \text{supp} \tilde{\varphi}_l \subset S_2 \), such that (see, e.g., [Ni1], Lemma 1.4.2)

\[
\lim_{l \to \infty} \tilde{\varphi}_l(x) = \tilde{\psi}(x) \quad \text{for almost all } x \in S_2 \quad \text{and} \quad |\tilde{\varphi}_l(x)| \leq \text{ess sup}_{y \in S_2} |\tilde{\psi}(y)| \quad \text{for almost all } x \in S_2.
\]

Therefore from inequality (3.31) by the Lebesgue dominated convergence theorem it follows that

\[
\int_{S_2}\{\pm \{\tau_s^{(n)}(G^+h_0)\}^+\cdot \psi - g|\psi|\}dS \leq 0 \quad \forall \psi \in [L_\infty(S_2)]^3.
\]

In the place of \( \psi \) we can put here \( \chi(S_2^*) \psi \) where \( \psi \in [L_\infty(S_2)]^3 \) and \( \chi(S_2^*) \) is the characteristic function of an arbitrary measurable subset \( S_2^* \subset S_2 \). As a result we arrive at the inequality

\[
\pm \{\tau_s^{(n)}(G^+h_0)\}^+\cdot \psi - g|\psi| \leq 0 \quad \text{almost everywhere on } S_2 \quad \text{for all } \psi \in [L_\infty(S_2)]^3 \quad \text{and consequently by choosing } \psi = \{\tau_s^{(n)}(G^+h_0)\}^+ \quad \text{we finally get (3.32).} \]
By virtue of (3.30) and (3.32) we obtain
\[
\int_{S_2} g |\theta_{0s}| \, dS \leq - \int_{S_2} \{\tau_s^{(n)}(G^+ h_0)\}^+ \cdot \theta_{0s} \, dS \\
\leq \int_{S_2} |\{\tau_s^{(n)}(G^+ h_0)\}^+| |\theta_{0s}| \, dS \leq \int_{S_2} g |\theta_{0s}| \, dS,
\]
hence (3.30) holds with the equality sign, further by (3.32), the integrand in (3.30) is nonnegative. Thus we arrive at
\[
r_{S_2}\{\tau_s^{(n)}(G^+ h_0)\}^+ \cdot \theta_{0s} + g|\theta_{0s}| = 0. \tag{3.33}
\]
If \(|r_{S_2}\{\tau_s^{(n)}(G^+ h_0)\}^+| < g\), then (3.33) yields \(\theta_{0s} = 0\). Hence \(r_{S_2}h_{0s}^{(1)} = \varphi_0\) that is \(r_{S_2}\{u_s\}^+ = \varphi_0\) and (3.9) holds. Otherwise, \(|r_{S_2}\{\tau_s^{(n)}(G^+ h_0)\}^+| = g\). Then we can rewrite (3.33) as
\[
g|\theta_{0s}| (\cos \alpha + 1) = 0,
\]
where \(\alpha\) is the angle between the vectors \(r_{S_2}\{\tau_s^{(n)}(G^+ h_0)\}^+\) and \(r_{S_2}\{\theta_{0s}\}^+(x), x \in S_2\). Now, it is clear that there are functions \(\lambda_1(x) \geq 0\) and \(\lambda_2(x) \geq 0\) with \(\lambda_1(x) + \lambda_2(x) > 0\), such that
\[
\lambda_1 \theta_{0s} = -\lambda_2 r_{S_2}\{\tau_s^{(n)}(G^+ h_0)\}^+,
\]
i.e.,
\[
\lambda_1 r_{S_2}h_{0s}^{(1)} = -\lambda_2 r_{S_2}\{\tau_s^{(n)}(G^+ h_0)\}^+ + \lambda_1 \varphi_0.
\]
Therefore
\[
\lambda_1 r_{S_2}\{u_s\}^+ = -\lambda_2 r_{S_2}\{\tau_s^{(n)}(G^+ h_0)\}^+ + \lambda_1 \varphi_0,
\]
and (3.10) holds. The proof is complete. \(\square\)

4 Existence and uniqueness of solutions and their dependence on the data of the problem

4.1 Uniqueness

We start with the following uniqueness result.

THEOREM 4.1 Suppose, the Dirichlet boundary part \(S_1\) has positive measure. Then the boundary variational inequality (3.22) has at most one solution.

Proof. Let \(h_0 = (h_0^{(1)}(1), h_0^{(2)}_0) \top \in \mathbb{H}\) and \(h_0^* = (h_0^{(1)*}(1), h_0^{(2)*}_0) \top \in \mathbb{H}\) be two arbitrary solutions of the variational inequality (3.22). Then from (3.22) we have:
\[
\langle A^+ h_0^*, h_0 - h_0 \rangle_{S_2} + j(h_0^{(1)*}) - j(h_0^{(1)}) \geq \int_{S_2} F_0 (h_0^{(1)*} - h_0^{(1)}) \, dS + \langle \varphi, r_{S_2} (h_0^{(2)*} - h_0^{(2)}) \rangle_{S_2},
\]
\[
\langle A^+ h_0^*, h_0 - h_0 \rangle_{S_2} + j(h_0^{(1)*}) - j(h_0^{(1)*}) \geq \int_{S_2} F_0 (h_0^{(1)*} - h_0^{(1)}) \, dS + \langle \varphi, r_{S_2} (h_0^{(2)*} - h_0^{(2)*}) \rangle_{S_2}.
\]
By summing up these inequalities we obtain
\[ \langle A^+(h_0 - h_0^*), h_0 - h_0^* \rangle_S \leq 0. \]

Hence, in view of the non-negativity of the operator $A^+$
\[ \langle A^+(h_0 - h_0^*), h_0 - h_0^* \rangle_S = 0. \]

By (2.16) we get
\[ 0 = \langle \{T(\partial, n)V(\mathcal{H}^{-1}(h_0 - h_0^*))\}^+, h_0 - h_0^* \rangle_S = \langle \{T(\partial, n)G^+(h_0 - h_0^*)\}^+, \{G^+(h_0 - h_0^*)\}^+ \rangle_S \]
\[ = B(V(\mathcal{H}^{-1}(h_0 - h_0^*)), V(\mathcal{H}^{-1}(h_0 - h_0^*))). \]

Thus
\[ G^+(h_0 - h_0^*) = V(\mathcal{H}^{-1}(h_0 - h_0^*)) = ([a \times x] + b, a) \quad \text{in} \quad \Omega^+. \]

Since $h_0$, $h_0^* \in \mathbb{H}$ we have
\[ r_{S_1}\{V(\mathcal{H}^{-1}(h_0 - h_0^*))\} = r_{S_1}(h_0 - h_0^*) = 0, \quad \text{i.e.,} \quad ([a \times x] + b, a) = 0 \quad \text{on} \quad S_1. \]

Therefore $a = b = 0$ and $V(\mathcal{H}^{-1}(h_0 - h_0^*)) = 0$ in $\Omega^+$. Hence we conclude that $h_0 = h_0^*$ on $S$. The proof is complete. \qed

4.2 Existence results

Consider the following functional on the closed subspace $\mathbb{H}$ (see (3.21))
\[ J(h) = \frac{1}{2}\langle A^+ h, h \rangle_S + j(h^{(1)}) - \int_{S_2} F_0 h_0^{(1)} dS + \langle \varphi, r_{S_2} h^{(2)} \rangle_{S_2} \quad \forall h = (h^{(1)}, h^{(2)})^\top \in \mathbb{H}. \]

(4.1)

It is easy to show that, due to the self-adjointness property of the operator $A^+$ (see Lemma 3.1. (a)), the solvability of the boundary variational inequality (3.22) is equivalent to the minimization problem for the functional (1.1) on the set $\mathbb{H}$.

Since $j(h^{(1)}) \geq 0$ and the operator $A^+$ is bounded from below on $\mathbb{H}$ (see Lemma 3.1. (d)) we have
\[ J(h) \geq c_1\|h\|_{[H^{1/2}(S)]^6}^2 - c_2\|h\|_{[H^{1/2}(S)]^6}^6 \quad \forall h \in \mathbb{H}. \]

Consequently, when $h \in \mathbb{H}$ and $\|h\|_{[H^{1/2}(S)]^6} \to \infty$, then $J(h) \to +\infty$. Therefore the functional $J$ in (1.1) is coercive on the closed subspace $\mathbb{H}$. Moreover, the functional $J$ is convex and continuous. Due to the general theory of variational inequalities (see [DMLi1], [GLT1]) we conclude that the variational inequality (3.22) has a unique solution. In turn this implies the existence and uniqueness theorems for Problems $(A_0)$ and $(A)$.

**THEOREM 4.2** Suppose, the Dirichlet boundary part $S_1$ has positive measure. Let $\varphi \in [H^{-1/2}(S_2)]^3$, $F_0 \in L_\infty(S_2)$, $\varphi_0 \in [H^{1/2}(S_2)]^3$. Then Problem $(A_0)$ is uniquely solvable in the space $[H^1(\Omega^+)]^6$ and the solution is representable in the form $U = G^+h_0$, where $h_0$ is a unique solution of the variational inequality (3.22).

**Proof.** It immediately follows from Theorem 3.2 and Theorem 4.1. \qed

**COROLLARY 4.3** Let $X \in [L_2(\Omega^+)]^6$, $\varphi \in [H^{-1/2}(S_2)]^3$, $F_0 \in L_\infty(S_2)$, $f \in [H^{1/2}(S_1)]^6$ and $J : S_2 \to [0, \infty)$ be a bounded measurable function. Then Problem $(A)$ has a unique solution in the space $[H^1(\Omega^+)]^6$.\n
4.3 Lipschitz continuous dependence of solutions on the problem data

Let $U \in [H^1(\Omega^+)]^6$ and $\bar{U} \in [H^1(\Omega^+)]^6$ be two solutions of Problem (A0) corresponding to the data $F_0, \varphi, g$ and $\bar{F}_0, \bar{\varphi}, \bar{g}$ respectively. Further, let $h_0 = (\tilde{h}_0^{(1)}, \tilde{h}_0^{(2)})^\top \in \mathbb{H} \subset [H^{1/2}(S)]^6$ and $\bar{h}_0 = (\bar{h}_0^{(1)}, \bar{h}_0^{(2)})^\top \in \mathbb{H} \subset [H^{1/2}(S)]^6$ be the traces of the vector-functions $U$ and $\bar{U}$ on the surface $S$. Then, by virtue of Theorem 3.2, the vectors $h_0$ and $\bar{h}_0$ will be two solutions of the variational inequality (3.22) corresponding to the above data. So we have two variational inequalities of the type (3.22), the first one for $h_0$ and the second one for $\bar{h}_0$. Substitute $h = \bar{h}_0$ in the first one and $h = h_0$ in the second one, and sum up to obtain

$$-\langle A^+(h_0 - \bar{h}_0), h_0 - \bar{h}_0 \rangle_S - \int_{S_2} (g - \bar{g})(|h_{0s}^{(1)} - \varphi_0| - |ar{h}_{0s}^{(1)} - \varphi_0|) \, dS$$

$$\geq -\int_{S_2} (F_0 - \bar{F}_0)(h_{0m}^{(1)} - \bar{h}_{0m}^{(1)}) \, dS - \langle \varphi - \bar{\varphi}, r_{S_2}(h_{0s}^{(2)} - \bar{h}_{0s}^{(2)}) \rangle_S.$$

From this inequality, taking into account (3.14) and also the property (d) of the operator $A^+$, we can easily derive the following Lipschitz estimate:

$$\|U - \bar{U}\|_{H^1(\Omega^+)}^6 \leq c_1 \|h_0 - \bar{h}_0\|_{H^{1/2}(S)}^6$$

$$\leq c_2 \left( \|g - \bar{g}\|_{L_2(S_2)} + \|F_0 - \bar{F}_0\|_{L_2(S_2)} + \|\varphi - \bar{\varphi}\|_{H^{-1/2}(S)} \right),$$

where the positive constants $c_1$ and $c_2$ do not depend on the data of the problem.

5 The semicoercive case

Let $S_1 = \emptyset$. Then $S_2 = S$ and for the corresponding Problem (A) we will have the following formulation. Assume that $X \in [L_2(\Omega^+)]^6$, $F_0 \in L_\infty(S)$, $\varphi \in [H^{-1/2}(S)]^3$, $\mathcal{F} : S \to [0, \infty)$ is a bounded measurable function and $g = \mathcal{F}[F_0]$.

Problem (B) (semicoercive case). Find a vector-function $U = (u, \omega)^\top \in [H^1(\Omega^+)]^6$ which is a weak solution of the equation

$$L(\partial) U + X = 0 \quad \text{in} \quad \Omega^+, \quad (5.1)$$

satisfying on $S$ the conditions $\{\tau_s^{(n)}(U)\}^+ \in [L_\infty(S)]^3$ and

(i) $\{\tau_n^{(n)}(U)\}^+ = F_0$;

(ii) $\{\mathcal{M}U\}^+ = \varphi$;

(iii) (a) if $|\{\tau_s^{(n)}(U)\}^+| < g$, then $\{u_s\}^+ = 0$,

(b) if $|\{\tau_s^{(n)}(U)\}^+| = g$, then there exist nonnegative functions $\lambda_1$ and $\lambda_2$ which do not vanish simultaneously and $\lambda_1 \{u_s\}^+ = -\lambda_2 \{\tau_s^{(n)}(U)\}^+$.

Let the boundary $S$ of $\Omega^+$ be neither rotational nor a ruled surface (see e.g. [Is1]). To reduce this problem to the boundary variational inequality we need to reduce equivalently the nonhomogeneous equation (5.1) to the homogeneous one. To this end, consider the following auxiliary problem: Find a weak solution $U_0 = (u_0, \omega_0)^\top \in [H^1(\Omega^+)]^6$ of the equation (5.1) in $\Omega^+$ satisfying on $S$ the following conditions:

$$\{u_{0n}\}^+ = 0, \quad \{\tau_s^{(n)}(U_0)\}^+ = 0, \quad \{\mathcal{M}U_0\}^+ = 0 \quad \text{on} \quad S.$$
As it is known (see [GGN1], Theorem 4.4) this problem is uniquely solvable, since $S$ is neither rotational nor ruled. If $W \in [H^1(\Omega^+)]^6$ is a solution of Problem $(B)$ and $U_0 \in [H^1(\Omega^+)]^6$ is a solution of the above auxiliary problem, then the difference $U := W - U_0$ will be a solution of the following problem.

**Problem $(B_0)$**. Find a vector-function $U = (u, \omega)^\top \in [H^1(\Omega^+)]^6$ which is a weak solution of the homogeneous equation

$$L(\partial) U = 0 \quad \text{in} \quad \Omega^+,$$

satisfying on $S$ the following conditions $\{\tau^{(n)}_s(U)\}^+ \in [L_\infty(S)]^3$ and

1. $\{\tau^{(n)}_s(U)\}^+ = \psi$;
2. $\{\mathcal{M}U\}^+ = \varphi$;
3. (a) if $||\tau^{(n)}_s(U)||^+ < g$, then $\{u_s\}^+ = \varphi_0$, (b) if $||\tau^{(n)}_s(U)||^+ = g$, then there exist nonnegative functions $\lambda_1$ and $\lambda_2$ which do not vanish simultaneously and

$$\lambda_1 \{u_s\}^+ = -\lambda_2 \{\tau^{(n)}_s(U)\}^+ + \lambda_1 \varphi_0,$$

where $\psi = F_0 - \{\tau^{(n)}_n(U_0)\}^+$ and $\varphi_0 = -\{u_{0s}\}^+$.

Analogously to the previous coercive case (see Theorem 3.2) it can be shown that Problem $(B_0)$ is equivalent to the following boundary variational inequality:

**Find** $h_0 = (h_0^{(1)}, h_0^{(2)})^\top \in [H^{1/2}(S)]^6$ such that the inequality

$$\langle \mathcal{A}^+ h_0, h - h_0 \rangle_S + j(h^{(1)}) - j(h_0^{(1)}) \geq \langle \psi, h^{(1)} - h_0^{(1)} \rangle_S + \langle \varphi, h^{(2)} - h_0^{(2)} \rangle_S$$

holds for all $h = (h^{(1)}, h^{(2)})^\top \in [H^{1/2}(S)]^6$, where now

$$j(h^{(1)}) = \int_S g \, |h^{(1)} - \varphi_0| \, dS.$$

Namely, the variational inequality (5.3) and Problem $(B_0)$ are equivalent in the following sense: If $U \in [H^1(\Omega^+)]^6$ is a solution of the Problem $(B_0)$, then $h = \{U\}^+ \in [H^{1/2}(S)]^6$ is a solution of the variational inequality (5.3) and, vice versa, if $h \in [H^{1/2}(S)]^6$ is a solution of the variational inequality (5.3), then $G^+ h \in [H^1(\Omega^+)]^6$ is a weak solution of Problem $(B_0)$. Here the operator $G^+$ is defined by the equality (3.16).

Unfortunately, the variational inequality (5.3) is not unconditionally solvable.

Now we derive the necessary condition of solvability of the variational inequality (5.3). Let $h_0 = (h_0^{(1)}, h_0^{(2)})^\top \in [H^{1/2}(S)]^6$ be a solution of the inequality (5.3). Take $h = (\varphi_0, h_0^{(2)})^\top \in [H^{1/2}(S)]^6$ and $h = (2h_0^{(1)} - \varphi_0, h_0^{(2)})^\top$ instead of $h = (h^{(1)}, h^{(2)})^\top$ in (5.3) and take into account that the normal component of $\varphi_0$ vanishes. We obtain

$$\langle \{\tau^{(n)}(G^+ h_0)\}^+, h_0^{(1)} - \varphi_0 \rangle_S + j(h_0^{(1)}) = \langle \psi, h_0^{(1)} \rangle_S.$$  

If we sum up the inequalities (5.3) and (5.4) we get

$$\langle \{\tau^{(n)}(G^+ h_0)\}^+, h_0^{(1)} - \varphi_0 \rangle_S + \langle \{\mathcal{M}(G^+ h_0)\}^+, h_0^{(2)} - h_0^{(2)} \rangle_S + j(h^{(1)}) \geq \langle \psi, h_0^{(1)} \rangle_S + \langle \varphi, h_0^{(2)} - h_0^{(2)} \rangle_S.$$  

Rewrite (5.5) as follows

$$\langle \psi, h_0^{(1)} \rangle_S + \langle \varphi, h_0^{(2)} - h_0^{(2)} \rangle_S + \langle \{\tau^{(n)}(G^+ h_0)\}^+, h_0^{(1)} - \varphi_0 \rangle_S - \langle \{\mathcal{M}(G^+ h_0)\}^+, h_0^{(2)} - h_0^{(2)} \rangle_S \leq j(h^{(1)}).$$
Take here $2\varphi_0 - h^{(1)}$ instead of $h^{(1)}$ and $2h^{(2)} - h_0^{(2)}$ instead of $h^{(2)}$ to obtain

$$
-\langle \psi, h_n^{(1)} \rangle_S - \langle \varphi, h^{(2)} - h_0^{(2)} \rangle_S + \langle \{\mathcal{M}(G^+h_0)\}^+, h^{(1)} - \varphi_0 \rangle_S \\
+ \langle \{\mathcal{M}(G^+h_0)\}^+, h^{(2)} - h_0^{(2)} \rangle_S \leq j(h^{(1)}).
$$

From (5.6) and (5.7) we get the inequality

$$
\left| \langle \{\tau^{(n)}(G^+h_0)\}^+, h^{(1)} - \varphi_0 \rangle_S + \langle \{\mathcal{M}(G^+h_0)\}^+, h^{(2)} - h_0^{(2)} \rangle_S \\
- \langle \psi, h_n^{(1)} \rangle_S - \langle \varphi, h^{(2)} - h_0^{(2)} \rangle_S \right| \leq j(h^{(1)}) \quad \forall h = (h^{(1)}, h^{(2)})^\top \in [H^{1/2}(S)]^6.
$$

(5.8)

Let $h^{(1)} - \varphi_0 = \vartheta$ and $h^{(2)} - h_0^{(2)} = a$, where $\vartheta = [a \times x] + b$ with arbitrary constant vectors $a$ and $b$. Set $\chi := (\vartheta, a)^\top$. Since $\langle \{T(\vartheta, n)(G^+h_0)\}^+, \chi \rangle_S = 0$, we have from (5.8)

$$
\left| \langle \psi, \vartheta_n \rangle_S + \langle \varphi, a \rangle_S \right| \leq \int_S g |\vartheta_s| \, dS \quad \forall \chi = (\vartheta, a)^\top \in \Lambda(S),
$$

(5.9)

where $\Lambda(S) = \ker \mathcal{A}^\top$. Thus, if $h_0 = (h_0^{(1)}, h_0^{(2)})^\top \in [H^{1/2}(S_2)]^6$ is a solution of the variational inequality (5.3), then (5.9) holds for all $\chi \in \Lambda(S)$, i.e., (5.9) is a necessary condition for solvability of (5.3).

Let us show that if (5.9) holds with strict inequality sign, i.e.,

$$
\int_S g |\vartheta_s| \, dS - \left| \langle \psi, \vartheta_n \rangle_S + \langle \varphi, a \rangle_S \right| > 0 \quad \forall \chi = (\vartheta, a)^\top \in \Lambda(S), \quad \chi \neq 0,
$$

(5.10)

then the variational inequality (5.3) is solvable. Since $\Lambda(S)$ is a finite dimensional space, (5.10) can be sharpened to

$$
\int_S g |\vartheta_s| \, dS - \left| \langle \psi, \vartheta_n \rangle_S + \langle \varphi, a \rangle_S \right| \geq M \|\chi\|_{L^2(S)}^6 \quad \forall \chi = (\vartheta, a)^\top \in \Lambda(S),
$$

(5.11)

with some positive constant $M > 0$.

Therefore it suffices to show that (5.11) is a sufficient condition of solvability of the variational inequality (5.3). We proceed as follows.

Let $P$ be the operator of orthogonal projection (in the sense of the space $[L^2(S)]^6$) of the space $[H^{1/2}(S)]^6$ on the space $\Lambda(S)$ and $Q = I - P$. For any $h \in [H^{1/2}(S)]^6$ we have the representation $h = Qh + Ph$, where $Ph = (\vartheta, a)^\top \in \Lambda(S)$, $Qh = (\psi^{(1)}, \psi^{(2)})^\top \in [\Lambda(S)]^\perp$.

Consider the functional $\mathcal{J}$ on the space $[H^{1/2}(S)]^6$:

$$
\mathcal{J}(h) = \frac{1}{2} \langle \mathcal{A}^\top h, h \rangle_S + j(h^{(1)}) - \langle \psi, h_n^{(1)} \rangle_S - \langle \varphi, h^{(2)} \rangle_S, \quad h = (h^{(1)}, h^{(2)})^\top \in [H^{1/2}(S)]^6.
$$

Since the operator $\mathcal{A}^\top$ is self-adjoint, as in the previous case, the solvability of the inequality (5.3) is equivalent to the minimization problem for the functional $\mathcal{J}$ on the space $[H^{1/2}(S)]^6$. Now we show that the functional $\mathcal{J}$ is coercive, i.e.,

$$
\mathcal{J}(h) \to +\infty \quad \text{as} \quad \|h\|_{H^{1/2}(S)}^6 \to \infty.
$$
Since $A^+$ is self-adjoint, we have $\langle A^+(Qh + Ph), Qh + Ph \rangle_S = \langle A^+ Qh, Qh \rangle_S$ and from Lemma 3.1 (e) we derive with some positive constant $c$,

\[
\mathcal{J}(h) = \mathcal{J}(Qh + Ph)
\]

\[
= \frac{1}{2} \langle A^+ Qh, Qh \rangle_S + \int_S g |\psi_s^{(1)} + \vartheta_s - \varphi_0| dS - \langle \psi, \psi_n^{(1)} + \vartheta_n \rangle_S - \langle \varphi, \psi^{(2)} + a \rangle_S
\]

\[
\geq c \|Qh\|^2_{[H^{1/2}(S)]^6} - \int_S g |\psi_s^{(1)} - \varphi_0| dS - \langle \psi, \psi_n^{(1)} \rangle_S - \langle \varphi, \psi^{(2)} \rangle_S + \int_S g |\vartheta_s| dS - \langle \psi, \vartheta_n \rangle_S - \langle \varphi, a \rangle_S.
\]

From this inequality, taking into account (5.11) we obtain for $\chi := Ph$

\[
\mathcal{J}(h) \geq c \|Qh\|^2_{[H^{1/2}(S)]^6} - c_1 \|Qh\|_{[H^{1/2}(S)]^6} + M \|\chi\|_{L_2(S)^6} - c_2,
\]

where $c_1$ and $c_2$ are positive constants.

It is easy to see that for $h = Qh + Ph$ the norm $\|h\|_{[H^{1/2}(S)]^6}$ is equivalent to the norm $\|Qh\|_{[H^{1/2}(S)]^6}$. Therefore, from (5.12) we see that, if $\|h\|_{[H^{1/2}(S)]^6} \to \infty$, then $\mathcal{J}(h) \to +\infty$ which proves the coercivity of the functional $\mathcal{J}$. Due to the general theory of variational inequalities (see [GLT1], [DaL1]) we conclude that the convex continuous functional $\mathcal{J}(h)$ has a minimum on $[H^{1/2}(S)]^6$ and the minimizing function is a solution of the boundary variational inequality (5.3).

Let $h, h^* \in [H^{1/2}(S)]^6$ be two arbitrary solutions of the boundary variational inequality (5.3). It is easy to show that then

\[
\langle A^+(h - h^*), h - h^* \rangle_S = 0
\]

and consequently

\[
h - h^* = ([a \times x] + b, a) \in \Lambda(S).
\]

Thus, from the above results we can formulate the following assertion.

**THEOREM 5.1** Suppose, $S_1 = \emptyset$. Let $X \in [L_2(\Omega^+)]^6$, $F_0 \in L_\infty(S)$, $\varphi \in [H^{-1/2}(S)]^3$ and $g = \mathcal{F}|F_0|$ with $\mathcal{F} : S \to [0, \infty)$ being a bounded measurable function, and let (5.11) hold. Then the boundary variational inequality (5.3) is solvable and the solutions are determined modulo a generalized rigid displacement vector.

Due to the equivalence of the boundary variational inequality (5.3) and Problem (B) the counterpart of Theorem 5.1 holds for Problem (B) as well.

Analogously to the semicoercive case, in the same way we can investigate the contact Problem (C) when instead of the Dirichlet condition (3.3) the Neumann condition

\[
r_{S_1} \{T(\partial, n)U\}^+ = \Psi
\]

is given on the part $S_1$ of the boundary, where $\Psi \in [H^{-1/2}(S_1)]^6$; in contrast to the previous case, here the vector $\varphi$ in the condition (3.3) must be in $[H^{-1/2}(S_2)]^3$ and all other conditions on $S_2$ remain the same. Now, we need a solution to the following auxiliary problem: Find a vector-function $U_0 = (u_0, \omega_0)^T \in [H^1(\Omega^+)]^6$ which is a weak solution in $\Omega^+$ of the equation

\[
L(\partial) U_0 + X = 0
\]
and satisfies the following conditions:
\[
\begin{align*}
  r_{s_1} \{ T(\partial, n)U \}^+ &= 0 \quad \text{on} \quad S_1 \\
  r_{s_2} \{ u_{0n} \}^+ &= 0, \quad r_{s_2} \{ r_s^{(n)}(U_0) \}^+ &= 0, \quad r_{s_2} \{ MU_0 \}^+ &= 0 \quad \text{on} \quad S_2
\end{align*}
\]
As it is known (see [GGN1], Theorem 4.4) if the part \( S_2 \) is neither rotational nor ruled surface, this problem has a unique solution.

Let \( W \in [H^1(\Omega^+)]^6 \) be a solution of the above Problem (C) and \( U_0 \in [H^1(\Omega^+)]^6 \) be a solution of the auxiliary problem, then the difference \( U := W - U_0 \) will be a solution of the following problem. Problem \( (C_0) \). Find a weak solution \( U = (u, \omega)^T \in [H^1(\Omega^+)]^6 \) of the equation
\[
L(\partial) U = 0 \quad \text{in} \quad \Omega^+,
\]
satisfying the inclusion \( r_{s_2} \{ r_s^{(n)}(U) \}^+ \in [L^\infty(S_2)]^3 \) and the following conditions:
\[
\begin{align*}
  (i) \quad r_{s_1} \{ T(\partial, n)U \}^+ &= \Psi \quad \text{on} \quad S_1; \\
  (ii) \quad r_{s_2} \{ r_s^{(n)}(U) \}^+ &= \psi \quad \text{on} \quad S_2; \\
  (iii) \quad r_{s_2} \{ MU \}^+ &= \varphi \quad \text{on} \quad S_2; \\
  (iv) \quad (a) \quad &\text{if} \quad |r_{s_2} \{ r_s^{(n)}(U) \}^+| < g, \quad \text{then} \quad r_{s_2} \{ u_s \}^+ = \varphi_0 \quad \text{on} \quad S_2, \\
  (b) \quad &\text{if} \quad |r_{s_2} \{ r_s^{(n)}(U) \}^+| = g, \quad \text{then} \quad \text{there exist nonnegative functions} \ \lambda_1 \ \text{and} \ \lambda_2 \ \text{which do not vanish simultaneously and} \\
  &\lambda_1 r_{s_2} \{ u_s \}^+ = -\lambda_2 r_{s_2} \{ r_s^{(n)}(U) \}^+ + \lambda_1 \varphi_0 \quad \text{on} \quad S_2,
\end{align*}
\]
where \( g \) is defined by formula (3.1), \( \psi = F_0 - r_{s_2} \{ r_s^{(n)}(U_0) \}^+ \) and \( \varphi_0 = -r_{s_2} \{ u_{0s} \}^+ \).

The equivalent boundary variational inequality to this problem is the following: Find \( h_0 = (h_0^{(1)}, h_0^{(2)})^T \in [H^{1/2}(S)]^6 \) such that the inequality
\[
\langle A^+ h_0, h - h_0 \rangle_S + j(h^{(1)}) - j(h_0^{(1)}) \geq \langle \psi, r_{s_2}(h^{(1)} - h_0^{(1)}) \rangle_S + \langle r_{s_2} \varphi, r_{s_2}(h^{(2)} - h_0^{(2)}) \rangle_S + \langle r_{s_1} \Psi, r_{s_1}(h - h_0) \rangle_S 
\]
holds for all \( h = (h^{(1)}, h^{(2)})^T \in [H^{1/2}(S)]^6 \).

The variational inequality (5.13) and Problem \( (C_0) \) are equivalent in the sense described in Theorem 3.2. The necessary condition of solvability of the variational inequality (5.13) reads
\[
\left| \langle \psi, r_{s_2} \{ \partial_n \}^+ \rangle_{S_2} + \langle r_{s_2} \varphi, a \rangle_{S_2} + \langle r_{s_1} \Psi, r_{s_1} \{ \chi \}^+ \rangle_{S_1} \right| \leq \int_{S_2} g |\{ \vartheta_s \}^+| dS,
\]
where \( \chi = (\vartheta, a)^T \in \Lambda(S) \), \( \vartheta = [a \times x] + b \), \( a, b \in \mathbb{R}^3 \).

If (5.14) holds with the strict inequality sign, then we can sharpen it to
\[
\int_{S_2} g |\{ \vartheta_s \}^+| dS - |\langle \psi, r_{s_2} \{ \partial_n \}^+ \rangle_{S_2} + \langle r_{s_2} \varphi, a \rangle_{S_2} + \langle r_{s_1} \Psi, r_{s_1} \{ \chi \}^+ \rangle_{S_1} | \geq M \| \chi \|_{L^2(S)}^6
\]
for \( \forall \chi = (\vartheta, a)^T \in \Lambda(S) \) with some positive constant \( M \), since \( \Lambda(S) \) is a finite dimensional space. By the same arguments as above it can be shown that this condition is sufficient for the solvability of the variational inequality (5.13). Finally we arrive at the following theorem.

**THEOREM 5.2** If \( \Psi \in [\bar{H}^{-1/2}(S_1)]^6 \), \( \varphi \in [\bar{H}^{-1/2}(S_2)]^3 \), \( \psi \in \bar{H}^{-1/2}(S_2) \), \( \mathcal{F} : S_2 \to [0, \infty) \) is a bounded measurable function, \( g = \mathcal{F}|F_0| \) and (5.15) holds, then there exists a solution \( h_0 \) of the variational inequality (5.13) and \( G^+ h_0 \) solves Problem \( (C_0) \). Solutions of the variational inequality (5.13) and Problem \( (C_0) \) are defined modulo generalized rigid displacement vectors.
6 Exterior problems

First of all let us observe that the bilinear form

$$B(U,V) := \int_{\Omega^-} E(U,V) \, dx$$

is well defined for vectors $U = (u, \omega)^\top \in [H^1_{\text{loc}}(\Omega^-)]^6$ and $V = (v, w)^\top \in [H^1_{\text{loc}}(\Omega^-)]^6$ satisfying the decay conditions (Z) at infinity.

Let $X \in [L_{2,\text{comp}}(\Omega^-)]^6$, $f \in [H^{1/2}(S_1)]^6$, $F_0 \in L_{\infty}(S_2)$, $\varphi \in [H^{-1/2}(S_2)]^3$, and $g = \mathcal{F}|F_0|$ with $\mathcal{F} : S_2 \to [0, \infty)$ being a bounded measurable function. Consider the following bilateral contact problem with friction.

Problem (D). Find a weak solution $U = (u, \omega)^\top \in [H^1_{\text{loc}}(\Omega^-)]^6$ of equation

$$L(\partial) U + X = 0 \quad \text{in} \quad \Omega^-,$$

satisfying the decay conditions (Z) at infinity, the inclusion $r_{s_2}\{\tau_s^{(n)}(U)\}^- \in [L_{\infty}(S_2)]^3$ and the boundary conditions

(i) $r_{s_1}\{U\}^- = f$ on $S_1$;
(ii) $r_{s_2}\{\tau_s^{(n)}(U)\}^- = F_0$ on $S_2$;
(iii) $r_{s_2}\{LU\}^- = \varphi$ on $S_2$;
(iv) (a) if $|r_{s_2}\{\tau_s^{(n)}(U)\}^-| < g$, then $r_{s_2}\{u_s\}^- = 0$,
     (b) if $|r_{s_2}\{\tau_s^{(n)}(U)\}^-| = g$, then there exist nonnegative
         functions $\lambda_1$ and $\lambda_2$ which do not vanish simultaneously and
         $\lambda_1 r_{s_2}\{u_s\}^- = -\lambda_2 r_{s_2}\{\tau_s^{(n)}(U)\}^-$ on $S_2$.

To reduce this problem to the boundary variational inequality, as a first step, again we have to reduce the nonhomogeneous equation (6.1) and the nonhomogeneous Dirichlet condition (6.2) to the homogeneous ones. For this purpose consider the following auxiliary problem: Find a weak solution $U_0 \in [H^1_{\text{loc}}(\Omega^-)]^6$ of the equation

$$L(\partial) U_0 + X = 0 \quad \text{in} \quad \Omega^-$$

satisfying the decay conditions (Z) at infinity and the boundary conditions on $S$:

$$r_{s_1}\{U_0\}^- = f \quad \text{on} \quad S_1, \quad r_{s_2}\{T(\partial,n)U_0\}^- = 0 \quad \text{on} \quad S_2.$$

By the same approach as in the case of the interior problem, with the help of the solution vector $U_0 = (u_0, \omega_0)^\top$ the original Problem (D) can be reduced to the following one.

Problem (D_0). Find a weak solution $U = (u, \omega)^\top \in [H^1_{\text{loc}}(\Omega^-)]^6$ of the homogeneous equation

$$L(\partial) U = 0 \quad \text{in} \quad \Omega^-$$

satisfying the decay conditions (Z) at infinity, the inclusion $r_{s_2}\{\tau_s^{(n)}(U)\}^- \in [L_{\infty}(S_2)]^3$ and bound-
ary conditions

(i) \( r_{S_1}\{U\}^- = 0 \) on \( S_1 \);
(ii) \( r_{S_2}\{\tau_n(U)\}^- = F_0 \) on \( S_2 \);
(iii) \( r_{S_2}\{MU\}^- = \varphi \) on \( S_2 \);
(iv) (a) if \( |r_{S_2}\{\tau_s(U)\}^-| < g \), then \( r_{S_2}\{u_s\}^- = \varphi_0 \);
(b) if \( |r_{S_2}\{\tau_s(U)\}^-| = g \), then there exist nonnegative functions \( \lambda_1 \) and \( \lambda_2 \) which do not vanish simultaneously and

\[
\lambda_1 r_{S_2}\{u_s\}^- = -\lambda_2 r_{S_2}\{\tau_s(U)\}^- + \lambda_1 \varphi_0 \text{ on } S_2,
\]

where \( \varphi_0 = -r_{S_2}\{u_0\}^+ \).

To reduce this exterior problem equivalently to the boundary variational inequality we apply the following representation of solution of the equation (6.4) from the class \([H^1_{\text{loc}}(\Omega^-)]^6\) satisfying the decay condition (Z) at infinity (see \([NGST]\))

\[
U(x) = (G^{-1}h)(x) := V(H^{-1}h)(x) = \int_S \Gamma(x-y) (H^{-1}h)(y) \, dS_y, \quad x \in \Omega^-,
\]

where \( \Gamma \) is the fundamental solution of the operator \( L(\partial) \), \( H \) is defined by \([3.11]\) and \( h = \{U\}^- \in [H^{1/2}(S)]^6 \).

Define the Steklov-Poincaré type operator \( A^- \) with the help of the formula

\[
A^- h := \{T(\partial, n)(G^{-1}h)\}^- = \{T(\partial, n)V(H^{-1}h)\}^-.
\]

Due to the properties of the single layer potential this operator can be represented as

\[
A^- = (2^{-1}I_6 + K) H^{-1},
\]

where \( K \) is defined by \([3.15]\).

The operator \( A^- \) possesses almost the same properties as \( A^+ \).\( A^- \) possesses almost the same properties as \( A^+ \). Namely,

(a) \( \langle A^- h', h'' \rangle_S = \langle A^- h'', h' \rangle_S \quad \forall h' \in [H^{1/2}(S)]^6 \) and \( \forall h'' \in [H^{1/2}(S)]^6 \);
(b) \( A^- : [H^{1/2}(S)]^6 \to [H^{-1/2}(S)]^6 \) is a continuous operator;
(c) there is a constant \( c_0 > 0 \), such that \( \langle A^- h, h \rangle_S \geq c_0 \|h\|^2_{[H^{1/2}(S)]^6} \forall h \in [H^{1/2}(S)]^6 \).

Further, let us recall that \( \mathbb{H} = \{h = (h^{(1)}, h^{(2)})^\top \in [H^{1/2}(S)]^6 : r_{S_1} h = 0 \} \) and consider the variational inequality on \( \mathbb{H} \): Find a vector function \( h_0 = (h_0^{(1)}, h_0^{(2)})^\top \in \mathbb{H} \), such that the inequality

\[
\langle A^- h_0, h-h_0 \rangle_S + j(h^{(1)}) - j(h_0^{(1)}) \geq \int_{S_2} F_0 (h_n^{(1)} - h_{0n}^{(1)}) \, dS + \langle \varphi, r_{S_2} (h^{(2)} - h_0^{(2)}) \rangle_{S_2}, \quad (6.5)
\]

holds for all \( h = (h^{(1)}, h^{(2)})^\top \in \mathbb{H} \). Here \( j(\cdot) \) is defined by the relation \([3.20]\).

Using the same arguments as in Theorems 3.2 and 4.1 we can prove the equivalence of the variational inequality \([6.5]\) and Problem \((D_0)\), and the uniqueness theorem of solution to the variational inequality \([6.5] \). To prove the existence of solutions, we consider the following functional on the closed subspace \( \mathbb{H} \)

\[
\mathcal{J}(h) = \frac{1}{2} \langle A^- h, h \rangle_S + j(h^{(1)}) - \int_{S_2} F_0 h_n^{(1)} \, dS - \langle \varphi, r_{S_2} (h^{(2)}) \rangle_{S_2}, \quad h = (h^{(1)}, h^{(2)})^\top \in \mathbb{H}.
\]


Due to the symmetry property (a) of the operator $A^-$, the solvability of the variational inequality \[(6.5)\] is equivalent to the minimization problem for the functional $J(h)$ on the set $H$. It is easy to see that from the coercivity property (c) of the operator $A^-$ and from the inequality $j(h^{(1)}) \geq 0$ we obtain the estimate from below for the functional $J(h)$

$$J(h) \geq c_1\|h\|^2_{[H^{1/2}(S)]^6} - c_2\|h\|^6_{[H^{1/2}(S)]^6} \quad \forall h \in H.$$

Hence the coercivity of the functional $J(h)$ follows, i.e., $J(h) \to +\infty$ if $\|h\|^6_{[H^{1/2}(S)]^6} \to \infty$. Due to the theory of variational inequalities (see [GLTII, DuLiI]) we conclude that the convex continuous functional $J(h)$ has a minimum on $[H^{1/2}(S)]^6$ and the minimizing function $h_0$ is a unique solution of the variational inequality \[(6.5)\]. Consequently, the unique solution of Problem $(D_0)$ can be represented in the form $U = G^-h_0$. Finally, we arrive at the following existence result.

**THEOREM 6.1** Let $X \in [L_2, \text{comp}(\Omega^-)]^6$, $\varphi \in [H^{-1/2}(S_2)]^3$, $F_0 \in L_\infty(S_2)$, $f \in [H^{1/2}(S_1)]^6$ and $F : S_2 \to [0, \infty)$ be a bounded measurable function. Then Problem $(D)$ has a unique solution in the space $[H^1_{\text{loc}}(\Omega^-)]^6$ satisfying the decay conditions $(Z)$ at infinity.

**REMARK 6.2** By the same approach one can investigate the problem when either

(a) $S_1 = \emptyset$ and the friction conditions are considered on the whole boundary

or

(b) $S_1 \neq \emptyset$ and instead of the Dirichlet condition \[(6.2)\] there is given the Neumann condition

$$r_{S_1}\{T(\partial, n)U\}^+ = \Psi,$$

where $\Psi \in [\tilde{H}^{-1/2}(S_1)]^6$. Note that the vector $\varphi$ involved in the boundary condition \[(6.3)\] now should be from the space $[\tilde{H}^{-1/2}(S_2)]^3$.

Both these problems are uniquely solvable.

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