Gromov hyperbolicity and the Kobayashi metric on "convex" sets

Nikolai Nikolov and Maria Trybułka

Abstract. In this paper we study the global geometry of the Kobayashi metric on "convex" sets. We provide new examples of non-Gromov hyperbolic domains in \( \mathbb{C}^n \) of many kinds: pseudoconvex and non-pseudoconvex, bounded and unbounded. Our first aim is to prove that if \( \Omega \) is a bounded weakly linearly convex domain in \( \mathbb{C}^n \), \( n \geq 2 \), and \( S \) is an affine complex hyperplane intersecting \( \Omega \), then the domain \( \Omega \setminus S \) endowed with the Kobayashi metric is not Gromov hyperbolic (Theorem 1.3). Next we localize this result on Kobayashi hyperbolic convex domains. Namely, we show that Gromov hyperbolicity of every open set of the form \( \Omega \setminus S' \), where \( S' \) is relatively closed in \( \Omega \) and \( \Omega \) is a convex domain, depends only on that how \( S' \) looks near the boundary, i.e., whether \( S' \) near \( \partial \Omega \) (Theorem 1.7). We close the paper with a general remark on Hartogs type domains. The paper extends in an essential way results in [6].

Mathematics Subject Classification (2010). 32F45, 53C23.

Keywords. Kobayashi distance, Kobayashi metric, Gromov hyperbolicity, weakly linearly convex set, quasi-geodesic.

1. Introduction

The Kobayashi distance was introduced in late 60’s. Since then much has been told about the behavior of the infinitesimal Kobayashi metric (cf. [8] and the references therein). However, the global behavior of the Kobayashi distance function is still out of scope our possibilities.

Generally the Kobayashi metric is not Riemannian, for instance one can consider complex ellipsoids in \( \mathbb{C}^2 \), and thus no longer has a curvature. Instead one can consider a coarser notion of non-positive curvature from geometric group theory that has the origins in the fundamental work of Alexandrov (see a survey article [1]). He gave several equivalent definitions of what it means

The last author was supported by National Center of Science (Poland), grant no. 2013/10/A/ST1/00091.
for a metric space to have curvature bounded above by a real number $\kappa$. It was observed by M. Gromov that one of them, given by the co-called CAT($\kappa$) inequality, allows one to faithfully reflect the same concept in a much wider setting - that of geodesic metric spaces (see [5]). Motived by this Gromov introduced the notion of $\delta$-hyperbolicity, known also as Gromov hyperbolicity.

Gromov hyperbolic metric spaces indicate a number of regularities. We shall mention here only a few of them. For instance, it is know that every quasi-geodesic is within a bounded distance of an actual geodesic (cf. [4, Chapter III.H, Theorem 1.7]), what can be very helpful in many situation because it is often easy to construct quasi-geodesics but difficult to find an actual geodesic. Moreover, every quasi-isometry $f : \Omega_1 \to \Omega_2$ between Gromov hyperbolic spaces have a continuous extension to natural compactifications of $\Omega_1$ and $\Omega_2$ (cf. [4, Chapter III.H, Theorem 3.9]).

In [10], Thomas and the authors started considering the sensitivity of Gromov hyperbolic metric spaces to removing some part from it. Roughly speaking, they observed that under certain conditions on a set $\Omega_2$, a space $(\Omega_1 \setminus \Omega_2, k_{\Omega_1 \setminus \Omega_2})$ is hyperbolic if and only if $(\Omega_1, k_{\Omega_1})$ is hyperbolic (see [10, Proposition 5 and 6]).

Proposition 1.1. ([10 Proposition 6]) Let $\Omega$ be a bounded domain in $\mathbb{C}^n$, $n \geq 2$. Assume that $K$ is a linearly convex compact subset of $\Omega$, i.e., through any point $z \in \mathbb{C}^n \setminus K$ passes a complex line disjoint from $K$. Then $\Omega \setminus K$ is a domain such that $k_{\Omega \setminus K}$ is quasi-isometrically equivalent to $k_{\Omega \setminus (\Omega \setminus K) \times (\Omega \setminus K)}$.

In particular, if $\Omega$ is Gromov hyperbolic, then so is $\Omega \setminus K$.

A situation when we cut out a non-compact subset is completely different.

Theorem 1.2. ([6 Theorem 3.1]) Let $\Omega$ be a bounded convex open set in $\mathbb{C}^n$ and $S$ be a complex affine hyperplane such that $\Omega \cap S$ is not empty. Then $\Omega \setminus S$ is not Gromov hyperbolic.

On the other hand, if $A$ is relatively closed in $\Omega$ and so that $A$ is negligible with respect to the $(2n - 2)$-dimensional Hausdorff measure, then

$$k_{\Omega \setminus A} = k_{\Omega \setminus (\Omega \setminus A) \times (\Omega \setminus A)}$$

(cf. [8 Theorem 3.4.2]). Hence, the case considered by Haggui and Chrih is the essential one.

One of our goal is to refine the construction used in [6], and show (purely geometrically) that what is really needed in Theorem 1.2 is a weak linearly convexity. Namely we have:

Theorem 1.3. Let $\Omega$ be a bounded weakly linearly convex open set in $\mathbb{C}^n$, $n \geq 2$, and $S$ be a complex affine hyperplane such that $\Omega \cap S$ is not empty. Then $\Omega \setminus S$ is not Gromov hyperbolic.

Observe that $\Omega \setminus S$ is weakly linearly convex. Thus, in fact one may carry on the cutting procedure, and the resulting set is not Gromov hyperbolic as well.
Corollary 1.4. Let $\Omega$ be a bounded weakly linearly convex open set in $\mathbb{C}^n$, $n \geq 2$, and $S \neq \emptyset$ be a finite family of complex affine hyperplans intersecting $\Omega$. Then $\Omega \setminus \bigcup S$ is not Gromov hyperbolic.

Clearly, Corollary 1.4 applies to $\Omega$ convex domains, and this is highly non obvious if one wants to deduce it directly from Theorem 1.2.

Corollary 1.5. Let $\Omega$ be a bounded convex open set in $\mathbb{C}^n$, $n \geq 2$, and $S \neq \emptyset$ be a finite family of complex affine hyperplans intersecting $\Omega$. Then $\Omega \setminus \bigcup S$ is not Gromov hyperbolic.

The key ingredient in the proof of Theorem 1.3 is that every weakly linearly convex set contains ”sufficiently” big convex set (we make it more precise in §3). Roughly speaking, this simple observation with some other observations which are of an independent interest allow us to apply the construction of quasi-triangles considered in [6]. However, Theorem 1.3 does not follow directly from [6].

On the other hand, since Riemann Singularity Removable Theorems for bounded holomorphic functions and square integrable holomorphic functions hold, the phenomena described above is not observed for the Bergman and the Carathéodory distances (cf. [8, Appendix]). Namely:

Proposition 1.6. Let $\Omega$ and $S'$ are as in Theorem 1.3. Then $(\Omega, d_{\Omega})$ is Gromov hyperbolic if and only if $(\Omega \setminus S', d_{\Omega'} S')$ is Gromov hyperbolic, where $d_{\Omega'}$ denotes the Bergman or the Carathéodory distance on $\Omega'$.

It is an intriguing and very natural at the same time to ask whether Theorem 1.2 or 1.3 can be localized, i.e., if the intersection $\Omega \cap S$ might be replaced by a different set. The case when $S'$ is compact in $S \cap \Omega$ is solved by Proposition 1.1. Hence, the essential case is when $S'$ touches $\partial \Omega$. However, the situation in that case is totally different from the original one since there is no pseudoconvexity nor $k$-completness (see Proposition 5.1). Even though, under additional conditions we are able to obtain a partial result.

Theorem 1.7. Let $\Omega \subset \mathbb{C}^n$, $n \geq 2$, be a convex domain containing no complex line. Suppose there is a closed subset $S'$ of $\Omega$ such that there are a complex affine line $S$, a point $\zeta \in \partial \Omega \cap S$, and $R > 0$ so that

$$\Omega \cap B(\zeta, R) \cap S' = \Omega \cap B(\zeta, R) \cap S,$$

and $\partial \Omega$ is strongly convex near $\zeta$, i.e., $\partial \Omega$ contains no segment in some neighbourhood of $\zeta$. Then $\Omega \setminus S'$ is not Gromov hyperbolic.

In some sense the proof of the just stated result is unusual. We observe there an interesting pheonemana that the conglomerate of quasi-geodesics gives again a quasi-geodesic parametrized on the whole real line (!). To our knowledge it is the first time when one consider quasi-geodesics defined on $\mathbb{R}$, instead of quasi-rays in a context of studying the Kobayashi metric on general domains. The proof is purely geometric, relays doubtedly on the local strict convexity and does not use the boundness of $\Omega$ (only its $k$-hyperbolicity).
It would be interesting to know whether the condition of the local strict convexity is superfluous, and could be removed.

Before we close the present section we would like to make one remark. Haggui and Chrih in [6] also investigated some Hartogs type domains. In the present paper we easily obtain the following generalization of [6, Theorem 3.2]:

**Theorem 1.8.** Let $\Omega \subset \mathbb{C}^n$, $n \geq 1$, be a bounded weakly linearly convex domain, and $\varphi$ be a bounded function on $\Omega$ so that the set $\{(z, w) : z \in \Omega, \|w\| < e^{-\varphi(z)}\}$ is open, i.e., for instance $\varphi$ is an upper semicontinuous. Then, the open set $\Omega_\varphi = \{(z, w) : z \in \Omega, 0 < \|w\| < e^{-\varphi(z)}\}$ is not Gromov hyperbolic.

Observed that above we assumed only that $\varphi$ is bounded on $\overline{\Omega}$, no assumption about its continuity or (strict) plurisubharmonicity (!). The rest of this paper is organized as follows. In §2, we collect definitions and basic facts that we shall need. §3 is supplementary for §, §5, and §6 where we present proofs of Theorem 1.3, Theorem 1.7, and Theorem 1.8 respectively.

2. Preliminaries

2.1. Basic notation

- For any set $X \subset \mathbb{C}$ let $X_\ast = X \setminus \{0\}$
- If $A \subset \mathbb{R}$ put $A_{>0} = \{t \in A : t > 0\}$, $A_{\geq 0} = \{t \in A : t \geq 0\}$
- If $X$ is a linear space over $\mathbb{C}$ and $B$, $p \in X$, $C \subset \mathbb{C}$ let $p + B = \{p + x : x \in B\}$ and $Cp = \{\lambda p : \lambda \in C\}$
- If $p$, $q \in \mathbb{C}^n$ let $[p, q] = \{(1 - t)p + tq : t \in [0, 1]\}$, $[p, q]^{\ast} = \{(1 - t)p + tq : t \in [0, 1]\}$
- For $r > 0$ let $D(r) = \{z \in \mathbb{C} : |z| < r\}$, $D(1) = D$
- Let $H_+ = \{z \in \mathbb{C} : \text{Im} z > 0\}$
- For $z \in \mathbb{C}^n$ let $\|z\|$ denote the standard Euclidean norm of $z$
- For $\zeta \in \mathbb{C}^n$, $r > 0$ let $B(\zeta, r) = \{z \in \mathbb{C}^n : |z - \zeta| < r\}$
- Given an open set $\Omega \subset \mathbb{C}^n$ and $z \in \Omega$, $X \in \mathbb{C}^n$ let

$$\text{dist}_\Omega(z) = \inf \{\|z - w\| : w \in \partial \Omega\}$$

$$\text{dist}_\Omega(z; X) = \inf \{\|z - w\| : w \in (z + \mathbb{C}X) \cap \partial \Omega\}$$

2.2. The Kobayashi metric and distance

Given a domain $\Omega \subset \mathbb{C}^n$ the **Kobayashi metric** is defined as follows

$$\kappa_\Omega(z; X) = \inf \{||\lambda|| : \text{there exists } f \in H(D, \Omega) \text{ so that } f(0) = z, \lambda f'(0) = X\}.$$ 

If $\alpha : [a, b] \to \Omega$ is a $C^1$ piecewise curve its **Kobayashi length** is equal

$$l_{\kappa_\Omega}(\alpha) = \int_a^b \kappa_\Omega(\alpha(t); \alpha'(t)) \, dt.$$
For two points $p, q \in \Omega$ one can now define the \textit{Kobayashi pseudodistance} to be

$$k_{\Omega}(p, q) = \inf \left\{ l_{\kappa_{\Omega}}(\alpha) : \alpha : [0, 1] \to \Omega \text{ is a piecewise } C^1 \text{ smooth} \right\}$$

with $\alpha(0) = p, \alpha(1) = q$.

Among many regularities indicated by the Kobayashi distance is the so-called \textit{holomorphic contractibility}. More precisely, suppose $f : \Omega_1 \to \Omega_2, \Omega \subset \mathbb{C}^{n_j}, n_j \geq 1, j = 1, 2$, is a holomorphic map. Then for every $z, w \in \Omega_1$:

$$k_{\Omega_2}(f(z), f(w)) \leq k_{\Omega_1}(z, w).$$

In particular $k_{\Omega} \geq k_{\Omega'}$ on $\Omega \times \Omega$ if $\Omega \subset \Omega'$.

We say that a domain $\Omega \subset \mathbb{C}^n$ is $k$-\textit{hyperbolic} if $k_{\Omega}$ is a distance.

\textbf{Proposition 2.1.} ([8, Proposition 7.2.9]) Suppose that $\Omega$ is a $k$-hyperbolic domain in $\mathbb{C}^n$ and let $U \subset \Omega$ be any subdomain. Then

$$\kappa_U(z; X) \leq \inf \left\{ \coth k_{\Omega}(z, w) : w \in \Omega \setminus U \right\} k_{\Omega}(z; X), \quad z \in U, X \in \mathbb{C}^n.$$

We shall need later the following:

\textbf{Theorem 2.2.} [3] Let $\Omega$ be a geometrically convex domain in $\mathbb{C}^n$ containing no complex affine line. Then the Carathéodory pseudodistance is a distance, and every closed ball with respect to it is compact. In particular, $\Omega$ is $k$-hyperbolic.

For further information on the Kobayashi metric/distance we refer the reader to [8], or [9].

\section{2.3. Complex convexity}

An open set $\Omega \subset \mathbb{C}^n$ is said to be:

- \textit{C-convex} if any non-empty intersection with a complex line is a simply connected domain.
- \textit{linearly convex} (respectively: \textit{weakly linearly convex}) if each point in its complement (respectively: boundary) belongs to a complex hyperplane disjoint from the domain.

The following implications hold:

$$\text{C-convexity} \Rightarrow \text{linear convexity} \Rightarrow \text{weak linear convexity}.$$ 

Let us note that all three notions of complex convexity are different. However, for bounded domains with $C^1$-smooth boundaries they coincide. In the general case their place is between convexity and pseudoconvexity.

More properties of complex convex domains can be found in [2] or [7].

\section{2.4. Gromov hyperbolicity}

Suppose $(X, d)$ is a metric space. If $[a, b] \subset \mathbb{R}$ is an interval, a curve $\alpha : [a, b] \to X$ is a $(A, B)$-quasi-geodesic if

$$A^{-1}|s - t| - B \leq d(\alpha(s), \alpha(t)) \leq A|s - t| + B$$

for all $s, t \in [a, b]$. An $(A, B)$-\textit{quasi-triangle} in the metric space is a choice of three points in $X$ and $(A, B)$-quasi-geodesic segments connecting these
points. A geodesic triangle is said to be \( M \)-\textit{thin} if any point on any of the sides of the triangle is within distance \( M \) of the other two sides.

A curve \( \alpha : [a, b] \to X \) has a \textit{length parametrization} if
\[
d(\alpha(s), \alpha(t)) = |s - t|
\]
for every \( s, t \in [a, b] \).

A metric space is a \textit{length} space if the distance between every pair of points is equal to the infimum of the length of rectifiable curves joining them.

Example. For every \( \Omega \subset \mathbb{C}^n, n \geq 1 \), the space \((\Omega, k_\Omega)\) is a length space.

Moreover, if for every two points the distance \( d(x, y) \) is realized by some curve, then we say that \((X, d)\) is a \textit{geodesic} space.

If closed balls in \((X, d)\) are closed, we shall say \((X, d)\) is a \textit{proper} space. If \( d = k_\Omega \) for an open set \( \Omega \subset \mathbb{C}^n, n \geq 1 \), and \((\Omega, d_\Omega)\) is a proper space, then we say \( \Omega \) is \( k \)-\textit{finite complete}.

\textbf{Theorem 2.3 (Hopf-Rinow Theorem \cite{4}).} Let \( X \) be a length space. If \( X \) is a complete and locally compact, then

1. \( X \) is a proper space;
2. \( X \) is a geodesic space.

\textbf{Corollary 2.4.} Suppose \( \Omega \subset \mathbb{C}^n, n \geq 1 \) is an open set, and \( x, y \in \Omega \). If \((\Omega, k_\Omega)\) is complete, then \((\Omega, k_\Omega)\) is a geodesic space.

\textit{Proof.} Follows directly from Hopf-Rinow’s Theorem and \cite[Theorem 7.3.2]{8} \( \square \)

\textbf{Definition 2.5.} A proper geodesic metric space \((X, d)\) is called \( M \)-\textit{hyperbolic} if every geodesic triangle is \( M \)-thin. If \((X, d)\) is \( M \)-hyperbolic for some \( M \geq 0 \) then \((X, d)\) is called \textit{Gromov hyperbolic}.

The book by Bridson and Haefliger is one of the standard references for Gromov hyperbolic metric spaces.

\section{3. The Kobayashi Distance On Weakly Linearly Convex Sets}

We open the section with a simple observation that turns out to be an essential ingredient in the sequel.

\textbf{Lemma 3.1.} For any \( z, w \in \mathbb{D}_* \) we have
\[
k_{\mathbb{D}_*}(z, w) \geq \frac{1}{2} \left| \log \frac{\log |w|}{\log |z|} \right|.
\]
The equality holds if and only if \( \frac{w}{z} \in \mathbb{R}_{>0} \).

\textit{Proof.} Using a rotation we may assume that \( z \in (0, 1) \) and \( w = |w|e^{i\varphi} \) for some \( \varphi \in [-\pi, \pi) \). Now recall that the map
\[
\pi : \mathbb{H}_+ \to \mathbb{D}_* \\
\pi(z) = e^{iz}
\]
is a holomorphic covering. Hence
\[ k_{\mathbb{D}}(z, w) = \inf \left\{ k_H( -i \log z, -i \log |w| + \varphi + 2k\pi) : k \in \mathbb{Z} \right\} \]
(see for instance [8, Theorem 3.3.7])
\[ = \inf \left\{ k_D(\sigma( -i \log z), \sigma( -i \log |w| + \varphi + 2k\pi)) : k \in \mathbb{Z} \right\}, \]
where \( \sigma(u) = \frac{u + i \log z}{u - i \log z} \). \( \sigma \) maps the line \( \{ s \in \mathbb{C} : \text{Im } s = \log |w| \} \) onto the circle symmetric with respect to the real axis and orthogonal to \( \partial \mathbb{D} \). Consequently, from the formula for the Kobayashi distance on the disc we get:
\[ k_{\mathbb{D}}(z, w) = k_D(\sigma( -i \log z), \sigma( -i \log |w|)) = \frac{1}{2} \left| \log \frac{\log |w|}{\log z} \right|. \]
□

From the last lemma easily follows:

**Lemma 3.2.** Suppose \( \Omega \subset \mathbb{C}^n \) is a bounded open weakly linearly convex set, and \( p, q \in \Omega \) are distinct. Let \( L \) be the complex line containing \( p \) and \( q \). If \( \zeta \in L \cap \partial \Omega \), then
\[ k_{\Omega}(p, q) \geq \frac{1}{2} \left| \log \frac{\|p - \zeta\|}{\|q - \zeta\|} \right|, \]
where \( d = \text{diam} \Omega \).

**Proof.** Since \( \Omega \) is weak linearly convex there exists a complex hyperplane \( H \) such that \( \Omega \cap (\zeta + H) = \emptyset \) (cf. [2]). Using an affine transformation we may assume that \( \zeta = 0 \). Now consider the projection \( \pi : \mathbb{C}^n \to \mathbb{C} \) onto \( L \) in direction \( H \). Then \( \pi(\Omega) \subset \mathbb{D}(r) \), for some \( r > 0 \). But every projection is a contraction. Consequently, we may take \( r = d \). By the holomorphic contractibility of the Kobayashi distance
\[ k_{\Omega}(p, q) \geq k_{\mathbb{D}(d)}(p, q) \geq \frac{1}{2} \left| \log \frac{\|p - \zeta\|}{\|q - \zeta\|} \right| \]
(in the last inequality we applied Lemma 3.1). □

The estimates in Lemma 3.2 is not optimal even in dimension 1. However, it is sufficient for demonstrating the \( k \)-completeness of every bounded weak linearly convex open set what seems to our knowledge not to be mentioned anywhere in the literature.

**Proposition 3.3.** Let \( \Omega \subset \mathbb{C}^n \) be a weakly linearly bounded domain. Then \( \Omega \) is \( k \)-finitely compact. In particular, \( \Omega \) is \( k \)-complete.

**Proof.** Fix \( \zeta \in \partial \Omega \) and \( p \in \Omega \). Without loss of generality we may assume that \( \zeta = 0 \). By Lemma 3.2
\[ k_{\Omega}(p, z) \geq k_{\mathbb{D}(d)}(p, \pi(\zeta)) \]

for some \( r > 0 \). Hence, by the completeness of the punctured disc, we conclude that

\[ k_\Omega(p, z) \to \infty \text{ if } z \to \zeta. \]

Since \( p \) was arbitrary chosen we conclude that Kobayashi balls with finite radii are relatively compact in \( \Omega \). From this easily follows the second part (see also [8, Theorem 7.3.2]).

In the next proposition we prove that for weakly linearly convex sets one may easily construct a quasi-geodesic. Namely, segments with one endpoint in the boundary can be parametrized as quasi-geodesics.

**Proposition 3.4.** Suppose a weakly linearly convex domain \( \Omega \) in \( \mathbb{C}^n \) contains \( n \) discs with center at a point \( q \in \Omega \) lying in the complex lines parallel to coordinate lines. Fix \( \zeta \in \partial \Omega \cap (q + \partial \mathbb{D}(r_1)e_1) \). For every non-empty \( \omega \supseteq \text{conv} \left( \bigcup_{j>1} q + \mathbb{D}(r_j)e_j \right) \) there exist \( A, B > 0 \) so that for every \( p \in \omega \) the curve

\[ \alpha : [0, \infty) \ni u \mapsto p^u \in \Omega \]

is a \((A, B)\)-quasi-geodesic in \((\Omega, k_\Omega)\).

Since weak linearity convexity is invariant under projective transformations, it is not indispensable that discs in Proposition 3.4 are orthogonal.

**Proof.** Before we proceed to the proof let us recall that by [12] the convex hull of the union

\[ \bigcup_{j=1}^n q + \mathbb{D}(r_j)e_j \]

is contained in \( \Omega \). Consequently, there are \( c, \epsilon > 0 \) so that

\[ \text{dist}_\Omega(\zeta + u(p - \zeta)) \geq cu. \]

Let \( H \) denote an arbitrary complex hyperplane such that \((H + \zeta) \cap \Omega = \emptyset\). Fix positive numbers \( s, t \) so that \( s > t \). Using the fact that the Kobayashi distance is an integrated form of the Kobayashi metric, we have the following

\[ k_\Omega(p^s, p^t) \leq l_{k_\Omega}(\alpha|_{[t,s]}) \leq \int_t^s \frac{2\|p - \zeta\|}{c} du = \frac{2\|p - \zeta\|}{c} |s - t|. \]

It remains to indicate the opposite inequality with some \( c' > 0 \). For that purpose let \( \alpha : [0, 1] \to \Omega \) be a \( C^1 \)-smooth curve such that \( \alpha(0) = p^s, \alpha(1) = p^t \). Next, there is an \( r > 0 \) so that the following holds:

\[ l_{k_\Omega}(\alpha) \geq l_{k_\pi(\Omega)}(\pi \circ \alpha) \geq l_{k_\pi(p) + \mathbb{D}(r)\ast}(\pi \circ \alpha) \]

\[ \geq k_{\pi(\zeta) + \mathbb{D}(r)\ast}(\pi(p^s), \pi(p^t)) \geq \frac{1}{2} \log \frac{\log \frac{e^{-2r\|p - \zeta\|}}{r}}{\log \frac{e^{-2r\|p - \zeta\|}}{r}} \]

where \( \pi \) is the projection onto the complex line passing through \( p, \zeta \) in the direction \( H \). \( \square \)
4. Proof of Theorem 1.3

We can assume that \( S = \{ z_n = 0 \} \), the origin 0 is contained in \( \Omega \) and that \( \Omega \) is a domain. Fix a positive \( r \) so that \( r B_n \subset \Omega \). Next, choose \( s \in (0, r) \) and put \( p = (0, \ldots, 0, s) \). Let \( d = \text{diam} \, \Omega \), \( r_k = \text{dist} \, (0; e_k) \), \( k = 1, \ldots, n \). Define

\[
\omega = \text{conv} \left( \bigcup_{k=1}^{n} \mathbb{D}(r_k)e_k \right).
\]

By a weak linearly convexity \( \omega \subset \Omega \) (cf. [12]).

**Step I - \( \Omega \setminus S \) is complete.**

Proof. Observe that \( \Omega \setminus S \) is a weakly linearly convex domain. Hence, the statement follows directly from Proposition 3.3. \( \square \)

In the next three steps we construct some quasi-geodesics in \( \Omega \setminus S \).

**Step II - For every \( \zeta \in \partial \omega \cap \partial \Omega \cap S \) the segment \( [p, \zeta) \) parametrized as follows

\[
\mathbb{R}_{\geq 0} \ni u \mapsto \zeta + e^{-2u}(p - \zeta)
\]

is a quasi-geodesic. Moreover, if \( K \subset \omega \setminus S \), then there are \( A, B > 0 \) so that the segment \( [q, \zeta) \) is an \( (A, B) \)-quasi-geodesic for every \( q \in K \).

Proof. Choose \( u_1, u_2 \geq 0 \). From the holomorphic contractibility of the Kobayashi distance we have

\[
k_{\Omega}(p^{u_1}, p^{u_2}) \leq k_{\Omega \setminus S}(p^{u_1}, p^{u_2}) \leq k_{\omega \setminus S}(p^{u_1}, p^{u_2}).
\]

By Proposition 3.3 and [6] Claim 2 we get the first part. Now notice that since \( K \subset \omega \setminus S \) we may find a positive \( c > 0 \) so that

\[
\text{dist} \, \omega \setminus S \, (q; q - \zeta) \geq c \| q - \zeta \| \quad \text{for} \ q \in K.
\]

Hence, for the remaining part it suffices to repeat the first part of Proposition 3.4. \( \square \)

**Step III - The real segment \( [p, 0) \)

\[
\alpha : \mathbb{R}_{\geq 0} \ni u \mapsto e^{-2u}p \in [p, 0)
\]

is a quasi-geodesic.

Here and below let \( p^u = e^{-2u}p \), \( u > 0 \).

Proof. Clearly \( \{0\}^{n-1} \times \mathbb{D}(r)_* \subset \Omega \setminus S \subset \mathbb{C}^{n-1} \times \mathbb{D}(d)_* \). Now if one combines Lemma 3.1 and the holomorphic contractibility of the Kobayashi metric, then gets the conclusion. Indeed,

\[
\frac{1}{2} \left| \log \frac{2u + \log s/d}{2t + \log s/d} \right| = k_{\mathbb{D}(d)_*}(\alpha_n(s), \alpha_n(t)) \leq k_{\mathbb{C}^{n-1} \times \mathbb{D}(d)_*}(\alpha(s), \alpha(t))
\]

\[
\leq k_{\Omega \setminus S}(\alpha(s), \alpha(t)) \leq k_{\mathbb{D}(r)_*}(\alpha_n(s), \alpha_n(t)) = \frac{1}{2} \left| \log \frac{2u + \log s/r}{2t + \log s/r} \right|.
\]

\( \square \)
Fix a point \( \zeta \in \partial \Omega \cap \partial \Omega \cap S \) and \( T > 0 \). Let \( \zeta^T \) be a unique point that lies in the intersection \( \partial \Omega \cap p^T + \mathbb{R}_{>0} \zeta \). Consider the curve \( \beta \) defined as follows

\[
\beta : \mathbb{R}_{\geq 0} \ni u \rightarrow \zeta^T + e^{-2u}(p^T - \zeta^T) \in \Omega.
\]

Clearly segments: \([p, \zeta], [p^T, \zeta^T]\) intersect at exactly one point, say \( \eta^T \).

**Step IV** - There exist \( A, B > 0 \) so that the curve \( \beta|_{[0, \beta^{-1}(\eta^T)]} \) is a \((A, B)\)-quasi-geodesic.

**Proof.** The proof goes along the same lines as the proof of Proposition 3.4. Indeed, it is enough to use the fact that \( \omega \subset \Omega \), and consequently

\[
\dist_{\Omega}(\cdot; \zeta) = \dist_{\Omega \setminus S}(\cdot; \zeta) \geq \dist_{\omega}(\cdot; \zeta) = \dist_{\omega \setminus S}(\cdot; \zeta)
\]
on \([p^T, \eta^T]\). \( \square \)

Finally:

**Step V** - \( \Omega \setminus S \) is not Gromov hyperbolic.

**Proof.** Assume the contrary. We will show that for any \( M > 0 \) there exists \( T > 0 \) so that the \((A, B)\)-quasi-triangle \([p, p^T], [p^T, \eta^T], [\eta^T, p]\) is not \( M\)-thin. By Proposition 3.3 we can choose \( T_0 > 0 \) so that \( k_{\Omega \setminus S}(\eta^{T_0}, [p, 0]) > M \). Now Lemma 3.2 implies that there is \( T > T_0 \) so that \( k_{\Omega \setminus S}(\eta^{T_0}, [p^T, \eta^T]) > M \). Consequently, \([p, p^T], [p^T, \eta^T], [\eta^T, p]\) is not \( M\)-thin, the contradiction and so \( \Omega \setminus S \) is not Gromov hyperbolic (see Corollary 2.4). \( \square \)

5. Proof of Theorem 1.7

**Proposition 5.1.** Suppose \( \Omega \subset \mathbb{C}^n \), \( n \geq 2 \), is a bounded convex domain and \( S \) is an affine hyperplane so that \( S \cap \Omega \) has non empty interior. If \( S' \subsetneq S \cap \Omega \) then

\[
\limsup_{z \to \zeta} k_{\Omega \setminus S'}(z, z_0) < \infty
\]

for every \( \zeta \in \Omega \cap S' \). In particular, \( \Omega \setminus S' \) is not pseudoconvex.

**Proof.** Using a linear transformation we may assume that \( S = \{z_1 = 0\} \). Fix \( \zeta \in \Omega \cap S' \) and \( z_0 \in \Omega \setminus S' \). We shall need as well the orthogonal projection \( \pi : \mathbb{C}^n \to \zeta + \{z_1 = 0\} \). Put \( q = \pi(z_0) \). Then

\[
k_{\Omega \setminus S'}(z, z) \leq k_{\Omega \setminus S'}(z_0, \pi(z_0)) + k_{\Omega \setminus S'}(\pi(z_0), \pi(z)) + k_{\Omega \setminus S'}(\pi(z), z).
\]

(5.1)

Now observe that

\[
k_{\Omega \setminus S'}(\pi(z_0), \pi(z)) \leq k_{\Omega \cap \zeta + \{z_1 = 0\}}(\pi(z_0), \pi(z)),
\]

and by the continuity of the Kobayashi distance the right side is bounded when \( z \) is near \( z_0 \). The last summand in (5.1) can be treated as follows. Put \( 2\varepsilon = \dist_{\Omega \setminus S}(\zeta) \). Since the set

\[
K' = \{w \in \Omega \cap S : \dist_{\Omega \setminus S}(w) \geq \varepsilon\} \subset \Omega
\]
is compact there is a $\delta > 0$ so that $\mathbb{D}(\delta) \times \text{int} \ K' \subset \Omega$. Hence
\[
k_{\Omega \setminus S'}(\pi(z), z) \leq k_{\text{int} \ K'}(\pi(z), z) \\
\leq \sup \left\{ k_{\text{int} \ K'}(w, w') : \text{dist}_{\text{int} \ K'}(w), \text{dist}_{\text{int} \ K'}(w') \leq \frac{\varepsilon}{2} \right\} < \infty.
\]
Putting altogether the inequalities, we get the first part. The last part holds due to the [8, Corollary 14.5.2].

Corollary 5.2. Assume that $\Omega$ and $S'$ are as in Theorem 1.7. Then $\Omega \setminus S'$ is not $k$-complete, neither it is pseudoconvex.

Proof of Theorem 1.7. In the following $A$, $B$ will be constants depending only on $\Omega$. The actual values of $A$ and $B$ do not matter and may change within the lines.

Assume the contrary, i.e., $\Omega \setminus S'$ is Gromov hyperbolic. Similarly as in the proof of Theorem 1.3 we construct a family of $(A, B)$-quasi-triangles which is not $M$-thin for every $M > 0$.

Using a rotation, we may assume that $S = \{z_1 = 0\}$ and $\zeta$ is a unique point lying on the intersection $\partial \Omega \cap \mathbb{R}_{\geq 0} \times \mathbb{C}^{n-1}$. Moreover, taking smaller $R$ we may assume that every point in $\partial \Omega \cap \mathbb{B}(\zeta, R)$ is strongly convex. Fix $0 < r \ll R$. Next define
\[
p \in \Omega \cap \partial \mathbb{B}(\zeta, r) \cap \{\text{Im } z_1 = 0\}
\]
and
\[
\eta \in \partial \Omega \cap \partial \mathbb{B}(\zeta, r) \cap \{\text{Im } z_1 > 0\}.
\]

Case: $S' \subset S$

If we consider the parametrization of $(p, \eta)$ given by
\[
\alpha : \mathbb{R}_{>0} \ni u \rightarrow \tilde{\eta}^u = \eta + e^{-2u}(p - \eta) \in \Omega
\]
then Proposition 5.1 with the proof of Proposition 3.4 imply that $\alpha$ is a $(A, B)$-quasi-geodesic in $(\Omega \setminus S', k_{\Omega \setminus S'})$ for some $A, B > 0$.

Fix $u > 0$. Put $p^u = p + e^{-2u}(\eta - p)$. Let $\zeta^u, \eta^u \in (\eta, \zeta)$ be so that $\text{Im } \zeta^u = \text{Im } p^u, \text{Im } \eta^u = \text{Im } \tilde{\eta}^u$.

Proposition 5.1 and Step IV in the proof of Theorem 1.3 imply that for $u \gg 1$ the segment $[p^u, \zeta^u]$ can be parametrized (as the $\alpha$) to be a $(A, B)$-quasi-geodesic - possibly after increasing $A$ and $B$. Moreover, the strong convexity and the proof of Proposition 3.4 allow us to assume that $[p^u, \eta^u]$ is a $(A, B)$-quasi-geodesic (see also 11, Lemma 4.7).

In aim to end the construction of the quasi-triangles we shall need the following:

Lemma 5.3. For $u \gg 1$ the segment $[\eta^u, \zeta^u]$ can be parametrized to be $(A, B)$-quasi-geodesic.

Proof. Fix $0 < s < s' \ll \frac{\varepsilon}{2}$. Let $\eta', \zeta' \in [\eta, \zeta]$ be so that $\|\eta - \eta'\| = s$, $\|\zeta - \zeta'\| = s$. 
Clearly, \([\eta', \eta] \) and \([\zeta', \zeta] \) after parametrization are \((A', B')\)-quasi-geodesics for some \(A', B' > 0 \). Let
\[
\alpha_{\eta} : [0, \infty) \to [\eta', \eta], \\
\alpha_{\zeta} : [0, \infty) \to [\zeta', \zeta]
\]
be these parametrizations. Our goal is to show that they can be conglomerated with \([\eta', \zeta'] \) into a quasi-geodesic.

Put
\[
2M = l_{k_{\Omega \setminus S'}[\eta', \zeta']},
\]
and let \(\beta : [0, 2M] \to \Omega \setminus S' \) be the length-parametrization of \([\eta', \zeta'] \). Define the curve
\[
\gamma : \mathbb{R} \to (\eta, \zeta) \subset \Omega \setminus S'
\]
as follows
\[
\gamma(t) = \begin{cases} \\
\alpha_{\eta}(-t - M) & \text{if } t \leq -M, \\
\beta(t - M) & \text{if } -M \leq t \leq M, \\
\alpha_{\zeta}(t - M) & \text{if } t \geq M.
\end{cases}
\]
Clearly, it suffices to consider the case when \(t_1 \ll -M \leq M \ll t_2 \). From above we easily compute
\[
k_{\Omega \setminus S'}(\gamma(t_1), \gamma(t_2)) \leq k_{\Omega \setminus S'}(\alpha_{\eta}(-t_1 - M), \eta') + k_{\Omega \setminus S'}(\eta', \zeta') + \\
k_{\Omega \setminus S'}(\zeta', \alpha_{\eta}(t_2 - M)) \leq A'|t_2 - t_1| + 2B' + +k_{\Omega \setminus S'}(\eta', \zeta').
\]

It remains to prove the estimates from below for \(k_{\Omega \setminus S'}(\gamma(t_1), \gamma(t_2)) \). Fix a geodesic \(a : [0, 1] \to \Omega \setminus S' \) such that \(a(0) = \gamma(t_1), a(1) = \gamma(t_2) \). Define
\[
s_1 = \inf \{u : a((u,1)) \subset \Omega \setminus (S' \cup \mathbb{B}(\eta, s))\}
\]
and
\[
s_2 = \sup \{u : a((0,u)) \subset \Omega \setminus (S' \cup \mathbb{B}(\zeta, s))\}
\]
Then
\[
l_{k_{\Omega \setminus S'}(a)} \geq k_{\Omega \setminus S'}(\gamma(t_1), a(s_1)) + k_{\Omega \setminus S'}(\gamma(t_2), a(s_2)) = (*).
\]
By the strong convexity we may assume that
\[
\inf \{|z - w| : z \in \partial \mathbb{B}(\zeta, s) \cap \Omega, w \in H + \zeta\} = \varepsilon > 0, \quad (5.2)
\]
where \(H \) is a supporting hyperplane of \(\Omega \) at \(\zeta \). Hence, if \(H_{\Omega} \) denotes the halfspace which contains \(\Omega \) and \(\partial H_{\Omega} = H + \zeta \), then
\[
k_{\Omega \setminus S'}(\gamma(t_2), a(s_2)) \geq \inf \{k_{H_{\Omega}}(\gamma(t_2), z) : z \in \partial \mathbb{B}(\eta, s) \cap \Omega\} \geq -\frac{1}{2} \log \frac{\text{dist}_{H_{\Omega}}(\gamma(t_2))}{\varepsilon}.
\]
Employing again the strict convexity of \(\Omega \), notice that there is a constant \(c > 0 \) so that
\[
\text{dist}_{H_{\Omega}}(\gamma(t)) \geq c||\gamma(t) - \zeta|| \quad \text{for } t \gg 1.
\]
Consequently,
\[
k_{\Omega \setminus S'}(\gamma(t_2), a(s_2)) \geq -\frac{1}{2} \log \frac{c||\gamma(t_2) - \zeta||}{\varepsilon} = t_2 - \frac{1}{2} \log \frac{c||\zeta - \eta||}{\varepsilon}.
\]
We spot that the above reasoning works for the first compound of \((\ast)\) as well, and so the proof is derived. \(\Box\)

We shall need also
\[
\inf_{z \in (p^u, \eta^u)} k_{\Omega \setminus S'}(\zeta^v, z) \geq \inf_{z \in (p^u, \eta^u)} k_{H_{\Omega}}(\zeta^v, z). \quad (5.3)
\]

On the other hand we claim that

**Lemma 5.4.** For every \(M > 0\) there exist \(v_0\) so that for every \(v \geq v_0\)
\[
\inf_{z \in (p^u, \eta^u)} k_{\Omega \setminus S'}(\zeta^v, z) \geq M
\]
for \(u \gg v\).

**Proof.** Before we proceed to the main part let us recall that by the localization of the Kobayashi metric:
\[
\kappa_B(\zeta, s') \cap (\Omega \setminus S')(z; X) \leq C \kappa_{\Omega \setminus S'}(z; X), \quad z \in B(\zeta, s) \cap \Omega \setminus S', \quad X \in \mathbb{C}^n
\]
for some \(C > 0\) (see Proposition 2.1). Observe that the usage of the localization is justified since \(\Omega\) does not contain any complex line and so is \(k\)-hyperbolic.

Fix \(u \gg v\) and \(z \in (p^u, \zeta^u)\). Choose a smooth curve \(i : [0, 1] \to \Omega \setminus S'\) so that \(i(0) = \zeta^v, i(1) = z\). Thus:
\[
l_{k_{\Omega \setminus S'}}(i) \geq \begin{cases} l_{k_{H_{\Omega}}}(i) & \text{if } i \not\subset B(\zeta, s) \cap (\Omega \setminus S') \\ \frac{1}{C} l_{\kappa_B(\zeta, s') \cap (\Omega \setminus S')}(i) & \text{if } i \subset B(\zeta, s) \cap (\Omega \setminus S') \end{cases}
\]
\[
\geq \begin{cases} -\frac{1}{2} \log \text{dist}_{H_{\Omega}}(\zeta^v) - \frac{1}{2} \log \text{dist}_{H_{\Omega}}(\zeta^u) + \log \varepsilon & \text{if } i \not\subset B(\zeta, s) \cap (\Omega \setminus S') \\ -\frac{1}{2C} \log (\log \|\zeta^v - \zeta\| - \log \|\zeta^u - \zeta\|) & \text{if } i \subset B(\zeta, s) \cap (\Omega \setminus S') \end{cases}
\]

(5.4)

(in the last inequality we applied Lemma 3.2). Since the last estimate does not depend on the choice of the \(i\), letting \(u \to \infty\) we obtain the statement. \(\Box\)

Fix \(M > 0\). Choose \(u\) so big that \(\eta^u \in B(\eta, s)\). By (5.3) there is \(v_0\) so that
\[
k_{\Omega \setminus S'}(\zeta^v, (p^u, \eta^u)) > M
\]
for \(v \geq v_0\). On the other hand, Lemma 5.4 says that possibly increasing \(v_0\) and \(u\) we have
\[
k_{\Omega \setminus S'}(\zeta^v, (p^u, \zeta^u)) > M.
\]
Hence \([p^u, \zeta^u, \eta^u]\) is not \(M\)-thin, and that contradicts with the hyperbolicity of \(\Omega \setminus S'\).

**Case:** \(S'\) is an arbitrary

The only thing that requires a comment in the previous case is to show that \([p, \eta]\) is an \((A, B)\)-quasi-geodesic.
Lemma 5.5. Assume $\Omega \subset \mathbb{C}^n$ is a convex domain, $L \subset \Omega$ closed and $\mu \in \partial \Omega \setminus L$ is a strongly convex point of $\partial \Omega$. Then, if the segment $[z, \mu) \subset \Omega \setminus L$, where $z \in \Omega \setminus L$, then $[z, \mu)$ can be parametrized as an $(A, B)$-quasi-geodesic in $(\Omega \setminus L, k_{\Omega \setminus L})$ for some $A, B > 0$.

Proof. Consider the standard parametrization $\alpha$ of $[z, \mu)$ (cf. Step II). We must show that

$$A^{-1}|s - t| - B \leq k_{\Omega \setminus L}(\alpha(t), \alpha(s)) \leq A|s - t| + B, \quad s, t \geq 0.$$ 

Since $\Omega$ is convex, an estimate from below follows by considering an appropriate supporting hyperplane at $\eta$. The estimates from below follows from $(5.4)$.

The proof is completed.

6. Proof of Theorem 1.8

Proof. Without loss of generality we may assume that $0 \in \Omega$. Putting $R = e^{-\inf \varphi}$, $r = e^{-\sup \varphi}$, we get

$$\Omega_r = \mathbb{C}^n \times D(r)_* \subset \Omega_{\varphi} \subset \mathbb{C}^n \times D(R)_* = \Omega_R.$$ 

Since $\Omega_r$ and $\Omega_R$ are weakly linearly convex, the statement follows directly from Theorem 1.3.

Acknowledgment.

Part of this work was done while the stay of the author in Singapore in May 2017 during Complex Geometry, Dynamical Systems and Foliation Theory. She would like to thank the Organizators for their hospitality and excellent working conditions. The author also would like to thank John Erik Fornaess for many helpful remarks that essentially improved the presentaion of the paper.

References

[1] A.D. Alexandrov, Über eine Verallgemeinerung der Riemannschen Geometrie, Schriftenreihe des Forschsinstituts für Mathematik 1(1957), Berlin, 33-84.
[2] M. Andersson, M. Passare, R. Sigurdsson, Complex convexity and analytic functionals, Progress in Mathematics, vol. 225, Birkhäuser Verlag, Basel, 2004.
[3] T.J. Barth, Convex domains and Kobayashi hyperbolicity, Proc. Amer. Math. Soc. 79(1980), 556-558.
[4] M.R. Bridson, A. Haefliger, Metric spaces of non-positive curvature, Grundlehren der Mathematischen Wissenschaften, vol. 319, Springer-Verlag, Berlin, 1999.
[5] M. Gromov, Hyperbolic groups. Essays in group theory, 75-263, Math. Sci. Res. Inst. Publ., 8, Springer, New York, 1987.
[6] F. Haggui, A. Chrih, *On the Gromov hyperbolicity of certain domains in $\mathbb{C}^n$*, to appear in C.R. Acad. Sci. Paris, [http://dx.doi.org/10.1016/j.crma.2017.03.013](http://dx.doi.org/10.1016/j.crma.2017.03.013).

[7] L. Hörmander, *Notions of Convexity*, Birkhäuser, Basel, 1994.

[8] M. Jarnicki, P. Pflug, *Invariant distances and metrics in complex analysis*, extended ed., de Gruyter Expositions in Mathematics, vol. 9, Walter de Gruyter & Co. KG, Berlin, 2013.

[9] S. Kobayashi, *Hyperbolic complex spaces*, Springer-Verlag, Berlin, 1998.

[10] N. Nikolov, P.J. Thomas, M. Trybula, *Gromov (non)hyperbolicity of certain domains in $\mathbb{C}^2$*, Forum Math., 2016, 28(4), 783-794.

[11] A.M. Zimmer, *Gromov hyperbolicity, the Kobayashi metric, and $\mathbb{C}$-convex sets* to appear in Trans. Amer. Math. Soc., [https://doi.org/10.1090/tran/6909](https://doi.org/10.1090/tran/6909).

[12] S. V. Znamenskii, L. N. Znamenskaya, *Spiral connectedness of the sections and projections of $\mathbb{C}$-convex sets*, Math. Notes 59 (1996), 253-260.

Nikolai Nikolov  
Institute of Mathematics and Informatics  
Bulgarian Academy of Sciences  
Acad. G. Bonchev 8, 1113 Sofia, Bulgaria  
Faculty of Information Sciences  
State University of Library Studies and Information Technologies  
Shipchenski prohod 69A, 1574 Sofia, Bulgaria  
e-mail: nik@math.bas.bg

Maria Trybula  
Faculty of Mathematics and Informatics  
Adam Mickiewicz University  
Umultowska 87, 61-614 Poznań, Poland  
e-mail: maria.h.trybula@gmail.com