HEISENBERG ACTION IN THE EQUIVARIANT K-THEORY OF HILBERT SCHEMES VIA SHUFFLE ALGEBRA.

BORIS FEIGIN, ALEKSANDER TSYMBALIUK

Abstract. In this paper we construct the action of Ding-Iohara and shuffle algebras in the sum of localized equivariant K–groups of Hilbert schemes of points on $\mathbb{C}^2$. We show that commutative elements $K_i$ of shuffle algebra act through vertex operators over positive part $\{K_i\}_{i>0}$ of the Heisenberg algebra in these K–groups. Hence we get the action of Heisenberg algebra itself. Finally, we normalize the basis of the structure sheaves of fixed points in such a way that it corresponds to the basis of Macdonald polynomials in the Fock space $k[h_1, h_2, \ldots]$.

1. Introduction

The Heisenberg algebra $\{h_i\}_{i \in \mathbb{Z} \setminus 0}$ is known (see [7]) to act through natural correspondences in the sum of localized equivariant cohomology rings $R = \oplus_n H^2_T(X[n]) \otimes H_T(pt) \operatorname{Frac}(H_T(pt))$, which is isomorphic to the Fock space $\Lambda_F := \mathbb{C}(\hbar, \hbar')[h_1, h_2, \ldots]$. Here $X$ is any surface and $X[n]$ stands for the Hilbert scheme of $n$ points on this surface. It was also shown in [5] that after certain normalization, there is an isomorphism $\Delta : R \to \Lambda_F$, which sends the basis of fixed points to Jack polynomials and $\{h_i\}_{i>0}$ are sent to operators of multiplication by $p_i$.

Till the end of this paper we are interested only in the case $X = \mathbb{C}^2$. In this paper we construct the action of operator $1 + \sum_{i>0} \tilde{K}_i z^i := \exp\left(\frac{(-1)^{i-1}}{i} h_i z^i\right)$ in the sum of localized equivariant K–groups $M = \oplus_n K^T_T(X[n]) \otimes K_T(pt) \operatorname{Frac}(K_T(pt))$ in geometric terms. This determines the action of Heisenberg algebra itself. We find an isomorphism $\Theta : M \to \Lambda_F$ which takes normalized fixed points basis $\{\langle \lambda \rangle\}$ to Macdonald polynomials $\{P_\lambda\}$. Isomorphism $\Theta$ takes operators $\tilde{K}_i$ to operators of multiplication by $e_i$ acting in the Macdonald polynomials basis through Pieri formulas.

For achieving this result and for its own sake we construct representations of two algebras: $A$ and $S$ (called Ding-Iohara and shuffle algebras correspondingly) in $M$. In fact, the subalgebra $S$ of shuffle algebra $S$, generated by $S_1$ is of particular interest to us. It is a trigonometric analog of Feigin-Odesskii algebra, studied in [4].

We also found that the same operators appear in [8], where the action of the Hall algebra of an elliptic curve is constructed in $R$. We expect there exists a surjective homomorphism from shuffle algebra to elliptic Hall algebra, lifting the representation in [8] to our one.

In section 2 we define Ding-Iohara and shuffle algebras and remind some properties of them. In section 3 we construct the action of Ding-Iohara algebra $A$ in $M$. In section 4 we verify that the constructed operators do give representation of $A$. In section 5 we define the action of shuffle algebra in $M$. Finally in section 6 we present operators $\tilde{K}_i$, normalization of the fixed points basis $\{\langle \lambda \rangle\}$ and an isomorphism $\Theta : M \to \Lambda_F$ with the above mentioned properties.

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2. Ding-Iohara and Shuffle Algebras

Let us fix any parameters $q_1$, $q_2$, $q_3$. Now we define Ding-Iohara algebra $A$. This is an associative algebra generated by $e_i$, $f_i$, $\psi_j^+$ ($i \in \mathbb{Z}, j \in \mathbb{Z}_+$) with the following defining relations:

1. $e(z)e(w)(z - q_1 w)(z - q_2 w)(z - q_3 w) = -e(w)e(z)(w - q_1 z)(w - q_2 z)(w - q_3 z)$

2. $f(z)f(w)(w - q_1 z)(w - q_2 z)(w - q_3 z) = -f(w)f(z)(z - q_1 w)(z - q_2 w)(z - q_3 w)$

3. $[e(z), f(w)] = \frac{\delta(z/w)}{(1 - q_1)(1 - q_2)(1 - q_3)}(\psi^+(w) - \psi^-(z))$

4. $\psi^\pm(z)e(w)(z - q_1 w)(z - q_2 w)(z - q_3 w) = -e(w)\psi^\pm(z)(w - q_1 z)(w - q_2 z)(w - q_3 z)$

5. $\psi^\pm(z)f(w)(w - q_1 z)(w - q_2 z)(w - q_3 z) = -f(w)\psi^\pm(z)(z - q_1 w)(z - q_2 w)(z - q_3 w)$

where the generating series are defined as follows:

$e(z) = \sum_{i=-\infty}^{\infty} e_i z^{-i}$, $f(z) = \sum_{i=-\infty}^{\infty} f_i z^{-i}$, $\psi^+(z) = \sum_{j \geq 0} \psi_j^+ z^{-j}$, $\psi^-(z) = \sum_{j \geq 0} \psi_j^- z^j$, $\delta(z) = \sum_{i=-\infty}^{\infty} z^i$.

Remark: These relations are very similar to the relations of quantum affine algebras (except Serre relations).

We denote by $A_+ (A_-)$ the subalgebra of $A$ generated by $e_i$ ($f_i$) correspondingly.

Following [3] we define a shuffle algebra $S$ depending on $q_1$, $q_2$, $q_3$. Fix a function

$\lambda(x, y) = \frac{(x - q_1 y)(x - q_2 y)(x - q_3 y)}{(x - y)^3}$.

The algebra $S$ is an associative graded algebra $S = \oplus_{n \geq 0} S_n$. Each graded component $S_n$ consists of rational functions of the form $F(x_1, \ldots, x_n) = \frac{f(x_1, \ldots, x_n)}{\prod_{1 \leq i < j \leq n} (x_i - x_j)^2}$, where $f(x_1, \ldots, x_n)$ is a symmetric Laurent polynomial. For $F \in S_m$ and $G \in S_n$, the product $F \ast G \in S_{m+n}$ is defined by the formula

$F \ast G(x_1, \ldots, x_{n+m}) = \text{Sym} \left( F(x_1, \ldots, x_m) G(x_{m+1}, \ldots, x_{m+n}) \prod_{1 \leq i < j \leq m+n} \lambda(x_i, x_j) \right)$.

Here the symbol Sym stands for the symmetrization. This endows $S$ with a structure of an associative algebra.

Now we formulate some known properties of shuffle algebras:

**Theorem 2.1.** For general parameters $q_1$, $q_2$, $q_3$ there is a natural isomorphism $\Xi : A_+ \to S$, which takes $e_n \in A_+$ into $x^n \in S$. Particularly, the whole algebra $S$ is generated by $S_1$.

**Theorem 2.2.** In case $q_1$, $q_2$ are generic and $q_1 q_2 q_3 = 1$ the subalgebra $S$ generated by $S_1$ consists of rational functions of the form $F(x_1, \ldots, x_n) = \frac{f(x_1, \ldots, x_n)}{\prod_{1 \leq i < j \leq n} (x_i - x_j)^2}$, where $f(x_1, \ldots, x_n)$ is a symmetric Laurent polynomial satisfying $f(x_1, \ldots, x_n) = 0$ if $\frac{x_1}{x_2} = q_1$, $\frac{x_2}{x_3} = q_2$ for $j = 2, 3$. 


Remark: For any parameters the connection between Ding-Iohara algebra $A$ and shuffle algebra $S$ can be established in the following way (here algebras $A$ and $S$ are considered with the same parameters $q_1, q_2, q_3$). Let $I_+$ be the kernel of the map $\Xi$ from Theorem 2.3 (this theorem claims $I_+$ is trivial for generic parameters). Denote by $I_T$ the transposed ideal of $A_-$, i.e. $I_T$ is obtained from $I_e$ by the change $e_i \mapsto f_{-i}$. Then the factor of $A$ by ideals $I_T, I_e$ is just what we are most interested in. It may be viewed over as a double of shuffle algebra $S$.

Theorem 2.3. If $q_1q_2q_3 = 1$ the following elements $K_n \in S_n$ commute

$$K_1(z) = 1, \quad K_2(z_1, z_2) = \frac{(z_1 - q_1 z_2)(z_2 - q_1 z_1)}{(z_1 - z_2)^2}, \quad K_n(z_1, \ldots, z_n) = \prod_{1 \leq i < j \leq n} K_2(z_i, z_j).$$

Subalgebra generated by $K_i$ is studied in [2], in particular, Theorem 2.3 is proved here.

This work was motivated by [1], [9].

3. Construction of operators

3.1. Correspondences. We recall that through out all this paper $X = \mathbb{C}^2$. In this case the Hilbert scheme of $n$ points $X^{[n]}$ as a set is identified with the set of all ideals in $\mathbb{C}[x, y]$ of codimension $n$. Let us remind correspondences used by H. Nakajima to construct a representation of Heisenberg algebra in $\oplus_n \mathbb{H}^*(X^{[n]})$. This action is constructed through correspondences $P[i] = \bigsqcup_n X^{[n]} \times X^{[n+i]}$. Though in future we will need only $P[1], P[2]$ let us mention the general definition of $P[i]$ for any $i$. For $i > 0$: $P[i] = \bigsqcup_n X^{[n]} \times X^{[n+i]}$ consists of all pairs of ideals $(J_1, J_2)$ of $\mathbb{C}[x, y]$ of codimension $n, n+i$ correspondingly, such that $J_2 \subset J_1$ and the factor $J_1/J_2$ is supported at a single point (for $i = 1$ the factor is always supported at one point). For $i < 0$ $P[i]$ is transposed to $P[-i]$. Let $L$ be a tautological line bundle on $P[1]$ whose fiber at any point $(J_1, J_2) \in P[1]$ equals to $J_1/J_2$. There are natural projections $p, q$ from $P[1]$ to $X^{[n]}$ and $X^{[n+1]}$ correspondingly.

3.2. Fixed Points. There is a natural action of $T = \mathbb{C}^* \times \mathbb{C}^*$ on each $X^{[n]}$ induced from the one on $X$ given by the formula $(t_1, t_2)(x, y) = (t_1 x, t_2 y)$. The set $(X^{[n]})^T$ of $T$-fixed points in $X^{[n]}$ is finite and all these fixed points are parameterized by Young diagrams of size $n$. Namely for each diagram $\lambda = (\lambda_1, \ldots, \lambda_k)$ we have an ideal $(t_1^{\lambda_1}, t_2^{\lambda_2}, \ldots, t_1^{\lambda_{k-1}}, t_2^k) =: J_\lambda \in (X^{[n]})^T$.

3.3. We denote by $'M$ the direct sum of equivariant (complexified) $K$-groups: $'M = \oplus_n K^T(X^{[n]})$. It is a module over $K^T(pt) = \mathbb{C}[T] = \mathbb{C}[t_1, t_2]$. We define $M = 'M \otimes_{K^T(pt)} \text{Frac}(K^T(pt)) = 'M \otimes_{\mathbb{C}[t_1, t_2]} \mathbb{C}(t_1, t_2)$.

We have an evident grading $M = \oplus_n M_n, \quad M_n = K^T(X^{[n]}) \otimes_{K^T(pt)} \text{Frac}(K^T(pt))$.

According to the Thomason localization theorem, restriction to the $T$-fixed point set induces an isomorphism

$$K^T(X^{[n]}) \otimes_{K^T(pt)} \text{Frac}(K^T(pt)) \rightarrow K^T((X^{[n]})^T) \otimes_{K^T(pt)} \text{Frac}(K^T(pt))$$

The structure sheaves \{\lambda\} of the $T$-fixed points $\lambda$ (see 3.2) form a basis in $\oplus_n K^T((X^{[n]})^T) \otimes_{K^T(pt)} \text{Frac}(K^T(pt))$. The embedding of a point $\lambda$ into $X^{[n]}$ is a proper morphism, so the direct image in the equivariant $K$-theory is well defined, and we will denote by $[\lambda] \in M_n$ the direct image of the structure sheaf $\{\lambda\}$. The set $[\lambda]$ forms a basis of $M$. 


3.4. Let us now consider the tautological vector bundle $\mathfrak{F}$ on $X^{[n]}$, whose fiber at the point corresponding to an ideal $J$ equals to $\mathbb{C}[x,y]/J$. We introduce generating series $a(z)$, $c(z)$ as follows:

$$a(z) := \Lambda_{-1/2}(\mathfrak{F}) = \sum_{i\geq 0} \Lambda^i(\mathfrak{F})(-1/2)^i$$

$$c(z) := a(zt_1)a(zt_2)a(t_{1}^{-1}t_{2}^{-1})a(zt_{1}^{-1})^{-1}a(zt_{2}^{-1})^{-1}a(zt_{1}t_{2})^{-1}.$$ 

We also define the operators

$$e_i = q_*(L'^{\otimes i} \otimes p^*) : M_n \to M_{n+1}$$

$$f_i = p_*(L'^{(i-1)} \otimes q^*) : M_n \to M_{n-1}$$

So $e_i$ is a composition of pull-back by $P[1] \to M_n$, tensoring by $L'^{\otimes i}$ and finally taking the direct image by $P[1] \to M_{n+1}$, while $f_{i+1}$ is received by the inverse order of these operations.

We consider the following generating series of operators acting in $M$:

$$e(z) = \sum_{r=-\infty}^{\infty} e_r z^{-r} : M_n \to M_{n+1}[[z,z^{-1}]]$$

$$f(z) = \sum_{r=-\infty}^{\infty} f_r z^{-r} : M_n \to M_{n-1}[[z,z^{-1}]]$$

$$\psi^+(z) |_{M_n} = \sum_{r=0}^{\infty} \psi^+_r z^{-r} := \left( -\frac{1-t_1^{-1}t_2^{-1}z^{-1}}{1-z^{-1}} c(z) \right)^+ \in M_n[[z^{-1}]]$$

$$\psi^-(z) |_{M_n} = \sum_{r=0}^{\infty} \psi^-_r z^{-r} := \left( -\frac{1-t_1^{-1}t_2^{-1}z^{-1}}{1-z^{-1}} c(z) \right)^- \in M_n[[z]]$$

where $(\cdot)^\pm$ denotes the expansion at $z = \infty, 0$ respectively.

Formulas (10) and (11) should be understood as follows: $\psi^\pm(z)$ acts by multiplication in $K$-theory by $\left( -\frac{1-t_1^{-1}t_2^{-1}z^{-1}}{1-z^{-1}} c(z) \right)^\pm$ and $\psi^\pm$ are defined as the coefficients of these series.

**Theorem 3.5.** The operators $e_i$, $f_i$, $\psi^+_j$, $\psi^-_j$ satisfy relations (7)–(12) with parameters $q_1 = t_1$, $q_2 = t_2$, $q_3 = t_1^{-1}t_2^{-1}$, i.e. they give a representation of algebra $A$ in the sum of localized equivariant $K$–groups of Hilbert schemes of points on $\mathbb{C}^2$.

This theorem will be proved in the next chapter.

Now we compute the matrix coefficients of the operators $e_i$, $f_i$ and the eigenvalues of $\psi^\pm(z)$ in the fixed points basis. Let us take any diagram $\lambda = (\lambda_1, \ldots, \lambda_k)$ and its box $[i,j]$, with the coordinates $(i,j)$ (i.e. it stands in the $i$th row and $j$th column), where $1 \leq i \leq k$, $1 \leq j \leq \lambda_i$. We introduce functions $l([\Box])$, $a([\Box])$ called legs and arms correspondingly:

$$l([\Box]) := \lambda_i - j$$

$$a([\Box]) := \max\{k|\lambda_k \geq j\} - i$$

We also denote by $\Sigma_1([\Box])$ all boxes of $\lambda$ with the coordinates $(i,k < j)$ and by $\Sigma_2([\Box])$ all boxes of $\lambda$ with the coordinates $(k < i,j)$. Sometimes we will write $\lambda + j$ for the diagram.
Lemma 3.6. a) The matrix coefficients of the operators $e_i, f_i$ in the fixed points basis $[\lambda]$ of $M$ are as follows:

$$
\begin{align*}
e_{i[\lambda,\lambda+k]} &= (1-t_1)^{-1}(1-t_2)^{-1} \left( t_1^{s_1} t_2^{s_2} \right)^i \\
& \prod_{s \in \Sigma_1(\mathbb{D}_k,\lambda_k+1)} \left( t_1^{s_1} t_2^{s_2(1)} \right)^{-1} \left( 1-t_1^{s_1+1} t_2^{a(s)} \right)^{-1} \left( 1-t_1^{s_1+1} t_2^{a(s)+1} \right) ;

f_{i[\lambda,\lambda-k]} &= \left( t_1^{s_1-1} t_2^{s_2} \right)^{i-1} \times \\
& \prod_{s \in \Sigma_1(\mathbb{D}_k,\lambda_k)} \left( 1-t_1^{s_1} t_2^{a(s)} \right)^{-1} \left( 1-t_1^{s_1+1} t_2^{a(s)+1} \right) \prod_{s \in \Sigma_2(\mathbb{D}_k,\lambda_k)} \left( 1-t_1^{s_1} t_2^{a(s)} \right)^{-1} \left( 1-t_1^{s_1} t_2^{a(s)+1} \right). 
\end{align*}
$$

All the other matrix coefficients of $e_i, f_i$ vanish.

b) The eigenvalue of $e_i^{(k)}(z)$ on $[\lambda]$ equals to

$$
\left( \frac{1-t_1^{s_1} t_2^{s_2} z^{-1}}{1-z^{-1}} \prod_{s \in \Lambda} \left( 1-t_1^{s_1} t_2^{s_2(1)} \right)^{-1} \left( 1-t_1^{s_1+1} t_2^{a(s)} \right)^{-1} \left( 1-t_1^{s_1+1} t_2^{a(s)+1} \right) \right)^{\pm},
$$

where $\chi(\mathbb{D}) = t_1^{s_1-1} t_2^{s_2}$. 

Proof. a) For $(\lambda, \lambda') \in P[1]$ let $\rho : J_{\lambda'} \to J_\lambda$, $\pi : k[x,y]/J_{\lambda'} \to k[x,y]/J_\lambda$ be the natural maps. The tangent space $T_{\lambda}(J_{\lambda'}, J_{\lambda'})$ is a kernel of the map $\text{Hom}(J_{\lambda'}, k[x,y]/J_{\lambda'}) \oplus \text{Hom}(J_\lambda, k[x,y]/J_\lambda)$, which sends $(\alpha, \beta) \mapsto \pi \circ \alpha - \beta \circ \rho$. Further we will write simply $\lambda$ instead of $\lambda'$.

Let us denote by $\chi(\lambda, \lambda')$ the character of $\mathbb{T}$ in the tangent space $T_{\lambda}(J_{\lambda}, J_{\lambda'})$ and by $\chi(L_{\lambda, \lambda'})$ the character of $\mathbb{T}$ in the fiber $L$ at the point $(\lambda, \lambda')$. We write $S_{\chi(\lambda)}$ (respectively $S_{\chi(\lambda, \lambda')}^*$) for the character of $\mathbb{T}$ in the symmetric algebra $\text{Sym}^* T_{\lambda}(\lambda, \lambda') X[n]$ (respectively $\text{Sym}^* T_{\lambda}(\lambda, \lambda') P[1]$).

According to the Bott-Lefschetz fixed point formula, the matrix coefficient $p_{\lambda}(L^{\otimes i} \otimes q^*)^{[\lambda]}$ of $p_{\lambda}(L^{\otimes i} \otimes q^*)^{[\lambda]} : M_{n+1} \to M_n$ with respect to the basis elements $[\lambda] \in K^T(X[n], \lambda') \in K^T(X^{n+1})$ equals $\chi(L_{\lambda, \lambda'})^{[\lambda]} S_{\chi(\lambda, \lambda')}$. Similarly, the matrix coefficient $q_{\lambda}(L^{\otimes i} \otimes p^*)^{[\lambda]}$ of $q_{\lambda}(L^{\otimes i} \otimes p^*)^{[\lambda]} : M_n \to M_{n+1}$ with respect to the basis elements $[\lambda] \in K^T(X[n], \lambda') \in K^T(X^{n+1})$ equals $\chi(L_{\lambda, \lambda'})^{[\lambda]} S_{\chi(\lambda, \lambda')}$. 

Now it is straightforward to check the formulas.

b) Follows from the exactness of $\Lambda_i(F) := \sum_{i \geq 0} \Lambda^i(F) z^i$ on the category of coherent sheaves and the fact that $\{ \chi(\square) \in \lambda \}$ is a set of characters of $\mathbb{T}$ at the fiber $\mathfrak{g}$. 

Sometimes we will use another expressions for the matrix coefficients of operators $e_i, f_i$:

Proposition 3.7.

$$
\begin{align*}
e_r[\lambda-i,\lambda] &= \prod_{j=1}^r \frac{1-t_1^{\lambda_j-i+1} t_2^{i-j-1}}{1-t_1^{\lambda_j-i+1} t_2^{i-j}} \\
f_r[\lambda+i,\lambda] &= \prod_{j=1}^r \frac{1-t_1^{\lambda_j-i} t_2^{j-1}}{1-t_1^{\lambda_j-i} t_2^{j}} 
\end{align*}
$$

Proof. It is straightforward to get these formulas from Lemma 3.6. 

□
4. PROOF OF THEOREM 3.5

Definition We denote by $\sigma_1, \sigma_2, \sigma_3$ the elementary symmetric polynomials in $q_1, q_2, q_3$, i.e. $\sigma_1 := q_1 + q_2 + q_3$, $\sigma_2 := q_1q_2 + q_1q_3 + q_2q_3 = q_1^2 + q_2^2 + q_3^2$, $\sigma_3 := q_1q_2q_3 = 1$.

Convention: In this section we check equations (13) explicitly in the fixed points basis. While comparing expressions of LHS and RHS we denote by $P$, the mutual factor.

4.1. Let us check the equation (1) firstly.

Proof. For any integers $i, j$ we have to prove the following equation:

$$
\begin{align*}
& e_{i+3}e_j - \sigma_1 e_{i+2}e_{j+1} + \sigma_2 e_{i+1}e_{j+2} - \sigma_3 e_i e_{j+3} = \sigma_3 e_i e_{j+3} - \sigma_2 e_{i+j+1}e_{i+2} + \sigma_1 e_{j+2}e_{i+1} - e_{j+3}e_i.
\end{align*}
$$

Let us compare the matrix elements of LHS and RHS on any pair $[\lambda, \lambda' = \lambda + \Box_{i_1, j_1} + \Box_{i_2, j_2}]$.

a) $i_1 = i_2$, i.e. the added two boxes lie in the same row.

$$(e_{i+3}e_j - \sigma_1 e_{i+2}e_{j+1} + \sigma_2 e_{i+1}e_{j+2} - \sigma_3 e_i e_{j+3})|_{[\lambda, \lambda]} = (\ldots)(1 - \sigma_1 t_1^{-1} + \sigma_2 t_2^{-2} - \sigma_3 t_3^{-3}) = 0,$$

since $t_1^{-1}$ is a root of $1 - \sigma_1 t + \sigma_2 t^2 - \sigma_3 t^3$.

Similarly $(\sigma_3 e_i e_{j+3} - \sigma_2 e_{i+j+1}e_{i+2} + \sigma_1 e_{j+2}e_{i+1} - e_{j+3}e_i)|_{[\lambda, \lambda]} = 0$.

b) $j_1 = j_2$, i.e. the added two boxes lie in the same column.

This case in entirely similar since $t_2^{-1}$ is also a root of $1 - \sigma_1 t + \sigma_2 t^2 - \sigma_3 t^3$.

c) $i_1 < i_2, j_1 > j_2$.

The only difference occurs in the box $\Box_{i_1, j_2}$. Let us denote $a := j_1 - j_2, b := i_2 - i_1, \chi_1 := t_1^{i_1}t_2^{i_2} - 1, \chi_2 := t_1^{i_1}t_2^{i_2} - 1$. Then

$$
\begin{align*}
& (e_{i+3}e_j - \sigma_1 e_{i+2}e_{j+1} + \sigma_2 e_{i+1}e_{j+2} - \sigma_3 e_i e_{j+3})|_{[\lambda, \lambda]} = \\
& P_1(1-t_1^{-a}t_2^{-b})(1-t_1^{-a+1}t_2^{-b})(1-t_1^{-a+1}t_2^{-b+1})(1-t_1^{-a+1}t_2^{-b+1})\chi_1^{-i_1}\chi_2^{-i_2} \left(1 - \sigma_1 \left(\frac{\chi_1}{\chi_2}\right) + \sigma_2 \left(\frac{\chi_1}{\chi_2}\right)^2 - \sigma_3 \left(\frac{\chi_1}{\chi_2}\right)^3\right) + \\
& P_1(1-t_1^{-a}t_2^{-b})(1-t_1^{-a+1}t_2^{-b+1})(1-t_1^{-a+1}t_2^{-b+1})\chi_1^{-i_1}\chi_2^{-i_2} \left(1 - \sigma_1 \left(\frac{\chi_2}{\chi_1}\right) + \sigma_2 \left(\frac{\chi_2}{\chi_1}\right)^2 - \sigma_3 \left(\frac{\chi_2}{\chi_1}\right)^3\right).
\end{align*}
$$

Similarly $(\sigma_3 e_i e_{j+3} - \sigma_2 e_{i+j+1}e_{i+2} + \sigma_1 e_{j+2}e_{i+1} - e_{j+3}e_i)|_{[\lambda, \lambda]} = 0$.

Denote $u := t_1^a$, $v := t_2^b$. So the first summand of LHS equals

$$
\begin{align*}
P_1 \chi_1^{-i_1} \chi_2^{-i_2} (u - v)^{-1}(u - t_1 v)(v - t_1 u)(v - t_1 u)(v - t_1 t_2 u)(v - t_1 u)(v - t_1 t_2 u)(v - t_1 t_2 u) &= \\
P_1 \chi_1^{-i_1} \chi_2^{-i_2} (u - v)^{-1}(u - t_1 v)(v - t_1 t_2 u)(v - t_1 t_2 u)(v - t_1 t_2 u)(v - t_1 t_2 u)
\end{align*}
$$

while the first summand of RHS equals

$$
\begin{align*}
P_1 \chi_1^{-i_1} \chi_2^{-i_2} (u - v)^{-1}(u - t_2 u)(v - t_2 u)(v - t_2 u)(u - t_2 u)(u - t_2 u)(u - t_2 u)(u - t_2 u)
\end{align*}
$$

So we get the same expressions for the 1st summands of LHS and RHS. In the same way we check the equality of the 2nd summands. This completes the proof in this case.

d) $i_1 > i_2, j_1 < j_2$. Follows from c).

The equation (2) is entirely similar to the one above, so we omit it.
4.2. Now we compute \([e(z), f(w)]\). We prove the following proposition at first:

**Proposition 4.3.** The coefficients of the the series \([e(z), f(w)]\) are diagonalizable in the fixed points basis \([\lambda]\).

**Proof.** We have to check \((e_i f_j)_{\lambda, \lambda'} = (f_j e_i)_{\lambda, \lambda'}\) for any pair \([\lambda, \lambda'] = \lambda + \square_{i_1, j_1} - \square_{i_2, j_2}\) \((i_1, j_1) \neq (i_2, j_2)\) of Young diagrams.

Let us consider the case \(i_1 < i_2, j_1 > j_2\) (the case \(i_1 > i_2, j_1 < j_2\) is analogous). We define \(a := j_1 - j_2, b := i_2 - i_1\). Then

\[
(e_i f_j)_{\lambda, \lambda'} = P_2(1-t_1^{1-a} t_2^{b+1}) (1-t_1^{-a} t_2^{b})^{-1} (1-t_1^{1-a} t_2^{b})^{-1} (1-t_1^{-a} t_2^{b+1})^{-1},
\]

\[
(f_j e_i)_{\lambda, \lambda'} = P_2(1-t_1^{1-a} t_2^{b+1})^{-1} (1-t_1^{-a} t_2^{b})^{-1} (1-t_1^{1-a} t_2^{b+1})^{-1} (1-t_1^{-a} t_2^{b})^{-1} = P_2(1-t_1^{1-a} t_2^{b+1}) (1-t_1^{-a} t_2^{b+1})^{-1}.
\]

So \((e_i f_j)_{\lambda, \lambda'} = (f_j e_i)_{\lambda, \lambda'}\) \(\square\).

4.4. Now we introduce the operators \(\Phi^+ (z) = \sum_{i=0}^{\infty} \phi_i^+ z^{-i}\), \(\Phi^- (z) = \sum_{i=0}^{\infty} \phi_i^- z^i\) diagonalizable in the fixed points basis and satisfying the equation

\[
[e(z), f(w)] = \frac{\delta(z/w)}{(1-t_1)(1-t_2)(1-t_1^{-1} t_2^{-1})} (\Phi^+(w) - \Phi^-(z)).
\]

We show that \(\phi_i^\pm\) are determined uniquely by the conditions \(\phi_0^+ = -1, \phi_0^- = -\frac{1}{t_1 t_2}\). Next we check

\[
\phi^\pm (z)e(w)(z-q_1 w)(z-q_2 w)(z-q_3 w) = e(w)\phi^\pm (z)(w-q_1 z)(w-q_2 z)(w-q_3 z)
\]

\[
\phi^\pm (z)f(w)(w-q_1 z)(w-q_2 z)(w-q_3 z) = f(w)\phi^\pm (z)(z-q_1 w)(z-q_2 w)(z-q_3 w)
\]

Finally by showing that \(\psi_i^\pm = \phi_i^\pm\) we get equations \(12, 13\) from equations \(10, 11\). And so the Theorem \(\square\) will be proved.

From Proposition \(\square\) and the formulas of Lemma \(\square\) one gets that \([e(z), f(w)]\) is diagonalizable in the fixed points basis and moreover its eigenvalue on \([\lambda]\) equals to

\[
\sum_{a, b \in \mathbb{Z}} z^{-a} w^{-b} \gamma_{a+b},
\]

where

\[
\gamma_i = (1-t_1)^{-1}(1-t_2)^{-1} \times
\]

\[
\sum_{\square-\text{corner}} \left( \prod_{s \in \Sigma_1(\square)} \left[ (1-t_1^{l(s)} t_2^{-a(s)})^{-1} (1-t_1^{-l(s)} t_2^{a(s)+1})^{-1} (1-t_1^{l(s)} t_2^{-a(s)+1})^{-1} \right] \right) \times
\]

\[
\prod_{s \in \Sigma_2(\square)} \left[ (1-t_1^{-l(s)} t_2^{-a(s)+1}) (1-t_1^{l(s)} t_2^{a(s)+1})^{-1} (1-t_1^{-l(s)} t_2^{a(s)})^{-1} \right] \chi^{-1}(\square) \times
\]

\[
(1-t_1)^{-1}(1-t_2)^{-1} \times
\]

\[
\sum_{\square-\text{hole}} \left( \prod_{s \in \Sigma_1(\square)} \left[ (1-t_1^{-l(s)} t_2^{a(s)+1})^{-1} (1-t_1^{l(s)} t_2^{-a(s)+1})^{-1} (1-t_1^{-l(s)} t_2^{a(s)})^{-1} \right] \right) \times
\]
\[ \prod_{s \in \Sigma_2(\square)} \left[ (1 - t_1^{-l(s)}t_2^a(s))^{-1}(1 - t_1^{-l(s)-1}t_2^{-a(s)-1}) \right] \chi^{s-1}(\square). \]

So as we want an equality \[ f(z, e(w)) = \frac{1}{1 - c(z) + e(2z)} \right) (\phi^+(w) - \phi^-(z)) = \sum_{a,b | a + b = 0} z^{-a} w^{-b} \phi^+_{a,b} - \sum_{a,b | a + b > 0} z^{-a} w^{-b} \phi^-_{a,b} + \sum_{a,b | a + b < 0} z^{-a} w^{-b} \phi^+_{a,b} \]
to hold, we determine \( \phi^+_i \) uniquely as they are equal to the corresponding \( (1 - t_1)(1 - t_2)(1 - t_1^{-1}t_2^{-1}) \gamma_s \). So to determine all \( \phi^+_i \) we need to specialize \( \phi^+_0, \phi^-_0 \).

Next lemma is crucial:

**Lemma 4.6.** \([e_0, f_0] | \lambda = - \frac{1}{(1 - t_1)(1 - t_2)} \); \([e_0, f_1] | \lambda = - \frac{1}{(1 - t_1)(1 - t_2)} + \sum_{\square \in \lambda} \chi(\square). \]

**Corollary 4.6.** \([e_0, f_1 - f_0] \) is the operator of multiplication by \( \text{det}(\square) \).

**Proof of Lemma 4.6.** In the proof below we use another expression for \( \gamma_s = [e_0, f_s] \), which is received by using Proposition 3.7 instead of Lemma 3.6a. With this purpose for any Young diagram \( \lambda = (\lambda_1, \lambda_2, \ldots) \) we define \( \chi_i := t_i^{-1} \). Let us notice that due to the finiteness of \( \lambda \): \( \chi_i = t_i^{-1} \) for all \( i >> 1 \), which will be called 'stabilizing condition'.

\[ \gamma_s = (1 - t_1)^{-2} \sum_{i=1}^{l(\lambda)+1} \chi_i^{s-1} \prod_{i < j} \left( 1 - t_1^{\lambda_i - 1}t_2^{-j+1} \right) \left( 1 - t_1^{\lambda_j - 1}t_2^{-i+1} \right) - \sum_{i=1}^{l(\lambda)+1} (t_1 \chi_i)^{s-1} \prod_{i < j} \left( 1 - t_1^{\lambda_i - 1}t_2^{-i+j} \right) \left( 1 - t_1^{\lambda_j - 1}t_2^{-i-j} \right) \]

a) Firstly we prove \([e_0, f_0] = - \frac{1}{(1 - t_1)(1 - t_2)} \) for any \( \lambda \). This is obvious for an empty diagram (and straightforward for any diagram \( \lambda \) consisting of 1 row). So it is enough to prove that \([e_0, f_0] \) does not depend on the diagram. Let \( \lambda_{k-1} = 0 \) (we do not need \( \lambda_{k-2} \neq 0 \)). Then for \( i \geq k - 1 \) we have \( \chi_i = t_i^{-1} \). Hence, according to formula (14) and 'stabilizing condition':

\[ \gamma_0 = (1 - t_1)^{-2} \sum_{i=1}^{k} \chi_i^{s-1} \left( \frac{1}{t_1^{\lambda_i - 1}t_2^{-i}} \chi_i \right) \prod_{1 \leq j \neq i} \left( \frac{1}{t_1^{\lambda_j - 1}t_2^{-i+j}} \right) \left( \frac{1}{t_1^{\lambda_i - 1}t_2^{-i-j}} \right) \]

So we have a rational expression in \( \chi_i \), \( 1 \leq i \leq k - 2 \). Moreover the degree of numerator is not greater than that of denominator. The possible poles of this function can occur only at \( \chi_i = \chi_j \), \( \chi_i = t_1 \chi_j \) or at \( \chi_i = t_2^{-1} \chi_j \). In case \( \chi_i = t_2^{-1} \), \( t_1^{-1} \) the poles will be compensated by zeros of \( \chi_i - t_1 \chi_{k-1} \) or \( \chi_i - t_2 \chi_{k-1} \) correspondingly. All the poles \( \chi_i = \chi_j \), \( \chi_i = t_1 \chi_j \) are simple. We see easily that the principal part at these points vanish. We conclude that this rational function \( \gamma_0 \) is constant. This completes the proof of \([e_0, f_0] = - \frac{1}{(1 - t_1)(1 - t_2)} \).

b) Let us check \([e_0, f_1] = - \frac{1}{(1 - t_1)(1 - t_2)} + \sum_{\square \in \lambda} \chi(\square) \) for any \( \lambda \). By the definition of \( \chi_i \) we have \( \sum_{\square \in \lambda} \chi(\square) = \sum_{i=1}^{\infty} \frac{t_2^{i-1} - t_1^{i-1}}{1 - t_1 t_2} \). So we have to prove: \([e_0, f_1] = - \frac{1}{(1 - t_1)(1 - t_2)} + \sum_{i=1}^{\infty} \frac{t_2^{i-1} - t_1^{i-1}}{1 - t_1} \).

It is obvious for an empty diagram and it is straightforward to check this for any diagram \( \lambda \).
consisting of 1 row. So it is enough to prove that \([e_0, f_1] - \sum_{i=1}^{\infty} \frac{t_i^{-1} t_i^{x_1}}{1-t_1^{x_1}}\) does not depend on the diagram.

Let \(\lambda_{k-1} = 0\). Then for \(i \geq k - 1\) we have \(\chi_i = t_1^{-1} t_2^{i-1}\). Hence, according to formula (14):

\[
\gamma_1 = (1 - t_1)^{-2} \sum_{i=1}^{k} \frac{\chi_i(1-t_1 t_2^{-k} \chi_i)}{\chi_i - t_2^{k-1}} \prod_{1 \leq j \neq i}^{k-1} \frac{(\chi_j - t_2 \chi_i)(\chi_i - t_1 t_2 \chi_j)}{(\chi_j - \chi_i)(\chi_i - t_1 \chi_j)}.
\]

So we have a rational expression in \(\chi_i\), \(1 \leq i \leq k - 2\). Moreover the degree of numerator is not greater than that of denominator plus 1. The possible poles of this function can occur only at \(\chi_i = \chi_j\), \(\chi_i = t_1 \chi_j\) or at \(\chi_i = t_2^{k-1}, t_1^{-1} t_2^{k-1}\). In case \(\chi_i = t_2^{k-1}, t_1^{-1} t_2^{k-1}\) the poles do not really occur (see argument in part a)). All the poles \(\chi_i = \chi_j\), \(\chi_i = t_1 \chi_j\) are simple. We see easily that the principal part at these points vanish. So \(\gamma_1\) is a linear function in \(\chi_i(1 \leq i \leq k - 2)\). Finally one checks that the principal part of \(\gamma_1\) equals to \(\sum_{i=1}^{k-2} \frac{1}{t_i^{x_1}} \chi_i\).

We conclude that \([e_0, f_1] - \sum_{i=1}^{\infty} \frac{t_i^{-1} t_i^{x_1}}{1-t_1^{x_1}}\) is constant. This completes the proof of \([e_0, f_1] = -\frac{1}{(1-t_1)(1-t_2)} + \sum_{\square \in \lambda} \chi(\square)\).

**Corollary 4.7.** It follows from Lemma 4.3 that \(\gamma_0 \equiv -\frac{1}{(1-t_1)(1-t_2)}\). So we can define

\[
\phi_0^+ := -1, \quad \phi_0^- := -\frac{1}{t_1 t_2}.
\]

All the operators \(\phi_j^\pm\) are diagonalizable in the fixed points basis according to Proposition 4.3.

Now we compute the matrix elements of LHS and RHS of (12) in the fixed points basis. It is enough to check for any Young diagrams \(\lambda, \lambda' = \lambda + i_1\) the following equality:

\[
(\phi_{i_1+3}^+ e_{j_1+2}^+ \phi_{i_2+1}^+ e_{j_2+1}^+ \phi_{i_3+2}^+ e_{j_3+2}^+ - 3 \phi_{i_1+3}^+ e_{j_1+3}^+)(\lambda, \lambda') = (\phi_{i_1+3}^+ e_{j_1+3}^+ - 3 \phi_{i_1+3}^+ e_{j_1+3}^+)(\lambda, \lambda').
\]

Let us denote \(\chi_1 := t_1^{i_1} t_2^{i_2-1}\), where \(j_2 := \lambda_3 + 1\). Taking into account the equality \(e_{j+1}^+ (\lambda, \lambda') = \chi_1^j e_{j}^+ (\lambda, \lambda')\) and the diagonalizability of \(\phi_i\) we reduce the above equation to the following:

\[
(\phi_{i_1+3}^+ - \sigma_1 \chi_1 \phi_{i_2+1}^+ + \sigma_2 \chi_2 \phi_{i_3+2}^+ - 3 \phi_{i_1+3}^+)(\lambda, \lambda') \mid \lambda' = (\sigma_3 \phi_{i_3}^+ - 3 \phi_{i_1+3}^+) \mid \lambda,
\]

where \(\phi_j^+ = 0\) whenever \(j < 0\).

Firstly we prove the analogous equation for \(\gamma_i\):

\[
(\gamma_{i+3} - \sigma_1 \gamma_{i+2} + 2 \chi_2 \gamma_{i+1} - 3 \chi_1^3)(\lambda, \lambda') \mid \lambda' = (\sigma_3 \gamma_{i+3} - 3 \chi_1^3)(\lambda, \lambda') \mid \lambda,
\]

**Proof.** 1-st case. Summand in the expression for \(\gamma_i\) corresponds to the corner \(\square_{i_2, j_2}\) which appears in the both sides of (15).

- \(i_1 < i_2, \quad j_1 > j_2\).

Let us denote \(a := j_1 - j_2, \quad b := i_2 - i_1, \quad u := t_1^a, \quad v := t_2^b, \quad \chi_2 := t_2^{a-1} t_1^{b-1}\). So \(\chi_3 = uv^{-1}\). Then

\[
(\gamma_{i+3} - \sigma_1 \gamma_{i+2} + 2 \chi_2 \gamma_{i+1} - 3 \chi_1^3)(\lambda, \lambda') = P_3(1-t_1^{-a} t_2^{b-1})(1-t_1^{a-1} t_2^{-b})^{-1}(1-t_1^{a} t_2^{b+1})^{-1}(1-t_1^{a+1} t_2^{-b+1})^{-1}\times
\]

\[
\left(1 - \sigma_1 \left(\frac{\chi_1}{\chi_2}\right)^2 + \sigma_2 \left(\frac{\chi_1}{\chi_2}\right)^3 - 3 \left(\frac{\chi_1}{\chi_2}\right)^3\right) = P_3(u-v)^{-1}(u-t_2 v)(v-t_1 u)^{-1}(v-t_1 t_2 u)(v-t_1 u)(v-t_2 u)\times
\]
\[(v - t_1^{-1}t_2^{-1}u)v^3 = P_1(u - v)^{-1}(u - t_2v)(v - t_1t_2u)(v - t_2u)(v - t_1^{-1}t_2^{-1}u)v^3, \]

\[
(\sigma_3\gamma_{i+3} - \sigma_2\chi_{i+2} + \sigma_1\chi_{i+1} - \chi^2 \gamma_{i+1}) |\lambda = P_3(1 - t_1^{a+1}t_2^{b+1})(1 - t_1^{-a+1}t_2^{b+1})(1 - t_1^2t_2^{b+1})(1 - t_1^3t_2^{b+1}) \times \\
\left( \sigma_3 - \sigma_2 \left( \frac{\chi_1}{\chi_2} \right) + \sigma_1 \left( \frac{\chi_1}{\chi_2} \right)^2 - \left( \frac{\chi_1}{\chi_2} \right)^3 \right) = P_3(u - t_1v)^{-1}(u - t_1t_2v)(v - u)^{-1}(v - t_2u)(u - t_1v)(u - t_2v) \times \\
(u - t_1^{-1}t_2^{-1}v)(-u^3v^3)u^{-3} = P_3(u - t_1t_2v)(u - v)^{-1}(u - t_2u)(u - t_1v)(u - t_2v)(u - t_1^{-1}t_2^{-1}v)v^3.
\]

We received the same expressions.

b) \(i_1 > i_2, j_1 < j_2\). This case is completely analogous to a).

2–d case. Summand in the expression for \(\gamma_i\) corresponds to the hole \(\Box_{i_2,j_2}\) which appears in the both sides of (16). In this case everything is analogous as the expression for the summands in \(\gamma\) corresponding to a corner and a hole differ only by the sign.

3–d case. Let us finally consider the summands occurring only in one side of (16). In this case the summands corresponding to deleting \(\Box_{i_1,j_1}\) in LHS of (16) and to inserting \(\Box_{i_1,j_1}\) in RHS of (16) are equal. All other summands are zero (we use the argument that \(t_1^{-1}, t_2^{-1}\) are roots of polynomial \(1 - \sigma_1t + \sigma_2t^2 - \sigma_3t^3\) again). 

Let us prove equation (16) now.

Proof. If \(i > 0\) then (15) follows directly from (16). So let us consider the remaining cases: \(i = -3, 2, -1, 0\) (in case \(i < -3\) all summands are zero). According to (16) and the relation between \(\gamma_i\) and \(\phi_i^{\pm}\) we have to check only the following equalities:

\[\phi_0^+ |\lambda = \phi_0^+ |\lambda, \phi_0^- |\lambda = \phi_0^- |\lambda, \phi_i^+ |\lambda = (\phi_i^+ + (\sigma_1 - \sigma_2)\chi_1\phi_0^+) |\lambda, \phi_i^- |\lambda = (\phi_i^- + (\sigma_2 - \sigma_1)\chi_1^{-1}\phi_0^-) |\lambda.\]

The first two of them are obvious since \(\phi_0^+, \phi_0^-\) are constant. It follows from Lemma 4.5 that \(\gamma_i |\mu = [e_0, f_1] |\mu = \frac{1}{(1 - t_1)(1 - t_2)} + \sum_{\Box \in \mu} \chi(\Box). \)

So \(\phi_i^+ |\lambda = (1 - t_1)(1 - t_2)(1 - t_1^{-1}t_2^{-1})(\gamma_i^+ |\lambda) + (1 - \sigma_1 - \sigma_2)(1 - t_2^{-1}t_1^{-1})\phi_0^+ |\lambda\), since \((\sigma_1 - \sigma_2) = -(1 - t_1)(1 - t_2)(1 - t_1^{-1}t_2^{-1})\) and \(\phi_0^+ = -1\).

The equation \(\phi_i^+ |\lambda = (\phi_i^- + (\sigma_2 - \sigma_1)\chi_1^{-1}\phi_0^-) |\lambda\) is proved in the same way. 

The proof of equation (13) is entirely similar to the one of (12) and so we omit it.

Finally let us prove \(\Phi^+(z) = \psi^+(z)\). From equation (15) we get

\[\Phi^+(z)(1 - \sigma_1\chi_1z^{-1} + \sigma_2\chi_1^2z^{-2} - \sigma_3\chi_1^3z^{-3}) |\lambda = \Phi^+(z)(\sigma_3 - \sigma_2\chi_1z^{-1} + \sigma_1\chi_1^2z^{-2} - \chi_1^3z^{-3}) |\lambda,\]

Thus:

\[\Phi^+(z) |\lambda = \Phi^+(z) |\lambda \frac{(1 - t_1^{-1}z^{-1})(1 - t_2^{-1}z^{-1})}{(1 - t_1^{-1}ch_1z^{-1})(1 - t_2^{-1}ch_1z^{-1})} \frac{(1 - t_1t_2\chi_1^{-1})}{(1 - t_1\chi_1^{-1})(1 - t_2\chi_1^{-1})}.\]

So by induction \(\Phi^+(z) |\lambda = A \cdot c(z)\), where \(A\)–coefficient of proportionality, which is equal to

\[A = \Phi^+(z) \mid_{\text{empty}} \phi_0^+ - (1 - t_1^{-1}t_2^{-1}) \sum_{i < 0} e^i = -1 - \frac{(1 - t_1^{-1}t_2^{-1})z^{-1}}{1 - z^{-1}} = -1 - \frac{t_1^{-1}t_2^{-1}z^{-1}}{1 - z^{-1}}.\]

So \(\Phi^+(z) = \psi^+(z)\) and analogously one gets \(\Phi^-(z) = \psi^-(z)\).

Theorem 3.5 is proved.
5. The action of shuffle algebra in $M$

In the previous section we have constructed the action of the Ding-Iohara algebra $A$ in $M$. Unfortunately, the parameters $q_1, q_2, q_3$ were not generic (we had $q_1 q_2 q_3 = 1$), so the Theorem 2.1 does not give the representation of $S$ automatically. However, if we write the formulas in the same way we will get the representation of $S$ in $M$.

Namely, we define the action of $S$ in $M$ in the following way. For any $F \in S_n$ we say that for any Young diagrams $\lambda, \lambda' = \lambda + i_1 + \ldots + i_n$ ($i_1 \leq i_2 \leq \ldots \leq i_n$) the matrix element

$$F \mid [\lambda, \lambda'] := \frac{F(\chi_1, \ldots, \chi_n)}{\prod_{1 \leq a < b \leq n} \lambda(\chi_{ia}, \chi_{ib})} \prod_{k=1}^{n} e_{0[\lambda+i_1+\ldots+i_{k-1}, \lambda+i_1+\ldots+i_k]},$$

where $\chi_{ik}$ is the character of the $k$th added box to $\lambda$. All other matrix elements are zero.

Now we prove the following theorem:

**Theorem 5.1.** Formula (17) gives a representation of the shuffle algebra $S$ in $M$.

**Proof.** Firstly, we note the following proposition holds:

**Proposition 5.2.** If $\lambda, \lambda + j_1, \lambda + j_1 + j_2, \ldots, \lambda' = \lambda + j_1 + \ldots + j_n$ are Young diagrams then $F \mid [\lambda, \lambda'] = \frac{F(\chi_1, \ldots, \chi_n)}{\prod_{1 \leq a < b \leq n} \lambda(\chi_{ia}, \chi_{ib})} \prod_{k=1}^{n} e_{0[\lambda+j_1+\ldots+j_{k-1}, \lambda+j_1+\ldots+j_k]}$, where $\chi_{jk}$ is the character of the $k$th added box to $\lambda$ (we are adding boxes in the order: $j_1$, then $j_2$ and so on). So the formula for the matrix elements does not depend on the order of adding the boxes.

**Proof.** As the symmetric group is generated by transpositions it is enough to check the statement only for them. But the case of transpositions follows from relation (1). This completes the proof of proposition.

Now we prove the theorem. Let $F \in S_m, G \in S_n$ and $\lambda, \lambda' = \lambda + j_1 + \ldots + j_{m+n}$ be the Young diagrams. Then by Proposition 5.2

$$(F \circ G) \mid [\lambda, \lambda'] = \text{Sym} \left( \frac{G(\chi_1, \ldots, \chi_n)}{\prod_{1 \leq a < b \leq n} \lambda(\chi_{ia}, \chi_{ib})} \prod_{k=1}^{n} e_{0[\lambda+j_1+\ldots+j_{k-1}, \lambda+j_1+\ldots+j_k]} \times \frac{F(\chi_1, \ldots, \chi_n)}{\prod_{1 \leq a < b \leq n} \lambda(\chi_{ia}, \chi_{ib})} \prod_{k=1}^{n} e_{0[\lambda+j_1+\ldots+j_{k-1}, \lambda+j_1+\ldots+j_k]} \times \frac{G(\chi_1, \ldots, \chi_n)}{\prod_{1 \leq a < b \leq n} \lambda(\chi_{ia}, \chi_{ib})} \prod_{k=1}^{n} e_{0[\lambda+j_1+\ldots+j_{k-1}, \lambda+j_1+\ldots+j_k]} \right).$$

On the other hand:

$$(G * F)(\chi_1, \ldots, \chi_{n+m}) = \text{Sym} \left( G(\chi_1, \ldots, \chi_n) F(\chi_{n+1}, \ldots, \chi_{n+m}) \prod_{1 \leq a < b \leq n+m} \lambda(\chi_{ia}, \chi_{ib}) \right).$$

Thus applying Proposition 5.2 we get:

$$(F \circ G) \mid [\lambda, \lambda'] = (G * F) \mid [\lambda, \lambda'].$$

This completes the proof of theorem. □
Lemma 6.3. is proved in Section VI.6 of [6]:

If ψ(18) 

Corollary 5.3. If i_1 < i_2 < ... < i_n and λ + i_1 + ... + i_n is a Young diagram the matrix element

K_n |[λ_{, λ'} = λ + i_1 + ... + i_n] = \prod_{1 \leq a < b \leq n} \frac{(χ_a - χ_b)(χ_b - t_1 χ_a)}{(χ_a - t_2 χ_b)(χ_a - t_1 t_2^{-1} χ_b)} \prod_{1 \leq r \leq n} c_0[λ + i_1 + ... + i_r - 1, λ + i_1 + ... + i_r],

where χ_a = t_1^{λ_i} t_2^{i-1}. All other matrix elements are zero.

Remark 5.4. So we have constructed the actions of Ding-Iohara and shuffle algebras in M. While the action of Ding-Iohara algebra is purely geometric (it is given by operators e_i, f_i, ψ^±_i), the action of shuffle algebra is algebraic unfortunately. Nevertheless, according to the criteria of Theorem 2.2 elements K_i belong to the subalgebra generated by S_1 and so they are geometrically represented.

6. Macdonald polynomials. Heisenberg algebra and vertex operators over it

6.1. Macdonald polynomials. In this subsection, we review basic facts about Macdonald polynomials. Our basic reference is Macdonald’s book [6].

Recall that algebra Λ_F of symmetric functions over F = ℚ(q, t) is freely generated by the power-sum symmetric functions p_k, where k ∈ ℕ that is

Λ = F[p_1, p_2, ...].

For any diagram λ = (λ_1, ..., λ_k) = (1^{m_1} 2^{m_2} ...) we define

p_λ := p_{λ_1} ... p_{λ_k}, z_λ := \prod_{r \geq 1} r^{m_r} r!

Consider the Macdonald inner product (·, ·)_{q, t}, s.t. (p_λ, p_μ)_{q, t} = δ_{λ, μ} z_λ \prod_{1 \leq i \leq k} \frac{1 - q^{λ_i}}{1 - t^{λ_i}}.

Definition 6.2. Macdonald polynomials P_λ are characterized by two conditions:

a) P_λ = m_λ + lower terms.

b) (P_λ, P_μ)_{q, t} = 0 if λ ≠ μ.

Here by the lower terms we mean m_μ for μ < λ.

Let e_r be the rth elementary symmetric function. The following result called Pieri formula is proved in Section VI.6 of [6]:

Lemma 6.3. P_μ e_r = \sum_λ ψ_{λ/μ} P_λ, where the sum is taken over λ such that λ/μ is a vertical r-strip. Here

ψ_{λ/μ} = \prod (1 - q^{μ_i - μ_j} t^{i-j-1})(1 - q^{λ_i - λ_j} t^{i-j+1}) \prod (1 - q^{μ_i - μ_j} t^{i-j})(1 - q^{λ_i - λ_j} t^{i-j}),

where the product is taken over all pairs (i, j) such that i < j and λ_i = μ_i, λ_j = μ_j + 1.
In particular

\[
\psi_{\mu+j/\mu} = \prod_{i=1}^{j-1} \frac{(1 - q^{\mu_i - \mu_j} t^{i-1}) (1 - q^{\mu_i - \mu_j} t^{-i+1})}{(1 - q^{\mu_i - \mu_j} t^{i-1}) (1 - q^{\mu_i - \mu_j} t^{-i+1})}.
\]

6.4. Fixed points via Macdonald polynomials. Now we prove that the basis \([\lambda]\) of \(M\) can be normalized in such a way that normalized \(K_i \in S_i\) will act as \(e_i\) in the basis of Macdonald polynomials in \(\Lambda_F\). The normalization will be found through comparing the matrix elements of \(K_1\) with the matrix elements of \(e_1\) in the basis of Macdonald polynomials.

We define

\[
c_\lambda := (-1 - t_2 - t_2^{-1})^{-\lambda} \sum_1^{\lambda} t^{\lambda_1 - 1} \prod_{\square \in \Lambda} \left( 1 - t_1(\square) t_2^{-a(\square)-1} \right)^{-1}.
\]

We define the normalized vectors \((\lambda) := c_\lambda \cdot [\lambda]\). Firstly we check the following lemma

**Lemma 6.5.** Let \(\lambda = (\lambda_1, \ldots, \lambda_k)\). Then \((1 - t_1)(1 - t_2)K_1(\lambda, \lambda + j) = \psi_{\lambda + j/\lambda} \mod{q = t_1, t_2 = t_2^{-1}}\).

*Remark:* This lemma means that \((1 - t_1)(1 - t_2)K_1\) acts in a normalized basis like \(e_1\) in the basis of Macdonald polynomials. Moreover, this condition defines a normalization uniquely to a mutual factor

**Proof.**

\[
\frac{c_{\lambda+j}}{c_\lambda} = \frac{-t_1 t_2^{-1} (1 - t_2)}{1 - t_1 \sum_{\square \in \Sigma_1(\square, \lambda+j+1)} t^{\lambda_1 - 1} \prod_{\square \in \Sigma_2(\square, \lambda+j+1)} \left( 1 - t_1(\square) t_2^{-a(\square)-1} \right)^{-1}}.
\]

Now we compute the products above:

\[
\prod_{\square \in \Sigma_2(\square, \lambda+j)} \frac{1 - t_1(\square) t_2^{-a(\square)-1}}{1 - t_1(\square) t_2^{-a(\square)-2}} = \prod_{i<j} \frac{1 - t_1(\square) t_2^{-a(\square)-2}}{1 - t_1(\square) t_2^{-a(\square)-1}} t_2^{-j-1},
\]

\[
\prod_{\square \in \Sigma_1(\square, \lambda_j+i+1)} t_2 \frac{1 - t_1(\square) t_2^{-a(\square)-1}}{1 - t_1(\square) t_2^{-a(\square)-1}} = (1 - t_2^{-1}) \prod_{i>j} t_1^{\lambda_j - \lambda_i} t_2^{-i-1} t_2^{-j-1} = (1 - t_2^{-1}) \prod_{i>j} t_1^{\lambda_j - \lambda_i} t_2^{-i-1} t_2^{-j-1} = -(1 - t_2^{-1}) \prod_{i>j} \frac{1 - t_1^{\lambda_i - \lambda_j} t_2^{i+j+1}}{1 - t_1^{\lambda_i - \lambda_j} t_2^{i+j}} t_2^{-j} t_1^{-\lambda_j}.
\]

Thus:

\[
\frac{c_{\lambda+j}}{c_\lambda} = (1 - t_2) \prod_{i>j} \frac{1 - t_1^{\lambda_i - \lambda_j} t_2^{i+j+1}}{1 - t_1^{\lambda_i - \lambda_j} t_2^{i+j}} \prod_{i<j} \frac{1 - t_1^{\lambda_i - \lambda_j} t_2^{i+j-1}}{1 - t_1^{\lambda_i - \lambda_j} t_2^{i+j-1}}.
\]

On the other hand it follows from Proposition 3.7 that

\[
e_{0[\lambda, \lambda+j]} = (1 - t_1)^{-1} \prod_{1 \leq i \neq j} \frac{1 - t_1^{\lambda_i - \lambda_j} t_2^{i+j+1}}{1 - t_1^{\lambda_i - \lambda_j} t_2^{i+j}}.
\]
After specializing of formula (19) for parameters \( q := t_1 \), \( t := t_2^{-1} \) we have

\[
\psi_{\lambda+j/\lambda} = \prod_{i=1}^{j-1} \frac{(1 - t_1^{-\lambda_i-\lambda_{j-i}+1})(1 - t_2^{-\lambda_{j-i}+1})}{(1 - t_1^{-\lambda_i})(1 - t_2^{-\lambda_{j-i}})}.
\]

Now it is straightforward to check that \( \psi_{\lambda+j/\lambda} = (1 - t_1)(1 - t_2)e_0[\lambda, \lambda+j] \cdot \frac{e_\lambda}{e_{\lambda+j}} \). This completes the proof of Lemma.

We denote \( d_n := \frac{(-t_1)^{n-1}}{(1-t_1)(1-t_2)}. \)

**Theorem 6.6.** For any Young diagrams \( \mu \subset \lambda \) such that \( \lambda/\mu \) is a vertical \( n \)-strip with the boxes located in the rows \( j_1 < \ldots < j_n \) we have: \( \frac{1}{d_1 \ldots d_n} K_{n(\mu, \lambda)} = \psi_{\lambda/\mu} \big|_{q:=t_1, t:=t_2^{-1}}. \)

**Proof.** According to Corollary [3],

\[
K_{n(\mu, \lambda)} = \prod_{1 \leq a < b \leq n} \frac{(\chi_a - \chi_b)(\chi_b - t_1\chi_a)}{(\chi_a - t_2\chi_b)(\chi_a - t_1^{-1}t_2^{-1}\chi_b)} \prod_{1 \leq r \leq n} e_0(\lambda - j_r - \ldots - j_n, \lambda - j_{r+1} - \ldots - j_n),
\]

where \( \chi_a = t_1^{\lambda_a-1}t_2^{\lambda_a-1} \). In the normalized basis we have the same formula:

\[
K_{n(\mu, \lambda)} = \prod_{1 \leq a < b \leq n} \frac{(\chi_a - \chi_b)(\chi_b - t_1\chi_a)}{(\chi_a - t_2\chi_b)(\chi_a - t_1^{-1}t_2^{-1}\chi_b)} \prod_{1 \leq r \leq n} e_0(\lambda - j_r - \ldots - j_n, \lambda - j_{r+1} - \ldots - j_n).
\]

After specializing \( q := t_1 \), \( t := t_2^{-1} \) the coefficients in the formula (18) looks as following:

\[
(21) \quad \psi_{\lambda/\mu} = \prod_{j_1, \ldots, j_{n-1}, \neq i < j_n} (1 - t_1^{-\mu_i - \mu_{j_n} + i - j_n + 1})(1 - t_2^{-\lambda_i - \lambda_{j_n} + i - j_n + 1}) (1 - t_1^{-\lambda_i - \lambda_{j_n} + i - j_n})(1 - t_2^{-\mu_i - \mu_{j_n} + i - j_n - 1}) = \prod_{a < b} (1 - t_1^{\lambda_a - \lambda_b + 1} t_2^{j_a - j_b})(1 - t_1 t_2^{j_a - j_b - 1}) \times
\]

\[
\left( \prod_{j_1, \ldots, j_{n-1}, \neq i < j_n} (1 - t_1^{-\mu_i - \mu_{j_n} + i - j_n + 1})(1 - t_2^{-\lambda_i - \lambda_{j_n} + i - j_n + 1}) (1 - t_1^{-\lambda_i - \lambda_{j_n} + i - j_n})(1 - t_2^{-\mu_i - \mu_{j_n} + i - j_n - 1}) = \prod_{a < b} (1 - t_1^{\lambda_a - \lambda_b + 1} t_2^{j_a - j_b})(1 - t_1 t_2^{j_a - j_b - 1}) \times \frac{e_0(\lambda - j_{n-1} - \ldots - j_n, \lambda - j_{n+1} - \ldots - j_n)}{d_1(d_2)}, \right.
\]

The last equality follows from Lemma [5].

Finally:

\[
\prod_{a < b} (1 - t_1^{\lambda_a - \lambda_b + 1} t_2^{j_a - j_b})(1 - t_1^{\lambda_a - \lambda_b} t_2^{j_a - j_b - 1}) = \prod_{a < b} \frac{(\chi_b - t_1\chi_a)(\chi_b - t_1\chi_a)}{(\chi_a - t_2\chi_b)(\chi_a - t_1 t_2^{-1}\chi_b)} (-t_1)^{1-b}
\]

This completes the proof of Theorem. \( \square \)
6.7. **Heisenberg action on M through vertex operators.** In the previous section we constructed an isomorphism $\Theta : M \to \Lambda_F$ which takes $\langle \lambda \rangle$ to Macdonald polynomial $P_\lambda$ and which sends operators $\tilde{K}_i := \frac{1}{d_1 \ldots d_i} K_i$ to operators of multiplication by $e_i$. 

On the other hand there is a well known identity of generalized functions:

$$1 + \sum_{i>0} e_i z^i = \exp \left( \sum_{i>0} \frac{(-1)^i-1}{i} p_i z^i \right).$$

Hence the operators $\tilde{K}_i$ acting on $M$, which may be viewed as a Fock space over $p_i$, are vertex operators over half of the Heisenberg algebra: $\{h_i\}_{i>0}$. The isomorphism $\Theta$ takes $h_i$ to operators of multiplication by $p_i$ (for $i > 0$). As a result an action of the positive part of the Heisenberg algebra is received. Obviously starting from $f_i$ instead of $e_i$ we will get in the analogous way the vertex operators over the negative half of the Heisenberg algebra. So the whole Heisenberg algebra is acting in $M$.

**Remark 6.8.** The disadvantage of our approach is that we do not know explicit formulas for $K_i$ in terms of $x_1^j \ast x_2^j \ast \ldots \ast x_i^j$.

6.9. In paper [5] the authors studied the action of the Heisenberg algebra in $R := \oplus_n H^n_{\mathbb{P}^1}(X \otimes H_2(pt)) \otimes H_T(pt)$ they proved that under certain normalization of the fixed points basis there is an isomorphism $\Delta : M \to \Lambda_F$, which sends the basis of fixed points to Jack polynomials and $\{h_i\}_{i>0}$ are sent to operators of multiplication by $p_i$. It is also known (see [6]) that Jack polynomials $J^{(\alpha)}_\lambda$ can be received from the Macdonald polynomials $P^{(q,t)}_\lambda$ by specializing $q := t^\alpha$, $t \to 1$. So by the above mentioned specialization of our normalization (20) we get the same formulas (they differ only in some scalars) for normalization as those of [5] (see formulas (2.12) and (2.14) of loc.cit. and mind you that $l(\Box)$, $a(\Box)$ are interchanged with our notations). So as the formulas in the fixed points basis in $H^\bullet$ are additive analogues of the formulas for $K^\bullet$, our approach gives the same action of the Heisenberg algebra in $R$ as the approach using higher correspondences $P[i]_{i \in \mathbb{Z}}$.

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Address:
B.F.: Landau Institute for Theoretical Physics, Kosygina st 2, Moscow 117940, Russia
A.T.: IMU, MSU; Independent Moscow University, Bolshoj Vlasievskij Pereulok, Dom 11, Moscow 121002 Russia
E-mail address: bfeigin@gmail.com, sasha_ts@mail.ru