The Sorting Buffer Problem is NP-hard

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September 23, 2010

Abstract

We consider the offline sorting buffer problem. The input is a sequence of items of different types. All items must be processed one by one by a server. The server is equipped with a random-access buffer of limited capacity which can be used to rearrange items. The problem is to design a scheduling strategy that decides upon the order in which items from the buffer are sent to the server. Each type change incurs unit cost, and thus, the cost minimizing objective is to minimize the total number of type changes for serving the entire sequence. This problem is motivated by various applications in manufacturing processes and computer science, and it has attracted significant attention in the last few years. The main focus has been on online competitive algorithms. Surprisingly little is known on the basic offline problem.

In this paper, we show that the sorting buffer problem with uniform cost is NP-hard and, thus, close one of the most fundamental questions for the offline problem. On the positive side, we give an $O(1)$-approximation algorithm when the scheduler is given a buffer only slightly larger than double the original size. We also give a dynamic programming algorithm for the special case of buffer size two that solves the problem exactly in linear time, improving on the standard DP which runs in cubic time.

1 Introduction

The sorting buffer problem results from the following scenario. The input is a sequence $\sigma$ of $n$ items of different types. W.l.o.g., we represent different types by different colors, i.e., each item $i$ is associated with a color $c(i)$. The total number of distinct colors in the sequence is denoted by $C$. All items must be processed by a server. The server is equipped with a random-access buffer of limited capacity which can be used to rearrange the items. The items are moved one after another into the buffer that can hold at most $k$ items. At any step, a scheduling algorithm chooses a color, say $c$, and then all items in the buffer associated with color $c$ are removed from the buffer and processed by the server. This creates space in the buffer, and the next items in the sequence will be moved to the buffer. If some of these new items have color $c$, they will be removed and processed immediately and it continues until no item in the buffer is associated with color $c$. The scheduling algorithm then chooses a new color and repeats, until all items in the sequence are removed for processing. The goal is to

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design a scheduling algorithm that minimizes the total number of color changes. The buffer has no color initially.

While the sorting buffer problem looks simple, it models a number of important problems in manufacturing processes, hardware design, computer graphics, file servers and information retrieval. For example, consider the sequencing problem in an automotive paint shop [9], where cars are painted in different colors. The cars traverse this production stage consecutively, and whenever a color change is necessary, this causes setup and cleaning cost. The goal is to minimize the total cost for changing colors. For an extended discussion on various applications and more references, we refer the readers to, e.g., [3, 6, 12].

1.1 Previous work

The sorting buffer problem (also known as buffer reordering problem) has attracted significant attention since it was first proposed by Räcke, Sohler, and Westermann [12]. The original focus was on competitive analysis of online algorithms. Räcke et al. [12] proposed an $O(\log^2 k)$-competitive algorithm and showed that some simple heuristics like First In First Out (FIFO) and Least Recently Used (LRU) are $\Omega(\sqrt{k})$-competitive, where $k > 0$ is the buffer size. Englert and Westermann [7] improved these results and gave an $O(\log k)$-competitive algorithm for a more general non-uniform cost function, where the cost of a color change depends on the final color. To obtain this result, they first relate their algorithm’s solution to an optimal offline solution using a buffer of size $k/4$. Then, they prove that an offline optimum with buffer size $k/4$ is $O(\log k)$-competitive against an offline optimum with buffer size $k$. The first result translates into a constant competitiveness result using resource augmentation, i.e., their algorithm is 4-competitive when given a buffer with size 4 times the original size. The currently best known result was derived very recently by Avigdor-Elgrabli and Rabani [3]; they gave an $O(\log k/\log\log k)$-competitive deterministic online algorithm for the sorting buffer problem with non-uniform costs.

Considerable work has been done for the problem when the cost function is a metric and the cost of a color change depends on both the original and final colors. We do not review the results here and refer the readers to the summary in [3].

In order to develop good online methods, one of the most natural steps is to investigate the offline sorting buffer problem and identify its structural properties. Even if the offline problem is less relevant in practice, its analysis should be easier and give new insight to the problem. However, only little is known on the offline problem. It is easy to see that there are dynamic programming algorithms that solve the problem optimally in $O(n^{k+1})$ or $O(n^k)$ time; see also [10][11]. Aiming at polynomial time algorithms, the above mentioned online algorithms already provide the best known offline approximation guarantees (which are non-constant). A constant approximate algorithm for the offline case on the line metric has been derived by Khandekar and Pandit [10]; however, it runs in quasi-polynomial time.

There has been research on the complementary variant of the sorting buffer problem, where the objective is to maximize the number of avoided color changes in the input stream. This problem is more successful in terms of approximation algorithms. Kohrt and Pruhs [11] gave a polynomial time 20-approximate algorithm. This was later improved by Bar-Yehuda and Laserson [4] who gave a 9-approximation algorithm for non-uniform cost. Note that the maximization and minimization problems have the same optimal solution, but they may be very different in terms of approximation.
1.2 Our results

We give a concise NP-hardness proof for the sorting buffer problem by a reduction from 3-Partition \cite{8}, and hence close one of the fundamental open questions on this problem \cite{3}. Clearly, this implies that both variants, the minimization and the maximization problem, are NP-hard. Independently, an NP-hardness proof was given by Asahiro et al. \cite{1}. However, their proof is much longer than ours and turned out to be incorrect. Recently, they gave a new, though still very long, proof \cite{2}.

We also note, that increasing the number of servers does not make the problem easier. The idea of modeling more servers leads to an intuitive generalization of (or joint model for) the sorting buffer problem and the somewhat related well-known paging problem. In the latter problem, there is given a cache of \( m \) colors while a request from an online request sequence must be served immediately without intermediate buffering. We could interpret the cache as \( m \) servers that may immediately serve a current request. This leads to a generalized sorting buffer problem in which we have a buffer of size \( k \) and \( m \) servers. In this general formulation, the sorting buffer problem corresponds to the special case with \( m = 1 \), while the paging problem has \( k = 1 \). Yet, the earlier problem is NP-hard, as we show in this paper, while the later problem is polynomial-time solvable \cite{5}.

Naturally, we also consider an immediate adaption of the optimal paging algorithm \cite{5} Longest Forward Distance (LFD) as a candidate for the sorting buffer problem. However, we show that it is \( \Omega(k^{1/3}) \)-approximate, hence a different strategy is needed to derive constant approximate algorithms for sorting buffers. This negative result is in line with similar observations for several other natural (online) strategies in Räcke et al. \cite{12}.

On the positive side, we give an \( O(n \log C) \)-time optimal algorithm for the special case in which the size of the buffer is \( k = 2 \). The algorithm uses a somewhat special dynamic programming approach with a non-trivial combination of data structures that guarantee the linear running time in the input size. Note that it is straightforward to obtain a dynamic programming algorithm that runs in \( O(n^{k+1}) \) time; our algorithm improves this significantly.

Finally, we consider the setting with resource augmentation, where the algorithm is given a larger buffer than the optimal one. We give a new LP formulation for the sorting buffer problem and show that it can be rounded using a larger buffer size. This gives an \( O(1/\epsilon) \)-approximate algorithm using a buffer of size \((2 + \epsilon)\) times that of optimal.

**Organization.** In Section \cite{2} we show that the sorting buffer problem and its generalization are NP-hard. In Section \cite{5} we provide the LP and the constant factor approximation algorithm using a larger buffer size, and finally we give in Section \cite{4} the dynamic programming algorithm for \( k = 2 \). We present the lower bound on the approximation ratio of LFD in Section \cite{5}. We conclude with some open problems in Section \cite{6}.

2 Complexity

**Theorem 1** The sorting buffer problem is strongly NP-hard.

**Proof** We reduce from 3-Partition which is known to be strongly NP-hard \cite{8}: Given \( 3q \) positive integers \( a_1, a_2, \ldots, a_{3q} \) and an integer \( A \) such that \( a_1 + a_2 + \ldots + a_{3q} = qA \) can we partition \( \{1, 2, \ldots, 3q\} \) into \( q \) sets \( I_i \) such that \( \sum_{j \in I_i} a_j = A \) for all \( i \in \{1, 2, \ldots, q\} \)?

Given an instance of 3-Partition, we construct an instance \( \sigma \) for the sorting buffer problem as follows. We multiply all numbers by a large number \( L \). Let \( b_j = La_j \) for all \( j \)
and $B = LA$. We define the buffer size as $V = qB + \varepsilon$, with $q^2A \leq \varepsilon \leq L/2$. We see the buffer as having a main part of capacity $qB$ and an extra part of capacity $\varepsilon$. For each $j \in \{1, 2, \ldots, 3q\}$ we define a color $j$. We call these the primary colors. The sequence contains many other colors but we do not label those explicitly. We call these the secondary colors. The input sequence is defined by $3q + 4$ subsequences:

$$\sigma = \beta \gamma_1 \delta_1 \alpha_1 \gamma_2 \delta_2 \alpha_2 \ldots \gamma_q \delta_q \alpha_q \gamma_{q+1} \delta_{q+1} \alpha_{q+1}.$$ 

The subsequences are defined as follows.

- $\beta$ contains $b_j$ items of color $j$ for each $j \in \{1, 2, \ldots, 3q\}$. Items are given in arbitrary order. Note that the total number of items equals $qB$ which is the size of the main part of the buffer.

- $\alpha_i$ $(i=1\ldots q+1)$ contains $a_j$ items of color $j$ for each $j \in \{1, 2, \ldots, 3q\}$. Again, the order is arbitrary.

- $\gamma_i$ $(i=1\ldots q+1)$ We distinguish between $i \leq q$ and $i = q + 1$. For $i \leq q$ it starts with $iB$ items of different colors followed by again one item of each of these colors. Any color used in $\gamma_i$ is unique in the sense that it appears twice in $\gamma_i$ and nowhere else in the sequence $\sigma$. Sequence $\gamma_{q+1}$ is defined exactly the same but the number of colors is now two times $V - M$, where $M = \frac{1}{2}q(q+1)A$.

- $\delta_i$ $(i=1\ldots q+1)$ contains $V$ items of the same color. This color is not used anywhere else in $\sigma$.

Let us, just for clarity, count the number of colors in $\sigma$. The subsequences $\beta$ and $\alpha_i$ contain the $3q$ primary colors. The sequences $\delta_i$ together contain $q + 1$ colors. A sequence $\gamma_i$ contains $iB$ colors for $i \leq q$ and $V - M$ colors for $i = q + 1$ (each color twice). The total number of colors in the sequence $\sigma$ is $C = 3q + q + 1 + (\sum_{i=1}^d iB) + V - M$.

We list some properties that any optimal solution $OPT$ has.

**Lemma 2** Before the first item of $\alpha_i$ enters the buffer, $OPT$ must have used the color of $\delta_i$.

**Proof** The length of $\delta_i$ is equal to the buffer size. 

We remark that the reduction would be valid without the subsequences $\delta_1, \ldots, \delta_q$. However, these subsequences gives us separations of the server sequence which enhance the analysis. Let us call the moment that the server switches to color $\delta_i$ simply as time $\delta_i$.

**Lemma 3** We may assume that $OPT$ serves $\gamma_i$ completely before time $\delta_i$.

**Proof** Since the items of $\gamma_i$ that remain in the buffer when $OPT$ switches to color $\delta_i$ cannot be combined with items arriving later, we may as well serve these remaining items before switching to $\delta_i$.

It follows from the preceding two lemmas that we may assume that

**Lemma 4** $OPT$ serves the sequences $\gamma_i$ and $\delta_i$ in the order $\gamma_1 \delta_1 \gamma_2 \delta_2 \ldots \gamma_{q+1} \delta_{q+1}$.

Assume that $\sigma$ can be served with at most $C + 3q$ color switches. We shall prove that a 3-Partition exists. For $1 \leq i \leq q$, let $H_i$ be the set of primary colors used before time $\delta_i$. 

4
Lemma 5
\[
\sum_{j \in H_i} a_j \geq iA \text{ for all } i \in \{1, 2, \ldots, q\}.
\] (1)

Proof Assume that \(\sum_{j \in H_i} a_j \leq iA - 1\) for some \(i\). Then, \(\sum_{j \in H_i} b_j \leq L(iA - 1) = iB - L\). This means that from the \(qB\) items of \(\beta\) at most \(iB - L\) items are removed before time \(\delta\). Hence, the free space we have to serve \(\gamma\) is certainly no more than \(iB - L + \varepsilon \leq iB - L/2\). But then at least \(L/2\) colors of \(\gamma\) must be used more than once. The total number of color switches will be at least \(C + L/2 > C + 3q\) (for \(q \geq 2\)). \(\square\)

Lemma 6 Every primary color is used exactly twice: once before time \(\delta_q\) and once after time \(\delta_{q+1}\). Every secondary color is used exactly once.

Proof Taking \(i = q\) in Lemma 5 we see that all 3q primary colors must be used before time \(\delta_q\). Further, each primary color must also be chosen at least once after switching to \(\delta_{q+1}\) since \(\alpha_{q+1}\) contains all primary colors and is served after the switch to \(\delta_{q+1}\). We see that the bound of \(C + 3q\) can only be reached if the statement in the lemma holds. \(\square\)

Let \(I_1 = H_1\) and \(I_i = H_i \setminus H_{i-1}\) for \(2 \leq i \leq q\), i.e., the set of primary colors used between \(\delta_{i-1}\) and \(\delta_i\). Consider the buffer contents at time \(\delta_{q+1}\). If \(j \in I_2\) then the buffer contains at least \((q + 1 - i)a_j\) items of color \(j\). The total number of primary colored items in the buffer is at least
\[
\sum_{i=1}^{q} \sum_{j \in I_i} (q - i + 1)a_j = \sum_{i=1}^{q} \sum_{j \in H_i} a_j \geq q \sum_{i=1}^{q} iA = \frac{1}{2}q(q + 1)A = M. \tag{2}
\]

If (2) holds with strict inequality, then at least one color of \(\gamma_{q+1}\) is used twice which contradicts Lemma 6. Hence, equality holds and this can only be true if equality in (1) holds for all \(i\). This implies that \(\sum_{j \in H_i} a_j = A\) for all \(i\). Hence, a 3-Partition exists.

The other direction is easily verified. Assume that a 3-Partition \(I_1, \ldots, I_q\) exist. Then we have equality in (1). The server takes the colors in \(I_i\) just before \(\gamma_i\). Note that all items from the \(\alpha_i's\) fit together in the extra part of the buffer. Hence, the free space in the buffer just before \(\gamma_i\) is at least \(IB\) for all \(i \leq q\) and \(\gamma_i\) can be served such that each color is used only once. All primary colors are used exactly twice. \(\square\)

The NP-hardness of buffer sorting extends to the generalized sorting buffer problem with multiple servers \(m\), even if \(m\) is constant.

Theorem 7 The generalized sorting buffer problem is NP-hard for any number of servers \(m \geq 1\).

Proof The idea is to show that the problem with \(m\) servers and buffer size \(V\) can be reduced to the problem with \(m + 1\) servers and the same buffer size. The theorem then follows by induction with the induction basis (case \(m = 1\) is NP-hard) given in Theorem 1.

Assume \(m = \ell\) is NP-hard for some integer \(\ell \geq 1\). Consider an arbitrary sequence \(\rho\) for the case \(m = \ell\). We take a color \(x\) not used in \(\rho\) and add \(k\) items of color \(x\) between any two consecutive items in \(\rho\). Let the resulting sequence be \(\rho'\). We claim that it is optimal to \(\rho'\) to let one server serve only color \(x\) and the other ones the remaining colors. Suppose this were true, then \(\rho\) can be served using \(\ell\) servers with minimum cost \(z\) if and only if \(\rho'\) can be served using \(\ell + 1\) servers with minimum cost \(z + 1\). Hence, the case for \(m = \ell + 1\) is also NP-hard.
It is left to prove the claim. For the sake of contradiction, suppose there is an optimal solution $\text{OPT}$ to sequence $\rho'$ which does not serve all items of color $x$ by the same server. Let $m_1$ be the server which serves the first item of color $x$ in $\rho'$. Consider the first moment in which an item preempts the sequence of consecutively serving color $x$ by $m_1$, i.e., an item $i$ of color $c(i) \neq x$ is assigned to $m_1$. Let $S$ be the set of items that are in the buffer at that moment. We can assume that the next item $j$ that enters the buffer is the first of $k$ consecutively incoming color-$x$ items. (Whenever we remove one color-$x$ item from the buffer, then we can serve all of them without extra cost.) Hence, with the buffer capacity $k$, $\text{OPT}$ must serve at least one (and thus w.l.o.g. all) items of color $x$ before an new item with color different from $x$ can enter the buffer.

Consider the schedule after $\text{OPT}$ served the color $x$ items by some server, say $m_2$. Suppose $m_1 \neq m_2$. While the current color of $m_2$ is $x$, server $m_1$ might have served after $i$ some items of the same or other colors from $S$; let $i'$ be the last item assigned to $m_1$ so far. Now, we simply exchange the current output sequence on server $m_1$ from item $i$ up to $i'$, with the sequence of color-$x$ items on $m_2$. This is feasible since we only swap output positions of items in $S$ that are in the buffer or enter with the same color $x$. Note, that the currently active colors of the servers are not changed. Moreover, the cost of the schedule can only decrease: Moving the color-$x$ items to $m_1$ reduces the cost by one and moving the sequence starting with item $i$ to $m_2$ does not cause a new color change. Thus, $\text{OPT}$ was not an optimal solution.

If $m_1 = m_2$, then we extract from the output sequence on $m_1$ the subsequence $i$ up to $i'$, and assign it to the end of the current sequence of some server, say $m_2$. Clearly, the current color of $m_2$ changes and may cause an additional unit of cost when $\text{OPT}$ assigns the next item to $m_2$. However, we reduce the cost by one unit when removing the color change on $m_1$ for switching back to color $x$. Thus, the cost do not increase. This exchange can be applied iteratively to an optimal solution until no items of a color different from $x$ is assigned to $m_1$.

\[\square\]

3 Resource Augmentation

In this section, we give an LP-based algorithm which yields an $O(1/\epsilon)$-approximation with respect to the optimal solution that uses no more than $1/2 - 2\epsilon$ times the original buffer size. By scaling up the buffer size by a factor of $2 + O(\epsilon)$, it gives an $O(1/\epsilon)$-approximate algorithm using a buffer size of $2 + \epsilon$ times that of optimal.

We first introduce a new LP relaxation, followed by a rounding scheme. We consider that the buffer is empty initially. For each time step $i = 1, 2, \ldots, n$, the following three events occur. (1) The $i$-th item is moved to the buffer, (2) the algorithm chooses $c(i)$ to be the color of the buffer, and (3) all items in the buffer with color $c(i)$ are removed. Call an interval a $c$-interval if the color of the buffer is $c$ throughout the interval and call it non-$c$ if the color is not $c$ throughout the interval. The cost for serving a color $c$ is the number of maximal $c$-intervals. Note that the cost over all colors is exactly $2 - C$ plus the number of maximal non-$c$ intervals. One observation is that after each time step $i = 1, 2, \ldots, n$, the number of items in the buffer should be at most $k - 1$. It motivates the following IP.

We define a variable $y_{c,s,t}$ for every color $c$ and time steps $s, t$ with $1 \leq s \leq t \leq n$. $y_{c,s,t}$ should be one if $[s, t]$ is a maximal non-$c$ interval; and it is zero otherwise. For each color $c$ and time
step \( s \leq i \), let \( A_{s,t}^c \) be the number of items with color \( c \) moved into the buffer during \([s,i]\).

\[
\text{minimize} \quad 2 - C + \sum_{c} \sum_{s,t: \ s \leq t} y_{s,t}^c
\]

subject to

\[
\sum_{s,t: \ s \leq t} y_{s,t}^c \leq 1 \quad \text{for all } c \text{ and } i = 1, 2, \ldots, n + k - 1 \quad (3)
\]

\[
\sum_{c} \sum_{s,t: \ s \leq t} y_{s,t}^c = C - 1 \quad \text{for all } i = 1, 2, \ldots, n + k - 1 \quad (4)
\]

\[
\sum_{c} \sum_{s,t: \ s \leq t} A_{s,t}^c y_{s,t}^c \leq k - 1 \quad \text{for all } i = 1, 2, \ldots, n - 1 \quad (5)
\]

\[
\sum_{c} A_{s,t}^c y_{s,t}^c = 0 \quad \text{for all } c \text{ and } i = n + k - 1 \quad (6)
\]

\[
y_{s,t}^c \in \{0, 1\} \quad \text{for all } c, s, t \in \{1, 2, \ldots, n + k - 1\}. \quad (7)
\]

The first constraint (3) ensures two things: (i) for any color \( c \) and time \( i, i \) is included in at most one maximal non-\( c \) interval and (ii) maximal non-\( c \) intervals are really maximal, i.e. if \( y_{s,t}^c = y_{u,v}^c = 1 \) then \( t \leq u + 2 \) or \( v \leq s + 2 \). By (i), each color \( c \) contributes at most 1 to the left hand side of the second constraint (4). Hence this constraint ensures that at any time \( i \), the color of the buffer is different from exactly \( C - 1 \) colors. Constraint (5) ensures that by the end of each time step \( i \leq n - 1 \), the number of items remaining in the buffer is at most \( k - 1 \) and constraint (6) ensures that the buffer is empty at the end. It is easy to verify that for any valid schedule we can set the values of \( y_{s,t}^c \) according to whether it is a maximal non-\( c \) interval and this satisfies all the constraints. Reversely, any IP-solution corresponds with a feasible coloring sequence with the same cost. The LP-relaxation is obtained by replacing (7) with \( y_{s,t}^c \geq 0 \). It is easy to verify that any LP-solution has value at least \( C \). We can round the LP to get an \( O(1/\epsilon) \)-approximation against an optimal solution that uses no more than \( 1/2 - 2\epsilon \) times the buffer size. Define

\[
x_i^c = \sum_{s,t: \ s \leq t} y_{s,t}^c.
\]

Intuitively, \( 1 - x_i^c \) is the fraction of color \( c \) on the machine at step \( i \). Further, define

\[
z_i^c = \sum_{s,t: \ 1 \leq s \leq t} y_{s,t}^c, \quad \text{and} \quad Z_i^c = \sum_{j=1}^{i} z_j^c.
\]

The variable \( z_i^c \) sums over all intervals ending in \( i \) and the variable \( Z_i^c \) sums over all intervals ending in \( i \) or before that. In particular, \( Z_n^c \) is the LP-cost for color \( c \). The value \( Z_i^c \) is non-decreasing in \( i \). We mark every step that \( Z_i^c \) increases by another \( \epsilon \). More precisely, mark the first step \( i \) for which \( Z_i^c \geq \epsilon \) and mark every next step \( i' \) for which \( Z_i^c \) has increased by at least \( \epsilon \) since the last marking.

A feasible integral solution is found by the following rounding scheme.
LP Rounding. Start with an arbitrary buffer color. For $i = 1$ to $n + k - 1$ do:

(i) Remove all items with the current color (state) of the buffer.

(ii) For each marked color $c$, remove all its items.

(iii) If $x_j^i \leq 1/2 - \varepsilon$ for some $c'$, then switch the color to $c'$ and remove all items with color $c'$.

Theorem 8 The LP Rounding Algorithm applied to an optimal LP solution yields an $O(1/\varepsilon)$-approximate solution for the sorting buffer problem when the optimum is using a buffer of size at most $1/2 - 2\varepsilon$ times the original buffer size $k$.

Proof First we argue that (iii) is well defined. Constraint 4 states that $\sum_c x_j^i \geq C - 1$ and 3 states $x_j^i \leq 1$. Hence, there is at most one $c'$ for which $x_j^i \leq 1/2 - \varepsilon$. (\*)

The first step (i) is done for free, and one can easily verify that only the just entered item is possibly removed in this step. Clearly, the number of markings is $O(1/\varepsilon)$ times the LP cost. Consider two consecutive switches. If at least one of the two is due to a marking then we charge both to the marking. To prove that the total number of switches is $O(1/\varepsilon)$ times the LP cost we only need to bound the number of pairs of consecutive switches in which both are of type (iii). Assume the buffer switches to $c'$ in step $i$ and subsequently switches to another color $c''$ in step $j > i$ and both are of type (iii). We have $x_j^i \leq 1/2 - \varepsilon$ and $x_j'' \leq 1/2 - \varepsilon$. The first implies that $x_j'' \geq 1/2 + \varepsilon$; see (\*). Hence, $x_j'' - x_j^i \leq -2\varepsilon$.

Notice that for every $j > i$ and $c$ holds that

$$x_j^i - x_j^c = \sum_{s \leq i} y^c_{s,j} - \sum_{s \leq i} y^c_{s,j} = \sum_{s \leq i} y^c_{s,j} - \sum_{i+1 \leq s \leq j} y^c_{s,j} \geq 0 - \sum_{s \leq i \leq j-1} y^c_{s,j}$$

$$= -(Z_{j-1}^c - Z_{i-1}^c).$$

Therefore, $2\varepsilon \leq x_j'' - x_j^i \leq Z_{j-1}^{c''} - Z_{i-1}^{c''}$. Thus, for color $c''$ there is an increase of the $Z$-variable of $2\varepsilon$ between two switches of the third type. We conclude that the total cost due to switches of the third type is also $O(1/\varepsilon)$ times the LP cost.

Now we bound the capacity needed. Consider any $c$ and step $j$ and let $i < j$ be the last time before $j$ that $c$ was removed from the buffer in the rounded solution. We may assume that $c$ was not removed at step $j$ since otherwise there are no items of color $c$ at the end of step $j$. Denote the term for color $c$ in constraint 5 by $a_j^c$.

$$a_j^c = \sum_{s \leq j \leq t} A_{s,j}^c y^c_{s,t}.$$

Intuitively, $a_j^c$ is the amount of color $c$ in the buffer at step $j$ in the LP-solution. On the other hand, the number of items of color $c$ in the buffer at step $j$ in the rounded solution is $A_{i+1,j}^c$.

To relate the rounded solution to the LP-solution we are interested in the variables that correspond to $(s,t)$-intervals with $s \leq i + 1 \leq j \leq t$. For these $(s,t)$-intervals we have $A_{s,t}^c \geq A_{i+1,j}^c$. 


Since we have not picked color \( c \) in steps \( i + 1, \ldots, j \), we have \( Z_j^c - Z_i^c < \varepsilon \). Note further that \( \sum_{s : t, s \leq i + 1 \leq j \leq t} y_{s,t}^c > 1/2 - \varepsilon - Z_{j-1}^c + Z_i^c \). Since \( c \) is not removed at step \( i + 1 \) we have \( x_{i+1}^c > 1/2 - \varepsilon \). Using additionally \( Z_j^c \leq Z_j \) we conclude that

\[
\sum_{s : t, s \leq i + 1 \leq j \leq t} y_{s,t}^c > \frac{1}{2} - \varepsilon - Z_{j-1}^c + Z_i^c \geq \frac{1}{2} - \varepsilon - Z_j^c + Z_i^c > \frac{1}{2} - 2\varepsilon.
\]

Finally, we can relate the amount of \( c \) in the LP-buffer with the number of \( c \) in the buffer of the rounded solution.

\[
a_j^c = \sum_{s : t, s \leq j \leq t} A_{s,t}^c y_{s,t}^c \geq \sum_{s : t, s \leq i + 1 \leq j \leq t} A_{s,t}^c y_{s,t}^c \geq A_{i+1,j}^c \sum_{s : t, s \leq i + 1 \leq j \leq t} y_{s,t}^c \geq A_{i+1,j}^c \left( \frac{1}{2} - 2\varepsilon \right).
\]

Hence, the total number of items in the buffer after step \( j \) is \( A_{i+1,j}^c \leq a_j^c/(1/2 - 2\varepsilon) \leq (k - 1)/(1/2 - 2\varepsilon) \). Moreover, when \( j = n + k - 1 \), we have \( a_j^c = 0 \) by constraint (6). This implies that the buffer is empty at the end. \( \square \)

### 4 Dynamic programming

Straightforward dynamic programming algorithms solve the sorting buffer problem optimally in running time \( O(n^{k+1}) \) or \( O(n^{C+1}) \); see also [10],[11]. In this section we consider the special problem setting with a buffer of size \( k = 2 \), and give an algorithm with linear running time for this special case. This is optimal since the size of the input is \( O(n \log C) \).

**Theorem 9** There is an optimal algorithm solving the sorting buffer problem with buffer size \( k = 2 \) in time \( O(n \log C) \).

In our dynamic programming algorithm, we maintain the optimal cost \( \text{Opt}_i \), a set \( S_i \) of colors, and the sizes of those colors. A color \( c \) is in \( S_i \) if there exists an optimal way to serve the first \( i \) items in the sequence such that an item of color \( c \) is served last. The size of a color is the (or, a possible) number of items of this color that are served together if this color is served last. In order to use only linear time, from one step to the next we only store the changes in \( S_i \) and in the sizes of the colors. This works because the number of these changes is amortized constant per step.

We can initialize \( S_1 = \{ c_1 \} \) and \( \text{Opt}_1 = 1 \). The cost \( \text{Opt}_i \) increases as soon as \( c_i \neq c_1 \) for some \( i \); as long as \( c_1 = c_2 = \cdots = c_i \), we have \( \text{Opt}_i = 1 \) and \( \text{size}(c_i) = i \).

**Definition 1** For any step \( i > 1 \), let \( j < i \) be the most recent step such that \( c_j \neq c_1 \). If there is no such step, set \( j = 0 \).

**Observation 1** For each \( i > 1 \), we have \( |S_i| \leq |S_{i-1}| + 1 \).

The only color that could possibly enter the set of optimal finishing colors is the color of the most recent item; any other color would have been optimal before.

**Lemma 10** At any step \( i \), there can be at most one color \( c \) such that \( \text{size}(c) > 1 \); this is color \( c_1 \).
Proof Suppose there is any other color \( c \) in \( S_i \) with \( \text{size}(c) > 1 \). Then the last two items in some optimal serving order have color \( c \neq c_i \). But then item \( i \) is served in step \( i - 2 \) or before, i.e. before it entered the buffer, a contradiction.

We are now ready to present our dynamic program. As stated, it begins processing as soon as a color different from \( c_1 \) appears in the input. In each step, it determines the current set \( S_i \) and the sizes of all the colors, based on this information of the previous step.

1. Before processing the \( i \)th item, store the answers to the following questions using \( S_{i-1} \) and the current sizes of colors (i.e., as they are after processing step \( i - 1 \)):
   (a) \( c_i \in S_{i-1} \)?
   (b) \( c_{i-1} \in S_{i-1} \)?
   (c) If so, do we have \( \text{size}(c_{i-1}) = 1 \)?
   (d) \( c_{i+1} \in S_{i-1} \)? (We may need this information in step \( i + 1 \); hence, we need to remember this bit for one step)

Finally, if \( c_i \neq c_{i-1} \), set \( j = i - 1 \) and update the bits indicating whether \( c_j = c_{i-1} \in S_{i-1} \) and \( c_i \in S_{j+1} = S_{i-2} \) (using the answer to (d) that was stored in the previous step: if this is the first step that the dynamic program is executed, we have \( c_i \notin S_{i-2} \)). If \( c_i = c_{i-1} \), keep those bits unchanged.

2. If \( c_i \in S_{j-1} \), then \( \text{OPT}_{j} = \text{OPT}_{j-1} \) and \( \text{size}(c_i) \) increases by 1. If \( c_i = c_{i-1} \), \( S_i \) remains unchanged. Else, \( S_i \) consists of at most two colors: \( c_i \) and possibly \( c_{i-1} \). This color \( c_{i-1} \) is only in \( S_i \) if \( c_{i-1} \in S_{i-1} \) and \( \text{size}(c_{i-1}) = 1 \).

3. If \( c_i \notin S_{i-1} \), there are two cases.
   (a) If \( c_i \in S_{j-1} \), \( c_j \in S_{j-1} \) and \( \text{size}(c_j) = 1 \), then \( S_i = \{ c_j \} \), \( \text{size}(c_j) = 1 \), and \( \text{OPT}_i = \text{OPT}_{j-1} \).
   (b) Else, add \( c_i \) to \( S_{i-1} \) to get \( S_i \), let \( \text{size}(c_i) = 1 \), and \( \text{OPT}_i = \text{OPT}_{i-1} + 1 \). If \( c_i \in S_{i-2} \), \( c_{i-1} \in S_{i-1} \) and \( \text{size}(c_{i-1}) \geq 2 \), then we still have \( c_{i-1} \in S_i \), but now with \( \text{size}(c_{i-1}) = 1 \). There are no other changes between \( S_{i-1} \) and \( S_i \).

Lemma 11 This dynamic program calculates the optimal cost.

Proof We use induction. For the base case \( i = 1 \), the optimal cost is 1, and \( S_1 = \{ c_1 \} \).
Consider a later step \( i \). For item \( i \), there are only two options: the item must be served last or next-to-last (because it does not enter the buffer until \( i - 2 \) items have been served).

The easiest case occurs when \( c_i \in S_{i-1} \): if it was optimal to serve the first \( i - 1 \) items and finish in color \( c_i \), this is now still optimal, and we can just serve the new item last. (Step 2.) However, we still need to determine the other colors in \( S_i \). Now, if \( c_i = c_{i-1} \), consider any color \( c \in S_{i-1}, c \neq c_i \). By induction, an optimal way to serve the first \( i - 1 \) items is to end with colors \( c_{i-1}, c \) in this order (item \( i - 1 \) cannot be served before step \( i - 2 \)). It follows that we can now still use this order without increasing the cost, since \( c_i = c_{i-1} \). Hence, we find that \( S_i = S_{i-1} \).

If \( c_i \neq c_{i-1} \), \( c_{i-1} \in S_{i-1} \), and \( \text{size}(c_{i-1}) = 1 \), then item \( i - 1 \) can be served last as well. This holds because \( c_i \in S_{i-2} \) (a color can only enter \( S_i \) when it is being requested), so an optimal way to serve the first \( i - 1 \) items is to end with colors \( c_i, c_{i-1} \) in this order, and we can still use this order now without extra cost.

Consider the case \( c_i \notin S_{i-1} \). There are two possible reasons for not serving item \( i \) last:
• Item $i$ can be combined with an earlier item of the same color (but only if item $i$ is not served last). To find out whether this is the case, we consider the set $S_{j-1}$ of optimal finishing colors just before item $j$ arrived. If $c_i \in S_{j-1}$, then color $c_i$ was dropped from the set of finishing colors when item $j$ arrived. In this case we can combine items $j+1, \ldots, i$ (that all have color $c_i$) with a previous item of color $c_i$ if and only if $\text{size}(c_j) = 1$ (because that allows us to keep item $j$ in the buffer while serving color $i$ without increasing the total cost for color $c_j$). (Step 3(a).)

Else, it is optimal to serve item $i$ last, but the optimal cost increases by 1 compared to the previous step.

• Some other item can be combined with a future item of the same color. (This is the reason why we keep track of optimal finishing colors.) It can be seen that any color in $S_{i-1}$ with size 1 can also be delayed for one more step without increasing the cost further (note that we are in the case where the optimal cost has increased compared to the previous step). A color $c$ in $S_{i-1}$ of size at least 2 could be served last by splitting it into two parts and keeping one item until the end. But in order for this to be optimal, we must have $c_i \in S_{i-2}$, since we now pay 1 more for color $c$ and hence must serve color $c_i$ with the same total cost as before, i.e., item $i$ must be served together with a previous item of the same color, while also being served in step $i-1$ (or $i$).

We complete the proof by showing that $S_i = \{c_j\}$ in Case 3(a). This is a case in which the optimal cost for serving the first $i$ items is the same as it was for serving the first $i-1$ items. In particular, the cost to serve color $c_j$ did not increase, although $c_i \notin S_{i-1}$. Hence, in any optimal serving order, item $i$ must be combined with at least one previous item of the same color. In particular, item $i$ must be served as the penultimate item (since it cannot be served earlier, and if it is served last, we must use a suboptimal way to serve the first $i-1$ items if we want to combine item $i$ with a previous one of the same color). In fact, the items served in all steps $j-1, \ldots, i-1$ must be of color $c_i$. This is clear if $j = i-1$. Else, all items following $j$ have color $c_i$, and yet $c_i \notin S_{i-1}$. This can only happen if the items $j+1, \ldots, i$ are served in steps $j, \ldots, i-1$, following another item of color $c_i$ which is served in step $j-1$. This means that item $j$, which is not of color $i$, must be served in step $i$, thus fixing $S_i = \{c_j\}$ (and $\text{size}(c_j) = 1$).

Hence, we maintain for each step the optimal cost so far, whether color $c_j$ enters the set of optimal finishing colors, which colors leave, and which unique color has size more than 1 (if any). It is a nontrivial task to maintain these things in only linear time, and in particular to do this in such a way that an actual optimal solution can be constructed afterwards (and not just the optimal cost). We are going to use three objects:

• An array $S$ of size $C \leq n$, where $C$ is the number of different colors. In this array, $S[c]$ indicates the current size of color $c$ in $S_i$. (We assume the colors are given by numbers from 1 to $C$.) Also, with each item $S[c]$ we associate a pointer to $c$ in the list $L$ below. (If $S[c] = 0$, it is a null pointer.)

• A doubly-linked list $L$ which at all times has size $|S_i|$ and contains links from each item $c \in S_i$ to $S[c]$ to indicate which colors are nonzero (we need this in order to efficiently remove items from $S$ whenever needed).

• An array $H$ of size at most $7n$ in which we store the entire history of changes in $S$ and OPT. For each $i$, the first number indicates whether $\text{OPT}_i > \text{OPT}_{i-1}$. Then, we have
a sequence of pairs (color, change), followed by a zero to mark the end of processing for this $i$.

Regarding the size of $H$, in each step $i$ at most one color can enter $S_i$, and many may leave. However, the latter ones must have entered before. Since each item in the input may cause only its color to enter $S_i$ (and this happens at most once for each item), and each item may cause only preceding colors that entered $S_i$ to leave $S_i$, the total number of these changes is at most $2n$. Finally, each item $i$ may cause one size of one other color $c' \in S_i$ to drop to 1 (in Step 3(b)); we have at most $n/2$ such events, since the size of $c'$ must first have increased to above 1.

In total we have at most $5n/2$ changes that can be stored in an array of length $n + 5n + n = 7n$, where for each $i$ we first store the possible change in $\text{OPT}_i$, then use two places for each change in $S$ indicating the color and the amount of change (positive or negative), and finally a separator bit.

Note that by doing it in this way, we need to store numbers up to $n$ (the possible decrease of a color size in one step), which takes $\log n$ place, for a total space requirement of $n \log n$. However, we could also encode a decrease of $d$ for color $c$ by using $d$ successive entries $c$. Naturally we do need to specify colors, which takes $\log C$ bits, so the overall space and time requirement can be limited to $O(n \log C)$, i.e., linear in the size of the input, which is a list of $n$ colors.

The questions in 1(a–d) can be answered in $O(\log C)$ time by checking the array $S$. In fact, all operations in the dynamic program take $O(\log C)$ or constant time apart from clearing the set $S_i$ in Step 2 and 3(a) when needed, the cost of which however can be amortized as argued above.

To find an optimal way to serve the sequence, we can finish with any color in the array $S$ as it is when step $n$ has been processed. We then search the input for this color, starting from the end. As soon as we find it, say at position $i$, we know that it is optimal to keep item $i$ in the buffer in the end, and therefore to serve items $i+1, \ldots, n$ at places $i, \ldots, n-1$. We can then reconstruct $S_{i-1}$ from $S_n$ using the changes that we stored, take any color from $S_{i-1}$, and repeat. This also takes only linear time, and thus, Theorem 9 follows.

In Appendix II, we will also give an example to illustrate the algorithm and the storage and access of information in the described data structures.

## 5 A lower bound for LFD

The well-known paging problem has several offline algorithms that solve it to optimality. One of those is the Longest Forward Distance (LFD) algorithm [5]. With the mentioned relation to the sorting buffer problem, it is reasonable to consider a natural adaption of this algorithm for sorting buffer. In the following we give a negative result that rules out LFD as a candidate for a constant approximation algorithm.

### Longest Forward Distance (LFD).
If no item can be served without a color change, then choose the color of item $i$ that has its next occurrence $j > i$ farthest in the future of the sequence. If no more items $j$ with the same color as $i$ exist, the distance is infinity.

### Theorem 12
LFD has an approximation ratio of at least $\Omega(k^{1/3})$.  

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Proof Consider the following input instance. Given is a buffer of size $M + n$, where $M \geq n^3$. The sequence of items is as follows; we describe each item by its color (natural number), and we denote by $a^b$ that the item with color $a$ appears $b$ times consecutively.

\[
\begin{array}{c}
0^M \\
123 \ldots n \ [ 2 \ 3^{2} \ 4^{3} \ \ldots \ n^{n-1} ] \\
0123 \ldots n - 1 \ [ 2 \ 3^{2} \ 4^{3} \ \ldots \ (n - 1)^{n-2} ] \\
0123 \ldots n - 2 \ [ 2 \ 3^{2} \ 4^{3} \ \ldots \ (n - 2)^{n-3} ] \\
\vdots \\
0123 \ [ 2 \ 3^{2} ] \\
012 \ [ 2 ] \\
01 \\
\end{array}
\]

The sequence consists of $n + 1$ lines; let us denote them as $L_0, L_1, \ldots, L_n$. Initially, the buffer contains all items of line $L_0$ and the first block (in brackets) of $L_1$. An optimal solution chooses color 0 first; it can serve all items of this color and by the end, all remaining items of the sequence are in the buffer. Thus, there are no more than $n + 1$ color changes necessary.

LFD chooses color 1 first, moving the next item of color 2 into the buffer. Then it picks 2, moving two items of color 3 into the buffer and repeats until it chooses $n$ and moves the first block of $L_2$ into the buffer. Then the process repeats. This way, LFD causes $n - i$ color changes serving the first block of line $L_i$. Thus, it has total cost $n(n + 1)/2$.

The ratio of LFD’s cost and the optimal cost for this sequence are $n/2$. Hence, LFD has an approximation ratio bounded by $\Omega(k^{1/3})$ for a given buffer of size $k$. \qed

6 Open problems

Now that NP-hardness has been settled, the main open problem is to design a polynomial time constant factor approximation. In the introduction we listed several partial results on this. Given our LP-rounding result, a natural next step is to design an algorithm that gives an $O(1/\varepsilon)$-approximation against an offline solution using only $(1 - \varepsilon)k$ capacity, instead of $(1/2 - 2\varepsilon)k$.

We gave a dynamic program for $k = 2$ which has a significantly better running time than the straightforward DP. It would be interesting to give an exact algorithm for with a running time that is much less than $O(n^{k+1})$.

Our NP-completeness proof is not approximation preserving. It remains a question whether the buffer sorting problem is APX-hard or not.

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