Words and Dominions
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Abstract. A necessary and sufficient condition for an element of an algebra (in the sense of
Universal Algebra) to be in the dominion of a subalgebra is given, in terms of transferable
sets. This criterion is then used to formulate a more wieldy sufficient condition. Finally,
some connections to a purely combinatorial setting are outlined.

Section 1. Introduction

Let \( C \) be a full subcategory of the category of all algebras (in the sense of Universal
Algebra) of a fixed type, which is closed under passing to subalgebras. Let \( A \in C \),
and let \( B \) be a subalgebra of \( A \). Recall that, in this situation, Isbell (see [1])
defines the domain of \( B \) in \( A \) (in the category \( C \)) to be the intersection of all
equalizer subalgebras of \( A \) containing \( B \). Explicitly,

\[
\text{dom}^C_A(B) = \{ a \in A \mid \forall C \in C, \forall f, g: A \to C, \text{ if } f|_B = g|_B \text{ then } f(a) = g(a) \}.
\]

Note that \( \text{dom}^C_A(B) \) always contains \( B \). If \( B \) is properly contained in its dominion,
we will say that the dominion of \( B \) in \( A \) is nontrivial, and call it trivial otherwise.
A category \( C \) has instances nontrivial dominions if there is an algebra \( A \in C \),
and a subalgebra \( B \) of \( A \) such that the dominion of \( B \) in \( A \) (in the category \( C \))
is nontrivial.

In this work we will present a general result on words and dominions in a given
variety of \( \Omega \)-algebras. The result is contained in Theorem 2.6. From it we deduce
a particular, but more manageable, sufficient condition in Section 3. Finally, in
Section 4, we relate the criteria in Lemma 2.5 to a purely combinatorial setting,
and raise some related open questions.

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Section 2. Transferable words

Let $\Omega$ be a type, and consider a variety $\mathcal{V}$ of $\Omega$-algebras.

**Convention 2.1.** When we form the amalgamated $\mathcal{V}$-coproduct of $A$ with $A$, amalgamated over $B$, denoted by $A \amalg_B^\mathcal{V} A$, the universal pair of maps from $A$ into $A \amalg_B^\mathcal{V} A$ will be written $(\lambda, \rho)$. We will refer to the maps $\lambda$ and $\rho$ as the left and right embeddings, respectively.

**Lemma 2.2.** Let $\mathcal{V}$ be a variety of $\Omega$-algebras, and let $A, B \in \mathcal{V}$, and $B$ a subalgebra of $A$. Then $\text{dom}^\mathcal{V}_A(B)$ is the equalizer of the two canonical embeddings $\lambda$ and $\rho$ of $A$ into $A \amalg_B^\mathcal{V} A$. Furthermore, we have

$$\text{dom}^\mathcal{V}_A(B) = \lambda^{-1}(\lambda(A) \cap \rho(A)) = \rho^{-1}(\lambda(A) \cap \rho(A)).$$

*Proof:* Clearly the dominion is contained in the equalizer of $\lambda$ and $\rho$. Conversely, any pair of maps $f, g: A \to C$, with $C \in \mathcal{V}$, such that $f|_B = g|_B$ must factor through $A \amalg_B^\mathcal{V} A$ and $(\lambda, \rho)$; hence the equalizer of $f$ and $g$ contains the equalizer of $\lambda$ and $\rho$, giving the reverse inclusion.

For the second assertion, note that both $\lambda$ and $\rho$ are embeddings, so the pullback $\lambda^{-1}$ simply identifies the subgroup $\lambda(A) \cap \rho(A)$ with its counterpart inside of $A$. We also know that the dominion consists precisely of the elements $a \in A$ which are identified with their counterparts, that is, for which $\lambda(a) = \rho(a)$. Clearly these elements lie in $\lambda(A) \cap \rho(A)$. Conversely, suppose that $\lambda(a) = \rho(a') \in \lambda(A) \cap \rho(A)$.

Let $\phi: A \amalg_B^{\mathcal{V}} A \to A$ be the map induced by the identity map on $A$ (that is, the universal map obtained from the pair $(\text{id}_A, \text{id}_A: A \to A$). In particular,

$$\phi \circ \lambda = \text{id}_A = \phi \circ \rho.$$

Therefore, $a = \text{id}_A(a) = \phi(\lambda(a)) = \phi(\rho(a')) = a'$. Hence, $a$ is identified with its counterpart, which establishes the lemma. \qed

Let $W$ be a derived operation in a set $S$ of variables (that is, $W$ corresponds to an equivalence class of elements in the countably generated free $\mathcal{V}$-algebra, which involves only variables in $S$). Let $x = (x_s)_{s \in S}$ be an $S$-tuple of elements of an algebra $A \in \mathcal{V}$, and let $B$ be a subalgebra of $A$.

**Definition 2.3.** For each partition of $S$ into disjoint sets $S_1$ and $S_2$ (a situation we will denote by writing $S = S_1 \amalg S_2$), we define $W_{S_1,S_2}(x)$ to be the element of $A \amalg_B^{\mathcal{V}} A$ obtained by substituting for the $s$-th argument of $W$ the element $\lambda(x_s)$ for all $s \in S_1$, and $\rho(x_s)$ for all $s \in S_2$.

**Definition 2.4.** A subset $T \subseteq S$ is called transferable over $B$ with respect to $W(x)$, $A$, and $\mathcal{V}$ if and only if for every partition of the form

$$S = S_1 \amalg T \amalg S_2,$$
we have
\[ W_{S_1 \cup T, S_2}(x) = W_{S_1, T \cup S_2}(x). \]

We will usually omit the mention of \( A \) and \( V \), taking them as given.

We recall that for any word \( W(x_1, \ldots, x_n) \) and any algebras \( R \) and \( S \), and morphism \( f: R \to S \), for any \( r_1, \ldots, r_n \in R \) we have
\[ f(W(r_1, \ldots, r_n)) = W(f(r_1), \ldots, f(r_n)). \]

**Lemma 2.5.** With notation as in the preceding two paragraphs,

(i) \( \emptyset \) is transferable over \( B \) with respect to \( W(x) \).

(ii) If \( T_1 \) and \( T_2 \) are subsets of \( S \), each transferable over \( B \) with respect to \( W(x) \), and their intersection \( T_1 \cap T_2 \) is also transferable over \( B \) with respect to \( W(x) \), then their union is transferable over \( B \) with respect to \( W(x) \). In particular, if \( T_1 \cap T_2 = \emptyset \) and each is transferable over \( B \) with respect to \( W(x) \), then so is their union.

(iii) Suppose \( T_1 \) and \( T_2 \) are disjoint subsets of \( S \), and that both \( T_1 \) and \( T_1 \cup T_2 \) are transferable over \( B \) with respect to \( W(x) \). Suppose further that for every proper subset \( U \) of \( T_1 \), either \( U \) or \( T_1 \setminus U \) is transferable over \( B \) with respect to \( W(x) \). In this case, \( T_2 \) is transferable over \( B \) with respect to \( W(x) \).

(iv) If \( W(x) = W'(W''((x_s)_{s \in T}), (x_s)_{s \in S \setminus T}) \) for some words \( W' \) and \( W'' \) and subset \( T \subseteq S \), and \( W'' \) evaluated at the \( T \)-tuple \((x_s)_{s \in T}\) gives an element of \( B \), then \( T \) is transferable over \( B \) with respect to \( W(x) \).

(v) If \( U \subset S \) is such that for every \( s \in U \), \( x_s \in B \), then for every subset \( T \) of \( S \), \( T \) is transferable over \( B \) with respect to \( W(x) \) if and only if \( T \cup U \) is transferable over \( B \) with respect to \( W(x) \).

**Proof:** (i) is clear.

(ii) Let \( S_1 \amalg (T_1 \cup T_2) \amalg S_2 \) be a partition of \( S \). Then
\[
W_{S_1 \cup (T_1 \cup T_2), S_2}(x) = W_{S_1 \cup (T_1 \setminus T_2), T_2 \cup S_2}(x) \\
\text{(since } T_2 \text{ is transferable)} \\
= W_{S_1 \cup (T_1 \setminus T_2) \cup (T_1 \setminus T_2) \cup (T_2 \setminus T_1) \cup S_2}(x) \\
\text{(since } T_1 \cap T_2 \text{ is transferable)} \\
= W_{S_1 \cup T_1 \cup (T_2 \setminus T_1) \cup S_2}(x) \\
= W_{S_1, T_1 \cup T_2 \cup S_2}(x) \\
\text{(since } T_1 \text{ is transferable)}
\]

so \( T_1 \cup T_2 \) is transferable. The last assertion now follows from (i) and the general case we just proved.
(iii) Let \( S_1 \coprod T_2 \coprod S_2 \) be a partition of \( S \). Let \( T_{11} = T_1 \cap S_1 \) and \( T_{12} = T_1 \cap S_2 \). If \( T_{12} \) is transferable, then

\[
W_{S_1 \cup T_2, S_2}(x) = W_{S_1 \cup T_2 \cup T_{12}, S_2 \setminus T_{12}}(x) \\
\text{(since } T_{12} \text{ is transferable)}
\]

\[
= W_{(S_1 \setminus T_{11}) \cup T_{12}, S_2 \setminus T_{12}}(x)
\]

\[
= W_{S_1 \setminus T_{11}, T_2 \cup (S_2 \setminus T_{12})}(x) \\
\text{(since } T_1 \cup T_2 \text{ is transferable)}
\]

\[
= W_{(S_1 \setminus T_{11}) \cup T_1, T_2 \cup (S_2 \setminus T_{12})}(x) \\
\text{(since } T_1 \text{ is transferable)}
\]

\[
= W_{S_1 \cup T_2, T_2 \cup (S_2 \setminus T_{12})}(x)
\]

\[
= W_{S_1 \cup T_2, S_2}(x) \]

so \( T_2 \) is transferable.

If \( T_{12} \) is not transferable, then by hypothesis \( T \setminus T_{12} = T_{11} \) must be transferable.

Therefore,

\[
W_{S_1 \cup T_2, S_2}(x) = W_{(S_1 \setminus T_{11}) \cup T_2, T_1 \cup S_2}(x) \\
\text{(since } T_{11} \text{ is transferable)}
\]

\[
= W_{(S_1 \setminus T_{11}) \cup T_2, T_1 \setminus S_2 \setminus T_{12}}(x) \\
\text{(since } T_1 \text{ is transferable)}
\]

\[
= W_{(S_1 \setminus T_{11}), (T_1 \cup T_2) \cup (S_2 \setminus T_{12})}(x) \\
\text{(since } T_1 \cup T_2 \text{ is transferable)}
\]

\[
= W_{(S_1 \setminus T_{11}), T_1 \cup T_2 \cup (S_2 \setminus T_{12})}(x) \\
\text{(since } T_{11} \text{ is transferable)}
\]

\[
= W_{S_1 \cup T_2, S_2}(x)
\]

so again \( T_2 \) is transferable.

(iv) Let \( S_1 \coprod T \coprod S_2 \) be a partition of \( S \). By definition, we have

\[
W_{S_1 \cup T, S_2} = W'(W''(\lambda(x_t)_{t \in T}), \lambda(x_s)_{s \in S_1 \cup \rho(x_s)_{s \in S_2}}).
\]

By hypothesis, \( W''((x_t)_{t \in T}) \in B \), and since \( \lambda|_B = \rho|_B \), we have

\[
W''(\lambda(x_t)_{t \in T}) = W''(\rho(x_t)_{t \in T}).
\]
Therefore,
\[
W_{S_1 \cup T, S_2}(x) = W'(W''(\lambda(x_t)_{t \in T}), \lambda(x_s)_{s \in S_1 \cup \rho(x_s)_{s \in S_2})
\]
\[
= W'(W''(\rho(x_t)_{t \in T}), \lambda(x_s)_{s \in S_1 \cup \rho(x_s)_{s \in S_2})
\]
\[
= W_{S_1, T \cup S_2}(x)
\]

which means that $T$ is transferable over $B$ with respect to $W(x)$, as claimed.

(v) Clearly, $U$ and every subset of $U$ are transferable over $B$ with respect to $W(x)$. Therefore, if $T$ is transferable, then transferability of $T \cup U$ follows from (iv), and transferability of $T$ when $T \cup U$ is transferable follows from (iii). □

From Theorem 2.6 we obtained the promised necessary and sufficient condition:

**Theorem 2.6.** Let $\mathcal{V}$ be a variety of $\Omega$-algebras for a fixed type $\Omega$. Let $W$ be a derived word in a set $S$ of variables, let $x = (x_s)_{s \in S}$ be an $S$-tuple of elements of an algebra $A$, and let $B$ be a subalgebra of $A$. Then $W(x)$ lies in $\text{dom}^\mathcal{V}_A(B)$ if and only if $S$ is transferable over $B$ with respect to $W(x_s)$.

**Proof:** If $S$ is transferable over $B$ with respect to $W(x)$, then necessarily

\[
W_{S, \emptyset}(x) = W_{\emptyset, S}(x),
\]

that is, $W(\lambda(x)) = W(\rho(x))$, so $\lambda(W(x)) = \rho(W(x))$ in $A \Pi^\mathcal{V}_B A$, hence $W(x)$ lies in the dominion.

Since each of the implications above is reversible, this completes the proof. □

**Section 3. A corollary on words**

In this section we will present a consequence of Theorem 2.6 which is a bit easier to use than the general case.

Let $\mathcal{V}$ be a variety of $\Omega$-algebras, let $A \in \mathcal{V}$, and let $B$ be a subalgebra of $A$. Let $W_1, \ldots, W_m$ be words in $m$ letters each, and let

\[
(w_{ij})_{i,j \in \{1, \ldots, m\}}
\]

be words, with $w_{ij}$ a word in $n_j$ letters for each $i$.

**Definition 3.7.** We will say that \{\{W_i, w_{ij} \mid 1 \leq i, j \leq m\}\} is an equational array in $\mathcal{V}$ if the following identities in $n_1 + \cdots + n_m$ indeterminates hold in $\mathcal{V}$:

\[
W_1(w_{11}(x_{11}, \ldots, x_{1n_1}), w_{12}(x_{21}, \ldots, x_{2n_2}), \ldots, w_{1m}(x_{m1}, \ldots, x_{mn_m}))
\]
\[
= W_2(w_{21}(x_{11}, \ldots, x_{1n_1}), w_{22}(x_{21}, \ldots, x_{2n_2}), \ldots, w_{2m}(x_{m1}, \ldots, x_{mn_m}))
\]
\[
\vdots
\]
\[
= W_m(w_{m1}(x_{11}, \ldots, x_{1n_1}), w_{m2}(x_{21}, \ldots, x_{2n_2}), \ldots, w_{mm}(x_{m1}, \ldots, x_{mn_m})).
\]
In that case, we say the size of the array is \( m \), and the signature of the array is the sequence \((n_1, \ldots, n_m)\) (that is, the number of arguments of the words in the “columns”).

We will often abbreviate \( x_j = (x_{j1}, \ldots, x_{jn_j}) \), so that

\[
w_{ij}(x_j) = w_{ij}(x_{j1}, \ldots, x_{jn_j}).
\]

In what follows, by abuse of notation we shall use the symbols \( x_{jk} \) to mean both the indeterminates used in writing the words \( w_{ij} \), and the values we substitute for those indeterminates. The context will make it clear which of the two meanings we are using, and it should prevent even more proliferation of notation.

**Lemma 3.8.** Let notation be as in the two preceding paragraphs. Suppose that

\[
\{W_i, w_{ij} \mid 1 \leq j \leq n_i; 1 \leq i \leq m\}
\]

is an equational array in \( V \) of size \( m \) and signature \((n_1, \ldots, n_m)\). Let \( A \in V \), and let \( B \) be a subalgebra of \( A \). Suppose that for a particular choice of \( x_{ij} \in A \), we have

\[
w_{ii}(x_{i1}, \ldots, x_{in_i}) \in B \quad 1 \leq i \leq m.
\]

Then \( W_1(w_{11}(x_1), \ldots, w_{1m}(x_m)) \) lies in \( \text{dom}^V_A(B) \).

**Proof:** We want to apply Theorem 2.6, so we let

\[
S = \{(i, j) \mid 1 \leq j \leq n_i; 1 \leq i \leq m\},
\]

and \( x_S = (x_{ij})_{(i, j) \in S} \). For each \( i, 1 \leq i \leq m \), let \( T_i = \{(i, 1), \ldots, (i, n_i)\} \). Then \( T_i \) is transferable over \( B \) with respect to \( W_i \) by Lemma 2.5(iv). Since all the \( W_i(\cdots) \) (where by \( (\cdots) \) we mean the system of arguments for \( W_i \) occurring in the given equational array) have the same value at \( x \), it follows from Lemma 2.5(ii) that \( T_1 \cup \cdots \cup T_m \) is transferable over \( B \) with respect to \( W_i(\cdots) \) for any \( i \); but \( S = T_1 \cup \cdots \cup T_m \), so \( S \) itself is transferable over \( B \) with respect to \( W_i \). By Theorem 2.6

\[
W_1(w_{11}(x_{11}, \ldots, x_{1n_1}), \ldots, w_{1m}(x_{m1}, \ldots, x_{mn_m}))
\]

lies in \( \text{dom}^V_A(B) \), as claimed.

**Remark 3.9.** Here is a more direct way of verifying that the value of \( W \) lies in the dominion. Let \( C \in V \), and let \( f, g : A \to C \) be two morphisms with \( f|_B = g|_B \). Then
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\[ f \left( W_1(w_{11}(x_1), w_{12}(x_2), \ldots, w_{1m}(x_m)) \right) \]

\[ = W_1 \left( w_{11}(f(x_1)), w_{12}(f(x_2)), \ldots, w_{1m}(f(x_m)) \right) \]

\[ = W_1 \left( w_{11}(g(x_1)), w_{12}(f(x_2)), \ldots, w_{1m}(f(x_m)) \right) \]

(since \( w_{11}(x_1) \) lies in \( B \))

\[ = W_2\left( w_{21}(g(x_1)), w_{22}(f(x_2)), \ldots, w_{2m}(f(x_m)) \right) \]

(using our equational array)

\[ = W_2\left( w_{21}(g(x_1)), w_{22}(g(x_2)), \ldots, w_{2m}(f(x_m)) \right) \]

(since \( w_{22}(x_2) \) lies in \( B \))

\[ : \]

\[ = W_m\left( w_{m1}(g(x_1)), w_{m2}(g(x_2)), \ldots, w_{mm}(g(x_m)) \right) \]

\[ = g\left( W_m\left( w_{m1}(x_1), w_{m2}(x_2), \ldots, w_{mm}(x_m) \right) \right) \]

\[ = g\left( W_1\left( w_{11}(x_1), w_{m2}(x_2), \ldots, w_{mm}(x_m) \right) \right) \]

(using our equational array)

Therefore, \( W_1(w_{11}(x_{11}, \ldots, x_{1n_1}), \ldots, w_{1m}(x_{m1}, \ldots, x_{mn_m})) \) lies in \( \text{dom}_A^V(B) \), as we claimed. \( \square \)

**Example 3.10.** Let \( V = N_2 \) be the variety of all nilpotent groups of class at most two; that is, groups satisfying the identity \([ [x, y], z] = e \). One can deduce from this the identity \([x, y^n] = [x, y]^n = [x^n, y] \). Now observe that the identity \([x, y^n] = [x^n, y] \) may be written as an equational array, namely an equational array of size 2 and signature \((1, 1)\) given by

\[ W_1(x, y) = W_2(x, y) = [x, y] \]

\[ w_{11}(z) = w_{22}(z) = z^n \]

\[ w_{12}(z) = w_{21}(z) = z. \]

This yields a class of sufficient conditions for elements to lie in the dominion of a subgroup in this variety; namely, if \( G \in V \) and \( H \) is a subgroup, and for some \( x, y \in G \) and \( n > 0 \) we have that \( x^n \) and \( y^n \) both lie in \( H \), then \([x, y]^n \) lies in the dominion of \( H \) in \( G \) in the variety \( V \). Dominions in the variety \( N_2 \) are discussed at length in [3], Section 3.

Let \( V \) be a variety, and let \( A \in V \) and \( B \) a subalgebra of \( A \). As we let the \( \{W_i, w_{ij}\} \) vary over all equational arrays in \( V \), and all possible assignments satisfying the hypothesis of Lemma 3.8, we obtain a collection of elements of \( A \)
(namely, the ones for which the conclusion of Lemma 3.8 tells us lie in $\text{dom}^y_A(B)$). Denote this collection by $B^*$.

**Proposition 3.11.** Let $A$, $B$, and $B^*$ be as in the preceding paragraph. Then $B \subseteq B^* \subseteq \text{dom}_A(B)$, and $B^*$ is a subalgebra of $A$.

**Proof:** To show that $B \subseteq B^*$, we consider the word $W_1(x) = w_{11}(x) = x$, and the equational array of size 1 given by $\{W_1, w_{11}\}$. Then, for all $x$ in $B$, this gives $x \in B^*$. The inclusion $B^* \subseteq \text{dom}_A(B)$ follows from Lemma 3.8.

Finally, we want to show that $B^*$ is a subalgebra of $A$, so we want to prove that $B^*$ is closed under the operations of the type $\Omega$. Any zeroary operations give elements that already lie in $B$ (as the latter is assumed a subalgebra), so they also lie in $B^*$. Let $\tau$ be a $k$-ary operation, with $k > 0$, and let

$$\{W^{(1)}_i; w^{(1)}_{ij} | 1 \leq j \leq n^{(1)}_i; 1 \leq i \leq m^{(1)}\}$$

$$\{W^{(2)}_i; w^{(2)}_{ij} | 1 \leq j \leq n^{(2)}_i; 1 \leq i \leq m^{(2)}\}$$

$$\vdots$$

$$\{W^{(k)}_i; w^{(k)}_{ij} | 1 \leq j \leq n^{(k)}_i; 1 \leq i \leq m^{(k)}\}$$

be equational arrays corresponding to $k$ elements of $B^*$. By introducing dummy variables, we may assume that all the $m^{(\ell)}$ are equal; we therefore write them simply as $m$. (This last step is not strictly necessary, but it will simplify the notation a bit; it will be clear, from the proof, what to do when the $m^{(\ell)}$ are different from each other).

We want to show that $\tau(W^{(1)}_1, \ldots, W^{(k)}_1) \in B^*$. Consider the $km$ words in $km$ letters given by

$$V_1(x_1, \ldots, x_{km}) = \tau(W^{(1)}_1(x_1, \ldots, x_m), \ldots, W^{(k)}_1(x_{km}))$$

$$V_2(x_1, \ldots, x_{km}) = \tau(W^{(1)}_2(x_1, \ldots, x_m), \ldots, W^{(k)}_1(x_{km}))$$

$$\vdots$$

$$V_m(x_1, \ldots, x_{km}) = \tau(W^{(1)}_m(x_1, \ldots, x_m), \ldots, W^{(k)}_1(x_{km}))$$

$$V_{m+1}(x_1, \ldots, x_{km}) = \tau(W^{(1)}_m(x_1, \ldots, x_m), \ldots, W^{(k)}_2(x_{km}))$$

$$\vdots$$

$$V_{km}(x_1, \ldots, x_{km}) = \tau(W^{(1)}_m(x_1, \ldots, x_m), \ldots, W^{(k)}_m(x_{km}))$$

Then, if we consider the equational array given by $\{V_i, v_{ij}\}$, where $v_{11} = w^{(1)}_{11}$, $v_{12} = w^{(1)}_{12}$, $\ldots$, $v_{1km} = w^{(k)}_{1m}$, $v_{21} = w^{(1)}_{21}$, etc. we obtain that

$$\tau(W^{(1)}_1, W^{(2)}_1, \ldots, W^{(k)}_1) \in B^*$$
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as claimed. So \( B^* \) is closed under the operations, and it is indeed a subalgebra of \( A \).

There is at least one refinement that can be made to this proposition. For simplicity, we will state and prove it only in its simplest version (when the size of the arrays is 2). The generalization to larger arrays is easy to prove.

**Lemma 3.12.** Let \( \mathcal{V} \) be a variety of algebras, and let

\[
\begin{align*}
\mathbf{x}_1 &= (x_{11}, x_{12}, \ldots, x_{1m_1}) \\
\mathbf{x}_2 &= (x_{21}, x_{22}, \ldots, x_{2m_2}) \\
\mathbf{y} &= (y_1, y_2, \ldots, y_{m_3})
\end{align*}
\]

be indeterminates. Let \( w_{11}, w_{21} \) be words in \( m_1 + m_3 \) indeterminates; \( w_{12}, w_{22} \) words in \( m_2 + m_3 \) indeterminates; and \( W_1, W_2 \) words in two indeterminates. Suppose that in \( \mathcal{V} \) the following identity holds:

\[
W_1\left(w_{11}(\mathbf{x}_1, \mathbf{y}), w_{12}(\mathbf{x}_2, \mathbf{y})\right) = W_2\left(w_{21}(\mathbf{x}_1, \mathbf{y}), w_{22}(\mathbf{x}_2, \mathbf{y})\right),
\]

Let \( A \in \mathcal{V}, B \) a subalgebra of \( A \). If for some choice of \( x_{ij} \in A, y_{k\ell} \in B \) we have

\[
w_{11}(\mathbf{x}_1, \mathbf{y}), w_{22}(\mathbf{x}_2, \mathbf{y}) \in B
\]

then \( W_1(w_{11}(\mathbf{x}_1, \mathbf{y}), w_{12}(\mathbf{x}_2, \mathbf{y})) \in \text{dom}^\mathcal{V}_A(B) \).

**Proof:** Let

\[
\begin{align*}
T_1 &= \{(1, j) \mid 1 \leq j \leq m_1\} \\
T_2 &= \{(2, j) \mid 1 \leq j \leq m_2\} \\
U &= \{1, 2, \ldots, m_3\},
\end{align*}
\]

and let \( S = T_1 \cup T_2 \cup U \). Clearly, \( U \) and every subset of \( U \) are transferable over \( B \) with respect to \( W_1 \), by Lemma 2.5(iv). By hypothesis, both \( T_1 \cup U \) and \( T_2 \cup U \) are transferable over \( B \) with respect to \( W_1 \). Hence, by Lemma 2.5(v), both \( T_1 \) and \( T_2 \) are transferable. By Lemma 2.5(ii), the union \( S = T_1 \cup T_2 \cup U \) is therefore transferable. By Theorem 2.6, we conclude that

\[
W_1(w_{11}(\mathbf{x}_1, \mathbf{y}), w_{12}(\mathbf{x}_2, \mathbf{y}))
\]

lies in \( \text{dom}^\mathcal{V}_A(B) \), as claimed.

One can also use a trick similar to that of Proposition 3.11 to show that the analogous subset obtained using Lemma 3.12 is also a subalgebra of \( A \), containing \( B \) and contained in \( \text{dom}^\mathcal{V}_A(B) \).
Example 3.13. Let $V$ be the variety of semigroups. Let

$$
W_1(x_1, x_2) = W_2(x_1, x_2) = x_1 x_2,
$$

$$
w_{11}(x, z, y) = xy, \quad w_{22}(x, z, y) = yz,
$$

$$
w_{12}(x, z, y) = z, \quad w_{21}(x, z, y) = x.
$$

Then the equational array, with $m_1 = m_2 = 2$, $m_3 = 1$, says that in $V$ we have $(xy)z = x(yz)$. Then Lemma 3.12 gives the smallest case of Isbell’s Zigzag Lemma [1]: If $A$ is a semigroup, and $B$ a subsemigroup of $A$, then an element $d \in A$ lies in the dominion of $B$ if you can write $d = xyz$ with $x, z \in A$ and $y, (xy), (yz) \in B$.

Section 4. Pre-transfer systems and transfer systems

Looking at the properties listed in Lemma 2.5, we note that (i), (ii) and (iii) are purely set-theoretic; following a suggestion of George Bergman, we present some definitions:

Definition 4.14. Let $S$ be a set. We will call a set $T$ of subsets of $S$ a transfer system on $S$ if there exists an equivalence relation $\sim$ on subsets of $S$, such that a set $T$ belongs to $T$ if and only if for every subset $U$ of $S \setminus T$, we have $U \sim U \cup T$.

Lemma 4.15. Let $V$ be a variety of $\Omega$-algebras, $A \in V$, and let $B$ be a subalgebra of $A$. Let $W$ be a derived operation in a set $S$ of variables, and let $x = (x_s)_{s \in S}$ be an $S$-tuple of elements of $A$. Let $T$ be the collection of all subsets of $S$ which are transferable over $B$ with respect to $W(x)$. Then $T$ is a transfer system, associated to the equivalence relation $\sim$ on subsets of $S$ which makes $U \sim V$ if and only if $W_{U,S \setminus U}(x) = W_{V,S \setminus V}(x)$.

Proof: That $\sim$ is an equivalence relation is immediate. To prove the lemma, first let $T \in T$. That it, $T$ is transferable over $B$ with respect to $W(x)$. Let $U \subseteq S \setminus T$. We need to show that $U \sim U \cup T$.

Let $V = S \setminus (U \cup T)$. Then $S = U \cup T \cup V$ is a partition of $S$, and by definition, we have

$$
W_{U \cup T, V}(x) = W_{U, T \cup V}(x).
$$

In particular, $U \cup T \sim U$, which is what we needed to show.

Conversely, suppose that $T \subseteq S$ is such that for every $U \subseteq S \setminus T$, we have $U \sim U \cup T$. Let

$$
S = S_1 \cup T \cup S_2
$$

be a partition of $S$. Then $S_1 \subseteq S \setminus T$, so $S_1 \sim S_1 \cup T$. By definition of $\sim$, this means that

$$
W_{S_1, T \cup S_2}(x) = W_{S_1 \cup T, S_2}(x).
$$

Since this is true for arbitrary $S_1$, it follows that $T$ is transferable over $B$ with respect to $W(x)$, which proves the lemma.
We also have a following partial converse:

**Lemma 4.16.** Let \( S \) be a set, and let \( \mathcal{T} \) be a transfer system of subsets of \( S \). Then

(i) \( \emptyset \in \mathcal{T} \).

(ii) If \( T_1 \) and \( T_2 \) are both in \( \mathcal{T} \), and \( T_1 \cap T_2 \) is also in \( \mathcal{T} \), then \( T_1 \cup T_2 \in \mathcal{T} \). In particular, if \( T_1 \cap T_2 = \emptyset \) and each lies in \( \mathcal{T} \), then so does their union.

(iii) Suppose that \( T_1 \) and \( T_2 \) are disjoint subsets of \( S \), and that both \( T_1 \) and \( T_1 \cup T_2 \) lie in \( \mathcal{T} \). Suppose further that for every proper subset \( U \) of \( T_1 \), either \( U \) or \( T_1 \setminus U \) lies in \( \mathcal{T} \). In this case, \( T_2 \) also lies in \( \mathcal{T} \).

**Proof:** Let \( \sim \) be an equivalence relation on the subsets of \( S \) which makes \( \mathcal{T} \) a transfer system. Since \( \sim \) is reflexive, \( \emptyset \in \mathcal{T} \), which proves (i).

To prove (ii), let \( U \subseteq S \setminus (T_1 \cup T_2) \). Then

\[
U \sim U \cup T_2 \quad \text{(since } T_2 \in \mathcal{T})
\]

\[
\sim (U \cup T_2) \setminus (T_1 \cap T_2) \quad \text{(since } T_1 \cap T_2 \in \mathcal{T})
\]

\[
\sim ((U \cup T_2) \setminus (T_1 \cap T_2)) \cup T_1 \quad \text{(since } T_1 \in \mathcal{T})
\]

\[
= U \cup (T_1 \cup T_2).
\]

Therefore, \( T_1 \cup T_2 \in \mathcal{T} \). The last assertion in (ii) now follows from the general case and (i).

Note the parallels between this proof and the proof of Lemma 2.5(ii). It is easy to prove (iii) using similar parallels to the proof of Lemma 2.5(iii), and so we will omit the proof.

Let us make the following definition:

**Definition 4.17.** Let \( S \) be a set, and let \( \mathcal{T} \) be a collection of subsets of \( S \). If \( \mathcal{T} \) satisfies (i), (ii), and (iii) of Lemma 4.16, we will say that \( \mathcal{T} \) is a pre-transfer system on \( S \).

By Lemma 4.16, if \( \mathcal{T} \) is a transfer system on \( S \), then it is a pre-transfer system on \( S \). By Lemma 4.15, if \( \mathcal{T} \) consists of all sets transferable with respect to some particular variety \( \mathcal{V} \), and particular \( A, B, (x_s), \) and \( W \), then \( \mathcal{T} \) is a transfer system on \( S \), and hence also a pre-transfer system on \( S \). We have the following implications:

\[
\begin{align*}
\text{all sets transferable} & \quad \Rightarrow \quad \mathcal{T} \text{ is a transfer system on } S. \\
\text{with respect to some particular } \mathcal{V}, A, B, (x_s), \text{ and } W. & \quad \Rightarrow \quad \mathcal{T} \text{ is a pre-transfer system on } S.
\end{align*}
\]
If the first implication above is reversible, then the set-theoretic properties of subsets which are transferable over $B$ with respect to some $W$ (that is, properties which, unlike Lemma 2.5(iv) do not refer to the specific algebra structures), can be studied in terms of the purely combinatoric model of transfer systems.

Unfortunately, the second implication is not reversible as stated. We will provide a counterexample below, in Example 4.21. If it becomes reversible after an appropriate strengthening of (i), (ii), and (iii), then this would provide an axiomatization of transfer systems. Some global properties may also make the implications reversible. For example, we will next prove that under additional hypothesis, all three concepts become equivalent.

For simplicity, if a collection $T$ consists of all the sets which are transferrable with respect to some algebra $A$, subalgebra $B$, variety $V$, word $W$, and tuple $(x_s)$, let us say that $T$ comes from dominions.

**Definition 4.18.** Given a finite set $S$ and a subset $V$ of $S$, let

$$T(V) = \{ \emptyset \} \cup \{ A \subseteq S | V \subseteq A \}.$$ 

Assume for the remainder of this section that $S$ has at least two elements.

**Lemma 4.19.** Let $S$ be a finite set, and let $V \subseteq S$. Then $T(V)$ is a pre-transfer system if and only if $V$ does not have exactly one element.

**Proof:** Clearly, $T(V)$ satisfies (i) and (ii), regardless of whether $V$ has exactly one element or not. If $V$ has at least two elements, then $T(V)$ satisfies (iii) vacuously: for given $x \in V$, and sets $T_1$ and $T_2$ as in (iii), then $\{x\} \subseteq T_1$, but neither $\{x\}$ nor $T_1 \setminus \{x\}$ lie in $T(V)$, so the hypothesis of (iii) are never satisfied. Thus, if $V$ has at least two elements then $T(V)$ is a pre-transfer system. If $V = \emptyset$, then $T(V)$ is the power set of $S$, which is also clearly a pre-transfer system. Finally, it is easy to see that if $V$ is a singleton, the least pre-transfer system which contains $T(V)$ is also the entire power set of $S$.

Let us say that a pre-transfer system $T$ is principal if $T = T(V)$ for some subset $V$ of $S$. Note in particular that if $T$ is a principal pre-transfer system corresponding to $V$, then $V$ is not a singleton.

**Theorem 4.20.** If $T$ is a principal pre-transfer system on $S$, then $T$ comes from dominions, and in particular $T$ is a transfer system.

**Proof:** Let $S$ be a set, and let $T = T(V)$, for some subset $V$ which is not a singleton.

If $V = \emptyset$, then $T$ is the power set of $S$; to see it comes from dominions, pick any variety, and any algebra $A$, and let $B = A$. So we may assume that $V$ is nonempty.

Let $V$ be the variety of commutative semigroups, and let $A$ be the multiplicative semigroup of positive integers. Assume, without loss of generality, that we have $S = \{1, 2, \ldots, n\}$ and that $V = \{1, 2, \ldots, m\}$, with $m \geq 2$. 

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Let $x_i$ be the $i$-th prime number for $1 \leq i \leq n$, and let $B$ be the subsemigroup consisting of all multiples of $M = x_1 \cdots x_m$. Finally, let

$$W(y_1, \ldots, y_n) = y_1 \cdots y_n.$$ 

We claim that the collection of subsets of $S$ which are transferable over $B$ with respect to $W$ and the tuple $(x_s)$ are precisely those subsets of $S$ which contain $V$, together with the empty set.

First, we note that $\text{dom}^V_A(B) = B$. Indeed, consider the canonical map from $A$ to the integers modulo $M$, and compare it with the zero map. Both maps agree on $B$, but disagree everywhere else. So the dominion of $B$ must be contained in $B$, hence is equal to $B$.

Next, note that the amalgamated coproduct $A \amalg_V B A$ may be described as a quotient of $A \times A$ by the congruence relation that identifies elements of $B$.

Suppose that $T \subseteq S$ is transferable and nonempty. Therefore, for every partition $S = S_1 \amalg T \amalg S_2$ we have $W_{S_1 \cup T, S_2}(x_s) = W_{S_1, T \cup S_2}(x_s)$. In $A \times A$, this corresponds to saying that the elements

$$\left( \prod_{i \in S_1 \cup T} x_i, \prod_{j \in S_2} x_j \right) \quad \text{and} \quad \left( \prod_{i \in S_1} x_i, \prod_{j \in T \cup S_2} x_j \right)$$

map onto the same element in $A \amalg_V B A$. But, since $A$ is a cancellation semigroup, this is equivalent to saying that the elements

$$\left( \prod_{i \in T} x_i, 1 \right) \quad \text{and} \quad \left( 1, \prod_{j \in T} x_j \right)$$

map onto the same element in $A \amalg_V B A$. This in turn is equivalent to saying that $\prod_{i \in T} x_i \in B$, which is true if and only if $V \subseteq T$ or $T = \emptyset$. Therefore, the collection of sets which are transferable over $B$ is exactly equal to $\mathcal{T}(V)$.

That $\mathcal{T}(V)$ is also a transfer system now follows. \hfill \qed

Finally, we present a pre-transfer system which is not a transfer system.

**Example 4.21.** Let $S = \{a, b, c, d, e\}$, and let $\mathcal{T}$ be the following collection of subsets of $S$:

$$\mathcal{T} = \{ \emptyset, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}, \{a, b, e\} \}.$$ 

It is not hard to verify that $\mathcal{T}$ satisfies both (i) and (ii), and that (iii) is true vacuously. Hence, $\mathcal{T}$ is a pre-transfer system on $S$. 

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To see that it is not a transfer system, let \( \sim \) be the least equivalence relation on subsets of \( S \) such that for every \( T \in \mathcal{T} \), and for every \( U \subseteq S \setminus T \), \( U \sim U \cup T \).

It is not hard to verify that under this equivalence, \( U \sim V \) if and only if it is possible to obtain \( V \) from \( U \) by successively taking disjoint unions with sets in \( \mathcal{T} \) and taking proper differences with sets in \( \mathcal{T} \).

Let \( \mathcal{T}^* \) be the transfer system on \( S \) associated to \( \sim \). It will suffice to show that \( S \in \mathcal{T}^* \). Showing that \( S \) is in \( \mathcal{T}^* \) is equivalent to showing that \( \emptyset \sim S \), and this is indeed the case:

\[
\begin{align*}
\emptyset & \sim \{a, b, c, d\} \quad \text{(since \( \{a, b, c, d\} \in \mathcal{T} \))} \\
& \sim \{d\} \quad \text{(since \( \{a, b, c\} \in \mathcal{T} \))} \\
& \sim \{a, b, d, e\} \quad \text{(since \( \{a, b, e\} \in \mathcal{T} \))} \\
& \sim \{e\} \quad \text{(since \( \{a, b, d\} \in \mathcal{T} \))} \\
& \sim \{a, b, c, d, e\} \quad \text{(since \( \{a, b, c, d\} \in \mathcal{T} \))}
\end{align*}
\]

so \( \emptyset \sim S \), hence \( S \in \mathcal{T}^* \), which shows that \( \mathcal{T} \) is not a transfer system.

At the moment, I do not know if the first implication is reversible, or if there are any additional “local” conditions on a pre-transfer system that will make it a transfer system. Therefore I leave the following open questions:

**Question 4.22.** Is the first implication above reversible? That is, does every transfer system come from dominions?

**Question 4.23.** What is an axiomatization of the notion of transfer system? Equivalently, what additional properties on a pre-transfer system will make it a transfer system?

Finally, let us note some questions directly related to the concept of dominions. Note that in Proposition 3.11, we defined a subalgebra \( B^* \) of the dominion of \( B \). It is not hard to verify, using slight variations of Example 3.10 and the classification of dominions in \( N_2 \) (see Theorem 3.29 in [3]), that in the case of the variety of groups \( N_2 \), the subgroup \( B^* \) is actually equal to the dominion of \( B \). It would be interesting if this holds in general, so we ask:

**Question 4.24.** Does the subalgebra \( B^* \) defined in Proposition 3.11 always equal \( \text{dom}_{N_2}(B) \)? If not, what are the conditions under which equality will hold? Alternatively, is there some class of varieties in which it will hold?

**References**

[1] Isbell, J. R. *Epimorphisms and dominions* in *Proc. of the Conference on Categorical Algebra, La Jolla 1965*, pp. 232–246. Lange and Springer, New York 1966. MR:35#105a (The statement of the Zigzag Lemma for rings in this paper is incorrect. The correct version is stated in [2]).
A lemma on words

[2] Isbell, J. R. *Epimorphisms and dominions IV*. Journal London Math. Society (2), 1 (1969) pp. 265–273. MR:41#1774

[3] Magidin, Arturo. *Dominions in varieties of nilpotent groups*. In preparation.

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