Spin-Charge Separation in the $t-J$ Model:
Magnetic and Transport Anomalies

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Abstract

A real spin-charge separation scheme is found based on a saddle-point state of the $t-J$ model. In one-dimensional (1D) case, such a saddle-point reproduces the correct asymptotic correlations at the strong-coupling fixed-point of the model. In two-dimensional (2D) case, the transverse gauge field confining spinon and holon is shown to be gapped at finite doping so that a spin-charge deconfinement is obtained for its first time in 2D. The gap in the gauge fluctuation disappears at half-filling limit, where a long-range antiferromagnetic order is recovered at zero temperature and spinons become confined. The most interesting features of spin dynamics and transport are exhibited at finite doping where exotic residual couplings between spin and charge degrees of freedom lead to systematic anomalies with regard to a Fermi-liquid system. In spin dynamics, a commensurate antiferromagnetic fluctuation with a small, doping-dependent energy scale is found, which is characterized in momentum space by a Gaussian peak at $(\pi/a, \pi/a)$ with a doping-dependent width ($\propto \sqrt{\delta}$, $\delta$ is the doping concentration). This commensurate magnetic fluctuation contributes a non-Korringa behavior for NMR spin-lattice relaxation rate. There also exits a characteristic temperature scale below which a
pseudo-gap behavior appears in the spin dynamics. Furthermore, an incommensurate magnetic fluctuation is also obtained at a finite energy regime. In transport, a strong short-range phase-interference leads to an effective holon Lagrangian which can give rise to a series of interesting phenomena including linear-$T$ resistivity and $T^2$ Hall-angle. We discuss the striking similarities of these theoretical features with those found in the high-$T_c$ cuprates and give a consistent picture for the latter. Electronic properties like Fermi-surface and superconducting pairing in this framework are also discussed.

71.27.+a, 74.20.Mn, 74.72.-h, 75.10.Jm
I. INTRODUCTION

The normal state of high-$T_c$ cuprate superconductors has shown peculiar properties in both charge and spin channels. In transport aspect, the resistivity exhibits a linear-temperature dependence up to 1000 K and down to a temperature $\simeq T_c$, which can be as low as 10 K. This temperature dependence has been related to a scattering rate that behaves like $\eta \frac{k_B T}{\hbar}$ with $\eta \sim 2$ for all the optimally doped cuprates with $T_c$ ranging from 10 up to over 100 K (a linear-frequency dependence of the scattering rate at frequency $> k_B T / \hbar$ is also found in the infrared spectroscopy up to 0.15 eV). Such a linear-$T$ longitudinal resistivity is also accompanied by a Hall coefficient which implies hole-like charge carriers and shows a $1/T$ dependence in contrast to a $T$-independent Fermi liquid case. A Hall-angle experimental analysis has demonstrated that the $1/T$ behavior in the Hall coefficient is due to an additional scattering rate in the transverse channel, which behaves like $T^2$. Most recently, the magneto-resistance has been found to have a $T^{-4}$ temperature dependence, at variance with the Kohler’s rule. Furthermore, the thermopower has shown a strong doping-dependence which decreases with the increase of doping and even changes sign in the overdoped regime. All of these transport results are anomalous with regard to the canonical phenomena in a conventional Fermi liquid (FL) system.

Spin magnetic properties have also exhibited a number of anomalies, which persist into the superconducting phases. Two powerful probes of spin dynamics in the cuprates are nuclear-magnetic-resonance (NMR) and neutron-scattering. In the cuprates, NMR spin-lattice relaxation rate $T_1^{-1}$, which probes a very small energy scale ($\sim 10^{-4} \text{ meV}$), has shown a strong enhancement at low temperature for planar $Cu$ nuclei, in contrast to the usual Korringa behavior in a Fermi liquid. On the other hand, $T_1^{-1}$ for planar $O$ nuclei is more or less close to the Korringa law. The sharp contrast of the NMR $T_1^{-1}$ between the planar $Cu$ and $O$ nuclei strongly suggests that the non-Korringa behavior at $Cu$ sites should be caused by antiferromagnetic (AF) correlations among the $Cu$ spins whose effect could be effectively cancelled out at $O$ sites. AF spin fluctuations have been also directly measured
by the inelastic neutron-scattering experiments\textsuperscript{9} at higher energies in $La_{2-x}[Sr, Ba]_xCuO_4$ ($LSCO$) [Ref. \textsuperscript{16}–\textsuperscript{18}] and $YBa_2Cu_3O_{7-\delta}$ ($YBCO$) [Ref. \textsuperscript{19}–\textsuperscript{21}] materials. However, in contrast with more or less universal behaviors in NMR spin relaxation rates, neutron-scattering data have shown distinctive characteristics among these compounds. For example, commensurate AF fluctuations are observed in the insulating $LSCO$ systems\textsuperscript{16,17} and underdoped $YBCO$ compounds\textsuperscript{19,20} with characteristic energy scales much smaller than the exchange energy $J \simeq 120$ $meV$. But incommensurate AF correlations have been found\textsuperscript{18} in the metallic $LSCO$ down to an energy scale as low as 1 $meV$. Nevertheless, recent analyses have revealed\textsuperscript{22,23} that the NMR relaxation rates in $LSCO$ are inconsistent with an incommensurate AF structure at $\omega \simeq 0$. This is because the effect of incommensurate fluctuations could not be well cancelled out at $O$ sites, and a large non-Korringa signal would leak to the latter, as in contradiction to experiment. In the optimally-doped $YBCO$, the absence of low-energy AF fluctuation in neutron-scattering experiment\textsuperscript{21} is also inconsistent with the NMR measurements\textsuperscript{8}. Thus a consistent explanation of the NMR data and a reconciliation of NMR and neutron-scattering are two essential issues in spin dynamics.

Given these anomalous transport and magnetic properties, one would naturally question if a single type of elementary excitations carrying both charge and spin can explain them without leading to any intrinsic contradiction. Since the sharp Fermi edge has been well identified in the cuprates by the angle-resolved photo-emission measurements\textsuperscript{24,25}, such low-lying electron-like excitations, if exist, should always locate near the Fermi surface, no matter whether it is a Fermi liquid or a marginal Fermi liquid (MFL) [Ref. \textsuperscript{26}] system. The main difference between FL and MFL lies in their energy spaces (the broadening of the quasi-particle in MFL is linear in energy\textsuperscript{24} as compared to $\omega^2$ in FL case). But in the phase space (momentum space), they are facing essentially the same problems\textsuperscript{27} Low-lying states as labelled by momenta are uniformly distributed in momentum space around Fermi surface. The only conceivable structure would come from the topology of Fermi surface. The question is whether such a structure of Fermi surface (maybe with nesting and van Hove singularity) is capable of explaining an army of abundant experimental anomalies.
transport channel, one of the key problems is how to get hole-type charge carriers, since the elementary excitation is of electron-type. Even though a special curvature of Fermi surface may explain the hole-like sign indicated by the Hall effect, the total charge carrier number could by no means coincide with the hole number as measured with regard to the half-filled insulating parent compound. Many other problems could arise within such a framework. For example, a monotonic decrease of the thermopower from over $100 \mu V/K$ down to 0 and even changing sign, when doping is increased from zero to an overdoped level, is difficult to understand here because the phase-space is too limited. Even with the presence of a van Hove singularity, an overall change as estimated by theory ($\sim 8 \mu V/K$ [Ref. 28]) is too small as compared to the experimental value. Without additional structure, a second scattering rate implied in the Hall-angle measurements is even more difficult to comprehend, especially the different ways by which it gets into the transverse and longitudinal channels.

In the spin channel, a number of FL and MFL theories have been proposed in terms of the geometry of Fermi surface. The topology of Fermi surface determines essential characteristics of spin dynamics through the Lindhard-type response function. These theories provide an explanation of incommensurate AF structure in the metallic LSCO, but fail to account for other equally important issues. One is about the NMR relaxation rates whose aforementioned distinctive behaviors at planar Cu and O nuclei are difficult to reconcile. The fundamental reason is that a Lindhard response function cannot produce a sharp enough AF peak near ($\pi/a$, $\pi/a$). Furthermore, the overall width of the AF peak in these theories is generally not sensitive to doping concentration, and one cannot find NMR anomalies at low temperature to be sufficiently enhanced with the decrease of doping as indicated by the experiments. Most seriously, the itinerant picture fails to provide a small AF energy scale, as the only characteristic scale is the Fermi energy $\epsilon_F$. Although a peak at small energy may be introduced by van Hove singularity, it is hard to be related to a magnetic energy. The exchange energy scale $\sim J$ as indicated in various magnetic measurements, and an even smaller doping-dependent magnetic energy scale found by inelastic neutron-scattering is a strong indication that the local spin description of AF fluctuation may be more ap-
propriate than an itinerant picture. This is further supported by the high temperature $T^{-1}$ measurement in LSCO [Ref. 30] which shows that doping effect is basically evaporated when $T > 600K$, where $T^{-1}$ at finite doping coincides with that of an insulating antiferromagnet, described by the localized spins under Heisenberg model.  

Therefore, the phase-space limitation in an itinerant description, which becomes inevitable in a system where fermionic elementary excitations carry both spin and charge, prevents a consistent explanation of rich phenomena in the high-$T_c$ cuprates. This implies that an electron in the cuprates may be a composite particle, consisting of more elementary excitations which carry spin and charge quanta (spinon and holon) separately. In this way, spinon and holon may find enough phase-spaces of their own to account for experimental anomalies. A spin-charge separation scenario for the cuprates has been first proposed by Anderson. Under this idea, a strong on-site Coloumb repulsion can lead to a split of energy band with a large Hubbard gap, and thus real charge carriers (holons) in the conduction band will have equal number as that of doped holes controlled by experiments. It is a significant first step towards understanding the transport properties. However, in order to reconcile with a sharp Fermi edge found by the angle-resolved photo-emission, a fermionic spinon with a similar Fermi surface has to be assumed. Thus at least in the spin channel, basic properties should not be very different from conventional FL or MFL theory, which is a serious set back. Later it has been further realized that there actually exists a strong coupling among spinons and holons, known as gauge interaction, and such a gauge force plays a role essentially to confine spinon and honon together, like the quark confinement in quantum chromodynamics. In other words, whether there is a real spin-charge separation in a 2D strongly-correlated model is still unclear.

So we are in a paradox situation. Experimental measurements, on one hand, point to a possibility of spin-charge separation. Strong-correlation theories, on the other hand, lead to spin-charge confinement. This awkward situation is actually caused by an oversimplified choice to let spinons to have a large Fermi surface, which undermines a local description
for spins. One may argue that in order to recover a correct electron Fermi surface—an experimental constraint—a large spinon Fermi surface would be necessary. But this is not the case even in one dimension, where a real spin-charge separation exists in the Hubbard model and an electron Fermi surface (points) still satisfies the Luttinger volume theorem. Even though usually people may be used to the thinking that a spinon Fermi surface is the reason for an electron Fermi surface in 1D, the various correlations have never been correctly derived under such a picture. In fact, a FL description of spinons is a too rough approximation in 1D. In the strong-coupling (large $U$) regime, a correct spin-charge separation description has been established in a path-integral formalism, where an electron is described as a composite particle of a spinon and a holon, together with a nonlocal phase-shift field. It is this phase-shift field that helps to recover the right Fermi surface position. In this formalism, spinons are described by a local spin representation without a Fermi surface, and both spin and density, as well as various electronic correlation functions have been correctly obtained. An important lesson learned here is that the electronic Fermi surface is no longer essential in a spin-charge separation scheme, and it can be reproduced from a phase-shift in a correct spin and charge separation scheme. (We note that in 1D another powerful method is the bosonization method in weak-coupling regime. There the non-Fermi-liquid like low-lying spin and charge processes can be directly described near the electronic Fermi points. But its possible generalization to two dimensions has many difficulties and is still under investigation.)

This leads to a new 2D spin-charge scenario for the cuprates. In this scenario, spinons would be free of “duty” to be responsible for electron Fermi surface, so that they could get sufficient phase-space freedom to describe anomalous spin dynamics. Electronic Fermi surface would be produced by extra phase-shift field associated with the fermion-statistics of electron, and should not be directly related to holon and spinon. This provides a fundamental reason for the Luttinger theorem to be valid even in strong correlations. It also means a Fermi-surface topology is no longer crucial in determining spin and charge dynamics. That is, the shape of a Fermi surface due to the detailed band structure, which may vary from
one material to others in the cuprates, should not be so relevant to the basic spin and charge anomalies in a spin-charge separation framework. Therefore it may well justify using a simple one-band $t - J$ model to describe the realistic cuprates if a spin-charge separation is indeed present there. It is noted that several interesting experimental features have been already known for the $t - J$ model. For example, spin degree of freedom is described in a local spin representation, which reduces to the Heisenberg Hamiltonian at half-filling and well describes the magnetic behaviors in the insulating cuprates. On the other hand, charge carriers in this system are naturally holes as measured from the half-filled insulating phase, which is also a strong indication of the experimental relevance for the model. A spin-charge separation is already well-known for such a model in 1D case.

It is the purpose of the present paper to establish the above-described spin-charge separation scheme within the $t - J$ framework. Like other approximations, we cannot directly prove that this 2D spin-charge separation state is the solution of the $t - J$ model. But such a state will satisfy the following important criteria. The transverse gauge fluctuation will be found suppressed in the long-wavelength and low-energy regime so that spinon and holon are indeed deconfined at finite doping (i.e., spin-charge separation). This is in contrast to singular infrared gauge fluctuation in the case of uniform RVB state. At zero doping limit, the gap of the gauge fluctuation will vanish such that the spinons become confined again to form spin-wave excitations. In this case, a long-range AF order will emerge at zero temperature. Furthermore, the present state can also reproduce the correct results in 1D case, which is an important check of the theory because only in 1D one has exact solution.

In the present paper, the effective Hamiltonian describing the spin-charge separation saddle-point will be derived in the following form:

$$H_{\text{eff}} = H_s + H_h,$$

(1.1)

where $H_s$ and $H_h$ are the spinon and holon Hamiltonians, respectively, defined by

$$H_s = -J_s \sum_{<ij>,\sigma} \left( e^{i\sigma A_{ij}^h} \right) b_{i\sigma}^+ b_{j\sigma} + h.c.,$$

(1.2)
\[ H_h = -t_h \sum_{<ij>} \left( e^{i(\phi_{ij}^0 + A_{ij}^h)} \right) h_i^+ h_j + h.c. \]  

(1.3)

Here \( b_{i\sigma} \) and \( h_i \) are called spinon and holon annihilation operators, respectively, which are connected to electron operator in Eq.(1.6) below. \( H_s \) and \( H_h \) in Eqs.(1.2) and (1.3) would resemble a standard tight-binding Hamiltonian if there are no phase fields \( A_{ij}^h, \phi_{ij}^0 \) and \( A_{ij}^s \) (\(<ij>\) refers to two nearest-neighboring sites on the lattice). Here \( \phi_{ij}^0 \) [as defined in Eq.(2.45)] represents a \( \pi \)-flux threading through each plaquette. A tight-binding model under \( \phi_{ij}^0 \) can be easily diagonalized. The most prominent feature in Eqs.(1.2) and (1.3) is the presence of \( A_{ij}^s \) and \( A_{ij}^h \), the so-called topological phases. \( A_{ij}^h \) is defined by

\[ A_{ij}^h = \frac{1}{2} \sum_{l \neq i,j} \text{Im} \ln \left( \frac{z_i - z_l}{z_j - z_l} \right) n_l^h, \]  

(1.4)

with complex coordinate \( z_i = x_i + iy_i \) in 2D, and \( n_l^h = h_i^+ h_l \) as the holon number operator. \( A_{ij}^s \) is given by

\[ A_{ij}^s = \frac{1}{2} \sum_{l \neq i,j} \text{Im} \ln \left( \frac{z_i - z_l}{z_j - z_l} \right) \left( \sum_\sigma \sigma n_l^b \right), \]  

(1.5)

with \( n_l^b = b_{l\sigma}^+ b_{l\sigma} \) as the spinon number operator.

The usual gauge coupling, which otherwise would confine spinon and holon together, will be shown suppressed and has been neglected in Eq.(1.1). Thus the residual interaction between spinon and holon are solely mediated through \( A_{ij}^s \) and \( A_{ij}^h \) in Eqs.(1.2) and (1.3). In 1D case, the complex coordinate \( z_i \) is reduced to an 1D variable, and it is easy to show that \( A_{ij}^h \) and \( A_{ij}^s \) vanish in Eqs.(1.4) and (1.5). Thus spinon and holon are decoupled in 1D, and behave just like free-particles on their own tight-binding lattices. In 2D case, however, \( A_{ij}^h \) and \( A_{ij}^s \) can no longer be gauged away. \( A_{ij}^h \) in Eq.(1.4) describes fictitious \( \pi \)-flux quanta attached to holons which are seen only by spinons in \( H_s \). Hence \( A_{ij}^h \) will play the role to introduce doping effect into spin degree of freedom. On the other hand, \( A_{ij}^s \) represents \( \pi \)-flux quanta bound to spinons which can be only seen by holons. \( A_{ij}^s \) will then play the role of a scattering source in holon transport. So in the spin-charge separation scheme, new types of scatterings are present, and we will show in this paper that it is due to these unconventional
forces that magnetic and transport anomalies are produced in consistence with those found in the high-$T_c$ cuprates.

Finally, let us outline how an electron is composed of the elementary excitations, holon and spinon, in this scheme. For both 1D and 2D, the electron annihilation operator $c_{i\sigma}$ will be rewritten as

$$c_{i\sigma} = h_i^+b_{i\sigma} \left[ e^{-\frac{i}{2} \sum_{l \neq i} \theta(l)(\sigma n_{i\sigma}^l - \sum_\alpha \alpha n_{i\alpha}^l + 1)} (-\sigma)_i \right], \quad (1.6)$$

where $\theta_i(l)$ is defined in Eqs. (2.19) and (2.20). Decomposition (1.6) can be understood as that annihilating an electron is equivalent to creating a holon and annihilating a spinon, and, at the same time, inducing overall nonlocal phase-shifts [in the brackets of Eq.(1.6)]. There are several ways to interpret the involvement of phase shift fields in Eq.(1.6). Since the nonlocal fields appearing in Eq.(1.6) resemble the Jordan-Wigner transformation in both 1D and 2D cases[54,55] one may interpret them as statistical transmutations. Here $h_i^+$ is a hard-core bosonic operator, and $b_{i\sigma}$ also satisfies bosonic commutation relation for the same spin index $\sigma$ but anticommutation relation for opposite spins [for details see Sec. II]. Thus these nonlocal fields in Eq.(1.6) are to guarantee $c_{i\sigma}$ obeying an electronic commutation relation. A deeper physics behind this is related to the phase-shift idea, whose important role in a strongly-correlated system has been first realized by Anderson[32,41,47] In 1D it has been explicitly shown[56] that an overall adjustment of the system occurs when a hole is doped into a large-$U$ Hubbard chain, in order to retain the Marshall sign rule[57] in the spin degree of freedom which is decoupled from the charge degree of freedom. This adjustment is found to be just represented by a phase-shift shown in Eq.(1.6), and it can lead to the correct Luttinger liquid behavior of the single electron Green’s function. Furthermore, a phase-shift in 1D can also be interpreted as reflecting the fact that each holon carries a spin domain wall [see Ref. [52] and Sec.III]. Here one important distinction from Anderson’s original phase-shift idea is that the many-body phase-shift in 1D will not only give rise to the right non-Fermi liquid behavior, but also shift the Fermi surface to the position satisfying the Luttinger theorem. In other words, there is no need to assume a right Fermi surface at
the very beginning. In the present scheme, the form of Eq.(1.6) can ensure a large electron Fermi surface for both 1D and 2D. But we shall discuss these electronic properties in a separated paper for the sake of clarity. In that same paper, we also show that when holons and spinons are both Bose-condensed, the phase-shift field in Eq.(1.6) will lead to a long-range \textit{pairing} correlation of electrons (i.e., off-diagonal long-range order), and determine the symmetry of the superconducting order parameter. In the present paper, we mainly focus on the spin-charge separation formalism and explore the corresponding spin dynamics and transport properties.

The remainder of the paper will be organized as the following. In section II, we construct the above-described spin-charge separation scheme based on the \( t - J \) model. At half-filling limit, a long-range AF order can be recovered in 2D, while the well-known spin-spin correlation is also obtained in 1D. At finite doping, the gauge fluctuation is shown to be suppressed so that one has a real spin-charge separation. In section III, the spin dynamics in this scheme is studied at finite doping. In 1D, the correct spin-correlation is to be reproduced. In 2D, a number of interesting properties are discovered which provide a consistent picture for the anomalous spin properties in the cuprates. In section IV, the transport properties is investigated. We show that there exists an interesting scattering mechanism within the present framework which leads an effective long-wavelength, low-energy Lagrangian. Such a Lagrangian can give rise to a series of exotic transport properties, in excellent agreement with the experimental measurements. Finally, a conclusive summery will be given in section V.

**II. SPIN-CHARGE SEPARATION: FORMALISM**

In this section, a mathematical formalism of the spin-charge separation state will be constructed based on the \( t - J \) model.
A. The slave-boson formalism and mean-field states

According to Anderson, Zhang and Rice, low-energy physics in the cuprates layers may be properly described by a single band $t - J$ model. The $t$-$J$ Hamiltonian reads

$$H_{t-J} = H_t + H_J,$$

(2.1)

with a hopping term

$$H_t = -t \sum_{\langle ij \rangle \sigma} c_{i\sigma}^+ c_{j\sigma} + H.c.,$$

(2.2)

and a superexchange term

$$H_J = J \sum_{\langle ij \rangle} \left( S_i \cdot S_j - \frac{n_i n_j}{4} \right),$$

(2.3)

in which $S_i = \frac{1}{2} \sum_{\sigma} c_{i\sigma}^+ (\hat{\sigma})_{\sigma\sigma'} c_{i\sigma'}$ ( $\hat{\sigma}$ is the Pauli matrix), and $n_i = \sum_{\sigma} c_{i\sigma}^+ c_{i\sigma}$. The Hilbert space of the Hamiltonian Eq.(2.1) is restricted by the no-double-occupancy constraint

$$\sum_{\sigma} c_{i\sigma}^+ c_{i\sigma} \leq 1,$$

(2.4)

which imposes a strong correlation on electrons.

By using the slave-boson decomposition of the electron operator,

$$c_{i\sigma} = h_{i\sigma}^+ f_{i\sigma},$$

(2.5)

the no-double-occupancy constraint (2.4) can be replaced by an equality constraint

$$h_{i\sigma}^+ h_{i\sigma} + \sum_{\sigma} f_{i\sigma}^+ f_{i\sigma} = 1,$$

(2.6)

where $h_{i\sigma}^+$ is a bosonic creation operator and $f_{i\sigma}$ is a fermionic annihilation operator, known as holon and spinon operators, respectively. In this formulation, the hopping and exchange terms of the $t - J$ Hamiltonian become

$$H_t = -t \sum_{\langle ij \rangle \sigma} h_{j\sigma}^+ h_{i\sigma} f_{i\sigma}^+ f_{j\sigma} + H.c.,$$

(2.7)

and

$$H_J = J \sum_{\langle ij \rangle} \left( S_i \cdot S_j - \frac{n_i n_j}{4} \right).$$

(2.8)
\[ H_J = -\frac{J}{2} \sum_{<ij>,\sigma,\sigma'} f_{i\sigma}^+ f_{j\sigma} f_{j\sigma'}^+ f_{i\sigma'}. \]  

(2.8)

In obtaining Eq.(2.8), a term \( \propto J\delta^2 \) (\( \delta \) is the doping concentration) has been neglected as usual for simplicity.

The advantage of using the slave-particle formulation for the \( t-J \) model is that once the initial state satisfies the constraint (2.6), the no-double-occupancy condition can be always preserved under the Hamiltonian (2.7) and (2.8). Thus the constraint (2.6) in the slave-particle representation will not play a crucial role as the constraint (2.4) does in the original Hamiltonians (2.2) and (2.3). Furthermore, in the slave-boson formulation the natural mean-field decouplings usually have simpler structure at finite doping than those in the so-called slave-fermion, Schwinger-boson formalism.

Various RVB-type mean-field states have been obtained in the slave-boson formulation. The mean-field decoupling of the Hamiltonians (2.7) and (2.8) may be realized by introducing the following mean-fields

\[ \chi_{ij}^{\sigma} = \langle f_{i\sigma'}^+ f_{j\sigma} \rangle; \]  

(2.9)

\[ H_{ij} = \langle h_{i}^+ h_{j} \rangle. \]  

(2.10)

By neglecting higher order fluctuations around \( \chi_{ij}^{\sigma} \) and \( H_{ij} \), the mean-field versions of \( H_{t-J} \) can be obtained in the form of

\[ H_{t-J}^{MF} = H_{h}^{MF} + H_{s}^{MF}, \]

where

\[ H_{h}^{MF} = -t \sum_{<ij>,\sigma} \chi_{ij}^{\sigma} h_{i}^+ h_{j} + H.c.; \]  

(2.11)

\[ H_{s}^{MF} = -\frac{J}{2} \sum_{<ij>,\sigma,\sigma'} (\chi_{ij}^{\sigma'} + \frac{t}{J} H_{ji}) f_{i\sigma}^+ f_{j\sigma} + H.c. \]  

(2.12)

The constraint (2.6) should be simultaneously relaxed at this level to a mean-field one \( \langle h_{i}^+ h_{i} \rangle + \sum_{\sigma} \langle f_{i\sigma}^+ f_{i\sigma} \rangle = 1 \).

As shown by Eqs.(2.11) and (2.12), both \( H_{h}^{MF} \) and \( H_{s}^{MF} \) have similar hopping forms decided basically by one mean-field \( \chi_{ij}^{\sigma} \) (usually \( H_{ij} \propto \chi_{ij}^{\sigma} \) at the saddle-point). Since
\( \chi^\sigma_{ij} \) is responsible for both the hopping and antiferromagnetic superexchange strengths in Eqs.(2.11) and (2.12), an optimization of these two competing charge and spin processes may be relatively easy within this framework.

A simplest mean-field state can be obtained by choosing \( \chi^\sigma_{ij} = \chi_0 \) and \( H_{ij} = H_0 \), known as the uniform RVB state.\(^{22}\) Such a state recently has attracted intensive attention, as one has been able to go beyond the mean-field (or saddle-point) approximation by including the phase fluctuations of \( \chi^\sigma_{ij} \) and \( H_{ij} \) in terms of the gauge-field description.\(^{35-37}\) A lesson learned from the gauge theory approach is that a singular transverse gauge fluctuation could become very important, which implies that spinon and holon are usually confined by gauge field, in opposing to the artificial decomposition of electron operator in scheme (2.5). The non-conventional transport phenomena have been discussed\(^{36,37}\) in this framework which are both theoretically and experimentally interesting.

The uniform RVB state is presumably energetically favorable at a sufficiently large doping. On the other hand, when it is close to half-filling and the doping concentration \( \delta \) is low, the uniform RVB state may not be an appropriate saddle-point state because the antiferromagnetic energy could be underestimated. For example, at half-filling limit with \( \delta \to 0 \), the variational superexchange energy of the uniform RVB state, which can be computed by using the variational Monte Carlo (VMC) method with the no-double-occupancy condition (2.6) being exactly implemented, is found\(^{23}\) to be \(-0.53J\) (per site), about 20\% higher than the best ground state energy \(-0.6692J\) [Ref. \(^{14}\)]. Furthermore, no AF long-range order would appear in this limit as it should be present in the true ground state of \( H_{t-J} \).

A substantial improvement of the variational ground-state energy at half-filling can be realized by introducing a phase to \( \chi^\sigma_{ij} \) in Eq.(2.9), i.e.,

\[
\chi^\sigma_{ij} = \chi_0 e^{i \theta_{ij}}. \tag{2.13}
\]

(Note that \( H_{ij} = 0 \) at \( \delta = 0 \).) Here the phase \( \theta_{ij} \) can not be simply reduced to a difference like \( \theta_i - \theta_j \), or in other words, a summation of \( \theta_{ij} \) along the closed bonds of a plaquette,

\[
\Phi_{\square} = \sum_{\square} \theta_{ij} \tag{2.14}
\]
is none-zero. $\Phi_\square$ may be regarded as some fictitious flux threading through the plaquette. At half-filling, the variational energy is optimized at $\Phi_\square = \pi$, known as the $\pi$-flux phase\cite{Ref. 28}, whose VMC value of the exchange energy is $-0.623J$ [Ref. 23], only 5% higher than the exact ground-state energy. However, the absence of a long-range AF order in the $\pi$-flux phase implies that the long-wavelength, low-energy AF correlations are still underestimated which should predominantly account for the 5% energy missing at this saddle-point state. On the other hand, the high-energy properties of the Heisenberg model, like the temperature dependence of the specific heat may be well explained in terms of the $\pi$-flux phase, as discussed in Ref. 65 (In Ref. 65, the $\pi$-flux phase is obtained from a different formulation, involving a 2D Jordan-Wigner transformation\cite{Ref. 55} of the spin-1 operators, instead of the spin-$1/2$ operators used in the present framework.)

Thus, by comparing with the uniform RVB state at half-filling, the short-range AF correlation in the $\pi$-flux phase has been substantially improved as a result that the transverse gauge fluctuation in the former is condensed as the static uniform flux in the latter case. Nevertheless, lack of long-range AF order in the latter means that the long-range AF correlation is still not appropriately accounted here. In the following, we will consider a new type of saddle-point states which could retain the high-energy characteristics of the $\pi$-flux phase, while properly include the long-wavelength AF correlation in low-energy regime.

### B. Flux binding: new saddle-point

In the $\pi$-flux phase discussed above, the mean-field Hamiltonian describes a non-interacting fermion gas under some uniform fictitious magnetic field. However, according to the exact diagonalizations on small lattice\cite{Ref. 66} such a type of states is generally not energetically favorable as compared to a state in which the uniformly-distributed lattice flux is quantized into infinitesimal-size flux tubes, and each of them is bound to an individual particle. The former may be regarded as a “mean-field” version of the latter. In general, the latter is known as an anyon system\cite{Ref. 67} as the new composites of particles and flux-tubes
obey different statistics, depending on the flux strength of each flux-tube, as compared to the statistics of the underline particles.

This numerical result is very suggestive here, as in the slave-boson formalism there exists a gauge (phase) degree of freedom which can allow the above flux binding procedure to happen if it is indeed energetically favorable. This gauge freedom is manifested in the decomposition (2.5), where one may always associate phases $e^{i\theta}$ and $e^{-i\theta}$ to $h_i^+$ and $f_{i\sigma}$, respectively, without changing $c_{i\sigma}$. Such a freedom reflects the arbitrariness of the decomposition (2.5). A gauge field will then play a role to confine any non-physical spinon and holon together as an electron. Only when one finds a correct spin-charge decomposition, the gauge fluctuation can be expected to get suppressed. Our strategy in the following is to optimize the flux phase by exploiting such a gauge freedom. Main procedure is to regard the uniform fictitious flux as a “mean-field” version of the system where fluxes are quantized and bound to the particles. This latter system may be generally called as a flux-binding state. Given the above-discussed uniform $\pi$-flux, there can be two ways to construct the corresponding flux-binding states at half-filling, as will be outlined below.

**Scheme one** All of $f_{i\sigma}$, no matter their spins, are bound to the same type of flux quanta. When these flux quanta are uniformly smeared out in space, one should recover the uniform flux in $\pi$-flux phase. At half-filling, the total number of spinons is one per plaquette on average. Thus each flux tube has to be quantized at $\pi$, such that

$$\sum_{\square} \theta_{ij} = \pi \sum_{l \in \square} \sum_{\sigma} n_{i\sigma}^l$$

with $n_{i\sigma}^l = f_{i\sigma}^+ f_{i\sigma}$. On average $< \sum_{\square} \theta_{ij} > = \pi$. In this scheme, since each flux-tube is quantized at $\pi$, spinons are effectively turned into semions (an exchange of two semions gives rise to a phase $\pm i$). **Scheme one** has been already discussed in detail in Ref. 68. Away from half-filling, the mean-field version of this flux-binding state corresponds to the so-called commensurate flux-phase (CFP) [Ref. 69]. And the gauge theories based on CFP has a close connection with such a flux-binding state. In both cases, superconducting condensation is found in the ground state, and is related to the semionic condensation. It has been shown
in the flux-binding state that there is no explicit time-reversal and parity symmetry broken due to the cancellation between charge and spin degrees of freedom. The most interesting features show up in the normal state, which can well explain the anomalous in-plane transport properties in cuprates, including resistivity, the Hall effect, and thermopower. But such a state still has trouble to recover the AF ordering at half-filling.

**Scheme two** In this scheme, the spinons with opposite spins may not see the flux-tubes carried by each other. Thus the flux quantum of each flux-tube has to be doubled in order to recover the \(\pi\)-flux phase in the mean-field limit. In this case, the phase \(\theta_{ij}\) as seen by the spinons with spin index \(\sigma\) has to satisfy the following condition

\[
\sum_\square \theta_{\sigma ij} = 2\pi \sum_{l \in \square} n_{l\sigma},
\]

such that \(\langle \sum_\square \theta_{\sigma ij} \rangle = \pi\) at half-filling. Equation (2.16) means that each spinon is bound to a \(2\pi\) flux-tube, which would effectively change the spinon statistics from fermion into (hard-core) boson, for spinons with the same spin index \(\sigma\). In this second flux-binding scheme, the experimental features of the transport properties found in **Scheme one** will be essentially retained, while interesting magnetic properties can be also obtained here. As an example, the AF long-range order can be recovered at half-filling. Furthermore, the correct 1D behaviors is to be reproduced naturally within this framework. All of these suggest that **Scheme two** may be a better approximation than **Scheme one** for the same physical state. Therefore, we shall fully focus on the saddle-point of **Scheme two** later, and explore its various properties in the remainder of the paper.

**1. Mathematical scheme: Half-filling**

At half-filling, \(H_t\) has no contribution. One may rewrite \(H_J\) in Eq.(2.8) in a form

\[
H_J = -\frac{J}{2} \sum_{<ij>\sigma\beta} \left( e^{iA_{ij}^\sigma} f_i^\dagger \sigma f_j^\dagger \sigma \right) \left( e^{iA_{ij}^\beta} f_j^\dagger \beta f_i^\dagger \beta \right),
\]

where a phase \(A_{ij}^\sigma\), which will play a role similar to \(\theta_{ij}^\sigma\) in Eq.(2.16), is introduced.
\[ A_{ij}^{\sigma} = \sigma \sum_{l \neq i,j} [\theta_i(l) - \theta_j(l)] \left( n_{i\sigma}^l - \delta_{\sigma,\uparrow} \right), \quad (2.18) \]

which depends on the positions of spinons nonlocally. This nonlocality is introduced through the multiple-valued phase \( \theta_i(l) \):

\[ \theta_i(l) = Im \ln(z_i - z_l), \quad (2.19) \]

with \( z_i \equiv x_i + iy_i \) in 2D. A vorticity \( \pm 2\pi \) can be obtained in \( \theta_i(l) \) if the coordinate \( z_i \) has been continuously changed around \( z_l \) once. In 1D case, \( \theta_i(l) \) in Eq.(2.19) will reduce to

\[ \theta_i(l) = \pm \pi \theta(l - i), \quad (2.20) \]

where \( \theta(x) \) on the right-hand-side is the step function. In obtaining Eq.(2.17), the constraint condition (2.6) has been used, i.e.,

\[ \sum_{\sigma} n_{i\sigma}^l = 1, \quad (2.21) \]

at \( \delta = 0 \).

The identical transformation in Eq.(2.17) has no real physical meaning until the mean-field decoupling scheme is introduced. One may define a hopping operator

\[ \hat{\chi}_{ij}^{\sigma} = e^{iA_{ij}^{\sigma}} f_{i\sigma}^{+} f_{j\sigma}. \quad (2.22) \]

Then under the mean-field \( \chi_0 = < \hat{\chi}_{ij}^{\sigma} > \), a saddle-point Hamiltonian of Eq.(2.17) is found as

\[ H_s = -J\chi_0 \sum_{<ij>\sigma} e^{iA_{ij}^{\sigma}} f_{i\sigma}^{+} f_{j\sigma} + H.c. \quad (2.23) \]

Now the nonlocal field \( A_{ij}^{\sigma} \) appears as a phase in the hopping matrix, which is equivalent to the role of \( \theta_{ij} \) in Sec.A.

*Two-dimensional case* \quad In terms of Eqs.(2.18) and (2.19), a summation of the phase \( A_{ij}^{\sigma} \) along a plaquette gives

\[ \sum_{\square} A_{ij}^{\sigma} = 2\pi \sigma \sum_{l \in \square} n_{i\sigma}^l - 2\pi \delta_{\sigma,\uparrow}. \quad (2.24) \]
Besides a \((-2\pi)\) lattice flux for \(\sigma = \uparrow\), which has no real physical meaning, the right-hand-side of Eq.\((2.24)\) implies that each spinon of spin \(\sigma\) carries a flux-tube quantized at \(2\pi\), with a sign \(\sigma = \pm 1\). Since \(<n^f_{i\sigma}> = 1/2\), one has \(\sum_{\Box} <A^\sigma_{ij}> = -\pi\). That is, the “mean-field” version of the saddle-point Hamiltonian \((2.23)\) just corresponds to a \(\pi\)-flux phase Hamiltonian as required (note that \(-\pi\) and \(\pi\) fluxes per plaquette are equivalent).

As pointed out before, the flux binding procedure discussed here is related to a statistics-transmutation which is easy to see after introducing a new operator \(\bar{b}_{i\sigma}\)

\[
\bar{b}_{i\sigma} = f_{i\sigma} e^{-i\sigma \sum_{l \neq i} \theta_l (n^f_l - \delta_{\sigma \uparrow})}.
\]

Then \(H_s\) is reduced to

\[
H_s = -J\chi_0 \sum_{<ij>,\sigma} (-\sigma) \bar{b}^+_{i\sigma} \bar{b}_{j\sigma} + H.c.,
\]

(2.26)

One can easily check that \(\bar{b}_{i\sigma}\) satisfies the hard-core boson commutation relations (for the same index \(\sigma\)): \([\bar{b}_{i\sigma}, \bar{b}^+_{j\sigma}] = 0\) \((i \neq j)\), etc. Of course, for opposite spins one still finds anti-commutation relations.

Equation \((2.25)\) resembles the Jordan-Wigner transformation in both 1D and 2D cases which changes the statistics of a fermion into that of a hard-core boson. Thus the presence of a uniform lattice flux in the \(\pi\)-flux phase is a precursor for a fermionic spinon to become a boson. Such a “bosonization” tendency for spinons in flux phase has been already noted before. Bosonic representation has been known generally to be superior in the description of spin AF correlations.

Equation \((2.26)\) can be further written in the form of a standard hopping Hamiltonian

\[
H_s = -J\chi_0 \sum_{<ij>,\sigma} b^+_{i\sigma} b_{j\sigma} + H.c.
\]

(2.27)

by redefining

\[
b_{i\sigma} = (-\sigma)^i \bar{b}_{i\sigma},
\]

(2.28)
in which \(\sigma = \pm 1\), and \((-1)^i = +1\) for the even-sublattice and \(-1\) for the odd-sublattice of a bipartite lattice. The spin operator like \(S^+_{i\uparrow} = f^+_i f_{i\downarrow}\) can then expressed in terms \(b_{i\sigma}\) as follows
\[ S_i^+ = (-1)^i b_i^+ b_i^- . \]  

One also has \( S_i^- = (S_i^+)^\dagger \), and \( S_i^z = \frac{1}{2} \sum_\sigma \sigma b_i^\sigma b_i^- \). 

The Bose-condensation of \( b_i^\sigma \) at zero-temperature as determined by Eq.(2.27) means 

\[ < S_i^+ > = (-1)^i < b_i^+ > < b_i^- > \neq 0 . \]  

In other words, an antiferromagnetic long-range order in the x-y plane can be indeed obtained in the present flux-binding scheme. The direction of the magnetization in x-y plane will be determined by the relative phase between the up and down spinons in Eq.(2.30), and could be arbitrary. The evaluation of the magnetization value in Eq.(2.30) needs a detailed knowledge of the hard-core boson behavior governed by \( H_s \). Since we do not know the exact solution of the hard-core boson Hamiltonian (2.27), we can not directly estimate the variational ground state energy of the Heisenberg Hamiltonian \( H_J \) in the present saddle-point state. Nevertheless, the ground state energy for a hard-core boson system like Eq.(2.27) is generally lower as compared to the uniform \( \pi \)-flux system in the fermion representation, which has been discussed by Gros et al. in Ref. 76. As pointed out by these authors, the ground state energy of a hard-core boson system described by \( H_s \) could be as lower as 11% than that in the \( \pi \)-flux-phase. Of course, one still needs to check the variational energy of the original Hamiltonian \( H_J \) in the present saddle-point state. By using a trial bosonic wavefunction for \( H_s \) given in Ref. 76, which is obtained by taking the absolute value of the known fermionic \( \pi \)-flux-phase wave function and gives 7.4% higher variational energy than the best estimate for \( H_s \), we found analytically the variational energy of \( H_J \) is just identical to that in the \( \pi \)-flux phase. The latter is already known from the VMC calculation, i.e., 5% higher than the exact value of the Heisenberg Hamiltonian. Since the above-mentioned bosonic trial wavefunction does not have a long-range order (Bose-condensation), one may attribute the 5% higher ground state energy to it. A better bosonic trial wavefunction of \( H_s \) should further improve the variational ground state energy of \( H_J \).

We note that beyond the present mean-field approximation, a gauge fluctuation should be considered. As will be discussed in Sec. C, such a gauge fluctuation is not gapped at
half-filling and thus could become very important. In fact, spin-1/2 spinons will presumably be confined by the gauge field to form spin-1 excitation (spin wave) at half-filling. This bosonic description is different from the Schwinger boson representation. The latter has been found to well describe the low-lying spin excitation in 2D half-filling, even at a mean-field level. In contrast, the present scheme will become more powerful in spinon deconfined cases. For instance, it will be shown later that the gauge fluctuation is to be suppressed at finite doping, and with the presence of spin-charge separation there, the present bosonic representation can become quite convenient to include doping effect. An another spinon deconfinement case is in 1D, which will be discussed below.

One dimensional case As a further check of the present saddle-point state, we now turn to the one dimensional case where the asymptotic spin-spin correlation of the Heisenberg chain has been known for many years.

Under the saddle-point Hamiltonian (2.27), one can write

\[
<S_i^+(t)S_j^-(0)> = (-1)^{i-j} <b_i^+(t)b_j^-(0)> <b_i^-(t)b_j^+(0)>
\]

(2.31)

In order to evaluate the averages on the right-hand-side, the behavior of the hard-core boson described by \(b_i(t)\) has to be determined first. A trick which has been often used in 1D is to transform the bosonic operator \(b_i\) into fermionic operator again. By using the expression (2.20) for \(\theta_i(l)\) in 1D, \(b_{i\sigma}\) is expressed in terms of \(f_{i\sigma}\) in Eqs.(2.25) and (2.28) as follows

\[
b_{i\sigma} = f_{i\sigma} e^{\pm i\pi \sum_{l > i} n_{l\sigma}}.
\]

(2.32)

One also finds that \(A_{ij}\) = 0 in Eq.(2.23), and thus in the fermionic representation \(H_s\) simply describes a free lattice fermion gas. Then the correlation function like \(<b_i^+(t)b_j^+(0)>\) in Eq.(2.31) can be straightforwardly calculated in long-time and -distance as follows

\[
< b_i^+(t)b_j^+(0) > = <f_i^+(t)e^{\pm i\pi \sum_{l > i} n^f_{l\sigma}(t)}e^{\mp i\pi \sum_{l < j} n^f_{l\sigma}(0)}f_j^+(0)>
\]

\[
\propto \frac{1}{[(x_i - x_j)^2 - v_f^2 t^2]^{1/4}},
\]

(2.33)

with \(v_f = J\chi_0a\). The average \(<b_i^-(t)b_j^+(0)>\) shows the same asymptotic form. Finally by using \((-1)^{i-j} = \cos \frac{\pi}{a}(x_i - x_j)\) and the rotational invariance, one obtains
\[
< S_i(t) \cdot S_j(0) > \propto \frac{\cos \frac{\pi}{a} (x_i - x_j)}{[(x_i - x_j)^2 - v_f^2t^2]^{1/2}}
\]

which describes the correct asymptotic spin-spin correlation in 1D and was first derived by Luther and Peschel from XXZ model.\textsuperscript{77}

Thus the present flux-binding saddle-point state has produced a correct result in 1D, even though it is originally constructed in 2D. This is in contrast to the Schwinger-boson approach,\textsuperscript{51} which works rather successfully for the 2D Heisenberg model, but could not predict the correct behavior for the 1D spin 1/2 case. This may be attributed to the fact that spin-1/2 excitations exist in 1D Heisenberg chain, but not in 2D. The present bosonic representation properly describes these spin-1/2 spinons.

We note that in the present approach the spin-spin correlation in \(z\)-axis can not be evaluated at the same level of approximation as in the \(x - y\) plane. The reason is that

\[
< S_i^z \cdot S_j^z > \sim \frac{1}{4} \sum_\sigma < n_i^b n_j^b >
\]

basically involves the density-density correlation of spinons with the same spin index, while \(< S_i^+ \cdot S_j^- >\) is related to the single-particle propagators of spinons under the saddle-point (2.27). The former is generally much harder to evaluate because of the hard-core condition of \(b_{i\sigma}\). The 1D correlation

\[
< S_i^z \cdot S_j^z > \sim (-1)^{i-j}/|x_i - x_j|
\]

could be obtained with the hard-core condition being accurately treated through the Jordan-Wigner transformation.

One final point we like to stress here is that \(H_s\) in Eq.(2.23) is formally the same as that in the uniform RVB saddle-point, since \(A_{ij}^\sigma = 0\) in 1D. In other words, the flux-binding, or the statistics-transmutation, has no effect on the Hamiltonian in 1D as is well known. What is different between these two saddle-point states is the boundary condition for the fermion \(f_{i\sigma}\): in the present saddle-point, \(f_{i\sigma}\) appearing in Eq.(2.32) will become multiple-valued at the boundary once the 1D chain is closed as a loop. As discussed by Shastry,\textsuperscript{79} such a boundary condition decides why a hard-core boson system is energetically more favorable than a fermionic gas.
2. Flux-binding: finite doping

For a small doping concentration, one expects the antiferromagnetic correlation still to remain dominant, even though it may be strongly modified by doped holes. It is very interesting to see the evolution of the flux-binding saddle-point state in this finite doping case.

We may follow the procedure shown in Sec. 1, starting with the identical transformation of $H_J$ in Eq.(2.17). At finite doping, the form of $A_{ij}^\sigma$ in Eq.(2.18) will be slightly modified because the constraint Eq.(2.21) now is replaced by the full equation (2.6). It is easy to show that $A_{ij}^\sigma$ is given as follows:

$$A_{ij}^\sigma = \sigma \sum_{l \neq i, j} [\theta_i(l) - \theta_j(l)] \left( n_{l\sigma}^f - \delta_{\sigma,\uparrow} + \frac{1}{2} n_{l\sigma}^h \right),$$

(2.35)

which keeps the identical transformation in Eq.(2.17) unchanged. In Eq.(2.35), an additional factor $n_{l\sigma}^h / 2$ is introduced, with $n_{l\sigma}^h = h_i^+ h_l$ as the holon number operator.

At finite doping, the hopping term $H_t$ in the $H_{t-J}$ will also contribute, and it may be rewritten in a form

$$H_t = -t \sum_{<ij>\sigma} e^{iA_{ij}^f} h_i^+ h_j\chi_{ij}^\sigma,$$

(2.36)

where $\chi_{ij}^\sigma$ is defined by Eq.(2.22) with $A_{ij}^\sigma$ given in Eq.(2.35). $A_{ij}^f$ in Eq.(2.36) is a re-arranged form of $A_{ij}^\sigma$ after using the constraint (2.6):

$$A_{ij}^f = \frac{1}{2} \sum_{l \neq i, j} [\theta_i(l) - \theta_j(l)] \left( \sum_{\sigma} \sigma n_{l\sigma}^f - 1 \right).$$

(2.37)

Note that $A_{ij}^f$ in Eq.(2.37) has no explicit $\sigma$-dependence.

Now we may take the mean-field decoupling of $H_J$ in Eq.(2.17) and $H_t$ in Eq.(2.36), by introducing the mean-fields $\chi_0 = <\chi_{ij}^\sigma>$ and $H_0 = <\hat{H}_{ij}>$ (here $\hat{H}_{ij} = e^{iA_{ij}^f} h_i^+ h_j$). At the same time, the constraint (2.6) has to be relaxed up to a global level, as outlined before. Then a saddle-point Hamiltonian at finite doping is found to be

$$H_{t-J}^{MF} = H_s + H_h,$$

(2.38)
in which the spinon degree of freedom has the same form as in the half-filling case:

\[ H_s = -J_s \sum_{<ij>,\sigma} e^{iA_{ij}^s} f_{i\sigma}^+ f_{j\sigma} + H.c., \]  

(2.39)

with \( J_s = J \chi_0 + t H_0 \). The holon degree of freedom is described by:

\[ H_h = -t \sum_{<ij>} e^{iA_{ij}^h} h_i^+ h_j + H.c., \]  

(2.40)

with \( t_h = 2t \chi_0 \). \( t_h \) and \( J_s \) have to be determined self-consistently. Since the averages \( \chi_0 \) and \( H_0 \) actually do not depend on \( t_h \) and \( J_s \), the self-consistency should be always satisfied here, even though an actual determination of the values for \( t_h \) and \( J_s \) is nontrivial. Generally one can estimate \( t_h \sim t \) and \( J_s \sim J \) at small doping. In the bosonic representation of Eqs.(2.25) and (2.28), \( H_s \) can be further expressed as

\[ H_s = -J_s \sum_{<ij>,\sigma} e^{iA_{ij}^s} b_{i\sigma}^+ b_{j\sigma} + H.c., \]  

(2.41)

where the topological phase \( A_{ij}^h \) is defined by

\[ A_{ij}^h = \frac{1}{2} \sum_{l \neq i,j} [\theta_i(l) - \theta_j(l)] n_l^h. \]  

(2.42)

In one dimension, one will find \( A_{ij}^h = A_{ij}^f = 0 \) in Eqs. (2.40) and (2.41) according to Eq.(2.20). The holon (described by \( h_i \)) and spin (described by \( b_{i\sigma} \)) degrees of freedom are thus decoupled at this saddle-point state. In two dimensions, however, the holon and spinon are coupled with each other through the nonlocal fields \( A_{ij}^f \) and \( A_{ij}^h \) in \( H_h \) and \( H_s \). The topological gauge field \( A_{ij}^h \) can be interpreted as describing fictitious \( \pi \) flux quanta attached to holons, but only seen by spinons. This becomes clear if one considers an arbitrary loop \( C \), and counts the flux enclosed, which is given by

\[ \sum_C A_{ij}^h = \pi \sum_{l \in C} n_l^h. \]  

(2.43)

\( A_{ij}^f \) in Eq.(2.37) may be rewritten as \( A_{ij}^f = A_{ij}^h - \phi_{ij}^0 \), where

\[ A_{ij}^h = \frac{1}{2} \sum_{l \neq i,j} [\theta_i(l) - \theta_j(l)] \left( \sum_{\sigma} n_{l\sigma}^b \right). \]  

(2.44)
and $\phi_{ij}^0$ is defined by

$$\phi_{ij}^0 = \frac{1}{2} \sum_{l \neq i, j} [\theta_i(l) - \theta_j(l)], \quad (2.45)$$

which describes a lattice $\pi$-flux with $\sum_\square \phi_{ij}^0 = \pi$. The topological gauge field $A_{ij}^s$ describes fictitious $\pi$ flux-tubes carried by spinons which are seen by holons. In Secs. III and IV, we will investigate how the couplings induced by nonlocal fields $A_{ij}^h$ and $A_{ij}^s$ will lead to highly nontrivial spin and charge properties in 2D case.

To end this section, we would like to give a different perspective about the construction of the flux-binding state. Recall that one has a gauge degree of freedom in a slave-boson decomposition: $c_i = h_i^+ b_i = (h_i^+ e^{i\theta_h}) \left( f_i e^{i\theta_f} \right)$, where $\theta_h$ and $\theta_f$ can be any phase satisfying $\theta_h + \theta_f = 0$. Particularly, if $\theta_h$ and $\theta_f$ are chosen as the topological phases in the Jordan-Wigner transformation, statistics-transmutation can happen so that one may end up with a slave-fermion, or more generally, slave-anyon decomposition. This is not surprising because all of these formalisms are mathematically equivalent. Much complicated decomposition can be also constructed. Distinctive physics is involved here only when one makes the mean-field decoupling. In this way, different decomposition means different saddle-point state. The usual slave-boson and slave-fermion formalisms are the simplest ones, but not necessarily the appropriate ones for the correct saddle-point. The present saddle-point state corresponds to $\theta_h = 1/2 \sum_{l \neq i} \theta_i(l) [\sigma(1 - n_i^h) + \sum_\alpha \alpha n_i^f_\alpha]$ and $\theta_f = -i\sigma \sum_{l \neq i} \theta_i(l) n_i^f_\sigma$. It is easy to check that $\theta_h + \theta_f = 0$ in terms of the constraint (2.6). Correspondingly, $c_i\sigma$ may be rewritten as in Eq.(1.6) where $b_i\sigma$ is defined by Eqs.(2.25) and (2.28). In this new decomposition, an electron is composed of two bosonic holon and spinon, together with some nonlocal fields. The spin operator like $S_i^+ = f_i^+ f_i^-$ can also be expressed as

$$S_i^+ = (-1)^i b_i^+ b_i e^{i \sum_{l \neq i} \theta_i(l) n_i^h}. \quad (2.46)$$

Its physical meaning will be explored in Sec. III. In the following, we will go beyond the mean-field approximation and show that a real spin-charge separation can be indeed realized with the decomposition (1.6).
C. Gauge-theory description: Spin-charge separation

It is now a well-recognized fact\cite{36,37} that any saddle-point-state properties of the $t-J$ model could be substantially modified by the low-lying gauge fluctuation around the saddle-point. So it is very important to check the gauge fluctuations around the present saddle-point. Here gauge fluctuations are to be related to phase fluctuations of the mean-fields $\langle \hat{\chi}_{ij} \rangle$ and $\langle \hat{H}_{ij} \rangle$. The main idea behind the construction of flux-binding saddle-point states is to incorporate the most singular transverse gauge fluctuations into the saddle-point via flux binding so that additional phase fluctuations become insignificant. These singular gauge fields are represented by the topological gauge fields $A_{ij}^h$ and $A_{ij}^s$ in the saddle-point Hamiltonians $H_s$ and $H_h$. In the following, we will show that the transverse gauge fluctuation around this saddle-point is indeed suppressed at finite doping, and therefore we have a real spin-charge separation whose low-energy physics will be determined in terms of the effective Hamiltonians $H_s$ and $H_h$, defined by Eqs.\(1.1)-(1.5)\) in the Introduction.

An effective Lagrangian, with the constraint (2.6) enforced and the (phase) gauge fluctuation around the present flux-binding saddle-point state included, can be written down as\cite{36,37}

\[
\mathcal{L}_{\text{eff}} = \sum_i \left( \sum_\sigma b_{i\sigma}^+(\partial_{\tau} - \mu)b_{i\sigma} + h_{i}^+(\partial_{\tau} - \mu)h_{i} \right) + \tilde{H}_{\text{eff}}, \tag{2.47}
\]

where

\[
\tilde{H}_{\text{eff}} = -J_s \sum_{\langle ij \rangle \sigma} e^{i(a_{ij} + \sigma A_{ij}^h)}b_{i\sigma}b_{j\sigma} + c.c. - t_h \sum_{\langle ij \rangle} e^{i(a_{ij} + A_{ij}^s - \phi_{ij})}h_i^+h_j + c.c. \tag{2.48}
\]

Here $a_{ij}$ in Eq.\(2.47\) describe an internal gauge field. Originally the no-double-occupancy constraint Eq.\(2.6\) [with $f_{i\sigma}$ changed to $b_{i\sigma}$ by transformations Eqs. (2.25) and (2.28)] is implemented through a Lagrangian multiplier field, whose fluctuating part is then absorbed by the longitudinal part of $a_{ij}$ in Eq.\(2.48\) due to the gauge invariance\cite{36} with $\mu$ left in Eq.\(2.47\) enforcing the constraint at a global level.
The existence of a gauge freedom in $\mathcal{L}_{\text{eff}}$ will guarantee the following current constraint between spinon and holon:

$$J_s = -J_h.$$  \hspace{1cm} (2.49)

It is noted that Eq. (2.49) in longitudinal channel is a simple reflection of the density constraint (2.6), but its transverse channel has nothing directly to do with the no-double-occupancy constraint, and is related to the property of the $t - J$ Hamiltonian. It is the transverse gauge fluctuation that becomes singular in long-wavelength, low-energy regime in the uniform RVB state. This transverse field would serve as a confine force in a usual non-spin-charge-separation state. The current constraint (2.49) is also connected to the Ioffe-Larkin combination rule of the response to an external electromagnetic field

$$K_e = [K_s^{-1} + K_h^{-1}]^{-1},$$  \hspace{1cm} (2.50)

where $K_s$ and $K_h$ are the response matrices of spinon and holon systems under effective Hamiltonians $H_s$ and $H_h$ in Eqs. (1.2) and (1.3).

In gauge theory, the dynamics of the gauge field $a_{ij}$ is determined after the spinon and holon degrees of freedom are integrated out. When the gauge fluctuation is weak, one may use the Gaussian approximation, and the gauge field propagator $D^a_{\mu\nu} = -\langle T_{\tau} a_{\mu} a_{\nu} \rangle$ in imaginary-time can be determined by

$$D^a = -[K_s + K_h]^{-1},$$  \hspace{1cm} (2.51)

in imaginary-frequency space. In the present system, holons are under some fluctuating flux described by $A_{ij}^h$ [cf. Sec. IV], and its response function will follow a usual metallic $q$ and $\omega$ dependances: $i\omega \sigma_h - \chi_h q^2$. The most interesting behavior will come from the spinon part $K_s$ as discussed below.

In the spinon part of $\tilde{H}_{\text{eff}}$, there is a sign $\sigma = \pm 1$ in front of the topological phase $A_{ij}^s$. This sign means that spinons with $\uparrow$ and $\downarrow$ spins see the fictitious flux quanta carried by holons in opposite directions. Since there is no such a sign in front of the gauge field $a_{ij}$, a
non-zero $a_{ij}$ will then polarize the spinon system with regard to spin $\sigma$. This polarization is generally energetically unfavorable, and we expect a suppression of the gauge field $a_{ij}$ to stabilize the system.

A mathematical demonstration is straightforward. As will be discussed in Sec. III, one may rewrite $A^h_{ij}$ as $A^h_{ij} = \bar{A}^h_{ij} + \delta A^h_{ij}$, where $\bar{A}^h_{ij}$ is a “mean-field” one with the flux-quanta uniformly smeared out in space. A uniform fictitious magnetic field $B_h = \pi \delta / a^2$ will correspond to such a vector potential $\bar{A}^h_{ij}$. On the other hand, $\delta A^h_{ij}$ will be correlated with the holon density fluctuation. If one integrates out the spinon degree of freedom (under the “mean-field” $\bar{A}^h_{ij}$) in $L_{\text{eff}}$, an effective action for quadratic fluctuation of the gauge field is found by

$$S_s[a, \delta A^h] = \frac{1}{2} \sum_{\sigma} \left( a + \sigma \delta A^h \right) \Pi_\sigma \left( a + \sigma \delta A^h \right), \quad (2.52)$$

where $\Pi_\sigma$ is the response matrix of spinon $\sigma$ under a fictitious magnetic field $\sigma B^h$. $K_s$ will be determined through $S_s = \frac{1}{2} a K_s a$ [Ref. 70], and one needs to further integrate out $\delta A^h$ in Eq.(2.52). At first step, if one neglects $\delta A^h$’s own dynamics, by directly integrating out $\delta A^h$ in Eq.(2.52), the following expression for $K_s$ can be obtained

$$K_s = 4 \left[ \Pi_T^{-1} + \Pi_\perp^{-1} \right]^{-1}. \quad (2.53)$$

The key thing here is that the up and down spinons see the opposite fictitious magnetic fields, and the Hall conductance terms in $\Pi_T$ and $\Pi_\perp$ will have opposite signs. As shown by Wiegmann$^3$ in a generic situation, due to the same amplitude but opposite signs of the Hall conductances, $K_s$ in Eq.(2.53) must exhibit the “Meissner effect” in the transverse channel: i.e., $K_s^\perp = 1/4 \pi \lambda^2 \neq 0$ at $\omega = 0$, and $q \to 0$ limit. Here we find $1/\lambda^2 = \pi B_h J_s$. Such a “Meissner effect” implies that the gauge field will get suppressed in the spinon system. In the long-wavelength and low-energy limit, $K_s$ becomes dominant in the transverse channel of $D^a$ as $K_h \sim 0$, and a non-zero $K_s^\perp$ then leads to a gap $\propto \delta J$ in the transverse gauge fluctuation of Eq.(2.51) (Anderson-Higgs mechanism). This gap means a spin-charge deconfinement as pointed out before. In contrast, a finite $K_s^{-1}$ will play a negligible role in the electromagnetic response function $K_e$ in Eq.(2.50) at small $q$ and $\omega$, and one finds
$K_e \simeq K_h,$ \hfill (2.54)

in such a limit. So the holon degree of freedom solely determines the long-wavelength, low-energy response to an external electromagnetic field in this spin-charge separation system. Obviously, all these features will disappear at $\delta \to 0$ as $B_h$ vanishes.

The above conclusion is still true if one includes $\delta A^h$'s own dynamics by adding a term \(\frac{1}{2}\delta A^h D^A^{-1} \delta A^h\) in Eq.(2.52). Here $D^A$ is the free propagator for $\delta A^h$ which is related to holon density fluctuation as follows:

\[
D^A_{\mu\nu} = \left(\delta_{\mu\nu} - \frac{q_{\mu}q_{\nu}}{q^2}\right) \left(\frac{\pi^2}{q^2} D^h_{\rho}\right),
\] \hfill (2.55)

with $D^h_{\rho}$ as the holon density-density correlation function. Then Eq.(2.53) is modified as

\[
K_s = \left[\left(\Pi^\uparrow + D^A^{-1}\right)^{-1} + \Pi^{-1}\right]^{-1} + \left[\left(\Pi^\downarrow + \Pi^\downarrow + D^A^{-1}\right)^{-1}\right]^{-1} - 1
\]

\[
+ \left[\left(\Pi^{-1}_\uparrow + \Pi^{-1}_\downarrow + (D^A)^{-1}\Pi^{-1}_\uparrow\right)^{-1} + \left[\left(\Pi^{-1}_\uparrow + \Pi^{-1}_\downarrow + (D^A)^{-1}\Pi^{-1}_\uparrow\right)^{-1} + \left(\Pi^{-1}_\downarrow + (D^A)^{-1}\Pi^{-1}_\downarrow\right)^{-1}\right]^{-1},
\] \hfill (2.56)

Since $D^A$ does not have a transverse-longitudinal mixed term (the Hall term), the above conclusion about $K_s$ can be easily found to remain unchanged as long as $D^A$ approaches to a constant or vanishes slower than $q^2$ when $q \to 0$. (Note that the density-density correlation function of a hard-core boson gas is very similar to a free-fermion gas, and in the latter case one has $D^h_{\rho} \sim \text{const.}$ at $\omega = 0$ and $q \to 0$.)

**III. SPIN DYNAMICS AT FINITE DOPING**

Spin dynamics in the insulating cuprates has been well understood within the framework of the Heisenberg model. A real important issue is how the antiferromagnet is affected under doping. In this section, we will explore such a doping effect under the spin-charge separation scheme, and compare its unique features with those found in the cuprates. We first consider the 1D case, where the exact analytical results are available for comparison.
A. Spin-spin correlation in one dimension

The present case corresponds to the large-$U$ limit of the Hubbard model, whose exact solution has been obtained a long time ago. However, only recently the underlying physics and various correlation functions have been clarified, following Anderson and other authors.

The spin-spin correlation function is one of important correlation functions in 1D. According to the numerical calculation based on the exact solution, a $2k_f$ oscillation is present in the spin-spin correlation function. One may naively relate this incommensurate structure with the existence of a large electron Fermi surface (points at $k_f$ and $-k_f$). Similar argument may also be applied to a 2D case. This observation should be physically reasonable, but by itself is not sufficient to get the correct spin dynamics. Much rich effects will be involved here due to strong electron-electron correlations. In fact, one has a spin-charge separation. In this case, the spinons are presumably responsible for the spin dynamics. But at the same time, holons as solitons of doped holes bound with spin domain-walls will also contribute to spin properties. A combination of these decides a peculiar doped spin-spin correlation, which is highly non-trivial from a Fermi-liquid point of view. In the following, we will show how a correct spin-spin correlation can be obtained without directly involving the electron Fermi-surface.

In the present scheme, the spin flip operator $S_i^+$ in Eq.(2.46) can be expressed in 1D in terms of Eq.(2.20) as

$$S_i^+ = (-1)^i b_i^+ b_{i+1} e^{\pm i \pi \sum_{l>i} n_l^h}. \quad (3.1)$$

Here a prominent feature is a Jordan-Wigner-type nonlocal phase $\pi \sum_{l>i} n_l^h$ which involves holon number operator. At first glimpse, one may feel strange for a holon number operator to appear in a spin expression. But this is not new, and has already been found in Ref. by a path-integral approach, as a result of spin-charge separation. It actually describes the effect of spin domain-walls carried by holons mentioned above. This is easy to see if one
freezes the dynamics of spinon $b_{i\sigma}$ by letting it “condensed” in Eq.(3.1). Then one would have a Néel order in spin $x-y$ plane when holons are absent. For each added holon at $l$ site, one finds an extra sign $e^{\pm i\pi} = -1$ for every $S_i^z$ at $i < l$, which means a flip for all those spins. In other words, there is indeed a spin domain-wall accompanying with each doped hole, and that is the physical reason why the holon number operator enters into Eq.(3.1) nonlocally.

We note that the nonlocal field appearing in Eq.(3.1) can be also related to the phase-shift field that leads to the Luttinger liquid behavior and an electron Fermi-surface satisfying the Luttinger liquid theorem in the single electron propagator, which is to be discussed elsewhere. In fact, a $2k_f$ oscillation in the spin correlation function will arise naturally due to this nonlocal field as shown below.

Due to $A_{ij}^+ = A_{ij}^h = 0$ in 1D, holon and spinon are decoupled in the Hamiltonians (1.2) and (1.3). The nonlocal phase in Eq.(3.1) will then solely determine the doping effect. One may write

$$< S_i^z(t)S_j^z(0) > = (-1)^{i-j} < b_{i\uparrow}^+(t)b_{j\uparrow}(0) > < b_{j\downarrow}(t)b_{j\downarrow}^+(0) > \times < e^{\pm i\pi \sum_{l>i} n_l^h(t)} e^{\mp i\pi \sum_{l>j} n_l^h(0)} > .$$

(3.2)

Since $b_{i\sigma}$ and $h_i$ as described by Eqs.(1.2) and (1.3) are hard-core bosons, each average on the right-hand-side of Eq.(3.2) can be evaluated straightforwardly in the asymptotic limit (see Sec. II B, and Ref. [42]). For example

$$< e^{\pm i\pi \sum_{l>j} n_l^h(t)} e^{\mp i\pi \sum_{l>j} n_l^h(0)} > \propto \frac{\cos(\pi \delta x)}{(x^2 - v_h^2 t^2)^{1/4}},$$

(3.3)

with $x = x_i - x_j$, and $v_h = 2t_h a \sin(\pi \delta)$. The final result is

$$< S_i^z(t)S_j^z(0) > \propto \frac{\cos(2k_f x)}{(x^2 - v_h^2 t^2)^{1/2}(x^2 - v_h^2 t^2)^{1/4}},$$

(3.4)

where $k_f = (1 - \delta)\pi/2a$. The doping effect as described by the phase-shift in Eq.(3.1) enters Eq.(3.4) by changing the oscillation from $2\pi/a$ to the incommensurate $2k_f$, and at the same time contributing an additional power $(x^2 - v_h^2 t^2)^{-1/4}$. It is noted that $< S_i^z(t)S_j^z(0) >$ can not be directly computed at the same level of approximation as explained in Sec. II, but it should follows the same behavior due to the rotational invariance.
Equation (3.4) recovers the correct spin behavior at the strong-coupling fixed point of the model (i.e., the large-$U$ Hubbard model). It suggests that the present state indeed catches the correct characteristics of charge-spin separation in 1D, even though such a scheme is originally constructed for 2D.

**B. Doping effect in two dimensions**

In the one-dimensional case, spinon and holon are decoupled, and the doping effect on spin dynamics is solely contributed by the spin domain walls carried by holons. In 2D, this nonlocal doping effect takes a different form, because spinon and holon will no longer behave like free solitons. As noted before, the topological phase $A_{ij}^h$ in $H_s$ cannot be gauged away in 2D, and it will represent a new type of nonlocal influence of doping on spinons.

1. **New length and energy scales introduced by doping**

As shown in Sec.II, the gauge field confining spinon and holon are suppressed at finite doping. But there are still residual interactions between spinons and holons, and spinons can always feel the existence of holons nonlocally by seeing the flux-quanta bound to the latter.

Let us first see how this exotic interaction can change the topology of a holon. In the spin flip operator $S_i^+$ (Eq.(2.46)), one has a nonlocal phase $[\sum_{l \neq i} \text{Im} \ln (z_i - z_l)n_l^h]$ involving holon number operator. In resemblance to the domain-wall picture in 1D, one may interpret it as describing a spin vortex in $x - y$ plane with vorticity $2\pi$ bound to each holon. This could be seen if one freezes the dynamics of $b_{i\sigma}^+$ and treats it as a number in Eq.(2.46). But $b_{i\sigma}^+$ here can no longer be regarded as a single-valued quantity because it is under the topological phase $A_{ij}^h$ in $H_s$. A Berry-phase counting shows that the phase in $b_{i\sigma}^+$ cancels out the effect of the nonlocal phase-shift in Eq.(2.46) such that there is actually no $2\pi$-vortex topological texture formed around a holon. Nevertheless, if one goes along a line across such a holon, one can still find a domain-wall-like spin singularity at holon site similar to that
in 1D. Thus, a holon in 2D is no longer associated with a spin topological object. Instead, it may carry a Shraiman-Siggia-type dipolar texture with vorticity = 0. Such an object will not break the T- and P-symmetries. One expects a strong dynamical renormalization to be involved in determining the profile of each holon. However, we are not interested in a single-doped-hole problem here. We shall focus on a finite doping concentration in the following, where a useful mathematical description becomes available.

For simplicity, we are going to use a continuum version of the spinon Hamiltonian $H_s$ by taking the lattice constant $a \to 0$. This continuum approximation may be justified at small doping and in the low-energy, long-wavelength regime, where the amplitude of $A_{ij}^h$ is small and the spinons as bosons mainly stay near the bottom of energy band. Such a continuum version for $H_s$ in Eq.(1.2) can be easily written down as [cf. Ref. 68]

$$\tilde{H}_s = \sum_\sigma \int d^2r \ b_\sigma^+(r) \left( -i \nabla - \sigma A_h^\sigma \right)^2 b_\sigma(r),$$  (3.5)

with $m_s = (2J_s a^2)^{-1}$ and $A_h^\sigma(r)$ defined by

$$A_h^\sigma(r) = \frac{1}{2} \int d^2r' \hat{z} \times \frac{(r - r')}{|r - r'|^2} \rho_h(r').$$  (3.6)

Here $\rho_h(r)$ is the holon density $\rho_h = h^+(r)h(r)$.

We note that Eqs.(3.5) and (3.6) would describe a semion problem if the holon density $\rho_h(r')$ were replaced by the spinon density. The mathematical similarity suggests that one may borrow the method developed in the anyon problem [82]. The idea is to rewrite $\rho_h = \bar{\rho}_h + (\rho_h - \bar{\rho}_h)$ in Eq.(3.6) by introducing an average holon density $\bar{\rho}_h = \delta/a^2$. Then $A_h^\sigma(r)$ is rewritten as

$$\vec{A}_h(r) = \vec{A}_h(r) + \delta \vec{A}_h(r).$$  (3.7)

Here $\vec{A}_h(r)$ corresponds to $\bar{\rho}_h$, and in the symmetric gauge it may be expressed by

$$\vec{A}_h(r) = \frac{B_h}{2} (\hat{z} \times r)$$ with $B_h = \pi \bar{\rho}_h$. Physically, the vector potential $\vec{A}_h$ describes a mean-field magnetic field $B_h$ obtained after the flux quanta bound to holons are uniformly smeared out in space. $\delta \vec{A}_h(r)$ in Eq.(3.7) is the fluctuation part.
\[ \delta A^h(r) = \frac{1}{2} \int d^2 r \frac{\hat{z} \times (r-r')}{|r-r'|^2} [\rho_h(r') - \bar{\rho}_h], \tag{3.8} \]

which can be treated perturbatively. We emphasize that in an anyon problem, \( \delta A^h \) would correlate with anyon density fluctuation and represent a long-range interaction among anyons. Here \( \delta A^h \) is determined by the density fluctuation of holons instead of spinons themselves, which belong to an independent degree of freedom. Thus one expects \( \delta A^h(r) \) to provide an independent dynamical scattering source in Eq. (3.5) just like phonon in a usual electron system.

The separation (3.7) is meaningful when the hole density is not too low. With the presence of a fictitious magnetic field \( B_h \), a new length scale is introduced to the spinon system, which is the magnetic cyclotron length

\[ l_c = \frac{1}{\sqrt{B_h}} = \frac{a}{\sqrt{\pi \delta}}. \tag{3.9} \]

\( l_c \) will later be connected to the antiferromagnetic correlation length. At this mean-field level, a Landau-level structure appears in the spinon energy spectrum, and a basic energy scale is the cyclotron energy

\[ \omega_c = \frac{B_h}{m_s} = 2\pi \delta J_s. \tag{3.10} \]

Another energy scale measuring the broadening \( \Gamma_s \) of each Landau level caused by the fluctuation \( \delta A^h \) should be also correlated with the doping concentration. \( \Gamma_s \) will be related to an important low-energy scale in spin dynamics. It can be estimated to be in an order of magnitude \( \sim \delta J \) if the energy scale of the fluctuating \( \delta A^h \) is sufficiently small. \( \Gamma_s \) could be even sharper when the fluctuating \( \delta A^h \) has a higher energy scale as holons become more mobile at finite doping. Of course, different Landau levels generally could have different broadening widths, and a further discussion will be given later. Similar broadening problem in a semion system has been recently studied by Levy and Laughlin, where the dynamics of \( \delta A^h \) is already known from the RPA calculation. In the present case, the dynamics of \( \delta A^h \) is directly related to that of \( \delta \rho_h = \rho_h - \bar{\rho}_h \), which will be in turn determined by coupling with
the spinon degree of freedom. So a self-consistent treatment of the Landau level broadening could be much complicated here. Nevertheless, for the purpose of understanding basic characteristics for spin dynamics, only a general shape of the Landau level broadening is needed, and the detailed structure will not be crucial.

Besides the length and energy scales, we point out that there also exists a basic temperature scale which is related to the Bose condensation temperature $T^*_c$ of the bosonic spinon $b_{i\sigma}$. Recall that for a 2D free boson gas, a 2D Bose condensation is always suppressed by the thermal excitations at any finite temperature, due to finite density of states at low energy. In the present case, due to the broadening of the lowest-Landau-level (LLL), the low-energy density of states presumably will fall off continuously to zero at the LLL energy bottom. Thus the low-lying thermal excitation has vanishing weight at the low-energy tail, which could not kill the Bose-condensation of spinons at a sufficient low temperature. As will be demonstrated later, $T^*_c$ will represent an important temperature scale in spin dynamics.

2. Spin susceptibility function

The spin dynamic structure factor $S(q, \omega)$ is defined as a Fourier transformation of the spin-spin correlation function

$$S_{\alpha\beta}(q, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt e^{i\omega t} \int d^2r e^{-i\mathbf{q} \cdot \mathbf{r}} < S_\alpha(\mathbf{r}, t) \cdot S_\beta(0, 0) >. \quad (3.11)$$

$S_{\alpha\beta}(q, \omega)$ can be directly measured in neutron-scattering experiment, and is related to the dynamic spin susceptibility function $\chi_{\alpha\beta}(q, \omega)$ through the so-called fluctuation-dissipation theorem

$$S_{\alpha\beta}(q, \omega) = \frac{1}{\pi} \left[1 - e^{-\beta \omega}\right]^{-1} \chi''_{\alpha\beta}(q, \omega), \quad (3.12)$$

where $\beta = 1/k_BT$, and $\chi''_{\alpha\beta}(q, \omega) \equiv Im\chi_{\alpha\beta}(q, \omega + i0^+)$ is the imaginary part of the retarded spin susceptibility function.

In the Matsubara representation, the transverse spin susceptibility is defined by
\[ \chi(\mathbf{q}, i\omega_n) = \int_0^\beta d\tau e^{i\omega_n \tau} \int d^2r e^{-i\mathbf{q}\cdot\mathbf{r}} < T_\tau S^+(\mathbf{r}, \tau) S^-(0, 0) >, \]  

with \( \omega_n = 2\pi n/\beta \). In terms of a continuum version of Eq.(2.46), one may write

\[ < T_\tau S^+(\mathbf{r}, \tau) S^-(0, 0) > = \frac{a^2}{4} \left( \sum_{\mathbf{Q}_0} e^{i\mathbf{Q}_0\cdot\mathbf{r}} \right) < T_\tau b^+_\uparrow(\mathbf{r}, \tau) b^\downarrow_\uparrow(0, 0) e^{i\int_{(0,0)}^{(r,\tau)} \mathbf{A}^h \cdot d\mathbf{r}} > \times < T_\tau b^\downarrow_\uparrow(\mathbf{r}, \tau) b^+\downarrow_\uparrow(0, 0) e^{-i\int_{(0,0)}^{(r,\tau)} \mathbf{A}^h \cdot d\mathbf{r}} >, \]  

in which \( \mathbf{Q}_0 = (\pm \frac{\pi}{a}, \pm \frac{\pi}{a}) \) are the AF wave vectors. \( \mathbf{A}^h \) in Eq.(3.14) is from the nonlocal phase in the spin expression (2.46). In the mean-field approximation with \( \mathbf{A}^h \) replaced by \( \bar{\mathbf{A}}^h \), Eq.(3.14) may be rewritten as

\[ < T_\tau S^+(\mathbf{r}, \tau) S^-(0, 0) > = \frac{a^2}{4} \left( \sum_{\mathbf{Q}_0} e^{i\mathbf{Q}_0\cdot\mathbf{r}} \right) e^{2i\int_0^\tau \bar{\mathbf{A}}^h \cdot d\mathbf{r}} G_\uparrow^\uparrow(-\mathbf{r}, -\tau) G_\downarrow^\downarrow(\mathbf{r}, \tau), \]  

where the line integration on the right-hand side is chosen along a straight line connecting \((x, y)\) and \((0, 0)\) on 2D plane. The Green’s function \( G_\sigma^\sigma \) is defined by

\[ G_\sigma^\sigma(\mathbf{r}, \tau) = - < T_\tau b_\sigma(\mathbf{r}, \tau) b^\uparrow_\sigma(0, 0) >. \]  

\( b_\sigma(\mathbf{r}) \) may be expressed in the representation of the Landau levels as \( b_\sigma(\mathbf{r}) = \sum_{nk} < \mathbf{r}|nk \> b^\sigma_{nk} \), where \( n = 0, 1, 2 \cdots \) is the Landau level index and \( k \) is the quantum number inside each Landau level. Correspondingly,

\[ G^\sigma(\mathbf{r}, \tau) = - \sum_{n,k} \sum_{n',k'} < \mathbf{r}|nk \> < n'k'|0 > < T_\tau b^\sigma_{nk}(\tau) b^\sigma_{n'k'}(0) > \]

\[ = - \sum_n \sum_{k} < \mathbf{r}| \left( \sum_{k'} |nk'\> < nk| \right) |0 > < T_\tau b^\sigma_{nk}(\tau) b^\sigma_{nk}(0) > \]

\[ = \sum_n \Pi_n^\sigma(\mathbf{r}, 0) G_\sigma^\sigma(n, \tau). \]  

In obtaining last line of Eq.(3.17), the \( k \)-dependence of the Green’s function \( G_\sigma = - < T_\tau b_{nk}(\tau) b^\uparrow_{nk}(0) > \) has been neglected. Note that \( k \) represents the center of each cyclotron orbital in the present symmetric gauge and \( k \)-dependence of \( G_\sigma \) should not be important due to the translational invariance of the system. \( \Pi_n^\sigma(\mathbf{r}, 0) \) in Eq.(3.17) is given by

\[ \Pi_n^\sigma(\mathbf{r}, 0) = L_n \left( \frac{r^2}{2l_c^2} \right) \Pi_0^\sigma(\mathbf{r}, 0) \]  

(3.17)
with $L_n(t)$ as the Laguerre polynomials: $L_0(t) = 1$, $L_1(t) = 1 - t$, etc., and

$$
\Pi_0^r(r, 0) = \frac{1}{2\pi l_c^2} \exp \left[ -r^2/4l_c^2 \right].
$$

(3.19)

Then one gets

$$
\chi(q, i\omega_n) = \sum_{l,m=0}^\infty K_{lm}(q) \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \rho_b(l, \omega') \rho_b(m, \omega'') \frac{n(\omega'') - n(\omega')}{i\omega_n + \omega' - \omega''},
$$

(3.20)

where

$$
K_{lm}(q) = \frac{a^2}{32\pi^3 l_c^5} \sum_{Q_0} \int d^2r L_l \left( \frac{r^2}{2l_c^2} \right) L_m \left( \frac{r^2}{2l_c^2} \right) e^{-\frac{r^2}{2l_c^2}} e^{-i|q-Q_0| r} = \frac{\delta}{16\pi} \sum_{Q_0} \int_0^\infty dy L_l(y) L_m(y) e^{-y J_0(\sqrt{2l_c^2 y} |q - Q_0|)}
$$

(3.21)

with $J_0(x)$ as the Bessel function, and the spectral function $\rho_b$ is defined through

$$
G_0(l, i\omega_n) = \int \frac{d\omega'}{2\pi} \rho_b(l, \omega') \left[ n(\omega + \omega') - n(\omega') \right]
$$

(3.22)

in which the spin index $\sigma$ has been omitted for simplicity since $\rho_b$ does not explicitly depend on it in the unpolarized case. $n(\omega) = 1/(e^{\beta \omega} - 1)$ in Eq.(3.20) is the Bose distribution function. Note that the spectral function $\rho_b(l, \omega)$ ensures $\omega \geq \omega_0 - \mu \geq 0$, where $\omega_0$ is the energy minimum of spinon spectrum and $\mu$ is the chemical potential determined by $
\sum_l (2\pi l_c^2)^{-1} \int (d\omega/2\pi) \rho(l, \omega)n(\omega) = (1 - \delta)/2a^2$.

Finally $\chi''(q, \omega)$ is found by

$$
\chi''(q, \omega) = -\pi \sum_{l,l'} K_{ll'}(q) \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \rho_b(l, \omega') \rho_b(l', \omega + \omega') \left[ n(\omega + \omega') - n(\omega') \right].
$$

(3.23)

In the next sections, we shall examine the $q$-dependence, $\omega$-dependence and temperature-dependence of $\chi''(q, \omega)$ and compare them with neutron-scattering and NMR experimental measurements in cuprates.

### 3. Basic characteristics of spin dynamics in 2D

Spin susceptibility $\chi''(q, \omega)$ in Eq.(3.23) will characterize the basic features of spin dynamics. The real part of spin susceptibility can be also determined from $\chi''(q, \omega)$ through
the Kramers-Kronig relation. Here we shall mainly focus on $\chi''(q, \omega)$ in the vicinity of the AF wave vector $Q_0$ and low energy $\omega$.

As discussed in Sec. 1, doping creates a Landau-level structure in the spinon spectrum. So the low-lying spin fluctuations are expected to be sensitive to doping. Due to the particular Landau-level structure, one can distinguish two types of contributions to $\chi''(q, \omega)$, which correspond to intra-Landau-level transition and inter-Landau-level transition, respectively. We will show below that these two processes are related to the so-called commensurate and incommensurate AF spin fluctuations in the present state.

**Commensurate AF fluctuation**  
First we consider $\chi''(q, \omega)$ at small $\omega$ such that only the intra Landau-level transition contributes. At low temperature with spinons staying in the LLL, the $q$ dependence of $\chi''(q, \omega)$ [Eq.(3.23)] is solely decided by $K_{00}(q)$:

$$K_{00}(q) = \frac{\delta}{16\pi} \sum_{Q_0} \exp \left(-\frac{|q - Q_0|^2}{2l_c^{-2}}\right),$$

(3.24)

in terms of Eq.(3.21). Eq.(3.24) shows that $\chi''(q, \omega)$ will be peaked at the AF wave vector $Q_0$’s. The corresponding spin fluctuation is known as the commensurate AF fluctuation. With the increase of temperature, spinons can be thermally excited to higher Landau levels such that $K_{11}(q), K_{22}(q)$, etc., will contribute. For example, $K_{11}(q)$ has the following form:

$$K_{11}(q) = \frac{\delta}{16\pi} \sum_{Q_0} \left(1 - \frac{|q - Q_0|^2}{2l_c^{-2}}\right)^2 \exp \left(-\frac{|q - Q_0|^2}{2l_c^{-2}}\right),$$

(3.25)

which is still peaked at $Q_0$ with a width essentially the same as $K_{00}(q)$ in Eq.(3.24). $K_{ll}$ at higher $l$ can be checked by numerical calculation and generally a Gaussian (3.24) is well satisfied when $l$ is not too big. This means that the width is not sensitive to temperature.

Thus, when the intra-level transition dominates, $\chi''(q, \omega)$ generally follows a Gaussian:

$$\chi''(q, \omega) \simeq \exp \left(-\frac{|q - Q_0|^2}{(2\sigma^2)}\right) A(\omega),$$

(3.26)

around $Q_0$, where $\sigma = 1/l_c$ and $A(\omega) \equiv \chi''(Q_0, \omega)$. The Gaussian (3.26) with a width $\sigma$ determines a spin-spin correlation in real space: $\cos(Q_0 \cdot r) \exp[-|r|^2/\xi^2]$, where the correlation length

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is in the same order of the average hole-hole distance. Namely, the doped holes break up the long-range AF correlation into short-range AF fragments with a length scale $\sim \xi$.

The energy scale of this AF fluctuation will be characterized by the $\omega$ dependence of $A(\omega) = \chi''(Q_0, \omega)$ in Eq.(3.26). Since only the intra-level transition is involved here, the magnetic energy scale will be basically decided by the broadenings of the Landau levels. The expression of $\chi''(Q_0, \omega)$ can be found as

$$\chi''(Q_0, \omega) = -\frac{\delta}{16} \int \frac{d\omega'}{2\pi} \sum_l \rho_b(l, \omega') \rho_b(l, \omega + \omega') \left[ n(\omega + \omega') - n(\omega') \right].$$

(3.28)

Here the broadening width $\Gamma_s$ of the spectral function $\rho_b$ for a given Landau level is generally of the order $\delta J$, as caused by the fluctuation $\delta A_h$. The detailed broadenings will be sensitive to $\delta A_h$ and other factors, which is to be further discussed later. Equation (3.28) has been obtained from Eq.(3.23) by using

$$K_{ll'}(Q_0) = \frac{\delta}{16\pi} \delta_{l,l'},$$

(3.29)

in terms of Eq.(3.21) and the orthogonality of the Laguerre functions

$$\int_0^\infty dy L_l(y)L_{l'}(y)e^{-y} = \delta_{l,l'},$$

(3.30)

[The contribution to $K_{ll'}(Q_0)$ from the other three $Q_0$’s is exponentially small and has been neglected.]

The typical $\omega$ dependence of $\chi''(Q_0, \omega)$ is shown in Fig.1 at various temperatures. The choice of the spectral function $\rho_b$ and other parameters in Fig.1 is to describe the underdoped YBa$_2$Cu$_3$O$_{6.6}$, which will be explained in the next section. Here we mainly focus on the general features shown in Fig.1. Besides an overall small energy scale decided by the Landau-level broadening, Fig.1 shows an interesting temperature effect characterized by the Bose-condensation temperature $T^*$. When $T > T^*$, one finds $\chi''(Q_0, \omega) \propto \omega/T$ at small $\omega$. The slope of the linear $\omega$ dependence increase with the decrease of $T$. However, this increase gets
arrested at $T = T^*_c$, and when $T < T^*_c$ the low-energy part of $\chi''(Q_0, \omega)$ becomes continuously suppressed instead. This feature resembles a typical “spin gap” behavior. When $T < T^*_c$, the spinons begin to condensate into the bottom state of the lowest Landau level. The contribution due to the transition from such a condensate state to the rest of the quantum states in the broadened LLL will then emerge in $\chi''(Q_0, \omega)$. This process effectively will map out the shape of the spectral function of the same Landau level, and becomes dominant with the decrease of temperature. In fact, at $T = 0$ when all the spinons are condensed, $\chi''(Q_0, \omega)$ in Eq.(3.28) is simply reduced to

$$\chi''(Q_0, \omega)_{T=0} = \left(\frac{n_s}{16}\right) \rho_b(l = 0, \omega)_{T=0},$$

with $n_s = 1 - \delta$, which is directly proportional to the spectral function $\rho_b(0, \omega)$ of the LLL.

If $\rho_b(0, \omega)$ has a small Lifshitz tail before it vanishes at the low-energy end $\omega = 0$, a gap feature is exhibited in $\chi''(Q_0, \omega)$ as illustrated in Fig.1. Even in the case that $\rho_b(0, \omega)$ does not approach to zero very fast at $\omega = 0$, $\chi''(Q_0, \omega)$ could still generically show a pseudo-gap trend below $T < T^*_c$ as described above. This spin gap behavior and the temperature scale $T^*_c$ are the unique, and important features of the present spin state.

Another interesting property of $\chi''(q, \omega)$ is its behavior at $\omega \to 0$ limit. This behavior can be probed by NMR measurement. The NMR spin-lattice relaxation rate of nuclear spin due to the coupling to spin fluctuation described by $\chi''(q, \omega)$ is given by

$$\frac{1}{T_1} = \frac{k_B T}{N} \sum_q A^2(q) \left| \frac{\chi''(q, \omega)}{\omega} \right|_{\omega \to 0},$$

where the form factor $A^2(q)$ is from the hyperfine coupling between nuclear spin and the spin fluctuation. For $^{63}Cu(2)$ nuclear spin in the cuprates, with the applied field perpendicular to the $CuO_2$ plane, the form factor $A^2(q)$ is found to be

$$A^2(q)|_{^{63}Cu} = [A_\perp + 2B(\cos q_x a + \cos q_y a)]^2,$$

where the hyperfine couplings $A_\perp$ and $B$ are estimated as $A_\perp / B \simeq 0.84$, $B \simeq 3.8 \times 10^{-4} meV$ [These coefficients may slightly vary among $YBCO$ and $LSCO$ compounds]. For $^{17}O(2)$ nuclear spin, one has

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\[ A^2(q)|_{17O} = 2C^2 [1 + \cos(q \cdot a)] , \]

with \( C \simeq 0.87B \) [Ref. 4]. \( A^2(q)|_{17O} \) in Eq. (3.34) vanishes at \( q = Q_0 \). Thus a combined measurement of \( 1/^{63}T_1 \) and \( 1/^{17}T_1 \) can provide a \( q \)-dependent information about \( \chi''(q, \omega) \) at \( \omega \to 0 \).

In the present framework, \( \chi''(q, \omega) \) have been already obtained in Eq.(3.23). So by substituting it into the above \( 1/T_1 \) expression, it is straightforward to get

\[
\frac{1}{^{63}T_1} = \pi \sum_l C_l \int \frac{d\omega}{2\pi} n(\omega)(1 + n(\omega)) [\rho_b(l, \omega)]^2 ,
\]

where

\[
C_l = \frac{\delta^2 B^2}{8\pi} \left[ \left( \frac{A_l^2}{B^2} + 4 \right) - 8 \frac{A_l}{B} L_i^2 \left( \frac{a^2}{\xi^2} \right) e^{-a^2/\xi^2} + 8L_i^2 \left( \frac{2a^2}{\xi^2} \right) e^{-2a^2/\xi^2} + 4 \left( \frac{4a^2}{\xi^2} \right) e^{-4a^2/\xi^2} \right] .
\]

In obtaining Eqs.(3.35) and (3.36) only the intra-level transition (i.e, \( l = l' \)) is involved.

Using the same \( \chi''(q, \omega) \) whose \( \omega \)-dependence at \( q = Q_0 \) is illustrated in Fig.1, one finds the corresponding \( 1/^{63}T_1 \) vs. \( T \) as shown in Fig.2. In contrast to a usual Korringa rule for a Fermi liquid: \( 1/T_1 \sim const. \), Fig.2 clearly depicts a non-Korringa behavior which falls off like \( \propto 1/T \) at high temperature. And \( 1/^{63}T_1 \) is peaked at the characteristic temperature \( T^*_c \), below which \( \chi''(Q_0, \omega) \) begins to develop a “spin gap” behavior.

On the other hand, at oxygen \( ^{17}O(2) \) site, the coefficient \( C_l \) has to be replaced by \( C_l^O \) in Eq.(3.35):

\[
C_l^O = \frac{\delta^2 C^2}{4\pi} \left[ 1 - L_i^2 \left( \frac{a^2}{\xi^2} \right) e^{-a^2/\xi^2} \right] .
\]

The difference between \( 1/^{63}T_1 \) and \( 1/^{17}T_1 \) is due to the different weight functions [Eqs.(3.33) and (3.34)], which pick up the commensurate magnetic fluctuation around \( q = Q_0 \) at \(^{63}Cu(2)\) site, but suppresses such a contribution at \(^{17}O(2)\) site. \( C_l^O \) in Eq. (3.37) simply vanishes at \( \xi \to \infty \): the long-range AF limit where the whole contribution is from \( q = Q_0 \). At \( \xi = 4a \), for example, we find a reduction of \( \sim 1/93 \) for \( 1/^{17}T_1 \) as compared to \( 1/^{63}T_1 \). Thus
the commensurate AF fluctuation as described by Eq.(3.26) has negligible contribution to
the spin relaxation for planar oxygen nuclei of the cuprates. In contrast, non-magnetic
incoherent contributions should not be suppressed so strongly by the form factor, and would
become dominant contribution in $1/17T_1$ as well as in the Knight shift, which measures the
real part of the spin susceptibility near $q = 0$.

Hence, $\chi''(q, \omega)/\omega$ at $\omega \to 0$ gives rise to a non-Korringa behavior of the NMR spin
relaxation rate. This $1/T$ law of $1/T_1T$ can be obtained analytically, if the broadening $\Gamma_s^0$
of the spectral function $\rho_b$ for the LLL is small as compared to temperature. That is, when
$\Gamma_s^0 \ll T \ll \omega_c$ (and of course $T > T_c^*$), one has

$$\frac{1}{63T_1T} \simeq \frac{D}{T},$$

in terms of Eq.(3.35), where $D = \pi C_0 n_s/\Gamma_s^0 \delta^2$. Next let us consider the case when tempera-
ture is further increased such that more Landau levels are involved. Here we have to assume
a general Lorentz-like broadening for each Landau level with $\Gamma_s = 0.4 \omega_c \propto \delta$. Then $1/63T_1$
at high temperature is shown in Fig.3, where the curves become very flat and not sensitive
to the doping concentration. This is in contrast to the low temperature regime, where a
strong doping-dependence can emerge (in Eq.(3.38), $D \sim 1/\delta$ if one simply takes $\Gamma_s^0 \propto \delta$).

Therefore, a full picture for the low-lying commensurate AF correlation is formed in
the present spin-charge separation state. This picture is rather unique, and is drastically
different from those of Fermi liquid and local antiferromagnetic descriptions. Doping effect
plays a key role here. It decides a doping-dependent energy scale for the AF fluctuation,
which can be much smaller than the characteristic energies in a Fermi liquid ($\epsilon_f$) and an
effective local antiferromagnet ($[1 - \delta]J$). It also leads to a doping-dependent correlation
length, comparable with the average spacing of holes. Furthermore, the Bose condensation
of spinons determines a new characteristic temperature scale, below which the low-lying
spin fluctuation is suppressed and non-Korringa behavior of the spin relaxation rate gets
interrupted, similar to a spin gap effect.

Incommensurate AF fluctuation

Now let us consider higher energy regime. If $\omega$ is
increased such that the inter Landau-level transition gets involved, the simple Gaussian-like $q$-dependence in Eq.(3.26) will be modified. A typical $\chi''(q,\omega)$ vs. $q$ is shown in Fig.4 at different $\omega$'s. For the purpose of illustration, we have chosen the same broadening $\Gamma_s = 0.4\omega_c = 0.4\delta\omega_0$ for each Landau level, and $T = 0.1\omega_0$, $\delta = 0.15$. It shows that the width of the Gaussian is broadened at first with the increase of $\omega$. Then the commensurate peak at $Q_0$ is split and two incommensurate peaks emerge at some fixed positions as the inter-level transition becomes dominant. In fact, this incommensurate structure is due to $K_{01}(q)$ in Eq.(3.23). According to Eq.(3.21), one finds

$$K_{01}(q) = \frac{\delta}{16\pi} \sum_{Q_0} \frac{|q - Q_0|^2}{2\sigma^2} \exp \left( -\frac{|q - Q_0|^2}{2\sigma^2} \right),$$

(3.39)

which leads to incommensurate peaks at a ring circling $Q_0$ by a radius of $\sqrt{2\pi\delta}/a$. We stress that the doping concentration is presumably small here so that the lattice effect is negligible. At a finite doping where spin-correlation length becomes comparable with the lattice spacing, the lattice effect is expected to become important, which could strongly affect the positions of the incommensurate peaks in $q$ space. In this case, both flux and lattice have to be treated on the same footing like in the incommensurate-flux-phase.

With the further increase of $\omega$, more complicated structure will show up as $K_{02}$, $K_{03}$, etc., are involved. The weight of $\chi''(q)$ will be further shifted towards whole Brillouin zone, i.e., non-magnetic regime. At such a high energy, the present continuum approximation may no longer be appropriate. Nevertheless, one could still get some generic feature. By integrating $q$ within the whole $q$ space, we get

$$\chi''_{total}(\omega) = \frac{1}{N} \sum_q \chi''(q,\omega)$$

$$= \frac{\delta^2}{8\pi} \int \frac{d\omega'}{2\pi} \left[ \sum_l \rho_b(l,\omega') \right] \left[ \sum_m \rho_b(m,\omega + \omega') \right] (n(\omega + \omega') - n(\omega')).$$

(3.40)

Equation (3.40) predicts that at high energy, the $q$-integrated $\chi''_{total}(\omega)$ will be saturated at a constant level, modulated by a Landau-level-like oscillation. We note that the higher energy part is contributed by those fluctuations with less magnetic character. Thus it is less easy for elastic neutron-scattering to fully collect data for $\chi''_{total}(\omega)$. On the other hand, at low
energy, finite widths of the Landau levels will lead to a more broadened first peak in \( \chi''_{\text{total}}(\omega) \) as compared to the overall energy peak of \( \chi''(Q_0, \omega) \) where only intra-level transitions are involved.

C. Comparison with experimental measurements in cuprates

The normal-state spin dynamics represents an important characteristics for the high-\( T_c \) cuprates. A lot of measurements have been done on these materials, especially \( YBCO \) and \( LSCO \) compounds. Experimental results have revealed a rather rich phenomenon, and also seem to suggest that material-dependent effects, like double-layer structure in \( YBCO \), may play major roles in the spin dynamics. Thus, whether there is an underlying universal mechanism for spin dynamics in the cuprates is not transparent in terms of the experiments alone. In the following, we argue that the present spin-charge separation scheme can relate a variety of anomalous spin properties together, and provide a consistent picture for these materials.

In the underdoped cuprates, like \( YBa_2Cu_3O_{6.6} \) and insulating \( La_{1.95}Ba_{0.5}CuO_4 \), a commensurate AF fluctuation has been verified by neutron scattering,\(^{19,20,17}\) which is peaked at \( Q_0 \) in \( \chi'' \). These neutron data have been well fitted\(^{20,17}\) by a Gaussian form (3.26), and the width \( \sigma \) is indeed roughly independent of temperature. The corresponding spin-spin correlation length follows a \( a/\sqrt{5} \) rule similar to Eq.(3.27), as shown\(^{16}\) in \( LSCO \) at small doping. Thus the momentum feature of the commensurate AF fluctuation at small doping is well described by the theory. In the underdoped \( YBa_2Cu_3O_{6+x} \), the energy scale of such a commensurate AF fluctuation has been systematically investigated.\(^{19,86,87}\) It is found to be doping dependent and small as compared to the exchange energy \( J \simeq 120 \text{ meV} \). As discussed in the last section, a small, doping-sensitive energy scale for the commensurate AF fluctuation is one of intrinsic features of the present scheme, and is distinguished from the usual Fermi liquid as well as the local spin descriptions. To our knowledge, so far there is no other alternative theory could be able to obtain such a small energy scale (\( \ll J \)) for the
commensurate AF fluctuation. Furthermore, a prominent feature has been exhibited\textsuperscript{19,20} at the low energy part of the spectroscopy, where the weight of \( \chi'' \) is continuously suppressed below some characteristic temperature, resembling an opening of a spin gap. In the present theory, even though there is no real gap in the spin fluctuation spectrum, a characteristic temperature \( T^*_c \) is found to be naturally associated with a spin-gap-like phenomenon. In Fig. 1, a typical energy and temperature dependence of \( \chi''(Q_0, \omega) \) has been shown, where the spectral function is chosen to describe \( YBa_2Cu_3O_{6.6} \) (see below). And the overall energy and temperature features in Fig.1, particularly the “spin gap” behavior, are in good agreement with those found in the underdoped \( YBa_2Cu_3O_{6+x} \)\textsuperscript{19,20} In the underdoped cuprates, the NMR spin relaxation rate of planar copper nuclei\textsuperscript{88} manifests a non-Korringa behavior, which is suppressed when the spin gap feature shows up in the neutron scattering at low temperature. In contrast, a conventional Korringa temperature behavior\textsuperscript{88} is found for planar oxygen nuclei. The theoretical \( 1/T_1 T \) has been presented in Fig.2, which is calculated by using the same spectral function as used in Fig. 1 for \( YBa_2Cu_3O_{6+x} \). Fig. 2 shows a non-Korringa behavior at \( T > T^*_c \) as well as a “spin gap” feature below \( T^*_c \). All of them are also qualitatively consistent with the experimental measurements [the contribution of the present \textit{commensurate} AF fluctuation to \( 1/T_1 T \) is dramatically suppressed (\( \sim 1/93 \) at \( \xi \sim 4a \) for \( YBa_2Cu_3O_{6.6} \)) so that the non-Korringa signal does not leak to the oxygen sites]. Therefore, the \( q \)-, \( \omega \)- and \( T \)- dependence of the present low-lying magnetic fluctuation at small doping are all in agreement with the main experimental features found in the underdoped cuprates.

Technically, let us discuss the spectral function used in the theoretical calculation. According to Eq.(3.31), one may use the experimental measurement of \( \chi''(Q_0, \omega) \) at low temperature to determine \( \rho_b(0, \omega) \), instead of a first-principal calculation which would involve much complicate factors here like inter-layer coupling. As \( \rho_b(0, \omega) \) satisfies the normalized condition, if its broadening at low temperature is quite large as compared to the temperature scale which we are interested, the temperature dependence of \( \rho_b(0, \omega) \) may not be important. Then \( \chi''(Q_0, \omega) \) in a whole temperature range can be determined. In Fig.1, we have
chosen \( \rho_b(0, \omega) \) such that to give a right energy scale of \( \chi''(Q_0, \omega) \) for \( YBa_2Cu_3O_{6.6} \) at low-temperature [Fig.7 in Ref. 20]. The contribution from the higher Landau levels are neglected because the temperature range considered is comparatively smaller than the broadening of the LLL. Besides the shape of \( \rho_b(0, \omega) \), an overall strength of \( \chi''(Q_0, \omega) \) has also been adjusted, in order to compare with experiment, by reducing the spinon concentration from \( n_s = 1 - \delta \) to \( n_s^* < n_s \). This is because in the present approximation the spinons are treated as ideal bosons instead of hard-core bosons. At half-filling, this approximation could lead to a too big magnetization. In the present doped case, it would also give rise to a too strong magnetic fluctuation. The hard-core effect as well as the incoherent band in the spinon spectrum should reduce the effective number of spinons contributing to the AF correlation. We find a reduction of \( n_s^*/n_s \sim 1/3 \) at \( \delta = 0.10 \) giving a \( T_c^* \sim 175K \), in consistence with the corresponding experimental characteristic temperature (\( \sim 160K \)) and at the same time, leading to a spin relaxation rate whose magnitude quantitatively agreeing with the NMR measurement.

To end discussion of the underdoped materials, we give several remarks below. Some authors have attributed the spin gap phenomenon in the underdoped \( YBCO \) to their peculiar bi-layer structure. In contrast, one may have noticed that the theory here is purely two dimensional. Nevertheless, it has been noted that the spinon spectral function \( \rho_b(0, \omega) \) for the LLL has been determined directly from the experiments, in which the inter-layer coupling could have been already included and may play a key role in the broadening. Recall that in the present scheme, there is no real gap opened in the spin spectrum, and the “gap” feature is very sensitive to the detailed broadening of the LLL. Theoretically, for a pure 2D system one would expect the broadening of the LLL to be much narrower at low temperature. The reason is that a spinon in the LLL could not be scattered into a lower energy state while emits a phonon-like excitation of \( \delta A^h \), because it is already at the energy bottom. It cannot jump up to higher Landau-levels either, due to the energy conservation. Such a feature has been indeed found in a similar problem. If this were true, the energy scale in the underdoped material might have been much sharper than observed.
But in the insulating cuprates like $La_{1.95}Ba_{0.5}CuO_4$ [Ref. [17]], holons should be localized and randomly distributed such that $\delta A^h$ describes a random flux with energy scale $\simeq 0$, instead of a phonon-like dynamic mode. Then the degeneracy of the LLL can be lifted under the static random vector potential, and consequently a much broadened LLL could appear. And mixing with other levels is also expected here. For the underdoped $YBCO$, however, one is in the metallic phase, and holes must be mobile which would lead to a well-defined dynamics for $\delta A^h$. Nonetheless, one may still expect a strong localization tendency of holes at low temperature, as suggested by the transport measurements which could in turn lead to a more broadened LLL than in the optimally-doped regime. Furthermore, the bi-layer coupling in $YBCO$ can split a sharp LLL into two peaks (symmetric and antisymmetric states), but for a larger broadened LLL, the bi-layer coupling may well result in a single, much broadened peak like the one shown in $YBa_2Cu_3O_{6.6}$. Finally, we point out that in these underdoped materials, incommensurate structure has not been observed yet at high energy. But the width $\sigma$ in Eq.(3.26) has been found to be increased with the energy, which is an indication of the involvement of inter-level transition in the present theory.

Next we consider optimally doped cuprates, like $YBa_2Cu_3O_{7-y}$ and metallic $La_{2-x}Sr_xCuO_4$. A striking common feature in these materials is the lack of a commensurate AF fluctuation at low energy in neutron measurements. The spin-polarized neutron-scattering measurement of $YBa_2Cu_3O_{7-y}$ has only revealed a sharp high energy AF peak around $41 \text{ meV}$. In the metallic $LSCO$ ($x = 0.075, 0.14$ and $0.15$), an incommensurate structure has been found down to an energy scale $\simeq 1 \text{ meV}$, and no commensurate AF correlation is observed within the experimental resolution. However, the absence of a low-energy AF fluctuation or the presence of a low-energy incommensurate fluctuation would be both in sharp conflict with the NMR measurements which probes $\omega \simeq 0$ regime and implies a strong commensurate AF correlations in these materials. For example, by extrapolating the spin susceptibility with the incommensurate structure observed by neutron-scattering down to the NMR frequency ($\sim 10^{-4} \text{ meV}$), it has been found that a large magnetic contribution could leak to $^{17}\text{O}(2)$ sites, in contrast to the measured spin relaxation rate.
$1/T_1$ which appears\cite{footnote1} to be completely dominated by a nonmagnetic ($\sim q$ independent) contribution. As a matter of fact, the spin relaxation rates in $LSCO$ have shown canonical behaviors, which basically are the same as those found in $YBCO$ compounds. As elaborated in Sec. B3, only a commensurate contribution can lead to a non-Korringa behavior of $1/T_1$ at $^{63}Cu(2)$ sites, and at the same time, be strongly cancelled out at $^{17}O(2)$ sites. This fact has been the basis for the so-called nearly antiferromagnetic-Fermi-liquid theory\cite{footnote2,footnote14}. Therefore, in order to reconcile neutron and NMR experiments, one is led to the conclusion that a commensurate AF fluctuation should re-emerge in the optimally-doped cuprates within an energy scale beyond the experimental resolution.

However, it is hard to perceive such a characteristic scale in a conventional theory. So far, there have been a number of theoretical conjectures for the mechanism of incommensurate magnetic fluctuations in the metallic $LSCO$. In the Fermi-liquid-like framework, the incommensurability is directly connected\cite{footnote27,footnote29} with the Fermi-surface shape. But it lacks a small energy scale, within which a commensurate structure could be recovered. The spiral state of Shraiman and Siggia\cite{footnote93} based on the $t-J$ model also provides an incommensurate structure. The original spiral state is a long-ranged state, and some short-range versions have been proposed.\cite{footnote94,footnote95} But a low-energy commensurability is still hard to be comprehended here. Some non-intrinsic mechanism for the incommensurability due to the inhomogeneity in the $LSCO$ system has been also proposed\cite{footnote23} in order to reconcile the neutron and NMR data, where the commensurate AF fluctuation is assumed intrinsic in the metallic $LSCO$. Nevertheless, the details still need to be carried out in order to make comparison with the experiments.

The present spin-charge separation scheme is unique to have a small characteristic energy scale of the commensurate AF fluctuation, as already discussed in the underdoped case. As pointed out there, an even smaller energy scale could be present at optimal regime as the LLL broadening becomes sharper when holons are much mobile in this larger doping regime. Due to the narrowness of the energy range, the amplitude of $\chi''$ will be also small due to the cancellation of the Bose functions inside Eq.(3.23) [one has $\chi'' \rightarrow 0$ at $\omega/T \rightarrow 0$]. Therefore,
this commensurate AF correlation may well be beyond the experimental resolution to be
directly observed by neutron-scattering. In $YBa_2Cu_3O_{7-y}$, the bi-layer coupling could also
split the sharp LLL into two peaks as mentioned before. It then explains the sharp 41 $meV$
AF peak found by neutron-scattering as a result of the spinon transition between these two
split peaks. On the other hand, such an AF fluctuation can give rise to a non-Korringa law of
the NMR spin relaxation rate [Eq.(3.38)], which is consistent with the NMR measurements
in $YBCO$ and $LSCO$. Imai et al. have measured $1/63T_1$ in $La_{2-x}Sr_xCu_2O_4$ system up
to 900K. At high temperature, all data ($x = 0 - 0.15$) seem to converge and saturate to
the same temperature-independent value $2700 \pm 150 sec^{-1}$. The theoretical results shown in
Fig.3 agree well with this tendency. And if we choose, say, $\omega_0 = 1000 K$, we find that the
saturation value of $1/63T_1$ is about twice larger than the measured one, which is a reasonable
value, considering no adjustment of the spectral function has been made to fit the data.

In our theory, the incommensurate fluctuation in the metallic $LSCO$ will be attributed
to the dominance of the inter-Landau-level transition in the experimental energy-transfer
regime, as discussed in Sec.B3. To be consistent with neutron-scattering measurements, the
LLL broadening has to be very sharp as explained above, while the second level broadening
is relatively larger, which should be centered around $\omega_c \sim 10 \ meV$ in $x = 0.15$ case. When
the temperature is increased such that $k_B T \sim \omega_c$, a large amount of spinons are expected
to be thermally excited to the second Landau level. Then in the experimental-observable
energy regime, the intra-level transition could emerge again due to a larger broadening in
the second Landau level. Correspondingly, the incommensurate structure should be replaced
once again by the commensurate peak around $Q_0$. This has been indeed observed in the
neutron-scattering measurement where a broad commensurate peak is found to re-emerge
around $T \sim 100K$.

The characteristic energy scale $\omega_c$ of $YBCO$ seems to be several times larger than that of
$LSCO$ for some unknown reason. This decides a main distinction of spin dynamics between
$YBCO$ and $LSCO$, in terms of the present theory. In other words, one should expect that at
a sufficient high energy, incommensurate structures could also show up in $YBCO$ system. Of
course, many factors, especially the inter-layer coupling, may complicate the details. Due to these incommensurate contributions at high energy, the $q$-integrated susceptibility function $\chi''_{\text{total}}$ [Eq.(3.40)] will stretch up to an overall energy scale $\sim J_s$ with a roughly-constant amplitude (modulated by the oscillation). Such a unique $\chi''_{\text{total}}$ behavior is a high energy prediction of the present theory for the cuprates.

In the optimally-doped $LSCO$ and $YBCO$, the spin gap feature is absent in the commensurate spectrum due to the fact that the energy scale is too small. But according to the theory, a pseudo-gap should be also present in the $LSCO$ compounds below $T^*_c$, which involves the inter-level transition instead of intra-level transition in the commensurate fluctuation case. This pseudo-gap phenomenon has been clearly shown by the neutron scattering. In these optimally-doped cases, the spin characteristic temperature $T^*_c$ should become very close to the superconducting transition temperature $T_c$, to be consistent with neutron scattering and NMR measurements. In the present theory, superconducting transition will occur when spinons and holons are both condensed. Usually one finds $T^*_c > T_c$ at small doping. But when the spinons are condensed, the frustration effect on holons from the spin part (see next section) will be reduced too, which in turn is in favor of the Bose condensation of holons. In other words, $T_c$ and $T^*_c$ may correlate with each other. An optimal regime in the present theory may be properly defined as when $T^*_c$ and $T_c$ coincides. A further discussion of the superconducting transition will be presented elsewhere.

IV. TRANSPORT PROPERTIES

Due to spin-charge separation, holon degree of freedom is to be solely responsible for the electron transport phenomenon in this system. As noted before, the nonlocal phase $A^s_{ij}$ in the holon Hamiltonian Eq.(1.3) will provide an unconventional scattering mechanism in two dimensions, as in contrast to the 1D case where it vanishes and holons simply behave like free particles. In the following, we will explore 2D transport properties under such an effective Hamiltonian $H_h$. Similar to the magnetic properties discussed in Sec. III, transport will be
an another crucial test for the experimental relevance of the present spin-charge separation theory.

For simplicity, we may consider $H_h$ in the continuum limit at small doping. The validity of the continuum approximation will be discussed later. The corresponding continuum version of Eq.(1.3) has the following form:

$$\tilde{H}_h = \int d^2r \ h^+(r)\frac{(-i\nabla - A^s)^2}{2m_t}h(r),$$

(4.1)

where the holon mass $m_t = (\sqrt{2t_h}a^2)^{-1}$, and $A^s$ is defined by $A^s = A^s_+ + A^s_-$, with

$$A^s_\sigma = \frac{\sigma}{2} \int d^2r' \hat{\mathbf{z}} \times \frac{(r - r')}{|r - r'|^2}\rho^s_\sigma(r').$$

(4.2)

Here $\rho^s_\sigma(r) = b^\dagger_\sigma(r)b_\sigma(r)$ is the local spinon density with spin index $\sigma$. Similar to the lattice version [Eq.(1.5)], equation (4.2) describes fictitious $\pi$-flux quanta bound to spinons, and the sign $\sigma = \pm 1$ in front of the integration means that spinons with different spins carry flux tubes in opposite directions (see Fig.5). In an unpolarized system, one has $<\rho^\uparrow_\sigma >= <\rho^\downarrow_\sigma >$ so that on average $<A^s> = 0$. In other words, the scattering source as contributed by spinons may be regarded as a sort of fluctuating gauge field, and no $T$- and $P$-symmetry violations occur here.

Gauge-fluctuation-related scattering mechanisms have been intensively studied within the framework of the uniform RVB state. So we need to distinguish the present gauge field $A^s$ and those studied in the literature. Since the fictitious flux quanta are bound to spinons in the present case, let us first consider the local density distribution of spinons. At small doping, spinons are forming fluctuating AF domains, as discussed in Sec. III, within the spin correlation length $\xi > a$. For the purpose of illustration, an extreme case is shown in Fig.6, where $\xi \sim a$ and only a pair of $\uparrow$ and $\downarrow$ spinons are bound together like a valence bond. $+$ and $-$ signs in Fig.6 represent a spin configuration, or equivalently, directions of the $\pi$-flux quanta attached to spinons. If a holon moves about a closed path $C$ as shown in Fig.6, the accumulated flux will obviously have a large cancellation within the loop, with only the contribution from those near the path $C$ whose pairs are cut by the loop. Then the
mean-square accumulated flux will follow a perimeter law: \( \propto L_C \) (\( L_C \) is the perimeter of the loop C). It is easy to observe that this perimeter law is generally true as long as there exists a short-range AF correlation, which is present at small doping and within a temperature range \( k_B T < J \). This is in contrast to the gauge fluctuation in the uniform RVB state where fluctuating flux satisfies an area law instead of a perimeter law. And the present flux problem is more similar to a random flux problem in the so-called “Meissner phase” where a perimeter law is present. But in the conventional case, the strength of the fluctuating flux is presumably weak so that the accumulated flux enclosed in a loop will become negligible when the loop is small enough. In this case only the long-distance behavior matters, and a perturbative approach may be applicable which is similar to a usual phonon-scattering problem.

However, in the present case, each flux tube attached to a spinon is quantized at \( \pi \). It means that a slight deformation of the path C, with one spinon enclosed or excluded, could lead to an additional Berry-phase \( \pm \pi \), or a sign change of the wavefunction. A strong phase interference is therefore expected at a short-distance due to the high density of spinons at small doping. Such a short-distance effect can drastically change the nature of scattering mechanism which in the usual case would only involve long-wavelength processes. This is going to be a key distinction between the present theory and the usual gauge theories. We will see that such a short-range interference can lead to an exotic “localization” effect of holons, and result in a set of very interesting transport anomalous in longitudinal and transverse channels.

We begin by considering a single holon problem. For an one-body problem, a Feynman path-integral formulation will become very useful. The transition amplitude for a holon to travel from \( a \) to \( b \) in space under the vector potential \( A^s \) is given by

\[
K(b, a) = \int_a^b D\mathbf{r}(t) e^{i\mathcal{S}[b,a]}, \tag{4.3}
\]

with the action

\[
\mathcal{S}[b, a] = \int_{t_a}^{t_b} dt \left[ \frac{m}{2} \dot{\mathbf{r}}^2 - \mathbf{r} \cdot (A^s_+ + A^s_-) \right]. \tag{4.4}
\]
By introducing the following identities

$$\int \frac{d^2 \lambda}{(2\pi)^2} e^{i \lambda \cdot (r - r_p)} = \delta(r - r_p),$$

(4.5a)

$$\int \frac{d^2 \beta}{(2\pi)^2} e^{i \beta \cdot (r - r_q)} = \delta(r - r_q),$$

(4.5b)

$K(b, a)$ may be rewritten as

$$K(b, a) = \int_a^b D\mathbf{r}(t) D\mathbf{r}_p(t) D\mathbf{r}_q(t) \int D\lambda(t) \int D\beta(t) e^{i S_{\lambda\beta}[b, a]},$$

(4.6)

in which

$$S_{\lambda\beta}[b, a] = \int_{t_a}^{t_b} dt \left[ \frac{m_h}{2} \dot{r}^2 + \frac{m_p}{2} \dot{r}_p^2 + \frac{m_q}{2} \dot{r}_q^2 \right] - \int_{t_a}^{t_b} d\mathbf{r}_p \cdot \mathbf{A}_+^s(r_p) - \int_{t_a}^{t_b} d\mathbf{r}_q \cdot \mathbf{A}_-^s(r_q)
+ \int_{t_a}^{t_b} dt \left[ \lambda(t) \cdot (r - r_p) + \beta(t) \cdot (r - r_q) \right],$$

(4.7)

where $m_h + m_p + m_q = m_t$. Extra degrees of freedom are thus introduced in Eq.(4.6). Without the last term involving the Lagrangian multipliers $\lambda$ and $\beta$, $S_{\lambda\beta}[b, a]$ in Eq.(4.7) would simply describe three independent particles: a free holon with mass $m_h$ known as $h$ species, $p$ and $q$ species which interact with the vector potentials $\mathbf{A}_+^s$ and $\mathbf{A}_-^s$, respectively. The fields $\lambda$ and $\beta$ play the role to recombine these three species together as a real holon, and thus effectively eliminate the additional degrees of freedom in the end.

So far no approximation has been made. By introducing $p$ and $q$ degrees of freedom, the effects of $\mathbf{A}_+^s$ and $\mathbf{A}_-^s$ are separated. In terms of Eq.(4.2), $\mathbf{A}_\sigma^s$ may be rewritten as

$$\mathbf{A}_\sigma^s = \sigma \bar{\mathbf{A}}^s + \delta \mathbf{A}_\sigma^s,$$

where

$$\bar{\mathbf{A}}^s = \frac{B_s}{2} (\hat{z} \times \mathbf{r}),$$

(4.8)

with $B_s = (1 - \delta)\pi/2$, and $\delta \mathbf{A}_\sigma^s$ has the same form as Eq.(4.2) but with $\rho_\sigma^s$ replaced by $\delta \rho_\sigma^s = \rho_\sigma^s - < \rho_\sigma^s >$. Similar procedure has been used in dealing with the topological phase $\mathbf{A}_h$ in spinon system (cf. Sec. III), and is familiar in an anyon problem. Here $\sigma \bar{\mathbf{A}}^s$ describes a mean-field effect with the flux-quanta smeared out uniformly in space. And $\delta \mathbf{A}_\sigma^s$ represents the fluctuation of the flux-quanta due to the density fluctuation of spinons with
spin index $\sigma$. In the following, we shall show that, to a leading order of approximation, the effect of $\delta A_\sigma$ in the action (4.7) could be effectively represented by a relaxation of the binding constraint enforced by $\lambda$ and $\beta$ within a scale of the cyclotron length $d_c$ of $p$ and $q$ species (under the fictitious field $B_s$). Under such an approximation a renormalized holon will be a composite particle of a bare holon and the $p$ and $q$ species which are in the cyclotron orbitals. Below we elaborate this approximation.

Due to the symmetry, one only needs to focus on the $p$ species. Let us consider an arbitrary closed path $C$ on a $2D$ plane (Fig.7). The contribution of $A_\sigma^s$ to the transition amplitude in Eq. (4.6) for such a path will be a gauge-invariant phase:

$$\oint_C dr_p \cdot A_\sigma^s = \oint_{C_p} dr_p \cdot \bar{A}^s + \oint_C dr_p \cdot \delta A_\sigma^s.$$  

Eq. (4.9) may be further rewritten as

$$\oint_C dr_p \cdot A_\sigma^s = \oint_{C_p} dr_p \cdot \bar{A}^s,$$

in which a path $C_p$ is introduced such that

$$\left( \oint_{C_p} - \oint_C \right) (dr_p \cdot \bar{A}^s) = \oint_C dr_p \cdot \delta A_\sigma^s = \pi \Delta N_c^{\uparrow}.$$  

Eq. (4.10) in which a path $C_p$ is introduced such that

$$\left( \oint_{C_p} - \oint_C \right) (dr_p \cdot \bar{A}^s) = \oint_C dr_p \cdot \delta A_\sigma^s = \pi \Delta N_c^{\uparrow}.$$  

Here the path $C_p$ as shown in Fig.7 is a deformation of the path $C$ to account for the fluctuation of $\delta A_\sigma^+ \uparrow$ in terms of the mean-field $\bar{A}^s$. In second line of Eq.(4.11), $\Delta N_c^{\uparrow}$ represents the total fluctuating $\uparrow$ spinon number (with regard to the average one) enclosed by the path $C$, and such a term describes the total fluctuating fluxes bound to $\uparrow$ spinons as deduced from the right-hand-side integral in the first line. So the flux enclosed between the paths $C$ and $C_p$ under a uniform field $B_s$ will represent the fluctuation of the flux quanta attached to $\uparrow$ spinons inside the path $C$. In the following we shall point out that $\pi \Delta N_c^{\uparrow}$ should satisfy a perimeter law discussed at the beginning of this section.

In the spin background, there is no density fluctuation due to the no-double-occupancy constraint, and locally an increase of $\uparrow$ spinons is always compensated by a decrease of $\downarrow$
spinons. Thus the distribution of $\uparrow$ spinons will actually reflect that of all spins in the hole-absent region. With the presence of a short-range AF correlation (the correlation length $\xi > a$), excess and deficit of $\uparrow$ spinons are neighboring to each other (a $\xi \sim a$ case is shown in Fig.6). In such a “Meissner phase”, the net contribution to $\Delta N_c^\uparrow$ mainly comes from $\uparrow$ spinons close to the path $C$, and $\Delta N_c^\uparrow$ should satisfy a perimeter law instead of an area law as explained before. Correspondingly, the path $C_p$ should be always near the path $C$ to account for $\Delta N_c^\uparrow$ in terms of Eq. (4.11), as shown in Fig.7. An average separation $d_p$ of $C_p$ from $C$ may be estimated as follows: $\pi d_p^2 \times B_s \sim$ one flux quanta $= \pi$, which leads to $d_p \sim 1/\sqrt{B_s} = d_c$ — the cyclotron length under $B_s$.

So after one replaces $A_s^\times$ by $\bar{A}^s$ in Eq.(4.7), the fluctuation effect can be taken into account by deforming the path of $p$ species from $C$ to $C_p$. The deviation of $C_p$ from $C$ depends on the detailed fluctuation of $A_s^\times$. But if temperature is sufficiently higher than the spinon characteristic energy scale (the broadening of the Landau level in the spinon spectrum), the path $C_p$ shown in Fig.7 may be reasonably regarded as a random one, with an average separation $d_c$ from the path $C$. This effectively means that the binding constraints of $h$, $p$ and $q$ implemented by the Lagrangian fields $\lambda$ and $\beta$ have been relaxed within a scale $\sim d_c$. Here the kinetic energy cost of the deformation path $C_p$ is neglected due to the smallness of $d_c$. In the weak gauge-fluctuation case, $d_c$ could be too large for the present approximation to be valid.

Thus, the motion of a holon under the influence of the gauge field $A^s$ can be effectively described as a bare holon bound to a pair of auxiliary species, $p$ and $q$, which are undergoing cyclotron motions in opposite directions. These $p$ and $q$ species reflect a “localized” effect caused by the phase interference at short-distances. As $p$ and $q$ are confined in the Landau levels, a large degeneracy is involved here. In real space, such a renormalized holon would look like a polaron, and behave like a diffusive particle in the absence of external fields. This peculiar structure will decide a unique transport phenomenon.

The generalization of the above scheme to a many-body case is straightforward. Here one has to be cautious about the statistics of each species in the many-body case. A holon as
A composite particles of $h, p,$ and $q$ species has to satisfy the (hard-core) bosonic statistics. Thus a symmetric choice would be that the $h$ species corresponds to a boson while the $p$ and $q$ species are spinless fermions (so that the hard-core condition can be automatically realized). Then the effective many-body Lagrangian as a generalization of the one-body approximation discussed above can be written down in the functional-integral formalism as

$$
\mathcal{L}_h = \int d^2r \left\{ \hbar^+ \partial_r h + p^+ \partial_r p + q^+ \partial_r q + \lambda(p^+ p - h^+ h) + \beta(q^+ q - h^+ h) \right\} \\
+ \int d^2r \left\{ h^+(r) \frac{(-i \nabla + a^{ext})^2}{2 m_h} h(r) + p^+(r) \frac{(-i \nabla - \vec{A}s)^2}{2 m_p} p(r) + q^+(r) \frac{(-i \nabla + \vec{A}s)^2}{2 m_q} q(r) \right\}, \tag{4.12}
$$

in which $a^{ext}$ is an external vector potential. And the fields $h(r), p(r)$ and $q(r)$ are in the coherent representation. The Lagrangian multipliers $\lambda(r)$ and $\beta(r)$ enforce the following binding constraint

$$
h^+(r)h(r) = p^+(r)p(r) = q^+(r)q(r), \tag{4.13}
$$

in the length scale larger than $d_c$. An ultraviolet momentum cut-off $\Lambda = 1/d_c$ is then implied in $\lambda$ and $\beta$ fields in the Lagrangian $\mathcal{L}_h$. The masses $m_h, m_p$ and $m_q$ are determined up to that their total summation is equal to $m_t$. The value of each individual mass has to be decided beyond the present approximation, and this uncertainty will not affect the general conclusions we shall draw from the Lagrangian $\mathcal{L}_h$.

Therefore, we obtain an effective long-wavelength Lagrangian (4.12) for the holons after the short-distance phase-interference is carefully treated. This is a rather unusual Lagrangian because there are three auxiliary species involved. But it is not derived in its first time here. A similar one has already been found in the Scheme One flux-binding state (see Sec. II) through different method. Even though the approximations involved in these two states are different, their origins are quite similar and it is not of surprise finding their normal states to be so close. In fact, the structures of two effective Lagrangians for holons are identical, except that $B_s$ is twice larger in Ref. and the $h$ field is a fermionic one there. A larger $B_s$ in Ref. is due to the $\pi$-flux phase $\phi_{ij}$ has been incorporated into $B_s$ at continuum limit. These differences do not change the canonical behaviors described below.
The effective Lagrangian $L_h$ determines an anomalous transport phenomenon which amazingly matches the essential characteristics of the optimally doped cuprates. We shall outline the main results in the following. For detailed discussions, one is referred to Ref. \[74\]. $L_h$ can be treated by the standard gauge-theory approach. Here it is even simpler for lacking the transverse fields. It is easy to show that the longitudinal fields $\lambda$ and $\beta$ will enforce the following current constraint among $h$, $p$, and $q$ species:

$$J^l_h = J^l_p = J^l_q, \quad (4.14)$$

where the superscript $l$ implies the longitudinal channel. This constraint is consistent with the density constraint [Eq.(4.13)]. It is important to note that there is no similar current constraint in the transverse channel due to the absence of the transverse gauge fields. The total response to an external electromagnetic field will be related to each species through Eq.(4.14). And different combination rules will thus be found in the longitudinal and transverse transport channels, which will lead to distinctive Hall angle behavior.

The scattering rates of $h$, $p$ and $q$ are decided by their coupling with the longitudinal gauge fields $\lambda$ and $\beta$ in Eq.(4.12), whose dynamics are in turn determined by coupling to $h$, $p$, and $q$ species. A self-consistent treatment is required here. Due to the peculiar feature that $p$ and $q$ stay in the Landau level, one finds that the scattering rate for $p$ and $q$ goes like linear in temperature, i.e., $\frac{\hbar}{\tau_s} = 2\kappa k_B T$, where $\kappa \sim O(1)$ is independent of the coupling constant (the masses) and only has a weak doping-dependence (and $\frac{1}{\tau_s} \propto \omega$ is also found when $\hbar \omega > k_B T$).

And for the $h$ species, the scattering rate has a $T^2$ behavior: $\frac{\hbar}{\tau_h} \propto \frac{(k_B T)^2}{t_h}$. In terms of the constraint Eq.(4.14), the total longitudinal resistivity can be determined and at $k_B T \ll t_h$ it is dominant by $1/\tau_s$ so that $\rho \propto T$. The linear-$T$ resistivity in the cuprates is indeed found to be related to a linear-$T$ relaxation rate. Particularly, the coefficient is roughly around a numerical factor of 2 for all the optimal materials. This is very interesting feature and is quantitatively consistent with the present theory, where the coupling-independent coefficient $2\kappa \sim 2$ is determined by the unique structure in the scheme. The transverse resistivity $\rho_{xy}$ can be also obtained, and as noted before the distinctive combination rules in
the longitudinal and transverse channels will lead to new consequences. Namely, the Hall angle $\Theta$ as defined by $\cot \Theta = \rho / \rho_{yx}$ is found to be related to the second scattering rate $1/\tau_h$: $\cot \Theta \propto 1/\tau_h \propto T^2$. Consequently the Hall coefficient $R_H$ follows a $1/T$ behavior.

The involvement of a second scattering rate $\propto T^2$ in the Hall angle for high-$T_c$ cuprates was first pointed out by Princeton group [Ref. 4,5] based on the analysis of the experimental data. The effective Lagrangian in Eq.(4.12) provides a microscopic theory for it in the first time. A magneto-resistance with $\Delta \rho / \rho \sim T^{-4}$ dependence has been also predicted in the longitudinal channel, which has been recently observed in $YBCO$ (We note that the overall sign is uncertain in the theory while experimentally it is found to be positive). Due to the “localized” effect of holon, a strongly doping-dependent thermopower has been obtained in the present framework, which also agrees well with the experimental measurements in the high-$T_c$ cuprates.

The transport properties determined by the effective Lagrangian $L_h$ may be called as the canonical ones, which well account for the optimally doped cuprates. In the following we briefly discuss the condition for deviation from such a canonical case in the present theory. The key assumption involved in deriving Eq.(4.12) from $\tilde{H}_h$ is the randomness of the flux quanta movement near the path $C$ shown in Fig.7, which leads to uncorrelated segments along the path $C_p$ so that $p$ and $q$ can be effectively treated as detached from $h$ at a scale of $d_c$. In this case, the detailed spin dynamics becomes irrelevant. In the optimal doped cuprates, the spinon LLL broadening is very sharp (cf. Sec. III) so that the above condition may be always satisfied at the normal-state temperature. For the underdoped $YBCO$, however, the LLL broadening is quite large, and when the thermal energy is less than it the detailed spin dynamics could get involved in $C_p$, especially when $T \sim T_c^*$, and spinons begin to condensate into the energy bottom. In this case, $C_p$ may not be treated as a random one with regard to $C$, and the present approximation could break down. As a consequence, a deviation from the canonical transport behaviors is expected below some temperature scale, which should be correlated with $T_c^*$. Experimentally, such a correlation between the transport and the “spin gap” is indeed found in the underdoped $YBCO$
system. Furthermore, at sufficiently small doping, the transport properties described by Eq. (4.12) itself can also deviate from the canonical behavior as discussed in Ref. 74.

Finally, we would like to make a comment on the continuum approximation of $H_h$. At small doping, the continuum approximation can be well justified for the spinon Hamiltonian $H_s$ because $A^h_{ij}$ in it is vanishingly small. In the present holon case, however, the strong fluctuation effect at short-distance is very important, even though $< A^h_{ij} >= 0$. Thus one would expect the lattice effect to be involved even at small doping. So far we have not been able to include such a lattice effect. Nevertheless, as the phase interference at short-distances is not a coherent effect, the role of the lattice effect may not be really crucial. Furthermore, $p$ and $q$ are just auxiliary particles to take care of the phases interference, which can be introduced in the continuum space even when lattice is included. A further study is still needed.

V. CONCLUSION

In this paper, we have obtained the spin-charge separation scheme based on a saddle-point state of the $t−J$ model. In this saddle-point state, we find a deconfinement of spin and charge degree of freedom at finite doping in 2D case, where the transverse gauge fluctuation as the confinement force is gapped. Such a gap disappears at half-filling, where spinons are presumably confined to form spin-1 excitations, and a long-range AF order has been recovered. This saddle-point state has been also shown to reproduce the known asymptotic spin-spin correlation in one dimension at both half-filling and finite-doping. Thus, in some important limits where the behaviors of the $t−J$ model are known, the present state has produced the right results. These constitute important check for a strongly-correlated model where the conventional approximation breaks down.

The most interesting properties have been found at finite doping in 2D case, when spin and charge become deconfined. In this regime, the saddle-point state becomes meaningful due to the suppression of the gauge fluctuation, and spinons and holons can be appropriately
treated as separated systems in terms of conventional approaches. We have shown that in this saddle-point state there still exist some exotic residual couplings between spin and charge degrees of freedom, in spite of the spin-charge deconfinement. These couplings are nonlocal in the sense that spinons can feel the existence of holons nonlocally by seeing some fictitious flux-quanta bound to the latter, and \textit{vice versa}. Different from a usual electron-phonon system where the coupling is a single \textit{interactive} term, spinons and holons here are scattered by each other in distinctive forms. These scattering forces lead to anomalous spin dynamics and transport properties in the present system. For example, a sharp AF peak centered at $Q_0$ is exhibited in the imaginary dynamic spin susceptibility with an energy scale much smaller than the exchange energy $J$. The width of such an AF peak in $q$-space is determined by doping concentration in a form $\propto \sqrt{\delta}$, which is temperature-independent but increases with the energy transfer. The NMR spin relaxation rate of nuclear spin due to coupling with such a magnetic fluctuation shows a non-Korringa behavior for planar $Cu$ nuclei and a strong suppression for planar $O$ nuclei. Furthermore, we have found a characteristic temperature $T_{c^*}$ below which all these AF anomalies get suppressed, in resemblance to a “spin gap” system if the superconducting temperature $T_c < T_{c^*}$ and $T_c < T < T_{c^*}$. Incommensurate AF fluctuation has also been found in this system at a higher energy scale. For the charge degree of freedom, we have demonstrated that the scattering from the spinon background leads to a strong phase interference at short-distance for holons, and an effective long-wavelength Lagrangian is derived. Such a Lagrangian has been found to give the following canonical transport phenomena: resistivity $\rho \sim T$ with $\hbar/\tau \simeq 2k_BT$; the Hall angle $\cot \theta_H \propto 1/\tau_h \propto T^2$; the magneto-resistance $\Delta \rho/\rho \propto T^{-4}$; and a strong doping-dependence of the thermopower.

The magnetic and transport properties of the present spin-charge separation state share amazing similarities with those found in the high-$T_c$ cuprates, as discussed in the context of the paper. Based on the theory, a consistent picture can be conjecturally formed for the normal-state of the cuprates. The optimally-doped materials can be defined as $T_c \sim T_{c^*}$, where the \textit{commensurate} AF fluctuation energy scale becomes very small, maybe unde-
etectable in terms of the present neutron-scattering resolution. Such a small energy scale is
a unique feature of the present state, and as discussed in the context, NMR and neutron-
scattering data in both metallic $LSCO$ and $YB_2Cu_3O_7$ can be reconciled here. And in the
transport channel, the canonical behaviors are exhibited in agreement with the transport
measurements. A larger commensurate AF fluctuation energy scale can be realized at a
smaller doping regime, where holons tend to be localized, and the interlayer coupling in
$YBCO$ could further help the broadening of the energy scale. In this regime, a commen-
surate AF fluctuation may become observable in neutron-scattering like in the underdoped
$YBCO$, and pseudo spin gap behaviors will also show up in both neutron-scattering and
NMR spin relaxation rate below $T^*_c (> T_c)$. In this case, a deviation from the canonical
behaviors are expected in the transport channel below a temperature scale around $T^*_c$. We
note that in the present theory, there is no direct experimental input except that the under-
lying $t-J$ model is widely perceived as a simplified description of the copper-oxide layer in
the cuprates. This becomes a compelling fact as so many important experimental features
are naturally exhibited in the present state. We have pointed out in the introduction that
the spin-charge separation is a key for all these delicate magnetic and transport anomalies
to appear.

Two other important issues remain to be clarified in the present spin-charge separation
scheme. One is about the single-electron properties, particularly the location of Fermi
surface, and the other is about the superconducting condensation in this framework. Since
an electron is described as composed of two bosonic holon and spinon, the accompanied
nonlocal phase fields in Eq.(1.6) will play a central role to restore the fermionic properties
of the electron. As outlined in the introduction, the nonlocal phase fields will be responsible
for an electronic Fermi surface satisfying the Luttinger volume theorem as well as a finite
pairing order parameter $< c^+_i c^+_j >$ (and its symmetry) when both holons and spinons are
Bose condensed. We shall discuss these important problems in the follow-up papers.
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As usual, \( \exp(iA_{ij}^s) \) in Eq.(1.3) will be expanded up to the quadratic term of \( A_{ij}^s \). The leading term in \( \tilde{H}_h \) will be a tight-binding model with the presence of \( \pi \)-flux per plaquette. A continuum expansion near the band bottom will yield an additional \( \sqrt{2} \) mass enhancement due to the flux \( \phi_{ij}^0 \) in \( H_h \).

It is interesting to note that a characteristic short-distance scale under a uniform magnetic field is the cyclotron length which can be defined as the radius of a circle which encloses a flux just equal to \( \pi \): 

\[
\pi \times (\text{cyclotron length})^2 \times \text{magnetic field} = \pi.
\]

Beyond the cyclotron length, phase destruction could greatly suppress the amplitude of the wavepacket.

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CAPTIONS

Fig. 1. $\chi''(Q_0, \omega)$ vs. $\omega$. A pseudo spin-gap behavior is exhibited at $T < T^*_c$, where the low-energy part ($\approx 5meV$) is continuously suppressed with the decrease of the temperature. The spectral function in $\chi''(Q_0, \omega)$ is chosen to describe $YBa_2Cu_3O_6.6$ (see context).

Fig. 2. The NMR spin-lattice relaxation rate for the planar $Cu$ nuclei calculated in terms of the same spectral function used in Fig. 1.

Fig. 3. High-temperature behaviors of the $Cu$ nuclear spin relaxation rate at various doping concentration. Notice that the spin relaxation rates are leveled off at high temperature and essentially indistinguishable at small doping ($\delta < 0.2$).

Fig. 4. Typical $\chi''(q, \omega)$ vs. $q$ at various $\omega$’s. With the increase of $\omega$, the Gaussian form first gets broadened (dotted curves with $\omega = 0.10$ and $0.15$ [\$\omega_0$ as the unit]), and then an incommensurate split occurs when the inter-level transition becomes dominant (solid curves with $\omega = 0.18$, $0.20$ and $0.30$).

Fig. 5. Fictitious flux-tubes attached to the spinons (solid circles) which are seen by the holon (open circle) on a 2D plane.

Fig. 6. An extreme case of short-range AF correlation. The + and − signs represent a spin configuration, or equivalently, an array of fictitious flux-tubes bound to the spinons. The solid closed path cuts through those dimer pairs of spins and the total flux enclosed satisfies a perimeter law (see Ref. 98).

Fig. 7. An arbitrary closed path $C$ of a holon (solid one) and a corresponding path $C_p$ for $p$ species (dashed one).