LINEAR STATISTICS FOR ZEROS OF RIEMANN’S ZETA FUNCTION

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ABSTRACT. We consider a smooth counting function of the scaled zeros of the Riemann zeta function, around height $T$. We show that the first few moments tend to the Gaussian moments, with the exact number depending on the statistic considered.

1. Introduction

In this paper we will examine linear statistics of zeros of the Riemann zeta function. Denote its nontrivial zeros by $1/2 + i\gamma_j$, $j = \pm 1, \pm 2, \ldots$ with $\gamma_{-j} = -\gamma_j$ and $\Re(\gamma_1) \leq \Re(\gamma_2) \leq \ldots$. Let $N(T)$ denote the number of zeros in the strip $0 < \Re(\gamma) \leq T$, then $N(T) = \mathcal{N}(T) + S(T)$ where

$$\mathcal{N}(T) = 1 + \frac{1}{\pi} \text{Im} \log \left( \frac{\pi - iT}{2} \Gamma \left( \frac{1}{4} + \frac{1}{2} iT \right) \right)$$

$$= \frac{T}{2\pi} \log \frac{T}{2\pi e} + \frac{7}{8} + O(1/T)$$

Selberg [6] has studied the distribution of the remainder term in the counting function, $S(t) = N(t) - \mathcal{N}(t)$, as $t$ varies between $T$ and $T + H$, where $H = Ta$ with $1/2 < a \leq 1$. He showed that $S(t)$ has Gaussian moments in the sense that when $T \to \infty$,

$$\frac{1}{H} \int_T^{T+H} \left| \frac{S(t)}{\sqrt{(\log \log t)/2\pi^2}} \right|^2 dt \to \frac{(2k)!}{k!2^k}$$

Fujii [1], among others, has studied the distribution of $N(t + h) - N(t)$ around $t$ near $T$. Since $\frac{1}{H} \int_T^{T+H} S(t) dt \to 0$, the mean of this is asymptotic to $\mathcal{N}(T+h) - \mathcal{N}(T)$, and the error term, $S(t+h) - S(t)$, (which is thus asymptotically centered) has Gaussian moments, so long as $h$ is larger than the mean spacing of zeros at that height. That is, if $h \log T \to \infty$ but $h \ll 1$ then

$$\frac{1}{T} \int_T^{2T} \left( \frac{S(t+h) - S(t)}{\sigma} \right)^{2k} dt = \frac{(2k)!}{2^k k!} + O \left( \frac{1}{\sigma} \right)$$

where

$$\sigma^2 = \frac{1}{\pi^2} \int_0^{h \log T} \frac{1 - \cos t}{t} \, dt$$

He has similar results for when $h \to \infty$ subject to $h \ll T$, when $\sigma^2$ is replaced by $\sigma^2' = \frac{1}{\pi^2} (\log \log T - \log |\zeta(1 + ih)|)$. 

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Note that if $h$ is of the order of the mean spacing, that is if $h = \mathcal{O}(1/\log T)$, then the main term is the same size as the error term (that is, both are $\mathcal{O}(1)$), and we may no longer conclude the distribution is Gaussian. This is not surprising, since for $h = \mathcal{O}(1/\log T)$ the distribution of $N(t + h) − N(t)$ in the large $T$ limit is discrete.

In this paper we will study the counting function in that critical scaling. Rather than study $N(t)$ itself, instead we will investigate the distribution of a smooth version of the counting function in intervals of size comparable to the mean spacing, $2\pi/\log T$. In particular, for a real-valued even function $f$, and real numbers $\tau$ and $T > 1$, set

$$N_f(\tau) := \sum_{j = \pm 1, \pm 2, \ldots} f(\log T/2\pi (\gamma_j - \tau)).$$

If $f$ is the characteristic function of an interval $[-1, 1]$ and if all the $\gamma_j$ are real, then $N_f(\tau)$ counts the number of zeros in the interval $[\tau - 2\pi/\log T, \tau + 2\pi/\log T]$. However, we will take $f$ so that its Fourier transform, $\hat{f}(u) := \int_{-\infty}^{\infty} f(x)e^{-2\pi i xu}dx$, is smooth and of compact support, and will not assume the Riemann Hypothesis.

As $T \to \infty$, we consider the fluctuations of $N_f(\tau)$ as $\tau$ varies near $T$ in an interval of size about $H = T^a$, where $0 < a \leq 1$. More precisely, given a weight function $w \geq 0$, with $\int_{-\infty}^{\infty} w(x)dx = 1$, and $\hat{w}$ compactly supported, we define an averaging operator

$$\langle W \rangle_{T,H} := \int_{-\infty}^{\infty} W(\tau)w(\frac{\tau - T}{H}) \frac{d\tau}{H}.$$

We will show that the expected value of $N_f$ is

$$\langle N_f \rangle_{T,H} = \int_{-\infty}^{\infty} f(x)dx + \mathcal{O}(\frac{1}{\log T}).$$

We will also show that for $\hat{f} \in C^\infty_c(\mathbb{R})$ the first few moments of $\langle (N_f)^m \rangle_{T,H}$ of $N_f$ are Gaussian:

**Theorem 1.1.** Let $H = T^a$ with $0 < a \leq 1$, and let $\hat{f} \in C^\infty_c(\mathbb{R})$ be such that $\text{supp} \hat{f} \subseteq (-2a/m, 2a/m)$. Then the first $m$ moments of $N_f$ converge as $T \to \infty$ to those of a Gaussian random variable with expectation $\int_{-\infty}^{\infty} f(x)dx$ and variance

$$\sigma_f^2 = \int_{-\infty}^{\infty} \min(|u|, 1)\hat{f}(u)^2du.$$  

The local statistics of the critically scaled zeros of the Riemann zeta function around height $T$ (that is, zeros scaled by the mean density, $\frac{\log T}{2\pi}$) are believed \cite{3, 4} to behave like eigenangles of a random unitary matrix, when scaled by $N/2\pi$, which is their mean density. Indeed, a similar result to the theorem above holds in random matrix theory \cite{3}. Since the $\theta_n$ are angles, we consider the $2\pi$–periodic function

$$F_N(\theta) := \sum_{j = -\infty}^{\infty} f\left(\frac{N}{2\pi}(\theta + 2\pi j)\right)$$

and model $N_f$ by

$$Z_f(U) := \sum_{j = 1}^{N} F_N(\theta_j)$$

where $U$ is an $N \times N$ unitary matrix with eigenangles $\theta_1, \ldots, \theta_N$. 
Writing $E$ to denote the average over the unitary group with Haar measure, then without any restrictions on the support of the function $f$, we found in [2] that $E\{Z_f(U)\} = \int_{-\infty}^{\infty} f(x) \, dx$, and that the variance is

$$\sigma(f)^2 = \int_{-\infty}^{\infty} \min(1,|u|) \hat{f}(u)^2 \, du$$

Observe that this is in complete agreement with the mean and variance of $N_f$ if $\hat{f}$ has the same support restrictions. Furthermore, we showed that for any integer $m \geq 2$, if $\text{supp} \hat{f} \subseteq [-2/m,2/m]$, then

$$\lim_{N \to \infty} E\{(Z_f - E\{Z_f\})^m\} = \begin{cases} 0 & \text{if } m \text{ odd} \\ \frac{m!}{2^{m/2}(m/2)!} \sigma(f)^m & \text{if } m \text{ even} \end{cases}$$

where $\sigma(f)^2$, the variance, is given in (2).

These are the moments of a normal random variable. However, the higher moments are not Gaussian, and we called this “mock-Gaussian behaviour”.

The random matrix results suggests that theorem 1.1 is not the complete truth. We expect the variance of $N_f$, (1), to hold without any restriction on the support of $\hat{f}$, and the $m$-th moment of $N_f$ to be Gaussian so long as $\text{supp} \hat{f} \subseteq [-2/m,2/m]$.

Random unitary matrices can also be used to model the low-lying zeros of Dirichlet $L$–functions, and mock-Gaussian behaviour was found there too [2]. Other classical groups (symplectic, special orthogonal) are believed to model different classes of $L$–functions (like the quadratic $L$–functions), and they also show mock-Gaussian behaviour [3].

2. Proofs

2.1. The explicit formula. Set $\Omega(r) = \frac{1}{2} \Psi(\frac{1}{4} + \frac{1}{2}ir) + \frac{1}{2} \Psi(\frac{1}{4} - \frac{1}{2}ir) - \log \pi$, where $\Psi(s) = \frac{\Gamma'(s)}{\Gamma(s)}$ is the polygamma function. We need the following version of Riemann’s explicit formula:

**Lemma 2.1.** Let $g(u) \in C_c^\infty(\mathbb{R})$ and let $h(r) = \int_{-\infty}^{\infty} g(u)e^{iru} \, du$. Then

$$\sum \delta\{\gamma_j\} = h(-\frac{i}{2}) + h(\frac{i}{2}) + \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r)\Omega(r) \, dr - \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} (g(\log n) + g(-\log n))$$

where $\Lambda(n)$ is the von Mangoldt function.

For $\hat{f} \in C_c^\infty(\mathbb{R})$, setting

$$h(r) = f(\frac{\log T}{2\pi}(r - \tau)),$$  
$$g(u) = \frac{e^{-iru}}{\log T} \hat{f}(\frac{u}{\log T})$$

we have $N_f(\tau) = \overline{N_f}(\tau) + S_f(\tau)$ where

$$\overline{N_f}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f\left(\frac{\log T}{2\pi}(r - \tau)\right) \Omega(r) \, dr$$  
$$+ f\left(\frac{\log T}{2\pi}(\frac{i}{2} - \tau)\right) + f\left(\frac{\log T}{2\pi}(\frac{i}{2} - \tau)\right)$$
\[ S_f(\tau) = -\frac{1}{\log T} \sum_{n \geq 2} \Lambda(n) \frac{j(\log T)}{\sqrt{n}} \left( e^{i \tau \log n} + e^{-i \tau \log n} \right) \] 

**Remark.** The conditions on \( f \) (which are determined by the explicit formula), that its Fourier transform has compact support and is infinitely differentiable, can be considerably weakened to requiring that \( f(r) \) is analytic in the strip \(-c \leq \Im(r) \leq 1 + c \) \( (c > 0) \) such that \( f(r) \ll (1 + |r|)^{-1+\delta} \) \( (\delta > 0, r \in \mathbb{R}) \).

### 2.2. The mean.

**Lemma 2.2.** For all \( A > 1 \),

\[ \int_{-\infty}^{\infty} f\left( \frac{\log T}{2\pi} (r - \tau) \right) \Omega(r) \frac{dr}{2\pi} = \frac{\Omega(\tau)}{\log T} \int_{-\infty}^{\infty} f(x)dx + O\left( \frac{1}{1 + |\tau| (\log T)^2} \right) + O\left( \frac{\log(1 + |\tau|)}{(\log T)^A} \right) \]

**Proof.** To evaluate the integral, we change variables and split the domain of integration into two parts:

\[ \int_{-\infty}^{\infty} f\left( \frac{\log T}{2\pi} (r - \tau) \right) \Omega(r) \frac{dr}{2\pi} = \frac{1}{\log T} \int_{-\infty}^{\infty} f(x)\Omega(\tau + \frac{2\pi x}{\log T})dx \]

\[ = \frac{1}{\log T} \left( \int_{|x| \leq Y} f(x)\Omega(\tau + \frac{2\pi x}{\log T})dx + \int_{|x| > Y} f(x)\Omega(\tau + \frac{2\pi x}{\log T})dx \right) \]

where \( Y \to \infty \) but \( Y = o(\log T) \), say \( Y = \sqrt{\log T} \).

For the bulk of the integral \( x/\log T \) is small, and so we expand

\[ \Omega(\tau + \frac{2\pi x}{\log T}) = \Omega(\tau) + O\left( \frac{1}{1 + |\tau| (\log T)} \right) \]

to find

\[ \frac{1}{\log T} \int_{|x| \leq Y} f(x)\Omega(\tau + \frac{2\pi x}{\log T})dx \]

\[ = \frac{\Omega(\tau)}{\log T} \int_{|x| \leq Y} f(x)dx + O\left( \frac{1}{1 + |\tau| (\log T)} \int_{|x| \leq Y} f(x)dx \right) \]

\[ = \frac{\Omega(\tau)}{\log T} \int_{-\infty}^{\infty} f(x)dx + O\left( \frac{1}{1 + |\tau|} \int_{-\infty}^{\infty} f(x)dx \right) \]

For the tail of the integral we use \( f(x) \ll |x|^{-N} \) for any \( N \gg 1 \) (which follows from \( f \in C_c^\infty(\mathbb{R}) \)) and Stirling’s formula (which yields \( \Omega(r) = \log(1 + |r|) + O(1) \) for all \( r \in \mathbb{R} \)) to find that it is dominated by

\[ \frac{1}{\log T} \int_{|x| > Y} |x|^N \log(1 + |\tau| + |x|/\log T) dx \]

\[ = \frac{2N}{Y^{N-1}\log T} \log(1 + |\tau| + Y/\log T) + \frac{2N}{\log T} \int_{x > Y} \frac{1}{x^{N-1}} \left( x + (1 + |\tau|)/\log T \right) dx \]

\[ \ll \log(1 + |\tau|) \]

\[ Y^{N-1}\log T \]
Thus we have
\[
\int_{-\infty}^{\infty} f \left( \frac{\log T}{2\pi} (r - \tau) \right) \Omega(r) \frac{dr}{2\pi} = \frac{\Omega(\tau)}{\log T} \int_{-\infty}^{\infty} f(x) dx + \mathcal{O}(\frac{\log(1 + |\tau|)}{(\log T)^2}) + \mathcal{O}(\frac{\log T}{YN - 1 \log T})
\]

Taking \( Y = \sqrt{\log T} \) gives (3).

**Lemma 2.3.** For all \( \hat{f} \in C_c^{\infty}(\mathbb{R}) \),
\[
\langle NF_f \rangle_{T,H} = \int_{-\infty}^{\infty} f(x) dx + \mathcal{O}(\frac{1}{\log T}), \quad T \to \infty
\]

**Proof.** Since \( \Omega(\tau) = \log(1 + |\tau|) + \mathcal{O}(1) \) for all \( \tau \), we have
\[
\langle \frac{\Omega(\tau)}{\log T} \int_{-\infty}^{\infty} f(x) dx \rangle_{T,H} = \int_{-\infty}^{\infty} f(x) dx \left( \int_{-\infty}^{\infty} \log(1 + |\tau|) w(\frac{T - \tau}{H}) \frac{d\tau}{H} + \mathcal{O}(1) \right)
\]
and since the average of the error term in lemma 2.2 is similarly \( \mathcal{O}(1/\log T) \), we have that
\[
\langle 2\pi f \left( \frac{\log T}{2\pi} (r - \tau) \right) \Omega(r) \frac{dr}{2\pi} \rangle_{T,H} = \int_{-\infty}^{\infty} f(x) dx + \mathcal{O}(\frac{1}{\log T})
\]

The averages of the polar terms, \( f \left( \frac{\log T}{2\pi} (\frac{i}{2} - \tau) \right) + f \left( \frac{\log T}{2\pi} (-\frac{i}{2} - \tau) \right) \), in \( NF_f \) are bounded by \( \mathcal{O}(1/\log T) \) since by Parseval
\[
\int_{-\infty}^{\infty} f \left( \frac{\log T}{2\pi} (\frac{i}{2} - \tau) \right) w(\frac{T - \tau}{H}) \frac{d\tau}{H} = \int_{-\infty}^{\infty} \frac{2\pi}{\log T} \hat{f}(\frac{2\pi y}{\log T}) e^{\pi y} \tilde{w}(H y) e^{-2\pi iTy} dy
\]
and since \( \hat{w} \) has compact support, the integral is over \( |y| \ll 1/H \) and is bounded by \( \mathcal{O}(1/H \log T) \).

**Proposition 2.4.** For \( f \) with \( \hat{f} \in C_c^{\infty}(\mathbb{R}) \), if \( H \to \infty \) then the mean value of \( N_f \) is given by
\[
\langle NF_f \rangle_{T,H} = \int_{-\infty}^{\infty} f(x) dx + \mathcal{O}(\frac{1}{\log T}), \quad T \to \infty
\]

**Proof.** In view of Lemma 2.3, it suffices to show that the mean value of \( S_f \) is zero as \( H \to \infty \). Indeed, we have
\[
\langle S_f \rangle_{T,H} = -\frac{1}{\log T} \sum_{n} \frac{\Lambda(n)}{\sqrt{n}} \hat{f} \left( \frac{\log n}{\log T} \right) \left( \hat{w}(\frac{H}{2\pi} \log n) e^{-iT \log n} + \hat{w}(\frac{-H}{2\pi} \log n) e^{iT \log n} \right)
\]
Since \( \hat{w} \) has compact support, and the prime powers \( n \) are at least 2, the summands vanish once \( H \) is larger than a certain constant which depends upon the support of \( \hat{w} \).
2.3. The centered moments.

Proof of Theorem 1.1. From lemma 2.2 and proposition 2.4 we have
\[
\left\langle \left( N_f - \langle N_f \rangle_{T,H} \right)^m \right\rangle_{T,H} = \sum_{n=0}^{m} \binom{m}{n} \left\langle S^{m-n}_f (N_f - \langle N_f \rangle_{T,H})^n \right\rangle_{T,H} = \left\langle S^m_f \right\rangle_{T,H} \left( 1 + O\left( \frac{1}{\log T} \right) \right)
\]
and so it is sufficient to show that the \( m \)th moment of \( S^f \) is the same as that of a centered normal random variable with variance given by (1). This is achieved in the theorem 2.6.

Before we calculate the \( m \)-th moment of \( S^f \), as a warm-up we will consider the variance.

Proposition 2.5. If \( H = T^a \) with \( 0 < a \leq 1 \) and \( \text{supp} \hat{f} \subseteq (-a, a) \) then we have
\[
\left\langle (S^f)^2 \right\rangle_{T,H} = \int_{-\infty}^{\infty} \min(|u|, 1) \hat{f}(u)^2 \, du + O\left( \frac{1}{\log T} \right)
\]

Proof. Using the expression (5), multiplying out \((S^f)^2\) and integrating we find
\[
\left\langle (S^f)^2 \right\rangle_{T,H} = \frac{1}{\log T} \sum_{\epsilon_1, \epsilon_2 = \pm 1} \sum_{n_1, n_2} \frac{\Lambda(n_1) \Lambda(n_2)}{\sqrt{n_1} \sqrt{n_2}} \hat{f}\left( \frac{\log n_1}{\log T} \right) \hat{f}\left( \frac{\log n_2}{\log T} \right) \\
\times \hat{w}\left( \frac{H}{2\pi} (\epsilon_1 \log n_1 + \epsilon_2 \log n_2) \right) e^{-iT(\epsilon_1 \log n_1 + \epsilon_2 \log n_2)}
\]

In order to get a nonzero contribution we need \( \epsilon_1 = -\epsilon_2 \) (since once \( H \) is larger than a certain constant which depends upon the support of \( \hat{w} \) all the \( \epsilon_1 = \epsilon_2 \) terms vanish). Furthermore, since \( \text{supp} \hat{f} \subseteq (-a, a) \) we have \( n_1 \leq T^a - \epsilon \) for some \( \epsilon > 0 \), and therefore
\[
\hat{w}\left( \frac{T^a}{2\pi} \log \frac{n_1}{n_2} \right) = 0
\]
(for large enough \( T \)) unless \( n_1 = n_2 \).

Therefore, taking into account that \( \hat{w}(0) = \int_{-\infty}^{\infty} w(x) \, dx = 1 \) we find as soon as \( T \) is sufficiently large,
\[
\left\langle (S^f)^2 \right\rangle_{T,H} = \frac{1}{\log T} \sum_n 2 \frac{\Lambda(n)^2}{n} \hat{f}\left( \frac{\log n}{\log T} \right)^2
\]
We note that by the Prime Number Theorem, as \( T \to \infty \)
\[
\frac{1}{\log T} \sum_n 2 \frac{\Lambda(n)^2}{n} \hat{f}\left( \frac{\log n}{\log T} \right)^2 \sim 2 \int_{0}^{\infty} u \hat{f}(u)^2 \, du + O\left( \frac{1}{\log T} \right)
\]
\[
= \sigma_f^2 + O\left( \frac{1}{\log T} \right)
\]
where \( \sigma_f^2 \) is given by (1), this being true since \( \text{supp} \hat{f} \subseteq (-1, 1) \) by assumption of the theorem. \( \square \)
Theorem 2.6. If $H = T^a$ with $0 < a \leq 1$ and supp $\tilde{f} \subset (-\alpha, \alpha)$ then for $2 \leq m < 2a/\alpha$
we have

$$\langle (S_f)^m \rangle_{T,H} \sim \begin{cases} \frac{(2k)!}{2} \sigma_j^2, & m = 2k \\ 0, & m = 2k + 1 \end{cases} + O\left(\frac{1}{\log T}\right)$$

where the variance $\sigma_j^2$ is given by (9).

Proof. Using the expression (5), multiplying out $(S_f)^m$ and integrating we find

$$\langle (S_f)^m \rangle_{T,H} = \left(\frac{-1}{\log T}\right)^m \sum_{\epsilon_1, \ldots, \epsilon_m = \pm 1} \sum_{n_1, \ldots, n_m} \prod_{j=1}^m \Lambda(n_j) \tilde{f}\left(\log \frac{n_j}{\log T}\right)$$

$$\times \tilde{w}\left(\frac{H}{2\pi} \sum_{j=1}^m \epsilon_j \log n_j\right) e^{-iT \sum_{j=1}^m \epsilon_j \log n_j}$$

Since $\tilde{w}$ has compact support, in order to get a non-zero contribution we need

$$|\sum_{j=1}^m \epsilon_j \log n_j| \ll \frac{1}{H}$$

Set $M = \prod_{\epsilon_j = +1} n_j$ and $N = \prod_{\epsilon_j = -1} n_j$. If $M \neq N$ then assume w.l.o.g. that $M > N$, say $M = N + u$ with $u \geq 1$. Thus for a non-zero contribution we need

$$\frac{1}{H} \gg \log \frac{M}{N} = \log(1 + \frac{u}{N}) \gg \frac{1}{N}$$

and hence $T^a = H \ll N \leq \sqrt{MN} \leq T^{m(\alpha-\epsilon)/2}$ since $n_j \ll T^{a-\epsilon}$ by assumption on the support of $\tilde{f}$. But $\alpha < 2a/m$, so this is a contradiction. Therefore $M = N$, and

$$\sum \epsilon_j \log n_j = 0.$$

Thus for $T \gg 1$, we find (taking into account that $\tilde{w}(0) = \int_{-\infty}^\infty w(x)dx = 1$)

$$\langle (S_f)^m \rangle_{T,H} = \left(\frac{-1}{\log T}\right)^m \sum_{\epsilon_1, \ldots, \epsilon_m = \pm 1} \sum_{n_1, \ldots, n_m \geq 2} \prod_{j=1}^m \Lambda(n_j) \tilde{f}\left(\log \frac{n_j}{\log T}\right)$$

$$\sum_{E \subseteq \{1, \ldots, m\}} J(E)$$

(8)

where

$$J(E) = \left(\frac{-1}{\log T}\right)^m \sum_{n_1, \ldots, n_m \geq 2} \prod_{j \in E} \Lambda(n_j) \tilde{f}\left(\log \frac{n_j}{\log T}\right)$$

and the subset of indices $E$ denotes the $\epsilon_j$ which are positive. That is $j \in E$ iff $\epsilon_j = +1$.

Fix a subset $E \subseteq \{1, \ldots, m\}$. The sum in $J(E)$ is over tuples $(n_1, \ldots, n_m)$ which satisfy $\prod_{j \in E} n_j = \prod_{j \notin E} n_j$. We say that there is a perfect matching of terms if there is a bijection $\sigma$ of $E$ onto its complement $E^c$ in $\{1, \ldots, m\}$ so that $n_j = n_{\sigma(j)}$, for all $j \in E$. This can happen only if $m = 2k$ is even and $\#E = \#E^c = k$.

Decompose

$$J(E) = J_{diag}(E) + J_{non}(E)$$

(10)
where $J_{\text{diag}}(E)$ is the sum of matching terms - the diagonal part of the sum (nonexistent for most $E$), and $J_{\text{non}}(E)$ is the sum over the remaining, nonmatching, terms.

2.3.1. Diagonal terms. Assume that $m = 2k$ is even. There are $\binom{2k}{k}$ subsets $E \subset \{1, \ldots, 2k\}$ of cardinality $k = m/2$, and for each such subset $E$, $J_{\text{diag}}(E)$ is the sum over all $k!$ bijections $\sigma : E \rightarrow E^c$ of $E$ onto its complement, of terms

\[
\frac{1}{(\log T)^2} \sum_n \frac{\Lambda(n)^2}{n} \hat{f}(\frac{\log n}{\log T})^2
\]

We evaluate each factor by using the Prime Number Theorem:

\[
\frac{1}{(\log T)^2} \sum_n \frac{\Lambda(n)^2}{n} \hat{f}(\log n)^2 \sim \frac{1}{(\log T)^2} \int_{2}^{\infty} \frac{\log t}{t} \hat{f}(\frac{\log t}{\log T})^2 \, dt
\]

\[
\sim \int_{0}^{\infty} u \hat{f}(u)^2 \, du
\]

Since our function is even and supported inside $(-\alpha, \alpha)$ and $\alpha < 2a/m \leq 1$, we can rewrite this as

\[
\frac{1}{2} \int_{-\infty}^{\infty} \min(1, |u|) \hat{f}(u)^2 \, du =: \sigma(f)^2/2
\]

This shows that for $m = 2k$ even we have as $T \rightarrow \infty$ that

\[
\sum_{E \subset \{1, \ldots, m\}} J_{\text{diag}}(E) \rightarrow \frac{(2k)!}{2^{k!}} \sigma(f)^{2k}
\]

Below we will show that the nondiagonal terms $J_{\text{non}}(E)$ are negligible, and hence by (8) and (10) we will have thus proved Theorem 2.6.

2.3.2. Bounding the off-diagonal terms $J_{\text{non}}(E)$. We will show that

\[
J_{\text{non}}(E) \ll \frac{1}{\log T}
\]

Proof. Since

\[
\frac{1}{\log T} \sum_p \sum_{k \geq 3} \frac{\log p}{p^{k/2}} \ll \frac{1}{\log T} \sum_p \frac{\log p}{p^{3/2}} \ll \frac{1}{\log T}
\]

the contribution of cubes and higher prime powers to (9) is negligible, and we may assume in $J_{\text{non}}(E)$ that the $n_i$ are either prime or squares of primes (upto a remainder of $O(1/\log T)$). By the Fundamental Theorem of Arithmetic, an equality $\prod_{j \in E} n_j = \prod_{i \in E^c} n_i$ forces some of the terms to match, and unless there is a perfect matching of all terms, the remaining integers satisfy equalities of the form $n_1 n_2 = n_3$ with $n_1 = n_2 = p$ prime and $n_3 = p^2$ a square of that prime. Thus upto a remainder of $O(1/\log T)$, $J_{\text{non}}(T)$ is a sum of terms of the form

\[
\left( \frac{1}{(\log T)^2} \sum_p \frac{(\log p)^2}{p^{2k}} \hat{f}(\frac{\log p}{\log T})^2 \right)^u \cdot \left( \frac{1}{(\log T)^3} \sum_p \frac{(\log p)^3}{p^{2k}} \hat{f}(\frac{\log p}{\log T})^2 \hat{f}(\frac{\log p^2}{\log T}) \right)^v
\]

with $2u + 3v = m$, and $v \geq 1$. 

We showed (11) that the matching terms have an asymptotic value, hence are bounded. We bound the second type of term by

\[
\frac{1}{(\log T)^3} \sum_p \frac{(\log p)^3}{p^2} \hat{f}(\frac{\log p}{\log T})^2 \hat{f}(\frac{\log p^2}{\log T}) \ll \frac{1}{(\log T)^3} \sum_p \frac{(\log p)^3}{p^2} \ll \frac{1}{(\log T)^3}
\]

Thus as long as \( v \geq 1 \) (that is if there is no perfect matching of all terms), we get that the contribution is \( \mathcal{O}(1/\log T) \).

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