NOTE ON EQUIVARIANT $\mathcal{I}$-FUNCTION OF LOCAL $\mathbb{P}^n$

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ABSTRACT. Several properties of a hypergeometric series related to Gromov-Witten theory of some Calabi-Yau geometries was studied in [8]. These properties play basic role in the study of higher genus Gromov-Witten theories. We extend the results of [8] to equivariant setting for the study of higher genus equivariant Gromov-Witten theories of some Calabi-Yau geometries.

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0. Introduction

0.1. Local $\mathbb{P}^n$ geometries. Equivariant local $\mathbb{P}^n$ theories can be constructed as follows. Let the algebraic torus

$$T_{n+1} = (\mathbb{C}^*)^{n+1}$$

act with the standard linearization on $\mathbb{P}^n$ with weights $\lambda_0, \ldots, \lambda_n$ on the vector space $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. Let $\overline{M}_g(\mathbb{P}^n, d)$ be the moduli space of stable maps to $\mathbb{P}^n$ equipped with the canonical $T_{n+1}$-action, and let

$$C \to \overline{M}_g(\mathbb{P}^n, d), \ f : C \to \mathbb{P}^n, \ S = f^*\mathcal{O}_{\mathbb{P}^n}(-1) \to C$$

be the standard universal structures.

The equivariant Gromov-Witten invariants of the local $\mathbb{P}^n$ are defined via the equivariant integrals

$$N_{g,d}^{GW,n} = \int_{[\overline{M}_g(\mathbb{P}^n, d)]^{vir}} e\left(-R\pi_*f^*\mathcal{O}_{\mathbb{P}^n}(-n-1)\right).$$

The integral (1) defines a rational function in $\lambda_i$

$$N_{g,d}^{GW,n} \in \mathbb{C}(\lambda_0, \ldots, \lambda_n).$$

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We associate Gromov-Witten generating series by

\[ \mathcal{F}_{g,n}^\text{GW}(Q) = \sum_{d=0}^{\infty} \tilde{N}_{g,d} Q^d \in \mathbb{C}(\lambda_0, \ldots, \lambda_n)[[Q]]. \]

Motivated by mirror symmetry ([1] [2] [7]), we can make the following predictions about the genus \( g \) generating series \( \mathcal{F}_{g,n}^\text{GW} \).

(A) There exist a finitely generated subring \( G \subset \mathbb{C}(\lambda_0, \ldots, \lambda_n)[[Q]] \) which contains \( \mathcal{F}_{g,n}^\text{GW} \) for all \( g \).

(B) The series \( \mathcal{F}_{g,n}^\text{GW} \) satisfy holomorphic anomaly equations, i.e. recursive formulas for the derivative of \( \mathcal{F}_{g,n}^\text{GW} \) with respect to some generators in \( G \).

0.2. \textit{I}-function. \textit{I}-function defined by

\[ I_n = \sum_{d=0}^{\infty} \prod_{k=1}^{(n+1)d-1} \frac{(- (n + 1)H - k z)}{\prod_{i=0}^{n} \prod_{k} (H + k z - \lambda_i)} q^d \in H^*_T(\mathbb{P}^n, \mathbb{C})[[q]], \]

is the central object in the study of Gromov-Witten invariants of local \( \mathbb{P}^n \) geometry. See [5], [6] for the arguments. Several important properties of the function \( I_n \) was studied in [8] after the specialization \( \lambda_i = \zeta_{n+1}^i \)

where \( \zeta_{n+1} \) is primitive \( (n + 1) \)-th root of unity. For the study of full equivariant Gromov-Witten theories, we extend the result of [8] without the specialization (2).

0.3. Picard-Fuchs equation and Birkhoff factorization. Define differential operators

\[ D = q \frac{d}{dq}, \quad M = H + z D. \]

The function \( I_n \) satisfies following Picard-Fuchs equation

\[ \left( \prod_{i=0}^{n} (M - \lambda_i) - q \prod_{k=0}^{n} \left( - (n + 1)M - k z \right) \right) I_n = 0. \]

The restriction \( I_n|_{H=\lambda_i} \) admits following asymptotic form

\[ I_n|_{H=\lambda_i} = e^{\mu_i} \left( R_{0,i} + R_{1,i} z + R_{2,i} z^2 + \ldots \right) \]

with series \( \mu_i, R_{k,i} \in \mathbb{C}(\lambda_0, \ldots, \lambda_n)[[q]]. \)

A derivation of (3) is obtained from [3, Theorem 5.4.1] and the uniqueness lemma in [3, Section 7.7]. The series \( \mu_i \) and \( R_{k,i} \) are found
by solving differential equations obtained from the coefficient of $z^k$. For example,

$$\lambda_i + D\mu_i = L_i,$$

where $L_i(q)$ is the series in $q$ defined by the root of following degree $(n + 1)$ polynomial in $L$

$$\prod_{i=0}^{n}(L - \lambda_i) - (-1)^{n+1}qL^{n+1}.$$

with initial conditions,

$$L_i(0) = \lambda_i.$$

Let $f_n$ be the polynomial of degree $n$ in variable $x$ over $\mathbb{C}(\lambda_0, \ldots, \lambda_n)$ defined by

$$f_n(x) := \sum_{k=0}^{n}(-1)^kks_kx^{n-k},$$

where $s_k$ is $k$-th elementary symmetric function in $\lambda_0, \ldots, \lambda_n$. The ring

$$G_n := \mathbb{C}(\lambda_0, \ldots, \lambda_n)[L_0^{\pm 1}, \ldots, L_n^{\pm 1}, f_n(L_0)^{-\frac{1}{2}}, \ldots, f_n(L_n)^{-\frac{1}{2}}]$$

will play a basic role.

The following Conjecture was proven under the specialization (2) in [8, Theorem 4].

**Conjecture 1.** For all $k \geq 0$, we have

$$R_{k,i} \in G_n.$$}

Conjecture [1] for the case $n = 1$ will be proven in Section [2]. Conjecture [1] for the case $n = 2$ will be proven in Section [3] under the specialization [11]. In fact, the argument in Section [3] proves Conjecture [1] for all $n$ under the specialization which makes $f_n(x)$ into power of a linear polynomial.

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1. **Admissibility of Differential Equations**

Let $R$ be a commutative ring. Fix a polynomial $f(x) \in R[x]$. We consider a differential operator of *level* $n$ with following forms.

$$\mathcal{P}(A_{lp}, f)[X_0, \ldots, X_{n+1}] = DX_{n+1} - \sum_{n \geq l \geq 0, p \geq 0} A_{lp}D^pX_{n-l},$$

(4)
where $D := \frac{d}{dx}$ and $A_{lp} \in \mathbb{R}[x]_f := \mathbb{R}[x][f^{-1}]$. We assume that only finitely many $A_{lp}$ are not zero.

**Definition 2.** Let $R_i$ be the solutions of the equations for $k \geq 0$,
\begin{equation}
P(A_{lp}, f)[X_{k+1}, \ldots, X_{k+n}] = 0,
\end{equation}
with $R_0 = 1$. We use the conventions $X_i = 0$ for $i < 0$. We say differential equations $\text{(5)}$ is admissible if the solutions $R_k$ satisfies for $k \geq 0$,

\[ R_k \in \mathbb{R}[x]. \]

**Remark 3.** Note that the admissibility of $\mathcal{P}(A_{lp}, f)$ in Definition $2$ do not depend on the choice of the solutions $R_k$.

**Lemma 4.** Let $f$ be a degree one polynomial in $x$. Each $A \in \mathbb{R}[x]_f$ can be written uniquely as

\[ A = \sum_{i \in \mathbb{Z}} a_i f^i \]

with finitely many non-zero $a_i \in \mathbb{R}$. We define the order $\text{Ord}(A)$ of $A$ with respect to $f$ by smallest $i$ such that $a_i$ is not zero. Then

\[ \mathcal{P}(A_{lp}, f)[X_0, \ldots, X_{n+1}] := DX_{n+1} - \sum_{n \geq l \geq 0, p \geq 0} A_{lp} D^p X_{n-l} = 0, \]

is admissible if following condition holds:

\[ \text{Ord}(A_0) \leq -2, \]
\[ \text{Ord}(A_1) \leq 0, \]
\[ \text{Ord}(A_{lp}) \leq p + 1 \text{ for } p \geq 2. \]

**Proof.** The proof follows from simple induction argument. □

**Lemma 5.** Let $f$ be a degree two polynomial in $x$. Denote by
\[ \mathbb{R}_f \]
the subspace of $\mathbb{R}[x]_f$ generated by $f^i$ for $i \in \mathbb{Z}$. Each $A \in \mathbb{R}_f$ can be written uniquely as

\[ A = \sum_{i \in \mathbb{Z}} a_i f^i \]

with finitely many non-zero $a_i \in \mathbb{R}$. We define the order $\text{Ord}(A)$ of $A \in \mathbb{R}_f$ with respect to $f$ by smallest $i$ such that $a_i$ is not zero. Then

\[ \mathcal{P}(A_{lp}, f)[X_0, \ldots, X_{n+1}] := DX_{n+1} - \sum_{n \geq l \geq 0, p \geq 0} A_{lp} D^p X_{n-l} = 0, \]

is admissible if following condition holds:
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\[ A_{lp} = B_{lp} \quad \text{if } p \text{ is odd}, \]
\[ A_{lp} = \frac{df}{dx} \cdot B_{lp} \quad \text{if } p \text{ is even}, \]

where $B_{lp}$ are elements of $\mathbb{R}_f$ with
\[ \text{Ord}(B_{l0}) \leq -2, \]
\[ \text{Ord}(B_{lp}) \leq \left\lfloor \frac{p - 1}{2} \right\rfloor \text{ for } p \geq 1. \]

Proof. Since $f$ is degree two polynomial in $x$, we have
\[ \frac{d^2f}{dx^2}, \left( \frac{df}{dx} \right)^2 \in \mathbb{R}_f. \]

Then the proof of Lemma follows from simple induction argument. \qed

2. Local $\mathbb{P}^1$

2.1. Overview. In this section, we prove Conjecture $\Box$ for the case $n = 1$. Recall the $I$-function for $K\mathbb{P}^1$,

\[ I_1(q) = \sum_{d=0}^{\infty} \frac{\prod_{k=0}^{d-1} (-2H - kz)}{\prod_{i=0}^{d} \prod_{k=1}^{d} (H - \lambda_i + kz)} q^d. \]

The function $I_1$ satisfies following Picard-Fuchs equation

\[ \left( (M - \lambda_0)(M - \lambda_1) - 2qM(2M + z) \right) I_1 = 0. \]

Recall the notation used in above equation,
\[ D = q \frac{d}{dq}, \quad M = H + zD. \]

The restriction $I_1|_{H=\lambda_i}$ admits following asymptotic form

\[ I_1|_{H=\lambda_i} = e^{\mu_i/z} \left( R_{0,i} + R_{1,i}z + R_{2,i}z^2 + \ldots \right) \]

with series $\mu_i, R_{k,i} \in \mathbb{C}(\lambda_0, \lambda_1)[[q]]$. The series $\mu_i$ and $R_{k,i}$ are found by solving differential equations obtained from the coefficient of $z^k$ in (8). For example, we have for $i = 0, 1$,
\[ \lambda_i + D\mu_i = L_i, \]
\[ R_{0,i} = \left( \frac{\lambda_i \prod_{j \neq i} (\lambda_i - \lambda_j)}{f_1(L_i)} \right)^{\frac{1}{2}}, \]
\[ R_{1,i} = \left( \frac{\lambda_i \prod_{j \neq i} (\lambda_i - \lambda_j)}{f(L_i)} \right)^{\frac{1}{2}} \cdot \left( \frac{-16s_1^2s_2^2 + 88s_2^2 + (27s_1^2s_2 - 132s_1s_2^2)L_i + (-12s_1^4 + 54s_1^2s_2)L_i^2}{24s_1(L_is_1 - 2s_2)^3} + \frac{12\lambda_i^2 - 9\lambda_i\lambda_{i+1} + \lambda_{i+1}^2}{24(\lambda_i^2 - \lambda_i\lambda_{i+1})} \right). \]

Here \( s_1 = \lambda_0 + \lambda_1 \) and \( s_2 = \lambda_0\lambda_1 \). In the above expression of \( R_{1,i} \), we used the convention \( \lambda_2 = \lambda_0 \).

### 2.2. Proof of Conjecture [1]

We introduce new differential operator \( D_i \) defined by for \( i = 0, 1, \)

\[ D_i = (DL_i)^{-1}D. \]

By definition, \( D_i \) acts on rational functions in \( L_i \) as the ordinary derivation with respect to \( L_i \). If we use following normalizations,

\[ R_{k,i} = f_1(L_i)^{-\frac{1}{2}}\Phi_{k,i} \]

the Picard-Fuchs equation [13] yields the following differential equations,

\[ D_i\Phi_{p,i} - A_{00,i}\Phi_{p-1,i} - A_{01,i}D_i\Phi_{p-1,i} - A_{02,i}D_i^2\Phi_{p-1,i} = 0, \]

with

\[ A_{00,i} = \frac{-s_1^2s_2^2 + (-s_1^2s_2 + 8s_1s_2^2)L_i + (2s_1^4 - 9s_1^2s_2)L_i^2}{4(L_is_1 - 2s_2)^4}, \]

\[ A_{01,i} = \frac{2s_1s_2^2 + (-s_1^2s_2 - 8s_2^2)L_i + (-s_1^2 + 10s_1s_2)L_i^2 - s_1^2L_i^3}{2(L_is_1 - 2s_2)^3}, \]

\[ A_{02,i} = \frac{s_2^2 - 2(s_1s_2)L_i + (s_1^2 + s_2)L_i^2 - s_1L_i^3}{(L_is_1 - 2s_2)^2}. \]

Here \( s_k \) is the \( k \)-th elementary symmetric functions in \( \lambda_0, \lambda_1 \). Since the differential equations [10] satisfy the condition [6], we conclude the differential equations [10] is admissible.

### 2.3. Gomov-Witten series

By the result of previous subsection, we obtain the following result which verifies the prediction (A) in Section 0.1.

**Theorem 6.** For the Gromov-Witten series of \( K\mathbb{P}^1 \), we have

\[ \mathcal{F}_{g}^{GW,1}(Q(q)) \in G_1, \]
where $Q(q)$ is the mirror map of $K\mathbb{P}^1$ defined by

$$Q(q) := q \cdot \exp\left(2 \sum_{d=1}^{\infty} \frac{(2d-1)!}{(d!)^2} q^d\right).$$

Theorem 6 follows from the argument in [5]. The prediction (B) in Section 0.1 is trivial statement for $K\mathbb{P}^1$.

3. Local $\mathbb{P}^2$

3.1. Overview. In this section, we prove Conjecture 1 for the case $n = 2$ with following specializations,

$$(\lambda_0 \lambda_1 + \lambda_1 \lambda_2 + \lambda_2 \lambda_0)^2 - 3\lambda_0 \lambda_1 \lambda_2 (\lambda_0 + \lambda_1 + \lambda_2) = 0 .$$

The predictions (A) and (B) in Section 0.1 are studied in [4] based on the result of this section. For the rest of the section, the specialization (11) will be imposed. Recall the $I$-function for $K\mathbb{P}^2$.

$$I_2(q) = \sum_{d=0}^{\infty} \frac{\prod_{k=0}^{3d-1} (-3H - k\bar{z})}{\prod_{i=0}^{2} \prod_{k=1}^{d} (H - \lambda_i + k\bar{z})} q^d .$$

The function $I_2$ satisfies following Picard-Fuchs equation

$$\left((M - \lambda_0)(M - \lambda_1)(M - \lambda_2) + 3qM(3M + \bar{z})(3M + 2\bar{z})\right)I_2 = 0$$

Recall the notation used in above equation,

$$D = q \frac{d}{dq}, \quad M = H + zD .$$

The restriction $I_2|_{H=\lambda_i}$ admits following asymptotic form

$$I_2|_{H=\lambda_i} = e^{\mu_i/z} \left(R_{0,i} + R_{1,i}z + R_{2,i}z^2 + \ldots \right)$$

with series $\mu_i, R_{k,i} \in \mathbb{C}(\lambda_0, \lambda_1, \lambda_2)[[q]]$. The series $\mu_i$ and $R_{k,i}$ are found by solving differential equations obtained from the coefficient of $z^k$ in (13). For example,

$$\lambda_i + D\mu_i = L_i ,$$

$$R_{0,i} = \left(\frac{\lambda_i \prod_{j\neq i} (\lambda_i - \lambda_j)}{f_2(L_i)}\right)^{\frac{1}{2}} .$$
3.2. Proof of Conjecture 1. We introduce new differential operator $D_i$ defined by

$$D_i = (DL_i)^{-1}D.$$ 

If we use following normalizations,

$$R_{k,i} = f_2(L_i)^{-\frac{1}{2}}\Phi_{k,i}$$

the Picard-Fuchs equation (13) yields the following differential equations,

$$(15) \quad D_i\Phi_{p,i} - A_{00,i}\Phi_{p-1,i} - A_{01,i}D_i\Phi_{p-1,i} - A_{02,i}D_i^2\Phi_{p-1,i}$$

$$- A_{10,i}\Phi_{p-2,i} - A_{11,i}D_i\Phi_{p-2,i} - A_{12,i}D_i^2\Phi_{p-2,i} - A_{13,i}D_i^3\Phi_{p-2,i} = 0,$$

with $A_{jl,i} \in \mathbb{C}(\lambda_0, \lambda_1, \lambda_2)[L_i, f_2(L_i)^{-1}]$. We give the exact values of $A_{jl,i}$ for reader’s convenience.

$$A_{00,i} = \frac{s_1}{9(s_1L_i - s_2)^5} \left( s_1^3 + (-4s_1^2s_2 + 3s_2^2)L_i ight.$$ 

$$+ (-s_1^3s_2 + 12s_1s_2^3)L_i^2 + (11s_1^4 - 36s_1^2s_2)L_i^3 \left),

$$A_{01,i} = \frac{-s_1}{3(s_1L_i - s_2)^4} \left( s_1^3 - 4(s_1s_2^2)L_i + (3s_1^2s_2 + 9s_2^2)L_i^2 ight.$$ 

$$+ (3s_1^3 - 21s_1s_2)L_i^3 + 3s_1^2L_i^4 \left),

$$A_{02,i} = \frac{-1}{3(s_1L_i - s_2)^3} \left( s_1^3 - 5(s_1s_2^2)L_i + 9s_1^2s_2L_i^2 + (-6s_1^3 - 3s_1s_2)L_i^3 ight.$$ 

$$+ 6s_1^2L_i^4 \left),

$$A_{10,i} = \frac{s_2^2L_i}{27(s_1L_i - s_2)^9} \left( (8s_1^2s_2^5 - 21s_2^6) + (-48s_1^3s_2^4 + 126s_1s_2^5)L_i + (120s_1^4s_2^3 ight.$$ 

$$- 315s_1^2s_2^4)L_i^2 + (-124s_1^5s_2^2 + 264s_1^3s_2^3 + 144s_1^4s_2^2)L_i^3 

+ (12s_1^6s_2 + 153s_1^4s_2^2 - 432s_1^2s_2^3)L_i^4 + (60s_1^7 - 342s_1^5s_2 

+ 432s_1^3s_2^2)L_i^5 + (-33s_1^6 + 108s_1^4s_2)L_i^6 \right).
Here that the differential equations (15) is admissible. The differential equations (15) satisfy the condition (6), we conclude

\[ A_{11,i} = \frac{-s_1 L_i}{27(s_1 L_i - s_2)^8} \left( (8s_1^2 s_2^5 - 21s_2^6) + (-48s_1^3 s_2^4 + 126s_1 s_2^5) L_i 
+ (120s_1^4 s_2^3 - 315s_1^2 s_2^4) L_i^2 + (-124s_1^5 s_2^2 + 264s_1^3 s_2^3 + 144s_1 s_2^4) L_i^3 
+ (12s_1^6 s_2 + 153s_1^4 s_2^2 - 432s_1^2 s_2^3) L_i^4 + (60s_1^7 - 342s_1^5 s_2 - 432s_1^3 s_2^2 + (120s_1^2 - 124s_1 s_2 + 315s_1^4) L_i^5 + (-33s_1^6 + 108s_1^4 s_2)L_i^6 \right) \]

\[ A_{12,i} = \frac{s_1}{9(s_1 L_i - s_2)^7} \left( -s_2^6 + 9s_1 s_2^5 L_i + (-32s_1^2 s_2^4 - 9s_2^5) L_i^2 
+ (57s_1^3 s_2^3 + 60s_1 s_2^4) L_i^3 + (-48s_1^4 s_2^2 - 171s_1^2 s_2^3) L_i^4 
+ (9s_1^5 s_2 + 237s_1^3 s_2^2 + 27s_1^2 s_2^3) L_i^5 + (9s_1^6 - 144s_1^4 s_2 - 90s_1^2 s_2^2) L_i^6 
+ (9s_1^5 + 108s_1^3 s_2) L_i^7 - 18s_1^4 L_i^8 \right) \],

\[ A_{13,i} = -\frac{(3L_i^2 s_2^3 - 3L_i s_1 s_2 + s_2)}{27(s_1 L_i - s_2)^6} \left( -3L_i^2 s_2^3 + 3L_i^2 s_2^3 - 3L_i s_1 s_2 + s_2 \right)^2 \].

Here \( s_k \) is the \( k \)-th elementary symmetric functions in \( \lambda_0, \lambda_1, \lambda_2 \). Since the differential equations (15) satisfy the condition (6), we conclude that the differential equations (15) is admissible.

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