Doubles of Associative Algebras and Their Applications

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Abstract—For a couple of associative algebras we define the notion of their double and give a set of examples. Also, we discuss applications of such doubles to representation theory of certain quantum algebras and to a new type of Noncommutative Geometry.

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1. INTRODUCTION

In this letter by a double of associative algebras we mean an ordered couple \((A, B)\) of associative unital algebras and such that their tensor products \(A \otimes B\) can be also endowed with an associative product by means of a permutation map \(\sigma: A \otimes B \to B \otimes A\). If the algebra \(A\) is equipped with a counit (an algebra homomorphism) \(\varepsilon: A \to \mathbb{C}\), then under some natural conditions on \(\sigma\) and \(\varepsilon\) the algebra \(A\) can be represented in the algebra \(B\).

The simplest example of such a double is a Heisenberg–Weyl (HW) algebra. The smash-product of a bi-algebra \(A\) and an \(A\)-module \(M\) is another example of a double. In this case the role of the algebra \(B\) can be played by the free tensor algebra \(A \otimes M\) or by some of its quotient algebras. We are mainly interested in doubles related to braidings.

Let \(V\) be a finite dimensional complex vector space, \(\dim V = N\). An invertible operator \(R: V^{\otimes 2} \to V^{\otimes 2}\) is called a \textit{braiding}, if it is subject to the \textit{braid relation}

\[
R_{12}R_{34}R_{12} = R_{34}R_{12}R_{34}, \quad \quad R_{12} = R \otimes I, \quad R_{34} = I \otimes R.
\]

Hereafter, \(I\) stands for the identity operator or its matrix. A braiding \(R\) is called respectively an \textit{involutive} or a \textit{Hecke symmetry} if it is subject to a supplementary condition

\[
R^2 = I \quad \text{or} \quad (qI - R)(q^{-1}I + R) = 0 \quad q \notin \{0, \pm 1\}.
\]

The best known examples of Hecke symmetries come from the Drinfeld–Jimbo Quantum Groups (QG) \(U_q(sl(N))\). They are deformations of the usual flips. Nevertheless, there exist involutive and Hecke symmetries, which are neither deformations of the flips nor super-flips.

All symmetries, we are dealing with, are assumed to be skew-invertible (see [1]). We mainly deal with Hecke symmetries \(R = R(q)\) at a \textit{generic} value of the parameter \(q\). To any such a Hecke symmetry \(R\) we associate \(R\)-analogs of the symmetric and skew-symmetric algebras of the space \(V\) by respectively setting

\[
\text{Sym}_R(V) = T(V)/\langle \text{Im}(qI - R)\rangle, \quad \Lambda_R(V) = T(V)/\langle \text{Im}(q^{-1}I + R)\rangle.
\]

Besides, we consider the so-called RTT and Reflection Equation (RE) algebras defined respectively by

\[
R_{12}L_{12}L_{23} - L_{23}L_{12}R_{12} = 0, \quad (1.1)
\]

\[
R_{12}L_{12}L_{23} - L_{23}L_{12}R_{12} = 0, \quad (1.2)
\]

where \(T = \|\mathbf{t}\|_{\mathbb{S}_{i,j,N}}, \ L = \|\mathbf{y}\|_{\mathbb{S}_{i,j,N}}, \ T_i = T \otimes I\) and \(T_2 = I \otimes T\) etc.

In Section 3 we exhibit examples of doubles \((A, B)\), where the algebra \(A\) is an RTT or an RE algebra. In a number of papers there were considered doubles with \(RE\) algebras playing the role of \(A\) and the corresponding RTT algebras, playing the role of \(B\). By contrast, in [2, 3] we considered doubles, where \(B\) was another copy of the \(RE\) algebra. Combining the generating matrices of the algebras \(A\) and \(B\) we constructed other doubles, giving rise to the notion of partial derivatives in the noncommutative generators of \(B\).

Since the \(RE\) algebra in its modified form tends to the algebra \(U(gl(N))\), provided the Hecke symmetry
$R = R(q)$ tends to the usual flip $P$ as $q \to 1$, we obtained partial derivatives on the algebra $U(gl(N))$.

These partial derivatives gives rise to a new noncommutative (NC) differential calculus which is $GL(N)$-covariant and which turns into the usual calculus on the algebra $\text{Sym}(gl(N))$ as $h \to 0$.

In a particular case $N = 2$ we treat the compact form $U(u(2))_h$ of the algebra $U(gl(2))_h$ as an NC version of the polynomial algebra on the Minkowski space. Given a differential operator with polynomial coefficients on the classical Minkowski space, we quantize the coefficients and replace the usual partial derivatives with their “quantum counterparts”. In this way we get an operator, defined on the algebra $U(u(2))_h$ which turns into the initial one as $h \to 0$. We call this procedure the quantization with an NC configuration space. In [4, 5] we extend this procedure on some operators with non-polynomial coefficients.

In the present letter we reproduce some elements of this NC calculus. However, the main our objective is comparing the doubles $(A, B)$ where $A$ is an RTT or RE algebra, and constructing the corresponding representations of these algebras. In the last section we consider an example of the mentioned quantization with an NC configuration space.

2. REPRESENTATIONS VIA DOUBLES, FIRST EXAMPLES

Let $A$ and $B$ be two associative unital algebras endowed with a linear map $\sigma : A \otimes B \to B \otimes A$, such that

$$\sigma \circ (\mu_A \otimes id_B) = \mu_A \circ \sigma_{12} \circ \sigma_{23} \text{ on } A \otimes A \otimes B,$$

$$\sigma \circ (id_A \otimes \mu_B) = \mu_B \circ \sigma_{12} \circ \sigma_{23} \text{ on } A \otimes B \otimes B,$$

$$\sigma(1_A \otimes b) = b \otimes 1_A, \quad \sigma(\alpha \otimes 1_B) = \alpha \otimes a$$

where $\mu_A : A \otimes A \to A$ is the product in the algebra $A$, $1_A$ is its unit, and similarly for $B$.

Under these assumptions the space $B \otimes A$ can be equipped with a bilinear map $*$:

$$(B \otimes A)^{\otimes 2} \xrightarrow{\ast} B \otimes A : (b \otimes a) \ast (b' \otimes a') := (\mu_B \otimes \mu_A) \circ (id_B \otimes \sigma_{23} \otimes id_A)(b \otimes a \otimes b' \otimes a').$$

**Proposition 1.** The map $*$ endows the space $B \otimes A$ with the structure of a unital associative algebra with the unit element $1_B \otimes 1_A$.

We call the corresponding algebra the double of associative algebras $A$ and $B$ and denote it as $B \otimes_A A$.

If the algebra $A$ is equipped with a counit (an algebra homomorphism) $\epsilon_A : A \to C$, then we define an action of the algebra $A$ onto $B$ by the rule

$$a \triangleright b = (\text{id}_B \otimes \epsilon_A) \circ \sigma(a \otimes b), \quad \forall a \in A, \quad \forall b \in B.$$

Identifying $b \otimes 1_C$ and $b$, we get that $a \triangleright b \in B$, so each element $a \in A$ defines a linear operator

$$Op(a) : B \to B.$$

**Proposition 2.** The map $a \mapsto Op(a)$ defines a representation of the algebra $A$ in the algebra $B$:

$$Op(ab) = Op(a)Op(b), \quad Op(1_A) = I.$$

Note that if $\sigma = P$, this representation becomes trivial $a \mapsto \epsilon(a) I$.

(1) As an example we consider an HW algebra, generated by two polynomial subalgebras $A = \mathbb{C}[x_1, \ldots, x^n]$ and $B = \mathbb{C}[x_1, \ldots, x_m]$. Introduce the following permutation relations

$$x'^i x_i = x_i x' + \delta_i^j 1_B \otimes 1_A.$$  

Then, by setting $\epsilon(x') = 0$, we get a double $(A, B)$, such that the corresponding operators $\delta' = Op(x')$ are the partial derivatives defined on the polynomial algebra $B$. Below, we omit the factors $1_A$ and $1_B$ in the permutation relations similar to (2.1).

**Remark 3.** In the particular case $N = 1$ by slightly modifying the permutation relations as

$$yx = qxy + 1, \quad q \in \mathbb{C}, \quad q \not\in \{0, \pm 1\},$$

we get the well-known Jackson derivative.

(2) Especially, we are interested in a matrix version of the permutation relations (2.1). Consider $N \times N$ matrices $M = ||m||$ and $D = ||d||$. Define the algebra $B \otimes_A A$, where $A = A(D), B = B(M)$ (in the brackets we put the generating matrix of the algebra) by the following system

$$D_1 D_2 = D_2 D_1, \quad M_1 M_2 = M_2 M_1,$$

$$D_1 M_2 = M_2 D_1 + P_{12}.$$

Two first equalities of this system mean that the algebras $A$ and $B$ are commutative. The last equality (the permutation relations) together with the counit $\epsilon(D) = 0$ leads to the action

$$D_1 \triangleright M_2 = P_{12} \iff \partial_i^j \triangleright M_k^l = \delta_i^j \delta_k^l,$$

where $\partial_i^j = Op(d_i^j)$.

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1 Sometimes we will deal with the algebra $U(gl(N))_h$, where $h$ is a numerical multiplier introduced in the bracket of the Lie algebra $gl(N)$. This rescaling of the bracket enables us to treat the algebra $U(gl(N))_h$ as a quantization of the commutative algebra $\text{Sym}(gl(N))$ with respect to the linear Poisson bracket.

2 By permutation relations we mean equalities $a \otimes b = \sigma(a \otimes b)$, $a \in A, b \in B$. All the doubles $(A, B)$ below are defined via relations on generators of each component and the permutation relations.
The Leibniz rule for the matrix \( Op(D) = \|Op(d_i^j)\| \) can be expressed via the coproduct 
\[
\Delta(\text{Op}(D)) = \text{Op}(D) \otimes I + I \otimes \text{Op}(D).
\]

(3) Now, consider the double \((A, B)\), with \( A = U(gl(N)) \) and \( B = T(V) \), where \( V \) is the space of the covariant representation of the algebra \( A \). Let \( \{x_1, ..., x_N\} \) be a basis of \( V \) and \( \{l_i^j\} \) be the corresponding basis of \( U(gl(N)) \), i.e. such that \( l_i^j \triangleright x_k = x_i^j \delta_k^k \). Then the relations between the generators \( l_i^j \) can be cast in the following matrix form
\[
L_1L_2 - L_2L_1 = L_1P - L_2P, \quad L = \|l_i^j\| \quad (2.2)
\]
We impose no relation on the generators \( x_i \). The permutation relation are defined as follows
\[
L_2x_1 = x_1L_2 + P_{12}x_1 \iff l_i^j x_k = x_l^j l_i^j + x_j^j \delta_k^k.
\]

Hereafter, \( x \) stands for the column \( \{x_1, ..., x_N\} \). Note that in this double the algebra \( B = T(V) \) can be replaced by one of the algebras \( \text{Sym}(V) \) or \( \Lambda(V) \). Also, there exist similar doubles with the dual space \( V^* \) instead of \( V \).

(4) Let \( B \) be another copy of the algebra \( U(gl(N)) \) with a similar basis \( m_i^j \) and \( M = \|m_i^j\| \) be the corresponding generating matrix. It meets the system of relations similar to \( (2.2) \). We define two types of the permutation relations by the following formulae
\[
\text{(i)} : L_iM_2 = M_2L_1 + M_1P_{12} - M_2P_{12} \quad \text{or} \quad \text{(ii)} : L_iM_2 = M_2L_1 + M_1P_{12}.
\]
Then, taking the counit \( \epsilon(L) = 0 \), we get the corresponding actions
\[
\text{(i)} : L_i \triangleright M_2 = M_1P_{12} - M_2P_{12} \quad \text{or} \quad \text{(ii)} : L_i \triangleright M_2 = M_1P_{12}.
\]
The above algebra \( B = B(M) \) can be replaced by the commutative algebra \( \text{Sym}(gl(N)) \). Then the relations in the algebra \( B \) become \( M_1M_2 = M_2M_1 \). All other relations remain unchanged. In this case the algebra \( A = U(gl(N)) \) is respectively represented by the adjoint and left vector fields onto the algebra \( B = \text{Sym}(gl(N)) \).

(5) The following double was constructed in [3] as a limit case of a double, considered in the next section. Namely, introduce a double \((A(D), B(N))\), where the generating matrices \( D = \|d_i^j\| \) and \( N = \|n_i^j\| \) satisfies the following systems
\[
D_iD_2 = D_2D_i, \quad N_1N_2 - N_2N_1 = h(N_1P_{12} - N_2P_{12}), \quad D_iN_2 = N_2D_i + P_{12} + hD_1P_{12}.
\]
Thus, the algebra \( A = A(D) \) is commutative and \( B = U(gl(N)) \).

The algebra \( B \otimes_o A \) is an NC analog of the HW algebra from the example 2 above. Namely, this algebra is the main ingredient of our NC \( GL(N) \)-covariant calculus.

It is convenient to introduce the matrix \( \tilde{D} = D + h^{-1}I \) and simplify the permutation relations to the form:
\[
\tilde{D}_iN_2 = N_2\tilde{D}_i + h\Delta\tilde{P}_{12}.
\]

By setting \( \epsilon(D) = 0 \) and therefore \( \epsilon(\tilde{D}) = h^{-1}I \), we get the action of operators \( \partial_i = \text{Op}(d_i^j) \) on all elements of the algebra \( U(gl(N)) \). Note that this action is classical on the generators of the algebra \( B(N) \):
\[
\partial_i \triangleright n_k^j = \delta_k^j \partial_k^j. \quad \text{Its extension on the higher monomials can be done by means of the coproduct}
\]
\[
\Delta(\partial_i) = \partial_i \otimes 1 + 1 \otimes \partial_i - h\sum \partial_i \otimes \partial_i.
\]

Observe that our partial derivatives turn into the usual ones on \( \text{Sym}(gl(N)) \) as \( h \to 0 \).

3. DOUBLES RELATED TO BRAIDINGS

(1) Let \( A = A(T) \), \( T = \|t_i^j\| \) and \( B = B(M) \), \( M = \|M_i^j\| \) be two RTT algebras, corresponding to a Hecke symmetry \( R \). Let us define the permutation relations by the rule
\[
R_iT_1M_2 = M_1T_1R_2 \iff T_1M_2 = R_2^{-1}M_1T_1R_2.
\]
Defining the counit \( \epsilon(T) = I \), we get to the following action
\[
T_i \triangleright M_2 = R_2^{-1}M_1T_1R_2.
\]

(2) Now, we set \( B = T(V) \) and define the permutation relations as follows:
\[
T_i x_2 = R_{12}P_{12}x_2T_i \iff t_i^j x_k = R_{mk}^{ij} x_m t_i^j,
\]
where the summation over repeated indices is understood. With the same counit, we have the action
\[
T_i \triangleright x_2 = R_{12}P_{12}x_2 \iff t_i^j \triangleright x_k = R_{mk}^{ij} x_m.
\]

Computing the action of the elements \( t_i^j \) on higher elements from \( T(V) \), we arrive to the representations, which can be constructed via the fusion procedure. If a Hecke symmetry \( R = R(q) \to P \) at \( q \to 1 \), in this limit we get the trivial representations \( t_i^j \to \epsilon(t_i^j)I \) in the both examples above. Note that the algebra \( B = T(V) \) in this construction can be replaced by \( R \)-symmetric or \( R \)-skew-symmetric algebras of the space \( V \).
(3) The differential calculus from [6] (see Section 7), which is a generalization of the Wess-Zumino calculus on the quantum planes [7], can be also presented in terms of a double. Consider a double \((A, B)\) where \(B = \text{Sym}_R(V)\) and \(A = \text{Sym}_R(V^*)\). Here the space \(V^*\) is endowed with the right dual basis \(\{x_1, \ldots, x^n\}\), i.e. such that \(\langle x_i, x^j \rangle = \delta_i^j\). Let us put together all defining relations of this double:

\[
q \partial x_j = R_{ij}^k x_k x_j, \quad q x^j x^i = R_{ij}^k x^i x^k, \quad x^j R_{jk} x_i = h \delta_k^i + q^{-1} x_k x^i.
\] (3.1)

Setting \(q(x') = 0\), we get \(R\)-analogs \(\partial' = Op(x')\) of the partial derivatives multiplied by \(h\). The above permutation relations together with the counit \(\varepsilon(x') = 0\) play the role of the Leibniz rule for the operators \(\partial'\).

The algebra \(\text{Sym}_R(V)\) endowed with these operators is an \(R\)-counterpart of the bosonic Fock space. In a similar manner an \(R\)-analog of the fermionic Fock space can be constructed.

(4) Let us consider elements \(k_i^j = x_j x^i\) and compose the matrix \(K = \|k_i^j\|\).

**Proposition 4.** In virtue of (3.1) the matrix \(K\) is subject to the following relation:

\[
R_{2} K_1 R_2 K_1 - K_1 R_2 K_1 R_2 = h (R_{2} K_1 - K_1 R_2).
\] (3.2)

We call the algebra defined by (3.2) the modified RE algebra. If \(R\) is an involutive symmetry, the claim above is still valid. However, only if \(R\) is a Hecke symmetry, this algebra is isomorphic to the RE algebra defined by (1.2). This isomorphism can be defined as follows

\[
L = h I - (q - q^{-1}) K.
\] (3.3)

Now, consider a double \((A, B)\), where \(A = A(K)\) is a modified RE algebra (3.2) and \(B\) is one of the algebras \(T(V), \text{Sym}_R(V), \Lambda_R(V)\). Taking into account (3.1) and the identification \(k_i^j = x_j x^i\) we get the following permutation relations between these algebras

\[
R_{2} K_1 R_2 x_1 = x_1 K_2 + h R_{2} x_1.
\]

The counit \(\varepsilon(K) = 0\) leads to the action

\[
R_{2} K_1 R_2 \triangleright x_1 = h R_{2} x_1.
\]

Assuming that \(R = R(q) \rightarrow P\) as \(q \rightarrow 1\), we get the limit action \(\triangleright x_1 = h x_1 \delta_k^i\) which coincides with the covariant representation of the algebra \(U(gl(N)_s)\).

In a similar manner it is possible to define a double with the space \(V^*\) instead of \(V\) and thus to get the contravariant representation of the modified RE algebra \(A\).

In [1] there was described a way of constructing a category of finite dimensional \(A\)-modules similar to \(U(gl(N)_s)\)-module. In that construction we used the “braided bi-algebra structure” of the modified RE algebra and the categorical morphisms transposing the objects \(\text{span}(k_i^j) \equiv V \otimes V^*\) and \(M\), where \(M\) is an arbitrary object of the mentioned category. More precisely, the corresponding permutation relations are

\[
\sigma(a \otimes b) = (\triangleright_{12} \otimes id) \circ \mathcal{P}_{23}(a \otimes a_2 \otimes b),
\]

where \(a_1 \otimes a_2 = \Delta(a)\), and \(\Delta\) is the coproduct in the modified RE algebra defined in [1], while \(\mathcal{P}\) stands for the braiding (a categorical morphism), transposing the objects \(\text{span}(k_i^j)\) and \(M\). Note that in the case related to the quantum group \(U_q(sl(N))\), \(\mathcal{P}\) is the product of the usual flip and the image of the corresponding universal \(R\)-matrix. However, in general, the mentioned categorical morphism can be constructed via the initial symmetry \(R\) without any quantum group.

(5) Let \(A\) be again a modified RE algebra, defined by (3.2). The role of \(B\) is often attributed to the corresponding RTT algebra. We consider two doubles where the role of \(B = B(M), M = \|m_i^j\|\) is also played by another copy of the RE algebra (in its non-modified form). Define the two types of permutation relations:

(i) \(R_{2} K_1 R_2 M_1 = M_1 R_{2} K_1 R_2 + h (R_{2} M_1 - R_2 M_2)\).

(ii) \(R_{2} K_1 R_2 M_1 = M_1 R_{2} K_1 R_2 + h R_2 M_1\).

The first system of permutation relations defines braided analogs of the adjoint vector fields. The second one defines braided analogs of the left vector fields (see [2]).

Turn to the double, defined by (3.5). As was shown in [3], the matrix \(D = M^{-1} K\) (the matrix \(M^{-1}\) can be found via the Cayley–Hamilton identity) and \(K\) generate a double \((A(D), B(M))\), where \(D = \|d_i^j\|\) and \(M = \|m_i^j\|\), with the following defining system

\[
R_{2} D_1 R_2 R_{2} = D_1 R_2 R_{2} D_1, \quad R_{2} M_1 R_2 M_1 = M_1 R_2 M_1 R_2, \quad D_1 R_2 M_1 R_2 = R_2 M_1 R_2 D_1 + R_2.
\]

Now, in this double we replace the matrix \(M\) by \(N\) where \(M = h I - (q - q^{-1}) N\) and get a double \((A(D), B(N))\). The matrix \(N = \|n_i^j\|\) generates the modified RE algebra and the permutation relations are as follows

\[
D_1 R_2 N_1 R_2 - R_2 N_1 R_2 D_1 = R_2 + h D_1 R_2.
\]

Note that if \(R = R(q)\) tends to \(P\), in the limit we get the double, exhibited at the end of the previous section.
It is possible to construct similar doubles associated with generalized (braided) Yangians introduced in [6]. We plan to consider them elsewhere.

4. EXAMPLE OF QUANTIZATION WITH NC CONFIGURATION SPACE

Consider the last example from Section 2 for the case \( N = 2 \) and pass by a change of basis to the algebra \( B = U(u(2)_a) \). Making the corresponding change of basis in the algebra \( A \), we get a double \((A, B)\), where \( B \) is generated by the elements \( t, x, y, z \), subject to the relations

\[
\begin{align*}
[t, x] &= [t, y] = [t, z] = 0, \quad [x, y] = h z, \\
[y, z] &= h x, \quad [z, x] = h y.
\end{align*}
\]

The commutative algebra \( A \) is generated by \( \partial_t, \partial_x, \partial_y, \partial_z \) and the permutation relations read

\[
\begin{align*}
[\partial_t, t] &= \frac{h}{2} \partial_t + 1, \quad [\partial_t, x] = -\frac{h}{2} \partial_x \\
[\partial_x, x] &= \frac{h}{2} \partial_x, \quad [\partial_x, y] = \frac{h}{2} \partial_y + 1 \\
[\partial_y, y] &= \frac{h}{2} \partial_y, \quad [\partial_y, z] = -\frac{h}{2} \partial_y \\
[\partial_z, z] &= \frac{h}{2} \partial_z, \quad [\partial_z, x] = \frac{h}{2} \partial_x \\
[\partial_t, y] &= \frac{h}{2} \partial_t, \quad [\partial_t, y] = \frac{h}{2} \partial_y \\
[\partial_y, t] &= \frac{h}{2} \partial_y, \quad [\partial_y, z] = \frac{h}{2} \partial_z \\
[\partial_z, t] &= \frac{h}{2} \partial_z, \quad [\partial_z, x] = \frac{h}{2} \partial_x
\end{align*}
\]

where \( \mathbf{H} = (H_1, H_2, H_3) \) is the magnetic field, \( \mathbf{r} = (x, y, z) \), \( \text{rot} \) and \( \text{div} \) are the curl and divergence respectively. We succeeded in finding an NC solution of this system:

\[
\mathbf{H} = \frac{g}{r_c(r_c^2 - v^2)} \mathbf{r}.
\]

Note that it tends to the usual Dirac monopole as \( v \to 0 \).

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