PRESCRIBING SIGN-CHANGING MEAN CURVATURE CANDIDATES
ON THE $n+1$-DIMENSIONAL UNIT BALL

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Abstract. This paper focuses on the problem of prescribing mean curvature on the unit ball. Assume that $f$, which is allowed to change sign, satisfies Morse index counting condition or certain kind of symmetry condition. By using a negative gradient flow method, we then prove that $f$ can be realized as the boundary mean curvature of some conformal metric.

1. Introduction

In this paper, we consider the prescribing boundary mean curvature problem on the $n+1$-dimensional unit ball with $n \geq 2$. Such a problem is a natural analogue of the problem of prescribing scalar curvature on the sphere and it can be stated as follows. Let $(B^{n+1}, g_e)$ be the $n+1$-dimensional unit ball with the Euclidean metric $g_e$. Assume that $f$ is a smooth function on the boundary $\partial B^{n+1} = S^n$. Then, one may ask if there exists a scalar-flat metric $g$ point-wisely conformally related to $g_e$, i.e., $g = u^{4/(n-1)}g_e$ for some positive and smooth function $u$, such that $f$ can be realized as the mean curvature of $g$. It is well known that this geometric problem is equivalent to finding a positive solution of the boundary value problem

$$
\begin{align*}
\Delta_g u &= 0, & \text{in } B^{n+1}, \\
\frac{2}{n-1} \frac{\partial u}{\partial n} + u &= f(x)u^{n+1}, & \text{on } S^n,
\end{align*}
$$

(1.1)

where $\Delta_g$ and $\partial/\partial n$ are, respectively, the Laplace operator and the out normal derivative of the metric $g_e$.

Many research works have dealt with the Eq. (1.1) during the past few decades, see, for instance [1, 2, 5–8, 13], and the references therein. Among them, Xu and the author [13], recently, obtained the following result

Theorem 1.1 (Xu & Zhang). Let $n \geq 2$ and $f > 0 : S^n \to \mathbb{R}$ be a smooth Morse function satisfying the non-degeneracy condition: $|\nabla f|_{g_{S^n}}^2 + |\Delta_{g_{S^n}} f|^2 \neq 0$ and the simple bubble condition: $\max_{S^n} f / \min_{S^n} f < \delta_n$, where $\delta_n = 2^{1/n}$, $n = 2$ and $\delta_n = 2^{1/(n-1)}$, $n \geq 3$. Moreover, define the numbers associated with $f$ as follows

$$
m_i = \#\{\theta \in S^n; \nabla_{S^n} f(\theta) = 0, \Delta_{g_{S^n}} f(\theta) < 0, \text{ind}_f(\theta) = n - i\},
$$

where $\text{ind}_f(\theta)$ denotes the Morse index of $f$ at critical point $\theta$. If the following algebraic system has no non-trivial solutions,

$$
m_0 = 1 + k_0, m_i = k_{i-1} + k_i, 1 \leq i \leq n, k_n = 0,
$$

where coefficients $k_i \geq 0$, then the Eq. (1.1) admits at least one positive solution.

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For the prescribed function \( f \) possesses some kind of symmetry, Ho [8] proved a similar result as Leung & Zhou did in [9]. Before stating his result, let us describe two types of symmetries

(Sym1) A mirror reflection upon a hyperplane \( \mathcal{H} \subset \mathbb{R}^{n+1} \) passing through the origin. As the situation is invariant under a rotation, we assume that \( \mathcal{H} \) is perpendicular to the \( x^1 \)-axis. In this way, a mirror reflection \( \sigma : S^n \mapsto S^n \) is given by \( \sigma(x^1, x^2, \ldots, x^{n+1}) = (-x^1, x^2, \ldots, x^{n+1}) \), for \((x^1, x^2, \ldots, x^{n+1}) \in S^n \). As a result, \( \Sigma = \{(0,x^2,\ldots,x^{n+1}) \in S^n \} = \mathcal{H} \cap S^n \) is the fixed point set.

(Sym2) A rotation of angle \( \theta/k \) with axis being a straight line in \( \mathbb{R}^{n+1} \) passing through the origin and \( k > 1 \) being an integer. We may assume that the straight line is the \( x^{n+1} \)-axis. In this case \( \Sigma = [N,S] \) is the fixed point set, where \( N \) is the north pole and \( S \) is the south pole.

Now, Ho’s result can be stated as

**Theorem 1.2 (Ho).** Suppose \( f > 0 \) is a smooth function on \( S^n \) which is invariant under the symmetry (Sym1) or (Sym2). Assume that there exists a point \( y \in \Sigma \) with \( f(y) = \max_{\Sigma} f \) and \( \Delta_{S^n} f(y) > 0 \). If, in addition, there holds

\[
\max_{\Sigma} f / \max_{\Sigma} f < 2^{1/(n-1)}, \tag{1.2}
\]

then there exists a smooth positive solution of Eq.(1.1).

In this paper, inspired by the results in [14], we aim to extend the Theorems 1.1 and 1.2 to the case that the prescribed function \( f \) is allowed to change sign. Our first result reads

**Theorem 1.3.** Suppose \( f(x) \) is a smooth Morse function on \( S^n \) satisfying the following conditions:
(i) \( \int_{S^n} f \, d\mu_{S^n} > 0 \);
(ii) \((\max_{S^n} |f|)/(\int_{S^n} f \, d\mu_{S^n}) < 2^{1/n}\);
(iii) \(|\nabla f|_{g^n}^2 + (\Delta_{S^n} f)^2 \neq 0\);
(iv) The following algebraic system has no non-trivial solutions

\[
m_0 = 1 + k_0, m_i = k_{i-1} + k_i, 1 \leq i \leq n, k_n = 0, \tag{1.3}
\]

with coefficients \( k_i \geq 0 \) and \( m_i \) defined as

\[
m_i = \# \{x \in S^n; f(x) > 0, \nabla_{g^n} f(x) = 0, \Delta_{g^n} f(x) < 0, \text{ind}_f(x) = n - i\}, \tag{1.4}
\]

where \( \text{ind}_f(x) \) denotes the Morse index of \( f \) at critical point \( x \), then \( f \) can be realized as the boundary mean curvature of some metric \( g \) in the conformal class of \( g_e \) on the unit ball, i.e., Eq.(1.1) possesses a positive solution.

One will see that, later, the index counting condition (1.5) below is, indeed, a special case of the Morse index condition (1.3). Hence, we have the following corollary

**Corollary 1.4.** Suppose \( f(x) \) is a smooth Morse function on \( S^n \) satisfying the following conditions:
(i) \( \int_{S^n} f \, d\mu_{S^n} > 0 \);
(ii) \((\max_{S^n} |f|)/(\int_{S^n} f \, d\mu_{S^n}) < 2^{1/n}\);
(iii) \(|\nabla f|_{g^n}^2 + (\Delta_{S^n} f)^2 \neq 0\);
(iv) \[ \sum_{\{x \in S^n : f(x) > 0, \nabla S_n f(x) = 0 \text{ and } \Delta S_n f(x) < 0\}} (-1)^{\text{ind}(x)} \neq (-1)^n, \] (1.5)

then \( f \) can be realized as the boundary mean curvature of some metric \( g \) in the conformal class of \( g_e \) on the unit ball, i.e., Eq.(1.1) possesses a positive solution.

For the case that the prescribed function \( f \) possesses some kind of symmetry, we first consider a little more general situation. To do so, let us set up some notations first. Let \( G \) be a subgroup of isometry group of \( S^n \). Then, a function \( f \) is \( G \)-invariant if
\[ f(\theta(x)) = f(x), \quad \text{for } \forall \theta \in G \text{ and } \forall x \in S^n. \]

In addition, we define \( \Sigma \) to be the fixed point set under the group \( G \) as follow
\[ \Sigma = \{ x \in S^n : \theta(x) = x, \quad \text{for all } \theta \in G \}. \]

our third result reads

**Theorem 1.5.** Let \( G \) be a subgroup of isometry group of \( S^n \). Assume that \( f \) is a \( G \)-invariant function satisfying
\begin{enumerate}
\item \( \int_{S^n} f \, d\mu_{S^n} > 0; \)
\item \( (\max_{S^n} |f|/\int_{S^n} f \, d\mu_{S^n}) < 2^{1/n}; \)
\end{enumerate}
If there holds either
\begin{enumerate}
\item [(a)] \( \Sigma = \emptyset \) or \( \Sigma \neq \emptyset \) and \( \max_{S^n} f \leq \int_{S^n} f \, d\mu_{S^n} \),
\end{enumerate}
then \( f \) can be realized as the boundary mean curvature of some metric \( g \) in the conformal class of \( g_e \) on the unit ball, i.e., Eq.(1.1) possesses a positive \( G \)-invariant solution.

Finally, with the help of Theorem 1.5, we can extend Theorem 1.2 to be as follows

**Theorem 1.6.** Assume that \( f(x) \) is a smooth function on \( S^n \) satisfying
\begin{enumerate}
\item [(i)] \( \int_{S^n} f \, d\mu_{S^n} > 0, \) and
\item [(ii)] \( (\max_{S^n} |f|/\int_{S^n} f \, d\mu_{S^n}) < 2^{1/n}. \)
\end{enumerate}
Moreover, suppose that \( f \) is invariant under the symmetry (Sym1) or (Sym2). If there exists a point \( y \in \Sigma \) with \( f(y) = \max_{\Sigma} f \) such that \( \Delta g f(y) > 0 \). Then Eq.(1.1) possesses a positive smooth solution.

The paper is organized as follows: In §2, we describe the evolution equations and derive some elementary estimates; In §3, we focus ourself on the global existence of our evolution equations; In §4, we try to perform the blow-up analysis and describe th asymptotic behavior of the flow in the case of divergence; In the final section §5, we will prove the main results.

2. THE FLOW EQUATION AND SOME ELEMENTARY ESTIMATES

We consider, as in Xu & Zhang [13], the conformal mean curvature flow as follows. Let \( g(t) \) be a family of time-dependent metrics on the unit ball \( B^{n+1} \) conformal to \( g_e \). For simplicity, we denote by, respectively, \( R(t) \) and \( H(t) \) the scalar
curvature and boundary mean curvature of the metric \( g(t) \). Then, the evolution equation reads as

\[
\begin{aligned}
\frac{\partial}{\partial t} g(t) &= -(H(t) - \lambda(t)f) g(t) \quad \text{on } \partial B^{n+1}, \\
R(t) &\equiv 0 \quad \text{in } B^{n+1}, \\
g(0) &= g_0,
\end{aligned}
\]  \quad (2.1)

where \( \lambda(t) \) is to be determined later and \( g_0 \) is the initial metric conformal to \( g_e \). We would like to point out that such evolution equation as above was firstly considered by Brendle [3] where the prescribed function \( f \) is constant.

Now, by substituting \( g(t) = u(t)^{4/(n-1)} g_e \) and \( g_0 = u_0^{4/(n-1)} g_e \) into (2.1), we can obtain the evolution equation for the conformal factor \( u(t) \)

\[
\begin{aligned}
\Delta_e u &= 0 \quad \text{in } B^{n+1}, \\
\frac{\partial}{\partial t} u(t) &= -\frac{n-1}{4}(H(t) - \lambda(t)f) u(t) \quad \text{on } \partial B^{n+1}, \\
u(0) &= u_0,
\end{aligned}
\]  \quad (2.2)

where \( H(t) \) can be written, in terms of \( u \), as

\[
H = u^{2n-1} \left( a_n \frac{\partial u}{\partial \eta_e} + u \right).
\]  \quad (2.3)

The terms \( \Delta_e \) and \( \partial/\partial \eta_e \) above means, respectively, the Laplace-Beltrami operator and the outer normal derivative of the metric \( g_E \).

It is well known that the prescribed mean curvature problem has the associated energy functional given by

\[
E_f[u] = \frac{E[u]}{\left( \int_{S^n} f u^{2n} \, d\mu_{S^n} \right)^{2n/2}}.
\]  \quad (2.4)

where \( E[u] \) can be expressed as

\[
E[u] = \frac{1}{\omega_n} \int_{B^{n+1}} a_n |\nabla u|_{g_e}^2 \, dV_{g_e} + \int_{S^n} u^2 \, d\mu_{S^n}.
\]

Here, \( \omega_n \) and \( d\mu_{S^n} \) are, respectively, the volume and volume form of the unit \( n \)-sphere. Moreover, we make a convention here that the sign ‘\( \int_{S^n} \)’ means \( \frac{1}{\omega_n} \int_{S^n} \) from now on.

Observe that, if \( \Delta_e u = 0 \), then we have, by the divergence theorem and (2.3), that

\[
E[u] = \int_{S^n} a_n \frac{\partial u}{\partial \eta_e} u + u^2 \, d\mu_{S^n} = \int_{S^n} H u^{2n} \, d\mu_{S^n} = \int_{S^n} H \, d\mu_g
\]  \quad (2.5)

Hence, \( E[u] \) is nothing but the average of the mean curvature \( H \) if the metric \( g \) is scalar-flat.

For the sake of convenience, we choose the factor \( \lambda(t) \) in such a way that the boundary volume of \( B^{n+1} \) w.r.t. the metric \( g \) is preserved during the evolution, that is,

\[
0 = \frac{d}{dt} \text{vol}(S^n, g(t)) = \frac{d}{dt} \int_{S^n} u^{2n} \, d\mu_{S^n} = \frac{n}{2} \int_{S^n} \lambda(t) f - H \, d\mu_g.
\]
Thus, the choice of $\lambda(t)$ would be
\[
\lambda(t) = \frac{E[u]}{\int_{S^n} f u^2 \, d\mu_{S^n}},
\]
where we have used (2.5) in the calculation.

To end this section, we collect some useful formulae which have appeared in [13]. We will omit their detailed derivation.

**Lemma 2.1.** (i) The mean curvature satisfies the evolution equation
\[
(\lambda f - H)_t = -\frac{1}{2} \frac{\partial}{\partial \eta} (\lambda f - H) + \frac{1}{2} (\lambda f - H) H + \lambda' f
\]
on $S^n$. Here, the function $\lambda f - H$ is extended to the interior of $B^{n+1}$ such that
\[
\Delta g_{\eta} (\lambda f - H) = 0, \quad \text{in } B^{n+1}.
\]
(ii) Let $u$ be any positive smooth solution of the flow (2.2). Then one has
\[
\frac{d}{dt} E_f[u] = -\frac{n-1}{2} \left( \int_{S^n} f \, d\mu_g \right)^{\frac{2}{n}} \int_{S^n} (\lambda f - H)^2 \, d\mu_g.
\]
In particular, the energy functional $E_f[u]$ is decay along the flow.

3. Global existence of the flow

In this section, we will show that our flow is globally well-defined. To achieve this purpose, the critical step is to show that positive property of the quantity $\int_{S^n} f u^2 \, d\mu_{S^n}$ will be preserved along the flow if we, initially, have $\int_{S^n} f u^2 \, d\mu_{S^n} > 0$.

**Lemma 3.1.** Assume that $\int_{S^n} f u^2 \, d\mu_{S^n} > 0$ and $u$ is a smooth solution of the flow (2.2) on $[0, T)$ for some $T > 0$. Then one has
\[
\int_{S^n} f u^2 \, d\mu_{S^n} > 0,
\]
for all $t \in [0, T)$.

**Proof.** Since $\int_{S^n} f u^2 \, d\mu_{S^n} > 0$, it is easy to see, by the definition of $E_f[u]$, that $E_f[u_0] > 0$. From the sharp trace Sobolev inequality and the boundary volume-preserving property of our flow that
\[
E[u] \geq \left( \int_{S^n} u_0^2 \, d\mu_{S^n} \right)^{2/2^d}.
\]
On the other hand, it follows from Lemma (2.1) (ii) that
\[
\frac{E[u]}{\left( \int_{S^n} f u(t)^2 \, d\mu_{S^n} \right)^{2/2^d}} \leq E_f[u_0].
\]
Combining the two inequalities above gives
\[
\int_{S^n} f u(t)^2 \, d\mu_{S^n} \geq \frac{\int_{S^n} u_0^2 \, d\mu_{S^n}}{(E_f[u_0])^{2/2^d}} > 0.
\]
\[\square\]

With help of (3.2), we can, immediately, obtain the boundedness of $\lambda(t)$. 

Lemma 3.2. Along the flow (2.2), \( \lambda(t) \) remains bounded. To be precise, we have
\[
\lambda_1 \leq \lambda(t) \leq \lambda_2,
\]
where
\[
\lambda_1 = \frac{1}{\max_{S^*} f} \left( \int_{S^*} u_0^{2g} \, d\mu_{S^*} \right)^{-1/n} \quad \text{and} \quad \lambda_2 = \frac{(E_f[u_0])^{n/(n-1)}}{\left( \int_{S^*} u_0^{2g} \, d\mu_{S^*} \right)^{1/n}}.
\]

Proof. In view of (2.6) and (2.4), we may rewrite \( \lambda(t) \) as
\[
\lambda(t) = E_f[u] \left( \int_{S^*} f u^{2g} \, d\mu_{S^*} \right)^{-1/n}.
\]
Now, by using (3.2) and the decay property of \( E_f[u] \), we get
\[
\lambda(t) \leq E_f[u_0] \left( \int_{S^*} u_0^{2g} \, d\mu_{S^*} \right)^{-1/n} \leq \frac{1}{\left( \int_{S^*} u_0^{2g} \, d\mu_{S^*} \right)^{1/n}} \lambda_1.
\]
On the other hand, the boundary volume-preserving property and (3.1) yields
\[
\lambda(t) = \frac{E[u]}{\int_{S^*} f u^{2g} \, d\mu_{S^*}} \geq \frac{\left( \int_{S^*} u_0^{2g} \, d\mu_{S^*} \right)^{2/2g}}{(\max_{S^*} f) \left( \int_{S^*} u_0^{2g} \, d\mu_{S^*} \right)^{1/2g}} = \lambda_2.
\]
To continue, we set
\[
F_2(t) = \int_{S^*} (\lambda f - H)^2 \, d\mu_g.
\]
Then we conclude that \( \lambda'(t) \) is bounded by \( F_2(t) \).

Lemma 3.3. Let \( u \) be a smooth solution of the flow (2.2). Then there holds
\[
\lambda'(t) = -\left( \int_{S^*} f \, d\mu_g \right)^{-1} \left[ \frac{n-1}{2} \int_{S^*} (\lambda f - H)^2 \, d\mu_g + \frac{1}{2} \int_{S^*} \lambda f (\lambda f - H) \, d\mu_g \right]. \tag{3.3}
\]
In particular,
\[
|\lambda'(t)| \leq C \left( F_2(t) + \sqrt{F_2(t)} \right), \tag{3.4}
\]
where \( C > 0 \) is a universal constant.

Proof. The proof of (3.3) follows from a direct computation. As for (3.4), it follows from the Hölder’s inequality, Lemma 3.2 and (3.2). □

To bound \( \lambda'(t) \), in view of the lemma above, it suffices to bound the quantity \( F_2(t) \).

Lemma 3.4. One can find a universal constant \( C > 0 \) such that
\[
F_2(t) \leq C,
\]
for all \( t \geq 0 \).
Proof. From (2.7) and (2.2), it follows that
\[ \frac{d}{dt} F_2(t) = \frac{d}{d\mu_g} \left( \int_{S^n} (\lambda f - H)^2 \right) = 2 \int_{S^n} (\lambda f - H) \left[ -\frac{1}{2} \frac{\partial}{\partial \eta} (\lambda f - H) + \frac{1}{2} (\lambda f - H) H + \lambda' f \right] d\mu_g \]
\[ + \frac{n}{2} \int_{S^n} (\lambda f - H)^3 d\mu_g \]
\[ = -\frac{1}{\omega_n} \int_{B^{n+1}} |\nabla (\lambda f - H)|^2 dV_g + \frac{2}{n} \int_{S^n} H (\lambda f - H)^2 d\mu_g \]
\[ + 2\lambda \int_{S^n} f (\lambda f - H) d\mu_g + \frac{n}{2} \int_{S^n} \lambda f (\lambda f - H)^2 d\mu_g \]
\[ = -\frac{n-2}{2} \left[ \frac{1}{\omega_n} \int_{B^{n+1}} a_n |\nabla (\lambda f - H)|^2 dV_g + \int_{S^n} H (\lambda f - H)^2 d\mu_g \right] \]
\[ - \frac{1}{2\omega_n} \int_{B^{n+1}} a_n |\nabla (\lambda f - H)|^2 dV_g + 2\lambda \int_{S^n} f (\lambda f - H) d\mu_g \]
\[ + \frac{n}{2} \int_{S^n} \lambda f (\lambda f - H)^2 d\mu_g. \]

Using the sharp Sobolev trace inequality, (3.4) and Hölder’s inequality, we obtain
\[ \frac{d}{dt} F_2(t) \leq - \frac{(n-2)}{2} \left( \int_{S^n} |\nabla (\lambda f - H)|^2 d\mu_g \right)^{\frac{1}{2}} - \frac{1}{2\omega_n} \int_{B^{n+1}} a_n |\nabla (\lambda f - H)|^2 dV_g \]
\[ + 2\lambda \int_{S^n} f (\lambda f - H) d\mu_g + \frac{n}{2} \int_{S^n} \lambda f (\lambda f - H)^2 d\mu_g \]
\[ \leq CF_2(t) \left( 1 + \sqrt{F_2(t)} \right). \]

Set
\[ \alpha(t) = \int_0^{F_2(t)} \frac{1}{1 + \sqrt{s}} ds. \]

Then,
\[ \frac{d\alpha}{dt} \leq CF_2(t). \tag{3.5} \]

From Lemma 2.1 (ii) and the fact that \( E_f[u] \geq 0 \), it follows that
\[ \int_0^t F_2(s) ds \leq CE_f[u_0]. \]

Hence, by integrating (3.5) from 0 to \( t \) with \( t > 0 \), we can get
\[ \alpha(t) \leq \alpha(0) + C \int_0^t F_2(s) ds \leq F_2(0) + CE_f[u_0]. \tag{3.6} \]

It is easy to see by the definition of \( \alpha(t) \) that
\[ \alpha(t) \geq \frac{F_2(t)}{1 + \sqrt{F_2(t)}} \begin{cases} \frac{F_2(t)}{1 + \sqrt{F_2(t)}}, & F_2(t) \leq 1, \\ \frac{\sqrt{F_2(t)}}{2}, & F_2(t) > 1. \end{cases} \tag{3.7} \]

Combining (3.6) and (3.7) yields the conclusion. \( \square \)

Up to here, Lemma 3.4 and (3.4) immediately imply the corollary.
Corollary 3.5. There exists a universal constant $\Lambda_0 > 0$ such that

$$|\lambda'(t)| \leq \Lambda_0,$$

for all $t \geq 0$.

Now, with the help of the boundedness of $\lambda'(t)$, we are able to show the mean curvature $H$ is uniformly bounded. Before doing so, we define

$$\gamma := \min \left\{ \min_{S^n} H(0) - \lambda_2 \max_{S^n} |f|, -\sqrt{\frac{1}{4}(\lambda_2 \max_{S^n} |f|)^2 + \frac{8}{\pi} \Lambda_0 \max_{S^n} |f|} \right\}$$

Lemma 3.6. The mean curvature function $H$ of $g$ satisfies

$$H - \lambda(t)f \geq \gamma,$$

for all $t \geq 0$.

Proof. Set

$$\mathcal{L} = \begin{cases} \Delta_g, & \text{in } B^{n+1}, \\ \partial_t + \frac{1}{2} \frac{\partial}{\partial \eta} + \frac{1}{2}(\lambda f - \gamma), & \text{on } S^n. \end{cases}$$

A simple calculation and our choice of $\gamma$ gives

$$\mathcal{L}(\lambda f - H + \gamma) = 0,$$

in $B^{n+1}$ and on $S^n$

$$\mathcal{L}(\lambda f - H + \gamma) = \partial_t (\lambda f - H) + \frac{1}{2} \frac{\partial}{\partial \eta} (\lambda f - H)$$

$$+ \frac{1}{2}(\lambda f - \gamma)(\lambda f - H + \gamma)$$

$$= \frac{1}{2}(\lambda f)^2 - \frac{1}{2}\gamma^2 + \frac{1}{2}(\gamma - H)H + \lambda' f$$

$$\leq \frac{1}{2} (\lambda_2 \max_{S^n} |f|)^2 + \Lambda_0 \max_{S^n} |f| - \frac{3}{8} \gamma^2$$

Moreover, it is easy to see that $\lambda f - \gamma \geq 0$ and $(\lambda f - H + \gamma)(0) \leq 0$ due to our choice. Hence, we can apply the maximum principle to operator $\mathcal{L}$ to obtain $\lambda f - H + \gamma \leq 0$, which proves the assertion.

Now, with the help of Lemmas 3.1, 3.2 and 3.6, the proof of the global existence of the flow (2.2) will be exactly the same as that in [13]. We thus omit the detail. Before we state this result, let us define

$$X_* = \left\{ 0 < u \in C^{\infty}(\overline{B^{n+1}}) : \int_{S^n} f u^{-2*} d\mu_{S^n} > 0 \right\}.$$

Proposition 3.7. If the initial data $u_0 \in X_*$, then the flow (2.2) has a unique smooth solution which is well defined on $[0, +\infty)$. 

4. Blow-up Analysis

From this section onward, we dealt with the convergence of the flow (2.2). To realize this goal, one has to bound the conformal factor $u$ uniformly. However, one, in general, can not obtain this uniform bound directly. Instead, we assume the contrary, that is, the flow (2.2) is divergent. In this way, one can expect that the blow-up phenomenon will appear and the blow-up analysis has to come into play. As an initial step, we notice the following $L^p$ convergence which is one of the key ingredients.

**Proposition 4.1.** For $0 < p < +\infty$ there holds
\[
\int_{S^n} |\lambda(t)f - H|^p \, d\mu_g \to 0, \quad \text{as } t \to +\infty.
\]

**Proof.** The proof is the same as [13, Lemma 3.2]. We omit it here. \qed

4.1. Compactness-Concentration. To derive the asymptotic behavior of the flow (2.2) in the case of divergence, the following compactness-concentration theorem in [13] serves good for our purpose. One can find its proof in [13, Lemma 4.1]. We remark that such a compactness-concentration theorem is an analogue of that by Schwetlick & Struwe [11]. Before we state the theorem, let us set some notations. For $r > 0$, $x_0 \in S^n$, set
\[
B^+_r(x_0) = \{x \in B^{n+1} : d_{g_k}(x, x_0) < r\} \quad \text{and} \quad \partial B^+_r(x_0) = \partial B^+_r(x_0) \cap S^n.
\]

**Theorem 4.2 (Xu & Zhang).** Let $g_k = u_k^{4/(n-1)}g$, where $0 < u_k \in C^\infty(B^{n+1}, g)$, $k \in \mathbb{N}$, be a sequence of conformal metrics on $B^{n+1}$ with $R_{g_k} \equiv 0$ and $\text{vol}(S^n, g_k) = \omega_n$. Assume that the associated boundary mean curvature $H_{g_k}$ satisfies
\[
\int_{S^n} H_{g_k} \, d\mu_{g_k} \leq C_0, \quad \int_{S^n} \left| H_{g_k} - \int_{S^n} H_{g_k} \, d\mu_{g_k} \right|^p \, d\mu_{g_k} \leq C_0 \quad (4.1)
\]
for all $k$ and some $p > n$. Then, either
(i) the sequence $(u_k)_k$ is uniformly bounded in $W^{1,p}(S^n, g_{S^n})$; or
(ii) there exists a subsequence of $(u_k)_k$ and finitely many points $x_1, \ldots, x_L \in S^n$ such that for any $r > 0$ and any $l \in \{1, \ldots, L\}$ there holds
\[
\liminf_{k \to +\infty} \left( \int_{\partial B^+_r(x_l)} |H_{g_k}|^p \, d\mu_{g_k} \right)^{\frac{1}{p}} \geq \omega_n^{\frac{1}{p}}. \quad (4.2)
\]

To go further, let us define the so-called normalized metric and function. It is well known that for every smoothly varying family of metrics $g(t) = u(t)^{4/(n-1)}g$, there exists a family of conformal transformations $\phi(t) : (B^{n+1}, S^n) \mapsto (B^{n+1}, S^n)$ which taken $S^n$ into itself such that
\[
\int_{S^n} x \, d\mu_{g} = 0, \quad \text{for } t \geq 0, \quad (4.3)
\]
where $h = \phi^*g$ and $x = (x^1, x^2, \ldots, x^{n+1})$. Here the pull-back metric $h$ is called the normalized metric such that the scalar curvature $R_h \equiv 0$ in $B^{n+1}$ and the boundary mean curvature is given by $H_h = H \circ \phi$. In fact, $h$ can be expressed as $h = u^{4/(n-1)}g$ with
\[
v = (u \circ \phi)|\det(\phi)|^{\frac{n-1}{2(n+1)}}. \quad (4.4)
\]
which satisfies the equation

\[
\begin{aligned}
\Delta_{g_0} v &= 0 & \text{in } B^{n+1} \\
\partial_n v &= H_0 |\nabla v|^{2n-2} & \text{on } S^n
\end{aligned}
\]  

(4.5)

Since we only focus ourselves on the boundary most of time, it will be helpful to specify the formula of the conformal transformation restricted on the boundary. As a matter of fact, it is well-known that their restriction on the boundary can be written as

\[
\varphi(t) := \phi(t)|_{S^n} = \pi^{-1} \circ \delta_{q(t), r(t)} \circ \pi,
\]

where \( \pi : S^n \setminus \{0, 0, \ldots, 0, -1\} \mapsto \mathbb{R}^n \) is the stereographic projection from the south pole to \( n \)-plane, \( \delta_{q,r}(z) = g + rz \) for \( q \in \mathbb{R}^n, r > 0 \) and \( z \in \mathbb{R}^n \). In consequence, we always use \( \varphi \) as the conformal transformation without specific explanation. By following the idea in [10], for \( t_0 \geq 0 \) fixed and \( t \geq 0 \) close to \( t_0 \), we let

\[
\varphi_t(t) = \varphi(t_0)^{-1} \varphi(t).
\]

Then

\[
\varphi_t(t) \circ \psi = \psi_{q(t), r(t)},
\]

where \( \psi = \pi^{-1} \) and \( \psi_{q,r} = \psi \circ \delta_{q,r} \). This implies that at \( t = t_0 \), \( \delta_{q(t_0), r(t_0)}(z) = z \).

Hence, \( q(t_0) = 0, r(t_0) = 1 \). It is equivalent to saying that we make a translation and a dilation such that \( q(t_0) = 0, r(t_0) = 1 \) at each fixed \( t_0 \).

Now, given \( t_0 \geq 0 \), we consider a rotation mapping some \( p = p(t_0) \in S^n \) into the north pole \( N = (0, 0, \ldots, 1) \). Then \( \varphi(t_0) \) can be expressed as \( \varphi(t_0) = \psi_r \circ \pi \) for some \( \varepsilon = \varepsilon(t_0) > 0 \), where \( \psi_r(z) = \psi(\varepsilon z) = \psi_{0, \varepsilon}(z) \) by the notation above. Hence, in stereographic coordinates, \( \varphi(t) := \varphi_{p(t), \varepsilon(t)} \) is given by

\[
\varphi(t) \circ \psi = \varphi(t_0) \circ \varphi_t(t) \circ \psi = \psi_r \circ \delta_{q,t}.
\]

So, our proceeding calculations, involving any transformation, are always at each fixed time \( t_0 \). In this way, the conformal transformation \( \varphi \) has the expression:

\[
\varphi(t) = \psi_r \circ \pi.
\]

Now, we can obtain a considerably sharpen version of the previous result if we make some additional assumptions on the associated mean curvature \( H_k \). We should point out that the proof of the following theorem is inspired by Struwe [12].

**Theorem 4.3.** Let \( \{u_k\} \) be a sequence of smooth functions on \( \overline{B}^{n+1} \) with associated scalar-flat metrics \( g_k = u_k^{4/(n-1)} g_0 \) and the associated mean curvatures \( H_k \), \( k \in \mathbb{N} \). Assume that \( \text{vol}(S^n, g_k) = \omega_n \) and there exists a smooth function \( H_\infty \) on \( S^n \) satisfying

\[
\max_{S^n} |H_\infty| \leq 2^{1/2}, \quad \text{(4.6)}
\]

for some positive number \( \tau \), such that

\[
||H_k - H_\infty||_{L^p(S^n, g_k)} \to 0 \quad \text{as } k \to +\infty, \quad \text{(4.7)}
\]

for some \( p > n \). In addition, suppose that there exists some constant \( C \), such that

\[
H_k \geq C. \quad \text{(4.8)}
\]

Also, let \( h_k = \phi_k^* g_k = u_k^{4/(n-1)} g_0 \) be the associated sequence of normalized metrics satisfying (4.3). Here \( \phi_k \) is the conformal transformation on the unit ball and its restriction on the boundary are given by \( \varphi_k = \phi_k|_{S^n} = \varphi_{p_k, \varepsilon_k} \) with \( p_k = p(t_k) \) and \( \varepsilon_k = \varepsilon(t_k) \). Then, up to a subsequence, either
(i) $u_k$ is uniformly bounded in $W^{1,p}(S^n, g_{S^n})$. In addition, $u_k \to u_\infty$ in $W^{1,p}(S^n, g_{S^n})$ as $k \to +\infty$. If we let $g_\infty = u_\infty^{4/(n-1)} g_e$, then $g_\infty$ has mean curvature $H_\infty$; or

(ii) there exists a unique point $Q \in S^n$ such that

$$\mu_{\mathcal{H}_k} \to \omega_n \delta_Q,$$

weakly in the sense of measure as $k \to +\infty$. Moreover, in the latter case, we have

(a) $H_\infty(Q) = 1$, and (b) as $k \to +\infty$, there hold

$$\|u_k - 1\|_{C^\alpha(S^n)} \to 0, \|H_k - 1\|_{L^p(S^n, g_k)} \to 0, \text{ and } \|\varphi_k - Q\|_{L^2(S^n, g_{S^n})} \to 0,$$

(4.10)

Here $0 < \alpha < 1 - n/p$.

**Proof.** In order to apply Theorem 4.2, we need to verify

(a) $\int_{S^n} H_k \, d\mu_{\mathcal{H}_k}$ is bounded, and

(b) $\int_{S^n} |H_k - \int_{S^n} H_k \, d\mu_{\mathcal{H}_k}|^p \, d\mu_{\mathcal{H}_k}$ is bounded.

To see this, from (4.6) and (4.7) it follows that

$$\left| \int_{S^n} H_k \, d\mu_{\mathcal{H}_k} \right| \leq \omega_n \|H_k - H_\infty\|_{L^p(S^n, g_k)} + \tau \omega_n \leq C.$$  

(4.11)

and

$$\left[ \int_{S^n} |H_k - \int_{S^n} H_k \, d\mu_{\mathcal{H}_k}|^p \, d\mu_{\mathcal{H}_k} \right]^{\frac{1}{p}} \leq \left[ \int_{S^n} |H_k - H_\infty|^p \, d\mu_{\mathcal{H}_k} \right]^{\frac{1}{p}} + \left[ \int_{S^n} \left| \int_{S^n} H_k \, d\mu_{\mathcal{H}_k} - \int_{S^n} H_\infty \, d\mu_{\mathcal{H}_k} \right|^p \, d\mu_{\mathcal{H}_k} \right]^{\frac{1}{p}} \leq C.$$  

(4.12)

So, (a) and (b) hold and then all conditions in Theorem 4.2 are satisfied. Thus, we can apply Theorem 4.2 to the sequence $(u_k)_k$.

(i) If $u_k$ is uniformly bounded in $W^{1,p}(S^n, g_{S^n})$ for $p > n$, then there exists $u_\infty \in W^{1,p}(S^n, g_{S^n})$ such that, up to a subsequence, $u_k \to u_\infty$ weakly in $W^{1,p}(S^n, g_{S^n})$ and strongly in $C^\alpha(S^n)$ for $0 < \alpha < 1 - \frac{2}{p}$ as $k \to +\infty$. In addition, by Sobolev embedding theory, we conclude that $\|u_k\|_{C^\alpha(S^n)} \leq C$. Let $P = 1 + \sup_{x \in S^n} \sup_{k \in \mathbb{N}} \{ -C, u_k^{2(n-1)} \}$. Then $P$ is bounded. From the fact that $u_k > 0$ and (4.8), it follows that

$$0 \leq (H_k - C) u_k^{\frac{n+1}{n}} = a_n \frac{\partial u_k}{\partial \eta_e} + u_k - C u_k^{\frac{n+1}{n}} = a_n \frac{\partial u_k}{\partial \eta_e} \left( 1 - C u_k^{\frac{1}{n-1}} \right) u_k \leq a_n \frac{\partial u_k}{\partial \eta_e} + P u_k.$$  

Since $\Delta_{g_k} u_k = 0$ and $\text{vol}(S^n, g_k) = \omega_n$, we may apply [13, Theorem 7.2] to obtain that $u_k \geq C^{-1}$. Now, it follows from the assumption $\|H_k - H_\infty\|_{L^p(S^n, g_k)} \to 0$ that $u_\infty$ weakly solves

$$\left\{ \begin{array}{ll}
\Delta_{g_k} u_\infty = 0, & \text{in } B^{e+1}, \\
\frac{\partial u_\infty}{\partial \eta_e} + u_\infty = H_\infty u_\infty^{\frac{n+1}{n}}, & \text{on } S^n.
\end{array} \right.$$
By standard elliptic regularity and bootstrapping, we conclude that $u_\infty$ is smooth since $H_\infty$ is smooth. Moreover, we have $C^{-1} \leq u_\infty \leq C$ and from 

$$d_\eta (\frac{\partial u_k}{\partial \eta} - \frac{\partial u_\infty}{\partial \eta}) = u_\infty - u_k + (H_k - H_\infty)u_k^{\frac{1}{2}} + H_\infty(u_k^{\frac{1}{2}} - u_\infty^{\frac{1}{2}}),$$

it follows that $u_k \to u_\infty$ strongly in $W^{1,p}(S^n, g_\infty)$. Since $u_\infty > 0$, we may let $g_\infty = u_\infty^{4/(n-1)} g_\eta$. Then the metric $g_\infty$ has mean curvature $H_\infty$.

(ii) If the second case of Theorem 4.2 occurs, we then prove that this leads to the concentration behavior as described in the Theorem 4.3. The proof will be divided into several claims.

**Claim 1:** There exists only one concentration point in the sense of (4.2).

**Proof of Claim 1:** By (4.6) and the fact that $\text{vol}(S^n, g_k) = \omega_n$, we can estimate

$$\|H_k\|_{L^p(S^n, g_k)} \leq \|H_k - H_\infty\|_{L^p(S^n, g_k)} + \tau \omega_n^{\frac{2}{3}}.$$

This together with (4.7) implies that

$$\liminf_{k \to +\infty} \int_{S^n} |H_k|^p \, d\mu_{g_k} \leq \tau^p \omega_n < 2 \omega_n.$$

Now, suppose $\{x_1, \ldots, x_m\}$, defined in the Theorem 4.2, are concentration points with $m \geq 2$. Let $0 < r < \frac{1}{2} \text{dist}(x_i, x_j); 1 \leq i < j \leq m$. It follows from (4.2) and the estimate above that

$$m \leq \sum_{i=1}^m \liminf_{k \to +\infty} \omega_n^{-1} \int_{\partial B_r(x_i)} |H_k|^\frac{p}{2} \, d\mu_k$$

$$\leq \liminf_{k \to +\infty} \left[ \sum_{i=1}^m \omega_n^{-1} \int_{\partial B_r(x_i)} |H_k|^p \, d\mu_k \right]$$

$$\leq \liminf_{k \to +\infty} \omega_n^{-1} \int_{\bigcup_{i=1}^m \partial B_r(x_i)} |H_k|^p \, d\mu_k$$

$$\leq \liminf_{k \to +\infty} \int_{S^n} |H_k|^p \, d\mu_k < 2,$$

which implies that $m < 2$ and thus contradicts with $m \geq 2$. This shows that $m = 1$, i.e. concentration point is unique.

**Claim 2:** There exists a constant $C > 0$ such that $C^{-1} \leq v_k \leq C$.

**Proof of Claim 2:** For the normalized sequence $(v_k)_k$, we note that

$$\int_{S^n} H_k \circ \varphi_k \, d\mu_{h_k} = \int_{S^n} H_k \, d\mu_{g_k},$$

$$\int_{S^n} |H_k \circ \varphi_k - \int_{S^n} H_k \circ \varphi_k \, d\mu_{h_k}|^p \, d\mu_{h_k} = \int_{S^n} |H_k - \int_{S^n} H_k \, d\mu_{g_k}|^p \, d\mu_{g_k},$$

$$R_{h_k} = R_k \circ \phi_k \equiv 0 \quad \text{and} \quad \text{vol}(S^n, h_k) = \text{vol}(S^n, g_k) = \omega_n.$$

Therefore, it follows from (4.11) and (4.12) that all the conditions in Theorem 4.2 hold for $(h_k)_k$. This means that we can apply Theorem 4.2 to the sequential metrics $(h_k)_k$. We claim that the case (ii) in Theorem 4.2 can never occur to $(h_k)_k$. Suppose the contrary. Then, we may follow the exact same proof as that in Claim 1 above.
to get that there exists a unique point $Q$ such that (4.2) holds for $(h_k)_k$. So, for sufficiently large $k$ and any $r > 0$, we have, by (4.6) and (4.7), that
\[
\frac{1}{\omega_n^2} + o(1) \leq \left( \int_{\partial B_r(Q)} |H_{h_k}|^n \, d\mu_{h_k} \right)^{\frac{1}{n}} 
\leq \left( \int_{\partial B_r(Q)} |H_{h_k} - H_{\infty} \circ \varphi_k|^{\ast n} \, d\mu_{h_k} \right)^{\frac{1}{n}} + \max_{3^n} |H_{\infty}| \left( \int_{\partial B_r(Q)} d\mu_{h_k} \right)^{\frac{1}{n}}
\leq o(1) + \tau \left( \int_{\partial B_r(Q)} d\mu_{h_k} \right)^{\frac{1}{n}},
\]
which implies that
\[
\int_{\partial B_r(Q)} d\mu_{h_k} \geq \tau^{-n} \omega_n + o(1).
\]
Since $\text{vol}(S^n, h_k) = \omega_n$, we get
\[
\int_{S^n \cap \partial B_r(Q)} d\mu_{h_k} \leq (1 - \tau^{-n}) \omega_n + o(1).
\]
(4.13)

From (4.13), it follows that
\[
\left| \int_{S^n} x \, d\mu_{h_k} - 1 \right| = \left| \int_{S^n} x \, d\mu_{h_k} - |Q| \right|
\leq \left| \int_{S^n} x \, d\mu_{h_k} - Q \right| \leq \int_{S^n} |x - Q| \, d\mu_{h_k}
= \omega_n^{-1} \int_{S^n \cap \partial B_r(Q)} |x - Q| v_k^{\ast} \, d\mu_{S^n} + \omega_n^{-1} \int_{\partial B_r(Q)} |x - Q| v_k^{\ast} \, d\mu_{S^n}
\leq 2(1 - \tau^{-n}) + r + o(1).
\]
Notice that $\tau < 2^{1/n}$. This implies that $2\tau^{-n} - 0$. Now, by choosing $r = (2\tau^{-n} - 1)/2$ and $k$ large enough, we then have
\[
\left| \int_{S^n} x \, d\mu_{h_k} \right| \geq 1 - 2(1 - \tau^{-n}) - r + o(1)
= \frac{2\tau^{-n} - 1}{2} + o(1) > 0.
\]
However, this contradicts with the fact that $h_k$ satisfies (4.3). Such a contradiction shows that case (i) in Theorem 4.2 will happen to $(h_k)_k$, that is, $v_k$ is uniformly bounded in $W^{1,p}(S^n, g_{S^n})$ for $p > n$. By Sobolev embedding theory, we conclude that there exists a positive constant $C$ such that $\|v_k\|_{C^\omega(S^n)} \leq C$ for $0 < \alpha < 1 - n/p$. Let
\[
P := 1 + \sup_{k \in \mathbb{N}} \sup_{S^n} \|Cv_k^{1/(n-1)}\|.
\]
Then by using the facts that $v_k > 0$ and $C_k \leq H_k \circ \varphi_k$, we follow the same proof as before to get
\[
\begin{cases}
\Delta_{g_k} v_k = 0, & \text{in } B^{n+1}, \\
a_n \frac{\partial v_k}{\partial n} + rv_k \geq 0, & \text{on } S^n.
\end{cases}
\]
From [13, Theorem 7.2] and the fact that $\int_{S^n} v_k^{\alpha} \, d\mu_{S^n} = \omega_n$, it follows that $v_k \geq C^{-1}$.

Claim 3: $v_k \to 1$ in $C^\alpha(S^n)$ with $0 < \alpha < 1 - n/p$.

Proof of Claim 3: Since $v_k$ is bounded in $W^{1,p}(S^n, g_{S^n})$ by the proof of Claim 2, it follows from the Sobolev embedding theory that there exists a function $v_\infty \in$
$W^{1,p}(S^n, g_{S^n})$ such that, up to a subsequence, $v_k \to v_\infty$ weakly in $W^{1,p}(S^n, g_{S^n})$ and strongly in $C^\alpha(S^n)$ with $0 < \alpha < 1 - n/p$.

Since $(p_k) \subset S^n$, we may assume that $p_k \to Q_*$ for some $Q_* \in S^n$. Up to here, we claim that the parameter $\varepsilon_k$ in the conform transformation $\varphi_k$ satisfies $\varepsilon_k \to 0$ as $k \to +\infty$. If not, we may assume that $\varepsilon_k \to \varepsilon_0 > 0$, then $\varphi_k \to \varphi_{Q_*,\varepsilon_0}$ and thus $\det(d\varphi_k) \to \det(d\varphi_{Q_*,\varepsilon_0})$ which is bounded away from zero. From this fact, the assertion of Claim 2 and (4.4), it follows that $u_k$ is bounded from both below and above by a positive constant. Now, in view of the equation

$$a_n \frac{\partial u_k}{\partial \eta} + u_k = H_k u_k^{n-1},$$

we can get that $u_k$ is uniformly bounded in $W^{1,p}(S^n, g_{S^n})$, which contradicts with our assumption. Hence, we have $\varepsilon_k \to 0$. By the definition of $\varphi_k$, we conclude that $\varphi_k \to Q_*$ for $x \in S^n \setminus \{Q_*\}$. This fact together with (4.7), Claim 2 and the dominated convergence theorem implies that

$$\|H_{h_k} - H_{v_\infty(Q_*)}\|_{L^p(S^n, h_k)} \leq \|H_{h_k} - H_{v_\infty \circ \varphi_k}\|_{L^p(S^n, h_k)} + \|H_{v_\infty \circ \varphi_k} - H_{v_\infty(Q_*)}\|_{L^p(S^n, h_k)} \to 0 \quad (4.14)$$

as $k \to +\infty$. This implies that $v_\infty$ weakly solves

$$\begin{cases}
\Delta_g v_\infty = 0 & \text{in } B^{n+1},
\frac{\partial v_\infty}{\partial \eta} + v_\infty = H_{v_\infty(Q_*)}^{n-1} & \text{on } S^n.
\end{cases}$$

Since we have $\int_{S^n} x v_k^n \, d\mu_{S^n} = 0$ and $\int_{S^n} v_k^n \, d\mu_{S^n} = \omega_n v_\infty$ will satisfy $\int_{S^n} x v_k^n \, d\mu_{S^n} = 0$ and $\int_{S^n} v_k^n \, d\mu_{S^n} = \omega_n$. It follows, by the Escobar’s uniqueness theorem, that $v_\infty$ must be a constant and hence $v_\infty \equiv 1$. Moreover, plugging $v_\infty \equiv 1$ into the equation above yields $H_{v_\infty(Q_*)} = 1$. Finally, substituting $H_{v_\infty(Q_*)} = 1$ into (4.14) gives

$$\|H_k - 1\|_{L^p(S^n, g_k)} \to 0, \quad \text{as } k \to +\infty. \quad (4.15)$$

Claim 4: There exists a unique point $Q \in S^n$ such that $d\mu_{g_k} \to \omega_n \delta_Q$ weakly in the sense of measure as $k \to +\infty$.

Proof of Claim 4: It follows from (4.15), Claim 1 and (4.2) that there exists a unique point $Q \in S^n$ such that, for $k$ large enough and any $r > 0$, there holds

$$\omega_n^{\frac{1}{n}} + o(1) \leq \left( \int_{\partial^r B_1(Q)} |H_k|^{n} \, d\mu_k \right)^{\frac{1}{n}} \leq \left( \int_{\partial^r B_1(Q)} |H_k - 1|^{n} \, d\mu_k \right)^{\frac{1}{n}} + \left( \int_{\partial^r B_1(Q)} d\mu_k \right)^{\frac{1}{n}} \leq \omega_n^{\frac{1}{n}} + o(1).$$

Hence, $d\mu_k \to \omega_n \delta_Q$, weakly in the sense of measure as $k \to +\infty$.

Claim 5: There hold (a) $H_{v_\infty(Q)} = 1$ and (b) $\|\varphi_k - Q\|_{L^2(S^n, g_{S^n})} \to 0$ as $k \to +\infty$.

Proof of Claim 5: In view of the proof of Claim 3, it suffices to show that $Q_* = Q$.

Indeed, on one hand, from Claim 4 it follows that

$$\int_{S^n} x \, d\mu_k \to Q, \quad \text{as } k \to +\infty.$$
On the other hand, it follows from the fact that $u_k$ is uniformly bounded and the dominated convergence theorem that
\[
\left| \int_{S^n} \varphi_k d\mu_{h_k} - Q_+ \right| \leq \int_{S^n} |\varphi_k - Q_+| d\mu_{h_k} \to 0,
\]
as $k \to +\infty$. Notice that, by the change of variables, one has
\[
\int_{S^n} x d\mu_k = \int_{S^n} \varphi_k d\mu_{h_k}.
\]
Now, it is easy to see that $Q_+ = Q$. \qed

4.2. **Blow-up analysis.** The principal goal of this subsection is to describe the sequential asymptotic behavior of the flow (2.2). Firstly, let us recall that $f$ satisfies $\max_{S^n} |f|/(\int_{S^n} f d\mu_{S^n}) < 2^{1/n}$. Hence, we can choose
\[
\sigma = \frac{1}{2} \left[ 2^\frac{n}{\max_{S^n} |f|} - 1 \right] > 0,
\]
and set
\[
\beta = (1 + \sigma)^\frac{1}{\max_{S^n} |f|} \int_{S^n} f d\mu_{S^n}. \tag{4.16}
\]
With all notations above settled, we finally define the set
\[
X_f = \left\{ u \in X; \int_{S^n} u^2 d\mu_{S^n} = \omega_n \text{ and } E_f[u] \leq \beta \right\}.
\]

**Remark 4.4.** Notice that $X_f \neq \emptyset$. In fact, when $u \equiv 1$ we have $\int_{S^n} u^2 = \omega_n$, $\int_{S^n} f u^2 d\mu_{S^n} = \int_{S^n} f d\mu_{S^n} > 0$ and $E_f[1] = (\int_{S^n} f d\mu_{S^n})^{(1-n)/n} < \beta$. Hence, $u \equiv 1 \in X_f$.

Then for any $u_0 \in X_f$, it follows from Proposition 3.7 that the flow (2.2) has a unique smooth solution $u(t)$ well defined on $[0, +\infty)$. For an arbitrary time sequence $(t_k)_k \subset [0, +\infty)$ with $t_k \to +\infty$ as $k \to +\infty$, we set
\[
u_k = u(t_k), \quad g_k = g(t_k), \quad H_k = H_{g_k} \quad \text{and} \quad d\mu_k = d\mu_{g_k}.
\]
In order to apply Theorem 4.3 to our flow, we have to verify all conditions in that theorem are satisfied.

**Lemma 4.5.** For the sequence $(u_k)_k$ with the associated mean curvatures $H_k$ defined as above, there hold
(i) $\text{vol}(S^n, g_k) = \omega_n$ and $R_k \equiv 0$.
(ii) there exists a smooth function $H_\infty$ with $\max_{S^n} |H_\infty| < \tau < 2^{1/n}$ such that $\|H_k - H_\infty\|_{L^p(S^n, g_k)} \to 0$, for some $p > n$;
(iii) there exists a sequence of smooth functions $\sigma_k(x)$ with $\sup_{k \in \mathbb{N}} \|\sigma_k\|_{C^0(S^n)} < C_*$ for some positive number $C_*$ such that $\sigma_k \leq H_k$.

**Proof.** (i) from the choice of the initial data $u_0$ and the volume-preserving property of the flow (2.2), it follows that $\text{vol}(S^n, g_k) = \omega_n$. Moreover, the flow equation (2.1) shows that $R_k \equiv 0$.
(ii) By Lemma 3.2, we may assume that, up to a subsequence, $\lambda(t_k) \to \lambda_\infty$ as $k \to +\infty$, where $\lambda_\infty \in [\lambda_1, \lambda_2]$. Now, by the definition of $\lambda_2$ and the choices of initial data $u_0$ and $\beta$, we can estimate
\[
\lambda_\infty |f| \leq \lambda_2 \max_{S^n} |f| \leq (E_f[u_0])^{\frac{1}{\max_{S^n} |f|}}.
\]
to the sequential metrics (4.3), we obtain \( \lambda_{\infty} \) is well defined for all \( t \in (0, +\infty) \). Moreover, from Proposition 4.1, it follows that

\[
\|H_k - \lambda_{\infty} f \|_{L^p(S^n, g_k)} \leq \|H_k - \lambda(t_k) f \|_{L^p(S^n, g_k)} + |\lambda(t_k) - \lambda_{\infty}| \|f\|_{L^p(S^n, g_k)} \rightarrow 0,
\]

for some \( p > n \). By setting \( H_{\infty} = \lambda_{\infty} f \), we thus complete the proof of (i).

(ii) By Lemma 3.6, we have \( H_k \geq \lambda(t_k) f + \gamma \). From Lemma 3.2, we conclude that \( \lambda(t_k) \geq -\lambda_2 \max_{S^n} |f| \). By setting \( C_\ast = -\lambda_2 \max_{S^n} |f| + \gamma \), we thus obtain \( H_k \geq C_\ast \). 

From this lemma, a direct consequence of Theorem 4.3 reads

**Corollary 4.6.** Let \( u(t) \) be the smooth solution of the flow (2.2) with initial data \( u_0 \in X_1 \). Associated with the sequential metrics \( g_k = u_k^{4/(n-1)} g_e \), we let \( h_k = \phi_k^\gamma g_k = v_k^{4/(n-1)} g_e \) be the sequence of corresponding normalized metrics, where \( \phi_k \) is the conformal transformation on the unit ball and its restriction on the boundary are given by \( \phi_k = \phi_k|_{S^n} = \phi_{p_k, x_k} \) with \( p_k = p(t_k) \) and \( x_k = x(t_k) \). Then there hold either (i) there exists a positive function \( u_\infty \in W^{1,p}(S^n, g_{\infty}) \) such that \( u_k \rightarrow u_\infty \) in \( W^{1,p}(S^n, g_{\infty}) \). In addition, if we let \( g_{\infty} = u_\infty^{4/(n-1)} g_e \) then \( g_{\infty} \) has mean curvature \( \lambda_{\infty} \); or

(ii) there exists a uniqueness point \( Q \) (depending only on \( u_0 \)) such that \( d\mu_k \rightarrow \omega_n \delta_Q \) weakly in the sense of measure as \( k \rightarrow +\infty \). Moreover, in the latter case, one has (a) \( \lambda_{\infty} f(Q) = 1 \). In particular, \( f(Q) > 0 \), and (b) \( \|\phi_k - 1\|_{C^\alpha(S^n)} \rightarrow 0 \) with \( 0 < \alpha < 1 - n/p \), \( \|H_k - 1\|_{L^p(S^n, g_k)} \rightarrow 0 \) and \( \|\phi_k - Q\|_{L^2(S^n, g_{\infty})} \rightarrow 0 \) as \( k \rightarrow +\infty \).

### 4.3. Asymptotic behavior of the flow

In view of Corollary 4.6, to prove our theorems, it suffices to prove that there exists a time sequence \( (t_k) \) such that case (i) occurs to the corresponding metrics \( (g_k) \). However, this assertion is generally hard to be obtained directly. Here, we adopt the contradiction argument. So, we assume that \( f \) cannot be realized as a mean curvature of any conformal metric. In other words, the flow will be divergent and then case (ii) happens to the metrics \( (g_k) \) for arbitrary time sequence \( (t_k) \). In this scenario, we can pass the sequential behavior to the uniform one of the flow (2.2). For \( t > 0 \), let

\[
S = S(t) = \int_{S^n} x \, d\mu_g
\]

be the center of mass of \( g = g(t) \). Then we have

**Lemma 4.7.** \( S(t) \rightarrow Q \) as \( t \rightarrow +\infty \). In particular, \( S(t) \neq 0 \) for all large \( t \).

**Proof.** For an arbitrary time sequence \( (t_k) \subset [0, +\infty) \) with \( t_k \rightarrow +\infty \) as \( k \rightarrow +\infty \), we can apply Corollary 4.6 to the sequential metrics \( (g(t_k)) \) to get that \( d\mu_{g_k} \rightarrow \omega_n \delta_Q \) as \( k \rightarrow +\infty \). Hence \( S(t_k) \rightarrow Q \) as \( k \rightarrow +\infty \). By the arbitrariness of the sequence \( (t_k) \), we thus conclude that \( S(t) \rightarrow Q \) as \( t \rightarrow +\infty \).

In view of this lemma, we may assume that \( S(t) \neq 0 \) for all \( t > 0 \). Then the image of \( S(t) \) under radial projection

\[
Q(t) = S / |S| \in S^n
\]

is well defined for all \( t > 0 \).
Proposition 4.8. Suppose that $f$ can not be realized as the mean curvature of any conformal metric on the boundary $S^n$. Let $u(t)$ be the smooth solution of (2.2), $\varphi(t)$ the restriction of the conformal transformation $\phi(t)$ on the boundary, $v(t)$ the corresponding normalized flow and $h(t)$ the normalized metric. Then, as $t \to +\infty$, there hold

(i) $\max_{S^n} u(\cdot, t) \to +\infty$;
(ii) $v(t) \to 1$ in $C^\alpha(S^n)$ for $\alpha \in (0, 1)$, $\varphi(t) \to Q$ in $L^2(S^n, g_{S^n})$, and $\|H(t) - 1\|_{L^p(S^n, g(t))} \to 0$ for $p > n$;
(iii) $Q(t) \to Q$ and $\lambda(t)f(Q(t)) \to 1$;
(iv) $E_f[u(t)] \to (f(Q))^{-\frac{n}{n-1}}$, and furthermore, $\nabla_{S^n} f(Q) = 0$ and $\Delta_{S^n} f(Q) < 0$.

Proof. (i) Assume that $\|u(t)\|_{C^\alpha(S^n)} \leq C$ for some positive number $C$ and all $t \geq 0$. Observe that
$$a_n \frac{\partial}{\partial \eta_n} u(t) + u(t) = (H(t) - \lambda(t)f)u(t) \frac{\partial}{\partial \eta_n} + \lambda(t)f u(t) \frac{\partial}{\partial \eta_n}.$$ 

Hence, by Lemma 3.2 and Proposition 4.1, it is easy to see that $u(t)$ is uniformly bounded in $W^1, f(S^n, g_{S^n})$. Then by Corollary 4.6, we can obtain that, up to a constant, $f$ can be realized as a mean curvature of some conformal metric. But this contradicts with our assumption.

(ii) By way of contradiction, we assume that there exists a time sequence $t_k \to +\infty$ such that
$$\liminf_{k \to +\infty} (\|u_k - 1\|_{C^\alpha(S^n)} + \|\varphi_k - Q\|_{L^2(S^n, g)} + \|H_k - 1\|_{L^p(S^n, g)}) > 0.$$ 

But then, by Corollary 4.6, a subsequence $u_k \to u_\infty$ with $g_\infty = u_\infty^{4/(n-1)} g_\infty$. The metric $g_\infty$, up to a constant, has the mean curvature $f$, contrary to our assumption.

(iii) It follows from Lemma 4.7 and the definition of $Q(t)$ that $Q(t) \to Q$. This together with the fact that $\int_{S^n} H_h d\mu_h \to 1$, and Proposition 4.1 implies that
$$\lim_{t \to +\infty} \left(1 - \lambda(t)f(Q(t))\right) = \lim_{t \to +\infty} \left[ \int_{S^n} H_h d\mu_h - \lambda(t)f(Q(t)) \right]$$
$$= \lim_{t \to +\infty} \left[ \int_{S^n} H_h - \lambda(t)f \circ \varphi(t) d\mu_h + \lambda(t) \int_{S^n} f \circ \varphi(t) - f(Q) d\mu_h + \lambda(t) \left(f(Q) - f(Q(t))\right) \right] = 0$$

(iv) Recall that
$$E_f[u(t)] = \frac{\int_{S^n} H(t) d\mu_{g(t)}}{\left(\int_{S^n} f d\mu_{g(t)}\right)^{\frac{n}{n-1}}}.$$ 

On one hand, $\int_{S^n} H(t) d\mu_{g(t)} \to 1$; On the other hand,
$$\left| \int_{S^n} f d\mu_{g(t)} - f(Q) \right| \leq \int_{S^n} \left| f \circ \varphi(t) - f(Q) \right| d\mu_h \to 0.$$ 

Hence, the assertion holds.

(v) Since the proof is exactly the same as that in [13, Proposition 5.7], we omit the details. \qed
5. Proof of Theorems 1.3 and 1.5

5.1. **Proof of Theorem 1.3.** Recall the notations in subsection 4.2, for $p \in \mathbb{S}^n, 0 < \varepsilon < +\infty$, if we put the point $p$ at the origin in stereographic coordinates, then the restriction of conformal transform $\phi$ on the boundary $\mathbb{S}^n$ is given by $\varphi_{p,\varepsilon} = \psi_{\varepsilon} \circ \pi$.

Let $g_{p,\varepsilon}|_{\mathbb{S}^n} = u_{p,\varepsilon}^{4/(n-1)} g_{\mathbb{S}^n}$ with $u_{p,\varepsilon} = |\det(d\varphi_{p,\varepsilon})|^{1/2\varepsilon}$. Then

$$d\mu_{g_{p,\varepsilon}} \to \omega_n \delta_p, \quad \text{as } \varepsilon \to 0.$$ 

Now, for $\rho \in \mathbb{R}_+$, we denote the sub-level set of $E_f$ by

$$L_\rho = \{ u \in X_f : E_f[u] \leq \rho \}.$$

It follows from Proposition 4.8 that the concentration phenomenon can only occur at the critical points of $f$ where it takes positive values. For convenience, we label all these critical points $p_1, \ldots, p_N$ of $f$ so that $0 < f(p_i) < f(p_j)$ for $1 \leq i < j \leq N$ and let

$$\beta_i = (f(p_i))^{1/n} = \lim_{\varepsilon \to 0} E_f[u_{p_i,\varepsilon}], 1 \leq i \leq N.$$

We may assume, w.l.o.g., that all positive critical levels $f(p_i), 1 \leq i \leq N$ are distinct. By choosing $s_0 = \frac{1}{2} \min_{1 \leq i \leq N} (f(p_i) - f_{p_i-1}) > 0$, we then have $\beta_i - 2s_0 > \beta_{i+1}$ for all $i$. Now, we are ready to characterize the homotopy on $L_\rho$. We remark here that the contraction mapping given by Xu & Zhang [13] does not work anymore. We need a new construction for such a mapping.

**Proposition 5.1.**

(i) If $\max \left\{ \beta_1, \left( \int_{\mathbb{S}^n} f \, d\mu_{\mathbb{S}^n}(1-n)/n \right) \right\} < \gamma$ where $\beta$ has been chosen in (4.16), then $L_{\beta_1}$ is contractible.

(ii) For $0 < s \leq s_0$ and each $i$, the set $L_{\beta_i-s}$ is homotopy equivalent to the set $L_{\beta_{i-1}+s}$.

(iii) For each critical point $p_i$ of $f$ with $\Delta_{\mathbb{S}^n}f(p_i) > 0$, the set $L_{\beta_i+s_0}$ is homotopy equivalent to the set $L_{\beta_{i-1}-s_0}$.

(iv) For each critical point $p_i$ of $f$ with $\Delta_{\mathbb{S}^n}f(p_i) < 0$, the set $L_{\beta_i+s_0}$ is homotopy equivalent to the set $L_{\beta_{i-1}-s_0}$ with $(n - \text{ind}_f(p_i))$-cell attached.

**Proof.** (i). For each $u_0 \in L_{\beta_0}$, we fix a sufficiently large $T > 0$ and set $\zeta = \zeta(T) = (\max_{\mathbb{S}^n} u(T, u_0))$. It follows from Proposition 4.8 (i) that

$$\lim_{T \to +\infty} \zeta = 0. \quad (5.1)$$

In view of the proof of [13, Lemma 6.2], $T$ can be chosen continuously depending on the initial data $u_0$. So, $\zeta$ is continuously depending on $u_0$ too. Define

$$u_s = \begin{cases} u(2sT, u_0), & 0 \leq s \leq \frac{1}{2}, \\ \frac{1}{(2-2s)(2\varepsilon u(T,u_0))^{1/2} + 2s - 1} & , \frac{1}{2} < s \leq 1. \end{cases}$$

Then, we have the claim

**Claim:** The function $u_s$ satisfies $1^n \int_{\mathbb{S}^n} u_s^{2\varepsilon} \, d\mu_{\mathbb{S}^n} = 1, 2^n \int_{\mathbb{S}^n} f u_s^{2\varepsilon} \, d\mu_{\mathbb{S}^n} > 0$ and $3^nE_f[u_s] \leq \beta_0$.

**Proof of Claim:** From the choice of the initial data $u_0$, the volume-preserving property of the flow (2.2), Lemma 3.1 and the decay property of the energy functional $E_f[u]$, it follows that $u_s$ fulfills the said properties in the claim for $0 \leq s \leq \frac{1}{2}$. Thus, we are left to check that for $\frac{1}{2} < s \leq 1$. 

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1°. By a direct computation and the volume-preserving property of the flow (2.2), we conclude that

$$\int_{S^2} u_s^{2s} \, d\mu_{S^*} = \frac{(2 - 2s)\zeta^{2s} \int_{S^2} u(T, u_0)^{2s} \, d\mu_{S^*} + 2s - 1}{(2 - 2s)\zeta^{2s} + 2s - 1} = 1.$$  

2°. It follows from a direct computation, Lemma 3.1 and assumption (i) in Theorem that

$$\int_{S^2} f u_s^{2s} \, d\mu_{S^*} = \frac{(2 - 2s)\zeta^{2s} \int_{S^2} f u(T, u_0)^{2s} \, d\mu_{S^*} + (2s - 1)\int_{S^2} f \, d\mu_{S^*}}{(2 - 2s)\zeta^{2s} + 2s - 1} > 0.$$  

3°. We set

$$w_s = \left[ (2 - 2s)(\zeta u(T, u_0))^{2s} + (2s - 1) \right]^{\frac{1}{2s}}.$$  

Since the energy functional $E_f[u]$ is scale-invariant, we have

$$E_f[u_x] = E_f[w_x] = \frac{1}{\mu_{S^2}} \int_{S^2} a_n |\nabla w |^2 \, dV_{S^2} + \int_{S^2} w_s \, d\mu_{S^*} = \frac{I}{(II)^{\frac{s}{n}}}.$$  

**Estimate of I:** A simple calculation shows that

$$|\nabla w_s |^2_{S^2} = \zeta^2 (2 - 2s) \frac{n+1}{n} \frac{(2 - 2s)(\zeta u(T, u_0))^{2s}}{(2 - 2s)(\zeta u(T, u_0))^{2s} + (2s - 1)} \frac{n+1}{n} |\nabla u(T, u_0) |^2_{S^0},$$

and by an elementary inequality, we get that

$$w_s^2 = \left[ (2 - 2s)(\zeta u(T, u_0))^{2s} + (2s - 1) \right]^{\frac{1}{2s}} \leq \zeta^2 (2 - 2s) \frac{n+1}{n} u^2(T, u_0) + (2s - 1) \frac{n+1}{n}.$$  

Combining the two estimates above yields

$$I \leq \zeta^2 (2 - 2s) \frac{n+1}{n} E[u(T, u_0)] + (2s - 1) \frac{n+1}{n}. \quad (5.3)$$

**Estimate of II:** Notice that

$$II = \int_{S^2} f w_s^{2s} \, d\mu_{S^*} = \zeta^2 (2 - 2s) \int_{S^2} f u^{2s} (T, u_0) \, d\mu_{S^*} + (2s - 1) \int_{S^2} f \, d\mu_{S^*}. \quad (5.4)$$  

Plugging (5.3) and (5.4) into (5.2) gives

$$E_f[u_x] = E_f[w_x] \leq \frac{\zeta^2 (2 - 2s) \frac{n+1}{n} E[u(T, u_0)] + (2s - 1) \frac{n+1}{n}}{\left( \zeta^2 (2 - 2s) \int_{S^2} f u^{2s} (T, u_0) \, d\mu_{S^*} + (2s - 1) \int_{S^2} f \, d\mu_{S^*} \right) \frac{n+1}{n}}.$$  

From the volume-preserving property of flow (2.2) and (3.2), it follows that

$$\alpha_1 := \left( \frac{1}{E_f[u_0]} \right)^\frac{n}{n+1} \leq \int_{S^2} f u^{2s} (T, u_0) \, d\mu_{S^*} \leq \operatorname{max} |f| := \alpha_2.$$  

Moreover, by the fact that $E_f[u(T, u_0)] \leq \beta_0$, we get

$$E[u(T, u_0)] \leq \beta_0 \left( \int_{S^2} f u^{2s} (T, u_0) \, d\mu_{S^*} \right)^\frac{n+1}{n} \leq \beta_0 \alpha_2.$$
Substituting all the estimates above into (5.5) yields
\[ E_f[u_s] \leq \frac{\zeta^2(2 - 2s)^{\frac{m+n}{n}} \beta_0 \alpha_{2k}^{\frac{m+n}{n}} + (2s - 1)^{\frac{m+n}{n}}}{(\zeta^2(2 - 2s)\alpha_1 + (2s - 1)\int_{S^n} f \, d\mu_{S^n})^{\frac{m+n}{(n-2)\alpha_1}}}. \]

By letting \( T \to +\infty \) in the estimate above, observing the fact (5.1) and the choice of \( \beta_0 \), we have
\[ \lim_{T \to +\infty} E_f[u_s] \leq \left( \int_{S^n} f \, d\mu_{S^n} \right)^{\frac{m+n}{n}} < \beta_0. \]

By choosing \( T > 0 \) even larger, we thus complete the proof of claim.

Therefore, we can conclude by the claim that \( u_s \in L_{\beta_0} \) for all \( 0 \leq s \leq 1 \). Moreover, by the definition of \( u_s \), it is easy to see that \( u_s = u_0 \in L_{\beta_0} \) for \( s = 0 \) and \( u_s \equiv 1 \) for \( s = 1 \). Hence, \( u_s \) induces a contraction within \( L_{\beta_0} \) and we complete the proof of (i).

Recall that \( f(p_i) > 0 \) at each critical point \( p_i \), we have that \( f(x) > 0 \) in a small neighborhood of \( p_i \). Since the proof of (ii), (iii) and (iv) only requires the local information around each critical point \( p_i \), it would have no difference from the case that \( f \) is strictly positive. Hence, one can follow the exact same proof as that in [13, Proposition 6.1], we omit the details. \( \square \)

**Proof of Theorem 1.3 and Corollary 1.4:** Suppose the contrary, namely, \( f \) cannot be realized as the boundary mean curvature of any conformal metric \( g \) on the unit ball. A suitable choice of \( \beta_0 \) in part (i) of Proposition 5.1 shows that \( L_{\beta_0} \) is contractible. In addition, the flow (2.2) defines a homotopy equivalence of the set \( \mathcal{E}_0 = L_{\beta_0} \) with a set \( \mathcal{E}_\infty \), whose homotopy type is that of a point with \( n - \text{ind}_f(x) \) dimensional cells attached for every critical point \( x \) of \( f \) on \( S^n \) where \( f(x) > 0 \) and \( \Delta_{S^n}f(x) < 0 \). It then follows from [4, Theorem 4.3] that
\[ \sum_{i=0}^{n} s^i m_i = 1 + (1 + s) \sum_{i=0}^{n} s^i k_i \] (5.6)
holds for the Morse polynomials of \( \mathcal{E}_0 \) and \( \mathcal{E}_\infty \), where \( k_i \geq 0 \) and \( m_i \) are given in (1.4). By equating the coefficients in the polynomials on the left and right hand side of (5.6), we obtain a set of non-trivial solutions of (1.3), which violates the hypothesis in Theorem. We thus obtain the desired contradiction and the proof of Theorem 1.3 is completed. Furthermore, by setting \( s = -1 \) in (5.6) we can obtain (1.5) and thus the assertion in Corollary 1.4 holds. \( \square \)

5.2. **Proof of Theorem 1.5.** In view of [8, Lemma 2.1], we see that \( u \) is a \( G \)-invariant function if the initial data \( u_0 \in X_f \) is a \( G \)-invariant function. Fix any \( G \)-invariant initial data \( u_0 \in X_f \), by the uniqueness of the solution of the flow (2.2) and the decay of \( E_f[u] \), we can assume that
\[ E_f[u(t)] < E_f[u_0], \quad \text{for } t \in (0, +\infty), \] (5.7)
Since we have assumed that case (ii) in Theorem 4.2 occurs to the corresponding sequential metrics \( (g(t_k))_k \), the blow-up behavior of \( (g(t_k))_k \) in Proposition 4.6 will happen. In particular, it follows from Proposition 4.6 that
\[ \lim_{k \to +\infty} \omega_n^{-1} \int_{\partial^* B_i(Q)} f u_k^\alpha \, d\mu_{S^n} = f(Q) = \frac{1}{\lambda_\infty}, \] (5.8)
where \( r > 0 \) is arbitrary and \( Q \), depending on the choice of \( u_0 \), is the unique concentration point in Proposition 4.6, which also implies that for any \( y \in S^n \), if \( Q \notin B_r(y) \) for some \( r > 0 \), then
\[
\lim_{k \to +\infty} \int_{\partial B_r(y)} f u_k^2 \, d\mu_{S^n} = 0. \tag{5.9}
\]

Now, we split our argument into two cases

**Case 1.** \( \Sigma = \emptyset \). If this case happens, then we can find \( \theta \in G \) such that \( \theta(Q) \neq Q \).

Since \( f \) and \( u_k \) are \( G \)-invariant, we conclude, by (5.8) and change of variables, that
\[
\lim_{k \to +\infty} \omega_n^{-1} \int_{\partial B_r(\theta(Q))} f u_k^2 \, d\mu_{S^n} = \lim_{k \to +\infty} \omega_n^{-1} \int_{\partial B_r(Q)} (f \circ \theta(y))(u_k \circ \theta(y))^2 \, d\mu_{S^n} = \frac{1}{\lambda_\infty} \neq 0.
\]

On the other hand, as \( \theta(Q) \neq Q \), we can find \( r > 0 \) small enough such that \( Q \notin \partial B_r(\theta(Q)) \). Then, by (5.9), we have
\[
\lim_{k \to +\infty} \omega_n^{-1} \int_{\partial B_r(\theta(Q))} f u_k^2 \, d\mu_{S^n} = 0,
\]
which is a contradiction.

**Case 2.** \( \Sigma \neq \emptyset \) and \( \max_{\Sigma} f < \frac{\int_{S^n} f \, d\mu_{S^n}}{\int_{S^n} f^2 \, d\mu_{S^n}} \). By Remark 4.4, we can choose the initial data \( u_0 \equiv 1 \) which is obviously a \( G \)-invariant function. If \( Q \notin \Sigma \), then we can obtain a contradiction by repeating the argument in Case 1. Hence, we must have \( Q \in \Sigma \). To proceed, we need a refined estimate upon the number \( \lambda_\infty \). By the decay of \( E[f[u]] \), we have
\[
\frac{E[u_k]}{(\int_{S^n} f u_k^2 \, d\mu_{S^n})^{\frac{1}{n}}} \leq E[f[u_1]],
\]
for all \( k \geq 1 \), which implies, by sharp Sobolev trace inequality and volume-preserving property, that
\[
\int_{S^n} f u_k^2 \, d\mu_{S^n} \geq \left( \frac{1}{E[f[u_1]]} \right)^{\frac{2}{n}}.
\]
From this, it follows that
\[
\lambda_k = E[f[u_k]]\left( \int_{S^n} f u_k^2 \, d\mu_{S^n} \right)^{-\frac{1}{2}} \leq E[f[u_1]]\left( \frac{1}{E[f[u_1]]} \right)^{\frac{1}{n}} = (E[f[u_1]])^{\frac{n}{n+1}}.
\]

By letting \( k \to +\infty \) in the inequality above and (5.7), we have
\[
\lambda_\infty \leq (E[f[u_1]])^{\frac{n}{n+1}} < (E[f[u_0]])^{\frac{n}{n+1}} = \frac{1}{\int_{S^n} f \, d\mu_{S^n}}.
\]
where we have used the fact $u_0 \equiv 1$ in the last equality. On the other hand, we know that $\lambda_\infty f(Q) = 1$. Hence, we conclude that

$$f(Q) > \int_{S^n} f \, d\mu_{S^n},$$

which contradicts with our assumption in Theorem 1.5.

### 5.3. Proof of Theorem 1.6

Notice that $\Sigma \neq \emptyset$. If $\max_{\Sigma} f \leq \int_{S^n} f \, d\mu_{S^n}$, then we can repeat the argument in the case 2 of Theorem 1.5 to obtain the assertion. While $\max_{\Sigma} f > \int_{S^n} f \, d\mu_{S^n}$, from [8, Lemma 3.1], we can choose an initial data $u_0$ which is invariant under (Sym1) or (Sym2) such that 1) $\int_{S^n} f u_0^2 \, d\mu_{S^n} > 0$; 2) for sufficiently small $\varepsilon > 0$ there holds

$$E_f[u_0] < \frac{1}{(\max_{\Sigma} f)^\frac{n+1}{n}} + \varepsilon,$$

which implies, by the choice of $\beta$, that

$$E_f[u_0] < \frac{1}{\left(\int_{S^n} f \, d\mu_{S^n}\right)^\frac{n+1}{n}} + \varepsilon \leq \beta.$$

This shows that $u_0 \in X_f$. Once we have this fact, the conclusions in Proposition 4.8 will hold. In particular, at the corresponding concentration point $Q$, there holds $\Delta_{S^n} f(Q) \leq 0$. In view of [8, Proof of Theorem 1.2], we can obtain that there exists a constant $\alpha > 0$ such that

$$f(y) = \max_{\Sigma} f > f(Q) + \alpha.$$

Substituting this inequality into (5.10) yields

$$E_f[u_0] < \frac{1}{(f(Q) + \alpha)^\frac{n+1}{n}} + \varepsilon = \frac{1}{(f(Q))^\frac{n+1}{n}} \left(\frac{f(Q)}{(f(Q) + \alpha)}\right)^\frac{n+1}{n} + \varepsilon.$$

Since $\varepsilon$ is sufficiently small, we have

$$\varepsilon \leq \frac{1}{(f(Q))^\frac{n+1}{n}} \left[1 - \left(\frac{f(Q)}{(f(Q) + \alpha)}\right)^\frac{n+1}{n}\right],$$

which implies, by Proposition 4.8, that

$$E_f[u_0] < (f(Q))^\frac{1-n}{n} = \lim_{k \to +\infty} E_f[u_k].$$

But this contradicts with the decay property of the energy functional $E_f[u]$. \hfill \square

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