On the constrained KP hierarchy

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Abstract

An explanation for the so-called constrained hierarchies is presented by linking them with the symmetries of the KP hierarchy. While the existence of ordinary symmetries (belonging to the hierarchy) allows one to reduce the KP hierarchy to the KdV hierarchies, the existence of additional symmetries allows to reduce KP to the constrained KP.

There are several papers published lately concerning the so-called “constrained KP hierarchies”, [1-4]. Closely related to this topic is the “two-boson representation” of the KP hierarchy also given in a series of papers [5-7].

In this note we are going to explain, in a not very formal way, our viewpoint on the origin of the special form of those constraints linking them with well-known subjects such as symmetries of the KP hierarchy.

Roughly speaking, in the same way as the existence of ordinary symmetries (belonging to the hierarchy) allows one to reduce the KP hierarchy to the KdV hierarchies, the existence of additional symmetries allows to reduce KP to the “constrained KP”. More than that, the possibility of this type of constraints is equivalent to the existence of the additional symmetries.

The proof that the above mentioned constraints are compatible with the hierarchy is rather simple, see quoted papers or the last paragraph of this note, while our main goal is to put the problem into a wider context giving an interpretation of the constrained hierarchy from the point of view of the symmetries. It is based on a new formula for the generator of additional symmetries [8]. For convenience, since that paper is not published yet, the proof of the formula is also enclosed into this note as an Appendix.

We are not discussing Hamiltonian properties of the equations; some of the above cited works were devoted to their study.

The constrained hierarchy is the KP hierarchy restricted to pseudo-differential operators of the form

\[ L_n = \partial^n + U_{n-2}\partial^{n-2} + \ldots + U_0 + w\partial^{-1}w^*, \quad \partial = d/dx \]

where \( w \) and \( w^* \) satisfy the equations

\[ \partial_m w = L_{m+}^m w, \quad \partial_m w^* = -L_{m+}^{m*} w^* \]

and \( L_{m+}^{m*} \) is the operator adjoint to \( L_{m+}^m \). The equations are exactly the same as those for the Baker and the adjoint Baker functions. However, \( w \) and \( w^* \) are not supposed to be necessarily the Baker and the adjoint functions (i.e., eigenfunctions of the operators \( L \) and \( L^* \)), just any solutions.

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Recall some well-known definitions and results. Let

\[ L = \partial + u_1 \partial^{-1} + u_2 \partial^{-2} + \ldots \]

be a pseudo-differential operator (ΨDO), the coefficients \( u_k \) being taken as independent generators of a differential algebra. The KP-hierarchy is the set of equations

\[ \partial_m L = [L^m_+, L], \quad \partial_m = \partial/\partial t_m \]  

(1)

where \( t_m, m = 1, 2, \ldots \) are some variables, and the subscript + symbolizes preserving only terms with non-negative powers of \( \partial \) (similarly, the subscript − below for the negative part). All the equations commute and can be solved simultaneously. It follows from (1) that

\[ \partial_m L^n = [L^m_+, L^n] \quad \text{and} \quad \partial_m L^n_- = [L^m_+, L^n_-] = [L^m_+, L^n_-]. \]

This implies that if \( L^n_- \) initially vanishes then it remains zero along the trajectory. This allows to restrict the hierarchy to the operators \( L \) such that \( L^n_- \) is identically zero, i.e., \( L^n \) is a differential operator. We call this the \( n \)-restricted KP or the \( n \)th KdV-hierarchy.

In many cases it is convenient to represent the KP operator \( L \) in a “formal dressing” form:

\[ L = \phi \partial \phi^{-1} \]

where \( \phi \) is a ΨDO \( \phi = \sum_0^\infty w_i \partial^{-i} \) with \( w_0 = 1 \). This yields expressions of \( u_i \) as differential polynomials in terms of \( w_i \). The dressing operator \( \phi \) is determined up to a multiplication on the right by a series in \( \partial^{-1} \) with constant coefficients starting with 1. In terms of \( \phi \), the equations of the hierarchy are

\[ \partial_m \phi = -L^m_- \phi. \]

Let \( \xi(t, z) = \sum_1^\infty t_k z^k \). Then

\[ w(t, z) = \phi \exp \xi(t, z) = \sum_0^\infty w_i z^{-i} \exp \xi(t, z) = \hat{w}(t, z) \exp \xi(t, z) \]

is called the (formal) Baker, or wave, function. The Baker function satisfies equations

\[ Lw = zw, \quad \partial_m w = L^m_+ w. \]

Let \( \phi^* \) be the formal conjugate to \( \phi \) (by definition, \( (f \partial)^* = -\partial \circ f \)). The function \( w^*(t, z) = (\phi^*)^{-1} \exp(-\xi(t, z)) = \hat{w}^*(t, z) \exp(-\xi(t, z)) \) is called the adjoint Baker function. The adjoint Baker function satisfies the equations

\[ L^* w^* = zw^*, \quad \partial_m w^* = -L^m_+ w^*. \]

Nothing prevents introducing constraints more general than \( L^n_- = 0 \), namely, linear combinations of them,

\[ L^n_- - \sum a_l L^l_- = 0, \]

which is a very natural generalization (of course, the term \( L^n_- \) can be included into the linear combination, however, it is more convenient to write the constraint as above). The
proof of compatibility of those constraints with the hierarchy remains the same. In particular, we can take, as \( \sum a_i L_i \), a generating function of equations of the hierarchy, \( T(\lambda) = \sum_{-\infty}^{\infty} L_i^{\lambda^{i-1}} \), the so-called resolvent (see [11]). It can be proven ([11], (7.6.2)) that \( T(\lambda) = w(t, \lambda) \partial^{-1} w^*(t, \lambda) \). Thus, the hierarchy can be restricted to operators \( L \) such that

\[
L^n = L_+^n + w(t, \lambda) \partial^{-1} w^*(t, \lambda). \tag{2}
\]

Note also the following useful representation of the resolvent ([11], (7.6.6), (7.6.3))

\[
T(\lambda) = (\partial - \chi)^{-1} S = \sum_{0}^{\infty} \partial^{-1-k} P_k(\chi) S \tag{3}
\]

where \( \chi = w'/w \) and \( S = ww^* \). They have an advantage being, in contrast to \( w \) and \( w^* \), local expressions in terms of coefficients of the operator \( L \); \( P_k \) are the Faà di Bruno differential polynomials.

Those formulas do not make much sense yet since \( w \) and \( w^* \) are formal series in \( \lambda \). However, \( w \) and \( w^* \) can be constructed as actual functions on a Riemann surface while the series appear only as their asymptotic expansions in a neighborhood of an essential singular point. Then \( w \) and \( w^* \) in Eq.(2) are values of these functions at a particular point \( \lambda \). The equations \( Lw = \lambda w \) and \( L^* w^* = -\lambda w^* \) become meaningless since the action of a \( \Psi DO \) \( L \) on functions is not defined. The equations

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\partial_m w = L_+^m w, \quad \partial_m w^* = -L_+^m w^*, \tag{4}
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this is all what remains as a definition of \( w \) and \( w^* \). Can we understand \( w \) and \( w^* \) in (2) as arbitrary solutions of (4) just disregarding the argument \( \lambda \)? The proof in the last paragraph will give a positive answer to this question. However, the preceding analysis cannot explain this fact. Of course, linear combinations of all \( w(t, \lambda) \) with various \( \lambda \) give spectral representation of all solutions of the equation \( \partial_m w = L_+^m w \) (no one has ever proven that every solution of the initial value problem can be represented as a series in eigenfunctions, though it is a plausible conjecture; this is why we call this reasoning not very formal). However, we cannot use this in (2) because the last term is quadratic. To overcome this difficulty, we need a bilinear expression \( w(t, \mu) \partial^{-1} w^*(t, \lambda) \) in (2), rather than quadratic \( w(t, \lambda) \partial^{-1} w^*(t, \lambda) \).

And this is precisely the generator of the so-called additional symmetries, what we are going to discuss next.

Commutativity of flows generated by the equations of the KP hierarchy means that each of them is a symmetry for all the others. There are, however, symmetries which do not belong to the hierarchy itself. They are called additional symmetries. They do not commute between themselves. The hallmark of these symmetries is their explicit dependence on the variables \( t_i \). We use additional symmetries in a form given them by Orlov and Schulman [10].

Dressing an obvious relation \([\partial_k - \partial^k, \partial] = 0\) we obtain the equation of the hierarchy \([\partial_k - L_+^k, L] = 0\). There is another operator commuting with \( \partial_k - \partial^k \). This is

\[
\Gamma = \sum_{i=1}^{\infty} t_i i \partial^{-1}. \tag{3}
\]
Dressing the relation \([\partial_k - \partial^k, \Gamma] = 0\) one obtains \([\partial_k - L^k_+, M] = 0\), or
\[
\partial_k M = [L^k_+, M], \text{ where } M = \phi \Gamma \phi^{-1}.
\] (5)

From (5) it follows that
\[
\partial_k (L^l M^m) = [L^k_+, L^l M^m].
\] (6)

An additional symmetry is a differential equation
\[
\partial^*_{lm} \phi = -(M^m L^l) - \phi
\]
where \(\partial^*_{lm}\) symbolizes a derivative with respect to some additional variable \(t^*_{lm}\). In terms of the operator \(L\) this definition becomes
\[
\partial^*_{lm} L = -[(M^m L^l) - L].
\]

The operators \(\partial^*_{lm}\) commute with all \(\partial_k\), i.e., they determine symmetries, indeed.

One can introduce a generating function of these symmetries
\[
Y(\lambda, \mu) = \sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{l=-\infty}^{\infty} \lambda^{l-m-1} (M^m L^{m+l})_-. \] (7)

The following theorem holds ([8], see also the Appendix to this note).

**Theorem.** The operator \(Y(\lambda, \mu)\) is equal to
\[
Y(\lambda, \mu) = w(t, \mu) \partial^{-1} w^*(t, \lambda).
\] (8)

For \(\lambda = \mu\) we obtain resolvents.

Let us consider the following constraint:
\[
L^n_+ = (M^m L^{m+l})_-.
\]

We have
\[
\partial_k (L^n_+ - (M^m L^{m+l})_-) = [L^k_+, L^n_+ - M^m L^{m+l}]_+ = [L^k_+, (L^n_+ - M^m L^{m+l})_-]_-.
\]

Hence, if \((L^n_+ - M^m L^{m+l})_-\) is zero at the initial moment, it remains zero along the whole trajectory, and the constraint \(L^n_+ = (M^m L^{m+l})_-\) is compatible with the hierarchy. Actually, what we have done, we restricted the hierarchy to operators
\[
L^n = L^n_+ + (M^m L^{m+l})_-.
\] (9)

Moreover, \((M^m L^{m+l})_-\) can be replaced by any linear combination of such expressions with different \(m\) and \(l\). The same proof remains valid. Our claim is that in this manner the constrained hierarchy can be obtained.

One can take the generating function \(Y(\lambda, \mu)\) as a particular case of a linear combination. According to the above theorem this is \(w(t, \mu) \partial^{-1} w^*(t, \lambda)\).
Thus, we obtain a restriction to the operators of the form
\[ L^n = L^n_+ + w(t, \mu)\partial^{-1}w^*(t, \lambda). \] (10)

If the Baker and the adjoint functions exist as genuine functions of \( \lambda \) and \( \mu \), then one can take their values at fixed points. Now, \( w(t, \mu) \) can be replaced by any linear combination of them with different \( \mu \), the same with \( w^* \). This allows to consider restrictions to operators
\[ L^n = L^n_+ + w(t)\partial^{-1}w^*(t) \] (11)
where \( w(t) \) and \( w^*(t) \) are solutions to Eq.(4) with arbitrary initial conditions.

The above discussion was not a rigorous proof. The proof of possibility of restriction (11) is much simpler, see below. Our goal was to show that the appearance of the expression \( w(t)\partial^{-1}w^*(t) \) is not accidental, it is a consequence of deep connections with the symmetries.

The proof follows from the equation
\[ \partial_k(w(t, \mu)\partial^{-1}w^*(t, \lambda)) = [L^k_+, w(t, \mu)\partial^{-1}w^*(t, \lambda)]_ - \]
which is easy to verify. Indeed, for an arbitrary function \( f \), \( L^k_+ f = L^k_+ \circ f + A\partial \) where \( A \) is a differential operator. The obvious equality \( f\partial = \partial \circ f + (-\partial f) \) implies that \( L^k_+ f = fL^k_+ + \partial \circ B \) where \( B \) is another differential operator. Now,
\[ \partial_k(w(t, \mu)\partial^{-1}w^*(t, \lambda)) = \partial_k(w(t, \mu)\partial^{-1}w^*(t, \lambda))_ - \]
\[ = (L^k_+ w(t, \mu)\partial^{-1}w^*(t, \lambda) - w(t, \mu)\partial^{-1}L^k_+ w^*(t, \lambda))_ - \]
\[ = (L^k_+ \circ w(t, \mu)\partial^{-1}w^*(t, \lambda) - w(t, \mu)\partial^{-1}w^*(t, \lambda)L^k_+)_ - = [L^k_+, w(t, \mu)\partial^{-1}w^*(t, \lambda)]_ -. \]
The terms with differential operators \( A \) and \( B \) vanish since they do not contribute to the negative part of the whole expression, and the subscript “–” kills them. The proof can be accomplished as we did above. Relation of thus constrained KP to the so-called “two-boson representation” one can find in an article by Depireux and Schiff [5], Eq.(2.10). Apparently, it must be connected with the representation of the resolvent in the form (3).

**Appendix.** Proof of Eq.(8).

**Lemma.** Let \( P \) and \( Q \) be two \( \Psi \)DO, then
\[ \text{res}_z[(P e^{xz}) \cdot (Q e^{-xz})] = \text{res}_\partial PQ^* \]
where \( Q^* \) is the formal adjoint to \( Q \).

The proof is in a straightforward verification.

It is quite obvious that for every \( \Psi \)DO \( P \) the equality \( P_- = \sum_1^\infty \partial^{-i}\text{res}_\partial \partial^{-i}P \) holds. We have
\[ (M^m L^{m+l})_- = (\phi \Gamma^m \partial^{m+l} \phi^{-1})_- = \sum_1^\infty \partial^{-i}\text{res}_\partial \phi^{-i} \phi \Gamma^m \partial^{m+l} \phi^{-1}. \]

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According to Lemma this can be written as
\[
(M^m L^{m+l})_+ = \sum_1^{\infty} \partial^{-i} \text{res}_z \partial^{i-1} \phi \Gamma^m \partial^{m+l} e^{\xi(t,z)} (\phi^*)^{-1} e^{-\xi(t,z)}.
\]
Taking into account that
\[
\Gamma \exp \xi(t,z) = \sum_1^{\infty} t_i i z^{i-1} \exp \xi(t,z) = \partial_z \exp \xi(t,z)
\]
and that \(\phi\) commutes with \(\partial_z\) we have
\[
(M^m L^{m+1})_+ = \text{res}_z \sum_1^{\infty} \partial^{-i} (z^{m+l} \partial_z^{m} w)^{(i-1)} \cdot w^* = \text{res}_z z^{m+l} \partial_z^m w \cdot \partial^{-1} \cdot w^*.
\]
Now,
\[
Y(\lambda, \mu) = \text{res}_z \sum_1^{\infty} \sum_{m=0}^{\infty} \sum_{l=-\infty}^{\infty} \frac{z^{m+l}}{(m+l+1)!} (\mu - \lambda)^{-m-l-1} f(z) = f(\lambda),
\]
and
\[
Y(\lambda, \mu) = \exp((\mu - \lambda) \partial_\lambda) w(t, \lambda) \cdot \partial^{-1} \cdot w^*(t, \lambda) = w(t, \mu) \cdot \partial^{-1} \cdot w^*(t, \lambda).
\]

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