Enumeration of quarter-turn symmetric alternating-sign matrices of odd order

A. V. Razumov, Yu. G. Stroganov
Institute for High Energy Physics
142281 Protvino, Moscow region, Russia

Abstract

It was shown by Kuperberg that the partition function of the square-ice model related to the quarter-turn symmetric alternating-sign matrices of even order is the product of two similar factors. We propose a square-ice model whose states are in bijection with the quarter-turn symmetric alternating-sign matrices of odd order, and show that the partition function of this model can be also written in a similar way. This allows to prove, in particular, the conjectures by Robbins related to the enumeration of the quarter-turn symmetric alternating-sign matrices.

1 Introduction

An alternating-sign matrix is a matrix with entries 1, 0, and −1 such that the 1 and −1 entries alternate in each column and each row and such that the first and last nonzero entries in each row and column are 1. Starting from the famous conjectures by Mills, Robbins and Rumsey [1, 2] a lot of enumeration and equinumeration results on alternating-sign matrices and their various subclasses were obtained. Most of the results were proved using bijections between alternating-sign matrices and states of different variants of the statistical square-ice model. For the first time such a method to solve enumeration problems was used by Kuperberg [3], see also the rich in results paper [4].

Our previous paper [5] is devoted to enumerations of the half-turn symmetric alternating-sign matrices of odd order on the base of the corresponding square-ice model. In the present paper we again treat matrices of odd order. But this time we consider the quarter-turn symmetric alternating-sign matrices.

In Section 2 we discuss first the square-ice model related to the quarter-turn symmetric alternating-sign matrices of even order proposed by Kuperberg [4]. Then a square-ice model whose states are in bijection with the quarter-turn symmetric alternating-sign matrices of odd order is introduced. In contrast with the case of the matrices of even order, the usual recursive relations are not enough to determine the partition function of the model recursively by Lagrange interpolation.

In Section 3 we obtain some important additional recursive relations involving the special spectral parameter that is attached to the middle line of the graph describing the states of the model.

In Section 4 we show that the partition function of the model is the product of two factors closely related to the Pfaffians used by Kuperberg to write an expression for the partition function of the square-ice model corresponding to quarter-turn symmetric alternating-sign matrices of even order [4].
In Section 5 we consider an important special case of the overall parameter of the model that allow to prove, in particular, the enumeration conjectures by Robbins [6] on the quarter-turn symmetric alternating-sign matrices of odd order.

We denote $\bar{x} = x^{-1}$ and use the following convenient abbreviations

$$\sigma(x) = x - \bar{x},$$
$$\alpha(x) = \sigma(ax)\sigma(a\bar{x}),$$

proposed by Kuperberg [4]. Here $a$ is some parameter, which will be introduced below.

## 2 Square-ice models related to quarter-turn symmetric alternating-sign matrices

An alternating-sign $n \times n$ matrix $A$ is said to be quarter-turn symmetric if

$$(A)_{j,n+1-i} = (A)_{ij}, \quad i, j = 1, \ldots n.$$  

It can be shown that quarter-turn symmetric alternating-sign matrices of an even order $n$ exist only when $n$ is a multiple of 4. A quarter-turn symmetric alternating-sign matrix of order $n = 2m + 1$ has $-1$ in the center if $m$ is odd, and it has $1$ in the center if $m$ is even.

To enumerate a symmetry class of the alternating-sign matrices Kuperberg proposed to start with a square-ice model whose states are in bijection with the elements of the symmetry class under consideration [4]. The next step is to find the partition function of the model, defined as the sum of the weights of all possible states. It appears that for many symmetry classes of alternating-sign matrices a determinant or Pfaffian representation of the partition function of the corresponding square-ice model can be found. Using such a representation and specifying in an appropriate way the parameters of the model one finds desired enumerations [4, 7, 3, 8, 5].

To describe the states of a square-ice model it is convenient to use a graphical pattern. For example, the states of the square-ice model corresponding to the quarter-turn symmetric alternating-sign matrices of an even order are described by the graph given in Figure 1. The labels $x_i$ are the spectral parameters which are used to define the partition function of the model.

![Figure 1: Square-ice corresponding to the quarter-turn symmetric alternating-sign matrices of even order](image-url)
To get a concrete state of the model one chooses an orientation for each of the unoriented edges in such a way that two edges enter and leave every tetravalent vertex, and either two edges enter or two edges leave every bivalent vertex.\footnote{Kuperberg uses a dashed line crossing an edge to say that its orientation reverses as it crosses the line \cite{Kuperberg}. It is convenient for our purposes to treat reversal of the orientation as a special type of a vertex.} Certainly, we draw a pattern for a fixed order of matrices, but a generalisation to the case of an arbitrary possible order is always evident.

The weight of a state is the product of the weights of the vertices. The choice for the weights of tetravalent vertices used in the present paper is as given in Figure 2. The parameter $a$ is common for all tetravalent vertices. All bivalent vertices have weight 1. If a vertex is unlabelled and formed by intersection of two labeled lines, then the value of the vertex label is set to $x\bar{y}$ if it is in the quadrant which is swept by the line with the spectral parameter $x$ when it is rotated anticlockwise to the line with the spectral parameter $y$. One may move a vertex label $x$ one quadrant to an adjacent one changing it to $\bar{x}$.

A graph, similar to the one given in Figure 1, denotes also the corresponding function. Here the summation over all possible orientations of internal edges is implied. It can be easily understood that our conventions make the formalism invariant under rotations and orientation preserving smooth deformation of graphs. If we reflect a graph over a line and overline the line labels we obtain a graph which describes the same function as the initial graph. After all, reversing orientation of all oriented edges we obtain a graph which again gives the same function as the initial graph.

If we have unoriented boundary edges, then the graph represents the set of the quantities corresponding to their possible orientations. Usually, graphs with such edges arise when we give a graphical representation of equality of functions. In such a case, if it is needed, we should rotate both sides of an equality simultaneously.

As a useful example one can take the graph corresponding to the well-known Yang–Baxter equation

$$\begin{align*}
\sigma(a^2) & \sigma(a^2) \\
\sigma(a) & \sigma(a x) \\
\sigma(a x) & \sigma(a \bar{x}) \\
\sigma(a \bar{x}) & \sigma(a \bar{x})
\end{align*}$$

Figure 2: The weights of the tetravalent vertices

This equation is satisfied if $xyz = a$.

The procedure described above to find enumerations of quarter-turn symmetric alternating-sign matrices of even order was realised by Kuperberg \cite{Kuperberg}. In the present paper we treat the case of quarter-turn symmetric alternating-sign matrices of odd order. The graphical pattern for the state space of the corresponding square-ice model depends on the order of the matrices. For the order $2m + 1$ with $m$ even we have the pattern given at Figure 3, and for the order $2m + 1$ with $m$ odd we have the pattern given in Figure 4. The difference actually is in the orientation.
of the boundary edges belonging to the ‘middle’ line. It is not difficult to get convinced that there is a bijection between the states of the square-ice models described by Figures 3 and 4 and the corresponding subsets of the alternating-sign matrices.

The partition function of the model depends on $m + 1$ spectral parameters $x_1, x_2, \ldots, x_{m+1}$. We denote it $Z_{QT}(2m + 1; \mathbf{x})$, where $\mathbf{x} = (x_1, \ldots, x_{m+1})$ is the $(m + 1)$-dimensional vector formed by the spectral parameters. Using Yang–Baxter equation (1) and an evident equality

$$ y 
\begin{array}{c}
\rightarrow
\end{array}
\begin{array}{c}
\rightarrow
\end{array}

= 
\begin{array}{c}
\rightarrow
\end{array}
\begin{array}{c}
\rightarrow
\end{array}

x $$

one can show that the function $Z_{QT}(2m + 1; \mathbf{x})$ is symmetric in the variables $x_1, x_2, \ldots, x_m$.

It is not difficult to get convinced that the following equality

$$ y 
\begin{array}{c}
\rightarrow
\end{array}
\begin{array}{c}
\rightarrow
\end{array}
\begin{array}{c}
\rightarrow
\end{array}

= 
\begin{array}{c}
\rightarrow
\end{array}
\begin{array}{c}
\rightarrow
\end{array}
\begin{array}{c}
\rightarrow
\end{array}

x $$

(2)

is valid. Actually there are similar equalities with different orientations of the oriented edges in the left hand side and reversed orientations of the corresponding edges in the right hand side. Reflect now the graph in Figure 4 over the line which is drawn as a dotted line in Figure 5, overline all the labels, and reverse the orientations of the oriented edges. As follows from the remarks made above, the resulting graph which is given in Figure 5 corresponds to the same function as the graph given in Figure 4. Using equality (2), we transform Figure 5 to Figure 6 which again corresponds to the same function as the graph given in Figure 4. Thus, we proved the equality

$$ Z_{QT}(2m + 1; \mathbf{x}) = Z_{QT}(2m + 1; \bar{x}), $$

(3)

where $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_{m+1})$. 

4
Following the usual procedure (see, for example, the proof of Lemma 13 in paper [4]) we obtain $2m - 2$ recursive relations

$$Z_QT(2m + 1; \mathbf{x})|_{x_i = x_j} = \sigma^2(a)\sigma^2(a^2) \times \prod_{k=2}^{m+1} \sigma^2(a^2\bar{x}_k x_j)\sigma^2(a\bar{x}_j x_k)Z_QT(2m - 3; \mathbf{x}\setminus x_1\setminus x_j),$$

$$Z_QT(2m + 1; \mathbf{x})|_{x_i = \bar{x}_j} = \sigma^2(a)\sigma^2(a^2) \times \prod_{k=2}^{m+1} \sigma^2(a\bar{x}_k x_j)\sigma^2(a^2\bar{x}_j x_k)Z_QT(2m - 3; \mathbf{x}\setminus x_1\setminus x_j),$$

where $j = 2, \ldots, m$. Since the partition function $Z_QT(2m + 1; \mathbf{x})$ is symmetric in the variables $x_1, \ldots, x_m$, we actually have $m^2 - m$ recursive relations

$$Z_QT(2m + 1; \mathbf{x})|_{x_i = x_j} = \sigma^2(a)\sigma^2(a^2) \times \prod_{k=1}^{m+1} \sigma^2(a^2\bar{x}_k x_j)\sigma^2(a\bar{x}_j x_k)Z_QT(2m - 3; \mathbf{x}\setminus x_i\setminus x_j),$$

where $i, j = 1, \ldots, m$, and $i \neq j$.

Consider some fixed value of the index $i$ such that $1 \leq i \leq m$. The partition function $Z_QT(2m + 1; \mathbf{x})$ is a centered Laurent polynomial of width $2m - 2$ in the square of the variable $x_i$. Therefore, if we know $2m - 1$ values of $Z_QT(2m + 1; \mathbf{x})$ for $2m - 1$ values of $x_i^2$ we know $Z_QT(2m + 1; \mathbf{x})$ completely. Recursive relations (4) supply us with the expressions for $Z_QT(2m + 1; \mathbf{x})$ via $Z_QT(2m - 3; \mathbf{x}\setminus x_i\setminus x_j)$ for $2m - 2$ values of $x_i$. It is not enough to determine $Z_QT(2m + 1; \mathbf{x})$ recursively by the Lagrange interpolation. Instead, the partition function $Z_QT(2m + 1; \mathbf{x})$ is a centered Laurent polynomial in $x_{m+1}$ of width $m - 1$ if $m$ is odd, and of width $m$ if $m$ is even. It appears that there are enough recursive relations involving the variable $x_{m+1}$ to determine $Z_QT(2m + 1; \mathbf{x})$ via Lagrange interpolation.

\footnote{Note that for $Z_QT(4k; \mathbf{x})$ the recursive relations similar to (4) give enough data for Lagrange interpolation [4].}
3 Recursive relations involving variable \( x_{m+1} \)

Multiply the partition function \( Z_{QT}(2m + 1; x) \) by \( \sigma(az_m) \), where \( z_m \) is a parameter which is not specified yet. One can easily understand that the resulting function can be represented by Figure 7. Put \( z_m = ax_{m+1} \bar{x}_m \), and transform the graph in Figure 7 to the graph in Figure 8 using the Yang–Baxter equation (1). Repeating this procedure we see that Figure 9, where \( z_i = ax_{m+1} \bar{x}_i \) corresponds to the partition function \( Z_{QT}(2m + 1; x) \) multiplied by the product \( \prod_{i=1}^{m} \sigma(a^2x_{m+1} \bar{x}_i) \). Put now \( x_{m+1} = \bar{a}x_1 \). One can easily see that after that some vertices become fixed (see Figure 10). If we remove these vertices we will come to the function described by Figure 11. Removal of a fixed vertex from a graph describing a function is equivalent to the division of the function by the weight of the vertex. Taking into account all multiplications and divisions we made, we obtain an important recursive relation

\[
Z_{QT}(2m + 1; x)|_{x_{m+1} = \bar{a}x_1} = \sigma(a)\sigma(a^2) \\
\times \prod_{k=2}^{m} \sigma(a^2 \bar{x}_1 x_k)\sigma(a \bar{x}_k x_1) Z_{QT}(2m - 1; (x_2, \ldots, x_m, x_1)).
\]
Using the symmetricity of the partition function $Z_{QT}(2m + 1; x)$ in the variables $x_1, \ldots, x_m$ we obtain $m$ recursive relations

$$Z_{QT}(2m + 1; x)|_{x_{m+1}=\hat{x} x_j} = \sigma(a)\sigma(a^2)$$

$$\times \prod_{k=1}^{m} \sigma(a^2 \bar{x}_k x_k)\sigma(a \bar{x}_k x_k) Z_{QT}(2m - 1; (x_1, \ldots, \hat{x}_j, \ldots, x_m, x_j)), \quad (5)$$

where the hat means omission of the corresponding argument. Taking into account the inversion symmetry (3), we obtain $m$ additional recursive relations

$$Z_{QT}(2m + 1; x)|_{x_{m+1}=x_j} = \sigma(a)\sigma(a^2)$$

$$\times \prod_{k=1}^{m} \sigma(a^2 \bar{x}_k x_j)\sigma(a \bar{x}_j x_k) Z_{QT}(2m - 1; (x_1, \ldots, \hat{x}_j, \ldots, x_m, x_j)). \quad (6)$$

Hence we have $2m$ specializations in the square of the variable $x_{m+1}$. It is more than enough to reconstruct the partition function by recursion. Certainly, we have to use also the initial value

$$Z_{QT}(3; x) = \sigma(a)\sigma(a^2). \quad (7)$$

## 4 Kuperberg’s pfaffians and partition function

Following Kuperberg for any positive integer $r$ introduce an antisymmetric $2l \times 2l$ matrix $M^{(r)}(l; x)$ with the matrix elements

$$M_{ij}^{(r)}(l; x) = \frac{\sigma(\bar{x}_i x_j^r)}{\alpha(\bar{x}_i x_j)}, \quad i, j = 1, \ldots, 2l.$$

Recall that the Pfaffian of an antisymmetric $2l \times 2l$ matrix $A$ can be defined as

$$\text{Pf } A = \frac{1}{2^l} \sum_{s \in S_{2l}} \text{sgn}(s) A_{s(1)s(2)} A_{s(3)s(4)} \cdots A_{s(2l-1)s(2l)},$$

where $S_{2l}$ is the symmetric group of degree $2l$.

Again following Kuperberg define the following functions of $2l$ variables $x_1, x_2, \ldots, x_{2l}$:

$$Z_{QT}^{(r)}(l; x) = \prod_{1 \leq i < j \leq 2l} \frac{\alpha(\bar{x}_i x_j)}{\sigma(\bar{x}_i x_j)} \text{Pf } M^{(r)}(l; x).$$
The functions $Z_{QT}^{(r)}(l; x)$ are symmetric in the variables $x_1, \ldots, x_{2l}$, and one can verify the validity of the following recursive relations

$$Z_{QT}^{(r)}(l; x) \bigg|_{x_i = a x_j} = \frac{\sigma(a^r)}{\sigma(a)} \prod_{k=1 \atop k \neq i, j}^{2l} \left[ \sigma(a^2 x_k x_j) \sigma(a x_j x_k) \right] Z_{QT}^{(r)}(l - 1; x \setminus x_i \setminus x_j), \quad (8)$$

where $i, j = 1, \ldots, 2l$ and $i \neq j$. On the basis of these recursive relations Kuperberg proved [4] that the partition function of the square-ice model corresponding to the quarter-turn symmetric alternating-sign matrices of even order can be represented as

$$Z_{QT}(4l; x) = [\sigma^{3l}(a) \sigma^l(a^2)] Z_{QT}^{(1)}(l; x) Z_{QT}^{(2)}(l; x).$$

It appears that the partition function for the case of the quarter-turn symmetric alternating-sign matrices of odd order can be also written in a similar way.

The function $Z_{QT}^{(1)}(l; x)$ is a centered Laurent polynomial of width $2l - 2$ in the square of each of the variables $x_1, \ldots, x_{2l}$, and for the function $Z_{QT}^{(2)}(l; x)$ we have

$$Z_{QT}^{(2)}(l, x) = \sum_{i=1}^{2l} c_i(l; x ; x_{2l}) x_{2l}^{2i - 2l - 1}.$$ 

Introduce the function

$$\tilde{Z}_{QT}^{(2)}(l, x) = \left[ \prod_{k=1}^{2l - 1} x_k \right] c_{2l}(l; x),$$

which is a centered Laurent polynomial of width $2l - 2$ in the square of each of the variables $x_1, \ldots, x_{2l - 1}$. It follows from (8) that

$$\tilde{Z}_{QT}^{(2)}(l; x) \bigg|_{x_i = a x_j} = -\frac{\sigma(a^2)}{\sigma(a)} \prod_{k=1 \atop k \neq i, j}^{2l - 1} \left[ \sigma(a^2 x_k x_j) \sigma(a x_j x_k) \right] \tilde{Z}_{QT}^{(2)}(l - 1; x \setminus x_i \setminus x_j), \quad (9)$$

where $i, j = 1, \ldots, 2l - 1$ and $i \neq j$. Now we can write the following expressions for the partition function of the square-ice model corresponding to the quarter-turn symmetric alternating-sign matrices of odd order

$$Z_{QT}(4l + 1; x) = [(-1)^l \sigma^{3l}(a) \sigma^l(a^2)] Z_{QT}^{(1)}(l, x \setminus x_{2l+1}) \tilde{Z}_{QT}^{(2)}(l + 1; x), \quad (10)$$

$$Z_{QT}(4l - 1; x) = [(-1)^{l+1} \sigma^{3l-2}(a) \sigma^l(a^2)] Z_{QT}^{(1)}(l, x) \tilde{Z}_{QT}^{(2)}(l; x \setminus x_{2l}). \quad (11)$$

Using recursive relations (8) and (9) and the initial values

$$Z_{QT}^{(1)}(1; x) = 1, \quad \tilde{Z}_{QT}^{(2)}(1; x) = 1,$$

it is not difficult to check that the right-hand sides of (10) and (11) satisfy the initial condition (7) and recursive relations (5) and (6).
5 Special value of the parameter $a$ and enumerations

It turns out, that in the special case $a = \exp(i\pi/3)$ one can relate the functions $Z^{(1)}_{QT}(l; \mathbf{x})$ and $Z^{(2)}_{QT}(l; \mathbf{x})$ to the partition functions of the square-ice models corresponding to all alternating-sign matrices and to the half-turn symmetric alternating-sign matrices of odd order.

If $a = \exp(i\pi/3)$, then

$$\sigma(a^2 x) = -\sigma(ax) = \sigma(\bar{a}x),$$

and recursive relations (8) for $r = 1$ becomes

$$Z^{(1)}_{QT}(l; \mathbf{x}) \bigg|_{x_i=ax_j} = \prod_{k=1}^{2l} \sigma^2(a\bar{x}_j x_k) Z^{(1)}_{QT}(l-1; \mathbf{x} \setminus x_i \setminus x_j).$$

Recall that the partition function $Z(l; \mathbf{x}, \mathbf{y})$ of the square-ice model, corresponding to all alternating-sign matrices depends on $2l$ parameters $x_1, \ldots, x_l$ and $y_1, \ldots, y_l$ (see, for example [4]). It is a centered Laurent polynomial of width $l - 1$ in the square of each of the variables $x_1, \ldots, x_l$ and $y_1, \ldots, y_l$, satisfying the recursive relations

$$Z(l; \mathbf{x}, \mathbf{y}) \big|_{y_i=ax_j} = \sigma(a^2) \prod_{k=1}^{l} \sigma(a\bar{x}_j y_k) \prod_{k=1}^{l} \sigma(a^2 \bar{x}_k x_j) Z(l-1; \mathbf{x} \setminus x_i \setminus y_j),$$

where $i, j = 1, \ldots, l$ and $i \neq j$, with the initial value $Z(1; \mathbf{x}, \mathbf{y}) = \sigma(a^2)$. It was shown in paper [9] that in the case $a = \exp(i\pi/3)$ this function is symmetric in the union of the variables $x_1, \ldots, x_l$ and $y_1, \ldots, y_l$ (see also [10] and references therein). Introducing the symmetric notations

$$x_i = x_i, \quad x_{i+l} = y_i, \quad i = 1, 2, \ldots, l,$$

and taking into account identity (12) we write the recursive relations for $Z(l; \mathbf{x})$ as

$$Z(l; \mathbf{x}) \bigg|_{x_i=ax_j} = \sigma(a^2) \prod_{k=1}^{2l} \sigma(a\bar{x}_j x_k) Z(l-1; \mathbf{x} \setminus x_i \setminus x_j).$$

Comparing relations (13) with relations (15) and taking into account the initial values for $Z^{(1)}_{QT}(l; \mathbf{x})$ and $Z^{(2)}_{QT}(l; \mathbf{x})$, we find that

$$Z^{(1)}_{QT}(l; \mathbf{x}) = \sigma^{-2l}(a^2) Z^{(2)}_{QT}(l; \mathbf{x}).$$

This equality was also obtained by Okada [10].

The function $Z^{(2)}_{QT}(l; \mathbf{x})$ in the case $a = \exp(i\pi/3)$ satisfies the recursive relations

$$Z^{(2)}_{QT}(l; \mathbf{x}) \bigg|_{x_i=ax_j} = -\prod_{k=1}^{2l} \sigma^2(a\bar{x}_j x_k) Z^{(2)}_{QT}(l-1; \mathbf{x} \setminus x_i \setminus x_j),$$

which are rather different from (13).

In our recent paper [5] we considered the partition function for the square-ice model corresponding to the half-turn symmetric alternating-sign matrices of odd order (see Figure 13). The corresponding partition function $Z_{HTT}(2l-1; \mathbf{x}, \mathbf{y})$ depends here on $2l$ spectral parameters
Figure 13: Square-ice corresponding to the half-turn symmetric alternating-sign matrices of odd order

$x_1, \ldots, x_l$ and $y_1, \ldots, y_l$. We proved that in the case $a = \exp(i\pi/3)$ and $y_{m+1} = x_{m+1}$ the partition function is a symmetric function in all $2l - 1$ variables. Again introducing symmetric notations

$$x_i = x_i, \quad i = 1, \ldots, l, \quad x_{i+l} = y_i, \quad i = 1, \ldots, l - 1,$$

and using for the function under consideration the same notation $Z_{HT}(2l - 1; x)$, one sees that this function satisfies the recursive relations

$$Z_{HT}(2l - 1; x)|_{x_i = a x_j} = \sigma^2(a^2) \prod_{k=1}^{2l-1} \sigma^2(a \bar{x}_j x_k) Z_{HT}(2l - 3; x \setminus x_i \setminus x_j),$$

(18)

where $i, j = 1, \ldots, 2l - 1$ and $i \neq j$. Comparing relations (17) with relations (18) and taking into account the initial values for $Z_{QT}^{(2)}(l; x)$ and $Z_{HT}(2l - 1; x)$, we see that

$$\tilde{Z}_{QT}^{(2)}(l; x) = (-1)^{l+1} \sigma^2 - 2i(a^2) Z_{HT}(2l - 1; x).$$

(19)

Relations (16) and (19) allow us to write equalities (10) and (11) as

$$Z_{QT}(4l + 1; x) = Z^2(l; x \setminus x_{2l+1}) Z_{HT}(2l + 1; x),$$

$$Z_{QT}(4l - 1; x) = Z^2(l; x) Z_{HT}(2l - 1; x \setminus x_{2l}).$$

It is natural to recall here the similar equality

$$Z_{QT}(4l; x) = Z^2(l; x) Z_{HT}(2l; x),$$

obtained by Okada [10].

Considering the last equalities at $x = (1, \ldots, 1)$, one comes to the relations

$$A_{QT}(4l + 1) = A^2(l) A_{HT}(2l + 1), \quad A_{QT}(4l - 1) = A^2(l) A_{HT}(2l - 1),$$

where $A$ instead of $Z$ means the number of alternating-sign matrices of the corresponding kind. Combining these relations with the result obtained by Kuperberg for the matrices of even order, we have

$$A_{QT}(4l + \epsilon) = A^2(l) A_{HT}(2l + \epsilon), \quad \epsilon = -1, 0, 1.$$

Thus, the Robbins conjecture [6] on the enumeration of the quarter-turn symmetric alternating-sign matrices is proved.

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