Normal subgroups and relative centers of linearly reductive quantum groups

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ABSTRACT

We prove a number of structural and representation-theoretic results on linearly reductive quantum groups, i.e., objects dual to those of cosemisimple Hopf algebras: (a) a closed normal quantum subgroup is automatically linearly reductive if its squared antipode leaves invariant each simple subcoalgebra of the underlying Hopf algebra; (b) for a normal embedding $\mathbb{H} \leq \mathbb{G}$ there is a Clifford-style correspondence between two equivalence relations on irreducible $\mathbb{G}$- and, respectively, $\mathbb{H}$-representations; and (c) given an embedding $\mathbb{H} \leq \mathbb{G}$ of linearly reductive quantum groups, the Pontryagin dual of the relative center $Z(\mathbb{G}) \cap \mathbb{H}$ can be described by generators and relations, with one generator $g_V$ for each irreducible $\mathbb{G}$-representation $V$ and one relation $g_U = g_V g_W$ whenever $U$ and $V \otimes W$ are not disjoint over $\mathbb{H}$.

This latter center-reconstruction result generalizes and recovers Müger’s compact-group analogue and the author’s quantum-group version of that earlier result by setting $\mathbb{H} = \mathbb{G}$.

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1. Introduction

The quantum groups in the title are as in [25, Section 1.2]: objects $\mathbb{G}$ dual to corresponding Hopf algebras $\mathcal{O}(\mathbb{G})$, with the latter regarded as the algebra of regular functions on (the otherwise non-existent) linear algebraic quantum group $\mathbb{G}$. Borrowing standard linear-algebraic-group terminology (e.g. [23, Chapter 1, Section 1, Definition 1.4]), the linear reductivity condition then simply means that the Hopf algebra $\mathcal{O}(\mathbb{G})$ is cosemisimple.

The unifying thread through the material below is the concept of a (closed) normal quantum subgroup. In the present non-commutative setting normality can be defined in a number of ways that are frequently equivalent [34, Theorem 2.7]. We settle here on the concept introduced in [25, Section 1.5] (and recalled in Definition 3.1): a quotient Hopf algebra $\mathcal{O}(\mathbb{G}) \rightarrow \mathcal{O}(\mathbb{H})$ dual to a closed quantum subgroup $\mathbb{H} \leq \mathbb{G}$ is normal if that quotient is an $\mathcal{O}(\mathbb{G})$-comodule under both adjoint coactions $\mathcal{O}(\mathbb{G}) \rightarrow (\mathcal{O}(\mathbb{G}))^{\otimes 2}$:

$$x \mapsto x_2 \otimes S(x_1)x_3 \quad \text{and} \quad x \mapsto x_1S(x_3) \otimes x_2$$

One piece of motivation for the material is the observation (cf. Remark 3.9) that classically, normal closed subgroups of linearly reductive algebraic groups are again linearly reductive. The non-commutative version of this remark, appearing as Theorem 3.2, can be phrased (in somewhat weakened but briefer form) as follows.

**Theorem 1.1.** A normal quantum subgroup $\mathbb{H} \leq \mathbb{G}$ of a linearly reductive quantum group is again linearly reductive, provided the squared antipode of $\mathcal{O}(\mathbb{H})$ leaves invariant all simple subcoalgebras of the latter.
In particular, this recovers the classical version: in that case the squared antipode is trivial.

Keeping with the theme of what is (or isn’t) afforded by normality, another motivating strand is that of Clifford theory (so named for [12], where the relevant machinery was introduced). This is a suite of results relating the irreducible representations of a (finite, compact, etc.) group and those of a normal subgroup via induction/restriction functors; the reader can find a brief illuminating summary in [7, Section 2] (in the context of finite groups).

Hopf-algebra analogues (both purely algebraic and analytic) abound. Not coming close to doing the literature justice, we will point to a selection: [6, 30, 33, 36, 37], say, and the references therein. [13, Section 5, especially Theorem 5.4] provides a version for compact quantum groups [38], which are (dual to) cosemisimple complex Hopf $^*$-algebras with positive Haar integral (the CQG algebras of [14, Definition 2.2]); they thus fit within the confines of the present paper.

The following result paraphrases and summarizes Theorems 4.4, 4.5 and Proposition 4.6. To make sense of it:

- In the language of Section 4, the surjection $O(G) \to O(H)$ of Theorem 1.2 is $H \to B$.
- As explained in Section 2, for a quantum group $G$ the symbol $\hat{G}$ denotes its category of irreducible representations (i.e. simple right $O(G)$-comodules).
- $\text{Ind}_H^G$ and $\text{Res}_H^G$ denote the induction and restriction functors respectively, as discussed in Section 2.1.

**Theorem 1.2.** Let $H \subset G$ be a normal embedding of linearly reductive quantum groups, and consider the binary relation $\sim$ on $\hat{G} \times \hat{H}$ defined by

$$\hat{G} \ni V \sim W \in \hat{H} \iff \text{hom}_H(\text{Res}_H^G V, W) \neq 0 \iff \text{hom}_G(\text{V, Ind}_H^G W) \neq 0.$$  

The following statements hold.

(a) The left-hand slices

$$\text{slice}_W := \{V \in \hat{G} \mid V \sim W\}, \ W \in \hat{H}$$

of $\sim$ are the classes of an equivalence relation $\sim_G$, given by

$$V \sim_G V' \iff \text{Res}_H^G V \text{ and } \text{Res}_H^G V' \text{ have the same simple constituents.}$$

(b) The right-hand slices

$$V_{\text{slice}} := \{W \in \hat{H} \mid V \sim W\}, \ V \in \hat{G}$$

are the finite classes of an equivalence relation.

A third branch of the present discussion has to do with the relative centers of the title: having defined the center $Z(G)$ of a linearly reductive quantum group (Definition 5.3), and given a closed linearly reductive quantum subgroup $H \leq G$, one can then make sense of the relative center $Z(G, H)$ as the intersection $H \cap Z(G)$; see Definition 5.4.

Though not immediately obvious, it follows from [11, Section 3] (cited more precisely in the text below) that for embeddings $H, K \leq G$ of linearly reductive quantum groups, operations such as the intersection $H \cap K$ and the quantum subgroup $HK$ generated by the two are well defined and behave as usual when $K$, say, is normal (hence the relevance of normality, again).

The initial spark of motivation for Section 5 was provided by the main result of [22] (Theorem 3.1 therein), reconstructing the center of a compact group $G$ as a universal grading group for the category of $G$-representations. This generalizes to linearly reductive quantum groups [8, Proposition 2.9], and, as it turns out, goes through in the relative setting; per Theorem 5.5:

**Theorem 1.3.** Let $H \leq G$ be an embedding of linearly reductive quantum groups, and define the relative chain group $C(G, H)$ by generators $g_V, \ V \in \hat{G}$ and relations $g_U = g_V g_w$ whenever $U$ and $V \otimes W$ have common simple constituents over $H$. 


Then, the map
\[ C(G, H) \ni gV \mapsto W \in \hat{Z}(G, H) \] where \( \text{Res}_{Z(G, H)}^G \cong \text{sum of copies of } W \)
is a group isomorphism.

Or, in words: mapping \( gV \) to the “central character” of \( V \) restricted to \( Z(G, H) \) gives an isomorphism \( C(G, H) \cong Z(G, H) \). The “plain” (non-relative) version [8, Proposition 2.9] (and hence also its classical compact-group counterpart [22, Theorem 3.1]) are recovered by setting \( H = G \).

Although strictly speaking outside the scope of the present paper, some further remarks, suggestive of an intriguing connection to semisimple-Lie-group representation theory, will perhaps serve to further motivate the relative chaingroups discussed in Theorem 1.3.

Definition 5.1 was inspired by the study of plain (non-relative) chaingroups of connected, semisimple Lie groups \( G \) with finite center, studied in [10, Section 4]; specifically, the problem of whether
\[ \text{hom}_H(\sigma''', \sigma \otimes \sigma') \neq 0, \quad \sigma, \, \sigma', \, \sigma''' \in \hat{M} \quad (1.1) \]
for a compact-group embedding \( H \leq M \) arises naturally while studying the direct-integral decomposition of a tensor product of two principal-series representations of such a Lie group \( G \). To summarize, consider the setup of [20] (to which we also refer, along with its own references, for background on the following).

- a connected, semisimple Lie group \( G \) with finite center, with its Iwasawa decomposition
  \[ G = KAN \]
  \((K \leq G \text{ maximal compact}, \, A \text{ abelian and simply-connected}, \, N \text{ nilpotent and simply-connected})\);
- the corresponding decomposition
  \[ P = MAN \]
of a minimal parabolic subgroup, with \( M \leq K \) commuting with \( A \);
- the resulting principal-series unitary representations
  \[ \pi_{\sigma,\nu} := \text{Ind}_P^G(\sigma \otimes \nu \otimes \text{triv}), \]
  where \( \sigma \in \hat{M} \) and \( \nu \in \hat{A} \) unitary irreducible representations over those groups.

One is then interested in which \( \pi_{\sigma''',\nu''} \) are weakly contained [3, Definition F.1.1] in tensor products \( \pi_{\sigma,\nu} \otimes \pi_{\sigma',\nu'} \) (i.e. feature in a direct-integral decomposition of the latter); we write
\[ \pi_{\sigma''',\nu''} \preceq \pi_{\sigma,\nu} \otimes \pi_{\sigma',\nu'}. \]
It turns out that in the cases worked out in the literature there is a closed subgroup \( H \leq M \) that determines this weak containment via (1.1). Examples:

- When the (connected, etc.) Lie group \( G \) is complex, one can simply take \( H = Z(G) \) (the center of \( G \), which is always automatically contained in \( M \)). This follows, for instance, from [35, Theorem 3.5.5] in conjugation with [20, Theorems 1 and 2].
- For \( G = SL(n, \mathbb{R}), \, n \geq 2 \) one can again set \( H = Z(G) \): [27, Section 4] for \( n = 2 \) and [20, p.210, Theorem] for the rest.
- Finally, for real-rank-one \( G \) the main result of [20], Theorem 16 of that paper, provides such an \( H \leq M \) (denoted there by \( M_0 \); it is in general non-central, and in fact not even normal).

The phenomenon presumably merits some attention in its own right.

2. Preliminaries

Everything in sight (algebras, coalgebras, etc.) will be linear over a fixed algebraically closed field \( k \). We assume some background on coalgebras and Hopf algebras, as covered by any number of good sources such as [1, 21, 26, 31].
Notation 2.1. A number of notational conventions will be in place throughout.

- \( \Delta, \varepsilon \) and \( S \) denote, respectively, coproducts, counits and antipodes. They will occasionally be decorated with letters indicating which coalgebra, Hopf algebra, etc. they are attached to; \( S_H \), for instance, is the antipode of the Hopf algebra \( H \).

- We use an un-parenthesized version of Heyneman-Sweedler notation ([21, Notation 1.4.2] or [26, Section 2.1]):
  \[
  \Delta(c) = c_1 \otimes c_2, \quad (((\Delta \otimes \text{id}) \circ \Delta)(c) = c_1 \otimes c_2 \otimes c_3
  \]
  and so on for coproducts and
  \[
  c \mapsto c_0 \otimes c_1, \quad c \mapsto c_{-1} \otimes c_0
  \]
  for right and left comodule structures respectively.

- \( \mathcal{O}(G), \mathcal{O}(H) \), and so on denote Hopf algebras over a fixed algebraically closed field \( k \); they are to be thought of as algebras of representative functions on linear algebraic quantum groups \( G, H \), etc.

- An embedding \( \mathbb{H} \leq G \) of quantum groups means a Hopf algebra surjection \( \mathcal{O}(G) \to \mathcal{O}(\mathbb{H}) \) and more generally, a morphism \( \mathbb{H} \to G \) is one of Hopf algebras in the opposite direction \( \mathcal{O}(G) \to \mathcal{O}(\mathbb{H}) \).

- Categories of (co)modules are denoted by \( \mathcal{M} \), decorated with the symbol depicting the (co)algebra, with the left/right position of the decoration matching the chirality of the (co)module structure. Examples: \( A\mathcal{M} \) means left \( A \)-modules, \( \mathcal{M}^C \) denotes right \( C \)-comodules, etc. Comodule structures are right unless specified otherwise.

- These conventions extend to relative Hopf modules ([21, Section 8.5] or [26, Section 9.2]): if, say, \( A \) is a right comodule algebra [21, Definition 4.1.2] over a Hopf algebra \( H \) with structure
  \[
  A \ni a \mapsto a_0 \otimes a_1 \in A \otimes H
  \]
  then \( \mathcal{M}^H_A \) denotes the category of right \( A \)-modules internal to \( \mathcal{M}^H \); that is, right \( A \)-modules \( M \) that are also right \( H \)-comodules via
  \[
  m \mapsto m_0 \otimes m_1
  \]
  such that
  \[
  (ma)_0 \otimes (ma)_1 = m_0 a_0 \otimes m_1 a_1.
  \]
  There are analogues \( \mathcal{M}^C_H \), say, for right \( H \)-module coalgebras \( C \), left- or half-left-handed versions thereof, and so on.

- An additional ‘\( f \)’ adornment on one of the above-mentioned categories means finite-dimensional (co)modules: \( \mathcal{M}^C_H \) is the category of finite-dimensional right \( C \)-comodules, for instance.

- Reprising a convention common in the operator-algebra literature (e.g. [15, Section 2.3.2, Section 18.1.1]), \( \hat{G} \) denotes the isomorphism classes of simple and hence finite-dimensional [21, Theorem 5.1.1] (right, unless specified otherwise) \( C \)-comodules and \( \widehat{\mathbb{G}} = \hat{\mathcal{O}(G)} \).

  The purely-algebraic and operator-algebraic notations converge when \( G \) is compact and \( \mathcal{O}(G) \) denotes the Hopf algebra of representative functions on \( G; \widehat{\mathbb{G}} \) as defined above can then be identified with the set of isomorphism classes of irreducible unitary \( G \)-representations.

- In the same spirit, it will also occasionally be convenient to write
  \[
  \text{Rep}(G) := \mathcal{M}^{\mathcal{O}(G)}.
  \]

The linear algebraic quantum groups \( G \) in the sequel will frequently be linearly reductive, in the sense that the Hopf algebra \( \mathcal{O}(G) \) is cosemisimple [21, Section 2.4]: \( \text{Rep}(G) \) is a semisimple category, i.e. every comodule is a direct sum of simple subcomodules. Equivalently ([21, Definition 2.4.1]), \( \mathcal{O}(G) \) is a direct sum of simple subcoalgebras.

Cosemisimple Hopf algebras \( H \) are equipped with unique unital integrals \( f : H \to k \) [21, Theorem 2.4.6] and hence have bijective antipodes (by [16, Corollary 5.4.6], say); more is true, though. Still assuming \( H \) cosemisimple, for a simple comodule \( V \in H \) the canonical coalgebra morphism
\[
\text{End}(V)^* \cong V^* \otimes V \to H
\]
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A.conceptually dual to the analogous map \( A \to \text{End}(V) \) giving \( V \) a module structure over an algebra \( A \) is one-to-one and gives the direct-sum decomposition

\[
H = \bigoplus_{V \in \hat{H}} (V^* \otimes V) = \bigoplus_{V \in \hat{H}} C_V
\]  

(2.1)

into simple subcoalgebras \( C_V := V^* \otimes V \) (the Peter-Weyl decomposition, in compact-group parlance: [14, Definition 2.2], [17, Theorem 27.40], etc.) that makes \( H \) cosemisimple to begin with. With this in place, not only is the antipode \( S := S_H \) bijective but in fact its square leaves every \( C_V, V \in \hat{H} \) invariant and acts as an automorphism thereon [16, Theorem 7.3.7].

We refer to \( C_V = V^* \otimes V \) as the coefficient coalgebra of the simple \( H \)-comodule \( V \). This is the coalgebra associated to \( V \) in [16, Proposition 2.5.3], and is the smallest subcoalgebra \( C \leq H \) for which the comodule structure \( V \to V \otimes H \) factors through \( V \otimes C \).

2.1. Restriction, induction and the like

Given a coalgebra morphism \( C \to D \), the cotensor product ([21, Definition 8.4.2] or [5, Section 10]) \( - \square_D C \) is right adjoint to the natural "scalar corestriction" functor \( \mathcal{M}^C \to \mathcal{M}^D \):

\[
\mathcal{M}^C \overset{\text{cores}}{\longrightarrow} \mathcal{M}^D \quad \quad - \square_D C
\]  

the central symbol indicating that the top functor is the left adjoint. When \( \mathbb{H} \leq G \) is, say, an inclusion of compact groups and \( C \to D \) the corresponding surjection \( \mathcal{O}(\mathbb{G}) \to \mathcal{O}(\mathbb{H}) \) of algebras of representative functions, the cotensor functor

\[
- \square_{\mathcal{O}(\mathbb{H})} \mathcal{O}(\mathbb{G}) : \text{Rep}(\mathbb{H}) \to \text{Rep}(\mathbb{G})
\]

is naturally isomorphic with the usual induction \( \text{Ind}_{\mathbb{H}}^{\mathbb{G}} \) [28, p. 82]. For that reason we repurpose this same notation for the general setting of quantum-group inclusions, writing

\[
\text{Ind}_{\mathbb{H}}^{\mathbb{G}} := - \square_{\mathcal{O}(\mathbb{H})} \mathcal{O}(\mathbb{G}) : \text{Rep}(\mathbb{H}) \to \text{Rep}(\mathbb{G})
\]

for any quantum-group inclusion \( \mathbb{H} \leq \mathbb{G} \); for consistency, we also occasionally also denote the rightward functor in (2.2) by

\[
\text{Res}_{\mathbb{G}}^{\mathbb{H}} : \text{Rep}(\mathbb{G}) \to \text{Rep}(\mathbb{H}).
\]

3. Normal subgroups and automatic reductivity

Consider a quantum group embedding \( \mathbb{H} \leq \mathbb{G} \), expressed as a surjective Hopf-algebra morphism \( \pi : \mathcal{O}(\mathbb{G}) \to \mathcal{O}(\mathbb{H}) \). As is customary in the literature on quantum homogeneous spaces (e.g. [34, proof of Theorem 2.7]), we write

\[
\mathcal{O}(\mathbb{G}/\mathbb{H}) := \{ x \in \mathcal{O}(\mathbb{G}) \mid (\text{id} \otimes \pi) \Delta(x) = x \otimes 1 \}
\]

\[
\mathcal{O}(\mathbb{H}\backslash \mathbb{G}) := \{ x \in \mathcal{O}(\mathbb{G}) \mid (\pi \otimes \text{id}) \Delta(x) = 1 \otimes x \}.
\]

According to [2, Definition 1.1.5] a quantum subgroup \( \mathbb{H} \leq \mathbb{G} \) would be termed normal provided the two quantum homogeneous spaces \( \mathcal{O}(\mathbb{G}/\mathbb{H}) \) and \( \mathcal{O}(\mathbb{H}\backslash \mathbb{G}) \) coincide. This will not quite do for our purposes (see Example 3.8), so instead we follow [25, Section 1.5] (also, say, [34, Definition 2.6], relying on the same source) in the following
Definition 3.1. The quantum subgroup $\mathbb{H} \leq G$ cast as the surjection $\pi : \mathcal{O}(G) \to \mathcal{O}(\mathbb{H})$

- **left-normal** if $\pi$ is a morphism of left $\mathcal{O}(G)$-comodules under the left adjoint coaction

$$\text{ad}_l := \text{ad}_{l, G} : x \mapsto x_1 S(x_3) \otimes x_2.$$  

- **right-normal** if similarly, $\pi$ is a morphism of right $\mathcal{O}(G)$-comodules under the right adjoint coaction

$$\text{ad}_r := \text{ad}_{r, G} : x \mapsto x_2 \otimes S(x_1)x_3. \quad (3.1)$$

- **normal** if it is both left- and right-normal.

The following result is essentially a tautology in the framework of [11, Section 1.2], but only because in that paper the definition of a normal quantum subgroup is more restrictive (see [11, Definition 1.2.3], which makes an additional (co)flatness requirement).

**Theorem 3.2.** Let $\mathbb{H} \leq G$ be a left- or right-normal quantum subgroup of a linearly reductive group such that $S^2$ leaves invariant every simple subcoalgebra of $\mathcal{O}(\mathbb{H})$.

$\mathbb{H}$ is then linearly reductive and normal.

**Remark 3.3.** The condition that $S^2$ leave invariant the simple subcoalgebras is certainly necessary for cosemisimplicity [16, Theorem 7.3.7], but I do not know if it is redundant as a hypothesis in the context of Theorem 3.2.

In particular, the squared-antipode condition of Theorem 3.2 is automatic when $S^2 = \text{id}$ (i.e. when $\mathcal{O}(G)$, or $G$, is involutory or involutive [26, Definition 7.1.12]). We thus have

**Corollary 3.4.** Left- or right-normal quantum subgroups of involutive linearly reductive quantum groups are normal and linearly reductive.

The proof of Theorem 3.2 requires some preparation. First, a simple remark for future reference.

**Lemma 3.5.** Let $\pi : H \to K$ be a surjective morphism of Hopf algebras with $H$ cosemisimple. $K$ then has bijective antipode, and hence $\pi$ intertwines antipode inverses.

**Proof.** That a morphism of bialgebras intertwines antipodes or antipode inverses as soon as these exist is well known, so we focus on the claim that $S_K$ is bijective.

By the very definition of cosemisimplicity $H$ is the direct sum of its simple (hence finite-dimensional [21, Theorem 5.1.1]) subcoalgebras $C_i \leq H$. The assumption is that $\pi$ is a morphism of Hopf algebras, so the antipode $S := S_H$ restricts to maps

$$S : \ker(\pi|_{C_i}) \to \ker(\pi|_{S(C_i)}), \quad (3.2)$$

injective because $S$ is bijective. On the other hand though, for cosemisimple Hopf algebras the squared antipode leaves every subcoalgebra invariant [16, Theorem 7.3.7], so

$$S^2 : \ker(\pi|_{C_i}) \to \ker(\pi|_{S^2(C_i)}) = \ker(\pi|_{C_i}),$$

being a one-to-one endomorphism of a finite-dimensional vector space, must be bijective. Since that map decomposes as (3.2) followed by its (similarly one-to-one) analogue defined on $S(C_i)$, (3.2) itself must be bijective, and hence the inverse antipode $S^{-1}$ leaves $\ker(\pi)$ invariant. This, in essence, was the claim. □
The conclusion of Lemma 3.5 is by no means true of arbitrary bijective-antipode Hopf algebras $H$:

**Example 3.6.** [29, Theorem 3.2] gives an example of a Hopf algebra $H$ with bijective antipode and a Hopf ideal $I \triangleleft H$ that is not invariant under the inverse antipode. In other words, even though $H$ has bijective antipode, the quotient Hopf algebra $H \rightarrow H/I$ does not.

**Proof of Theorem 3.2.** The proof proceeds gradually.

**Step 1: normality.** According to Lemma 3.5 the antipode $S := \Lambda_{\mathcal{O}(G)}$ and its inverse both leave the kernel $\mathcal{K}$ of the surjection

$$\pi : \mathcal{O}(G) \rightarrow \mathcal{O}(\mathbb{H})$$

invariant, so $S(\mathcal{K}) = \mathcal{K}$. The fact that left- and right-normality are equivalent now follows from [25, Proposition 1.5.1].

**Step 2: The homogeneous spaces $G/\mathbb{H}$ and $\mathbb{H}\backslash G$ coincide.** This means that

$$\mathcal{O}(\mathbb{H}\backslash G) = \mathcal{O}(G/\mathbb{H}) =: A,$$  \hspace{1cm} (3.3)

and follows from [2, Lemma 1.1.7].

**Step 3: Reduction to trivial $G/\mathbb{H}$.** The subspace $A \leq \mathcal{O}(G)$ of (3.3) is in fact a Hopf subalgebra [2, Lemma 1.1.4]. $A$ is also invariant under the right adjoint action

$$\mathcal{O}(G) \otimes \mathcal{O}(G) \ni x \otimes y \mapsto S(y_1)xy_2 \in \mathcal{O}(G)$$

([11, Lemma 1.20]), so by [2, Lemma 1.1.11] the left ideal

$$\mathcal{O}(G)A^- \leq \mathcal{O}(G)$$

where $A^- := \ker(\epsilon|_A)$

is bilateral. The quotient $\mathcal{O}(G)/\mathcal{O}(G)A^-$ must then be a *cosemisimple* quotient Hopf algebra [9, Theorem 2.5] $\mathcal{O}(G) \rightarrow \mathcal{O}(\mathbb{K})$, and we have an exact sequence

$$\begin{array}{c}
\mathbb{k} \rightarrow \mathcal{O}(G/\mathbb{K}) \rightarrow \mathcal{O}(G) \rightarrow \mathcal{O}(\mathbb{K}) \rightarrow \mathbb{k}
\end{array}$$

of quantum groups in the sense of [2, Section 1.2], with everything in sight cosemisimple. Since furthermore $A^-$ is annihilated by the original surjection $\mathcal{O}(G) \rightarrow \mathcal{O}(\mathbb{H})$, $\mathbb{H}$ can be thought of as a quantum subgroup of $\mathbb{K}$ (rather than $G$):

$$\mathcal{O}(\mathbb{K}) \rightarrow \mathcal{O}(\mathbb{H}).$$

I now claim that the corresponding homogeneous space is trivial:

$$\mathcal{O}(\mathbb{K}/\mathbb{H}) = \mathcal{O}(\mathbb{H}\backslash \mathbb{K}) = \mathbb{k}. \hspace{1cm} (3.4)$$

To see this, consider a simple representation $V \in \hat{\mathbb{K}}$ that contains invariant vectors over $\mathbb{H}$. Because $\mathcal{O}(\mathbb{K})$ is cosemisimple, $V$ is a subcomodule (rather than just a subquotient) of a simple comodule $W \in \hat{G}$, and it follows that

$$W|_{\mathbb{H}} \supseteq V|_{\mathbb{H}}$$

contains invariant vectors. The fact that (3.3) is a Hopf subalgebra means that it is precisely

$$\bigoplus_U C_U, \ U \in \hat{G} \text{ and } U|_{\mathbb{H}} \text{ has invariant vectors},$$
so \( C_W \leq A \) and the restriction \( W|_K \) decomposes completely as a sum of copies of the trivial comodule \( k \). But then \( V \leq W|_K \) itself must be trivial, proving the claim (3.4). Now simply switch the notation back to \( G := K \) to conclude Step 3:

\[
\mathcal{O}(G/H) = \mathcal{O}(H\backslash G) = k. 
\] (3.5)

This latter condition simply means that for an \( \mathcal{O}(G) \)-comodule \( V \) its \( G \) - and \( H \)-invariants coincide:

\[
\text{hom}_G(k, V) = \text{hom}_H(k, V).
\]

Equivalently, since

\[
\text{hom}_G(V, W) = \text{hom}_G(k, W \otimes V^*),
\]

this simply means that the restriction functor

\[
\text{Rep}(G) \ni V \mapsto V|_H \in \text{Rep}(H)
\] (3.6)

is full (for both left and right comodules, but here we focus on the latter).

**Step 4: Wrapping up.** Because the restriction functor (3.6) is full, simple, non-isomorphic \( G \)-representations that remain simple over \( H \) also remain non-isomorphic.

Now, assuming \( H \leq G \) is not an isomorphism (or there would be nothing to prove), some irreducible \( V \in \hat{G} \) must become reducible over \( H \). There are two possibilities to consider:

(a) All simple subquotients of the reducible representation \( V|_H \) are isomorphic. We then have (in \( \text{Rep}(H) \)) a surjection of \( V \) onto a simple quotient thereof, which then embeds into \( V \) again. All in all this gives a non-scalar endomorphism of \( V \) over \( H \), contradicting the fullness of the restriction functor (3.6).

(b) \( V \) acquires at least two non-isomorphic simple subquotients \( V_i, i = 1, 2 \) over \( H \). Then, the image of the coefficient coalgebra \( C_V = V^* \otimes V \) of (2.1) through \( \pi : \mathcal{O}(G) \to \mathcal{O}(H) \) will contain both

\[
C_{V_i} = V^*_i \otimes V_i \leq \mathcal{O}(H), \quad i = 1, 2
\]

as (simple) subcoalgebras.

The requirement that \( S^2(C_{V_i}) = C_{V_i} \) means that the simple comodules \( V_i \) are isomorphic to their respective double duals \( V_i^{**} \) (as \( \mathcal{O}(H) \)-comodules, not just vector spaces). But then

\[
C_{V_i} = V^*_i \otimes V_i \cong V^*_i \otimes V_i^{**}
\]

contains an \( H \)-invariant vector, namely the image of the coevaluation [19, Definition 9.3.1]

\[
\text{coev}_{V_i^*} : k \to V^*_i \otimes V_i^{**}.
\]

It follows that the space of \( H \)-invariants of the \( \mathcal{O}(G) \)-comodule \( \pi(C_V) \) is at least 2-dimensional, whereas that of \( G \)-invariants is at most 1-dimensional (because the same holds true of \( C_V = V^* \otimes V \)). This contradicts the fullness of (3.6) and hence our assumption that \( H \leq G \) is not an isomorphism.

The proof of the theorem is now complete. \( \square \)

**Remark 3.7.** Left and right normality are proven equivalent to an alternative notion ([34, Definition 2.3]) in [34, Theorem 2.7] in the context of \( CQG \) algebras, i.e. complex cosemisimple Hopf \(*\)-algebras with positive unital integral (this characterization is equivalent to [14, Definition 2.2]).

The substance of **Theorem 3.2**, however, is the cosemisimplicity claim; this is of no concern in the \( CQG \)-algebra case, as a Hopf \(*\)-algebra that is a quotient of a \( CQG \) algebra is automatically again \( CQG \) (as follows, for instance, from [14, Proposition 2.4]), and hence cosemisimple.
Example 3.8. The weaker requirement that $O(G/H) = O(H\backslash G)$ for normality would render Theorem 3.2 false.

Let $G$ be a semisimple complex algebraic group and $B \leq G$ a Borel subgroup [18, Part II, Section 1.8]. The restriction functor

$$\text{Res} : \text{Rep}(G) \to \text{Rep}(B)$$

is full [18, Part II, Corollary 4.7], so in particular

$$O(G/B) = \text{hom}_B(\text{triv}, O(G)) = \text{hom}_G(\text{triv}, O(G)) = \mathbb{C}$$

and similarly for $O(B\backslash G)$. This means that $O(G/B) = O(B\backslash G)$, but $B$ is nevertheless not reductive.

Remark 3.9. The classical (as opposed to quantum) analogue of Theorem 3.2 admits an alternative, more direct proof relying on the structure of reductive groups:

- In characteristic zero linear reductivity is equivalent (by [24, p.88 (2)], for instance) to plain reductivity [4, Section 11.21], i.e. the condition that the unipotent radical $R_u(G)$ of $G$ (the largest normal connected unipotent subgroup) be trivial.

  Assuming $G$ is reductive, for any normal $K \triangleleft G$ the corresponding unipotent radical $R_u(K)$ is characteristic in $N$ and hence normal in $G$, meaning that

  $$R_u(K) \leq R_u(G) = \{1\}$$

  and hence $N$ is again reductive (so linearly reductive, in characteristic zero).

- On the other hand, in positive characteristic $p$ [24, p.88 (1)] says that the linearly reductive groups $G$ are precisely those fitting into an exact sequence

  $$\{1\} \to K \to G \to G/K \to \{1\}$$

  with $K$ a closed subgroup of a torus and $G/K$ finite of order coprime to $p$. Clearly then, normal subgroups of $G$ have the same structure.

4. Clifford theory

We work with an exact sequence (4.1)

$$k \to A \to H \to B \to k$$

of cosemisimple Hopf algebras in the sense of [2, p. 23]. Note that we additionally know that $H$ is left and right coflat over $B$ (simply because the latter is cosemisimple) and left and right faithfully flat over $A$ (by [9, Theorem 2.1]).

We will make frequent use of [32, Theorem 1], to the effect that

$$\mathcal{M}^H_A \xrightarrow{\text{corestrict}} \mathcal{M}^B$$

is an equivalence, where the $-\superscript{\text{co}}$ superscript denotes kernels of counits.

Upon identifying $\mathcal{M}^B$ with $\mathcal{M}^H_A$ via (4.2), the adjunction

$$\mathcal{M}^H \xrightarrow{\text{corestrict}} \mathcal{M}^B$$

becomes

$$\mathcal{M}^H \xrightarrow{-\otimes A} \mathcal{M}^H_A.$$
We will freely switch points of view between the two perspectives provided by (??). Consider the following binary relation ∼_B on ˆB.

**Definition 4.1.** For V, W ∈ ˆB, V ∼_B W provided there is a simple H-comodule U such that V and W are both constituents of the corestriction of U to B.

Similarly, we will study the following relation on ˆH:

**Definition 4.2.** For V, W ∈ ˆH we set V ∼_H W provided \( \text{hom}^B(V, W) \neq 0 \).

**Remark 4.3.** In other words, ∼_H signifies the fact that the corestrictions of V and W to \( \mathcal{M}^B \) have common simple constituents.

Our first observation is that ∼_H is an equivalence relation, and provides an alternate characterization for it.

**Theorem 4.4.** ∼_H is an equivalence relation on ˆH, and moreover, for V, W ∈ ˆH the following conditions are equivalent

1. V ∼_H W;
2. as B-comodules, V and W have the same simple constituents;
3. V embeds into W ⊗ A ∈ \( \mathcal{M}^H \).

**Proof.** Note first that (2) clearly defines an equivalence relation on ˆH, so the first statement of the theorem will be a consequence of

\( (1) \Leftrightarrow (2) \Leftrightarrow (3) \).

We prove the latter result in stages.

\( (1) \Leftrightarrow (3). \) By definition, V ∼_H W if and only if

\[ \text{hom}^B(V, W) \neq 0. \]

Via (4.2) and the hom-tensor adjunction (4.4), this hom space can be identified with

\[ \text{hom}^H_A(V ⊗ A, W ⊗ A) \cong \text{hom}^H(V, W ⊗ A). \] (4.5)

The simplicity of V ∈ ˆH now implies that every nonzero element of the right hand side of (4.5) is an embedding, hence finishing the proof of the equivalence of (1) and (3).

\( (1) \Leftrightarrow (2). \) Let us denote by \( \text{const}(\bullet) \) the set of simple constituents of a B-comodule \( \bullet \).

By definition V ∼_H W means that some of the simple constituents of V and W as objects in \( \mathcal{M}^B \) coincide, so (2) is clearly stronger than (1). Conversely, note that by the equivalence \( (1) \Rightarrow (3) \) proven above, whenever V ∼_H W we have

\[ \text{const}(V) \subseteq \text{const}(W ⊗ A), \] (4.6)

where the respective objects are regarded as B-comodules via the corestriction functor \( \mathcal{M}^H \to \mathcal{M}^B \).

In turn however, given that A ∈ \( \mathcal{M}^H \) breaks up as a sum of copies of k in \( \mathcal{M}^B \) (because of the exactness of (4.1)), the right hand side of (4.6) is simply \( \text{const}(W) \). All in all, we have

\[ V ∼_H W \Rightarrow \text{const}(V) \subseteq \text{const}(W). \]

This together with the symmetry of ∼_H (obvious by definition from the semisimplicity of \( \mathcal{M}^B \)) finishes the proof of \( (1) \Rightarrow (2) \) and of the theorem. \( \square \)
Theorem 4.5. \( \sim_B \) is an equivalence relation on \( \hat{B} \) with finite classes.

Proof. We know from Theorem 4.4 that as \( U \) ranges over \( \hat{H} \), the sets \( \text{const}(U) \) of constituents of \( U \in \mathcal{M}^B \) partition \( \hat{B} \), thus defining an equivalence relation on the latter set.

The definition of \( \sim_B \) ensures that \( V \sim_B W \) if and only if \( V \) and \( W \) fall in the same set \( \text{const}(U) \), and hence \( \sim_B \) coincides with the equivalence relation from the previous paragraph.

Finally, the statement on finiteness of classes is implicit in their description given above: an equivalence class is the set of simple constituents of a simple \( H \)-comodule \( U \) viewed as a \( B \)-comodule, and it must be finite because \( \dim(U) \) is.

Theorems 4.4 and 4.5 establish a connection between the equivalence relations \( \sim_H \) and \( \sim_B \) on \( \hat{H} \) and \( \hat{B} \) respectively. We record it below.

Before getting to the statement, recall the notation \( \text{const}(\bullet) \subseteq \hat{B} \) for the set of simple summands of an object \( \bullet \in \mathcal{M}^B \). With that in mind, we have the following immediate consequence of Theorems 4.4 and 4.5.

Proposition 4.6. The range of the map

\[ \hat{H} \rightarrow \text{finite subsets of} \hat{B} \]

sending \( V \in \hat{H} \) to \( \text{const}(V) \) consists of the equivalence classes of \( \sim_B \), and its fibers are the classes of \( \sim_H \).

5. Relative chain groups and centers

Definition 5.1. Let \( \mathbb{H} \leq \mathbb{G} \) be an inclusion of linearly reductive quantum groups. The (relative) chain group \( C(\mathbb{G}, \mathbb{H}) \) is defined by

- generators \( g_V \) for simple comodules \( V \in \hat{\mathbb{G}} \);
- relations

\[
\text{hom}_{\mathbb{H}}(U, V \otimes W) \neq 0 \Rightarrow g_U = g_V g_W; \tag{5.1}
\]

that is, one such relation whenever the restrictions of \( U \) and \( V \otimes W \) to \( \mathbb{H} \) have non-trivial common summands (i.e. \( U \) and \( V \otimes W \) are not disjoint over \( \mathbb{H} \)).

We write \( C(\mathbb{G}) := C(\mathbb{G}, \mathbb{G}) \).

Remark 5.2. For chained inclusions \( \mathbb{K} \leq \mathbb{H} \leq \mathbb{G} \) we have a map \( C(\mathbb{G}, \mathbb{K}) \rightarrow C(\mathbb{G}, \mathbb{H}) \) sending the class of \( V \in \hat{\mathbb{G}} \) in the domain to the class of the selfsame \( V \) in the codomain. This is easily seen to be well-defined and a group morphism.

Recall [8, Definition 2.10].

Definition 5.3. Let \( \mathbb{G} \) be a linearly reductive quantum group. Its center \( Z(\mathbb{G}) \leq \mathbb{G} \) is the quantum subgroup dual to the largest Hopf algebra quotient

\[
\pi : \mathcal{O}(\mathbb{G}) \rightarrow \mathcal{O}(Z(\mathbb{G}))
\]

that is central in the sense of [8, Definition 2.1]:

\[
\pi(x_1) \otimes x_2 = \pi(x_2) \otimes x_1 \in \mathcal{O}(Z(\mathbb{G})) \otimes \mathcal{O}(\mathbb{G}), \forall x \in \mathcal{O}(\mathbb{G}).
\]

The relative version of this construction, alluded to in the title, is as follows.

Definition 5.4. Let \( \mathbb{H} \leq \mathbb{G} \) be an embedding of linearly reductive quantum groups. The corresponding relative center \( Z(\mathbb{G}, \mathbb{H}) \) is the intersection \( Z(\mathbb{G}) \cap \mathbb{H} \) denoted by \( Z(\mathbb{G}) \wedge \mathbb{H} \) in [11, Definition 1.15].
This is a quantum subgroup of both $\mathbb{H}$ and $Z(G)$ (and hence also of $G$), and is automatically linearly reductive by [11, Proposition 3.1].

Each irreducible $G$-representation breaks up as a sum of mutually isomorphic (one-dimensional) representations over the center $Z(G)$, and hence gets assigned an element of $\widehat{Z(G)}$: its central character. Two such irreducible representations that are not disjoint over $\mathbb{H}$ must have corresponding central characters agreeing on

$$Z(G, \mathbb{H}) := Z(G) \cap \mathbb{H}$$

(the relative center associated to the inclusion $\mathbb{H} \leq G$), so we have a canonical morphism

$$\text{can} : C(G, \mathbb{H}) \to \widehat{Z(G, \mathbb{H})} \quad (5.2)$$

**Theorem 5.5.** For any embedding $\mathbb{H} \leq G$ of linearly reductive quantum groups (5.2) is an isomorphism.

**Proof.** Consider the commutative diagram

$$C(G) \xrightarrow{\text{can}} C(G, \mathbb{H}) \xrightarrow{\text{can}} \widehat{Z(G)} \xrightarrow{\text{can}} \widehat{Z(G, \mathbb{H})} \quad (5.3)$$

where

- the upper left-hand morphism is an instance of the maps noted in Remark 5.2;
- the bottom right-hand map is the (plain) group surjection dual to the quantum-group inclusion $Z(G) \cap \mathbb{H} \leq Z(G)$;
- and the fact that the bottom left-hand map is an isomorphism is a paraphrase of [8, Proposition 2.9] in conjunction with [8, Definition 2.10].

The surjectivity of the bottom composition entails that of (5.2), so it remains to show that the latter is one-to-one.

Let $V \in \widehat{G}$ be a simple comodule where $Z(G, \mathbb{H})$ operates with trivial character, i.e. one whose class in $C(G, \mathbb{H})$ is annihilated by (5.2). We can then form the quantum subgroup

$$Z(G) \mathbb{H} := Z(G) \vee \mathbb{H} \leq G$$

generated by $Z(G)$ and $\mathbb{H}$ as in [11, Definition 1.15] (the ‘$\vee$’ notation is used there; we suppress the symbol here for brevity), which then satisfies, according to [11, Theorem 3.4], a quantum-flavored isomorphism theorem:

$$\mathbb{H}/Z(G, \mathbb{H}) \xrightarrow{\cong} Z(G) \mathbb{H}/Z(G)$$

via the canonical map induced from $\mathbb{H} \to Z(G) \mathbb{H}$. Since $V$ (or rather its restriction $V|_{\mathbb{H}}$) is a representation of the former group because $Z(G, \mathbb{H})$ operates trivially, it lifts to a $Z(G) \mathbb{H}$-representation with $Z(G)$ acting trivially. In summary:

The restriction $V|_{\mathbb{H}}$ extends to a $Z(G) \mathbb{H}$-representation $W$ with trivial $Z(G)$-action.

But then the induced representation $\text{Ind}^G_{Z(G) \mathbb{H}} W$ again has trivial central character, and hence so do all of its simple summands $V_1$. The adjunction (2.2) yields

$$\text{hom}_{Z(G) \mathbb{H}}(V_1|_{Z(G) \mathbb{H}}, W) \cong \text{hom}_G \left( V_1, \text{Ind}^G_{Z(G) \mathbb{H}} W \right) \neq \{0\},$$

meaning that $V_1$ fails to be disjoint from $W$ over $Z(G) \mathbb{H}$ and hence also from

$$V|_{\mathbb{H}} = W|_{\mathbb{H}} \text{ over } \mathbb{H}.$$
To conclude, observe that
• (5.2) agrees on $V$ and $V_1$ due to the noted non-disjointness
  \[ \text{hom}_{\mathbb{H}}(V_1, V) \neq 0; \]
• while the bottom left-hand map $\text{can} : C(G) \to \widehat{Z(G)}$ of (5.3) annihilates $V_1$ because the latter has trivial central character;
• and hence the top right-hand $\text{can}$ map in (5.3) must also annihilate $V$.

This being the desired conclusion, we are done.

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