The supersymmetry method for chiral random matrix theory with arbitrary rotation-invariant weights

Vural Kaymak\textsuperscript{1}, Mario Kieburg\textsuperscript{2} and Thomas Guhr\textsuperscript{1}

\textsuperscript{1} Fakultät für Physik, Universität Duisburg-Essen, Lotharstraße 1, D-47048 Duisburg, Germany
\textsuperscript{2} Fakultät für Physik, Universität Bielefeld, Postfach 100131, D-33501 Bielefeld, Germany

E-mail: vural.kaymak@uni-due.de, mkieburg@physik.uni-bielefeld.de and thomas.guhr@uni-due.de

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Abstract

In the past few years, the supersymmetry method has been generalized to real symmetric, Hermitian, and Hermitian self-dual random matrices drawn from ensembles invariant under the orthogonal, unitary, and unitary symplectic groups, respectively. We extend this supersymmetry approach to chiral random matrix theory invariant under the three chiral unitary groups in a unifying way. Thereby we generalize a projection formula providing a direct link and, hence, a ‘short cut’ between the probability density in ordinary space and that in superspace. We emphasize that this point was one of the main problems and shortcomings of the supersymmetry method, since only implicit dualities between ordinary space and superspace were known before. To provide examples, we apply this approach to the calculation of the supersymmetric analogue of a Lorentzian (Cauchy) ensemble and an ensemble with a quartic potential. Moreover, we consider the partially quenched partition function of the three chiral Gaussian ensembles corresponding to four-dimensional continuum quantum chromodynamics. We identify a natural splitting of the chiral Lagrangian in its lowest order into a part for the physical mesons and a part associated with source terms generating the observables, e.g. the level density of the Dirac operator.

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1. Introduction

Chiral random matrix theory is the oldest of all random matrix ensembles. It was introduced by Wishart [1] in the 1920s to model generic properties of correlation matrices. Since then chiral random matrix theory has been applied to many other fields of physics and beyond, because of its versatility. One important application is in the study of correlation matrices in time series analysis [2–4, 10–14]. Chiral random matrix theory serves as a benchmark model for empirical correlation matrices and is used to extract the system-specific correlations from the generic statistical fluctuations. Another famous development is the introduction of chiral random matrix theory to quantum chromodynamics (QCD) by Shuryak and Verbaarschot [15–17]. They showed the equivalence of the microscopic limit of the QCD Dirac operator with chiral random matrix theory. In particular, chiral random matrix theory explained the statistical fluctuations of the smallest eigenvalues of the Dirac operator and predicted relations between low energy constants and observables which are confirmed by lattice QCD data [18, 19]. Recent applications of chiral random matrix theory can also be found in condensed matter theory [20], telecommunications [21–23], and quantum information theory [24], but its range is far from being restricted to those examples.

For the sake of simplicity, a Gaussian function is often used within the context of random matrix theory. Due to universality [25–28], this choice is quite often legitimate as long as the interest lies in correlations on the local scale of the mean level spacing. To prove universality as well as to modify random matrix theory in order to describe particular systems, many technical tools have been developed. For example, the supersymmetry method, originally introduced for Gaussian weights [29–32], is established as a versatile tool in the field of random matrix theory because of its broad applicability to non-Gaussian ensembles. For the history of the supersymmetry method and its variants, we refer the reader to [32].

Moreover, one is not always interested in the local scale; e.g., see the analysis of universality on macroscopic scales as discussed with free probability [33, 34]. Hence a generalization to arbitrary statistical weights is of particular interest. Other important techniques are the methods based on orthogonal polynomials [6], Toda lattice structures [7], free probability theory [35] and maps to Hamiltonian systems [8, 9]. For a comprehensive overview, see [5, 51, 64] and references therein.

Here, we focus on the supersymmetry method, not on aspects related to other methods such as orthogonal polynomial ones. We start from a close connection between matrix invariants in ordinary space and in superspace which was first observed in [36]. In particular, for chiral random matrix models we investigate how probability densities which only depend on matrix invariants (but are otherwise arbitrary) are uniquely mapped from ordinary space to superspace. This is the issue at hand.

An exact map from ordinary space to superspace for arbitrary isotropic ensembles for real symmetric, Hermitian and Hermitian self-dual matrices was provided in two different but related approaches, a few years ago. Isotropy is invariance under the orthogonal, unitary or unitary symplectic group; see [35]. One approach pursues the idea to generalize the original Hubbard–Stratonovich transformation in superspace for Gaussian weights [29–32] to that for arbitrary weights [36, 37]. In another approach, one tries to find a direct, exact identity between integrals over dyadic supermatrices and integrals over cosets. This second approach is known as the superbosonization formula method [38, 39]. The two approaches are completely equivalent [40] and both have their advantages as well as disadvantages. One crucial disadvantage that they share is that they do not directly relate the probability density in ordinary space to that in superspace. They only become explicit when the characteristic function (Fourier transform of the probability density) is known in a closed form. Hence one has to calculate the statistical
weight for each random matrix ensemble, separately. This is exactly the problem that we want to address.

The problem of the extension of the generalized Hubbard–Stratonovich transformation as well as the superbosonization formula to the other seven classes in the tenfold classification via the Cartan scheme [41, 42] is still unsolved. We address three of these seven classes in a unifying way, namely with chiral random matrices generated by non-Gaussian probability densities. In particular, we derive a projection formula explicitly relating the probability density in ordinary space with the one in superspace. Thus we present a resolution for the disadvantage of the generalized Hubbard–Stratonovich transformation and the superbosonization formula that one has to study each ensemble separately. Such a projection formula was already accomplished for real symmetric, Hermitian, and Hermitian self-dual matrices; see [43]. In section 2, we briefly summarize the idea behind such a projection formula for ensembles in the original classification given by Dyson [41] and contrast it with the well established generalized Hubbard–Stratonovich transformation and the superbosonization formula. In section 3, we generalize this approach to the three chiral random matrix theories of real, complex and quaternion rectangular matrices in a unifying way.

To underline that the projection formula is a powerful tool, we apply it to a selection of ensembles encountered in different fields of random matrix theory, in section 4. Some of these ensembles, such as the Lorentz (Cauchy)-like ensembles and the ensemble with a quartic potential, are not at all trivial and it is not immediately clear what their supersymmetric counterpart will look like. The other examples are the norm-dependent ensembles without and with empirical correlations and the unquenched chiral Gaussian random matrix ensembles modelling QCD with quarks. In particular, for the partially quenched partition function we derive a representation whose microscopic limit agrees with QCD and shows a natural splitting into physical mesons and those corresponding to the source term generating observables like the level density and higher order correlations. The explicit calculation of this result is presented in the appendix. The article is concluded with a summary in section 5.

2. The main idea of a projection formula

The supersymmetry method is essentially based on a general relation between partition functions in ordinary space,

\[ Z(\kappa) = \int d[H] P(H) \frac{\prod_{j=1}^{k_2} \det(H - \kappa(2)_j I_N)}{\prod_{j=1}^{k_1} \det(H - \kappa(1)_j I_N)}, \]

(2.1)

and partition functions in superspace, which we expect to be of the form

\[ Z(\kappa) = \int d[\sigma] Q(\sigma) \det^{\mu(N)}(\sigma - \kappa). \]

(2.2)

The \( N \times N \) matrix \( H \) is distributed through \( P \) and drawn from one of the Hermitian ensembles classified in the tenfold way via the Cartan classification scheme [41, 42]. The exponent \( \mu \) is some affine linear function in the former ordinary dimension \( N \). The supermatrix \( \sigma \) has a dimension related to the number of determinants in equation (2.1). It fulfils certain symmetries depending on those of the ordinary matrix \( H \), and is drawn from a probability density \( Q \) in superspace. The source variables

\[ \kappa := \begin{cases} \text{diag}(\kappa_1, \kappa_2), & \beta = 2, \\ \text{diag}(\kappa_1, \kappa_1, \kappa_2, \kappa_2), & \beta = 1, 4, \end{cases} \]

(2.3)
with
\[ \kappa_1 := \text{diag}(\kappa^{(1)}_1, \ldots, \kappa^{(1)}_{k_1}), \quad \kappa_2 := \text{diag}(\kappa^{(2)}_1, \ldots, \kappa^{(2)}_{k_2}), \]
are distinguished by the Dyson index \( \beta = 1, 2, 4 \). We notice that \( \kappa \) is always a supermatrix. In the context of QCD, it comprises masses of the physical fermions as well as masses of the valence fermions usually denoted by \( m_j \) [17]. The masses of the valence fermions consist of source variables for differentiation to generate the matrix Green functions often denoted by \( \sigma \) \( \in \) real axis. \( \beta \) have to assume that \( \kappa(\beta) \) has a non-zero imaginary part, since the spectrum of \( H \) lies on the real axis.

The main task is to derive two things. First of all, the corresponding supermatrix space, \( \sigma \in \mathcal{M}_{\text{SUSY}} \), has to be identified, which is independent of the probability density \( P \). This identification was already done in [42]. Second, one has to calculate the probability distribution \( Q \), which crucially depends on the ordinary matrix space, \( H \in \mathcal{M}_{\text{ord}} \), and on the probability density \( P \). The second task is the hardest one and the solution is up to now only known in a closed form when \( H \) is real symmetric, Hermitian, or Hermitian self-dual [43].

After recalling the standard supersymmetry method in subsection 2.1, we briefly rederive a projection formula for ensembles of real symmetric, Hermitian, and Hermitian self-dual matrices in subsection 2.2, to indicate the main idea of such a projection formula.

2.1. The standard supersymmetry approach

Let us introduce three abbreviations:
\[ U^{(\beta)}(n) := \begin{cases} 
O(n), & \beta = 1, \\
U(n), & \beta = 2, \\
USp(2n), & \beta = 4, \end{cases} \]
\[ \text{Herm}^{(\beta)}(n) := \begin{cases} 
\text{Gl}(n, \mathbb{R})/O(n) \cong U(n)/O(n), & \beta = 1, \\
\text{Gl}(n, \mathbb{C})/U(n), & \beta = 2, \\
\text{Gl}(n, \mathbb{H})/USp(2n) \cong U(2n)/USp(2n), & \beta = 4, \end{cases} \]
and
\[ \gamma := \begin{cases} 
1, & \beta = 1, 2, \\
2, & \beta = 4, \end{cases} \quad \text{and} \quad \tilde{\gamma} := \begin{cases} 
2, & \beta = 1, \\
1, & \beta = 2, 4, \end{cases} \]
such that we can deal with all three Dyson indices \( \beta = 1, 2, 4 \) in a unifying way. Equation (2.6) is an abbreviation for the set of real symmetric, Hermitian, and Hermitian self-dual matrices. Here, \( \mathbb{H} \) is the quaternion number field which we represent via the Pauli matrices and the two-dimensional unit matrix \( I_2 \) throughout the work.

The aim is to identify a partition function in superspace starting from a partition function in ordinary space,
\[ Z(\kappa) := \int d[H] P(H) \frac{\prod_{j=1}^{k_1} \det (H - \kappa_j^{(2)} I_{\gamma n})}{\prod_{j=1}^{k_2} \det (H - \kappa_j^{(1)} I_{\gamma n})} \]
\[ = \int d[H] P(H) s \det^{-1/(\tilde{\gamma} \gamma)} (H \otimes I_{\gamma n} \otimes I_{\gamma n} - I_{\gamma n} \otimes I_{\gamma n} \otimes \kappa), \]
with \( H \in \text{Herm}^{(\beta)}(n) \) and \( P \) fulfilling the rotation invariance (also known as the isotropy [35])
\[ P(H) = P(UHU^{-1}), \quad \forall U \in U^{(\beta)}(n). \]
Let, for simplicity, \( \text{Im} \kappa_1 > 0 \) in this subsection. We will weaken this condition later on.
In the original supersymmetry method one introduces a rectangular complex supermatrix $V$ [32] of dimension $(\gamma n) \times (\gamma' k_1 | \gamma' k_2)$ and uses the crucial identity

$$\mathrm{sdet}^{-1/(\gamma \gamma')} (H \otimes I_{\gamma' k_1 | \gamma' k_2} - I_{\gamma n} \otimes \kappa) = \int d[V] \exp[\tau^{\dagger} V V^\dagger \kappa - \tau V^\dagger H V] \int d[V] \exp[-\tau^{\dagger} V V^\dagger].$$

Recall the definition of $\kappa$ in equation (2.3) and those of $\gamma$ and $\gamma'$ in equation (2.7). The rescaling by the imaginary unit $i$ is needed to ensure the convergence of the integral over $V$. The supermatrix $V$ consists of independent complex random variables as well as complex Grassmann variables (anti-commuting variables) and fulfills some symmetries under complex conjugation if the Dyson index is $\beta = 1, 4$, i.e. the complex conjugate of $V$ is

$$V^* = \begin{cases} V \text{diag}(I_{\gamma 2k_1}, \tau_2 \otimes I_{k_2}), & \beta = 1, \\ (-\tau_2 \otimes I_n) V \text{diag}(\tau_2 \otimes I_{k_1}, I_{2k_2}), & \beta = 4, \end{cases}$$

where $\tau_2$ is the second Pauli matrix. The case $\beta = 1$ is some kind of reality condition and for $\beta = 4$ it is some kind of generalization of quaternions.

On plugging equation (2.10) into the partition function (2.8), the integration over $H$ reduces to a Fourier transform of the probability density $P$. We assume that the Fourier transform,

$$\Phi(A) := \int d[H] P(H) \exp[-i \text{tr} HA],$$

exists for any $(\gamma n) \times (\gamma n)$ matrix $A$ sharing the same symmetries as $H$ apart from relations involving complex conjugations. The invariance property (2.9) of $P$ carries over to one of $\Phi$, i.e.,

$$\Phi(A) = \Phi(UAU^{-1}) \quad \forall U \in U(\beta)(n),$$

meaning that the function $\Phi$ can be written as a function of the traces of $A$. Identifying the matrix $A = VV^\dagger$, one can show that there is a superfunction $\tilde{\Phi}$, which is far from being unique (see [37]), such that another essential identity of the supersymmetry method holds [32]:

$$\Phi(VV^\dagger) = \tilde{\Phi}(V^\dagger V).$$

Note that the tilde emphasizes that $\tilde{\Phi}$ is not the same as, but is related to the function $\Phi$. The partition function reads

$$Z(\kappa) = \int d[V] \exp[i \text{tr} V^\dagger V \kappa] \tilde{\Phi}(V^\dagger V) \int d[V] \exp[-i \text{tr} V^\dagger V]$$

which is already a representation in superspace.

Two different routes can be pursued from this point. One approach is using the superbosonization formula [38, 39]. With the help of the superbosonization formula the integral over $V^\dagger V$ is replaced by an integral over a $(\gamma' \gamma k_1 | \gamma' \gamma k_2) \times (\gamma' \gamma k_1 | \gamma' \gamma k_2)$ supermatrix $U$ fulfilling some symmetries under the transposition if $\beta = 1, 4$, i.e.,

$$U^T = \begin{cases} \text{diag}(I_{\gamma 2k_1}, -\tau_2 \otimes I_{k_2}) U \text{diag}(I_{\gamma 2k_1}, \tau_2 \otimes I_{k_2}), & \beta = 1, \\ \text{diag}(-\tau_2 \otimes I_{k_1}, I_{2k_2}) U \text{diag}(\tau_2 \otimes I_{k_1}, I_{2k_2}), & \beta = 4, \end{cases}$$

which means that $U_{BB}$ is symmetric (self-dual) and $U_{FF}$ is self-dual (symmetric) for $\beta = 1$ ($\beta = 4$). Additionally, the matrix $U$ consists of four blocks,

$$U = \begin{bmatrix} U_{BB} & \eta^\dagger \\ \eta & U_{FF} \end{bmatrix},$$

whose off-diagonal blocks $\eta$ and $\eta^\dagger$ contain independent Grassmann variables apart from what is specified by the condition (2.16), the boson–boson block is positive definite, $U_{BB} > 0$, and
Hermitian, $U^{\dagger}_{BB} = U_{BB}$, and the fermion–fermion block is unitary, $U^{\dagger}_{FF} = U^{-1}_{FF}$. Hence the supermatrix $U$ is in one of the three cosets \cite{38–40}

$$\text{Herm}^{(\beta)}(\overline{\gamma_1}k_1|\gamma k_2) := \begin{cases} U(2k_1|2k_2)/\text{USp}^{(\beta)}(2k_1|2k_2), & \beta = 1, \\
\text{Gl}(k_1|k_2)/U(k_1|k_2), & \beta = 2, \\
U(2k_1|2k_2)/\text{USp}^{-1}(2k_1|2k_2), & \beta = 4,\end{cases} \quad (2.18)$$

where $U(p,q)$ is the unitary supergroup and $\text{Gl}(p,q)$ is the general linear, complex supergroup. The two supergroups $\text{USp}^{(\pm)}(2k_1|2k_2)$ for $\beta = 1, 4$ are the two independent matrix representations of the unitary ortho-symplectic supergroup $\text{USp}(2k_1|2k_2)$. Matrices in this group are real in the boson–boson block and quaternion in the fermion–fermion block for $\beta = 1$ denoted by the superscript ‘(+’ and vice versa for $\beta = 4$ denoted by the superscript ‘(−)’; see \cite{37}. The subscript ‘⊙’ refers to the kind of embedding of the coset which is a contour integral around the origin for the fermion–fermion block $U_{FF}$ in the case of the superbosonization formula.

The superbosonization formula can be summarized as the following simple equation:

$$Z(\kappa) = \int \frac{\mu(U)}{\text{vol}(k_1+k_2)} \frac{d\mu(U)}{\text{vol}(k_1+k_2)} \text{sdet}(U) \exp[-\text{str}(U)] \Phi(U); \quad (2.19)$$

see \cite{38, 39}. The measure $d\mu(U)$ is the Haar measure of the corresponding coset.

The second supersymmetric approach is the generalized Hubbard–Stratonovich transformation \cite{36, 37}. Instead of replacing $V^\dagger V$ by a supermatrix, one assumes that the superfunction $\Phi$ is a Fourier transform of another superfunction $Q$ as well, i.e.,

$$\tilde{\Phi}(B) = \int d[\sigma] Q(B) \exp[-\text{str} \sigma B], \quad (2.20)$$

for some supermatrix $B$. The integration domain of $\sigma$ is very important. First of all it fulfills the same symmetries under transposition as $U$ in the superbosonization formula (see equation \eqref{2.16}), i.e.,

$$\sigma = \begin{bmatrix} \sigma_{BB} & \eta^\dagger \\
\eta & \sigma_{FF} \end{bmatrix} \quad (2.21)$$

which is again equivalent to saying that $\sigma_{BB}$ is symmetric (self-dual) and $\sigma_{FF}$ is self-dual (symmetric) for $\beta = 1$ ($\beta = 4$). However the blocks of

$$\sigma = \begin{bmatrix} \sigma_{BB} & \eta^\dagger \\
\eta & \sigma_{FF} \end{bmatrix} \quad (2.22)$$

are drawn from different supports, as for $U$. The off-diagonal blocks $\eta$ and $\eta^\dagger$ are again independent Grassmann variables apart from what is specified by the condition \eqref{2.21}, while the boson–boson block is now only Hermitian, $\sigma_{BB} \equiv \sigma_{BB}$. The fermion–fermion block can be diagonalized by $\tilde{U} \in U^{(\beta)}(\gamma k_2)$, i.e. $\sigma_{FF} = \tilde{U} \sigma_{FF} \tilde{U}^\dagger$. The eigenvalues $\{s_{FF}\}_j$ live on contours such that the integral over them converges. For a Gaussian ensemble, the standard Wick rotation, i.e. $\{s_{FF}\}_j \in i\mathbb{R}$, does the job. For other polynomial potentials, one has to choose other Wick rotations, e.g. for $P(H) \propto \exp[-\text{tr} H^{2m}]$ it is $\{s_{FF}\}_j \in \exp[i\pi/(2m)] \mathbb{R}$. Therefore the supermatrix $\sigma$ lies also in an embedding of the cosets \eqref{2.18}, but the set will now be denoted by $\text{Herm}^{(\beta)}(\overline{\gamma_1}k_1|\gamma k_2)$ where the subscript ‘Wick’ reflects the nature of the integration domain.

Reading off $B = V^\dagger V$ and integrating over $V$, one obtains the final result for the generalized Hubbard–Stratonovich transformation:

$$Z(\kappa) = \int d[\sigma] Q(\sigma) \text{sdet}^{-\gamma/k_2}(\sigma - \kappa); \quad (2.23)$$

see \cite{36, 37}. The measure $d[\sigma]$ is the flat one, i.e. the product of the differentials of all independent matrix elements.
Both approaches, the superbosonization formula and the generalized Hubbard–Stratonovich transformation, have a crucial weakness. Without an explicit knowledge of the Fourier transform $\Phi$, no direct functional relation between the probability density $P$, the superfunction $\Phi$, and the superfunction $Q$ is known. The reason is the duality relation (2.13) between ordinary space and superspace. In particular, for the generalized Hubbard–Stratonovich transformation, the dyadic matrices $VV^\dagger$ and $V^\dagger V$ are in different matrix spaces. Hence, one cannot expect the Fourier transforms (2.12) and (2.20) to yield the same functional dependence for $P$ and $Q$. The projection formula [43], briefly rederived in subsection 2.2, circumvents this problem.

2.2. The projection formula for Dyson’s threefold approach

The key idea for finding a direct relation between $P$ and $Q$ is to extend the original matrix set $H \in \text{Herm}^{(\beta)}(n)$ to a larger matrix set also containing the target set $\sigma \in \text{Herm}_{\text{Wick}}^{(\beta)}(\gamma k_1 | \gamma k_2)$. Let $\bar{\gamma} k_1 \leq \gamma k_2 + n$, to keep the calculation as simple as possible; otherwise we have to give a case by case discussion. This condition is usually the case when applying supersymmetry to random matrix theory. Nevertheless, we underline that this condition is not at all a restriction, since the other case can be taken care by slightly modifying the ensuing discussion; see [43].

The idea of our approach is based on a Cauchy-like integration formula for supermatrices in the coset $\text{Herm}_{\text{Wick}}^{(\beta)}(p|q)$ with $p, q \in \mathbb{H}_0$ which was first derived by Wegner [44]; see also [45–47] for slightly modified versions. Let $l \in \mathbb{N}$ be a positive integer and $f$ be an integrable and smooth superfunction on the set of supermatrices $\text{Herm}_{\text{Wick}}^{(\beta)}(p + \bar{\gamma} l | q + \gamma l)$ and invariant under

$$f(\tilde{U} \Sigma \tilde{U}^{-1}) = f(\Sigma)$$

(2.24)

for all $\Sigma \in \text{Herm}_{\text{Wick}}^{(\beta)}(p + \bar{\gamma} l | q + \gamma l)$ and

$$\tilde{U} \in U^{(\beta)}(p + \bar{\gamma} l | q + \gamma l) := \begin{cases} U^{(\beta)}(p + 2l | 2q + 2l), & \beta = 1, \\ U(p + l | q + l), & \beta = 2, \\ U^{(\beta)}(2p + 2l | q + 2l), & \beta = 4. \end{cases}$$

(2.25)

Employing the following splitting of

$$\Sigma = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} + \hat{\Sigma} \quad \text{with} \quad \hat{\Sigma} = \begin{bmatrix} 0 & \hat{V} \\ \hat{V}^\dagger & \sigma \end{bmatrix}$$

(2.26)

such that $\hat{\Sigma} \in \text{Herm}_{\text{Wick}}^{(\beta)}(p|q)$ and $\sigma \in \text{Herm}_{\text{Wick}}^{(\beta)}(\gamma l | \gamma l)$, the Cauchy-like integral identity [44–47] reads

$$\frac{f(d[\hat{\Sigma}]) f(\Sigma)}{f(d[\Sigma]) \exp[-\text{str} \Sigma^2]} = f\left(\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}\right).$$

(2.27)

reducing a large supermatrix, $\Sigma$, to a smaller one, $\hat{\Sigma}$, independently of the concrete form of the superfunction $f$. The notation $\Sigma, \hat{\Sigma}, \tilde{\Sigma}$ has no deeper meaning. It only underlines that all three matrices are essentially of the same form apart from their different dimensions.

Equation (2.27) is at the heart of our approach. Let us consider the partition function (2.8) in ordinary space. From now on, we lift the condition $\text{Im} \kappa_1 > 0$ to emphasize that our idea works in general and define $L = \text{sign} \text{Im} \kappa$. We assume that the probability density $P$ is rotation invariant; see equation (2.9). Moreover we assume that a contour like the Wick rotation and an extension of $P$, denoted by $\tilde{P}$, from the ordinary matrix set $\text{Herm}^{(\beta)}(n)$ to the supermatrix set $\text{Herm}_{\text{Wick}}^{(\beta)}(n + \bar{\gamma} k_2 | \gamma k_2)$ exist such that the superfunction $\tilde{P}$ is integrable and smooth on
Herm$_{Wick}^{(\beta)}(n + \tilde{\gamma}k_2|y k_2)$. Then we can extend the integral (2.8) to an integral in superspace, i.e.,

$$Z(\kappa) = \frac{\int d[\Sigma] \tilde{P}(\Sigma) \det^{-1/(\gamma \tilde{\gamma})}(H \otimes \mathbb{1}_{y k_1|y k_2} - \mathbb{1}_{y n} \otimes \kappa)}{\int d[\Sigma] \exp[-\text{str } \Sigma^2]},$$

(2.28)

where we employ a splitting similar to that of equation (2.26), i.e.,

$$\Sigma = \begin{bmatrix} 0 & 0 \\ 0 & \sigma \end{bmatrix} + \Sigma'$$

and with \( H \in \text{Herm}_{Wick}^{(\beta)}(n|0) = \text{Herm}_{Wick}^{(\beta)}(n + \tilde{\gamma}(k_2 - k_1)|0) = \text{Herm}_{Wick}^{(\beta)}(n + \tilde{\gamma}(k_2 - k_1)|y k_2), \tilde{\sigma} \in \text{Herm}_{Wick}^{(\beta)}(y k_2|y k_2), \) and \( \sigma \in \text{Herm}_{Wick}^{(\beta)}(y k_1|y k_2). \) The second splitting becomes more important later on. Notice that we extended \( \hat{H} \rightarrow \Sigma \) in the probability density \( \tilde{P} \), only.

From now on we pursue the ideas of the standard supersymmetry method; see subsection 2.1. We introduce the same rectangular supermatrix \( V \) as in equation (2.10), i.e.,

$$\det^{-1/(\gamma \tilde{\gamma})}(H \otimes \mathbb{1}_{y k_1|y k_2} - \mathbb{1}_{y n} \otimes \kappa) = \frac{\int d[V] \exp[\text{str } V^\dagger V L - \text{str } V^\dagger H V L]}{\int d[V] \exp[-\text{str } V^\dagger V]}$$

(2.30)

In terms of \( \Sigma \), the partition function reads

$$Z(\kappa) = \frac{\int d[\Sigma] \tilde{P}(\Sigma) \int d[V] \exp[\text{str } V^\dagger V L - \text{str } \Sigma \tilde{\Lambda}]}{\int d[V] \exp[-\text{str } V^\dagger V]}$$

(2.31)

with

$$\tilde{\Lambda} = \begin{bmatrix} V L V^\dagger & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & V \sqrt{L} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \sqrt{L} V^\dagger \\ \sqrt{L} V^\dagger & 0 \end{bmatrix}$$

(2.32)

and \( \sqrt{L} \) the positive root of the diagonal elements of \( L \). The block structure of \( \tilde{\Lambda} \) corresponds to the first splitting of \( \Sigma \) in equation (2.29). The Fourier–Laplace transform

$$\tilde{\Phi}(\tilde{\Lambda}) = \int d[\Sigma] \tilde{P}(\Sigma) \exp[-\text{str } \Sigma \tilde{\Lambda}]$$

(2.33)

is assumed to exist such that we can interchange the integrals over \( \Sigma \) and \( V \). Employing the same symmetry arguments as in equation (2.13), we have

$$\tilde{\Phi}(\tilde{\Lambda}) = \tilde{\Phi}(\tilde{\Lambda}) \quad \text{with} \quad \tilde{\Lambda} = \begin{bmatrix} 0 & 0 \\ \sqrt{L} V^\dagger & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{L} V^\dagger \sqrt{L} \end{bmatrix}.$$ 

(2.34)

The block structure of \( \tilde{\Lambda} \) is that of the second splitting of \( \Sigma \) in equation (2.29). The advantage of equation (2.34), in comparison to equation (2.14), is that the superfunction \( \tilde{\Phi} \) is still the same, since \( \tilde{\Lambda} \) and \( \tilde{B} \) are in the same supermatrix set. Hence the inverse Fourier transform is still \( \tilde{P} \) and not some new superfunction.

The only technical difficulty grows from a non-trivial \( L \), because we cannot simply exchange the integrations over \( \Sigma \) and \( V \) again. To overcome this problem, we introduce an auxiliary supermatrix \( \sigma_{aux} \in \text{Herm}_{Wick}^{(\beta)}(y k_1|y k_2) \) drawn from a Gaussian distribution where the subscript ‘\( \prime \)’ denotes the standard Wick rotation [29–32] by the imaginary unit. This Gaussian models some kind of Dirac \( \delta \)-function, i.e. we can ‘simplify’ as follows:

$$\exp[-\text{str } \sqrt{L} V \sqrt{L} V^\dagger V] = \lim_{t \rightarrow 0} \frac{\int d[\sigma_{aux}] \exp[-\text{str } (\sigma_{aux} - \sqrt{L} V \sqrt{L} V^\dagger V)]}{\int d[\sigma_{aux}] \exp[-\text{str } \sigma_{aux}^2/2]}$$

(2.35)

where \( t/2 \) is the variance of the Gaussian distribution. Assuming that the integral of \( \tilde{P} \) multiplied with \( \exp[\text{str } \sigma^2] \) exists, we are allowed to interchange the integrations over \( \Sigma, V, \) and \( \sigma_{aux} \). We underline that the integrability of \( \tilde{P} \) with \( \exp[\text{str } \sigma^2] \) is a weak restriction which can be
lifted at the end of the consideration; for example a modification of \( P(H) \) to \( P(H) \exp[-\delta \operatorname{tr} H^4] \) \((\delta > 0)\) does the job, and we can take \( \delta \to 0 \) at the end.

After introducing \( \sigma_{\text{aux}} \), we interchange the integrals and integrate over \( V \) first. Shifting \( \sigma_{\text{aux}} \) by \( \sqrt{L} \sigma \sqrt{L} \), we can take the limit \( t \to 0 \). Finally the partition function takes the simple form

\[
Z(\kappa) = \frac{\int d[\Sigma] \tilde{P}(\Sigma) \operatorname{sdet}^{-n/\gamma}(\sigma - \kappa)}{\int d[\Sigma] \exp[-\operatorname{str} \Sigma^2]}. \tag{2.36}
\]

Notice that the superdeterminant only depends on \( \sigma \) and no longer on the ordinary matrix \( H \).

In the last step, we identify the superfunction \( Q \) by comparing the result (2.36) with the result of the generalized Hubbard–Stratonovich transformation (2.23), yielding the final result of this section which is the projection formula

\[
Q(\sigma) = \frac{\int d[\Sigma'] \tilde{P}\left(\begin{bmatrix} 0 & 0 \\ 0 & \sigma \end{bmatrix} + \Sigma'\right)}{\int d[\Sigma] \exp[-\operatorname{str} \Sigma^2]}. \tag{2.37}
\]

We integrate over different splittings of \( \Sigma \) in the numerator and the denominator. Recall the definition (2.29) of the matrices \( \Sigma \) and \( \Sigma' \). The superfunction \( \Phi \) in the superbosonization formula (2.19) can be obtained through the Fourier transformation (2.20) of \( Q \).

We underline that the projection formula also holds if the source \( \kappa \) is chosen non-diagonal, as sometime happens in QCD [48], or if we add an external operator \( H_0 \) to the original random matrix \( H \) often considered in transition ensembles [49, 50]. In both cases the integral (2.36) is slightly modified, but the fundamental functional relation (2.37) still remains the same.

The projection formula (2.37) has one big advantage which the results from the superbosonization formula (2.19) and the generalized Hubbard–Stratonovich transformation (2.23) are lacking: with the aid of the projection formula, one can study deformations of the probability weight in quite an elegant way. This is exactly the sort of advantage that we want to achieve for the chiral ensembles, too.

Finally, we emphasize that the projection formula (2.37), after extending \( P \) to \( \tilde{P} \), yields one of infinitely many probability weights in superspace corresponding to the same partition function in ordinary space (2.8). This ambiguity of the weight in superspace is well known [37]. Moreover, other extensions of \( P \) to superspace certainly result in other superfunctions \( Q \).

Thus an interesting mathematical question is: when varying over all possible extensions \( \tilde{P} \) of \( P \), do we get all possible probability weights \( Q \) in superspace obtained through the generalized Hubbard–Stratonovich transformation, agreeing with exactly the same partition functions in ordinary space?

### 3. The projection formula for chiral ensembles

The aim is to generalize the projection formula (2.37) to chiral ensembles. We introduce the chiral matrix

\[
H_\chi = \begin{bmatrix} 0 & W \\ W^\dagger & 0 \end{bmatrix}. \tag{3.1}
\]

where the matrix entries of \( W \) are real, complex, or quaternion independent random variables for \( \beta = 1, 2, 4 \), respectively. The chiral matrix \( H_\chi \) is related to the anti-Hermitian, chiral random matrix

\[
\mathcal{D} \longrightarrow D = \begin{bmatrix} 0 & W \\ -W^\dagger & 0 \end{bmatrix} = \gamma_5 H_\chi, \quad \text{with} \quad \gamma_5 = (\mathbb{I}_n, -\mathbb{I}_{n+1}) \tag{3.2}
\]

modelling the Euclidean Dirac operator \( \mathcal{D} \) in four dimensions [15–17]. The modulus of the index \( v \in \{-n, 1-n, 2-n, \ldots\} \) is equal to the number of generic zeros of \( H_\chi \) which can...
be identified with the topological charge in continuum theory. The random matrix \( W \) is drawn from the coset

\[
G^{(\beta)}(n; n + \nu) := U^{(\beta)}(2n + \nu)/[U^{(\beta)}(n) \times U^{(\beta)}(n + \nu)]
\]

(3.3)
distributed via \( P_\chi \) such that \( H_\chi \in \text{Herm}^{(\beta)}(2n + \nu) \). The probability density is assumed to be invariant under

\[
P_\chi(W) = P_\chi(UW), \quad \forall U \in U^{(\beta)}(n).
\]

(3.4)

Notice that we do not assume invariance under right transformations as well, which is usually the case \([17, 51]\). The reason is that we also want to study correlated random matrix ensembles, as they naturally appear in the analysis of one-sided correlated Wishart ensembles where the invariance is broken by an empirical correlation matrix; see \([2–4, 11–14]\).

Due to the invariance (3.4), we can reduce the functional dependence of \( P_\chi \) on \( W \) to one on \( WW^\dagger \). Thus there is a function \( P \) such that

\[
P_\chi(W) = P(W^\dagger W).
\]

(3.5)

Moreover, we assume that the chiral partition function,

\[
Z_\chi(\kappa) := \int d[W] P_\chi(H_\chi) \prod_{j=1}^{k_1} \det(H_\chi - \kappa_j^{(2)} I_{\gamma(2n+\nu)}) \prod_{j=1}^{k_2} \det(H_\chi - \kappa_j^{(1)} I_{\gamma(2n+\nu)})
\]

\[= \int d[W] P_\chi(H_\chi) \text{sdet}^{-1/(\nu \gamma)}(H_\chi \otimes I_{\gamma \gamma^k_1 | \gamma \gamma^k_2} - I_{\gamma(2n+\nu) \otimes \kappa}),
\]

(3.6)
can be reduced to one for \( WW^\dagger \) or/and \( W^\dagger W \),

\[
Z_\chi(\kappa) = (-1)^{\nu(n+\nu)(k_2-k_1)} \text{sdet}^{-\nu/(\nu \gamma)} \times \int d[W] P(W^\dagger W) \text{sdet}^{-1/(\nu \gamma)}(WW^\dagger \otimes I_{\gamma \gamma^k_1 | \gamma \gamma^k_2} - I_{\gamma \otimes \kappa^2}),
\]

\[= (-1)^{\nu(n+\nu)(k_2-k_1)} \text{sdet}^{-\nu/(\nu \gamma)} \times \int d[W] P(W^\dagger W) \text{sdet}^{-1/(\nu \gamma)}(W^\dagger W \otimes I_{\gamma \gamma^k_1 | \gamma \gamma^k_2} - I_{\gamma(n+\nu) \otimes \kappa^2}).
\]

(3.7)

One has to understand that those partition functions do not cover all interesting spectral correlation functions. For example, QCD with finite chemical potential or/and finite temperature cannot be modelled with this restriction; cf \([17, 52–54]\). For those partition functions, the approach of a projection formula can be modified. Unluckily, this modified approach only works for the case \( \beta = 2 \). We will elaborate on this problem further in a forthcoming publication \([55]\).

To make contact with the projection formula (2.37) for the original ensembles in Dyson’s threefold way, we notice that the second representation of the partition function in equation (3.7) can be expressed in terms of an integral over \( H \in \text{Herm}^{(\beta)}(n) \) if \( \nu \leq 0 \):

\[
Z_\chi(\kappa) \propto \text{sdet}^{\nu/(\nu \gamma)} \int d[H] \Theta(H) \text{det}^{\nu/(\nu \gamma + 1)} HP(H) \times \text{sdet}^{-1/(\nu \gamma)}(H \otimes I_{\gamma \gamma^k_1 | \gamma \gamma^k_2} - I_{\gamma(n+\nu) \otimes \kappa^2}),
\]

(3.8)

with the matrix version of the Heaviside \( \Theta \) function. It is unity if \( H \) is positive definite and otherwise vanishes. Apart from the similarity of equation (3.8) with equation (2.8), identifying \( \Theta(H) \text{det}^{\nu/(\nu \gamma + 1)} HP(H) \) as the new probability density, the crucial differences are the non-isotropy of \( P \), i.e. equation (2.9) does not necessarily apply, and the Heaviside \( \Theta \) function,
which is far from being smooth. Thus the original projection formula (2.37) is no longer applicable.

In subsection 3.1, we pursue an idea similar to that presented in subsection 2.2 in order to find a projection formula for partition functions of the form (3.7). This formula is simplified via a combination with the superbosonization formula in subsection 3.2.

3.1. The projection formula

The key idea for deriving a projection formula is again to apply one of the Cauchy-like integration theorems for supermatrices first derived by Wegner [44]; see also [45–47]. This time we need a Cauchy-like integration theorem for extending the set of rectangular matrices $GL^{(β)}(n; n + v)$ to a space of rectangular supermatrices, which is the coset

$$GL^{(β)}(n + γl| γl; n + v) := U^{(β)}(2n + v + γl| γl)/[U^{(β)}(n + γl| γl) × U^{(β)}(n + v)]$$

(3.9)

with $l \in \mathbb{N}$.

Let $p_1$, $p_2$, $q$, $l$ ∈ $\mathbb{N}_0$. We split a rectangular $(p_1 + γl| q + γl) \times p_2$ supermatrix $Ω$ in the following way:

$$Ω = \left[\begin{array}{c}
\tilde{Ω} \\
\hat{Ω}
\end{array}\right] \in GL^{(β)}(p_1 + γl| q + γl; p_2)$$

(3.10)

with $\tilde{Ω} \in GL^{(β)}(p_1| q; p_2)$ and $\hat{Ω} \in GL^{(β)}(γl| γl; p_2)$. Assuming a smooth superfunction $f$ integrable on the set $GL^{(β)}(p_1 + γl| q + γl; p_2)$ and invariant under $f(Ω) = f(UΩ)$, $∀ U ∈ U^{(β)}(p_1 + γl| q + γl)$ and $Ω ∈ GL^{(β)}(p_1 + γl| q + γl; p_2)$,

(3.11)

the Cauchy-like integration theorem for rectangular supermatrices [44–47] reads

$$\frac{\int dΩ f(Ω)}{\int dΩ \exp[-\text{tr} Ω^TΩ]} = f\left(\begin{bmatrix} \tilde{Ω} \\ 0 \end{bmatrix}\right).$$

(3.12)

We notice that no Wick rotation is needed for this theorem, in contrast to equation (2.27), simplifying the derivation by getting rid of one technical detail.

We apply the identity (3.12) to the partition function

$$Z_γ(κ) = (-1)^{γ(n + v)(k_2 - k_1)} \text{sdet}^{-1/γγ} \times \int d[W] P(W^T W) \text{sdet}^{-1/γγ}(WW^T ⊗ \mathbb{I}_{γγk_1 | γγk_2} - \mathbb{I}_{γγ} ⊗ \kappa^2).$$

(3.13)

We have chosen the first version of equation (3.7); the reason for this choice becomes clearer later on. The product $W^T W$ and, hence, the function $P(W^T W)$ are obviously invariant under left multiplication of $W$ with unitary matrices and can, thus, generally be extended to $Ω^T Ω$ and $P(Ω^T Ω)$, respectively, by the integration theorem (3.12). The only thing that we assume is that $P(Ω^T Ω)$ has to be smooth and integrable on $GL^{(β)}(n + γk_2| γk_2; n + v)$, where we again restrict ourselves to the case $γk_1 ≤ γk_2 + n$. The other, usually less interesting, case $γk_1 ≥ γk_2 + n$ can be derived in a slightly modified discussion.

In the first step we apply the Cauchy-like integration theorem to the partition function to extend the integral over the ordinary space $GL^{(β)}(n; n + v)$ to an integral over the superspace $GL^{(β)}(n + γk_2| γk_2; n + v)$, i.e.,

$$Z_γ(κ) = (-1)^{γ(n + v)(k_2 - k_1)} \text{sdet}^{-1/γγ} \times \frac{\int d[Ω] P(Ω^T Ω) \text{sdet}^{-1/γγ}(WW^T ⊗ \mathbb{I}_{γγk_1 | γγk_2} - \mathbb{I}_{γγ} ⊗ \kappa^2)}{\int d[Ω] \exp[-\text{tr} Ω^T Ω]},$$

(3.14)
where we employ the following splitting of the rectangular supermatrix:

\[ \Omega = \begin{bmatrix} W \Omega \\ \Omega \end{bmatrix} = \begin{bmatrix} W \\ \Omega \end{bmatrix} \]

(3.15)

with \( W \in \text{Gl}(\beta)(n; n + v) \), \( W' \in \text{Gl}(\beta)(n + \gamma k_2 - \gamma k_1; n + v) \), \( \Omega \in \text{Gl}(\beta)(\gamma k_2; n + v) \), and \( \Omega' \in \text{Gl}(\beta)(\gamma k_1; n + v) \). The second splitting corresponds to the embedding of the superspace that we aim at.

Let \( \hat{L} = \text{sign Im} \kappa^2 \) be the sign of the squared source variables arrayed on a diagonal matrix. In the next step of our approach, we introduce Gaussian integrals over exactly the same rectangular supermatrix \( V \) as in equation (2.10), yielding

\[ Z_\chi(\kappa) = (-1)^\gamma(n + v)(k_2 - k_1) \text{sdet}^{-\gamma/2} \kappa \times \frac{\int d[\Omega] P(\Omega' \Omega) \int d[V] \exp[\text{str} V' V \hat{L} \kappa^2 - 1 \text{str} \Omega' \Omega \hat{A}]}{\sqrt{\nu n(k_2 - k_1)} \text{sdet}^{-n/2} \hat{L} \int d[V] \exp[-\text{str} V' V] \int d[\Omega] \exp[-\text{tr} \Omega' \Omega]} \]

(3.16)

with

\[ \hat{A} = \begin{bmatrix} \sqrt{L} \hat{V}^+ & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \]

(3.17)

cf equations (2.31) and (2.32). The dyadic matrix \( \hat{A} \) has again a dual matrix

\[ \hat{B} = \begin{bmatrix} 0 & 0 \\ \sqrt{L} \hat{V}^+ & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{L} \hat{V}^+ V \sqrt{L} \end{bmatrix} \]

(3.18)

cf equation (2.34). Interchanging the integrals over \( \Omega \) and \( V \) in equation (3.16), we arrive at the following integral transform of \( P \):

\[ \Psi(\hat{A}) = \int d[\Omega] P(\Omega' \Omega) \exp[-1 \text{str} \Omega' \Omega \hat{A}], \]

(3.19)

which plays the role of the Fourier–Laplace transform (2.33) in the case of Dyson’s threefold way. Now the invariance of \( \Omega \) under multiplication from the left with unitary supermatrices enters, implying

\[ \Psi(\hat{U} \hat{A} \hat{U}^{-1}) = \Psi(\hat{A}), \quad \forall \hat{U} \in \text{U}(\beta)(n + \gamma k_2 | \gamma k_2). \]

(3.20)

Hence, the following identity is true:

\[ \Psi(\hat{A}) = \Psi(\hat{B}), \]

(3.21)

connecting the ordinary matrix space with the superspace. This identity is remarkable, as it relates the two spaces with one and the same superfunction \( \Psi \). We notice that the supermatrices \( \hat{A} \) and \( \hat{B} \) are of the same size, corresponding to the first and second splittings of equation (3.15), respectively, while their non-zero blocks are not.

The duality relation (3.21) can be plugged into the partition function, which reads

\[ Z_\chi(\kappa) = (-1)^\gamma(n + v)(k_2 - k_1) \text{sdet}^{-\gamma/2} \kappa \times \frac{\int d[\Omega] P(\Omega' \Omega) \exp[\text{str} V' V \hat{L} \kappa^2 - 1 \text{str} \Omega' \Omega \hat{B}]}{\sqrt{\nu n(k_2 - k_1)} \text{sdet}^{-n/2} \hat{L} \int d[V] \exp[-\text{str} V' V] \int d[\Omega] \exp[-\text{tr} \Omega' \Omega]} \]

(3.22)

Due to the convergence of the integrals, we can again not easily switch the integration of \( \Omega \) and \( V \) unless the boson–boson block of \( \hat{L} \) is proportional to the identity. However, this problem can be circumvented, as was discussed in subsection 2.2, by introducing an auxiliary Hermitian supermatrix. We skip this here, because it is exactly the same procedure as was explained in subsection 2.2. Hence we end up with the partition function

\[ Z_\chi(\kappa) = (-1)^\gamma(n + v)(k_2 - k_1) \text{sdet}^{-\gamma/2} \kappa \frac{\int d[\Omega] P(\Omega' \Omega) \text{sdet}^{-n/2} \Omega' \Omega \hat{B}^+ - \kappa^2 \int d[\Omega] \exp[-\text{tr} \Omega' \Omega]}{\int d[\Omega] \exp[-\text{tr} \Omega' \Omega]} \]

(3.23)
which is one of the main results of this section. We emphasize a few things about this formula. The supermatrices $\Omega'$ in the numerator and $\Omega$ in the denominator have different sizes; see the splittings (3.15). Moreover, the index $\nu$ can take negative values as well, since we have not at any point used an assumption like $WW^\dagger$ being smaller than $W^\dagger W$. Equation (3.23) can be slightly modified such that the supermatrix $\kappa$ can be easily assumed to be non-diagonal—e.g. in QCD you need a non-diagonal $\kappa$ to generate mixed pion condensates [48]—or we can think of a symmetry breaking term in the determinant of equation (3.14), which may happen through circumventing the problem of a two-sided correlated Wishart ensemble, such as appears for modelling spatial–time correlation matrices [56–58]; see subsection 4.1.

The superdeterminant in equation (3.23) only depends on the product $\Omega'\Omega^\dagger$. Therefore the integral over $W$ defines a new probability distribution $\tilde{Q}$ on the superspace $\text{SU}(\beta)(\gamma k_1|\gamma k_2; n + \nu)$, i.e.,

$$\tilde{Q}(\Omega'\Omega^\dagger) = \frac{\int d[W] P(\Omega'\Omega)}{\int d[\Omega] \exp[-\text{tr} \Omega^\dagger \Omega]} = \frac{\int d[W] P(W^\dagger W + \Omega^\dagger \Omega)}{\int d[\Omega] \exp[-\text{tr} \Omega^\dagger \Omega]}, \quad \text{(3.24)}$$

Notice that there is one crucial disadvantage of this projection formula as compared to the one for Dyson’s threefold way; cf equation (2.37): the superfunction $\tilde{Q}$ is still a function depending on a matrix $\Omega'\Omega^\dagger$ with ordinary dimensions. It is easy to get rid of this flaw if the original probability density $P$ is also invariant under right multiplication of $W$. Such a restriction becomes a problem for two-sided correlated Wishart matrices. For a one-sided correlated Wishart matrix ensemble we can circumvent this problem; see subsection 4.1.

3.2. Rotation-invariant probability densities

In this subsection we further simplify the projection formula by assuming that the probability density $P$ is rotation invariant, i.e.,

$$P(W^\dagger W) = P(\tilde{U}W^\dagger W\tilde{U}^{-1}), \quad \forall \tilde{U} \in \text{U}^{(\beta)}(n + \nu) \text{ and } W \in \text{SU}(\beta)(n; n + \nu). \quad \text{(3.25)}$$

Then this invariance is obviously true, as seen by making the replacement $W \to \Omega$, too. Therefore there is certainly a supersymmetric extension of $P$ denoted by $\tilde{P}$ with

$$P(W^\dagger W + \Omega^\dagger \Omega) = \tilde{P} \left( \begin{array}{c} W^\dagger W \\ \Omega^\dagger \Omega \end{array} \right). \quad \text{(3.26)}$$

The reason is that we can write $P$ in terms of matrix invariants like traces, which is also a source of ambiguity when extending $P$ to the superspace [37].

For further calculations, we assume $\nu \geq 0$, which becomes important for the convergence of some integrals. Because of the invariance under independent left and right multiplication of $W$ with unitary matrices, this is not a restriction at all. One can simply choose $W$ such that it has the smaller dimension $n$ on its left side.

Since the integral (3.24) is invariant under the transformation $\Omega'\Omega^\dagger \to \tilde{U}\Omega'\Omega^\dagger \tilde{U}^{-1}$ for all $\tilde{U} \in \text{U}^{(\beta)}(n)$, too, we can define a probability density on the superspace:

$$Q(\Omega'\Omega^\dagger) = \frac{\int d[W] \tilde{P} \left( \begin{array}{c} W^\dagger W \\ \Omega^\dagger \Omega \end{array} \right)}{\int d[\Omega] \exp[-\text{tr} \Omega^\dagger \Omega]}, \quad \text{(3.27)}$$

The crucial difference of equations (3.24) and (3.27) is that $Q$, in contrast to $\tilde{Q}$, depends on a $(\gamma \tilde{k}_1|\gamma \tilde{k}_2) \times (\gamma \tilde{k}_1|\gamma \tilde{k}_2)$ supermatrix. Thus, there is a chance to get rid of a number of integration variables which scale with $n$. This is quite important when taking the limit of large matrices, as is the case when deriving the universal behaviour of the spectrum of $H_f$. 


The aim is to express the integral (3.27) in terms of the combination \( \Omega^\dagger \Omega^\dagger \) and some integration variables. For this purpose we introduce Dirac \( \delta \)-functions for the blocks depending on \( W^\dagger \):

\[
Q(\Omega^\dagger \Omega^\dagger) \propto \int d[W^\dagger] \int d[H_1] \int d[H_2] \int d[W_1] \int d[W_2] \tilde{P} \left( \begin{bmatrix} H_1 & W_1 \\ \Omega^\dagger & \Omega^\dagger \end{bmatrix} \right) \times \exp[\text{tr}(H_1 - W^\dagger W^\dagger)(H_2 + I_{(n+\tilde{\gamma}(k_2-k_1))})] \\
\times \exp[\text{tr}(W_1 - W^\dagger W^\dagger)W^\dagger_2 + 1 \text{ tr } W_2(W^\dagger_1 - \Omega^\dagger W^\dagger)].
\]

We drop the normalization constant for now and introduce it later on by fixing it with the Gaussian case. The matrices are drawn from \( H_1, H_2 \in \text{Herm}^{(\beta)}(n + \tilde{\gamma}(k_2 - k_1)) \) and \( W_1^\dagger, W_2^\dagger \in \text{Gl}^{(\beta)}(\tilde{\gamma} k_1 \mid \gamma k_2; n + \tilde{\gamma}(k_2-k_1)) \). Recall the definition of the cosets and the splitting of \( \Omega \) in equations (2.6), (3.9) and (3.15), respectively. The shift in \( H_2 \) guarantees the convergence of the integration over \( W^\dagger \), which is the first one that we perform, yielding

\[
Q(\Omega^\dagger \Omega^\dagger) \propto \lim_{\delta \to 0} \int d[H_1] \int d[H_2] \int d[W_1] \int d[W_2] \tilde{P} \left( \begin{bmatrix} H_1 & W_1 \\ \Omega^\dagger & \Omega^\dagger \end{bmatrix} \right) \times \exp[\text{tr} H_1((H_2 + I_{(n+\tilde{\gamma}(k_2-k_1))}) + 1 \text{ tr } W_1 W^\dagger_2 + 1 \text{ tr } W_2 W^\dagger_2)] \\
\times \exp[-\text{tr}(U_2 + I_{(n+\tilde{\gamma}(k_2-k_1))} - 1 W_2(\Omega^\dagger \Omega^\dagger + \delta I_{\tilde{\gamma} k_1 \mid \gamma k_2}) W^\dagger_2)] \\
\times \text{det}^{-((\nu + \tilde{\gamma})k_2 - k_1)}((H_2 + I_{(n+\tilde{\gamma}(k_2-k_1))}).
\]

The variable \( \delta \) is a regularization guaranteeing the convergence of the integrals, since \( \Omega^\dagger \Omega^\dagger \) is not invertible if it contains a fermion–fermion block, i.e. \( k_2 \neq 0 \). We perform the rescaling \( W_1 \to W_1 \sqrt{\Omega^\dagger \Omega^\dagger + \delta I_{\tilde{\gamma} k_1 \mid \gamma k_2}} \) and \( W_2 \to W_2/\sqrt{\Omega^\dagger \Omega^\dagger + \delta I_{\tilde{\gamma} k_1 \mid \gamma k_2}} \). The Jacobians of the transformations \( W_1 \) and \( W_2 \) cancel out and the limit of the regulator \( \delta \to 0 \) can be made exact.

The next integration that we perform is over \( W_2 \) and we find

\[
Q(\Omega^\dagger \Omega^\dagger) \propto \int d[H_1] \int d[H_2] \int d[W_1] \tilde{P} \left( \begin{bmatrix} H_1 \\ \sqrt{\Omega^\dagger \Omega^\dagger} W^\dagger_1 \end{bmatrix} \begin{bmatrix} W_1 \sqrt{\Omega^\dagger \Omega^\dagger} W^\dagger_1 \\ \Omega^\dagger \Omega^\dagger \end{bmatrix} \right) \times \exp[\text{tr}(H_1 - W_1 W^\dagger_1)(H_2 + I_{(n+\tilde{\gamma}(k_2-k_1))})] \\
\times \text{det}^{-((\nu + \tilde{\gamma})_k_2 - k_1)}((H_2 + I_{(n+\tilde{\gamma}(k_2-k_1))}).
\]

We notice that \( \tilde{P} \) depends on invariants only. Hence in an explicit representation of \( \tilde{P} \) we do not encounter the ill-defined matrix \( \sqrt{\Omega^\dagger \Omega^\dagger} \), but only the supermatrix \( \Omega^\dagger \Omega^\dagger \). The remaining integral over \( H_2 \) is an ordinary Ingham–Siegel integral [59, 60]. Making the shift \( H_1 \to H_1 + W_1 W^\dagger_1 \), the Ingham–Siegel integral tells us that \( H_1 \) has to be positive definite and yields a determinant of \( H_1 \) to the power \( \nu/\tilde{\gamma} + (\gamma - \tilde{\gamma})/2 \) (here we need \( \nu \geq 0 \)). The positivity constraint of \( H_1 \) is quite often hard to handle, so we replace \( H_1 \) by a rectangular matrix \( \tilde{W}_1 \in \text{Gl}^{(\beta)}(n + \tilde{\gamma}(k_2 - k_1); n + \nu + \tilde{\gamma}(k_2 - k_1)) \). Finally, we arrive at the main result of this section and the projection formula for rotation-invariant chiral ensembles:

\[
Q(\Omega^\dagger \Omega^\dagger) = C \int d[\tilde{W}_1] \int d[W_1] \tilde{P} \left( \begin{bmatrix} \tilde{W}_1 & \tilde{W}_1 \\ \sqrt{\Omega^\dagger \Omega^\dagger} W^\dagger_1 \end{bmatrix} \begin{bmatrix} \tilde{W}_1 W^\dagger_1 + W_1 W^\dagger_1 \\ \sqrt{\Omega^\dagger \Omega^\dagger} W^\dagger_1 \end{bmatrix} \right)
\]

with the normalization constant

\[
C = \frac{\int d[W^\dagger] \exp[-\text{tr } W^\dagger W^\dagger]}{\int d[W_1] \exp[-\text{tr } W_1 W^\dagger_1] \int d[W_1] \exp[-\text{tr } W_1 W^\dagger_1] \int d[\Omega] \exp[-\text{tr } \Omega^\dagger \Omega^\dagger]}
\]

The reason for fixing the normalization with Gaussian weights lies in the universality of the projection formula (3.31). The projection formula holds true for almost all ensembles depending on invariants of the rectangular matrix \( W \). Due to this broad applicability,
equation (3.31) is a powerful tool. In section 4, we will present some examples, often encountered in different fields of random matrix theory.

Additionally one can apply the superbosonization formula to the partition function
\[
Z_\chi(\kappa) = (-1)^{\nu(n+n)k_2k_1} \int d[\Omega] Q(\Omega, \Omega^\dagger) sdet^{-\nu/\gamma} \kappa \int d[\Omega'] \exp[-\str \Omega' \Omega'^\dagger] \, sdet^{-\nu/\gamma} \kappa, 
\]
which is justified since the whole integral depends on the dyadic supermatrix \( \Omega, \Omega^\dagger \). Thus we replace \( \Omega, \Omega^\dagger \) by the supermatrix \( \hat{U} \in \text{Herm}^{(\mathbb{P})}_{2}(\gamma k_1|\gamma k_2) \) which has the same structure as the supermatrix \( U \) in the original approach of the superbosonization formula (2.19). The partition function reads
\[
Z_\chi(\kappa) = (-1)^{\nu(n+n)k_2k_1} \int d[\Omega] \exp[-\str \Omega' \Omega'^\dagger] \, \int d[\Omega'] \exp[-\str \hat{U}] \, \int d[\hat{U}] \, Q(\hat{U}) \, sdet^{-\nu/\gamma}(\hat{U} - \kappa^2) \, sdet^{(n+n)/\gamma} \hat{U}, 
\]
with the superfunction
\[
Q(\hat{U}) = C \int d[\hat{W}] d[\hat{W}_1] \hat{F} \left( \begin{array}{ccc} \hat{W}_1 \hat{W}_1^\dagger + W_1W_1^\dagger & W_1 \sqrt{\hat{U}} \\ \sqrt{\hat{U}}W_1^\dagger & \hat{U} \end{array} \right). 
\]

Importantly, one should not confuse the superfunction \( \Phi \) of equation (2.19) with the superfunction \( Q \); we note the different terms in the integrands. The prefactor in equation (3.34) is the global normalization constant resulting from the superbosonization formula and strongly depends on the normalization of the Haar measure \( d\mu(\hat{U}) \) of the supersymmetric coset \( \text{Herm}^{(\mathbb{P})}_{2}(\gamma k_1|\gamma k_2) \).

4. Some examples

We apply the projection formula (3.35) to four non-trivial examples to illustrate how our approach works. In particular, it becomes clear what the advantages of the projection formula (3.35) are in comparison to the standard approaches with the generalized Hubbard–Stratonovich transformation [36, 37] and the superbosonization formula [38, 39].

In particular, we discuss norm-dependent ensembles and correlated Wishart ensembles in subsection 4.1, Lorentz-like (Cauchy) ensembles in subsection 4.2, the three unquenched chiral Gaussian ensembles in subsection 4.3, and a probability density with a quartic potential in subsection 4.4. The norm-dependent ensembles serve as a check since they can readily be calculated with the previous variants of the supersymmetry method. With the help of the correlated Wishart ensembles, we show that the projection formula can easily be extended to include a symmetry breaking constant term in the determinants; cf equation (3.7). The Lorentz-like (Cauchy) weight is another standard probability density, like the Gaussian weight. It has a particular property, namely it exhibits heavy tails, and thus not all moments exist. For the unquenched chiral Gaussian ensemble, we derive an alternative representation of the chiral Lagrangian; see [15–17] for the common representation. In this representation the physical mesons are split off from the artificial ones which result from introducing source terms to generate the desired observables. With the help of the quartic potential, we want to show that one can also study non-trivial potentials via the projection formula (3.35).

4.1. Norm-dependent ensembles and correlated Wishart ensembles

The first class of ensembles that we want to look at are the norm-dependent chiral ensembles [51, 61], i.e.,
\[
P(W^\dagger W) = p(\text{tr} W^\dagger W) 
\]
with an integrable function \( p \). A particular choice is a fixed-trace ensemble, namely \( p(\text{tr} W^1 W) \propto \delta(\text{tr} W^1 W - c n) \) with a constant \( c > 0 \). Such an ensemble naturally appears when modelling lattice QCD [62]. The lattice QCD Dirac operator is built up of unitary matrices and fulfills a fixed-trace condition. However, one can readily show that this condition has only a minor effect on the microscopic regime of the Dirac spectrum and is completely suppressed in the exact limit [62]. The choice \( p \propto \delta(\text{tr} W^1 W - c n) \) only enhances the \( 1/n \) correction. Also, in quantum information it plays an important role [63], since the density operator is normalized.

The corresponding superfunction of the probability density \( P \) for an arbitrary \( p \) can be simply read off from the projection formula (3.35) and is up to a constant

\[
Q(\hat{U}) \propto \int_0^\infty dr \, r(\text{str} \, \hat{U})^r \nu/(\gamma \nu + 1) (\text{tr} \, \hat{U}^2 - \kappa^2) \exp(-n \text{tr} \, \hat{U}).
\] (4.2)

The exponent of the integration variable \( r \) is the difference of the number of commuting real variables and that of anti-commuting Grassmann variables in the rectangular matrices \( W_1 \) and \( \hat{W}_1 \). Those matrices are of dimension \((\gamma \nu k_1 | \gamma \nu k_2) \times (\gamma n + \gamma \nu (k_2 - k_1))\) and \((\gamma n + \gamma \nu (k_2 - k_1)) \times (\gamma (n + \nu) + \gamma \nu (k_2 - k_1))\), respectively, and fulfill certain symmetries like equation (2.11).

A natural representative of a norm-dependent ensemble is the Gaussian one, i.e. \( p(\text{tr} W^1 W) \propto \exp(-n \text{tr} W^1 W) \). Then the integral over \( r \) factorizes in equation (4.2). This apparently yields again a Gaussian

\[
Q(\Omega, \Omega'^\dagger) \propto \exp\left(-\frac{n}{\gamma} \text{str} \, \Omega \Omega'^\dagger \right)
\] (4.3)

in terms of the dyadic supermatrix \( \Omega \Omega'^\dagger \) and reads, in terms of the supermatrix \( \hat{U} \in \text{Herm}_n(\gamma) \),

\[
Q(\hat{U}) \propto \exp\left(-\frac{n}{\gamma} \text{str} \, \hat{U} \right).
\] (4.4)

For a Gaussian weight, this result is not surprising, but it serves as a simple check for the projection formula (3.35). On plugging equation (4.4) into the partition function (3.34), we arrive at

\[
Z_\nu(\kappa) \propto \text{sdet}^{-n/\gamma} \int d\mu(\hat{U}) \exp(-n \text{str} \, \hat{U}) \text{sdet}^{-n/\gamma} (\hat{U} - \kappa^2) \text{sdet}^{n/\gamma} \hat{U}. \] (4.5)

The microscopic limit \((n \to \infty) \text{ while } \nu \text{ and } n \kappa \text{ are fixed}) \text{ connects chiral random matrix theory with QCD [17]} \text{ and is obtained from our expression via the rescaling } \hat{U} \to -\kappa \hat{U}. \text{ After taking the limit } n \to \infty, \text{ we find the well-known chiral Lagrangian [17]}

\[
Z_\nu(\kappa) \propto 1 \int d\mu(\hat{U}) \exp\left(\frac{n}{\gamma} \text{str} \, \hat{U} + U^{-1}\right) \text{sdet}^{\gamma/\gamma} \hat{U}. \] (4.6)

Surprisingly, we did not have to adopt any saddle-point approximation with our approach, which is usually the case in the other approaches of the supersymmetry method [15, 16]. The reason is that the projection formula already mapped the ordinary space to the correct coset describing the mesons of the chiral Lagrangian in QCD.

Other applications of norm-dependent ensembles are correlated Wishart matrices with a non-Gaussian weight. In section 3 we claimed that we can also study one-sided correlated Wishart ensembles with arbitrary weight. Those ensembles appear in many situations where one encounters time series analysis, like in finance [11, 12], telecommunications [21], etc. Thus we consider the following partition function:

\[
Z_\nu(\kappa) = (-1)^{\nu(n + \nu)(k_2 - k_1)} \text{sdet}^{\nu/\gamma} \kappa \times \int d[W] \, p(\text{tr} W^1 W) \text{sdet}^{-1/(\gamma \nu)} \left(W W^\dagger \otimes I_{\gamma \nu k_1 | \gamma \nu k_2} - I_{\gamma \nu} \otimes \kappa^2 \right). \] (4.7)
where the function \( p \) is, as before, arbitrary and \( C \) is an empirical correlation matrix and thus positive definite. In the first step we rescale \( W \to \sqrt{CW} \) and have

\[
Z_\gamma(\kappa) = (-1)^{p(p+n+\nu+|k_2-k_1|)}\frac{\text{sdet}^{-\nu/2}k\text{det}^{(n+\nu)/2+(k_2-k_1)}C}{\sqrt{\det(CW)}} \times \int d[W] \ p(\text{tr} \ W^\dagger W) \text{sdet}^{-1/(2\gamma)}(WW^\dagger \otimes I_{\nu+\nu_1+\nu_2} - C^{-1} \otimes \kappa^2). \tag{4.8}
\]

In the second step, we apply the projection formula (3.35) in combination with a slightly modified version of equation (3.34) and find

\[
Z_\gamma(\kappa) \propto \frac{\text{sdet}^{-\nu/2}k\text{det}^{(n+\nu)/2+(k_2-k_1)}C}{\sqrt{\det(CW)}} \times \int d[\tilde{U}] \ Q(\tilde{U}) \text{sdet}^{-1/(2\gamma)}(I_{\nu+\nu_1+\nu_2} - C^{-1} \otimes \kappa^2) \det^{(n+\nu)/2} \tilde{U}. \tag{4.9}
\]

The superfunction \( Q \) is the one from the onefold integral (4.2). For the case of a Gaussian weight the one-point correlation function has already been studied with the help of supersymmetry for \( \beta = 1, 2 \); see [13, 14]. Equation (4.9) is an alternative compact representation of this partition function.

### 4.2. Lorentz (Cauchy)-like ensembles

Another kind of probability density serving as a ‘standard candle’ in statistical physics is the Lorentz weight. In contrast to the Gaussian weight, almost all moments of the matrix \( W \) do not exist for the Lorentzian. In random matrix theory, one introduces this weight with a constant \( \Gamma \in \mathbb{R}_+ \) determining the width of the distribution and an exponent \( \mu \in \mathbb{N} \) indicating how rapidly the tails fall off, i.e. the Lorentzian ensemble is given by

\[
P(W^\dagger W) \propto \det^{-\mu}(\Gamma^2 I_{\nu+\nu_1+\nu_2} + W^\dagger W). \tag{4.10}
\]

The exponent \( \mu \) has to be large enough to guarantee the normalizability of the probability density. This ensemble is also known as the Cauchy ensemble [64, 65]. Of particular interest is its heavy-tailed behaviour, which has not been studied in such detail as the exponential cut-off from ensembles with polynomial potentials. Importantly, one can expect that the universal results may break down. Recent works on heavy tails of random matrices include [66–68] and references therein.

Again we are interested in the supersymmetric analogue of \( P \) which is given via the projection formula (3.35)

\[
Q(\tilde{U}) \propto \int d[\tilde{W}_1] \int d[W_1] \text{sdet}^{-\mu} \left( \Gamma^2 I_{\nu+\nu_1+\nu_2} + \left[ \tilde{W}_1 \tilde{W}_1^\dagger + W_1 W_1^\dagger \frac{W_1 \sqrt{\tilde{U}}}{\sqrt{\det(CW)}} \right] \right)
\]

\[
= \text{sdet}^{-\mu}(\Gamma^2 I_{\nu+\nu_1+\nu_2} + \tilde{U}) \int d[\tilde{W}_1] \int d[W_1] \times \text{sdet}^{-\mu}(\Gamma^2 I_{\nu+\nu_1+\nu_2} + \tilde{W}_1 \tilde{W}_1^\dagger + \Gamma^2 W_1 (\Gamma^2 I_{\nu+\nu_1+\nu_2} + \tilde{U})^{-1} W_1^\dagger). \tag{4.11}
\]

In the second line we pulled out the lower right block of the superdeterminant. Here, we once more observe that one can often calculate with the superdeterminant as if it were a determinant; see [69]. After performing the rescaling \( W_1 \to W_1 (\Gamma^2 I_{\nu+\nu_1+\nu_2} + \tilde{U})^{1/2} \), the integrals over \( \tilde{W}_1 \) and \( W_1 \) factorize and yield a constant. The projection formula leads to the superfunction (up to a normalization constant)

\[
Q(\tilde{U}) \propto \text{sdet}^{\nu/2+(k_2-k_1)-\mu}(\Gamma^2 I_{\nu+\nu_1+\nu_2} + \tilde{U}). \tag{4.12}
\]

Thus the counterpart of the Lorentzian weight (4.10) is also Lorentzian in superspace. Only the exponent changes. Notice that the fermion–fermion block of \( \tilde{U} \) is a compact integral, such
that we do not have any problems of convergence if $n/\gamma + (k_2 - k_1) - \mu \leq 0$. The exponent $\mu$ has only to be large enough that the corresponding partition function,

$$Z_\chi(\kappa) \propto \text{sdet}^{-\nu/\gamma} \kappa \int d\mu(\hat{U}) \text{sdet}^{n/\gamma + (k_2 - k_1) - \mu} (\Gamma^2 \mathbb{1}_{\gamma k_1 \gamma k_2} + \hat{U}) \times \text{sdet}^{-n/\gamma} (\hat{U} - \kappa^2) \text{sdet}^{(n + \nu)/\gamma} \hat{U},$$  

(4.13)

exists, namely it has to be larger than $\mu > (n + \nu)/\gamma$ for this integral. To guarantee the integral of the partition function in ordinary space, the exponent has to fulfill $\mu > (n + \nu)/\gamma + k_2 - k_1$. Therefore one has only to take $\mu > (n + \nu)/\gamma + \max\{0, k_2 - k_1\}$ to guarantee the convergence of both integrals.

Interestingly, from equation (4.13) it immediately follows that in the microscopic limit $n \to \infty$ ($\nu, \eta \to 2$, and $n\kappa$ fixed) for $\mu/\gamma + \tilde{\mu}$ with $\tilde{\mu}$ fixed, we do not find the universal result (4.6). We already expected that something may change, i.e. the partition function becomes

$$Z_\chi(\kappa) \propto \int d\mu(\hat{U}) \text{sdet}^{(k_2 - k_1) - \tilde{\mu}} (\eta \Gamma^2 \mathbb{1}_{\gamma k_1 \gamma k_2} + n\kappa \hat{U}) \text{sdet}^{\nu/\gamma} \hat{U} \exp \left[ \frac{n}{\gamma} \text{str} \kappa \hat{U}^{-1} \right].$$  

(4.14)

However one can find the universal result at the hard edge of the spectrum, as the microscopic limit is also known, if $\tilde{\mu}/n$ and $\Gamma^2$ are fixed instead.

4.3. The unquenched chiral Gaussian ensemble

The unquenched partition function is in QCD a statistical weight, where additionally to the gauge action we have an interaction with fermionic quarks [17]. They are equivalent with

$$P(\tilde{W}) = \exp(-n \text{tr} \tilde{W}^\dagger \tilde{W} / \gamma) \prod_{j=1}^{N_f} \text{det}(\tilde{W}^\dagger \tilde{W} + m_j^2 \mathbb{1}_{\gamma (n + \nu)})$$  

(4.15)

with the quark masses $m = \text{diag}(m_1, \ldots, m_{N_f})$ of the $N_f$ flavours. This time we explicitly wrote the normalization constant, since it is mass dependent and is, thus, quite essential.

The partition function (3.7) with the probability density (4.15), i.e. the partially quenched partition function

$$Z_\chi(\kappa, m) = \frac{(-1)^{\nu(n + \nu)(k_2 - k_1)} \text{sdet}^{-\nu/\gamma} \kappa}{\int d[W] \exp(-n \text{tr} \tilde{W}^\dagger \tilde{W} / \gamma) \prod_{j=1}^{N_f} \text{det}(\tilde{W}^\dagger \tilde{W} + m_j^2 \mathbb{1}_{\gamma (n + \nu)})} \times \int d[W] \exp \left( -\frac{n}{\gamma} \text{tr} \tilde{W}^\dagger \tilde{W} \right) \prod_{j=1}^{N_f} \text{det}(\tilde{W}^\dagger \tilde{W} + m_j^2 \mathbb{1}_{\gamma (n + \nu)}) \times \text{sdet}^{-(\nu/\gamma)} (\tilde{W}^\dagger \otimes \mathbb{1}_{\gamma k_1 \gamma k_2} - \mathbb{1}_{\gamma k_1} \otimes \kappa^2),$$  

(4.16)

can be dealt with in two different ways. Either the additional determinants and the determinants generating the correlation functions are computed on an equal footing or one can consider the additional determinants to be part of the probability density $P$. We decide to opt for the latter choice, since we aim at a separation of the physical quarks from the artificial ones, which are also known as valence quarks.

In the appendix we calculate the partially quenched partition function at finite $n$. It is a double integral over an ordinary matrix $U_\pi \in \text{Herm}^{\nu}_{\gamma} \times \gamma N_f = U^{(k)/\beta}(\gamma N_f)$ and the supermatrix $\hat{U} \in \text{Herm}^{\nu}_{\gamma} \otimes \gamma \kappa$:
depending on the relation of the two constants \( \alpha > 0 \) and \( \tilde{\alpha} \in \mathbb{R} \). This probability density is the standard one for the analysis of multicritical behaviour [72–75]. Depending on the relation of the two constants \( \alpha \) and \( \tilde{\alpha} \), the macroscopic level density of \( WW^\dagger \) can exhibit a one-cut or a two-cut solution, which also influences the universality on the local scale of the mean level density where the two cuts are merging to one.

### 4.4. The probability density with a quartic potential

In the last example we want to consider the probability density with a quartic potential

\[
P(\WW^\dagger) \propto \exp[-\alpha \text{tr}(\WW^\dagger)^2 - \tilde{\alpha} \text{tr} \WW^\dagger],
\]

\( \alpha > 0 \) and \( \tilde{\alpha} \in \mathbb{R} \).
We are aiming at a supersymmetric representation of the partition function with the probability density (4.20).

The superfunction $Q$ corresponding to the probability density (4.20) is via the projection formula (3.35)

$$Q(\hat{U}) \propto \int d[\hat{W}_1] \int d[W_1] \exp[-\alpha (\text{tr}(\hat{W}_1 \hat{W}_1^\dagger + W_1 W_1^\dagger)^2 + \text{tr} W_1 \hat{U} W_1^\dagger + \text{str} \hat{U}^2)]$$

$$\times \exp[-\alpha \text{tr}(\hat{W}_1 \hat{W}_1^\dagger + W_1 W_1^\dagger + \text{str} \hat{U})].$$

The quartic term $\text{tr}(\hat{W}_1 \hat{W}_1^\dagger + W_1 W_1^\dagger)^2$ can be traced back to a quadratic structure by introducing a Gaussian over an auxiliary matrix $H \in \text{Herm}^{(\beta)}(n + \gamma(k_2 - k_1))$. Then the integrations over $\hat{W}_1$ and $W_1$ are purely Gaussian and can be performed without any problem, leading to

$$Q(\hat{U}) \propto \exp[-\alpha \text{str} \hat{U}^2 - \tilde{\alpha} \text{str} \hat{U}] \int d[H] \exp \left[ -\frac{1}{4\alpha} \text{tr}(H - \text{i}(\tilde{\alpha} - 1)E_{\gamma n + \gamma \tilde{\gamma}(k_2 - k_1)})^2 \right]$$

$$\times \det^{-1/(\gamma n + k_1 - k_2)}(iH - E_{\gamma n + \gamma \tilde{\gamma}(k_2 - k_1)} \otimes (\alpha \hat{U} + E_{\gamma n + \gamma \tilde{\gamma}(k_2 - k_1)} \otimes \hat{U} \gamma \tilde{\gamma}(k_2 - k_1))).$$

The determinant results from the integration over $\hat{W}_1 \in \text{Gl}^{(\beta)}(n + \gamma(k_2 - k_1); n + \gamma \tilde{\gamma}(k_2 - k_1))$ while the superdeterminant results from the integration over $W_1 \in \text{Gl}^{(\beta)}(\gamma \tilde{\gamma}k_1)(n + \gamma \tilde{\gamma}(k_2 - k_1)))$. We recall the definition of the cosets in equations (2.6), (3.3), and (3.9). The shift in the Gaussian of the auxiliary ordinary matrix $H$ also guarantees the convergence of the integrals over $\hat{W}_1$ and $W_1$ for negative $\tilde{\alpha}$.

For $\beta = 2$ the integral (4.22) can be further simplified via various techniques in random matrix theory [51, 76–78]. In one of these techniques [51, 78], one constructs the orthogonal polynomials of the weight $g(E) = \exp[-(E - \text{i}(\tilde{\alpha} - 1))^2/(4\alpha)]/\text{i}(E - 1)$. Then one obtains a quotient of two determinants of $\max[k_1, k_2] \times \max[k_1, k_2]$ matrices where the determinant in the numerator depends on the orthogonal polynomials and their Cauchy transform with respect to the weight $g(E)$ whose arguments are the eigenvalues of the supermatrix $(\alpha \hat{U} + E_{\gamma n + \gamma \tilde{\gamma}(k_2 - k_1)} \otimes \hat{U} \gamma \tilde{\gamma}(k_2 - k_1))$. The determinant in the denominator is the square root of the Berezinian (the Jacobian in superanalysis) resulting from a diagonalization of the supermatrix $\hat{U}$ [77]. See [51, 78] and references therein for an introduction to the application of orthogonal polynomials.

For $\beta = 1$, 4 the situation is not as simple. Though the ordinary matrix $H$ is decoupled from the supermatrix $\hat{U}$ and no unknown group integrals make the calculation insurmountable, the square root of the superdeterminant hinders the application of orthogonal polynomial theory. The obvious way out of this dilemma is via the expansion of the integral (4.22) in the matrix $\hat{U}$. Then one can calculate each of the expansion coefficients. Since $H$ and $\hat{U}$ are decoupled, such an expansion is trivial. The non-trivial task is to perform the integration over $H$ to find the coefficients. We emphasize that such an expansion is finite if $k_1 = 0$ because the superdeterminant becomes a determinant in the numerator and, thus, a polynomial in $\hat{U}$.

What is the benefit of $Q$ (see equation (4.22)), in particular when there is no explicit, simple expression? The advantage of the result (4.22) with the corresponding partition function in superspace is revealed when considering the correlated situation, meaning that we destroy the invariance of $W$ under the multiplication from the right (or left) with unitary matrices by an external correlation matrix $C$. In contrast to the case for the partition function in ordinary space with the probability weight (4.20), we do not encounter large group integrals (if $k_1$ and $k_2$ are small) when diagonalizing $H$. The resulting partition function is equation (4.9) where we replace the norm-dependent superfunction by the superfunction (4.22). In particular, the calculation of the level density in this way is appropriate for all three Dyson indices; see [13, 14] for the Gaussian ensemble.
5. Summary and conclusions

We presented a new variant of the supersymmetry method which directly relates the probability density in ordinary space with that in superspace via a projection formula. Thereby we briefly rederived this formula (see equation (2.37)) for the ensembles originally included in Dyson’s threefold way [41], namely real symmetric, Hermitian, and Hermitian self-dual matrices; this was first done in [43]. In a second step, we extended the idea behind such a projection formula to the three chiral ensembles. Hereby we found a formula for ensembles whose invariance of rectangular matrices under multiplication from the right (or left) is broken; see equation (3.24). This formula is quite convenient for those situations where one is introducing empirical correlation matrices on both sides of the rectangular random matrix, as is the case in spatial–temporal correlations [56–58].

The result (3.24) is not as compact as the further simplified formula (3.35), which can only be produced if we ensure the invariance of the probability density under left and right multiplication of the rectangular random matrix with unitary matrices. The supersymmetric integral in the partition function is over one of the three coset integrals depending on the Dyson index $\beta$ which already play a crucial role in the standard approach with the superbosonization formula [38, 39]. Nevertheless, one should not confuse our approach with the one in [38, 39].

The projection formula (2.37) for the three non-chiral ensembles agrees with the result of the generalized Hubbard–Stratonovich transformation [36, 37] in the integration domain as well as in the form of the integrand. This is not the case for the chiral ensembles, where the projection formula shares the integration domain with the original superbosonization formula [38, 39] while the integrand is of a completely different form and resembles more closely that of the generalized Hubbard–Stratonovich transformation [36, 37]. Therefore the projection formulae (3.24) and (3.35) for chiral ensembles represent an alternative approach to the standard supersymmetry methods in random matrix theory.

We applied the projection formula (3.35) to the relatively simple example of norm-dependent ensembles and found a quite compact and explicit dependence of the probability density in superspace on that in ordinary space, which reduces to a onefold integral (4.2). For the Gaussian case we recovered the well-known chiral Lagrangian of QCD [15–17] in the microscopic limit; see equation (4.6). Furthermore we showed how to generalize the projection formula in the case of one-sided correlated random matrices; see equation (4.9). This underlines that the projection formula (3.35) is not at all restricted to rotation-invariant (‘isotropic’ [35]) ensembles, and can also cover a simple, but also the most popular, kind of symmetry breaking.

Another ensemble to which we applied the projection formula (3.35) is of Lorentz (Cauchy) type. Surprisingly, it is not only the Gaussian weight that is form invariant under mapping the probability density in ordinary space to that in superspace; the Lorentz weight is too. Only the exponent of the determinant changes and has to be taken care of. With the help of the representation in superspace, we showed that, depending on the exponent of the determinant, the Lorentzian may show universal behaviour in the microscopic limit or not. Hence the projection formula (3.35) provides a new tool for investigating universality issues in chiral random matrix theory, as well.

Moreover, we considered the standard application of chiral random matrix theory to QCD. With the help of the projection formula (3.35), we split the chiral Lagrangian of the partially quenched theory in QCD into two parts; see equation (4.19). One part consists of the lowest order of the unquenched theory in the physical mesons (the pions for two flavours), which is the well-known linear term in the quark masses [15–17]. We refer the reader to the expansion scheme of the microscopic limit (the limit of large space–time volume, $V \to \infty$, with fixed
rescaled quark masses, $V_m = \text{const.}$; see [17]), which is one kind of low energy expansion. The part represents the interaction with the source terms which are artificially introduced to generate the observables. This is some kind of natural splitting into the physical system and the measurement. It would be quite interesting if such a splitting is also applicable to the kinetic modes of the mesons which are not included in the lowest order description by random matrix theory. Perhaps chiral perturbation theory can shed light on this.

In a fourth example, we considered a probability density with a quartic potential, emphasizing that the projection formula (3.35) can also deal with more complicated situations. We derived a representation of the probability density in superspace, which is still an integral over a Hermitian matrix $H$; see equation (4.22). However, the coupling of the ordinary matrix $H$ with the supermatrix $\hat{U}$ occurs in an invariant way, meaning that $H$ and $\hat{U}$ are independently invariant under unitary transformations. In the case of the Dyson index $\beta = 2$, this allows us to apply the machinery of orthogonal polynomials [51, 78] and other techniques [76, 77] (whereby [76] is not limited to $\beta = 2$) to calculate an explicit expression of the probability density $Q$; see equation (4.22). For an elaborate presentation of the calculation methods, we refer the reader to [5]. In the other two cases, $\beta = 1, 4$, the situation is not as simple. Nevertheless, we showed how to circumvent unknown group integrals via the projection formula (3.35) in the supersymmetry method if one considers one-sided correlated rectangular random matrices drawn from an ensemble with a quartic potential. For the Gaussian case, two of the authors have already applied the supersymmetry method to the correlated Wishart ensemble and derived a compact expression for the level density; see [13, 14]. The projection formula (3.35) opens a way to perform this calculation for other probability densities as well.

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Appendix. The derivation of equation (4.17)

Considering the partially quenched partition function (4.16) with the probability density (4.15), the integration that has to be performed via the projection formula (3.35) is

$$Q(\hat{U}) \propto \frac{1}{\int \text{d}[W] \exp(-n \text{tr } W^\dagger W/\gamma) \prod_{j=1}^{N_f} \det(WW^\dagger + m_j^2 \mathbb{1} \gamma_n)}$$

$$\times \int \text{d}[\hat{W}_1] \text{d}[W_1] \exp \left( -\frac{n}{\gamma} \left[ \text{tr } \hat{W}_1 W_1^\dagger + \text{tr } W_1 W_1^\dagger + \text{str } \hat{U} \right] \right)$$

$$\times \prod_{j=1}^{N_f} \text{sdet} \left( \begin{bmatrix} \hat{W}_1 W_1^\dagger + W_1 W_1^\dagger & W_1 \sqrt{\hat{U}} \\ \sqrt{\hat{U}} W_1^\dagger & m_j^2 \mathbb{1} \gamma_n + \gamma \gamma_k \gamma_k \end{bmatrix} \right)$$

$$\exp[-n \text{str } \hat{U}/\gamma] \prod_{j=1}^{N_f} \text{sdet} \left( \hat{U} + m_j^2 \mathbb{1} \gamma_k \gamma_k \right)$$

$$= \frac{\int \text{d}[W] \exp(-n \text{tr } W^\dagger W/\gamma) \prod_{j=1}^{N_f} \det(WW^\dagger + m_j^2 \mathbb{1} \gamma_n)}$$
\[
\times \int d[\hat{W}_1] \int d[W_1] \exp \left( -\frac{n}{y} \left[ \text{tr} \, \hat{W}_1^\dagger \hat{W}_1 + \text{tr} \, W_1 W_1^\dagger \right] \right) \times \prod_{j=1}^N \det \left( \hat{W}_1 \hat{W}_1^\dagger + m_j^2 W_1 \left( \hat{U} + m_j^2 \mathbb{1}_{y \gamma(k_j) \gamma(k_j)} \right)^{-1} W_1^\dagger + m_j^2 \mathbb{1}_{y \gamma(k_j) \gamma(k_j-1)} \right). \quad (A.1)
\]

In the second step, we pushed out the block matrices \( \hat{U} + m_j^2 \mathbb{1}_{y \gamma(k_j) \gamma(k_j)} \) for each mass \( m_j \). The product of the determinants can be rewritten as

\[
\det \left( \hat{W}_1 \hat{W}_1^\dagger + m_j^2 W_1 \left( \hat{U} + m_j^2 \mathbb{1}_{y \gamma(k_j) \gamma(k_j)} \right)^{-1} W_1^\dagger + m_j^2 \mathbb{1}_{y \gamma(k_j) \gamma(k_j-1)} \right) = m_j^{-2y} \, \sdet^{-1} \left( \hat{U} + m_j^2 \mathbb{1}_{y \gamma(k_j) \gamma(k_j)} \right) \sdet \left( \hat{W}_1 \hat{W}_1^\dagger + \hat{U}' \hat{W}_1 + m_j^2 \mathbb{1}_{y \gamma(k_j) \gamma(k_j)} \right) \quad (A.2)
\]

with

\[
\hat{W}_1 = [\hat{W}_1 | W_1]. \quad (A.3)
\]

such that \( \hat{W}_1 \in \text{Gl}^y(n + v + \gamma k_2; n + \gamma (k_2 - k_1)) \), and with the supermatrix

\[
\hat{U}' = \begin{bmatrix} 0 & 0 \\ 0 & \hat{U} \end{bmatrix}. \quad (A.4)
\]

The superfunction \( Q \) reads

\[
Q(\hat{U}) \propto \frac{\exp[-n \, \text{str} \, \hat{U} / \gamma]}{\int d[W] \exp(-n \, \text{tr} \, W^\dagger W / \gamma) \prod_{j=1}^N \det(W^\dagger W + m_j^2 \mathbb{1}_{y(n+v)}) \times \int d[\hat{W}_1] \exp \left( -\frac{n}{\gamma} \, \text{str} \, \hat{W}_1 \hat{W}_1^\dagger \right) \times \sdet^{1/\gamma} \left( [\hat{W}_1 \hat{W}_1^\dagger + \hat{U}'] \otimes \mathbb{1}_{y \gamma N_1} \otimes \mathbb{1}_{y \gamma(k_2) \gamma(k_2) \gamma(k_2-1) \otimes m^2} \right). \quad (A.5)
\]

The integral over the supermatrix \( \hat{W}_1 \) resembles the partition function (4.8) with an external matrix \( \hat{U}' \). One can easily show that the projection formula (3.35) can be generalized to a partition function with rotation-invariant probability density in superspace. Thus we apply the projection formula for norm-dependent ensembles (see equation (4.9)) to replace the dyadic supermatrix \( \hat{W}_1 \hat{W}_1 \) with a \( \gamma \gamma N_1 \times \gamma \gamma N_1 \) unitary matrix \( U_\gamma \in \text{Herm}^y(\gamma \gamma N_1) = \text{U}^{(y)}(\gamma \gamma N_1) \) in the second tensor space in the superdeterminant (A.5). We recall the definitions (2.5) and (2.18). The subscript ‘\( \gamma \)’ on the unitary matrix \( U \) refers to physical mesons, as they do indeed agree with the mesons (Goldstone bosons) in the microscopic limit. For \( N_1 = 2 \) the mesons are the pions which are usually denoted by \( \pi \).

Applying the projection formula (3.35) to the expression (A.5), the superfunction \( Q \) takes the form

\[
Q(\hat{U}) \propto \frac{\exp[-n \, \text{str} \, \hat{U} / \gamma]}{\int d\mu(U_\gamma) \exp(n \, \text{tr} \, U_\gamma / \gamma) \det(m^2 / \gamma) \det(N_1^2 / \gamma) \det(U_\gamma + m_1^2 \mathbb{1}_{y \gamma N_1}) \times \int d\mu(U_\gamma) \exp \left( -\frac{n}{\gamma} \, \text{tr} \, U_\gamma \right) \det^{-n/\gamma} U_\gamma \times \sdet^{1/\gamma} \left( U_\gamma \otimes \mathbb{1}_{y \gamma N_1} + \mathbb{1}_{y \gamma(k_2) \gamma(k_2-1) \otimes (U_\gamma + m_2^2)} \right) \exp[-n \, \text{str} \, \hat{U} / \gamma] \right) = \int d\mu(U_\gamma) \exp(n \, \text{tr} \, U_\gamma / \gamma) \det(m^2 / \gamma) \det(U_\gamma + m_1^2 \mathbb{1}_{y \gamma N_1}) \times \int d\mu(U_\gamma) \exp \left( -\frac{n}{\gamma} \, \text{tr} \, U_\gamma \right) \det^{-n/\gamma} U_\gamma \times \sdet^{1/\gamma} \left( U_\gamma \otimes \mathbb{1}_{y \gamma N_1} + \mathbb{1}_{y \gamma(k_2) \gamma(k_2-1) \otimes (U_\gamma + m_2^2)} \right) \quad (A.6)
\]
We also replaced the integral in the denominator via the projection formula. The superfunction $Q$ can be plugged into the partition function (4.16) and we find equation (4.17).

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