ASYMPTOTIC EXPANSION OF GENERALIZED WITTEN INTEGRALS FOR CIRCLE ACTIONS ON SYMPLECTIC MANIFOLDS

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Abstract. On a symplectic manifold with a Hamiltonian $S^1$-action we derive a complete asymptotic expansion for generalized Witten integrals via singular stationary phase asymptotics, characterizing the coefficients in the expansion as integrals over the symplectic strata of the corresponding reduced space. In particular, we exhibit singular contributions of the lower-dimensional strata in an explicit way.

1. INTRODUCTION

Let $(M, \omega)$ be a $2n$-dimensional symplectic manifold with a Hamiltonian action of a compact connected Lie group $K$ and momentum map $J : M \to t^*$, and assume that $0$ is a regular value of $J$, which is equivalent to the fact that the stabilizer of each point of $J^{-1}(\{0\})$ is finite. In this case, the corresponding Marsden-Weinstein reduced space, or symplectic quotient

$$\mathcal{M}^0 := J^{-1}(\{0\}) / K$$

is an orbifold. If $0$ is not a regular value of $J$, $\mathcal{M}^0$ becomes a stratified space with serious singularities. In case that $M$ is compact, the geometry and topology of $\mathcal{M}^0$ have been extensively studied during the last decades [16, 17, 17, 17, 17, 17], one of the major tools being the Witten integral and its asymptotic expansion, whose coefficients carry important geometric and topological information.

In this paper, we study the Witten integral in the case where $K = T := SO(2) \simeq S^1$ is the circle group, $M$ is not necessarily compact, and $0$ is not necessarily a regular value. More precisely, we derive a complete singular stationary phase expansion of generalized Witten integrals of the form

$$I_{\zeta}^\varepsilon (\varepsilon) := \int_M \left( \int_M e^{i(J(p) - \zeta(x)) / \varepsilon} a(p, x) \, dM(p) \right) \, dx, \quad \mathbb{R}_+^* \ni \varepsilon \to 0, t^* \ni \zeta \to 0,$$
supported on \(M\) Brummelhuis, Paul, and Uribe [2]. Concretely, consider for an arbitrary asymptotic expansions for the corresponding linearized Witten integrals following an approach of Morse-Bott function and the usual stationary phase theorem yields a complete asymptotic expansion of \(I_\varepsilon^F(\varepsilon)\). Nevertheless, serious difficulties arise when \(\zeta\) is a singular value, since then the stationary phase principle cannot be applied. To overcome this, we linearize the problem and derive complete asymptotic expansions for the corresponding linearized Witten integrals following an approach of Brummelhuis, Paul, and Uribe [2]. Concretely, consider for an arbitrary \(\zeta \in t^*\) the stratification of the quotient \(\mathcal{M}^\zeta := \mathcal{J}^{-1}(\{\zeta\})/T\) by infinitesimal orbit types

\[
\mathcal{M}^\zeta = \mathcal{M}^\zeta_{\text{top}} \sqcup \mathcal{M}^\zeta_{\text{sing}}, \quad \mathcal{M}^\zeta_{\text{R}} := (\mathcal{J}^{-1}(\{\zeta\}) \cap M_{(h_{\text{top}})})/T,
\]

where \(M_{(h_{\text{top}})}\) denotes the stratum of \(M\) of infinitesimal orbit type \((h_{\text{top}})\) with \(h_{\text{top}} = \{0\}\) and \(h_{\text{sing}} = t\). The stratum \(\mathcal{M}^\zeta_{\text{top}}\) is dense, an orbifold, and called the top stratum. Writing \(M^T\) for the space of fixed-points of the \(T\)-action on \(M\) and \(\mathcal{F}\) for the set of all connected components of \(M^T\), the singular stratum \(\mathcal{M}^\zeta_{\text{sing}}\) is the union of all those \(F \in \mathcal{F}\) with \(\mathcal{J}(F) = \zeta\). Each \(F \in \mathcal{F}\) is a symplectic submanifold of \((M, \omega)\), and the singular values of \(\mathcal{J}\) are \(\{\mathcal{J}(F) : F \in \mathcal{F}\} \subset t^*\). In addition, we introduce the finite set \(\mathcal{F}_0 := \{F \in \mathcal{F} : F \cap K_a \neq \emptyset\}\), where \(K_a\) is the compact set from the footnote on this page. Let \(\omega_{\text{top}}\) be the unique symplectic form on \(\mathcal{M}^\zeta_{\text{top}}\) characterized by \(i^*\omega = \pi^*\omega_{\text{top}}, \) where \(i : \mathcal{J}^{-1}(\{\zeta\}) \cap M_{(h_{\text{top}})} \rightarrow M\) is the inclusion and \(\pi : \mathcal{J}^{-1}(\{\zeta\}) \cap M_{(h_{\text{top}})} \rightarrow \mathcal{M}^\zeta_{\text{top}}\) is the canonical projection. Then

\[
d\mathcal{M}^\zeta_{\text{top}} := \omega_{\text{top}}^{n-1}/(n-1)!, \quad dF := \omega^\text{dim } F/((\text{dim } F)/2)!
\]

are the symplectic volume forms on \(\mathcal{M}^\zeta_{\text{top}}\) and \(F \in \mathcal{F}\), respectively.

Our main result is Theorem 4.15 in which we show with respect to a suitable partition of unity \(\{\chi_{\text{top}}; \chi_F\}_{F \in \mathcal{F}}\) on \(M\) that there are \(\delta_0, \varepsilon_0 > 0\) such that for all \(\zeta \in t^*\) and all \(\delta \in (0, \delta_0)\) one has with \(\zeta_F := \zeta - \mathcal{J}(F)\) an asymptotic expansion

\[
I_\varepsilon^F(\varepsilon) \sim \sum_{j=0}^{\infty} \varepsilon^{1+j} \int \mathcal{M}^\zeta_{\text{top}} Q_j(\psi, a\chi_{\text{top}}) d\mathcal{M}^\zeta_{\text{top}} + \sum_{F \in \mathcal{F}_0, \delta \leq \|\cdot\|} \int \mathcal{M}^\zeta_{\text{top}} Q_j(\psi, a\chi_F) d\mathcal{M}^\zeta_{\text{top}}
\]

\[
+ \sum_{l=0}^{\infty} \left(\sum_{F \in \mathcal{F}_0, \|\cdot\| \leq \delta} \zeta_F^l \int \mathcal{M}^\zeta_{\text{top}} \mathcal{R}_{j,l}(a\chi_F) d\mathcal{M}^\zeta_{\text{top}}(F)
\]

\[
+ \sum_{F \in \mathcal{F}_0, \|\cdot\| \leq \delta} \zeta_F^l \int F S_{j,l}(a, \zeta_F/\varepsilon) dF\right), \quad 0 < \varepsilon < \varepsilon_0,
\]

where all coefficients are explicitly given. They are smooth and bounded in \(\zeta\) and \(\zeta_F/\varepsilon\), respectively, and the singular contributions in the third line of the formula are independent of the partition of unity.

In [3, Section 9], an asymptotic expansion of the Witten integral is derived using the localization instead of the stationary phase principle. Accordingly, the coefficients in that expansion are given in terms of residues, but not in terms of integrals on the symplectic strata of the reduced space. In a forthcoming paper, we shall give first applications of the results of this paper, and relate the cohomology of the reduced space \(\mathcal{M}^0\) to the equivariant cohomology of the underlying Hamiltonian space \(M\) and the fixed-point data of the \(S^1\)-action via the asymptotic expansion for \(I_\varepsilon^F(\varepsilon)\) derived in this

\footnote{More precisely, for each amplitude \(a\) there is a compact set \(K_a \subset M\) with \(\text{supp } a(\cdot, x) \subset K_a\) for all \(x \in t\) and \(a(p, \cdot) \in L_2(t)\) for all \(p \in K_a\), where \(L_2(t)\) is the space of Schwartz functions on \(t\).}
paper, extending previous work of Lerman-Tolman and Jeffrey-Kiem-Kirwan-Woolf to non-compact situations.

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2. Background and setup

2.1. Hamiltonian manifolds and reduced spaces. Let $M$ be a $2n$-dimensional symplectic manifold with symplectic form $\omega$. Assume that $M$ carries a Hamiltonian action of a compact connected Lie group $K$ of dimension $d$ with Lie algebra $\mathfrak{k}$, and denote the corresponding Kostant-Souriau momentum map by

$$\mathcal{J} : M \to \mathfrak{k}^*,$$

$$\mathcal{J}(p)(X) = J(X)(p),$$

which is characterized by the property

$$dJ(X) = \iota_X\omega \quad \forall \, X \in \mathfrak{k},$$

(2.1)

where $\widetilde{X}$ denotes the fundamental vector field on $M$ associated to $X$, $d$ is the de Rham differential and $\iota$ denotes contraction. Note that $\mathcal{J}$ is $K$-equivariant in the sense that $\mathcal{J}(k^{-1}p) = \text{Ad}^*(k)\mathcal{J}(p)$.

Let $(\Omega^*_K(M)_c, D)$ be the complex of compactly supported equivariant differential forms on $M$. The elements in $\Omega^*_K(M)_c$ can be regarded as $K$-equivariant polynomial maps $\mathfrak{k} \to \Omega^*_c(M)$, where $K$ acts on $\mathfrak{k}$ by the adjoint action $\text{Ad}(K)$ and on the algebra $\Omega^*_c(M)$ of compactly supported differential forms by the pullbacks associated to the $K$-action on $M$. The differential $D$ is then defined by

$$D(\alpha)(X) := d(\alpha(X)) + \iota_X(\alpha(X)), \quad \alpha \in \Omega^*_c(M)_c.$$

We denote the cohomology of the complex $(\Omega^*_K(M)_c, D)$, which is called the equivariant cohomology of $M$, by $H^*_K(M)_c$. Further, let

$$\mathcal{J} := \omega - \mathcal{J}$$

be the equivariantly closed extension $\mathcal{J}$ of the symplectic form $\omega$. The approach used here is usually called the Cartan model.

Remark 2.1 (Sign convention). The sign convention in the definition of $D$ (and hence $\mathcal{J}$) varies in the literature. We define $D$ in coherence with [11], while in [7] one has $D(\alpha)(X) := d(\alpha(X)) - \iota_X(\alpha(X))$, which leads to $\mathcal{J} = \omega + \mathcal{J}$ as opposed to our definition (2.2).

From the definition of the momentum map it is clear that the kernel of its derivative is given by

$$\ker d\mathcal{J}|_p = (\mathfrak{k} \cdot p)\omega^*, \quad p \in M,$$

(2.3)

where we denoted the symplectic complement of a subspace $V \subset T_p M$ by $V\omega$, and $\mathfrak{k} \cdot p := \{\widetilde{X}_p : X \in \mathfrak{k}\}$, Consequently, if $\zeta \in \mathcal{J}(M)$ is a regular value of $\mathcal{J}$, the level set $\mathcal{J}^{-1}(\zeta)$ is a (not necessarily connected) manifold of codimension 1, and $T_p(\mathcal{J}^{-1}(\zeta)) = \ker d\mathcal{J}|_p = (\mathfrak{k} \cdot p)\omega^*$, which is equivalent to

$$\widetilde{X}_p \neq 0 \quad \forall \, p \in \mathcal{J}^{-1}(\zeta), \quad 0 \neq X \in \mathfrak{k},$$

compare [13] Chapter 8]. The latter condition means that all stabilizers of points $p \in \mathcal{J}^{-1}(\zeta)$ are finite, and therefore either of exceptional or principal type, so that $\mathcal{J}^{-1}(\zeta)/K\zeta$ is an orbifold. In addition, in view of the exact sequence

$$0 \to T_p(\mathcal{J}^{-1}(\zeta)) \xrightarrow{d\zeta} T_p M \xrightarrow{d\mathcal{J}} T\mathfrak{k}^* \to 0, \quad p \in \mathcal{J}^{-1}(\zeta),$$

where $d\zeta : \mathcal{J}^{-1}(\zeta) \hookrightarrow M$ denotes the inclusion, and the corresponding dual sequence, $\mathcal{J}^{-1}(\zeta)$ is orientable because $M$ is orientable, compare [11] Chapter XV.6].
If $\zeta$ is not a regular value, both $\mathcal{J}^{-1}(\{\zeta\})$ and $\mathcal{J}^{-1}(\{\zeta\})/K^\zeta$ are singular and given by Whitney stratified spaces, compare [16]. In particular, for any $\zeta \in \mathfrak{t}^*$ the space $\mathcal{J}^{-1}(\{\zeta\})$ has a decomposition into smooth manifolds given by

$$
\mathcal{J}^{-1}(\{\zeta\}) = \bigcup_{H < K} \mathcal{J}^{-1}(\{\zeta\})(H),
$$

where $\mathcal{J}^{-1}(\{\zeta\})(H) := \mathcal{J}^{-1}(\{\zeta\}) \cap M(H)$ denotes the union of orbits in $\mathcal{J}^{-1}(\{\zeta\})$ of isotropy type $(H)$. Furthermore, there is a stratum $\mathcal{J}^{-1}(\{\zeta\})(H_{\text{reg}})$, called the regular stratum, which is open and dense in $\mathcal{J}^{-1}(\{\zeta\})$. To this decomposition corresponds a stratification of $\mathcal{M}^\zeta := \mathcal{J}^{-1}(\{\zeta\})/K^\zeta$ into a union of disjoint symplectic manifolds

$$
\mathcal{M}^\zeta = \bigcup_{H < K} \mathcal{J}^{-1}(\{\zeta\})(H)/K^\zeta,
$$

see [16] Theorem 2.1, $\mathcal{J}^{-1}(\{\zeta\})(H_{\text{reg}})/K^\zeta$ being the regular stratum. Note that $\mathcal{J}^{-1}(\{\zeta\})$ and the strata $\mathcal{J}^{-1}(\{\zeta\})(H)$ might not be connected.

Let us now restrict to the case where $K = T = S^1$, in which case only the isotropy types $H = \{e\}$, cyclic subgroup of $S^1$, can occur, $e$ being the identity element. Let $M^T$ denote the set of fixed-points of the $T$-action. The connected components $F$ of $M^T$ are submanifolds of possibly different dimensions, and we denote the set of these components by $\mathcal{F}$. For our purposes, it will actually be more convenient to work with the stratification of the quotient $\mathcal{M}_T = \mathcal{J}^{-1}(\{\zeta\})/T$ by infinitesimal orbit types [12] Section 3

$$
\mathcal{M}_T^\zeta = \mathcal{M}_\text{top} \cup \mathcal{M}_\text{sing}, \quad \mathcal{M}_\text{top}^\zeta := (\mathcal{J}^{-1}(\{\zeta\}) \cap M(h_\text{top}))/T,
$$

where $M(h_\text{top})$ denotes the stratum of $M$ of infinitesimal orbit type $(h_\text{top})$ with $h_\text{top} = \{0\}$ and $h_\text{sing} = \mathbb{R}$. Since $\omega$ is non-degenerate, we see from (2.3) that

$$
p \in M^T = M(h_\text{sing}) \iff d\mathcal{J}|_p = 0.
$$

Since $\mathcal{J}$ is constant on each $F$ we have

**Lemma 2.2.** The momentum map $\mathcal{J} : M \to \mathfrak{t}^*$ has no critical points in $M(h_\text{top})$ and its singular values are $\{\mathcal{J}(F) : F \in \mathcal{F}\}$. □

As already mentioned in the introduction on page 2, the stratum $\mathcal{M}_\text{top}^\zeta$ is dense, an orbifold, and called the top stratum, while $\mathcal{M}_\text{sing}^\zeta$ is a manifold, each of its components being identical to some $F \in \mathcal{F}$. Furthermore, recall from page 2 that each $F$ is a symplectic submanifold of $M$ and there is also a natural symplectic form $\omega_\text{top}$ on $\mathcal{M}_\text{top}^\zeta$.

### 2.2. The Witten integral

As explained in the introduction, the central objects of our study are generalized Witten integrals of the form [14], and our main tools will be Fourier analysis and singular stationary phase expansion. Let $T = S^1$, $dx$ and $d\zeta$ be measures on $t$ and $\mathfrak{t}^*$ that correspond to the Ad$(T)$-invariant inner product on $t$, and agree with Lebesgue measure under the identification $t \cong \mathfrak{t}^* \cong \mathbb{R}$. Denote by

$$
\mathcal{F}_t : S(t^*) \to S(t), \quad \mathcal{F}_t : S'(t) \to S'(t^*)
$$

the $t$-Fourier transform on the Schwartz space and the space of tempered distributions, respectively, given by\(^3\)

$$
\hat{\psi}(x) := (\mathcal{F}_t \psi)(x) := \int_{t^*} e^{-i \langle \zeta, x \rangle} \psi(\zeta) \, d\zeta, \quad \langle \zeta, x \rangle := \zeta(x), \quad x \in t, \quad \psi \in S(t^*),
$$

\(^3\)Regarding normalization conventions, see also [8] footnotes on p. 125.
and recall that $\omega = \omega - J$. Consider now the generalized Duistermaat-Heckman integral
\begin{equation}
L_\varrho : t \to \mathbb{C}, \quad L_\varrho(x) := \int_M e^{-i\mathcal{W}(x)} \varrho(x), \quad \varrho \in \Omega^*_T(M)_c,
\end{equation}
regarded as a tempered distribution in $\mathcal{S}'(t)$, compare [17, 23, 68]. If $\varrho = 1$, $L_\varrho$ is the classical Duistermaat-Heckman integral, whose $t$-Fourier transform is given by the pushforward $J_\varrho(\omega^n/n!)$ of the Liouville form along $J$, which is a piecewise polynomial measure on $t^*$. Motivated by this, we shall examine the behavior of the Fourier transform of $L_\varrho$ near the origin, and for this sake consider an approximation of the Dirac $\delta$-distribution centered at $\zeta \in t^*$ given by
\[ \phi_\zeta(\xi) := \phi((\xi - \zeta)/\varepsilon) / \varepsilon, \quad \varepsilon > 0, \]
where $\phi \in \mathcal{C}_c^\infty(t^*)$ is a test function satisfying $\hat{\phi}(0) = 1$. We are then interested in the limit
\begin{equation}
\lim_{\varepsilon \to 0^+} \left( L_\varrho, \phi_\zeta \right) = \lim_{\varepsilon \to 0^+} \int_M L_\varrho(x) \hat{\phi}_\zeta(x) \, dx = \lim_{\varepsilon \to 0^+} \int_M L_\varrho(x) e^{-i\zeta(x)} \hat{\varrho}(x) \, dx \equiv 0,
\end{equation}
which might not exist in general, and its dependence on $\zeta$ in a neighbourhood of $0 \in t^*$, where we took into account that $\hat{\phi}(x) = e^{-i(\zeta(x)}) \hat{\varrho}(x)$. Thus, we are led to the definition of the Witten integral
\begin{equation}
W^\zeta_{\varrho, \phi}(\varepsilon) := \int_M \int_M e^{i(\zeta(x) - \zeta(y))} e^{-i\xi(x)} \varrho(y) \hat{\varrho}(x) \, dx \, dy, \quad \varrho \in \Omega^*_T(M)_c, \phi \in \mathcal{S}(t^*), \varepsilon > 0, \zeta \in t^*,
\end{equation}
and to the investigation of its asymptotic behavior as $\varepsilon \to 0^+$ and $\zeta \to 0$. Note that in this notation, $\langle L_\varrho, \phi_\zeta \rangle = W^\zeta_{\varrho, \phi}(\varepsilon)$. Furthermore, if $\varrho$ is equivariantly closed, $W^\zeta_{\varrho, \phi}(\varepsilon)$ actually only depends on the cohomology class of $\varrho$ in view of Lemma 1.

**Remark 2.3.** The original Witten integral considered in [17] reads in our setting
\[ \frac{1}{(2\pi i)^t} \int_M \int_M (e^{-i\xi(x)} \varrho(x)) e^{-\xi^2} \, dx, \quad \nu > 0, \varrho \in \Omega^*_T(M)_c. \]
Writing $\varepsilon := \sqrt{\nu}$ we see that this equals $((2\pi)^t)^{-1}$ times $W^\zeta_{\varrho, \phi}(\varepsilon)$ with $\hat{\varrho}(x) = e^{-\xi^2}$ and $\zeta = 0$.

To formulate (2.8) more explicitly, write $\varrho$ as a finite linear combination
\begin{equation}
\varrho(x) = \sum_{e,f} \varrho_{e,f} x^e, \quad \varrho_{e,f} \in \Omega^f(M)_c, \quad e, f \in \mathbb{N}.
\end{equation}
For those $\varrho_{e,f}$ which are differential forms of odd degree, there is no appropriate power $k \in \mathbb{N}$ such that $\omega^k \wedge \varrho_{e,f}$ is a volume form, therefore only the $\varrho_{e,f}$ with $f$ even contribute to $W^\zeta_{\varrho, \phi}(\varepsilon)$. Thus,
\begin{equation}
W^\zeta_{\varrho, \phi}(\varepsilon) = \sum_{\varrho_{e,f} \text{even}} \varepsilon^{-e-1} \int_M \int_M e^{i(\zeta(x) - \zeta(y))} e^{-(\omega^2)/2} \varrho_{e,f} \frac{(\xi)^n-f/2}{(n-f/2)!} x^e \hat{\varrho}(x) \, dx.
\end{equation}
We associate to each $\varrho_{e,f}$ a $T$-invariant function $b_{e,f} \in \mathcal{C}_c^\infty(M)$ by the relation
\begin{equation}
\frac{(-i\omega)^{n-f/2} \varrho_{e,f}}{(n-f/2)!} = b_{e,f} \, dp,
\end{equation}
where $dp := \omega^n/n!$ is the symplectic volume form on $M$. In this way, we are reduced to study the asymptotic behavior of the generalized Witten integral
\begin{equation}
J^\zeta_\varrho(\varepsilon) := \int_M e^{i\zeta(p,x)/\varepsilon} a(p,x) \, dp \, dx, \quad \zeta \in t^*, \varepsilon \to 0^+, \quad a(p,x) = b(p) \sigma(x), \quad b \in \mathcal{C}_c^\infty(M), \sigma \in \mathcal{S}(t),
\end{equation}
\[^3\]Jeffrey and Kirwan use the notation $\Pi_{\varrho}(\omega^e/e^n)$ for our map $L_\varrho$, see [2] p. 299.
and the phase function $\psi^\zeta \in C^\infty(M \times t)$ is given by

\begin{equation}
\psi^\zeta(p, x) := J(p)(x) - \zeta(x).
\end{equation}

Now, when trying to describe the asymptotic behavior of the integrals $I_\delta^\zeta(\varepsilon)$ by means of the generalized stationary phase principle, one faces the serious difficulty that the critical set of the phase function $\psi^\zeta$ is in general not smooth. Indeed, due to the linear dependence of $J(x)$ on $x$ we obtain

$$\partial_x \psi^\zeta(p, x) = J(p) - \zeta,$$

and because of the non-degeneracy of $\omega$,

$$dJ(x) = \iota_{\tilde{x}}\omega = 0 \iff \tilde{x} = 0,$$

where $\tilde{x}$ is the fundamental vector field on $M$ associated to $x$. Hence, the critical set reads

\begin{equation}
\text{Crit}(\psi^\zeta) := \{(p, x) \in M \times t : d\psi^\zeta(p, x) = 0\} = \{(p, x) \in J^{-1}(\{\zeta\}) \times t : \tilde{x}_p = 0\}.
\end{equation}

Let us first assume that $\zeta$ is a regular value. As was discussed in Section 2.1, $J^{-1}(\{\zeta\})$ is an orientable manifold, and all stabilizers of points in $J^{-1}(\{\zeta\})$ are finite. Consequently, $\text{Crit}(\psi^\zeta) = J^{-1}(\{\zeta\}) \times \{0\}$; in particular, it is an orientable manifold. Even further, the critical set of the phase function $\psi^\zeta$ is clean \cite{BSS} Proposition 2, and the generalized stationary phase theorem \cite{BSS} Theorem C can be applied, yielding a complete asymptotic expansion for $I_\delta^\zeta(\varepsilon)$. Slightly more generally, we have the following

**Proposition 2.4.** Assume that $J$ is regular on the support of the amplitude $a$. Then, for each $N \in \mathbb{N}$ there exists a constant $C_{N, \psi^\zeta, a} > 0$ such that

$$\left| I_\delta^\zeta(\varepsilon) - \varepsilon \sum_{j=0}^{N+1} \varepsilon^j Q_j(\psi^\zeta, a) \right| \leq C_{N, \psi^\zeta, a} \varepsilon^{N+1} \quad \forall \varepsilon > 0,$$

where the coefficients $Q_j(\psi^\zeta, a) \in \mathbb{C}$ can be expressed explicitly in terms of measures on $J^{-1}(\{\zeta\})$.

Moreover, if $\mathcal{Y} \subset t^*$ is an open set consisting entirely of regular values of $J$ on the support of $a$, then the functions $\mathcal{Y} \ni \zeta \mapsto Q_j(\psi^\zeta, a) \in \mathbb{C}$ are smooth.

*Proof.* See \cite{BSS} Proposition 2. \hfill $\square$

If $\zeta$ is not a regular value, both $J^{-1}(\{\zeta\})$ and $J^{-1}(\{\zeta\})/\mathbb{T}$ are singular. Consequently, Crit($\psi^\zeta$) is no longer clean and the usual stationary phase theorem cannot be applied. Instead, we shall normalize the generalized Witten integral in suitable coordinates and derive a complete asymptotic expansion by generalizing a stationary phase lemma of Brummellhuis, Paul, and Uribe \cite{BPU}.

### 3. Asymptotic Expansion of the Generalized Witten Integral for Circle Actions

As before, consider a $2n$-dimensional symplectic manifold $(M, \omega)$ carrying a Hamiltonian action of $T = S^1$ with momentum map $J : M \to t^*$. We commence our study of the integrals \ref{eq:generalized_integral} by introducing suitable coordinates on $M$ near the set of fixed-points $M^T = \{p \in M : t \cdot p = p \quad \forall t \in T\}$. The connected components of $M^T$ are symplectic submanifolds of $M$ of possibly different dimensions. Recall that we denote the set of these components by $\mathcal{F}$.

#### 3.1 A local normal form for the momentum map

Let $F \in \mathcal{F}$ and consider the symplectic perpendicular bundle $E_F := TF^\omega$ of the tangent bundle $TF$ of $F$ in the tangent bundle $TM$ of $M$. Since $F$ is symplectic, $TM|_F = TF \oplus E_F$, so that $E_F$ corresponds to the symplectic normal bundle $NF$ of $F$ and carries a symplectic structure. In particular, the total space of $E_F$ becomes a symplectic manifold. Furthermore, the group $T = S^1$ acts on $E_F$ fiberwise, and we may choose an $S^1$-invariant complex structure on $E_F$ compatible with the symplectic one. Each fiber of the complexified bundle $E_F$ then splits into a direct sum of complex 1-dimensional representations of $S^1$, so that with dim $F = 2n_F$

\begin{equation}
E_F = \bigoplus_{j=1}^{n-n_F} \mathcal{E}_j^F,
\end{equation}
the $\mathcal{E}_j^F$ being complex line bundles over $F$. The Lie algebra $\mathfrak{t}$ acts on them by

$$(\mathcal{E}_j^F)_p \ni v \mapsto i\lambda_j^F(x)v \in (\mathcal{E}_j^F)_p, \quad p \in F, \; x \in \mathfrak{t}, \; \lambda_j^F \in \mathfrak{t}^*,$$

where $\lambda_1^F, \ldots, \lambda_{n-F}^F \in \mathbb{Z}$ are the weights of the $T$-action on $(E_p)_F$. They do not depend on the point $p \in F$ because $F$ is connected, and they can be grouped into positive weights $\lambda_1^F, \ldots, \lambda_\ell^F$ and negative weights $\lambda_{\ell+1}^F, \ldots, \lambda_{n-F}^F$. The codimension of $F$ in $M$ is given by $\text{codim} \; F = 2(n - n_F) = 2(\ell^+ + \ell^-)$. We shall now make use of the local normal form theorem for the momentum map $\mathcal{J}$ due to Guillemin-Sternberg \cite{Guillemin:1982} and Marle \cite{Marle:1978}, which in our situation reads as follows.

**Proposition 3.1.** For each component $F \in \mathcal{F}$, there exist

1. a faithful unitary representation $\Pi_F : S^1 \to (S^1)^{\ell^+ + \ell^-} \subset U(\ell^+ + \ell^-)$ with positive weights $\lambda_1^F, \ldots, \lambda_\ell^F \in \mathbb{Z}$ and negative weights $\lambda_{\ell+1}^F, \ldots, \lambda_{n-F}^F \in \mathbb{Z}$,

2. a principal $K_F$-bundle $P_F \to F$, where $K_F$ is a subgroup of $U(\ell^+) \times U(\ell^-)$ commuting with $\Pi_F(S^1)$, such that

$$E_F \cong P_F \times_{K_F} \mathbb{C}^{\ell^+ + \ell^-},$$

where $P_F \times_{K_F} \mathbb{C}^{\ell^+ + \ell^-} \to F$ is the vector bundle associated to $P_F$. Furthermore, there is a symplectomorphism $\Phi_F : U \to V_F$ from an $S^1$-invariant neighborhood $U_F$ of $F$ in $M$ onto an $S^1$-invariant neighborhood $V_F$ of the zero section in $E_F$, which is equivariant with respect to the $S^1$-action on $E_F \cong P_F \times_{K_F} \mathbb{C}^{\ell^+ + \ell^-}$ given by $\Pi_F$, and

$$(\mathcal{J} \circ \Phi_F^{-1})([\varphi, \psi]) = \frac{1}{2} \sum_{j=1}^{\ell^+ + \ell^-} \lambda_j^F |w_j|^2 + \mathcal{J}(F), \quad w = (w_1, \ldots, w_{\ell^+ + \ell^-}), \; [\varphi, \psi] \in P_F \times_{K_F} \mathbb{C}^{\ell^+ + \ell^-}.$$

In particular, $2\ell^-$ and $2\ell^+$ are the dimensions of the negative and positive eigenspaces of the Hessian of $\mathcal{J}$ at a point of $F$, respectively.

**Proof.** See \cite[Lemma 3.1]{ASymptoticExpansion}. \qed

By making the neighborhoods $U_F$ smaller if necessary, we can and will assume that the set

$$(3.3) \quad \mathcal{F}_a := \{ F \in \mathcal{F} : F \cap \text{supp} \; b \neq \emptyset \} \subset \mathcal{F}$$

is finite, given that $\text{supp} \; b = \text{supp}_M a$ is compact. Next, consider a partition of unity $\{ \chi_{\text{top}}, \chi_F \}_{F \in \mathcal{F}}$ consisting of $T$-invariant functions and subordinated to the cover

$$(3.4) \quad M = M_{(\text{top})} \cup \bigcup_{F \in \mathcal{F}} U_F,$$

where we may assume that each $\chi_F$ satisfies $\chi_F \equiv 1$ in a neighborhood of the component $F$.

**Remark 3.2.** There are Hamiltonian $S^1$-spaces $M$ for which the local normal form neighborhoods $U_F$ can be chosen large enough that $\bigcup_{F \in \mathcal{F}} U_F$ already covers all of $M$. A family of examples is given by finite products $S^2 \times \cdots \times S^2$ of 2-spheres on which $S^1$ acts diagonally, the action on each $S^2$ being given by rotation around the axis through the north and south poles. In such a case one can put $\chi_{\text{top}} := 0$, avoiding the individual treatment of the principal stratum.

The generalized Witten integral \eqref{eq:genWittenIntegral} can now be written as a sum

$$(3.5) \quad I^\xi_a(\varepsilon) = I^\xi_{\chi_{\text{top}}}(\varepsilon) + \sum_{F \in \mathcal{F}_a} I^\xi_a(\chi_F)\varepsilon.$$

We shall focus our attention in the following on the integrals $I^\xi_a(\chi_F)\varepsilon$. In terms of the coordinates provided by $\Phi_F$ we obtain with \eqref{eq:genWittenIntegral2}

$$(3.6) \quad I^\xi_{a\chi_F}(\varepsilon) = \int_{\mathcal{J}^{-1}} \int_{V_F} e^{i(\mathcal{J} \circ \Phi_F^{-1}([\varphi, \psi]) - \varepsilon/\varepsilon)(x)} (a\chi_F)(\Phi_F^{-1}([\varphi, \psi], x)) d([\varphi, \psi], x) dx$$

$$= \int_{\mathbb{R}} \int_{V_F} e^{i\frac{\varepsilon}{\varepsilon}(Q(\varphi, w) - 2\zeta_F)} (a\chi_F)(\Phi_F^{-1}([\varphi, \psi], x)) d([\varphi, \psi], x) dx,$$
where we identified \( t \) with \( \mathbb{R} \), \( d([\varphi, w]) \) denotes the pullback of the symplectic volume form \( dp \) on \( M \) under the symplectomorphism \( \Phi_F^{-1} \) and coincides precisely with the symplectic volume form on \( V_F \),

\[
\zeta_F := \zeta - \mathcal{J}(F),
\]

and we introduced on \( \mathbb{C}^{\ell^+ + \ell^-} \) the non-degenerate quadratic form

\[
\langle Q_F w, w \rangle := \sum_{j=1}^{\ell^+ + \ell^-} \lambda_j^F |w_j|^2 = \sum_{j=1}^{\ell^+ + \ell^-} \lambda_j^F ((\text{Re} \, w_j)^2 + (\text{Im} \, w_j)^2).
\]

Since \( \mathcal{J} \circ \Phi_F^{-1}([\varphi, w]) \) actually defines a function on \( P_F \times \mathbb{C}^{\ell^+ + \ell^-} \), we shall lift \( \mathcal{J}_\chi^\ell \) accordingly. For this, let us note that since \( \pi_F : P_F \times \mathbb{C}^{\ell^+ + \ell^-} \to P_F \times_K \mathbb{C}^{\ell^+ + \ell^-} \) is a principal \( K_F \)-bundle there exists for every volume density \( d_{\text{vol}} \) on \( P_F \times \mathbb{C}^{\ell^+ + \ell^-} \) a form \( \eta \) on the same space such that \( d_{\text{vol}} = |\pi_F^* (d([\varphi, w])) \wedge \eta| \) and the restriction of \( \eta \) to each fiber \( \pi_F^{-1}([\varphi, w]) \) defines a volume density denoted by \( \eta_{[\varphi, w]} \) (cf [11, p. 430]). Then by [11, Theorem 4.8], we have for any continuous function \( f \) with compact support in \( V_F \) the equality

\[
\int_{P_F \times \mathbb{C}^{\ell^+ + \ell^-}} \pi_F^* (f) d_{\text{vol}} = \int_{V_F} \int_{\pi_F^{-1}([\varphi, w])} \pi_F^* (f) \eta_{[\varphi, w]} d([\varphi, w]) = \int_{V_F} f([\varphi, w]) \mathcal{V}([\varphi, w]) d([\varphi, w]),
\]

where \( \mathcal{V}([\varphi, w]) := \int_{\pi_F^{-1}([\varphi, w])} \eta_{[\varphi, w]} \) is the volume of the fiber over \([\varphi, w]\) with respect to \( \eta_{[\varphi, w]} \). Note that \( \pi_F^* (f) \) has still compact support since the fibers are compact. Now, let \( d\varphi \) be a volume density on \( P_F \) and \( dw \) the canonical symplectic volume form on \( \mathbb{C}^{\ell^+ + \ell^-} \). Applying the previous considerations to our integral \( \mathcal{J}_\chi^\ell \) with respect to the volume density \( d_{\text{vol}} = d\varphi \, dw \) yields Fubini

\[
\mathcal{J}_\chi^\ell(F) = \int_{\mathbb{R}^{\ell^+ + \ell^-}} \int_{\mathbb{R}^{\ell^+ + \ell^-}} e^{i\mathcal{J}(Q_F w, w - 2\zeta_F)} \, (a_F(F)(\Phi_F^{-1}(\pi_F([\varphi, w])), x) \, \mathcal{V}(\pi_F([\varphi, w])))^{-1} \, d\varphi \, dw.
\]

where we identified \( \mathbb{C}^{\ell^+ + \ell^-} \) with \( \mathbb{R}^{2(\ell^+ + \ell^-)} \). With respect to this identification, denote by

\[
n_F^+ := 2\ell^+, \quad n_F^- := 2\ell^-
\]

the real dimensions of the positive and negative eigenspaces of \( Q_F \), and assume first that \( n_F^+ \neq 0 \) and \( n_F^- \neq 0 \). Introducing polar coordinates \( w^+ = (w_1, \ldots, w_{\ell^+}) = r \theta^+ \in \mathbb{R}^{2\ell^+} \) and \( w^- = (w_{\ell^+ + 1}, \ldots, w_{\ell^+ + \ell^-}) = s \theta^- \in \mathbb{R}^{2\ell^-} \) in these directions, the integral \( \mathcal{J}_\chi^\ell(F) \) reads

\[
\mathcal{J}_\chi^\ell(F) = \int_{-\infty}^{\infty} \int_{0}^{\infty} e^{i\mathcal{J}(r^2 + s^2 - 2\zeta_F)} \, \alpha_F(r, s, x) \, dr \, ds \, dx,
\]

where

\[
\alpha_F(r, s, x) := r^{n_F^+ - 1} s^{n_F^- - 1} \int_{S^{n_F^+ - 1}} \int_{S^{n_F^- - 1}} \bar{a}_F(r \theta^+, s \theta^-), x) \, d\theta^+ \, d\theta^-,
\]

With the notation \( (2.12) \) the amplitude \( \alpha_F \) can be written in the form

\[
\alpha_F(r, s, x) = \beta_F(r, s) \sigma(x),
\]
where
\[
\beta_F(r, s) := R_F(r, s)S_F(r, s),
\]
(3.10)
\[
R_F(r, s) := r^{n_F^+ - 1}s^{n_F^- - 1},
\]
\[
S_F(r, s) := \int_{S^{n_F^- - 1}} \int_{S^{n_F^- - 1}} \tilde{b}_F(r\theta^+, s\theta^-) d\theta^+ d\theta^-
\]
with
\[
\tilde{b}_F(w) := \int_{p_F} (b\chi_F)(\Phi_F^{-1}(\pi_F(\varphi, w))) V(\pi_F(\varphi, w))^{-1} d\varphi.
\]
The double spherical mean \(S_F(r, s)\) is symmetric in \(r\) and \(s\). If \(n_F^+ = 0\) and \(n_F^- \neq 0\) or \(n_F^+ \neq 0\) and \(n_F^- = 0\), the integral (3.10) can be written as
\[
I_{\alpha_F}^c(\varepsilon) = \int_{-\infty}^{\infty} \int_0^\infty e^{\frac{\varepsilon}{2\pi}(\pm r^2 - 2\varepsilon\rho)} \alpha_F(r, x) \, dr \, dx, \quad n_F^- = 0,
\]
where
\[
\alpha_F(r, x) := r^{2n_F^- - 1} \int_{S^{2n_F^- - 1}} \tilde{a}_F(r\theta, x) d\theta = \beta_F(r)\sigma(x),
\]
and
\[
\beta_F(r) := R_F(r)S_F(r),
\]
(3.11)
\[
R_F(r) := r^{2n_F^- - 1},
\]
\[
S_F(r) := \int_{S^{2n_F^- - 1}} \tilde{b}_F(r\theta) d\theta.
\]
Note that \(S_F(r)\) is symmetric in \(r\).

### 3.2. Regular asymptotics.

We are now ready to give an asymptotic expansion of the generalized Witten integral, and begin with the regular contributions.

#### 3.2.1. Contribution of the top chart \(M_{\text{top}}\).

Recall that by Lemma 2.22 the momentum map is regular on \(M_{\text{top}}\). Therefore Proposition 2.4 yields a complete stationary phase expansion for \(I_{\alpha_F}^c(\varepsilon)\). The coefficients \(Q_j(\psi, a\chi)\) do have a geometric interpretation in terms of measures on \(\mathcal{J}^{-1}(\{\zeta\})\) and are smooth in \(\zeta\).

#### 3.2.2. Contributions of the charts \(U_F\) with \(\mathcal{J}(F) \neq \zeta\).

For a component \(F \in \mathcal{F}_a\), consider the corresponding chart \(U_F\). By Lemma 2.22 and the fact that \(F\) is the only component of \(M^T\) contained in \(U_F\), the only singular value of \(\mathcal{J} : U_F \to t^*\) is \(\mathcal{J}(F)\), so that, again, Proposition 2.4 yields a complete stationary phase expansion for \(I_{\alpha_F}^c(\varepsilon)\) if \(\mathcal{J}(F) \neq \zeta\). The obtained coefficients \(Q_j(\psi, a\chi)\) are smooth in \(\zeta\) away from \(\mathcal{J}(F)\) and given in terms of measures on \(\mathcal{J}^{-1}(\{\zeta\})\). In what follows we shall calculate them in terms of the normal form of the momentum map, based on the expressions (3.9) and (3.11).

**Proposition 3.3.** For each \(\delta > 0\) and \(N \in \mathbb{N}_0\) there is a constant \(C_{\delta, N} > 0\) such that for all \(\zeta \in \mathbb{R}\) with \(|\zeta - \mathcal{J}(F)| > \delta\) one has
\[
\left| I_{\alpha_F}^c(\varepsilon) - 2\pi \varepsilon \sum_{k=0}^N (2\varepsilon)^k \frac{(-i \text{sgn} \mathcal{J}(F))^{k} \sigma^{(k)}(0) Q_{F,a}^k(\zeta)}{k!} \right| \leq C_{\delta, N} \varepsilon^{N+2} \quad \forall \varepsilon > 0,
\]
where the coefficients $Q^k_{F,a}(\zeta)$ are given by the expressions

$$
\begin{align*}
&\frac{d^k}{ds^k}\left[n^{-1} S_F(\sqrt{s})\right](\pm 2\zeta_F), \\
&\int_0^\infty t^{n_{\zeta}} \frac{d^k}{ds^k}\left[(t^2 + s)^{n_{\zeta}}/2 - 1 S_F(\sqrt{t^2 + s}, t)\right](2\zeta_F)\frac{dt}{t}, \\
&\int_0^\infty t^{n_{\zeta}} \frac{d^k}{ds^k}\left[(t^2 + s)^{n_{\zeta}}/2 - 1 S_F(t, \sqrt{t^2 + s})\right](-2\zeta_F)\frac{dt}{t},
\end{align*}
$$

In particular, the map $\mathbb{R} \setminus [J(F) - \delta, J(F) + \delta] \ni \zeta \mapsto Q^k_{F,a}(\zeta)$ is smooth.

Proof. For $\zeta \in \mathbb{R} \setminus [J(F) - \delta, J(F) + \delta]$ we have $|\zeta_F| > \delta$. Let us begin with the simpler case $n_{\zeta} = 0$. We first linearize the phase function via the substitution $r^2 \mapsto t$, yielding

$$
I_{a,\chi}^\zeta(\varepsilon) = \frac{1}{2} \int_{-\infty}^\infty \int_{-\infty}^\infty e^{i \sqrt{t} (\pm t - 2\zeta_F)} t^{n-1} S_F(\sqrt{t}) \, dt \, \sigma(x) \, dx.
$$

Next, take a cutoff function $\chi \in C^\infty_c(\mathbb{R})$ with $\text{supp } \chi \subset (0, \infty)$ and $\chi \equiv 1$ on $[2|\zeta_F| - \delta, 2|\zeta_F| + \delta]$. We then define the smooth function

$$
\Gamma_{F,\chi}(t) := \begin{cases} 
  t^{n-1} S_F(\sqrt{t}) \chi(t), & t > 0, \\
  0, & t \leq 0,
\end{cases}
$$

and get for $\zeta \in \mathbb{R} \setminus [J(F) - \delta, J(F) + \delta]$ and all $N \in \mathbb{N}_0$ the equality

$$
\begin{align*}
I_{a,\chi}^\zeta(\varepsilon) &= \frac{1}{2} \int_{-\infty}^\infty \int_{-\infty}^\infty e^{i \sqrt{t} (\pm t - 2\zeta_F)} t^{n-1} S_F(\sqrt{t}) \chi(t) \, dt \, \sigma(x) \, dx \\
&\quad + \frac{1}{2} \int_{-\infty}^\infty \int_{-\infty}^\infty e^{i \sqrt{t} (\pm t - 2\zeta_F)} t^{n-1} S_F(\sqrt{t}) (1 - \chi(t)) \, dt \, \sigma(x) \, dx \\
&\quad - \frac{1}{2} \int_{-\infty}^\infty \int_{-\infty}^\infty e^{i \sqrt{t} (\pm t - 2\zeta_F)} \Gamma_{F,\chi}(t) \, dt \, \sigma(x) \, dx \\
&\quad + \varepsilon^N (-1)^N 2^{N-1} \int_0^\infty \int_0^\infty \frac{d^N}{dx^N} \left[e^{i \sqrt{t} (\pm t - 2\zeta_F)}\right] \sigma(x) \, dx \frac{t^{n-1} S_F(\sqrt{t}) (1 - \chi(t))}{(\pm t - 2\zeta_F)^N} \, dt \\
&= \frac{1}{2} \int_{-\infty}^\infty \int_{-\infty}^\infty e^{i \sqrt{t} (\pm t - 2\zeta_F)} \Gamma_{F,\chi}(t) \, dt \, \sigma(x) \, dx \\
&\quad + \varepsilon^N (-1)^N 2^{N-1} \int_0^\infty \int_{-\infty}^\infty e^{i \sqrt{t} (\pm t - 2\zeta_F)} \sigma^{(N)}(x) \, dx \frac{t^{n-1} S_F(\sqrt{t}) (1 - \chi(t))}{(\pm t - 2\zeta_F)^N} \, dt,
\end{align*}
$$

(3.13)
the last summand being of order $O_{δ,χ}(ε^N)$ uniformly in $ζ$. Performing the substitutions $x \mapsto 2εx$ and $t \mapsto ±t + 2ζ ± 2J(F)$ yields

$$
\frac{1}{2} \int_{-∞}^{∞} \int_{-∞}^{∞} e^{iζ}(±t - 2ζF) Γ_{F,χ}(t) dt = ε \int_{-∞}^{∞} \int_{-∞}^{∞} e^{-ixt} Γ_{F,χ}(±t ± 2ζF) dt σ(2εx) dx
$$

$$
= ε \int_{-∞}^{∞} F[Γ_{F,χ}(±t ± 2ζF)](x)σ(2εx) dx.
$$

The last integral can be rewritten by expanding $σ$ into a Taylor series at 0, with dominated convergence applied to the Taylor remainder term. In this way we obtain for all $N ∈ \mathbb{N}_0$

$$
\int_{-∞}^{∞} F[Γ_{F,χ}(±t ± 2ζF)](x)σ(2εx) dx
$$

$$
= \sum_{k=0}^{N} \frac{ε^k k^2 σ(k)(0)}{k!} \int_{-∞}^{∞} F[Γ_{F,χ}(±t ± 2ζF)](x)x^k dx + O_{σ,Γ_{F,χ}}(ε^{N+1})
$$

$$
= \sum_{k=0}^{N} \frac{ε^k (-i)^k 2k^2 σ(k)(0)}{k!} \int_{-∞}^{∞} F[Γ_{F,χ}(±t ± 2ζF)](0)(x)dx + O_{σ,Γ_{F,χ}}(ε^{N+1})
$$

$$
= 2π \sum_{k=0}^{N} \frac{ε^k (-i)^k 2k^2 σ(k)(0)}{k!} \frac{d^k}{dt^k} (Γ_{F,χ}(±t ± 2ζF))(0) + O_{σ,Γ_{F,χ}}(ε^{N+1})
$$

$$
= 2π \sum_{k=0}^{N} \frac{ε^k (±i)^k 2k^2 σ(k)(0)}{k!} \frac{d^k}{dt^k} (ln^{-1} S_F(√t))(±2ζF) + O_{σ,Γ_{F,χ}}(ε^{N+1})
$$

Taking everything together, we arrive for $n_F = 0$ and $ζ ∈ \mathbb{R} \setminus [J(F) − δ, J(F) + δ]$ at the expansion

$$
(3.14) \quad \Gamma_{αχ_F}^ζ(ε) ∼ ε \sum_{k≥0} Q_{F,a}^k(ζ)ε^k
$$

where the estimates in the asymptotic are uniform in $ζ ∈ \mathbb{R} \setminus [J(F) − δ, J(F) + δ]$ but depend on $δ$, and

$$
(3.15) \quad Q_{F,a}^k(ζ) = 2π \frac{ε^k (±i)^k 2k^2 σ(k)(0)}{k!} \frac{d^k}{dt^k} (ln^{-1} S_F(√t))(±2ζF).
$$

Let us now turn to the case $n_F ≠ 0, n_F ≠ 0$, and first assume that $ζ_F > 0$. As in the previous case, we linearize the phase function to obtain

$$
\Gamma_{αχ_F}^ζ(ε) = \frac{1}{4} \int_{-∞}^{∞} \int_{0}^{∞} e^{iζ(t−u−2ζF)} ln_F^k/2-1 u^{n_F/2-1} S_F(√t, √u) dt du σ(x) dx,
$$
and get for \( \zeta \in \mathbb{R} \setminus [\mathcal{J}(F) - \delta, \mathcal{J}(F) + \delta] \)

\[
I_{\alpha F}^{\zeta}(\varepsilon) = \frac{1}{4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i \frac{\pi}{2} (t-u-2\zeta F)} t^{n_{F}/2-1} u^{n_{F}/2-1} S_F(\sqrt{t}, \sqrt{u}) \chi(t-u) \, dt 
\frac{d}{du} \sigma(x) \, dx + O_\delta(\varepsilon^\infty)
\]

\[
= \frac{1}{4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i \frac{\pi}{2} (t-2\zeta F)} (t+u)^{n_{F}/2-1} u^{n_{F}/2-1} S_F(\sqrt{t+u}, \sqrt{u}) \chi(t) \, dt 
\frac{d}{du} \sigma(x) \, dx + O_\delta(\varepsilon^\infty)
\]

\[
= \frac{1}{4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i \frac{\pi}{2} (t-2\zeta F)} \Gamma_{F, \chi}^+(t) \, dt \sigma(x) \, dx + O_\delta(\varepsilon^\infty),
\]

where

\[
\Gamma_{F, \chi}^+(t) := \begin{cases} 
\int_{0}^{\infty} (t+u)^{n_{F}/2-1} u^{n_{F}/2-1} S_F(\sqrt{t+u}, \sqrt{u}) \, du \chi(t), & t > 0, \\
0, & t \leq 0.
\end{cases}
\]

Comparing this with \ref{eq:3.13}, we can apply the above computations to obtain

\[
I_{\alpha F}^{\zeta}(\varepsilon) \sim \varepsilon \sum_{k \geq 0} Q_{F,a}^k(\zeta) \varepsilon^k, \quad \zeta \in \mathbb{R} \setminus [\mathcal{J}(F) - \delta, \mathcal{J}(F) + \delta],
\]

where now

\[
Q_{F,a}^k(\zeta) = \pi \frac{\varepsilon^k (1-i)^k 2^k \sigma(k)(0)}{k!} \frac{d^k}{du^k} (\Gamma_{F, \chi}^+(t+2\zeta F))(0)
\]

\[
= \pi \frac{\varepsilon^k (1-i)^k 2^k \sigma(k)(0)}{k!} \int_{0}^{\infty} \frac{d^k}{du^k} \left[ (t+u)^{n_{F}/2-1} S_F(\sqrt{t+u}, \sqrt{u}) \right] (2\zeta F) u^{n_{F}/2-1} \, du.
\]

Finally, assume that \( \zeta F < 0 \). Then we get for \( \zeta \in \mathbb{R} \setminus [\mathcal{J}(F) - \delta, \mathcal{J}(F) + \delta] \)

\[
I_{\alpha F}^{\zeta}(\varepsilon) = \frac{1}{4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i \frac{\pi}{2} (t-u-2\zeta F)} t^{n_{F}/2-1} u^{n_{F}/2-1} S_F(\sqrt{t}, \sqrt{u}) \chi(u-t) \, dt 
\frac{d}{du} \sigma(x) \, dx + O_\delta(\varepsilon^\infty)
\]

\[
= \frac{1}{4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i \frac{\pi}{2} (u-2\zeta F)} t^{n_{F}/2-1} (t+u)^{n_{F}/2-1} S_F(\sqrt{t+u}, \sqrt{u}) \chi(u) \, dt 
\frac{d}{du} \sigma(x) \, dx + O_\delta(\varepsilon^\infty)
\]

\[
= \frac{1}{4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i \frac{\pi}{2} (u-2\zeta F)} \Gamma_{F, \chi}^-(u) \, du \sigma(x) \, dx + O_\delta(\varepsilon^\infty),
\]

where

\[
\Gamma_{F, \chi}^-(u) := \begin{cases} 
\int_{0}^{\infty} (t+u)^{n_{F}/2-1} (t+u)^{n_{F}/2-1} S_F(\sqrt{t+u}, \sqrt{u}) \, dt \chi(u), & u > 0, \\
0, & u < 0.
\end{cases}
\]

Comparing again with \ref{eq:3.13}, we arrive at the expansion

\[
I_{\alpha F}^{\zeta}(\varepsilon) \sim \varepsilon \sum_{k \geq 0} Q_{F,a}^k(\zeta) \varepsilon^k, \quad \zeta \in \mathbb{R} \setminus [\mathcal{J}(F) - \delta, \mathcal{J}(F) + \delta],
\]

where

\[
Q_{F,a}^k(\zeta) = \pi \frac{\varepsilon^k (1-i)^k 2^k \sigma(k)(0)}{k!} \frac{d^k}{du^k} (\Gamma_{F, \chi}^-(u-2\zeta F))(0)
\]

\[
= \pi \frac{\varepsilon^k (1-i)^k 2^k \sigma(k)(0)}{k!} \int_{0}^{\infty} \frac{d^k}{du^k} \left[ (t+u)^{n_{F}/2-1} S_F(\sqrt{t+u}, \sqrt{u}) \right] (-2\zeta F) u^{n_{F}/2-1} \, dt,
\]

and renaming \((u, t)\) to \((t, u)\) and substituting \(u \to v^2\) finishes the proof.
3.3. Singular asymptotics. In what follows, we shall derive an asymptotic expansion for the general-
ized Witten integral in the charts $U_F$ in the singular case when $|\zeta_F| \ll 1$.

3.3.1. The contributions of the indefinite charts. Let us begin by considering a chart $U_F$ with $F \in F_a$
in which $Q_F$ is indefinite, that is, $n_F^+ \neq 0$ and $n_F^- \neq 0$. In this case, complete asymptotic expansions
for integrals of the type $I_{\chi_F}^\zeta(3.19)$ were derived by Brunmehluis, Paul, and Uribe [2] Section 3] if $\zeta_F = 0$,
and it is not too difficult to extend their analysis to cover also the case $|\zeta_F| \ll 1$. Indeed, since $n_F^+$
and $n_F^-$ are even, a computation along the lines of [2, Section 3] shows that there are $\delta_0, \varepsilon_0 > 0$ such
that for $|\zeta_F| < \delta_0$, $0 < \varepsilon < \varepsilon_0$ the asymptotic expansion

$$I_{\chi_F}^\zeta(3.19) \sim \sum_{j=0}^{\infty} (2\varepsilon)^{1+j} \left[ (B_j(\beta_F) + q_2(\beta_F)) \Xi_j(\zeta_F/\varepsilon) + (B_j(\beta_F) + q_2(\beta_F)) \tilde{\Xi}_j(\zeta_F/\varepsilon) \right]$$

holds, where

$$B_j(\beta_F) := -\frac{1}{(2j)!} \int_0^\infty \left[ \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right) \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial s^2} \right) \beta_F \right] (t,t) \log(2t) \, dt$$

$$q_j(\beta_F) := \frac{1}{2j} \sum_{k=0}^{j} A_{j,k} \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial s} \right) \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right) \left( 2k - j \right) \left( 2k - j \right) \beta_F(0,0),$$

the $A_{j,k}$ being certain computable\footnote{Here we corrected [2, (79) on p. 497] by inserting an overall factor $\frac{1}{j!}$} combinatorial coefficients fulfilling $A_{j,j-k} = -A_{j,k}$, while

$$B_j(\beta_F) := B_j(\beta_F, s, r) = (-1)^j B_j(\beta_F), \quad q_j(\beta_F) := q_j(\beta_F, s, r),$$

with $\beta_F(s, r), \beta_F, q_j(\beta_F)$, and for $t \in \mathbb{R}$ one has

$$\Xi_j(t) := \int_{-t}^t (t + \xi)^j \tilde{\sigma}(-\xi) \, d\xi = \sum_{l=0}^{j} \left( \begin{array}{c} j \varepsilon \\ l \end{array} \right) \int_{-t}^t \xi^{j-l} \tilde{\sigma}(-\xi) \, d\xi =: \sum_{l=0}^{j} t^l \Xi_{j,l}(t),$$

$$\tilde{\Xi}_j(t) := \int_{-t}^t (t - \xi)^j \tilde{\sigma}(-\xi) \, d\xi = \sum_{l=0}^{j} \left( \begin{array}{c} j \varepsilon \\ l \end{array} \right) \int_{-t}^t (-\xi)^{j-l} \tilde{\sigma}(-\xi) \, d\xi =: \sum_{l=0}^{j} t^l \tilde{\Xi}_{j,l}(t).$$

Note that if $\sigma(x) = e^x \phi(x)$ for some $\varepsilon, \sigma \in \mathbb{N}$ and $\phi \in \mathcal{S}(\mathbb{R})$,

$$\tilde{\sigma}(-\xi) = \left( i \partial_t \right)^k \hat{\phi}(\xi) = 2\pi (i \partial_t)^k \hat{\phi}(\xi) = 2\pi (i \partial_t)^k \hat{\phi}(\xi) \phi(\xi).$$

Clearly, the limits $\lim_{t \to \pm \infty} \Xi_{j,l}(t)$, $\lim_{t \to \pm \infty} \tilde{\Xi}_{j,l}(t)$ exist. In particular, $\Xi_{j,l}(t)$ and $\tilde{\Xi}_{j,l}(t)$ are bounded
in $t$, with bounds depending on $\sigma, j, l$. Moreover,

$$B_j(\beta_F) \Xi_{j,l}(t) + B_j(\beta_F) \tilde{\Xi}_{j,l}(t) = \left( \begin{array}{c} j \varepsilon \\ l \end{array} \right) B_j(\beta_F) \int_{-\infty}^{\infty} \xi^{j-l} \tilde{\sigma}(-\xi) \, d\xi$$

as a consequence of (3.19). Collecting everything we arrive at

\footnote{See 3.4.1 for a computation of the first few values.}
Proposition 3.4. Assume that $Q_F$ is indefinite. Then, there are $\delta_0, \varepsilon_0 > 0$ such that for $|\zeta_F| < \delta_0$ one has an asymptotic expansion

$$I^0_{\alpha \times F}(\varepsilon) \sim \sum_{j=0}^{\infty} (2\varepsilon)^{1+j} \sum_{l=0}^{\infty} \Theta_{l+j,l}^F(\zeta_F / \varepsilon) (2\varepsilon)^l,$$

where the coefficients

$$\Theta_{j,l}^F(t) := \left( \frac{1}{l!} \right) B_j(\beta_F) \int_{-\infty}^{\infty} \xi^{l-1} \xi_0(-\xi) d\xi + q_{2j}(\beta_F) \Xi_{j,l}(t) + \tilde{q}_{2j}(\beta_F) \tilde{\Xi}_{j,l}(t)$$

can be computed explicitly, and the limits $\lim_{t \to \pm \infty} \Theta_{j,l}^F(t)$ exist. In particular, $\Theta_{j,l}^F(t)$ is bounded in $t$.

Remark 3.5. The asymptotics in the previous proposition can also be obtained by more direct methods that do not use [2] and were suggested to us recently by Michèle Vergne, and which we will outline in a future version of this paper.

Thus, we have obtained a complete asymptotic expansion for the integrals $I^0_{\alpha \times F}(\varepsilon)$ simultaneously in both parameters $\zeta$ and $\varepsilon$. In the following, we shall examine the $B_j$-coefficients from (3.15) more closely. To this end, recall (3.11) and introduce the notation

$$\partial_\pm := \partial_r \pm \partial_s, \quad n_F^0 := n_F^+ + n_F^- - 2 = 2(\ell^+ + \ell^- - 1).$$

Lemma 3.6. The coefficients $B_j(\beta_F)$ are given by the following expressions:

1. If $j < n_F^0 / 2$,

$$B_j(\beta_F) = \sum_{k=0}^{\frac{j}{2}} \mu_k^j \int_0^\infty t^{n_F^0 - 1} \left( \partial_k^j S_F(t) \right) \frac{dt}{t} \quad \text{=: } B_j^{\text{top}}(\beta_F);$$

2. if $n_F^0 / 2 \leq j < n_F^0$,

$$B_j(\beta_F) = \sum_{k=2j-n_F^0+1}^{\frac{j}{2}} \mu_k^j \int_0^\infty t^{n_F^0 - 1} \left( \partial_k^j S_F(t) \right) \frac{dt}{t} + \sum_{k=0}^{2j-n_F^0} \omega_k^j \int_0^\infty T_{k,j}^F(t) dt \quad \text{=: } B_j^{\text{top}}(\beta_F) + B_j^{\text{sing}}(\beta_F);$$

3. if $j \geq n_F^0$,

$$B_j(\beta_F) = \sum_{k=j-n_F^0}^{j} \omega_k^j \int_0^\infty T_{k,j}^F(t) dt + \sum_{k=j-n_F^0+1}^{\frac{j}{2}} \eta_k^j \left( \partial_k^j S_F(t) \right) \left( 0, 0 \right) \quad \text{=: } B_j^{\text{top}}(\beta_F) + B_j^{\text{sing}}(\beta_F);$$

Here

$$\eta_k^j := (-1)^k \sum_{l=0}^{n_F^0 - j - k - 1} (n_F^0 - j - k - l - 1)! C_{k,l}, \quad \mu_k^j := \sum_{l=0}^{j} (j - l)! C_{k,l}, \quad \omega_k^j := (-1)^k C_{k,n_F^0 - j + k},$$

$$\delta_0, \varepsilon_0 > 0$$
where
\[ C_{k,l}^{j} := \frac{(n_{F}^{+} - 1)!((n_{F}^{-} - 1)!)^{-1}}{(j!)^{k}(j!)^{l}} \binom{j}{k} \binom{j}{l} \sum_{w=\text{max}(0, -k-l-n_{F}^{-} + 1)}^{\min(j-k+l, n_{F}^{-} - 1)} \frac{\sum_{v=\text{max}(0, w-l)}^{\min(j-k, w)} \binom{l}{1} \binom{k-v}{v} (-1)^{k+v} v^{k-v} d_{r}^{l} d_{s}^{-u} d_{r}^{k-l} d_{s}^{v}}{(n_{F}^{+} - 1 - w)!(n_{F}^{-} - 1 - j + k - l + w)!} \cdot \]

and the functions \( T_{F}^{j,k} \in C^{\infty}((0, \infty)) \) are defined as
\[ T_{F}^{j,k}(t) := \frac{1}{2t} \left( \partial_{+}^{2j-n_{F}^{+} - k} \partial_{-}^{k} - \partial_{+}^{j} \partial_{-}^{j-k-n_{F}^{+} - k} \right) S_{F}(t, t), \]
where the expression that is multiplied with \( \frac{1}{2t} \) is of order \( O(t) \) as \( t \to 0^{+} \).

**Proof.** To begin, we notice for \( f, g \in C^{\infty}(\mathbb{R}^{2}) \) the equalities
\[ \partial_{\pm}(f \cdot g) = (\partial_{\pm} f) g + f \partial_{\pm} g, \quad (\partial_{r}^{2} - \partial_{s}^{2}) = (\partial_{+} \partial_{-}) f = (\partial_{-} \partial_{+}) f, \]
as well as
\[ \partial_{t}[f(t, t)] = (\partial_{+} f)(t, t). \]
In view of the above relations we can re-write the \( B_{j} \)-coefficients as
\[ B_{j}(\beta_{F}) = \frac{1}{(j!)^{2} 22j+1} \int_{0}^{\infty} \left| (\partial_{+} \partial_{-})^{j} \beta_{F}(t, t) \right| \frac{dt}{t} = \frac{1}{(j!)^{2} 22j+1} \int_{0}^{\infty} \partial_{t}^{j} \left[ \partial_{-}^{2} \beta_{F}(t, t) \right] \frac{dt}{t}. \]
If the amplitude \( b \) were supported away from the fixed-point \( F \), the integrand would be 0 in a neighborhood of \( r = 0 \), and partial integration would yield
\[ B_{j}(\beta_{F}) = \int_{0}^{\infty} \frac{1}{j!} \partial_{t}^{j} \beta_{F}(t, t) \frac{dt}{t}, \]
up to multiplication by a constant, which corresponds to the coefficient of order \( j \) in the usual stationary phase expansion. In general, it is not possible to proceed like this, since non-integrable terms would arise. Instead, notice that
\[ (\partial_{+} \partial_{-})^{j} \beta_{F}(t, t) = \sum_{k,l=0}^{j} c_{k,l}^{j} \left( \partial_{+}^{k} \partial_{-}^{l} R_{F}(t, t) \right) \left( \partial_{t}^{j-k-l} \partial_{-}^{k} S_{F}(t, t) \right), \]
where \( c_{k,l}^{j} = \binom{j}{k} \binom{j}{l} \). Furthermore,
\[ (\partial_{+} \partial_{-})^{j} \beta_{F}(t, t) = \begin{cases} d_{k,l} t^{n_{F}^{+} - k - l}, & k + l \leq n_{F}^{+}, \\ 0, & \text{otherwise}, \end{cases} \]
where
\[ d_{k,l} = (n_{F}^{+} - 1)!(n_{F}^{-} - 1)! \sum_{w=\text{max}(0, k+l-n_{F}^{-} - 1)}^{\min(k+l, n_{F}^{-} - 1)} \sum_{v=\text{max}(0, w-l)}^{\min(k, w)} \binom{l}{v} \binom{k-v}{v} (-1)^{k-v} v^{k-v} d_{r}^{l} d_{s}^{-u} d_{r}^{k-l} d_{s}^{v}. \]
To see this, we compute
\[ \partial_{+}^{k} \partial_{-}^{l} = (\partial_{+} \partial_{-})^{l}(\partial_{+} \partial_{-})^{k} = \sum_{u=0}^{l} \sum_{v=0}^{k} \binom{l}{u} \binom{k}{v} (-1)^{k-v} v^{k-v} d_{r}^{l} d_{s}^{-u} d_{r}^{k-l} d_{s}^{v}. \]
\[ = \sum_{u=0}^{l} \sum_{v=0}^{k} \binom{l}{u} \binom{k}{v} (-1)^{k-v} v^{k-v} d_{r}^{i} d_{s}^{-u} d_{r}^{k-l} d_{s}^{v}, \]
\[ = \sum_{w=0}^{k+l} \sum_{v=\text{max}(0, w-l)}^{\min(k, w)} \binom{l}{v} \binom{k-v}{v} (-1)^{k-v} v^{k-v} d_{r}^{l} d_{s}^{-u} d_{r}^{k-l} d_{s}^{v}. \]
proving (3.27) and (3.28). In particular, if \( k \) from this we infer \( j < (3.30) \)

Now, from (3.25) - (3.28) we get \( (3.29) \)

which follows from (3.29) by substituting \( v \) by \( n_F^+ - 1 - v \). Note that \( C_{k,l}^j = (-1)^{j+l} c_{j-k,l}^j n_F^+-1 v \)

yielding the relation \( (3.30) \)

which follows from (3.29) by substituting \( v \) by \( n_F^+-1 v \). Note that \( C_{k,l}^j = (-1)^{j+l} c_{j-k,l}^j n_F^+-1 v \)

Now, from (3.25) - (3.28) we get \( 6 \)

Let us turn first to the case \( j < \frac{n_F^+}{2} \). Then (3.24) and partial integration give \( (j!)^2 2^{j+1} B_j(\beta_F) = \sum_{0 \leq k,l \leq j} c_{k,l}^j d_{k,l} \int_0^\infty t^{n_F^+-k-l-1} \left( \partial_+^{j-l} \partial_-^{j-k} S_F \right) (t,t) dt \)

6Note that this expression contains non-integrable summands which, nevertheless, cancel out each other, see below.
yielding (1). Assume next that \( j \geq n_F^0 \). In this case,

\[
(j!)^2 2^{2j+1} B_j(\beta_F) = \sum_{k+l \leq n_F^0} c_{k,l}^j dk,li \int_0^\infty e^{i(k-\alpha_{k,l}^j)} (\partial^j_+ \partial^{j-k}_- S_F)(t,t) dt
\]

\[
= \sum_{k+l \leq n_F^0} c_{k,l}^j dk,li (-1)^{n_F^0-k-l} (n_F^0-k-l-1)! \int_0^\infty (\partial^j_+ \partial^{j-k}_- S_F)(t,t) dt
\]

\[
+ \sum_{k+l=n_F^0} c_{k,l}^j dk,li \int_0^\infty (\partial^j_+ \partial^{j-k}_- S_F)(t,t) dt.
\]

Taking into account (3.24) this can be simplified further, yielding

\[
(j!)^2 2^{2j+1} B_j(\beta_F) = \sum_{k=0}^{n_F^0} \left[ - \sum_{l=0}^{n_F^0-k-1} \alpha_{k,l}^j \right] (\partial^j_+ \partial^{j-k}_- S_F)(0,0)
\]

\[
+ \sum_{k=0}^{n_F^0} \omega_{k,n_F^0-k}^j \int_0^\infty (\partial^j_+ \partial^{j-k}_- S_F)(t,t) dt.
\]

The last sum contains non-integrable contributions which, nevertheless, must cancel out each other, since all other terms in the above equality are finite. To see this, and identify the integrable contributions, note that (3.30) and \( c_{k,l}^j = c_{l,k}^j \) imply with \( L := 2j - n_F^0 \)

\[
(3.31) \quad \sum_{k=j-n_F^0}^{j} \omega_k^j \int_0^\infty (\partial_{\omega}^{L-k} \partial S_F)(t,t) dt = \frac{1}{2} \sum_{k=j-n_F^0}^{j} \omega_k^j \int_0^\infty [(\partial_{\omega}^{L-k} \partial S_F)(t,t) dt.
\]

Assume \( L - k \leq k \) as we may, and write \( k - (L - k) = 2\mu \) for some natural number \( \mu \), taking into account that \( n_F^0 \) is even. Then

\[
(3.32) \quad \partial_{\omega}^{L-k} \partial S_F = (\partial_{\omega}^{2\mu} - \partial_{\omega}^{2\mu}) S_F.
\]

Note that \( (\partial_{\omega} S_F)^{L-k} = (\partial_{\omega}^2 - \partial_{\omega}^2)^{L-k} \) is a sum of derivatives in the variables \( r \) and \( s \) of even order, respectively, while

\[
(3.33) \quad \partial_{\omega}^{2\mu} - \partial_{\omega}^{2\mu} = - \sum_{\nu=0}^{2\mu} 2(2\mu)! \partial_{\omega}^{\nu} \partial_{\omega}^{2\mu-\nu}.
\]

involves only derivatives of odd order in \( r \) and \( s \), respectively. Since \( S_F(r,s) \) is even in \( r \) and \( s \), applying \( (\partial_{\omega} S_F)^{L-k} \) to \( S_F \) yields an even function in \( r \) and \( s \). If we now also apply \( \partial_{\omega}^{2\mu} \) we obtain an odd function in \( r \) and \( s \). Consequently, \( (\partial_{\omega}^{L-k} \partial S_F)(r,s) \) must be even in \( r \) and vanish of order \( O(r^2) \) as \( r \to 0 \), so that each of the integrands on the right-hand side of (3.31) is integrable.
Treating the case $L - k \geq k$ alike we have shown (3). Finally, in the intermediate case $\frac{n_F^0}{2} \leq j < n_F^0$, one computes

\[
(j!)^2 2^{2j+1} B_j(\beta_F) = \sum_{0 \leq j, l \leq k, k + l \leq n_F^0} c_{k,l}^j d_{k,l} \int_0^\infty t^{n_F^0 - k - l - 1} \left( \partial^j_{\sigma^k} \partial^l_{S_F} \right)(t,t) dt
\]

\[
+ \sum_{k = n_F^0 - j}^{n_F^0 - j - 1} \sum_{l=0}^{n_F^0 - k - l - 1} c_{k,l}^j d_{k,l} \int_0^\infty t^{n_F^0 - j - l - 1} \left( \partial^j_{\sigma^k} \partial^l_{S_F} \right)(t,t) dt
\]

\[
+ \sum_{k = n_F^0 - j}^{2j} \sum_{l=0}^{n_F^0 - j - k - 1} c_{k,l}^j d_{k,l} \int_0^\infty t^{n_F^0 + k - 2j - 1} \left( \partial^k_{\sigma^l} \partial^j_{S_F} \right)(t,t) dt
\]

\[
(n!)^2 2^{2j+1} \left( \sum_{k = 2j - n_F^0 + 1}^{n_F^0 - j - 1} \mu_k^j \int_0^\infty t^{n_F^0 + k - 2j} \left( \partial^k_{\sigma^l} \partial^j_{S_F} \right)(t,t) dt + \sum_{k = 0}^{2j - n_F^0} \eta_k^j \left( \partial^2j - n_F^0 - k \partial^k_{S_F} \right)(0,0) \right)
\]

\[
+ \sum_{k = 0}^{2j - n_F^0} \omega_k^j \int_0^\infty \left( \partial^j_{\sigma^k} \partial^k_{S_F} \right)(t,t) dt
\]

Again, the final sum is finite by the arguments given above and we obtain (2). \qed

In the relevant case where $n_F^+ = n_F^-$, half of the coefficients in Lemma 3.6 vanish. Indeed, we have the following

**Lemma 3.7.** Assume that $n_F^+ = n_F^-$. If $j - k$ is odd,

\[
\mu_k^j = \omega_k^j = \eta_k^j = 0.
\]

**Proof.** Suppose that $k$ is odd. Clearly,

\[
\partial^k_{\sigma^l} R(r,s) = \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} \partial^i_{\sigma^l} \partial^{k-i}_{S_F} (rs)^{n_F^0}.
\]

Now, note that

\[
\left( \partial^i_{\sigma^l} \partial^{k-i}_{S_F} (rs)^{n_F^0} \right)_{r=\infty} = \left( \partial^{k-i}_{\sigma^l} (rs)^{n_F^0} \right)_{r=\infty}
\]

and

\[
\binom{k}{i} (-1)^{k-i} = -\binom{k}{k-i} (-1)^i.
\]
Indeed, if both \( k \) and \( i \) are odd the difference \( k - i \) is even. On the other hand, if \( i \) is even then \( k - i \) is odd. Since there is an even number of summands which are cancelling each other by the above relations we conclude that \( \partial_k^k R(t, t) = 0 \). From this we infer
\[
(\partial_k^k \partial_k^k R)(t, t) = \partial_k^k \partial_k^k R(t, t) = 0,
\]
which implies that \( d_k, t = 0 \) if \( k \) is odd. Finally, since each of the coefficients \( \mu_k^j, \omega_k^j \) and \( n_k^j \) is given by an expression of the form \( \sum_i \lambda_i \cdot d_{j-k, t} \), each summand is zero in case that \( j - k \) is odd.

In addition, for particular amplitudes there might be cancellations between contributions coming from different charts.

**Lemma 3.8.** Assume that \( S_F(r, s) = f(r)g(s) \) and \( S_{-F}(r, s) = g(r)f(s) \) for a pair of charts \((U_F, \Phi_F)\) and \((U_{-F}, \Phi_{-F})\). Then
\[
(\partial_k^k S_F)(t, t) = (-1)^k (\partial_k^k S_{-F})(t, t)
\]

**Proof.** With \((k_i) = \binom{k}{k-1}\) one clearly has
\[
(\partial_r - \partial_s)^k S_F(r, s) = \sum_{v=0}^k \binom{k}{v} \partial_r^{k-v} (-\partial_s)^v S_F(r, s) = \sum_{v=0}^k (-1)^v \binom{k}{v} f^{(k-v)}(r)g^{(v)}(s)
\]
and the assertion follows.

In what follows, we shall interpret the coefficients \( B_j^{\text{top}}(\beta_F) \) and \( B_j^{\text{sing}}(\beta_F) \) geometrically. The derivatives of \( S_F \) at \( r = s = 0 \) occurring in the terms \( B_j^{\text{sing}}(\beta_F) \) can clearly be associated with derivatives of the Dirac delta distribution \( \delta_0 \) centered at 0 in \( \mathbb{R}^{\text{codim} F} \). Regarding the remaining terms, note that for \( n \geq 2 \) (which is true when \( n_F > 0 \) and \( n_{-F} > 0 \)) there is a natural locally finite measure \( \mu_{Q_F} \) on the quadric \( \Sigma_{Q_F} := \{ w \in \mathbb{R}^{\text{codim} F} : \langle Q_F w, w \rangle = 0 \} \) determined by the condition
\[
dw = dq_F \wedge d\mu_{Q_F},
\]
where we wrote \( q_F(w) := \langle Q_F w, w \rangle \). In terms of the inertial polar coordinates introduced in Section 3.1 one computes
\[
d\mu_{Q_F} = r^{n_F - 1}s^{n_{-F} - 1} d\theta^+ \wedge d\theta^- \wedge \frac{r \, ds + s \, dr}{2(r^2 + s^2)}
\]
since \( q_F(w) \equiv r^2 - s^2 \) and
\[
dq_F \wedge \frac{r \, ds + s \, dr}{2(r^2 + s^2)} = (r \, dr - s \, ds) \wedge \frac{r \, ds + s \, dr}{r^2 + s^2} = dr \wedge ds,
\]
compare [2] Page 498. Consequently, if
\[
S_f(r, s) := \int_{S^{n_F-1}} \int_{S^{n_{-F}-1}} f(r\theta^+, s\theta^-) \, d\theta^+ \, d\theta^-,
\]
denotes the double spherical mean value of a function \( f \in C^\infty(\mathbb{R}^{\text{codim} F}) \),
\[
\frac{1}{2} \int_0^\infty \int_{S^{n_F-1}} r^{n_F} S_f(t, t) \frac{dt}{t} = \frac{1}{2} \int_{\{r=s\}} r^{n_F} s^{n_{-F}-1} S_f(r, s) \, ds + r^{n_F - 1}s^{n_{-F}} S_f(r, s) \, dr
\]
\[
= \int_{\Sigma_{Q_F}} f(w) \, d\mu_{Q_F}(w).
\]
The first step in the direction of the geometric interpretation of the coefficients \( B_j^{\text{top}}(\beta_F) \) is now the following
Lemma 3.9. Let \( f \in C_c^\infty(\mathbb{R}^{\text{codim} F}) \) be a double spherical mean value as in (3.35). Then, for arbitrary \( j, k \in \mathbb{N}_0 \), one has

\[
\int_0^\infty t^{n_f^+} \left( \frac{1}{r} \frac{\partial}{\partial r} \right)^j \left( \frac{1}{s} \frac{\partial}{\partial s} \right)^k S_f \left( t, \frac{r}{t} \right) dt \]

\[
= \begin{cases}
2^{3-j-k} \frac{(j-1)!(k-1)!}{(j-1)!(k-1)!} \int_{\Sigma_{Q_F}} \int_0^1 \int_0^1 \left[ \Delta_{w^+}^j \Delta_{w^+}^k f \right](t+ w^+, t- w^-) t^{n_f^+ - 1} t^{n_f^+ - 1} dt_+ dt_- d\mu_{Q_F}(w), & j, k \geq 1, \\
2^{-j} \frac{(k-1)!}{(j-1)!} \int_{\Sigma_{Q_F}} \int_0^1 \int_0^1 \left[ \Delta_{w^+}^j f \right](t+ w^+, w^-) t^{n_f^+ - 1} dt_+ d\mu_{Q_F}(w), & j \geq 1, k = 0, \\
2^{1-k} \frac{(j-1)!}{(k-1)!} \int_{\Sigma_{Q_F}} \int_0^1 \int_0^1 \left[ \Delta_{w^+}^k f \right](w^+, t- w^-) t^{n_f^+ - 1} dt_- d\mu_{Q_F}(w), & j = 0, k \geq 1, \\
2 \int_{\Sigma_{Q_F}} f(w) d\mu_{Q_F}(w), & j = k = 0,
\end{cases}
\]

where \( \Delta_{w^\pm} = \sum_{i=1}^{n_f^\pm} \frac{d^2}{d(w_i^\pm)^2} \) denotes the Euclidean Laplacian with respect to the variables \( w^\pm \).

Proof. By the formula of Ostrogradski one has

\[
(3.37) \quad \int_{S^{l-1}} \partial_r [h(r\theta)] \, d\theta = \int_{S^{l-1}} \langle \text{grad} h(r\theta), \theta \rangle \, d\theta = \frac{1}{r} \int_{B^{l}} \Delta h_r(w) \, dw = r \int_{B^{l}} \Delta h(rw) \, dw
\]

for an arbitrary function \( h \in C^\infty(\mathbb{R}^l) \), where we wrote \( h_r(w) := h(rw) \) and \( B^l := \{ w \in \mathbb{R}^l \mid \|w\| \leq 1 \} \). Consequently,

\[
\left[ \left( \frac{1}{r} \frac{\partial}{\partial r} \right) S_f \right](r, s) = \int_{S^{n_f^+ - 1}} \int_{B^{n_f^+}} \left[ \Delta_{w^+}^j f \right](rw^+, s\theta^-) \, dw^+ \, d\theta^- \\
= \int_{S^{n_f^+ - 1}} \int_{S^{n_f^+ - 1}} \int_0^1 \left[ \Delta_{w^+}^j f \right](t_1 r\theta^+, s\theta^-) t_1^{n_f^+ - 1} dt_1 \, d\theta^+ \, d\theta^- \\
=: S_{f_{1,0}}(r, s)
\]

is again a spherical mean value with respect to the new function \( f_{1,0} \in C_c^\infty(\mathbb{R}^{\text{codim} F}) \) given by

\[
f_{1,0}(w) := \int_0^1 \left[ \Delta_{w^+}^j f \right](t_1 w^+, w^-) t_1^{n_f^+ - 1} dt_1.
\]

A similar formula holds for the operator \( s^{-1} \partial_s \), so that iterating we obtain for arbitrary \( j, k \in \mathbb{N}_0 \) the relation

\[
\left[ \left( \frac{1}{r} \frac{\partial}{\partial r} \right)^j \left( \frac{1}{s} \frac{\partial}{\partial s} \right)^k S_f \right](r, s) = S_{f_{j,k}}(r, s),
\]

where \( f_{j,k} \in C_c^\infty(\mathbb{R}^{\text{codim} F}) \) is given by

\[
f_{j,k}(w) := \int_{(0,1)^j} \int_{(0,1)^k} \left[ \Delta_{w^+}^j \Delta_{w^+}^k f \right](t_1 \cdots t_j w^+, \sigma_1 \cdots \sigma_k w^-) \\
t_1^{n_f^+ + 2j - 3} \sigma_1^{n_f^+ + 2k - 3} \cdots t_j^{n_f^+ - 1} \sigma_1^{n_f^+ + 2k - 3} \cdots \sigma_k^{n_f^+ - 1} d\sigma.
\]
If \( j, k \geq 1 \), the substitutions \( t_j \mapsto t_j^{-1} \cdots t_{j-1}^{-1} \) and \( \sigma_k \mapsto \sigma_k \sigma_1^{-1} \cdots \sigma_{k-1}^{-1} \) transform \( f_{j,k}(w) \) into

\[
\int_{(0,1)^j} \int_{(0,1)^k} \left[ \Delta_w^j \Delta_w^k \right] (t_j w^+, \sigma_k w^-) t_1^{-1} \cdots t_{j-1}^{-1} n_p^+ + 2j-3 t_2^+ \cdots (t_1 \cdots t_{j-1} t_j)^n_p^{-1} \\
\sigma_1^{-1} \cdots \sigma_{k-1}^{-1} n_p^+ + 2k-3 \sigma_2^+ \cdots (\sigma_1^{-1} \cdots \sigma_{k-1}^{-1} \sigma_k)^n_p^{-1} dt \, d\sigma
\]

\[
= \int_{(0,1)^j} \int_{(0,1)^k} \left[ \Delta_w^j \Delta_w^k \right] (t_j w^+, \sigma_k w^-) t_1^{2j-3} t_2^{2j-5} \cdots t_{j-1} t_j^n_p^{-1} \\
\sigma_1^{2k-3} \sigma_2^{2k-5} \cdots \sigma_{k-1}^{-1} \sigma_k n_p^{-1} dt \, d\sigma
\]

\[
= \frac{1}{2^{j+k-2} (j-1)! (k-1)!} \int_{(0,1)^j} \int_{(0,1)^k} \left[ \Delta_w^j \Delta_w^k \right] (t_j w^+, t_- w^-) t_1^{2j-3} t_2^{2j-5} \cdots t_{j-1} t_j^n_p^{-1} dt \, d\sigma
\]

where we took into account that

\[
\int_{(0,1)^j} t_1^{2j-3} t_2^{2j-5} \cdots t_{j-1} dt \int_{(0,1)^{k-1}} \sigma_1^{2k-3} \sigma_2^{2k-5} \cdots \sigma_{k-1}^{-1} \sigma_k dt = \frac{1}{2^{j+k-2} (j-1)! (k-1)!}
\]

The assertion now follows in this case with (3.36). The intermediate cases \( j \geq 1, k = 0 \) and \( j = 0, k \geq 1 \) are treated analogously. \( \square \)

As a next step, we need certain formulae for the derivatives of smooth even functions. For this, we introduce the notation

\[
\delta_r := \frac{1}{r} \frac{d}{dr}
\]

Lemma 3.10. (1) Let \( f \in C^\infty(\mathbb{R}) \) be an even function. Then \( \delta_r f \in C^\infty(\mathbb{R}) \) is even and the \( j \)-th derivative of \( f \) can be expressed as

\[
(3.38) \quad f^{(j)} = \sum_{i=\lfloor \frac{j}{2} \rfloor}^j p_{ij} \cdot \delta_r^i f
\]

with \( p_{ij}(r) = c_{ij} r^{2j-i} \) and \( c_{ij} \in \mathbb{N} \).

(2) Let \( g \in C^\infty(\mathbb{R} \times \mathbb{R}) \) be such that \( g = g(r,s) \) is even in both variables \( r \) and \( s \). Then \( \delta_r \delta_s g \in C^\infty(\mathbb{R} \times \mathbb{R}) \) is even. Furthermore, the \( j \)-th derivative of \( g \) with respect to \( \partial_+ = \partial_r + \partial_s \) can be expressed as

\[
(\partial^+_+ g)(t,t) = \sum_{i=\lfloor \frac{j}{2} \rfloor}^j p_{ij}(t) \left[ (\delta_r + \delta_s)^i \right](t,t),
\]

where \( p_{ij} \) is above.

Proof. Let us first prove (1). Since \( f \) is even, \( f' \in C^\infty(\mathbb{R}) \) is odd, so that \( f'(0) = 0 \). Moreover,

\[
f'(r) = \int_0^r f''(s) ds = r \int_0^1 f''(rs) ds.
\]

Consequently, \( \frac{1}{r} \cdot f'(r) = \int_0^1 f''(rs) ds \), yielding \( \delta_r f \in C^\infty(\mathbb{R}) \). Since \( \delta_r f \) is even, we inductively obtain \( \delta_r^i f \in C^\infty(\mathbb{R}) \). Next, we prove (3.38) by induction on \( j \). Cearly,

\[
f'(r) = r \cdot (\delta_r f)(r).
\]
Suppose now that (3.38) is true for some $j \in \mathbb{N}$. If $j$ is even,

\begin{align*}
  f^{(j+1)}(r) &= \sum_{i=\frac{j}{2}}^{j} \frac{d}{dr} \left[ c_{ij} r^{2i-j} (\delta_r f)(r) \right] \\
  &= \sum_{i=\frac{j}{2}}^{j} \left[ c_{ij} (2i-j) r^{2i-j-1} (\delta_r f)(r) \right] + \sum_{i=\frac{j}{2}}^{j} \left[ c_{ij} r^{2(i+1)-j-1} (\delta_r f)(r) \right] \\
  &= \sum_{i=\left\lfloor \frac{j+1}{2} \right\rfloor}^{j+1} c_{i(j+1)} r^{2i-j-1} (\delta_r f)(r)
\end{align*}

since $c_{ij} (2i-j) = 0$ for $i = j/2$. If $j$ is odd,

\begin{align*}
  f^{(j+1)}(r) &= \sum_{i=\left\lfloor \frac{j}{2} \right\rfloor}^{j} \frac{d}{dr} \left[ c_{ij} r^{2i-j} (\delta_r f)(r) \right] \\
  &= \sum_{i=\left\lfloor \frac{j}{2} \right\rfloor}^{j} \left[ c_{ij} (2i-j) r^{2i-j-1} (\delta_r f)(r) \right] + \sum_{i=\left\lfloor \frac{j}{2} \right\rfloor}^{j} \left[ c_{ij} r^{2(i+1)-j-1} (\delta_r f)(r) \right] \\
  &= \sum_{i=\left\lfloor \frac{j+1}{2} \right\rfloor}^{j+1} c_{i(j+1)} r^{2i-j-1} (\delta_r f)(r)
\end{align*}

since $\left\lfloor \frac{j}{2} \right\rfloor = \left\lfloor \frac{j+1}{2} \right\rfloor$, and (1) follows. To see (2), note that the first claim follows directly from (1). To prove the formulae for the derivatives, note that for $f(t) := g(t, t)$ we have $f^{(j)}(t) = (\partial^j_r g)(t, t)$. Moreover, $f \in C^\infty(\mathbb{R})$ is an even function, and applying (1) to $f$ we obtain

\begin{align*}
  (\partial^j_r g)(t, t) &= f^{(j)}(t) = \sum_{i=\left\lfloor \frac{j}{2} \right\rfloor}^{j} p_{ij}(t) \delta^t_i f(t) = \sum_{i=\left\lfloor \frac{j}{2} \right\rfloor}^{j} p_{ij}(t) \left[ (\delta_r + \delta_s)^i g \right](t, t).
\end{align*}

\[\square\]

As a consequence, we now deduce

**Lemma 3.11.** The integrals

\begin{equation}
  \int_0^\infty t^{n-2j+k} (\partial^k_r S_F)(t, t) \frac{dt}{t}
\end{equation}

occurring in the expressions for the $B_j$-coefficients in Lemma 3.6 can be interpreted geometrically as integrals over $\Sigma_{Q_F}$ with respect to the measure $\mu_{Q_F}$.

**Proof.** Let $f = f(t) \in C^\infty([0, \infty))$ be an even function with compact support. By partial integration we have the identity

\begin{equation}
  \int_0^\infty r^k f(r) \, dr = -\frac{1}{k+1} \int_0^\infty t^{k+2} \delta_r f(r) \, dr,
\end{equation}

provided $k \geq 0$. Furthermore, with Lemma 3.10 one computes

\begin{align*}
  (\partial^k_r S_F)(r, s) &= \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} (\partial^i_r \partial^{k-i}_s S_F)(r, s) \\
  &= \sum_{i=0}^{k} \sum_{\mu=\left\lfloor \frac{i}{2} \right\rfloor \nu=\left\lfloor \frac{k-i}{2} \right\rfloor} \binom{k}{i} (-1)^{k-i} c_{\mu \nu} c_{\nu, k-i} r^{2\nu-i} s^{2\nu-k+i} \delta_{\delta_r} \delta_{\delta_s} S_F(r, s).
\end{align*}

Now, note that in Lemma 3.6 we either have
\[ n_F^0 - 2j > 0 \text{ and } 0 \leq k \leq j, \text{ or} \]
\[ n_F^0/2 \leq j < n_F^0 \text{ and } 2j - n_F^0 + 1 \leq k \leq j. \]

In both cases, \[ n_F^0 - 2j + 2(\mu + \nu) \geq n_F^0 - 2j + 2 \min_{0 \leq i \leq k} \left( (i/2) + \left[ (k - i)/2 \right] \right) \geq n_F^0 - 2j + k > 0, \]
as well as \[ j - (\mu + \nu) \geq j - k \geq 0. \] With (3.40) we therefore obtain
\[ \int_0^\infty t^{n_F^0 - 2j + k} (\partial_k^\mu S_F)(t, t) \frac{dt}{t} = \sum_{i, \mu, \nu} c_{i, \mu, \nu, k} \int_0^\infty t^{n_F^0 - 2j + k} (2(\mu + \nu) - k) \delta_\mu^\nu \delta_s S_F(t, t) \frac{dt}{t} = \sum_{i, \mu, \nu} c_{i, \mu, \nu, k} \int_0^\infty t^{n_F^0} \delta_l^{-(\mu + \nu)} \left[ (\delta_\mu^\nu \delta_s S_F)(t, t) \right] \frac{dt}{t} = \sum_{i, \mu, \nu} c_{i, \mu, \nu, k} \int_0^\infty t^{n_F^0} \left( \delta_r + \delta_s \right) j^{-(\mu + \nu)} \delta_\mu^\nu \delta_s S_F(t, t) \frac{dt}{t} \]
for suitable constants \( c_{i, \mu, \nu, k}, c_{i, \mu, \nu, k} \in \mathbb{Z}. \) Lemma 3.9 then implies that all summands above can be interpreted geometrically. \[ \square \]

Similarly, one shows

**Lemma 3.12.** The integrals
\[ \int_0^\infty T_F^j(t) \frac{dt}{t} \]
 occurring in the expressions for the \( B_j \)-coefficients in Lemma 3.6 can be interpreted geometrically as integrals over \( \Sigma_Q, \) with respect to the measure \( \mu_{Q, \ell}. \)

**Proof.** By (3.23), the relevant integrals to be considered are
\[ \int_0^\infty \left[ (\partial_L^L \partial_r^\alpha - \partial_L^L \partial_r^\beta) S_F \right](t, t) \frac{dt}{t} \]
where \( L := 2j - n_F^0, \) and either
\[
(\text{I}) \quad 0 \leq k \leq 2j - n_F^0 \quad \text{if } n_F^0/2 \leq j < n_F^0, \quad \text{or}
(\text{II}) \quad j - n_F^0 \leq k \leq j \quad \text{if } n_F^0 \leq j.
\]
Assume as we may that \( L - k \leq k. \) The arguments in (3.32)–(3.33) imply that
\[ \partial_L^L \partial_r^\alpha - \partial_L^L \partial_r^\beta = (\partial_+ \partial_-)^L - \partial_+^\mu - \partial_-^\nu = \sum_{l=0, l \equiv 1(2)} L c_l \partial_+^l \partial_-^{L-l} \]
with \( 2\mu := 2k - L \) even, and with Lemma 3.10 (1) we obtain
\[ \left[ (\partial_L^L \partial_r^\alpha - \partial_L^L \partial_r^\beta) S_F \right](t, t) = \sum_{l=0, l \equiv 1(2)} \sum_{\alpha=[\frac{L}{2}]} \sum_{\beta=[\frac{L}{2}]} c_{l, \alpha, \beta} t^{2(\alpha + \beta) - L} (\delta_\mu^\nu \delta_s S_F)(t, t). \]
Note that the highest exponent of \( t \) in the sum above is given by \( 2 \left( \left[ \frac{L}{2} \right] + \left[ \frac{L}{2} \right] \right) - L = (L + 2) - L = 2 \) since \( w \) is odd and \( L \) is even. In this way we see that (3.41) is given by a linear combination of integrals of the form
\[ \int_0^\infty t^{2(\alpha + \beta) - L - 1} (\delta_\mu^\nu \delta_s S_F)(t, t) \frac{dt}{t}, \quad 2(\alpha + \beta) - L - 1 \geq 1. \]
To proceed, we need a more refined description of the integrals that can occur in (3.42). For this notice that, when applied to functions which are even in \( r \) and \( s, \)
\[ \partial_2^{2\mu} - \partial_2^{2\nu} = \sum_{\nu=0}^{2\mu} c_{\mu, \nu} \delta_2^{2\mu-\nu} = \sum_{\nu=0}^{2\mu} \sum_{\mu=2\nu} c_{\mu, \nu} \delta_2^{2\mu-2\nu} \delta_2^{2\nu} S, \]

\[ a \mu + b \nu \]
by Lemma \textbf{3.10} (1). In both cases (I) and (II), the degree of the monomial $r^{2u - \nu} s^{2v - 2\mu + \nu}$ is even and at most $2\mu \leq n_F^0$. Similarly one computes with Leibniz’ rule

$$\frac{\partial^M}{\partial \mu} \frac{\partial^N}{\partial \nu} = \frac{\partial^M}{\partial \mu} \sum_{n=0}^{N} \binom{N}{n} \frac{\partial^n}{\partial \nu^{N-n}} = \frac{\partial^M}{\partial \mu} \sum_{n=0}^{N} \sum_{n=0}^{N} \sum_{n=0}^{N-n} c_{u'=n} \frac{\partial^M}{\partial \mu} (r^{2u'-n} s^{2v'-n}) \frac{\partial^n}{\partial \nu} \frac{\partial^n}{\partial \nu}$$

(3.44)

where $\frac{\partial^M}{\partial \mu} (r^{2u'-n} s^{2v'-n})$ is a sum of monomials of degree 0, \ldots, $N - M + m$ and parity equal to the parity of $N - M + m$. If we now apply $(\partial_+ \partial_-)^{L-k}$ to (3.43) we are reduced to the study of

$$\frac{\partial^M}{\partial \mu} (r^{2u'-n} s^{2v'-n}) \frac{\partial^n}{\partial \nu} \frac{\partial^n}{\partial \nu}$$

Here $\frac{\partial^M}{\partial \mu} (r^{2u'-n} s^{2v'-n})$ is a sum of monomials of degree 0, \ldots, $2\mu - 2(L-k) + M + N$ and parity equal to the parity of $N - M$. Evaluating $\frac{\partial^M}{\partial \mu} (r^{2u'-n} s^{2v'-n})$ with the aid of (3.44), which we are entitled to do since it will be applied to $S_F$, an even function in $r$ and $s$, we see that $\frac{\partial^M}{\partial \mu} (r^{2u'-n} s^{2v'-n})$ is given by a linear combination of terms of the form

$$ P_{mu',nu'}(r,s) \frac{\partial^m}{\partial r} \frac{\partial^n}{\partial s} S_F,$$

where $P_{mu',nu'}(r,s)$ is a sum of monomials of degree less or equal $n_F^0$ and parity equal to the parity of $m$. In view of Lemma \textbf{3.10} (2) we conclude that $[(\partial_+^L - \partial_-^L) S_F](t,t)$ is given by a linear combination of terms of the form

$$ P_{mu',nu'}(t,t) \frac{\partial^m}{\partial r} \frac{\partial^n}{\partial s} S_F(t,t) = P_{mu',nu'}(t,t) \sum_{i=0}^{m} c_{im} t^{2i-m} \left[ (\delta_r + \delta_s)^i \frac{\partial^m}{\partial r} \frac{\partial^n}{\partial s} S_F(t,t) \right]$$

where, again, we took into account that $\frac{\partial^m}{\partial r} \frac{\partial^n}{\partial s} S_F(t,t)$ is an even function in $r$ and $s$. Thus, (3.41) is given by a linear combination of integrals of the form

$$ \int_0^\infty t^{p+2i-m-1} \left[ (\delta_r + \delta_s)^i \frac{\partial^m}{\partial r} \frac{\partial^n}{\partial s} S_F(t,t) \right] dt, \quad i = \left[ m/2 \right], \ldots, m, \quad p = 0, \ldots, n_F^0,$$

with $p - m$ even. Moreover, taking into account (3.42) we must have $p + 2i - m - 1 \geq 1$. Integrating partially with respect to $\delta_r$ the latter integrals can be brought into the form of those in Lemma \textbf{3.10}.

In fact, if $D := n_F^0 - p - 2i + m > 0$,

$$ \int_0^\infty t^{p+2i-m-1} \left[ (\delta_r + \delta_s)^i \frac{\partial^m}{\partial r} \frac{\partial^n}{\partial s} S_F(t,t) \right] dt = C \int_0^\infty t^{p+2i-m-1+D/2} \delta_r^D \left[ (\delta_r + \delta_s)^i \frac{\partial^m}{\partial r} \frac{\partial^n}{\partial s} S_F(t,t) \right] dt$$

$$ = C \int_0^\infty t^{p+2i-m-1} \left[ (\delta_r + \delta_s)^i \frac{\partial^m}{\partial r} \frac{\partial^n}{\partial s} S_F(t,t) \right] dt$$

for a certain constant $C \in \mathbb{R}$. On the other hand, if $D < 0$,

$$ \int_0^\infty t^{p+2i-m-1} \left[ (\delta_r + \delta_s)^i \frac{\partial^m}{\partial r} \frac{\partial^n}{\partial s} S_F(t,t) \right] dt = \int_0^\infty t^{p+2i-m-1} \delta_r^D \left[ (\delta_r + \delta_s)^i \frac{\partial^m}{\partial r} \frac{\partial^n}{\partial s} S_F(t,t) \right] dt$$

$$ = C' \int_0^\infty t^{p+2i-m-1} \left[ (\delta_r + \delta_s)^i \frac{\partial^m}{\partial r} \frac{\partial^n}{\partial s} S_F(t,t) \right] dt$$

for some $C' \in \mathbb{R}$. Since the case $L - k > k$ can be treated alike, the assertion follows with Lemma \textbf{3.9}.
To conclude our analysis in the indefinite case, let us proceed to a closer description of the $q_j$-coefficients that will be convenient in applications. We have

$$q_j(\beta_F) = \frac{1}{2^j} \sum_{k=0}^{[j/2]-1} \left[ \frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial s^2} \right]_k \left( (\frac{\partial}{\partial r} - \frac{\partial}{\partial s})^{j-2k} - (\frac{\partial}{\partial r} + \frac{\partial}{\partial s})^{j-2k} \right) \beta_F(0,0),$$

where taking derivatives at $r = s = 0$ amounts to localizing at the fixed-point $F$. Expanding the $k$-th power and performing some substitutions in the sums, one finds

$$q_{2j}(\beta_F) = \sum_{N=1}^{j} \sum_{k=0}^{j-1} (-1)^k C_{j,k,N} \frac{\partial^{2(j-N)+1}}{\partial r^{(j-N)+1} \partial s^{2N-1}} \beta_F(0,0),$$

$$q_{2j}(\beta_F) = \sum_{N=1}^{j} \sum_{k=0}^{j-1} C_{j,k,N} \frac{\partial^{2(j-N)+1}}{\partial r^{(j-N)+1} \partial s^{2N-1}} \beta_F(0,0),$$

where

$$C_{j,k,N} = -2^{1-2j} A_{2j,k} \sum_{m=\max(0,k-N+1)}^{\min(k,j-N)} (-1)^m \binom{k}{m} \frac{2(j-k)}{2(m+N-k) - 1}.$$

Finally, computing the constants $A_{j,k}$ according to [2, p. 496], one finds that the first three values read

$$A_{2,0} = \frac{1}{2^3}, \quad A_{4,0} = \frac{1}{3 \cdot 2^6}, \quad A_{4,1} = \frac{1}{3 \cdot 2^3}.$$

### 3.3.2. Contributions of the definite charts

It remains to study the less difficult case of a chart $U_F$ with $F \in \mathcal{F}_a$ in which $Q_F$ is definite, so that either $n_F^+ = \text{codim} F$ or $n_F^- = \text{codim} F$. Again, a complete asymptotic expansions for integrals of the type $I_{\alpha \chi}^F(\varepsilon)$ can be derived along the lines of [2, Section 3]. In fact, for $|\zeta_F| \ll 1$ and $n_F = 0$ one has the asymptotic expansion

$$I_{\alpha \chi}^F(\varepsilon) \sim \sum_{j=0}^{\infty} (2\varepsilon)^{\text{codim} F+j+1} \sum_{m=\text{codim} F+j-1}^{\text{codim} F+1} \zeta_{\text{codim} F+j-1}(\zeta_F/\varepsilon)$$

and for $n_F^+ = 0$

$$I_{\alpha \chi}^F(\varepsilon) \sim \sum_{j=0}^{\infty} (2\varepsilon)^{\text{codim} F+j+1} \sum_{m=\text{codim} F+j-1}^{\text{codim} F+1} \zeta_{\text{codim} F+j-1}(\zeta_F/\varepsilon),$$

where $\Xi_j(t)$ and $\Xi_j(t)$ are as in (3.20). Clearly, the coefficients in these expansions are given in terms of derivatives of the $\delta_0$-distribution in $\mathbb{R}^{\text{codim} F}$. Thus, we obtain

**Proposition 3.13.** Assume that $Q_F$ is definite. Then, there are $\delta_0, \varepsilon_0 > 0$ such that for $|\zeta_F| < \delta_0$ one has an asymptotic expansion

$$I_{\alpha \chi}^F(\varepsilon) \sim \sum_{j=0}^{\infty} (2\varepsilon)^{1+j} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \Delta_{j,l}(\zeta_F/\varepsilon)(2\zeta_F)^l, \quad 0 < \varepsilon < \varepsilon_0,$$

where the coefficients $\Delta_{j,l}(t)$ can be computed explicitly, and the limits $\lim_{t \to \pm \infty} \Delta_{j,l}(t)$ exist. In particular, $\Delta_{j,l}(t)$ is bounded in $t$. □
3.4. The asymptotic expansion. The analysis carried out in Sections 3.2 and 3.3 yields a complete description of the asymptotic behavior of the generalized Witten integral $I^\alpha_\varepsilon(z)$ defined in (2.11) simultaneously in $\varepsilon \in \mathbb{R}^+$ and $\varepsilon \ll 1$, together with a geometric description of the coefficients in the local charts. In what follows, we shall give global expressions for them in terms of the stratification (2.4)

$$\mathcal{M}_{\text{reg}} = \mathcal{M}_{\text{top}} \sqcup \mathcal{M}_{\text{sing}}, \quad \varepsilon \in \mathbb{R}^+,$$

of the reduced spaces by infinitesimal orbit types. Let $\omega_\mathbb{R}$ be the unique symplectic form on $\mathcal{M}_{\text{reg}}$ characterized by the condition $i^*\omega = \pi^*\omega_\mathbb{R}$, where $i : \mathcal{J}^{-1}(\{\varepsilon\}) \cap M(\mathbb{R}) \to M$ is the inclusion and $\pi : \mathcal{J}^{-1}(\{\varepsilon\}) \cap M(\mathbb{R}) \to \mathcal{M}_{\text{reg}}$ is the canonical projection. Also, recall the partition of unity $\{\chi_{\text{top}}, \chi_F\}_{F \in F}$ subordinated to the cover (3.4) of $M$.

**Lemma 3.14.** Fix a chart $(U_F, \Phi_F)$, and define for $f \in C_c(U_F)$ the average

$$\bar{f}(T, p) := \int_T f(g \cdot p) \, dg, \quad p \in M,$$

where $dg$ is the Haar measure on $T$ fixed by our identification $t \cong \mathbb{R}$.

1. Let $\mu_{Q_F}$ be the measure introduced in (3.3.3) on the pointed quadric $\Sigma^*_F := \{w \in \mathbb{R}^{\text{codim } F} \setminus \{0\} : \langle Q_F \ell w, w \rangle = 0\}$. Then

$$\int_{\mathcal{M}_{\text{top}}(F)} \bar{f} \, d\mathcal{M}_{\text{top}}(F) = \int_{U_F} \int_{\Sigma^*_F} f(\Phi^{-1}_F(\pi_F(\varphi, w))) \, d\nu_F(\pi_F(\varphi, w)) \, d\mu_{Q_F}(w),$$

where $d\mathcal{M}_{\text{top}}(F) = \omega_{\text{top}} n^{-1}/(n-1)!$ denotes the symplectic volume form on the top stratum.

2. Similarly, we have with $dF = \omega_{\text{dim } F}/(\text{dim } F)!$

$$\int_F f \, dF = \int_{U_F} \int_{\Sigma^*_F} f(\Phi^{-1}_F(\pi_F(\varphi, 0))) \, d\nu_F(\pi_F(\varphi, 0)).$$

**Proof.** To begin, note that from (3.2) and (3.7) it is clear that

$$\Phi_F(\mathcal{J}^{-1}(\{\mathcal{J}(F)\})_{\text{top}} \cap U_F) = \{w, w \in \Phi_F(U_F) \mid w \in \Sigma^*_F \setminus \{0\}\}.$$

By slight abuse of notation, let us write $q_F(\varphi, w) := \langle Q_F \ell w, w \rangle = : q_F(\varphi, w)$, so that we deduce from (3.7) that $(\Phi_F^{-1})^*d\mathcal{J} = dq_F$. Furthermore, recall that we have equalities of smooth densities

$$d\nu_F = dq_F \wedge d\omega,$$

and that by (3.3.3) we have $d\nu_F \wedge dw = d\nu_F \wedge (d(p^*q_F)) \wedge d\mu_{Q_F}$. On the other hand, the Liouville measure $d\mathcal{J}^{-1}(\{\mathcal{J}(F)\})_{\text{top}}$ on $\mathcal{J}^{-1}(\{\mathcal{J}(F)\})_{\text{top}}$ is characterized by the condition

$$d\mathcal{J}^{-1}(\{\mathcal{J}(F)\})_{\text{top}} = \frac{1}{n!} \omega_{\text{top}}(p) \, d\mathcal{J}_{\text{top}}(\{\mathcal{J}(F)\})_{\text{top}}$$

for any $p \in \mathcal{J}^{-1}(\{\mathcal{J}(F)\})_{\text{top}}$. Pulling back (3.48) along $\Phi_F^{-1} \circ \pi_F$ yields

$$(\Phi_F^{-1} \circ \pi_F)^* d\mathcal{J}^{-1}(\{\mathcal{J}(F)\})_{\text{top}} = d\nu_F \wedge d\mu_{Q_F}.$$

This proves that

$$\int_{\mathcal{J}^{-1}(\{\mathcal{J}(F)\})_{\text{top}}} f \, d\mathcal{J}^{-1}(\{\mathcal{J}(F)\})_{\text{top}} = \int_{U_F} \int_{\Sigma^*_F} f(\Phi^{-1}_F(\pi_F(\varphi, w))) \, d\nu_F(\pi_F(\varphi, w)) \, d\mu_{Q_F}(w).$$

It remains to show that

$$\int_{\mathcal{M}_{\text{top}}(F)} \bar{f} \, d\mathcal{M}_{\text{top}}(F) = \int_{\mathcal{J}^{-1}(\{\mathcal{J}(F)\})_{\text{top}}} f \, d\mathcal{J}^{-1}(\{\mathcal{J}(F)\})_{\text{top}}.$$

To this end, recall that $\omega_{\text{top}}$ is characterized by $\pi^*\omega_{\text{top}} = i^*\omega$, where $i : \mathcal{J}^{-1}(\{\mathcal{J}(F)\})_{\text{top}} \to M$ is the inclusion and $\pi : \mathcal{J}^{-1}(\{\mathcal{J}(F)\})_{\text{top}} \to \mathcal{M}_{\text{top}}(F)$ the orbit projection. Also notice that our identification of $t$ with $\mathbb{R}$ corresponds to a choice of an element $x_0 \in t$ that is identified with 1, and leads to an
identification of $\mathcal{J}$ with $J(x_0)$. On the top stratum $M_{(b_{top})} \subset M$, the fundamental vector field $\tilde{x}_0$ is nowhere-vanishing, so it has a dual one-form $\eta_{x_0}$. One then computes on $M_{(b_{top})}$

$$\iota_{\tilde{x}_0}(\eta_{x_0} \wedge d\mathcal{J}) = d\mathcal{J} = dJ(x_0) = \iota_{\tilde{x}_0}\omega,$$

where the first equality uses the $T$-invariance of $\mathcal{J}$, the middle equality is the remark above, and the last equality is the defining property of the momentum map $\mathcal{J}$. Consequently,

$$(\eta_{x_0} \wedge d\mathcal{J})|_{M_{(b_{top})}} + \eta = \omega|_{M_{(b_{top})}}$$

for some $\eta \in \Omega^2(M_{(b_{top})})$ that fulfills $\iota_{\tilde{x}_0}\eta = 0$. Thus, on $M_{(b_{top})}$ we have

$$\frac{1}{n!}\omega^n = \frac{1}{n!} \omega^{n-1} \wedge \eta_{x_0} \wedge d\mathcal{J} + \frac{1}{n!}\omega^{n-1} \wedge \eta$$

$$= \frac{1}{n!} \left( \omega^{n-1} \wedge \eta_{x_0} \wedge d\mathcal{J} + \frac{1}{n!} \omega^{n-2} \wedge \eta \right)$$

$$= \frac{1}{n!} \left( \sum_{j=1}^{n} \omega^{n-j} \wedge \eta^{j-1} \right) \wedge \eta_{x_0} \wedge d\mathcal{J} + \frac{1}{n!} \eta^n$$

$$= \frac{1}{n!} \left( \sum_{j=1}^{n} \omega^{n-j} \wedge \eta^{j-1} \right) \wedge \eta_{x_0} \wedge d\mathcal{J}$$

since $\eta^n = 0$ because $\eta$ is degenerate. Now, if we insert $\eta = \omega - \eta_{x_0} \wedge d\mathcal{J}$, then all non-zero powers of $\eta_{x_0} \wedge d\mathcal{J}$ get killed by the wedge product with $\eta_{x_0} \wedge d\mathcal{J}$, and we arrive at

$$\frac{1}{n!}\omega^n = \frac{1}{n!} \left( \sum_{j=1}^{n} \omega^{n-1} \right) \wedge \eta_{x_0} \wedge d\mathcal{J} = \frac{1}{(n-1)!}\omega^{n-1} \wedge \eta_{x_0} \wedge d\mathcal{J}.$$ 

Inserting this in (3.48) gives us

$$d\mathcal{J}^{-1}(\{\mathcal{J}(F)\})_{top} = \frac{1}{(n-1)!} i^*\omega^{n-1} \wedge i^*(\eta_{x_0}).$$

On the other hand, we compute

$$\pi^* d\mathcal{M}_{top}(F) = \pi^*(\omega^\text{top}_{1}/(n-1)!)) = \frac{1}{(n-1)!} \pi^*\omega^\text{top}_{1} = \frac{1}{(n-1)!} i^*\omega^{n-1},$$

so that we find

$$d\mathcal{J}^{-1}(\{\mathcal{J}(F)\})_{top} = \pi^*(d\mathcal{M}_{top}(F)) \wedge \iota^*_\eta(\eta_{x_0}),$$

$$\int_{\mathcal{J}^{-1}(\{\mathcal{J}(F)\})_{top}} f \ d\mathcal{J}^{-1}(\{\mathcal{J}(F)\})_{top} = \int_{\mathcal{M}_{top}(F)} \left( \int_{\mathcal{O}} f \eta_{x_0} \right) \ d\mathcal{M}_{top}(F)(\mathcal{O}).$$

We are left with comparing $\int_{\mathcal{O}} f \eta_{x_0}$ with $\hat{f}(\mathcal{O})$ for any orbit $\mathcal{O} = T \cdot p$, where $p \in \mathcal{J}^{-1}(\{\mathcal{J}(F)\})_{top}$. The map $\Psi : S^1 \ni g \mapsto g \cdot p \in \mathcal{O}$ is an $S^1$-equivariant diffeomorphism, so $\Psi^*\eta_{x_0}$ is a Haar measure on $T$ and thus a constant multiple of $dg$. To determine the constant, we note that the derivative of $\Psi$ at the identity fulfills $D\Psi_1(x) = \tilde{x}_0$, $x \in t = T_1S^1$. Consequently,

$$\Psi^*\eta_{x_0}|_{\{x_0\}} = \eta_{x_0}(D\Psi_1(x_0)) = \eta_{x_0}(\tilde{x}_0) = 1,$$

proving $\Psi^*\eta_{x_0} = dg$. This finishes the proof of (3.49) and Assertion (1). Assertion (2) follows along the same arguments taking into account that $\mathcal{J}^{-1}(\{\mathcal{J}(F)\})_{\text{sing}} \cap U_F = F$ and

$$\Phi_F(F) = \{[\varphi, w] \in \Phi(F) | \{w = 0\}. \}

As a consequence of the previous lemma, we see from Lemmas 3.11 and 3.12 that the $B^\text{top}_{q_{2j}}(\beta_F)$-coefficients from Section 3.3.1 have a geometric interpretation as integrals over the top stratum $\mathcal{M}_{top}(F)$ with respect to $d\mathcal{M}_{top}(F)$, while the $q_{2j}(\beta_F)$- and the $B^\text{sing}_{q_{2j}}(\beta_F)$-coefficients, and those from Section
reduce to integrals over $F$. Similarly, the coefficients $Q_j(\psi^\zeta, a\chi_{\text{top}})$ and $Q_j(\psi^\zeta, a\chi_F)$ in Sections 3.2.1 and 3.2.2 respectively, have a natural interpretation in terms of integrals on $\mathcal{M}_{\text{top}}^{\zeta}$ with respect to $d\mu_{\text{top}}^{\zeta}$. Taking everything together we arrive at

**Theorem 3.15 (Asymptotic expansion of the generalized Witten integral).** Let $(M, \omega)$ be a symplectic manifold with a Hamiltonian action of $T = S^1$ and momentum map $\mathcal{J} : M \to \mathfrak{t}^*$, and consider for each $\zeta \in \mathfrak{t}^*$ the stratification of the symplectic quotient

$$\mathcal{M}^{\zeta} = \mathcal{J}^{-1}(\{\zeta\})/T = \mathcal{M}_{\text{top}}^{\zeta} \sqcup \mathcal{M}_{\text{sing}}^{\zeta}$$

by infinitesimal orbit types. Let further $I_\alpha^\zeta(\varepsilon)$ denote the generalized Witten integral introduced in (2.11) and consider its decomposition $\mathcal{D}$ with respect to the partition of unity $\{\chi_{\text{top}}, \chi_F\}_{F \in \mathcal{F}}$ on $M$ introduced there, involving the finite set $\mathcal{F}_\alpha \subset \mathcal{F}$. Then, there are $\delta_0, \varepsilon_0 > 0$ such that for all $\zeta \in \mathfrak{t}^*$ and all $\delta \in (0, \delta_0)$ one has with $\zeta_F := \zeta - \mathcal{J}(F)$ the asymptotic expansion

$$I_\alpha^\zeta(\varepsilon) \sim \sum_{j=0}^{\infty} \varepsilon^{1+j} \left[ \int_{\mathcal{M}_{\text{top}}^{\zeta}} Q_j(\psi^\zeta, a\chi_{\text{top}}) \, d\mu_{\text{top}}^{\zeta} + \sum_{F \in \mathcal{F}_\alpha} \int_{\mathcal{M}_{\text{top}}^{\zeta}} Q_j(\psi^\zeta, a\chi_F) \, d\mu_{\text{top}}^{\zeta} \right. + \left. \sum_{l=0}^{\infty} \left( \sum_{F \in \mathcal{F}_\alpha, |\zeta_F| < \varepsilon} \mathcal{R}_{j+l}(a\chi_F) \, d\mu_{\text{top}}^{\zeta} \right) \right] + \sum_{F \in \mathcal{F}_\alpha, |\zeta_F| < \delta} \zeta_F^{j-l} \left[ \int_{\mathcal{M}_{\text{top}}^{\zeta}} S_{j,l}^F(a\chi_F, \zeta_F/\varepsilon) \, dF \right], \quad 0 < \varepsilon < \varepsilon_0,$$

where all coefficients are explicitly given. They are smooth and bounded in $\zeta$ and $\zeta_F/\varepsilon$, respectively, and the singular contributions in the third line of the formula are independent of the partition of unity.

**Proof.** Collecting all contributions from Sections 3.2.1 and 3.2.2 gives, with $\delta_0, \varepsilon_0$ denoting the minima of the individual $\delta_0, \varepsilon_0$ obtained for each of the finitely many $F \in \mathcal{F}_\alpha$: If $0 < \varepsilon < \varepsilon_0$, $0 < \delta < \delta_0$, then

$$I_\alpha^\zeta(\varepsilon) \sim \sum_{j=0}^{\infty} \varepsilon^{1+j} \left[ 2\pi Q_j(\psi^\zeta, a\chi_{\text{top}}) + 2\pi \sum_{F \in \mathcal{F}_\alpha} Q_j(\psi^\zeta, a\chi_F) \right. + \left. \sum_{l=0}^{\infty} \left( \sum_{F \in \mathcal{F}_\alpha, |\zeta_F| < \varepsilon} \mathcal{R}_{j+l}(a\chi_F) \right) \right] + \sum_{F \in \mathcal{F}_\alpha, |\zeta_F| < \delta} \zeta_F^{j-l} \left[ \int_{\mathcal{M}_{\text{top}}^{\zeta}} S_{j,l}^F(a\chi_F, \zeta_F/\varepsilon) \, dF \right],$$

where the coefficients $Q_j$ were introduced in Proposition 3.4, the coefficients $\Theta_{j+l,\delta}^{\zeta}$ in Proposition 3.11 and the coefficients $\Delta_{j,l}^{\zeta}$ in Proposition 3.13. The coefficients $Q_j(\psi^\zeta, a\chi_{\text{top}})$ and $Q_j(\psi^\zeta, a\chi_F)$ sum up to contributions denoted by $\int_{\mathcal{M}_{\text{top}}^{\zeta}} Q_j(\psi^\zeta, a\chi_{\text{top}}) \, d\mu_{\text{top}}^{\zeta}$ and $\int_{\mathcal{M}_{\text{top}}^{\zeta}} Q_j(\psi^\zeta, a\chi_F) \, d\mu_{\text{top}}^{\zeta}$, respectively. By the considerations preceding this theorem, the coefficients $B_{j,l}^{\text{top}}(\beta_F)$ give rise to the contributions $\int_{\mathcal{M}_{\text{top}}^{\zeta}} \mathcal{R}_{j,l}(a\chi_F) \, d\mu_{\text{top}}^{\zeta}$, while the coefficients $q_{j,l}(\beta_F)$, $\mathcal{R}^{\text{sing}}(\beta_F)$ and $\Delta_{j,l}^{\zeta}(\zeta_F/\varepsilon)$ give rise to contributions $\int_F S_{j,l}^F(a\chi_F, \zeta_F/\varepsilon) \, dF$. In the latter, $\zeta/\varepsilon$ appears only as a bound of integration in integrals of compactly supported functions which are independent of $\zeta$ and $\varepsilon$, so that these integrals are bounded and smooth in $\zeta/\varepsilon$. Moreover, we can replace $a\chi_F$ by $a$ in $S_{j,l}^F(a\chi_F, \zeta_F/\varepsilon)$ because $\chi_F$ evaluates by construction to 1 near $F$. \qed

Notice that the previous theorem in particular describes the limit of $I_\alpha^\zeta(\varepsilon)$ as $\zeta$ approaches any of the singular values $\mathcal{J}(F)$, $F \in \mathcal{F}_\alpha$, of $\mathcal{J}$. Indeed, let $F_0 \in \mathcal{F}_\alpha$ and choose the number $\delta$ in Theorem 3.15 small enough that no singular values of $\mathcal{J}$ except $\mathcal{J}(F_0)$ lie in $(\mathcal{J}(F_0) - \delta, \mathcal{J}(F_0) + \delta)$. Then we
obtain the result
\[
\lim_{\zeta \to J} I^J_0(\zeta) \sim \sum_{j=0}^{\infty} \varepsilon^{1+j} \left[ \int_{\mathcal{M}^J_{\text{top}}(F_0)} Q_j(\psi^J(F_0), a\chi_{\text{top}}) \, d\mathcal{M}^J_{\text{top}}(F_0) \right. \\
+ \sum_{F \in \mathcal{F}_a, J(F) \neq J(F_0)} \int_{\mathcal{M}^J_{\text{top}}(F_0)} Q_j(\psi^J(F_0), a\chi_F) \, d\mathcal{M}^J_{\text{top}}(F_0) \\
+ \sum_{F \in \mathcal{F}_a, J(F) = J(F_0)} \int_{\mathcal{M}^J_{\text{top}}(F_0)} \mathcal{R}_{J,0}(a\chi_F) \, d\mathcal{M}^J_{\text{top}}(F_0) \\
+ \sum_{F \in \mathcal{F}_a, J(F) = J(F_0)} \int_F \mathcal{S}_{J,0}(a, 0) \, dF \right].
\]

This describes how the regular stationary phase expansion described in Proposition 2.4 bursts as \( \zeta \) approaches a singular value of \( J \), giving rise to new singular contributions that are independent of the choice of partition of unity.

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