SHARP INEQUALITIES FOR ANTI-ININVARIANT RIEMANNIAN
SUBMERSIONS FROM SASAKIAN SPACE FORMS

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Abstract. We obtain sharp inequalities involving the Ricci curvature and
the scalar curvature for anti-invariant Riemannian submersions from Sasakian
space forms onto Riemannian manifolds.

1. Introduction

To find relationship between the extrinsic and intrinsic invariants of a subman-
ifold have been very popular problems in the recent twenty five years. The first
study in this direction was started by B.-Y. Chen in 1993. He established some
inequalities between the main extrinsic (the squared mean curvature) and main
intrinsic invariants (the scalar curvature and the Ricci curvature) of a subman-
ifold in a real space form [11]. In 1999, Chen also established a relation between
the Ricci curvature and the squared mean curvature for a submanifold [12]. After
that, many papers have been published by various authors in different ambient
spaces. In 2011, Chen published a book which consists of the all studies doing
in these directions [13]. The topic is still very popular and there are many new
papers related to the inequalities which are introduced by Chen. For example see
[2], [10], [12], [15], [16], [17], [18], [19] and [21].

Let \((M, g)\) and \((B, g')\) be \(m\) and \(b\)-dimensional Riemannian manifolds, respec-
tively. A Riemannian submersion \(\pi : M \to B\) is a mapping of \(M\) onto \(B\) such
that \(\pi\) has a maximal rank and the differential \(\pi_\ast\) preserves the lengths of the
horizontal vectors [8]. In [4] and [5], Chen proved a simple optimal relationship be-
tween Riemannian submersions and minimal immersions [4]. In [1], Alegre, Chen
and Munteanu established a sharp relationship between the \(\delta\)-invariants and Rie-
mannian submersions with totally geodesic fibers. In [7], Gülbahar, Meriç and
Kılıç obtained sharp inequalities involving the Ricci curvature for Riemannian

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submersions. In [20], Şahin introduced anti-invariant Riemannian submersions from almost Hermitian manifolds onto Riemannian manifolds.

Motivated by the above studies, in the present study, we consider anti-invariant Riemannian submersions from Sasakian manifolds onto Riemannian manifolds. We obtain sharp inequalities involving the Ricci curvature and the scalar curvature.

The paper is organized as follows. In Section 2, we give brief introduction about Sasakian manifolds and submersions. We give some lemmas which will be used in Section 3 and Section 4. In Section 3, we obtain some inequalities involving the Ricci curvature and the scalar curvature on the vertical and horizontal distributions for anti-invariant Riemannian submersions from Sasakian space forms. The equality cases are also discussed. In Section 4, we prove Chen-Ricci inequalities on the vertical and horizontal distributions for anti-invariant Riemannian submersions from Sasakian space forms. We find relationships between the intrinsic and extrinsic invariants using fundamental tensors. The equality cases are also considered.

2. Preliminaries

Let \( \pi : M \to B \) be a Riemannian submersion. We put \( \dim M = 2m + 1 \) and \( \dim B = b \). For \( x \in B \), Riemannian submanifold \( \pi^{-1}(x) \) with the induced metric \( \overline{g} \) is called a fiber and denoted by \( \overline{M} \). We notice that the dimension of each fiber is always \((2m + 1 - b) = r \) and dimension of the horizontal distribution is \( n = (2m + 1 - r) \). In the tangent bundle \( TM \) of \( M \), the vertical and horizontal distributions are denoted by \( \mathcal{V}(M) \) and \( \mathcal{H}(M) \), respectively. We call a vector field \( X \) on \( M \) projectable if there exists a vector field \( X_* \) on \( B \) such that \( \pi_*(X_p) = X_{*\pi(p)} \) for each \( p \in M \). In this case, we call that \( X \) and \( X_* \) are \( \pi \)-related. A vector field \( X \) on \( M \) is called basic if it is projectable and horizontal ([8] and [9]).

The tensor fields \( T \) and \( A \) of type \((1,2)\) are defined by

\[
T_E F = h \nabla_{vE} vF + v \nabla_{vE} hF,
\]

\[
A_E F = h \nabla_{hE} vF + v \nabla_{hE} hF.
\]

Denote by \( R, R', \hat{R} \) and \( R^* \) the Riemannian curvature tensor of Riemannian manifolds \( M, B \), the vertical distribution \( \mathcal{V} \) and the horizontal distribution \( \mathcal{H} \),
respectively. Then the Gauss-Codazzi type equations are given by

\[ R(U, V, F, W) = \hat{R}(U, V, F, W) + g(T_U W, T_V F) - g(T_V W, T_U F), \]

\[ R(X, Y, Z, H) = R^*(X, Y, Z, H) - 2g(A_X Y, A_Z H) \]
\[ \quad + g(A_Y Z, A_X H) - (A_X Z, A_Y H), \]

\[ R(X, V, Y, W) = g(\nabla_X (T_V (V, W), Y) + g(\nabla_V A (X, Y), W) \]
\[ \quad - g(T_V X, T_W Y) + g(A_Y W, A_X V), \]

where

\[ \pi_\ast (R^* (X, Y) Z) = R' (\pi_\ast X, \pi_\ast Y) \pi_\ast Z \]

for any \( X, Y, Z, H \in \chi^h(M) \) and \( U, V, F, W \in \chi^v(M) \) \[1\].

Moreover, the mean curvature vector field \( H \) of any fibre of Riemannian submersion \( \pi \) is given by

\[ H = rN, \quad N = \sum_{j=1}^r T_{U_j} U_j \]

where \( \{U_1, ..., U_r\} \) is an orthonormal basis of the vertical distribution \( \mathcal{V} \). Furthermore, \( \pi \) has totally geodesic fibers if \( T \) vanishes on \( \chi^h(M) \) and \( \chi^v(M) \).

Now we give the following lemmas:

**Lemma 2.1.** \[6\] Let \((M, g)\) and \((B, g')\) be Riemannian manifolds admitting a Riemannian submersion \( \pi : M \to B \). For \( E, F, G \in \chi(M) \), we have

\[ g(T_E F, G) = -g(F, T_E G), \]

\[ g(A_E F, G) = -g(F, A_E G). \]

That is, \( A_E \) and \( T_E \) are anti-symmetric with respect to \( g \).

**Lemma 2.2.** \[6\] Let \((M, g)\) and \((B, g')\) be Riemannian manifolds admitting a Riemannian submersion \( \pi : M \to B \).

(i) For \( U, V \in \chi^v(M) \),

\[ T_U V = T_V U, \]

(ii) For \( X, Y \in \chi^h(M) \),

\[ A_X Y = -A_Y X. \]
Let \( M \) be a \((2m + 1)\)-dimensional manifold and \( \phi, \xi, \eta \) a tensor field of type (1, 1), a vector field, a 1-form on \( M \), respectively. If \( \phi, \xi \) and \( \eta \) satisfy the following conditions
\[
\eta(\xi) = 1, \quad \phi^2 X = -X + \eta(X)\xi
\]
for \( X \in TM \), then \( M \) is said to have an almost contact structure \((\phi, \xi, \eta)\) and \((M, \phi, \xi, \eta)\) is called an almost contact manifold. If
\[
\nabla_X \xi = -\phi X, \quad (\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X,
\]
then \((M, \nabla, g, \phi, \xi, \eta)\) is called a Sasakian manifold \([3]\), where \( \nabla \) denotes the Levi-Civita connection of \( g \). \( \phi \) is anti-symmetric with respect to \( g \), that is, for \( X, Y \in TM \)
\[
g(\phi X, Y) + g(X, \phi Y) = 0.
\]
A plane section \( \pi \) in \( TM \) is called a \( \phi \)-section if it is spanned by \( X \) and \( \phi X \), where \( X \) is a unit tangent vector field orthogonal to \( \xi \). The sectional curvature of a \( \phi \)-section is called a \( \phi \)-sectional curvature. A Sasakian manifold with constant \( \phi \)-sectional curvature \( c \) is said to be a Sasakian space form \([3]\) and is denoted by \( M(c) \). The curvature tensor \( R \) of \( M(c) \) is expressed by
\[
R(X, Y)Z = \frac{c+3}{4}[g(Y, Z)X - g(X, Z)Y] + \frac{c-1}{4}[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z].
\]

**Definition 2.1.** \([14]\) Let \((M, \nabla, g, \phi, \xi, \eta)\) be a Sasakian manifold and \((B, g')\) a Riemannian manifold. A Riemannian submersion \( \pi : M \to B \) is called anti-invariant if \( \mathcal{V}(M) \) is anti-invariant with respect to \( \phi \), i.e. \( \phi(\mathcal{V}(M)) \subseteq \mathcal{H}(M) \).

Let \( \pi : (M, \nabla, g, \phi, \xi, \eta) \to (B, g') \) be an anti-invariant Riemannian submersion from a Sasakian manifold \((M, \nabla, g, \phi, \xi, \eta)\) to a Riemannian manifold \((B, g')\). From Definition 2.1, we have \( \phi(\mathcal{V}(M)) \cap \mathcal{H}(M) \neq \{0\} \). We denote the complementary orthogonal distribution to \( \phi(\mathcal{V}(M)) \) in \( \mathcal{H}(M) \) by \( \mu \). Then we have
\[
\mathcal{H}(M) = \phi(\mathcal{V}(M)) \oplus \mu.
\]
Suppose that \( \xi \) is vertical. It is easy to see that \( \mu \) is an invariant distribution of \( \mathcal{H}(M) \) under the endomorphism \( \phi \). Thus for \( X \in \chi^h(M) \), we write
\[
\phi X = BX + CX,
\]
where $BX \in \chi^v(M)$ and $CX \in \chi(\mu)$ \[14\].

Suppose that $\xi$ is horizontal. It is easy to see that $\mu = \phi\mu + \{\xi\}$. Thus for $X \in \chi^h(M)$, we write

$$\phi X = BX + CX,$$

where $BX \in \chi^v(M)$ and $CX \in \chi(\mu)$ \[14\].

**Lemma 2.3.** \[14\] Let $\pi : M \to B$ be an anti-invariant Riemannian submersion from a Sasakian manifold $(M, \nabla, g, \phi, \xi, \eta)$ to a Riemannian manifold $(B, g^\prime)$.

(i) If $\xi$ is vertical, then $C^2X = -X - \phi BX$

(ii) If $\xi$ is horizontal, then $C^2X = -X + \eta(X) \xi - \phi BX$.

**Example 2.1.** \[3\] Let us take $M = \mathbb{R}^{2m+1}$ with the standard coordinate functions $(x_1, \ldots, x_m, y_1, \ldots, y_m, z)$, the contact structure $\eta = \frac{1}{2}(dz - \sum_{i=1}^{m} y_i dx_i)$, the characteristic vector field $\xi = \frac{2}{\partial z}$ and the tensor field $\varphi$ given by

$$\varphi = \begin{bmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & y_j & 0 \end{bmatrix}.$$ 

The Riemannian metric is $g = \eta \otimes \eta + \frac{1}{2} \sum_{i=1}^{m} ((dx_i)^2 + (dy_i)^2)$. Then $(M^{2m+1}, \varphi, \xi, \eta, g)$ is a Sasakian space form with constant $\varphi$-sectional curvature $c = -3$ and it is denoted by $\mathbb{R}^{2m+1}(-3)$. The vector fields

$$E_i = 2 \frac{\partial}{\partial y_i}, \quad E_{i+m} = \varphi X_i = 2 \left( \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial z} \right), \quad 1 \leq i \leq m, \quad \xi = 2 \frac{\partial}{\partial z},$$

form a $g$-orthonormal basis for the contact metric structure.

**Example 2.2.** \[14\] We consider $M = \mathbb{R}^5(-3)$ with the structure given in Example 2.1. The Riemannian metric $g_{\mathbb{R}^2}$ is given by

$$g_{\mathbb{R}^2} = \frac{1}{8} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

on $\mathbb{R}^2$. Let $\pi : \mathbb{R}^5(-3) \to \mathbb{R}^2$ be a map defined by

$$\pi(x_1, x_2, y_1, y_2, z) = (x_1 + y_1, x_2 + y_2).$$

Then

$$\mathcal{V}(M) = sp \{ V_1 = E_1 - E_3, V_2 = E_2 - E_4, V_3 = E_5 = \xi \}$$
and
\[ \mathcal{H}(M) = \text{sp} \{ H_1 = E_1 + E_3, H_2 = E_2 + E_4 \}. \]
So \( \pi \) is a Riemannian submersion. Moreover, \( \phi V_1 = H_1, \phi V_2 = H_2, \phi V_3 = 0 \) imply
that \( \phi(\mathcal{V}(M)) = \mathcal{H}(M) \). Hence \( \pi \) is an anti-invariant Riemannian submersion
such that \( \xi \) is vertical.

Example 2.3. [14] We consider \( M = \mathbb{R}^5(-3) \) with the structure given in Example 2.1. Let
\( N = \mathbb{R}^3 - \{ (y_1, y_2, z) \in \mathbb{R}^3 \mid y_1^2 + y_2^2 \leq 2 \} \). The Riemannian metric tensor
\( g_N \) is given by
\[
g_N = \frac{1}{4} \begin{bmatrix}
\frac{1}{2} & \frac{y_1 y_2}{2} & -\frac{y_1}{2} \\
\frac{y_1 y_2}{2} & \frac{1}{2} & -\frac{y_2}{2} \\
-\frac{y_1}{2} & -\frac{y_2}{2} & 1
\end{bmatrix}
\]
on \( N \). Let \( \pi : \mathbb{R}^5(-3) \to N \) be a map defined by
\[
\pi(x_1, x_2, y_1, y_2, z) = \left( x_1 + y_1, x_2 + y_2, \frac{y_1^2}{2} + \frac{y_2^2}{2} + z \right).
\]
Then
\[ \mathcal{V}(M) = \text{sp} \{ V_1 = E_1 - E_3, V_2 = E_2 - E_4 \} \]
and
\[ \mathcal{H}(M) = \text{sp} \{ H_1 = E_1 + E_3, H_2 = E_2 + E_4, H_3 = E_5 = \xi \}. \]
So \( \pi \) is a Riemannian submersion. Moreover, \( \phi V_1 = H_1, \phi V_2 = H_2 \) imply that
\( \phi(\mathcal{V}(M)) \subset \mathcal{H}(M) = \phi(\mathcal{V}(M)) \oplus \{ \xi \} \). Hence \( \pi \) is an anti-invariant Riemannian
submersion such that \( \xi \) is horizontal.

3. Inequalities for anti-invariant Riemannian submersions

In the present section, we aim to obtain some inequalities involving the Ricci
curvature and the scalar curvature on the vertical and horizontal distributions
for anti-invariant Riemannian submersions from Sasakian space forms. We shall
also consider the equality cases of these inequalities.

Let \( (M(c), g), (B, g') \) be a Sasakian space form and a Riemannian manifold,
respectively and \( \pi : M(c) \to B \) an anti-invariant Riemannian submersion. Fur-
thermore, let \( \{ U_1, ..., U_r, X_1, ..., X_n \} \) be an orthonormal basis of \( TM(c) \) such that
\( \mathcal{V} = \text{span} \{ U_1, ..., U_r \}, \mathcal{H} = \text{span} \{ X_1, ..., X_n \} \). Then using (2.4) and (2.1), we have
\[
\hat{R}(U, V, F, W) = \frac{c+3}{4} \left\{ g(V, F) g(U, W) - g(U, F) g(V, W) \right\} \\
+ \frac{c-1}{4} \left\{ \eta(U) \eta(F) g(V, W) - \eta(V) \eta(F) g(U, W) \right\} \\
+ \frac{c-1}{4} \left\{ \eta(U) g(V, F) - \eta(V) g(U, F) + g(\phi V, F) g(\phi U, W) \\
- g(\phi V, W) g(\phi U, F) - 2g(W, \phi F) g(\phi U, V) \right\} \\
- g(T_U W, T_V F) + g(T_V W, T_U F) .
\]

Similarly, from (2.4) and (2.2), we get
\[
R^*(X, Y, Z, H) = \frac{c+3}{4} \left\{ g(Y, Z) g(X, H) - g(X, Z) g(Y, H) \right\} \\
+ \frac{c-1}{4} \left\{ \eta(X) \eta(Z) g(Y, H) - \eta(Y) \eta(Z) g(X, H) \right\} \\
+ \frac{c-1}{4} \left\{ \eta(Y) g(X, Z) - \eta(X) g(Y, Z) + g(\phi Y, Z) g(\phi X, H) \\
- g(\phi Y, H) g(\phi X, Z) - 2g(H, \phi Z) g(\phi X, Y) \right\} \\
+ 2g(A_X Y, A_Z H) - g(A_Y Z, A_X H) + (A_X Z, A_Y H) .
\]

**Case I:** Assume that \( \xi \) is vertical.

For the vertical distribution, in view of (3.1), since \( \pi \) is anti-invariant and \( \xi \) is vertical, we find
\[
\hat{\text{Ric}}(U) = \frac{c+3}{4} (r-1) g(U, U) + \frac{c-1}{4} \left\{ (2-r) \eta(U)^2 - g(U, U) \right\} \\
- r g(T_U U, H) + \sum_{j=1}^{r} g(T_{U_j} U, T_{U_j}) .
\]

Hence we obtain the following theorem:

**Theorem 3.1.** Let \( \pi : M(c) \to B \) be an anti-invariant Riemannian submersion from a Sasakian space form \( (M(c), g) \) onto a Riemannian manifold \( (B, g') \) such that \( \xi \) is vertical. Then
\[
\hat{\text{Ric}}(U) \geq \frac{c+3}{4} (r-1) - \frac{c-1}{4} \left\{ (r-2) \eta(U)^2 + 1 \right\} - r g(T_U U, H) .
\]

The equality case of the inequality holds for a unit vertical vector field \( U \in \chi^V(M(c)) \) if and only if each fiber is totally geodesic.
Similarly in view of (3.1), using the symmetry of $T$, we have

$$2\hat{\tau} = \frac{c+3}{4} r (r - 1) + \frac{c-1}{4} (2 - 2r) - r^2 \|H\|^2 + \sum_{i,j=1}^{r} g(T_{U_i}U_j, T_{U_i}U_j),$$

where $\hat{\tau} = \sum_{1 \leq i < j \leq r} \hat{R}(U_i, U_j, U_j, U_i)$. Then we can write

$$2\hat{\tau} \geq \frac{c+3}{4} r (r - 1) - \frac{c-1}{2} (r - 1) - r^2 \|H\|^2.$$

The equality case of the inequality holds if and only if $T = 0$, which means that each fiber is totally geodesic. Thus we can state the following theorem:

**Theorem 3.2.** Let $\pi : M(c) \to B$ be an anti-invariant Riemannian submersion from a Sasakian space form $(M(c), g)$ onto a Riemannian manifold $(B, g')$ such that $\xi$ is vertical. Then

$$2\hat{\tau} \geq \frac{c+3}{4} r (r - 1) - \frac{c-1}{2} (r - 1) - r^2 \|H\|^2.$$

The equality case of the inequality holds if and only if each fiber is totally geodesic.

For the horizontal distribution, in view of (3.2), since $\pi$ is anti-invariant and $\xi$ is vertical, using the anti-symmetry of $A$, we find

$$2\tau^* = \frac{c+3}{4} n (n - 1)$$

$$+ \sum_{i,j=1}^{n} \left[ \frac{3(c-1)}{4} g(CX_i, X_j) g(CX_i, X_j) - 3g(A_n, X_j, A_n, X_j) \right].$$

By the use of Lemma 2.3, we obtain

$$2\tau^* = \frac{c+3}{4} n (n - 1) + \frac{3}{4} (c-1) (n + tr (\phi B)) - \sum_{i,j=1}^{n} 3g(A_n, X_j, A_n, X_j).$$

Then we can write

$$2\tau^* \leq \frac{c+3}{4} n (n - 1) + \frac{3}{4} (c-1) (n + tr (\phi B)),$$

where $\tau^* = \sum_{1 \leq i < j \leq n} R^* (X_i, X_j, X_j, X_i)$. The equality case of (3.4) holds if and only if $A = 0$, which means that the horizontal distribution is integrable. So we can state the following theorem:
Theorem 3.3. Let \( \pi : M(c) \to B \) be an anti-invariant Riemannian submersion from a Sasakian space form \((M(c), g)\) onto a Riemannian manifold \((B, g')\) such that \(\xi\) is vertical. Then
\[
2\tau^* \leq \frac{c + 3}{4} n (n - 1) + \frac{3}{4} (c - 1) (n + tr (\phi B)).
\]
The equality case of (3.4) holds if and only if \(\mathcal{H}(M)\) is integrable.

Case II: Assume that \(\xi\) is horizontal. From (3.1), since \(\pi\) is anti-invariant submersion, after some computations, we have
\[
2\tilde{\tau} = \frac{c + 3}{4} r (r - 1) - r^2 \|H\|^2 + \sum_{i,j=1}^{r} g(T_U i U_j, T_U U_j).
\]
Hence we can state the following theorem:

Theorem 3.4. Let \( \pi : M(c) \to B \) be an anti-invariant Riemannian submersion from a Sasakian space form \((M(c), g)\) onto a Riemannian manifold \((B, g')\) such that \(\xi\) is horizontal. Then
\[
2\tilde{\tau} \geq \frac{c + 3}{4} r (r - 1) - r^2 \|H\|^2.
\]
The equality case of the inequality holds if and only if each fiber is totally geodesic.

For the horizontal distribution, from (3.2), since \(\xi\) is horizontal and \(A\) is anti-symmetric, after some computations, we have
\[
2\tau^* = \frac{c + 3}{4} n (n - 1) + \sum_{i,j=1}^{n} \left[ \frac{c - 1}{4} \{2 - 2n + 3g (CX_i, X_j) g (CX_i, X_j)\} - 3g (A X_i X_j, A X_i X_j) \right].
\]
Then using Lemma 2.3, we obtain
\[
2\tau^* = \frac{c + 3}{4} n (n - 1) + \frac{c - 1}{4} (3tr \phi B + n - 1)
- \sum_{i,j=1}^{n} 3g (A X_i X_j, A X_i X_j),
\]
where \(\tau^* = \sum_{1 \leq i < j \leq n} R^* (X_i, X_j, X_j, X_i)\).

So we can state the following theorem:
Theorem 3.5. Let $\pi : (M(c), g) \to (B, g')$ be an anti-invariant Riemannian submersion from a Sasakian space form $(M(c), g)$ onto a Riemannian manifold $(B, g')$ such that $\xi$ is horizontal. Then
\[ 2\tau^* \leq \frac{c+3}{4} n (n-1) + \frac{(c-1)}{4} (3tr(\phi B) + n - 1). \]
The equality case of the inequality holds if and only if $H(M)$ is integrable.

4. Chen-Ricci inequalities for anti-invariant Riemannian submersions

In the present section, we aim to obtain Chen-Ricci inequality on the vertical and horizontal distributions for anti-invariant Riemannian submersions from a Sasakian space forms onto a Riemannian manifold. The equality cases will be also considered.

Let $(M(c), g)$ be a Sasakian space form and $(B, g')$ a Riemannian manifold. Assume that $\pi : (M(c), g) \to (B, g')$ is an anti-invariant Riemannian submersion and $\{U_1, \ldots, U_r, X_1, \ldots, X_n\}$ is an orthonormal basis of $TM(c)$ such that $\mathcal{V} = \text{span}\{U_1, \ldots, U_r\}$, $\mathcal{H} = \text{span}\{X_1, \ldots, X_n\}$. Now we denote $T^s_{ij}$ by
\[ T^s_{ij} = g(T_{U_i U_j} X_s), \quad (4.1) \]
where $1 \leq i, j \leq r$ and $1 \leq s \leq n$ (see [7]).

Similarly, we denote $A^\alpha_{ij}$ by
\[ A^\alpha_{ij} = g(A_{X_i X_j} U_\alpha), \quad (4.2) \]
where $1 \leq i, j \leq n$ and $1 \leq \alpha \leq r$. From [7], we use
\[ \delta (N) = \sum_{i=1}^{n} \sum_{k=1}^{r} g((\nabla_{X_i} T)_{U_k} U_k, X_i). \quad (4.3) \]

Case I: Assume that $\xi$ is vertical.

Then from [31], we have
\[ 2\hat{\tau} = \frac{c+3}{4} r (r - 1) - \frac{c-1}{2} (r - 1) - r^2 \|H\|^2 + \sum_{i,j=1}^{r} g(T_{U_i U_j}, T_{U_i U_j}). \]

Using (4.4) in the last equality and the symmetry of $T$, we can write
\[ 2\hat{\tau} = \frac{c+3}{4} r (r - 1) - \frac{c-1}{2} (r - 1) - r^2 \|H\|^2 + \sum_{s=1}^{n} \sum_{i,j=1}^{r} (T^s_{ij})^2. \quad (4.4) \]
For a local orthonormal frame \( \{X_i, U_j\}_{1 \leq i \leq n, 1 \leq j \leq r} \) on \( M(c) \), such that the horizontal and vertical distributions are spanned by \( \{X_i\}_{1 \leq i \leq n} \) and \( \{U_j\}_{1 \leq j \leq r} \), respectively, we know from [7] that

\[
\sum_{s=1}^{n} \sum_{i,j=1}^{r} (T_{ij}^s)^2 = \frac{1}{2} r^2 \|H\|^2 + \frac{1}{2} \sum_{s=1}^{n} [T_{11}^s - T_{22}^s - \ldots - T_{rr}^s]^2 \\
+ 2 \sum_{s=1}^{n} \sum_{j=2}^{r} (T_{1j}^s)^2 - 2 \sum_{s=1}^{n} \sum_{2 \leq i < j \leq r} \left[ T_{ii}^s T_{jj}^s - (T_{ij}^s)^2 \right].
\] (4.5)

So using the above equality in (4.4), we get

\[
2 \hat{\tau} = \frac{c + 3}{4} r (r - 1) - \frac{c - 1}{2} (r - 1) \\
\frac{1}{2} r^2 \|H\|^2 + \frac{1}{2} \sum_{s=1}^{n} [T_{11}^s - T_{22}^s - \ldots - T_{rr}^s]^2 \\
+ 2 \sum_{s=1}^{n} \sum_{j=2}^{r} (T_{1j}^s)^2 - 2 \sum_{s=1}^{n} \sum_{2 \leq i < j \leq r} \left[ T_{ii}^s T_{jj}^s - (T_{ij}^s)^2 \right].
\] (4.6)

Then from the last equality, we have

\[
2 \hat{\tau} \geq \frac{c + 3}{4} r (r - 1) - \frac{c - 1}{2} (r - 1) \\
\frac{1}{2} r^2 \|H\|^2 - 2 \sum_{s=1}^{n} \sum_{2 \leq i < j \leq r} \left[ T_{ii}^s T_{jj}^s - (T_{ij}^s)^2 \right].
\] (4.7)

Furthermore, from (2.1), taking \( U = W = U_i, V = F = U_j \) and using (4.1) we can write

\[
2 \sum_{2 \leq i < j \leq r} R(U_i, U_j, U_j, U_i) = 2 \sum_{2 \leq i < j \leq r} \mathcal{R}(U_i, U_j, U_j, U_i) \\
+ 2 \sum_{s=1}^{n} \sum_{2 \leq i < j \leq r} \left[ T_{ii}^s T_{jj}^s - (T_{ij}^s)^2 \right].
\]
Then using the equality
\[ 2\hat{\tau} = 2 \sum_{2 \leq i < j \leq r} \hat{R}(U_i, U_j, U_j, U_i) + 2 \sum_{j=1}^{r} \hat{R}(U_1, U_j, U_j, U_1), \] (4.8)
in view of (4.7), we have
\[ 2\hat{\text{Ric}}(U_1) \geq \frac{c + 3}{4} r (r - 1) - \frac{c - 1}{2} (r - 1)\]
\[ - \frac{1}{2} r^2 \|H\|^2 - 2 \sum_{2 \leq i < j \leq r} R(U_i, U_j, U_j, U_i). \]

Since \( M \) is a Sasakian space form, its curvature tensor \( R \) satisfies the equality (2.4). So we obtain
\[ \hat{\text{Ric}}(U_1) \geq \frac{c + 3}{4} (r - 1) + \frac{c - 1}{4} \{ (2 - r) \eta(U_1)^2 - 1 \} - \frac{1}{4} r^2 \|H\|^2. \]

Hence we state the following theorem:

**Theorem 4.1.** Let \( \pi : M(c) \to B \) be an anti-invariant Riemannian submersion from a Sasakian space form \((M(c), g)\) onto a Riemannian manifold \((B, g')\) such that \( \xi \) is vertical. Then
\[ \hat{\text{Ric}}(U_1) \geq \frac{c + 3}{4} (r - 1) - \frac{c - 1}{4} \{ (r - 2) \eta(U_1)^2 + 1 \} - \frac{1}{4} r^2 \|H\|^2. \]
The equality case of the inequality holds if and only if
\[ T_{s11}^s = T_{s22}^s + \ldots + T_{sr}^s, \]
\[ T_{ij} = 0, \ j = 2, \ldots, r. \]

On the other hand, using (4.2) and Lemma 2.3, the equation (3.3) can be rewritten as

\[ 2\tau^* = \frac{c + 3}{4} n (n - 1) + \frac{3}{4} (c - 1) (n + tr(\phi B)) - 3 \sum_{\alpha=1}^{r} \sum_{i,j=1}^{n} (A_{ij}^\alpha)^2. \]

Since \( A \) is anti-symmetric on \( \chi^H(M(c)) \), the above equality turns into
\[ 2\tau^* = \frac{c + 3}{4} n (n - 1) + \frac{3}{4} (c - 1) (n + tr(\phi B)) \]
\[ - 6 \sum_{\alpha=1}^{r} \sum_{j=2}^{n} (A_{1j}^\alpha)^2 - 6 \sum_{\alpha=1}^{r} \sum_{2 \leq i < j \leq n} (A_{ij}^\alpha)^2. \] (4.9)
Furthermore, from (2.2), taking \( X = H = X_i, Y = Z = X_j \) and using (4.2), we have
\[
2 \sum_{2 \leq i < j \leq n} R(X_i, X_j, X_j, X_i) = 2 \sum_{2 \leq i < j \leq n} R^*(X_i, X_j, X_j, X_i) + 6 \sum_{\alpha=1}^{r} \sum_{2 \leq i < j \leq n} (A^\alpha_{ij})^2.
\]
(4.10)

If we consider the last equality in (4.9), then we get
\[
2 \tau^* = \frac{c+3}{4} n(n-1) + \frac{3}{4} (c-1) (n+tr(\phi B)) - 6 \sum_{\alpha=1}^{r} \sum_{j=2}^{n} (A^\alpha_{1j})^2
\]
\[
+2 \sum_{2 \leq i < j \leq n} R^*(X_i, X_j, X_j, X_i) - 2 \sum_{2 \leq i < j \leq n} R(X_i, X_j, X_j, X_i).
\]
Since \( M \) is a Sasakian space form, its curvature tensor \( R \) satisfies the equality (2.4). Then we have
\[
2Ric^*(X_1) = \frac{c+3}{2} (n-1) + \frac{3}{4} (c-1) \|CX_1\|^2
\]
\[
-6 \sum_{\alpha=1}^{r} \sum_{j=2}^{n} (A^\alpha_{1j})^2.
\]
So we can write
\[
Ric^*(X_1) \leq \frac{c+3}{2} (n-1) + \frac{3}{4} (c-1) \|CX_1\|^2.
\]

Hence we obtain the following theorem:

**Theorem 4.2.** Let \( \pi : M(c) \to B \) be an anti-invariant Riemannian submersion from a Sasakian space form \((M(c), g)\) onto a Riemannian manifold \((B, g')\) such that \( \xi \) is vertical. Then
\[
Ric^*(X_1) \leq \frac{c+3}{4} (n-1) + \frac{3}{4} (c-1) \|CX_1\|^2.
\]
The equality case of the inequality holds if and only if
\[
A_{1j} = 0, \quad j = 2, ..., n.
\]

Since
\[
2\tau = \sum_{s=1}^{n} Ric(X_s, X_s) + \sum_{k=1}^{r} Ric(U_k, X_k),
\]
\[
2\tau = \sum_{j,k=1}^{r} R(U_j, U_k, U_k, U_j) + \sum_{i=1}^{n} \sum_{k=1}^{r} R(X_i, U_k, U_k, X_i) \\
+ \sum_{i,s=1}^{n} R(X_i, X_s, X_s, X_i) + \sum_{s=1}^{n} \sum_{j=1}^{r} R(U_j, X_s, X_s, U_j),
\]
(4.11)

where \(\tau\) is the scalar curvature of \(M(c)\). Since \(M(c)\) is a Sasakian space form, using (4.11) and (2.4), we find
\[
2\tau = \frac{c + 3}{4} \left( r(r - 1) + n(n - 1) + 2nr \right) + \frac{c - 1}{4} \left( 4(r - 1) + n + 3tr\phi B \right).
\]
(4.12)

On the other hand, from the Gauss-Codazzi type equations (2.1), (2.2) and (2.3), we have
\[
2\tau = 2\hat{\tau} + 2\tau^* + r^2 \|H\|^2 + \sum_{k,j=1}^{r} g(T_{U_k} U_j, T_{U_k} U_j) \\
+ 3 \sum_{i,s=1}^{n} g(A_X X_s, A_X X_s) - \sum_{i=1}^{n} \sum_{k=1}^{r} g((\nabla X_i T)_{U_k} U_k, X_i) \\
+ \sum_{s=1}^{n} \sum_{j=1}^{r} \{ g(T_{U_j} X_s, T_{U_j} X_s) - g(A_X U_j, A_X X_s) \}.
\]
(4.13)

Using (4.8), (4.10) and (4.12) in the last equality, we obtain
\[
2\tau = 2\hat{\tau} + 2\tau^* + \frac{1}{2} r^2 \|H\|^2 - \frac{1}{2} \sum_{s=1}^{n} \left[ T_{11}^s - T_{22}^s - ... - T_{rr}^s \right]^2 \\
- 2 \sum_{s=1}^{n} \sum_{j=2}^{r} (T_{1j}^s)^2 + 2 \sum_{s=1}^{n} \sum_{2 \leq j < k \leq r} \left[ T_{22}^s T_{kk}^s - (T_{jk}^s)^2 \right] + 6 \sum_{\alpha=1}^{n} \sum_{s=2}^{r} (A_{1s}^\alpha)^2 \\
+ 6 \sum_{\alpha=1}^{n} \sum_{2 \leq i < s \leq n} (A_{is}^\alpha)^2 + \sum_{i=1}^{n} \sum_{k=1}^{r} \{ g(T_{U_k} X_i, T_{U_k} X_i) - g(A_X U_k, A_X X_i) \} \\
- 2\delta(N) + \sum_{s=1}^{n} \sum_{j=1}^{r} \{ g(T_{U_j} X_s, T_{U_j} X_s) - g(A_X U_j, A_X X_j) \}.
\]

By making use of (4.8), (4.10) and (4.12) in the last equality, we obtain
\[
\frac{c + 3}{2} nr + \frac{c - 1}{2} (3 (r - 1) - n)
\]

\[
+ 2 \sum_{k=1}^{r} R(U_1, U_k, U_k, U_1) + 2 \sum_{s=1}^{n} R(X_1, X_s, X_s, X_1)
\]

\[
= 2 \widehat{\text{Ric}}(U_1) + 2 \text{Ric}^*(X_1) + \frac{1}{2} r^2 \|H\|^2 - \frac{1}{2} \sum_{s=1}^{n} [T_{11}^s - T_{22}^s - \ldots - T_{rr}^s]^2
\]

\[
- 2 \sum_{s=1}^{n} \sum_{j=2}^{r} (T_{ij}^s)^2 + 6 \sum_{\alpha=1}^{r} \sum_{\beta=1}^{n} (A_{\alpha}^s)^2 + \sum_{i=1}^{n} \sum_{k=1}^{r} \{ g(T_{U_k} X_i, T_{U_k} X_i) - g(A X_i U_k, A X_i U_k) \}
\]

\[
- 2 \delta (N) + \sum_{s=1}^{n} \sum_{j=1}^{r} \{ g(T_{U_j} X_s, T_{U_j} X_s) - g(A X_s U_j, A X_s U_j) \}
\]

We denote

\[
\|T^V\|^2 = \sum_{i=1}^{n} \sum_{k=1}^{r} g(T_{U_k} X_i, T_{U_k} X_i)
\]

and

\[
\|A^H\|^2 = \sum_{i=1}^{n} \sum_{k=1}^{r} g(A X_i U_k, A X_i U_k)
\]

(see \[7\]).

Since \((M(c), g)\) is a Sasakian space form, from \[2.4\], we obtain the following theorem:

**Theorem 4.3.** Let \(\pi : M(c) \rightarrow B\) be an anti-invariant Riemannian submersion from a Sasakian space form \((M(c), g)\) onto a Riemannian manifold \((B, g')\) such that \(\xi\) is vertical. Then

\[
\frac{c + 3}{4} \{nr + n + r - 2\} + \frac{c - 1}{4} \{3r - 4 - n\}
\]

\[
- (r - 2) \eta (U_1)^2 + 3 \|CX_1\| \leq \widehat{\text{Ric}}(U_1) + \text{Ric}^*(X_1) + \frac{1}{4} r^2 \|H\|^2
\]

\[
+ 3 \sum_{\alpha=1}^{r} \sum_{\beta=2}^{n} (A_{\alpha}^s)^2 - \delta (N) + \|T^V\|^2 - \|A^H\|^2.
\]

The equality case of the inequality holds if and only if

\[
T_{11}^s = T_{22}^s = \ldots = T_{rr}^s,
\]

\[
T_{ij} = 0, \quad j = 2, \ldots, r.
\]
Case II: Assume that $\xi$ is horizontal.

From (3.1), similar to Theorem 4.1, we can state the following theorem:

**Theorem 4.4.** Let $\pi : M(c) \to B$ be an anti-invariant Riemannian submersion from a Sasakian space form $(M(c), g)$ onto a Riemannian manifold $(B, g')$ such that $\xi$ is horizontal. Then

$$\overline{\text{Ric}}(U_1) \geq \frac{c+3}{4} (r-1) - \frac{1}{4} r^2 \|H\|^2.$$

The equality case of the inequality holds if and only if

$$T_{11}^s = T_{22}^s + \ldots + T_{rr}^s,$$

$$T_{1j} = 0, \ j = 2, \ldots, r.$$

From (3.2), similar to Theorem 4.2, we have the following theorem:

**Theorem 4.5.** Let $\pi : M(c) \to B$ be an anti-invariant Riemannian submersion from a Sasakian space form $(M(c), g)$ onto a Riemannian manifold $(B, g')$ such that $\xi$ is horizontal. Then

$$\text{Ric}^*(X_1) \leq \frac{c+3}{4} (n-1) + \frac{c-1}{4} \left\{ (2-n) \eta(X_1)^2 - 1 + 3 \|CX_1\|^2 \right\}.$$

The equality case of the inequality holds if and only if

$$A_{1j} = 0, \ j = 2, \ldots, n.$$

Since $\xi$ is horizontal, from (4.11), we find

$$2\tau = \frac{c+3}{4} [r (r-1) + n (n-1) + 2nr] + \frac{c-1}{4} [n + 3tr\phi B + 4r - 7].$$

Using the above equation, (4.13), (4.3), (4.8), (4.10) and (4.3), we get

$$\frac{c+3}{2} nr + \frac{c-1}{2} (2r-3) + 2 \sum_{k=1}^{r} R(U_1, U_k, U_k, U_1) + 2 \sum_{s=1}^{n} R(X_1, X_s, X_s, X_1)$$

$$= 2\overline{\text{Ric}}(U_1) + 2\text{Ric}^*(X_1) + \frac{1}{2} r^2 \|H\|^2 - \frac{1}{2} \sum_{s=1}^{n} [T_{11}^s - T_{22}^s - \ldots - T_{rr}^s]^2$$

$$- 2 \sum_{s=1}^{n} \sum_{j=2}^{r} (T_{1j}^s)^2 + 6 \sum_{\alpha=1}^{r} \sum_{s=2}^{n} (A_{1s}^\alpha)^2 - 2\delta (N)$$
\[ + \sum_{s=1}^{n} \sum_{j=1}^{r} \left\{ g(T_{Uj}X_s, T_{Uj}X_s) - g(A_{Xj}U_s, A_{Xj}U_s) \right\} \]
\[ + \sum_{i=1}^{n} \sum_{k=1}^{r} \left\{ g(T_{Uk}X_i, T_{Uk}X_i) - g(A_{Xi}U_s, A_{Xi}U_s) \right\}. \]

Hence in view of (2.4), we obtain the following theorem:

**Theorem 4.6.** Let \( \pi : M(c) \to B \) be an anti-invariant Riemannian submersion from a Sasakian space form \( (M(c), g) \) onto a Riemannian manifold \( (B, g') \) such that \( \xi \) is horizontal. Then

\[ \frac{c + 3}{4} \left\{ nr + n + r - 2 \right\} + \frac{c - 1}{4} \left\{ 2r - 4 - (n - 2) \eta(X_1)^2 \right\} \]
\[ + 3 \| CX_1 \|^2 \leq \tilde{\text{Ric}}(U_1) + \text{Ric}^*(X_1) + \frac{1}{4} r^2 \| H \|^2 \]
\[ + 3 \sum_{\alpha=1}^{r} \sum_{s=2}^{n} (A_{\alpha s})^2 - \delta(N) + \| T^V \|^2 - \| A^H \|^2. \]

The equality case of the inequality holds if and only if

\[ T_{11}^s = T_{22}^s + \ldots + T_{rr}^s, \]
\[ T_{1j} = 0, \quad j = 2, \ldots, r. \]

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