Biorthogonal Polynomials for Potentials of two Variables and External Sources at the Denominator

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April 10, 2004

Abstract

We construct biorthogonal polynomials for a measure over the complex plane which consists in the exponential of a potential \(-V(z, z^*)\) and in a set of external sources at the numerator and at the denominator. We use the pseudonorm of these polynomials to calculate the resolvent integral for correlation functions of traces of powers of complex matrices (under certain conditions).

1 Introduction

In several domains of physics like, for instance, in string theory [1] and in quantum chromodynamics [2] we are interested in the spectral properties of random matrices; most results in this domain are included in the correlation functions for the invariants of the matrices like determinants, traces...Although most interesting results are expected from infinitely large matrices, exact results can be obtained in appropriate cases for finite matrices using the technique of orthogonal polynomials. The appropriate cases are when the integrals over the matrix elements can be transformed into integrals over the eigenvalues of the matrices; this is the case for hermitian matrices where a unitary transformation reduces the integrals to the real line for each eigenvalue. When the matrices are complex, there exists a small class of potentials [3] (which includes the Gaussian potential) where the integration can be reduced to the complex plane.
(or a compact domain of the complex plane) for each eigenvalue. Moreover this is valid only for a restricted class of correlation functions which is essentially

\[ < \prod_{i=1}^{p} \left[ \text{Tr} \: M^{I_i} \right] \prod_{i=1}^{q} \left[ \text{Tr} \: (M^+)^{K_i} \right] > J_i \text{ and } K_i > 0 \]  

(1)

in a potential \( V(M, M^+) \), which becomes

\[ < \prod_{i=1}^{p} \left[ \sum_j z_j^{I_i} \right] \prod_{i=1}^{q} \left[ \sum_j (z_j^*)^{K_i} \right] > \]  

(2)

in a potential \( V(z, z^*) \). In this context, given a potential \( V(z, z^*) \) such that the partition function is defined as

\[ Z = \int \prod_{i=1}^{N} d^2z_i \prod_{i<j} |z_i - z_j|^2 \cdot e^{-\sum_{i=1}^{N} V(z_i, z_i^*)} \]  

(3)

we wish to calculate the following integrals

\[ I_N = \frac{1}{Z} \int \prod_{i=1}^{N} d^2z_i \prod_{i<j} |z_i - z_j|^2 \prod_{j=1}^{N} \left\{ \frac{L_2}{M_2} \left( \prod_{i=1}^{n} \left( z_j - \eta_i \right) \right) \prod_{i=1}^{L_1} \left( z_j - \xi_i \right) \right\} \cdot e^{-\sum_{i=1}^{N} V(z_i, z_i^*)} \]  

(4)

In the special case where \( L_1 = M_1 \) and \( L_2 = M_2 \) the application of

\[ \prod_{i=1}^{M_1} \left( -\frac{\partial}{\partial \eta_i} \right) x_i = \eta_i, \prod_{i=1}^{M_1} \left( -\frac{\partial}{\partial \xi_i} \right) y_i = \xi_i, \]  

on \( I_N \) defines the resolvent

\[ J_N = \frac{1}{Z} \int \prod_{i=1}^{N} d^2z_i \prod_{i<j} |z_i - z_j|^2 \prod_{i=1}^{M_2} \left( \sum_{j=1}^{M_2} \frac{1}{z_j - \eta_i} \right) \prod_{i=1}^{M_1} \left( \sum_{j=1}^{M_1} \frac{1}{z_j - \xi_i} \right) \cdot e^{-\sum_{i=1}^{N} V(z_i, z_i^*)} \]  

(5)

Clearly enough, the large \( \xi_i, \eta_i \) expansion of \( J_N \) is a formal power series the coefficients of which are the correlation functions (2).

In this publication, we calculate \( I_N \) and \( J_N \); the formal power series of \( J_N \) will be done somewhere else. We now describe the method which is based on the existence of biorthogonal polynomials for the potential \( V(z, z^*) \) in the presence of external sources.

We consider a real potential \( V(z, z^*) \) which admits an infinite set \( \{ p_n(z) \} \) of orthogonal monic polynomials

\[ \int d^2z \: p^*_m(z) \: p_n(z) \: e^{-V(z, z^*)} = h_n \: \delta_{nm} \]  

(6)
where $p_m^*(z)$ is a short notation for $[p_m(z)]^*$. More generally, we introduce a positive Borel measure $\mu(z,z^*)$ on the complex plane and write (6) as

$$\int d\mu(z,z^*) \ p_m^*(z) p_n(z) = h_n \delta_{nm} \tag{7}$$

We introduce four operations:

1. \(\text{operation 1} : d\mu_1(z,z^*) = (z - \xi) \ d\mu(z,z^*)\)
2. \(\text{operation 2} : d\mu_2(z,z^*) = (z^* - \eta^*) \ d\mu(z,z^*)\)
3. \(\text{operation 3} : d\mu_3(z,z^*) = \frac{1}{z-y} \ d\mu(z,z^*)\)
4. \(\text{operation 4} : d\mu_4(z,z^*) = \frac{1}{z^*-x} \ d\mu(z,z^*)\)

Although these four measures are not real, each of them admits an infinite set of monic biorthogonal polynomials. The successive iterations of the four operations construct an infinite set of monic biorthogonal polynomials for the measure

$$d\mu(z,z^*; \xi_i, \eta_i; y_i, x_i^*) = \frac{L_2}{M_2} \prod_{i=1}^{L_2} (z^* - \eta_i^*) \prod_{i=1}^{L_1} (z - \xi_i) \ d\mu(z,z^*) \tag{8}$$

These monic biorthogonal polynomials are denoted by $q_n(z; \xi_i, \eta_i; y_i, x_i^*)$ and $q_n^*(z; \eta_i, \xi_i^*; x_i, y_i^*)$ and satisfy

$$\int d\mu(z,z^*; \xi_i, \eta_i; y_i, x_i^*) \ q_m^*(z; \eta_i, \xi_i^*; x_i, y_i^*) \ q_n(z; \xi_i, \eta_i^*; y_i, x_i^*) = ||q_n||^2 \delta_{nm} \tag{9}$$

Clearly, $q_n^*(z; \eta_i, \xi_i^*; x_i, y_i^*)$ is deduced from $q_n(z; \xi_i, \eta_i; y_i, x_i^*)$ by complex conjugation and by the exchange $\xi_i \leftrightarrow \eta_i$, $y_i \leftrightarrow x_i$, which implies $L_1 \leftrightarrow L_2$, $M_1 \leftrightarrow M_2$. The unicity of the biorthogonal polynomials requires that we stay away from the surfaces of the $(\xi_i, \eta_i^*; y_i, x_i^*)$ space where $||q_n|| = 0$.

This construction is a generalization of the well-known result by Christoffel [4] which shows that, given a positive Borel measure of one variable $d\mu(x)$ on the real line and its infinite set of orthogonal polynomials, it is possible to construct an infinite set of orthogonal polynomials for the measure

$$d\mu(x; \xi_i) = \prod_{i=1}^{L} (x - \xi_i) \ d\mu(x) \tag{10}$$

The integrals $I_N$ corresponding to this measure have been calculated by Brezin and Hikami [5].
Then, Uvarov [6] in 69 and recently Fyodorov and Strahov [7–8] constructed an infinite set of orthogonal polynomials for the measure

\[ d\mu(x;\xi_i, y_i) = \prod_{i=1}^{M} \frac{(x - \xi_i)}{(x - y_i)} \]

if the variables \( y_i \) are chosen away from the real axis. In (2003), Akemann and Vernizzi [9] (see also ref.[10]) constructed an infinite set of (bi)orthogonal polynomials corresponding to the measure

\[ d\mu(z, z^*; \xi_i; \eta_i^*) = \prod_{i=1}^{L_2} (z^* - \eta_i^*) \prod_{i=1}^{L_1} (z - \xi_i) \quad d\mu(z, z^*) \]

where \( d\mu(z, z^*) \) is a positive Borel measure over the complex plane.

In the following sections we construct the biorthogonal polynomials corresponding to the measure (8). We now explain how to calculate the integrals \( I_N \) and \( J_N \) from the biorthogonal polynomials. As usual, we write the expression

\[ \Delta(z) = \prod_{i<j} (z_i - z_j) = \begin{vmatrix} \pi_{N-1}(z_1) & \ldots & \pi_{N-1}(z_N) \\ \ldots & \ldots & \ldots \\ \pi_0(z_1) & \ldots & \pi_0(z_N) \end{vmatrix} \]

where the polynomials \( \pi_n(z) \) are any set of monic polynomials. If moreover, the polynomials \( \pi_n(z) \) are (bi)orthogonal, the variables of integration factorize in the integrals (3–4) and the (bi)orthogonality can be used to obtain

\[ Z = N! \prod_{i=0}^{N-1} h_i \]
\[ Z I_N = N! \prod_{i=0}^{N-1} ||q_i||^2 \]

In the general case, the pseudonorms \( ||q_i||^2 \) are given in section 3; in the special case where \( L_1 = M_1 \) and \( L_2 = M_2 \), we have

\[ ||q_i||^2 = h_i \frac{D_i}{D_{i-1}} \quad i \geq 0 \]

where \( D_i \) is defined in (19) and \( D_{-1} \) is given in (108-109); consequently

\[ I_N = \frac{D_{N-1}}{D_{-1}} \]
or from (108-109)

\[ I_N = (-)^{M_3(M_1-1)} (-)^{M_2(M_2-1)} \prod_{i,j=1}^{M_2} (x_i^* - \eta_j^*) \prod_{i,j=1}^{M_2} (y_i - \xi_j) \frac{\Delta(x^*) \Delta(y) \Delta(\eta^*) \Delta(\xi)}{\Delta(y)} D_{N-1} \]  

(18)

where \( \Delta(y) \) is the Vandermonde determinant of the variables \( y_i \) and \( D_n \) is a determinant expressed in terms of four kernels \( K_n, N_n, A_n, N_n^* \) defined in section 2.

\[
D_n = \begin{vmatrix}
N_n(\xi_1, y_1) & \cdots & N_n(\xi_{M_1}, y_1) & A_n(x_1^*, y_1) & \cdots & A_n(x_{M_2}^*, y_1) \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
N_n(\xi_1, y_M) & \cdots & N_n(\xi_{M_1}, y_M) & A_n(x_1^*, y_M) & \cdots & A_n(x_{M_2}^*, y_M) \\
K_n(\xi_1, \eta_1^*) & \cdots & K_n(\xi_{M_1}, \eta_1^*) & N_n^*(x_1, \eta_1) & \cdots & N_n^*(x_{M_2}, \eta_1) \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
K_n(\xi_1, \eta_{M_2}^*) & \cdots & K_n(\xi_{M_1}, \eta_{M_2}^*) & N_n^*(x_1, \eta_{M_2}) & \cdots & N_n^*(x_{M_2}, \eta_{M_2}) \\
\end{vmatrix}
\]  

(19)

Although \( N_n(\xi, y) \) has a single pole at \( \xi = y \), this pole is cancelled by the prefactor in \( I_N \). Let us mention that this type of determinant has also been found in Ref. [11] for the correlation functions of several hermitian matrices coupled by a chain of potentials. Finally, the resolvant \( J_N \) is calculated in section 4 as the determinant

\[ J_N = "D_{N-1}" \]  

(20)

where "\( D_n \)" is

\[
" \text{det} " = \begin{vmatrix}
H_n(\xi_1, \xi_1) & \cdots & N_n(\xi_{M_1}, \xi_1) & A_n(\eta_1^*, \xi_1) & \cdots & A_n(\eta_{M_2}^*, \xi_1) \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
N_n(\xi_1, \xi_{M_1}) & \cdots & H_n(\xi_{M_1}, \xi_{M_1}) & A_n(\eta_{M_2}^*, \xi_{M_1}) & \cdots & A_n(\eta_{M_2}^*, \xi_{M_1}) \\
K_n(\xi_1, \eta_1^*) & \cdots & K_n(\xi_{M_1}, \eta_1^*) & H_n^*(\eta_1, \eta_1) & \cdots & N_n^*(\eta_{M_2}, \eta_1) \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
K_n(\xi_1, \eta_{M_2}^*) & \cdots & K_n(\xi_{M_1}, \eta_{M_2}^*) & N_n^*(\eta_1, \eta_{M_2}) & \cdots & H_n^*(\eta_{M_2}, \eta_{M_2}) \\
\end{vmatrix}
\]  

(21)

The kernel \( H_n(\xi, y) \) is defined in (26) and consists in subtracting the pole of \( N_n(\xi, y) \) at \( \xi = y \). The determinant "\( D_n \)" is obtained from the determinant \( D_n \) by changing all \( N_n \) on the diagonal into \( H_n \), changing all \( y_k \) into \( \xi_k \) and finally ignoring all double poles at \( \xi_j = \xi_k \) and at \( \eta_j^* = \eta_k^* \) as we develop the determinant (which we denote by "\( \text{det} " \)). In fact, there is no single poles either at \( \xi_j = \xi_k \) or at \( \eta_j^* = \eta_k^* \) since the residues are zero.

We finally mention that the large \( \xi_i, \eta_k^* \) expansion of \( J_N \) (which will be done somewhere else [12]) is related, in the case of the Gaussian potential, to the number of graphs obtained from (1) by Wick’s contraction. The large \( N \)
expansion of this number classifies the graphs according to their genus (t’Hooft expansion); another large \(N\) limit which is considered in [12] is the so called BMN limit [13–14–15] where, in (1), \(\frac{1}{\sqrt{N}}\) and \(\frac{K}{\sqrt{N}}\) are kept constant.

2 The functions and the kernels

In this section we describe the biorthogonal polynomials for the four operations described in the introduction; then we give several simple examples of measures before describing the general case and the proofs in section 3.

- operation1:

\[
p_n(z) \rightarrow q_n(z; \xi; \Phi, \Phi) = \frac{1}{z - \xi} \begin{vmatrix} p_{n+1}(z) & p_{n+1}(\xi) \\ p_n(z) & p_n(\xi) \end{vmatrix} \frac{1}{p_n(\xi)}
\]

\[
p_n^*(z) \rightarrow q_n^*(z; \Phi, \xi^*; \Phi, \Phi) = K_n^*(z; \xi^*) \frac{h_n}{p_n(\xi)}
\]

\[
h_n \rightarrow \|q_n\|^2 = -h_n \frac{p_{n+1}(\xi)}{p_n(\xi)}
\]

In (22), the symbol \(\Phi\) means an empty set of variables and the kernel \(K_n(z; \xi^*)\) is

\[
K_n(z, \xi^*) = \sum_{i=0}^{n} \frac{p_i(z)}{h_i} \frac{p_i^*(\xi)}{p_i^*(\eta)}
\]

The iteration of operation1 to several sources \(\xi_i\) is the generalization to potentials of two variables of the Christoffel’s construction with potential of one variable (in the case of one real variable \(x\), \(q_n(x; \xi)\) and \(q_n^*(x; \xi)\) have similar forms as in (22) and are equal by Christoffel-Darboux relation; such a relation does not exist with two variables \(z\) and \(z^*\)).

- operation2:

\[
p_n(z) \rightarrow q_n(z; \eta^*; \Phi, \Phi) = K_n(z; \eta^*) \frac{h_n}{p_n^*(\eta)}
\]

\[
p_n^*(z) \rightarrow q_n^*(z; \eta, \Phi; \Phi, \Phi) = \frac{1}{z^* - \eta^*} \begin{vmatrix} p_{n+1}(z) & p_{n+1}(\eta) \\ p_n(z) & p_n(\eta) \end{vmatrix} \frac{1}{p_n(\eta)}
\]

\[
h_n \rightarrow \|q_n\|^2 = -h_n \frac{p_{n+1}(\eta)}{p_n^*(\eta)}
\]
We introduce the function
\[
t_n(y) = \int d\mu(z,z^*) \frac{p_n^*(z)}{z - y} 
\]  
which, in the case of potentials of one variable is called the Cauchy-Hilbert transform of the polynomial \( p_n(x) \). We also introduce the kernels
\[
H_n(z, y) = \sum_{i=0}^{n} \frac{p_i(z)}{h_i} t_i(y) 
\]
\[
N_n(z, y) = \frac{1}{y - z} + H_n(z, y) 
\]

Then, 
\[
p_n(z) \rightarrow q_n(z; \Phi, \Phi; y, \Phi) = (z - y) N_{n-1}(z, y) \frac{h_{n-1}}{t_{n-1}(y)} \quad n > 0
\]
\[
p_n^*(z) \rightarrow q_n^*(z; \Phi, \Phi, y^*) = \left| \sum_{i=0}^{n} \frac{p_i^*(z)}{p_{n-1}^*(z)} t_i^*(y) \frac{1}{t_{n-1}(y)} \right| \quad n > 0
\]
\[
h_n \rightarrow ||q_n||^2 = -h_{n-1} \frac{t_{n}(y)}{t_{n-1}(y)} \quad n > 0, \quad ||q_0||^2 = t_0(y)
\]  

- operation3:

We see from these four operations that the biorthogonal polynomials take different forms according to the measure. Before proceeding to the general case and to the general proofs, let us give some simple examples which make the general result easier to understand. We introduce the function
\[
Q(x^*, y) = \int d\mu(z, z^*) \frac{1}{(z^* - x^*) (z - y)} 
\]  
and the kernel
\[
A_n(x^*, y) = \sum_{i=0}^{n} \frac{t_i^*(x)}{h_i} t_i(y) - Q(x^*, y)
\]
We consider the measure
\[ d\mu(z, z^*; \xi, \eta^*; y, x^*) = \frac{(z^* - \eta^*)}{(z^* - x^*)} \frac{(z - \xi)}{(z - y)} d\mu(z, z^*) \] (32)

Then, it is easy to find that
\[ p_n(z) \rightarrow q_n(z; \xi, \Phi; \eta, x^*) \]
\[ D_n = \begin{vmatrix} p_n(z) & p_n(\xi) & t_n^*(x) \\ N_n(z, y) & N_n(\xi, y) & A_n(x^*, y) \\ K_n(z, \eta^*) & K_n(\xi, \eta^*) & N_n^*(\eta, x) \end{vmatrix} \frac{1}{D_{n-1}} \quad n > 0 \]
\[ \|q_n\|^2 = h_n \frac{D_n}{D_{n-1}} \quad n > 0, \quad \|q_0\|^2 = h_0 (x^* - \eta^*) (y - \xi) D_0 \] (33)

We let the reader calculate \( q_n^* \) from \( q_n \) by complex conjugation and by the exchange \( \xi \leftrightarrow \eta, x \leftrightarrow y \).

Now, if we let the variable \( \eta^* \rightarrow \infty \) in (32-33), we obtain the biorthogonal polynomials for the measure
\[ d\mu(z, z^*; \xi, \Phi; y, x^*) = \frac{1}{(z^* - x^*)} \frac{(z - \xi)}{(z - y)} d\mu(z, z^*) \] (34)

Performing the corresponding expansion in the kernels, we obtain
\[ p_n(z) \rightarrow q_n(z; \xi, \Phi; y, x^*) \]
\[ D_n = \begin{vmatrix} p_n(\xi) & t_n^*(x) \\ N_n(\xi, y) & A_n(x^*, y) \end{vmatrix} \frac{1}{D_{n-1}} \quad n > 0 \]
\[ \|q_n\|^2 = -h_{n-1} \frac{D_n}{D_{n-1}} \quad n > 0, \quad \|q_0\|^2 = -(y - \xi) D_0 \] (35)

If we let the variable \( \xi \rightarrow \infty \) in (32-33), we obtain the biorthogonal polynomials for the measure
\[ d\mu(z, z^*; \xi, \Phi; y, x^*) = \frac{(z^* - \eta^*)}{(z^* - x^*)} \frac{1}{(z - y)} d\mu(z, z^*) \] (36)

Performing the corresponding expansion in the kernels, we obtain
\[ p_n(z) \rightarrow q_n(z; \xi, \Phi; y, x^*) \]
\[ = (z-y) \begin{vmatrix} n_{n-1}(z, y) & A_{n-1}(x^*, y) \\ K_{n-1}(z, \eta^*) & N_{n-1}^*(\eta, x) \end{vmatrix} \frac{h_{n-1}}{D_{n-1}} \quad n > 0 \]
\[ D_n = \begin{vmatrix} t_n(y) & A_n(x^*, y) \\ p_n^*(\eta) & N_n^*(\eta, x) \end{vmatrix} \]
\[ \|q_n\|^2 = -h_{n-1} \frac{D_n}{D_{n-1}} \quad n > 0, \quad \|q_0\|^2 = (x^* - \eta^*) D_0 \quad (37) \]

Also, if we let in (32-33) the variable \( x^* \rightarrow \infty \) and if we use the following result proved in Appendix A
\[ t_n(y) \sim -\frac{h_n}{y^{n+1}} \quad \text{as} \quad y \rightarrow \infty \quad (38) \]
\[ N_n(\xi, y) \sim \frac{p_{n+1}(\xi)}{y^{n+2}} \quad \text{as} \quad y \rightarrow \infty \quad (39) \]
\[ A_n(x^*, y) \sim \frac{t_{n+1}^*(x)}{y^{n+2}} \quad \text{as} \quad y \rightarrow \infty \quad (40) \]
we obtain the biorthogonal polynomials for the measure
\[ d\mu(z, z^*: \xi, \eta^*: y, \Phi) = (z^* - \eta^*) \frac{(z - \xi)}{(z-y)} \, d\mu(z, z^*) \quad (41) \]

as
\[ p_n(z) \rightarrow q_n(z; \xi, \eta^*: y, \Phi) \]
\[ = \frac{z-y}{z-\xi} \begin{vmatrix} n_n(z, y) & A_n(x^*, y) \\ K_n(z, \eta^*) & N_n^*(\eta, x) \end{vmatrix} \frac{h_n}{D_{n-1}} \quad n > 0 \]
\[ D_n = \begin{vmatrix} t_{n+1}(y) & A_{n+1}(\xi, y) \\ p_{n+1}^*(\eta) & N_{n+1}^*(\eta, x) \end{vmatrix} \]
\[ \|q_n\|^2 = -h_n \frac{D_n}{D_{n-1}} \quad n > 0, \quad \|q_0\|^2 = -h_0 (y - \xi) \quad D_0 \quad (42) \]

Finally, if we let in (32-33) the variable \( y \rightarrow \infty \) we obtain the biorthogonal polynomials for the measure
\[ d\mu(z, z^*: \xi, \eta^*: y, \Phi) = \frac{(z^*-\eta^*)}{(z^*-x^*)} (z - \xi) \, d\mu(z, z^*) \quad (43) \]

as
\[ p_n(z) \rightarrow q_n(z; \xi, \eta^*: \Phi, x^*) \]
\[ = \frac{1}{z-\xi} \begin{vmatrix} p_{n+1}(z) & p_{n+1}(\xi) & t_{n+1}(x) \\ p_n(z) & p_n(\xi) & t_n^*(x) \\ K_n(z, \eta^*) & K_n(\xi, \eta^*) & N_n^*(\eta, x) \end{vmatrix} \frac{1}{D_{n-1}} \quad n > 0 \]
\( D_n = \left\| p_{n+1}(\xi) - K_n(\xi, \eta^*) N_n^*(\eta, x) \left| n \geq 0 \right. \right. \)

\[ \|q_n\|^2 = h_n \frac{D_n}{D_{n-1}} \quad n > 0, \quad \|q_0\|^2 = -h_0 (x^* - \eta^*) D_0 \quad (44) \]

The rest of this section is devoted to six other simple examples:

1°)

\[ d\mu(z, z^*; \xi, \Phi; y, \Phi) = \frac{(z - \xi)}{(z - y)} d\mu(z, z^*) \quad (45) \]

which gives

\[ p_n(z) \rightarrow q_n(z; \xi, \Phi; y, \Phi) = \frac{z - y}{z - \xi} \left| \begin{array}{c|c|c} p_n(z) & p_n(\xi) & 1 \\ \hline N_n(z, y) & N_n(\xi, y) & N_{n-1}(\xi, y) \end{array} \right| n > 0 \]

\[ \|q_n\|^2 = h_n \frac{N_n(\xi, y)}{N_{n-1}(\xi, y)} \quad n > 0, \quad \|q_0\|^2 = h_0 (y - \xi) N_0(\xi, y) \quad (46) \]

2°)

\[ d\mu(z, z^*; \Phi, \eta^*; \Phi, x^*) = \frac{(z^* - \eta^*)}{(z^* - x^*)} d\mu(z, z^*) \quad (47) \]

which gives

\[ p_n(z) \rightarrow q_n(z; \xi, \Phi; y, \Phi) = \frac{z - y}{z - \xi} \left| \begin{array}{c|c|c} p_n(z) & p_n(\xi) & 1 \\ \hline K_n(z, \eta^*) & N_n^*(\eta, x) & N_{n-1}^*(\eta, x) \end{array} \right| n > 0 \]

\[ \|q_n\|^2 = h_n \frac{N_n^*(\eta, x)}{N_{n-1}^*(\eta, x)} \quad n > 0, \quad \|q_0\|^2 = h_0 (x^* - \eta^*) N_0^*(\eta, x) \quad (48) \]

3°)

\[ d\mu(z, z^*; \xi, \eta^*; \Phi, \Phi) = (z^* - \eta^*) (z - \xi) d\mu(z, z^*) \quad (49) \]

which gives

\[ p_n(z) \rightarrow q_n(z; \xi, \eta^*; \Phi, \Phi) = \frac{1}{z - \xi} \left| \begin{array}{c|c|c} p_{n+1}(z) & p_{n+1}(\xi) & 1 \\ \hline K_n(z, \eta^*) & K_n(\xi, \eta^*) & K_{n-1}(\xi, \eta^*) \end{array} \right| n \geq 0 \]

\[ \|q_n\|^2 = h_{n+1} \frac{K_{n+1}(\xi, \eta^*)}{K_n(\xi, \eta^*)} \quad n \geq 0 \quad (50) \]

4°)

\[ d\mu(z, z^*; \Phi, \Phi; y, x^*) = \frac{1}{(z^* - x^*) (z - y)} d\mu(z, z^*) \quad (51) \]
which gives

\[ p_n(z) \rightarrow q_n(z; \Phi; y, x^\ast) \]

\[ = (z - y) \begin{vmatrix} p_{n+1}(z) & t_{n+1}^*(x) \\ N_{n+1}(z, y) & A_{n+1}(x^\ast, y) \end{vmatrix} \frac{1}{D_{n-1}} n > 0 \]

\[ D_n = A_{n-1}(x^\ast, y) n > 0, \quad D_0 = -Q(x^\ast, y) \]

\[ \|q_n\|^2 = h_{n-1} \frac{D_n}{D_{n-1}} n > 0, \quad \|q_0\|^2 = -D_0 \] (52)

5°)

\[ d\mu(z, z^\ast; \xi, \Phi; \Phi^\ast) = \frac{(z - \xi)}{(z^\ast - x^\ast)} d\mu(z, z^\ast) \] (53)

which gives

\[ p_n(z) \rightarrow q_n(z; \xi, \Phi; x^\ast) \]

\[ = \frac{1}{z - \xi} \begin{vmatrix} p_{n+1}(z) & t_{n+1}^*(x) \\ p_n(z) & t_n^*(x) \end{vmatrix} \frac{1}{D_{n-1}} n > 0 \]

\[ D_n = \begin{vmatrix} p_{n+1}(\xi) & t_{n+1}^*(x) \\ p_n(\xi) & t_n^*(x) \end{vmatrix} \frac{1}{D_{n-1}} n > 0 \]

\[ \|q_n\|^2 = h_{n-1} \frac{D_n}{D_{n-1}} n > 0, \quad \|q_0\|^2 = -D_0 \] (54)

6°)

\[ d\mu(z, z^\ast; \Phi, \eta^\ast; y, \Phi) = \frac{(z^\ast - \eta^\ast)}{(z - y)} d\mu(z, z^\ast) \] (55)

which gives

\[ p_n(z) \rightarrow q_n(z; \xi, \Phi; x^\ast) \]

\[ = (z - y) \begin{vmatrix} N_{n-1}(z, y) & t_n(y) \\ K_{n-1}(z, \eta^\ast) & p_n(\eta) \end{vmatrix} \frac{h_{n-1}}{D_{n-1}} n > 0 \]

\[ D_n = \begin{vmatrix} t_n(y) & t_{n+1}(y) \\ p_n(\eta) & p_{n+1}(\eta) \end{vmatrix} n \geq 0 \]

\[ \|q_n\|^2 = -h_{n-1} \frac{D_n}{D_{n-1}} n > 0, \quad \|q_0\|^2 = -D_0 \] (56)
3 The biorthogonal polynomials

We now consider the measure (8). We learned from the previous section how to construct the biorthogonal polynomials; clearly, it is of the form

\[ q_n(z; \xi_i, \eta_i^*; y_i, x_i^*) = \frac{\prod_{i=1}^{M_1} (z - y_i)}{\prod_{i=1}^{L_1} (z - \xi_i)} \cdot \frac{\prod_{i=1}^{M_1} (z - \xi_i)}{\prod_{i=1}^{L_1} (z - y_i)} \cdot \frac{a_{d,\gamma}}{D_{d,\gamma}} \] (57)

where \( \frac{a_{d,\gamma}}{D_{d,\gamma}} \) is a factor which makes the polynomial \( q_n \) monic. The form of the determinant depends of two parameters

\[ d = n + L_1 - M_1 \] (58)
\[ \gamma = n + L_2 - M_2 \] (59)

and its construction is always related to the linear properties of its columns.

Clearly, the expression \( \prod_{i=1}^{M_1} (z - y_i) \cdot \prod_{i=1}^{L_1} (z - \xi_i) \cdot \frac{a_{d,\gamma}}{D_{d,\gamma}} \) must be a polynomial in \( z \) of degree \( (n + L_1) \) which vanishes when \( z = \xi_i \) for any \( i \). In the determinant, the first column is the only \( z \) dependant column; then, some of the \( L_1 \) columns at the right of the first one are chosen to be identical to the first one with \( z \) successively replaced by \( \xi_1, ..., \xi_{L_1} \). Consequently, the polynomial \( \prod_{i=1}^{M_1} (z - y_i) \cdot \prod_{i=1}^{L_1} (z - \xi_i) \) vanishes for \( z = \xi_i \) for any \( i \), and can be divided by \( \prod_{i=1}^{L_1} (z - \xi_i) \) to give a polynomial of degree \( n \).

If \( d \geq 0 \) we may introduce in the first column the polynomial \( p_d(z) \), the kernels \( K_d(z, \eta^*) \), \( N_d(z, y) \). If \( d < 0 \) it is convenient to introduce

\[ N_{k<0}(z, y) = -\frac{1}{z - y} \] (60)
\[ K_{k<0}(z, \eta^*) = 0 \] (61)
\[ A_{k<0}(x^*, y) = -Q(x^*, y) \] (62)

so that if \( d = -1 \) the expression \( \prod_{i=1}^{M_1} (z - y_i) \cdot N_{-1}(z, y_k) \) is a polynomial in \( z \) of degree \( (n + L_1) \). Finally, if \( d < -1 \) we introduce in the determinant a line

\[ \begin{vmatrix} N_{-1}(z, y) & p_0(y) & ... & p_{-d-2}(y) \end{vmatrix} \sim z^d \quad \text{as} \quad z \to \infty \] (63)
The remaining columns are constructed in such a way that

\[ \int d\mu (z, z^*; \xi_i, \eta_i; y_i, x_i) \ p_m^* (z) \ q_n (z; \xi_i, \eta_i; y_i, x_i) = 0 \quad \text{for } m < n \]  

or equivalently

\[ \int d^2 z \ p_m^* (z) \ \prod_{i=1}^{L_2} (z^* - \eta_i^*) \ \prod_{i=1}^{M_2} (z^* - x_i^*) \ | \ . \ . \ . \ | e^{-V(z, z^*)} = 0 \quad \text{for } m < n \]  

The square pseudonorm is calculated by

\[ \int d^2 z \ p_n^* (z) \ \prod_{i=1}^{L_2} (z^* - \eta_i^*) \ \prod_{i=1}^{M_2} (z^* - x_i^*) \ | \ . \ . \ . \ | e^{-V(z, z^*)} = \| q_n \|^2 \]  

We use the property

\[
p_m^* (z) \ \prod_{i=1}^{L_2} (z^* - \eta_i^*) \ \prod_{i=1}^{M_2} (z^* - x_i^*) = \frac{1}{\Delta (x_1)} \left[ p_{m+L_2-M_2} (x_1) \varphi_{M_2} (\eta_i^*) - p_{m} (x_1) \varphi_{L_2-M_2} (\eta_i^*) \right] \]

where

\[
\varphi_j (\eta_i^*) = \prod_{i=1}^{L_2} (x_j^* - \eta_i^*)
\]

and where the polynomial

\[
\pi_{m+L_2-M_2} (z^*) = \sum_{i=0}^{m+L_2-M_2} \alpha_i \ p_i^* (z) \quad \text{with } \alpha_{m+L_2-M_2} = 1
\]

is monic of degree \( (m + L_2 - M_2) \) in \( z^* \) and is 0 if \( (m + L_2 - M_2) < 0 \).
If the first column of the determinant contains a term \( p_k(z) \), the integrals (65-66) generate a term

\[
 p_k(z) = \left\{ \frac{\alpha_k h_k}{0} \right\} + \frac{1}{\Delta(x^*_i)} \\
 p_m^*(x_1) \varphi_i (\eta^*_i) t^*_k (x_1) ... p_m^*(x_{M_2}) \varphi_i (\eta^*_i) t^*_k (x_{M_2}) \\
 p_{M_2-2}^* (x_1) ... p_{M_2-2}^* (x_{M_2}) \\
 ... ... ... \\
 p_0^* (x_1) ... p_0^* (x_{M_2}) \\
 \right]
\]

where \( \alpha_k h_k \) requires \( 0 \leq k \leq (m + L_2 - M_2) \) and 0 otherwise. Similarly, the integrals (65-66) generate the terms

\[
 N_k (z, y) = - \sum_{i=\text{Sup}(0, k+1)}^{m+L_2-M_2} \alpha_i t_i (y) \frac{1}{\Delta(x^*_i)} \\
 p_m^*(x_1) \varphi_i (\eta^*_i) A_k (x^*_i, y) ... p_m^*(x_{M_2}) \varphi_i (\eta^*_i) A_k (x_{M_2}, y) \\
 p_{M_2-2}^* (x_1) ... p_{M_2-2}^* (x_{M_2}) \\
 ... ... ... \\
 p_0^* (x_1) ... p_0^* (x_{M_2}) \\
 \right]
\]

\[
 K_k (z, \eta^*_i) = - \sum_{i=\text{Sup}(0, k+1)}^{m+L_2-M_2} \alpha_i p_i^* (\eta^*_i) \frac{1}{\Delta(x^*_i)} \\
 p_m^*(x_1) \varphi_i (\eta^*_i) N_k^* (\eta^*_i, x_1) ... p_m^*(x_{M_2}) \varphi_i (\eta^*_i) N_k^* (\eta^*_i, x_{M_2}) \\
 p_{M_2-2}^* (x_1) ... p_{M_2-2}^* (x_{M_2}) \\
 ... ... ... \\
 p_0^* (x_1) ... p_0^* (x_{M_2}) \\
 \right]
\]

where the sum \( \sum \) is absent if \( k \geq (m + L_2 - M_2) \) or if \((m + L_2 - M_2) < 0 \). The equations (71-72) are still valid if \( k < 0 \).

### 3.1 \( \gamma = n + L_2 - M_2 < 0 \)

Since, in that case, there is no polynomial \( \pi_{m+L_2-M_2} (z^*) \) in (67) for \( m \leq n \), the calculation of (65-66) as given by (70-71-72), may be written

\[
 \begin{align*}
 p_k (z) & \\
 N_k (z, y) & \\
 K_k (z, \eta^*_i) & \\
 \end{align*}
\]

\[
 \begin{align*}
 \sum_{i=1}^{M_2} \beta_i \left\{ \begin{array}{c}
 t^*_k (x_i) \\
 A_k (x^*_i, y) \\
 N_k^* (\eta^*_i, x_i) \\
 \end{array} \right. \\
 \end{align*}
\]

where the coefficients \( \beta_i \) are the same in the three cases. Consequently, a determinant containing

\[
 \begin{vmatrix}
 p_k (z) & t^*_k (x_i) \\
 N_k (z, y) & A_k (x^*_i, y) \\
 K_k (z, \eta^*_i) & N_k^* (\eta^*_i, x_i) \\
 \end{vmatrix}
\]

\[
 (74)
\]
where the pseudonorm is generated in (76) from the terms $D_{d, \gamma}$ are such that

$$
\sum_{i=1}^{M_1} \beta_i \begin{cases} p^*_i (x_i) \\ p^*_c (x_i) \end{cases} = \begin{cases} 0 & \text{if } c < -(m + L_2 - M_2 + 1) \\ 1 & \text{if } c = -(m + L_2 - M_2 + 1) \end{cases} (75)
$$

$a$) if $d \geq 0$, then $q_n (z; \xi_i, \eta^*_i; y_i, x_i^*)$ contains a determinant of size $(L_1 + M_2 + 1) \times (L_2 + M_1 + d - \gamma + 1)$

$$
q_n (z; \xi_i, \eta^*_i; y_i, x_i^*) = \prod_{i=1}^{M_1} \frac{p_d (z)}{p_0 (z)} \frac{p_d (\xi_j)}{p_0 (\xi_j)} \frac{t_d^* (x_i)}{t_0^* (x_i)} \frac{1}{D_{d, \gamma}}
$$

where $D_{d, \gamma}$ is written in Appendix B ((B1) for $d > 0$ and (B2) for $d = 0$). The pseudonorm is generated in (76) from the terms $p^*_{-\gamma-1} (x_i)$; from (75) we find

$$
\|q_n\|^2 = (-)^{d-\gamma-1} \frac{D_{d+1, \gamma+1}}{D_{d, \gamma}} (77)
$$

where $D_{d+1,0}$ is given in (B4).

$b$) if $d = -1$, then $q_n (z; \xi_i, \eta^*_i; y_i, x_i^*)$ contains a determinant of size $(L_1 + M_2 + 1) \times (L_2 + M_1 - \gamma)$

$$
q_n (z; \xi_i, \eta^*_i; y_i, x_i^*) = \prod_{i=1}^{M_1} \frac{(z - y_i)}{(z - \xi_i)} \frac{1}{D_{-1, \gamma}}
$$

$$
\begin{vmatrix}
0 & 0 & p_0^* (x_i) \\
\vdots & \vdots & \vdots \\
0 & 0 & p^*_{-\gamma-1} (x_i) \\
N_{-1} (z, y_i) & N_{-1} (\xi_j, y_i) & A_{-1} (x_i^*, y_i) \\
0 & 0 & N_{-1}^* (\eta_k, x_i) \\
\end{vmatrix} (78)
$$

where $D_{-1, \gamma}$ is given in (B3). The pseudonorm is still obtained from the term $p^*_{-\gamma-1} (x_i)$ as

$$
\|q_n\|^2 = (-)^{1-\gamma} \frac{D_{0, \gamma+1}}{D_{-1, \gamma}} (79)
$$
where $D_{0, \gamma + 1}$ is given in (B2) if $\gamma < -1$ and in (B5) if $\gamma = -1$.

$$c^0) \text{ if } d < -1, \text{ then } q_n(z; \xi_i, \eta_i^*; y_i, x_i^*) \text{ contains a determinant of size}$$

$$(L_1 + M_2 - d) \times (L_2 + M_1 - \gamma)$$

$$q_n(z; \xi_i, \eta_i^*; y_i, x_i^*)$$

$$= \prod_{i=1}^{M_1} (z - y_i) \prod_{i=1}^{L_2} (z - \xi_i)$$

$$\begin{vmatrix}
0 & 0 & \ldots & 0 & 0 & p_0^* (x_i) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0 & \ldots \\
N_{-1} (z, y_i) & p_0 (y_i) & \ldots & p_{-d-2} (y_i) & N_{-1} (\xi_j, y_i) & A_{-1} (x_i^*, y_i) \\
0 & 0 & \ldots & 0 & 0 & N_{-1}^* (\eta_k, x_i) \\
\end{vmatrix}$$

(80)

where $D_{d, \gamma}$ is given in (B3) The pseudo norm is still obtained from the term $p_{-\gamma-1}^* (x_i)$ as

$$\|q_n\|^2 = (-)^{d-\gamma} \frac{D_{d+1, \gamma+1}}{D_{d, \gamma}}$$

(81)

where $D_{d+1, 0}$ is given in (B8).

### 3.2 $\gamma = n + L_2 - M_2 \geq 0$

$a^0)$ we suppose $d \geq \gamma$.

In this case, for $m \leq n$, equations (65-66) give from (70-71-72)

$$\begin{cases}
 p_k (z) \\
 N_d (z, y) \\
 K_d (z, \eta_k^*)
\end{cases} \Rightarrow \sum_{i=1}^{M_2} \beta_i \begin{cases}
 t_i^* (x_i) \\
 A_d (x_i^*, y) \\
 N_d^* (\eta_k, x_i)
\end{cases} \quad k \geq \gamma + 1 \quad (82)$$

while these equations give for $k = \gamma$

$$p_\gamma (z) \Rightarrow \begin{cases}
 \sum_{i=1}^{M_2} \beta_i t_i^* (x_i) \\
 h_\gamma + \sum_{i=1}^{M_2} \beta_i t_i^* (x_i)
\end{cases} \quad \begin{array}{ll}
m < n & m = n
\end{array} \quad (83)$$

Consequently, the polynomial is a determinant of size
\[(L_1 + M_2 + 1) \times (L_2 + M_1 + d - \gamma + 1)\]

\[
q_n (z; \xi_i, \eta^*_i; y_i, x^*_i) = \prod_{i=1}^{M_1} (z - y_i) \prod_{i=1}^{L_1} (z - \xi_i)
\]

\[
\begin{array}{ccc}
N_d (z, y_i) & N_d (\xi_j, y_i) & A_d (x^*_i, y_i) \\
K_d (z, \eta^*_k) & K_d (\xi_j, \eta^*_k) & N^*_d (\eta_k, x_i)
\end{array}
\]

\[
\frac{1}{D_{d,\gamma}}
\]

\[
(84)
\]

where \(D_{d,\gamma}\) is given in (B4) if \(d > \gamma\) and in (B5) if \(d = \gamma\). The pseudonorm comes from the term \(p_\gamma (z)\) and is found from (83) to be

\[
\|q_n\|^2 = (-d - \gamma) h_{\gamma} \frac{D_{d+1,\gamma+1}}{D_{d,\gamma}}
\]

\[
(85)
\]

where \(D_{d+1,\gamma+1}\) is also given in (B4) if \(d > \gamma\) and in (B5) if \(d = \gamma\).

b°) we suppose \(d = \gamma - 1\)

In this case, for \(m < n\), we still have in (65)

\[
\left\{ \begin{array}{c}
N_d (z, y) \\
K_d (z, \eta^*_k)
\end{array} \right\} \Rightarrow \sum_{i=1}^{M_2} \beta_i \left\{ \begin{array}{c}
A_d (x^*_i, y) \\
N^*_d (\eta_k, x_i)
\end{array} \right\}
\]

\[
(86)
\]

however, for \(m = n\)

\[
\left\{ \begin{array}{c}
N_d (z, y) \\
K_d (z, \eta^*_k)
\end{array} \right\} \Rightarrow - \left\{ \begin{array}{c}
t_\gamma (y) \\
p_\gamma (\eta)
\end{array} \right\} + \sum_{i=1}^{M_2} \beta_i \left\{ \begin{array}{c}
A_d (x^*_i, y) \\
N^*_d (\eta_k, x_i)
\end{array} \right\}
\]

\[
(87)
\]

Consequently, the polynomials contain a determinant of size

\[(L_1 + M_2 + 1) \times (L_2 + M_1)\]

\[
q_n (z; \xi_i, \eta^*_i; y_i, x^*_i)
\]

\[
\prod_{i=1}^{M_1} (z - y_i) \prod_{i=1}^{L_1} (z - \xi_i)
\]

\[
\begin{array}{ccc}
N_{-1} (z, y_i) & N_{-1} (\xi_j, y_i) & A_{-1} (x^*_i, y_i) \\
0 & 0 & N^*_{-1} (\eta_k, x_i)
\end{array}
\]

\[
\frac{(-1)}{D_{-1,0}}
\]

\[
(88)
\]

for \(d > -1\)

\[
q_n (z; \xi_i, \eta^*_i; y_i, x^*_i)
\]

\[
\prod_{i=1}^{M_1} (z - y_i) \prod_{i=1}^{L_1} (z - \xi_i)
\]

\[
\begin{array}{ccc}
N_{-1} (z, y_i) & N_{-1} (\xi_j, y_i) & A_{-1} (x^*_i, y_i) \\
0 & 0 & N^*_{-1} (\eta_k, x_i)
\end{array}
\]

\[
\frac{(-1)}{D_{-1,0}}
\]

\[
(89)
\]
where \( D_{d,\gamma} \) is given in (B6) if \( d > -1 \) and \( D_{-1,0} \) is given in (B8). The pseudonorm is obtained from (87) as

\[
\|q_n\|^2 = -h_d \frac{D_{d+1,\gamma+1}}{D_{d,\gamma}} \quad d > -1
\]

\[
\|q_n\|^2 = \frac{D_{0,1}}{D_{-1,0}} \quad \text{for} \quad d = -1
\]

\( D_{d+1,\gamma+1} \) is also given in (B6).

we suppose \(-1 \leq d < \gamma - 1 \Rightarrow \gamma > 0\)

We are now in the domain where equations (65-66) give from (70-71-72)

\[
\( \{ N_d (z, y) \} \) \Rightarrow - \sum_{i=d+1}^{m-n+\gamma} \{ t_i (y) \} + \sum_{i=1}^{M_2} \beta_i \{ A_d (x_i^*, y) \}
\]

This relation leads to biorthogonal polynomials which contains a determinant of size

\( (L_1 + M_2 + \gamma - d) \times (L_2 + M_1) \)

\[
q_n (z; \xi_i, \eta_i^*; y_i, x_i^*) = \prod_{i=1}^{M_1} (z - y_i) \prod_{i=1}^{L_1} (z - \xi_i) (-)^{\gamma-d-1} h_d \frac{D_{d,\gamma}}{D_{d+1,\gamma+1}}
\]

\[
\begin{bmatrix}
N_d (z, y_i) & t_{[\gamma-1,d+1]} (y_i) & N_d (\xi_j, y_i) & A_d (x_j^*, y_i) \\
K_d (z, \eta_k^*) & p_{[\gamma-1,d+1]} (\eta_k) & K_d (\xi_j, \eta_k^*) & N_d^* (\eta_k, x_i)
\end{bmatrix}
\]

for \( d > -1 \)

\[
q_n (z; \xi_i, \eta_i^*; y_i, x_i^*) = \prod_{i=1}^{M_1} (z - y_i) \prod_{i=1}^{L_1} (z - \xi_i) (-)^{\gamma-1} \frac{D_{-1,\gamma}}{D_{d+1,\gamma+1}}
\]

\[
\begin{bmatrix}
N_{-1} (z, y_i) & t_{[\gamma-1,0]} (y_i) & N_{-1} (\xi_j, y_i) & A_{-1} (x_j^*, y_i) \\
0 & p_{[\gamma-1,0]} (\eta_k) & 0 & N_{-1}^* (\eta_k, x_i)
\end{bmatrix}
\]

for \( d = -1 \)

where

\[
t_{[\gamma-1,d+1]} (y_i) = t_{\gamma-1} (y_i) \quad t_{\gamma-2} (y_i) \ldots t_{d+1} (y_i)
\]

\[
p_{[\gamma-1,d+1]} (\eta_k) = p_{\gamma-1}^* (\eta_k) \quad p_{\gamma-2}^* (\eta_k) \ldots p_{d+1}^* (\eta_k)
\]
and $D_{d,\gamma}$ is given in (B6) for $d > -1$ and $D_{-1,\gamma}$ is given in (B7). The pseudonorm is found from (92), with $m = n$, as

$$
\|q_n\|^2 = (-)^{\gamma-d} h_d \frac{D_{d+1,\gamma+1}}{D_{d,\gamma}} \quad d > -1
$$

(97)

$$
\|q_n\|^2 = (-)^{\gamma} \frac{D_{0,\gamma+1}}{D_{-1,\gamma}} \quad d = -1
$$

(98)

where $D_{d+1,\gamma+1}$ is also given in (B6).

$d^c$) we suppose $d < -1, \gamma \geq 0$

Equations (92) are still valid but we have to apply (63). The determinant is of size

$$(L_1 + M_2 + \gamma - d) \times (L_2 + M_1)$$

$$q_n (z; \xi_i, \eta_i^*; y_i, x_i^*) = \prod_{i=1}^{M_1} (z - y_i) \prod_{i=1}^{L_1} (z - \xi_i) (-)^{\gamma-d} \frac{1}{D_{d,\gamma}}$$

$$| N_{-1} (z, y_i) \quad t_{[\gamma-1,0]} (y_i) \quad p_{[0,-d-2]} (y_i) \quad N_{-1} (\xi_j, y_i) \quad A_{-1} (x_i^*, y_i) |$$

$$| 0 \quad p_{[\gamma-1,0]} (\eta_k) \quad 0 \quad A_{-1} (x_i^*, y_i) |$$

for $\gamma > 0$

(99)

$$\|q_n\|^2 = (-)^{-d} \frac{1}{D_{d,0}}$$

$$q_n (z; \xi_i, \eta_i^*; y_i, x_i^*) = \prod_{i=1}^{M_1} (z - y_i) \prod_{i=1}^{L_1} (z - \xi_i) (-)^{-d} \frac{1}{D_{d,0}}$$

$$| N_{-1} (z, y_i) \quad t_{[\gamma-1,0]} (y_i) \quad p_{[0,-d-2]} (y_i) \quad N_{-1} (\xi_j, y_i) \quad A_{-1} (x_i^*, y_i) |$$

$$| 0 \quad p_{[\gamma-1,0]} (\eta_k) \quad 0 \quad N_{-1}^* (\eta_k, x_l) |$$

for $\gamma = 0$

(100)

where

$$p_{[0,-d-2]} (y_i) = p_0 (y_i) \quad p_1 (y_i) \ldots p_{-d-2} (y_i)$$

(101)

and where $D_{d,\gamma}$ is given in (B7) if $\gamma > 0$ and $D_{d,0}$ is given in (B8). Finally, the pseudonorm in that case is obtained from (92), with $m = n$, as

$$\|q_n\|^2 = (-)^{\gamma-d-1} \frac{D_{d+1,\gamma+1}}{D_{d,\gamma}}$$

(102)
where $D_{d+1, \gamma+1}$ is given in (B7).

We note that the pseudonorm can be written, for all values of $d$, and $\gamma$ as

$$\|q_n\|^2 = (-)^{\gamma-d} h_{d, \gamma} \frac{D_{d+1, \gamma+1}}{D_{d, \gamma}}$$

(103)

with the convention

$$h_{d, \gamma} = h_{inf(d, \gamma)} \quad d \text{ and } \gamma \geq 0$$

$$h_{d, \gamma} = -1 \quad d \text{ or } \gamma < 0$$

$$h_{d, \gamma} = 1 \quad d \text{ and } \gamma < 0$$

(104)

4 The integrals $I_N$ and $J_N$

As shown in (14-15), the expression for $I_N$ is related to the product of the pseudonorms from 0 to $N - 1$. Since the pseudonorms are essentially the ratio of two determinants $\frac{D_{d+1, \gamma+1}}{D_{d, \gamma}}$, we obtain from (58-59)

$$I_N = (-)^{N(L_1-M_1-L_2+M_2)} \prod_{i=0}^{N-1} h_i \frac{D_{N+L_1-M_1, N+L_2-M_2}}{D_{L_1-M_1, L_2-M_2}}$$

(105)

with the convention (104) and where the determinants $D_{L_1-M_1, L_2-M_2}$ and $D_{N+L_1-M_1, N+L_2-M_2}$ are given in the previous section depending of the values of $N, L_1 - M_1$ and $L_2 - M_2$. In the special case where $L_1 = M_1$ and $L_2 = M_2$ the integrals $I_N$ are simply

$$I_N = \frac{D_{N-1}}{D_{-1}}$$

(106)

where

$$D_{N-1} = D_{N,N} = \begin{vmatrix} N_{N-1} (\xi_j, y_i) & A_{N-1} (x_i^*, y_i) \\ K_{N-1} (\xi_j, \eta_k) & N^*_{N-1} (\eta_k, x_i) \end{vmatrix}$$

$$D_{-1} = D_{0,0} = \begin{vmatrix} N_{-1} (\xi_j, y_i) & A_{-1} (x_i^*, y_i) \\ 0 & N^*_{-1} (\eta_k, x_i) \end{vmatrix}$$

$$= N_{-1} (\xi_j, y_i) \quad N^*_{-1} (\eta_k, x_i)$$

(107)

(108)
with

\[
N_{-1}(\xi_j, y_i) = \begin{vmatrix}
\frac{1}{y_1 - \xi_1} & \ldots & \frac{1}{y_1 - \xi_{M}} \\
\ldots & \ldots & \ldots \\
\frac{1}{y_{M} - \xi_1} & \ldots & \frac{1}{y_{M} - \xi_{M}}
\end{vmatrix} = (-\frac{M_1(M_1-1)}{2}) \Delta(y) \Delta(\xi) \prod_{i,j} (y_i - \xi_j) \tag{109}
\]

The result (106-107-108-109) is nothing but equation (18).

In order to obtain the resolvent \( J_N \) from \( I_N \), we must perform the derivatives

\[
\begin{pmatrix}
\prod_{i=1}^{M_2} \left( -\frac{\partial}{\partial \eta^*_i} \right) 
\prod_{i=1}^{M_1} \left( -\frac{\partial}{\partial \xi^*_i} \right) 
\frac{\prod_{i,j} (x^*_i - \eta^*_j)}{\Delta(x^*)} \Delta(y) \Delta(\eta^*) \Delta(\xi) 
D_n
\end{pmatrix}
\tag{110}
\]

This operation can be decomposed into the derivatives in \( \xi^*_i \) involving the \( M_1 \) first column of \( D_n \) while the derivatives in \( \eta^*_i \) involve the \( M_2 \) last rows of \( D_n \). Both operations are similar and we describe here the derivatives in \( \xi^*_i \). We define

\[
R(y_k, \xi_j) = \frac{\prod_{i\neq j} (y_k - \xi_j)}{\Delta(y) \Delta(\xi)} \tag{111}
\]

while the factors \((y_i - \xi_i)\) are distributed over the columns of \( D_n \). Given \( I \subset \{ \xi_1, ..., \xi_{M_1} \} \) we write

\[
\sum_I \left[ \prod_{i \notin I} \left( -\frac{\partial}{\partial \xi^*_i} \right) R(y_k, \xi_j) \right]_{y_i = \xi_i} \left[ \prod_{i \in I} \left( -\frac{\partial}{\partial \xi^*_i} \right) \left\{ \prod_{k=1}^{M_1} (y_k - \xi_k) D_n \right\} \right]_{y_i = \xi_i} \tag{112}
\]

The right \([.]\) is easily calculated since for the columns \( i \notin I \) we obtain 1 on the diagonal and 0 otherwise, and for the columns \( i \in I \) we obtain \( H_n(\xi_i, \xi_i) \) on the diagonal, the remainder being unchanged except for \( y_k = \xi_k \). We call \( D_n(I) \) the corresponding subdeterminant with the indices \( i \in I \). Now, we calculate the left \([.]\); for any \( J \subset \{ \xi_1, ..., \xi_{M_1} \} \) we have

\[
\left[ \prod_{i \notin I} \left( -\frac{\partial}{\partial \xi^*_i} \right) R(y_k, \xi_j) \right]_{y_i = \xi_i} = 0 \quad \text{if card}(J) \text{ is odd} \tag{113}
\]

and if card\((J)\) is even

\[
\left[ \prod_{i \notin I} \left( -\frac{\partial}{\partial \xi^*_i} \right) R(y_k, \xi_j) \right]_{y_i = \xi_i} = (-\frac{M_1(M_1-1)}{2}) \sum \text{all pairings in } J(j,k) \left( \frac{-1}{(\xi_k - \xi_j)^2} \right) \tag{114}
\]
Altogether, we have obtained

\[ (-\frac{M_1(M_1-1)}{2}) \sum_{l \text{ even}} \left[ \sum_{\text{all pairings in } J_{(i,j)}} \prod \left( \frac{-1}{(\xi_i - \xi_j)^2} \right) \right] D_n(I) \]

\[ = (-\frac{M_1(M_1-1)}{2}) "D_n" \]  \hspace{1cm} (115)

where ""D_n"" is the determinant obtained from D_n after the following changes:
the diagonal is H_n(ξ_i, ξ_i), all variables y_i are changed into ξ_i, finally "" means
that all double poles at ξ_k = ξ_j are ignored as we develop the determinant. In
fact, there is no single poles at ξ_k = ξ_j either, since the residues are zero.

The same result applies for the derivatives
\[ \prod_{i=1}^{M_2} \left( -\frac{\partial}{\partial \eta_i^*} \right) x_i^* = \eta_i^* \] and this gives
J_N as in (20).

5 Appendix A

We compute the large y behaviour of the function t_n(y) and of the kernels
N_n(ξ, y) and A_n(x*, y) defined in (27) and in (31). The formal power series for
t_n(y) when y → ∞ can be written as

\[ t_n(y) = -\sum_{p=0}^{\infty} a_p \frac{1}{y^{p+1}} \]  \hspace{1cm} (A1)

\[ a_p = \int d\mu(z, z^*) \ p_n^*(z) \ z^p \]  \hspace{1cm} (A2)

Clearly, \( a_p = 0 \) for \( p < n \). Let us write

\[ z^p = \sum_{k=0}^{p} \alpha_{pk} \ p_k(z), \quad \alpha_{pk} = 0 \text{ if } k > p, \quad \alpha_{pp} = 1 \]  \hspace{1cm} (A3)

Consequently,

\[ t_n(y) = -h_n \sum_{p=n}^{\infty} \alpha_{pn} \frac{1}{y^{p+1}} \]  \hspace{1cm} (A4)

\[ t_n(y) \sim -\frac{h_n}{y^{n+1}} \text{ as } y \to \infty \]  \hspace{1cm} (A5)
Now, if we insert the expansion (A4) for \( t_n(y) \) and if we use (A3), we obtain
\[
\sum_{n=0}^{\infty} \frac{p_n(\xi)}{h_n} t_n(y) = \frac{1}{\xi - y} \tag{A6}
\]
If this expression is reported in the definition of \( N_n(\xi, y) \) we obtain
\[
N_n(\xi, y) = -\sum_{p=n+1}^{\infty} \frac{p_p(\xi)}{h_p} t_p(y) \tag{A7}
\]
\[
N_n(\xi, y) \sim \frac{p_{n+1}(\xi)}{y^{n+2}} \text{ as } y \to \infty \tag{A8}
\]
Similarly, we find for \( Q(x^*, y) \) defined in (30)
\[
Q(x^*, y) = \sum_{n=0}^{\infty} \frac{t_n^*(x)}{h_n} t_n(y) \tag{A9}
\]
so that
\[
A_n(x^*, y) = -\sum_{p=n+1}^{\infty} \frac{t_p^*(x)}{h_p} t_p(y) \tag{A10}
\]
\[
A_n(x^*, y) \sim \frac{t_{n+1}^*(x)}{y^{n+2}} \text{ as } y \to \infty \tag{A11}
\]

6 Appendix B

We collect the various forms of the determinant \( D_{d,\gamma} \) according to the values of \( d \) and \( \gamma \).

6.1 \( \gamma < 0 \)

1a°) \( d > 0 \)

\[
D_{d,\gamma} = \begin{vmatrix}
p_{d-1}(\xi_j) & t_{d-1}^*(x_i) \\
0 & p_0^*(x_i) \\
& \ddots & \ddots & \ddots \\
0 & 0 & p_{d-1}^*(x_i) \\
N_{d-1}(\xi_j, y_i) & A_{d-1}(x_i^*, y_i) \\
K_{d-1}(\xi_j, \eta_k) & N_{d-1}^*(\eta_k, x_i) \\
\end{vmatrix} \tag{B1}
\]
1b°) $d = 0$

$$D_{0,\gamma} = \begin{bmatrix} 0 & p_0^*(x_i) \\ \vdots & \vdots \\ 0 & p_{-\gamma-1}^*(x_i) \\ N_{-1}(\xi_j, y_i) & A_{-1}(x_i^*, y_i) \\ 0 & N_{-1}^*(\eta_k, x_i) \end{bmatrix} \quad (B2)$$

1c°) $d < 0$

$$D_{d,\gamma} = \begin{bmatrix} 0 & \ldots & 0 & 0 & p_0^*(x_i) \\ \vdots & \ldots & \ldots & \ldots & \vdots \\ 0 & \ldots & 0 & 0 & p_{-\gamma-1}^*(x_i) \\ p_0(y_i) & \ldots & p_{-d-1}(y_i) & N_{-1}(\xi_j, y_i) & A_{-1}(x_i^*, y_i) \\ 0 & \ldots & 0 & 0 & N_{-1}^*(\eta_k, x_i) \end{bmatrix} \quad (B3)$$

6.2 $\gamma \geq 0$

2a°) $d > \gamma$

$$D_{d,\gamma} = \begin{bmatrix} p_{d-1}(\xi_j) & t_{d-1}^*(x_i) \\ \vdots & \vdots \\ p_{\gamma}(\xi_j) & t_{\gamma}^*(x_i) \\ N_{d-1}(\xi_j, y_i) & A_{d-1}(x_i^*, y_i) \\ K_{d-1}(\xi_j, \eta_k) & N_{d-1}^*(\eta_k, x_i) \end{bmatrix} \quad (B4)$$

2b°) $d = \gamma$

$$D_{d,\gamma} = \begin{bmatrix} N_{d-1}(\xi_j, y_i) & A_{d-1}(x_i^*, y_i) \\ K_{d-1}(\xi_j, \eta_k) & N_{d-1}^*(\eta_k, x_i) \end{bmatrix} \quad (B5)$$

2c°) $0 \leq d < \gamma$

$$D_{d,\gamma} = \begin{bmatrix} t_{[\gamma-1,d]}(y_i) & N_d(\xi_j, y_i) \\ p_{[\gamma-1,d]}(\eta_k) & K_d(\xi_j, \eta_k) \\ A_d(x_i^*, y_i) \\ N_d^*(\eta_k, x_i) \end{bmatrix} \quad (B6)$$

where $t_{[\gamma-1,d]}(y_i)$ and $p_{[d,\gamma-1]}(\eta_k)$ are defined in (95-96).

2d°) $d \leq -1, \gamma \geq 1$

$$D_{d,\gamma} = \begin{bmatrix} t_{[\gamma-1,0]}(y_i) & p_{[0,-d-1]}(y_i) \\ p_{[\gamma-1,0]}(\eta_k) & 0 \\ N_{-1}(\xi_j, y_i) & A_{-1}(x_i^*, y_i) \\ 0 & N_{-1}^*(\eta_k, x_i) \end{bmatrix} \quad (B7)$$

where $p_{[d-1,0]}(y_i)$ is defined in (101).

2e°) $d \leq -1, \gamma = 0$

$$D_{d,\gamma} = \begin{bmatrix} p_{[0,-d-1]}(y_i) & N_{-1}(\xi_j, y_i) \\ 0 & 0 \\ A_{-1}(x_i^*, y_i) \\ N_{-1}^*(\eta_k, x_i) \end{bmatrix} \quad (B8)$$
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