REWORKING ON AFFINE EXTERIOR ALGEBRA OF GRASSMANN: PEANO AND HIS SCHOOL

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Abstract. In this paper a construction of affine exterior algebra of Grassmann, with a special attention to the revisitation of this subject operated by Peano and his School, is examined from a historical viewpoint. Even if the exterior algebra over a vector space is a well known concept, the construction of an exterior algebra over an affine space, in which points and vectors coexist, has been neglected. This paper wants to fill this lack.

Some attention is given to the introduction of defining by abstraction (today called definition by quotienting or by equivalence relation), a procedure due to and used by Peano to define geometric forms, basic elements of an affine exterior algebra. This Peano’s innovative way of defining, is a relevant contribution to mathematics.

It is observed that in the construction of an affine exterior algebra on the Euclidean three-dimensional space, Grassmann and Peano make use of metric concepts: an accurate analysis shows that, in some cases, the metric aspects can be eliminated, putting into evidence the sufficiency of the underlying affine structure of the Euclidean space.

In the final part of the paper some geometrical and mechanical applications and interpretations of the affine exterior algebra given by Grassmann and Peano are presented.

1. Introduction

Calcolo geometrico [36] of 1888 marks strong interest of Giuseppe Peano (1858-1932) in Extension Theory (Ausdehnungslehre) of Hermann Grassmann (1809-1887): despite difficulties related to a philosophical aura of the book of Grassmann, Peano was attracted by the applications of the new calculus proposed by Grassmann.

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Nowadays a philosophical prejudice is nothing but a pretext to keep away from Ausdehnungslehre. This prejudice, shared also by some prominent mathematicians, have been originated by the fact that, following the habit of the time, Grassmann received a philosophical-theological education. Enriques in [21] (1911) p. 71] says: “... le système philosophique [of J. F. Herbart] a exercé [...] une grande influence sur le développement des idées de H. Grassmann et de B. Riemann”. Moreover, Weyl says in [71] (1918) p. 319]: “In forming the conception of a manifold of more than three dimensions, Grassmann as well as Riemann was influenced by the philosophic ideas of Herbart.”.

Lewis [28] (1977) provides further connections between theology and Ausdehnungslehre. It is worth saying, however, that a comparative analysis of the works of Grassmann, Schleiermacher, Herbart and Fichte shows that this philosophical influence is not perceptible to us.

Peano in [44] (1896) p. 953] says: “La teoria di Grassmann è oggigiorno, dai vari autori che l’hanno risposta ed applicata, giudicata col più grandi elogi. [...] se tanto tardò quest’opera a farsi conoscere, e se tanta difficolta presenta tuttora nel diffonderli, la ragione ci dev’essere, e, secondo
Geometric calculus of Grassmann consists of an algebraic calculus on geometric entities (points, oriented segments, oriented triangles and so on), differently from what happens in analytical geometry, that is based on an algebraic manipulation of coordinates. This distinguished aspect of the geometrical calculus of Grassmann is emphasized by Peano in the introduction of his *Calcolo geometrico*:

Il calcolo geometrico presenta analogie con la geometria analitica; ne differisce in ciò, che, mentre nella geometria analitica i calcoli si fanno sui numeri che determinano gli enti geometrici, in questa nuova scienza i calcoli si fanno sugli enti stessi.

Among the members of the School of Peano [27, p. 187], who share with him the interest for Grassmann, we have to mention Burali-Forti, Castellano, Boggio, Bottasso, Pensa, Marcolongo and Burgatti, the last two mainly oriented to mathematical-physics. All them took an active role in development and diffusion of the “vector calculus à la Grassmann” hereafter referred to as “geometric calculus”.

Burali-Forti, in collaboration with Marcolongo, published many papers and books on vector calculus (see for instance [7, (1897)], [9, (1909)], [15, (1932)]). The first book of rational mechanics, in terms of concepts introduced by Grassmann and revisited by Peano was written by Castellano [19, (1894)]. Boggio wrote some books and papers on vector calculus and differential geometry. Bottasso must be mentioned for his book *Astatiche* [5, (1915)]. Pensa extended in [60, 59, (1919)] the results of *Calcolo geometrico* of Peano from 3-dimensional to arbitrary n-dimensional spaces.

*Calcolo geometrico* of Peano revisits in an original way the ideas of Grassmann; in particular he fruitfully introduces the modern notion of vector space. In construction of an exterior algebra on Euclidean three-dimensional space Grassmann and Peano make use of metric concepts: an accurate analysis shows that, in a large part of the geometric calculus, the metric attribute of the space can be eliminated, putting into evidence the sufficiency of the underlying affine structure alone.

A concise presentation of the geometric calculus is given by Peano in an essay *Elementi di calcolo geometrico* [39, (1891)] and in a book for university students *Lezioni di analisi infinitesimale* [40, (1893)] pp. 16-41; a wider comprehensive presentation is given in an essay *Saggio di calcolo geometrico* [44, (1895)]; an axiomatic exposition is given in a paper *Analisi della teoria dei vettori* [45, (1897)] and in the celebrated *Formulario mathematico* (5th edition [54, (1908)] pp. 188-201, 269-270), 4th edition [49, (1902-03) pp. 277-285]). In his seminal book *Applicazioni geometriche* [35, (1887)], in a paper *Teoremi sui massimi e minimi* [37, (1888)] and...
in a book for students *Lezioni di analisi infinitesimale* [40] (1893), Peano gives applications of Grassmann’s exterior algebra.

The aim of this paper, focussed on historical facts, is to give a brief presentation of the main ideas of affine aspects of Grassmann’s exterior algebra, a subject that has not yet received enough attention from the mathematical community. In our exposition, reorganizing Peano’s contribution, we offer a reader a logical path in order to make the comprehension of geometric calculus easier.

From a methodological point of view, we are focussed on primary sources, and not on the secondary ones, that is, on mathematical facts, and not on opinions or interpretations of other scholars of history of mathematics.

In section 2 we examine concepts of affine and vector space and of affine volume, in view of the introduction of an affine exterior algebra.

In Section 3 we present definitions by abstraction, today called definition by quotienting or by equivalence relation. This procedure is due to and used by Peano to define geometric forms. In our opinion, the Peano’s innovative way of definition through an equivalence relation, is a relevant contribution to Mathematics.

In sections 4, 5, 6 and 7, devoted to affine exterior algebra (definition, properties, geometric and mechanical interpretation respectively), we present the reconstruction of the theory of Grassmann due to Peano.

2. AFFINE AND VECTOR SPACES, AFFINE VOLUME AND LINEAR EXTENSIONS OF AFFINE SPACES

In Grassmann’s work, points and vectors coexist distinctly in a common structure, together with other objects, like exterior products of points (for instance, bipoints, tripoints, k-points) and vectors (for instance, bivectors, trivectors, k-vectors). This subtle distinction was very demanding in comparison with today habits of mathematicians.

The conceptual distinction between points and vectors, and consequently between affine and vector spaces, is a necessary and fundamental precondition in order to grasp the geometrical ideas of Grassmann. In the second edition of *Ausdehnungslehre* (1862) the structure of the exterior algebra, as usual at the present day, is built on vectors; on the contrary, in the first edition of *Ausdehnungslehre* (1844) it is built on points.

Peano maintains firmly the distinction between points and vectors and so on. In several papers he applies the geometric calculus of Grassmann, for instance in *Applicazioni geometriche* (see [35] (1887) p. 164) and in [38] (1890) where he defines the area of a surface, and in [45] (1898) he gives an axiomatic re-foundation (today

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4 This paper is not a definitive word on the revisitation of the School of Peano of the geometric calculus of Grassmann: only a part of that will be reworked in this paper. In particular the so called regressive product of Grassmann, the differential and integral aspects and, in general, all the “metric” concepts, might deserve a specific paper. Another lack of this paper is the absence of comparison between different ways of reception of Grassmann’s geometric calculus by Peano and other authors (for instance, Hankel, Schlegel, Whitehead, Cartan, Feir, Forder, Hyde, . . . ). Moreover the contributions of the members of the School of Peano to the geometric calculus should require more attention and further investigations. We hope to fill these gaps in forthcoming papers.

5 In *Applicazioni geometriche*, published by Peano one year before *Calcolo geometrico*, differential and integral calculus for functions with values in an exterior algebra is developed too.
standard) of affine spaces and Euclidean geometry, based on the three primitive
notions of point, vector (i.e., difference of points) and scalar product.

In addition to the axiomatization of affine space, Peano in \textit{Calcolo geometrico} \cite{Peano1888} (Cap. IX, pag. 141–142) provides a modern definition of vector space.

In Grassmann there is no explicit definition of abstract vector space. Nevertheless the following properties (2.1)-(2.5), based on sum, difference and multiplication by scalar (see \cite{Grassmann1862} pp. 4–5 and p. 129) are recognized fundamental by Grassmann in order to obtain all the other algebraic properties depending on the three operations: sum, difference and multiplication. It is easy verifying that these properties describe exactly a vector spaces, as stated in the following proposition.

**Proposition 2.1.** Let \( V \) be a nonempty set, endowed with two binary operations “+”, “−” such that, for every \( a, b, c \in V \), the following two conditions hold:

\[
\begin{align*}
(2.1) \quad a + b &= b + a, \quad a + (b + c) = (a + b) + c \\
(2.2) \quad (a + b) - b &= a, \quad (a - b) + b = a.
\end{align*}
\]

Then the element \( 0 := a - a \) is well defined (it does not depend on the choice of \( a \in V \)) and \( (V, +) \) is a commutative group whose null element is \( 0 \).

If, in addition, \( V \) is endowed with a left and right multiplication by real numbers (denoted merely by juxtaposition) such that, for every \( a, b \in V \) and \( \beta, \gamma \) real numbers, the following three conditions hold:

\[
\begin{align*}
(2.3) \quad a\beta &= \beta a, \quad (a\beta)\gamma = a(\beta\gamma), \quad a1 = a, \\
(2.4) \quad (a + b)\gamma &= a\gamma + b\gamma, \quad a(\beta + \gamma) = a\beta + a\gamma, \\
(2.5) \quad a\beta &= 0 \quad \text{if and only if} \quad a = 0 \text{ or } \beta = 0,
\end{align*}
\]

then \( V \) is a (real) vector space with respect to + and multiplication by scalars.

Peano’s definition of vector space is based on the following four primitive ingredients: 1) null element, 2) equivalence relation, 3) addition and 4) multiplication by real numbers.

**Definition 2.2.** (Peano’s definition of a vector space \cite{Peano1888} §72) A (nonempty) set \( V \), endowed with a binary relation \( \equiv \), a binary operation + and a left multiplication by real numbers (denoted merely by juxtaposition), is said to be a vector space, if there is an element \( 0 \in V \) (called null element) and the following properties hold:

\[
\begin{align*}
(2.6) \quad a \equiv b \iff b \equiv a, \\
(2.7) \quad (a \equiv b \text{ and } b \equiv c) \implies a \equiv c, \\
(2.8) \quad a \equiv b \implies a + c \equiv b + c, \\
(2.9) \quad a + b \equiv b + a, a + (b + c) \equiv (a + b) + c, \\
(2.10) \quad a \equiv b \implies ma \equiv mb, \\
(2.11) \quad m(a + b) \equiv ma + mb, (m + n)a \equiv ma + na, m(na) \equiv (mn)a, \\
(2.12) \quad 1a \equiv a, 0a \equiv 0.
\end{align*}
\]

\textsuperscript{6} In the first edition of \textit{Ausdehnungslehre} \cite{Grassmann1844} pp. 33-40 Grassmann analyzes in details the two operations of addition ‘+’ and subtraction ‘−’. It is worth noticing that properties (2.3)-(2.5) characterize an abelian group.

\textsuperscript{7} Peano used the name “linear system” in place of our “vector space”.

for every \( a, b, c \in V \) and \( m, n \) real numbers.

When the equivalence is in fact an identity, Peano’s definition \( 2.2 \) agrees completely with the modern definition of vector space. The presence of an equivalence relation, instead of an identity, makes the definition more powerful, allowing him, for instance, to define a structure of vector spaces on a set whose elements are freely generated by fixed rules acting on some given \( \text{\textipa{a-priori}} \) elements. Basic examples are given by the definition of a “linear extension” of an affine space (see Peano \( 45 \) (1898), pp. 525-526) and by the definition of “geometric forms” in Calcolo geometrico \( 36 \) (1888 n. 5).

The totality of geometric forms of Peano is a vector space: its basic four ingredients mentioned above being defined through the notion of volume of the Euclidean 3-space. An accurate analysis puts into evidence that the metric aspects of the space are not strictly necessary for a large amount of geometric calculus.

This “non-metric” approach may be pursued by introducing a notion of affine volume. Affine volume over an arbitrary \( n \)-dimensional affine space, is a function defined on the ordered \((n+1)\)-tuples of vertices of a (possibly degenerate) \( n \)-simplex, and it is characterized by the following four properties: non-triviality (i.e. volume of some \( n \)-simplex is non-zero), translational invariance, change of sign under an odd permutation of its vertices, multi-affine dependence on the vertices.

In a modern language this non-metric approach can be formulated in terms of a Peano space, namely of a pair \((\mathbb{A}_n, \text{vol}_n)\) where \( \mathbb{A}_n \) is an \( n \)-dimensional affine space and \( \text{vol}_n \) is an affine volume over \( \mathbb{A}_n \). As usual, the affine space \( \mathbb{A}_n \) will be regarded as a triple \((\mathbb{P}_n, \mathbb{V}_n, +)\), where \( \mathbb{P}_n \) is the underlying set of points of \( \mathbb{A}_n \), \( \mathbb{V}_n \) is the associated \( n \)-dimensional vector space and \(+: \mathbb{P}_n \times \mathbb{V}_n \to \mathbb{P}_n \) is the usual translation operation.

**Definition 2.3.** The affine volume \( \text{vol}_n : (\mathbb{P}_n)^{n+1} \to \mathbb{R} \) is characterized the following properties:

\begin{align*}
(2.13) \quad \text{vol}_n & \quad \text{is not vanishing}, \\
(2.14) \quad \text{vol}_n(p_1 + v, \ldots, p_{n+1} + v) & = \text{vol}_n(p_1, \ldots, p_{n+1}), \\
(2.15) \quad \text{vol}_n(p_{\sigma(1)}, \ldots, p_{\sigma(n+1)}) & = \text{sgn}(\sigma) \text{vol}_n(p_1, \ldots, p_{n+1}), \\
(2.16) \quad \text{vol}_n(\alpha p_1 + \beta p_1', p_2, \ldots, p_{n+1}) & = \alpha \text{vol}_n(p_1, p_2, \ldots, p_{n+1}) + \beta \text{vol}_n(p_1', p_2, \ldots, p_{n+1})
\end{align*}

for arbitrary \( p_1', p_1, \ldots, p_{n+1} \in \mathbb{P}_n \), \( v \in \mathbb{V}_n \), \( \beta := 1 - \alpha \), \( \alpha \in \mathbb{R} \) and permutations \( \sigma \) of \( 1, \ldots, n+1 \).

It is worth observing that in the definition of geometric forms, Peano involves the “ratios” between volumes of simplexes: in this way Peano’s definition is independent of the choice of a specific volume. This independence is due to the fact that the ratio between the volumes of two non-degenerate \( n \)-simplexes is invariant under affine transformations.

Geometric forms allowed Peano to give, in different ways, linear extensions of affine spaces. \textsuperscript{9} Given an affine space \( \mathbb{A} \) and a vector space \( W \), we say that \( W \) is

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\textsuperscript{8} It is worth noticing that reflexivity property of the binary relation \( \cong \) follows by (2.6), (2.7), and (2.12).

\textsuperscript{9} For given points \( p, q \in \mathbb{P}_n \) the difference \( p - q \) denotes the usual difference in an affine space, namely the unique vector \( v \in \mathbb{V}_n \) such that \( p = q + v \).

\textsuperscript{10} Berger in [33] (1987), p. 68] introduced the notion of universal vector space. Berger says: “the construction of this universal space may at first appear to come out of the blue”. On the
a linear extension of \( \mathbb{A} \) with respect to an injective application \( j : \mathbb{A} \rightarrow W \) if \( j \) is an affine application and its image is a hyperplane \( H \) of \( W \) with \( 0 \notin H \).

The approach of Burali-Forti \[11\] (1915) to geometric forms suggests a way of constructing a linear extension (denoted by \( \mathbb{B}(\mathbb{A}_n) \) and called the Burali-Forti space of \( \mathbb{A}_n \)) using the notion of affine volume introduced above. In this sense, we regard points of \( \mathbb{P}_n \) as functions on sets of \( n \)-tuples of points of \( \mathbb{P}_n \) through the application \( \text{vol}_n \). More precisely, for all \( p \in \mathbb{P}_n \), we denote by \( j(p) : (\mathbb{P}_n)^n \rightarrow \mathbb{R} \) the function defined by

\[
j(p)(p_1, p_2, \ldots, p_n) := \text{vol}_n(p, p_1, p_2, \ldots, p_n).
\]

Then the vector space \( \mathbb{B}(\mathbb{A}_n) \) is defined to be the totality of functions \( j(p) \). By properties \[2.18\] - \[2.19\] the Burali-Forti space \( \mathbb{B}(\mathbb{A}_n) \) is a linear extension of the affine space \( \mathbb{A}_n \) with respect to \( j \).

It is not surprising that linear relations in Burali-Forti space \( \mathbb{B}(\mathbb{A}_n) \) can be recovered as affine relations in \( \mathbb{A}_n \) (and conversely \[1\]), in virtue of the following proposition.

**Proposition 2.4.** Let \( \{\alpha_i\}_{i=1}^m \subset \mathbb{R} \) and \( \{p_i\}_{i=1}^m \subset \mathbb{P}_n \). The following properties are equivalent:

\[
\begin{align*}
\text{(2.18)} & \quad \sum_{i=1}^m \alpha_i (p_i - p) = 0 \quad \text{for any} \quad p \in \mathbb{P}_n \\
\text{(2.19)} & \quad \sum_{i=1}^m \alpha_i \text{vol}_n(p_i, q_2, \ldots, q_{n+1}) = 0 \quad \text{for any} \quad q_2, \ldots, q_{n+1} \in \mathbb{P}_n.
\end{align*}
\]

Other examples of linear extensions are obtained by introducing Möbius spaces. A pair formed by a vector space \( W \) and a non-vanishing linear form \( \mu : W \rightarrow \mathbb{R} \) is called a Möbius space with mass \( \mu \), in honor of Möbius. To a Möbius space is associated an affine space \( \mathbb{A} := (\mathbb{P}, \mathbb{V}, +) \), where \( \mathbb{P} := \{ x \in W : \mu(x) = 1 \} \), \( \mathbb{V} := \{ x \in W : \mu(x) = 0 \} \) and the translation operation \( + \) is the restriction to \( \mathbb{P} \times \mathbb{V} \) of the sum operation \( + \) on \( W \). It is clear that a Möbius space \( W \) is a linear extension of the affine space \( \mathbb{A} \) with respect to the canonical injection from \( \mathbb{P} \) to \( W \).

3. **Definitions in Mathematics**

A relevant subject of the research activity of Peano and his School concerned definitions in Mathematics, a subject that received and till now receives more attention by philosophers than by mathematicians. \[10\]

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11 For instance, through a relation of type \[2.13\] it is possible to characterize a “barycenter”: given a (finite) system of weighted points \( \{(p_i, \alpha_i)\} \) such that \( \sum_i \alpha_i = 1 \), its barycenter is the point \( G \) uniquely determined by the condition \( \sum_i \alpha_i(p_i - p) = G - p \), for all point \( p \in \mathbb{P}_n \). Therefore, by \[2.19\], the barycenter \( G \) may be characterized through the condition

\[
\begin{align*}
\text{(2.17)} & \quad \sum_{i=1}^m \alpha_i \text{vol}_n(p_1, q_2, \ldots, q_{n+1}) = \text{vol}_n(G, q_2, \ldots, q_{n+1}) \quad \text{for any} \quad q_2, \ldots, q_{n+1} \in \mathbb{P}_n.
\end{align*}
\]

Condition \[2.17\] motivates the characterization of the barycenter given by Carnot \[13\] (1801) p. 154 as “centre des moyennes distances”, i.e., the center of the signed distance of the point \( p_i \) (weighted with \( \alpha_i \)) by arbitrary planes.

12 The proof can be based on the use of the vector volume \( \text{mlt}_n : (\mathbb{V}_n)^n \rightarrow \mathbb{R} \) associated to the affine volume \( \text{vol}_n \) and well defined by \( \text{mlt}_n(p_1, \ldots, v_n) := \text{vol}_n(p, p + v_1, \ldots, p + v_n) \) for arbitrary \( p \in \mathbb{P}_n \) and \( v_1, \ldots, v_n \in \mathbb{V}_n \). Observe that the application \( \text{mlt}_n \) is multilinear and alternating.

13 For instance, recently (in the year 2005) the philosophical Journal Mind has celebrated with a special issue (1222 pages) the Centenary of Bertrand Russell’s landmark essay On Denoting
The problem of defining has been treated by ARISTOTLE in Analytica Posteriora (II, 10) and in Topica (I, 5). Among the philosophers who proposed a theory of definition we mention HOBBES and LEIBNIZ (see COUTURAT [20] (1901) cap. VI, n. 7).

In 1903 VAILATI, a member of the School of Peano, discovered (see [65]) an interesting booklet: Logica demonstrativa of SACCHERI [14]. Saccheri, in studying the Organon (or logical works) of ARISTOTLE, recognized in his Logica demonstrativa two modalities of definitions [64] (1697) pars II cap. IV: nominal (quid nominis) and real (quid rei), giving examples of mistakes originating by confusing these two types. A definition quid rei denotes an existing object; on the contrary a definition quid nominis gives a meaning to a word. Concerning a definition quid nominis, its correctness simply relies on the fact that the word has not been previously defined. In order for the correctness of a definition quid rei, SACCHERI considers two cases: if the definition is incomplexa (i.e., concerning only one property), it is necessary to postulate or to give a proof of the existence of definitum; if the definition is complexa (i.e., concerning more properties), it is necessary to show the compatibility of the properties themselves.

In [66] (1899) VACC, another member of the School of Peano, praises the theory of definitions of GERGONNE [22] (1818-19), and gives rich comments about implicit definitions.

PEANO is aware of the historical development of definition theory [46] (1900)]. In 1894 BURALI-FORTI in his booklet Logica matematica [6] gives a classification of various types of definitions: 1) nominal definitions without any hypothesis (i.e., the definiens is made only of constant symbols whose meaning is fixed by previous definitions), 2) nominal definitions with hypothesis (in which the definiens contains variables whose meaning is determined by the hypothesis), 3) definition by induction, 4) implicit definitions and 5) definition by abstraction.

The term definition by abstraction has been introduced in mathematics by PEANO [41] (1894) §38-39]. Such way of defining, starting from a given set A and a reflexive, symmetric and transitive relation \(\approx\) on A, provides a pair \((X, \varphi)\) made of a set \(X\) and a surjective function \(\varphi : A \to X\) such that

\[
\varphi(a) = \varphi(b) \iff a \approx b.
\]

In the review paper [51] (1915) PEANO gives a rich and significant list, starting with EUCLIDES, of mathematical entities constructed through the definition by abstraction (for example, rational numbers, real numbers [15], cardinality of sets, sets of points, and so on).
directions, vectors and so on). In the same paper Peano exposes “alternative ways” of formulating a definition by abstraction: in particular he considers the case in which \( \varphi \) is defined by

\[
\varphi(a) := \text{“the set of element } b \in A \text{ such that } b \approx a”.
\]

The definition (3.2), ascribed by Peano to Russell, corresponds to the modern concept of quotient space. Russell introduced the definition (3.2) in [63, (1901), p. 320] (see also [61, (1903) §108-111]); he observed its relevance in providing nominal definitions whenever definitions by abstraction are possible, and in avoiding the plurality of pairs \((X, \varphi)\) involved in definition (3.1).

Definitions by abstraction have been analyzed from different viewpoints (correctness, history, avoidance of logical drawbacks, practical use) by members of the School of Peano: Burali-Forti [8, (1899)], Padoa [34, (1908)], Vailati [69, (1908)], Burali-Forti [10, (1912)], Maccaferri [30, (1913)] and Burali-Forti [11, (1915), [12, (1924), [13, (1925)].

Definitions by abstraction play a basic role in this paper. We will see in the next section that the spaces of geometric forms were obtained by Peano in *Calcolo geometrico* by quotienting a freely generated vector space with respect to a suitable subspace. This way of defining geometric forms marks the explicit origin of the modality of definition by abstraction in mathematics. According to Vailati [69, (1908) p. 105], Grassmann with *Ausdehnungslehre* paved the way for definitions by abstraction.

4. AFFINE EXTERIOR ALGEBRA: DEFINITIONS AND NOTATION

Nowadays an exterior algebra is built on a given vector space. Grassmann, in the second edition of *Ausdehnungslehre*, built the exterior algebra by imposing an anticommutative product (called combinatorial product by him) between the linear independent generators, called unit elements.

The elements of his exterior algebra are called by Grassmann extensive magnitudes. In his construction the product is not explicitly defined. His procedure, restated in modern terms, consists in quotienting with respect to a suitable equivalence relation (accounting for anticommutativity) the associative algebra freely generated by unit elements.

In the first edition of *Ausdehnungslehre* [25] Grassmann built an affine exterior algebra from points. A notion of mass of elementary magnitudes allows him to distinguish points by vectors. Besides, Grassmann introduced an operator (called divergence by Grassmann, denoted by \( \omega \) by Peano, and corresponding to the modern boundary operator \( \partial \)) that extends the notion of mass to the whole exterior algebra, enabling him to separate products of points and products of vectors.

Let \( G(\mathbb{A}_n) \) denote the affine exterior algebra related to an \( n \)-dimensional affine space \( \mathbb{A}_n \). Following Grassmann, in \( G(\mathbb{A}_n) \) there are elements of zero degree (i.e., real numbers), elements of degree 1 (i.e., points of \( \mathbb{A}_n \) and their linear combinations), elements of degree 2 (i.e., products of two points of \( \mathbb{A}_n \) and their linear combinations) and so on.

For \( k = 1, 2, \ldots, \) the symbol \( F_k(\mathbb{A}_n) \) denotes the linear combinations of products of \( k \) points of \( \mathbb{A}_n \), called geometric forms of degree \( k \) by Peano. \( F_0 \) denotes the set of real numbers.
The operator \( \omega \) is defined on all geometric forms. The linearity and the following equalities characterize \( \omega \) uniquely: for every points \( P_0, P_1, \ldots, P_k, \ldots \) of the affine space \( \mathbb{A}_n \)

\[
\begin{align*}
\omega(1) &= 0 \\
\omega(P_0) &= 1 \\
\omega(P_0 P_1) &= P_1 - P_0 \\
\omega(P_0 P_1 P_2) &= (P_1 - P_0)(P_2 - P_0) \\
\omega(P_0 P_1 P_2 P_3) &= (P_1 - P_0)(P_2 - P_0)(P_3 - P_0) \\
&\quad \vdots \\
\omega(P_0 P_1 P_2 \cdots P_k) &= (P_1 - P_0)(P_2 - P_0) \cdots (P_k - P_0)
\end{align*}
\]

(4.1)

Let \( \omega_r \) denote the restriction of \( \omega \) to \( \mathbb{F}_r(\mathbb{A}_n) \). The operator \( \omega_1 \), the so-called mass, allows us to recover in \( \mathbb{F}_1(\mathbb{A}_n) \) an affine copy of the affine space \( \mathbb{A}_n \) and to deduce that \( \mathbb{F}_1(\mathbb{A}_n) \) has dimension \( n + 1 \). In fact, \( \mathbb{F}_1(\mathbb{A}_n) \) is a linear extension of the affine space \( \mathbb{A}_n \) with respect to the canonical injection; in other words, the pair \( (\mathbb{F}_1(\mathbb{A}_n), \omega_1) \) is a Möbius space with \( \mathbb{F}_n = \{ x \in \mathbb{F}_1(\mathbb{A}_n) : \omega_1(x) = 1 \} \).

Now denote by \( \mathbb{V}_k(\mathbb{A}_n), k = 1, 2, \ldots \) the linear combinations of products of \( k \) vectors of \( \mathbb{G}(\mathbb{A}_n) \), called the space of \( k \)-vectors. \( \mathbb{V}_0(\mathbb{A}_n) \) denotes the set of real numbers.

Clearly, from (4.1) it follows that the operators \( \omega_r \) maps the geometric forms of degree \( r \) onto \( \mathbb{V}_{r-1}(\mathbb{A}_n) \).

The elements of \( \mathbb{F}_n \) (resp. of \( \mathbb{V}_1(\mathbb{A}_n) \)) are called points (resp. vectors) of the affine exterior algebra \( \mathbb{G}(\mathbb{A}_n) \); the latter may be “identified” with the vectors of the affine space \( \mathbb{A}_n \).

Following Peano, the product of two, three and four vectors of \( \mathbb{G}(\mathbb{A}_n) \) are called bivector, trivector and quadri-vector, respectively; similarly, the product of two, three and four points of \( \mathbb{G}(\mathbb{A}_n) \) are called bipoint, tripoint and quadri-point, respectively. Moreover, for any geometric form \( x \) of degree \( r = 2, 3, 4 \), the value \( \omega_r(x) \) is called by Peano a vector (resp. bivector, trivector) of the geometric form \( x \).

The exterior product is anticommutative, namely

\[
xy = (-1)^{rs}yx \quad \text{for every} \quad x \in \mathbb{F}_r(\mathbb{A}_n), y \in \mathbb{F}_s(\mathbb{A}_n).
\]

In particular, the product of two geometric forms of degree 1 is null if the factors are linearly dependent. Therefore \( \mathbb{V}_k(\mathbb{A}_n) \) contains only the zero geometric form, for \( k > n \). Analogously, \( \mathbb{F}_k(\mathbb{A}_n) \) contains only the zero geometric form, for \( k > n + 1 \); in symbols

\[
\mathbb{G}(\mathbb{A}_n) = F_0(\mathbb{A}_n) \oplus F_1(\mathbb{A}_n) \oplus \cdots \oplus F_n(\mathbb{A}_n) \oplus F_{n+1}(\mathbb{A}_n).
\]

The main innovation operated by Peano in this context can be found in the construction of an affine exterior algebra based on the notion of affine volume. Concerning \( \mathbb{G}(\mathbb{A}_3) \), Peano says in Calcolo geometrico [6] (1888) p. VI:

Le definizioni introdotte per le formazioni di quarta specie \([\mathbb{F}_4(\mathbb{A}_3)]\), o volumi, sono già comuni in geometria analitica; le definizioni per le formazioni delle tre prime specie \([\mathbb{F}_1(\mathbb{A}_3), \mathbb{F}_2(\mathbb{A}_3), \mathbb{F}_4(\mathbb{A}_3)]\) sono ridotte con metodo uniforme a quelle date per volumi \([\ldots]\).

\[\text{[16]}\] In the sequel we will see that the linear structure of \( \mathbb{F}_1(\mathbb{A}_n) \) allows also to recover the barycentric calculus of Möbius [8] (1827).
Let $\text{vol}_n$ be an affine volume on $\mathbb{A}_n$. Following Peano, for $1 \leq k \leq n + 1$, the space $F_k(\mathbb{A}_n)$ of geometric forms of degree $k$ is defined by quotiening the vector space of the homogeneous polynomials of the same degree $k$ (freely generated by points) by the subspace of homogeneous polynomials

\begin{equation}
\sum_i \alpha_i P_{(i,1)} \cdots P_{(i,k)}
\end{equation}

such that, for any choice of the points $P_{k+1}, P_{k+2}, \ldots, P_{n+1}$, the following equality holds true:

\begin{equation}
\sum_i \alpha_i \text{vol}_n(P_{(i,1)}, \ldots, P_{(i,k)}, P_{k+1}, \ldots, P_{n+1}) = 0.
\end{equation}

To complete the definition of affine exterior algebra given by Peano notice that:

\begin{enumerate}
\item[(4.5)] the exterior product of forms of degree $k$ by forms of degree $s$ (when $0 \leq k + s \leq n + 1$) is defined by Peano starting from the formal product of polynomials, verifying that the definition is compatible with the operation of quotiening defined above, and satisfies the properties of associativity, distributivity and anticommutativity.
\item[(4.6)] Peano, following Grassmann, indicates the exterior product by juxtaposition. Notice that this notation does not create any ambiguity with the “polynomial” notation for geometric forms, because a “monomial” geometric form $P_{(i,1)} \cdots P_{(i,k)}$ is equal to the exterior product of the points $P_{(i,1)} \cdots P_{(i,k)}$.
\end{enumerate}

The notion of volume, that stands at the basis of the construction of the affine exterior algebra, leads quite naturally Peano to describe linear forms on the space $F_k(\mathbb{A}_n)$ of the geometric forms of degree $k$ by geometric forms of supplementary degree $s := (n + 1) - k$ (see [36, (1888) n. 82]). For instance, given a geometric form $\varphi := \sum_i \alpha_i Q_{(i,1)} \cdots Q_{(i,s)}$ of degree $s$, a linear form $\varphi^*$ on $F_k(\mathbb{A}_n)$ can be uniquely defined through the relation

\begin{equation}
\varphi^*(P_1, \ldots, P_k) := \sum_i \alpha_i \text{vol}_n(P_1, \ldots, P_k, Q_{(i,1)}, \ldots, Q_{(i,s)}),
\end{equation}

for any point $P_1, \ldots, P_k$.

It is worth observing that Burali-Forti [11, (1915)] defines the elements (points, bipoints and so on) that generate the exterior algebra, as functions on elements of supplementary degree. More precisely he uses formula (4.7) to define these functions.

Burali-Forti’s approach avoids the necessity of introducing the spaces of geometric forms as quotients (as Peano did in his construction). Indeed, in Burali-Forti the geometric forms are regarded as functions, thus inheriting the usual vector space structure. See the Burali-Forti spaces of Section 2.

Peano’s approach to exterior algebra through the notion of volume is emphasized in section 35 “Begründung der Punktrechnung durch G. Peano” of [29] (1923), pp. 1543-1545 in the celebrated Encyklopädie der mathematischen Wissenschaften.

\footnote{17For $k = n + 1$, the quantified variables $P_{k+1}, \ldots, P_{n+1}$ disappear in formula (4.3).}

\footnote{18In the various editions of Formulario Mathematico Peano uses the letter $\alpha$ to denote the exterior product; for example $P_1 \alpha Q$, $P_1 \alpha P_2 \alpha Q_3$ in place of $P_1 Q$, $P_1 Q_3$ respectively.}

\footnote{19It is surprising to perceive in (4.7) a trace of the idea of Hopf algebra and Hodge dual. In this context, it is also worth mentioning the metric Hodge $*$ operator that corresponds to the Grassmann’s index.}
In several papers Peano introduces the geometric forms of degree 1 directly, without using the notion of volume \textit{vol}. See, for instance, Peano \cite{peano1898} (1898), p. 525-526, \cite{peano1899} (1899), p. 155, \cite{peano1901} (1901), p. 196. More clearly, Peano describes the space \( \mathbb{F}_1(\mathbb{A}_n) \) by quotienting the vector space freely generated by points of \( \mathbb{A}_n \) with respect to the subspace of the free linear combinations \( \sum_i \alpha_i P_i \) such that
\begin{equation}
\sum_i \alpha_i (P_i - P) = 0 \quad \text{for all} \quad P \in \mathbb{P}_n.
\end{equation}

The equivalence between this description of the geometric forms of degree 1 and the previous one given in terms of volume is guaranteed by Proposition \textit{2.4}.

5. AFFINE EXTERIOR ALGEBRA \( \mathbb{G}(\mathbb{A}_3) \): PROPERTIES AND REDUCTION FORMULAE

In accordance with the presentation given by Grassmann in \textit{Ausdehnungslehre} and by Peano in \textit{Calcolo geometrico}, we restrict our presentation to the affine exterior algebra \( \mathbb{G}(\mathbb{A}_3) \), related to a 3-dimensional affine space \( \mathbb{A}_3 \):
\begin{equation}
\mathbb{G}(\mathbb{A}_3) = \mathbb{F}_0(\mathbb{A}_3) \oplus \mathbb{F}_1(\mathbb{A}_3) \oplus \mathbb{F}_2(\mathbb{A}_3) \oplus \mathbb{F}_4(\mathbb{A}_3).
\end{equation}

As planned in the Introduction, in this section (and in the subsequent ones) we will present affine exterior algebra as revisited by Peano, focussing our attention on the aspects and properties that, in our opinion, are characteristic and reveal the richness, fecundity and clarity of the geometric calculus. For convenience of the reader some significant proofs of the simplicity of geometric calculus and some comments are confined in footnotes \textit{20}.

BASES, DIMENSION AND COORDINATES (see Peano \cite{peano1888} (1888) \S 60, \cite{peano1908} (1908) p. 194-195). Tetrahedrons suggest a way to construct bases for the forms of different degrees of the affine exterior algebra \( \mathbb{G}(\mathbb{A}_3) \).

Let fix a (non-degenerate) tetrahedron with vertices \( A, B, C, D \). Its 4 vertices are a basis of \( \mathbb{F}_1(\mathbb{A}_3) \). The 6 bipoints \( AB, AC, AD, BC, BD, CD \) which correspond to its 6 edges, are a basis of \( \mathbb{F}_2(\mathbb{A}_3) \). To its faces there correspond 4 tripoints \( ABC, ACD, ABD, BCD \) which are a basis of \( \mathbb{F}_3(\mathbb{A}_3) \). Finally, the quadri-point \( ABCD \) is a basis of the 1-dimensional \( \mathbb{F}_4(\mathbb{A}_3) \).

The coordinates of a geometric form of degree 1 with respect to the basis of \( \mathbb{F}_1(\mathbb{A}_3) \) formed by the four vertices of a tetrahedron are called \textit{baricentric coordinates}. The coordinates with respect to an arbitrary basis are referred to as \textit{projective coordinates}. Choosing as a basis of \( \mathbb{F}_1(\mathbb{A}_3) \) the point \( A \) and the 3 vectors \( B - A, C - A, D - A \), one has a (non-orthogonal) \textit{Cartesian coordinate system} with origin \( A \) and axes directed along the three vectors.

Let \( x_1, x_2, x_3 \) and \( x_4 \in \mathbb{F}_1(\mathbb{A}_3) \) be arbitrary geometric forms of degree 1. They are a basis of \( \mathbb{F}_1(\mathbb{A}_3) \) if and only if \( x_1x_2x_3x_4 \neq 0 \).

Now assume that \( x_1, x_2, x_3, x_4 \in \mathbb{F}_1(\mathbb{A}_3) \) is a basis of \( \mathbb{F}_1(\mathbb{A}_3) \).

\textit{20} In some cases we have indicated some concepts present in the works of Peano with a new terminology (for instance, applied vector, applied bivector, applied trivector, Poinset pair, boundary-cycle).

\textit{21} Similarly, in an \( n \)-dimensional affine space \( \mathbb{A}_n \) one has
\[ \dim(\mathbb{F}_k(\mathbb{A}_n)) = \binom{n+1}{k}, \]
for \( 0 \leq k \leq n + 1 \). If \( T \) is an arbitrary \( n \)-simplex of \( \mathbb{A}_n \), a basis of \( \mathbb{F}_k(\mathbb{A}_n) \) corresponds to \( \binom{n+1}{k} \) \( k \)-dimensional facets of \( T \).
(5.2) Then the product \( x_1x_2x_3x_4 \) is a basis of the 1-dimensional space \( \mathbb{F}_4(\mathbb{A}_3) \). For all \( z \in \mathbb{F}_4(\mathbb{A}_3) \) let us denote by

\[
\frac{z}{x_1x_2x_3x_4}
\]

the real number \( \lambda \) such that \( z = \lambda x_1x_2x_3x_4 \).

(5.3) If \( \alpha_1, \alpha_2, \alpha_3 \) and \( \alpha_4 \) are the coordinates of an element \( x \in \mathbb{F}_4(\mathbb{A}_3) \) with respect to the basis \( x_1, x_2, x_3 \) and \( x_4 \), then the following formulae hold true:

\[
\alpha_1 = \frac{x_1x_2x_3x_4}{x_1x_2x_3x_4}, \quad \alpha_2 = \frac{x_1x_3x_4}{x_1x_2x_3x_4}, \quad \alpha_3 = \frac{x_1x_2x_4}{x_1x_2x_3x_4}, \quad \alpha_4 = \frac{x_1x_2x_3}{x_1x_2x_3x_4}.
\]

(5.4) Moreover, \( x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4 \) is a basis of \( \mathbb{F}_2(\mathbb{A}_3) \); the corresponding coordinates \( \alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{23}, \alpha_{24}, \alpha_{34} \) of an element \( s \in \mathbb{F}_2(\mathbb{A}_3) \) satisfy the following formulae:

\[
\alpha_{12} = \frac{x_1x_3}{x_1x_2x_3x_4}, \quad \alpha_{13} = \frac{x_1x_2}{x_1x_2x_3x_4}, \quad \alpha_{14} = \frac{x_1x_4}{x_1x_2x_3x_4},
\alpha_{23} = \frac{x_2x_4}{x_1x_2x_3x_4}, \quad \alpha_{24} = \frac{x_2x_3}{x_1x_2x_3x_4}, \quad \alpha_{34} = \frac{x_3x_4}{x_1x_2x_3x_4}.
\]

(5.5) The set of geometric forms \( x_2x_3x_4, x_1x_3x_4, x_1x_2x_4 \) and \( -x_1x_2x_3 \) is a basis of \( \mathbb{F}_3(\mathbb{A}_3) \); the corresponding coordinates \( \beta_1, \beta_2, \beta_3, \beta_4 \) of an element \( y \in \mathbb{F}_3(\mathbb{A}_3) \) satisfy the following formulae:

\[
\beta_1 = \frac{x_1y}{x_1x_2x_3x_4}, \quad \beta_2 = \frac{x_2y}{x_1x_2x_3x_4}, \quad \beta_3 = \frac{x_3y}{x_1x_2x_3x_4}, \quad \beta_4 = \frac{x_4y}{x_1x_2x_3x_4}.
\]

THE OPERATOR \( \omega \) AND THE FORMULA OF REDUCTION. In addition to linearity, properties of the operator \( \omega \) on products of geometric forms of any degree are contained in the following exhaustive list [50] (1908) p. 197]:

(5.6)

\[
\begin{align*}
\omega(x_1x_2) & = \omega(x_1)x_2 - x_1\omega(x_2), \\
\omega(x_1x_2x_3) & = \omega(x_1)x_2x_3 + \omega(x_2)x_3x_1 + \omega(x_3)x_1x_2, \\
\omega(xs) & = \omega(s) = \omega(x)s + \omega(s)x, \\
\omega(xy) & = \omega(x)y - x\omega(y), \\
\omega(s_1s_2) & = \omega(s_1)s_2 + s_1\omega(s_2), \\
\omega(x_1x_2x_3x_4) & = \omega(x_1)x_2x_3x_4 - \omega(x_2)x_1x_3x_4 + \omega(x_3)x_1x_2x_4 - \omega(x_4)x_1x_2x_3
\end{align*}
\]

for every \( x, x_1, x_2, x_3, x_4 \in \mathbb{F}_4(\mathbb{A}_3) \), \( s, s_1, s_2 \in \mathbb{F}_2(\mathbb{A}_3) \), \( y \in \mathbb{F}_3(\mathbb{A}_3) \).

---

22 These formulae are obtained multiplying the two members of the equality \( x = \alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3 + \alpha_4x_4 \) by the tripints \( x_2x_3x_4, x_1x_3x_4, x_1x_2x_4, \) and \( x_1x_2x_3 \), respectively. A similar calculus provides coordinate formulae \([64]\) and \([83]\).

23 Nowadays it is relevant the notion of differential graded algebra. Such an algebra is endowed with a linear operator \( \partial \) satisfying the following properties:

\[
\begin{align*}
(a) \quad \partial(x) &= 0, \\
(b) \quad \partial(xy) &= \partial(x)y + (-1)^{x}\partial(y)
\end{align*}
\]

for any \( x, y \) belonging to the algebra, with \( x \) of degree \( r \).

It is worth observing that last two properties are satisfied by the operator \( \omega \). The first one \( (a) \) is merely a consequence of \([50]\). We found no explicit evidence of the second property \( (b) \) in the works of Grassmann and Peano, but it is evident that properties \([64]\) are nothing but instances of \( (b) \).
Concerning the problem of reducing geometric forms, the following fundamental reduction formula holds:

\[(5.7)\]
\[x = P\omega_1(x) + \omega_2(Px)\]

for every point \(P\) and every geometric form \(x\), whose proof follows immediately from (5.6) or, alternatively, by the linearity of \(\omega\). \(^{24}\)

A consequence of the reduction formula is

\[(5.8)\]
\[\nabla_r(\mathbb{A}_3) = \text{im}\omega_{r+1} = \ker\omega_r.\]

![Figure 1. Geometric forms in a 3-dimensional affine space](image)

The equality (5.8) summarizes the three basic properties of operator \(\omega\):

\[(5.9)\]
\[\nabla_r(\mathbb{A}_3) = \text{im}\omega_{r+1},\]
that is: \(\omega_1\) sends (weighted) points into scalars (their weight), \(\omega_2\) sends bipoints into vectors, \(\omega_3\) send tripoints into bivectors, and \(\omega_4\) send quadri-points into trivectors.

\[(5.10)\]
\[\nabla_r(\mathbb{A}_3) \subset \ker\omega_r,\]
that is: the mass of a vector, the vector of a bivector, the bivector of a trivector are all null.

\[(5.11)\]
\[\nabla_r(\mathbb{A}_3) \supset \ker\omega_r,\]
that is: a geometric form \(x\) of degree \(r = 1\) (resp. 2 and 3) is a vector (resp. bivector and trivector), whenever its mass \(\omega_1(x)\) (resp. vector \(\omega_2(x)\) and bivector \(\omega_3(x)\)) is null.

**Geometric forms of degree 1** (see Peano \[36\] (1888) chap.II], \[50\] (1908) p.196], \[10\] (1893) nn. 265-266]). An instance of the reduction formula (5.7) is the following reduction formula for a form \(x\) of degree 1:

\[(5.12)\]
\[x = P\omega_1(x) + \omega_2(Px)\]

for every point \(P\).

Consequently, for every point \(P\), a form \(x \in \mathbb{F}_1(\mathbb{A}_3)\) can be reduced to a sum of a weighted point \(P\omega_1(x)\) and a vector \(\omega_2(Px)\). If \(x = \sum \alpha_i P_i\), the equality (5.12) becomes:

\[(5.13)\]
\[\sum_i \alpha_i P_i = (\sum \alpha_i) P + \sum_i \alpha_i (P_i - P).\]

If \(x \neq 0\) and the vector \(x - P\omega_1(x)\), called deviation by Grassmann, is null, then the point \(P\) is said to be a barycenter of \(x\).

A geometric form \(x\) of degree 1 is a vector, if its mass \(\omega_1(x)\) is null; in such a case, for every point \(P\), the element \(Q := x + P\) is a point and

\[(5.14)\]
\[x = Q - P;\]

---

\(^{24}\) Instances of this formula, that encompasses all reduction formulae present in Peano, will be detailed in the following. We note that reduction formula is a straightforward consequence of the signed Leibniz rule (b) of footnote 23.
otherwise, if $\omega_1(x) \neq 0$, $x$ is a weighted point, that is
\begin{equation}
(5.15) \quad x = \omega_1(x) \left( \frac{x}{\omega_1(x)} \right).
\end{equation}
and the element $\frac{x}{\omega_1(x)}$ is a point: the barycenter of $x$.

Geometric forms of degree 2. (see Peano [30] (1888) chap. III, [50] (1908) p. 197], [40, (1893) nn. 267-270]). An instance of the reduction formula (5.7) is the following reduction formula for a form $x$ of degree 2:
\begin{equation}
(5.16) \quad x = P\omega_2(x) + \omega_3(Px).
\end{equation}
Consequently, a form $x \in \mathbb{F}_2(\mathbb{A}_3)$ can be reduced to a sum of a bipoint $P\omega_2(x)$ and a bivector $\omega_3(Px)$. If $x = \sum_i Q_i P_i$, then the equality (5.16) becomes:
\begin{equation}
(5.17) \quad \sum_i Q_i P_i = P \sum_i (P_i - Q_i) + \sum_i (Q_i - P)(P_i - P).
\end{equation}
There are different useful ways of representing bipoints and bivectors. Name applied vector every product $Av$ of a point $A$ and a vector $v$. The difference
$$Av - Bv$$
is said to be a Poinsot pair, whenever $v$ is a vector and $A, B$ are two points. A geometric form $x$ of degree 2 is said to be a boundary-cycle of a triangle \footnote{A boundary-cycle of a triangle can be viewed as oriented boundary of a triangle.}, if there exists three points $A, B, C$ such that
\begin{equation}
(5.18) \quad x = AB + BC + CA
\end{equation}
Moreover
\begin{equation}
(5.19) \quad \text{The bipoints are the applied vectors}\footnote{For every bipoint $AB$ it is clear that $B - A$ is a vector and $AB = A(B - A)$. Conversely, let $A$ be a point and $v$ a vector, then $Av = AB$, where $B := A + v$.}
\end{equation}
\begin{equation}
(5.20) \quad \text{The bivectors are the Poinsot pairs}\footnote{If $v$, $w$ are vectors and $A$ is an arbitrary point, then $vw = Bw - Aw$ where $B := A + v$. Conversely, a Poinsot pair $Av - Bv$ may be rewritten as $(A - B)v$.}
\end{equation}
\begin{equation}
(5.21) \quad \text{The bivectors are the boundary-cycles of triangles}\footnote{If $v, w$ are vectors and $A$ is an arbitrary point, then $vw = (B - A)(C - A) = AB + BC + CA$ where $B := A + v$ and $C := A + w)$. Conversely, a boundary-cycle $AB + BC + CA$ may be rewritten as $(B - A)(C - A)$.}
\end{equation}
\begin{equation}
(5.22) \quad \text{Every couple of bivectors can be reduced to a couple of bivectors with a common factor}\footnote{That is: if $v_1, v_2, w_1, w_2$ are vectors, then there exist vectors $y, v, w$ such that $v_1v_2 = yv$ and $w_1w_2 = yw$.}.
\end{equation}
\begin{equation}
(5.23) \quad \text{A linear combination of bivectors is a bivector (Poinset rule).}
\end{equation}
\begin{equation}
(5.24) \quad \text{A geometric form $x \in \mathbb{F}_2(\mathbb{A}_3)$ is a bivector if and only if $\omega_2(x) = 0$.}
\end{equation}
\begin{equation}
(5.25) \quad \text{The vector space $\mathbb{V}_2(\mathbb{A}_3)$ has dimension 3. Given a basis $v_1, v_2, v_3$ of $\mathbb{V}_1(\mathbb{A}_3)$, the exterior products $v_1v_2$, $v_2v_3$, $v_3v_1$, form a basis for $\mathbb{V}_2(\mathbb{A}_3)$.}
\end{equation}
From the reduction formula (5.16) and the description (5.18) of a bivector, it follows that, for every geometric form $x$ of degree 2, there exist four points $P, B, C, D$ such that
\begin{equation}
(5.26) \quad x = PB + PC + CD + DP.
\end{equation}
Hence every geometric form of degree 2 is one of the following three types:

\[
\begin{cases}
(i) \text{ a bipoint:} & PB \\
(ii) \text{ a bivector:} & PC + CD + DP \\
(iii) \text{ neither bipoint nor bivector:} & PB + PC + CD + DP \text{ with } P, B, C, D \text{ vertices of a non-degenerate tetrahedron}
\end{cases}
\]

In particular, every geometric form of degree 2 is either a bipoint or a bivector, if and only if
\[
x x = 0.
\]

Geometric forms of degree 3 (see Peano [5] (1888) chap. IV], [50] (1908) p. 197], [50] (1893) nn. 271]). Another instance of the reduction formula \((5.7)\) is the cycle of a tetrahedron

\[
\sum_i \alpha_i Q_i P_i R_i = \sum_i P(P_i - Q_i)(R_i - Q_i) + \sum_i (Q_i - P)(P_i - P)(R_i - P).
\]

Name applied bivector every product \(A y\) of a point \(A\) and a bivector \(y\). The difference \(A y - B y\) is said to be a Poinsot triangle pair, whenever \(y\) is a bivector and \(A, B\) are two points. A geometric form \(x\) of degree 3 is said to be a boundary-cycle of a tetrahedron [30], if there exist four points \(A, B, C, D\) such that

\[
x = BCD - ACD + ABD - ABC
\]

(5.31) The tripoints are the applied bivectors [31]
(5.32) The trivectors are the Poinsot triangle pairs [32]
(5.33) The trivectors are the boundary-cycles of tetrahedron [33]
(5.34) A geometric form \(x \in F_3(A_3)\) is a trivector if and only if \(\omega_3(x) = 0\).
(5.35) The vector space \(V_3(A_3)\) has dimension 1. Given a basis \(v_1, v_2, v_3\) of \(V_1(A_3)\), the exterior product \(v_1 v_2 v_3\) is a basis for \(V_3(A_3)\).
(5.36) A geometric form \(x \in F_3(A_3)\) is either a tripoint or a trivector.

Geometric forms of degree 4. Any form \(x\) of degree 4 is a quadri-point, that may be regarded as a product of an arbitrary point \(P\) by a trivector:

\[
x = P \omega_4(x)
\]

Quadri-points are related to volume forms and allow to check linear independence of forms \(x_1, x_2, x_3, x_4\) of degree 1:

\[
x_1 x_2 x_3 x_4 \neq 0 \iff x_1, x_2, x_3, x_4 \text{ are linearly independent.}
\]

Due to anticommutativity of the exterior product, the vector space \(V_4(A_3)\) is 0-dimensional, that is

\[
(5.40) \text{ the exterior product of four vectors is null.}
\]

---

30 A boundary-cycle of a tetrahedron can be viewed as oriented boundary of a tetrahedron.
31 For every tripoint \(ABC\) it is clear that \((B - A)(C - A)\) is a bivector and \(ABC = A(B - A)(C - A)\). Conversely, let \(A\) be a point and \(v, w\) vectors, then \(A vw = ABC\), where \(B := A + v\) and \(C := A + w\).
32 To prove this property use the following equality: \((B - A)wz = Bwz - Awz\).
33 To prove this property use the following equality: \((B - A)(C - A)(D - A) = BCD - ACD + ABD - ABC\).
This dimensionality restriction can be restated in terms of points: for every points $P, A, B, C, D$, the following equality holds:

\[(5.41) \quad ABCD - PBCD + PACD - PABD + PABC = 0,\]

since $(A - P)(B - P)(C - P)(D - P) = ABCD - PBCD + PACD - PABD + PABC$.

**Relevant examples of reductions** (see Peano [36] (1888) nn. 27, 31], [35] (1887) n.19, 24], [40] (1893) nn. 269, 273]). In various moments of his activity, for the purpose of evaluating areas and volumes, Peano considered reduction formulae for (plane or not) polygons and for (open or closed) polyhedral surfaces. Two significant examples of reductions, due to the characterizations of bivectors and trivectors (see (5.24) and (5.35), respectively) are:

\[(5.42) \quad \text{Let } s := \sum_i A_i B_i \text{ denote a geometric form of degree 2 by } A_i, B_i \text{ points of a plane } \pi \text{ and with } \omega(s) = 0. \text{ There exist } X, Y, Z \text{ in } \pi \text{ such that } \sum_i P A_i B_i = XYZ \text{ for any choice of the point } P \text{ in } \pi.\]

\[(5.43) \quad \text{Let } x := \sum_{i=1}^n A_i B_i C_i \text{ denote a geometric form of order 3 such that } \omega(x) = 0. \text{ There exist four points } X, Y, Z, T \text{ such that } \sum_i P A_i B_i C_i = XYZT \text{ for any choice of the point } P.\]

### 6. Affine Exterior Algebra $\mathcal{G}(A_3)$: Geometrical Interpretation

Accordingly to the following citation of Peano, the nature of the relation between algebraic and geometric objects is made clear by examining the equalities between geometric forms:

Il calcolo geometrico, in generale, consiste in un sistema di operazioni a eseguirsi su enti geometrici, analoghe a quelle che l’algebra fa sopra i numeri. Esso permette di esprimere con formule i risultati di costruzioni geometriche, di rappresentare con equazioni proposizioni di geometria, e di sostituire una trasformazione di equazioni ad un ragionamento.

Following Grassmann, Peano associates a same geometric concept to different elements of the affine exterior algebra: this is why different attributes (length, area, volume, direction, orientation, position and so on) of a geometric object may be taken into account in the various cases. For instance, Peano associates to points, bipoles, tripoints, quadri-points, the geometric objects: points, segments, 

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\[^34\] The left side of equality (5.40) can be viewed as oriented boundary of a 4-simplex.

\[^35\] For the proof, the condition $\omega(s) = 0$ implies that $s$ is a bivector; therefore there exist three points $X, Y, Z$ in $\pi$ such that $s = (Y - X)(Z - X)$. Multiplying by $P$ we get the equality $Ps = P(Y - X)(Z - X)$; moreover, being $P(Y - X)(Z - X) = XYZ$ (because $X, Y, Z, P$ belong to the same plane $\pi$) and $Ps = \sum P A_i B_i$ we have the required equality.

\[^36\] For the proof, the condition $\omega(x) = 0$ implies that $x$ is a trivector; therefore there exist four points $X, Y, Z, T$ such that $x = (Y - X)(Z - X)(T - X)$. Multiplying by $P$ we get the equality $Px = P(Y - X)(Z - X)(T - X)$; moreover, being $P(Y - X)(Z - X)(T - X) = XYZT$ and $Px = \sum_i P A_i B_i C_i$, we have the required equality.
tri-angles, tetrahedra respectively; similarly to vectors, bivectors, trivectors, he associates segments, triangles, tetrahedra.

In what follows, by segment, triangle, tetrahedron we will denote the geometric sets of three-dimensional affine space $A_3$ convexly generated by their vertices (two, three, four respectively). Moreover, with oriented segment, triangle, tetrahedron we will denote the pair formed by the geometric sets with an order of their vertices. We will denote an oriented geometric object of this type through the ordered list of its vertices. It is evident that the notion of affine volume given above in section 2 concerns oriented tetrahedra.

For a better understanding of the point of view of Peano (that is, in a pre-vectorial epoch) it is necessary to regard the usual affine space $A_3$ as a set of points endowed with the action of translations. The notions of parallelism between two straight lines, between a straight line and a plane and between two planes, direction of straight lines and of planes) and affine transformations are well defined in this context.

Concerning orientation, Peano introduces the following definitions:

(6.1) two (non-degenerate) oriented tetrahedra have the same orientation if the affine volumes of the corresponding tetrahedra have the same sign.

(6.2) two (non-degenerate) oriented triangles of vertices $A, B, C$ and $Q, R, S$ belonging to the same plane $\pi$ have the same orientation if, for any point $P \notin \pi$, the tetrahedra of vertices $A, B, C, P$ and $Q, R, S, P$ have the same orientation.

(6.3) two (non-degenerate) oriented segments of vertices $A, B$ and $Q, R$ belonging to the same straight line $r$ have the same orientation if, for any points $P, S \notin r$, the tetrahedra of vertices $A, B, P, S$ and $Q, R, P, S$ have the same orientation.

Concerning extension (i.e., length, area, volume), Peano retains the metric definitions that are standard in Euclidean geometry (choice of a unit measure, orthogonality and so on). A deeper analysis of Peano’s geometric calculus reveals that his definitions relies only on purely affine properties of the underlying space. Consequently, we will assume, as a starting point, the following purely affine definitions, that are consistent with the use of the metric notion of extension given by Peano:

(6.4) two oriented tetrahedra have the same extension if their affine volumes have the same absolute value.

(6.5) two oriented triangles of vertices $A, B, C$ and $Q, R, S$ belonging to the same plane $\pi$ have the same extension if, for any point $P \notin \pi$, the oriented tetrahedra $A, B, C, P$ and $Q, R, S, P$ have the same extension.

(6.6) two oriented segments of vertices $A, B$ and $Q, R$ belonging to the same straight line $r$ have the same extension if, for any points $P, S \notin r$, the oriented tetrahedra $A, B, P, S$ and $Q, R, P, S$ have the same extension.

Following Peano, equalities between elements of the affine exterior algebra $G(A_3)$ may be easily interpreted geometrically in $A_3$, as it is evident by the definitions of geometric forms through the notion of affine volume. More explicitly, we have the following facts:

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37 In a pre-vectorial epoch were regarded as collinearities.
(6.7) \[ \sum_i \alpha_i A_i = \sum_j \beta_j Q_j \text{ if and only if for arbitrary points } R, S, T \text{ the sums of the affine volumes of the two systems of oriented tetrahedra } A_i, R, S, T \text{ and } Q_j, R, S, T \text{ coincide, that is} \]
\[ \sum_i \alpha_i A_i RST = \sum_j \beta_j Q_j RST. \]

(6.8) \[ \sum_i \alpha_i A_i B_i = \sum_j \beta_j Q_j R_j \text{ if and only if for arbitrary points } S, T \text{ the weighted sums of the affine volumes of the two systems of oriented tetrahedra } A_i, B_i, S, T \text{ and } Q_j, R_j, S, T \text{ coincide.} \]

(6.9) \[ \sum_i \alpha_i A_i B_i C_i = \sum_j \beta_j Q_j R_j S_j \text{ if and only if for arbitrary point } T \text{ the weighted sums of the affine volumes of the two systems of oriented tetrahedra } A_i, B_i, C_i, T \text{ and } Q_j, R_j, S_j, T \text{ coincide.} \]

(6.10) \[ \sum_i \alpha_i A_i B_i C_i D_i = \sum_j \beta_j Q_j R_j S_j T_j \text{ if and only if the weighted sums of the affine volumes of the two systems of oriented tetrahedra } A_i, B_i, C_i, D_i \text{ and } Q_j, R_j, S_j, T_j \text{ coincide.} \]

As stated by Peano, the correspondence between elements in the affine exterior algebra \( G(\mathbb{A}_3) \) and geometric concepts in the usual affine tri-space \( \mathbb{A}_3 \) is summarized in the following list of equivalences, of direct proof using the properties and definitions listed above.

(6.11) Two points \( A \) and \( B \) coincide if and only if \( AB = 0 \).

(6.12) Three points \( A, B, C \) are collinear in \( \mathbb{A}_3 \) if and only if \( ABC = 0 \).

(6.13) Four points \( A, B, C, D \) are co-planar in \( \mathbb{A}_3 \) if and only if \( ABCD = 0 \).

(6.14) Two oriented segments of vertices \( A, B \) and \( C, D \) lie on a same straight line and have a same extension and orientation if and only if \( AB = CD \).

(6.15) Two oriented segments of vertices \( A, B \) and \( C, D \) lie on parallel straight line and have a same extension and orientation if and only if \( B - A = D - C \).

(6.17) Two (non-degenerate) oriented triangles of vertices \( A, B, C \) and \( P, Q, R \) lie on the same plane and have the same extension and orientation if and only if \( ABC = PQR \).

(6.18) Two (non-degenerate) oriented tetrahedra of vertices \( A, B, C, D \) and \( P, Q, R, S \) have the same extension and orientation if and only if \( ABCD = PQR S \).

The geometric meaning of the equalities between vectors, bivector, and trivectors is deduced from the corresponding equalities between bipoint, and tripoints and quadri-points since, for arbitrary vectors \( v, w, u, t, r, z \) and a point \( P \), the following properties hold:

- \( v = w \text{ if and only if } P v = P w \)
- \( v w = u t \text{ if and only if } P v w = P u t \)
- \( v w z = u t r \text{ if and only if } P v w z = P u t r \).

Several identities may be straightforwardly generated by algebraic manipulations. In the following we list some of them, geometrically significant, present in *Calcolo geometrico* of Peano:

\[ B - A = D - C \iff (B + C)/2 = (D + A)/2. \]

This shows that the quadrilateral of vertices \( A, B, D, C \) is a parallelogram because the two diagonals meet in their middle point. The property (6.10) was and is frequently assumed as a starting point of the definition of vector as an equivalence class (see [15 (1898)]).
(6.19) A straight line determined by two different points $A$ and $B$ is parallel to the plane determined by a non-null tripoint $\pi$ if and only if
\[(B - A)\pi = 0.\]

(6.20) A straight line determined by two different points $A$ and $B$ is parallel to the straight line determined by a non-null bipoint $a$ if and only if
\[(B - A)a = 0.\]

(6.21) For every point $P$ belonging to the straight line determined by two different points $A$ and $B$, the following equality holds:
\[AB = P(B - A).\]

(6.22) For every point $P$ belonging to the plane determined by three non-collinear points $A, B, C$, the following equality holds:
\[ABC = P(B - A)(C - A).\]

(6.23) $ABCD = P(B - A)(C - A)(D - A)$ for every point $P$ in $S$. \[41\]

Expanding the products in the equalities listed above we have the following description of point belonging to a straight line and to a plane (see Peano [36]):

(6.24) Let $A, B$ be two different points and $P$ an arbitrary point. Then
\[AB + BP + PA = 0\]
if and only if $P$ belongs to the straight line determined by $A, B$.

(6.25) Let $A, B, C$ be three non-collinear points and $P$ an arbitrary point. Then
\[ABC + BAP + CBP + ACP = 0\]
if and only if $P$ belongs to the plane determined by $A, B, C$.

In previous section, using the reduction formula, we presented two statements \[21\] and \[22\], which concern polygons and polyhedral surfaces, respectively.

The geometric interpretation of the first statement corresponds to the following proposition:

(6.26) Let $\{A_i\}_{i=1}^{n+1}$ denote a set of points belonging to the plane $\pi$, with $A_1 = A_{n+1}$.

The sum of the areas of the oriented triangles of vertices $P, A_i, A_{i+1}$, for
\(1 \leq i \leq n\), does not depend on the choice of the point $P$ in the given plane $\pi$.

With respect to the second statement \[26\] the corresponding geometric interpretation is:

(6.27) Let $\{A_iB_iC_i\}_{i=1}^{n+1}$ denote a set of tripoints belonging to the affine tridimensional space, with $\sum_{i=1}^{n+1}(B_i - A_i)(C_i - A_i) = 0$. \[42\] The sum of the volumes of the oriented tetrahedra of vertices $P, A_i, B_i, C_i$, for $1 \leq i \leq n$, does not depend on the choice of the point $P$ in the given tri-space.

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39 The equality follows by reduction formula \[15\]: $AB = P\omega(AB) + \omega(PAB) = P(B - A) + \omega(PAB) = P(B - A)$ being $PAB = 0$ due to collinearity.

40 The equality follows by reduction formula \[24\]: $ABC = P\omega(ABC) + \omega(PABC) = P(B - A)(C - A) + \omega(PABC) = P(B - A)(C - A)$ being $PABC = 0$ due to co-planarity.

41 The equality follows by reduction formula \[35\]: $ABCD = P\omega(ABCD) = P(B - A)(C - A)(D - A)$.

42 Whenever the set of oriented triangles $A_iB_iC_i$ for $1 \leq i \leq n$ forms an oriented closed polyhedral surface, the condition $\sum_{i=1}^{n+1}(B_i - A_i)(C_i - A_i) = 0$ is satisfied, since $\sum_{i=1}^{n+1}(B_i - A_i)(C_i - A_i) = \sum_{i=1}(A_iB_i + B_iC_i + C_iA_i)$. 
As noticed in [23], these two properties have been proved with slight differences by Bellavitis and Möbius.

7. Affine exterior algebra $\mathcal{G}(\mathbb{A}_3)$: Mechanical interpretation

In this section we present Peano’s point of view on elements of the exterior algebra $\mathcal{G}(\mathbb{A}_3)$ in terms of mechanical concepts.

Systems of material points and barycenters. (See Peano [40 (1893), v. II §263]).

Geometric forms of degree 1 are related to systems $S$ of material points.

Let $\{(P_i, \alpha_i)\}$ denote a set of material points $P_i$ of mass $\alpha_i$. The notion of static moment with respect to a given plane $\pi$ of a system of points is well known in Physics and is given by $\sum \alpha_i \text{dist}(P_i, \pi)$, sum of the weighted signed distance of points $P_i$ from $\pi$. It is evident that this static moment is proportional to the quantity

\[
\sum \alpha_i \text{vol}_n(P_i, A, B, C)
\]

for arbitrary (not collinear, of course) points $A, B, C$ belonging to $\pi$.

Moreover two material systems of points are said to be (mechanically) equivalent if their static moments with respect to any given plane are the same. A material system having static moment null with respect to any plane, is called a null system.

In accordance with the definition of forms of degree 1 given by Peano (see 6.7), we have that

\textit{two systems of material points are equivalent if and only if the corresponding geometric forms of degree 1 coincide.}

In particular to a zero geometric form of degree 1 corresponds a null system.

On the basis of the definition of equivalence of material systems of points, we can observe that a geometric form of degree 1 with a non-vanishing mass $\alpha$ coincides with a weighted point $G$ of mass $\alpha$; therefore $G$ is the physical barycenter.

Systems of (applied) forces and equilibrium. Following Grassmann, Peano relates geometric forms $s := \sum \alpha_i P_i Q_i$ of degree 2 to systems of applied forces $\{F_i\}$.

In Mechanics the equivalence of two systems of forces relies on the equality of the so-called axial moments of the two systems with respect to an arbitrary axis. It is evident that the axial moment is proportional to the quantity

\[
\sum \alpha_i \text{vol}_n(P_i, Q_i, A, B)
\]

where $A, B$ are two distinct points of the axis. Therefore

\textit{two systems of applied forces are (mechanically) equivalent if and only if the corresponding geometric forms of degree 2 coincide.}

This fact follows easily by the definition of geometric form.

As a consequence of this correspondence between systems of applied forces and geometric forms, Grassmann and Peano derive some well known statements concerning equivalence of systems of applied forces, reducibility and equilibria:

\footnote{More precisely to the bipoint $\alpha_i P_i Q_i$, it is associated the force $F_i$ given by the vector $\alpha_i (Q_i - P_i)$ applied in $P_i$, and, consequently, to a geometric form $\sum \alpha_i P_i Q_i$, the system of applied forces $\{F_i\}$.}

\footnote{It is worth noticing that the effective value of axial moment is a scalar which depends on the choice of a metric structure of the space.}
(7.3) a system of applied forces is equivalent (in infinite ways) to a combination of a force and a Poinsot pair of forces (Poinsot theorem); 45
(7.4) the sum of finitely many Poinsot pair of forces is a Poinsot pair; 46
(7.5) a planar system of applied forces is equivalent to a force or to a Poinsot pair of forces;
(7.6) a system of parallel applied forces is equivalent to a single force or to a Poinsot pair of forces;
(7.7) a system of applied forces is equivalent to a force or to a Poinsot pair of forces if and only if the scalar invariant is null; 47
(7.8) a system of applied forces is reducible to the sum of two forces;
(7.9) a system of applied forces is reducible in a unique way to a system of forces acting along the edges of a given (non-degenerate) tetrahedron; 49
(7.10) characterization of the barycenter $G$ of a material system of points $\{ (P_i, \alpha_i) \}_i$ of total mass $\sum_i \alpha_i = 1$, through the introduction of a system of parallel forces (*) or a system of concurrent forces (**):

\[ \sum_i \alpha_i P_i = G \iff (*) \sum_i \alpha_i P_i u = Gu \text{ for any vector } u \]
\[ \iff (** \sum_i \alpha_i OP_i = OG \text{ for any point } O. \]

It is surprising that the statements listed above concerning equivalence, reducibility of systems of applied forces (and in general in problems of static equilibrium), do not require any metric concept (i.e., scalar product and vectorial moment). The object of Grassmann-Peano calculus corresponding to the “modern” vectorial moment is the Poinsot pair. In Grassmann-Peano calculus applied forces and their moments are seen homogeneously: they are represented by bipoints and Poinsot pairs, namely both of them as forms of degree 2. Unfortunately there is no trace of this attitude in the modern presentation of the theory of applied forces: forces and moments are (at least for their physical dimensions) inhomogeneous objects.

8. Appendix

All articles of Peano are collected in Opera Omnia 58, a CD-ROM, edited by Roero. Selected works of Peano were assembled and commented in Opere scelte 55 by Cassina, a disciple of Peano. A few have English translations in Selected

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45 Statement is a consequence of the reduction formula 5.16 for geometric forms. Here and in the following “Poinsot pair of forces” means “pair of forces with null resultant”. The concept of “pair of forces with null resultant” is due to Poinsot.

46 It is worth noticing that geometric result 6.20 was proved in a mechanical way by Bellavitis (see 2 (1834) p. 262) using the notion of Poinsot’s pair of forces.

47 In the case of gravitational forces the system of applied forces reduces to a single force applied in the center of mass.

48 Denoting with $s$ the geometric form of degree 2 corresponding to a given system of applied forces, by reduction formula 5.16 there exist a bipoint $a$ and a bivector $\iota$ such that $s = a + \iota$. Observe that $ss = (a + \iota)(a + \iota) = 2a\iota$ and $a\iota$ is proportional to the well known scalar invariant of the system of applied forces; therefore $ss = 0$ if and only if $a = 0$ or $\iota = 0$.

49 This fact follows easily by observing that the space $\mathbb{F}_2$ of geometric forms of degree 2 is 6-dimensional and a basis is provided by the 6 bipoints corresponding to the 6 edges of a given tetrahedron.
Regrettably, even fewer Peano’s articles have a public URL and are freely downloadable. One finds the following articles of Peano:

| in *Opere scelte*, vol. 1: | 38 (1890) |
|--------------------------|----------|
| in *Opere scelte*, vol. 2: | 41 (1894), 40 (1900), 51 (1915), 53 (1917), 51 (1921) |
| in *Opere scelte*, vol. 3: | 57 (1888), 39 (1891), 42 (1894), 43 (1895), 41 (1896), 43 (1898), 52 (1915) |
| in *Selected Works*: | 38 (1890), 44 (1896), 52 (1915), 54 (1915) |

For reader’s convenience, we provide a chronological list of some mathematicians mentioned in the paper, together with biographical sources. The html file with biographies of mathematicians listed below with an asterisk can be attained at University of St Andrews’s web-page http://www-history.mcs.st-and.ac.uk/history/{Name}.html

| Name                | Year       |
|---------------------|------------|
| Arnauld, Antoine    | 1612-1694  |
| Nicole, Pierre      | 1625-1695  |
| Huygens, Christiaan | 1629-1695  |
| Leibniz, Gottfried Wilhelm | 1646-1716 |
| Saccheri, Girolamo Giovanni | 1667-1733 |
| Carnot, Lazare Nicolas Marguérite | 1753-1823 |
| Gergonne, Joseph Diaz | 1771-1859 |
| Poinsot, Louis      | 1777-1859  |
| Möbius, August Ferdinand | 1790-1868 |
| Bellavitis, Giusto   | 1803-1880  |
| Grassmann, Hermann  | 1809-1877  |
| Hankel, Hermann     | 1839-1873  |
| Schlegel, Victor    | 1843-1905  |
| Hyde, Edward Wyllys | 1843-1930  |
| Klein, Felix        | 1849-1925  |
| Peano, Giuseppe     | 1858-1932  |
| Castellano, Filiberto | 1860-1919 |
| Whitehead, Alfred North | 1861-1947 |
| Burali-Forti, Cesare | 1861-1931 |
| Marcolongo, Roberto  | 1862-1943  |
| Vailati, Giovanni   | 1863-1909  |
| Couturat, Louis     | 1868-1914  |
| Burgatti, Pietro    | 1868-1938  |
| Cartan, Élie        | 1869-1951  |
| Maccaferri, Eugenio | 1870-1953  |
| Fehr, Henri         | 1870-1954  |
| Enriques, Federigo  | 1871-1946  |
| Russell, Bertrand Arthur William | 1872-1970 |
| Pensa, Angelo       | 1875-1960  |
| Vacca, Giovanni     | 1875-1953  |
| Boggio, Tommaso     | 1877-1963  |
BOTTASSO, Matteo (1878-1918) (*)
WEYL, Hermann (1885-1955) (*)
FORDER, Henry George (1889-1981), see Newsletter NZ Math. Soc. 27 (1983)
CASSINA, Ugo (1897-1964), see KENNEDY [27]

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