Geometry and probability on the noncommutative 2-torus in a magnetic field

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Abstract
In this work, we describe the geometric and probabilistic properties of a noncommutative 2−torus in a magnetic field. We study the volume invariance, integrated scalar curvature and volume form by using the method of perturbation by inner derivation of the magnetic Laplacian in the noncommutative 2−torus. Then, we analyze the magnetic stochastic process describing the motion of a particle subject to a uniform magnetic field on the noncommutative 2−torus, derive and discuss the related main properties.

Keywords. Noncommutative 2−torus; magnetic Laplacian; quantum stochastic process.

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1 Introduction

Noncommutative geometry (NG), initiated by Alain Connes [5], is an exciting dynamic research area of mathematics with applications in physics, greatly developed over the last decades (see [15], [21] and references therein). The Gelfand Naimark theorem [24] which gives an anti-equivalence between the category of locally compact Hausdorff spaces and the one of commutative $C^*$-algebra $A$, is the starting point of noncommutative topology [23], [24]. The correspondence is given by the map $X \longrightarrow C_0(X)$, where $C_0$ is the algebra of continuous complex valued functions [36]. Thus, one thinks of noncommutative $C^*$-algebras as noncommutative topological spaces and tries to apply topological methods to understand them [36].

Sakamoto and Tanimura in [34], extended the Fourier analysis for the noncommutative $n$-torus in a magnetic field. In particular, they studied the solutions of the Schrödinger equation in a uniform magnetic field for the noncommutative $n$-torus. They defined a magnetic algebra and showed that, when the space of functions is an irreducible representation space of this algebra, it characterizes the quantum mechanics in the magnetic torus. In addition, they solved the eigenvalue problem of the Laplacian for the magnetic torus, and provided simple forms for all eigenfunctions.

In [3], Chakraborty, Goswami, and Sinha studied, in a noncommutative 2-torus, the effect of perturbation by inner derivation on the quantum stochastic process and geometric parameters like the volume and the scalar curvature. Their cohomological calculations showed that the above perturbation produces new spectral triples. They also obtained, for the Weyl $C^*$-algebra $A_{\theta}$, (algebra of noncommutative 2-torus developed in [32]), the Laplacian associated with a natural stochastic process and computed the associated volume form.

The purpose of our paper is to elucidate the geometric and stochastic properties of a noncommutative 2-torus in a uniform magnetic field. Especially, we define a 2-torus noncommutative magnetic Laplacian, and derive the related magnetic quantum stochastic differential equation on the line of the construction given in [3], [17], [18], and [35]. Further, from the Weyl asymptotics for the constructed $C^*$-algebra $D_{\theta}$, we obtain the volume and integrated scalar curvature for the 2-torus, and investigate their invariance under perturbation by inner derivation of the noncommutative magnetic Laplacian. The volume form invariance is also analyzed. Finally, using the magnetic position and momentum operators in a noncommutative 2d space, we establish the associated magnetic quantum stochastic differential equation, and discuss the properties of its solution.

The paper is organized as follows. In section 2, we briefly recall the basic definitions and properties of the magnetic Laplacian. In section 3, we construct and discuss the associated magnetic quantum stochastic diffusion equation. In section 4, we introduce the volume, the integrated scalar curvature using Weyl asymptotics for the $C^*$-algebra of the noncommutative 2-torus, and study their invariance under the perturbation of the noncommutative magnetic Laplacian. In section 5, we investigate the invariance of the volume form under the above perturbation and explore the spectral triple on $D_{\theta}$. The magnetic quantum stochastic differential equation in the case of a noncommutative 2d-dimensional space is investigated and analyzed in Section 6. Section 7 is devoted to some concluding remarks.

2 Basic properties

We consider a finite dimensional real vector space $V$ equipped with a positive defined inner product $(\cdot, \cdot)$. Let $\theta$ be an irrational number in $[0, 1]$. The irrational rotation $C^*$-algebra $D_{\theta}$ is, by
definition, the $C^*$—algebra generated by two unitaries symbols $X$ and $Y$ satisfying [3]

$$XY = e^{2\pi i\theta} Y X.$$  
(2.1)

Let $\mathcal{H}$ be the Hilbert space of square integrable functions $L^2$, and $T^1$ the circle.

There are many important representations $\pi_1$ and $\pi_2$ of the $C^*$—algebra $\mathcal{D}_\theta$, given by the following relations [3,36]:

- Let $\mathcal{H} = L^2(T^1)$, $g \in \mathcal{H}$, $\lambda \in \mathbb{C}$, and $v \in T^1$, then

$$\langle \pi_1(X)g(v) \rangle = g(\lambda v), \quad \langle \pi_1(Y)g(v) \rangle = vg(v).$$  
(2.2)

- Let $\mathcal{H} = L^2(T^1)$, $g \in \mathcal{H}$, and $v \in T^1$, then

$$\langle \pi_2(Y)g(v) \rangle = g(\overline{\lambda}v), \quad \langle \pi_2(X)g(v) \rangle = vg(v).$$  
(2.3)

- Let $\mathcal{H} = L^2(\mathbb{R})$, $g \in \mathcal{H}$, and $u \in \mathbb{R}$, then

$$\langle \pi_3(X)g(u) \rangle = g(u + 1), \quad \langle \pi_3(Y)g(u) \rangle = \lambda^u g(u).$$  
(2.4)

There is a continuous action of $T^2$, $T = \mathbb{R}/2\pi\mathbb{Z}$, on $\mathcal{D}_\theta$ by $C^*$—algebra automorphism $\{\alpha_x\}$, $x \in \mathbb{R}^2$, defined as follows [10]

$$\alpha_x(X^{n_1}Y^{n_2}) = e^{ix(n_1,n_2)}X^{n_1}Y^{n_2},$$  
(2.5)

where $n_1, n_2 \in \mathbb{Z}$.

The space of smooth elements for this action, that is those elements $a \in \mathcal{D}_\theta$ for which the map $x \mapsto \alpha_x(a)$ is $C^\infty$, will be denoted by $\mathcal{D}_\theta^\infty := C^\infty(T^2_\theta)$. It is a dense $*$-subalgebra of $\mathcal{D}_\theta$ and can alternatively be described as the space of elements of the form $\sum_{n_1,n_2 \in \mathbb{Z}} a_{n_1,n_2}X^{n_1}Y^{n_2}$, $a_{n_1,n_2} \in \mathbb{R}^*$ with rapidly decreasing coefficients [13]:

$$\mathcal{D}_\theta^\infty \equiv \left\{ \sum_{n_1,n_2 \in \mathbb{Z}} a_{n_1,n_2}X^{n_1}Y^{n_2}; \quad \sup_{m,n \in \mathbb{Z}} (|n_1|^k |n_2|^q |a_{n_1,n_2}|) < \infty, \forall k,q \in \mathbb{Z} \right\}.$$  
(2.6)

There exists a unique faithful trace $\varphi$ on $\mathcal{D}_\theta$ defined as follows

$$\varphi\left( \sum_{n_1,n_2 \in \mathbb{Z}} a_{n_1,n_2}X^{n_1}Y^{n_2} \right) = a_{00}. $$  
(2.7)

We consider the inner product

$$\langle u, v \rangle = \varphi(v^* u), \quad u,v \in \mathcal{D}_\theta.$$  
(2.8)

The derivations $d_j : \mathcal{D}_\theta^\infty \rightarrow \mathcal{D}_\theta^\infty$, $j \in \{1,2\}$ associated to the group of automorphisms $\{\alpha_x\}$ are given by the following relations [3],

$$d_1(X) = X, \quad d_1(Y) = 0,$$  
(2.9)
\[ d_2(X) = 0, \quad d_2(Y) = Y. \quad (2.10) \]

A theorem by Bratteli, Elliot, and Jorgensen [2] describes all the derivations of \( \mathcal{D}_\theta \) which map \( \mathcal{D}_\theta^\infty \) to itself: for almost all \( \theta \) (Lebesgue), a derivation is of the form
\[ d = \sum_{j=1}^{2} b_j d_j + [r, \cdot], \quad \text{with} \quad r \in \mathcal{D}_\theta^\infty, \quad b_j \in \mathbb{C}. \quad (2.11) \]

Consider an orthonormal basis \( e_1, e_2 \) for \( V \). Then the restriction of the above map \( d \) defines commuting derivations \( d_i := d(e_i): \mathcal{D}_\theta^\infty \to \mathcal{D}_\theta^\infty, i \in \{1, 2\} \) satisfying
\[ d_i(X_j) = d_{ij}X_i, \quad (i, j) \in \{1, 2\}. \quad (2.12) \]

The \( d_j \)'s are the analogues of the differential operators \( \frac{1}{i} \partial / \partial u_j \) acting on smooth functions on the ordinary torus. We have
\[ d_j(a^\ast) = -d_j(a)^\ast, \quad (2.13) \]

for \( j \in \{1, 2\} \) and \( a \in \mathcal{D}_\theta^\infty \). Moreover, since \( \varphi \circ d_j = 0 \), for all \( j \), we have the analogue of the formula for integration by parts:
\[ \varphi(ad_j(b)) = -\varphi(d_j(a)b), \quad a, b \in \mathcal{D}_\theta^\infty. \quad (2.14) \]

Using all these derivations, we can define the Laplacian \( \triangle : \mathcal{D}_\theta^\infty \to \mathcal{D}_\theta^\infty \) such that
\[ \triangle \equiv \sum_{i=1}^{2} d_i^2. \quad (2.15) \]

We note that the Laplacian \( \triangle \) is independent of the orthonormal basis's choice \((e_1, e_2)\). All the above derivations are studied in the Hilbert space \( \mathcal{H} = L^2(\mathcal{D}_\theta, \varphi) \) and the family \( \{V^l\}_{l \in \mathbb{Z}_n} \) forms an orthonormal basis in \( \mathcal{H} \) [28, 36].

**Theorem 2.1** [3] The canonical derivations \( d_j, j \in \{1, 2\} \) are self-adjoint on their domains:
\[ \text{Dom}(d_1) = \left\{ \sum a_{n_1 n_2} X^{n_1} Y^{n_2} \mid \sum (1 + n_1^2)|a_{n_1 n_2}|^2 < \infty \right\} \quad (2.16) \]
\[ \text{Dom}(d_2) = \left\{ \sum a_{n_1 n_2} X^{n_1} Y^{n_2} \mid \sum (1 + n_2^2)|a_{n_1 n_2}|^2 < \infty \right\}, \quad (2.17) \]
where \( \text{Dom}(d) \) is the domain of \( d \).

Moreover if we denote by \( d_r := [r, \cdot] \) with \( r \in \mathcal{D}_\theta \subset L^\infty(\mathcal{D}_\theta, \varphi) \) acting as left multiplication in \( \mathcal{H} \), then \( d_r^\ast = d_r \in \mathcal{B}(\mathcal{H}) \), the space of bounded functions on \( \mathcal{H} \).
3 Noncommutative magnetic Laplacian and diffusion on \( D_\theta \)

3.1 Construction of the magnetic quantum stochastic diffusion

Let us now derive the quantum stochastic differential equation on the \( C^* \)– algebra \( D_\theta \) following the canonical construction of quantum stochastic flow or diffusion on a von Neumann or a \( C^* \)– algebra \( A \) associated with a completely positive semigroup on \( A \). For more details on such a canonical construction, see [17], [18] and references therein. Sauvageot in [35] studied semi-groups which have a local generator \( L \) characterized by:

- \( D \subseteq \text{Dom}(L) \subseteq A \subseteq B(H) \), dense in \( A \) such that \( D \) itself is a \(*\)-algebra;
- A \(*\)-representation \( \pi \) in some Hilbert space \( H_1 \) and an associated \( \pi \) derivation \( \delta \) such that \( \delta(x) \in B(H, H_1) \) and \( \delta(xy) = \delta(x)y + \pi(x)\delta(y) \);
- A second order cocycle relation: \( \mathcal{L}(x^*y) - \mathcal{L}(x)^*y - x^*\mathcal{L}(y) = \delta(x)^*\delta(y), \) for \( x, y \in \mathcal{D} \).

By analogy to the classical case, \( \mathcal{L} \) is called the noncommutative Laplacian or Lindbladian.

In this work concerning the magnetic field in the noncommutative 2–torus, we consider the notation \( d = \frac{\partial}{\partial t} \), the coordinate \((t_1, t_2)\) and \( d_j = \frac{\partial}{\partial t_j} \), for each \( j \in \{1, 2\} \). We denote by \( U(1) \) the unitary symmetric group.

**Definition 3.1** The component of the \( U(1) \) gauge field is defined as follows:

\[
G_k = \frac{1}{2} \sum_{j=1}^{2} \psi_{jk} t_j + \beta_k, \tag{3.1}
\]

where \( k \in \{1, 2\} \), \( \{\beta_k\} \) are real numbers and \( \{\psi_{jk}\} \) are integers such that \( \{\psi_{jk} = -\psi_{kj}\} \).

The gauge \( G = \sum_{k=1}^{2} G_k dt_k \) generates a uniform magnetic field

\[
B = \frac{1}{2} \sum_{j=1}^{2} \sum_{k=1}^{2} \psi_{jk} dt_j \wedge dt_k, \tag{3.2}
\]

**Definition 3.2** The magnetic Laplacian in a noncommutative 2–torus, \( \Delta^m : D^\infty_\theta \rightarrow D^\infty_\theta \) is defined by:

\[
\Delta^m := \sum_{j,k=1}^{2} g^{jk} (d_j - 2\pi i G_j) (d_k - 2\pi i G_k), \tag{3.3}
\]

where \( g^{jk} \) is a metric.

We denote by \( \mathcal{L}^m \) the noncommutative magnetic Laplacian.

**Definition 3.3** The unperturbed and perturbed noncommutative magnetic Laplacians in a noncommutative 2–torus are given, respectively, by:

\[
\mathcal{L}^m_0 := \sum_{j,k=1}^{2} g^{jk} (d_j - 2\pi i G_j) (d_k - 2\pi i G_k), \tag{3.4}
\]
and
\[ L^m := \sum_{j,k=1}^{2} g^{jk} (\delta_j - 2\pi iG_j) (\delta_k - 2\pi iG_k), \] (3.5)

where \( \delta_j = d_j + d_{r_j} \) with \( j \in \{1, 2\}; r \in D_\theta \) and \( d_j \) is the canonical derivation.

The following Lemma gives the relation between the noncommutative magnetic Laplacian \((NCML)\), \( L^m_0 \) (resp. \( L^m \)), and the ordinary noncommutative Laplacian \((NCL)\), \( L_0 \) (resp. \( L \)).

**Lemma 3.4** The unperturbed and perturbed noncommutative magnetic Laplacians are given, respectively, by:

- \[ L^m_0 = L_0 + T^0_{jk}, \] (3.6)
  
  where
  \[ T^0_{jk} = \sum_{j=1}^{2} (\pi iG_j d_j + 2\pi^2 G^2_j) + \sum_{j \neq k} g^{jk} (d_j - 2\pi iG_j)(d_k - 2\pi iG_k), \] (3.7)

and

- \[ L^m = L + T_{jk}, \] (3.8)
  
  with
  \[ T_{jk} = \sum_{j=1}^{2} (\pi iG_j \delta_j + 2\pi^2 G^2_j) + \sum_{j \neq k} g^{jk} (\delta_j - 2\pi iG_j)(\delta_k - 2\pi iG_k). \] (3.9)

**Proof:**

- Let \( f \) be the test function. Then, according to the relation (3.4), and taking the metric \( g^{jj} = -1/2 \), we obtain
  \[ \sum_{j=1}^{2} g^{jj} (d_j - 2\pi iG_j)^2 f = L_0 f + \sum_{j=1}^{2} (\pi iG_j d_j + 2\pi^2 G^2_j) f. \] (3.10)

Thus,

\[ L^m_0 f = L_0 f + \sum_{j=1}^{2} (\pi iG_j d_j + 2\pi^2 G^2_j) f + \sum_{j \neq k} g^{jk} (d_j - 2\pi iG_j)(d_k - 2\pi iG_k) f, \] (3.11)

yielding the relation (3.7).

- By Definition 3.5 we get
  \[ \sum_{j=1}^{2} g^{jj} (\delta_j - 2\pi iG_j)^2 f = \sum_{j=1}^{2} g^{jj} \left( \delta_j (\delta_j f) - \delta_j (2\pi iG_j f) - 2\pi iG_j \delta_j f - 4\pi^2 G^2_j f \right). \] (3.12)
The computation of the terms in the right hand side of the above equation gives
\[
\sum_{j=1}^{2} g^{jj} (\delta_j - 2\pi i G_j)^2 f = \sum_{j=1}^{2} g^{jj} \left( \delta_j^2 f - 2\pi i G_j \delta_j f - 4\pi^2 G_j^2 f \right)
\]  
(3.13)
and the result holds. \[\square\]

The next Proposition gives the construction of the unperturbed and perturbed magnetic quantum processes.

**Proposition 3.5** Let \( \mathcal{D}_0^\infty \) be a \(*\)-subalgebra of \( \mathcal{D}_\theta \). Consider a Hilbert space \( \mathcal{H} \) and a \(*\)-representation \( \pi \) in \( \mathcal{H}_1 = \mathcal{H} \otimes \mathbb{C}^2 \cong \mathcal{H} \oplus \mathcal{H} \) defined by \( \pi(y) = y \otimes I \). Then, the unperturbed and perturbed magnetic quantum processes are driven by the triples \((\pi, \delta_0^m, L_0^m)\) and \((\pi, \delta^m, L^m)\), respectively.

**Proof**: 

(i) Unperturbed magnetic quantum diffusion

- It is obvious to have \( \mathcal{D}_0^\infty \subseteq \text{Dom}(L_0^m) \subseteq \mathcal{D}_\theta \subseteq \mathcal{B}(\mathcal{H}) \).
- Taking \( \delta_0^m = \sum_{j=1}^{2} d_j \), one proves that, for \( u, v \in \mathcal{D}_0^\infty \), \( \delta_0^m(u) \in \mathcal{B}(\mathcal{H}, \mathcal{H}_1) \) and
  \[
  \delta_0^m(uv) = \delta_0^m(u)v + \pi(u)\delta_0^m(v). 
  \]  
(3.14)
- Moreover, using Lemma 3.4 and according to [3], we get
  \[
  L_0^m(u^*v) = u^*L_0(v) + L_0(u)^*v + \delta_0^m(u)^*v + T_{jk}^0(u^*v). 
  \]  
(3.15)
It suffices to verify \( T_{jk}^0(u^*v) \). Using the same idea as in the above part, we get
\[
\sum_{j \neq k} g^{jk} (d_j - 2\pi i G_j)(d_k - 2\pi i G_k) = \sum_{j \neq k} g^{jk} \left( d_j d_k - \Gamma_{jk} - 2\pi i G_j d_k - 4\pi^2 G_j G_k \right). 
\]  
(3.16)
with \( \Gamma_{jk} = \pi i \sum_{j=1}^{n} \psi_{jk} f \). Setting \( C_1 = 4\pi^2 G_j^2 \) and \( C_2 = 4\pi^2 G_j G_k + \Gamma_{jk} \) and using the assumption that \( C_1 \) and \( C_2 \) satisfy the Leibniz rule, we obtain
\[
T_{jk}^0(u^*v) = T_{jk}^0(u^*v) + u^*T_{jk}^0(v) 
\]  
(3.17)
and therefore, the result holds.

(ii) Perturbed magnetic quantum diffusion

For the first relation, we perform the same computation as above and obtain:

- \( \mathcal{D}_0^\infty \subseteq D(L^m) \subseteq \mathcal{D}_\theta \subseteq \mathcal{B}(\mathcal{H}) \).

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• It is obvious to have \( \delta^m(u) \in \mathcal{B}(\mathcal{H}, \mathcal{H}_1) \) and
\[
\delta^m(uv) = \delta^m(u)v + \pi(u)\delta^m(v),
\]
with \( \delta^m = \sum_{j=1}^{2}(d_j + dr_j) \).

• According to the expression of the perturbed magnetic Laplacian \( \mathcal{L}^m \), the Lemma 3.4 and [3], we have
\[
\mathcal{L}^m(u^*v) = u^*\mathcal{L}(v) + \mathcal{L}(u^*)v + \delta(u)^*\delta(v) + T_{jk}(u^*v) \quad \text{for} \quad (u, v) \in \mathcal{D}_\theta^\infty. \tag{3.19}
\]
Besides,
\[
T_{jk}(u^*v) = \sum_{j=1}^{2} \left( \pi i G_j \delta_j + 2\pi^2 G_j^2 \right)(u^*v) + \sum_{j \neq k} g^{jk} \left( \delta_j - 2\pi i G_j \right) \left( \delta_k - 2\pi i G_k \right)(u^*v). \tag{3.20}
\]
As \( d_j \) is a derivation and \( d_r = [r, \cdot] \), \( r \in \mathcal{D}_\theta \), we obtain the same result as in equation (3.16). Thus, considering the assumption as in the case of \( \mathcal{L}^m_0 \), and after some algebra, we obtain
\[
T_{jk}(u^*v) = T_{jk}(u^*)v + u^*T_{jk}(v). \tag{3.21}
\]
Therefore, the result holds.

Both the unperturbed and perturbed triples \((\pi, \delta^m_0, \mathcal{L}^m_0)\) and \((\pi, \delta^m, \mathcal{L}^m)\) satisfy previous properties (1), (2), (3) giving rise to two quantum stochastic processes as stated by the theorem established in the following section.

### 3.2 Magnetic quantum stochastic differential equation

In this section, we derive the magnetic quantum stochastic differential equation arising from the unperturbed and perturbed magnetic quantum stochastic processes, respectively. More details on the quantum stochastic process can be found in [7, 8, 22, 29]. Let us first recall the following important statement used in the sequel:

**Theorem 3.6 [3]**

1. The quantum stochastic differential equation for \( y \in \mathcal{D}_\theta^\infty \)
\[
\begin{cases}
  d^0_j(y) = \sum_{k=1}^{2} j^0_k(id_k(y))dw_k(t) + j^0_t(\mathcal{L}_0(y))dt, \\
  j^0_0(y) = y \otimes I,
\end{cases} \tag{3.22}
\]

has unique solution \( j^0_t \) which is a \( * \)-homomorphism from \( \mathcal{D}_\theta \) to \( \mathcal{D}_\theta \otimes \mathcal{B}(\Gamma(L^2(R_+ \otimes C^2)) \). Furthermore \( j^0_t(y) = \alpha_{(exp2\pi i(w_1(t), exp2\pi i w_2(t))(y)) \text{ where} (w_1, w_2)(t) \text{ is the classical Brownian motion in dimension two and} E j^0_t(y) = e^{t\mathcal{L}_0}(y) \text{ where} E \text{ is the vacuum expectation in the Fock space} \Gamma(L^2(R_+) \otimes C^2). \)
2. The quantum stochastic differential equation in $\mathcal{H} \otimes \Gamma$:
\[
\begin{array}{l}
\begin{align*}
\frac{dU_t}{dt} &= \sum_{k=1}^{2} U_t \left\{ ij_t^0 (r_k) dA_k^\dagger + ij_t^0 (r_k^*) dA_k - \frac{1}{2} j_t^0 (r_k^* r_k) dt \right\}, \\
U_0 &= I,
\end{align*}
\end{array}
\tag{3.23}
\]
has a unique unitary solution $[8]$. Putting $j_t(y) = U_t j_0(y) U_t^*$, we obtain the quantum stochastic differential equation:
\[
\frac{dj_t}{dt}(y) = \sum_{k=1}^{2} \left\{ j_t(i \delta_k(y)) dA_k^\dagger + j_t(i \delta_k^*(y)) dA_k \right\} + j_t(\mathcal{L}(y))
\tag{3.24}
\]
and $E j_t(y) = e^{t\mathcal{L}(y)}$.

Then our main result in this section can be formulated as follows:

**Theorem 3.7**

(i) Let $j_0^0(x)$ be the solution of the quantum stochastic differential equation (3.22). Then, the magnetic quantum stochastic differential equation for $x \in \mathcal{D}_\theta$:
\[
\begin{array}{l}
\begin{align*}
\frac{df_t^0}{dt}(x) &= df_t^0(x) + f_t^0(\mathcal{T}_{j_k^0}(x)) dt, \quad (j, k) \in \{1, 2\} \\
f_t^0(x) &= x \otimes I,
\end{align*}
\end{array}
\tag{3.25}
\]
admits a unique solution $f_t^0$ which is a $*$-homomorphism defined as follows:
\[
f_t^0 : \mathcal{D}_\theta \longrightarrow \mathcal{D}_\theta \otimes \mathcal{B} \left( \Gamma(L^2(\mathbb{R}^+)) \otimes \mathbb{C}^2 \right)
\]
\[
f_t^0(x) = j_t^0(x) + F_t^0(\mathcal{T}_{j_k^0}(x)),
\tag{3.26}
\]
where $F_t^0$ is a primitive of $f_t^0$.

Moreover, the vacuum expectation in the Fock space $\Gamma(L^2(\mathbb{R}^+)) \otimes \mathbb{C}^2)$ is given by:
\[
Ef_t^0(x) = \exp(t \mathcal{L}_0)(x) + F_t^0(\mathcal{T}_{j_k^0}(x)), \quad (j, k) \in \{1, 2\}.
\tag{3.27}
\]

(ii) Let $U_t$ be the solution of the quantum stochastic differential equation (3.23). Then, the magnetic quantum stochastic differential equation in $\mathcal{H} \otimes \Gamma$:
\[
\begin{array}{l}
\begin{align*}
\frac{dV_t}{dt} &= dU_t + V_t f_t^0(R_{lk}) dt \\
V_0 &= I,
\end{align*}
\end{array}
\tag{3.28}
\]
where
\[
R_{lk} = -1/2 \sum_{l=1}^{2} (r_l^*(2\pi i G_l) + 2\pi i G_l r_l) + \sum_{l \neq k} (r_l^* - 2\pi i G_l)(r_k - 2\pi i G_k),
\tag{3.29}
\]
has a unique unitary solution.

Putting $f_t(x) = V_t f_t^0(x) V_t^*$ leads to the following magnetic quantum stochastic differential equation:
\[
\frac{df_t}{dt}(x) = df_t(x) + f_t(\tilde{R}_{lk}(x)) dt, \quad (l, k) \in \{1, 2\},
\tag{3.30}
\]
where $\tilde{R}_{lk}(x) = R_{lk}x + xR^*_{lk}$, and $g_t(x)$ satisfies the quantum stochastic differential equation.

Furthermore, the solution of (3.30) and the vacuum expectation are given by:

$$f_t(x) = j_t(x) + F_t(\tilde{R}_{lk}(x)), \quad (3.31)$$

and

$$Ef_t(x) = \exp(t\mathcal{L})(x) + F_t(\tilde{R}_{lk}(x)), \quad (3.32)$$

respectively; $F_t$ is a primitive of $f_t$.

Proof:

(i) From the construction of the unperturbed magnetic quantum process, we get

$$
\begin{cases}
    df_t^0(x) = \sum_{k=1}^{2} f_t^0(id_k(x))dw_k(t) + f_t^0(\mathcal{L}_0^m(x))dt \\
    f_0^0(x) = x \otimes I,
\end{cases}

(3.33)$$

where $(w_k)_{k \in \{1, 2\}}$ is a classical 2-dimensional Brownian motion.

Since the noncommutative Brownian motion is $dw = dA + dA^\dagger$, thus the equation (3.33) is equivalent to

$$
\begin{cases}
    df_t^0(x) = f_t^0(\delta^\dagger(x))dA + f_t^0(\delta(x))dA^\dagger + f_t^0(\eta(x))dt, \\
    f_0^0(x) = x \otimes I,
\end{cases}

(3.34)$$

where $\delta$ and $\eta$ are the map structures satisfying

$$\eta(xy) = \eta(x)y + x\eta(y) + \delta^\dagger(x)\delta(y). \quad (3.35)$$

Moreover, if $\delta$ and $\eta$ are bounded in $\mathcal{B}(\mathcal{H})$, then there exists a unique quantum diffusion $f^0(t)$ which satisfies (3.34) [8]. In our case,

$$\delta(x) = \sum_{k=1}^{2} id_k(x) \quad \text{and} \quad \eta(x) = \mathcal{L}_0^m(x), \quad (3.36)$$

both satisfying

$$\mathcal{L}_0^m(x^*y) - \mathcal{L}_0^m(x^*)y - x^*\mathcal{L}_0^m(y) = \delta(x)^*\delta(y). \quad (3.37)$$

since $d_j : \mathcal{D}_\theta^\infty \rightarrow \mathcal{D}_\theta^\infty$, $j \in \{1, 2\}$ are the canonical derivations. By definition of $\mathcal{L}_0^m$, we deduce that both the two operators are bounded. Therefore, the equation (4.48) admits a unique quantum diffusion $f^0_t$.

According to the Lemma (3.4), the equation (3.33) can be also written as follows:

$$
\begin{cases}
    df_t^0(x) = \sum_{k=1}^{2} f_t^0(id_k(x))dw_k(t) + f_t^0(\mathcal{L}_0(x))dt + f_t^0(T^0_{jk}(x))dt \\
    f_0^0(x) = x \otimes I.
\end{cases}

(3.38)$$

According to [3], the equation (3.25) holds, since $dg_t^0(x)$ admits a unique solution [3]. Therefore, the equation (3.25) has a unique solution.
Using the relation (3.33) and the linearity of the vacuum expectation, we obtain

\[ Ef_0^0(x) = f_0^0(x) + \int_0^t E \left( h_0^0(L_0^m(x)) \right) ds. \] (3.39)

A similar computation as in [29] implies that

\[ Ef_0^0(x) = e^{tL_0^m}(x). \] (3.40)

Besides, as \( f_0^0(x) = g_0^0(x) \), the integral form of (3.25) is given by:

\[ f_0^t(x) = j_0^t(x) + \int_0^t f_0^s(T_{jk}^0(x)) ds, \] (3.41)

and considering the primitive \( F_0^t \) of \( f_0^t \), we have (3.26). Using the linearity of the vacuum expectation, we obtain

\[ Ef_0^0(x) = Ej_0^0(x) + EF_0^0(T_{jk}^0(x)), \] (3.42)

and, according to [3], we obtain the result.

(ii) By definition of the quantum Brownian motion and setting \( d_l = r_l \), for \( l \in \{1, 2\} \), we obtain, from the relation (3.33), the following equation:

\[
\begin{cases}
  dV_t = \sum_{l=1}^{2} V_t \left( if_0^l(r_l) dA_l^\dagger + if_0^l(r_l^*) dA_l + f_0^l(L_0^m) dt \right) \\
  V_0 = I,
\end{cases}
\] (3.43)

Setting

\[ dV_t^k = V_t \left\{ ih_0^0(r_k) dA_k^\dagger + ih_0^0(r_k^*) dA_k + f_0^0(L_0^m) dt \right\}, \] (3.44)

where \( k \in \{1, 2\} \), then

\[ dV_t = dV_t^1 + dV_t^2. \] (3.45)

But, the equation \( dV_t^1 \) is equivalent to

\[ dV_t^1 = V_t f_0^0(\lambda_1^k) d\Lambda_1^\mu, \] (3.46)

and using [8], we conclude that (3.46) admits a unique solution, and for \( k \in \{1, 2\} \)

\[ dV_t^k = V_t h_0^0(\lambda_1^k) d\Lambda_1^\mu \] (3.47)

admits a unique solution. Therefore, (3.43) admits a unique solution.

Now, we show the unitarity of the solution. As \( V_t \) is solution of the (3.43), the adjoint \( V_t^\ast \) is also solution of the equation (3.43), and we have

\[ dV_t^\ast = \sum_{l=1}^{2} (-if_0^l(r_l) dA_l - if_0^l(r_l^*) dA_l^\dagger + f_0^0(L_0^m) V_t^\ast. \] (3.48)

Using Itô’s formula and table, we have \( d(V_t V_t^\ast) = 0 \). Thus, \( V_t V_t^\ast = V_0 \), but \( V_0 = I \), and, therefore, the result follows.
Furthermore, taking the metric \( g^l = -1/2 \), for \( l \in \{1, 2\} \) and after computation, we obtain
\[
dV_t = dU_t + V_t f^0_t(R_{lk}) dt.
\]
Thus, we obtain the relation (3.28). Using [3], the result holds.

Putting \( f_t(x) = V_t f^0_t(x) V^*_t \), and according to Itô’s formula and table [8]
\[
df_t(x) = 2 \sum_{l=1}^2 f_t(i[r_l, x]) dA_l^t + 2 \sum_{l=1}^2 f_t(L^m(x)) dt,
\]
with \( L^m(x) = x H_{lk} - H^*_k x + x r^*_l r_l \). According to [3], we obtain the relation (3.30)
Moreover, using the integral form and the linearity of the vacuum expectation, we get (3.31) and (3.32), respectively.

\[\square\]

4 Weyl asymptotics for \( D_\theta \)

Let us define the volume and the integrated scalar curvature of \( D_\theta \) using the Weyl asymptotics, and their invariance under the perturbation from \( L^m_0 \) to \( L^m \). For a general discussion on Weyl asymptotics for torus, see [35], [36], and [5] (and references therein).

The magnetic expectation of the Brownian motion on the manifold is considered as the magnetic heat semigroup \( T^m_t \) with the magnetic Laplace-Beltrami operator \( \Delta^m \) as its generator. According to [33], we consider that \( T^m_t \) is a magnetic integral operator on the space of square integrable functions on \( M \) to \( dvol, (L^2(M, dvol)) \), and has the magnetic integral kernel \( T^m_t(s, h) \).

The volume of the manifold is defined as follows [36]:
\[
Vol(M) := \int_M T^m_t(0, e, e) dvol(e).
\]

Using [33], the smooth integral kernel admits an asymptotic expansion as \( t \to 0^+ \). Thus,
\[
Vol(M) = \int_M \left( T^m_t(e, e) t^{d/2} - \sum_{l=1}^\infty T^m_l(e, e) t^{-d/2+l} \right) dvol(e) = \lim_{t \to 0^+} t^{d/2} \int_M T^m_t(e, e) dvol(e).
\]

Taking the trace in the space \( (L^2(M, dvol)) \), we obtain
\[
Vol(M) = \lim_{t \to 0^+} t^{d/2} T_r T^m_t.
\]

Moreover, the scalar curvature at \( e \in M \) is given by \( s(e) = 1/6 T^m_t(1, e, e) \).

Thus, the integrated scalar curvature is defined as follows:
\[
s = \int_M s(e) dvol(e) = \frac{1}{6} \lim_{t \to 0^+} t^{d/2-1} [T_r T^m_t - t^{-d/2} Vol(M)].
\]
In the classical case, the magnetic heat semigroup $T^m_t$ is replaced by the unperturbed magnetic expectation $e^{tL^0_m}$ and the perturbed magnetic expectation $e^{tL^m}$, respectively. According to Lemma 3.4 and the fact that $L^0_m$ and $T^0_{jk}$ are two commuting operators, we have

$$T^m_t = e^{T^0_{jk} t} T^m_t = e^{tT_{jk} T_t}, \quad (4.5)$$

For the estimation of the magnetic expectations, we need to study both the unperturbed and perturbed magnetic Laplacians. In the sequel, we denote by $B_p$ the Schatten ideals in $B(H)$.

Before giving the Proposition summarizing the properties of the unperturbed and perturbed noncommutative magnetic operators in the case of the noncommutative $2-$ torus, let us recall the following important definition and theorem.

**Definition 4.1** [31] Let $T : \text{Dom}(T) \to H$ be a self-adjoint operator and let $F : \text{Dom}(F) \to H$ be symmetric. We say that $F$ is $T-$ bounded with bound $\alpha > 0$ if $\text{Dom}(T) \subset \text{Dom}(F)$ and there exists $\beta > 0$ so that:

$$\|Fh\| \leq \alpha \|Th\| + \beta \|h\|, \quad (4.6)$$

for all $h \in \text{Dom}(T)$.

**Theorem 4.2 (Kato-Rellich theorem)** [31] Suppose that $T$ is self-adjoint, $F$ is symmetric, and $F$ is $T-$ bounded with relative bound $\alpha \leq 1$. Then, $T + F$ is self-adjoint on $\text{Dom}(T)$ and essentially self-adjoint on any core of $T$. Further, if $T$ is bounded below by $M$, then, $T + F$ is bounded below by $M - \text{Max}\{\frac{\beta}{1-\alpha}, \alpha |M| + \beta\}$.

**Proposition 4.3**

1. $L^m_0$ is a self-adjoint operator in $L^2(\varphi)$ and for $n_1, n_2 \in \mathbb{Z}$, we have

$$L^m_0(X^{n_1} Y^{n_2}) = \left(\frac{1}{2}(n_1^2 + n_2^2 + \eta_{jk}(n_1, n_2))\right)(X^{n_1} Y^{n_2}), \quad (4.7)$$

where

$$\eta_{jk}(n_1, n_2) = \sum_{j=1}^{2} \left(\pi iG_j n_j + 2\pi^2 G^2_j\right) + \sum_{j \neq k} g^{jk}(n_j - 2\pi iG_j)(n_k - 2\pi iG_k). \quad (4.8)$$

2. For $r_j \in D^\infty_\theta$, $j \in \{1, 2\}$, and self-adjoint, then

(i) $$L^m = L^m_0 + T^1_{jk} + T^2_{jk}, \quad (4.9)$$

with

$$T^1_{jk} = \sum_{j=1}^{2} g^{jj}(d^2_{r_j} - 2\pi iG_j d_{r_j}) - \sum_{j \neq k} g^{jk}(2\pi iG_k d_{r_j} + 2\pi iG_j d_{r_k}) \quad (4.10)$$

or

$$T^1_{jk} = \sum_{j=1}^{2} g^{jj}(d^2_{r_j} + d_{d_{(r_j)}} - 2\pi iG_j d_{r_j}) - 2\pi i \sum_{j \neq k} g^{jk}(G_k d_{r_j} + G_j d_{r_k}), \quad (4.11)$$
and

\[
T_{jk}^2 = \begin{cases} 
  \sum_{j=1}^{2} g^{ij} (d_{r_j} d_j + d_j d_{r_j}) + \sum_{j \neq k} g^{jk} (d_j d_{r_k} + d_{r_j} d_k + d_{r_j} d_{r_k}) \\
  \text{or} \\
  2 \sum_{j=1}^{2} g^{ij} (d_{r_j} d_j + \sum_{j \neq k} g^{jk} (d_j d_{r_k} + d_{r_j} d_k + d_{r_j} d_{r_k})) .
\end{cases}
\tag{4.12}
\]

(ii) \( T_{jk}^2 \) is compact relative to \( \mathcal{L}_0^m \).
(iii) \( \mathcal{L}^m \) is self-adjoint on \( \text{Dom}(\mathcal{L}_0^m) \) and has a compact resolvent.

Proof:

1. According to Lemma 3.4 and the fact that \( d_j, j \in \{1, 2\} \) are self-adjoint on their domains, we deduce that for \( (j, k) \in \{1, 2\} \), \( T_{jk}^0 \) is self-adjoint on \( \mathcal{D}_0^\infty \subseteq L^2(\varphi) \) and using [3], we conclude that \( \mathcal{L}_0^m \) is self-adjoint. Besides, according to [3], we know the expression of \( \mathcal{L}_0(X^{n_1}Y^{n_2}) \) for \( n_1, n_2 \in \mathbb{Z} \) and \( (X, Y) \in \mathcal{D}_\theta \). It is obvious to compute \( T_{jk}^0(X^{n_1}Y^{n_2}) \) and the relation (4.7) holds.

2. (i). Since \( \delta = d + d_r \), and according to (3.5), we obtain, respectively,

\[
\sum_{j=1}^{2} g^{ij} (\delta_j - 2\pi i G_j)^2 f = \sum_{j=1}^{2} g^{ij} \left( d_j^2 + d_{r_j}^2 + d_j d_{r_j} + d_{r_j} d_j \right)
- \ 2\pi i G_j d_j - 2\pi i G_j d_{r_j} - 4\pi^2 G_j^2 f \tag{4.13}
\]

and

\[
\sum_{j \neq k} g^{jk} (\delta_j - 2\pi i G_j) (\delta_k - 2\pi i G_k) f = \sum_{j \neq k} g^{jk} \left( d_j d_k + d_j d_{r_k} \\
+ d_{r_j} d_k + d_{r_j} d_{r_k} \\
- \ 2\pi i d_j G_k - 2\pi i G_j d_k \\
- \ 4\pi^2 G_j G_k - 2\pi i d_{r_j} G_k \\
- \ 2\pi i G_j d_{r_k} \right) f . \tag{4.14}
\]

Using the relations (4.13) and (4.14), the result holds.

(ii) Taking \( g^{11} = g^{22} = -1/2 \), we get

\[
T_{jk}^2 (-\mathcal{L}_0^m + n^2) = A(-\mathcal{L}_0^m + n^2) + K_{jk}(-\mathcal{L}_0^m + n^2) . \tag{4.15}
\]

According to Lemma 3.4, we have

\[
A(-\mathcal{L}_0^m + n^2) = A(-\mathcal{L}_0 + N^2) , \tag{4.16}
\]

with \( N^2 = n^2 - T_{jk}^0 \). Using [3], \( A(-\mathcal{L}_0 + N^2) \) is compact relative to \( \mathcal{L}_0 \). As the operator \( K_{jk} \) is similar to \( A \), we deduce that \( T_{jk}^2 \) is compact relative to \( \mathcal{L}_0^m \).
3. Since $\mathcal{L}_0^n$ is self-adjoint in $L^2(\varphi)$ and
\[ (\mathcal{L}_0^n - \mathcal{L}_0^m)(n^2 - \mathcal{L}_0^m)^{-1} \rightarrow 0 \quad \text{in operator norm as} \quad n \rightarrow \infty, \quad (4.17) \]
according to Lemma 3.4 and Kato-Rellich theorem, $\mathcal{L}^m$ is self-adjoint. Furthermore,
\[ (-\mathcal{L}^m + n^2)^{-1} = (-\mathcal{L} + f(n))^{-1}, \quad (4.18) \]
where $f(n) = n^2 - T_{jk}^1$. Using [3], $\mathcal{L}$ has a compact resolvent. Thus, we deduce that $\mathcal{L}^m$ has a compact resolvent.

Let us show that $\left(\mathcal{L}^m - n^2\right)^{-1} - \left(\mathcal{L}_0^m - n^2\right)^{-1}$ is a trace class. For $u \in \rho(\mathcal{L}^m) \cap \rho(\mathcal{L}_0^m)$, we have
\[ (\mathcal{L}^m - u)^{-1} - (\mathcal{L}_0^m - u)^{-1} = (\mathcal{L} - g(u))^{-1} - (\mathcal{L}_0 - g_0(u))^{-1}, \quad (4.19) \]
where $g(u) = u - T_{jk}$ and $g_0(u) = u - T_{jk}^0$.

Since $u \in \rho(\mathcal{L})$, then $g(u) \in \rho(\mathcal{L}) \subset \rho(\mathcal{L}^m)$. Similarly, we show that $g_0(u) \in \rho(\mathcal{L}_0^m)$. Thus,
\[ (\mathcal{L}^m - u)^{-1} - (\mathcal{L}_0^m - u)^{-1} = (\mathcal{L}^m - \mathcal{L}_0^m)^{-1}[(\mathcal{L}^m - \mathcal{L}_0^m)(\mathcal{L}_0^m - u)^{-1} + 1]^{-1}, \quad (4.20) \]
since $(\mathcal{L}^m - \mathcal{L}_0^m)(\mathcal{L}_0^m + n^2)^{-\frac{1}{2}}$ is bounded. In fact, using the Proposition 2.7, we have
\[ (\mathcal{L}^m - \mathcal{L}_0^m)(\mathcal{L}_0^m + n^2)^{-\frac{1}{2}} = (T_{jk}^1 + T_{jk}^2)(\mathcal{L}_0^m + n^2)^{-\frac{1}{2}}, \quad (4.21) \]
and taking the norm, we get the result. It follows that $(\mathcal{L}_0^m + n^2)^{-\frac{1}{2}} \in \mathcal{B}_3(\mathcal{L}^2(\varphi))$. Moreover, $(\mathcal{L}_0^m - u)^{-1} \in \mathcal{B}_3/2(\mathcal{L}^2(\varphi))$. Therefore, we deduce that $(\mathcal{L}^m - n^2)^{-1} - (\mathcal{L}_0^m - n^2)^{-1}$ is a trace class.

In the next theorem, we investigate the invariance of volume and integrated scalar curvature under the perturbation of $\mathcal{L}_0^m$ from $\mathcal{L}^m$.

**Definition 4.4** The volume and integrated scalar curvature in noncommutative 2-torus are given, respectively, by :
\[ V(D_\theta) := \lim_{t \rightarrow 0^+} t \text{Tr} T_t^m, \quad (4.22) \]
\[ s(D_\theta) := \frac{1}{6} \lim_{t \rightarrow 0^+} [\text{Tr} T_t^m - t^{-1}V]. \quad (4.23) \]

**Theorem 4.5** Let $V$ and $s$ be the volume and integrated scalar curvature, respectively, on $D_\theta$. Then,

1. The volume $V$ of $D_\theta$ is invariant under the perturbation from $\mathcal{L}_0^m$ to $\mathcal{L}^m$.

2. The integrated scalar curvature is not invariant under the same perturbation.

**Proof:**
1. Using the relation (4.22), we have

\[ V(L^m) - V(L^m_0) = \lim_{t \to 0^+} t Tr(e^{tL^m} - e^{tL^m_0}). \]  

(4.24)

Then, we need to compute \( Tr(e^{tL^m} - e^{tL^m_0}) \). For that, we distinguish two cases of \( r_j \).

(i) For \( r_j \in D^\infty_0, j \in \{1, 2\} \) and \( 0 < v \leq t \leq 1 \), we have

\[ e^{tL^m} - e^{tL^m_0} = \int_0^t e^{(t-v)L^m} (L^m - L^m_0)e^{vL^m_0} \, dv, \]  

(4.25)

and using 2 iterations, we get

\[
\begin{align*}
& e^{tL^m} - e^{tL^m_0} = \int_0^t e^{(t-v)L^m} (L^m - L^m_0)e^{vL^m_0} \, dv - \int_0^t dt_1 e^{(t-t_1)L^m} (L^m - L^m_0) \\
& \quad \times \int_0^{t_1} dt_2 e^{(t_1-t_2)L^m_0} (L^m - L^m_0)e^{t_2L^m_0} \\
& \quad - \int_0^t dt_1 e^{(t-t_1)L^m} (L^m - L^m_0) \int_0^{t_1} dt_2 e^{(t_1-t_2)L^m_0} (L^m - L^m_0) \\
& \quad \times \int_0^{t_2} dt_3 e^{(t_2-t_3)L^m_0} (L^m - L^m_0)e^{t_3L^m_0} \\
& \quad \equiv J_1(t) + J_2(t) + J_3(t). \quad (4.26)
\end{align*}
\]

Using the trace norm, we estimate all these terms. For that, we need to get the \( B_p \)-norm of \((L^m - L^m_0)e^{tL^m_0}\). For \( 0 \leq v \leq 1 \), we have

\[
\left\| (L^m - L^m_0)e^{vL^m_0} \right\|_p \leq \left\| e^{vT_{jk}^0} \right\| v^{-\frac{1}{p} - \frac{1}{2}} , \quad (j, k) \in \{1, 2\}. \quad (4.27)
\]

We estimate the terms in (4.26), using the Hölder inequality for Schatten norms and the estimation of \( \|I_1(t)\|_1, \|I_2(t)\|_1 \), and \( \|I_3(t)\|_1 \) in [3]. We get,

\[
\|J_1(t)\|_1 \leq \int_0^t \left\| e^{(t-v)L^m_0} \right\|_{p_1} \left\| (L^m - L^m_0)e^{vL^m_0} \right\|_{p_2} \, dv. \quad (4.28)
\]

According to (4.27), and [3,35], we get the following relation

\[
\|J_1(t)\|_1 \leq \alpha_1 \beta_1 t^{-1/2}, \quad (4.29)
\]

where \( \alpha_1 = \left\| e^{(t-v)T_{jk}^0} \right\| \) and \( \beta_1 = \left\| e^{vT_{jk}^0} \right\| \). Similarly,

\[
\|J_2(t)\|_1 \leq \alpha_2 \beta_2 \gamma_2 c, \quad (4.30)
\]

where \( c \in \mathbb{R}_+^* \), \( \alpha_2 = \left\| e^{(t-t_1)T_{jk}^0} \right\|, \beta_2 = \left\| e^{(t_1-t_2)T_{jk}^0} \right\|, \) and \( \gamma_2 = \left\| e^{t_2T_{jk}^0} \right\| \).

Considering the following relation,

\[
\| (L^m - n^2)^{-1} \| \leq \frac{2}{n} , \quad n \in \mathbb{N}^*, \quad (4.31)
\]

we obtain the following estimation for \( J_3(t) \)

\[
\|J_3(t)\|_1 \leq \alpha_3 \beta_2 \gamma_3 \|I_3(t)\|_1, \quad (4.32)
\]

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where \( \alpha_3 = \left\| e^{(t_2-t_3)T_j^0} \right\| \) and \( \gamma_3 = \left\| e^{t_3 T_j^0} \right\| \).

Since, \( \| I_3(t) \|_1 \rightarrow 0 \) as \( t \rightarrow 0^+ \) (see [3]), then, we deduce that \( \| J_3(t) \|_1 \rightarrow 0 \) as \( t \rightarrow 0^+ \).

Therefore,

\[
V(L^m) = V(L_0^m).
\]  

(ii) For \( r_j \in D_\theta \), we have

\[
L^m - L_0^m = T_{jk}^1 + \sum_{j=1}^{2} g^{ij} (d_j B_j + C_j d_j) + \sum_{j \neq k} g^{jk} (d_j B_k + C_j d_k + C_j B_k),
\]  

where \( T_{jk}^1, B_j, C_j, \) and \( B_k \) are bounded.

Let us compute all the integrals in the equation (4.26).

- According to the expression of \( J_1(t) \) and the relation (4.34), we have
  \[
e^{(t-v)L_0^m} (L^m - L_0^m) e^{vL_0^m} = e^{(t-v)L_0^m} (T_{jk}^1 + K_1 + K_2) e^{vL_0^m} = E_1 + E_2 + E_3.
\]  

Thus,

\[
J_1(t) = \int_0^t E_1 \, dv + \int_0^t E_2 \, dv + \int_0^t E_3 \, dv.
\]  

Besides,

\[
\int_0^t E_1 \, dv = T_{jk}^1 I_1(t),
\]  

where \( I_1(t) \) is given in [3, 36].

Thus,

\[
\left\| \int_0^t E_1 \, dv \right\|_1 \leq \| T_{jk}^1 \| \| I_1(t) \|_1.
\]  

Moreover,

\[
\int_0^t E_2 \, dv = \sum_{j=1}^{2} g^{ij} \int_0^t e^{(t-v)L_0^m} d_j B_j e^{vL_0^m} \, dv + \sum_{j=1}^{2} g^{ij} \int_0^t e^{(t-v)L_0^m} C_j d_j e^{vL_0^m} \, dv
\]  

and, after computation, we get

\[
\left\| \int_0^t E_2 \, dv \right\|_1 \leq \mu \sum_{j=1}^{2} g^{ij} \alpha_j (\| B_j \| + \| C_j \|) t,
\]  

with \( \mu \in \mathbb{R}_+^* \).
Furthermore,
\[
\int_0^t E_3 \, dv = \int_0^t e^{(t-v)\mathcal{L}_0^m} \sum_{j \neq k} g^{jk}(d_j B_k + C_j d_k + C_j B_k) e^{v\mathcal{L}_0^m} \, dv. \tag{4.41}
\]

Analogously to the above calculation, we obtain
\[
\left\| \int_0^t E_3 \, dv \right\|_1 \leq \nu \sum_{j \neq k} g^{jk} \alpha_j ||B_k|| t + \alpha_k ||C_j|| t + ||C_j B_k||, \tag{4.42}
\]
with \(\nu \in \mathbb{R}_+^*\).

Therefore,
\[
\|J_1(t)\|_1 \leq \mu \sum_{j=1}^2 g^{j} \alpha_j \left( ||B_j|| + ||C_j|| \right) t
+ \nu \sum_{j \neq k} g^{jk} \left( (\alpha_j ||B_k|| + \alpha_k ||C_j||) t \right)
+ \nu \sum_{j \neq k} ||C_j B_k|| + \left\| T_{jk}^1 \right\| \left\| I_1(t) \right\|_1. \tag{4.43}
\]

Similarly,
\[
\|J_2(t)\|_1 \leq \gamma \sum_{j=1}^2 g^{j} \alpha_j \left( ||B_j||^2 + ||C_j||^2 \right) t^3
+ \gamma \sum_{j \neq k} g^{jk} \left( \alpha_j^2 ||B_k||^2 + \alpha_k^2 ||C_j||^2 \right) t^3
+ \gamma \sum_{j \neq k} ||C_j B_k||^2 + \left\| T_{jk}^1 \right\| \left\| I_2(t) \right\|_1, \tag{4.44}
\]

and
\[
\|J_3(t)\|_1 \leq \eta \sum_{j=1}^2 g^{j} \alpha_j \left( ||B_j||^3 + ||C_j||^3 \right) t^4
+ \eta \sum_{j \neq k} g^{jk} \left( \alpha_j^3 ||B_k||^3 + \alpha_k^3 ||C_j||^3 \right) t^4
+ \eta \sum_{j \neq k} ||C_j B_k||^3 + \left\| T_{jk}^1 \right\| \left\| I_3(t) \right\|_1, \tag{4.45}
\]

where \((\gamma, \eta) \in \mathbb{R}_+^*\).

Thus,
\[
V(\mathcal{L}^m) - V(\mathcal{L}_0^m) = \lim_{t \to 0^+} t Tr \left( \|J_1(t)\|_1 + \|J_2(t)\|_1 + \|J_3(t)\|_1 \right)
= 0. \tag{4.46}
\]

Therefore, for \(r_j \in \mathcal{D}_0\), the volume is invariant under perturbation from \(\mathcal{L}_0^m\) to \(\mathcal{L}^m\).
2. The integrated scalar curvature is given by (4.23) and according to the perturbation of the volume from $L^m_0$ to $L^m$, we have

$$s(L^m) - s(L^m_0) = \frac{1}{6} \lim_{t \to 0^+} [Tr(e^{tL^m} - e^{tL^m_0})]. \quad (4.47)$$

- Let us start with the case $r_j \in D^\infty_\theta$, where $j \in \{1, 2\}$.

Using the relation (4.30) and [3], we deduce that $TrJ_2(t) \to 0$ as $t \to 0^+$. Since $J_3(t)$ vanishes, thus the computation of $s(L^m) - s(L^m_0)$ concerns only $J_1(t)$.

According to the relation (4.43), we get

$$s(L^m) - s(L^m_0) = \frac{1}{6} \lim_{t \to 0^+} t \left( Tr(T^1_{jk} e^{tL^m_0}) + Tr(T^2_{jk} e^{tL^m_0}) \right). \quad (4.48)$$

Let us now compute $Tr(T^1_{jk} e^{tL^m_0})$ and $Tr(T^2_{jk} e^{tL^m_0})$, for all $t > 0$.

For $j = 1, k = 2$, and $r \in D^\infty_\theta$, we have $Tr(d_r d_1 e^{tL^m_0}) = Tr(d_r d_2) = 0$. Then, for $(j, k) \in \{1, 2\}$, we deduce $Tr(T^1_{jk} e^{tL^m_0}) = 0$.

Moreover,

$$Tr(T^2_{jk} e^{tL^m_0}) = Tr \left( \sum_{j=1}^2 g^{ij}(d^2_{r_j} - 2\pi i G_j G_{r_j}) e^{tL^m_0} \right)$$

$$- Tr \left( \sum_{j \neq k} g^{jk}(2\pi i G_k G_{r_j} + 2\pi i G_j G_{r_k}) e^{tL^m_0} \right). \quad (4.49)$$

For $j \neq k$ and $r \in D^\infty_\theta$, we have $Tr(d_r e^{tL^m_0}) = Tr(d_r e^{tL^m_0}) = 0$. Thus, $Tr(T^2_{jk} e^{tL^m_0}) = Tr \left( \sum_{j=1}^2 g^{ij} d^2_{r_j} e^{tL^m_0} \right)$ and according to Lemma [3.4], the relation (4.48) becomes

$$s(L^m) - s(L^m_0) = \frac{1}{6} \lim_{t \to 0^+} t Tr \left( \sum_{j=1}^2 g^{ij} d^2_{r_j} e^{tL^m_0} e^{tL^m_0} \right), \quad (4.50)$$

where $k \in \{1, 2\}$. We perform the same computation as in [3], and deduce that there exists $K \in \mathbb{R}_+$ such that

$$s(L^m) - s(L^m_0) \geq K. \quad (4.51)$$

- For $r_j \in D_\theta$, with $j \in \{1, 2\}$, we have

$$s(L^m) - s(L^m_0) = \frac{1}{6} \lim_{t \to 0^+} Tr \left( J_1(t) + J_2(t) + J_3(t) \right). \quad (4.52)$$

Furthermore, according to the relation (4.43), we get

$$\lim_{t \to 0^+} TrJ_1(t) = \nu \sum_{j \neq k} g^{jk} \| C_j B_k \| + \| T^1_{jk} \| \| J_1(t) \|_1. \quad (4.53)$$

From [3], it follows that $\lim_{t \to 0^+} TrJ_1(t) = \nu \sum_{j \neq k} g^{jk} \| C_j B_k \|$. Similarly, we obtain

$$\lim_{t \to 0^+} TrJ_2(t) = \gamma \sum_{j \neq k} g^{jk} \| C_j B_k \|^2 \quad (4.54)$$
and
\[ \lim_{t \to 0^+} Tr J_3(t) = \eta \sum_{j \neq k} g^{jk} \|C_j B_k\|^3. \] (4.55)

Thus,
\[ s(\mathcal{L}^m) - s(\mathcal{L}^m_0) = \sum_{j \neq k} g^{jk} \left( \nu \|C_j B_k\| + \gamma \|C_j B_k\|^2 + \eta \|C_j B_k\|^3 \right), \] (4.56)

and, therefore, the result holds. \[ \square \]

5 Spectral triple on \( \mathcal{D}^\infty_\theta \)

5.1 Volume form on \( \mathcal{D}^\infty_\theta \)

Now, we investigate the invariance of the volume form under the perturbation from \( \mathcal{L}^m_0 \) to \( \mathcal{L}^m \).

We denote by \( T_1 \) the self-adjoint operator with a compact resolvent \( \tilde{T}_1 = T_1 |\mathcal{N}(T_1)^\perp \), where \( \mathcal{N}(T_1) \) is the kernel of \( T_1 \).

**Definition 5.1** \([5]\) The volume form on \( \mathcal{D}^\infty_\theta \) is a linear functional given by:
\[ v(x) := \frac{1}{2} Tr_w(x|\tilde{D}|^{-2} P), \] (5.1)

where, \( Tr_w \) is the Dixmier trace, \( D \) an operator, \( x \in \mathcal{D}^\infty_\theta \), and \( P \) is the projection on \( \mathcal{N}(T)^\perp \).

**Lemma 5.2** \([36]\) Let \( T_1 \) be a self-adjoint operator with a compact resolvent such that \( \tilde{T}_1^{-1} \) is Dixmier trace-able. Then, for \( x \in \mathcal{D}^\infty_\theta \) and every \( b \in \rho(T_1) \),
\[ Tr_w(x\tilde{T}_1^{-1} P) = Tr_w(x(T_1 - z)^{-1}), \] (5.2)

where \( \rho \) is a resolvent.

**Lemma 5.3** The operator \( \partial : \mathcal{D}^\infty_\theta \to \mathcal{D}^\infty_\theta \) defined as follows:
\[ \partial = \sum_{j=1}^{2} (d_j - 2\pi iG_j), \] (5.3)

is a self-adjoint operator on \( \mathcal{D}^\infty_\theta \).

**Proof:** Since \( d : \mathcal{D}^\infty_\theta \to \mathcal{D}^\infty_\theta \) is self-adjoint, thus, we deduce that for \( j \in \{1, 2\} \), \( d_j - 2\pi iG_j \) : \( \mathcal{D}^\infty_\theta \to \mathcal{D}^\infty_\theta \) is self-adjoint. Therefore, we get the result. \[ \square \]

**Definition 5.4**
1. The unperturbed magnetic operator \( D^m_0 : \mathcal{D}^\infty_\theta \to \mathcal{D}^\infty_\theta \) is given by:
\[ D^m_0 = \begin{pmatrix} 0 & d^m_1 + i d^m_2 \\ d^m_1 - i d^m_2 & 0 \end{pmatrix}. \] (5.4)

2. For \( r \in \mathcal{D}^\infty_\theta \), the perturbed magnetic operator \( D^m : \mathcal{D}^\infty_\theta \to \mathcal{D}^\infty_\theta \) is defined as follows:
\[ D^m = D^m_0 + \begin{pmatrix} 0 & d_r \\ d_r^* & 0 \end{pmatrix}. \] (5.5)
**Definition 5.5** The unperturbed and perturbed volume forms associated with a magnetic Laplacian are given by:

\[ v_m^0 (x) := \frac{1}{2} Tr_w (x |\bar{D}_m^0|^{-2} P) \quad \text{and} \quad v^m (x) := \frac{1}{2} Tr_w (x |\bar{D}_m^0|^{-2} P), \]  

(5.6)

where \( x \in D^\infty_\theta \).

**Remark 5.6**

1. The unperturbed magnetic operator is given by:

\[ D^m_0 = D_0 + G \]  

(5.7)

where

\[ G = -2\pi i \begin{pmatrix} 0 & G_1 + i G_2 \\ G_1 - i G_2 & 0 \end{pmatrix}. \]  

(5.8)

2. The perturbed magnetic operator is also written as follows:

\[ D^m = D + G. \]  

(5.9)

3. When \( G_j = 0 \), for \( j \in \{1, 2\} \), we obtain the unperturbed and perturbed operators defined by Chakraborty and co-workers [3].

**Theorem 5.7** The volume form associated with the magnetic Laplacian is invariant under the perturbation from \( L^m_0 \) to \( L^m \).

**Proof:** It is obvious to have:

\[ (D^m_0)^2 = \begin{pmatrix} \tilde{L}^m_0 & 0 \\ 0 & \hat{L}^m_0 \end{pmatrix}, \]  

(5.10)

where \( \tilde{L}^m_0 \) and \( \hat{L}^m_0 \) are given, respectively, by:

\[ \tilde{L}^m_0 = L^m_0 + i d^m_2 d^m_1 - i d^m_1 d^m_2 - \sum_{j \neq k} g^{jk} d^m_j d^m_k \]  

(5.11)

and

\[ \hat{L}^m_0 = L^m_0 + i d^m_1 d^m_2 - i d^m_2 d^m_1 - \sum_{j \neq k} g^{jk} d^m_j d^m_k. \]  

(5.12)

Furthermore,

\[ (D^m)^2 = \begin{pmatrix} L^m_1 & 0 \\ 0 & L^m_2 \end{pmatrix}, \]  

(5.13)

with

\[ L^m_1 = \tilde{L}^m_0 + (d^m_1 + i d^m_2) d_r + d_r (d^m_1 - i d^m_2) + d_r d_r. \]  

(5.14)

and

\[ L^m_2 = \hat{L}^m_0 + d_r (d^m_1 + i d^m_2) + (d^m_1 - i d^m_2) d_r + d_r d_r. \]  

(5.15)

Let us consider \( P_1 \) and \( P_2 \), two projections on \( \mathcal{N}(L^m_1)^\perp \) and \( \mathcal{N}(L^m_2)^\perp \), respectively. By Theorem 5.3 (see [3]), \( L^m_1 \) and \( L^m_2 \) have compact resolvent and using the above Lemma [5.2] for \( Imy \neq 0 \), we have

\[ v_0^m (x) = Tr_w (x (\bar{L}^m_0 - y)^{-1} + x (\bar{L}^m_0 - y)^{-1}). \]  

(5.16)
Since \((-L_i^m - y)^{-1} - (L_i^m - y)^{-1}\) is a trace class for \(i = 1, 2\), then

\[
v^m(x) = Tr_w(x(-L_1^m - y)^{-1} + x(-L_2^m - y)^{-1}) = Tr_w(x(-L_1^m - y)^{-1} + x(\tilde{\lambda}_0^m - y)^{-1}) + Tr_w(x(-L_2^m - y)^{-1} + x(\tilde{\lambda}_0^m - y)^{-1}) = v_0^m(x).
\]

(5.17)

Therefore, the volume form associated with the magnetic Laplacian is invariant under our perturbation. □

5.2 Spectral triple

Let us recall some basic notions concerning the spectral triples.

**Definition 5.8** [36] An even spectral triple for a \(*-\) algebra \(A\) is a triple \((\pi, H, D)\) together with a \(\mathbb{Z}_2\) grading \(\gamma\) on \(H\) such that

1. The map \(\pi : A \to \mathcal{L}(H)\) is a \(*-\) representation such that \(\pi(x)\gamma = \gamma\pi(x)\) for \(x \in A\).

2. The operator \(D\) is an unbounded operator with compact resolvent such that \(D\gamma = -\gamma D\).

3. The commutator \([D, \pi(x)]\) is bounded for every \(x \in A\).

**Definition 5.9** Let \(B\) be a \(*-\) algebra, \(K\) a Hilbert space, and \(D\) an operator. Then, the spectral triples \((B_1, K_1, D_1)\) and \((B_2, K_2, D_2)\) are unitarily equivalent if there exists a unitary operator \(V\) and a representation \(\pi_l\), for \(l \in \{1, 2\}\) defined as follows:

\[
V : K_1 \to K_2 \quad \text{such that} \quad D_2 = V D_1 V^*,
\]

and

\[
\pi_l : B_2 \to K_l, \quad \pi_2(x) = V \pi_1(x) V^*,
\]

respectively.

In the rest of this work, we call the magnetic even spectral triple, the even spectral triple associated with the magnetic Laplacian.

The next Lemma characterizes the magnetic even spectral triple. We consider the Hilbert space \(H = L^2(\varphi) \oplus L^2(\varphi)\).

**Lemma 5.10** Let \(\Gamma\) be the grading operator. Then,

1. The triple \((D_\infty^0, L^2(\varphi) \oplus L^2(\varphi), D_0^m)\) is an unperturbed magnetic even spectral triple.

2. The triple \((D_\infty^0, L^2(\varphi) \oplus L^2(\varphi), D_0^m)\) is a perturbed magnetic even spectral triple.

**Proof:** For the proof, we consider the grading operator given as follows: \(\Gamma = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\) and the \(*-\) representation \(\pi(x) = x\) where \(x \in D_\infty^0\).
1. It is obvious to see that $x \Gamma = \Gamma x$, $\Gamma^* = \Gamma = \Gamma^{-1}$ and $\Gamma G = -\Gamma G$. Since $D_0^m = D_0 + G$, and according to [36], we have, $\Gamma D_0^m = -D_0^m \Gamma$. Furthermore, the dimension of $\ker D_0^m$ is 2, and according to the relation (5.10), we conclude that the unperturbed magnetic $D_0^m$ has a compact resolvent.

In addition, for $x \in D_\infty^\infty$, $[G, x]$ is bounded, and using [3], we conclude that $[D_0^m, x] = [D_0, x] + [G, x]$ is bounded

2. Similarly, we obtain the result.

From [5], the next definition is formulated.

**Definition 5.11** Let $\Omega^1(D_\infty)$ be the universal space of 1-forms. Then, the representation $\pi : \Omega^1(D_\infty) \rightarrow \mathcal{K}$ is defined as follows:

$$\pi(x) = x \quad \text{and} \quad \pi(\delta^m(x)) = [D^m, x]. \quad (5.18)$$

**Lemma 5.12** Let $D_0^m$ and $D^m$ two operators both mapping to $D_\infty^\infty$ from $D_\infty^\infty$. Then, the following relations hold:

1.

$$D_0^m = i(d_1 \gamma_1 + d_2 \gamma_2) - 2\pi i(G_1 \gamma_1 + iG_2 \gamma_2). \quad (5.19)$$

2.

$$[D^m, x] = [D, x] - 2\pi i(G_1 \gamma_1 + iG_2 \gamma_2), \quad (5.20)$$

where $\gamma_1, \gamma_2$ are the $2 \times 2$ Clifford matrices.

**Proof:**

1. According to the relation (5.7) and using [3], we have $D_0^m = i(\gamma_1 d_1 + \gamma_2 d_2) - G$. Moreover, the matrix $G$ is expressed as $G = 2\pi iG_1 \gamma_1 - 2\pi G_2 \gamma_2$. Thus, the relation (5.19) holds.

2. Using the relations (5.5), (5.7), and after computation, we get the result.

Now, we study the unitarity of the unperturbed and perturbed magnetic spectral triples.

**Definition 5.13** ([20]) Let $(\pi_1, \mathcal{H}_1, D_1)$ and $(\pi_2, \mathcal{H}_2, D_2)$ be two even spectral triples such that $\pi = \pi_1 + \pi_2$, $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and $D = D_1 + D_2$. Then, $(\pi, \mathcal{H}, D)$ is an even spectral triple, the so called the direct sum of $(\pi_1, \mathcal{H}_1, D_1)$ and $(\pi_2, \mathcal{H}_2, D_2)$.

We arrive at the following Proposition.

**Proposition 5.14** Let $X \in D_\theta$. Then, for $r = X^{n_1}$, and $n_1 \in \mathbb{Z}$, we have:

1. $\pi(\Omega^1) = D_\theta^\infty \oplus D_\theta^\infty$.

2. $\Omega^2(D_\theta^\infty) = 0$.

**Proof:**
1. According to the relations (5.18) and (5.20), we get

\[ \pi(\delta^m(x)) = \pi(\delta(x)) - 2\pi i (G_1 \gamma_1 + i G_2 \gamma_2). \]  \hspace{1cm} (5.21)

Setting \( \Omega^1(D_\theta^\infty) = -2\pi i (G_1 \gamma_1 + i G_2 \gamma_2) \) and using the Lemma\textsuperscript{3.4} we obtain

\[ \Omega_{1,m}^1(D_\theta^\infty) = \Omega_{D}(D_\theta^\infty) + \Omega_{T,jk}^1(D_\theta^\infty). \hspace{1cm} (5.22) \]

Since \( \Omega_{1,m}^1(D_\theta^\infty) = D_\theta^\infty \oplus D_\theta^\infty \), performing the same calculation as in \textsuperscript{3}, we deduce that \( \Omega_{T,jk}^1(D_\theta^\infty) = D_\theta^\infty \oplus D_\theta^\infty \). Therefore, \( \Omega_{1,m}^1(D_\theta^\infty) = D_\theta^\infty \oplus D_\theta^\infty \).

2. It is obvious. \hspace{1cm} \qed

Thus, we obtain the following result.

**Proposition 5.15** For \( r = X^{n_1} \) with \( n_1 \in \mathbb{Z} \) and \( X \in D_\theta \), the unperturbed magnetic and perturbed magnetic spectral triples \((D_\theta^\infty, L^2(\varphi) \oplus L^2(\varphi), D_\theta^m)\) and \((D_\theta^\infty, L^2(\varphi) \oplus L^2(\varphi), D^m)\) are not unitarily equivalent.

**Proof:** Using Lemma\textsuperscript{3.4} and Theorem (4.4) \textsuperscript{3}, it suffices to study the unitarity of the even spectral triples \((D_\theta^\infty, L^2(\varphi) \oplus L^2(\varphi), T_{\theta}^0)\) and \((D_\theta^\infty, L^2(\varphi) \oplus L^2(\varphi), T_{\theta}^i)\). According to the above Proposition, \( \Omega^2(D^m) = 0 \), but \( \Omega^2(D^m) = 2 \Omega^2(D) + 2 \Omega^2(T_{\theta}^i) \). Thus, using \textsuperscript{36}, we deduce that \( \Omega^2_{T_{\theta}^i}(D_\theta^\infty) = 0 \). Since \( \Omega^2_{T_{\theta}^i}(D_\theta^\infty) = D_\theta^\infty \), thus, the result holds. \hspace{1cm} \qed

Let \( \mathcal{K}^m \) be the vector space of all magnetic derivations \( d^m : D_\theta^\infty \to D_\theta^\infty \). \( \mathcal{K}^m \) is identified as the space of derivations given by \textsuperscript{2}, i.e as the space of the form

\[ \left\{ \sum_{j=1}^{2} c_j d_j^m + [r, \cdot], r \in D_\theta^\infty, c_j \in \mathbb{C} \right\}. \]

Let \( \mathcal{B} \) be any normed \( D_\theta^\infty \)-module. For \( \delta^m \in \mathcal{K}^m \), we consider the contraction \( C_{\delta^m} : \mathcal{B} \otimes \mathcal{K}^m \to \mathcal{B} \).

**Definition 5.16** The magnetic connection is a complex linear map defined by: \( \nabla^m : \mathcal{B} \otimes \mathcal{K}^m \to \mathcal{B} \) such that

\[ C_{\delta^m}(\nabla^m(x)) = C_{\delta^m}(\nabla^m(x)) + \xi \delta^m(x), \quad \forall \delta^m \in \mathcal{K}^m. \hspace{1cm} (5.23) \]

In addition, we define a magnetic connection as follows:

\[ C_{\delta^m}(\nabla^m(x)) = C_{\theta}(\nabla(x)) - \xi \alpha_j(x), \hspace{1cm} (5.24) \]

where \( \alpha_j = 2 \pi i \sum_{j=1}^{2} c_j G_j \).

We assume that the maps \( \nabla_j : \mathcal{B} \to \mathcal{B} \) for \( j \in \{1, 2\} \) are given by the following relation

\[ \nabla_j^m(x) = \nabla_j^m(x) + \xi d_j^m(x). \hspace{1cm} (5.25) \]

**Proposition 5.17** The map \( \nabla^m : \mathcal{B} \to \mathcal{B} \) given by:

\[ C_{\delta^m}(\nabla^m(\xi)) = \sum_{j=1}^{2} \nabla_j^m \otimes d_j^m - \sum \xi X^{n_1} Y^{n_2} \otimes \delta_{n_1 n_2}. \hspace{1cm} (5.26) \]

is a magnetic connection if \( \nabla_j \otimes G_j = \xi (G_j \otimes d_j - G_j \otimes G_j) \).
Proof: Since $d^m_j = d_j - \beta_j$, where $\beta_j = 2\pi i G_j$, thus

\[
\sum_{j=1}^{2} \nabla^m_j \otimes d^m_j = \sum_{j=1}^{2} \nabla^m_j \otimes d_j - 2\pi i \sum_{j=1}^{2} \nabla^m_j \otimes G_j. \tag{5.27}
\]

Moreover,

\[
\nabla^m_j(\xi x) = \nabla_j(\xi x) - \xi \beta_j(x). \tag{5.28}
\]

Thus, using the relation (5.27), we obtain

\[
\nabla^m(\xi) = \nabla(\xi) + \gamma(\xi), \tag{5.29}
\]

where

\[
\gamma(\xi) = 2\pi i \sum_{j=1}^{2} \left( \xi G_j \otimes d_j - \nabla_j \otimes G_j - \xi G_j \otimes G_j \right). \tag{5.30}
\]

According to [3], the map $\nabla$ is a connection. It follows that $\nabla^m$ is a connection. \hfill $\Box$

Considering the definition of the connection, we consider the following relations.

\[
\begin{aligned}
C_\beta \nabla &= C_d \beta = 0 \\
C_{d_\nu} \nabla &= \nabla_r \\
C_{d_j} \nabla &= \nabla_j
\end{aligned} \tag{5.31}
\]

We consider $\mathcal{L}(\mathcal{B})$, the space of linear forms on $\mathcal{B}$.

**Definition 5.18** The curvature 2-form $R : \mathcal{K}^m \otimes \mathcal{K}^m \to \mathcal{L}(\mathcal{B})$ associated with the magnetic connection $\nabla^m$ is defined by:

\[
R(\delta^m_1, \delta^m_2) = C_{[\delta^m_1, \delta^m_2]} \nabla^m - [C_{\delta^m_1} \nabla^m, C_{\delta^m_2} \nabla^m]. \tag{5.32}
\]

**Theorem 5.19** The curvature 2-form associated with the magnetic connection is not invariant under the perturbation by inner derivation.

**Proof:** It suffices to show the relation between $R(d^m_1, d^m_2)$ and $R(\delta^m_1, \delta^m_2)$. Using the relations (5.31) and (5.28), we obtain, respectively,

\[
[C_{\delta_1} \nabla^m, C_{\delta_2} \nabla^m] = [C_{\delta_1} \nabla, C_{\delta_2} \nabla] \tag{5.33}
\]

and

\[
[C_{d^m_1} \nabla^m, C_{d^m_2} \nabla^m] = [C_{d_1} \nabla, C_{d_2} \nabla]. \tag{5.34}
\]

Moreover,

\[
[\delta^m_1, \delta^m_2] = [\delta_1, \delta_2] + 2\pi i (G_2 \delta_1 - G_1 \delta_2 + \psi_{21}), \tag{5.35}
\]

\[
[d^m_1, d^m_2] = [d_1, d_2] + 2\pi i (G_2 d_1 - G_1 d_2 + \psi_{21}). \tag{5.36}
\]
Thus, using once (5.28), we obtain
\[ C[\delta_1^m, \delta_2^m] \nabla^m = C[\delta_1, \delta_2] \nabla + 2\pi i (G_2(\nabla_1 + \nabla_{r_1}) - G_1(\nabla_2 + \nabla_{r_2})), \] (5.37)
and
\[ C[d_1^m, d_2^m] \nabla^m = C[d_1, d_2] \nabla + 2\pi i (G_2 - G_1). \] (5.38)
Therefore, according to the invariance condition of the curvature 2–form \[3, 36\], we have
\[ R(\delta_1^m, \delta_2^m) = R(d_1^m, d_2^m) + 2\pi i (G_2(\nabla_1 + \nabla_{r_1}) - G_1(\nabla_2 + \nabla_{r_2})). \] (5.39)
Futhermore,
\[ R(d_1^m, d_2^m) = R(d_1, d_2) + 2\pi i (G_2 - G_1). \] (5.40)
Therefore, using the relations (5.39) and (5.40), we get
\[ R(\delta_1^m, \delta_2^m) = R(d_1^m, d_2^m) + 2\pi i (G_2(\nabla_{r_1} - G_1(\nabla_{r_2})). \] (5.41)

6 Noncommutative 2d-dimensional space

In this section, we study the magnetic quantum stochastic differential equation (mqsde) in 2d-dimensional noncommutative space and some properties of its solution. For more information on the geometry of the simplest of noncompact manifolds (Euclidean 2d-dimensional space), see [3] and [14].

6.1 Basic tools

For the integer \( d \geq 1 \), we consider \( C_0(\mathbb{R}^{2d}) \) as the \( C^* \)- algebra of complex valued continuous functions on \( \mathbb{R}^{2d} \) whic vanish at infinity. The magnetic partial derivative on \( C_0(\mathbb{R}^{2d}) \) in the k-th direction is the operator \( \partial_k^m = \partial_k - 2\pi i G_k \) with \( k \in \{1, 2, \cdots, 2d\} \).

The space of smooth complex valued functions on \( \mathbb{R}^{2d} \) with compact support is denoted by \( \mathcal{B}_c^\infty := C_c^\infty(\mathbb{R}^{2d}) \). Let us consider the Hilbert space of square integrable functions, \( L^2(\mathbb{R}^{2d}) \). Then, the symmetric linear magnetic operator which maps on \( L^2(\mathbb{R}^{2d}) \) with the domain \( \mathcal{B}_c^\infty \) is defined by
\[ i\partial_k^m, i\partial_k^m = i\partial_k + 2\pi G_k. \]

We consider \( (\cdot, \cdot) \) as the Euclidean inner product of \( \mathbb{R}^d \).

**Definition 6.1** [3] The Fourier transform on the Hilbert space \( L^2(\mathbb{R}^{2d}) \) is defined by:
\[ \hat{g}(\epsilon) = (2\pi)^{-d} \int e^{i\langle \epsilon, u \rangle} g(u) \, du. \] (6.1)

The operator of multiplication by the function \( \psi \) is defined by \( M_\psi \), and setting \( \hat{M}_\psi = \mathcal{F}^{-1} M_\psi \mathcal{F} \), we get
\[ i\partial_j^m = i\partial_j + \left\langle \hat{M}_\psi, T_{jk}^0 \right\rangle. \]
Definition 6.2 The $2-d$ dimensional magnetic operator is given by:

$$i \partial_j^m = i \partial_j + \hat{M}_f(x_{jk}) \quad (j, k) \in \{1, 2\}$$ \hfill (6.2)

where

$$f(x_{jk}) = \sum_{j=1}^{2d} (\pi i G_j x_j + 2\pi^2 G_j^2) + \sum_{j \neq k} (x_j - 2\pi i G_j)(x_k - 2\pi i G_k).$$ \hfill (6.3)

We assume that the differential operator $\sum_{j, k=1}^{2d} (d_j - 2\pi i G_j)(d_k - 2\pi i G_k)$ is the restriction of the magnetic Laplacian $\Delta^m$ on $\mathcal{B}_c^\infty$.

Definition 6.3 The continuous groups of unitaries in $\mathcal{H} := L^2(\mathbb{R}^d)$, the space of square integrable functions on $\mathbb{R}^d$, are defined as follows:

$$\begin{align*}
(X_\alpha g)(u) &= g(u + \alpha) \\
(Y_\mu g)(u) &= \exp i \langle u, \mu \rangle g(u),
\end{align*}$$ \hfill (6.4)

where $\alpha, \mu, u \in \mathbb{R}^d$, and $g \in C_c^\infty(\mathbb{R}^d)$.

It is straightforward to see that $X_\alpha$ and $Y_\mu$ verify the following relations:

$$\begin{align*}
X_{\alpha_1} X_{\alpha_2} &= X_{\alpha_1 + \alpha_2} \\
Y_{\mu_1} Y_{\mu_2} &= Y_{\mu_1 + \mu_2} \\
X_{\alpha_1} Y_{\mu_1} &= \exp i \langle \alpha_1, \mu_1 \rangle Y_{\mu_1} X_{\alpha_1}
\end{align*}$$ \hfill (6.5)

and the equation (6.5) takes the form:

$$Z_u Z_v = Z_{u+v} e^{(i/2)f(u,v)},$$ \hfill (6.6)

with $f(u, v) = u_1.v_2 - u_2.v_1$ such that $u = (u_1, u_2)$ and $v = (v_1, v_2)$.

Let $B(\mathcal{H})$ be the space of bounded operators in $\mathcal{H}$. For $\hat{g} \in L^1(\mathbb{R}^{2d})$, we set

$$a(g) = \int_{\mathbb{R}^{2d}} \hat{g}(u) Z_u du.$$ \hfill (6.7)

We denote by $B^\infty$ the $*$- algebra of elements of the form $\{a(g) \quad \text{s.t.} \quad g \in C_c^\infty(\mathbb{R}^d)\}$, and $B$ the $C^*$-algebra generated by $B^\infty$. For $(g, h) \in C_c^\infty(\mathbb{R}^{2d})$, we set

$$(\hat{g} \ast \hat{h})(u) = \int \hat{g}(u-v) \hat{h}(v) e^{(i/2)f(u,v)} dv, \quad g^N(u) = \bar{g}(-u).$$ \hfill (6.8)

Using the relation (6.5), it is obvious to verify that

$$a(g)a(h) = a(g \ast h) \quad \text{and} \quad a(g)^* = a(g^N).$$ \hfill (6.9)

Corollary 6.4 [14] The linear functional $\phi$ on $B^\infty$ defined by

$$\phi(a(g)) = (2\pi)^{-d} \int g(u) du$$ \hfill (6.10)

is a faithful trace on $B^\infty$.  

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Definition 6.5 The magnetic 2d-parameter group of automorphisms of $\mathcal{B}$ is given by
$$\psi_\alpha(a(g)) = \int_{\mathbb{R}^{2d}} e^{i(\alpha, w)} \hat{g}(u) Zu\, du,$$  \hspace{1cm} (6.11)
where $\alpha \in \mathbb{R}^{2d}$ and $g \in C_c^\infty(\mathbb{R}^{2d})$.

It is straightforward to see that for $a(g) \in \mathcal{B}^\infty$, the map $\alpha \rightarrow \psi_\alpha(a(g))$ is smooth.

Definition 6.6 For $j \in \{1, \cdots, 2d\}$, the magnetic derivation is defined as follows:
$$\delta_j^m(a(g)) = a(\partial_j(g)) - 2\pi i a(G_j(g)),$$  \hspace{1cm} (6.12)
with $g \in C_c^\infty(\mathbb{R}^{2d})$.

Let us now look for the case of a Riemannian manifold. We study the volume form associated with the magnetic 2d-dimensional Laplacian $\Delta^m$. Let us consider $T^m_t$ as the contractive $C_0$ semigroup generated by the magnetic Laplacian, and called the magnetic heat semigroup on $\mathbb{R}^{2d}$.

It is easy to verify that, for $(j, k) \in \{1, \cdots, 2d\}$, $T^m_t = T_t e^{(t/2)^m}$, where $T_t$ is the classical heat semigroup.

Proposition 6.7 Let $T^m_t$ be the magnetic heat semigroup and $M_g$ be the multiplication operator by the function $g$. Then,
1. $M_g T^m_t$ is a trace-class.
2. $Tr\left(M_g T^m_t\right) = (2\pi)^d t^{-d} f(G_1, G_2)$.

Proof:
1. Since $M_g T^m_t = F M_g F^{-1} M e^{-(t/2) \sum x_j^2 + f(x_j)}$, then,
$$Tr\left(M_g T^m_t\right) = Tr\left(F M_g F^{-1} M e^{-(t/2) \sum x_j^2 + f(x_j)}\right),$$  \hspace{1cm} (6.13)
and the kernel of the integral operator is given by
$$K^m_t(u, v) = \hat{g}(u - v) e^{-(t/2) w(v)},$$  \hspace{1cm} (6.14)
with $w(v) = \sum v_j^2 + f(v_j)$. It is easy to see that $K^m_t(u, v)$ is continuous in $u$ and $v$ and $\int |K^m_t(u, u)|\, du < \infty$. Following [16], we deduce the result.

2.
$$Tr\left(M_g T^m_t\right) = \int K^m_t(u, u)\, du$$
$$= \hat{g}(0) e^{-2\pi^2 G_1 G_2} \int e^{-(t/2) w(u)}\, du.$$  \hspace{1cm} (6.15)
Moreover, $w(u) = (u_1 + \tau)^2 + (u_2 + \sigma)^2 - \sigma^2 - \tau^2$ where $\tau = \frac{1}{2} \pi i G_1 - \pi i G_2$ and $\sigma = \frac{1}{2} \pi i G_2 - \pi i G_1$. Setting $f(G_1, G_2) = \exp\left((t/2)(4\pi^2 G_1 G_2 + \sigma^2 + \tau^2)\right)$, we obtain
$$Tr\left(M_g T^m_t\right) = \hat{g}(0) f(G_1, G_2) \int e^{-(t/2) \left((u_2 + \sigma)^2 + (u_1 + \tau)^2\right)}\, du_1\, du_2$$
$$= f(G_1, G_2) Tr\left(M_g T_t\right).$$  \hspace{1cm} (6.16)
According to [36], we get
$$Tr\left(M_g T^m_t\right) = (2\pi)^d t^{-d} f(G_1, G_2).$$  \hspace{1cm} (6.17)
In particular, we define the volume as follows:

\[ v^m(g) := t^d Tr(M_g T^m_t). \]  
(6.18)

As in previous section, we give the analogue expression of the volume form using the Dixmier trace.

Lemma 6.8 [3] For \( h, g_1 \in L^p(\mathbb{R}^2) \) with \( 2 \leq p < \infty \), then \( M_h \hat{M}_{g_1} \) is a compact operator in \( L^p(\mathbb{R}^2) \).

Lemma 6.9 [36] Let \( T \) be a square in \( \mathbb{R}^2 \) and \( h \) be a smooth function such that \( \text{Supp}(h) \subseteq \text{int}(T) \). Let \( \Delta_T \) be the Laplacian on \( T \). Then, for \( \nu > 0 \),

\[ Tr_w(M_h(-\Delta_T + \nu))^{-1} = \pi \int h(u)du. \]

Proposition 6.10 Let \( \Delta^m \) be the magnetic Laplacian and \( M_g \) be the multiplication operator by the function \( g \). Then, for \( \nu > 0 \),

1. \( M_g(-\Delta^m + \nu)^{-d} \) is of Dixmier trace class.
2. \( Tr_w(M_g(-\Delta^m + \nu))^{-d} = \pi^d v^m(g) \).

Proof: Let \( h \in Dom(\Delta^m) \subseteq L^2(\mathbb{R}^2) \). We get \( gh \in Dom(\Delta_T) \) and according to Lemma 3.4 we have

\[ (\Delta^m T M_g - M_g \Delta^m_T)(h) = (\Delta_T M_g - M_g \Delta)(h) + (T^0_{jk} M_g - M_g T^0_{jk})(h). \]  
(6.19)

Using the Proof of Theorem 5.2 [3], we have

\[ (\Delta^m_T M_g - M_g \Delta^m_T)(h) = Bh + Ah, \]  
(6.20)

where \( B \) is given as in [3], and

\[
A = -M^T g + 2i \left( \sum_{j=1}^{2} \pi i G_j M_{\partial j} g + 2\pi^2 G^2_j \right) \\
+ 2ig^{jk} \sum_{j \neq k} (M_{\partial j} g - 2\pi i G_j)(\partial_k - 2\pi i G_k). 
\]  
(6.21)

From the relation (5.2) [3], we have

\[ M_g(-\Delta^m + \nu)^{-1} - (-\Delta^m_T + \nu)^{-1} M_g = (-\Delta_T + \nu)^{-1} B(-\Delta + \nu)^{-1} \\
+ [T^0_{jk}, M_g](h). \]  
(6.22)

Using the fact that the derivations \( d_j \) for \( j \in \{1, 2\} \) commute, we obtain

\[ M_g(-\Delta^m + \nu)^{-1} - (-\Delta^m_T + \nu)^{-1} M_g = (-\Delta_T + \nu)^{-1} B(-\Delta + \nu)^{-1}. \]  
(6.23)

The result is obtained following step by step [3].

Let us consider now the noncommutative case. Using the Lemma 3.4 in Section 2, we get the following definition:
Def 6.11 The magnetic Lindbladian $L^m_0$ generated by the magnetic derivation on $B$ is defined as follows:

$$L^m_0(a(g)) = L_0(a(g)) + a(T^0_{jk} g),$$

where $(j, k) \in \{1, 2\}$, $g \in C_c^\infty(\mathbb{R}^{2d})$ and $L_0$ is the Lindbladian on $\mathbb{R}^{2d}$.

Since $L^m_0 = \Delta + T^0_{jk}$, and the operator $T^0_{jk}$ is self-adjoint in $L^2(\mathbb{R}^{2d})$, according to [4], the operator $\Delta$ has a self-adjoint extension in $L^2(\mathbb{R}^{2d})$. We deduce that $L^m_0$ is a self-adjoint operator in the same space. Thus, in this case, the magnetic heat semigroup is given by $T^m_t = e^{tL^m_0}$.

It is obvious to see that

$$It is similar to the Proof of the Proposition 6.7. The magnetic integral operator $a(g)T^m_t$ in the Hilbert space $\mathcal{H}$ has the kernel given by

$$K^m_t(u, v) = g(u - v)e^{-(1/2)t}e^{-i(p(u, v)/2).}

It is obvious to see that $K^m_t(u, v) = K^m_t(u, u)$. As $K^m_t$ is continuous in $\mathbb{R}^{2d}$, we deduce the continuity of $K^m_t$ in the same space. Since

$$Tr(a(g)T^m_t) = f(G_1, G_2)Tr(a(g)T_t),$$

and using [14], we have

$$Tr(a(g)T^m_t) = (2\pi)^d t \cdot f(G_1, G_2).$$

Thus, we get the result.

Prop 6.12 The volume form is given by

$$v^m(a(g)) = f(G_1, G_2) \int g du$$

Proof: It is similar to the Proof of the Proposition 6.7. The magnetic integral operator $a(g)T^m_t$ in the Hilbert space $\mathcal{H}$ has the kernel given by

$$K^m_t(u, v) = g(u - v)e^{-(1/2)t}e^{-i(p(u, v)/2)\}.\)

It is obvious to see that $K^m_t(u, v) = K^m_t(u, u)$. As $K^m_t$ is continuous in $\mathbb{R}^{2d}$, we deduce the continuity of $K^m_t$ in the same space. Since

$$Tr(a(g)T^m_t) = f(G_1, G_2)Tr(a(g)T_t),$$

using [14], we have

$$Tr(a(g)T^m_t) = (2\pi)^d t \cdot f(G_1, G_2).$$

Thus, we get the result.

6.2 Magnetic quantum stochastic process

The stochastic process associated with the heat semigroup in the case of classical (or noncommutative $C^*$-algebra) of $C_0(\mathbb{R}^{2d})$ is the well known standard Brownian motion [36]. For the noncommutative $C^*$-algebra $A$, Sinha and co-workers [3] studied the stochastic process associated with the heat semigroup in the case of $B(L^2(\mathbb{R}))$ by the Stone-von Neumann theorem on the representation of the Weyl relation [14]:

$$(U_\alpha f)(x) = f(x + \mu), \quad (V_\mu f)(x) = e^{i\alpha \cdot x} f(x),$$

with $q_k, p_k (k = 1, 2, \ldots, n)$ playing the role of generators of $V_\mu$ and $U_\alpha$, respectively. They are the position and momentum operators in the Schrodinger representation. We consider the case $d = 1$ and the following quantum stochastic differential equation in $L^2(\mathbb{R}) \otimes \Gamma(L^2(\mathbb{R}^+, C^2))$ [3]:

$$\begin{cases}
    \frac{dX_t}{dt} = X_t[-ip \, dw_1(t) - iq \, dw_2(t) - \frac{1}{2}(p^2 + q^2)dt], \\
    X_0 = I
\end{cases}$$

where $w_1, w_2$ are independent Brownian motions. The next Corollary gives the magnetic Laplacian in the noncommutative 2-dimentional space.
**Corollary 6.13** Let $q$ and $p$ be the magnetic momentum and position operators, respectively. Then, the unperturbed noncommutative magnetic Laplacian is given by:

$$L^m_0 = L_0 + 1/2 \sum_{j=1}^{2} \left( p_j^2 + 2\pi G_j q_j - 4\pi^2 G_j^2 \right) - \sum_{j \neq k} g^{jk} (q_j - 2\pi G_j)(q_k - 2\pi G_k). \quad (6.31)$$

**Proof:** It is straightforward. □

In this work, let us consider the magnetic quantum stochastic differential equation (m.q.s.d.e) in $L^2(\mathbb{R}) \otimes \Gamma(L^2(\mathbb{R}^+, C^2))$:

$$\begin{cases} dY_t = dX_t + Y_t \tilde{T}_{jk} dt, \quad (j, k) \in \{1, 2\} \\ Y_0 = I, \end{cases} \quad (6.32)$$

where $dX_t$ is the quantum stochastic differential equation (6.30).

**Theorem 6.14**

1. The magnetic quantum stochastic differential equation (m.q.s.d.e) (6.32) has a unique unitary solution.

2. If we set $h_t(y) = Y_t(y \otimes I_t)Y^*_t$, then for all $y \in D^\infty_\theta$, $h_t$ satisfies the magnetic quantum stochastic differential equation (m.q.s.d.e):

   $$dh_t(y) = h_t(-i[p, y])dw_1(t) + h_t(-i[q, y])dw_2(t) + h_t(L^m(y))dt. \quad (6.33)$$

Furthermore,

$$dh_t(y) = dj_t(y) + h_t(\hat{R}_{jk}(y))dt, \quad (j, k) \in \{1, 2\}, \quad (6.34)$$

where

$$\hat{R}_{jk}(y) = \tilde{T}_{jk}y + y\tilde{T}_{jk}. \quad (6.35)$$

Moreover,

$$Eh_t(y) = e^{L_t(y)} + H_t(\hat{R}_{jk}(y)), \quad (6.36)$$

where $H_t$ is a primitive of $h_t$.

**Proof:**

1. According to [3], the quantum stochastic differential equation $dX_t$ has a unique unitary solution. Then, we deduce that $dY_t$ has a unique unitary solution.

2. Using Itô’s formula and table [8][9], we have

   $$dh_t(y) = h_t(-i[p, y])dw_1(t) + h_t(-i[q, y])dw_2(t) + h_t(L^m(y))dt, \quad (6.37)$$

where

$$L^m(y) = L^m_0 y + yL^m_0 + y(p^2 + q^2). \quad (6.38)$$

Moreover, $L^m_0 = -\frac{1}{2}(p^2 + q^2) + \tilde{T}_{jk}$ and according to [3], the equation (6.34) yields. The integral form of equation (6.34) implies

$$h_t(y) = j_t(y) + H_t(\hat{R}_{jk}(y)). \quad (6.39)$$

Performing the same computation as in Section 1, we get the vacuum expectation. Using the linearity of vacuum expectation, and [3], we obtain the result. □
6.3 Properties of the mqsde solution

The goal of this section is to derive different moments (first, second, and \( r \)th moments) and the variance of the mqsde solution \([6.34]\). We start by recalling the following Proposition, while the main result is contained in the theorem below.

Proposition 6.15 \([29]\) For every \( X \in \mathcal{B}_0 \), there exists a sequence \( \{v_t^{(n)}(X)\} \), \( t \geq 0 \) of \((\xi, v_0, \mathcal{M})\)-adapted processes satisfying

1. \( v_t^{(0)}(X) = X \),
2. \( v_t^{(n)}(X) = X + \int_0^t \sum_{i,j \geq 0} v_s^{(n-1)}(\theta_s^i(X))d\Lambda_t^v(s) \).

Theorem 6.16 Let \( h_t(x) \) be the solution of the magnetic quantum stochastic differential equation \([6.34]\) and \( j_t(x) \) the solution of the quantum stochastic differential equation \([3.22]\). Then, for \( x \in \mathcal{D}_\infty^\theta \), the \( r \)th-moment and the variance are given by:

\[
\begin{align*}
\mathbb{E}h_t^{(r)}(x) &= \mathbb{E}h_0(x) + \int_0^t \mathbb{E}h_s^{(r-1)}(\mathcal{L}^m(x))ds, \\
\text{Var}h_t(x) &= (\mathcal{L}^m)^{-1}(x)e^{t\mathcal{L}^m}.\mathcal{L}^m(x)(1 - e^{-t\mathcal{L}^m} - e^{t\mathcal{L}^m}(x)).
\end{align*}
\]

In addition, for \((j, k) \in \{1, 2\}\), we get

\[
\begin{align*}
\mathbb{E}h_t^{(r)}(x) &= \mathbb{E}j_t^{(r)}(x) + \int_0^t \mathbb{E}j_s^{(r-1)}(\mathcal{T}_{jk})ds, \\
\text{Var}h_t(x) &= \text{Var}j_t(x) - m(G_t(\mathcal{T}_{jk}(x))), \mathcal{D}_\infty^\theta,
\end{align*}
\]

with

\[ m(y) = y^2 + 2y(\mathbb{E}j_t(x) + 1). \]

Proof: Using the integral form of \([6.34]\), the linearity of the vacuum expectation, and the following relation

\[
\int_0^t \mathbb{E}h_s(-i \sum_{k=1}^2 [p_k, x])dw_{2k-1}(s) = \int_0^t \mathbb{E}h_s(-i \sum_{k=1}^2 [q_k, x])dw_{2k}(s) = 0,
\]

we have

\[ \mathbb{E}h_t(x) = \mathbb{E}h_0(x) + \int_0^t \mathbb{E}h_s(\mathcal{L}^m(x))ds. \]

The solution of the above equation \([6.44]\) is given by:

\[ \mathbb{E}h_t(x) = \exp(t\mathcal{L}^m_0)(x). \]

According to Lemma \([3.4]\) and \([3]\), we obtain

\[ h_t(x) = j_t(x) + G_t(\mathcal{T}_{jk}(x)), \ (j, k) \in \{1, 2\}, \]

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where $G_t$ is the primitive of $h_t$, and the expectation of $h_t$ is
\begin{equation}
Eh_t(x) = Ej_t(x) + G_t(\hat{T}_{jk}(x)), \quad (j, k) \in \{1, 2\}.
\end{equation}
(6.47)

According to the above Proposition and the relation (6.43), we get
\begin{equation}
Eh_t^{(r)}(x) = Eh_0(x) + \int_0^t Eh_s^{(r-1)}(L^m(x))ds.
\end{equation}
(6.48)

For $r = 2$ and once (6.43), we have
\begin{equation}
Eh_t^{(2)}(x) = Eh_0(x) + \int_0^t Eh_s^{(1)}(L^m(x))ds,
\end{equation}
(6.49)

and the relation (6.44) implies $Eh_t^{(2)}(x) = L^{m-1}(x)e^{tL^m(L^m(x))} - I$. By definition of the variance of $h_t$ and after calculation, we obtain the second equation of the system (6.40).

Besides, using the relation between $L_0$ and $T_{jk}^0$, we obtain
\begin{equation}
h_t^{(r)}(x) = j_t^{(r)}(x) + \int_0^t h_s^{(r-1)}(T_{jk}(x))ds,
\end{equation}
(6.50)

and the linearity of the vacuum expectation gives
\begin{equation}
Eh_t^{(r)}(x) = Ej_t^{(r)}(x) + \int_0^t Eh_s^{(r-1)}(\hat{T}_{jk}(x))ds.
\end{equation}
(6.51)

For $r = 2$, we obtain $Eh_t^{(2)}(x) = Ej_t^{(2)}(x) + G_t(\hat{T}_{jk}(x))$. Thus,
\begin{equation}
\begin{aligned}
\text{Var} h_t(x) &= Var j_t(x) - G_t(\hat{T}_{jk}(x))^2 - 2Ej_t(x)G_t(\hat{T}_{jk}(x)) \\
&- G_t(\hat{T}_{jk}(x)).
\end{aligned}
\end{equation}
(6.52)

Setting $m(y) = y^2 + 2y(Ej_t(x) + 1)$ yields the result. \qed

7 Concluding Remarks

The magnetic quantum stochastic differential equation associated to the noncommutative magnetic Laplacian in a noncommutative 2−torus has been derived. It has been shown that the volume and the volume form of a noncommutative 2−torus remain invariant under a perturbation by inner derivation of the noncommutative magnetic Laplacian. The vacuum expectation of the solution to the magnetic quantum stochastic differential equation in a noncommutative 2d-dimensional space has also been computed and discussed.

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