Existence of Solution for a Conformable Fractional Cauchy Problem with Nonlocal Condition

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In this work, we prove the existence and uniqueness of mild solution of the fractional conformable Cauchy problem with nonlocal condition. We obtained these results by applying the fixed point theorems precisely to the fixed point theorem of Krasnoselskii and Banach’s fixed point theorem. At the end, we provide application.

1. Introduction

Many dynamic processes in physics, biology, economics, and other fields of application can be governed by differential evolution equations of neutral type of the following form:

\[
\frac{d}{dt} \left( x(t) - F\left( t, x\left( h_1(t) \right) \right) \right) = -A\left( x(t) - F\left( t, x\left( h_1(t) \right) \right) \right) + G\left( t, x\left( h_2(t) \right) \right). \tag{1}
\]

We replace the partial derivative by a fractional derivative [1,2] because fractional derivatives have been proven to be a very good way to model many phenomena with memory in various fields of science and engineering [1–9]. Consequently, several researchers are working to form the best definition of the fractional derivative. A new definition called conformal fractional derivative is introduced in [7].

This new fractional derivative becomes the subject of many contributions in several areas of science [9]. Motivated by the better effect of the fractional derivative and simple properties of the conformable fractional derivative, we consider model (1) in this work; we are going to study Cauchy problem with fractional derivative. Precisely, we consider fractional Cauchy problem of the following form:

\[
\begin{aligned}
\frac{d^n}{dt^n} \left( x(t) - F\left( t, x\left( h_1(t) \right) \right) \right) &= -A\left( x(t) - F\left( t, x\left( h_1(t) \right) \right) \right) + G\left( t, x\left( h_2(t) \right) \right), \\
x(0) + g(x) &= x_0 \in X = D(A^n),
\end{aligned} \tag{1.1}
\]
where \( \frac{d^n}{dt^a} \) is the conformable fractional derivative of \( a \in (0, 1) \). A is a sectorial operator which generates a strongly analytic semigroup \( (T(t)) \) on a Banach space \( X \). For more details about semigroup theory, we refer to [7]. We denote by \( \mathcal{C} \left( [0, a], D(A^a) \right) \) the banach space of continuous function from \( [0, a] \) into \( D(A^a) \) with the norm \( \|x\|_a = \sup_{t \in [0,a]} \|x(t)\|_a \). For our study, the functions \( F: [0, a] \times D(A^a) \rightarrow D(A^a) \) and \( G: [0, a] \times D(A^a) \rightarrow D(A^a) \) and \( \mathcal{F}([0, a], D(A^a)) \rightarrow D(A^a) \) and also \( h_1, h_2 \in \mathcal{C}([0, a], D(A^a)) \).

The nonlocal condition joined the main equation instead of the classical condition:wT_hemainresultsarebasedonsemigrouptheory

mathematically, physical phenomena like in electronics, in mechanics, thus making it more realistic. The nonlocal problems has brought a lot of improvement at the level of modeling, thus making it more realistic. The nonlocal condition joined the main equation instead of the classical initial condition which is necessary to model well and write mathematically, physical phenomena like in electronics, in mechanics of materials, or in biomathematics in the way closest to the reality of many phenomena in multiple disciplines. The nonlocal condition means that the initial condition depends on some future times.

In this paper, we prove the existence of mild solution of conformable fractional differential equations with nonlocal condition. The main results are based on semigroup theory combined with the Krasnoselskii fixed point theorem.

The content of this paper is organized as follows. In Section 2, we recall some preliminary facts on the conformable fractional calculus and Section 3 is devoted to prove the main result.

In this section, we recall some concepts on conformable fractional calculus.

2. Preliminaries

Definition 1. The conformable fractional derivative of \( x \) of order \( a \) at \( t > 0 \) is defined as

\[
\frac{d^ax(t)}{dt^a} = \lim_{h \to 0} \frac{x(t + he^{(a-1)h}) - x(t)}{h}. \tag{3}
\]

When the limit exists, we say that \( x \) is \((a)\)-differentiable at \( t \).

If \( x \) is \((a)\)-differentiable and \( \lim_{t \to 0^-} \frac{d^ax(t)}{dt^a} \) exists, then define

\[
\frac{d^ax(0)}{dt^a} = \lim_{t \to 0^-} \frac{d^ax(t)}{dt^a}. \tag{4}
\]

The \((a)\)-fractional integral of a function \( x \) is defined by

\[
I^a(f)(t) = \int_0^t s^{a-1} f(s) ds. \tag{5}
\]

Theorem 1. If \( x \) is a continuous function in the domain of \( I^a \), then

\[
\frac{d^a(I^a(x(t)))}{dt^a} = x(t). \tag{6}
\]

Definition 2. The fractional Laplace transform of order \( a \) of \( x \) is defined by

\[
\mathcal{L}_a(x(t))(\lambda) = \int_0^{\infty} t^{a-1} e^{-\lambda t} x(t) dt. \tag{7}
\]

The fractional Laplace transform of conformable fractional derivative is given by the following proposition.

Proposition 1. If \( x(t) \) is differentiable, then

\[
\mathcal{L}_a\left(\frac{d^ax(t)}{dt^a}\right)(\lambda) = \lambda \mathcal{L}_a(x(t))(\lambda) - x(0). \tag{8}
\]

Fractional powers of an operator.

Definition 3. Let \( A \) be a sectorial operator defined on a Banach space \( X \), such that \( \Re \text{e}(A) > 0 \); for \( a > 0 \), we note by \( A^{-a} \) the operator defined by

\[
A^{-a} = \frac{1}{\Gamma(a)} \int_0^{\infty} t^{a-1} T(t) dt. \tag{9}
\]

Definition 4. Let \( A \) be a sectorial operator defined on a Banach space \( X \), such that \( \Re \text{e}(A) > 0 \). We define the family of operators \( (A^a)_{a \geq 0} \) as follows: \( A^0 = I_X \), and for \( a > 0 \),

\[
A^a = (A^{-a})^{-1}, D(A^a) = \text{Im}(A^{-a}). \tag{10}
\]

Theorem 2. If \( (-A) \) is the infinitesimal generator of an analytic semigroup \( (T(t)) \) and if \( a \in \rho(A) \), then

1. \( D(A^a) \) is a Banach space with the norm \( \|x\|_a = \|A^a x\| \) for every \( x \in D(A^a) \).

2. \( T(t): X \rightarrow D(A^a) \) for all \( t > 0 \) and \( a \geq 0 \).

3. For every \( x \in D(A^a) \), we have \( T(t)A^ax = A^a T(t)x \).

We assume that \( M \) is a closed bounded convex subset of a Banach space \( E \):

(i) \( Ax + By \in M \) for each \( x, y \in M \).

(ii) \( A \) is continuous and compact.

(iii) \( B \) is contraction.

Then, there exists \( y \in M \) such that \( y = Ay + By \). We end these preliminaries with the notion of sectorial operator.

Definition 3: \( A: D(A) \subset X \rightarrow X \) is said to be sectorial operator of type \((M, \omega, \Theta)\) if there exists \( M > 0 \), \( \omega \in \mathbb{R} \), and \( 0 < \Theta < \pi/2 \) as follows:

1. \( A \) is closed and linear operator.

2. \( \forall \lambda \notin w + S_0 \), the resolvent \((\lambda I - A)^{-1}\) of \( A \) exists.

3. \( \forall \lambda \notin w + S_0 \), \( |(\lambda I - A)^{-1}| \leq M/|\lambda - w| \),

where \( w + S_0 = \{w + \lambda \lambda \in \mathbb{C} \text{ave}[\text{Arg}(-\lambda)] < \Theta\} \).

Theorem 2. A densely sectorial operator generates a strongly analytic semigroup \( (T(t))_{t \geq 0} \). Moreover,
\[ T(t) = \frac{1}{2\pi i} \int_{\gamma} e^{\lambda t} (\lambda I - A)^{-1} d\lambda , \] (11) with gamma being a suitable path \( \lambda \notin w + S_0 \). Now, we give the main contribution results.

3. Main Results

Before presenting our main results, we introduce the following assumptions:

(H1) \( F: [0, a] \times D(A^a) \rightarrow D(A^a) \) is continuous and there exists a constant \( L_1 > 0 \) such that \( \| F(t, x_1) - F(t, x_2) \|_a \leq L_1 \| x_1 - x_2 \|_a \) for all \( x_1, x_2 \in D(A^a) \) and for \( 0 \leq t \leq a \).

(H2) \( G: [0, a] \times D(A^a) \rightarrow D(A^a) \) is continuous and for all \( r > 0 \), there exists a function \( \mu_r \in L^\infty ([0, a], \mathbb{R}^+) \) such that \( \sup_{t \in [0, a]} \| G(t, x) \|_a \leq \mu_r(t) \).

\[
\mathcal{L}_a \left( \frac{d}{dt} x(t) - F(t, x(h(t))) \right)(\lambda) = -\mathcal{L}_a (A(x(t) - F(t, x(h(t))))) + \mathcal{L}_a (G(t, x(h(t))))(\lambda)
\]
(13)

Then,

\[
\mathcal{L}_a ((x(t) - F(t, x(h(t)))))(\lambda) = (\lambda + A)^{-1} (x(0) - F(0, x(h(0)))) + (\lambda + A)^{-1} \mathcal{L}_a (G(t, x(h(t))))(\lambda).
\] (14)

Hence,

\[
\mathcal{L}_a ((x(t) - F(t, x(h(t)))))(\lambda) = (\lambda + A)^{-1} ((x_0 - g(x) - F(0, x(h(0)))) + (\lambda + A)^{-1} \mathcal{L}_a (G(t, x(h(t))))(\lambda).
\] (15)

Then,

\[
(\lambda + A)^{-1} ((x_0 - g(x) - F(0, x(h(0)))) = \mathcal{L}_a \left( \frac{t^a}{a} \right)(\lambda)(x_0 - g(x) - F(0, x(h(0))))
\] (16)

\[
(\lambda + A)^{-1} \mathcal{L}_a (G(t, x(h(t))))(\lambda) = \mathcal{L}_a \left( \int_0^{t/a} T \left( \frac{t^a}{a} - s \right) G(s, x(h(s))) \right) ds (\lambda)
\] (17)
Therefore,

\[ \mathcal{L}_\alpha \left( (x(t) - F(t, x(h_1(t)))) \right)(\lambda) = \mathcal{L}_\alpha \left( T \left( \frac{t^\alpha}{\alpha} \right) (x_0 - g(x) - F(0, x(h_1(0)))) \right)(\lambda) 
+ \mathcal{L}_\alpha \left( \int_0^t s^{\alpha-1} T \left( \frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha} \right) G(s, x(h_2(s))) ds \right)(\lambda). \]  

(18)

According to inverse fractional Laplace transform, we find the formula

\[ x(t) - F(t, x(h_1(t))) = T \left( \frac{t^\alpha}{\alpha} \right) (x_0 - g(x) - F(0, x(h_1(0)))) 
+ \mathcal{L}_\alpha \left( \int_0^t s^{\alpha-1} T \left( \frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha} \right) G(s, x(h_2(s))) ds \right)(\lambda). \]

(19)

Then, we obtain

\[ x(t) = F(t, x(h_1(t))) + T \left( \frac{t^\alpha}{\alpha} \right) (x_0 - g(x) - F(0, x(h_1(0)))) 
+ \int_0^t s^{\alpha-1} T \left( \frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha} \right) G(s, x(h_2(s))) ds. \]

(20)

**Definition 1.** We say that \( x \in \mathcal{C}([0, a], D(A^\alpha)) \) is a mild solution of equation (2) if the following assertion is true:

\[ x(t) = F(t, x(h_1(t))) + T \left( \frac{t^\alpha}{\alpha} \right) (x_0 - g(x) - F(0, x(h_1(0)))) 
+ \int_0^t s^{\alpha-1} T \left( \frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha} \right) G(h_1s, x(v)) ds. \]

(21)

**Theorem 3.** If \( (T(t))_{t \geq 0} \) is compact and \( (H1) - (H4) \) are satisfied, then problem (2) has at least one mild solution, provided that

\[ L_1 + \sup_{t \in [0, a]} \left\| T \left( \frac{t^\alpha}{\alpha} \right) \right\| (L_2 + L_1) < 1. \]

(22)

**Proof.** Choosing

\[ r \geq \left\| F(t, x(h_1(t))) \right\|_a + \sup_{t \in [0, a]} \left\| T \left( \frac{t^\alpha}{\alpha} \right) \right\| \left( \left\| (A^\alpha) \right\|_{L^{\infty}} \right) \left\| x_0 \right\|_a + \left\| g(x) \right\|_a + \left\| F(t, x(h_1(t))) \right\|_a \
1 - \left( L_1 + \sup_{t \in [0, a]} \left\| T \left( \frac{t^\alpha}{\alpha} \right) \right\| (L_2 + L_1) \right) \]

(23)

let \( B_r = \{ x \in X \| x \|_a \leq r \} \), for \( x \in B_r \), and \( t \in [0, a] \) define the operators \( \Gamma_1 \) and \( \Gamma_2 \) by

\[ \Gamma_1(x)(t) = F(t, x(h_1(t))) + T \left( \frac{t^\alpha}{\alpha} \right) (x_0 - g(x) - F(0, x(h_1(0)))) \]

\[ \Gamma_2(x)(t) = \int_0^t s^{\alpha-1} T \left( \frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha} \right) G(s, x(h_2(s))) ds. \]

(24)
Claim 1. We prove that $\Gamma_1$ is contraction on $B_r$. We have

$$
\| \Gamma_1(x)(t) - \Gamma_1(y)(t) \|_a \leq \left\| A^a \left( F(t, x(h_1(t))) - F(t, y(h_1(t))) \right) \right\|
+ \left\| T \left( \frac{t^a}{\alpha} \right) A^a (g(y) - g(x)) \right\|
\quad \quad \quad + \left\| T \left( \frac{t^a}{\alpha} \right) A^a (F(0, y(h_1(0))) - F(0, y(h_1(0)))) \right\|
\leq L_1 \sup_{0 \leq s \leq a} \| x(s) - y(s) \|_a + \sup_{t \in [0, a]} \left\| T \left( \frac{t^a}{\alpha} \right) \right\| (L_3 + L_1) \sup_{0 \leq s \leq a} \| x(s) - y(s) \|_a.
$$

(25)

Then,

$$
\| \Gamma_1(x) - \Gamma_1(y) \|_a \leq \left( L_1 + \sup_{t \in [0, a]} \left\| T \left( \frac{t^a}{\alpha} \right) \right\| (L_3 + L_1) \right) \| x - y \|_a.
$$

(26)

Since $L_1 + \sup_{t \in [0, a]} \left\| T \left( \frac{t^a}{\alpha} \right) \right\| (L_3 + L_1) < 1$, so $\Gamma_1$ is a contraction on $B_r$.

Claim 2. We prove that $\Gamma_1(x) + \Gamma_1(y) \in B_r$ for every $x, y \in B_r$.

$$
\| \Gamma_1(x)(t) + \Gamma_1(y)(t) \|_a \leq \left\| F(t, x(h_1(t))) \right\|_a + \frac{a^a}{\alpha} \sup_{t \in [0, a]} \left\| T \left( \frac{t^a}{\alpha} \right) \right\| [\mu_r]_{\mathbb{R}^a}
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad + \sup_{t \in [0, a]} \left\| T \left( \frac{t^a}{\alpha} \right) \right\| \| x_0 \|_a + \| g(x) \|_a + \left\| F(0, x(h_1(0))) \right\|_a
\leq \left( 1 + \sup_{t \in [0, a]} \left\| T \left( \frac{t^a}{\alpha} \right) \right\| (L_3 + L_1) \right) \| x_0 \|_a + \| g(x) \|_a + \left\| F(0, x(h_1(0))) \right\|_a
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad + \frac{a^a}{\alpha} [\mu_r]_{\mathbb{R}^a}
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \leq r \left( 1 + \sup_{t \in [0, a]} \left\| T \left( \frac{t^a}{\alpha} \right) \right\| (L_3 + L_1) \right).
$$

(27)

Claim 3. We will prove that $\Gamma_2$ is continuous on $B_r$. Let $x_n \in B_r$ such that $\lim_{n \to \infty} x_n = x$. We will prove that $\lim_{n \to \infty} \Gamma_2(x_n) = \Gamma_2(x)$. We have
\[
\|\Gamma_2(x_n) - \Gamma_2(x)\|_a \leq \int_0^1 s^{\alpha - 1}\left\|T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)\right\|G(s, x_n(h_2(s))) - G(s, x(h_2(s)))\|_a ds \\
\leq \sup_{t \in [0,a]}\left\|T\left(\frac{t^\alpha}{\alpha}\right)\right\| \int_0^1 s^{\alpha - 1}\left\|G(s, x_n(h_2(s))) - G(s, x(h_2(s)))\right\|_a ds.
\]

(28)

By (H2), we have \(\|s^{\alpha - 1}(G(s, x_n(h_2(v))) - G(s, x(h_2(s))))\|_a \leq 2\mu_s s^{\alpha - 1}\) and \(\lim_{n \to \infty} G(s, x_n(h_2(s))) = G(s, x(h_2(s)))\). According to Lebesgue dominated convergence theorem, we obtain

\[
\lim_{n \to \infty} \|\Gamma_2(x_n) - \Gamma_2(x)\|_a = 0.
\]

(29)

Claim 4. We prove that \(\Gamma_2\) is compact.

Step 1. We prove that \(\{\Gamma_2(x)(t)|x \in B_1\}\) is relativatively compact in \(X\).

For some fixed \(t \in [0,a]\), let \(\epsilon \in (0,t]\) and define the operator \(\Gamma_2^\epsilon\) by

\[
\Gamma_2^\epsilon(x)(t) = \int_0^{(t-s)^{\alpha - 1} \alpha} s^{-1}T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)G(s, x(h_2(s)))ds \\
= T\left(\frac{t^\alpha}{\alpha}\right) \int_0^{(t-s)^{\alpha - 1} \alpha} s^{-1}T\left(\frac{t^\alpha - s^\alpha - \epsilon^\alpha}{\alpha}\right)G(s, x(h_2(s)))ds.
\]

(30)

According to compactness of \((T(t))_{t \geq 0}\), the set \(\{\Gamma_2^\epsilon(x)(t)|x \in B_1\}\) is relatively compact in \(X\).

We have

\[
\|\Gamma_2^\epsilon(x)(t) - \Gamma_2(x)(t)\|_a \leq \sup_{t \in [0,a]}\left\|T\left(\frac{t^\alpha}{\alpha}\right)\right\| \|\Gamma_2^\epsilon(x) - \Gamma_2(x)\|_{L^\infty([0,a],R^\alpha)} \frac{\epsilon^\alpha}{\alpha}.
\]

(31)

Therefore, \(\{\Gamma_2^\epsilon(x)(t)|x \in B_1\}\) is relatively compact in \(X\). It is clear that \(\{\Gamma_2^\epsilon(x)(0)|x \in B_1\}\) is compact.

Finally, \(\{\Gamma_2(x)(t)|x \in B_1\}\) is relativatively compact in \(X\) for all \(t \in [0,a]\).

Step 2. We prove that \(\Gamma_1(B_0)\) is equicontinuous.

Let \(t_1, t_2 \in [0,a]\) such that \(t_1 < t_2\). We have

\[
\Gamma_2(x)(t_2) - \Gamma_2(x)(t_1) = \int_0^{t_2} s^{-1}T\left(\frac{t_2^\alpha - s^\alpha}{\alpha}\right)G(s, x(h_2(s)))ds - \int_0^{t_1} s^{-1}T\left(\frac{t_1^\alpha - s^\alpha}{\alpha}\right)G(s, x(h_2(s)))ds \\
= \int_0^{t_1} s^{-1}\left(T\left(\frac{t_1^\alpha - s^\alpha}{\alpha}\right) - T\left(\frac{t_2^\alpha - s^\alpha}{\alpha}\right)\right)G(s, x(h_2(s)))ds \\
+ \int_0^{t_1} s^{-1}T\left(\frac{t_1^\alpha - s^\alpha}{\alpha}\right)G(s, x(h_2(s)))ds.
\]

(32)

Then,

\[
\Gamma_2(x)(t_2) - \Gamma_2(x)(t_1) = T\left(\frac{t_1^\alpha - t_2^\alpha}{\alpha}\right) - I \int_0^{t_1} s^{-1}T\left(\frac{t_1^\alpha - s^\alpha}{\alpha}\right)G(s, x(h_2(s)))ds \\
+ \int_0^{t_1} s^{-1}T\left(\frac{t_1^\alpha - s^\alpha}{\alpha}\right)G(s, x(h_2(s)))ds.
\]

(33)

Then,
\[ \| \Gamma_2(x)(t_2) - \Gamma_2(x)(t_1) \|_a \leq \left\| T\left( \frac{t_2^a - t_1^a}{\alpha} \right) - I \right\| \left\| \int_0^t s^{a-1} \left\| T\left( \frac{t_2^a - s^a}{\alpha} \right) \right\| G(s,x(h_2(s))) \, ds \right\| _a + \left\| \int_{t_1}^{t_2} s^{a-1} \left\| T\left( \frac{t_2^a - s^a}{\alpha} \right) \right\| G(s,x(h_2(s))) \, ds \right\| _a \]

\[ \leq \sup_{t \in [0,a]} \left\| T\left( \frac{t^a}{\alpha} \right) \right\| |\mu|_{L^\infty([0,a],R^+)} \left( \left\| T\left( \frac{t_2^a - t_1^a}{\alpha} \right) - I \left\| a^a + \frac{t_2^a - t_1^a}{\alpha} \right\| \right) \]

\[ \leq \frac{\sup_{t \in [0,a]} \left\| T\left( \frac{t^a}{\alpha} \right) \right\| |\mu|_{L^\infty([0,a],R^+)} \left( \left\| T\left( \frac{t_2^a - t_1^a}{\alpha} \right) - I \left\| a^a + \frac{t_2^a - t_1^a}{\alpha} \right\| \right)}{\alpha} < 1. \]  

We conclude that \( \Gamma_2(x), x \in B_r \) are equicontinuous at \( t \in [0,a] \). By using Arzela Ascoli theorem, we obtain that \( \Gamma_2 \) is compact. Finally, the Krasnosel’skii theorem helps us to complete the proof.

Uniqueness of mild solution.

**Theorem 4.** Assume that \((H1) - (H5)\) hold, then the Cauchy problem (2) has a unique solution, provided that

\[(Px)(t) = F(t,x(h_1(t))) + T\left( \frac{t^a}{\alpha} \right)(x_0 - g(x) - F(0,x(h_1(0))) \]

\[ + \int_0^t s^{a-1} T\left( \frac{s^a}{\alpha} \right) G(s,x(h_2(s))) \, ds. \]  

Next, let \( x, y \in C([0,a],D(A^a)) \), we have

\[(Px)(t) - (Py)(t) = F(t,x(h_1(t))) - F(t,y(h_1(t))) \]

\[ + T\left( \frac{t^a}{\alpha} \right)(g(y) - g(x) + F(0,y(h_1(0))) - F(0,x(h_1(0)))) \]

\[ + \int_0^t s^{a-1} T\left( \frac{s^a}{\alpha} \right)(G(s,x(h_2(s))) - (G(s,y(h_2(s)))) \, ds. \]  

Then,

\[ \| (Px)(t) - (Py)(t) \|_a \leq A^a(F(t,x(h_1(t))) - F(t,y(h_1(t))) \]

\[ + \| T\left( \frac{t^a}{\alpha} \right) A^a(g(y) - g(x)) \| + \| T\left( \frac{t^a}{\alpha} \right) A^a(F(0,y(h_1(0))) - F(0,x(h_1(0)))) \| \]

\[ + \int_0^t s^{a-1} \left\| T\left( \frac{s^a}{\alpha} \right) A^a(G(s,x(h_2(s))) - (G(s,y(h_2(s)))) \, ds. \]
Hence,

\[
\| (P_\alpha x)(t) - (P_\alpha y)(t) \|_A \leq \left( 1 + \sup_{t \in [0,a]} \left\| T \left( \frac{t^n}{\alpha} \right) \right\| \right) L_1 \sup_{0 \leq s \leq a} \| x(s) - y(s) \|_A
\]

\[
+ L_3 \sup_{t \in [0,a]} \left\| T \left( \frac{t^n}{\alpha} \right) \right\| \sup_{0 \leq s \leq a} \| x(s) - y(s) \|_A + \sup_{t \in [0,a]} \left\| T \left( \frac{t^n}{\alpha} \right) \right\| \frac{a^n}{\alpha} L_2 \sup_{0 \leq s \leq a} \| x(s) - y(s) \|_A.
\]

Therefore,

\[
\| (P_\alpha x) - (P_\alpha y) \|_A \leq \left( L_1 + \left( L_1 + L_3 + \frac{a^n}{\alpha} \right) \right) \sup_{t \in [0,a]} \left\| T \left( \frac{t^n}{\alpha} \right) \right\| \| x - y \|_A.
\]

Since \( L_1 + \left( L_1 + L_3 + \frac{a^n}{\alpha} \right) \sup_{t \in [0,a]} \| T (t^n/\alpha) \| < 1 \), as a consequence, \( P_\alpha \) has a unique fixed point in \( C^1([0,a], D(A^\alpha)) \).

4. Application

We consider the conformable fractional Cauchy problem of the following form:

\[
\begin{aligned}
&\frac{d^{1/2}}{dt^{1/2}} \left( u(t, \xi) - \int_0^\pi \int_0^\pi b(t, \xi, \sigma) u(\sin t, \sigma) d\sigma \right) = \frac{\partial^2}{\partial x^2} \left( u(t, \xi) - \int_0^\pi \int_0^\pi b(t, \xi, \sigma) u(\sin t, \sigma) d\sigma \right) + \psi(t, \xi), \\
&u(t, 0) = u(t, \pi) = 0, \\
&u(0, \xi) - \int_0^\pi k(\xi, \sigma) u(0, \sigma) d\sigma = u_0(\xi), \quad 0 \leq \xi \leq 1.
\end{aligned}
\]

With \( X = L^2([0,\pi], \mathbb{R}) \), we define operator \( A \) by \( Af = -f' \) and \( D(A) = H^2_0([0,\pi]) = \{ f \in X: f', f'' \in X \text{ and } f(0) = f(\pi) = 0 \} \) with the norm \( 2 \).

(i) \( b: [0,1] \times [0,\pi] \times [0,\pi] \rightarrow \mathbb{R} \) is class \( C^1 \) and \( b(t,.0) = b(t,.\pi) \).

(ii) \( \psi: [0,1] \times \mathbb{R} \rightarrow \mathbb{R} \) is continuous and there exists a constant \( a_0 > 0 \) such that

\[
|\psi(t, \xi_1) - \psi(t, \xi_2)| \leq a_0 |\xi_1 - \xi_2|, \quad 0 \leq t \leq 1.
\]

(iii) \( k: [0,\pi] \times [0,\pi] \rightarrow \mathbb{R} \) is class \( C^1 \) and \( k(.,0) = k(.,\pi) \).

40 For \( \alpha = 1/2, \tau \in [0,1], v \in D(A^{1/2}) \) with \( D(A^{1/2}) = \{ f \in X/ \sum_{n=0}^{\infty} n(f, e_n) \} \) and \( w \in E = C^1([0,1], D(A^{1/2})) \), we define \( F(t,v)(.) = \int_0^\pi b(t,.,\sigma) v(\sigma) d\sigma \), \( G(t,v)(.) = G(t,\sigma), \quad g(w)(.) = \int_0^\pi k(\sigma,\xi) w(\sigma) d\sigma \), and \( h_1(t) = h_2(t) = \sin t \quad A = \partial^2 / \partial x^2 \).

The problem takes the following form:

\[
\begin{aligned}
&\frac{d^\alpha}{dt^\alpha} \left( x(t) - F(t, x(h_1(t))) \right) = -A(x(t) - F(t, x(h_1(t)))) + G(t, x(h_2(t))) \\
&x(0) + g(x) = x_0.
\end{aligned}
\]
which implies that hypothesis $(H1)$ is verified.

Let $v_1, v_2 \in D(A^{1/2})$ and $\xi \in [0, \pi]$. We know that
\[
\|G(t, v_1) - G(t, v_2)\| \leq (t, v_1)(\xi) - G(t, v_1)(\xi)]^2d\xi \\
\leq a_0^2 \int_0^\pi |v_1'(\xi) - v_2'(\xi)|^2d\xi \\
\leq a_0\|v_1 - v_2\|_{1/2}^2,
\]
and hypothesis $(H5)$ is verified.

From the equality of Poincare, we have
\[
\|g(v)\|_{1/2} \leq \left( \int_0^\pi \left( \int_0^\pi \left( \frac{\partial}{\partial \xi} k(\xi, \sigma) \right)^2 d\sigma \right)^2 d\xi \right)^{1/2},
\]
\[\|g(v)\|_{1/2} \leq \left( \int_0^\pi \left( \int_0^\pi \left( \frac{\partial}{\partial \xi} k(\xi, \sigma) \right)^2 d\sigma \right)^2 d\xi \right)^{1/2}. \quad (47)
\]

Then, $(H3)$ is verified.

Moreover, \( \sup_{0 \leq \xi \leq 1} (\int_0^\pi \left( \frac{\partial}{\partial \xi} b(\xi, \sigma) \right)^2d\xi + (\int_0^\pi (\partial/\partial \xi k(\xi, \sigma))^2d\xi + 2a_0)M = 1 \). As a consequence, (41) has a unique fixed point in \( C([0, a], D(A^a)). \)

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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