1 Introduction

The theory of complex dynamics started in the 1870’s with works by Schroder ([S]) in one complex dimension. Complex dynamics in $\mathbb{C}^2$ began with Fatou ([Fa]) in the second decade of this century. There has been a lot of work in finite-dimensional complex dynamics, especially in the last two decades, see for example ([F]) and references therein.
In this paper we will make a first step towards a theory of complex dynamics in infinite dimension. The main initial problem is to define the basic concepts such as Chaotic behaviour etc. For this we will center our discussion around a concrete example. We also need to find an example which is sufficiently elementary, mathematically simple and easily modified to create distinct dynamical behaviour, so that it can be used for testing concepts. Hence we will find an infinite dimensional complex manifold $M$, a holomorphic map $F : M \to M$ and investigate chaotic features, or lack thereof. One natural way to measure the dynamics is to compare with a finite-dimensional approximation, $f : N \to N$ since finite-dimensional systems have been extensively studied. Another advantage to this comparison is that infinite dimensional manifolds are not locally compact while comparisons with the finite dimensional space provides a useful substitute.

The maps $f$ and $F$ can be compared by defining a map $\Lambda : M \to N$ and comparing $f^n \circ \Lambda$ and $\Lambda \circ F^n$. We call such $\Lambda$ quasiconjugacies.

We choose our example from the field of Quantum Chaos. This is very natural because solutions of the Schrodinger equation are holomorphic maps $F : M \to M$ on infinite-dimensional complex projective Hilbert space, and can be compared with finite-dimensional maps $f : N \to N$ describing the classical case. Quantum Chaos is also a reasonable topic since it is a field without well defined dynamical concepts and often without mathematical rigour.

To be as specific as possible, we will focus on one of the most basic problems of dynamics, namely whether orbits of $F$ and $f$ are bounded. The phenomenon that some orbits of $f$ are unbounded while all orbits of $F$ are bounded is called localization. Our example is of this type, see Theorem 4.5 and Corollary 4.8.

2 Quasiconjuagies

Suppose $F : M \to M$ and $f : N \to N$ are two dynamical systems. Recall that $F$ and $f$ are semi-conjugate if there is a map $\lambda : M \to N$ for which $f \circ \lambda = \lambda \circ F$.

It might happen that $dim(M) > dim(N)$ and that $F$ mathematically models a situation more precisely than $f$. In that case we introduce the
notion of quasiconjugacy, \( \Lambda : M \rightarrow N \). For that, we decompose \( M \) into three disjoint subsets, depending on the (assumed) size of the difference between \( f \circ \Lambda \) and \( \Lambda \circ F \). So we write, inspired by physics, \( M = M_Q \cup M_{SC} \cup M_C \), and call these the Quantum, Semi-Classical and Classical regions. We assume that the differences are largest in the Quantum region and smallest in the Classical region. The system \((N, f)\) is called Classical and \((M, F)\) is said to be the Quantization of \((N, f)\).

We say that \( \Lambda \) satisfies the Correspondence Principle if the difference between \( f^n \circ \Lambda \) and \( \Lambda \circ F^n \) is within some small bound (to be decided) on \( M_C \) while \( f^n \circ \Lambda \) and \( \Lambda \circ F^n \) have sufficiently close qualitative features on \( M_{SC} \) (again in some sense to be made precise). Here we might also put bounds on the number of iterates.

In discussing quasiconjugacies, one often should think of quasiconjugacy as a working hypothesis. In fact, the conclusion of a research might rather be that a supposed quasiconjugacy actually is not a quasiconjugacy or at least fails to be for certain parameters.

Let \( u : M \rightarrow \mathbb{R} \) (or more generally \( \mathbb{R}^n \)) and \( v : N \rightarrow \mathbb{R} \) be given functions, we call them observables. The underlying working hypothesis is then usually that \( u \) and \( v \) measure the same phenomenon on the respective manifolds. We say that \( F \) localizes (with respect to \( u, v \)) if for every \( x \in M \) the sequence \( \{u(F^n(x))\} \) is bounded, while there is a \( y \in N \) for which \( \{v(f^n(y))\} \) is unbounded. This phenomenon has been experimentally and numerically observed in some quantum systems \( M \) for which the classical system \( N \) is chaotic according to numerical experiments. This then is considered a failure of the Correspondence Principle. Another problem for the correspondence principle has been found in resonances, where \( u \) can be rigorously proved to go to infinity at a faster pace than indicated numerically for \( v \).

Here we discuss briefly an example arising from quantum mechanics where \( N \) is of complex dimension 1 and \( M \) is an infinite-dimensional complex manifold. Here \( f \) is a piecewise holomorphic map and \( F \) is holomorphic.

For other rigorous works on related systems, see Combescure ([Co]).

Our example has the advantage that it is sufficiently simple that the classical and quantized versions can be analyzed rigorously with elementary means, as opposed to the more complicated standard map ([Ch]) and its
quantized counterpart, the kicked rotor ([IS]). Our main result is a mathematically rigorous proof that localization takes place. We believe this is the first such rigorous proof, previous results have been based on computer experiments. Our main results is for the case which is considered furthest from resonance, namely, with Golden Mean spacing between kicks. Of course, it would be very interesting to extend this proof to the kicked rotor case.

3 The two dynamical systems

3.1 The 1−dim complex dynamical system \((N, f)\)

We will define a piecewise holomorphic map \(f : N := \mathbb{C}^* \to \mathbb{C}^*\), the punctured complex plane, \(\mathbb{C}^* = \mathbb{C} \setminus \{0\}\). Here \(f\) will be the composition of two maps

\[ f = \Phi \circ \kappa \]

where \(\Phi\) is a rotation and \(\kappa\) is a piecewise defined scaling.

We set \(\Phi(z) = e^{i\lambda} z\) for some real parameter \(\lambda\) and

\[
\begin{align*}
\kappa(z) &= e^{-K} z \text{ if } \text{Im}(z) \geq 0, \\
\kappa(z) &= e^{K} z \text{ if } \text{Im}(z) < 0
\end{align*}
\]

where \(K\) is a nonzero real parameter. The set of points with bounded orbit, i.e. bounded away from 0 and \(\infty\), depends on \(\lambda\) but not on \(K\).

Before introducing \(M\) and \(F\), we rewrite \(f\) using variables \(\theta = \arg z\) and \(P = \log |z|\) and consider \(\Phi\) and \(\kappa\) as maps given by Hamiltonian flows on the cylinder \(T^1 \times \mathbb{R}\) also called \(N\). This allows us to set up the corresponding Schrödinger equations.

3.2 \((N, f)\) as a Hamiltonian system on a cylinder

Physically speaking, we model a particle moving on a circular, concentric orbit in an evenly charged spherical plasma and being kicked by an external field at given evenly spaced times.
We use the variables $\theta$ and $P$ to denote angle $\theta \in [0,2\pi)$ and angular momentum $P \in \mathbb{R}$ respectively of the particle. Our manifold $N$ is $T^1 \times \mathbb{R}$. The particle moves with constant speed between kicks with motion governed by the Hamiltonian $C(\theta, P) = \omega P$ where $\omega$ is the angular velocity. So the equations of motion are

$$\frac{d\theta}{dt} = \frac{\partial C}{\partial P} = \omega$$
$$\frac{dP}{dt} = -\frac{\partial C}{\partial \theta} = 0$$

If the time between kicks is $T$, this gives rise to a time-$T$ map on $N$,

$$\Phi(\theta, P) = (\theta + \omega T, P).$$

This corresponds to setting $\lambda = \omega T$.

We assume that the motion of the particle is governed by a Hamiltonian $C'(\theta, P) = \frac{KH(\theta)}{\epsilon}$ during kicks, where $H : T^1 \times \mathbb{R}$ is continuous and piecewise smooth, $K$ is a parameter called the kick strength and $\epsilon$ is the duration of the kick. This gives rise to a time-$\epsilon$ map $\kappa : N \rightarrow N$ given by $\kappa(\theta, P) = (\theta, P - KH'(\theta))$. Our example corresponds then to the choice of $H(\theta)$ being the tent map, normalized so that $\int_0^{2\pi} H(\theta) = 0$.

$$H(\theta) = \theta - \frac{\pi}{2} \text{ if } 0 \leq \theta \leq \pi,$$
$$H(\theta) = \frac{3\pi}{2} - \theta \text{ if } \pi \leq \theta \leq 2\pi$$

### 3.3 The $\infty$-dim complex dynamical system $(M, F)$

The complex manifold $M$ consists of projective Hilbert space $\mathbb{P}L^2([0,2\pi])$, where $L^2([0,2\pi])$ is the complex Hilbert space of complex valued $L^2$ functions. We can identify $M$ with the boundary of the unit ball in $L^2([0,2\pi])$ when we identify $\psi$ and $e^{i\theta}\psi$. 

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The map \( \Phi : N \to N \) corresponds to a map \( \Phi' : M \to M \) obtained by solving the Schrödinger equation with Hamiltonian \( C(\theta, P) = \omega P \). Since \( P \) corresponds to the operator \(-i\hbar \frac{\partial}{\partial \theta}\) on \( L^2 \), we get for \( L^2 \) functions on \([0, 2\pi)\):

\[
\begin{align*}
    i\hbar \frac{\partial \psi}{\partial t} &= -i\hbar \omega \frac{\partial \psi}{\partial \theta}, \text{ so} \\
    \psi(\theta, t) &= \psi(\theta - \omega t) \\
    (\Phi'(\psi))(\theta) &= \psi(\theta - \omega T)
\end{align*}
\]

Similarly the map \( \kappa : N \to N \) corresponds to the map \( \kappa' : M \to M \) obtained by solving the Schrödinger equation with Hamiltonian \( C' = \frac{KH(\theta)}{\epsilon} \), so

\[
\begin{align*}
    i\hbar \frac{\partial \psi}{\partial t} &= \frac{KH(\theta)}{\epsilon} \psi, \text{ hence} \\
    \psi(\theta, t) &= e^{-i\frac{KH(\theta)}{\epsilon} t} \psi(\theta) \\
    (\kappa'(\psi))(\theta) &= e^{-i\frac{KH(\theta)}{\kappa} t} \psi(\theta)
\end{align*}
\]

Finally we define \( F : M \to M \) by \( F := \Psi' \circ \kappa' \).

### 3.4 The Quasiconjugacy

It is convenient to write \( \psi \in M \) as Fourier series, \( \psi = \sum_{n=-\infty}^{n=\infty} a_n e^{in\theta} \). The \( \{e^{in\theta}\} \) are then eigenfunctions for the operator corresponding to \( P \), with eigenvalues \( \{n\hbar\} \). Since by the uncertainty principle \( \Delta \theta \Delta P \geq \hbar \), if \( \psi \) has most of its \( L^2 \) norm in an angle \( \Delta \theta \leq \frac{1}{20} \) say, then \( \Delta P \geq 20\hbar \). We use this to define the Quantum region to consist of those \( \psi \) for which \( \sum_{|n|\leq 20} |a_n|^2 > .9 \). At the other extreme, when the particle is in an eigenstate when \( |n| > 200 \) say, the particle is easily ionized by an external field, and one defines the Classical region to consist of those \( \psi \) for which \( \sum_{|n|>200} > .9 \). The remaining \( \psi \) belong to the Semi-Classical region. These definitions are of course somewhat arbitrary, and should be modified as appropriate in various cases.
If the particle is kicked with a change in angular momentum on the order of \( \hbar \), and the kicks are random, then one can expect that the angular momentum grows like \( \sqrt{m} \) where \( m \) is the number of iterates. This means that a particle in the Quantum region can be expected to ionize, i.e. reach the Classical region after about \( 4 \times 10^4 \) iterations. This then is a reasonable upper bound on the number of iterations one ought to consider.

Next we define the semi-conjugacy \( \Lambda : M \to N \). So, given \( \psi \in M \), we need to associate a \( (\theta, P) = (\theta(\psi), P(\psi)) = \Lambda(\psi) \in N \). This is not canonical, for example, if \( \psi \equiv \frac{1}{\sqrt{2\pi}} \), the mass is evenly distributed on the circle. (There is no canonical center on a compact manifold.) We define

\[
\theta(\psi) = \sup_{0 < \zeta < 2\pi} \left\{ \int_{0}^{\zeta} |\psi|^2 < \frac{1}{2} \right\},
\]

so the particle can be found with equal probability in \((0, \theta(\psi))\) and in \((\theta(\psi), 2\pi)\). Of course this selects \( \theta = 0 \) as a privileged point, which is rather arbitrary.

It is easier to define \( P(\psi) \). We set

\[
P(\psi) = \sup_m \left\{ \sum_{n \leq m} |a_n|^2 \leq \frac{1}{2} \right\}.
\]

We notice then that the map \( \Lambda : M \to N \) is not onto since the values of \( P(\psi) \) always is an integer.

Next we define the observables \( u : M \to \mathbb{R} \) and \( v : N \to \mathbb{R} \) using the angular momentum: For every \( \psi \in M \), set \( u(\psi) = \sup_m \{ \sum_{|n| \leq m} |a_n|^2 \leq \frac{1}{2} \} \) and for \( (\theta, P) \in N \), set \( v(\theta, P) = |P| \).

**Remark 3.1** In our example we will be able to study the long-term behaviour of \( u, v \) directly without a detailed analysis of \( \Lambda \). The example suffers from a common defect, namely we let \( n \to \infty \), instead of restricting to physically meaningful \( n \) as indicated above. This aspect should be analyzed. Note also that analysis is often done by letting \( \hbar \to 0 \). This is similarly physically meaningless since \( \hbar \) is a constant. We keep \( \hbar \) fixed here.
3.5 The iterates.

We can write down formulas for the \(n\)th iterates.

\[
H_n(\theta) := \sum_{k=1}^{n} H(\theta - k\omega T)
\]

\[
(F^n(\psi))(\theta) = e^{-\frac{1}{n}KH_n(\theta)}\psi(\theta - n\omega T)
\]

\[
\tilde{H}'_n(\theta) := \sum_{k=0}^{n-1} H'(\theta + k\omega T)
\]

\[
f^n(\theta, P) = (\theta + n\omega T, P - K\tilde{H}'_n(\theta))
\]

We will consider the case when \(\frac{\omega T}{2\pi} = \frac{\sqrt{5}+1}{2}\), the case of the Golden Mean rotation. This is the case where one might perhaps expect the highest amount of diffusion in the angular momentum.

4 The Main Theorems

4.1 Remarks on the Golden Mean

To carry out our investigation of the localization properties we will at first recall some simple properties of the Golden Mean. For ease of reference we include proofs. We write \(r = \frac{\sqrt{5}+1}{2}\) as the limit of the fractions \(r_n = \frac{p_n}{q_n}\) where \(\{q_n\}_{n=1}^{\infty} = 2, 3, 5, 8, \ldots\) is a Fibonacci sequence and \(p_n = q_{n+1}\). Note that \(q_{3n+1}\) is even, the others are odd and \(r_{n+1} - r_n = \frac{(-1)^{n+1}}{q_{n+1}q_n}\). Notice that for large \(n\) we have the estimate \(|r_n - r| < c/(q_n)^2\) for some constant \(c < 1/2\).

We have that \(r_n < r < r_k\) for \(n\) even and \(k\) odd.

Consider the map on the unit circle, \(T: f(\theta) = \theta + 2\pi r\).

**Lemma 4.1** Let \(\theta \in T\). Then for each large \(n\) the number of iterates \(\{f^k(\theta)\}_{k=1}^{q_n}\) in the upper half circle \([0, \pi]\) (or \((0, \pi))\) is in the interval \([\frac{q_n}{2} - 3, \frac{q_n}{2} + 3]\).

**Proof:** Divide the interval \([0, 2\pi]\) into \(q_n\) equal intervals \(I_j = [\frac{2\pi j}{q_n}, \frac{2\pi (j+1)}{q_n})\), \(j = 0, \ldots q_n - 1\). Notice that if we ignore the error between \(r_n\) and \(r\), the map simply is a permutation of the \(I_j\). If there was no error,
then the number of points in the orbit in the upper half circle is exactly half, except that $q_n$ is sometimes odd and some points land exactly on the endpoints of the intervals. However, the errors between $r$ and $r_n$ are very small and even after $q_n$ iterates they are less than half the length of the intervals, i.e. at most $2\pi c/q_n$. This makes at most an error of $\pm 1$ in the counting.

\section*{Lemma 4.2}
Any positive integer $k$ can be written as a strictly increasing sum $q_{j_1} + \ldots + q_{j_\ell}$ of Fibonacci numbers where $\ell \leq \frac{\ln k}{\ln 1.5}$.

\textbf{Proof:} We prove this estimate by induction. Note that $1.5q_n \leq q_{n+1} \leq 2q_n$. Let $q_\ell$ be the largest Fibonacci number less than or equal to $k$. Then

\begin{align*}
k - q_\ell & \leq q_{\ell+1} - q_\ell \\
& = q_{\ell-1} \\
& \leq \frac{1}{1.5} q_\ell \\
& \leq \frac{1}{1.5} k.
\end{align*}

Hence we can immediately extend the above estimate on the number of points on an orbit in the upper half circle:

\section*{Lemma 4.3}
Let $k$ be any positive integer. Then for any $\theta \in [0, 2\pi]$ the number of points in $[0, \pi]$ (or $(0, \pi)$) of the orbit $\{f^n(\theta)\}_{1 \leq n \leq k}$ differs from $k/2$ by at most $3\frac{\ln k}{\ln 1.5}$.

Our next goal is to estimate the functions $H_{q_n}$ and their derivatives.

\section*{Lemma 4.4}
$H_{q_n}$ is a Lip 1 function, $|H'_{q_n}| \leq 3$ and $|H_{q_n}(\theta)| \leq C \frac{n}{1.5^n}$ for some fixed constant $C$. 

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Proof: We carry out the details in the case $r_n < r$ and $q_n$ even, the other cases are similar. For this, observe that at least for large $n$ there exist $0 < \alpha_j, \beta_j < \frac{1}{2q_n^2}$ and for each $j = 1, \ldots, q_n$ corresponding $0 \leq \ell = \ell(j), k = k(j) \leq q_n - 1$ so that

\[
\begin{align*}
  f^\ell\left(\frac{\pi(j + 1)}{q_n} - \alpha_j, \frac{\pi(j + 1)}{q_n}\right) &= [\pi, \pi + \alpha_j] \\
  f^k\left(\frac{\pi(j + 1)}{q_n} - \beta_j, \frac{\pi(j + 1)}{q_n}\right) &= [0, \beta_j]
\end{align*}
\]

Next we calculate the difference between $\ell$ and $k$. But note that since $r = \frac{q_n}{q_n} + \delta$ with very small $\delta$, we can write $\frac{q_n}{q_n} = \frac{q_n}{q_n} + \delta'$ where $\frac{q_n}{q_n} = \frac{1}{2}$ mod 1 since necessarily $p_n$ is odd. Hence we know that $|\ell - k| = \frac{q_n}{q_n}$. In particular this means that $\alpha_j - \beta_j = \pm \frac{q_n}{q_n} (r_n - r)$. Also, there must necessarily be equally many positive as negative ones since on the average equally many points are in the upper as lower half circle.

If $\alpha_j > \beta_j$, it means that $\ell = k + \frac{q_n}{q_n}$ and that on the interval $\hat{I}_j := \left(\frac{\pi(j + 1)}{q_n} - \alpha_j, \frac{\pi(j + 1)}{q_n} - \beta_j\right)$, the function $H_{q_n}' = -2$. We say that $\hat{I}_j$ is negative. If $\alpha_j < \beta_j$, it means that $k = \ell + \frac{q_n}{q_n}$ and that on the interval $\hat{I}_j := \left(\frac{\pi(j + 1)}{q_n} - \beta_j, \frac{\pi(j + 1)}{q_n} - \alpha_j\right)$, the function $H_{q_n}' = 2$. We say that $\hat{I}_j$ is positive.

We want to prove the estimate of $H_{q_n}$. Since $\int |H_{q_n}'| \sim 1$, which is too large for our purposes, we need to show that there is a lot of cancellations in order to prove that $\max_{\tilde{\theta}} \int_0^{\tilde{\theta}} H_{q_n}'$ is small. For this, define a string $S(i, j), 0 \leq i < j < q_n$ to be the collection of intervals $I_i, \ldots, I_j$.

We want to show that for any string there is about the same number of positive and negative intervals. For this we need to understand the behaviour under iteration of nearest neighbors in a string.

Note that $r_n - r_n > r$. Consider the iterate

\[
f^{q_n-1}(\left[\pi, \pi + \frac{2\pi}{q_n}\right]) = \left[\pi - 2\pi q_n(r_n - r), \pi + \frac{2\pi}{q_n} - 2\pi q_n(r_n - r)\right].\]
Observe that \( \pi - 2\pi q_{n-1}(r_{n-1} - r) = \pi - \frac{2\pi}{q_n} + 2\pi q_{n-1}(r - r_n) \). This implies that \( f^{q_{n-1}} \) maps \( I_j \) to \( I_{j-1} \) with the "usual" error coming from \( r_n \neq r \). Note that \( \hat{I}_j \) is negative when \( \ell(j) \in \left[ \frac{q_n}{2} + 1, \ldots, q_n \right] \) and \( \hat{I}_j \) is positive when \( \ell(j) \in [1, \ldots, \frac{q_n}{2}] \).

Hence \( \hat{I}_j \) is negative if \( \frac{2\pi}{q_n} \ell(j) \in (\pi, 2\pi] \) and \( \hat{I}_j \) is positive if \( \frac{2\pi}{q_n} \ell(j) \in (0, \pi] \).

Next we compare \( \frac{2\pi}{q_n} \ell(j) \) and \( \frac{2\pi}{q_n} \ell(j - 1), 0 \leq j < q_n \).

\[
\begin{align*}
f^{\ell(j)} \left( \frac{\pi(j + 1)}{q_n} \right) &= \pi + \alpha_j \\
f^{\ell(j-1)} \left( \frac{\pi j}{q_n} \right) &= \pi + \alpha_{j-1}.
\end{align*}
\]

Also \( f^{q_{n-1}} \) maps \( I_j \) to \( I_{j-1} \). Hence as points on the unit circle,

\[
\begin{align*}
\frac{2\pi[\ell(j - 1) + q_{n-1}]}{q_n} &= \frac{2\pi}{q_n} \ell(j) \\
\frac{2\pi\ell(j - 1)}{q_n} + 2\pi r_n &= \frac{2\pi}{q_n} \ell(j).
\end{align*}
\]

Therefore, given a string \( S(i, j) \), we can count positive and negative intervals as numbers of points on the orbit of \( \frac{2\pi \ell(i)}{q_n} \) in the upper and lower half circle of rotation by the Golden Mean. Hence Lemma 4.3 implies that the difference between the number of positive and negative intervals in a string \( S(i, j) \) is at most \( \frac{3 \ln q_n}{\ln 1.5} \). Therefore

\[
\left| \int_{S(i,j)} H_{q_n} \right| \leq \frac{3 \ln q_n}{1.5} \frac{1}{2q_n}
\leq C \frac{n}{(1.5)^n}
\]

\[\blacksquare\]
4.2 Localization in the Quantized Case

Now we are ready to prove that the quantized map has an empty basin of attraction at infinity in the topology defined by $u$, i.e. that the map localizes:

**THEOREM 4.5** For any $\psi \in L^2([0,1])$, $\|\psi\| = 1$, and any $\epsilon > 0$ there exists an integer $N$ so that for any iterate $F^n(\psi) = \sum_m c_m e^{im\theta}$ we have $\sum_{|m|<N} |c_m|^2 > 1 - \epsilon$. In particular, $\{u(F^n(\psi))\}_n$ is a bounded sequence for any $\psi$.

**Proof:** Pick any $\delta > 0$. Write $\psi = \sum_m c_m e^{im\theta}$. Let $N_\psi$ be chosen so that $\sum_{|m|<N_\psi} |c_m|^2 > 1 - \delta$. Suppose that $h(\theta)$ is any continuous real function on $T^1$, and suppose $\lambda = \{\lambda_m\}$ is any sequence of real numbers. We denote by $\psi_\lambda$ the function $\sum_m c_m e^{i\lambda_m} e^{im\theta}$. Set $\tilde{\psi} = e^{ih}\psi_\lambda = \sum_m \tilde{c}_m e^{im\theta}$. Then

$$
\tilde{c}_m = \frac{1}{2\pi} \int_0^{2\pi} \tilde{\psi} e^{-im\theta} d\theta = \frac{1}{2\pi} \int_0^{2\pi} e^{ih}\psi_\lambda e^{-im\theta} d\theta = c_m e^{i\lambda_m} + \frac{1}{2\pi} \int_0^{2\pi} (e^{ih} - 1)\psi_\lambda e^{-im\theta},
$$

so

$$
|\tilde{c}_m - c_m e^{i\lambda_m}| \leq \sup |h| \|\psi_\lambda\|_{L^1} \leq \sqrt{2\pi} \sup |h| \|\psi_\lambda\| = \sqrt{2\pi} \sup |h| \|\psi\| = \sqrt{2\pi} \sup |h|
$$

Hence if $\sup |h| < \eta$ for some constant $\eta = \eta(\delta) > 0$

$$
\sum_{|m|<N_\psi} |\tilde{c}_m|^2 \geq 1 - 2\delta.
$$

There is an integer $k = k(\delta)$ so that $\frac{k}{h} \sum_{\ell=k}^\infty \sup |H_{q\ell}| < \eta$. 

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Next let \( n \) be any integer. By Lemma 4.2 we can write

\[
  n = q_{n_1} + \cdots + q_{n_j} + q_{n_{j+1}} + \cdots + q_{n_i} \\
  = q_{n_1} + \cdots + q_{n_j} + n' \\
  = n' + n'
\]

where \( q_{n_1} < q_{n_2} < \cdots \) are Fibonacci numbers and \( n_j < k, n_{j+1} \geq k \). We can then write

\[
  F^n(\psi) = F^\prime(\psi) \equiv F^{n_j}(F^{n_j'}(\psi)) \\
  \psi = F^{n_j'}(\psi). \text{ From Section 3.5:} \\
  \tilde{\psi} = e^{-\sum_{\ell=j+1} H_q(\theta - \sum_{r=j+1}^{\ell-1} q_n \omega T)} \sum_m c_m e^{-i q_n \omega T} e^{im\theta}
\]

Setting

\[
  h = -K \sum_{\ell=j+1}^i H_q(\theta - \sum_{r=j+1}^{\ell-1} q_n \omega T) \\
  \lambda = \{\lambda_m\} = \{-mn'\omega T\} \\
  \tilde{\psi} = e^{ih}\psi \lambda \\
  = \sum \tilde{c}_m e^{im\theta} \text{ we get:} \\
  \sum_{|m| < N_{\psi}} |\tilde{c}_m|^2 \geq 1 - 2\delta.
\]

Note that the collection of all possible operators \( F^{q_{n_1} + \cdots + q_{n_j}} \) is finite.

We write \( \tilde{\psi} = \tilde{\psi}_1 + \tilde{\psi}_2 \) where \( \tilde{\psi}_1 = \sum_{|m| < N_{\psi}} \tilde{c}_m e^{im\theta} \). Then \( ||\tilde{\psi}_2|| \leq \sqrt{2\delta} \) and \( ||F^{q_{n_1} + \cdots + q_{n_j}} (\tilde{\psi}_2)|| \leq \sqrt{2\delta} \).
Also, by compactness of the space \( \{ \{ s_m \}_{|m| \leq N} : 1 - 2\delta \leq \sum |s_m|^2 \leq 1 \} \), it follows that for some \( N \), if \((F^{q_{n_1}+\cdots+q_{n_j}})(\tilde{\psi}_1) = \sum \hat{c}_m e^{im\theta} \) then \( \sum_{|m|<N} |\hat{c}_m|^2 \geq 1 - 3\delta \).

It follows that

\[
\left( \sum_{|m|>N} |\hat{c}_m|^2 \right)^{\frac{1}{2}} < \sqrt{3\delta},
\]

so

\[
\left( \sum_{|m|>N} |c_m^n|^2 \right)^{\frac{1}{2}} < \sqrt{3\delta} + \sqrt{2\delta}
\]

and hence

\[
\sum_{|m|<N} |c_m^n|^2 \geq 1 - 12\delta.
\]

Set \( \delta = \epsilon/12 \) to complete the proof.

\[ \blacksquare \]

### 4.3 Diffusion in the Classical Case

Next we show that the classical dynamics does not localize. We show (slow) diffusion to infinity, which is stronger than finding one unbounded orbit. Diffusion is sometimes considered as Chaos in the Classical System, while the Localization in the previous subsection is considered as a Suppression of Chaos in the Quantized Case, assumed to be a result of Destructive Interference.

**THEOREM 4.6** Let \( I = \{(0, 2\pi) \times (0)\} \) be a set of initial states for the classical system. Let \( \epsilon > 0 \) and \( N > 0 \) be given. Then there exists an integer \( n \) so that \( \{ \theta : f^n(\theta, 0) \in \{|P|<N\} \} \) has linear measure at most \( \epsilon \).

**Proof:** We will first discuss a situation where we consider that \( \frac{|n\alpha - \beta|}{n} = \delta \) is independent of \( n \) and then we correct for the error at the end.

So suppose at first that \( 0 < \delta < 1/2 \) is given. Let \( \chi_n(\theta) \) be an integer-valued step function with values in \( \{-n, \ldots, n\} \). Set \( E_{n,k} = \{ \theta : \chi_n(\theta) = k \} \). We assume that \( \chi_{n+1}|_{E_{n,k}} \) is a step function with values in \( \{k-1, k, k+1\} \).
The set of points with value $k$ in $E_{n,k}$ has measure $(1 - 2\delta)|E_{n,k}|$ and the set of points with value $k \pm 1$ both have measure $\delta|E_{n,k}|$. We get inductively that $|E_{n,k}| = \sum_{i-j=k} a_{i,j}$ where the coefficients $a_{i,j}$ are given by the binomial formula:

$$(1 - 2\delta + \delta x + \delta y)^n = \sum a_{i,j} x^i y^j.$$ 

In this simplified case, the theorem follows from a simple calculation which shows:

**Lemma 4.7** For any given $k$, $\lim_{n \to \infty} |E_{n,k}| = 0$.

Finally to take into account that $\epsilon$ actually will vary, observe that

$$\frac{|\alpha_j - \beta_j|}{2\pi} = \frac{q_n^2 |r_n - r|}{2\pi} =: \delta_n$$

stays bounded and bounded away from zero. We may assume by taking a subsequence that $\delta_n \to \delta$ arbitrarily fast. Pick $m_0$ large enough that $|E_{n,k}| < \frac{\epsilon}{3N}$ whenever $k < N$ and $m \geq m_0$. Then we choose Fibonacci numbers $1 << q_{k_1} << q_{k_2} << \cdots << q_{k_{m_0}}$ for which $r_{k_j} < r$ and the $q_k$ are even (as we discussed in the proof of Lemma 4.4). Then $n = \sum q_{k_j}$ will work.

**Corollary 4.8** Almost every sequence $\{v(f^n(\theta, P))\}$ is unbounded. In particular, there does exist a $y \in N$ for which all $f^n(y)$ are well defined and $\{v(f^n(y))\}$ is unbounded.

**References**

[Ch] Chirikov, B. V; Phys. Rep. 52 (1979), 263.

[Co] Combescure, M; *Recurrent versus diffusive quantum behavior for time dependent Hamiltonians*, Operator Theory: Advances and Applications, Vol 57 (1992) Birkhauser
[Fa] Fatou, P; *Sur les équations fonctionelles*, Bull. Soc. Math. France 47 (1919), 161-271.

[F] Fornæss, J. E; *Dynamics in Several Complex Variables*, CBMS lecture notes 87 (1996).

[IS] Izrailev, F. M, Shepelyanski, D. L; Theor. Math. Phy. 43 (1980) 553.

[S] Schroder, E; *Ueber unendlich viele Algorithmen zur Auflösung der Gleichungen*, Math. Ann. 2 (1870), 317-365.

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