Boson Stars with Vector Meson Exchange Repulsive Interaction

M Fitrah Alfian Rangga Sakti, Anto Sulaksono
Department of Physics, Universitas Indonesia, Depok, Indonesia 16424
E-mail: fitrahalfian@sci.ui.ac.id

Abstract. Spherically symmetric static boson stars are solutions of the system of equations of Klein-Gordon equation which is coupled to the Einstein and Proca equation with complex scalar field with $U(1)$ gauge symmetry. In this work, we do not solve these equations directly but first we solve simultaneous equations Klein-Gordon and Proca in flat space-time numerically to obtain interacting boson equation of state (EOS), then we "boost" the corresponding EOS to curved space-time so that, we can solve Einstein equations. If we assume that the distribution of boson in boson stars is inhomogeneous, the boosted EOS is anisotropic in the sense that the pressure to the tangential direction is not the same as the one in the radial direction. We find numerically solutions to see the EOS which are formed in boson stars as the consequence of inhomogeneous assumption. We have found that there is no physically stable solution for inhomogeneous EOS. However, if we assume that the distribution of bosons in matter is homogeneous, we can get a stable solution for static boson stars.

1. Introduction

Study on compact objects has been done since some decades ago. White dwarfs, neutron, and quark stars are kinds of compact object. White dwarfs exist before the invention of other stars. It’s stability is caused by Fermi degeneracy pressure. Chandrasekhar has been studied about white dwarfs maximum mass [1]. After the invention of the neutron by J. Chadwick, the prediction of the neutron stars had begun. In 1967, J. Bell and A. Hewish found a radio pulse that is indicated as pulsar (neutron stars) [2]. Neutron stars are stable due to the gravitational collapse because of the repulsive interaction between nucleons. Quarks which are stabler than nuclear matter [3] are also believed to exist as composer of quark stars.

In the following, we study the other compact object, i.e. boson stars. Different with neutron stars, boson stars have no observable evidence, yet. However, the existence of one of scalar boson particle, as the main constituent of boson stars, has been proved by Large Hadron Collider experiment [4], i.e. Higgs boson. It has not be proved yet that Higgs boson can compose the boson stars but this invention can be the beginning of the new research of boson stars.

Kaup used complex scalar field for modeling boson stars in semi-classical manner [5]. Ruffini and Bonazzola quantisized real scalar field and consider the ground state energy to find the maximum mass of boson stars [6]. Then Jetzer found there is a critical value of the coupling constant of scalar field and potential vector of boson stars [7]. In the following, we use repulsive self-interactions of vector meson for modeling the interactions between bosons.

Analytically, we derive the Einstein-Klein-Gordon-Proca equation in the curved space-time and in the flat space-time. We have found that there is no physically stable solution...
for inhomogeneous EOS. However, if we assume that the distribution of boson in matter is homogeneous, we can get a stable solution for static boson stars by varying the interaction parameter y and boson mass. Then from homogeneous assumption, we can get the maximum mass and radius of the boson stars.

2. Meson Exchange Model for Interacting Boson

We describe boson stars which consist of scalar field ($\Phi$). We model the interactions between bosons by meson exchange ($A_\mu$). The Lagrangian density of the matter, in natural units with $\hbar = c = 1$, is

$$L_M = \sqrt{-g} \left[ g^{\mu\nu}(D_\mu \Phi)^*(D_\nu \Phi) - m_b^2 \Phi^* \Phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m_v^2 A_\mu A^\mu \right]. \tag{1}$$

Here $g \equiv \det[g_{\mu\nu}]$, $m_b$ is the scalar field mass, $m_v$ is the vector meson mass and $D_\mu = \partial_\mu + iq A_\mu$, where $q$ is coupling constant of the interaction strength. Then $A_\mu$ is the electromagnetic vector potential, while $F_{\mu
u} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor. Then the total Lagrangian density is

$$L = \sqrt{-g} \frac{R}{16\pi G} + L_M. \tag{2}$$

In the case of the spherically symmetry, the general line element can be written in Schwarzschild-like coordinates ($t, r, \theta, \varphi$) as

$$ds^2 = e^{2\nu} dt^2 - e^{2\lambda} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2). \tag{3}$$

where $v = v(r)$ and $\lambda = \lambda(r)$. Then we set stationary scalar field ansatz [6, 7, 10]

$$\Phi(r, t) = \phi(r) e^{-i\omega t}. \tag{4}$$

Equation (4) describes a spherically symmetric bound state of scalar fields with eigen frequency $\omega$. Accordingly, the electromagnetic four-potential is $A_\mu = (A_t(r) = A(r), 0)$. Then we get the following conserved current

$$J_\mu = \sqrt{-g} \left[ i(\Phi^* \partial_\mu \Phi - \Phi \partial_\mu \Phi^*) - 2q A_\mu \Phi^* \Phi \right], \tag{5}$$

and energy-momentum tensor $T^\mu_\nu$ becomes

$$T^\mu_\nu = g^{\alpha\mu} \left( (D_\sigma \Phi)^*(D_\nu \Phi) + (D_\nu \Phi)^*(D_\sigma \Phi) - g^{\alpha\beta} F_{\alpha\sigma} F_{\beta\nu} + m_v^2 A_\sigma A_\nu \right) - \delta^\mu_\nu L_M. \tag{6}$$

From equation (5), we obtain the boson number density as

$$n_b = J_0 = 2\sqrt{-g}(\omega - qA)\phi^2. \tag{7}$$

The non-zero components of the energy-momentum tensor are

$$T^{\mu}_0 = \left[ m_b^2 + e^{-2\nu}(\omega - qA)^2 \right] \phi^2 + \frac{e^{-(\nu + \lambda)} A^2}{2} + \phi^2 e^{-2\lambda} + \frac{1}{2} m_v^2 A^2 e^{-2\nu}, \tag{8}$$

$$T^{\mu}_1 = \left[ m_b^2 - e^{-2\nu}(\omega - qA)^2 \right] \phi^2 + \frac{e^{-(\nu + \lambda)} A^2}{2} - \phi^2 e^{-2\lambda} - \frac{1}{2} m_v^2 A^2 e^{-2\nu}, \tag{9}$$

$$T^{\mu}_2 = T^{\mu}_3 = \left[ m_b^2 - e^{-2\nu}(\omega - qA)^2 \right] \phi^2 - \frac{e^{-(\nu + \lambda)} A^2}{2} + \phi^2 e^{-2\lambda} - \frac{1}{2} m_v^2 A^2 e^{-2\nu}. \tag{10}$$
Then for the system described by Eqs. (1) and (2), the set of the Euler-Lagrange equations gives the two following independent equations for the metric components:

\[ \lambda' = \frac{1 - e^{2\lambda}}{2r} - 4\pi G r e^{2\lambda} T^0_0 \] (11)

\[ v' = -\frac{1 + e^{2\lambda}}{2r} + 4\pi G r e^{2\lambda} T^1_1. \] (12)

Those two equations are equivalent to the Einstein equation [8]

\[ G_{\mu\nu} = 8\pi G T_{\mu\nu}. \] (13)

From the Eq. (1), we also get the following Proca equation for massive meson

\[ A'' + A' \left( \frac{2}{r} - v' - \lambda' \right) + 2 q e^{2\lambda} (\omega - qA) \phi^2 - m^2 v e^{2\lambda} A = 0, \] (14)

and the Klein-Gordon equation is

\[ \phi'' + \phi' \left( \frac{2}{r} + v' - \lambda' \right) + e^{2\lambda} \left[ e^{-2v} (\omega - qA)^2 - m^2_b \right] \phi = 0. \] (15)

The general line element for the flat space-time is

\[ ds^2 = dt^2 - dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2). \] (16)

Because of the Eq. (16) the Klein-Gordon and Proca equations becomes

\[ \phi'' + \frac{2}{r} \phi' + \left[ (\omega - qA)^2 - m^2_b \right] \phi = 0, \] (17)

\[ A'' + \frac{2}{r} A' + 2 q (\omega - qA) \phi^2 - m^2 v A = 0. \] (18)

The non-zero components of the energy-momentum tensor \( T^\mu_\nu \) becomes

\[ T^0_0 = \left[ m^2_b + (\omega - qA)^2 \right] \phi^2 + \frac{A^2}{2} + \phi^2 + \frac{1}{2} m^2 v A^2, \] (19)

\[ T^1_1 = \left[ m^2_b - (\omega - qA)^2 \right] \phi^2 + \frac{A^2}{2} - \phi^2 - \frac{1}{2} m^2 v A^2, \] (20)

\[ T^2_2 = T^3_3 = \left[ m^2_b - (\omega - qA)^2 \right] \phi^2 - \frac{A^2}{2} + \phi^2 - \frac{1}{2} m^2 v A^2. \] (21)

The non-zero energy-momentum tensors are not dependent on metric explicitly any more. Thus, the Euler-Lagrange equations for the metric (11) and (12) will not be suitable in the flat space-time. So, we use the TOV equations for anisotropic fluid

\[ \frac{dp_r}{dr} = -\frac{GM \rho}{r^2} \left( 1 + \frac{p_r}{\rho} \right) \left( 1 + \frac{4\pi r^3 p_r}{M} \right) \left( 1 - \frac{2GM}{r} \right)^{-1} + \frac{2(p_\perp - p_r)}{r}, \] (22)

\[ \frac{dM}{dr} = 4\pi r^2 \rho. \] (23)

where the energy density \( \rho \), radial pressure \( p_r \), and tangential pressure \( p_\perp \) are equal to \( T^0_0 \), \(-T^1_1\), and \(-T^2_2\) in Eqs. (19) - (21). \( M \) denotes the mass of the boson stars.
We can also use homogeneous assumption in the energy-momentum tensor so, the derivative terms will be equal to zero. Then the Eqs. (19) - (21) will be

\[
T_{00}^0 = \left[ m_b^2 + (\omega - qA)^2 \right] \phi^2 + \frac{1}{2} m_v^2 A^2, \tag{24}
\]

\[
T_{11}^1 = T_{22}^2 = T_{33}^3 = \left[ m_b^2 - (\omega - qA)^2 \right] \phi^2 - \frac{1}{2} m_v^2 A^2. \tag{25}
\]

Then the equation (22) will be

\[
\frac{dp}{dr} = - \frac{GM\rho r^2}{r^2} \left( 1 + \frac{p}{\rho} \right) \left( 1 + \frac{4\pi r^3 p}{M} \right) \left( 1 - \frac{2GM}{r} \right)^{-1}, \tag{26}
\]

for the isotropic boson stars. From Eqs. (17), (18) and (27) for the homogeneous assumption we get

\[
\omega = qA + m_b, \quad 2qm_b \phi^2 = m_v^2 A, \tag{27}
\]

so, the boson number density will be

\[
n_b = 2m_b \phi^2. \tag{28}
\]

From energy-momentum tensor (24) and (25) then by using relation (28), we get EOS such as

\[
\rho = m_b n_b + \frac{q^2 m_b^2}{2m_v^2}, \quad p = \frac{q^2 m_b^2}{2m_v^2}. \tag{29}
\]

These two expressions are thermodynamically consistent and suitable with thermodynamic relation [9]

\[
p = n_b^2 \frac{d(\rho/n_b)}{dn_b}. \tag{30}
\]

3. Variation for Numerical Method

In the following, we study the solutions of the KGP equations in the flat space-time by using inhomogeneous and homogeneous assumption. Then we try to find maximum mass and radius of the boson stars from TOV equations. Before we get the characteristics of boson stars, we have to solve the KGP equation in flat space-time using numerical method. For the homogeneous assumption, we just plot the equations as the function of \( \phi \).

To solve the KGP equations, we need the boundary conditions which shows a localized particle distribution. We impose the following boundary conditions \([7, 10]\):

\[
\phi(0) = \text{constant}, \quad \phi(\infty) = 0, \quad \phi'(0) = 0, \quad \text{and} \quad \phi'(\infty) = 0. \tag{31}
\]

We also impose the vector field to be vanishing at the origin so that

\[
A'(0) = 0, \tag{32}
\]

and we demand that

\[
A(\infty) = 0, A'(\infty) = 0. \tag{33}
\]

We try to find the solutions that is decreasing and fulfil all the boundary conditions.

We vary some variables to find stable solutions, i.e. scalar boson mass \((m_b)\) and interaction parameter \((y)\) like in this Ref. [9]. We vary the scalar boson mass in the order of Higgs boson mass and nucleon mass, i.e. 100 GeV and 1 GeV. The parameter \(y\) is equal to \(m_b/m_i\) where \(m_i = \sqrt{2}m_v/q\). The variation of the interaction are:
1. Free case, $m_i = M_{Pl}$ (Planck mass),
2. Landau limit, $m_i = m_b$,
3. Weak interaction, $m_i = 100$ GeV, and
4. QCD, $m_i = 100$ MeV.

Besides that, we set some constant parameters, i.e. initial value of scalar boson and the vector meson mass. Then we also find the ground state eigenvalues ($\omega$).

4. Solutions of KGP Equations, Stability, and Star Mass

4.1. Solutions of KGP equations and equation of state of inhomogeneous assumption

From the numerical calculation, first we find the solutions of scalar field ($\phi$) and vector potential ($A_\mu$) as we can see in Figs. 1 - 5. We also get the ground state eigenvalues ($\omega$) in Table 1.

![Figure 1](image1.png)

**Figure 1.** Scalar field solutions for inhomogeneous assumption and free case (a) $m_b = 100$ GeV and (b) $m_b = 1$ GeV.

![Figure 2](image2.png)

**Figure 2.** Scalar field solutions for inhomogeneous assumption and Landau limit (a) $m_b = 100$ GeV and (b) $m_b = 1$ GeV.

We can see the solutions of the scalar field and it’s derivative in Figs. 1 - 4 for all variation of interaction parameter and boson mass where the red line shows the scalar field and blue line shows it’s derivative. We get unstable solutions for all variation. Boson mass also has no impact to the solution magnitudes. The magnitudes of the scalar field and it’s derivative for variation are almost similar to each other except Fig. 4 for QCD where the solutions do not decrease smoothly along with the radius that become bigger. It’s difference can be caused by the vector potential solutions which are too big for QCD parameter. We also get the eigenvalue which are almost the same as shown in Table 1.
Figure 3. Scalar field solutions for inhomogeneous assumption and weak interaction (a) $m_b = 100$ GeV and (b) $m_b = 1$ GeV.

Figure 4. Scalar field solutions for inhomogeneous assumption and QCD (a) $m_b = 100$ GeV and (b) $m_b = 1$ GeV.

The solutions of the vector potential and field show in Fig. 5 for all variation of interaction parameter where (a) free case, (b) Landau limit, (c) weak interaction, (d) QCD and boson mass where the red line for $m_b = 100$ GeV and blue line for $m_b = 1$ GeV. The lines show the vector potential and the dashed lines show the vector field. We get stable solutions for all variation but they do not fulfill the boundary condition in Eq. (33). It can be caused by we that didn’t use exponentially explicit decreasing vector potential ansatz. But we can see from the inset graphics, before the radius reach 4 GeV$^{-1}$, the vector filed still fulfill the boundary condition.

After getting the solutions, we get the EOS and boson number density shows in Figs. 6 - 7. We get unstable EOS and boson number density for all variation. We also get unphysical EOS for Landau limit and QCD because the vector potentials are too big to balance the scalar field. That EOS also violate the causality principle [11]. Therefore, we can not solve TOV equations to get the characteristics of the s for inhomogeneous assumption and for all variation.

Table 1. Eigenvalue ($\omega$) of the ground state in GeV unit for inhomogeneous assumption.

| $m_b$ (GeV) | Free case | Landau limit | Weak interaction | QCD |
|------------|-----------|--------------|------------------|-----|
| 100        | 141.463   | 141.461      | 141.461          | 141.400 |
| 1          | 100.064   | 100.065      | 100.063          | 100.000 |
Figure 5. Vector potential and field solutions for inhomogeneous assumption where (a) free case, (b) Landau limit, (c) weak interaction, and (d) QCD.

Figure 6. EOS and boson number density for inhomogeneous, (1) free case, and (2) Landau limit.
4.2. Solutions of KGP equations and equation of state of homogeneous assumption

In homogeneous assumption, the scalar field is independent on the position because we do not have to solve some differential equations. We just plot the vector potential, EOS, and boson number density as a function of scalar field and the eigenvalue explicitly depends on the vector potential as shown in Eq. (27) and implicitly depends on the scalar field. We show the vector potential, EOS, and boson number density for homogeneous assumption in Figs. 8 - 9.

All vector potential which we obtain are stable and decrease smoothly as long as scalar fields decrease till reach the boundary conditions as shown in Figs. 8.1a - 9.2a. But in homogeneous assumption, we neglect the derivative boundary conditions for either vector field or scalar field. We obtain the EOS in Eq. (29) which are stable as shown in Figs. 8.1b - 9.2b for the energy density and in Figs. 8.1c - 9.2c for the isotropic pressure, and also the boson number density in Figs. 8.1d - 9.2d. The EOS are also suitable with the causality principle [11]. By using that stable EOS and boson number density, we can solve TOV equations (23) and (26) for isotropic pressure to get the maximum mass and radius of boson stars.

Figure 8. Vector potential, EOS, and boson number density for homogeneous, (1) free case, and (2) Landau limit.
4.3. Mass-radius relation of boson stars

From inhomogeneous assumption in flat space-time, we can not get stable solutions and EOS to be inserted to the TOV equations but from homogeneous we can. From the EOS as shown in Figs. 8 - 9, we can explicitly get the relation between maximum mass and radius as shown in Fig. 10. The dashed blue line shows total boson mass and the red line shows proper star mass. The real mass of the boson stars is the proper star mass. The difference between total boson mass and proper star mass is caused by the gravitational binding energy. In the equation of the total boson mass such in this Ref. [8], the gravitational binding energy is not included so, the star mass is bigger. We show the maximum mass and radius of boson stars for all variation and for homogeneous assumption in Table 2. Besides the mass-radius relations as shown in Fig. 10, we get other mass-radius relation by using scaling relation in Ref. [9]. From that table we get the biggest mass and radius for QCD and $m_b = 1$ GeV besides, the smallest which is free case and $m_b = 100$ GeV.

Figure 10. Mass-radius relation for (a) QCD with $m_b = 1$ GeV, (b) Landau limit with $m_b = 1$ GeV, and (c) QCD with $m_b = 100$ GeV.
Table 2. Proper star mass ($M_{pr}$), total boson mass ($M_{bos}$), and radius of boson stars for all variation.

| Interaction          | Maximum Value | 100 GeV | 1 GeV |
|----------------------|---------------|---------|-------|
|                      | $M_{pr}(M_{\odot})$ | $2.7 \times 10^{-22}$ | $2.7 \times 10^{-20}$ | Free Case |
|                      | $M_{bos}(M_{\odot})$ | $2.9 \times 10^{-22}$ | $2.9 \times 10^{-20}$ | $m_b = M_{Pl}$ |
|                      | $R$(km)       | $1.8 \times 10^{-21}$ | $1.8 \times 10^{-19}$ | |
|                      | $M_{pr}(M_{\odot})$ | $2.7 \times 10^{-8}$  | 0.27 | Landau Limit |
|                      | $M_{bos}(M_{\odot})$ | $2.9 \times 10^{-5}$ | 0.29 | $y = 1$ |
|                      | $R$(km)       | $1.8 \times 10^{-4}$  | 1.8 | |
|                      | $M_{pr}(M_{\odot})$ | $2.7 \times 10^{-2}$  | $2.7 \times 10^{-3}$ | Weak Interaction |
|                      | $M_{bos}(M_{\odot})$ | $2.9 \times 10^{-5}$ | $2.9 \times 10^{-3}$ | $m_b = 100$ GeV |
|                      | $R$(km)       | $1.8 \times 10^{-4}$  | $1.8 \times 10^{-2}$ | |
|                      | $M_{pr}(M_{\odot})$ | $2.7 \times 10^{-2}$  | $2.7$ | QCD |
|                      | $M_{bos}(M_{\odot})$ | $2.9 \times 10^{-2}$ | $2.9$ | $m_b = 100$ MeV |
|                      | $R$(km)       | 0.18                 | 18 | |

5. Conclusions
We obtain that solutions of the Klein-Gordon-Proca equations have unstable scalar field solutions and vector field solutions which are stable but do not fulfil the boundary conditions. The interaction parameter and boson mass do not affect the stability of the solutions. So, we also get the unstable EOS that can not be used to solve TOV equations to get the mass and radius of the boson stars. Different with that, in the homogeneous assumption we obtain stable solutions and fulfil the non-derivative boundary conditions because the derivatives are equal to zero. Boson mass affects the magnitude of the solution where the bigger boson mass, the bigger solutions we get. The effect of the interaction parameters is also the same with the effect of mass. Then we also get stable EOS where the pressure is isotropic. Because of that, we can solve TOV equations to get the mass and radius of boson stars as shown in Table 2. From Table 2, we know the biggest mass and radius for QCD and $m_b = 100$ GeV besides the smallest is free case and $m_b = 100$ MeV. We also find that all proper masses are multiplication of 2.7 and all radius are multiplication of 1.8.

Acknowledgments
We would like to thank people who support us for this work. We thank for the discussion and help to all people in Theoretical Physics Laboratory of Department of Physics, Universitas Indonesia. We acknowledge support from the Universitas Indonesia through Research Cluster Grant 2015 on "Nonperturbative Phenomena in Nuclear Astrophysics and Cosmology” No. 1862/UN.R12/HKP.05.00/2015.

References
[1] Chandrasekhar S 1931 Astrophys. J. 74 81
[2] Hewish A, Bell S J, Pilkington J D H, Scott P F, and Collins R A 1968 Nature 217 709
[3] Weber F 2005 Prog. Part. Nucl. Phys. 54 193 arXiv: astro-ph/0407155v2
[4] CMS Collaboration 2012 Phys. Rev. Lett. B 716 30
[5] Kaup D J 1968 Phys. Rev. 172 1331
[6] Ruffini R and Bonazzola S 1969 Phys. Rev. 187 1767
[7] Jetzer P and van der Bij J J 1989 Phys. Lett. B 227 341
[8] Ryder L 2009 Introduction to General Relativity (Cambridge: Cambridge University Press)
[9] Agnihotri P, Bielich J S, and Mishustin I N 2009 Phys. Rev. D 79 084033
[10] Pugliese D, Quevedo H, Rueda J A, and Ruffini R 2013 Phys. Rev. D 88 024053
[11] Ellis G F R, Maartens R, and MacCallum M A H 2007 Gen. Rel. Grav. 39 arXiv:gr-qc/0703121.