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Completeness of coherent state subsystems for nilpotent Lie groups

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Abstract. Let $G$ be a nilpotent Lie group and let $\pi$ be a coherent state representation of $G$. The interplay between the cyclicity of the restriction $\pi|_\Gamma$ to a lattice $\Gamma \leq G$ and the completeness of subsystems of coherent states based on a homogeneous $G$-space is considered. In particular, it is shown that necessary density conditions for Perelomov’s completeness problem can be obtained via density conditions for the cyclicity of $\pi|_\Gamma$.

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1. Introduction

Let $G$ be a connected unimodular Lie group and let $(\pi, \mathcal{H}_\pi)$ be an irreducible unitary representation of $G$. For a unit vector $\eta \in \mathcal{H}_\pi$, consider its orbit under the action $\pi$ on $\mathcal{H}_\pi$,

$$\pi(G)\eta = \{\pi(g)\eta : g \in G\}. \quad (1)$$

As $\pi$ is irreducible, $\pi(G)\eta$ is complete in $\mathcal{H}_\pi$. Two elements $\pi(g_1)\eta$ and $\pi(g_2)\eta$ differ from one another up to a phase factor, i.e. determine the same state or ray, only if $\pi(g_2^{-1}g_1)\eta \in \mathbb{C}\eta$.

Let $H \leq G$ be a closed subgroup that stabilises the state defined by $\eta \in \mathcal{H}_\pi$, i.e.

$$\pi(h)\eta = \chi(h)\eta, \quad h \in H, \quad (2)$$

where $\chi : H \rightarrow \mathbb{T}$ is a unitary character of $H$. Denote by $X = G/H$ the associated homogeneous $G$-space and let $\sigma : X \rightarrow G$ be a cross-section for the canonical projection $p : G \rightarrow X$. Then the system of coherent vectors

$$\{|\eta_x\rangle : x \in X\} = \{|\pi(\sigma(x))\eta\rangle : x \in X\}, \quad (3)$$

determine a $\pi$-system of coherent states based on $X$, in the sense of [24, 29].
It will be assumed that \( X = G/H \) is unimodular, i.e. \( X \) admits a \( G \)-invariant positive Radon measure \( \mu_X \), and that \( \eta \) is admissible, that is,
\[
\int_X |\langle \eta, \eta_X \rangle|^2 \, d\mu_X(x) < \infty.
\] (4)
Then there exists an admissibility constant \( d_{\pi, \eta} > 0 \) such that
\[
\int_X |\langle f, \eta_X \rangle|^2 \, d\mu_X(x) = d_{\pi, \eta}^{-1} \| f \|^2_{L^2(\mathcal{H}_\pi)}, \quad \text{for all } f \in \mathcal{H}_\pi.
\] (5)
The identity (5) implies, in particular, that the system (3) is overcomplete, i.e. the system \( \{ \eta_x \}_{x \in X} \) contains proper subsystems which are complete in \( \mathcal{H}_\pi \).

For an irreducible representation \((\pi, \mathcal{H}_\pi)\) of \( G \) that is square-integrable modulo the center \( Z = Z(G) \) (resp. the kernel \( K = \ker(\pi) \)), any vector \( \eta \in \mathcal{H}_\pi \) satisfies (2) and (4) for \( H = Z \) (resp. \( H = K \)). Another common choice \([12, 22, 26, 29]\) for the index space \( X = G/H \) is a symplectic \( G \)-space or a homogeneous Kähler manifold that arises as a phase space in geometric quantization \([34]\).

Subgroups \( H \leq G \) defining such a phase space do not need to satisfy (2) for all \( \eta \in \mathcal{H}_\pi \) and might not be contained in the isotropy group of a chosen \( \eta \).

In \([24, 26]\), a particular focus is on coherent states for which the stabilising subgroup \( H \leq G \) is assumed to be maximal with the property (2), that is, \( H = G_{[\eta]} \), where
\[
G_{[\eta]} := \{ g \in G : \pi(g)\eta = e^{i\theta(g)}\eta \}
\] (6)
is the stabiliser of \( \eta \) for the \( G \)-action in the projective Hilbert space \( P(\mathcal{H}_\pi) \). The associated coherent states are so-called Perelomov-type coherent states; see Section 4.

Perelomov’s completeness problem \([24, 26]\) concerns the completeness of subsystems arising from discrete subgroups \( \Gamma \leq G \) for which the volume of \( \Gamma \setminus X \) is finite. More explicitly, subsystems parametrised by an orbit \( \Gamma' = \Gamma \cdot \alpha \) of the base point \( \alpha = eH \in X \),
\[
\{ \eta_{\gamma'} \}_{\gamma' \in \Gamma'} = \{ \pi(\sigma(\gamma'))\eta \}_{\gamma' \in \Gamma'}.
\] (7)
Criteria for the completeness of subsystems (7) involving the volume of the coset space \( \Gamma \setminus X \) and the admissibility constant \( d_{\pi, \eta} > 0 \) were posed as a problem in \([24, \text{p.} 226]\) and \([26, \text{p.} 44]\). Note that if \( H = G_{[\eta]} \), then \( X = G/G_{[\eta]} \) depends on \( \eta \) and so does the volume of \( \Gamma \setminus G/G_{[\eta]} \).

The classical example of coherent states arises from the Heisenberg group \( G = \mathbb{H}^1 \) and the Schrödinger representation \((\pi, L^2(\mathbb{R}))\) of \( \mathbb{H}^1 \). For any \( \eta \in L^2(\mathbb{R}) \setminus \{ 0 \} \), the stabiliser \( G_{[\eta]} \) defined in (6) coincides with the centre \( Z(\mathbb{H}^1) \) of \( \mathbb{H}^1 \), and \( X = G/G_{[\eta]} \equiv \mathbb{R}^2 \). Therefore, the coherent state system (3) is parametrised by the classical phase space \( \mathbb{R}^2 \) and the subsystem (7) associated to \( \Gamma \subset \mathbb{H}^1 \) is parametrised by a lattice \( \Gamma' \subset \mathbb{R}^2 \). If the square-integrable representation \( \pi \) is treated as a projective representation \( \rho \) of \( G/G_{[\eta]} \equiv \mathbb{R}^2 \), then the coherent vectors (3) and the subsystem (7) arise as orbits of \( \mathbb{R}^2 \) and \( \Gamma' \), respectively. In particular, a subsystem \( \{ \pi(\sigma(\gamma'))\eta \}_{\gamma' \in \Gamma'} \) is complete in \( L^2(\mathbb{R}) \) if, and only if, \( \eta \) is a cyclic vector for \( \rho|_{\Gamma'} \), i.e. the linear span of \( \rho(\Gamma')\eta \) is dense in \( L^2(\mathbb{R}) \). This shows that Perelomov’s completeness problem for the Heisenberg group is equivalent to determining whether a vector is cyclic for the restriction \( \rho|_{\Gamma'} \). If \( \eta \) is the Gaussian, the cyclicity of \( \eta \) has been completely characterised in \([2, 23]\) (see also \([21]\)) in terms of the co-volume or density of the lattice. The necessity of these density conditions have been shown to hold for arbitrary vectors and in arbitrary dimensions \([28]\), but a density condition alone is not sufficient for describing the cyclicity of the Gaussian in higher-dimensions \([7, 27]\). The criteria \([2, 23, 28]\) coincide with the density conditions characterising the cyclicity of the restricted projective representations as obtained in, e.g. \([3, 30]\).

In other settings than the Heisenberg group, the stabilisers \( G_{[\eta]} \) defined in (6) do not need to be normal subgroups and could depend crucially on the vector \( \eta \in \mathcal{H}_\pi \setminus \{ 0 \} \). For example, this occurs for the holomorphic discrete series \( \pi \) of \( G = \text{PSL}(2, \mathbb{R}) \), where \( G_{[\eta]} = \text{PSO}(2) \) for a class of rotation-invariant vectors \( \eta \). Hence, the coherent vectors (3) do not arise as orbits of a (projective)
representation of $G/G_{\eta}$ and the subsystems (7) are not parametrised by an associated discrete subgroup. Perelomov’s problem for the highest weight vector has been studied for this setting in [9, 10, 25], and the criteria for the cyclicity of $\pi|_\Gamma$ are quite different from the completeness of coherent state subsystems; see [31, Section 9.1] for an overview.

Of particular interest are representations and vectors that support a system of coherent states based on an index manifold $X = G/H$ with additional properties, such as a symplectic [16, 17] or complex structure [13, 18]. For nilpotent Lie groups, another common choice (cf. [26, Section 10]) is the manifold $X$ to be the corresponding coadjoint orbit $\mathcal{O}_\pi$ of the representation $\pi$, which forms the classical phase space, like in the special case of the Heisenberg group.

The purpose of this note is to combine characterisations of coherent state representations [13, 16, 18] and criteria for the cyclicity of restricted representations [3, 31] to obtain necessary density conditions for (variants of) Perelomov’s completeness problem on nilpotent Lie groups.

The first result on the completeness of subsystems concerns $\pi$-systems of coherent states based on the coadjoint orbit $\mathcal{O}_\pi$. (cf. Section 2 for the precise definitions.)

**Theorem 1.** Let $G$ be a connected, simply connected nilpotent Lie group and let $\Gamma \leq G$ be a discrete, co-compact subgroup. Suppose $(\pi, \mathcal{H}_\pi)$ is an irreducible representation of $G$ that admits an admissible vector $\eta \in \mathcal{H}_\pi \setminus \{0\}$ defining a $\pi$-system of coherent states based on a homogeneous $G$-space $X = G/H \cong \mathcal{O}_\pi$, with admissibility constant $d_{\pi, \eta} > 0$. Then

(i) $H = \{ g \in G : \pi(g) \in \mathbb{C} \cdot I_{\mathcal{H}_\pi} \}$;

(ii) If $|\pi(\sigma(\gamma'))| \eta|_{\gamma' \in \Gamma_0}$ is complete in $\mathcal{H}_\pi$, then $\text{covol}(p(\Gamma))d_{\pi, \eta} \leq 1$.

(The value $\text{covol}(p(\Gamma))d_{\pi, \eta}$ is independent of the normalisation of $G$-invariant measure on $X$.)

Theorem 1 considers $\pi$-systems of coherent states parametrised by the canonical phase space $\mathcal{O}_\pi$ (cf. [26, Section 10]), and provides a necessary condition for the completeness of associated subsystems. The representations satisfying the hypothesis of Theorem 1 are called coherent state representations in [16], and are characterised as those being an irreducible representation whose associated coadjoint orbit is a linear variety. The considered representations are therefore essentially square-integrable, like in the special case of the Heisenberg group.

The second result concerns $\pi$-systems of coherent states associated to vectors yielding a symplectic projective orbit (cf. Section 4 for the precise definitions.)

**Theorem 2.** Let $G$ be a connected, simply connected nilpotent Lie group and let $\Gamma \leq G$ be a discrete, co-compact subgroup. Suppose $(\pi, \mathcal{H}_\pi)$ is an irreducible representation of $G$ that admits an admissible vector $\eta \in \mathcal{H}_\pi \setminus \{0\}$ yielding a symplectic orbit and defines a $\pi$-system of coherent states based on $X = G/G_{\eta}$, with admissibility constant $d_{\pi, \eta} > 0$. Then

(i) $G_{\eta} = \{ g \in G : \pi(g) \in \mathbb{C} \cdot I_{\mathcal{H}_\pi} \}$;

(ii) If $|\pi(\sigma(\gamma'))| \eta|_{\gamma' \in \Gamma_0}$ is complete in $\mathcal{H}_\pi$, then $\text{covol}(p(\Gamma))d_{\pi, \eta} \leq 1$.

In contrast to Theorem 1, the index manifold $X = G/G_{\eta}$ in Theorem 2 is selected via the maximal subgroup (6) stabilising the state determined by $\eta \in \mathcal{H}_\pi \setminus \{0\}$. The vectors $\eta \in \mathcal{H}_\pi$ yielding a symplectic orbit play a distinguished role in geometric quantization [12, 22]. Theorem 2 applies, in particular, to smooth vectors of a square-integrable representation (see Proposition 10) and to so-called highest weight vectors (see Remark 12).

The proofs of Theorem 1 and Theorem 2 are relatively simple and short, but they hinge on a combination of several non-trivial statements on coherent state representations [13, 16, 18] and density conditions for restricted discrete series [3, 31]. More explicitly, exploiting results of [13, 16, 18], it will be shown that the completeness of coherent state subsystems is equivalent to the admissible vector being a cyclic vector for a restricted projective representation; the necessary density conditions then being a direct consequence of [31].
Notation

For a complex vector space $\mathcal{H}$, the notation $P(\mathcal{H})$ will be used for its projective space, i.e. the space of all one-dimensional subspaces. The subspace or ray generated by $\eta \in \mathcal{H} \setminus \{0\}$ will be denoted by $[\eta] := \mathbb{C}\eta$. Henceforth, unless stated otherwise, $G$ is a connected, simply connected nilpotent Lie group with exponential map $\exp : \mathfrak{g} \to G$. Haar measure on $G$ is denoted by $\mu_G$. If $\Lambda \leq G$ is a discrete subgroup, then the co-volume is defined as $\text{covol}(\Lambda) := \mu_{G/\Lambda}(G/\Lambda)$, where $\mu_{G/\Lambda}$ denotes $G$-invariant Radon measure on $G/\Lambda$.

2. Coherent state representations of nilpotent Lie groups

This section provides preliminaries on irreducible representations of nilpotent Lie groups and associated coherent states. References for these topics are the books [6] and [1, 26].

2.1. Coadjoint orbits

Let $\mathfrak{g}^*$ denote the dual vector space of $\mathfrak{g}$. The coadjoint representation $\text{Ad}^* : G \to \text{GL}(\mathfrak{g}^*)$ is defined by $\text{Ad}^*(g)^{\ell} = \ell \circ \text{Ad}(g)^{-1}$ for $g \in G$ and $\ell \in \mathfrak{g}^*$. The stabiliser of $\ell \in \mathfrak{g}^*$ is the connected closed subgroup $G(\ell) = \{g \in G : \text{Ad}^*(g)^{\ell} = \ell\}$, its Lie algebra is the annihilator subalgebra $\mathfrak{g}(\ell) = \{X \in \mathfrak{g} : [\ell(X), X] = 0, \forall Y \in \mathfrak{g}\}$.

For $\ell \in \mathfrak{g}^*$, its coadjoint orbit is denoted by $\Theta_\ell := \text{Ad}^*(G)^{\ell}$ and endowed with the relative topology from $\mathfrak{g}^*$. The orbit $\Theta_\ell$ is homeomorphic to $G/G(\ell)$; in notation: $\Theta_\ell \cong G/G(\ell)$.

2.2. Irreducible representations

A Lie subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is subordinated to $\ell \in \mathfrak{g}^*$ if $\ell(X) = 0$ for every $X \in [\mathfrak{p}, \mathfrak{p}]$. If $\mathfrak{p}$ is subordinate to $\ell$, then the map $\chi_\ell : \exp(\mathfrak{p}) \to \mathbb{T}$, $\chi_\ell(\exp(X)) = e^{2\pi i \ell(X)}$ defines a unitary character of $\mathfrak{p} = \exp(\mathfrak{p})$. The associated induced representation of $G$ is denoted by $\pi_\ell = \pi(\ell, \mathfrak{p}) = \text{ind}^G_{\mathfrak{p}}(\chi_\ell)$.

For every $\pi$ in the unitary dual $\hat{G}$ of $G$, there exists $\ell \in \mathfrak{g}^*$ and a subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$, subordinate to $\ell$, such that $\pi$ is unitarily equivalent to $\pi_\ell = \pi(\ell, \mathfrak{p})$. A representation $\pi_\ell = \pi(\ell, \mathfrak{p})$, with $\mathfrak{p}$ subordinate to $\ell \in \mathfrak{g}^*$, is irreducible if, and only if, $\mathfrak{p}$ is a maximal subalgebra subordinated to $\ell \in \mathfrak{g}^*$ satisfying $\dim(\mathfrak{p}) = \dim(\mathfrak{g}) - \dim(\Theta_\ell)/2$, a so-called (real) polarisation.

Two irreducible induced representations $\text{ind}^G_{\exp(\mathfrak{p})}(\chi_\ell)$ and $\text{ind}^G_{\exp(\mathfrak{p}')}(\chi_{\ell'})$ are unitarily equivalent if and only if the linear functionals $\ell, \ell' \in \mathfrak{g}^*$ belong to the same coadjoint orbit. The orbit associated to the equivalence class $\pi \in \hat{G}$ will also be denoted by $\Theta_\pi$.

2.3. Moment set

Let $(\pi, \mathcal{H}_\pi)$ be an irreducible unitary representation of $G$. Denote by $\mathcal{H}_\pi^\infty$ the space of smooth vectors for $\pi$, i.e. the space of $\eta \in \mathcal{H}_\pi$ for which $g \to \pi(g)\eta$ is smooth.

The derived representation $d\pi : \mathfrak{g} \to L(\mathcal{H}_\pi^\infty)$ is defined by

$$d\pi(X)\eta := \frac{d}{dt} \bigg|_{t=0} \pi(\exp(tX))\eta, \quad X \in \mathfrak{g}, \; \eta \in \mathcal{H}_\pi^\infty. \tag{8}$$

It can be extended complex linearly to a representation of the complexification $\mathfrak{g}_C$ of $\mathfrak{g}$.

The moment map of $\pi$ is the mapping $J_\pi : \mathcal{H}_\pi^\infty \to \mathfrak{g}^*$ defined by

$$J_\pi(\eta)(X) = \frac{1}{i} \langle d\pi(X)\eta, \eta \rangle, \quad X \in \mathfrak{g}, \; \eta \in \mathcal{H}_\pi^\infty. \tag{9}$$

Note that the right-hand side of (9) only depends on the ray $[\eta]$ generated by $\eta \in \mathcal{H}_\pi^\infty \setminus \{0\}$. 

The moment map $I_\pi$ is equivariant with respect to the canonical $G$-actions on $\mathcal{H}_\pi^\infty$ and $\mathfrak{g}^*$, i.e. $I_\pi(\pi(g)\eta)(X) = (\text{Ad}(g)^* J_\pi(\eta))(X)$ for $g \in G$, $X \in \mathfrak{g}$ and $\eta \in \mathcal{H}_\pi^\infty$. In particular, $I_\pi(G \cdot \eta)$ is the coadjoint orbit $\Theta_\pi(\eta)$ of $I_\pi(\eta) \in \mathfrak{g}^*$.

The moment set $I_\pi$ of $\pi$ is the closure $I_\pi := \overline{\text{conv}(\Theta_\pi)}$ in $\mathfrak{g}^*$. Its relation to the coadjoint $\Theta_\pi$ of $\pi \in \hat{G}$ is

$$I_\pi = \overline{\text{conv}(\Theta_\pi)},$$

where $\overline{\text{conv}}$ denotes the closed convex hull; see [33, Theorem 4.2].

### 2.4. Coherent state representations

Henceforth, it is assumed that $(\pi, \mathcal{H}_\pi)$ is non-trivial. Let $\eta \in \mathcal{H}_\pi$ be a unit vector and let $H \leq G$ be a closed subgroup such that there exists a unitary character $\chi : H \rightarrow T$ satisfying

$$\pi(h)\eta = \chi(h)\eta, \quad h \in H.$$  

Denote $X := G/H$ and let $\mu_X$ be $G$-invariant Radon measure on $X$, which is unique up to scalar multiplication. Fix a Borel cross-section $\sigma : X \rightarrow G$ for the quotient map $p : G \rightarrow X$. The vector $\eta$ is called admissible if

$$\int_X |\langle \eta, \pi(\sigma(x))\eta \rangle|^2 \, d\mu_X(x) < \infty.$$  

A pair $(\eta, \chi)$ satisfying (11) and (12) is said to define a $\pi$-system of coherent states based on $X = G/H$. The condition (12) is independent of the particular choice of section $\sigma$.

For a $\pi$-system of coherent states, there exists an admissibility constant $d_{\pi,\eta} > 0$ such that, for all $f \in \mathcal{H}_\pi$,

$$\int_X |\langle f, \pi(\sigma(x))\eta \rangle|^2 \, d\mu_X(x) = d_{\pi,\eta}^{-1} \|f\|_{\mathcal{H}_\pi}^2.$$  

For further properties on square-integrability modulo a subgroup, see, e.g. [17, 19].

An irreducible representation $(\pi, \mathcal{H}_\pi)$ is called a coherent state representation if it admits a $\pi$-system of coherent states based on connected, simply connected homogeneous $G$-space $X$.

### 3. Completeness of coherent state subsystems

This section considers the relation between subsystems of coherent states parametrised by a simply connected $G$-space and lattice orbits of an associated projective representation.

#### 3.1. Projective kernel

The kernel and projective kernel of a unitary representation $(\pi, \mathcal{H}_\pi)$ of $G$ are defined by

$$\ker(\pi) = \{ g \in G : \pi(g) = I_{\mathcal{H}_\pi} \} \quad \text{and} \quad \text{pker}(\pi) = \{ g \in G : \pi(g) \in \mathbb{C} \cdot I_{\mathcal{H}_\pi} \},$$

respectively. If $(\pi, \mathcal{H}_\pi)$ is non-trivial and irreducible, then $\text{pker}(\pi) \leq G$ is a connected, closed normal subgroup, and there exists $\chi_\pi : \text{pker}(\pi) \rightarrow T$ such that $\pi(g) = \chi_\pi(g) I_{\mathcal{H}_\pi}$ for $g \in \text{pker}(\pi)$.

The following observation plays a key role in the sequel. Its proof hinges on [16, Lemma 3.5], which characterises coherent state representations $\pi$ in terms of their coadjoint orbit $\Theta_\pi$.

**Proposition 3.** Let $H \leq G$ be a connected subgroup. Suppose $\pi$ admits a $\pi$-system of coherent states based on $G/H$. Then $H = \text{pker}(\pi)$. In particular, $H \leq G$ is normal.

---

1 The definition of a coherent state representation used here is the same as in [16,17,19], but differs from the definition in [13,14,18], where the square-integrability assumption (12) is not part of the definition.
Proof. If \( \pi \) admits a pair \((\eta, \chi)\) satisfying (11) and (12), then \( \pi \) is unitarily equivalent to a subrepresentation of the induced representation \( \text{ind}^G_H \chi \), see, e.g. [16, Proposition 1.2]. Since \( H \leq G \) is assumed to be connected, it follows by [16, Lemma 3.5] that \( H = G(\ell) \) for any \( \ell \in \mathcal{O}_\pi \).

By [4, Theorem 2.1], the projective kernel of an arbitrary irreducible representation \( \pi \) of \( G \) is given by \( \text{pker}(\pi) = \bigcap_{\ell \in \mathcal{O}_\pi} G(\ell) \). Therefore, \( \text{pker}(\pi) = \bigcap_{\ell \in \mathcal{O}_\pi} G(\ell) = H \).

The conclusion of Proposition 3 may fail for disconnected subgroups \( H \leq G \) whenever \( \pi \) has a discrete kernel:

Remark 4. Let \((\pi, \mathcal{H}_\pi)\) be an irreducible unitary representation of \( G \).

(a) If \( \pi \) is square-integrable modulo \( K = \ker(\pi) \), then \( \pi|_K \) satisfies (11) for the trivial character \( \chi \equiv 1 \) and any vector \( \eta \in \mathcal{H}_\pi \) defines a \( \pi \)-system of coherent states based on \( G/K \).

(b) If \( \pi \) is square-integrable modulo \( Z = Z(G) \), then \( \pi\big|_Z \) satisfies (11) for the central character \( \chi \in \hat{Z} \) and any vector \( \eta \in \mathcal{H}_\pi \) defines a \( \pi \)-system of coherent states based on \( G/Z \). Moreover, \( \text{pker}(\pi) = Z(G) \) by [6, Corollary 4.5.4].

3.2. Necessary density conditions

A uniform subgroup \( \Gamma \leq G \) is a discrete subgroup such that \( \Gamma \backslash G \) is compact. For a nilpotent Lie group \( G \), the uniformity of a discrete subgroup \( \Gamma \leq G \) is equivalent to \( \Gamma \) being a lattice, i.e. having finite co-volume; see [6, Corollary 5.4.6].

The following result provides a criterium for cyclicity of restricted (projective) representations in terms of the lattice co-volume or density (cf. [31, Theorem 7.4]).

Theorem 5 ([31]). Let \((\pi, \mathcal{H}_\pi)\) be an irreducible, square-integrable projective unitary representation of a unimodular group \( G \), with formal dimension \( d_\pi > 0 \). Let \( \Gamma \leq G \) be a lattice. If there exists \( \eta \in \mathcal{H}_\pi \) such that \( \pi(\Gamma)\eta \) is complete in \( \mathcal{H}_\pi \), then \( \text{covol}(\Gamma)d_\pi \leq 1 \).

For a genuine representation \( \pi \) of \( G \) that is square-integrable modulo the centre \( Z(G) \), a version of Theorem 5 can also be deduced from [3, Theorem 5]; see also [3, Theorem 3] for a converse in the setting of nilpotent Lie groups. However, in order to treat a representation \( \pi \) that is merely square-integrable modulo \( \ker(\pi) \) (equivalently, \( \text{pker}(\pi) \)), the projective version of Theorem 5 is particularly convenient for the purposes of the present note.

The following completeness result for coherent state subsystems can simply be obtained by combining Proposition 3 and Theorem 5.

Theorem 6. Let \( H \leq G \) be a connected subgroup. Suppose \((\pi, \mathcal{H}_\pi)\) is an irreducible representation that admits an admissible vector \( \eta \in \mathcal{H}_\pi \) defining a \( \pi \)-system of coherent states based on \( X = G/H \), with admissibility constant \( d_{\pi,\eta} > 0 \). Then

(i) \( H = \text{pker}(\pi) \);

(ii) If \( \Gamma \leq G \) is uniform and \( |\pi(\sigma(\gamma'))\eta|_{\gamma' \in \Gamma \cdot o} \) is complete, then \( \text{covol}(\Gamma)d_{\pi,\eta} \leq 1 \).

Proof. By Proposition 3, the admissibility of \( \pi \) implies that \( H = \text{pker}(\pi) \leq G \) is normal. Hence, the induced mapping \( \pi' : G/H \to \mathcal{H}(\mathcal{H}_\pi) \), \( x \mapsto \pi(\sigma(x)) \) forms an irreducible projective representation of \( G/H \). Since the measure \( \mu_X \) is Haar measure on \( X = G/H \), it follows that \( \pi' \) is square-integrable on \( G/H \) by the admissibility condition (12). In particular, the constant \( d_{\pi,\eta} > 0 \) in (13) coincides with the (unique) formal dimension \( d_{\pi'} > 0 \) of the projective representation \((\pi', \mathcal{H}_\pi)\) normalised according to the \( G \)-invariant measure \( \mu_X \).

Suppose \( \Gamma \leq G \) is a uniform subgroup. As in the proof of Proposition 3, the admissibility of \( \pi \) implies that \( \text{pker}(\pi) = G(\ell) \) for any \( \ell \in \mathcal{O}_\pi \). A combination of [6, Proposition 5.2.6] and [6, Theorem 5.1.11] therefore yields that \( \Gamma \cap H \) is a uniform subgroup of \( H = \text{pker}(\pi) \). Hence, the image \( p(\Gamma) \) is a uniform subgroup of \( G/H \) by [6, Lemma 5.1.4 (a)].

In combination, applying Theorem 5 to \((\pi', \mathcal{H}_\pi)\) and \( p(\Gamma) \leq G/H \) yields the result.
Remark 7. The constant \( d_{\pi, \eta} > 0 \) coincides with the formal dimension \( d_{\pi'} > 0 \) of the projective representation \((\pi', \mathcal{H}_\pi')\) of \( X = G / \ker(\pi) \). In particular, the product \( \text{covol}(\rho(\Gamma))d_{\pi'} \) is independent of the choice of \( G \)-invariant measure \( \mu_X \): if \( \mu_X' = c \cdot \mu_X \) for \( c > 0 \), then \( \text{covol}'(\rho(\Gamma)) = c \cdot \text{covol}(\rho(\Gamma)) \) and \( d_{\pi'}' = d_{\pi'} / c \).

Theorem 1 follows directly from Proposition 3 and Theorem 6:

**Proof of Theorem 1.** By assumption, there exists an admissible \( \eta \in \mathcal{H}_\pi \) and associated character \( \chi : H \to \mathbb{T} \) defining a \( \pi \)-system of coherent states based on \( G / H \cong O_\pi \). Since \( O_\pi \) is simply connected, it follows that \( H < G \) is connected, see, e.g. [11, Proposition 1.94]. The conclusions are therefore a direct consequence of Proposition 3 and Theorem 6. 

\[ \square \]

4. Perelomov-type coherent states

Let \((\pi, \mathcal{H}_\pi)\) be an irreducible representation of \( G \). Then \( \pi \) yields an action of \( G \) on the projective spaces \( P(\mathcal{H}_\pi) \) and \( P(\mathcal{H}_\pi^\infty) \) by \( g \cdot [\eta] = [\pi(g)\eta] \).

A system of *Perelomov-type coherent states* is a \( G \)-orbit in \( P(\mathcal{H}_\pi) \),

\[ G \cdot [\eta] = \{ [\pi(g)\eta] : g \in G \}. \]

Let \( G_{[\eta]} \) be the isotropy group of \( \eta \in \mathcal{H}_\pi \setminus \{0\} \) in the projective space \( P(\mathcal{H}_\pi) \),

\[ G_{[\eta]} := \{ g \in G : \pi(g)\eta \in C\eta \}. \]

(14)

Denote by \( X = G / G_{[\eta]} \) the associated homogeneous space and let \( \sigma : X \to G \) be a Borel section for the quotient map \( p : G \to X \). Then a Perelomov-type coherent state system is determined by the system of vectors,

\[ \{ [\eta_x] \}_{x \in X} = \{ [\pi(\sigma(x))\eta] \}_{x \in X}. \]

See [24, Section 2] and [26, Chapter 2] for the basic properties of Perelomov-type states.

Let \( \chi_{[\eta]} : G_{[\eta]} \to \mathbb{T} \) be the unitary character of \( G_{[\eta]} \), such that \( \pi(g)\eta = \chi_{[\eta]}(g)\eta \) for all \( g \in G_{[\eta]} \). Note that \( G_{[\eta]} \) is the maximal subgroup satisfying the property (11) for a chosen \( \eta \).

The following sections consider Perelomov-type coherent states of vectors \( \eta \in \mathcal{H}_\pi^\infty \setminus \{0\} \) with the property that \( G / G_{[\eta]} \) has a symplectic or complex structure. Such systems are of particular interest for geometric quantization, see [22] and [26, Section 16].

4.1. Symplectic projective orbits

Following [12, 13], an orbit \( G \cdot [\eta] = \{ [\pi(g)\eta] : g \in G \} \) is called *symplectic* if \( [\eta] \in P(\mathcal{H}_\pi^\infty) \) and \( G \cdot [\eta] \) is a symplectic submanifold of \( P(\mathcal{H}_\pi) \).

The following simple characterisation of symplectic orbits will be used below, see, e.g. [8, Theorem 26.8] or [5, Proposition 2.1] for proofs.

**Lemma 8 ([18]).** Let \( [\eta] \in P(\mathcal{H}_\pi^\infty) \) and let \( J_\pi : P(\mathcal{H}_\pi^\infty) \to \mathfrak{g}^* \) be the momentum map of \( \pi \). The orbit \( G \cdot [\eta] \) is symplectic if, and only if, the stabiliser \( G_{[\eta]} \) is an open subgroup of \( G(J_\pi(\eta)) \).

For the purposes of this note, the significance of a symplectic orbit is that its stabiliser subgroups coincides with the projective kernel, and hence does not depend on the chosen vector. This is demonstrated by the following proposition.

**Proposition 9.** Suppose \( \eta \in \mathcal{H}_\pi^\infty \setminus \{0\} \) is such that \( G \cdot [\eta] \) is symplectic. Then \( G_{[\eta]} \) is connected. In particular, if \( \eta \) is an admissible vector defining a \( \pi \)-system of coherent states based on \( G / G_{[\eta]} \), then \( G_{[\eta]} = \ker(\pi) \).
Proof. If $G \cdot [\eta]$ is symplectic, then $G \cdot [\eta]$ forms a Hamiltonian $G$-space, with momentum map $J_{\pi} : G \cdot [\eta] \rightarrow \mathfrak{g}^*$ given as in (9), see, e.g. [13, Section 2.5]. Set $\ell := J_{\pi}(\eta)$. Then, by Lemma 8, the stabiliser $G_{[\eta]}$ is an open subgroup of $G(\ell)$. Since $G(\ell)$ is connected (cf. Section 2.1), it follows that $G_{[\eta]} = G(\ell)$ is connected. The last assertion follows from Proposition 3. □

The following provides a partial converse to Proposition 9.

Proposition 10. Suppose $(\pi, \mathcal{H}_\pi)$ is square-integrable modulo $\text{pker}(\pi)$. Then, for any $[\eta] \in \mathcal{P}(\mathcal{H}_\pi^\infty)$, the orbit $G \cdot [\eta]$ is symplectic and $G_{[\eta]} = \text{pker}(\pi)$.

Proof. Let $\eta \in \mathcal{H}_\pi^\infty \setminus \{0\}$ be fixed. The inclusion $\text{pker}(\pi) \subseteq G_{[\eta]}$ is immediate. Conversely, if $g \in G_{[\eta]}$, then

$$J_\pi([\pi(g)\eta]) = \frac{1}{i} \langle \pi(g)\eta, d\pi(X)\pi(g)\eta \rangle = \frac{1}{i} \langle \eta, d\pi(X)\eta \rangle = J_\pi([\eta]), \quad X \in \mathfrak{g},$$

so that by the $G$-equivariance of $J_\pi$ it follows that $\text{Ad}^*(g)J_\pi([\eta]) = J_\pi([\eta])$. This means that $g \in G(J_\pi([\eta]))$, and it remains to show that $G(J_\pi([\eta])) \subseteq \text{pker}(\pi)$. Since $\pi \in G$ is square-integrable modulo $\text{pker}(\pi)$, it is also square-integrable modulo $\text{ker}(\pi)$, see, e.g., [4, Corollary 2.1]. It follows therefore by [6, Theorem 4.5.2] and [6, Theorem 3.2.3] that $\mathcal{O}_\pi$ is a linear variety of the form $\mathcal{O}_\pi = \ell + \mathbb{T}$ for $\ell \in \mathcal{O}_\pi^\perp$, with $\mathbb{T}$ being the Lie algebra of $\text{pker}(\pi)$. In addition, [6, Theorem 3.2.3] yields that $\mathcal{O}_\pi$ is connected, so that $G(\ell) = \text{pker}(\pi)$ for $\ell \in \mathcal{O}_\pi$. By [33, Theorem 4.2] (see also Equation (10)) it follows, in particular, that

$$J_\pi([\eta]) \in J_\pi(\mathcal{P}(\mathcal{H}_\pi^\infty)) \subseteq I_\pi = \overline{\text{conv}}(\mathcal{O}_\pi) = \mathcal{O}_\pi,$$

where $I_\pi := \overline{J_\pi(\mathcal{H}_\pi^\infty)}$ denotes the moment set of $\pi$. Therefore, $G(J_\pi([\eta])) = \text{pker}(\pi)$.

Lastly, since $G_{[\eta]} = \text{pker}(\pi) = G(J_\pi([\eta]))$ by the arguments above, the orbit $G \cdot [\eta]$ is symplectic by Lemma 8. □

Proof of Theorem 2. If $G \cdot [\eta]$ is symplectic, then $G_{[\eta]}$ is connected by Proposition 9. Therefore, if $\eta$ determines a $\pi$-system of coherent states based on $G/G_{[\eta]}$, the conclusions of Theorem 2 follow directly from Theorem 6. □

4.2. Highest weight vectors

In [13,18], an orbit $G \cdot [\eta] = \{[\pi(g)\eta] : g \in G\}$ is called complex if $[\eta] \in \mathcal{P}(\mathcal{H}_\pi^\infty)$ and $G \cdot [\eta]$ is a complex submanifold of $\mathcal{P}(\mathcal{H}_\pi)$.

The following lemma characterises complex orbits in terms of a (complex) stabiliser; cf. [13, Proposition 2.8] and [20, Lemma XV.2.3].

Lemma 11 ([13]). Let $s = (g)_{\mathcal{C}}$. For $[\eta] \in \mathcal{P}(\mathcal{H}_\pi^\infty)$, let $s_{[\eta]} = \{X \in s : d\pi(X)\eta \in \mathcal{C} \cdot \eta\}$.

The following assertions are equivalent:

(i) The orbit $G \cdot [\eta]$ is complex;
(ii) $s_{[\eta]} = \overline{s_{[\eta]}} = s$.

A stabiliser $s_{[\eta]}$ satisfying part (ii) of Lemma 11 is called maximal in [26, Section 2.4], where it is part of a principle for selecting coherent states that minimise the uncertainty principle. Such vectors and associated orbits play an important role in Berezin’s quantization, see [26, Section 16]. In addition, vectors of this type are intimately related to highest weight modules and representations (cf. [18,20]) and are also referred to as highest weight vectors.

Remark 12. By [13, Proposition 2.8], any complex orbit is automatically symplectic in the sense of Section 4.1. Theorem 2 applies therefore to highest weight vectors.
Remark 13. The significance of a complex orbit $G \cdot [\eta]$ is that the quotient manifold $G / G_{[\eta]}$ admits a complex structure (cf. [20, Section XV.2]). In turn, for certain (classes of) representations admitting highest weight vectors, the representation space may be realised as a space of holomorphic functions (see [26, Section 2.4] and [32]); in particular, see [15, Section 5] for complex orbits for the Heisenberg group. For nilpotent Lie groups, the existence of complex orbits appears to be restrictive, i.e. [14, Theorem 1] asserts that the only irreducible representations with a discrete kernel admitting complex orbits are those of Heisenberg groups. In contrast, symplectic orbits do exist for all groups admitting square-integrable representations by Proposition 10.

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