Dirac Spinors and Representations of GL(4) Group in GR

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Abstract

Transformation properties of Dirac equation correspond to Spin(3,1) representation of Lorentz group SO(3,1), but group GL(4,\mathbb{R}) of general relativity does not accept a similar construction with Dirac spinors. On the other hand, it is possible to look for representation of GL(4,\mathbb{R}) in some bigger space, there Dirac spinors are formally situated as some “subsystem.” In the paper is described construction of such representation, using Clifford and Grassmann algebras of 4D space.

1 Introduction

Let us consider the Dirac equation \[ D\psi = m\psi, \quad D = \sum_{\mu=0}^{3} \gamma_{\mu} \frac{\partial}{\partial x_{\mu}}, \] (1)

where \( \psi \in \mathbb{C}^{4} \) and \( \gamma_{\mu} \in \text{Mat}(4, \mathbb{C}), \mu = 0, \cdots, 3 \) are four Dirac matrices with usual property

\[ \gamma_{\mu} \gamma_{\nu} + \gamma_{\nu} \gamma_{\mu} = 2g_{\mu\nu} \mathbb{1}, \]

(2)

where \( g_{\mu\nu} \) is Minkowski metric.

More generally, Eq. (2) corresponds to definition of Clifford algebra \( \mathfrak{C}(g) \) for arbitrary quadratic form \( g_{\mu\nu} \) and dimension. Dirac matrices are four generators for (complexified) representation of \( \mathfrak{C}(3,1) \) in space of \( 4 \times 4 \) matrices. For general universal Clifford algebra with \( n \) generators \( \dim \mathfrak{C}(n) = 2^n \).

It is possible to write Dirac equation for arbitrary metric, but covariant transformation between two solutions \( \psi \) exists only for isometries of coordinates \( A: g(Ax, Ay) = g(x, y) \) for given fixed metric \( g \). Such isometries produce some subgroup of GL(4,\mathbb{R}), isomorphic to Lorentz group (for diagonal form, i.e. Minkowski metric), it is usual Lorentz group). It is briefly recollected in Sec. 2. A matrix extension of Dirac equation is considered in Sec. 3. It is analyzed using Clifford (Sec. 4) and Grassmann representation (Sec. 5) of such equation. Main proposition about representation of GL(4,\mathbb{R}) is formulated at end of Sec. 5 and discussed with more details in Sec. 6.
2 Spinor Representation of Lorentz Group

Any transformation \( v' = Av \) of coordinate system corresponds to new set of gamma matrices:

\[
\gamma'_\mu = \sum_{\nu=0}^{3} A^\nu_\mu \gamma_\nu, \tag{3}
\]

but only for isometries \( A \). Eq. (3) may be rewritten as internal isomorphism of algebra:

\[
\gamma'_\mu = \Sigma_{(A,g)} \gamma_\mu \Sigma_{(A,g)}^{-1}, \quad \text{(there is no sum on } \mu) \tag{4}
\]

where matrix \( \Sigma_{(A,g)} \) is same for any \( \gamma_\mu \) and depends only on transformation \( A \) and metric \( g \). For Minkowski metric \( g \), it is usual form of spinor \( 2 \rightarrow 1 \) representation of Lorentz group.

Similar with general case, spin group is implemented here as subset of Clifford algebra. More precisely, it is subset of \( \mathbb{C}l \), \( i.e. \) even subalgebra of Clifford algebra generated by all possible products with even number of generators \( 2 \).

In particular case of Lorentz group \( SO(3,1) \), it is \( Spin(3,1) \) group and isomorphic with \( SL(2,\mathbb{C}) \), group of \( 2 \times 2 \) complex matrices with determinant unit.

To demonstrate transformation properties of spinor \( \psi \), it is enough to rewrite Eq. (1) as

\[
(\Sigma D \Sigma^{-1}) \Sigma \psi = m \Sigma \psi. \tag{5}
\]

The Eq. (5) also shows, why Eq. (3) with general \( A \in GL(4,\mathbb{R}) \) may not correspond to such kind of covariant transformation \( \psi(x) \rightarrow \Sigma_A \psi(Ax) \).

3 An Extension of Spinor Space

One way to overcome problem with representation of general coordinate transformation of \( GL(4,\mathbb{R}) \) group may be extension of linear space \( \mathbb{C}^4 \) of Dirac spinors.\(^1\)

There is also more technical way to explain idea of extension. It is not possible to build some specific symmetry between two solutions of Dirac equation with different metric, but may be it is possible to construct a new one using a few solutions?

Let us consider instead of \( \psi \in \mathbb{C}^4 \) some \( 4 \times 4 \) complex matrix \( \Psi \) and write equation

\[
D\Psi = m\Psi, \quad D \equiv \sum_{\mu=0}^{3} \gamma_\mu \frac{\partial}{\partial x_\mu}, \quad \Psi \in Mat(4,\mathbb{C}). \tag{6}
\]

Formally it may be considered as set of four usual Dirac equations for each row of matrix \( \Psi \).

\(^1\)Say, two-component spinors may not represent \( P \)-inversion, but extension from \( \mathbb{C}^2 \) to \( \mathbb{C}^4 \), from Pauli to Dirac spinors resolves this problem \( \text{[1]} \).
Maybe such extension from 4D to 16D space is not minimal, but it is appropriate for purpose of present paper, i.e. for representation of transformation properties of such equation with respect to GL(4, \mathbb{R}) group of coordinate transformations. For 16D space all possible linear transformations may be represented by 256D space, but application of algebraic language may simplify consideration. The construction uses both Clifford and Grassmann algebras of 4D space and described below.

4 Clifford Algebra

Complexified Clifford algebra of Minkowski quadratic form is isomorphic with space of $4 \times 4$ complex matrices. So it is reasonable to try consider Eq. (6) as equation on Clifford algebra $\gamma_{\mu}, \Psi \in \mathfrak{Cl}(3, 1)$. Really, as it will be shown below, complete and rigour solution of discussed problem with GL(4, \mathbb{R}) representation may not be based only on Dirac equation on Clifford algebra, but this construction discussed here, because provides an essential step.

If to consider Eq. (6) as equation on Clifford algebra, then initial Dirac equation may be compared first with restriction of such equation on left ideal of Clifford algebra.

Left ideal of algebra $\mathcal{A}$ by definition is linear subspace $\mathcal{I} \subset \mathcal{A}$ with property $\mathcal{A}\mathcal{I} \subset \mathcal{I}$, i.e. any element of algebra after multiplication on element of an ideal produces again element of the ideal. Simplest example of left ideal in matrix algebra is set of matrices $M \psi$ with only one nonzero column $\psi$ and it provides reason for consideration of Dirac equation on such ideal as an analogue of usual case Eq. (1).

It was already discussed above, that spin group may be naturally implemented as subspace of Clifford algebra, e.g. transformations of $\Psi$ are also elements of Clifford algebra, $\Sigma \in \mathfrak{Cl}(3, 1)$. On the other hand, construction with ideals of Clifford algebra have some problems with interpretations of symmetries of Dirac equation, necessary for purposes of given paper. Even for Lorentz transformation, simply represented as isomorphisms of Clifford algebra Eq. (4), instead of Eq. (5) we must have the same transformation law for all elements of algebra

$$(\Sigma D \Sigma^{-1}) \Sigma \Psi \Sigma^{-1} = m \Sigma \Psi \Sigma^{-1}. \quad (7)$$

Really it makes consideration a bit more difficult, but does not change it much. Resolution of the problem, is additional symmetry of Eq. (6): if some $\Psi$ is solution of Eq. (6), then $\Psi R$ is also solution, for arbitrary element $R$ of Clifford algebra.

In such a case right multiplication on $\Sigma^{-1}$ does not changes anything and may be ignored. The same property of equation requires consideration not only element $M_{\psi}$ of some left ideal, but also all $M_{\psi}R$, i.e. matrices with columns proportional to same vector.

If $\psi \in \mathbb{C}^4$ is initial vector (spinor, solution of Dirac equation), and $\alpha \in \mathbb{C}^4$ is arbitrary vector of coefficients, then any matrices with proportional columns may be expressed as $M_{ij} = \psi_i \alpha_j$. 3
So instead of left ideals discussed above it is necessary to consider matrices

\[ M_{ij} = \psi_i \alpha_j, \quad M = \psi \alpha^T = \psi \otimes \alpha. \]  

(8)

For arbitrary matrices \( L, R \)

\[ LMR = L(\psi \otimes \alpha)R = (L\psi) \otimes (R^T \alpha), \]

(9)

so multiplication saves “the product structure,” but it is not an ideal, because space of such matrices is not linear subspace, e.g. sum of elements does not necessary may be presented as tensor product of two vectors like Eq. (8). Really linear span of the “singular” space coincides with whole algebra.

On the other hand, fixed \( \psi \) corresponds to a linear subspace, right ideal \( R_\psi \) of the algebra. It is similar with interpretation of physical solution of usual Dirac equation Eq. (1) as a ray in Hilbert space.

But such construction still not produce covariant transformation of Dirac equation in matrix form Eq. (6) with respect to general element of \( \text{GL}(4, \mathbb{R}) \). It is necessary to use slightly different construction with Grassmann algebra described in next section.

5 Grassmann Algebra

Formally Grassmann (or exterior) algebra \( \Lambda_n \) is defined by \( n \) generators \( d_i \), associative operation denoted as \( \wedge \) and property

\[ d_\mu \wedge d_\nu + d_\nu \wedge d_\mu = 0. \]  

(10)

So \( \dim \Lambda_n = 2^n \), similarly with Clifford algebra. Linear subspace of Grassmann algebra generated by \( \wedge \)-product of \( k \) different elements \( d_i \) are usually denoted as \( \Lambda^n_k \ (“k$-forms”) \). For convenience here is used complex Grassmann algebra.

On the other hand, Clifford algebra, Dirac operator and spin group also may be expressed using Grassmann algebra \[2\]. Let us consider Grassmann algebra \( \Lambda_n \) of \( n \)-dimensional vector space \( V \) and metric \( g \) on \( V \). Algebra of linear transformations of Grassmann algebra is denoted here as \( \mathcal{L}(\Lambda_n) \) and for any vector \( v \in V \), it is possible to construct linear transformations \( \delta_v, \delta^*_v \in \mathcal{L}(\Lambda_n) \):

\[ \delta_v : v_1 \wedge \cdots \wedge v_k \mapsto v \wedge v_1 \wedge \cdots \wedge v_k, \]

(11)

\[ \delta_v^* : v_1 \wedge \cdots \wedge v_k \mapsto \sum_{l=1}^{k} (-1)^l g(v, v_l) v_1 \wedge \cdots \hat{v}_l \cdots \wedge v_k, \]

(12)

where \( \hat{v} \) means, that term \( v \) must be omitted. Let \( v_i \) is basis of \( V \), then operators

\[ \hat{\gamma}_i = \delta_i + \delta^*_i \quad (\delta_i \equiv \delta_{v_i}, \delta^*_i \equiv \delta^*_{v_i}) \]

(13)

satisfy usual relations with anticommutators Eq. \(3\) for \( n \) generators of Clifford algebra with quadratic form \( g \), and so \( \mathcal{C}(g) \) may be represented as subspace of \( \mathcal{L}(\Lambda_n) \). Let us denote this representation

\[ \mathcal{C}_E : \mathcal{C}(n) \rightarrow \mathcal{L}(\Lambda_n). \]  

(14)
Let us consider also canonical isomorphism of Grassmann and Clifford algebras \( \Lambda : \Lambda_n \rightarrow \mathcal{C}(n) \) as linear spaces defined on basis as
\[
\Lambda : d_{i_1} \wedge \cdots \wedge d_{i_k} \mapsto \gamma_{i_1} \cdots \gamma_{i_k}, \quad i_1 < i_2 < \cdots < i_k, \tag{15}
\]
together with inverse one \( \Lambda^{-1} \).

It is possible to express basic property of formal constructions above as
\[
\Lambda^{-1}(LM) = \mathcal{E}_L(L)\left(\Lambda^{-1}_C(M)\right), \quad L, M \in \mathcal{C}. \tag{16}
\]

It should be mentioned, that using Hodge operator \( \star : \Lambda^k_n \rightarrow \Lambda^{n-k}_n \) it is possible to write \( \delta^* = \star \delta \star \). By using dual Grassmann operation \( \vee = \star \wedge \star \), it is possible to simplify Eq. (11), Eq. (12)
\[
\delta_v(\omega) = v \wedge \omega, \quad \delta^*_v(\omega) = v \vee \omega, \quad \omega \in \Lambda_n. \tag{17}
\]

Similarly, it is possible to introduce right actions
\[
\delta_v(\omega) = \omega \wedge v, \quad \delta^*_v(\omega) = \omega \vee v, \quad \gamma_i = \delta_i + \delta^*_i, \tag{18}
\]
and right representation \( \mathcal{E}_R \) of Clifford algebra in \( \mathcal{L}(\Lambda_n) \) with property
\[
\Lambda^{-1}(LMR) = \left(\mathcal{E}_L(L) \circ \mathcal{E}_R(R)\right)(\Lambda^{-1}_C(M)), \quad L, M, R \in \mathcal{C}. \tag{19}
\]

The Dirac equation also has natural representation here [2]. It is possible to express Dirac operator using exterior differential for forms \( d \) and its Hodge dual \( d^* \)
\[
d = \sum_{i=1}^n \delta_i \frac{\partial}{\partial x_i}, \quad d^* = \sum_{i=1}^n \delta^*_i \frac{\partial}{\partial x_i}, \quad \mathcal{D} = d + d^*. \tag{20}
\]

It is also possible to use standard representation of spin group via left representation \( \mathcal{E}_L \) of Clifford algebra Eq. (14). Usually it is restricted on spaces of odd and even forms [2].

On the other hand, unlike of Clifford space, Grassmann space also accept external \( GL(4, \mathbb{R}) \) isomorphism induced by extension of transformation
\[
d'_\mu = \sum_{\nu=0}^3 A'_{\nu} d_{\nu}. \tag{21}
\]

to general form \( d_{i_1} \wedge \cdots \wedge d_{i_k} \). Existence of such kind of isomorphism is common property of antisymmetric forms, and Grassmann algebra may be represented as exterior algebra of the forms.

Representation of Dirac operator in form Eq. (20) ensures proper transformation property. It is clear for exterior differential \( d \), and for \( d^* \) it is also so, because terms with \( g(v, v_i) \) in Eq. (12) are transformed to \( g(Av, Av_i) \), i.e. in agreement with change of metric \( g' = A^T g A \) and also demonstrate desired property of \( \mathcal{D} = d + d^* \) with respect to map \( \mathcal{E}_L \) and Eq. (3).
The main proposition needs for some explanation. Transformation of space of differential forms induced by general linear coordinate transformation is really valid representation of GL(4,R), but that is relation with usual spinor representation in case of restriction to SO(3,1) ⊂ GL(4,R)? For example, exterior space in respect to GL(4,R) have five irreducible subspaces corresponding to spaces $\Lambda_k^4$, $k = 0, \ldots, 4$.

On the other hand, from a naive point of view Dirac spinors in such classification should correspond to some (fictitious) index like $k = 1/2$, because exterior form with index $k = 1$ corresponds to (co)vector, i.e. “spin one.”

To justify the proposition, let us show first, that for Minkowski metric and Lorentz group SO(3,1) ⊂ GL(4,R) suggested transformation corresponds to $\Psi \mapsto \Sigma \Psi \Sigma^{-1}$ used for description of transformation property Eq. (7) of Dirac equation in matrix representation Eq. (6).

Let us consider subspaces $\mathfrak{Cl}_k(n) \subset \mathfrak{Cl}(n)$, $k = 0, \ldots, n$ produced by products of $k$ matrices $\gamma_\mu$. Only for element $\Sigma$ from spin group internal isomorphism Eq. (22) of Clifford algebra maps subspaces $\mathfrak{Cl}_k(n)$ to itself. On the other hand, such isomorphism may be expressed using substitution of generators like Eq. (3) with isometry $A$, i.e. Lorentz group for particular case under consideration.

For Lorentz group transformation law for $\mathfrak{Cl}_k(3,1)$ due to Eq. (3) is the same as for $\Lambda_A^4$ and Eq. (21) respectively. More formally, in such a case it is possible to express transformation suggested in proposition as

$$(\mathcal{C}_L(\Sigma_A) \circ \mathcal{C}_R(\Sigma_A^{-1}))(M), \quad M \in \Lambda_A, \quad \Sigma_A \in \text{Spin}(3,1), \quad A \in \text{SO}(3,1).$$

Where is also other way to represent constructions suggested above. Let we have Dirac algebra of $4 \times 4$ complex matrices and some fixed basis of Dirac matrices $\gamma_\mu$. Let us now together with usual matrix multiplication introduce other associative and distributive operation “$\wedge$” induced by structure of Grassmann algebra due to linear map $\Lambda_\xi$ Eq. (15). Formally, it would be necessary to write $16^2 = 256$ products for elements of basis, but really the operation “$\wedge$” is unique defined by expression with two matrices

$$\gamma_\mu \wedge \gamma_\mu \equiv 0, \quad \gamma_\mu \wedge \gamma_\nu \equiv -\gamma_\nu \wedge \gamma_\mu \equiv \gamma_\mu \gamma_\nu \quad (\mu < \nu),$$

used for description of transformation property Eq. (7) of Dirac equation in matrix representation Eq. (6).
associativity and property \( \gamma_{i_1} \wedge \cdots \wedge \gamma_{i_k} \equiv \gamma_{i_1} \cdots \gamma_{i_k} \) for \( i_1 < i_2 < \cdots < i_k \).

Now linear map Eq. (3) with arbitrary \( A \in \text{GL}(4, \mathbb{R}) \) may be extended to representation of \( \text{GL}(4, \mathbb{R}) \) on full 16D space using “\( \wedge \)" products of generators. It is also valid for \( A \in \text{SO}(3, 1) \), but in such a case difference between “\( \wedge \)" and usual matrix (Clifford) products does not matter, because due to property of spin group usual product does not produce “junk" terms with \( g_{\mu \nu} \mathbb{1} \) (unit of algebra multiplied on some coefficient of metric). The other property of \( A \in \text{SO}(3, 1) \) is possibility to express considered representation as internal isomorphism with respect to usual product, \( \Psi \mapsto \Sigma_A \Psi \Sigma_A^{-1} \), Eq. (22).

It was already discussed earlier, how Eq. (22) corresponds to usual spinor representation. Because of Eq. (8) matrix function \( \Psi \) may be associated with tensor product of usual spinor \( \psi \) on some “auxiliary spinor \( \alpha \)” and then due to Eq. (9), it is possible for Lorentz group to rewrite Eq. (22) as

\[
\psi \otimes \alpha \mapsto (\Sigma \psi) \otimes (\Sigma^{-1} T \alpha),
\]

but state of auxiliary system for such a product does not matter due to possibility to apply arbitrary transformation \( R : \alpha \mapsto R^T \alpha \) (see Sec. 4). So it is possible to take into account only transformation of first term \( \psi \mapsto \Sigma \psi \) in tensor product, i.e. spinor representation of Lorentz group.

On the other hand, general \( \text{GL}(4, \mathbb{R}) \) transformation does not correspond to map between “product states” like \( \psi \otimes \alpha \). Using some quantum mechanical jargon it could be possible to say, that general \( \text{GL}(4, \mathbb{R}) \) transformation “entangles” state \( \psi \) and auxiliary state \( \alpha \), and transformation \( R \) on second state used above may not improve situation (“disentangle” states), because it is “local”. With same analogy, relation between \( \Psi \) and usual Dirac spinor \( \psi \) may be compared with conception of subsystem in quantum mechanics. But really such jargon should be considered only as some hint, because more detailed consideration may use also real representation of Clifford algebra and constructions used above may correspond to real or even quaternionic matrices and tensor products and, after all, transformations used here correspond to some finite-dimensional unitary representations only for \( \text{SO}(3) \) subgroup of \( \text{GL}(4, \mathbb{R}) \).

References

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