We present and test a coupled wake boundary layer (CWBL) model that describes the distribution of the power output in a wind-farm. The model couples the traditional, industry-standard wake expansion/superposition approach with a top-down model for the overall wind-farm boundary layer structure. The wake expansion/superposition model captures the effect of turbine positioning, while the top-down portion adds the interaction between the wind-turbine wakes and the atmospheric boundary layer. Each portion of the model requires specification of a parameter that is not known a-priori. For the wake model the wake expansion coefficient is required, while the top-down model requires an effective span-wise turbine spacing within which the model’s momentum balance is relevant. The wake expansion coefficient is obtained by matching the predicted mean velocity at the turbine from both approaches, while the effective span-wise turbine spacing depends on turbine positioning and thus can be determined from the wake expansion/superposition model. Coupling of the constitutive components of the CWBL model is achieved by iterating these parameters until convergence is reached. We illustrate the performance of the model by applying it to both developing wind-farms including entrance effects and to fully developed (deep-array) conditions. Comparisons of the CWBL model predictions with results from a suite of large eddy simulations (LES) shows that the model closely represents the results obtained in the high-fidelity numerical simulations. A comparison with measured power degradation at the Horns Rev and Nysted wind-farms shows that the model can also be successfully applied to real wind-farms.

I. INTRODUCTION

It is well known that wakes created by upstream wind-turbines can significantly influence the power production of downstream turbines in wind-farms. Depending the relative positioning of the turbines within the wind-farm, power output reductions of 40 – 50% due to wake effects are not uncommon [1, 2]. Modeling these wake effects is important in order to estimate the power production of different wind-farm layouts [3]. Especially for large wind-farms, the two-way coupling of the relevant dynamics to the overall structure of the atmospheric boundary layer is another important factor that affects the performance of wind-farms [1]. Analytical modeling of these two main aspects of the problem have traditionally relied on two quite different approaches. The first approach is based on modeling of wind-turbine wakes, in which the wake diameter is assumed to expand (typically linearly) behind the turbine and the velocity deficit is obtained assuming mass (or linearized momentum) conservation. This procedure can be considered a “bottom-up” approach. The resulting model, which is sometimes referred to as the Jensen/PARK model [4–11], is built into typical commercial packages that are used to predict wind-farm performance. When many wakes are superposed in large wind-farms, additional complexities arise due to the vertical structure of the atmospheric boundary layer and the associated wake-atmosphere interactions are not typically captured by wake models.

The second analytical approach for modeling wind-farms consists of representing the flow in an entire wind-turbine array region based on horizontal averaging. In this method, which can be considered a “top-down” or single-column modeling approach, the turbines are seen as roughness elements. In this framework, the average velocity profile at hub-height can be obtained based on the assumption of the existence of two logarithmic regions, one above the turbine hub-height and one below [12–16]. The top-down approach can predict the effective roughness height of the wind-farm. In the Calaf et al. model [17] some wake effects are also included, although the results and predictions depend only on the area-averaged turbine spacing. Therefore, the specific spatial arrangement of the wind-turbines, e.g. distinguishing between aligned and staggered configurations, is not possible. Later this work has been extended to include predictions for the power development in large wind-farms by Meneveau [18] and Stevens [19]. These models have also been used to predict the optimal (average) turbine spacing, by taking the cost of the turbines and the land into account [19, 20]. Extensions towards different atmospheric stability conditions have been developed [8, 21, 22].

Ideally then, one would wish to combine both approaches and allow each to predict complementary features
Effective span-wise spacing, $s_{ve}$

Topdown model

Wake expansion superposition model

Effective wake expansion coefficient, $k_w$

FIG. 1: Conceptual sketch of the coupling between the wake expansion/superposition and the top-down model. The top-down part captures the deep farm effects and is used, via iterations, to determine the wake expansion coefficient needed in the wake model in order to accurately capture the fully developed regime of the wind-farm. Conversely, the top-down model requires specification of an effective span-wise spacing. This distance depends on the turbine positioning and is determined with the wake expansion/superposition model. Convergence to a consistent CWBL system is obtained by iterating until the mean stream-wise velocity at turbine hub-height is the same in both models for the fully developed region of the wind-farm. The method is described in detail in §IV.

of the flow. Prior efforts at combining both approaches include the original works of Frandsen [16], in which three regimes are identified. In regime 1 of that model the wakes are expected to expand axisymmetrically. In regime 2 the wakes merge and specific expansion rates for the wakes are proposed. Further downstream, in regime 3, the wind-farm performance is estimated with a top-down approach like the one presented in Ref. [14]. This model has led to using the top-down model as an “upper limit” in commercial codes [23, 24]. Another commonly used approach is to set the wake expansion coefficient based on the turbulence intensity of the incoming flow. This was first proposed by Lissaman [4], and similar ideas can be found in Frandsen [16] as well as in Yang, Kang & Sotiropoulos [25]. More recently, Peña and Rathmann [8, 26] evaluated the effects of atmospheric stratification by using a top-down modeling framework to develop predictions for turbulence intensities as well as an expansion coefficient in terms of stratification.

Wake models are practical and easy to implement, however their ability to make realistic predictions degrades in the fully developed region [1]. On the other hand the top-down model captures the interaction with a fully developed wind-farm and the ABL well, but does not know the effect of the relative turbine positioning. To-date both the wake expansion/superposition model and the top-down model have been applied without two-way coupling, as we will propose in the present work in an effort to combine the positive aspects of each.

Evaluating models based on field data from operational wind-farms is sometimes possible but it is generally very difficult due to the limited availability and control over the flow parameters for the field sites. Conversely, high-fidelity numerical simulations can provide data that can be used to test simplified engineering models under idealized and well-controlled conditions. State-of-the-art Large Eddy Simulations (LES), which only require parameterizations of the smallest turbulent scales, can be utilized for this purpose. Recently, LES have been used to obtain parameterizations of the roughness height of wind-farms with an improved top-down model approach [17, 19], thus describing the entire wind-farm as a roughened surface with increased momentum flux and kinetic energy extraction.

As LES requires a significant computational effort, industry still relies on less expensive methods in order to design and optimize wind-farm layouts. For example the wake expansion/superposition (Jensen/PARK) model described above [5, 6] is used in several optimization studies [27, 30]. It was shown by Nygaard [11] that with a simple version of the Jensen model close to that implemented in the WAsP model (without image wakes) power degradation data from various wind-farms (e.g. London Array and Nysted) could be predicted well, including in the deep-array region for the aligned cases considered. Other examples include the use
of parabolized forms of the Reynolds-Averaged Navier Stokes (RANS) equations such as the Ainslie model \[31\] and UPMPARK, which uses a $k - \varepsilon$ turbulence model, and was later improved into WakeFarm (see e.g. Schepers and van der Pijl \[32\] [33]) and Farmflow (see e.g. Eecen and Bot \[34\], Schepers \[35\], and Özdemir et al. \[36\]), or models that are based on a parametrization of the internal boundary layer growth coupled with some eddy viscosity model, e.g. the Deep-Arraw Wake Model of Openwind \[24\]. Other approaches include the Large Array wind-farm model in WindFarmer \[37\] [38] and linearized CFD models such as FUGA \[39\], Windmodeller \[23\], Ellipsys \[40\], and the Advanced Regional Prediction System (ARPS) \[41\] [42]. The above is not a comprehensive list. For reviews of these and different methods we refer to Refs. \[43\] [46] [47].

In this paper we introduce the Coupled Wake Boundary Layer (CWBL) model, which provides a method of coupling the wake expansion/superposition model \[5\] [6] [8] and the top-down model \[17\] to provide improved predictions of the mean velocity distributions in a wind-farm and to estimate the associated wind-turbine power outputs. The wake expansion/superposition model within the CWBL model ensures that the relative positioning of the turbines is represented, while the fully developed (deep wake array) wind-farm’s vertical structure is captured with the top-down portion of the model. Both the top-down and the wake portions of the CWBL system each contain a parameter that is not known a priori. These two parameters can be obtained from the complementary part of the CWBL model using an iterative procedure as shown schematically in figure 1. Here the wake growth coefficient required for the wake expansion/superposition model is obtained by matching the predicted mean velocities or mean power with the predictions from the top-down model. Similarly the effective span-wise spacing needed by the top-down model is specified using the wake expansion/superposition model. Being an analytical model (as opposed to the differential equations-based models such as RANS or LES), the CWBL model inherits the practical advantages of the Jensen/PARK - type approaches.

As an initial step, the model only considers wind-farms in which the turbines are placed on a regular lattice. Before the coupling between both models is presented, we first briefly review the basic concepts of the wake expansion/superposition model (section II) and the top-down model (section III), and illustrate their merits and drawbacks by comparing their respective predictions with LES data. In section V the two-way coupling of the models is discussed in detail. This is followed by detailed comparisons of the model results with LES data, in section VI. The LES data we use are for wind-farms with 10 or more downstream turbine rows with different combinations of span-wise and stream-wise spacings. For details about the simulations we refer the reader to Refs. \[48\] [50]. In section VI D the model is compared to measurements in Horns Rev and Nysted. Section VI provides general conclusions and an outlook to future work.

II. WAKE EXPANSION/SUPERPOSITION MODEL

The classic wake expansion/superposition model has been developed based on successive contributions by Lissaman \[4\], Jensen \[5\] and Katíc et al. \[6\]. It assumes that wind-turbine wakes grow linearly (based on the notion that the background turbulence provides a spatially constant level of transverse velocity fluctuations \[4\]). In the far wake, conservation of linear momentum (in addition to a linearization valid in the far field) implies that the defect velocity evolves according to \[4\] [5]:

$$u = u_{\text{free}} \left(1 - \frac{1 - \sqrt{1 - C_T}}{(1 + k_w x/R)^2}\right) = u_{\text{free}} \left(1 - \frac{2a}{(1 + k_w x/R)^2}\right),$$  \hspace{1cm} (1)$$

where $u_{\text{free}}$ is the incoming free stream velocity, $k_w$ is the wake expansion coefficient, $R$ the rotor radius, and $C_T = 4a (1 - a)$ the thrust coefficient with a flow induction factor $a$. Here $x$ is the downstream distance with respect to the turbine.

If several turbines are located upstream of a given turbine of interest, their wake effects accumulate. It was proposed by Katíc et al. \[6\] (in fact also by Lissaman in 1979 \[4\]) that the kinetic energy deficit of the mixed wake is the same as the sum of the energy deficits of upstream wakes that are modeled as if they were each exposed to the unperturbed free-stream velocity $u_{\text{free}}$. Thus, Katíc et al. \[6\] proposed to model the wake effects by adding the squared velocity deficits of the individual wakes. The velocity deficit at position $\mathbf{x} = (x, y, z)$ due to some upstream turbine (turbine $j$) centered at position $(x_j, y_j, z_h)$, where $z_h$ is the turbine hub-height, is defined according to

$$\delta u(\mathbf{x}; j) = u_{\text{free}} - u(\mathbf{x}; j) = \frac{2 a u_{\text{free}}}{(1 + k_w (x-x_j)/R)^2}.$$  \hspace{1cm} (2)$$

where $u_{\text{free}}$ is the incoming free stream velocity, $k_w$ is the wake expansion coefficient, $R$ the rotor radius, and $C_T = 4a (1 - a)$ the thrust coefficient with a flow induction factor $a$. Here $x$ is the downstream distance with respect to the turbine.

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An upstream turbine only generates a velocity deficit at point \( x \) if its wake intersects this point, i.e. as long as
\[
(y - y_j)^2 + (z - z_h)^2 \leq [R + k_w (x - x_j)]^2 \quad \text{for} \quad x > x_j, \tag{3}
\]
Here \((x - x_j)\) indicates the downstream, \((y - y_j)\) the ‘transverse’, and \((z - z_h)\) the vertical distance with respect to the turbine hub-height \( z_h \).

The interaction of the wakes with the ground is modeled by incorporating “ghost” or “image” turbines under the ground surface based on the procedure in Lissaman [4]. That is to say, to each turbine \( j \) at position \((x_j, y_j, z_h)\) we associate an image turbine at \((x_j, y_j, -z_h)\). The interaction of the wakes originating from the “ghost” turbines with the wakes originating from the actual turbines is assumed to model the reduced rate of wake recovery (and thus larger velocity deficit) due to ground effects. Thus it is assumed that the following four types of upstream wakes interact when modeling the velocity at some turbine location \( x \):

1. Turbines directly upstream of point \( x \) [5] (the set of turbines, to be denoted as turbine set \( J_U \), that are in front of the point \( x \)),
2. Turbines in adjacent rows whose wakes grow sufficiently to overlap with position \( x \) (turbine set \( J_S \)) ,
3. Underground “ghost” turbines directly upstream of \( x \) (turbine set \( J_G \)), and
4. “Ghost” turbines associated with the turbines in adjacent rows (turbine set \( J_{GS} \)).

The corresponding superposition of velocity defect kinetic energies leads to the following model for the velocity at the point \( x \)
\[
u(x) = \nu_{\text{free}} - \sqrt{\sum_{j \in J_A} \delta u_2(x; j)}, \tag{4}
\]
where \( J_A = J_U \cup J_S \cup J_G \cup J_{GS} \) is the union of all the four sets of wake effects that are seen at point \( x \) according
FIG. 3: Power output ratio $P_r/P_1$ in the fully developed regime according to the wake expansion/superposition model (equations (5)-(7)) with $k_w = 0.0579$ as function of the (geometric mean) turbine spacing $s = \sqrt{s_x s_y}$. The lines indicate the model results for $s_x = 3.49$ (black), 5.23 (red), and 7.85 (blue) and varying stream-wise spacings to yield variations as function of $s$. $P_r$ is the power output per turbine in the fully developed wind-farm while $P_1$ is the reference power output of a single turbine without effects from other turbines. The panels show the results for the (a) aligned and (b) staggered configuration compared to LES results [48–50]. The dashed lines indicate the sum approximations (equations (25)-(30)) for the wake expansion/superposition model given in appendix 1.

to the condition in equation (3). Next, consider points that are located on the disk associated with a particular wind-turbine $T$. Suppose that there are $N_d$ such points (on some suitably chosen spatial “lattice”). The mean velocity at a particular point $\mathbf{x}_{T,k}$ (with $k = 1, 2, \ldots, N_d$) on the turbine disk is given by evaluating equation (4) at the position $\mathbf{x} = \mathbf{x}_{T,k}$. The ratio of the velocity at that point divided by the incoming unperturbed velocity $u_{\text{free}}$ is thus given by

$$\frac{u(\mathbf{x}_{T,k})}{u_{\text{free}}} = 1 - \sqrt{2a \sum_{j \in J_{T,k}} [1 + k_w(x_{T,k} - x_j)/R]^{-4}}. \quad (5)$$

Note that in this equation the set $J_{T,k}$ depends on the specific point $\mathbf{x}_{T,k}$ since different locations on the disk may intersect different wakes from different sets of upstream turbines. The velocity of turbine $T$ with respect to the incoming wind is obtained by computing the average velocity over all points in the turbine disk area using

$$\frac{u_T}{u_{\text{free}}} = \frac{1}{N_d} \sum_{k=1}^{N_d} \frac{u(\mathbf{x}_{T,k})}{u_{\text{free}}}. \quad (6)$$

The power $P_T$ of that turbine normalized with the power of a free-standing turbine $P_1$ (or the first row of the wind-farm) is given by

$$\frac{P_T}{P_1} = \left(\frac{u_T}{u_{\text{free}}}\right)^3. \quad (7)$$

In this model the wind-speed reduction at a particular turbine $T$ is therefore a function of (1) the assumed spatial distribution of the upstream and adjacent turbines, and (2) of the wake decay parameter $k_w$. Frandsen [14] proposed a relationship between this parameter and the atmospheric turbulence characteristics. Following a reasoning that was also articulated in Lissaman [1], the growth rate can be assumed to be on the order of the ratio of transverse velocity fluctuations to the mean velocity. Assuming that the former is on the order of the friction velocity, the ratio defining the wake decay parameter becomes $k_w = \zeta/\log(z_h/z_{0,lo})$, where $z_h$ is the height of the turbine, $z_{0,lo}$ is the roughness length of the ground surface and $\zeta = 0.4$ is the von Karman constant.

With this assumed wake coefficient the wake model can be shown to capture the velocity deficits in the beginning of the wind-farm quite well. However, the fully developed regime (deep array) is not described well with $k_w$, see also Ref. [1]. As will be shown later, an important ingredient of the coupled model is to adjust the wake expansion coefficient in the fully developed regime of the wind-farm based on parameters obtained from the top-down model.

The turbine velocity and power output of the turbines in the fully developed region of the wind-farm can be obtained by applying the wake expansion/superposition model to predict the stream-wise velocity field...
$u(x)$ at all points on a three-dimensional mesh. For the calculation presented here we use a resolution of $\Delta x = \Delta y = \Delta z = 6$ meters. To determine the velocity field in the fully developed regime we consider the effect of a very large number of upstream rows. Specifically, we consider 100 upstream rows with up to 4 columns of turbines on the left and the right side (including the corresponding “ghost” turbines). These parameters can be shown to lead to fully converged results for the wake expansion/superposition model. That is to say, adding more turbines upstream or to the sides did not make any difference in the results. In fact, for most cases only a fraction of the turbines used in this study are necessary to reach convergence.

Figure 2(a) gives a three-dimensional representation of the predicted mean velocity in the fully developed staggered wind-farm for a dimensionless stream-wise spacing (in units of rotor diameter) of $s_x = 7.85$ and a span-wise spacing of $s_y = 5.23$. The geometric average of the spacing is defined as $s = \sqrt{s_x s_y}$ and is $s = 6.41$ in this case. In appendix 1 we present some practically relevant simplifications that can be used for calculating the velocity at wind-turbine locations more efficiently in the aligned and staggered configurations. In figure 2(b) the results from a corresponding LES run, averaged in time, are also shown for a qualitative comparison. The parameters for both the LES and the wake expansion/superposition model used here are: $D = 100m$ and $z_h = 100m$. The surface roughness height in the LES was $z_{0.0} = 0.1m$ and $C_T = 0.75$.

Next, in order to highlight some advantages and drawbacks of the wake expansion/superposition model, it is applied (without coupling with the top-down model) to predict wind-turbine power output for various wind-farm configurations consisting of different stream-wise and span-wise turbine spacings. In figure 3 the wake expansion/superposition model results are compared with the LES results. For the LES the power ratio $P_w/P_1$ is determined by measuring $\langle u_1^3 \rangle / \langle u_1 \rangle$, where $u_1$ is the velocity averaged over a turbine disk for turbines in the end of the wind-farm and $u_1$ is the velocity averaged over the disk for turbines in the first row. We have verified from the LES that the difference in results using this model versus the $\langle u_1^3 \rangle$ implied in the wake model are negligible. The actual power will be higher using $\langle u^2 \rangle$ than using $\langle u^3 \rangle$ due to the fluctuations. However, for the power ratio $P_w/P_1$ most of these differences cancel out due to the normalization.

Figure 3 shows that the model predicts correctly that $P_w/P_1 \to 1$ as $s \to \infty$. The figure also shows that for aligned wind-farms with the same geometric mean turbine spacing $s$, the power is greater for the cases in which the stream-wise distance $s_x$ is increased while the span-wise distance $s_y$ is smaller. The LES results show that all cases tend to collapse onto a single curve, i.e. the dependence is mainly on the geometric mean spacing $s$. However, the model also predicts strong dependencies upon the turbine spacings in both stream-wise and span-wise directions even in the fully developed limit. The results shown in figure 3(b) indicate that for the staggered configuration, the wake expansion/superposition approach does not accurately represent the power output in the fully developed region of the wind-farm.

III. TOP-DOWN MODEL

The top-down wind-farm model traces its origins to Lissaman [4]. It was further developed and presented in an updated form by Frandsen [14][16]. The model is a single-column model of the atmospheric boundary layer based on momentum theory. It postulates the existence of two constant momentum flux layers, one above the turbine hub-height and one below. Each has a characteristic friction velocity and roughness length. Detailed analysis and comparisons with LES [17] showed that the assumption inherent in the Frandsen derivation, namely that two logarithmic layers would meet at hub-height needed to be corrected in order to account for the horizontally averaged effects of turbine wakes. The top-down model by Calaf et al. [17] accounts for such a layer by increasing the eddy-viscosity in this region. This augmented model was shown to predict roughness heights that agree well with results from LES. In this section we first describe this top-down model in section III A. Subsequently we discuss in III B the specific role of span-wise spacing in the top-down model, see also figure 1 and how the wake expansion/superposition model can be used to determine it.

A. Model description

The objective of the top-down model is to predict the horizontally averaged velocity profile $\langle \bar{u}(z) \rangle$ in the wind-turbine array boundary layer. The model assumes the presence of two constant stress layers, one above and one below the turbine region [14][16][19]. First, as a reference, if there is no wind-farm, then the flow can
be assumed to be undisturbed, and we have the traditional logarithmic law:

$$\langle u_0 \rangle (z) = \frac{u_*}{\kappa} \ln \left( \frac{z}{z_{0,lo}} \right) \quad \text{for} \quad z_{0,lo} \leq z \leq \delta,$$

(8)

above a surface with roughness length $z_{0,lo}$ and friction velocity $u_*$. In the cases with a wind-farm, a logarithmic region above the wind-turbine array is characterized by an upper friction velocity $u_{*hi}$ and the lower logarithmic region by a friction velocity $u_{*lo}$. Next, one considers the horizontally averaged momentum balance, in which the vertical momentum flux above each turbine in the array (see figure 6) is equal to the stress and this wake layer, and matching the velocities at $z_h$. In the fully developed region of the wind-farm the difference between these two quantities must be the thrust force at the turbine, which is modeled using the thrust coefficient $C_T$ and the horizontally averaged mean velocity at hub-height $\langle \bar{\nu} \rangle (z_h)$ according to $\frac{1}{2} C_T \langle \bar{\nu} \rangle (z_h) \frac{D^2}{2}$. As a result, we can write

$$u_{*hi}^2 = u_{*lo}^2 + \frac{1}{2} C_T [\langle \bar{\nu} \rangle (z_h)]^2,$$

(9)

where $c_h = C_T / (4 s_s s_v)$.

The modeling of the momentum flux using an appropriate eddy-viscosity allows one to write an equation for the mean velocity $(\kappa z u_{*lo}) d[\langle \bar{\nu} \rangle] / dz = u_{*lo}^2$ inside an assumed constant flux layer below the turbine area that can be integrated from the ground up and to yield:

$$\langle u \rangle (z) = \frac{u_{*lo}}{\kappa} \ln \left( \frac{z}{z_{0,lo}} \right) \quad \text{for} \quad z_{0,lo} \leq z \leq z_h - \frac{D}{2},$$

(10)

A similar integration of $(\kappa z u_{*hi}) d[\langle \bar{\nu} \rangle] / dz = u_{*hi}^2$ in the layer above the turbine area in which one assumes a roughness length $z_{0,hi}$ representing the entire wind-farm yields

$$\langle u \rangle (z) = \frac{u_{*hi}}{\kappa} \ln \left( \frac{z}{z_{0,hi}} \right) \quad \text{for} \quad z_h + \frac{D}{4} \leq z \leq \delta,$$

(11)

where $\delta$ is the upper scale which in the fully developed boundary layer case is on the order of the height of the atmospheric boundary layer (here the top-down model is only used to model the fully developed region of the wind-farm, although generalizations to the developing case are possible [18,19]). Inside the wake region $z_h - D/2 \leq z \leq z_h + D/4$ and the horizontally averaged velocity profiles can be obtained by assuming that the eddy viscosity is increased by an additional wake eddy viscosity $\nu_w$. We note that in Refs. [17,18] the wake region was assumed to reach up to $z_h + D/2$ whereas here we follow the recommendation made in Ref. [19] based on LES, which showed that in most cases the “wake layer” extends only up to about $z_h + D/4$. This gives

$$(\kappa u_* + \nu_w) \frac{d[\langle \bar{\nu} \rangle]}{dz} = u_*^2 \rightarrow (1 + \nu_w^*) \frac{d[\langle \bar{\nu} \rangle]}{d\ln(z/z_h)} = \frac{u_*}{\kappa} \quad \text{for} \quad z_h - \frac{D}{2} < z < z_h + \frac{D}{4}.$$  

(12)

where $\nu_w^* = \nu / (\kappa u_* z) \approx \sqrt{\frac{1}{2} c_h} \langle \bar{\nu}(z_h) \rangle / (\kappa u_* z_h)$. Since the value of $\nu_w^*$ depends on the roughness height and the downstream position in the wind-farm, this value should in principle be determined by iteration [19].

In the wake layer the friction velocity is assumed to be $u_{*lo}$ for $z < z_h$ and $u_{*hi}$ for $z > z_h$. Vertically integrating this wake layer, and matching the velocities at $z = z_h - D/2$ and $z = z_h + D/4$ gives

$$\langle u \rangle (z) = \frac{u_{*lo}}{\kappa} \ln \left[ \left( \frac{z}{z_h} \right)^{1 + \nu_w^*} \left( \frac{z_h}{z_{0,lo}} \right) \left( 1 - \frac{D}{2z_h} \right)^\beta \right] \quad \text{for} \quad z_h - \frac{D}{2} \leq z \leq z_h,$$

(13)

and

$$\langle u \rangle (z) = \frac{u_{*hi}}{\kappa} \ln \left[ \left( \frac{z}{z_h} \right)^{1 + \nu_w^*} \left( \frac{z_h}{z_{0,hi}} \right) \left( 1 + \frac{D}{4z_h} \right)^\beta \right] \quad \text{for} \quad z_h \leq z \leq z_h + \frac{D}{4}.$$  

(14)
In both (13) and (14) the exponent $\beta$ is defined as $\beta = \nu^{*}_{w}/(1 + \nu^{*}_{w})$. Enforcing continuity between equation (13) and (14) at $z = z_{h}$ gives

$$\frac{u^{*}_{hi}}{u^{*}_{lo}} = \left( \ln \frac{z_{h}}{z_{0,lo}} + \beta \ln \left[ 1 - \frac{D}{2z_{h}} \right] \right) / \left( \ln \frac{z_{h}}{z_{0,hi}} + \beta \ln \left[ 1 + \frac{D}{4z_{h}} \right] \right).$$

Substituting this relationship in the momentum balance (equation (9)) and replacing the mean velocity at hub-height one can obtain the roughness height $z_{0,hi}$, as provided later in the paper (equation (23)). Also, matching the velocity at $z = \delta$ between the wind-farm case and the free atmosphere situation (assuming that at this height the velocity assumes a reference value such as that of the geostrophic wind) one has

$$u^{*}_{hi} = u^{*} \frac{\ln \left( \frac{\delta}{z_{0,lo}} \right)}{\ln \left( \frac{\delta}{z_{0,hi}} \right)}.$$

Combining this with equation (14) allows us to write the velocity from the top-down model at hub-height as

$$\langle u \rangle (z_{h}) = \frac{u^{*}}{\kappa} \frac{\ln \left( \frac{\delta}{z_{0,lo}} \right)}{\ln \left( \frac{\delta}{z_{0,hi}} \right)} \ln \left( \frac{z_{h}}{z_{0,hi}} \right) \left[ 1 + \frac{D}{4z_{h}} \right]^{\beta}.$$

The ratio mean velocity to the reference case without wind-farms is then given by

$$\frac{\langle u \rangle (z_{h})}{\langle u \rangle (z_{h})} = \ln \left( \frac{\delta}{z_{0,lo}} \right) / \ln \left( \frac{\delta}{z_{0,hi}} \right) \ln \left( \frac{z_{h}}{z_{0,hi}} \right) \left[ 1 + \frac{D}{4z_{h}} \right]^{\beta} \left[ \ln \left( \frac{z_{h}}{z_{0,lo}} \right) \right]^{-1}.$$
FIG. 6: Wind-farm parameters for the top-down model for the fully developed (deep array) case and the control volumes used in the momentum analysis. (a) For small span-wise spacings the control volume used coincides with the actual spacing $s_y = s_y$. (b) For widely spaced cases, the control volume (dashed region) uses a smaller span-wise length $s_y = s_y < s_y$ which can be determined using the wake expansion/superposition model.

The corresponding power ratio is given by the ratio of cubed mean velocity at hub-height with wind-turbines compared to the reference case without wind-farms:

$$\frac{P_\infty}{P_1} = \left( \frac{\langle u \rangle (z_h)}{\langle u_0 \rangle (z_h)} \right)^3.$$

Figure 4 shows a comparison of the vertical velocity profile for the stream-wise velocity obtained from the top-down model, i.e. equations (10) - (14), with the stream-wise 'turbine' velocity measured in an infinitely long staggered wind-farm simulation with $s_x = 7.85$ and $s_y = 5.23$. The figure shows that the top-down model correctly captures the turbine velocity at hub-height, see details in appendix 2.

Figure 5 compares the top-down model predictions with results from LES. As expected, the results only depend upon the geometric mean of the turbine spacing ($s$) and no distinction can be made between the aligned or staggered cases. Remarkably, the predictions for the staggered cases appear in very good agreement with the results of Refs. [19, 20]. However, for the aligned cases significant differences can be seen, especially in those cases when the span-wise spacing is large. These large spacings lead to the power degradation being underestimated by the model. In examining the outputs from the LES, we observe that in the cases in which the span-wise spacing between turbines is large, there is little sideways interactions among the turbines even for the fully developed case. There remains significant span-wise inhomogeneity even in the fully developed case. At large span-wise spacings, the top-down model is less accurate but its predictions can be improved by including knowledge about the wake expansion, as discussed in the next subsection.

B. Effective span-wise spacing in the top-down model

The LES results indicate that, depending on the span-wise spacing, the velocity deficit due to the turbine wakes can be confined into narrow “channels”. This confinement is most likely to occur in an aligned configuration, where high velocity wind channels are formed in between the turbine rows. The top-down model, in relating the horizontally averaged velocity with the friction velocity, which depends upon the stresses that are directly affected by the wind-farm near the turbines, assumes that the momentum balance is averaged over the entire horizontal plane. However, when the span-wise spacing between turbines becomes larger than some threshold spacing which we will denote by $s_y$, then this assumed association between the mean velocity and the mean momentum fluxes is no longer valid. The limiting case of small $s_x$ and $s_y \to \infty$ in the top-down model that only depends upon $s$ is obviously unrealistic since even for a single line of turbines aligned in the wind direction significant power degradation is to be expected. Hence, we propose to apply the momentum analysis of the top-down model to a more limited area which is directly affected by the turbine wakes (this region is the shaded area in figure 6). For each wind-turbine the area has length $s_x D$ as before, but the span-wise length becomes $s_y D$ where $s_y = \min(s_y, s_y^*)$ is the “effective span-wise distance” between turbines. Then in general, we consider the vertical momentum flux above and below the turbine to be $u_{bl}^2 (s_x s_y D^2)$ and $u_{lb}^2 (s_x s_y D^2)$
In order to determine $s_y^*$, information about the strength of span-wise interactions among the turbines is required. Such information is not available within the context of the horizontally averaged top-down model but it is available from the wake expansion/superposition model and is a crucial ingredient in the coupled approach.

**IV. THE COUPLED WAKE BOUNDARY LAYER (CWBL) MODEL**

In the previous section we have seen the requirement to determine the effective span-wise spacing $s_{ye}$ needed for the top-down model. In this section we explain the two-way coupling between the top-down and the wake expansion/superposition model. We begin by discussing the fully developed regime in section IV A and extend the approach to the entrance region of the wind-farms in section IV B.

**A. The fully developed regime**

A sketch of the coupling between the wake expansion/superposition and the top-down model is given in figure 1. The procedure requires an initial guess for the wake expansion coefficient $k_w$ which is used to determine the effective span-wise spacing $s_{ye}$ according to the following procedure: $s_y^*$ is determined from the wake expansion/superposition model by finding the span-wise distance for which the wake effect are negligible, i.e. for which the velocity at wind-turbines differs only by $\frac{1}{3}$ (a fraction of 0.0033) of the velocity...
obtained for a single line of turbines (the \(1/3\) % maps into a \(1\sim 1\) % difference in predicted power). To explain the procedure, consider the predictions of the wake expansion/superposition model applied for the “infinite” (very large) wind-farm for a given \(s_y\) and \(s_x\) are shown in Figure 7. It is apparent that the turbine velocity increases with \(s_y\) as well as with \(s_x\), but the latter effect saturates after a particular value of \(s_x^\ast\). For larger span-wise spacings above this value, the turbine velocities are no longer dependent upon the span-wise spacing. The \(1/3\)% (0.991/3 \(\approx\) 0.9967) threshold is indicated by the dashed lines in each case.

Figure 8 shows how \(s_y^\ast\) depends on the stream-wise distance (\(s_x\)) and the wake decay coefficient (\(k_w\)). Figure 8(a) shows that \(s_x^\ast \approx 3.5\) for the aligned case and \(s_x^\ast \approx 7\) for the staggered case. We find that \(s_x^\ast\) depends weakly on the stream-wise distance and the wake decay coefficient. Note that especially for the staggered configuration, the values of \(s_x^\ast\) do not collapse to a single curve for large turbine spacings. The reason is that for these very large turbine spacings the wakes are very weak and \(s_y^\ast\) defined based on the threshold can vary significantly, especially when plotted as function of the logarithm of stream-wise spacing. In the limit of large \(s_x\), the predictions are almost independent of spacing, hence these features do not have noticeable impact in practice.

The effective span-wise spacing \(s_{ye} = \min (s_y, s_x^\ast)\) from the top-down model is used to predict the mean horizontal velocity at hub-height, normalized by the reference inflow velocity \(\langle u \rangle / \langle u_0 \rangle\) at \(z_k\) according to equation \(18\). Using the same initial guess for the wake expansion coefficient, the wake expansion/superposition model is used to predict the velocity ratio using equation \(4\) applied to turbines deep into a very large array. Since the assumed wake expansion coefficient \(k_w\) may not appropriately reflect the asymptotic effects of turbulence in the boundary layer, there is no guarantee that the two predictions will be the same, i.e. typically we find that \(\langle u \rangle / \langle u_0 \rangle(z_k) \neq u_T / u_0\) for \(k_w = k_w(0) = \kappa \log(z/h_1)\) and using the actual span-wise spacing \(s_x\) in the top-down model. Therefore the wake expansion \(k_w\) and the effective span-wise spacing \(s_{ye}\) are iterated until convergence is reached, see figure 1.

The details of the coupled model can be summarized as follows:

1. For the current value of \(k_{w,\infty}\) determine \(s_x^\ast\) from wake expansion/superposition model, by finding the value of \(s_x^\ast\) that solves

\[
\frac{u_T}{u_{free}}(s_y, s_x, k_{w,\infty}, \ldots) = \frac{u_T}{u_{free}}(s_y \rightarrow \infty, s_x, k_{w,\infty}, \ldots) - \varepsilon, \tag{20}
\]

where

\[
\frac{u_T}{u_{free}}(s_y, s_x, k_{w,\infty}, \ldots) = \frac{1}{N_d} \sum_{k=1}^{N_d} \left[ 1 - \left( 2a \sum_{j \in \rho_{k}} \left[ 1 + k_{w,\infty} \frac{x_{T,k} - x_j}{R} \right]^{-4} \right) \right]^{1/2} \tag{21}
\]

when applying the wake expansion/superposition model to a very large wind-farm in the deep array limit. In practice, the limit \(s_y \rightarrow \infty\) is replaced by \(s_y = 200\) and the threshold \(\varepsilon\) is chosen as \(\varepsilon = 1 - 0.991/3 \approx 0.0033\).
2. Use the result above to compute \( s_{y,e} = \min(s_y, s_x) \).

3. Calculate \( \langle u \rangle / \langle u_0 \rangle \) at \( z = z_h \) with the top-down model and find the wake expansion \( k_{w,0} \) that makes it consistent with the wake expansion/superposition model. Equating equations (18) and (6), and replacing the expression for \( z_{0,hi} \) leads to a single equation for \( k_{w,0} \):

\[
\frac{u_T}{u_{free}} (s_{y,e}, s_x, k_{w,0}, \ldots) = \ln \left( \frac{\delta}{\delta/z_{0,lo}} \right) \ln \left[ \left( \frac{z_h}{z_{0,hi}} \right) \left( 1 + \frac{D}{4z_h} \right)^{\beta} \right] \left[ \ln \left( \frac{z_h}{z_{0,lo}} \right) \right]^{-1},
\]

(22)

where

\[
z_{0,hi} = z_h \left( 1 + \frac{D}{4z_h} \right)^{\beta} \exp \left( - \left[ \frac{C_T}{8s_x s_{y,e} k^2} + \left( \ln \left[ \frac{z_h}{z_{0,lo}} \left( 1 - \frac{D}{2z_h} \right)^{\beta} \right] \right)^{-2} \right]^{1/2} \right)
\]

(23)

with \( \beta = \frac{v'_w}{(1 + v'_w)} \), and \( v'_w \approx 28 \sqrt{\frac{C_T}{8s_x s_{y,e}}} \). This estimate for \( v'_w \) was obtained by Calaf et al. [17] for \( z_h/z_{0,lo} = 100 \). As indicated before the actual value for \( v'_w \) should be obtained by iteration. However, for simplicity, we use the above approximation as we find that using \( v'_w \approx 28 \sqrt{\frac{C_T}{8s_x s_{y,e}}} \) seems to give almost the same answer for the top-down model as is obtained through the iterative procedure. Note that with this approximation the right hand side of equation (22) can be easily evaluated using \( \kappa = 0.4 \) and the appropriate \( C_T, z_h, s_x, s_{y,e}, z_{0,lo}, D, \) and \( \delta \) (the internal boundary layer height \( \delta \) is set to the measured value in the LES, i.e. 850m [19]) parameters based on the particular wind-farm. The left hand side, i.e. the wake expansion/superposition part of the model, takes the relative turbine positions into account.

We iterate steps 1 to 3 until equation (22) is satisfied to within some prescribed accuracy. For the results shown here we use a tolerance of 0.05%.

B. The entrance region of the wind-farm

The entrance region of the wind-farms can be considered by using the wake portion of the coupled model and assuming that the wake expansion coefficient at the entrance of the wind-farm is equal to the free stream value \( k_{w,0} = \kappa / \log(z_h/z_{0,lo}) \) (in our case we use \( \kappa = 0.4 \), \( z_h/z_{0,lo} = 100 \) for \( z_h = 100m \) and \( z_{0,lo} = 0.1m \), i.e. \( k_{w,0} = 0.0579 \)). This approach is chosen as the free stream wake expansion coefficient seems to describe the entrance region of the wind-farm well. We assume that the wake expansion coefficient merges continuously towards the value of \( k_{w,\infty} \) found using the analysis presented in [18] for the fully developed region of the wind-farm. The following empirical interpolation function is used to determine the expansion coefficient for the turbines in the wind-farm:

\[
k_{w,T} = k_{w,\infty} + (k_{w,0} - k_{w,\infty}) \exp(-\theta n),
\]

(24)

where \( n \) is the number of turbine wakes that overlap with the turbine of interest and \( \theta \) is an empirical parameter determining the rate at which the asymptotic behavior is reached. Based on an analysis of our results a good choice is \( \theta = 1 \). Note that this approach means that the wake expansion/superposition part of the model dominates in the entrance region of the wind-farm, while for the wake development further downstream is primarily determined by the top-down model.

V. RESULTS

In this section we compare the predictions of power degradation using the CWBL approach with LES results from Refs. [48, 50]. We first focus on the comparisons in the fully developed regime (section V A) and in section V B we perform a comparison of the model and LES at the entrance of the wind-farm. A more detailed comparison of the downstream development of the entire mean velocity field from both CWBL and LES for several cases is given in section V C.
FIG. 9: Comparison of the model (lines) and LES (symbols) \[48-50\] results for the power output ratio in the fully developed regime \((P_s/P_a)\) for the (a) aligned and (b) staggered configuration as function of the geometric mean turbine spacing \(s = \sqrt{s_x s_y}\). The line colors correspond to the span-wise spacing of the datapoints, i.e. \(s_y = 3.49\) (solid line, black), \(s_y = 5.23\) (dashed line, red), \(s_y = 7.85\) (dotted line, blue).

FIG. 10: Comparison of the CWBL model and LES \[48-50\] results for the ratio of the power output of the staggered and the aligned configuration \((P_s/P_{a,∞})\) in the fully developed regime of the wind-farm as a function of the geometric mean turbine spacing \(s = \sqrt{s_x s_y}\). The line colors correspond to the span-wise spacing of the datapoints, i.e. \(s_y = 3.49\) (solid line, black), \(s_y = 5.23\) (dashed line, red), \(s_y = 7.85\) (dotted line, blue).

A. The fully developed regime

Figures 9 and 10 and figure 10 compare the power output in the fully developed regime (deep array) of the wind-farm obtained from LES with the CWBL model results. In these and all remaining figures in this section, the model results are shown for four span-wise spacings and stream-wise spacings that vary between \(s_x = 2.5\) and \(s_x = 35\). The figures also reveal that the model accurately captures the main trends observed in the LES data. A comparison with figures 3 and 5 reveals that the CWBL model reproduces the LES data better than the individual, uncoupled models.

For the wind-farm configurations considered with span-wise spacings up to \(\sim 8D\), in both the LES and the model the power output in the fully developed regime depends mainly on the geometric mean turbine spacing when the configuration is staggered, while for the aligned case the ratio between the span-wise and stream-wise spacing is also very important. Figure 11 shows that the ratio of the power output of the staggered and the aligned configuration depends on the span-wise spacing. For small span-wise spacings the power output in the fully developed regime is nearly the same in both configurations. A significantly higher power output in the fully developed regime is obtained when the span-wise spacing in the wind-farm is larger than 4D.

Figure 11a and 11b show the wake expansion coefficient obtained after the iterations in the CWBL model for the fully developed regime of the wind-farm. The results show that the wake expansion coefficient is larger for the aligned than for the staggered configuration. This means that the wakes are recovering faster when the turbines are aligned compared to when they are staggered. This captures the faster wake recovery that has been observed for an aligned wind-farm configuration compared to a staggered one \[50\]. This faster recovery means that aligned wind-farms with short stream-wise turbine spacings perform better than one would expect.
FIG. 11: Panels (a) and (b) give the wake expansion coefficient $k_w$ computed from the CBWL model and panel (c) and (d) the corresponding effective span-wise spacing $s_{ye}$ as function of the geometric mean turbine spacing $s = \sqrt{s_x s_y}$ for the aligned and staggered cases. The line colors correspond to the span-wise spacing of the datapoints, i.e. $s_y = 3.49$ (solid line, black), $s_y = 5.23$ (dashed line, red), $s_y = 7.85$ (dotted line, blue).

Note that for large $s_x$ the $k_w,\infty$ obtained for the fully developed regime is different than the free stream value. The reason for this is that for large $s_x$ the wake recovery of the wake expansion/superposition model is matched to the recovery predicted by the top-down model. The wake recovery in the top-down model is slower than that of the wake expansion/superposition model, therefore using the free wake expansion value $k_w,0$ a lower $k_w,\infty$ value is obtained for large $s_x$. The wake expansion/superposition model is inherently less accurate in the fully developed regime when $s_x$ is large. This inaccuracy is due in part to the following factors: (1) the wake expansion may not be linear in the fully developed region, (2) adding wake interactions using equation (4) could miss some effects, (3) the wake expansion in the vertical direction assumed in this model is not limited by the maximum internal boundary layer thickness. The panels (c) and (d) of figure 11 show the effective span-wise spacing $s_{ye}$ obtained with the CWBL model. For the aligned configuration we see that $s_{ye} \approx 3.5$ for most cases. For this reason increasing the spacing beyond this value does not increase the power output in downstream turbine rows for the aligned configuration. Figure 9a shows that this is in agreement with the LES data.

B. The entrance region of the wind-farms

In this section the results of the CWBL model for the entrance region of the wind-farm are compared with LES. Figure 12 shows the downstream power development for aligned and staggered wind-farms with different combinations of the span-wise and stream-wise turbine distances. From the figure we can see that the power output as function of the stream-wise distance is captured well by the model. The differences observed for the fully developed state are in agreement with the differences seen in section 5A. Figure 13 shows the development of the wake expansion coefficient and the effective span-wise spacing of the cases shown in figure 12. The development of the wake expansion coefficient is given by equation (24). This figure shows that the
FIG. 12: Comparison of the power development as function of downstream distance obtained from LES [48–50] and the model. Panels (a) and (b) indicate the results for two different combinations of the stream-wise and span-wise spacings as indicated. The symbols and dashed lines indicate the results from LES and the model respectively. The blue/black (top pair) lines indicate the results for the staggered configuration and the red/magenta lines (bottom pair) the results for the aligned configuration.

FIG. 13: Development of the wake expansion coefficient for the cases shown in figure 12. Panels (a) and (b) indicate the results for two different combinations of span-wise and stream-wise spacing as indicated.

FIG. 14: Same data presented in figure 9a and 9b for the power ratio output ratio for the third row $P_3/P_1$ and including for the model the additional case $s_y = 15.0$ (dot-dashed green line).

Main changes in the wake expansion coefficient are in the beginning of the wind-farm. In each case the change from the free stream value to that of the fully developed regime and the evolution of this value determines the absolute changes in the wake expansion coefficient that are observed.

In figure 14 and figure 15 we show the CWBL model results compared to LES for more cases. These figures
FIG. 15: Same data presented in figure 10 with the power ratio output ratio $P_{a,3}/P_{s,3}$ for the third turbine row and including for the model the additional case $s_y = 15.0$ (dot-dashed green line).

show the ratio of the power at turbines in row 3 compared to the first one. Again we see the model predicts the trends in the observed data well for aligned and staggered configurations. Comparing the results with the result for the fully developed state reveals that the benefit of the staggered over the aligned configuration is larger at the entrance of the wind-farm than in the fully developed state of the wind-farm. This observation is consistent with expectations and the results obtained from LES. A close look at figure 14 reveals a small decrease of the power output with increasing stream-wise distance when $s_y = 3.49$ and the stream-wise distance $s_x \gtrsim 20$. This is a feature of the regular wake expansion/superposition model and is a results of span-wise wake expansion that effect the turbine of interest.

C. Comparisons of entire hub-height velocity field

Both the CWBL model and LES allow one to study the downstream development of velocities in the entire wind-farm. Figures 16 to figure 19 compare the velocity at hub-height obtained from the model with the LES for different cases. In agreement with what we have seen before we see that the CWBL model captures the main features of the LES. However, there are certain differences such as the exact wake recovery rate as function of the downstream distance and the precise way the velocity deficits progress further inside the farm. This can be made more quantitative by extracting the mean velocity at hub-height and the velocity in one of the
FIG. 17: Comparison between the (a) LES and (b) model averaged hub-height velocity in an staggered wind-farm with a stream-wise spacing of $s_x = 7.85$ and a span-wise spacing $s_y = 5.23$.

FIG. 18: Comparison between the (a) LES and (b) model averaged hub-height velocity in an aligned wind-farm with a stream-wise spacing of $s_x = 3.49$ and a span-wise spacing $s_y = 7.85$.

turbine rows as function of the downstream position. Figure 20 shows that the recovery of the wind velocity in the turbine rows is somewhat different in LES than in the model. We believe this is an effect of the wake-wake interactions that are not fully captured in the CWBL framework. As a consequence the horizontally averaged mean velocity at hub-height predicted by the model is not always accurate.

D. Comparison with Horns Rev and Nysted data

In this section we briefly illustrate how the model can be applied to an operational wind-farm using two well-known test cases, i.e. the aligned configuration for the Horns Rev and Nysted wind-farms. We apply the CWBL model to these wind-farms and compare the power degradation data for aligned flow from Ref.
Fig. 19: Comparison between (a) LES and (b) model averaged hub-height velocity in an staggered wind-farm with a stream-wise spacing of \( s_x = 3.49 \) and a span-wise spacing \( s_y = 7.85 \).

Fig. 20: A comparison of the horizontally averaged mean and velocity in the turbine row as function of the downstream position for four different cases as indicated obtained from the LES and the model.

Specifically, for Horns Rev we use \( s_x = 7.00, s_y = 6.95 \) as the layout parameters for the aligned flow configuration (270°) and \( s_x = 10.4 \) and \( s_y = 5.8 \) for the aligned configuration of Nysted at 278°. Horns Rev
FIG. 21: Power degradation in the Horns Rev and Nysted wind-farms. The field data (from Ref. [1], their figure 2) are shown as circles while the prediction from the CWBL model is shown by the squares. Panels (a) and (b) indicate the results for Horns Rev and (c) and (d) for Nysted. Panels (b) and (d) give the velocity at hub-height predicted by the CWBL model for Horns Rev and Nysted, for wind directions of 270° and 278° respectively.

consists of Vestas V-80 2 MW turbines each with a hub-height of $z_h = 70$ m and a rotor diameter $D = 80$ m. The turbines at Nysted have the parameters $z_h = 69$ m and $D = 82.4$ m. We use a standard value of $C_T = 0.70$, as this value is usually assumed to be representative of good operating conditions, set $z_{0,lo} = 0.01$ m and assume an internal boundary layer height of $\delta = 5.5D$.

The predicted power degradation with stream-wise distance is shown in figure 21. As there is an uncertainty in the field measurement data of $\pm 2^\circ$ we average results from the CWBL model over the same range of angles. Figure 21 shows reasonably good agreement between the CWBL model and the field data. These results are promising but further work needs to be done such as contrasting these predictions with those obtained using other models as summarized in Ref. [2, 11, 46, 48, 51]. More cases and further tests, including a comparison with the LES study of Horns Rev provided by Porté-Agel, Wu and Chen [52], will be considered elsewhere and are not included here for sake of brevity.

VI. CONCLUSIONS

In this paper we have introduced a modeling framework for predicting the power output in both the entrance and fully developed regions of wind-farms. The method combines two well-known approaches, the wake expansion/superposition model and the top-down boundary layer model thus resulting in the proposed coupled model. Both of the constitutive approaches have one parameter that needs to be determined. For the wake model this is the wake expansion coefficient $k_w$ and for the top-down model this is the effective span-wise spacing $s_{ye}$. In the CWBL model, the effective span-wise spacing is obtained from the wake model and is then used in the top-down model. These results are then coupled through an iterative procedure to obtain the wake expansion coefficient $k_w$ that ensures that the turbine velocity is matched in both models. A detailed comparison with LES results for a variety of cases reveals that the model represents the LES data quite well.
for both the fully developed region and the entrance region of the wind-farm. The final part of the work illustrates application of the CWBL model to field-scale wind-farm data by comparing the power degradation measurements for the Horns Rev and Nysted wind-farms to those estimated using the CWBL model. Good agreement has been obtained. By combining relevant wake growth and boundary layer physics, the coupled model is promising and can be explored in further tests and applications.

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Appendix 1: Computationally efficient methods for wake model

For computational reasons it can be convenient to have an approximation of the results obtained by the wake model in the fully developed regime. It has been shown by Peña and Rathmann \[8\] that such an approximation can be obtained for an aligned wind-farm configuration while assuming that a turbine experiences the full wake effects when the wake has reached the turbine center. This appendix present a generalization of this approach.

Following Peña and Rathmann \[8\] we define that the total wake deficit \( \delta_T \) is given by

\[
\delta_T^2 = \delta_I^2 + \delta_{II}^2 \tag{25}
\]

where \( \delta_I \) indicate the contributions from the turbines directly upstream (the turbines above ground as well as the ‘ghost’ turbines, i.e. turbines \( J_U \) and \( J_G \)) and \( \delta_{II} \) the contributions from adjacent turbines (again for the turbines above and below the ground, i.e. turbines \( J_S \) and \( J_{GS} \)). The initial wake deficit \( \delta_0 \) is given by

\[
\delta_0 = 2a = 1 - \sqrt{1-C_T}. \tag{26}
\]

The total wake contributions for the aligned and staggered case can then be approximated by determining \( \delta_I^2 \) and \( \delta_{II}^2 \) as indicated below.

Parenthetically, we note that it is assumed implicitly that the initial area of the wake corresponds to the turbine disk area rather than the slightly enlarged area appropriate for the velocity reduction \( 2a \). If the wake is assumed to begin at the enlarged stream tube area behind the turbine, the denominator \( (1 + k_w x / R)^2 \) should be replaced by \( (\gamma + k_w x / R)^2 \) where \( \gamma = [(1-a)/(1-2a)]^{1/2} \). For typical values of \( a \), \( \gamma \) can in fact be quite a bit larger than 1. The usual wake expansion/superposition model \[5\] assumes instead that the wake with a deficit \( 2a \ u_{free} \) begins at the smaller turbine area \( \pi R^2 \). Here we have followed the same standard approach but keep in mind that future improvements may be required to make the entire approach more internally self-consistent.

**Aligned configuration**

For an aligned wind-farm \( \delta_I^2 \) can be approximated as

\[
\delta_I^2 = \delta_0^2 \sum_{j=1}^{\infty} (1 + c) (1 + 2k_w s_j)^{-4} \quad \text{with}
\]

\[
a = (k_w s_j) - (2z_h/D - 1), \quad b = \min(a, 1), \quad c = \max(b, 0). \tag{27}
\]

Here the term \( (1 + 2k_w s_j)^{-4} \) indicates the squared velocity deficit resulting from an upstream turbine. The \( (1 + c) \) term indicates whether the wakes have reached the turbine of interest. The turbine of interest will always be completely in the wake of directly upstream turbines (which is represented by the 1). The \( c \) estimates the fraction of the turbine of interest that is covered by wakes originating from the directly upstream ‘ghost’ turbines and is defined such that it is between 0 and 1.
The wake contributions for the adjacent and adjacent 'ghost' turbines, i.e. $\delta^2_{\text{II}}$, are approximated as

$$\delta^2_{\text{II}} = \delta^2_0 \sum_{\text{rows}} \sum_{j=1}^{\infty} 2(f + g) (1 + 2k ws_j)^{-4} \quad \text{with}$$

$$a = (k ws_j) - (2z_h/D - 1), \quad b = \min(a, 1), \quad c = \max(b, 0)$$

$$d = (k ws_j) - (s_y - 1 + (\text{rows} - 1)s_y), \quad e = \min(d, 1), \quad f = \max(e, 0)$$

$$g = cf.$$  \hspace{1cm} (28)

Just as above $c$ gives the fraction of the turbine that is covered by wakes created from upstream 'ghost' turbines, while $f$ determines the fraction of the turbine that is covered by wakes created from adjacent turbines. The factor $g$ determines the fraction of the turbine that is covered by wakes created from the adjacent 'ghost' turbines. The factor $2(f + g)$ adds the effects of the adjacent and adjacent 'ghost' turbine rows on the left and right side of the turbine of interest. We note that figure 22 shows this is a very good approximation for the aligned configuration.

**Staggered configuration**

For a staggered wind-farm $\delta^2_{\text{I}}$ and $\delta^2_{\text{II}}$ can be approximated in a similar way as for the aligned case. For $\delta^2_{\text{I}}$ it becomes

$$\delta^2_{\text{I}} = \delta^2_0 \sum_{j=1}^{\infty} (1 + c) (1 + 2k ws_j)^{-4} \quad \text{with}$$

$$a = (k ws_j) - (2z_h/D - 1), \quad b = \min(a, 1), \quad c = \max(b, 0).$$

Here the term $2j$ makes sure that we have a staggered configuration (direct upstream turbines every other row). Similarly, $\delta^2_{\text{II}}$ is approximated as

$$\delta^2_{\text{II}} = \delta^2_0 \sum_{\text{rows} = 1}^{\infty} \sum_{j=1}^{\infty} 2(f + g) (1 + 2k ws_j(2j - 1))^{-4} \quad \text{with}$$

$$a = (k ws_j(2j - 1)) - (2z_h/D - 1), \quad b = \min(a, 1), \quad c = \max(b, 0)$$

$$d = (k ws_j(2j - 1)) - (s_y - 1 + (\text{rows} - 1)s_y), \quad e = \min(d, 1), \quad f = \max(e, 0)$$

$$g = cf.$$  \hspace{1cm} (30)

where the term $2j - 1$ selects that we only have adjacent and adjacent 'ghost' turbines every other row.

**Results**

In figure 22 the results of the 3D wake expansion/superposition solver are compared with the approximation given in this appendix and the results from Peña and Rathmann [8]. The figure reveals that our approximation reproduces the results from the wake model very well in the fully developed regime. A comparison with the sum approximation by Peña and Rathmann [8] reveals good agreement between the two methods although our approximation is smoother when partial wake overlaps are important, because our method makes less assumptions. We note that in the above approximations it is assumed that the turbines and wakes are square, just as Peña and Rathmann [8]. Figure 22 shows this is a reasonable assumption as a comparison of both cases with our 3D solver only shows small differences due to this approximation. The approximations given in this appendix can be useful for an efficient implementation of the model coupled with the top-down model, whereas the use of the 3D solver offers more flexibility.

**Appendix 2: Additional details on top-down model**

As the top-down model uses horizontal averaging it only knows one velocity scale. This implies that the model assumes that the velocity in front of the turbines $u_{\text{turb}}$ should be equal to the horizontally averaged mean
FIG. 23: Comparison of the vertical velocity profile obtained using the top-down model with velocities obtained from an infinitely large staggered wind-farm with $s_x = 7.85$ and $s_y = 5.23$. Panels (a) and (b) show the vertical profiles of the stream-wise velocity averaged over the complete horizontal ($s_y = 5.23$, squares), averaged over a $s_y = 3.5$ region around the turbines (diamonds), and turbine velocity (circles). Panels (c) and (d) compare the turbine velocity obtained from the LES data with the top-down model predictions with the appropriate $s_y$.

Velocity $u_{\text{mean}}$ when averaging over the appropriate span-wise $s_{xy}$ region. It is not obvious that this condition is always met. From figure 23 we see that the top-down model predicts the power output of the staggered case very well, i.e. cases in which $s_y^*$ is larger than the actual $s_y$ such that it does not influence the top-down model calculations. This observation indicates that the top-down model predicts the velocity in front of the turbines very well. Below we show with results from LES that this observation is consistent with the measured mean velocity profiles from LES. We think the agreement stems from the use of the velocity scale in equation (9) to calculate the momentum loss leading to predictions of the mean velocity profile closer to the turbine velocity than to the mean velocity.

The results indicated below are from simulations of infinitely large wind-farms, as the available symmetries there allow for more averaging and therefore better comparisons then the developing cases. For the cases here the stream-wise spacing $s_x = 7.85$ and the span-wise spacing is $s_y = 5.23$. The results in figure 23 show $u_{\text{turb}}$, $u_{\text{mean}}$, and $u_{\text{mean}}$ averaged using a smaller span-wise span-wise distance of 3.49$D$. For the aligned case the smaller span-wise area is roughly equal to $s_{xy}$ and over this region $u_{\text{turb}}$ and the local mean are almost the same. As a result the predicted velocity by the top-down model agrees at hub-height with both velocities. For the staggered case the situation is more complicated. Here $s_y^*$ is larger than the actual span-wise spacing, so the relevant averaging interval should be the whole horizontal area. However, the figure shows that using this interval $u_{\text{turb}}$ and $u_{\text{mean}}$ are not the same. A comparison with the predicted top-down velocities shows its prediction is much closer to $u_{\text{turb}}$ than to $u_{\text{mean}}$ as argued above.