MIRROR OF VOLUME FUNCTIONALS ON MANIFOLDS WITH SPECIAL HOLONOMY

KOTARO KAWAI AND HIKARU YAMAMOTO

Abstract. We can define the “volume” $V$ for Hermitian connections on a Hermitian complex line bundle over a Riemannian manifold $X$, which can be considered to be the “mirror” of the standard volume for submanifolds. This is called the Dirac-Born-Infeld (DBI) action in physics.

In this paper, (1) we introduce the negative gradient flow of $V$, which we call the line bundle mean curvature flow. Then, we show the short-time existence and uniqueness of this flow. When $X$ is Kähler, we relate the negative gradient of $V$ to the angle function and deduce the mean curvature for Hermitian metrics on a holomorphic line bundle defined by Jacob and Yau.

(2) We relate the functional $V$ to a deformed Hermitian Yang–Mills (dHYM) connection, a deformed Donaldson–Thomas connection for a $G_2$-manifold (a $G_2$-dDT connection), a deformed Donaldson–Thomas connection for a Spin(7)-manifold (a Spin(7)-dDT connection), which are considered to be the “mirror” of special Lagrangian, (co)associative and Cayley submanifolds, respectively. When $X$ is a compact Spin(7)-manifold, we prove the “mirror” of the Cayley equality, which implies the following. (a) Any Spin(7)-dDT connection is a global minimizer of $V$ and its value is topological. (b) Any Spin(7)-dDT connection is flat on a flat line bundle. (c) If $X$ is a product of $S^1$ and a compact $G_2$-manifold $Y$, any Spin(7)-dDT connection on the pullback of the Hermitian complex line bundle over $Y$ is the pullback of a $G_2$-dDT connection modulo closed 1-forms.

We also prove analogous statements for $G_2$-manifolds and Kähler manifolds of dimension 3 or 4.

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Let $X$ be a Calabi–Yau, $G_2$- or Spin(7)-manifold. If $X$ is the total space of a torus fiber bundle, we can send geometric objects on $X$ to those on the “mirror” manifold via the real Fourier–Mukai transform. For example, special Lagrangian, (co)associative and Cayley submanifolds (more generally, cycles) are sent to deformed Hermitian Yang–Mills (dHYM) connections, deformed Donaldson–Thomas connections for a $G_2$-manifold ($G_2$-dDT connections), deformed Donaldson–Thomas connections for a Spin(7)-manifold (Spin(7)-dDT connections), respectively. These are defined without a torus fiber bundle structure and they are Hermitian connections on a Hermitian line bundle $L$ over $X$ defined by fully nonlinear PDEs. As the names indicate, dHYM and $G_2$, Spin(7)-dDT connections can also be considered as analogues of Hermitian Yang–Mills (HYM) connections and Donaldson–Thomas connections ($G_2$, Spin(7)-instantons), respectively.

\[
\begin{align*}
\{ \text{calibrated submanifolds} \} & \xrightarrow{\text{mirror}} \{ \text{dHYM, } G_2, \text{Spin(7)-dDT connections} \} & \xrightarrow{\text{analogue}} \{ \text{HYM, } G_2, \text{Spin(7)-instantons} \}
\end{align*}
\]
Readers may wonder why the discussion is limited to line bundles. The reason for the dHYM case is explained in [4, Section 8.1]. For holomorphic bundles with non-abelian gauge group, the analogue of the dHYM equation is not known. Though there is a natural guess for the higher rank dHYM equation, the resulting equations are fully nonlinear systems, and the analytic difficulties in addressing them are formidable. We focus on line bundles in the dDT case for the same reason.

In this paper, we study the “volume” for Hermitian connections, which can be considered to be a “mirror” of the standard volume for submanifolds via the real Fourier–Mukai transform. This is called the Dirac-Born-Infeld (DBI) action in physics. Then, we show that dHYM, $G_2$, Spin(7)-dDT connections indeed have similar properties to calibrated submanifolds and HYM, $G_2$, Spin(7)-instantons in addition to the similarities of moduli spaces given in [11, 13]. More explicitly, we mainly study the following two topics.

1.1. **The “mirror” of the mean curvature flow.** Let $(X, g)$ be a compact oriented $n$-dimensional Riemannian manifold and $L \to X$ be a smooth complex line bundle with a Hermitian metric $h$. Let $\mathcal{A}_0$ be the space of Hermitian connections on $(L, h)$. We regard the curvature 2-form $F_\nabla$ of $\nabla \in \mathcal{A}_0$ as a $\sqrt{-\text{I}}\mathbb{R}$-valued closed 2-form on $X$. Define the volume functional $V : \mathcal{A}_0 \to \mathbb{R}$ by

$$V(\nabla) = \int_X \sqrt{\det(\text{id}_{TX} + (-\sqrt{-\text{I}}F_\nabla)^\sharp)} \, \text{vol}_g,$$

where $(-\sqrt{-\text{I}}F_\nabla)^\sharp \in \Gamma(X, \text{End} \, TX)$ is defined by $u \mapsto (-\sqrt{-\text{I}}i(u)F_\nabla)^\sharp$ and $\text{vol}_g$ is the volume form defined by the Riemannian metric $g$. Note that $\det(\text{id}_{TX} + (-\sqrt{-\text{I}}F_\nabla)^\sharp) \geq 1$ since $(-\sqrt{-\text{I}}F_\nabla)^\sharp$ is skew-symmetric. See Lemma A.2 for details.

As the negative gradient of $V$, we define the mean curvature $H(\nabla) \in \Omega^1$ for $\nabla \in \mathcal{A}_0$, which is described explicitly in (3.3). Then, following [8], we say that a smooth family $\{\nabla_t\}_{t \in [0, T]} \subset \mathcal{A}_0$, where $T \in (0, \infty]$, satisfies the line bundle mean curvature flow if

$$\frac{\partial}{\partial t} \left( \frac{\nabla_t}{\sqrt{-\text{I}}} \right) = H(\nabla_t).$$

We first show the short-time existence and uniqueness of (1.1).

**Theorem 1.1 (Theorem 3.7).** (1) For any $\nabla_0 \in \mathcal{A}_0$, there exist $\varepsilon > 0$ and a smooth family of Hermitian connections $\{\nabla_t\}_{t \in [0, \varepsilon]}$ satisfying (1.1) and $\nabla_0|_{t=0} = \nabla_0$.

(2) Suppose that $\{\nabla^1_t\}_{t \in [0, \varepsilon]}$ and $\{\nabla^2_t\}_{t \in [0, \varepsilon]}$ satisfy (1.1) for $\varepsilon > 0$. If $\nabla^1_t|_{t=0} = \nabla^2_t|_{t=0}$, then $\nabla^2_t - \nabla^1_t$ is a (pure imaginary-valued) time-dependent exact 1-form for any $t \in [0, \varepsilon]$.

Since $L$ is a line bundle, the curvature is invariant under the addition of closed 1-forms, that is, $F_{\nabla + a} = F_{\nabla}$ for any $\nabla \in \mathcal{A}_0$ and $a \in Z^1$, where $Z^1$ is the space of closed 1-forms. This implies that $V$ is invariant under the addition of closed
1-forms, and hence, $H$ degenerates in the direction of $\sqrt{-1}Z^1$. In particular, (1.1) is not strongly parabolic.

However, giving a “nice” expression of $H(\nabla)$ in Proposition 3.2, we show that the principal symbol of $H$ degenerates only in this direction, just as in the case of the mean curvature for submanifolds. This motivates us to use DeTurck’s trick to prove the short-time existence and uniqueness of (1.1). By the similarity between mean curvature flows and line bundle mean curvature flows, we can introduce a modified parabolic flow to obtain Theorem 1.1.

When $X$ is a Kähler manifold, we consider the space $A$ of $\nabla \in A_0$ with $F^{0,2}_\nabla = 0$, where $F^{0,2}_\nabla$ is the $(0,2)$-part of $F_\nabla$. The space $A$ can be considered as the set of holomorphic bundle structures in $L$ by [14, Section 7.1]. For $\nabla \in A$, we relate $H(\nabla)$ to the exterior derivative of the angle function $\theta_\nabla$ defined by (6.5) as in the case of Lagrangian submanifolds.

**Theorem 1.2** (Theorem 6.10). Suppose that $X$ is compact and connected Kähler manifold. For any $\nabla \in A$, we have

$$H(\nabla) = -(\det G_\nabla)^{1/4}(G_\nabla^{-1} Jd\theta_\nabla).$$

Here, $G_\nabla = \text{id}_{TX} - (\sqrt{-1} F_\nabla)^{\sharp} \circ (\sqrt{-1} F_\nabla)^{\sharp} \in \Gamma(X, \text{End } TX)$, where $(\sqrt{-1} F_\nabla)^{\sharp} \in \Gamma(X, \text{End } TX)$ is defined by $u \mapsto (\sqrt{-1} i(u) F_\nabla)^{\sharp}$, $J$ is the complex structure on $X$ and we set $Jd\theta_\nabla = d\theta_\nabla(J(\cdot))$.

In particular, for $\nabla \in A$, $H(\nabla) = 0$ if and only if $\nabla$ is a $dHYM$ connection with phase $e^{\sqrt{-1} \theta}$ for some $\theta \in \mathbb{R}$.

Jacob and Yau [8] considered a similar volume functional in [8, Definition 3.1] for Hermitian metrics on a holomorphic line bundle and defined the mean curvature and the line bundle mean curvature flow. In Remark 6.12, we explain the relation between the volume functional $V$ in this paper and that in [8] and show that we can recover the first variation formula in [8, Proposition 3.4] from Theorem 1.2.

Due to the introduction of the flow (1.1) for Hermitian connections, many issues arise simultaneously. For instance, are all $\nabla_t$ integrable if the initial connection $\nabla_t|_{t=0}$ is integrable (i.e. $\nabla_t|_{t=0} \in A$) along the line bundle mean curvature flow? It seems to be true comparing the Lagrangian mean curvature flow with a slogan that “integrable” corresponds to “Lagrangian” in mirror symmetry. Actually, in the case of Lagrangian mean curvature flows, the Lagrangian condition is preserved along the mean curvature flow in a Kähler-Einstein manifold (see [22]). Following this slogan, the authors expect that studies of Lagrangian mean curvature flows can be imported into the study of our line bundle mean curvature flow (1.1). Moreover, the authors presume that the Lagrangian deformation corresponds to the deformation in $A$, and the Hamiltonian deformation corresponds to the deformation in the space of the Chern connections of Hermitian metrics of a line bundle with a fixed holomorphic structure. From this perspective, the authors also expect that studies
of the Hamiltonian stability of Lagrangian submanifolds can be also imported into
the study of Hermitian connections.

1.2. The “mirror” of special Lagrangian, associator, Cayley equalities. Special Lagrangian, associator, Cayley equalities are fundamental identities in calibrated geometry introduced in [6]. Using these, we can show that the real part of a holomorphic volume, a $G_2$-structure, a Spin(7)-structure are calibrations and characterize special Lagrangian, associative, Cayley submanifolds, respectively, by the vanishing of certain tensors. We first show the “mirrors” of these equalities, which are predicted by the real Fourier–Mukai transform in [12, Lemmas 4.3 and 5.5]. The following is considered to be the “mirror” of the Cayley equality.

**Theorem 1.3** (Theorem 4.2). Let $X^8$ be an 8-manifold with a Spin(7)-structure $\Phi$ and $L \to X$ be a smooth complex line bundle with a Hermitian metric $h$. Let $\mathcal{A}_0$ be the space of Hermitian connections of $(L, h)$. We regard the curvature 2-form $F_\nabla$ of $\nabla$ as a $\sqrt{-1}$-$\mathbb{R}$-valued closed 2-form on $X$. Denote by $\pi^{k\ell}_k : \Omega^k \to \Omega^{k\ell}_\ell$ the projection onto the Spin(7)-irreducible component of rank $\ell$. Then, for any $\nabla \in \mathcal{A}_0$, we have

$$\left(1 + \frac{1}{2} \langle F_{\nabla}^2, \Phi \rangle + \frac{\ast F_{\nabla}^4}{24}\right)^2 + 4 \left|\pi^{2}_{7} \left( F_{\nabla} + \frac{1}{6} \ast F_{\nabla}^3 \right) \right|^2 + 2 \left|\pi^{4}_{7} (F_{\nabla}^2) \right|^2 = \det(\text{id}_{TX} + (-\sqrt{-1} F_{\nabla})^\sharp),$$

where $(-\sqrt{-1} F_{\nabla})^\sharp \in \Gamma(X, \text{End} TX)$ is defined by $u \mapsto (-\sqrt{-1} i (u) F_{\nabla})^\sharp$. In particular,

$$\left|1 + \frac{1}{2} \langle F_{\nabla}^2, \Phi \rangle + \frac{\ast F_{\nabla}^4}{24}\right| \leq \sqrt{\det(\text{id}_{TX} + (-\sqrt{-1} F_{\nabla})^\sharp)}$$

for any $\nabla \in \mathcal{A}_0$ and the equality holds if and only if $\nabla$ is a Spin(7)-dDT connection.

Note that the Spin(7)-structure $\Phi$ does not have to be torsion-free and $X^8$ is not necessarily compact. We need some tricky and complicated computations for the proof and we would not have found this without the prediction by the real Fourier–Mukai transform.

Theorem 1.3 is the core of results in Section 1.2. From Theorem 1.3, we obtain the “mirror” of the associator equality in the $G_2$-case (Theorem 5.1) and that of the special Lagrangian equality in the Kähler case of dimension 3 or 4 (Theorems 6.1 and 6.2). Moreover, there are many applications of these equalities. The first application is the following Theorem 1.4 together with the corresponding statements for the $G_2$-case (Theorem 5.3, Corollaries 5.4 and 5.6) and the Kähler case of dimension 3 or 4 (Theorem 6.4, Corollaries 6.5 and 6.6). By integrating the “mirror” of the Cayley equality, we can relate the volume functional $V$ to a Spin(7)-dDT connection and we see that the volume of a Spin(7)-dDT connection is topological. This further implies some interesting results.

**Theorem 1.4** (Theorem 4.4, Corollaries 4.5 and 4.6). In addition to the assumptions of Theorem 1.3, suppose that $X^8$ is compact and connected.
(1) For any $\nabla \in A_0$, we have
\[
\left| \int_X \left(1 + \frac{1}{2}\langle F_2^\nabla, \Phi \rangle + \frac{\ast F_4^\nabla}{24} \right) \text{vol}_g \right| \leq V(\nabla)
\]
and the equality holds if and only if $\nabla$ is a Spin(7)-dDT connection.

(2) If the Spin(7)-structure $\Phi$ is torsion-free, the left hand side of (1.2) is given by
\[
\left| \text{Vol}(X) + \left(-2\pi^2 c_1(L)^2 \cup [\Phi] + \frac{2}{3}\pi^4 c_1(L)^4 \right) \cdot [X] \right|,
\]
where $c_1(L)$ is the first Chern class of $L$. In particular, for any Spin(7)-dDT connection $\nabla$, $V(\nabla)$ is topological and any Spin(7)-dDT connection is a global minimizer of $V$.

(3) Suppose that the Spin(7)-structure $\Phi$ is torsion-free and $L$ is a flat line bundle. Then, any Spin(7)-dDT connection is a flat connection. In particular, the moduli space of Spin(7)-dDT connections is $H^1(X, \mathbb{R})/2\pi H^1(X, \mathbb{Z})$.

It is known that there are similar results to (1) and (2) for calibrated submanifolds and HYM, $G_2$, Spin(7)-instantons. That is, each calibrated submanifold is homologically volume minimizing and the volume is topological by [6]. Similarly, any HYM, $G_2$, Spin(7)-instanton is a global minimizer of Yang–Mills functional and the value is topological by [23]. These make us confirm more the similarities explained at the beginning of the introduction.

Theorem 1.4 (3) states that the moduli space of Spin(7)-dDT connections on a flat line bundle is completely determined. Note that there are analogous results to (3) for HYM, $G_2$, Spin(7)-instantons on line bundles. (For example, see [20, Section 7] for the case of Spin(7)-instantons. We can also prove in the other cases similarly.) However, these results depend on the fact that the conditions of HYM, $G_2$, Spin(7)-instantons on line bundles are linear. Since the defining equation of the Spin(7)-dDT is nonlinear, (3) is a much more nontrivial result (though results look similar).

Next, we give another application of the “mirror” of Cayley and associator equalities. That is, we study the relation between Spin(7)-dDT connections and $G_2$-dDT connections. Let $(Y^7, \varphi)$ be a $G_2$-manifold and $L \to Y$ be a smooth complex line bundle with a Hermitian metric $h$. Then, $X^8 = S^1 \times Y^7$ admits a canonical torsion-free Spin(7)-structure. Denote by $\pi_Y : X^8 = S^1 \times Y^7 \to Y^7$ the projection.

In [12, Lemma 7.1], we prove that a Hermitian connection $\nabla$ of $(L, h)$ is a $G_2$-dDT connection if and only if the pullback $\pi_Y^\ast \nabla$ is a Spin(7)-dDT connection of $\pi_Y^\ast L$. However, there may be a Spin(7)-dDT connection of $\pi_Y^\ast L$ which is not of the form $\pi_Y^\ast \nabla$. In fact, as an application of the “mirror” of Cayley and associator equalities, we can show that any Spin(7)-dDT connection of $\pi_Y^\ast L$ is essentially the pullback of a $G_2$-dDT connection if $Y^7$ is compact and connected.

**Theorem 1.5** (Theorem 7.1). Suppose that $Y^7$ is a compact and connected $G_2$-manifold.
(1) The pullback $\pi^* Y \rightarrow X^8$ admits a $\text{Spin}(7)$-dDT connection if and only if $L \rightarrow Y^7$ admits a $G_2$-dDT connection.

(2) For any $\text{Spin}(7)$-dDT connection $\tilde{\nabla}$ of $\pi^* Y L$, there exist a $G_2$-dDT connection $\nabla$ of $L$ and a closed 1-form $\xi \in \sqrt{-1}\Omega^1(X^8)$ such that $\tilde{\nabla} = \pi^* Y \nabla + \xi$.

(3) Denote by $\mathcal{M}_{\text{Spin}(7)}$ the moduli space of $\text{Spin}(7)$-dDT connections of $\pi^* Y L$ and denote by $\mathcal{M}_{G_2}$ the moduli space of $G_2$-dDT connections of $L$ as defined by (4.3) and (5.3). Then, $\mathcal{M}_{\text{Spin}(7)}$ is homeomorphic to $S^1 \times \mathcal{M}_{G_2}$.

Results similar to Theorem 1.5 hold between $G_2$-dDT and dHYM connections (Theorem 7.3) and between $\text{Spin}(7)$-dDT and dHYM connections (Theorem 7.5).

In (2), we show that any $\text{Spin}(7)$-dDT connection of $\pi^* Y L$ is essentially the pullback of a $G_2$-dDT connection. We can prove (3) from (2) and it implies that the topology of $\mathcal{M}_{\text{Spin}(7)}$ is determined by that of $\mathcal{M}_{G_2}$. Theorem 1.5 (1) might have interesting implications. In the case between $G_2$-dDT and dHYM connections (Theorem 7.3), we show that the existence of a $G_2$-dDT connection is equivalent to that of a dHYM connection. It is conjectured in [3, Conjecture 1.5] that the existence of a dHYM connection is equivalent to a certain stability condition and the conjecture is partially proved in [2]. This implies that the existence of a $G_2$-dDT connection of $\pi^* Y L \rightarrow X^7$ would be equivalent to this stability condition. More generally, the existence of a $G_2$- or $\text{Spin}(7)$-dDT connection of a general line bundle over a general $G_2$- or $\text{Spin}(7)$-manifold might be related to a certain stability condition.

There are similar results for calibrated submanifolds and HYM, $G_2$, Spin(7)-instantons. That is, the moduli space of irreducible Spin(7)-instantons of $\pi^* Y L$ is homeomorphic to the product of $S^1$ and the moduli space of irreducible $G_2$-instantons of $L$ by [24, Theorem 1.17]. By [21, Proposition 5.20], we see that the moduli space of all local Cayley deformations of $S^1 \times A^3$, where $A^3$ is a given compact associative submanifold in $Y^7$, is identified with the moduli space of all local associative deformations of $A^3$. These also make us confirm more the similarities explained at the beginning of the introduction.

One of the problems of the geometry of $G_2$, Spin(7)-dDT connections is that few explicit examples are known. (Recently, Lotay and Oliveira [16] constructed nontrivial examples of $G_2$-dDT connections on the trivial complex line bundle over a manifold with a coclosed $G_2$-structure.) Theorems 1.4 (3), 1.5 and 7.5 impose several restrictions to construct nontrivial examples on a compact and connected Spin(7)-manifold.

**Organization of this paper.** This paper is organized as follows. Section 2 gives basic identities in $G_2$- and Spin(7)-, and 3,4-dimensional Calabi–Yau geometry that are used in this paper. Section 3 is devoted to the study of the “mirror” of the mean curvature flow. We prove Theorem 1.1 (Theorem 3.7) here. In Section 4, reviewing the definition of Spin(7)-dDT connections we state Theorem 1.3 (Theorem 4.2) with some remarks and prove Theorem 1.4 (Theorem 4.4). We state and prove the corresponding statement on $G_2$-manifolds (Theorems 5.1, 5.3, Corollaries 5.4 and 5.6) in Section 5 and that on 3, 4-dimensional Kähler manifolds (Theorems 6.1,
6.2, 6.4, Corollaries 6.5 and 6.6) in Section 6. In Section 6, we also prove related results that hold for any dimension including Theorem 1.2 (Theorem 6.10) under an additional assumption. In Section 7, we study the relation between Spin(7)-dDT and $G_2$-dDT connections and prove Theorem 1.5 (Theorem 7.1) here. We also prove the corresponding statement between $G_2$-dDT and dHYM connections (Theorem 7.3) and that between Spin(7)-dDT and dHYM connections (Theorem 7.5). In Appendix A, we prove Theorems 4.2, 5.1, 6.1 and 6.2. Appendix B is the list of notation in this paper.

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2. Basics on $G_2$-, Spin(7)-, SU(3)- and SU(4)-geometry

In this section, we collect some basic definitions and equations on $G_2$-, Spin(7)-, SU(3)- and SU(4)-geometry which we need in the calculations in this paper.

2.1. The Hodge star operator. Let $V$ be an $n$-dimensional oriented real vector space with an inner product $g$. Denote by $\langle \cdot, \cdot \rangle$ the induced inner product on $\Lambda^k V^*$ from $g$. Let $\ast$ be the Hodge star operator. The following identities are frequently used throughout this paper.

For $\alpha, \beta \in \Lambda^k V^*$ and $v \in V$, we have

$$i(v) \ast \alpha = (-1)^k \ast (v^\flat \wedge \alpha) \quad \text{and} \quad \ast (i(v) \alpha) = (-1)^{k+1} v^\flat \wedge \ast \alpha.$$

2.2. $G_2$-geometry. Let $V$ be an oriented 7-dimensional vector space. A $G_2$-structure on $V$ is a 3-form $\varphi \in \Lambda^3 V^*$ such that there is a positively oriented basis $\{e_i\}_{i=1}^7$ of $V$ with the dual basis $\{e_i^\flat\}_{i=1}^7$ of $V^*$ satisfying

$$(2.1) \quad \varphi = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356},$$

where $e^{i_1 \cdots i_k}$ is short for $e^{i_1} \wedge \cdots \wedge e^{i_k}$. Setting $\text{vol} := e^{1 \cdots 7}$, the 3-form $\varphi$ uniquely determines an inner product $g_\varphi$ via

$$(2.2) \quad g_\varphi(u, v) \text{vol} = \frac{1}{6} i(u) \varphi \wedge i(v) \varphi \wedge \varphi$$

for $u, v \in V$. It follows that any oriented basis $\{e_i\}_{i=1}^7$ for which (2.1) holds is orthonormal with respect to $g_\varphi$. Thus, the Hodge-dual of $\varphi$ with respect to $g_\varphi$ is given by

$$(2.3) \quad \ast \varphi = e^{4567} + e^{2367} + e^{2345} + e^{1357} - e^{1346} - e^{1256} - e^{1247}. $$

The stabilizer of $\varphi$ is known to be the exceptional 14-dimensional simple Lie group $G_2 \subset \text{GL}(V)$. The elements of $G_2$ preserve both $g_\varphi$ and vol, that is, $G_2 \subset \text{SO}(V, g_\varphi)$.

We summarize important well-known facts about the decomposition of exterior powers of $G_2$-modules into irreducible summands. Denote by $V_k$ the $k$-dimensional
irreducible $G_2$-module if there is a unique such module. For instance, $V_7$ is the irreducible 7-dimensional $G_2$-module $V$ from above, and $V_7^* \cong V_7$. For its exterior powers, we obtain the decompositions

\begin{equation}
\Lambda^0 V^* \cong \Lambda^7 V^* \cong V_1, \quad \Lambda^2 V^* \cong \Lambda^5 V^* \cong V_7 \oplus V_{14}, \\
\Lambda^1 V^* \cong \Lambda^6 V^* \cong V_7, \quad \Lambda^3 V^* \cong \Lambda^4 V^* \cong V_1 \oplus V_7 \oplus V_{27},
\end{equation}

where $\Lambda^k V^* \cong \Lambda^{7-k} V^*$ due to the $G_2$-invariance of the Hodge isomorphism $*: \Lambda^k V^* \to \Lambda^{7-k} V^*$. We denote by $\Lambda^k V^* \subset \Lambda^k V^*$ the subspace isomorphic to $V_k$. Let $\pi_k^G : \Lambda^k V^* \to \Lambda^k V^*$ be the canonical projection. Note that we have the following.

\begin{equation}
\begin{aligned}
\Lambda^2 V^* &= \{ i(u) \varphi \mid u \in V \} = \{ \alpha \in \Lambda^2 V^* \mid *(\varphi \wedge \alpha) = 2\alpha \}, \\
\Lambda^2 V^* &= \{ \alpha \in \Lambda^2 V^* \mid \varphi \wedge \alpha = 0 \} = \{ \alpha \in \Lambda^2 V^* \mid *(\varphi \wedge \alpha) = -\alpha \}, \\
\Lambda^3 V^* &= \mathbb{R} \varphi, \\
\Lambda^3 V^* &= \{ i(u) \star \varphi \in \Lambda^3 V^* \mid u \in V \}.
\end{aligned}
\end{equation}

The following equations are well-known and useful in this paper.

**Lemma 2.1.** For any $u \in V$, we have the following identities.

\[
\varphi \wedge i(u) \star \varphi = -4 \star u^b,
\]

\[
\star \varphi \wedge i(u) \varphi = 3 \star u^b,
\]

\[
\varphi \wedge i(u) \varphi = 2 \star (i(u) \varphi) = 2 u^b \wedge \star \varphi.
\]

**Definition 2.2.** Let $X$ be an oriented 7-manifold. A $G_2$-structure on $X$ is a 3-form $\varphi \in \Omega^3$ such that at each $p \in X$ there is a positively oriented basis $\{ e_i \}_{i=1}^7$ of $T_p X$ such that $\varphi_p \in \Lambda^3 T^*_p X$ is of the form (2.1). As noted above, $\varphi$ determines a unique Riemannian metric $g = g_\varphi$ on $X$ by (2.2), and the basis $\{ e_i \}_{i=1}^7$ is orthonormal with respect to $g$. A $G_2$-structure $\varphi$ is called torsion-free if it is parallel with respect to the Levi-Civita connection of $g = g_\varphi$. A manifold with a torsion-free $G_2$-structure is called a $G_2$-manifold.

A manifold $X$ admits a $G_2$-structure if and only if its frame bundle is reduced to a $G_2$-subbundle. Hence, considering its associated subbundles, we see that $\Lambda^* T^* X$ has the same decomposition as in (2.4). The algebraic identities above also hold.

### 2.3. Spin(7)-geometry.

Let $W$ be an 8-dimensional oriented real vector space. A Spin(7)-structure on $W$ is a 4-form $\Phi \in \Lambda^4 W^*$ such that there is a positively oriented basis $\{ e_i \}_{i=0}^7$ of $W$ with dual basis $\{ e^i \}_{i=0}^7$ of $W^*$ satisfying

\begin{equation}
\Phi := e^{0123} + e^{0145} + e^{0167} + e^{0246} - e^{0257} - e^{0347} - e^{0356} \\
+ e^{4567} + e^{2367} + e^{2345} + e^{1357} - e^{1346} - e^{1256} - e^{1247}.
\end{equation}
where \( e_{i_1 \cdots i_k} \) is short for \( e_{i_1} \wedge \cdots \wedge e_{i_k} \). Defining forms \( \varphi \) and \( *_7 \varphi \) on \( V := \text{span}\{ e_i \}_{i=1}^7 \subset W \) as in (2.1) and (2.3), where \( *_7 \) stands for the Hodge star operator on \( V \), we have

\[
\Phi = e^0 \wedge \varphi + *_7 \varphi.
\]

Note that \( \Phi \) is self-dual, that is, \( *_8 \Phi = \Phi \), where \( *_8 \) is the Hodge star operator on \( W \).

It is known that \( \Phi \) uniquely determines an inner product \( g_\Phi \) and the volume form and the subgroup of \( \text{GL}(W) \) preserving \( \Phi \) is isomorphic to \( \text{Spin}(7) \). As in Definition 2.2, we can define an 8-manifold with a \( \text{Spin}(7) \)-structure and a \( \text{Spin}(7) \)-manifold.

Denote by \( W_k \) the \( k \)-dimensional irreducible \( \text{Spin}(7) \)-module if there is a unique such module. For example, \( W_8 \) is the irreducible 8-dimensional \( \text{Spin}(7) \)-module from above, and \( W_8^* \cong W_8 \). The group \( \text{Spin}(7) \) acts irreducibly on \( W_7 \cong \mathbb{R}^7 \) as the double cover of \( \text{SO}(7) \). For its exterior powers, we obtain the decompositions

\[
\begin{align*}
\Lambda^0 W^* &\cong \Lambda^8 W^* \cong W_1, \\
\Lambda^1 W^* &\cong \Lambda^7 W^* \cong W_8, \\
\Lambda^2 W^* &\cong \Lambda^6 W^* \cong W_7 + W_{21}, \\
\Lambda^3 W^* &\cong \Lambda^5 W^* \cong W_8 + W_{48}, \\
\Lambda^4 W^* &\cong W_1 + W_7 + W_{27} + W_{35}
\end{align*}
\]

(2.7)

where \( \Lambda^k W^* \cong \Lambda^{8-k} W^* \) due to the \( \text{Spin}(7) \)-invariance of the Hodge isomorphism \( * = *_8 : \Lambda^k W^* \to \Lambda^{8-k} W^* \). Again, we denote by \( \Lambda^k W_\pi \subset \Lambda^k W^* \) the subspace isomorphic to \( W_\pi \) in the above notation.

The space \( \Lambda^k W^* \) for \( k = 2, 4, 6 \) is explicitly given as follows. For the explicit descriptions of the other irreducible summands, see for example [10, (4.7)].

**Lemma 2.3** ([10, Lemma 4.2], [12, Lemma 3.4]). Let \( e^0 \in W^* \) be a unit vector. Set \( V^* = (\mathbb{R} e^0)^\perp \), the orthogonal complement of \( \mathbb{R} e^0 \). The group \( \text{Spin}(7) \) acts irreducibly on \( V^* \) as the double cover of \( \text{SO}(7) \), and hence, we have the identification \( V^* \cong W_7 \). Then, the following maps are \( \text{Spin}(7) \)-equivariant isometries.

\[
\lambda^2(\alpha) := \frac{1}{2} (e^0 \wedge \alpha + i(\alpha^2) \varphi),
\]

(2.8)

\[
\lambda^k : V^* \longrightarrow \Lambda^k W^*,
\]

\[
\lambda^4(\alpha) := \frac{1}{\sqrt{8}} (e^0 \wedge i(\alpha^2) *_7 \varphi - \alpha \wedge \varphi),
\]

\[
\lambda^6(\alpha) := \frac{1}{3} \Phi \wedge \lambda^2(\alpha) = *_8 \lambda^2(\alpha).
\]

Here, \( * = *_8 \) and \( *_7 \) are the Hodge star operators on \( W^* \) and \( V^* \), respectively.

The following decomposition is also well-known.

\[
\begin{align*}
\Lambda^2 W^* &= \{ \alpha \in \Lambda^2 W^* \mid \alpha \wedge \Phi = 3 * \alpha \}, \\
\Lambda^2_{21} W^* &= \{ \alpha \in \Lambda^2 W^* \mid \alpha \wedge \Phi = - * \alpha \}.
\end{align*}
\]

Denote by

\[
\pi^k : \Lambda^k W^* \to \Lambda^k W^*
\]

the canonical projection. Since \( \alpha = \pi^2_2(\alpha) + \pi^2_{21}(\alpha) \) and \( *(\Phi \wedge \alpha) = 3\pi^2_7(\alpha) - \pi^2_{21}(\alpha) \) for a 2-form \( \alpha \in \Lambda^2 W^* \), it follows that

\[
\pi^2_2(\alpha) = \frac{\alpha + *(\Phi \wedge \alpha)}{4}, \quad \pi^2_{21}(\alpha) = \frac{3\alpha - *(\Phi \wedge \alpha)}{4}.
\]

(2.9)
For the wedge product of two 2-forms, we have the following by [20, Proposition 2.1].

\begin{align}
\Lambda^2_i W^* \wedge \Lambda^2_j W^* &= \Lambda^4_i W^* \oplus \Lambda^4_{27} W^*, \\
\Lambda^2_7 W^* \wedge \Lambda^2_{21} W^* &= \Lambda^4_7 W^* \oplus \Lambda^4_{35} W^*, \\
\Lambda^2_{21} W^* \wedge \Lambda^2_{21} W^* &= \Lambda^4_{21} W^* \oplus \Lambda^4_{27} W^* \oplus \Lambda^4_{35} W^*.
\end{align}

(2.10)

The following is important in the proof of Proposition A.1.

**Proposition 2.4** ([13, Proposition 2.7]). For any \( \beta \in \Lambda^2_7 W^* \) and \( \gamma \in \Lambda^2_{21} W^* \), we have

\begin{align}
\beta^3 &= \frac{3}{2} |\beta|^2 \ast \beta, \\
|\beta|^4 &= \frac{2}{3} |\beta^2|^2, \\
|\gamma|^4 &= |\gamma^2|^2 - \frac{1}{3} \ast \gamma^4, \\
|\beta|^2 |\gamma|^2 &= 2 |\beta \wedge \gamma|^2.
\end{align}

The following lemma is very useful, which is used in the proof of Lemmas A.3 and A.10 several times. Note that the following identity depends only on the metric and the orientation and is independent of the Spin(7)-structure.

**Lemma 2.5.** For any \( F \in \Lambda^2 W^* \) and \( \xi \in \Lambda^4 W^* \), we have

\[ * (\xi \wedge (\ast F^3)^2) = \frac{3}{2} \langle F^2, \xi \rangle \ast F^4. \]

**Proof.** Set \( F_{ij} = F(e_i, e_j) \). Then, \( F = (1/2) \sum_{i,j} F_{ij} e^{ij} \). We use

\[ F^{k+1} = \frac{1}{2} \sum_{i,j} F_{ij} e^{ij} \wedge F^k \text{ and } * ((i(e_j)F) \wedge F^3) = * \left( \frac{i(e_j)F^4}{4} \right) = - \left( \frac{* F^4}{4} \right) e^j \]

several times to prove the statement. Since \( F^3 = (1/2) \sum_{i,j} F_{ij} e^{ij} \wedge F^2 \), we have

\begin{align}
* (\xi \wedge (\ast F^3)^2) &= \langle F^3, (\ast F^3) \wedge \xi \rangle \\
&= \frac{1}{2} \sum_{i,j} F_{ij} \langle F^2, i(e_j)i(e_i)((\ast F^3) \wedge \xi) \rangle = I_1 + I_2 + I_3,
\end{align}

(2.11)
where

\[ I_1 = \frac{1}{2} \sum_{i,j} F_{ij} \langle (*F^3)(e_i, e_j)\xi, F^2 \rangle, \]

\[ I_2 = -\sum_{i,j} F_{ij} \langle i(e_i)(*F^3) \wedge i(e_j)\xi, F^2 \rangle, \]

\[ I_3 = \frac{1}{2} \sum_{i,j} F_{ij} \langle (*F^3) \wedge i(e_j)i(e_i)\xi, F^2 \rangle. \]

We compute \( I_1, I_2 \) and \( I_3 \). Since

\[ \sum_{i,j} F_{ij}(*F^3)(e_i, e_j) = \sum_{i,j} F_{ij} (e^{ij} \wedge F^3) = 2 * F^4, \tag{2.12} \]

we have

\[ I_1 = \langle F^2, \xi \rangle * F^4. \tag{2.13} \]

Since

\[ \sum_{i} F_{ij}i(e_i)(*F^3) = \sum_{i} F_{ij} (e^i \wedge F^3) = - * ((i(e_j)F) \wedge F^3) = \frac{F^4}{4} e^j, \tag{2.14} \]

we obtain

\[ I_2 = -\frac{F^4}{4} \sum_{j} \langle e^j \wedge i(e_j)\xi, F^2 \rangle = -\langle F^2, \xi \rangle * F^4. \tag{2.15} \]

Since \((1/2) \sum_{i,j} F_{ij}i(e_j)i(e_i)\xi = *(F \wedge *\xi)\) and \( F^2 = (1/2) \sum_{i,j} F_{ij}e^{ij} \wedge F \), we compute

\[ I_3 = \langle (*F^3) \wedge *(F \wedge *\xi), F^2 \rangle \]

\[ = (1/2) \sum_{i,j} F_{ij} \langle i(e_j)i(e_i) ((*F^3) \wedge *(F \wedge *\xi)), F \rangle = I_{3,1} + I_{3,2} + I_{3,3}, \]

where

\[ I_{3,1} = \frac{1}{2} \sum_{i,j} F_{ij}(*F^3)(e_i, e_j) \langle *(F \wedge *\xi), F \rangle, \]

\[ I_{3,2} = -\sum_{i,j} F_{ij} \langle i(e_i)(*F^3) \wedge i(e_j) \ast (F \wedge *\xi), F \rangle, \]

\[ I_{3,3} = \frac{1}{2} \left( \sum_{i,j} F_{ij}i(e_j)i(e_i) \ast (F \wedge *\xi) \right) \langle *F^3, F \rangle. \]

By (2.12), we have

\[ I_{3,1} = \langle F^2, \xi \rangle * F^4. \]
By (2.14), we have
\[
I_{3,2} = -\frac{F^4}{4} \sum_j \langle e^j \wedge i(e_j) \ast (F \wedge \ast \xi), F \rangle = -\frac{1}{2} \langle F^2, \xi \ast F^4 \rangle.
\]
Since \(\sum_{i,j} F_{ij} i(e_j)i(e_i) \ast (F \wedge \ast \xi) = 2 * (F^2 \wedge \ast \xi)\), we have
\[
I_{3,3} = *(F^2 \wedge \ast \xi) \langle F^3, F \rangle = \langle F^2, \xi \rangle \ast F^4.
\]
Hence, it follows that
\[
I_3 = \frac{3}{2} \langle F^2, \ast \xi \rangle \ast F^4. \tag{2.16}
\]
By (2.11), (2.13), (2.15) and (2.16), the proof is completed. \(\square\)

2.4. SU(3)-geometry. Set \(U = \mathbb{R}^6 \cong \mathbb{C}^3\). Denote by \(\{e_i\}_{i=1}^7\) and \(\{e^j\}_{j=1}^7\) the standard basis of \(U\) and its dual, respectively. Denote by \(g\) the standard inner product on \(U\). This together with the standard orientation induces the Hodge star operator \(\ast\). Denote by \(\omega = e^{23} + e^{45} + e^{67}\) the standard Kähler form on \(U \cong \mathbb{C}^3\). Set \(f^2 = e^2 + \sqrt{-1}e^3, f^3 = e^4 + \sqrt{-1}e^5\) and \(f^4 = e^6 + \sqrt{-1}e^7\). The holomorphic volume form \(\Omega\) is given by \(\Omega = f^{234} := f^2 \wedge f^3 \wedge f^4\). It is well-known that
\[
\ast \omega = \frac{1}{2} \omega^2, \quad \ast \text{Re} \Omega = \text{Im} \Omega, \quad \ast \text{Im} \Omega = - \text{Re} \Omega. \tag{2.17}
\]
The standard complex structure \(J\) on \(U \cong \mathbb{C}^3\) induces the decomposition \(U \otimes \mathbb{C} = U^{1,0} \oplus U^{0,1}\), where \(U^{1,0}\) and \(U^{0,1}\) are \(\sqrt{-1}\)- and \(-\sqrt{-1}\)-eigenspaces of \(J\), respectively. Set \(\Lambda^{p,q} = \Lambda^p(U^{1,0})^* \otimes \Lambda^q(U^{0,1})^*\). Define real vector spaces \([\Lambda^{p,q}]\) for \(p \neq q\) and \([\Lambda^{p,p}]\) by
\[
[\Lambda^{p,q}] = (\Lambda^{p,q} \oplus \Lambda^{q,p}) \cap \Lambda^{p+q}U^* = \{ \alpha \in \Lambda^{p,q} \oplus \Lambda^{q,p} \mid \bar{\alpha} = \alpha \},
\]
\[
[\Lambda^{p,p}] = \Lambda^{p,p} \cap \Lambda^{2p}U^* = \{ \alpha \in \Lambda^{p,p} \mid \bar{\alpha} = \alpha \}.
\]
The Kähler form \(\omega\) is contained in \([\Lambda^{1,1}]\) and denote by \([\Lambda^{0,1}]\) the orthogonal complement of \(\mathbb{R}\omega\) in \([\Lambda^{1,1}]\). Then, we have the decomposition
\[
\Lambda^2U^* = [\Lambda^{2,0}] \oplus [\Lambda^{1,1}] \oplus \mathbb{R}\omega
\]
and \([\Lambda^{1,1}], [\Lambda^{0,1}]\) are identified with \(u(3), su(3)\), respectively. For a subspace \(S \subset \Lambda^kU^*\), denote by \(\pi_S : \Lambda^kU^* \to S\) the orthogonal projection. For simplicity, set
\[
\pi^{[p,q]} = \pi_{[\Lambda^{p,q}]} : \Lambda^{p+q}U^* \to [\Lambda^{p,q}], \quad \pi^{[p,p]} = \pi_{[\Lambda^{p,p}]} : \Lambda^{p+q}U^* \to [\Lambda^{p,p}].
\]
Since \(\ast \Lambda^{p,q} = \Lambda^{3-q,3-p}\), we see that
\[
\ast \circ \pi^{[p,q]} = \pi^{[3-p,3-q]} \circ \ast. \tag{2.18}
\]
Lemma 2.6. For any $u \in U$, we have
\[ *(i(u)\text{Re}\Omega \wedge \text{Re}\Omega) = 2(Ju)^{\flat}, \]
\[ *(i(u)\text{Re}\Omega \wedge \text{Im}\Omega) = -2u^{\flat}, \]
\[ |i(u)\text{Re}\Omega|^{2} = 2|u|^{2}. \]

Proof. The first two equations follow from [18, (2), (3)]. By the second equation and (2.17), we have
\[ |i(u)\text{Re}\Omega|^{2} = *(i(u)\text{Re}\Omega \wedge *(i(u)\text{Re}\Omega)) = *(i(u)\text{Re}\Omega \wedge u^{\flat} \wedge \text{Im}\Omega) = 2|u|^{2}. \]

\[ \square \]

Lemma 2.7.
\[ [\Lambda^{2,0}] = \{ i(u)\text{Re}\Omega \mid u \in U \} = \{ \beta \in \Lambda^{2}U^{*} \mid *(\omega \wedge \beta) = \beta \}, \]
\[ [\Lambda_{0}^{1}] = \{ \beta \in \Lambda^{2}U^{*} \mid *(\omega \wedge \beta) = -\beta \}, \]
\[ \mathbb{R}\omega = \{ \beta \in \Lambda^{2}U^{*} \mid *(\omega \wedge \beta) = 2\beta \}. \]

Proof. The equations for $\mathbb{R}\omega$ and $[\Lambda_{0}^{1}]$ hold by (2.17) and [18, (13)], respectively. The first equation for $[\Lambda^{2,0}]$ follows from [18, p.59]. By the equation $\omega \wedge \text{Re}\Omega = 0$ and (2.17), we have for $u \in U$
\[ *(\omega \wedge i(u)\text{Re}\Omega) = -*(i(u)\omega \wedge \text{Re}\Omega) \]
\[ = i(Ju)\text{Im}\Omega = \text{Im} (i(Ju)\Omega) = \text{Im} (\sqrt{-1}i(u)\Omega) = i(u)\text{Re}\Omega, \]
which implies the last equation for $[\Lambda^{2,0}]$.

By Lemmas 2.6 and 2.7, we obtain the following.

Corollary 2.8. For any $\beta \in \Lambda^{2}U^{*}$, we have
\[ |\beta \wedge \text{Re}\Omega|^{2} = |\beta \wedge \text{Im}\Omega|^{2} = 2|\pi^{[2,0]}(\beta)|^{2} \]

Proof. Since $\beta \wedge \text{Re}\Omega = \pi^{[2,0]}(\beta) \wedge \text{Re}\Omega$ and $\pi^{[2,0]}(\beta) = i(u)\text{Re}\Omega$ for some $u \in U$ by Lemma 2.7, Lemma 2.6 implies that
\[ |\beta \wedge \text{Re}\Omega|^{2} = 4|u|^{2} = 2|\pi^{[2,0]}(\beta)|^{2}. \]

We can compute $|\beta \wedge \text{Im}\Omega|^{2}$ similarly and the proof is completed. \[ \square \]

2.5. SU(4)-geometry. Set $W = \mathbb{R}^{8} \cong \mathbb{C}^{4}$. Denote by $\{ e_{i} \}_{i=0}^{7}$ and $\{ e^{i} \}_{i=0}^{7}$ the standard basis of $W$ and its dual, respectively. Denote by $g$ the standard inner product on $W$. This together with the standard orientation induces the Hodge star operator $\ast$. Denote by $\omega = e^{01} + e^{23} + e^{45} + e^{67}$ the standard Kähler form on $W \cong \mathbb{C}^{4}$. Set $f^{1} = e^{0} + \sqrt{-1}e^{1}, f^{2} = e^{2} + \sqrt{-1}e^{3}, f^{3} = e^{4} + \sqrt{-1}e^{5}$ and $f^{4} = e^{6} + \sqrt{-1}e^{7}$. The holomorphic volume form $\Omega$ is given by $\Omega = f^{1234} := f^{1} \wedge f^{2} \wedge f^{3} \wedge f^{4}$. Using the standard complex structure $J$ on $W \cong \mathbb{C}^{4}$, we can define spaces $[\Lambda^{p,q}]$ and $[\Lambda^{p,p}]$ as in Section 2.4. For a subspace $S \in \Lambda^{k}W^{*}$, denote by
\[ \pi_{S} : \Lambda^{k}W^{*} \rightarrow S \]
the orthogonal projection. For simplicity, set
\[ \pi^{[p,q]} = \pi^{[\Lambda_{p,q}]} : \Lambda^{p+q}W^* \to \Lambda^{p+q}, \quad \pi^{[p,p]} = \pi^{[\Lambda_{p,p}]} : \Lambda^{p+q}W^* \to \Lambda^{p+q}. \]
Since \( *\Lambda^{p,q} = \Lambda^{4-q,p-q} \), we see that
\[ (2.19) \quad \ast \circ \pi^{[p,q]} = \pi^{[4-q,4-p]} \circ \ast. \]
It is well-known that
\[ (2.20) \quad \ast \omega = \frac{1}{6} \omega^3, \quad \ast \omega^2 = \omega^2, \quad \ast \Re \Omega = \Re \Omega, \quad \ast \Im \Omega = \Im \Omega. \]
and
\[ (2.21) \quad [\Lambda^{4,0}] = \mathbb{R} \Re \Omega \oplus \mathbb{R} \Im \Omega. \]
The Kähler form \( \omega \) is contained in \([\Lambda^{1,1}]\) and denote by \([\Lambda_0^{1,1}]\) the orthogonal complement of \( \mathbb{R} \omega \) in \([\Lambda^{1,1}]\). We have the following irreducible decomposition with respect to the standard U(4)-action
\[ \Lambda^2 W^* = [\Lambda^{2,0}] \oplus [\Lambda_0^{1,1}] \oplus \mathbb{R} \omega, \]
where \([\Lambda^{1,1}], [\Lambda_0^{1,1}]\) are identified with \(u(4), su(4)\), respectively. Note that
\[ \pi_{\mathbb{R} \omega}(\beta) = \frac{\langle \beta, \omega \rangle}{4} \omega. \]
These spaces are characterized as follows.

**Lemma 2.9.**

- \([\Lambda^{2,0}] = \{ \beta \in \Lambda^2 W^* | * (\omega^2 \wedge \beta) = 2\beta \}\),
- \([\Lambda_0^{1,1}] = \{ \beta \in \Lambda^2 W^* | * (\omega^2 \wedge \beta) = -2\beta \}\),
- \(\mathbb{R} \omega = \{ \beta \in \Lambda^2 W^* | * (\omega^2 \wedge \beta) = 6\beta \}\).

**Proof.** The equation for \( \mathbb{R} \omega \) holds by (2.20). Set \( \beta = \Re(e^{12}) = e^{02} - e^{13} \in [\Lambda^{2,0}] \) and \( \beta' = e^{01} - e^{23} \in [\Lambda_0^{1,1}] \). Since \( \omega^2 = 2 (e^{0123} + e^{0145} + e^{0167} + e^{2345} + e^{2367} + e^{4567}) \), we see that
\[ * (\omega^2 \wedge \beta) = 2 * (e^{024567} - e^{134567}) = 2\beta, \quad * (\omega^2 \wedge \beta') = 2 * (e^{014567} - e^{234567}) = -2\beta'. \]
Since \( *(\omega^2 \wedge \cdot) : \Lambda^2 W^* \to \Lambda^2 W^* \) is U(4)-equivariant, the proof is completed by Schur’s lemma. \( \square \)

The spaces \([\Lambda_0^{1,1}]\) and \(\mathbb{R} \omega\) are also irreducible with respect to the standard SU(4)-action. As a representation of SU(4), \([\Lambda^{2,0}]\) further decomposes as follows.

**Lemma 2.10.** We have the following irreducible decomposition with respect to the standard SU(4)-action:
\[ [\Lambda^{2,0}] = A_+ \oplus A_-, \]
where
\[ A_+ = \{ \beta \in \Lambda^2 W^* | * (\Re \Omega \wedge \beta) = 2\beta \}, \]
\[ A_- = \{ \beta \in \Lambda^2 W^* | * (\Re \Omega \wedge \beta) = -2\beta \}. \]
Proof. The decomposition \([\Lambda^{2,0}] = A_+ \oplus A_-\) is given by [19, (3)]. Our definition of \(A_\pm\) may look different from that of [19], but we see the equivalence as follows. As we see below, the SU(4)-structure induces the Spin(7)-structure \(\Phi\) given by (2.22). By [19, Proposition 2], \(A_+ \subset \Lambda_7^2 W^*\) and \(A_- \subset \Lambda_7^2 W^*\). Then, for \(a_+ \in A_+\), we have
\[
3a_+ = *(\Phi \wedge a_+) = \frac{1}{2} *(\omega^2 \wedge a_+) + *(\text{Re} \Omega \wedge a_+).
\]
By Lemma 2.9, we have \(*(\omega^2 \wedge a_+) = 2a_+.\) Hence, we obtain \(*(\text{Re} \Omega \wedge a_+) = 2a_-\) for \(a_- \in A_-\). Since \(*(\text{Re} \Omega \wedge \cdot) : \Lambda^2 W^* \to \Lambda^2 W^*\) is the zero map on \([\Lambda^1,1]\), the proof is completed. \(\square\)

Since SU(4) \(\subset\) Spin(7), the SU(4)-structure induces the Spin(7)-structure. Explicitly, the 4-form in (2.6) is given by
\[
\Phi = \frac{1}{2} \omega^2 + \text{Re} \Omega.
\]
(2.22)
The relation among the irreducible representations of SU(4) and Spin(7) is studied in detail in [19, Proposition 2]. The following decomposition is useful in this paper.
\[
\Lambda_7^4 W^* = \mathbb{R} \omega \oplus A_+, \quad \Lambda_7^4 W^* = \mathbb{R} \text{Im} \Omega \oplus (\omega \wedge A_-).
\]
(2.23)
We use the following in the proof of Proposition A.9.

Lemma 2.11. For any 4-form \(\xi \in \Lambda^4 W^*\), we have
\[
2|\pi_{\omega \wedge A_-}(\xi)|^2 = |\pi_{A_-}(*(\omega \wedge \xi))|^2.
\]
Proof. Let \(\{\beta_j\}_{j=1}^6\) be an orthonormal basis of \(A_-\). Since any element of \(\Lambda_7^4 W^*\) is self dual, \(\omega \wedge \beta_j\) is self dual by (2.23). Then, by Lemma 2.9, we have
\[
\langle \omega \wedge \beta_j, \omega \wedge \beta_k \rangle = *(\beta_j \wedge \beta_k \wedge \omega^2) = 2 *(\beta_j \wedge * \beta_k) = 2 \delta_{jk}.
\]
Thus, we see that \(\{\omega \wedge \beta_j/\sqrt{2}\}_{j=1}^6\) is an orthonormal basis of \(\omega \wedge A_-\). Hence,
\[
2|\pi_{\omega \wedge A_-}(\xi)|^2 = \sum_j \langle \xi, \omega \wedge \beta_j \rangle^2 = \sum_j *(\xi \wedge \omega \wedge \beta_j)^2 = \sum_j \langle * (\omega \wedge \xi), \beta_j \rangle^2 = \left| \pi_{A_-} (* (\omega \wedge \xi)) \right|^2.
\]
\(\square\)

3. The volume functional
In this section, we introduce the volume functional \(V\), which corresponds to the volume functional for submanifolds via the real Fourier–Mukai transform and is called the Dirac-Born-Infeld (DBI) action in theoretical physics [17]. The relation to the similar volume functional in [8, Definition 3.1] for Hermitian metrics on a holomorphic line bundle is explained in Remark 6.12.
Let \((X, g)\) be a compact oriented \(n\)-dimensional Riemannian manifold and \(L \to X\) be a smooth complex line bundle with a Hermitian metric \(h\). Set
\[
\mathcal{A}_0 = \{ \text{Hermitian connections of } (L, h) \} = \nabla + \sqrt{-1} \Omega^1 : \text{id}_L,
\]
where \(\nabla \in \mathcal{A}_0\) is any fixed connection. We regard the curvature 2-form \(F_\nabla\) of \(\nabla\) as a \(\sqrt{-1}\mathbb{R}\)-valued closed 2-form on \(X\). For simplicity, set
\[
E_\nabla := -\sqrt{-1} F_\nabla \in \Omega^2.
\]
Define the *volume functional* \(V : \mathcal{A}_0 \to \mathbb{R}\) by
\[
V(\nabla) = \int_X \sqrt{\det(\text{id}_{TX} + E_\nabla)} \, \text{vol}_g,
\]
where \(E_\nabla \in \Gamma(X, \text{End} T X)\) is defined by \(u \mapsto (i(u)E_\nabla)^t\) and \(\text{vol}_g\) is the volume form defined by the Riemannian metric \(g\). Note that \(\det(\text{id}_{TX} + E_\nabla) \geq 1\) by the argument in Lemma A.2. We see that the functional \(V\) corresponds to the volume functional for submanifolds via the real Fourier–Mukai transform by the proof of [12, Lemma 4.3].

**Remark 3.1.** It is useful to introduce \(G_\nabla : TX \to TX\) for \(\nabla \in \mathcal{A}_0\) defined by
\[
G_\nabla := \text{id}_{TX} - E_\nabla^t \circ E_\nabla^t = \left(\text{id}_{TX} - E_\nabla^t\right) \circ \left(\text{id}_{TX} + E_\nabla^t\right) = t\left(\text{id}_{TX} + E_\nabla^t\right) \circ \left(\text{id}_{TX} + E_\nabla^t\right),
\]
where we denote by \(t \text{T}\) the transpose of a linear map \(T : TX \to TX\). We see that \(G_\nabla\) is positive definite. Since \(E_\nabla^t \circ G_\nabla = G_\nabla \circ E_\nabla^t\), we have
\[
E_\nabla^t \circ G_\nabla^{-1} = G_\nabla^{-1} \circ E_\nabla^t,
\]
which we use frequently. We see that \(G_\nabla^{-1} \circ E_\nabla^t\) is skew-symmetric by (3.2).

The volume functional \(V\) is rewritten as
\[
V(\nabla) = \int_X (\det G_\nabla)^{1/4} \, \text{vol}_g.
\]
In this section, we mainly use this expression for \(V\), which enables us to deduce the following results. Note that setting \(K_\nabla = G_\nabla^{1/4} : TX \to TX\), we see that \(V(\nabla) = \int_X \text{vol}_{K_\nabla^* g}\), where \(\text{vol}_{K_\nabla^* g}\) is the volume form defined by the metric \(K_\nabla^* g\).

We compute the first variation of \(V\).

**Proposition 3.2.** Let \(\delta_\nabla V : T_\nabla \mathcal{A}_0 = \sqrt{-1} \Omega^1 \to \mathbb{R}\) be the linearization of \(V : \mathcal{A}_0 \to \mathbb{R}\) at \(\nabla \in \mathcal{A}_0\). Then, we have
\[
(\delta_\nabla V)(\sqrt{-1} a) = -\langle a, H(\nabla) \rangle_{L^2}.
\]
for \(a \in \Omega^1\). Here, \(\langle \cdot, \cdot \rangle_{L^2}\) is the \(L^2\) inner product with respect to the metric \(g\) and \(H(\nabla) \in \Omega^1\) is defined by
\[
H(\nabla) = -d^* \left( (\det G_\nabla)^{1/4} \left( G_\nabla^{-1} \circ E_\nabla^t \right)^b \right),
\]
(3.3)
where for a skew-symmetric endomorphism $K$, $K^\flat$ is a 2-form defined by $K^\flat(u, v) = g(K(u), v)$ for $u, v \in TX$. We call $H(\nabla)$ the mean curvature 1-form of $\nabla$.

In [8, Definition 2.3], a similar notion of the mean curvature 1-form is defined. We will explain the relation in Remark 6.12.

**Proof.** Set $\nabla_t = \nabla + t\sqrt{-1}a$ for $t \in \mathbb{R}$. Then,

$$
(\delta_\nabla) V(\sqrt{-1}a) = \left. \frac{d}{dt} \right|_{t=0} V(\nabla_t)
= \frac{1}{4} \int_X \text{tr} \left( G_{\nabla}^{-1} \circ \left( (da)^2 \circ E^e_{\nabla} + E^\sharp_{\nabla} \circ (da)^2 \right) \right) \left( \text{det } G_{\nabla} \right)^{1/4} \text{vol}_g,
$$

where we use (3.2) and $(da)^2 \in \Gamma(X, \text{End}(TX))$ is defined by $g((da)^2(u), v) = da(u, v)$ for $u, v \in TX$. Since $\text{tr}(A^2 \circ B^2) = -2 \langle A, B \rangle$ for any 2-forms $A, B$, where $\langle \cdot, \cdot \rangle$ on the right hand side is the inner product for 2-forms, we have

$$
\text{tr} \left( G_{\nabla}^{-1} \circ E^\sharp_{\nabla} \circ (da)^2 \right) = -2 \left( da, \left( G_{\nabla}^{-1} \circ E^\sharp_{\nabla} \right)^\flat \right).
$$

Multiplying (3.4) by $(-1/2)(\text{det } G_{\nabla})^{1/4}$, integrating it over $X$ with respect to $\text{vol}_g$ and using the integration by parts, we have $(\delta_\nabla) V(a) = -\langle a, H(\nabla) \rangle_{L^2}$ with

$$
H(\nabla) = -d^* \left( \left( \text{det } G_{\nabla} \right)^{1/4} \left( G_{\nabla}^{-1} \circ E^\sharp_{\nabla} \right)^\flat \right),
$$

and the proof is completed. \hfill \Box

**Proposition 3.3.** Define $\mathcal{L} : \Omega^1 \to \Omega^1$ by $\mathcal{L}(a) = \delta_\nabla H(\sqrt{-1}a)$, where $\delta_\nabla H$ is the linearization of the operator $H : \mathcal{A}_0 \to \Omega^1$ at $\nabla \in \mathcal{A}_0$. Then, the principal symbol $\sigma_\mathcal{L}(\xi) : T^*_pX \to T^*_pX$ of $\mathcal{L}$ satisfies

$$
\langle \sigma_\mathcal{L}(\xi)(a), a \rangle
= \left( \text{det } G_{\nabla} \right)^{1/4} \left( a \left( G_{\nabla}^{-1}(a^2) \right) \xi \left( G_{\nabla}^{-1}(\xi^2) \right) - \{ a \left( G_{\nabla}^{-1}(\xi^2) \right) \}^2 \right)
$$

for $\xi, a \in T^*_pX$ at each $p \in X$.

**Proof.** First, we compute $\mathcal{L}$. Put $\nabla_t := \nabla + t\sqrt{-1}a$. Then,

$$
\mathcal{L}(a) = \left. \frac{d}{dt} \right|_{t=0} d^* \left( \left( \text{det } G_{\nabla_t} \right)^{1/4} \left( G_{\nabla_t}^{-1} \circ E^\sharp_{\nabla_t} \right)^\flat \right).
$$

Since

$$
0 = \left. \frac{d}{dt} \right|_{t=0} G_{\nabla_t} \circ G_{\nabla_t}^{-1} = \left( -(da)^2 \circ E^e_{\nabla} - E^\sharp_{\nabla} \circ (da)^2 \right) \circ G_{\nabla}^{-1} + G_{\nabla} \circ \frac{dG_{\nabla_t}^{-1}}{dt} \bigg|_{t=0},
$$
we have

\[ \frac{dG^{-1}}{dt} \bigg|_{t=0} = G^{-1} \circ (da)^2 \circ E^{-1}_G + E^{-1}_G \circ (da)^2 \circ G^{-1} \]  

Then, we compute

\[ \mathcal{L}(a) = J_1(a) + J_2(a) + J_3(a) + J_4(a), \]

where

\[
J_1(a) = \frac{1}{2} d^* \left( (\det G)^{1/4} \tr \left( G^{-1} \circ E^2_G \circ (da)^2 \right) \left( G^{-1} \circ E^2_G \right)^b \right),
\]

\[
J_2(a) = -d^* \left( (\det G)^{1/4} \left( G^{-1} \circ (da)^2 \circ G^{-1} \circ E^2_G \right)^b \right),
\]

\[
J_3(a) = -d^* \left( (\det G)^{1/4} \left( G^{-1} \circ E^2_G \circ (da)^2 \circ G^{-1} \circ E^2_G \right)^b \right),
\]

\[
J_4(a) = -d^* \left( (\det G)^{1/4} \left( G^{-1} \circ (da)^2 \right)^b \right).
\]

Denote by \( \sigma_{J_k}(\xi) \) the principal symbol of the linear differential operator \( J_k : \Omega^1 \to \Omega^1 \) \((k = 1, \ldots, 4)\) for \( \xi \in T_p^* X \) at \( p \in X \). Then, for \( a \in T_p^* X \), we have

\[
\frac{\langle \sigma_{J_k}(\xi)(a), a \rangle}{(\det G)^{1/4}} = -\frac{\tr \left( G^{-1} \circ E^2_G \circ (\xi \wedge a)^2 \right)}{2} \left\langle i(\xi^2) \left( G^{-1} \circ E^2_G \right)^b, a \right\rangle
\]

\[
= \left\langle \left( G^{-1} \circ E^2_G \right)^b, \xi \wedge a \right\rangle^2 = \left\{ a \left( G^{-1} \circ E^2_G(\xi^2) \right) \right\}^2,
\]

where we use the same argument as in (3.4) for the second equality.

Next, we work on \( J_2 \). We have

\[
\langle \sigma_{J_2}(\xi)(a), a \rangle = (\det G)^{1/4} \left\langle \left( G^{-1} \circ (\xi \wedge a)^2 \circ E^2_G \circ G^{-1} \circ E^2_G \right)^b, \xi \wedge a \right\rangle.
\]

By (3.2) and \( G_T = \text{id}_{T_X} - E^2_G \circ E^2_G \), we have

\[
E^2_G \circ G^{-1} \circ E^2_G = G^{-1} \circ E^2 \circ E^2_G = -\text{id}_{T_X} + G^{-1}.
\]

This together with \( (\xi \wedge a)^2 = \xi \otimes a^2 - a \otimes \xi^2 \) implies that

\[
(\det G_{\xi}^{-1})^{-1/4} \langle \sigma_{J_2}(\xi)(a), a \rangle
\]

\[
= \left\langle -\left( G^{-1} \circ (\xi \wedge a)^2 \right)^b + \left( G^{-1} \circ (\xi \wedge a)^2 \circ G^{-1} \right)^b, \xi \wedge a \right\rangle
\]

\[
= \left\langle -\left( G^{-1} \circ (\xi \wedge a)^2 \right)(\xi^2) + \left( G^{-1} \circ (\xi \wedge a)^2 \circ G^{-1} \right)(\xi^2), a^2 \right\rangle
\]

\[
= -a(\xi^2(\xi^2))\xi^2 + a(\xi^2(\xi^2))a(\xi^2) + a(\xi^2(\xi^2))\xi(\xi^2) - \left\{ a(\xi^2(\xi^2)) \right\}^2.
\]
Next, we work on $J_3$. By $(\xi \wedge a)^\sharp = \xi \otimes a^\sharp - a \otimes \xi^\sharp$, we compute
\begin{equation}
(\det G_V)^{-1/4} \langle \sigma_{J_3}(\xi)(a), a \rangle \\
= \left\langle \left( G_V^{-1} \circ E_V^\sharp \circ (\xi \wedge a)^\sharp \circ G_V^{-1} \circ E_V^\sharp \right)^\flat, \xi \wedge a \right\rangle \\
= \left\langle \left( G_V^{-1} \circ E_V^\sharp \circ (\xi \wedge a)^\sharp \circ G_V^{-1} \circ E_V^\sharp \right)(\xi^\sharp), a^\sharp \right\rangle \\
= \left\langle \left( G_V^{-1} \circ E_V^\sharp \right)(a^\sharp), a^\sharp \right\rangle \xi \left( \left( G_V^{-1} \circ E_V^\sharp \right)(\xi^\sharp) \right) \\
- \left\langle \left( G_V^{-1} \circ E_V^\sharp \right)(\xi^\sharp), a^\sharp \right\rangle a \left( \left( G_V^{-1} \circ E_V^\sharp \right)(\xi^\sharp) \right) = - \left\{ a \left( G_V^{-1} \circ E_V^\sharp(\xi^\sharp) \right) \right\}^2,
\end{equation}
where use $\left\langle \left( G_V^{-1} \circ E_V^\sharp \right)(a^\sharp), a^\sharp \right\rangle = 0$ since $G_V^{-1} \circ E_V^\sharp$ is skew-symmetric.

Finally, we work on $J_4$. We have
\begin{equation}
(\det G_V)^{-1/4} \langle \sigma_{J_4}(\xi)(a), a \rangle = \left\langle \left( G_V^{-1} \circ (\xi \wedge a)^\sharp \right)^\flat, \xi \wedge a \right\rangle \\
= \left\langle \left( G_V^{-1} \circ (\xi \wedge a)^\sharp \right)(\xi^\sharp), a^\sharp \right\rangle \\
= a \left( G_V^{-1}(a^\sharp) \right) |\xi|^2 - a \left( G_V^{-1}(\xi^\sharp) \right) a(\xi^\sharp).
\end{equation}
Then, by (3.7), (3.8), (3.10), (3.11) and (3.12), we obtain the desired formula. \hfill \Box

Since $G_V$ is positive definite, a bilinear form defined by
$$\langle a, b \rangle_V := a \left( G_V^{-1}(b^\sharp) \right)$$
for $a, b \in T^*_pX$ is an inner product. Introducing the notation $|a|^2_V := \langle a, a \rangle_V$, (3.5) is represented as
\begin{equation}
\langle \sigma_L(\xi)(a), a \rangle = (\det G_V)^{1/4} \left( |a|^2_V |\xi|^2_V - \langle a, \xi \rangle^2_V \right).
\end{equation}
Then, by the Cauchy–Schwarz inequality, the following proposition is clear.

**Proposition 3.4.** The principal symbol $\sigma_L(\xi) : T^*_pX \to T^*_pX$ of $L$ has only one zero eigenvalue with eigenvector $\xi$ and the other eigenvalues are positive.

Since $L$ is a line bundle, the curvature is invariant under the addition of closed 1-forms, that is, $F_{V + \sqrt{-1}a} = F_V$ for any $V \in A_0$ and $a \in Z^1$, where $Z^1$ is the space of closed 1-forms. This implies that $V$ is invariant under the addition of closed 1-forms, and hence, $H$ degenerates in the direction of $\sqrt{-1}Z^1$. Proposition 3.4 implies that the principal symbol of $H$ degenerates only in this direction, just as in the case of the mean curvature for submanifolds. The statement for submanifolds is given, for example, in [25, Chapter 2]. This motivates us to prove the short-time existence and uniqueness of the following flow by DeTurck’s trick.
Fix $T \in (0, \infty]$. Assume that $\{\nabla_t\}_{t \in [0, T]}$ is a continuous path in $A_0$ and smooth on $(0, T)$. For such $\{\nabla_t\}$, we may define the line bundle mean curvature flow if it satisfies
\begin{equation}
\frac{\partial}{\partial t} \left( \frac{\nabla_t}{\sqrt{-1}} \right) = H(\nabla_t).
\end{equation}
for all $t \in (0, T)$ and call $\nabla_0$ the initial connection of the flow. We remark that the left hand side is a real valued 1-form.

Since the functional $V$ corresponds to the volume functional for submanifolds via the real Fourier–Mukai transform, the flow (3.14) can be considered as the “mirror” of the mean curvature flow for submanifolds. Note that this type of flow is introduced in [8, Definition 3.1] for Hermitian metrics on a holomorphic line bundle to find dHYM metrics.

We prove the short-time existence and uniqueness of the flow (3.14) using DeTurck’s trick. DeTurck’s trick is known as an easy way to prove the short-time existence and uniqueness of the Ricci flow, and it also works for the mean curvature flow. In general, instead of some original (degenerate) flow, a kind of modified flow is studied first and its result is reduced to the original flow. In this case, by the similarity between mean curvature flows and line bundle mean curvature flows, we can introduce the following modified flow.

Fix $\nabla_0 \in A_0$. Instead of (3.14), we consider
\begin{equation}
\frac{\partial}{\partial t} \left( \frac{\nabla_t}{\sqrt{-1}} \right) = H(\nabla_t) - d \left( (\det G_{\nabla_0})^{1/4} d^* a_t \right).
\end{equation}
We make some remarks. The $d^*$ in the second term is defined by the fixed background metric $g$ on $X$. The connection to define $G_{\nabla_0}$ in the second term is the fixed $\nabla_0$ for all time. The form $a_t$ in the second term is defined by $a_t := (\nabla_t - \nabla_0)/\sqrt{-1}$, in other words, $\nabla_t = \nabla_0 + \sqrt{-1} a_t$. Thus, $a_t$ is a real valued 1-form on $X$.

Denote by $\tilde{H}(\nabla_t)$ the right hand side of (3.15). Define $\tilde{\mathcal{L}} : \Omega^1 \to \Omega^1$ by $\tilde{\mathcal{L}}(\xi) = \delta_{\nabla_0} \tilde{H}(\sqrt{-1}a)$, where $\delta_{\nabla_0} \tilde{H}$ is the linearization of $\tilde{H} : \mathcal{A}_0 \to \Omega^1$ at $\nabla_0 \in \mathcal{A}_0$. Let $\sigma_{\tilde{\mathcal{L}}}(\xi) : T_p^* X \to T_p^* X$ be the principal symbol of $\tilde{\mathcal{L}}$ for $\xi \neq 0$.

**Lemma 3.5.** *All eigenvalues of $\sigma_{\tilde{\mathcal{L}}}(\xi)$ are positive.*

**Proof.** For $a \in \Omega^1$, we have
\[ \tilde{\mathcal{L}}(\xi) = \mathcal{L}(\xi) - d \left( (\det G_{\nabla_0})^{1/4} d^* a \right), \]
where we set $\mathcal{L} = \delta_{\nabla_0} H(\sqrt{-1}(\cdot))$. Then, by (3.13), we have
\[ \langle \sigma_{\tilde{\mathcal{L}}}(\xi)(\sqrt{-1}a), a \rangle = \langle \sigma_{\mathcal{L}}(\xi)(\sqrt{-1}a), a \rangle + (\det G_{\nabla_0})^{1/4} \langle \xi \cdot i(\xi^2) a, a \rangle \]
\[ = (\det G_{\nabla_0})^{1/4} \left( |a|_0^2 |\xi|_0^2 - \langle a, \xi \rangle_0^2 + \langle a, \xi \rangle_0^2 \right). \]
Assume that $a \neq 0$. By the Cauchy–Schwarz inequality, $|a|_0^2 |\xi|_0^2 - \langle a, \xi \rangle_0^2 \geq 0$. Thus, it is clear that $|a|_0^2 |\xi|_0^2 - \langle a, \xi \rangle_0^2 + \langle a, \xi \rangle_0^2$ is nonnegative. Suppose that this is
zero for some $a \in T_p^*X$. Then, we should have $|a|_{\nu_0}^2|\xi|_{\nu_0}^2 - \langle a, \xi \rangle_{\nu_0}^2 = 0$ and $\langle a, \xi \rangle = 0$. The first equality implies that $a = \alpha \xi$ for some $\alpha \neq 0$. But, this contradicts to the second one, $\langle a, \xi \rangle = 0$. Thus, we have proved that $|a|_{\nu_0}^2|\xi|_{\nu_0}^2 - \langle a, \xi \rangle_{\nu_0}^2 + \langle a, \xi \rangle^2$ is strictly positive for $a, \xi \neq 0$. Thus, the proof is completed. □

By Lemma 3.5, the linearization of (3.15) at $\nabla_0 \in A_0$ is strongly parabolic. Thus, by the standard theory of partial differential equations, we obtain the following.

**Proposition 3.6.**

1. For any $\nabla_0 \in A_0$, there exist $\varepsilon > 0$ and a smooth family of Hermitian connections $\{\nabla_t\}_{t \in [0, \varepsilon]}$ satisfying (3.15) and $\nabla_t|_{t=0} = \nabla_0$.
2. Suppose that $\{\nabla^1_t\}_{t \in [0, \varepsilon]}$ and $\{\nabla^2_t\}_{t \in [0, \varepsilon]}$ satisfy (3.15) for $\varepsilon > 0$. If $\nabla^1_t|_{t=0} = \nabla^2_t|_{t=0}$, then $\nabla^1_t = \nabla^2_t$ for any $t \in [0, \varepsilon]$.

Using Proposition 3.6, we show the following.

**Theorem 3.7.**

1. For any $\nabla_0 \in A_0$, there exist $\varepsilon > 0$ and a smooth family of Hermitian connections $\{\nabla_t\}_{t \in [0, \varepsilon]}$ satisfying (3.14) and $\nabla_t|_{t=0} = \nabla_0$.
2. Suppose that $\{\nabla^1_t\}_{t \in [0, \varepsilon]}$ and $\{\nabla^2_t\}_{t \in [0, \varepsilon]}$ satisfy (3.14) for $\varepsilon > 0$. If $\nabla^1_t|_{t=0} = \nabla^2_t|_{t=0}$, then $\nabla_t^2 - \nabla_t^1$ is a (pure imaginary-valued) time-dependent exact 1-form for any $t \in [0, \varepsilon]$.

**Proof.** First, we prove the existence. Let $\tilde{\nabla}_t := \nabla_0 + \sqrt{-1}a_t$ be the unique short-time solution of (3.15). Its existence is ensured by Proposition 3.6. Define a time-dependent 1-form $\eta$ by

$$
\eta_t := \int_0^t d \left( (\det G_{\nu_0})^{1/4} d^*a_s \right) ds.
$$

We remark that $d\eta_t = 0$ for all $t$ and $\eta_0 = 0$. Let $\tilde{\nabla}_t := \tilde{\nabla}_t + \sqrt{-1}\eta_t$. Then, we have $\nabla_t|_{t=0} = \nabla_0$ and $F_{\tilde{\nabla}_t} = F_{\nabla_t}$ since $d\eta_t = 0$. Moreover, we have

$$
\frac{\partial}{\partial t} \left( \frac{\nabla_t}{\sqrt{-1}} \right) = \frac{\partial}{\partial t} \left( \frac{\tilde{\nabla}_t}{\sqrt{-1}} \right) + \frac{\partial}{\partial t} \eta_t
$$

$$
= H(\tilde{\nabla}_t) - d \left( (\det G_{\nu_0})^{1/4} d^*a_t \right) + d \left( (\det G_{\nu_0})^{1/4} d^*a_t \right)
$$

$$
= H(\nabla_t),
$$

where the last equality follows from the fact that $H(\tilde{\nabla}) = H(\nabla)$ for $\tilde{\nabla}$ and $\nabla$ with $F_{\tilde{\nabla}} = F_{\nabla}$. Thus, $\nabla_t$ is a solution of the line bundle mean curvature flow with initial connection $\nabla_0$.

Next, we consider the uniqueness of the solution. Let $\nabla^1_t$ and $\nabla^2_t$ be solutions of the line bundle mean curvature flow with initial connection $\nabla_0$. Let $f^i_t$ ($i = 1, 2$) be the unique solution of the following parabolic PDE:

$$
\frac{\partial}{\partial t} f^i_t = (\det G_{\nu_0})^{1/4} \left( -d^*df_t^i \right) - (\det G_{\nu_0})^{1/4} d^* \left( \frac{\nabla_t^i - \nabla_0}{\sqrt{-1}} \right)
$$
with the initial condition \( f^i_0 = 0 \) for \( i = 1, 2 \). The uniqueness follows from the fact that the PDE is strongly parabolic. Put \( \tilde{\nabla}^i_t := \nabla^i_t + \sqrt{-1} df^i_t \) for \( i = 1, 2 \). Then, we have

\[
\frac{\partial}{\partial t} \left( \sqrt{-1} \nabla^i_t \right) = H(\nabla^i_t) + d \left( (\det G_{\nabla_0})^{1/4} \left( -d^* df^i_t \right) \right) - d \left( (\det G_{\nabla_0})^{1/4} d^* \left( \nabla^i_t - \nabla^i_0 \right) \right)
\]

\[
= H(\nabla^i_t) - d \left( (\det G_{\nabla_0})^{1/4} d^* \left( \nabla^i_t - \nabla^i_0 \right) \right)
\]

\[
= H(\tilde{\nabla}^i_t) - d \left( (\det G_{\nabla_0})^{1/4} d^* \left( \tilde{\nabla}^i_t - \nabla^i_0 \right) \right),
\]

where we use \( H(\nabla^i_t) = H(\tilde{\nabla}^i_t) \). Thus, \( \tilde{\nabla}^i_t \) is the solution of (3.15) with initial condition \( \nabla_0 \). Since the solution is unique, we have \( \tilde{\nabla}^1_t = \tilde{\nabla}^2_t \). Thus, we have \( \tilde{\nabla}^i_t = \nabla^i_t + \sqrt{-1} df^i_t - f^i_t \) and the proof is completed. \( \square \)

4. The “mirror” of the Cayley equality

In this section, we first recall the definition of deformed Donaldson–Thomas connections for a manifold with a Spin(7)-structure (Spin(7)-dDT connections) and its moduli space from [13] together with a detailed description of the orbit of the unitary gauge group. Then, we show a “mirror” of the Cayley equality. Using this, if the Spin(7)-structure is torsion-free, we show that Spin(7)-dDT connections are globally volume-minimizing. As an application of this, we show that any Spin(7)-dDT connection of a flat line bundle over a compact connected Spin(7)-manifold is a flat connection.

4.1. Preliminaries for the moduli space. Use the notation (and identities) of Subsection 2.3. Let \( X^8 \) be an 8-manifold with a Spin(7)-structure \( \Phi \) and \( L \to X \) be a smooth complex line bundle with a Hermitian metric \( h \). Let \( \mathcal{A}_0 \) be the space of Hermitian connections on \( (L, h) \). We regard the curvature 2-form \( F_{\nabla} \) of \( \nabla \) as a \( \sqrt{-1}\mathbb{R} \)-valued closed 2-form on \( X \). Define maps \( \mathcal{F}^1_{\text{Spin}(7)} : \mathcal{A}_0 \to \sqrt{-1} \Omega^2_7 \) and \( \mathcal{F}^2_{\text{Spin}(7)} : \mathcal{A}_0 \to \Omega^4_7 \) by

\[
\mathcal{F}^3_{\text{Spin}(7)}(\nabla) = \pi_7^2 \left( F_{\nabla} + \frac{1}{6} * F_{\nabla}^3 \right), \quad \mathcal{F}^4_{\text{Spin}(7)}(\nabla) = \pi_7^4 (F_{\nabla}^2).
\]

Each element of

\[
(4.1) \quad \tilde{\mathcal{M}}_{\text{Spin}(7)} := (\mathcal{F}^4_{\text{Spin}(7)})^{-1}(0) \cap (\mathcal{F}^3_{\text{Spin}(7)})^{-1}(0)
\]

is called a deformed Donaldson–Thomas connection. We call this a Spin(7)-dDT connection for short. It is known by [13] that for \( \nabla \in \mathcal{A}_0 \) satisfying \( * F_{\nabla}^4 / 24 \neq 1 \), \( \mathcal{F}^1_{\text{Spin}(7)}(\nabla) = 0 \) implies \( \mathcal{F}^2_{\text{Spin}(7)}(\nabla) = 0 \).

The moduli space of Spin(7)-dDT connections is defined as follows. Let \( \mathcal{G}_h = \{ f \cdot \text{id}_L \mid f \in C^\infty(X, S^1) \} \) be the group of unitary gauge transformations of \( (L, h) \),
where we consider $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ and $C^\infty(X, S^1)$ is the space of smooth maps from $X$ to $S^1$. The group $G_U$ acts canonically on $A_0$ by $(\lambda, \nabla) \mapsto \lambda^{-1} \circ \nabla \circ \lambda$ for $\lambda \in G_U$ and $\nabla \in A_0$. When $\lambda = f \cdot \text{id}_L$, we have $\lambda^{-1} \circ \nabla \circ \lambda = \nabla + f^{-1} df \cdot \text{id}_L$. Thus, the $G_U$-orbit through $\nabla \in A_0$ is given by $\nabla + K_U \cdot \text{id}_L$, where
\[
K_U = \left\{f^{-1} df \in \sqrt{-1} \Omega^1 \mid f \in C^\infty(X, S^1)\right\}.
\]
Since the curvature 2-form $F_\nabla$ is invariant under the action of $G_U$, the moduli space $\mathcal{M}_{\text{Spin}(7)}$ of Spin(7)-dDT connections of $(L, h)$ is given by
\[
\mathcal{M}_{\text{Spin}(7)} := \widetilde{\mathcal{M}}_{\text{Spin}(7)}/G_U.
\]
The space $K_U$ is described more explicitly as follows if $X$ is compact and connected. It would be well-known for experts, but we give the proof for completeness. We use this in Section 7. Note that the following holds for any smooth compact connected oriented Riemannian manifold $X$ of any dimension.

**Lemma 4.1.** Suppose that $X$ is compact and connected. Then, we have
\[
\left\{[f^{-1} df] \in \sqrt{-1} H^1_{dR}(X) \mid f \in C^\infty(X, S^1)\right\} = 2 \pi \sqrt{-1} H^1(X, \mathbb{Z}),
\]
where we identify $H^1(X, \mathbb{Z})$ with its image in $H^1_{dR}(X)$, that is,
\[
H^1(X, \mathbb{Z}) = \left\{[\alpha] \in H^1_{dR}(X) \mid \int_A \alpha \in \mathbb{Z} \text{ for any } A \in H_1(X, \mathbb{Z})\right\}.
\]
In particular, we have
\[
K_U = \sqrt{-1} \left(2 \pi \mathcal{H}_Z^1(X) \oplus d\Omega^0(X)\right)
\]
\[
:= \left\{\sqrt{-1}(2 \pi \alpha z + df_0) \mid \alpha \in \mathcal{H}_Z^1(X), f_0 \in \Omega^0(X)\right\},
\]
where $\mathcal{H}_Z^1(X)$ is the space of harmonic forms representing $H^1(X, \mathbb{Z})$.

**Proof.** We first prove (4.4). Define a 1-form $\alpha_{S^1} \in \Omega^1(S^1)$ on $S^1$ by
\[
(\alpha_{S^1})_z = \frac{1}{2 \pi \sqrt{-1}} \frac{dz}{z} = \frac{d\theta}{2 \pi} \quad \text{for } z = e^{\sqrt{-1} \theta} \in S^1.
\]
Then, $\alpha_{S^1}$ is closed and $[\alpha_{S^1}] \in H^1(S^1, \mathbb{Z})$. Take any $f \in C^\infty(X, S^1)$. Since $f^{-1} df = 2 \pi \sqrt{-1} f_* \alpha_{S^1}$, we see that $[f^{-1} df] \in 2 \pi \sqrt{-1} H^1(X, \mathbb{Z})$.

Conversely, take any $[\alpha] \in H^1(X, \mathbb{Z})$. Define a homomorphism $T_{[\alpha]} : \pi_1(X) \to \mathbb{Z}$ by
\[
T_{[\alpha]}([g]) = \int_{g_*[S^1]} \alpha
\]
where $g : S^1 \to X$ is a continuous map representing $[g] \in \pi_1(X)$ and $[S^1]$ is the fundamental class of $S^1$. Recall that the isomorphism $\text{deg} : \pi_1(S^1) \to \mathbb{Z}$ is given by
\[
\text{deg}([h]) = \int_{h_*[S^1]} \alpha_{S^1},
\]
where $h : S^1 \to S^1$ is a continuous map representing $[h] \in \pi_1(S^1)$. Then, by [7, Proposition 1B.9], there is a continuous map $f : X \to S^1$ such that $f_* = \text{deg}^{-1} \circ T_{[\alpha]}$.
\[ \pi_1(X) \to \pi_1(S^1). \] By [1, Proposition 17.8], we may assume that \( f \in C^\infty(X, S^1) \).

Then, we see that
\[
T_\alpha([g]) = (\deg \circ f_*)([g]) \iff \int_{g_*[S^1]} \alpha = \int_{f_*g_*[S^1]} \alpha_{S^1} = \int_{g_*[S^1]} f^*\alpha_{S^1}
\]
for any continuous map \( g : S^1 \to X \). Hence, we obtain
\[
2\pi \sqrt{-1}[\alpha] = 2\pi \sqrt{-1}[f^*\alpha_{S^1}] = [f^{-1}df],
\]
which implies (4.4).

Next, we prove (4.5). By (4.4), we see that \( K_U \subset \sqrt{-1}(2\pi \mathcal{H}_Z^1(X) \oplus d\Omega^0(X)) \).

Conversely, take any \( \alpha_Z \in \mathcal{H}_Z^1(X) \) and \( f_0 \in \Omega^0(X) \). By (4.4), there exist \( f_Z \in C^\infty(X, S^1) \) and \( g_z \in \Omega^0(X) \) such that \( 2\pi \sqrt{-1}\alpha_Z = f_Z^{-1}df_Z + \sqrt{-1}d\sigma_Z \). Then, \( f = e^{\sqrt{-1}(g_z + f_0)}f_Z \) satisfies \( f^{-1}df = \sqrt{-1}(2\pi \alpha_Z + df_0) \) and the proof is completed. \( \square \)

### 4.2 Statements.

The following statement is considered to be a “mirror” of the Cayley equality [6, Chapter IV, Theorem 1.28] via the real Fourier–Mukai transform as stated in [12, Lemma 5.5]. Theorem 4.2 is the core of Sections 4-7. The proof is given in Proposition A.1.

**Theorem 4.2.** For any \( \nabla \in \mathcal{A}_0 \), we have
\[
\left( 1 + \frac{1}{2}(F_\nabla^2, \Phi) + \frac{\ast F_\nabla^4}{24} \right)^2 + 4 \left| \pi_i^2 \left( F_\nabla + \frac{1}{6} \ast F_\nabla^3 \right) \right|^2 + 2 \left| \pi_i^4 \left( F_\nabla^4 \right) \right|^2
\]

where \( (-\sqrt{-1}F_\nabla)^2 \in \Gamma(X, \text{End } TX) \) is defined by \( u \mapsto (-\sqrt{-1}\text{ii}(u)F_\nabla)^2 \). In particular,
\[
\left| 1 + \frac{1}{2}(F_\nabla^2, \Phi) + \frac{\ast F_\nabla^4}{24} \right| \leq \sqrt{\det(\text{id}_{TX} + (-\sqrt{-1}F_\nabla)^2)}
\]
for any \( \nabla \in \mathcal{A}_0 \) and the equality holds if and only if \( \nabla \) is a Spin(7)-dDT connection.

**Remark 4.3.** By Theorem 4.2, it is immediate to see that \( 1 + \langle F_\nabla^2, \Phi \rangle/2 + \ast F_\nabla^4/24 \) is nowhere vanishing for any Spin(7)-dDT connection \( \nabla \). This is also proved in [13, Remark A.10].

By integrating the “mirror” of the Cayley equality, we can relate the volume functional \( V \) to a Spin(7)-dDT connection and we see that the volume of a Spin(7)-

**Theorem 4.4.** Suppose that \( X \) is compact and connected. For any \( \nabla \in \mathcal{A}_0 \), we have
\[
\left| \int_X \left( 1 + \frac{1}{2}(F_\nabla^2, \Phi) + \frac{\ast F_\nabla^4}{24} \right) \text{vol}_g \right| \leq V(\nabla)
\]
and the equality holds if and only if \( \nabla \) is a Spin(7)-dDT connection.
If the $\text{Spin}(7)$-structure $\Phi$ is torsion-free, the left hand side of (4.6) is given by
\[
\left| \text{Vol}(X) + \left( -2\pi^2 c_1(L)^2 \cup [\Phi] + \frac{2}{3} \pi^4 c_1(L)^4 \right) \cdot [X] \right|
\]
where $c_1(L)$ is the first Chern class of $L$. In particular, for any $\text{Spin}(7)$-dDT connection $\nabla$, $V(\nabla)$ is topological.

Proof. By Theorem 4.2, we have
\[
\left| \int_X \left( 1 + \frac{1}{2} \langle F_\nabla^2, \Phi \rangle + \frac{*F_\nabla^4}{24} \right) \text{vol}_g \right| \leq \int_X \left| 1 + \frac{1}{2} \langle F_\nabla^2, \Phi \rangle + \frac{*F_\nabla^4}{24} \right| \text{vol}_g \leq V(\nabla).
\]
Then, if the equality holds in (4.6), Theorem 4.2 implies that $\nabla$ is a $\text{Spin}(7)$-dDT connection.

Conversely, if $\nabla$ is a $\text{Spin}(7)$-dDT connection, we have
\[
V(\nabla) = \int_X \left| 1 + \frac{1}{2} \langle F_\nabla^2, \Phi \rangle + \frac{*F_\nabla^4}{24} \right| \text{vol}_g
\]
by Theorem 4.2. By Remark 4.3 and the assumption that $X$ is connected, $1 + \langle F_\nabla^2, \Phi \rangle/2 + *F_\nabla^4/24$ has constant sign for any $\text{Spin}(7)$-dDT connection $\nabla$. Hence, the equality holds in (4.6).

If the $\text{Spin}(7)$-structure $\Phi$ is torsion-free, we have
\[
\left( -2\pi^2 c_1(L)^2 \cup [\Phi] + \frac{2}{3} \pi^4 c_1(L)^4 \right) \cdot [X] = \int_X \left( \frac{1}{2} \langle F_\nabla^2, \Phi \rangle + \frac{*F_\nabla^4}{24} \right) \text{vol}_g
\]
by $c_1(L) = [\sqrt{-1} F_\nabla/2\pi]$, and the proof is completed. \hfill $\Box$

Then, Theorem 4.4 implies the following.

Corollary 4.5. If $X$ is compact and connected and the $\text{Spin}(7)$-structure $\Phi$ is torsion-free, any $\text{Spin}(7)$-dDT connection is a global minimizer of $V$.

Proof. Let $\nabla$ be a $\text{Spin}(7)$-dDT connection and $\nabla' \in A_0$ be any Hermitian connection. Then, by Theorem 4.4, we have
\[
V(\nabla) = \left| \text{Vol}(X) + \left( -2\pi^2 c_1(L)^2 \cup [\Phi] + \frac{2}{3} \pi^4 c_1(L)^4 \right) \cdot [X] \right| \leq V(\nabla'),
\]
and the proof is completed. \hfill $\Box$

By Corollary 4.5, we see that $H(\nabla) = 0$ for any $\text{Spin}(7)$-dDT connection $\nabla$ if $X$ is a compact connected $\text{Spin}(7)$-manifold, where $H(\nabla)$ is the mean curvature at $\nabla$ defined by (3.3).

By Theorems 4.2 and 4.4, we see the following.

Corollary 4.6. Let $(X, \Phi)$ be a compact connected $\text{Spin}(7)$-manifold and $L \to X$ be a smooth complex line bundle with a Hermitian metric $h$. Then, we have the following.
MIRROR OF VOLUME FUNCTIONALS ON MANIFOLDS WITH SPECIAL HOLONOMY

(1) Suppose that there is a Spin(7)-dDT connection $\nabla_0$ such that
\[ 1 + \langle F_2^2, \Phi \rangle/2 + *F_4^4/24 > 0 \quad (\text{resp. } < 0). \]
Then, for any Spin(7)-dDT connection $\nabla$, we have
\[ 1 + \langle F_2^2, \Phi \rangle/2 + *F_4^4/24 > 0 \quad (\text{resp. } < 0). \]

(2) Suppose that $L$ is a flat line bundle. Then, any Spin(7)-dDT connection is a flat connection. In particular, the moduli space of Spin(7)-dDT connections is $H^1(X, \mathbb{R})/2\pi H^1(X, \mathbb{Z})$.

Proof. By the proof of Theorem 4.4,
\[
\int_X \left( 1 + \frac{1}{2} \langle F_2^2, \Phi \rangle + \frac{*F_4^4}{24} \right) \text{vol}_g
\]
is independent of $\nabla \in A_0$. By Remark 4.3 and the assumption that $X$ is connected, $1 + \langle F_2^2, \Phi \rangle/2 + *F_4^4/24$ has constant sign for any Spin(7)-dDT connection $\nabla$. Thus, we obtain (1).

Next, we prove (2). By Theorem 4.4, $V(\nabla_0) = V(\nabla)$ for Spin(7)-dDT connections $\nabla_0$ and $\nabla$. Let $\nabla_0$ be a flat connection. Then, by Lemma A.2, we have
\[
\int_X \text{vol}_g = \int_X \sqrt{1 + |F_3^3|^2 + \frac{|F_4^4|^2}{2!} + \frac{|F_4^4|^2}{3!} + \frac{|F_4^4|^2}{4!}} \text{vol}_g,
\]
which implies that $F_3^3 = 0$. Hence, we have $\nabla = \nabla_0 + \sqrt{-1}a$ for $a \in Z^1(X)$, where $Z^1(X)$ is the space of closed 1-forms on $X$. Then, by Lemma 4.1, we see that
\[
\mathcal{M}_{\text{Spin(7)}} \cong (\sqrt{-1}Z^1(X))/K_U \cong H^1(X, \mathbb{R})/2\pi H^1(X, \mathbb{Z}).
\]

5. The “mirror” of the associator equality

In this section, we first recall the definition of deformed Donaldson–Thomas connections for a manifold with a $G_2$-structure ($G_2$-dDT connections) and its moduli space from [11]. Then, we show a “mirror” of the associator equality. Using this, if the $G_2$-structure is closed, we show that $G_2$-dDT connections are global minimizers of the volume functional $V$, just as associative submanifolds are homologically volume-minimizing. As an application of this, we show that any $G_2$-dDT connection of a flat line bundle over a compact connected manifold with a closed $G_2$-structure is a flat connection.

5.1. Preliminaries for the moduli space. First, recall some definitions. Let $X^7$ be a 7-manifold with a $G_2$-structure $\varphi \in \Omega^3$ and $L \to X$ be a smooth complex line bundle with a Hermitian metric $h$. Let $A_0$ be the space of Hermitian connections of $(L, h)$. We regard the curvature 2-form $F_3^3$ of $\nabla$ as a $\sqrt{-1}\mathbb{R}$-valued closed 2-form on $X$. Define a map $F_{G_2} : A_0 \to \sqrt{-1}\Omega^6$ by
\[
F_{G_2}(\nabla) = \frac{1}{6} F_3^3 + F_3^3 \wedge \varphi.
\]
Each element of
\[ \hat{M}_{G_2} = \mathcal{F}_{G_2}^{-1}(0) \]
is called a deformed Donaldson–Thomas connection. We call this a \(G_2\)-dDT connection for short. As in Section 4, the group \(G_U\) of unitary gauge transformations of \((L, h)\) acts on \(\mathcal{A}_0\) preserving the curvature 2-forms. Note that \(G_U\)-orbits are described explicitly as in Lemma 4.1 when \(X\) is compact and connected. The moduli space \(\mathcal{M}_{\text{Spin}(7)}\) of \(G_2\)-dDT connections of \((L, h)\) is given by
\[ \mathcal{M}_{G_2} := \hat{M}_{G_2} / G_U. \]

5.2. Statements. The following statement is considered to be a “mirror” of the associator equality [6, Chapter IV, Theorem 1.6] via the real Fourier–Mukai transform as stated in [12, Lemma 4.3]. The proof is given in Corollary A.5.

**Theorem 5.1.** For any \(\nabla \in \mathcal{A}_0\), we have
\[
\left( 1 + \frac{1}{2} \langle F^2_{\nabla}, *\varphi \rangle \right)^2 + \left| *\varphi \land F_{\nabla} + \frac{1}{6} F^3_{\nabla} \right|^2 + \frac{1}{4} \left| \varphi \land *F^2_{\nabla} \right|^2 = \det \left( \text{id}_T X + (-\sqrt{-1} F_{\nabla})^\sharp \right),
\]
where \((-\sqrt{-1} F_{\nabla})^\sharp \in \Gamma(X, \text{End} T X)\) is defined by \(u \mapsto (-\sqrt{-1} i(u) F_{\nabla})^\sharp\). In particular,
\[
\left| 1 + \frac{1}{2} \langle F^2_{\nabla}, *\varphi \rangle \right| \leq \sqrt{\det \left( \text{id}_T X + (-\sqrt{-1} F_{\nabla})^\sharp \right)}
\]
for any \(\nabla \in \mathcal{A}_0\) and the equality holds if and only if \(\nabla\) is a \(G_2\)-dDT connection.

**Proof.** The first equation is proved in Corollary A.5. By [11, Remark 3.3], \(\varphi \land *F^2_{\nabla} = 0\) if \(\nabla\) is a \(G_2\)-dDT connection. Then, the last statement holds. \qed

**Remark 5.2.** By Theorem 5.1, it is immediate to see that \(1 + \langle F^2_{\nabla}, *\varphi \rangle / 2\) is nowhere vanishing for any \(G_2\)-dDT connection \(\nabla\). This is also implied by [11, Section 3.3].

We might similarly consider a “mirror” of the coassociator equality, but the resulting equation would be the same as in Theorem 5.1. It is because the real Fourier–Mukai transform of a coassociative submanifold is again a \(G_2\)-dDT connection, which is obtained by the real Fourier–Mukai transform of an associative submanifold. See [12, Propositions 4.1 and 6.1]. Since \(G_2\)-dDT connections behave like associative submanifolds in deformation theory as shown in [11], it would be appropriate to call the identity in Theorem 5.1 a “mirror” of the associator equality.

By Theorem 5.1, we can prove the following as in Theorem 4.4, Corollaries 4.5 and 4.6.

**Theorem 5.3.** Suppose that \(X\) is compact and connected. For any \(\nabla \in \mathcal{A}_0\), we have
\[
\left| \int_X \left( 1 + \frac{1}{2} \langle F^2_{\nabla}, *\varphi \rangle \right) \text{vol}_g \right| \leq V(\nabla)
\]
and the equality holds if and only if $\nabla$ is a $G_2$-dDT connection. If the $G_2$-structure $\varphi$ is closed, the left hand side of (5.4) is given by

$$\left| \text{Vol}(X) + (-2\pi^2 c_1(L)^2 \cup [\varphi]) \cdot [X] \right|,$$

where $c_1(L)$ is the first Chern class of $L$. In particular, for any $G_2$-dDT connection $\nabla$, $V(\nabla)$ is topological.

**Corollary 5.4.** If $X$ is compact and connected and the $G_2$-structure $\varphi$ is closed, any $G_2$-dDT connection is a global minimizer of $V$.

**Remark 5.5.** Karigiannis and Leung [9] constructed a Chern-Simons type functional whose critical points are $G_2$-dDT connections.

By Corollary 5.4, we see that $H(\nabla) = 0$ for any $G_2$-dDT connection $\nabla$ if $X$ is compact and connected and the $G_2$-structure $\varphi$ is closed, where $H(\nabla)$ is the mean curvature at $\nabla$ defined by (3.3). The functional $V$ can have critical points other than $G_2$-dDT connections, just as associative submanifolds are not only critical points of the (standard) volume functional.

We give an example of a connection $\nabla$ that satisfies $H(\nabla) = 0$ but is not a $G_2$-dDT connection on a noncompact manifold $X$. (Note that we can define $H(\nabla)$ on a noncompact manifold by the formula (3.3).) Set $X = \mathbb{R}^3 \times T^4$. Denote by $(x^1, \ldots, x^7)$ the coordinates on $X = \mathbb{R}^3 \times T^4$. Set $\nabla = d + \sqrt{-1}x^1 dx^4$, which is a Hermitian connection of the trivial line bundle $X \times \mathbb{C} \to X$. Note that $\nabla$ is obtained by the real Fourier–Mukai transform of $\{(x^1, x^2, x^3, x^4, 0, 0, 0) \in \mathbb{R}^3 \times T^4 \mid x^j \in \mathbb{R}\}$, which corresponds to a plane that is not associative. (For more details about the real Fourier–Mukai transform, see [12].) Set $e^j = dx^j, e_j = \partial/\partial x^j$ and we may assume that $G_2$-structure $\varphi$ is given by (2.1). Then, since $F_\nabla = \sqrt{-1}e^{123467} \neq 0$, we see that

$$F_\nabla^3 = 0, \quad F_\nabla \wedge * \varphi = \sqrt{-1}e^{123467} \neq 0,$$

which implies that $\nabla$ is not a $G_2$-dDT connection.

Next, we show that $H(\nabla) = 0$. Recall the notation in Section 3. Since $E_\nabla^2 = e^1 \otimes e_4 - e^4 \otimes e_1$, we see that $G_\nabla := \text{id} + e^1 \otimes e_1 + e^4 \otimes e_4$. These equations imply that $\det G_\nabla$ is constant and $(G_\nabla^{-1} \circ E_\nabla^2)^b$ is a parallel 2-form. Then, by the definition of $H(\nabla)$ in (3.3), we see that $H(\nabla) = 0$.

**Corollary 5.6.** Let $X^7$ be a compact connected manifold with a $G_2$-structure $\varphi \in \Omega^3$ with $d\varphi = 0$. Let $L \to X$ be a smooth complex line bundle with a Hermitian metric $h$. Then, we have the following.

1. Suppose that there is a $G_2$-dDT connection $\nabla_0$ such that $1 + \langle F^2_{\nabla_0}, * \varphi \rangle/2 > 0$ (resp. $< 0$). Then, for any $G_2$-dDT connection $\nabla$, we have $1 + \langle F^2_{\nabla}, * \varphi \rangle/2 > 0$ (resp. $< 0$).
2. Suppose that $L$ is a flat line bundle. Then, any $G_2$-dDT connection is a flat connection. In particular, the moduli space of $G_2$-dDT connections is $H^1(X, \mathbb{R})/2\pi H^1(X, \mathbb{Z})$. 
Remark 5.7. Corollary 5.6 (2) implies that we need to consider a non-flat line bundle to construct non-trivial examples of $G_2$-dDT connections on a compact connected 7-manifold with a closed $G_2$-structure.

Recently, Lotay and Oliveira [16] constructed nontrivial examples of $G_2$-dDT connections on the trivial complex line bundle over a 3-Sasakian 7-manifold. This does not contradict Corollary 5.6 (2) since the $G_2$-structure in [16] is not closed but coclosed.

6. The “mirror” of the special Lagrangian equality

In this section, we first recall the definition of deformed Hermitian Yang–Mills (dHYM) connections for a Kähler manifold and its moduli space from [11]. Then, we show a “mirror” of the special Lagrangian equality on a 3 or 4 dimensional Kähler manifold. We limit ourselves to these dimensions since we use the “mirror” of the Cayley equality in Theorem 4.2 and reduce from there. Using this “mirror” of the special Lagrangian equality, we show that dHYM connections with phase $e^{\sqrt{-1}\theta}$ are global minimizers of the volume functional $V$, just as special Lagrangian submanifolds are homologically volume-minimizing. As an application of this, we show that any dHYM connection with phase 1 of a flat line bundle over a compact connected Kähler manifold is a flat connection. Assuming that a Hermitian connection $\nabla$ satisfies $F_{\nabla}^{0,2}=0$, where $F_{\nabla}^{0,2}$ is the $(0,2)$-part of the curvature $F_{\nabla}$, we show that some of these results also hold in any dimension. We also relate $H(\nabla)$ in (3.3) to the angle function and explain the relation to [8].

6.1. Preliminaries for the moduli space. First, recall some definitions. Let $X$ be an $n$-dimensional Kähler manifold with a Kähler form $\omega$. Set $\Lambda^{p,q}=\Lambda^p(T^{1,0}X)^*\otimes \Lambda^q(T^{0,1}X)^*$. Define real vector bundles $[\Lambda^{p,q}]$ for $p\neq q$ and $[\Lambda^{p,p}]$ by

$[\Lambda^{p,q}]=((\Lambda^p+\Lambda^q)\cap \Lambda^{p+q}T^*X=\{\alpha\in \Lambda^p+\Lambda^q\mid \bar{\alpha}=\alpha\}$,

$[\Lambda^{p,p}]=\Lambda^{p,p}\cap \Lambda^{2p}T^*X=\{\alpha\in \Lambda^{p,p}\mid \bar{\alpha}=\alpha\}$.

Denote by $[\Omega^{p,q}]$ and $[\Omega^{p,p}]$ the space of sections of $[\Lambda^{p,q}]$ and $[\Lambda^{p,p}]$, respectively. Denote by $\pi^{[p,q]}$ the projection $\Omega^{p+q}\to [\Omega^{p,q}]$.

Let $L\to X$ be a smooth complex line bundle with a Hermitian metric $h$. Let $A_0$ be the space of Hermitian connections of $(L,h)$. We regard the curvature 2-form $F_{\nabla}$ of $\nabla$ as a $\sqrt{-1}\mathbb{R}$-valued closed 2-form on $X$. Fixing $\theta\in \mathbb{R}$, define a map $\mathcal{F}=(\mathcal{F}_1,\mathcal{F}_2) : A_0 \to [\Omega^{2,0}]\otimes \Omega^{2n}$ by

$$\mathcal{F}_{dHYM}(\nabla) = (\mathcal{F}_{dHYM}^1(\nabla), \mathcal{F}_{dHYM}^2(\nabla))$$

$$= (\pi^{[2,0]}(-\sqrt{-1}F_{\nabla}), \text{Im} (e^{-\sqrt{-1}\theta}(\omega+F_{\nabla}^2))) .$$

Each element of

$$\widehat{\mathcal{M}}_{dHYM} = \mathcal{F}_{dHYM}^{-1}(0)$$

is called a deformed Hermitian Yang–Mills (dHYM) connection with phase $e^{\sqrt{-1}\theta}$. As in Section 4, the group $G_U$ of unitary gauge transformations of $(L,h)$ acts on $A_0$. 

preserving the curvature 2-forms. Note that $G_U$-orbits are described explicitly as in Lemma 4.1 when $X$ is compact and connected. The moduli space $\mathcal{M}_{dHYM}$ of dHYM connections of $(L, h)$ is given by

\[ \mathcal{M}_{dHYM} := \hat{\mathcal{M}}_{dHYM} / G_U. \]

By [11, Theorem 1.2 (1)], $\mathcal{M}_{dHYM}$ is an empty set or a $b^1$-dimensional torus, where $b^1$ is the first Betti number of $X$.

6.2. Statements. If $\dim \mathbb{C} X = 3$, we have the following. The proof is given in Corollary A.7.

**Theorem 6.1.** For any $\nabla \in \mathcal{A}_0$, we have

\[
\left( \frac{\omega + F_\nabla}{3!} \right)^3 \left( \frac{(\omega + F_\nabla)^2}{2!} \right) = \det(id_{TX} + (-\sqrt{-1}F_\nabla)^2),
\]

where $(-\sqrt{-1}F_\nabla)^2 \in \Gamma(X, \text{End} TX)$ is defined by $u \mapsto (-\sqrt{-1}i(u)F_\nabla)^2$. In particular,

\[
\left| \text{Re} \left( e^{-\sqrt{-1}\theta} \left( \frac{\omega + F_\nabla}{3!} \right)^3 \right) \right| \leq \sqrt{\det(id_{TX} + (-\sqrt{-1}F_\nabla)^2)}
\]

for any $\theta \in \mathbb{R}$ and $\nabla \in \mathcal{A}_0$. The equality holds if and only if $\nabla$ is a dHYM connection with phase $e^{\sqrt{-1}\theta}$.

If $\dim \mathbb{C} X = 4$, we have the following. The proof is given in Proposition A.9.

**Theorem 6.2.** For any $\nabla \in \mathcal{A}_0$, we have

\[
\left( \frac{\omega + F_\nabla}{4!} \right)^4 \left( \frac{(\omega + F_\nabla)^3}{3!} \right)^2 + 8 \pi^{4,2} \left( \frac{(\omega + F_\nabla)^3}{3!} \right)^2 = \det(id_{TX} + (-\sqrt{-1}F_\nabla)^2),
\]

where $(-\sqrt{-1}F_\nabla)^2 \in \Gamma(X, \text{End} TX)$ is defined by $u \mapsto (-\sqrt{-1}i(u)F_\nabla)^2$. In particular,

\[
\left| \text{Re} \left( e^{-\sqrt{-1}\theta} \left( \frac{\omega + F_\nabla}{4!} \right)^4 \right) \right| \leq \sqrt{\det(id_{TX} + (-\sqrt{-1}F_\nabla)^2)}
\]

for any $\theta \in \mathbb{R}$ and $\nabla \in \mathcal{A}_0$. The equality holds if and only if $\nabla$ is a dHYM connection with phase $e^{\sqrt{-1}\theta}$.

**Remark 6.3.** By Theorems 6.1 and 6.2, it is immediate to see that for $n = 3, 4$, $\text{Re} \left( e^{-\sqrt{-1}\theta} (\omega + F_\nabla)^n / n! \right)$ is nowhere vanishing for any $\theta \in \mathbb{R}$ and dHYM connection $\nabla$ with phase $e^{\sqrt{-1}\theta}$.

By Theorems 6.1 and 6.2, we can prove the following as in Theorem 4.4, Corollaries 4.5 and 4.6.
Theorem 6.4. Let $X$ be a compact connected $n$-dimensional Kähler manifold for $n = 3, 4$. For any $\theta \in \mathbb{R}$ and $\nabla \in A_0$, we have
\[
\left| \text{Re} \left( e^{-\sqrt{-1}\theta} \left( [\omega] - 2\pi \sqrt{-1}c_1(L) \right)^n \right) \cdot [X] \right| = \left| \int_X \text{Re} \left( e^{-\sqrt{-1}\theta} (\omega + F_\nabla)^n \right) \right| \leq V(\nabla),
\]
where $c_1(L)$ is the first Chern class of $L$. The equality holds if and only if $\nabla$ is a dHYM connection with phase $e^{\sqrt{-1}\theta}$. In particular, for any dHYM connection $\nabla$ with phase $e^{\sqrt{-1}\theta}$, $V(\nabla)$ is topological.

Corollary 6.5. If $X$ is a compact connected $n$-dimensional Kähler manifold for $n = 3, 4$, any dHYM connection with phase $e^{\sqrt{-1}\theta}$ is a global minimizer of $V$.

Corollary 6.6. Let $X$ be a compact connected $n$-dimensional Kähler manifold for $n = 3, 4$. Let $L \to X$ be a smooth complex line bundle with a Hermitian metric $h$. Then, we have the following.

1. Suppose that there is a dHYM connection $\nabla_0$ with phase $e^{\sqrt{-1}\theta}$ such that $\text{Re} \left( e^{-\sqrt{-1}\theta} (\omega + F_{\nabla_0})^n / n! \right) > 0$ (resp. $< 0$). Then, for any dHYM connection $\nabla$ with phase $e^{\sqrt{-1}\theta}$, we have $\text{Re} \left( e^{-\sqrt{-1}\theta} (\omega + F_\nabla)^n / n! \right) > 0$ (resp. $< 0$).

2. Suppose that $L$ is a flat line bundle. Then, any dHYM connection with phase 1 is a flat connection. In particular, the moduli space of dHYM connections with phase 1 is $H^1(X, \mathbb{R}) / 2\pi H^1(X, \mathbb{Z})$.

Remark 6.7. A flat connection $\nabla$ satisfies
\[
\text{Im} \left( e^{-\sqrt{-1}\theta} (\omega + F_\nabla)^n \right) = \text{Im} \left( e^{-\sqrt{-1}\theta} \omega^n \right) = -\sin \theta \cdot \omega^n.
\]
Thus, a flat connection $\nabla$ is dHYM if and only if $\sin \theta = 0$. Then, we see that any dHYM connection with phase $\pm 1$ is a flat connection. Since the set of dHYM connections with phase 1 is the same as that with phase $-1$, we only state the phase 1 case in Corollary 6.6 (2). Though Corollary 6.6 (2) is a special case of [11, Theorem 1.2 (1)], we put this to emphasize the correspondence with Corollary 5.6 (2).

Theorems 6.1 and 6.2 are considered to be a “mirror” of the special Lagrangian equality [6, p.90, Remark]. We can also deduce these identities on a trivial torus bundle via the real Fourier–Mukai transform as in [12, Lemmas 4.3 and 5.5]. For general dimensions, we conjecture the following from the special Lagrangian equality [6, p.90, Remark] and Theorems 6.1 and 6.2.

1. When $\dim_{\mathbb{C}} X = 2p+1$, $\det(id_{\nabla} + (-\sqrt{-1}F_\nabla)^2)$ will be described by a linear combination of $\left| \pi_{[2p+1,2(p-k)+1]}^{[p]} ((\omega + F_\nabla)^{p-k} / (p-k)!) \right|^2$ for $0 \leq k \leq p$.

2. When $\dim_{\mathbb{C}} X = 2p$, $\det(id_{\nabla} + (-\sqrt{-1}F_\nabla)^2)$ will be described by a linear combination of $\left| \pi_{[2p,2(p-k)]}^{[p]} ((\omega + F_\nabla)^{p-k} / (p-k)!) \right|^2$ for $0 \leq k \leq p$.

We did not try to prove them because the proof will require a large amount of computations and we could not find an interesting application, while there are applications
for 3 and 4 dimensional cases in Section 7. However, if we assume from the beginning that the $(0,2)$-part $F_{\nabla}^{0,2}$ of $F_{\nabla}$ vanishes, which is equivalent to $\pi^{[2,0]}(F_{\nabla}) = 0$, computations become easier considerably and we obtain the similar results as we see in the next subsection.

6.3. The case $F_{\nabla}^{0,2} = 0$. In this subsection, we consider Hermitian connections $\nabla \in \mathcal{A}_0$ with $F_{\nabla}^{0,2} = 0$. The condition $F_{\nabla}^{0,2} = 0$ will correspond to the Lagrangian condition via mirror symmetry. We first recall some definitions and identities from [11, Section 4.1]. Set

\[(6.4) \quad \mathcal{A} = \{ \nabla \in \mathcal{A}_0 \mid F_{\nabla}^{0,2} = 0 \} = \nabla_0 + \sqrt{-1}(Z^1 \oplus d_c^0) \cdot \text{id}_L,\]

where $\nabla_0 \in \mathcal{A}$ is any fixed connection, $Z^1$ is the space of closed 1-forms on $X$ and $d_c$ is the complex differential. The second equation follows from [11, Lemma A.2].

For $\nabla \in \mathcal{A}$, define a Hermitian metric $\eta_{\nabla}$ and its associated 2-form $\omega_{\nabla}$ by

\[\eta_{\nabla} = (\text{id}_{TX} + (-\sqrt{-1} F_{\nabla})^2) g, \quad \omega_{\nabla} = (\text{id}_{TX} + (-\sqrt{-1} F_{\nabla})^2)^* \omega\]

and define $\zeta_{\nabla} : X \to \mathbb{C}$ by

\[(\omega + F_{\nabla})^n = \zeta_{\nabla} \omega^n.\]

By [11, Lemma 4.4], we have $|\zeta_{\nabla}| \geq 1$. Then, there exist $r_{\nabla} : X \to [1, \infty)$ and $\theta_{\nabla} : X \to \mathbb{R}/2\pi \mathbb{Z}$ such that

\[(6.5) \quad \zeta_{\nabla} = r_{\nabla} e^{\sqrt{-1} \theta_{\nabla}}.\]

We first prove the following.

**Proposition 6.8.** Let $(X, \omega, g, J)$ be a Kähler manifold with $\dim_{\mathbb{C}} X = n$ and $L \to X$ be a smooth complex line bundle with a Hermitian metric $h$. Then, for any Hermitian connection $\nabla \in \mathcal{A}$, we have

\[
\left| \frac{\omega + F_{\nabla}}{n!} \right|^2 = \det(\text{id}_{TX} + (-\sqrt{-1} F_{\nabla})^2). \]

**Proof.** We prove this using computations in [11, Section 4.1]. Since the statement is pointwise, we fix a point $p \in X$ arbitrarily. Then, there exists an orthonormal basis $u_1, \ldots, u_n, v_1, \ldots, v_n$ satisfying $v_i = J u_i$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that $(-\sqrt{-1} F_{\nabla})^2 = \sum_{i=1}^n \lambda_i (u^i \otimes v_i - v^i \otimes u_i)$ at $p$, where $\{u^i, v^i\}_{i=1}^n$ is the dual basis of $\{u_i, v_i\}_{i=1}^n$. Then, by [11, (4.6)], we have

\[
\left| \frac{\omega + F_{\nabla}}{n!} \right|^2 = r_{\nabla}^2 = \prod_{i=1}^n (1 + \lambda_i^2) = \det(\text{id}_{TX} + (-\sqrt{-1} F_{\nabla})^2). \]

\[\square\]

Theorems 6.1 and 6.2 imply Proposition 6.8 if $\dim_{\mathbb{C}} X = 3$ or 4. Indeed, if $\nabla \in \mathcal{A}$, we have $F_{\nabla} \in \Omega^{1,1}$. This implies that $\omega + F_{\nabla} \in \Omega^{1,1}$, and hence, $\pi^{[3,1]}((\omega + F_{\nabla})^2) = 0$ and $\pi^{[4,2]}((\omega + F_{\nabla})^2) = \pi^{[14,0]}((\omega + F_{\nabla})^2) = 0$.

**Remark 6.9.** Proposition 6.8 is essentially proved in [8, p.875]. We explain the minor difference from [8].
(1) The operator \(-\sqrt{-1}K : T^{1,0}X \to T^{1,0}X\) as in [8] is given by extending \((-\sqrt{-1}F_\nabla)^2\) complex linearly and restricting it to \(T^{1,0}X\). Since \(F_\nabla^0,2 = 0\), \((-\sqrt{-1}F_\nabla)^2\) commutes with \(J\), and hence, \(K(T^{1,0}X) \subset T^{1,0}X\). Then, we see that

\[
\det_R(\text{id}_{T_X} + (-\sqrt{-1}F_\nabla)^2) = \det_C(\text{id}_{T_X} + (-\sqrt{-1}F_\nabla)^2 : TX \otimes \mathbb{C} \to TX \otimes \mathbb{C})
\]

\[
= \det_C(\text{id}_{T^{1,0}X} - \sqrt{-1}K) \cdot \det_C(\text{id}_{T^{1,0}X} - \sqrt{-1}K) = |\det_C(\text{id}_{T^{1,0}X} - \sqrt{-1}K)|^2.
\]

(2) In [8], \(\nabla \in \mathcal{A}_0\) is called a dHYM connection with phase \(e^{\sqrt{-1}\theta}\) if \(F_\nabla^0,2 = 0\) and \(\Im \left( e^{-\sqrt{-1}\theta}(\omega - F_\nabla)^n \right) = 0\), which is different from (6.1) by sign of \(F_\nabla\).

This causes the difference in the sign of \(K\) from [8]. In particular, the angle function \(\theta\) in [8, (2.4)] corresponds to \(-\theta_\nabla\) in our paper.

By Proposition 6.8, we see that the following statements hold for compact and connected Kähler manifolds \(X\) of any dimension.

(1) For any \(\theta \in \mathbb{R}\) and \(\nabla \in \mathcal{A}\), we have

\[
\left| \Re \left( e^{-\sqrt{-1}\theta} \left( \frac{[\omega] - 2\pi \sqrt{-1}c_1(L)}{n!} \right)^n \right) \cdot [X] \right| = \left| \int_X \Re \left( e^{-\sqrt{-1}\theta} \left( \frac{[\omega] + F_\nabla}{n!} \right)^n \right) \right| \leq V(\nabla) .
\]

The equality holds if and only if \(\nabla\) is a dHYM connection with phase \(e^{\sqrt{-1}\theta}\).

(2) For any dHYM connection \(\nabla\) with phase \(e^{\sqrt{-1}\theta}\), \(V(\nabla)\) is topological.

(3) Any dHYM connection with phase \(e^{\sqrt{-1}\theta}\) is a minimizer of \(V|_\mathcal{A} : \mathcal{A} \to \mathbb{R}\).

(4) The statement of Corollary 6.6 holds for any dimension of \(X\).

We can regard Corollary 6.5 as a generalization of (3) above. That is, any dHYM connection with phase \(e^{\sqrt{-1}\theta}\) minimizes \(V\) not only on \(\mathcal{A}\) but also on \(\mathcal{A}_0\) in the case of dimension 3 or 4.

Next, we relate our mean curvature \(H_\nabla\) in (3.3) to \(d\theta_\nabla\). They are expected to be related closely as in the case of Lagrangian submanifolds. Indeed, Jacob and Yau [8] considers the space of Chern connections of Hermitian metrics on a holomorphic line bundle and define the mean curvature by the exterior derivative of the angle function.

**Theorem 6.10.** Suppose that \(X\) is compact and connected Kähler manifold. For any \(\nabla \in \mathcal{A}\), we have

\[
H(\nabla) = -(\det G_\nabla)^{1/4} (G_\nabla^{-1})^*(Jd\theta_\nabla),
\]

where \(G_\nabla\) is defined by (3.1), \(J\) is the complex structure on \(X\) and we set \(Jd\theta_\nabla = d\theta_\nabla(J(\cdot))\).

In particular, for \(\nabla \in \mathcal{A}\), \(H(\nabla) = 0\) if and only if \(\nabla\) is a dHYM connection with phase \(e^{\sqrt{-1}\theta}\) for some \(\theta \in \mathbb{R}\).

**Proof.** In the proof, we use Greek indices \(\mu, \nu, \xi\) to run over 1, \cdots, 2n and Latin indices \(i, j\) range over 1, \cdots, \(n\). Use the notation of Section 3. We denote by \(D\) the Levi-Civita connection with respect to \(g\) to distinguish from elements in \(\mathcal{A}_0\). Fix
$p \in X$. Take an orthonormal frame $\{e_{\mu}\}_{\mu=1}^{2n}$ of $TX$ around $p$ with respect to $g$ such that

$$D_{e_{\mu}}e_{\nu} = 0, \quad e_{2i} = J(e_{2i-1}), \quad E_{\nabla} = \sum_{i=1}^{n} \lambda_i e^{2i-1} \wedge e^{2i}$$

for $\lambda_i \in \mathbb{R}$ at $p \in X$. Denote by $\{e^{\mu}\}_{\mu=1}^{2n}$ the dual. Then, we see that at $p \in X$

$$E_{\nabla}^{\mu} = \sum_{i=1}^{n} \lambda_i (e^{2i-1} \otimes e_{2i} - e^{2i} \otimes e_{2i-1}),$$

(6.6)

$$G_{\nabla} = \sum_{i=1}^{n} (1 + \lambda_i^2) (e^{2i-1} \otimes e_{2i-1} + e^{2i} \otimes e_{2i}),$$

$$G_{\nabla}^{-1} = \sum_{i=1}^{n} \frac{1}{1 + \lambda_i^2} (e^{2i-1} \otimes e_{2i-1} + e^{2i} \otimes e_{2i}).$$

Set

$$D E_{\nabla} = \frac{1}{2} \sum_{\mu, \nu, \xi=1}^{2n} F_{\mu \nu \xi} e^{\xi} \otimes e^{\mu} \wedge e^{\nu}$$

with $F_{\mu \nu \xi} = -F_{\nu \mu \xi}$. Note that $F_{\mu \nu \xi} \in \mathbb{R}$. For simplicity, set $D_{\xi} E_{\nabla} = D_{e_{\xi}} E_{\nabla}$ and $D_{\xi} E_{\nabla}^{\xi} = D_{e_{\xi}} E_{\nabla}^{\xi}$. Then,

$$D_{\xi} E_{\nabla}^{\xi} = \sum_{\mu, \nu=1}^{2n} F_{\mu \nu \xi} e^{\mu} \otimes e_{\nu}.$$

Since $D_{\xi} E_{\nabla}^{\xi} \in \Omega^{1,1}$, we have at $p \in X$

$$F_{2i-1,2j-1;\xi} = F_{2i,2j;\xi}, \quad F_{2i-1,2j;\xi} = -F_{2i,2j-1;\xi}.$$

(6.8)

By the Bianchi identity, we have

$$0 = \sum_{\xi=1}^{2n} e^{\xi} \wedge D_{\xi} E_{\nabla} = \sum_{\mu, \nu, \xi=1}^{2n} (F_{\mu \nu \xi} / 2) e^{\mu} \wedge e^{\nu} \wedge e^{\xi},$$

and hence,

$$F_{\mu \nu;\xi} + F_{\nu \xi;\mu} + F_{\xi;\mu \nu} = 0.$$

(6.9)

Using these equations, we compute $H(\nabla)$ at $p \in X$. By (3.3), we have

$$H(\nabla) = -d^* \left( (\det G_{\nabla})^{1/4} \left( G_{\nabla}^{-1} \circ E_{\nabla}^{\xi} \right)^{\flat} \right)$$

$$= \sum_{\xi=1}^{2n} e_{\xi} (\det G_{\nabla})^{1/4} \cdot i(\xi) \left( G_{\nabla}^{-1} \circ E_{\nabla}^{\xi} \right)^{\flat} + (\det G_{\nabla})^{1/4} i(\xi) D_{\xi} \left( G_{\nabla}^{-1} \circ E_{\nabla}^{\xi} \right)^{\flat}. $$

(6.10)
At \( p \in X \), we have

\[
e_\xi (\det G_\nabla)^{1/4} = \frac{(\det G_\nabla)^{1/4}}{4} \text{tr} \left( G_\nabla^{-1} \circ D_\xi G_\nabla \right)
\]

\[
= \frac{(\det G_\nabla)^{1/4}}{4} \text{tr} \left( G_\nabla^{-1} \circ \left( -(D_\xi E_\nabla^\xi) \circ E_\nabla^\xi - E_\nabla^\xi \circ (D_\xi E_\nabla^\xi) \right) \right)
\]

\[
= - \frac{(\det G_\nabla)^{1/4}}{2} \text{tr} \left( (D_\xi E_\nabla^\xi) \circ G_\nabla^{-1} \circ E_\nabla^\xi \right),
\]

where we use (3.2). Substituting (6.6) and (6.7) into this, it follows that

\[
e_\xi (\det G_\nabla)^{1/4} = - \frac{(\det G_\nabla)^{1/4}}{2} \sum_{i=1}^{n} \left\{ g \left( \left( (D_\xi E_\nabla^\xi) \circ G_\nabla^{-1} \circ E_\nabla^\xi \right) (e_{2i-1}), e_{2i-1} \right) 
+ g \left( \left( (D_\xi E_\nabla^\xi) \circ G_\nabla^{-1} \circ E_\nabla^\xi \right) (e_{2i}), e_{2i} \right) \right\}
\]

\[
= (\det G_\nabla)^{1/4} \sum_{i=1}^{n} \frac{\lambda_i}{1 + \lambda_i^2} F_{2i-1,2i;\xi}.
\]

Hence, we obtain at \( p \in X \)

(6.11)

\[
\sum_{\xi=1}^{2n} e_\xi (\det G_\nabla)^{1/4} \cdot i(e_\xi) \left( G_\nabla^{-1} \circ E_\nabla^\xi \right)^b
\]

\[
= \sum_{j=1}^{n} e_{2j-1} (\det G_\nabla)^{1/4} \cdot \left( (G_\nabla^{-1} \circ E_\nabla^\xi) (e_{2j-1}) \right)^b + e_{2j} (\det G_\nabla)^{1/4} \cdot \left( (G_\nabla^{-1} \circ E_\nabla^\xi) (e_{2j}) \right)^b
\]

\[
= (\det G_\nabla)^{1/4} \sum_{i,j=1}^{n} \frac{\lambda_i \lambda_j}{(1 + \lambda_i^2)(1 + \lambda_j^2)} \left( F_{2i-1,2i;2j-1} e^{2j-1} - F_{2i-1,2i;2j} e^{2j-1} \right).
\]

Next, we compute

\[
\sum_{\xi=1}^{2n} i(e_\xi) D_\xi \left( G_\nabla^{-1} \circ E_\nabla^\xi \right)^b = \sum_{\xi=1}^{2n} i(e_\xi) \left( (D_\xi G_\nabla^{-1}) \circ E_\nabla^\xi + G_\nabla^{-1} \circ (D_\xi E_\nabla^\xi) \right)^b.
\]

By the same computation as in (3.6), we have

\[
D_\xi G_\nabla^{-1} = G_\nabla^{-1} \circ \left( (D_\xi E_\nabla^\xi) \circ E_\nabla^\xi + E_\nabla^\xi \circ (D_\xi E_\nabla^\xi) \right) \circ G_\nabla^{-1}.
\]

Then, by (3.9) and (3.2), it follows that

\[
i(e_\xi) D_\xi \left( G_\nabla^{-1} \circ E_\nabla^\xi \right)^b
\]

\[
= \left( \left( G_\nabla^{-1} \circ (D_\xi E_\nabla^\xi) \circ G_\nabla^{-1} + E_\nabla^\xi \circ G_\nabla^{-1} \circ (D_\xi E_\nabla^\xi) \circ G_\nabla^{-1} \circ E_\nabla^\xi \right) (e_\xi) \right)^b.
\]
By (6.6) and (6.7), we have at \( p \in X \)

\[
(G^{-1}_p \circ (D_\xi E^2_p) \circ G^{-1}_p)(e_{2i-1}) = \sum_{j=1}^{n} \frac{F_{2i-1,2j-1;2i}e_{2j-1} + F_{2i-1,2j;2i}e_{2j}}{(1 + \lambda_i^2)(1 + \lambda_j^2)},
\]

\[
(G^{-1}_p \circ (D_\xi E^2_p) \circ G^{-1}_p)(e_{2i}) = \sum_{j=1}^{n} \frac{F_{2i,2j-1;2i}e_{2j-1} + F_{2i,2j;2i}e_{2j}}{(1 + \lambda_i^2)(1 + \lambda_j^2)},
\]

which implies that

\[
(E^2_p \circ G^{-1}_p \circ (D_\xi E^2_p) \circ G^{-1}_p \circ E^2_p)(e_{2i-1}) = \sum_{j=1}^{n} \frac{\lambda_i \lambda_j (F_{2i,2j-1;2i}e_{2j} - F_{2i,2j;2i}e_{2j-1})}{(1 + \lambda_i^2)(1 + \lambda_j^2)},
\]

\[
(E^2_p \circ G^{-1}_p \circ (D_\xi E^2_p) \circ G^{-1}_p \circ E^2_p)(e_{2i}) = \sum_{j=1}^{n} \frac{-\lambda_i \lambda_j (F_{2i-1,2j-1;2i}e_{2j} - F_{2i-1,2j;2i}e_{2j-1})}{(1 + \lambda_i^2)(1 + \lambda_j^2)}.
\]

Hence, it follows that at \( p \in X \)

\[
\sum_{\xi=1}^{2n} i(e_\xi) D_\xi \left( G^{-1}_p \circ E^2_p \right) = \sum_{i,j=1}^{n} \frac{F_{2i-1,2j-1,2i}e_{2j-1} + F_{2i-1,2j,2i}e_{2j} + F_{2i,2j-1,2i}e_{2j-1} + F_{2i,2j,2i}e_{2j}}{(1 + \lambda_i^2)(1 + \lambda_j^2)}
\]

\[
+ \sum_{i,j=1}^{n} \frac{\lambda_i \lambda_j (F_{2i,2j-1,2i}e_{2j} - F_{2i,2j,2i}e_{2j-1} - (F_{2i-1,2j-1,2i}e_{2j} - F_{2i-1,2j,2i}e_{2j-1}))}{(1 + \lambda_i^2)(1 + \lambda_j^2)}.
\]

By (6.8) and (6.9), we have

\[
F_{2i-1,2j-1,2i} + F_{2i,2j-1,2i} = F_{2i-1,2j,2i} - F_{2i-1,2j,2i} = -F_{2i-1,2j;2i},
\]

\[
F_{2i-1,2j;2i} + F_{2i,2j;2i} = -F_{2i-1,2j-1;2i} + F_{2i-1,2j-1;2i}
\]

\[
= F_{2i-1,2j-1;2i} + F_{2i-1,2j-1;2i} = F_{2i-1,2j-1;2i},
\]

\[
F_{2i,2j-1,2i} - F_{2i,2j-1,2i} = F_{2i,2j-1,2i} + F_{2j-1,2i-1,2i} = -F_{2i,2j-1,2i},
\]

\[
-F_{2i,2j-1,2i} + F_{2i,2j-1,2i} = F_{2j,2i-1,2i} + F_{2i-1,2j,2i} = F_{2i-1,2j-1;2i}.
\]

Hence, we obtain at \( p \in X \)

(6.12)

\[
\sum_{\xi=1}^{2n} i(e_\xi) D_\xi \left( G^{-1}_p \circ E^2_p \right) = \sum_{i,j=1}^{n} \left( \frac{-F_{2i-1,2j,2i}e_{2j-1} + F_{2i-1,2j,2i}e_{2j}}{(1 + \lambda_i^2)(1 + \lambda_j^2)} + \frac{\lambda_i \lambda_j (-F_{2i-1,2j,2i}e_{2j} + F_{2i-1,2j,2i}e_{2j-1})}{(1 + \lambda_i^2)(1 + \lambda_j^2)} \right).
\]
Then, by (6.10), (6.11) and (6.12), we obtain at $p \in X$

$$H(\nabla) = \sum_{i,j=1}^{n} \frac{(\det G_{ij})^{1/4} (-F_{2i-1,2i,j+1} e^{2j} + F_{2i-1,2i,j-1} e^{2j})}{(1 + \lambda_i^2)/(1 + \lambda_j^2)}.$$  

(6.13)

Now, we relate (6.13) to $d\theta_\nabla$. We first prove the following.

**Lemma 6.11.** At $p \in X$, we have

$$e_\xi(\theta_\nabla) = \sum_{i=1}^{n} \frac{F_{2i-1,2i;\xi}}{1 + \lambda_i^2}.$$  

**Proof.** Since $\zeta_\nabla = r_\nabla e^{\sqrt{-1}\phi_\nabla}$, we have

$$e_\xi(\theta_\nabla) = e_\xi(\text{Im}(\log \zeta_\nabla)) = \text{Im} \left( \frac{e_\xi(\zeta_\nabla)}{\zeta_\nabla} \right).$$

Differentiating $(\omega + \sqrt{-1}E_\nabla)^n = \zeta_\nabla \omega^n$, we have

$$n \sqrt{-1} D_\xi E_\nabla \wedge (\omega + \sqrt{-1}E_\nabla)^{n-1} = e_\xi(\zeta_\nabla) \omega^n.$$  

Dividing both hand sides by $\zeta_\nabla$ and take the imaginary part, we obtain

$$n D_\xi E_\nabla \wedge \text{Re} \left( \frac{(\omega + \sqrt{-1}E_\nabla)^{n-1}}{\zeta_\nabla} \right) = e_\xi(\theta_\nabla) \omega^n.$$  

By [11, (4.3)], we have $\text{Re} \left( (\omega + \sqrt{-1}E_\nabla)^{n-1}/\zeta_\nabla \right) = \omega^{n-1} r_\nabla^2 = (n-1)! r_\nabla^{n-2} \omega^{n}$, where $*_{\nabla}$ is the Hodge star defined by $\eta_{\nabla}$. Using $\omega_{\nabla}^{n} = r_\nabla^{2} \omega^{n}$ by [11, (4.2)], we obtain

$$\langle D_\xi E_\nabla, \omega_{\nabla} \rangle_{\nabla} = e_\xi(\theta_\nabla),$$  

(6.14)

where $\langle \cdot, \cdot \rangle_{\nabla}$ is the metric on $T^*X$ induced from $\eta_{\nabla}$.

We write the left hand side of (6.14) at $p \in X$. Denote by $\langle \cdot, \cdot \rangle$ is the metric on $T^*X$ induced from $g$. We compute at $p \in X$

$$\langle D_\xi E_\nabla, \omega_{\nabla} \rangle = \langle ((\text{id}_{TX} + E_{\nabla}^{*})^{-1}) D_\xi E_\nabla, \omega \rangle$$

$$= \sum_{i=1}^{n} \langle D_\xi E_\nabla ((\text{id}_{TX} + E_{\nabla}^{*})^{-1}(e_{2i-1}), (\text{id}_{TX} + E_{\nabla}^{*})^{-1}(e_{2i}) \rangle.$$  

By (6.6), we have

$$(\text{id}_{TX} + E_{\nabla}^{*})^{-1}(e_{2i-1}) = \frac{e_{2i-1} - \lambda_i e_{2i}}{1 + \lambda_i^2}, \quad (\text{id}_{TX} + E_{\nabla}^{*})^{-1}(e_{2i}) = \frac{\lambda_i e_{2i-1} + e_{2i}}{1 + \lambda_i^2}.$$  

Hence, we obtain

$$\langle D_\xi E_\nabla, \omega_{\nabla} \rangle = \sum_{i=1}^{n} \frac{\langle D_\xi E_\nabla (e_{2i-1}, e_{2i}) \rangle}{1 + \lambda_i^2} = \sum_{i=1}^{n} \frac{F_{2i-1,2i;\xi}}{1 + \lambda_i^2}.$$  

By (6.14), the proof of Lemma 6.11 is completed.  

\square
By Lemma 6.11, we compute at $p \in X$
\[ Jd\theta_\nabla = \sum_{j=1}^{n} e_{2j-1}(\theta_\nabla)e^{2j-1}(J(\cdot)) + e_{2j}(\theta_\nabla)e^{2j}(J(\cdot)) = \sum_{i,j=1}^{n} \frac{-F_{2i-1,2j-1}e^{2j} + F_{2i-1,2j}e^{2j-1}}{1 + \lambda_i^2}. \]

Then, by (6.6), we obtain
\[ (G_\nabla^{-1})^*(Jd\theta_\nabla) = \sum_{i,j=1}^{n} \frac{-F_{2i-1,2j-1}e^{2j} + F_{2i-1,2j}e^{2j-1}}{(1 + \lambda_i^2)(1 + \lambda_j^2)}. \]

Then, by (6.13) and (6.15), the proof is completed. \(\Box\)

**Remark 6.12.** We can recover [8, Proposition 3.4] from (3.3) and Theorem 6.10.

Suppose that $L \rightarrow X$ is a holomorphic line bundle and the holomorphic structure is fixed. Set
\[ \mathcal{A}_{met} = \{ h \mid \text{a Hermitian metric of } L \}. \]

Fixing $h \in \mathcal{A}_{met}$, we have $\mathcal{A}_{met} = \{ e^{-f}h \mid f \in \Omega^0 \}$. Denote by $\mathcal{A}_0$ the space of Hermitian connections of $(L, h)$. Define $\mathcal{I} : \mathcal{A}_{met} \rightarrow \mathcal{A}_0$ by
\[ \mathcal{I}(e^{-f}h) = \nabla - \frac{\sqrt{-1}}{2} d_c f, \]

where $\nabla$ is the Chern connection of $h$. Since $2\partial\bar{\partial} = -\sqrt{-1}dd_c$ in our notation given in [11, Appendix A.2], we see that $F_{h'} = F_{\mathcal{I}(h')}$ for any $h' \in \mathcal{A}_{met}$, where $F_{h'}$ is the curvature 2-form of the Chern connection of $h' \in \mathcal{A}_{met}$. Then, the volume functional $V_{JY} : \mathcal{A}_{met} \rightarrow \mathbb{R}$ in [8, Definition 3.1] is given by
\[ V_{JY} = V \circ \mathcal{I}. \]

Thus,
\[ \left. \frac{d}{dt} V_{JY}(e^{-tf}h) \right|_{t=0} = \delta_\nabla V \left( -\frac{\sqrt{-1}}{2} d_c f \right) \]

is computed in [8, Proposition 3.4]. By Theorem 6.10, we have
\[ \delta_\nabla V \left( -\frac{\sqrt{-1}}{2} d_c f \right) = \frac{1}{2} \langle d_c f, H(\nabla) \rangle_{L^2} = -\frac{1}{2} \int_{X} \langle (G_\nabla^{-1})^*(Jd\theta_\nabla), d_c f \rangle (\det G_\nabla)^{1/4} \text{vol}_g. \]

Denote by $^tT$ the transpose of a linear map $T : T^*X \rightarrow T^*X$ with respect to $g$. By (3.1), we have
\[ (G_\nabla^{-1})^* = \quad \left( \left( \text{id}_{T^*X} + E^g_\nabla \right)^{-1} \right)^* \left( \left( \text{id}_{T^*X} + E^g_\nabla \right)^{-1} \right)^*. \]

Then,
\[ \langle (G_\nabla^{-1})^*(Jd\theta_\nabla), d_c f \rangle = \left\langle \left( \left( \text{id}_{T^*X} + E^g_\nabla \right)^{-1} \right)^* (Jd\theta_\nabla), \left( \left( \text{id}_{T^*X} + E^g_\nabla \right)^{-1} \right)^* d_c f \right\rangle 
= \langle Jd\theta_\nabla, d_c f \rangle_{\nabla}, \]
where $\langle \cdot, \cdot \rangle_\nabla$ is the metric on $T^*X$ induced from $\eta_\nabla$. Since $\eta_\nabla$ is Hermitian and $d_c f = -J df$, we have

$$\langle J d \theta_\nabla, d_c f \rangle_\nabla = -\langle d \theta_\nabla, df \rangle_\nabla.$$

Hence, by (3.1) again, we obtain

$$\frac{d}{dt} V_{JY} (e^{-tf} h) \bigg|_{t=0} = \delta \nabla V \left( -\frac{\sqrt{-1}}{2} d_c f \right) = \frac{1}{2} \int_X \langle d \theta_\nabla, df \rangle_\nabla \sqrt{\det (id_{TX} + E^2_\nabla)} \text{vol}_g.$$

Note that the formula above differs from [8, Proposition 3.4] by a constant because we use the convention that $|a + \bar{a}|^2 = 2|a|^2$ for $a \in (T^{1,0}X)^*$ (See also [8, (3.7)]) and adopt the different definition for $d$HYM connections as explained in Remark 6.9 (2).

### 7. Holonomy reductions

If a Spin(7)- or $G_2$-manifold admits a further reduction of the holonomy group, we have Spin(7)- or $G_2$-dDT connections from $G_2$-dDT or dHYM connections, respectively. In this section, as applications of results proved in previous sections, we show that all of Spin(7)- or $G_2$-dDT connections essentially arise in this way if the holonomy group reduces and the manifold is compact and connected.

#### 7.1. Reduction from Spin(7) to $G_2$

Let $(Y^7, \varphi)$ be a $G_2$-manifold and $L \to Y^7$ be a smooth complex line bundle with a Hermitian metric $h$. Set $X^8 = S^1 \times Y^7$ and denote by

$$\pi_{S^1} : X^8 = S^1 \times Y^7 \to S^1, \quad \pi_Y : X^8 = S^1 \times Y^7 \to Y^7$$

the projections. The 8-manifold $X^8 = S^1 \times Y^7$ is a Spin(7)-manifold with the Spin(7)-structure

$$\Phi = \pi_{S^1}^* dx \wedge \pi_Y^* \varphi + \pi_Y^* \ast_7 \varphi,$$

where $x$ is a coordinate of $S^1$ and $\ast_7$ is the Hodge star on $Y^7$. In [12, Lemma 7.1], we prove that a Hermitian connection $\nabla$ of $(L, h)$ is a $G_2$-dDT connection if and only if the pullback $\pi_Y^* \nabla$ is a Spin(7)-dDT connection of $\pi_Y^* L$. We can improve this if $Y^7$ is compact and connected as follows.

**Theorem 7.1.** Suppose that $Y^7$ is a compact and connected $G_2$-manifold.

1. The pullback $\pi_Y^* L \to X^8$ admits a Spin(7)-dDT connection if and only if $L \to Y^7$ admits a $G_2$-dDT connection.
2. For any Spin(7)-dDT connection $\nabla$ of $\pi_Y^* L$, there exist a $G_2$-dDT connection $\nabla$ of $L$ and a closed 1-form $\xi \in \sqrt{-1} \Omega^1(X^8)$ such that $\nabla = \pi_Y^* \nabla + \xi$.
3. Denote by $\mathcal{M}_{\text{Spin}(7)}$ the moduli space of Spin(7)-dDT connections of $\pi_Y^* L$ and denote by $\mathcal{M}_{G_2}$ the moduli space of $G_2$-dDT connections of $L$ as defined by (4.3) and (5.3). Then, $\mathcal{M}_{\text{Spin}(7)}$ is homeomorphic to $S^1 \times \mathcal{M}_{G_2}$.

**Remark 7.2.** An analogous result for Spin(7)-instantons is given in [24, Theorem 1.17], where the moduli space of irreducible Spin(7)-instantons of $\pi_Y^* L$ is homeomorphic to the product of $S^1$ and the moduli space of irreducible $G_2$-instantons of
L. An analogous result for compact Cayley submanifolds is given in [21, Proposition 5.20], where submanifolds are allowed to have boundaries. This implies that the moduli space of all local Cayley deformations of $S^1 \times A^3$, where $A^3$ is a given compact associative submanifold in $Y^7$, is identified with the moduli space of all local associative deformations of $A^3$.

**Proof.** We first prove (1). By [12, Lemma 7.1], for any $G_2$-dDT connection $\nabla$ of $(L,h)$, $\pi_Y^* \nabla$ is a Spin(7)-dDT connection of $\pi_Y^* L$. Thus, we only have to prove the converse.

Suppose that a Spin(7)-dDT connection $\tilde{\nabla}$ of $\pi_Y^* L$ is given. By Remark 4.3 and the connectedness of $X^8$, we may assume that $1 + \langle F^2_{\tilde{\nabla}}, \Phi \rangle / 2 + * F^4_{\tilde{\nabla}} / 24 > 0$ everywhere without loss of generality. Then, Theorem 4.4 implies that

$$V(\tilde{\nabla}) = \int_X \left( 1 + \frac{1}{2} \langle F^2_{\tilde{\nabla}}, \Phi \rangle + \frac{* F^4_{\tilde{\nabla}}}{24} \right) \text{vol}_g$$

$$= \int_X \left( 1 + \frac{1}{2} \langle F^2_{\tilde{\nabla}}, \pi_Y^* dx \wedge \pi_Y^* \varphi + \pi_Y^* *_7 \varphi \rangle \right) \text{vol}_g + \frac{F^4_{\tilde{\nabla}}}{24}.$$ 

We compute

$$\int_X F^4_{\tilde{\nabla}} = \left( -2 \pi \sqrt{-1} c_1(\pi_Y^* L)^4 \right) \cdot [X] = 16 \pi^4 \left( \pi_Y^* c_1(L)^4 \right) \cdot [X] = 0$$

since $c_1(L)^4 \in H^8_{dR}(Y^7) = \{0\}$. Similarly, since $\langle F^2_{\tilde{\nabla}}, \pi_Y^* dx \wedge \pi_Y^* \varphi \rangle \text{vol}_g = F^2_{\tilde{\nabla}} \wedge \pi_Y^* *_7 \varphi$, we have

$$\int_X \langle F^2_{\tilde{\nabla}}, \pi_Y^* dx \wedge \pi_Y^* \varphi \rangle \text{vol}_g = \left( -2 \pi \sqrt{-1} c_1(\pi_Y^* L)^2 \cup [\pi_Y^* *_7 \varphi] \right) \cdot [X]$$

$$= -4 \pi^2 \pi_Y^* \left( c_1(L)^2 \cup [*_7 \varphi] \right) \cdot [X] = 0$$

since $c_1(L)^2 \cup [*_7 \varphi] \in H^8_{dR}(Y^7) = \{0\}$. Hence, we obtain

$$\int_X \left( 1 + \frac{1}{2} \langle F^2_{\tilde{\nabla}}, \pi_Y^* *_7 \varphi \rangle \right) \text{vol}_g = V(\tilde{\nabla}).$$

By Lemma A.6, (7.1) implies that $1 + \langle F^2_{\tilde{\nabla}}, \pi_Y^* *_7 \varphi \rangle / 2 = \sqrt{\det(id_{TX} + (-\sqrt{-1} F^2_{\tilde{\nabla}})^2)}$ and

$$i \left( \frac{\partial}{\partial x} \right) F_{\tilde{\nabla}} = 0 \quad \text{and} \quad F_{\tilde{\nabla}} \wedge \pi_Y^* *_7 \varphi + \frac{1}{6} F^3_{\tilde{\nabla}} = 0.$$ 

Now, fix a Hermitian connection $\nabla_0$ of $L$ and set $\tilde{\nabla} = \pi_Y^* \nabla_0 + \tilde{\gamma}$ for $\tilde{\gamma} \in \sqrt{-1} \Omega^1(X^8)$. Then, $F_{\tilde{\nabla}} = \pi_Y^* F_{\nabla_0} + d\tilde{\gamma}$. Since

$$0 = i \left( \frac{\partial}{\partial x} \right) F_{\tilde{\nabla}} = i \left( \frac{\partial}{\partial x} \right) d\tilde{\gamma} \quad \text{and} \quad L^g d\tilde{\gamma} = 0,$$
implies (2).

Since \( \pi^*_Y : H^2_{dR}(Y^7) \to H^2_{dR}(X^8) \) is injective, there exists a 1-form \( \gamma \in \sqrt{-1}\Omega^1(Y^7) \) such that \( d\tilde{\gamma} = \pi^*_Y d\gamma \). Thus, there exists a closed 1-form \( \xi \in \sqrt{-1}\Omega^1(X^8) \) such that \( \tilde{\gamma} = \pi^*_Y \gamma + \xi \). Hence, we obtain

\[
\tilde{\nabla} = \pi^*_Y \nabla_0 + \pi^*_Y \gamma + \xi.
\]

By (7.2), the fact that \( F_{\tilde{\nabla}} = \pi^*_Y F_{\nabla_0 + \gamma} \) and \( \pi^*_Y \) is injective, we see that \( \nabla_0 + \gamma \) is a \( G_2 \)-dDT connection. Then, the proof of (1) is completed. This argument also implies (2).

Next, we prove (3). Let \( \widehat{\mathcal{M}}_{\text{Spin}(7)} \) and \( \widehat{\mathcal{M}}_{G_2} \) be the set of \( \text{Spin}(7) \)-dDT connections of \( \pi^*_Y L \) and \( G_2 \)-dDT connections of \( L \) as defined by (4.1) and (5.2), respectively. Define a continuous map \( \Theta : \sqrt{-1}Z^1(S^1) \times \widehat{\mathcal{M}}_{G_2} \to \widehat{\mathcal{M}}_{\text{Spin}(7)} \) by

\[
\Theta(\alpha, \nabla) = \pi^*_S \alpha + \pi^*_Y \nabla,
\]

where \( Z^1(S^1) \) is the space of closed 1-forms on \( S^1 \). By Lemma 4.1, it induces a continuous map

\[
\Theta : \frac{\sqrt{-1}Z^1(S^1)}{\sqrt{-1}(2\pi \mathcal{H}^1_{Z}(S^1) \oplus d\Omega^0(S^1))} \times \mathcal{M}_{G_2} \to \mathcal{M}_{\text{Spin}(7)}.
\]

We show that \( \Theta \) is a homeomorphism by constructing the inverse map. Fixing \( pt_{S^1} \in S^1 \) and \( pt_Y \in Y^7 \), define \( \iota_{S^1} : S^1 \hookrightarrow S^1 \times Y^7 \) and \( \iota_Y : Y^7 \hookrightarrow S^1 \times Y^7 \) by \( \iota_{S^1}(z) = (z, pt_Y) \) and \( \iota_Y(y) = (pt_{S^1}, y) \). Define a continuous map \( \widehat{\Xi} : \widehat{\mathcal{M}}_{\text{Spin}(7)} \to \sqrt{-1}Z^1(S^1) \times \widehat{\mathcal{M}}_{G_2} \) by

\[
\widehat{\Xi}(\tilde{\nabla}) = \left( \iota^*_{S^1} \left( \tilde{\nabla} - \pi^*_Y \iota^*_Y \tilde{\nabla} \right) , \iota^*_Y \tilde{\nabla} \right),
\]

where we regard \( \tilde{\nabla} - \pi^*_Y \iota^*_Y \tilde{\nabla} \) as a \( \sqrt{-1}\mathbb{R} \)-valued 1-form on \( X^8 \). First, we show that \( \widehat{\Xi} \) is well-defined. For any \( \tilde{\nabla} \in \widehat{\mathcal{M}}_{\text{Spin}(7)} \), there exist a \( G_2 \)-dDT connection \( \nabla \in \widehat{\mathcal{M}}_{G_2} \) and a closed 1-form \( \xi \in \sqrt{-1}\Omega^1(X^8) \) such that

\[
(7.3) \quad \tilde{\nabla} = \pi^*_Y \nabla + \xi
\]

by (2). Then,

\[
\iota^*_{S^1} \left( \tilde{\nabla} - \pi^*_Y \iota^*_Y \tilde{\nabla} \right) = \iota^*_{S^1} \left( \pi^*_Y (\nabla - \iota^*_Y \tilde{\nabla}) + \xi \right) = \iota^*_{S^1} \xi,
\]

and hence, \( \iota^*_{S^1} \left( \tilde{\nabla} - \pi^*_Y \iota^*_Y \tilde{\nabla} \right) \in \sqrt{-1}Z^1(S^1) \). We also have

\[
\iota^*_Y \tilde{\nabla} = \iota^*_Y \pi^*_Y \nabla + \iota^*_Y \xi = \nabla + \iota^*_Y \xi.
\]

Since \( F_{\pi^*_Y \tilde{\nabla}} = F_{\nabla_0 + \gamma} \), we see that \( \pi^*_Y \tilde{\nabla} \in \widehat{\mathcal{M}}_{G_2} \).

The map \( \widehat{\Xi} \) induces the continuous map

\[
\Xi : \mathcal{M}_{\text{Spin}(7)} \to \frac{\sqrt{-1}Z^1(S^1)}{\sqrt{-1}(2\pi \mathcal{H}^1_{Z}(S^1) \oplus d\Omega^0(S^1))} \times \mathcal{M}_{G_2}.
\]
Indeed, if Spin(7)-dDT connections \(\tilde{\nabla}, \tilde{\nabla}'\) are in the same \(G_U\)-orbit, Lemma 4.1 implies that \(\tilde{\nabla}' - \tilde{\nabla} = \eta\) for \(\eta \in \sqrt{-1}(2\pi \mathcal{H}_2^1(X^8) \oplus d\Omega^0(X^8))\). Then, since
\[
i_{S^1}^\ast(\eta - \pi_Y^\ast \iota_Y \eta) = i_{S^1}^\ast \eta \in \sqrt{-1} \left( 2\pi \mathcal{H}_2^1(S^1) \oplus d\Omega^0(S^1) \right),
\]
\[
i_{S^1}^\ast \eta \in \sqrt{-1} \left( 2\pi \mathcal{H}_2^1(Y^7) \oplus d\Omega^0(Y^7) \right),
\]
we see that \(\tilde{\Xi}\) induces \(\Xi\).

Now, we show that \(\Theta\) and \(\Xi\) are mutually inverse. For any \((\alpha, \nabla) \in \sqrt{-1}Z^1(S^1) \times \tilde{\mathcal{M}}_{G_2}\), we compute
\[
(\Xi \circ \hat{\Theta})(\alpha, \nabla) = (i_{S^1}^\ast (\pi_{S^1}^\ast \alpha + \pi_Y^\ast \nabla - \pi_Y^\ast \iota_Y^\ast (\pi_{S^1}^\ast \alpha + \pi_Y^\ast \nabla)), \iota_Y^\ast (\pi_{S^1}^\ast \alpha + \pi_Y^\ast \nabla)) = (\alpha, \nabla),
\]
which implies that \(\Xi \circ \hat{\Theta}\) is the identity map. For any \(\tilde{\nabla} \in \tilde{\mathcal{M}}_{\text{Spin}(7)}\), we compute
\[
(\hat{\Theta} \circ \tilde{\Xi})(\tilde{\nabla}) = \pi_{S^1}^\ast \iota_{S^1}^\ast \eta + \pi_Y^\ast \nabla + \pi_Y^\ast \iota_Y^\ast \eta = \tilde{\nabla} + (\pi_{S^1}^\ast \iota_{S^1}^\ast \xi + \pi_Y^\ast \iota_Y^\ast \xi - \xi).
\]
Since \(H^1_{dR}(S^1) \oplus H^1_{dR}(Y^7) \cong H^1_{dR}(X^8)\) by the Künneth formula, we see that there exist \(\eta_{S^1}, \eta_Y \in \sqrt{-1}Z^1(S^1), \eta_Y \in \sqrt{-1}Z^1(Y^7)\) and \(f_0 \in \sqrt{-1}d\Omega^0(X^8)\) such that \(\xi = \pi_{S^1}^\ast \eta_{S^1} + \pi_Y^\ast \eta_Y + df\). Then,
\[
\pi_{S^1}^\ast \iota_{S^1}^\ast \xi + \pi_Y^\ast \iota_Y^\ast \xi - \xi = \pi_{S^1}^\ast \iota_{S^1}^\ast df + \pi_Y^\ast \iota_Y^\ast df - df \in \sqrt{-1}d\Omega^0(X^8).
\]
Thus, by Lemma 4.1, \((\hat{\Theta} \circ \tilde{\Xi})(\tilde{\nabla})\) and \(\tilde{\nabla}\) are in the same \(G_U\)-orbit, which implies that \(\Theta \circ \Xi\) is the identity map. Hence, we obtain the homeomorphism
\[
\mathcal{M}_{\text{Spin}(7)} \cong \frac{\sqrt{-1}Z^1(S^1)}{\sqrt{-1}(2\pi \mathcal{H}_2^1(S^1) \oplus d\Omega^0(S^1))} \times \mathcal{M}_{G_2} \cong \frac{H^1(S^1, \mathbb{R})}{2\pi H^1(S^1, \mathbb{Z})} \times \mathcal{M}_{G_2},
\]
and the proof is completed. \(\square\)

7.2. Reduction from \(G_2\) to \(SU(3)\). Let \((Y^6, \omega, g, J, \Omega)\) be a real 6 dimensional Calabi–Yau manifold and \(L \to Y\) be a smooth complex line bundle with a Hermitian metric \(h\). Set \(X^7 = S^1 \times Y^6\) and denote by
\[
\pi_{S^1} : X^7 = S^1 \times Y^6 \to S^1, \quad \pi_Y : X^7 = S^1 \times Y^6 \to Y^6
\]
the projections. The 7-manifold \(X^7 = S^1 \times Y^6\) is a \(G_2\)-manifold and the \(G_2\)-structure \(\varphi\) and its Hodge dual \(*\varphi\) are given by
\[
\varphi = \pi_{S^1}^\ast dx \wedge \pi_Y^\ast \omega + \pi_Y^\ast \text{Re} \Omega, \quad *\varphi = \frac{1}{2} \pi_Y^\ast \omega^2 - \pi_{S^1}^\ast dx \wedge \pi_Y^\ast \text{Im} \Omega,
\]
where \(x\) is a coordinate of \(S^1\). In [11, Lemma 3.5], we prove that a Hermitian connection \(\nabla\) of \((L, h)\) is a dHYM connection with phase 1 if and only if the pullback \(\pi_Y^\ast \nabla\) is a \(G_2\)-dDT connection of \(\pi_Y^\ast L\). We can improve this if \(Y^6\) is compact and connected as follows.
**Theorem 7.3.** Suppose that $Y^6$ is a compact and connected Calabi–Yau manifold.

1. The pullback $\pi_Y^*L \to X^7$ admits a $G_2$-dDT connection if and only if $L \to Y^6$ admits a dHYM connection with phase 1.
2. For any $G_2$-dDT connection $\tilde{\nabla}$ of $\pi_Y^*L$, there exist a dHYM connection $\nabla$ with phase 1 of $L$ and a closed 1-form $\xi \in \sqrt{-1}\Omega^1(X^7)$ such that $\tilde{\nabla} = \pi_Y^*\nabla + \xi$.
3. Denote by $M_{G_2}$ the moduli space of $G_2$-dDT connections of $\pi_Y^*L$ and denote by $M_{dHYM}$ the moduli space of dHYM connections with phase 1 of $L$ as defined by (5.3). Then, $M_{G_2}$ is homeomorphic to $S^1 \times M_{dHYM}$.

In particular, $M_{G_2}$ is a torus of dimension $1 + b^1(Y^6) = b^1(X^7)$ by [11, Theorem 1.2 (1)].

**Remark 7.4.** An analogous result for $G_2$-instantons is given in [24, Theorem 1.17], where the moduli space of irreducible $G_2$-instantons of $\pi^*L$ is homeomorphic to the product of $S^1$ and the moduli space of irreducible Hermitian Yang–Mills connections with 0-slope of $L$. An analogous result for compact associative submanifolds is given in [5, Proposition 4.6], where the moduli space of all local associative deformations of $\ast \times A^3$, where $A^3$ is a given compact special Lagrangian submanifold in $Y^6$, is identified with the product of $S^1$ and the moduli space of all local special Lagrangian deformations of $A^3$.

Corollary 5.6 (2) and Theorem 7.3 imply that we will need to consider a non-flat line bundle over a manifold with full holonomy $G_2$ to construct nontrivial examples on a compact and connected $G_2$-manifold.

It is conjectured in [3, Conjecture 1.5] that the existence of a dHYM connection is equivalent to a certain stability condition and the conjecture is partially proved in [2]. Theorem 7.3 (1) implies that the existence of a $G_2$-dDT connection of $\pi_Y^*L \to X^7$ would be equivalent to this stability condition. More generally, the existence of a $G_2$-dDT connection of a general line bundle over a general $G_2$-manifold might be related to a certain stability condition.

**Proof.** We first prove (1). The proof is also almost the same as for Theorem 7.1. By [11, Lemma 3.5], for any dHYM connection $\nabla$ with phase 1 of $(L, h)$, $\pi_Y^* \nabla$ is a $G_2$-dDT connection of $\pi_Y^*L$. Thus, we only have to prove the converse.

Suppose that a $G_2$-dDT connection $\tilde{\nabla}$ of $\pi_Y^*L$ is given. By Remark 5.2 and the connectedness of $X^7$, we may assume that $1 + \langle F^2_{\tilde{\nabla}}, *\varphi \rangle / 2 > 0$ everywhere without loss of generality. Denote by $\ast = \ast_7$ and $\ast_6$ the Hodge stars on $X^7$ and $Y^6$, respectively. Then, Theorem 5.3 implies that

$$V(\tilde{\nabla}) = \int_X \left( 1 + \frac{1}{2} \langle F^2_{\tilde{\nabla}}, *\varphi \rangle \right) \text{vol}_g$$

$$= \int_X \left( 1 + \frac{1}{2} \langle F^2_{\tilde{\nabla}}, \frac{1}{2} \pi_Y^* \omega^2 - \pi_Y^* dx \wedge \pi_Y^* \text{Im } \Omega \rangle \right) \text{vol}_g.$$
Since \(-\langle F^2, \pi_Y^* dx \wedge \pi_Y^* \text{Im } \Omega \rangle \text{vol}_g = -F^2 \wedge \pi_Y^* \text{Im } \Omega = F^2 \wedge \pi_Y^* \text{Re } \Omega\), where we use (2.17), it follows that
\[
-\int_X \langle F^2, \pi_Y^* dx \wedge \pi_Y^* \text{Im } \Omega \rangle \text{vol}_g = ((-2\pi \sqrt{-1} c_1(\pi_Y^* L))^2 \cup [\pi_Y^* \text{Re } \Omega]) \cdot [X]
\]
\[
= -4\pi^2 \pi_Y^* (c_1(L)^2 \cup [\text{Re } \Omega]) \cdot [X] = 0
\]
since \(c_1(L)^2 \cup [\text{Re } \Omega] \in H^7_{dR}(Y^6) = \{0\}\). Hence, we obtain
(7.4)
\[
\int_X \left(1 + \frac{1}{4} \langle F^2, \pi_Y^* \omega^2 \rangle \right) \text{vol}_X = V(\tilde{\nabla}).
\]
By Lemma A.8, (7.4) implies that \(1 + \langle F^2, \pi_Y^* \omega^2 \rangle / 4 = \sqrt{\det(\text{id}_X + (-\sqrt{-1} F^\nabla)^2)}\) and
\[
i \frac{\partial}{\partial x} F^\nabla = 0, \quad \text{Im} \left(\frac{\pi_Y^* \omega + F^\nabla}{3!}\right) = 0 \quad \text{and} \quad \pi^{[2,0]}(F^\nabla) = 0.
\]
Then, (1) follows by the same argument as in Theorem 7.1 (1). This argument also implies (2). Since (3) follows from (2) by the same argument as in Theorem 7.1, the proof is completed. □

7.3. Reduction from Spin(7) to SU(4). Let \((X^8, \omega, g, J, \Omega)\) be a real 8 dimensional Calabi–Yau manifold and \(L \to Y\) be a smooth complex line bundle with a Hermitian metric \(h\). Then, \(X^8\) has an induced Spin(7)-structure \(\Phi\) given by
\[
\Phi = \frac{1}{2} \omega^2 + \text{Re } \Omega.
\]
In [12, Lemma 7.2], if the \((0, 2)\)-part \(F^0_{\nabla}^2\) of the curvature of \(F^\nabla\) of a Hermitian connection \(\nabla\) of \((L, h)\) vanishes, we show that \(\nabla\) is a dHYM connection with phase 1 if and only if it is a Spin(7)-dDT connection. We can improve this if \(X^8\) is compact and connected as follows.

**Theorem 7.5.** Suppose that \(X^8\) is a compact and connected Calabi–Yau manifold.

1. The complex line bundle \(L \to X^8\) admits a holomorphic structure and a Spin(7)-dDT connection if and only if \(L \to X^8\) admits a dHYM connection with phase 1.

2. Suppose that there exists a dHYM connection \(\nabla_0\) with phase 1. Then, any Spin(7)-dDT connection is a dHYM connection with phase 1. Hence, the moduli space of Spin(7)-dDT connections agrees with that of dHYM connections with phase 1, which is a \(b_1\)-dimensional torus by [11, Theorem 1.2 (1)].

**Remark 7.6.** An analogous result for Spin(7)-instantons is given in [15, Theorem 3.1], where any Spin(7)-instanton is a Hermitian Yang–Mills connection if a vector bundle admits a Hermitian Yang–Mills connection. An analogous result for compact Cayley submanifolds is given in [21, Proposition 5.12], where submanifolds are allowed to have boundaries. This implies that the moduli space of all local Cayley
deformations of $C^4$, where $C^4$ is a given compact special Lagrangian submanifold in $X^8$, is identified with the moduli space of all local special Lagrangian deformations of $C^4$.

Corollary 4.6 (2), Theorems 7.1 and 7.5 imply that we will need to consider a non-flat line bundle over a manifold with full holonomy Spin(7) to construct nontrivial examples on a compact and connected Spin(7)-manifold.

**Proof.** We first prove (1). The proof is almost the same as for Theorems 7.1 and 7.3. By [12, Lemma 7.2], any dHYM connection $\nabla$ with phase 1 is a Spin(7)-dDT connection. The vanishing of the $(0,2)$-part $F^0_{\nabla}$ implies that there exists a holomorphic structure on $L$ by [14, Proposition 1.3.7]. Thus, we only have to prove the converse.

Suppose that a Spin(7)-dDT connection $\tilde{\nabla}$ of $L$ is given. By Remark 4.3 and the connectedness of $X^8$, we may assume that $1 + \langle F^2_{\tilde{\nabla}}, \Phi \rangle/2 + *F^4_{\tilde{\nabla}}/24 > 0$ everywhere without loss of generality. Then, Theorem 4.4 implies that

$$V(\tilde{\nabla}) = \int_X \left( 1 + \frac{1}{2} \langle F^2_{\tilde{\nabla}}, \Phi \rangle + \frac{*F^4_{\tilde{\nabla}}}{24} \right) \text{vol}_g$$

$$= \int_X \left( \text{vol}_g + \frac{1}{2} F^2_{\tilde{\nabla}} \wedge (\frac{1}{2} \omega^2 + \text{Re}\Omega) + \frac{F^4_{\tilde{\nabla}}}{24} \right).$$

Since $L$ admits a holomorphic structure, we have $c_1(L) \in H^{1,1}(X)$, which implies that

$$\int_X F^2_{\tilde{\nabla}} \wedge \text{Re}\Omega = ((-2\pi \sqrt{-1}) c_1(L))^2 \cup [\text{Re}\Omega] \cdot [X] = 0.$$ 

Hence, we obtain

$$(7.5) \quad V(\tilde{\nabla}) = \int_X \left( \text{vol}_g + \frac{1}{4} F^2_{\tilde{\nabla}} \wedge \omega^2 + \frac{F^4_{\tilde{\nabla}}}{24} \right) = \int_X \text{Re} \left( \frac{(\omega + F_{\tilde{\nabla}})^4}{4!} \right).$$

Then, by Theorem 6.2, $\tilde{\nabla}$ is a dHYM connection with phase 1. This argument also implies (2) and the proof is completed.$\square$

**Appendix A. The proof of Theorems 4.2, 5.1, 6.1 and 6.2**

In this appendix, we prove Theorems 4.2, 5.1, 6.1 and 6.2, which follow from Proposition A.1, Corollaries A.5, A.7 and Proposition A.9.

**A.1. The Cayley case.** Use the notation of Section 2.3. Set $W = \mathbb{R}^8$ with the standard basis $\{ e_i \}_{i=0}^7$ and its dual $\{ e^i \}_{i=0}^7$. Let $g$ be the standard inner product on $W$. For a 2-form $F \in \Lambda^2 W^*$, define $F^2 \in \text{End}(W)$ by

$$g(F^2(u), v) = F(u, v)$$

for $u, v \in W$. 

Proposition A.1. For any $F \in \Lambda^2 W^*$, we have
\[
\left(1 - \frac{1}{2} \langle F^2, \Phi \rangle + \frac{F^4}{24}\right)^2 + 4 \left| \pi^2_7 \left( F - \frac{1}{6} F^3 \right) \right|^2 + 2 \left| \pi^4_7 \left( F^2 \right) \right|^2 = \det(I_8 + F^\sharp),
\]
where $I_8$ is the identity matrix of dimension 8.

This statement is essential. Using Proposition A.1, we can show all the main statements in Appendix A (Corollaries A.5, A.7 and Proposition A.9).

Proposition A.1 follows from the following Lemmas A.2 and A.3. Note that a similar identity to Lemma A.2 holds for any dimension.

Lemma A.2. For any $F \in \Lambda^2 W^*$, we have
\[
\det(I_8 + F^\sharp) = 1 + |F|^2 + \left| \frac{F^2}{2!} \right|^2 + \left| \frac{F^3}{3!} \right|^2 + \left| \frac{F^4}{4!} \right|^2.
\]

Proof. Since $F^\sharp$ is skew-symmetric, there exist $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}$ and $h \in O(8)$ such that
\[
h^{-1} F^\sharp h = \begin{pmatrix} 0 & -\lambda_1 \\ \lambda_1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -\lambda_2 \\ \lambda_2 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -\lambda_3 \\ \lambda_3 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -\lambda_4 \\ \lambda_4 & 0 \end{pmatrix}.
\]

In other words, we have
\[
h^* F = \lambda_1 e^{01} + \lambda_2 e^{23} + \lambda_3 e^{45} + \lambda_4 e^{67}.
\]

Then, we obtain
\[
\det(I_8 + F^\sharp) = (1 + \lambda_1^2)(1 + \lambda_2^2)(1 + \lambda_3^2)(1 + \lambda_4^2)
\]
\[
= 1 + \sum_i \lambda_i^2 + \sum_{i<j} \lambda_i^2 \lambda_j^2 + \sum_{i<j<k} \lambda_i^2 \lambda_j^2 \lambda_k^2 + \lambda_1^2 \lambda_2^2 \lambda_3^2 \lambda_4^2
\]
\[
= 1 + |F|^2 + \left| \frac{F^2}{2!} \right|^2 + \left| \frac{F^3}{3!} \right|^2 + \left| \frac{F^4}{4!} \right|^2.
\]

Lemma A.3. For any $F \in \Lambda^2 W^*$, we have
\[
\langle F^2, \Phi \rangle + 4 \left| \pi^2_7 (F) \right|^2 = |F|^2,
\]
\[
\frac{1}{4} \langle F^2, \Phi \rangle^2 + \frac{F^4}{12} - \frac{4}{3} \langle \pi^2_7 (F), \pi^2_7 (F^3) \rangle + 2 \left| \pi^4_7 (F^2) \right|^2 = \frac{1}{4} |F^2|^2,
\]
\[
\langle F^2, \Phi \rangle \frac{F^4}{24} + \frac{1}{9} \left| \pi^2_7 (F^3) \right|^2 = \frac{1}{36} |F^3|^2.
\]

Proof. Set
\[
F = F_7 + F_{21} \in \Lambda^2 W^* \oplus \Lambda^2_{21} W^*.
\]
Then, we have
\[ -\langle F^2, \Phi \rangle = -\ast (F \wedge F \wedge \Phi) = -\ast (F \wedge \ast (3F_7 - F_{21})) = -3|F_7|^2 + |F_{21}|^2, \]
which implies (A.1).

Next, we show (A.3). By (A.1), we have
\[ |F^3|^2 = |\ast F^3|^2 = 4|\pi_7^2 (\ast F^3)|^2 - \ast ((\ast F^3) \wedge \Phi). \]
By Lemma 2.5, it follows that
\[ \ast ((\ast F^3) \wedge \Phi) = \frac{3}{2} \langle F^2, \Phi \rangle \ast F^4. \]
Hence, we obtain (A.3).

Next, we prove (A.2). Set
\[ \xi_1 = F_7^2, \quad \xi_2 = F_7 \wedge F_{21}, \quad \xi_3 = F_{21}^2, \]
\[ \eta = F^2, \quad \eta_j = \pi_4^j(\eta) \quad \text{for } j = 1, 7, 27, 35. \]
Recall the decomposition (2.10) and that \( \Lambda_4^4W^* \oplus \Lambda_4^4W^* \oplus \Lambda_2^4W^* \) and \( \Lambda_4^4W^* \) are the spaces of self dual 4-forms and anti self dual 4-forms, respectively. Then, we have \( F^2 = \xi_1 + 2\xi_2 + \xi_3 \) and
\[ \xi_1 = \ast \xi_1, \]
\[ \xi_1 \wedge \xi_2 = 0, \]
\[ \xi_2 \wedge (\xi_3 + \ast \xi_3) = 0, \]
\[ \ast \xi_2^2 = \ast (\xi_1 \wedge \xi_3) = \langle \xi_1, \xi_3 \rangle. \]
(A.4)
We first show that \( |\eta_j|^2 \) is described in terms of \( \xi_k \)'s.

**Lemma A.4.** We have
\[ |\eta_1|^2 = \frac{18|\xi_1|^2 - 36|\xi_2|^2 + 3|\xi_3|^2 - \ast \xi_3^2}{42}, \]
\[ |\eta_1|^2 + |\eta_{27}|^2 = |\xi_1|^2 + \frac{|\xi_3|^2 + \ast \xi_3^2}{2} + 2\langle \xi_1, \xi_3 \rangle, \]
\[ |\eta_7|^2 = 2|\xi_2|^2 + 2\langle \xi_1, \xi_3 \rangle, \]
\[ |\eta_{35}|^2 = 2|\xi_2|^2 + \frac{1}{2}|\xi_3|^2 - 2\langle \xi_1, \xi_3 \rangle - 4 \ast (\xi_2 \wedge \xi_3) - \frac{1}{2} \ast \xi_3^2. \]

**Proof.** Since \( |\Phi|^2 = 14 \), we have \( \eta_1 = \langle \eta, \Phi \rangle \Phi/14 \), and hence,
\[ |\eta_1|^2 = \frac{\langle \eta, \Phi \rangle^2}{14}. \]
(A.5)
Since \( \eta = F^2 \), it follows that
\[ \langle \eta, \Phi \rangle = \ast (F \wedge F \wedge \Phi) = \ast (F \wedge \ast (3F_7 - F_{21})) = 3|F_7|^2 - |F_{21}|^2. \]
and 
\[ |\eta_1|^2 = \frac{(3|F_7|^2 - |F_{21}|^2)^2}{14} = \frac{9|F_7|^4 - 6|F_7|^2|F_{21}|^2 + |F_{21}|^4}{14}. \]

By Proposition 2.4, it follows that
\[ |F_7|^4 = \frac{2}{3}|\xi_1|^2, \quad |F_7|^2|F_{21}|^2 = 2|\xi_2|^2, \quad |F_{21}|^4 = |\xi_3|^2 - \frac{1}{3} \ast \xi_3^2. \]

This gives the desired formula for \(|\eta_1|^2|.

Next, we compute \(|\eta_1|^2 + |\eta_{27}|^2|. By (2.10) and (A.4), we have
\[ \eta_1 + \eta_{27} = \pi_1^4(\xi_1 + \xi_3) + \pi_2^4(\xi_1 + \xi_3) = \xi_1 + \frac{\xi_3 + \ast \xi_3}{2}. \]

Then, by (A.4), it follows that
\begin{align*}
|\eta_1|^2 + |\eta_{27}|^2 &= |\eta_1 + \eta_{27}|^2 \\
&= |\xi_1|^2 + \frac{|\xi_3 + \ast \xi_3|^2}{4} + \langle \xi_1, \xi_3 + \ast \xi_3 \rangle \\
&= |\xi_1|^2 + \frac{|\xi_3|^2 + \ast \xi_3^2}{2} + 2\langle \xi_1, \xi_3 \rangle.
\end{align*}

Next, we compute \(|\eta_7|^2|. By (2.10) and (A.4), we compute
\[ \eta_7 = 2\pi_7^4(\xi_2) = \xi_2 + \ast \xi_2. \]

Hence, by (A.4), we obtain
\[ |\eta_7|^2 = 2|\xi_2|^2 + 2\ast \xi_2^2 = 2|\xi_2|^2 + 2\langle \xi_1, \xi_3 \rangle. \]

Finally, we compute \(|\eta_{35}|^2|. By (2.10) and (A.4), we have
\[ \eta_{35} = \pi_3^4(2\xi_2 + \xi_3) = \frac{2\xi_2 + \xi_3 - \ast(2\xi_2 + \xi_3)}{2}. \]

Then, by (A.4), we obtain
\begin{align*}
2|\eta_{35}|^2 &= |2\xi_2 + \xi_3|^2 - \ast(2\xi_2 + \xi_3)^2 \\
&= 4|\xi_2|^2 + |\xi_3|^2 + 4\langle \xi_2, \xi_3 \rangle - \ast(4\xi_2^2 + 4\xi_2 \land \xi_3 + \xi_3^2) \\
&= 4|\xi_2|^2 + |\xi_3|^2 - 4\langle \xi_1, \xi_3 \rangle - 8\ast(\xi_2 \land \xi_3) - \ast\xi_3^2.
\end{align*}

Then, we compute
\[ \langle \pi_7^2(F), \pi_7^2(\ast F^3) \rangle = \ast(F_7 \land F \land F^2) = \ast((\xi_1 + \xi_2) \land (\xi_1 + 2\xi_2 + \xi_3)). \]

By (A.4), we obtain
\[ (A.6) \quad \frac{4}{3} \langle \pi_7^2(F), \pi_7^2(\ast F^3) \rangle = \frac{4}{3} \left( |\xi_1|^2 + 3\langle \xi_1, \xi_3 \rangle + \ast(\xi_2 \land \xi_3) \right). \]

By the equation
\[ \ast F^4 = \ast(\eta \land \eta) = |\eta_1|^2 + |\eta_2|^2 + |\eta_{27}|^2 - |\eta_{35}|^2, \]

\[ \Box \]
and (A.5), it follows that
\[ \frac{1}{4} \langle F^2, \Phi \rangle^2 + \frac{F^4}{12} + 2 \left| \pi_{\mathbf{7}}^2 (F^2) \right|^2 - \frac{1}{4} \left| F^2 \right|^2 \]
\[ = \frac{7}{2} |\eta_1|^2 + \frac{1}{12} \left( |\eta_1|^2 + |\eta|_7|^2 + |\eta_{27}|^2 - |\eta_{35}|^2 \right) + 2 |\eta|^2 \]
\[ - \frac{1}{4} \left( |\eta_1|^2 + |\eta|^2 + |\eta_{27}|^2 + |\eta_{35}|^2 \right) \]
\[ = \frac{1}{6} \left( 21 |\eta_1|^2 - (|\eta_1|^2 + |\eta_{27}|^2) + 11 |\eta|^2 - 2 |\eta_{35}|^2 \right). \]

By Lemma A.4, we further compute
\[ = \frac{1}{6} \left( 9 |\xi_1|^2 - 18 |\xi_2|^2 + \frac{3 |\xi_3|^2 - \ast |\xi_3|^2}{2} \right) \]
\[ - \left( |\xi_1|^2 + \frac{|\xi_3|^2 + \ast |\xi_3|^2}{2} + 2 \langle \xi_1, \xi_3 \rangle \right) + 11 \left( 2 |\xi_2|^2 + 2 \langle \xi_1, \xi_3 \rangle \right) \]
\[ - 2 \left( 2 |\xi_2|^2 + \frac{1}{2} |\xi_3|^2 - 2 \langle \xi_1, \xi_3 \rangle - 4 \ast (\xi_2 \wedge \xi_3) - \frac{1}{2} \ast |\xi_3|^2 \right) \]
\[ = \frac{4}{3} \left( |\xi_1|^2 + 3 \langle \xi_1, \xi_3 \rangle + \ast (\xi_2 \wedge \xi_3) \right). \]

This together with (A.6) implies (A.2). \qed

A.2. The associator case. We can show the following by Proposition A.1. Set \( V = \mathbb{R}^7 \) and use the notation of Section 2.2.

Corollary A.5. For any \( F \in \Lambda^2 V^* \), we have
\[ \left( 1 - \frac{1}{2} \langle F^2, \ast \varphi \rangle \right)^2 + \left| \varphi \wedge F - \frac{1}{6} F^3 \right|^2 + \frac{1}{4} |\varphi \wedge \ast F^2|^2 = \det (I_7 + F^2), \]
where \( I_7 \) is the identity matrix of dimension 7.

Proof. By \( V^* \ni e^\mu \mapsto e^\mu \in W^* \) for \( \mu = 1, \cdots, 7 \), we identify \( F \in \Lambda^2 W^* \) with an element of \( \Lambda^2 V^* \). We rewrite Proposition A.1 to obtain Corollary A.5. For clarification, denote by \( \ast = \ast_8 \) and \( \ast_7 \) the Hodge star operators on \( W \) and \( V \), respectively.

Since \( \Phi = e^0 \wedge \varphi \wedge \ast_7 \varphi \), we have \( \langle F^2, \Phi \rangle = \langle F^2, \ast_7 \varphi \rangle \). Since \( F \in \Lambda^2 V^* \), we have \( F^4 = 0 \) and \( I_8 + F^2 = 1 \oplus (I_7 + F^2) \). Then,
\[ \left( 1 - \frac{1}{2} \langle F^2, \Phi \rangle + \frac{F^4}{24} \right)^2 = \left( 1 - \frac{1}{2} \langle F^2, \ast_7 \varphi \rangle \right)^2, \]
\[ \det(I_8 + F^2) = \det(I_7 + F^2). \]
Next, we rewrite \( 4 |\pi_7^2(F - *_{\tau}(F^3/6))^2 \). By Lemma 2.3, we have
\[
\pi_7^2(F) = \sum_{\mu=1}^7 \langle F, \lambda^2(e^\mu) \rangle \cdot \lambda^2(e^\mu).
\]
Since
\[
2\langle F, \lambda^2(e^\mu) \rangle = \langle F, i(e^\mu) \varphi \rangle = \langle e^\mu \wedge F, \varphi \rangle = \langle e^\mu, *_{\tau}(F \wedge *_{\tau} \varphi) \rangle,
\]
it follows that
\[
2\pi_7^2(F) = \lambda^2(*_{\tau}(F \wedge *_{\tau} \varphi)).
\]
Similarly, since \(*_{\tau} F^3 = e^0 \wedge *_{\tau} F^3\), we have
\[
2\pi_7^2(*_{\tau} F^3) = 2 \sum_{\mu=1}^7 \langle *_{\tau} F^3, \lambda^2(e^\mu) \rangle \cdot \lambda^2(e^\mu) = \sum_{\mu=1}^7 \langle *_{\tau} F^3, e^\mu \rangle \cdot \lambda^2(e^\mu) = \lambda^2(*_{\tau} F^3).
\]
Hence, we obtain
\[
(A.8) \quad 4 \left| \frac{\pi_7^2(F - \frac{1}{6} *_{\tau} F^3)}{\det(I + F^3)} \right|^2 = \left| F \wedge *_{\tau} \varphi - \frac{1}{6} F^3 \right|^2.
\]
Finally, we rewrite \( 2 |\pi_7^4(F^2)|^2 \). By Lemma 2.3, we have
\[
\pi_7^4(F^2) = \sum_{\mu=1}^7 \langle F^2, \lambda^4(e^\mu) \rangle \cdot \lambda^4(e^\mu).
\]
Since
\[
\sqrt{8} \langle F^2, \lambda^4(e^\mu) \rangle = -\langle F^2, e^\mu \wedge \varphi \rangle = -\langle e^\mu, *_{\tau}(\varphi \wedge *_{\tau} F^2) \rangle,
\]
it follows that
\[
\sqrt{8} \pi_7^4(F^2) = -\lambda^4(*_{\tau}(\varphi \wedge *_{\tau} F^2)).
\]
Hence, we obtain
\[
(A.9) \quad 2 \left| \pi_7^4(F^2) \right|^2 = \frac{1}{4} \left| \varphi \wedge *_{\tau} F^2 \right|^2.
\]
By (A.7), (A.8), (A.9) and Proposition A.1, we obtain Corollary A.5. □

Consider \(*_{\tau} \varphi\) as a 4-form of \( W = \mathbb{R}^8 = \mathbb{R}e_0 \oplus V \) by pullback. Then, we see the following.

**Lemma A.6.** For any \( F \in \Lambda^2 W^* \), we have
\[
\left| 1 - \frac{1}{2} \langle F^2, *_{\tau} \varphi \rangle \right| \leq \sqrt{\det(I + F^2)}.
\]
The equality holds if and only if
\[
i(e_0) F = 0, \quad F \wedge *_{\tau} \varphi - \frac{1}{6} F^3 = 0.
\]
Proof. Set \( F = e^0 \wedge F_1 + F_2 \), where \( F_1 \in V^* \) and \( F_2 \in \Lambda^2 V^* \). Then, by the fact that \( F^2 = 2e^0 \wedge F_1 \wedge F_2 + F_2^2 \) and Corollary A.5, we have
\[
\left| 1 - \frac{1}{2} \langle F^2, *\gamma \varphi \rangle \right| = \left| 1 - \frac{1}{2} \langle F_2^2, *\gamma \varphi \rangle \right| \leq \sqrt{\det(I_7 + F_2^2)}.
\]
The equality holds if and only if \( F_2 \wedge *\gamma \varphi - F_2^3/6 = 0 \). Here we use the fact that \( F_2 \wedge *\gamma \varphi - F_2^3/6 = 0 \) implies that \( \varphi \wedge *F_2^2 = 0 \) by [11, Corollary C.3].

By the proof of Lemma A.2, we have
\[
det(I_7 + F_2^2) = 1 + |F_2|^2 + \left| \frac{F_2^2}{2!} \right|^2 + \left| \frac{F_2^3}{3!} \right|^2,
\]
\[
det(I_8 + F_2^2) = 1 + |F|^2 + \left| \frac{F_2^2}{2!} \right|^2 + \left| \frac{F_3^2}{3!} \right|^2 + \left| \frac{F_4^2}{4!} \right|^2.
\]
Since \( F = e^0 \wedge F_1 + F_2, F^2 = 2e^0 \wedge F_1 \wedge F_2 + F_2^2 \) and \( F_3 = 3e^0 \wedge F_1 \wedge F_2^2 + F_3^2 \), we have \( |F|^2 = |F_1|^2 + |F_2|^2, |F^2|^2 = 4|F_1 \wedge F_2|^2 + |F_2^2|^2 \) and \( F_3^2 = 9|F_1 \wedge F_2^2|^2 + |F_2^3|^2 \).

Hence, we obtain
\[
det(I_7 + F_2^2) \leq det(I_8 + F_2^2) \]
and the equality holds if and only if \( F_1 = 0 \). Then, the proof is completed. \( \square \)

A.3. The 3-dimensional special Lagrangian case. We can show the following by Corollary A.5. Set \( U = \mathbb{R}^6 \) and use the notation of Section 2.4.

Corollary A.7. For any \( F \in \Lambda^2 U^* \), we have
\[
\left| \frac{(\omega + \sqrt{-1}F)^3}{3!} \right|^2 + 2 \left| \pi^{[3,1]} \left( \frac{(\omega + \sqrt{-1}F)^2}{2!} \right) \right|^2 = \det(I_6 + F^2),
\]
where \( I_6 \) is the identity matrix of dimension 6. In particular,
\[
\left| \text{Re} \left( e^{-\sqrt{-1}\theta} \frac{(\omega + \sqrt{-1}F)^3}{3!} \right) \right| \leq \sqrt{\det(I_6 + F^2)}
\]
for any \( F \in \Lambda^2 U^* \) and \( \theta \in \mathbb{R} \). The equality holds if and only if
\[
\text{Im} \left( e^{-\sqrt{-1}\theta} \frac{(\omega + \sqrt{-1}F)^3}{3!} \right) = 0 \quad \text{and} \quad \pi^{[2,0]}(F) = 0.
\]

Note that the identity depends only on the Kähler structure \( (\omega, g, J) \) and is independent of the holomorphic volume form \( \Omega \) (the Calabi–Yau structure).

Proof. By \( U^* \ni e^i \mapsto e^i \in V^* \) for \( i = 2, \ldots, 7 \), we identify \( F \in \Lambda^2 U^* \) with an element of \( \Lambda^2 V^* \). As in the proof of Corollary A.5, we rewrite Corollary A.5 to obtain Corollary A.7. For clarification, denote by \( * = *_7 \) and \( *_6 \) the Hodge star operators on \( V \) and \( U \), respectively.

Since \( \varphi = e^1 \wedge \omega + \text{Re}\Omega \) and \( *_7 \varphi = \omega^2/2 - e^1 \wedge \text{Im}\Omega \), we have
\[
\frac{1}{2} \langle F^2, *_7 \varphi \rangle = \frac{1}{4} \langle F^2, \omega^2 \rangle = \frac{1}{2} *_6 (F^2 \wedge \omega),
\]
and
where we use $\ast_6 \omega^2 = 2\omega$. Thus,

$$1 - \frac{1}{2} \langle F^2, \ast_7 \varphi \rangle = \ast_6 \left( \frac{\omega^3 - 3\omega \wedge F^2}{6} \right) = \ast_6 \text{Re}\left( \frac{\left(\omega + \sqrt{-1}F\right)^3}{6} \right).$$

We also compute

$$1 - \frac{1}{2} \langle F^2, \ast_7 \varphi \rangle \leq \sqrt{\text{det}(I_7 + F^2)}.$$

By Corollary 2.8, Lemma 2.7 and (2.18), we have

$$\text{Im} \frac{\left(\omega + \sqrt{-1}F\right)^3}{6} = 2 \left| \pi_{[2,0]}(F) \right|^2 = 2 \left| \pi_{[2,0]}(\ast_6 (\omega \wedge F)) \right|^2 = 2 \left| \pi_{[3,1]}(\omega \wedge F) \right|^2.$$

This together with (A.10) implies the identity of Corollary A.7. The last statement follows from (A.10) and the first equation of (A.11).

Consider $\omega$ as a 2-form on $V = \mathbb{R}^7 = \mathbb{R} e_1 \oplus U$ by pullback. Then, we can prove the following in the same way as Lemma A.6 using Corollary A.7.

**Lemma A.8.** For any $F \in \Lambda^2 V^*$, we have

$$\left| 1 - \frac{1}{4} \langle F^2, \omega^2 \rangle \right| \leq \sqrt{\text{det}(I_7 + F^2)}.$$

The equality holds if and only if

$$i(e_1)F = 0, \quad \text{Im}\left( \frac{\left(\omega + \sqrt{-1}F\right)^3}{3!} \right) = 0 \quad \text{and} \quad \pi_{[2,0]}(F) = 0.$$
A.4. **The 4-dimensional special Lagrangian case.** We can show the following by Proposition A.1. Set $W = \mathbb{R}^8$ and use the notation of Section 2.5.

**Proposition A.9.** For any $F \in \Lambda^2 W^*$, we have

$$
\left| \frac{\omega + \sqrt{1-F}}{4!} \right|^2 + 2 \left| \pi^{[4,2]} \left( \frac{\omega + \sqrt{1-F}}{3!} \right) \right|^2 + 8 \left| \pi^{[4,0]} \left( \frac{\omega + \sqrt{1-F}}{2!} \right) \right|^2
= \det(I_8 + F^2).
$$

In particular,

$$
\left| \text{Re} \left( e^{-\sqrt{-1}\theta} \frac{(\omega + \sqrt{1-F})^4}{4!} \right) \right| \leq \sqrt{\det(I_8 + F^2)}
$$

for any $F \in \Lambda^2 W^*$ and $\theta \in \mathbb{R}$. The equality holds if and only if

$$
\text{Im} \left( e^{-\sqrt{-1}\theta} \frac{(\omega + \sqrt{1-F})^4}{4!} \right) = 0 \quad \text{and} \quad \pi^{[2,0]}(F) = 0.
$$

As in Corollary A.7, note that the identity depends only on the Kähler structure $(\omega, g, J)$ and is independent of the holomorphic volume form $\Omega$ (the Calabi–Yau structure).

We prove Proposition A.9 by rewriting Proposition A.1 (Lemma A.3). We first prove the following.

**Lemma A.10.** For any $F \in \Lambda^2 W^*$, we have

$$
\text{Re} \left( e^{-\sqrt{-1}\theta} \frac{\omega + \sqrt{1-F}}{4!} \right) = 0
$$

and

$$
\pi^{[2,0]}(F) = 0.
$$

**Proof.** We rewrite Lemma A.3. Since $\Phi = \omega^2/2 + \text{Re} \Omega$ and $\pi^2_\omega(F) = \pi_{\omega, \omega}(F) + \pi_{A_+}(F) = \langle F, \omega \rangle \omega/4 + \pi_{A_+}(F)$ by (2.23), (A.1) is described as

$$
\text{Re} \left( e^{-\sqrt{-1}\theta} \frac{\omega + \sqrt{1-F}}{4!} \right) = 0
$$

and Lemma 2.10, we have

$$
\pi_{A_+}(F) = \langle F, \omega \rangle \omega/4 + \pi_{A_+}(F).
$$

This together with $\omega^3/6$ implies (A.12).

Next, we show (A.14). By (A.3), we have

$$
\pi^{[2,0]}(F) = \frac{1}{36} |F^3|^2.
$$

(A.14)
By Lemmas 2.5 and 2.10, it follows that
\[
\langle F^2, \Re \Omega \rangle \ast F^4 = \frac{2}{3} \ast \left( (\ast F)^3 \wedge \Re \Omega \right) = \frac{4}{3} \left( |\pi_{A_+} (\ast F^3)|^2 - |\pi_{A_-} (\ast F^3)|^2 \right),
\]
which implies that
\[
\frac{1}{9} |\pi_{A_+} (\ast F^3)|^2 - \frac{1}{24} \langle F^2, \Re \Omega \rangle \ast F^4 = \frac{1}{18} \left( |\pi_{A_+} (\ast F^3)|^2 + |\pi_{A_-} (\ast F^3)|^2 \right) = \frac{1}{18} |\pi^{[2,0]} (\ast F^3)|^2.
\]
Hence, we obtain (A.14).

Next, we prove (A.13). We rewrite (A.2). Since \( \Phi = \omega^2/2 + \Re \Omega \), we have
\[
\frac{1}{4} \langle F^2, \Phi \rangle^2 = \frac{1}{16} \langle F^2, \omega^2 \rangle^2 + \frac{1}{4} \langle F^2, \omega^2 \rangle \langle F^2, \Re \Omega \rangle + \frac{1}{4} \langle F^2, \Re \Omega \rangle^2.
\]
Since \( \pi_\Phi^2 (F) = \pi_{\Re \Omega} (F) + \pi_{A_+} (F) = \langle F, \omega \rangle / 4 + \pi_{A_+} (F) \), we have
\[
-\frac{4}{3} \langle \pi_\Phi^2 (F), \pi_\Phi^2 (\ast F^3) \rangle = -\frac{4}{3} \left( \frac{1}{4} \langle F, \omega \rangle \langle \ast F^3, \omega \rangle + \langle \pi_{A_+} (F), \pi_{A_+} (\ast F^3) \rangle \right)
\]
\[
= -\frac{1}{18} \ast (\omega^3 \wedge F) * (\omega \wedge F^3) - \frac{4}{3} \langle \pi_{A_+} (F), \pi_{A_+} (\ast F^3) \rangle.
\]
By (2.23), (2.21), \(|\Re \Omega|^2 = 8\) and Lemma 2.11, we have
\[
2 \left| \pi_\Phi^2 (F^2) \right|^2 = 2 \left| \pi_{\Re \Omega} (F^2) \right|^2 + 2 \left| \pi_{\omega \wedge A_-} (F^2) \right|^2
\]
\[
= 2 \left| \pi^{[4,0]} (F^2) \right|^2 - 2 \left| \pi_{[4] \Re \Omega} (F^2) \right|^2 + \left| \pi_{A_-} (\ast (\omega \wedge F^2)) \right|^2
\]
\[
= 2 \left| \pi^{[4,0]} (F^2) \right|^2 - \frac{1}{4} \langle F^2, \Re \Omega \rangle^2 + \left| \pi_{A_-} (\ast (\omega \wedge F^2)) \right|^2.
\]
Thus, by (A.15), (A.16) and (A.17), (A.2) is equivalent to
\[
\frac{1}{16} \langle F^2, \omega^2 \rangle^2 + \frac{F^4}{12} - \frac{1}{18} \ast (\omega^3 \wedge F) * (\omega \wedge F^3) + 2 \left| \pi^{[4,0]} (F^2) \right|^2
\]
\[
+ \frac{1}{4} \langle F^2, \omega^2 \rangle \langle F^2, \Re \Omega \rangle - \frac{4}{3} \langle \pi_{A_+} (F), \pi_{A_+} (\ast F^3) \rangle + \left| \pi_{A_-} (\ast (\omega \wedge F^2)) \right|^2 = \frac{1}{4} |F^2|^2.
\]
Now, we prove the following lemma.

**Lemma A.11.** We have
\[
\frac{1}{4} \langle F^2, \omega^2 \rangle \langle F^2, \Re \Omega \rangle = \frac{2}{3} \langle \pi_{A_+} (F), \pi_{A_+} (\ast F^3) \rangle - \frac{2}{3} \langle \pi_{A_-} (F), \pi_{A_-} (\ast F^3) \rangle
\]
\[
+ \frac{1}{2} \left| \pi_{A_+} (\ast (\omega \wedge F^2)) \right|^2 - \frac{1}{2} \left| \pi_{A_-} (\ast (\omega \wedge F^2)) \right|^2.
\]
Proof. Set $F(t) = F + t \omega$ for $t \in \mathbb{R}$. By Lemma 2.5, we have

\[(A.19) \quad \ast (\text{Re}\Omega \wedge (\ast F(t)^3)^2) = \frac{3}{2} \langle F(t)^2, \text{Re}\Omega \rangle \ast F(t)^4.\]

We compute

\[(A.20) \quad \left. \frac{d^2}{dt^2} \ast (\text{Re}\Omega \wedge (\ast F(t)^3)^2) \right|_{t=0} = 6 \left. \frac{d}{dt} \ast (\text{Re}\Omega \wedge (\ast F(t)^3) \wedge (\omega \wedge F(t)^2)) \right|_{t=0}
= 6 \ast (\text{Re}\Omega \wedge (3 \ast (\omega \wedge F^2) \wedge (\omega \wedge F^2) + 2(\ast F^3) \wedge (\omega^2 \wedge F)))\]

and

\[(A.21) \quad \frac{3}{2} \left. \frac{d^2}{dt^2} \langle F(t)^2, \text{Re}\Omega \rangle \ast F(t)^4 \right|_{t=0} = \frac{3}{2} \left. \frac{d}{dt} \langle 2\omega \wedge F(t), \text{Re}\Omega \rangle \ast F(t)^4 + 4\langle F(t)^2, \text{Re}\Omega \rangle \ast (\omega \wedge F(t)^3) \right|_{t=0}
= 6 \left. \frac{d}{dt} \langle F(t)^2, \text{Re}\Omega \rangle \ast (\omega \wedge F(t)^3) \right|_{t=0}
= 18 (F^2, \text{Re}\Omega) \langle F^2, \omega^2 \rangle,\]

where we use $\ast \text{Re}\Omega = \text{Re}\Omega, \ast \omega^2 = \omega^2$ and $\omega \wedge \text{Re}\Omega = 0$. Hence, (A.19), (A.20) and (A.21) imply that

\[(A.22) \quad 3 \ast (\text{Re}\Omega \wedge (\ast (\omega \wedge F^2))^2) + 2 \ast (\text{Re}\Omega \wedge (\ast (\omega^2 \wedge F) \wedge (\ast F^3))) = 3\langle F^2, \text{Re}\Omega \rangle \langle F^2, \omega^2 \rangle.\]

By Lemmas 2.9 and 2.10, we have

\[(A.23) \quad 3 \ast (\text{Re}\Omega \wedge (\ast (\omega \wedge F^2))^2) = 6 \left( |\pi_{A_+} (\ast (\omega \wedge F^2))|^2 - |\pi_{A_-} (\ast (\omega \wedge F^2))|^2 \right),\]

\[2 \ast (\text{Re}\Omega \wedge (\omega^2 \wedge F) \wedge (\ast F^3)) = 4 \ast \left( \text{Re}\Omega \wedge F \wedge (\ast F^3) \right)
= 8 \left( \langle \pi_{A_+} (F), \pi_{A_+} (\ast F^3) \rangle - \langle \pi_{A_-} (F), \pi_{A_-} (\ast F^3) \rangle \right).\]

Then, by (A.22) and (A.23), the proof is completed. \(\square\)
Then, by Lemma A.11, we obtain
\[
\frac{1}{4} \langle F^2, \omega^2 \rangle \langle F^2, \text{Re}\Omega \rangle - \frac{4}{3} \langle \pi_{A^+}(F), \pi_{A^+}(\ast F^3) \rangle + \left| \pi_{A_-} \left( \ast (\omega \wedge F^2) \right) \right|^2
\]
\[
= -\frac{2}{3} \left\{ \langle \pi_{A^+}(F), \pi_{A^+}(\ast F^3) \rangle + \langle \pi_{A^-}(F), \pi_{A^-}(\ast F^3) \rangle \right\}
+ \frac{1}{2} \left\{ \left| \pi_{A^+} \left( \ast (\omega \wedge F^2) \right) \right|^2 + \left| \pi_{A^-} \left( \ast (\omega \wedge F^2) \right) \right|^2 \right\}
\]
\[
= -\frac{2}{3} \langle \pi^{[2,0]}(F), \pi^{[2,0]}(\ast F^3) \rangle + \frac{1}{2} \left| \pi^{[2,0]} \left( \ast (\omega \wedge F^2) \right) \right|^2
\]
\[
= -\frac{2}{3} \langle \pi^{[2,0]}(F), \pi^{[2,0]}(\ast F^3) \rangle + \frac{1}{2} \left| \pi^{[4,2]} \left( \omega \wedge F^2 \right) \right|^2.
\]
Then, this together with (A.18) implies (A.13). □

**Proof of Proposition A.9.** Since
\[
\frac{1}{4!} (\omega + \sqrt{-1} F)^4 = \frac{1}{4!} \left( \omega^4 + 4 \sqrt{-1} \omega^3 \wedge F - 6 \omega^2 \wedge F^2 - 4 \sqrt{-1} \omega \wedge F^3 + F^4 \right)
\]
\[
= \left( 1 - \frac{1}{4} \ast (\omega^2 \wedge F^2) + \frac{F^4}{24} \right) + \sqrt{-1} \left( \frac{\omega^3 \wedge F - \omega \wedge F^3}{6} \right),
\]
we have
(A.24)
\[
\left| \frac{(\omega + \sqrt{-1} F)^4}{4!} \right|^2 = \left( 1 - \frac{1}{4} \ast (\omega^2 \wedge F^2) + \frac{F^4}{24} \right)^2 + \left( \frac{\omega^3 \wedge F - \omega \wedge F^3}{6} \right)^2
\]
\[
= 1 + \left( \frac{F^4}{24} \right)^2 + \left\{ -\frac{1}{2} \ast (\omega^2 \wedge F^2) + \frac{1}{36} \ast (\omega^3 \wedge F)^2 \right\}
\]
\[
+ \left\{ \frac{1}{16} \ast (\omega^2 \wedge F^2)^2 + \frac{F^4}{12} - \frac{1}{18} \ast (\omega^3 \wedge F) \ast (\omega \wedge F^3) \right\}
\]
\[
+ \left\{ -\frac{1}{48} \ast (\omega^2 \wedge F^2) \ast F^4 + \frac{1}{36} \ast (\omega \wedge F^3)^2 \right\}.
\]
Since
\[
\pi^{[4,2]} \left( \frac{(\omega + \sqrt{-1} F)^3}{3!} \right) = \pi^{[4,2]} \left( \frac{3 \sqrt{-1} \omega^2 \wedge F - 3 \omega \wedge F^2 - \sqrt{-1} F^3}{6} \right)
\]
\[
= -\frac{1}{2} \pi^{[4,2]} (\omega \wedge F^2) + \sqrt{-1} \pi^{[4,2]} \left( \frac{\omega^2 \wedge F}{2} - \frac{F^3}{6} \right)
\]
and
\[
\pi^{[4,2]} \left( \frac{\omega^2 \wedge F}{2} \right) = \frac{\omega^2 \wedge \pi^{[2,0]}(F)}{2} = \ast \pi^{[2,0]}(F)
\]
by Lemma 2.9, we have
\[(A.25)\]
\[2 \left| \pi^{[4,2]} \left( \frac{(\omega + \sqrt{-1}F)^3}{3!} \right) \right|^2 = \frac{1}{2} \left| \pi^{[4,2]}(\omega \wedge F^2) \right|^2 + 2 \left| \pi^{[2,0]} \left( F - \frac{\ast F^3}{6} \right) \right|^2 \]
\[= 2 \left| \pi^{[2,0]}(F) \right|^2 + \left\{ \frac{1}{2} \left| \pi^{[4,2]}(\omega \wedge F^2) \right|^2 - \frac{2}{3} \langle \pi^{[2,0]}(F), \pi^{[2,0]}(\ast F^3) \rangle \right\} + \frac{1}{18} \left| \pi^{[2,0]}(\ast F^3) \right|^2. \]

Since \(\pi^{[4,0]} \left( (\omega + \sqrt{-1}F)^2 \right) = -\pi^{[4,0]}(F^2)\), we have
\[(A.26)\]
\[8 \left| \pi^{[4,0]} \left( \frac{(\omega + \sqrt{-1}F)^2}{2!} \right) \right|^2 = 2 \left| \pi^{[4,0]}(F^2) \right|^2. \]

Then, by (A.24), (A.25), (A.26), Lemmas A.10 and A.2, we obtain the first identity of Proposition A.9.

By the first identity of Proposition A.9, \(\text{Im} \left( e^{-\sqrt{-1} \theta} (\omega + \sqrt{-1} F)^4/4! \right) = 0\) for \(\theta \in \mathbb{R}\) and \(\pi^{[2,0]}(F) = 0\) imply that \(\text{Re} \left( e^{-\sqrt{-1} \theta} (\omega + \sqrt{-1} F)^4/4! \right) = \sqrt{\text{det}(I_8 + F^2)}\).

We prove the converse. Suppose that \(\text{Re} \left( e^{-\sqrt{-1} \theta} (\omega + \sqrt{-1} F)^4/4! \right) = \sqrt{\text{det}(I_8 + F^2)}\) for \(\theta \in \mathbb{R}\). Then, by the first identity of Proposition A.9, (A.25) and (A.26), we have \(\text{Im} \left( e^{-\sqrt{-1} \theta} (\omega + \sqrt{-1} F)^4/4! \right) = 0\) and
\[(A.27)\]
\[\pi^{[2,0]} \left( F - \frac{\ast F^3}{6} \right) = 0, \quad \pi^{[4,2]}(\omega \wedge F^2) = 0, \quad \pi^{[4,0]}(F^2) = 0. \]

We show that (A.27) implies that \(\pi^{[2,0]}(F) = 0\). Set
\[F = F' + F'' \in [\Lambda^{1,1}] \oplus [\Lambda^{2,0}] \quad \text{and} \quad F'' = \beta + \bar{\beta} \in \Lambda ^{2,0} \oplus \Lambda ^{0,2}. \]

Since
\[F^2 = (F')^2 + 2F' \wedge F'' + (F'')^2 \in [\Lambda^{2,2}] \oplus [\Lambda^{3,1}] \oplus ([\Lambda^{4,0}] \oplus [\Lambda^{2,2}]), \]
\[F^3 = ((F')^3 + 3F' \wedge (F'')^2) + (3(F')^2 \wedge F'' + (F'')^3) \in [\Lambda^{3,3}] \oplus [\Lambda^{4,2}], \]
(A.27) is equivalent to
\[F'' - \frac{1}{6} (* 3(F')^2 \wedge F'' + (F'')^3) = 0, \quad \omega \wedge F' \wedge F'' = 0, \quad \pi^{[4,0]}((F'')^2) = 0. \]

Since \((F'')^3 = 3\beta^2 + \bar{\beta} + 3\beta \wedge \bar{\beta} \in \Lambda^{4,2} \oplus \Lambda^{2,4}\), (A.27) is equivalent to
\[\beta - \frac{1}{2} (* (F')^2 \wedge \beta + \beta^2 \wedge \bar{\beta}) = 0, \quad \omega \wedge F' \wedge \beta = 0, \quad \beta^2 = 0, \]
and hence,
\[(A.28) \quad \beta - \frac{1}{2}((F')^2 \wedge \beta) = 0, \quad \omega \wedge F' \wedge \beta = 0, \quad \beta^2 = 0.\]

Since all equations in \((A.28)\) are \(U(4)\) equivariant and \(u(4) \cong [\Lambda^{1,1}]\), we may assume that
\[
\begin{align*}
\beta &= \beta_1 f^{12} + \beta_2 f^{13} + \beta_3 f^{14} + \beta_4 f^{23} + \beta_5 f^{24} + \beta_6 f^{34}, \\
F' &= \mu_1 e^{01} + \mu_2 e^{23} + \mu_3 e^{45} + \mu_4 e^{67}, \\
\omega &= \frac{-1}{2} \left( \mu_1 f^1 \wedge f^1 + \mu_2 f^2 \wedge f^2 + \mu_3 f^3 \wedge f^3 + \mu_4 f^4 \wedge f^4 \right)
\end{align*}
\]
for \(\beta_j \in \mathbb{C}\) and \(\mu_k \in \mathbb{R}\), where \(e^i\) and \(f^j\) are defined at the beginning of Section 2.5.

Since \(\omega = (\sqrt{-1}/2) \sum f^i \wedge f^j\), we see that
\[
4\omega \wedge F' = (\mu_1 + \mu_2) f^{12} \wedge f^{12} + (\mu_1 + \mu_3) f^{13} \wedge f^{13} + (\mu_1 + \mu_4) f^{14} \wedge f^{14}
\]
\[
+ (\mu_2 + \mu_3) f^{23} \wedge f^{23} + (\mu_2 + \mu_4) f^{24} \wedge f^{24} + (\mu_3 + \mu_4) f^{34} \wedge f^{34}.
\]
Hence, \(\omega \wedge F' \wedge \beta = 0\) is equivalent to
\[(A.29) \quad \beta_1 (\mu_3 + \mu_4) = \beta_2 (\mu_2 + \mu_4) = \beta_3 (\mu_2 + \mu_3)
\]
\[
= \beta_4 (\mu_1 + \mu_3) = \beta_5 (\mu_1 + \mu_3) = \beta_6 (\mu_1 + \mu_2) = 0.
\]

Since
\[
2(F')^2 = \mu_1 \mu_2 f^{12} \wedge f^{12} + \mu_1 \mu_3 f^{13} \wedge f^{13} + \mu_1 \mu_4 f^{14} \wedge f^{14}
\]
\[
+ \mu_2 \mu_3 f^{23} \wedge f^{23} + \mu_2 \mu_4 f^{24} \wedge f^{24} + \mu_3 \mu_4 f^{34} \wedge f^{34},
\]
\[
2(F')^2 \wedge \beta = \beta_1 \mu_3 \mu_4 f^{1234} \wedge f^{1234} - \beta_2 \mu_2 \mu_4 f^{1234} \wedge f^{1234} + \beta_3 \mu_2 \mu_3 f^{1234} \wedge f^{1234}
\]
\[
+ \beta_4 \mu_1 \mu_4 f^{1234} \wedge f^{1234} - \beta_5 \mu_1 \mu_3 f^{1234} \wedge f^{1234} + \beta_6 \mu_1 \mu_2 f^{1234} \wedge f^{1234}
\]
and \(* (f^{1234} \wedge f^{34}) = 4 f^{12}\), etc., \(\beta - (1/2) * ((F')^2 \wedge \beta) = 0\) is equivalent to
\[(A.30) \quad \beta_1 (1 - \mu_3 \mu_4) = \beta_2 (1 - \mu_2 \mu_4) = \beta_3 (1 - \mu_2 \mu_3)
\]
\[
= \beta_4 (1 - \mu_1 \mu_3) = \beta_5 (1 - \mu_1 \mu_3) = \beta_6 (1 - \mu_1 \mu_2) = 0.
\]

Then, \((A.29)\) and \((A.30)\) imply that \(\beta = 0\). Indeed, Since \(\beta_1 (\mu_3 + \mu_4) = 0\), we have
\[
0 = \beta_1 (1 - \mu_3 \mu_4) = \beta_1 (1 + \mu_3^2).
\]
Since \(\mu_3 \in \mathbb{R}\), we obtain \(\beta_1 = 0\). Similarly, we obtain \(\beta_2 = \ldots = \beta_6 = 0\), and hence, \(\beta = 0\).

\[\square\]

**Appendix B. Notation**

We summarize the notation used in this paper. We use the following for a smooth manifold \(X\).
When \((X, g)\) is an oriented Riemannian manifold, we use the following.

| Notation | Meaning |
|----------|---------|
| \(v^\flat \in T^*X\) | \(v^\flat = g(v, \cdot)\) for \(v \in TX\) |
| \(\alpha^\flat \in TX\) | \(\alpha = g(\alpha^\flat, \cdot)\) for \(\alpha \in T^*X\) |
| \(\text{vol} = \text{vol}_g\) | The volume form induced from \(g\) |

When \((X, J)\) is a complex manifold, we use the following.

| Notation | Meaning |
|----------|---------|
| \(\Lambda^{p,q}\) | \(\Lambda^{p,q} = \Lambda^p(T^{1,0}X)^* \otimes \Lambda^q(T^{0,1}X)^*\) |
| \(\Omega^{p,q}\) | \(\Omega^{p,q} = \Gamma(X, \Lambda^{p,q})\) |
| \([\Lambda^{p,q}]\) (for \(p \neq q\)) | \(\{ \alpha \in \Lambda^{p,q} \oplus \Lambda^{q,p} \mid \bar{\alpha} = \alpha \} \subset \Lambda^{p+q}T^*X\) |
| \([\Lambda^{p,p}]\) | \(\{ \alpha \in \Lambda^{p,p} \mid \bar{\alpha} = \alpha \} \subset \Lambda^{2p}T^*X\) |
| \([\Omega^{p,q}]\) (for \(p \neq q\)) | \(\Gamma(X, [\Lambda^{p,q}])\) |
| \([\Omega^{p,p}]\) | \(\Gamma(X, [\Lambda^{p,p}])\) |
| \(\pi_S\) | The orthogonal projection \(\pi_S : \Lambda^kT^*X \to S\) or \(\Omega^k \to \Gamma(X, S)\) for a subbundle \(S \subset T^*X\) |
| \(\pi[p,q]\) | The projection \(\pi[p,q] : \Lambda^{p+q}T^*X \to [\Lambda^{p,q}]\) or \(\Omega^{p+q} \to [\Omega^{p,q}]\) |

When \(X\) is a manifold with a \(G_2\)- or \(\text{Spin}(7)\)-structure, we use the following.

| Notation | Meaning |
|----------|---------|
| \(\Lambda^{k,T^*X}_\ell\) | The subspace of \(\Lambda^kT^*X\) corresponding to an \(\ell\)-dimensional irreducible subrepresentation |
| \(\Omega^k_\ell\) | \(\Omega^k_\ell = \Gamma(X, \Lambda^{k,T^*X}_\ell)\) |
| \(\pi_k^\ell\) | The projection \(\Lambda^kT^*X \to \Lambda^kT^*X\) or \(\Omega^k \to \Omega^k_\ell\) |

**References**

[1] R. Bott, L. W. Tu. Differential forms in algebraic topology. Graduate Texts in Mathematics, 82. Springer-Verlag, New York-Berlin, 1982. xiv+331 pp. ISBN: 0-387-90613-4
[2] G. Chen. Supercritical deformed Hermitian-Yang-Mills equation. arXiv:2005.12202.
[3] T. C. Collins, A. Jacob and S.-T. Yau. (1, 1) forms with specified Lagrangian phase: a priori estimates and algebraic obstructions. Camb. J. Math. 8 (2020), no. 2, 407–452.
[4] T. C. Collins and S.-T. Yau. Moment maps, nonlinear PDE, and stability in mirror symmetry. arXiv:1811.04824.
[5] D. Gayet. Smooth moduli spaces of associative submanifolds. Q. J. Math. 65 (2014), no. 4, 1213–1240.
[6] R. Harvey and H. B. Lawson. Calibrated geometries, Acta Math. 148 (1982), 47–157.
[7] A. Hatcher. Algebraic topology. Cambridge University Press, Cambridge, 2002. xii+544 pp. ISBN: 0-521-79160-X; 0-521-79540-0
[8] A. Jacob and S.-T. Yau. A special Lagrangian type equation for holomorphic line bundle. Math. Ann. 369 (2017), no. 1-2, 869–898.
[9] S. Karigiannis and N.-C. Leung. Hodge theory for $G_2$-manifolds: intermediate Jacobians and Abel-Jacobi maps. Proc. Lond. Math. Soc. (3) 99 (2009), no. 2, 297–325.
[10] K. Kawai, H. V. Lê and L. Schwachhöfer. The Frölicher-Nijenhuis bracket and the geometry of $G_2$- and Spin(7)-manifolds. Ann. Mat. Pura Appl. (4) 197 (2018), no. 2, 411–432.
[11] K. Kawai and H. Yamamoto. Deformation theory of deformed Hermitian Yang-Mills connections and deformed Donaldson-Thomas connections. to appear in J. Geom. Anal., arXiv:2004.00532.
[12] K. Kawai and H. Yamamoto. The real Fourier–Mukai transform of Cayley cycles. Pure Appl. Math. Q. 17 (2021), no. 5, 1861–1898.
[13] K. Kawai and H. Yamamoto. Deformation theory of deformed Donaldson–Thomas connections for Spin(7)-manifolds. J. Geom. Anal. 31 (2021), no. 12, 12098–12154.
[14] S. Kobayashi. Differential geometry of complex vector bundles. Princeton Legacy Library, Princeton University Press, Princeton, NJ, (2014).
[15] C. Lewis. Spin(7) instantons. D.Phil. thesis Oxford University (1998).
[16] J. D. Lotay and G. Oliveira. Examples of deformed $G_2$-instantons/Donaldson-Thomas connections. arXiv:2007.11304.
[17] M. Mariño, R. Minasian, G. Moore and A. Strominger. Nonlinear instantons from supersymmetric $p$-branes. J. High Energy Phys. 2000, no. 1, Paper 5, 32 pp.
[18] A. Moroianu, P.-A. Nagy, U. Semmelmann. Deformations of nearly Kähler structures. Pacific J. Math. 235 (2008), no. 1, 57–72.
[19] V. Muñoz. Spin(7)-instantons, stable bundles and the Bogomolov inequality for complex 4-tori. J. Math. Pures Appl. (9) 102 (2014), no. 1, 124–152.
[20] V. Muñoz and C.S. Shahbazi. Transversality for the moduli space of Spin(7)-instantons. Rev. Math. Phys. 32 (2020), 2050013, 47 pp.
[21] M. Ohst. Deformations of Compact Cayley Submanifolds with Boundary. arXiv:1405.7886.
[22] K. Smoczyk. A canonical way to deform a Lagrangian submanifold. arXiv:dg-ga/9605005.
[23] G. Tian. Gauge theory and calibrated geometry. I. Ann. of Math. (2) 151 (2000), no. 1, 193–268.
[24] Y. Wang. Moduli spaces of $G_2$-instantons and Spin(7)-instantons on product manifolds. Ann. Henri Poincaré 21 (2020), no. 9, 2997–3033.
[25] X.-P. Zhu, Lectures on mean curvature flows. AMS/IP Studies in Advanced Mathematics, 32. American Mathematical Society, Providence, RI; International Press, Somerville, MA, 2002. x+150 pp. ISBN: 0-8218-3311-1
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