THE DIVISORS OF PRYM SEMICANONICAL PENCILS

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Abstract. In the moduli space \( \mathcal{R}_g \) of double étale covers of curves of a fixed genus \( g \), the locus of covers of curves with a semicanonical pencil decomposes as the union of two divisors \( T_{e g} \) and \( T_{o g} \). Adapting arguments of Teixidor for the divisor of curves having a semicanonical pencil, we prove that both divisors are irreducible and compute their divisor classes in the Deligne-Mumford compactification \( \overline{\mathcal{R}}_g \).

1. Introduction

Let \( \pi : \tilde{C} \to C \) be a double étale cover between smooth curves of genus \( g = g(C) \) and \( \tilde{g} = g(\tilde{C}) = 2g - 1 \), and denote by \( (P, \Xi) \) its (principally polarized) Prym variety.

In his fundamental work \([12]\), Mumford classified the singularities of the theta divisor \( \Xi \). More precisely, he considered a translation \( P^+ \) of the Prym variety to \( \text{Pic}^{2g-2} \tilde{C} = \text{Pic}^{\tilde{g}-1} \tilde{C} \), together with a canonical theta divisor \( \Xi^+ \subset P^+ := \{ M \in \text{Pic}^{2g-2} \tilde{C} \mid N_{\pi^*}(M) = \omega_C, \ h^0(\tilde{C}, M) \text{ even} \} \),

\[ \Xi^+ = \{ M \in P^+ \mid h^0(\tilde{C}, M) \geq 2 \} \]

Then every singular point of \( \Xi^+ \) is stable \((M \in \Xi^+ \text{ with } h^0(\tilde{C}, M) \geq 4)\) or exceptional \((M = \pi^*L \otimes A \in \Xi^+ \text{ such that } h^0(C, L) \geq 2 \text{ and } h^0(\tilde{C}, A) > 0)\).

Assume \( C \) has a semicanonical pencil, that is, an even theta-characteristic \( L \) with \( h^0(C, L) \geq 2 \) (in the literature, this is also frequently referred to as a vanishing theta-null). If \( h^0(\tilde{C}, \pi^*L) \) is furthermore even, then \( M = \pi^*L \in \Xi^+ \) is an example of exceptional singularity. In that case, \( L \) is called an even semicanonical pencil for the cover \( \pi \), and the Prym variety \( (P, \Xi) \) belongs to the divisor \( \theta_{null} \subset A_{g-1} \) of principally polarized abelian varieties whose theta divisor contains a singular 2-torsion point.

In the paper \([2]\), Beauville showed that the Andreotti-Mayer locus \( N_0 = \{ (A, \Xi) \in A_4 \mid \text{Sing} (\Xi) \text{ is non-empty} \} \)

in \( A_4 \) is the union of two irreducible divisors: the (closure of the) Jacobian locus \( J_4 \) and \( \theta_{null} \).

An essential tool for the proof is the extension of the Prym map \( P_g : \mathcal{R}_g \to \mathcal{A}_{g-1} \) to a proper map \( \tilde{P}_g : \tilde{\mathcal{R}}_g \to \tilde{\mathcal{A}}_{g-1} \), by considering admissible covers instead of only smooth covers. In the case \( g = 5 \), this guarantees that every 4-dimensional principally polarized abelian variety is a Prym variety (i.e. the dominant map \( P_5 \) is replaced by the surjective map \( \tilde{P}_5 \)).

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Then, one of the key points in Beauville’s work is an identification of the coverings whose (generalized) Prym variety is contained in $\theta_{\text{null}}$. Indeed, the results in [2, Section 7] together with [2, Theorem 4.10] show that

$$T^e = (\text{closure in } \bar{R}_5 \text{ of}) \left\{ [\pi : \bar{C} \to C] \in R_5 \mid \text{the cover } \pi \text{ has an even semicanonical pencil} \right\}$$

is irreducible and equals $\bar{P}_5^{-1}(\theta_{\text{null}})$. Therefore, the irreducibility of $\theta_{\text{null}} \subset A_4$ is obtained from the irreducibility of $T^e$; the proof of the latter starts by noticing that

$$T = \{ [C] \in M_5 \mid C \text{ has a semicanonical pencil} \}$$

is an irreducible divisor of $M_5$.

Now let us consider double étale covers of curves with a semicanonical pencil, in arbitrary genus. For a fixed $g \geq 3$, let $T_g \subset M_g$ denote the locus of (isomorphism classes of) curves with a semicanonical pencil (i.e. with an even, effective theta-characteristic). Note that $T_g$ is the divisorial part of the locus of curves admitting a theta-characteristic of positive (projective) dimension ([16]).

The general element of $T_g$ has a unique such theta-characteristic (which is a semicanonical pencil $L$ with $h^0(C, L) = 2$), and the pullback of $T_g$ to $R_g$ decomposes as a union $T^e_g \cup T^o_g$ according to the parity of $h^0(\bar{C}, \pi^*L)$. In other words, the general element of $T^e_g$ (resp. $T^o_g$) is a cover with an even semicanonical pencil (resp. an odd semicanonical pencil).

In view of Beauville’s work, it is natural to ask whether $T^e_g$ and $T^o_g$ are irreducible divisors, and to ask about the behaviour of the restricted Prym maps $\bar{P}_g|_{T^e_g}$ and $\bar{P}_g|_{T^o_g}$.

This paper exclusively deals with the first question, and studies the divisors $T^e_g$ and $T^o_g$ of even and odd semicanonical pencils. Aside from its independent interest, it provides tools for attacking the second question; a study of the restricted Prym maps $\bar{P}_g|_{T^e_g}$ and $\bar{P}_g|_{T^o_g}$ is carried out in the subsequent paper [9].

Coming back to the first question, the divisor $T_g \subset M_g$ was studied by Teixidor in [17]. Using the theory of limit linear series on curves of compact type developed by Eisenbud and Harris in [3], Teixidor proved the irreducibility of $T_g$ and computed the class of its closure in the Deligne-Mumford compactification $\bar{M}_g$.

In our case, we will work in the Deligne-Mumford compactification $\bar{R}_g$ of $R_g$ (first considered in [2, Section 6], during the construction of the proper Prym map). Following closely Teixidor’s approach, we obtain natural analogues of her results for the two divisors of Prym semicanonical pencils:

**Theorem A.** Let $[T^e_g], [T^o_g] \in \text{Pic}(\bar{R}_g)_{\mathbb{Q}}$ denote the classes of (the closures of) $T^e_g$, $T^o_g$ in the Deligne-Mumford compactification $\bar{R}_g$. Then, the following equalities hold:

$$[T^e_g] = a\lambda - b'_00' - b''_00'' - b'_00'_{\text{ram}}\delta_0^{\text{ram}} - \sum_{i=1}^{[g/2]} (b_i\delta_i + b_{-i}\delta_{-i}) + b_{i; g-1}\delta_{i; g-1}),$$

$$[T^o_g] = c\lambda - d'_00' - d''_00'' - d'_00'_{\text{ram}}\delta_0^{\text{ram}} - \sum_{i=1}^{[g/2]} (d_i\delta_i + d_{-i}\delta_{-i}) + d_{i; g-1}\delta_{i; g-1}),$$
where
\[ a = 2^{g-3}(2^{g-1} + 1), \quad c = 2^{2g-4}, \]
\[ b_0 = 2^{2g-7}, \quad d_0 = 2^{2g-7}, \]
\[ b_0^\text{ram} = 2^{g-5}(2^{g-1} + 1), \quad d_0^\text{ram} = 2^{g-5}(2^{g-1} - 1), \]
\[ b_i = 2^{g-3}(2^{g-i} - 1)(2^{i-1} - 1), \quad d_i = 2^{g+i-4}(2^{g-i} - 1), \]
\[ b_7 = 2^{g-3}(2^{g-1} - 2^{i-1} - 2^{g-i-1} + 1), \quad d_i \quad d_i,g-i = 2^{g-3}(2^{g-1} - 2^{g-i} - 2^{i-1}). \]

**Theorem B.** For every \( g \neq 4 \) the divisors \( T^e_g \) and \( T^o_g \) are irreducible.

A crucial role in the proofs is played by the intersection of \( T^e_g \) and \( T^o_g \) with the boundary divisors in \( \overline{\mathcal{R}}_g \) of covers of reducible curves. This is the content of Proposition 3.1. Then in section 3 we prove Theorem A by intersecting \( T^e_g \) and \( T^o_g \) with appropriate test curves in \( \overline{\mathcal{R}}_g \).

The proof of Theorem B for \( g \geq 5 \) is given in section 4, and combines monodromy arguments with the intersection of \( T^e_g \) and \( T^o_g \) with the boundary divisor \( \Delta_1 \subset \overline{\mathcal{R}}_g \). We point out that the irreducibility for \( g = 3 \) can be immediately checked in terms of hyperelliptic curves (Example 2.1), whereas the irreducibility of \( T^e_4 \) and \( T^o_4 \) is obtained in the paper [9] as a consequence of the study of the restricted Prym maps \( P_4|_{T^e_4} \) and \( P_4|_{T^o_4} \).

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### 2. Preliminaries

#### 2.1. The moduli space \( \overline{\mathcal{R}}_g \).

This part is a brief review of the Deligne-Mumford compactification \( \overline{\mathcal{R}}_g \) and its boundary divisors. We follow the presentation of [5, Section 1]; the reader is referred to it for further details.

Let \( \mathcal{M}_g \) be the moduli space of smooth curves of genus \( g \), and let \( \overline{\mathcal{M}}_g \) be its Deligne-Mumford compactification by stable curves. Following the standard notations, we denote by \( \Delta_i \) (\( i = 0, \ldots, \lfloor g/2 \rfloor \)) the irreducible divisors forming the boundary \( \overline{\mathcal{M}}_g \setminus \mathcal{M}_g \). The general point of \( \Delta_0 \) is an irreducible curve with a single node, whereas the general point of \( \Delta_i \) (for \( i \geq 1 \)) is the union of two smooth curves of genus \( i \) and \( g-i \), intersecting transversely at a point.

The classes \( \delta_i \) of the divisors \( \Delta_i \), together with the Hodge class \( \lambda \), are well known to form a basis of the rational Picard group \( \text{Pic}(\overline{\mathcal{M}}_g) \mathbb{Q} \).

We denote by \( \mathcal{R}_g \) the moduli space of connected double étale covers of smooth curves of genus \( g \). In other words, \( \mathcal{R}_g \) parametrizes isomorphism classes of pairs \((C, \eta)\), where \( C \) is smooth of genus \( g \) and \( \eta \in JC_2 \setminus \{O_C\} \). It comes with a natural forgetful map \( \pi : \mathcal{R}_g \to \mathcal{M}_g \) which
is étale of degree $2^g - 1$. Then, the Deligne-Mumford compactification $\overline{R}_g$  is obtained as the normalization of $\overline{\mathcal{M}}_g$ in the function field of $\mathcal{R}_g$. This gives a commutative diagram

$$
\begin{array}{ccc}
\mathcal{R}_g & \rightarrow & \overline{\mathcal{R}}_g \\
\pi \downarrow & & \downarrow \\
\mathcal{M}_g & \rightarrow & \overline{\mathcal{M}}_g
\end{array}
$$

where $\overline{\mathcal{R}}_g$ is normal and the morphism $\overline{\mathcal{R}}_g \rightarrow \overline{\mathcal{M}}_g$ (that we will denote by $\pi$ as well) is finite. Beauville’s partial compactification $\tilde{\mathcal{R}}_g$ by admissible covers admits a natural inclusion into $\overline{\mathcal{R}}_g$.

As proved in [1], the variety $\overline{\mathcal{R}}_g$ parametrizes isomorphism classes of Prym curves of genus $g$, that is, isomorphism classes of triples $(X, \eta, \beta)$ where:

- $X$ is a quasi-stable curve of genus $g$, i.e. $X$ is semistable and any two of its exceptional components are disjoint$^1$.
- $\eta \in \text{Pic}^0(X)$ is a line bundle of total degree 0, such that $\eta|_E = \mathcal{O}_E(1)$ for every exceptional component $E \subset X$.
- $\beta : \eta^\otimes 2 \rightarrow \mathcal{O}_X$ is generically nonzero over each non-exceptional component of $X$.

In case that $\beta$ is clear from the context, by abuse of notation the Prym curve $(X, \eta, \beta)$ will be often denoted simply by $(X, \eta)$.

Then the morphism $\pi : \overline{\mathcal{R}}_g \rightarrow \overline{\mathcal{M}}_g$ sends (the class of) $(X, \eta, \beta)$ to (the class of) the stable model $\text{st}(X)$, obtained by contraction of the exceptional components of $X$.

Using pullbacks of the boundary divisors of $\overline{\mathcal{M}}_g$, the boundary $\overline{\mathcal{R}}_g \setminus \mathcal{R}_g$ admits the following description (see [5, Examples 1.3 and 1.4]):

1. Let $(X, \eta, \beta)$ be a Prym curve, such that $\text{st}(X)$ is the union of two smooth curves $C_i$ and $C_{g-i}$ (of respective genus $i$ and $g-i$) intersecting transversely at a point $P$. In such a case $X = \text{st}(X)$, and giving a 2-torsion line bundle $\eta \in \text{Pic}^0(X)_2$ is the same as giving a nontrivial pair $(\eta_i, \eta_{g-i}) \in (JC_i)_2 \times (JC_{g-i})_2$.

Then the preimage $\pi^{-1}(\Delta_i)$ decomposes as the union of three irreducible divisors (denoted by $\Delta_i$, $\Delta_{g-i}$ and $\Delta_{i;g-i}$), which are distinguished by the behaviour of the 2-torsion bundle. More concretely, their general point is a Prym curve $(X, \eta)$, where $X = C_i \cup_P C_{g-i}$ is a reducible curve as above and the pair $\eta = (\eta_i, \eta_{g-i})$ satisfies:

- $\eta_{g-i} = \mathcal{O}_{C_{g-i}}$, in the case of $\Delta_i$,
- $\eta_i = \mathcal{O}_{C_i}$, in the case of $\Delta_{g-i}$,
- $\eta_i \neq \mathcal{O}_{C_i}$ and $\eta_{g-i} \neq \mathcal{O}_{C_{g-i}}$, in the case of $\Delta_{i;g-i}$.

2. Let $(X, \eta, \beta)$ be a Prym curve, such that $\text{st}(X)$ is the irreducible nodal curve obtained by identification of two points $p, q$ on a smooth curve $C$ of genus $g-1$.

If $X = \text{st}(X)$ and $\nu : C \rightarrow X$ denotes the normalization, then $\eta \in \text{Pic}^0(X)_2$ is determined by the choice of $\eta_C = \nu^*(\eta) \in JC_2$ and an identification of the fibers $\eta_C(p)$ and $\eta_C(q)$.

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$^1$Recall that a smooth rational component $E \subset X$ is called exceptional if $\sharp E \cap X \setminus E = 2$, namely if it intersects the rest of the curve in exactly two points.
If \( \eta_C = \mathcal{O}_C \), there is only one possible identification of \( \mathcal{O}_C(p) \) and \( \mathcal{O}_C(q) \) (namely identification by \(-1\)) giving a nontrivial \( \eta \in \text{Pic}^0(X)_2 \). The corresponding element \((X, \eta)\) may be regarded as a Wirtinger cover of \( X \).

If \( \eta_C \neq \mathcal{O}_C \), for each of the \( 2^{g-2} - 1 \) choices of \( \eta_C \) there are two possible identifications of \( \mathcal{O}_C(p) \) and \( \mathcal{O}_C(q) \). The \( 2(2^{g-2} - 1) \) corresponding Prym curves \((X, \eta)\) are non-admissible covers of \( X \).

If \( X \neq \text{st}(X) \), then \( X \) is the union of \( C \) with an exceptional component \( E \) through the points \( p \) and \( q \). The line bundle \( \eta \in \text{Pic}^0(X) \) must satisfy \( \eta|_E = \mathcal{O}_E(1) \) and \( \eta|_C \otimes 2 = \mathcal{O}_C(p - q) \), which gives \( 2^{2g-2} \) possibilities. The corresponding Prym curves \((X, \eta)\) give Beauville admissible covers of \( \text{st}(X) \).

It follows that \( \pi^{-1}(\Delta_0) = \Delta_0' \cup \Delta_0'' \cup \Delta_0^{ram} \), where \( \Delta_0' \) (resp. \( \Delta_0'' \), \( \Delta_0^{ram} \)) is an irreducible divisor whose general point is a non-admissible (resp. Wirtinger, resp. Beauville admissible) cover. Moreover, \( \Delta_0^{ram} \) is the ramification locus of \( \pi \) (see [5, Page 763] or [1, Section 3]).

In terms of divisor classes, we have equalities

\[
\pi^*(\delta_i) = \delta_i + \delta_{g-i} + \delta_{i-g-i}, \quad \pi^*(\delta_0) = \delta_0' + \delta_0'' + 2\delta_0^{ram}
\]

where of course \( \delta_i, \delta_{g-i}, \delta_{i-g-i} \) (\( 1 \leq i \leq \lfloor g/2 \rfloor \)) and \( \delta_0', \delta_0'', \delta_0^{ram} \) are the classes of the boundary divisors of \( \overline{R}_g \). These boundary classes, together with the pullback (also denoted by \( \lambda \)) of the Hodge class of \( \overline{M}_g \), form a basis of the rational Picard group \( \text{Pic}(\overline{R}_g)_{\mathbb{Q}} \).

2.2. Divisors of Prym semicanonical pencils. If \( C \) is a smooth curve of genus \( g \geq 3 \), by semicanonical pencil on \( C \) we mean an even, effective theta-characteristic. By dimension of a theta-characteristic \( L \) we mean the (projective) dimension \( h^0(C, L) - 1 \) of the linear system \( |L| \).

The locus of smooth curves admitting a semicanonical pencil is a divisor in \( \mathcal{M}_g \), whose irreducibility was proved in [17, Theorem 2.4]. In the same paper, the class of its closure \( \mathcal{T}_g \) in \( \overline{M}_g \) was computed.

Since the parity of theta-characteristics remains constant in families ([13]), the pullback of \( \mathcal{T}_g \) to \( \overline{R}_g \) decomposes as \( \pi^{-1}(\mathcal{T}_g) = \mathcal{T}_g^e \cup \mathcal{T}_g^o \), where \( \mathcal{T}_g^e \) (resp. \( \mathcal{T}_g^o \)) is the closure in \( \overline{R}_g \) of the set

\[
\{(C, \eta) \in \mathcal{R}_g \mid C \text{ has a semicanonical pencil } L \text{ with } h^0(C, L \otimes \eta) \text{ even}\}
\]

(resp. \( \{(C, \eta) \in \mathcal{R}_g \mid C \text{ has a semicanonical pencil } L \text{ with } h^0(C, L \otimes \eta) \text{ odd}\}\))

Note that both \( \mathcal{T}_g^e \) and \( \mathcal{T}_g^o \) have pure codimension 1 in \( \overline{R}_g \), since their union is the pullback by a finite map of an irreducible divisor. Furthermore, the restriction

\[
\pi|_{\mathcal{T}_g^e} : \mathcal{T}_g^e \to \mathcal{T}_g \quad (\text{resp. } \pi|_{\mathcal{T}_g^o} : \mathcal{T}_g^o \to \mathcal{T}_g)
\]

is surjective and generically finite of degree \( 2^{g-1}(2^g + 1) - 1 \) (resp. of degree \( 2^{g-1}(2^g - 1) \)). This follows from the fact that a general element of \( \mathcal{T}_g \) has a unique semicanonical pencil ([16, Theorem 2.16]), as well as from the number of even and odd theta-characteristics on a smooth curve.

Example 2.1. When \( g = 3 \) a semicanonical pencil is the same as a \( g_1^1 \), and thus the divisor \( \mathcal{T}_3 \subset \overline{M}_3 \) equals the hyperelliptic locus \( \mathcal{H}_3 \). Of course, the semicanonical pencil on every
smooth curve $C \in \mathcal{T}_3$ is unique. The 63 non-trivial elements of $JC_2$ can be represented by linear combinations of the Weierstrass points $R_1, \ldots, R_8$ as follows:

- Those represented as a difference of two Weierstrass points, $\eta = \mathcal{O}_C(R_i - R_j)$, form a set of $\binom{8}{2} = 28$ elements. Observe that in this case the theta-characteristic $g_1^2 \otimes \eta = \mathcal{O}_C(2R_j + R_i - R_j) = \mathcal{O}_C(R_i + R_j)$ is odd.
- Those expressed as a linear combination of four distinct Weierstrass points, $\eta = \mathcal{O}_C(R_i + R_j - R_k - R_l)$, form a set of $\binom{4}{2} = 35$ elements. According to the number of odd and even theta-characteristics on a genus 3 curve, in this case $g_1^2 \otimes \eta$ is even.

Hence we obtain

$$\mathcal{T}_3^e = (\text{closure of}) \{(C, \eta) \in \mathcal{R}_3 \mid C \text{ hyperelliptic, } \eta = \mathcal{O}_C(R_i - R_j)\} \subset \mathcal{R}_3$$

$$\mathcal{T}_3^o = (\text{closure of}) \{(C, \eta) \in \mathcal{R}_3 \mid C \text{ hyperelliptic, } \eta = \mathcal{O}_C(R_i + R_j - R_k - R_l)\} \subset \mathcal{R}_3$$

and, since monodromy on hyperelliptic curves acts transitively on tuples of Weierstrass points, it turns out that both divisors $\mathcal{T}_3^o$ and $\mathcal{T}_3^e$ are irreducible.

3. Proof of Theorem A

We denote by $[\mathcal{T}_g^e], [\mathcal{T}_g^o] \in \text{Pic}(\mathcal{R}_g)\mathbb{Q}$ the classes in $\mathcal{R}_g$ of the divisors $\mathcal{T}_g^e$ and $\mathcal{T}_g^o$. This section is entirely devoted to proving Theorem A.

First of all, observe that the pullback of the class $[\mathcal{T}_g] \in \text{Pic}(\mathcal{M}_g)\mathbb{Q}$ (computed in [17, Proposition 3.1]) expresses $[\mathcal{T}_g^e] + [\mathcal{T}_g^o]$ as

$$\pi^*[\mathcal{T}_g] = 2^{g-3} \left((2^g + 1)\lambda - 2^{g-3}(\delta_0^e + \delta_0^o + 2\delta_{\text{ram}}) - \sum_{i=1}^{\frac{g}{2}} (2^{g-i} - 1)(2^i - 1)(\delta_i + \delta_{g-i} + \delta_{i:g-i})\right).$$

This relation, together with the linear independence of the basic classes considered in $\mathcal{R}_g$, simplifies the computations: if we know a coefficient for one of the divisors, then we also know the coefficient corresponding to the same basic class for the other divisor. Keeping this in mind, the coefficients of Theorem A can be determined by essentially following three steps:

1. The pushforward $\pi_*[\mathcal{T}_g^e]$ easily gives the coefficient $a$ (hence $c$), as well as a relation between $b_0', b_0''$ and $b_0''_{\text{ram}}$ (hence between $d_0', d_0''$ and $d_0''_{\text{ram}}$).
2. We adapt an argument of Teixidor [17] to compute the coefficients $b_i, b_{g-i}$ and $b_{i:g-i}$ for every $i \geq 1$: first we describe the intersection of $\mathcal{T}_g^e$ with the boundary divisors $\Delta_i, \Delta_{g-i}$ and $\Delta_{i:g-i}$, and then we intersect $\mathcal{T}_g^o$ with certain test curves.
3. Finally, $d_0'$ and $d_0''$ are obtained intersecting $\mathcal{T}_g^o$ with test curves contained inside $\Delta_0'$ and $\Delta_0''$ respectively. The relation obtained in (1) determines $d_0''_{\text{ram}}$ as well.

For step (1), note that on the one hand

$$\pi_*[\mathcal{T}_g^e] = \deg(\mathcal{T}_g^e \rightarrow \mathcal{T}_g) \cdot [\mathcal{T}_g] = (2^{g-1}(2^g + 1) - 1)2^{g-3} \left((2^g + 1)\lambda - 2^{g-3}\delta_0 - \ldots\right)$$

$\text{Division by } 2 \text{ comes from the fact that any two complementary sets of four Weierstrass points induce the same two-torsion line bundle.}$
where \ldots is a expression involving only the classes \(\delta_1, \ldots, \delta_{[g/2]}\). On the other hand

\[
\pi_*[T_g^e] = \alpha \pi_* \lambda - b_0' \pi_* \delta_0' - b_0'' \pi_* \delta_0'' - b_{\text{ram}}' \pi_* \delta_{\text{ram}}' - \sum_{i=1}^{[g/2]} (b_i \pi_* \delta_i + b_{g-i} \pi_* \delta_{g-i} + b_{i,g-i} \pi_* \delta_{i,g-i})
\]

and, since \(\pi_* \lambda = \pi_*(\pi^* \lambda) = \deg \pi \cdot \lambda\) and the divisors \(\Delta_0', \Delta_0''\) and \(\Delta_{\text{ram}}'\) of \(\overline{\Theta}_g\) have respective degrees \(2(2g-2-1), 1\) and \(2^g-2\) over \(\Delta_0 \subset \overline{\mathcal{M}}_g\), we obtain

\[
\pi_*[T_g^e] = a(2^g-1)\lambda - (2(2^g-2-1)b_0' + b_0'' + 2^{g-2}b_{\text{ram}}')\delta_0 + \ldots
\]

where \ldots again denotes a linear combination of \(\delta_1, \ldots, \delta_{[g/2]}\).

Using that \(\lambda, \delta_0, \ldots, \delta_{[g/2]} \in \text{Pic}(\overline{\mathcal{M}}_g)_Q\) are linearly independent, we can compare the coefficients of \(\lambda\) and \(\delta_0\). Comparison for \(\lambda\) yields

\[
a = \frac{(2^g-1)(2^g+1) - 2^{g-3}(2^g+1)}{2^g-1} = 2^{g-3}(2^g-1),
\]

due to the relation \(a + c = 2^{g-3}(2^g+1)\).

Comparison for \(\delta_0\) gives

\[
(2^{g-1}-2)b_0' + b_0'' + 2^{g-2}b_{\text{ram}}' = 2^{g-6}(2^g-1) - 1,
\]

or equivalently

\[
(2^{g-1}-2)d_0' + d_0'' + 2^{g-2}d_{\text{ram}}' = 2^{3g-7}(2^g-1).
\]

In step (2), the key point is the following description of the intersection of \(T_g^e\) and \(T_g^o\) with the preimages \(\pi^{-1}(\Delta_i)\). It is nothing but an adaptation of [17, Proposition 1.2].

**Proposition 3.1.** For \(i \geq 1\), the general point of the intersection \(T_g^e \cap \pi^{-1}(\Delta_i)\) (resp. \(T_g^o \cap \pi^{-1}(\Delta_i)\)) is a pair \((C, \eta)\) where:

(i) The curve \(C\) is the union at a point \(P\) of two smooth curves \(C_i\) and \(C_{g-i}\) of respective genera \(i\) and \(g-i\), and satisfies one of these four conditions \((j = i, g-i)\):

\(\alpha_j\) \(C_j\) has a 1-dimensional (even) theta-characteristic \(L_j\). In this case, the 1-dimensional limit theta-characteristics on \(C\) are determined by the aspects \(|L_j| + (g-j)P\) on \(C_j\) and \(|L_{g-j} + 2P| + (j-2)P\) on \(C_{g-j}\), where \(L_{g-j}\) is any even theta-characteristic on \(C_{g-j}\).

\(\beta_j\) \(P\) is in the support of an effective (0-dimensional) theta-characteristic \(L_j\) on \(C_j\). The aspects of the 1-dimensional limit theta-characteristics on \(C\) are \(|L_j| + (g-j-1)P\) on \(C_j\) and \(|L_{g-j} + 2P| + (j-2)P\) on \(C_{g-j}\), where \(L_{g-j}\) is any odd theta-characteristic on \(C_{g-j}\).

(ii) \(\eta = (\eta_i, \eta_{g-i})\) is a non-trivial 2-torsion line bundle on \(C\), such that the numbers \(h^0(C_i, L_i \otimes \eta_i)\) and \(h^0(C_{g-i}, L_{g-i} \otimes \eta_{g-i})\) have the same (resp. opposite) parity.

**Proof.** First of all, note that item (i) describes the general element of the intersection \(T_g \cap \Delta_i\) in \(\overline{\mathcal{M}}_g\): this is exactly [17, Proposition 1.2].

Moreover, if \((C, \eta) \in T_g^e \cap \pi^{-1}(\Delta_i)\) (resp. \((C, \eta) \in T_g^o \cap \pi^{-1}(\Delta_i)\)), then there exists (a germ of) a 1-dimensional family \((\mathbb{C} \to S, H, \mathcal{L})\) of Prym curves \((C_s, H_s)\) endowed with a 1-dimensional theta-characteristic \(L_s\), such that:
(1) For every \( s \neq 0 \), \((C_s, H_s)\) is a smooth Prym curve such that \( L_s \otimes H_s \) is an even (resp. odd) theta-characteristic on \( C_s \).

(2) The family \((C \to S, H)\) specializes to \((C, \eta) = (C_0, H_0)\).

The possible aspects of the 1-dimensional limit series of \( L \) on \( C = C_0 \) are described by item (i). Now the result follows from the fact that, on the one hand, the aspects of the limit series of \( L \otimes H \) on \( C = C_0 \) are the same aspects as the limit of \( L \), but twisted by \( \eta = H_0 \); and on the other hand, the parity of a theta-characteristic on the reducible curve \( C \) is the product of the parities of the theta-characteristics induced on \( C_i \) and \( C_{g-i} \).

**Remark 3.2.** For a fixed general element \( C \) of the intersection \( T_g \cap \Delta_i \) (i.e. a curve \( C \) satisfying the condition (i) above), the number of \( \eta = (\eta_i, \eta_{g-i}) \) such that \((C, \eta) \in T_g^c\) can be easily computed. Indeed, the number of \( \eta \) giving parities (even,even) is the product of the number of even theta-characteristics on \( C_i \) and the number of even theta-characteristics on \( C_{g-i} \):

\[
2^{g-2}(2^i - 1)(2^{g-i} - 1) + 2^{g-2}(2^i - 1)(2^{g-i} - 1) - 1 = 2^{g-1}(2^g + 1) - 1,
\]

which indeed coincides with the degree of \( T_g^c \) over \( T_g \). Of course the configuration of the fiber \( \pi^{-1}_T(C) \) along the divisors \( \Delta_i, \Delta_{g-i} \) and \( \Delta_{i,g-i} \) will depend on whether \( C \) satisfies \( \alpha_j \) or \( \beta_j \).

**Lemma 3.3.** If \( C \) is a smooth curve of genus \( g \) and \( \eta \in JC_2 \) is a non-trivial 2-torsion line bundle, then there are exactly \( 2g(2g-1) \) odd theta-characteristics \( L \) on \( C \) such that \( L \otimes \eta \) is also odd.

**Proof.** This can be checked, for example, by considering how the group \( JC_2 \) of 2-torsion line bundles acts on the set \( S_g(C) \) of theta-characteristics. The associated difference map

\[
S_g(C) \times S_g(C) \longrightarrow JC_2, \quad (M, N) \longmapsto M \otimes N^{-1}
\]

can be restricted to the set of pairs of non-isomorphic odd theta-characteristics, that is,

\[
S_g^-(C) \times S_g^-(C) - \Delta \longrightarrow JC_2 - \{O_C\}.
\]

Since \( \#S_g^-(C) = 2^{g-1}(2^g - 1) \) and \( \#JC_2 = 2^{2g} \), the fibers of this restriction have order

\[
\#S_g^-(C) \cdot (\#S_g^-(C) - 1) \cdot (\#JC_2 - 1)^{-1} = 2^{g-1}(2^g-1),
\]

which reflects the number of odd theta-characteristics \( L \) such that \( L \otimes \eta \) is also odd.
Now, given an integer \( i \geq 1 \), we proceed to compute the coefficients \( b_i, b_{g-i} \) and \( b_{i:g-i} \) of the class \([\mathcal{T}_g^e]\). We follow the argument in [17, Proposition 3.1].

Fix two smooth curves \( C_i \) and \( C_{g-i} \) of respective genera \( i \) and \( g-i \) having no theta-characteristic of positive dimension, as a well as a point \( p \in C_i \) lying in the support of no effective theta-characteristic. We denote by \( F \) the curve (isomorphic to \( C_{g-i} \) itself) in \( \Delta_i \subset \mathcal{M}_g \), obtained by identifying \( p \) with a variable point \( q \in C_{g-i} \). This curve has the following intersection numbers with the basic divisor classes of \( \mathcal{M}_g^t \):

\[
F \cdot \lambda = 0, \quad F \cdot \delta_j = 0 \quad \text{for} \quad j \neq i, \quad F \cdot \delta_i = -2(g - i - 1)
\]

(for a justification of these intersection numbers, see [8, page 81]).

Since the curve \( F \subset \mathcal{M}_g \) does not intersect the branch locus of the morphism \( \pi \), it follows that the preimage \( \pi^{-1}(F) \) has \( 2g - 1 \) connected components; each of them is isomorphic to \( F \), and corresponds to the choice of a pair \( (\eta, \eta_{g-i}) \) of \( 2 \)-torsion line bundles on \( C_i \) and \( C_{g-i} \) being not simultaneously trivial.

Let \( \tilde{F}_i \) be one of the components of \( \pi^{-1}(F) \) contained in the divisor \( \Delta_i \) of \( \mathcal{R}_g \); it is attached to an element \( \eta = (\eta_i, \eta_{g-i}) \), for a fixed non-trivial \( \eta_i \in (JC_i)_2 \).

On the one hand, clearly \( \delta_i \) is the only basic divisor class of \( \mathcal{R}_g \) that intersects \( \tilde{F}_i \). The projection formula then says that the number \( \tilde{F}_i \cdot \delta_i \) in \( \mathcal{R}_g \) equals the intersection \( F \cdot \delta_i = -2(g - i - 1) \) in \( \mathcal{M}_g \). Therefore,

\[
\tilde{F}_i \cdot [\mathcal{T}_g^e] = \tilde{F}_i \cdot (a\lambda - b_i\delta_i - \ldots) = 2(g - i - 1)b_i.
\]

On the other hand, according to Proposition 3.1 an element \((C, \eta) \in \tilde{F}_i \) belongs to \( \mathcal{T}_g^e \) if and only if the two following conditions are satisfied:

(a) The point \( q \in C_{g-i} \) that is identified with \( p \) lies in the support of an effective theta-characteristic \( L_{g-i} \). That is, \( C \) satisfies \( \beta_{g-i} \).

(b) The odd theta-characteristic \( L_i \) of \( C_i \), when twisted by \( \eta_i \), remains odd.

This gives the intersection number

\[
\tilde{F}_i \cdot [\mathcal{T}_g^e] = (g - i - 1)2^{g-i-1}(2^{g-i} - 1)2^{i-1}(2^{i-1} - 1),
\]

where we use Lemma 3.3 to count the possible theta-characteristics \( L_i \).

Comparing both expressions for \( \tilde{F}_i \cdot [\mathcal{T}_g^e] \), it follows that \( b_i = 2^{g-3}(2^{g-i} - 1)(2^{i-1} - 1) \).

With a similar argument (considering a connected component of \( \pi^{-1}(F) \) contained in \( \Delta_{g-i} \) or \( \Delta_{i:g-i} \)), one can find the numbers

\[
b_{g-i} = 2^{g-3}(2^{g-i-1} - 1)(2^{i-1} - 1), \quad b_{i:g-i} = 2^{g-3}(2^{g-1} - 2^{i-1} - 2^{g-i-1} + 1).
\]

Remark 3.4. The transversality of these intersections can be shown by looking at the scheme \( X^e \) parametrizing pairs \( ((C, \eta), L) \), where \( (C, \eta) \) is a Prym curve and \( L \) is a semicanonical pencil on \( C \) such that \( L \otimes \eta \) is even. If we restrict the forgetful map \( X^e \to \mathcal{T}_g^e \) to the component \( \tilde{F}_i \), we obtain a scheme \( X \to \mathcal{T}_g^e |_{\tilde{F}_i} \) which is, by the above discussion, isomorphic to the scheme \( \mathfrak{g}_{g-1}^1(\tilde{F}_i) \) of limit linear series of type \( \mathfrak{g}_{g-1}^1 \) on Prym curves \( (C, \eta) \in \tilde{F}_i \) satisfying conditions (a)
and (b). Following the description of this moduli space given in [3, Theorem 3.3], we see that the scheme $\mathcal{Z}_{g-1}(F_i)$ splits as the product of two reduced 0-dimensional schemes, namely

$$\{(L_{g-i}, q) \text{ as in (a)}\} \times \{L_i \text{ as in (b)}\}.$$  

Therefore $\mathcal{Z}_{g-1}(F_i) \cong \mathcal{X} \to T_g|_{\tilde{F}_i}$ is everywhere reduced and the intersection between $\tilde{F}_i$ and $T_g$ is transverse. A breakdown of this argument may be found in [4, Theorem 2.2].

Now we proceed with step (3). We will determine the constants $d'_0, d''_0, d_{ram}$ of the class $[T_g]$ by using the test curve of [7, Example 3.137].

Fix a general smooth curve $D$ of genus $g - 1$, with a fixed general point $p \in D$. Identifying $p$ with a moving point $q \in D$, we get a curve $G$ (isomorphic to $D$) which lies in $\Delta_0 \subset \overline{M}_g$. As explained in [7], the following equalities hold:

$$G \cdot \lambda = 0, G \cdot \delta_0 = 2 - 2g, G \cdot \delta_1 = 1, G \cdot \delta_i = 0 \text{ for } i \geq 2,$$

where the intersection of $G$ and $\Delta_1$ occurs when $q$ approaches $p$; in that case the curve becomes reducible, having $D$ and a rational nodal curve as components.

Combining this information with the known divisor class $[T_g]$ in $\overline{M}_g$, we have

$$G \cdot [T_g] = 2^{g-3}((g - 3) \cdot 2^{g-2} + 1).$$

In order to compute $d''_0$, let $\tilde{G}''$ be the connected component of $\pi^{-1}(G)$ obtained by attaching to every curve $C = D_{pq}$ the 2-torsion line bundle $c = (\mathcal{O}_D)_{-1}$ (i.e. $\mathcal{O}_D$ glued by -1 at the points $p, q$). Indeed $e$ is well defined along the family $G$, so $\tilde{G}''$ makes sense and is isomorphic to $G$.

Then:

- By the projection formula, $\tilde{G}'' \cdot \lambda = 0$.
- Again by projection, $\tilde{G}'' \cdot (\pi^*\delta_0) = 2 - 2g$. Actually, since $\tilde{G}'' \subset \Delta'_0$ and $\tilde{G}''$ intersects neither $\Delta_0'$ nor $\Delta_{ram}'$, the following equalities hold:

$$\tilde{G}'' \cdot \delta_0'' = 2 - 2g, \tilde{G}'' \cdot \delta_0' = 0 = \tilde{G}'' \cdot \delta_{ram}.$$

- We have $\tilde{G}'' \cdot (\pi^*\delta_1) = 1$, with $\tilde{G}'' \cdot \delta_1 = 1$ and $\tilde{G}'' \cdot \delta_{g-1} = 0 = \tilde{G}'' \cdot \delta_1; g-1$.

   Indeed, the intersection $G \cap \Delta_1$ occurs when $p = q$; for that curve, the 2-torsion that we consider is trivial on $D$ but not on the rational component. Hence the lift to $\tilde{G}''$ of the intersection point $G \cap \Delta_1$ gives a point in $\tilde{G}'' \cap \Delta_1$.

- It is clear that $\tilde{G}'' \cdot \delta_i = \tilde{G}'' \cdot \delta_{g-1} = \tilde{G}'' \cdot \delta_{i; g-1} = 0$ for $i \geq 2$.

- Since twisting by $e$ changes the parity of any theta-characteristic in any curve of the family $G$ by [6, Theorems 2.12 and 2.14], it follows that all the intersection points of $G$ and $T_g$ lift to points of $\tilde{G}'' \cap T_g^o$.

All in all, we have

$$2^{g-3}((g - 3) \cdot 2^{g-2} + 1) = \tilde{G}'' \cdot [T_g^o] = (2g - 2)d''_0 - 2^{g-3}(2^{g-1} - 1)$$

and solving the equation we obtain $d''_0 = 2^{2g-6}$. 
For the computation of \( d'_0 \), we consider \( \tilde{G}' = \pi^{-1}(G) \cap \Delta'_0 \) in \( \overline{R}_g \). Note that for an element \((C = D_{pq}, \eta) \in \tilde{G}'\), \( \eta \) is obtained by gluing a non-trivial 2-torsion line bundle on \( D \) at the points \( p, q \). Then:

- \( \tilde{G}' \cdot \lambda = 0 \) by the projection formula.
- Again by projection, \( \tilde{G}' \cdot (\pi^* \delta_0) = \deg(\tilde{G}' \to G)(G \cdot \delta_0) = 2(2 - 2g)(2^{2g-2} - 1) \). Moreover, since \( \tilde{G}' \subset \Delta'_0 \) intersects neither \( \Delta'_0^{\text{ram}} \) nor \( \Delta'_0^{\text{am}} \) it follows that
  \[
  \tilde{G}' \cdot \delta_0' = 2(2 - 2g)(2^{2g-2} - 1), \quad \tilde{G}' \cdot \delta_0'' = 0 = \tilde{G}' \cdot \delta_0^{\text{am}}.
  \]
- \( \tilde{G}' \cdot (\pi^* \delta_1) = \deg(\tilde{G}' \to G)(G \cdot \delta_1) = 2(2^{2g-2} - 1) \). We claim that \( \tilde{G}' \cdot \delta_1 = 0 \) and \( \tilde{G}' \cdot \delta_{g-1} = 2^{2g-2} - 1 = \tilde{G}' \cdot \delta_{1:g-1} \).
  Indeed, \( G \cap \Delta_1 \) occurs when \( p = q \); when such a point is lifted to \( \tilde{G}' \), the 2-torsion is nontrivial on \( D \) (by construction). This gives \( \tilde{G}' \cdot \delta_1 = 0 \).
  Moreover, triviality on the rational nodal component will depend on which of the two possible gluings of the 2-torsion on \( D \) we are taking; in any case, since \( \tilde{G}' = \pi^{-1}(G) \cap \Delta'_0 \) considers simultaneously all possible gluings of all possible non-trivial 2-torsion line bundles on \( D \), we have \( \tilde{G}' \cdot \delta_{g-1} = \tilde{G}' \cdot \delta_{1:g-1} \). This proves the claim.
- Of course, \( \tilde{G}' \cdot (\pi^* \delta_i) = \tilde{G}' \cdot \delta_{g-1} = \tilde{G}' \cdot \delta_{1:g-1} = 0 \) whenever \( i \geq 2 \).
- Finally, we use again that the parity of a theta-characteristic on a nodal curve of the family \( G \) is changed when twisted by \( e = (O_D)_{-1} \). Since the two possible gluings of a non-trivial 2-torsion bundle on \( D \) precisely differ by \( e \), the intersection numbers \( \tilde{G}' \cdot [T_g^e] \) and \( \tilde{G}' \cdot [T_g^o] \) have to coincide, and at the same time add up to the total
  \[
  \tilde{G}' \cdot (\pi^*[T_g]) = \deg(\tilde{G}' \to G)(G \cdot [T_g]) = 2(2^{2g-2} - 1) \cdot 2^{g-3}(g - 3) \cdot 2^{g-2} + 1
  \]
  by the projection formula. That is,
  \[
  \tilde{G}' \cdot [T_g^e] = \tilde{G}' \cdot [T_g^o] = (2^{2g-2} - 1) \cdot 2^{g-3}(g - 3) \cdot 2^{g-2} + 1.
  \]
  Putting this together with the coefficients \( d_{g-1} = 2^{2g-5} \) and \( d_{1:g-1} = 2^{g-3}(2^{2g-2} - 1) \) obtained in step (2), we get
  \[
  (2^{2g-2} - 1) \cdot 2^{g-3}(g - 3) \cdot 2^{g-2} + 1 = \tilde{G}' \cdot [T_g^o] = 2(2g - 2)(2^{2g-2} - 1)d'_0 - 2^{2g-5}(2^{2g-2} - 1) - 2^{g-3}(2^{2g-2} - 1)(2^{2g-2} - 1)
  \]
  and therefore \( d'_0 = 2^{2g-7} \).

Finally, to compute \( d_0^{\text{ram}} \) we simply combine the relation
  \[
  (2^{2g-1} - 2)d'_0 + d_0'' + 2^{2g-2}d_0^{\text{ram}} = 2^{g-1}(2g - 1)2^{2g-6}
  \]
  obtained in step (1) with the coefficients \( d'_0, d_0'' \) just found, to obtain \( d_0^{\text{ram}} = 2^{g-5}(2^{g-1} - 1) \). This concludes step (3) and hence the proof of Theorem A.

**Remark 3.5.** The divisor \( T_g \) has a more natural interpretation in the compactification of the moduli space \( S^+_g \) of even spin curves (i.e. curves equipped with an even theta-characteristic). In the same way, it would be preferable to discuss the divisors \( T_g^e \) and \( T_g^o \) in a space of curves endowed with both a Prym and a spin structure. In particular, if a good compactification of
$R_g \times_{\mathcal{M}_g} S_g^+$ were constructed and studied, then the divisor classes of $\mathcal{T}_g^e$ and $\mathcal{T}_g^o$ could also be derived from the diagram

$$
\begin{array}{cccc}
R_g & \leftarrow & R_g \times_{\mathcal{M}_g} S_g^+ & \rightarrow S_g^+ \\
\end{array}
$$

and the fact that the class of (the closure in $S_g^+$ of) the divisor

$$
\{(C, L) \in S_g^+ \mid L \text{ is a semicanonical pencil on } C\}
$$

was computed by Farkas in [4, Theorem 0.2]. Following the ideas of [15], a candidate space for such a compactification is proposed in [10, Section 2.4], although it remains to check that this space is indeed a smooth and proper Deligne-Mumford stack. Under the assumption that it is, a study of its boundary reveals the same expressions obtained in Theorem A. Further details can be found in [10].

4. Proof of Theorem B

In this section we study the irreducibility of the divisors $\mathcal{T}_g^o$ and $\mathcal{T}_g^e$. Recall that for $g = 3$, we already saw in Example 2.1 that the divisors $\mathcal{T}_3^o$ and $\mathcal{T}_3^e$ are irreducible. In the general case ($g \geq 5$), our arguments are essentially an adaptation of those of Teixidor in [17, Section 2], used to prove the irreducibility of $\mathcal{T}_g$ in $\mathcal{M}_g$.

The idea for proving the irreducibility of $\mathcal{T}_g^o$ is the following (the proof for $\mathcal{T}_g^e$ will be similar, but some simplifications will arise). By using Proposition 3.1, first we will fix a Prym curve $(C, \eta)$ (degeneration of smooth hyperelliptic ones) lying in all the irreducible components of the intersection $\mathcal{T}_g^o \cap \Delta_1$. This reduces the problem to the local irreducibility of $\mathcal{T}_g^o$ in a neighborhood of $(C, \eta)$ (after checking that every irreducible component of $\mathcal{T}_g^o$ intersects $\Delta_1$). For the proof of the local irreducibility of $\mathcal{T}_g^o$, we can take advantage of the scheme of pairs $((C, \eta), L)$ introduced in Remark 3.4 and use the following observation:

**Remark 4.1.** In a neighborhood of a given point, the local irreducibility of $\mathcal{T}_g^o$ (resp. $\mathcal{T}_g^e$) is implied by the local irreducibility of the scheme $X^o$ (resp. $X^e$) parametrizing pairs $((C, \eta), L)$, where $(C, \eta)$ is a Prym curve and $L$ is a semicanonical pencil on $C$ such that $L \otimes \eta$ is odd (resp. even). This follows from the surjectivity of the forgetful map $X^o \to \mathcal{T}_g^o$ (resp. $X^e \to \mathcal{T}_g^e$).

Then the local irreducibility of $X^o$ (near our fixed $(C, \eta)$) will be argued by showing that monodromy interchanges the (limit) semicanonical pencils on $C$ that become odd when twisted by the 2-torsion bundle $\eta$. Let us recall, for later use in this monodromy argument, some features of theta-characteristics on hyperelliptic curves:

**Remark 4.2.** Let $C$ be a smooth hyperelliptic curve of genus $g$, with Weierstrass points $R_1, \ldots, R_{2g+2}$.

Then, it is well known (see e.g. [14, Proposition 6.1]) that the theta-characteristics on $C$ have the form $r \cdot g_2^+ + S$, $r$ being its dimension (with $-1 \leq r \leq \lfloor \frac{g-1}{2} \rfloor$) and $S$ being the fixed part of the linear system (which consists of $g - 1 - 2r$ distinct Weierstrass points).
Moreover, given a 2-torsion line bundle of the form \( \eta = \mathcal{O}_C(R_i - R_j) \), theta-characteristics changing their parity when twisted by \( \eta \) are exactly those for which \( R_i, R_j \in S \) (the dimension increases by 1) or \( R_i, R_j \notin S \) (the dimension decreases by 1).

For the proof of Theorem B we also need the following result, which will guarantee that every irreducible component of \( \mathcal{T}_g^\alpha \) and \( \mathcal{T}_g^\nu \) intersects the boundary divisor \( \Delta_1 \subset \overline{R}_g \):

**Lemma 4.3.** Let \( \mathcal{D} \subset \mathcal{R}_g \) be any divisor, where \( g \geq 5 \). Then the closure \( \overline{\mathcal{D}} \subset \overline{\mathcal{R}}_g \) intersects \( \Delta_1 \) and \( \Delta_{g-1} \).

**Proof.** We borrow the construction from [11, Section 4], where (a stronger version of) the corresponding result for divisors in \( \mathcal{M}_g \) is proved.

Fix a complete integral curve \( B \subset \mathcal{M}_{g-2} \) (whose existence is guaranteed by the assumption \( g \geq 5 \)), two elliptic curves \( E_1, E_2 \) and a certain 2-torsion element \( \eta \in JE_1 \setminus \{0\} \). If \( \Gamma_b \) denotes the smooth curve of genus \( g - 2 \) corresponding to \( b \in B \), one defines a family of Prym curves parametrized by \( \Gamma_b^2 \) as follows.

If \( (p_1, p_2) \in \Gamma_b^2 \) is a pair of distinct points, glue to \( \Gamma_b \) the curves \( E_1 \) and \( E_2 \) at the respective points \( p_1 \) and \( p_2 \) (this is independent of the chosen point on the elliptic curves). To this curve attach a 2-torsion bundle being trivial on \( \Gamma_b \) and \( E_2 \), and restricting to \( \eta \) on \( E_1 \).

To an element \( (p, p) \in \Delta_{g-2} \subset \Gamma_b^2 \), we attach the curve obtained by gluing a \( \mathbb{P}^1 \) to \( \Gamma_b \) at the point \( p \), and then \( E_1, E_2 \) are glued to two other points in \( \mathbb{P}^1 \). Of course, the 2-torsion bundle restricts to \( \eta \) on \( E_1 \), and is trivial on the remaining components.

Moving \( b \) in \( B \), this construction gives a complete threefold \( T = \bigcup_{b \in B} \Gamma_b^2 \) contained in \( \Delta_1 \cap \Delta_{g-1} \).

Let also \( S = \bigcup_{b \in B} \Delta_{g-2} \) be the surface in \( T \) given by the union of all the diagonals; it is the intersection of \( T \) with \( \Delta_2 \). Then, the following statements hold:

1. \( \delta_1|_S = 0 \) and \( \delta_{g-1}|_S = 0 \) (the proof of [11, Lemma 4.2] is easily translated to our setting).
2. \( \lambda|_{\Delta_{g-2}} = 0 \) for every \( b \in B \), since all the curves in \( \Delta_{g-2} \) have the same Hodge structure.
3. If \( a \in \mathbb{Q} \) is the coefficient of \( \lambda \) for the class \([\overline{\mathcal{D}}]\) in \( \text{Pic}(\overline{\mathcal{R}}_g)_\mathbb{Q} \), then \( a \neq 0 \). Indeed, \( 2g-1 a \in \mathbb{Q} \) is the coefficient of \( \lambda \) for the class \([\overline{\pi(D)}]\) in \( \text{Pic}(\mathcal{M}_g)_\mathbb{Q} \); then [11, Remark 4.1] proves the claim.

These are the key ingredients in the original proof of [11, Proposition 4.5]. The same arguments there work verbatim in our case and yield the analogous result: \( [\overline{\mathcal{D}}]|_T \neq m \cdot S \) for every \( m \in \mathbb{Q} \).

In particular, the intersection \( \overline{\mathcal{D}} \cap T \) is non-empty (and not entirely contained in \( S \)). \qed

**Proposition 4.4.** For \( g \geq 5 \), the divisor \( \mathcal{T}_g^\alpha \) is irreducible.

**Proof.** According to Proposition 3.1, the intersection \( \mathcal{T}_g^\alpha \cap \Delta_1 \) consists of two loci \( \alpha \) and \( \beta \). The general point of each of these loci is the union at a point \( P \) of a Prym elliptic curve \((E, \eta)\) and a smooth curve \( C_{g-1} \) (with trivial line bundle) of genus \( g - 1 \), such that:
In the case of $\alpha$, the curve $C_{g-1}$ has a 1-dimensional theta-characteristic, i.e., $C_{g-1} \in T_{g-1}$ in $\mathcal{M}_{g-1}$. Moreover, there is exactly one limit semicanonical pencil on $C_{g-1} \cup P E$ changing parity when twisted by the 2-torsion bundle; it induces the theta-characteristic $\eta$ on $E$.

It follows that $\alpha$ is irreducible (by irreducibility of $T_{g-1}$) and the intersection of $T_{g-1}^\circ$ and $\Delta_1$ along $\alpha$ is reduced. In particular, there is a unique irreducible component of $T_{g-1}^\circ$ (that we will denote by $T_{g,0}$) intersecting $\Delta_1$ along the whole locus $\alpha$.

- In the case of $\beta$, $P$ is in the support of a 0-dimensional theta-characteristic on $C_{g-1}$. Again, there is a unique limit semicanonical pencil changing parity, with induced theta-characteristic $O_E$ on $E$.

Now we consider a reducible Prym curve $(C, \eta) \in \Delta_1$ constructed as follows: $C$ is the join of an elliptic curve $E$ and a general smooth hyperelliptic curve $C'$ of genus $g - 1$ at a Weierstrass point $P \in C'$, whereas the line bundle $\eta$ is trivial on $C'$. Note that $(C, \eta)$ is the general point of the intersection $\tilde{H}_{g} \cap \Delta_1$, where $\tilde{H}_{g} \subset T_{g}^\circ$ denotes the locus of hyperelliptic Prym curves whose 2-torsion bundle is a difference of two Weierstrass points. Of course $(C, \eta)$ belongs to $\alpha$ and $\beta$; we claim that it actually belongs to any component of $\beta$.

To prove this claim, consider any irreducible component of $\beta$, and fix a general element of it. This general element admits the description given above: let us denote by $X$ (written $C_{g-1}$ above) the irreducible component of genus $g - 1$, and by $Q_X \in X$ the point connecting $X$ with the elliptic component. Recall that $Q_X$ lies in the support of a 0-dimensional theta-characteristic $L_X$ on $X$.

We deform the pair $(X, L_X)$ to a pair $(C', L)$ formed by our hyperelliptic curve $C'$ and a 0-dimensional theta-characteristic $L$ on it. According to the description of Remark 4.2, under this deformation the point $Q_X \in X$ specializes to a Weierstrass point $Q \in C'$.

Therefore, our irreducible component of $\beta$ contains a Prym curve which is the union of $C'$ (with trivial 2-torsion) and a Prym elliptic curve $(E', \eta')$ at the Weierstrass point $Q \in C'$. Since the monodromy on hyperelliptic curves acts transitively on the set of Weierstrass points, we may replace $Q$ by our original Weierstrass point $P$ without changing the component of $\beta$. Using that $R_1$ is connected we can also replace $(E', \eta')$ by $(E, \eta)$. This proves the claim.

Now, to prove the irreducibility of $T_{g}^\circ$ we argue as follows: since $T_{g}^\circ$ has pure codimension 1, we know by Lemma 4.3 that each of its irreducible components intersects $\Delta_1$. As our point $(C, \eta)$ belongs to all the irreducible components of $T_{g}^\circ \cap \Delta_1$, it suffices to check the local irreducibility of $T_{g}^\circ$ in a neighborhood of $(C, \eta)$.

To achieve this, in view of Remark 4.1 we will check the local irreducibility of the scheme $X^\circ$. In other words, we need to study the limit semicanonical pencils on $C$ changing parity when twisted by $\eta$. We do this in the rest of the proof, by checking that monodromy on $\tilde{H}_{g} \subset T_{g}^\circ$ connects any limit semicanonical pencil changing parity on $(C, \eta)$ of type $\beta$ with one of type $\alpha$, and checking that limits of type $\alpha$ are also permuted by monodromy on $T_{g,0}$.

Let $R_1, R_2, R_3$ be the points on $E$ differing from $P$ by 2-torsion, and let $R_4, \ldots, R_{2g+2}$ be the Weierstrass points on $C'$ that are different from $P$: reordering if necessary, we assume $\eta|_E = O_E(R_1 - R_2)$. Note that $R_1, \ldots, R_{2g+2}$ are the limits on $C$ of Weierstrass points on nearby smooth hyperelliptic curves, since they are the ramification points of the limit $g_{2}^{1}$ on $C$. 

With this notation, arguing as in the proof of Proposition 3.1, the possible aspects on $E$ of a limit semicanonical pencil changing parity on $(C, \eta)$ are:

- Those of type $\alpha$ have aspect on $E$ differing from the even theta-characteristic $\eta$ by $(g - 1)P$, hence $\mathcal{O}_E(R_3 + (g - 2)P) = \mathcal{O}_E(R_1 + R_2 + (g - 3)P)$.
- Those of type $\beta$ have aspect differing from the odd theta-characteristic $\mathcal{O}_E$ by $(g - 1)P$, hence $\mathcal{O}_E((g - 1)P) = \mathcal{O}_E(R_1 + R_2 + R_3 + (g - 4)P)$.

Given a family of semicanonical pencils changing parity on nearby smooth curves of $\tilde{\mathcal{H}}_g$, we can distinguish the type of its limit on $C$ by knowing how many of the $g - 1 - 2r$ fixed Weierstrass points in the moving theta-characteristic (recall Remark 4.2) specialize to $E$. If this number is 0 or 3 (resp. 1 or 2), then our limit is of type $\beta$ (resp. of type $\alpha$).

Hence, after using monodromy on smooth hyperelliptic curves to interchange the (limit) Weierstrass point $R_3$ with an appropriate (limit) Weierstrass point on $C'$, we obtain that monodromy on $\tilde{\mathcal{H}}_g \subset \mathcal{T}_g^\circ$ interchanges any limit semicanonical pencil changing parity of type $\beta$ with one of type $\alpha$ (of the same dimension). The only possible exception is a limit of $\beta_0$ or $3 \beta$ (resp. $1 \beta$ or $2 \beta$), then our limit is of type $\beta_0$ (mod 4), since in that case there are no fixed points to interchange with $R_3$.

In addition, monodromy on $\mathcal{T}_{g,\alpha}^\circ$ (the unique irreducible component of $\mathcal{T}_g^\circ$ containing $\alpha$) acts transitively on the set of limit semicanonical pencils changing parity of type $\alpha$. Indeed, if $X_\alpha^\circ$ denotes the preimage of $\mathcal{T}_{g,\alpha}^\circ$ in $X^\circ$, then the forgetful map $X_\alpha^\circ \to \mathcal{T}_{g,\alpha}^\circ$ is birational (by [16, Theorem 2.16]) and has finite fibers; consequently $X_\alpha^\circ$ is irreducible, which proves the assertion.

Therefore to conclude the proof of the local irreducibility of $X^\circ$ near $(C, \eta)$ it only remains to show that, if $g \equiv 3 \pmod{4}$, the monodromy on $\mathcal{T}_g^\circ$ interchanges the limit of $2^{g-1} \cdot g_2^1$ with a limit of theta-characteristics of lower dimension. This can be achieved exactly with the same family of limit theta-characteristics as in [17, Proposition 2.4] for certain reducible curves; let us include a few words about the geometry of this family.

First, one degenerates $C'$ to a reducible hyperelliptic curve obtained by identifying a point $P' \in E'$ ($E'$ elliptic curve) with a Weierstrass point $Q \in C''$ ($C'' \in \mathcal{M}_{g-2}$ hyperelliptic), such that the Weierstrass point $P \in C'$ specializes to a point of $E'$. This naturally induces a degeneration $C_{P'}$ of our Prym curve $(C, \eta)$, in which the 2-torsion bundle is non-trivial only along the component $E$. We will denote by $R_4, R_5$ the points of $E'$ differing by 2-torsion from $P$ and $P'$ (limits of the corresponding Weierstrass points of $C'$).

Consider the family of Prym curves $C_X$ obtained by glueing $E$ (the only component with non-trivial 2-torsion) and $E'$ at $P$, and by identifying $Q \in C''$ with a variable point $X \in E'$. Note that for $X = P'$, we indeed recover our deformation $C_{P'}$ of $(C, \eta)$. Every such Prym curve $C_X$ can be equipped with a limit semicanonical pencil changing parity of aspects $\mathcal{O}_E((g - 1)P)$ on $E$, $\mathcal{O}_{E'}(Q + (g - 2)X)$ on $E'$ and $\mathcal{O}_{C''}((g - 1)Q)$ on $C''$.

On $C_{P'}$, this corresponds to the limit of $2^{g-5} \cdot g_2^1$ on nearby smooth Prym curves of $\tilde{\mathcal{H}}_g$; on the other hand, $C_{R_5}$ is also hyperelliptic and we have a limit of theta-characteristics of the form $2^{g-5} \cdot g_2^1 + R_1 + R_2 + R_3 + R_4$.

Therefore, monodromy on $\mathcal{T}_g^\circ$ moves the limit of $2^{g-1} \cdot g_2^1$ to a limit theta-characteristic of type $\beta$ of lower dimension, which concludes the proof. □
Proposition 4.5. For $g \geq 5$, the divisor $T^e_g$ is irreducible.

Proof. The proof is similar to that of $T^o_g$, but with some simplifications (due to the fact that the intersection $T^e_g \cap \Delta_1$ consists only of a locus $\alpha$). Let us give an outline of the argument.

In virtue of Proposition 3.1, the general point of $\alpha$ is the union at a point $P$ of a Prym elliptic curve $(E, \eta)$ and a curve $C_{g-1} \in M_{g-1}$ (with trivial 2-torsion bundle) having a 1-dimensional theta-characteristic. Let us denote by $R_1, R_2, R_3$ the points of $E$ differing from $P$ by 2-torsion, so that $\eta = \mathcal{O}_E(R_1 - R_2)$.

Then there are exactly two limit semicanonical pencils on $E \cup_P C_{g-1}$ remaining even when twisted by $(\eta, \mathcal{O}_{C_{g-1}})$. For these limit semicanonical pencils, $\mathcal{O}_E(R_1 - R_3)$ and $\mathcal{O}_E(R_2 - R_3)$ are the induced theta-characteristics on $E$ (and hence $|R_2 + P| + (g - 3)P$ and $|R_1 + P| + (g - 3)P$ are the corresponding aspects on $E$).

It follows that the intersection of $T^e_g$ and $\Delta_1$ is irreducible (by irreducibility of $T_{g-1}$) but not reduced. We also deduce that $T^e_g$ will have at most two irreducible components, but we cannot directly derive the irreducibility of $T^e_g$.

To circumvent this problem, we consider (as in the proof of Proposition 4.4) a Prym curve $(C, \eta) \in T^e_g \cap \Delta_1$ obtained by taking $C_{g-1} = C'$ ($C'$ general smooth hyperelliptic curve) and $P \in C'$ a Weierstrass point. Recall that $(C, \eta)$ is the general point of the intersection $\tilde{H}_g \cap \Delta_1$ ($\tilde{H}_g \subset T^e_g$ being the locus of hyperelliptic Prym curves whose 2-torsion bundle is a difference of two Weierstrass points).

By using monodromy on smooth hyperelliptic curves to interchange the (limit) Weierstrass points $R_1$ and $R_2$, we obtain that monodromy on $\tilde{H}_g \subset T^e_g$ connects (locally around $(C, \eta)$) the two possible irreducible components of $T^e_g$. This finishes the proof.

All in all, we have showed the irreducibility of $T^o_g$ and $T^e_g$ for every $g \neq 4$. As explained in the introduction, the irreducibility of $T^o_g$ and $T^e_g$ can be deduced from a study of the Prym map $\mathcal{P}_4$ restricted to these divisors, which is contained in [9].

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