LOWER BOUNDS ON THE RADIUS OF SPATIAL ANALYTICITY FOR THE KAWAHARA EQUATION

JAESEOP AHN, JIMYEONG KIM AND IHYEOK SEO

Abstract. In this paper we obtain lower bounds on the radius of spatial analyticity of solutions to the Kawahara equation
\[ u_t + uu_x + \alpha u_{xxx} + \beta u_{xxxxx} = 0, \]
\( \beta \neq 0, \) given initial data which is analytic with a fixed radius. It is shown that the uniform radius of spatial analyticity of solutions at later time \( t \) can decay no faster than \( 1/|t| \) as \( |t| \to \infty. \)

1. Introduction

Consider the Cauchy problem for the Kawahara equation
\[
\begin{aligned}
&u_t + uu_x + \alpha u_{xxx} + \beta u_{xxxxx} = 0, \\
u(0, x) = u_0(x),
\end{aligned}
\]  
(1.1)
where \( u : \mathbb{R}^{1+1} \to \mathbb{R} \) and \( \alpha, \beta \) are real constants with \( \beta \neq 0. \) This fifth-order KdV type equation has been derived to model gravity-capillary waves on a shallow layer and magneto-sound propagation in plasmas (12, 14).

The well-posedness of the above Cauchy problem with initial data in Sobolev spaces \( H^s \) has been studied by several authors (see e.g. 2, 24, 7, 6). In particular, it was shown in 2 that (1.1) has a global solution when \( s = 0. \) This was improved by Wang, Cui and Deng 24 to \( s > -1/2 \) and then by Chen, Li, Miao and Wu 7 to \( s > -7/4. \) More recently, Chen and Guo 6 obtained the global well-posedness for \( s \geq -7/4. \)

In this paper, we are concerned with the persistence of spatial analyticity for the solutions of (1.1), given initial data in a class of analyticity functions. While the well-posedness theory in Sobolev spaces is well developed, nothing is known about the spatial analyticity for the Kawahara equation. From now on, we focus on the situation where we consider a real-analytic initial data with uniform radius of analyticity \( \sigma_0 > 0, \) so there is a holomorphic extension to a complex strip
\[ S_{\sigma_0} = \{ x + iy : x, y \in \mathbb{R}, |y| < \sigma_0 \}. \]
Now, it is natural to ask whether this property may be continued analytically to a complex strip \( S_{\sigma(t)} \) for all later times \( t, \) but with a possibly smaller and shrinking radius of analyticity \( \sigma(t) > 0. \)

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This type of problem was first introduced by Kato and Masuda \cite{10}, and it was shown by Bona and Grujić \cite{3} that the radius $\sigma(t)$ for the Korteweg-de Vries (KdV) equation can decay no faster than $e^{-t^2}$ as $t \to \infty$. This was improved greatly by Bona, Grujić and Kalisch \cite{4} to a decay rate of $|t|^{-12}$. Later, Selberg and da Silva \cite{17} obtained a further refinement, $\sigma(t) \geq |t|^{-2/3+\varepsilon}$, where the $\varepsilon$ exponent was also removed by Tesfahun \cite{23}. Most recently, it was improved by Huang and Wang \cite{9} to a lower bound $|t|^{-1/4}$.

In spite of these many works, there have been no results on this issue for the Kawahara equation \cite{11}. Our aim in this paper is to obtain the spatial analyticity for this equation which is a fifth-order KdV type equation, motivated by earlier works for the KdV equation mentioned above and a recent one for the quartic generalized KdV equation by Selberg and Tesfahun \cite{19}. We also mention some references for other dispersive equations like Schrödinger equations \cite{5, 22, 1}, Klein-Gordon equations \cite{15}, and Dirac-Klein-Gordon equations \cite{18, 16}.

The Gevrey space, denoted $G^{\sigma,s}(\mathbb{R})$, $\sigma \geq 0$, $s \in \mathbb{R}$, is a suitable function space to study analyticity of solution. In our case, it will be used with the norm

$$
\|f\|_{G^{\sigma,s}} = \|e^{\sigma|D|}\langle D \rangle^{s}f\|_{L^2},
$$

where $\langle D \rangle = 1 + |D|$ and $D$ denotes the derivative. According to the Paley-Wiener theorem \cite{12} (see e.g. \cite{11}, p. 209), a function $f$ belongs to $G^{\sigma,s}$ with $\sigma > 0$ if and only if it is the restriction to the real line of a function $F$ which is holomorphic in the strip $S_{\sigma} = \{x+iy : x, y \in \mathbb{R}, |y| < \sigma\}$ and satisfies $\sup_{|y|<\sigma} \|F(x+iy)\|_{H^{s}} < \infty$. Therefore, every function in $G^{\sigma,s}$ with $\sigma > 0$ has an analytic extension to the strip $S_{\sigma}$. Based on this property of the Gevrey space, which is the key to studying spatial analyticity of solution, our result below gives a lower bound $|t|^{-1}$ on the radius of analyticity $\sigma(t)$ of the solution to \cite{11} as the time $t$ tends to infinity.

**Theorem 1.1.** Let $u$ be the global $C^{\infty}$ solution of \cite{11} with $u_0 \in G^{\sigma_0,s}(\mathbb{R})$ for some $\sigma_0 > 0$ and $s \in \mathbb{R}$. Then, for all $t \in \mathbb{R}$

$$
u(t) \in G^{\sigma(t),s}(\mathbb{R})
$$

with $\sigma(t) \geq c|t|^{-1}$ as $|t| \to \infty$. Here, $c > 0$ is a constant depending on $\|u_0\|_{G^{\sigma_0,s}(\mathbb{R})}$.

It should be noted that the existence of the global $C^{\infty}$ solution in the theorem is always guaranteed. Indeed, observe that $G^{0,s}$ coincides with the Sobolev space $H^{s}$ and the embeddings

$$
G^{\sigma,s} \subset G^{\sigma',s'}
$$

(1.2)

hold for all $0 \leq \sigma' < \sigma$ and $s, s' \in \mathbb{R}$. As a consequence of this embedding with $\sigma' = 0$ and the existing global well-posedness theory in $H^{s'}(= G^{0,s'})$ for $s' \geq -7/4$, the Cauchy problem \cite{11} has a unique global well-posed solution for all time, given initial data $u_0 \in G^{\sigma_0,s}$ for any $\sigma_0 > 0$ and $s \in \mathbb{R}$.

The outline of this paper is as follows: In Section 2 we introduce some function spaces such as Bourgain and Gevrey-Bourgain spaces, and their basic properties which

\footnote{The proof given for $s = 0$ in \cite{11} applies also for $s \in \mathbb{R}$ with some obvious modifications.}
will be used in later sections. In Section 3 we present a bilinear estimate in Gevrey-Bourgain spaces. By making use of a contraction argument involving this estimate, we prove that in a short time interval \( 0 \leq t \leq \delta \) with \( \delta > 0 \) depending on the norm of the initial data, the radius of analyticity remains strictly positive. Next, we prove an approximate conservation law, although the conservation of \( G^{\sigma_s} \)-norm of the solution does not hold exactly, in order to control the growth of the solution in the time interval \([0, \delta]\), measured in the data norm \( G^{\sigma_s} \). Section 4 is devoted to the proofs of such a local result and the almost conservation law. In the final section, Section 5, we finish the proof of Theorem 1.1 by iterating the local result based on the conservation law.

Throughout this paper, the letter \( C \) stands for a positive constant which may be different at each occurrence. We also denote \( A \lesssim B \) and \( A \sim B \) to mean \( A \leq CB \leq A \leq CB \), respectively, with unspecified constants \( C > 0 \).

2. Function spaces

In this section we introduce some function spaces and their basic properties which will be used in later sections for the proof of Theorem 1.1.

For \( s, b \in \mathbb{R} \), we use \( X^{s,b} = X^{s,b}(\mathbb{R}^2) \) to denote the Bourgain space defined by the norm
\[
\|f\|_{X^{s,b}} = \|\langle \xi \rangle^s (\tau - p(\xi))^b \hat{f}(\tau, \xi)\|_{L^2_{\tau, \xi}},
\]
where \( p(\xi) = \alpha \xi^3 - \beta \xi^5 \), \( \langle \cdot \rangle = 1 + |\cdot| \) and \( \hat{f} \) denotes the space-time Fourier transform given by
\[
\hat{f}(\tau, \xi) = \int_{\mathbb{R}^2} e^{-i(\tau \tau + \xi x)} f(t, x) \, dt \, dx.
\]
The restriction of the Bourgain space, denoted \( X^{s,b}_\delta \), to a time slab \((0, \delta) \times \mathbb{R}\) is a Banach space when equipped with the norm
\[
\|f\|_{X^{s,b}_\delta} = \inf \left\{ \|g\|_{X^{s,b}} : g = f \text{ on } (0, \delta) \times \mathbb{R} \right\}.
\]
We also introduce the Gevrey-Bourgain space \( X^{\sigma_s,b} = X^{\sigma_s,b}(\mathbb{R}^2) \) defined by the norm
\[
\|f\|_{X^{\sigma_s,b}} = \|e^{\sigma |\partial_x|} f\|_{X^{s,b}}.
\]
Its restriction \( X^{\sigma_s,b}_\delta \) to a time slab \((0, \delta) \times \mathbb{R}\) is defined in a similar way as above, and when \( \sigma = 0 \) it coincides with the Bourgain space \( X^{s,b} \).

We now list some basic properties of these spaces. When \( \sigma = 0 \), the proofs of the first two lemmas below can be found in Section 2.6 of [21], and the third lemma follows by the argument used for Lemma 3.1 of [8]. But, by the substitution \( f \to e^{\sigma |\partial_x|} f \), the properties of \( X^{s,b} \) and its restrictions carry over to \( X^{\sigma_s,b} \).

**Lemma 2.1.** Let \( \sigma \geq 0, s \in \mathbb{R} \) and \( b > 1/2 \). Then, \( X^{\sigma_s,b} \subset C(\mathbb{R}, G^{\sigma_s}) \) and
\[
\sup_{t \in \mathbb{R}} \|f(t)\|_{G^{\sigma_s}} \leq C \|f\|_{X^{\sigma_s,b}},
\]
where \( C > 0 \) is a constant depending only on \( b \).
Lemma 2.2. Let $\sigma \geq 0$, $s \in \mathbb{R}$, $-1/2 < b < b' < 1/2$ and $\delta > 0$. Then
\[
\|f\|_{X^{\sigma,s,b}_{\delta}} \leq C_{b,b'} \delta^{b' - b}\|f\|_{X^{\sigma,s,b'}_{\delta}},
\]
where $C_{b,b'} > 0$ is a constant depending only on $b$ and $b'$.

Lemma 2.3. Let $\sigma \geq 0$, $s \in \mathbb{R}$, $-1/2 < b < 1/2$ and $\delta > 0$. Then, for any time interval $I \subset [0, \delta]$,
\[
\|\chi_I f\|_{X^{\sigma,s,b}_{\delta}} \leq C\|f\|_{X^{\sigma,s,b}_{\delta}},
\]
where $\chi_I(t)$ is the characteristic function of $I$, and $C > 0$ is a constant depending only on $b$.

3. Bilinear estimates in Gevrey-Bourgain spaces

In this section we present a bilinear estimate in Gevrey-Bourgain spaces, Lemma 3.1, which plays a key role in obtaining the local well-posedness and almost conservation law in the next section. With the aid of it, we shall also deduce an estimate, Lemma 3.2, which is another useful tool particularly in obtaining the almost conservation law.

Lemma 3.1. For all $\sigma \geq 0$ and $s > -7/4$, there exist $1/2 < b < 1$ and $\varepsilon > 0$ such that
\[
\|\partial_x (uv)\|_{X^{\sigma-1,s,b}_{\delta}} \leq C_{s,b,b'} \|u\|_{X^{\sigma,s,b}_{\delta}}\|v\|_{X^{\sigma,s,b}_{\delta}},
\]
for any $b'$ satisfying $b \leq b' < b + \varepsilon$. Here, $C_{s,b,b'} > 0$ is a constant depending only on $s$, $b$ and $b'$.

It is worth comparing this lemma with the analogous result by Chen, Li, Miao and Wu [7] (cf. Proposition 2.2) who proved (3.1) only in the case $b' = b$ to obtain their local well-posedness results in Sobolev spaces. But, it is crucial for the present issue to have $b'$ that can range over a small interval for which (3.1) holds. Chen, Li, Miao and Wu’s argument is based on Tao’s $[k;Z]$-multiplier norm method [20] and carries over to our case where $b \leq b' < b + \varepsilon$ by a direct modification. So we omit the details. Instead we derive the following lemma from Lemma 3.1, which, along with the function $f$ defined here, plays a crucial role in obtaining the almost conservation law in the next section.

Lemma 3.2. Let
\[
f(u) = \frac{1}{2} \partial_x \left((e^{\sigma|\partial_x|} u)^2 - e^{\sigma|\partial_x|} u^2\right).
\]
Given $0 \leq \rho \leq 1$, there exist $1/2 < b < 1$ and $C > 0$ such that
\[
\|f(u)\|_{X^{\sigma,0,b}_{\delta}} \leq C\sigma^\rho\|u\|_{X^{\sigma,0,b}_{\delta}}^2
\]
for all $\sigma > 0$ and $v \in X^{\sigma,0,b}_{\delta}$. 

**Proof.** Notice first that

\[ \hat{v}_1 \hat{v}_2(\tau, \xi) = \hat{v}_1 \ast \hat{v}_2(\tau, \xi) \]

\[ = \int_{\mathbb{R}^2} \hat{v}_1(\tau_1, \xi_1) \hat{v}_2(\tau - \tau_1, \xi - \xi_1) \, d\tau_1 \, d\xi_1 \]

\[ = \int_{\mathbb{R}^2} \hat{v}_1(\tau_1, \xi_1) \hat{v}_2(\tau_2, \xi_2) \, d\tau_1 \, d\xi_1 \]

with \( \tau_2 = \tau - \tau_1 \) and \( \xi_2 = \xi - \xi_1 \). Using this and the following estimate

\[ e^{\sigma|\xi_1|} e^{\sigma|\xi_2|} - e^{\sigma(|\xi_1| + |\xi_2|)} \leq [2\sigma \min(|\xi_1|, |\xi_2|)]^{b} e^{\sigma|\xi_1|} e^{\sigma|\xi_2|}, \]

where \( \sigma > 0, 0 \leq \rho \leq 1 \), and \( \xi_1, \xi_2 \in \mathbb{R} \), one can see that

\[ \|f(u)\|_{X^{0,b-1}} \]

\[ \approx \| \frac{\xi}{(\tau - p(\xi))^{1-b}} \int_{\mathbb{R}^2} e^{\sigma|\xi_1|} \hat{u}(\tau_1, \xi_1) e^{\sigma|\xi_2|} \hat{u}(\tau_2, \xi_2) - e^{\sigma|\xi_1|} \hat{u}(\tau_1, \xi_1) \hat{u}(\tau_2, \xi_2) \, d\tau_1 \, d\xi_1 \|_{L^2_{\tau,\xi}} \]

\[ \leq \| \frac{\xi}{(\tau - p(\xi))^{1-b}} \int_{\mathbb{R}^2} [2\sigma \min(|\xi_1|, |\xi_2|)]^{b} e^{\sigma|\xi_1|} e^{\sigma|\xi_2|} \hat{u}(\tau_1, \xi_1) \hat{u}(\tau_2, \xi_2) \, d\tau_1 \, d\xi_1 \|_{L^2_{\tau,\xi}}, \]

where \( \tau_2 = \tau - \tau_1 \) and \( \xi_2 = \xi - \xi_1 \). Here, from the triangle inequality,

\[ \min(|\xi_1|, |\xi_2|) \leq 2 \left( \frac{1 + |\xi_1|}{1 + |\xi_1 + \xi_2|} \right) = \frac{2}{\langle \xi \rangle}, \]

and therefore

\[ \|f(u)\|_{X^{0,b-1}} \lesssim \sigma^b \left\| \frac{\xi(\xi)^{-\rho}}{(\tau - p(\xi))^{1-b}} \int_{\mathbb{R}^2} e^{\sigma|\xi_1|} \langle \xi_1 \rangle^{\rho} \hat{u}(\tau_1, \xi_1) e^{\sigma|\xi_2|} \langle \xi_2 \rangle^{\rho} \hat{u}(\tau_2, \xi_2) \, d\tau_1 \, d\xi_1 \right\|_{L^2_{\tau,\xi}} \]

\[ = \sigma^b \left\| \partial_x \left( e^{\sigma|\xi_1|} \langle \partial_x \rangle^{\rho} u \cdot e^{\sigma|\xi_2|} \langle \partial_x \rangle^{\rho} u \right) \right\|_{X^{0,-\rho,b-1}} \]

\[ \lesssim \sigma^b \| e^{\sigma|\xi_1|} \langle \partial_x \rangle^{\rho} u \|^2_{X^{0,-\rho,b}} \]

\[ = \sigma^b \| u \|^2_{X^{s,0,b}} \]

as desired. Here we used Lemma 3.1 with \( \sigma = 0, s = -\rho \) and \( b' = b \) for the second inequality. \( \square \)

### 4. Local well-posedness and almost conservation law

In this section we first establish the local well-posedness and then the almost conservation law, by making use of the bilinear estimate in the previous section. They lie at the core of the proof of Theorem 1.1 in the next section.

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2 It can be found in Lemma 12 of [17].
4.1. Local well-posedness. Based on Picard’s iteration in the $X_{\sigma,s}^{b,\delta}$-space and Lemma 2.1 we establish the following local well-posedness in $G^{\sigma,s}$, with a lifespan $\delta > 0$. Equally the radius of analyticity remains strictly positive in a short time interval $0 \leq t \leq \delta$, where $\delta > 0$ depends on the norm of the initial data.

**Theorem 4.1.** Let $\sigma > 0$ and $s > -7/4$. Then, for any $u_0 \in G^{\sigma,s}$, there exist $\delta > 0$ and a unique solution $u$ of the Cauchy problem (1.1) on the time interval $[0, \delta]$ such that $u \in C([0, \delta], G^{\sigma,s})$ and the solution depends continuously on the data $u_0$. Here we have

$$\delta = c_0 (1 + \|u_0\|_{G^{\sigma,s}})^{-a}$$

for some constants $c_0 > 0$ and $a > 2$ depending only on $s$. Furthermore, if $1/2 < b < 1$, the solution $u$ satisfies

$$\|u\|_{X_{\sigma,s,b}^{\delta}} \leq C \|u_0\|_{G^{\sigma,s}}$$

with a constant $C > 0$ depending only on $b$.

**Proof.** Fix $\sigma > 0$, $s > -7/4$ and $u_0 \in G^{\sigma,s}$. By Lemma 2.1 we shall employ an iteration argument in the space $X_{\sigma,s,b}^{\delta}$ instead of $G^{\sigma,s}$. Consider first the Cauchy problem for the linearized Kawahara equation

$$\begin{cases}
u_t + \alpha u_{xxx} + \beta u_{xxxxx} = F(t, x), \\
u(0, x) = f(x).
\end{cases}$$

By Duhamel’s principle the solution can be then written as

$$u(t, x) = e^{itp(-i\partial_x)} f(x) + \int_0^t e^{i(t-t') p(-i\partial_x)} F(t', x) \, dt',$$

where the Fourier multiplier $e^{itp(-i\partial_x)}$ with symbol $e^{itp(x)}$ is given by

$$e^{itp(-i\partial_x)} f(x) = \frac{1}{(2\pi)^{\text{dim} x}} \int_{\mathbb{R}} e^{ix\xi p(\xi)} \hat{f}(\xi) \, d\xi.$$

(Recall from Section 2 that $p(\xi) = \alpha \xi^3 - \beta \xi^5$.) Then the following $X_{\delta}^{\sigma,s,b}$ energy estimate follows directly from Proposition 2.1 in [7] (see also [13]).

**Lemma 4.2.** Let $\sigma \geq 0$, $s \in \mathbb{R}$, $1/2 < b < 1$ and $0 < \delta \leq 1$. Then we have

$$\|e^{itp(-i\partial_x)} f\|_{X_{\delta}^{\sigma,s,b}} \leq C_b \|f\|_{G^{\sigma,s}}$$

and

$$\left\| \int_0^t e^{i(t-t') p(-i\partial_x)} F(t', x) \, dt' \right\|_{X_{\delta}^{\sigma,s,b-1}} \leq C_b \|F\|_{X_{\delta}^{\sigma,s,b-1}},$$

with a constant $C_b > 0$ depending only on $b$.

Now, let $\{u^{(n)}\}_{n=0}^{\infty}$ be the sequence defined by

$$\begin{cases}u^{(0)}_t + \alpha u^{(0)}_{xxx} + \beta u^{(0)}_{xxxxx} = 0, \\
u^{(0)}(0) = u_0(x),\end{cases}$$
and for $n \in \mathbb{Z}^+$
\[
\begin{align*}
  u^{(n)}_t + \alpha u^{(n)}_{txx} + \beta u^{(n)}_{xxxx} &= -\frac{1}{2} \partial_x \left( u^{(n-1)} u^{(n-1)} \right), \\
  u^{(n)}(0) &= u_0(x).
\end{align*}
\]

Applying (4.3), we first write
\[
  u^{(0)}(t, x) = e^{it \partial_x} u_0(x)
\]
and
\[
  u^{(n)}(t, x) = e^{it \partial_x} u_0(x) - \frac{1}{2} \int_0^t e^{i(t-t') \partial_x} \partial_x \left( \left( u^{(n-1)}(t') \cdot u^{(n-1)}(t') \right) \right) dt'.
\]

By Lemma 4.2 we have
\[
  \|u^{(0)}\|_{X^{\sigma,s,b}^r} \leq C_b \|u_0\|_{G^{\sigma,s}}, \tag{4.4}
\]
and Lemmas 4.2, 2.2, and 3.1 combined imply
\[
  \|u^{(n)}\|_{X^{\sigma,s,b}^r} \leq C_b \|u_0\|_{G^{\sigma,s}} + C_b \|\partial_x \left( u^{(n-1)} u^{(n-1)} \right)\|_{X^{\sigma,s,b-1}_r}
  \leq C_b \|u_0\|_{G^{\sigma,s}} + C_{b,b'} \delta^{b'-b} \|\partial_x \left( u^{(n-1)} u^{(n-1)} \right)\|_{X^{\sigma,s,b-1}_r}
  \leq C_b \|u_0\|_{G^{\sigma,s}} + C_{b,b'} \delta^{b'-b} \|u^{(n-1)}\|_{X^{\sigma,s,b}} \tag{4.5}
\]
with $1/2 < b < b' < 1$. By induction together with (4.4) and (4.5), it follows that for all $n \geq 0$
\[
  \|u^{(n)}\|_{X^{\sigma,s,b}^r} \leq 2C_b \|u_0\|_{G^{\sigma,s}}, \tag{4.6}
\]
if we choose $\delta$ sufficiently small so that
\[
  \delta^{b'-b} \|u_0\|_{G^{\sigma,s}} \leq \frac{1}{8C_{b,b'} C_{s,b',b'} C_b^2}. \tag{4.7}
\]

Using Lemmas 4.2, 2.2, and 3.1 together with (4.6) and (4.7) in that order, we therefore get
\[
  \|u^{(n)} - u^{(n-1)}\|_{X^{\sigma,s,b}_r}
  \leq C_b \|\partial_x \left( u^{(n-1)} u^{(n-2)} - u^{(n-2)} u^{(n-2)} \right)\|_{X^{\sigma,s,b-1}_r}
  \leq C_{b,b'} \delta^{b'-b} \|\partial_x \left( u^{(n-1)} u^{(n-2)} - u^{(n-2)} u^{(n-2)} \right)\|_{X^{\sigma,s,b-1}_r}
  \leq C_{b,b'} \delta^{b'-b} \|u^{(n-1)} + u^{(n-2)}\|_{X^{\sigma,s,b}_r} \|u^{(n-1)} - u^{(n-2)}\|_{X^{\sigma,s,b}_r}
  \leq 4C_{b,b'}^2 \delta^{b'-b} \|u_0\|_{G^{\sigma,s}} \|u^{(n-1)} - u^{(n-2)}\|_{X^{\sigma,s,b}_r}
  \leq \frac{1}{2} \|u^{(n-1)} - u^{(n-2)}\|_{X^{\sigma,s,b}_r},
\]
which guarantees the convergence of the sequence $\{u^{(n)}\}_{n=0}^\infty$ to a solution $u$ with the bound (4.6). Furthermore, (4.7) follows easily from (4.4) and $0 < b' < b < 1/2$.

Now assume that $u$ and $v$ are solutions to the Cauchy problem (1.1) for initial data $u_0$ and $v_0$, respectively. Then similarly as above, again with the same choice of $\delta$ and for any $\delta'$ such that $0 < \delta' < \delta$, we have
\[
  \|u - v\|_{X^{\sigma,s,b}_r} \leq C_b \|u_0 - v_0\|_{G^{\sigma,s}} + \frac{1}{2} \|u - v\|_{X^{\sigma,s,b}_r}.
\]
provided \( \|u_0 - v_0\|_{G^{\sigma,\delta}} \) is sufficiently small, which proves the continuous dependence of the solution on the initial data.

Finally, it remains to show the uniqueness of solutions. Assume \( u, v \in C_t G^{\sigma,\delta} \) are solutions to (1.1) for the same initial data \( u_0 \) and let \( w = u - v \). Then \( w \) satisfies
\[
w_t + \alpha w_{xxx} + \beta w_{xxxxx} + wu_x + vw_x = 0.
\]
Multiplying both sides by \( w \) and integrating in space yield
\[
\frac{1}{2} \int_{\mathbb{R}} (w^2)_t dx + \alpha \int_{\mathbb{R}} w w_{xxx} dx + \int_{\mathbb{R}} w w_{xxxxx} dx + \int_{\mathbb{R}} w^2 u_x dx + \int_{\mathbb{R}} w w_x dx = 0.
\]
Using \( 2wvw_x = (vw^2)_x - v_x w^2 \) and integrating by parts, we then have
\[
\frac{1}{2} \int_{\mathbb{R}} (w^2)_t dx - \frac{\alpha}{2} \int_{\mathbb{R}} (w_x w_x)_x dx + \frac{\beta}{2} \int_{\mathbb{R}} (w_{xx} w_{xx})_x dx + \int_{\mathbb{R}} w^2 u_x dx + \int_{\mathbb{R}} w v w_x dx = 0.
\]
We may here assume that \( w \) and its all spatial derivatives decay to zero as \( |x| \to \infty \).

It follows then that
\[
\frac{1}{2} \int_{\mathbb{R}} (w^2)_t dx = - \int_{\mathbb{R}} w^2 u_x dx + \frac{1}{2} \int_{\mathbb{R}} v_x w^2 dx.
\]
By Hölder’s inequality, this implies
\[
\frac{d}{dt} \|w(t)\|_{L^2}^2 \leq 2 (\|u_x(t)\|_{L^\infty} + \|v_x(t)\|_{L^\infty}) \|w(t)\|_{L^2}^2 \leq C \|w(t)\|_{L^2}^2.
\]
Here we used the fact that \( G^{\sigma,\delta} \subseteq G^{0.2} = H^2 \subseteq L^q \) for all \( 2 \leq q \leq \infty \) and \( \sigma > 0 \). By Grönwall’s inequality, we now conclude that \( w = 0 \).

4.2. Almost conservation law. We have established the existence of local solutions; we would like to apply the local result repeatedly to cover time intervals of arbitrary length. This, of course, requires some sort of control on the growth of the norm on which the local existence time depends. The following approximate conservation will allow us (see Section 5) to repeat the local result on successive short-time intervals to reach any target time \( T > 0 \), by adjusting the strip width parameter \( \sigma \) according to the size of \( T \).

**Theorem 4.3.** Let \( 0 \leq \rho \leq 1, \frac{1}{2} < b < 1 \) and \( \delta \) be as in Theorem 4.1. Then there exists \( C > 0 \) such that for any \( \sigma > 0 \) and any solution \( u \in X^{\sigma,0,b}_\delta \) to the Cauchy problem (1.1) on the time interval \([0, \delta]\), we have the estimate
\[
\sup_{t \in [0, \delta]} \|u(t)\|_{G^{\sigma,\rho}}^2 \leq \|u(0)\|_{G^{\sigma,\rho}}^2 + C \sigma^\rho \|u\|_{X^{\sigma,0,b}_\delta}^3. \quad (4.8)
\]

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3 This property can be shown by approximation using the monotone convergence theorem and the Riemann-Lebesgue lemma whenever \( u \in X^{1.1,b}_\delta \). See the argument in [17], p. 1018.
Proof. Let $0 \leq \delta' \leq \delta$. Setting \( v(t, x) = e^{\sigma|\partial_x|}u(t, x) \) and applying \( e^{\sigma|\partial_x|} \) to (1.1), we obtain
\[
v_t + \alpha v_{xxx} + \beta v_{xxxxx} + vv_x = f(u),
\]
where \( f(u) \) is as in (3.2):
\[
f(u) = \frac{1}{2} \partial_x \left( (e^{\sigma|\partial_x|}u)^2 - e^{\sigma|\partial_x|}u^2 \right).
\]
Multiplying both sides by \( v \) and integrating in space yield
\[
\int_R v v_t dx + \alpha \int_R v v_{xxx} dx + \beta \int_R v v_{xxxxx} dx + \int_R v^2 v_x dx = \int_R v f(u) dx.
\]
As before, we may here assume that \( v \) and its all spatial derivatives decay to zero as \(|x| \to \infty\). Using this fact and integration by parts, we have
\[
\frac{1}{2} \int_R (v^2)_t dx - \frac{\alpha}{2} \int_R (v_x v_x)_x dx + \frac{\beta}{2} \int_R (v_{xxx} v_{xxx}) x dx + \frac{1}{3} \int_R (v^3)_x dx = \int_R v f(u) dx,
\]
and furthermore, the second, third and fourth terms on the left side vanish. Subsequently integrating in time over the interval \([0, \delta']\), we obtain
\[
\|u(\delta')\|_{G^\sigma, 0}^2 \leq \|u(0)\|_{G^\sigma, 0}^2 + 2 \int_{R^2} \chi_{[0, \delta']} (t) v f(u) dt dx.
\]
Now by Hölder’s inequality, Lemma 2.3 and Lemma 3.2
\[
\left| \int_{R^2} \chi_{[0, \delta']} (t) v f(u) dt dx \right| \leq \|\chi_{[0, \delta']} (t) v\|_{X^{0,1-b}} \|\chi_{[0, \delta']} (t) f(u)\|_{X^{0,b-1}}
\leq C \|v\|_{X^{0,1-b}_{\sigma'}} \|f(u)\|_{X^{0,b-1}_{\sigma'}}
\leq C \|u\|_{X^{b,1-b}_{\sigma'}} \sigma^s \|u\|_{X^{s,0,b}}^3.
\]
Since \( 1 - b < b \), we therefore get
\[
\sup_{t \in [0, \delta]} \|u(t)\|_{G^\sigma, 0}^2 \leq \|u(0)\|_{G^\sigma, 0}^2 + C \sigma^s \|u\|_{X^{s,0,b}}^3
\]
as desired. \( \square \)

5. Proof of Theorem 1.1

By invariance of the Kawahara equation under the reflection \((t, x) \to (-t, -x)\), we may restrict to positive times. By the embedding (1.2), the general case \( s \in \mathbb{R} \) will reduce to \( s = 0 \) as shown in the end of this section.

5.1. The case \( s = 0 \). Combining (4.2) and (4.8), we first note that
\[
\sup_{t \in [0, \delta]} \|u(t)\|_{G^\sigma, 0}^2 \leq \|u(0)\|_{G^\sigma, 0}^2 + C \sigma^s \|u(0)\|_{G^\sigma, 0}^3.
\] (5.1)

Let \( u_0 = u(0) \in G^{\sigma_0,0} \) for some \( \sigma_0 > 0 \) and \( \delta \) be as in Theorem 4.1. For arbitrarily large \( T \), we want to show that the solution \( u \) to (1.1) satisfies
\[
u(t) \in G^{\sigma(t),0} \quad \text{for all} \quad t \in [0,T],
\]
Indeed, for \( k \) where we used the fact that \( \| u_0 \|_{G^{\sigma_0,0}} \) and \( \sigma_0 \).

Now fix \( T \) arbitrarily large. It suffices to show

\[
\sup_{t \in [0,T]} \| u(t) \|_{G^{\sigma,0}}^2 \leq 2 \| u_0 \|_{G^{\sigma_0,0}}^2 \tag{5.3}
\]

for \( \sigma \) satisfying (5.2), which in turn implies \( u(t) \in G^{\sigma(t),0} \) as desired.

To prove (5.3), we first choose \( n \in \mathbb{Z}^+ \) so that \( n\delta \leq T \leq (n+1)\delta \). Using induction we shall show for any \( k \in \{1, 2, \cdots, n+1\} \) that

\[
\sup_{t \in [0,k\delta]} \| u(t) \|_{G^{\sigma,0}}^2 \leq \| u_0 \|_{G^{\sigma_0,0}}^2 + kC\sigma^p 2^{\frac{3p}{2}} \| u_0 \|_{G^{\sigma_0,0}}^3 \tag{5.4}
\]

and

\[
\sup_{t \in [0,k\delta]} \| u(t) \|_{G^{\sigma,0}}^2 \leq 2 \| u_0 \|_{G^{\sigma_0,0}}^2, \tag{5.5}
\]

provided \( \sigma \) satisfies

\[
\sigma \leq \sigma_0 \quad \text{and} \quad \frac{2T}{\delta} C\sigma^p 2^{\frac{3p}{2}} \| u_0 \|_{G^{\sigma_0,0}} \leq 1. \tag{5.6}
\]

Indeed, for \( k = 1 \), we have from (5.1) that

\[
\sup_{t \in [0,\delta]} \| u(t) \|_{G^{\sigma,0}}^2 \leq \| u_0 \|_{G^{\sigma_0,0}}^2 + C\sigma^p \| u_0 \|_{G^{\sigma_0,0}}^3
\]

\[
\leq 2 \| u_0 \|_{G^{\sigma_0,0}}^2,
\]

where we used the fact that \( \| u_0 \|_{G^{\sigma,0}} \leq \| u_0 \|_{G^{\sigma_0,0}} \) and \( C\sigma^p \| u_0 \|_{G^{\sigma_0,0}} \leq 1 \) which are a direct consequence of (5.3). Now assume (5.4) and (5.5) hold for some \( k \in \{1, 2, \cdots, n\} \). Applying (5.1), (5.5) and (5.4), we then have

\[
\sup_{t \in [k\delta, (k+1)\delta]} \| u(t) \|_{G^{\sigma,0}}^2 \leq \| u(k\delta) \|_{G^{\sigma,0}}^2 + C\sigma^p \| u(k\delta) \|_{G^{\sigma,0}}^3
\]

\[
\leq \| u(k\delta) \|_{G^{\sigma,0}}^2 + C\sigma^p 2^{\frac{3p}{2}} \| u_0 \|_{G^{\sigma_0,0}}^3
\]

\[
\leq \| u_0 \|_{G^{\sigma_0,0}}^2 + C\sigma^p (k + 1) 2^{\frac{3p}{2}} \| u_0 \|_{G^{\sigma_0,0}}^3.
\]

Combining this with the induction hypothesis (5.4) for \( k \), we get

\[
\sup_{t \in [0, (k+1)\delta]} \| u(t) \|_{G^{\sigma,0}}^2 \leq \| u_0 \|_{G^{\sigma_0,0}}^2 + C\sigma^p (k + 1) 2^{\frac{3p}{2}} \| u_0 \|_{G^{\sigma_0,0}}^3 \tag{5.7}
\]

which proves (5.4) for \( k + 1 \). Since \( k + 1 \leq n + 1 \leq T/\delta + 1 \leq 2T/\delta \), from (5.6) we also get

\[
C\sigma^p (k + 1) 2^{\frac{3p}{2}} \| u_0 \|_{G^{\sigma_0,0}} \leq \frac{2T}{\delta} C\sigma^p 2^{\frac{3p}{2}} \| u_0 \|_{G^{\sigma_0,0}} \leq 1
\]

which, along with (5.7), proves (5.5) for \( k + 1 \).

Finally, the condition (5.6) is satisfied for

\[
\sigma = \left( \frac{\delta}{C2^{\frac{3p}{2}} \| u_0 \|_{G^{\sigma_0,0}}} \right)^{\frac{1}{p}}, \left( \frac{1}{T} \right)^{\frac{1}{p}}.
\]
Particularly when $\rho = 1$, the constant $c$ in (5.2) may be given as
$$c = \frac{\delta}{C2^\frac{2}{\sigma}} \|u_0\|_{G^{\sigma_0,0}}$$
which depends only on $\|u_0\|_{G^{\sigma_0,0}}$.

5.2. The general case $s \in \mathbb{R}$. Recall that (1.2) states
$$G^{\sigma,s} \subset G^{\sigma',s'} \text{ for all } \sigma > \sigma' \geq 0 \text{ and } s, s' \in \mathbb{R}.$$  

For any $s \in \mathbb{R}$ we use this embedding to get
$$u_0 \in G^{\sigma_0,0} \subset G^{\sigma_0/2,0}.$$  

From the local well-posedness result, there is a $\delta = \delta(\|u_0\|_{G^{\sigma_0/2,0}})$ such that
$$u(t) \in G^{\sigma_0/2,0} \text{ for } 0 \leq t \leq \delta.$$ 

Similarly as in the case $s = 0$, for $T$ fixed greater than $\delta$, we have $u(t) \in G^{\sigma'/0}$ for $t \in [0,T]$ and $\sigma' \geq c/T$ with $c > 0$ depending on $\|u_0\|_{G^{\sigma_0/2,0}}$ and $\sigma_0$. Applying the embedding again, we conclude
$$u(t) \in G^{\sigma,s} \text{ for } t \in [0,T]$$
where $\sigma = \sigma'/2$.

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Department of Mathematics, Sungkyunkwan University, Suwon 16419, Republic of Korea

E-mail address: j.ahn@skku.edu
E-mail address: jimkim@skku.edu
E-mail address: ihseo@skku.edu