Parameter identification problems for thin inclusions in elastic bodies

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Abstract. The paper concerns an identification of rigidity parameters for thin inclusions located inside elastic bodies. A delamination of the inclusions is assumed thus providing a crack between inclusions and the elastic matrix. Inequality type boundary conditions are imposed at the crack faces to exclude a mutual penetration. We consider elastic as well as rigid inclusions and solve an optimal control problem for finding a rigidity parameter minimizing a suitable cost functional. The cost functional characterizes a displacement of the inclusion, and a rigidity parameter serves as a control function. We prove a solution existence of the problems formulated.

1. Setting the problem
Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\Gamma$, and $\gamma_0 = (-1,0) \times \{0\}$, $\gamma_0 \subset \Omega$, $\Omega_\gamma = \Omega \setminus \gamma_0$, $\gamma_e = (0,1) \times \{0\}$, $\gamma = \gamma_0 \cup \gamma_e \cup \{(0,0)\}$. We assume that $\gamma_e$ does not belong to $\Omega$, see Figure 1. Denote by $\nu = (0,1)$ a unit normal vector to $\gamma$; $\tau = (1,0)$ is a tangential vector.

Let $B = \{b_{ijkl}\}, i,j,k,l = 1,2$, be a given elasticity tensor with the usual properties of symmetry and positive definiteness,

\[ b_{ijkl} = b_{jikl} = b_{klij}, \quad i,j,k,l = 1,2, \quad b_{ijkl} \in L^\infty(\Omega), \]

\[ b_{ijkl}\xi_{ij}\xi_{kl} \geq c_0|\xi|^2 \quad \forall \xi_{ij}, \quad c_0 = \text{const} > 0. \]

In what follows, we identify functions defined on $\gamma$ with functions of the variable $x_1 \in \mathbb{R}; x = (x_1,x_2) \in \mathbb{R}^2$. Summation convention over repeated indices is used; all functions with two lower indices are assumed to be symmetric in these indices.

Let $f = (f_1, f_2) \in L^2(\Omega)^2$, $g \in L^2(\gamma)$ be given functions corresponding to external forces acting on the elastic body and the inclusion, respectively. The domain $\Omega_\gamma$ is a region filled with an elastic material, and $\gamma$ corresponds to an inclusion crossing the boundary $\Gamma$ at the point $(0,0)$. We assume that a delamination of the inclusion takes place. This means that a crack is located between the inclusion and the elastic matrix. To fix a cracking we assume that the positive (with respect to $\nu$) side of $\gamma_0$ is delaminated.

We first provide a formulation of an equilibrium problem for the elastic body with the elastic inclusion $\gamma$. We want to find a displacement field $u^\lambda = (u^\lambda_1, u^\lambda_2)$, a stress tensor $\sigma^\lambda = \{\sigma^\lambda_{ij}\}, i,j = 1,2$, defined in $\Omega_\gamma$, and a thin inclusion displacement $v^\lambda$ defined on $\gamma$ such that

\[ -\text{div} \sigma^\lambda = f; \quad \sigma^\lambda - B\varepsilon(u^\lambda) = 0 \quad \text{in} \quad \Omega_\gamma, \quad (1) \]
\[ \lambda v_\lambda^{1111} = [\sigma^\lambda_{\nu}] + g \text{ on } \gamma_0; \quad \lambda v_\lambda^{1111} = g \text{ on } \gamma_c, \]  
\[ u^\lambda = 0 \text{ on } \Gamma; \quad v_\lambda^{11} = v_\lambda^{111} = 0 \text{ for } x_1 = -1, 1, \]  
\[ v^\lambda = u_\nu^\lambda, \quad [u_\nu^\lambda] \geq 0, \quad \sigma_\nu^\lambda \geq 0, \quad \sigma_\tau^\pm = 0 \text{ on } \gamma_0, \]  
\[ v^\lambda(0) = [v_\lambda^\lambda(0)] = [v_\lambda^{11}(0)] = 0. \]  

Here, \( \lambda \) is a positive rigidity parameter of the elastic inclusion, \([p] = p^+ - p^-\) is a jump of a function \( p \) on \( \gamma \), where \( p^\pm \) are the traces of \( p \) on the faces \( \gamma^\pm \). The signs \( \pm \) correspond to positive and negative directions of \( \nu \); \( u_\nu = u_\nu^\lambda \); \([w(0)] = w(0^+) - w(0^-)\); \( \varepsilon(u) = \{\varepsilon_{ij}(u)\} \) is a strain tensor,

\[ \sigma_\nu = \sigma_{ij} \nu_j \nu_i, \quad \sigma_\tau = \sigma_{ij} \nu_j \tau_i, \quad \varepsilon_{ij}(u) = \frac{1}{2} (u_{i,j} + u_{j,i}), \quad i, j = 1, 2. \]

We use notations \( p_{i,j} = \frac{dp}{dx_i} \), and write the equilibrium equations (2) for the elastic inclusion \( \gamma \).

The term \([\sigma^\lambda_{\nu}]\) in the first equation of (2) describes forces acting on \( \gamma_0 \) from the surrounding elastic body.

Relations (1) are equilibrium equations and Hooke’s law, respectively. According to the first relation of (4), the vertical (along the axis \( x_2 \)) displacements of the elastic body on \( \gamma_0^- \) coincide with displacements of the inclusion. The first inequality of (4) provides a mutual nonpenetration between the crack faces. All the rest relations of (4) are standard for contact problems with unknown set of a contact, see [1, 2, 3]. The second group of boundary conditions of (3) fits to zero moments and zero transverse forces at the tip points of the inclusion \( \gamma \). Relations (5) are junction conditions at the point \((0, 0)\), see, for example, [4, 5].

The problem (1)-(5) can be formulated as a variational one. To this end, introduce Sobolev spaces

\[ H^1_\gamma(\Omega_\gamma) = \{ \phi \in H^1(\Omega_\gamma) : \phi = 0 \text{ on } \gamma \}, \quad H^{2,0}_\gamma(\gamma) = \{ w \in H^2(\gamma) : w(0) = 0 \} \]

and a set of admissible displacements

\[ K = \{ (u, v) \in H^1_\gamma(\Omega_\gamma)^2 \times H^{2,0}_\gamma(\gamma) : [u_\nu] \geq 0 \text{ on } \gamma_0; \ u_\nu^-|_{\gamma_0} = v \}. \]
Consider the energy functional
\[ \pi_\lambda(u,v) = \frac{1}{2} \int_{\Omega_\gamma} \sigma(u) \varepsilon(u) - \int_{\Omega_\gamma} f u + \frac{\lambda}{2} \int_{\gamma} v_{11}^2 - \int_{\gamma} g v \]
and the minimization problem:
\[ \text{find } (u^\lambda, v^\lambda) \in K \text{ such that } \pi_\lambda(u^\lambda, v^\lambda) = \inf_{K} \pi_\lambda. \] (6)

Here and below we write \( \sigma(u) \varepsilon(u) \) instead of \( \sigma_{ij}(u) \varepsilon_{ij}(u) \). A solution of the problem (6) satisfies the variational inequality
\[ (u^\lambda, v^\lambda) \in K, \]
\[ \int_{\Omega_\gamma} \sigma(u^\lambda) \varepsilon(\bar{u} - u^\lambda) - \int_{\Omega_\gamma} f(\bar{u} - u^\lambda) + \]
\[ + \lambda \int_{\gamma} v_{11}^\lambda(\bar{v},11 - v_{11}^\lambda) - \int_{\gamma} g(\bar{v} - v^\lambda) \geq 0 \quad \forall (\bar{u}, \bar{v}) \in K. \] (8)

The problem (7)-(8) has a unique solution. Moreover, it can be checked that problem formulations (1)-(5) and (7)-(8) are equivalent provided that the solutions are smooth, see [3].

2. Rigid Inclusion
In this section, we formulate an equilibrium problem in the case when the inclusion \( \gamma \) is a rigid one, see Figure 1. This means that a displacement field on \( \gamma \) has a special structure. Introduce a space of infinitesimal rigid displacements
\[ L(\gamma) = \{ l : l(x_1) = cx_1, x_1 \in (-1,1), c \in \mathbb{R} \}. \]

An equilibrium problem for the elastic body occupying the domain \( \Omega_\gamma \) with the rigid inclusion \( \gamma \) can be formulated as follows [6]. Find functions \( u = (u_1, u_2) \), \( \sigma = \{ \sigma_{ij} \}, i,j = 1,2 \), \( l_0 \in L(\gamma) \), such that
\[ -\text{div } \sigma = f \text{ in } \Omega_\gamma, \]
\[ \sigma - B \varepsilon(u) = 0 \text{ in } \Omega_\gamma, \]
\[ u = 0 \text{ on } \Gamma; \quad l_0 = u_\nu^- \text{ on } \gamma_0, \]
\[ [u_\nu] \geq 0, \quad \sigma_\nu^+ \leq 0, \quad \sigma_\tau^\pm = 0, \quad [u_\nu] = 0 \text{ on } \gamma_0, \]
\[ \int_{\gamma_0} [\sigma_\nu]l + \int_{\gamma} gl = 0 \quad \forall l \in L(\gamma). \] (13)

According to (11), a displacement \( l_0 \) of the inclusion \( \gamma \) coincides with the vertical displacement of the elastic body on \( \gamma_0^- \). Non-local boundary condition (13) guarantees that a principal moment of forces acting on \( \gamma \) is equal to zero.

The problem (9)-(13) admits a variational formulation. To this end, we introduce a set of admissible displacements
\[ K_\infty = \{(u,l) \in H^1_\gamma(\Omega_\gamma)^2 \times L(\gamma) : [u_\nu] \geq 0 \text{ on } \gamma_0; \quad u_\nu^-|_{\gamma_0} = l \}. \]
Then it is possible to check that the problem (9)-(13) is equivalent to the following variational inequality (provided that the solutions are quite smooth)

\[(u, l_0) \in K, \quad \int \sigma(u) \varepsilon(\bar{u} - u) - \int f(\bar{u} - u) - \int g(\bar{l} - l_0) \geq 0 \quad \forall (\bar{u}, \bar{l}) \in K. \quad (14)\]

The problem (14)-(15) has a unique solution. We omit a checking of this fact; moreover, checking the equivalence of the problem formulations (14)-(15) and (9)-(13) is also omitted.

To conclude this section, we should underline that the problem (14)-(15) is a limit one for the family of problems (7)-(8) provided that \(\lambda \to \infty\). Therefore, for any fixed \(\lambda > 0\) the model (1)-(5) describes an equilibrium state of the elastic body with the elastic inclusion \(\gamma\). This model reduces to the model (9)-(13) provided that \(\lambda \to \infty\) which describes an equilibrium of the elastic body with the rigid inclusion \(\gamma\).

3. Optimal Control Problem

This section concerns an investigation of an optimal control problem for an elastic body with thin delaminated inclusions, and a rigidity parameter \(\lambda\) being a control function. For any fixed \(\lambda > 0\) we can find a solution of the problem

\[(u^\lambda, v^\lambda) \in K, \quad \int \sigma(u^\lambda) \varepsilon(\bar{u}^\lambda - u^\lambda) - \int f(\bar{u}^\lambda - u^\lambda) + \lambda \int g(\bar{v}^\lambda - v^\lambda) \geq 0 \quad \forall (\bar{u}, \bar{v}) \in K. \quad (16)\]

For \(\lambda = \infty\), we can find a solution of the problem (14)-(15). Hence, for a finite \(\lambda\) we have an elastic inclusion \(\gamma\), and \(\lambda = \infty\) corresponds to the rigid inclusion \(\gamma\).

A cost functional will characterize a displacement of the inclusion part \(\gamma_e\), and \(\lambda\) is a control parameter, \(\lambda \in [\lambda_0, \infty]\). Let \(v_* \in H^2(\gamma_e)\) be a given function. For each \(\lambda \in [\lambda_0, \infty]\) we can find a solution \((u^\lambda, v^\lambda)\) of the suitable boundary value problem. Define a cost functional for \(\lambda \in [\lambda_0, \infty]\),

\[G(\lambda) = ||v^\lambda - v_*||_{H^2(\gamma_e)}.\]

Optimal control problem is formulated as follows

\[\min_{\lambda \in [\lambda_0, \infty]} G(\lambda). \quad (18)\]

We see that a solution of the optimal control problem (18) allows us to find a rigidity parameter that minimizes a difference on \(\gamma_e\), in a suitable sense, between the solution \(v^\lambda\) and the given function \(v_*\). It is very important that we compare solutions of different models corresponding to finite and infinite values of \(\lambda\).

**Theorem.** There exists a solution of the optimal control problem (18).

**Proof.** Let \(\lambda_n \in [\lambda_0, \infty]\) be a minimizing sequence. For any \(\lambda_n\) we can find a unique solution of the problem like (16)-(17) provided \(\lambda_n\) is finite, or of the problem (14)-(15) for \(\lambda = \infty\). We can assume that the sequence is converging. There are two possible cases:

1. \(\lambda_n \to \lambda_*\), \(n \to \infty\), \(\lambda_* \in [\lambda_0, \infty)\), \(\lambda_* \in \mathbb{R}\);
2. \(\lambda_n \to \infty\), \(n \to \infty\), \(\lambda_n \in [\lambda_0, \infty)\).
If $\lambda_n = +\infty$ for $n \geq n_0$, then a solution of the problem (18) clearly exists. We consider two cases separately.

**Case 1.** Assume that $\lambda_n \to \lambda_*$, $n \to \infty$, $\lambda_n \in [\lambda_0, \infty)$, $\lambda_* \in \mathbb{R}$. For every $n$ we find a solution of the problem

\[
(u^n, v^n) \in K, \tag{19}
\]

\[
\int_{\gamma} \sigma(u^n)\varepsilon(\bar{u} - u^n) - \int_{\Omega_\gamma} f(\bar{u} - u^n) + \lambda_n \int_{\gamma} v_{11}^n (\bar{v}_{11} - v^n_{11}) - \int_{\gamma} g(\bar{v} - v^n) \geq 0 \quad \forall (\bar{u}, \bar{v}) \in K. \tag{20}
\]

It is possible to pass to the limit in (19)-(20) as $n \to \infty$.

The following estimates take place uniformly in $n \geq n_0$, see [7],

\[
\|u^n\|^2_{H^1_\gamma(\Omega_\gamma)^2} + \|v^n\|^2_{H^2,0(\gamma)} \leq c. \tag{21}
\]

By (21), we can assume that as $n \to \infty$

\[
 u^n \to u \text{ weakly in } H^1_\gamma(\Omega_\gamma)^2, \quad v^n \to v \text{ weakly in } H^2,0(\gamma).
\]

This convergence allows us to pass to the limit in (19)-(20) as $n \to \infty$ which gives

\[
(u, v) \in K, \tag{22}
\]

\[
\int_{\gamma} \sigma(u)\varepsilon(\bar{u} - u) - \int_{\Omega_\gamma} f(\bar{u} - u) + \lambda_* \int_{\gamma} v_{11} (\bar{v}_{11} - v_{11}) - \int_{\gamma} g(\bar{v} - v) \geq 0 \quad \forall (\bar{u}, \bar{v}) \in K. \tag{23}
\]

Relations (22)-(23) mean that $(u, v) = (u^{\lambda_*}, v^{\lambda_*})$. Thus, we have

\[
\inf_{\lambda \in [\lambda_0, \infty]} G(\lambda) = \lim G(\lambda_n) = \lim \inf \|v^n - v_*\|_{H^2(\gamma_\gamma)} \geq
\]

\[
\geq \|v - v_*\|_{H^2(\gamma_\gamma)} = G(\lambda_*) \geq \inf_{\lambda \in [\lambda_0, \infty]} G(\lambda),
\]

and the existence proof is complete in the case 1.

**Case 2.** Assume that $\lambda_n \to \infty$, $n \to \infty$, $\lambda_n \in [\lambda_0, \infty)$. In this case, for any $n$, the solution $(u^n, v^n)$ satisfies (19)-(20). From (19)-(20) we obtain the relation

\[
\int_{\gamma} \sigma(u^n)\varepsilon(u^n) - \int_{\Omega_\gamma} f(u^n) + \lambda_n \int_{\gamma} (v_{11}^n)^2 - \int_{\gamma} g v^n = 0. \tag{24}
\]

From (24) we can derive for $n \geq n_0$

\[
\|u^n\|^2_{H^1_\gamma(\Omega_\gamma)^2} \leq c, \quad \|v^n\|^2_{H^2,0(\gamma)} \leq c, \tag{25}
\]

and moreover, from (24) it follows for $n \geq n_0$

\[
\int_{\gamma} (v_{11}^n)^2 \leq \frac{c}{\lambda_n}. \tag{26}
\]
By (25)-(26), we assume that
\[ u^n \to u \text{ weakly in } H^1_0(\Omega)^2, \quad v^n \to v \text{ weakly in } H^{2,0}(\gamma), \]
\[ v(x_1) = a_1 x_1, \quad x_1 \in (-1, 1). \]  
(27)
(28)

Denote \( l_0(x_1) = a_1 x_1, \quad x_1 \in (-1, 1) \). We take any fixed element \((\bar{u}, \bar{l}) \in K_\infty\). It is clear that \((\bar{u}, \bar{l}) \in K\). Substitute \((\bar{u}, \bar{l})\) as a test function in (20) and pass to the limit as \( n \to \infty \). It provides
\[ (u, l_0) \in K_\infty, \]
\[ \int_{\Omega} \sigma(u)\varepsilon(\bar{u} - u) - \int_{\Omega} f(\bar{u} - u) - \int_{\gamma} g(\bar{l} - l_0) \geq 0 \quad \forall (\bar{u}, \bar{l}) \in K_\infty. \]

Thus, the limit functions \( u, v \) from (27)-(28) satisfy the condition \((u, l_0) = (u^\infty, v^\infty)\), compare with (14)-(15). By this, we obtain
\[ \inf_{\lambda \in [\lambda_0, \infty]} G(\lambda) = \lim G(\lambda_n) = \lim \inf \|v^n - v^*\|_{H^2(\gamma_e)} \geq \]
\[ \geq \|l_0 - v^*\|_{H^2(\gamma_e)} = G(\infty) \geq \inf_{\lambda \in [\lambda_0, \infty]} G(\lambda). \]

The proof of Theorem is complete.

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