Default Contagion with Domino Effect
—A First Passage Time Approach

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1 Introduction

The systematic risk and contagion play a crucial role in the financial crisis. Many recent researches have put attention on this object since the Asian banking crisis of the late 90s, and the more recent banking crisis of 2007-2008. Most of them used directed graphs (network) to model interdependencies of system finance. For example, in Elliott et al. (2014), values of organizations depend on each other - e.g., through cross-holdings of shares, debt, or other liabilities. By tracking how asset values and failure costs propagate through the network of interdependencies as domino effect, the authors show how the probability of cascades and their extent depend on two key aspects of cross-holdings: integration and diversification. Rogers & Veraart (2013) is interested in the role of linkage in interbank and provide condition when rescue consortia exists. In order to study the importance of institutions, one can use the Contagion Index Cont et al. (2010). CDS spreads or equity volatility Acharya et al. (2017), Huang et al. (2009)...

On the contrary, the global level of systemic risk in the entire network can be considered by many other authors. References on the global level is referred to Cont et al. (2010) for a short survey. In another approach, Acemoglu et al. (2015) is interested in the stability and resilience of the financial system under effecting of negative shocks. Fouque & Sun (2013) focuses on the number of components reaching a default level in a given time. The authors used the mean-field limit and a large deviation estimate for a simple linear model of lending and borrowing banks to illustrate systematic risk. The mean-field limit is also used in
Chong & Klüppelberg (2015) to prove a law of large numbers. Both articles investigate system of interacting stochastic differential equations.

The present paper introduces a structural framework to model dependent defaults, with a particular interest in their contagion. The idea is based on the modeling of dependent stochastic intensities in Jarrow & Yu (2001). By “structural”, we mean, as usual in the context, modeling bankruptcy or default of a contingent claim by an event that a firm value process reaches a level. The level can be either endogenous as the seminal paper Leland & Toft (1996), but at this stage it is given exogenously. Since we are concerned with mutual dependence of the default, we consider a vector valued process, each component of which is the value of a firm. In our model, to describe the contagion, the default level of each firm is assumed to be affected every time a default of another firm occurs. Such kind of model has also been studied in Chong & Klüppelberg (2016). In their paper Chong & Klüppelberg (2016), the authors use the Bayesian network methodology to characterize the joint default distribution of the financial system at a given maturity.

In our model, a default of a firm brings about a prescribed constant jump to the default level of other firms. One default can therefore cause other defaults, but each of the second order default may trigger third order ones, and so on. This gives a structural framework to Bayesian network type dependence of joint default probability.

Another difference is that Chong & Klüppelberg (2016) only considers the firms value at maturity time. The default or survive of a company is determined by its equity. It is different from our approach. We are not only interested in the number of default at given time but also in default time and number of default at default time. They depend on the state of firms value which hits some special zone, called contagion region.

2 Model

The model used in the present paper is similar to the one in Chong & Klüppelberg (2016). Let $X^i_t$ denote the firm value process of the $i$-th company, for $i = 1, 2, \ldots, n$ with $n \geq 2$. Define “default time” by

$$\tau_i := \inf\{s \geq 0 : X^i_s \leq K^i\},$$

with
where $k^i \in \mathbb{R}$ is a exogenously given default level for the $i$-th company. We assume that $X \equiv (X^1, \cdots, X^n)$ solves the following equation:

$$
X^i_t = x^i - \sum_{j \neq i} C^i_j 1_{\{\tau^j_t < \tau^i_t\}} + \int_0^{\tau^i_t} (\sigma^i(X^i_s) dW^i_s + \mu^i(X^i_s) dt) \\
+ \sum_{j \neq i} \int_{\tau^j_t \wedge \tau^i_t}^{\tau^i_t} (\sigma^j(X^i_s) dW^i_s + \mu^j(X^i_s) dt)
$$

(1)

for $i = 1, 2, \cdots, n$, where $W^i, i = 1, \cdots, n$ are independent Brownian motions, and for $i, j = 1, \cdots, n$, $C^i_j$ are non-negative constants, $\sigma^i, \mu^i, \sigma^j, \mu^j$, each defined on $\mathbb{R}^n$, are smooth function with at most linear growth.

In a more concise way of saying,

- each component is a diffusion process, independent to each other, for each interval from a default time to next one,
- the default of $i$-th company brings about a jump $C^i_j$ to $j$-th company,
- which may causes the default of $j$-th company,
- The $i$-th default may also affects the dynamics of the $j$-th firm value process in terms of its growth rate or the volatility.

Define the first contagion time by

$$
\tau(1) := \min\{\tau_i : i = 1, \cdots, n\},
$$

with the convention that, $\min \emptyset = \infty$. The $j$-th contagion time is defined recursively by

$$
\tau(j) := \min\{\tau_k : \tau_k > \tau(j - 1)\}, \quad j = 2, 3, \cdots, n^*,
$$

where $n^*$ is the random variable so defined that at $n^*$-the contagion time all the companies left default. Note that $n^* \leq n$ but each of $\tau(j)$ can be infinity.

To price credit derivatives such as CDO or CDS, the distribution of the number of defaulted companies by a fixed time, denoted by $N^t$, and the joint distribution of $\tau_i, i \in I_0 \subset \{1, \cdots, n\}$ are required.

These are in principle obtained from the joint distribution of

$$
(\tau(1), \cdots, \tau(n), d(\tau(1)), \cdots, d(\tau(n))),
$$

where

$$
d(\tau(k)) = \{i \in \{1, \cdots, n\} : \tau_i = \tau(k)\}, k = 1, \cdots, n^*.
$$
3 General Case

3.1 Key Ideas

The first key idea is that we regard \((\tau(i), X_{\tau(i)})\) as (something like) a “renewal-reward” process. We shall have a formula of the joint density of 
\[(d(\tau(1)), \tau(1), X_{\tau(1)})\]
conditioned by the starting point \(X_0\). Here we understand \(X_{\tau(1)+t}, t \geq 0\) to be an \(R^{d(\tau(1))}\)-valued process; we are only interested in the survived companies. Then, by replacing \(\{1, \ldots, n\}\) with \(d(\tau(1))\) and \(X_0\) with \(X_{\tau(1)}\), we obtain the joint distribution of 
\[(d(\tau(2)), \tau(2), X_{\tau(2)})\]
conditioned by \(X_{\tau(1)}\), thanks to Markov property of \(X\). We can repeat this procedure to get the desired joint distribution.

We can separate the problem of determining the joint distribution of 
\[(d(\tau(1)), \tau(1), X_{\tau(1)})\]
into three parts.

- We pretend that we are given the harmonic measure of \(X_{\tau(1)}\) (before the “artificial” jumps): in a simple Brownian case it is known.

- Then the problem reduces to the description of “contagion domain”, but it may not be in the form of disjoint union.

- To get a computable form, we rely on the independence and a recursive equation.

To take into account that we work on a “renewal” setting described as above, from now on we let the index set of \(X\) be arbitrary finite subset. In order to specify the initial index set, we put superscript \(I\) to the previously defined notations; \(\tau^I(1), d^I\), and so on. We then concentrate on the study of the joint distribution of 
\[(d^I(\tau^I(1)), \tau^I(1), X_{\tau^I(1)})\].

3.2 Contagion domain

Let \(I := \{i_1, \cdots, i_{|I|}\}\) and for a permutation \(\sigma \in \mathcal{S}_I\) over \(I\), we put
\[D_{I, \sigma} := \{ (x_{i_1}, \cdots, x_{i_{|I|}}) \in \mathbb{R}^I : x_{i_{\sigma(1)}} = K^{i_{\sigma(1)}}, x_{i_{\sigma(2)}} \in (K^{i_{\sigma(2)}}, K^{i_{\sigma(2)}} + C_{i_{\sigma(1)}}, i_{\sigma(2)}], \]
\[\cdots, x_{i_{\sigma(|I|)}} \in (K^{i_{\sigma(|I|)}}, K^{i_{\sigma(|I|)}} + \sum_{j=1}^{|I|-1} C_{i_{\sigma(j)}, i_{\sigma(|I|)}}] \} \]
Then, we have the following

**Lemma 1** For \( \emptyset \neq J \subseteq I \), we have that

\[
\{ \tau^I(1) = J \} = \left\{ X^I_{\tau^I(1) -} = (X^J_{\tau^I(1) -}, X^{I \setminus J}_{\tau^I(1) -}) \in \bigcup_{\sigma \in \mathcal{S}_J} D_{J, \sigma} \times \prod_{i \in I \setminus J} (K^i + \sum_{j \in J} C_{j, i}, \infty) \right\}.
\]

**Proof.** The relation is clear if one sees that \( d^I(\tau^I(1)) = J := \{ j_1, \ldots, j_m \} \) is equivalent to the following: there is a permutation \( \sigma \in \mathcal{S}_J \) such that

(i) \( X^j_{\sigma(1)} \) hits \( K^{j_{\sigma(1)}} \),

(ii) at the hitting time \( X^{j_{\sigma(2)}} \) is in the interval \( (K^{j_{\sigma(2)}}, K^{j_{\sigma(2)}} + C_{j_{\sigma(1)}}, j_{\sigma(2)}) \) so that the \( j_{\sigma(1)} \)-th company’s default caused \( j_{\sigma(2)} \)-th company,

(iii) \( X^{j_{\sigma(3)}} \) is in the interval \( (K^{j_{\sigma(3)}}, K^{j_{\sigma(2)}} + C_{j_{\sigma(1)}}, j_{\sigma(3)} + C_{j_{\sigma(2)}}, j_{\sigma(3)}) \) so that \( j_{\sigma(3)} \)-th company defaulted due to \( j_{\sigma(1)} \) and/or \( j_{\sigma(2)} \)-th company’s default,

\[
\vdots
\]

(m) \( X^{j_{\sigma(m)}} \in (K^{j_{\sigma(m)}}, K^{j_{\sigma(m)}} + \sum_{l=1}^{m-1} C_{j_{\sigma(l)}, j_{\sigma(m)}}) \),

and \((m+1)\) for \( i \in I \setminus J \), \( X^i \in (K^i + \sum_{j \in J} C_{j, i}, \infty) \) so that the default of the companies indexed by \( J \) did not cause the default of \( i \)-th company. 

We put

\[
D^I_J := \bigcup_{\sigma \in \mathcal{S}_J} D_{I, \sigma},
\]

and for non-empty \( J \subseteq I \), we put

\[
D^I_J := D^I_J \times A^I_J,
\]

where

\[
A^I_J := \prod_{i \in I \setminus J} (K^i + \sum_{j \in J} C_{j, i}, \infty).
\]

**Lemma 2** \( D^I_J \cap D^I_{J'} = \emptyset \) if \( J \neq J' \).
Proof. We may assume without loss of generality, \( J \setminus J' \) is non-empty. Let \( J \setminus J' = \{k_1, \ldots, k_l\} \) and \( J \cap J' = \{k_{i+1}, \ldots, k_{jJ}\} \). Then, for \( x \in D_f \), there is a permutation \( \sigma \in S_J \) such that

\[
\begin{align*}
(x_{k_1}, \ldots, x_{k_l}) \\
\in (K^{k_1}, K^{k_1} + \sum_{j: \sigma^{-1}(j) < \sigma^{-1}(1)} C_{j,k_1}) \times \cdots \times (K^{k_l}, K^{k_l} + \sum_{j: \sigma^{-1}(j) < \sigma^{-1}(l)} C_{j,k_l}),
\end{align*}
\]

(4)

where if \( \sigma(i) = 1 \), the sum is set to be zero. On the other hand, for \( x = (x_{i_1}, \ldots, x_{i_{tt}}) \in D^f_{J'} \), it holds that

\[
\begin{align*}
(x_{k_1}, \ldots, x_{k_l}) \in (K^{k_1} + \sum_{j \in J'} C_{j,k_1}, \infty) \times \cdots \times (K^{k_l} + \sum_{j \in J'} C_{j,k_l}, \infty)
\end{align*}
\]

(5)

since \( \{k_1, \ldots, k_l\} \subset I \setminus J' \).

Let

\[
j_* := \operatorname{argmin}_{j \in J} \sigma^{-1}(j).
\]

Then

\[
\sigma^{-1}(j) < \sigma^{-1}(j_*)
\]

implies \( j \in J' \cap J \) and therefore

\[
\{ j : \sigma^{-1}(j) < \sigma^{-1}(j_*) \} \subset J'.
\]

Now we see that (4) and (5) are not compatible, meaning that \( D^f_I \cap D^f_{J'} \) is empty. □

Further, we set

\[
D^I_f := \bigcup_{i \in I} \{K^i\} \times \prod_{j \neq i} (K^j, \infty).
\]

Then we have the following

**Lemma 3** Let \( J_1, \ldots, J_k \) be such that \( \emptyset \neq J_k \subseteq \cdots \subseteq J_1 \), and set

\[
H_{J_1, \ldots, J_k} := D^I_f \times A^{J_{k-1}} \times \cdots \times A^{J_1}.
\]

Then it holds that

\[
D^I_f = H_I \setminus \bigcup_{\emptyset \neq J_k \subseteq I} H_{I,J_1,J_2} \setminus \bigcup_{\emptyset \neq J_k \subseteq J_2} H_{H_{I,J_1,J_2}, \cdots}
\]

(6)

so that for any measure \( \mu \) on \( \mathcal{B}(D^I_f) \),

\[
\begin{align*}
\mu(D^I_f) = \mu(H_I) + \sum_{\emptyset \neq J_k \subseteq I} (-1) \mu(H_{I,J_k}) + \sum_{J_1 \subseteq I} \sum_{\emptyset \neq J_2 \subseteq J_1} (-1) \mu(H_{I,J_1,J_2}) \\
+ \cdots + \sum_{J_{t-1} \subseteq I} \cdots \sum_{\emptyset \neq J_{t-2} \subseteq J_{t-2}} (-1)^{t-1} \mu(H_{I,J_1, \ldots, J_{t-1}}).
\end{align*}
\]

(7)
Proof. We first prove
\[ D^I = \bigcup_{J \subseteq I} D^I_J. \] (8)
That \( D^I \) includes the right-hand-side is clear. Suppose that \( x \in D^I \). Then, \( x^i = K^i \) for a unique \( i \) and \( x^j > K^j \) for \( j \neq i \). If there exits \( j \neq i \) such that \( x^j \leq K^j + C_{i,j} \), we rename it as \( j_1 \). Otherwise, \( (x_j)_{j \in I \setminus \{i\}} \in A^I_{\{i\}} \) and so \( x \in D^I_{\{i\}} \). Among \( I \setminus \{i, j_1\} \), if we could find \( j \) such that \( x^j \leq K^j + C_{i,j} + C_{j_1,j} \), we rename it \( j_2 \). Otherwise \( x \in D^I_{\{i,j_1\}} \). This procedure can be repeated at most \( |I| - 1 \) times when we have \( x \in D^I_I \). So in any case \( x \in \bigcup_{J \subseteq I} D^I_J \).

Next, observe that
\[ D^I_I = D^I \setminus \bigcup_{J \subseteq I} D^I_{J_{\{i\}}} = D^I \setminus \bigcup_{J \subseteq I} (D^I_{J_{\{i\}}} \times A^I_{J_{\{i\}}}) \]
by (8) and Lemma 2. By applying (8) to \( D^I_{J_{\{i\}}} \), and by Lemma 2 we obtain that
\[ D^I_I = D^I \setminus \bigcup_{J \subseteq I} \left( (D^I_{J_{\{i\}}} \times A^I_{J_{\{i\}}}) \setminus \bigcup_{J_{\subseteq J_{\{i\}}} J_{\{i\}}} (D^I_{J_{\{i\}}} \times A^I_{J_{\{i\}}}) \right). \]
By applying (8) to \( D^I_{J_{\{i\}}} \) and Lemma 2 and so on, we finally reach (6).

3.3 The first main result
Let \( \emptyset \neq J \subseteq I \), and define a family of measures
\[ h^I_J(x, A, S) := P(d^I(\tau^I(1)) = J, X^I_{\tau^I(1)} \in A, \tau^I(1) \in S | X^I_0 = x), \]
\[ h^I(x, S) := P(d^I(\tau^I(1)) = I, \tau^I(1) \in S | X^I_0 = x), \]
and
\[ Q^I(x, A, S) = P(X^I_{\tau^I(1)-} \in A, \tau^I(1) \in S | X^I_0 = x) \]
for \( x \in \prod_{i \in I} (K^i, \infty) \), \( A \in \mathcal{B}(D^I) \) and \( S \in \mathcal{B}(0, \infty) \). The last one is the harmonic measure of the process \( X^I \) on the boundary \( \partial \prod_{i \in I} (K^i, \infty) = D^I \).

Our first main result relates the joint distribution of \((d^I(\tau^I(1)), \tau^I(1), X^I_{\tau^I(1)})\) to the harmonic measure \( Q \).

Theorem 4 We have that
\[ h^I_J(x, A, S) = Q^I(x, D^I_J \times s^I_J(A), S) \]
and
\[ h^I(x, S) = Q^I(x, D^I_I, S). \]
Here \( s^I_J \) is a shift on \( \mathbb{R}^I \setminus J \) defined by
\[ s^I_J((x^i, i \in I \setminus J)) = ((x^i + \sum_{j \in J} C^i_{j,i}). \] (11)
Proof. The formula (9) is almost clear from Lemma 1 since we have
\[
\{d^I(\tau^I(1)) = J, X^I_{\tau^I(1)} \in A, \tau^I(1) \in S\} \\
\quad = \left\{ X^I_{\tau^I(1)} \in \bigcup_{\sigma \in S^I} D_{J,\sigma} \times s^I_{J}(A) \right\} \cap \{ \tau^I(1) \in S\}
\]
for \( S \in \mathcal{B}(0, \infty) \). The formula (10) is also clear from Lemma 1. \( \blacksquare \)

4 Joint Distribution of Contagions and Formulas for Contagion Probabilities

In this section, we are interested in the distribution of the number of defaulted companies by a fixed time, denoted by \( N_t \), and the distribution of the default time of some specific firm. These distributions are useful for pricing of credit derivatives like CDO or CDS.

Denote
\[
\mathbb{I}_j := I \setminus \bigcup_{l=1}^{j-1} d(\tau^I(l)) = \mathbb{I}_{j-1} \setminus d(\tau^I(j))
\]
the random set of indices of the firms that survived after the \( j \)-th contagion time \( \tau^I(j) \), for \( j = 1, 2, \ldots, n^* \). Let \( U_1 = \tau^I(1), U_2 = \tau^I(2) - \tau^I(1), \ldots, U_k = \tau^I(k) - \tau^I(k-1) \) be inter-arrival times between consecutive defaults. We let \( U_i = \infty \) if \( \tau(i) = \infty \).

Thanks to the Markov property of the firm value evolution we get the following \( (X^I, \tau^I, \mathbb{I}_j) \).

Theorem 5 Let \( J_j \subset \mathbb{I}_{j-1} \) be a non-empty set, \( A_j \in \mathfrak{B}(\prod_{v \in J_j} [K^i, \infty)) \), and \( S_j \in \mathfrak{B}([0, \infty)) \), for \( j = 1, 2, \ldots, n^* \). (i) We have the “renewal Markov property”: for \( m = 1, \ldots, n^* \),
\[
P \left( X^I_{\tau^I(m)} \in A_m, d(\tau^I(m)) = J_m, \tau^I(m) \in S_m \mid \{ X^I_{\tau^I(j)}, \mathbb{I}_j, \tau^I(j) : j < m \} \right) \\
\quad = P \left( X^I_{\tau^I(m)} \in A_m, d(\tau^I(m)) = J_m, \tau^I(m) \in S_m \mid X^I_{\tau^I(m-1)}, \mathbb{I}_{m-1}, \tau^I(m-1) \right),
\]
and (ii) the transition probability is described by the harmonic measure as
\[
P \left( X^I_{\tau^I(m)} \in A_m, d(\tau(m)) = J_m, \tau^I(m) \in S_m \mid X^I_{\tau^I(m-1)}, \mathbb{I}_{m-1}, \tau^I(m-1) \right) \\
\quad = h^I_{J_{m-1}^{-1}(X^I_{\tau^I(m-1)}), A_m, S_m - \tau^I(m-1)} \circ P^I_{J_{m-1}^{-1}(X^I_{\tau^I(m-1)}), D^I_{J_m} \times S^I_{J_m}(A_m), S_m - \tau^I(m-1))}.
\]
(iii) Consequently, we have that

\[
P\left(\left\{ X_{\tau^I(j)}^I \in A_j, d(\tau^I(j)) = J_j, U_j \in S_j; j \leq m \right\} \mid X_0^I = x^I \right) = \int_{A_1 \times \cdots \times A_m} \prod_{j=1}^m h_{j_0_{j-1}}^{I_{j-1}} \left( x_{j-1}^{I_{j-1}}, dx_{j}^{I_j}, S_j \right)
\]

\[
= \int_{\prod_{j=1}^m s_{j_0_{j-1}}^{I_{j-1}}(A_j)} \prod_{j=1}^m Q_{j_{j-1}}^{I_{j-1}} \left( (s_{j_{j-1}}^{I_{j-2}})^{-1}(x_{j-1}^{I_{j-2}} \setminus J_{j-1} = I_{j-1}), D_{j_{j-1}}^{I_{j-1}} \times dx_{j}^{I_{j-1} \setminus J_{j-1} = I_{j-1}}, S_j \right),
\]

where

\[
I_0 = I, \quad I_j := I \setminus \bigcup_{l=1}^{j} J_l, \quad j = 1, 2, \cdots, m \leq n^*,
\]

and \( s_{j_{j-1}}^{I_{j-2}} \) is the shift defined in the previous section as \( \Pi \), with the convention that \( s_{i-1}^{I_{i-2}} = \text{the identity map}. \)

**Proof.** The first assertion (i) is clear from the Markov property of \( X \). The second one is also a direct consequence of the time-homogeneous property and Theorem 4. The third one is obtained by combining (ii) and the renewal Markov property (i) \( \blacksquare \).

As a consequence, we can get the “marginal distribution” concerning on inter-arrival time between two consecutive defaults and the set of the next default index.

**Corollary 6** For \( J_1, \cdots, J_m \subset I \) with \( \emptyset \neq J_m^c \subsetneq \cdots \subsetneq J_1^c \), and \( S_1, \cdots S_m \in \mathcal{B}([0, \infty)) \),

\[
P\left(\left\{ U_j \in S_j, d(\tau^I(j)) = J_j; j \leq m \right\} \right) = \prod_{j=1}^m \int_{A_j^{I_{j-1}}} Q_{j_{j-1}}^{I_{j-1}} \left( (s_{j_{j-1}}^{I_{j-2}})^{-1}(x_{j-1}), D_{j_{j-1}}^{I_{j-1}} \times dx_{j}^{I_{j-1} \setminus J_{j}, S_j} \right),
\]

where \( A_{j_{j-1}}^{I_{j-2}} \) is the set defined in the previous section as \( \Pi \).

Corollary 6 is an important key to find out some important distributions such as the number of defaulted firms given a fixed time, the time to default of a given firm, or the time to the \( m \)-th default. The following proposition provides the distribution for the number of defaulted firms given a fixed time.
Proposition 7 For $k = 1, \ldots, n$, we have
\[
P(N_t = k \mid X_0^I = x_0^I)
= \sum_{m=1}^{k} \sum_{\mathcal{J}_p \leq m} \int_{u_1 + \cdots + u_m \leq t} \prod_{j=1}^{m} \int_{A_{j}^{I - 1}} Q^{I \setminus \mathcal{J}_p - 1} \left( (s_{I_{j-1}}^{-2})^{-1}(x_{j-1}), D_{j}^{I} \times dx_{j}, du_{j} \right). \]

Proof. Since the event \( \{ N_t = k \} \) can be expressed as \( \bigcup_{m=1}^{k} \{ \tau^I(m) \leq t < \tau^I(m+1), \#(I \setminus \mathcal{J}_m) = \sum_{p=1}^{m} \#d(\tau^I(p)) = k \} \), we have that

\[
P(N_t = k \mid X_0^I = x_0^I)
= \sum_{m=1}^{k} P\left( \tau^I(m) \leq t < \tau^I(m+1), \sum_{p=1}^{m} \#d(\tau^I(p)) = k \mid X_0^I = x_0^I \right)
= \sum_{m=1}^{k} \sum_{\mathcal{J}_p \leq m} P\left( \sum_{j=1}^{m} U_j \leq t < \sum_{j=1}^{m+1} U_j, d(\tau^I(p)) = J_p, p \leq m \mid X_0^I = x_0^I \right).
\]

(12)

By applying Corollary \( \star \) we have that

(12)

\[
= \int_{u_1 + \cdots + u_m \leq t} \prod_{j=1}^{m} \int_{A_{j}^{I - 1}} Q^{I \setminus \mathcal{J}_p - 1} \left( (s_{I_{j-1}}^{-2})^{-1}(x_{j-1}), D_{j}^{I} \times dx_{j}, du_{j} \right). \]

We can also obtain the distribution function of the $m$-th contagion time.

Proposition 8 For $m = 1, \ldots, n$, we have
\[
P(\tau(m) > t \mid X_0^I = x_0^I)
= \sum_{k=1}^{m-1} \sum_{\#I_{-1} \leq m} \int_{u_1 + \cdots + u_m \leq t} \prod_{j=1}^{m} \int_{A_{j}^{I - 1}} Q^{I \setminus \mathcal{J}_p - 1} \left( (s_{I_{j-1}}^{-2})^{-1}(x_{j-1}), D_{j}^{I} \times dx_{j}, du_{j} \right). \]

(13)
Proof. We have
\[
P(\tau^I(k) > t \mid X_0^I = x_0^I)
\]
\[
= \sum_{m=1}^{k} P(\tau^I(m - 1) \leq t < \tau^I(m) \mid X_0^I = x_0^I)
\]
\[
= \sum_{m=1}^{k} \sum_{\{\emptyset \leq p \leq m\}} P(\tau^I(m - 1) \leq t < \tau^I(m), d(\tau^I(p)) = J_p, p \leq m \mid X_0^I = x_0^I).
\]
Using the same technique as we did for Proposition 7, we obtain the formula

Proposition 9 Let \( K \subset I \) be a non-empty set. Then we have that
\[
P(\tau_k > t, k \in K \mid X_0^I = x_0^I)
\]
\[
= \sum_{m=0}^{n-1} \sum_{\emptyset \leq J \leq K} \int_{u_1 + \cdots + u_m \leq t, u_1 + \cdots + u_{m+1} > t} \prod_{j=1}^{m} \int_{A_{J_j}} Q_{J_j}^{I \setminus \{\emptyset \} \setminus J_l} (s_{J_{j-1}}^{-1} J_l^{-1} (x_{j-1}), D_{J_j} J_j x, J_{j-1} dx, J_{j-1} du).
\]
Proof. The formula can be obtained in a similar way as the previous propositions by noting
\[
P(\tau_k > t, k \in K \mid X_0^I = x_0^I)
\]
\[
= \sum_{m=0}^{n-1} \sum_{\emptyset \leq J \leq K} P(\tau^I(m) \leq t < \tau^I(m + 1), d(\tau^I(p)) = J_p, p \leq m \mid X_0^I = x_0^I),
\]
where we understand \( J_0 = \emptyset \) when \( m = 0 \).

5 Models with Independence

5.1 The “harmonic measure”
As we have seen, the joint distributions we need is obtained from the “harmonic measure” \( Q^I \). In this section we impose independence among \( X^I \).
Then the it is in fact expressed in terms of the harmonic measures of \( X^i \) to \([K^i, \infty), i \in I\).

Let us be more precise. Let \( \tilde{X} \) be a kind of *Business As Usual* process given as

\[
\tilde{X}^i_t = x^i + \int_0^t (\sigma_i(\tilde{X}^i_s) dW^i + \mu^i(\tilde{X}^i_s) dt),
\]

and \( \tilde{\tau}_i \) be its default time:

\[
\tilde{\tau}_i := \inf \{s > 0 : \tilde{X}^i_s \leq K^i\}.
\]

We assume that each of the distribution of \((\tilde{\tau}_i, \tilde{X}^i)\) has a density, and put

\[
p_j(x^i, s) = \frac{P(\tilde{\tau}_i \in ds | \tilde{X}^i = x^i)}{ds},
\]

and

\[
q_i(x^i, y^i, s) = \frac{P(\tilde{\tau}_i > s, \tilde{X}^i_s \in dy_i | \tilde{X}^i = x^i)}{dy_i},
\]

for \( x^i, y^i \in [K^i, \infty) \) and \( s > 0 \).

The “harmonic measure”, the distribution of \( X^I_{\tilde{\tau}(1)} \) can be obtained by the following

**Lemma 10** For \( A \in \mathfrak{B}(G) \) and \( S \in \mathfrak{B}(0, \infty) \),

\[
Q(x, A, S) = \int_S \sum_i p_i(x^i, s) ds \int_A \delta_{K^i}(dy_i) \prod_{j \neq i} q_j(x^j, y_j, s) dy_j,
\]

(14)

where \( \delta_s \) is the Dirac delta at \( s \).

**Proof.** The left-hand-side of (14)

\[
= \int_S \sum_i P(\{\tilde{\tau}_i \in ds \} \cap_{j \neq i} \{\tilde{\tau}_j < \tilde{\tau}_i, \tilde{X}_{\tilde{\tau}_j} \in A\})
\]

\[
= \int_S \sum_i \int_A \delta_{K^i}(dy_i) P(\tilde{\tau}_i \in ds, s < \tilde{\tau}_j, \tilde{X}^i_s \in dy_j, \forall j \neq i).
\]

By the independence of \( \tilde{X}^i, i \in I \), we have the desired relation (14). ■

We put

\[
g^I_J(x^I, A, s) := \int_{\prod_{i \in I \setminus J} [K^i, \infty) \cap A} \prod_{i \in I \setminus J} q_i(x^i, y_i + \sum_{j \in J} C_{j,i}, s) dy_i,
\]

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and
\[ g_I^f (x^{\Gamma\setminus J}, s) := \int_{A^f_I} \prod_{i \in \Gamma \setminus J} q_i(x^i, y_i, s) \, dy_i \]

for \( s > 0 \) and \( A \in \mathcal{B}(\mathbb{R}^{\Gamma\setminus J}). \)

Since here \( h_I^f \) have a density, we will write, with a slight abuse of the notations,
\[ h_I^f (x^I, s) = \frac{\partial^f_I (\tau^I(1))}{ds} = \mathbb{P}(d^{I'}(\tau^I(1)) = I, \tau^I(1) \in ds|X_0^I = x_I^I), \]

for \( s > 0 \) and \( A \in \mathcal{B}(\mathbb{R}^{\Gamma\setminus J}). \)

**Theorem 11** (i) For a non-empty \( J \subset I \), \( S \in \mathcal{B}(0, \infty), \) and \( A \in \mathcal{B}(\mathbb{R}^{I \setminus J}), \)
\[ h_I^J (x = x_I^I, A, S) = \int_S h_I^f (x, s) g_I^f (s, x^{I \setminus J}, A) \, ds. \quad (15) \]

(ii) For \( s > 0, \)
\[ h_I^f (x = x_I^I, s) = \left( \sum_{i \in L} p_i(x^i, s) \right) \left( 1 + \sum_{m=1}^{m-1} \sum_{I_m \subset \ldots \subset I_1 \subset I_0} \prod_{l=1}^{l-1} g_{I_l}^{h_{l-1}^f} (x^{I_l \setminus I_l}, s) \right). \quad (16) \]

**Proof.** (i) It suffices to show when \( A = \prod_{i \in I \setminus J} (a_i, b_i). \) By combining (9) and Lemma 10 we see that
\[ h_I^J (S, \prod_{i \in I \setminus J} (a_i, b_i)) = \int_S \sum_{i \in I} p_i(s) \int_{D_J} \delta_{K_i}(dx_i) \prod_{j \in J \setminus \{i\}} q_j(s, x_j) \, dx_j \]
\[ \times \prod_{i \in I \setminus J} \int_{(a_i \setminus \sum_{j \in J} C_{j,i}, b_i \setminus \sum_{j \in J} C_{j,i}) \cap (K_i \setminus \sum_{j \in J} C_{j,i}, \infty)} q_i(s, x_i) \, dx_i \]
\[ = \int_S h_I^f (s) \prod_{i \in I \setminus J} \left( \int_{(a_i \setminus \sum_{j \in J} C_{j,i}, b_i \setminus \sum_{j \in J} C_{j,i}) \cap (K_i \setminus \sum_{j \in J} C_{j,i}, \infty)} q_i(s, x_i - \sum_{j \in J} C_{j,i}) \, dx_i \right) \, ds. \]

Thus we obtained (15).

(ii) The equation (16) is a direct consequence of (10) in Theorem 4 and (7) in Lemma 3, together with the relation
\[ g_I^f (s, x = x_I^I) = \int_{A^f_I} \prod_{i \in I \setminus J} q_i(s, x^i, y_i) \, dy_i. \]
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