Local Well and Ill Posedness for the Modified KdV Equations in Subcritical Modulation Spaces

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Abstract
We consider the Cauchy problem of the modified KdV equation (mKdV)
\[ u_t + u_{xxx} \pm (u^3)_x = 0, \quad u(0, x) = u_0(x). \] (0.1)
Local well-posedness of this problem is obtained in modulation spaces \( M_{2,q}^{1/4}(\mathbb{R}) \) \((2 \leq q \leq \infty)\). Moreover, we show that the data-to-solution map fails to be \( C^3 \) continuous in \( M_{s,2}^{1/4},q(\mathbb{R}) \) when \( s < 1/4 \). It is well-known that \( H^{1/4} \) is a critical Sobolev space of mKdV so that it is well-posed in \( H^s \) for \( s \geq 1/4 \) and ill-posed (in the sense of uniform continuity) in \( H^{s'} \) with \( s' < 1/4 \). Noticing that \( M_{2,q}^{1/4} \subset B_{2,q}^{1/4-1/4} \) is a sharp embedding and \( H^{-1/4} \subset B_{2,q}^{-1/4} \), our results contains all of the subcritical data in \( M_{2,q}^{1/4} \), which contains a class of functions in \( H^{-1/4} \backslash H^{1/4} \).

Keywords: Local well-posedness, Ill-posedness, Modified KdV equations, Modulation spaces.

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1 Introduction
In this paper we study the Cauchy problem of the modified Korteweg-de Vries (mKdV) equation on the real line \( \mathbb{R} \):
\[ u_t + u_{xxx} \pm (u^3)_x = 0, \quad u(0, x) = u_0(x), \quad x \in \mathbb{R}, \] (1.1)
where \( u = u(x, t) \in \mathbb{R} \) with \((x, t) \in \mathbb{R}^{1+1}\).

The scale invariant homogeneous Sobolev space for mKdV is \( \dot{H}^{-1/2} \). That is to say, for any solution \( u(x, t) \) of (1.1) with initial data \( u_0(x) \), the scaling function \( u_\lambda(x, t) := \lambda u(\lambda x, \lambda^3 t) \) is also a solution of (1.1) with initial data \( u_{0,\lambda} := \lambda u_0(\lambda x) \), and satisfies
\[ \|u_{0,\lambda}\|_{\dot{H}^{-1/2}} = \|u_0\|_{\dot{H}^{-1/2}}. \] (1.2)

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On the other hand, $H^{1/4}$ is the critical Sobolev space of mKdV so that it is globally well-posed in $H^{s}$ for $s \geq 1/4$ and ill-posed in $H^{s'}$ with $s' < 1/4$. The ill-posed result is in the sense that the data-to-solution map fails to be uniformly continuous on a fixed ball in $H^{s'}$ with $s' < 1/4$. The local well-posed result for $s \geq 1/4$ by using the contraction method and ill-posed result for the focusing equation ($+$ sign in front of the nonlinearity) were proved by Kenig, Ponce and Vega, see [22] and [23], respectively. The local well-posed result was extended to a global one for $s > 1/4$ due to Colliander, Keel, Staffilani, Takaoka and Tao by using I-method, see [10]. The global result for $s = 1/4$ was obtained by Guo in [16]. In addition, the ill-posed result for the defocusing equation ($-$ sign in front of the nonlinearity) was obtained by Christ, Colliander and Tao [6].

Therefore, there is 3/4 derivative gap between $H^{-1/2}$ and $H^{1/4}$ for the well-posedness result of mKdV. In order to discover the behavior of the solution out of $H^{1/4}$, Grünrock brought in the $\widehat{H}^{s}_{q}$ spaces for which the norm is defined by

$$\|u\|_{\widehat{H}^{s}_{q}} := \|\langle \xi \rangle^{s} \hat{u}\|_{L^{q}}, \quad 1/q + 1/q' = 1,$$

and he obtained the local well-posedness of (1.1) for data $u_{0} \in \widehat{H}^{s}_{q}(\mathbb{R}), 2 \leq q < 4, s \geq s(q) := 1/2q$ in [12]. In 2009, Grünrock and Vega broadened the range of $q$ to $2 \leq q < \infty$ by using the trilinear estimates in [13]. From the scaling point, the spaces $H^{s}_{q}$ behave like the Sobolev spaces $H^{s}$, if $s - 1/2 + 1/q = \sigma$. Thus, they can lower the regularity to $-1/2$ by taking $q$ tending to infinity, but there is no result for $q = \infty$. In this paper we consider the initial data in more general modulation spaces $M^{s}_{2,q}, 2 \leq q \leq \infty$ (Indeed, $\widehat{H}^{s}_{q} \subset M^{s}_{2,q}$).

Modulation space $M^{s}_{p,q}$ was introduced by Feichtinger [14] in 1983 and equivalently defined in the following way (cf. [28] [31] [32] [33]):

$$\|f\|_{M^{s}_{p,q}(\mathbb{R})} = \left( \sum_{k \in \mathbb{Z}} \langle k \rangle^{sq} \|\Box f\|_{L^{p}(\mathbb{R})}^{q} \right)^{1/q}, \quad (1.3)$$

where $\Box = \mathcal{F}^{-1} \chi_{[k-1/2,k+1/2]} \mathcal{F}$, $\mathcal{F}$ ($\mathcal{F}^{-1}$) denotes the (inverse) Fourier transform on $\mathbb{R}$, $\chi_{E}$ denotes the characteristic function on $E$ and $\langle k \rangle = (1 + |k|^{2})^{1/2}$. From Plancherel theorem and Hölder’s inequality, we know that $\widehat{H}^{s}_{q} \subset M^{s}_{2,q}$ ($2 \leq q \leq \infty$). Moreover, combining the sharp inclusions between Besov and modulation spaces, we have (cf. [28] [32])

$$\widehat{H}^{s}_{1/4} \subset M^{1/4}_{2,q} \subset B^{1/q-1/4}_{2,q}, \quad 2 \leq q \leq \infty,$$

where the inclusions are optimal. Therefore, our result in which the initial data belongs to $M^{1/4}_{2,\infty}$ can be certainly seen as an improvement. Our main theorem is as follows.

**Theorem 1.1** Let $2 \leq q \leq \infty$, $u_{0} \in M^{1/4}_{2,q}$. Then there exists a time $T > 0$ such that mKdV (1.1) is locally well posed in $C([0,T]; M^{1/4}_{2,q}) \cap X^{1/4}_{q,A}([0,T]),$ where $X^{1/4}_{q,A}$ is defined in next section. Moreover, the regularity index $1/4$ in $M^{1/4}_{2,q}$ is optimal. Specifically, if $s < 1/4$, the data-to-solution map in $M^{1/4}_{2,q}(\mathbb{R})$ is not $C^{3}$ continuous at origin.
Modulation spaces contain a class of initial data out of the critical Sobolev spaces $H^{s_c}$, for which the nonlinear PDE is well-posed for $s > s_c$ and ill-posed for $s < s_c$. Therefore, solving the nonlinear PDE in modulation spaces has absorbed some researchers’ attention, see [1, 2, 3, 4, 7, 8, 9, 18, 19, 20, 21, 29, 34]. We will use $U^p$ and $V^p$ spaces in our discussion, since the dual relation and other important properties are ideally to deal with the nonlinearity. $U^p$ and $V^p$ spaces are introduced to solving PDEs by Koch and Tataru, see [5, 17, 25, 26]. Combining $U^p$, $V^p$ and modulation spaces, Guo, Ren and the second author have considered the cubic and derivative non-linear Schrödinger equation, respectively, see [14, 15].

Let us list some notations. Let $c < 1$, $C > 1$ denote positive universal constants, which can be different at different places; $a \lesssim b$ stands for $a \leq Cb$, $a \sim b$ means that $a \lesssim b$ and $b \lesssim a$; $a \approx b$ means that $a \sim b$ and $b \sim a$; $a \gg b$ means that $a > b + C$; We write $a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$; $p'$ is the dual number of $p \in [1, \infty]$, i.e., $1/p + 1/p' = 1$.

2 Function spaces

2.1 Definitions

In this subsection, we review some function spaces used to obtain the well-posedness theory for non-linear dispersive equations. $U^p$ spaces were first applied by Koch and Tataru [5, 25, 26, 27], and $V^p$ spaces are due to Wiener [35].

Let $Z$ be the set of finite partitions $-\infty = t_0 < t_1 < \ldots < t_{K-1} < t_K = \infty$. In the following, we consider functions taking values in $L^2 := L^2(\mathbb{R}^d; \mathbb{C})$, but in the general $L^2$ may be replaced by an arbitrary Hilbert space or general Banach space.

**Definition 2.1** Let $1 \leq p < \infty$. For any \( \{t_k\}_{k=0}^{K-1} \in Z \) and \( \{\phi_k\}_{k=0}^{K-1} \subset L^2 \) with $\sum_{k=0}^{K-1} \| \phi_k \|_2^p = 1$, $\phi_0 = 0$. A step function $a : \mathbb{R} \to L^2$ given by

$$a = \sum_{k=1}^{K} \chi_{[t_{k-1},t_k)} \phi_{k-1}$$

is said to be a $U^p$-atom. All of the $U^p$ atoms is denoted by $\mathcal{A}(U^p)$. The $U^p$ space is

$$U^p := \left\{ u = \sum_{j=1}^{\infty} c_j a_j : a_j \in \mathcal{A}(U^p), \ c_j \in \mathbb{C}, \ \sum_{j=1}^{\infty} |c_j| < \infty \right\}$$

for which the norm is given by

$$\| u \|_{U^p} := \inf \left\{ \sum_{j=1}^{\infty} |c_j| : \ u = \sum_{j=1}^{\infty} c_j a_j, \ a_j \in \mathcal{A}(U^p), \ c_j \in \mathbb{C} \right\}.$$
Definition 2.2 Let \( 1 \leq p < \infty \). We define \( V^p \) as the normed space of all functions \( v : \mathbb{R} \to L^2 \) such that \( \lim_{t \to \pm \infty} v(t) \) exist and for which the norm

\[
\|v\|_{V^p} := \sup_{\{t_k\}_{k=1}^K} \left( \sum_{k=1}^K \|v(t_k) - v(t_{k-1})\|_{L^2} \right)^{1/p}
\]

is finite, where we use the convention that \( v(-\infty) = \lim_{t \to -\infty} v(t) \) and \( v(\infty) = 0 \) (here \( v(\infty) \) and \( \lim_{t \to -\infty} v(t) \) are different notations). Likewise, we denote by \( V^p \) the subspace of all \( v \in V^p \) so that \( v(-\infty) = 0 \). Moreover, we define the closed subspace \( V_{rc}^p (V_{-rc}^p) \) as all of the right continuous functions in \( V^p (V_1^p) \).

Definition 2.3 We define

\[
U_A^p := e^{-\partial_x^3} U^p, \quad \|u\|_{U_A^p} = \|e^{i\partial_x^3} u\|_{U^p},
\]

\[
V_A^p := e^{-\partial_x^3} V^p, \quad \|u\|_{V_A^p} = \|e^{i\partial_x^3} u\|_{V^p},
\]

and similarly for the definition of \( V_{rc,A}^p, V_{-A}^p, V_{-rc,A}^p \).

Definition 2.4 Besov type Bourgain’s spaces \( \hat{X}^{s,b,q} \) are defined by

\[
\|u\|_{\hat{X}^{s,b,q}} := \left\| \left| \chi_{[\tau - \xi \in [2^{-s}, 2^s)]} |\xi|^s \tau - \xi^3 |\hat{u}(\tau, \xi)| \right\|_{L^q_x}, \quad \xi \in \mathbb{Z}.
\]

Definition 2.5 The frequency-uniform localized \( U^q \)-spaces \( X_q^s(I) \) and \( V^q \)-spaces \( Y_q^s(I) \) are defined by

\[
\|u\|_{X_q^s} = \left( \sum_{\lambda \in I \cap \mathbb{Z}} (\lambda)^{sq} \|\lambda u\|_{U^2}^q \right)^{1/q}, \quad X_q^s := X_q^s(\mathbb{R}), \quad \text{(2.1)}
\]

\[
\|v\|_{Y_q^s} = \left( \sum_{\lambda \in I \cap \mathbb{Z}} (\lambda)^{sq} \|\lambda v\|_{V^2}^q \right)^{1/q}, \quad Y_q^s := Y_q^s(\mathbb{R}), \quad \text{(2.2)}
\]

\[
\|u\|_{X_{q,A}^s} := \|e^{i\partial_x^3} u\|_{X_q^s}, \quad \|v\|_{Y_{q,A}^s} := \|e^{i\partial_x^3} v\|_{Y_q^s}. \quad \text{(2.3)}
\]

2.2 Known results

The following known results about \( U^p \) and \( V^p \) can be found in \([14, 17, 25, 27]\).

Proposition 2.6 (Embedding) Let \( 1 \leq p < q < \infty \). We have the following results.

(i) \( U^p \) and \( V^p, V_{rc}^p, V_{-}^p, V_{rc,-}^p \) are Banach spaces.

(ii) \( U^p \subset V_{rc,-}^p \subset U^q \subset L^\infty(\mathbb{R}, L^2) \). Every \( u \in U^p \) is right continuous on \( t \in \mathbb{R} \).

(iii) \( V^p \subset V^q, V_{rc}^p \subset V_{rc}^q, V_{rc}^p \subset V_{rc}^q, V_{rc,-}^p \subset V_{rc,-}^q \).

(iv) \( \hat{X}^{0,1/2,1} \subset U_{\overline{A}}^2 \subset V_{\overline{A}}^2 \subset \hat{X}^{0,1/2,\infty} \).
Similar to the Schrödinger equation, whose dispersive modulation is $|\tau + \xi^2|$, the mKdV equation’s dispersive modulation is $|\tau - \xi^3|$. By the last inclusion of (iv) in Proposition 2.6, we see that

**Lemma 2.7 (Dispersion Modulation Decay)** Suppose that the dispersion modulation $|\tau - \xi^3| \gtrsim \mu$ for a function $u \in L^2_{x,t}$, then we have

$$
\|u\|_{L^2_{x,t}} \lesssim \mu^{-1/2} \|u\|_{V^2_x}.
$$

**Proposition 2.8 (Interpolation)** Let $1 \leq p < q < \infty$. There exists a positive constant $\epsilon(p, q) > 0$, such that for any $u \in V^p$ and $M > 1$, there exists a decomposition $u = u_1 + u_2$ satisfying

$$
\frac{1}{M} \|u_1\|_{U^p} + \epsilon M \|u_2\|_{V^q} \lesssim \|u\|_{V^p}.
$$

Let $I \subset \mathbb{R}$ be an interval with finite length. For the sake of simplicity, we denote

$$
u_\lambda = \Box \nu, \quad u_I = \sum_{\lambda \in I \cap \mathbb{Z}} u_\lambda.
$$

**Proposition 2.9 (orthogonality in $V^2$)** Take an interval $I \subset \mathbb{R}$, then for $u \in V^2$ the following orthogonality holds:

$$
\|u_I\|_{V^2} \lesssim \left( \sum_{\lambda \in I \cap \mathbb{Z}} \|u_\lambda\|^2_{V^2} \right)^{1/2}.
$$

**Proposition 2.10 (Duality)** Let $1 \leq p < \infty$, $1/p + 1/p' = 1$. Then $(U^p)^* = V^{p'}$ in the sense that

$$
T : V^{p'} \to (U^p)^*; \quad T(v) = B(\cdot, v),
$$

is an isometric mapping. The bilinear form $B : U^p \times V^{p'}$ is defined in the following way: For a partition $t := \{t_k\}_{k=0}^K \subset \mathbb{Z}$, we define

$$
B_t(u, v) = \sum_{k=1}^K \langle u(t_{k-1}), v(t_k) - v(t_{k-1}) \rangle.
$$

Here $\langle \cdot, \cdot \rangle$ denotes the inner product on $L^2$. For any $u \in U^p$, $v \in V^{p'}$, there exists a unique number $B(u, v)$ satisfying the following property. For any $\varepsilon > 0$, there exists a partition $t$ such that

$$
|B(u, v) - B_{t'}(u, v)| < \varepsilon, \quad \forall t' \supset t.
$$

Moreover,

$$
|B(u, v)| \leq \|u\|_{U^p} \|v\|_{V^{p'}}.
$$

In particular, let $u \in V^1$ be absolutely continuous on compact interval, then for any $v \in V^{p'}$,

$$
B(u, v) = \int \langle u'(t), v(t) \rangle dt.
$$
Proposition 2.11 \[15\] (Duality) Let \(1 \leq q < \infty\). Then \((X_q^s)^* = Y_q^{-s}\) in the sense that
\[
T : Y_q^{-s} \to (X_q^s)^*; \quad T(v) = B(\cdot, v),
\]
is an isometric mapping, where the bilinear form \(B(\cdot, \cdot)\) is defined in Proposition 2.10.
Moreover, we have
\[
|B(u, v)| \leq \|u\|_{X_q^s}\|v\|_{Y_q^{-s}}.
\]

3 Basic Estimates

Lemma 3.1 \[24\] (Strichartz Estimates) Let \((p, q)\) satisfy the admissibility condition
\[
\frac{2}{p} + \frac{1}{q} = \frac{1}{2}, \quad 4 \leq p \leq \infty, \quad 2 \leq q \leq \infty.
\]
Then
\[
\|D_x^{1/p}e^{-t\partial_3^3}\phi\|_{L_x^p L_t^q} \lesssim \|\phi\|_{L_x^2}.
\]
In particular, for \(N \geq 1\),
\[
\|P_N e^{-t\partial_3^3}\phi\|_{L_x^p L_t^q} \lesssim \langle N \rangle^{-1/8} \|\phi\|_{L_x^2}.
\]
By testing atoms in \(U_8^A\) space, we obtain
\[
\|P_N u\|_{L_x^p L_t^q} \lesssim \langle N \rangle^{-1/8} \|u\|_{U_8^A}.
\]

Lemma 3.2 (Bilinear Estimate) Suppose that \(\hat{u}_0, \hat{v}_0\) are localized in some compact intervals \(I_1, I_2\) with \(\text{dist}(I_1, I_2) \gtrsim \lambda, \text{dist}(I_1, -I_2) \gtrsim \mu\). Then,
\[
\|e^{-t\partial_3^3}u_0e^{-t\partial_3^3}v_0\|_{L_x^2} \lesssim (\lambda \mu)^{-1/2} \|u_0\|_{L_x^2} \|v_0\|_{L_x^2}.
\]
By testing atoms in \(U_2^A\) space, we obtain
\[
\|uv\|_{L_x^2} \lesssim (\lambda \mu)^{-1/2} \|u\|_{U_2^A} \|v\|_{U_2^A}.
\]

Applying the interpolation in Proposition 2.8, for any \(0 < \varepsilon < 1\) and \(0 < T \leq 1\), we get
\[
\|uv\|_{L_{x,t}^{2,\varepsilon}} \lesssim T^{\varepsilon/4}(\lambda \mu)^{-1/2} \varepsilon \|u\|_{V_4^A} \|v\|_{V_4^A}.
\]

Proof. Taking the Fourier transform in space, we have
\[
\mathcal{F}_x \left( e^{-t\partial_3^3}u_0e^{-t\partial_3^3}v_0 \right) (\xi, t) = \int e^{it(\xi^3 + 3\xi^2\xi_1 - 3\xi\xi_1^2)}\hat{u}_0(\xi - \xi_1)\hat{v}_0(\xi_1)d\xi_1.
\]
Then taking the Fourier transform in time, we obtain
\[
\mathcal{F}_{x,t} \left( e^{-t\partial_3^3}u_0e^{-t\partial_3^3}v_0 \right) (\xi, \tau) = \int \delta(\tau + 3\xi_1 - \xi^3 - 3\xi\xi_1^2)\hat{u}_0(\xi - \xi_1)\hat{v}_0(\xi_1)d\xi_1.
\]
Denote
\[ g(\xi_1) = \tau + 3\xi^2\xi_1 - \xi^3 - 3\xi^2, \]
we see that the zeros and the derivative are
\[ \xi_1^\pm = \frac{\xi}{2} \pm \sqrt{\frac{\xi^2}{4} - \frac{\xi^3 - \tau}{3\xi}} := \frac{\xi}{2} \pm y, \quad g'(\xi_1) = 3\xi^2 - 6\xi_1. \]

Recall that \( \delta(g(\xi_1)) = \delta(\xi_1 - \xi_1^+)/|g'(\xi_1^+)| + \delta(\xi_1 - \xi_1^-)/|g'(\xi_1^-)| = \delta(\xi_1 - \xi_1^+)/6|\xi|y + \delta(\xi_1 - \xi_1^-)/6|\xi|y, \) we have
\[ \mathcal{X}_{x,t}(e^{-\omega t}u_0e^{-\omega t}v_0)(\xi, \tau) = \frac{1}{6|\xi|y}\hat{u}_0\left(\frac{\xi}{2} - y\right)\hat{v}_0\left(\frac{\xi}{2} + y\right) + \frac{1}{6|\xi|y}\hat{u}_0\left(\frac{\xi}{2} + y\right)\hat{v}_0\left(\frac{\xi}{2} - y\right). \quad (3.10) \]

By symmetry, it suffices to estimate the first term in (3.10). Changing of variables \( y = \sqrt{\frac{\xi}{4} - \frac{\xi_1 - \tau}{\xi}} \) and considering \( d\tau = c|\xi|/|y|dy, \) we see that
\[
\left\| e^{-\omega t}u_0e^{-\omega t}v_0 \right\|_{L^2_{x,t}}^2 \lesssim \int_{\mathbb{R}^2} \frac{c}{|\xi||y|} \left| \hat{u}_0\left(\frac{\xi}{2} - y\right) \right|^2 \left| \hat{v}_0\left(\frac{\xi}{2} + y\right) \right|^2 dy d\xi \\
\lesssim \int_{\mathbb{R}^2} \frac{1}{|\xi_1 - \xi_2||\xi_1 + \xi_2|} |\hat{u}_0(\xi_1)|^2|\hat{v}_0(\xi_2)|^2 d\xi_1 d\xi_2 \\
\lesssim \lambda^{-1} \mu^{-1} \int_{\mathbb{R}^2} |\hat{u}_0(\xi_1)|^2|\hat{v}_0(\xi_2)|^2 d\xi_1 d\xi_2 \\
\lesssim \lambda^{-1} \mu^{-1} \left\| u_0 \right\|^2_{L^2_{x,t}} \left\| v_0 \right\|^2_{L^2_{x,t}}, \quad (3.11)
\]
where in the last inequality, we have applied \( \text{dist}(I_1, I_2) \geq \lambda \) and \( \text{dist}(I_1, -I_2) \geq \mu. \)

**Lemma 3.3** \((L^4 \text{ Estimates})\) Let \( I \subset [0, +\infty) \) or \((-\infty, 0] \) with \(|I| < \infty. For any \( \theta \in (0, 1), \beta > 0, \) we have
\[
\left\| u_I \right\|^2_{L^4_{x,t}([0,T] \times \mathbb{R})} \lesssim \left( T^{1/4} + T^{(1-\theta)/4}|I|^{2\beta + (1-\theta)/2} \right) \left\| u \right\|^2_{X^{1/4, 0}_{\lambda, A}(I)}. \quad (3.12)
\]

In particular, if \( 1 < |I| < \infty, 0 < T < 1, \) then for any \( 0 < \varepsilon \ll 1, 4 \leq q \leq \infty \)
\[
\left\| u_I \right\|^2_{L^4_{x,t}([0,T] \times \mathbb{R})} \lesssim T^{\varepsilon/4}|I|^{1/4-1/q+\varepsilon} \max_{\lambda \in I}\lambda^{-3/8} \left\| u \right\|^2_{X^{1/4, 0}_{\lambda, A}(I)}. \quad (3.13)
\]

**Proof.** Without loss of generality, we assume \( I \subset [0, +\infty). \)
\[
\left\| u_I \right\|^2_{L^4_{x,t}([0,T] \times \mathbb{R})} = \left\| (u_I)^2 \right\|_{L^2([0,T] \times \mathbb{R})} = \left\| \sum_{m,n \in \mathbb{Z}} u_{m,n} \right\|_{L^2([0,T] \times \mathbb{R})} \\
\lesssim \sum_{k \in \mathbb{N}} \sum_{m-n \sim 2^k} \left\| u_{m,n} \right\|_{L^2([0,T] \times \mathbb{R})}. \quad (3.14)
\]
Case $k = 0$, i.e. $m \approx n$:

$$
\left\| \sum_{n \in I \cap \mathbb{Z}} u_n^2 \right\|_{L^2([0,T] \times \mathbb{R})} \lesssim \left( \sum_{n \in I \cap \mathbb{Z}} \| u_n^2 \|_{L^2([0,T] \times \mathbb{R})}^2 \right)^{1/2}
\lesssim \left( \sum_{n \in I \cap \mathbb{Z}} \| u_n \|_{L^4([0,T] \times \mathbb{R})}^4 \right)^{1/2} \lesssim T^{1/4} \left( \sum_{n \in I \cap \mathbb{Z}} \| u_n \|_{L^8([0,T] \times \mathbb{R})}^4 \right)^{1/2}
\lesssim T^{1/4} \sum_{n \in I \cap \mathbb{Z}} \left( \langle n \rangle^{-1/8} \| u_n \|_{L^8} \right) \lesssim T^{1/4} \| u \|_{X^{0,-1/8}}^2,
$$

where the first step is by the orthogonality in $L^2$ and the last step follows from the Strichartz estimate.

Case $k > 0$: Notice that $k$ is summed for $\ln |I|$ times, we have $\sum_{k \in \mathbb{N}} \lesssim \ln |I|$. We split the other sum as follows

$$
\sum_{m-n \sim 2^k} u_m u_n = \sum_{n \in I \cap \mathbb{Z}} \sum_{m \in I \cap \mathbb{Z}, m-n \sim 2^k} u_m u_n = \sum_{j \in \mathbb{N}^+} \sum_{n \in I \cap \mathbb{Z}, n \sim j 2^k} \sum_{m \in I \cap \mathbb{Z}, m-n \sim 2^k} u_m u_n,
$$

where $j$ is chosen such that $j 2^k, (j + 1) 2^k \in I$. Hence for $u_n$ with $n \sim j 2^k$ and $u_m$ with $m-n \sim 2^k$, we have that the frequency of the function $u_m u_n$ will be close to $(2j + 1) 2^k$, which implies by orthogonality that

$$
\sum_{k \in \mathbb{N}} \left\| \sum_{m-n \sim 2^k} u_m u_n \right\|_{L^2([0,T] \times \mathbb{R})} \lesssim \sum_{k \in \mathbb{N}} \left( \sum_{j \in \mathbb{N}^+} \sum_{n \in I \cap \mathbb{Z}, n \sim j 2^k} \sum_{m \in I \cap \mathbb{Z}, m-n \sim 2^k} \left\| u_m u_n \right\|_{L^2([0,T] \times \mathbb{R})} \right)^{1/2}. \tag{3.14}
$$

Denote $u_{j,k} := \sum_{n \in I, n \sim j 2^k} u_n$, from proposition 2.8 we can write as a sum $u_{j,k} = u_{1,j,k} + u_{2,j,k}$ with the estimate

$$
\frac{1}{|I|^2} \| u_{1,j,k} \|_{L^2}^2 + e^{\| I \|/2} \| u_{2,j,k} \|_{L^2}^2 \lesssim \| u_{j,k} \|_{L^2}^2. \tag{3.15}
$$

Then the estimate (3.14) will be continued by four terms. For the term containing $u_{1,j,k}$ and $u_{1,j+1,k}$, which will be denoted as $I_1$.

$$
I_1 \lesssim \sum_{k \in \mathbb{N}} \left( \sum_{j \in \mathbb{N}^+} \left\| u_{1,j,k} u_{1,j+1,k} \right\|_{L^2}^2 \left\| u_{1,j,k} u_{1,j+1,k} \right\|_{L^2([0,T] \times \mathbb{R})}^{2(1-\theta)} \right)^{1/2}
\lesssim \sum_{k \in \mathbb{N}} \left( \sum_{j \in \mathbb{N}^+} \left\| u_{1,j,k} u_{1,j+1,k} \right\|_{L^2}^2 \left\| u_{1,j,k} \right\|_{L^2([0,T] \times \mathbb{R})} \left\| u_{1,j+1,k} \right\|_{L^2([0,T] \times \mathbb{R})}^{2(1-\theta)} \right)^{1/2}
$$

8
\[ T^{(1-\theta)/4} \sum_{k \in \mathbb{N}} \left( \sum_{j \in \mathbb{N}^+} \left( |j(j+1)|^{-(1-\theta)/4} 2^{-k\theta} \|u_{1,j,k}\|_{L_T^4}^2 \|u_{1,j+1,k}\|_{L_T^4}^2 \right) \right)^{1/2} \text{.} \]  

(3.16)

Since \(|m - n| \sim 2^k, |m + n| \sim (2j + 1)2^k\), we have the bilinear estimates

\[ \|u_{1,j,k}u_{1,j+1,k}\|_{L^2} \lesssim (j(j+1))^{-1/4} 2^{-k}\|u_{1,j,k}\|_{U^4_A} \|u_{1,j+1,k}\|_{U^4_A}. \]  

(3.17)

Combining with Strichartz estimate, (3.16) is dominated by

\[ T^{(1-\theta)/4} \sum_{k \in \mathbb{N}} \left( \sum_{j \in \mathbb{N}^+} (j^2 2^{k})^{-(1-\theta)/4} \|u_{1,j,k}\|_{U^8_A} \|u_{1,j+1,k}\|_{U^8_A} \right)^{1/2} \]

\[ \lesssim T^{(1-\theta)/4} \sum_{k \in \mathbb{N}} \left( \sum_{j \in \mathbb{N}^+} (j^2 2^{k})^{-(1+\theta)/4} \|u_{1,j,k}\|_{U^8_A} \|u_{1,j+1,k}\|_{U^8_A} \right)^{1/2}. \]

By applying (3.15) and the orthogonality in \(V^2\), it follows that

\[ T^{(1-\theta)/4} |I|^{2\beta} \sum_{k \in \mathbb{N}} \left( \sum_{j \in \mathbb{N}^+} (j^2 2^{k})^{-1/4} \|u_{1,j,k}\|_{V^2_A}^2 \|u_{1,j+1,k}\|_{V^2_A}^2 \right)^{1/2} \]

\[ \lesssim T^{(1-\theta)/4} |I|^{2\beta} \sum_{k \in \mathbb{N}} \left( \sum_{j \in \mathbb{N}^+} (j^2 2^{k})^{-1/4} \|u_{1,j,k}\|_{V^2_A}^2 \|u_{1,j+1,k}\|_{V^2_A}^2 \right)^{1/2} \]

\[ \lesssim T^{(1-\theta)/4} |I|^{2\beta} \sum_{k \in \mathbb{N}} \left( \sum_{j \in \mathbb{N}^+} (j^2 2^{k})^{-1/4} \|u_{1,j,k}\|_{V^2_A}^4 \|u_{1,j+1,k}\|_{V^2_A}^4 \right)^{1/2} \]

\[ \lesssim T^{(1-\theta)/4} |I|^{2(\beta + (1-\theta)/2)} \|u\|^2_{X_{\lambda,4}^{1/8}} \]

\[ \lesssim T^{(1-\theta)/4} |I|^{2(\beta + (1-\theta)/2)} \|u\|^2_{X_{\lambda,4}^{-\infty}}. \]  

(3.18)

where the last inequality is by using \(2^{k(1-\theta)/2} \lesssim |I|^{(1-\theta)/2}\) and Hölder’s inequality. For the rest three terms we will do in a uniform way. We take the term containing \(u_{2,j,k}\) and \(u_{2,j+1,k}\) for example, and denote it as \(I_2\).

\[ I_2 \lesssim \sum_{k \in \mathbb{N}} \left( \sum_{j \in \mathbb{N}^+} \|u_{2,j,k}u_{2,j+1,k}\|_{L^2((0,T) \times \mathbb{R})}^2 \right)^{1/2} \]
\[ T^{1/4} \sum_{k \in \mathbb{N}} \left( \sum_{j \in \mathbb{N}^+} \|u_{2,j,k}\|_{L^4_x}^2 \|u_{2,j+1,k}\|_{L^4_x}^2 \right)^{1/2} \]
\[ \lesssim T^{1/4} \sum_{k \in \mathbb{N}} \left( \sum_{j \in \mathbb{N}^+} (j2^k)^{-1/4} \|u_{2,j,k}\|_{L^4_x}^2 ((j+1)2^k)^{-1/4} \|u_{2,j+1,k}\|_{L^4_x}^2 \right)^{1/2}. \tag{3.19} \]

By applying (3.15) and the orthogonality in \( V^2 \) again, (3.19) follows that

\[ \lesssim T^{1/4} e^{-2e|I|/|I|} \sum_{k \in \mathbb{N}} \left( \sum_{j \in \mathbb{N}^+} (j2^k)^{-1/4} \|u_{j,k}\|_{V^2}^2 ((j+1)2^k)^{-1/4} 2^{-k\theta} \|u_{j+1,k}\|_{V^2}^2 \right)^{1/2} \]
\[ \lesssim T^{1/4} e^{-2e|I|/|I|} \sum_{k \in \mathbb{N}} \left( \sum_{j \in \mathbb{N}^+} (j2^k)^{-1/4} \left( \sum_{m \in \mathcal{I}, n \sim j2^k} \|u_m\|_{V^2}^2 \right) ((j+1)2^k)^{-1/4} \left( \sum_{m \in \mathcal{I}, m \sim (j+1)2^k} \|u_m\|_{V^2}^2 \right) \right)^{1/2} \]
\[ \lesssim T^{1/4} e^{-2e|I|/|I|} |I|^{1/2} \ln |I| \|u\|_{X^{4,q-1/8}_A}^2 \]
\[ \lesssim T^{1/4} \|u\|_{X^{4,q-1/8}_A}^2. \tag{3.20} \]

Thus we complete the proof of (3.12). In particular, for \( 1 \lesssim |I| < \infty \) and \( 0 < T < 1 \), taking \( \beta \) and \( 1 - \theta \) small sufficiently, we have

\[ \|u_I\|_{L^4_{x,t}([0,T])} \lesssim T^{\varepsilon/4} |I|^{\varepsilon} \|u\|_{X^{4,q-1/8}_A(I)}. \tag{3.21} \]

In the end we can obtain (3.13) by Hölder inequality. \( \square \)

**Lemma 3.4** Let \( I \subset \mathbb{R} \) with \( 1 \lesssim |I| < \infty \), \( 2 \leq q \leq \infty \), we have

\[ \|u_I\|_{L^4_{q,v} L^2_{x,v} \cap V^2_A} \lesssim |I|^{1/2 - 1/q} \max_{\lambda \in I} (\lambda)^{-1/4} \|u_{X^{4,q-1/8}_A(I)}\|. \tag{3.22} \]

**Proof.** Using \( V^2_A \subset L^\infty_{t,v} L^2_x \), the orthogonality in \( V^2 \) and Hölder’s inequality one by one, we have

\[ \|u_I\|_{L^\infty_{q,v} L^2_x \cap V^2_A} \lesssim \left( \sum_{\lambda \in I} \|u_{X}^\lambda\|_{V^2_A}^q \right)^{1/2} \lesssim \max_{\lambda \in I} (\lambda)^{-1/4} \left( \sum_{\lambda \in I} (\lambda)^{1/4} \|u_{X}^\lambda\|_{V^2_A}^2 \right)^{1/2} \]
\[ \lesssim |I|^{1/2 - 1/q} \max_{\lambda \in I} (\lambda)^{-1/4} \|u\|_{X^{4,q-1/8}_A(I)}. \tag{3.23} \]
4 Trilinear estimates

At first, we apply the duality to the norm calculation (Proposition 2.11) to the inhomogeneous part of the solution of mKdV in $X^{s}_{q,A}$. It is known that (4.1) is equivalent to the following integral equation:

$$u(x, t) = e^{-t\partial_x^3}u_0 - \mathcal{A}((u^3)_x),$$

where

$$e^{-t\partial_x^3} = \mathcal{F}^{-1} e^{it\xi^3} \mathcal{F}, \quad \mathcal{A}(f) = \int_0^t e^{-(t-\tau)\partial_x^3} f(\tau) d\tau.$$ 

By Propositions 2.10 and 2.11 we see that, for $\text{supp } v \subset \mathbb{R} \times [0, T], 1 \leq q < \infty$,

$$\|\mathcal{A}(f)\|_{X^{1/4}_{q,A}} = \|e^{t\partial_x^3} \mathcal{A}(f)\|_{X^{1/4}_{q}} = \sup \left\{ \left| B \left( \int_0^t e^{t\partial_x^3} f(\tau) d\tau, v \right) \right| : \|v\|_{Y^{-1/4}_{q'}} \leq 1 \right\}$$

$$\leq \sup_{\|v\|_{Y^{-1/4}_{q'}} \leq 1} \left| \int_{[0,T]} (e^{t\partial_x^3} f(t), v(t)) dt \right|$$

$$\leq \sup_{\|v\|_{Y^{-1/4}_{q'}} \leq 1} \left| \int_{[0,T]} (f(t), e^{-t\partial_x^3} v(t)) dt \right|$$

$$\leq \sup_{\|v\|_{Y^{-1/4}_{q',A}} \leq 1} \left| \int_{[0,T]} (f(t), v(t)) dt \right|. \quad (4.2)$$

For $q = \infty$, we have

$$\|\mathcal{A}(f)\|_{X^{1/4}_{\infty,A}} = \|e^{t\partial_x^3} \mathcal{A}(f)\|_{X^{1/4}_{\infty}}$$

$$= \sup_{\lambda \in \mathbb{Z}} \langle \lambda \rangle^{1/4} \left\| \int_0^t e^{t\partial_x^3} f(\tau) d\tau \right\|_{L^2}$$

$$\leq \sup_{\lambda \in \mathbb{Z}} \langle \lambda \rangle^{1/4} \sup_{\|v(\lambda)\|_{\dot{H}^1} \leq 1} \left| \int_{[0,T]} (\Box \lambda e^{t\partial_x^3} f(t), v(\lambda)(t)) dt \right|$$

$$\leq \sup_{\lambda \in \mathbb{Z}} \langle \lambda \rangle^{1/4} \sup_{\|v(\lambda)\|_{\dot{H}^1} \leq 1} \left| \int_{[0,T]} (f(t), \Box \lambda e^{-t\partial_x^3} v(\lambda)(t)) dt \right|$$

$$\leq \sup_{\lambda \in \mathbb{Z}} \langle \lambda \rangle^{1/4} \sup_{\|v(\lambda)\|_A \leq 1} \left| \int_{[0,T]} (f(t), \Box \lambda v(\lambda)(t)) dt \right|. \quad (4.3)$$

To prove Theorem 1.1, we need to control the second term of the integral equation (4.1) in $X^{1/4}_{q,A}(2 \leq q \leq \infty)$. More precisely, we want to prove the following lemma.
Lemma 4.1 For $2 \leq q \leq \infty$, there exists $\varepsilon > 0$ such that
\[
\left\| \int_0^1 e^{-(t-\tau)\partial_\xi^3 (u^3)_{\xi}(\tau)} d\tau \right\|_{X_{q,4}^{1/4}} \lesssim T^\varepsilon \|u\|_{Y_{q,4}^{1/4}}^3. \tag{4.4}
\]

Proof. When $2 \leq q < \infty$, in view of (4.2), it suffices to show that
\[
\left\| \int_{R \times [0,T]} \nabla u^2 \partial_x u \, dxdt \right\| \lesssim T^\varepsilon \|u\|_{X_{q,4}^{1/4}}^3 \|v\|_{Y_{q',4}^{-1/4}}. \tag{4.5}
\]

We perform a uniform decomposition with $u, v$ in the left hand side of (4.5), it suffices to prove that
\[
\sum_{\lambda_0, \ldots, \lambda_3} |\lambda\rangle^{1/4} \int_{[0,T] \times R} \nabla u \lambda_1 u \lambda_2 \partial_x u \lambda_3 \, dxdt \lesssim T^\varepsilon \|u\|_{X_{q,4}^{1/4}}^3 \|v\|_{Y_{q',4}^{-1/4}}. \tag{4.6}
\]

When $q = \infty$, in view of (4.3), it suffices to show that, for any fixed $\lambda \in Z$,
\[
\sum_{\lambda_1, \lambda_2, \lambda_3} |\lambda\rangle^{1/4} \int_{[0,T] \times R} \nabla \lambda u (\lambda) u \lambda_1 u \lambda_2 \partial_x u \lambda_3 \, dxdt \lesssim T^\varepsilon \|u\|_{X_{q,4}^{1/4}}^3 \|v(\lambda)\|_{Y_4^2}. \tag{4.7}
\]

4.1 $q = \infty$, Proof of (4.7).

For convenience, denote $\lambda$ as $\lambda_0$, $\Box \lambda u(\lambda) = v_\lambda = v_{\lambda_0}$. In order to keep the left hand side of (4.7) nonzero, we have the frequency constraint condition (FCC)
\[
\lambda_1 + \lambda_2 + \lambda_3 \approx \lambda_0 \tag{4.8}
\]

and dispersion modulation constraint condition (DMCC)
\[
\max_{0 \leq k \leq 3} |\xi_k^3 - \tau_k| \gtrsim \left( |\xi_0^3 - \tau_0| - \sum_{1 \leq k \leq 3} (|\xi_k^3 - \tau_k|) \right) \gtrsim |(\xi_0 - \xi_1)(\xi_0 - \xi_2)(\xi_0 - \xi_3)|. \tag{4.9}
\]

It suffices to consider the cases that $\lambda_0$ is maximal or secondly maximal number in $\lambda_0, \ldots, \lambda_3$ (In the opposite case, one can replace $\lambda_0, \ldots, \lambda_3$ with $-\lambda_0, \ldots, -\lambda_3$).

Step 1. We assume that $\lambda_0 = \max_{0 \leq k \leq 3} |\lambda_k|$. From the frequency constraint condition (FCC) $\lambda_0 \approx \lambda_1 + \lambda_2 + \lambda_3$, we know that the non-trivial case is that $\lambda_0 \gg 0$ (The case $\lambda_0 < 0$ never happens due to the condition (FCC). In addition, the case $|\lambda_0| \lesssim 1$, which leads to $\max_{0 \leq k \leq 3} |\lambda_k| \lesssim 1$, implies that the summation in (4.7) has at most finite terms). Furthermore, in view of $\lambda_0 = \max_{0 \leq k \leq 3} |\lambda_k|$, $\lambda_0 \approx \lambda_1 + \lambda_2 + \lambda_3$, and $\lambda_0 \gg 0$, we see that $\lambda_0 \approx \max_{0 \leq k \leq 3} |\lambda_k| \gg 0$. For convenience, we can take
\[
\lambda_0 = \max_{0 \leq k \leq 3} |\lambda_k| \gg 0. \tag{4.10}
\]
By the symmetry, we can assume \( \lambda_1 \geq \lambda_2 \). Then \( \lambda_0, \ldots, \lambda_3 \) have the following three orders:

- **Order 1**: \( \lambda_0 \geq \lambda_3 \geq \lambda_1 \geq \lambda_2 \);
- **Order 2**: \( \lambda_0 \geq \lambda_1 \geq \lambda_3 \geq \lambda_2 \);
- **Order 3**: \( \lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \lambda_3 \).

We just take Order 1 for example because the other two orders are similar and even more easier (noticing that the derivative located in \( u_{\lambda_3} \)).

**Order 1**: \( \lambda_0 \geq \lambda_3 \geq \lambda_1 \geq \lambda_2 \). For short, considering the higher and lower frequency of \( \lambda_k \), we use the following notations:

\[
\begin{align*}
\lambda_k \in h \iff & \lambda_k \in [3\lambda_0/4, \lambda_0]; \\
\lambda_k \in h^- \iff & \lambda_k \in [-\lambda_0, -3\lambda_0/4]; \\
\lambda_k \in l \iff & \lambda_k \in [0, 3\lambda_0/4]; \\
\lambda_k \in l^- \iff & \lambda_k \in [-3\lambda_0/4, 0].
\end{align*}
\]

Then we divide Order 1 into several cases.

**Case 1**: \( \lambda_3 \in h \) and \( \lambda_1 \in h \). In consideration of (FCC), we easily see that \( \lambda_0, \lambda_1, \lambda_2, \lambda_3 \) satisfy the following frequency constraint condition:

\[
\lambda_0 = \lambda_1 + \lambda_2 + \lambda_3 + l, \quad |l| \leq 10.
\]

We know that this case implies that \( \lambda_2 \in [-\lambda_0, -\lambda_0/2-l] \). We do dyadic decomposition for \( u_{\lambda_1}, u_{\lambda_2} \) and \( u_{\lambda_3} \), and keep using uniform decomposition for \( v_{\lambda_0} \). Let us denote \( I_0 = [0, 1), I_j = [2^{j-1}, 2^j), j \geq 1 \). We decompose \( \lambda_1, \lambda_2, \lambda_3 \) by:

\[
\lambda_k \in [3\lambda_0/4, \lambda_0] = \bigcup_{j_k \geq 0} \lambda_0 - I_{j_k}, \quad k = 1, 3; \quad \lambda_2 \in [-\lambda_0, -\lambda_0/2-l] = \bigcup_{j_2 \geq 0} -\lambda_0 + I_{j_2}.
\]

From \( \lambda_3 \geq \lambda_1 \) we know that \( j_3 \leq j_1 \). In view of condition (FCC), we see that \( j_2 \approx j_1 \). It follows that

\[
0 \leq j_3 \leq j_1 \approx j_2 \leq \log_2 \lambda_0.
\]

In the following discussion, we shall omit the condition \( j_k \in [0, \log_2 \lambda_0], k = 1, 2, 3 \), for convenience, but it is always satisfied in Step 1. We denote the left hand side of \([4.7]\) as \( \mathcal{L}_{hhh-}(u, v) \), and divide it into three parts:

\[
\begin{align*}
\mathcal{L}_{hhh-}(u, v) := & \sum_{j_3 \leq j_1 \approx j_2} \langle \lambda_0 \rangle^{1/4} \int_{[0,T] \times \mathbb{R}} \left| \tau_{\lambda_0} u_{\lambda_0 - I_1} u_{-\lambda_0 + I_2} \partial_x u_{\lambda_0 - I_3} \right| dxdt \\
= & \left( \sum_{j_3 \leq j_1 \approx j_2} + \sum_{j_3 \leq 1 \approx j_1 \approx j_2} + \sum_{1 \leq j_3 \leq j_1 \approx j_2} \right) \langle \lambda_0 \rangle^{1/4} \int_{[0,T] \times \mathbb{R}} \left| \tau_{\lambda_0} u_{\lambda_0 - I_1} u_{-\lambda_0 + I_2} \partial_x u_{\lambda_0 - I_3} \right| dxdt.
\end{align*}
\]
It is easy to see that in $L^h_{h\bar{h}h\bar{h}}(u, v)$, $\lambda_0 \approx \lambda_3 \approx \lambda_1 \approx -\lambda_2$ holds. Therefore, by Hölder inequality and Strichartz estimate, we have

$$L^h_{h\bar{h}h\bar{h}}(u, v) \lesssim (\lambda_0)^{5/4} \|\overline{v_0}\|_{L^4_x} \|u_{\lambda_0}\|_{L^2_{x,t}} \|u_{-\lambda_0}\|_{L^2_{x,t}}$$

$$\lesssim T^{1/2} \lambda_0^{1/4} \|\overline{v_0}\|_{L^4_x} \|u_{\lambda_0}\|_{L^2_{x,t} L^2_x} \|u_{-\lambda_0}\|_{L^2_{x,t} L^2_x}$$

$$\lesssim T^{1/2} \lambda_0^{1/4} \|\overline{v_0}\|_{H^2} \|u_{\lambda_0}\|_{H^2_{x,t} L^2_x} \|u_{-\lambda_0}\|_{H^2_{x,t}}$$

$$\lesssim T^{1/2} \lambda_0^{1/4} \|\overline{v_0}\|_{H^2} \|u_{\lambda_0}\|_{H^2_{x,t} L^2_x} \|u_{-\lambda_0}\|_{H^2_{x,t}}$$

$$\lesssim T^{1/2} \|v_0\|_{H^2} \|u\|_{X_{\infty,4}^1}.$$
For $\mathcal{L}_{hhh_h}^h(u, v)$, we can use the bilinear estimate due to $j_2 \gg j_3$. By Hölder’s inequality we have

$$\mathcal{L}_{hhh_h}^h(u, v) \lesssim \sum_{1 \leq j_1 < j_2 \leq j_3} \langle \lambda_0 \rangle^{1/4} \|v_{\lambda_0} - I_{j_1} \|_{L^2_t L^\infty_x} \|u_{\lambda_0 - I_{j_2}} \|_{L^\infty_t L^2_x} \langle 1 \| u_{\lambda_0 + I_{j_3}} - I_{j_1} \|_{L^2_{x,t}} \rangle.$$  

Using $\|v_{\lambda_0} \|_{L^\infty_x} \lesssim \|v_{\lambda_0} \|_{L^2_x}$, $V_A^2 \subset L^\infty_t L^2_x$, the dispersion modulation decay (2.4), the bilinear estimate (3.7) and Lemma 3.4 we have

$$\mathcal{L}_{hhh_h}^h(u, v) \lesssim \sum_{1 \leq j_1 < j_2 \leq j_3} \langle \lambda_0 \rangle^{5/4} \langle \lambda_0 \rangle^{-1/2} \langle 2^j \rangle^{-1/2} \langle 2^{2j} \rangle^{-1/2} \|v_{\lambda_0} \|_{V_A^2} \|u_{\lambda_0 - I_{j_1}} \|_{V_A^2}$$

$$\times T^{\varepsilon/4} \langle \lambda_0 \rangle^{-1/2 + \varepsilon} \langle 2^j \rangle^{-1/2 + \varepsilon} \|u_{\lambda_0 + I_{j_3}} - I_{j_1} \|_{V_A^2} \|v_{\lambda_0} \|_{V_A^2}$$

$$\lesssim T^{\varepsilon/4} \sum_{1 \leq j_1 < j_2 \leq j_3} \langle \lambda_0 \rangle^{1/4 + \varepsilon} \langle 2^j \rangle^{-1/2} \langle 2^{2j} \rangle^{-1/2} \langle 2^{3j} \rangle^{-1/2} \|v_{\lambda_0} \|_{V_A^2}$$

$$\times (2^j)^{1/2} (2^{2j})^{1/2} (2^{3j})^{1/2} \langle \lambda_0 \rangle^{-3/4} \|u\|_{X_{\infty,A}^{3/4}}^3.$$  

(4.13)

Noticing that $2^{3j} \lesssim \langle \lambda_0 \rangle$ and $2^j \lesssim (2^{2j})^\varepsilon$, we can take $\varepsilon \leq 1/6$ such that

$$\mathcal{L}_{hhh_h}^h(u, v) \lesssim T^{\varepsilon/4} \langle \lambda_0 \rangle^{-1/2 + 3\varepsilon} \|v_{\lambda_0} \|_{V_A^2} \|u\|_{X_{\infty,A}^{3/4}}^3.$$  

$$\lesssim T^{\varepsilon/4} \|v_{\lambda_0} \|_{V_A^2} \|u\|_{X_{\infty,A}^{3/4}}^3.$$  

For $\mathcal{L}_{hhh_h}^h(u, v)$, by Hölder’s inequality, $\|v_{\lambda_0} \|_{L^\infty_x} \lesssim \|v_{\lambda_0} \|_{L^2_x}$, $V_A^2 \subset L^\infty_t L^2_x$, the dispersion modulation decay (2.4), the $L^4$ estimate (3.13) and Lemma 3.4 we have

$$\mathcal{L}_{hhh_h}^h(u, v) \lesssim \sum_{1 \leq j_1 < j_2 \leq j_3} \langle \lambda_0 \rangle^{5/4} \|\nabla v_{\lambda_0} - I_{j_1} \|_{L^2_t L^\infty_x} \|u_{\lambda_0 - I_{j_2}} \|_{L^\infty_t L^2_x} \langle 1 \| u_{\lambda_0 + I_{j_3}} - I_{j_1} \|_{L^4_{x,t}} \rangle$$

$$\lesssim \sum_{1 \leq j_1 < j_2 \leq j_3} \langle \lambda_0 \rangle^{5/4} \langle \lambda_0 \rangle^{-1/2} \langle 2^j \rangle^{-1/2} \langle 2^{2j} \rangle^{-1/2} \|v_{\lambda_0} \|_{V_A^2} \|u_{\lambda_0 - I_{j_1}} \|_{V_A^2}$$

$$\times T^{-\varepsilon/4} \langle \lambda_0 \rangle^{-3/4} (2^j)^{1/4 + \varepsilon} (2^{2j})^{1/4 + \varepsilon} \|u\|_{X_{\infty,A}^{3/4}}^2$$

$$\lesssim T^{\varepsilon/2} \sum_{j_1 \leq j_2} \langle \lambda_0 \rangle^{-1/4} (2^j)^{1/4 + \varepsilon} (2^{2j})^{-1/2 + \varepsilon} \|v_{\lambda_0} \|_{V_A^2} \|u\|_{X_{\infty,A}^{3/4}}^3$$

$$\lesssim T^{\varepsilon/2} \|v_{\lambda_0} \|_{V_A^2} \|u\|_{X_{\infty,A}^{3/4}}^3.$$  

(4.14)

where the last inequality is by taking $\varepsilon \leq 1/8$.

If $u_{\lambda_0 - I_{j_1}}$ has the highest dispersion modulation, we just take $L^\infty_{x,t}$ and $L^2_{x,t}$ norms to $v_{\lambda_0}$ and $u_{\lambda_0 - I_{j_1}}$, respectively, then

$$\|\nabla v_{\lambda_0} - I_{j_1} \|_{L^2_{x,t}} \lesssim \|v_{\lambda_0} \|_{V_A^2} \langle \lambda_0 \rangle^{-1/2} \langle 2^j \rangle^{-1/2} \langle 2^{2j} \rangle^{-1/2} \|u_{\lambda_0 - I_{j_1}} \|_{V_A^2}.$$  

(4.15)
where we use the fact \( \|v_{\lambda_0}\|_{L^\infty_{x,t}} \lesssim \|v_{\lambda_0}\|_{L^\infty_t L^2_x} \lesssim \|v_{\lambda_0}\|_{V^2_4} \) and the dispersion modulation decay (2.4) to \( u_{\lambda_0-I_3} \). Then this case reduces to the same estimate as that when \( v_{\lambda_0} \) has the highest dispersion modulation.

If \( u_{-\lambda_0+I_2} \) has the highest dispersion modulation, we still divide \( \mathcal{L}_{hhhh}^h(u,v) \) into two parts as (4.12). For \( \mathcal{L}_{hhhh-}^h(u,v) \), we can take \( L^\infty_{x,t}, L^2_{x,t} \) and \( L^2_{x,t} \) norms to \( v_{\lambda_0}, u_{-\lambda_0+I_2} \) and \( u_{\lambda_0-I_3} \), \( \partial_x u_{\lambda_0-I_3} \), respectively. By Hölder’s inequality, the dispersion modulation decay (2.4), the bilinear estimate (3.7) and Lemma 3.4, we have

\[
\mathcal{L}_{hhhh-}^h(u,v) \lesssim \sum_{1 \leq j_3 < j_1 \approx j_2} \langle \lambda_0 \rangle^{1/4} \| v_{\lambda_0} \|_{L^\infty_{x,t}} \| u_{-\lambda_0+I_2} \|_{L^2_{x,t}} \| u_{\lambda_0-I_3} \|_{L^2_{x,t}} \| \partial_x u_{\lambda_0-I_3} \|_{L^2_{x,t}}
\]

\[
\lesssim \sum_{1 \leq j_3 < j_1 \approx j_2} \langle \lambda_0 \rangle^{1/2} \left( \langle 2^{j_3} \rangle \right)^{-1/2} \left( \langle 2^{j_1} \rangle \right)^{-1/2} \| v_{\lambda_0} \|_{V^2_4} \| u_{-\lambda_0+I_2} \|_{V^2_4} \times T^{1/4} \langle \lambda_0 \rangle^{-1/2 + \epsilon} \left( \langle 2^{j_1} \rangle \right)^{-1/2 + \epsilon} \| u_{\lambda_0-I_3} \|_{V^2_4} \| u_{-\lambda_0+I_2} \|_{V^2_4},
\]

which is the same as the right hand side of the first inequality in (4.13) (noticing that \( j_1 \approx j_2 \)).

For \( \mathcal{L}_{hhhh-}^h(u,v) \), we take \( L^\infty_{x,t}, L^2_{x,t}, L^2_{x,t} \) and \( L^4_{x,t} \) norms to \( v_{\lambda_0}, u_{-\lambda_0+I_2}, u_{\lambda_0-I_1} \) and \( u_{\lambda_0-I_3} \), respectively, then

\[
\mathcal{L}_{hhhh-}^h(u,v) \lesssim \sum_{1 \leq j_3 < j_1 \approx j_2} \langle \lambda_0 \rangle^{5/4} \left( \langle 2^{j_3} \rangle \right)^{-1/2} \left( \langle 2^{j_1} \rangle \right)^{-1/2} \| v_{\lambda_0} \|_{V^2_4} \| u_{-\lambda_0+I_2} \|_{V^2_4} \| u_{\lambda_0-I_1} \|_{L^4_{x,t}} \| u_{\lambda_0-I_3} \|_{L^4_{x,t}}
\]

\[
\lesssim \sum_{1 \leq j_3 < j_1 \approx j_2} \langle \lambda_0 \rangle^{5/4} \left( \langle 2^{j_3} \rangle \right)^{-1/2} \left( \langle 2^{j_1} \rangle \right)^{-1/2} \| v_{\lambda_0} \|_{V^2_4} \| u_{-\lambda_0+I_2} \|_{V^2_4} \times T^{1/4} \langle \lambda_0 \rangle^{-3/4} \left( \langle 2^{j_1} \rangle \right)^{1/4 + \epsilon} \left( \langle 2^{j_2} \rangle \right)^{1/4 + \epsilon} \| u \|_{X_{\infty,A}^{1/4}},
\]

which is the same as the right hand side of the second inequality in (4.4).

If \( u_{\lambda_0-I_3} \) has the highest dispersion modulation, we don’t need to divide \( \mathcal{L}_{hhhh-}^h(u,v) \). By Hölder’s inequality, we obtain that

\[
\mathcal{L}_{hhhh-}^h(u,v) \lesssim \sum_{1 \leq j_3 < j_1 \approx j_2} \langle \lambda_0 \rangle^{5/4} \left( \langle 2^{j_3} \rangle \right)^{-1/2} \left( \langle 2^{j_1} \rangle \right)^{-1/2} \| v_{\lambda_0} \|_{V^2_4} \| u_{-\lambda_0+I_2} \|_{V^2_4} \| u_{\lambda_0-I_3} \|_{L^4_{x,t}} \| u_{-\lambda_0+I_2} \|_{L^4_{x,t}}
\]

\[
\lesssim \sum_{1 \leq j_3 < j_1 \approx j_2} \langle \lambda_0 \rangle^{5/4} \left( \langle 2^{j_3} \rangle \right)^{-1/2} \left( \langle 2^{j_1} \rangle \right)^{-1/2} \| v_{\lambda_0} \|_{V^2_4} \| u_{-\lambda_0+I_2} \|_{V^2_4} \times T^{1/2} \langle \lambda_0 \rangle^{-3/4} \left( \langle 2^{j_1} \rangle \right)^{1/4 + \epsilon} \left( \langle 2^{j_2} \rangle \right)^{1/4 + \epsilon} \| u \|_{X_{\infty,A}^{1/4}}
\]

\[
\lesssim T^{1/2} \| v_{\lambda_0} \|_{V^2_4} \| u \|_{X_{\infty,A}^{1/4}},
\]

where the last inequality is by using \( 2^{j_1} \lesssim \langle \lambda_0 \rangle, j_1 \lesssim \langle 2^{j_1} \rangle^\epsilon \), and taking \( \epsilon \leq 1/12 \).

**Case 2**: \( \lambda_3 \in h \) and \( \lambda_1 \in l \). In view of (4.11), we see that \( \lambda_2 \in [-3\lambda_0/4 - l, \lambda_0/4 - l] \), i.e., \( \lambda_2 \in l \) or \( \lambda_2 \in L_- \). We denote by \( \langle \lambda_k \rangle \in hhll \) that all \( \lambda_0, \ldots, \lambda_3 \) satisfy the conditions...
\( \lambda_3 \in h, \quad \lambda_1, \lambda_2 \in l. \)

Taking the similar notations to \( hhll_\cdot \), then we divide Case 2 into two subcases.

Case \( hhll \). We decompose \( \lambda_1, \lambda_2, \lambda_3 \) by:

\[
\lambda_k \in [0, 3\lambda_0/4] = \bigcup_{j_k \geq 0} I_{j_k}, \quad k = 1, 2; \quad \lambda_3 \in [3\lambda_0/4, \lambda_0] = \bigcup_{j_3 \geq 0} \lambda_0 - I_{j_3}, \quad j_1, j_2, j_3 \leq \log_2 \lambda_0.
\]

In view of the condition (FCC) \( \lambda_1 + \lambda_2 + \lambda_3 \approx \lambda_0 \), we see that \( 2^{j_3} \approx 2^{j_2} + 2^{j_1} \). Moreover, we can get \( j_1 \geq j_2 \) from \( \lambda_1 \geq \lambda_2 \). Therefore, we know that \( j_3 \approx j_1 \geq j_2 \). It means that we need to estimate

\[
\mathcal{L}_{hhll}(u, v) := \sum_{0 \leq j_2 \leq j_1 \leq j_3 \leq \log_2 \lambda_0} (\lambda_0)^{1/4} \int_{[0, T] \times \mathbb{R}} |v_{\lambda_0} u_{I_{j_1}} u_{I_{j_2}} \partial_x u_{\lambda_0 - I_{j_3}}| \, dx \, dt.
\]

In view of (DMCC) (4.9), we have the highest dispersion modulation

\[
\max_{0 \leq k \leq 3} |\xi_k^3 - \tau_k| \approx (\lambda_0)^{2} \cdot 2^{j_3}.
\]

If \( v_{\lambda_0} \) has the highest dispersion modulation, by Hölder’s inequality we have

\[
\mathcal{L}_{hhll}(u, v) \lesssim \sum_{0 \leq j_2 \leq j_1 \leq j_3 \leq \log_2 \lambda_0} (\lambda_0)^{5/4} \|v_{\lambda_0}\|_{L^\infty_t L^2_x} \|u_{I_{j_1}}\|_{L^\infty_t L^2_x} \|u_{I_{j_2}}\|_{L^4_x} \|u_{\lambda_0 - I_{j_3}}\|_{L^4_x}.
\]

Using \( \|v_{\lambda_0}\|_{L^\infty_t L^2_x} \lesssim \|v_{\lambda_0}\|_{L^2_x} \), \( V_{\lambda_0} \subset L^\infty_t L^2_x \), the dispersion modulation decay (2.4), the \( L^4 \) estimate (3.13) and Lemma 3.3, we have

\[
\mathcal{L}_{hhll}(u, v) \lesssim \sum_{0 \leq j_2 \leq j_1 \leq j_3 \leq \log_2 \lambda_0} (\lambda_0)^{5/4} (\lambda_0)^{-1} (2^{j_3})^{-1/2} \|v_{\lambda_0}\|_{V_{\lambda_0}} \|u_{I_{j_1}}\|_{V_{\lambda_0}}^2 \times \|u\|_{X_{5/4}}^2.
\]

where the last but one inequality is obtained by summarizing over \( j_2, j_1 \) and taking \( \varepsilon < 1/8 \).

If \( u_{I_{j_1}} \) has the highest dispersion modulation, we have

\[
\mathcal{L}_{hhll}(u, v) \lesssim \sum_{0 \leq j_2 \leq j_1 \leq j_3 \leq \log_2 \lambda_0} (\lambda_0)^{5/4} \|v_{\lambda_0}\|_{L^\infty_t L^2_x} \|u_{I_{j_1}}\|_{L^2_x} \|u_{I_{j_2}}\|_{L^4_x} \|u_{\lambda_0 - I_{j_3}}\|_{L^4_x}.
\]

(4.16)
\[ \times T^{\varepsilon/2}(2^{j_2})^{-1/8+\varepsilon}(2^{j_1})^{1/4+\varepsilon}(\lambda_0)^{-3/8}\|u\|_{X_{\infty,4}}^2, \]

which is the same as the right hand side of the first inequality in (4.16).

If \( u_{I_{j_2}} \) has the highest dispersion modulation, we take \( L^\infty_{x,t}, L^2_{x,t}, L^4_{x,t} \) and \( L^4_{x,t} \) norms to \( v_{\lambda_0}, u_{I_{j_2}}, u_{I_{j_1}} \) and \( u_{\lambda_0-I_{j_3}} \), respectively. Then applying the dispersion modulation decay (2.4) to \( u_{I_{j_2}} \), we have

\[
\mathcal{L}_{hhll}(u,v) \lesssim \sum_{0<j_2\leq j_1 \approx j_3 \leq \log_2 \lambda_0} (\lambda_0)^{5/4} \|v_{\lambda_0}\|_{L^\infty_{x,t}} \|u_{I_{j_2}}\|_{L^2_{x,t}} \|u_{I_{j_1}}\|_{L^4_{x,t}} \|u_{\lambda_0-I_{j_3}}\|_{L^4_{x,t}} \times T^{\varepsilon/2}(2^{j_2})^{-1/8+\varepsilon}(2^{j_1})^{1/4+\varepsilon}(\lambda_0)^{-3/8}\|u\|_{X_{\infty,4}}^2
\]

\[
\lesssim T^{\varepsilon/2}(\lambda_0)^{-1/8} \sum_{0<j_2\leq j_1 \leq \log_2 \lambda_0} (2^{j_2})^{-3/8+\varepsilon}(2^{j_1})^{1/4}\|v_{\lambda_0}\|_{L^2_{x,t}}^2 \|u\|_{X_{\infty,4}}^3.
\]

Making the summation on \( j_2, j_1 \) in order, and taking \( \varepsilon < 1/16 \), we can obtain the desired estimate.

If \( u_{\lambda_0-I_{j_3}} \) has the highest dispersion modulation, by Hölder’s inequality we have

\[
\mathcal{L}_{hhll}(u,v) \lesssim \sum_{0<j_2\leq j_1 \approx j_3 \leq \log_2 \lambda_0} (\lambda_0)^{5/4} \|v_{\lambda_0}\|_{L^\infty_{x,t}} \|u_{\lambda_0-I_{j_3}}\|_{L^2_{x,t}} \|u_{I_{j_1}}\|_{L^4_{x,t}} \|u_{I_{j_2}}\|_{L^4_{x,t}}
\]

Using \( \|v_{\lambda_0}\|_{L^\infty_{x,t}} \lesssim \|v_{\lambda_0}\|_{L^2_{x,t}}, V^2_{\lambda_0} \subset L^\infty_{x,t}L^2_{x,t} \), the dispersion modulation decay (2.4), the \( L^4 \) estimate (3.14) and Lemma 3.4 we have

\[
\mathcal{L}_{hhll}(u,v) \lesssim \sum_{0<j_2\leq j_1 \approx j_3 \leq \log_2 \lambda_0} (\lambda_0)^{5/4} \|v_{\lambda_0}\|_{L^2_{x,t}}(\lambda_0)^{-1}(2^{j_3})^{-1/2}\|u_{\lambda_0-I_{j_3}}\|_{L^2_{x,t}} \times T^{\varepsilon/2}(2^{j_2})^{-1/8+\varepsilon}(2^{j_1})^{1/4+\varepsilon}\|u\|_{X_{\infty,4}}^2
\]

\[
\lesssim T^{\varepsilon/2}\|v_{\lambda_0}\|_{L^2_{x,t}} \sum_{0<j_2\leq j_1 \leq \log_2 \lambda_0} (2^{j_2})^{-1/8+\varepsilon}(2^{j_1})^{1/4+\varepsilon}\|u\|_{X_{\infty,4}}^2
\]

\[
\lesssim T^{\varepsilon/2}\|v_{\lambda_0}\|_{L^2_{x,t}} \|u\|_{X_{\infty,4}}^3.
\] (4.17)

where the last inequality is by taking \( \varepsilon < 1/8 \).

Case \( hhll_\ldots \). We decompose \( \lambda_1, \lambda_2, \lambda_3 \) by:

\[ \lambda_1 \in [0, 3\lambda_0/4] = \bigcup_{j_1 \geq 0} I_{j_1}; \quad \lambda_2 \in [-3\lambda_0/4, 0] = \bigcup_{j_2 \geq 0} -I_{j_2}; \quad \lambda_3 \in [3\lambda_0/4, \lambda_0] = \bigcup_{j_3 \geq 0} \lambda_0 - I_{j_3}. \]

In view of the condition (FCC) \( \lambda_1 + \lambda_2 + \lambda_3 \approx \lambda_0 \), we see that \( 2^{j_1} \approx 2^{j_2} + 2^{j_3} \). Thus, we know that \( j_1 \approx j_2 \lor j_3 \). If \( j_1 \approx j_3 \geq j_2 \) or \( j_1 \approx j_2 \approx j_3 \), it is the same as Case \( hhll \) to get the conclusion. So we only need to consider \( j_1 \approx j_2 \gg j_3 \), which means that we need to estimate

\[
\mathcal{L}_{hhll_\ldots}(u,v) := \sum_{j_3 \ll j_2 \approx j_1} (\lambda_0)^{1/4} \int_{[0,T] \times \mathbb{R}} |v_{\lambda_0} u_{I_{j_1}} u_{I_{j_2}} \partial_x u_{\lambda_0-I_{j_3}}| \, dxdt.
\]
In view of (DMCC) (4.13), we have the highest dispersion modulation satisfying
\[
\max_{0 \leq k \leq 3} |\xi_k^3 - \tau_k| \gtrsim (\lambda_0)^2 \cdot 2^{j_3}.
\]

If \(v_{\lambda_0}\) has the highest dispersion modulation, we can easily see that \(|\lambda_0 - 2^{j_3} + 2^{j_2}| \gtrsim (\lambda_0)\) and \(|\lambda_0 - 2^{j_3} - 2^{j_2}| \approx |\lambda_0 - 2^{j_1}| \gtrsim (\lambda_0)\), then we shall use the bilinear estimate to \(u_{-I_{j_2}} u_{\lambda_0 - I_{j_3}}\),
\[
\mathcal{L}_{hll} (u, v) \lesssim \sum_{j_3 < j_2 \leq j_1} \langle \lambda_0 \rangle^{5/4} \|v_{\lambda_0}\|_{L^2_t L_x^\infty} \|u_{I_{j_1}}\|_{L^\infty_t L^2_x} \|u_{-I_{j_2}} u_{\lambda_0 - I_{j_3}}\|_{L^2_t}.
\] (4.18)

Using \(\|v_{\lambda_0}\|_{L^\infty_t L^2_x} \lesssim \|v_{\lambda_0}\|_{L^2_t A^2} A^2 \subset L^\infty_t L^2_x\), the dispersion modulation decay (2.4), the bilinear estimate (3.7) and Lemma 3.4 we have
\[
\mathcal{L}_{hll} (u, v) \lesssim \sum_{j_3 < j_2 \leq j_1} \langle \lambda_0 \rangle^{5/4} \langle \lambda_0 \rangle^{-1} \|v_{\lambda_0}\| V_{\lambda_0}^2 \|u_{I_{j_1}}\| V_{\lambda_0}^2
\]
\[
\times T^{e/4} (\lambda_0)^{-1 + 2\varepsilon} \|u_{-I_{j_2}}\| V_{\lambda_0}^2 \|u_{\lambda_0 - I_{j_3}}\| V_{\lambda_0}^2
\]
\[
\lesssim T^{e/4} \sum_{j_3 < j_2 \leq j_1} \langle \lambda_0 \rangle^{-3/4 + 2\varepsilon} \|v_{\lambda_0}\| V_{\lambda_0}^2 \|u_{I_{j_1}}\| V_{\lambda_0}^2 \|u_{\lambda_0 - I_{j_3}}\| V_{\lambda_0}^2
\]
\[
\lesssim T^{e/4} \sum_{j_3 < j_2 \leq j_1} \langle \lambda_0 \rangle^{-1 + 2\varepsilon} \|v_{\lambda_0}\| V_{\lambda_0}^2 \|u_{I_{j_1}}\| V_{\lambda_0}^2 \|u_{\lambda_0 - I_{j_3}}\| V_{\lambda_0}^2
\]
\[
\lesssim T^{e/4} \|v_{\lambda_0}\| V_{\lambda_0}^2 \|u\| V_{\lambda_0}^{\frac{3}{4}},
\] (4.19)

where the last but one inequality is obtained by taking \(\varepsilon < 1/4\).

If \(u_{I_{j_1}}\) has the highest dispersion modulation, we have
\[
\mathcal{L}_{hll} (u, v) \lesssim \sum_{j_3 < j_2 \leq j_1} \langle \lambda_0 \rangle^{5/4} \|v_{\lambda_0}\|_{L^\infty_t L^2_x} \|u_{I_{j_1}}\|_{L^2_t L_x^\infty} \|u_{-I_{j_2}} u_{\lambda_0 - I_{j_3}}\|_{L^2_t}.
\]
\[
\lesssim \sum_{j_3 < j_2 \leq j_1} \langle \lambda_0 \rangle^{5/4} \|v_{\lambda_0}\| V_{\lambda_0}^2 \langle \lambda_0 \rangle^{-1} \|u_{I_{j_1}}\| V_{\lambda_0}^2
\]
\[
\times T^{e/4} (\lambda_0)^{-1 + 2\varepsilon} \|u_{-I_{j_2}}\| V_{\lambda_0}^2 \|u_{\lambda_0 - I_{j_3}}\| V_{\lambda_0}^2,
\]

which is the same as the right hand side of the first inequality in (4.19).

If \(u_{-I_{j_2}}\) has the highest dispersion modulation, noticing that \(j_1 \approx j_2\), we can take \(L^\infty_t L^2_x\) and \(L^2_t L_x^\infty\) norms to \(v_{\lambda_0}, u_{-I_{j_2}}\) and \(u_{I_{j_1}} u_{\lambda_0 - I_{j_3}}\), respectively. Then we can repeat the above proof to obtain the desired estimates.

If \(u_{\lambda_0 - I_{j_3}}\) has the highest dispersion modulation, comparing with Case \(hll\), the difference is the summation in (4.17) (taking \(\varepsilon < 1/8\))
\[
\sum_{j_3 < j_2 \leq j_1} (2^{j_1})^{-1/8 + \varepsilon} (2^{j_2})^{-1/8 + \varepsilon} \lesssim \sum_{0 \leq j_1 \leq \log_2 \lambda_0} (2^{j_1})^{-1/4 + 2\varepsilon} \times j_1 \lesssim 1.
\] (4.20)
Case 3: $\lambda_3 \in h$ and $\lambda_1 \in l_-$. It is easy to see that $\lambda_2 \in l_-$. We decompose $\lambda_1, \lambda_2, \lambda_3$ by:

$$\lambda_k \in [-c\lambda_0, 0] = \bigcup_{j_k \geq 0} -I_{j_k}, k = 1, 2; \quad \lambda_3 \in [c\lambda_0, \lambda_0] = \bigcup_{j_3 \geq 0} \lambda_0 - I_{j_3}.$$ 

In view of the condition (FCC) $\lambda_1 + \lambda_2 + \lambda_3 \approx \lambda_0$, we see that $2^{j_1} + 2^{j_2} + 2^{j_3} \approx 0$. It means that $0 \leq j_1, j_2, j_3 \leq 1$. Then using the dispersion modulation decay (2.4), the $L^4$ estimate (5.13) and Lemma 6.3 and noticing that the summation about $j_1, j_2, j_3$ is finite, we can get the result and the details are omitted.

Case 4: $\lambda_3 \in l$. This case is easy to estimate because the derivative locate in the low frequency, $\lambda_1, \lambda_2 \in \{l, l_- \}$ and the highest dispersion modulation satisfies

$$\max_{0 \leq k \leq 3} |\xi_k - \tau_k| \gtrsim (\lambda_0)^3.$$ 

We take Case $\text{hil}\text{l}(\lambda_1, \lambda_2, \lambda_3 \in l)$ as an example. When $v_{l_0}$ attains the highest dispersion modulation, using a similar way as above, we have

$$\mathcal{L}_{\text{hil}}(u, v) \lesssim \sum_{j_1, j_2, j_3} \langle \lambda_0 \rangle^{1/4} 2^{j_3} \|v_{l_0}\|_{L^4_t L^\infty_x} \|u_{I_1}\|_{L^\infty_t L^2_x} \|u_{I_2}\|_{L^4_t L^2_x} \|u_{I_3}\|_{L^4_t L^2_x}$$

$$\lesssim \sum_{j_1, j_2, j_3} \langle \lambda_0 \rangle^{1/2} 2^{j_3} \langle \lambda_0 \rangle^{-3/2} \|v_{l_0}\|_{V^2_A} (2^{j_1})^{1/4} T^{1/2 - 1/8 + \varepsilon} (2^{j_2})^{-1/8 + \varepsilon} \|u\|^3_{L^{1/4}_x A}$$

$$\lesssim T^{1/2} \langle \lambda_0 \rangle^{-5/4} \sum_{j_1, j_2, j_3} (2^{j_1})^{1/4} (2^{j_2})^{-1/8 + \varepsilon} (2^{j_3})^{7/8 + \varepsilon} \|v_{l_0}\|_{V^2_A} \|u\|^3_{L^{1/4}_x A}$$

$$\lesssim \|v_{l_0}\|_{V^2_A} \|u\|^3_{L^{1/4}_x A}.$$ 

When $u_{I_1}, u_{I_2}$, or $u_{I_3}$ attains the highest dispersion modulation, we can use an analogous way to get the result. In fact, we just need to take $L^\infty_{x,t}$ norm to $v_{l_0}, L^2_{x,t}$ norm to the item which has the highest dispersion modulation, and $L^4_{x,t}$ norm to the other two items.

Step 2. We consider the case that $\lambda_0$ is the secondly maximal integer in $\lambda_0, \ldots, \lambda_3$. By the symmetry, we can assume $\lambda_1 \geq \lambda_2$. Then $\lambda_0, \ldots, \lambda_3$ have the following three orders:

**Order 1:** $\lambda_3 \geq \lambda_0 \geq \lambda_1 \geq \lambda_2$;

**Order 2:** $\lambda_1 \geq \lambda_0 \geq \lambda_3 \geq \lambda_2$;

**Order 3:** $\lambda_1 \geq \lambda_0 \geq \lambda_2 \geq \lambda_3$.

Considering the derivative is located in $u_{\lambda_3}$, we take the Order 1 for example in the following proof (the other orders are similar). We divide the proof into three cases $|\lambda_0| \leq 1$, $\lambda_0 \ll 0$ and $\lambda_0 \gg 0$.

**Case 1:** $|\lambda_0| \leq 1$. We decompose $\lambda_1, \lambda_2, \lambda_3$ by:

$$\lambda_k \in (-\infty, \lambda_0] = \bigcup_{j_k \geq -1} -I_{j_k}, k = 1, 2; \quad \lambda_3 \in [\lambda_0, +\infty) = \bigcup_{j_3 \geq -1} I_{j_3}, \quad I_{-1} = [-|\lambda_0|, 0).$$
In view of $\lambda_0 \approx \lambda_1 + \lambda_2 + \lambda_3$ and $\lambda_1 \geq \lambda_2$, we have $j_3 \approx j_2 \geq j_1 \geq -1$. By DMCC (4.9) the highest dispersion modulation satisfies
\begin{equation}
\max_{\theta \in \mathbb{K} \in \mathbb{L}} |\xi_k^3 - \tau_k| \gtrsim 2^{j_1} \cdot 2^{j_2} \cdot 2^{j_3}.
\end{equation}
(4.21)

If $v_{\lambda_0}$ gains the highest dispersion modulation, we have
\begin{align*}
\sum_{j_3 \approx j_2 \geq j_1 \geq -1} \langle \lambda_0 \rangle^{1/4} \int_{[0,T] \times \mathbb{R}} |v_{\lambda_0} u_{\lambda_1} u_{\lambda_2} \partial_x u_{\lambda_3}| \, dxdt
&\lesssim \sum_{j_3 \approx j_2 \geq j_1 \geq -1} 2^{j_3} \|v_{\lambda_0}\|_{L^\infty_t L^\infty_x} \|u_{\lambda_1}\|_{L^\infty_t L^2_x} \|u_{\lambda_2}\|_{L^4_x} \|u_{\lambda_3}\|_{L^4_x} \\
&\lesssim \sum_{j_3 \approx j_2 \geq j_1 \geq -1} 2^{j_3} (2^{j_1})^{-1/2} (2^{j_2})^{-1/2} (2^{j_3})^{-1/2} \|v_{\lambda_0}\|_{V^2_A} \|u_{\lambda_1}\|_{V^4_A} \\
&\quad \times T^{\varepsilon/4} (2^{j_2})^{-1/8 + \varepsilon} T^{\varepsilon/4} (2^{j_1})^{-1/8 + \varepsilon} \|u\|^2_{X^{1/4}_{\infty,A}} \\
&\lesssim T^{\varepsilon/2} \sum_{j_3 \approx j_2 \geq j_1 \geq -1} (2^{j_3})^{-1/4 + 2\varepsilon} (2^{j_1})^{-1/4} \|v_{\lambda_0}\|_{V^2_A} \|u\|^3_{X^{1/4}_{\infty,A}} \\
&\lesssim T^{\varepsilon/2} \|v_{\lambda_0}\|_{V^2_A} \|u\|^3_{X^{1/4}_{\infty,A}} .
\end{align*}
(4.22)

If $u_{\lambda_1}$ has the highest dispersion modulation, we take $L^\infty_t L^2_x$, $L^4_x$ and $L^4_{\infty,t}$ norms to $v_{\lambda_0}$, $u_{\lambda_2}$, $u_{\lambda_3}$ and $u_{\lambda_3}$, respectively. Then applying the dispersion modulation decay (2.4) and the $L^4$ estimate Lemma (3.6) we can get the desired conclusion.

If $u_{\lambda_1}$ has the highest dispersion modulation, we divide the left hand side of (4.7) into two terms.
\begin{align*}
\sum_{j_3 \approx j_2 \geq j_1 \geq -1} \langle \lambda_0 \rangle^{1/4} \int_{[0,T] \times \mathbb{R}} |v_{\lambda_0} u_{\lambda_1} u_{\lambda_2} \partial_x u_{\lambda_3}| \, dxdt
&\leq \left( \sum_{j_3 \approx j_2 \geq j_1 \geq -1} + \sum_{j_3 \approx j_2 \geq j_1 \geq -1} \right) \langle \lambda_0 \rangle^{1/4} \int_{[0,T] \times \mathbb{R}} |v_{\lambda_0} u_{\lambda_1} u_{\lambda_2} \partial_x u_{\lambda_3}| \, dxdt \\
:= & I_1(u,v) + I_2(u,v).
\end{align*}
(4.23)

For $I_1(u,v)$, $L^4$ estimate (3.13) is enough.
\begin{align*}
I_1(u,v) &\lesssim \sum_{j_3 \approx j_2 \geq j_1 \geq -1} 2^{j_3} \|v_{\lambda_0}\|_{L^\infty_t L^\infty_x} \|u_{\lambda_2}\|_{L^2_x} \|u_{\lambda_1}\|_{L^4_x} \|u_{\lambda_3}\|_{L^4_x} \\
&\lesssim \sum_{j_3 \approx j_2 \geq j_1 \geq -1} 2^{j_3} \|v_{\lambda_0}\|_{V^2_A} (2^{j_1})^{-1/2} (2^{j_2})^{-1/2} (2^{j_3})^{-1/2} (2^{j_3})^{1/4} \|u\|_{X^{1/4}_{\infty,A}} \\
&\quad \times T^{\varepsilon/4} (2^{j_2})^{-1/8 + \varepsilon} T^{\varepsilon/4} (2^{j_1})^{-1/8 + \varepsilon} \|u\|^2_{X^{1/4}_{\infty,A}} \\
&\lesssim T^{\varepsilon/2} \sum_{j_3 \approx j_2 \geq j_1 \geq -1} (2^{j_3})^{-1/4 + 2\varepsilon} (2^{j_1})^{-1/4} \|v_{\lambda_0}\|_{V^2_A} \|u\|^3_{X^{1/4}_{\infty,A}} \\
&\lesssim T^{\varepsilon/2} \|v_{\lambda_0}\|_{V^2_A} \|u\|^3_{X^{1/4}_{\infty,A}} .
\end{align*}
(4.24)
For $I_2(u,v)$, we need to use the bilinear estimate \((3.7)\). 

$$I_2(u,v) \lesssim \sum_{j_3 \approx j_2 \gtrsim j_1 \gtrsim -1} 2^{j_3} \|v_{\lambda_0} \|_{L_x^\infty} \left\|u_I \right\|_{L_x^4} \left\|u_{-I_{j_3}^1} \right\|_{L_x^4} \left\|u_{-I_{j_3}^2} \right\|_{L_x^4}$$

$$\lesssim \sum_{j_3 \approx j_2 \gtrsim j_1 \gtrsim -1} 2^{j_3} \left\|v_{\lambda_0} \right\|_{V_\lambda^2} \left((2 j_3)^{-1/2} (2 j_2)^{-1/2} (2 j_3)^{-1/2} (2 j_1)^{-1/2}\right) \left\|u \right\|_{X_{\infty,A}^{1/4}}$$

$$\lesssim T^{\varepsilon/4} \sum_{j_3 \gtrsim j_1 \gtrsim -1} (2 j_3)^{-1/2} (2 j_1)^{-1/2} \left\|v_{\lambda_0} \right\|_{V_\lambda^2} \left\|u \right\|_{X_{\infty,A}^{1/4}}^3$$

$$\lesssim T^{\varepsilon/4} \left\|v_{\lambda_0} \right\|_{V_\lambda^2} \left\|u \right\|_{X_{\infty,A}^{1/4}}^3 \cdot \quad (4.25)$$

If $u_{-I_{j_2}}$ has the highest dispersion modulation, we can get the desired estimate by exchanging the positions of $u_{I_{j_2}}$ and $u_{-I_{j_2}}$ in the above discussion (noticing that $j_2 \approx j_3$).

**Case 2**: $\lambda_0 \ll 0$. We decompose $\lambda_1$ and $\lambda_2$ by:

$$\lambda_k \in (-\infty, \lambda_0] = \bigcup_{j_k \gtrsim 0} \lambda_0 - I_{j_k}, \quad k = 1, 2.$$ 

From the following frequency constraint condition

$$\lambda_0 = \lambda_1 + \lambda_2 + \lambda_3 + l, \quad |l| \leq 10,$$ 

we can decompose $\lambda_3$ as follows.

$$\lambda_3 \in [-\lambda_0 - l, +\infty] = \bigcup_{j_3 \gtrsim -1} -\lambda_0 + I_{j_3}, \quad I_{-1} = [-|l|, 0).$$

In view of $\lambda_0 \approx \lambda_1 + \lambda_2 + \lambda_3$ and $\lambda_1 \gtrsim \lambda_2$, we have $j_3 \approx j_2 \gtrsim j_1$. By DMCC \((4.1)\), we can see that the highest dispersion modulation satisfies

$$\max_{0 \leq k \leq 3} |\xi_k^3 - \tau_k| \gtrsim 2^j \cdot 2^j \cdot ((\lambda_0) + 2^j).$$

If the highest dispersion modulation is located in $v_{\lambda_0}$, from the dispersion modulation decay \((4.2)\), $L^4$ estimate \((5.13)\) and Lemma \((5.4)\) we have

$$\sum_{j_3 \approx j_2 \gtrsim j_1} \langle \lambda_0 \rangle^{1/4} \int_{[0,T] \times \mathbb{R}} \left| \overline{v_{\lambda_0}} \right| u_{\lambda_0 - I_3} u_{\lambda_0 - I_2} \partial_x u_{-\lambda_0 + I_{j_3}} \, dx dt$$

$$\lesssim \sum_{j_3 \approx j_2 \gtrsim j_1} \langle \lambda_0 \rangle^{1/4} \left(\langle \lambda_0 \rangle + 2^j \right) \left\|v_{\lambda_0} \right\|_{L_x^2 L_t^\infty} \left\|u_{\lambda_0 - I_3} \right\|_{L_x^\infty L_t^2} \left\|u_{\lambda_0 - I_2} \right\|_{L_x^\infty L_t^2} \left\|u_{-\lambda_0 + I_{j_3}} \right\|_{L_x^4}$$

$$\lesssim \sum_{j_3 \approx j_2 \gtrsim j_1} \langle \lambda_0 \rangle^{1/4} \left(\langle \lambda_0 \rangle + 2^j \right) \left(2^j \right)^{-1/2} \left(2^j \right)^{-1/2} \left(2^j \right)^{-1/2} \left\|v_{\lambda_0} \right\|_{V_\lambda^2} \left(\langle \lambda_0 \rangle + 2^j \right)^{-1/4}$$

$$\times T^{\varepsilon/4} (2^j)^{-1/4+\varepsilon} \left(\langle \lambda_0 \rangle + 2^j \right)^{-3/8} T^{\varepsilon/4} (2^j)^{1/4+\varepsilon} \left(\langle \lambda_0 \rangle + 2^j \right)^{-3/8} \left\|u \right\|_{X_{\infty,A}^{1/4}}^3$$

$$\lesssim T^{\varepsilon/2} \sum_{j_3 \gtrsim 0} \langle \lambda_0 \rangle^{-1/4} (2^j)^{2\varepsilon} \sum_{0 \leq j_1 \leq j_3} \langle \lambda_0 \rangle^{1/4} \left(\langle \lambda_0 \rangle + 2^j \right)^{-1/4} \left\|v_{\lambda_0} \right\|_{V_\lambda^2} \left\|u \right\|_{X_{\infty,A}^{1/4}}^3 \cdot \quad (4.28)$$
Making the summation on $j_1$, we see that the summation is controlled by $j_3$. Then one has that for $0 < \varepsilon < 1/8$,

$$
(4.28) \lesssim T^{\varepsilon/2} \left( \sum_{j_3 \geq 0} (2^{j_3})^{-1/4 + 2\varepsilon} \right) \|v \lambda_0\|_{L_x^\infty} \|u\|_{X_{\infty, A}^{1/4}}^3.
$$

If the highest dispersion modulation is located in $u_\lambda_0 - I_{j_1}$, we take $L_x^{\infty}$, $L_x^{3}$, $L_x^{4,t}$ norms to $v_\lambda_0$, $u_\lambda_0 - I_{j_1}$, $u_\lambda_0 - I_{j_2}$ and $u - \lambda_0 + I_{j_3}$, respectively. Then we can reduce the desired estimate as the above case, so the details are omitted.

If the highest dispersion modulation is located in $u_\lambda_0 - I_{j_2}$, we divide the left hand side of (4.7) into two terms.

$$
\sum_{j_3 \approx j_2 \geq j_1 \geq 0} (\lambda_0)^{1/4} \int_{[0,T] \times \mathbb{R}} |v_\lambda_0 u_{\lambda_0 - I_{j_1}} u_{\lambda_0 - I_{j_2}} \partial_x u - \lambda_0 + I_{j_3}| \, dx dt
\lesssim \left( \sum_{j_3 \approx j_2 \geq j_1 \geq 0} (\lambda_0)^{1/4} \int_{[0,T] \times \mathbb{R}} |v_\lambda_0 u_{\lambda_0 - I_{j_1}} u_{\lambda_0 - I_{j_2}} \partial_x u - \lambda_0 + I_{j_3}| \, dx dt \right) + (4.29)
$$

For $I_1(u, v)$, from the dispersion modulation decay (2.4), $L_x^4$ estimate (3.13) and Lemma 3.4, we have

$$
I_1(u, v) \lesssim \sum_{j_3 \approx j_2 \geq j_1 \geq 0} (\lambda_0)^{1/4} ((\lambda_0) + 2^{j_3}) \|v_\lambda_0\|_{L_x^\infty} \|u_{\lambda_0 - I_{j_1}}\|_{L_x^3} \|u_{\lambda_0 - I_{j_2}}\|_{L_x^4} \|u - \lambda_0 + I_{j_3}\|_{L_x^4}
\lesssim \sum_{j_3 \approx j_2 \geq j_1 \geq 0} (\lambda_0)^{1/4} ((\lambda_0) + 2^{j_3}) \|v_\lambda_0\|_{L_x^3} \|u_{\lambda_0 - I_{j_1}}\|_{L_x^3} \|u_{\lambda_0 - I_{j_2}}\|_{L_x^4} \|u - \lambda_0 + I_{j_3}\|_{L_x^4}
\lesssim T^{\varepsilon/4} (2^{j_3})^{1/4 + \varepsilon} ((\lambda_0) + 2^{j_3})^{-3/8} T^{\varepsilon/4} (2^{j_3})^{1/4 + \varepsilon} ((\lambda_0) + 2^{j_3})^{-3/8} \|u\|_{X_{\infty, A}^{1/4}}^3
\lesssim T^{\varepsilon/2} \sum_{j_3 \geq 0} (\lambda_0)^{1/4} ((\lambda_0) + 2^{j_3})^{-1/2} (2^{j_3})^{2\varepsilon} \|v_\lambda_0\|_{L_x^3} \|u\|_{X_{\infty, A}^{1/4}}^3. (4.30)
$$

Noticing

$$
(\lambda_0)^{1/4} ((\lambda_0) + 2^{j_3})^{-1/4} \leq 1, \quad ((\lambda_0) + 2^{j_3})^{-1/4} (2^{j_3})^{2\varepsilon} \leq (2^{j_3})^{-1/4 + 2\varepsilon},
$$

for $0 < \varepsilon < 1/8$, (4.30) is dominated by

$$
\lesssim T^{\varepsilon/2} \sum_{j_3 \geq 0} (2^{j_3})^{-1/4 + 2\varepsilon} \|v_\lambda_0\|_{L_x^3} \|u\|_{X_{\infty, A}^{1/4}}^3 \lesssim T^{\varepsilon/2} \|v(\lambda_0)\|_{L_x^3} \|u\|_{X_{\infty, A}^{1/4}}^3.
$$

For $I_2(u, v)$, from the dispersion modulation decay (2.4), the bilinear estimate (3.7) and Lemma 3.4, we have

$$
I_2(u, v) \lesssim \sum_{j_3 \approx j_2 \geq j_1 \geq 0} (\lambda_0)^{1/4} ((\lambda_0) + 2^{j_3}) \|v_\lambda_0\|_{L_x^\infty} \|u_{\lambda_0 - I_{j_1}}\|_{L_x^3} \|u_{\lambda_0 - I_{j_2}}\|_{L_x^4} \|u - \lambda_0 + I_{j_3}\|_{L_x^4}.
$$
We can easily get that the highest dispersion modulation satisfies three subcases:

\[\lambda_{1}\in[-c\lambda_{0},0]\] and \[\lambda_{2}\in[-c\lambda_{0},0]\] or \([0,c\lambda_{0}]\) and \([\lambda_{0},\lambda_{0}+c\lambda_{0}-l]\). From the dispersion modulation decay estimate (4.31), we have

\[\sum_{j_{3}}\langle\lambda_{0}\rangle^{1/4}(2^{j_{3}})^{-1/2}(2^{j_{2}})^{-1/2}\langle\lambda_{0}\rangle^{1/2}(2^{j_{3}})^{1/2}\|v_{\lambda_{0}}\|_{L_{t}^{2}}^{2}\|u_{\lambda_{0}-j_{2}}\|_{V_{\lambda_{0}}^{2}}^{2}\]

\[\times T^{e/4}(\langle\lambda_{0}\rangle^{1/2}(2^{j_{3}})^{-1/2+\varepsilon}(2^{j_{3}})^{-1/2+\varepsilon}\|u_{\lambda_{0}-j_{1}}\|_{V_{\lambda_{0}}^{2}}^{2}\|u_{-\lambda_{0}+I_{j_{3}}}\|_{V_{\lambda_{0}}^{2}}^{2}\]

\[\lesssim T^{e/4}\sum_{j_{3}\gg j_{1}\geq 0}\langle\lambda_{0}\rangle^{1/4}(\langle\lambda_{0}\rangle^{1/2}(2^{j_{3}})^{-1/2+\varepsilon}(2^{j_{3}})^{-1/2+\varepsilon}\|v_{\lambda_{0}}\|_{V_{\lambda_{0}}^{2}}^{2}\|u\|_{X_{\infty,\lambda_{0}}^{1/4}}^{3}\]

\[\lesssim T^{e/4}\|v_{\lambda_{0}}\|_{V_{\lambda_{0}}^{2}}^{2}\|u\|_{X_{\infty,\lambda_{0}}^{1/4}}^{3}\]

If the highest dispersion modulation is located in \(u_{-\lambda_{0}+I_{j_{3}}}\), noticing that \(j_{2}\approx j_{3}\) and \(|-\lambda_{0}+I_{j_{3}}|\approx |\lambda_{0}-I_{j_{2}}|\approx (\langle\lambda_{0}\rangle+2^{j_{3}})|\), we can get the desired estimate by exchanging the positions of \(u_{-\lambda_{0}+I_{j_{3}}}\) and \(u_{\lambda_{0}-j_{2}}\) in the above discussion.

**Case 3:** \(\lambda_{0}\gg 0\). From the frequency constraint condition \(\lambda_{0}=\lambda_{1}+\lambda_{2}+\lambda_{3}+l, \ |l|\leq 10\), we know that \(\lambda_{2}\) must be less than zero. Furthermore, one can divide this case into three subcases: \(\lambda_{2}\in[-c\lambda_{0},0]\), \(\lambda_{2}\in[-\lambda_{0},-c\lambda_{0}]\) and \(\lambda_{2}\in(-\infty,-\lambda_{0}]\).

**Case 3.1:** \(\lambda_{2}\in[-c\lambda_{0},0]\). From the frequency constraint condition we find that \(\lambda_{1}\in[-c\lambda_{0},0]\) or \([0,c\lambda_{0}]\) \((\lambda_{1}\in[c\lambda_{0},\lambda_{0}]\) will never happen\), and \(\lambda_{3}\) satisfies Table 1

| Case | \(\lambda_{2}\in[-c\lambda_{0},0]\) | \(\lambda_{1}\in[-c\lambda_{0},0]\) | \(\lambda_{3}\in[\lambda_{0},\lambda_{0}+2c\lambda_{0}-l]\) |
|------|-----------------|-----------------|-----------------|
| \(l\ldots h\) | \([-c\lambda_{0},0]\) | \([-c\lambda_{0},0]\) | \([\lambda_{0},\lambda_{0}+2c\lambda_{0}-l]\) |

\(|\lambda_{k}|\leq j_{k}, \ k=1,2;\)

\(\lambda_{3}\in[\lambda_{0},(1+2\varepsilon)\lambda_{0}-l]=\bigcup_{j_{3}\geq 0}\lambda_{0}+I_{j_{3}}, \ j_{1},j_{2},j_{3}\ll\log_{2}(\langle\lambda_{0}\rangle)+1.\)

From the frequency constraint condition (4.8), we know

\[2^{j_{1}}+2^{j_{2}}\approx 2^{j_{3}} \Rightarrow j_{3}\approx j_{1} \vee j_{2}.\] (4.31)

We can easily get that the highest dispersion modulation satisfies

\[\max_{0\leq k\leq 3}|\xi_{k}^{3}-\tau_{k}|\gtrsim\langle\lambda_{0}\rangle^{2}cdot 2^{j_{3}}.\] (4.32)

If \(v_{\lambda_{0}}\) attains the highest dispersion modulation, from the dispersion modulation decay (2.4), \(L^{4}\) estimate (3.13) and Lemma 3.4, we have

\[\sum_{j_{3}\approx j_{1} \vee j_{2}}\langle\lambda_{0}\rangle^{1/4}\int_{[0,T]\times \mathbb{R}}|v_{\lambda_{0}}u_{-j_{1}}u_{-j_{2}}\partial_{x}u_{\lambda_{0}+I_{j_{3}}}|dxdt\]
\[
\sum_{j_3 \approx j_1 \lor j_2} (\lambda_0)^{5/4} \| u_{\lambda_0} \|_{L^2 T^0} \| u - I_{j_1} \|_{L^2_t L^\infty_x} \| u - I_{j_2} \|_{L^4_t L^4_x} \| u_{\lambda_0 + I_{j_3}} \|_{L^4_t} \\
\lesssim T^{c/2} \sum_{j_3 \approx j_1 \lor j_2} (\lambda_0)^{-1/8} (2^{j_1})^{-1/4+\varepsilon} (2^{j_2})^{1/4} \| v_{\lambda_0} \|_{L^2} \| u \|^3_{X^{1/4}_{\infty, A}} \\
\lesssim T^{c/2} \| u_{\lambda_0} \|_{L^2} \| u \|^3_{X^{1/4}_{\infty, A}},
\]
where the last but one inequality is gained by summarizing over \(j_2, j_1\) and \(j_3\) in order. One just note that \(j_1 \leq j_3 \leq \log_2 (\lambda_0) + 1\) and take \(\varepsilon < 1/8\).

If \(u - I_{j_1}\) has the highest dispersion modulation, we take \(L^\infty_{x,t}, L^2_{x,t}, L^4_{x,t}\) and \(L^4_t\) norms to \(v_{\lambda_0}, u - I_{j_1}, u - I_{j_2}\) and \(u_{\lambda_0 + I_{j_3}}\), respectively. Then we can get the desired conclusion by the same way as above. If \(u - I_{j_2}\) gains the highest dispersion modulation, one can exchange the positions of \(j_1\) and \(j_2\) to obtain the desired estimate.

If \(u_{\lambda_0 + I_{j_3}}\) has the highest dispersion modulation, we have
\[
\sum_{j_3 \approx j_1 \lor j_2} (\lambda_0)^{1/4} \int_{[0,T] \times \mathbb{R}} \| u_{\lambda_0} u - I_{j_1} u - I_{j_2} \partial_x u_{\lambda_0 + I_{j_3}} \| dx dt \\
\lesssim \sum_{j_3 \approx j_1 \lor j_2} (\lambda_0)^{5/4} \| u_{\lambda_0} \|_{L^\infty_{x,t}} \| u_{\lambda_0 + I_{j_1}} \|_{L^2_{x,t}} \| u_{\lambda_0 + I_{j_3}} \|_{L^4_{x,t}} \| u - I_{j_1} \|_{L^2_{x,t}} \| u - I_{j_2} \|_{L^4_{x,t}} \\
\lesssim T^{c/2} \sum_{j_3 \approx j_1 \lor j_2} (2^{j_1})^{-1/8+\varepsilon} (2^{j_2})^{-1/8+\varepsilon} \| v_{\lambda_0} \|_{L^2} \| u \|^3_{X^{1/4}_{\infty, A}} \\
\lesssim T^{c/2} \| u_{\lambda_0} \|_{L^2} \| u \|^3_{X^{1/4}_{\infty, A}},
\]
(4.33)

**Case \(L.1h\).** One can use the dyadic decomposition:
\[
\lambda_1 \in [0, c \lambda_0] = \bigcup_{j_1 \geq 0} I_{j_1}, \quad \lambda_2 \in [-c \lambda_0, 0] = \bigcup_{j_2 \geq 0} -I_{j_2}, \\
\lambda_3 \in [\lambda_0, (1 + c) \lambda_0 - l] = \bigcup_{j_3 \geq 0} \lambda_0 + I_{j_3}, \quad j_1, j_2, j_3 \leq \log_2 (\lambda_0).
\]

From the frequency constraint condition (4.8), we get
\[
2^{j_1} + 2^{j_3} \approx 2^{j_2} \Rightarrow j_2 \approx j_1 \lor j_3.
\]
(4.34)

One can get that the highest dispersion modulation satisfies
\[
\max_{0 \leq k \leq 3} | \xi_k^3 - \tau_k | \gtrsim \langle \lambda_0 \rangle^2 2^{j_1}.
\]
(4.35)

If \(v_\lambda\) attains the highest dispersion modulation, from the dispersion modulation decay (2.4), the bilinear estimate and Lemma 3.4, we have
\[
\sum_{j_2 \approx j_1 \lor j_3} (\lambda_0)^{1/4} \int_{[0,T] \times \mathbb{R}} \| u_{\lambda_0} u_{\lambda_0} u_{\lambda_0} \| dx dt
\]

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Table 2: $\lambda_2 \in [-\lambda_0, -c_\lambda 0]$

$$\leq \sum_{j_2 \approx j_1 \vee j_3} \langle \lambda_0 \rangle^{5/4} \|v_{\lambda_0}\|_{L_t^2 L_x^\infty} \|u_{I_{j_1}}\|_{L_t^\infty L_x^2} \|u_{-I_{j_2}} u_{\lambda_0 + I_{j_3}}\|_{L_t^2 L_x^4}$$

$$\leq T^{e/4} \sum_{j_2 \approx j_1 \vee j_3} \langle \lambda_0 \rangle^{1/4} (2^{j_3})^{-1/2} \|v_{\lambda_0}\|_{V_A^2} \|u_{I_{j_1}}\|_{V_A^2} \langle \lambda_0 \rangle^{-1 + 2e} \|u_{-I_{j_2}}\|_{V_A^2} \|u_{\lambda_0 + I_{j_3}}\|_{V_A^2}$$

$$\leq T^{e/4} \sum_{j_2 \approx j_1 \vee j_3} \langle \lambda_0 \rangle^{-1 + 2e} (2^{j_1})^{1/4} (2^{j_2})^{1/4} \|v_{\lambda_0}\|_{V_A^2} \|u\|_{X_{\infty, A}^3}^3$$

$$\leq T^{e/4} \|v_{\lambda_0}\|_{V_A^2} \|u\|_{X_{\infty, A}^3}^3.$$ If $u_{I_{j_1}}$ has the highest dispersion modulation, we just take $L_t^\infty, L_x^2$ and $L_t^2 L_x^4$ norms to $v_{\lambda_0}, u_{I_{j_1}}$ and $u_{-I_{j_2}} u_{\lambda_0 + I_{j_3}}$, respectively. If $u_{-I_{j_2}}$ attains the highest dispersion modulation, one can further exchange the positions of $j_1$ and $j_2$ to obtain the desired estimate. If $u_{\lambda_0 + I_{j_3}}$ has the highest dispersion modulation, we can get the result by the same way as (4.33) in Case $L_{-h}$. 

**Case 3.2:** $\lambda_2 \in [-\lambda_0, -c_\lambda 0]$. We consider $\lambda_1 \in [c_\lambda 0, \lambda_0], [0, c_\lambda 0], [-c_\lambda 0, 0]$ and $[-\lambda_0, -c_\lambda 0]$, respectively. From the frequency constraint condition we can obtain the corresponding range of $\lambda_3$ (see Table 2).

**Case $h_{-hh}$.** We decompose $\lambda_1, \lambda_2, \lambda_3$ by:

$$\lambda_1 \in [c_\lambda 0, \lambda_0] = \bigcup_{j_1 \geq 0} \lambda_0 - I_{j_1}, \quad \lambda_2 \in [-\lambda_0, -c_\lambda 0] = \bigcup_{j_2 \geq 0} -\lambda_0 + I_{j_2},$$

$$\lambda_3 \in [\lambda_0, (2 - c_\lambda 0 - l] = \bigcup_{j_3 \geq 0} \lambda_0 + I_{j_3}, \quad j_1, j_2, j_3 \leq \log_2 (\lambda_0).$$

From the frequency constraint condition (4.8), we have

$$2^{j_1} \approx 2^{j_2} + 2^{j_3}. \quad (4.36)$$

It follows that $j_1 \approx j_2 \vee j_3$. When $j_1 \approx j_2 \geq j_3$, we can get the result by using the similar technique as that used in Case 1 of Step 1. When $j_1 \approx j_3 \geq j_2$, we just need to exchange the positions of $j_2$ and $j_3$ and use the similar way to obtain our conclusion. We omit the details.

**Case $h_{-lh}$.** We decompose $\lambda_1, \lambda_2, \lambda_3$ by:

$$\lambda_1 \in [0, c_\lambda 0] = \bigcup_{j_1 \geq 0} I_{j_1}, \quad \lambda_2 \in [-\lambda_0, -c_\lambda 0] = \bigcup_{j_2 \geq 0} -\lambda_0 + I_{j_2},$$

$$\lambda_3 \in [\lambda_0, (2 - c_\lambda 0 - l] = \bigcup_{j_3 \geq 0} \lambda_0 + I_{j_3}, \quad j_1, j_2, j_3 \leq \log_2 (\lambda_0).$$

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\[
\lambda_3 \in [\lambda_0, 2\lambda_0 - l] = \bigcup_{j_3 \geq 0} \lambda_0 + I_{j_3}, \quad j_1, j_2, j_3 \leq \log_2(\lambda_0).
\]

From the frequency constraint condition (4.8), we have

\[
2^{j_1} + 2^{j_2} + 2^{j_3} \approx \lambda_0.
\]  (4.37)

By DMCC (4.9) the highest dispersion modulation satisfies

\[
\max_{0 \leq k \leq 3} |s_k^3 - \tau_k| \gtrsim (\lambda_0)^2 \cdot 2^{j_1}.
\]  (4.38)

If \(v_{\lambda_0}\) has the highest dispersion modulation, we have dispersion modulation decay to \(v_{\lambda_0}\). For \(u_{I_{j_1}}\) and \(u_{\lambda_0 + I_{j_3}}\), we have \(|\lambda_0 + 2^{j_1} + 2^{j_3}| \gtrsim (\lambda_0)\) and \(|\lambda_0 + 2^{j_2} + 2^{j_3}| \gtrsim (\lambda_0)\). Thus we can use bilinear estimate (5.7) to \(u_{I_{j_1}} u_{\lambda_0 + I_{j_3}}\). To be specific, we have

\[
\sum_{j_1, j_2, j_3 \leq \log_2(\lambda_0)} \langle \lambda_0 \rangle^{1/4} \int_{[0,T] \times \mathbb{R}} |\nu_{\lambda_0} u_{I_{j_1}} u_{\lambda_0 + I_{j_2}} \partial_x u_{\lambda_0 + I_{j_3}}| \, dx \, dt
\]
\[
\lesssim \sum_{j_1, j_2, j_3 \leq \log_2(\lambda_0)} \langle \lambda_0 \rangle^{5/4} \|v_{\lambda_0}\|_{L^5_t L^\infty_x} \|u_{\lambda_0 + I_{j_2}}\|_{L^\infty_t L^2_x} \|u_{I_{j_1}} u_{\lambda_0 + I_{j_3}}\|_{L^2_x} \|u_{\lambda_0 + I_{j_2}}\|_{V^2_A}
\]
\[
\lesssim \sum_{j_1, j_2, j_3 \leq \log_2(\lambda_0)} \langle \lambda_0 \rangle^{5/4} \lambda_0^{-1/2} (2^{j_1})^{-1/2} \|v_{\lambda_0}\|_{V^2_A} \|u_{\lambda_0 + I_{j_2}}\|_{V^2_A}
\]
\[
\lesssim T^{k/4} \langle \lambda_0 \rangle^{-1/2 + 2^k} \log_2(\lambda_0) \|v_{\lambda_0}\|_{V^2_A} \|u\|_{X^{1/4}_{\infty, A}}^3.
\]  (4.39)

If \(u_{-\lambda_0 + I_{j_2}}\) has the highest dispersion modulation, we take \(L^\infty_t L^2_{x,t}\) and \(L^2_{x,t}\) norms to \(v_{\lambda_0}\), \(u_{-\lambda_0 + I_{j_2}}\), and \(u_{I_{j_1}} u_{\lambda_0 + I_{j_3}}\), respectively. Then applying the dispersion modulation decay estimate (2.3) to \(u_{-\lambda_0 + I_{j_2}}\) and the bilinear estimate (5.7) to \(u_{I_{j_1}} u_{\lambda_0 + I_{j_3}}\), we can get the desired conclusion.

If \(u_{I_{j_1}}\) has the highest dispersion modulation, we have dispersion modulation decay to \(u_{I_{j_1}}\). For \(u_{-\lambda_0 + I_{j_2}}\) and \(u_{\lambda_0 + I_{j_3}}\), we have \(|\lambda_0 + 2^{j_1} - \lambda_0 + 2^{j_2}| \gtrsim (2^{j_1} + 2^{j_2}) \approx \lambda_0 - 2^{j_1} \gtrsim (\lambda_0)\) and \(|\lambda_0 + 2^{j_2} + \lambda_0 - 2^{j_1}| \gtrsim (\lambda_0)\). Thus we can use bilinear estimate (5.7) to \(u_{-\lambda_0 + I_{j_2}} u_{\lambda_0 + I_{j_3}}\). Therefore, we have

\[
\sum_{j_1, j_2, j_3 \leq \log_2(\lambda_0)} \langle \lambda_0 \rangle^{1/4} \int_{[0,T] \times \mathbb{R}} |\nu_{\lambda_0} u_{I_{j_1}} u_{-\lambda_0 + I_{j_2}} \partial_x u_{\lambda_0 + I_{j_3}}| \, dx \, dt
\]
\[
\lesssim \sum_{j_1, j_2, j_3 \leq \log_2(\lambda_0)} \langle \lambda_0 \rangle^{5/4} \|v_{\lambda_0}\|_{L^5_t L^\infty_x} \|u_{I_{j_1}}\|_{L^2_{x,t}} \|u_{-\lambda_0 + I_{j_2}} u_{\lambda_0 + I_{j_3}}\|_{L^2_{x,t}}
\]
\[
\lesssim \sum_{j_1, j_2, j_3 \leq \log_2(\lambda_0)} \langle \lambda_0 \rangle^{5/4} \|v_{\lambda_0}\|_{V^2_A} \langle \lambda_0 \rangle^{-1/2} (2^{j_1})^{-1/2} \|u_{I_{j_1}}\|_{V^2_A}
\]

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\[ \times T^{\varepsilon/4}(\lambda_0)^{-1+2\varepsilon} \| u_{-\lambda_0+I_{j_2}} \|_{V^2_A}  \| u_{\lambda_0+I_{j_2}} \|_{V^2_A}, \] (4.40)

which is the same with the third line of (4.39), so we omit the details.

If \( u_{\lambda_0+I_{j_3}} \) has the highest dispersion modulation, from the dispersion modulation decay (4.4) and \( L^4 \) estimate (4.13), we have

\[ \sum_{j_1,j_2,j_3 \leq \log_2(\lambda_0)} (\lambda_0)^{1/4} \int_{[0,T] \times \mathbb{R}} |u_{\lambda_0} u_{I_{j_1}} u_{-\lambda_0+I_{j_2}} \partial_x u_{\lambda_0+I_{j_3}}| \, dxdt \]
\[ \lesssim \sum_{j_1,j_2,j_3 \leq \log_2(\lambda_0)} (\lambda_0)^{5/4} \| v_{\lambda_0} \|_{L^\infty_T} \| u_{\lambda_0+I_{j_3}} \|_{L^2_{x,t}} \| u_{I_{j_1}} \|_{L^4_{x,t}} \| u_{-\lambda_0+I_{j_2}} \|_{L^4_{x,t}} \]
\[ \lesssim \sum_{j_1,j_2,j_3 \leq \log_2(\lambda_0)} (\lambda_0)^{5/4} \| v_{\lambda_0} \|_{V^2_A} (\lambda_0)^{-1/2} (2^{j_1})^{-1/2} (2^{j_2})^{1/2} (\lambda_0)^{-1/4} \| u \|_{X_{1/4,A}} \]
\[ \times T^{\varepsilon/4} (2^{j_1})^{-1/8+\varepsilon} T^{\varepsilon/4} (2^{j_2})^{1/4+\varepsilon} (\lambda_0)^{-3/8} \| u \|_{X_{1/4,A}}^2 \]
\[ \lesssim T^{\varepsilon/2} \sum_{j_1,j_2,j_3 \leq \log_2(\lambda_0)} (\lambda_0)^{-3/8} (2^{j_1})^{-1/8+\varepsilon} (2^{j_2})^{1/4+\varepsilon} \| v_{\lambda_0} \|_{V^2_A} \| u \|_{X_{1/4,A}}^3. \] (4.41)

Taking \( 0 < \varepsilon < 1/8 \), the summation over \( j_1 \) is finite. The summation over \( j_2 \) and \( j_3 \) can be controlled, so (4.41) is continued by

\[ \lesssim T^{\varepsilon/2} (\lambda_0)^{-1/8+\varepsilon} \| v_{\lambda_0} \|_{V^2_A} \| u \|_{X_{1/4,A}}^3 \lesssim T^{\varepsilon/2} (\lambda_0)^{3/4} \| u \|_{X_{1/4,A}}^3. \] (4.42)

**Case h_{-h}.** We decompose \( \lambda_1, \lambda_2, \lambda_3 \) in the following way:

\[ \lambda_1 \in [-c\lambda_0, 0] = \bigcup_{j_1 \geq 0} -I_{j_1}, \quad \lambda_2 \in [-\lambda_0, -c\lambda_0] = \bigcup_{j_2 \geq 0} -\lambda_0 + I_{j_2}, \quad \lambda_3 \in [c\lambda_0 - l, 2\lambda_0 + c\lambda_0 - l] = \bigcup_{j_3 \geq \log_2(\lambda_0) - C} \lambda_0 + I_{j_3}, \quad j_3 \leq \log_2(\lambda_0) + 1. \]

From the frequency constraint condition (4.8), we have

\[ 2^{j_2} + 2^{j_3} \approx \lambda_0 + 2^{j_1}. \] (4.43)

It is easy to see that this case is similar to the above Case h_{-h}, so the details are omitted.

**Case h_{-h}.** We decompose \( \lambda_1, \lambda_2, \lambda_3 \) as follows:

\[ \lambda_k \in [-\lambda_0, -c\lambda_0] = \bigcup_{j_k \geq 0} -\lambda_0 + I_{j_k}, \quad j_k \leq \log_2(\lambda_0), \quad k = 1, 2; \] (4.44)

\[ \lambda_3 \in [\lambda_0 + 2c\lambda_0 - l, 3\lambda_0 - l] = \bigcup_{j_3 \geq \log_2(2c\lambda_0 - l)} \lambda_0 + I_{j_3}, \quad j_3 \leq \log_2(\lambda_0) + 1. \] (4.45)

From the dispersion modulation constraint condition (4.9), we know the highest dispersion modulation satisfies

\[ \max_{0 \leq k \leq 3} |\epsilon_k^3 - \tau_k^3| \gtrsim (\lambda_0)^2 \cdot 2^{j_1}. \] (4.46)
If \( v_{\lambda_0} \) has the highest dispersion modulation, we take dispersion modulation decay to \( v_{\lambda_0} \). For \( u_{-\lambda_0 + I_j^1} \) and \( u_{\lambda_0 + I_j^1} \), we have \( |\lambda_0 + 2^j_3 - \lambda_0| + 2^j_1 | \gtrsim 2^j_1 \) and \( |\lambda_0 + 2^j_3 + \lambda_0 - 2^j_1| \gtrsim (\lambda_0) \). Thus we can use bilinear estimate (3.7) to \( u_{-\lambda_0 + I_j^1}u_{\lambda_0 + I_j^1} \). Thus we have

\[
\sum_{j_1,j_2,j_3 \in \log_2(\lambda_0)+1} \lambda_0^{1/4} \int_{[0,T]} \langle u_{\lambda_0 + I_j^1}u_{-\lambda_0 + I_j^2} \partial_x u_{\lambda_0 + I_j^3} \rangle \, dx dt \\
\lesssim \sum_{j_1,j_2,j_3 \in \log_2(\lambda_0)+1} \lambda_0^{5/4} \| u_{\lambda_0 + I_j^1} \|_{L^\infty_t L^2_x} \| u_{-\lambda_0 + I_j^2} \|_{L^2_t L^\infty_x}\| u_{-\lambda_0 + I_j^3} \|_{L^2_t L^\infty_x} \\
\lesssim \sum_{j_1,j_2,j_3 \in \log_2(\lambda_0)+1} \lambda_0^{5/4} \langle \lambda_0 \rangle^{-1}(2^j_1)^{-1/2} \| v_{\lambda_0} \|_{V^2_A} \| u_{-\lambda_0 + I_j^2} \|_{V^2_A} \\
\times T^{e/4} \langle \lambda_0 \rangle^{-1/2 + \varepsilon} (2^j_1)^{1/2} \| v_{\lambda_0} \|_{V^2_A} \| u_{\lambda_0 + I_j^1} \|_{V^2_A}^{3/\lambda_{1/4} A} \\
\lesssim T^{e/4} \| u_{\lambda_0}(\lambda_0) \|_{V^2_A} \| u \|_{X_{1/4,1/4}^A}^{3/\lambda_{1/4} A}. 
\] (4.47)

If \( u_{-\lambda_0 + I_j^1} \) has the highest dispersion modulation, we take \( L^\infty_{x,t}, L^2_{x,t} \) and \( L^2_{x,t} \) norms to \( v_{\lambda_0} \), \( u_{-\lambda_0 + I_j^2} \), and \( u_{-\lambda_0 + I_j^3} \), respectively. Then we can get the desired estimate by using the analogue technique. If \( u_{-\lambda_0 + I_j^1} \) has the highest dispersion modulation, due to the symmetry between \( u_{-\lambda_0 + I_j^1} \) and \( u_{-\lambda_0 + I_j^2} \), the estimate is similar so we omit the details.

If \( u_{\lambda_0 + I_j^1} \) has the highest dispersion modulation, from the dispersion modulation decay (2.4) and \( L^4 \) estimate (3.13), we have

\[
\sum_{j_1,j_2,j_3 \in \log_2(\lambda_0)+1} \langle \lambda_0 \rangle^{1/4} \int_{[0,T]} \langle u_{\lambda_0 + I_j^1}u_{-\lambda_0 + I_j^2} \partial_x u_{\lambda_0 + I_j^3} \rangle \, dx dt \\
\lesssim \sum_{j_1,j_2,j_3 \in \log_2(\lambda_0)+1} \lambda_0^{5/4} \| u_{\lambda_0 + I_j^1} \|_{L^\infty_t L^2_x} \| u_{-\lambda_0 + I_j^2} \|_{L^2_t L^\infty_x}\| u_{-\lambda_0 + I_j^3} \|_{L^2_t L^\infty_x} \\
\lesssim \sum_{j_1,j_2,j_3 \in \log_2(\lambda_0)+1} \lambda_0^{5/4} \langle \lambda_0 \rangle^{-1} (2^j_3)^{-1/4} \| v_{\lambda_0} \|_{V^2_A} \langle \lambda_0 \rangle^{-1/4} \| u \|_{X_{1/4,1/4}^A} \\
\times T^{e/4}(2^j_1)^{1/4+\varepsilon} \langle \lambda_0 \rangle^{-3/8} T^{e/4}(2^j_2)^{1/4+\varepsilon} \langle \lambda_0 \rangle^{-3/8} \| u \|_{X_{1/4,1/4}^A}^{2} \\
\lesssim T^{e/4} \langle \lambda_0 \rangle^{-3/4} (2^j_3)^{1/4+\varepsilon} (2^j_2)^{1/4+\varepsilon} \| v_{\lambda_0} \|_{V^2_A} \| u \|_{X_{1/4,1/4}^A} \\
\lesssim T^{e/2} \langle \lambda_0 \rangle^{-1/4+2\varepsilon} \log_2(\lambda_0) \| v_{\lambda_0} \|_{V^2_A} \| u \|_{X_{1/4,1/4}^A} \\
\lesssim T^{e/2} \| u_{\lambda_0}(\lambda_0) \|_{V^2_A} \| u \|_{X_{1/4,1/4}^A}^{3/\lambda_{1/4} A}. 
\]  

Case 3.3: \( \lambda_2 \in (0, -\lambda_0) \). We consider \( \lambda_1 \in [c\lambda_0, \lambda_0], [0, c\lambda_0], [-\lambda_0, 0] \) and \((0, -\lambda_0)\), respectively. From the frequency constraint condition we can obtain the corresponding range of \( \lambda_3 \) (see Table 3).
When \( 0 \leq j_1 \leq j_2 \approx j_3 \), the summation in above inequality becomes

\[
\sum_{j_3 \geq 0} (\langle \lambda_0 \rangle + 2^{j_3})^{-1/4} (2^{j_3})^{2\varepsilon} \cdot j_3 \lesssim 1.
\]
When $0 \leq j_2 \leq j_1 \approx j_3$, noticing that $j_1 \leq \log_2 \langle \lambda_0 \rangle$, we can know that the summation satisfies

$$\sum_{0 \leq j_2 \leq j_3 \leq \log_2 \langle \lambda_0 \rangle} (\langle \lambda_0 \rangle)^{-1/4} (2^{j_2})^{-1/4+\varepsilon} (2^{j_3})^{1/4+\varepsilon} \cdot j_3 \lesssim 1.$$  \hspace{1cm} (4.50)

If $u_{\lambda_0-I_{j_1}}$ has the highest dispersion modulation, we take $L_{x,t}^\infty$, $L_{x,t}^2$, $L_{x,t}^4$ and $L_{x,t}^4$ norms to $v_{\lambda_0}$, $u_{\lambda_0-I_{j_1}}$, $u_{-\lambda_0-I_{j_2}}$ and $u_{\lambda_0+I_{j_3}}$, respectively. Then we can get the desired estimate by a similar way.

If $u_{-\lambda_0-I_{j_2}}$ has the highest dispersion modulation, we take the dispersion modulation decay estimate to $u_{-\lambda_0-I_{j_2}}$. For $u_{\lambda_0-I_{j_1}}$ and $u_{\lambda_0+I_{j_3}}$, we have $|\lambda_0 + 2^{j_3} - \lambda_0 + 2^{j_1}| \gtrsim 2^{j_3}$ and $|\lambda_0 + 2^{j_3} + \lambda_0 - 2^{j_1}| \gtrsim (\langle \lambda_0 \rangle + 2^{j_3})$. Thus we can use the bilinear estimate \[ \text{(3.7)} \] to $u_{\lambda_0-I_{j_1}} u_{\lambda_0+I_{j_3}}$. Thus we have

$$\sum_{j_1, j_2, j_3 \geq 0} (\langle \lambda_0 \rangle)^{1/4} \int_{[0,T] \times R} |\langle \lambda_0 \rangle u_{\lambda_0-I_{j_1}} u_{-\lambda_0-I_{j_2}} \partial_x u_{\lambda_0+I_{j_3}}| \, dx \, dt$$

$$\lesssim \sum_{j_1, j_2, j_3 \geq 0} (\langle \lambda_0 \rangle)^{1/4} \| \langle \lambda_0 \rangle L_{x,t}^\infty L_{x,t}^\infty \| u_{-\lambda_0-I_{j_2}} \| L_{x,t}^2 \| u_{\lambda_0+I_{j_3}} \| L_{x,t}^2$$

$$\lesssim \sum_{j_1, j_2, j_3 \geq 0} (\langle \lambda_0 \rangle)^{1/4} (\langle \lambda_0 \rangle + 2^{j_3})(\langle \lambda_0 \rangle + 2^{j_2})^{-1/2}(2^{j_1})^{-1/2}(2^{j_3})^{-1/2}(2^{j_2})^{-1/2}\| u_{\lambda_0+I_{j_3}} \| L_{x,t}^2$$

$$\times T^{\varepsilon/4} (\langle \lambda_0 \rangle + 2^{j_3})^{-1/2+\varepsilon} (2^{j_2})^{-1/2+\varepsilon} \| u_{\lambda_0-I_{j_1}} \| L_{x,t}^2 \| u_{\lambda_0+I_{j_3}} \| L_{x,t}^2$$

$$\lesssim T^{\varepsilon/4} \sum_{j_1, j_2, j_3 \geq 0} (\langle \lambda_0 \rangle + 2^{j_3})^{1/4+\varepsilon} (\langle \lambda_0 \rangle + 2^{j_2})^{-3/4}(2^{j_1})^{-1/2+\varepsilon} (2^{j_2})^{1/2}\| u_{\lambda_0+I_{j_3}} \| L_{x,t}^2.$$  \hspace{1cm} (4.51)

If $0 \leq j_1 \leq j_2 \approx j_3$, the summation in above inequality becomes

$$\sum_{j_3 \geq 0} (\langle \lambda_0 \rangle + 2^{j_3})^{-1/2+\varepsilon} (2^{j_3})^\varepsilon \cdot j_3 \lesssim 1.$$  \hspace{1cm} (4.52)

If $0 \leq j_2 \leq j_1 \approx j_3$, recalling that $j_1 \leq \log_2 \langle \lambda_0 \rangle$, we can get the summation satisfying

$$\sum_{0 \leq j_2 \leq j_3 \leq \log_2 \langle \lambda_0 \rangle} (\langle \lambda_0 \rangle)^{-1/2+\varepsilon} (2^{j_3})^{-1/2+\varepsilon} (2^{j_2})^{1/2} \cdot j_3 \lesssim 1.$$  \hspace{1cm} (4.53)

If $u_{\lambda_0+I_{j_3}}$ attains the highest dispersion modulation, noticing that for $u_{\lambda_0-I_{j_1}}$ and $u_{-\lambda_0-I_{j_2}}$, we have $|\lambda_0 + 2^{j_2} - \lambda_0 + 2^{j_1}| \gtrsim 2^{j_2}$ and $|\lambda_0 + 2^{j_2} + \lambda_0 - 2^{j_1}| \gtrsim (\langle \lambda_0 \rangle + 2^{j_2})$. We can use the bilinear estimate \[ \text{(3.7)} \] to $u_{\lambda_0-I_{j_1}} u_{-\lambda_0-I_{j_2}}$ to get our result by using the same way as above.

Case 2h-th. From (FCC) \[ \text{(4.8)} \], we see that $\lambda_2 \in [-\lambda_0 - c\lambda_0 - l, -\lambda_0]$. We decompose $\lambda_1, \lambda_2, \lambda_3$ in a dyadic way:

$$\lambda_1 \in [0, c\lambda_0] = \bigcup_{j_1 \geq 0} I_{j_1}, \quad \lambda_2 \in [-\lambda_0 - c\lambda_0 - l, -\lambda_0] = \bigcup_{j_2 \geq 0} (-\lambda_0 - I_{j_2},$$

$$\lambda_3 \in [2\lambda_0 - c\lambda_0 - l, 2\lambda_0] = \bigcup_{j_3 \geq 0} \lambda_0 + I_{j_3}, \quad j_1, j_2, j_3 \leq \log_2 \langle \lambda_0 \rangle.$$
From the frequency constraint condition (4.8), we have

\[ 2^{j_1} + 2^{j_3} - 2^{j_2} \approx \lambda_0. \]

By DMCC (4.9) the highest dispersion modulation satisfies

\[
\max_{0 \leq k \leq 3} |\xi_k^3 - \tau_k| \gtrsim (\lambda_0)^2 \cdot 2^{j_1}.
\]

Therefore, the approach to this case is similar to Case \( h_- lh \), and we omit it.

Case 2\( h_- lh2 \). We decompose \( \lambda_1, \lambda_2, \lambda_3 \) in the following way:

\[
\lambda_1 \in [0, c\lambda_0] = \bigcup_{j_1 \geq 0} I_{j_1}, \quad j_1 \leq \log_2 (\lambda_0);
\]

\[
\lambda_2 \in [-\infty, -\lambda_0] = \bigcup_{j_2 \geq 0} -\lambda_0 - I_{j_2}, \quad \lambda_3 \in [2\lambda_0, +\infty] = \bigcup_{j_3 \geq 0} 2\lambda_0 + I_{j_3}.
\]

From the frequency constraint condition (4.8), we have

\[
2^{j_2} \approx 2^{j_1} + 2^{j_3}, \quad \text{i.e.} \quad j_2 \approx j_1 \lor j_3.
\]

By DMCC (4.9) the highest dispersion modulation satisfies

\[
\max_{0 \leq k \leq 3} |\xi_k^3 - \tau_k| \gtrsim (\lambda_0)^2 \cdot (\lambda_0 + 2^{j_2}) \cdot (\lambda_0 + 2^{j_3}).
\]

If \( v_{\lambda_0} \) has the highest dispersion modulation decay to \( v_{\lambda_0} \). For \( u_{I_{j_1}} \) and \( u_{2\lambda_0 + I_{j_3}} \), we have \( |2\lambda_0 + 2^{j_3} \pm 2^{j_1}| \gtrsim (\lambda_0 + 2^{j_3}) \). Thus we can use bilinear estimate (3.7) to \( u_{I_{j_1}} u_{2\lambda_0 + I_{j_3}} \). Specifically, we have

\[
\sum_{j_2 = j_1 \lor j_3} (\lambda_0)^{1/4} \int_{[0,T] \times \mathbb{R}} |\tau_{\lambda_0} u_{I_{j_1}} u_{\lambda_0 - I_{j_2}} \partial_x u_{2\lambda_0 + I_{j_3}}| \, dx dt
\]

\[
\lesssim \sum_{j_2 = j_1 \lor j_3} (\lambda_0)^{1/4} (\lambda_0 + 2^{j_3}) \|\tau_{\lambda_0}\|_{L^2_t L^\infty_x} \|u_{\lambda_0 - I_{j_2}}\|_{L^\infty_t L^2_x} \|u_{I_{j_1}} u_{2\lambda_0 + I_{j_3}}\|_{L^2_t L^2_x}
\]

\[
\lesssim \sum_{j_2 = j_1 \lor j_3} (\lambda_0)^{1/4} (\lambda_0 + 2^{j_3})^{-1/2} (\lambda_0 + 2^{j_3})^{-1/2} \|v_{\lambda_0}\|_{V_D^2} \|u_{\lambda_0 - I_{j_2}}\|_{V_D^2}
\]

\[
\times T^{1/4} (\lambda_0 + 2^{j_3})^{-1+2\varepsilon} \|u_{I_{j_1}}\|_{V_D^3} \|u_{2\lambda_0 + I_{j_3}}\|_{V_D^2}
\]

\[
\lesssim T^{1/4} \sum_{j_2 = j_1 \lor j_3} (\lambda_0)^{-1/4} (\lambda_0 + 2^{j_3})^{-3/4+2\varepsilon} (\lambda_0 + 2^{j_3})^{-3/4}
\]

\[
\times (2^{j_1})^{1/4} (2^{j_3})^{1/2} (2^{j_3})^{1/2} \|v_{\lambda_0}\|_{V_D^2} \|u\|_{X_{\infty,A}^{3/4}}^3
\]

\[
\lesssim T^{1/4} \|v_{\lambda_0}\|_{V_D^2} \|u\|_{X_{\infty,A}^{3/4}}^3,
\]

where the last inequality is by summing over \( j_1, j_2 \) and \( j_3 \). Indeed we have the following estimates:

\[
\sum_{j_1 \leq \log_2 (\lambda_0)} (2^{j_1})^{1/4} \lesssim (\lambda_0)^{1/4}; \quad \sum_{j_2 \geq 0} (\lambda_0 + 2^{j_3})^{-3/4}(2^{j_3})^{1/2} \lesssim \sum_{j_2 \geq 0} (2^{j_3})^{-1/4} \lesssim 1;
\]

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\[
\sum_{j_3 \geq 0} \left( (\lambda_0) + 2^{j_3} \right)^{-3/4 + 2\varepsilon} \left( 2^{j_3} \right)^{1/2} \lesssim \sum_{j_3 \geq 0} \left( 2^{j_3} \right)^{-1/4 + 2\varepsilon} \lesssim 1, \quad 0 < \varepsilon < 1/8.
\]

If \( u_{-\lambda_0 - I_{j_2}} \) has the highest dispersion modulation, we take \( L^\infty_{x,t}, L^2_{x,t} \) and \( L^2_{x,t} \) norms to \( v_{\lambda_0}, u_{-\lambda_0 - I_{j_2}} \) and \( u_{I_{j_1}} u_{2\lambda_0 + I_{j_3}} \), respectively. Then it will be same with (4.53).

If \( u_{2\lambda_0 + I_{j_3}} \) has the highest dispersion modulation, we take dispersion modulation decay to \( u_{2\lambda_0 + I_{j_3}} \). For \( u_{I_{j_1}} \) and \( u_{-\lambda_0 - I_{j_2}} \), we have \( |\lambda_0 + 2^{j_2} + 2^{j_3}| \gtrsim (\lambda_0) + 2^{j_2} \) and \( |\lambda_0 + 2^{j_2} - 2^{j_3}| \gtrsim 2^{j_2} \). Thus we can use bilinear estimate (3.7) to \( u_{I_{j_1}} u_{-\lambda_0 - I_{j_3}} \). To be specific, we have

\[
\sum_{j_2 \approx j_1 \land j_3} \langle \lambda_0 \rangle^{1/4} \int_{[0, T] \times \mathbb{R}} |v_{\lambda_0} u_{I_{j_1}} u_{-\lambda_0 - I_{j_2}} \partial_x u_{2\lambda_0 + I_{j_3}}| \ dx dt \\
\lesssim \sum_{j_2 \approx j_1 \land j_3} \langle \lambda_0 \rangle^{1/4} (\langle \lambda_0 \rangle + 2^{j_3}) \|v_{\lambda_0}\|_{L^\infty_{x,t}} \|u_{2\lambda_0 + I_{j_3}}\|_{L^2_{x,t}} \|u_{I_{j_1}} u_{-\lambda_0 - I_{j_2}} u_{2\lambda_0 + I_{j_3}}\|_{L^2_{x,t}} \\
\lesssim \sum_{j_2 \approx j_1 \land j_3} \langle \lambda_0 \rangle^{1/4} \|v_{\lambda_0}\|_{L^2_{x,t}} (\langle \lambda_0 \rangle + 2^{j_3})^{-1/2} (\langle \lambda_0 \rangle + 2^{j_2})^{-1/2} (\langle \lambda_0 \rangle + 2^{j_3})^{1/2} \|u_{2\lambda_0 + I_{j_3}}\|_{L^2_{x,t}} \\
\times T^{\varepsilon/4} (\langle \lambda_0 \rangle + 2^{j_2})^{-1/2 + \varepsilon} (2^{j_2})^{-1/2 + \varepsilon} \|u_{I_{j_1}}\|_{L^\infty_{x,t}} \|u_{-\lambda_0 - I_{j_2}}\|_{L^\infty_{x,t}} V_A^2 \\
\lesssim T^{\varepsilon/4} \|v_{\lambda_0}\|_{L^2_{x,t}} \|u\|_{X_{\infty,A}^{1/4}}^3,
\]

where the last inequality is by summing over \( j_1, j_2 \) and \( j_3 \) in order.

If \( u_{I_{j_1}} \) has the highest dispersion modulation, from the dispersion modulation decay (2.4) and \( L^4 \) estimate (3.13), we have

\[
\sum_{j_2 \approx j_1 \land j_3} \langle \lambda_0 \rangle^{1/4} \int_{[0, T] \times \mathbb{R}} |v_{\lambda_0} u_{I_{j_1}} u_{-\lambda_0 - I_{j_2}} \partial_x u_{2\lambda_0 + I_{j_3}}| \ dx dt \\
\lesssim \sum_{j_2 \approx j_1 \land j_3} \langle \lambda_0 \rangle^{1/4} (\langle \lambda_0 \rangle + 2^{j_3}) \|v_{\lambda_0}\|_{L^\infty_{x,t}} \|u_{I_{j_1}}\|_{L^2_{x,t}} \|u_{-\lambda_0 - I_{j_2}}\|_{L^\infty_{x,t}} \|u_{2\lambda_0 + I_{j_3}}\|_{L^2_{x,t}} \\
\lesssim \sum_{j_2 \approx j_1 \land j_3} \langle \lambda_0 \rangle^{1/4} \|v_{\lambda_0}\|_{L^2_{x,t}} (\langle \lambda_0 \rangle + 2^{j_3})^{-1/2} (\langle \lambda_0 \rangle + 2^{j_2})^{-1/2} (\langle \lambda_0 \rangle + 2^{j_3})^{1/2} \|u\|_{X_{\infty,A}^{1/4}} \\
\times T^{\varepsilon/4} (2^{j_2})^{1/4 + \varepsilon} (\langle \lambda_0 \rangle + 2^{j_2})^{-3/8} T^{\varepsilon/4} (2^{j_3})^{1/4 + \varepsilon} (\langle \lambda_0 \rangle + 2^{j_3})^{-3/8} \|u\|_{X_{\infty,A}^{1/4}}^2 \\
\lesssim T^{\varepsilon/2} \sum_{j_2 \approx j_1 \land j_3} \langle \lambda_0 \rangle^{1/4} (\langle \lambda_0 \rangle + 2^{j_2})^{-1/8} (\langle \lambda_0 \rangle + 2^{j_2})^{7/8} \\
\times (2^{j_2})^{1/4 + \varepsilon} (2^{j_3})^{1/4 + \varepsilon} \|v_{\lambda_0}\|_{L^2_{x,t}} \|u\|_{X_{\infty,A}^{1/4}}^3 \\
\lesssim T^{\varepsilon/2} \|v_{\lambda_0}\|_{V_A^2} \|u\|_{X_{\infty,A}^{1/4}}^3,
\]

where the last inequality is by summing over \( j_1, j_2 \) and \( j_3 \) in order and noticing the condition \( j_1 \lesssim \log_2(\langle \lambda_0 \rangle) \), \( j_3 \lesssim j_2 \).
Case 2$h_\ldots h$. We decompose $\lambda_1, \lambda_2, \lambda_3$ as follows:

$$
\lambda_1 \in [-\lambda_0, 0] = \bigcup_{j_1 \geq 0} - I_{j_1}, \quad j_1 \leq \log_2(\lambda_0);
$$

$$
\lambda_2 \in [-\infty, -\lambda_0] = \bigcup_{j_2 > 0} - \lambda_0 - I_{j_2}, \quad \lambda_3 \in [2\lambda_0 - l, +\infty] = \bigcup_{j_3 \geq -1} 2\lambda_0 + I_{j_3}.
$$

From the frequency constraint condition (4.8), we have

$$
2^{j_3} \approx 2^{j_1} + 2^{j_2}, \quad \text{i.e.} \quad j_3 \approx j_1 \lor j_2.
$$

(4.54)

If $j_3 \approx j_2 \geq j_1$, the method of this case will be same with Case 2$h_\ldots h2$. If $j_3 \approx j_1 \geq j_2$, it is to say that $0 \leq j_2 \leq j_3$ holds, which can also ensure the convergence of the summation in Case 2$h_\ldots h2$. Therefore, the details are omitted.

Case 2$h_\ldots h$. We decompose $\lambda_1, \lambda_2, \lambda_3$ in the following way:

$$
\lambda_k \in [-\infty, -\lambda_0] = \bigcup_{j_k \geq 0} - \lambda_0 - I_{j_k}, \quad k = 1, 2; \quad \lambda_3 \in [3\lambda_0 - l, +\infty] = \bigcup_{j_3 \geq -1} 3\lambda_0 + I_{j_3}.
$$

From the frequency constraint condition (4.8) and $\lambda_1 \geq \lambda_2$, we have

$$
2^{j_3} \approx 2^{j_1} + 2^{j_2}, \quad j_1 \leq j_2, \quad \text{i.e.} \quad j_3 \approx j_2 \geq j_1.
$$

(4.55)

By DMCC (4.9) the highest dispersion modulation satisfies

$$
\max_{0 \leq k \leq 3} |\xi_k^3 - \tau_k| \gtrsim (\langle \lambda_0 \rangle + 2^{j_3}) \cdot (\langle \lambda_0 \rangle + 2^{j_3}) \cdot (\langle \lambda_0 \rangle + 2^{j_3}).
$$

(4.56)

If $\nu_{\lambda_0}$ has the highest dispersion modulation, from the dispersion modulation decay (4.4), $L^4$ estimate (3.11), and Lemma 3.21, we have

$$
\sum_{j_1 \approx j_2 \geq j_1} \langle \lambda_0 \rangle^{1/4} \int_{[0,T] \times \mathbb{R}^d} |\nu_{\lambda_0} u_{\lambda_0 - I_{j_1}} u_{\lambda_0 - I_{j_2}} \partial_x u_{3\lambda_0 + I_{j_3}}| \, dxdt
\lesssim \sum_{j_3 \approx j_2 \geq j_1} \langle \lambda_0 \rangle^{1/4} \langle \lambda_0 \rangle + 2^{j_3} \rangle^{1/4} \|\nu_{\lambda_0} \|_{L^2_t L^\infty_x} \|u_{\lambda_0 - I_{j_1}} \|_{L^\infty_t L^2_x} \|u_{\lambda_0 - I_{j_2}} \|_{L^\infty_t L^2_x} \|u_{3\lambda_0 + I_{j_3}} \|_{L^4_t L^4_x}
\lesssim \sum_{j_3 \approx j_2 \geq j_1} \langle \lambda_0 \rangle^{1/4} \langle \lambda_0 \rangle + 2^{j_3} \rangle^{1/4} \langle \lambda_0 \rangle + 2^{j_3} \rangle^{1/4} \langle \lambda_0 \rangle + 2^{j_3} \rangle^{1/4} \|\nu_{\lambda_0} \|_{L^2(2^{j_3})^{1/2}}
\times (\langle \lambda_0 \rangle + 2^{j_3} \rangle)^{-1/4} T^{\varepsilon/2} (2^{j_3})^{1/4} \langle \lambda_0 \rangle + 2^{j_3} \rangle^{-3/8} (2^{j_3})^{1/4} \langle \lambda_0 \rangle + 2^{j_3} \rangle^{-3/8} \|u\|_{X_{\infty,A}}^{3/4}
\lesssim T^{\varepsilon/2} \sum_{j_3 \approx j_2 \geq j_1} \langle \lambda_0 \rangle^{1/4} \langle \lambda_0 \rangle + 2^{j_3} \rangle^{3/4} (2^{j_3})^{1/2} + \langle \lambda_0 \rangle + 2^{j_3} \rangle^{1/2} \|\nu_{\lambda_0} \|_{L^2} \|u\|_{X_{\infty,A}}^{3/4}
\lesssim T^{\varepsilon/2} \left( \sum_{j_3 \approx j_2 \geq j_1} (2^{j_3})^{1/4} \|\nu_{\lambda_0} \|_{L^2_x} \|u\|_{X_{\infty,A}}^{3/4} \right)
\lesssim T^{\varepsilon/2} \|\nu_{\lambda_0} \|_{L^2} \|u\|_{X_{\infty,A}}^{3/4}.
$$

(4.57)
If \( u_{-\lambda_0-I_j} \) has the highest dispersion modulation, we take \( L^\infty_{x,t}, L^2_{x,t}, L^4_{x,t} \) and \( L^4_{x,t} \) norms to \( v_{\lambda_0}; u_{-\lambda_0-I_j}, u_{-\lambda_0-I_{j_2}} \) and \( u_{3\lambda_0+I_{j_3}} \), respectively. Then it will be same with (4.57).

If \( u_{3\lambda_0+I_{j_3}} \) has the highest dispersion modulation, we divide the left hand side of (4.7) into two terms.

\[
\sum_{j_3 \approx j_2 \gtrsim j_1} \langle \lambda_0 \rangle^{1/4} \int_{[0,T] \times \mathbb{R}} \left| v_{\lambda_0} u_{-\lambda_0-I_j} u_{-\lambda_0-I_{j_2}} \partial_x u_{3\lambda_0+I_{j_3}} \right| \, dx \, dt 
\leq \left( \sum_{j_3 \approx j_2 \gtrsim j_1} + \sum_{j_3 \approx j_2 \gtrsim j_1 \gtrsim 0} \right) \langle \lambda_0 \rangle^{1/4} \int_{[0,T] \times \mathbb{R}} \left| v_{\lambda_0} u_{-\lambda_0-I_j} u_{-\lambda_0-I_{j_2}} \partial_x u_{3\lambda_0+I_{j_3}} \right| \, dx \, dt
\]

\[
:= I_1(u, v) + I_2(u, v). 
\]

For \( I_1(u, v) \), from the dispersion modulation decay (2.31), \( L^4 \) estimate (3.13) and Lemma 3.4, we have

\[
I_1(u, v) \lesssim \sum_{j_3 \approx j_2 \gtrsim j_1 \gtrsim 0} \langle \lambda_0 \rangle^{1/4} \langle \langle \lambda_0 \rangle + 2^{j_3} \rangle \| v_{\lambda_0} \|_{L^\infty_{x,t}} \| u_{3\lambda_0+I_{j_3}} \|_{L^2_{x,t}} \| u_{-\lambda_0-I_{j_1}} \|_{L^4_{x,t}} \| u_{-\lambda_0-I_{j_2}} \|_{L^4_{x,t}} 
\]

\[
\lesssim \sum_{j_3 \approx j_2 \gtrsim j_1 \gtrsim 0} \langle \lambda_0 \rangle^{1/4} \| v_{\lambda_0} \|_{V^2_A} \langle \langle \lambda_0 \rangle + 2^{j_3} \rangle^{-1/2} (\langle \lambda_0 \rangle + 2^{j_3})^{-1/2} (\langle \lambda_0 \rangle + 2^{j_3})^{1/4} (2^{j_3})^{1/2} \times T^{\varepsilon/2} (2^{j_3})^{1/4+\varepsilon} (\langle \lambda_0 \rangle + 2^{j_3})^{-3/8} (2^{j_3})^{1/4+\varepsilon} (\lambda_0) + 2^{j_3})^{-3/8} \| u \|_{X^{14/4}_{\infty,A}}^{3/4} 
\]

\[
\lesssim T^{\varepsilon/2} \sum_{j_3 \gtrsim 0} \langle \lambda_0 \rangle^{1/4} \langle \langle \lambda_0 \rangle + 2^{j_3} \rangle^{-3/2} (2^{j_3})^{1/2+2\varepsilon} \| v_{\lambda_0} \|_{V^2_A} \| u \|_{X^{14/4}_{\infty,A}}^{3/4} 
\]

\[
\lesssim T^{\varepsilon/2} \| u^{(\lambda_0)} \|_{V^2_A} \| u \|_{X^{14/4}_{\infty,A}}^{3/4}. 
\]

For \( I_2(u, v) \), due to \( j_2 \gg j_1 \), we have \( | -\lambda_0 - 2^{j_2} - \lambda_0 - 2^{j_3} | \gtrsim (\langle \lambda_0 \rangle + 2^{j_3}) \) and \( | -\lambda_0 - 2^{j_2} + \lambda_0 + 2^{j_1} | \gtrsim 2^{j_2} \), so we can use bilinear estimate (3.7) to \( u_{-\lambda_0-I_{j_1}} \) and \( u_{-\lambda_0-I_{j_2}} \). To be specific,

\[
I_2(u, v) \lesssim \sum_{j_3 \gtrsim j_2 \gtrsim j_1 \gtrsim 0} \langle \lambda_0 \rangle^{1/4} \langle \langle \lambda_0 \rangle + 2^{j_3} \rangle \| v_{\lambda_0} \|_{V^2_A} \langle \langle \lambda_0 \rangle + 2^{j_3} \rangle^{-1/2} (\langle \lambda_0 \rangle + 2^{j_3})^{-1/2} (\langle \lambda_0 \rangle + 2^{j_3})^{1/2} \times (2^{j_3})^{1/2} (\lambda_0) + 2^{j_3})^{-1/4} | u |_{X^{14/4}_{\infty,A}} T^{\varepsilon/4} (2^{j_3})^{-1/2+\varepsilon} (\langle \lambda_0 \rangle + 2^{j_3})^{-1/2+\varepsilon} 
\]

\[
\times (2^{j_3})^{1/2} (\langle \lambda_0 \rangle + 2^{j_3})^{-1/4} (2^{j_2})^{1/2} (\langle \lambda_0 \rangle + 2^{j_2})^{-1/4} | u |_{X^{14/4}_{\infty,A}}^{2} 
\]

\[
\lesssim T^{\varepsilon/4} \sum_{j_3 \approx j_2 \gtrsim j_1 \gtrsim 0} \langle \lambda_0 \rangle^{1/4} \langle \langle \lambda_0 \rangle + 2^{j_3} \rangle^{-1+\varepsilon} (2^{j_3})^{1/2+\varepsilon} (\lambda_0) + 2^{j_3})^{-1/4} \| v_{\lambda_0} \|_{V^2_A} \| u \|_{X^{14/4}_{\infty,A}}^{3} 
\]

\[
\lesssim T^{\varepsilon/4} \left( \sum_{j_3 \approx j_2 \gtrsim j_1 \gtrsim 0} \langle \lambda_0 \rangle^{-1/2+2\varepsilon} \cdot j_3 \right) \| v_{\lambda_0} \|_{V^2_A} \| u \|_{X^{14/4}_{\infty,A}}^{3}. 
\]
If \( u - \lambda_0 - I_{j_2} \) has the highest dispersion modulation, we still divide the left hand side of (4.48) into two terms as (4.58). For \( I_1(u, v) \), because of \( j_3 \approx j_2 \approx j_1 \), the estimate is exactly same with (4.59). For \( I_2(u, v) \), we use the bilinear estimate (3.7) to \( u - \lambda_0 - I_{j_1} \) and \( u \lambda_0 + I_{j_3} \). Noticing \( |3\lambda_0 + 2j_2 \pm (\lambda_0 + 2j_2)| \gtrsim (\lambda_0 + 2j_2) \), we have

\[
I_2(u, v) \lesssim \sum_{j_3 \approx j_2 > j_1 \geq 0} (\lambda_0)^{1/4}(\lambda_0) + 2j_3) \|v\|_\infty \|u - \lambda_0 - I_{j_2}\|_{L^2_{x,t}} \|u - \lambda_0 - I_{j_1} u \lambda_0 + I_{j_3}\|_{L^2_{x,t}}
\]

\[
\lesssim \sum_{j_3 \approx j_2 > j_1 \geq 0} (\lambda_0)^{1/4} v_\lambda \|\lambda_0 + 2j_2\|^{-1/2}(\lambda_0) + 2j_2\|^{-1/2}(\lambda_0) + 2j_2) \|/4\|_{X_{\infty,A}^1} \|u\|_{X_{\infty,A}^1/4}^2
\]

\[
\lesssim T^{\varepsilon/4} \sum_{j_3 \approx j_2 > j_1 \geq 0} (\lambda_0)^{1/4}(\lambda_0) + 2j_3\|^{-3/2 + 2\varepsilon}2j_3(\lambda_0) + 2j_2\|^{-3/4}(2j_2)^{1/2}\|v\|_{V_{\infty,A}^1} \|u\|_{X_{\infty,A}^1/4}^3
\]

\[
\lesssim T^{\varepsilon/4} \sum_{j_3 \approx j_2 > j_1 \geq 0} (\lambda_0)^{1/4} v_\lambda \|v\|_{V_{\infty,A}^1} \|u\|_{X_{\infty,A}^1/4}^3
\]

\[
\lesssim T^{\varepsilon/4} \|v\|_{V_{\infty,A}^1} \|u\|_{X_{\infty,A}^1/4}^3.
\]

4.2 \( q < \infty, \) Proof of (4.6).

This subsection \( q < \infty \) is similar to the last subsection \( q = \infty \), the only difference is to deal with the summation of \( \lambda_0 \). The frequency constraint condition (FCC) and dispersion modulation constraint condition (DMCC) are same. Thus, we can use the exactly same assortment to \( \lambda_0, \ldots, \lambda_3 \). Next we take the Case 1 of Step 1 in last subsection for example.

We just denote the left hand side of (4.6) as \( \mathcal{L}_{hhhh} \), and divide it into three parts like the last subsection. For \( \mathcal{L}^l_{hhhh} \), \( \lambda_0 \approx \lambda_3 \approx \lambda_1 \approx -\lambda_2 \) holds. Thus, From Hölder inequality and Strichartz estimate, we have

\[
\mathcal{L}^l_{hhhh}(u, v) \lesssim \sum_{\lambda_0} (\lambda_0)^{1/4} \|v\|_{V_{\infty,A}^1} \|u\|_{X_{\infty,A}^1/4}^3
\]

\[
\lesssim T^{1/2} \sum_{\lambda_0} (\lambda_0)^{1/4} \|v\|_{X_{\infty,A}^1/4} \|u\|_{X_{\infty,A}^1/4}^3
\]

\[
\lesssim T^{1/2} \sum_{\lambda_0} (\lambda_0)^{1/4} \|v\|_{X_{\infty,A}^1/4} \|u\|_{X_{\infty,A}^1/4}^3
\]

\[
\lesssim T^{1/2} \|v\|_{V_{\infty,A}^1} \|u\|_{X_{\infty,A}^1/4}^3.
\]

For \( \mathcal{L}^m_{hhhh} \), we still use bilinear estimate (3.7), Lemma 3.4 and Hölder’s inequality to obtain that for \( 0 < \varepsilon < 1/4q \),

\[
\mathcal{L}^m_{hhhh}(u, v) \lesssim \sum_{\lambda_0, j_3 \leq j_2 < j_1} (\lambda_0)^{1/4} \|v\|_{V_{\infty,A}^1} \|u\|_{X_{\infty,A}^1/4}^3
\]

\[
\lesssim T^{1/2} \|v\|_{V_{\infty,A}^1} \|u\|_{X_{\infty,A}^1/4}^3.
\]
\[ \sum_{q,A} \langle \lambda \rangle^{5/4} \langle \lambda \rangle^{-1/2+\varepsilon} \langle 2^j \rangle^{-1/2+\varepsilon} \|v\lambda_0\|_{V}^2 \|u\lambda_0-I_j\|_{L^2_{x,t}}^2 \]
\[ \times \langle \lambda \rangle^{-1/2+\varepsilon} \langle 2^j \rangle^{-1/2+\varepsilon} \|u_{\lambda_0-I_1}\|_{V}^2 \|u\lambda_0-I_3\|_{V}^2 \]
\[ \lesssim T^{\varepsilon/2} \sum_{\lambda_0,j_1,j_2} \langle \lambda \rangle^{-1/2+2\varepsilon} \langle 2^j \rangle^{-1/2} \|u\lambda_0-I_{j_1}\|_{V}^2 \|u\lambda_0-I_{j_2}\|_{V}^2 \|u\lambda_0-I_{j_3}\|_{V}^2 \]
\[ \times \|v\lambda_0\|_{V} \|u\|_{X^{1/4,q}_{q,A}(\lambda_0-I_{j_3})}^2 \|u\|_{X^{1/4,q}_{q,A}}^2. \]

Making the summation on \( j_1, j_2 \), then applying Hölder’s inequality on \( \lambda_0 \), and finally summing on \( j_3 \), we obtain
\[ \mathcal{L}^h_{hhhh_+}(u, v) \lesssim T^{\varepsilon/2} \sum_{\lambda_0,j_3} \langle \lambda \rangle^{5/4} \langle \lambda \rangle^{-1/2+\varepsilon} \langle 2^j \rangle^{-1/2+\varepsilon} \|v\lambda_0\|_{V}^2 \|u\lambda_0-I_{j_3}\|_{V}^2 \]
\[ \|u\|_{X^{1/4,q}_{q,A}(\lambda_0-I_{j_3})}^2 \|u\|_{X^{1/4,q}_{q,A}}^2. \]

For \( \mathcal{L}^h_{hhhh_+}(u, v) \), we just take the case \( v\lambda_0 \) has the highest dispersion modulation for example and divide \( \mathcal{L}^h_{hhhh_+}(u, v) \) into two parts:
\[ \mathcal{L}^h_{hhhh_+}(u, v) = \mathcal{L}^h_{hhhh_+}^{1}(u, v) + \mathcal{L}^h_{hhhh_+}^{2}(u, v) \]
\[ := \left( \sum_{\lambda_0,1 \leq j_3 \leq j_1 \geq j_2} \sum_{\lambda_0,1 \leq j_3 \leq j_1 \geq j_2} \langle \lambda \rangle^{1/4} \int_{[0,T] \times \mathbb{R}} \left| v\lambda_0 u_{\lambda_0-I_{j_1}} u_{\lambda_0-I_{j_2}} \partial_x u_{\lambda_0-I_{j_3}} \right| dx dt. \]

For \( \mathcal{L}^h_{hhhh_+}^{1}(u, v) \), we have for \( 0 < \varepsilon < 1/4q \),
\[ \mathcal{L}^h_{hhhh_+}^{1}(u, v) \lesssim \sum_{\lambda_0,1 \leq j_3 \leq j_1 \geq j_2} \langle \lambda \rangle^{5/4} \langle \lambda \rangle^{-1/2+(2j_1)^{-1/2}+(2j_3)^{-1/2}} \|v\lambda_0\|_{V}^2 \|u\lambda_0-I_{j_1}\|_{V}^2 \]
\[ \times T^{\varepsilon/4} \langle \lambda \rangle^{-1/2+\varepsilon} \langle 2^j \rangle^{-1/2} \|u_{\lambda_0-I_{j_1}}\|_{V}^2 \|u\lambda_0-I_{j_2}\|_{V}^2 \]
\[ \lesssim T^{\varepsilon/4} \sum_{\lambda_0,1 \leq j_3 \leq j_1 \geq j_2} \langle \lambda \rangle^{-1/2+\varepsilon} \langle 2^j \rangle^{-1/2} \|u\lambda_0\|_{V}^2 \|u\lambda_0-I_{j_3}\|_{V}^2 \]
\[ \|u\|_{X^{1/4,q}_{q,A}(\lambda_0-I_{j_3})}^2 \|u\|_{X^{1/4,q}_{q,A}}^2 \]
\[ \lesssim T^{\varepsilon/4} \|v\|_{V} \|u\|_{X^{1/4,q}_{q,A}}^2 \]

For \( \mathcal{L}^h_{hhhh_+}^{2}(u, v) \), we know that for \( 0 < \varepsilon < 1/4q \),
\[ \mathcal{L}^h_{hhhh_+}^{2}(u, v) \lesssim \sum_{\lambda_0,1 \leq j_3 \leq j_1 \geq j_2} \langle \lambda \rangle^{5/4} \langle \lambda \rangle^{-1/2+(2j_1)^{-1/2}+(2j_3)^{-1/2}} \|v\lambda_0\|_{V}^2 \|u\lambda_0-I_{j_1}\|_{V}^2 \]
\[ \times T^{\varepsilon/2} \langle \lambda \rangle^{-3/4+(2j_2)\varepsilon} \langle 2^j \rangle^{-1/4-1/q+\varepsilon} \|u\|_{X^{1/4,q}_{q,A}}^2 \]

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\[
\lesssim T^{\varepsilon/2} \sum_{\lambda_0, j_1} (2^{j_1})^{\varepsilon - 3/4} \|v_{\lambda_0}\|_V \|u\|_{X_{q,A}^{1/4}} \|u\|_{X_{q,A}^{1/4}}^2
\]
\[
\lesssim T^{\varepsilon/2} \|v\|_V^{\varepsilon} \|v\|_{X_{q,A}^{1/4}}.
\]
Where the last inequality is by applying Hölder’s inequality on \(\lambda_0\). For other cases, we can take the similar calculation to get the desired estimates, thus we omit it.

## 5 Ill-posedness result

In this section we study the Cauchy problem of the defocusing mKdV equation (the focusing case can also be treated by our method):

\[
u_t + u_{xxx} - (u^3)_x = 0, \quad u(0, x) = \delta u_0.
\]

We have the ill-posedness result as follows.

**Theorem 5.1** Let \(s < 1/4\), \(2 \leq q \leq \infty\), \(0 < \delta \ll 1\). Then for the mKdV equation (5.1), the solution map \(\delta u_0 \to u(\delta, t)\) in \(M_{s,q}^2\) is not \(C^3\) continuous at the origin.

**Proof.** From (4.1) we can define the solution map as follows:

\[
\mathcal{T} : \delta u_0 \to u(\delta, t) = e^{-\partial_3^2} \delta u_0 + \int_0^t e^{-(t-\tau)\partial_3^2} (u^3)_x(\tau) d\tau.
\]

By straightforward calculations, we get

\[
u(\delta, t)|_{\delta=0} = 0; \quad u_1 := \frac{\partial u}{\partial \delta}|_{\delta=0} = e^{-\partial_3^2} u_0; \quad u_2 := \frac{\partial^2 u}{\partial \delta^2}|_{\delta=0} = 0; \quad u_3 := \frac{\partial^3 u}{\partial \delta^3}|_{\delta=0} = 6 \int_0^t e^{-(t-\tau)\partial_3^2} \partial_3 (e^{-\partial_3^2} u_0)^3 d\tau.
\]

It is well known that if the map \(\delta u_0 \to u(\delta)\) is of class \(C^3\) at the origin, the necessary condition is

\[
sup_{t \in [0,T]} \|u_3\|_{M_{s,q}^2} \leq C\|u_0\|_{M_{s,q}^2}^3.
\]

We choose a suitable \(u_0 \in M_{s,q}^2\), \(s < 1/4\) defined by

\[
\hat{u}_0(\xi) = N^{-s+1/4} \left( \chi_{[N,N+\frac{1}{\sqrt{N}}]}(\xi) + \chi_{[-N-\frac{1}{\sqrt{N}},-N]}(\xi) \right).
\]

Note that \(\|u_0\|_{M_{s,q}^2} \sim 1\).

We estimate the Fourier transform of \(u_3\) in (5.4) as follows

\[
\hat{u}_3(\xi) \simeq \int_0^t e^{i(t-\tau)\xi^3}(i\xi) \int_{\mathbb{R}^2} e^{i\xi_1(\xi - \xi_1 - \xi_2)} \hat{u}_0(\xi_1 - \xi_2) e^{i\xi_1 \hat{u}_0(\xi_2)} d\xi_1 d\xi_2 d\tau.
\]
\[\begin{align*}
\zeta e^{\xi^3} (\xi) \int_{\mathbb{R}^2} e^{i\Phi(\xi, \xi_1, \xi_2)} dt_0 (\xi - \xi_1 - \xi_2) \mu(\xi_1) \mu_0(\xi_2) d\xi_1 d\xi_2 \\
\zeta e^{\xi^3} (\xi) \int_{\mathbb{R}^2} e^{i\Phi(\xi, \xi_1, \xi_2)} - \frac{1}{16 \Phi(\xi, \xi_1, \xi_2)} \mu_0(\xi - \xi_1 - \xi_2) \mu(\xi_1) \mu_0(\xi_2) d\xi_1 d\xi_2,
\end{align*}\]

where \(\Phi(\xi, \xi_1, \xi_2) = -3(\xi - \xi_1)(\xi - \xi_2)(\xi_1 + \xi_2)\). Noticing that for \(\xi - \xi_1 - \xi_2, \xi_1, \text{ and } \xi_2\), if one or two items among them are located in \([N, N + 1/\sqrt{N}]\), we have \(|\Phi(\xi, \xi_1, \xi_2)| \lesssim 1\); if all three items are located in \([N, N + 1/\sqrt{N}]\) (or \([-N - 1/\sqrt{N}, -N]\)), we have \(|\Phi(\xi, \xi_1, \xi_2)| \sim N^3\), then \(5.6\) shall be much smaller. Therefore,

\[
\begin{align*}
\tilde{u}_3(\xi) \sim N^{-3s + 3/4} e^{\xi^3} (\xi) \int_{\mathbb{R}^2} e^{i\Phi(\xi, \xi_1, \xi_2)} - \frac{1}{16 \Phi(\xi, \xi_1, \xi_2)} \\
\times \chi_{[N, N + \frac{1}{\sqrt{N}}]}(\xi - \xi_1 - \xi_2) \chi_{[N, N + \frac{1}{\sqrt{N}}]}(\xi_1) \chi_{[-N - \frac{1}{\sqrt{N}}, -N]}(\xi_2) d\xi_1 d\xi_2 \\
\sim N^{-3s + 3/4} e^{\xi^3} (\xi) \int_{N}^{N + \frac{1}{\sqrt{N}}} \int_{-N - \frac{1}{\sqrt{N}}}^{-N} t e^{i\theta} \cdot \chi_{[N, N + \frac{1}{\sqrt{N}}]}(\xi - \xi_1 - \xi_2) d\xi_1 d\xi_2,
\end{align*}\]

where \(\theta \in [0, \Phi]\) or \([\Phi, 0]\), \(|\Phi(\xi, \xi_1, \xi_2)| = O(1), \xi \in [N - 1/\sqrt{N}, N + 2/\sqrt{N}]\). Thus there exists a small and fixed constant \(t\) such that

\[\|u_3\|_{M^2_t} \geq C N^{-2s + 1/2} \quad (2 \leq q \leq \infty).\]

We find that \(5.5\) leads to

\[-2s + 1/2 \leq 0 \quad \text{i.e.} \quad s \geq 1/4.\]

Now we complete the proof. \(\Box\)

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