A rapid restoration of the bath state is usually required to induce Markovian dynamics for an open quantum system, which typically can be realized only in limits such as weak system-bath coupling and infinitely large bath. In this work, we investigate the Markovianity of a qubit system coupled to a single-qubit bath with the qubit bath being continuously refreshed by quantum cooling. A surprising result is that there exists a finite threshold for the cooling rate at which the system transitions from non-Markovian dynamics to Markovian dynamics, which is in sharp contrast to the usual understanding that Markovian dynamics is an asymptotic behavior under the Born-Markov approximation. We also study the time correlation of the bath, and find that the decay rate of bath time correlation is of the same order as the system evolution speed. This suggests that quantum Markovian dynamics can exist beyond the usual short bath correlation limit.

I. INTRODUCTION

The dynamics of an open quantum system is very different from the unitary evolution of a closed quantum system, as it may lose information from the system to the environment and become physically irreversible. Such dynamics cannot be simply described by the ordinary Schrödinger equation, and needs more sophisticated tools such as the Feynman-Vernon influence functional [1]. The interest in open quantum systems has grown more intensive in recent years with the development of quantum information, as it is important to learn how quantum technologies can work in real environments.

Regardless of the physical details, the dynamics of open quantum systems can be roughly divided into two categories based on the memory effect of the bath: Markovian and non-Markovian dynamics. If the bath correlation time is much shorter than the time scale of the system evolution, the bath has almost no memory effect and the state of the system at any time is determined by its state at the immediate previous time step, and the evolution of the system can be effectively described by a dynamical semigroup [2–4]. This kind of open system dynamics is called Markovian. By contrast, if the bath correlation is enduring and lasts for a time comparable to the time scale of the system evolution, the instantaneous state of the system is determined by the entire history of the system evolution rather than its immediate predecessor, and the dynamics of the system do not form a semigroup generally. This kind of open system dynamics is called non-Markovian.

While Markovian quantum dynamics has been studied with various approximations for a long time, the exact quantification or even definition of Markovianity for quantum dynamics is not easy. Two main approaches to defining the Markovianity of quantum dynamics currently are completely positive (CP) divisibility [5–7] and distinguishability of quantum states [8, 9]. The CP-divisibility approach defines Markovian quantum dynamics as those which can be decomposed into a sequence of completely positive maps, representing the evolution for successive, arbitrarily-chosen intervals of time. The distinguishability of quantum states approach defines Markovian quantum dynamics as those under which the trace distance between two arbitrary quantum states always decreases with time, which can be interpreted as irreversible loss of information from the system into the environment. A notable recent advance is that these two approaches were proven to be equivalent, first for quantum states in special Hilbert spaces [10] and then universally [11, 12]. Meanwhile, many different measures have been proposed to witness or quantify the Markovianity of an open quantum system, among which are quantum Fisher information flow [13], quantum fidelity [14], quantum correlation flow [15], quantum channel capacity [16], geometry of dynamically accessible states [17], quantum interferometric power [18], etc; and the Lindblad master equation has been modified in various ways to include memory effects [19, 20]. We refer the readers to [9, 21–23] for reviews of recent progress in open quantum systems.

A common and widely held assumption in Markovian quantum dynamics is that fast restoration of the bath state is required, so that the evolution of the quantum system is memoryless, which is rooted in the Born-Markov approximation. If the bath state is not restored fast enough, the information of the system lost into the bath may flow back into the system, and the dynamics of the system becomes non-Markovian.

An interesting question is: how rapidly does the bath state need to be restored to make the dynamics of an open quantum system Markovian?
In this paper, we study this problem by a simple model with a qubit as the system and another qubit as the bath. The bath qubit is assumed to be in the ground state initially, and is continuously cooled down to that ground state to simulate the restoration process. In principle, the cooling process requires another large reservoir at zero temperature, and the interaction between the large reservoir and the bath qubit here can be complex. However, as we do not care about the details of the cooling process, we can trace out the degrees of freedom of the large reservoir and effectively formulate the cooling of the bath qubit as a simple energy decay process [24, 25].

As the dimension of bath is low in this problem, one may expect a strong memory effect from the bath on the system and an extremely fast restoration of the bath state needed to make the system dynamics Markovian. However, the result reveals that the bath does not need to be refreshed extremely fast to make the system dynamics Markovian, and interestingly there exists a sharp threshold in the bath cooling rate between the non-Markovian regime and the Markovian regime of the system dynamics, and the threshold is of the same order of the system evolution speed. Moreover, calculating the bath correlation function shows that the decay rate of bath correlation is just the bath cooling rate, implying that the dynamics of the system can become Markovian even when the bath correlation time is comparable to the time scale of system evolution. This is in marked contrast to the usual belief that the Markovian dynamics is an asymptotic behavior in the limit of short bath correlation times.

II. MARKOVIAN QUANTUM DYNAMICS

In an open quantum system, the system interacts with a bath. The total Hamiltonian of the system and the bath is

\[ H_{\text{tot}} = H_S + H_B + H_{\text{int}}, \]

where \( H_S, H_B \) are the free Hamiltonians of the system and the bath respectively and \( H_{\text{int}} \) is the interaction Hamiltonian between the system and the bath.

To derive the reduced dynamics of the system, one can transform into the interaction picture, in which the interaction Hamiltonian becomes

\[ \hat{H}_{\text{int}}(t) = e^{i(H_S + H_B)t} H_{\text{int}} e^{-i(H_S + H_B)t}, \]

and make the Born-Markov approximation. The Born approximation assumes that the joint state of the system and the bath at any time \( t \) can be written as \( \hat{\rho}^{SB}_t \approx \hat{\rho}^S_t \otimes \hat{\rho}^B_0 \), assuming that the coupling between the system and the bath is weak and the size of the bath is large, where \( \hat{\rho}^S_t \) is the reduced density matrix of the system at time \( t \) and \( \hat{\rho}^B_0 \) is the reduced density matrix of the bath at the initial time. One can then obtain the master equation for the system [24]:

\[
\partial_t \hat{\rho}^S_t = -\int_0^\infty \text{Tr}_B \left[ \hat{H}_{\text{int}}(t), \hat{H}_{\text{int}}(t-t'), \hat{\rho}^S_t \otimes \hat{\rho}^B_{t'} \right] dt', \]

which assumes \( \text{Tr}_B[\hat{H}_{\text{int}}(t), \hat{\rho}^B_t] = 0 \). If the interaction Hamiltonian \( \hat{H}_{\text{int}}(t) \) can be decomposed as \( \hat{H}_{\text{int}}(t) = \sum \alpha A_\alpha(t) \otimes B_\alpha(t) \), the partial trace of the bath degrees of freedom in Eq. (2) will generate the bath correlation functions

\[ \langle B^\alpha_\alpha(t) B^\alpha_\alpha(t-t') \rangle_B = \text{Tr}_B \left[ B^\alpha_\alpha(t) B^\alpha_\alpha(t-t') \hat{\rho}^B_0 \right]. \]

Eq. (2) shows that the evolution of \( \hat{\rho}^S_t \) generally depends on the history of \( \hat{\rho}^B_0 \). To remove the time integral in (2) and make the equation Markovian, the bath correlation functions (3) need to decay very fast with time so that the contributions from the state at times far from the current time \( t \) in the integral are negligible and the time variation of the system state depends only on the system state at time \( t \). In this case, the \( \hat{\rho}^B_0 \) in the integral in Eq. (2) can be replaced by \( \hat{\rho}^B_0 \), which is called the Markov approximation. The dynamics of the system described by such a Markovian master equation is generally coarse-grained in the time framework and cannot be resolved within the bath correlation time, but the faster the bath correlation functions decay, the more precisely the system dynamics can be described by the master equation.

Model.— In this work, we are interested in how fast the bath state needs to be refreshed in order to make the system dynamics Markovian. This is important to the understanding of quantum Markovian dynamics, as a fast restoration of the bath state is one of the key assumptions in the Born-Markov approximation reviewed above. We consider a simple model with a qubit as the system and another qubit as the bath. We suppose the system and the bath interact via the X-X coupling,

\[ H_{\text{int}} = \xi \sigma^S_x \otimes \sigma^B_x. \]

For simplicity, we drop the free Hamiltonian of both the system and the bath.

The dynamics of the system qubit is generally non-Markovian as it is coupled to a bath. To rapidly restore the bath state, we prepare the bath in the ground state initially and introduce a continuous cooling of the bath to that ground state. The cooling process can be described by a dissipation term \( \kappa D[\sigma^B_- | \hat{\rho}^S_t \rangle \langle \hat{\rho}^S_t | \sigma^B_+] \Delta t \), where \( \kappa \) is the cooling rate and \( \hat{\rho}^S_t \) is the density matrix of the joint state of the system and bath at time \( t \). Putting the Hamiltonian evolution and the cooling process together, we have the following master equation for the joint state of the system qubit and the bath qubit:

\[
\partial_t \hat{\rho}^{SB}_t = -i\xi [\sigma^S_x \otimes \sigma^B_x, \hat{\rho}^{SB}_t] + \kappa D[\sigma^B_- | \hat{\rho}^{SB}_t \rangle \langle \hat{\rho}^{SB}_t | \sigma^B_+],
\]

where \( D[\sigma^B_-] \) is performed on the bath qubit alone, defined as \( D[\sigma^B_- | \hat{\rho}^{SB}_t \rangle \langle \hat{\rho}^{SB}_t | = \sigma^B_- \hat{\rho}^{SB}_t \sigma^B_+ - 1/2 (\sigma^B_- \sigma^B_+ \hat{\rho}^{SB}_t - \hat{\rho}^{SB}_t \sigma^B_- \sigma^B_+) \). In principle, there should be another term \( D[\sigma^B_+ | \hat{\rho}^{SB}_t \rangle \langle \hat{\rho}^{SB}_t | \) in Eq. (5) representing the heating process, if the thermal reservoir in contact with the qubit bath is not at zero temperature [25], due to the detailed balance principle. Here, as we
want to cool the bath to the ground state, we assume the thermal reservoir to be at the zero temperature, and the heating term vanishes in this case.

To study the Markovianity of the system dynamics, we need to find how the system evolves. The derivation of the evolution of an open quantum system is generally non-trivial. However, in this problem, as the system and the bath are simple enough, we can obtain the exact evolution of the system by solving the master equation (5) exactly.

Let us assume the initial state of the system qubit is \( \rho_0 = \frac{1}{2} |S\rangle \langle S| + y_0 |S| 2 \rangle \langle y_2 | + z_0 |y_2 \rangle \langle z_0 | \) and the initial state of the bath is \( |0B\rangle \), which is also the state that the bath will be cooled down to. It is straightforward to obtain the solution to the joint state of the system and the bath at any time \( t \) from Eq. (5). If we trace out the degrees of freedom of the bath and focus on the state of the system alone, we can obtain the reduced density matrix of the system qubit at any time \( t \), which turns out to be

\[
\rho^S_t = \frac{1}{2} \left[ |S\rangle \langle S| + c_t (y_0 |S| 2 \rangle \langle y_2 | + z_0 |y_2 \rangle \langle z_0 | + \text{H.c.} \right],
\]

where \( c_t = e^{-\frac{4t}{\kappa^2} \left( 64\xi^2 - 64\xi^2 \right)} + \cosh \frac{4t}{\kappa^2} \). This implicitly assumes that \( \kappa^2 > 64\xi^2 \). If \( \kappa^2 < 64\xi^2 \), \( c_t \) should be \( e^{-\frac{4t}{\kappa^2} \left( 64\xi^2 - \kappa^2 \right)} + \cosh \frac{4t}{\kappa^2} \); and if \( \kappa^2 = 64\xi^2 \), \( c_t = e^{-\frac{4t}{\kappa^2} \left( 1 + \frac{4\kappa^2}{\kappa^2} \right)} \).

This is the solution to the system dynamics. It is simple, but will turn out to give an interesting characterization of the Markovianity of the system dynamics.

### III. COMPLETELY POSITIVE DIVISIBILITY

Based on the evolution of the system state above, we can study the Markovianity of the evolution of the system. We will first look at the CP-divisibility of the dynamics, and later consider quantum state distinguishability.

The CP-divisibility criterion [6] tells that a quantum dynamics \( \Lambda_{0\rightarrow t} \) is Markovian if each piece \( \Lambda_{t_i \rightarrow t_{i+1}} \) of an arbitrary division of \( \Lambda_{0\rightarrow t} \),

\[
\Lambda_{0\rightarrow t} = \Lambda_{t_n \rightarrow t_{n-1}} \Lambda_{t_{n-1} \rightarrow t_{n-2}} \cdots \Lambda_{0\rightarrow t_1},
\]

is completely positive. An intuitive idea behind this definition is that if the evolution of a quantum system between two arbitrary times is completely positive, and hence can be derived from an interaction with a memoryless bath, then the entire evolution is memoryless and depends only on the state of the system at the present time.

Determining the CP-divisibility of a quantum dynamics is highly non-trivial in general. However, it can be proven [7] that a quantum dynamics is CP-divisible if and only if the system evolution can be formulated as a master equation with the coefficients of the dissipation terms non-negative for all times. This gives a more convenient way to determine whether a quantum dynamics is CP-divisible or not. Below, we will use this theorem to figure out when the evolution of the system qubit in the current problem is CP-divisible.

We first need to find the master equation for the evolution of the system qubit. In the Appendix, it was obtained that the master equation for the system evolution corresponding to (6) is

\[
\partial_t \rho^S_t = \frac{-\dot{c}_t}{2c_t} D[\sigma^S_2] \rho^S_t.
\]

From the solution of \( c_t \) in the last section, one can see that if \( \kappa^2 > 64\xi^2 \), \( -\frac{\dot{c}_t}{2c_t} = \frac{8\xi^2}{\kappa^2 + \sqrt{\kappa^2 - 64\xi^2}} \cosh \left( \frac{4t}{\kappa^2} \sqrt{\kappa^2 - 64\xi^2} \right) \), which is always positive; if \( \kappa^2 < 64\xi^2 \), \( -\frac{\dot{c}_t}{2c_t} = \frac{8\xi^2}{\kappa^2 + \sqrt{64\xi^2 - 64\xi^2}} \cosh \left( \frac{4t}{\kappa^2} \sqrt{64\xi^2 - 64\xi^2} \right) \), which is negative when \( \cot \left( \frac{4t}{\kappa^2} \sqrt{64\xi^2 - 64\xi^2} \right) < -\frac{\kappa^2}{\sqrt{\kappa^2 - \kappa^2}} \). And if \( \kappa^2 = 64\xi^2 \),

\[
-\frac{\dot{c}_t}{2c_t} = \frac{\kappa^2_t}{2(16+4\kappa^2)}
\]

is non-negative for all times \( t \).

Therefore, we can immediately infer by the CP-divisibility theorem that the dynamics of the system qubit is Markovian when \( \kappa^2 \geq 64\xi^2 \), and is non-Markovian when \( \kappa^2 < 64\xi^2 \). We plot \( \partial_t |c_t| \) for different cases in Fig. 1. \( \partial_t |c_t| \) has the same sign as \( \frac{\dot{c}_t}{c_t} \), so it shows how the sign of \( \frac{\dot{c}_t}{c_t} \) varies with time for different cooling rates, which verifies the above results.

The interesting thing here is that there exists a threshold line \( \kappa^2 = 64\xi^2 \) that divides the Markovian and non-Markovian regimes of the system dynamics. When \( \kappa \) goes from above \( 8|\xi| \) to below \( 8|\xi| \), the system undergoes an abrupt transition from non-Markovian dynamics to Markovian dynamics. More importantly, this transition line lies at the same order of the \( \xi \), which means that the system dynamics becomes Markovian when the bath restoration rate is of the same order as the system evolution speed. This shows the existence of Markovian quantum dynamics beyond the traditional asymptotic regime of \( \kappa \gg |\xi| \) under the Born-Markov approximation.

### IV. DISTINGUISHABILITY OF QUANTUM STATES

The non-Markovianity of quantum dynamics can also be defined and measured in an information theoretic way, through the distinguishability of two different quantum states. The trace distance between two quantum states determines the distinguishability of two quantum states [26, 27], which can be considered as a measure of the information that can be learned from the two states. Breuer et al. pointed out that if a quantum dynamics is Markovian, the trace distance between any two
quantum states will decrease with time [24], due to the contractivity of completely positive quantum maps. The non-Markovianity of quantum dynamics can be defined as the existence of a pair of quantum states whose trace distance increases for some time under that dynamics [8]. An intuitive interpretation for this definition is that if the trace distance between two quantum states increases for some time during the evolution, it means that some of the information about the system that was lost into the environment flows back into the system. This implies that the environment has memory, and thus the dynamics of the quantum system is non-Markovian.

The trace distance between two quantum states \( \rho_1(t) \), \( \rho_2(t) \) is defined as

\[
d_{TV}(\rho_1(t), \rho_2(t)) = \frac{1}{2} \text{Tr}[|\rho_1(t) - \rho_2(t)|]. \tag{9}
\]

A relationship between the trace distance and the distinguishability of \( \rho_1(t) \), \( \rho_2(t) \) is that the maximum probability of correctly distinguishing \( \rho_1(t) \) and \( \rho_2(t) \) is \( \frac{1}{2}[1 + d_{TV}(\rho_1(t), \rho_2(t))] \). Therefore, the trace distance determines the distinguishability between two quantum states, and characterizes how much information one may extract from the two states.

Now, we can use this information theoretic definition of quantum non-Markovianity to examine the Markovianity of the quantum dynamics in our problem. Suppose we have two arbitrary initial states for the system qubit, \( \rho_{0b} = \frac{1}{2}I^S + x_0\sigma_x^S + y_0\sigma_y^S + z_0\sigma_z^S \) and \( \rho_{0s} = \frac{1}{2}(I^S + x'_0\sigma_x^S + y'_0\sigma_y^S + z'_0\sigma_z^S) \). It follows from Eq. (6) that \( \rho_{t} - \rho_{0s} = \frac{1}{2}(\Delta_0\sigma_x^S + c_t\Delta_\rho\sigma_y^S + c_t\Delta_\rho\sigma_z^S) \), and the trace distance between \( \rho_{0b} \) and \( \rho_{0s} \) is

\[
d_{TV}(\rho_s^0, \rho_s^0) = \frac{1}{2}\sqrt{\Delta_x^2 + c_t^2(\Delta_y^2 + \Delta_z^2)}, \tag{10}
\]

where \( \Delta_x = x_0 - x'_0 \), \( \Delta_y = y_0 - y'_0 \), \( \Delta_z = z_0 - z'_0 \). To see whether the trace distance between \( \rho_{0b} \) and \( \rho_{0s} \) increases or not during the evolution of the system, we can take the derivative of \( d_{TV}(\rho_s^0, \rho_s^0) \) with respect to \( t \), and it produces

\[
\partial_t d_{TV}(\rho_s^0, \rho_s^0) = \frac{c_t}{2\sqrt{\Delta_x^2 + c_t^2(\Delta_y^2 + \Delta_z^2)}}. \tag{11}
\]

Considering \( \text{sgn}(c_t c_t') = \text{sgn}(\Delta_t) \), the time derivative of trace distance gives the same characterization of the system dynamics as the CP-divisibility: when \( \kappa^2 \geq 64\xi^2 \), the trace distance always decreases with time and the system dynamics (6) is Markovian; when \( \kappa^2 < 64\xi^2 \), the trace distance with \( \Delta_y \neq 0 \) or \( \Delta_z \neq 0 \) can increase at times \( t \) that satisfies \( c_t^2 \sqrt{64\xi^2 - \kappa^2} < -\kappa^2 / \sqrt{64\xi^2 - \kappa^2} \), so the system dynamics is non-Markovian. Fig. 1 illustrates this result clearly.

Moreover, we can take the integral of the increase in trace distance during the whole evolution of the system as the measure of non-Markovianity, which was proposed by Breuer et al. [8],

\[
N = \max_{\rho_1(0), \rho_2(0)} \int_{t' = 0}^{4\pi} dt' \partial_t d_{TV}(\rho_1(t'), \rho_2(t')) dt', \tag{10}
\]

where the integral is taken over all time intervals during which the trace distance \( d_{TV} \) increases. This measure of non-Markovianity is non-zero only for the case \( \kappa^2 < 64\xi^2 \), as there is increase in the trace distance only in that region. The result turns out to be (see Appendix)

\[
N = \frac{1}{\exp\left(\frac{\kappa\pi}{\sqrt{64\xi^2 - \kappa^2}}\right) - 1}. \tag{11}
\]

It can be clearly seen from the measure of non-Markovianity (10) that when there is no cooling on the bath qubit, \( \kappa = 0 \), \( N \) is infinity, at its maximum, as there is periodic information backflow from the bath to the system and the amplitude of the backflow never decays, indicating the strongest non-Markovianity for this case. When \( \kappa \) becomes nonzero, \( N \) becomes finite, as the amplitude of the information backflow decays with time, indicating a weaker non-Markovianity. When \( \kappa^2 = 64\xi^2 \), \( N = 0 \), indicating the vanishing of non-Markovianity.

V. BATH CORRELATION TIME

Finally, we want to investigate the bath correlation for this problem. The bath correlation time plays a key role
in the Born-Markov approximation for Markovian quantum dynamics, as the main characteristic of Markovian dynamics is that the bath correlation decays extremely fast so that the system dynamics is memoryless. And in this case, the Lindblad master equation can be derived in the weak coupling limit [24] which is the most widely-used approach to describing Markovian quantum dynamics.

Below we study the bath correlation to compare the bath correlation time with the time scale of the system evolution. In fact, as the bath qubit is being continuously cooled to the ground state at rate \( \kappa \), one can expect that the bath correlation time is approximately \( 1/\kappa \). In the following, we will see this is indeed the case.

We first perform a non-unitary transformation of the bath state to eliminate the dissipation term in the master equation:

\[
\rho(t) \rightarrow e^{-\kappa \sigma^x} \rho(t) e^{\kappa \sigma^x}.
\]

In the new picture under this transformation, we can trace out the degrees of freedom of the bath and obtain a reduced master equation for the system qubit similar to (2). The period over which earlier states of the system can influence the current state of the system is determined by the time width of bath correlation functions.

In the Appendix, the relevant bath correlation function is derived in the weak coupling limit [24] which is the most widely-used approach to describing Markovian quantum dynamics beyond the limit of fast bath restoration or short bath correlation. Moreover, a sharp boundary exists in the bath restoration rate, over which the system undergoes an abrupt transition between non-Markovian and Markovian dynamics, which does not show under the Born-Markov approximation.

VI. CONCLUSIONS

In the usual Markovian dynamics under the Born-Markov approximation, the bath must restore much faster than the system evolves so that the bath is memoryless and the evolution of the system is determined only by its current state. In this work, however, a surprising result is that the system dynamics can be Markovian when the bath restoration rate is of the same order as the system evolution rate, which implies the existence of Markovian quantum dynamics beyond the limit of fast bath restoration or short bath correlation.

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of the bath qubit is can be represented by a system and bath at time $t$, and $D[\sigma^B]$ is the dissipator defined as

$$ D[\sigma^B]\rho_t^{SB} = \sigma^B\rho_t^{SB}\sigma^B + \frac{1}{2}\{\sigma^B\rho_t^{SB}, \rho_t^{SB}\}. $$

Putting the Hamiltonian evolution and the cooling process together, we have the following master equation for the joint state of the system qubit and the bath qubit:

$$ \partial_t \rho_t^{SB} = -i[\sigma^S \otimes \sigma^B, \rho_t^{SB}] + \kappa D[\sigma^B]\rho_t^{SB}. $$

Suppose the initial density matrix of the system qubit is $\rho_0^S = \frac{1}{2}(I^S + x_0\sigma^S_x + y_0\sigma^S_y + z_0\sigma^S_z)$, and the initial state of the bath qubit is $|0^B\rangle$. To solve the master equation (S3), we work in the operator space of the system and bath qubits, which has $\sigma^S \otimes \sigma^B$, $i, j = 0, x, y, z$, as its basis, where $\sigma_0 = I$. In this representation, the joint density matrix of the system and the bath can be represented by a $16 \times 1$ vector, and a superoperator on the system and the bath can be represented by a $16 \times 16$ matrix. The vector form of the initial density matrix of the system and the bath is

$$ \mathbf{v}_0^{SB} = \frac{1}{4}[1, 0, 0, -1, x_0, 0, 0, -x_0, y_0, 0, 0, -y_0, z_0, 0, 0, -z_0]^T. $$

The master equation (S3) can be written in a matrix form:

$$ \partial_t \mathbf{v}_t^{SB} = M \mathbf{v}_t^{SB}, $$

where

$$ M = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\kappa & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\kappa & 0 & 0 & 0 & 0 & 0 & -2\xi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\kappa & 0 & 0 & -\kappa & 0 & 2\xi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\kappa & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\kappa & 0 & 0 & 0 & 0 & -\kappa & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2\xi & 0 & 0 & -\kappa & 0 & 0 & -\kappa & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\kappa}{2} & 0 & 0 & -2\xi & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\kappa}{2} & 0 & 0 & -2\xi & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\kappa & 0 & 0 & -\kappa & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\xi & 0 & 0 & 0 & 0 & 0 & -\frac{\kappa}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\kappa \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\kappa & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\kappa
\end{pmatrix}. $$
The solution to $v_t^{SB}$ can be obtained by solving Eq. (S4) directly:

$$v_t^{SB} = \exp(Mt) v_0^{SB}.$$  \hfill (S6)

If we trace out the freedom degrees of the bath qubit, the reduced density matrix of the system qubit at time $t$ is

$$\rho_t^S = \frac{1}{2} \left[ I + x_0 \sigma_x^S + c_t \left( y_0 \sigma_y^S + z_0 \sigma_z^S \right) \right].$$  \hfill (S7)

where

$$c_t = e^{-\frac{\kappa t}{4}} \left( \frac{\kappa \sin \frac{t}{4} \sqrt{\kappa^2 - 64 \xi^2}}{\sqrt{\kappa^2 - 64 \xi^2}} + \cosh \frac{t}{4} \sqrt{\kappa^2 - 64 \xi^2} \right).$$  \hfill (S8)

Eq. (S8) implicitly assumes that $\kappa^2 > 64 \xi^2$. If $\kappa^2 < 64 \xi^2$, $c_t$ becomes

$$c_t = e^{-\frac{\kappa t}{4}} \left( \frac{\kappa \sin \frac{t}{4} \sqrt{64 \xi^2 - \kappa^2}}{\sqrt{64 \xi^2 - \kappa^2}} + \cos \frac{t}{4} \sqrt{64 \xi^2 - \kappa^2} \right).$$  \hfill (S9)

If $\kappa^2 = 64 \xi^2$, $c_t$ is

$$c_t = e^{-\frac{\kappa t}{4}} \left( 1 + \frac{1}{4} \kappa t \right).$$  \hfill (S10)

B. Master equation

If we want to determine the Markovianity of the system dynamics based on the CP-divisibility criterion, we need to obtain the master equation for the system qubit and check whether the coefficients of the dissipation terms are non-negative or not [7]. Generally, deriving the master equation for an open quantum system is not easy, but in this problem, as we have obtained the solution to the system evolution, we can infer the master equation of the system from that solution.

To derive the master equation of the system, we transform into the operator space of the system which has basis $\{I, \sigma_x^S, \sigma_y^S, \sigma_z^S\}$. The density matrix of the system can be represented by a $4 \times 1$ vector, and a superoperator on the system can be represented by a $4 \times 4$ matrix. In this representation, the vector form of the system qubit at time $t$ is

$$v_t^S = \frac{1}{2} [w_0, x_0, c_t y_0, c_t z_0]^T = Q_t v_0^S,$$  \hfill (S11)

where $v_0^S = \frac{1}{2} [w_0, x_0, y_0, z_0]^T$ is the vector form of the initial density matrix of the system qubit, and $Q_t$ is the matrix representation of the system qubit evolution from the initial time to the time $t$,

$$Q_t = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & c_t & 0 \\
0 & 0 & 0 & c_t
\end{pmatrix}. \hfill (S12)
$$

Note that $w_0 = 1$ for a normalized density matrix, but as we want to find the linear transformation corresponding to the time change of $v_t^S$ in the operator space of the system, we temporarily denote it as a variable. It will be restored to 1 when the linear transformation is derived.

The time change of $v_t^S$ is

$$\partial_t v_t^S = \dot{Q}_t v_0^S = \dot{Q}_t Q_t^{-1} v_t^S.$$  \hfill (S13)

Thus, $\dot{Q}_t Q_t^{-1}$ is the matrix representation of the linear transformation corresponding to the time derivative of the system density matrix, and

$$\dot{Q}_t Q_t^{-1} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \frac{c_t}{\kappa} & 0 \\
0 & 0 & 0 & \frac{c_t}{\kappa}
\end{pmatrix}. \hfill (S14)$$
Now we can find the superoperator corresponding to $\hat{Q}^{-1}_{\ell}$. In order to do this, we need to know the matrix representations $s_{ij}$ for the basis of the superoperator $\sigma_i^S |\sigma_j^S$. It is straightforward to obtain that

$$s_{00} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad s_{0x} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad s_{0y} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad s_{0z} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$s_{x0} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad s_{xx} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad s_{xy} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad s_{xz} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (S15)$$

By decomposing $\hat{Q}^{-1}_{\ell}$ in Eq. (S14) along $s_{ij}$, one can obtain that

$$\dot{\hat{Q}}^{-1}_{\ell} = \frac{\dot{c}_{\ell}}{2c_{\ell}} (s_{00} - s_{xx}). \quad (S16)$$

Therefore, we have

$$\partial_{t} \rho^{S}_{\ell} = \frac{\dot{c}_{\ell}}{2c_{\ell}} (\rho^{S}_{\ell} - \sigma_{x}^{S} \rho^{S}_{\ell} \sigma_{x}^{S}) = -\frac{\dot{c}_{\ell}}{2c_{\ell}} D[\sigma_{x}^{S}] \rho^{S}_{\ell}, \quad (S17)$$

which is the master equation for the system qubit.

II. NON-MARKOVIANITY MEASURE BASED ON TRACE DISTANCE

A useful measure of non-Markovianity of a quantum dynamics was proposed by Breuer et al. [8] from an information theoretic point of view. The measure is the integral of the increase in the trace distance between two quantum states under that quantum dynamics maximized over all possible pairs of initial quantum states.

Mathematically, if we denote the measure of non-Markovianity as $N$, and two quantum states under the quantum dynamics of interest as $\rho_{1}(t)$ and $\rho_{2}(t)$, the measure of non-Markovianity is

$$N = \max_{\rho_{1}, \rho_{2}(0)} \int_{\partial_{t} d_{Tr} > 0} \partial_{t} d_{Tr}(\rho_{1}(t'), \rho_{2}(t')) dt', \quad (S18)$$

where $d_{Tr}(\rho_{1}(t'), \rho_{2}(t'))$ is the trace distance between $\rho_{1}(t')$ and $\rho_{2}(t')$ defined as

$$d_{Tr}(\rho_{1}(t'), \rho_{2}(t')) = \frac{1}{2} \text{Tr} |\rho_{1}(t') - \rho_{2}(t')|, \quad (S19)$$

and the integral is over all time intervals where the trace distance between $\rho_{1}(t')$, $\rho_{2}(t')$ increases, i.e., $\partial_{t} d_{Tr}(\rho_{1}(t'), \rho_{2}(t')) > 0$.

In this problem, we assume we have two initial states for the system qubit,

$$\rho_{0}^{S} = \frac{1}{2} (I^{S} + x_{0} \sigma_{x}^{S} + y_{0} \sigma_{y}^{S} + z_{0} \sigma_{z}^{S}), \quad (S20)$$

Then, according to Eq. (S7), the difference between $\rho^{S}_{\ell}$ and $\rho^{S}_{\ell}$ at time $t$ is

$$\rho^{S}_{\ell} - \rho^{S}_{\ell} = \frac{1}{2} \left( \Delta_{x} \sigma_{x}^{S} + c_{\ell} \Delta_{y} \sigma_{y}^{S} + c_{\ell} \Delta_{z} \sigma_{z}^{S} \right), \quad (S21)$$
and the trace distance between $\rho^S_t$ and $\rho^S_{t'}$ is
\[
d_{\mathcal{Tr}}(\rho^S_t, \rho^S_{t'}) = \frac{1}{2} \sqrt{\Delta_x^2 + c_t^2 (\Delta_y^2 + \Delta_z^2)},
\]  
(S22)

where $\Delta_x = x_0 - x'_0$, $\Delta_y = y_0 - y'_0$, $\Delta_z = z_0 - z'_0$.

According to the result in the main paper, the trace distance between two quantum states under the dynamics in this problem may oscillate only when $\kappa^2 \leq 64\xi^2$, so we will focus on this case below.

To investigate when the trace distance between $\rho^S_t$ and $\rho^S_{t'}$ increases, we take the derivative of Eq. (S22) with respect to time $t$, which produces
\[
\frac{\partial_t d_{\mathcal{Tr}}(\rho^S_t, \rho^S_{t'})}{dt} = (\Delta_y^2 + \Delta_z^2) \frac{c_t^2}{d_{\mathcal{Tr}}(\rho^S_t, \rho^S_{t'})} \frac{\dot{c}_t}{c_t}.
\]  
(S23)

And when $\kappa^2 < 64\xi^2$,
\[
\frac{\dot{c}_t}{c_t} = \frac{-16\xi^2}{\kappa + \sqrt{64\xi^2 - \kappa^2}} \cot \left( \frac{1}{4} t \sqrt{64\xi^2 - \kappa^2} \right).
\]  
(S24)

Obviously, $\frac{\dot{c}_t}{c_t} > 0$ when $\cot \left( \frac{1}{4} t \sqrt{64\xi^2 - \kappa^2} \right) < -\frac{\kappa}{\sqrt{64\xi^2 - \kappa^2}}$, therefore, the trace distance $d_{\mathcal{Tr}}(\rho^S_t, \rho^S_{t'})$ increases for any
\[
t_n - \delta < t < t_n, \quad n = 1, 2, \ldots,
\]  
(S25)

where
\[
t_n = \frac{4n\pi}{\sqrt{64\xi^2 - \kappa^2}}, \quad \delta = \frac{4\arctan \sqrt{\frac{\kappa^2}{64\xi^2 - \kappa^2}}}{\sqrt{64\xi^2 - \kappa^2}}.
\]  
(S26)

It can be verified from Eq. (S9) that
\[
c_{t_n - \delta} = 0, \quad c_{t_n} = (-1)^n e^{-\frac{\kappa t_n}{4}}.
\]  
(S27)

Then it follows Eq. (S22), the increase in the trace distance in the time interval determined by Eq. (S25) for each $n$ is
\[
\Delta_n d_{\mathcal{Tr}}(\rho^S_t, \rho^S_{t'}) = \frac{1}{2} \left( \sqrt{\Delta_x^2 + e^{-\frac{\kappa}{4} n} (\Delta_y^2 + \Delta_z^2)} - |\Delta_x| \right).
\]  
(S28)

Now we need to maximize $\Delta_n d_{\mathcal{Tr}}(\rho^S_t, \rho^S_{t'})$ over all possible $\Delta_x$, $\Delta_y$, $\Delta_z$. Note that $\Delta_n d_{\mathcal{Tr}}(\rho^S_t, \rho^S_{t'})$ can be written as
\[
\Delta_n d_{\mathcal{Tr}}(\rho^S_t, \rho^S_{t'}) = \frac{e^{-\frac{\kappa}{4} n} (\Delta_y^2 + \Delta_z^2)}{2 \left( \sqrt{\Delta_x^2 + e^{-\frac{\kappa}{4} n} (\Delta_y^2 + \Delta_z^2)} + |\Delta_x| \right)}.
\]  
(S29)

If we fix $\Delta_y$ and $\Delta_z$, it is obvious that $\Delta_n d_{\mathcal{Tr}}(\rho^S_t, \rho^S_{t'})$ is maximized when $\Delta_x = 0$. And letting $\Delta_x = 0$ is possible, because we can always choose proper $x_0 = x'_0$ (at least we can choose $x_0 = x'_0 = 0$) without violating the positivity of either $\rho^S_0$ or $\rho^S_0'$ for any $\Delta_y$ and $\Delta_z$. Therefore,
\[
\max_{\Delta_x} \Delta_n d_{\mathcal{Tr}}(\rho^S_t, \rho^S_{t'}) = \frac{1}{2} e^{-\frac{\kappa}{4} n} \sqrt{\Delta_y^2 + \Delta_z^2}.
\]  
(S30)

To maximize Eq. (S30) over $\Delta_y$ and $\Delta_z$, we denote two vectors $r_0 = [x_0, y_0, z_0]$, $r'_0 = [x_0, y'_0, z'_0]$, and $\|r_0\| \leq 1$, $\|r'_0\| \leq 1$ due to the positivity of $\rho^S_0$ and $\rho^S_0'$. Note the $x$ components of $r_0$ and $r'_0$ have been explicitly chosen to be the same. Then,
\[
\max_{\Delta_x} \Delta_n d_{\mathcal{Tr}}(\rho^S_t, \rho^S_{t'}) = \frac{1}{2} e^{-\frac{\kappa}{4} n} \|r_0 - r'_0\|.
\]  
(S31)
And the maximization of \( \max_{\Delta_n} \Delta_n d_{T}(\rho^S_1, \rho^S_0) \) over \( \Delta_n \) and \( \Delta_x \) becomes a maximization over \( r_0 \) and \( r'_0 \). Obviously \( \|r_0 - r'_0\| \) is maximized when \( r_0 \) and \( r'_0 \) are antiparallel and \( \|r_0\| = \|r'_0\| = 1 \), which means \( r_0 = -r'_0 \) and thus \( x_0 = -x_0 \), implying \( x_0 = 0 \), and the maximum of \( \|r_0 - r'_0\| \) is 2. So, the maximum of \( \Delta_n d_{T}(\rho^S_1, \rho^S_0) \) is

\[
\max_{\Delta_n} \Delta_n d_{T}(\rho^S_1, \rho^S_0) = e^{-\frac{\pi}{4\xi^2}}. \tag{S32}
\]

Therefore, the measure of non-Markovianity \( \mathcal{N} \) for \( \kappa^2 \leq 64\xi^2 \) is

\[
\mathcal{N} = \sum_{n=1}^{\infty} e^{-\frac{n+\mu}{\kappa^2}} = \sum_{n=1}^{\infty} e^{-\frac{n+\mu}{\sqrt{64\xi^2-\kappa^2}}} = \frac{1}{\exp\left(\frac{\kappa}{\sqrt{64\xi^2-\kappa^2}}\right) - 1}. \tag{S33}
\]

It can be seen that when \( \kappa = 0 \), \( \mathcal{N} \) is infinity, indicating the maximum non-Markovianity, as the state of the system qubit always oscillates in this case which induces periodic information backflow from the bath to the system and the amplitude of the information backflow never decays. When \( \kappa \) decreases, \( \mathcal{N} \) decreases and becomes finite, indicating a weaker non-Markovianity, as the oscillation of the system state is weakened and the amplitude of the information backflow decays with time. When \( \kappa^2 = 64\xi^2 \), \( \mathcal{N} = 0 \), indicating no non-Markovianity in the system dynamics.

Of course, if \( \kappa^2 > 64\xi^2 \), as there is no increase in the trace distance between any two quantum states at any time in this problem, \( \mathcal{N} \) is zero by its definition for this case.

### III. Bath Correlation

In this section, we use the standard procedures under the Born-Markov approximation \[24\] to compute the time correlation functions for the bath qubit. The purpose is to find how fast the bath correlation must decay in order to validate the Born-Markov approximation and compare it to the threshold for the transition between Markovian and non-Markovian dynamics found in our problem.

We first make the following transformation to eliminate the dissipation term in the master equation (S3),

\[
\tilde{\rho}^S_B = e^{-\kappa D[\sigma^B]} \rho^S_B. \tag{S34}
\]

Then,

\[
\partial_t \tilde{\rho}^S_B = -i\xi e^{-\kappa D[\sigma^B]} [\sigma^S_B \otimes \gamma^S_B, \rho^S_B]. \tag{S35}
\]

In order to make it an equation for \( \tilde{\rho}^S_B \), we write \( \tilde{\rho}^S_B \) as \( \rho^S_B = e^{\kappa D[\sigma^B]} \tilde{\rho}^S_B \), then

\[
\partial_t \rho^S_B = -i\xi e^{-\kappa D[\sigma^B]} [\sigma^S_B \otimes \gamma^S_B, e^{\kappa D[\sigma^B]} \rho^S_B]. \tag{S36}
\]

If we integrate Eq. (S36) over the time, we have

\[
\tilde{\rho}^S_B = \rho^S_B - i\xi \int_0^t e^{-\kappa D[\sigma^B]} [\sigma^S_B \otimes \gamma^S_B, e^{\kappa D[\sigma^B]} \rho^S_B] dt'. \tag{S37}
\]

Plugging Eq. (S37) into Eq. (S35), we can obtain a master equation of \( \tilde{\rho}^S_B \) up to the second order of \( \xi \),

\[
\partial_t \tilde{\rho}^S_B = -i\xi e^{-\kappa D[\sigma^B]} [\sigma^S_B \otimes \gamma^S_B, e^{\kappa D[\sigma^B]} \rho^S_B] - \xi^2 e^{-\kappa D[\sigma^B]} \int_0^t [\sigma^S_B \otimes \gamma^S_B, e^{\kappa D[\sigma^B]} \rho^S_B] dt'. \tag{S38}
\]

In order to obtain the reduced master equation for the system qubit alone, we trace out the freedom degrees of the bath qubit for Eq. (S38). By invoking the Born approximation, we assume that

\[
\rho^S_B \approx \tilde{\rho}^S_B \otimes \rho^B_0, \tag{S39}
\]

where \( \rho^B_0 = |0^B \rangle \langle 0^B | \). Then, it can be verified that

\[
\text{Tr}_B e^{-\kappa D[\sigma^B]} [\sigma^S_B \otimes \gamma^S_B, e^{\kappa D[\sigma^B]} \rho^S_B] = 0. \tag{S40}
\]
So,

$$\partial_t \rho_i^S = \text{Tr}_B \partial_t \rho_{t'}^{SB} = -\xi^2 \int_0^t \text{Tr}_B [\sigma_x^S \otimes \sigma_x^B, e^{\kappa(t-t')D[\sigma_B^B]}(\sigma_x^S \otimes \sigma_x^B, e^{\kappa(t-t')D[\sigma_B^B]}\rho_{t'}^{SB})] \, dt' , \quad (S41)$$

where we have used $\text{Tr}_B e^{-\kappa t D[\sigma_B^B]} A = \text{Tr}_B A$ for an arbitrary operator $A$ of the bath qubit, as $\text{Tr}_B D[\sigma_B^B] = 0$ for $k \geq 1$. According to the transformation (S34) and the Born approximation (S39), Eq. (S41) can be simplified to

$$\partial_t \rho_i^S = -\xi^2 \int_0^t \text{Tr}_B [\sigma_x^S \otimes \sigma_x^B, e^{\kappa(t-t')D[\sigma_B^B]}(\sigma_x^S \otimes \sigma_x^B, \rho_{t'}^{SB} \otimes \rho_0^B)] \, dt' , \quad (S42)$$

followed by

$$\partial_t \rho_i^S = \xi^2 \int_0^t f_B(t, t') D[\sigma_x^S] \rho_i^S \, dt' , \quad (S43)$$

where $f_B(t, t')$ is the bath correlation function

$$f_B(t, t') = \text{Tr}_B [\sigma_x^B e^{\kappa(t-t')D[\sigma_B^B]}(\sigma_x^B \rho_0^B + \rho_0^B \sigma_x^B)] . \quad (S44)$$

As the initial state of the bath qubit is $|0^B\rangle$, $\rho_0^B = |0^B\rangle \langle 0^B|$, so $\sigma_x^B \rho_0^B + \rho_0^B \sigma_x^B = \sigma_x^B$. And it can be easily verified that $e^{\kappa(t-t')D[\sigma_B^B]} \sigma_x^B = e^{-\frac{\kappa}{2}(t-t')} \sigma_x^B$. Therefore,

$$f_B(t, t') = \exp \left[ -\frac{1}{2} \kappa(t-t') \right] . \quad (S45)$$

This means that the time width of the bath correlation is $2/\kappa$.

The Markov approximation is to further assume that the bath correlation $f_B(t, t')$ decays very fast in the time scale of the system evolution, so that the $\rho_{t'}^S$ in the integrand in Eq. (S43) can be replaced by $\rho_i^S$ and the evolution of $\rho_i^S$ does not depend on the history of $\rho_{t'}^S$ then, which leads a Markovian dynamics of the system. This essentially requires $\xi \ll \kappa$, in sharp contrast to the result in the main text that the system dynamics becomes Markovian when $\kappa \geq 8|\xi|$ where $\kappa$ is at the same order of magnitude of $\xi$. 