Flexural buckling of a revolving bar in a rigid pipe with a gap exposed to axial force and dead weight

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Abstract. The article proposes a solution to the stability problem of a revolving bar loaded with axial force and inserted into a rigid pipe with a gap by the Bubnov-Galerkin method. Two options as the deflection coordinate functions are considered. When calculating, the dead weight of the bar is taken into account. With a total load greater than the Euler load, lateral forces (reactions), touching the pipe wall, appear in the bar. At the end of the article, the formulas for determining the voltage after buckling are presented. The obtained formulas make it possible to determine the critical load of the system for each individual case, taking into account its dead weight and centrifugal forces from the revolving bar. The above-mentioned relations also give an opportunity to determine the reactive moment magnitude from the contact of the revolving bar and the pipe wall.

Introduction
During rotary wells’ drilling in bedrock at several thousand meters depths, when driving torque is applied to the upper end of the drill string (DS), and the force on the bit is created by its gravity force, the system functioning can be accompanied by the appearance of a number of mechanical phenomena, which negatively affects the entire working process. So, for example: DS stability loss in its lower part as a compressed-curved swirling bar.

In theoretical modeling of DS static behavior in the drilling process, it becomes necessary to integrate the differential equations of their equilibrium and oscillations. These tasks are associated with significant analytical and computational difficulties that arise at the initial stage of their formulation, and are caused by a complex combination of static force factors acting on DS at work. The main one is the inhomogeneous field of internal longitudinal forces in DS, formed by the gravity columns, bits.

Methods
Let us consider a long bar revolving at an angular speed $\omega$ and inserted with a gap $y_0$ into a rigid pipe. The design scheme is presented in Figure 1.

The shaft is exposed to axial force $F_0$ and centrifugal force proportional to the bar and its angular velocity deflection. Additive to the above-mentioned loads $qI$ is considered as the dead weight of the bar. Centrifugal load intensity is:

$$q^* = \frac{q}{g} \omega^2 y$$
where \( q \) is the bar’s dead weight intensity; \( g \) is the gravity acceleration; \( \omega \) is the bar angular velocity; \( y \) is the bar deflection.

Let us consider an elastic homogeneous, rectilinear bar located in a rigid pipe and revolving around its axis with a given angular velocity.

In the case of small deviations, the bar will take the form of a spatial curve. To investigate the supercritical bar state, it is necessary to use the twelve Kirchhoff – Clebsch equations [307, 279] and three dependences, introducing the hypothesis that the moments are directly proportional to changes in curvature and taking into account the influence of the well walls. This is a very difficult task.

From the obtained approximate results of spatial bending, it follows that the bending stresses in drill pipes, calculated by the well-known technique of a string flat bending [197], significantly exceed the stresses determined by spiral bending.

Consequently, the calculation results using the formulas of the string flat bending are obtained already with a safety factor. Therefore, considering a half-wave (Fig. 1), we assume that the bend is flat.

We consider the half-wave separately, replacing the inflection points with hinged supports (Figure 1. a). A statically undefined system was received once. To solve this problem, we compose the differential equations of the bar elastic line in the areas \( 0 \leq x \leq l \).

\[
\frac{d^2 v}{dx^2} + \frac{F \cdot v(x)}{EI_z} + \int_0^x \frac{q}{EI_z} \frac{dv}{dz} dz - \int_0^x \int_0^z \frac{q \omega^2}{gEI_z} v(\zeta) d\zeta dz = N_1 x \frac{N_1 x}{EI_z} \tag{1}
\]

\[
\frac{d^2 v}{dx^2} + \frac{F \cdot v(x)}{EI_z} + \int_0^x \frac{q}{EI_z} \frac{dv}{dz} dz - \int_0^x \int_0^z \frac{q \omega^2}{gEI_z} v(\zeta) d\zeta dz = N_1 x \frac{N_1 x}{EI_z} - N_2 \left( x - \frac{l}{2} \right) \tag{2}
\]

The deflection equal to the formula below is shorter in practice than the half-wave length \( l \)

\[
f = \frac{D - d}{2} \tag{3}
\]

It is possible to use the differential equation of the beam curved axis.

It is possible to reveal the static indeterminacy of the system (Fig. 1.) using the equations (1) and (2) only if the axial force \( F \) acts. This problem is solved in the work [94].

V. I. Feodosyev [246], solving this problem, considered only one section (1). This was enough for him due to the symmetry of the circuit and loading. Let us briefly consider his solution.

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**Figure 1.** The design scheme of the bar elastic equilibrium.
With the force $F = \frac{\pi^2EI}{l^2}$ the bar loses stability and then touches the pipe walls with its middle part. We assume, that there is a long bar $l_2$ abutment tight to the pipe walls for $F > F_e$. The equation (1) in the section $0 \leq x \leq l_1$ will take the form:

$$\frac{d^2v}{dx^2} + \frac{F \cdot v(x)}{EI_z} = \frac{N_1 x}{EI_z}$$

The solution has the form:

$$v = Asinax + Bcosax + \frac{N_1}{F}x$$

here $\left(\alpha^2 = \frac{F}{EI}\right)$

Using the boundary conditions, we define the constants $A$ and $B$.

At $x = 0 \quad v = 0, \quad B = 0$.

At $x = l_1 \quad v = f$, \quad $Asin\alpha l_1 + \frac{N_1}{F}l_1 = f$

At $x = l_1 \quad v' = 0$, \quad $A\alpha cos\alpha l_1 + \frac{N_1}{F} = 0$  

The bar remains straight on the section $l_2$. Accordingly, at a given span of the bar $M_{bend} = 0$. Consequently:

$$Ff - N_1x_1 + N_1(x - l_1) = 0,$$

hence

$$N_1 = \left(\frac{Ff}{l_1}\right)$$

From the equations (6) we determine:

$$A = \frac{f}{\pi}; \quad l_1 = \frac{\pi}{\alpha}; \quad F = \frac{\pi^2EI}{l_1^2}; \quad v = \frac{f}{\pi}(sinax + \alpha x);$$

From the expression (7) it follows that for $l_1 = 0,5l$

$$F = \frac{4\pi^2EI}{l^2}$$

This means that in the case of

$$\frac{\pi^2EI}{l^2} < F < \frac{4\pi^2EI}{l^2}$$

the bar touches the wall only at one point and only when

$$F > \frac{4\pi^2EI}{l^2}$$

Abutment to the section occurs.

**Results and Discussion**

Let us consider the case of a different version of the problem. The bar dead weight will be taken into account with the appropriate additive $ql$ to axial force. Differentiating twice the equation (1) by $x$ we have:
\[
\frac{d^4v}{dx^4} + \frac{F \cdot d^2v}{EI_x \cdot dx^2} - \frac{q\omega^2}{gEI_z} v = 0 \tag{8}
\]

The boundary conditions are:

\begin{align*}
\text{At } x &= 0 \quad v = 0 \text{ and } v'' = 0, \\
\text{At } x &= 0.5l \quad v = f \text{ and } v' = 0.
\end{align*}

The critical force is determined by the Bubnov-Galerkin method, defining the deflection function:

\[
v(x) = a_1 \varphi(x) = a_1 \sin \left(\frac{n\pi x}{l}\right);
\]

The second and fourth derivatives have the form:

\[
v''(x) = -a_1 \left(\frac{n\pi}{l}\right)^2 \sin \left(\frac{n\pi x}{l}\right); \quad v''''(x) = -a_1 \left(\frac{n\pi}{l}\right)^4 \sin \left(\frac{n\pi x}{l}\right); \tag{9}
\]

Due to the fact that the equation (8) is equal to zero, for any function \(v(x)\) the orthogonality condition will be satisfied:

\[
\int_0^l \left(\frac{d^4v}{dx^4} + \frac{F \cdot d^2v}{EI_x \cdot dx^2} - \frac{q\omega^2}{gEI_z} v\right) \varphi(x) = 0
\]

We define the integrals separately:

\[
\int_0^l \left[\frac{d^4v}{dx^4}\right] \varphi(x) = \int_0^l \left[a_1 \left(\frac{n\pi}{l}\right)^4 \sin \left(\frac{n\pi x}{l}\right)\right] \sin \left(\frac{n\pi x}{l}\right) dx = a_1 \left(\frac{n\pi}{l}\right)^4 \left(\frac{l}{2} - \frac{\sin(2\pi n)}{4\pi n}\right);
\]

\[
\int_0^l \left[\frac{d^2v}{dx^2}\right] \varphi(x) = \int_0^l -\frac{F}{EI_z} \left[a_1 \left(\frac{n\pi}{l}\right)^2 \sin \left(\frac{n\pi x}{l}\right)\right] \sin \left(\frac{n\pi x}{l}\right) dx = -a_1 \frac{F}{EI_z} \left(\frac{n\pi}{l}\right)^2 \left(\frac{l}{2} - \frac{\sin(2\pi n)}{4\pi n}\right);
\]

\[
\int_0^l [v(x)] \varphi(x) = -\int_0^l \frac{q\omega^2}{gEI_z} \left[a_1 \sin \left(\frac{n\pi x}{l}\right)\right] \sin \left(\frac{n\pi x}{l}\right) dx = -\frac{q\omega^2}{gEI_z} a_1 \left(\frac{n\pi}{l}\right)^4 \left(\frac{l}{2} - \frac{\sin(2\pi n)}{4\pi n}\right);
\]

Finally we get:

\[
a_1 \left(\frac{n\pi}{l}\right)^4 \left(\frac{l}{2} - \frac{\sin(2\pi n)}{4\pi n}\right) - a_1 \frac{F}{EI_z} \left(\frac{n\pi}{l}\right)^2 \left(\frac{l}{2} - \frac{\sin(2\pi n)}{4\pi n}\right) - \frac{q\omega^2}{gEI_z} a_1 \left(\frac{n\pi}{l}\right)^4 \left(\frac{l}{2} - \frac{\sin(2\pi n)}{4\pi n}\right) = 0
\]

or

\[
a_1 \left[\left(\frac{n\pi}{l}\right)^4 - \frac{F}{EI_z} \left(\frac{n\pi}{l}\right)^2 - \frac{q\omega^2}{gEI_z}\right] = 0
\]

Hence, taking the expression of compressive force \(F = F_0 + ql\) into consideration, it follows that the expression for critical strength is:

\[
F_{kp} = \left(\frac{n\pi}{l}\right)^2 EI_z - \frac{q\omega^2}{g} \left(\frac{l}{n\pi}\right)^2 - ql \tag{10}
\]

For the first form of stability loss \((n = 1)\) we get:

\[
F_{kp} = \left(\frac{\pi}{l}\right)^2 EI_z - \frac{q\omega^2}{g} \left(\frac{l}{\pi}\right)^2 - ql \tag{11}
\]

At \(q = 0\) and \(\omega = 0\) the critical force corresponds to Euler force \(F_{cr} = \left(\frac{\pi}{l}\right)^2 EI_z\).
The analysis of the expression (6) shows that the additional load in the axial direction (due to its dead weight and rotation) leads to a decrease in the critical load. This allows us to assume that the total load on the bar considered by the problem can be represented by the expression

\[ F_\Sigma = F_0 + ql + \frac{q\omega^2}{g} \left( \frac{l}{\pi} \right)^2 \]  

At \( F_\Sigma > F_{Eul} \) the bar loses its stability and its middle part touches the pipe wall, which causes the reaction \( N \). To determine the reaction of the pipe wall to the bar, we compose the equation for lateral forces:

\[ N = R_0 - \frac{q\omega^2}{g} \int_0^x v(\zeta) d\zeta \]  

Using the value for deflection \( v(x) = a_1 \sin \frac{n\pi x}{l} \) (where \( l = l_1, a_1 = v_{max} \)) we get for the first form of stability loss

\[ N = R_0 + \frac{q\omega^2}{g} v_{max} \frac{l}{\pi} \]  

Horizontal response \( N \) in the left support of the bar is determined from the condition that in the area of the bar adjacent to the well walls, when \( F_\Sigma > F_{Eul} \) the bending moment \( M_{bend} = 0 \),

Composing the moments equation relative to the section point \( x_1 \), adjacent to the pipe walls:

\[ Fv_{max} - R_0 x_1 + N(x_1 - l_1) - \frac{q\omega^2}{g} \int_0^x v(x_1 - \zeta) d\zeta = 0 \]  

Substituting the expression for the deflections \( v(x) = a_1 \sin \frac{n\pi x}{l} \), we obtain for the first form of stability loss and for \( x_1 = l_1 \)

\[ R_0 = v_{max} \left( \frac{F}{l_1} + 2 \frac{q\omega^2 l_1}{g \pi} \right) \]  

Substituting the expression (16) into (14) we have the expression for \( N \)

\[ N = \left( \frac{F}{l_1} + 2 \frac{q\omega^2 l_1}{g \pi} \right) \]  

Reaction \( N \) creates a reactive moment in the wall \( M_R \), directed against the engine driving torque, which is represented by the expression:

\[ M_R = 2\varphi r N \]  

Here \( \varphi \) is the friction coefficient of a revolving bar against a pipe wall, \( r \) is the bar cross section radius.

Bending moment in any bar section, taking into account the general solution has an extremum:

\[ EI \frac{d^2\psi}{dx^2} = -v_{max} \left( \frac{\pi}{l} \right)^2 \sin \frac{\pi x}{l} \]  

\[ M_{max} = EI v_{max} \left( \frac{\pi}{l} \right)^2 \]  

The greatest stress is the sum of the bending stress and the compression stress:
Summary

The obtained formulas make it possible to determine the critical force of the system, taking into account its dead weight and centrifugal forces from the bar revolving for each individual case.

The presented expressions make it possible to determine the magnitude of the reactive moment from the contact of the revolving bar and the pipe wall. And in conclusion, the expression for the maximum stresses in the bar after stability loss is presented.

\[
\sigma_{max} = \sigma_{comp} + \sigma_{bend} = \frac{F}{A} + \frac{EIv_{\text{max}}^2 (\pi T)^2}{W}
\]

\(21\)

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