On Why Disks Generate Magnetic Towers and Collimate Jets

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ABSTRACT

We show that accretion disks with magnetic fields in them ought to make jets provided that their electrical conductivity prevents slippage and there is an ambient pressure in their surroundings.

We study equilibria of highly wound magnetic structures. General Energy theorems demonstrate that they form tall magnetic towers whose height grows with every turn at a velocity related to the circular velocity in the accretion disk.

The pinch effect amplifies the magnetic pressures toward the axis of the towers whose stability is briefly considered.

We give solutions for all twist profiles $\Phi(P) = \Omega(P)t$ and for any external pressure distribution $p(z)$. The force–free currents are given by $j = \tilde{\alpha}(P)B$ and we show that the constant pressure case gives $\tilde{\alpha} \propto P^{-\frac{1}{2}}$ which leads to analytic solutions for the fields.

Key words: Jets, quasars, star–formation, MHD.

1 INTRODUCTION

To take some of the most dynamic objects in the universe in which ‘apparently superluminal’ motions have been seen and to work on such problems using statics may seem an indication of a seriously deranged scientist. Nevertheless, the potential energies involved in any problem can be studied statically and it is those potential energies that drive the motion. Thus, without studies of the way the potential energies operate, the basic understanding of why the motions occur in the way they do may be lost. As Eddington (1926) said ‘the chief aim of the physicist in discussing a theoretical problem is to obtain “insight” – to see which of the numerous factors are particularly concerned in any effect and how they work together to give it’.

Even a perfect model of a phenomenon, that gives all the observables correctly, is not good science until it is analysed to show which aspects are essential for the phenomenon.

An unnecessarily detailed model, which reproduces the phenomenon, can actually be a barrier to understanding. The lack of real understanding of what makes red–giants is a case in point! It may require the insight of an Eddington rather than the calculations of a Chandrasekhar to simplify the model to the bare essentials.

Within an accretion disk any radial magnetic field will be sheared and stretched by the differential rotation, so the resultant toroidal magnetic field will grow until it is strong enough to arch up out of the disk with the gas flowing back down.

Hoyle & Ireland (1960) were the first to describe this instability but the conditions for its occurrence were more precisely calculated by Parker (1965). In the resulting configuration the mass of gas in the accretion disk continues to anchor and to twist the feet of the flux tube. We show here that if there is an ambient pressure in the tenuous gas above the accretion disk then the flux tube will grow in height with every twist of its feet but will not expand laterally. Thus a tall tower of magnetic field is formed whose collimation or aspect ratio $Z/R$ increases linearly with the number of twists.

My thesis is that an important prerequisite to understanding the dynamics of the jets above accretion disks is a serious study of the magneto-statics of the magnetic fields that they twist into their coronae. My first studies in this direction ended in total failure. In 1979 I had a mechanism that would give a tall tower of magnetic field that grew more collimated with every twist. With much enthusiasm I made a more detailed and exact calculation but to my amazement and chagrin the field managed to expand to infinity and disconnect itself after just over half a turn! The hoped for collimation that got better with every turn ended with only half a turn when the degree of collimation was not a needle-like, one degree, but a full 120 degrees! See Figure 1.
Figure 1. After half a turn a field stretches to very large distances in the absence of a confining coronal pressure. After \( \frac{2}{3} \) turns = 207 deg the field stretches to infinity and further turning does no work. For more recent work on field opening see Uzdensky et al. (2002) and Uzdensky (2002).

Years later I wrote up this problem (Lynden-Bell & Boily, 1994 P[aper I]) for a conference celebrating Mestel’s work. It proved to be a fine example of a phenomenon long advocated by Aly (1984, 1993) in solar MHD, see also Sturrock (1991). This rekindled my interest, but it still took two years for me to realise an ambient coronal pressure external to the field would prevent it expanding to infinity. To do so would take too much energy. When the field cannot expand to infinity, the continual turning of the accretion disk does wind many turns into the corona, provided the conductivities are great enough for flux freezing at the feet. Then the original argument comes into operation and tall towers will be generated whose heights grow with each successive turn. Basic theorems on magneto-statistics and much simplified models of such magnetic towers in a constant external pressure were given in paper II (Lynden-Bell, 1996) which used a rigidly rotating inner disk and a fixed outer disk to which the field returned. Simplified models with realistically rotating accretion disks were derived in a conference paper Lynden-Bell (2001). The present paper is a development of those calculations to allow for a pressure that decreases with height. We also find a better approximation for the distribution of twist with height along each field line. This improves on the assumption of a uniform distribution made earlier.

We derive the magneto-statistics of force–free magnetic fields whose feet have been twisted by an accretion disk. We study the equilibria as a function of the twist angles. The magnetic field lines labelled \( P \) being twisted by an angle \( \Phi(P) = \Omega(P) t \) at their feet which are anchored in the disk. Here \( \Omega(P) \) is the differential rotation of the feet of the flux tube and \( t \) is a parameter that gives the amount of that twist. If we increase \( t \) we change our equilibrium along a Poincaré sequence. Thus, in the language of his catastrophe theory, \( t \) is our control parameter. Of course, if all inertia were unimportant and if a real accretion disk were dragging the feet of the flux tubes at relative angular velocity \( \Omega(P) \), then this sequence of equilibria would give us a film of how the field would evolve in the presence of the external pressure field \( p(z) \). This would not give a true picture just because the inertial terms are not always negligible, nevertheless such sequences are very instructive in that they show us how the field would like to change if inertia did not slow its accelerations. We find it possible to get a reasonable understanding of this problem for any specified \( \Omega(P) \) and any chosen external pressure field \( p(z) \). We emphasise that we have the simplest magnetically dominated model. Above the disk, magnetism so dominates that the field is force–free everywhere within the towers, but at their surfaces the magnetic forces balance the ambient hot gas pressure \( p(z) \). Where there is gas there is no field, where there is magnetic field there is no gas pressure but we nevertheless assume perfect electrical conductivity. When we view our sequences of equilibria parameterised by \( t \) they are so reminiscent of what is seen in radio galaxies, quasars, star–forming disks with Herbig–Haro objects, etc., that it is hard to resist the temptation of talking in the terminology of dynamics rather than statics. Please remember that the velocities we talk of are only the velocities that would occur in the absence of inertia when accelerations can become arbitrarily large without penalty. We consider some effects of inertia in paper IV (Lynden-Bell, in preparation).

We believe that the very simple calculation given in section 2.3 contains the answer to the question ‘Why do flat accretion disks produce needle like magnetic jets?’, but further refinements, given in paper IV on detailed field structure, give greater understanding of why jets act as giant linear accelerators and why their electric currents are so concentrated to the tower’s axis.
and mini–quasars (Lynden-Bell, 1969, 1971), and the galactic centre (Ekers & Lynden-Bell, 1971; Lynden-Bell & Rees, 1974). Fiel d tubes anchored in the accretion disk at
We consider force-free magnetic configurations with the feet of purely toroidal fields (Contopoulos, 1995b), Winds and similar solutions for jets and has considered the possibility of the flux tubes confined by an external coronal cavity occupied by magnetic field the greater is the topology of the magnetic cavity whose cross-sectional area at height 

2 MAGNETOSTATIC THEOREMS AND DEDUCTIONS

2.1 Theorems

We consider force-free magnetic configurations with the feet of the flux tubes anchored in the accretion disk at \( z = 0 \) and the magnetic configuration confined by an external coronal gas pressure \( p(z) \) which may depend on height \( z \). Where there is magnetic field, there is no gas pressure. Earlier versions of these theorems were proved only for the constant pressure case in paper II. The work done against the pressure in making a magnetic cavity whose cross-sectional area at height \( z \) is \( A(z) \) is

\[
W_p = \int p(z)A(z)dz.
\]

This may be thought of as a potential energy. The larger the cavity occupied by magnetic field the greater is the \( pV \) energy stored, an asset that could be drawn upon in recession by contraction. The total potential energy of the configuration stored in both magnetic energy and \( W_p \) is

\[
W = 8\pi^{-1} \int (B_x^2 + B_y^2 + B_z^2) dV + W_p
\]

\[
= W_x + W_y + W_z + W_p.
\]

Contributions to these energies from slices at different heights \( z \) are given by

\[
w_x = (8\pi)^{-1} \int B_x^2 dx dy,
\]

over the area \( A \), etc and

\[
w_p = p(z)A(z).
\]

Evidently, \( W_x = \int w_x dz \), etc, and \( W_p = \int w_p dz \). The minimum of the potential energy give stable equilibrium configurations. At any equilibrium (stable or not) the energy is stationary, so the work done by any small displacement consistent with the constraints is zero. We consider a small vertical displacement caused by expanding the slice of the configuration between \( z \) and \( z + \delta z \) (see Figure 2) so that it now occupies the region between \( z \) and \( z + \mu \delta z \). The region below \( z \) is unchanged; the region above \( z + \delta z \) is lifted by \( (\mu - 1)\delta z \). The work done by the pressure at the edge of the slice is

\[
\frac{1}{2}(\mu - 1)\delta z \delta z (-dA/dz)p.
\]

which is second order in \( \delta z \) and does not concern us; but significant work is done on the pressure elsewhere because the area \( A(z') \) that was initially at \( z' \) has been moved to \( z' + (\mu - 1)\delta z \), so the area at \( z' \) is now

\[
A - (\mu - 1)\delta zdA/dz'.
\]

Thus the change in \( W_p \) due to the displacement is

\[
\Delta W_p = (\mu - 1)\delta z \int_0^\infty p(z')(-dA/dz')dz'.
\]

To get the second equality we integrated by parts and used the fact that \( A \) vanishes at sufficiently great heights. This expression does indeed yield \( pdV \) when the pressure is constant. Magnetic fluxes must be conserved during the displacement so

\[
B_x \to \mu^{-1}B_x, \quad B_y \to \mu^{-1}B_y, \quad B_z \to B_z.
\]

within the expanded slice whose volume increases by a factor \( \mu \). Evidently

\[
8\pi \Delta W_x = \mu^{-2} \int B_x^2 dz dy d\mu \delta z - \int B_x^2 dz dy \delta z = (\mu^{-1} - 1) \int B_x^2 dz dy \delta z
\]

\[
8\pi \Delta W_z = (\mu - 1) \int B_z^2 dz dy \delta z.
\]

Collecting all the \( \Delta W \) terms dividing by \( (\mu - 1)\delta z \) and letting \( \mu \to 1 \) we have theorem I:

\[
-w_x + \mu w_y + w_z + p(z)A(z) + \int_0^\infty A(z')dp/dz'dz' = 0 .
\]

Figure 2. A slice of a force–free field structure. The twists about the axis are not shown. The theorem is true without assumptions of axial symmetry. To prove theorem I we consider a virtual displacement in which the slice \( dz \) is expanded by a factor of \( \mu \). At equilibrium an infinitesimal expansion will do no work.
If we integrate over all $z$ (by parts for the last term), we obtain
\[-(W_z + W_y) + W_z + W_p + \int_0^\infty z A(z) dp/dz dz = 0. \tag{4}\]

The integrals in both (3) and (4) are negative when $p$ decreases with height but vanish when $p$ is constant. If $d\ln p/dn_z = -s$, then the final integral in (3) is $-s W_p$. Our next theorem is obtained by taking our virtual displacement to be a uniform lateral expansion with no vertical shift. Here we can not apply the virtual work principle unchanged since one of the constraints is that the vertical magnetic flux is frozen at the anchor points all over the disk $z = 0$. Our lateral expansion would violate this. However, there is an extension of the principle in which the constraints are replaced by the forces of constraint in the equilibrium configuration and the virtual work then contains a contribution from the work done against the forces of constraint. In our case these forces are those due to the disk that balance the equilibrium Maxwell stresses.

\[-(4\pi)^{-1} \left[ B_R B_z \Phi + B_p B_\phi \Phi + \frac{1}{2} \left( B_x^2 + B_y^2 - B_z^2 \right) \frac{\Phi}{\Phi} \right] \]

per unit area of disk. The work done on the disk in a radial displacement $\xi = \mu R$ is

\[\mu W_0 = (4\pi)^{-1} \mu \int B_R B_z R^2 d\phi dR \]

evaluated in $z = 0$.

The change in $W_p$ is just $(\mu^2 - 1) \int p Adz$ since $A \rightarrow \mu^2 A$, similarly $B_z \rightarrow \mu^{-1} B_z$, $B_y \rightarrow \mu^{-1} B_y$ and $B_z \rightarrow \mu^{-2} B_z$. Since $dW \rightarrow \mu^2 dW$ we find

\[\Delta W = (\mu^2 - 1) W_z + (\mu^2 - 1) W_p \]

with NO contribution from $W_z$ or $W_p$ since they are unchanged. The condition of equilibrium is no longer $\Delta W = 0$ but rather that any decrease in $W$ must be balanced by the work done against the forces of constraint i.e.,

\[-d\Delta W/d\mu_{\mu=1} = W_0 \]

Thus we deduce Theorem II

\[W_z - W_p = \frac{1}{\Phi} W_0. \tag{5}\]

Many scientists brought up on the pinch effect find it strange that the toroidal component of the magnetic field has NO effect on the change of magnetic energy in such a global lateral expansion. In fact, there is no general tendency for the whole magnetic structure to shrink towards an axis. In axial symmetry $W_z + W_y = W_R + W_\phi$ and none of them changes in a uniform contraction transverse to the axis. We return to explain the relationship of this result to the pinch effect in section 2.2.

If, in place of the lateral expansion at all heights, we freeze the configuration below some height $z$ and laterally expand everything above that height, then it is the magnetic stresses across the plane at height $z$ that become our new $W_0$ so we call their contribution $W_0(z)$. Likewise, the $W_z$ and $W_p$ involved are integrated over the region above height $z$ only. We shall call these quantities $W_z(z)$ and $W_p(z)$. We may then generalise (3) to give the exact result (6).

\[W_z(z) - W_p(z) = \frac{1}{8\pi} \int B_R B_z R^2 d\phi dR. \tag{6}\]

To localise theorem II consider a uniform lateral expansion that varies slowly with $z$ so that $R \rightarrow [1 + \mu(z)] R$. We shall take $\mu$ to be zero in $z = 0$ and to climb to a maximum at $z_1$ before declining again to zero. Provided $\mu(z)$ varies slowly enough the shear $\mu(z)$ will not produce significant radial field from the displacement of vertical field. If we neglect this effect, the scalings are as for theorem II and we get a version of it localised near $z_1$ (which we then replace by $z$). Then our theorem III reads for $z/R \gg 1$ and $dB/dz \ll |B|/R$

\[w_z \simeq w_p \tag{7}\]

$W_0$ does not contribute as there is no displacement on $z = 0$. (6) may also be derived by differentiating the result (5) with respect to $z$. That derivation demonstrates that (6) is valid provided $d/dz (\Phi W_0(z))$, is small compared with $w_z$, which is so provided $B_R \ll B_z$ at height $z$. Unlike theorems I and II, theorem III is approximate and only true well away from the disk. If we integrated it down to the disk it would conflict with theorem II except in the special case when $W_0$ is zero.

### 2.2 Winding Makes Tall Towers

We shall use cylindrical polar coordinates but we shall not assume axial symmetry. Then theorem I may be written

\[W_R + W_\phi = W_z + (1 - s) W_p \]

adding $W_z + W_p$ and using (theorem II)

\[W = W_R + W_\phi + W_z + W_p = (4 - s) W_p + W_0 = (4 - s) W_z + \frac{4}{\Phi} (s - 2) W_0. \]

Remember that for the constant pressure case $s = 0$. Here we shall assume $(-d\ln p/dn_z) = s < 4$. We shall show elsewhere that when $(-d\ln p/dn_z) \geq 4$ the magnetic field balloons off to infinity as in the highly wound pressureless case of paper I. We assume that even after several turns the field structure resists being wound still further and that the work done per turn is asymptotically a constant. Furthermore, we shall assume that the boundary term $W_0$ which only involves the field on $z = 0$ tends to a limiting value. Then in each turn $W$ will be raised by a finite $\Delta W$ and

\[\Delta W = (4 - s) \Delta W_p = (4 - s) \Delta W_z. \]

Hence $W_p$ and $W_z$ must increase without limit as the winding continues. Now $W_p = \int p(z) A(z) dz$ can increase by increasing $A(z)$ at given $z$ or by increasing the height to which the whole configuration reaches. However, increase of $A(z)$ for given $z$ does not increase $W_z$ since $B_z$ is of order $F/A(z)$ where $F$ is the poloidal flux. Thus $\int B_z^2 dx dy \propto F^2/A(z)$. Since $W_p$ and $W_z$ have to increase together we deduce that the height of the whole structure increases for each turn of the flux anchor points on the disk. This argument is reinforced by the very crude estimates of field structure that follow and by the more accurate but more specific field models calculated later. When the configurations are tall, certain simplifications occur in the theorems. Defining $\langle B_z^2 \rangle = A^{-1} \int \int B_z^2 dx dy$ theorem III becomes

\[\langle B_z^2 \rangle = 8\pi p(z), \tag{8}\]

where the average is taken at height $z$. Secondly, there is only a finite poloidal flux and as the system gets taller and taller the radial flux through a cylinder gets more and more...
spread out. Thus \( B_R \) becomes very small compared with \( B_\phi \) and \( B_z \). If \( W_y \) is neglected in theorem I, we have
\[
 w_\phi = w_z + w_p + \int_z^\infty A(z') dp/dz' dz'.
\] (9)
In the pressure constant case we see from (8) and (11)
\[
 w_\phi = 2w_z = 2w_p 
\]
so
\[
 \langle B^2 \rangle = 2\langle B^2_z \rangle = 16\pi p.
\] (10)
When \( p \) varies we define
\[
 \sigma(z) = w_p^{-1} \int_z^\infty A(z') (-dp/dz') dz',
\]
then we have
\[
 \langle B^2 \rangle = (2 - \sigma) \langle B^2_z \rangle = (2 - \sigma) 8\pi p,
\] (11)
When the vertical Maxwell stresses \((8\pi)^{-1} (B^2_z - B^2_\phi - B^2)\)
are integrated over a cross section we find a net tension of
\( w_z - w_\phi - w_p \) which becomes a force driving longitudinal
expansion when that quantity is negative as in (8), (9) or
(11).

2.3 Crude Estimates Give Essential Understanding

Let the total poloidal magnetic field emerging from an accretion disk on \( z = 0 \) be \( F \). We shall assume that it returns to the disk at some larger radius so that it is anchored at both ends. Suppose that a typical field line reaches a height \( Z \) and that the tall magnetic structure has a radius \( R \). Now each turn of the poloidal flux generates an equal toroidal flux which must pass through the area \( RZ \). Hence after \( N \) turns of the feet relative to one another the typical \( B_\phi \) is of order \( NF/RZ \) and the volume is \( \pi R^2 Z \) so
\[
 8\pi W_\phi = N^2 F^2 \pi Z.
\]
The vertical flux \( F \) goes once up and once down so if it goes
up in the inner \( R/\sqrt{2} \) and down outside that \( |B_z| \approx 2F/\pi R^2 \)
and have
\[
 8\pi W_z = 4F^2 Z/\pi R^2
\]
Finally the flux passes through a cylinder of radius \( R/\sqrt{2} \)
radially so
\[
 B_R = F/\sqrt{2}\pi R Z,
\]
and hence
\[
 8\pi W_R = F^2 / (2\pi Z).
\]
For a constant pressure \( p \) we have
\[
 W_p = p\pi R^2 Z.
\]
Hence
\[
 W = \frac{F^2}{\pi} \left[ \left( N^2 \pi^2 + \frac{1}{4} \right)/Z + \left( 4R^{-2} + 8\pi^3 pF^{-2} R^2 \right) Z \right].
\] (12)
Minimising \( W \) over all values of \( R^2 \) we find
\[
 \pi R^2 = F (2p)^\frac{1}{2},
\] (13) so the area is determined by \( F \) and \( p \) whatever the minimising \( Z \) may be and the two terms that make up the coefficient of \( Z \) are equal at equilibrium. Minimising \( W \) over \( Z \) we find
\[
 Z = N \pi \left( R/\sqrt{3} \right) \left( 1 + \frac{1}{2} N^{-2} \pi^{-2} \right)
\]
with \( R \) already given by (13) independent of \( N \), this clearly
shows that the height \( Z \) increases linearly with \( N \gg 1 \). Indeed, if we write \( N = (2\pi)^{-1} \Omega \) where \( \Omega \) is the relative angular velocity of the flux feet, then
\[
 Z \to \frac{1}{4\sqrt{2}} \Omega Rt,
\] (14) so the height of a steadily wound magnetic structure will grow with a velocity \( \propto \Omega R \). Notice that \( R \) is typically larger than the \( R \) of the inner foot of the flux tube at whose radius \( R \) is determined. Thus even with this excessively crude model we can see why the velocities of growing magnetic towers are directly related to the velocities in the accretion disk.

In this section we have assumed thus far that our growing
towers do not have hollow cores with no field in them; consider however the top of a tower on axis. There the magnetic field must splay out radially before descending down the exterior of the tower. Normally such division of one field line only occurs at a neutral point but here the magnetic field must resist the external pressure \( p \) so \( B^2/8\pi \) cannot be zero. Thus on the axis there has to be a most interesting cusp point. The cusp points downward and has a vanishing opening angle. The axial field line comes to the cusp at finite field strength and there splays very gradually at first, but then more rapidly with the field strength \( B^2/8\pi \) balancing the ‘external’ pressure which has invaded the axis and its neighbourhood, down to the level of the cusp. For points above such a cusp the tower will be hollow. Of course, it may be that this cusp is a mere dimple in the top of the tower, however the MHD calculations of Li et al. (2001), who were only able to calculate the towers for the first two turns, suggest that the towers may be hollow over a significant fraction of their height. Let us recalculate the energy assuming that our tower has a hollow core radius of \( R_c \) and a maximum radius \( R_m \). Following our previous calculations
\[
 B_\phi = NF/ (|R_m - R_c| Z),
\]
\[
 8\pi W_\phi = N^2 F^2 \pi (|R_m + R_c| / (R_m - R_c) )/Z.
\]
\[
 |B_z| = 2F/ \left[ \pi (R_m^2 - R_c^2) \right],
\]
\[
 8\pi W_z = 4F^2 Z/ \left[ \pi (R_m^2 - R_c^2) \right],
\]
\[
 B_R = F/ \left[ 2\pi (R_m^2 + R_c^2)^\frac{1}{2} Z \right],
\]
\[
 8\pi W_R = F^2 / (2\pi Z) \left( R_m^2 - R_c^2 \right) / (R_m + R_c^2),
\]
\[
 8\pi W_p = 8\pi^2 p \left( R_m^2 - R_c^2 \right) Z.
\]
Writing \( x = R_c/R_m \) we deduce
\[
 W = F^{-1} \left[ \frac{N^2 \pi^2 (1 + x^2)}{(1 + x^2)^2} + \frac{1}{(1 + x^2)^2} \right] Z^{-1} + \left[ 4 \left( 1 - x^2 \right)^{-1} R_m^2 \right] \left( 1 - x^2 \right) Z
\] (15) MINIMISING OVER \( R_m \) keeping \( x \) and \( Z \) fixed we find
\[ R_m^2 = (2\pi^3 p)^{-\frac{3}{2}} F \left(1 - x^2\right)^{-1} \]

so at these minima we have for all \( x \) and \( Z \)

\[ W_0 = F^2 \pi^{-1} \left\{ \left[ N^2 \pi^2 \left(1 + x^2\right) + \frac{2}{3} \left(1 - x^2\right)^2 / \left(1 + x^2\right)\right] Z^{-1+} \right. \]
\[ \left. + 8 \left(2\pi^3 p\right)^2 F^{-1} Z \right\} . \]

Minimising over all \( x \) in the range \( 0 \leq x < 1 \) we find that the quantity in square brackets increases throughout the range so the least value occurs at \( x = R_0/R_m = 0 \). Thus, at the level of this crude approximation, the towers are not hollow but filled with magnetic field.

### 2.4 Stability

Potential energy minima are always stable but if the minimisation is carried out under the restriction of axial symmetry there may exist asymmetrical distortions that lead to lower energy configurations. A useful insight was taught me by Moreno Insertis. Magnetic towers in tension are stable to sideways bowing. Like Euler struts, tall towers in compression are unstable. Such ideas require modification in the presence of an external pressure. The tower of fluid giving the ambient pressure is not unstable although it is under compression. Thus a magnetic tower that merely gives ambient pressure will not be unstable this way, rather it is pressures above ambient that cause such instabilities. The modified stability criterion is \( w_0 - w_z \leq w_p \). This criterion is only marginally satisfied by our constant pressure configurations for which the equality holds, so those will be pretty floppy, but when \( p \) decreases with height the buoyancy term provides stability.

### 2.5 The Pinch Effect as a Pressure Amplifier

By showing that the energy in the \( B_9 \) and \( B_R \) components of the field did not change in a uniform lateral expansion we demonstrated that there is NO tendency for a highly wound magnetic structure to contract overall laterally. Where then does the Pinch Effect come from? To elucidate this consider a specified toroidal magnetic flux \( B_\phi \) contained between inner and outer cylinders of radii \( R_0 \) and \( R_1 \) both of height \( Z \). Minimising the total energy over all possible \( B_\phi \)(\( R \)) that give the flux, the minimum occurs with \( B_\phi \propto R^{-1} \) and the energy is then

\[ W_\phi = \frac{1}{4} F_\phi^2 Z^{-1}/[\ln (R_0/R_1)] . \]

In uniform lateral contractions or expansions \( R_0/R_1 \) remains constant, but if we fix \( R_0 \) then \( W_\phi \) can be decreased by making \( R_1 \) smaller.

It is this that gives the pinch effect. Since \( B_\phi^2 \propto R^{-2} \) the toroidal magnetic field acts as a pressure amplifier delivering on the cylinder at \( R_1 \) a pressure \( R_0^2/R_1^2 \) times the pressure it exerts on the outer cylinder. The whole pinch effect fails if there is nothing to push on at the outside and indeed the field would expand outward to larger radii reducing the pressure on \( R_0 \) to zero. The Pinch Effect is a pressure amplifier; amplifying nothing gives nothing.

In the presence of a \( B_z \) the amplifier has less gain. By our theorems \((8\pi)^{-1} \int B_z^2 2\pi RdR = \pi \left( R_0^2 p_0 - R_1^2 p_1 \right) \) and if \((8\pi)^{-1} \int B_z^2 2\pi RdR = \eta \pi R_0^2 p_0 \) then \( p_1 = (1 - \eta) p_0 R_0^2 / R_1^2 \) so the amplifier only amplifies the excess pressure unbalanced by \( B_z^2 \).

### 3 Confined Force–Free Magnetic Fields

#### 3.1 Basic Equations

Inside the tower the magnetic fields dominate over any other forces so the magnetic field takes up a configuration that delivers no body force inside the jet, i.e., a force-free configuration with \( j \times B = 0 \). Thus \( j \) is parallel to \( B \) and we may write \( j = \tilde{\alpha} B \) where \( \tilde{\alpha} \) is a scalar function of position. Now both \( j \) and \( B \) have no divergence so we deduce that \( B \cdot \nabla \tilde{\alpha} = 0 \) which implies that \( \tilde{\alpha} \) is constant along each line of force. We consider axially symmetrical systems in cylindrical polar coordinates \((R, \phi, z)\) in terms of the poloidal flux function \( P(R, z) \) which gives the flux through a circle radius \( R \) at height \( z \). By flux conservation

\[ B_z = (2\pi R)^{-1} \partial P/\partial R , \]

and

\[ B_R = -(2\pi R)^{-1} \partial P/\partial z . \]

We write

\[ B_\phi = (2\pi R)^{-1} \beta , \]

and then deduce

\[ B = \nabla P \times \left(\phi/2\pi\right) + \beta \nabla (\phi/2\pi) , \]

\[ \beta \] is an axially symmetrical scalar function of position. Evaluating the curl of \( B \) we find

\[ 4\pi j = \nabla \times B = \]
\[ - \left[ R \partial / \partial R \left( R^{-1} \partial P / \partial R \right) + \partial^2 P / \partial z^2 \right] \nabla \left( \phi/2\pi \right) + \nabla \beta \times \nabla \phi / 2\pi . \]

The force-free condition \( j = \tilde{\alpha} B \) now gives, cf (19), both

\[ \beta \nabla = 4\pi \alpha \nabla P , \]

\[ R \partial / \partial R \left( R^{-1} \partial P / \partial R \right) + \partial^2 P / \partial z^2 = -4\pi \alpha \beta . \]

From (22) it follows that the normals to the surfaces of constant \( \beta \) are the normals to the surfaces of constant \( P \) so \( \beta \) is a function of \( P \) and

\[ \beta \left( P \right) = 4\pi \alpha . \]

Inserting this expression for \( \tilde{\alpha} \) into (22) we get the basic equation for force–free equilibrium (we write \( Q(P) \) for \( \beta \left( P \right) \))

\[ R \partial / \partial R \left( R^{-1} \partial P / \partial R \right) + \partial^2 P / \partial z^2 = -\beta \beta = -Q(P) . \]

From (22) we see that \( B \cdot \nabla P = 0 \) so that \( P \) is constant along a line of force. Indeed the equations for the lines of force follow from the condition \( ds || B \) which gives

\[ d\frac{R}{R} = \frac{R d\phi}{B_\phi} = \frac{dz}{B_z} , \]

or using the above expressions for \( B \)

\[ - \frac{dR}{\partial P / \partial z} = \frac{R d\phi}{\beta (P)} = \frac{dz}{\partial P / \partial R} . \]

We shall use this later to work out the twists of field lines.
3.2 Field Structure Above a Differentially Rotating Disk

Consider the tube of flux that rises within an inner circle of radius $R_0(P)$ on an accretion disk and returns to it at some outer radius $R_0(P)$. There will be a differential twisting due to the fact that $\Omega_t > \Omega_\circ$ and both are radius dependent. We define $\Omega(P) = \Omega_t - \Omega_\circ$ so $\Omega(P)$ is the rate of differential twisting of the field lines labelled $P$. At any time $t$ when the twist angle has accumulated to be $\Phi(P) = \Omega(P)t$, the field lines labelled $P$ will rise to some maximum height $Z(P)$ before heading down to re-intersect the disk at $R_0(P)$. We are concerned to know how this total twist $\Phi$ is distributed over the height $z$. In [lynden-Bell (2001)] I found that great progress could be made by simply adopting the idea that each field line had a certain twist per unit height $f(P)/Z(P)$, however I did not use the full power of the variational principle to derive the equations. Equations (25) for the lines of force show that

$$\frac{d\phi}{dz} = \beta(P)/(R\partial P/\partial R)$$

whereas $P$ and $\beta(P)$ are constant along each line, $R\partial P/\partial R$ is not. Now as the magnetic tower grows, its cross-section at any height will have some dimensionless profile $f$ with

$$P = P_m(z)f(\lambda).$$

Here $P_m(z)$ is the maximum value $P$ achieves on a cross section of height $z$. Now $Z(P_m(z)) = z$, so $P_m(z)$ is the inverse function of $Z(P_m)$. $\lambda$ is the fractional area of the cross section at height $z$, $\lambda = \pi R^2/A(z) = R^2/R_m^2$, where $R_m(z)$ is the radius of the magnetic cavity at height $z$ and $A(z)$ is its area. From these definitions the maximum value of $f$ is one and since $P$ is zero on axis and at the tower’s surface $f$ will be zero at $\lambda = 0$ and 1. If $f$ is independent of height the tower is said to be self-similar, however, in general the form of the function $f(\lambda)$ may depend on height although such changes may be slow except at the tower tops and near their feet. From (25)

$$R\partial P/\partial R = P_m(z)2\lambda f'(\lambda) = 2P/(d\ln\lambda/d\ln f).$$

Thus from (26)

$$d\phi/dz = \frac{1}{2}P^{-1}\beta(P)(d\ln\lambda/d\ln f).$$

Within any small height interval $dz$ the twist of the line for force labelled $P$ will have two contributions, one $+\frac{1}{2}P^{-1}\beta(P)(d\ln\lambda_0/d\ln f)dz$ as the flux rises through $z$ at $\lambda_1$ and a second $-\frac{1}{2}P^{-1}\beta(d\ln\lambda_0/d\ln f)dz$ as field lines descend through $z$ at $\lambda_0$. Thus the total contribution to the twist of the line from this height interval will be

$$d\Phi = \frac{1}{2}P^{-1}\beta(P)\left[\ln\lambda_0/\lambda_1 (d\ln f)\right]dz = \left[\frac{d\phi}{dz}|_{\lambda_1} - \frac{d\phi}{dz}|_{\lambda_0}\right]dz,$$

as both $\lambda_1$ and $\lambda_0$ correspond to the same $P$ and are at the same height, $z$, the value of $P/P_m(z) = f$ is the same for each.

Thus we may regard $\lambda_1$ and $\lambda_0$ as the roots for $\lambda$ of the equation $f(\lambda) = f$. A simple example will illustrate this.

Suppose $f$ is given by

$$1/f = \lambda^2/\lambda + (1 - \lambda)^2/(1 - \lambda).$$

This $f$ is clearly zero at $\lambda = 0$ and 1, we have chosen the coefficients to ensure $f = 1$ at the maximum at $\lambda = \lambda_1$. Now

$$f' = (\lambda_1 - \lambda)(\lambda_1 + \lambda - 2\lambda_1)/(\lambda_1 + \lambda(1 - 2\lambda_1))^2,$$

and at $\lambda = \lambda_1$

$$f'' = -2/(\lambda_1(1 - \lambda_1)).$$

Hence near the maximum of $f$ we find it is well approximated by

$$f = 1 - (\lambda - \lambda_1)^2[1/(1 - \lambda_1)]^{-1},$$

which is actually exact for all $\lambda$ when $\lambda_1 = \frac{4}{5}$. The roots for $\lambda$ are $\lambda_0 = \lambda_1 \pm [\lambda(1 - \lambda)]^{1/2}(1 - f)^{1/2}$ and

$$d\ln\left(\frac{\lambda_1}{\lambda_0}\right)\left[\ln((1 - f))\frac{1}{f} - \ln\left(\frac{1}{1 - (1 - \lambda_1)^2f(1 + f)}\right)\right]^{-1}$$

the curly bracket reduces to 1 when $\lambda_1 = \frac{4}{5}$. Near the maximum $1 - f \ll 1$ so the expression becomes $(1 - f)^{1/2}[\lambda_0 - 1]^{1/2}$. Away from the maximum $\lambda_1 \simeq \lambda/3f[1 + (1 - \lambda_1)^2f]$ and $\lambda_0 \simeq 1 - (1 - \lambda_1)^2f(1 + f)$ so we find

$$d\ln\left(\frac{\lambda_1}{\lambda_0}\right)/d\ln f \simeq [1 - (1 - \lambda_1)^2f]^{-1} + + (1 - \lambda_1)^2f[1 + (1 - \lambda_1)^2f]^{-1}.$$

For $\lambda_1 < \frac{4}{5}$ this gives $\frac{1}{2}P^{-1}\beta(P)(d\ln\lambda_0/d\ln f)$ does not vary greatly. The integrable singularity at the top of each field line, $P = P_m$ is not a peculiarity of the example chosen. Returning to the general case and looking near the maximum $f = 1$ at $\lambda = \lambda_1$ the roots $\lambda_0$ and $\lambda_1$ must behave as $\lambda_1 \pm c\sqrt{1 - f}$ so quite generally $d\ln([\lambda_0/\lambda_1]/d\ln f)$ will contain the $(1 - f)^{-1/2}$ factor near $f = 1$. We shall show in paper IV that in one limit $f = -e\ln\lambda$ and in the opposite limit $f = 2\lambda$ or $2(1 - \lambda)$ depending on whether $\lambda$ is less than or greater than $\frac{4}{5}$. In both these cases $d\ln([\lambda_0/\lambda_1]/d\ln f)$ is near 1 far from the top of the field line and in the latter case it is $(1 - f/2)^{-1}$ which varies from 1 to 2 over the whole range $0 \leq f \leq 1$.

Notwithstanding such variations we shall start with the rough approximation that twist is distributed uniformly with height so that its distribution function $g(P, z)dz = dz/Z(P)$

$$g(P, z)dz = dz/Z(P).$$

This integrates to 1 over the range $z = 0$ to $Z(P)$ and gives so simple a result that it may be used as a starting point of a refined treatment that gives greater weight near the tops of field lines see section 4. With the distribution $g(P, z)$ we calculate the $B_\phi$ flux between $z$ and $z + dz$. Each turn around the axis of an element of poloidal flux $dP$ generates equal toroidal flux so the twist $\Phi(P, z)dz$, gives a toroidal flux $2\pi^{-1} \Phi(P)g(P, z)dPdz$ in the element of height $dz$. Adding the contributions over all the $P$ that reach height $z$, i.e., those with $P \geq P_m(z)$ we find that the total toroidal flux per unit $dz$ is
We shall find it convenient to work in terms of average fields at a given height, so that the left-hand-side we re-write as $B_\theta(z) R_m$. Our former $\langle B_\theta^2 \rangle$ will be related to $\langle B_\theta^2 \rangle$ by some structural constant related to the profile of $B_\theta$ with $\lambda$. We define the ratio $J^2$ by

$$J^2 = \frac{\langle B_\theta^2 \rangle}{\langle B_\theta^2 \rangle} = 4 \frac{\int \beta^2(P) \lambda^{-1} d\lambda}{\int \beta(P) \lambda^{-1} d\lambda}^2.$$  

We note the tower hollow for $R < R_e$ then multiply the right-hand-side of (33) by $(R_\theta - R_e)/(R_\theta + |B_\theta|)$. Similarly for the $z$ components we define $|B_z|$ by

$$|B_z| = A^{-1} \int_{R_\theta}^{R_{m}} 2\pi R |B_z| dR = 2 \pi A,$$  

and

$$I^2 = \langle \frac{df}{d\lambda} \rangle^2 \lambda = \langle B_\theta^2 \rangle \lambda / |B_\lambda| \geq 1.$$  

If the profile $f$ depends on height then $I^2$ and $J^2$ will in general depend on height too, but for the tall towers generated by continual winding we may expect that the profile form, $f$, to set down to a typical one except near the tower top and bottom. Thus $I$ and $J$ will not vary strongly and indeed for self-similar towers they are dimensionless constants. In paper IV we find for the pressure constant case $I = 1.359$, $J = 2.999$ and for the linear case $I = 1.179$, $J = 1.098$.

Combining (33) with (34) and (35) and setting $R_e = 0$

$$\frac{P_m}{A} = \frac{P_m}{\pi R_m} = \frac{m^2}{I} = \frac{\langle B_\theta^2 \rangle}{I},$$  

in which all the symbols may be functions of $z$. We use (33) to write $R_m$ in terms of $P_m$ and $p$. Using (34) to evaluate $B_\theta$ via (33), (35) gives us the average $\langle B_\theta^2 \rangle$ at given $z$.

$$\langle B_\theta^2 \rangle = \int_0^{P_m} \langle \Phi(P)/Z(P)dP \rangle^2 \pi R_m^2.$$  

This gives the two terms equal, that by equality merely reproduces what is already contained in equation (33) in the form $\pi R_m = (2\pi)^{1/2} P_m I$. Inserting this $R_m$ into the first term and doubling the result as the second is equal to it, those terms of $8\pi W$ now reduce to $\int 4\sqrt{8\pi p} dP dZ$. Later we find it useful to change the independent variable from $z$ to $P_m$.

We therefore write $dz = dZ/dP_m dP_m$ and on changing the limits appropriately we obtain

$$\int_0^F 4\sqrt{8\pi p(z)} (dz/dP_m) dP_m.$$

We now introduce $\Pi(Z) = \int_0^Z 4\sqrt{8\pi p(z)} dz$ and after an integration by parts in which $Z$ and $\Pi$ vanish at one or other end point we have just $\int_0^F \Pi(Z) dP_m$ in place of the first two terms of $8\pi W$.

Using $\Pi(Z)$ as our variable in the remaining terms we find that we do not have to vary both the function $P_m(z)$ and its inverse function $Z(P_m)$, but $W$ still contains both $Z(P_m)$ and $Z(P)$. They are the same function (albeit of a different variable). The third term of $8\pi W$ in (33) simplifies greatly when integrated by parts

$$\int_0^F \left[ \int_0^{P_m} (\Phi(Z)/Z) dP_m \right] (-dZ/dP_m) dP_m =$$

$$= 2 \int_0^F \Phi(P_m) \int_0^{P_m} \Phi(Z) dP_m dP_m.$$

Again, we used the fact that $Z(F)$ is zero in the boundary terms. We reverse the order of integration using $\int_0^F dP_m \int_0^{P_m} dP_m dP_m$ to obtain

$$2 \int_0^F \Phi(Z) dZ \int_0^{F} \Phi(P_m) dP_m dP_m.$$

Finally we exchange the dummy variables and $P_m$ and $P$ to write the third term of $W$ in terms of $Z(P_m)$

$$8\pi W_0 = 2 \int_0^F \Phi(P_m) [Z(P_m)]^{-1} \int_0^{P_m} \Phi(P) dP dP_m.$$

Inserting all these simplifications into $8\pi W$ and using $\zeta(Z) = Z^{-1}$ as our variable we find

$$8\pi W = \int_0^F \Pi dP_m +$$

$$+ [J^2/(2\pi)] \int_0^F \Phi \zeta \int_0^{P_m} dP dP_m +$$

$$+ \int (F - P_m)^2 (2\pi)^{-1} (d\zeta/dP_m) dP_m,$$

where $\Pi = \Pi(Z)$ may equally be considered as a function of $\zeta$.

### 3.3 Solution of the Variational Equations

Varying $\zeta(Z)$ and demanding that $\Pi$ be a minimum is now easy since the second and third terms are linear in $\zeta$ and $\delta \Pi/\delta \zeta = -4IZ^2 \sqrt{8\pi p(Z)}$. Thus the variational equation for $Z(P_m)$ reads

$$4IZ^2 \sqrt{8\pi p(Z)} = (2\pi)^{-1} J^2 \Phi(P_m) \int_0^{P_m} \Phi dP + \pi^{-1} (F - P_m).$$

Here $P_m(z)$ and $R_m(z)$ are functions of $z$ to be varied, $Z(P_m)$ is the inverse function of $P_m(z)$ while $p(z)$ and $\Phi(P)$ are given functions. $I$, $J$, and $F$ we treat as fixed constants. Only the first two terms involve $R_m$, varying it and demanding that $W$ be a minimum for all such variations.
Thus, the function $Z(P_m)$ is given by

$$Z [p(Z)] = \frac{p}{Z} \left[ \int_{\pi}^{\Omega(P_m)} \int_{P_m}^{\pi} \frac{p(Z)}{\Omega(P)dP} \right] \frac{1}{\Omega}.$$  

The final $F - P$ term arises from our rough estimate of the $B_R$ term; unlike the $t^2$ term next door, it does not grow with time, so after a few turns it is negligible as we expected. Neglecting it we define

$$\tilde{\Omega}(P_m) = \left[ \Omega(P_m) \int_{P_m}^{\pi} \frac{p(Z)}{\Omega(P)dP} \right] \frac{1}{\Omega}.$$  

and we obtain the very pleasing result

$$Z [p(Z)] = C_1 P_m \tilde{\Omega}(P_m)t,$$  

where

$$C_1 = \frac{(\pi/2)}{\Omega[\Omega(1/4\pi)]} I^{-1/2}.$$  

At given $P_m$ (43) gives $Z$ as a function of $t$. Knowing that function we know how $p^\frac{1}{\gamma}$ behaves with $t$. That then shows us how $Z/t$ behaves with $t$; (44) shows us that, at constant $P_m$, $R_m$ also behaves as $p^\frac{1}{\gamma}$. Since $p(z)$ is a given function this serves to define $Z(P_m)$ or for that matter $P_m(z)$. Of course we still have to derive the values of the dimensionless structure functions $I$ and $J$ that we gave earlier, but these only affect the time-scale of the evolution. Using (43) to re-express $P_m^{\frac{1}{\gamma}}$ in terms of $R_m$ (44) takes the pleasing form

$$Z = \frac{1}{\Omega} \frac{F}{R_m^0} \tilde{\Omega} t$$  

which is a more sophisticated version of our result (44). Notice that (44) holds at each value of $P_m$ thus the collimation $Z(R_m)/R(P_m)$ grows linearly with time even when the pressure depends on height.

The shapes of the magnetic cavities as functions of time follow directly by substituting the value of $P_m$ in terms of $R_m$ and $p(Z)$ from (44) viz $P_m = \left[ 2\pi^2 p(Z) \right]^{1/2} R_m^2 / I$ into $\tilde{\Omega}(P_m)$ in (44). After that (43) becomes an equation connecting the radius $R_m$ of the magnetic cavity to the height $Z$ at which the cavity has that radius. We draw some examples of the evolution of the cavity shapes as functions of time in Figures 3 and 4. When $p$ is independent of $z$, $R_m(P_m)$ is independent of time by (44) so all the expansion takes place along $z$ axis. However when $p$ decreases with height $R_m(P_m) \propto \left[ p(Z) \right]^{-1}$ and the last factor increases as $z$ increases so it does increase with time. In spite of this $Z/R$ grows linearly with $t$ as seen in (44). A simple model will illustrate this, we take $p(z) = p_0 (1+z/a)^{-s}$. We need $s \leq 4$ so that the field does not splay out as in paper I. Equation (44) for $Z$ now reads, writing

$$V = \frac{(2\pi)^{1/2}}{\Omega[\Omega(1/4\pi)]} P_m^{\frac{1}{\gamma}} \tilde{\Omega}(P_m) \left[ \frac{z^2}{(1+z/a)^{s/2}} \right] = V t.$$  

Taking $s = 2$ as a simple example
\[ Z = V t \left[ \frac{V t}{a} + \sqrt{1 + V^2 t^2 / a^2} \right]. \]

Thus, when the pressure falls asymptotically like \( z^{-2} \), \( Z(t) \) starts out at constant velocity \( V \) but then accelerates and asymptotically it behaves as \( 2V^2 t^2 / a \) of course ram pressure and inertia which are missing from our model may reduce this acceleration and ram-pressure could even reverse it, nevertheless, without such our model predicts acceleration when the pressure falls with height. The radius corresponding to \( P_m \) is now given by

\[ R_m \propto P_m^{\frac{1}{2}} \left[ \frac{V t}{a} + \sqrt{1 + V^2 t^2 / a^2} \right]. \]

For small \( V t / a \) this is proportional to \( 1 + V t / a \) but for large \( V t / a \) this factor changes to \( 2V t / a \) so asymptotically \( R_m \) is proportional to time when \( Z \) is proportional to \( t^2 \). These results are dependent on our choice of \( s = 2 \). For general \( s \), \( Z \) asymptotes to \( V t [(Vt/a)^{1/(4-s)}] \) while \( R_m \) asymptotes to \( [Vt/a]^{1/(4-s)} \). Both these expressions demonstrate the very rapid expansion as \( n \) approaches 1 i.e., when \( p \) falls almost as fast as \( z^{-4} \) and indeed our former considerations were limited to pressure that fell less fast than \( r^{-4} \) in spherical coordinates. The great extent of radio jets might however be an indication that we should be considering not merely enclosed fields but those configurations that “reach out to infinity” We shall consider these again in another paper, but we remark here that if a field carrying the poloidal flux \( F \) reaches out to region where the pressure behaves as \( Hr^{-4} \) then the field will escape to infinity with \( B_z^2 = 8 \pi H r^{-4} \) and with opening angle \( \theta \) given by \( 2\pi [1 - \cos(\theta/2)] \). For small \( F / \sqrt{H} \) such opening angles can be small

\[ \theta \approx \left[ 2F^2 / (\pi^2 H) \right]^{\frac{1}{4}}, \]

and in such configurations the field will be close to radial from the source. It is possible that we should be considering the large scale radio jets as examples of this phenomenon rather than taking the pressure to fall off less strongly than \( r^{-4} \), but currently I feel the other case to be more general.

### 3.4 Estimation of the Twist Function \( \beta(P) \).

In the discussion following equation (28) we showed that the concentration of twist toward the top of each field-line behaved as \( (1 - f)^{-\frac{1}{2}} \). If we substitute this behaviour for \( \Phi(\lambda_0 / \lambda_0) d\lambda_0 \) in equation (28) and integrate along each line we find

\[ \Phi(P) \propto \Phi^{-1} \beta(P) \int_P^E \left[ 1 - (P/P_m) \right]^{-\frac{1}{4}} (-dZ/dP_m) dP_m. \]

With \( Z(P_m) \) now known we can deduce the form of \( \beta(p) \)

\[ \beta(P) \propto 2 \Phi(P) \int_P^E \frac{P_m (-dZ/dP_m) dP_m}{(P_m - P)^{\frac{1}{2}}} \] (46)

We can not go further without specific models for \( \Omega(P) \) and \( p(z) \).

### 3.5 Displaced Power Law Models

\( \Omega(P) \) has to be zero at \( P = F \) and it must fall as \( P \) increases from 0 to \( F \). If \( \Omega \) is finite at \( P = 0 \) then the simplest ‘power law’ model is the doubly displaced one

\[ \Omega(P) = \Omega_0 P_0^{\gamma} \left( (P_0 + P)^{-\gamma} - (P_0 + F)^{-\gamma} \right). \] (47)

We calculate in the regime \( P_0 \ll F, F - P \gg P_0 \) in which circumstance the \( (P_0 + F)^{-\gamma} \) term is negligible.

\[ \int_{P_m}^E \Omega(P) dP = \Omega_0 P_0^{2\gamma} \int_{P_m}^E \left[ \frac{1}{\gamma - 1} (P_0 + P)^{-\gamma} - (P_0 + F)^{-\gamma} \right]. \] (48)

where \( \varpi = P_0 + P_m \). From (4) the above expression is proportional to \( Z[p(Z)]^{\frac{1}{4}} \). Two cases give power laws; \( \gamma > 1 \) gives \( Zp^{\frac{1}{4}} \propto \varpi^{-\gamma - 1} \) so while \( 0 < \gamma < 1 \) gives \( Zp^{\frac{1}{4}} \propto \varpi^{-\gamma - 2} \). We take our former displaced power law for the pressure with \( s < 4 \)

\[ p = p_0 ([z/a] + 1)^{-s}. \]

Notice that, for \( z < a/s \), the pressure is almost constant so that regime is equivalent to taking \( s = 0 \) but, when \( z \) is large, \( p \propto z^{-4} \) so \( Z[p(Z)]^{\frac{1}{4}} \propto Z^{\frac{1}{4} - s} / 4 \). Hence we have the power laws

\[ Z \propto \varpi^{-\frac{s}{4}}, \] (49)

where

\[ \Gamma = \{ \begin{array}{ll} (\gamma - \frac{1}{2})S & \text{for } \gamma > 1, \\ \frac{1}{2} \gamma S & \text{for } \gamma < 1. \end{array} \}

Rewriting (48) in terms of \( \varpi \) setting \( \varpi_* = p_0 + P \)

\[ \beta \propto 2 \Omega(P) t \int_{\varpi_*}^{(P_0 + F)} \frac{1}{(\varpi - \varpi_*)^{\frac{1}{2} \gamma - 1}} d\varpi. \]

Now write \( x = \varpi / \varpi_* \) and recall \( \Omega(P) \propto \varpi_*^{-\gamma} \)

\[ \beta \propto P \varpi_*^{\gamma - 1 - s} M^{1 - S}, \] (50)

where

\[ M = \Gamma \int_{x = 1}^{\infty} \frac{[x - (P_0 / \varpi_*)^{\frac{1}{2}} x^{-(\Gamma + 1)}}{(x - 1)^{\frac{1}{2}}}. \]

and we have assumed \( [(P + P_0)/(P + P_0)] \gg 1 \) to allow the upper limit to tend to infinity.

Notice that, for \( P \gg P_0 \), \( M \) is independent of \( P \) and indeed \( M = \pi^2 \Gamma / (\Gamma - \frac{1}{2}) \) which is \( 3\pi/4 \) for \( \Gamma = \frac{1}{2} \) and tends to 1 as \( \Gamma \to 1 \); for \( P = 0, M = 1 \).

For the constant pressure case \( s = 0, S = 1, \) and when \( \gamma > 1 \)

\[ \beta \propto P/(P_0 + P)^{\frac{1}{4}}, \] (51)

so when \( P_0 \) is small compared with \( P, \beta \propto P^{\frac{1}{4}} \)

In general \( \Gamma - \gamma = \{ \begin{array}{ll} \gamma(S - 1) - \frac{1}{2} S & \gamma > 1, \\ \frac{1}{2} \gamma (S - 2) & 0 < \gamma < 1. \end{array} \}

For \( s = 2, S = 2 \) so \( \Gamma - \gamma = \{ \begin{array}{ll} \gamma - 1 & \gamma > 0, \\ 0 & 0 < \gamma < 1. \end{array} \}

In paper IV we shall solve for the fields in such power law cases. Particular interest centres around cases in which \( \beta \) is proportional to \( P^{\frac{1}{4}} \) or \( P \) which give rise to linear partial
differential equations for $P$ c.f. equation (24). The former occurs when pressure is constant i.e., $S = 1$ when $\gamma > 1$ wherever $P_0$ is small compared with $P$. This has solution $f = -e^{\lambda n L}$. The latter ($\beta \propto P$) occurs when $S = 2\gamma/(2\gamma - 1)$ and $\gamma > 1$ i.e., when $s = 2/\gamma$ and with $S = s = 2$ with $\gamma < 1$. Although the other power laws give non-linear equations they too can be solved except near the top of the tower. Again we leave the discussion to paper IV. Rotation in a gravity–dominated disk cannot fall off more rapidly than $R > R_s$ where the field at radius $R_s$ of the flux feet with $\Omega_s$ will be a co–rotation point in the disk at some point. Of course the field at radius $R > R_s$ can be approximated as constant at the value $-dZ/dlnP$. Since most of the contribution to the integral comes from near there we may take $-dZ/dlnP$ constant at that value and move it outside the integration which then becomes writing $x = P_m/P$ and $1 - x^{-1} = y^2$

$$\Omega_s = \Omega_s \left\{ y^{-1} \left[ 1 - y - \frac{3}{2} \left( 1 - y^{\frac{3}{2}} \right) \right] \left[ 1 - y^{\frac{3}{2}} \right] \right\}^{\frac{1}{2}}$$

4 AN IMPROVEMENT IN ACCURACY

We showed earlier that the twist of the field line labelled $P$ is given by

$$d\Phi/dz = \frac{1}{2} \beta(P) P^{-1} dln (\lambda/\lambda_0) / dln f,$$

and that this last factor had an integrable infinity that behaved as $(1 - f)^{-\frac{1}{2}} = [1 - (P/P_m)]^{-\frac{1}{2}}$. Here we show how to take some account of such a distribution of twist with height. In the total twist we shall have the factor

$$\int_0^Z \left[ 1 - (P/P_m) \right]^{-\frac{1}{2}} dz = Z + \int_{ln P}^{ln P_m} \left[ 1 - (P/P_m) \right]^{-\frac{1}{2}} - 1 \left(-dZ/dlnP \right) dlnP .$$

The quantity within { } is small when $(P/P_m)$ is small but becomes large close to $P_m = P$. Near there $-dZ(dlnP)/dlnP$ can be approximated as constant at the value $-dZ/dlnP$. Since most of the contribution to the integral comes from near there we may take $-dZ/dlnP$ constant at that value and move it outside the integration which then becomes

$$\mathcal{L}(P) = \int_0^{F/P} \left[ (1 - f)^{-\frac{1}{2}} - 1 \right] x^{-1} dx = \int_{\sqrt{1-F/P}}^{\sqrt{1-F/P}} \left( y^2 \right) dy = 2 \ln \left[ 1 + \sqrt{1-F/P} \right].$$

The small region where $P$ is close to $F$ inhabits very little of the volume as these lines of force only just emerge from the disk before returning to it and, for $P$ small, this integral is close to $\ln 4 = 1.38$. We may take account of the increased weight near the top of each field line by taking the distribution of twist with height to be the sum of a uniform distribution and a $\delta$ function just below the top. Thus

$$g(P,z) dz = [1 + \delta(z - Z_\text{c})] \left(-dZ(dlnP)\mathcal{L}(P)\right) N_\text{c}^{-1} Z^{-1} dz ,$$

where $Z_\text{c}$ is $Z(P)$ minus a very small quantity so that integrating up to $Z(P)$ integrates over the $\delta$ function. $N_\text{c}$ is the normalising factor that ensures that $g$ integrates to 1.

$$N_\text{c} = [1 + ( -dZ(dlnP)\mathcal{L}(P))].$$

Now $Z$ decreases as $P$ increases so $-dZ(dlnP)$ is positive and will show no great variation. Since $\mathcal{L}(P)$ is around unity for much of the range and $N_\text{c}$ is bounded below by 1 we shall simply replace by $-dZ(dlnP)$ by $-dZ_0/dlnP$ in the expression for $N_\text{c}$, where $Z_0$ is the approximate solution given by (24). We insert the above $g$ into (28) and find in place of

$$(B^z_0)^2 = (J/(2\pi))^2 \left[ \Phi P_m \mathcal{L}(P_m) N_1^{-1} Z^{-1} + \int_0^{P_m} \Phi / (N_1 Z) dP \right]^2 R_{m}^{-2} .$$

Writing $\zeta$ for $Z^{-1}$ our expression for $8\pi W_\phi$ becomes

$$(4\pi)^{-1} J \int_0^{P_m} \left[ \Phi^2 P_m^2 \mathcal{L}^2 N_1^{-2} d\Phi dP + 2 \Phi P_m \mathcal{L} N_1^{-1} d\Phi dP + 2 \Phi N_1^{-1} dP \right] dP_m .$$

3.6 Models with a Central Dipole

If a central object of radius $R_s$ rotates uniformly at angular velocity $\Omega_s$ and carries a dipole of strength $D$ then initially the field at radius $R$ on the disk is $B_2 = D/R_0^3$ for $R > R_s$. If the accretion disk is dominated by the central object then in centrifugal force gravity balance

$$\Omega_d = \left( GM_\star / R_0^3 \right)^{\frac{1}{2}}$$

Now before it is twisted the flux function $P$ of the dipole is $P = 2\pi DR_0^{-1} \sin^2 \theta$ so on the disk $P = 2\pi DR^{-1}$ for $R > R_s$. Now suppose $\Omega_d < (GM_\star / R_0^3)^{\frac{1}{2}}$ then there will be a co–rotation point in the disk at some point $R = R_c = \left[\Omega_c^2 (GM_\star / R_0^3)^{\frac{1}{2}} \right]$ where $P = P_c = 2\pi DR_c^{-1}$

Then for $P > P_c$ we shall have a differential winding of the flux feet with $\Omega_s - \Omega_d$ negative while for $P > P_c$ we shall have a differential winding with $\Omega_s - \Omega_d$ positive.

So

$$\Omega(P) = \Omega_s - (GM_\star / R_0^3)^{\frac{1}{2}} \left[ P / (2\pi D) \right]^{\frac{1}{2}} = \Omega_s \left[ 1 - \left( P / P_c \right) \right]^{\frac{1}{2}} .$$

Assuming that all the flux is wound up there will no doubt be continued reconnection close to $P = P_c$ where the toroidal flux runs in opposite directions. It then becomes of interest to ask whether a greater poloidal flux crosses the equator in the range $R_s$ to $R_c$ or in the range $R_0$ to $\infty$. Since $P$ falls like $R^{-1}$ outside the star half the flux is within $R_s$ so if the star rotates at less than $\sqrt{E (GM_\star / R_0^3)}$ then the majority of the poloidal flux is dragged forward by the disk while if the star rotates at closer to breakup speed the disk will be dragging the majority of the flux forward. Of course the central object’s field may be so strong that it enforces co–rotation as the inner disk without being able to do so further out. There is yet another interesting radius in the problem, if we think of the oppositely wound fields then at what radius is the rate of generation of toroidal flux within balanced by the rate of generation of net toroidal flux of the other sign on the outside? A little thought shows this can only occur when $R_c = \left( \frac{4}{3} \right)^{\frac{1}{2}} R_s = 1.84 R_s$. On integration (43) gives, using (42) with $P_c = F$

$$Z^2 \sqrt{P(Z)} = \left( \frac{1}{4\pi} \right)^{\frac{1}{2}} \left( \frac{\pi}{2} \right)^{\frac{1}{2}} J^2 I^{-1} F y \Omega^2 ,$$

where $y = P_m / F$. 

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The first and last terms are linear in $\zeta$ so their variation is simple. The central term while homogeneous of degree 1 in $\zeta$ is not linear because of the $d\ln \zeta/dP_m$ term. Varying this central term inside the integral and writing $\Psi = d/dP(\Phi P L/N_{i})$ we find

$$
\int_{0}^{P} -2\delta \zeta \left\{ \Phi N^{-1} P_{m} L + + Z \Psi(P_m) \int_{P_{m}}^{P} \Phi/(N_{i} Z) dP - - \Phi N^{-1} \int_{P_{m}}^{P} \ln[Z(P)/Z(P_{m})] \Psi(P) dP \right\} dP_m.
$$

The coefficient of $\delta \zeta$ is homogeneous of degree zero in $Z$ but causes significant difficulties when one tries to solve the equations for the minimising $Z(P_m)$. We surmount these difficulties by evaluating these terms with our zero order solution $Z = Z_0(P_m)$ as given by (44). If greater accuracy is required we can then solve and put back the solution iteratively.

The variation of $8\pi W_{\phi}$ as given above leads us to the following equation for $\delta(8\pi W)$ in place of (43)

$$
4I \sqrt{8\pi Z^4 p(Z)} = (2\pi)^{-1} J \left\{ \frac{N^{-1}_{i} \Phi(P_m) \int_{P_{m}}^{P} \Phi/(N_{i} Z) dP}{-\Phi N^{-1} L P_m \Psi(P_m) - - \Phi^2 N^{-2} L P_m - - z_0 \Psi(P_m) \int_{P_{m}}^{P} \Phi/(N_{i} Z) dP + + \frac{1}{N_{i} Z_0(P_m)} \int_{P_{m}}^{P} \Psi(P) \times \times \ln \left[ \frac{Z_0(P)}{Z_0(P_m)} \right] dP \right\}. \tag{55}
$$

All terms on the right are known so this gives us a refined solution for $Z(P_m)$. Every variable outside an integral is a function of $P_m$ and, except where stated explicitly, variables inside an integral are functions of $P$. While the above expression may be more accurate it provides far less understanding than our former solution (44). As our aim should be insight rather than accuracy we have used (44) in preference to (55).

A method of improving on the solution (44) so that the variable twist with height is properly allowed for is given in the Appendix.

### 5 CONCLUSIONS

A sequence of static models can be more illuminating than detailed dynamical simulations. The statics should be understood before the dynamics is attempted. The continued winding of the magnetic field of an accretion disk will build up tall towers which make magnetic cavities, towering bubbles of magnetic field in the surrounding medium. Such a conclusion does not depend on axial symmetry.

Non–axysymmetric behaviour is observed in the pretty plasma experiments of Hsu & Bellan (2002).

When that symmetry is imposed we can calculate the tall tower shapes of the magnetic cavities for any prescribed winding angles $\Phi(P) = \Omega(P)$ and for any prescribed pressure distribution $p(z)$. Examples of these towers are shown in figures 3 and 4. The heights of these towers grow at a velocity closely related to the maximum circular velocity in the accretion disk. Our primary results are encapsulated in equations (44) and (45).

As stellar–mass black–holes form, the winding of the magnetic field of the collapsing core may cause jets to emerge from supernovae as first predicted by Le Blanc & Wilson (1970). Such ideas may have application to some $\gamma$–ray bursts, to the micro–quasars (Mirabel & Rodriguez, 1999) and to SS433 (Mason, 1984). The possibility that the elongated hour–glass planetary nebulae, some of the most delicate objects in the sky, may be magnetic towers arising from the accretion disks of their central binaries is particularly appealing.

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APPENDIX

APPENDIX A: PERTURBATION THEORY

FOR THE VARIABLE TWIST PROBLEM

We would like to give the field lines the correct distribution of twist with height approximately

\[ g(P, z) = [1 - (P/P_m)]^{-\frac{1}{2}} \frac{dz}{L(P)}, \]  

(A1)

where

\[ L(P) = \int_0^Z [1 - (P/P_m(z))]^{-\frac{1}{2}} dz \]

\[ = \int_0^P [1 - (P/P_m)]^{-\frac{1}{2}} (-dZ/P_m) dP_m, \]

putting this into \((A2)\) gives in place of \((B3)\)

\[ (B3) = \int_0^Z (2\pi)^{-\frac{1}{2}} \Phi(P) \times \left[ 1 - \left( \frac{P}{P_m} \right)^{-\frac{1}{2}} \right]^{-1} L^{-1} dP \]

which in turn gives an 8\(\pi\)\(W_\phi\)

\[ 8\pi W_\phi = (4\pi)^{-1} J^2 \times \int_0^P \left\{ \int_0^{P_m} \Phi(P)[1 - \left( \frac{P}{P_m} \right)^{-\frac{1}{2}}] L^{-1} dP \right\}^2 \times \left[ \frac{-dZ}{dP_m} \right] dP_m. \]  

(A2)

Here the function to be varied \(Z(P_m)\) is deeply embedded in the integrals that define \(L(P_m)\) as well as occurring explicitly. There is no simple way of solving the variational equations that result, that is why we chose to consider the simpler forms of \(g\) given in equations \((51)\) and \((52)\). However the variational principle \((51)\) based on \((50)\) is so simple that a perturbation theory based on it can be developed. We write \(W\) for the energy based on \((52)\) and \(\tilde{W}\) for the energy based on \((53)\) so that

\[ W[\zeta] = \tilde{W}[\zeta] + (W_\phi - \tilde{W}_\phi). \]  

(A3)

Now \(8\pi \tilde{W}\) minimises at \(\zeta_0\) and only the first II term is nonlinear so we may write

\[ 8\pi \tilde{W}[\zeta] = 8\pi \tilde{W}[\zeta_0] + \int_0^P \left[ \delta^2 \Pi/8\zeta_2^2 (\zeta - \zeta_0) + \delta^2 \Pi/8\zeta_3^3 (\zeta - \zeta_0)^3 \right] dP_m. \]  

(A4)

Here \(\zeta_0 = 1/Z_0\) which is the solution given by \((44)\). We believe that \(\zeta_1 = 1/Z_1\) where \(Z_1\) is given by \((52)\) will be close to the minimum of \(W\). Our problem is to find the correction of \(\zeta_1\) which will give \(\delta W/\delta \zeta = 0\) more accurately. Evidently from \((A3)\)

\[ \delta W/\delta \zeta = \delta W/\delta \zeta + \delta (W_\phi - \tilde{W}_\phi)/\delta \zeta. \]

Hence using \((A4)\) and evaluating \(\delta (W_\phi - \tilde{W}_\phi)\) etc., at the approximate solution \(\zeta_1\) we find

\[ \zeta - \zeta_1 = - \frac{8\pi \delta \left( W_\phi - \tilde{W}_\phi \right)}{\delta \zeta_1 + \delta^2 \Pi/\delta \zeta_1 \left( \zeta_1 - \zeta_0 \right) + \delta \delta^3 \Pi/\delta \zeta_0^3 (\zeta_1 - \zeta_0)^2}, \]

\[ \delta \delta^3 \Pi/\delta \zeta_0^3 = 4IZ^2 \partial/\partial Z \left[ Z^2 \sqrt{8\pi \zeta (Z)} \right], \]

and

\[ \delta^3 \Pi/\delta \zeta_2^2 = \delta^3 \Pi/\delta \zeta_3^3 = 4IZ^2 d/dZ \left[ \frac{Z^2 d/dZ (Z^2 \sqrt{8\pi \zeta (Z)})}{\sqrt{8\pi \zeta (Z)}} \right], \]

so all those quantities are quite simple functions of \(Z\) and we have to evaluate them at \(Z = Z_0\) the solution \((44)\). Thus all terms on the right are known except \(8\pi \delta (W_\phi - \tilde{W}_\phi)/\delta \zeta_1\) which we have to evaluate at \(\zeta_1\). We refrain from giving the gory details of the evaluation but

\[ 8\pi \delta \left( W_\phi - \tilde{W}_\phi \right)/\delta \zeta_1 = -2\pi J^2 EZ \left[ d(EZ)/dP_m + \delta Y/\delta \zeta - 2P_m \bar{\Omega}^2 \right], \]

where \(E(P_m) = \int_0^{P_m} \Phi(P)/(2\pi L) \left[ 1 - (P/P_m) \right]^{-\frac{1}{2}} dP\)

\[ \delta Z/\delta \zeta = Z^2 d/dZ \left[ Z^2 \sqrt{8\pi \zeta (Z)} \right], \]

\[ \delta Y/\delta \zeta = \delta^2 \Pi/\delta \zeta_1 \left( \zeta_1 - \zeta_0 \right) + \delta \delta^3 \Pi/\delta \zeta_0^3 (\zeta_1 - \zeta_0)^2. \]

In these terms we must put \(Z = Z_1\) the solution given in \((53)\). Thus we can in principle obtain by perturbation theory a solution to the variable twist problem.

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