ON THE DIRECT IMAGES OF PARABOLIC VECTOR BUNDLES AND PARABOLIC CONNECTIONS

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Abstract. Let \( \varphi : Y \to X \) be a finite surjective morphism between smooth complex projective curves, where \( X \) is irreducible but \( Y \) need not be so. Let \( V_* \) be a parabolic vector bundle on \( Y \). We construct a parabolic structure on the direct image \( \varphi_*V \) on \( X \), where \( V \) is the vector bundle underlying \( V_* \). The parabolic vector bundle \( \varphi_*V_* \) on \( X \) obtained this way has a ramified torus sub-bundle; it is a torus bundle of \( \text{Ad}(\varphi_*V) \) outside the parabolic divisor for \( \varphi_*V_* \) that satisfies certain conditions at the parabolic points. Conversely, given a parabolic vector bundle \( E_* \) on \( X \), and a ramified torus sub-bundle \( T \) for it, we construct a ramified covering \( Z \) of \( X \) and a parabolic vector bundle \( W_* \) on \( Z \), such that the parabolic bundle \( E_* \) is the direct image of \( W_* \). A connection on \( V_* \) produces a connection on \( \varphi_*V_* \). The ramified torus sub-bundle for \( \varphi_*V_* \) is preserved by the logarithmic connection on \( \text{End}(\varphi_*V) \) induced by this connection on \( \varphi_*V_* \). If the parabolic vector bundle \( E_* \) on \( X \) is equipped with a connection \( D \) such that the connection on the endomorphism bundle induced by it preserves the ramified torus sub-bundle \( T \), then we prove that the corresponding parabolic vector bundle \( W_* \) on \( Z \) has a connection that produces the connection \( D \) on the direct image \( E_* \).

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1. Introduction

Let $X$ be a smooth complex projective curve and $S$ an effective reduced divisor on $X$. A parabolic bundle on $X$ with parabolic divisor $S$ is a vector bundle on $X$ equipped with a filtration of subspaces of the fiber over each point of $S$ together with a system of weights for these filtrations. Parabolic bundles were introduced by Mehta and Seshadri [MS]; parabolic vector bundles on higher dimensional varieties were introduced by Maruyama and Yokogawa [MY].

Let $\varphi : Y \rightarrow X$ be a ramified covering. Take a parabolic vector bundle $V_*$ on $Y$ with underlying vector bundle $V$. We construct a parabolic structure on the direct image $\varphi_*V$ over $X$. This is done using the parabolic structure of $V_*$ and the ramifications of the covering map $\varphi$; the details are in Section 3. This construction is compatible with other structures on parabolic bundles. For example, a connection on $V_*$ produces a connection on the direct image $\varphi_*V$ (see Theorem 5.2).

Let $E_*$ be a parabolic vector bundle on $X$ with parabolic divisor $S$ and rational parabolic weights. Denote the vector bundle underlying $E_*$ by $E$. Let

$$\text{Ad}(E) \subset \text{End}(E) = E \otimes E^*$$

be the locus of invertible endomorphisms. A ramified torus sub-bundle for $E_*$ is a torus sub-bundle (fibers are tori)

$$\mathcal{T} \subset \text{Ad}(E)|_{X\setminus S}$$

satisfying a certain condition on $S$ (see Definition 4.2). We show that the above mentioned direct image $\varphi_*V_*$ has a natural ramified torus sub-bundle (Proposition 4.4). If $D$ is a connection on $V_*$, then the connection on $\varphi_*V_*$ given by $D$ is compatible with this ramified torus sub-bundle for $\varphi_*V_*$ (Lemma 5.4).

In the reverse direction, take a parabolic vector bundle $E_*$ on $X$ with rational parabolic weights. Take a ramified torus sub-bundle $\mathcal{T}$ for $E_*$. Then there is a ramified covering

$$\phi : Z \rightarrow X$$

and a parabolic vector bundle $W_*$ on $Z$, such that $\phi_*W_* = E_*$, and the ramified torus sub-bundle for the direct image $\phi_*W_*$ coincides with $\mathcal{T}$ (see Proposition 6.1). We prove the following in Theorem 6.3.

**Theorem 1.1.** Let $E_*$ be a parabolic vector bundle on a connected smooth complex projective curve $X$ with parabolic divisor $S$ and rational parabolic weights. Then there is a natural equivalence between the following two classes:

1. Triples of the form $(Y, \varphi, V_*)$, where $\varphi : Y \rightarrow X$ is a ramified covering map, and $V_*$ is a parabolic vector bundle on $Y$, such that $\varphi_*V_* = E_*$.
2. Ramified torus bundles for $E_*$.  

When the torus is maximal, which means that the parabolic bundle $V_*$ in Theorem 1.1 is a line bundle, then Theorem 1.1 was proved in [AB].
As before, \(E_s\) is a parabolic vector bundle on \(X\) with rational parabolic weights, and \(\mathcal{T}\) is a ramified torus sub-bundle for \(E_s\). Now take a connection on \(D\) that preserves \(\mathcal{T}\). We prove that \(D\) induces a connection \(D'\) on the corresponding parabolic vector bundle \(W_s\) over the covering \(Z\) such that the connection on \(\phi_sW_s = E_s\) given by \(D'\) coincides with \(D\) (see Proposition 6.4). We prove the following in Theorem 6.6.

**Theorem 1.2.** The equivalence in Theorem 1.1 takes a connection on the parabolic vector bundle \(V_s\) on \(Y\) to a connection on \(E_s\) that preserves the ramified torus sub-bundle for \(E_s\) corresponding to \((Y, \varphi, V_s)\). Conversely, a connection on \(E_s\) preserving a ramified torus sub-bundle \(\mathcal{T}\) for \(E_s\) is taken to a connection on the parabolic vector bundle \(V_s\) on \(Y\), where \((Y, \varphi, V_s)\) corresponds to \(\mathcal{T}\).

When there is no parabolic structure and the coverings are unramified, then Theorem 1.1 and Theorem 1.2 were proved in [BCW]. Direct images of vector bundles under finite maps of curves are also studied in [DePa].

2. **Parabolic vector bundles and connections**

2.1. **Parabolic vector bundles.** Let \(X\) be an irreducible smooth complex projective curve. Take an algebraic vector bundle \(E\) over \(X\) of rank \(r\). The fiber of \(E\) over any closed point \(z \in X\) will be denoted by \(E_z\).

A *quasi-parabolic* structure on \(E\) consists of the following data:

- a finite set of distinct closed points \(S = \{x_1, \cdots, x_n\} \subset X\), and
- for each \(x_i \in S\), a filtration of subspaces
  \[
  E_{x_i} = F_{x_i}^1 \supseteq F_{x_i}^2 \supseteq \cdots \supseteq F_{x_i}^{\ell_i} \supseteq F_{x_i}^{\ell_i+1} = 0.
  \]

The above subset \(S\) is called the *parabolic divisor*. A *quasi-parabolic bundle* is a vector bundle equipped with a quasi-parabolic structure. A *parabolic vector bundle* is a quasi-parabolic vector bundle \((E, S, \{\{F_j^i\}_{j=1}^{\ell_i}\}_{i=1}^n)\) as above together with real numbers \(\lambda_j^i, 1 \leq j \leq \ell_i, 1 \leq i \leq n\), such that

\[
0 \leq \lambda_1^i < \lambda_2^i < \cdots < \lambda_{\ell_i-1}^i < \lambda_{\ell_i}^i < \lambda_{\ell_i+1}^i = 1.
\]

These numbers \(\lambda_j^i\) are called the *parabolic weights*. For notational convenience, a parabolic vector bundle \((E, S, \{\{(F_j^i, \lambda_j^i)\}_{j=1}^{\ell_i}\}_{i=1}^n)\) is abbreviated as \(E_s\). See [MS], [MY] for more on parabolic vector bundles.

For any \(x_i \in S\), and any real number \(0 \leq c \leq 1\), define

\[
E_{x_i}^c := F_{j(c)}^i \subset E_{x_i},
\]

where \(1 \leq j(c) \leq \ell_i + 1\) is the smallest integer such that \(c \leq \lambda_{j(c)}^i\). Evidently, the parabolic structure of \(E_s\) at the parabolic point \(x_i\) is uniquely determined by the weighted subspaces \(\{E_{x_i}^c\}_{0 \leq c \leq 1}\) of \(E_{x_i}\). We note that in [MY] p. 80, the parabolic structure is in fact defined this way.

The parabolic degree of \(E_s\), which is denoted by \(\text{par-deg}(E_s)\), is defined to be

\[
\text{par-deg}(E_s) := \text{degree}(E) + \sum_{i=1}^n \ell_i \sum_{j=1}^{\ell_i} \lambda_j^i \cdot \dim F_j^i / F_{j+1}^i.
\]
The notions tensor product, dualization and homomorphism bundles for usual vector bundles extend to the context of parabolic vector bundles \[\text{Yo}, \text{Bi2}\]. More precisely, for any parabolic vector bundles \(E_\ast\) and \(E'_\ast\), there are the parabolic tensor product bundle \(E_\ast \otimes E'_\ast\), parabolic dual bundle \(E_\ast^*\) and parabolic homomorphism bundle \(\text{Hom}(E_\ast, E'_\ast) = E'_\ast \otimes E_\ast^*\); there are all parabolic vector bundles (see \[\text{Yo}, \text{Bi2}\] for their construction). However, we will recall below the description of the vector bundle underlying the parabolic bundle \(\text{End}(E_\ast) = \text{Hom}(E_\ast, E_\ast) = E_\ast \otimes E_\ast^*\).

As before, let \(E\) be the vector bundle underlying the parabolic bundle \(E_\ast\). Consider the vector bundle \(\text{End}(E) = \text{Hom}(E, E) = E \otimes E^*\).

Let
\[
\text{End}^p(E) \subset \text{End}(E)
\] (2.3)
be the sub-sheaf such that for any section
\[
s \in \Gamma(U, \text{End}^p(E)) \subset \Gamma(U, \text{End}(E))
\]
defined over an open subset \(U \subset X\), we have
\[
s(x_i)(F^i_j) \subset F^i_j
\]
for all \(x_i \in S \cap U\) and \(1 \leq j \leq \ell_i\), where \(S\) as before denotes the parabolic divisor for \(E_\ast\). Note that the inclusion of \(\text{End}^p(E)\) in \(\text{End}(E)\) is an isomorphism over the complement \(X \setminus S\).

The vector bundle underlying the parabolic bundle \(\text{End}(E_\ast)\) is \(\text{End}^p(E)\) constructed in (2.3).

Let \(Y\) be a smooth complex projective curve which need not be connected. By a vector bundle on \(Y\) we will mean a vector bundle on every connected component of \(Y\). Note that we do not assume that they all have the same rank independent of the components. Similarly, by a parabolic bundle on \(Y\) we will mean a parabolic vector bundle on every connected component of \(Y\); their rank is allowed to be different on different connected components.

### 2.2. Connections on a parabolic vector bundle.

The cotangent line bundle of \(X\) will be denoted by \(K_X\).

As before, let \(E_\ast\) be a parabolic vector bundle with parabolic divisor \(S\) such that the underlying vector bundle is \(E\). The line bundle \(K_X \otimes \mathcal{O}_X(S)\) will be denoted by \(K_X(S)\) for notational convenience.

We note that for any point \(y \in S\), the fiber \(K_X(S)_y\) is identified with \(\mathbb{C}\) by sending any meromorphic 1-form defined around \(y\) to its residue at \(y\). More precisely, for any holomorphic coordinate function \(z\) on \(X\) defined around the point \(y\), consider the homomorphism
\[
R_y : K_X(S)_y \longrightarrow \mathbb{C}, \quad c \cdot \frac{dz}{z} \longmapsto c.
\] (2.4)
This homomorphism is in fact independent of the choice of the above coordinate function \(z\).
A logarithmic connection on $E$ singular over $S$ is an algebraic differential operator of order one
\[ D : E \longrightarrow E \otimes K_X(S) \]
such that $D(fs) = fD(s) + s \otimes df$ for all locally defined algebraic function $f$ and locally defined algebraic section $s$ of $E$.

For a logarithmic connection $D$ on $E$ singular over $S$, and a point $y \in S$, consider the composition
\[ E \overset{D}{\longrightarrow} E \otimes K_X(-y) \overset{\text{Id} \otimes R_y}{\longrightarrow} E_y \otimes \mathbb{C} = E_y, \]
where $R_y$ is the homomorphism in (2.4). This composition homomorphism evidently vanishes on the sub-sheaf $E \otimes \mathcal{O}_X(-y) \subset E$, and hence it produces a homomorphism
\[ \text{Res}(D,y) : E/(E \otimes \mathcal{O}_X(-y)) = E_y \longrightarrow E_y. \quad (2.5) \]
The homomorphism $\text{Res}(D,y)$ in (2.5) is called the residue of the connection $D$ at the point $y$; see [De, p. 53].

Definition 2.1. A connection on $E_*$ is a logarithmic connection $D$ on $E$, singular over $S$, such that

1. $\text{Res}(D,x_i)(F^i_j) \subset F^i_j$ for all $1 \leq j \leq \ell_i$, $1 \leq i \leq n$, and
2. the endomorphism of $F^i_j/F^i_{j+1}$ induced by $\text{Res}(D,x_i)$ coincides with multiplication by the parabolic weight $\lambda^i_j$.

(See [BL, Section 2.2].)

If $D$ is a logarithmic connection on $E$ singular over $S$, then
\[ \text{degree}(E) = -\sum_{i=1}^{n} \text{trace}(\text{Res}(D,x_i)) \]
[Oh, p. 16, Theorem 3]. From this, and the definition of parabolic degree reproduced in (2.2), it follows immediately that if a parabolic bundle $E_*$ admits a connection, then
\[ \text{par-deg}(E_*) = 0. \]
Conversely, an indecomposable parabolic vector bundle of parabolic degree zero admits a connection [BL, Proposition 4.1].

A connection on $E_*$ induces a connection on the parabolic dual $E_*^\vee$. Also, if $E_*$ and $E'_*$ are equipped with connections, then there are induced connections on $E_* \otimes E'_*$ and $\text{Hom}(E_*, E'_*)$. In particular, a connection on $E$ induces a connection on the parabolic endomorphism bundle $\text{End}(E_*)$.

Let $Y$ be a smooth complex projective curve which need not be connected, and let $E_*$ be a parabolic vector bundle on $Y$. Then a connection on $E_*$ is defined to be a connection on the parabolic vector bundle over each connected component of $Y$.

3. Direct image of a parabolic vector bundle

Let $Y$ be a smooth complex projective curve which need not be connected. As before, the curve $X$ is irreducible. Let
\[ \varphi : Y \longrightarrow X \quad (3.1) \]
be a finite morphism.

Take a parabolic vector bundle $V_\ast$ on $Y$ of positive rank. The vector bundle underlying $V_\ast$ will be denoted by $V$. Note that the direct image $\varphi_\ast V \to X$ is a vector bundle, because $\varphi$ is a finite morphism so the first direct image vanishes. We will construct a parabolic structure on the vector bundle $\varphi_\ast V$.

Since $\varphi$ is a finite morphism, there are only finitely points of $Y$ where $\varphi$ fails to be smooth. Let

$$R \subset Y \quad \text{(3.2)}$$

be the finite subset where the map $\varphi$ is ramified. Let $P \subset Y$ be the parabolic divisor for the parabolic vector bundle $V_\ast$. The parabolic divisor for the parabolic structure on $\varphi_\ast V$ is the image $\varphi(R \cup P)$. Take a point $x \in \varphi(R \cup P) \setminus \varphi(R)$ in the complement of $\varphi(R) \subset \varphi(R \cup P)$. Then we have

$$(\varphi_\ast V)_x = \bigoplus_{y \in \varphi^{-1}(x)} V_y.$$ 

Since $x \in \varphi(R \cup P) \setminus \varphi(R)$, some points of $\varphi^{-1}(x)$ are parabolic points for $V_\ast$. If $y \in \varphi^{-1}(x)$ is not a parabolic point of $V_\ast$, equip $V_y$ with the trivial quasi-parabolic filtration

$$V_y = V^1_y \supset V^2_y = 0$$

and the trivial parabolic weight $\lambda_1 = 0$.

Now for any real number $0 \leq c \leq 1$, define

$$(\varphi_\ast V)^c_x := \bigoplus_{y \in \varphi^{-1}(x)} V^c_y \subset \bigoplus_{y \in \varphi^{-1}(x)} V_y = (\varphi_\ast V)_x,$$

where the subspace $V^c_y \subset V_y$ is defined in (2.1). As mentioned following (2.1), this filtration of subspaces $\{(\varphi_\ast V)^c_x\}_{0 \leq c \leq 1}$ of $(\varphi_\ast V)_x$ produces a parabolic structure on $\varphi_\ast V$ at the point $x$ (see [MY, p. 80]).

Now take any $x \in \varphi(R)$, where $R$ is the divisor in (3.2). Let

$$\varphi^{-1}(x)_{\text{red}} = \{y_1, \cdots, y_m\} \subset Y$$

be the reduced inverse image of $x$ under $\varphi$, so $y_1, \cdots, y_m$ are distinct points of $Y$. The multiplicity of $\varphi$ at $y_k$, $1 \leq k \leq m$, will be denoted by $b_k$; therefore, we have

$$\varphi^{-1}(x) = \sum_{k=1}^m b_k y_k.$$ 

The projection formula says that

$$\varphi_\ast (V \otimes O_Y(-\sum_{j=1}^m b_j y_j)) = (\varphi_\ast V) \otimes O_X(-x).$$

Hence the composition of homomorphisms

$$\varphi_\ast (V \otimes O_Y(-\sum_{j=1}^m b_j y_j)) \to \varphi_\ast V \to (\varphi_\ast V)_x \quad \text{(3.3)}$$
vanishes identically, where the first homomorphism is the natural inclusion map, and the second homomorphism is the restriction of the vector bundle $\varphi_*V$ to its fiber $(\varphi_*V)_x$ over $x$. For every $1 \leq k \leq m$, let

$$W_k \subset (\varphi_*V)_x$$

be the image of the composition homomorphism

$$\varphi_*(V \otimes \mathcal{O}_Y(- \sum_{j=1, j \neq k}^m b_j y_j)) \rightarrow \varphi_*V \rightarrow (\varphi_*V)_x,$$

where the first homomorphism is the natural inclusion map, and the second homomorphism is the one in (3.3). It is straight-forward to check that

$$\dim W_k = b_k \cdot \dim V_{y_k},$$

and

$$(\varphi_*V)_x = \bigoplus_{k=1}^m W_k. \quad (3.5)$$

We will construct a weighted filtration on each of these vector spaces $W_k$; after that, all these weighted filtrations combined together will give the weighted filtration of $(\varphi_*V)_x$ using (3.5), which in turn would define the parabolic structure on $\varphi_*V$ over the point $x$.

For each integer $0 \leq \ell \leq b_k$, let

$$F^k_{\ell} \subset (\varphi_*V)_x$$

be the image of the composition homomorphism

$$\varphi_*(V \otimes \mathcal{O}_Y(-\ell \cdot y_k - \sum_{j=1, j \neq k}^m b_j y_j)) \rightarrow \varphi_*V \rightarrow (\varphi_*V)_x,$$

where the first homomorphism is the natural inclusion map, and the second homomorphism is the one in (3.3). Note that $F^k_{b_k} = 0$ (as the composition homomorphism in (3.3) vanishes), and $F^k_0 = W_k$ (constructed in (3.4)); in particular, we have $F^k_{\ell} \subset W_k$ for all $0 \leq \ell \leq b_k$. Therefore, we have a filtration of subspaces

$$W_k = F^k_0 \supset F^k_1 \supset F^k_2 \supset \cdots \supset F^k_{\ell} \supset \cdots \supset F^k_{b_k-1} \supset F^k_{b_k} = 0. \quad (3.7)$$

We will now show that every $0 \leq \ell \leq b_k - 1$, the quotient space $F^k_{\ell}/F^k_{\ell+1}$ in (3.7) is canonically identified with the tensor product $V_{y_k} \otimes (K^*_Y)_{y_k}$, where $V_{y_k}$ and $(K^*_Y)_{y_k}$ are the fibers, over the point $y_k \in \varphi^{-1}(x)$, of $V$ and $K^*_Y$ respectively ($K_Y$ is the holomorphic cotangent bundle of $Y$).

Take a holomorphic function $\beta$ defined on a sufficiently small open neighborhood $U$ of $y_k$ in $Y$ such that $\beta(y_k) = 0$ and $d\beta(y_k) \neq 0$. We take $U$ such that $U \cap R = y_k$, where $R$ is the ramification divisor in (3.2). Take any $v \in V_{y_k}$. Let $s_v$ be a holomorphic section of $V$, defined on the neighborhood $U$ of $y_k$ in $Y$, such that

$$s_v(y_k) = v.$$
Now $\beta^\ell \cdot s_v$ defines a holomorphic section of $\varphi_*(V \otimes \mathcal{O}_Y(-\ell \cdot y_k - \sum_{j=1,j\neq k}^m b_jy_j))$ over the open subset $\varphi(U) \subset X$. Let

$$f'_{\beta, s_v} \in \varphi_*(V \otimes \mathcal{O}_Y(-\ell \cdot y_k - \sum_{j=1,j\neq k}^m b_jy_j))_x \quad (3.8)$$

be the evaluation, at the point $x \in X$, of this section of the vector bundle $\varphi_*(V \otimes \mathcal{O}_Y(-\ell \cdot y_k - \sum_{j=1,j\neq k}^m b_jy_j))$ over $X$. Since $F^k_\ell$ is the image of the composition in (3.6), it follows that $F^k_\ell$ is a quotient of the fiber $\varphi_*(V \otimes \mathcal{O}_Y(-\ell \cdot y_k - \sum_{j=1,j\neq k}^m b_jy_j))_x$. So $f'_{\beta, s_v}$, constructed in (3.8), produces an element of $F^k_\ell$. Let

$$f_{\beta, s_v} \in F^k_\ell / F^k_{\ell+1} \quad (3.9)$$

be the image of $f'_{\beta, s_v}$ in the quotient space $F^k_\ell / F^k_{\ell+1}$ in (3.7).

We will first show that $f_{\beta, s_v}$ in (3.9) is independent of the choice of the section $s_v$ passing through $v$.

If $t_v$ is another holomorphic section of $V$, defined on $U$, such that $t_v(y_k) = v$. Then $(s_v - t_v)(y_k) = 0$, and therefore, the section $\beta^\ell \cdot (s_v - t_v)$ vanishes at $y_k$ of order at least $\ell + 1$. This implies that the section of $\varphi_*(V \otimes \mathcal{O}_Y(-\ell \cdot y_k - \sum_{j=1,j\neq k}^m b_jy_j))$ over $\varphi(U)$ defined by $\beta^\ell \cdot (s_v - t_v)$ lies in the image of the inclusion map

$$\varphi_*(V \otimes \mathcal{O}_Y(-(\ell + 1) \cdot y_k - \sum_{j=1,j\neq k}^m b_jy_j))_x \hookrightarrow \varphi_*(V \otimes \mathcal{O}_Y(-\ell \cdot y_k - \sum_{j=1,j\neq k}^m b_jy_j)). \quad (3.10)$$

Consider the element of the fiber $\varphi_*(V \otimes \mathcal{O}_Y(-\ell \cdot y_k - \sum_{j=1,j\neq k}^m b_jy_j))_x$ obtained by evaluating, at the point $x$, of the above section of the vector bundle

$$\varphi_*(V \otimes \mathcal{O}_Y(-\ell \cdot y_k - \sum_{j=1,j\neq k}^m b_jy_j))$$

defined by $\beta^\ell \cdot (s_v - t_v)$. Since $\beta^\ell \cdot (s_v - t_v)$ lies in the sub-sheaf in (3.10), it follows that the image of this element of $\varphi_*(V \otimes \mathcal{O}_Y(-\ell \cdot y_k - \sum_{j=1,j\neq k}^m b_jy_j))_x$ in $F^k_\ell$ actually lies in the subspace $F^k_{\ell+1} \subset F^k_\ell$. Therefore, the element $f_{\beta, s_v}$ in (3.9) coincides with $f_{\beta, t_v}$ constructed similarly. Hence it follows that $f_{\beta, s_v}$ does not depend on the choice of the section $s_v$ passing through $v$.

Regarding the dependence of $f_{\beta, s_v}$ on $\beta$, we will now show that $f_{\beta, s_v}$ depends only on $(d\beta(y_k))^{\otimes \ell} \in (K^\otimes Y)_y$.

To prove this, take another holomorphic function $\beta_1$ defined around $y_k$ such that $\beta_1(y_k) = 0$ and $(d\beta_1(y_k))^{\otimes \ell} = (d\beta(y_k))^{\otimes \ell}$. Then the function $\beta^\ell - \beta_1^\ell$ vanishes at $y_k$ of order at least $\ell + 1$. Consequently, the local section $(\beta^\ell - \beta_1^\ell) \cdot s_v$ of $V$ vanishes at $y_k$ of order at least $\ell + 1$. Form this it is straight-forward to deduce that

$$f_{\beta_1, s_v} = f_{\beta, s_v} \in F^k_\ell / F^k_{\ell+1},$$

where $f_{\beta_1, s_v}$ is constructed as done in (3.9).

Consequently, for every $0 \leq \ell < b_k$, we get a homomorphism

$$\nu(y_k, \ell) : V_{y_k} \otimes (K^\otimes Y_y) \rightarrow F^k_\ell / F^k_{\ell+1}, \quad v \otimes w \mapsto f_{\beta, s_v}, \quad (3.11)$$
where \( w = d \beta(y_k)^{\otimes \ell} \) and \( s_v(y_k) = v \). The homomorphism \( \nu(y_k, \ell) \) in (3.11) is evidently an isomorphism.

Now, this \( y_k \) may be a parabolic point of \( \mathcal{V} \), in which case \( V_{y_k} \) would have a quasi-parabolic filtration. If \( y_k \) is a parabolic point of \( \mathcal{V} \), let

\[
V_{y_k} = V^1_{y_k} \supseteq V^2_{y_k} \supseteq \cdots \supseteq V^{l(y_k)}_{y_k} \supseteq V^{l(y_k)+1}_{y_k} = 0
\]

be the quasi-parabolic filtration of the fiber \( V_{y_k} \), and let

\[
0 \leq \lambda^1_{y_k} < \cdots < \lambda^{l(y_k)}_{y_k} < \lambda^{l(y_k)+1}_{y_k} = 1
\]

be the corresponding system of parabolic weights.

If \( y_k \) is not a parabolic point of \( \mathcal{V} \), then we equip \( V_{y_k} \) with the trivial quasi-parabolic filtration

\[
V_{y_k} = V^1_{y_k} \supseteq V^2_{y_k} = 0
\]

and the trivial parabolic weight \( \lambda^1_{y_k} = 0 \). This way we would not need to distinguish between parabolic \( y_k \) and non-parabolic \( y_k \).

Using the isomorphism \( \nu(y_k, \ell) \) in (3.11), the filtration in (3.12) produces a filtration of subspaces \( F^k_{\ell}/F^k_{\ell+1} \) for every \( 0 \leq \ell \leq b_k - 1 \), because \( (K^\otimes \ell)_{y_k} \) is a line, so linear subspaces of \( V_{y_k} \) are in a canonical bijection with the linear subspaces of \( V_{y_k} \otimes (K^\otimes \ell)_{y_k} \). This filtration of each \( F^k_{\ell}/F^k_{\ell+1}, 0 \leq \ell \leq b_k - 1 \), and the filtration of \( W_k \) in (3.7) together produce a finer filtration \( W_k \), while the parabolic weights in (3.13) produce weights for the terms of this finer filtration of \( W_k \).

More precisely, the length of this finer filtration of \( W_k \)

\[
W_k = S_1 \supseteq S_2 \supseteq \cdots \supseteq S_{l(y_k)b_k} \supseteq S_{l(y_k)b_k+1} = 0
\]

is \( l(y_k)b_k \), and for every \( 1 \leq i \leq l(y_k)b_k \), the \( i \)-th term \( S_i \), and its weight, are constructed as follows: First write \( i = c \cdot l(y_k) + d \), where \( c \) is a nonnegative integer and \( 1 \leq d \leq l(y_k) \).

- If \( d = 1 \), then set

\[
S_i := F^k_c
\]

(see (3.7)), and set the weight of \( S_i \) to be \( (c + \lambda^1_{y_k})/b_k \), where \( \lambda^1_{y_k} \) is the weight in (3.13).

- If \( 2 \leq d \leq l(y_k) \), then \( S_i \) satisfies the two conditions

\begin{enumerate}
\item \( F^k_{c+1} \subset S_i \subset F^k_c \), and
\item the subspace \( S_i/F^k_{c+1} \subset F^k_c/F^k_{c+1} = V_{y_k} \otimes (K^\otimes c) \) (it is the isomorphism \( \nu(y_k, c) \) in (3.11)) coincides with the subspace \( V_{y_k} \otimes (K^\otimes c) \subset V_{y_k} \otimes (K^\otimes c) \) (see (3.12) for \( V_{y_k} \)).
\end{enumerate}

The above two conditions evidently determine \( S_i \) uniquely. The weight of \( S_i \) is set to be

\[
(c + \lambda^d_{y_k})/b_k,
\]

where \( \lambda^d_{y_k} \) is the weight in (3.13).

Now that we have the weighted filtration of \( W_k \) constructed above, using the decomposition in (3.5) we construct a parabolic structure on \( (\varphi, V)_x \) in the obvious way: The subspace of \( (\varphi, V)_x \) corresponding to any \( 0 \leq c \leq 1 \) is simply the direct sum of the subspaces of \( W_k, 1 \leq k \leq m, \) corresponding to the same \( c \).
Therefore, we get a parabolic structure on the vector bundle $\varphi_* V$ over $X$. This parabolic bundle over $X$ will be denoted by $\varphi_* V$.

4. Direct image of parabolic bundles and ramified torus sub-bundles

4.1. Ramified torus sub-bundle. Let $V_0$ be a finite dimensional complex vector space. Consider the group $GL(V_0)$ consisting of all linear automorphisms of $V_0$; it is a complex affine algebraic group. A torus in $GL(V_0)$ is a complex algebraic subgroup of $GL(V_0)$ isomorphic to a product of copies of the multiplicative group $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. Given a torus $T_0 \subset GL(V_0)$ consider the isotypical decomposition of $V_0$ for the standard action of $T_0$ on $V_0$; it consists of finitely many distinct characters $\chi_1, \cdots, \chi_n$ of $T_0$ and a decomposition

$$V_0 = \bigoplus_{j=1}^n V_{0j}$$

such that any $t \in T_0$ acts on $V_{0j}$ as multiplication by $\chi_j(t) \in \mathbb{C}^\times$. The centralizer of $T_0$ in $GL(V_0)$ will be denoted by $C(T_0)$. It is straight-forward to see that $C(T_0)$ consists of all $\alpha \in GL(V_0)$ such that $\alpha(V_{0j}) = V_{0j}$ for every $1 \leq j \leq n$.

Conversely, given a direct sum decomposition

$$V_0 = \bigoplus_{j=1}^m B_j , \quad (4.1)$$

let $\mathbb{L} \subset GL(V_0)$ be the subgroup consisting of all $\alpha \in GL(V_0)$ such that $\alpha(B_j) = B_j$ for every $1 \leq j \leq m$. Then the center of $\mathbb{L}$ is a torus in $GL(V_0)$. The decomposition of $V_0$ corresponding to this torus coincides with the one in $[41]$.

Let $Z$ be a smooth complex algebraic curve; it need not be projective or connected. Take an algebraic vector bundle $W$ on $Z$. Consider the vector bundle $\text{ad}(W) = \text{End}(W) = W \otimes W^*$. We have a function on the total space of $\text{ad}(W)$ given by the determinant of endomorphisms. Let

$$\text{Ad}(W) \subset \text{ad}(W)$$

be the Zariski open subset where the determinant is nonzero, so $\text{Ad}(W)$ parametrizes the automorphisms. Let

$$p_0 : \text{Ad}(W) \longrightarrow Z$$

be the projection. So the fiber $\text{Ad}(W)_y = p_0^{-1}(y)$ of $\text{Ad}(W)$ over any $y \in Z$ is the space of all linear automorphisms of the fiber $W_y$. Note that $\text{Ad}(W)$ is smooth group-scheme over $Z$.

**Definition 4.1.** A torus sub-bundle of $\text{Ad}(W)$ is a smooth abelian semi-simple subgroup scheme over $Z$

$$\mathcal{T} \subset \text{Ad}(W) .$$

In other words, the restriction of $p_0$ to $\mathcal{T}$ is smooth, and for every $y \in Z$, the fiber $\mathcal{T}_y$ is a torus in $\text{Ad}(W)_y$.

Note that the dimension of $\mathcal{T}_y$ may depend on the component of $Z$ in which $y$ lies. In other words, the connected components of $\mathcal{T}$ are allowed to have different dimensions.

As mentioned earlier, we have the isotypical decomposition of $W_y$ for the standard action of $\mathcal{T}_y$ on $W_y$. It should be noted that these point-wise decomposition of the fibers of $W$
does not produce a decomposition of the vector bundle $W$ because the direct summands may get interchanged as $y$ runs over $Z$. To give such an example, take $X$ to be an irreducible smooth complex projective curve of sufficiently large genus. Let $P_m$ be the group of permutations of $\{1, 2, 3, \ldots, m\}$, with $m \geq 3$; it acts on $\mathbb{C}^m$ by permuting the elements of the standard basis of $\mathbb{C}^m$. Let

$$
\rho : \pi_1(X, z_0) \longrightarrow P_m
$$

be a surjective homomorphism, and let $W$ be the flat vector bundle on $X$ of rank $m$ associated to the composition homomorphism

$$
\pi_1(X, z_0) \xrightarrow{\rho} P_m \hookrightarrow \text{GL}(m, \mathbb{C}) .
$$

Each fiber of $W$ has a canonical decomposition into an unordered direct sum of lines. However, the vector bundle $W$ may get interchanged as $C$ elements of the standard basis of $\mathbb{C}^m
$ given by the $\Gamma \cdot \text{invariant part of the direct image. This vector bundle } (\gamma_1^0 W)^{\Gamma_0} \subset \gamma_1^0 W \longrightarrow Z$
given by the $\Gamma_0$-invariant part of the direct image. This vector bundle $(\gamma_1^0 W)^{\Gamma_0}$ has a filtration of sub-sheaves given by the invariant direct images

$$
(\gamma_i^0 (W \otimes \mathcal{O}_{Z_1}(-i \cdot D)))^{\Gamma_0} \subset \gamma_i^0 (W \otimes \mathcal{O}_{Z_1}(-i \cdot D)) \longrightarrow Z , \quad i \geq 1 .
$$

This filtration of sub-sheaves of $(\gamma_1^0 W)^{\Gamma_0}$ produces a parabolic structure on the vector bundle $(\gamma_1^0 W)^{\Gamma_0}$. The parabolic weights are rational numbers. Conversely, given a a parabolic bundle $E_*$ on $Z$ with rational parabolic weights, there is a (possibly) ramified Galois covering $\gamma^0 : Z_1 \longrightarrow Z$, and a Gal$(\gamma^0)$-equivariant vector bundle $W$ on $Z_1$, such that the parabolic vector bundle on $Z$ corresponding to $W$ is isomorphic to $E_*$. It should be mentioned that the above equivalence is local in $Z$, both in analytic and Zariski topology. In other words, $Z$ can be replaced by an open subset of it in this equivalence of categories. We also note that outside the parabolic points, the equivariant bundle corresponding to a parabolic bundle is simply the pullback of the vector bundle underlying the parabolic bundle.

The complement $Z \setminus \{z_1, \ldots, z_n\}$ will be denoted by $U$. 

We assume that all the parabolic weights $\lambda_j^i$ are rational numbers.
Fix a point \( y \in \{ z_1, \cdots, z_n \} \), and take an open neighborhood \( y \in U_y \subset Z \) of \( y \) in \( Z \), in analytic or Zariski topology. Consider an equivariant bundle \( (\bar{U}_y, \gamma, W^y) \) corresponding to the parabolic bundle \( W_*|_{U_y} \). This means that
\[
\gamma : \bar{U}_y \longrightarrow U_y
\]
is a ramified Galois covering morphism, and \( W^y \) is a \( \text{Gal}(\gamma) \)-equivariant vector bundle on \( \bar{U}_y \) that corresponds to \( W_*|_{U_y} \). As mentioned before, the restriction of \( W^y \) to \( \gamma^{-1}(U_y \cap U) \) is identified with the restriction of \( \gamma^*W \) to \( \gamma^{-1}(U_y \cap U) \). We also note that \( \gamma^*\text{Ad}(W) \) is canonically identified with \( \text{Ad}(\gamma^*W) \).

**Definition 4.2.** A ramified torus sub-bundle for \( W_* \) is a torus sub-bundle \( T \subset \text{Ad}(W)|_U \) over \( U \) satisfying the following condition: For every point \( y \in \{ z_1, \cdots, z_n \} \), there is an open neighborhood \( y \in U_y \subset Z \) of \( y \) in \( Z \), such that if \( (\bar{U}_y, \gamma, W^y) \) is an equivariant bundle corresponding to the parabolic bundle \( W_*|_{U_y} \), then the torus sub-bundle \( \gamma^*T \subset \text{Ad}(\gamma^*W) = \text{Ad}(W^y) \) on \( \gamma^{-1}(U_y \cap U) \) extends across \( \gamma^{-1}(y) \) to produce a torus sub-bundle of \( \gamma^*\text{Ad}(W^y) \) over the open subset \( \gamma^{-1}(U_y \cap (U \cup \{\gamma^{-1}(y)\})) \).

We have used above that \( W^y \) and \( \gamma^*W \) are identified over \( \gamma^{-1}(U_y \cap U) \).

In the above definition it is equivalent to take the open neighborhood \( U_y \) to be in Zariski topology or in analytic topology. More precisely, if there is an analytic open subset \( U_y \) satisfying the above conditions, then there is a also Zariski open subset \( U_y \) satisfying the above conditions. Indeed, the only issue is the order of ramification of \( \gamma \) over \( y \). Any given order can be achieved by coverings over both analytic and Zariski neighborhoods of \( y \). In fact, \( U_y \) can be taken to be a ramified covering of \( Z \) which works simultaneously for all points of \( Z \setminus U \). To explain this, we recall a result on coverings of surfaces.

Let \( S \) be a connected compact oriented \( C^\infty \) surface and
\[
\{ x_1, \cdots, x_d \} \subset S
\]
a finite subset. For each \( 1 \leq i \leq d \), fix an integer \( n_i \geq 2 \) for the point \( x_i \). Assume that least one of the following three conditions are satisfied:

1. \( \text{genus}(S) \geq 1 \),
2. \( d \notin \{ 1, 2 \} \),
3. if \( d = 2 \), then \( n_1 = n_2 \).

So only two cases are ruled out by this assumption: \( \text{genus}(S) = 0 = d - 1 \) and \( \text{genus}(S) = 0 = d - 2 \neq n_1 - n_2 \). A theorem due to Bundgaard–Nielsen and Fox says that there is a finite Galois covering
\[
\beta : \bar{S} \longrightarrow S
\]
such that \( \beta \) is unramified over \( S \setminus \{ x_1, \cdots, x_d \} \), and for every \( 1 \leq i \leq d \), the order of ramification at every point of \( \beta^{-1}(x_i) \) is \( n_i \) [Na, p. 26, Proposition 1.2.12]. (We call the order of ramification at 0 of the map \( z \mapsto z^k \) to be \( k \).)
In view of the above theorem, given a ramified torus sub-bundle $T \subset \text{Ad}(W)_U$ for $W_*$, there is a ramified Galois covering
$$
\gamma : \tilde{Z} \rightarrow Z
$$
étale over $U$, and a $\text{Gal}(\gamma)$-equivariant vector bundle $\tilde{W}$ on $\tilde{Z}$ that corresponds to $W_*$, such that the torus sub-bundle
$$
\gamma^* T \subset \text{Ad}(\gamma^* W)
$$
over $\gamma^{-1}(U)$ extends to entire $\tilde{Z}$ as a torus sub-bundle of $\text{Ad}(\tilde{W})$.

If we are in a situation of the two exceptions in the above theorem of Bundgaard–Nielsen and Fox, simply introduce either one or two extra parabolic points with trivial parabolic structure over those points (meaning trivial quasi-parabolic filtration and zero parabolic weight).

It should be mentioned that given a ramified torus sub-bundle $T \subset \text{Ad}(W)_U$ for $W_*$, this $T$ does not, in general, extend to some torus of $\text{Ad}(W)_{z_i}$ on a parabolic point $z_i$. However $T$ produces a parabolic subgroup of $\text{Ad}(W)_{z_i}$.

4.2. Ramified torus sub-bundle for direct image of a vector bundle. As in Section 4.1, let $Z$ be a smooth complex algebraic curve, which may not be projective or connected. Let $Y$ be a smooth complex algebraic curve and $\phi : Y \rightarrow Z$ an étale covering; so $Y$ also need not be connected or projective. Take a vector bundle $V$ on $Y$, and consider the vector bundle $W = \phi_* V$ on $Z$. For any $x \in Z$, the fiber $W_x$ has the direct sum decomposition
$$
W_x = (\phi_* V)_x = \bigoplus_{y \in \phi^{-1}(x)} V_y. \tag{4.2}
$$
This produces a torus sub-bundle $T$ of $\text{Ad}(W)$.

To describe $T$ in another way, note that the direct image $\phi_* \text{Ad}(V)$ is a subgroup scheme $\phi_* \text{Ad}(V) \subset \text{Ad}(W)$.

In fact, we have $\phi_* \text{Ad}(V) = (\phi_* \text{End}(V)) \bigcap \text{Ad}(W)$. The center of $\phi_* \text{Ad}(V)$ is the torus sub-bundle
$$
T \subset \phi_* \text{Ad}(V) \subset \text{Ad}(W). \tag{4.3}
$$
Note that the centralizer of $T$ in $\text{Ad}(W)$ is $\phi_* \text{Ad}(V)$. Related constructions can be found in [DG], [DoPa].

Now assume that $Z$ is connected and projective, and allow the covering $\phi$ to be ramified, but assume that $Y$ is also connected; note that $Y$ is projective because $Z$ is so. Let $U \subset Z$ be the Zariski open dense subset over which $\phi$ is étale.

As before, $W = \phi_* V$ is a vector bundle on $Z$, because $\phi$ is a finite morphism. As constructed in (4.3), we have a torus sub-bundle
$$
T \subset \text{Ad}(\phi_* V)|_U = \text{Ad}(W)|_U \tag{4.4}
$$
over $U$. 

In Section 3 we saw that $W$ has a parabolic structure on the complement of $U$; we are taking the trivial parabolic structure on $V$. Let $W_\ast$ be the parabolic vector bundle defined by this parabolic structure with $W$ as the underlying vector bundle. Note that all the parabolic weights of $W_\ast$ are rational numbers.

**Proposition 4.3.** The torus sub-bundle $\mathcal{T}$ in (4.4) is a ramified torus sub-bundle for $W_\ast$.

**Proof.** There is a ramified covering $\widetilde{\phi} : \widetilde{Y} \rightarrow Y$ such that the composition

$$\phi \circ \widetilde{\phi} : \widetilde{Y} \rightarrow Z$$

is Galois. Indeed, this follows immediately from the fact that any finite index subgroup of a finitely presented group $G$ contains a normal subgroup of $G$ of finite index. Note that this implies that the covering $\widetilde{\phi}$ is Galois. Let $G$ (respectively, $H$) denote the Galois group of $\phi \circ \widetilde{\phi}$ (respectively, $\widetilde{\phi}$). The quotient map $G \rightarrow G/H$ will be denoted by $q$. Fix a finite subset $C$ of $G$ such that the composition of maps

$$C \hookrightarrow G \xrightarrow{q} G/H$$

is a bijection. The left–translation action of $G$ on $G/H$ produces an action of $G$ on $C$ using the above bijection.

Consider the vector bundle

$$\widetilde{V} := \bigoplus_{\sigma \in C} (\sigma^{-1})^* \widetilde{\phi}^* V \rightarrow \widetilde{Y}.$$  \hfill (4.5)

Using the above action of $G$ on $C$, the vector bundle $\widetilde{V}$ in (4.5) has the structure of a $G$–equivariant vector bundle on $\widetilde{Y}$. The parabolic vector bundle on $\widetilde{Y}/G = Z$ corresponding to this $G$–equivariant vector bundle $\widetilde{V}$ is in fact $W_\ast$. (This is straight-forward to deduce from the correspondence between parabolic bundles and equivariant bundles.) In particular, we have

$$((\phi \circ \widetilde{\phi})^* W)|_{(\phi \circ \widetilde{\phi})^{-1}(U)} = \widetilde{V}|_{(\phi \circ \widetilde{\phi})^{-1}(U)}.$$  \hfill (4.6)

Over $U \subset Z$, consider the decomposition of the fibers of $W|_U = (\phi_* V)|_U$ given be the torus sub-bundle $\mathcal{T}$ in (4.4). We recall that for any $x \in U$, this decomposition is the natural direct sum decomposition of $W_x$ in (4.2). Using the isomorphism in (4.6), the pullback of this decomposition of the fibers of $W|_U$ produces a decomposition of the fibers of $\widetilde{V}$ over $(\phi \circ \widetilde{\phi})^{-1}(U)$. It is straight-forward to check that this decomposition of the fibers of $\widetilde{V}|_{(\phi \circ \widetilde{\phi})^{-1}(U)}$ coincides with the decomposition of $\widetilde{V}|_{(\phi \circ \widetilde{\phi})^{-1}(U)}$ in (4.5). But the decomposition of $\widetilde{V}$ in (4.5) is over entire $\widetilde{Y}$. This immediately implies that the torus sub-bundle

$$(\phi \circ \widetilde{\phi})^* \mathcal{T} \subset (\phi \circ \widetilde{\phi})^*(\text{Ad}(W)|_U) = \text{Ad}(\widetilde{V})|_{(\phi \circ \widetilde{\phi})^{-1}(U)}$$

(see (4.6) and (4.4)) extends to a torus sub-bundle of $\text{Ad}(\widetilde{V})$ over entire $\widetilde{Y}$. Hence $\mathcal{T}$ is a ramified torus sub-bundle for $W_\ast$ (see Definition 4.2). \hfill \square
4.3. Ramified torus sub-bundle for direct image of a parabolic bundle. As before, $Y$ and $Z$ are smooth connected projective complex curves, and $\phi$ is a finite morphism from $Y$ to $Z$. Let $V_*$ be a parabolic vector bundle over $Y$ with parabolic divisor $\{y_1, \cdots, y_n\} \subset Y$; for each $1 \leq i \leq n$, the quasi-parabolic filtration on $V_{y_i}$ is

$$V_{y_i} = F_1^i \supseteq F_2^i \supseteq \cdots \supseteq F_{\ell_i}^i \supseteq F_{\ell_i+1}^i = 0,$$

while the corresponding parabolic weights are

$$0 \leq \lambda_1^i < \lambda_2^i < \cdots < \lambda_{\ell_i-1}^i < \lambda_{\ell_i}^i < \lambda_{\ell_i+1}^i = 1.$$

We assume that all the parabolic weights $\lambda_j^i$ are rational numbers.

Let

$$W_* := \phi_*V_*$$

be the parabolic vector bundle over $Z$ (constructed as done in Section 3). Let

$$U \subset Z \setminus \phi(\{y_1, \cdots, y_n\}) \subset Z$$

be the Zariski open dense subset over which the restriction of $\phi$ to $\phi^{-1}(Z\setminus\phi(\{y_1, \cdots, y_n\}))$ is étale. We have a torus sub-bundle

$$\mathcal{T} \subset \text{Ad}(\phi_*V)|_U = \text{Ad}(W)|_U,$$

where $V$ and $W$ are the vector bundle underlying $V_*$ and $W_*$ respectively.

The following is a generalization of Proposition 4.3.

**Proposition 4.4.** The torus sub-bundle $\mathcal{T}$ in (4.3) is a ramified torus sub-bundle for the parabolic bundle $W_* := \phi_*V_*$. 

*Proof.* Let $\psi : \hat{Z} \longrightarrow Z$ be a ramified Galois covering and $\hat{W}$ a $\text{Gal}(\psi)$–equivariant vector bundle on $\hat{Z}$ that corresponds to the parabolic vector bundle $W_*$; such a pair $(\psi, \hat{W})$ exists because all the parabolic weights of $W_*$ are rational numbers, which is in fact ensured by the assumption that all the parabolic weights of $V_*$ are rational numbers. The Galois group $\text{Gal}(\psi)$ for $\psi$ will be denoted by $G$.

Let $\tilde{Y}$ be the normalization of the fiber product $\hat{Z} \times_Z Y$ over $Z$. Let

$$\phi : \tilde{Y} \longrightarrow \hat{Z} \quad \text{and} \quad \tilde{\psi} : \tilde{Y} \longrightarrow Y$$

be the natural projections. The above morphism $\tilde{\psi}$ is a ramified Galois covering with Galois group $G$. On the other hand, the morphism $\phi$ is étale.

There is a $G$–equivariant vector bundle $\tilde{V}$ on $\tilde{Y}$ that corresponds to the parabolic vector bundle $V_*$ on $Y = \tilde{Y}/G$ (the Galois group for $\tilde{\psi}$ is $G$). The action of $G$ on $\tilde{V}$ produces an action of $G$ on the direct image $\phi_*\tilde{V} \longrightarrow \hat{Z}$. This action of $G$ on $\phi_*\tilde{V}$ commutes with the action of $G = \text{Gal}(\psi)$ on $\hat{Z}$.

Consider the open subset $U \subset Z$ in (4.7). It is straightforward to check that the restriction of $\phi_*\tilde{V}$ to $\psi^{-1}(U)$ is identified with the pullback $\psi^*(W_*|_U) \longrightarrow \psi^{-1}(U)$; also this identification is $G$–equivariant for the natural action of $G = \text{Gal}(\psi)$ on $\psi^*(W_*|_U)$. On the other hand, the vector bundle $\hat{W}|_{\psi^{-1}(U)}$ is identified with $\psi^*(W_*|_U) \longrightarrow \psi^{-1}(U)$, because the parabolic vector bundle $W_*$ corresponds to the $G$–equivariant vector bundle $\hat{W}$ on $\hat{Z}$; this identification is also $G$–equivariant.
The resulting isomorphism between the two $G$-equivariant vector bundles $\widetilde{W}|_{\psi^{-1}(U)}$ and $(\phi_{*}\widetilde{V})|_{\psi^{-1}(U)}$ over $\psi^{-1}(U)$ extends to an isomorphism of $G$-equivariant vector bundles between $\widetilde{W}$ and $\phi_{*}\widetilde{V}$ over $\widetilde{Z}$. In fact, both the $G$-equivariant vector bundles $\widetilde{W}$ and $\phi_{*}\widetilde{V}$ correspond to the parabolic vector bundle $W_{*}$ on $\widetilde{Z}/G = Z$.

Since the morphism $\phi$ is étale, for the direct image $\phi_{*}\widetilde{V}$ there is a torus sub-bundle $\widetilde{T} \subset \text{Ad}(\phi_{*}\widetilde{V})$ over $\widetilde{Z}$.

Consider $\mathcal{T}$ over $U$ in (4.8). The above isomorphism between $\widetilde{W}$ and $\phi_{*}\widetilde{V}$ takes $\psi^{*}\mathcal{T}$ over $\psi^{-1}(U)$ to $\widetilde{T}|_{\psi^{-1}(U)}$. Consequently, $\psi^{*}\mathcal{T}$ extends to a torus sub-bundle of $\text{Ad}(\widetilde{W})$ over entire $\widetilde{Z}$. \hfill \Box

5. Direct image, connection and ramified torus sub-bundle

5.1. Direct image of connection on parabolic bundle. Let $\varphi : Y \rightarrow X$ is a finite morphism between smooth complex projective curves, with $X$ connected. Let $V_{*}$ be a parabolic vector bundle on $Y$. Consider the parabolic vector bundle $\varphi_{*}V_{*}$ on $X$ constructed in Section 3. We will show that a connection on the parabolic vector bundle $V_{*}$ induces a connection on $\varphi_{*}V_{*}$. We begin with a general observation.

Lemma 5.1. Let $W$ be a vector bundle over $Y$ equipped with a logarithmic connection $D$ with singular divisor $S_{0}$. Then $D$ induces a logarithmic connection on the vector bundle $\varphi_{*}W \rightarrow X$ whose singular divisor is $\varphi(S_{0} \cup R)$, where $R \subset Y$ as in (3.2) is the ramification divisor for $\varphi$.

Proof. For notational convenience, the reduced divisor $\varphi(S_{0} \cup R)_{\text{red}}$ will be denoted by $S_{X}$. The reduced divisor $\varphi^{-1}(S_{X})_{\text{red}}$ will be denoted by $S_{Y}$. It is straight-forward to check that

$$\varphi^{*}(K_{X} \otimes \mathcal{O}_{X}(S_{X})) = K_{Y} \otimes \mathcal{O}_{Y}(S_{Y}).$$

Therefore, by the projection formula, we have

$$\varphi_{*}(W \otimes K_{Y} \otimes \mathcal{O}_{Y}(S_{Y})) = (\varphi_{*}W) \otimes K_{X} \otimes \mathcal{O}_{X}(S_{X}). \quad (5.1)$$

Consider the composition homomorphism

$$W \xrightarrow{D} W \otimes K_{Y} \otimes \mathcal{O}_{Y}(S_{0}) \hookrightarrow W \otimes K_{Y} \otimes \mathcal{O}_{Y}(S_{Y});$$

note that $S_{0} \subset S_{Y}$. Now taking direct image of this composition, and using (5.1), we have

$$\varphi_{*}W \rightarrow \varphi_{*}(W \otimes K_{Y} \otimes \mathcal{O}_{Y}(S_{Y})) = (\varphi_{*}W) \otimes K_{X} \otimes \mathcal{O}_{X}(S_{X}),$$

which will be denoted by $\varphi_{*}D$. Since $D$ satisfies the Leibniz identity, it follows that $\varphi_{*}D$ also satisfies the Leibniz identity. Hence $\varphi_{*}D$ is a logarithmic connection on $\varphi_{*}D$ singular over $S_{X}$. \hfill \Box

Theorem 5.2. Let $V_{*}$ be a parabolic vector bundle over $Y$ equipped with a connection $D$. Then the logarithmic connection $\varphi_{*}D$ on $\varphi_{*}V$ is a connection on the parabolic vector bundle $\varphi_{*}V_{*}$, where $V$ is the vector bundle underlying $V_{*}$.
Proof. We need to show that the residues of \( \varphi_* D \) have the required properties (see Definition 2.1). Since it is a purely local question, we may restrict \( V_* \) to a neighborhood of a point of \( S_0 \cup R \); note that the parabolic structure on \( \varphi_* V_* \) over a point \( x \in S_X \) was constructed using the decomposition in (3.5), so we can treat the points of \( \varphi^{-1}(x) \) separately. Therefore, we may assume that \( \varphi^{-1}(x) \) has only one point. Also, locally, around a parabolic point, a parabolic bundle can be expressed as a direct sum of parabolic line bundles.

Recall that the quasi-parabolic filtration of \( \varphi_* V_* \) at \( x \) is constructed by combining the filtrations of \( F^k/L_k \) and \( F^k_{t+1}/L_k \), obtained from the quasi-parabolic filtration in (3.12), with the filtration in (3.7). Since \( D \) is a connection on the parabolic bundle \( V_* \), its residue at \( y_k \in \varphi^{-1}(x) \) preserves the filtration in (3.12).

In view of the above observations, we may assume that \( V \) is a line bundle, and \( x \) is a point \( S_X \) such that \( \varphi^{-1}(x) \) is a point; the Riemann surfaces \( X \) and \( Y \) are not compact anymore because we do not need this assumption for local computations.

We need to show that the residue of the connection \( \varphi_* D \) at the point \( x \) preserves the quasi-parabolic filtration of \( (\varphi_* V)_x \), and on each successive quotient of this filtration the residue acts as multiplication by the corresponding parabolic weight.

Let \( Z \) be a Riemann surface and \( y \in Z \) a point. Let \( L \) be a holomorphic line bundle on \( Z \), and let \( D_L \) be a logarithmic connection on \( L \) singular over \( y \) such that the residue of \( D_L \) at \( y \) is \( \tau \in \mathbb{C} \). For any integer \( n \), consider the holomorphic line bundle \( L_n := L \otimes \mathcal{O}_Z(n \cdot y) \). So \( L \) and \( L_n \) are canonically identified over the complement \( Z \setminus \{ y \} \). The connection \( D_L \) on \( L_n \) over \( Z \setminus \{ y \} \) extends to a logarithmic connection on \( L_n \). Indeed, for a holomorphic section \( s \) of \( L \) defined on an open neighborhood \( U_y \) of \( y \), and a holomorphic function \( f \), defined on \( U_y \), such that \( f(y) = 0 \) and \( df(y) \neq 0 \), we have

\[
D_L(\frac{1}{f^n} s) = \frac{1}{f^n} \cdot D_L(s) - \frac{n}{f} \cdot \frac{df}{f^n} s.
\]

This shows that the connection \( D_L \) on \( L_n \) is logarithmic. Moreover, the residue, at \( y \), of this logarithmic connection on \( L_n \) is \( \tau - n \).

Let \( \psi : Z' \longrightarrow Z \) be a ramified covering of degree \( \delta \) which is totally ramified over \( y \). Denote the point \( \psi^{-1}(y) \) by \( y' \). Then \( \psi^* D_L \) is a logarithmic connection on \( \psi^* L \) singular over \( y' \). The residue of \( \psi^* D_L \) at \( y' \) is \( \delta \cdot \tau \).

Using the above observations the theorem follows for the case of rank one. Note that \( c \) in (3.14) arises because of the tensor product with the line bundle \( \mathcal{O}_V(-cy_k) \), and \( b_k \) occurs in (3.14) because the degree of ramification of \( \varphi \) at \( y_k \) is \( b_k \).

As noted before, it suffices to prove the rank one case. \( \square \)

5.2. Ramified torus sub-bundle and connection. Let \( E_* \) be a parabolic vector bundle on \( X \) equipped with a connection \( D \). The parabolic divisor for \( E_* \) is \( S \subset X \); the complement \( X \setminus S \) is denoted by \( U \). The logarithmic connection \( D \) on the underlying vector bundle \( E \) induces a logarithmic connection on the endomorphism bundle \( \text{ad}(E) = \text{End}(E) \). Recall that the connection \( D \) is regular on \( U \).
The above regular connection on $\text{ad}(E)|_U$ produces a connection on the sub-fiber bundle $\text{Ad}(E)|_U \subset \text{ad}(E)|_U$. This connection on $\text{Ad}(E)|_U$ will be denoted by $D^{\text{Ad}}$. Let

$$\mathcal{T} \subset \text{Ad}(E)|_U$$

be a torus sub-bundle for the parabolic vector bundle $E_*$. 

**Definition 5.3.** We will say that $D$ preserves $\mathcal{T}$ if the above connection $D^{\text{Ad}}$ on $\text{Ad}(E)|_U$ preserves the sub-bundle $\mathcal{T}$.

Now consider the set-up of Theorem 5.2. Let $S_X \subset X$ be the parabolic divisor for the parabolic vector bundle $\varphi_*V$; recall that it the union of the ramification points for $\varphi$ and the image of the parabolic divisor for $V_*$. The complement $X \setminus S_X$ will be denoted by $U$.

The parabolic vector bundle $\varphi_*V_*$ has a ramified torus sub-bundle

$$\mathcal{T} \subset \text{Ad}(\varphi_*V)|_U$$

by Proposition 4.4.

**Lemma 5.4.** The connection $\varphi_*D$ on $\varphi_*V$ preserves the ramified torus sub-bundle $\mathcal{T}$ in (5.2).

**Proof.** It is a straight-forward consequence of the definition of the push-forward of a connection by an étale morphism. \qed

6. Characterization of direct images of parabolic bundles and connections

6.1. Characterization of direct images of parabolic bundles. In Section 3 we constructed a parabolic structure on the direct image of a vector bundle with parabolic structure, and in Proposition 4.4 it was shown that the parabolic bundle thus obtained is equipped with a ramified torus sub-bundle, provided all the parabolic weights of the initial parabolic bundle are rational numbers. We will now prove a converse of it.

Let $X$ be a connected smooth complex projective curve and $S := \{x_1, \cdots, x_n\} \subset X$ a reduced effective divisor. Let $E_*$ be a parabolic vector bundle over $X$, with parabolic structure over $S$ and underlying vector bundle $E$, such that for every $x_i \in S$,

$$E_{x_i} = F_1^i \supseteq F_2^i \supseteq \cdots \supseteq F_{\ell_i}^i \supseteq F_{\ell_i+1}^i = 0,$$

is the quasi-parabolic filtration and

$$0 \leq \lambda_{i}^1 < \lambda_{i}^2 < \cdots < \lambda_{i}^\ell_{i-1} < \lambda_{i}^\ell_{i} < \lambda_{i}^\ell_{i+1} = 1,$$

are the corresponding parabolic weights.

All $\lambda_{j}^i$, $1 \leq j \leq \ell_i$, $1 \leq i \leq n$, are assumed to be rational numbers.

**Proposition 6.1.** Let $\mathcal{T} \subset \text{Ad}(E)|_U$ be a ramified torus sub-bundle for $E_*$, where $U := X \setminus S$. Then there is a finite ramified covering

$$\varphi : Y \longrightarrow X$$

and a parabolic vector bundle $V_*$ on $Y$ with rational parabolic weights, such that the parabolic vector bundle $\varphi_*V_*$ is isomorphic to $E_*$. Moreover the isomorphism between $\varphi_*V_*$ and $E_*$ can be so chosen that it takes $\mathcal{T}$ to the ramified torus sub-bundle for $\varphi_*V_*$ given by Proposition 4.4.
Proof. There is a (ramified) Galois covering
\[ \phi : \tilde{X} \rightarrow X \]
and a Gal(\(\phi\))-equivariant vector bundle \(\tilde{E}\) on \(\tilde{X}\), such that
- the parabolic vector bundle \(E_x^*\) corresponds to the Gal(\(\phi\))-equivariant vector bundle \(\tilde{E}\), and
- the torus sub-bundle \(\phi^*T \subset \text{Ad}(\tilde{E})\) extends to a torus sub-bundle
  \[ \tilde{T} \subset \text{Ad}(\tilde{E}) \] (6.1)
over entire \(\tilde{X}\).

For notational convenience, the Galois group Gal(\(\phi\)) will be denoted by \(G\).

The Lie algebra bundle on \(\tilde{X}\) for the group scheme \(\tilde{T}\) will be denoted by \(A\). So \(A\) is also a sub-algebra bundle of the associative algebra bundle on \(\tilde{X}\)
\[ A = \text{Lie}(\tilde{T}) \subset \text{Lie}(\text{Ad}(\tilde{E})) = \text{End}(\tilde{E}) ; \]
the Lie algebra structure is given by the commutator. The total space of the associative algebra bundle \(A\) will also be denoted by \(A\). Let
\[ Z_0 \subset A \] (6.3)
be the sub scheme defined by the equation \(z^2 - z = 0\). So \(Z_0\) is the locus of all idempotent elements in the associative algebra bundle \(A\). It is straight-forward to check that \(Z_0\) is an étale cover of \(\tilde{X}\). This \(Z_0\) is smooth but not connected; note that the zero section of \(A\) is a connected component of \(Z_0\).

Let
\[ Z \subset Z_0 \]
be the locus of nonzero elements that are not nontrivial sum of other elements of \(Z_0\). This \(Z\) is in fact a union of some connected components of \(Z_0\). Let
\[ \psi : Z \rightarrow \tilde{X} \] (6.4)
be the restriction of the natural projection of \(A\) to \(\tilde{X}\).

Note that for any \(x \in \tilde{X}\), we have the direct sum decomposition
\[ \tilde{E}_x = \bigoplus_{z \in \psi^{-1}(x)} z(\tilde{E}_x), \] (6.5)
where \(\psi\) is the projection in (6.4); recall that \(z\) is an idempotent endomorphism of \(\tilde{E}_x\).

From (6.5) it follows that the pulled back vector bundle \(\psi^*\tilde{E}\) on \(Z\) has a tautological sub-bundle
\[ W \subset \psi^*\tilde{E} \rightarrow Z \] (6.6)
whose fiber over any \(z \in Z\) is the image \(z(\tilde{E}_{\psi(z)}) \subset \tilde{E}_{\psi(z)} = (\psi^*\tilde{E})_z\). To describe \(W\) in another way, note that since \(Z\) is contained in the total space of \(\text{End}(\tilde{E})\), there is a tautological homomorphism
\[ \psi^*\tilde{E} \rightarrow \psi^*\tilde{E}. \]
The sub-bundle \(W\) in (6.6) is the image of this tautological homomorphism.
The sub-bundle $W$ of $\psi^*\tilde{E}$ is a direct summand. In fact, it has a tautological complement $W^c$ whose fiber over any $z \in Z$ is

$$W^c_z = \ker(z) = \bigoplus_{y \in \psi^{-1}(\psi(z)) \setminus \{z\}} \psi(y) \cdot (\psi^*\tilde{E})_z.$$ 

It should be clarified that, in general, the decomposition in (6.5) does not produce any decomposition of $W^c_z$, because the direct summands in the fiber $W^c_z$ may get interchanged as $z$ runs over a loop in $Z$.

Consider the action of $\text{Gal}(\phi) = G$ on $\text{End}(\tilde{E})$ induced by the action of $G$ on $\tilde{E}$. It can be shown that both $\tilde{T}$ and $A$ (see (6.2)) are preserved by this action of $G$ on $\text{End}(\tilde{E})$. Indeed, this follows immediately from the fact that $\tilde{T}|_{\phi^{-1}(U)}$ is pulled back from $X$. The action of $G$ on $A$ evidently preserves $Z_0$ in (6.3). The action of $G$ on $Z_0$ clearly preserves $Z$ in (6.4). The map $\psi$ in (6.4) is evidently $G$–equivariant.

The action of $G$ on $\tilde{E}$ pulls back to an action of $G$ on $\psi^*\tilde{E}$ such that the projection map from $\psi^*\tilde{E}$ to $Z$ intertwines the actions of $G$. The sub-bundle $W$ in (6.6) is clearly preserved by this action of $G$ on $\psi^*\tilde{E}$. Consequently, the vector bundle $W$ on $Z$ is $G$–equivariant.

The quotient $Z/G$ will be denoted by $Y$. It was noted above $\psi$ is $G$–equivariant. Therefore, the map $\psi$ descends to a map

$$\varphi : Y \longrightarrow X.$$ 

More precisely, we have a commutative diagram of morphisms

$$\begin{array}{ccc} Z & \xrightarrow{\psi} & \bar{X} \\ \downarrow q & & \downarrow \phi \\ Y & \xrightarrow{\varphi} & X \end{array}$$

where $q$ is the quotient map to $Z/G$.

Let $V_\ast$ be the parabolic vector bundle on $Y$ associated to the above $G$–equivariant vector bundle $W$ on $Z$ [B1], [B5]. Note that the parabolic points of $V_\ast$ are contained in $\varphi^{-1}(S)$.

Let any point $x \in U = X \setminus S$. The fibers of both $\varphi_*V_\ast$ and $E_\ast$ over $x$ are identified with

$$\bigoplus_{z \in \psi^{-1}(y)} z(\tilde{E}_y),$$

where $y$ is any point of $\phi^{-1}(x)$; note that for different choices of $y$, the corresponding direct sums get identified using the action of $G$. In other words, on $U$, the vector bundles $E_\ast$ and $\varphi_*V_\ast$ are canonically identified. This isomorphism clearly takes the torus sub-bundle $\mathcal{T} \subset \text{Ad}(E)|_U$ to the torus over sub-bundle of $\text{Ad}(\varphi_*V_\ast)|_U$ given by Proposition 4.4.

Using the above constructions, it is straight-forward to check that this isomorphism between $E_\ast$ and $\varphi_*V_\ast$ over $U$ extends to an isomorphism between the parabolic vector bundles $E_\ast$ and $\varphi_*V_\ast$ over $X$. It was already noted that this isomorphism takes $\mathcal{T}$ to the torus sub-bundle of $\text{Ad}(\varphi_*V_\ast)|_U$ given by Proposition 4.4. $\square$

**Remark 6.2.** Let $\phi^1 : Z \longrightarrow X$ be a ramified covering map, where $X$ is a connected smooth complex projective curve. Let $V^*_\ast$ be a parabolic vector bundle on $Z$ with rational
parabolic weights. We have the parabolic vector bundle $\phi_*^1 V_*^1$ on $X$ constructed in Section 3. Since all the parabolic weights of $V_*^1$ are rational numbers, the parabolic weights of $\phi_*^1 V_*^1$ are rational too. Let

$$S \subset X$$

be the parabolic divisor for $\phi_*^1 V_*^1$.

The ramified torus sub-bundle for $\phi_*^1 V_*^1$ constructed in Proposition 4.4 will be denoted by $T^1$.

Consider the pair $(\phi_*^1 V_*^1, T^1)$. Using it, Proposition 6.1 produces a ramified covering map

$$\varphi : Y \rightarrow X$$

and a parabolic bundle $V_*$ on $Y$ (see the proof of Proposition 6.1). Consider the two coverings of $X \setminus S$

$$\phi^1_0 : (\phi^1)^{-1}(X \setminus S) \rightarrow X \setminus S \quad \text{and} \quad \varphi_0 : \varphi^{-1}(X \setminus S) \rightarrow X \setminus S,$$

where $\phi^1_0$ (respectively, $\varphi_0$) is the restriction of $\phi^1$ (respectively, $\varphi$) to the Zariski open subset $(\phi^1)^{-1}(X \setminus S)$ (respectively, $\varphi^{-1}(X \setminus S)$) of $Z$ (respectively, $Y$). It is straightforward to check that these two coverings of $X \setminus S$ are canonically identified. In other words, we have commutative diagram of maps

$$
\begin{array}{ccc}
(\phi^1)^{-1}(X \setminus S) & \xrightarrow{\delta} & \varphi^{-1}(X \setminus S) \\
\downarrow \phi^1_0 & & \downarrow \varphi_0 \\
X \setminus S & \xrightarrow{\text{Id}} & X \setminus S
\end{array}
$$

where $\delta$ is an isomorphism. Since $(\phi^1)^{-1}(X \setminus S)$ and $\varphi^{-1}(X \setminus S)$ are Zariski open dense in $Z$ and $Y$ respectively, the above map $\delta$ extends to an isomorphism

$$\hat{\delta} : Z \rightarrow Y$$

such that the diagram

$$
\begin{array}{ccc}
Z & \xrightarrow{\hat{\delta}} & Y \\
\downarrow \phi^1 & & \downarrow \varphi \\
X & \xrightarrow{\text{Id}} & X
\end{array}
$$

(6.7)

is commutative.

The pulled back vector bundle $\delta^*(V_*|_{\varphi^{-1}(X \setminus S)})$ is identified with $V_*^1|_{(\phi^1)^{-1}(X \setminus S)}$. Note that the parabolic divisor for the parabolic vector bundle $V_*$ (respectively, $V_*^1$) is contained in $\varphi^{-1}(S)$ (respectively, $(\phi^1)^{-1}(S)$). It can checked that this identification extends to an isomorphism between the parabolic vector bundles $\hat{\delta}^* V_*$ and $V_*^1$ over $Z$, where $\hat{\delta}$ is the isomorphism in (6.7).

Combining the construction in Section 3 with Proposition 4.4, Proposition 6.1 and Remark 6.2, we have the following theorem.

**Theorem 6.3.** Let $E_*$ be a parabolic vector bundle on a connected smooth complex projective curve $X$ with parabolic divisor $S$ and rational parabolic weights. Then there is a natural equivalence between the following two classes:
6.2. Characterization of direct images of connections.} As in Section 6.1, X is a connected smooth complex projective curve and S ⊂ X a reduced effective divisor. Let E* be a parabolic vector bundle over X, with parabolic structure over S and underlying vector bundle E, such that all the parabolic weights are rational numbers. Let \( T \subset \text{Ad}(E)|_U \) be a ramified torus sub-bundle for \( E_* \), where \( U := X \setminus S \). Proposition 6.1 says that there is a finite ramified covering \( \varphi : Y \longrightarrow X \) and a parabolic vector bundle \( V_* \) on \( Y \) with rational parabolic weights, such that

- the parabolic vector bundle \( \varphi_* V_* \) is isomorphic to \( E_* \), and
- the isomorphism between \( \varphi_* V_* \) and \( E_* \) can be so chosen that it takes \( T \) to the ramified torus sub-bundle of \( \varphi_* V_* \) given by Proposition 6.4.

**Proposition 6.4.** Let \( D \) be a connection on the parabolic vector bundle \( E_* \) such that \( D \) preserves the ramified torus sub-bundle \( T \) in (6.8). Then \( D \) produces a connection \( D' \) on the above parabolic vector bundle \( V_* \) such that the connection \( \varphi_* D' \) on \( \varphi_* V_* \) (see Theorem 5.2 for \( \varphi, D' \)) coincides with the connection \( D \) on \( E_* \).

*Proof.* Since \( E_* \) corresponds to the \( \text{Gal}(\phi) \)-equivariant vector bundle \( \tilde{E} \longrightarrow \tilde{X} \) in (6.1), the connection \( D \) on \( E_* \) produces a \( \text{Gal}(\phi) \)-invariant connection on \( \tilde{E} \). This connection on \( \tilde{E} \) given by \( D \) will be denoted by \( \tilde{D} \). On \( \phi^{-1}(U) \), the vector bundle \( \tilde{E}|_{\phi^{-1}(U)} \) is identified with the pullback \( \phi^*(E|_U) \). Recall that this isomorphism over \( \phi^{-1}(U) \) takes \( \phi^* T \) to \( \tilde{T} \) in (6.1).

Let \( \tilde{D}' \) be the connection on the vector bundle \( \text{End}(\tilde{E}) \) induced by the above connection \( \tilde{D} \). Since \( \tilde{D} \) preserves the ramified torus sub-bundle \( T \) in (6.8), it follows that \( \tilde{D}' \) preserves \( \tilde{T} \) over \( \phi^{-1}(U) \). As \( \phi^{-1}(U) \) is Zariski dense in \( \tilde{X} \), this implies that \( \tilde{D}' \) preserves \( \tilde{T} \) over \( \tilde{X} \). Therefore, the connection \( \tilde{D}' \) preserves \( A \) in (6.2).

Since \( \tilde{D}' \) preserves \( A \), we conclude that \( \tilde{D}' \) preserves the sub-scheme \( Z_0 \) in (6.3), meaning for every \( z \in Z_0 \), the horizontal subspace of \( T_z \text{End}(\tilde{E}) \) for the connection \( \tilde{D}' \) actually lies inside the subspace

\[ T_z Z_0 \subset T_z \text{End}(\tilde{E}). \]

Note that this implies that the above horizontal subspace of \( T_z \text{End}(\tilde{E}) \) coincides with \( T_z Z_0 \), because \( Z \) is an étale cover of \( \tilde{X} \). We now conclude that \( Z \) in (6.4) is also preserved by \( \tilde{D}' \), because \( Z \) is a union of some connected components of \( Z_0 \).

Moreover, the connection \( \psi^* \tilde{D} \) on \( \psi^* \tilde{E} \) preserves the tautological sub-bundle \( W \) in (6.6). Indeed, this follows from the fact that the connection \( \tilde{D}' \) preserves \( A \). Let \( D^W \) denote the connection on \( W \) given by \( \psi^* \tilde{D} \).

Recall that the connection \( \tilde{D} \) is preserved by the action of the Galois group \( \text{Gal}(\phi) \) on \( \tilde{E} \). Hence the connection \( \tilde{D}' \) on \( \text{End}(\tilde{E}) \) is also \( \text{Gal}(\phi) \)-invariant. Consequently, the above
connection $D^W$ on $W$ in preserved by the action $\text{Gal}(\phi)$ on $W$. This implies that $D^W$ produces a connection on the parabolic vector bundle $V_*$ on $Z/\text{Gal}(\phi) = Y$.

The above connection on $V_*$ given by $D^W$ produces a connection on the direct image $\varphi_*V_*$ (see Theorem 5.2). This connection on $\varphi_*V_*$ will be denoted by $\hat{D}$.

The identification between $\varphi_*V_*$ and $E_*$ takes the restriction of the connection $D$ to $E|_U$ to the restriction of $\hat{D}$ to $(\varphi_*V_*)|_U$. Therefore, the connection $D$ on $E_*$ coincides with the connection $\hat{D}$ on $\varphi_*V_*$.

□

Remark 6.5. Consider the set-up of Remark 6.2. Let $D^1$ be a connection on the parabolic vector bundle $V^1_*$ over $Z$. Using Theorem 5.2, this $D^1$ produces a connection on the parabolic vector bundle $\phi^1_*V^1_*$ given by $D^1$ will be denoted by $\phi^1_*D^1$. From Lemma 5.4 we know that this connection $\phi^1_*D^1$ preserves the ramified torus sub-bundle $T^1$ in Remark 6.2 for $\phi^1_*V^1_*$. Now from Proposition 6.4 and Remark 6.2 we know that $\phi^1_*D^1$ produces a connection on $V^1_*$; this connection on $V^1_*$ will be denoted by $D^2$. The two connections $D^1$ and $D^2$ on the parabolic vector bundle $V^1_*$ coincide over the dense open subset $(\phi^1)^{-1}(X \setminus S)$. This implies that $D^1$ and $D^2$ coincide over $Z$.

Combining Lemma 5.4, Proposition 6.4 and Remark 6.5, we have the following theorem.

Theorem 6.6. The equivalence in Theorem 6.3 takes a connection on the parabolic vector bundle $V_*$ on $Y$ to a connection on $E_*$ that preserves the ramified torus sub-bundle for $E_*$ corresponding to $(Y, \varphi, V_*)$. Conversely, a connection on $E_*$ preserving a ramified torus sub-bundle $T$ for $E_*$ is taken to a connection on the parabolic vector bundle $V_*$ on $Y$, where $(Y, \varphi, V_*)$ corresponds to $T$.

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