ON LIPSCHITZ APPROXIMATIONS IN SECOND ORDER SOBOLEV SPACES 
AND THE CHANGE OF VARIABLES FORMULA

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ABSTRACT. In this paper we study approximations of functions of Sobolev spaces $W^2_{p, \text{loc}}(\Omega)$, $\Omega \subset \mathbb{R}^n$, by Lipschitz continuous functions. We prove that if $f \in W^2_{p, \text{loc}}(\Omega)$, $1 \leq p < \infty$, then there exists a sequence of closed sets $\{A_k\}_{k=1}^\infty$, $A_k \subset A_{k+1} \subset \Omega$, such that the restrictions $f|_{A_k}$ are Lipschitz continuous functions and $\text{cap}_p(S) = 0$, $S = \Omega \setminus \bigcup_{k=1}^\infty A_k$. Using these approximations we prove the change of variables formula in the Lebesgue integral for mappings of Sobolev spaces $W^2_{p, \text{loc}}(\Omega; \mathbb{R}^n)$ with the Luzin capacity-measure $N$-property.

1. INTRODUCTION

In this paper we study approximations of the second order Sobolev spaces $W^2_{p, \text{loc}}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is an open set, by Lipschitz continuous functions. The Lipschitz approximations in Sobolev spaces $W^1_{p, \text{loc}}(\Omega)$ have significant applications in geometric measure theory [10, 12, 15] and in the analysis on metric measure spaces [10, 12, 15].

In [9] it was proved that if $f \in W^1_{1, \text{loc}}(\Omega)$, then there exists a sequence of closed sets $\{A_k\}_{k=1}^\infty$, $A_k \subset A_{k+1} \subset \Omega$, for which the restrictions $f^*|_{A_k}$ are Lipschitz continuous functions on the sets $A_k$ and $|S| = 0$, $S = \Omega \setminus \bigcup_{k=1}^\infty A_k$, where we denote by $|S|$ the $n$-dimensional Lebesgue measure of the set $S \subset \mathbb{R}^n$. Here $f^*$ stands for the precise representative of the function $f$ (see [24] below).

In the case of the second order Sobolev spaces $W^2_{1, \text{loc}}(\Omega)$ we prove the following refined version of this Lipschitz approximation: 

Let $f \in W^2_{1, \text{loc}}(\Omega)$. Then there exists a sequence of closed sets $\{A_k\}_{k=1}^\infty$, $A_k \subset A_{k+1} \subset \Omega$, such that the restrictions $f^*|_{A_k}$ are Lipschitz continuous functions $\mathcal{H}^{n-1}$-a.e. in $A_k$ and $\mathcal{H}^{n-1}(S) = 0$, $S = \Omega \setminus \bigcup_{k=1}^\infty A_k$, where $\mathcal{H}^{n-1}$ is the $(n-1)$-Hausdorff measure.

Using this Lipschitz approximation we obtain the following change of variables formula in the Lebesgue integral for twice weakly differentiable mappings: Let $\varphi : \Omega \to \mathbb{R}^n$ be a mapping which belongs to the Sobolev space $W^2_{1, \text{loc}}(\Omega; \mathbb{R}^n)$. Then there exists a Borel set $S \subset \Omega$, $\mathcal{H}^{n-1}(S) = 0$, such that the mapping $\varphi : \Omega \setminus S \to \mathbb{R}^n$ has the Luzin $N$-property (an image of a set of Lebesgue measure zero has Lebesgue measure zero) and the change of variables formula

\begin{equation}
\int_A u \circ \varphi(x) |J(x, \varphi)| \, dx = \int_{\mathbb{R}^n \setminus \varphi(S)} u(y) N_\varphi(A, y) \, dy,
\end{equation}

where $N_\varphi(A, y)$ is the multiplicity function which is defined as the number of preimages of $y$ in $A$ under $\varphi$ and $J(x, \varphi) = \det (D\varphi(x))$ is the Jacobian of $\varphi$ at the point $x$, holds for every measurable set $A \subset \Omega$ and every nonnegative measurable function $u : \mathbb{R}^n \to \mathbb{R}$.

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If the mapping $\varphi$ possesses the Luzin Hausdorff-Lebesgue measure $N$-property (an image of a set of $\mathcal{H}^{n-1}$-Hausdorff measure zero has $n$-dimensional Lebesgue measure zero), then $|\varphi(S)| = 0$ and the integral on the right hand side of (1.1) can be rewritten as an integral on $\mathbb{R}^n$.

In the case of the second order Sobolev spaces $W^2_{p,\text{loc}}(\Omega)$, $1 < p < \infty$, we prove the refined version of the Lipschitz approximation in capacitary terms: Let $f \in W^2_{p,\text{loc}}(\Omega)$, $1 < p < \infty$. Then there exists a sequence of closed sets $\{A_k\}_{k=1}^{\infty}$, $A_k \subset A_{k+1} \subset \Omega$, such that the restrictions $f^*|_{A_k}$ are Lipschitz continuous functions $p$-quasieverywhere in $A_k$ (means that outside a set of cap$_p$ zero) and cap$_p(S) = 0$, $S = \Omega \setminus \bigcup_{k=1}^{\infty} A_k$, where cap$_p$ is the $p$-capacity.

Using this Lipschitz approximation we obtain the following capacitory version of the change of variables formula in the Lebesgue integral for twice weakly differentiable mappings: Let $\varphi : \Omega \to \mathbb{R}^n$ be a mapping which belongs to the Sobolev space $W^2_{p,\text{loc}}(\Omega; \mathbb{R}^n)$, $1 < p < \infty$. Then there exists a Borel set $S \subset \Omega$, cap$_p(S) = 0$, such that the mapping $\varphi : \Omega \setminus S \to \mathbb{R}^n$ has the Luzin $N$-property and the change of variables formula

$$
\int_A u \circ \varphi(x) |J(x, \varphi)| \, dx = \int_{\mathbb{R}^n \setminus \varphi(S)} u(y) N_{\varphi}(A, y) \, dy,
$$

holds for every measurable set $A \subset \Omega$ and every nonnegative measurable function $u : \mathbb{R}^n \to \mathbb{R}$.

If the mapping $\varphi$ possesses the Luzin capacity-measure $N$-property (an image of a set of capacity zero has Lebesgue measure zero), then $|\varphi(S)| = 0$ and the integral on the right hand side of (1.2) can be rewritten as an integral on $\mathbb{R}^n$. Since for any $E \subset \mathbb{R}^n$ we have the inequality $|E| \leq \text{cap}_p(E)$, $1 < p < \infty$ [14], then mappings which possess the Luzin measure $N$-property have the Luzin capacity-measure $N$-property.

Note, mappings which possess the Luzin capacity-capacity $N$-property (an image of a set of capacity zero has capacity zero) and so possess the Luzin capacity-measure $N$-property arise, in particular, in the geometric theory of composition operators on Sobolev spaces, see, for example, [8, 21, 26]. The geometric theory of composition operators on Sobolev spaces have applications in the spectral theory of elliptic operators, see, for example, [7]. We note also the generalized quasiconformal mappings, so-called $Q$-mappings, which possess the Luzin capacity-capacity $N$-property. The theory of $Q$-mappings is intensively developed in the last decades. See, for example, [19].

On the base of the refined Lipschitz approximation we obtain the Luzin type theorem for capacity for second order Sobolev spaces. It is known [1, 4, 13, 20] that if $f \in W^1_{p,\text{loc}}(\Omega)$, then for each $\varepsilon > 0$ there exists an open set $U_\varepsilon$ of $p$-capacity less than $\varepsilon$ such that $f^*$ is continuous on the set $\Omega \setminus U_\varepsilon$. We prove the following refined version: If $f \in W^2_{p,\text{loc}}(\Omega)$, then for each $\varepsilon > 0$ there exists an open set $U_\varepsilon$ of $p$-capacity less than $\varepsilon$ such that $f^*$ is Lipschitz continuous on the set $\Omega \setminus U_\varepsilon$.

The refined Luzin theorem was obtained in [16]. It was proved that Sobolev functions coincide with Hölder continuous functions on the complement of a set of arbitrary small capacity. This result plays a crucial role in the refined versions of the change of variables formula for the Sobolev mappings [17, 18]. In the case of higher order Sobolev spaces the Luzin type theorem considered in [27]. The refined Luzin theorem on the Hölder continuity of higher order Sobolev spaces was proved in [2]. In the recent work [3] the Luzin type theorem for functions of $W^n_1(\mathbb{R}^n)$ in the terms of the Hausdorff content was proved.

We prove the following refined version of the Luzin type theorem: If $f \in W^2_{p,\text{loc}}(\Omega)$ and $B \subset \Omega$ is a Borel set such that $\mathcal{H}^{n-1}(B) < \infty$, then for every $\varepsilon > 0$ there exists a Borel set $B_\varepsilon \subset B$ such that $f^*|_{B_\varepsilon}$ is Lipschitz continuous $\mathcal{H}^{n-1}$-a.e in $B_\varepsilon$ and $\mathcal{H}^{n-1}(B \setminus B_\varepsilon) < \varepsilon$. 
In the case of the second order Sobolev spaces $W^2_{p,\text{loc}}(\Omega)$, $1 < p < \infty$, we prove: If $f \in W^2_{p,\text{loc}}(\Omega)$, $1 < p < \infty$, and $A \subset \Omega$ is a compact set with an additional assumption (see Corollary 2.10), then for every $\varepsilon > 0$ there exists a Borel set $A_\varepsilon \subset A$ such that $f|_{A_\varepsilon}$ is Lipschitz continuous $p$–quasieverywhere in $A_\varepsilon$ and $\text{cap}_p(A \setminus A_\varepsilon) < \varepsilon$.

The suggested methods are based in the non-linear potential theory [13, 20] and the Chebyshev type inequality for capacity [4, 22, 23]. This paper is organized as follows: Section 2 contains definitions and we consider Lipschitz approximations in Sobolev spaces $W^2_{p,\text{loc}}(\Omega)$ and prove Luzin type theorems for Lipschitz $p$–quasicontinuity (see Definition 2.5). In Section 3 we prove the change of variables formula for twice weakly differentiable mappings which have the Luzin capacity-measure $N$–property.

2. Lipschitz approximations of Sobolev functions

Let $E \subset \mathbb{R}^n$ be a measurable set. Recall that the Lebesgue space $L^p(E), 1 \leq p < \infty$, is defined as the space of $p$–integrable functions with the norm

$$\|f \mid L^p(E)\| = \left( \int_E |f(x)|^p \, dx \right)^{1/p}, \quad 1 \leq p < \infty.$$ 

By $L^p_{p,\text{loc}}(E)$ we denote the space of locally $p$–integrable functions, means that, $f \in L^p_{p,\text{loc}}(E)$ if and only if $f \in L^p(F)$ for every compact subset $F \subset E$.

Let $\Omega \subset \mathbb{R}^n$ be an open set. The Sobolev space $W^m_p(\Omega)$, $m \in \mathbb{N}, 1 \leq p < \infty$, is defined as the normed space of functions $f \in L^p(\Omega)$ such that the partial derivatives of order less than or equal to $m$ exist in the weak sense and belong to $L^p(\Omega)$. The space is equipped with the norm

$$\|f \mid W^m_p(\Omega)\| = \sum_{|\alpha| \leq m} \left( \int_{\Omega} |D^\alpha f(x)|^p \, dx \right)^{1/p} < \infty,$$

where $D^\alpha f$ is the weak derivative of order $\alpha$ of the function $f$ which is defined by the following formula:

$$\int_{\Omega} f D^\alpha \eta \, dx = (-1)^{|\alpha|} \int_{\Omega} (D^\alpha f) \eta \, dx, \quad \forall \eta \in C_0^\infty(\Omega),$$

where $\alpha := (\alpha_1, \alpha_2, ..., \alpha_n)$ is a multiindex, $\alpha_i = 0, 1, ..., |\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$. The space $C_0^\infty(\Omega)$ is the space of smooth functions with compact support in $\Omega$. The Sobolev space $W^m_{p,\text{loc}}(\Omega)$ is defined as follows: $f \in W^m_{p,\text{loc}}(\Omega)$ if and only if $f \in W^m_p(U)$ for every open and bounded set $U \subset \Omega$ such that $\overline{U} \subset \Omega$, where $\overline{U}$ stands for the topological closure of the set $U$.

Let us recall the definition of the capacity [1, 13, 20]. Suppose $\Omega$ is an open set in $\mathbb{R}^n$ and $F \subset \Omega$ is a compact set. The $p$–capacity of $F$ with respect to $\Omega$ is defined by

$$\text{cap}_p(F; \Omega) = \inf \{ \|\nabla f\|_{L^p(\Omega)}^p : f \geq 1 \text{ on } F \}. \quad (2.1)$$

where the inferior is taken over all $f \in C_0^\infty(\Omega)$ such that $f \geq 1$ on $F$. If $U \subset \Omega$ is an open set, we define

$$\text{cap}_p(U; \Omega) = \sup \{ \text{cap}_p(F; \Omega) : F \subset U, \text{ } F \text{ is compact} \}. \quad (2.2)$$

In the case of an arbitrary set $E \subset \Omega$ we define

$$\text{cap}_p(E; \Omega) = \inf \{ \text{cap}_p(U; \Omega) : E \subset U \subset \Omega, \text{ } U \text{ is open} \}. \quad (2.3)$$
The p-capacity is an outer measure on \( \Omega \), i.e. it is a set function which is defined on every subset of \( \Omega \) such that \( \text{cap}_p(\emptyset; \Omega) = 0 \) and it is \( \sigma \)-subadditive, means that, if \( E \subset \bigcup_{i \in \mathbb{N}} E_i \), then \( \text{cap}_p(E; \Omega) \leq \sum_{i \in \mathbb{N}} \text{cap}_p (E_i; \Omega) \). We write for short \( \text{cap}_p(E) = \text{cap}_p(E; \mathbb{R}^n), E \subset \mathbb{R}^n \).

The notion of the p-capacity permits us to refine the notion of Sobolev functions [11, 20]. Let \( f \in L_{1, \text{loc}}(\Omega) \). The precise representative of \( f \) is defined by

\[
(2.4) \quad f^* : \Omega \to \mathbb{R}, \quad f^*(x) := \begin{cases} \lim_{\epsilon \downarrow 0} f_{B(x, \epsilon)}, & \text{if the limit exists;} \\ 0, & \text{otherwise;} \end{cases}
\]

where

\[
(2.5) \quad f_{B(x, \epsilon)} := \int_{B(x, \epsilon)} f(y) \, dy = \frac{1}{|B(x, \epsilon)|} \int_{B(x, \epsilon)} f(y) \, dy.
\]

Recall that a function \( f \) is termed \textbf{p-quasicontinuous} in an open set \( \Omega \) if and only if for every \( \epsilon > 0 \) there exists an open set \( U_\epsilon \subset \Omega \) such that the \( p \)-capacity of \( U_\epsilon \) is less than \( \epsilon \) and on the set \( \Omega \setminus U_\epsilon \) the function \( f \) is continuous. If \( f \in W_{1, \text{loc}}^1(\Omega) \), then \( f^* \) is \( p \)-quasicontinuous and there exists a Borel set \( E \) such that \( \text{cap}_p(E) = 0 \) and \( f^*(x) = \lim_{\epsilon \to 0^+} f_{B(x, \epsilon)}, \forall x \in \Omega \setminus E \) [4, 11, 13, 20]. In case \( f \in W_{p, \text{loc}}^1(\Omega) \) the refined function \( f^* \) is called the unique quasicontinuous representation (or the canonical representation) of \( f \).

2.1. Lipschitz \( p \)-quasicontinuous functions. Let us recall the following two theorems from the theory of Sobolev spaces, see for example [11, 13, 20]:

**Theorem 2.1.** (The local Poincaré inequality for functions of \( W_{1, \text{loc}}^1(\mathbb{R}^n) \)).
There exists a constant \( C = C(n) \) such that

\[
(2.6) \quad \int_{B(x, r)} |f(y) - f_{B(x, r)}| \, dy \leq Cr \int_{B(x, r)} |\nabla f(y)| \, dy,
\]
for every ball \( B(x, r) \subset \mathbb{R}^n \) and every \( f \in W_{1, \text{loc}}^1(B(x, r)) \).

**Theorem 2.2.** (The Chebyshev type inequality for capacity).
Assume \( p \geq 1 \), \( f \in W_{p}^1(\mathbb{R}^n) \) and let \( \alpha > 0 \). Define

\[
R_\alpha := \left\{ x \in \mathbb{R}^n \mid \sup_{\epsilon > 0} f_{B(x, \epsilon)} \leq \alpha \right\}.
\]

Then

\[
\text{cap}_p(\mathbb{R}^n \setminus R_\alpha) \leq \frac{C(n, p)}{\alpha^p} \|\nabla f\|_{L_p(\mathbb{R}^n)}^p.
\]

The following theorem gives us an approximation of twice weakly differentiable functions by Lipschitz continuous functions.

**Theorem 2.3.** Let \( \Omega \subset \mathbb{R}^n \) be an open set and \( f \in W_{p, \text{loc}}^2(\Omega), 1 \leq p < \infty \). Then there exists a sequence of closed sets \( \{C_k\}_{k=1}^\infty \) such that for every \( k = 1, 2, \ldots \), \( C_k \subset C_{k+1} \subset \Omega \), the restriction \( f^*|_{C_k} \) is a Lipschitz continuous function defined \( p \)-quasieverywhere in \( C_k \) and

\[
(2.7) \quad \text{cap}_p \left( \Omega \setminus \bigcup_{k=1}^\infty C_k \right) = 0.
\]
Proof. Assume first \( f \in W^2_p(\mathbb{R}^n) \). Then for any \( \alpha > 0 \) we define a set

\[
R_{\alpha} = \left\{ x \in \mathbb{R}^n \mid \sup_{r > 0} |\nabla f|_{B(x,r)} \leq \alpha \right\}.
\]

The set \( R_{\alpha} \) is closed: fix an arbitrary number \( \varepsilon > 0 \) and let \( \{x_i\}_{i=1}^{\infty} \subseteq R_{\alpha} \) be a sequence of points such that \( x_i \to x \) as \( i \to \infty \). Since for every \( r > 0 \) the function \( z \mapsto |\nabla f|_{B(z,r)} \) is continuous, there exists a number \( r_0 > 0 \) such that

\[
||\nabla f|_{B(z,r)} - |\nabla f|_{B(x,r)}| \leq \varepsilon, \quad \forall z \in B(x,r_0).
\]

Thus, for sufficiency large numbers \( i \) such that \( x_i \in B(x,r_0) \) we have

\[
|\nabla f|_{B(x,r_0)} \leq \varepsilon + |\nabla f|_{B(x,r_0)} \leq \varepsilon + \alpha.
\]

Hence \( \sup_{r > 0} |\nabla f|_{B(x,r)} \leq \varepsilon + \alpha \) and since \( \varepsilon > 0 \) is arbitrary we get \( x \in R_{\alpha} \) and so \( R_{\alpha} \) is a closed set. Note, in addition, that by the definition of the sets \( R_{\alpha}, \alpha > 0 \), we have \( R_{\alpha_1} \subseteq R_{\alpha_2} \) for numbers \( \alpha_2 \geq \alpha_1 > 0 \). Now, let \( x \in R_{\alpha} \). Since \( |\nabla f| \in L_p(\mathbb{R}^n) \), then by the local Poincaré inequality we have

\[
\int_{B(x,r)} |f(y) - f_B(x,r)| dy \leq C(n,p)r \int_{B(x,r)} |\nabla f(y)| dy \leq C r \alpha.
\]

Hence

\[
|f_B(x,r/2^{k+1}) - f_B(x,r/2^k)| \leq \int_{B(x,r/2^{k+1})} |f(y) - f_B(x,r/2^k)| dy \leq 2^n \int_{B(x,r/2^k)} |f(y) - f_B(x,r/2^k)| dy \leq \frac{C r \alpha}{2^k},
\]

where \( C \) is a constant which is dependent on \( n \) and \( p \) only. Since \( f \in W^2_p(\mathbb{R}^n) \), then \( f \in W^1_p(\mathbb{R}^n) \) and [11] [20] there exists a Borel set \( E \subseteq \mathbb{R}^n \) such that \( \text{cap}_p(E) = 0 \) and

\[
\lim_{r \to 0^+} f_B(x,r) = f^*(x) \quad \text{for all} \quad x \in \mathbb{R}^n \setminus E.
\]

Therefore, for every \( x \in R_{\alpha} \setminus E \) it follows that

\[
|f^*(x) - f_B(x,r)| = \sum_{k=0}^{\infty} |f_B(x,r/2^{k+1}) - f_B(x,r/2^k)| \leq C r \alpha.
\]

Now, we take arbitrary points \( x, y \in R_{\alpha} \) such that \( x \neq y \) and we set \( r = |x - y| \). Then

\[
|f_B(x,r) - f_B(y,r)| \leq \int_{B(x,r) \setminus B(y,r)} (|f_B(x,r) - f(z)| + |f(z) - f_B(y,r)|) dz \leq 2^n \left( \int_{B(x,r)} |f_B(x,r) - f(z)| dz + \int_{B(y,r)} |f(z) - f_B(y,r)| dz \right) \leq C r \alpha.
\]

By the results (2.12) and (2.13) and the triangle inequality we obtain

\[
|f^*(x) - f^*(y)| \leq C \alpha |x - y|, \quad \text{for any} \quad x, y \in R_{\alpha} \setminus E.
\]
In particular, for every integer \( k \geq 1 \) the function \( f^* \) is Lipschitz continuous on \( R_k \setminus E, \text{cap}(E) = 0 \). Now, since \( f \in W^2_p(\mathbb{R}^n) \), then \( \nabla f \in W^1_p(\mathbb{R}^n, \mathbb{R}^n) \) and by the Chebyshev type inequality for capacity we have the following estimate

\[
\text{cap}_p(\mathbb{R}^n \setminus R_\alpha) \leq \frac{C(n, p)}{\alpha^p} ||\nabla^2 f| \, L_p(\mathbb{R}^n)||^p, \quad \text{for any number } \alpha > 0.
\]

Therefore, by (2.15)

\[
\text{cap}_p(\mathbb{R}^n \setminus \bigcup_{k=1}^\infty R_k) = 0.
\]

It completes the proof in the case \( f \in W^2_p(\mathbb{R}^n) \).

Now, assume \( f \in W^2_{p,\text{loc}}(\Omega) \). Let \( \Omega_k \subset \Omega_{k+1} \subset \Omega \) be a nested sequence for \( \Omega \), i.e., \( \forall k, \Omega_k \) is an open set such that \( \Omega_k \subset \Omega_{k+1}, \Omega_k \) is compact and the union of the sets \( \Omega_k \) equals to \( \Omega \).

For each \( k \) we choose a function \( \zeta_k \in C^\infty(\Omega_{k+1}) \) such that \( \zeta_k \equiv 1 \) on \( \Omega_k \).

It follows that \( f_k := f \zeta_k \in W^2_p(\Omega_{k+1}), \text{Supp}(f_k) \subset \Omega_{k+1} \), where \( \text{Supp}(f_k) \) denotes the support of \( f_k \). We extend \( f_k \) from \( \Omega_{k+1} \) to \( \mathbb{R}^n \) by zero and denote the extension again by \( f_k \). Then \( f_k \in W^2_p(\mathbb{R}^n) \) and we get by the previous case a non decreasing sequence of closed sets \( \{A^k_l\}_{l=1}^\infty \) such that \( f_k|_{A^k_l} \) is Lipschitz continuous in \( A^k_l \) up to a set of \( p \)-capacity zero and \( \text{cap}_p(\mathbb{R}^n \setminus \bigcup_{l=1}^\infty A^k_l) = 0 \). Let us choose a sequence of numbers \( \alpha_k \in \mathbb{N} \) such that

\[
\alpha_k^p \geq 2^k ||\nabla^2 f_k| \, L_p(\mathbb{R}^n)||^p.
\]

Using the Chebyshev type inequality for capacity we have

\[
\text{cap}_p(\mathbb{R}^n \setminus A^k_{\alpha_k}) \leq \frac{C(n, p)}{\alpha_k^p} ||\nabla^2 f_k| \, L_p(\mathbb{R}^n)||^p \leq \frac{C(n, p)}{2^k}.
\]

For a sequence \( \{\alpha_k\}_{k=1}^\infty \) as above let us define a sequence of sets

\[
\{B_l\}_{l=1}^\infty, \quad B_l = \bigcap_{k=l}^\infty A^k_{\alpha_k}.
\]

It follows that

\[
\mathbb{R}^n \setminus \bigcup_{l=1}^\infty B_l = \mathbb{R}^n \setminus \liminf_{l \to \infty} A^l_{\alpha_l} = \limsup_{l \to \infty} (\mathbb{R}^n \setminus A^l_{\alpha_l}) = \bigcap_{j=1}^\infty \bigcup_{k=j}^\infty (\mathbb{R}^n \setminus A^k_{\alpha_k}).
\]

Therefore, for any \( j \geq 1 \) we obtain

\[
\text{cap}_p \left( \mathbb{R}^n \setminus \bigcup_{l=1}^\infty B_l \right) \leq \sum_{k=j}^\infty \text{cap}_p \left( \mathbb{R}^n \setminus A^k_{\alpha_k} \right) \leq C(n, p) \sum_{k=j}^\infty \frac{1}{2^k}.
\]

Thus,

\[
\text{cap}_p \left( \mathbb{R}^n \setminus \bigcup_{l=1}^\infty B_l \right) = 0.
\]

Now, define a sequence of sets

\[
\{C_l\}_{l=1}^\infty, \quad C_l = B_l \cap \Omega_l.
\]
The sequence $\{C_l\}_{l=1}^\infty$ is monotone increasing since $\{B_l\}_{l=1}^\infty, \{\overline{B}_l\}_{l=1}^\infty$ are monotone increasing. $C_l$ is closed as an intersection of closed sets $B_l, \overline{B}_l$. Since

$$\Omega \setminus \bigcup_{l=1}^\infty C_l = \Omega \setminus \bigcup_{l=1}^\infty B_l \cap \overline{B}_l = \Omega \setminus \bigcup_{l=1}^\infty B_l \cap \Omega = \Omega \cap \left(\mathbb{R}^n \setminus \bigcup_{l=1}^\infty B_l\right) \subset \mathbb{R}^n \setminus \bigcup_{l=1}^\infty B_l,$$

then

$$\operatorname{cap}_p \left(\Omega \setminus \bigcup_{l=1}^\infty C_l\right) = 0.$$

At last, since $C_l \subset A^\alpha_{\Omega_l} \cap \overline{B}_l$ and $f^*|_{A^\alpha_{\Omega_l} \cap \overline{B}_l} = f^*|_{A^\alpha_{\Omega_l} \cap \overline{B}_l}$ is Lipschitz continuous in $A^\alpha_{\Omega_l} \cap \overline{B}_l$ up to a set of $p$-capacity zero, then $f^*|_{C_l}$ is a Lipschitz continuous function in $C_l$ up to a set of $p$-capacity zero. It completes the proof of the theorem. \hfill \Box

We denote by $\mathcal{H}^s, s \geq 0$, the $s$-dimensional Hausdorff measure. For Borel sets $B \subset \mathbb{R}^n$ such that $\mathcal{H}^{n-1}(B) < \infty$ it follows that

$$\mathcal{H}^{n-1}(B) = 0 \iff \operatorname{cap}_1(B) = 0.$$  

For a proof of this relation between the (outer) measures $\mathcal{H}^{n-1}, \operatorname{cap}_1$ see for example Theorem 3 in Section 5.6 in the book [1].

The following corollary is an immediate consequence of Theorems 2.18 and 2.20.

**Corollary 2.4.** Let $\Omega \subset \mathbb{R}^n$ be an open set and $f \in W^{2,1, \text{loc}}(\Omega)$. Then there exists a sequence of closed sets $\{C_k\}_{k=1}^\infty$ such that for every $k = 1, 2, ..., C_k \subset C_{k+1} \subset \Omega$, the restriction $f^*|_{C_k}$ is a Lipschitz continuous function defined 1-quasieverywhere in $C_k$ and

$$\mathcal{H}^{n-1}\left(\Omega \setminus \bigcup_{k=1}^\infty C_k\right) = 0.$$  

Now we define the notion of Lipschitz $p$-quasicontinuous functions.

**Definition 2.5.** Let $\Omega \subset \mathbb{R}^n$ be an open set and let $f : \Omega \to \mathbb{R}$ be a function. The function $f$ is called $p$-**Lipschitz quasicontinuous** if for each $\varepsilon > 0$ there exists an open set $V \subset \Omega$ such that

$$\operatorname{cap}_p(V) \leq \varepsilon$$

and

$$|f|_{\Omega \setminus V} \text{ is Lipschitz continuous.}$$

The following corollary is an immediate consequence of the previous theorem.

**Corollary 2.6.** Let $f \in W^{2,p}(\mathbb{R}^n), 1 \leq p < \infty$. Then $f^*$ is $p$-Lipschitz quasicontinuous.

**Proof.** Fix $\varepsilon > 0$. By (2.24) there exists a big enough $\alpha > 0$ such that $\operatorname{cap}_p(\mathbb{R}^n \setminus (R_\alpha \setminus E)) \leq \varepsilon$ and $f^*$ is Lipschitz continuous in $R_\alpha \setminus E$, where $R_\alpha$ and $E$ are the same as in the proof of Theorem 2.21. By the outer regularity of $p$-capacity (2.23) there exists an open set $V$ such that $\mathbb{R}^n \setminus (R_\alpha \setminus E) \subset V$, $\operatorname{cap}_p(V) \leq \varepsilon$ and $f^*$ is Lipschitz continuous in $\mathbb{R}^n \setminus V$. \hfill \Box
2.2. The refined Luzin type theorem for functions of second order Sobolev spaces.

**Theorem 2.7.** Let \( \Omega \subset \mathbb{R}^n \) be an open set and \( f \in W_{1,\text{loc}}^2(\Omega) \). Let \( B \subset \Omega \) be a Borel set such that \( \mathcal{H}^{n-1}(B) < \infty \). Then for any real number \( \varepsilon > 0 \) there exists a Borel set \( B_{\varepsilon} \subset B \) such that \( f^*|_{B_{\varepsilon}} \) is Lipschitz continuous in \( B_{\varepsilon} \) up to a set of 1-capacity zero and \( \mathcal{H}^{n-1}(B \setminus B_{\varepsilon}) < \varepsilon \).

**Proof.** By Theorem 2.3 since \( f \in W_{1,\text{loc}}^2(\Omega) \), there exists a sequence of closed sets \( B_k \subset B_{k+1} \subset \Omega \) such that \( f^*|_{B_k} \) is Lipschitz continuous in \( B_k \) up to a set of 1-capacity zero and \( \text{cap}_1(\Omega \setminus \bigcup_{k=1}^{\infty} B_k) = 0 \). Since

\[
\text{cap}_1\left(B \setminus \bigcup_{k=1}^{\infty} B_k\right) = 0,
\]

then

\[
\mathcal{H}^{n-1}\left(B \setminus \bigcup_{k=1}^{\infty} B_k\right) = 0.
\]

By the assumption that \( \mathcal{H}^{n-1}(B) < \infty \), we have

\[
\lim_{l \to \infty} \mathcal{H}^{n-1}\left(B \setminus \bigcup_{k=1}^{l} B_k\right) = 0.
\]

Choose a big enough natural number \( l \) such that \( \mathcal{H}^{n-1}\left(B \setminus \bigcup_{k=1}^{l} B_k\right) < \varepsilon \) and set \( B_{\varepsilon} := \bigcup_{k=1}^{l} B_k \cap B \). It follows that \( B_{\varepsilon} \subset B \) is a Borel set for which \( f^*|_{B_{\varepsilon}} \) is Lipschitz continuous in \( B_{\varepsilon} \) up to a set of 1-capacity zero and \( \mathcal{H}^{n-1}(B \setminus B_{\varepsilon}) < \varepsilon \). \( \square \)

**Remark 2.8.** If we assume in Theorem 2.7 that the set \( B \) is closed, then we see from the proof of this theorem that we can also choose the set \( B_{\varepsilon} \) to be closed.

Before we introduce another capacity version of Luzin’s theorem we recall that for a monotone decreasing sequence of compact sets \( C_{k+1} \subset C_k \subset \mathbb{R}^n \) we have

\[
\lim_{k \to \infty} \text{cap}_p(C_k) = \text{cap}_p\left(\bigcap_{k=1}^{\infty} C_k\right).
\]

The proof of this formula can be found, for example, in [4]. We denote by \( \text{Int}(B) \) the topological interior of a set \( B \).

**Theorem 2.9.** Let \( \Omega \subset \mathbb{R}^n \) be an open set and \( f : \Omega \to \mathbb{R} \) be a function. Assume the existence of a sequence of closed sets \( \{B_k\}_{k=1}^{\infty} \), \( B_k \subset B_{k+1} \subset \Omega \) for which the restrictions \( f|_{B_k} \) are Lipschitz continuous functions on the sets \( B_k \) and

\[
\text{cap}_p\left(\Omega \setminus \bigcup_{k=1}^{\infty} B_k\right) = 0.
\]

Let \( A \) be a compact set such that \( A \subset \bigcup_{k=1}^{\infty} \text{Int}(B_k) \). Then for every \( \varepsilon > 0 \) there exists a Borel set \( A_{\varepsilon} \subset A \) such that \( f|_{A_{\varepsilon}} \) is Lipschitz continuous and \( \text{cap}_p(A \setminus A_{\varepsilon}) \leq \varepsilon \).
Proof. By assumption there exists a monotone increasing sequence of closed sets $B_k$ such that $f|_{B_k}$ is Lipschitz continuous for any integer $k \geq 1$ and $\text{cap}_p (\Omega \setminus \bigcup_{k=1}^{\infty} B_k) = 0$. Thus,

$$0 = \text{cap}_p \left( A \setminus \bigcup_{k=1}^{\infty} \text{Int}(B_k) \right) = \lim_{l \to \infty} \text{cap}_p \left( A \setminus \bigcup_{k=1}^{l} \text{Int}(B_k) \right).$$

Fix $\varepsilon > 0$. There exists a big enough integer $l_0$ such that

$$\text{cap}_p \left( A \setminus \bigcup_{k=1}^{l} \text{Int}(B_k) \right) \leq \varepsilon, \quad \forall l \geq l_0.$$

Set $A_\varepsilon := \bigcup_{k=1}^{l_0} \text{Int}(B_k) \cap A$. It follows that $A_\varepsilon \subset A$ is a Borel set such that $f|_{A_\varepsilon}$ is Lipschitz continuous and $\text{cap}_p (A \setminus A_\varepsilon) \leq \varepsilon$. \qed

The following Corollary is an immediate consequence of Theorem 2.8 and Theorem 2.10.

**Corollary 2.10.** Let $\Omega \subset \mathbb{R}^n$ be an open set and $f \in W_{p,\text{loc}}^2(\Omega), 1 \leq p < \infty$. There exists a sequence of closed sets $\{B_k\}_{k=1}^{\infty}, B_k \subset B_{k+1} \subset \Omega$ for which the restrictions $f^{|B_k}$ are Lipschitz continuous in $B_k$ up to a set of $p$-capacity zero and

$$\text{cap}_p \left( \Omega \setminus \bigcup_{k=1}^{\infty} B_k \right) = 0.$$

Moreover, if $A$ is a compact set such that $A \subset \bigcup_{k=1}^{\infty} \text{Int}(B_k)$, then for every $\varepsilon > 0$ there exists a Borel set $A_\varepsilon \subset A$ such that $f^{|A_\varepsilon}$ is Lipschitz continuous in $A_\varepsilon$ up to a set of $p$-capacity zero and $\text{cap}_p (A \setminus A_\varepsilon) \leq \varepsilon$.

3. **The change of variables formula**

In this section we will derive from the results of the previous section the change of variables formula for mappings of the class $W_{p,\text{loc}}^2(\Omega; \mathbb{R}^n)$.

**Remark 3.1.** Because differential and geometric properties of mappings $\varphi : \Omega \to \mathbb{R}^n$ are defined by their coordinate functions, the previous results on Lipschitz approximations of Sobolev functions are valid for mappings $\varphi : \Omega \to \mathbb{R}^n$. More precisely, one can generalize the results of Section 2 from the case of functions $f : \Omega \to \mathbb{R}$ to the case of mappings $\varphi : \Omega \to \mathbb{R}^n$ using that a mapping $\varphi : \Omega \to \mathbb{R}^n$ is a Sobolev mapping if and only if its coordinate functions are Sobolev functions and it is a Lipschitz mapping if and only if its coordinate functions are Lipschitz functions.

The following theorem refines the formula of change of variables in the Lebesgue integral in the terms of the non-linear potential theory. The change of variables formula for Lipschitz and Sobolev mappings can be found, for example, in [3 9 24].

**Theorem 3.2.** Let $\Omega \subset \mathbb{R}^n$ be an open set and let $\varphi : \Omega \to \mathbb{R}^n$ be a measurable mapping such that there exists a collection of closed sets $\{A_k\}_{k=1}^{\infty}, A_k \subset A_{k+1} \subset \Omega$ for which the restrictions $\varphi|_{A_k}$ are Lipschitz continuous mappings on the sets $A_k$ and

$$\text{cap}_p \left( \Omega \setminus \bigcup_{k=1}^{\infty} A_k \right) = 0.$$

Then there exists a Borel set $S \subset \Omega, \text{cap}_p (S) = 0$, such that the mapping $\varphi : \Omega \setminus S \to \mathbb{R}^n$ has the Luzin $N$-property and the change of variables formula
Let $\varphi : \Omega \to \mathbb{R}$ be a Sobolev mapping of the class $W^{2}_{p,\text{loc}}(\Omega; \mathbb{R}^{n})$. Then there exists a Borel set $S \subset \Omega$, $\mathcal{H}^{n-1}(S) = 0$, such that the mapping $\varphi : \Omega \setminus S \to \mathbb{R}^{n}$ has the Luzin $N$-property and the change of variables formula (3.6) holds for every measurable set $A \subset \Omega$ and every nonnegative Lebesgue measurable function $u : \mathbb{R}^{n} \to \mathbb{R}$.
If the mapping \( \varphi \) possesses the Luzin capacity-measure \( N \)-property (the image of a set of capacity zero has Lebesgue measure zero), then \( |\varphi(S)| = 0 \) and the integral on the right hand side of (3.1) can be rewritten as an integral on \( \mathbb{R}^n \). We summarize it in the following corollary:

**Corollary 3.4.** Let \( \Omega \subset \mathbb{R}^n \) be an open set and let \( \varphi \in W^2_{p,loc}(\Omega; \mathbb{R}^n) \) which has the Luzin capacity-measure \( N \)-property. Then the change of variables formula

\[
\int_A u \circ \varphi(x) |J(x, \varphi)| \, dx = \int_{\mathbb{R}^n} u(y) N_\varphi(A, y) \, dy,
\]

holds for every measurable set \( A \subset \Omega \) and every nonnegative Lebesgue measurable function \( u : \mathbb{R}^n \to \mathbb{R} \).

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