Determination of $\alpha_s$
from the QCD static energy

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(1) A. Bazakov, N. Brambilla, X. Garcia i Tormo, P. Petreczky, J. Soto and A. Vairo
Determination of \( \alpha_s \) from the QCD static energy: an update
Phys. Rev. D90 (2014) 7, 074038 arXiv:1407.8437

(2) X. Garcia i Tormo
Review on the determination of \( \alpha_s \) from the QCD static energy
Mod. Phys. Lett. A28 (2013) 1330028 arXiv:1307.2238

(3) A. Bazakov, N. Brambilla, X. Garcia i Tormo, P. Petreczky, J. Soto and A. Vairo
Determination of \( \alpha_s \) from the QCD static energy
Phys. Rev. D86 (2012) 114031 arXiv:1205.6155

(4) N. Brambilla, X. Garcia i Tormo, J. Soto and A. Vairo
Precision determination of \( r_0 \mathcal{A}_{\overline{MS}} \) from the QCD static energy
Phys. Rev. Lett. 105 (2010) 212001 arXiv:1006.2066

(5) N. Brambilla, X. Garcia i Tormo, J. Soto and A. Vairo
The QCD static energy at NNLL
Phys. Rev. D80 (2009) 034016 arXiv:0906.1390

(6) N. Brambilla, X. Garcia i Tormo, J. Soto and A. Vairo
The logarithmic contribution to the QCD static energy at N^4LO
Phys. Lett. B647 (2007) 185 arXiv:hep-ph/0610143
$\alpha_s$ in 2014
| Authors | $\alpha_s$ (MeV) | Method | Reference |
|---------|-----------------|--------|-----------|
| Alkhaton [2001] | 0.140 | 0.0013 | DBS (4) |
| BBG [2004] | 0.113 | 0.0011 | valence analysis, NNLO [10] |
| CRQ | 0.140 | 0.0011 | valence analysis, NNLO [9] |
| ABKEM | 0.125 | 0.0014 | IQCD FNS $N_f = 3$ [8] |
| JLQ | 0.128 | 0.0008 | dynamical approach [8] |
| MSTW | 0.171 | 0.0014 | including NLO QCD [7] |
| Thorne | 0.136 | 0.0014 | (2009) [6] |
| ABSMU [1] | 0.125 ± 0.0016 | 0.0012 | Evt-q curves (NLQCD inc. [5]) |
| ABSMU [3] | 0.125 ± 0.0001 | 0.0001 | Evt-q curves [4] |
| ABM9 [4] | 0.140 ± 0.0001 | 0.0001 | Evt-q curves [3] |
| CTEQ | 0.140 | 0.0001 | Evt-q curves [2] |
| NNLO | 0.135 ± 0.0002 | 0.0001 | Evt-q curves [1] |

| Authors | $\alpha_s$ (MeV) | Method | Reference |
|---------|-----------------|--------|-----------|
| Gehrmann et al. | 0.135 | 0.0001 | Evt-q curves [1] |
| Abelen et al. | 0.140 | 0.0011 | Evt-q curves [2] |
| CMS | 0.131 | 0.0014 | Evt-q curves [3] |
| NLO q- f AS, ATLAS | 0.140 | 0.0001 | Evt-q curves [4] |
| Z-QCD | 0.119 | 0.0001 | Evt-q curves [5] |
| Z-decay rate | 0.139 | 0.0001 | Evt-q curves [6] |
| Z-decay width | 0.132 | 0.0001 | Evt-q curves [7] |
| Z-decay width | 0.126 | 0.0001 | Evt-q curves [8] |
| Z-decay width | 0.134 | 0.0001 | Evt-q curves [9] |

| Authors | $\alpha_s$ (MeV) | Method | Reference |
|---------|-----------------|--------|-----------|
| Lattice | 0.130 | 0.0010 | PACS-CS 2000 (2+4) [24] |
| Lattice | 0.140 | 0.0006 | ENMC 2002a [25] |
| Lattice | 0.140 | 0.0001 | ENMC 2002 [25] |
| Lattice | 0.150 | 0.0022 | HERA2000 [25] |
| Lattice | 0.150 | 0.0010 | RIC(U)QCD (phenomen.). [25] |

Moch et al arXiv:1405.4781
Current uncertainties of $\alpha_s$ are not fully reflected in the PDG average.
Theory
Static energy

\[ E_0(r) = \lim_{T \to \infty} i T \ln \langle \varphi \rangle; \quad \varphi = \exp \left\{ i g \oint dz \mu A_\mu \right\} \]

Perturbation theory describes \( E_0(r) \) in the short range \( r \Lambda \ll 1, \alpha_s(1/r) < 1 \):

\[ E_0(r) = \Lambda_s - C_F \alpha_s \left( 1 + \# \alpha_s + \# \alpha_s^2 + \# \alpha_s^3 \ln \alpha_s + \# \alpha_s^4 \ln^2 \alpha_s + \# \alpha_s^5 \ln \alpha_s + \ldots \right) \]

- \( E_0(r) \) is known at three loops.
  
  Anzai Kiyo Sumino PRL 104 (2010) 112003  
  A.Smirnov V.Smirnov Steinhauser PRL 104 (2010) 112002

- \( \ln \alpha_s \) signals the cancellation of contributions coming from different energy scales:

\[ \ln \alpha_s = \ln \frac{\mu}{1/r} + \ln \frac{\alpha_s/r}{\mu} \]

Brambilla Pineda Soto Vairo PRD 60 (1999) 091502
Energy scales

In the short range the static Wilson loop is characterized by a hierarchy of energy scales:

\[
\frac{1}{r} \gg V_o - V_s \gg \Lambda; \quad V_s \approx -C_F \frac{\alpha_s}{r}, \quad V_o \approx \frac{1}{2N} \frac{\alpha_s}{r}
\]
Effective Field Theories

It is convenient to factorize the contributions from the different scales with EFTs:

\[
E_0(r) = \Lambda_s + V_s(r, \mu) - \frac{g^2}{N} \int_0^\infty dt e^{-it(V_0 - V_s)} (\text{Tr} \, r \cdot E(t) \cdot E(0)) (\mu) + \ldots
\]

res. mass potential ultrasoft contribution

\[V_s \sim \ln r \mu, \ln^2 r \mu, \ldots\]

ultrasoft contribution \(\sim \ln(V_0 - V_s)/\mu, \ln^2(V_0 - V_s)/\mu, \ldots\) \(\ln r \mu, \ln^2 r \mu, \ldots\)
Static singlet potential and energy at $N^3LL$

\[ V_s(r, \mu) = V_s(r, 1/r) - \frac{C_F C_A^3}{6\beta_0} \frac{\alpha_s^3(1/r)}{r} \left\{ \left( 1 + \frac{3}{4} \frac{\alpha_s(1/r)}{\pi} \beta_1 \right) \ln \frac{\alpha_s(1/r)}{\alpha_s(\mu)} \right. \\
\left. - \left( \frac{\beta_1}{4\beta_0} - 6 \right) \left[ \frac{\alpha_s(\mu)}{\pi} - \frac{\alpha_s(1/r)}{\pi} \right] \right\} \]

Summed to the ultrasoft contribution at two loops, it provides the static energy at $N^3LL$. 
Mass renormalon

The perturbative expansion of $V_s$ is affected by a renormalon ambiguity of order $\Lambda$. This ambiguity does not affect the slope of the potential (and the extraction of $\alpha_s$).

It may be eliminated from the perturbative series

- either by subtracting a (constant) series in $\alpha_s$ to $V_s$ and reabsorb it in a redefinition of the residual mass,
- or by considering the force:

\[ F(r, \alpha_s(\nu)) = \frac{d}{dr} E_0(r, \alpha_s(\nu)) \]

- The force $F(r, \alpha_s(1/r))$ could be directly compared with lattice,
- or integrated and compared with the static energy

\[ E_0(r) = \int_{r_s}^{r} dr' F(r', \alpha_s(1/r')) \]

up to an irrelevant constant fixed by the overall normalization of the lattice data.

Note that there are no $\ln \nu r$ ($\nu =$ renormalization scale).
Analysis
We use 2+1-flavor lattice QCD obtained from tree-level improved gauge action and Highly-Improved Staggered Quark (HISQ) action by the HotQCD collaboration. $m_s$ was fixed to its physical value, while $m_l = m_s/20$. This corresponds to a pion mass of about 160 MeV in the continuum limit.

| $\beta$  | 7.373 | 7.596 | 7.825 |
|----------|-------|-------|-------|
| $r_1/a$  | 5.172(34) | 6.336(56) | 7.690(58) |
| Volume   | $48^3 \times 64$ | $64^4$ | $64^4$ |

The largest gauge coupling, $\beta = 7.825$, corresponds to lattice spacings of $a = 0.041$ fm. 

Bazakov et al PRD 90 (2014) 094503

The lattice spacing was fixed using the $r_1$ scale defined as $r_1^2 \frac{dE_0(r)}{dr} \bigg|_{r=r_1} = 1.0$; $r_1 = 0.3106 \pm 0.0017$ fm from the pion decay constant $f_\pi$.

Bazakov et al PoS LATTICE 2010 (2010) 074
Procedure

We use data for each value of the lattice spacing separately, and at the end perform an average of the different obtained values of $\alpha_s$ with the following procedure.

- Perform fits to the lattice data for the static energy $E_0(r)$ at different orders of perturbative accuracy. The parameter of the fits is $\Lambda_{\text{MS}}$.
- Repeat the above fits for each of the following distance ranges: $r < 0.75r_1$, $r < 0.7r_1$, $r < 0.65r_1$, $r < 0.6r_1$, $r < 0.55r_1$, $r < 0.5r_1$, and $r < 0.45r_1$.
- Use ranges where the reduced $\chi^2$ either decreases or does not increase by more than one unit when increasing the perturbative order, or is smaller than 1.
- To estimate the perturbative uncertainty of the result, repeat the fits
  - by varying the scale in the perturbative expansion, from $\nu = 1/r$ to $\nu = \sqrt{2}/r$ and $\nu = 1/(\sqrt{2}r)$,
  - by adding/subtracting a term $\pm (C_F/r^2)\alpha_s^{n+2}$ to the expression at $n$ loops. Take the largest uncertainty.
Data ranges

![Graph showing data ranges with markers for different values of \( \beta \).]
$\chi^2$/d.o.f. for $\beta = 7.825$

Fits for $r < 0.6r_1$ are acceptable. In the final result we will use only fits for $r < 0.5r_1$. The fitting curve has been normalized on the 7th, 8th and 9th lattice point respectively.
$a\Lambda_{\overline{MS}}$ at different orders of perturbative accuracy for $\beta = 7.825$
$r_1\Lambda_{\text{MS}}$ at three-loop accuracy

The band shows the determination of 2012.
The statistical error is estimated by taking values of $\Lambda_{\text{SM}}$ at one $\chi^2$ unit above minimum.
Short-distance points vs long-distance points

The band shows the determination of 2012.
Looking for condensates

By repeating the fits adding a monomial term proportional to $r^3$ and $r^2$, which could be associated with gluon and quark local condensates, and also a term proportional to $r$, we do not find evidence for a significant non-perturbative term at short distances and the value of $\Lambda_{\text{QCD}}$ remains unchanged.
Results
Results at three-loop plus leading-ultrasoft resummation for the $r < 0.5 r_1$ fit range.

The final result is the weighted average of different $\beta$s with linearly added errors.

| $r_1 \Lambda_{\text{MS}}$; range spanned | $r_1 \Lambda_{\text{MS}}$; range spanned | $r_1 \Lambda_{\text{MS}}$; range spanned |
|----------------------------------------|----------------------------------------|----------------------------------------|
| $\beta = 7.373$                        | $0.0097_{-0.0055}^{+0.0060}$           | $0.0097_{-0.0055}^{+0.0060}$           |
|                                       | $+0.0017$                              | $+0.0017$                              |
| $\beta = 7.596$                        | $0.078_{-0.006}^{+0.007}$              | $0.078_{-0.006}^{+0.007}$              |
|                                       | $+0.0010$                              | $+0.0010$                              |
| $\beta = 7.825$                        | $0.064_{-0.006}^{+0.006}$              | $0.064_{-0.006}^{+0.006}$              |
|                                       | $+0.0008$                              | $+0.0008$                              |

Average $r_1 \Lambda_{\text{MS}} = 0.495_{-0.018}^{+0.028}$

which converts to $\Lambda_{\text{MS}} = 315_{-12}^{+18}$ MeV
Note the agreement between perturbation theory and lattice data up to about 0.2 fm.
Lattice data with $\beta$ from 6.664 to 7.825 are displayed.
The red error bars correspond to the errors of the lattice data (include normalization).
\[ \alpha_s(1.5 \text{ GeV}, n_f = 3) = 0.336^{+0.012}_{-0.008} \]

which corresponds to

\[ \alpha_s(M_Z, n_f = 5) = 0.1166^{+0.0012}_{-0.0008} \]

from four-loop running, \( m_c = 1.6 \text{ GeV} \) and \( m_b = 4.7 \text{ GeV} \).
Comparison with other determinations

For $\tau$ decays (ALEPH + OPAL) see also

$$\alpha_s(M_Z, n_f = 5) = 0.1165 \pm 0.0012 \, \text{(FOPT)}, \quad \alpha_s(M_Z, n_f = 5) = 0.1185 \pm 0.0015 \, \text{(CIPT)}$$

Boito et al. arXiv:1410.3528