A Bargmann-Wightman-Wigner Type Quantum Field Theory

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Abstract

We show that the \((j,0) \oplus (0,j)\) representation space associated with massive particles is a concrete realisation of a quantum field theory, envisaged many years ago by Bargmann, Wightman and Wigner, in which bosons and antibosons have opposite relative intrinsic parities. Demonstration of the result requires a careful \textit{ab initio} study of the \((j,0) \oplus (0,j)\) representation space for massive particles, introducing a wave equation with well defined transformation properties under \(C, P\) and \(T\), and addressing the issue of nonlocality required of such a theory by the work of Lee and Wick.

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While most of the specific theoretical questions of hadronic structure and interactions must be decided within the framework of quantum chromodynamics, there remain certain aspects which depend only on the constraints imposed by Poincaré covariance. Many years ago, Wigner [1] provided the basic framework for the Poincaré covariant considerations. The essential elements of these considerations are the kinematical symmetries (continuous Poincaré symmetries and space time reflections) and the behaviour of quantum mechanical states under these transformations. From these follow certain general characteristics such as equal masses and relative intrinsic parities of particle and antiparticle pairs. Such an approach may therefore have utility in establishing the general framework of an effective field theory of hadrons, and such considerations have, in fact, motivated the pioneering work of Weinberg [2] on field theories in a specific Lorentz group representation, \((j, 0) \oplus (0, j)\), of spin-j particles, as well as recent extensions of this work [3].

Although there has been considerable work on the \((j, 0) \oplus (0, j)\) representation with specific applications in mind, additional insight into this representation can be obtained by considering it as a special case of the general classification of quantum field theories by Wigner [1b], in which he distinguishes four classes. The class of theories in which a boson and anti-boson have opposite intrinsic parities are informally known as “Wigner-type,” but in view of Wigner’s note [1b, p. 38] that “much of the” work in Ref. [1b] “was taken from a rather old but unpublished manuscript by V. Bargmann, A. S. Wightman and myself,” we take the liberty of calling this type of theory Bargmann-Wightman-Wigner type (BWW-type) quantum field theory. Even though the generality of BWW’s arguments is remarkable, at present no explicit construction of a BWW-type quantum field theory is known to exist. Nor has it been realised that the \((j, 0) \oplus (0, j)\) representation for massive particles is a realisation of the BWW-type quantum field theory. In this paper we show that this is the case. We do this by considering the special case of the \((1, 0) \oplus (0, 1)\) field and working out explicitly its properties under C, P and T.

We begin with a brief review of \((j, 0) \oplus (0, j)\) representation space. In the notation of Refs. [3,4] \((j, 0)\) and \((0, j)\) spinors have the following Lorentz transformation properties
\( (j, 0) : \quad \phi_R(\vec{p}) = \Lambda_R \phi_R(\vec{0}) = \exp\left( + \vec{J} \cdot \vec{\varphi} \right) \phi_R(\vec{0}) \),

\( (0, j) : \quad \phi_L(\vec{p}) = \Lambda_L \phi_L(\vec{0}) = \exp\left( - \vec{J} \cdot \vec{\varphi} \right) \phi_L(\vec{0}) \).

(1)

(2)

The \( \vec{J} \) are the standard \((2j + 1) \times (2j + 1)\) angular momentum matrices, and \( \vec{\varphi} \) is the boost parameter defined as

\[
\cosh(\varphi) = \gamma = \frac{1}{\sqrt{1 - v^2}} = \frac{E}{m}, \quad \sinh(\varphi) = v \gamma = \frac{|\vec{p}|}{m}, \quad \hat{\varphi} = \frac{\vec{p}}{|\vec{p}|},
\]

with \( \vec{p} \) the three-momentum of the particle. In order to stay as close as possible to the standard treatments of the \((1/2, 0) \oplus (0, 1/2)\) Dirac field, we now introduce a generalised canonical representation \([3a,3f]\) \((j, 0) \oplus (0, j)\)-spinor

\[
\psi(\vec{p}) = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_R(\vec{p}) + \phi_L(\vec{p}) \\ \phi_R(\vec{p}) - \phi_L(\vec{p}) \end{pmatrix},
\]

(4)

In the \((j, 0) \oplus (0, j)\) representation space there are \((2j + 1)\) “\( u_\sigma(\vec{p}) \) spinors” and \((2j + 1)\) “\( v_\sigma(\vec{p}) \) spinors.” As a consequence of the transformation properties (1,2), and the definition (4), these spinors transform as

\[
\psi(\vec{p}) = M(j, \vec{p}) \psi(\vec{0}) = \begin{pmatrix} \cosh(\vec{J} \cdot \vec{\varphi}) & \sinh(\vec{J} \cdot \vec{\varphi}) \\ \sinh(\vec{J} \cdot \vec{\varphi}) & \cosh(\vec{J} \cdot \vec{\varphi}) \end{pmatrix} \psi(\vec{0}).
\]

(5)

If we work in a representation of the \( \vec{J} \) matrices in which \( J_z \) is diagonal, then the rest spinors \( u_\sigma(\vec{0}) \) and \( v_\sigma(\vec{0}) \), \( \sigma = j, j - 1, \ldots, -j \), can be written in the form of \(2(2j + 1)\) single column matrices each with \(2(2j + 1)\) elements as follows

\[
u_{+j}(\vec{0}) = \begin{pmatrix} N(j) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad u_{j-1}(\vec{0}) = \begin{pmatrix} 0 \\ N(j) \\ \vdots \\ 0 \end{pmatrix}, \quad v_{-j}(\vec{0}) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ N(j) \end{pmatrix}.
\]

(6)

For convenience, and so that the rest spinors vanish in the \( m \to 0 \) limit, we choose the normalisation factor \( N(j) = m^j \). With this choice of the normalisation the spinors \( u_\sigma(\vec{p}) \) and \( v_\sigma(\vec{p}) \) are normalised as follows: \( \overline{u}_\sigma(\vec{p}) u_{\sigma'}(\vec{p}) = m^{2j} \delta_{\sigma\sigma'} \) and \( \overline{v}_\sigma(\vec{p}) v_{\sigma'}(\vec{p}) = -m^{2j} \delta_{\sigma\sigma'} \),
with $\overline{\psi}_\sigma(\vec{p}) = \psi^\dagger_\sigma(\vec{p}) \Gamma^0$. Here $\Gamma^0$ is a block diagonal matrix with $(2j + 1) \times (2j + 1)$ identity matrix $I$ on the upper left corner and $-I$ on the bottom right corner. The reader would immediately realise that for $\vec{J} = \vec{\sigma}/2$, $\vec{\sigma}$ being Pauli matrices, the boost $M(1/2, \vec{p})$ is identical with the canonical representation boost $[4]$ for the $(1/2, 0) \oplus (0, 1/2)$ Dirac spinors. While investigating the $C$, $P$ and $T$ properties we will consider the $(1, 0) \oplus (0, 1)$ field as an example case. Hence, we present the explicit expressions for the $(1, 0) \oplus (0, 1)$ spinors (in the generalised canonical representation introduced above). The three $u_{0,\pm 1}(\vec{p})$ and the three $v_{0,\pm 1}(\vec{p})$ spinors read:

$$
u_{+1}(\vec{p}) = \begin{pmatrix}
m + [(2p_x^2 + p_x p_\perp)/2(E + m)] \\
p_x p_+ / \sqrt{2}(E + m) \\
p_\perp^2 / 2(E + m) \\
p_z \\
p_+ / \sqrt{2} \\
p_\perp^2 / 2(E + m) \\
- p_x p_- / \sqrt{2}(E + m)
\end{pmatrix}, 
\nu_{-1}(\vec{p}) = \begin{pmatrix}
m + [(2p_x^2 + p_x p_\perp)/2(E + m)] \\
p_x p_- / \sqrt{2}(E + m) \\
p_\perp^2 / 2(E + m) \\
p_z \\
p_- / \sqrt{2} \\
p_\perp^2 / 2(E + m) \\
- p_x p_+ / \sqrt{2}(E + m)
\end{pmatrix},$$

$$\nu_0(\vec{p}) = \begin{pmatrix} p_x p_+ / \sqrt{2}(E + m) \\
p_+ / \sqrt{2} \\
p_- / \sqrt{2} \end{pmatrix}, \nu_{\sigma}(\vec{p}) = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \nu_\sigma(\vec{p}).$$

In the above equation we have defined $p_\pm = p_x \pm ip_y$.

Once we obtain the $(j, 0) \oplus (0, j)$ spinors we make use Weinberg’s [2] observation that the general form of a field operator is dictated upon us by arguments of Poincaré covariance without any explicit reference to a wave equation. The $(j, 0) \oplus (0, j)$ field operator reads:

$$\Psi(x) = \sum_{\sigma=+j} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2 \omega_{\vec{p}}} [u_\sigma(\vec{p}) a_\sigma(\vec{p}) e^{-ip \cdot x} + v_\sigma(\vec{p}) b_\sigma(\vec{p}) e^{ip \cdot x}],$$

where $\omega_{\vec{p}} = \sqrt{m^2 + \vec{p}^2}$; and $[a_\sigma(\vec{p}), a_{\sigma'}(\vec{p})] = \delta_{\sigma \sigma'} \delta(\vec{p} - \vec{p}')$.

Since so far no wave equation has been invoked it is important to see how one may obtain a wave equation. To derive the wave equation satisfied by the $(j, 0) \oplus (0, j)$ spinors
we observe that the general structure of the rest spinors given by Eq. (6) implies that:

$$\phi_R(\vec{0}) = \varphi_{u,v} \phi_L(\vec{0})$$

with $$\varphi_{u,v} = +1$$ for the $$u$$-spinors and $$\varphi_{u,v} = -1$$ for the $$v$$-spinors.

It may be parenthetically noted that in a similar context for spin one half, Ryder [4, p.44] assumes the validity of an equation which reads, $$\phi_R(\vec{0}) = \phi_L(\vec{0})$$, on the grounds that “when a particle is at rest, one cannot define its spin as either left- or right-handed.” However, as we note this is simply a consequence of the general structure of our theory — moreover, in the process we find an additional minus sign (in the $$\varphi_{u,v}$$ factor). This factor would be found to have profound significance for the internal consistency and consequences of our study.

When we couple the relations $$\phi_R(\vec{0}) = \varphi_{u,v} \phi_L(\vec{0})$$ with the transformations properties (1) and (2) we [3f] obtain

$$\left[ \gamma_{\mu\nu...\lambda} p^\mu p^\nu ... p^\lambda - \varphi_{u,v} m^{2jI} \right] \psi(\vec{p}) = 0.$$ This equation, except for the factor of $$\varphi_{u,v}$$ attached to the mass term, is identical to the Weinberg equation [2] for the $$(j,0) \oplus (0,j)$$ spinors. The $$(2j+1) \times (2j+1)$$ dimensional $$\gamma_{\mu\nu...\lambda}$$ matrices, with $$2j$$ Lorentz indices, which appear here can be found in Ref. [2], or in more explicit form in Ref. [3f]. For $$j = 1/2$$ case, this wave equation is found to be identical to the Dirac equation in momentum space. For the $$(1,0) \oplus (0,1)$$ configuration-space-free-wave-functions $$\psi(x) \equiv \psi(\vec{p}) \exp(-i\varphi_{u,v} \vec{p} \cdot x)$$, this wave equation becomes

$$\left( \gamma_{\mu\nu} \partial^\mu \partial^\nu + \varphi_{u,v} m^2 \right) \psi(x) = 0,$$

(9)

with generalised canonical representation expressions for the $$6 \times 6$$ ten $$\gamma_{\mu\nu}$$-matrices given by

$$\gamma_{00} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma_{00} = \gamma_{0\ell} = \begin{pmatrix} 0 & -J_\ell \\ J_\ell & 0 \end{pmatrix},$$

$$\gamma_{\ell\ell} = \gamma_{j\ell} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \eta_{\ell\ell} + \begin{pmatrix} \{J_\ell,J_\ell\} & 0 \\ 0 & -\{J_\ell,J_\ell\} \end{pmatrix}.$$ (10)

1Note that the $$(1/2,0) \oplus (0,1/2)$$ wave functions satisfy $$(i\gamma_{\mu} \partial^\mu - mI) \psi(x) = 0$$, without the $$\varphi_{u,v}$$ attached to the mass term. It is readily seen that this “cancellation” of the $$\varphi_{u,v}$$ when going from momentum space $$\rightarrow$$ configuration space occurs for fermions, but not for bosons, in general.
Here $\eta_{\mu\nu}$ is the flat space time metric with $\text{diag}(1, -1, -1, -1)$; and $\ell, j$ run over the spacial indices 1, 2, 3.

Equation (9) has three $u_\sigma(\vec{p})$ and three $v_\sigma(\vec{p})$ solutions given by Eq. (7). However these solutions can also be interpreted as associated with not only Einsteinian $E = \pm \sqrt{p^2 + m^2}$ but also with tachyonic $[3e,3f] E = \pm \sqrt{p^2 - m^2}$. This can be readily inferred $[3e,3f]$ by studying the 12th order polynomial in $E$: $\text{Det} (\gamma_{\mu\nu} p^\mu p^\nu + \varphi_{u,v} m^2 I) = 0$. The tachyonic solutions are at this stage unphysical and can be ignored as long as interactions are introduced in such a manner that they do not induce transitions between the physical and unphysical solutions.

In this context, for the spin one spinors we introduce

$$P_u = \frac{1}{m^2} \sum_{\sigma = 0, \pm 1} u_\sigma(\vec{p}) \Upsilon_\sigma(\vec{p}) , \quad P_v = -\frac{1}{m^2} \sum_{\sigma = 0, \pm 1} v_\sigma(\vec{p}) \Upsilon_\sigma(\vec{p}) \quad ,$$

and verify that $P_u^2 = P_u, P_v^2 = P_v$ and $P_u P_v = 0$.

In order to establish that the field operator defined by Eq. (8) describes a quantum field theory of the BWW-type we now show that bosons and antibosons, within the $(j, 0) \oplus (0, j)$ framework developed above, indeed have opposite intrinsic parity and well defined $C$ and $T$ characteristics. We begin with the classical considerations similar to the ones found for the $(1/2, 0) \oplus (0, 1/2)$ Dirac field in the standard texts, such as Ref. [6]. As the simplest example case we study the $(1, 0) \oplus (0, 1)$ field in detail. As such we seek a parity-transformed wave function $\psi'(t', \vec{x}') = S(P) \psi(t, \vec{x})$; $x'^\mu = [\Lambda_P]^\mu_\nu, x^\nu$, [Here: $\Lambda_P = \text{diag}(1, -1, -1, -1)$ so that $t' = t$ and $\vec{x}' = -\vec{x}$] such that Eq. (3) holds true for $\psi'(t', \vec{x}')$. That is: $(\gamma_{\mu\nu} \partial^\mu \partial^\nu + \varphi_{u,v} m^2 I) \psi'(t', \vec{x}') = 0$. It is a straight forward algebraic exercise to find that $S(P)$ must simultaneously satisfy the following requirements

$$S^{-1}(P) \gamma_{00} S(P) = \gamma_{00}, \quad S^{-1}(P) \gamma_{0j} S(P) = -\gamma_{0j} \quad ,$$
$$S^{-1}(P) \gamma_{j0} S(P) = -\gamma_{j0}, \quad S^{-1}(P) \gamma_{\ell j} S(P) = \gamma_{\ell j} \quad .$$

Referring to Eqs. (11), we now note that while $\gamma_{00}$ commutes with $\gamma_{\ell j}$ it anticommutes with $\gamma_{0j}$

$$[\gamma_{00}, \gamma_{\ell j}] = [\gamma_{00}, \gamma_{j\ell}] = 0, \quad \{\gamma_{00}, \gamma_{0j}\} = \{\gamma_{00}, \gamma_{j0}\} = 0 \quad .$$

6
As a result, confining to the norm preserving transformations (and ignoring a possible global phase factor), we identify $S(P)$ with $\gamma_{00}$, yielding: $\psi'(t', \vec{x}') = \gamma_{00} \psi(t, \vec{x}) \iff 
abla \psi'(t', \vec{x}') = \gamma_{00} \psi(t', -\vec{x}')$. This prepares us to proceed to the field theoretic considerations. The $(1,0) \oplus (0,1)$ matter field operator is defined by letting $\sigma = 0, \pm 1$ in the general expression (8). The transformation properties of the states $|\vec{p}, \sigma\rangle^u = a^\dagger_\sigma(\vec{p}) |\text{vac}\rangle$ and $|\vec{p}, \sigma\rangle^v = b^\dagger_\sigma(\vec{p}) |\text{vac}\rangle$ are obtained from the condition

$$U(P) \Psi(t', \vec{x}') U^{-1}(P) = \gamma_{00} \Psi(t', -\vec{x}') \quad ,$$

where $U(P)$ represents a unitary operator which governs the operation of parity in the Fock space. Using the definition of $\gamma_{00}$, Eqs. (10), and the explicit expressions for the $(1,0) \oplus (0,1)$ spinors $u_\sigma(\vec{p})$ and $v_\sigma(\vec{p})$ given by Eqs. (7), we find

$$\gamma_{00} u_\sigma(p') = + u_\sigma(p) , \quad \gamma_{00} v_\sigma(p') = - v_\sigma(p) \quad ,$$

with $p'$ the parity-transformed $p$ — i.e. for $p^\mu = (p^0, \vec{p})$, $p'^\mu = (p^0, -\vec{p})$. The observation (15) when coupled with the requirement (14) immediately yields the transformation properties of the particle-antiparticle creation operators:

$$U(P) a^\dagger_\sigma(\vec{p}) U^{-1}(P) = + a^\dagger_\sigma(-\vec{p}) , \quad U(P) b^\dagger_\sigma(\vec{p}) U^{-1}(P) = - b^\dagger_\sigma(-\vec{p}) .$$

Under the assumption that the vacuum is invariant under the parity transformation, $U(P) |\text{vac}\rangle = |\text{vac}\rangle$, we arrive at the result that the “particles” (described classically by the $u$-spinors) and “antiparticles” (described classically by the $v$-spinors) have opposite relative intrinsic parities: $U(P) |\vec{p}, \sigma\rangle^u = + | -\vec{p}, \sigma\rangle^u$, $U(P) |\vec{p}, \sigma\rangle^v = - | -\vec{p}, \sigma\rangle^v$. This is precisely what we set out to prove. That is, the $(1,0) \oplus (0,1)$ boson and anti-boson have opposite relative intrinsic parity. As a consequence the number of physical states, in comparison to the description of a

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2Such a global phase factor acquires crucial importance for constructing internally consistent theory Majorana-like $(j,0) \oplus (0,j)$ fields.

3The fuller justification for the terminology “particle” and “antiparticle,” apart from the convention of what we call “particle,” will be realised when we consider the operation of $C$. 


massive spin one particle by the Proca vector potential $A^\mu(x)$, are doubled from $(2j + 1) = 3$ to $2(2j + 1) = 6$ in agreement with BWW’s work [1b].

Next we consider the operation of C. The charge conjugation operation C must be carried through with a little greater care for bosons than for fermions within the $(j,0) \oplus (0,j)$ framework developed here because of $\varphi_{u,v}$ factor in the mass term. For the $(1,0) \oplus (0,1)$ case, at the classical level we want

$$ C: \ (\gamma_{\mu\nu} D^\mu_{(+)} D^\nu_{(+)} + m^2) u(x) = 0 \longrightarrow (\gamma_{\mu\nu} D^\mu_{(-)} D^\nu_{(-)} - m^2) v(x) = 0 \ , \quad (16) $$

where the local $U(1)$ gauge covariant derivatives are defined as: $D^\mu_{(+)} = \partial^\mu + i q A^\mu(x)$, $D^\mu_{(-)} = \partial^\mu - i q A^\mu(x)$. Again a straight forward algebraic exercise yields the result that for (16) to occur we must have: $\psi(t,\vec{x}) \rightarrow C \psi^*(t,\vec{x})$; where $C$ satisfies,

$$ C \gamma^*_{\mu\nu} C^{-1} = - \gamma_{\mu\nu} \ . $$

We find that $C = \eta_\sigma A \gamma_{00}$, with

$$ A = \begin{pmatrix} 0 & \Theta_{[1]} \\ \Theta_{[1]} & 0 \end{pmatrix} , \quad \Theta_{[1]} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} , \quad (17) $$

and for convenience we choose $\eta_{\pm 1} = +1$ and $\eta_0 = -1$; and $\Theta_{[1]}$ is Wigner’s time reversal operator [7] for spin-1. The effect of charge conjugation in the Fock space is now immediately obtained by using the easily verifiable identities: $C u_{++}^*(\vec{p}) = v_{--}(\vec{p})$, $C u_0^*(\vec{p}) = u_0(\vec{p})$, $C u_{+-}^*(\vec{p}) = v_{-+}(\vec{p})$; and the requirement $U(C) \Psi(x) U^{-1}(C) = \Psi^c(x)$, where

$$ \Psi^c(x) = \sum_{\sigma=0,\pm 1} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} \left[ S(C) u_\sigma^*(\vec{p}) a_\sigma^\dagger(\vec{p}) e^{i p \cdot x} + S(C) v_\sigma^*(\vec{p}) b_\sigma(\vec{p}) e^{-i p \cdot x} \right] . \quad (18) $$

While our agreement with BWW [1b] is complete, we differ with Weinberg’s [2, footnote 13] claim that the $(j,0) \oplus (0,j)$ fields have same relative intrinsic parity for bosons. The disagreement with Ref. [3] arises because it’s author did not realise that the bosonic $\nu$-spinors are not solutions of the equation which he proposed. The factor $\varphi_{u,v}$ in Eq. [4] is required for internal consistency in the theory.
These considerations yield: $U(C) a_\sigma^\dagger(\vec{p}) U^{-1}(C) = b_\sigma^\dagger(\vec{p})$, $U(C) b_\sigma(\vec{p}) U^{-1}(C) = a_\sigma(\vec{p})$.

We thus see that the definition of charge conjugation operation $C$ as given by (16) indeed yields the correct picture in the Fock space: $U(C) |\vec{p}, \sigma\rangle^u = |\vec{p}, \sigma\rangle^v$ and $U(C) |\vec{p}, \sigma\rangle^v = |\vec{p}, \sigma\rangle^u$.

Finally, following Nachtmann [8], we define the operation of time reversal as a product of an operation, $S'(\Lambda_T)$, $\psi(t, \vec{x}) \rightarrow S'(T) \psi(-t, \vec{x})$, which preserves the form of Eq. (8) under $x^\mu \rightarrow [\Lambda_T]^\mu\nu x'^\nu$, $\Lambda_T = \text{diag}(-1, 1, 1, 1)$, and, based on Stückelberg-Feynman [8] arguments, the operation of charge conjugation. So, classically, under $T$ we have: $\psi(t, \vec{x}) \rightarrow \psi'(t, \vec{x}) = S'(\Lambda_T) S(C) \psi^*(-t, \vec{x})$. We find that $S(T) \equiv S'(\Lambda_T) S(C)$ is given by: $S(T) = (A \text{ global phase factor}) \times \gamma_{00} \eta_\sigma A \gamma_{00}$. Taking note of the fact that $A$ anticommutes with $\gamma_{00}$, $\{A, \gamma_{00}\} = 0$, and dropping the resultant global phase factor, we obtain $S(T) = \eta_\sigma A$. In the Fock space above considerations are implemented by finding the effect of an anti-unitary operator on the creation and annihilation operators via: $[V(T) \Psi(t, \vec{x}) V^{-1}(T)]^\dagger = \Psi'(t, \vec{x})$, where

$$
\Psi'(t, \vec{x}) = \sum_{\sigma = 0, \pm 1} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} \left[ S(T) u_\sigma^*(\vec{p}) a_\sigma^\dagger(\vec{p}) e^{ip\cdot x'} + S(T) v_\sigma^*(\vec{p}) b_\sigma(\vec{p}) e^{-ip\cdot x'} \right],
$$

with $x'^\mu = (-t, \vec{x})$. Exploiting the identities $S(T) u_\sigma^*(\vec{p}) = u_{-\sigma}(-\vec{p})$, $S(T) v_\sigma^*(\vec{p}) = v_{-\sigma}(-\vec{p})$ we arrive at the result: $V(T) a_\sigma^\dagger(\vec{p}) V^{-1}(T) = a_{-\sigma}^\dagger(-\vec{p})$, $V(T) b_\sigma(\vec{p}) V^{-1}(T) = b_{-\sigma}(\vec{p})$. Therefore if the vacuum is invariant under $T$, the physical states transform as $V(T) |\vec{p}, \sigma\rangle = | -\vec{p}, -\sigma\rangle$ and observables $\mathcal{O} \rightarrow \mathcal{O}' = [V(T) \mathcal{O} V^{-1}(T)]^\dagger$.

We thus see that the $(1, 0) \oplus (0, 1)$ field theory constructed above is indeed of BWW-type. What is left, in view of the already cited work of Lee and Wick [8], is to explicitly show that the $(j, 0) \oplus (0, j)$ field operator [8] for spin one describes a nonlocal theory (in the sense to

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5The order of the operations in the product which follows is not important because the two operations are found to anticommute, and therefore the ambiguity of ordering only involves an overall global phase factor.
(1, 0) ⊕ (0, 1) field operator associated with massive particles

\[
\begin{align*}
\left[ \Psi_\alpha(t, \vec{x}), \overline{\Psi}_\beta(t, \vec{x}') \right] &= \left( \frac{1}{2\pi} \right)^6 \int \frac{d^3\vec{p}}{2E(\vec{p})} \sum_{\sigma=\pm 1} \left( u_\sigma(\vec{p}) \overline{n}_\sigma(\vec{p}) + v_\sigma(-\vec{p}) \overline{\tau}_\sigma(-\vec{p}) \right)_{\alpha\beta} e^{i\vec{p} \cdot (\vec{x} - \vec{x}')} .
\end{align*}
\]

(20)

The nonlocality is now immediately inferred. Using the explicit forms (7) of \( u_{0,\pm 1}(\vec{p}) \) and \( v_{0,\pm 1}(\vec{p}) \), we find

\[
\sum_{\sigma=0,\pm 1} \left( u_\sigma(\vec{p}) \overline{n}_\sigma(\vec{p}) + v_\sigma(-\vec{p}) \overline{\tau}_\sigma(-\vec{p}) \right) = \begin{pmatrix} M & 0 \\ 0 & -M \end{pmatrix},
\]

(21)

with

\[
M = \begin{pmatrix} E^2 + p_z^2 & \sqrt{2}p_-p_z & p_z^2 \\ \sqrt{2}p_+p_z & E^2 + p_-p_z - 2p_z^2 & -\sqrt{2}p_-p_z \\ p_z^2 & -\sqrt{2}p_+p_z & E^2 + p_z^2 \end{pmatrix}.
\]

(22)

In Eq. (22) \( p_\pm = p_x \pm ip_y \). Consequently: \( \left[ \Psi_\alpha(t, \vec{x}), \overline{\Psi}_\beta(t, \vec{x}') \right] \neq (\text{const.}) \times \delta^3(\vec{x} - \vec{x}') \). For comparison we note that a similar calculation for the spin half Dirac case yields \( \left\{ \Psi_\alpha(t, \vec{x}), \overline{\Psi}_\beta(t, \vec{x}') \right\} = (\text{const.}) \times \gamma^{0}_{\alpha\beta} \delta^3(\vec{x} - \vec{x}') \). The crucial property of the \((1/2, 0) \oplus (0, 1/2)\) representation space which enters in obtaining this result is:

\[
\sum_{\sigma=\pm 1/2} \left( u_\sigma(\vec{p}) \overline{n}_\sigma(\vec{p}) + v_\sigma(-\vec{p}) \overline{\tau}_\sigma(-\vec{p}) \right)_{\alpha\beta} \sim (\gamma^0)^{0}_{\alpha\beta} p_0 .
\]

No corresponding overall factor of \( E \) appears in \( M \). If it did, the \( E(\vec{p})^{-1} \) factor in the invariant element of phase space could be cancelled leading to a \( \delta^3(\vec{x} - \vec{x}') \) and its derivatives (Schwinger terms [14]) on the rhs of Eq. (20) thus restoring the locality. For an alternate derivation of the nonlocality the reader may wish to refer to Ref. [3b]. For the sake of completeness we note that the momentum conjugate to the field operator \( \Psi(x) \) for spin one is given by \( \Pi_\mu(x) = \overline{\Psi}(x) \gamma_\mu \partial^\mu \), and \( \left[ \Psi_\alpha(t, \vec{x}), \left( \Pi_\mu \right)_\beta(t, \vec{x}') \right] = -\delta^3(\vec{x} - \vec{x}') \partial^\mu \Psi_\alpha(t, \vec{x}) \overline{\Psi}_\xi(t, \vec{x}) (\gamma_\mu)^{\xi\beta} \). The physical interpretation of this last result requires further study.

To summarise we note the \((j, 0) \oplus (0, j)\) representation space associated with massive particles is a concrete realisation of a quantum field theory, envisaged many years ago by
Bargmann, Wightman and Wigner, in which bosons and antibosons have opposite relative intrinsic parities. It is our hope that our detailed analysis of the \((j, 0) \oplus (0, j)\) representation space would supplement the canonical Bargmann-Wigner/Rarita-Schwinger [9] formalism (where a boson and anti-boson have same intrinsic parity) and open new experimentally observable possibilities for the Poincaré covariant aspects of hadrons and their propagation in nuclei.

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