Fractional and semi-local non-Abelian Chern-Simons vortices

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Abstract

In this paper we study fractional as well as semi-local Chern-Simons vortices in $G = U(1) \times SO(2M)$ and $G = U(1) \times USp(2M)$ theories. The master equations are solved numerically using appropriate Ansätze for the moduli matrix field. In the fractional case the vortices are solved in the transverse plane due to the broken axial symmetry of the configurations (i.e. they are non-rotational invariant). It is shown that unless the fractional vortex-centers are all coincident (i.e. local case) the ring-like flux structure, characteristic of Chern-Simons vortices, will become bell-like fluxes – just as those of the standard Yang-Mills vortices. The asymptotic profile functions are calculated in all cases and the effective size is identified.

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1 Introduction

Solitons play a crucial role in a vast area of physics, from solid state physics, through high-energy physics, string theory, condensed matter physics to cosmology. One of the most well-known solitons is the Abrikosov-Nielsen-Olesen (ANO) vortex \[1, 2\] which is measurable in the laboratory forming a so-called Abrikosov lattice in type II superconducting materials in the presence of an external magnetic field.

In the recent years, there has been developed a genuine non-Abelian generalization of this vortex \[3, 4\], which possesses orientational modes and a moduli space of solutions. These vortices correspond to $1/2$ BPS objects in $\mathcal{N} = 2$ gauge theories and the BPS properties have as a consequence that all static inter-vortex forces cancel out exactly – just as in the Abelian case. This development started out with the unitary gauge groups, which first in the last three years has been extended to arbitrary gauge groups \[5\] and in particular the orthogonal \[6, 7\] and symplectic groups \[7\], viz. $G = U(1) \times SO(N)$ and $G = U(1) \times USp(2M)$. The overall $U(1)$ factor is of utter importance for the topological construction, giving rise to the stability supported by $\pi_1(G) \sim \mathbb{Z}^1$.

The moduli space of these vortices has been studied much in detail in Refs. \[3, 4, 8, 9, 10, 11\] for $U(N)$ and in Ref. \[7\] for $U(1) \times SO(N), U(1) \times USp(2M)$. The vacuum of the $U(N)$ theory is a unique color-flavor locked phase when the minimal number of flavors (in order to break completely the gauge symmetry) is present: $N_f = N$, in this case the vacuum is simply just a point. On the other hand, when the number of flavors is increased, the vacuum becomes in general a manifold and it depends on the details of the matter content, such as the charges etc.

A crucial difference between the aforementioned theories, which possess local vortices\[2\], and this theory with extra matter, is that so-called semi-local zero-modes appear. They bear the property that they are not normalizable – they tend to zero obeying a power-law instead of exponentially as the local profile functions. They were first found in the (extended) Abelian Higgs model \[12, 13, 14\] and later also in its non-Abelian extension \[3, 15, 16\]. An interesting fact about the vortices in $U(1) \times SO(N)$ and $U(1) \times USp(2M)$ theories, is that they are in general of the semi-local type \[17, 7\]. There is a very interesting relation between the vortices of the semi-local kind and non-linear sigma model (NL$\sigma$M) lumps. The gauge theories become NL$\sigma$Ms and the vortices become lumps \[18, 19, 20, 17, 16\]. This is technically done by taking the strong gauge

\[1\]In the $SO$ case, there is an additional $\mathbb{Z}_2 = \pi_1((U(1) \times SO(N))/\mathbb{Z}_n_0)$ factor (with $n_0 = 2, 1$ for even and odd $N$, respectively), i.e. a topological charge not being the vortex number, which has important consequences for the connectedness properties of the moduli spaces of vortices \[7\].

\[2\]By local vortices, we define the localized topological objects which has exponential transverse cut-offs, as opposed to polynomial tales, which will be termed semi-local.
coupling limit: \( g \to \infty \), which corresponds to very low energies or very long distances. The lumps are supported topologically by \( \pi_2(M) \) with \( M \) being the vacuum manifold.

The theories with fundamental matter content all charged with respect to an overall \( U(1) \) factor of the gauge group have been embedded into a high-energy theory with a simple gauge group for \( SU(N+1) \to U(N) \) in Ref. [4]. For \( SO(N) \) theories this embedding has been made in Ref. [6] using adjoint matter content at high energies giving rise to fundamental matter with the following breaking \( SO(N+2) \to U(1) \times SO(N) \). Similarly, for \( USp(2M) \) theories, the embedding has been made explicit in Ref. [21] with the breaking \( USp(2M+2) \to U(1) \times USp(2M) \) again using adjoint matter at high energies as in the \( SO \) case. These symmetry breaking patterns are relevant for non-Abelian monopoles. Eventually, at much lower energies, all these theories get their gauge symmetry completely broken by a Fayet-Iliopoulos term. The interest in these systems lies in the exact homotopy sequences that relate the monopole properties to the vortex properties in this kind of system. For a review see Ref. [22].

Some of the recent results and studies on the topic of non-Abelian vortices include the following: A multiple layer structure has been recognized in the nature of the standard non-Abelian vortex with a \( U(N) \) gauge group in Ref. [23]. Non-Abelian global vortices still with a \( U(N) \) group has been studied in Refs. [24, 25, 26] giving rise to a somewhat surprising non-rational \( U(1) \)-winding going like \( 1/\sqrt{N} \). Non-Abelian vortices have also been studied on a torus in Ref. [27]. Non-Abelian vortices in dense QCD have been studied in a series of papers [28, 29, 30, 31, 32]. The metric of the moduli space of the non-Abelian vortices with a \( U(N) \) gauge group has been calculated in Ref. [33] and for well-separated vortices in Ref. [34]. Using a non-linear realization method the low-energy effective action including the spatial fluctuations of the string as well as its internal orientations and their higher-order mixed terms have been derived in Ref. [35]. In Ref. [36] a vortex description of a quantum Hall ferromagnet has been made, considering an effective theory for an incompressible fluid corresponding to the moduli space of a vortex theory. The Refs. [37, 38] considered the non-BPS corrections to the non-Abelian vortices finding new types of vortices with interactions depending on both relative distance and internal group orientation. A non-Abelian vortex in the mass deformed ABJM model has also been considered in Refs. [39, 40]. The quantum phases of the vortex in \( \mathcal{N} = 1^* \) theory have been studied in Ref. [41]. A new duality between a \( U(N) \) theory at strong coupling and a \( U(N - N_1) \) theory at weak coupling, relating extended objects (solitons) to elementary objects of either side of the duality, has been found in Refs. [42, 43, 44].

Many results for the non-Abelian vortices, with especially the \( U(N) \) gauge group, are summarized in the excellent reviews [45, 46, 47].
One property of the vortices and lumps which has not been noticed until recently and is only seen when there is a non-trivial vacuum manifold present in the theory – namely that the minimal vortex or lump can be composed by seemingly unconfined sub-objects which, however, cannot be infinitely separated. If one tries to do so, the configuration will either dilute completely or become singular \[48\]. This type has been named fractional vortices and lumps. They can be engineered in many ways. For instance, one can alter the charge assignment of the different flavors. Another possibility is that there resides a singular submanifold in the vacuum manifold, which is exactly the case in the theories with $U(1) \times SO(N)$ and $U(1) \times USp(2M)$ gauge groups \[17\]. In fact, the first fractional lump solution was found in Ref. \[17\].

Soon after a fractional lump was constructed in the Taub-NUT space \[49\] with a smooth interpolation from a fractional lump to the standard $\mathbb{C}P^1$ lump. It was made by turning on a finite gauge coupling in an appropriate quiver theory which in the strong gauge coupling limit reduces to the standard $\mathbb{C}P^1$ NL$\sigma$M.

Subsequently, a classification of fractional vortices was made in Ref. \[48\]. The first type is constructed with a conical singularity on the vacuum manifold, which in practice is easily made by a non-equal relatively prime and rational charge assignment for the different flavors of squarks in the theory. The second type, however, does not rely on a singularity on the vacuum manifold, but it suffices to deform the geometry in some way, and it has been conjectured that a strong positive scalar curvature in at least two (separate) regions of the vacuum manifold gives rise to a fractional vortex or lump \[48\].

So far the story has nothing to do with the Chern-Simons vortices. The vortices can be thought of as string-like objects in for instance four space-time dimensions or as particle-like objects in three space-time dimensions. The latter option is interesting per se for several reasons. The physics has radically different characteristics, first of all, because the spin is not quantized in half-integers as in four space-time dimensions, admitting possibility for anyons, objects possessing fractional charge and statistics \[50\]. This can be realized by introducing the Chern-Simons term, which has been widely used, e.g. in the theory of the fractional quantum Hall effect \[51\]. Furthermore, Chern-Simons theories provide a topological and gauge invariant mechanism for mass generation, not relying on the Higgs mechanism \[52\]. Abelian vortices in the Chern-Simons Higgs model were studied in Refs. \[53\] \[54\] (see e.g. Refs. \[55\] \[56\] for excellent reviews). In the non-Abelian case, the Chern-Simons vortices of genuinely non-Abelian kind, that is, possessing orientational moduli and a moduli space of solutions were found in the Refs. \[57\] \[58\] in the case of a $U(N)$ gauge group. Aldrovandi and Schaposnik identified the moduli space of a single vortex solution as $\mathbb{C} \times \mathbb{C}P^{N-1}$ (which equals that of a single vortex in Yang-Mills-Higgs theory.
with a $U(N)$ gauge group) and furthermore found the vortex world-line theory, i.e. the sigma model living on the vortex. Furthermore, Refs. [59, 60, 61] have considered packaging together the Yang-Mills and the non-Abelian Chern-Simons terms for $U(N)$ gauge groups. In Ref. [59] the dynamics of the vortices has been studied and they found a modification to the adiabatic motion of linear order proportional to the Chern-Simons coupling as well as the usual free kinetic term. In Ref. [60], in addition to the topological charge, conserved Noether charges associated with a $U(N-1)$ flavor symmetry of the theory, due to inclusion of a mass term for the squarks, were found. A dimensional reduction of this theory to $1+1$ dimensions gives rise to so-called trions [62]. Numerical solutions were provided in Ref. [61] for the Yang-Mills-Chern-Simons $U(N)$ theory and in Ref. [58] for the Chern-Simons $U(N)$ theory and in both cases they have found that the semi-local moduli parameter, when non-zero, destroys the ring-like characteristic of the magnetic flux that the vortex solutions in Chern-Simons theories usually possess. Thus the magnetic flux turns into bell-like structures similar to those of Yang-Mills theories.

In Ref. [63], the program of non-Abelian vortices with arbitrary gauge groups has been extended to the Chern-Simons vortices and furthermore, the identification of the moduli spaces using the moduli matrix formalism was pursued. It was conjectured that the $k$ moduli space of the vortices with gauge group $G = U(1) \times G'$, with $G'$ being a simple gauge group, is the same in the case of only the Yang-Mills kinetic term as in the case of only the Chern-Simons kinetic term for the gauge fields. The vortices constructed in Ref. [63] and studied numerically with gauge group $U(1) \times SO(2M)$ and $U(1) \times USp(2M)$, provide the base for fractional vortices due to the singular submanifolds in the vacuum manifolds. Their existence is quite easily guessed, as the moduli matrix generating them, is exactly the one giving rise to the fractional lumps of Ref. [17].

Now we want to pose a very simple question, what happens to the planar structure of the flux when the fractional sub-vortices are moved apart? Do we get $M$ rings of flux or $M$ Gauss bells of flux. The analysis of the holomorphic invariants tells us that the position moduli of the fractional sub-vortices should be interpreted as a kind of semi-local moduli. Then we would guess the answer is the last option. The flux spreads out to $M$ Gaussian bells of flux as we confirm with a numerical study. We furthermore study the semi-local vortex with gauge groups $G = U(1) \times SO(2M)$ and $G = U(1) \times USp(2M)$ and in addition to the previous studies, we change the relative coupling constants $\kappa, \mu$ of the $U(1)$ and $G'$ part of the gauge fields, respectively. In the local case studied in Ref. [63], this gave rise to negative Abelian magnetic flux and positive non-Abelian flux at the origin of the vortex in the case of $\kappa > \mu$ (and vice versa for $\kappa < \mu$). We expect and confirm numerically, that this effect vanishes when semi-local size moduli are turned on. For completeness, we give also an example of the fractional semi-local vortex in our model.
2 The model

We will consider the model studied in Ref. [63] in the limit that it reduces to the non-Abelian Chern-Simons-Higgs theory. It is an $\mathcal{N} = 2$ supersymmetric gauge theory in $2 + 1$ dimensions with gauge group $G = U(1) \times G'$ where $G'$ is a simple group. The theory contains the gauge fields, an adjoint scalar and $N_f$ complex scalar squark fields (hypermultiplets). Taking the strong coupling limit of the Yang-Mills couplings $e, g \to \infty$, being the Maxwell and Yang-Mills coupling respectively, we obtain the following non-Abelian $\mathcal{N} = 2$ Chern-Simons-Higgs theory

$$\mathcal{L}_{CSH} = -\frac{\mu}{8\pi} \epsilon^{\mu\nu\rho} \left( A_\mu^a \partial_\nu A_\rho^a - \frac{1}{3} f^{abc} A_\mu^a A_\nu^b A_\rho^c \right) - \frac{\kappa}{8\pi} \epsilon^{\mu\nu\rho} \left( A_\mu^0 \partial_\nu A_\rho^0 \right) + \text{Tr} \left( D_\mu H \right)^\dagger \left( D^n H \right)$$

$$- 4\pi^2 \text{Tr} \left\{ \frac{1}{N} \left( \text{Tr} \left( H H^\dagger \right) - \xi \right) + \frac{2}{\mu} \text{Tr} \left( H H^\dagger t^a \right) \right\} H^2. \quad (1)$$

The adjoint scalars are infinitely massive in the limit considered here, so they have been integrated out. The remaining fields are the complex scalar fields $H$ which are combined into a $\text{dim}(R_G) \times N_f$ matrix, consisting of $N_f$ matter multiplets, $A_\mu = A_\mu^a t^a$, $\alpha = 0, 1, 2, \ldots, \text{dim}(G')$, is the gauge potential and finally $t^a$ and $f^{abc}$ are the generators and the structure constants, respectively, of the non-Abelian gauge group $G'$. $N \equiv \text{dim}(R_G)$ and the Abelian generator is defined as $t^0 = 1_{N}/\sqrt{2N}$. The index 0 is for the Abelian part of the gauge group, while $a, b, c = 1, 2, \ldots, \text{dim}(G')$ are the non-Abelian indices. The space-time indices are denoted by Greek letters $\mu, \nu = 0, 1, 2$ where we adopt the metric $\eta^{\mu\nu} = \text{diag}(+, -, -)$. We will use the following conventions

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i [A_\mu, A_\nu], \quad D_\mu H = (\partial_\mu + i A_\mu) H, \quad D_\mu \phi = \partial_\mu \phi + i [A_\mu, \phi]. \quad (2)$$

In this paper we will fix the gauge group to $G = U(1) \times SO(2M)$ and $G = U(1) \times USp(2M)$ and treat them on equal footing with just the change of the invariant rank-two tensor $J$ (in the following $N = 2M$). It is defined as $J^T = \epsilon J$, $J^T J = 1_{2M}$ and explicitly, we will choose the basis

$$J = \begin{pmatrix} 0 & 1_{M} \\ \epsilon 1_{M} & 0 \end{pmatrix}, \quad (3)$$

where $\epsilon = +1$ for $SO$, while $\epsilon = -1$ for $USp$. Furthermore, we will only consider the fundamental representation of the matter fields here, hence $R_G := \Box$. The remaining parameters of the theory are the Chern-Simons couplings $\kappa \in \mathbb{R}, \mu \in \mathbb{Z}$, being the Abelian (tracepart) and the non-Abelian (traceless part) couplings, respectively. $\xi > 0$ is the Fayet-Iliopoulos parameter, putting the theory on the Higgs branch. The last parameter governing the vortex solutions, which is not explicitly present in the Lagrangian density, is $k$ being the vorticity or winding number. The $U(1)$ winding, is however given by $\nu = k/n_0$, \footnote{where $n_0$ denotes the greatest common divisor}.
(gcd) of the Abelian charges of the holomorphic invariants of $G'$, see [5]. For simple groups this coincides with the center as $\mathbb{Z}_{n_0}$. We will take $k > 0$, corresponding to vortices as opposed to anti-vortices.

There are three different phases of the theory at hand. An unbroken phase with $\langle H \rangle = 0$ and a broken phase with $\langle H \rangle = \frac{1}{2M} \sqrt{\xi/(2M)}$. In between there are partially broken phases. Furthermore, there are plenty of other possible vacua breaking the gauge symmetry, which however will break (partially) also the global color-flavor symmetry, which is only present (in its entirety) in the latter vacuum. This will be our reason for choosing this particular vacuum.

The masses are generated by the Higgs mechanism (not topologically, as in the case with also a Yang-Mills term present in the Lagrangian) and are given by

$$m_\kappa = \frac{2\pi \xi}{M\kappa}, \quad m_{\mu} = \frac{2\pi \xi}{M\mu},$$

which by supersymmetry are the same for the gauge fields as the scalar fields, Abelian and non-Abelian, respectively.

The tension, defined by the integral on the plane over the time-time component of the energy-momentum tensor, is given by

$$T = \int_C \text{Tr} \left\{ |D_0 H|^2 + |D_i H|^2 + 4\pi^2 \left( \frac{1_{2M}}{2M\kappa} \left( \text{Tr} (HH^\dagger) - \xi \right) + \frac{2}{\mu} \text{Tr} (HH^\dagger t^a) t^a \right) H \right\}. \quad (5)$$

As shown in more detail in Ref. [63], by a Bogomol’nyi completion, the BPS-equations

$$\bar{D} H = 0, \quad D_0 H = i2\pi \left( \frac{1_{2M}}{2M\kappa} \left( \text{Tr} (HH^\dagger) - \xi \right) + \frac{2}{\mu} \text{Tr} (HH^\dagger t^a) t^a \right) H, \quad (6)$$

can be combined with the Gauss law

$$F_{12}^{\alpha} = -\frac{i4\pi}{\mu} \text{Tr} \left[ H^\dagger t^a D_0 H - (D_0 H)^\dagger t^a H \right], \quad F_{12}^{0} = -\frac{i4\pi}{\kappa} \text{Tr} \left[ H^\dagger t^0 D_0 H - (D_0 H)^\dagger t^0 H \right], \quad (7)$$

yielding the following system

$$\bar{D} H = 0, \quad (8)$$

$$F_{12}^{\alpha} = \frac{2\pi^2}{M\kappa \mu} \left( \text{Tr} (HH^\dagger) - \xi \right) \left( HH^\dagger - J^\dagger \left( HH^\dagger \right)^T J \right) + \frac{2\pi^2}{\mu^2} \left( \left( HH^\dagger \right)^2 - J^\dagger \left( \left( HH^\dagger \right)^2 \right)^T J \right),$$

$$F_{12}^{0} = \frac{2\pi^2}{M^2\kappa^2} \text{Tr} (HH^\dagger) \left( \text{Tr} (HH^\dagger) - \xi \right) 1_{2M} + \frac{2\pi^2}{M\kappa \mu} \text{Tr} \left( HH^\dagger - J^\dagger \left( HH^\dagger \right)^T J \right) \right) 1_{2M},$$

and by using the moduli matrix Ansatz $H = S^{-1} H_0(z)$, $S = s S'$, the master equations can be written down

$$\bar{\partial} \left[ \Omega' \partial \Omega'^{-1} \right] = \frac{\pi^2}{M\kappa \mu} \frac{1}{\omega} \text{Tr} \left( \Omega_0 \Omega'^{-1} - \xi \right) \left( \Omega_0 \Omega'^{-1} - J^\dagger \left( \Omega_0 \Omega'^{-1} \right)^T J \right) + \frac{\pi^2}{\mu^2} \frac{1}{\omega^2} \left[ \left( \Omega_0 \Omega'^{-1} \right)^2 - J^\dagger \left( \Omega_0 \Omega'^{-1} \right)^T J \right], \quad (9)$$
\[ \partial \bar{\partial} \log \omega = -\frac{\pi^2}{M^2 \kappa^2 \omega} \text{Tr} \left( \Omega_0 \Omega'^{-1} \right) \left( \frac{1}{\omega} \text{Tr} \left( \Omega_0 \Omega'^{-1} \right) - \xi \right) \]
\[ -\frac{\pi^2}{M \kappa \mu \omega^2} \text{Tr} \left( \Omega_0 \Omega'^{-1} \left( \Omega_0 \Omega'^{-1} - J \left( \Omega_0 \Omega'^{-1} \right)^T J \right) \right), \]

where we have defined \( \Omega_0 \equiv H^\dagger_0 H_0 \) as well as the gauge invariant quantity \( \Omega = SS^\dagger = \omega \Omega' \), which splits into the Abelian part \( \omega = |s|^2 \) and the non-Abelian part \( \Omega' = S'(S')^\dagger \). These fields are conjectured to be uniquely determined by the master equation \( s \) applying the appropriate boundary conditions for a given moduli matrix \( H_0(z) \) – up to \( V \)-equivalence \[8\]

\[ \{H_0, S\} \sim V \{H_0, S\}, \]

with the transformation matrix

\[ V = vV', \quad v \in \mathbb{C}^*, \quad V' \in G'^\mathbb{C}. \]

Writing the energy density in terms of our new variables, including the boundary term

\[ E = 2\xi \partial \bar{\partial} \log \omega + 2 \partial \bar{\partial} \left( \frac{1}{\omega} \text{Tr} \Omega_0 \Omega'^{-1} \right), \]

we obtain by integration, the total energy

\[ E = \int \mathcal{E} = 2\pi \xi \nu = \frac{2\pi \xi k}{n_0}, \]

which is simply proportional to the topological charge. The Abelian and non-Abelian magnetic flux densities are, respectively

\[ B = F^{0}_{12} = -4\sqrt{M} \partial \bar{\partial} \log \omega, \quad F^{a}_{12} = 2S'^{-1} \partial \left[ \Omega' \partial \Omega'^{-1} \right] S', \]

while the Abelian and non-Abelian electric field densities read, respectively

\[ E_i = F^{0}_{i0} = \frac{2\pi}{\kappa \sqrt{M}} \partial_i \left( \frac{1}{\omega} \text{Tr} \left( \Omega_0 \Omega'^{-1} \right) \right), \]
\[ E^{a}_{i} = F^{a}_{i0} = \frac{\pi}{\mu} \partial_i \left[ \frac{1}{\omega} S'^{-1} \left( \Omega_0 \Omega'^{-1} - J \left( \Omega_0 \Omega'^{-1} \right)^T J \right) S' \right]. \]

Finally, we need only to specify the boundary conditions, which have been obtained in Refs. \[17, 7\]

\[ \Omega' = H_0(z) \frac{1}{\sqrt{M^\dagger M}} H^\dagger_0(\bar{z}), \quad \omega = \frac{1}{\xi} \text{Tr} \sqrt{M^\dagger M}, \]

with \( M \equiv H^\dagger_0(z) J H_0(z) \) being the meson field.

For concreteness, we will consider the cases \( G' = SO(4) \) and \( G' = USp(4) \) as the main result of the paper will be made using numerical methods. However, when the generalization to
\( G' = SO(2M), USp(2M) \) is straightforward, we will work out the equations in the generalized case and at the end of the day set \( M = 2 \). The question we are addressing, is the study of the semi-local and “fractional” moduli parameters of this single non-Abelian Chern-Simons vortex \((k = 1)\). Now, consider the given moduli matrix
\[
H_0(z) = \begin{pmatrix}
  z1_M - A & C_{S,A} \\
  B_{A,S} & 1_M
\end{pmatrix},
\]
which is the most general matrix for \( k = 1 \) in a particular patch of the moduli space. The subscript \( S, A \) denotes symmetric in the case of \( \epsilon = +1 \) (\( SO \) case) and anti-symmetric for \( \epsilon = -1 \) (\( USp \) case) (and vice versa for \( A, S \)). The matrix obeys the weak condition on the holomorphic invariants
\[
H_0^T(z)JH_0(z) = (z - z_0)J + \mathcal{O}(z^0),
\]
and by insertion of the moduli matrix (19) into the weak condition, we obtain
\[
H_0^T(z)JH_0(z) = (z - a_0)J + \begin{pmatrix}
  B_{A,S} \hat{A} - \hat{A}^T B_{A,S} & -\hat{A}^T - B_{A,S} C_{S,A} \\
  \epsilon B_{A,S} C_{S,A} - \epsilon \hat{A} & \epsilon 2 C_{S,A}
\end{pmatrix},
\]
where we have defined \( A \equiv a_01_M + \hat{A} \) with \( \text{Tr} \hat{A} = 0 \). The presence of \( \hat{A} \) (even for \( B = 0 \)) in the constant matrix (i.e. not proportional to \( J \)) tells us that the fractional position moduli are semi-local moduli. The moduli parameters are classified into two types. Normalizable orientational zero-modes and non-normalizable semi-local “size” parameters. \( B \) contains orientational modes and does not change the Abelian flux density contribution to the energy density. The center of mass is given by \( a_0 = \text{Tr}A/M \) which is the sum of the eigenvalues of the complex matrix \( A \). We will fix this parameter to the origin. The eigenvalues of \( A \) are the positions of the “fractional vortices” and if they all (in this case both) coincide, they compose a normal non-Abelian vortex, but not necessarily local. It becomes local only if \( \hat{A} = C = 0 \). Finally, there are the semi-local “size” parameters \( C \). What we would like to show is that the fractional vortex has semi-independent sub-structures which should be interpreted as an \( M \)-th vortex (flux) with a certain size parameter (even when \( C = 0 \)).

We will proceed in three steps. First we will in Sec. 3 study the fractional vortices, choosing \( A \) to be diagonal and setting \( B = C = 0 \). Subsequently, in Sec. 4 we will study the semi-local parameters \( C \), setting \( B = A = 0 \). Finally, in Sec. 5 we will give an example of non-vanishing \( A \) and \( C \). We will conclude with a discussion in Sec. 6 and finally give a brief review of the asymptotic properties of the Abelian semi-local vortex in Appendix A.
3 Fractional vortices

In this Section, we will study the moduli matrix (19) with $B = C = 0$ and take $A$ to be diagonal

$$A = \text{diag} \left( z_1, \ldots, z_M \right), \quad a_0 = \frac{1}{M} \sum_{n=1}^{M} z_n = 0. \quad (22)$$

With this moduli matrix there will be no difference between $G' = SO(2M)$ and $G' = USp(2M)$, so we will apply the same yardstick to both of them. As the matrix $A$ is not proportional to the unit matrix $1_M$, the vortex will be of the semi-local type, but a special semi-local type — viz. the fractional vortex. We will denote it the pure fractional vortex. Hence, we are left with the moduli matrix of the following form

$$H_0(z) = \text{diag} \left( z_1 - z, \ldots, z_M - z \right), \quad (23)$$

for which we can choose the Ansatz

$$\Omega' = \text{diag} \left( e^{\chi_1}, \ldots, e^{\chi_M}, e^{-\chi_1}, \ldots, e^{-\chi_M} \right), \quad (24)$$

with the $\det \Omega' = 1$ being manifest and we define $\omega \equiv e^\psi$. Inserting this Ansatz into the master equations (9)-(10) for the non-Abelian Chern-Simons vortex leaves us with the following system of partial differential equations

$$\bar{\partial} \partial \chi_m = -\frac{\pi^2}{M \kappa \mu} \left[ \sum_{n=1}^{M} \left( |z - z_n|^2 e^{-\psi - \chi_n} + e^{-\psi + \chi_n} \right) - \xi \left[ |z - z_m|^2 e^{-\psi - \chi_m} - e^{-\psi + \chi_m} \right] \right]$$

$$- \frac{\pi^2}{\mu^2} \left[ (|z - z_m|^2 e^{-\psi - \chi_m})^2 - (e^{-\psi + \chi_m})^2 \right], \quad m = 1, 2, \ldots, M, \quad (25)$$

$$\bar{\partial} \partial \psi = -\frac{\pi^2}{M^2 \kappa^2} \left[ \sum_{n=1}^{M} \left( |z - z_n|^2 e^{-\psi - \chi_n} + e^{-\psi + \chi_n} \right) - \xi \right] \times \sum_{n'=1}^{M} \left( |z - z_{n'}|^2 e^{-\psi - \chi_{n'}} + e^{-\psi + \chi_{n'}} \right)$$

$$- \frac{\pi^2}{M \kappa \mu} \sum_{n=1}^{M} \left( |z - z_n|^2 e^{-\psi - \chi_n} - e^{-\psi + \chi_n} \right)^2. \quad (26)$$

The boundary conditions for $|z| \to \infty$ are

$$\psi^\infty = \log \left( \frac{2}{\xi} \sum_{n=1}^{M} |z - z_n| \right), \quad \chi_m^\infty = \log |z - z_m|. \quad (27)$$

The energy density comprises the contribution from the Abelian magnetic field strength

$$F_{12}^0 = -4 \sqrt{M} \bar{\partial} \partial \psi, \quad (28)$$
and a boundary term summing up to

$$E = 2\zeta \partial \bar{\partial} \psi + 2 \sum_{n=1}^{M} \partial \bar{\partial} (|z - z_n|^2 e^{-\psi - \chi_n} + e^{-\psi + \chi_n}) ,$$  \hspace{1cm} (29)$$

however, when integrated, only the Abelian magnetic flux contributes as it is the topological charge of the vortex. The non-Abelian magnetic field strength is given by

$$F^m_{12} = -4\bar{\partial} \partial \chi_m , \hspace{0.5cm} m = 1, \ldots, M .$$  \hspace{1cm} (30)$$

The Abelian and non-Abelian electric field strengths are, respectively

$$E^0_i = \frac{2\pi}{\kappa \sqrt{M}} \partial_i \left[ \sum_{n=1}^{M} (|z - z_n|^2 e^{-\psi - \chi_n} + e^{-\psi + \chi_n}) \right] , \hspace{0.5cm} E^m_i = \frac{2\pi}{\mu} \partial_i \left[ |z - z_m|^2 e^{-\psi - \chi_m} - e^{-\psi + \chi_m} \right] ,$$

where we conveniently have defined the relevant generators for the non-Abelian field strengths as

$$(t^m)^j_i = \frac{1}{2} (\delta^m_i \delta^m_j - \delta^m_i \delta^{m+M} \delta^{m+M,j}) ,$$  \hspace{1cm} (31)$$

and the labels $m = 1, \ldots, M$ denote the generators corresponding to normalized generators spanning the Cartan subalgebra of $SO(2M)$ and $USp(2M)$.

First let us make some qualitative calculations. We consider some small fluctuations around the boundary conditions (27) as follows

$$\chi_m = \chi_m^\infty + \delta \chi_m , \hspace{0.5cm} \psi = \psi^\infty + \delta \psi .$$  \hspace{1cm} (32)$$

Plugging them into the master equations (25)-(26) yields to linear order in the fluctuations

$$\bar{\partial} \partial \delta \chi_m = \frac{M^2 \mu^2}{4} \frac{|z - z_m|^2}{\left( \sum_{n=1}^{M} |z - z_n| \right)^2} \delta \chi_m ,$$  \hspace{1cm} (33)$$

$$\bar{\partial} \partial \delta \psi + \bar{\partial} \partial \psi^\infty = \frac{m^2 \partial_i}{4} \delta \psi .$$  \hspace{1cm} (34)$$

The lump solution for $\chi_m$ vanishes when acted upon by the Laplacian operator, hence the fluctuation is in some sense local, up to the corrections of the rational function multiplying the right hand side of Eq. (33) which asymptotically will be of order 1. Thus asymptotically, the solution will be the modified Bessel function of the second kind $K_0(\mu |z|)$. This is not the case for the profile function $\psi$, governing the Abelian magnetic flux. The Laplacian operator on the lump solution does not vanish. Expanding in $z^{-1}, \bar{z}^{-1}$ (and using an appropriate Kähler
transformation)

\[ \hat{\partial} \partial \log \left( \sum_{n=1}^{M} |z - z_n| \right) \simeq \frac{1}{4M} \left( \sum_{n=1}^{M} \left| z_n \right|^2 - \frac{1}{M} \left| \sum_{n=1}^{M} z_n \right|^2 \right) |z|^{-4} \]

\[ + \frac{1}{4M} \left[ \frac{1}{2} \sum_{n=1}^{M} |z_n|^2 \left( \frac{z_n}{z} + \bar{z}_n \right) - \frac{1}{2M} \left( \sum_{n=1}^{M} \frac{z_n}{z} \sum_{n'=1}^{M} \bar{z}_{n'} + \sum_{n=1}^{M} z_n \sum_{n'=1}^{M} \frac{z_{n'}}{\bar{z}} \right) \right. 
\[ \left. + \frac{1}{M} \left( \sum_{n=1}^{M} |z_n|^2 - \frac{1}{M} \left| \sum_{n=1}^{M} z_n \right|^2 \right) \sum_{n'=1}^{M} \left( \frac{z_{n'}}{z} + \bar{z}_{n'} \right) \right] |z|^{-4} + O \left( |z|^{-6} \right) \quad , \tag{35} \]

we obtain the power behavior for \( \delta \psi \), well-known for the semi-local vortex profile functions:

\[ \delta \psi = \frac{1}{M m_v^2} \left( \sum_{n=1}^{M} |z_n|^2 - \frac{1}{M} \left| \sum_{n=1}^{M} z_n \right|^2 \right) |z|^{-4} \]

\[ + \frac{1}{M m_v^2} \left[ \frac{1}{2} \sum_{n=1}^{M} |z_n|^2 \left( \frac{z_n}{z} + \bar{z}_n \right) - \frac{1}{2M} \left( \sum_{n=1}^{M} \frac{z_n}{z} \sum_{n'=1}^{M} \bar{z}_{n'} + \sum_{n=1}^{M} z_n \sum_{n'=1}^{M} \frac{z_{n'}}{\bar{z}} \right) \right. 
\[ \left. + \frac{1}{M} \left( \sum_{n=1}^{M} |z_n|^2 - \frac{1}{M} \left| \sum_{n=1}^{M} z_n \right|^2 \right) \sum_{n'=1}^{M} \left( \frac{z_{n'}}{z} + \bar{z}_{n'} \right) \right] |z|^{-4} + O \left( |z|^{-6} \right) \quad , \tag{36} \]

This clearly demonstrates the semi-local nature of the fractional vortex also in the Chern-Simons theory. It is easily seen that when all the centers of the fractional vortices coincide, \( z_n = z_0 \forall n \), the solution \[ \text{(36)} \] vanishes and should be replaced by the “local” solution \( K_0(m_\kappa |z|) \) which is easily found from Eq. \[ \text{(34)} \]. Note also that the solution is radially symmetric only to lowest order: \( |z|^{-4} \). In Eq. \[ \text{(33)} \] it is easy to see that the rational function becomes simply \( 1/M^2 \) when the centers \( z_n \) coincide.

We will now solve the equations numerically. For simplicity, as already mentioned, we will focus on the groups \( G' = SO(4) \) and \( G' = USp(4) \), which will give rise to a fractional vortex possessing two subpeaks. The solution is local in the sense that there are no size moduli when \( z_1 = z_2 \), however when the “centers” are non-coincident: \( z_1 \neq z_2 \) the vortex should be interpreted as a semi-local vortex.

An important difference between this configuration and the configurations we normally consider, is that the rotational symmetry in the \( \mathbb{C} \)-plane has been lost and we are forced to consider the full partial differential equations in the \( \mathbb{C} \)-plane instead of simplified ones in the radial direction which reduce to ordinary differential equations that we easily can solve.

Furthermore, the fact that the non-Abelian Chern-Simons vortex has some crucial differences with respect to the corresponding lump obtained in the weak Chern-Simons coupling limit, for instance the Abelian magnetic flux being a ring instead of a Gaussian bell, suggests us to find the
finite Chern-Simons coupling solutions in the $\mathbb{C}$-plane. Thus, we will now compute the numerical solutions in the $\mathbb{C}$-plane using a relaxation method.

In Fig. 1 is shown a matrix of graphs of the non-Abelian Chern-Simons vortex configuration for $\kappa = \mu = 2$, where the rows correspond to the energy density, the Abelian magnetic flux $F^0_1$, the non-Abelian magnetic flux $F^2_1$, the magnitude of the Abelian electric field $|E^0_1|$ and finally the magnitude of the non-Abelian electric field $|E^2_1|$, whereas the columns correspond to the separation distance $2d = 2\{0, 1, 2, 10\}$. The centers of the fractional sub-peaks are placed at $z_1 = d$ and $z_2 = -d$, respectively. Note that the symmetry of the configuration has allowed us to show only one of the non-Abelian fields, viz. $F^2_1$ for which the other field is given by the reflection in the $x$-axis $F^1_1(x, y) = F^2_1(-x, y)$. Analogously for the non-Abelian electric field strengths. If we look at the $d = 0$ column, we see that all fields are ring-like, except the energy density which has a bell-like shape with a tiny valley on the top. For the non-zero but small separation $2d = 2$ (the second column in the Figure), we see that the magnetic flux densities are no more ring-like structures but (distorted) bell-like ones. The energy density has also become a distorted bell-like structure and finally the Abelian electric field strength is a squashed ring, while the non-Abelian electric field strength is a ring-like structure (a bit distorted though). The tendency for larger separation, is that the energy is described by two bell-like shapes and so is the Abelian magnetic flux density. Each Abelian peak of magnetic flux has a single corresponding peak of non-Abelian flux (each with a different Cartan generator). The electric field strengths, Abelian and non-Abelian retain their ring-like status. The non-Abelian ones however split into rings at the same positions as the non-Abelian magnetic flux peaks, whereas the Abelian electric field has a ring on all the sites (i.e. at each point where the determinant of $H_0$ vanishes).

To understand the structure of the configuration, the contour plots of Fig. 1 are convenient, however, to get a more detailed look at the solution, we show slices of the configuration as function of $x$ for $y = 0$ for the values of the separation modulus parameter $d = \{0, 1, 2, 5, 10\}$. In Fig. 2 is shown the energy density as function of $x$ for $y = 0$, while in Fig. 3 are shown the Abelian magnetic and electric field strengths on the same slice. There is a transition from a ring-like structure to separate bell-like structures in the Abelian magnetic flux density. We show in Fig. 4a, a detailed graph of the Abelian magnetic flux with various values of the separation modulus parameter $d$ from $d = 0$ to $d = 2$ and in Fig. 4b the corresponding non-Abelian magnetic field strength $F^2_1$. We observe that the Abelian magnetic flux goes quite fast from being a ring-like structure to become a single peak, which eventually spreads into two sub-peaks that will depart from each other and dilute as the separation modulus parameter is increased. For the non-Abelian magnetic flux a similar situation is happening. The ring-structure becomes a single
Figure 1: The non-Abelian Chern-Simons fractional vortex with $G' = SO(4)$ and $G' = USp(4)$ for $\kappa = \mu = 2$, where the rows of the figure correspond to the energy density, the Abelian magnetic flux $F_{12}^0$, the non-Abelian magnetic flux $F_{12}^2$, the magnitude of the Abelian electric field $|E_0^i|$ and finally the magnitude of the non-Abelian electric field $|E_2^i|$, whereas the columns of the figure correspond to the separation distance $2d = 2\{0, 1, 2, 10\}$. We have set $\xi = 2$. 

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Figure 2: The energy density, rescaled by \((1+d)\) with \(2d\) being the separation distance as function of \(d\). We have here set \(\kappa = \mu = 2\) and \(\xi = 2\).

Figure 3: The Abelian (a) magnetic and (b) electric field strength density, rescaled by \((1 + d)\) with \(2d\) being the separation distance as function of \(d\). We have set \(\kappa = \mu = 2\) and \(\xi = 2\).
Figure 4: The (a) Abelian and (b) non-Abelian magnetic field strength density with $2d$ being the separation distance as function of $d = \{0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, 1.4, 1.6, 1.8, 2\}$. We have set $\kappa = \mu = 2$ and $\xi = 2$.

Figure 5: The Abelian magnetic field strength density, rescaled by $(1 + d)$ with $2d$ being the separation distance as function of $d$. Here the couplings are $\kappa = 4$ and $\mu = 2$, while we have set $\xi = 2$.

peak at the origin, which then moves (and gets distorted) as $d$ is increased.

### 3.1 Negative Abelian magnetic flux density at the origin: $\kappa > \mu$

For completeness, let us repeat the calculation with the different Chern-Simons couplings chosen as in Ref. [63]. First we consider the case with $\kappa = 4, \mu = 2$. The main difference between the configurations with $\kappa = \mu$ and those with $\kappa \neq \mu$ lies in the magnetic field strengths. Therefore we will focus on those. In Fig. 5 is shown the Abelian magnetic flux density on the slice $(x, 0)$ of the configuration for different values of the separation modulus parameter $d = \{0, 1, 2, 5, 10\}$, while
Figure 6: The (a) Abelian and (b) non-Abelian magnetic field strength density with 2d being the separation distance as function of d = {0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, 1.4, 1.6, 1.8, 2}. Here the couplings are κ = 4 and µ = 2, while we have set ξ = 2.

in Fig. 6 are shown the Abelian and non-Abelian magnetic flux densities in detail for various values of d ranging from d = 0 to d = 2. We observe a situation quite similar to the case with equal couplings, except from the fact that the Abelian magnetic flux density starts out being negative at the origin for d = 0, while the non-Abelian one remains positive at the origin. The end result for large separation 2d is analogous to the equal coupling case.

3.2 Positive Abelian magnetic flux density at the origin: κ < µ

Let us now turn to the case with κ = 1, µ = 2. In Fig. 7 is shown the Abelian magnetic flux density on the slice (x, 0) of the configuration for different values of the separation modulus parameter d = {0, 1, 2, 5, 10}, while in Fig. 8 are shown the Abelian and non-Abelian magnetic flux densities in details for various values of d ranging from d = 0 to d = 2. We observe a situation quite similar to the case with equal couplings, except from the fact that the Abelian magnetic flux density starts out being positive at the origin for d = 0, while the non-Abelian one is negative at the origin. The end result for large separation 2d is analogous to the equal coupling case.

3.3 Bell-like structures: −κ = µ = 2

The last case studied in Ref. [63], was the vortex with negative Abelian Chern-Simons coupling and positive non-Abelian coupling: k = −2 and µ = 2.
Figure 7: The Abelian magnetic field strength density, rescaled by \((1 + d)\) with \(2d\) being the separation distance as function of \(d\). Here the couplings are \(\kappa = 1\) and \(\mu = 2\), while we have set \(\xi = 2\).

Figure 8: The (a) Abelian and (b) non-Abelian magnetic field strength density with \(2d\) being the separation distance as function of \(d = \{0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, 1.4, 1.6, 1.8, 2\}\). Here the couplings are \(\kappa = 1\) and \(\mu = 2\), while we have set \(\xi = 2\).
Figure 9: The non-Abelian Chern-Simons fractional vortex with $G' = SO(4)$ and $G' = USp(4)$ for $-\kappa = \mu = 2$, where the rows of the Figure correspond to the energy density, the Abelian magnetic flux $F_{12}^0$, the non-Abelian magnetic flux $F_{12}^2$, the magnitude of the Abelian electric field $|E_0^1|$, and finally the magnitude of the non-Abelian electric field $|E_2^3|$, whereas the columns of the Figure correspond to the separation distance $2d = 2\{0, 1, 2, 10\}$. We have set $\xi = 2$. 
In Fig. 9 is shown a matrix of graphs of the non-Abelian Chern-Simons vortex configuration for couplings with opposite signs: \(-\kappa = \mu = 2\), where the rows correspond to the energy density, the Abelian magnetic flux \(F_{12}^0\), the non-Abelian magnetic flux \(F_{12}^2\), the magnitude of the Abelian electric field \(|E_1^0|\) and finally the magnitude of the non-Abelian electric field \(|E_2^0|\), whereas the columns correspond to the separation distance \(2d = 2\{0, 1, 2, 10\}\). The centers of the fractional sub-peaks are placed at \(z_1 = d\) and \(z_2 = -d\), respectively. Note that the symmetry of the configuration has allowed us to show only one of the non-Abelian fields, i.e. \(F_{12}^2\) for which the other field is given by the reflection in the \(x\)-axis \(F_{12}^1(x, y) = F_{12}^2(-x, y)\) and analogously for the non-Abelian electric field strengths. Unlike the case with equal couplings, there is basically not changing much when separating the fractional sub-peaks in this case with opposite signs of the couplings. The energy density and the Abelian magnetic flux density are both single peaked bells which split into two sub-peaks for non-zero separation modulus parameter \(d\). The electric field strengths on the other hand retain their structure as rings. The non-Abelian one simply moves with \(d\) (though it gets a bit distorted), while the Abelian density has a transition where it gets distorted until it splits into two ring-like structures. For the intermediate value \(d = 2\), the Abelian electric field strength density has in fact six maxima, demonstrating the complexity of this soliton.

### 4 Semi-local vortices

In this Section, we will study the moduli matrix (19) with \(A = B = 0\) and only have a non-vanishing \(C\). We will thus denote this vortex a purely semi-local vortex. We will however restrict ourselves to configurations with a single size modulus \(c\). For both \(SO(2M)\) and \(USp(2M)\) with \(M = 2m, m \in \mathbb{Z}_{>0}\), we can parametrize the moduli matrix (19) in this case as follows

\[
H_0(z) = \begin{pmatrix} z1_{2m} & C_{S,A} \\ 0 & 1_{2m} \end{pmatrix}, \quad C_{S,A} = cJ_{2m} = c \begin{pmatrix} 0 & 1_m \\ \epsilon & 0 \end{pmatrix},
\]

with \(\epsilon = +1\) for \(SO\) and \(\epsilon = -1\) for \(USp\) as usual. For \(USp(4)\) it is the most generic case for \(C\), while it is a simplified case for \(SO(4)\) as the diagonal of \(C\) does not have to vanish. For \(m > 1\) the above matrix is a simplified example, but it turns out that the equations have a very simple dependence on \(m\), hence we will keep \(m\) as a parameter. In the case of \(SO(2M = 4m + 2)\), we can use the parametrization as follows

\[
C = c1_M.
\]
In both cases it turns out that we can still use a diagonal Ansatz

$$\Omega' = \text{diag} \left( e^{\chi} \mathbf{1}_M, e^{-\chi} \mathbf{1}_M \right),$$

(39)
even though $\Omega_0$ no longer is diagonal due to the fact that the master equations remain diagonal. The master equations are thus

$$\bar{\partial} \partial \chi = -\frac{\pi^2}{\kappa \mu} \left[ (|z|^2 + |c|^2) e^{-\psi - \chi} + (|z|^2 + |c|^2) e^{-\psi + \chi} - \frac{\xi}{M} \right]$$

$$- \frac{\pi^2}{\mu^2} \left[ \left( (|z|^2 + |c|^2) e^{-\psi - \chi} \right)^2 - \left( (|z|^2 + |c|^2) e^{-\psi + \chi} \right)^2 \right],$$

(40)

$$\bar{\partial} \partial \psi = -\frac{\pi^2}{\kappa \mu} \left[ (|z|^2 + |c|^2) e^{-\psi - \chi} + (|z|^2 + |c|^2) e^{-\psi + \chi} - \frac{\xi}{M} \right]$$

$$- \frac{\pi^2}{\kappa \mu} \left[ \left( (|z|^2 + |c|^2) e^{-\psi - \chi} \right)^2 - \left( (|z|^2 + |c|^2) e^{-\psi + \chi} \right)^2 \right],$$

(41)

which has the typical form of semi-local equations. Note that the dependence of $M$ can be eliminated by a rescaling of $\xi$.

The boundary conditions for $|z| \to \infty$, which are also the lump solution, that is, the solution in the weak coupling limit $\kappa = \mu \to 0$, read

$$\psi^\infty = \log \left\{ \frac{2M}{\xi} \sqrt{|z|^2 + |c|^2} \right\}, \quad \chi^\infty = \log \sqrt{|z|^2 + |c|^2}. \quad (42)$$

The Abelian and non-Abelian magnetic field strengths read respectively

$$F_{12}^0 = -4\sqrt{M} \bar{\partial} \partial \psi, \quad F_{12}^a t^a \equiv F_{NA} t_{12} = -4\sqrt{M} \bar{\partial} \partial \chi t,$$

(43)

where the Abelian one contributes together with a boundary term to the energy density as

$$E = 2\xi \bar{\partial} \partial \psi + 2M \bar{\partial} \partial \left[ (|z|^2 + |c|^2) e^{-\psi - \chi} + e^{-\psi + \chi} \right]. \quad (44)$$

The Abelian and non-Abelian electric field strengths are, respectively

$$E^r_{12} = \frac{2\sqrt{M}}{\kappa} \partial_r \left[ (r^2 + |c|^2) e^{-\psi - \chi} + e^{-\psi + \chi} \right],$$

(45)

$$E^r_{12} t^a \equiv E_{r12}^a = \frac{2\sqrt{M}}{\mu} \partial_r \left[ (r^2 + |c|^2) e^{-\psi - \chi} - e^{-\psi + \chi} \right] t,$$

(46)

where the radial coordinate is $r \equiv |z|$ and we have defined the normalized generator

$$t \equiv \frac{1}{2\sqrt{M}} \text{diag} \left( \mathbf{1}_M, -\mathbf{1}_M \right). \quad (47)$$

Let us now make some qualitative calculations. We consider some small fluctuations around the boundary conditions (42) as follows

$$\chi = \chi^\infty + \delta \chi, \quad \psi = \psi^\infty + \delta \psi. \quad (48)$$
Plugging them into the master equations (40)-(41) yields to linear order

\[
\ddel \partial \delta \chi + \ddel \partial \delta \chi = \frac{m^2}{4} \delta \chi , \tag{49}
\]

\[
\ddel \partial \delta \psi + \ddel \partial \delta \psi = \frac{m^2}{4} \delta \psi . \tag{50}
\]

Expanding in small \(|c|/|z|\), we obtain

\[
\ddel \partial \psi = \ddel \partial \chi \approx \frac{|c|^2}{2|z|^4} - \frac{|c|^4}{|z|^6} , \tag{51}
\]

which has the consequence that both the field fluctuations attain a power-law behavior

\[
\delta \chi = \frac{2|c|^2}{m^2} |z|^{-4} + \frac{4|c|^2}{m^2} \left( \frac{8}{m^2} - |c|^2 \right) |z|^{-6} + \mathcal{O} \left( |z|^{-8} \right) , \tag{52}
\]

\[
\delta \psi = \frac{2|c|^2}{m^2} |z|^{-4} + \frac{4|c|^2}{m^2} \left( \frac{8}{m^2} - |c|^2 \right) |z|^{-6} + \mathcal{O} \left( |z|^{-8} \right) . \tag{53}
\]

This behavior is the typical and well-known behavior of a semi-local non-Abelian vortex. Next we will show that in the case of equal Chern-Simons couplings \(\kappa = \mu\), the Abelian magnetic flux and the non-Abelian magnetic flux are equal as well as that the Abelian electric field equals the non-Abelian electric field. Subtracting the two master equations (40)-(41), we obtain

\[
\ddel \partial (\psi - \chi) = -\frac{\pi^2}{\kappa^2} \left[ \left( 1 - \frac{\kappa^2}{\mu^2} \right) \left( (|z|^2 + |c|^2) e^{-\psi - \chi} \right)^2 + \left( 1 + \frac{\kappa}{\mu} \right)^2 \left( e^{-\psi + \chi} \right)^2 \right.
\]

\[
+ 2 \left( 1 - \frac{\kappa}{\mu} \right) (|z|^2 + |c|^2) e^{-\psi - \chi} \left( e^{-\psi + \chi} - \frac{\xi}{2M} \right) + \left( 1 + \frac{\kappa}{\mu} \right) \frac{\xi}{M} e^{-\psi + \chi} \right] , \tag{54}
\]

which in the case \(\kappa = \mu\) clearly reduces to

\[
\ddel \partial (\psi - \chi) = -\frac{4\pi^2}{\kappa^2} e^{-\psi + \chi} \left( e^{-\psi + \chi} - \frac{\xi}{2M} \right) , \tag{55}
\]

being independent of \(z, \bar{z}\) and thus is satisfied by the vacuum solution

\[
e^{-\psi + \chi} = \frac{\xi}{2M} . \tag{56}
\]

Now it is easy to show that

\[
F^{0}_{12} = F^{NA}_{12} = \frac{8\sqrt{M} \pi^2}{\kappa^2} \left( |z|^2 + |c|^2 \right) e^{-\psi - \chi} \left( (|z|^2 + |c|^2) e^{-\psi - \chi} - \frac{\xi}{2M} \right) , \tag{57}
\]

and we can readily observe that for vanishing size modulus \(c = 0\), the magnetic fields are zero at the center of the vortex (the local case) [63]. This statement gets modified by the presence of the size modulus. The exact value of the magnetic fields at the center of the semi-local vortex is not so easy to calculate and we demonstrate numerically, that it is indeed non-vanishing at
the center of the vortex. Inserting the vacuum solution (56) into the electric field strengths, it is easily seen that the Abelian and the non-Abelian one are equal in the case of \( \kappa = \mu \)

\[
E^0_r = E^\text{NA}_r = \frac{2\sqrt{M\pi}}{\kappa} \partial_r \left[ (r^2 + |c|^2) e^{-\psi - \chi} \right].
\]  

(58)

It furthermore turns out that the Abelian and non-Abelian field strengths are equal also in the opposite coupling case. This can be seen from equation (54) by inserting \( \mu = -\kappa \) giving rise to

\[
\bar{\partial} (\psi - \chi) = -\frac{4\pi^2}{\kappa^2} (|z|^2 + |c|^2) e^{-\psi - \chi} \left( e^{-\psi + \chi} - \frac{\xi}{2M} \right),
\]

which does depend on \( z, \bar{z} \) but still allows the vacuum solution (56) and the consequence is then the same as in the case of \( \kappa = \mu \); the Abelian magnetic and electric fields equal their non-Abelian counterparts.

Let us now compute some numerical solutions as examples of semi-local non-Abelian vortices. First we take the equal coupling case and show in Fig. 10 the energy density for thirteen different values of the semi-local size parameter. In Fig. 11 are shown the magnetic flux densities and electric field densities for the Abelian fields (which equals the non-Abelian ones as demonstrated above). We observe that there is a transition from the ring-like magnetic flux into a Gauss-bell-like magnetic flux. The electric field does not change qualitatively, but only spreads out as the vortex size increases.

### 4.1 Negative magnetic flux density at the origin

For completeness, let us repeat the calculation with the different Chern-Simons couplings as we did in the fractional case, considering first \( \kappa = 4, \mu = 2 \). In Fig. 12 are shown the magnetic field strengths; Abelian in (a) and non-Abelian in (b), as function of the size modulus \( c \). It is observed that the effect of negative Abelian magnetic flux at the origin of the vortex vanishes quickly as the size is turned on. This can be understood from the fact that both the Abelian and the non-Abelian fluxes become bell-like structures. Furthermore, they become equal even for different Chern-Simons couplings \( \kappa \neq \mu \) in the limit of large \( |c| \). This can qualitative be shown in terms of the lump solution (42) by calculating the field strengths in the limit \( |c| \gg 1/(g\sqrt{\xi}) \) (i.e. the size modulus being much larger than the local size of the vortex)

\[
F^0_{12} = F^{\text{NA}}_{12} = -2\sqrt{M} \frac{|c|^2}{(|z|^2 + |c|^2)^2}.
\]

(60)

Now we set \( \kappa = 1, \mu = 2 \) and show the corresponding magnetic flux densities in Fig. 13.
Figure 10: The energy density for the semi-local vortex solution for various values of the semi-local modulus $c = \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.2, 1.5, 2.0\}$. The couplings are chosen as $\kappa = \mu = 2$ and $\xi = 2$.

Figure 11: The Abelian and non-Abelian (a) magnetic flux density and (b) electric field density for the semi-local vortex solution for various values of the semi-local modulus $c = \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.2, 1.5, 2.0\}$. The couplings are chosen as $\kappa = \mu = 2$ and $\xi = 2$. 
Figure 12: The (a) Abelian and (b) non-Abelian magnetic flux density for the semi-local vortex solution for various values of the semi-local modulus $c = \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.2, 1.5, 2.0\}$. The couplings are chosen as $\kappa = 4, \mu = 2$ and $\xi = 2$.

Figure 13: The (a) Abelian and (b) non-Abelian magnetic flux density for the semi-local vortex solution for various values of the semi-local modulus $c = \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.2, 1.5, 2.0\}$. The couplings are chosen as $\kappa = 1, \mu = 2$ and $\xi = 2$. 

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Figure 14: The energy density for the semi-local vortex solution for various values of the semi-local modulus $c = \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.2, 1.5, 2.0\}$. The couplings are chosen as $-\kappa = \mu = 2$ and $\xi = 2$.

Figure 15: The Abelian and non-Abelian (a) magnetic flux density and (b) electric field density for the semi-local vortex solution for various values of the semi-local size modulus $c = \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.2, 1.5, 2.0\}$. The couplings are chosen as $-\kappa = \mu = 2$ and $\xi = 2$.

4.2 Bell-like structures: $-\kappa = \mu = 2$

Finally, we will consider the case of opposite signs of the Chern-Simons couplings $\kappa = -\mu$. The energy is shown in Fig. 14, the magnetic flux densities in Fig. 15a and the electric field densities in Fig. 15b, all as function of the semi-local size modulus. We can confirm numerically that the Abelian magnetic field equals the non-Abelian one as well as the Abelian electric field equals minus the non-Abelian one (as shown above).
5 Semi-local fractional vortices

We will now consider the fractional vortices with non-zero size parameter $C$ in the moduli matrix \([19]\) turned on. It turns out that it is not possible to obtain diagonal master equations in this case for $USp$. Hence, we will consider only $G' = SO(2M)$ here and take the moduli matrix as the one of Refs. \([17, 48]\), namely

$$H_0 = \begin{pmatrix}
  z - z_1 & c_1 \\
  \vdots & \ddots & \ddots & \ddots \\
  z - z_M & c_M \\
  0 & \ddots & 1 \\
  \vdots & \ddots & \ddots & \ddots \\
  0 & \ddots & \ddots & 1
\end{pmatrix}, \quad (61)$$

while we still use the Ansatz for the moduli matrix field of Eq. \((24)\). This leads to the master equations which remain diagonal

$$\bar{\partial} \partial \chi_m = -\frac{\pi^2}{M\kappa\mu} \left[ \sum_{n=1}^{M} (|z - z_n|^2 + |c_n|^2) e^{-\psi - \chi_n} + \sum_{n=1}^{M} e^{-\psi + \chi_n} - \xi \right]$$

$$\times \left[ (|z - z_m|^2 + |c_m|^2) e^{-\psi - \chi_m} - e^{-\psi + \chi_m} \right]$$

$$- \frac{\pi^2}{\mu^2} \left[ (|z - z_m|^2 + |c_m|^2)^2 (e^{-\psi - \chi_m})^2 - (e^{-\psi + \chi_m})^2 \right], \quad m = 1, 2, \ldots, M, \quad (62)$$

$$\bar{\partial} \partial \psi = -\frac{\pi^2}{M^2\kappa^2} \left[ \sum_{n=1}^{M} (|z - z_n|^2 + |c_n|^2) e^{-\psi - \chi_n} + \sum_{n=1}^{M} e^{-\psi + \chi_n} - \xi \right]$$

$$\times \left[ \sum_{n'=1}^{M} (|z - z_{n'}|^2 + |c_{n'}|^2) e^{-\psi - \chi_{n'}} + \sum_{n'=1}^{M} e^{-\psi + \chi_{n'}} \right]$$

$$- \frac{\pi^2}{M\kappa\mu} \sum_{n=1}^{M} \left[ (|z - z_n|^2 + |c_n|^2) e^{-\psi - \chi_n} - e^{-\psi + \chi_n} \right]^2. \quad (63)$$

Setting to zero all $c_n$ we get back the fractional master equations \((25)-(26)\). The boundary conditions which are also the lump solution (i.e. corresponding to the weak coupling limit $\kappa \to 0$ and $\mu \to 0$) read

$$\psi^\infty = \log \left( \frac{2}{\xi} \sum_{n=1}^{M} \sqrt{|z - z_n|^2 + |c_n|^2} \right), \quad \chi_m^\infty = \log \sqrt{|z - z_m|^2 + |c_m|^2}. \quad (64)$$
The magnetic fluxes remain those of Eqs. (28) and (30), respectively, while the energy density changes according to

\[ \mathcal{E} = 2\xi \bar{\partial} \partial \psi + 2 \sum_{n=1}^{M} \bar{\partial} \partial \left[ (|z - z_n|^2 + |c_n|^2) e^{-\psi - \chi_n} + e^{-\psi + \chi_n} \right]. \] (65)

The Abelian and non-Abelian electric field strengths also change and are now, respectively

\[ E_0^i = \frac{2\pi}{\kappa \sqrt{M}} \sum_{n=1}^{M} \partial_i \left[ (|z - z_n|^2 + |c_n|^2) e^{-\psi - \chi_n} + e^{-\psi + \chi_n} \right], \] (66)

\[ E_m^i = \frac{2\pi}{\mu} \partial_i \left[ (|z - z_m|^2 + |c_m|^2) e^{-\psi - \chi_m} - e^{-\psi + \chi_m} \right], \quad m = 1, \ldots, M, \] (67)

where the relevant generators \( t_m \) are those of Eq. (31).

Let us now repeat some qualitative calculations in this case. We again consider some small fluctuations around the boundary conditions (64) as follows

\[ \chi_m = \chi_m^\infty + \delta \chi_m, \quad \psi = \psi^\infty + \delta \psi. \] (68)

Plugging them into the master equations (62)-(63) yields to linear order

\[ \bar{\partial} \partial \delta \chi_m + \bar{\partial} \partial \chi_m^\infty = \frac{M^2 m^2}{4} \frac{|z - z_m|^2 + |c_m|^2}{\left( \sum_{n=1}^{M} \sqrt{|z - z_n|^2 + |c_n|^2} \right)^2} \delta \chi_m, \] (69)

\[ \bar{\partial} \partial \delta \psi + \bar{\partial} \partial \psi^\infty = \frac{m^2}{4} \delta \psi. \] (70)

We first expand the Laplacian of the lump solution in \( z^{-1}, \bar{z}^{-1} \) as

\[ \bar{\partial} \partial \chi_m^\infty \approx \frac{|c_m|^2}{2|z|^4} + \left( \frac{z_m}{z} + \frac{\bar{z}_m}{\bar{z}} \right) \frac{|c_m|^2}{|z|^4} + O \left( |z|^{-6} \right), \] (71)

\[ \bar{\partial} \partial \psi^\infty \approx \frac{1}{4M} \left( \sum_{n=1}^{M} (|z_n|^2 + 2|c_n|^2) - \frac{1}{M} \left( \sum_{n=1}^{M} |z_n|^2 \right) \right) |z|^{-4} \]

\[ + \frac{1}{4M} \left[ \frac{1}{2} \sum_{n=1}^{M} (|z_n|^2 + 4|c_n|^2) \left( \frac{z_n}{z} + \frac{\bar{z}_n}{\bar{z}} \right) - \frac{1}{2M} \left( \sum_{n=1}^{M} \frac{z_n^2}{z} \sum_{n'=1}^{M} \frac{\bar{z}_{n'}}{\bar{z}} + \sum_{n=1}^{M} \frac{z_n}{z} \sum_{n'=1}^{M} \frac{\bar{z}_{n'}}{\bar{z}} \right) \right] |z|^{-4} + O \left( |z|^{-6} \right), \]

and then we expand also the right hand side of Eq. (69) to obtain the following asymptotic
solutions

\[
\delta \chi_m = \frac{2|c_m|^2}{m^2 \mu} |z|^{-4} + \frac{2|c_m|^2}{m^2 \mu} \left[ 3 \left( \frac{z_m}{z} + \frac{\bar{z}_m}{\bar{z}} \right) - \frac{1}{M} \sum_{n=1}^{M} \left( \frac{z_n}{z} + \frac{\bar{z}_n}{\bar{z}} \right) \right] |z|^{-4} + O(|z|^{-6}) , \quad (73)
\]

\[
\delta \psi = \frac{1}{M m^2 \kappa} \left( \sum_{n=1}^{M} \left( |z_n|^2 + 2|c_n|^2 \right) - \frac{1}{M} \left( \sum_{n=1}^{M} z_n \right)^2 \right) |z|^{-4}
+ \frac{1}{M m^2 \kappa} \frac{1}{2} \sum_{n=1}^{M} \left( |z_n|^2 + 4|c_n|^2 \right) \left( \frac{z_n}{z} + \frac{\bar{z}_n}{\bar{z}} \right) - \frac{1}{2M} \left( \sum_{n=1}^{M} \frac{z_n^2}{z} \sum_{n'=1}^{M} \bar{z}_{n'} + \sum_{n=1}^{M} z_n \sum_{n'=1}^{M} \frac{z_{n'}^2}{\bar{z}} \right)
+ \frac{1}{M} \left( \sum_{n=1}^{M} \left( |z_n|^2 + 2|c_n|^2 \right) - \frac{1}{M} \left( \sum_{n=1}^{M} z_n \right)^2 \right) \left( \sum_{n'=1}^{M} \left( \frac{z_{n'}}{z} + \frac{\bar{z}_{n'}}{\bar{z}} \right) \right) |z|^{-4} + O(|z|^{-6}) , \quad (74)
\]

The solution \( \delta \psi \) is similar to that of the fractional case (34) now with a dependence on \( c_m \).
Interestingly, it was already noted in Eq. (33) that \( \chi_m \) in the “purely” fractional case is kind of “local” and in fact we now observe the power-behaved tail depending only on \( c_m \). This is consistent with the results of [17, 48], viz. the lump solution becomes singular if any of \( c_m \) vanishes. We can also see that if the fractional vortex centers coincide, \( z_n = z_0, \forall n \), then the solution (74) coincides to lowest order with that of the purely semilocal case, Eq. (50) by identification of \(|c|^2\) and \( \sum_{n=1}^{M} |c_n|^2 / M \).

Let us close this section with showing this semi-local fractional vortex in Fig. 16 for equal Chern-Simons couplings \( \kappa = \mu = 2 \) and size parameters \( c_1 = c_2 = 1 \). The graph is a matrix as in the previous sections with the columns representing the energy density, the Abelian magnetic flux \( F_{12}^0 \), the non-Abelian magnetic flux \( F_{12}^2 \), the magnitude of the Abelian electric field \(|E_0^0|\) and finally the magnitude of the non-Abelian electric field \(|E_2^2|\), whereas the rows represent the relative distance \( 2d = 2\{0, 1, 2, 10\} \). The difference between this semi-local fractional vortex with equal couplings and is purely fractional counterpart (i.e. with \( c = 0 \)) is obviously quite big. The magnetic fluxes have a completely different nature for vanishing relative distance \( d = 0 \), whereas the difference is almost absent for large \( d \). On the other hand, there is seemingly not much difference between the pure fractional vortex (\( c = 0 \)) with opposite couplings \(-\kappa = \mu \) and the semi-local fractional vortex with equal couplings, except some differences in the electric field densities. We will not exhaust the reader with further graphs but just mention what happens by changing the individual sizes \( c_n \). Each sub-peak, when well separated has the size controlled by its corresponding size modulus.
Figure 16: The non-Abelian Chern-Simons semi-local fractional vortex with $G' = SO(4)$ and $G' = USp(4)$ for $\kappa = \mu = 2$ with sizes $c_1 = c_2 = 1$, where the rows of the figure correspond to the energy density, the Abelian magnetic flux $F_{12}^0$, the non-Abelian magnetic flux $F_{12}^3$, the magnitude of the Abelian electric field $|E_1^0|$, and finally the magnitude of the non-Abelian electric field $|E_1^3|$, whereas the columns of the figure correspond to the separation distance $2d = 2\{0, 1, 2, 10\}$. We have set $\xi = 2$. 

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6 Discussion

In this paper we have studied the fractional as well as semi-local Chern-Simons vortices in $G = U(1) \times SO(2M)$ and $G = U(1) \times USp(2M)$ theories. It is known that the fractional-vortex positions are semi-local moduli and it is shown that unless they are all coincident (local case) the ring-like flux structure, characteristic of Chern-Simons vortices, will become bell-like fluxes – just as those of the standard Yang-Mills vortices. The asymptotic profile functions have been calculated and it is shown that the vortex becomes (a kind of) semi-local for non-coincident positions of the fractional vortices. The calculation was repeated for the purely semi-local vortex (i.e. with non-vanishing size modulus but coincident zeroes of the squark fields), which however differs in the fact that the non-Abelian profile function $\chi$ approaches its VEV exactly as the Abelian profile function $\psi$, whereas in for the fractional vortex the non-Abelian profile function $\chi$ remains approximately local. Note however, that in the classic way of describing the squark fields with a matrix $q = \text{diag}(f, g)$, we should identify the profile functions as

$$f^2 = \text{diag}(|z - z_1|^2 e^{-\psi - \chi_1}, \ldots, |z - z_M|^2 e^{-\psi - \chi_M}), \quad g^2 = \text{diag}(e^{-\psi + \chi_1}, \ldots, e^{-\psi + \chi_M}),$$

hence it suffices that just $\psi$ has a power-law behavior in order for $f, g$ to both have it as well.

In order to study the fractional vortices we have solved the (BPS) vortex equations (master equations) numerically in the plane, as the rotational symmetry obviously is lost when the fractional position moduli are non-coincident. A peculiar effect of the non-Abelian Chern-Simons vortices was discovered in Ref. [63] which is observed only for different Chern-Simons couplings $\kappa \neq \mu$ where the local vortex possesses a negative (positive) Abelian and positive (negative) non-Abelian magnetic flux for $\kappa > \mu$ ($\kappa < \mu$). This effect is destroyed by the semi-local size moduli which is easily seen from the lump solution that has both the Abelian and the non-Abelian flux concentrated at the origin in same amounts. The fractional vortices have the same fate for large relative distance as each fractional peak can be thought of as an effective semi-local vortex having power-like tails in its profile functions.

Comparing the asymptotic expansions (52)-(53) of the semi-local vortex to the asymptotic expansions of the semi-local fractional vortex (73)-(74), we can identify the effective size of a $k = 1$ semi-local fractional Chern-Simons vortex with sizes $c_n$

$$|c_{\text{effective}}|^2 = \frac{1}{2M} \sum_{n=1}^{M} \left( |z_n|^2 + 2|c_n|^2 \right) - \frac{1}{2M^2} \left| \sum_{n=1}^{M} z_n \right|^2,$$

valid for the Abelian fields, while the non-Abelian fields have the effective size $c_n$. If we now set $c_n$ to zero on the right hand side, we see the effective size of the Abelian field explicitly in terms
of the fractional vortex centers $z_n$ (i.e. the case of the pure fractional Chern-Simons vortex). Note that the next-to-leading order is not radially symmetric.

In order to understand this type of fractional vortex better, one should understand the second homotopy group of the moduli space of vacua in the theory at hand \[17\]. This has not been pursued in this paper and remains as a future task.

The substructures of the non-Abelian vortices are very interesting. For many reasons it would be very interesting to study the non-Abelian vortices in a symmetric phase (i.e. the Chern-Simons phase) – these vortices are known as non-topological vortices. They have been studied in Refs. \[64, 65, 66, 67\] in the Abelian case and are still not very well-understood in the following sense. Their moduli have not been written down explicitly. Their existence has been proven in Ref. \[68\] and their topology on compact manifolds has been studied in Ref. \[69\]. To the best of our knowledge, the closest attempt to classify the moduli has been done by Lee-Min-Rim in Ref. \[70\] where the non-topological moduli have been written down in the asymptotic profile functions. The generalization of these studies to the non-Abelian case would be quite interesting.

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**A Review of the Abelian semi-local vortex**

Let us recall the asymptotic behavior of the Abelian $G = U(1)$ semi-local vortex. The master equation is

$$\bar{\partial}\partial\psi = -\frac{m^2}{4\xi} \left[ H_0(z)H_0^\dagger(\bar{z})e^{-\psi} - \xi \right], \quad (77)$$

with $m \equiv \sqrt{\xi}e$. We can formally calculate the lump solution by sending the mass (the gauge coupling constant $e$) to infinity and obtain

$$\psi_\infty = \log \left[ \frac{1}{\xi} H_0(z)H_0^\dagger(\bar{z}) \right]. \quad (78)$$

The lump solution is in the vacuum manifold and can for a finite lump size be considered as the (approximate) long distance behavior of the vortex fields. It is however singular in the local vortex case (i.e. the small lump singularity). If we now consider a small fluctuation around the
lump solution setting $\psi = \psi^\infty + \delta \psi$ we have

$$\bar{\partial} \partial \delta \psi + \bar{\partial} \partial \psi^\infty = \frac{m^2}{4} \delta \psi ,$$

(79)

and we can identify two cases. Since the vacuum manifold is Kähler, the lump solution can formally be a Kähler transformation of the VEV, in which case the lump solution is singular, however the Laplacian will vanish (apart from some delta functions at each zero of the squark fields). This is the local vortex case and we can also calculate the asymptotic profile function as

$$\delta \psi = c_{\text{local}} K_0 (m |z|) .$$

(80)

In semi-local case, in contradistinction, we observe that the fluctuation will have a power-law behavior. For simplicity we consider $N_f = 2$ with the following moduli matrix

$$H_0 (z) = \begin{pmatrix} z \\ c \end{pmatrix} ,$$

(81)

for which the fluctuation equation reads

$$\bar{\partial} \partial \delta \psi + \bar{\partial} \partial \log \left[ |z|^2 + |c|^2 \right] = \frac{m^2}{4} \delta \psi .$$

(82)

Since the vacuum manifold is Kähler, we make a Kähler transformation yielding

$$\bar{\partial} \partial \log \left[ |z|^2 + |c|^2 \right] = \bar{\partial} \partial \log \left[ 1 + \frac{|c|^2}{|z|^2} \right] \simeq \bar{\partial} \partial \left[ \frac{|c|^2}{|z|^2} - \frac{|c|^4}{2 |z|^4} + O (|z|^{-6}) \right] ,$$

(83)

where we are calculating the asymptotic function valid when $|z| \gg |c|$. Then we find using a series expansion

$$\delta \psi = \sum_{n=2}^{\infty} b_n |z|^{-2n} = \frac{4 |c|^2}{m^2} |z|^{-4} + \left( \frac{64 |c|^2}{m^4} - \frac{8 |c|^4}{m^2} \right) |z|^{-6} + O (|z|^{-8}) .$$

(84)

These are of course well-known facts about the Abelian semi-local vortex solution (see Refs. [12, 13, 14]).

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