AN ASSOCIATIVE ANALOGY OF LIE $H$-PSEUDOBIALGEBRA

LINLIN LIU* AND ZHITAO GUO

Abstract. The purpose of this paper is to study infinitesimal $H$-pseudobialgebra, which is an associative analogy of Lie $H$-pseudobialgebra. We first define the infinitesimal $H$-pseudobialgebra and investigate some properties of this new algebraic structure. Then we consider the coboundary infinitesimal $H$-pseudobialgebra, which is the subclass of infinitesimal $H$-pseudobialgebra and we obtain the associative Yang-Baxter equation over an associative $H$-pseudoalgebra. Finally, we found the connection between the (coboundary) infinitesimal $H$-pseudobialgebra and the (coboundary) Lie $H$-pseudobialgebra. Meanwhile, the relationship between the associative Yang-Baxter equation and the classical Yang-Baxter equation (over an $H$-pseudoalgebra) is established.

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1. Introduction

The notion of conformal algebra ( [21]) was introduced by Kac as an axiomatic description of the operator product expansion (OPE) of chiral fields in conformal field theory, and it came to be useful for investigation of vertex algebras. Recall that a Lie conformal algebra $L$ is defined as a $\mathbb{C}[\partial]$-module ($\partial$ is an indeterminate), endowed with a $\mathbb{C}$-linear map $L \otimes L \rightarrow \mathbb{C}[\lambda] \otimes L$, $a \otimes b \mapsto [a, b]$ satisfying axioms similar to those of Lie algebra (see [11, 21]). Later, Bakalov, D’Andrea and Kac replaced the above polynomial algebra $\mathbb{C}[\partial]$ with any cocommutative Hopf algebra $H$ in [6], and found that this high-dimensional conformal algebra is actually the algebra in a pseudotensor

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*Corresponding author.

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category, which is called (Lie) $H$-pseudoalgebra. So far, the classification problems, cohomology theory and representation theory of $H$-pseudoalgebras have been considered in [6–8]. As a natural generation of conformal algebras, (Lie) $H$-pseudoalgebras are closely related to the differential Lie algebras of the Ritt and Hamiltonian formalism in the theory of nonlinear evolution equations (see [12,16,17]). However, their roles in many fields of mathematical physics are not yet completely understood since they are relatively new algebraic structures. The fact that the annihilation algebra of the associative $H$-pseudoalgebra $\text{Cend} H$ is nothing else but the Drinfeld Double of the Hopf algebra $H$, which leads us to believe that there should be a deep connection between the theory of $H$-pseudoalgebras and quantum groups. Recently there has been some interest in the theory of $H$-pseudoalgebras (see, for example, [9,23–25,28]).

Infinitesimal bialgebras (also called $\varepsilon$-bialgebras) were introduced by Join and Rota in order to provide an algebraic framework for the calculus of divided differences ([20]). More precisely, an infinitesimal bialgebra $(A,\mu,\Delta)$ is an associative algebra $(A,\mu)$ and a coassociative coalgebra $(A,\Delta)$ such that the comultiplication $\Delta$ is a 1-cocycle in algebra cohomology (i.e., a derivation) with coefficient in $A \otimes A$. Moreover, the notions of coboundary and quasitriangular infinitesimal bialgebra, infinitesimal Hopf algebra, and the basic theory were established by Aguiar in [1,2]. Further research can be found in [4,27].

In addition to the widely researched in combinatorics ([3,15,18]), infinitesimal bialgebras are also closely related to Lie bialgebras ([13,14,20]). Indeed, a Lie bialgebra is a Lie algebra and a Lie coalgebra, in which the cobracket is a 1-cocycle in Lie algebra cohomology. Thus the cocycle condition in an infinitesimal bialgebra can be seen as an associative analogy of that in a Lie bialgebra. The necessary and sufficient conditions for infinitesimal bialgebra to be a related Lie bialgebra were given in [2]. Furthermore, it was shown that the solutions of associative Yang-Baxter equation ([2,19]) can induce the solutions of classical Yang-Baxter equation ([5,10]).

Lie $H$-pseudobialgebra appeared in [9] as a generalization of conformal bialgebra ([22]) and Lie bialgebra. Compared with Lie bialgebra, Lie $H$-pseudobialgebra is defined on the $H$-modules rather than the vector spaces. More precisely, Lie $H$-pseudobialgebra is both a Lie $H$-coalgebra and a Lie $H$-pseudoalgebra satisfying the cocycle condition. Similar to the relationship between infinitesimal bialgebra and Lie bialgebra, we want to find an associative analog of Lie $H$-pseudobialgebra, which is our motivation for defining infinitesimal $H$-pseudobialgebra. In addition, there is a natural, but not obvious way to establish the connections between infinitesimal $H$-pseudobialgebra and Lie $H$-pseudobialgebra. The current paper will devote to these questions.

The paper is organized as follows.

In Section 2, we recall some basic notions about associative $H$-pseudoalgebra and present some properties of its representation.

In Section 3, we mainly define the notion of infinitesimal $H$-pseudobialgebra (see Definition 3.4). Specifically, an infinitesimal $H$-pseudobialgebra $(A,\ast,\Delta)$ consists of an associative $H$-pseudoalgebra $(A,\ast)$ and a coassociative $H$-coalgebra $(A,\Delta)$ such that $\Delta$ is a 1-cocycle in associative $H$-pseudoalgebra cohomology (see Section 4) with coefficient in $A \otimes^2$, which is an associative analogy of the cocycle condition in a Lie $H$-pseudobialgebra. Moreover, some examples and basic properties are given.
In Section 4, we study the cohomology theory of associative $H$-pseudoalgebra and the coboundary infinitesimal $H$-pseudobialgebra (see Definition 4.2), which is an important subclass of infinitesimal $H$-pseudobialgebra. In particular, we consider the construction of coboundary infinitesimal $H$-pseudobialgebra and the corresponding associative pseudo-Yang-Baxter equation (pseudo-AYBE) (see Theorem 4.4).

In Sections 5, we mainly study the relationship between infinitesimal $H$-pseudobialgebras and Lie $H$-pseudobialgebras (see Theorem 5.6). We first find the necessary and sufficient conditions for infinitesimal $H$-pseudobialgebra to be related Lie $H$-pseudobialgebra (see Corollary 5.7), then we give a sufficient condition under which a coboundary infinitesimal $H$-pseudobialgebra gives rise to the corresponding coboundary Lie $H$-pseudobialgebra (see Theorem 5.10).

In the last sections, we show that under some suitable conditions, a solution of the pseudo-AYBE is also a solution of the classical Yang-Baxter equation (pseudo-CYBE) in related Lie $H$-pseudobialgebra (see Theorem 6.1).

Throughout this paper, $k$ is a fixed algebraically closed field of characteristic zero, $H$ is a cocommutative Hopf algebra over $k$, and $X = H^* = \text{Hom}_k(H, k)$ denotes the dual of $H$. As usual, we adopt Sweedler’s notations in [26]. For a coalgebra $C$, we write its comultiplication as $\Delta(c) = c_1 \otimes c_2$, $\forall c \in C$. For any vector space $V$, we will define $\sigma(f \otimes g) = g \otimes f$, $(12)(f \otimes g \otimes h) = g \otimes f \otimes h$ and $(13)(f \otimes g \otimes h) = h \otimes g \otimes f$, for all $f, g, h \in V$.

2. Associative $H$-pseudoalgebras and their representations

We first recall some definitions and notations of associative $H$-pseudoalgebras (see [6, 9] for more details).

**Definition 2.1** An $H$-pseudoalgebra is a left $H$-module $A$ together with a map (called the pseudoproduct):

$$*: A \otimes A \rightarrow (H \otimes H) \otimes_A A, \ a \otimes b \mapsto a * b$$

satisfying $H$-bilinearity: for all $a, b \in A$ and $f, g \in H$, one has

$$fa \ast gb = (f \otimes g \otimes 1)(a \ast b).$$

If $a \ast b = \sum_i (f_i \otimes g_i) \otimes_H e_i$, then we have $fa \ast gb = \sum_i (f f_i \otimes g g_i) \otimes_H e_i$.

Note that the $H$-pseudoalgebra $(A, \ast)$ is associative if it satisfies $(a \ast b) \ast c = a \ast (b \ast c)$ in $H^{\otimes 3} \otimes_A A$, for any $a, b, c \in A$, and $(A, \ast)$ is commutative if $a \ast b = (\sigma \otimes H id)(b \ast a)$ holds.

Moreover, $H$-pseudoalgebra $(A, \ast)$ is called finite, if it is finitely generated as $H$-module.

In particular, for the one-dimensional Hopf algebra $H = k$, an associative $H$-pseudoalgebra is just an ordinary algebra over the field $k$.

For an arbitrary Hopf algebra $H$, we recall that the map $\mathcal{F}: H \otimes H \rightarrow H \otimes H$ defined by the formula

$$\mathcal{F}(f \otimes g) = (f \otimes 1)(S \otimes id)\Delta(g) = f S(g_1) \otimes g_2$$

is called the Fourier transform. Observe that $\mathcal{F}$ is a vector space isomorphism with an inverse given by

$$\mathcal{F}^{-1}(f \otimes g) = (f \otimes 1)\Delta(g) = f g_1 \otimes g_2.$$
In order to reformulate the definition of associative $H$-pseudoalgebra, the authors in [6] introduced another product $\{a, b\} \in H \otimes A$ as the Fourier transform of $a * b$:

$$\{a, b\} = \sum_i \mathcal{F}(f_i \otimes g_i)(1 \otimes e_i) = \sum_i f_i S(g_{i1}) \otimes g_{i2} e_i; \text{ if } a * b = \sum_i (f_i \otimes g_i) \otimes_H e_i.$$ 

In other words,

$$\{a, b\} = \sum_i h_i \otimes c_i, \text{ if } a * b = \sum_i (h_i \otimes 1) \otimes_H c_i.$$ 

For $x \in X := H^*$, the $x$-product in $A$ is as follows:

$$a \circ_x b = \langle S(x), \cdot \rangle \otimes id[a, b] = \sum_i < S(x), f_i S(g_{i1}) > g_{i2} e_i = \sum_i < S(x), h_i > c_i,$$

if $a * b = \sum_i (f_i \otimes g_i) \otimes_H e_i = \sum_i (h_i \otimes 1) \otimes_H c_i$.

Using properties of the Fourier transform, an equivalent definition of an associative $H$-pseudoalgebra is as follows.

**Definition 2.2** An associative $H$-conformal algebra is a left $H$-module $A$ endowed with a product $\{\cdot, \cdot\} : A \otimes A \to H \otimes A$, satisfying the following properties (for all $a, b, c \in A$ and $h \in H$):

- **$H$-sesquilinearity:**
  $$\{ha, b\} = (h \otimes 1)\{a, b\}, \quad \{a, hb\} = (1 \otimes h_2)\{a, b\}(S(h_1) \otimes 1).$$

- **Associativity:**
  $$\{a, \{b, c\}\} = (\mathcal{F}^{-1} \otimes id)\{\{a, b\}, c\}$$ 

in $H \otimes H \otimes A$, where $\{a, \{b, c\}\} = (\sigma \otimes id)(id \otimes \{a, \cdot\})\{b, c\}, \{\{a, b\}, c\} = (id \otimes \{\cdot, c\})\{a, b\}$.

Definition 2.2 can also be reformulated in terms of the $x$-product. Formally, we will use the same notation for $X$ as for $H$.

**Definition 2.3** An associative $H$-conformal algebra is a left $H$-module $A$ equipped with $x$-product $\circ_x : A \otimes A \to A, a \otimes b \mapsto a \circ_x b$ for all $a, b \in A$ and $x \in X$, satisfying the following properties:

- **Locality:** for any basis $\{x_i\}$ of $X$, $a \circ_{x_i} b \neq 0$ for only a finite number of $i$.
- **$H$-sesquilinearity:** $\langle ha, b\rangle \circ_x = a \circ_x b, \quad a \circ_x (hb) = h_2(a \circ_{S(h_1)x} b)$, for all $a, b \in A$ and $h \in H$.
- **Associativity:** $a \circ_x (b \circ_y c) = (a \circ_{x_2} b) \circ_{yx_1} c$, for all $a, b, c \in A$ and $x, y \in X$.

Let $A, B$ be two associative $H$-pseudoalgebras. A left $A$-module is a left $H$-module $M$ together with an operation $\rho_l \in Hom_{H \otimes H}(A \otimes M, (H \otimes H) \otimes_H M)$, we denote $\rho_l(a \otimes m) = a * m$, which satisfies $a * (b * m) = (a * b) * m$, for all $a, b \in A$ and $m \in M$. Similarly, we can define a right $B$-module. If $M$ is a left $A$-module and right $B$-module such that $(a * m) * b = a * (m * b)$ for all $a \in A, b \in B$ and $m \in M$, then $M$ is called an $A$-$B$-bimodule. An $A$-$A$-bimodule is simply called an $A$-bimodule.

Next we show some properties that will be used later. Prior to this, we introduce the following notations:
Suppose that $A$ is an associative $H$-pseudoalgebra and $M$ is a left $A$-module. For all $a \in A$ and $m \in M$, we have $a \ast m = \sum_i f_i \otimes g_i \otimes_H m_i = \sum_i f_i S(g_i) \otimes 1 \otimes_H g_2 m_i \in (H \otimes H) \otimes_H M$. By Lemma 2.3 in [6], $a \ast m$ can be written uniquely in the form $\sum_i (h_i \otimes 1) \otimes_H c_i$, where $\{h_i\}$ is a fixed $k$-basis of $H$. Throughout this paper, we write

$$a \ast m = \sum_{(a,m)} h^{a,m} \otimes 1 \otimes_H c_{a,m} = h^{a,m} \otimes 1 \otimes_H c_{a,m}$$

for convenience. Similarly, for a right $A$-module $N$, we set

$$n \ast a = \sum_{(n,a)} 1 \otimes l^{n,a} \otimes_H e_{n,a} = 1 \otimes l^{n,a} \otimes_H e_{n,a}, \forall a \in A, n \in N.$$

**Proposition 2.4** Let $(A, \ast)$ be an associative $H$-pseudoalgebra. Suppose that $M$ is a left $A$-module and $N$ is a right $A$-modules. Then $M \otimes N$ is an $A$-bimodule with the following structures:

$$a \ast (m \otimes n) = a \ast m \otimes n = (h^{a,m} \otimes 1) \otimes_H (c_{a,m} \otimes n) \quad (2.1)$$

and

$$(m \otimes n) \ast a = m \otimes n \ast a = (1 \otimes l^{n,a}) \otimes_H (m \otimes e_{n,a}), \quad (2.2)$$

where $a \ast m = (h^{a,m} \otimes 1) \otimes_H c_{a,m}, n \ast a = (1 \otimes l^{n,a}) \otimes_H e_{n,a}$, for all $a \in A, m \in M$ and $n \in N$.

**Proof.** For all $h \in H, m \in M$, and $n \in N$, it is easy to prove that $M \otimes N$ is a left $H$-module with the action $h \cdot (m \otimes n) = h_1 m \otimes h_2 n$. We first check that $M \otimes N$ is a left $A$-module. Observe that

$$fa \ast gm = ((f \otimes g) \otimes_H 1)(a \ast m) = (fh^{a,m} \otimes g) \otimes_H c_{a,m} = (fh^{a,m} S(g_1) \otimes 1) \otimes_H g_2 c_{a,m}$$

for all $f, g \in H, a \in A$ and $m \in M$. Then we obtain

$$fa \ast g(m \otimes n) = fa \ast (g_1 m \otimes g_2 n) = (fh^{a,m} S(g_1) \otimes 1) \otimes_H (g_2 c_{a,m} \otimes g_3 n) = (fh^{a,m} S(g_1) \otimes 1) \Delta(g_2) \otimes_H (c_{a,m} \otimes n) = (fh^{a,m} \otimes g) \otimes_H (c_{a,m} \otimes n) = ((f \otimes g) \otimes_H 1)(a \ast (m \otimes n)),$$

which proving the $H$-bilinearity. Now we check the associativity. On the one hand,

$$a \ast (b \ast (m \otimes n)) = a \ast ((h^{b,m} \otimes 1) \otimes_H (c_{b,m} \otimes n)) = (1 \otimes h^{b,m} \otimes 1)(id \otimes \Delta)(h^{a,c_{b,m}} \otimes 1) \otimes_H (c_{a,c_{b,m}} \otimes n) = (h^{a,c_{b,m}} \otimes h^{b,m} \otimes 1) \otimes_H (c_{a,c_{b,m}} \otimes n).$$

On the other hand, suppose that $a \ast b = f^{a,b} \otimes g^{a,b} \otimes_H t_{a,b}$ for all $a, b \in A$, then we have

$$(a \ast b) \ast (m \otimes n) = f^{a,b} h_1^{a,b,m} \otimes g^{a,b} h_2^{a,b,m} \otimes 1 \otimes_H (c_{i,a,b,m} \otimes n).$$
Since $M$ is a left $A$-module, we have that $a \ast (b \ast m) = (a \ast b) \ast m$, which is equivalent to
\[ h^{a,c,b,m} \otimes h^{b,m} \otimes 1 \otimes_H c_{a,c,b,m} = f^{a,b} h^{a,b,m}_1 \otimes g^{a,b} h^{a,b,m}_2 \otimes 1 \otimes_H c_{a,b,m}. \]

It follows that $a \ast (b \ast (m \otimes n)) = (a \ast b) \ast (m \otimes n)$. $M \otimes N$ is also a right $A$-module by a similar calculation. Since
\[
(a \ast (m \otimes n)) \ast b = ((h^{a,m} \otimes 1) \otimes_H (c_{a,m} \otimes n)) \ast b = (h^{a,m} \otimes 1 \otimes 1)((\Delta \otimes \text{id})(1 \otimes l^{n,b}) \otimes_H (c_{a,m} \otimes e_{n,b})
\]
and
\[
a \ast ((m \otimes n) \ast b) = a \ast ((1 \otimes l^{n,b}) \otimes_H (m \otimes e_{n,b})) = (1 \otimes 1 \otimes l^{n,b})(\text{id} \otimes \Delta)(h^{a,m} \otimes 1) \otimes_H (c_{a,m} \otimes e_{n,b}) = h^{a,m} \otimes 1 \otimes l^{n,b} \otimes_H (c_{a,m} \otimes e_{n,b}),
\]
we have that $(a \ast (m \otimes n)) \ast b = a \ast ((m \otimes n) \ast b)$. Then the conclusion holds. \hfill \qed

**Remark 2.5**

(1) Let $(A, \ast)$ be an associative $H$-pseudoalgebra. Then $A \otimes A$ is an $A$-bimodule by setting $M = N = A$ in Proposition 2.4. More generally, $A^{\otimes n}(n > 2)$ is an $A$-bimodule with the following structures:
\[
a \ast (b_1 \otimes \cdots \otimes b_n) = (h^{a,b_1} \otimes 1) \otimes_H (c_{a,b_1} \otimes b_2 \otimes \cdots \otimes b_n),
\]
\[
(b_1 \otimes \cdots \otimes b_n) \ast a = (1 \otimes l^{b_n,a}) \otimes_H (b_1 \otimes \cdots \otimes b_{n-1} \otimes e_{b_n,a})
\]
for all $a, b_i (i = 1, 2, \cdots, n) \in A$. We write
\[
a \bullet (b_1 \otimes \cdots \otimes b_n) = h^{a,b_1} \otimes (c_{a,b_1} \otimes b_2 \otimes \cdots \otimes b_n),
\]
\[
(b_1 \otimes \cdots \otimes b_n) \bullet a = l^{b_n,a} \otimes (b_1 \otimes \cdots \otimes b_{n-1} \otimes e_{b_n,a}).
\]

(2) Let $(A, \ast)$ be an associative $H$-pseudoalgebra. Suppose that $M$ is a left $A$-module and $N$ is a left $H$-module. Similar to the proof of Proposition 2.4, $M \otimes N$ is a left $A$-module with condition (2.1). In addition, if $M$ is a left $H$-module and $N$ is a left $A$-module, then $M \otimes N$ is also a left $A$-module with the action
\[
a \ast (m \otimes n) = m \otimes a \ast n = (h^{a,n} \otimes 1) \otimes_H (m \otimes c_{a,n}),
\]
where $a \ast n = (h^{a,n} \otimes 1) \otimes_H c_{a,n}$.

Let $V$ and $W$ be two left $H$-modules. Recall that an $H$-pseudolinear map from $V$ to $W$ is a $k$-linear map $\phi : V \rightarrow (H \otimes H) \otimes_H W$ such that
\[
\phi(hv) = ((1 \otimes h) \otimes_H 1)\phi(v), \quad \forall h \in H, v \in V.
\]
The vector space of all such $\phi$ is denoted by $\text{Chom}(V,W)$ and the left action of $H$ on $\text{Chom}(V,W)$ is defined by
\[
(h\phi)(v) = ((1 \otimes h) \otimes_H 1)\phi(v).
\]
In the special case $V = W$, we write $\text{Cend}(V) = \text{Chom}(V,V)$. Everywhere in the paper, unless otherwise specified, we always set $V^* = \text{Chom}(V,k)$. 


Consider the map $\rho : Chom(V, W) \otimes V \to (H \otimes H) \otimes_H W$ given by $\rho(\phi \otimes v) = \phi(v)$. By definition it is $H$-bilinear, therefore it is a polylinear map in $M^*(H)$ (see [6]). Sometimes, we will use the notation $\phi \ast v := \phi(v)$ and consider this as a pseudoproduct or pseudoaction.

Suppose that $A$ is an associative $H$-pseudoalgebra, $U$ and $V$ are finite $A$-modules. Then the formula

$$(a \ast \phi) \ast u = a \ast (\phi \ast u), \quad \forall a \in A, u \in U, \phi \in Chom(U, V)$$

provides $Chom(U, V)$ with the structure of a left $A$-module. In particular, when $V$ is the base field $k$, the dual module of $M$ is $M^* = Chom(M, k)$, where $k$ is a trivial $A$-module with $h \cdot 1 = \varepsilon(h)1$ for all $h \in H$.

**Proposition 2.6** Let $M$ and $N$ be two left $A$-modules. Suppose that $M$ is a finitely generated free module (as an $H$-module). Then $M^* \otimes N \simeq Chom(M, N)$ as left $A$-modules, where the correspondence $\phi : M^* \otimes N \to Chom(M, N)$ is given by

$$\phi(f \otimes n) = (1 \otimes S(g_{f,m})) \otimes_H n, \forall f \in M^*, m \in M, n \in N$$

if $f \ast m = (g_{f,m} \otimes 1) \otimes_H 1 \in (H \otimes H) \otimes_H k$.

**Proof.** Firstly, we check that $\psi(f \otimes n) \in Chom(M, N)$. For all $f \in M^* = Chom(M, k)$, we have

$$f(hm) = ((1 \otimes h) \otimes_H 1)f(m) = (g_{f,m} \otimes h) \otimes_H 1 = (g_{f,m}S(h_1) \otimes 1) \otimes_H h_2 \cdot 1 = (g_{f,m}S(h) \otimes 1) \otimes_H 1.$$ 

It follows that

$$\psi(f \otimes n) \ast (hm) = (1 \otimes hS(g_{f,m})) \otimes_H n = ((1 \otimes h) \otimes_H 1)((1 \otimes S(g_{f,m})) \otimes_H n) = ((1 \otimes h) \otimes_H 1)(\psi(f \otimes n) \ast m).$$

Secondly, we show that $\psi$ is a morphism of left $A$-module. By Remark 2.5(2), $M^* \otimes N$ is a left $A$-module via $a \ast (f \otimes n) = (h^{a,n} \otimes 1) \otimes_H (f \otimes c_{a,n})$ if $a \ast n = (h^{a,n} \otimes 1) \otimes_H c_{a,n}$. Since

$$\psi(h(f \otimes n)) \ast m = \psi(h_1f \otimes h_2n) \ast m = (1 \otimes S(h_1g_{f,m})) \otimes_H h_2n = (1 \otimes S(g_{f,m}))(1 \otimes S(h_1)) \Delta(h_2) \otimes_H n = (h \otimes S(g_{f,m})) \otimes_H n = ((h \otimes 1) \otimes_H 1)(\psi(f \otimes n) \ast m) = (h\psi(f \otimes n)) \ast m,$$

we have that $\psi$ is an $H$-linear map. For all $m \in M$, we have

$$(a \ast \psi(f \otimes n)) \ast m = a \ast (\psi(f \otimes n) \ast m) = a \ast ((1 \otimes S(g_{f,m})) \otimes_H n) = h^{a,n} \otimes 1 \otimes S(g_{f,m}) \otimes_H c_{a,n}.$$
On the other hand,
\[
\psi(a \ast (f \otimes n)) \ast m = \psi((h^{a,n} \otimes 1) \otimes_H (f \otimes c_{a,n})) \ast m
\]
\[
= ((h^{a,n} \otimes 1) \otimes (\Delta \otimes id)(1 \otimes S(g_{f,m})) \otimes_H c_{a,n}
\]
\[
= h^{a,n} \otimes 1 \otimes S(g_{f,m})) \otimes_H c_{a,n}.
\]
It follows that \( a \ast \psi(f \otimes n) = \psi(a \ast (f \otimes n)) \).

Finally, it suffices to prove that \( \psi \) is both injective and surjective. The proofs are similar to Proposition 4.2 in [9] and we omit the details. \( \square \)

3. Infinitesimal \( H \)-pseudobialgebras

We start with the following definition.

**Definition 3.1** A coassociative \( H \)-coalgebra \( C \) is a left \( H \)-module, endowed with an \( H \)-linear map \( \Delta : C \rightarrow C \otimes C \) \((\Delta(c) = \sum c_1 \otimes c_2)\) satisfying the coassociativity
\[
(id \otimes \Delta) \circ \Delta = (\Delta \otimes id) \circ \Delta,
\]
that is, for all \( c \in C \),
\[
\sum c_1 \otimes c_21 \otimes c_22 = \sum c_{11} \otimes c_{12} \otimes c_2.
\]
where \( C \otimes C \) is a left \( H \)-module via \( h \cdot (c \otimes d) = h_1 c \otimes h_2 d \), for all \( h \in H \) and \( c,d \in C \).

The coassociative \( H \)-coalgebra \( C \) is cocommutative if it satisfies \( \Delta = \Delta^{op} \), where \( \Delta^{op}(c) = \sum c_2 \otimes c_1 \) for all \( c \in C \). For convenience, we omit the summation symbols.

** Remark 3.2** This is nothing but the standard definition of coassociative coalgebra when \( H = k \).

Let \( L \) be a finite free \( H \)-module with a basis \( \{a_i\}^n_{i=1} \). The dual basis to \( \{a_i\}^n_{i=1} \) in \( L^* = Chom(L,k) \) is defined as the set \( \{a^j\}^n_{j=1} \), where each \( a^j \in L^* \) is given by
\[
a^j \ast a_i = (1 \otimes 1) \otimes_H \delta_{ij}.
\]
Obviously, \( \{a^j\}^n_{j=1} \) is a linearly independent set that \( H \)-generates \( L^* \).

**Theorem 3.3** (1) Let \( (A = \bigoplus_{i=1}^N Ha_i, \ast) \) be a finite free associative \( H \)-pseudoalgebra with the following pseudoproduct:
\[
a_i \ast a_j = \sum_{k=1}^N (f_{ij}^k \otimes g_{ij}^k) \otimes_H a_k.
\]
Let \( A^* = Chom(A,k) = \bigoplus_{i=1}^N Ha^i \) be the dual of \( A \), where \( \{a^i\} \) is the dual basis corresponding to \( \{a_i\} \). Define \( \Delta : A^* \rightarrow A^* \otimes A^* \) as follows:
\[
\Delta(a^k) = \sum_{i,j} S(f_{ij}^k)a^i \otimes S(g_{ij}^k)a^j
\]
and extend it \( H \)-linearly, i.e., \( \Delta(ha^k) = h\Delta(a^k) \). Then \( (A^*, \Delta) \) is a coassociative \( H \)-coalgebra.
(2) Conversely, suppose \((C, \Delta)\) is a finite coassociative \(H\)-coalgebra, then the left \(H\)-module \(C^* = Chom(C, k)\) is an associative \(H\)-conformal algebra with the \(x\)-product defined by

\[(f \circ_x g) \circ_y (c) = f \circ_{x_2} (c_1) g \circ_{y S(x_1)} (c_2)\]

for all \(f, g \in C^*, c \in C\) and \(x, y \in X\).

**Proof.** (1) For all \(a^* \in L^*\), on the one hand,

\[(id \otimes \Delta) \Delta(a^*) = (id \otimes \Delta)(\sum_{i,k} S(f_{ik}^a) a^i \otimes S(g_{ik}^a) a^k)\]

\[= \sum_{i,j,l,k} S(f_{ik}^a) a^i \otimes \Delta(S(g_{ik}^a))(S(f_{lj}^b) a^j) \otimes S(g_{lk}^a) a^l\]

\[= \sum_{i,j,l,k} (S \otimes S \otimes S)(f_{ik}^a \otimes f_{lj}^b g_{ik}^a)_{1} \otimes g_{lk}^a_{2}(a^i \otimes a^j \otimes a^l).\]

On the other hand,

\[(\Delta \otimes id) \Delta(a^*) = (\Delta \otimes id)(\sum_{k,l} S(f_{kl}^a) a^k \otimes S(g_{kl}^a) a^l)\]

\[= \sum_{i,j,l,k} \Delta(S(f_{kl}^a))(S(f_{lj}^b) a^i \otimes S(g_{lk}^a) a^j) \otimes S(g_{kl}^a) a^l\]

\[= \sum_{i,j,l,k} (S \otimes S \otimes S)(f_{lj}^b \otimes f_{ik}^a g_{ik}^a)_{1} \otimes g_{kl}^a_{2}(a^i \otimes a^j \otimes a^l).\]

Using the associativity of \(A\), we have \((a_i * a_j) * a_l = a_i * (a_j * a_l)\), which is equivalent to

\[\sum_{k,s} f_{ik}^a (f_{k}^{ij} g_{ik}^a)_{1} \otimes g_{ik}^a (f_{k}^{ij})_{2} \otimes H a_s = \sum_{k,s} f_{ik}^a \otimes f_{k}^{ij} (g_{ik}^a)_{1} \otimes g_{ik}^a (g_{ik}^a)_{2} \otimes H a_s. \quad (3.2)\]

From (3.2) it follows that \((id \otimes \Delta) \Delta(a^*) = (\Delta \otimes id) \Delta(a^*)\). Hence \((A^*, \Delta)\) is a coassociative \(H\)-coalgebra.

(2) We only prove the associativity of \(C^*\), the remaining part is similar to Theorem 4.5 in [9] and we omit the details. For all \(f, g, l \in C^*\) and \(c \in C\), we have

\[(f \circ_x (g \circ_y l)) \circ_z (c) = f \circ_{x_2} (c_1)(g \circ_y l) \circ_{z S(x_1)} (c_2) = f \circ_{x_2} (c_1)g \circ_{y_2} (c_2) \circ_{l S(x_1) S(y_1)} (c_2) \quad (3.3)\]

and

\[((f \circ_x g) \circ_{y_1} l) \circ_z (c) = (f \circ_x g) \circ_{(y_1)_2} (c_1) \circ_{z S((y_1)_1)} (c_2) = f \circ_{x_2} (c_1)g \circ_{(y_1)_{2 S(x_1)}} (c_1) \circ_{l S((y_1)_1)} (c_2) \quad (3.4)\]

By using the coassociativity of \(\Delta\) and comparing (3.3) and (3.4), we only need to prove that

\[x_{22} \otimes (y_{x_1})_2 S(x_{21}) \otimes z S((y_{x_1})_1) = x_2 \otimes y_2 \otimes (z S(x_1)) S(y_1) \quad (3.5)\]

for all \(x, y, z \in X\). Since \(\Delta(xy) = x_1 y_1 \otimes x_2 y_2\), we have

\[x_{22} \otimes (y_{x_1})_2 S(x_{21}) \otimes z S((y_{x_1})_1) \]

\[= (1 \otimes y_2 \otimes z S(y_1)) (x_{22} \otimes x_{12} S(x_{21}) \otimes S(x_{11})) \]

\[= x_2 \otimes y_2 \otimes (z S(x_1)) S(y_1),\]
finishing the proof. \(\square\)

Now we introduce the notion of infinitesimal \(H\)-pseudobialgebra, which is an associative analogy of Lie \(H\)-pseudobialgebra.

**Definition 3.4** An infinitesimal \(H\)-pseudobialgebra is a triple \((A, *, \Delta)\) such that \((A, *)\) is an associative \(H\)-pseudoalgebra, \((A, \Delta)\) is a coassociative \(H\)-coalgebra and they satisfy the compatible condition

\[
\Delta(a * b) = a * \Delta(b) + \Delta(a) * b, \quad \forall \ a, b \in A,
\]

where

\[
a * \Delta(b) = \sum_i (f_i \otimes g_i) \otimes_H (e_i \otimes b_2),
\]

\[
\Delta(a) * b = \sum_j (k_j \otimes l_j) \otimes_H (a_1 \otimes t_j),
\]

if \(a * b_1 = \sum_i (f_i \otimes g_i) \otimes_H e_i, a_2 * b = \sum_j (k_j \otimes l_j) \otimes_H t_j\).

**Remark 3.5** In particular, for the one dimensional Hopf algebra \(H = k\), an infinitesimal \(H\)-pseudobialgebra is an ordinary infinitesimal bialgebra ([1]) over the field \(k\).

**Example 3.6** Let \(A = H\{e_1, e_2\}\) be a free associative \(H\)-pseudoalgebra with the pseudo-product given by \(e_1 * e_2 = e_2 * e_1 = e_2 * e_2 = 0, e_1 * e_1 = (f \otimes g) \otimes_H e_2, \forall f, g \in H\). Define \(\Delta : A \rightarrow A \otimes A\) as follows:

\[
\Delta(e_1) = e_1 \otimes e_2, \quad \Delta(e_2) = e_2 \otimes e_2,
\]

and extend it \(H\)-linearity, i.e., \(\Delta(h e_i) = h \Delta(e_i)\) for \(i = 1, 2\). Then \((A, \Delta)\) is a coassociative \(H\)-coalgebra. Furthermore, \((A, *, \Delta)\) is an infinitesimal \(H\)-pseudobialgebra. Suppose that \(r = e_i \otimes e_j, i, j = 1, 2\) except for \(i = j = 1\), then \((A, *, \Delta_r)\) is a coboundary infinitesimal \(H\)-pseudobialgebra, which will be defined in the next section.

**Proposition 3.7** Let \(H'\) be a Hopf subalgebra of \(H\) and \((A, *, \Delta)\) an infinitesimal \(H'\)-pseudobialgebra. Then \((\text{Cur}(A) = H \otimes_{H'} A, *, \delta)\) is an infinitesimal \(H\)-pseudobialgebra with the following structures \((\forall f, g \in H, a, b \in A)\):

\[
(f \otimes_{H'} a) \hat{*} (g \otimes_{H'} b) = \sum_i (f f_i \otimes gg_i) \otimes_H (1 \otimes_{H'} e_i),
\]

\[
\delta(f \otimes_{H'} a) = (f_1 \otimes_{H'} a_1) \otimes (f_2 \otimes_{H'} a_2),
\]

where \(a * b = \sum_i (f_i \otimes g_i) \otimes_{H'} e_i\).

**Proof.** By [6], \((\text{Cur}(A) = H \otimes_{H'} A, \hat{*})\) is an associative \(H\)-pseudoalgebra. For all \(f, g \in H\) and \(a, b \in A\), we have

\[
\delta(f \cdot (g \otimes_{H'} b)) = \delta(f g \otimes_{H'} b) = (f_1 g_1 \otimes_{H'} b_1) \otimes (f_2 g_2 \otimes_{H'} b_2) = f \cdot \delta(g \otimes_{H'} b)
\]

and

\[
(id \otimes \delta)(f \otimes_{H'} a) = (id \otimes \delta)((f_1 \otimes_{H'} a_1) \otimes (f_2 \otimes_{H'} a_2))
\]

\[
= ((f_1 \otimes_{H'} a_1) \otimes ((f_2 \otimes_{H'} a_{21})) \otimes ((f_3 \otimes_{H'} a_{22}))
\]
We compute:

\[ (\delta \otimes \text{id})\delta(f \otimes_{H'} a) = ((f_1 \otimes_{H'} a_{11}) \otimes ((f_2 \otimes_{H'} a_{12})) \otimes ((f_3 \otimes_{H'} a_2)) = \]

It follows that \((\text{Cur}(A), \delta)\) is a coassociative \(H\)-coalgebra. In what follows, we verify the compatible condition (3.6). Suppose

\[ a \ast b = \sum_i (f_i \otimes g_i) \otimes_{H'} e_i, \quad a \ast b_1 = \sum_i (l_i \otimes k_i) \otimes_{H'} t_i, \quad a_2 \ast b = \sum_i (m_i \otimes n_i) \otimes_{H'} r_i. \]

We compute:

\[ \delta((f \otimes_{H'} a) \ast (g \otimes_{H'} b)) = \delta(\sum_i (f f_i \otimes g g_i) \otimes_{H} (1 \otimes_{H'} e_i)) \]

\[ = \sum_i (f f_i \otimes g g_i) \otimes_{H} ((1 \otimes_{H'} e_{i1}) \otimes (1 \otimes_{H'} e_{i2})), \]

\[ (f \otimes_{H'} a) \ast \delta(g \otimes_{H'} b) = (f \otimes_{H'} a) \ast (g_1 \otimes_{H'} b_1) \otimes (g_2 \otimes_{H'} b_2) \]

\[ = \sum_i (f l_i \otimes g k_i) \otimes_{H} ((1 \otimes_{H'} t_i) \otimes (g_2 \otimes_{H'} b_2)) \]

\[ = \sum_i (f l_i \otimes g k_i) \otimes_{H} ((1 \otimes_{H'} t_i) \otimes (1 \otimes_{H'} b_2)) \]

and

\[ \delta(f \otimes_{H'} a) \ast (g \otimes_{H'} b) = (f_1 \otimes_{H'} a_1) \otimes (f_2 \otimes_{H'} a_2) \ast (g \otimes_{H'} b) \]

\[ = \sum_i (f_2 m_i \otimes g n_i) \otimes_{H} ((f_1 \otimes_{H'} a_1) \otimes (1 \otimes_{H'} r_i)) \]

\[ = \sum_i (f_2 m_i \otimes g n_i) \otimes_{H} ((1 \otimes_{H'} a_1) \otimes (1 \otimes_{H'} r_i)). \]

Since \((A, \ast, \Delta)\) is an infinitesimal \(H'-\)pseudobialgebra, we have

\[ \Delta(a \ast b) = \Delta(a) \ast b + a \ast \Delta(b), \]

that is,

\[ \sum_i (f_i \otimes g_i) \otimes_{H'} (e_{i1} \otimes e_{i2}) = \sum_i (l_i \otimes k_i) \otimes_{H'} (t_i \otimes b_2) + \sum_i (m_i \otimes n_i) \otimes_{H'} (a_1 \otimes r_i), \]

which implies that \(\delta((f \otimes_{H'} a) \ast (g \otimes_{H'} b)) = (f \otimes_{H'} a) \ast \delta(g \otimes_{H'} b) + \delta(f \otimes_{H'} a) \ast (g \otimes_{H'} b).\) This completes the proof.

**Remark 3.8** More generally, let \(\phi : H' \rightarrow H\) be a homomorphism of Hopf algebras, and \((A, \ast, \Delta)\) be an infinitesimal \(H'-\)pseudobialgebra. Then \((H \otimes_{H'} A, \ast, \delta)\) is an infinitesimal \(H\)-pseudobialgebra with the strucutres:

\[ (f \otimes_{H'} a) \ast (g \otimes_{H'} b) = \sum_i (f \phi(f_i) \otimes g \phi(g_i)) \otimes_{H} (1 \otimes_{H'} e_i), \]

\[ \delta(f \otimes_{H'} a) = (f_1 \otimes_{H'} a_1) \otimes (f_2 \otimes_{H'} a_2), \]

for all \(f, g \in H\) and \(a, b \in A.\)
Corollary 3.9 Let \((A, \mu, \Delta)\) be an infinitesimal bialgebra. Then \((\text{Cur}(A) = H \otimes A, \ast, \delta)\) is an infinitesimal \(H\)-pseudobialgebra with the following structures:

\[
(f \otimes a) \ast (g \otimes b) = (f \otimes g) \otimes_H (1 \otimes ab),
\]

\[
\delta(f \otimes a) = (f_1 \otimes a_1) \otimes (f_2 \otimes a_2),
\]

for all \(f, g \in H\) and \(a, b \in A\).

**Proof.** It can be obtained directly by taking \(H' = k\) in Proposition 3.7. \(\square\)

4. COBOUNDARY INFINITESIMAL \(H\)-PSEUDOBIALGEBRAS

In this section, we study an important subclass of infinitesimal \(H\)-pseudobialgebra, for which the coalgebra structure comes from a 1-coboundary in associative \(H\)-pseudoalgebra cohomology. First we introduce the cohomology of associative \(H\)-pseudoalgebras.

Let \(A\) be an associative \(H\)-pseudoalgebra and \(M\) an \(A\)-bimodule. Define \(C^n(A, M)(n \geq 1)\), consisting of all cochains

\[
\gamma \in \text{Hom}_{H^{\otimes n}}(A^{\otimes n}, H^{\otimes n} \otimes_H M).
\]

Explicitly, \(\gamma\) has the following defining property: \(H\)-polylinearity,

\[
\gamma(h_1a_1 \otimes \cdots \otimes h_na_n) = ((h_1 \otimes \cdots \otimes h_n) \otimes_H 1)\gamma(a_1 \otimes \cdots \otimes a_n)
\]

where \(h_i \in H\) and \(a_i \in A\) for \(i = 1, 2, \ldots, n\).

For \(n = 0\), we put \(C^0(A, M) = k \otimes_H M \simeq M/H_M\), where \(H_M = \{h \in H|\varepsilon(h) = 0\}\) is the augmentation ideal of \(H\). The differential \(d : C^0(A, M) \longrightarrow C^1(A, M) = \text{Hom}_H(A, M)\) is given by

\[
(d(1 \otimes_H m))(a) = \sum_i (1 \otimes \varepsilon)(h_i)\, m_i - \sum_j (\varepsilon \otimes 1)(f_j)\, n_j, \forall a \in A, m \in M,
\]

if \(a \ast m = \sum_i h_i \otimes_H m_i \in H^{\otimes 2} \otimes_H M\) and \(m \ast a = \sum_j f_j \otimes_H n_j \in H^{\otimes 2} \otimes_H M\).

For \(n \geq 1\), the differential \(d : C^n(A, M) \longrightarrow C^{n+1}(A, M)\) is given by

\[
d\gamma(a_1 \otimes \cdots \otimes a_{n+1}) = \rho_l(a_1 \otimes \gamma(a_2 \otimes \cdots \otimes a_{n+1}))
\]

\[
+ \sum_{i=1}^n (-1)^i \gamma(a_1 \otimes \cdots \otimes a_{i-1} \otimes \mu(a_i \otimes a_{i+1}) \otimes a_{i+2} \otimes \cdots \otimes a_{n+1})
\]

\[
+ (-1)^{n+1} \rho_r(\gamma(a_1 \otimes \cdots \otimes a_n) \otimes a_{n+1}). \tag{4.1}
\]

We also use the following convention in the above equation. If \(a \ast m = \sum_i h_i \otimes_H m_i \in H^{\otimes 2} \otimes_H M\), \(m \ast a = \sum_j f_j \otimes_H n_j \in H^{\otimes 2} \otimes_H M\) for all \(a \in A\) and \(m \in M\), then for any \(g \in H^{\otimes n}\), we set

\[
a \ast (g \otimes m) = \sum_i (1 \otimes g\Delta^{(n-1)})(h_i) \otimes_H m_i \in H^{\otimes n+1} \otimes_H M
\]

and

\[
(g \otimes m) \ast a = \sum_j (g\Delta^{(n-1)} \otimes 1)(f_j) \otimes_H n_j \in H^{\otimes n+1} \otimes_H M,
\]

where \(\Delta^{(n-1)} = (id \otimes \cdots \otimes id \otimes \Delta) \cdots (id \otimes \Delta) : H \longrightarrow H^{\otimes n}\) is the iterated comultiplication \((\Delta^{(0)} = id)\). Note that equation (4.1) holds also for \(n = 0\) if we define \(\Delta^{(-1)} = \varepsilon\).
Equation (4.1) is illustrated in Figure 1.

\[
\begin{array}{c}
1 \ldots n+1 \\
\downarrow \, A \\
M
\end{array}
\quad =
\begin{array}{c}
1 \ldots n+1 \\
\downarrow \, A \\
\downarrow \, r \\
M
\end{array}
\quad + \sum_{j \neq i} (-1)^{n+1-i} \\
\begin{array}{c}
1 \ldots n+1 \\
\downarrow \, A \\
\downarrow \, r \\
M
\end{array}
\quad + (-1)^{n+1} \\
\begin{array}{c}
1 \ldots n+1 \\
\downarrow \, A \\
\downarrow \, r \\
M
\end{array}
\]

**Figure 1.** The definition of differential

One can verify that \( d^2 = 0 \) by using the same argument as in the usual associative algebra case. \( \gamma \in C^n(A, M) \) is called an \( n \)-cocycle if \( d\gamma = 0 \). If \( \gamma = d\alpha \) for some \( \alpha \in C^{n-1}(A, M) \), then \( \gamma \) is called an \( n \)-coboundary. The cohomology of the resulting complex \( C^*(A, M) \) is called the reduced cohomology of \( A \) with coefficient in \( M \) and is denoted by \( H^*(A, M) \).

**Remark 4.1** For \( n = 1 \), \( \gamma \in C^1(A, M) \) is a 1-cocycle if \( \gamma(a_1 * a_2) = a_1 * \gamma(a_2) + \gamma(a_1) * a_2 \). Now if \( A \) is an infinitesimal \( H \)-pseudobialgebra with comultiplication \( \Delta \), then the compatible condition (3.6) is indeed the condition that \( \Delta \) is a 1-cocycle of \( A \) with coefficient in \( A \otimes A \) in the reduced complex.

Among all the 1-cocycles of \( A \) with values in \( A \otimes A \), we have 1-coboundary \( \Delta_r \) that comes from the differential of an element \( r \in A \otimes A \), that is \( \Delta_r(a) = (d(1 \otimes_H r))(a) \) for all \( a \in A \). More precisely, suppose \( r = \sum_i u_i \otimes v_i \in A \otimes A \), then

\[
\Delta_r(a) = d(1 \otimes_H r)(a) = h^{a,u_i} \cdot (c_{a,u_i} \otimes v_i) - l^{v_i,a} \cdot (u_i \otimes e_{v_i,a}),
\]

where \( a * u_i = h^{a,u_i} \otimes 1 \otimes_H c_{a,u_i}, v_i * a = 1 \otimes l^{v_i,a} \otimes_H e_{v_i,a} \).

Now, we introduce the subclass of infinitesimal \( H \)-pseudobialgebras.

**Definition 4.2** A coboundary infinitesimal \( H \)-pseudobialgebra is a quadruple \((A, *, \Delta_r, r)\), with \( r \in A \otimes A \), such that \((A, *, \Delta_r)\) is an infinitesimal \( H \)-pseudobialgebra.

**Example 4.3** Let \( A = H\{e_1, e_2\} \) be a free associative \( H \)-pseudobialgebra with pseudoproduct given by

\[
e_1 * e_1 = e_1 * e_2 = 0, \quad e_2 * e_1 = 1 \otimes 1 \otimes_H e_1, \quad e_2 * e_2 = 1 \otimes 1 \otimes_H e_2.
\]

Define \( r = e_2 \otimes e_1 - e_1 \otimes e_2 \in A \otimes A \). Then by straightforward computations, \( \delta_r(e_1) = e_1 \otimes e_1, \delta_r(e_2) = e_2 \otimes e_1 \), and \((A, *, r)\) is a coboundary infinitesimal \( H \)-pseudobialgebra.

Let \((A, *)\) be an associative \( H \)-pseudobialgebra and \( r = \sum_i u_i \otimes v_i \in A \otimes A \). We define the associative pseudo-Yang-Baxter equation (pseudo-AYBE) on \( A \) as

\[
A(r) = \mu_{-1}^1(u_i \otimes u_j \otimes v_i) - \mu_2^3(u_i \otimes \{v_i, u_j\} \otimes v_j) + \mu_3^1(u_i \otimes u_j \otimes \{v_j, v_i\}).
\]
in $A \otimes A \otimes A$, where $\mu^l_k$ means that the element of $H$ that appears in its argument in the $k$-th place acts via the antipode on the element of $A$ located in the $l$-th entry, $\mu^r_s$ means that the element of $H$ located in the $k$-th place in its argument acts on the elements of $A \otimes A$ formed by the elements in the $r$-th and $s$-th places. For example, $\mu^4_{-1}(h \otimes a \otimes b \otimes c) = a \otimes b \otimes S(h)c$, $\mu^4_3(a \otimes b \otimes h \otimes c) = h_1 a \otimes b \otimes h_2 c$ for all $h \in H$ and $a, b, c \in A$.

We say that $r$ is a solution of the pseudo-AYBE if $A(r) = 0$. Moreover, the pseudo-AYBE is exactly the usual AYBE when $H = k$.

In a coboundary infinitesimal $H$-pseudobialgebra, the comultiplication $\Delta_r$ is determined by $r$. If $(A, \ast)$ is an associative $H$-pseudoalgebra and $r \in A \otimes A$. We discuss what conditions $(A, \ast, \Delta_r, r)$ is a coboundary infinitesimal $H$-pseudobialgebra in the following.

**Theorem 4.4** Let $(A, \ast)$ be an associative $H$-pseudoalgebra and $r = \sum_{i} u_i \otimes v_i \in A \otimes A$. Then $(A, \ast, \Delta_r, r)$ is a coboundary infinitesimal $H$-pseudobialgebra if and only if

$$
\mu_3(a \ast A(r) - A(r) \ast a) = 0, \quad (4.4)
$$

where $\mu_3(h \otimes a \otimes b \otimes c) = ((\Delta \otimes id)\Delta(h))(a \otimes b \otimes c)$ for all $h \in H$ and $a, b, c \in A$.

**Proof.** By the definition of $\Delta_r$, we have that $\Delta_r$ satisfies the compatible condition (3.6) regardless of whether condition (4.4) holds or not. So we only need to prove that $\Delta_r$ is coassociativity if and only if $\mu_3(a \ast A(r) - A(r) \ast a) = 0$.

Suppose $a \ast u_i = h^{a,u_i} \otimes 1 \otimes e_{v_i,a}$. Then $\Delta_r(a) = h^{a,u_i} \cdot (c_{a,u_i} \otimes v_i) - l^{v_i,a} \cdot (u_i \otimes e_{v_i,a})$. On the one hand, we compute:

\[
\begin{align*}
& (\Delta_r \otimes id)\Delta_r(a) - (id \otimes \Delta_r)\Delta_r(a) \\
=& (\Delta_r \otimes id)(h^{a,u_i} \cdot (c_{a,u_i} \otimes v_i)) - l^{v_i,a} \cdot (u_i \otimes e_{v_i,a})) \\
& - (id \otimes \Delta_r)(h^{a,u_i} \cdot (c_{a,u_i} \otimes v_i)) - l^{v_i,a} \cdot (u_i \otimes e_{v_i,a})) \\
= & h^{a,u_i} \cdot \Delta_r(c_{a,u_i}) \otimes h^{a,u_i} \cdot \Delta_r(v_i) - l^{v_i,a} \cdot \Delta_r(u_i) \otimes l^{v_i,a} \cdot \Delta_r(e_{v_i,a}) \\
& - h^{a,u_i} \cdot \Delta_r(c_{a,u_i}) \otimes l^{v_i,a} \cdot \Delta_r(e_{v_i,a}) + l^{v_i,a} \cdot \Delta_r(u_i) \otimes l^{v_i,a} \cdot \Delta_r(e_{v_i,a}) \\
= & h^{a,u_i} \cdot (c_{a,u_i} \otimes v_i) - l^{v_i,a} \cdot (u_i \otimes e_{v_i,a}) \\
& - h^{a,u_i} \cdot (c_{a,u_i} \otimes v_i) - l^{v_i,a} \cdot (u_i \otimes e_{v_i,a}) \\
& + l^{v_i,a} \cdot (u_i \otimes e_{v_i,a}) \\
& - h^{a,u_i} \cdot (c_{a,u_i} \otimes v_i) - l^{v_i,a} \cdot (u_i \otimes e_{v_i,a}) \\
& + l^{v_i,a} \cdot (u_i \otimes e_{v_i,a}) \\
& - l^{v_i,a} \cdot (u_i \otimes e_{v_i,a}) \\
& (c_{v_i,a,u_i} \otimes v_j) - l^{v_j,a} \cdot (u_j \otimes e_{v_j,e_{v_i,a}})) \\
= & h^{a,u_i} \cdot (c_{a,u_i} \otimes v_i) - l^{v_i,a} \cdot (u_i \otimes e_{v_i,a}) \\
& - l^{v_i,a} \cdot (u_i \otimes e_{v_i,a}) \\
& l^{v_i,a} \cdot (u_i \otimes e_{v_i,a}) \\
& (c_{v_i,a,u_i} \otimes v_j) - l^{v_j,a} \cdot (u_j \otimes e_{v_j,e_{v_i,a}}))(4.5) \\
& h^{a,u_i} \cdot (c_{a,u_i} \otimes v_i) - l^{v_i,a} \cdot (u_i \otimes e_{v_i,a}) \\
& l^{v_i,a} \cdot (u_i \otimes e_{v_i,a}) \\
& (c_{v_i,a,u_i} \otimes v_j) - l^{v_j,a} \cdot (u_j \otimes e_{v_j,e_{v_i,a}}))(4.6) \\
& h^{a,u_i} \cdot (c_{a,u_i} \otimes v_i) - l^{v_i,a} \cdot (u_i \otimes e_{v_i,a}) \\
& l^{v_i,a} \cdot (u_i \otimes e_{v_i,a}) \\
& (c_{v_i,a,u_i} \otimes v_j) - l^{v_j,a} \cdot (u_j \otimes e_{v_j,e_{v_i,a}}))(4.7) \\
& h^{a,u_i} \cdot (c_{a,u_i} \otimes v_i) - l^{v_i,a} \cdot (u_i \otimes e_{v_i,a}) \\
& l^{v_i,a} \cdot (u_i \otimes e_{v_i,a}) \\
& (c_{v_i,a,u_i} \otimes v_j) - l^{v_j,a} \cdot (u_j \otimes e_{v_j,e_{v_i,a}}))(4.8) \\
& h^{a,u_i} \cdot (c_{a,u_i} \otimes v_i) - l^{v_i,a} \cdot (u_i \otimes e_{v_i,a}) \\
& l^{v_i,a} \cdot (u_i \otimes e_{v_i,a}) \\
& (c_{v_i,a,u_i} \otimes v_j) - l^{v_j,a} \cdot (u_j \otimes e_{v_j,e_{v_i,a}}))(4.9) \\
& h^{a,u_i} \cdot (c_{a,u_i} \otimes v_i) - l^{v_i,a} \cdot (u_i \otimes e_{v_i,a}) \\
& l^{v_i,a} \cdot (u_i \otimes e_{v_i,a}) \\
& (c_{v_i,a,u_i} \otimes v_j) - l^{v_j,a} \cdot (u_j \otimes e_{v_j,e_{v_i,a}}))(4.10) \\
& h^{a,u_i} \cdot (c_{a,u_i} \otimes v_i) - l^{v_i,a} \cdot (u_i \otimes e_{v_i,a}) \\
& l^{v_i,a} \cdot (u_i \otimes e_{v_i,a}) \\
& (c_{v_i,a,u_i} \otimes v_j) - l^{v_j,a} \cdot (u_j \otimes e_{v_j,e_{v_i,a}}))(4.11) \\
& h^{a,u_i} \cdot (c_{a,u_i} \otimes v_i) - l^{v_i,a} \cdot (u_i \otimes e_{v_i,a}) \\
& l^{v_i,a} \cdot (u_i \otimes e_{v_i,a}) \\
& (c_{v_i,a,u_i} \otimes v_j) - l^{v_j,a} \cdot (u_j \otimes e_{v_j,e_{v_i,a}}))(4.12)
\end{align*}
\]
On the other hand, we have

\[ \mu_3(a \cdot A(r) - A(r) \cdot a) \]

\[ \begin{align*}
&= \mu_3(a \cdot (\mu_1^1(\{u_i, u_j\} \otimes v_j \otimes v_i))) \\
&\quad - \mu_3(a \cdot (\mu_2^3(\{u_i \otimes \{v_i, u_j\} \otimes v_j))) \\
&\quad + \mu_3(a \cdot (\mu_3^1(\{u_i \otimes \{v_i, u_j\} \otimes v_j})) \\
&\quad - \mu_3((\mu_4^1(\{u_i \otimes \{v_i, u_j\} \otimes v_j})) \cdot a) \\
&\quad + \mu_3((\mu_5^4(\{v_i, u_j\} \otimes v_j)) \cdot a) \\
&\quad - \mu_3((\mu_6^4(\{u_i, u_j \otimes \{v_i, v_j\}))) \cdot a).
\end{align*} \]  

(4.13)

(4.14)

(4.15)

(4.16)

(4.17)

(4.18)

Next, we only need to verify that

\[ (\Delta_r \otimes id)\Delta_r(a) - (id \otimes \Delta_r)\Delta_r(a) - \mu_3(a \cdot A(r) - A(r) \cdot a) = 0. \]  

(4.19)

Using the property of Fourier transform, we obtain

\[ \{\{a, u_i\}, u_j\} = (\mathcal{F} \otimes id)\{\{a, u_i, u_j\}\} \]

\[ = (\mathcal{F} \otimes id)(h_{a,c_{u_i,u_j}} \otimes h_{u_i,u_j} \otimes H c_{a,c_{u_i,u_j}}) \]

\[ = h_{a,c_{u_i,u_j}} S(h_{u_i,u_j}^{1}) \otimes h_{u_i,u_j}^{2} \otimes H c_{a,c_{u_i,u_j}}. \]

So we have

\[ h_{a,u_i} \cdot (h_{c_{a,u_i,u_j}} \cdot (c_{a,c_{u_i,u_j}} \otimes v_j) \otimes v_i) \]

\[ = \mu_3(\mu_2^4(\{\{a, u_i\}, u_j\} \otimes v_j \otimes v_i)) \]

\[ = \mu_3(\mu_2^4((\mathcal{F} \otimes id)\{\{a, u_i, u_j\}\} \otimes v_j \otimes v_i)) \]

\[ = (h_{a,c_{u_i,u_j}} S(h_{u_i,u_j}^{1})) \cdot (h_{u_i,u_j}^{2} \cdot (c_{a,c_{u_i,u_j}} \otimes v_j) \otimes v_i) \]

\[ = h_{a,c_{u_i,u_j}} \cdot (c_{a,c_{u_i,u_j}} \otimes v_j \otimes S(h_{u_i,u_j})v_i) \]

\[ = \mu_3(a \cdot (\mu_1^1(\{u_i, u_j\} \otimes v_j \otimes v_i))). \]

Hence (4.5) \(-\) (4.13) = 0. Since

\[ S(h_{u_i,u_j})v_i \ast a = (S(h_{u_i,u_j}) \otimes 1)(1 \otimes l^{v_i,a}) \otimes H e_{v_i,a} \]

\[ = S(h_{u_i,u_j}) \otimes l^{v_i,a} \otimes H e_{v_i,a} \]

\[ = 1 \otimes l^{v_i,a} h_{u_i,u_j}^{1} \otimes H S(h_{u_i,u_j})e_{v_i,a}, \]

we have

\[ l^{v_i,a} \cdot (h_{u_i,u_j}^{1} \cdot (c_{a,u_j} \otimes v_j) \otimes e_{v_i,a}) \]

\[ = (l^{v_i,a} h_{u_i,u_j}^{1}) \cdot (c_{a,u_j} \otimes v_j \otimes S(h_{u_i,u_j})e_{v_i,a}) \]

\[ = \mu_3((c_{u_i,a_j} \otimes v_j \otimes S(h_{u_i,u_j})v_i) \cdot a) \]

\[ = \mu_3((\mu_4^1(\{u_i, u_j\} \otimes v_j \otimes v_i)) \cdot a). \]

It follows that (4.7) \(-\) (4.16) = 0. Similarly, we get

\[ l^{v_i,a} \cdot (l^{v_j,u_i} \cdot (u_j \otimes e_{v_j,u_i}) \otimes e_{v_i,a}) \]

\[ = (l^{v_i,a} l^{v_j,u_i}) \cdot (u_j \otimes e_{v_j,u_i} \otimes S(l^{v_j,u_i})e_{v_i,a}) \]

\[ = \mu_3((\mu_5^4(\{u_i, u_j \otimes \{v_j, u_i\}) \otimes \{v_j, u_i\}))) \cdot a). \]

\[ = \mu_3((\mu_6^4(\{u_i, u_j \otimes \{v_j, u_i\}) \otimes \{v_j, u_i\}) \cdot a). \]
Using the associativity of $\varepsilon$ 

Applying $*\otimes \mu$ 

Then we have 

So we have $(4.9) - (4.14) = 0$. Using the associativity of $A$, we have $(v_i * a) * u_j = v_i * (a * u_j)$. Suppose 

Then we have 

Applying $\varepsilon \otimes id \otimes S \otimes id$ to the above equation, we obtain 

Thus we get that $(4.6) + (4.11)$ is 

Using the associativity of $A$ again, we have $v_j * (v_i * a) = (v_j * v_i) * a$. Suppose 


Then we obtain
\[
1 \otimes l_1^{v_j,e_{v_i,a}} \otimes l_1^{v_i,a} l_2^{v_j,e_{v_i,a}} \otimes_H e_{v_j,e_{v_i,a}} \\
= 1 \otimes l_1^{v_j,v_i} \otimes l_1^{e_{v_j,v_i,a}} \otimes_H e_{v_j,v_i,a}.
\]

It follows that
\[
l_1^{v_i,a} \cdot (u_i \otimes l_1^{v_j,v_i}) = (l_1^{v_i,a})_1 \cdot (S(l_1^{v_j,v_i})u_i \otimes u_j \otimes e_{v_j,v_i,a})
\]
\[
= l_1^{v_j,v_i,a} \cdot (S(l_1^{v_j,v_i})u_i \otimes u_j \otimes e_{v_j,v_i,a})
\]
\[
= \mu_3((S(l_1^{v_j,v_i})u_i \otimes u_j \otimes e_{v_j,v_i}) \ast a)
\]
\[
= \mu_3((\mu_3^A(u_i \otimes u_j \otimes \{v_j,v_i\}) \ast a).
\]

So (4.12) – (4.18) = 0. Finally, it is easy to check that we have canceled all the terms of the
left-hand side of equation (4.19). This completes the proof. \(\square\)

5. FROM (COBOUNDARY) INFINITESIMAL H-PSEUDOBIALGEBRAS TO (COBOUNDARY) Lie
H-PSEUDOBIALGEBRAS

We first recall some definitions about Lie H-pseudobialgebras (see [9]).

Definition 5.1 A Lie H-pseudoalgebra \((L, [\ast])\) is a left \(H\)-module \(L\) endowed with a map
(called the pseudobracket):
\[
[\ast]: \quad L \otimes L \rightarrow (H \otimes H) \otimes_H L, \quad a \otimes b \mapsto [a \ast b]
\]
satisfying (\(\forall a, b, c \in L, f, g \in H\))

- **H-bilinearity:** \([fa \ast gb] = (f \otimes g) \otimes_H 1[a \ast b] \).
- **Skew-commutativity:** \([a \ast b] = -(\sigma \otimes id)[b \ast a] \).
- **Jacobi identity:** \([a \ast [b \ast c]] = [a \ast [b \ast c]] - ((\sigma \otimes id) \otimes_H id)[b \ast [a \ast c]] \).

Definition 5.2 A Lie H-coalgebra is a left \(H\)-module, endowed with an \(H\)-linear map \(\delta : L \rightarrow L \otimes L\) such that
\[
\tau \circ \delta = -\delta,
\]
\[
(id \otimes \delta)\delta - \sigma_{12}(id \otimes \delta)\delta = (\delta \otimes id)\delta,
\]
where \(\tau : L \otimes L \rightarrow L \otimes L\) is the permutation \(\tau(a \otimes b) = b \otimes a\), \(\forall a, b \in L\).

Definition 5.3 A Lie H-pseudobialgebra is a triple \((L, [\ast], \delta)\) such that \((L, [\ast])\) is a Lie H-pseudoalgebra, \((L, \delta)\) is a Lie H-coalgebra, and they satisfy the cocycle condition
\[
\delta([a \ast b]) = [a \ast \delta(b)] - (\sigma \otimes_H id)[b \ast \delta(a)], \quad (5.1)
\]
where \([a \ast \delta(b)] = \sum_i (f_i S(g_{i1}) \otimes 1) \otimes_H (g_{i2} a_i \otimes b_2) + \sum_j (k_j S(l_{j1}) \otimes 1) \otimes_H (b_1 \otimes l_{j2} b_j)\), if
\[
[a \ast b_1] = \sum_i (f_i \otimes g_i) \otimes_H a_i = \sum_i (f_i S(g_{i1}) \otimes 1) \otimes_H g_{i2} a_i
\]
and

\[ [a \ast b]_2 = \sum_j (k_j \otimes l_j) \otimes_H b_j = \sum_j (k_j S(l_{j1}) \otimes 1) \otimes_H l_{j2} b_j. \]

**Definition 5.4** A coboundary Lie \( H \)-pseudobialgebra \( (L,[*],\delta,\tau) \) consists of a Lie \( H \)-pseudobialgebra \( (L,[*],\delta) \) and an element \( r = \sum_i u_i \otimes v_i \in L \otimes L \) such that

\[ \delta(a) = \sum_i \mu([a,u_i] \otimes v_i + \sigma_12(u_i \otimes [a,v_i])), \]

(5.2)

where \( \mu : H \otimes (L \otimes L) \rightarrow L \otimes L \) is given by \( \mu(h \otimes m \otimes n) = \Delta(h)(m \otimes n) \), and \([a,b]\) is the Fourier transform of \([a \ast b]\).

To state the main result of this section, we need the following notation.

Let \( A \) be an infinitesimal \( H \)-pseudobialgebra. Define the \(*\)-bracket

\[ [a,b]_\ast = a \ast b - (\sigma \otimes id)(b \ast a), \]

for all \( a \in A \) and \( b = b_1 \otimes \cdots b_n \in A^{\otimes n} \).

**Definition 5.5** Let \((A,\ast,\Delta)\) be an infinitesimal \( H \)-pseudobialgebra. Define the map \( \mathcal{B} : A \otimes A \rightarrow (H \otimes H) \otimes_H (A \otimes A) \) by

\[ \mathcal{B}(a,b) = [a,\Delta^{op}(b)]_\ast + (\sigma \otimes \tau)([b,\Delta^{op}(a)]_\ast) \]

where \( \tau(a \otimes b) = b \otimes a \), for all \( a,b \in A \). We call \( \mathcal{B} \) the \( H \)-balanceator of \( A \). More precisely, we have

\[ \mathcal{B}(a,b) = f^{a,b_2} \otimes g^{a,b_2} \otimes_H (t_{a,b_2} \otimes b_1) - g^{b_2,a} \otimes f^{b_2,a} \otimes_H (b_2 \otimes t_{b_1,a}) \]

\[ + g^{b,a_2} \otimes f^{b,a_2} \otimes_H (a_1 \otimes t_{b,a_2}) - f^{a_1,b} \otimes g^{a_1,b} \otimes_H (t_{a_1,b} \otimes a_2), \]

if \( a \ast b = f^{a,b} \otimes g^{a,b} \otimes_H t_{a,b} \) for all \( a,b \in A \).

The \( H \)-balanceator \( \mathcal{B} \) is said to be symmetric if \( \mathcal{B}(a,b) = (\sigma \otimes_H id)\mathcal{B}(b,a) \) for all \( a,b \in A \).

**5.1 From infinitesimal \( H \)-pseudobialgebras to Lie \( H \)-pseudobialgebras.**

Recall that, if \((A,\ast)\) is an associative \( H \)-pseudobialgebra, then \((A,\ast)_{lie}\) is a Lie \( H \)-pseudobialgebra obtained from the associative one, where \([a \ast b]_{lie} = a \ast b - (\sigma \otimes_H id)(b \ast a) \) for all \( a,b \in A \). And if \((A,\Delta)\) is a coassociative \( H \)-coalgebra, then \((A,\delta_{lie} = (id - \tau)\Delta)\) is a Lie \( H \)-coalgebra by using the same argument as in the usual Lie coalgebra case. But, in general, we can not get that \((A,\ast)_{lie},\delta_{lie}\) is a Lie \( H \)-pseudobialgebra due to \((A,\ast,\Delta)\) is an infinitesimal \( H \)-pseudobialgebra. However, we have the following result.

**Theorem 5.6** Let \((A,\ast,\Delta)\) be an infinitesimal \( H \)-pseudobialgebra. Then we have

\[ \delta_{lie}([a \ast b]_{lie}) = [a \ast \delta_{lie}(b)]_{lie} - (\sigma \otimes_H id)[b \ast \delta_{lie}(a)]_{lie} + \mathcal{B}(a,b) \]

\[ - (\sigma \otimes_H id)\mathcal{B}(b,a) \]

(5.3)

for all \( a,b \in A \).

**Proof.** We use the following shorthand: if \( f(x,y) \) is an expression involving \( x \) and \( y \), then

\[ (f(x,y))^\wedge = f(x,y) - (\sigma \otimes id)f(y,x). \]
In particular, the right-hand side of equation (5.3) becomes \([a * \delta_t(b)]_{\text{tie}} + B(a, b)\)\(^\wedge\).

Suppose \(a * b = f^{a,b} \otimes H t_{a,b}\) for all \(a, b \in A\). On the one hand, we have
\[
\delta_{\text{tie}}(a * b) = (id - \tau)(a * b) - (\sigma \otimes id)(b * a)
\]
\[
= (id - \tau)(a * b) - (id - \tau)(\sigma \otimes \text{id})(b * a)
\]
\[
= f^{a,b_1} \otimes g^{a,b_1} \otimes H (t_{a,b_1} \otimes b_2) + f^{a_2,b} \otimes g^{a_2,b} \otimes H (a_1 \otimes t_{a_2,b})
\]
\[
- f^{a,b_1} \otimes g^{a,b_1} \otimes H (b_2 \otimes t_{a,b_1}) - f^{a_2,b} \otimes g^{a_2,b} \otimes H (t_{a_2,b} \otimes a_1)
\]
\[
- g^{b_1,a} \otimes f^{b_1,a} \otimes H (t_{b_1,a} \otimes a_2) - g^{b_2,a} \otimes f^{b_2,a} \otimes H (b_1 \otimes t_{b_2,a})
\]
\[
+ g^{b_1,a} \otimes f^{b_1,a} \otimes H (a_2 \otimes t_{b,a_1}) + g^{b_2,a} \otimes f^{b_2,a} \otimes H (t_{b_2,a} \otimes b_1)
\]
\[
= f^{a,b_1} \otimes g^{a,b_1} \otimes H (t_{a,b_1} \otimes b_2) - g^{b_2,a} \otimes f^{b_2,a} \otimes H (b_1 \otimes t_{b_2,a})
\]
\[
+ g^{b_2,a} \otimes f^{b_2,a} \otimes H (t_{b_2,a} \otimes b_1) - f^{a,b_1} \otimes g^{a,b_1} \otimes H (b_2 \otimes t_{a,b_1})\)\(^\wedge\). (5.4)
\]

On the other hand, observe that \(A \otimes A\) is an \(A\)-module (here we view \(A\) as a Lie \(H\)-pseudoalgebra with the pseudobracket \([*]_{\text{tie}}\) and the action is defined as in Lemma 4.1 in [9]. So we have
\[
[a * \delta_t(b)]_{\text{tie}} = [a * (b_1 \otimes b_2 - b_2 \otimes b_1)]
\]
\[
= [a * b_1] \otimes b_2 + b_1 \otimes [a * b_2] - [a * b_2] \otimes b_1 - b_2 \otimes [a * b_1]
\]
\[
= a \otimes b_1 \otimes b_2 - (\sigma \otimes \text{id})(b_1 \otimes a) \otimes b_2) + b_1 \otimes (a \otimes b_2) - b_1 \otimes (\sigma \otimes \text{id})(b_2 \otimes a)
\]
\[
- b_2 \otimes b_1 + (\sigma \otimes \text{id})(b_2 \otimes a) \otimes b_1) - b_2 \otimes a \otimes b_1 + b_2 \otimes (\sigma \otimes \text{id})(b_1 \otimes a)
\]
\[
= f^{a,b_1} \otimes g^{a,b_1} \otimes H (t_{a,b_1} \otimes b_2) - g^{b_1,a} \otimes f^{b_1,a} \otimes H (b_1 \otimes t_{b_1,a}) + f^{a,b_2} \otimes g^{a,b_2} \otimes H (b_1 \otimes t_{b_2,a})
\]
\[
- g^{b_2,a} \otimes f^{b_2,a} \otimes H (b_1 \otimes t_{b_2,a}) - f^{a,b_2} \otimes g^{a,b_2} \otimes H (t_{b_2,a} \otimes b_1) + g^{b_2,a} \otimes f^{b_2,a} \otimes H (t_{b_2,a} \otimes b_1)
\]
\[
- f^{a,b_1} \otimes g^{a,b_1} \otimes H (b_2 \otimes t_{a,b_1}) + g^{b_1,a} \otimes f^{b_1,a} \otimes H (b_2 \otimes t_{b_1,a})
\]
\[
= f^{a,b_1} \otimes g^{a,b_1} \otimes H (t_{a,b_1} \otimes b_2) - g^{b_2,a} \otimes f^{b_2,a} \otimes H (b_1 \otimes t_{b_2,a}) + g^{b_2,a} \otimes f^{b_2,a} \otimes H (t_{b_2,a} \otimes b_1)
\]
\[
- f^{a,b_1} \otimes g^{a,b_1} \otimes H (b_2 \otimes t_{a,b_1}) + (\sigma \otimes \text{H} \tau)([a, \Delta_{\text{op}}(b)]_*) - [a, \Delta_{\text{op}}(b)]_*
\] (5.5)

Using equations (5.4) and (5.5), we obtain
\[
([a * \delta_t(b)]_{\text{tie}} + B(a, b))\wedge
\]
\[
= (f^{a,b_1} \otimes g^{a,b_1} \otimes H (t_{a,b_1} \otimes b_2) - g^{b_2,a} \otimes f^{b_2,a} \otimes H (b_1 \otimes t_{b_2,a}) + g^{b_2,a} \otimes f^{b_2,a} \otimes H (t_{b_2,a} \otimes b_1)
\]
\[
- f^{a,b_1} \otimes g^{a,b_1} \otimes H (b_2 \otimes t_{a,b_1}) + \sigma \otimes \text{H} \tau)([a, \Delta_{\text{op}}(b)]_*) - [a, \Delta_{\text{op}}(b)]_*
\]
\[
+ [a, \Delta_{\text{op}}(b)]_* + (\sigma \otimes \text{H} \tau)([b, \Delta_{\text{op}}(a)]_*)\wedge
\]
\[
= (f^{a,b_1} \otimes g^{a,b_1} \otimes H (t_{a,b_1} \otimes b_2) - g^{b_2,a} \otimes f^{b_2,a} \otimes H (b_1 \otimes t_{b_2,a}) + g^{b_2,a} \otimes f^{b_2,a} \otimes H (t_{b_2,a} \otimes b_1)
\]
\[
- f^{a,b_1} \otimes g^{a,b_1} \otimes H (b_2 \otimes t_{a,b_1}) + \sigma \otimes \text{H} \tau)([a, \Delta_{\text{op}}(b)]_*) - (\sigma \otimes \text{H} \tau)([b, \Delta_{\text{op}}(a)]_*)
\]
\[
+ (\sigma \otimes \text{H} \tau)([b, \Delta_{\text{op}}(a)]_*)\wedge
\]
\[
= \delta_{\text{tie}}(a * b)_{\text{tie}}
\]

which completes the proof. \(\square\)

The following result can be obtained directly by Theorem 5.6.
Corollary 5.7 Let \((A, \ast, \Delta)\) be an infinitesimal \(H\)-pseudobialgebra. Then \((A, [\ast]_{\text{lie}}, \delta_{\text{lie}})\) is a Lie \(H\)-pseudobialgebra if and only if the \(H\)-balanceator \(\mathfrak{B}\) is symmetric.

Next, we discuss the construction of infinitesimal \(H\)-pseudobialgebras whose \(H\)-balanceators are symmetric, then we can get related Lie \(H\)-pseudobialgebras by Corollary 5.7.

Proposition 5.8 Let \((A, \ast, \Delta)\) be an infinitesimal \(H\)-pseudobialgebra that is both commutative and cocommutative. Then its \(H\)-balanceator \(\mathfrak{B} = 0\).

Proof. For all \(a, b \in A\), we denote \(a \ast b = (f^{a,b} \otimes g^{a,b}) \otimes_H t_{a,b}\). Using the commutativity and cocommutativity of \(A\), we compute:

\[
\begin{align*}
& f^{a_1,b} \otimes g^{a_1,b} \otimes_H (t_{a_1,b} \otimes a_2) + g^{b_1,a} \otimes f^{b_1,a} \otimes_H (b_2 \otimes t_{b_1,a}) \\
& = g^{b_1,a} \otimes f^{b_1,a} \otimes_H (t_{b_1,a} \otimes a_2) + g^{b_2,a} \otimes f^{b_2,a} \otimes_H (b_1 \otimes t_{b_2,a}) \\
& = (\sigma \otimes \text{id}) \Delta(a \ast b) = \Delta(a \ast b) \\
& = f^{a_2,b} \otimes g^{a_2,b} \otimes_H (a_1 \otimes t_{a_2,b}) + f^{a_1,b} \otimes g^{a_1,b} \otimes_H (t_{a_1,b} \otimes b_2) \\
& = g^{b_2,a} \otimes f^{b_2,a} \otimes_H (a_1 \otimes t_{b,a_2}) + f^{a_1,b} \otimes g^{a_1,b} \otimes_H (t_{a_1,b} \otimes a_2) \\
& = 0,
\end{align*}
\]

so we have

\[
\mathfrak{B}(a, b) = f^{a_1,b} \otimes g^{a_2,b} \otimes_H (t_{a_1,b} \otimes b_1) - g^{b_1,a} \otimes f^{b_1,a} \otimes_H (b_2 \otimes t_{b_1,a}) \\
+ f^{a_2,b} \otimes g^{a_1,b} \otimes_H (t_{a_1,b} \otimes b_2) - f^{a_1,b} \otimes g^{a_1,b} \otimes_H (t_{a_1,b} \otimes a_2) \\
= 0,
\]
as required. \(\square\)

5.2 From coboundary infinitesimal \(H\)-pseudobialgebras to coboundary Lie \(H\)-pseudobialgebras.

In this subsection, we give a sufficient condition under which a coboundary infinitesimal \(H\)-pseudobialgebra gives rise to a coboundary Lie \(H\)-pseudobialgebra.

Recall that a 2-tensor \(r \in A \otimes A\) is said to be symmetric (resp, anti-symmetric) if \(r = r^{op}\) (resp, \(r = -r^{op}\)), where \(r^{op} = \tau(r)\). We start with a useful proposition in the following.

Proposition 5.9 Let \((A, \ast, \Delta_r, r)\) be a coboundary infinitesimal \(H\)-pseudobialgebra with \(r\) anti-symmetric. Then the \(H\)-balanceator \(\mathfrak{B}\) of \(A\) is 0.

Proof. For all \(b \in A\), we have

\[
\Delta_r(b) = h^{b,u_i} \cdot (c_{b,u_i} \otimes v_i) - l^{v_i,b} \cdot (u_i \otimes c_{b,u_i}),
\]
where \(b \ast u_i = h^{b,u_i} \otimes 1 \otimes_H c_{b,u_i}\) and \(v_i \ast b = 1 \otimes l^{v_i,b} \otimes_H e_{v_i,b}\). It follows that

\[
\begin{align*}
[a, \Delta_r^{op}(b) \ast] & = a \ast \Delta_r^{op}(b) - (\sigma \otimes \text{id})(\Delta_r^{op}(b) \ast a) \\
& = a \ast (h^{b,u_i} \cdot (v_i \otimes c_{b,u_i})) - a \ast (l^{v_i,b} \cdot (e_{v_i,b} \otimes u_i)) - (\sigma \otimes \text{id})(h^{b,u_i} \cdot (v_i \otimes c_{b,u_i}) \ast a) \\
& \quad + (\sigma \otimes \text{id})(l^{v_i,b} \cdot (e_{v_i,b} \otimes u_i) \ast a) \\
& = (1 \otimes h^{b,u_i} \otimes_H 1)(a \ast (v_i \otimes c_{b,u_i})) - (1 \otimes l^{v_i,b} \otimes_H 1)(a \ast (e_{v_i,b} \otimes u_i)) \\
& \quad - (\sigma \otimes \text{id})(h^{b,u_i} \otimes 1 \otimes_H 1)((v_i \otimes c_{b,u_i}) \ast a)) + (\sigma \otimes \text{id})(l^{v_i,b} \otimes 1 \otimes_H 1)((e_{v_i,b} \otimes u_i) \ast a))
\end{align*}
\]
Interchanging the roles of \(A\) and \(B\) in the above equation, we obtain

\[
[b, \Delta^p_r(a)]_* = h^{b,v_i} \otimes h^{a,u_i} \otimes H (c_{a,u_i} \otimes c_{b,u_i}) - h^{a,e_{v_i,b}} \otimes l^{v_i,b} \otimes H (e_{v_i,b} \otimes e_{u_i,a})
\]

Using equations (5.6) and (5.7), we compute:

\[
\mathcal{B}(a, b) = [a, \Delta^p_r(b)]_* + (\sigma \otimes \tau)([b, \Delta^p_r(a)]_*)
\]

Using the associativity of \(A\), we have \(a \ast (u_i \ast b) = (a \ast u_i) \ast b\) and \((b \ast v_i) \ast a = b \ast (v_i \ast a)\), which are equivalent to

\[
h^{a,m_{u_i,b}} \otimes 1 \otimes g^{u_i,b} \otimes H c_{a,m_{u_i,b}} = h^{a,u_i} \otimes 1 \otimes g^{a,u_i,b} \otimes H m_{c_{a,u_i,b}}
\]

and

\[
h^{b,v_i} \otimes 1 \otimes g^{b,c_{u_i,v_i}} \otimes H m_{c_{b,m_{v_i,a}}} = h^{b,m_{v_i,a}} \otimes 1 \otimes g^{b,v_i,a} \otimes H c_{b,m_{v_i,a}}.
\]
that is, on the element of a solution of the Yang-Baxter equation (Theorem 6.1)

\[ f_{a,u} g_{a,u} = g_{a,u} f_{a,u} \]

\[ (f_{a,u} g_{a,u}) \cdot (g_{a,u} f_{a,u}) = (g_{a,u} f_{a,u}) \cdot (f_{a,u} g_{a,u}) \]

\[ + f_{a,u} g_{a,u} = g_{a,u} f_{a,u} \]

\[ (f_{a,u} g_{a,u}) \cdot (g_{a,u} f_{a,u}) = (g_{a,u} f_{a,u}) \cdot (f_{a,u} g_{a,u}) \]

\[ \text{On the other hand, since} \]

\[ [a * u_i] = a * u_i - (\sigma \otimes \text{id})(u_i * a) \]

\[ = h^{a,u_i} \otimes g^{a,u_i} \otimes h_{u_i,a} = g^{u_i,a} \otimes h_{u_i,a} \otimes h_{u_i,a} \]

\[ = h^{a,u_i} S(g_{a,u_i}) \otimes 1 \otimes h_{u_i,a} \]

\[ + f_{a,u} g_{a,u} = g_{a,u} f_{a,u} \]

\[ (f_{a,u} g_{a,u}) \cdot (g_{a,u} f_{a,u}) = (g_{a,u} f_{a,u}) \cdot (f_{a,u} g_{a,u}) \]

\[ = h^{a,u_i} S(g_{a,u_i}) \otimes 1 \otimes h_{u_i,a} \]

we have

\[ [a, u_i] = f_{a,u} g_{a,u} = g_{a,u} f_{a,u} \]

Similarly, we get

\[ [a, v_i] = f_{a,v_i} g_{a,v_i} = g_{a,v_i} f_{a,v_i} \]

Now we compute:

\[ \mu([a, u_i] \otimes v_i + \sigma_{12}(u_i \otimes [a, v_i]) = h^{a,u_i} \otimes g^{a,u_i} \otimes h_{u_i,a} \]

\[ + f_{a,u} g_{a,u} = g_{a,u} f_{a,u} \]

\[ (f_{a,u} g_{a,u}) \cdot (g_{a,u} f_{a,u}) = (g_{a,u} f_{a,u}) \cdot (f_{a,u} g_{a,u}) \]

Combining (5.11) with (5.12), we get \( \delta_{r}(a) = \mu([a, u_i] \otimes v_i + \sigma_{12}(u_i \otimes [a, v_i])) \), as desired.

6. FROM ASSOCIATIVE PSEUDO-YANG-BAXTER EQUATION TO THE CLASSICAL TYPE

Let \((L, [\ast])\) be a Lie \(H\)-pseudoalgebra and \(r = \sum u_i \otimes v_i \in L \otimes L\). Recall that the classical Yang-Baxter equation (pseudo-CYBE) on \(L\) has the form

\[ [[r, r]] = \mu_{-1}([u_j, u_i] \otimes v_j \otimes v_i) - \mu_{-2}^{1}(u_i \otimes [u_j, v_i] \otimes v_j) - \mu_{-3}^{1}(u_i \otimes [u_j \otimes [v_j, v_i]) = 0, \]

where \(\mu_{-k}\) means that the element of \(H\) in the \(k\)-th place in its argument acts via the antipode on the element of \(L\) located in the \(t\)-th place.

**Theorem 6.1** Let \((A, \ast)\) be an associative \(H\)-pseudoalgebra and \(r = \sum u_i \otimes v_i \in A \otimes A\) a solution of the pseudo-AYBE. Suppose that \(r\) is either symmetric or anti-symmetric. Then \(r\) is a solution of the pseudo-CYBE in the Lie \(H\)-pseudoalgebra \(A_{\text{lie}} = (A, [\ast]_{\text{lie}})\) for all \(x, y \in A\).

**Proof.** For all \(a, b \in A\), we write \(a \ast b = (f_{a,b} \otimes g^{a,b}) \otimes H t_{a,b}\). Then \([a \ast b]_{\text{lie}} = a \ast b - (\sigma \otimes H \text{id})(b \ast a) = (f_{a,b} \otimes g^{a,b}) \otimes H t_{a,b} - (g^{b,a} \otimes f^{b,a}) \otimes H t_{b,a}\). Define

\[ A(r') = \mu_{1}^{A}([u_j, u_i] \otimes v_j \otimes v_i) - \mu_{2}^{1,3}(u_i \otimes \{u_j, v_i\} \otimes v_i) + \mu_{3}^{1}(u_i \otimes u_j \otimes \{v_i, v_j\}, \]

that is,

\[ A(r') = f_{u_j, u_i} g_{u_i, u_j} \otimes v_j \otimes u_i \otimes v_j + g_{v_i, v_j} S(g_{u_i, u_j} v_i - f_{u_i, v_j} g_{u_i, v_j} u_j \otimes f_{v_i, v_j} t_{u_i, v_j} \otimes v_i \]

We first check that \(A(r') = \sigma_{13}(A(r))\). Using the (anti-)symmetry of \(r\), we have

\[ \sigma_{13}(A(r)) = \sigma_{13}(\mu_{-1}^{A}([u_i, u_j] \otimes v_j \otimes v_i) - \mu_{3}^{A}([u_i, u_j] \otimes v_j \otimes v_i)\]
Now, we compute:

\[ + \mu_3^{14}(u_i \otimes u_j \otimes \{v_j, v_i\}) \]

\[ = \sigma_{13}(g_{2}^{u_i,u_j} t_{u_i,u_j} \otimes v_j \otimes g_1^{u_i,u_j} S(f_{u_i,u_j}) v_i - u_i \otimes f_1^{v_i,u_j} t_{v_i,u_j} \otimes f_2^{v_i,u_j} S(g_{v_i,u_j}) v_j + f_1^{v_i,u_j} S(g_{v_i,u_j}) u_i \otimes u_j \otimes f_2^{v_i,u_j} v_j,v_i) \]

\[ = g_1^{u_i,u_j} S(f_{u_i,u_j}) v_i \otimes v_j \otimes g_2^{u_i,u_j} t_{u_i,u_j} \otimes v_j - f_2^{v_i,u_j} S(g_{v_i,u_j}) v_j \otimes f_1^{v_i,u_j} t_{v_i,u_j} \otimes u_i + f_2^{v_i,u_j} t_{v_i,u_j} \otimes u_j \otimes f_1^{v_i,u_j} v_j, v_i) \]

\[ = g_1^{v_i,v_j} S(f_{v_i,v_j}) u_i \otimes u_j \otimes g_2^{v_i,v_j} t_{v_i,v_j} \otimes v_i - f_2^{v_i,v_j} S(g_{v_i,v_j}) u_i \otimes u_j \otimes f_1^{v_i,v_j} v_j, v_i) \]

\[ = A(r)^{\prime}. \]

Now, we compute:

\[ [[r, r]] = \mu_{-1}^{3}(u_i \otimes u_j \otimes v_j \otimes v_i) - \mu_{-2}^{3}(u_i \otimes [u_j, v_i] \otimes v_i) - \mu_{-2}^{3}(u_i \otimes [u_j, v_i] \otimes v_i) \]

\[ = \mu_{-1}^{3}(f_{u_i,u_j} S(g_{u_i,u_j}) \otimes g_{2}^{u_i,u_j} t_{u_i,u_j} \otimes v_j \otimes v_i) - g_{2}^{u_i,u_j} t_{u_i,u_j} \otimes v_j \otimes v_i - f_2^{v_i,u_j} S(g_{v_i,u_j}) v_j \otimes f_1^{v_i,u_j} t_{v_i,u_j} \otimes v_j + u_i \otimes u_j \otimes v_i \otimes f_2^{v_i,u_j} S(g_{v_i,u_j}) v_j \]

\[ = g_{2}^{u_i,u_j} t_{u_i,u_j} \otimes v_j \otimes f_1^{v_i,u_j} S(g_{v_i,u_j}) v_j - u_i \otimes g_2^{v_i,u_j} t_{v_i,u_j} \otimes v_j \otimes g_1^{v_i,u_j} S(f_{v_i,u_j}) v_j + u_i \otimes u_j \otimes g_2^{v_i,u_j} S(f_{v_i,u_j}) v_j \]

\[ = g_1^{v_i,v_j} S(f_{v_i,v_j}) u_i \otimes u_j \otimes g_2^{v_i,v_j} t_{v_i,v_j} \otimes v_j - f_2^{v_i,v_j} S(g_{v_i,v_j}) u_i \otimes u_j \otimes f_1^{v_i,v_j} v_j, v_i) \]

\[ = A(r)^{\prime} - A(r) \]

\[ = (\sigma_{13} - id)(A(r)) \]

\[ = 0. \]

So the conclusion holds. \(\square\)

**Example 6.2** Consider the free associative \(H\)-pseudoalgebra \(A = H\{e_1, e_2\}\) in Example 4.3 with pseudoproduct

\[ e_1 \ast e_1 = e_1 \ast e_2 = 0, \quad e_2 \ast e_1 = 1 \otimes 1 \otimes H e_1, \quad e_2 \ast e_2 = 1 \otimes 1 \otimes H e_2. \]

Then \(r = e_2 \otimes e_1 - e_1 \otimes e_2\) is an anti-symmetric solution of pseudo-AYBE. By Theorem 6.1, \(r\) induces a solution of pseudo-CYBE in \(A_{\text{Lie}}\), where the pseudobracket in \(A_{\text{Lie}}\) is determined by \([e_1 \ast e_1]_{\text{Lie}} = [e_2 \ast e_1]_{\text{Lie}} = [e_2 \ast e_2]_{\text{Lie}} = 0, [e_1 \ast e_2]_{\text{Lie}} = -1 \otimes 1 \otimes H e_1.\)

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Department of Science, Henan Institute of Technology, Xinxiang 453003, China
E-mail address: liulinlin2016@163.com

Department of Science, Henan Institute of Technology, Xinxiang 453003, China
E-mail address: guotao60698@163.com