The Construction of Quantum Field Operators: Something of Interest

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Abstract

We draw attention to some tune problems in constructions of the quantum-field operators for spins 1/2 and 1. They are related to the existence of negative-energy and acausal solutions of relativistic wave equations. Particular attention is paid to the chiral theories, and to the method of the Lorentz boosts.
1 The Dirac Equation.

First of all, I would like to remind you some basic things in the quantum field theory.

The Dirac equation has been considered in detail in a pedagogical way \cite{Sakurai, Ryder}:

\[
+i\gamma^\mu \partial_\mu - m] \Psi(x) = 0 .
\]  

At least, 3 methods of its derivation exist:

- the Dirac one (the Hamiltonian should be linear in $\partial/\partial x^\mu$, and be compatible with $E^2 - p^2 c^2 = m^2 c^4$);
- the Sakurai one (based on the equation $(E - \sigma \cdot p)(E + \sigma \cdot p)\phi = m^2 \phi$);
- the Ryder one (the relation between 2-spinors at rest is $\phi_R(0) = \pm \phi_L(0)$).

The $\gamma^\mu$ are the Clifford algebra matrices

\[
\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} .
\]  

Usually, everybody uses the following definition of the field operator \cite{Itsykson}:

\[
\Psi(x) = \frac{1}{(2\pi)^3} \sum_\sigma \int \frac{d^3p}{2E_p} [u_\sigma(p)a_\sigma(p)e^{-ip \cdot x} + v_\sigma(p)b_\sigma^\dagger(p)e^{+ip \cdot x}] ,
\]  

as given \textit{ab initio}.

I studied in the previous works \cite{Dvoeglazov1, Dvoeglazov2, Dvoeglazov3}:

- $\sigma \rightarrow h$ (the helicity basis);
• the modified Sakurai derivation (the additional \(m_2\gamma^5\) term in the Dirac equation);

• the derivation of the Barut equation [Barut] from the first principles, namely based on the generalized Ryder relation, \((\phi_L^h(0) = \hat{A}\phi_L^{-h^*}(0) + \hat{B}\phi_L^{h*}(0))\). In fact, we have the second mass state (\(\mu\)-meson) from that equation:

\[
[i\gamma^\mu\partial_\mu - \alpha\partial_\mu\partial^\mu/m - \beta]\psi = 0; \quad (4)
\]

• the self/anti-self charge-conjugate Majorana 4-spinors [Majorana, Bilenky] in the momentum representation.

The Wigner rules [Wigner] of the Lorentz transformations for the \((0, S)\) left- \(\phi_L(p)\) and the \((S, 0)\) right- \(\phi_R(p)\) spinors are:

\[
\begin{align*}
(S, 0) : & \quad \phi_R(p) = \Lambda_R(p \leftarrow 0) \phi_R(0) = \exp(\mathbf{S} \cdot \varphi) \phi_R(0), \quad (5) \\
(0, S) : & \quad \phi_L(p) = \Lambda_L(p \leftarrow 0) \phi_L(0) = \exp(-\mathbf{S} \cdot \varphi) \phi_L(0), \quad (6)
\end{align*}
\]

with \(\varphi = n\varphi\) being the boost parameters:

\[
cosh(\varphi) = \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}, \quad \sinh(\varphi) = \beta\gamma = \frac{v/c}{\sqrt{1 - v^2/c^2}}; \quad (7)
\]

\[
tanh(\varphi) = \frac{v}{c}. \quad (8)
\]

They are well known and given, e.g., in [Wigner, Faustov, Ryder].

On using the Wigner rules and the Ryder relations we can recover the Dirac equation in the matrix form:

\[
\begin{pmatrix}
\mp m 1 \\
\mp m 1
\end{pmatrix}
\begin{pmatrix}
p_0 + \sigma \cdot p \\
p_0 - \sigma \cdot p
\end{pmatrix}
\psi(p^\mu) = 0,
\]

or \((\gamma \cdot p - m)u(p) = 0\) and \((\gamma \cdot p + m)v(p) = 0\). We have used the property \([\Lambda_{L,R}(p \leftarrow 0)]^{-1} = [\Lambda_{R,L}(p \leftarrow 0)]^\dagger\) above, and that
both $S$ and $\Lambda_{R,L}$ are Hermitian for the finite $(S = 1/2, 0) \oplus (0, S = 1/2)$ representation of the Lorentz group. Introducing $\psi(x) \equiv \psi(p) \exp(\mp ip \cdot x)$ and letting $p_\mu \rightarrow i\partial_\mu$, the above equation becomes the Dirac equation (1).

The solutions of the Dirac equation are denoted by $u(p) = \text{column}(\phi_R(p) \ \phi_L(p))$ and $v(p) = \gamma^5 u(p)$. Let me remind that the boosted 4-spinors in the common-used basis (the standard representation of $\gamma$ matrices) are

$$u_{\frac{1}{2}, \frac{1}{2}} = \sqrt{\frac{(E + m)}{2m}} \begin{pmatrix} 1 \\ 0 \\ p_z/(E + m) \\ p_r/(E + m) \end{pmatrix},$$

$$u_{\frac{1}{2}, -\frac{1}{2}} = \sqrt{\frac{(E + m)}{2m}} \begin{pmatrix} 0 \\ 1 \\ p_l/(E + m) \\ -p_z/(E + m) \end{pmatrix},$$

$$v_{\frac{1}{2}, \frac{1}{2}} = \sqrt{\frac{(E + m)}{2m}} \begin{pmatrix} p_z/(E + m) \\ p_r/(E + m) \\ 1 \\ 0 \end{pmatrix},$$

$$v_{\frac{1}{2}, -\frac{1}{2}} = \sqrt{\frac{(E + m)}{2m}} \begin{pmatrix} p_l/(E + m) \\ -p_z/(E + m) \\ 0 \\ 1 \end{pmatrix}.$$  \hspace{1cm} (10)

They are the parity eigenstates with the eigenvalues of $\pm 1$. In the parity operator the matrix $\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ was used as usual. They also describe eigenstates of the charge operator, $Q$, if at
rest
\[ \phi_R(0) = \pm \phi_L(0) \]  
(12)

(otherwise the corresponding physical states are no longer the charge eigenstates). Their normalizations are:

\[ \bar{u}_\sigma(p)u_{\sigma'}(p) = +\delta_{\sigma\sigma'}, \]  
(13)

\[ \bar{v}_\sigma(p)v_{\sigma'}(p) = -\delta_{\sigma\sigma'}, \]  
(14)

\[ \bar{u}_\sigma(p)v_{\sigma'}(p) = 0. \]  
(15)

The bar over the 4-spinors signifies the Dirac conjugation.

Thus in this Section we have used the basis for charged particles in the \((S, 0) \oplus (0, S)\) representation (in general)

\[
\begin{align*}
\begin{bmatrix}
1 \\
0 \\
\vdots \\
0
\end{bmatrix}, & u_{\sigma - 1}(0) = N(\sigma) \begin{bmatrix}
0 \\
1 \\
\vdots \\
0
\end{bmatrix}, & \ldots & v_{-\sigma}(0) = N(\sigma) \begin{bmatrix}
0 \\
0 \\
\vdots \\
1
\end{bmatrix}
\end{align*}
\]

(16)

Sometimes, the normalization factor is convenient to choose \(N(\sigma) = m^\sigma\) in order the rest spinors to vanish in the massless limit.

However, other constructs are possible in the \((1/2, 0) \oplus (0, 1/2)\) representation.

### 2 Majorana Spinors in the Momentum Representation.

During the 20th century various authors introduced self/anti-self charge-conjugate 4-spinors (including in the momentum rep-
presentation), see [Majorana, Bilenky, Ziino, Ahluwalia]. Later [Lounesto, Dvoeglazov1, Dvoeglazov2, Kirchbach] etc studied these spinors, they found corresponding dynamical equations, gauge transformations and other specific features of them. The definitions are:

\[ C = e^{i\theta} \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \]

\[ \mathcal{K} = -e^{i\theta} \gamma^2 \mathcal{K} \quad (17) \]

is the anti-linear operator of charge conjugation. \( \mathcal{K} \) is the complex conjugation operator. We define the self/anti-self charge-conjugate 4-spinors in the momentum space

\[ C\lambda^{S,A}(p) = \pm \lambda^{S,A}(p), \quad (18) \]
\[ C\rho^{S,A}(p) = \pm \rho^{S,A}(p). \quad (19) \]

Thus,

\[ \lambda^{S,A}(p^\mu) = \begin{pmatrix} \pm i\Theta\phi_L^*(p) \\ \phi_L(p) \end{pmatrix}, \quad (20) \]

and

\[ \rho^{S,A}(p) = \begin{pmatrix} \phi_R(p) \\ \mp i\Theta\phi_R^*(p) \end{pmatrix}. \quad (21) \]

The Wigner matrix is

\[ \Theta_{1/2} = -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (22) \]

and \( \phi_L, \phi_R \) can be boosted with \( \Lambda_{L,R} \) matrices.\(^1\)

\(^1\)Such definitions of 4-spinors differ, of course, from the original Majorana definition in \( x \)-representation:

\[ \nu(x) = \frac{1}{\sqrt{2}}(\Psi_D(x) + \Psi_D^c(x)), \quad (23) \]
The rest $\lambda$ and $\rho$ spinors are:

$$\lambda^S_{\uparrow}(0) = \sqrt{\frac{m}{2}} \begin{pmatrix} 0 \\ i \\ 1 \\ 0 \end{pmatrix}, \lambda^S_{\downarrow}(0) = \sqrt{\frac{m}{2}} \begin{pmatrix} -i \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{24}$$
$$\lambda^A_{\uparrow}(0) = \sqrt{\frac{m}{2}} \begin{pmatrix} 0 \\ -i \\ 1 \\ 0 \end{pmatrix}, \lambda^A_{\downarrow}(0) = \sqrt{\frac{m}{2}} \begin{pmatrix} i \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{25}$$
$$\rho^S_{\uparrow}(0) = \sqrt{\frac{m}{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -i \end{pmatrix}, \rho^S_{\downarrow}(0) = \sqrt{\frac{m}{2}} \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix}, \tag{26}$$
$$\rho^A_{\uparrow}(0) = \sqrt{\frac{m}{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ i \end{pmatrix}, \rho^A_{\downarrow}(0) = \sqrt{\frac{m}{2}} \begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix}. \tag{27}$$

Thus, in this basis the explicit forms of the 4-spinors of the second kind $\lambda^{S,A}_{\uparrow\downarrow}(p)$ and $\rho^{S,A}_{\uparrow\downarrow}(p)$ are

$$\lambda^S_{\uparrow}(p) = \frac{1}{2\sqrt{E + m}} \begin{pmatrix} i p_l \\ i(p^- + m) \\ p^- + m \\ -p_r \end{pmatrix}, \lambda^S_{\downarrow}(p) = \frac{1}{2\sqrt{E + m}} \begin{pmatrix} -i(p^+ + m) \\ -i p_l \\ -p_l \\ (p^+ + m) \end{pmatrix}, \tag{28}$$

$C\nu(x) = \nu(x)$ that represents the positive real $C-$parity field operator. However, the momentum-space Majorana-like spinors open various possibilities for description of neutral particles (with experimental consequences, see [Kirchbach]). For instance, "for imaginary $C$ parities, the neutrino mass can drop out from the single $\beta$ decay trace and reappear in $0\nu\beta\beta$, a curious and in principle experimentally testable signature for a non-trivial impact of Majorana framework in experiments with polarized sources."
\begin{align}
\lambda^A_{\uparrow}(p) &= \frac{1}{2\sqrt{E+m}} \begin{pmatrix}
-ip_l \\
-i(p^-+m) \\
(p^-+m) \\
-p_r
\end{pmatrix}, \quad \lambda^A_{\downarrow}(p) = \frac{1}{2\sqrt{E+m}} \begin{pmatrix}
i(p^++m) \\
-p_r \\
-p_l \\
(p^++m)
\end{pmatrix}, 

(29) \\
\rho^S_{\uparrow}(p) &= \frac{1}{2\sqrt{E+m}} \begin{pmatrix}
p^+ + m \\
p_r \\
-ip_l \\
-i(p^++m)
\end{pmatrix}, \quad \rho^S_{\downarrow}(p) = \frac{1}{2\sqrt{E+m}} \begin{pmatrix}
p_l \\
(p^-+m) \\
i(p^-+m) \\
-ip_r
\end{pmatrix}, 

(30) \\
\rho^A_{\uparrow}(p) &= \frac{1}{2\sqrt{E+m}} \begin{pmatrix}
p^+ + m \\
p_r \\
-ip_l \\
i(p^++m)
\end{pmatrix}, \quad \rho^A_{\downarrow}(p) = \frac{1}{2\sqrt{E+m}} \begin{pmatrix}
p_l \\
(p^-+m) \\
-i(p^-+m) \\
-ip_r
\end{pmatrix}. 

(31)
\end{align}

As we showed \(\lambda\) and \(\rho\) 4-spinors are NOT the eigenspinors of the helicity. Moreover, \(\lambda\) and \(\rho\) are NOT the eigenspinors of the parity (in this representation \(P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} R\)), as opposed to the Dirac case. The indices \(\uparrow\downarrow\) should be referred to the chiral helicity quantum number introduced in the 60s, \(\eta = -\gamma^5 h\). While

\[ P u_\sigma(p) = +u_\sigma(p), \quad P v_\sigma(p) = -v_\sigma(p), \quad (32) \]

we have

\[ P\lambda^{S,A}(p) = \rho^{A,S}(p), \quad P\rho^{S,A}(p) = \lambda^{A,S}(p), \quad (33) \]

for the Majorana-like momentum-space 4-spinors on the first quantization level. In this basis one has

\[ \rho^S_{\uparrow}(p) = -i\lambda^A_{\downarrow}(p), \quad \rho^S_{\downarrow}(p) = +i\lambda^A_{\uparrow}(p), \quad (34) \]

\[ \rho^A_{\uparrow}(p) = +i\lambda^S_{\downarrow}(p), \quad \rho^A_{\downarrow}(p) = -i\lambda^S_{\uparrow}(p). \quad (35) \]
The normalization of the spinors $\lambda_{\uparrow\downarrow}^{S,A}(p)$ and $\rho_{\uparrow\downarrow}^{S,A}(p)$ are the following ones:

\[
\begin{align*}
\overline{\lambda}_\uparrow^S(p)\lambda_\uparrow^S(p) &= -im, & \overline{\lambda}_\downarrow^S(p)\lambda_\downarrow^S(p) &= +im, \\
\overline{\lambda}_\uparrow^A(p)\lambda_\uparrow^A(p) &= +im, & \overline{\lambda}_\downarrow^A(p)\lambda_\downarrow^A(p) &= -im, \\
\overline{\rho}_\uparrow^S(p)\rho_\uparrow^S(p) &= +im, & \overline{\rho}_\downarrow^S(p)\rho_\downarrow^S(p) &= -im, \\
\overline{\rho}_\uparrow^A(p)\rho_\uparrow^A(p) &= -im, & \overline{\rho}_\downarrow^A(p)\rho_\downarrow^A(p) &= +im.
\end{align*}
\]

(36)\hspace{0.5cm}(37)\hspace{0.5cm}(38)\hspace{0.5cm}(39)

All other conditions are equal to zero.

The dynamical coordinate-space equations are:

\[
\begin{align*}
i\gamma^\mu \partial_\mu \lambda^S(x) - m \rho^A(x) &= 0, \\
i\gamma^\mu \partial_\mu \rho^A(x) - m \lambda^S(x) &= 0, \\
i\gamma^\mu \partial_\mu \lambda^A(x) + m \rho^S(x) &= 0, \\
i\gamma^\mu \partial_\mu \rho^S(x) + m \lambda^A(x) &= 0.
\end{align*}
\]

(40)\hspace{0.5cm}(41)\hspace{0.5cm}(42)\hspace{0.5cm}(43)

These are NOT the Dirac equation. However, they can be written in the 8-component form as follows:

\[
\begin{align*}
[i\Gamma^\mu \partial_\mu - m] \Psi_{(+)}(x) &= 0, \\
[i\Gamma^\mu \partial_\mu + m] \Psi_{(-)}(x) &= 0,
\end{align*}
\]

(44)\hspace{0.5cm}(45)

with

\[
\begin{align*}
\Psi_{(+)}(x) &= \left(\begin{array}{c}
\rho^A(x) \\
\lambda^S(x)
\end{array}\right), \\
\Psi_{(-)}(x) &= \left(\begin{array}{c}
\rho^S(x) \\
\lambda^A(x)
\end{array}\right), \text{ and } \Gamma^\mu = \begin{pmatrix} 0 & \gamma^\mu \\ \gamma^\mu & 0 \end{pmatrix}.
\end{align*}
\]

(46)

One can also re-write the equations into the two-component form. Similar formulations have been presented by M. Markov [Markov], and A. Barut and G. Ziino [Ziino]. The group-theoretical basis for such doubling has been given in the papers by Gelfand, Tsetlin and Sokolik [Gelfand].
The Lagrangian is
\[ L = \frac{i}{2} \left[ \bar{\lambda}^S \gamma^\mu \partial_\mu \lambda^S - (\partial_\mu \bar{\lambda}^S) \gamma^\mu \lambda^S + \bar{\rho}^A \gamma^\mu \partial_\mu \rho^A - (\partial_\mu \bar{\rho}^A) \gamma^\mu \rho^A + \bar{\lambda}^A \gamma^\mu \partial_\mu \lambda^A - (\partial_\mu \bar{\lambda}^A) \gamma^\mu \lambda^A + \bar{\rho}^S \gamma^\mu \partial_\mu \rho^S - (\partial_\mu \bar{\rho}^S) \gamma^\mu \rho^S - m(\lambda^S \rho^A + \bar{\lambda}^S \rho^A - \bar{\lambda}^S \rho^A) \right] \] (47)

The connection with the Dirac spinors has been found. For instance,
\[
\begin{pmatrix}
\lambda^S_\uparrow(p) \\
\lambda^S_\downarrow(p) \\
\lambda^A_\uparrow(p) \\
\lambda^A_\downarrow(p)
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
1 & i & -1 & i \\
-i & 1 & -i & -1 \\
1 & -i & -1 & -i \\
i & 1 & i & -1
\end{pmatrix} \begin{pmatrix}
u_{+1/2}(p) \\
u_{-1/2}(p) \\
v_{+1/2}(p) \\
v_{-1/2}(p)
\end{pmatrix}. \tag{48}
\]

See also ref. [Gelfand, Ziino].

The sets of \( \lambda \) spinors and of \( \rho \) spinors are claimed to be bi-orthonormal sets each in the mathematical sense [Ahluwalia], provided that overall phase factors of 2-spinors \( \theta_1 + \theta_2 = 0 \) or \( \pi \). For instance, on the classical level \( \bar{\lambda}^S_\uparrow \lambda^S_\downarrow = 2iN^2 \cos(\theta_1 + \theta_2) \).

Few remarks have been given in the previous works:

- While in the massive case there are four \( \lambda \)-type spinors, two \( \lambda^S \) and two \( \lambda^A \) (the \( \rho \) spinors are connected by certain relations with the \( \lambda \) spinors for any spin case), in a massless case \( \lambda^S_\uparrow \) and \( \lambda^A_\uparrow \) identically vanish, provided that one takes into account that \( \phi^\pm_{L} \) are eigenspinors of \( \sigma \cdot \hat{n} \), the \( 2 \times 2 \) helicity operator.

- It was noted the possibility of the generalization of the concept of the Fock space, which leads to the “doubling” Fock space [Gelfand, Ziino].

\[ ^{2}\text{We used above } \theta_1 = \theta_2 = 0. \]
It was shown [Dvoeglazov1] that the covariant derivative (and, hence, the interaction) can be introduced in this construct in the following way:

\[ \partial_{\mu} \rightarrow \nabla_{\mu} = \partial_{\mu} - igL^5 A_{\mu} \]  

(49)

where \( L^5 = \text{diag}(\gamma^5 - \gamma^5) \), the \( 8 \times 8 \) matrix. With respect to the transformations

\[ \lambda'(x) \rightarrow (\cos \alpha - i \gamma^5 \sin \alpha) \lambda(x) \]  

(50)

\[ \overline{\lambda}'(x) \rightarrow \overline{\lambda}(x)(\cos \alpha - i \gamma^5 \sin \alpha) \]  

(51)

\[ \rho'(x) \rightarrow (\cos \alpha + i \gamma^5 \sin \alpha) \rho(x) \]  

(52)

\[ \overline{\rho}'(x) \rightarrow \overline{\rho}(x)(\cos \alpha + i \gamma^5 \sin \alpha) \]  

(53)

the spinors retain their properties to be self/anti-self charge conjugate spinors and the proposed Lagrangian [Dvoeglazov1, p.1472] remains to be invariant. This tells us that while self/anti-self charge conjugate states have zero eigenvalues of the ordinary (scalar) charge operator but they can possess the axial charge (cf. with the discussion of [Ziino] and the old idea of R. E. Marshak).

In fact, from this consideration one can recover the Feynman-Gell-Mann equation (and its charge-conjugate equation). It is re-written in the two-component form

\[
\begin{align*}
\left[ \frac{\pi^-_{\mu} \pi^\mu_- - m^2 - \frac{g}{2} \sigma^{\mu\nu} F_{\mu\nu}}{\pi^+_{\mu} \pi^\mu_+ - m^2 + \frac{g}{2} \tilde{\sigma}^{\mu\nu} F_{\mu\nu}} \right] \chi(x) &= 0, \\
\left[ \frac{\pi^+_{\mu} \pi^\mu_+ - m^2 + \frac{g}{2} \tilde{\sigma}^{\mu\nu} F_{\mu\nu}}{\pi^-_{\mu} \pi^\mu_- - m^2 - \frac{g}{2} \sigma^{\mu\nu} F_{\mu\nu}} \right] \phi(x) &= 0,
\end{align*}
\]

(54)

where already one has \( \pi^\pm_{\mu} = i \partial_{\mu} \pm g A_{\mu} \), \( \sigma^0_i = -\tilde{\sigma}^0_i = i \sigma^i \), \( \sigma^{ij} = \tilde{\sigma}^{ij} = \epsilon_{ijk} \sigma^k \) and \( \nu^{DL}(x) = \text{column}(\chi \phi) \).
Next, because the transformations

$$
\lambda'_S(p) = \begin{pmatrix} \Xi & 0 \\ 0 & -\Xi \end{pmatrix} \lambda_S(p) \equiv \lambda_A^*(p),
$$

(55)

$$
\lambda''_S(p) = \begin{pmatrix} i\Xi & 0 \\ 0 & -i\Xi \end{pmatrix} \lambda_S(p) \equiv -i\lambda_S^*(p),
$$

(56)

$$
\lambda'''_S(p) = \begin{pmatrix} 0 & i\Xi \\ i\Xi & 0 \end{pmatrix} \lambda_S(p) \equiv i\gamma^0\lambda_A^*(p),
$$

(57)

$$
\lambda^{IV}_S(p) = \begin{pmatrix} 0 & \Xi \\ -\Xi & 0 \end{pmatrix} \lambda_S(p) \equiv \gamma^0\lambda_S^*(p)
$$

(58)

with the $2 \times 2$ matrix $\Xi$ defined as ($\phi$ is the azimuthal angle related with $p \rightarrow 0$)

$$
\Xi = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}, \quad \Xi\Lambda_{R,L}(p \leftarrow 0)\Xi^{-1} = \Lambda_{R,L}^*(p \leftarrow 0),
$$

(59)

and corresponding transformations for $\lambda^A$ do not change the properties of bispinors to be in the self/anti-self charge conjugate spaces, the Majorana-like field operator ($b_\dagger \equiv a_\dagger$) admits additional phase (and, in general, normalization) transformations:

$$
\nu^{ML'}(x^\mu) = [c_0 + i(\tau \cdot c)] \nu^{ML}(x^\mu),
$$

(60)

where $c_\alpha$ are arbitrary parameters. The $\tau$ matrices are defined over the field of $2 \times 2$ matrices and the Hermitian conjugation operation is assumed to act on the $c$-numbers as the complex conjugation. One can parametrize $c_0 = \cos \phi$ and $c = n \sin \phi$ and, thus, define the $SU(2)$ group of phase transformations. One can select the Lagrangian which is composed from the both field operators (with $\lambda$ spinors and $\rho$ spinors) and which remains to be invariant with respect to this kind of transformations. The conclusion is: it is permitted a non-Abelian construct which
is based on the spinors of the Lorentz group only (cf. with the old ideas of T. W. Kibble and R. Utiyama). This is not surprising because both the $SU(2)$ group and $U(1)$ group are the sub-groups of the extended Poincaré group (cf. [Ryder]).

The Dirac-like and Majorana-like field operators can be built from both $\lambda^{S,A}(p)$ and $\rho^{S,A}(p)$, or their combinations. For instance,

$$\Psi(x^\mu) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_\eta \left[ \lambda_\eta^S(p) a_\eta(p) \exp(-ip \cdot x) + \lambda_\eta^A(p) b_\eta(p) \exp(+ip \cdot x) \right].$$

The anticommutation relations are the following ones (due to the bi-orthonormality):

$$[a_\eta^\dagger(p'), a_\eta(p)] = (2\pi)^3 2E_p \delta(p - p') \delta_{\eta,-\eta'}$$

and

$$[b_\eta^\dagger(p'), b_\eta(p)] = (2\pi)^3 2E_p \delta(p - p') \delta_{\eta,-\eta'}$$

Other (anti)commutators are equal to zero: $([a_\eta^\dagger(p'), b_\eta(p)] = 0)$.

In the Fock space operations of the charge conjugation and space inversions can be defined through unitary operators such that:

$$U^c_{[1/2]} \Psi(x^\mu)(U^c_{[1/2]})^{-1} = C_{[1/2]} \Psi_{[1/2]}^\dagger(x^\mu),$$

$$U^s_{[1/2]} \Psi(x^\mu)(U^s_{[1/2]})^{-1} = \gamma^0 \Psi(x'^\mu),$$

the time reversal operation, through an antiunitary operator

$$[V^T_{[1/2]} \Psi(x^\mu)(V^T_{[1/2]})^{-1}]^\dagger = S(T) \Psi^\dagger(x'^\mu),$$

Let us remind that the operator of hermitian conjugation does not act on c-numbers on the left side of the equation (66). This fact is connected with the properties of an antiunitary operator: $[V^T \lambda A(V^T)^{-1}]^\dagger = [\lambda^* V^T A(V^T)^{-1}]^\dagger = \lambda [V^T A^\dagger(V^T)^{-1}]$. 

13
with \( x'' \equiv (x^0, -x) \) and \( x''' = (-x^0, x) \). We further assume the vacuum state to be assigned an even \( P \)- and \( C \)-eigenvalue and, then, proceed as in ref. \[Itsykson\].

As a result we have the following properties of creation (annihilation) operators in the Fock space:

\[
U_s^{1/2}a_\uparrow(p)(U_s^{1/2})^{-1} = -ia_\downarrow(-p),
\]

\[
U_s^{1/2}a_\downarrow(p)(U_s^{1/2})^{-1} = +ia_\uparrow(-p)
\]

and

\[
U_s^{1/2}b_\uparrow(p)(U_s^{1/2})^{-1} = +ib_\downarrow(-p),
\]

\[
U_s^{1/2}b_\downarrow(p)(U_s^{1/2})^{-1} = -ib_\uparrow(-p),
\]

what signifies that the states created by the operators \( a_\uparrow(p) \) and \( b_\uparrow(p) \) have very different properties with respect to the space inversion operation, comparing with Dirac states (the case also regarded in \[Ziino\]):

\[
U_s^{1/2}|p, \uparrow\rangle^+ = +i|\mp, \uparrow\rangle, U_s^{1/2}|p, \downarrow\rangle^+ = +i|\mp, \downarrow\rangle
\]

\[
U_s^{1/2}|p, \downarrow\rangle^+ = -i|\mp, \uparrow\rangle, U_s^{1/2}|p, \uparrow\rangle^- = -i|\mp, \downarrow\rangle.
\]

For the charge conjugation operation in the Fock space we have two physically different possibilities. The first one, \emph{e.g.},

\[
U_c^{1/2}a_\uparrow(p)(U_c^{1/2})^{-1} = +b_\uparrow(p), U_c^{1/2}a_\downarrow(p)(U_c^{1/2})^{-1} = +b_\downarrow(p),
\]

\[
U_c^{1/2}b_\uparrow(p)(U_c^{1/2})^{-1} = -a_\uparrow(p), U_c^{1/2}b_\downarrow(p)(U_c^{1/2})^{-1} = -a_\downarrow(p),
\]

in fact, has some similarities with the Dirac construct. However, the action of this operator on the physical states are

\[
U_c^{1/2}|p, \uparrow\rangle^+ = +|p, \uparrow\rangle, U_c^{1/2}|p, \downarrow\rangle^+ = +|p, \downarrow\rangle.
\]
\[ U_{[1/2]}^c |\mathbf{p}, \uparrow\rangle - = - |\mathbf{p}, \uparrow\rangle^+, \quad U_{[1/2]}^c |\mathbf{p}, \downarrow\rangle - = - |\mathbf{p}, \downarrow\rangle^+ . \quad (74) \]

But, one can also construct the charge conjugation operator in the Fock space which acts, e.g., in the following manner:

\[ \bar{U}_{[1/2]}^c a_{\downarrow}(\mathbf{p}) (\bar{U}_{[1/2]}^c)^{-1} = - b_{\downarrow}(\mathbf{p}), \quad \bar{U}_{[1/2]}^c b_{\downarrow}(\mathbf{p}) (\bar{U}_{[1/2]}^c)^{-1} = + a_{\downarrow}(\mathbf{p}), \quad (75) \]

and, therefore,

\[ \bar{U}_{[1/2]}^c |\mathbf{p}, \uparrow\rangle = - |\mathbf{p}, \downarrow\rangle, \quad \bar{U}_{[1/2]}^c |\mathbf{p}, \downarrow\rangle = + |\mathbf{p}, \uparrow\rangle . \quad (77) \]

Investigations of several important cases, which are different from the above ones, are required a separate paper to. Next, it is possible a situation when the operators of the space inversion and charge conjugation commute each other in the Fock space [Foldy]. For instance,

\[ U_{[1/2]}^c U_{[1/2]}^s |\mathbf{p}, \uparrow\rangle^+ = + i U_{[1/2]}^c |\mathbf{p}, \downarrow\rangle^+ = + i |\mathbf{p}, \downarrow\rangle, \quad \quad (79) \]
\[ U_{[1/2]}^s U_{[1/2]}^c |\mathbf{p}, \uparrow\rangle^- = - U_{[1/2]}^s |\mathbf{p}, \downarrow\rangle^- = + i |\mathbf{p}, \downarrow\rangle . \quad (80) \]

The second choice of the charge conjugation operator answers for the case when the \( \bar{U}_{[1/2]}^c \) and \( U_{[1/2]}^s \) operations anticommute:

\[ \bar{U}_{[1/2]}^c U_{[1/2]}^s |\mathbf{p}, \uparrow\rangle = - i U_{[1/2]}^c |\mathbf{p}, \downarrow\rangle, \quad \quad (81) \]
\[ U_{[1/2]}^s \bar{U}_{[1/2]}^c |\mathbf{p}, \uparrow\rangle = - U_{[1/2]}^s |\mathbf{p}, \downarrow\rangle . \quad (82) \]

Next, one can compose states which would have somewhat similar properties to those which we have become accustomed. The states \(|\mathbf{p}, \uparrow\rangle^+ \pm i|\mathbf{p}, \downarrow\rangle^+\) answer for positive (negative)
parity, respectively. But, what is important, the antiparticle states (moving backward in time) have the same properties with respect to the operation of space inversion as the corresponding particle states (as opposed to \( j = 1/2 \) Dirac particles). The states which are eigenstates of the charge conjugation operator in the Fock space are

\[
U_{[1/2]}^c (|p, \uparrow>^+ \pm i |p, \uparrow>^-) = \mp i (|p, \uparrow>^+ \pm i |p, \uparrow>^-). \tag{83}
\]

There is no any simultaneous set of states which would be eigenstates of the operator of the space inversion and of the charge conjugation \( U_{[1/2]}^c \).

Finally, the time reversal anti-unitary operator in the Fock space should be defined in such a way that the formalism to be compatible with the CPT theorem. If we wish the Dirac states to transform as \( V(T)|p, \pm 1/2 > = \pm |-p, \mp 1/2 > \) we have to choose (within a phase factor), ref. [Itsykson]:

\[
S(T) = \begin{pmatrix} \Theta_{[1/2]} & 0 \\ 0 & \Theta_{[1/2]} \end{pmatrix}. \tag{84}
\]

Thus, in the first relevant case we obtain for the \( \Psi(x^\mu) \) field, Eq. (61):

\[
V^T a^\dagger_\uparrow(p)(V^T)^{-1} = a^\dagger_\downarrow(-p), \quad V^T a^\dagger_\downarrow(p)(V^T)^{-1} = -a^\dagger_\uparrow(-p) \tag{85}
\]

\[
V^T b^\dagger_\uparrow(p)(V^T)^{-1} = b^\dagger_\downarrow(-p), \quad V^T b^\dagger_\downarrow(p)(V^T)^{-1} = -b^\dagger_\uparrow(-p). \tag{86}
\]

Thus, this construct has very different properties with respect to \( C, P \) and \( T \) comparing with the Dirac construct.

But, at least for mathematicians, the dependence of the physical results on the choice of the basis is a bit strange thing. Somewhat similar things have been presented in [Dvoeglazov3] when compared the Dirac-like
constructs in the parity and helicity bases. It was shown that the helicity eigenstates \((\sigma \cdot n) \otimes I\) are NOT the parity eigenstates (and the \(S_3\) eigenstates), and vice versa, in the helicity basis (cf. with [Berestetskii, Lifshitz, Pitaevskii]), while they obey the same Dirac equation. The bases are connected by the unitary transformation. And, the both sets of 4-spinors form the complete system in a mathematical sense.

3 The Spin 1.

3.1 Maxwell Equations as Quantum Equations.

In refs. [Gersten, Dvoeglazov4] the Maxwell-like equations have been derived\(^4\) from the Klein-Gordon equation. Here they are:

\[
\nabla \times E = -\frac{1}{c} \frac{\partial B}{\partial t} + \nabla \text{Im} \chi, \quad (87)
\]

\[
\nabla \times B = \frac{1}{c} \frac{\partial E}{\partial t} + \nabla \text{Re} \chi, \quad (88)
\]

\[
\nabla \cdot E = -\frac{1}{c} \frac{\partial}{\partial t} \text{Re} \chi, \quad (89)
\]

\[
\nabla \cdot B = \frac{1}{c} \frac{\partial}{\partial t} \text{Im} \chi. \quad (90)
\]

Of course, similar equations can be obtained in the massive case \(m \neq 0\), i.e., within the Proca-like theory. We should then consider

\[
(E^2 - c^2 p^2 - m^2 c^4) \Psi^{(3)} = 0. \quad (91)
\]

\(^4\)I call them ”Maxwell-like” because an additional gradient of a scalar field \(\chi\) can be introduced therein.
In the spin-1/2 case the equation (91) can be written for the two-component spinor \((c = \hbar = 1)\)

\[(EI^{(2)} - \sigma \cdot p)(EI^{(2)} + \sigma \cdot p)\Psi^{(2)} = m^2\Psi^{(2)}\], \hspace{1cm} (92)

or, in the 4-component form

\[
[i\gamma_\mu \partial_\mu + m_1 + m_2\gamma^5]\Psi^{(4)} = 0. \hspace{1cm} (93)
\]

In the spin-1 case we have

\[(EI^{(3)} - S \cdot p)(EI^{(3)} + S \cdot p)\Psi^{(3)} - p(p \cdot \Psi^{(3)}) = m^2\Psi^{(3)}. \hspace{1cm} (94)\]

These lead to (87-90), when \(m = 0\) provided that the \(\Psi^{(3)}\) is chosen as a superposition of a vector (the electric field) and an axial vector (the magnetic field)\(^5\). When \(\chi = 0\) we recover the common-used Maxwell equations.

Otherwise, we can start with \((c = \hbar = 1)\)\(^6\)

\[
\frac{\partial E}{\partial t} = curl B, \quad \frac{\partial B}{\partial t} = -curl E. \hspace{1cm} (95)
\]

Then,

\[
\frac{\partial (E + iB)}{\partial t} - curl(B - iE) = 0, \hspace{1cm} (96)
\]

\[
\frac{\partial (E - iB)}{\partial t} - curl(B + iE) = 0. \hspace{1cm} (97)
\]

In the component form:

\[
\frac{\partial (E + iB)^i}{\partial t} + i\epsilon^{ijk}\partial_j(E + iB)^k = 0, \hspace{1cm} (98)
\]

\[
\frac{\partial (E - iB)^i}{\partial t} - i\epsilon^{ijk}\partial_j(E - iB)^k = 0. \hspace{1cm} (99)
\]

\(^5\)We can continue writing down equations for higher spins in a similar fashion.

\(^6\)The question of both explicite and implicite dependences of the fields on the time (and, hence, the "whole-partial derivative") has been studied in [Brownstein, Dvoeglazov5].
Since the spin-1 matrices can be presented in the form: \((S^i)^{jk} = -i\epsilon^{ijk}\), we have

\[
\frac{\partial (E + iB)^i}{\partial t} + (S \cdot \nabla)^{ik}(E + iB)^k = 0, \quad (100)
\]

\[
\frac{\partial (E - iB)^i}{\partial t} - (S \cdot \nabla)^{ik}(E - iB)^k = 0. \quad (101)
\]

Finally, on using that \(\hat{p} = -i\hbar\nabla\) we have

\[
i\frac{\partial \phi}{\partial t} = (S \cdot \hat{p})\phi, \quad i\frac{\partial \xi}{\partial t} = -(S \cdot \hat{p})\xi. \quad (102)
\]

In the following we show that these equations can also be considered as the massless limit of the Weinberg \(S = 1\) quantum-field equation.

Meanwhile, we can calculate the determinants of the above equations, \(Det[E \mp (S \cdot p)] = 0\), and we can find that we have both the causal \(E = \pm |p|\) and acausal \(E = 0\) solutions.\(^7\) These results will be useful in analyzing the spin-1 quantum-field theory below.

### 3.2 The Weinberg 2(2S + 1) Theory for Spin-1

It is based on the following postulates [Wigner,Weinberg]:

- The fields transform according to the formula:

\[
U[\Lambda, a]\Psi_n(x)U^{-1}[\Lambda, a] = \sum_m D_{nm}[\Lambda^{-1}]\Psi_m(\Lambda x + a), \quad (103)
\]

where \(D_{nm}[\Lambda]\) is some representation of \(\Lambda\); \(x^\mu \rightarrow \Lambda^\mu_{\nu} x^\nu + a^\mu\), and \(U[\Lambda, a]\) is a unitary operator.

\(^7\)The possible interpretation of the \(E = 0\) solutions are the stationary fields.
• For \((x - y)\) spacelike one has
\[
[\Psi_n(x), \Psi_m(y)]_{\pm} = 0 \quad (104)
\]
for fermion and boson fields, respectively.

• The interaction Hamiltonian density is said by S. Weinberg to be a scalar, and it is constructed out of the creation and annihilation operators for the free particles described by the free Hamiltonian \(H_0\).

• The \(S\)-matrix is constructed as an integral of the \(T\)-ordering product of the interaction Hamiltonians by the Dyson’s formula.

In this talk we shall be mainly interested in the free-field theory. Weinberg wrote: “In order to discuss theories with parity conservation it is convenient to use \(2(2S + 1)\)-component fields, like the Dirac field. These do obey field equations, which can be derived as... consequences of \((103,104)\).” In such a way he proceeds to form the \(2(2S + 1)\)-component object
\[
\Psi = \begin{pmatrix} \Phi_\sigma \\ \Xi_\sigma \end{pmatrix}
\]
transforming according to the Wigner rules. They are the following ones (see also above, Eqs. (5,6)):
\[
\Phi_\sigma(p) = \exp (+\Theta \hat{p} \cdot \mathbf{S}) \Phi_\sigma(0), \quad (105)
\]
\[
\Xi_\sigma(p) = \exp (-\Theta \hat{p} \cdot \mathbf{S}) \Xi_\sigma(0) \quad (106)
\]
from the zero-momentum frame. \(\Theta\) is the boost parameter, \(\tanh\ \Theta = |p|/E\), \(\hat{p} = p/|p|\), \(p\) is the 3-momentum of the particle, \(\mathbf{S}\) is the angular momentum operator. For a given representation the matrices \(\mathbf{S}\) can be constructed. In the Dirac case (the

\[8\]In the \((2S + 1)\) formalism fields obey only the Klein-Gordon equation, according to the Weinberg wisdom.
(1/2, 0) ⊕ (0, 1/2) representation) \( S = \sigma / 2; \) in the \( S = 1 \) case (the (1, 0) ⊕ (0, 1) representation) we can choose \((S_i)_{jk} = -i\epsilon_{ijk}, \) etc. Hence, we can explicitly calculate \([105,106].\)

The task is now to obtain relativistic equations for higher spins. Weinberg uses the following procedure. Firstly, he defined the scalar matrix

\[
\Pi^{(s)}_{\sigma'\sigma}(q) = (-)^{2s} t_{\sigma'\sigma}^{\mu_1\mu_2...\mu_{2s}} q_{\mu_1} q_{\mu_2} \cdots q_{\mu_{2s}} \tag{107}
\]

for the \((S, 0)\) representation of the Lorentz group \((q_{\mu} q_{\mu} = -m^2),\) with the tensor \( t \) being defined by \([\text{Weinberg, Eqs.}(A4-A5)].\) Hence,

\[
D^{(s)}[\Lambda] \Pi^{(s)}(q) D^{(s)\dagger}[\Lambda] = \Pi^{(s)}(\Lambda q) \tag{108}
\]

Since at rest we have \([S^{(s)}, \Pi^{(s)}(m)] = 0,\) then according to the Schur’s lemma \(\Pi_{\sigma\sigma'}^{(s)}(m) = m^{2s} \delta_{\sigma\sigma'}\). After the substitution of \(D^{(s)}[\Lambda]\) in Eq. \((108)\) one has

\[
\Pi^{(s)}(q) = m^{2s} \exp(2\Theta \hat{q} \cdot S^{(s)}). \tag{109}
\]

One can construct the analogous matrix for the \((0, S)\) representation by the same procedure:

\[
\Pi^{(s)}(q) = m^{2s} \exp(-2\Theta \hat{q} \cdot S^{(s)}). \tag{110}
\]

Finally, by the direct verification one has in the coordinate representation

\[
\Pi_{\sigma\sigma'}(-i\partial)\Phi_{\sigma'} = m^{2s}\Xi_{\sigma'}, \tag{111}
\]

\[
\Pi_{\sigma\sigma'}(-\partial)\Xi_{\sigma'} = m^{2s}\Phi_{\sigma}, \tag{112}
\]

provided that \(\Phi_{\sigma}(0)\) and \(\Xi_{\sigma}(0)\) are indistinguishable.\(^9\)

\(^9\)Later, this fact has been incorporated in the Ryder book \([\text{Ryder}].\) Truely speaking, this is an additional postulate. It is possible that the zero-momentum-frame \(2(2S + 1)\)-component objects (the 4-spinor in the \((1/2, 0) \oplus (0, 1/2)\) representation, the bivector in the \((1, 0) \oplus (0, 1)\) representation, etc.) are connected by an arbitrary phase factor \([\text{Dvoeglazov6}].\)
As a result one has

\[ [\gamma^{\mu_1\mu_2...\mu_{2s}} \partial_{\mu_1} \partial_{\mu_2} \ldots \partial_{\mu_{2s}} + m^{2s}] \Psi(x) = 0, \quad (113) \]

with the Barut-Muzinich-Williams covariantly-defined matrices [BMW] Sankaranarayanan,Good. For the spin-1 they are:

\[ \gamma_{44} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_{4i} = \gamma_{4i} = \begin{pmatrix} 0 & iS_i \\ -iS_i & 0 \end{pmatrix}, \quad (114) \]

\[ \gamma_{ij} = \begin{pmatrix} 0 & \delta_{ij} - S_iS_j - S_jS_i \\ \delta_{ij} - S_iS_j - S_jS_i & 0 \end{pmatrix}. \quad (115) \]

Later Sankaranarayanan and Good considered another version of this theory [Sankaranarayanan,Good] (see also [Ahluwalia2]). For the $S = 1$ case they introduced the Weaver-Hammer-Good sign operator, ref. [Weaver], $m^2 \rightarrow m^2 (i\partial/\partial t)/E$, which led to the different parity properties of an antiparticle with respect to a boson particle. Next, Tucker and Hammer et al [Tucker,Hammer] introduced another higher-spin equations. In the spin-1 case it is:

\[ [\gamma_{\mu\nu} \partial^{\mu} \partial_{\nu} + \partial_{\mu} \partial_{\mu} - 2m^2] \Psi^{(s=1)} = 0 \quad (116) \]

(Euclidean metric is now used). In fact, they added the Klein-Gordon equation to the Weinberg equation. One can add the Klein-Gordon equation with arbitrary multiple factor to the Weinberg equation. So, we can study the generalized Weinberg-Tucker-Hammer equation ($S = 1$), which is written ($p_\mu = -i\partial/\partial x^\mu$):

\[ [\gamma_{\alpha\beta} p_\alpha p_\beta + Ap_\alpha p_\alpha + Bm^2] \Psi = 0. \quad (117) \]

It has solutions with relativistic dispersion relations $E^2 - p^2 = m^2$, $(c = \hbar = 1)$ provided that

\[ \frac{B}{A+1} = 1, \quad \text{or} \quad \frac{B}{A-1} = 1. \quad (118) \]
This can be proven by considering the algebraic equation
\[ \text{Det}[\gamma_{\alpha\beta}p_\alpha p_\beta + Ap_\alpha + Bm^2] = 0. \]
It is of the 12th order in \( p_\mu \). Solving it with respect to energy one obtains the conditions (118). Unlike the Maxwell equations there are NO any \( E = 0 \) solutions.

The solutions in the momentum representation have been explicitly presented by Ahluwalia2:

\[
\begin{align*}
\mathbf{u}_{+1}(\mathbf{p}) & = \begin{pmatrix}
m + \left(2p_z^2 + p_+ p_- / 2(E + m)\right) \\
p_+ p_- / \sqrt{2}(E + m) \\
p_+^2 / 2(E + m) \\
p_z \\
p_+ / \sqrt{2} \\
0
\end{pmatrix}, \quad (119) \\
\mathbf{u}_0(\mathbf{p}) & = \begin{pmatrix}
p_+ p_- / \sqrt{2}(E + m) \\
m + [p_+ p_- / (E + m)] \\
-p_+ p_- / \sqrt{2}(E + m) \\
p_- / \sqrt{2} \\
0 \\
p_+ / \sqrt{2} \\
p_+^2 / 2(E + m) \\
-p_+ p_- / \sqrt{2}(E + m) \\
m + \left(2p_z^2 + p_+ p_- / 2(E + m)\right) \\
0 \\
p_- / \sqrt{2} \\
-p_z
\end{pmatrix}, \quad (120) \\
\mathbf{u}_{-1}(\mathbf{p}) & = \begin{pmatrix}
m + \left(2p_z^2 + p_+ p_- / 2(E + m)\right) \\
p_+ p_- / \sqrt{2}(E + m) \\
p_+^2 / 2(E + m) \\
p_+ p_- / \sqrt{2}(E + m) \\
m + \left(2p_z^2 + p_+ p_- / 2(E + m)\right) \\
0 \\
p_- / \sqrt{2} \\
-p_z
\end{pmatrix}, \quad (121)
\end{align*}
\]

and
\[
\mathbf{v}_\sigma(\mathbf{p}) = \gamma_5 \mathbf{u}_\sigma(\mathbf{p}) = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} \mathbf{U}_\sigma(\mathbf{p})
\]

in the standard representation of \( \gamma_{\mu\nu} \) matrices. If the 6-component \( \mathbf{v}(\mathbf{p}) \) are defined in such way, we inevitably would get the ad-
ditional energy-sign operator [Weaver, Sankaranarayanan, Good] 
\[ \epsilon = i \partial_t / E = \pm 1 \] in the dynamical equation, and the different parities of the corresponding boson and antiboson, \( \hat{P} u_\sigma(p) = +u_\sigma(p) \) and \( \hat{P} v_\sigma(p) = -v_\sigma(p) \).

4 The Construction of Field Operators.

The method for constructions of field operators has been given in [Bogoliubov, Shirkov]:

\[
\phi(x) = \frac{1}{(2\pi)^{3/2}} \int dk e^{ikx} \tilde{\phi}(k).
\] (123)

From the Klein-Gordon equation we know:

\[
(k^2 - m^2)\tilde{\phi}(k) = 0.
\] (124)

Thus,

\[
\tilde{\phi}(k) = \delta(k^2 - m^2)\phi(k).
\] (125)

Next,

\[
\phi(x) = \frac{1}{(2\pi)^{3/2}} \int dk e^{ikx} \delta(k^2 - m^2)(\theta(k_0) + \theta(-k_0))\phi(k) =
\]

\[
= \frac{1}{(2\pi)^{3/2}} \int dk \left[e^{ikx} \delta(k^2 - m^2)\phi^+(k) + e^{-ikx} \delta(k^2 - m^2)\phi^-(k)\right],
\] (126)

where

\[
\phi^+(k) = \theta(k_0)\phi(k), \text{ and } \phi^-(k) = \theta(k_0)\phi(-k).
\] (127)

\[10\] In this book a bit different notation for positive- (negative-) energy solutions has been used comparing with the general accepted one.
\[ \phi^+(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{2E_k} e^{i k x} \phi^+(k), \quad (128) \]

\[ \phi^-(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{2E_k} e^{-i k x} \phi^-(k). \quad (129) \]

In the spinor case (the \((1/2, 0) \oplus (0, 1/2)\) representation space) we have more components. Instead of the equation (124) we have

\[ (\hat{k} + m)\psi(k)|_{k^2=m^2} = 0. \quad (130) \]

However, again

\[ \psi(x) = \frac{1}{(2\pi)^{3/2}} \int dk \, e^{i k x} \delta(k^2 - m^2)(\theta(k_0) + \theta(-k_0))\psi(k), \quad (131) \]

and

\[ \psi(x) = \frac{1}{(2\pi)^3} \int \frac{d^3k}{2E_k} \left[ e^{i k x} \theta(k_0)\psi(k) + e^{-i k x} \theta(k_0)\psi(-k) \right], \quad (132) \]

where \(k_0 = E = \sqrt{k^2 + m^2}\) is positive in this case. Hence:

\[ (\hat{k} + m)\psi^+(k) = 0, \quad (-\hat{k} + m)\psi^-(k) = 0. \quad (133) \]

Everything is OK? However, please note that the momentum-space Dirac equations \((\hat{k} - m)u = 0, (\hat{k} + m)v = 0\) have solutions \(k_0 = \pm \sqrt{k^2 + m^2}\), both for \(u\)- and \(v\)-spinors. This can be checked by calculating the determinants. Usually, one chooses \(k_0 = E = \sqrt{k^2 + m^2}\) in the \(u\)- and in the \(v\)-. This is because on the classical level (better to say, on the first quantization level) the negative-energy \(u\)- can be transformed in the positive-energy \(v\)-, and vice versa. This is not precisely so, if
we go to the secondary quantization level. The introduction of creation/annihilation noncommutating operators gives us more possibilities in constructing generalized theory even on the basis of the Dirac equation.

Various-type field operators are possible in the \( (1/2, 1/2) \) representation. During the calculations below we have to present \( 1 = \theta(k_0) + \theta(-k_0) \) (as previously) in order to get positive- and negative-frequency parts.

\[
A_\mu(x) = \frac{1}{(2\pi)^3} \int d^4k \delta(k^2 - m^2) e^{+ik \cdot x} A_\mu(k) =
\]

\[
= \frac{1}{(2\pi)^3} \sum_\lambda \int d^4k \delta(k^2_0 - E^2_k) e^{+ik \cdot x} \epsilon_\mu(k, \lambda)a_\lambda(k) =
\]

\[
= \frac{1}{(2\pi)^3} \int \frac{d^4k}{2E_k} \left[ \delta(k_0 - E_k) + \delta(k_0 + E_k) \right] \theta(k_0) + \theta(-k_0)] e^{+ik \cdot x} A_\mu(k) =
\]

\[
\left[ \theta(k_0) A_\mu(k)e^{+ik \cdot x} + \theta(k_0) A_\mu(-k)e^{-ik \cdot x} \right] =
\]

\[
= \frac{1}{(2\pi)^3} \int \frac{d^3k}{2E_k} \theta(k_0)[A_\mu(k)e^{+ik \cdot x} + A_\mu(-k)e^{-ik \cdot x}] =
\]

\[
= \frac{1}{(2\pi)^3} \sum_\lambda \int \frac{d^3k}{2E_k}[\epsilon_\mu(k, \lambda)a_\lambda(k)e^{+ik \cdot x} + \epsilon_\mu(-k, \lambda)a_\lambda(-k)e^{-ik \cdot x}].
\]

In general, due to theorems for integrals and for distributions the presentation \( 1 = \theta(k_0) + \theta(-k_0) \) is possible because we use this in the integrand. However, remember, that we have the \( k_0 = E = 0 \) solution of the Maxwell equations\[11] Moreover, it has the experimental confirmation (for instance, the stationary mag-

\[11\]Of course, the same procedure can be applied in the construction of the quantum field operator for \( F_{\mu\nu} \).
netic field \( \text{curl} \mathbf{B} = 0 \). Meanwhile the \textit{theta} function is NOT defined in \( k_0 = 0 \). Do we not loose this solution in the above construction of the quantum field operator? Mathematicians did not answer me in a straightforward way.

Moreover, we should transform the second part to \( \epsilon^\ast_\mu(k, \lambda) b^\dagger_\lambda(k) \) as usual. In such a way we obtain the charge-conjugate states.\(^{12}\) Of course, one can try to get \( P \)-conjugates or \( CP \)-conjugate states too.

In the Dirac case we should assume the following relation in the field operator:

\[
\sum_\lambda v_\lambda(k) b^\dagger_\lambda(k) = \sum_\lambda u_\lambda(-k) a_\lambda(-k) .
\] (135)

We know that [Ryder, Itsykson]

\[
\bar{u}_\mu(k) u_\lambda(k) = +m \delta_{\mu\lambda} , \quad (136)
\]
\[
\bar{u}_\mu(k) u_\lambda(-k) = 0 , \quad (137)
\]
\[
\bar{v}_\mu(k) v_\lambda(k) = -m \delta_{\mu\lambda} , \quad (138)
\]
\[
\bar{v}_\mu(k) u_\lambda(k) = 0 , \quad (139)
\]

but we need \( \Lambda_{\mu\lambda}(k) = \bar{v}_\mu(k) u_\lambda(-k) \). By direct calculations, we find

\[
-m b^\dagger_\mu(k) = \sum_\nu \Lambda_{\mu\lambda}(k) a_\lambda(-k) .
\] (140)

Hence, \( \Lambda_{\mu\lambda} = -im(\sigma \cdot \mathbf{n})_{\mu\lambda} \) and

\[
b^\dagger_\mu(k) = i(\sigma \cdot \mathbf{n})_{\mu\lambda} a_\lambda(-k) .
\] (141)

Multiplying (135) by \( \bar{u}_\mu(-k) \) we obtain

\[
a_\mu(-k) = -i(\sigma \cdot \mathbf{n})_{\mu\lambda} b^\dagger_\lambda(k) .
\] (142)

\(^{12}\)In the certain basis it is considered that the charge conjugation operator is just the complex conjugation operator for 4-vectors \( A_\mu \).
Thus, the above equations are self-consistent.

In the $(1, 0) \oplus (0, 1)$ representation we have somewhat different situation. Namely,

$$a_\mu(k) = [1 - 2(S \cdot n)^2]_{\mu\lambda}a_\lambda(-k) . \tag{143}$$

This signifies that in order to construct the Sankaranarayanan-Good field operator (which was used by Ahluwalia, Johnson and Goldman [Ahluwalia2], it satisfies $[\gamma_{\mu\nu}\partial_\mu\partial_\nu - \frac{(i\partial/\partial t)}{E}m^2]\Psi = 0$, we need additional postulates.

We can set for the 4-vector field operator:

$$\sum_\lambda \epsilon_\mu(-k, \lambda)a_\lambda(-k) = \sum_\lambda \epsilon^*_\mu(k, \lambda)b^\dagger_\lambda(k) , \tag{144}$$

multiply both parts by $\epsilon_\nu[\gamma_{44}]_{\nu\mu}$, and use the normalization conditions for polarization vectors.

However, in the $(1/2, 1/2)$ representation we can also expand (apart the equation (144)) in the different way:

$$\sum_\lambda \epsilon_\mu(-k, \lambda)a_\lambda(-k) = \sum_\lambda \epsilon_\mu(k, \lambda)a_\lambda(k) . \tag{145}$$

From the first definition we obtain (the signs $\mp$ depends on the value of $\sigma$):

$$b^\dagger_\sigma(k) = \mp \sum_{\mu\nu\lambda} \epsilon_\nu(k, \sigma)[\gamma_{44}]_{\nu\mu}\epsilon_\mu(-k, \lambda)a_\lambda(-k) , \tag{146}$$

or

$$b^\dagger_\sigma(k) = \begin{pmatrix} 1 + \frac{k^2}{E_k^2} & \sqrt{2}\frac{k_i}{E_k} & -\sqrt{2}\frac{k_i}{E_k} & -\frac{2k_3}{E_k} \\ -\sqrt{2}\frac{k_i}{E_k} & \frac{k^2}{k^2} & -\frac{m^2k_3^2}{E_k^2k^2} + \frac{k_\lambda k_\mu}{E_k^2} & \frac{\sqrt{2}k_3 k_\lambda}{k^2} \\ \sqrt{2}\frac{k_i}{E_k} & -\frac{m^2k_3^2}{E_k^2k^2} + \frac{k_\lambda k_\mu}{E_k^2} & \frac{k^2}{k^2} & -\frac{2k_3}{k^2} \\ \frac{2k_3}{E_k} & \frac{\sqrt{2}k_3 k_\lambda}{k^2} & -\frac{2k_3}{k^2} & \frac{m^2}{E_k^2} - \frac{2k_3}{k^2} \end{pmatrix} \begin{pmatrix} a_{00}(-k) \\ a_{11}(-k) \\ a_{1-1}(-k) \\ a_{10}(-k) \end{pmatrix} . \tag{147}$$
From the second definition $\Lambda_{\sigma\lambda}^2 = \mp \sum_{\nu\mu} \epsilon^*_\nu(k, \sigma) [\gamma_{4\mu}]_{\nu\mu} \epsilon_\mu(-k, \lambda)$ we have:

$$a_{\sigma}(k) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{k^2}{k^2} & \frac{k^2}{k^2} & \frac{\sqrt{2}k_3 k_4}{k^2} \\ 0 & \frac{k^2}{k^2} & \frac{k^2}{k^2} & -\frac{\sqrt{2}k_3 k_4}{k^2} \\ 0 & -\frac{\sqrt{2}k_3 k_4}{k^2} & \frac{\sqrt{2}k_3 k_4}{k^2} & 1 - \frac{2k^2}{k^2} \end{pmatrix} \begin{pmatrix} a_{00}(-k) \\ a_{11}(-k) \\ a_{1-1}(-k) \\ a_{10}(-k) \end{pmatrix} \quad (148)$$

It is the strange case: the field operator will only destroy particles (like in the $(1, 0) \oplus (0, 1)$ case). Possibly, we should think about modifications of the Fock space in this case, or introduce several field operators for the $(\frac{1}{2}, \frac{1}{2})$ representation.

However, other way is possible: to construct the left- and right- parts of the $(1, 0) \oplus (0, 1)$ field operator separately each other. In this case the commutation relations may be more complicated.

Finally, going back to the rest $(S, 0) \oplus (0, S)$ objects. Bogoliubov constructs them introducing the products with delta functions like $\delta(k_0 - m)$. Then, he makes the boost of the "spinors" only, and changes by hand the $\delta$ to $\delta(k^2 - m^2)$ (where we already have $k_0 = E = \sqrt{k^2 + m^2}$). Mathematicians did not answer me, how can it be possible to make the boost of the delta functions consistently in such a way.

The conclusion is: we still have few questions unsolved in the bases of the quantum field theory, which open a room for generalized theories.

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