LOWER BOUNDS FOR THE FIRST LAPLACIAN EIGENVALUE OF GEODESIC BALLS OF SPHERICALLY SYMMETRIC MANIFOLDS

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ABSTRACT. We obtain lower bounds for the first Laplacian eigenvalues of geodesic balls of spherically symmetric manifolds. These lower bounds are only $C^0$ dependent on the metric coefficients.

1. INTRODUCTION

Let $B(r)$ be a geodesic ball of radius $r$ in the $n$-dimensional sphere $S^n(1)$ of sectional curvature $+1$. Although the sphere is a well studied manifold, the values of the first Laplacian eigenvalue $\lambda_1(r)$ on $B(r)$, (Dirichlet boundary data if $r < \pi$) are pretty much unknown, exceptions are $\lambda_1(\pi/2) = n$ and $\lambda_1(\pi) = 0$. Among the various types of bounds for $\lambda_1(r)$, see [1], [7], [8] in dimension two, see [4] in dimension three, we would like to emphasize the following bounds due to Betz, Camera and Gzyl they obtained in [2].

(1) \[ \left( \frac{c_n}{r} \right)^2 > \lambda_1(r) \geq \frac{1}{\int_0^\pi \left[ \frac{\sin^{n-1}(\sigma)}{\sin^{n-1}(\sigma)} \cdot \int_0^\sigma \sin^{n-1}(s)ds \right] d\sigma}, \]

Where $c_n$ is the first zero of the $J_{(n-2)/2}$ Bessel function. The upper bound is just Cheng’s eigenvalue comparison theorem [3] and it is due to the fact that the Ricci curvature of the sphere is positive (need only to be non-negative). The interesting part is the lower bound that they obtained with probabilistic method. Denoting by $V(r)$ the $n$-volume of the geodesic ball $B(r)$ and by $S(r)$

2000 Mathematics Subject Classification. Primary 35B40. Secondary 35J40.

Key words and phrases. First eigenvalue, lower bounds, elliptic equations, fixed points.

The first author is grateful for the financial support by CAPES - PRODOC.
the \((n-1)\)-volume of the boundary \(\partial B(r)\) we can rewrite Betz-Camera-Gzyl lower bound as

\[
\lambda_1(r) \geq \frac{1}{\int_0^r \frac{V(\sigma)}{S(\sigma)} d\sigma}.
\]

In this note, using a fixed point theorem approach, we extend Betz-Camera-Gzyl’s lower bound to \(\lambda_1(r)\) of geodesic balls \(B(r)\) of complete spherically symmetric manifolds.

A spherically symmetric manifold is a quotient space \(M = ([0, R] \times S^{n-1})/\sim\), with \(R \in (0, \infty]\), where

\[
(t, \theta) \sim (s, \alpha) \iff \begin{cases} 
    t = s \text{ and } \theta = \alpha \\
    \text{or} \\
    s = t = 0.
\end{cases}
\]

endowed with a Riemannian metric of this form \(dt^2 + f^2(t) d\theta^2\), \(f(0) = 0\), \(f'(0) = 1\), \(f(t) > 0\) for all \(t \in (0, R]\). The class of spherically symmetric manifolds includes the canonical space forms \(\mathbb{R}^n\), \(S^n(1)\) and \(H^n(-1)\). A spherically symmetric manifold has a pole (at \(p = \{0\} \times S^{n-1}\)) if and only if \(R = \infty\).

**Theorem 1.1.** Let \(M = [0, R] \times S^{n-1}\) be a spherically symmetric manifold with Riemannian metric \(dt^2 + f^2(t) d\theta^2\), \(f(0) = 0\), \(f'(0) = 1\), \(f(t) > 0\) for all \(t \in (0, R]\) and \(B(r) \subset M\) a geodesic ball of radius \(r\). Then

\[
\lambda_1(r) \geq \frac{1}{\int_0^r \frac{V(\sigma)}{S(\sigma)} d\sigma}.
\]

**Definition 1.1.** Let \(M\) be a spherically symmetric manifold with a pole. The fundamental tone \(\lambda^*(M)\) is defined by

\[
\lambda^*(M) = \lim_{r \to \infty} \lambda_1(r)
\]
Corollary 1.1. Let $M = [0, \infty) \times S^{n-1}$ be a spherically symmetric manifold with a pole. Then

$$\lambda^*(M) \geq \frac{1}{\int_0^\infty \frac{V(\sigma)}{S(\sigma)} d\sigma}.$$  

This corollary is closely related to certain property of the Brownian motions on $M$. Denote by $p(t, x, y) \in C^\infty((0, \infty) \times M \times M)$ the heat kernel of $M$ and let $X_t$ be a Brownian motion on $M$ and denote by $P_x$ the corresponding measure in the space of paths emanating from a point $x$. See more details in [5].

Definition 1.2. A Brownian motion $X_t$ on a complete manifold $M$ is recurrent if for any $x \in M$ and any non-empty open set $\Omega \subset M$

$$P_x \left( \{\text{There is a sequence } t_k \to \infty \text{ such that } X_{t_k} \in \Omega \} \right) = 1.$$  

Otherwise is transient.

Definition 1.3. A Brownian motion $X_t$ on a complete manifold $M$ is stochastically complete if for all $x \in M$ and $t > 0$.

$$\int_M p(t, x, y) d\mu(y) = 1$$

Otherwise $X_t$ is incomplete.

We say that a complete manifold $M$ is recurrent, transient, stochastically complete, incomplete if the Brownian motion has this property. The following test is well known, see [5] and references there in.

Test for Stochastically Completeness: Let $M$ a spherically symmetric manifold with a pole. Then $M$ is stochastically complete if and only if

$$\int_0^\infty \frac{V(r)}{S(r)} dr = \infty.$$
Remark 1.2.

i. Let $M$ be a complete Riemannian manifold. If $\lambda^*(M) > 0$ then $M$ is transient.

ii. There are examples of complete, stochastically incomplete (therefore transient) Riemannian manifolds $M$ with $\lambda^*(M) = 0$, see [6].

The following corollary follows from the test for stochastically completeness and Corollary (1.1).

**Corollary 1.2.** Let $M$ be a spherically symmetric manifold with a pole. If $M$ is stochastically incomplete then $\lambda^*(M) > 0$. If $\lambda^*(M) = 0$ then $M$ is stochastically complete.

2. Proof of the results

Consider the space $X$ of all continuous functions on $[0, r]$ with the usual topology defined by the norm $\|u\| = \sup_{0 \leq t \leq r} |u(t)|$. For $a \in \mathbb{R}$ and $\Theta > 0$ let $T = T_{a, \Theta}$ be the operator in $X$ defined by

$$ T u(t) = \Theta - \int_0^t \int_0^\sigma \left( \frac{f_n(s)}{f_n(\sigma)} \right) [a + \lambda_1(r)] u(s) \, ds \, d\sigma, \quad 0 \leq t \leq r $$

Let $B(r) \subset M$ be a geodesic ball of radius $r < R$ in a spherically symmetric manifold $M = [0, R] \times \mathbb{S}^{n-1}$ with metric $dt^2 + f^2(t)d\theta^2$. The Laplacian operator $\Delta_M$ at a point $(t, \theta)$ is given by

$$ \Delta_M = \frac{\partial^2}{\partial t^2} + (n - 1) \frac{f'(t)}{f(t)} \frac{\partial}{\partial t} + \frac{1}{f^2(t)} \Delta_{\mathbb{S}^{n-1}} $$

Given $u \in X$, we can extend (radially) $u$ and $Tu$ to continuous functions $\tilde{u}$ and $\tilde{T}u$ on $B(r)$ respectively by $\tilde{u}(t, \theta) = u(t)$ and $\tilde{T}u(t, \theta) = Tu(t)$, for all $\theta \in \mathbb{S}^{n-1}$, $t \in [0, r)$. A straightforward computation shows that

$$ \Delta \tilde{T}u(t, \theta) + (a + \lambda_1(r)) \tilde{u}(t, \theta) = 0 $$

for all $t \in [0, r]$ and all $\theta \in \mathbb{S}^{n-1}$.
Let \( C(r) = \int_{0}^{r} \left[ \frac{1}{f^{n-1}(s)} \int_{0}^{s} f^{n-1}(t) dt \right] ds = \int_{0}^{r} \frac{V(\sigma)}{S(\sigma)} d\sigma \). Suppose that \( \lambda_1(r) < C(r)^{-1} \) and choose \( a > 0 \) such that \( \lambda_1(r) + a < C(r)^{-1} \). We will show that the operator \( T_{a,\Theta} \) has a fixed point \( u_{a,\Theta} \) in the closed convex subset \( F = \{ u \in X : 0 \leq u \leq \Theta \} \) of \( X \). If \( Tu_{a,\Theta} = u_{a,\Theta} \) then the radial extensions \( \tilde{u}_{a,\Theta} \) and \( \tilde{T}u_{a,\Theta} \) satisfies the following identity.

\[
\Delta \tilde{u}_{a,\Theta}(t, \theta) + (a + \lambda_1(r)) u_{a,\Theta}(t, \theta) = 0
\]

for all \( t \in [0, r] \) and all \( \theta \in S^{n-1} \). But this contradicts the following well known lemma.

**Lemma 2.1.** There is no non-trivial smooth solution to the problem

\[
\begin{align*}
\Delta u + (a + \lambda_1(r)) u &= 0 \quad \text{in } B(r) \\
u &\geq 0 \quad \text{in } \overline{B(r)},
\end{align*}
\]

if \( a > 0 \).

Thus we have that \( \lambda_1(r) \geq C(r)^{-1} \), proving (3).

To finish the proof of Theorem (1.1) we need to show that \( T_{a,\Theta} : F \to F \) has a fixed point. In order to get a fixed point for \( T_{a,\Theta} \), we are going to use the following well known Schauder-Tychonoff fixed point theorem.

**Theorem 2.1.** Let \( F \) be a nonempty closed convex subset of a separated locally convex topological vector space \( X \). Suppose that \( T : F \to F \) is a continuous map such that \( T(F) \) is relatively compact. Then \( T \) has a fixed point.

We are going to show that \( T_{a,\Theta} \) satisfies the hypotheses of Theorem (2.1) if \( \lambda_1(r) + a < C(r)^{-1} \). We start we few lemmas.

**Lemma 2.2.** Let \( F \) be the set

\[
F = \{ u \in X : 0 \leq u(r) \leq \Theta \}
\]

Then \( T \) maps \( F \) into itself.
Proof. Let \( u \in F \) be arbitrary. Clearly, \( Tu \) is continuous. Since \((a + \lambda_1)u \geq 0\), we have that \( \int_0^t \int_0^\sigma \left( \frac{f^{n-1}(s)}{f^{n-1}(\sigma)} \right) [a + \lambda_1(r)]u(s) \, ds \, d\sigma \geq 0 \) thus \( (Tu)(t) \leq \Theta \), for all \( 0 \leq t < r \). On the other hand, since \((a + \lambda_1(r)) < C(r)^{-1} \) and \( 0 \leq u(t) \leq \Theta \), we have that,

\[
(Tu)(t) = \Theta - \int_0^t \int_0^\sigma \left( \frac{f^{n-1}(s)}{f^{n-1}(\sigma)} \right) [a + \lambda_1(r)]u(s) \, ds \, d\sigma,
\]

for all \( 0 \leq t < r \). This proves that \( T(F) \subset F \).

**Lemma 2.3.** The map \( T = T_{a,\Theta} : F \rightarrow F \) is continuous and \( T(F) \) is relatively compact.

**Proof.** Note that \( F \) is closed and convex. Let \( \{u_m\} \subset F \) be a sequence such that \( u_m \rightarrow u \), for some \( u \in F \), (recall that \( \|u\| = \sup_{0 \leq s \leq r} |u(s)| \)). Thus, we have

\[
|Tu_m(t) - Tu(t)| \leq \|u_m - u\| [a + \lambda_1(r)] \int_0^t \int_0^\sigma \left( \frac{f^{n-1}(s)}{f^{n-1}(\sigma)} \right) ds \, d\sigma.
\]

We can conclude that \( Tu_m \) converges uniformly to \( Tu \). Moreover,

\[
|Tu'(t)| \leq \frac{\Theta C^{-1}(r)}{f^{n-1}(t)} \int_0^t f^{n-1}(s)ds = h(t)
\]

Observe that \( h(t) \) is a continuous function on \([0, r]\) thus \( |Tu'(t)| \leq \sup_{[0, r]} h(t) \) which implies that each \( T(F) \) is equicontinuous. Since \( T(F) \) is uniformly bounded, the Ascoli-Arzela theorem implies that \( T(F) \) is relatively compact. \[\square\]
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