VSPs of cubic fourfolds and the Gorenstein locus of the Hilbert scheme of 14 points on $\mathbb{A}^6$

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Abstract

To a cubic fourfold one may associate a geometric object (a hyperkähler manifold) via the theory of VSP and an algebraic object (a finite Gorenstein algebra) via apolarity. We prove that the associated algebra is smoothable if and only if the fourfold lies on the Iliev-Ranestad divisor (which parameterizes certain cubics whose VSP is isomorphic to the Hilbert scheme of two points on a K3 surface). This bridge allows us to give a detailed description of the algebraic side, i.e., the Gorenstein locus of 14 points on $\mathbb{A}^6$ and also to identify the equation of the Iliev-Ranestad divisor as the unique degree 10 invariant of $\text{SL}_6$. As our main technical tool, we develop a relative version of apolarity, building on ideas of Elias-Rossi.

1 Introduction

Let $V$ be a six-dimensional $\mathbb{C}$-vector space and let $F \in \text{Sym}^3 V$ be a cubic.

Let $(F = 0) \subset \mathbb{P}V^* \simeq \mathbb{P}^5$ be a cubic fourfold. It is classically known that for general $F$ the Fano variety of lines on $(F = 0)$ is a hyperkähler fourfold. In [IR01, IR07, RV17] another hyperkähler fourfold is constructed and investigated. Namely, one considers all 10-tuples $\lambda_1, \ldots, \lambda_{10}$ of points of $\mathbb{P}V$ such that $F \in \text{span}(\lambda_1, \ldots, \lambda_{10})$. Such tuple $\Gamma = \{\lambda_1, \ldots, \lambda_{10}\}$ gives a point of the Hilbert scheme of 10 points in $\mathbb{P}V$ and the closure of the set of $\Gamma$’s is denoted by $\text{VSP}(F, 10)$. For general $F$ the variety $\text{VSP}(F, 10)$ is a smooth hyperkähler fourfold [IR01]. The name VSP stands for Variety of Sums of Powers. To prove that $\text{VSP}(F, 10)$ is hyperkähler, Ranestad and Iliev construct a divisor $D_{IR}$ in the space of cubic fourfolds, such that for $[F] \in D_{IR}$ the variety $\text{VSP}(F, 10)$ is isomorphic to a Hilbert scheme of two points on a K3 surface, see [IR01, Theorem 3.7, Proposition 3.11].

In a more algebraic flavour, we have the apolarity (or contraction) action $(-) \triangleright (-) : \text{Sym}V^* \otimes \text{Sym} V \to \text{Sym} V$. This action is easier to describe after a choice of basis $x_1, \ldots, x_6$ of $V$. Then we also have a dual basis $\alpha_1, \ldots, \alpha_n$ of $V^*$ and $\text{Sym} V = \mathbb{C}[x_1, \ldots, x_n]$, $\text{Sym} V^* = \mathbb{C}[\alpha_1, \ldots, \alpha_n]$. We define $(-) \triangleright (-)$ on monomials by the rule

$$(\alpha_1^{a_1} \cdots \alpha_n^{a_n}) \triangleright (x_1^{b_1} \cdots x_n^{b_n}) = \begin{cases} x_1^{b_1-a_1} \cdots x_n^{b_n-a_n} & \text{if all } b_i \geq a_i \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

and extend it bi-linearly to all polynomials. For $G \in \text{Sym} V$, we define $\text{Ann}(G) := \{P \mid P \triangleright G = 0\}$ and $\text{Apolar}(G) := \text{Sym} V^* / \text{Ann}(G)$. This is a finite local Gorenstein algebra, corresponding to a scheme $\text{Spec Apolar}(G) \subset V$ supported at the origin. In particular for a cubic $F$ which is not a cone the algebra

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Spec Apolar \((F)\) has degree 14, hence gives a point on the Hilbert scheme of 14 points on \(A^6\). Since Apolar \((F)\) is Gorenstein, the point Spec Apolar \((F)\) lies in the Gorenstein locus of this Hilbert scheme.

In this paper we investigate the geometry of the Gorenstein locus near Spec Apolar \((F)\) and in fact we obtain a fairly full description of the Gorenstein locus for 14 points of \(A^6\). This is one of the very few cases, when such description is known; indeed little is known about the Hilbert schemes of points on higher dimensional affine or projective spaces, despite much research on their singularities \[IE78, IK99, Erm12\] and components \[CN09, CN11, BdSHJ12, Šiv12, DJNdT17, Jel17\]. The interest in the Hilbert schemes of points and especially their Gorenstein loci is recently renewed by their connections with secant varieties and thus with classical projective geometry, see \[BB14, LO13, IK99, Gał17\]. Note that the Hilbert scheme of \(\mathbb{P}^6\) is covered by the Hilbert schemes of \(A^6\) where \(A^6\) are standard open subsets. Hence the difference between \(A^6\) and \(\mathbb{P}^6\) is not crucial for our considerations. In this paper we have chosen to use \(A^6\).

Recall that \(V\) is a 6-dimension vector space with its natural affine space structure. Denote by \(\mathcal{H}\) the Gorenstein locus of the Hilbert scheme of 14 points on \(V\). Topologically, the scheme \(\mathcal{H}\) has two irreducible components: the smoothable component \(\mathcal{H}_{gen}\) of dimension 84 and another component \(\mathcal{H}_{1661}\) of dimension 76, see \[CJN15\]. A general point of \(\mathcal{H}_{gen}\) corresponds to a set of 14 points on \(V\). Therefore finite schemes \(Z\) corresponding to points in \(\mathcal{H}_{gen}\) are called smoothable.

By \[CJN15\] the points of \(\mathcal{H}_{1661}\) correspond bijectively to finite irreducible subschemes \(Z \subset V\) such that the local algebra \(H^0(Z, O_Z)\) has Hilbert function \((1, 6, 6, 1)\). Let \(\mathcal{H}_{1661}^{gr} \subset \mathcal{H}_{1661}\) be the set corresponding to \(Z\) supported at the origin of \(V\) and invariant under the dilation action of \(\mathbb{C}^*\), i.e., such that \(H^0(V, I_Z)\) is homogeneous. Then \(\mathcal{H}_{1661}^{gr} \subset \mathcal{H}_{1661}\) is a closed subset. We endow \(\mathcal{H}_{1661}\) with a scheme structure, which we formally define as the scheme-theoretic closure \(\mathcal{H}_{1661} = \mathcal{H} \setminus \mathcal{H}_{gen}\). Under this definition it is not clear whether \(\mathcal{H}_{1661}\) is reduced, although we show that indeed it is (we do not know whether \(\mathcal{H}_{gen}\) is reduced). We endow \(\mathcal{H}_{1661}^{gr}\) with a reduced scheme structure.

**Theorem 1.1** (description of the Gorenstein locus of Hilbert scheme of 14 points on \(A^6\)). With notation as above, the following hold.

1. The component \(\mathcal{H}_{1661}\) is connected and smooth (hence reduced).
2. There is a morphism
   \[
   \pi : \mathcal{H}_{1661} \to \mathcal{H}_{1661}^{gr}
   \]
   which makes \(\mathcal{H}_{1661}\) the total space of a rank 21 vector bundle over \(\mathcal{H}_{1661}^{gr}\).
3. The set \(\mathcal{H}_{1661}^{gr}\) is canonically isomorphic to an open subset of \(\mathbb{P}(\text{Sym}^3 V)\) consisting precisely of cubics which are not cones. We denote this subset by \(\mathbb{P}(\text{Sym}^3 V)_{1661}\).
4. The set theoretic intersection \(\mathcal{H}_{gen} \cap \mathcal{H}_{1661}\) is a prime divisor inside \(\mathcal{H}_{1661}\) and it is equal to \(\pi^{-1}(D_{IR})\), where \(D_{IR} \subset \mathcal{H}_{1661}^{gr} \subset \mathbb{P}(\text{Sym}^3 V)_{1661}\) is the restriction of the Iliev-Ranestad divisor. We get the following diagram of vector bundles:

\[
\begin{array}{ccc}
\mathcal{H}_{gen} \cap \mathcal{H}_{1661} & \subset & \mathcal{H}_{1661} \\
\downarrow & & \downarrow \\
D_{IR} & \subset & \mathbb{P}(\text{Sym}^3 V)_{1661}
\end{array}
\]

The most surprising part of this theorem is that the intersection of \(\mathcal{H}_{gen} \cap \mathcal{H}_{1661}\) — which parameterizes smoothable subschemes from \(\mathcal{H}_{1661}\) — is essentially given by the divisor \(D_{IR}\), i.e., by a geometric condition. This, along with the proof of reducedness of \(\mathcal{H}_{1661}\), are the most nontrivial parts of the proof.
The map $\pi$ is defined at the level of points as follows. We take $[Z] \in \mathcal{H}_{1661}$. Translating, we fix its support at $0 \in V$. Then we replace $H^0(Z, O_Z)$ by its associated graded algebra which in this very special case is also Gorenstein. We take $\pi([Z])$ to be the point corresponding to $\text{Spec } \text{gr} H^0(Z, O_Z)$ supported at the origin of $V$. The identification of $\mathcal{H}_{1661}$ with $\mathbb{P} (\text{Sym}^3 V)_{1661}$ in Point 3 is done canonically using Macaulay’s inverse systems. Note that the complement of $\mathbb{P} (\text{Sym}^3 V)_{1661}$ has codimension greater than one, hence divisors on $\mathbb{P} (\text{Sym}^3 V)$ and $\mathcal{H}_{1661}$ are identified via restriction and closure. We describe a set-theoretic equation of $D_{IR} \subset \mathbb{P} (\text{Sym}^3 V)$ explicitly as an $120 \times 120$ determinant, see the discussion before Lemma 3.6. We also note that $D_{IR} \subset \mathbb{P} (\text{Sym}^3 V)$ is the unique $\text{SL}(V)$-invariant divisor of degree 10.

The major part of this paper is the proof of Theorem 1.1, which is abstracted into three Claims, see Remark 3.11.

It also follows from Theorem 1.1 that both components of $\mathcal{H}$ are rational. It is an open question whether all components of Hilbert schemes of points are rational. Roggero and Lella prove that components satisfying strong smoothness assumptions are rational, see [LR11]. The component $\mathcal{H}_{1661}$ is also the unique known Gorenstein component with dimensional smaller than $\mathcal{H}_{gen}$, see Remark 3.12.

The major part of this paper is the proof of Theorem 1.1 which is abstracted into three Claims, see Section 3. The main underlying tool is given by relative Macaulay’s inverse systems, presented in Section 2 which we now discuss. Recall the definition of the apolarity action from (1). It is classically known (see [Ems78, IK99]) that this action parameterizes all finite Gorenstein subschemes as follows.

| Finite Gorenstein subschemes | Polynomials/power series |
|-----------------------------|---------------------------|
| $Z \subset \mathbb{C}^n$ is equal to | $\text{Spec } \text{Apolar} (F)$ for some $F \in \mathbb{C}_{\text{dp}}[[x_1, \ldots, x_n]]$ |
| $Z \subset \mathbb{C}^n$ supported at the origin is equal to | $\text{Spec } \text{Apolar} (F)$ for some $F \in \mathbb{C}_{\text{dp}}[x_1, \ldots, x_n]$ |
| $Z \subset \mathbb{C}^n$ supported at the origin, invariant under dilation is equal to | $\text{Spec } \text{Apolar} (F)$ for some homogeneous $F \in \mathbb{C}_{\text{dp}}[x_1, \ldots, x_n]$ |

Inspired by [Ems78, ER17, MT18], we generalize the above description to families over an affine Noetherian base $A$, as follows (see Section 2.2 for details).

| Families of Gorenstein subschemes | Polynomials/power series |
|-----------------------------------|---------------------------|
| Family of $Z \subset \mathbb{C}^n$ is locally | $\text{Spec } \text{Apolar} (F)$ for some $F \in A[[x_1, \ldots, x_n]]$ |
| Family of $Z \subset \mathbb{C}^n$, each $Z$ supported at the origin is locally | $\text{Spec } \text{Apolar} (F)$ for some $F \in A[x_1, \ldots, x_n]$ |
| Family of $Z \subset \mathbb{C}^n$, each $Z$ supported at the origin and invariant under dilation is locally | $\text{Spec } \text{Apolar} (F)$ for some homogeneous $F \in A[x_1, \ldots, x_n]$ |

The assumptions that the base is affine and that the description holds only locally can be removed at the expense of changing language to a more geometric one, for example one needs to replace $F$ by a line bundle.

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The computations were made using Magma \cite{BCP97}, Macaulay2 \cite{GS} and LiE \cite{LiE}.

2 Preliminaries on finite schemes

Let $Z$ be an irreducible finite scheme over $\mathbb{C}$. Then $R = H^0(Z, O_Z)$ is a local $\mathbb{C}$-algebra with maximal ideal $m_R$. The associated graded algebra of $R$ is $\text{gr} R = \bigoplus_{i \geq 0} m_R^i/m_R^{i+1}$. The Hilbert function of $Z$ (or $R$) is defined as $H(i) := \dim_{\mathbb{C}} m_R^i/m_R^{i+1}$. The algebra $R$ is Gorenstein if and only if the annihilator of $m_R$ is a one-dimensional $\mathbb{C}$-vector space. If $R$ is Gorenstein then the socle degree of $R$ is the largest $d$ such that $H(d) > 0$. Let $R$ be Gorenstein of socle degree $d$. We will use the following properties of the Hilbert function (see \cite{BB14, Lar94}):

1. If $R$ is graded, then $H(d - i) = H(i)$ for all $i$.
2. If $H(d - i) = H(i)$ for all $i$, then $\text{gr} R$ is also Gorenstein.
3. If $H = (1, n, n, 1)$ then $\text{gr} R$ is isomorphic to $R$, see \cite{ER12, ER15}.

2.1 Macaulay’s inverse systems

See \cite{ER12, IK99, Jel16} for general facts about Macaulay’s inverse systems. In this section we work over an arbitrary algebraically closed field $k$ and with a finite dimensional $k$-vector space $V$ with $n = \dim V$. For the main part of the article, in Section \ref{sec:examples} we only use the case $k = \mathbb{C}$ and 6-dimensional $V$.

Let $S := \text{Sym} V^*$ be a polynomial ring. Then we may view $P = \text{Hom}_k(S, k)$ as an $S$-module via $(s_1 f)(s_2) = f(s_1 s_2)$. It is common to introduce dual bases $\alpha_1, \ldots, \alpha_n$ on $V^*$ and $x_1, \ldots, x_n$ on $V$. Then $S$ becomes $k[\alpha_1, \ldots, \alpha_n]$ and $P = k_{dp}[x_1, \ldots, x_n]$ the divided powers power series ring, with the action

$$\left(\alpha_1^{a_1} \cdots \alpha_n^{a_n}\right) \cdot (x_1^{b_1} \cdots x_n^{b_n}) = \begin{cases} x_1^{b_1 - a_n} \cdots x_n^{b_n - a_n} & \text{if all } b_i \geq a_i \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

In fact $\alpha_i$ acts on $P$ as a derivation. By abuse of notation we will denote the subring $k_{dp}[x_1, \ldots, x_n] \subset P$ by $\text{Sym} V$. The abuse here is that $\text{Sym} V$ is a divided power ring and not a polynomial ring. Eventually we work over fields of characteristic zero, where these notions agree in a non-obvious way, see \cite[Example A.5, p. 266]{IK99}. In this paper the ring structure of $P$ is used only a few times and the $S$-module structure is far more important for our purposes.

If $I \subset S$ is homogeneous then for every $d$ by $I_d^+ \subset \text{Sym}^d V$ we denote the perpendicular space.

Let $F \in P$. By $\text{Ann} (F)$ we denote its annihilator in $S$ and by $\text{Apolar} (F) = S/\text{Ann} (F)$ its apolar algebra. Note that $S$-modules $\text{Apolar} (F)$ and $S \cdot F$ are isomorphic. As a special case, we may take an apolar algebra of $F \in \text{Sym} V = k_{dp}[x_1, \ldots, x_n] \subset P$. The crucial result by Macaulay is the following theorem.

**Theorem 2.1.** The algebra $\text{Apolar} (F)$ is a local artinian Gorenstein algebra supported at $0 \in \text{Spec} S$. Conversely, every such algebra is equal to $\text{Apolar} (F)$ for a (non-unique) $F \in \text{Sym} V$.  

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**Proof-sketch.** We include one implication of this well-known proof to motivate the relative version given in the next section. Namely for an artinian Gorenstein algebra $R = S/I$ supported at the origin we find an $F \in \text{Sym} V$ such that $R = \text{Apolar}(F)$.

Since $R$ is artinian, its dualising module $\omega_R$ is isomorphic to $\text{Hom}_k(R, k)$. Since $R$ is Gorenstein, this dualising module is principal (see [Eis95, Chapter 21]). Take a generator $F$ of this module. The surjection $S \to R$ gives an inclusion $\omega_R = \text{Hom}_k(R, k) \subset \text{Hom}_*(S, k) = I$. In this way $\omega_R$ becomes an $S$-module, whose annihilator is $I$. Since $\omega_R$ is principal generated by $F$, we get that $\text{Ann}(F) = I$. Therefore $R = \text{Apolar}(F)$. We have $m^{\text{len} R} \subset I$, where $m$ is the ideal of $0 \in \text{Spec} S$. Hence $m^{\text{len} R} \subset \text{Ann}(F)$ which forces $F \in \text{Sym} V$. 

The freedom of choice of $F$ is well understood, see for example [Jel16].

If the algebra in question is naturally graded, we may take $F$ to be homogeneous. This is the case of our main interest. By $S_e$ we denote forms of degree $r$. Let $F$ be homogeneous of degree $d$ and $I = \text{Ann}(F)$. Then $S_e \cdot F$ is annihilated by $I_{d-r}$ and in fact $S_e \cdot F = I_{d-r}^d$. The symmetry of the Hilbert function of $\text{Apolar}(F)$ shows that $\dim I_{d-r}^d = \dim I_{d-r}^1$ for all $r$.

Numerous examples of polynomials and corresponding artinian Gorenstein algebras can be found in the literature, for instance in [BBKT15, IK99, CN11].

**Remark 2.2.** If $F \in \text{Sym} V$ is arbitrary, but such that gr $\text{Apolar}(F)$ is also Gorenstein, then gr $\text{Apolar}(F)$ is isomorphic to $\text{Apolar}(F_d)$, where $F_d$ is the top degree form. In particular, if $F$ is of degree three and $H_{\text{Apolar}(F)} = (1, n, n, 1)$ then gr $\text{Apolar}(F) = \text{Apolar}(F_3)$. Also it follows that $H_{\text{Apolar}(F)} = (1, n, n, 1)$ if and only if $H_{\text{Apolar}(F_3)} = (1, n, n, 1)$, see [Iar94, Chapter 1].

**Remark 2.3.** Let $F \in \text{Sym} V$ be a polynomial of degree three such that $H_{\text{Apolar}(F)} = (1, n, n, 1)$ where $n = \dim V$; in other words the Hilbert function is maximal. It will be important in the following to note that $\text{Ann}(F)$ and $\text{Apolar}(F)$ only partially depend on the lower homogeneous components of $F$. To see this, it is enough to write explicitly the $S$-module $S \cdot F$.

$$S \cdot F = \text{span} \{ F, \{ \alpha_i \cdot F \mid i \}, \{ \alpha_i \alpha_j \cdot F \mid i, j \}, \{ \alpha_i \alpha_j \alpha_k \cdot F \mid i, j, k \} \} =$$

$$= \text{span} \{ F, \{ \alpha_i \cdot F \mid i \} \} \oplus \text{Sym}^{\leq 1} V = \text{span} (F_3 + F_2) \oplus \text{span} (\alpha_i \cdot F_3) \oplus \text{Sym}^{\leq 1} V.$$

Therefore $S \cdot F$ is uniquely defined by giving $F_3$ and $F_2$ mod $\text{span} (\alpha_i \cdot F_3) \mid i \}$, up to multiplication by a constant.

Finally we should give at least one explicit example.

**Example 2.4.** Let $F = x_1^3 + \ldots + x_n^3 \in k_{d_p}[x_1, \ldots, x_n]$. Then $\text{Ann}(F) = (\alpha_i \alpha_j \mid i \neq j) + (\alpha_i^2 - \alpha_j^2 \mid i, j)$. Hence $H_{\text{Apolar}(F)} = (1, n, n, 1)$.

### 2.2 Relative Macaulay’s inverse systems

The main result of this section is a local description of this universal family over the Gorenstein locus or, equivalently, any finite flat family with Gorenstein fibers (see Proposition 2.10 and Proposition 2.12). This local description involves certain choices. This local description, in a special case of $A = k[[t]]$, first appeared in [Eis78, Proposition 18], unfortunately without a proof.

With respect to the setting of the previous section there are two generalisations: first, we consider families instead of single subschemes and second, we consider all finite subschemes, in particular reducible ones (similar description of finite subschemes appeared independently in [Mou17]). We will see that both of there generalisations are not too hard, but in technical terms they force us to work with (divided powers) power series rings and base change, which accounts for some intransparency in the proofs. To
help the reader separate the results of Proposition 2.10 from the technicalities we provide the following intuitive comparison between families and apolar algebras. Here by dilation we mean the diagonal action of the torus \( k^* \). All \( Z \) considered are finite. All families are over an affine, Noetherian base \( A \).

| Finite Gorenstein subschemes | Polynomials/power series |
|-----------------------------|--------------------------|
| \( Z \subset \mathbb{C}^n \) | is equal to \( \text{Apolar}(F) \) for some \( F \in \mathbb{C}_{dp}[x_1, \ldots, x_n] \) |
| \( Z \subset \mathbb{C}^n \) supported at the origin | is equal to \( \text{Apolar}(F) \) for some \( F \in \mathbb{C}_{dp}[x_1, \ldots, x_n] \) |
| \( Z \subset \mathbb{C}^n \) supported at the origin, invariant under dilation | is equal to \( \text{Apolar}(F) \) for some homogeneous \( F \in \mathbb{C}_{dp}[x_1, \ldots, x_n] \) |

| Families of Gorenstein subschemes | Polynomials/power series |
|-------------------------------|--------------------------|
| Family of \( Z \subset k^n \) | is locally \( \text{Apolar}(F) \) for some \( F \in A[[x_1, \ldots, x_n]] \) |
| Family of \( Z \subset k^n \), each \( Z \) supported at the origin | is locally \( \text{Apolar}(F) \) for some \( F \in A[[x_1, \ldots, x_n]] \) |
| Family of \( Z \subset k^n \), each \( Z \) supported at the origin and invariant under dilation | is locally \( \text{Apolar}(F) \) for some homogeneous \( F \in A[x_1, \ldots, x_n] \) |

Conversely, given \( F \in A[[x_1, \ldots, x_n]] \) as above one can wonder, whether \( \text{Apolar}(F) \) is a finite and flat family. This is a subtle question, but over a reduced ring \( A \) it is necessary and enough that for each \( t \in \text{Spec} \, \mathbb{A} \) the algebra \( \text{Apolar}(F_t) \) is finite of length independent of \( t \). See Proposition 2.12.

### 2.2.1 Apolar families

Let \( k \) be an arbitrary algebraically closed field. Let \( A \) be a Noetherian \( k \)-algebra and \( V \) be a finite dimensional \( k \)-vector space. We define contraction action identically as in the previous section, but using coefficients from \( A \) instead of \( k \). Denote \( \text{Sym}_A(\_):= A \otimes_k \text{Sym}(\_) \). Denote by \( S_A \) the space \( \text{Sym}_A V^* \) and by \( P_A \) the space \( \text{Hom}_A(S_A, A) \). We have a natural action of \( S_A \) on \( P_A \) by precomposition: for \( s \in S_A \) and \( f : S_A \to A \) we define \((sf)(t) = f(st)\). Abusing notation, we write \( \text{Sym} V \) for the divided powers ring. Then we may identify \( \text{Sym} V^* \) with \( (\text{Sym} V)^* \) and view elements of \( \text{Sym}_A V \) as elements of \( P_A \). In coordinates, the above description is much easier and down to earth:

\[
S_A = A[\alpha_1, \ldots, \alpha_n] \quad \text{and} \quad \text{Sym}_A V = A[x_1, \ldots, x_n] \subset P_A = A[[x_1, \ldots, x_n]].
\]

The precomposition action is the \( A \)-linear action described by Formula (2) in the case \( A = k \). Note that this action is \( A \)-linear, hence behaves well with respect to a base change.

**Definition 2.5.** For \( F \in P_A \) we define an ideal \( \text{Ann}(F) \subset S_A \) by \( \text{Ann}(F) := \{ \sigma \in S_A \mid \sigma \cdot F = 0 \} \) and call it the **apolar ideal** of \( F \). The algebra \( \text{Apolar}(F) := S_A / \text{Ann}(F) \) is called the **apolar algebra** of \( F \). It gives an affine **apolar family**

\[
a_F : \text{Spec} \, \text{Apolar}(F) \to \text{Spec} \, A,
\]

which it is naturally embedded in \( V \times \text{Spec} \, A \to \text{Spec} \, A \).

Note that

1. if \( F \in \text{Sym}_A V \), then \( a_F \) is finite,
2. \( S_A \)-modules \( \text{Apolar}(F) \) and \( S_A \cdot F \) are isomorphic.

Different choices of \( F \) lead to the same apolar family. But it is crucial for our purposes to keep track of \( F \). So we always assume it to be fixed.

**Convention 2.6.** When speaking of \( \text{Apolar}(F) \) we always implicitly assume that \( F \) is fixed.
Let us see some examples.

Example 2.7. Assume \( \text{char } k \neq 2 \). Take \( A = k[t] \) and \( F = tx^2 + x_1x_2 \in A[x_1, x_2] \). Then \( S_A \bowtie F = \text{span}(F, x_1, x_2, 1) \) so that \( \text{Ann}(F) = (\alpha_1^2 - t\alpha_1\alpha_2, \alpha_2^2) \). The fiber of \( \text{Spec Apolar}(F) \) over \( t = \lambda \) is \( k[\alpha_1, \alpha_2]/(\lambda^2 - \lambda\alpha_1\alpha_2, \alpha_2^2) = \text{Apolar}(\lambda x_1^2 + x_1x_2) = \text{Apolar}(F_\lambda) \).

Example 2.8. Take \( A = k[t] \) and \( F = tx \in A[x] \). Then \( \text{Ann}(F) = (\alpha^2) \), so that the fibers of the associated apolar family are all equal to \( k[\alpha]/\alpha^2 \). For \( t = \lambda \) non-zero we have \( k[\alpha]/\alpha^2 = \text{Apolar}(F_\lambda) \), but for \( t = 0 \) we have \( F_0 = 0 \), so the fiber is not the apolar algebra of \( F_0 \).

Intuitively, we think of \( \text{Apolar}(F) \) as a family of apolar algebras \( \text{Apolar}(F_t) \), where \( t \in \text{Spec } A \). Example 2.8 shows that the fiber of \( \text{Apolar}(F) \) over a point \( t \) may differ from \( \text{Apolar}(F_t) \). To avoid such pathology we introduce an appropriate notion of continuity of apolar family \( \text{Spec Apolar}(F) \). This notion, among other things, will guarantee that the fibers are apolar algebras of corresponding restrictions of \( F \).

Definition 2.9. An apolar family \( \text{Spec Apolar}(F) \) is flatly embedded if the \( A \)-module \( P_A/(S_A \bowtie F) \) is flat.

Note that if an apolar family is flatly embedded, then \( S_A \bowtie F \) is a flat \( A \)-module, thus also the isomorphic module \( \text{Apolar}(F) \) is flat. Note also that in pathological Example 2.8 the \( A \)-module \( S_A \bowtie F = \text{Apolar}(F) \) is flat. Thus the flatness of \( S_A \bowtie F \) is strictly weaker than being flatly embedded.

If \( \text{Spec Apolar}(F) \) is flatly embedded, then its fiber over \( t \in \text{Spec } A \) is isomorphic to \( \text{Spec Apolar}(F_t) \), see Proposition 2.10 below.

Definition 2.9 is very similar to the definition of a subbundle, where we require that the cokernel is locally free. The technical difference is that the \( A \)-module \( P_A/(S_A \bowtie F) \) is usually very far from being finitely generated.

2.2.2 Local description

We state the two main results of this section below. On one hand, we show that each finite flat family with Gorenstein fibers is locally a flatly embedded apolar family. On the other hand, we give descriptions of such families and their constructions over a reduced base.

Proposition 2.10 (Local description of families). Let \( T \) be a locally Noetherian scheme. Let \( X \subset V \times T \rightarrow T \) be an embedded finite flat family of Gorenstein algebras. This family is isomorphic, after possibly shrinking \( T \) to a neighborhood of a given closed point, to a flatly embedded finite family \( a_F : \text{Spec Apolar}(F) \rightarrow \text{Spec } A \) for \( F \in P_A \). Moreover

1. if all fibers are irreducible supported at \( 0 \in V \) then we may take \( F \in \text{Sym}_A V \).

2. if all fibers are invariant under the dilation action of \( k^* \) and \( T \) is reduced, then we may take a homogeneous \( F \in \text{Sym}_A V \).

Example 2.11. Let \( V = \mathbb{A}^1 = \text{Spec } k[x] \). Consider a branched double cover

\[ X = \text{Spec } k[\alpha, t]/(\alpha^2 - t) \rightarrow T = \text{Spec } k[t]. \]

Then \( X_0 = \text{Spec } k[\alpha]/\alpha^2 \) and \( \omega_0 \) is generated by \( F_0 = x \). This \( F_0 \) lifts to an element \( F = x \cdot \sum_{n \geq 0} (tx^2)^n \) of \( k[t][[x]] \) and one checks that indeed \( X = \text{Apolar} \left( x \cdot \sum_{n \geq 0} (tx^2)^n \right) \). In this case no shrinking of \( T \) is necessary.
Before we state the next result we explain one more piece of notation. Let \( \varphi : A \to B \) be any homomorphism and \( F \in P_A \). By \( \varphi(F) \in P_B \) we denote the series obtained by applying \( \varphi \) to coefficients of \( F \).

**Proposition 2.12.** Let \( F \in P_A \). Let \( a_F : \text{Spec Apolar}(F) \to \text{Spec } A \) be an apolar family and suppose it is finite. Then the following holds:

1. If the apolar family is flatly embedded, then its base change via any homomorphism \( \varphi : A \to B \) is equal to \( \text{Spec Apolar}(\varphi(F)) \to \text{Spec } B \). In particular, the fibre of \( a_F \) over \( t \in \text{Spec } A \) is naturally \( \text{Spec Apolar}(F_t) \). Thus the length of \( \text{Apolar}(F_t) \) is independent of the choice of \( t \).

2. If \( A \) is reduced and the length of \( \text{Apolar}(F_t) \) is independent of the choice of \( t \in \text{Spec } A \), then \( a_F \) is flatly embedded.

**Example 2.13.** Let \( w \in V \) and denote \( \exp_{dp}(w) := \sum_{i \geq 0} w^i \in P \).

Pick a polynomial \( f \in \text{Sym} V \). It gives a subscheme \( \text{Spec Apolar}(f) \subset V \) supported at zero. We will now construct a family which moves the support of this subscheme along the line spanned by \( w \). The line is chosen for clarity only, the same procedure works for arbitrary subvariety of \( V \).

For every linear form \( \alpha \in V^* \) we have

\[
\alpha \cdot (f \exp_{dp}(w)) = (\alpha \cdot f) \exp_{dp}(w) + c \cdot f \exp_{dp}(w) = ((\alpha + c \cdot f) \cdot \exp_{dp}(w),
\]

where \( c = \alpha \cdot w \in k \). Hence for every polynomial \( \sigma(x) \in \text{Sym} V^* \) we have \( \sigma(x) \cdot (f \exp_{dp}(w)) = 0 \) if and only if \( \sigma(x + w) \cdot f = 0 \). It follows that \( \text{Spec Apolar}(f \exp_{dp}(w)) \) is the translate of \( \text{Spec Apolar}(f) \) by the vector \( w \), in particular it is supported on \( w \) and abstractly isomorphic to \( \text{Apolar}(f) \). Hence \( \text{Apolar}(f) \) and \( \text{Apolar}(f \exp_{dp}(w)) \) have the same length. By Proposition 2.12 the family \( \text{Spec Apolar}(f \exp_{dp}(tw)) \to \text{Spec } \mathbb{k}[t] \) is flatly embedded and geometrically corresponds to deformation by moving the support of \( \text{Spec Apolar}(f) \) along the line spanned by \( w \). Its restriction to \( \mathbb{k}[\varepsilon] = \mathbb{k}[t]/t^2 \) gives \( \text{Spec Apolar}(f + \varepsilon w f) \) corresponding to the tangent vector pointing towards this deformation.

### 2.2.3 Proofs of Proposition 2.10 and Proposition 2.12

**Proof of Proposition 2.10.** Consider the \( \mathcal{O}_X \)-sheaf \( \omega = \mathcal{H}om_T(\pi_* \mathcal{O}_X, \mathcal{O}_T) \), where the multiplication is by precomposition. Since the map \( X \to T \) is finite and flat, the base change to a fiber over \( t \in T \) is equal to \( \mathcal{H}om_{\kappa(t)}(\mathcal{O}_{X_t}, \kappa(t)) \), where \( \kappa(t) \) is the residue field in \( t \). This is the canonical module of the fibre, see [Eis95, Chapter 21]. Note also that \( \mathcal{O}_X \) and \( \omega \) taken as \( \mathcal{O}_T \)-sheaves are locally free of the same rank.

Choose a \( k \)-point \( 0 \in T \). The \( \mathcal{O}_{X_0} \)-module \( \omega|_{X_0} = \mathcal{H}om_{\kappa(0)}(\mathcal{O}_{X_0}, \kappa(0)) \) is principal because the fibre \( X_0 \) is Gorenstein. Choose a generator \( F \) of this module and lift it to \( \omega \). By Nakayama’s Lemma, after possibly shrinking \( T \), we may assume that \( F \) generates \( \omega \) as an \( \mathcal{O}_X \)-module. We get a surjection \( \mathcal{O}_X \to \mathcal{O}_X \cdot F = \omega \). This is in particular a surjection of locally free \( \mathcal{O}_T \)-sheaves of the same rank. Hence it is an isomorphism, so that \( \omega \) is a trivial \( \mathcal{O}_X \)-invertible sheaf, trivialised by \( F \).

Shrinking \( T \) even further, we may assume that \( T = \text{Spec } A \) is affine. Now, \( X \subset V \times T \) is embedded as a \( T \)-subscheme. Therefore \( \omega \subset \mathcal{H}om_T(\mathcal{O}_{V \times T}, \mathcal{O}_T) \) as \( \mathcal{O}_{V \times T} \)-modules. Since \( \omega \) is annihilated by the ideal \( I_X \subset \mathcal{O}_{V \times T} \) and is torsion free as the \( \mathcal{O}_X \)-module, we have \( \text{Ann}(\omega) = I_X \). Furthermore \( F \in H^0(X, \omega) \subset \mathcal{H}om_A(S_A, A) = P_A \) generates \( \omega \) as an \( \mathcal{O}_X \)-module or as an \( \mathcal{O}_{V \times T} \)-module, hence \( \text{Ann}(\omega) = \text{Ann}(F) \) and \( X = \text{Spec Apolar}(F) \) as families embedded into \( V \times T \).

Since \( X \to T \) is flat, it follows that \( \text{Apolar}(F) \) is \( A \)-flat. The sequence

\[
0 \to \text{Ann}(F) \to S_A \to \text{Apolar}(F) \to 0
\]

(3)
shows that $\text{Ann}(F)$ is also $A$-flat. Moreover the $A$-module $\text{Apol}(F)$ is flat and finitely generated, hence is projective, so that $S_A \simeq \text{Ann}(F) \oplus \text{Apol}(F)$ as $A$-modules. Since $A$ is Noetherian the module $\text{Hom}_A(S_A, A)$ is flat by [Lam99, 4.47, p. 139]. Then $\text{Hom}_A(\text{Ann}(F), A)$ is flat as well, as a direct summand of a flat module. Applying $\text{Hom}_A(-, A)$ to the sequence (3), which splits, we get

$$0 \to \text{Hom}_A(\text{Apol}(F), A) \to \text{Hom}_A(S_A, A) \to \text{Hom}_A(\text{Ann}(F), A) \to 0.$$ 

By definition $\text{Hom}_A(S_A, A) = P_A$ and $\text{Hom}_A(\text{Apol}(F), A) = H^0(\omega)$, which it is generated by $F$. Therefore the sequence can be written as

$$0 \to S_A \cdot F \to P_A \to \text{Hom}_A(\text{Ann}(F), A) \to 0.$$ 

Hence $\text{Hom}_A(\text{Ann}(F), A) \simeq P_A/(S_A \cdot F)$ so this last module is $A$-flat. By definition, $\text{Apol}(F)$ is flatly embedded. This concludes the general part of the proof.

Now we pass to the proof of Points 1. and 2.

Point 1. Suppose, that all fibres of $X \to T$ are supported at the origin of $V$. Take the ideal $m$ of the origin in $V \times T$ and its restriction to $X$, denoted $m_X$. Each fiber $X_t$ over a closed point is a scheme of length $d$ supported at the origin, so $m^d_X$ is zero on each fiber of $X \to T$. Thus $m^d_X$ is contained in the radical. This radical is nilpotent, as $X$ is Noetherian being finite over Noetherian $T$. Therefore $m^d_X$ is zero for $e$ large enough. In other words, $m^d$ annihilates $F$ obtained above, so in particular all monomials of degree at least $de$ annihilate it. Hence the corresponding coefficients of $F$ are zero and so $F \in \text{Sym}_A^{de} V \subset \text{Sym}_A V$. Therefore $F$ is a polynomial.

Point 2. By assumption $A$ is reduced. We already know that $m^d$ lies in $\text{Ann}(F)$, where $d$ is large enough (in fact, by reducedness $d = \deg(X \to T)$ is sufficient), so that $F \in \text{Sym}_A V$. Consider the image of the ideal $\text{Ann}(F)$ in $S_A/m^d$. We want to show that it is homogeneous. We know that it is homogeneous when restricting to each fiber. Pick an element of this ideal and its homogeneous coordinate modulo the ideal. Such coordinate vanishes on each fiber, hence is nilpotent. As $A$ is reduced, it is zero. This shows that $\text{Ann}(F)$ is homogeneous. Therefore $S_A \cdot F$ is spanned by homogeneous elements. Now we may pick a homogeneous generator of the fiber and lift it (after possibly shrinking as above) to a homogeneous generator of $S_A \cdot F$. \hfill \qed

Proposition 2.10 underlies our philosophy in the analysis of Hilbert scheme: we think of $P_A$ as a local version of a versal family (no uniqueness though).

Remark 2.14. The above proof shows that, without restricting $T$, each finite flat embedded and Gorenstein family $\pi : X \to T$ is isomorphic to $\text{Spec}_T \text{Apol}(\omega) \to T$, where $\omega \subset O_T \otimes \text{Sym} V$ is an invertible sheaf. We will not use this fact.

The $A$-module $P_A$ is far from being finitely generated and this technical problem needs to be dealt with before the next proof is conducted. In this following Lemma 2.15 we show that we can harmlessly replace $P_A$ by a suitable finitely generated $A$-submodule of $P_A$.

Lemma 2.15. Let $N \subset P_A$ be a $S_A$-module, which is finitely generated as an $A$-module. Then there exist a finitely generated flat $A$-submodule $P_{A,N} \subset P_A$ and a projection $P_A \to P_{A,N}$, such that

1. The composition $N \hookrightarrow P_A \to P_{A,N}$ is injective.

2. The module $P_{A,N}/N$ is flat over $A$ if and only if the module $P_A/N$ is flat over $A$.

3. For every homomorphism $A \to B$ the induced map $B \otimes P_{A,N} \to P_B$ is injective and its image is a direct summand of $P_B$ as a $B$-module.
Proof. Since $N$ is finitely generated over $A$, also the $A$-module $\text{Hom}_A(N, N)$ is finitely generated. The $S_A$-module structure gives a map $S_A \to \text{Hom}_A(N, N)$. Take a finite set of monomials in $S_A$ which generate the image as $A$-module and denote this set by $\mathcal{M}$. For every monomial $m$ not in $\mathcal{M}$ there are monomials $m_i \in \mathcal{M}$ and coefficients $a_i \in A$ such that $m - \sum a_i m_i$ annihilates $N$. Pick $f \in N$, then $(m - \sum a_i m_i) \cdot f = 0$. But $(m - \sum a_i m_i) \cdot f$ is a functional, whose value on $1 \in S_A$ is $f(m - \sum a_i m_i)$. Hence $f(m - \sum a_i m_i) = 0$, so $f(m) = \sum a_i f(m_i)$. Thus the values of $f$ on monomials from $\mathcal{M}$ determine $f$ uniquely. In other words, the composition $N \hookrightarrow P_A \to \text{Hom}_A(A(\mathcal{M}), A)$ is injective, where $A(\mathcal{M})$ is the $A$-submodule of $S_A$ with basis $\mathcal{M}$. Let

$$P_{A,N} = \text{Hom}_A(A(\mathcal{M}), A)$$

with the projection $P_A \to P_{A,N}$ coming from restriction of functionals to $A(\mathcal{M})$. Then Point 1 follows by construction of $P_{A,N}$. The restriction $P_A \to \text{Hom}_A(A(\mathcal{M}), A)$ has a natural section $\text{Hom}_A(A(\mathcal{M}), A) \hookrightarrow \text{Sym}_A V \subset P_A$. We take this section to obtain an inclusion $P_{A,N} \subset P_A$. This section also proves that we have an inner direct product $P_A \simeq P_{A,N} \oplus K$ for an $A$-module $K$. Since $A$ is Noetherian the $A$-module $P_A$ is flat by [Lam99, 4.47, p. 139]. Hence also $P_{A,N}$ and $K$ are flat. Snake Lemma applied to the diagram

$$\begin{array}{cccccc}
0 & \to & N & \to & P_A & \to & P_A/N & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & N & \to & P_{A,N} & \to & P_{A,N}/N & \to & 0
\end{array}$$

gives an extension

$$0 \to K \to P_A/N \to P_{A,N}/N \to 0,$$

whereas the projection $P_A \to K$ trivialises it to a direct sum $P_A/N \simeq P_{A,N}/N \oplus K$. Since $K$ is flat over $A$ we get that $P_{A,N}/N$ is flat over $A$ if and only if $P_A/N$ is flat over $A$; this proves Point 2.

For the Point 3 note that the homomorphism $B \otimes P_{A,N} \to B \otimes P_A \to P_B$ sends $B \otimes \text{Hom}_A(A(\mathcal{M}), A)$ isomorphically to the direct summand $\text{Hom}_B(B(\mathcal{M}), B) \subset P_B$, so this homomorphism is injective. \hfill \square

Proof of Proposition 2.12. The $A$-module $N = S_A \cdot F$ is finite by assumption. By Lemma 2.15 for $N = S_A \cdot F$ we get a module $P_{A,N}$. The Lemma says that:

1. $P_{A,N}$ is flat and finite over $A$,
2. The module $P_A/N$ is flat if and only if the module $P_{A,N}/N$ is flat,
3. the homomorphism $B \otimes P_{A,N} \to P_B$ is injective.

Proof of Point 1. Suppose that the apolar family is flatly embedded. Then $P_{A,N}/N$ is flat and $B \otimes N \to B \otimes P_{A,N}$ is injective, so also $B \otimes N \to P_B$ is injective. This morphism sends $F$ to $\varphi(F) \in P_B$, hence sends $B \otimes N$ to $S_B \cdot \varphi(F)$. Thus $B \otimes N$ is isomorphic to $S_B \cdot \varphi(F)$, which in turn is isomorphic to Apolar $(\varphi(F))$.

Proof of Point 2. As noted above, the family is flatly embedded if and only if $M := P_{A,N}/N$ is flat. This module is finitely generated, hence it is flat if and only if it is locally free. Now $A$ is reduced, so this happens if and only if $t \in \text{Spec } A$ has constant rank: the length of $M \otimes \kappa(t)$ is independent of the choice of $t \in \text{Spec } A$. But this length is $\text{len } P_{A,N} - \text{len } S_B \cdot \varphi(F) = \text{len } P_{A,N} - \text{Apolar } (\varphi(F))$. Hence this length is constant directly by assumption. \hfill \square
3 Analysis of the component $\mathcal{H}_{1661}$

We now return to working over the field $\mathbb{C}$ with a fixed $6$-dimensional vector space $V$ and we use the notation stated in the introduction.

We begin by defining the Iliev-Ranestad divisor $D_{IR}$. The Grassmannian $\text{Gr}(2, V) \subset \mathbb{P}(\Lambda^2 V)$ is non-degenerate, arithmetically Gorenstein and of degree $14$. A $5$-dimensional projective subspace $\mathbb{P}W \subset \mathbb{P}(\Lambda^2 V)$ is admissible if it does not intersect $\text{Gr}(2, V)$. A general $\mathbb{P}W$ is admissible. For an admissible $\mathbb{P}W$ the cone $Z = W \cap \text{cone}(\text{Gr}(2, V)) \subset \Lambda^2 V$ is a standard graded finite Gorenstein scheme $Z$ supported at the origin. For a general $\mathbb{P}^6$ containing a general admissible $\mathbb{P}W$, the intersection $\mathbb{P}^6 \cap \text{Gr}(2, V)$ is a set of $14$ points and $Z$ is a hyperplane section of the cone over these points, thus $Z$ is smoothable. Since $Z$ spans $\Lambda^6$ and is of length $14$, one checks, using the symmetry of the Hilbert function, that $Z$ has Hilbert function $(1, 6, 6, 1)$. Therefore $Z = \text{Spec Apolar}(F_Z)$ for a cubic $F_Z \in \text{Sym}^3 W^* \simeq \text{Sym}^3 \mathbb{C}^6$, unique up to scalars and $Z$ gives a well defined element $[F_Z] \in \mathbb{P}(\text{Sym}^3 V) \setminus \mathbb{P}(\text{GL}(V))$.

The set $D_{IR} \subset \mathbb{P}(\text{Sym}^3 V)$ is defined as the closure of the preimage of the set of such $[F_Z]$ obtained from all admissible $\mathbb{P}^5 = \mathbb{P}W$. It is called the Iliev-Ranestad divisor, see [IR01, Lemma 1.4] that $D_{IR}$ is a divisor and in [RV17, Lemma 2.4] that $F$ lies in $D_{IR}$ if and only if it is apolar to a quartic surface scroll.

We will now rigorously prove several claims which together lead to the proof of Theorem 1.1. Our approach is partially based on the natural method of [CEVV09]. Additional crucial steps are to prove that $\mathcal{H} \setminus \mathcal{H}_{\text{gen}}$ is smooth and that $\mathcal{H}_{\text{gen}} \cap \mathcal{H}_{1661}$ is irreducible.

In the first two steps we use the following abstract but trivial fact.

**Lemma 3.1.** Let $X$ and $Y$ be reduced, finite type schemes over $\mathbb{C}$. Let $X \to Y$, $Y \to X$ be two morphisms, which are bijective on closed points. If the composition $X \to Y \to X$ is equal to identity, then $X \to Y$ is an isomorphism.

**Proof.** Denote the morphisms by $i : X \to Y$ and $\pi : Y \to X$. The scheme-theoretical image of $i$ contains all closed points, hence is the whole $Y$. Therefore the pullback of functions via $i$ is injective. It is also surjective, since the pullback via composition $\pi \circ i$ is the identity. Hence $i$ is an isomorphism.

In our setting, $X$ is a subset of the Hilbert scheme, $Y$ a subspace of polynomials and the maps are constructed using relative Macaulay’s inverse systems.

**First, we identify $\mathcal{H}^{gr}_{1661}$ with an open subset of $\mathbb{P}(\text{Sym}^3 V)$.** We prove that associating to a general enough degree three polynomial its apolar algebra induces an isomorphism from $\mathbb{P}(\text{Sym}^3 V)_{1661}$ to $\mathcal{H}^{gr}_{1661}$.

**Construction 3.2.** Let $i : \text{Sym}_{\text{max}}^{\leq 3} V \hookrightarrow \text{Sym}^{\leq 3} V$ denote the set of $F$ such that $\dim(\text{Sym} V^* \oplus F)$ is equal to $1 + 6 + 6 + 1 = 14$. Since $14$ is maximal possible, this set is open. Recall that by Remark 2.7 a degree three polynomial $F$ lies in $\text{Sym}_{\text{max}}^{\leq 3} V$ if and only if its leading form lies in $\text{Sym}_{\text{max}}^{\leq 3} V$. Hence $\text{Sym}_{\text{max}}^{\leq 3} V = \text{Sym}_{\text{max}}^{\leq 2} V \times \text{Sym}_{\text{max}}^{3} V$.

Denote for brevity $U = \text{Sym}_{\text{max}}^{\leq 3} V$ and by $\Gamma(U)$ the global coordinate ring of this (affine) subvariety. Take an universal cubic $F \in \Gamma(U) \otimes \text{Sym}^{\leq 3} V$, so that for every $u \in U$ we have $F(u) = i(u)$. Then by Proposition 2.12 the family $\alpha_F : \text{Spec Apolar}(F) \to U = \text{Sym}_{\text{max}}^{\leq 3} V$ is flatly embedded and gives a morphism

$$\varphi : \text{Sym}_{\text{max}}^{\leq 3} V \to \mathcal{H}$$

to the Gorenstein locus of the Hilbert scheme. The points of the image are just $\text{Spec Apolar}(F(u))$, where $F(u) \in U$. By Macaulay’s inverse systems they correspond precisely to irreducible schemes with Hilbert function $(1, 6, 6, 1)$ and supported at the origin.
Let \( \mu : \text{Sym}^3 V \rightarrow \mathbb{P}(\text{Sym}^3 V) \) be the projectivisation. Denote \( \mu(U) \) by \( \mathbb{P}(\text{Sym}^3 V)_{1661} \). The restriction of \( \varphi \) to homogeneous cubics factors through \( \mu \) and gives a morphism

\[
\varphi : \mathbb{P}(\text{Sym}^3 V)_{1661} \rightarrow \mathcal{H}^g_{1661},
\]

which is bijective on points.

**CLAIM 1.** The map \( \varphi : \mathbb{P}(\text{Sym}^3 V)_{1661} \rightarrow \mathcal{H}^g_{1661} \) is an isomorphism.

**Proof.** Let \( [Z] \in \mathcal{H}^g_{1661} \). Each fibre of the universal family over \( \mathcal{H}^g_{1661} \) is \( \mathbb{C}^* \)-invariant thus the whole family is \( \mathbb{C}^* \)-invariant. By Local Description of Families (Proposition 2.10) near \( [Z] \) this family has the form \( \text{Spec Apolar}(F) \rightarrow \text{Spec } A \) for some \( F \in A \otimes \text{Sym}^3 V \), so that \( [F] \) gives a morphism \( \text{Spec } A \rightarrow \mathbb{P}(\text{Sym}^3 V)_{1661} \) which is locally an inverse to \( \mathbb{P}(\text{Sym}^3 V)_{1661} \rightarrow \mathcal{H}^g_{1661} \). The claim follows from Lemma 3.1.

We will abuse notation and speak about elements of \( \mathbb{P}(\text{Sym}^3 V)_{1661} \) being smoothable etc.

**Next, we construct the bundle** \( (\mathcal{H}_{1661})_{\text{red}} \rightarrow \mathcal{H}^g_{1661} \). We now prove the following claim, which informally reduces the questions about \( \mathcal{H}_{1661} \) to the questions about \( \mathcal{H}^g_{1661} \). Note that we will work on the reduced scheme \( (\mathcal{H}_{1661})_{\text{red}} \), which eventually turns out to be equal to \( \mathcal{H}_{1661} \).

**CLAIM 2.** \( (\mathcal{H}_{1661})_{\text{red}} \) is a rank 21 vector bundle over \( \mathcal{H}^g_{1661} \) via a map \( \pi : (\mathcal{H}_{1661})_{\text{red}} \rightarrow \mathcal{H}^g_{1661} \). This map on the level of points maps \( [R] \) to \( \text{Spec gr } H^0(R, O_R) \) supported at the origin of \( V \). The schemes corresponding to points in the same fibre of \( \pi \) are isomorphic.

**Proof.** First we recall the support map, as defined in [CEVV09, 5A]. Consider the universal family \( U \rightarrow \mathcal{H} \), which is flat. The multiplication by \( V^* \) on \( O_U \) is \( O_{\mathcal{H}} \)-linear. The relative trace of such multiplication defines a map \( V^* \rightarrow H^0(\mathcal{H}, O_{\mathcal{H}}) \), thus a morphism \( \mathcal{H} \rightarrow V \). We restrict this morphism to \( \mathcal{H}_{1661} \rightarrow V \) and compose it with multiplication by \( \frac{1}{14} \) on \( V \) to obtain a map denoted supp. If \( [Z] \in \mathcal{H}_{1661} \) corresponds to a scheme supported at \( v \in V \), then for every \( v^* \in V^* \) the multiplication by \( v^* - v^*(v) \) is nilpotent on \( Z \), hence traceless. Thus on \( Z \), we have \( \text{tr}(v^*) = \text{tr}(v^*(v)) = 14v^*(v) \) and \( \text{supp}([Z]) = v \) as expected.

The support morphism \( \text{supp} : \mathcal{H}_{1661} \rightarrow V \) is \( (V,+) \) equivariant, thus it is a trivial vector bundle:

\[
\mathcal{H}_{1661} \simeq V \times \text{supp}^{-1}(0).
\]

Restrict supp to \( (\mathcal{H}_{1661})_{\text{red}} \) and consider the fibre \( \mathcal{H}^0_{1661} := \text{supp}^{-1}(0) \). Since \( (\mathcal{H}_{1661})_{\text{red}} \) is reduced, also \( \mathcal{H}^0_{1661} \) is reduced. We will now use this in an essential way. By Local Description (Proposition 2.10) the universal family over this scheme locally has the form \( a_F : \text{Spec Apolar}(F) \rightarrow \text{Spec } A \) for some \( F \in A \otimes \text{Sym} V \). For every \( p \in \text{Spec } A \) we have \( \deg F(p) \leq 3 \). In other words, \( F_{\geq 4} = 0 \) for all points in \( A \). Since \( A \) is reduced, we have \( F_{\geq 4} = 0 \), so that \( \deg F \leq 3 \). Let \( F_3 \) be the leading form. The fibres of \( \text{gr } a_F : \text{Spec Apolar}(F_3) \rightarrow \text{Spec } A \) and \( a_F \) are isomorphic. Since \( A \) is reduced, by Proposition 2.12 the family \( \text{gr } a_F \) is also flat and gives a morphism \( \text{Spec } A \rightarrow \mathcal{H}^g_{1661} \). These morphisms glue to give a morphism

\[
\text{gr} : \mathcal{H}^0_{1661} \rightarrow \mathcal{H}^g_{1661}.
\]

Now we prove that \( \text{gr} \) makes \( \mathcal{H}^0_{1661} \) a vector bundle over \( \mathcal{H}^g_{1661} \). Let \( U = \text{Sym}^{\leq 3}_{\text{max}} V \). By Construction 3.2 we have a \( \varphi : U \rightarrow \mathcal{H}^0_{1661} \) which is a surjection on points. This surjection comes from a flatly embedded apolar family \( \text{Spec Apolar}(F) \rightarrow U \), where \( F \in \Gamma(U) \otimes \text{Sym}^{\leq 3}_{\text{max}} V \) is a universal cubic.

Let \( [Z] \in \mathcal{H}^0_{1661} \) and \( u \in U \) be a point in the preimage. Then \( Z = \text{Apolar}(\mathcal{F}(u)) \) and \( \text{gr}([Z]) \) is apolar to the cubic form \( [\mathcal{F}(u)] = [F_3(u)] \). Therefore \( \text{gr } \circ \varphi(\mathcal{F}(u)) = [F_3(u)] \). In this way \( U = \text{Sym}^{\leq 3}_{\text{max}} V \) becomes a trivial vector bundle of rank \( 1 + 6 + \binom{3}{2} = 28 \) over the cone \( \text{Sym}^{\leq 3}_{\text{max}} V \) over \( \mathbb{P}(\text{Sym}^3 V)_{1661} \).
We will prove that $\mathcal{H}_0^{1661}$ is a projectivisation of a quotient bundle of this bundle. Take a subbundle $\mathcal{K}$ of $U$ whose fiber over $F_3 \in \text{Sym}^3_{\text{max}} V$ is $(\text{Sym}^2 V^*) \hookrightarrow F_3$. Then as in Remark 2.3 we see that the family $\text{Spec Apolar}(F)$ over $U$ is pulled back from the quotient bundle $U/\mathcal{K}$, which we denote by $\mathcal{E}$. Hence also the associated morphism $U \to \mathcal{H}_0^{1661}$ factors as $U \to \mathcal{E} \to \mathcal{H}_0^{1661}$. Finally we may projectivise these bundles: we replace the polynomials in $\mathcal{E}$ by their classes, obtaining a bundle over $\mathbb{P}(\text{Sym}^3 V)_{1661}$ which we denote, abusing notation, by $\mathbb{P}\mathcal{E}$. The morphism $\mathcal{E} \to \mathcal{H}_0^{1661}$ factors as $\mathcal{E} \to \mathbb{P}\mathcal{E} \to \mathcal{H}_0^{1661}$. Finally we obtain
\[ \bar{\varphi} : \mathbb{P}\mathcal{E} \to \mathcal{H}_0^{1661}. \]
It is bijective on points (Remark 2.3).

By the Local Description of Families for every $[Z] \in \mathcal{H}_0^{1661}$ we have a neighborhood $U$ so that the universal family is $\text{Spec Apolar}(F) \to U$ for $F \in H^0(U, \mathcal{O}_U) \otimes \text{Sym}^3_{\text{max}} V$. Then $F$ gives a map $U \to \text{Sym}^3_{\text{max}} V$, thus $U \to \mathbb{P}\mathcal{E}$. This is a local inverse of $\bar{\varphi}$. Hence by Lemma 3.1 the variety $\mathcal{H}_0^{1661}$ is isomorphic to the bundle $\mathbb{P}\mathcal{E}$ over $\mathcal{H}_0^{1661}$. To prove Claim 2 we define $\pi$ to be the composition of projection $V \times \mathcal{H}_0^{1661} \to \mathcal{H}_0^{1661}$ and $\text{gr}$. Since the former is a trivial vector bundle and the latter is a vector bundle the composition is a vector bundle as well.

Finally note that $\pi([Z])$ is isomorphic to the scheme $\text{Spec} \text{gr} H^0(Z, \mathcal{O}_Z)$, which in turn is (abstractly) isomorphic to $Z$ by the discussion of Section 2. Hence all the schemes corresponding to points in the same fibre are isomorphic.

**Corollary 3.3.** The locus $\mathcal{H}_0^{1661} \cap \mathcal{H}_{\text{gen}} \subset \mathcal{H}_0^{1661}$ contains a divisor, which is equal to $\pi^{-1}(D_{1R})$, where $D_{1R} \subset \mathbb{P}(\text{Sym}^3 V)_{1661}$ is the restriction of the Iliev-Ranestad divisor.

**Proof.** By its construction, the divisor $D_{1R} \subset \mathbb{P}(\text{Sym}^3 V)_{1661} \cong \mathcal{H}_{gr}^{1661}$ parametrises smoothable schemes. By Claim 2 also schemes in $\pi^{-1}(D_{1R})$ are smoothable, hence $\pi^{-1}(D_{1R})$ is contained in $\mathcal{H}_0^{1661} \cap \mathcal{H}_{\text{gen}}$. Again by Claim 2 this preimage is divisorial in $\mathcal{H}_0^{1661}$.

**Now we prove that $\mathcal{H}_0^{1661} \setminus \mathcal{H}_{\text{gen}}$ is smooth, so $\mathcal{H}_0^{1661}$ is reduced.** Let $Z \subset V$ be a finite irreducible (locally) Gorenstein scheme with Hilbert function $(1, 6, 6, 1)$. Let $S := H^0(V, \mathcal{O}_V) = \text{Sym} V^*$ and $R = H^0(Z, \mathcal{O}_Z)$, then $R = S/I$. The tangent space to $\mathcal{H}$ at $[Z]$ is isomorphic to $\text{Hom}_{S/1}(I/I^2, S/I)$. Since $Z$ is Gorenstein, this space is dual to $I/I^2$. Note that $Z$ is isomorphic to $Z_0 = \text{Spec} \text{gr} R$ and $[Z_0] \in \mathcal{H}_{gr}^{1661}$.

**CLAIM 3.** $\mathcal{H}_0^{1661} \cap \text{Sing} \mathcal{H} = \mathcal{H}_0^{1661} \cap \mathcal{H}_{\text{gen}} = \pi^{-1}(D_{1R})$ as sets. Therefore $\mathcal{H}_0^{1661}$ is reduced. Moreover $\mathcal{H}_0^{1661} \cap \mathcal{H}_{\text{gen}} \subset \mathcal{H}_0^{1661}$ is a prime divisor.

Being a singular point of $\mathcal{H}$ and lying in $\mathcal{H}_{\text{gen}}$ are both independent of the embedding of a finite scheme. Hence all three sets appearing in the equality of Claim 3 are preimages of their images in $\mathcal{H}_{gr}^{1661}$. Therefore it is enough to prove the claim for elements of $\mathcal{H}_{gr}^{1661}$.

Take $[Z_0] \in \mathcal{H}_{gr}^{1661}$ with corresponding homogeneous ideal $I$. Take $F \in \text{Sym}^3 V$ so that $I = \text{Ann}(F)$. The point $[Z_0]$ is smooth if and only if $\dim S/I^2 = 76 + 14 = 90$. Consider the Hilbert series $H$ of $S/I^2$. By degree reasons $I^2$ annihilates $\text{Sym}^{\leq 3} V$. We now show that it annihilates also a 6-dimension space of quartics. Notabene, by Example 2.13 this space is the tangent space to deformations of $S/I$ obtained by moving its support in $V$.

**Lemma 3.4.** The ideal $I^2$ annihilates the space $V \cdot F \subset \text{Sym}^4 V$.

**Proof.** Let $\alpha \in V^*$ and $x \in V$ be linear forms. Then $\alpha$ acts on $\text{Sym} V$ as a derivation, so that $\alpha \cdot (xF) = (\alpha \cdot x)F + x(\alpha \cdot F) \equiv x(\alpha \cdot F) \mod (S \cdot F)$. Take any element $i \in I_2$ and write it as $i = \sum \beta_i \beta_j$ with $\beta_i$ linear. Then
\[ i \cdot (xF) = \sum \beta_i \cdot (\beta_j \cdot xF) \equiv \sum x(\beta_i \beta_j \cdot F) = x(i \cdot F) = 0 \mod (S \cdot F). \]
The condition \( q \) thus we have \( \sum H_{Z_q} = 84 + r \) and this equals 90 if and only if \( r = 6 \) and \(*\) consists of zeros. Now we show that if \( r = 6 \) then \( *\) consists of zeros. See [IE78, Lemma 2.31] for related statement.

**Lemma 3.5.** Let \( F \in \text{Sym}^3 V \) and \( I = \text{Ann}(F) \subset S \) be as above. Suppose that \( \dim (I^2)_{4} \leq 6 \). Then \( \text{Sym}^5 V^* \subset I^2 \). In particular \( H_{S/I_{12}} = (1, 6, 21, 56, 6, 0) \), so that the tangent space to \( H \) at \([S/I]\) has dimension 76. As a corollary, \([S/I]\) is singular if and only if \( \dim (I^2)_{4} > 6 \).

**Proof.** Suppose \( \text{Sym}^5 V^* \not\subset I^2 \) and take non-zero \( G \in \text{Sym}^5 V^* \) annihilated by this ideal. By assumption \( \dim (I^2)_{4} = 6 \) and by Lemma 3.4 the 6-dimensional space \( V F \) is perpendicular to \( I^2 \). Therefore \( (I^2)_{4} = V F \) and hence \( V^* \cdot G \subset V F \).

We first show that all linear forms are partials of \( G \), in other words that \( V \subset S G \). Clearly \( 0 \neq V^* \cdot G \subset V F \). Take a non-zero \( x \in V \) such that \( x F \) is a partial of \( G \). Let \( W^* = (x^1) \subset V^* \) be the space perpendicular to \( x \). Let \( x F = x^{e+1} F \), where \( F \) is not divisible by \( x \). Then there exists an element \( \sigma \) of \( \text{Sym} W^* \) such that \( \sigma \cdot (x F) = x^{e+1} F \). In particular \( x \in S \cdot (x F) \). Moreover \( S_{3}(xF) = S_{2}F = V \mod \mathbb{C} x \). Therefore, \( V \subset S \cdot G \), so \( V = S_{1} \cdot G \). By symmetry of the Hilbert function, \( \dim V^* G = \dim S_{2} G = 6 \). Since \( V^* G \) is annihilated by \( I^2 \), by comparing dimensions we conclude that

\[
V^* G = V F.
\]

Since \( I_3 \cdot (I_2 \cdot G) = 0 \) and \( I_4^\perp = CF \), we have \( I_3 \cdot G \subset CF \), so \( \dim (S_{2} \cdot G + CF) \leq 6 + 1 = 7 \). For every linear \( \alpha \in V^* \) and \( y \in V \) we have \( \alpha \cdot (y F) = (\alpha \cdot y) F + y(\alpha \cdot F) \equiv y(\alpha \cdot F) \mod CF \). Therefore we have

\[
S_{2} \cdot G = V^* \cdot (V^* \cdot G) = V^* \cdot (V F) \equiv V(V^* \cdot F) \mod CF,
\]

thus \( \dim V(V^* \cdot F) \leq 7 \). Take any two quadrics \( q_1, q_2 \in V^* F \). Then \( V q_1 \cap V q_2 \) is non-zero, so that \( q_1 \) and \( q_2 \) have a common factor. We conclude that \( V^* F = y V \) for some \( y \), but then \( \dim V(V^* F) = \dim y \text{Sym}^2 V > 7 \), a contradiction.

As explained in Claim 1, the map \([F] \to \text{Spec Apolar}(F)\) is an isomorphism \( \mathbb{P}(\text{Sym}^3 V)_{1661} \to H_{1661}^\text{gr} \). Thus we may consider the statements \([Z_0] \in H_{\text{gen}} \) and \([Z_0] \) is singular as conditions on the form \( F \in \mathbb{P}(\text{Sym}^3 V)_{1661} \). For a family \( F \) of forms we get a rank 120 bundle \( \text{Sym}^2 I_2 \) with an evaluation morphism

\[
ev : \text{Sym}^2 I_2 \to (V F)^\perp \subset \text{Sym}^4 V^*.
\]

The condition \( (I^2)_{4} > 6 \) is equivalent to degeneration of \( \ev \) on the fibre and thus it is divisorial on \( \mathbb{P}(\text{Sym}^3 V)_{1661} \). We now check that the associated divisor \( E = (\det \ev = 0) = \text{Sing} H_{1661} \cap H_{1661}^{\text{gr}} \) is prime of degree 10.

**Lemma 3.6.** Fix a basis \( x_0, \ldots, x_5 \) of \( V \) and let \( F = x_0 x_1 x_3 - x_0 x_4^2 + x_1 x_2^2 + x_2 x_4 x_5 + x_3 x_5^2 \). The line between \( F \) and \( x_5^2 \) intersects \( E \) in a finite scheme of degree 10 supported at \( x_5^2 \).

**Proof.** First, one checks that every linear form is a partial of \( F \), so that \( F \in \mathbb{P}(\text{Sym}^3 V)_{1661} \). A minor technical remark: in this example we use contraction action. The same polynomials will work when considering partial differentiation action, but the apolar ideals will differ. Let

\[
\text{fixed} := \text{span}(a_0^2, a_0 a_2, -a_0 a_3 + a_2^2, a_0 a_4 + a_2 a_5, a_0 a_5, a_1^2, a_1 a_2 - a_4 a_5, a_1 a_3 + a_4^2, a_1 a_4, a_1 a_5, a_2 a_3, a_2 a_4 - a_3 a_5, a_3^2, a_3 a_4)
\]

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be a 14-dimensional space. Let \( F(u, v) = uF + vx_5^3 \). Then
\[
\text{fixed} \oplus \mathbb{C} \left( \nu \alpha_3 \alpha_5 + w \alpha_0 \alpha_4 - w a_3^2 \right) \subset \text{Ann} \left( F_2 \right)
\]
and equality holds for a general choice of \((u : v) \in \mathbb{P}^1\). One verifies that the determinant of \( ev \) restricted to this line is equal, up to unit, to \( u^{10} \). This can be conveniently checked near \( x_3^2 \) by considering \( \text{Sym}^2 I \rightarrow \text{Sym}^4 V^*/Vx_3^2 \) and near \( F \) by \( \text{Sym}^2 I \rightarrow \text{Sym}^4 V^*/Vx_3^2 V^* \). We note that the same equality holds in any characteristic other than 2, 3.

**Remark 3.7.** For \( F \) as in Lemma 3.6 above Spec Apolar \((F)\) is an example of a non-smoothable scheme. Other examples may be easily obtained by choosing “random” degree three polynomials \( F \).

**Proposition 3.8.** The divisor \( E = \text{Sing} \mathcal{H}_{1661} \cap \mathcal{H}_{1661}^{\text{gr}} \) is prime of degree 10. We have \( E = \mathcal{H}_{\text{gen}} \cap \mathcal{H}_{1661}^{\text{gr}} = D_{IR} \cap \mathbb{P} \left( \text{Sym}^3 V \right)_{1661} \) as sets.

**Proof.** Take two forms \([F_1], [F_2] \in \mathbb{P} \left( \text{Sym}^3 V \right)\) and consider the intersection of \( E \) with the line \( \ell \) spanned by them. By Lemma 3.8 the restriction to \( \ell \) of the evaluation morphism from Equation (4) is finite of degree 10. Hence also \( E \) is of degree 10.

Note that \( E \) is SL\((V)\)-invariant. By a direct check, e.g. conducted with the help of computer (e.g. LiE), we see that there are no SL\((V)\)-invariant polynomials in \text{Sym}^2 \text{Sym}^3 V^* of degree less than ten. Therefore \( E \) is prime. Now smoothable schemes are singular, hence we have \( D_{IR} \cap \mathbb{P} \left( \text{Sym}^3 V \right)_{1661} \subset \mathcal{H}_{1661}^{\text{gr}} \cap \mathcal{H}_{\text{gen}} \subset E \). Since \( D_{IR} \cap \mathbb{P} \left( \text{Sym}^3 V \right)_{1661} \) is also a non-zero divisor, we have equality of sets.

**Remark 3.9.** In the proof of Proposition 3.8 (and so Lemma 3.6) we could avoid calculating the precise degree of \( ev \) restricted to \( \ell \), provided that we prove that \( (\det ev)_\ell = 0 \). Namely, let \( \mathcal{I} \subset \text{Sym}^2 V^* \) be the relative apolar ideal sheaf on \( \ell \). We look at \( \mathcal{I}_2 \). By the proof of Lemma we have \( \mathcal{I}_2 \simeq O^{14} \oplus O(-1) \). Hence \( \text{Sym}^2 \mathcal{I}_2 \) has determinant \( O(-16) \). Similarly \( V \mathcal{F} \simeq O(-1)^6 \), hence \( (V \mathcal{F})^\perp = O^{14} \oplus O(-1)^6 \) and thus \( \det ev|_\ell : O(-16) \to O(6) \) is zero on a degree 10 divisor. We conclude that \( E \) has degree 10.

**Proof of Claim 3** Schemes corresponding to different elements in the fiber of \( \pi \) are abstractly isomorphic. Therefore, we have Sing \( \mathcal{H}_{1661} = \pi^{-1} \pi(\text{Sing} \mathcal{H}_{1661}) \) and \( \mathcal{H}_{\text{gen}} \cap \mathcal{H}_{1661} = \pi^{-1} \pi(\mathcal{H}_{\text{gen}} \cap \mathcal{H}_{1661}) \). Hence the equality in Proposition 3.8 implies the equality in the Claim. The reducedness follows, because the scheme was defined via the closure: \( \mathcal{H}_{1661} = \mathcal{H}_{1661} \setminus \mathcal{H}_{\text{gen}} \). The last claim follows because \( D_{IR} \) is prime and of codimension one thus its preimage under \( \pi \) is also such.

**Remark 3.10.** If \([Z] \in \mathcal{H}_{1661} \) lies in \( \mathcal{H}_{\text{gen}} \), then the tangent space to \( \mathcal{H} \) at \([Z] \) has dimension at least 85 = 76 + 9. This is explained geometrically by an elegant argument of [IR01], which we sketch below. Recall that we have an embedding \( Z \subset \mathbb{A}^6 \cap \Lambda^2 V \). Define a rational map \( \varphi : \mathbb{P}(\Lambda^2 V) \dashrightarrow \mathbb{P}(\Lambda^2 V^*) \) as the composition \( \Lambda^2 V \to \Lambda^4 V \simeq \Lambda^2 V^* \) where the first map is \( w \to w \wedge w \) and the second is an isomorphism coming from a choice of element of \( \Lambda^6 V \). Then \( \varphi \) is defined by the 15 quadrics vanishing on \( \text{Gr}(2, V) \) and it is birational with a natural inverse given by quadrics vanishing on \( \text{Gr}(2, V^*) \). The 15 quadrics in the ideal of \( Z \) are the restrictions of the 15 quadrics defining \( \varphi \). Thus the map \( \varphi : Z \to \Lambda^2 V^* \) is defined by 15 quadrics in the ideal of \( Z \). Since \( \varphi^{-1}(\varphi(Z)) \) spans at most an \( \mathbb{A}^6 \) the coordinates of \( \varphi^{-1} \) give 15 − 6 = 9 quadratic relations between those quadrics. Therefore \( \dim I_4 \leq \dim \text{Sym}^2 I_2 - 9 \leq 126 - 15 \) and \( r \) from the discussion above Lemma 3.5 is at least 15 so that the tangent space dimension is at least 85.

We conclude by making a formal proof of the main theorem and adding some remarks.

**Proof of Theorem 1.1** 2. By Claim 3 the component \( \mathcal{H}_{1661} \) is reduced. Hence this part follows by Claim 2.

3. This is proved in Claim 1.
1. $\mathcal{H}_{1661}$ is smooth and connected, being a vector bundle over $\mathbb{P}(\text{Sym}^3 V)_{1661}$.

4. This is proved in Claim $3$. □

**Remark 3.11.** In [CHC+], Chiantini, Hauenstein, Ikenmeyer, Landsberg and Ottaviani give very strong numerical evidence that $\sigma_{9\nu_3}(\mathbb{P}V)$ is an ideal-theoretic complete intersection of a degree 10 and degree 28 divisors. Since the degree 10 divisor is $\text{SL}(V)$-invariant, it is equal to $D_{IR}$. It would be interesting to know whether the second divisor is $D_{V,\text{gen}}$ introduced in [RV17].

The outline of the argument of [CHC+], as communicated by the authors, is to use [HS10] to show that $\sigma_{9\nu_3}(\mathbb{P}V)$ has codimension 2 and degree 280. Then [DH16a, DH16b] are used to show that it is arithmetically Gorenstein. Finally, the Hilbert function of the 280 witness points allows one to conclude that it is actually a complete intersection of a degree 10 and a degree 28 polynomial.

**Remark 3.12.** The dimension of $\mathcal{H}_{1661}$ equal to 76 is smaller than the dimension of $\mathcal{H}_{\text{gen}}$ equal to $14 \cdot 6 = 84$. This is the only known example of a component $Z$ of the Gorenstein locus of Hilbert scheme of $d$ points on $\mathbb{A}^n$ such that $\dim Z \leq dn$ and points of $Z$ correspond to irreducible subschemes. It is an interesting question whether other examples exist.

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