A remark on the existence of entire large and bounded solutions to a \((k_1,k_2)\)-Hessian system with gradient term

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Abstract

In this paper, we study the existence of positive entire large and bounded radial positive solutions for the following nonlinear system

\[
\begin{aligned}
S_{k_1} \left( \lambda \left( D^2 u_1 \right) \right) + a_1 \left( |x| \right) |\nabla u_1|^{k_1} &= p_1 \left( |x| \right) f_1 \left( u_2 \right) \quad \text{for} \quad x \in \mathbb{R}^N, \\
S_{k_2} \left( \lambda \left( D^2 u_2 \right) \right) + a_2 \left( |x| \right) |\nabla u_2|^{k_2} &= p_2 \left( |x| \right) f_2 \left( u_1 \right) \quad \text{for} \quad x \in \mathbb{R}^N.
\end{aligned}
\]

Here \( S_{k_i} \left( \lambda \left( D^2 u_i \right) \right) \) is the \( k_i \)-Hessian operator, \( a_1, p_1, f_1, a_2, p_2 \) and \( f_2 \) are continuous functions. Our results give an answer of the question raised in [11].

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1 Introduction

Explosive or bounded radial solutions of elliptic systems have been studied intensively in the last few decades (see among others Alves and Holanda [1], Bandle and Giarrusso [2], Cirstea and Rădulescu [3], Clément-Manásevich and Mitidieri [6], the author [8], De Figueiredo and Jianfu [9], Galaktionov and Vázquez [10], Ghergu and Rădulescu [11], Lair and Wood [20], Lair [21, 22], Peterson and Wood [26], Quittner [27] and Zhang and Zhou [32]). Most of these works have studied the following system

\[
\begin{aligned}
\Delta u_1 &= p_1 \left( |x| \right) f_1 \left( u_2 \right) \quad \text{for} \quad x \in \mathbb{R}^N \quad (N \geq 3), \\
\Delta u_2 &= p_2 \left( |x| \right) f_2 \left( u_1 \right) \quad \text{for} \quad x \in \mathbb{R}^N \quad (N \geq 3).
\end{aligned}
\]

Cirstea and Rădulescu [3] (Theorem 1, p. 828), proved that the system (1) has infinitely many positive entire large solutions provided that \( f_1, f_2 \) are positive locally Hölder with exponent \( \beta \in (0,1) \), non-decreasing and satisfying

\[
\lim_{t \to \infty} \frac{f_2 \left( af_1 \left( t \right) \right)}{t} = 0 \quad \text{for all constants} \quad a > 1.
\]
In their work, the weights \( p_1, p_2 \) are positive, symmetric locally Hölder functions in \( \mathbb{R}^N \) and such that

\[
\int_0^\infty r^{1-N} \int_0^r t^{N-1} p_1(t) \, dt \, dr = \int_0^\infty r^{1-N} \int_0^r t^{N-1} p_2(t) \, dt \, dr = \infty. \tag{3}
\]

In a subsequent paper, Ghergu and Rădulescu [11] (Theorem 3, p. 438) established the existence of solutions for

\[
\begin{align*}
\Delta u_1 + |\nabla u_1| &= p_1(x) f_1(u_2) \quad \text{for } x \in \mathbb{R}^N \quad (N \geq 3), \\
\Delta u_2 + |\nabla u_2| &= p_2(x) f_2(u_1) \quad \text{for } x \in \mathbb{R}^N \quad (N \geq 3).
\end{align*} \tag{4}
\]

They replaced the condition (3) with

\[
\int_1^\infty e^{-r^{1-N}} \int_0^r e^{t^{N-1}} p_1(t) \, dt \, dr = \int_1^\infty e^{-r^{1-N}} \int_0^r e^{t^{N-1}} p_2(t) \, dt \, dr = \infty, \tag{5}
\]

while keeping all other conditions on \( p_1, f_1, p_2 \) and \( f_2 \). Moreover, they noticed although

\[
f_1(t) = \sqrt{t}, \quad f_2(t) = t, \quad p_1(r) = 4 \frac{r^3 + (N + 2)r^2}{\sqrt{r^2 + 1}}, \quad p_2(r) = 2 \frac{r + N}{\sqrt{r^2 + 1}}
\]

doesn’t satisfy (4), the corresponding system has the positive entire large solution \((|x|^1 + 1, |x|^2 + 1)\). Therefore it is only natural to weaken the assumptions (3). Inspired by this, our analysis, is developed for the more general nonlinear systems

\[
\begin{align*}
S_{k_1} (\lambda (D^2 u_1)) + a_1 (|x|) |\nabla u_1|^{k_1} &= p_1 (|x|) f_1 (u_2) \quad \text{for } x \in \mathbb{R}^N \quad (N \geq 3), \\
S_{k_2} (\lambda (D^2 u_2)) + a_2 (|x|) |\nabla u_2|^{k_2} &= p_2 (|x|) f_2 (u_1) \quad \text{for } x \in \mathbb{R}^N \quad (N \geq 3),
\end{align*} \tag{6}
\]

where \( a_1, p_1, f_1, a_2, p_2, f_2 \) are continuous functions satisfying certain monotonicity properties, \( k_1, k_2 \in \{1, 2, \ldots, N\} \) and \( S_{k_i} (\lambda (D^2 u_i)) \) stands for the \( k_i \)-Hessian operator defined as the sum of all \( k_i \times k_i \) principal minors of the Hessian matrix \( D^2 u_i \). The following well known operators

| Operator: | Laplacian | Monge–Ampère |
|-----------|-----------|-------------|
| \( S_1 (\lambda (D^2 u_i)) \) | \( \Delta u_i = \text{div} (\nabla u_i) \) | \( S_N (\lambda (D^2 u_i)) = \text{det} (D^2 u_i) \), |

are special \( k_i \)-Hessian operator.

The system (6) has been the subject of rather deep investigations since it appears in many branches of applied mathematics (for more on this see the papers of Bao-Ji and Li [3], Salani [29], Ji and Bao [19], Viaclovsky [30, 31] and Zhang and Zhou [32]).

2 The main results

We work under the following assumptions:

(P1) \( a_1, a_2 : [0, \infty) \to [0, \infty) \) and \( p_1, p_2 : [0, \infty) \to (0, \infty) \) are spherically symmetric continuous functions (i.e., \( p_i (x) = p_i (|x|) \) and \( a_i (x) = a_i (|x|) \) for \( i = 1, 2 \));

(C1) \( f_1, f_2 : [0, \infty) \to [0, \infty) \) are continuous, increasing, \( f_i (0) \geq 0 \) and \( f_i (s) > 0 \) for all \( s > 0 \) with \( i = 1, 2 \);

(C2) there exist positive constants \( \overline{r}_1, \overline{r}_2 \), the continuous and increasing functions \( h_1, h_2 : [0, \infty) \to [0, \infty) \) and the continuous functions \( \overline{\nu}_1, \overline{\nu}_2 : [0, \infty) \to [0, \infty) \) such that

\[
\begin{align*}
f_1 (t_1 \cdot w_1) &\leq \overline{\nu}_1 h_1 (t_1) \cdot \overline{\nu}_1 (w_1) \quad \forall \, w_1 \geq 1 \quad \text{and} \quad \forall \, t_1 \geq M_1 \cdot f_1^{1/k_2} (a), \tag{7} \\
f_2 (t_2 \cdot w_2) &\leq \overline{\nu}_2 h_2 (t_2) \cdot \overline{\nu}_2 (w_2) \quad \forall \, w_2 \geq 1 \quad \text{and} \quad \forall \, t_2 \geq M_2 \cdot f_2^{1/k_1} (b). \tag{8}
\end{align*}
\]
Remark 1  with central value in 

Assume that Theorem 1 2.) If in addition, 

and $a, b \in (0, \infty)$;

(C3) there are some constants $\xi_1, \xi_2 \in (0, \infty)$ and the continuous functions $\varphi_1, \varphi_2 : [0, \infty) \to [0, \infty)$ such that

$$f_1 (m_1 w_1) \geq \varphi_1 (w_1) \forall w_1 \geq 1, \quad (9)$$

$$f_2 (m_2 w_2) \geq \varphi_2 (w_2) \forall w_2 \geq 1, \quad (10)$$

where $m_1 = \min \{ b, f_2^{1/k_2} (a) \}$ and $m_2 = \min \{ a, f_1^{1/k_1} (b) \}$.

Throughout the paper, we use the following notations

$$C_0 = (N - 1)!/[k_1!(N - k_1)!], C_{00} = (N - 1)!/[k_2!(N - k_2)!]$$

$$G_1 (\xi) = \frac{\xi^{k_2 - N}}{C_0} e^{-\int_0^\xi \frac{1}{t_0} a_2 (t) dt}, \quad G_1^+ (\xi) = \xi^{N - 1} e^{-\int_0^\xi \frac{1}{t_0} a_2 (t) dt} G_2 (\xi), \quad G_2 (\xi) = \frac{\xi^{k_1 - N}}{C_0} e^{-\int_0^\xi \frac{1}{t_0} a_1 (t) dt}, \quad G_2^+ (\xi) = \xi^{N - 1} e^{-\int_0^\xi \frac{1}{t_0} a_1 (t) dt} G_1 (\xi),$$

$$\bar{P}_{1,2} (r) = \int_0^r |G_2 (y)| \int_0^y |G_2^+ (t) \varphi_1 \left(1 + \int_0^t (G_1^- (z) \int_0^z G_1^+ (\xi) d\xi \right) dt |^{1/k_1} dy,$$

$$\bar{P}_{2,1} (r) = \int_0^r |G_1 (y)| \int_0^y |G_1^+ (t) \varphi_2 \left(1 + \int_0^t (G_2^- (z) \int_0^z G_2^+ (\xi) d\xi \right) dt |^{1/k_2} dy,$$

$$H_{1,2} (r) = \int_a^b h_1^{1/k_1} (M_{1,2}^{1/k_2} (t)) dt, \quad H_{2,1} (r) = \int_b^r \frac{1}{h_2^{1/k_2} (M_{1,2}^{1/k_1} (t))} dt,$$

$$\bar{T}_{1,2} (\infty) = \lim_{r \to \infty} \bar{P}_{1,2} (r), \quad \bar{T}_{2,1} (\infty) = \lim_{r \to \infty} \bar{P}_{2,1} (r), \quad P_{1,2} (\infty) = \lim_{r \to \infty} P_{1,2} (r), \quad P_{2,1} (\infty) = \lim_{r \to \infty} P_{2,1} (r),$$

$$H_{1,2} (\infty) = \lim_{s \to \infty} H_{1,2} (s) \text{ and } H_{2,1} (\infty) = \lim_{s \to \infty} H_{2,1} (s).$$

Our main results are summarized by the following theorems.

Theorem 1  Assume that $H_{1,2} (\infty) = H_{2,1} (\infty) = \infty$ and (P1), hold. Furthermore, if $f_1$ and $f_2$ satisfy the hypotheses (C1) and (C2) then the system (6) has at least one positive radial solution

$$(u_1, u_2) \in C^2 ([0, \infty)) \times C^2 ([0, \infty))$$

with central value in $(a, b)$. Moreover, the following hold:

1.) If $\bar{T}_{1,2} (\infty) < \infty$ and $\bar{T}_{2,1} (\infty) < \infty$ then $\lim_{r \to \infty} u_1 (r) < \infty$ and $\lim_{r \to \infty} u_2 (r) < \infty$.

2.) If in addition, $f_1$ and $f_2$ satisfy the hypothesis (C3), $P_{1,2} (\infty) = \infty$ and $P_{2,1} (\infty) = \infty$ then

$$\lim_{r \to \infty} u_1 (r) = \infty \text{ and } \lim_{r \to \infty} u_2 (r) = \infty.$$
The following situations improve our theorems:

3.) If in addition, $f_2$ satisfy the condition (14), $P_{1,2}(\infty) < \infty$ and $P_{2,1}(\infty) = \infty$ then $\lim_{r \to \infty} u_1(r) < \infty$ and $\lim_{r \to \infty} u_2(r) = \infty$.

4.) If in addition, $f_1$ satisfy the condition (15), $P_{1,2}(\infty) = \infty$ and $P_{2,1}(\infty) < \infty$ then $\lim_{r \to \infty} u_1(r) = \infty$ and $\lim_{r \to \infty} u_2(r) < \infty$.

**Theorem 2** Assume that the hypothesis (P1) holds. Then, the following hold:

i.) If (C1), (C2), (C3), $P_{1,2}(\infty) < H_{1,2}(\infty) < \infty$ and $P_{2,1}(\infty) < H_{2,1}(\infty) < \infty$ are satisfied, then system (3) has one positive bounded radial solution

$$(u_1, u_2) \in C^2([0, \infty)) \times C^2([0, \infty)),$$

with central value in $(a, b)$, such that

$$\left\{ \begin{array}{l}
a + c_1^{1/k_1} P_{1,2}(r) \leq u_1(r) \leq H_{1,2}^{-1} \left( \frac{r^{1/k_1} P_{1,2}(r)}{c_1} \right), \\
b + c_2^{1/k_2} P_{2,1}(r) \leq u_2(r) \leq H_{2,1}^{-1} \left( \frac{r^{1/k_2} P_{2,1}(r)}{c_2} \right). 
\end{array} \right.$$

ii.) If (C1), (C2), (10), $H_{1,2}(\infty) = \infty$, $P_{1,2}(\infty) = \infty$ and $P_{2,1}(\infty) < H_{2,1}(\infty) < \infty$ are satisfied, then system (3) has one positive radial solution

$$(u_1, u_2) \in C^2([0, \infty)) \times C^2([0, \infty)),$$

with central value in $(a, b)$, such that $\lim_{r \to \infty} u_1(r) = \infty$ and $\lim_{r \to \infty} u_2(r) < \infty$.

iii.) If (C1), (C2), (10), $P_{2,1}(\infty) = \infty$, $H_{2,1}(\infty) = \infty$ and $P_{1,2}(\infty) < H_{1,2}(\infty) < \infty$ are satisfied, then system (3) has one positive radial solution

$$(u_1, u_2) \in C^2([0, \infty)) \times C^2([0, \infty)),$$

with central value in $(a, b)$, such that $\lim_{r \to \infty} u_1(r) < \infty$ and $\lim_{r \to \infty} u_2(r) = \infty$.

**Remark 2** Let

$$M_1^+ = \sup_{t \in [0, \infty)} \int_0^t (G_1^-(z) \int_0^z G_1^+(s) \, ds \, \frac{dz}{r^2} \, dz) \, dt \text{ and } M_2^+ = \sup_{t \in [0, \infty)} \int_0^t (G_2^-(z) \int_0^z G_2^+(s) \, ds \, \frac{dz}{r^2} \, dz).$$

The following situations improve our theorems:

a) If $M_1^+ \in (0, \infty)$ then the condition (7) is not necessary but $H_{1,2}(r)$ must be replaced by

$$H_{1,2}(r) = \int_a^r \frac{1}{f_1^{1/k_1} \left( M_1 (1 + M_1^+) f_1^{1/k_2}(t) \right)} \, dt.$$

b) If $M_2^+ \in (0, \infty)$ then the condition (8) is not necessary but $H_{2,1}(r)$ must be replaced by

$$H_{2,1}(r) = \int_b^r \frac{1}{f_2^{1/k_1} \left( M_2 (1 + M_2^+) f_2^{1/k_2}(t) \right)} \, dt.$$

c) If $M_1^+ \in (0, \infty)$ and $M_2^+ \in (0, \infty)$ then the conditions (7) and (8) are not necessary but $H_{1,2}(r)$ and $H_{2,1}(r)$ must be replaced by

$$H_{1,2}(r) = \int_a^r \frac{1}{f_1^{1/k_1} \left( M_1 (1 + M_1^+) f_1^{1/k_2}(t) \right)} \, dt \text{ and } H_{2,1}(r) = \int_b^r \frac{1}{f_2^{1/k_1} \left( M_2 (1 + M_2^+) f_2^{1/k_2}(t) \right)} \, dt.$$

d) If $m_1 \geq 1$ then $\varphi_1 = 1$ and $\varphi_2 = f_1$.

e) If $m_2 \geq 1$ then $\varphi_2 = 1$ and $\varphi_1 = f_2$.

f) If $m_1 \geq 1$ and $m_2 \geq 1$ then $\varphi_1 = \varphi_2 = 1$, $\varphi_1 = f_1$ and $\varphi_2 = f_2$. 

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Remark 3 We note that we can have the following situation

\[ H_{1,2}(\infty) = \infty \text{ if and only if } H_{2,1}(\infty) = \infty, \]  

(11)
as the nonlinearities satisfying \( \mathbb{F} \) show. On the other hand we can construct an example such that \( \mathbb{F} \) is not true.

Remark 4 Using the reference \( \mathbb{F} \) and working as above we can obtain similar results, as in Theorem 4 and Theorem 5, if the nonlinearities \( f_1, f_2, h_1 \) and \( h_2 \) are assumed to satisfy the conditions of the Keller-Osserman type

\[
\int_a^\infty \frac{1}{k_1+1/\sqrt{(k_1+1)\int_a^s h_1^{-1/k_1}(c_1 f_1^{1/k_2}(s)) ds}} ds \leq \infty, \quad \int_0^\infty \frac{1}{k_2+1/\sqrt{(k_2+1)\int_0^s h_2^{-1/k_2}(c_2 f_1^{1/k_1}(s)) ds}} ds \leq \infty, \quad \]  

(12)
but the results will not be so strong as here. There are, some differences!

3 Proofs of the main results

In this section we give the proofs of Theorems 1 and 2. The first important tool in our proof is a variant of the Arzelà–Ascoli Theorem.  

3.1 The Arzelà–Ascoli Theorem

Let \( r_1, r_2 \in \mathbb{R} \) with \( r_1 \leq r_2 \) and \( (K = [r_1, r_2], d_K (x, y)) \) be a compact metric space, with the metric \( d_K (x, y) = |x - y| \), and let

\[ C ([r_1, r_2]) = \{ g : [r_1, r_2] \rightarrow \mathbb{R} | g \text{ is continuous on } [r_1, r_2] \} \]
denote the space of real valued continuous functions on \([r_1, r_2]\) and for any \( g \in C ([r_1, r_2]) \), let

\[ \|g\|_\infty = \max_{x \in [r_1, r_2]} |g(x)| \]
be the maximum norm on \( C ([r_1, r_2]) \).

Remark 5 If \( d (g^1(x), g^2(x)) = \|g^1(x) - g^2(x)\|_\infty \) then \((C ([r_1, r_2]), d)\) is a complete metric space.

Definition 1 We say that the sequence \( \{g_n\}_{n \in \mathbb{N}} \) from \( C ([r_1, r_2]) \) is bounded if there exists a positive constant \( C < \infty \) such that \( \|g_n(x)\| \leq C \) for each \( x \in [r_1, r_2] \). (Equivalently: \( |g_n(x)| \leq C \) for each \( x \in [r_1, r_2] \) and \( n \in \mathbb{N}^\ast \).

Definition 2 We say that the sequence \( \{g_n\}_{n \in \mathbb{N}} \) from \( C ([r_1, r_2]) \) is equicontinuous if for any given \( \varepsilon > 0 \), there exists a number \( \delta > 0 \) (which depends only on \( \varepsilon \)) such that

\[ |g_n(x) - g_n(y)| < \varepsilon \text{ for all } n \in \mathbb{N} \text{ whenever } d_K (x, y) < \delta \text{ for every } x, y \in [r_1, r_2]. \]

Definition 3 Let \( \{g_n\}_{n \in \mathbb{N}} \) be a family of functions defined on \([r_1, r_2] \). The sequence \( \{g_n\}_{n \in \mathbb{N}} \) converges uniformly to \( g(x) \) if for every \( \varepsilon > 0 \) there is an \( N \) (which depends only on \( \varepsilon \)) such that

\[ |g_n(x) - g(x)| < \varepsilon \text{ for all } n > N \text{ and } x \in [r_1, r_2]. \]

Theorem 3 (Arzelà–Ascoli theorem) If a sequence \( \{g_n\}_{n \in \mathbb{N}} \) in \( C ([r_1, r_2]) \) is bounded and equicontinuous then it has a subsequence \( \{g_{n_k}\}_{k \in \mathbb{N}} \) which converges uniformly to \( g(x) \) on \( C ([r_1, r_2]) \).

5
Proof of the Theorems [1] and [2]. Recall that
\[
\lambda(D^2u(x)) = \begin{cases} 
(\xi''(r), \xi'(r), \xi(r)) & \text{for } r \in (0, R), \\
(\xi''(0), \xi''(0), \xi'(0)) & \text{for } r = 0 
\end{cases}
\]
\[
S_k(\lambda(D^2u(x))) = \begin{cases} 
C^{k-1}_{N-1} \xi''(r) \left( \frac{\xi'(r)}{r} \right)^{k-1} + C^{k-1}_{N-1} \frac{N-k}{k} \left( \frac{\xi'(r)}{r} \right)^{k} & \text{for } r \in (0, R), \\
C^{k}_{N} \xi''(0)^{k} & \text{for } r = 0, 
\end{cases}
\]
where for \( r = |x| < R, u(x) = \xi(r) \in C^2[0, R] \) is radially symmetric with \( \xi'(0) = 0 \) and \( C^{k-1}_{N-1} = (N-1)!/[(k-1)!(N-k)!] \) (see, for example, [19] or [29]).

We start by showing that the system (6) has positive radial solutions. Therefore it can be reduced to
\[
\begin{align*}
C^{k-1}_{N-1} \int_{k_1}^{r^k} \int_{0}^{\rho_0} e^t \rho_1 \xi_1(t) dt \left( \frac{\xi_1(t)}{r} \right)^{k-1} \left( \frac{\xi_1(t)}{r} \right)^{k} & = r^{N-1} \int_{0}^{\rho_0} e^t \rho_1 \xi_1(t) dt \left( \frac{\xi_1(t)}{r} \right)^{k-1} \left( \frac{\xi_1(t)}{r} \right)^{k} p_1(r) f_1(u_2(r)) \text{ for } r > 0, \\
C^{k-1}_{N-1} \int_{k_2}^{r^k} \int_{0}^{\rho_0} e^t \rho_2 \xi_2(t) dt \left( \frac{\xi_2(t)}{r} \right)^{k-1} \left( \frac{\xi_2(t)}{r} \right)^{k} & = r^{N-1} \int_{0}^{\rho_0} e^t \rho_2 \xi_2(t) dt \left( \frac{\xi_2(t)}{r} \right)^{k-1} \left( \frac{\xi_2(t)}{r} \right)^{k} p_2(r) f_2(u_1(r)) \text{ for } r > 0,
\end{align*}
\]
(13)
\[
\begin{cases} 
u_1'(r) \geq 0 \text{ and } \nu_2'(r) \geq 0 \text{ for } r \in [0, \infty), \\
u_1(0) = a \text{ and } u_2(0) = b.
\end{cases}
\]
The solutions of (13) can be obtained by using successive approximation. We define the sequences \( \{u_1^m\}_{m \geq 1} \) and \( \{u_2^m\}_{m \geq 1} \) on \([0, \infty)\) in the following way:
\[
\begin{align*}
u_1^0 & = a, \nu_2^0 = b \text{ for } r \geq 0, \\
u_1^m(r) & = a + \int_{0}^{r} \left[ G_2^+ (t) \int_{0}^{t} G_2^+ (s) f_1(u_2^{m-1}(s)) ds \right]^{1/k_1} dt, \quad (14) \\
u_2^m(r) & = b + \int_{0}^{r} \left[ G_1^+ (t) \int_{0}^{t} G_1^+ (s) f_2(u_1^{m}(s)) ds \right]^{1/k_2} dt.
\end{align*}
\]
We can see that \( \{u_1^m\}_{m \geq 1} \) and \( \{u_2^m\}_{m \geq 1} \) are non-decreasing on \([0, \infty)\). To do this, let us consider
\[
\begin{align*}
u_1^1(r) & = a + \int_{0}^{r} \left[ G_2^+ (t) \int_{0}^{t} G_2^+ (s) f_1(u_2^0(s)) ds \right]^{1/k_1} dt \\
& = a + \int_{0}^{r} \left[ G_2^+ (t) \int_{0}^{t} G_2^+ (s) f_1(b) ds \right]^{1/k_1} dt \\
& \leq a + \int_{0}^{r} \left[ G_2^+ (t) \int_{0}^{t} G_2^+ (s) f_1(u_2^1(s)) ds \right]^{1/k_1} dt = u_2^1(r).
\end{align*}
\]
This implies that
\[
u_1^1(r) \leq u_2^1(r) \text{ which further produces } u_2^1(r) \leq u_2^2(r).
\]
A mathematical induction argument applied to (13) show that for any \( r \geq 0 \)
\[
u_1^m(r) \leq u_1^{m+1}(r) \text{ and } u_2^m(r) \leq u_2^{m+1}(r) \text{ for any } m \in \mathbb{N},
\]
i.e., \( \{u_1^m\}_{m \geq 1} \) and \( \{u_2^m\}_{m \geq 1} \) are non-decreasing on \([0, \infty)\). We now prove that the non-decreasing sequences \( \{u_1^m\}_{m \geq 1} \) and \( \{u_2^m\}_{m \geq 1} \) are bounded from above on bounded sets. Indeed, by the monotonicity of \( \{u_1^m\}_{m \geq 1} \) and \( \{u_2^m\}_{m \geq 1} \) we have the inequalities
\[
C^{k-1}_{N-1} \int_{k_1}^{r^k} \int_{0}^{\rho_0} e^t \rho_1 \xi_1(t) dt \left[ (u_1^m(r))^{k_1} \right]^{1/k_1} = G_2^+ (r) f_1(u_2^{m-1}(r)) \leq G_2^+ (r) f_1(u_2^m(r)), \quad (15)
\]
\[
C^{k-1}_{N-1} \int_{k_2}^{r^k} \int_{0}^{\rho_0} e^t \rho_2 \xi_2(t) dt \left[ (u_2^m(r))^{k_2} \right]^{1/k_2} = G_1^+ (r) f_2(u_1^m(r)) = G_1^+ (r) f_2(u_1^m(r)). \quad (16)
\]
Integrating (15) leads to

\[(u_1^n(r))' = \left[ G_2 (r) \int_0^r G_2^\top (t) f_1 (u_2^{m-1} (t)) dt \right]^\top\]

\[\leq \left[ G_2 (r) \int_0^r G_2^\top (t) f_1 (u_2^m (t)) dt \right]^\top\]

\[= G_2 (r) \int_0^r G_2^\top (t) f_1 (b + \int_0^t (G_1^\top (z) \int_0^z G_1^\top (s) f_2 (u_2^m (s)) ds) \right] dt \right] \}

\[\leq (G_2 (r) \int_0^r G_2^\top (t) f_1 \left( f_2^{1/k_2} (u_2^m (t)) \int_0^t (G_1^\top (z) \int_0^z G_1^\top (s) ds) \right] \right) dt \}

\[\leq (G_2 (r) \int_0^r G_2^\top (t) f_1 \left( f_2^{1/k_2} (u_2^m (t)) \int_0^t (G_1^\top (z) \int_0^z G_1^\top (s) ds) \right] \right) dt \}

Arguing as above, but now with the second equation (16), we obtain

\[(u_2^m(r))' = \left[ G_1 (r) \int_0^r G_1^\top (t) f_2 (u_2^m (t)) ds \right]^{1/k_2}

\[\leq h_2 \left( M_2 f_1^{1/k_2} (u_2^m (r)) \right) \left( G_1 (r) \int_0^r G_1^\top (t) \int_0^t (G_1^\top (z) \int_0^z G_1^\top (s) ds) \right) dt \}

Combining the previous relations we obtain

\[\frac{(u_1^n(r))'}{h_1^{1/k_1} (M_1 f_2^{1/k_2} (u_1^n (r)))} \leq \tau_1^{1/k_1} \left( G_2 (r) \int_0^r G_2^\top (t) \int_0^t (G_1^\top (z) \int_0^z G_1^\top (s) ds) \right) dt \]

\[\frac{(u_2^n(r))'}{h_2^{1/k_2} (M_2 f_1^{1/k_1} (u_2^n (r)))} \leq \tau_2^{1/k_2} \left( G_1 (r) \int_0^r G_1^\top (t) \int_0^t (G_2^\top (z) \int_0^z G_2^\top (s) ds) \right) dt \]

\[
\int_a^{u_1^n(r)} \frac{1}{h_1^{1/k_1} (M_1 f_2^{1/k_2} (t))} \right) dt \leq \tau_1^{1/k_1} \left( V_{1.2} (r) \right)

\int_b^{u_2^n(r)} \frac{1}{h_2^{1/k_2} (M_2 f_1^{1/k_1} (t))} \right) dt \leq \tau_2^{1/k_2} \left( V_{2.1} (r) \right).

We then may write

\[H_{1.2} (u_1^n (r)) \leq \tau_1^{1/k_1} \left( V_{1.2} (r) \right) and H_{2.1} (u_2^n (r)) \leq \tau_2^{1/k_2} \left( V_{2.1} (r) \right),

which plays a basic role in the proof of our main results. Since \( H_{oo} \) is a bijection with the inverse function \( H_{oo}^{-1} \) strictly increasing on \([0, \infty)\), the inequalities (21) can be reformulated as

\[u_1^n (r) \leq H_{1.2}^{-1} \left( \tau_1^{1/k_1} \left( V_{1.2} (r) \right) \right) and u_2^n (r) \leq H_{2.1}^{-1} \left( \tau_2^{1/k_2} \left( V_{2.1} (r) \right) \right).\]
So, we have found upper bounds for 
\[ \{u_1^m\}_{m \geq 1} \quad \text{and} \quad \{u_2^m\}_{m \geq 1} \]
which are dependent of \( r \). We are now ready to give a complete proof of the Theorems [12].

**Proof of Theorem 1 completed:** We prove that the sequences \( \{u_1^m\}_{m \geq 1} \) and \( \{u_2^m\}_{m \geq 1} \) are bounded and equicontinuous on \([0, c_0]\) for arbitrary \( c_0 > 0 \). We take

\[
C_1 = H_{1,2}^{-1}(\mathcal{P}_1^{k_1} \mathcal{T}_{1,2}(c_0)) \quad \text{and} \quad C_2 = H_{2,1}^{-1}(\mathcal{P}_2^{k_2} \mathcal{T}_{2,1}(c_0))
\]
and since \( (u_1^m(r))' \geq 0 \) and \( (u_2^m(r))' \geq 0 \) it follows that

\[
u_1^m(r) \leq u_1^m(c_0) \leq C_1 \quad \text{and} \quad u_2^m(r) \leq C_2.
\]

We have proved that \( \{u_1^m(r)\}_{m \geq 1} \) and \( \{u_2^m(r)\}_{m \geq 1} \) are bounded on \([0, c_0]\) for arbitrary \( c_0 > 0 \). Using this fact in [13] and [18] we show that the same is true for \( (u_1^m(r))' \) and \( (u_2^m(r))' \). By construction we verify that

\[
(u_1^m(r))' = \left( \frac{r^{k_1} e^{-\int_0^r \frac{1}{u_1^m(s)} ds} \int_0^r t^{N-1} e^{\int_0^r \frac{1}{u_1^m(s)} ds} f_1(t) \, dt}{C_0} \right)^{1/k_1}
\]

\[
\leq \left( \frac{r^{k_1} e^{-\int_0^r \frac{1}{u_1^m(s)} ds} \int_0^r p_1(t) \, dt}{C_0} \right)^{1/k_1}
\]

\[
\leq \left( \int_0^r f_1(u_2^m(t)) \, dt \right)^{1/k_1}
\]

\[
c_0^{-1/k_1} \int_0^r f_1(u_2^m(t)) \, dt \geq c_0 \quad \text{on} \quad [0, c_0].
\]

Now we turn to \( (u_2^m(r))' \). A similar argument shows that

\[
(u_2^m(r))' \leq C_0^{-1/k_2} \int_0^r f_1(u_2^m(t)) \, dt \geq c_0 \quad \text{on} \quad [0, c_0].
\]

Finally, it remains to prove that \( \{u_1^m(r)\}_{m \geq 1} \) and \( \{u_2^m(r)\}_{m \geq 1} \) are equicontinuous on \([0, c_0]\) for arbitrary \( c_0 > 0 \). Let \( \varepsilon_1, \varepsilon_2 > 0 \) be arbitrary. By the mean-value formula we then deduce that

\[
|u_1^m(x) - u_1^m(y)| = |(u_1^m(\xi_1))'| |x - y| \leq C_0^{1/k_1} \int_0^r f_1(u_2^m(t)) \, dt \leq C_0^{-1/k_2} \|p_2\|_{L^\infty} f_1^{1/k_2} \|C_2\| c_0 |x - y|,
\]

\[
|u_2^m(x) - u_2^m(y)| = |(u_2^m(\xi_2))'| |x - y| \leq C_0^{-1/k_2} \|p_2\|_{L^\infty} f_1^{1/k_2} \|C_2\| c_0 |x - y|,
\]

for all \( n \in \mathbb{N} \) and all \( x, y \in [0, c_0] \). So it suffices to take

\[
\delta_1 = \frac{C_0^{1/k_1} \varepsilon_1}{\|p_2\|_{L^\infty} f_1^{1/k_1} \|C_2\| c_0} \quad \text{and} \quad \delta_2 = \frac{C_0^{1/k_2} \varepsilon_2}{\|p_2\|_{L^\infty} f_1^{1/k_2} \|C_2\| c_0}
\]

to see that \( \{u_1^m(r)\}_{m \geq 1} \) and \( \{u_2^m(r)\}_{m \geq 1} \) are equicontinuous on \([0, c_0]\). We now conclude immediately with the help of Arzelà-Ascoli theorem, possibly after passing to a subsequence, that the sequences \( \{u_j^m\}_{j=1,2} \) converges uniformly to \( \{u_j\}_{j=1,2} \) on \([0, c_0]\), in the norm \( C[0, c_0] \). At the end of this process, we conclude, by the arbitrariness of \( c_0 > 0 \), that \( \{u_1, u_2\} \) is a positive entire solution of system (6). The solution constructed in this way is radially symmetric. Going back to the system (5), the radial solutions of (13) are solutions of the ordinary differential equations system (6). We conclude that radial solutions of (5) with \( u_1(0) = a \), \( u_2(0) = b \) satisfy:

\[
u_1(r) = a + \int_0^r \left( G_2^-(y) \int_0^y G_2^+(t) f_1(u_2(t)) \, dt \right)^{1/k_1} dy, \quad r \geq 0,
\]

\[
u_2(r) = b + \int_0^r \left( G_1^-(y) \int_0^y G_1^+(t) f_2(u_1(t)) \, dt \right)^{1/k_2} dy, \quad r \geq 0.
\]
Next we prove that all four statements hold true.

1.) When $\mathcal{T}_{1,2}(\infty) < \infty$ and $\mathcal{T}_{2,1}(\infty) < \infty$, using the same arguments as in (22), we find from (23) and (24)
that
$$u_1(r) \leq H_{1,2}^{-1}\left(\tau_1^{1/k_1} \mathcal{T}_{1,2}(\infty)\right) < \infty \quad \text{and} \quad u_2(r) \leq H_{2,1}^{-1}\left(\tau_2^{1/k_2} \mathcal{T}_{2,1}(\infty)\right) < \infty \quad \text{for all} \quad r \geq 0,$$
and so $(u_1, u_2)$ is bounded, which completes the proof. We next consider:

2.) In the case $\mathcal{P}_{1,2}(\infty) = \mathcal{P}_{2,1}(\infty) = \infty$, we observe that
$$u_1(r) = a + \int_0^r \left( G_2^-(t) \int_0^t G_2^+(s) f_1(u_2(s)) \right) \frac{1}{t} \, dt$$
$$= a + \int_0^r \left( G_2^-(y) \int_0^y G_2^+(t) f_1 \left( b + \int_0^t (G_1^-)(z) \int_0^z G_1^+(s) f_2(u_1(s)) \right) \frac{1}{t} \, dy \right) dt$$
$$\geq a + \int_0^r \left( G_2^-(y) \int_0^y G_2^+(t) f_1 \left( 1 + \int_0^t (G_1^-)(z) \int_0^z G_1^+(s) \right) \frac{1}{t} \, dy \right) dt \tag{25}$$
$$\geq a + \int_0^r \left( G_2^-(y) \int_0^y G_2^+(t) \varphi_1 \left( 1 + \int_0^t (G_1^-)(z) \int_0^z G_1^+(s) \varphi_2 \right) \right) dt \frac{1}{t} \, dy$$
$$= a + c_1^{1/k_1} \mathcal{P}_{1,2}(r).$$

The same computations as in (25) yields
$$u_2(r) \geq b + \int_0^r \left( G_1^-(y) \int_0^y G_1^+(t) \varphi_2 \left( 1 + \int_0^t (G_2^-)(z) \int_0^z G_2^+(s) \right) \frac{1}{t} \, dy \right) dt \frac{1}{t} \, dy$$
$$= b + c_2^{1/k_2} \mathcal{P}_{2,1}(r),$$
and passing to the limit as $r \to \infty$ in (25) and in the above inequality we conclude that
$$\lim_{r \to \infty} u_1(r) = \lim_{r \to \infty} u_2(r) = \infty,$$
which yields the result.

3.) In the spirit of 1.) and 2.) above, we have
$$u_1(r) \leq H_{1,2}^{-1}\left(\tau_1^{1/k_1} \mathcal{T}_{1,2}(\infty)\right) < \infty \quad \text{and} \quad u_2(r) \geq b + c_2^{1/k_2} \mathcal{P}_{2,1}(r).$$

So, if
$$\mathcal{T}_{1,2}(\infty) < \infty \quad \text{and} \quad \mathcal{P}_{2,1}(\infty) = \infty$$
we have that
$$\lim_{r \to \infty} u_1(r) < \infty \quad \text{and} \quad \lim_{r \to \infty} u_2(r) = \infty.$$

In order, to complete the proof of Theorem 1 it remains to proceed to the

4.) In this case, we invoke the proof of 3.). We observe that
$$u_1(r) \geq a + c_1^{1/k_1} \mathcal{P}_{1,2}(r) \quad \text{and} \quad u_2(r) \leq H_{2,1}^{-1}\left(\tau_2^{1/k_2} \mathcal{T}_{2,1}(r)\right). \tag{26}$$

Our conclusion follows now by letting $r \to \infty$ in (26).

Proof of Theorem 2 completed:

i.) Combining (21) and the conditions of the theorem, we are led to
$$H_{1,2}(u_1^m(r)) \leq \tau_1^{1/k_1} \mathcal{T}_{1,2}(\infty) < \tau_1^{1/k_1} H_{1,2}(\infty) < \infty,$$
$$H_{2,1}(u_2^m(r)) \leq \tau_2^{1/k_2} \mathcal{T}_{2,1}(\infty) < \tau_2^{1/k_2} H_{2,1}(\infty) < \infty.$$
On the other hand, since $H^{-1}_{\circ}$ is strictly increasing on $[0,\infty)$, we find that
\[
 u^m_1(r) \leq H^{-1}_{1,2}\left(\frac{r^{1/k_1}}{r^{1/k_2}}\right) < \infty \quad \text{and} \quad u^m_2(r) \leq H^{-1}_{2,1}\left(\frac{r^{1/k_2}}{r^{1/k_1}}\right) < \infty,
\]
and then the non-decreasing sequences $\{u^m_1(r)\}_{m \geq 1}$ and $\{u^m_2(r)\}_{m \geq 1}$ are bounded above for all $r \geq 0$ and all $m$. Combining these two facts, we conclude that
\[
 (u^m_1(r), u^m_2(r)) \to (u_1(r), u_2(r)) \quad \text{as} \quad m \to \infty
\]
and the limit functions $u_1$ and $u_2$ are positive entire bounded radial solutions of system (6).

ii.) and iii.): For the proof, we follow the same steps and arguments as in the proof of Theorem 1.

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References

[1] C.O. Alves and ARF de Holanda, *Existence of blow-up solutions for a class of elliptic systems*, Differential Integral Equations, Volume 26, Number 1/2, Pages 105-118, 2013.

[2] C. Bandle and E. Giarrusso, *Boundary blow-up for semilinear elliptic equations with nonlinear gradient terms*, Advances in Differential Equations, Volume 1, Pages 133–150, 1996.

[3] L. Bieberbach, *$\Delta u = e^u$ und die automorphen Funktionen*, Mathematische Annalen, Volume 77, Pages 173-212, 1916.

[4] J. Bao, X. Ji and H. Li, *Existence and nonexistence theorem for entire subsolutions of k-Yamabe type equations*, J. Differential Equations 253 (2012) 2140–2160.

[5] F.L. Cirstea and V. R˘ adulescu, *Entire solutions blowing up at infinity for semilinear elliptic systems*, Journal de Mathématiques Pures et Appliquées, Volume 81, Pages 827–846, 2002.

[6] P. Clément, R. Manásevich and E. Mitidieri, *Positive solutions for a quasilinear system via blow up*, Communications in Partial Differential Equations, No. 12, Volume 18, Pages 2071–2106, 1993.

[7] D.-P. Covei, *The existence of entire radial solutions to a semilinear elliptic system*, http://arxiv.org/pdf/1509.01968.pdf.

[8] D.-P. Covei, *Boundedness and blow-up of solutions for a nonlinear elliptic system*, International Journal of Mathematics, Volume 25, No. 9, Pages 1-12, 2014.

[9] D. G. De Figueiredo and Y. Jianfu, *Decay, symmetry and existence of solutions of semilinear elliptic systems*, Nonlinear Analysis: Theory, Methods & Applications, Volume 33, Pages 211–234, 1998.

[10] V. Galaktionov and J.-L. Vázquez, *The problem of blow-up in nonlinear parabolic equations*, Discrete and Continuous Dynamical Systems - Series A, Volume 8, Pages 399–433, 2002.

[11] M. Ghergu and V. Rădulescu, *Explosive solutions of semilinear elliptic systems with gradient term*, RACSAM Revista Real Academia de Ciencias (Serie A, Matemáticas), Volume 97, Pages 437-445, 2003.

[12] E. Giarrusso, *On blow up solutions of a quasilinear elliptic equation*, Mathematische Nachrichten, Volume 213, Pages 89–104, 2000.

[13] H. Grosse and A. Martin, *Particle Physics and the Schrödinger Equation*, Cambridge Monographs on Particle Physics’s, Nuclear Physics and Cosmology, 1997.

[14] J. Gustavsson, L. Maligranda and J. Peetre, *Submultiplicative function*, Indagationes Mathematicae (Proceedings), Volume 92, Issue 4, Pages 435-442, 1989.
[15] D. D. Hai and R. Shivaji, An existence result on positive solutions for a class of semilinear elliptic systems, Proceedings of the Royal Society of Edinburgh, 134A, Pages 137-141, 2004.

[16] J.B. Keller, On solution of \( \Delta u = f(u) \), Communications on Pure and Applied Mathematics, 10 (1957), 503-510.

[17] J. B. Keller, Electrohydrodynamics I. The Equilibrium of a Charged Gas in a Container, Journal of Rational Mechanics and Analysis, Volume 5, Number 4, 1956.

[18] M. A. Krasnosel’skii and YA. B. Rutickii, Convex functions and Orlicz spaces, Translated from the first Russian edition by Leo F. Bo Ron, P. Noordhoff LTD. - Groningen - the Netherlands, 1961.

[19] X. Ji and J. Bao, Necessary and sufficient conditions on solvability for Hessian inequalities, Proceedings of the American Mathematical Society, Volume 138, Number 1, January 2010, Pages 175–188.

[20] A. V. Lair and A. W. Wood, Existence of Entire Large Positive Solutions of Semilinear Elliptic Systems, Journal of Differential Equations, Volume 164, Issue 2, 1 July 2000, Pages 380-394.

[21] A.V. Lair, A necessary and sufficient condition for the existence of large solutions to sublinear elliptic systems, Journal of Mathematical Analysis and Applications, Volume 365, Issue 1, 1 May 2010, Pages 103-108.

[22] A.V. Lair, Entire large solutions to semilinear elliptic systems, Journal of Mathematical Analysis and Applications, Volume 382, Issue 1, 1 October 2011, Pages 324-333.

[23] A. C. Lazer and P. J. McKenna, On a problem of Bieberbach and Rademacher, Nonlinear Analysis, Volume 21, Pages 327-335, 1993.

[24] A. W. Leung, Positive solutions for large elliptic systems of interacting species groups by cone index methods, Journal of Mathematical Analysis and Applications, Volume 291, Issue 1, 1 March 2004, Pages 302–321.

[25] R. Osserman, On the inequality \( \Delta u \geq f(u) \), Pacific Journal of Mathematics, 7, Pages 1641-1647, 1957.

[26] J. Peterson and A. W. Wood, Large solutions to non-monotone semilinear elliptic systems, Journal of Mathematical Analysis and Applications, Volume 384, Issue 2, 15 December 2011, Pages 284-292.

[27] P. Quittner, Blow-up for semilinear parabolic equations with a gradient term, Mathematical Methods in the Applied Sciences, Volume 14, Pages 413–417, 1991.

[28] H. Rademacher, Finie besondere probleme partieller Differentialgleichungen, in : Die Differential und Integralgleichungen der Mechanick und Physik, I, 2nd ed., Rosenberg, New York, Pages 838-845, 1943.

[29] P. Salani, Boundary blow-up problems for Hessian equations, Manuscripta Mathematica, Volume 96, Pages 281 – 294, 1998.

[30] J.A. Viaclovsky, Conformal geometry, contact geometry, and the calculus of variations, Duke Mathematical Journal, Volume 101, Pages 283–316, 2000.

[31] J.A. Viaclovsky, Estimates and existence results for some fully nonlinear elliptic equations on Riemannian manifolds, Communications in Analysis and Geometry, Volume 10, Pages 815–846, 2002.

[32] Z. Zhang and S. Zhou, Existence of entire positive k-convex radial solutions to Hessian equations and systems with weights, Applied Mathematics Letters, Volume 50, December 2015, Pages 48–55.