POSITIVE SCALAR CURVATURE AND STRONGLY INESSENTIAL MANIFOLDS

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Abstract. We prove that a closed $n$-manifold $M$ with positive scalar curvature and abelian fundamental group admits a finite covering $M'$ which is strongly inessential. The latter means that a classifying map $u: M' \to K(\pi_1(M'), 1)$ can be deformed to the $(n-2)$-skeleton. This is proven for all $n$-manifolds with the exception of 4-manifolds with spin universal coverings.

1. Introduction

The notion of macroscopic dimension was introduced by M. Gromov [G2] to study topology of manifolds with a positive scalar curvature metric.

1.1. Definition. A metric space $X$ has the macroscopic dimension $\dim_{mc} X \leq k$ if there is a uniformly cobounded proper map $f: X \to K$ to a $k$-dimensional simplicial complex. Then $\dim_{mc} X = m$ where $m$ is minimal among $k$ with $\dim_{mc} X \leq k$.

A map of a metric space $f: X \to Y$ is uniformly cobounded if there is a uniform upper bound on the diameter of preimages $f^{-1}(y)$, $y \in Y$.

Gromov’s Conjecture. The macroscopic dimension of the universal covering $\tilde{M}$ of a closed positive scalar curvature $n$-manifold $M$ satisfies the inequality $\dim_{mc} \tilde{M} \leq n - 2$ for the metric on $\tilde{M}$ lifted from $M$.

The main examples supporting Gromov’s Conjecture are $n$-manifolds of the form $M = N \times S^2$. They admit metrics with PSC in view of the formula $Sc_{x_1, x_2} = S\hat{c}_{x_1} + \hat{c}_{x_2}$ for the Cartesian product ($X_1 \times \ldots \times X_n \times S^2$).
of two Riemannian manifolds \((X_1, \mathcal{G}_1)\) and \((X_2, \mathcal{G}_2)\) and the fact that while \(Sc_N\) is bounded \(Sc_{S^2}\) can be chosen to be arbitrary large. Clearly, the projection \(p : \hat{M} = \tilde{N} \times S^2 \to \tilde{N}\) is a proper uniformly cobounded map to a \((n-2)\)-dimensional manifold which can be triangulated. Hence, \(\dim_{mc} \hat{M} \leq n - 2\).

Since \(\dim_{mc} X = 0\) for every compact metric space, the Gromov Conjecture holds trivially for manifolds with finite fundamental groups. Therefore, Gromov’s conjecture is about manifolds with infinite fundamental groups. We note that even a weaker version of the Gromov Conjecture that predicts the inequality \(\dim_{mc} \hat{M} \leq n - 1\) for positive scalar curvature manifolds is out of reach, since it implies perhaps the famous Gromov-Lawson’s Conjecture: A closed aspherical manifold cannot carry a metric of positive scalar curvature. The latter is known as a twin sister of the famous Novikov Higher Signature conjecture. Both conjectures are proven only for some tame classes of groups. The Gromov Conjecture is proven in far fewer cases \([B2],[Dr1],[BD2],[DD]\).

We note that in the study of topology of positive scalar curvature manifolds it makes sense to consider three different cases: the case of spin manifolds, almost spin manifolds, and totally non-spin manifolds. The reason for that is the existence of the index theory in the first two cases. Thus, the case of totally non-spin manifolds is the most difficult. Here we adopt the names almost spin for manifolds with the spin universal covering and totally non-spin for manifolds whose universal coverings are non-spin.

Gromov defined inessential manifolds \(M\) as those for which a classifying map \(u : M \to B\Gamma\) of the universal covering \(\hat{M}\) can be deformed to the \((n-1)\)-skeleton \(B\Gamma^{(n-1)}\) where \(n = \dim M\). Clearly, for an inessential \(n\)-manifold \(M\) we have \(\dim_{mc} \hat{M} \leq n - 1\). In the case of spin manifolds Rosenberg’s vanishing index theorem \([R1]\) implies that a positive scalar curvature manifold with the fundamental group \(\Gamma\) satisfying the Analytic Novikov conjecture and the Rosenberg-Stolz condition on injectivity of the real K-theory periodization map \(\text{per} : KO_{\ast}(B\Gamma) \to KO_{\ast}(B\Gamma)\) is inessential \([BD1]\).

Having this in mind we introduce even a stronger version of Gromov’s Conjecture. For that we extend Gromov’s definition of inessentiality to the following. We call an \(n\)-manifold strongly inessential if a classifying map of its universal covering \(u : M \to B\Gamma\) can be deformed to the \((n-2)\)-skeleton. Answering Gromov’s question Bolotov constructed an example of an inessential manifold which is not strongly inessential \([B1]\).
**Strong Gromov’s Conjecture.** A closed positive scalar curvature manifold $M$ admits a strongly inessential finite covering $M'$.

We note that, since the universal cover of $M'$ coincides with the universal cover of $M$, the Strong Gromov Conjecture implies the original one.

In this paper we prove the Strong Gromov Conjecture in the case of abelian fundamental groups. We have one exception here: Our proof does not work for almost spin 4-dimensional manifolds. In this case we can show the existence of an inessential finite cover but we cannot prove its strong inessentiality. In the totally non-spin case we were dealing with an opposite problem: We have a technique [BD2],[DD] to derive the strong inessentiality from the inessentiality but proving the latter became possible only after recent work of Schoen and Yau [SY].

1.2. **Remark.** We note that for spin manifolds the Strong Gromov Conjecture in the case of abelian fundamental group follows from the main result of [BD1]. Unfortunately, Lemma 4.1 in [BD1], which is essential for the main result, has a gap in its proof and the attempt to fix it in [Dr2], Lemma 6.2 failed as well. The problem there can be reduced to the following question about stable homotopy groups:

1.3. **Question.** For which finitely presented groups $\Gamma$ the natural homomorphism from the coinvariants $\xi : \pi^s_n(E\Gamma^{(n-1)}) \to \pi^s_n(B\Gamma^{(n-1)})$ is injective for all $n > 4$?

2. **Preliminaries**

We recall that for every spectrum $E$ there is a connective cover $e \to E$, that is the spectrum $e$ with the morphism $e \to E$ that induces isomorphisms of homotopy groups $\pi_i(e) \to \pi_i(E)$ for $i \geq 0$ and with $\pi_i(e) = 0$ for $i < 0$. By $KO$ we denote the spectrum for real $K$-theory, by $ko$ its connective cover, and by $per : ko \to KO$ the corresponding morphism of spectra. We use the standard notation $\pi^s$ for the stable homotopy groups. For a spectrum $E$ we will use the old-fashioned notation $E_*(X)$ for the $E$-homology of a space $X$.

The following proposition is taken from [BD1].

2.1. **Proposition.** The natural transformation $\pi^s_*(pt) \to ko_*(pt)$ induces an isomorphism $\pi^s_n(K/K^{(n-2)}) \to ko_n(K/K^{(n-2)})$ for any CW complex $K$.

We recall that the group of oriented relative bordisms $\Omega_n(X,Y)$ of the pair $(X,Y)$ consists of the equivalence classes of pairs $(M,f)$ where $M$ is an oriented $n$-manifold with boundary and $f : (M,\partial M) \to (X,Y)$
is a continuous map. Two pairs \((M, f)\) and \((N, g)\) are equivalent if there is a pair \((W, F)\), \(F : W \to X\) called a bordism where \(W\) is an orientable \((n+1)\)-manifold with boundary such that \(\partial W = M \cup W' \cup N\), \(W' \cap M = \partial M\), \(W' \cap N = \partial N\), \(F|_M = f\), \(F|_N = g\), and \(F(W') \subset Y\).

In the special case when \(X\) is a point, the manifold \(W\) is called a bordism between \(M\) and \(N\). The following proposition is proven in [BD2].

2.2. Proposition. For any CW complex \(K\) there is an isomorphism
\[
\Omega_n(K, K^{(n-2)}) \cong H_n(K, K^{(n-2)}).
\]

We recall the following classical result Corollary 6.10.3, [TtD]:

2.3. Theorem. Suppose that a CW-complex pair \((X, A)\) satisfies the conditions \(\pi_i(X) = 0\) for \(i < m\) and \(\pi_i(A) = 0\) for \(i < m-1\) with \(m \geq 2\). Then the quotient map \(q : (X, A) \to (X/A, *)\) induces isomorphisms \(q_* : \pi_i(X, A) \to \pi_i(X/A, *)\) for \(i \leq 2m - 1\).

3. INESSENTIAL MANIFOLDS

3.1. Definition. An \(n\)-manifold \(M\) with fundamental group \(\Gamma\) is called inessential if a classifying map \(u_M : M \to B\Gamma\) of its universal covering can be deformed into the \((n-1)\)-skeleton \(B\Gamma^{(n-1)}\).

Note that for an inessential \(n\)-manifold \(M\) we have \(\dim_{mc} \tilde{M} \leq n-1\). Indeed, a lift \(\tilde{u}_M : \tilde{M} \to ET^{(n-1)}\) of a classifying map is a uniformly cobounded proper map to an \((n-1)\)-complex.

Establishing inessentiality of positive scalar curvature manifolds is the first step in a proof of the strong Gromov conjecture. We recall that the inessentiality of a manifold can be characterized as follows [Ba] (see also [BD1], Proposition 3.2).

3.2. Theorem. Let \(M\) be a closed oriented \(n\)-manifold. Then the following are equivalent:
1. \(M\) is inessential;
2. \(\left( u_M^*([M]) = 0 \right)\) in \(H_n(B\Gamma)\) where \([M]\) is the fundamental class of \([M]\).

In [BD1] we proved the following addendum to Theorem 3.2.

3.3. Proposition ([BD1], Lemma 3.5). For an inessential manifold \(M\) with a CW complex structure a classifying map \(u : M \to B\Gamma\) can be chosen such that
\[
u(M, M^{(n-1)}) \subset (B\Gamma^{(n-1)}, B\Gamma^{(n-2)}).
\]
We recall that for a manifold to be spin is equivalent to the orientability in any of the K-theories: complex \(KU\), real \(KO\), or their connective covers \(ku\) or \(ko\) [Ru].

J. Rosenberg connected the realm of positive scalar curvature manifolds to the Novikov Higher Signature conjecture by proving the following [R1]:

3.4. **Theorem.** Suppose that the fundamental group \(\Gamma\) of a positive scalar curvature spin manifold \(M\) satisfies the Strong Novikov conjecture. Then \(u_\ast([M]_{KO}) = 0\) where \(u : M \to B\Gamma\) is a classifying map.

Below we slightly reformulate Theorem 1.3 from [SY].

3.5. **Theorem** (Schoen-Yau). Suppose that a compact oriented \(n\)-manifold has 1-dimensional integral cohomology classes \(\alpha_i, i = 1, \ldots, n-1\) with nontrivial cup product \(\alpha_1 \smile \cdots \smile \alpha_{n-1} \neq 0\). Then \(M\) cannot carry a metric of positive scalar curvature.

3.6. **Proposition.** A closed orientable \(n\)-manifold \(M\) carrying a metric of positive scalar curvature with \(\pi_1(M) = \mathbb{Z}^m\) is inessential.

**Proof.** Since the case \(m < n\) is trivial, we assume that \(m \geq n\). Assume the contrary, \(u_\ast([M]) \neq 0\) in \(H_\ast(T^n; \mathbb{Z})\) where \(u : M \to T^n\) is the classifying map. By Proposition 4.6 in [BD2] there is a projection onto the factor \(q : T^n \to T^n\) such that \(q_\ast u_\ast([M]) \neq 0\). Hence \(q_\ast u_\ast([M]) = \ell[T^n]\) with \(\ell \neq 0\). Let \(\beta_1, \ldots, \beta_n\) be generators of \(H^1(T^n; \mathbb{Z})\) with \(\beta_1 \smile \cdots \smile \beta_n \neq 0\). Denote by \(\alpha_i = (qu)_\ast(\beta_i)\). We show that \(\alpha_1 \smile \cdots \smile \alpha_n \neq 0\) to get a contradiction with Schoen-Yau theorem. Note that

\[
(qu)_\ast((\alpha_1 \smile \cdots \smile \alpha_n) \cap [M]) = (\beta_1 \smile \cdots \smile \beta_n) \cap q_\ast u_\ast([M]) \neq 0.
\]

Since \((qu)_\ast\) is an isomorphism of 0-dimensional homology groups, we obtain \(\alpha_1 \smile \cdots \smile \alpha_n \neq 0\). \(\square\)

3.7. **Definition.** An \(n\)-manifold \(M\) with fundamental group \(\Gamma\) is called **strongly inessential** if its classifying map \(u_M : M \to B\Gamma\) can be deformed into the \((n-2)\)-skeleton \(B\Gamma^{(n-2)}\).

4. **Spin and almost spin manifolds**

4.1. **Definition.** We call a discrete group \(G\) **\(p\)-tame** if there is a finite covering \(\beta : B' \to BG\) that induces zero homomorphism

\[
\beta^* : H^2(BG; \mathbb{Z}_p) \to H^2(B'; \mathbb{Z}_p).
\]

**EXAMPLE.** The group \(\mathbb{Z}^n\) is \(p\)-tame for all \(p\). Moreover, any finitely generated abelian group is \(p\)-tame.
4.2. **Proposition.** For any closed almost spin manifold $M$ with 2-tame fundamental group there is a finite cover $p : M' \to M$ with spin $M'$.

**Proof.** Let $G = \pi_1(M)$ and let $u_M : M \to BG$ be a classifying map. Let $p : M \to M$ be the pull-back of $\beta : B' \to BG$ with respect to $u_M$. Then for the Stiefel-Whitney classes we have $w_2(M') = p^*(w_2)$. It suffices to show that $w_2(M) = u_M^*(\omega)$ for some $\omega \in H^*(BG; \mathbb{Z}_2)$, then we obtain $w_2(M') = (u_M')^*\beta^*(\omega) = 0$. Since the universal cover $\tilde{M}$ is spin, it follows that the evaluation of $w_2(M)$ on every spherical cycle is trivial. In view of the short exact sequence

$$
\pi_2(M) \to H_2(M) \to H_2(G) \to 0
$$

it follows that the homomorphism $\cap w_2(M) : H_2(M) \to \mathbb{Z}_2$ lies in the image of the homomorphism $\text{Hom}(H_2(G), \mathbb{Z}_2) \to \text{Hom}(H_2(M), \mathbb{Z}_2)$. Thus, in the diagram generated by the universal coefficient theorem exact sequences

$$
0 \to \text{Ext}(H_1(M), \mathbb{Z}_2) \xrightarrow{i} H^2(M; \mathbb{Z}_2) \xrightarrow{j} \text{Hom}(H_2(M), \mathbb{Z}_2) \to 0
$$

$$
0 \to \text{Ext}(H_1(G), \mathbb{Z}_2) \xrightarrow{i'} H^2(G, \mathbb{Z}_2) \xrightarrow{j'} \text{Hom}(H_2(G), \mathbb{Z}_2) \to 0
$$

$j(w_2(M)) = u^*(\phi)$ for some $\phi$. Then the diagram chasing implies that $w_2(M) = u_M^*(\omega)$ for $\omega = i'(\alpha) + \tilde{\phi}$ where $\tilde{\phi}$ is arbitrary with $j'(\tilde{\phi}) = \phi$ and $\alpha = (u_M^*)^{-1}(\tilde{\omega})$ where $\tilde{\omega} = j^{-1}(w_2(M) - u_M^*(\tilde{\omega}))$. \qed

4.3. **Lemma.** Suppose that for a closed spin $n$-manifold $M$, $n > 4$, there is a map $u : M \to B\Gamma^{(n-1)}$ that classify its universal cover and has the properties: $u(M^{(n-1)}) \subset B\Gamma^{(n-2)}$ and $j_*u_*([M]_{\text{co}}) = 0$ where $j : B\Gamma^{(n-1)} \to B\Gamma^{(n-1)}/B\Gamma^{(n-2)}$ is the quotient map. Then $M$ is strongly inessential.

**Proof.** We may assume that $M$ has a CW complex structure with one $n$-dimensional cell. Let $\psi : D^n \to M$ be its characteristic map. By Proposition 3.3, we may assume that the classifying map $u$ satisfies the condition $u(M^{(n-1)}) \subset B\Gamma^{(n-2)}$. Note that the homotopy groups of the $(n-1)$-homotopy fiber $F$ of the inclusion $B\Gamma^{(n-2)} \to B\Gamma^{(n-1)}$ equal the relative $n$-homotopy groups, $\pi_{n-1}(F) = \pi_n(B\Gamma^{(n-1)}, B\Gamma^{(n-2)})$. Then the first and the only obstruction to deform $u$ to $B\Gamma^{(n-2)}$ is defined by the cocycle $c_u : C_n(M) \to \pi_{n-1}(F)$ represented by the composition

$$
C_n(M) = \pi_n(D^n, \partial D^n) \xrightarrow{\psi_*} \pi_n(M, M^{(n-1)}) \xrightarrow{u_*} \pi_n(B\Gamma^{(n-1)}, B\Gamma^{(n-2)})
$$
with the cohomology class \( o_u = [c_u] \in H^n(M; \pi_n(B\Gamma^{(n-1)}, B\Gamma^{(n-2)})) \). By the Poincare duality with local coefficients, \( o_u \) is dual to the homology class

\[
PD(o_u) \in H_0(M; \pi_n(B\Gamma^{(n-1)}, B\Gamma^{(n-2)})) = \pi_n(B\Gamma^{(n-1)}, B\Gamma^{(n-2)})_{\Gamma}
\]

represented by \( q_*u_*\psi_*(1) \) where

\[
q_* : \pi_n(B\Gamma^{(n-1)}, B\Gamma^{(n-2)}) \to \pi_n(B\Gamma^{(n-1)}, B\Gamma^{(n-2)})_{\Gamma}
\]

is the projection onto the group of coinvariants. Note that \( \pi_n(B\Gamma^{(n-1)}, B\Gamma^{(n-2)}) = \pi_n(\Gamma^{(n-1)}, \Gamma^{(n-2)}) \). Since \( n \leq 2(n-2) - 1 \), by Theorem 2.3

\[
\pi_n(\Gamma^{(n-1)}, \Gamma^{(n-2)}) = \pi_n(\Gamma^{(n-1)}/\Gamma^{(n-2)}).
\]

It is easy to see that

\[
\pi_n(\Gamma^{(n-1)}/\Gamma^{(n-2)})_{\Gamma} = \pi_n(\Gamma^{(n-1)}/\Gamma^{(n-2)}).
\]

Denote by \( \bar{u} : M/M^{(n-1)} = S^n \to B\Gamma/B\Gamma^{(n-2)} \) the induced map. The commutative diagram

\[
\begin{array}{ccc}
\pi_n(M, M^{(n-1)}) & \xrightarrow{u_*} & \pi_n(B\Gamma^{(n-1)}, B\Gamma^{(n-2)}) \xrightarrow{q_*} \pi_n(B\Gamma^{(n-1)}, B\Gamma^{(n-2)})_{\Gamma} \\
\pi_n(D^n/\partial D^n) & \xrightarrow{=} & \pi_n(M/M^{(n-1)}) \xrightarrow{\bar{u}_*} \pi_n(B\Gamma^{(n-1)}/B\Gamma^{(n-2)})
\end{array}
\]

implies that \( \bar{u}_*(1) = \bar{\rho}_*q_*u_*\psi_* (1) \). Thus, \( \bar{u}_*(1) = 0 \) if and only if the obstruction \( o_u \) vanishes.

We show that \( \bar{u}_*(1) = 0 \). The restriction \( n > 4 \) and Proposition 2.1 imply that \( \bar{u}_*(1) \) survives to the \( ko \)-homology group:

\[
\begin{array}{ccc}
\pi_n(B\Gamma^{(n-1)}/B\Gamma^{(n-2)}) & \xrightarrow{=} & \pi_n^k(B\Gamma^{(n-1)}/B\Gamma^{(n-2)}) \xrightarrow{=} \ko_n(B\Gamma^{(n-1)}/B\Gamma^{(n-2)}).
\end{array}
\]

Then the commutative diagram

\[
\begin{array}{ccc}
\pi_n(S^n) & \xrightarrow{=} & \ko_n(S^n) \\
\bar{\bar{u}}_* & & \bar{u}_*
\end{array}
\]

implies that \( \bar{u}_*(1) = 0 \) for \( \ko_n \) if and only if \( \bar{u}_*(1) = 0 \) for \( \pi_n \).

From the assumption and the diagram defined by the quotient maps \( j' : M \to M/M^{(n-1)} = S^n \) and \( j : B\Gamma^{(n-1)} \to B\Gamma^{(n-1)}/B\Gamma^{(n-2)} \)

\[
\begin{array}{ccc}
k\ko_n(M) & \xrightarrow{u_*} & k\ko_n(B\Gamma^{(n-1)}) \\
j'_* & & j_*
\end{array}
\]

\[
\begin{array}{ccc}
k\ko_n(S^n) & \xrightarrow{\bar{\bar{u}}_*} & k\ko_n(B\Gamma^{(n-1)}/B\Gamma^{(n-2)})
\end{array}
\]

it follows that \( \bar{\bar{u}}_*(1) = \bar{u}^*j'_*([M]_{ko}) = j_*u_*([M]_{ko}) = 0. \) \( \square \)
5. **K-theory Injectivity Conditions**

The following is well-known (see 4C, [Ha]).

5.1. **Proposition.** Let $X$ be an $(n - 1)$-connected $(n + 1)$-dimensional CW complex. Then $X$ is homotopy equivalent to the wedge of spheres of dimensions $n$ and $n + 1$ together with the Moore spaces $M(\mathbb{Z}_m, n)$.

We consider the following condition on K-theory of a group $\Gamma$ which appear in our proof of the Strong Gromov Conjecture.

(*) There is a classifying CW-complex $B\Gamma$ such that the inclusion homomorphism

$$(\phi_n)_* : KO_*(B\Gamma^{(n)}) \to KO_*(B\Gamma)$$

is injective for all $n > 4$.

(I) There is a classifying CW-complex $B\Gamma$ such that the inclusion homomorphism

$$(\phi_n)_* : KO_*(B\Gamma^{(n)}) \to KO_*(B\Gamma)$$

restricted to the image of $KO_*(B\Gamma^{(n-1)})$ is injective for all $n > 4$.

(\bar{I}) There is a classifying CW-complex $B\Gamma$ such that the inclusion homomorphism

$$(\bar{\phi}_n)_* : KO_*(B\Gamma^{(n)}/B\Gamma^{(n-2)}) \to KO_*(B\Gamma/B\Gamma^{(n-2)})$$

restricted to the image of $KO_*(B\Gamma^{(n-1)})$ is injective for all $n > 4$.

(\bar{I}_2) There is a classifying CW-complex $B\Gamma$ such that the inclusion homomorphism

$$(\bar{\phi}_n)_* : KO_*(B\Gamma^{(n)}/B\Gamma^{(n-2)}) \otimes \mathbb{Z}_2 \to KO_*(B\Gamma/B\Gamma^{(n-2)}) \otimes \mathbb{Z}_2$$

restricted to the image of $KO_*(B\Gamma^{(n-1)}) \otimes \mathbb{Z}_2$ is injective for all $n > 4$.

5.2. **Proposition.** There are implications

$$(*) \Rightarrow I \Rightarrow \bar{I} \Rightarrow \bar{I}_2.$$  

**Proof.** $(*) \Rightarrow I$. Obvious.
I \Rightarrow \tilde{I}. Consider the commutative diagram defined by exact sequence of pairs

\[
\begin{array}{cccc}
KO_*(B\Gamma^{(n-2)}) & \xrightarrow{i_\ast} & KO_*(B\Gamma^{(n)}) & \xrightarrow{j_\ast} & KO_*(B\Gamma^{(n)}/B\Gamma^{(n-2)}) \\
\downarrow & & \downarrow & & \downarrow \\
KO_*(B\Gamma^{(n-2)}) & \xrightarrow{i_\ast} & KO_*(B\Gamma) & \xrightarrow{j_\ast} & KO_*(B\Gamma/B\Gamma^{(n-2)}) \\
\end{array}
\]

\[KO_*(B\Gamma^{(n)}) \xrightarrow{i_\ast} KO_*(B\Gamma) \xrightarrow{j_\ast} KO_*(B\Gamma/B\Gamma^{(n-2)})\]

Suppose that \(a \in KO_*(B\Gamma^{(n)})\) lies in the image of \(KO_*(B\Gamma^{(n-1)})\) and \((\bar{\phi}_n)_\ast(j'_\ast(a)) = 0\). We need to show that \(j'_\ast(a) = 0\). By exactness, there is \(b'\) such that \(i_\ast(b') = (\bar{\phi}_n)_\ast(a)\). Then \(a - i'_\ast(b')\) lies in the image of \(KO_*(B\Gamma^{(n-1)})\). Since \((\bar{\phi}_n)_\ast(a - i'_\ast(b')) = 0\), by the condition \(I\), \(a = i'_\ast(b')\). By exactness, \(j'_\ast(a) = 0\).

\(\tilde{I} \Rightarrow \tilde{I}_{(2)}\). Straightforward.

5.3. Proposition. Suppose that for a group \(\Gamma\) satisfying the condition \(\tilde{I}\) a classifying map \(u : M \to B\Gamma\) of a closed spin \(n\)-manifold \(M\) takes the \(KO\) fundamental class to 0. Then \(M\) is inessential.

Proof. We may assume that \(u(M) \subset B\Gamma^{(n)}\). Then by \(\tilde{I}\), \(u_\ast([M]_{KO}) = 0\) in \(KO_n(B\Gamma^{(n)})\) and, hence, in \(KO_n(B\Gamma^{(n+1)}/B\Gamma^{(n-1)})\). Let \(u' = q \circ u\) where \(q : B\Gamma^{(n+1)} \to B\Gamma^{(n+1)}/B\Gamma^{(n-1)}\) is the quotient map. In the commutative diagram

\[
\begin{array}{ccc}
ko_n(M) & \xrightarrow{u'_\ast} & ko_n(B\Gamma^{(n+1)}/B\Gamma^{(n-1)}) \\
\downarrow & & \downarrow \cong \\
KO_n(M) & \xrightarrow{u'_\ast} & KO_n(B\Gamma^{(n+1)}/B\Gamma^{(n-1)})
\end{array}
\]

the homomorphism \(\text{per}\) is an isomorphism in view of Proposition 5.1. This implies that \(u'_\ast([M]_{ko}) = 0\) in \(KO_n(B\Gamma^{(n)}/B\Gamma^{(n-1)})\). In view of the natural transformation of homology theories \(ko_\ast \to H_\ast(\ ; \mathbb{Z})\) it follows that \(u'_\ast([M]) = 0\) in \(H_n(B\Gamma^{(n+1)}/B\Gamma^{(n-1)})\). Since the homomorphism \(q_\ast : H_n(B\Gamma^{(n+1)}) \to H_n(B\Gamma^{(n+1)}/B\Gamma^{(n-1)})\) is injective, we obtain that \(u_\ast([M]) = 0\). Theorem 5.2 completes the proof.

5.4. Proposition. Suppose that an inessential manifold \(M\) has the fundamental group with property \(\tilde{I}_{(2)}\). Then \(M\) is strongly inessential.

Proof. We may assume that \(M\) has a CW complex structure with one \(n\)-dimensional cell. Let \(\psi : D^n \to M\) be its characteristic map. By Proposition 5.3 we may assume that the classifying map \(u\) satisfies the
condition $u(M^{(n-1)}) \subset B\Gamma^{(n-2)}$ and $u(M) \subset B\Gamma^{(n-1)}$. We will show that the lifting problem

$$
\begin{array}{c}
M^{(n-1)} \longrightarrow B\Gamma^{(n-2)} \\
\downarrow \subset \longrightarrow \subset \\
M \longrightarrow B\Gamma^{(n)} \ni iu
\end{array}
$$

has a solution. Here $i : B\Gamma^{(n-1)} \to B\Gamma^{(n)}$. It would mean that there is a homotopy lift $\hat{u} : M \to B\Gamma^{(n-2)}$ of $i \circ u$ which agrees with $u$ on $M^{(n-2)}$. Since $n \geq 4$, the map $\hat{u}$ induces an isomorphism of the fundamental groups and, hence, is classifying map.

We note that a ‘simpler’ lifting problem

$$
\begin{array}{c}
M^{(n-1)} \longrightarrow B\Gamma^{(n-2)} \\
\downarrow \subset \longrightarrow \subset \\
M \longrightarrow B\Gamma^{(n-1)}
\end{array}
$$

might have no solution.

Note that the homotopy groups of the $(n - 1)$-homotopy fiber $F$ of the inclusion $B\Gamma^{(n-2)} \to B\Gamma^{(n)}$ equal the relative $n$-homotopy groups, $\pi_{n-1}(F) = \pi_n(B\Gamma^{(n)}, B\Gamma^{(n-2)})$. Then the first and the only obstruction to lift $iu$ to $B\Gamma^{(n-2)}$ is defined by the cocycle $c_u : C_n(M) \to \pi_{n-1}(F)$ represented by the composition

$$
\pi_n(D^n, \partial D^n) \xrightarrow{\psi_u} \pi_n(M, M^{(n-1)}) \xrightarrow{u_*} \pi_n(B\Gamma^{(n-1)}, B\Gamma^{(n-2)}) \xrightarrow{i_*} \pi_n(B\Gamma^{(n)}, B\Gamma^{(n-2)})
$$

with the cohomology class $o_u = [c_u] \in H^n(M; \pi_n(B\Gamma^{(n)}, B\Gamma^{(n-2)}))$. By the Poincare Duality with local coefficients, the cohomology class $o_u$ is dual to the homology class

$$
P\check{D}(o_u) \in H_0(M; \pi_n(B\Gamma^{(n)}, B\Gamma^{(n-2)})) = \pi_n(B\Gamma^{(n)}, B\Gamma^{(n-2)})_\Gamma
$$

represented by $q_\ast i_\ast u_\ast \check{\psi}_\ast(1)$ where

$$
q_\ast : \pi_n(B\Gamma^{(n)}, B\Gamma^{(n-2)}) \to \pi_n(B\Gamma^{(n)}, B\Gamma^{(n-2)})_\Gamma
$$

is the projection onto the group of coinvariants.

Note that $\pi_n(B\Gamma^{(n-1)}, B\Gamma^{(n-2)}) = \pi_n(E\Gamma^{(n-1)}, E\Gamma^{(n-2)})$. Below we will identify these groups. Denote by

$$
\check{i} : (E\Gamma^{(n-1)}, E\Gamma^{(n-2)}) \to (E\Gamma^{(n)}, E\Gamma^{(n-2)})
$$

is the inclusion induced by $i$.

Since $n \leq 2(n - 2) - 1$, by Theorem 2.3,

$$
\pi_n(E\Gamma^{(n-1)}, E\Gamma^{(n-2)}) = \pi_n(E\Gamma^{(n-1)}/E\Gamma^{(n-2)}).$$
It is easy to see that
\[ \pi_n(EG^{(n-1)}/EG^{(n-2)})_\Gamma = \pi_n(B\Gamma^{(n-1)}/B\Gamma^{(n-2)}). \]

Similarly,
\[ \pi_n(EG^{(n)}/EG^{(n-1)})_\Gamma = \pi_n(B\Gamma^{(n)}/B\Gamma^{(n-1)}). \]

The homotopy exact sequence of the triple \((EG^{(n)}, EG^{(n-1)}, EG^{(n-2)})\) brings the following commutative diagram

\[
\begin{array}{ccc}
\pi_{n+1}(EG^{(n)}, EG^{(n-1)}) & \longrightarrow & \pi_n(EG^{(n-1)}, EG^{(n-2)}) \\
\; & \downarrow & \; \\
\pi_{n+1}(EG^{(n)}, EG^{(n-1)})_\Gamma & \longrightarrow & \pi_n(EG^{(n-1)}, EG^{(n-2)})_\Gamma \\
\; & \downarrow & \; \\
\pi_{n+1}(BG^{(n)}/BG^{(n-1)}) & \longrightarrow & \pi_n(BG^{(n-1)}/BG^{(n-2)}) \\
\end{array}
\]

where the row in the middle is exact as obtained by tensor product of
the first row with \(\mathbb{Z}\) over \(\mathbb{Z}\). By the Five Lemma the homomorphism \(\xi\) is an isomorphism.

Denote by \(\bar{u} : M/M^{(n-1)} = S^n \to BG^{(n-1)}/BG^{(n-2)}\) the induced map.

The commutative diagram

\[
\begin{array}{ccc}
\pi_n(M, M^{(n-1)}) & \xrightarrow{\bar{i}_* u_*} & \text{im}(\bar{i}_*) \\
\psi_* \uparrow & & \downarrow q_* \\
\pi_n(D^n, \partial D^n) & \xrightarrow{=} & \pi_n(M/M^{(n-1)}) \\
\end{array}
\]

implies that \(i_* \bar{u}_*(1) = \xi q_* \bar{i}_* u_* \psi_*(1)\). Thus, \(i_* \bar{u}_*(1) = 0\) if and only if
the obstruction \(o_n\) vanishes.

We show that \(i_* \bar{u}_*(1) = 0\). The restriction \(n > 4\) and Proposition 2.1 imply that \(\bar{u}_*(1)\) survives to the \(ko\)-homology group:

\[
\pi_n(BG^{(n)}/BG^{(n-2)}) \xrightarrow{\cong} \pi_n^k(BG^{(n)}/BG^{(n-2)}) \xrightarrow{\cong} ko_n(BG^{(n)}/BG^{(n-2)}).
\]

Then the commutative diagram

\[
\begin{array}{ccc}
\pi_n(S^n) & \xrightarrow{\cong} & ko_n(S^n) \\
\bar{u}_* \downarrow & & \downarrow u_* \\
\pi_n(BG^{(n)}/BG^{(n-2)}) & \xrightarrow{\cong} & ko_n(BG^{(n)}/BG^{(n-2)}) \\
\end{array}
\]

implies that \(i_* \bar{u}_*(1) = 0\) for \(ko_n\) if and only if \(i_* \bar{u}_*(1) = 0\) for \(\pi_n\).
Note that in the diagram

\[
k_0(M/M^{(n-1)}) \xrightarrow{\bar{u}_*} k_0(B\Gamma^{(n)}(M)/B\Gamma^{(n-2)}) \xrightarrow{i_*} k_0(B\Gamma^{(n)}(M)/B\Gamma^{(n-2)}) \\
\cong \downarrow \quad \cong \quad \cong \downarrow
\]

the right vertical arrow is an isomorphism by Proposition 5.1. Since the group \( KO_n(B\Gamma^{(n-1)}/B\Gamma^{(n-2)}) \) is 2-torsion, from the property \( I_2 \) it follows that \( i_*\bar{u}_*(1) = 0 \) for \( KO \). The above diagram implies that \( i_*\bar{u}_*(1) = 0 \) for \( k_0 \). \( \square \)

5.5. **Theorem.** Suppose that a group \( \Gamma \) has the property \( \bar{I} \) and satisfies the Strong Novikov conjecture. Then the Strong Gromov conjecture holds for spin \( n \)-manifolds, \( n > 4 \), with the fundamental group \( \Gamma \).

**Proof.** Let \( M \) be a positive scalar curvature spin \( n \)-manifold. By Rosenberg’s theorem (Theorem 3.4) \( u_*([M]_{KO}) = 0 \). By Proposition 5.3 \( M \) is inessential. By Proposition 5.4 \( M \) is strongly inessential. \( \square \)

6. **Totally non-spin manifolds**

Let \( \nu_M : M \to BSO \) denote a classifying map for the stable normal bundle of a manifold \( M \).

The following theorem was proven in [BD2].

6.1. **Theorem.** Let \( M \) be a totally non-spin closed orientable inessential \( n \)-manifold, \( n \geq 5 \), whose fundamental group is of the type \( FP_3 \). Then \( M \) is strongly inessential.

The proof of Theorem 6.1 uses Wall’s theorem [W] on the cell structure of cobordisms which is known only in dimension \( \geq 5 \). In this section we extend this result to \( n = 4 \) using obstruction theory.

We recall that a map \( f : X \to Y \) is called \( k \)-equivalence if induces an isomorphism \( f_* : \pi_i(X) \to \pi_i(Y) \) for \( i < k \) and an epimorphism for \( i = k \).

6.2. **Proposition.** Let \( (X,Y) \) be a CW pair such that the inclusion \( Y \to X \) is \( 2 \)-equivalence. Then \( H_2(X,Y;F) = 0 \) for any \( \pi \)-module \( F \) where \( \pi = \pi_1(X) = \pi_1(Y) \).

**Proof.** We will be using two well-known facts:

(1) A \( k \)-equivalence, \( k > 1 \), between CW complexes induces an isomorphism of homology groups in dimensions \( < k \) for any local coefficients.
(2) By attaching cells of dimension $k+1, \ldots, n$ to $Y$ one can construct a CW complex $A$ and $n$-connected map $f : A \to X$ extending the inclusion $Y \to X$ (Theorem 8.6.1 in [TITD]).

We consider such $A$ and $f : A \to X$ for $n = 3$. Then the commutative diagram for homology with coefficients in $F$ generated by $f : (A, Y) \to (X, Y)$,

$$
\begin{array}{cccccc}
H_2(Y) & \longrightarrow & H_2(A) & \longrightarrow & H_2(A, Y) & \longrightarrow & H_1(Y) & \longrightarrow & H_1(A) \\
\downarrow & & \cong & & f_* & & \downarrow & & \cong \\
H_2(Y) & \longrightarrow & H_2(X) & \longrightarrow & H_2(X, Y) & \longrightarrow & H_1(Y) & \longrightarrow & H_1(Y)
\end{array}
$$

and the Five Lemma imply that $f_*$ is an isomorphism. Since the inclusion $(A^{(2)}, Y^{(2)}) \to (A, Y)$ induces an epimorphism of 2-homology with any coefficients, it follows that $H_2(X, Y; F) = H_2(A, Y; F) = H_2(A^{(2)}, Y^{(2)}; F) = H_2(Y^{(2)}, Y^{(2)}; F) = 0$. \hfill \Box

We recall that a finitely presented group $\Gamma$ is of type $\text{FP}_3$ if and only if there is a classifying space $B\Gamma$ with finite 3-skeleton $B\Gamma^{(3)}$.

6.3. Theorem. Let $M$ be a totally non-spin closed orientable inessential 4-manifold, whose fundamental group is of the type $\text{FP}_3$. Then $M$ is strongly inessential.

Proof. The proof can be broken into four steps:

(1). Let $\Gamma = \pi_1(M)$. We may assume that $M$ has a CW structure with one 4-dimensional cell. Since $M$ is inessential, by Proposition 3.3 it has a classifying map $u : M \to B\pi^{(3)}$ such that $u(M \setminus D) \subset B\pi^{(2)}$, where $D$ is a closed 4-ball $D$ in the 4-dimensional cell of $M$.

(2). Note that the restriction of $u$ to $D$ defines a zero element in $H_4(B\Gamma, B\Gamma^{(2)})$. By Proposition 2.2 $u|_D$ defines a zero element in $\Omega_4(B\Gamma, B\Gamma^{(2)})$. Thus, there is a relative stationary on the boundary bordism $W'$, $q : W' \to B\Gamma$, between $(D, u|_D), u|_D : (D, \partial D) \to (B\Gamma, B\Gamma^{(2)})$ and some pair $(N', q'), N' \to B\Gamma^{(2)}$. We extend $W'$ by the stationary bordism to a bordism $(W, q)$ between $(M, u)$ and $(N, q|_N)$. Then $q(x, t) = u(x)$ for all $x \in M \setminus D$ and all $t \in [0, 1]$.

(3). First we note that by applying 1-surgery to $\text{int} W$ we may assume that the inclusion $M \to W$ induces an isomorphism of the fundamental groups $\pi_1(M) \to \pi_1(W)$.

The induced homomorphism

$$(\nu_W)_* : \pi_2(W) \to \pi_2(BSO) = \mathbb{Z}_2.$$ 

is surjective in view of the total non-spin assumption. Note that every 2-sphere $S$ that generates an element of the kernel of $(\nu_W)_*$ has a trivial stable normal bundle. Since $\pi_1(W) \cong \pi_1(M)$ is a group of
type $FP_3$, $\pi_2(W)$ is a finitely generated $\pi_1(W)$-module (see [Br], VIII (4.3)). It follows from Proposition 3.2 and Proposition 3.3 that the kernel of $(\nu_W)_*$ is finitely generated. Hence we can perform 2-surgery on the 5-manifold $\text{Int}W$ to obtain a bordism $\hat{W}$ between $M$ and $N$ and a map $\nu_{\hat{W}} : \hat{W} \to BSO$ which induces an isomorphism of 2-dimensional homotopy groups. Let $i : M \to \hat{W}$ denote the inclusion map. Then $(\nu_{\hat{W}})_* \circ i_* = (\nu_M)_*$. Since $(\nu_M)_*$ is surjective and $(\nu_{\hat{W}})_*$ is an isomorphism, it follows that $i_* : \pi_2(M) \to \pi_2(\hat{W})$ is surjective. Since $B\Gamma$ is aspherical, there is a map $\hat{q} : \hat{W} \to B\Gamma$ with $\hat{q} = q$ on $\partial\hat{W}$.

(4) We want to extend the map $\hat{q}|_N : N \to B\Gamma^{(2)}$ to $\hat{W}$.

By the exact sequence of the pairs $(\hat{W}, M)$, we get

$$\pi_1(\hat{W}, M) = \pi_2(\hat{W}, M) = 0.$$ 

Thus, the inclusion $M \to \hat{W}$ is a 2-equivalence. We fix a CW complex structure on $\hat{W}$. By the cellular approximation theorem we may assume that $\hat{q}(\hat{W}) \subset B\Gamma^{(2)}$.

The first obstruction for this extension lives in $H^3(\hat{W}, N; \pi_2(B\Gamma^{(2)}))$. By the Poincare-Lefschetz Duality with twisted coefficients,

$$H^3(\hat{W}, N; F) = H_2(\hat{W}, M; F) = 0$$

in view of Proposition 6.2. Similarly, the second obstruction is trivial, since $H^4(\hat{W}, N; F) = H_1(\hat{W}, M; F) = 0$ for any coefficient system. And the third obstruction lives in

$$H^5(\hat{W}, N; \pi_4(B\Gamma^{(2)})) = H_0(\hat{W}, M; \pi_4(B\Gamma^{(2)})) = 0.$$ 

Thus, there is an extension of $\hat{q}|_N$ to a map $g : \hat{W} \to B\Gamma^{(2)}$. The commutative diagram

$$\begin{array}{ccc}
\pi_1(\hat{W}) & \cong & \pi_1(M) \\
g_* \downarrow & & \downarrow \cong \\
\pi_1(B\Gamma^{(2)}) & \cong & \pi_1(B\Gamma)
\end{array}$$

implies that the restriction $g|_M : M \to B\Gamma^{(2)}$ is a classifying map. □

7. ABELIAN FUNDAMENTAL GROUP

For a CW complex $X$ we denote by $\text{cell}_k(X)$ the set of $k$-dimensional cells. Let $\text{cell}(X) = \bigsqcup_{k \geq 0} \text{cell}_k(X)$. We call a cellular map $f : X \to Y$ between CW-complexes \textit{bijective cellular} if there is a preserving dimension bijection $\beta : \text{cell}(X) \to \text{cell}(Y)$ such that for each cell $e \in \text{cell}(X)$
there is a closed ball \( B \subset e \) such that the restriction \( f|_{\text{Int}(B)} : \text{Int}B \to \beta(e) \) is a homeomorphism.

**7.1. Proposition.** A bijective cellular map \( f : X \to Y \) is a homotopy equivalence with a bijective cellular homotopy inverse \( g : Y \to B \). If \( \beta \) is the cell bijection for \( f \), then \( \beta^{-1} \) is the cell bijection for \( g \).

**Proof.** Induction on dimension of \( X \).

Moreover, the following holds true:

**7.2. Corollary.** A bijective cellular map \( f : X \to Y \) between \( n \)-dimensional CW complexes is a stratified homotopy equivalence

\[
f : (X, X^{(n-1)}, \ldots, X^{(1)}, X^{(0)}) \to (Y, Y^{(n-1)}, \ldots, Y^{(1)}, Y^{(0)}).
\]

**7.3. Proposition.** A bijective cellular map \( f_0 : X^k \to Y^k \) between \( k \)-dimensional complexes extends to a bijective cellular map \( f : X \cup_{\phi} D^{k+1} \to Y \cup_{\phi} D^{k+1} \) provided \( f_0 \circ \phi \) is homotopic to \( \phi' \).

**Proof.** We define the map \( f \) on \( D^{k+1} \) to be a homeomorphism of interior of a ball \( D^{k+1} \subset D^{k+1} \) to \( \text{int} D^{k+1} \) extended to \( D^{k+1} \) by the homotopy between \( f_0 \circ \phi \) and \( \phi' \).

We consider the minimal CW complex structure on spheres \( S^k \).

**7.4. Proposition.** Suppose that the attaching maps for all cells in a connected CW-complex \( X \) are null-homotopic. Then \( X \) there is a bijective cellular map \( f : X \to \bigvee_{k=1}^{\dim X} \bigvee E_k S^k \).

**Proof.** We apply Proposition 7.3 and induction on \( \dim X \).

Let \( T^m \) denote the \( n \)-dimensional torus with the CW-complex structure induced from the CW-complex structure on \( S^1 = e^0 \cup e^1 \). By \( \Sigma^r \) we denote the \( r \)-times iterated reduces suspension. Note that

\[
\Sigma^r X = e^0 \cup \bigcup_{\alpha} e^{r+1}_{e\alpha} \cup \bigcup_{\beta} e^{r+2}_{e\beta} \cup \ldots
\]

for the cellular decomposition of \( X \),

\[
X = e^0 \cup \bigcup_{\alpha} e^1_{e\alpha} \cup \bigcup_{\beta} e^2_{e\beta} \ldots
\]

**7.5. Proposition.** The complex \( \Sigma^{r-1}T^r \) admits a strict homotopy equivalence

\[
f : \Sigma^{r-1}T^r \to \bigvee_{k=1}^{\binom{m}{k}} S^{k+r-1}.
\]
Proof. By induction on $r$ we show that all the attaching maps in $\Sigma^{\ell} T^r$ for $\ell \geq r - 1$ are null-homotopic. In view of the fact that $\Sigma(X \times Y)$ is homotopy equivalent to $\Sigma X \vee \Sigma Y \vee \Sigma (X \wedge Y)$ [11a], we obtain that $\Sigma^{\ell} (T^{r-1} \times S^1)$ is homotopy equivalent to $\Sigma^{\ell} T^{r-1} \vee S^{\ell+1} \vee \Sigma^{\ell+1} T^{r-1}$.

Then the induction assumption completes the proof.

Let $T^m_n$ denote the $n$-dimensional skeleton of $T^m$ with respect to the standard CW-complex structure.

7.6. Corollary. The pair $(T^m_n, T^m_\ell)$, $n \geq \ell$, is stably homotopy equivalent to the pair

$$\left(\bigvee_{k=1}^n S^k, \bigvee_{k=1}^\ell S^k\right).$$

Proof. Follows from Proposition 7.5 and Corollary 7.2. 

7.7. Corollary. A finitely generated free abelian group $\mathbb{Z}^m$ satisfies the $K$-theory condition $(\ast)$. 

7.8. Theorem. The Strong Gromov’s Conjecture holds true for closed $n$-manifolds $M$ with abelian $\pi_1(M)$ for all $n \neq 4$. For $n = 4$ it holds when $M$ is totally no-nspin.

Proof. Let $M$ be an almost spin manifold with positive scalar curvature with abelian $\pi_1(M)$. Taking a finite cover of $M$ we may assume that $\pi_1(M)$ is free abelian. In view of Proposition 4.2, by taking a finite cover, we may assume that $M$ is spin. By Corollary 7.7 and Theorem ?? $M$ is strongly inessential.

Let $M$ is totally non-spin. By Proposition 3.6 $M$ is inessential. By Theorem 6.3 and Theorem 6.1 $M$ is strongly inessential.

7.9. Question. Does the Strong Gromov Conjecture hold for spin 4-manifolds with abelian fundamental group ?

We recall that Bolotov’s example $M_b$ of inessential 4-manifold which is not strongly inessential is spin and has the fundamental group $\mathbb{Z} \ast Z^3$ [B1]. In view of Question 7.9 it is natural to ask if there is such example with the free abelian fundamental group. If the answer is no, then the restriction $n \neq 4$ in Theorem 7.8 can be dropped. We note [B1] that Bolotov’s manifold $M_b$ preserves its property after crossing with a circle $S^1$, i.e., the product $M_b \times S^1$ is inessential but not strongly inessential. In view of the following Proposition and Theorem ??, it cannot carry a metric of positive scalar curvature.
7.10. **Proposition.** (a) Let groups $\Gamma_1$ and $\Gamma_2$ satisfy the property (*), then the free product $\Gamma_1 \ast \Gamma_2$ satisfies (*).

(b) Let $\Gamma$ satisfy (*), then $\Gamma \ast \mathbb{Z}$ satisfies (*).

**Proof.** (a) Since $B\Gamma = B\Gamma_1 \vee B\Gamma_2$ for $\Gamma = \Gamma_1 \ast \Gamma_2$, the fact is obvious.

(b) We consider the product CW structure on $B\Gamma \times S^1 = B(\Gamma \times \mathbb{Z})$ with the CW complex structure on $B\Gamma$ satisfying (*) and the standard CW complex structure on $S^1$. In view of the suspension isomorphism, it suffices to check the condition (*) for the reduced suspension $\Sigma(B\Gamma \times S^1)$. Note that there are homotopy equivalences

$$\Sigma(B\Gamma \times S^1) = \Sigma^2 B\Gamma \vee \Sigma B\Gamma \vee \Sigma S^1 \quad \text{and} \quad \Sigma((B\Gamma \times S^1)^{(k)}) = \Sigma(B\Gamma^{(k-1)} \times S^1) \vee \Sigma B\Gamma^{(k)} = \Sigma^2 B\Gamma^{(k-1)} \vee \Sigma B\Gamma^{(k)} \vee \Sigma S^1.$$

Moreover, the inclusion $\Sigma((B\Gamma \times S^1)^{(k)}) \to \Sigma(B\Gamma \times S^1)$ is the wedge of the inclusions $\Sigma^2(B\Gamma^{(k-1)}) \to \Sigma^2(B\Gamma)$ and $\Sigma(B\Gamma^{(k)}) \to \Sigma(\Gamma B)$ plus the identity map on $\Sigma S^1$ where each of the inclusions is injective in the KO-homology. $\square$

We note that the proof in [Dr1] of the original Gromov’s conjecture for manifolds with the duality fundamental group implicitly assumes that $n > 4$. So the Question 7.9 is open for the original Gromov’s conjecture as well. The latter can be derived from a positive answer to the following

7.11. **Question.** Does the formula

$$\dim_{mc}(X \times \mathbb{R}) = \dim_{mc} X + 1$$

hold for metric spaces $X$?

In view of a counter-example for the asymptotic dimension [Dr3], this formula may not hold for general metric spaces. The spaces of interest here are the universal covers of closed manifolds.

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