1D area law with ground-state degeneracy

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Abstract

An area law is proved for 1D spin systems with ground-state degeneracy. In particular, consider a chain of \(d\)-dimensional spins governed by a Hamiltonian with nearest-neighbor interaction. Suppose the ground states are \(n\)-fold (exactly) degenerate with \(n = O(1)\), and are separated from excited states by an energy gap \(\epsilon\). Then for any given cut there exists a ground state such that the entanglement entropy across the cut is upper bounded by \(\tilde{O}(\log d/\epsilon)\).

1 Introduction

It is conjectured that the entanglement of a region scales as its boundary (area law) in the ground state of a gapped quantum system. Hastings rigorously proves the area law in 1D (one-dimensional) spin systems by giving the upper bound \(\exp(O((\log d)/\epsilon))\) on entanglement entropy \([2]\), where \(d\) is the dimension of each local spin and \(\epsilon\) is the energy gap. The bound is recently improved to \(\tilde{O}(\log^3 d/\epsilon^{3/2})\) \([1]\), where \(\tilde{O}(x) = O(x \text{ polylog}(x))\) hides a polylogarithmic factor.

The aforementioned proofs of the 1D area law assume a unique (nondegenerate) ground state. This paper proves an area law for 1D spin systems with ground-state degeneracy. Suppose the ground states are \(n\)-fold (exactly) degenerate with \(n = O(1)\). Then for any given cut there exists a ground state such that the entanglement entropy across the cut is upper bounded by \(\tilde{O}(\log^3 d/\epsilon)\), which even slightly improves the best known result (see above) for nondegenerate systems \((n = 1)\).

One major technical contribution of this paper is a stronger version (Theorem 1) of the robustness theorem (Theorem 6.1 in \([1]\)), and then the improved bound follows. All complications due to degeneracy need to be taken care of. To simplify the presentation, assume \(n = 2\) (generalization to \(n = O(1)\) is straightforward).

2 Approximate ground-state projector

This section generalizes the notion of approximate ground-state projector (AGSP) to nearly degenerate systems. Let \(H\) be the Hamiltonian of a 1D spin system. Let \(0 \leq \epsilon_0 \leq \epsilon_1 \leq \epsilon_2\) be the energies of the ground state \(|\psi_0\rangle\), the first excited state \(|\psi_1\rangle\), and the second excited state of \(H\), respectively. Suppose \(H\) is nearly 2-fold degenerate in the sense that \(\epsilon_0 \approx \epsilon_1\), and define \(\epsilon = \epsilon_2 - \epsilon_0\) to be the energy gap. Let \(G\) be the 2-dimensional space spanned by \(|\psi_{0,1}\rangle\). Fix a cut, and \(R(|\psi\rangle)\) denotes the Schmidt rank of a state \(|\psi\rangle\) across the cut.

\[1\] The bound claimed in \([1]\) is \(\tilde{O}(\log^3 d/\epsilon)\). In my opinion, the proof in \([1]\) of this claim is incomplete. See Appendix for details.
A linear operator $K$ is a $(\delta, \Delta, D)$-AGSP if (i) for any normalized state $|\psi\rangle \in G$,

$$K|\psi\rangle \in G, \ 1 - \delta \leq \|K|\psi\rangle\|^2 \leq 1;$$  \hspace{1cm} (1)

(ii) for any normalized state $|\psi\rangle \perp G$,

$$K|\psi\rangle \perp G, \ \|K|\psi\rangle\|^2 \leq \Delta;$$  \hspace{1cm} (2)

(iii) for any state $|\psi\rangle$,

$$R(K|\psi\rangle) \leq DR(|\psi\rangle).$$ \hspace{1cm} (3)

**Lemma 1.** Let

$$\mu^2 = |\langle \psi_0 | \psi \rangle |^2 + |\langle \psi_1 | \psi \rangle |^2.$$ \hspace{1cm} (4)

A $(\delta, \Delta, D)$-AGSP with $D\Delta \leq 1/10$ implies the existence of a normalized product state $|\psi\rangle$ across the cut such that

$$\mu^2 \geq (9/10 - \delta)/D.$$ \hspace{1cm} (5)

**Proof.** The proof is similar to that of Lemma 2.2 in [1]. Let $K$ be such an AGSP and $|\psi\rangle$ be the optimal normalized product state in the sense that $|\mu_{\psi}| \geq |\mu_{\psi'}|$ for any normalized product state $|\psi'\rangle$. The optimal state $|\psi\rangle$ exists as the set of all normalized product states is compact. The states $|\psi\rangle$ and $|\phi\rangle = K|\psi\rangle$ can be decomposed as

$$|\psi\rangle = \mu_{\psi}|\psi\rangle + \sqrt{1 - \mu_{\psi}^2}|\psi^\perp\rangle, \ |\phi\rangle = \mu'_{\psi}|\psi\rangle + \mu_{\psi}^\perp|\phi^\perp\rangle,$$ \hspace{1cm} (6)

where $|\psi\rangle, |\phi\rangle \in G, |\psi^\perp\rangle, |\phi^\perp\rangle \in G^\perp$ are normalized states. The definition of $K$ implies

$$(1 - \delta)\mu_{\psi}^2 \leq |\mu_{\psi}'|^2 \leq \mu_{\psi}^2, \ |\mu_{\psi}^\perp|^2 \leq \Delta, \ R(|\phi\rangle) \leq D.$$ \hspace{1cm} (7)

Let $|\Phi\rangle = |\phi\rangle/||\phi||$. Its Schmidt decomposition is given by $|\Phi\rangle = \sum_{i=1}^D \lambda_i |L_i\rangle |R_i\rangle$ with $\sum_i \lambda_i^2 = 1$. Then the Cauchy-Schwarz inequality implies

$$|\mu'_{\psi}/||\phi||| = |\langle \phi' | \Phi \rangle| \leq \sqrt{\sum_{i=1}^D \langle \phi' | L_i \rangle \langle L_i | R_i \rangle} \leq \sqrt{\sum_{i=1}^D \langle \phi' | L_i \rangle |R_i\rangle|^2}.$$ \hspace{1cm} (8)

There exists an index $i$ such that

$$\mu_{\psi}^2 \geq \mu_{\psi}^2 |L_i\rangle |R_i\rangle \geq |\langle \phi' | L_i \rangle |R_i\rangle|^2 \geq |\mu_{\psi}' D^{-1} |\phi\rangle|^{-2} = |\mu_{\psi}' D^{-1} (|\mu_{\psi}'|^2 + |\mu_{\psi}^\perp|^2)^{-1} \geq (1 - \delta) D^{-1} (\mu_{\psi}^2 + \Delta)^{-1} \geq (1 - \delta) (D \mu_{\psi}^2 + 1/10)^{-1} \Rightarrow \mu_{\psi}^2 \geq (9/10 - \delta)/D.$$ \hspace{1cm} (9)

\[ \square \]

Let $u$ be an upper bound on $\|H\|$ (the maximum eigenvalue of $H$).

**Lemma 2.** There exists a polynomial $C_\ell$ of degree $\ell$ such that (i) $C_\ell(\epsilon_0) = 1$; (ii) $1 - \delta \leq C_\ell(\epsilon_1)^2 \leq 1$,

$$\delta = 4\ell^2 (\epsilon_1 - \epsilon_0)/(u - \epsilon_2).$$ \hspace{1cm} (10)

(iii) $|C_\ell(x)|^2 \leq \Delta$ for $\epsilon_2 \leq x \leq u$, where

$$\Delta = 4 \exp(-4\ell \sqrt{\epsilon/u});$$ \hspace{1cm} (11)
Proof. The Chebyshev polynomial $T_ℓ(x)$ is defined as

$$T_ℓ(\cos θ) = \cos (ℓθ).$$

(12)

Clearly $|T_ℓ(x)| \leq 1$ for $|x| \leq 1$. For $x \geq 1$, the definition reduces to $T_ℓ(x) = \cosh(ℓt)$, where $t = \arccosh x \geq 0$. Then,

$$T_ℓ(x) \geq \exp(ℓt)/2 \geq \exp(2ℓ \tanh(t/2))/2 \geq \exp(2ℓ\sqrt{(x-1)/(x+1)})/2.$$  

(13)

Moreover,

$$|T_ℓ(x)/T_ℓ(x) = ℓ \tanh(ℓt)(dt/dx) = ℓ \tanh(ℓt)/\sinh t \leq ℓ(ℓt)/t = ℓ^2.$$  

(14)

Define

$$C_ℓ(x) = T_ℓ(f(x))/T_ℓ(f(ε_0)),$$

(15)

where $f(x) = (u + ε_2 - 2x)/(u - ε_2)$. Clearly $|C_ℓ(x)| \leq 1/T_ℓ(f(ε_0))$ for $ε_2 \leq x \leq u$, and

$$T_ℓ(f(ε_0)) \geq \exp(2ℓ\sqrt{(f(ε_0) - 1)/(f(ε_0) + 1)})/2 = \exp(2ℓ\sqrt{ε/(u - ε_0)})/2 \geq \exp(2ℓ\sqrt{ε/u})/2.$$  

(16)

Furthermore, there exists $ε_0 \leq ξ \leq ε_1$ such that

$$C_ℓ(ε_1) = C_ℓ(ε_0) - (ε_1 - ε_0)C_ℓ(ξ) \geq 1 - (ε_1 - ε_0)C_ℓ(ξ)/C_ℓ(ξ) \geq 1 - 2ℓ^2(ε_1 - ε_0)/(u - ε_2),$$  

(17)

as $C_ℓ(x)$ is a decreasing function of $x$ for $x < ε_2$. □

Lemma 3. For any normalized state $|ψ⟩$ satisfying $⟨ψ|H|ψ⟩ \leq ε_0 + b$, there exists a normalized state $|φ⟩ ∈ G$ such that

$$|||ψ⟩ - |φ⟩||^2 \leq 2b/ε.$$  

(18)

Proof. The proof is similar to that of the Markov lemma (Lemma 6.4 in [1]). The state $|ψ⟩$ can be decomposed as

$$|ψ⟩ = a|φ⟩ + \sqrt{1 - a^2}|φ^⊥⟩,$$

(19)

where $|φ⟩ ∈ G, |φ^⊥⟩ ⊥ G$ are normalized states, and $a ≥ 0$. Then,

$$ε_0 + (1 - a^2)ε = a^2ε_0 + (1 - a^2)e_2 \leq ⟨ψ|H|ψ⟩ \leq ε_0 + b ⇒ 1 - a^2 ≤ b/ε$$

$$⇒ |||ψ⟩ - |φ⟩||^2 = 2 - 2a ≤ 2(1 - a^2) ≤ 2b/ε.$$  

(20)

□

3 Perturbation theory

Suppose the original Hamiltonian is $H' = \sum_{i=1}^n H'_i$, where $0 \leq H'_i \leq 1$ acts on the spins $i$ and $i+1$. Define a new Hamiltonian

$$H = H_L + H_1 + H_2 + \cdots + H_s + H_R$$

(21)

as (i)

$$H_L = \sum_{i=1}^m H'_i - c, \quad H_R = \sum_{i=m+s+1}^n H'_i - c',$$

(22)
where \( c, c' \) are the ground state energies of \( \sum_{i=1}^{m} H_i', \sum_{i=m+s+1}^{n} H_i' \), respectively; (ii) \( H_i = H_{m+i}' \) for \( i = 1, s \); (iii)

\[
H_i = H_{m+i}' - c'/((s-2)
\]

(23)

for \( 2 \leq i \leq s-1 \), where \( c' \) is the ground state energy of \( \sum_{i=2}^{s-1} H_i' \).

Clearly, (a) \( H_{L,R} \geq 0 \), and the ground state energies of \( H_{L,R} \) are 0; (b) \( 0 \leq H_i \leq 1 \) for \( i = 1, s \); (c) \( 0 \leq \sum_{i=2}^{s-1} H_i \leq s-2 \), and the ground state energy of \( \sum_{i=2}^{s-1} H_i \) is 0; (c) the ground states (2-fold exactly degenerate) and the energy gaps of \( H \) and \( H' \) are the same as \( H - H' \) is a multiple of identity.

Let \( \epsilon_0 = \epsilon_1 \leq \epsilon_2 \leq \cdots \) be the lowest eigenenergies of \( H \), and define \( \epsilon = \epsilon_2 - \epsilon_0 \) to be the energy gap of \( H \). Let \( G \) be the 2-dimensional ground state space of \( H \). Let \( P_{tL,R} \) be the projection onto the space spanned by the eigenstates of \( H_{L,R} \) with energies at most \( t \). Define

\[
H_{L,R}^{\leq t} = P_{tL,R} H_{L,R} P_{tL,R} + t(1 - P_{tL,R}).
\]

(24)

Let \( \epsilon'_0 \leq \epsilon'_1 \leq \epsilon'_2 \leq \cdots \) be the lowest eigenenergies of

\[
H(t) = H_{L}^{\leq t} + H_1 + H_2 + \cdots + H_s + H_{R}^{\leq t},
\]

(25)

and define \( \epsilon' = \epsilon'_2 - \epsilon'_0 \) to be the energy gap of \( H(t) \). Let \( |\phi_0^{(t)}\rangle, |\phi_1^{(t)}\rangle, |\phi_2^{(t)}\rangle \) be the ground state, the first excited state, and the second excited state of \( H(t) \). Clearly \( H(t) \leq H \) and \( H(t) \leq 2t + s \).

**Lemma 4.**

\[
\epsilon'_0 \leq \epsilon_0 \leq 2, \quad \epsilon'_2 \leq \epsilon_2 \leq 20.
\]

(26)

**Proof.** Let \( |\psi_L\rangle, |\psi_M\rangle, |\psi_R\rangle, |\phi_R\rangle \) be the ground states of \( H_{L}, \sum_{i=2}^{s-1} H_i, H_{R}, \sum_{i=4}^{s} H_i + H_{R} \), respectively. Let \( |\Psi\rangle \in G \) be a ground state of \( H \), and

\[
|\psi_{LMR}\rangle = |\psi_L\rangle|\psi_M\rangle|\psi_R\rangle, \quad |\psi_{L,R}^{0,1}\rangle = |\psi_L\rangle|\phi_{M,R}^{0,1}\rangle.
\]

(27)

where

\[
|\phi_{M}^{0}\rangle = |00\rangle, \quad |\phi_{M}^{1}\rangle = |01\rangle, \quad |\phi_{M}^{2}\rangle = |10\rangle
\]

(28)

are pairwise orthogonal states of the spins \( m + 2 \) and \( m + 3 \). Then,

\[
\epsilon_0 \leq \langle \psi_{LMR} | H | \psi_{LMR} \rangle
\]

\[
= \langle \psi_L | H_L | \psi_L \rangle + \langle \psi_M | \sum_{i=2}^{s-1} H_i | \psi_M \rangle + \langle \psi_R | H_R | \psi_R \rangle + \langle \psi_{LMR} | A | \psi_{LMR} \rangle \leq |A| \leq 2,
\]

\[
\langle \psi_{L,R}^{0,1,2} | H | \psi_{L,R}^{0,1,2} \rangle = \langle \psi_L | H_L | \psi_L \rangle + \langle \psi_{L,R}^{0,1,2} | H_1 + H_2 + H_3 | \psi_{L,R}^{0,1,2} \rangle + \langle \phi_R | \sum_{i=4}^{s} H_i + H_R | \phi_R \rangle
\]

\[
\leq \langle \Psi | H_L | \Psi \rangle + \langle \Psi | H_1 + H_2 + H_3 | \Psi \rangle + 3 + \langle \Psi | \sum_{i=4}^{s} H_i + H_R | \Psi \rangle \leq \langle \Psi | H | \Psi \rangle + 3 = \epsilon_0 + 3
\]

(29)

Lemma 3 implies that \( |\psi_{L,R}^{0,1,2}\rangle \) are close to the 2-dimensional space \( G \) (this is impossible because they are pairwise orthogonal) unless \( \epsilon_2 \leq 20 \).

Let

\[
A = H_1 + H_s.
\]

(30)

Clearly the operators \( H_{L,R}, H - A, H^{(t)} - A \) have a common complete set of eigenstates. Let \( P_t \) be the projection onto the space spanned by the eigenstates of \( H - A \) with energies at most \( t \). Then,

\[
H_{L,R} P_t = H_{L,R}^{\leq t} P_t \Rightarrow H^{(t)} P_t = H P_t.
\]

(31)

Let \( |\phi^{(r)}\rangle \) be an eigenstate of \( H^{(r)} \) with energy \( \epsilon^{(r)} \).
Lemma 5. \[ ||(1 - P_t)|\phi^{(r)}||^2 \leq |\langle \phi^{(r)} | (1 - P_t)AP_t|\phi^{(r)} ||/(\min\{r, t\} - \epsilon^{(r)}). \] (32)

Proof.

\[ e^{(r)} = \langle \phi^{(r)} | H^{(r)} |\phi^{(r)} \rangle = \langle \phi^{(r)} | P_t H^{(r)} |\phi^{(r)} \rangle + \langle \phi^{(r)} | (1 - P_t) H^{(r)} (1 - P_t) |\phi^{(r)} \rangle + \langle \phi^{(r)} | (1 - P_t) H^{(r)} - A |(1 - P_t) |\phi^{(r)} \rangle + \langle \phi^{(r)} | (1 - P_t) AP_t |\phi^{(r)} \rangle \]

\[ \geq \epsilon^{(r)} ||P_t |\phi^{(r)}||^2 + \min\{r, t\} ||(1 - P_t) |\phi^{(r)}||^2 + \langle \phi^{(r)} | (1 - P_t) AP_t |\phi^{(r)} \rangle \]

\[ \geq \epsilon^{(r)} (1 - ||(1 - P_t) |\phi^{(r)}||^2) + \min\{r, t\} ||(1 - P_t) |\phi^{(r)}||^2 - |\langle \phi^{(r)} | (1 - P_t) AP_t |\phi^{(r)} ||. \] (33)

The proof is completed by rearrangement. \[ \square \]

Let \( r, t > 100 \) and \( \epsilon^{(r)} < 30. \)

Lemma 6. \[ ||(1 - P_t)|\phi^{(r)}|| \leq 2^{-\Omega(t)}. \] (34)

Proof. The proof is similar to that of the truncation lemma (Lemma 6.7 in [1]). In particular, it is an induction on \( t \) by keeping \( r \) fixed. Following the notations of [1], I just point out the difference (the remainder of the two proofs is identical). Lemma 5 implies

\[ ||(1 - P_t)|\phi^{(r)}||^2 \leq |\langle \phi^{(r)} | (1 - P_t)AP_t|\phi^{(r)} ||/(\min\{r, t\} - \epsilon^{(r)}), \] (35)

where \( t = s + n_0d. \) Clearly the denominator on the right-hand side is larger than 50. \[ \square \]

Let

\[ |\phi^{(t), t} | = P_t |\phi^{(t)} ||/|| P_t |\phi^{(t)} ||. \] (36)

Lemma 7. \[ \langle \phi^{(t), t} | H |\phi^{(t), t} \rangle \leq \epsilon^{(t)} + 2^{-\Omega(t)}. \] (37)

Proof.

\[ e^{(t)} = \langle \phi^{(t)} | H^{(t)} |\phi^{(t)} \rangle \geq \langle \phi^{(t)} | P_t H^{(t)} P_t |\phi^{(t)} || + \langle \phi^{(t)} | P_t H^{(t)} (1 - P_t) |\phi^{(t)} \rangle + \langle \phi^{(t)} | (1 - P_t) H^{(t)} P_t |\phi^{(t)} \rangle - 2 ||AP_t |\phi^{(t)} || \cdot ||(1 - P_t) |\phi^{(t)} |\rangle \]

\[ \geq \langle \phi^{(t)} | P_t H^{(t)} P_t |\phi^{(t)} || - 2^{-\Omega(t)} \Rightarrow \langle \phi^{(t), t} | H |\phi^{(t), t} \rangle \leq (\epsilon^{(t)} + 2^{-\Omega(t)})/|| P_t |\phi^{(t)} ||^2 \]

\[ = (\epsilon^{(t)} + 2^{-\Omega(t)})/(1 - 2^{-\Omega(t)}) = \epsilon^{(t)} + 2^{-\Omega(t)}. \] (38)

\[ \square \]

Theorem 1. Let \( t \geq c \log(1/\epsilon) \) for sufficiently large \( c = O(1). \) (a)

\[ |\epsilon_{0, 1} - \epsilon_0 | \leq 2^{-\Omega(t)}; \] (39)

(b) there exist \( |\psi_{0, 1}^{0} \rangle \in G \) such that

\[ |||\psi_{0, 1}^{0} \rangle - |\phi_{0, 1}^{(t)} ||^2 \leq 2^{-\Omega(t)}; \] (40)

(c) \( \epsilon' \geq \epsilon/10. \)
Proof. Lemma 3 implies
\[ \epsilon'_{0,1} \leq \epsilon_0 \leq \langle \phi_{0,1}, t | H | \phi_{0,1}, t \rangle \leq \epsilon'_0 + 2^{-\Omega(t)} \] (41)
\[ \langle \phi_{2,2}, t | H | \phi_{2,2}, t \rangle \leq \epsilon'_2 + 2^{-\Omega(t)} = \epsilon'_0 + \epsilon'_2 + 2^{-\Omega(t)}. \] (42)

(a) follows from (41). Lemma 8 implies the existence of \( |\psi_{0,1,2}^0 \rangle \in G \) such that
\[ \| \phi_{0,1}, t - |\psi_{0,1,2}^0 \rangle \|^2 \leq 2^{-\Omega(t)}/\epsilon = 2^{-\Omega(t) - \log(1/\epsilon)}, \| \phi_{2,2}, t - |\psi_{0,1,2}^0 \rangle \|^2 \leq \epsilon'/\epsilon + 2^{-\Omega(t)}/\epsilon. \] (43)

Lemma 6 implies
\[ \| \phi_{0,1}, t - |\psi_{0,1,2}^0 \rangle \|^2 \leq 2^{-\Omega(t)}. \] (44)

(b) follows from (43), (44) as \( t \geq c \log(1/\epsilon) \). Moreover, as \( |\psi_{0,1,2}^0 \rangle \) are pairwise orthogonal, \( |\psi_{0,1,2}^0 \rangle \) are approximately pairwise orthogonal (this is impossible because they are in the 2-dimensional space \( G \)) unless \( \epsilon' \geq \epsilon/10 \).

4 Area law

Lemma 8 (I). Let \( K = C_\ell(H) \), where \( C_\ell \) is an arbitrary polynomial of degree \( \ell \). For any state \( |\psi\rangle \),
\[ R(K|\psi\rangle) \leq (d\ell)^{O(\max(\ell/s, \sqrt{\ell}))} R(|\psi\rangle). \] (45)

To construct an AGSP for \( H^{(t_0)} \), let \( \ell = s^2 \) and \( u = 2t_0 + s \). Lemmas 28 imply
\[ D\Delta \leq 4(ds)^{O(s)} \exp(-4s^2 \sqrt{\epsilon/u}) \leq 4(ds)^{O(s)} \exp(-4s^2 \sqrt{\epsilon/(2t_0 + s)}). \] (46)

A little algebra shows that the condition \( D\Delta < 1/100 \) for \( t_0 = O(\log(1/\epsilon)) \) can be satisfied with \( s = \tilde{O}((\log^2 d)/\epsilon) \) so that \( \log D = \tilde{O}((\log^3 d)/\epsilon) \). Let \( t_0 = O(\log(1/\epsilon) + \log \log d) \) so that
\[ \delta \leq O(\ell^2 2^{-\Omega(t_0)}/(2t_0 + s)) \leq \tilde{O}(2^{-\Omega(t_0)} \log^6 d/\epsilon^3) < 1/10. \] (47)

Let \( t_i = ic + t_0 \) with \( c = O(1) \) so that \( \Omega(t_i) \geq i + 4 \) for the \( \Omega \) in Theorem I

Lemma 9. There exists a convergent sequence of states \( |\psi_0\rangle, |\psi_1\rangle, \ldots \) satisfying:

(i) \[ \| \langle \phi_{0,1}^{(t_i)} | \psi_i \rangle \|^2 + \| \langle \phi_{0,1}^{(t_i)} | \psi_i \rangle \|^2 \geq 1 - 2^{-i-4}; \] (48)

(ii) Letting \( \ell_j = O(\sqrt{(t_j + s)/\epsilon}), \log R(|\psi_0\rangle) = \tilde{O}((\log^3 d)/\epsilon), \log R(|\psi_i\rangle) = \log R(|\psi_0\rangle) + \sum_{j=1}^{i} \tilde{O}(\ell_j \log d); \] (49)

(iii) Letting \( |\psi_\infty\rangle = \lim_{i \to +\infty} |\psi_i\rangle \in G, \| \langle \psi_i | \psi_\infty \rangle \| \geq 1 - O(2^{-i}). \) (50)
Proof. Theorem 1 implies the existence of \( |\psi_{(t_i)}^{(t_1)} \rangle \in G \) such that

\[
|||\psi_{0,1}^{(t_i)}\rangle - |\phi_{0,1}^{(t_i)}\rangle||^2 \leq 2^{-(i+4)}.
\]

(51)

for all \( i \). The AGSP \( K = C_\ell(H^{(t_0)}) \) constructed above implies the existence of a normalized product state \( |\psi\rangle \) such that

\[
|||\langle\phi_0^{(t_0)}|\psi\rangle|^2 + |||\langle\phi_1^{(t_0)}|\psi\rangle|^2 \geq (9/10 - \delta)/D.
\]

(52)

The condition \( D\Delta < 1/100 \) implies that \( |\psi_0\rangle = K|\psi\rangle/||K|\psi\rangle|| satisfies

\[
||\langle\phi_0^{(t_0)}|\psi_0\rangle|^2 + ||\langle\phi_1^{(t_0)}|\psi_0\rangle|^2 \geq 15/16, \log R(|\psi_0\rangle) = \tilde{O}((\log^3 d)/\epsilon).
\]

(53)

Thus the sequence \( |\psi_i\rangle \) and (i) can be defined and proved by induction. (51) is a quantitative statement that the space spanned by \( |\phi_{0,1}^{(t_i)}\rangle \) is close to \( G \). It implies that the spaces spanned by \( |\phi_{0,1}^{(t_i)}\rangle \) and spanned by \( |\phi_{0,1}^{(t_i-1)}\rangle \) are also close. Thus the induction hypothesis implies

\[
||\langle\phi_0^{(t_i-1)}|\psi_{i-1}\rangle|^2 + ||\langle\phi_1^{(t_i-1)}|\psi_{i-1}\rangle|^2 \geq 1 - 2^{-i-3} \Rightarrow ||\langle\phi_0^{(t_i)}|\psi_{i-1}\rangle|^2 + ||\langle\phi_1^{(t_i)}|\psi_{i-1}\rangle|^2 \geq 1 - 2^{-i}.
\]

(54)

Let \( \ell_i = O(\sqrt{(t_i + s)/\epsilon}) \). Lemmas 2,8 imply the existence of a \((\delta,1/100,D)\)-AGSP \( K \) for \( H^{(t_i)} \) with \( \log D = \tilde{O}(\ell_i \log d) \) and

\[
\delta \leq O(\ell^2 2^{-\Omega(t_i)}/(t_i + s)) = O(2^{-\Omega(t_0)}/\epsilon)2^{-i} \leq 2^{-i}/10.
\]

(55)

Let \( |\psi_i\rangle = K|\psi_{i-1}\rangle/||K|\psi_{i-1}\rangle||. \) Clearly,

\[
||\langle\phi_0^{(t_i)}|\psi_i\rangle|^2 + ||\langle\phi_1^{(t_i)}|\psi_i\rangle|^2 \geq 1 - 2^{-i-4},
\]

(56)

and (ii) follows. (54) and \( \delta = O(2^{-i}) \) imply

\[
|||\psi_i\rangle - |\psi_{i-1}\rangle||^2 = O(2^{-i}).
\]

(57)

Thus the sequence \( |\psi_0\rangle, |\psi_1\rangle, \ldots \) is convergent, and

\[
|||\psi_i|\psi_\infty|| \geq 1 - |||\psi_i\rangle - |\psi_\infty\rangle||^2/2 \geq 1 - O(2^{-i}).
\]

(58)

\[ \square \]

Theorem 2. For any given cut there exists a ground state such that the entanglement entropy across the cut is upper bounded by \( \tilde{O}((\log^3 d)/\epsilon) \).

Proof. Let \( \lambda_i \) be the Schmidt coefficients of \( |\psi_\infty\rangle \) across the cut. Then

\[
1 - p_i = \sum_{i=1}^{R(|\psi_{i\rangle})} \lambda_i^2 \geq ||\langle\psi_i|\psi_\infty\rangle||^2 \geq 1 - O(2^{-i}).
\]

(59)

The entanglement entropy of \( |\psi_\infty\rangle \) is upper bounded by

\[
\sum p_i \log(R(|\psi_i\rangle) - R(|\psi_{i-1}\rangle)) - \sum p_i \log p_i \leq \sum p_i \log R(|\psi_i\rangle) + O(1)
\]

\[
= \sum p_i \sum_{j=1}^{\ell_i} \tilde{O}(\ell_j \log d) + \sum p_i \log R(|\psi_0\rangle) = \sum p_i \sum_{j=1}^{\ell_i} \tilde{O}(\sqrt{(t_j + s)/\epsilon \log d})
\]

\[
+ O(\log R(|\psi_0\rangle)) = \tilde{O}(\sqrt{(t_0 + s)/\epsilon \log d}) + O(\log R(|\psi_0\rangle)) = \tilde{O}((\log^3 d)/\epsilon).
\]

(60)

\[ \square \]
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References

[1] I. Arad, A. Kitaev, Z. Landau, and U. Vazirani. An area law and sub-exponential algorithm for 1D systems. arXiv:1301.1162v1, 2013.

[2] M. B. Hastings. An area law for one-dimensional quantum systems. J. Stat. Mech., 2007(08):P08024, 2007.

Appendix: Details on footnote [1]

For nondegenerate systems ($n = 1$), the upper bound claimed in [1] on entanglement entropy is $\tilde{O}((\log^3 d)/\epsilon)$. In my opinion, the proof in [1] of this claim is incomplete. In particular, in Lemma 6.3 $t_0$ should be at least $O(\epsilon_0/\epsilon^2 + 1/\epsilon)$ in order that the robustness theorem (Theorem 6.1) applies to $H(t_0)$ (the robustness theorem does not guarantee that $H(t_0)$ is gapped for $t_0 = O(1)$). Then $s = \tilde{O}((\log^2 d)/\epsilon)$ (and $\ell = s^2$) does not give an AGSP for $H(t_0)$ with $D\Delta \leq 1/2$, but $s = \tilde{O}((\log^2 d)/\epsilon^{3/2})$ does. A straightforward calculation shows that the upper bound $\tilde{O}((\log^3 d)/\epsilon^{3/2})$ follows from the proof in [1].

However, I have shown that the claim in [1] is correct (Theorem 2 also holds for $n = 1$). This is because Theorem 1 (as a stronger version of the robustness theorem) only requires $t \geq c \log(1/\epsilon)$ for $c = O(1)$. 

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