On the Carathéodory rank of polymatroid bases

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Abstract

In this paper we prove that the Carathéodory rank of the set of bases of a (poly)matroid is upper bounded by the cardinality of the ground set.

Keywords: Carathéodory rank, matroid, integer decomposition.

MSC: 90C10 (52B40).

1 Introduction

Let $H \subseteq \mathbb{R}^n$ be a finite set and denote by

$$\text{int.cone}(H) := \{\lambda_1 x_1 + \cdots + \lambda_k x_k \mid x_1, \ldots, x_k \in H, \lambda_1, \ldots, \lambda_k \in \mathbb{Z}_{\geq 0}\}$$

the integer cone generated by $H$. The Carathéodory rank of $H$, denoted $\text{cr}(H)$, is the least integer $t$ such that every element in $\text{int.cone}(H)$ is the nonnegative integer combination of $t$ elements from $H$.

The set $H$ is called a Hilbert base if $\text{int.cone}(H) = \text{cone}(H) \cap \text{lattice}(H)$, where $\text{cone}(H)$ and $\text{lattice}(H)$ are the convex cone and the lattice generated by $H$, respectively.

Cook et al. [3] showed that when $H$ is a Hilbert base generating a pointed cone, the bound $\text{cr}(H) \leq 2n - 1$ holds. This bound was improved to $2n - 2$ by Sebő [9]. In the same paper, Sebő conjectured that $\text{cr}(H) \leq n$ holds for any Hilbert base generating a pointed cone. A counterexample to this conjecture was found by Bruns et al. [1].

Here we consider the case that $H$ is the set of incidence vectors of the bases of a matroid on $n$ elements. In his paper on testing membership in matroid polyhedra, Cunningham [4] first asked for an upper bound on the number of different bases needed in a representation of a vector as a nonnegative integer sum of bases. It follows from Edmonds matroid partitioning theorem [5] that the incidence vectors of matroid bases form a Hilbert base for the pointed cone they generate. Hence the upper bound of $2n - 2$ applies. This bound was improved.

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by de Pina and Soares [7] to \( n + r - 1 \), where \( r \) is the rank of the matroid. Chaourar [2] showed that an upper bound of \( n \) holds for a certain minor closed class of matroids.

In this paper we show that the conjecture of Sebő holds for the bases of (poly)matroids. That is, the Carathéodory rank of the set of bases of a matroid is upper bounded by the cardinality of the ground set. More generally, we show that for an integer valued submodular function \( f \), the Carathéodory rank of the set of bases of \( f \) equals the maximum number of affinely independent bases of \( f \).

2 Preliminaries

In this section we introduce the basic notions concerning submodular functions. For background and more details, we refer the reader to [6, 8].

Let \( E \) be a finite set and denote its power set by \( \mathcal{P}(E) \). A function \( f : \mathcal{P}(E) \to \mathbb{Z} \) is called submodular if \( f(\emptyset) = 0 \) and for any \( A, B \subseteq E \) the inequality \( f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \) holds. The set

\[
\mathcal{E}P_f := \{ x \in \mathbb{R}^E \mid x(U) \leq f(U) \text{ for all } U \subseteq E \}
\]

is called the extended polymatroid associated to \( f \), and

\[
B_f = \{ x \in \mathcal{E}P_f \mid x(E) = f(E) \}
\]

is called the base polytope of \( f \). Observe that \( B_f \) is indeed a polytope, since for \( x \in B_f \) and \( e \in E \), the inequalities \( f(E) - f(E - e) \leq x(e) \leq f(\{e\}) \) hold, showing that \( B_f \) is bounded.

A submodular function \( f : \mathcal{P}(E) \to \mathbb{Z} \) is the rank function of a matroid \( M \) on \( E \) if and only if \( f \) is nonnegative, nondecreasing and \( f(U) \leq |U| \) for every set \( U \subseteq E \). In that case, \( B_f \) is the convex hull of the incidence vectors of the bases of \( M \).

Let \( f : \mathcal{P}(E) \to \mathbb{Z} \) be submodular. We will construct new submodular functions from \( f \). The dual of \( f \), denoted \( f^* \), is defined by

\[
f^*(U) := f(E \setminus U) - f(E).
\]

It is easy to check that \( f^* \) is again submodular, that \((f^*)^* = f \) and that \( B_{f^*} = -B_f \). For \( a : E \to \mathbb{Z} \), the function \( f + a \) given by \((f + a)(U) := f(U) + a(U)\) is submodular and \( B_{f + a} = a + B_f \). The reduction of \( f \) by \( a \), denoted \( f|a \) is defined by

\[
(f|a)(U) := \min_{T \subseteq U} (f(T) + a(U \setminus T)).
\]

It is not hard to check that \( f|a \) is submodular and that \( \mathcal{E}P_{f|a} = \{ x \in \mathcal{E}P_f \mid x \leq a \} \). Hence we have that \( B_{f|a} = \{ x \in B_f \mid x \leq a \} \) when \( B_f \cap \{ x \mid x \leq a \} \) is nonempty. We will only need the following special case. Let \( e_0 \in E \) and \( c \in \mathbb{Z} \) and define \( a : E \to \mathbb{Z} \) by

\[
a(e) := \begin{cases} c & \text{if } e = e_0, \\ f(\{e\}) & \text{if } e \neq e_0. \end{cases}
\]
Denote \( f|(e_0,c) := f|a \). If \( x_{e_0} \leq c \) for some \( x \in B_f \), we obtain
\[
B_f|(e_0,c) = \{ x \in B_f \mid x(e_0) \leq c \}. \tag{7}
\]

Our main tool is Edmonds’ \cite{Edmonds} polymatroid intersection theorem which we state for the base polytope.

**Theorem 1.** Let \( f, f' : \mathcal{P}(E) \to \mathbb{Z} \) be submodular. Then \( B_f \cap B_{f'} \) is an integer polytope.

We will also use the following corollary (see \cite{Edmonds}).

**Theorem 2.** Let \( f : \mathcal{P}(E) \to \mathbb{Z} \) be submodular. Let \( k \) be a positive integer and let \( x \in (kB_f) \cap \mathbb{Z}^E \). Then there exist \( x_1, \ldots, x_k \in B_f \cap \mathbb{Z}^E \) such that \( x = x_1 + \cdots + x_k \).

**Proof.** By the above constructions, the polytope \( x - (k - 1)B_f \) is the base polytope of the submodular function \( f' = x + (k - 1)f^* \). Consider the polytope \( P := B_f \cap B_{f'} \). It is nonempty, since \( \frac{1}{k}x \in P \) and integer by Theorem 1. Let \( x_k \in P \) be an integer point. Then \( x - x_k \) is an integer point in \( (k - 1)B_f \) and we can apply induction. \( \square \)

Important in our proof will be the fact that faces of the base polytope of a submodular function are themselves base polytopes as the following proposition shows.

**Proposition 1.** Let \( f : \mathcal{P}(E) \to \mathbb{Z} \) be submodular and let \( F \subseteq B_f \) be a face of dimension \( |E| - t \). Then there exist a partition \( E = E_1 \cup \cdots \cup E_t \) and submodular functions \( f_i : \mathcal{P}(E_i) \to \mathbb{Z} \) such that \( F = B_{f_1} \oplus \cdots \oplus B_{f_t} \). In particular, \( F \) is the base polytope of a submodular function.

A proof was given in \cite{Polytope}, but for convenience of the reader, we will also give a proof here.

**Proof.** Let \( \mathcal{T} \subseteq \mathcal{P}(E) \) correspond to the tight constraints on \( F \):
\[
\mathcal{T} = \{ U \subseteq E \mid x(U) = f(U) \text{ for all } x \in F \}.
\]
It follows from the submodularity of \( f \) that \( \mathcal{T} \) is closed under taking unions and intersections. Observe that the characteristic vectors \( \{ \chi^A \mid A \in \mathcal{T} \} \) span a \( t \)-dimensional space \( \mathcal{V} \). Let \( \emptyset = A_0 \subset A_1 \subset \cdots \subset A_{t'} = E \) be a maximal chain of sets in \( \mathcal{T} \). We claim that \( t' = t \). Observe that the characteristic vectors \( \chi^{A_1}, \ldots, \chi^{A_{t'}} \) are linearly independent and span a \( t' \)-dimensional subspace \( \mathcal{V}' \subseteq \mathcal{V} \). Hence \( t' \leq t \).

To prove equality, suppose that there exists an \( A \in \mathcal{T} \) such that \( \chi^A \notin \mathcal{V}' \). Take such an \( A \) that is inclusionwise maximal. Now let \( i \geq 0 \) be maximal, such that \( A_i \subseteq A \). Then \( A_i \subseteq A_{i+1} \cap A \not\subseteq A_{i+1} \). Hence by maximality of the chain, \( A_{i+1} \cap A = A_i \). By maximality of \( A \), we have \( \chi^{A \cup A_{i+1}} \in \mathcal{V}' \) and hence, \( \chi^A = \chi^{A \setminus A_{i+1}} + \chi^{A \cup A_{i+1}} - \chi^{A_{i+1}} \in \mathcal{V}' \), contradiction the choice of \( A \). This shows that \( t' = t \).
Define $E_i = A_i \setminus A_{i-1}$ for $i = 1, \ldots, t$. Define $f_i : \mathcal{P}(E_i) \to \mathbb{Z}$ by $f_i(U) := f(A_{i-1} \cup U) - f(A_{i-1})$ for all $U \subseteq E_i$. We will show that
\[
F = B_{f_1} \oplus \cdots \oplus B_{f_t}.
\tag{8}
\]
To see the inclusion ‘$\subseteq$’, let $x = (x_1, \ldots, x_t) \in F$. Then $x(A_i) = f(A_i)$ holds for $i = 0, \ldots, t$. Hence for any $i = 1, \ldots, t$ and any $U \subseteq E_i$ we have
\[
x_i(U) = x(A_{i-1} \cup U) - x(A_{i-1}) \leq f(A_{i-1} \cup U) - f(A_{i-1}) = f_i(U),
\tag{9}
\]
and equality holds for $U = E_i$.

To see the converse inclusion ‘$\supseteq$’, let $x = (x_1, \ldots, x_t) \in B_{f_1} \oplus \cdots \oplus B_{f_t}$. Clearly
\[
x(A_k) = \sum_{i=1}^k x_i(E_i) = \sum_{i=1}^k (f(A_i) - f(A_{i-1})) = f(A_k),
\tag{10}
\]
in particular $x(E) = f(E)$. To complete the proof, we have to show that $x(U) \leq f(U)$ holds for all $U \subseteq E$. Suppose for contradiction that $x(U) > f(U)$ for some $U$. Choose such a $U$ inclusionwise minimal. Now take $k$ minimal such that $U \subseteq A_k$. Then we have
\[
x(U \cup A_{k-1}) = x(A_{k-1}) + x_k(E_k \cap U) \\
\leq f(A_{k-1}) + f_k(E_k \cap U) = f(U \cup A_{k-1}).
\tag{11}
\]
Since $x(A_{k-1} \cap U) \leq f(A_{k-1} \cap U)$ by minimality of $U$, we have
\[
x(U) = x(A_{k-1} \cup U) + x(A_{k-1} \cap U) - x(A_{k-1}) \\
\leq f(A_{k-1} \cup U) + f(A_{k-1} \cap U) - f(A_{k-1}) \leq f(U).
\tag{12}
\]
This contradicts the choice of $U$. \hfill \Box

## 3 The main theorem

In this section we prove our main theorem. For $B_f \subseteq \mathbb{R}^E$, denote $\text{cr}(B_f) := \text{cr}(B_f \cap \mathbb{Z}^E)$.

**Theorem 3.** Let $f : \mathcal{P}(E) \to \mathbb{Z}$ be a submodular function. Then $\text{cr}(B_f) = \dim B_f + 1$.

We will need the following lemma.

**Lemma 1.** Let $B_{f_1}, \ldots, B_{f_t}$ be base polytopes. Then $\text{cr}(B_{f_1} \oplus \cdots \oplus B_{f_t}) \leq \text{cr}(B_{f_1}) + \cdots + \text{cr}(B_{f_t}) - (t - 1)$.

**Proof.** It suffices to show the lemma in the case $t = 2$.

Let $k$ be a positive integer and let $w = (w_1, w_2)$ be an integer vector in $k \cdot (B_{f_1} \oplus B_{f_2})$. Let $w_1 = \sum_{i=1}^s m_i x_i$ and $w_2 = \sum_{i=1}^r n_i y_i$, where the $n_i, m_i$ are positive integers, the $x_i \in B_{f_1}$ and $y_i \in B_{f_2}$ integer vectors. Denote
\[
\{0, m_1, m_1 + m_2, \ldots, m_1 + \cdots + m_r\} \cup \{0, n_1, n_1 + n_2, \ldots, n_1 + \cdots + n_s\} = \{l_0, l_1, \ldots, l_g\},
\tag{13}
\]
where \( 0 = l_0 < l_1 < \cdots < l_q = k \). Since \( m_1 + \cdots + m_r = n_1 + \cdots + n_s = k \), we have \( q \leq r + s - 1 \). For any \( i = 1, \ldots, q \), there exist unique \( j, j' \) such that \( m_1 + \cdots + m_{j-1} < l_i \leq m_1 + \cdots + m_j \) and \( n_1 + \cdots + n_{j'-1} < l_i \leq n_1 + \cdots + n_{j'} \). Denote \( z_i := (x_j, y_{j'}) \). We now have the decomposition \( w = \sum_{i=1}^q (l_i - l_{i-1})z_i \).

We conclude this section with a proof of Theorem [3].

**Proof of Theorem [3]** The inequality \( \text{cr}(B_f) \geq \dim B_f + 1 \) is clear. We will prove the converse inequality by induction on \( \dim B_f + |E| \), the case \( |E| = 1 \) being clear. Let \( E \) be a finite set, \( |E| \geq 2 \) and let \( f : \mathcal{P}(E) \to \mathbb{Z} \) be submodular.

Let \( k \) be a positive integer and let \( w \in kB_f \cap \mathbb{Z}^E \). We have to prove that \( w \) is the positive integer combination of at most \( \dim B_f + 1 \) integer points in \( B_f \).

We may assume that
\[
\dim B_f = |E| - 1. \tag{14}
\]

Indeed, suppose that \( \dim B_f = |E| - t \) for some \( t \geq 2 \). Then by Proposition [1] there exist a partition \( E = E_1 \cup \cdots \cup E_t \) and submodular functions \( f_i : \mathcal{P}(E_i) \to \mathbb{Z} \) such that \( B_f = B_{f_1} \oplus \cdots \oplus B_{f_t} \). By induction, \( \text{cr}(B_{f_i}) = \dim B_{f_i} + 1 \) for every \( i \). Hence by Lemma [4]
\[
\text{cr}(B_f) \leq \text{cr}(B_{f_1}) + \cdots + \text{cr}(B_{f_t}) - (t - 1) = \dim B_{f_1} + \cdots + \dim B_{f_t} + 1 = \dim B_f + 1. \tag{15}
\]

Fix an element \( e \in E \). Write \( w(e) = kq + r \) where \( r, q \) are integers and \( 0 \leq r \leq k - 1 \). Let \( f' = f(e, q + 1) \). By Theorem [2] we can find integer vectors \( y_1, \ldots, y_k \in B_{f'} \) such that \( w = y_1 + \cdots + y_k \). We may assume that \( y_i(e) = q + 1 \) for \( i = 1, \ldots, r \). Indeed, if \( y_i(e) \leq q \) would hold for at least \( k - r + 1 \) values of \( i \), then we would arrive at the contradiction \( w(e) \leq (k - r + 1)q + (r - 1)(q + 1) \leq kq + r - 1 < w(e) \).

Let \( f'' := f'(e, q) \). Denote \( w' := y_1 + \cdots + y_r \). So we have decomposed \( w \) into integer vectors
\[
w' \in rB_{f'}, \quad w - w' = y_{r+1} + \cdots + y_k \in (k-r)B_{f''} = B_{(k-r)f''}. \tag{16}
\]

We may assume that \( r \neq 0 \), since otherwise \( w \in kF \), where \( F \) is the face \( B_{f''} \cap \{ x \mid x(e) = q, x(E) = f(E) \} \) of dimension \( \dim F \leq |E| - 2 \) (since \( |E| \geq 2 \)). Then by induction we could write \( w \) as a nonnegative integer linear combination of at most \( 1 + (\dim F) < \dim B_f + 1 \) integer vectors in \( B_{f''} \subset B_f \).

Consider the intersection
\[
P := B_{rf'} \cap B_{w+(k-r)(f'')} \tag{17}
\]

Observe that \( P \) is nonempty, since it contains \( w' \). Furthermore, by Theorem [1] \( P \) is an integer polytope. Hence taking an integer vertex \( x' \) of \( P \) and denoting \( x'' := w - x' \), we have that \( x' \) is an integer vector of \( B_{rf'} \) and \( x'' \) is an integer vector of \( B_{(k-r)f''} \).
Let $F'$ be the inclusionwise minimal face of $B_{r,t}$ containing $x'$ and let $F''$ be the inclusionwise minimal face of $B_{w+(k-r)(f'')}$, containing $x'$. Denote $H' := \text{aff.hull}(F')$ and $H'' := \text{aff.hull}(F'')$. Since $x'$ is a vertex of $P$, we have

$$H' \cap H'' = \{x'\}. \quad (18)$$

Indeed, every supporting hyperplane of $B_{r,t}$ containing $x'$ also contains $F'$ by minimality of $F'$, and hence contains $H'$. Similarly, every supporting hyperplane of $B_{w+(k-r)(f'')}$, containing $x'$ also contains $H''$. Since $x'$ is the intersection of supporting hyperplanes for the two polytopes, the claim follows.

Observe that both $F'$ and $F''$ are contained in the affine space

$$\{x \in \mathbb{R}^n \mid x(E) = rf(E), \ x(e) = r(q+1)\}, \quad (19)$$

which has dimension $n-2$ since $|E| \geq 2$. It follows that

$$\dim F' + \dim F'' = \dim H' + \dim H'' = \dim(\text{aff.hull}(H' \cup H'')) + \dim(H' \cap H'') \leq n - 2. \quad (20)$$

Since $F''$ is a face of $B_{w+(k-r)(f'')}$, containing $x'$, we have that $w - F''$ is a face of $B_{(k-r)f''}$ containing $x''$. By induction we see that

$$\text{cr}(F') + \text{cr}(w - F'') \leq (\dim F' + 1) + (\dim(w - F'') + 1) = \dim F' + \dim F'' + 2 \leq n. \quad (21)$$

This gives a decomposition of $w = x' + x''$ using at most $n$ different bases of $B_f$, completing the proof.

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