BACKSTEPPING DESIGN FOR INCREMENTAL STABILITY OF STOCHASTIC HAMILTONIAN SYSTEMS WITH JUMPS

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Abstract. Incremental stability is a property of dynamical systems ensuring the uniform asymptotic stability of each trajectory rather than a fixed equilibrium point or trajectory. Here, we introduce a notion of incremental stability for stochastic control systems and provide its description in terms of existence of a notion of so-called incremental Lyapunov functions. Moreover, we provide a backstepping controller design scheme providing controllers along with corresponding incremental Lyapunov functions rendering a class of stochastic control systems, namely, stochastic Hamiltonian systems with jumps, incrementally stable. To illustrate the effectiveness of the proposed approach, we design a controller making a spring pendulum system in a noisy environment incrementally stable.

1. Introduction

The notion of incremental stability focuses on the convergence of trajectories with respect to each other rather than with respect to an equilibrium point or a fixed trajectory. This notion of stability has gained significant attention in recent years due to its potential applications in the study of nonlinear systems. Examples of such applications include synchronization of cyclic feedback systems [HSSG12]; construction of symbolic models [PGT08, MZ12]; modeling of nonlinear analog circuits [BML+10]; and synchronization of interconnected oscillators [SS07].

In the past few years, there have been several results characterizing the notion of incremental stability for nonprobabilistic dynamical systems using notions of so-called incremental Lyapunov functions and contraction metric. The interested readers may consult the results in [Ang02, PWN06, LS98, ZvdWM13, and references therein] for more detailed information about different characterizations of incremental stability. Furthermore, there have been several results on the construction of state feedback controllers enforcing a class of nonprobabilistic control systems incrementally stable. Examples include results on smooth strict-feedback form systems [ZT11] and a class of (not-necessarily smooth) control systems [ZvdWM13].

In recent years, similar notions of incremental stability have been introduced for different classes of stochastic systems including stochastic control systems [ZMEM+14], stochastic switched systems [ZAG15], randomly switched stochastic systems [ZA14], and their descriptions using some Lyapunov-like functions. In addition, there have been several results in the literature studying incremental stability of stochastic systems using a notion of contraction metric. Examples include the results on stochastic dynamical systems [PTS09] and a class of stochastic hybrid systems [ZCA13].

There exists a plethora of results for designing controllers enforcing some classes of stochastic systems stable with respect to an equilibrium point or a fixed trajectory. Examples include the results based on backstepping and inverse optimality [DK99], on strict-feedback form stochastic systems [KYMY13], based on passivity for stochastic port-Hamiltonian systems [SF13], on a backstepping approach for stochastic Hamiltonian systems [WCS12], on input-to-state stability of stochastic retarded systems [HM09], and finally on stabilization of jump stochastic systems [LLN12]. Unfortunately, to the best of our knowledge, there is no work available in the literature on the synthesis of state feedback controllers rendering a class of nonlinear stochastic systems incrementally stable. This is unfortunate because, based on our motivation from symbolic control, incremental
stability is a key property enabling the construction of bisimilar finite abstractions of continuous-time stochastic systems [ZMEM+14] [ZAG15] [ZAM13].

The main objective of this work is to propose a state feedback design scheme providing controllers enforcing a class of stochastic systems incrementally stable. The paper is divided in two major parts. In the first part, we introduce a notion of incremental stability for stochastic control systems with jumps and provide its description in terms of existence of a notion of so-called incremental Lyapunov functions. In the second part, we provide a feedback controller design approach based on backstepping scheme providing controllers together with the corresponding incremental Lyapunov functions enforcing a class of stochastic control systems, namely, stochastic Hamiltonian systems with jumps, incrementally stable. Further, we illustrate the effectiveness of the proposed results by designing a feedback controller making a spring pendulum subjected to stochastically vibrating ceiling with jumps incrementally stable.

The rest of the paper is organized as follow. Section 2 provides some mathematical preliminaries, the definition of stochastic control systems, and a notion of incremental input-to-state stability. In Section 3, we propose a feedback controller design scheme based on backstepping approach for a class of stochastic Hamiltonian systems. Section 4 demonstrates the efficacy of our results on a physical case study. Finally, the paper is concluded in Section 5.

2. Stochastic Control Systems

2.1. Notations. The symbols $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{R}^+_0$ denote the set of real, positive and non-negative real numbers, respectively. We use $\mathbb{R}^{n \times m}$ to denote a vector space of real matrices with $n$ rows and $m$ columns. The identity matrix in $\mathbb{R}^{n \times n}$ is denoted by $I_n$ and zero matrix in $\mathbb{R}^{n \times m}$ is denoted by $0_{n \times m}$. The $e_i \in \mathbb{R}^n$ denotes the vector whose all elements are zero, except the $i^{th}$ element, which is one. Given a vector $x \in \mathbb{R}^n$, $\|x\|$ denotes its euclidean norm. Given a matrix $A \in \mathbb{R}^{n \times m}$, $A^T$ represents transpose of matrix $A$ and $\|A\|_F$ represents the Frobenius norm of $A$ defined as $\|A\|_F = \sqrt{\text{Tr}(AA^T)}$, where $\text{Tr}(\cdot)$ denotes the trace of a square matrix. For all $x_i \in \mathbb{R}^n$, $[x_1; x_2; \ldots; x_N]$ represents a vector in $\mathbb{R}^n$ where $n = \sum_{i=1}^N n_i$. The symbol $A \otimes B$ represents a Kronecker product of matrices $A$ and $B$. The diagonal set $\Delta \subset \mathbb{R}^{2n}$ is defined as $\Delta = \{(x, x) | x \in \mathbb{R}^n\}$. A continuous function $\gamma : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ belongs to class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$; it belongs to class $\mathcal{K}_\infty$ if $\gamma \in \mathcal{K}$ and $\gamma(r) \rightarrow \infty$ as $r \rightarrow \infty$. A continuous function $\beta : \mathbb{R}_0^+ \times \mathbb{R}^n \rightarrow \mathbb{R}_0^+$ belongs to class $\mathcal{K}\mathcal{L}$ if for each fixed $s$, the map $\beta(r, s)$ belongs to $\mathcal{K}$ with respect to $r$ and, for each fixed $r \neq 0$, the map $\beta(r, s)$ is decreasing with respect to $s$ and $\beta(r, s) \rightarrow 0$ as $s \rightarrow \infty$. For any $x, y, z \in \mathbb{R}^n$, a function $d : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_0^+$ is a metric on $\mathbb{R}^n$ with conditions: (i) $d(x, y) = 0$ if and only if $x = y$; (ii) (symmetry property) $d(x, y) = d(y, x)$; and (iii) (triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$. Given a measurable function $f : \mathbb{R}_0^+ \rightarrow \mathbb{R}^n$, the (essential) supremum of $f$ is denoted by $\|f\|_\infty$; we recall that $\|f\|_\infty := \langle \text{ess}\sup\{\|f(t)\|, t \geq 0\} \rangle$.

2.2. Stochastic control systems. Let the triplet $(\Omega, F, P)$ denote a probability space with a sample space $\Omega$, filtration $F$, and the probability measure $P$. The filtration $F = (F_s)_{s \geq 0}$ satisfies the usual conditions of right continuity and completeness [Øks02]. Let $(W_s)_{s \geq 0}$ be an $r$-dimensional $\mathbb{F}$-Brownian motion and $(P_s)_{s \geq 0}$ be an $r$-dimensional $\mathbb{F}$-Poisson process. We assume that the Poisson process and the Brownian motion are independent of each other. The Poisson process $P_s := [P^1_s; \ldots; P^n_s]$ models $r$ kinds of events whose occurrences are assumed to be independent of each other.

**Definition 2.1.** A stochastic control system is a tuple $\Sigma_s = (\mathbb{R}^n, \mathbb{R}^m, \mathcal{U}, f, \sigma, \rho)$, where:
- $\mathbb{R}^n$ is the state space;
- $\mathbb{R}^m$ is the input space;
- $\mathcal{U}$ is a subset of the set of all $\mathbb{F}$-progressively measurable processes with values in $\mathbb{R}^m$; see [KS91] Def. 1.11;
- $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ satisfies the following Lipschitz assumption: there exist constants $L_x, L_u \in \mathbb{R}^+$ such that: $\|f(x, u) - f(x', u')\| \leq L_x\|x - x'\| + L_u\|u - u'\|$, $\forall x, x' \in \mathbb{R}^n$ and $\forall u, u' \in \mathbb{R}^m$;
A stochastic process $\xi : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}^n$ is said to be a solution process of $\Sigma_s$ if there exists $v \in \mathcal{U}$ satisfying
\begin{equation} \label{2.1}
d\xi = f(\xi, v) \, dt + \sigma(\xi) \, dW_t + \rho(\xi) \, dP_t,
\end{equation}
P-almost surely (P-a.s.), where $f(\cdot, \cdot)$, $\sigma(\cdot)$, and $\rho(\cdot)$ are the drift, diffusion, and reset terms, respectively. We emphasize that postulated assumptions on $f$, $\sigma$, and $\rho$ ensure the existence and uniqueness of the solution process $\xi$ \cite{OS05}. Throughout the paper, we use the notation $\xi_{\alpha}(t)$ to denote the value of a solution process at time $t \in \mathbb{R}_0^+$ under the input signal $v$ and with initial condition $\xi_{\alpha}(0) = a$, a P-a.s., in which $a$ is a random variable that is measurable in $\mathcal{F}_0$. Here, we assume that the Poisson processes $P_t^k$, for any $k \in \{1, \ldots, \bar{r}\}$, have the rates of $\lambda_i$.

### 2.3 Incremental stability for stochastic control systems.

This subsection introduces a notion of incremental input-to-state stability for stochastic control systems. The stability notion discussed here is the generalization of the ones defined in \cite{ZvdWM13, ZT11} for non-probabilistic control systems.

**Definition 2.2.** A stochastic control system $\Sigma_s = (\mathbb{R}^n, \mathbb{R}^m, \mathcal{U}, f, \sigma, \rho)$ is incrementally input-to-state stable (\textit{$\delta_3$-ISS-M}_k) in the $k$th moment, where $k \geq 1$, if there exist a metric $d$, a $\mathcal{K}\mathcal{L}$ function $\beta$, and a $\mathcal{K}_\infty$ function $\gamma$ such that for any $t \in \mathbb{R}_0^+$, any $\mathbb{R}^n$-valued random variables $a$ and $a'$ that are measurable in $\mathcal{F}_0$, and any $v, v' \in \mathcal{U}$, the following condition is satisfied:
\begin{equation} \label{2.2}
\mathbb{E}[\|d(\xi_{\alpha}(t), \xi_{\alpha'}(t))\|^k] \leq \beta(\mathbb{E}[\|d(a, a')\|^k], t) + \gamma(\|v - v'\|^k_{\infty}).
\end{equation}

Whenever the metric $d$ is the natural Euclidean one, \textit{$\delta_3$-ISS-M}_k becomes $\delta$-ISS-M$_k$ as defined in \cite{ZMEM+14}. One can describe $\delta_3$-ISS-M$_k$ in terms of existence of $\delta_3$-ISS-M$_k$ Lyapunov functions as defined next.

**Definition 2.3.** Consider a stochastic control system $\Sigma_s$ and a continuous function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$ that is twice continuously differentiable on $\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$. The function $V$ is called a $\delta_3$-ISS-M$_k$ Lyapunov function for $\Sigma_s$, if it has polynomial growth rate and there exist a metric $d$, $\mathcal{K}_\infty$ functions $\alpha, \beta, \varphi$, and a constant $\kappa \in \mathbb{R}^+$, such that:
\begin{enumerate}
\item [(i)] $\alpha$ (resp. $\pi$ and $\varphi$) is a convex (resp. concave) function;
\item [(ii)] $\forall x, x' \in \mathbb{R}^n$, $\alpha((d(x, x'))^k) \leq V(x, x') \leq \overline{\alpha}((d(x, x'))^k);
\item [(iii)] $\forall x, x' \in \mathbb{R}^n, x \neq x'$, and $\forall u, u' \in \mathbb{R}^m$, 
\begin{align*}
\mathcal{L}V(x, x') := & \left[ \partial_x V \quad \partial_{x'} V \right] \begin{bmatrix} f(x, u) \\ f(x', u') \end{bmatrix} + \frac{1}{2} \text{Tr} \left( \begin{bmatrix} \sigma(x) \\ \sigma(x') \end{bmatrix} \begin{bmatrix} \sigma^T(x) & \sigma^T(x') \end{bmatrix} \begin{bmatrix} \partial_{x, x} V \partial_{x', x} V \\ \partial_{x', x} V \partial_{x', x'} V \end{bmatrix} \right) \\
& + \sum_{i=1}^{\bar{r}} \lambda_i (V(x + \rho(x)e_i, x' + \rho(x')e_i) - V(x, x')) \leq -\kappa V(x, x') + \varphi(||u - u'||^k),
\end{align*}
\end{enumerate}

where $\mathcal{L}$ is the infinitesimal generator of the stochastic process $\xi$ in \eqref{2.1} acting on function $V$ \cite{Oks02} and the symbols $\partial_x$ and $\partial_{x,x}$ represents first and second-order partial derivatives with respect to $x$ and $x'$, respectively. The following theorem describes $\delta_3$-ISS-M$_k$ in terms of existence of $\delta_3$-ISS-M$_k$ Lyapunov functions.

**Theorem 2.4.** A stochastic control system $\Sigma_s$ is $\delta_3$-ISS-M$_k$ if it admits a $\delta_3$-ISS-M$_k$ Lyapunov function.
Proof. For any time instance \( t \geq 0 \), any \( v, v' \in \mathcal{U} \), and any random variable \( a \) and \( a' \) that are \( \mathcal{F}_t \)-measurable, one obtains

\[
\mathbb{E}[V(\xi_{av}(t), \xi_{a'v'}(t))] = \mathbb{E}\left[V(\xi_{av}(0), \xi_{a'v'}(0)) + \int_0^t \mathcal{L}V(\xi_{av}(s), \xi_{a'v'}(s)) \, ds\right]
\leq \mathbb{E}[V(\xi_{av}(0), \xi_{a'v'}(0))] + \mathbb{E}\left[\int_0^t (\kappa_1 V(\xi_{av}(s), \xi_{a'v'}(s)) + \varphi(||v(s) - v'(s)||_\infty)) \, ds\right]
\leq \mathbb{E}[V(\xi_{av}(0), \xi_{a'v'}(0))] + \int_0^t (\kappa_1 \mathbb{E}[V(\xi_{av}(s), \xi_{a'v'}(s))] + \mathbb{E}[\varphi(||v - v'||_\infty)]) \, ds,
\]

where the first equality is an application of the Itô’s formula for jump diffusions thanks to the polynomial rate of the function \( V \) \([Oko05], \text{Theorem 1.24}\) and the first inequality is because of condition iii) in Definition 2.3. By virtue of Gronwall’s inequality, one obtains

\[
(2.3) \quad \mathbb{E}[V(\xi_{av}(t), \xi_{a'v'}(t))] \leq \mathbb{E}[V(a, a')] e^{-\kappa t} + \frac{1}{\kappa} \mathbb{E}[\varphi(||v - v'||_\infty)] \leq \mathbb{E}[V(a, a')] e^{-\kappa t} + \frac{1}{\kappa} \mathbb{E}[\varphi(||v - v'||_\infty)],
\]

where the last inequality follows from Jensen’s inequality due to the concavity assumption on the function \( \varphi \) \([Oko02], \text{p. 310}\). In view of Jensen’s inequality, inequality (2.3), the convexity of \( \alpha \), the concavity of \( \overline{\alpha} \), and condition ii) in Definition 2.3, we have the following chain of inequalities

\[
\alpha\left(\mathbb{E}\left[(d(\xi_{av}(t), \xi_{a'v'}(t)))^k\right]\right) \leq \overline{\alpha}\left(\mathbb{E}\left[(d(\xi_{av}(t), \xi_{a'v'}(t)))^k\right]\right) \leq \mathbb{E}[V(\xi_{av}(t), \xi_{a'v'}(t))]
\leq \mathbb{E}[V(a, a')] e^{-\kappa t} + \frac{1}{\kappa} \mathbb{E}[\varphi(||v - v'||_\infty)]
\leq \overline{\alpha}\left(\mathbb{E}\left[(d(\xi_{av}(t), \xi_{a'v'}(t)))^k\right]\right) e^{-\kappa t} + \frac{1}{\kappa} \mathbb{E}[\varphi(||v - v'||_\infty)],
\]

which in conjunction with the fact that \( \underline{\alpha} \in \mathcal{K}_{\infty} \) leads to

\[
\mathbb{E}\left[(d(\xi_{av}(t), \xi_{a'v'}(t)))^k\right] \leq \alpha^{-1}\left(\overline{\alpha}\left(\mathbb{E}\left[(d(\xi_{av}(t), \xi_{a'v'}(t)))^k\right]\right) e^{-\kappa t} + \frac{1}{\kappa} \mathbb{E}[\varphi(||v - v'||_\infty)]\right)
\leq \alpha^{-1}\left(2\overline{\alpha}\left(\mathbb{E}\left[(d(\xi_{av}(t), \xi_{a'v'}(t)))^k\right]\right) e^{-\kappa t} + \frac{1}{\kappa} \mathbb{E}[\varphi(||v - v'||_\infty)]\right)
\]

Therefore, by introducing functions \( \beta \) and \( \gamma \) as

\[
(2.4) \quad \beta(y, t) := \alpha^{-1}\left(2\overline{\alpha}(y) e^{-\kappa t}\right), \quad \gamma(y) := \alpha^{-1}\left(\frac{2}{\kappa} \mathbb{E}[\varphi(||v - v'||_\infty)]\right),
\]

for any \( y, t \in \mathbb{R}^+_0 \), inequality (2.2) is satisfied. Note that if \( \alpha^{-1} \) satisfies the triangle inequality (i.e., \( \alpha^{-1}(a+b) \leq \alpha^{-1}(a) + \alpha^{-1}(b) \)), one can remove the coefficients 2 in the expressions of \( \beta \) and \( \gamma \) in (2.4) to get a less conservative upper bound in (2.2). \( \square \)

3. Backstepping Design Procedure

In this section, we propose a backstepping control design scheme for a class of stochastic control systems, namely, stochastic Hamiltonian systems with jumps. The proposed methodology provides controllers ensuring \( \delta_{\text{ISS-M}_k} \) of the closed loop system. A stochastic Hamiltonian system is a stochastic control system \( \Sigma = (\mathbb{R}^{2n}, \mathbb{R}^n, \mathcal{H}, f, \sigma, \rho) \) described by stochastic differential equations

\[
(3.1) \quad \Sigma: \quad \begin{cases} \dot{q} = \partial_p H(q, p) \, dt, \\ \dot{p} = -\partial_q H(q, p) + b(q, p) + G(q)v \, dt + \sigma(q) \, dW_t + \rho(q) \, dP_t, \end{cases}
\]

where \( q = q(\omega, t) \in \mathbb{R}^n, \forall t \in \mathbb{R}^+_0 \) and \( \forall \omega \in \Omega \), is a generalized coordinate vector of \( n \)-degree-of-freedom system; \( p = p(\omega, t) \in \mathbb{R}^n, \forall t \in \mathbb{R}^+_0 \) and \( \forall \omega \in \Omega \), represents a vector of generalized momenta and defined as \( dP_t = M(q) \, dq \), where \( M(q) \) is a symmetric, nonsingular, and positive definite inertia matrix; \( b(q, p) \) is a smooth damping term; \( G(q)v \) is the control force caused by \( G(q) \), a nonsingular smooth square matrix, and by control input \( v \) acting on the system; \( \sigma(q) \) is the diffusion term; \( \rho(q) \) is the reset term; and \( \partial_q \) and \( \partial_p \)
represents first order partial derivative of function $H(q, p)$ with respect to $q$ and $p$, respectively, where $H(q, p)$ is a continuous differentiable Hamiltonian function represented in terms of total energy of the system as the following

$$(3.2) \quad H(q, p) = \frac{1}{2} p^T M^{-1}(q)p + \Xi(q),$$

where $\Xi(q)$ represents potential energy of the system. By substituting (3.2) into (3.1), the dynamics of $\Sigma$ can be rewritten as

$$(3.3) \quad \Sigma: \begin{cases} \quad d\xi = M^{-1}(q)p \, dt, \\ \quad d\eta = (\eta(q, p) + G(q)v) \, dt + \sigma(q) \, dW_t + \rho(q) \, dP_t, \end{cases}$$

where $\eta(q, p) = -\partial_t H(q, p) + b(q, p)$.

By considering the Lipschitz assumption on the drift term in (3.3) (cf. Definition 2.1), ensuring the existence and uniqueness of the solution process of $\Sigma$, one has

$$(3.4) \quad \|M^{-1}(q)p - M^{-1}(q')p'\| \leq L_1\|q - q'\| + L_2\|p - p'\|,$$

for some $L_1, L_2 \in \mathbb{R}^+$. We can now state the main result of the paper on the backstepping controller design scheme providing controllers rendering the considered class of stochastic control systems $\delta_3$-ISS-$M_k$ for any $k \geq 2$.

**Theorem 3.1.** Consider the stochastic control system $\Sigma$ of the form (3.3). The state feedback control law

$$v = G^{-1}(q) \left( -\eta(q, p) + \kappa_1 \frac{dM(q)}{dt} - \kappa_2 M(q)q \right) - \left( \lambda \left( \frac{2^{k-1} - 1}{k} \right) + \frac{L_2}{s_1 \xi_{11}} + \frac{\min\{n, r\} L_2^2 (k-1)}{2s_2 \xi_2^2} \right) (p + \kappa_1 M(q)q + \hat{v}),$$

renders the closed-loop stochastic control system $\Sigma \delta_3$-ISS-$M_k$ for $k > 2$ with respect to the input $\hat{v}$, for all

$$\kappa_1 > L_1 + \max\{L_2, 1\} \epsilon_{11}^{r_1} + \frac{\min\{n, r\} L_2^2 \epsilon_2^k (k-1)}{2s_2 \xi_2^2} + \frac{2^{k-1} \xi_2^k \lambda}{k},$$

where $r_1 = \frac{k-1}{k-2}$, $s_1 = k$, $r_2 = \frac{k}{k-2}$, $s_2 = \frac{k}{k-2}$, $\epsilon_{11}$ and $\epsilon_2$ are positive constants which can be chosen arbitrarily, $\lambda = \sum_{i=1}^r \lambda_i$, and $L_1, L_2, L_\sigma$, and $L_{\rho}$ are the Lipschitz constants introduced in (3.4) and Definition 2.1 respectively.

Note that the term $\frac{dM(q)}{dt}$ in the control law (3.5) can be computed by using the definition of derivative of matrix $\mathbf{WCS12}$ as

$$\frac{dM(q)}{dt} = \frac{\partial M(q)}{\partial q} \times \left( \frac{d}{dt} \otimes I_n \right) - \frac{\partial M(q)}{\partial p} \times (M^{-1}(q)p \otimes I_n),$$

where $\frac{\partial M(q)}{\partial q} := \left[ \frac{\partial M(q)}{\partial q_1}, \frac{\partial M(q)}{\partial q_2}, \ldots, \frac{\partial M(q)}{\partial q_n} \right]_{n \times n^2}$.

**Proof.** Consider a coordinate transformation as

$$(3.6) \quad \zeta = \psi(\xi) = \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = \begin{bmatrix} q \\ p - \alpha(q) \end{bmatrix},$$

where $\xi = [q^T \ p^T]^T$ and $\alpha(q) = -\kappa_1 M(q)q$ for some $\kappa_1 > 0$. The dynamics of the stochastic control system $\Sigma$ in (3.3) after the change of coordinates can be written by using Ito’s differentiation $\mathbf{Oks02}$ as

$$(3.7) \quad \Sigma: \begin{cases} \quad d\zeta_1 = M^{-1}(\zeta_1)(\zeta_2 + \alpha(\zeta_1)) \, dt, \\ \quad d\zeta_2 = \left( \eta(\zeta_1, \zeta_2 + \alpha(\zeta_1)) + G(\zeta_1)v - \kappa_1 \frac{dM(\zeta_1)}{dt} \xi_1 - \kappa_1 (\zeta_2 + \alpha(\zeta_1)) \right) \, dt + \sigma(\zeta_1) \, dW_t + \rho(\zeta_1) \, dP_t. \end{cases}$$
Now consider a candidate Lyapunov function $V_1(z_1, z'_1), \forall z_1, z'_1 \in \mathbb{R}^n$, for the $\zeta_1$-subsystem as follows

\begin{equation}
V_1(z_1, z'_1) = \frac{1}{k} \left( (z_1 - z'_1)^T (z_1 - z'_1) \right)^{\frac{1}{2}}.
\end{equation}

The corresponding infinitesimal generator along $\zeta_1$-subsystem is given by

\begin{equation}
\mathcal{L}V_1(z_1, z'_1) = (z_1 - z'_1)^T \left( (z_1 - z'_1) (z_1 - z'_1) \right)^{\frac{1}{2}} - 1
\end{equation}

\[ \left[ (M^{-1}(z_1)z_2 - M^{-1}(z'_1)z_2) + (M^{-1}(z_1)\alpha(z_1) - M^{-1}(z'_1)\alpha(z'_1)) \right]. \]

Now by using the definition of $\alpha(z_1)$, consistency of norm, and (3.4), the infinitesimal generator reduces to

\begin{equation}
\mathcal{L}V_1(z_1, z'_1) \leq (L_1 - \kappa_1) \left( (z_1 - z'_1)^T (z_1 - z'_1) \right)^{\frac{1}{2}} + L_2 \left( (z_1 - z'_1)^T (z_1 - z'_1) \right)^{\frac{1}{2}} - 1 \|z_1 - z'_1\| \|z_2 - z'_2\|.
\end{equation}

To handle the second term, we use Young’s inequality \[\text{KKK95}\] as

\begin{equation}
ab \leq \frac{a^2}{r} + \frac{b^2}{s},
\end{equation}

where $\varepsilon > 0$, constants $a, b > 1$ satisfying condition $(r - 1)(s - 1) = 1$, and $a, b \in \mathbb{R}$. Now by using the consistency of norms and applying Young’s inequality (3.11), we can reduce the second term in (3.10) to

\begin{equation}
\mathcal{L}V_1(z_1, z'_1) \leq (L_1 + \frac{L_2^2}{r_1} - \kappa_1) \left( (z_1 - z'_1)^T (z_1 - z'_1) \right)^{\frac{1}{2}} + \frac{L_2}{s_1 \varepsilon_1} \left( (z_2 - z'_2)^T (z_2 - z'_2) \right)^{\frac{1}{2}},
\end{equation}

where,

\begin{equation}
r_1 = \frac{k}{k-1}, s_1 = k,
\end{equation}

and $\varepsilon_1$ is any positive constant. After substituting inequality (3.12) in (3.10), one obtains

\begin{equation}
\mathcal{L}V_1(z_1, z'_1) \leq (L_1 + \frac{L_2^2}{r_1} - \kappa_1) \left( (z_1 - z'_1)^T (z_1 - z'_1) \right)^{\frac{1}{2}} + \frac{L_2}{s_1 \varepsilon_1} \left( (z_2 - z'_2)^T (z_2 - z'_2) \right)^{\frac{1}{2}}.
\end{equation}

One can readily verify that $V_1$ is a $\delta_2$-ISS-M$_k$ function for $\zeta_1$-subsystem with respect to $z_2$ as the input provided that $L_1 + \frac{L_2^2}{r_1} - \kappa_1 < 0$. Function $V_1$ satisfies the conditions in Definition 2.3 with $\varTheta(y) = \varTheta(y) = \frac{1}{k} y$, $d$ is the natural Euclidean matrix, $\kappa = \kappa_1 - L_1 - \frac{L_2^2}{r_1}$, and $\varphi(y) = \frac{L_2}{s_1 \varepsilon_1} y$, for any $y \in \mathbb{R}^n$. Now consider a Lyapunov function $V_2(z_2, z'_2), \forall z_2, z'_2 \in \mathbb{R}^n$, for the $\zeta_2$-subsystem as

\begin{equation}
V_2(z_2, z'_2) = \frac{1}{k} \left( (z_2 - z'_2)^T (z_2 - z'_2) \right)^{\frac{1}{2}}.
\end{equation}

The respective infinitesimal generator is given by

\begin{equation}
\mathcal{L}V_2(z_2, z'_2) = (z_2 - z'_2)^T \left( (z_2 - z'_2) (z_2 - z'_2) \right)^{\frac{1}{2}} - 1
\end{equation}

\[ \left[ (G u + \eta - \kappa_1 \frac{d M}{dt} z_1 - \kappa_2 (z_2 - \kappa_1 M z_1)) - (G' u' + \eta' - \kappa_1 \frac{d M'}{dt} z'_1 - \kappa_1 (z'_2 - \kappa_1 M' z'_1)) \right]
\end{equation}

\[ + \frac{1}{2} \text{Tr} \left( \left( \sigma(z_1) - \sigma(z'_1) \right) \left( \sigma(z_1) - \sigma(z'_1) \right)^T (z_2 - z'_2) V_2(z_2, z'_2) \right) \]

\[ + \frac{1}{k} \sum_{i=1}^{\tilde{r}} \lambda_i \left( \left( (z_2 + \rho(z_1) e_i) - (z'_2 + \rho(z'_1) e_i) \right)^T (z_2 + \rho(z_1) e_i) - (z'_2 + \rho(z'_1) e_i) \right)^{\frac{1}{2}} \]

\[ - \frac{1}{k} \sum_{i=1}^{\tilde{r}} \lambda_i \left( (z_2 - z'_2)^T (z_2 - z'_2) \right)^{\frac{1}{2}}, \]
where \( G = G(z_1), \eta = \eta(z_1, z_2 + \alpha(z_1)), M = M(z_1), G' = G(z_1'), \eta' = \eta(z_1', z_2' + \alpha(z_1')), \) and \( M' = M(z_1'). \) The same abbreviation will be used in the rest of the proof. The first term can be simply handled by selecting proper control input \( u \) and the second term can be reduced using consistency of norm, Lipschitz assumption on the diffusion term \( \sigma(\cdot) \) and the Young’s inequality as follows

\[
\frac{1}{2} \text{Tr} \left( (\sigma(z_1) - \sigma(z_1'))(\sigma(z_1) - \sigma(z_1'))^T \partial_{zz} V_2(z_2, z_2') \right) \\
= \frac{1}{2} \text{Tr} \left( (\sigma(z_1) - \sigma(z_1'))(\sigma(z_1) - \sigma(z_1'))^T \left[ (z_2 - z_2')^T(z_2 - z_2') \right]^{\frac{p}{2} - 1} I_n \right) \\
+ (k - 2)(z_2 - z_2')(z_2 - z_2')^T \left( (z_2 - z_2')^T(z_2 - z_2') \right)^{\frac{p}{2} - 2} \right) \\
\leq \frac{k - 1}{2} ||\sigma(z_1) - \sigma(z_1')||_F^2 ||z_2 - z_2'||^{k - 2} \\
\leq \frac{\min\{n, r\}(k - 1)}{2} ||\sigma(z_1) - \sigma(z_1')||_F^2 ||z_2 - z_2'||^{k - 2} \\
\leq \frac{\min\{n, r\}L_0^2(k - 1)}{2} ||z_1 - z_1'||^2 ||z_2 - z_2'||^{k - 2} \\
\leq \frac{\min\{n, r\}L_0^2(k - 1)}{2} \frac{\varepsilon_2^2}{r_2} \left( (z_1 - z_1')^T(z_1 - z_1') \right)^{\frac{p}{2}} + \frac{1}{s_2 z_2'} \left( (z_2 - z_2')^T(z_2 - z_2') \right)^{\frac{p}{2}} \right) \\
\tag{3.17}
\]

where \( s_2 = \frac{k - 1}{2}, r_2 = \frac{k}{2}, \) and \( \varepsilon_2 \) is any positive constant. With the help of Jenson’s inequality for convex functions [AS03] and of Lipschitz assumption on the reset term \( \rho(\cdot) \) (cf. Definition 2.1), the third term in (3.16) can be reduced as

\[
\frac{1}{k} \sum_{i=1}^n \lambda_i \left[ ||(z_2 - z_2') + (\rho(z_1)e_i - \rho(z_1')e_i)||^k - (z_2 - z_2')^T(z_2 - z_2') \right]^{\frac{p}{2}} \\
\leq \frac{1}{k} \sum_{i=1}^n \lambda_i \left[ 2^{k - 1} ||z_2 - z_2'||^k + 2^{k - 1} L_0^k ||z_1 - z_1'||^k - (z_2 - z_2')^T(z_2 - z_2') \right]^{\frac{p}{2}} \\
\leq \left( (z_2 - z_2')^T(z_2 - z_2') \right)^{\frac{k - 1}{2}} \frac{(2^{k - 1} - 1)\lambda}{k} + \left( (z_1 - z_1')^T(z_1 - z_1') \right)^{\frac{k - 1}{2}} \frac{2^{k - 1} L_0^k \lambda}{k} \\
\tag{3.18}
\]

where \( \lambda = \sum_{i=1}^n \lambda_i. \) Finally, the infinitesimal generator (3.16) corresponding to \( V_2(z_2, z_2') \) can be reduced with the help of (3.17) and (3.18) to

\[
\mathcal{L}V_2(z_2, z_2') \leq \left( (z_1 - z_1')^T(z_1 - z_1') \right)^{\frac{k}{2}} \frac{\min\{n, r\}L_0^2 \varepsilon_2^2}{2 r_2} \left( (2^{k - 1} - 1)\lambda \right) + \frac{2^{k - 1} L_0^k \lambda}{k} \\
+ (z_2 - z_2')^T(z_2 - z_2') \left( (2^{k - 1} - 1)\lambda \right) + \frac{\min\{n, r\}L_0^2(k - 1)}{2 s_2 z_2'} \\
\left[ \left( G u + \eta - \kappa_1 \frac{d M}{dt} z_1 - \kappa_1 (z_2 - \kappa_1 M z_1) + \left( \frac{(2^{k - 1} - 1)\lambda}{k} + \frac{\min\{n, r\}L_0^2(k - 1)}{2 s_2 z_2'} \right) z_2 \right) \\
- \left( G' u' + \eta' - \kappa_1 \frac{d M'}{dt} z_1' - \kappa_1 (z_2' - \kappa_1 M' z_1') + \left( \frac{(2^{k - 1} - 1)\lambda}{k} + \frac{\min\{n, r\}L_0^2(k - 1)}{2 s_2 z_2'} \right) z_2' \right) \right] \\
\tag{3.19}
\]
Now consider the Lyapunov function $V$ for the overall system as $V(z, z') = V_1(z_1, z'_1) + V_2(z_2, z'_2)$ and the respective infinitesimal generator can be obtained by using (3.14) and (3.19) as
\[
\mathcal{L}V(z, z') \leq \left( L_1 + \frac{L_2\varepsilon_1^r}{r_1} + \frac{\min\{n, r\}L_2^2\varepsilon_2^r(z - 1) + 2k^{-1}L_k^k\lambda}{2r_2} - \kappa_1 \right) \left( (z_1 - z'_1)^T (z_1 - z'_1) \right)^{\frac{1}{2^k}}
\]
\[
+ (z_2 - z'_2)^T \left( (z_2 - z'_2)^T (z_2 - z'_2) \right)^{\frac{1}{2^k} - 1}
\]
\[
\left[ \left( Gu + \eta - \kappa_1 \frac{dM}{dt} z_1 - \kappa_1 (z_2 - \kappa_1 M z_1) + \frac{(2k^{-1} - 1)\lambda}{k} + \frac{L_2}{s_1\varepsilon_1^r} + \frac{\min\{n, r\}L_2^2(k - 1)}{2s_2\varepsilon_2^r} \right) z_2 \right]
\]
\[
- \left( G'u' + \eta' - \kappa_1 \frac{dM'}{dt} z'_1 - \kappa_1 (z'_2 - \kappa_1 M' z'_1) + \frac{(2k^{-1} - 1)\lambda}{k} + \frac{L_2}{s_1\varepsilon_1^r} + \frac{\min\{n, r\}L_2^2(k - 1)}{2s_2\varepsilon_2^r} \right) z'_2 \right].
\]
If we choose the state feedback control law $u(z_1, z_2)$ as
\[
u(z_1, z_2) = G^{-1} \left( - \eta + \kappa_1 \frac{dM}{dt} z_1 - \kappa_1 M z_1 - \frac{(2k^{-1} - 1)\lambda}{k} + \frac{L_2}{s_1\varepsilon_1^r} + \frac{\min\{n, r\}L_2^2(k - 1)}{2s_2\varepsilon_2^r} z_2 + \hat{u} \right),
\]
the infinitesimal generator (3.20) reduces to
\[
\mathcal{L}V(z, z') \leq - \left( \kappa_1 - \left( L_1 + \frac{L_2\varepsilon_1^r}{r_1} + \frac{\min\{n, r\}L_2^2\varepsilon_2^r(z - 1) + 2k^{-1}L_k^k\lambda}{2r_2} \right) \left( (z_1 - z'_1)^T (z_1 - z'_1) \right)^{\frac{1}{2^k}}
\]
\[
- \kappa_1 \left( (z_2 - z'_2)^T (z_2 - z'_2) \right)^{\frac{1}{2^k} - 1} (\hat{u} - \hat{u}').
\]
Now the third term can further be reduced by applying Young’s inequality to
\[
(z_2 - z'_2)^T \left( (z_2 - z'_2)^T (z_2 - z'_2) \right)^{\frac{1}{2^k} - 1} (\hat{u} - \hat{u}') \leq \|z_2 - z'_2\|^{k-1}\|\hat{u} - \hat{u}'\|^{\frac{1}{k}}
\]
\[
\leq \frac{\varepsilon_1^r}{r_1} \left( (z_2 - z'_2)^T (z_2 - z'_2) \right)^{\frac{1}{2^k} - 1} + \frac{1}{s_1\varepsilon_1^r}\|\hat{u} - \hat{u}'\|^{\frac{k}{k}},
\]
where the parameters $\varepsilon_1, s_1$ and $r_1$ are the same as the ones in (3.12) and (3.13). Using (3.23) inequality (3.22) reduces to
\[
\mathcal{L}V(z, z') \leq - c_1 \left( (z_1 - z'_1)^T (z_1 - z'_1) \right)^{\frac{1}{2^k}} - c_2 \left( (z_2 - z'_2)^T (z_2 - z'_2) \right)^{\frac{1}{2^k}} + c_3\|\hat{u} - \hat{u}'\|^{\frac{k}{k}},
\]
where $c_1 = \left( \kappa_1 - \left( L_1 + \frac{L_2\varepsilon_1^r}{r_1} + \frac{\min\{n, r\}L_2^2\varepsilon_2^r(z - 1) + 2k^{-1}L_k^k\lambda}{2r_2} \right) \right)$, $c_2 = \left( \kappa_1 - \frac{\varepsilon_1^r}{r_1} \right)$, $c_3 = \frac{1}{s_1\varepsilon_1^r}$, all required to be positive. By choosing the design parameter $\kappa_1$ as
\[
\kappa_1 > L_1 + \frac{\max\{L_2, 1\}\varepsilon_1^r}{r_1} + \frac{\min\{n, r\}L_2^2\varepsilon_2^r(z - 1) + 2k^{-1}L_k^k\lambda}{2r_2} + \frac{2k^{-1}L_k^k\lambda}{k},
\]
one obtains $c_1, c_2, c_3 > 0$. If $\kappa = \min\{k\kappa_1, k\kappa_2\}$, the inequality (3.24) can further be reduced to
\[
\mathcal{L}V \leq - \kappa V(z, z') + \varphi(\|\hat{u} - \hat{u}'\|^{\frac{k}{k}}),
\]
where $\varphi(y) = c_3 y, \forall y \in \mathbb{R}^+_0$, which satisfies condition (iii) of Definition 2.3. One can readily verify that conditions (i) and (ii) in Definition 2.3 are satisfied by defining metric $d$ as the natural Euclidean one, and defining $\alpha(y) = \frac{1}{2^{\frac{1}{2^k} - 1}} y$, and $\overline{\alpha}(y) = \frac{y}{k}, \forall y \in \mathbb{R}^+_0$. Now with the help of Theorem 2.4, one obtains
\[
\mathbb{E}[\|\xi; \xi(t) - \xi(t); (t)\|^{\frac{k}{k}} \leq \beta(\mathbb{E}[\|z - z'|^{\frac{k}{k}}], t) + \gamma(\mathbb{E}[\|\hat{u} - \hat{u}'|^{\frac{k}{k}}], t)],
\]
where \( \zeta_{z\psi}(t) \) denotes the value of the solution process of \( \hat{\Sigma} \) in (3.7) at time \( t \in \mathbb{R}_0^+ \) under the input signal \( \hat{\nu} \) and from the initial condition \( \zeta_{x\psi}(0) = z \) P-a.s. The \( KL \) function \( \beta \), and the \( \mathcal{K}_\infty \) function \( \gamma \) can be defined as

\[
\beta(y, t) = \alpha^{-1}(\sigma(y)e^{-\kappa t}) = 2^{2-1}e^{-\kappa t}y,
\]

\[
\gamma(y) = \alpha^{-1}(\varphi(y)) = 2^{1-1}k_1\text{c}_a y,
\]

for all \( y \in \mathbb{R}_0^+ \). Now by applying the change of coordinate \( \zeta = \psi(\xi) \), where \( \xi = [q^T, p^T]^T \), the control law \( v \) reduces to

\[
v = G^{-1}(q) \left( -\eta(q, p) + \kappa_1 \frac{dM(q)}{dt} q - \kappa_1^2 M(q)q \right) - \left( \frac{(2k-1)\lambda}{k} + \frac{L_2}{s_1 \varepsilon_1^2} + \frac{\min\{n, r\} L_4^2(k-1)}{2s_2 \varepsilon_2^2} \right) (p + \kappa_1 M(q)q + \hat{\nu}),
\]

and (3.27) can be rewritten as

\[
E[\psi(\xi_{z\psi}(t)) - \psi(\xi_{x\psi}(t))] \leq (\beta(E[\psi(x) - \psi(x')]^k), t) + \gamma(E[\|\hat{\nu} - \nu'\|_\infty]),
\]

where \( x = [q^T, p^T]^T \). By defining a metric \( \mathbf{d}(x, x') = \|\psi(x) - \psi(x')\| \), we can rewrite (3.30) as

\[
E[\mathbf{d}(\xi_{x\psi}(t), \xi_{x'\psi}(t))] \leq \beta(E[\mathbf{d}(x, x')]^k), t) + \gamma(E[\|\hat{\nu} - \nu'\|_\infty]),
\]

which satisfies condition (2.2) for original \( \Sigma \). Hence, \( \Sigma \) in (3.3) equipped with the feedback control law (3.29) is \( \delta_3\text{-ISS-M}_k \) for any \( k > 2 \).

The next corollary provides the same results as the ones in Theorem 3.1 but for \( k = 2 \).

**Corollary 3.2.** Consider the stochastic control system \( \Sigma \) in (3.3). The state feedback control law

\[
v = G^{-1}(q) \left( -\eta(q, p) + \kappa_1 \frac{dM(q)}{dt} q - \kappa_1^2 M(q)q \right) - \left( \frac{(2k-1)\lambda}{k} + \frac{L_2}{s_1 \varepsilon_1^2} + \frac{\min\{n, r\} L_4^2(k-1)}{2s_2 \varepsilon_2^2} \right) (p + \kappa_1 M(q)q + \hat{\nu}),
\]

renders the closed-loop stochastic control system \( \delta_3\text{-ISS-M}_2 \) with respect to input \( \hat{\nu} \), for all

\[
\kappa_1 > L_1 + \max\{L_2, 1\} \varepsilon_2^2 \frac{2}{2} + \frac{\min\{n, r\} L_4^2}{2} + L_\rho \lambda,
\]

where \( \varepsilon_1 \) is any positive constant which can be chosen arbitrarily, and \( L_1, L_2, L_\sigma, \) and \( L_\rho \) are the Lipschitz constants introduced in (3.4) and Definition 2.1 respectively.

**Proof.** The corollary is a particular case of Theorem 3.1. The proof is almost similar to that of Theorem 3.1 by substituting \( k = 2 \). The only difference will appear while handling the trace term (3.17) in \( \zeta_2\)-subsystem which is now given by

\[
\frac{1}{2} \text{Tr} \left( \left( \sigma(z_1) - \sigma(z_1') \right) \left( \sigma(z_1) - \sigma(z_1') \right)^T \partial_{z_2 z_2} V_2(z_2, z_2') \right) \leq \frac{1}{2} \text{Tr} \left( \left( \sigma(z_1) - \sigma(z_1') \right) \left( \sigma(z_1) - \sigma(z_1') \right)^T \right)
\]

\[
\leq \frac{\min\{n, r\} L_4^2}{2} (z_1 - z_1')^T (z_1 - z_1').
\]

The rest of the proof follows similarly to the one of Theorem 2.4.

**Remark 3.3.** Assume that for all \( x, x' \in \mathbb{R}^n \), the change of coordinate map \( \psi \) in (3.6) satisfies

\[
\lambda(\|x - x'\|^k) \leq \psi(x) - \psi(x') \leq \lambda(\|x - x'\|^k),
\]

for some \( \mathcal{K}_\infty \) convex function \( \lambda \) and \( \mathcal{K}_\infty \) concave function \( \chi \). Then, inequality (3.31) for the original system \( \Sigma \) reduces to

\[
E[\psi(\xi_{z\psi}(t) - \xi_{x'\psi}(t))] \leq \beta(E[\|x - x'\|^k], t) + \gamma(E[\|\hat{\nu} - \nu'\|_\infty]),
\]

\[\text{Since } \psi \text{ is a bijective function, } \mathbf{d} \text{ satisfies all the requirements of a metric.}\]
for the KL function \( \hat{\beta}(y, t) = \chi^{-1}(2\beta(\chi(y), t)) \) and the \( K_\infty \) function \( \hat{\gamma}(y) = \chi^{-1}(2\gamma(y)) \), for any \( y, t \in \mathbb{R}_0^+ \).

Note that if \( \chi^{-1} \) satisfies the triangle inequality (i.e., \( \chi^{-1}(a + b) \leq \chi^{-1}(a) + \chi^{-1}(b) \)), one can remove the coefficients 2 in the expressions of \( \hat{\beta} \) and \( \hat{\gamma} \). Particularly, if the inertia matrix \( (M) \) is constant, on has

\[
\| (q - q') \| = \| A(x - x') \|,
\]

where \( A \) is a constant matrix given by

\[
A = \begin{bmatrix} I_n & 0_n \\ \kappa_1 M & I_n \end{bmatrix}.
\]

Therefore, one obtains

\[
(\lambda_{\min}(A^T A))^{\frac{1}{2}} \| x - x' \|^k \leq \| \psi(x) - \psi(x') \|^k = \| A(x - x') \|^k \leq (\lambda_{\max}(A^T A))^{\frac{1}{2}} \| x - x' \|^k,
\]

where \( \lambda_{\min}(A^T A) \) and \( \lambda_{\max}(A^T A) \) denote minimum and maximum eigenvalues of \( A^T A \), respectively.

4. Case Study

To verify the efficacy of the control design framework proposed in this paper, we illustrate the results on a spring pendulum attached to stochastically vibrating ceiling and subjected to jump. The nonlinear dynamics of the considered system is borrowed from [WCS12], and schematically shown in Figure 1. Let us define the generalized coordinate vector as \( q = [q_1, q_2]^T \), where \( q_1 \) represents change of arm length as a difference between the dynamic length \( (l_d) \) and static length \( (l) \) of a spring pendulum; and \( q_2 \) is the angle of pendulum with vertical axis. The corresponding generalized momenta vector is given by \( p = [m \frac{dq_1}{dt}, m(l + q_1)^2 \frac{dq_2}{dt}]^T \), where \( m \) is the mass of the ball, which gives the inertia matrix \( M(q) \) as

\[
M(q) = \begin{bmatrix} m & 0 \\ 0 & m(l + q_1)^2 \end{bmatrix}.
\]

The Hamiltonian function \( H(q, p) \) is given by a total energy of the system as

\[
H(q, p) = \frac{p_1^2}{2m} + \frac{p_2^2}{2m(l + q_1)^2} + \frac{k_s q_1^2}{2} + mg(l + q_1)(1 - \cos q_2),
\]
where $k_s$ is an elasticity coefficient of spring and $g$ is the acceleration due to gravity. Now $\eta(q, p) = -\frac{\partial H}{\partial q}(q, p) + b(q, p)$ can be calculated as

$$
\eta(q, p) = \left[ \frac{p^2}{m(l+q_1)^3} - k_s q_1 - mg(1 - \cos q_2) + \frac{b_1 p_1}{m} \right] + \left[ \frac{b_2 p_2}{2m} \right],
$$

where $b_1$ is a damping coefficient of piston and $b_2$ is an air damping coefficient. By considering a 2-dimensional Brownian motion, the diffusion function $\sigma(q)$ can be determined with the help of notion of relative kinematics by considering point $O$ in Figure 1 stochastically vibrating [WCS12] which is given by

$$
\sigma(q) = \begin{bmatrix}
-m \sin q_2 \\
-m(l + q_1) \cos q_2 \\
m \cos q_2 \\
-m(l + q_1) \sin q_2
\end{bmatrix}.
$$

To introduce abrupt jumps in the system, we consider a one dimensional Poisson process with the rate $\lambda = 1$ and linear reset function $\rho(q) = q$. The term $\frac{dM(q)}{dt}$ can be obtained as

$$
\frac{dM(q)}{dt} = \frac{\partial M(q)}{\partial q^T} \times \left( \frac{d}{dt} \otimes I_2 \right) = \begin{bmatrix}
0 & 0 \\
0 & \frac{2(1+q_1)p_1}{m}
\end{bmatrix}.
$$

As control input $u = [v_1, v_2]^T$, it itself acting on the mass, one gets $G(q) = I_2$. Now with the help of (4.1), (4.3), (4.5), and Theorem 3.1 we can obtain the final state feedback control input $v$ with $k = 2$ for the considered stochastic control system as follows

$$
v_1(q, p) = -\frac{p^2}{m(l+q_1)^3} + k_s q_1 + mg(1 - \cos q_2) + \frac{b_1 p_1}{m} - \kappa_1 \left( \frac{\lambda}{2} + \frac{L_2}{2\varepsilon_1} - \kappa_1 \right) m q_1,
$$

$$
\dot{v}_1(t) = \dot{v}_2(t) = 0.5 \sin t.
$$

$$
v_2(q, p) = mg(l + q_1) \sin q_2 + \frac{b_2 p_2}{m} + \frac{2\kappa_1(1+q_1)p_1 q_2}{m} - \kappa_1 \left( \frac{\lambda}{2} + \frac{L_2}{2\varepsilon_1} - \kappa_1 \right) m q_2(l + q_1)^2,
$$

rendering the closed-loop system $\delta_2$-ISS-M$_2$ with respect to input $[\dot{v}_1, \dot{v}_2]^T$ for any arbitrarily chosen $\varepsilon_1 > 0$ and appropriately chosen $\kappa_1$.

For the simulation purpose, we consider system parameters as $m = 0.8$, $l = 1.5$, $g = 9.8$, $k_s = 15$, $b_1 = 1$, and $b_2 = 1$; all the constants and the variables are considered in SI units; the Lipschitz constants are computed as $L_1 = 1$, $L_2 = 2$, $\varepsilon_1 = 1$, and $\kappa = 1$, and the design parameters are chosen as $\varepsilon_1 = 0.5$ and $\kappa_1 = 4$. We choose inputs $\dot{v}_1(t) = \dot{v}_2(t) = 0.5 \sin t$. Figure 2 shows the evolution of the closed-loop trajectories $q$ and $p$ in presence of Brownian noise and Poisson jumps started from two different initial conditions $[q; p] = [0.5; -0.4; -2.5; 3]$ and $[q'; p'] = [-0.5; 0.6; 1; -0.5]$ and the evolution of the corresponding input trajectories $v_1$ and $v_2$. Figure 2 shows that indeed, by virtue of the $\delta_2$-ISS-M$_2$ property, both trajectories converge to each other. To verify the bound on $\mathbb{E}[\|\zeta_{\dot{v}}(t) - \zeta_{\dot{v}'}(t)\|^2]$ as given in (3.27), we simulated the closed-loop system for 5000 realizations, two fixed initial conditions, and the same input for both trajectories (i.e $\dot{v} = \dot{v}'$). The inequality (3.27) reduces to

$$
\mathbb{E}[\|\zeta_{\dot{v}}(t) - \zeta_{\dot{v}'}(t)\|^2] \leq \beta(z, z', t),
$$

where the $KC$ function $\beta$ is given in (4.28) and computed as $\beta(y, t) = e^{-\kappa t} y$ with $\kappa = 0.75$. The average value of the squared distance of two trajectories of $\Sigma$ started from two different initial conditions $z = [0.5; -0.4; -0.9; -2.12]$ and $z' = [-0.5; 0.6; -0.6; 1.42]$ together with computed theoretical bound are shown in Figure 3. One can readily verify that the simulated distance is always lower than the computed theoretical one in (4.8).
Figure 2. Two trajectories $q$ (top two plots), two trajectories $p$ (middle two plots) started from two different initial conditions $[q; p] = [0.5; -0.4; -2.5; 3]$ and $[q'; p'] = [-0.5; 0.6; 1; -0.5]$, and the two corresponding input trajectories $v_1$ and $v_2$ (bottom two plots).

5. Conclusion

We introduced a notion of incremental input-to-state stability for stochastic control systems with jumps and provided its description in terms of existence of a notion of so-called incremental Lyapunov functions. Furthermore, a backstepping controller design scheme was proposed for a class of nonlinear stochastic Hamiltonian systems with jumps which provides controllers rendering the close-loop systems $\delta_2$-ISS-M$_k$. We illustrated the effectiveness of the results on a nonlinear stochastic Hamiltonian system.

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FIGURE 3. The average value of the squared distance of two trajectories of $\bar{\Sigma}$ started from two different initial conditions $z = [0.5; -0.4; -0.9; -2.12]$ and $z' = [-0.5; 0.6; -0.6; 1.42]$. The black dotted curve indicates corresponding bound given by (4.8).

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