Infinitely many non-conservative solutions for the three-dimensional Euler equations with arbitrary initial data in $C^{1/3-\varepsilon}$

Calvin Khor∗ Changxing Miao† Weikui Ye‡

Abstract

Let $0 < \beta < \bar{\beta} < 1/3$. We construct infinitely many distributional solutions in $C^{\beta}_{x,t}$ to the three-dimensional Euler equations that do not conserve the energy, for a given initial data in $C^{\bar{\beta}}$. We also show that there is some limited control on the increase in the energy for $t > 1$.

1 Introduction

This paper studies the Cauchy problem for the 3D incompressible Euler equations,

$$\begin{cases} 
\partial_t v + (v \cdot \nabla)v + \nabla p = 0, \\
\nabla \cdot v = 0, \\
v|_{t=0} = v^{in},
\end{cases}$$

considered on the spatially periodic domain $\mathbb{T}^3 := (\mathbb{R}/2\pi\mathbb{Z})^3$. For $x \in \mathbb{T}^3$ and $t \geq 0$, $v(x,t) \in \mathbb{R}^3$ is the fluid velocity, and $p(x,t) \in \mathbb{R}$ is the pressure field. We are specifically interested in the scenario of low regularity initial data $v^{in} \in C^{\bar{\beta}}$ for $\bar{\beta} < 1/3$, in relation to the following celebrated result:

**Theorem 1.1** (Onsager’s Conjecture). Consider weak solutions of (1.1) in $C^{\beta}(\mathbb{T}^3 \times [0,T])$. 

---

∗Beijing Computational Science Research Center, Beijing 100193, China, calvin_khor@hotmail.com
†Institute of Applied Physics and Computational Mathematics, P.O. Box 8009, Beijing 100088, P. R. China, miao.changxing@iapcm.ac.cn
‡Institute of Applied Physics and Computational Mathematics, P.O. Box 8009, Beijing 100088, P. R. China, 904817751@qq.com
1. [CET94] If $\beta > 1/3$, then the energy $\int_{\mathbb{T}^3} |v|^2 \, dt$ is conserved in time.

2. [Ise18] If $\beta < 1/3$, then there exist solutions that do not conserve energy.

The work [Ise18] proving the negative part of Onsager’s conjecture is the culmination of many refinements over many papers over many years, starting with the constructions of Scheffer [Sch93] and Shnirelman [Shn03], followed by the discovery [DS09] of the deep connection between a counterintuitive result in geometry (the Nash–Kuiper theorem [Nas54; Kui55a; Kui55b]), and turbulence. In this seminal paper of De Lellis and Székelyhidi [DS09], they proved the existence of distributional solutions to the Euler equations in $L^\infty$ with compact support in spacetime.

After nearly half a decade, the authors switched from a ‘soft analysis’ approach to ‘hard analysis’ and managed [DS13] to show the existence of continuous solutions. This led to a flood of papers with the above mentioned work of Isett as a crowning result. One should also mention the streamlined proof of [Buc+19] that also allows you to prescribe an arbitrary positive energy profile. Our account of the history is extremely abridged; we refer readers to the surveys [BV19a; De 18; DS19] for more detailed information.

Roughly in parallel with the above developments, a version of Onsager’s conjecture for the Cauchy problem was also studied. As is well known, for sufficiently smooth initial data, there exists a unique smooth solution for short times. The question whether such a solution is in fact global is a classical open problem. The above results were proven by showing that there are $L^\infty$ [DS09] and $C^{\beta}$ ($\beta < 1/3$) [Ise18] solutions with compact support in time, and hence in particular that (so long as we are considering distributional solutions) there are multiple solutions starting from the identically zero flow. In fact, one expects convex integration to produce a set of solutions that is Baire-typical due to the h-principle (see e.g. [DS17]).

It is natural to predict that the same is true for other initial data. Daneri [Dan14], later with Szekelyhidi [DS17] and Runa [DRS21] proved (among other things) that the set of initial data in $C^{\bar{\beta}}$ without uniqueness of $C^{\beta}$ solutions ($0 < \beta < \bar{\beta} < 1/3$) is $L^2(\mathbb{T}^3)$-dense. Rosa and Haffter [RH21] also showed that any $C^\infty$ initial data gives rise to uncountably many solutions.

The main result of our paper is the following theorem, which generalises the very strong ill-posedness above to all initial data in $C^{\bar{\beta}}$:

**Theorem 1.2.** Let $v^{\text{in}} \in C^{\bar{\beta}}(\mathbb{T}^3)$ be divergence-free. Fix $0 \leq \beta < \bar{\beta} < 1/3$, and $T \in (1, \infty]$. Then there exists infinitely many smooth functions $e : [1, T] \to (\|v^{\text{in}}\|_{L^2}^2, \infty)$ with a corresponding a weak solution $v$ to (1.1) on
\[0, T] \text{ with initial data } v|_{t=0} = v^{\text{in}}, \text{ such that } v \in C^\beta_{x,t}, \text{ and for all } t \in [1, T], \]
\[\int_{T^3} |v(x, t)|^2 \, dx = e(t).\]

We achieve this by adjusting the convex integration scheme to take an approximate initial data, which recovers any chosen initial data in the limit. This shows the strongest possible ill-posedness for an equation that has solutions.

We note however that there are significant differences between Theorem 1.2 and the earlier mentioned works [Dan14; DS17; DRS21]. Recall that weak-strong uniqueness (see [Wie18]) holds for the Euler equations: energy dissipative weak solutions (called ‘admissible solutions’) with smooth initial data are classical. The above works of Daneri and collaborators show that weak-strong uniqueness does not hold for all \(C^\beta\) data. In contrast, the solutions that we construct must increase the energy: we have in effect traded some control the energy in exchange for control on the initial data. This energy increase is also present in the construction of [RH21].

We still manage some control over the energy, which is better than the energy control at a single point achieved by [DRS21], for instance. More precisely, we have the following quantitative version of Theorem 1.2:

**Theorem 1.3.** For the solutions constructed in Theorem 1.2 with chosen parameters \(a, b, \beta, \bar{\beta}\), there exist constants \(c, C > 0\) depending on the parameters of Proposition 3.1 (in particular diverging to \(+\infty\) as \(a \to \infty\), \(b \to 1\), or \(\beta \to \bar{\beta}\)) such that for each \(v^{\text{in}} \in C^\beta\), the energy profile \(e\) for times \(t \geq 1\) can be chosen to be an arbitrary smooth function satisfying

\[c < e(t) - \int_{T^3} |v^{\text{in}}|^2 \, dx < C \text{ for all } t \in [1, T].\]

We manage this control of energy by transitioning at some arbitrary positive time (we have chosen \(t = 1\) for simplicity) from the construction of [Ise18] (which allows velocity perturbations supported away from the initial time) to the squiggling cutoffs of [Buc+19].

It would be very interesting to know if each non-smooth initial data in \(C^\beta\) has energy dissipative solutions, and to what extent can the energy be controlled, but these questions are beyond the scope of the current paper.

We finish this introductory section with a non-exhaustive list of papers where convex integration has been adapted to show non-uniqueness for other equations, even equations with dissipation. For instance, there are results for Euler on \(\mathbb{R}^3\) [IO16], MHD [BBV20; DF21], active scalars [IV15], 2D Euler [Cho13], SQG [IM21], Navier–Stokes [BV19b; BCV20; LT20], transport
equations [CL21], stationary Navier–Stokes [Luo19], Boussinesq [LTZ20; TZ17; TZ18]. Recently, Colombo [BC21] showed the non-uniqueness of solutions to 2D Euler with vorticity in the Lorentz space $L^{1,p}, p > 2$. We also mention the paper [DK20] showing the non-uniqueness of globally dissipative solutions, which additionally solve the local energy inequality and strictly dissipate the energy (but at the moment, they are unable to achieve Hölder regularity beyond $C^{1/7}$).

The remainder of this paper is organised as follows. In Section 2, we fix some notation. In Section 3, we give the main iterative proposition, sketch its proof, and show how Theorems 1.2 and 1.3 follow from it. Section 4 carries out the convex integration scheme (in particular that (3.11) holds at step $q + 1$ in Corollary 4.15). In Section 5, we explain how the stress error terms are controlled, and in Section 6, we show that we have the claimed control on the energy. Finally we collect some classical estimates on Hölder spaces in the Appendices.

2 Notation

For $\alpha \in (0,1)$, and $N \in \mathbb{Z}_{\geq 0}$, $C^{N+\alpha}(X)$ denotes the usual Hölder space which we have defined in Appendix A. We will write $C^{N+\alpha}$ as shorthand for $C^{N+\alpha}(\mathbb{T}^3)$. For functions in $L^{\infty}(0,T;C^{\alpha}) =: L^{\infty}C^{\alpha}$, we define the norm

$$\|f\|_{N+\alpha} := \|f\|_{L^{\infty}C^{N+\alpha}}.$$ 

We also write $[f]_\alpha = \sup_{t \leq T}|f(t,\cdot)|_{C^\alpha}$ for the Hölder seminorm. Some classical estimates for Hölder functions are collected in Appendix A.

We fix for the remainder of the paper an even non-negative bump function $\tilde{\phi} \in C^\infty_c([-1, 1]; [0, \infty))$, i.e. $\tilde{\phi}(-t) = \tilde{\phi}(t)$, and define for each $\varepsilon > 0$ two sequences of mollifiers with integral 1:

$$\phi(t) := \frac{\tilde{\phi}(t)}{\int_{[-1,1]} \tilde{\phi}(\tau) \, d\tau}, \quad \phi_\varepsilon(t) := \frac{1}{\varepsilon} \phi \left( \frac{t}{\varepsilon} \right), \quad (2.1)$$

$$\psi(x) := \frac{\tilde{\phi}(|x|)}{\int_{B(0)} \tilde{\phi}(|y|) \, dy}, \quad \psi_\varepsilon(x) := \frac{1}{\varepsilon^3} \psi \left( \frac{x}{\varepsilon} \right). \quad (2.2)$$

We use the following distributional notion of a weak solution:

**Definition 2.1.** Let $T \geq 0$ and let $v^{\text{in}} \in C^{\bar{\beta}}$ for some $\bar{\beta} > 0$ be divergence-free (in the sense of distributions) and have zero mean, i.e. $\int_{\mathbb{T}^3} v^{\text{in}} \, dx = 0$. We say that $v \in C^0(T^3 \times [0,T])$ is a weak solution on $[0,T]$ to (1.1) with initial
data $v^\text{in}$ if $v(\cdot, t)$ has zero mean, is divergence-free, and for all divergence-free test functions $f \in C^\infty_c([0, T] \times \mathbb{T}^3; \mathbb{R}^3),$

$$\int_0^T \int_{\mathbb{T}^3} v \cdot (\partial_t + v \cdot \nabla) f \, dx \, dt = -\int_{\mathbb{T}^3} v^\text{in}(x) f(0, x) \, dx.$$  

The zero mean condition above is for convenience, and is easily removed via the Gallilean symmetry $\bar{v}(x, t) = v(x + t\bar{v}, t) - \bar{v}$ with $\bar{v} = \frac{1}{\mathbb{T}^3} v \, dx \in \mathbb{R}^3.$  
The pressure $p$ which does not appear in the weak formulation is recovered by solving the Poisson equation $-\Delta p = \text{div} \, \text{div}(v \otimes v).$  
Further background on fluid mechanics can be found for instance in [MB02].

3 Outline of the convex integration scheme

3.1 Parameters and their restrictions

For all $q \geq 0$ and given parameters $\alpha \in (0, 1), a \in (2, \infty), b \in (1, 2), 0 < \beta < \beta < 1/3$, we define ($\lceil x \rceil$ denotes the ceiling function)

$$\lambda_q := 2\pi \left\lceil a b^q \right\rceil > 1, \quad \delta_q := \lambda_q^{-2\beta} < 1.$$  

Note that $2\pi \leq \frac{\lambda_q}{a b^q} \leq 4\pi$. We restrict $b < 2$ so that $\frac{1}{50} \leq \frac{\lambda_q + 1}{b^q} \leq 2$. We will require that $\alpha \ll 1$ and $a \gg 1$ so that

$$2\alpha < \beta(b - 1), \quad a > 50^{3/\alpha}. \quad (3.1)$$  

These imply that $\lambda_q^{3\alpha} \leq \frac{\delta_q^{3/2}}{\delta_q^{3/2+1}} \leq \frac{\lambda_q + 1}{\lambda_q}$, as in [Buc+19]:

$$\lambda_q^{3\alpha} \leq \frac{\lambda_q^{6\alpha}}{50^{3\beta}} \leq \frac{\lambda_q^{3\beta(b-1)}}{50^{3\beta}} \leq \left( \frac{\lambda_q + 1}{\lambda_q} \right)^{3\beta} = \frac{\delta_q^{3/2}}{\delta_q^{3/2+1}} \leq \frac{\lambda_q + 1}{\lambda_q}.$$  

We also define the mollification parameter $\ell_q$ by

$$\ell_q := \frac{\delta_q^{1/2}}{\delta_q^{1/2+3\alpha/2}} \in \left( \frac{1}{2} \lambda_q^{-1 - \frac{3\alpha}{2} - (b-1)\beta}, \lambda_q^{-1 - \frac{3\alpha}{2}} \right). \quad (3.2)$$  

The lower bound on $\ell_q$ follows from $\frac{1}{2} \lambda_q^{-1-b} \leq \lambda_q^{-1}$. If only a rough bound on $\ell_q$ is needed, then we will use $\ell_q \in (\lambda_q^{-2}, \lambda_q^{-1})$, which follows from the mild inequalities $\alpha < 1/10, (b - 1)\beta < 1/5$ and $a^{3/5} > 2$. See also the definition of $\tau_q$ in (4.5).
We will use the notation $X \preceq Y$ to denote that $X \leq CY$ for some constant $C$ that depends on the various parameters $a, b, \alpha, \beta, \bar{\beta}$, but not on $q$. By routine calculations similar to the above, we will freely use inequalities like $\lambda_{q+1} \preceq \lambda_q^b$ and $\lambda_q^b \preceq \lambda_{q+1}$, so long as we can lose some small positive power of $\lambda_q$ to absorb the implicit constants by taking $a \gg 1$. The implicit constant may also depend on a certain number of derivatives $N$, but this is chosen to be a fixed number depending on $b$ and $\beta$ in Section 5.

3.2 Euler–Reynolds flow with smoothed initial data

As is usual in convex integration schemes, we consider a modification of (1.1) with a stress tensor error term $\hat{\mathcal{R}}_q$ that tends to 0 in the sense of distributions. However, we propagate an initial condition as well. That is, we require $(v_q, p_q, \hat{\mathcal{R}}_q)$ to solve

$$\begin{cases}
\partial_t v_q + \text{div}(v_q \otimes v_q) + \nabla p_q = \text{div} \hat{\mathcal{R}}_q, \\
\nabla \cdot v_q = 0, \\
v_q|_{t=0} = v_{in} \ast \psi_{q-1}.
\end{cases}$$

(3.3)

where $a \otimes b := a b^T = (a_i b_j)_{i,j=1}^3$, and the divergence of a 2-tensor $M = (M_{ij})_{i,j=1}^3$ is defined to be the vector $\text{div} M$ with components (using the Einstein summation convention)

$$(\text{div} M)_i := \partial_j M_{ij}.$$ 

In particular, since $v_q$ is divergence-free, $\text{div}(v_q \otimes v_q) = (v_q \cdot \nabla) v_q$. The tensor $\hat{\mathcal{R}}_q$ is required to be a symmetric, trace-free $3 \times 3$ matrix, i.e.

$$\hat{\mathcal{R}}_q = \hat{\mathcal{R}}_q^T, \quad \text{Tr} \hat{\mathcal{R}}_q = \sum_{i=1}^3 (\hat{\mathcal{R}}_q)_{ii} = 0.$$ 

(3.4)

We enforce mean zero conditions on $v_q$ and $p_q$:

$$\int_{T^3} v_q \, dx = 0, \quad \int_{T^3} p_q \, dx = 0.$$ 

(3.5)

3.3 The main iterative proposition

We will choose $a \gg 1$, and $b > 1$ very close to 1. We then choose a smooth function $e : [0, \infty) \rightarrow (\|v_{in}\|_{L^2}^2, \infty)$ such that

$$t \in [1 - \tau_0, T] \implies \delta_2 \lambda_1^{-\alpha} \leq e(t) - \int_{T^3} |v_{in}|^2 \, dx \leq \delta_2.$$ 

(3.6)
The estimates we propagate inductively are:

\[ \|v_q\|_0 \leq 1 - \delta_q , \]  
\[ \|v_q\|_1 \leq M \delta_{q+1}^{1/2} \lambda_q , \]  
\[ \|\hat{R}_q\|_0 \leq \delta_{q+1} \lambda_q^{-3\alpha} , \]  
\[ t \in [1 - \tau_{q-1}, T] \implies \delta_{q+1} \lambda_q^{-\alpha} \leq e(t) - \int_{\mathbb{T}^3} |v_q(x,t)|^2 \, dx \leq \delta_{q+1} . \]

The constant \( M \) in (3.8) is universal (in particular independent of \( q \)): it is defined in (4.53).

Note that the above estimates cannot be satisfied when using arbitrary initial data \( v^\text{in} \), but this restriction is easily removed in the proof of Theorem 1.2 by a scaling argument.

**Proposition 3.1.** Let \( M \) be the universal constant defined in (4.53). Let \( T \in [1, \infty] , 0 < \beta < \bar{\beta} < 1/3 \), and let \( v^\text{in} \in C^3(\mathbb{T}^3) \) be a periodic function with zero mean, such that \( \|v^\text{in}\|_{C^3} \leq 1 \), and let \( e : [0, T] \to [0, \infty) \) be any smooth function. Let \( b \in (1, b_0) \) where \( b_0 = b_0(\beta, \bar{\beta}) \) is given by the inequalities (4.12), (4.17), and (5.2).

\[ b_0 := 1 + \min \left( \frac{1}{\lambda_q + 1}, \frac{1}{\lambda_q + 1}, \frac{1}{\lambda_q + 1} \right) . \]

There exist \( a_0 \) and \( \alpha_0 \) such that for \( \alpha < \alpha_0(\beta, \bar{\beta}, b) \) and \( a > a_0(\beta, \bar{\beta}, b, \alpha) \), the following holds. Let \( (v_q, p_q, \hat{R}_q) \) solve (3.3) and satisfy (3.7)–(3.10). Then there exists smooth functions \( (v_{q+1}, p_{q+1}, \hat{R}_{q+1}) \), satisfying (3.3), (3.7)–(3.10) with \( q \) replaced by \( q + 1 \), and such that

\[ \|v_{q+1} - v_q\|_0 + \frac{1}{\lambda_{q+1}} \|v_{q+1} - v_q\|_1 \leq M \delta_{q+1}^{1/2} . \]

The upper bound \( \alpha_0 \) is given by the inequalities (3.1), (4.12), (4.13), (4.17), (4.60), and (5.2), and \( \alpha_0 \) is taken sufficiently large to remove various universal constants throughout the argument.

**3.4 Proof of Theorems 1.2 and 1.3 using Proposition 3.1**

Let \( 0 < \beta < \bar{\beta} < 1/3 \) and choose \( b > b_0(\beta, \bar{\beta}) \), \( \alpha < \alpha_0(\beta, \bar{\beta}, b) \) and \( a > a_0(\beta, \bar{\beta}, b, \alpha) \) as in Proposition 3.1. We will later impose some further mild conditions on \( a \) and \( b \).
To start the iteration, we define \((v_1, p_1, \tilde{R}_1)\) by

\[
v_1 := v^{in} * \psi_{t_0}, \quad p_1 := |v_1|^2 - \int_{\mathbb{T}^3} |v_1|^2 \, dx, \quad \tilde{R}_1 := v_1 \otimes v_1 - |v_1|^2 I_{5 \times 3}.
\]

It is easy to check that they solve (3.3), (3.4), and (3.5). In addition, we first assume that they satisfy \(\|v^{in}\|_{C^\beta} \leq 1,\) (3.7)–(3.9). We take a smooth function \(e : [0, T] \to [0, \infty)\) satisfying

\[
t \in [1 - \tau_0, T] \implies e(t) \in (\|v^{in}\|_{L^2}^2 + \delta_2 \lambda_1^{-\alpha}, \|v^{in}\|_{L^2}^2 + \delta_2)
\]

which therefore satisfies (3.6). As \(1 - \tau_0, T\) is compact, there exists a constant \(c = c(v^{in}) \in (1, 2)\) such that

\[
t \in [1 - \tau_0, T] \implies e(t) \in (\|v^{in}\|_{L^2}^2 + c \delta_2 \lambda_1^{-\alpha}, \|v^{in}\|_{L^2}^2 + \delta_2/c).
\]

In order to ensure that (3.10) holds for \(q = 1\), we need to ensure that \(\int_{\mathbb{T}^3} |v^{in}|^2 \, dx - \int_{\mathbb{T}^3} |v_1|^2 \, dx \ll \delta_2\). For this, we use the Constantin–E–Titi estimate [CET94]:

\[
\left| \int_{\mathbb{T}^3} |v^{in}|^2 \, dx - \int_{\mathbb{T}^3} |v_1|^2 \, dx \right| \leq \left| \int_{\mathbb{T}^3} |v^{in}|^2 * \psi_{t_0} \, dx - \int_{\mathbb{T}^3} |v^{in} * \psi_{t_0}|^2 \, dx \right|
\]

\[
\lesssim \|v^{in}|^2 - |v^{in} * \psi_{t_0}|^2\|_{C^0}
\]

\[
\lesssim c^{\frac{1}{2\beta}} \|v^{in}|^2_{C^\beta} \ll \lambda_0^{-\frac{2\beta}{\alpha}}
\]

which can be made smaller than \(\delta_2 \sim \lambda_0^{-\frac{2\beta a}{\alpha}} = \lambda_0^{-2\beta(b-1)^2 - 4\beta(b-1)^2} \) by further restricting \(b - 1 < \frac{\beta - \beta}{4\tilde{\beta}}\) and then increasing \(\alpha\) (at the beginning of the proof) to absorb the implicit constants, if necessary. Then, Proposition 3.1 applies inductively, giving rise to a \(C^0\) convergent sequence of functions \(v_\ell \to v\), that solve (1.1), with \(\|v\|_{L^2}^2(t) = e(t) > \|v^{in}\|_{L^2}^2\) for \(t \in [1, T]\). A standard interpolation argument with the \(C^1\) estimate in (3.11) shows that \(v \in C^\beta_{x,t}\) (see [Buc+19]).

To allow initial data of arbitrary size, we apply the theorem to

\[
\tilde{v}^{in} := \frac{v^{in}}{\Gamma}, \quad \Gamma := \max \left( \frac{\|v^{in}\|_{C^\beta}}{1 - \delta_1}, \frac{\|v^{in}\|_{C^0}}{M \delta_2^{1/2} \lambda_1^{-3\alpha/2}} \right).
\]

The scaling is chosen precisely so that (3.7) and (3.9) hold. As before, we set \(v_1 = \tilde{v}^{in} * \psi_{t_0}\). Note that (since \(\lambda_0 < \lambda_1\) and \(1 + 3\alpha/2 - \beta > 0\))

\[
\|v_1\|_1 \leq \frac{1}{\Gamma} \|v^{in}\|_{C^0} \ell_0^{-1} \leq M \ell_0^{-1} \delta_2^{1/2} \lambda_1^{-3\alpha/2} \leq M \delta_2^{1/2} \lambda_1,
\]

8
and hence (3.8) holds. We then take an arbitrary choice of \( \bar{e} : [0, \Gamma T] \to [0, \infty) \) that satisfies (3.6) with \( \Gamma T \) in place of \( T \), and construct a solution \( \tilde{v} \) on the time interval \([0, \Gamma T]\), with \( \|\tilde{v}\|_{L_2}^2(t) = \bar{e}(t) > \|v^{in}\|_{L_2}^2 \) for \( t \in [1, \Gamma T] \). Then we define
\[
v(x, t) := \Gamma \tilde{v}(x, \Gamma t), \quad \text{and} \quad e(t) := \Gamma^2 \bar{e}(\Gamma t).
\]
Due to the symmetry of the Euler equations, \( v \) also solves Euler, but on the time interval \([0, T]\), and with the initial data \( v^{in} \). In addition, since \( 1 > \Gamma^{-1} \), we also obtain that
\[
\|v\|_{L_2}^2(t) = e(t) > \|v^{in}\|_{L_2}^2 \text{ for times } t \geq 1.
\]

A closer inspection of the proof gives Theorem 1.3: for an initial data \( v^{in} \in C^\beta \), we can choose for times \( t \geq 1 \) an energy profile with image in \((\|v^{in}\|_{L_2}^2 + \Gamma^2 \delta_2 \lambda_1^{-\alpha}, \|v^{in}\|_{L_2}^2 + \Gamma^2 \delta_2)\). Since for a fixed \( v^{in} \), \( \|v^{in}\|_{C^\beta}/(1 - \delta_1) \) is bounded uniformly in the parameters \( a, b, \alpha \), we should generically have when \( a \gg 1 \) that
\[
\Gamma^2 = \frac{\|v^{in}\|_{L_2}^2}{M \delta_2 \lambda_1^{3\alpha}}.
\]
In such a situation, we would have
\[
M \|v^{in}\|_{C^0}^2 \alpha \lambda_1^2 \leq e(t) - \int_{T^3} |v^{in}|^2 \, dx \leq M \|v^{in}\|_{C^0}^2 \alpha \lambda_1^{3\alpha}.
\]
In particular, there is a large lower bound (as \( \alpha > 0 \) and \( a \gg 1 \)), and in order to achieve larger energies, we need the lower bound to increase as well.

The remainder of the paper is devoted to the proof of Proposition 3.1.

### 3.5 Proof sketch for Proposition 3.1

Starting from a tuple \( (v_q, p_q, \bar{R}_q) \) satisfying the estimates as in Proposition 3.1, the broad scheme of the iteration is as follows.

1. \((v_{\ell_q}, p_{\ell_q}, \bar{R}_{\ell_q})\) are defined by mollification. This step differs from previous works because we include a mollified initial data in the iteration scheme.

2. Next we define a family of exact solutions to Euler \((v_i, p_i), i \geq 1\), by exactly solving the Euler equations with smooth initial data \( v_i|_{t=t_i} = v_{\ell_q}(t_i) \), where \( t_i = i \tau_q \) defines an evenly spaced partition of \([0, T]\). For \( i = 0 \), we improve the initial data from \( v_0|_{t=0} = v^{in} * \psi_{\ell_q-1} * \psi_{\ell_q} \) to \( v_0|_{t=0} = v^{in} * \psi_{\ell_q} \), but this creates a small mismatch. For the estimates to work at this step, we require \( \bar{\beta} > \beta \).

3. These solutions are glued together using a partition of unity, resulting in the tuple \((\bar{v}_q, \bar{p}_q, \bar{R}_q)\). The stress error term is zero when \( \bar{v}_q = v_i \) for some \( i \geq 0 \).
4. We define $v_{q+1}$ by constructing a perturbation $w_{q+1}$ and setting $v_{q+1} = \bar{v}_q + w_{q+1}$. In order to achieve the optimal regularity, we construct some key cutoffs functions $\eta_i$ (see (4.25)) and use the Mikado flows from [DS17] to create the principal part $w_{q+1}^{(p)}$ of the perturbation. We also define an incompressibility corrector term $w_{q+1}^{(c)}$ so that $w_{q+1} := w_{q+1}^{(p)} + w_{q+1}^{(c)}$ is divergence-free.

5. Once we have fixed the definition of $v_{q+1}$, the term $\text{div } \hat{R}_{q+1}$ is fixed since $(v_{q+1}, \hat{R}_{q+1})$ has to solve (1.1). This leads to the natural definition of $\hat{R}_{q+1}$ using the inverse divergence operator $\mathcal{R}$.

6. We prove that the inductive estimates (3.7)–(3.10) hold with $q$ replaced with $q + 1$.

Step 4 is moderately involved, and breaks into the following sub-steps, where we have combined the approach of [Ise18] and [Buc+19] in order to achieve control of the energy:

4a. For times $t \geq 1$, we use the ‘squiggling’ cutoffs $\eta_i$ from [Buc+19] that allow energy to be added at such times, even outside the support of $\hat{R}_q$, while cancelling a large part of $\hat{R}_q$ norm.

4b. For times $t < 1$, we instead use the straight cutoffs of [Ise18], so that we can ensure we do not adjust the solution near $t = 0$. We leave some space in order to transition between the two approaches. Together with the modification of Step 2, this ensures that we have

$|v_{q+1}|_{t=0} = |\bar{v}_q|_{t=0} + |w_{q+1}|_{t=0} = v^*_{\psi} \psi_{\ell_q}$, so that the initial data is propagated inductively.

4c. Then we create the principal part $w_{q+1}^{(p)}$ of the perturbation using Mikado flows and the back-to-labels map of $\tilde{v}_q$, and add a small corrector term $w_{q+1}^{(c)}$ to enforce the incompressibility constraint.

4 The iterative step

In this section, we give some details for the construction of $(v_{q+1}, p_{q+1}, \hat{R}_{q+1})$ from $(v_q, p_q, \hat{R}_q)$. Our construction is a modification of [Buc+19], so we direct the reader there at a number of points for details that are not elaborated here.
4.1 Mollification

We recall that $\ell_q$ is defined in (3.2). The functions $v_{\ell_q}$ and $\hat{R}_{\ell_q}$ are defined with the spatial mollifier (2.2),

$$v_{\ell_q} := v_q \ast \psi_{\ell_q}, \quad \hat{R}_{\ell_q} := \hat{R}_q \ast \psi_{\ell_q} - (v_q \hat{\otimes} v_q) \ast \psi_{\ell_q} + v_{\ell_q} \hat{\otimes} v_{\ell_q},$$

where for vectors $a, b \in \mathbb{R}^3$, we have written $a \hat{\otimes} b := a \otimes b - (a \cdot b)I_{3 \times 3}$, which is a trace-free version of $a \otimes b$. $v_{\ell_q}$ is also divergence-free, and $\hat{R}_{\ell_q}$ is also symmetric and trace-free. They solve the initial value problem

$$\begin{cases}
\partial_t v_{\ell_q} + \text{div}(v_{\ell_q} \otimes v_{\ell_q}) + \nabla p_{\ell_q} = \text{div} \hat{R}_{\ell_q}, \\
\nabla \cdot v_{\ell_q} = 0, \\
v_{\ell_q}|_{t=0} = v_{\text{in}} \ast \psi_{\ell_q-1} \ast \psi_{\ell_q},
\end{cases}$$

where $p_{\ell_q} := p_q \ast \psi_{\ell_q} - |v_q|^2 - |v_{\ell_q}|^2$, and we have used the identity $\text{div}(fI_{3 \times 3}) = \nabla f$ for scalar fields $f$. Standard mollification estimates and the quadratic Constantin–E–Titi estimate [CET94] give the following proposition.

**Proposition 4.1** (Estimates for mollified functions).

\[
\|v_{\ell_q} - v_q\|_0 \lesssim \delta_{q+1}^{1/2} \lambda_q^{-\alpha} \tag{4.1}
\]

\[
\|v_{\ell_q}\|_{N+1} \lesssim \delta_q^{1/2} \lambda_q \ell_q^{-N} \quad \forall N \geq 0, \tag{4.2}
\]

\[
\|\hat{R}_{\ell_q}\|_{N+\alpha} \lesssim \delta_q \ell_q^{-N+\alpha} \quad \forall N \geq 0, \tag{4.3}
\]

\[
\left| \int_{\mathbb{T}^3} |v_q|^2 - |v_{\ell_q}|^2 \, dx \right| \lesssim \delta_q \ell_q^\alpha \quad \forall t \in [0, T]. \tag{4.4}
\]

**Proof.** See [Buc+19, Proposition 2.2]. \qed

4.2 Classical Exact flows

We define $\tau_q \in (0, \ell_q^\alpha)$ and the sequence of initial times $t_i$ ($i \in \mathbb{Z}_{\geq 0}$) by

$$\tau_q := \frac{\ell_q^{2\alpha}}{\delta_q^{1/2} \lambda_q}, \quad t_i := i \tau_q. \tag{4.5}$$

Note $\|v_{\ell_q}\|_1 \tau_q \lesssim \ell_q^{2\alpha}$, so $\|v_{\ell_q}\|_1 \leq \tau_q^{-1}$. That is, $\tau_q$ is chosen such that

$$\delta_q^{1/2} \tau_q^{\ell_q^{-1}} = \ell_q^{2\alpha} \lambda_q^{3\alpha/2} \leq \lambda_q^{-\alpha/2} \leq 1.$$
We define \((v_i, p_i)\) for \(i \geq 1\) to be the unique (exact) solutions to the Euler equations, defined on some interval around \(t_i\), with initial data \(v_i \mid_{t = t_i} = v_{\ell q}(t_i)\). That is,

\[
\begin{aligned}
\partial_t v_i + (v_i \cdot \nabla) v_i + \nabla p_i &= 0, \\
\nabla \cdot v_i &= 0, \\
v_i \mid_{t = t_i} &= v_{\ell q}(\cdot, t_i)
\end{aligned}
\]

For \(i = 0\), we define \((v_0, p_0)\) to be the classical solution to the Euler equations starting from \(v_{\text{in}} \ast \psi_{\ell q}\):

\[
\begin{aligned}
\partial_t v_0 + (v_0 \cdot \nabla) v_0 + \nabla p_0 &= 0, \\
\nabla \cdot v_0 &= 0, \\
v_0 \mid_{t = 0} &= v_{\text{in}} \ast \psi_{\ell q}
\end{aligned}
\]

We note that there is a mismatch of initial data: \(v_0 \mid_{t = 0} \neq v_{\ell q} \mid_{t = 0} = v_{\text{in}} \ast \psi_{\ell q} \ast \psi_{\ell q-1}\). This difference needs to be estimated below, and relies on the parameter inequality \(\beta < \bar{\beta}\).

**Proposition 4.2** (Estimates for classical exact solutions to Euler [Buc+19, Prop 3.1]). Let \(\bar{N} \geq 1\) and \(\bar{\alpha} \in (0, 1)\). The unique \(C^{\bar{N} + \bar{\alpha}}\) solution \(V\) to (1.1) with initial data \(V_0 \in C^{\bar{N} + \bar{\alpha}}\) is defined at least for \(t \in [-\bar{T}, \bar{T}]\), where \(\bar{T} = \frac{c}{||V_0||_{1+\bar{\alpha}}}\) for some universal \(c > 0\), and satisfies the following derivative estimates for \(0 \leq N \leq \bar{N}\),

\[
\|V\|_{C^0([-\bar{T}, \bar{T}]; C^{N+\alpha})} \lesssim \|V_0\|_{N+\alpha}.
\]

Note that the definition of \(\tau_q\) was chosen so that \(v_i\) is well-defined on \([t_{i-1}, t_{i+1}]\), once \(a > 1\) is sufficiently large.

**Proposition 4.3** (Stability). Let \(D_{t,v} := \partial_t + v \cdot \nabla\) denote the material derivative. Suppose \(\beta, \alpha\) satisfy the constraints (4.12), (4.13) below. For \(i \geq 0\), \(|t - t_i| \lesssim \tau_q\), and \(N \geq 0\), we have

\[
\begin{aligned}
\|v_i - v_{\ell q}\|_{N+\alpha} &\lesssim \tau_q \delta_q + \epsilon_q^{N-1+\alpha}, \\
\|\nabla p_i - \nabla p_{\ell q}\|_{N+\alpha} &\lesssim \delta_q + \epsilon_q^{N-1+\alpha}, \\
\|D_{t,v_{\ell q}}(v_i - v_{\ell q})\|_{N+\alpha} &\lesssim \delta_q + \epsilon_q^{N-1+\alpha}.
\end{aligned}
\]

**Proof.** The proof is similar to that of [Buc+19, Prop. 3.3], but we need to treat \(v_0\) separately. The equation for the difference is

\[
\partial_t(v_i - v_{\ell q}) + v_{\ell q} \cdot \nabla(v_i - v_{\ell q}) + (v_i - v_{\ell q}) \cdot \nabla v_i + \nabla(p_i - p_{\ell q}) = - \text{div} \hat{R}_{\ell q}.
\]
Taking the divergence of the above equation, we obtain a Poisson equation for \( p_i - p_{\ell q} \), leading to the estimates \((N \geq 0, i \geq 1)\)

\[
\| \nabla p_i - \nabla p_{\ell q} \|_{N+\alpha}
\]

\[
= \| \Delta^{-1} \nabla \left[ v_{\ell q} \cdot \nabla (v_i - v_{\ell q}) + (v_i - v_{\ell q}) \cdot \nabla v_i + \text{div} \tilde{R}_{\ell q} \right] \|_{N+\alpha}
\]

\[
\lesssim \| \nabla v_i \|_{N+\alpha} \| v_i - v_{\ell q} \|_\alpha + \| \nabla v_{\ell q} \|_\alpha \| v_i - v_{\ell q} \|_{N+\alpha} + \| \tilde{R}_{\ell q} \|_{N+\alpha}
\]

\[
\lesssim \delta_q^{1/2} \lambda_q \ell_q^{-N-\alpha} \| v_i - v_{\ell q} \|_\alpha + \delta_q^{1/2} \lambda_q \ell_q^{-\alpha} \| v_i - v_{\ell q} \|_{N+\alpha} + \delta_{q+1} \ell_q^{N-1+\alpha},
\]

(4.10) where we have used (4.2), (4.3), Schauder estimates (B.1), the identity \( \nabla \text{div}(V \cdot \nabla) = \nabla \text{div}((W \cdot \nabla)V) \) to move the gradient off of \( v_i - v_{\ell q} \), and the estimate \( \| v_i \|_{N+\alpha} \leq \| v_{\ell q} \|_{N+\alpha} \) from Proposition 4.2.

For \( i = 0 \), Proposition 4.2 instead gives \( \| v_0 \|_{N+\alpha} \leq \| v_{\ell q} \|_{N+\alpha} \) since this implies \( \| v_{\ell q} \|_{N+\alpha} \leq \| v_{\ell q} \|_{N+\alpha} \) from Proposition 4.2.

In order to have (4.10) for \( i = 0 \), we therefore require

\[
\ell_q^{-N-1-\alpha+\beta} \lesssim \delta_q^{1/2} \lambda_q \ell_q^{-N-\alpha} \iff \ell_q^{\frac{\beta-\alpha}{\beta}} \lesssim (\lambda_q \ell_q)^{1-\beta}.
\]

(4.11) For this, as \( \lambda_q \ell_q < 1 \), it suffices to make \( \lambda_q^{-\frac{\beta-\alpha}{\beta}} \lesssim \lambda_q \ell_q = \delta_q^{1/2} \delta_q^{-1/2} \lambda_q^{-3\alpha} \), and this is true if we impose

\[
b - 1 \leq \min \left( 1, \frac{\beta - \beta}{4\beta} \right), \quad \text{and} \quad 3\alpha < \beta(b - 1),
\]

(4.12) since this implies

\[
\lambda_q^{-\frac{\beta-\alpha}{\beta}} \lesssim \lambda_q^{-4\beta(b-1)} \ll \lambda_q^{-\beta(b-1)-3\alpha} \lesssim 2\delta_q^{1/2} \delta_q^{-1/2} \lambda_q^{-3\alpha}.
\]

Hence, from (4.9) we obtain for \( N = 0 \), using Proposition 4.1

\[
\| D_{t,v_{\ell q}} (v_i - v_{\ell q}) \|_\alpha \lesssim \delta_q^{1/2} \lambda_q \ell_q^{-\alpha} \| v_i - v_{\ell q} \|_\alpha + \| \tilde{R}_{\ell q} \|_{1+\alpha}
\]

\[
\lesssim \tau_q^{-1} \| v_i - v_{\ell q} \|_\alpha + \ell_q^{-1+\alpha} \delta_{q+1}. \]

From standard estimates [Buc+15, Appendix D] for the transport equation, it follows that

\[
\| v_{\ell q} - v_i \|_\alpha (t) \lesssim \| v_{\ell q} - v_i \|_\alpha (t_i) + \tau_q \delta_{q+1} \ell_q^{-1+\alpha} + \int_{t_i}^t \tau_q^{-1} \| v_{\ell q} - v_i \|_\alpha (s) \, ds.
\]

13
For $i \geq 1$, $v_{t_i}(t_i) = v_i(t_i)$. For $i = 0$, it suffices to control the initial data by $\tau_q \delta_{q+1} \ell_{q-1}^{-1+\alpha}$. By (A.2) and $\|v^{in}\|_{\beta} \leq 1$, we have
\[
\|
u_{t_i} - v_0\|_{\alpha} \big|_{t=0} = \|v^{in} \ast \psi_{t_i} - v^{in} \ast \psi_{t_i} \ast \psi_{t_i-1}\|_{\alpha} \lesssim \|v^{in}\|_{\beta} \ell_{q-1}^{-\alpha},
\]
and we need $\ell_{q-1}^{-\alpha} \leq \tau_q \delta_{q+1} \ell_{q-1}^{-1+\alpha}$, i.e. $\ell_{q-1}^{-\alpha} \leq \lambda_q^{-\beta} \ell_{q-1}^{\beta} \lambda_q^{\alpha/2} \ell_{q-1}^{\beta}$. It suffices to take $b-1 \leq \min(1, \frac{\beta-\beta}{3b})$ as in (4.12), and
\[
8\alpha < \frac{\beta(b-1)}{b},
\]
so that $-\bar{\beta} \leq -4\beta(b-1) - \beta$, and hence (since $b < 2$ and $\lambda_q^{-2} \leq \ell_{q}$)
\[
\lambda_{q-1}^{-\beta} \leq \lambda_{q-1}^{-\beta} \lambda_q^{-b(b-1)/b} \leq \lambda_{q-1}^{-\beta} \lambda_q^{-8\alpha} \leq \lambda_{q-1}^{-\beta} \ell_{q}^{8\alpha} \ll \lambda_{q+1}^{-\beta} \lambda_q^{\alpha/2} \ell_{q-1}^{\beta},
\]
which implies the wanted estimate. Hence by Grönwall’s inequality,
\[
\|v_i - v_{t_i}\|_{\alpha} \leq \tau_q \delta_{q+1} \ell_{q}^{-1+\alpha} e^{\tau_q^{-1}(t-t_i)} \lesssim \tau_q \delta_{q+1} \ell_{q}^{-1+\alpha},
\]
which is (4.6) for $N = 0$ and all $i \geq 0$. The estimates for $N > 0$ are similar: Let $\sigma$ be a multiindex with $|\sigma| = N$. Then we write
\[
D_{t,v_{t_i}} D^{\sigma}(v_i - v_{t_i}) = [D_{t,v_{t_i}}, D^{\sigma}](v_i - v_{t_i}) + D^{\sigma} D_{t,v_{t_i}}(v_i - v_{t_i}).
\]
The commutator term cancels the highest derivative on $v_i - v_{t_i}$:
\[
\|[D_{t,v_{t_i}}, D^{\sigma}](v_i - v_{t_i})\|_{\alpha} \lesssim \|v_{t_i}\|_{N+\alpha} \|v_i - v_{t_i}\|_{1+\alpha} + \|v_{t_i}\|_{1+\alpha} \|v_i - v_{t_i}\|_{N+\alpha} \lesssim \|v_{t_i}\|_{N+\alpha} \|v_i - v_{t_i}\|_{1+\alpha} + \|v_{t_i}\|_{1+\alpha} \|v_i - v_{t_i}\|_{N+\alpha} \lesssim \delta_q^{1/2} \lambda_q^{N-\alpha} \cdot \tau_q \delta_{q+1} \ell_{q}^{-1+\alpha} + \delta_q^{1/2} \lambda_q^{N-\alpha} \|v_i - v_{t_i}\|_{N+\alpha}.
\]
(We used Hölder interpolation in the second inequality.) For the other term $D^{\sigma} D_{t,v_{t_i}}(v_i - v_{t_i})$, we use the equation (4.9) to write
\[
\|D^{\sigma} D_{t,v_{t_i}}(v_i - v_{t_i})\|_{\alpha} = \|D^{\sigma}(\text{div} \hat{R}_{t_i} + \nabla(p_i - p_{t_i}) - (v_i - v_{t_i}) \cdot \nabla v_i)\|_{\alpha}.
\]
With the estimates (4.3), (4.10), (4.11) and the estimate
\[
\|D^{\sigma}(v_i - v_{t_i}) \cdot \nabla v_i\|_{\alpha} \lesssim \|v_i - v_{t_i}\|_{\alpha} \|v_i\|_{N+1+\alpha} + \|v_i - v_{t_i}\|_{N+\alpha} \|v_i\|_{1+\alpha} \lesssim \tau_q \delta_{q+1} \ell_{q}^{-1+\alpha} \cdot \delta_q^{1/2} \lambda_q^{N-\alpha} + \delta_q^{1/2} \lambda_q^{\alpha} \|v_i - v_{t_i}\|_{N+\alpha},
\]
14
we obtain the bound
\[ \|D_{t,v_{\ell q}} D^\sigma (v_i - v_{\ell q})\| \leq \tau_q \delta_q + \delta_q 1/2 + \lambda_q \ell_q^{-1-N} + \delta_q 1/2 + \lambda_q \ell_q^{-\alpha} \|v_i - v_{\ell q}\|_{N+\alpha} + \delta_q 1/2 + \lambda_q \ell_q^{-N-1+\alpha} \]
\[ \leq \delta_q + \lambda_q \ell_q^{-N-1+\alpha} + \tau_q^{-1} \|v_i - v_{\ell q}\|_{N+\alpha}. \]

By estimates for the transport equation again and summing over \(\sigma\), then applying Grönwall’s inequality, we obtain
\[ \|v_i - v_{\ell q}\|_{N+\alpha} \leq \tau_q \delta_q + \lambda_q \ell_q^{-N-1+\alpha}, \]
which gives (4.6) and (4.8) for \(N > 0\), and hence (4.7) for all \(N > 0\) and \(i \geq 0\). \hfill \(\square\)

### 4.3 Estimates for Vector potentials

By the well-known Helmholtz decomposition for smooth functions on the torus, a divergence-free field \(V\) of zero mean is in fact a curl, i.e. \(V = \nabla \times Z\) for an incompressible field \(Z =: BV\) called the vector potential of \(V\). The operator \(B = (-\Delta)^{-1}\) curl is the ‘Biot–Savart operator’. Here we note some estimates for the vector potential of \(v_i\). These are used in estimating the stress error \(\hat{R}_q\) that arises from gluing different Euler flows together: see Proposition 4.6.

**Proposition 4.4.** Define \(z_i := BV_i\). Let \(\beta\) and \(\alpha\) satisfy the constraints (4.17) below. For \(|t - t_i| \leq \tau_q\),
\[ \|z_i - z_{i+1}\|_{N+\alpha} \leq \tau_q \delta_q + \lambda_q \ell_q^{-N-1+\alpha}, \]
\[ \|D_{t,v_{\ell q}} (z_i - z_{i+1})\|_{N+\alpha} \leq \delta_q + \lambda_q \ell_q^{-N-1+\alpha}. \]

**Proof.** For \(N \geq 1\), (4.14) directly follows from the analogous estimates (4.6) for \(v_i\), and the boundedness of the zero-order operator \(\nabla B\) on Hölder spaces. For \(N = 0\), we first prove the intermediate estimate (4.16) that will also lead to (4.15). One can check that (4.9) can be written in terms of \(\tilde{z}_i = B(v_i - v_{\ell q})\) as
\[ \text{curl}(\partial_t \tilde{z}_i + v_{\ell q} \cdot \nabla \tilde{z}_i) = -\partial_x^k \left( (\tilde{z}_i \times \nabla)^j v_{\ell q}^j + (\tilde{z}_i \times \nabla)^j v_i^j - \nabla(p_i - p_{\ell q}) - \div \hat{R}_q, \right) \]

1. A minor difference from the iteration scheme in [Buc+19] is that we propagate the zero mean condition, and hence that \(Bv_i\) is well-defined.

2. Here, \((a \times \nabla)^j b^k := \varepsilon_{jkl} a^l \partial_x^k b^j\). The identities we need follow from \(\nabla \cdot \tilde{z}_i = 0\). They are \(v_{\ell q} \cdot \nabla (v_i - v_{\ell q}) = \text{curl}((v_{\ell q} \cdot \nabla) \tilde{z}_i) + \sum_{j,k=1}^3 \partial_x^k \left( (\tilde{z}_i \times \nabla)^j v_{\ell q}^j \right) + ((v_i - v_{\ell q}) \cdot \nabla)v_i = \sum_{j,k=1}^3 \partial_x^k \left( (\tilde{z}_i \times \nabla)^k v_i^j \right).\)
where there is an implicit sum over \( j, k = 1, 2, 3 \). On taking the curl and using the identity \( \text{curl curl} = -\Delta + \nabla \text{div} \), we find that \( D_{t,v_i} \tilde{z}_i \) solves the Poisson equation

\[
-\Delta (\partial_t \tilde{z}_i + (v_i \cdot \nabla) \tilde{z}_i) = -\nabla \text{div}(\tilde{z}_i \cdot \nabla) v_i - \text{curl} \partial_{xk} \left( (\tilde{z}_i \times \nabla)^j v_i^k + (\tilde{z}_i \times \nabla)^k v_i^j \right) - \text{curl div} \hat{R}_{t,q}.
\]

Schauder estimates for the Poisson equation and the boundedness of singular integrals on Hölder spaces therefore give

\[
\|D_{t,v_i} \tilde{z}_i\|_{N+\alpha} \lesssim \|(\tilde{z}_i \cdot \nabla)v_i\|_{N+\alpha} + \sum_{j,k=1}^3 \|(\tilde{z}_i \times \nabla)^j v_i^k + (\tilde{z}_i \times \nabla)^k v_i^j\|_{N+\alpha} + \|\hat{R}_{t,q}\|_{N+\alpha}
\]

\[
\lesssim \left( \|v_i\|_{N+1+\alpha} + \|v_i\|_{N+1+\alpha} \right) \|\tilde{z}_i\|_{\alpha} + \left( \|v_i\|_{1+\alpha} + \|v_i\|_{1+\alpha} \right) \|\tilde{z}_i\|_{N+\alpha} + \delta_{q+1} \ell_q^{-N+\alpha}.
\]

(4.16)

In the case \( i \neq 0 \), we can now easily prove both inequalities for all \( N \). Then the \( N = 0 \) case of (4.16) and standard estimates on the transport equation (since \( \tilde{z}_i|_{t=0} = 0 \)) give

\[
\|	ilde{z}_i\|_{\alpha} \lesssim \delta_{q+1}^{1/2} \lambda_q \ell_q^{-\alpha} \int_{t_i}^t \|	ilde{z}_i\|_{\alpha} \, ds + \delta_{q+1} \ell_q \|t - t_i\| \lesssim \tau_q^{-1} \int_{t_i}^t \|	ilde{z}_i\|_{\alpha} \, ds + \delta_{q+1} \ell_q \tau_q,
\]

and hence Grönwall’s inequality proves (4.14) for \( N = 0 \). Then (4.15) follows from (4.16) for all \( N \geq 0 \).

For \( i = 0 = N \), the initial data for \( \tilde{z}_0 \) is

\[
\tilde{z}_0|_{t=0} = B(v^i \ast \psi_{t_q} - v^i \ast \psi_{t_{q-1}}) \ast \psi_{t_q} - (Bv^i) \ast \psi_{t_{q-1}} \ast \psi_{t_q}.
\]

For this we use (A.3) to find

\[
\|Bv^i \ast \psi_{t_q} - (Bv^i) \ast \psi_{t_{q-1}} \ast \psi_{t_q}\|_{\alpha} \leq [\nabla Bv^i \ast \psi_{t_q}]_{\beta} \ell_q^{1+\beta-\alpha} \lesssim \ell_q^{1+\beta-\alpha}.
\]

To match with the estimate for \( i > 0 \), we require

\[
\ell_q^{1+\beta-\alpha} \leq \delta_{q+1} \ell_q^\alpha \tau_q \iff \ell_q^{\beta} \leq \delta_{q+1}^{1/2} \ell_q^{3\alpha/2} \lambda_q \ell_q \tau_q \iff \ell_q^{1+\beta-\alpha} \leq \ell_q^{1+\beta-\alpha}.
\]

Since \( \ell_q+1 = \ell_q^b \) up to universal multiplicative constants, \( \lambda_q^{-2(b-1)} \leq \ell_q^{b-1} \lesssim \frac{\ell_q}{\ell_{q+1}} \) and it suffices to assume that

\[
b - 1 < \min \left( 1, \frac{\beta - \beta}{4\beta + 3} \right) \text{ and } 8\alpha \leq \frac{b - 1}{b},
\]

(4.17)
so that
\[ \lambda_{q-1}^{-\beta} \leq \lambda_{q-1}^{-\beta - (2(b-1)\beta - 3(b-1))} \ll \delta_{q+1}^{1/2} \lambda_{q-1}^{-2(b-1)/b} \leq \delta_{q+1}^{1/2} \ell_q \lambda_{q-1}^{-8\alpha}. \]

This is enough because \( \lambda_q^{-8\alpha} \leq \ell_q^4 \ll \ell_q^3 \lambda_q^{-3\alpha/2} \), giving the required inequality for \( a \gg 1 \). This means that we can bound as before,

\[
\| \tilde{z}_0 \|_a \lesssim \| B v^{in} - (B v^{in}) \ast \psi_{t_q} \|_a + \tau_q^{-1} \int_0^t \| \tilde{z}_0 \|_a \, ds + \delta_q \ell_q \tau_q \lesssim \tau_q^{-1} \int_0^t \| \tilde{z}_0 \|_a \, ds + \delta_q \ell_q \tau_q.
\]

Using Grönwall again gives (4.14), and (4.15) again follows from (4.16) for all \( N \geq 0 \).

4.3.1 Gluing exact flows

This subsection is taken from [Buc+19, Section 4], so we omit some routine calculations and proofs: despite our modifications, we have shown that the same estimates hold for \( v_i, v_q, \) and \( v_{\ell_q} \) as in that paper, so the proofs for the results here need no modification.

Define the intervals \( I_i, J_i \) \((i \geq -1)\) by \( I_i := [t_i + \frac{\tau_q}{3}, t_i + \frac{2\tau_q}{3}] \), and \( J_i := (t_i - \frac{\tau_q}{3}, t_i + \frac{\tau_q}{3}) \). They partition \( \mathbb{R} \). Define \( i_{\text{max}} \) to be the smallest number so that \([0, T] \subseteq J_0 \cup I_0 \cup J_1 \cup I_1 \cup \cdots \cup J_{i_{\text{max}}} \cup I_{i_{\text{max}}} \), i.e.

\[
i_{\text{max}} := \sup \{ i \geq 0 : (J_i \cup I_i) \cap [0, T] \neq \emptyset \} \leq \left\lfloor \frac{T}{\tau_q} \right\rfloor.
\]

Also let \( \{ \chi_i \}_{i=0}^{i_{\text{max}}} \) be a partition of unity such that for all \( N \geq 0 \),

\[
\text{supp } \chi_i = I_{i-1} \cup J_i \cup I_i, \quad \chi_i|_{J_i} = 1, \quad \| \partial_t^N \chi_i \|_{C^0_p} \lesssim \tau_q^{-N}. \tag{4.18}
\]

In particular, for \( |i - j| \geq 2 \), \( \text{supp } \chi_i \cap \text{supp } \chi_j = \emptyset \). By a slight abuse of notation, we write \( \chi_i v_i := 0 \) for \( t \) outside the support of \( \chi_i \) even when \( v_i \) is not defined, and similarly for other functions. Then we define the glued velocity and pressure \((\bar{v}_q, \bar{p}_q)\) by

\[
\bar{v}_q(x, t) := \sum_{i=0}^{i_{\text{max}}} \chi_i(t) v_i(x, t),
\]

\[
\bar{p}_q(x, t) := \sum_{i=0}^{i_{\text{max}}} \chi_i(t) p_i(x, t).
\]
The definition (4.18) of $\chi_i$ implies that $\bar{v}_q$ is still divergence-free, and

$$
t \in I_i \implies \bar{v}_q(x, t) = v_i(x, t)\chi_i(t) + v_{i+1}(x, t)\chi_{i+1}(t),
$$
$$
t \in J_i \implies \bar{v}_q(x, t) = v_i(x, t).
$$

Hence, $\bar{v}_q$ is an exact Euler flow for the times $t \in J_i$. For $t \in I_i$, we define

$$
\bar{R}_q := \partial_t \chi_i \mathcal{R}(v_i - v_{i+1}) - \chi_i(1 - \chi_i)(v_i - v_{i+1}) \otimes (v_i - v_{i+1}),
$$
$$
\bar{p}_q := \bar{p}_q - \chi_i(1 - \chi_i)(|v_i - v_{i+1}|^2 - \int_{\mathbb{T}^3} |v_i - v_{i+1}|^2 \, dx),
$$

where we have used the inverse divergence operator $\mathcal{R}$ from Section B. It is easy to check that $v_i - v_{i+1}$ has mean zero (so that $\bar{R}_q$ is well-defined), and furthermore, $\bar{p}_q$ has mean zero, while $\bar{R}_q$ is symmetric and trace-free. A routine computation shows that the functions $(\bar{v}_q, \bar{p}_q, \bar{R}_q)$ solve:

$$
\begin{cases}
\partial_t \bar{v}_q + \text{div}(\bar{v}_q \otimes \bar{v}_q) + \nabla \bar{p}_q = \text{div} \bar{R}_q, \\
\nabla \cdot \bar{v}_q = 0, \\
\bar{v}_q|_{t=0} = v^\text{in} * \psi_{\ell_q}.
\end{cases}
$$

(4.19)

**Proposition 4.5** (Estimates for $\bar{v}_q$). For all $N \geq 0$,

$$
\|\bar{v}_q - v_{\ell_q}\|_\alpha \lesssim \delta_{q+1}^{1/2} \ell_q^\alpha,
$$

(4.20)

$$
\|\bar{v}_q - v_{\ell_q}\|_{N+\alpha} \lesssim \tau_q \delta_{q+1} \ell_{q-1}^{-N-\alpha},
$$

(4.21)

$$
\|\bar{v}_q\|_{1+N} \lesssim \delta_{q}^{1/2} \lambda_q \ell_{q}^{-N},
$$

(4.22)

**Proof.** Since the gluing functions $\chi_i$ do not depend on the space variable, (4.6) immediately implies (4.21). The estimate (4.20) follows because $\delta_{q}^{1/2} \tau_q \ell_{q}^{-1} \leq 1$, and (4.22) follows by the simple bound $\|\bar{v}_q\|_{1+N} \leq \|v_{\ell_q}\|_{1+N} + \|v_{\ell_q} - \bar{v}_q\|_{1+N+\alpha}$ since $\ell_{q}^{-1} \leq \lambda_q$. \hfill $\square$

**Proposition 4.6** (Estimates for $\hat{R}_q$). For all $N \geq 0$,

$$
\|\hat{R}_q\|_{N+\alpha} \lesssim \delta_{q+1} \ell_{q}^{-N-\alpha},
$$

(4.23)

$$
\|D_t \hat{v}_q \hat{R}_q\|_{N+\alpha} \lesssim \delta_{q+1} \delta_{q}^{1/2} \lambda_q \ell_{q}^{-N-\alpha}.
$$

(4.24)

**Proof.** See [Buc+19, Proposition 4.4]. \hfill $\square$

**Proposition 4.7** (Energy of $\bar{v}_q$). For all $t \in [0, T]$, $\left| \int_{\mathbb{T}^3} \bar{v}_q^2 - |v_{\ell_q}|^2 \, dx \right| \lesssim \delta_{q+1} \ell_{q}^{\alpha}$, and hence, for $a \gg 1$, and for $t \in [1 - \tau_{q-1}, T]$, $\delta_{q+1} \frac{a}{\lambda_{q}} \leq e(t) - \int_{\mathbb{T}^3} |\bar{v}_q|^2 \, dx \leq 2 \delta_{q+1}$.

**Proof.** See [Buc+19, Proposition 4.5]. \hfill $\square$
4.4 Definition of velocity increment $w_{q+1}$

4.4.1 Space-time cutoffs

Define the index

$$i_q := \left\lfloor \frac{1}{\tau_q} \right\rfloor - 2 \in \mathbb{Z}_{\geq 0}.$$ 

We will define the cutoffs $\eta_i$ by

$$\eta_i(x, t) := \begin{cases} \bar{\eta}_i(t) & 0 \leq i < i_q, \\ \tilde{\eta}_i(x, t) & i \geq i_q, \end{cases}$$

(4.25)

where $\bar{\eta}_i$ are ‘squiggling space-time cutoffs’ we will define shortly, and $\tilde{\eta}_i$ are ‘straight cutoffs’ defined as follows: let $\bar{\eta}_0 \in C_c^\infty(J_0 \cup I_0 \cup J_1; [0, 1])$ satisfy

$$\text{supp } \bar{\eta}_0 = I_0 + \left[\frac{-\tau_q}{6}, \frac{\tau_q}{6}\right] = \left[\frac{\tau_q}{3} - \frac{\tau_q}{6}, \frac{2\tau_q}{3} + \frac{\tau_q}{6}\right] \supset I_0,$$

be identically 1 on $I_0$, and satisfy the derivative estimates for $N \geq 0$:

$$\|\partial^N_t \bar{\eta}_0\|_{C_0^\infty} \lesssim \tau_q^{-N}.$$ 

Then we set $\tilde{\eta}_i(t) := \bar{\eta}_0(t - t_i)$ for $0 \leq i \leq i_q$. Next, we define $\tilde{\eta}_i$.

4.4.2 Squiggling space-time cutoffs

These cutoffs are adapted from [Buc+19, Section 5.2]. Let $\varepsilon \in (0, \frac{1}{3})$, $\varepsilon_0 \ll 1$ and define for $i_q \leq i \leq i_{\text{max}}$ using the mollifiers (2.1), (2.2),

$$I'_{i} := I_i + \left[\frac{- (1 - \varepsilon) \tau_q}{3}, \frac{1 - \varepsilon \tau_q}{3}\right] = \left[i\tau_q + \frac{\varepsilon \tau_q}{3}, i\tau_q + \frac{(3 - \varepsilon) \tau_q}{3}\right],$$

$$I''_{i} := \left\{(x, t + \frac{2\varepsilon \tau_q}{3} \sin(2\pi x_1)) : x \in \mathbb{T}^3, t \in I'_{i}\right\} \subset \mathbb{T}^3 \times \mathbb{R},$$

$$\tilde{\eta}_i(x, t) := \mathbb{1}_{I''_{i}}\ast x \psi_{\varepsilon_0} \ast t \phi_{\varepsilon_0 \tau_q} = \frac{1}{\varepsilon_0 \varepsilon_0 \tau_q} \int_{I''_{i}} \psi_{\varepsilon_0}(x - y) \phi_{\varepsilon_0 \tau_q}(t - s) \, dy \, ds.$$ 

Lemma 4.8. The functions $\eta_i$ satisfy for all $i = 0, 1, \ldots, i_{\text{max}}$:

1. $\eta_i \in C_c^\infty(\mathbb{T}^3 \times (J_i \cup I_i \cup J_{i+1}); [0, 1])$, with the estimates for $n, m \geq 0$:

$$\|\partial^N_t \eta_i\|_{L^\infty C_x^m} \lesssim_{n, m} \tau_q^{-n}.$$ 

2. $\eta_i(\cdot, t) \equiv 1$ for $t \in I_i$. 

19
Figure 1: The set $I''_i$ (light blue). To ensure that $I_i \times \mathbb{T}^3$ (blue) is a subset of $I''_i$, we require the leftmost point of the right boundary, $p = (t_{i+1} - \frac{2\tau_q}{3}, \frac{3}{4})$ to be outside $I_i \times \mathbb{T}^3$: this is ensured by $\varepsilon < 1/3$. The triangle (red) is a visual proof that any vertical slice of $I''_i$ for $t \in [t_i, t_{i+1}]$ has length at least $1/4$. Point 4 of Lemma 4.8 then follows by taking $\varepsilon_0 \ll 1$.

3. supp $\eta_i$ are pairwise disjoint.

4. For all $t \in [t_i, T]$, $c_\eta \leq \sum_{i=i_q}^{i_{i_q}} \int_{\mathbb{T}^3} \eta_i^2(x, t) \, dx \leq 1$ for a constant $c_\eta$ independent of $q$ (one can take $c_\eta = 1/5$).

5. For all $0 \leq i < i_q$, $\eta_i$ does not depend on $x$, and is supported in time on the $\frac{\tau_q}{6}$-neighbourhood of $I_i$.

4.4.3 Energy gap decomposition

Let $\zeta$ be a smooth cutoff function such that

\begin{align*}
t \leq t_{i_q} & \implies \zeta \equiv 1, \\
t \geq t_{i_q} + \frac{\tau_q}{3} & \implies \zeta \equiv 0, \\
t \in (t_{i_q}, t_{i_q} + \frac{\tau_q}{3}) & \implies |\partial_t N \zeta| \lesssim N \tau_q^{-N}.
\end{align*}

Observe that $t_{i_q} = \tau_q \lfloor \tau_q^{-1} \rfloor - 2\tau_q \in (1 - 3\tau_q, 1 - 2\tau_q)$. This function is needed for because we do not have the estimate (3.10) for small times. For $a \gg 1$, $1 - 3\tau_q > 1 - \tau_q - 1$, so we define the energy gap $\rho_q$ as

$$
\rho_q(t) := \frac{\delta_{q+1}}{2} \zeta(t) + \frac{1 - \zeta(t)}{3} \left( e(t) - \frac{\delta_{q+2}}{2} - \int_{\mathbb{T}^3} |\tilde{v}_q|^2 \, dx \right).
$$
Note that
\[ t \in [t_{q+1}, T] \implies \rho_q(t) = \frac{1}{3} \left( e(t) - \frac{\delta_{q+2}}{2} - \int_{\mathbb{T}^3} |\bar{v}_q(x, t)|^2 \, dx \right). \] (4.26)

We next use the cutoff functions \( \zeta \) and \( \eta \) to split \( \rho_q \) into functions \( \rho_{q,i} \) supported on disjoint regions of space-time. Note \( t_{q+1} = \tau_q([\tau_q^{-1}] - 1) \leq 1 - \tau_q \), so that \( t \geq 1 - \tau_q \) implies \( \zeta(t) = 0 \). We decompose \( \rho_q \) by setting \( (i = 0, 1, \ldots, i_{\text{max}}) \)
\[ \rho_{q,i}(x, t) := \frac{\eta_i^2(x, t)}{\zeta(t) + \sum_{i = i_q}^{i_{\text{max}}} \eta_i^2(\tilde{x}, t) \, d\tilde{x}} \rho_q(t). \] (4.27)

Note that for \( t \in I_i, i \geq 0, \)
\[ \zeta(t) + \sum_{i = i_q}^{i_{\text{max}}} \int_{\mathbb{T}^3} \eta_i^2(\tilde{x}, t) \, d\tilde{x} \equiv 1, \]
and for \( t \in J_i \) there is the strictly positive lower bound
\[ \zeta(t) + \sum_{i = i_q}^{i_{\text{max}}} \int_{\mathbb{T}^3} \eta_i^2(\tilde{x}, t) \, d\tilde{x} \geq c_\eta. \]

In particular, \( \rho_{q,i} \) is well-defined even for times \( t \) where \( \sum_{i = i_q}^{i_{\text{max}}} \int_{\mathbb{T}^3} \eta_i^2(\tilde{x}, t) \, d\tilde{x} = 0 \). By construction, \( \rho_q = \int_{\mathbb{T}^3} \sum_{i = 0}^{i_{\text{max}}} \rho_{q,i} \, dx \) for times \( t \geq 1 - \tau_q \).

**Proposition 4.9 (Estimates for \( \rho_q \) and \( \rho_{q,i} \)).** For all \( t \in [0, T], \)
\[ \frac{\delta_{q+1}}{8\lambda_q^a} \leq \rho_q(t) \leq \delta_{q+1}, \] (4.28)
\[ \|\rho_{q,i}\|_0 \leq \delta_{q+1}/c_\eta, \] (4.29)
\[ \|\rho_{q,i}\|_N \lesssim \delta_{q+1}, \] (4.30)
\[ \|\rho_{q,i}^{1/2}\|_N \lesssim \delta_{q+1}^{1/2}, \] (4.31)
\[ \sup_{t \in [0, T]} |\partial_t \rho_q(t)| \lesssim \delta_{q+1}\tau_q^{-1}, \] (4.32)
\[ \sup_{t \in \cup_{i \geq 0} I_i} |\partial_t \rho_q(t)| \lesssim \delta_{q+1}\tau_q^{1/2}\lambda_q, \] (4.33)
\[ \|\partial_t \rho_{q,i}\|_N \lesssim \delta_{q+1}\tau_q^{-1}. \] (4.34)

**Proof.** For \( t \geq 1 - \tau_q, (4.28) \) follows from because of \( 4\delta_{q+2} \leq \delta_{q+1} \), which holds for \( a \gg 1 \). Then, \( (4.29) \) follows from point 4 of Lemma 4.8 and the
Figure 2: At time $t_{i_q}$, we switch from using straight cutoffs $\eta_i$ to the squiggling cutoffs $\tilde{\eta_i}$. We give ourselves a little time to ‘turn off’ $\zeta$ since we no longer need it to ensure $\rho_{i,q}$ is well-defined. At time $t_{i_q+1} = \tau_q(\lfloor \tau_q^{-1} \rfloor - 1) < 1$, we begin controlling the energy.

definition of $\zeta$. The implicit constant in (4.30) depends only on $\varepsilon_0$ which is a fixed universal constant, and not on $\tau_q$ due to the definition of $\tilde{\eta_i}$. For (4.31) we emphasise that this norm is $L^\infty$ in time and $C^N$ in space. The remaining estimates follow from the definition (4.5) of $\tau_q$.

For $t < 1 - \tau_q$, all of the estimates are even more straightforward. We stress that (4.29) still holds, precisely because we added the term $\zeta$ in the denominator.

We remark that the analogous estimate of (4.32) is worse than (4.33) (which is [Buc+19, (5.14)]) by a small $\alpha$ power. This estimate is only used to show (4.42). We kept (4.42) the same as the corresponding estimate [Buc+19, (5.38)] for the convenience of the reader.

4.4.4 Inverse flow map

The inverse flow (also called ‘back-to-labels’) maps $\Phi_i$ are defined as the solutions to the vector transport equation

$$$(\partial_t + \vec{v}_q \cdot \nabla) \Phi_i = 0, \quad \Phi_i|_{t=t_i} = x.$$$$ (4.35)
For each fixed \( t \), \( \Phi_i \) is the inverse mapping for the flow \( \Xi = \Xi(x,t) \), i.e. the solution to \( \partial_t \Xi(x,t) = \tilde{v}_q(\Xi(x,t), t) \) with initial data \( \Xi|_{t=t_i} = x \).

In preparation for the use of the geometric property (4.44) in Lemma 4.12, we define
\[
R_{q,i} := I_{3 \times 3} - \frac{\eta_i^2 \overset{\cdot}{R}_q}{\rho_{q,i}}
\]
In the above, we multiply by \( \eta_i^2 \) to select a component of \( \text{supp} \overset{\cdot}{R}_q \); note that \( \eta_i^2 \overset{\cdot}{R}_q \) is compactly supported in \( I_i \), where it is identically equal to \( \overset{\cdot}{R}_q \).

In order to use \( C_k \cdot k = 0 \) from (4.46), we have to conjugate with \( \nabla \Phi \):
\[
\tilde{R}_{q,i} := \nabla \Phi R_{q,i} \nabla \Phi^T = \nabla \Phi \nabla \Phi^T - \frac{1}{\rho_{q,i}} \eta_i^2 \nabla \Phi \overset{\cdot}{R}_q \nabla \Phi^T.
\] (4.36)

See the algebraic computation in (5.7).

**Proposition 4.10 (Estimates for \( \Phi_i \)).** For \( a \gg 1 \),
\[
t \in J_i = \text{supp} \eta_i \implies \| \nabla \Phi_i - I_{3 \times 3} \|_0 \leq \frac{1}{10},
\]
\[
\| (\nabla \Phi_i)^{-1} \|_N + \| \nabla \Phi_i \|_N \lesssim \ell_q^{-N},
\]
\[
\| D_{t,\tilde{v}_q} \nabla \Phi_i \|_N \lesssim \delta_q^{1/2} \lambda_q \ell_q^{-N}.
\] (4.39)

**Proof.** The first two follow because there exists \( C = C_N \) such that
\[
\| \nabla \Phi_i - I_{3 \times 3} \|_0 \leq e^{(t-t_i)\|\tilde{v}_q\|_1} - 1 \leq \tau_q \|\tilde{v}_q\|_1 \lesssim \ell_q^{2a} \ll 1,
\]
\[
[\Phi_i - I_{3 \times 3}]_N \leq C(t-t_i)[\tilde{v}_q]_N e^{C(t-t_i)\|\tilde{v}_q\|_1},
\]
as in [Buc+15, App. D], \( a \gg 1 \) is used to control constants.

For the material derivative, we compute
\[
(D_{t,\tilde{v}_q} \nabla \Phi_i)_a = D_{t,\tilde{v}_q} \partial_b (\Phi_i)_a = \partial_b \partial_b (\Phi_i)_a + (\tilde{v}_q)_c \partial_c (\Phi_i)_a
\]
\[
= \partial_b (\partial_t (\Phi_i)_a + (\tilde{v}_q)_c \partial_c (\Phi_i)_a) - \partial_b (\tilde{v}_q)_c \partial_c (\Phi_i)_a
\]
\[
= \partial_b (D_{t,\tilde{v}_q} (\Phi_i)_a) - (\nabla \tilde{v}_q^T \nabla \Phi_i)_a,
\]
and hence by (4.35), \( D_{t,\tilde{v}_q} \nabla \Phi_i = -\nabla \tilde{v}_q^T \nabla \Phi_i \). So (4.39) follows from (4.38) and (4.22). \( \square \)

**Proposition 4.11 (Estimates for \( \tilde{R}_{q,i} \)).** For \( a \gg 1 \),
\[
\| \tilde{R}_{q,i} - I_{3 \times 3} \|_0 \leq \frac{1}{2},
\]
\[
\| \tilde{R}_{q,i} \|_N \lesssim \ell_q^{-N},
\]
\[
\| D_{t,\tilde{v}_q} \tilde{R}_{q,i} \|_N \lesssim \tau_q^{-1} \ell_q^{-N}.
\] (4.42)
Proof. For the first two estimates, we write

\[ \tilde{R}_{q,i} - I_{3 \times 3} = \nabla \Phi_i (R_{q,i} - I_{3 \times 3}) \nabla \Phi + \nabla \Phi \nabla \Phi^T - I_{3 \times 3} \]

= \nabla \Phi \frac{\tilde{R}_q}{\rho_{q,i}} \nabla \Phi^T + (\nabla \Phi - 3 \times 3)(\nabla \Phi^T + I_{3 \times 3}) + (\nabla \Phi - 3 \times 3) - (\nabla \Phi^T - I_{3 \times 3}), \]

so that when \( a \gg 1 \) so that \( \lambda_\alpha q \rho_{q,i} < \frac{9}{100} \), by (4.23) and the fact that either \( \tilde{R}_q \equiv 0 \) or \( \rho_{q,i} > \lambda_\alpha q \delta_{q+1} \) by (4.28),

\[ \| \tilde{R}_{q,i} - I_{3 \times 3} \|_{C^0} \leq \lambda_\alpha q \rho_{q,i} + \frac{1}{10} (2 + \frac{1}{10}) + \frac{1}{10} + \frac{1}{10} < \frac{1}{2}. \]

The second estimate (4.40) follows by routine direct computation. For the estimate on the material derivative, as \( \eta_\nu^2 \equiv 1 \) and \( \rho_{q,i} = \rho_q \) on a neighbourhood of the support of \( R_{q,i} \), we have

\[ D_t, \tilde{v}_q R_{q,i} = D_t, \tilde{v}_q (\rho_q^{-1} \tilde{R}_q) = -\frac{\partial \rho_q}{\rho_q} \tilde{R}_q + \rho_q D_t, \tilde{v}_q \tilde{R}_q, \]

Hence, from the estimates (4.24), (4.28), and (4.33) (since \( \tilde{R}_q \) is supported on \( T^3 \times \bigcup_{i \geq 0} I_i \)), we obtain

\[ \| D_t, \tilde{v}_q R_{q,i} \|_N \leq \tau_{q}^{-1} \ell_q^{-N}. \]

As \( \| \nabla \Phi_i \|_0 \leq 1 \) by (4.37), the estimate (4.39) follows:

\[ \| D_t, \tilde{v}_q \tilde{R}_{q,i} \|_N \]

\[ \lesssim \| D_t, \tilde{v}_q \nabla \Phi^T R_{q,i} \nabla \Phi \|_N + \| \nabla \Phi D_t, \tilde{v}_q R_{q,i} \nabla \Phi^T \|_N + \| \nabla \Phi R_{q,i} D_t, \tilde{v}_q \nabla \Phi \|_N \]

\[ \lesssim \| D_t, \tilde{v}_q \nabla \Phi \|_N \| R_{q,i} \|_0 + \| D_t, \tilde{v}_q \nabla \Phi \|_0 \| R_{q,i} \|_N \]

\[ + \| D_t, \tilde{v}_q \nabla \Phi \|_0 \| R_{q,i} \|_0 \| \nabla \Phi \|_N + \| \nabla \Phi \|_N \| D_t, \tilde{v}_q R_{q,i} \|_0 \| D_t, \tilde{v}_q R_{q,i} \|_N \]

\[ \lesssim \tau_{q}^{-1} \ell_q^{-N}. \]

\[ \square \]

4.5 Mikado flows

In this subsection, we summarize the Mikado flows as presented in [Buc+19] as a lemma. Mikado flows were first introduced in [DS17]. Write \( S_{3 \times 3}^+ \) for the space of symmetric positive definite 3 \times 3 matrices, and \( \mathbb{B}_{1/2}(I_{3 \times 3}) \subset S_{3 \times 3}^+ \) for the closed ball of radius 1/2 around the identity matrix.

Lemma 4.12. There exists a smooth map

\[ W : \mathbb{B}_{1/2}(I_{3 \times 3}) \times T^3 \rightarrow \mathbb{R}^3 \]
such that, for every fixed $R \in \mathbb{B}_{1/2}(I_{3 \times 3})$, $W(R, \cdot)$ is a pressureless solution to the stationary incompressible Euler equations,

\[
\text{div}_\xi(W(R, \xi) \otimes W(R, \xi)) = 0, \\
\text{div}_\xi W(R, \xi) = 0,
\]

and the low frequencies of $W$ are such that

\[
\int_{T^3} W(R, \xi) d\xi = 0, \quad (4.43)
\]

\[
\int_{T^3} W(R, \xi) \otimes W(R, \xi) d\xi = R. \quad (4.44)
\]

It follows from the smoothness of $W$ that there exist $a_k \in C^\infty(\mathbb{B}_{1/2}(I_{3 \times 3}); \mathbb{R}^3)$ such that

\[
W(R, \xi) = \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} a_k(R)e^{i k \cdot \xi}, \quad a_k(R) \cdot k = 0, \quad (4.45)
\]

and there exist $C_k \in C^\infty(\mathbb{B}_{1/2}(I_{3 \times 3}); \mathbb{R}^{3 \times 3})$ such that

\[
W(R, \xi) \otimes W(R, \xi) = R + \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} C_k(R)e^{i k \cdot \xi}, \quad C_k(R)k = 0. \quad (4.46)
\]

For all $N, m \geq 0$ and $k \in \mathbb{Z}^3 \setminus \{0\}$, there exist constants $C_{N,m}$ such that:

\[
\sup_{R \in \mathbb{B}_{1/2}(I_{3 \times 3})} \left| D^N_R a_k(R) \right| + \sup_{R \in \mathbb{B}_{1/2}(I_{3 \times 3})} \left| D^N_R C_k(R) \right| \leq \frac{C_{N,m}}{|k|^m}, \quad (4.47)
\]

Here, $D_R$ denotes the derivative with respect to $R$.

### 4.5.1 Principal part and incompressibility corrector

Define the principal part of the perturbation

\[
w^{(p)}_{q+1} := \sum_{i=0}^{i_{\text{max}}} w^{(p)}_{q+1,i} := \sum_{i=0}^{i_{\text{max}}} \rho_{q,i}^{1/2} \nabla \Phi_i^{-1} W(\tilde{R}_{q,i}, \lambda_{q+1} \Phi_i).
\]

We need to define a small corrector $w^{(c)}_{q+1}$ so that $w_{q+1} = w^{(p)}_{q+1} + w^{(c)}_{q+1}$ is divergence-free, i.e. that $w_{q+1}$ is a curl. We start by using the Fourier decomposition (4.45) to write

\[
w^{(p)}_{q+1} = \sum_{k,i} \rho_{q,i}^{1/2} \nabla \Phi_i^{-1} a_k(\tilde{R}_{q,i}) e^{i \lambda_{q+1} k \cdot \Phi_i},
\]
where the sum is over \( k \in \mathbb{Z}^3 \setminus \{0\} \) and \( i \in \{0, 1, \ldots, i_{\text{max}}\} \). From \( a_k \cdot k = 0 \) in (4.45), it follows that \( a_k = -\frac{k \times a_k}{|k|^2} \times k \). Also recall the standard cross product identity\(^3\) \( M^{-1}(a \times b) = (\det M)(M^T a) \times (M^T b) \). As \( \nabla \Phi_i = 1 \), it follows that

\[
\sum_{k,i} w_{q+1} = -i \lambda_q \sum_{k,i} \nabla \Phi_i k \times a_k (\tilde{R}_{q,i}) |k|^2 \times (\nabla e^{i \lambda_{q+1} k \cdot \Phi_i}).
\]

The above considerations show that we can define a divergence-free \( v_{q+1} \) with zero mean by setting

\[
w_{q+1} = w_{q+1}^p + w_{q+1}^c, \quad v_{q+1} = \bar{v}_{q+1} + \bar{v}_{q+1}.
\]

Note that \( w_{q+1} \) is supported away from zero. Hence,

\[
v_{q+1} |_{t=0} = \bar{v}_{q+1} |_{t=0} = v^\text{in} * \psi_{q+1}.
\]

We define \( b_{i,k} = b_{i,k}(x,t) \) and \( c_{i,k} = c_{i,k}(x,t) \) by

\[
b_{i,k}(x,t) := \rho_{q,i}^{1/2} a_k (\tilde{R}_{q,i}), \quad c_{i,k}(x,t) := -i \lambda_{q+1} \nabla \Phi_i k \times a_k (\tilde{R}_{q,i}) |k|^2 \times (\nabla e^{i \lambda_{q+1} k \cdot \Phi_i}).
\]

Proposition 4.13 (\( C^0 \) Estimate for \( a_k \) and \( b_{i,k} \)). There is a universal constant \( M \) such that

\[
\|b_{i,k}\|_0 \leq M \frac{|k|^5}{|k|^5} |q+1|^{1/2}.
\]

\(^3\)This easily follows from \( \det M = \varepsilon_{ijk} M_{ij} M_{jk} M_{kl}, \quad (\text{cof } M)_{ij} = \frac{1}{2} \varepsilon_{ijk} \varepsilon_{jkl} M_{pk} M_{ql} \) and \( A^{-1} = \frac{1}{\det A} \text{ cof } A^T \).
Proof. Putting $N = 0$ and $m = 5$ in (4.47) gives $\|a_k(\vec{R}_{q,i})\|_0 \leq \frac{C_0 \delta}{|k|^3}$. From the definition of $b_{i,k}$ and the upper bound (4.29) $\rho_{q,i} \leq \delta_{q+1}/c_q$, we can set $\overline{M} := C_{0.5}c_q^{-1/2}$.

We set the constant $M$ in the estimate (3.8) as

$$M := 64 \overline{M} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \frac{1}{|k|^3} < \infty. \quad (4.53)$$

**Proposition 4.14 (Estimates for $b_{i,k}$ and $c_{i,k}$).** For $t \in J_i \cup I_i \cup J_{i+1}$,

$$\|b_{i,k}\|_N \lesssim \delta_{q+1}^{1/2} |k|^{-6} \ell_q^{-N}, \quad (4.54)$$

$$\|c_{i,k}\|_N \lesssim \delta_{q+1}^{1/2} |k|^{-6} \ell_q^{-N-1} \lambda_{q+1}^{-1}, \quad (4.55)$$

$$\|D_t \vec{v}_q c_{i,k}\|_N \lesssim \delta_{q+1}^{1/2} \|a_k(\vec{R}_{q,i})\|_0 + \delta_{q+1}^{1/2} \|a_k(\vec{R}_{q,i})\|_N \lesssim \delta_{q+1}^{1/2} C_{N,6} \ell_q^{-N}. \quad (4.56)$$

Proof. Similarly to Proposition 4.13, the first estimate (4.54) follows from the definition (4.48) and the estimates (4.47), (4.31), (4.38), and (4.41):

$$\|b_{i,k}\|_N \lesssim \delta_{q+1}^{1/2} \|a_k(\vec{R}_{q,i})\|_0 + \|a_k(\vec{R}_{q,i})\|_N \lesssim \delta_{q+1}^{1/2} \frac{C_{N,6} \ell_q^{-N}}{|k|^3}. $$

A similar calculation gives (4.55) from the definition (4.49),

$$\|c_{i,k}\|_N \lesssim \delta_{q+1}^{1/2} \lambda_{q+1}^{-1} \left| \eta_q \nabla \Phi_i \frac{T_k x a_k(\vec{R}_{q,i})}{|k|^2} \right|_{N+1} \lesssim \lambda_{q+1}^{-1} \delta_{q+1} \ell_q^{-N-1} C_{N,6,5} |k|^6. $$

For (4.56), writing $c_{i,k} = \nabla \times \vec{c}_{i,k}$, we commute the curl and compute using (4.32), (4.39), (4.42) and (4.47):

$$\|D_t \vec{v}_q c_{i,k}\|_N \lesssim \|D_t \vec{v}_q \vec{c}_{i,k}\|_{N+1} + \|\vec{v}_q\|_1 \|\nabla \vec{c}_{i,k}\|_N + \|\vec{v}_q\|_{N+1} \|\nabla \vec{c}_{i,k}\|_0$$

$$\lesssim \lambda_{q+1}^{-1} \left| \frac{\rho_{q,i}}{\eta_q} \frac{\nabla \Phi_i T a_k(\vec{R}_{q,i})}{|k|} \right|_N + \lambda_{q+1}^{-1} \left| \frac{\rho_{q,i}}{\eta_q} D_t \vec{v}_q \nabla \Phi_i T a_k(\vec{R}_{q,i})}{|k|} \right|_N$$

$$+ \lambda_{q+1}^{-1} \left| \frac{1}{\rho_{q,i}} \nabla \Phi_i T \frac{D_t a_k(\vec{R}_{q,i})}{|k|} D_t \vec{v}_q \vec{R}_{q,i} \right|_N + \delta_{q+1} \lambda_q |k|^{-6} \ell_q^{-N-1} \lambda_{q+1}^{-1}$$

$$\lesssim \delta_{q+1}^{1/2} \ell_q^{-N-1} \lambda_{q+1}^{-1} |k|^{-6} + \delta_{q+1} \lambda_q \ell^{-N} \lambda_{q+1}^{-1} |k|^{-6}$$

$$+ \delta_{q+1}^{1/2} \ell_q^{-N-1} \lambda_{q+1}^{-1} |k|^{-6} + \delta_{q+1}^{1/2} \ell_q^{-N-1} \lambda_{q+1}^{-1} |k|^{-6}$$

$$\lesssim \delta_{q+1}^{1/2} \ell_q^{-N-1} \lambda_{q+1}^{-1} |k|^{-6}. $$
Corollary 4.15 (Estimates for $u_{q+1}^{(p)}$ and $w_{q+1}^{(c)}$).

\[
\|u_{q+1}^{(p)}\|_0 + \frac{1}{\lambda_{q+1}} \|u_{q+1}^{(p)}\|_1 \leq \frac{M}{4} \delta_{q+1}^{1/2}, \tag{4.57}
\]

\[
\|w_{q+1}^{(c)}\|_0 + \frac{1}{\lambda_{q+1}} \|w_{q+1}^{(c)}\|_1 \leq \ell_q^{-1} \lambda_{q+1}^{-1} \delta_{q+1}^{1/2}, \tag{4.58}
\]

\[
\|w_{q+1}\|_0 + \frac{1}{\lambda_{q+1}} \|w_{q+1}\|_1 \leq \frac{M}{2} \delta_{q+1}^{1/2}. \tag{4.59}
\]

**Proof.** These follow from the estimates of $b_{i,k}$ and $c_{i,k}$, as $w_{q+1}^{(p)} = \sum_i w_{q+1,i}^{(p)} = \sum_{i,k} \nabla \Phi_i^{-1} b_{i,k} e^{1+1k-\Phi_i}$, $w_{q+1}^{(c)} = \sum_i w_{q+1,i}^{(c)} = \sum_{k,i} c_{i,k} e^{1+k-\Phi_i}$, and $\nabla \Phi_i \sim I_{3x3}$. Specifically, by the disjoint support of $w_{q+1,i}^{(p)}$ (see Figure 2),

\[
\|w_{q+1}^{(p)}\|_0 \leq \sum_{k \neq 0} \sup_{i \geq 0} \|\nabla \Phi_i^{-1} b_{i,k}\|_0 \quad \text{(4.37),(4.52)} \leq \quad 2M \delta_{q+1}^{1/2} \sum_{k \neq 0} \frac{1}{|k|^6} \leq \frac{M}{32} \delta_{q+1}^{1/2}.
\]

For the $C^1$ norm, since $|\nabla e^{1+1k-\Phi_i}| \leq 2 \lambda_{q+1} |k|$ by (4.37),

\[
\|w_{q+1}^{(p)}\|_1 \leq \frac{M}{32} \delta_{q+1}^{1/2} + \sum_{k \neq 0} \sup_{i \geq 0} \|\nabla \Phi_i^{-1} b_{i,k} e^{1+1k-\Phi_i}\|_1 \leq \frac{M}{32} \delta_{q+1}^{1/2} + \sum_{k \neq 0} \ell_q^{-2} \sup_{i \geq 0} \|b_{i,k}\|_0 + 2 \sup_{i \geq 0} \|b_{i,k}\|_1 + 4 \lambda_{q+1} |k| \sup_{i \geq 0} \|b_{i,k}\|_0 \leq \frac{M}{32} \delta_{q+1}^{1/2} + \frac{(2C_0 \ell_q^{-2} + 2C_1) \delta_{q+1}^{1/2}}{|k|^4} + 4 \lambda_{q+1} \sum_{k \neq 0} |k|^{-6} \left( \frac{1}{32 \lambda_{q+1}} + \frac{2C_0 + 2C_1}{4 \ell_q \lambda_{q+1}} \right) + \frac{1}{16},
\]

where $C_0, C_1$ are the implicit constants coming from the $N = 0, 1$ cases of (4.54). Note that $\ell_q \lambda_{q+1} \geq \frac{1}{30} \ell_q \lambda_q \geq \frac{1}{100} \lambda_q^{(1-\beta)(b-1)}$ by the estimates in Section 3.1. Hence, (4.57) will follow by choosing $\alpha$ so that

\[
3 \alpha < \frac{(1-\beta)(b-1)}{2} \quad \text{(4.60)}
\]

and then taking $a \gg 1$ to control constants, for instance one could take $a > 1 + (1600 + (C_0 + C_1) \sum_{k \neq 0} |k|^{-6})^{1/(b-1)}$.

The same choices of constants easily implies the estimate (4.58), and the inequality $\lambda_{q+1} \ell_q > 4$, from which (4.59) follows. \qed
4.6 Derivation of new stress error term

By defining $v_{q+1} = \bar{v}_q + w_{q+1}$, we have already implicitly defined the error term $\text{div} \, \hat{R}_{q+1}$ through the equation (3.3) with $q$ replaced by $q + 1$:

$$\partial_t v_{q+1} + \text{div}(v_{q+1} \otimes v_{q+1})$$

$$= \partial_t \bar{v}_q + \text{div}(\bar{v}_q \otimes \bar{v}_q) + \nabla \bar{p}_q - \text{div} \hat{R}_q$$

$$+ \text{div} \hat{R}_q + \text{div}(w_{q+1} \otimes w_{q+1}) - \nabla \bar{p}_q$$

$$+ \partial_t w_{q+1} + \bar{v}_q \cdot \nabla w_{q+1} + \text{div}(w_{q+1} \otimes \bar{v}_q).$$

The first line of the right-hand side vanishes by (4.19); for the other terms we group as follows:

$$p_{q+1}(x,t) := \bar{p}_q(x,t) - 3 \sum_{i=0}^{i_{\text{max}}} \rho_{q,i}(x,t) + \rho_q(t),$$

$$R_{q+1}^{\text{osc}} := \mathcal{R}(\text{div}(w_{q+1} \otimes w_{q+1}) + \text{div} \hat{R}_q - \nabla \bar{p}_{q+1})$$

$$R_{q+1}^{\text{trans}} := \mathcal{R}(\partial_t w_{q+1} + \bar{v}_q \cdot \nabla w_{q+1}) = \mathcal{R}(D_t \bar{v}_q w_{q+1})$$

$$R_{q+1}^{\text{Nash}} := \mathcal{R} \text{div}(w_{q+1} \otimes \bar{v}_q) = \mathcal{R}(w_{q+1} \cdot \nabla \bar{v}_q).$$

The term $3 \sum_i \rho_{q,i}$ comes from the divergence of $\hat{R}_{q,i}$ (see (5.6)), and $\rho_q$ is subtracted so that $p_{q+1}$ is mean-free; it does not depend on $x$ so disppears on taking the gradient. Then we set

$$\hat{R}_{q+1} := R_{q+1}^{\text{osc}} + R_{q+1}^{\text{trans}} + R_{q+1}^{\text{Nash}}.$$

5 Estimates of the stress error terms

In this section, we show how to achieve the estimate

$$\|\hat{R}_{q+1}\|_{\alpha} \lesssim \frac{\delta_{q+1}^{1/2} \lambda_{q+1}^{1/2}}{\lambda_{q+1}^{\alpha - 4}}.$$ 

The calculations that we omit can be found in [Buc+19, Subsection 6.1]. We claim that we can choose parameters such that

$$\frac{\delta_{q+1}^{1/2} \lambda_{q+1}^{1/2}}{\lambda_{q+1}^{\alpha - 4}} \lesssim \frac{\delta_{q+2}^{1/2} \lambda_{q+2}^{1/2}}{\lambda_{q+2}^{8\alpha / 5}}.$$ 

(5.1)
which therefore implies \( \| \tilde{R}_{q+1} \|_\alpha \lesssim \delta_q^{-4} \lambda_{q+1}^{-\alpha} \). Since we have one extra copy of \( \lambda_q^{-\alpha} \ll 1 \), \( a \gg 1 \) gives (3.9) from (5.1).

Since (5.1) follows if we have \( 2\beta(b-1)^2 - (1-3\beta)(b-1) + 8ab < 0 \), it suffices to take
\[
0 < b - 1 < \frac{1 - 3\beta}{2\beta}, \quad 8\alpha < -\frac{2\beta(b-1)^2 - (1-3\beta)(b-1)}{2b},
\]
so that \( 2\beta(b-1)^2 - (1-3\beta)(b-1) + 8ab < \frac{2\beta(b-1)^2 - (1-3\beta)(b-1)}{2} < 0 \).

We will also need the estimate
\[
\frac{1}{\lambda_{q+1}^{N-\alpha} \ell_q^{N-\alpha}} \leq \frac{1}{\lambda_{q+1}^{1-\alpha}}
\]
which holds for sufficiently small \( \alpha \), \( N = N(b, \beta) \) sufficiently large and large \( a \), independently of \( q \) as follows. It is equivalent to the estimate \( \lambda_{q+1}^{1-\alpha} \ell_q^{N-\alpha} \leq 1 \). Since \( \ell_q^{-\alpha} \leq \lambda_{q+1}^{2\alpha} \), it suffices to show the stronger estimate \( \lambda_{q+1}^{1-\alpha} \ell_q^{N-\alpha} \lambda_{q+1}^{2\alpha} \leq 1 \). Expanding the definition of \( \ell_q \) gives
\[
\lambda_{q+1}^{1-\alpha} \lambda_{q+1}^{(1-\beta)+(1-\beta+3\alpha/2)+2\alpha} \leq 1.
\]
Since the exponent of \( \lambda_{q+1} \) is negative for \( N > 2 > \frac{1}{1-\beta} \), and \( \lambda_{q+1} \gtrsim \lambda_{q+1}^{\lambda_{q+1}} \), it is enough to enforce
\[
b - N(1-\beta)(b-1) + 1 < 0 \iff N > \frac{b + 2}{(1-\beta)(b-1)}.
\]
This means that in the entire iteration, we only ever use \( N \)th derivative estimates for a fixed \( q \)-independent \( N \).

**Proposition 5.1 (Estimate for \( R_{q+1}^{\text{trans}} \)).**

\[
\| R_{q+1}^{\text{trans}} \|_\alpha \lesssim \frac{\delta_{q+1}^{1/2} \delta_q^{1/2} \lambda_q}{\lambda_{q+1}^{1-\alpha}}.
\]

**Proof.** We separately estimate \( D_t, \tilde{v}_q w_{q+1}^{(p)} \) and \( D_t, \tilde{v}_q w_{q+1}^{(c)} \). For the first term, we use the fact that \( U_{ik} := (\nabla \Phi_i)^{-1} \rho_{q,i}^{1/2} a_k(\tilde{R}_{q+1,i})e^{i\lambda_{q+1} k \cdot \Phi_i} \) is Lie-advected up to an error term,
\[
D_t, \tilde{v}_q U_{ik} = (U_{ik} \cdot \nabla) \tilde{v}_q + (\nabla \Phi_i)^{-1} (D_t, \tilde{v}_q (\rho_{q,i}^{1/2} a_k(\tilde{R}_{q+1,i})) e^{i\lambda_{q+1} k \cdot \Phi_i}).
\]
Importantly, the term $D_{t,\bar{v}_q}e^{i\lambda_{q+1}k\cdot\Phi_i}$ does not contribute a bad power of $\lambda_{q+1}$. As $w_{q+1}^{(p)} = \sum_{i,k} U_{ik}$,

$$D_{t,\bar{v}_q}w_{q+1}^{(p)} = \sum_{i\geq 0, k \neq 0} ((\nabla \Phi_i)^{-1} \rho_{q,i}^{1/2} a_k(\bar{R}_{q,i},) \cdot \nabla) \bar{v}_q e^{i\lambda_{q+1}k\cdot\Phi_i}$$

$$+ \sum_{i\geq 0, k \neq 0} (\nabla \Phi_i)^{-1}(D_{t,\bar{v}_q}(\rho_{q,i}^{1/2} a_k(\bar{R}_{q,i}))) e^{i\lambda_{q+1}k\cdot\Phi_i}.$$  

Both these sums are treated similarly. For the first, by Lemma B.1, (4.38), (4.34), and (4.47),

$$\|\mathcal{R} \left( (\nabla \Phi_i)^{-1} \rho_{q,i}^{1/2} a_k(\bar{R}_{q,i}) \cdot \nabla) \bar{v}_q e^{i\lambda_{q+1}k\cdot\Phi_i} \right) \|_\alpha$$

$$\lesssim \frac{\lambda_q \delta_{q+1}^{1/2} \delta_{q}^{1/2}}{\lambda_{q+1}^{1-\alpha} |k|^6} + \frac{\lambda_q \delta_{q+1}^{1/2} \delta_{q}^{1/2}}{\lambda_{q+1}^{2-\alpha} |k|^6} \lesssim \frac{\lambda_q \delta_{q+1}^{1/2} \delta_{q}^{1/2}}{\lambda_{q+1}^{1-\alpha} |k|^6}.$$  

For the second sum, by Lemma B.1, (4.38), (4.34), and (4.47),

$$\|\mathcal{R} \left( (\nabla \Phi_i)^{-1}(D_{t,\bar{v}_q}(\rho_{q,i}^{1/2} a_k(\bar{R}_{q,i}))) e^{i\lambda_{q+1}k\cdot\Phi_i} \right) \|_\alpha$$

$$\lesssim \frac{\lambda_q \delta_{q+1}^{1/2} \delta_{q}^{1/2} \lambda_q^{-2\alpha}}{\lambda_{q+1}^{1-\alpha} |k|^6} \lesssim \frac{\lambda_q \delta_{q+1}^{1/2} \delta_{q}^{1/2} \lambda_q^{-2\alpha}}{\lambda_{q+1}^{1-\alpha} |k|^6}.$$  

For the last term $D_{t,\bar{v}_q}w_{q+1}^{(c)}$ we instead use the estimate (4.56) on $D_{t,\bar{v}_q}c_{i,k}$ since $w_{q+1}^{(c)} = \sum_{k \neq 0} c_{i,k} e^{i\lambda_{q+1}k\cdot\Phi_i}$:

$$\|\mathcal{R}(D_{t,\bar{v}_q}c_{i,k})e^{i\lambda_{q+1}k\cdot\Phi_i}\|_\alpha$$

$$\lesssim \frac{\|D_{t,\bar{v}_q}c_{i,k}\|_0}{\lambda_{q+1}^{1-\alpha}} + \frac{\|D_{t,\bar{v}_q}c_{i,k}\|_{N+\alpha}}{\lambda_{q+1}^{N-\alpha}}$$

$$\lesssim \frac{\lambda_q^{1/2} |k|^{-6} \delta_{q+1}^{1/2} \lambda_{q+1}^{-\alpha}}{\lambda_{q+1}^{1-\alpha}} + \frac{\lambda_q^{1/2} |k|^{-6} \delta_{q+1}^{1/2} \lambda_{q+1}^{-\alpha-1}}{\lambda_{q+1}^{N-\alpha}}$$

$$\lesssim \frac{\lambda_q^{1/2} \delta_{q+1}^{1/2}}{\lambda_{q+1}^{1-\alpha} |k|^6} \lesssim \frac{\lambda_q^{1/2} \delta_{q+1}^{1/2} \lambda_q^{-2\alpha}}{\lambda_{q+1}^{1-\alpha} |k|^6}.$$

The required estimate follows by summing in $k$ and the disjointness of the supports of $w_{q+1,i}^{(c)}$.

**Proposition 5.2** (Estimate for $R_{q+1}^{Nash}$),

$$\|R_{q+1}^{Nash}\|_{\alpha} \lesssim \frac{\delta_{q+1}^{1/2} \delta_{q}^{1/2} \lambda_q}{\lambda_{q+1}^{1-\alpha}}.$$

31
Proof. We write \(w_{q+1} = w_{q+1}^{(p)} + w_{q+1}^{(c)}\) and expand \(R_{q+1}^{\text{Nash}}\) with (4.50) and (4.51):

\[
R_{q+1}^{\text{Nash}} = R \left( \sum_{k \neq 0} \left( (\nabla \Phi_i)^{-1} b_{i,k} \cdot \nabla \tilde{v}_q \right) e^{i \lambda_{q+1} k \cdot \Phi_i} + \sum_{k \neq 0} (c_{i,k} \cdot \nabla \tilde{v}_q) e^{i \lambda_{q+1} k \cdot \Phi_i} \right).
\]

These sums can be estimated by using Lemma B.1, (4.22), (4.38), (4.54) and (4.55):

\[
\|R((\nabla \Phi_i)^{-1} b_{i,k} \cdot \nabla \tilde{v}_q)e^{i \lambda_{q+1} k \cdot \Phi_i}\|_\alpha \\
\lesssim \|\nabla \Phi_i^{-1} b_{i,k} \cdot \nabla \tilde{v}_q\|_0 + \|\nabla \Phi_i^{-1} b_{i,k} \cdot \nabla \tilde{v}_q\|_{N+\alpha} + \|\nabla \Phi_i^{-1} b_{i,k} \cdot \nabla \tilde{v}_q\|_0 \Phi_i\|_{N+\alpha} \\
\lesssim \lambda_q^{-1/2} \epsilon_{q+1}^{1/2} \delta_q^{1/2} + \lambda_q^{-\alpha} \| \nabla \tilde{v}_q \|_{N+\alpha} \| \Phi_i \|_{N+\alpha} \\
\lesssim \frac{\delta_{q+1}^{1/2} \delta_q^{1/2} \lambda_q}{\lambda_q^{1-\alpha} |k|^6}, \quad \text{and} \quad \sum_{k \neq 0} \left( (\nabla \Phi_i)^{-1} b_{i,k} \cdot \nabla \tilde{v}_q \right)
\]

Summing in \(k\) gives the claimed inequality. Details can be found in [Buc+19, Subsection 6.1].

**Proposition 5.3** (Estimate for \(R_{q+1}^{\text{osc}}\)).

\[\|R_{q+1}^{\text{osc}}\|_\alpha \lesssim \frac{\delta_{q+1}^{1/2} \delta_q^{1/2} \lambda_q}{\lambda_q^{1-3\alpha}}.\]

**Proof.** Note that \(w_{q+1} \otimes w_{q+1} = w_{q+1}^{(p)} \otimes w_{q+1}^{(p)} + (w_{q+1}^{(p)} \otimes w_{q+1}^{(c)} + w_{q+1}^{(c)} \otimes w_{q+1}^{(p)} + w_{q+1}^{(c)} \otimes w_{q+1}^{(c)}).\) We will momentarily show

\[
\left\| R \div \left( w_{q+1}^{(p)} \otimes w_{q+1}^{(p)} - \tilde{R}_q - 3 \sum_{i=0}^{i_{\max}} \nabla \rho_{q,i} \right) \right\|_{\alpha} \lesssim \frac{\delta_{q+1}^{1/2} \delta_q^{1/2} \lambda_q}{\lambda_q^{1-3\alpha}}. \tag{5.4}
\]

The other terms are sufficiently small because \(w_{q+1}^{(c)}\) is small: interpolating the estimates (4.57) and (4.58) gives \(\|w_{q+1}^{(p)}\|_\alpha \lesssim \delta_{q+1}^{1/2} \lambda_q^{1}, \|w_{q+1}^{(c)}\|_\alpha \lesssim \delta_{q+1}^{1/2} \lambda_q^{1}, \|w_{q+1}^{(c)}\|_\alpha \lesssim \delta_{q+1}^{1/2} \lambda_q^{1}\).
\( \delta_{q+1}^{1/2} \ell_q^{-1} \lambda_{q+1}^{-1+\alpha} \). Therefore,
\[
\| w_{q+1} \otimes w_{q+1} - w_{q+1}^{(p)} \otimes w_{q+1}^{(p)} \|_\alpha \\
\lesssim \| w_{q+1}^{(p)} \|_\alpha \| w_{q+1}^{(p)} \|_\alpha + \| w_{q+1}^{(c)} \|_\alpha \| w_{q+1}^{(c)} \|_\alpha + \| w_{q+1}^{(p)} \|_\alpha \| w_{q+1}^{(c)} \|_\alpha \\
\leq \delta_{q+1} \ell_q^{-1} \lambda_{q+1}^{-1+\alpha} + \delta_{q+1}^{1/2} \ell_q^{-2} \lambda_{q+1}^{-2+\alpha} \\
\lesssim \frac{\delta_{q+1}^{1/2} \ell_q^{-1/2} \lambda_{q+1}^1 \alpha/2}{\lambda_{q+1}^{-\alpha}} \leq \frac{\delta_{q+1}^{1/2} \ell_q^{-1/2} \lambda_q}{\lambda_q^{-\alpha}}.
\]

To show (5.4), we expand out \( w_{q+1}^{(p)} \otimes w_{q+1}^{(p)} = \sum_j w_{q+1,i}^{(p)} \otimes w_{q+1,j}^{(p)} \). Since \( \eta_i \) have disjoint supports, it follows that \( w_{q+1,i}^{(p)} \otimes w_{q+1,j}^{(p)} = 0 \) if \( i \neq j \). Hence
\[
w_{q+1,i}^{(p)} \otimes w_{q+1,j}^{(p)} = w_{q+1,i}^{(p)} (w_{q+1,i}^{(p)})^T \\
= \rho_{q,i} \nabla \Phi_i^{-1} (W \otimes W) (\tilde{R}_{q,i}, \lambda_{q+1} \Phi_i) \nabla \Phi_i^{-T} \\
(4.46) = \rho_{q,i} \nabla \Phi_i^{-1} \tilde{R}_{q,i} \nabla \Phi_i^{-T} + \sum_{k \neq 0} \rho_{q,i} \nabla \Phi_i^{-1} C_k (\tilde{R}_{q,i}) \nabla \Phi_i^{-T} e^{i \lambda_{q+1} k \cdot \Phi_i} \\
(4.36) = \rho_{q,i} I_{3 \times 3} - \tilde{R}_{q} + \sum_{k \neq 0} \rho_{q,i} \nabla \Phi_i^{-1} C_k (\tilde{R}_{q,i}) \nabla \Phi_i^{-T} e^{i \lambda_{q+1} k \cdot \Phi_i},
\]

(5.5)

We stress that the above step works despite our modified definition of \( \rho_{q,i} \). We need to compute \( \text{div}(w_{q+1,i}^{(p)} \otimes w_{q+1,i}^{(p)}) \). Observe that \( \text{div}(M f) = (\text{div} M) f + M \nabla f \), valid for matrix valued \( M \) and scalar \( f \), and the chain rule \( \nabla (g(k \cdot \Phi_i)) = \nabla \Phi_i^T k g' \), when \( g : \mathbb{R} \to \mathbb{R} \). Putting \( f = e^{i \lambda_{q+1} k \cdot \Phi_i} \), \( g = \exp \), we see that when the derivative falls on \( e^{i \lambda_{q+1} k \cdot \Phi_i} \), this left factor of \( \nabla \Phi_i^T \) cancels with the right factor of \( \nabla \Phi_i^{-T} \), allowing the use of \( C_k \cdot k = 0 \) in (5.7):
\[
\text{div}(w_{q+1,i}^{(p)} \otimes w_{q+1,i}^{(p)}) + \text{div} \tilde{R}_{q} \\
= 3 \nabla \rho_{q,i} + \sum_{i \geq 0, k \neq 0} \text{div} \left( \rho_{q,i} \nabla \Phi_i^{-1} C_k (\tilde{R}_{q,i}) \nabla \Phi_i^{-T} \right) e^{i \lambda_{q+1} k \cdot \Phi_i} \\
+ \lambda_{q+1} \sum_{i \geq 0, k \neq 0} \rho_{q,i} \nabla \Phi_i^{-1} C_k (\tilde{R}_{q,i}) \nabla \Phi_i^{-T} \nabla \Phi_i^{-T} k e^{i \lambda_{q+1} k \cdot \Phi_i} \\
= C_k (\tilde{R}_{q,i}) k \equiv 0
\]

(5.6)

The first term of (5.6) is specifically subtracted in (5.4). To finish the calculation, we set \( a := \text{div}(\rho_{q,i} \nabla \Phi_i^{-1} C_k (\tilde{R}_{q,i}) \nabla \Phi_i^{-T}) \) and apply Lemma B.1 to find (5.4), as needed:
\[
\left\| \text{R div} \left( w_{q+1}^{(p)} \otimes w_{q+1}^{(p)} - \tilde{R}_{q} - 3 \sum_{i=0}^{\ell_{\max}} \nabla \rho_{q,i} \right) \right\|_\alpha
\]

33
\[
\sum_{i \geq 0, k \neq 0} \|a\|_{N+\alpha} \lambda_{q+1}^{1-\alpha} + \sum_{i \geq 0, k \neq 0} \|a\|_0 \|\Phi_i\|_{N+\alpha} \lambda_{q+1}^{N-\alpha}
\]
\[\lesssim \sum_{i \geq 0, k \neq 0} \frac{\delta_{q+1}}{\lambda_{q+1}^{1-\alpha} |k|^\alpha} \lesssim \frac{\delta_{q+1}^{1/2} \delta_q^{1/2} \lambda_q^{1/2}}{\lambda_{q+1}}.\]

6 Energy iteration

Proposition 6.1 (Energy estimate for \(v_{q+1}\)). For \(t \in [1 - \tau_q, T]\),

\[\left| e(t) - \int_{T^3} |v_{q+1}|^2 \, dx - \frac{\delta_{q+2}}{2} \right| \lesssim \frac{\delta_q^{1/2} \delta_{q+1}^{1/2} \lambda_q^{1+2\alpha} \lambda_{q+1}}{\lambda_q} \leq \delta_{q+2}. \quad (6.1)\]

Proof. Notice that we only want to control the energy for times \(t \geq 1 - \tau_q\). For such times, the \(\zeta\) term (defined in Section 4.4.3) in the energy gap \(\rho_{q,i}\) is identically zero, i.e. we can write (4.27) as

\[\rho_{q,i}(x,t) = \frac{\eta_i^2(x,t)}{\sum_{i=0}^{\max} \int_{T^3} \eta_i(x,t) \, dx} \rho_q(t) .\]

Hence, for \(t \in [1 - \tau_q, T]\), our \(\rho_{q,i}\) matches the formula in [Buc+19, Section 5.2], and therefore we can simply use their proof, which we paraphrase here for the reader’s convenience.

We decompose the total energy into three parts,

\[\int_{T^3} |v_{q+1}|^2 \, dx = \int_{T^3} |\tilde{v}_q|^2 \, dx + 2 \int_{T^3} w_{q+1} \cdot \tilde{v}_q \, dx + \int_{T^3} |w_{q+1}|^2 \, dx =: I_1 + I_2 + I_3 .\]

Note that \(I_2\) is sufficiently small by integration by parts, since \(w_{q+1}\) is a curl, and we have the estimates (4.38), (4.54), and (4.22):

\[\left| \int_{T^3} w_{q+1}^{(p)} \cdot \tilde{v}_q \, dx \right| \lesssim \frac{1}{\lambda_{q+1}} \sum_{i \geq 0, k \neq 0} \|\nabla \Phi_i\|_{T^3} \|k\| |k|^\alpha \|\tilde{v}_q\|_0 \lesssim \frac{\delta_q^{1/2} \delta_{q+1}^{1/2} \lambda_q}{\lambda_{q+1}}.\]

Now we will show that \(I_3 \approx \int_{T^3} |w_{q+1}^{(c)}|^2 \, dx \approx 3 \rho_q\), which was designed in (4.26) specifically to cancel with \(I_1 = \int_{T^3} |\tilde{v}_q|^2 \, dx\), leaving behind an error term of size \(\approx \delta_{q+2}\).

To estimate \(I_3\), we rewrite

\[I_3 = \int_{T^3} |w_{q+1}^{(p)}|^2 \, dx + \int_{T^3} w_{q+1}^{(c)} \cdot (w_{q+1}^{(p)} + w_{q+1}^{(c)}) \, dx =: I_{31} + I_{32} .\]
$I_{32}$ is an error term, from the estimates (4.58) and (4.59) and the definition (3.2) of $\ell_q$:

$$I_{32} \lesssim \frac{\delta_{q+1}^{1/2} \lambda_{q+1}^{1+2\alpha}}{\delta_{q+1}^{1/2} \lambda_{q+1}^{1+2\alpha}}. $$

For $I_{31}$, we take the trace of (5.5) and use $\text{Tr} \tilde{R}_q = 0$ to find

$$\int_{\mathbb{T}^3} |w_{q+1}^{(p)}|^2 \, dx = \int_{\mathbb{T}^3} \text{Tr}(w_{q+1}^{(p)} \otimes w_{q+1}^{(p)}) \, dx$$

$$= 3\rho_q + \text{Tr} \left( \int_{\mathbb{T}^3} \sum_{i \geq 0, k \neq 0} \rho_{q,i} \nabla \Phi_i^{-1} C_k(\tilde{R}_{q,i}) \nabla \Phi_i^{-1} e^{i\lambda_{q+1} k \cdot \Phi_i} \, dx \right),$$

since $\rho_q = \int_{\mathbb{T}^3} \sum_{i \geq 0} \rho_{q,i} \, dx$. From (4.29), (4.41), (4.55), and (4.38), we see that

$$\| \nabla \Phi_i^{-1} C_k(\tilde{R}_{q,i}) \nabla \Phi_i^{-1} \|_N \lesssim \delta_{q+1} \ell_{q}^{-N}.$$

Since $\ell_q \ll \lambda_{q+1}$, repeated integration by parts (formalised by Lemma B.1) shows that this term is arbitrarily small:

$$\left| \int_{\mathbb{T}^3} |w_{q+1}^{(p)}|^2 \, dx - 3\rho_q \right| \lesssim \sum_{k \neq 0} \frac{\delta_{q+1} \ell_{q}^{-N}}{\lambda_{q+1}^{N} |k|^N},$$

which is finite for $N \geq 4$. Taking $N = N(b, \beta) \gg 1$ as in Section 5 so that

$$\frac{\delta_{q+1} \ell_{q}^{-N}}{\lambda_{q+1}^{N}} \leq \frac{\delta_{q+1} \ell_{q}^{-N}}{\lambda_{q+1}^{N}},$$

gives the required bound (6.1), since $3\rho_q = e(t) - \frac{\delta_{q+2}}{2} - I_1$ by (4.26).

### A Hölder spaces and estimates

For $N \in \mathbb{Z}_{\geq 0}$, $C^N(X)$ denotes the space of $N$ times differentiable functions with the norm $\|f\|_{C^N(X)} := \|f\|_{L^\infty(X)} + \sum_{|\sigma| \leq N} \|D^\sigma f\|_{L^\infty(X)}$. For $N \in \mathbb{Z}_{\geq 0}$ and $\alpha \in (0, 1)$, $C^{N+\alpha}(X)$ denotes the subspace of $C^N(X)$ whose $N$th derivatives are $\alpha$-Hölder continuous, with the norm

$$\|f\|_{C^{N+\alpha}(X)} := \|f\|_{C^N} + \sum_{|\sigma| = N} \|D^\sigma f\|_{C^\alpha(X)},$$

where $[f]_{C^\alpha(X)} := \sup_{x, y \in X: x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}$ is the Hölder seminorm.
Proposition A.1. Let $f$ be a smooth function $f : \mathbb{T}^3 \to \mathbb{R}$, and let the mollifier $\psi_\varepsilon$ be as defined in (2.2). For $\varepsilon \ll 1$, $\alpha \in (0,1)$, $\beta \in (0,1)$, $N \geq 0$, we have the estimates

\[
\|f * \psi_\varepsilon\|_{C^{N+\alpha}} \lesssim \|f\|_{C^\alpha} \varepsilon^{-N}, \tag{A.1}
\]

\[
\|f - f * \psi_\varepsilon\|_{C^\alpha} \lesssim [f]_{C^\beta} \varepsilon^{\beta - \alpha} \quad \text{(if $\alpha \leq \beta$)}, \tag{A.2}
\]

\[
\|f - f * \psi_\varepsilon\|_{C^\alpha} \lesssim [\nabla f]_{C^\beta} \varepsilon^{1 + \beta - \alpha}. \tag{A.3}
\]

Proof. These estimates are standard; for completeness, we give their short proofs. For (A.1), we have for $\varepsilon \ll 1$,

\[
\|f * \psi_\varepsilon\|_{C^{N+\alpha}} \leq \|f\|_{C^\alpha} + \|f * \nabla^N \psi_\varepsilon\|_{C^\alpha} \lesssim \varepsilon^{-N} \|f\|_{C^\alpha}.
\]

For (A.2), we first have the $L^\infty$ estimate

\[
|f(x) - f * \psi_\varepsilon(x)| \leq \int_{B_\varepsilon(0)} |f(x) - f(x - y)| \psi_\varepsilon(y) \, dy \lesssim [f]_{C^\beta} \varepsilon^\beta.
\]

We also have $|f(x) - f * \psi_\varepsilon(x) - (f(z) - f * \psi_\varepsilon(z))| \lesssim [f]_{C^\beta} |x - z|$, so that $[f - f * \psi_\varepsilon]_{C^\beta} \lesssim [f]_{C^\beta}$ and hence that $\|f - f * \psi_\varepsilon\|_{C^\alpha} \lesssim [f]_{C^\beta}$. So (A.2) follows by interpolating $\|f - f * \psi_\varepsilon\|_{C^\alpha} \leq [f - f * \psi_\varepsilon]_{C^\beta}^{(\beta - \alpha)/\beta} [f - f * \psi_\varepsilon]_{C^\alpha}^{\alpha/(1 + \beta)}$.

The estimate (A.3) similarly follows from the interpolation inequality

\[
\|f - f * \psi_\varepsilon\|_{C^\alpha} \leq \|f - f * \psi_\varepsilon\|_{C^{(1 + \beta - \alpha)/(1 + \beta)}} \|f - f * \psi_\varepsilon\|_{C^{1 + \beta}}^{\alpha/(1 + \beta)}
\]

once we prove the corresponding $L^\infty$ bound. For this, we use the fact that symmetry of the mollifier implies that $\nabla f(x) \cdot \int_{B_\varepsilon(0)} y \psi_\varepsilon(y) \, dy = 0$. Hence,

\[
f(x) - f * \psi_\varepsilon(x) = \int_{B_\varepsilon(0)} \left( f(x) - \nabla f(x) \cdot y - f(x - y) \right) \psi_\varepsilon(y) \, dy.
\]

Since the fundamental theorem of calculus implies $|f(x + ty) - f(x) - \nabla f(x) \cdot y| \leq \int_0^1 |\nabla f(x + ty) - \nabla f(x)| |y| \, dt \leq |y|^{1 + \beta} [\nabla f]_{C^\beta} \frac{1}{1 + \beta}$, we obtain

\[
\|f - f * \psi_\varepsilon\|_{C^\alpha} \lesssim [\nabla f]_{C^\beta} \varepsilon^{1 + \beta}.
\]

Since by (A.2) we also have $\|\nabla f - \nabla f * \psi_\varepsilon\|_{C^\beta} \lesssim [\nabla f]_{C^\beta}$, the inequality (A.3) follows. \qed
B Inverse Divergence Operator

We recall [Buc+15, Def. 1.4] the following operator of order $-1$,

$$R u = -(-\Delta)^{-1}(\nabla u + \nabla u^T) - \frac{1}{2}(-\Delta)^{-2}\nabla^2 \cdot u + \frac{1}{2}(-\Delta)^{-1}(\nabla \cdot u) I_{3 \times 3},$$

where $U = (-\Delta)^{-1}v$ is the mean-free solution to $-\Delta U = u - \int_{\mathbb{T}^3} u$. It is a matrix-valued right inverse of the divergence operator for mean-free vector fields, in the sense that

$$\text{div } R u = u - \int_{\mathbb{T}^3} u.$$

In addition, $R u$ is traceless and symmetric, satisfies the Schauder estimate

$$\|R u\|_{C^{1+\alpha}} \lesssim \|u\|_{C^\alpha}, \quad (B.1)$$

and the following non-stationary phase type estimates:

**Lemma B.1.** If $a \in C^\infty(\mathbb{T}^3)$ and $\Phi \in C^\infty(\mathbb{T}^3; \mathbb{R}^3)$ such that

$$1 \lesssim |\nabla \Phi| \lesssim 1,$$

then

$$\left| \int_{\mathbb{T}^3} a(x)e^{ik \cdot \Phi} \, dx \right| \lesssim \frac{\|a\|_{C^N} + \|a\|_{C^0} \|\Phi\|_{C^N}}{|k|^N},$$

and

$$\|R(a e^{ik \cdot \Phi})\|_{C^\alpha} \lesssim \frac{\|a\|_{C^\alpha}}{|k|^{1-\alpha}} + \frac{\|a\|_{C^{N+\alpha}} + \|a\|_{C^0} \|\Phi\|_{C^{N+\alpha}}}{|k|^{N-\alpha}}.$$

Acknowledgements

This work was supported by the National Key Research and Development Program of China (No. 2020YFA0712900) and NSFC Grant 11831004.

References

[BBV20] Rajendra Beekie, Tristan Buckmaster, and Vlad Vicol. “Weak solutions of ideal MHD which do not conserve magnetic helicity”. English. Ann. PDE 6.1 (2020). Id/No 1, p. 40. ISSN: 2524-5317. DOI: 10.1007/s40818-020-0076-1.
Elia Brué and Maria Colombo. *Nonuniqueness of solutions to the Euler equations with vorticity in a Lorentz space*. 2021. arXiv: 2108.09469 [math.AP].

Tristan Buckmaster, Maria Colombo, and Vlad Vicol. *Wild solutions of the Navier-Stokes equations whose singular sets in time have Hausdorff dimension strictly less than 1* (to appear in *JEMS*, DOI: 10.4171/JEMS/1162). 2020. arXiv: 1809.00600 [math.AP].

Tristan Buckmaster, Camillo De Lellis, Philip Isett, and László Székelyhidi. “Anomalous dissipation for 1/5-Hölder Euler flows”. English. *Ann. Math. (2)* 182.1 (2015), pp. 127–172. issn: 0003-486X. doi: 10.4007/annals.2015.182.1.3.

Tristan Buckmaster, Camillo De Lellis, László Székelyhidi, and Vlad Vicol. “Onsager’s conjecture for admissible weak solutions”. English. *Commun. Pure Appl. Math.* 72.2 (2019), pp. 229–274. issn: 0010-3640. doi: 10.1002/cpa.21781.

Tristan Buckmaster and Vlad Vicol. “Convex integration and phenomenologies in turbulence”. English. *EMS Surv. Math. Sci.* 6.1-2 (2019), pp. 173–263. issn: 2308-2151. doi: 10.4171/EMSS/34.

Tristan Buckmaster and Vlad Vicol. “Nonuniqueness of weak solutions to the Navier-Stokes equation”. English. *Ann. Math. (2)* 189.1 (2019), pp. 101–144. issn: 0003-486X. doi: 10.4007/annals.2019.189.1.3.

Peter Constantin, Weinan E, and Edriss S. Titi. “Onsager’s conjecture on the energy conservation for solutions of Euler’s equation”. English. *Commun. Math. Phys.* 165.1 (1994), pp. 207–209. issn: 0010-3616. doi: 10.1007/BF02099744.

Antoine Choffrut. “h-Principles for the Incompressible Euler Equations”. *Archive for Rational Mechanics and Analysis* 210.1 (2013), pp. 133–163. issn: 0003-9527. doi: 10.1007/s00205-013-0639-3.

Alexey Cheskidov and Xiaoyutao Luo. “Nonuniqueness of weak solutions for the transport equation at critical space regularity”. English. *Ann. PDE* 7.1 (2021). Id/No 2, p. 45. issn: 2524-5317. doi: 10.1007/s40818-020-00091-x.
[Dan14] S. Daneri. “Cauchy problem for dissipative Hölder solutions to the incompressible Euler equations”. English. Commun. Math. Phys. 329.2 (2014), pp. 745–786. issn: 0010-3616. doi: 10.1007/s00220-014-1973-5.

[De 18] Camillo De Lellis. “The Onsager theorem”. English. Celebrating the 50th anniversary of the Journal of Differential Geometry. Lectures given at the Geometry and Topology Conference at Harvard University, Cambridge, MA, USA, April 28 – May 2, 2017. Somerville, MA: International Press, 2018, pp. 71–101. isbn: 978-1-57146-361-6. doi: 10.4310/SDG.2017.v22.n1.a3.

[DF21] László Székelyhidi Daniel Faraco Sauli Lindberg. “Bounded Solutions of Ideal MHD with Compact Support in Space-Time”. Archive for Rational Mechanics and Analysis 239.1 (2021), pp. 51–93. issn: 0003-9527. doi: 10.1007/s00205-020-01570-y.

[DK20] Camillo De Lellis and Hyunju Kwon. On Non-uniqueness of Hölder continuous globally dissipative Euler flows. 2020. arXiv: 2006.06482 [math.AP].

[DRS21] Sara Daneri, Eris Runa, and László Székelyhidi. “Non-uniqueness for the Euler equations up to Onsager’s critical exponent”. English. Ann. PDE 7.1 (2021), Id/No 8, p. 44. issn: 2524-5317. doi: 10.1007/s40818-021-00097-z.

[DS09] Camillo De Lellis and László jun. Székelyhidi. “The Euler equations as a differential inclusion”. English. Ann. Math. (2) 170.3 (2009), pp. 1417–1436. issn: 0003-486X. doi: 10.4007/annals.2009.170.1417.

[DS13] Camillo De Lellis and László jun. Székelyhidi. “Dissipative continuous Euler flows”. English. Invent. Math. 193.2 (2013), pp. 377–407. issn: 0020-9910. doi: 10.1007/s00222-012-0429-9.

[DS17] Sara Daneri and László jun. Székelyhidi. “Non-uniqueness and h-principle for Hölder-continuous weak solutions of the Euler equations”. English. Arch. Ration. Mech. Anal. 224.2 (2017), pp. 471–514. issn: 0003-9527. doi: 10.1007/s00205-017-1081-8.

[DS19] Camillo De Lellis and László jun. Székelyhidi. “On turbulence and geometry: from Nash to Onsager”. English. Notices Am. Math. Soc. 66.5 (2019), pp. 677–685. issn: 0002-9920. doi: 10.1090/noti1868.
Philip Isett and Andrew Ma. “A direct approach to nonuniqueness and failure of compactness for the SQG equation”. English. *Nonlinearity* 34.5 (2021), pp. 3122–3162. ISSN: 0951-7715. DOI: 10.1088/1361-6544/abe732.

Philip Isett and Sung-Jin Oh. “On nonperiodic Euler flows with Hölder regularity”. English. *Arch. Ration. Mech. Anal.* 221.2 (2016), pp. 725–804. ISSN: 0003-9527. DOI: 10.1007/s00205-016-0973-3.

Philip Isett. “A proof of Onsager’s conjecture”. English. *Ann. Math. (2)* 188.3 (2018), pp. 871–963. ISSN: 0003-486X. DOI: 10.4007/annals.2018.188.3.4.

Philip Isett and Vlad Vicol. “Hölder continuous solutions of active scalar equations”. English. *Ann. PDE* 1.1 (2015). Id/No 2, p. 77. ISSN: 2524-5317. DOI: 10.1007/s40818-015-0002-0.

Nicolaas H. Kuiper. “On C1-isometric imbeddings. I” (1955), pp. 545–556.

Nicolaas H Kuiper. “On C1-isometric imbeddings. II” (1955), pp. 683–689.

Tianwen Luo and Edriss S. Titi. “Non-uniqueness of weak solutions to hyperviscous Navier-Stokes equations: on sharpness of J.-L. Lions exponent”. English. *Calc. Var. Partial Differ. Equ.* 59.3 (2020). Id/No 92, p. 15. ISSN: 0944-2669. DOI: 10.1007/s00526-020-01742-4.

Tianwen Luo, Tao Tao, and Liqun Zhang. “Finite energy weak solutions of 2d Boussinesq equations with diffusive temperature”. English. *Discrete Contin. Dyn. Syst.* 40.6 (2020), pp. 3737–3765. ISSN: 1078-0947. DOI: 10.3934/dcds.2019230.

Xiaoyutao Luo. “Stationary solutions and nonuniqueness of weak solutions for the Navier-Stokes equations in high dimensions”. English. *Arch. Ration. Mech. Anal.* 233.2 (2019), pp. 701–747. ISSN: 0003-9527. DOI: 10.1007/s00205-019-01366-9.

Andrew J. Majda and Andrea L. Bertozzi. *Vorticity and incompressible flow*. English. Cambridge: Cambridge University Press, 2002, pp. xii + 545. ISBN: 0-521-63057-6; 0-521-63948-4. DOI: 10.1017/CBO9780511613203.

John Nash. “$C^1$ isometric imbeddings”. English. *Ann. Math. (2)* 60 (1954), pp. 383–396. ISSN: 0003-486X. DOI: 10.2307/1969840.
Luigi De Rosa and Silja Haffter. *Dimension of the singular set of wild Hölder solutions of the incompressible Euler equations*. 2021. arXiv: 2102.06085v1 [math.AP].

Vladimir Scheffer. “An inviscid flow with compact support in space-time”. English. *J. Geom. Anal.* 3.4 (1993), pp. 343–401. issn: 1050-6926. DOI: 10.1007/BF02921318.

A. Shnirelman. “Weak solutions of incompressible Euler equations”. English. *Handbook of mathematical fluid dynamics*. Vol. II. Amsterdam: North-Holland, 2003, pp. 87–116. isbn: 0-444-51287-X.

Tao Tao and Liqun Zhang. “Hölder continuous solutions of Boussinesq equation with compact support”. English. *J. Funct. Anal.* 272.10 (2017), pp. 4334–4402. issn: 0022-1236. DOI: 10.1016/j.jfa.2017.01.013.

Tao Tao and Liqun Zhang. “Hölder continuous periodic solution of Boussinesq equation with partial viscosity”. English. *Calc. Var. Partial Differ. Equ.* 57.2 (2018). Id/No 51, p. 55. isbn: 0944-2669. DOI: 10.1007/s00526-018-1337-7.

Emil Wiedemann. “Weak-strong uniqueness in fluid dynamics”. English. *Partial differential equations in fluid mechanics. Based on the workshop “PDEs in Fluid Mechanics”, Warwick, UK, September 26–30, 2016* (2018), pp. 289–326.