PAPER

Dc to ac field conversion due to leaky-wave excitation in a plasma slab behind an ionization front

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Abstract

We present a way for generating coherent tunable electromagnetic radiation through dc to ac field conversion by an ionization front. The conversion is caused by the excitation of leaky waves behind the transversely limited ionization front propagating in a uniform electrostatic field. This differs significantly from the well-known dc-to-ac-radiation-converter models which consider Doppler-like frequency conversion by a transversely unlimited ionization front propagating in a spatially periodic electric field. We explore the dispersion properties and excitation of these leaky waves radiated through the transverse plasma boundary at the Cherenkov angle to the direction of propagation of a superluminal ionization front as dependent on the parameters of the plasma produced and on the speed of the ionization front. It is shown that not only the center frequency but also the duration and waveform of the generated pulse may significantly depend on the speed of the ionization front. The results indicate the possibility of using such converters based on planar photoconductive antennas to create sources of microwave and terahertz radiation with controllable waveforms that are transformed from video to radio pulse when the angle of incident ionizing radiation is tuned.

1. Introduction

Conversion of static electric fields by their interaction with moving ionization fronts has attracted considerable attention due to the possibilities of its use for the creation of frequency-tunable sources of electromagnetic radiation [1–12]. In the context of such a conversion, one-dimensional Doppler-like effects are often considered when waves interact with a transversely unlimited ionization front [1–7, 13–23]. A spatially periodic electric field produced by a system of capacitors is usually considered a wave to be converted. Such sources, called dc to ac radiation converters, are proposed for the generation of microwave and terahertz radiation [1–7]. Another type of problem is related to the conversion of a uniform static electric field interacting with a transversely limited superluminal ionization front [8–12]. An important role is played here by non-one-dimensional effects associated with a transverse (relative to the direction of the ionization front propagation) structure of the plasma produced, and the conversion is due to the excitation of plasma eigenmodes (so-called leaky waves) radiating through the transverse plasma boundaries. Ionization fronts behind which cylindrically shaped plasma is formed have been considered, and most studies have been carried out in the framework of the quasistatic approximation, in which the transverse dimensions of the plasma are supposed to be small. In this case, only the radiating mode is excited at the frequency of the so-called geometric resonance [10–12] of the forming plasma object, and the generated radiation spectrum does not depend on the ionization front speed.

In this paper, on the basis of the exact solutions to Maxwell’s equations, we examine the features of conversion of a uniform static electric field to electromagnetic radiation, features that are associated with the excitation of leaky waves behind the non-one-dimensional ionization front. For this purpose, we consider the interaction of a uniform electrostatic field with a moving ionization front confined in one of the transverse directions relative to the direction of the front propagation. Such ionization fronts are usually formed by oblique incidence of the ionizing
laser pulse on a photoconductive material flat boundary [5, 6, 20, 21]. In this case, the speed of the generated ionization front is greater than the speed of light and is determined by the incident angle of the ionizing radiation, which makes it easy to control the front speed through the adjustment of the angle. Available femtosecond laser systems are capable of securing the propagation of such fronts over sufficiently long distances and the formation of subsurface plasma over sufficiently large areas (tens of square centimeters) [24–27]. Photoconductive devices (including large-area ones) that use ultrafast ionization have been used for generation, detection, frequency conversion, phase shifting, and fast switching of terahertz or microwave radiation (see, e.g., [5–7, 20–31]).

Since the speed of propagation of ionization fronts is greater than the speed of light, fast leaky polarization waves are excited behind the front and radiate energy through the transverse plasma boundaries at an angle to the direction of front propagation. Leaky waves have been actively studied for more than half a century (see, e.g., [32–43]) and are now attracting attention in the context of the development of emitting and receiving devices with large apertures, including those which use metamaterials and photonic crystals [39–43]. In our case, the excited leaky waves are of a rarely studied type; they have complex values of frequency and wavenumber whose ratio has a real value equal to the speed of the ionization front. Here lies a distinction between our problem and similar ones regarding the conversion of plane waves interacting with an instantly generated plasma slab or half-space [44–47]. In the latter problems, both slow surface and fast leaky waves can be excited: their wavenumbers are determined by the longitudinal wavenumber of the wave to be converted, and the phase velocity is not fixed. The dispersion properties of leaky waves of fixed phase velocity, such as those we consider here, are practically unexplored for plasma slabs. In this paper, we for the first time pay special attention to the dispersion properties of waves excited in the propagation of the ionization front in an external electrostatic field directed perpendicular to the transverse boundaries of the forming plasma. The approach we develop allows us to find the excitation amplitudes of these waves, fully calculate the electromagnetic fields behind the front, and determine their dependences on all parameters of the front and the plasma formed behind it. We show for the first time that the parameters of the generated radiation, including its waveform and duration, may strongly depend not only on the transverse dimensions and density of the plasma produced but also on the speed of the ionization front, which gives one more control over the spectrum of the generated radiation. The control limits may have an extremely wide range, from multicycle (radio-pulse) to few-cycle (video-pulse) radiation.

The paper is organized as follows. In section 2, we describe the problem statement and initial equations. Then, in section 3, we solve these equations using the Laplace transform and find the images of electromagnetic fields and free-carrier current density. In section 4, using these Laplace images we find the temporal waveforms of the generated radiation. In section 4.1, we provide a solution in the form of a superposition of leaky modes guided along the plasma slab, and we describe the general properties of these modes. In the following sections, we consider various limiting cases in which we carry out a more detailed analytical examination of the dispersion properties of leaky modes and obtain analytical expressions for the generated waveforms. The obtained analytical solutions clearly demonstrate strong dependences of the generated waveforms on the ionization front speed and its transverse size. Sections 4.2 and 4.3 consider important limiting cases of producing, respectively, sufficiently thin or thick plasma slabs. In sections 4.4 and 4.5, we deal with limiting cases of slightly and strongly superluminal ionization fronts. In section 5, we find analytical expressions for the spectrum of the generated radiation and investigate the dependences of the spectra and radiated energy on the parameters of the generated plasma and the ionization front. Section 6 summarizes the results and makes conclusions.

2. Problem statement and assumptions

In this paper, we consider a superluminal ionization front that propagates inside a flat slab of an ionizable medium biased by an external uniform electrostatic field (see figure 1). The ionization front is produced at oblique incidence of a femtosecond laser pulse on one or both sides of this slab. The ionizable slab may be a ribbon-shaped supersonic gas jet or a solid-state photoconductive film. The slab is surrounded by a medium which is more difficult to ionize. The external electric field is perpendicular to the slab boundaries.

We should note that the presented theoretical approach to calculating the parameters of the generated electromagnetic radiation can be easily extended to a more general situation. And the ionization front can propagate in the surface layer of a bulk photoconductive sample or in a photoconductive film that is part of a layered solid structure, for example, film on a conducting substrate. The electric field can also be directed quite arbitrarily with respect to the direction of ionization front propagation and the transverse boundaries of the ionizable slab. We confine ourselves to the aforementioned separate slab of the ionized medium with a perpendicular external field only for reasons of simplicity.

The ionization front speed is determined by the laser pulse grazing angle $\theta$: $V_\parallel = V_I/\cos \theta$, where $V_I$ is the propagation speed of the laser pulse outside the slab. Only superluminal ionization fronts are considered, i.e., fronts moving with a speed $V_\parallel$ greater than the speed of light both in the ionizable slab and in the surrounding
non-ionizing medium. The time of plasma creation (ionization time) is thought to be small compared with the characteristic times of electric field transformation. For our calculations of the generated radiation, this means the instantaneous formation of plasma at the ionization front. We also do not consider slabs that are too thick in order to neglect the difference in laser radiation arrival times at different depths, and we assume the ionization front is flat and perpendicular to the boundaries of the ionizable slab.

Thus, the flat ionization front shaped as an infinite strip of width $2a$, $z = V_0 t$ and $|x| \leq a$, is moving along the $z$-axis of the Cartesian coordinate system in the presence of the external electrostatic field $E_0$. This external electric field is directed along the $x$-axis perpendicular to the direction of ionization front propagation and the transverse boundaries of the ionizable slab. Outside the slab, at $|x| > a$, the medium is non-ionized and has a constant and uniform dielectric permittivity $\varepsilon_x > 0$. Inside the slab before the ionization front, at $|x| \leq a$ and $\xi < 0$, the medium is also non-ionized and has the dielectric permittivity $\varepsilon_1 > 0$. Here $\xi = t - z/V_0$ is the shifted time. The speed of light outside and inside the slab is less than the ionization front speed: $c\varepsilon_x^{-1/2} < V_0$ and $c\varepsilon_1^{-1/2} < V_0$; $c$ is the speed of light in a vacuum. The external field is uniform in the Cartesian coordinates $y$ and $z$, so $E_0 = D_0/\varepsilon_x(x)$, where $D_0 = D_0\hat{x}$ is the external electric displacement, which does not depend on space coordinates and time, and $\varepsilon_x$ is the dielectric permittivity of the non-ionized medium: $\varepsilon_x = \varepsilon_n$ at $|x| > a$ and $\varepsilon_1 = \varepsilon_1$ at $|x| \leq a$. Plasma of density $N$ is formed instantly at the ionization front. The plasma density $N$ is uniform in the slab behind the ionization front and remains constant over time. In this way, we neglect the carrier loss processes (such as recombination, diffusion, electron attachment in gases, and trapping of carriers by impurities and defects in solids) since we assume the characteristic times of these processes to be large compared with the characteristic times of external field transformation.

To calculate the electric $E$ and magnetic $B$ fields behind the ionization front, we employ Maxwell's equations

$$\nabla \times E = -\frac{1}{c} \frac{\partial B}{\partial t}, \quad \nabla \times B = \frac{4\pi}{c} j + \frac{\varepsilon_0}{c} \frac{\partial E}{\partial t} \tag{2.1}$$

and the linear equation for the free-carrier current density $j$ in cold plasma,

$$\frac{\partial j}{\partial t} + \nu j = \frac{Ne^2}{m_{\text{eff}}} E, \quad |x| \leq a, \tag{2.2}$$

where $e = |e|$ is the elementary charge, $m_{\text{eff}}$ is the effective mass of free carriers, and $\nu$ is the effective collision frequency (the inverse time of current relaxation). In gas plasma, $m_{\text{eff}}$ is equal to the electron mass, and $\nu$ is defined by the frequency of collisions between electrons and heavy particles. In solids, $m_{\text{eff}}$ is equal to the reduced mass of the electrons and holes, and $\nu$ is determined by electron and hole mobilities. In what follows, we focus on the case of small $\nu$ compared with the plasma frequency $\omega_p = (4\pi Ne^2/\varepsilon_0 m_{\text{eff}})^{1/2}$, although some results and formulas hereafter are applicable to sufficiently large enough $\nu$. The small ratios $\nu/\omega_p$ can be inherent in the

![Figure 1. Schematic of conversion of a uniform electrostatic field to electromagnetic radiation. A laser pulse is obliquely incident from a medium with dielectric permittivity $\varepsilon_x$ on a dc-biased flat slab with a dielectric permittivity $\varepsilon_n$ and a half-thickness of $a$ and creates a superluminal ionization front with the speed $V_0$ inside the slab. The dc-bias field $E_0$ is perpendicular to the slab transverse borders. A fast polarization wave is excited behind the front and radiates energy into the surrounding space through the transverse boundaries of the slab at the angle $\theta = \arccos(c/V_0\varepsilon_x^{1/2})$ to the direction of the front propagation and the slab boundaries.](image)

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photoconductors excited by ultrafast laser pulses (for example, experimentally measured electron mobilities for GaAs in [48]) correspond to $\nu/\alpha_p$ less than 0.1 as well as in the laser gas plasma (for fully ionized atmospheric air with $N = 3 \times 10^{19}$ cm$^{-3}$ and a typical electron temperature of 15 eV, we have $\nu/\alpha_p \sim 0.02$ according to formulas in [49]). We neglect the time dispersion associated with the movement of bound charges (ions and bound electrons) since we consider it to be small at our frequencies. In particular, in the case of binary semiconductors, the radiation is calculated only at frequencies lower than the frequency of the optical phonon. The spatial dispersion of the plasma is also neglected. Thus, we study only the excitation of surface plasmons with charge-density perturbations localized at the plasma boundaries and do not consider the excitation of so-called volume or bulk plasmonic modes (see, for example, [50–52]) with charge-density perturbations in the bulk volume of the plasma.

Since the front velocity is greater than the speed of light in the non-ionized medium, the front has no precursors, and in the area before the front, at $\xi < 0$, the electric field is equal to $E_0$, and the magnetic field is zero. Assuming the propagation path of the front to be large enough, we are looking for the solution to (2.1) and (2.2) in the form of a steady-state wave. In such a wave, the electric and magnetic fields and the free-carrier current density depend only on the shifted time $\xi$ and the transverse coordinate $x$. To calculate the field and current density behind the ionization front, at $\xi > 0$, the initial conditions for the electric and magnetic fields and the current density in plasma are set at the ionization front, at $\xi = 0$. These conditions correspond to the continuity of the electric and magnetic fields and to the absence of current density at the ionization front:

$$E \mid_{\xi=0} = E_0, \quad B \mid_{\xi=0} = 0, \quad j \mid_{\xi=0} = 0. \quad (2.3)$$

The problem is finalized by radiation conditions at $|x| \to \infty$ that are equivalent to the conditions following from the principle of causality: the electric and magnetic fields are not disturbed and are equal to their initial values at points separated from the ionization front points by space-like intervals, i.e., at $|x| > a + (\epsilon_0 c^2 - V_{it}^2)^{-1/2} \xi$. Under this condition, the electromagnetic field is a superposition of a plane wave propagating at the angle $\theta = \arccos(\epsilon/e_i^{1/2} V_{it})$ to the plane $yz$ and a static electric field $E_0$ in each of the external areas $a < x < a + (\epsilon_0 c^2 - V_{it}^2)^{-1/2} \xi$ separated from the ionization front by time-like intervals.

Substitution of $E = e_i^{1/2} E'$, $B = B'$, $j = j', \xi = e_i^{1/2} \xi'$, $x = x'$ reduces the initial problem to a similar problem with primed variables in which $e_i' = 1$, $\epsilon_0' = \epsilon/e_i$, $\nu' = e_i^{1/2} \nu$, $V_{it}' = e_i^{1/2} V_{it}$. Therefore, without loss of generality, we further assume $e_i = 1$. This makes it convenient to characterize the superluminal ionization front speed by the angle $\theta$. Note also that $e_i$ can be less than unity but should be greater than $\cos^2 \theta$, i.e., $e_i > \cos^2 \theta$, since we assume superluminal propagation of the ionization front.

In general, the ionizing laser pulse produces a finite residual current density at the ionization front [53–58], which should be taken into account in the initial conditions (2.3). This residual current density may arise due to the acceleration of freed carriers by the ponderomotive force of the laser pulse or by the electric field of the few-cycle or bichromatic laser pulse itself. Since the stated problem is linear in the initial conditions, the contribution to the generated radiation from the external electric field and from freed-carrier acceleration by the ionizing laser field can be treated independently. In this article, we focus only on the contribution induced by the external electric field. In [53, 54], the contribution induced by the residual current density was studied for the case of the axisymmetric superluminal ionization front created by an axicon-focused femtosecond laser pulse in a gas.

Note that electric and magnetic fields at the plane of symmetry $x = 0$ are perpendicular to this plane at every instant. Thus, the solution to the stated problem also provides the solution to the problem regarding a superluminal ionization front propagating in a slab disposed on an ideally conducting metallic substrate.

### 3. Laplace images of fields and current density

The initial conditions (2.3) excite a stationary transverse magnetic wave behind the ionization front. This transverse magnetic wave has only nonzero $E_x$ and $B_z$ electric field components and a $B_y$ magnetic field component. We write out (2.1) and (2.2) for the steady-state transverse magnetic wave and apply the Laplace transform with respect to the variable $\xi$ to the obtained system, taking into account the initial conditions (2.3). The Laplace transform of function $F(\xi)$ is defined as $\hat{F}(q) = \int_{0}^{\infty} F(\xi) \exp(-q \xi) \, d\xi$, where $q$ is the Laplace variable [59]. We reduce the obtained system of equations to the equation for the Laplace image $\hat{B}_y$ of the magnetic field:

$$\frac{\partial}{\partial x} \left( \frac{1}{\epsilon} \frac{\partial \hat{B}_y}{\partial x} \right) - s^2 \frac{1}{\epsilon} \hat{B}_y = \frac{D_0}{\epsilon} \cos \theta \hat{q} \left( \frac{1}{\epsilon_0} - \frac{1}{\epsilon} \right). \quad (3.1)$$

Here $s^2 = q^2 \left( 1 - \cos^2 \theta \right)/c^2$ is the square of the transverse propagation constant, $\epsilon$ is the dielectric permittivity which relates the images $D$ and $E$ of the electric displacement and field vectors, respectively,
\[ D = \varepsilon \mathbf{E}, \quad \varepsilon = \begin{cases} 1 & \text{for } |x| > a, \\ \varepsilon_p & \text{for } |x| \leq a, \end{cases} \]

and \( \varepsilon_p = \varepsilon_i[1 + \omega_p^2/(q^2 + \nu q)] \) is the dielectric permittivity in plasma. The Laplace images \( \tilde{E}_z \) and \( \tilde{E}_x \) of the longitudinal and transverse components of the electric field are expressed in terms of \( \tilde{B}_y \) as follows:

\[ \tilde{E}_z = \frac{\varepsilon}{\nu q} \frac{\partial \tilde{B}_y}{\partial x}, \quad \tilde{E}_x = \frac{\cos \theta}{\varepsilon} \tilde{B}_y + \frac{D_b}{\nu q}. \tag{3.2} \]

The Laplace image of the current density vector in plasma \( \tilde{j} \), at \(|x| \leq a\), relates to the image of the electric field vector \( \tilde{E} \) as

\[ \tilde{j} = \frac{\varepsilon_i \omega_p^2}{4\pi (q + \nu)} \tilde{E}. \tag{3.3} \]

We solve (3.1) for the homogeneous areas \( x < -a, \ |x| < a, \) and \( x > a \) and satisfy the boundary conditions of continuity of the tangential electric and magnetic fields at the slab boundaries \( |x| = a \). In each of external areas \( x > a \) and \( x < -a \), radiation conditions allow us to leave only one of two exponential terms in the general solution to (3.1). The resultant Laplace image for the magnetic field has the following form:

\[ \tilde{B}_y = \begin{cases} \tilde{B}_h \exp[-s_p(|x| - a)] & \text{for } |x| \geq a; \\ \tilde{B}_h + \left( \tilde{B}_h - \tilde{B}_b \right) \frac{\cosh s_p x}{\cosh s_p a} & \text{for } |x| < a. \end{cases} \tag{3.4} \]

Here \( s_p = (q/c) \sin \theta \) and \( s_p = (q/c)(\varepsilon_p - \cos^2 \theta)^{1/2} \) are the values of \( s \) inside and outside the slab (the sign before the root in \( s_p \) can be chosen arbitrarily);

\[ \tilde{B}_h = -\frac{D_b q \cos \theta}{c^2 s_p} \left( \frac{\varepsilon_p}{\varepsilon_i} - 1 \right) \tag{3.5} \]

is the Laplace image of the magnetic field \( B_h(\xi) = \lim_{x \to -\infty} B_y(\xi, x = 0) \) behind the one-dimensional superluminal ionization front (i.e., behind the front of infinite width \( a \to \infty \)); and

\[ \tilde{B}_b = -\frac{D_b q \cos \theta}{c^2 \Delta} \left( \frac{\varepsilon_p}{\varepsilon_i} - 1 \right) \tag{3.6} \]

is the Laplace image of the magnetic field \( B_b(\xi) = B_y(\xi, x = a) \) at the slab boundary, where

\[ \Delta = s_p^2 + \varepsilon_p s_p \coth s_p a. \tag{3.7} \]

### 4. Inversion of Laplace images

#### 4.1. Expansion in leaky modes

Time-dependent originals of the fields and the current density can be obtained via the inversion integral

\[ F(\xi) = (1/2\pi i) \int_{a - i\infty}^{a + i\infty} \tilde{F}(q) \exp q^2 dq, \]

where the integration is performed along the vertical line \( \text{Re } q = \sigma \) in the complex plane of the variable \( q \), and \( \sigma \) is the real number greater than the abscissa of absolute convergence of \( F(\xi) \), i.e., a number such that the integration contour lies entirely within the convergence domain of the image function \( \tilde{F}(q) \) (see [59] for reference). For the fields and current density in our problem, it is sufficient that \( \sigma > 0 \). All the obtained images have single-valued analytic continuation from the convergence domain of the Laplace transform to the entire complex plane of the variable \( q \). These analytic continuations are also described by (3.2)–(3.7) and are regular functions of \( q \) everywhere except for a countable set of singularities. This set consists only of poles; i.e., the image functions are meromorphic if \( \nu = 0 \). If \( \nu > 0 \), the set of singularities also contains a non-isolated singularity which is a limit point of poles. Thus, for \( \nu = 0 \), we can calculate the inversion integral for the originals of the fields and the current density inside the slab \(|x| \leq a\) using the residue theorem [59], and we can write the originals as a series of residues as follows:

\[ [\mathbf{E}, \mathbf{B}, \mathbf{j}] = \sum_{k=0}^{\infty} \text{Res}_{q_k} \left( [\tilde{\mathbf{E}}, \tilde{\mathbf{B}}, \tilde{\mathbf{j}}] \exp q_k^2 \right), \quad \xi \geq 0. \tag{4.1} \]

Here \( q_k \) are the poles enumerated in nondecreasing order of their absolute values, and \( \text{Res} \) denotes the residue at a point. Using the obtained magnetic field image (3.4), the expressions (3.2) for the electric field images, and the translation theorem (see [59]), we can obtain the field outside the slab, at \(|x| > a\), in terms of the original \( B_h(\xi) \),...
where \( q \) is the signum; here it also taken into account that \( B_0(\xi < 0) \equiv 0 \). Expressions (4.2) and (4.3) demonstrate that the fields in each of the areas \( \pm x > a \) outside the slab are a superposition of the initial static field \( E_0 \) and the plane wave propagating at the angle \( \theta \) to the plane \( xz \). Thus, since expression (4.1) holds for the magnetic field at the slab boundary, at \( |x| = a \) and \( \xi > 0 \), this expression is also valid for the fields outside the slab at points separated from the ionization front by time-like intervals, i.e., at \( a < |x| < a + 2 \xi \sin \theta \).

If collisions are absent and \( v = 0 \), the set of poles includes zeros of function the \( \Delta(q) \) and also the point \( q = q_0 = 0 \). The point \( q = 0 \) is a pole only for the magnetic field image \( \tilde{B}_y \) and the longitudinal current density image \( \tilde{j}_z \) inside the slab, at \( |x| < a \), and for the image \( \tilde{E}_x \) of the transverse electric field outside the slab, at \( |x| > a \). The respective residues at \( q = 0 \) are constant over time and are described by the following expressions:

\[
\text{Res}_{q=0} \tilde{B}_y = -\frac{D_0 \cos \theta}{\epsilon_1} \left[ 1 - \frac{\cos(\sqrt{\epsilon_1 \omega_p a/c})}{\sin(\sqrt{\epsilon_1 \omega_p a/c})} \right],
\]

\[
\text{Res}_{q=0} \tilde{j}_z = \frac{i \omega_p D_0 \cos \theta}{4\pi \sqrt{\epsilon_1}} \frac{\sinh(\sqrt{\epsilon_1 \omega_p a/c})}{\cosh(\sqrt{\epsilon_1 \omega_p a/c})},
\]

at \( |x| < a \) and

\[
\text{Res}_{q=0} \tilde{E}_x = D_0
\]

at \( |x| > a \).

All other poles have a negative real part, and their respective residue terms in (4.1) vanish at \( \xi \to +\infty \). Each of these terms satisfies the original equations (2.1) and (2.2) and thus presents some eigensolution to the electrodynamic problem for the plasma slab. These eigensolutions can be referred to as leaky eigenmodes guided along the slab. In these modes, both the frequency \( \omega_k = \omega_0 \) and the longitudinal wavenumber \( h_k = \omega_k/\nu_0 = i \cos \theta \) are not real, whereas their ratio is real and is equal to the ionization front speed \( \nu_0 \) (in the expressions for \( \omega_k \) and \( h_k \), the signs correspond to \( e^{-i\omega t} \) processes). Substituting \( q = -i\omega_p \cos \theta = h/c \) converts the equation for the poles \( \Delta = 0 \) to the known dispersion relation for even transverse magnetic modes propagating along the plasma slab [32–35]. If a pole \( q_\epsilon \) has a nonzero imaginary part, the complex conjugate point \( \gamma_\epsilon \) is also a pole. In (4.1), two terms corresponding to complex conjugate poles constitute a purely real field of the leaky mode having a frequency with the real part the real part \( \Omega_k = |\Re \omega_\epsilon| = |\Im \omega_\epsilon| \) and the decrement \( \gamma_k = -\Im \omega_\epsilon = -\Re \epsilon_\epsilon \). Since there are no internal losses at \( \nu = 0 \), the attenuation of a leaky mode is due entirely to the radiation through the slab boundaries. Thus, the mode decrement \( \gamma_k \) can be considered as the decrement \( \gamma_k^{(1)} \equiv \gamma_k \) associated with radiative losses. Besides the complex conjugate poles, up to four real negative poles may exist that respond to the leaky modes with zero real part of the frequency, \( \Omega_k = 0 \), and the decrement \( \gamma_k = i\omega_k = -q_\epsilon \). The corresponding terms in (4.1) are purely real. Thus, at \( \nu = 0 \), the fields are superpositions of static fields described by (4.4) and (4.6) and of an infinite number of leaky eigenmodes inside the wedge \( |x| < a + 2 \xi \sin \theta, \xi > 0 \).

As far as we know, the dispersion properties of such leaky modes (fast waves with a real ratio of complex frequency and wavenumber) in a plasma slab have not been systematically studied before. Therefore, to determine the dependence of complex frequencies of leaky modes on the parameters of the plasma slab and the speed of the ionization front, we solved the equation \( \Delta = 0 \) numerically for a wide range of parameters. Figure 2 presents the obtained real parts \( \Omega_k \) of the eigenfrequencies and decrements \( \gamma_k \) of the radiation losses of the lowest leaky modes as functions of the angle \( \theta \) for different values of the dimensionless slab half-thickness \( \alpha_0 a/c \) and the initial dielectric permittivity \( \epsilon_i \) of the slab. It is seen that the real parts of the frequencies and decrements depend strongly on the parameters of the ionization front: its speed, its transverse size, the density of the forming plasma, and the initial dielectric permittivity of the ionized slab. These dependences can be rather complicated. In particular, there are parameter values from which the dependences become extremely sharp and a small change in one of the parameters leads to a strong change in the frequencies and decrements of the leaky modes. This can happen, for example, around the degeneracy points, i.e., the points at which a high-order pole is present. In the case of a purely real second-order pole (marked by filled circles in figure 2), two complex conjugate poles merge and then turn into two different purely real poles as one of the parameters changes continuously. In this case, one mode with a frequency having a nonzero real part turns into two modes with purely imaginary frequencies. This corresponds to the transition from oscillating damped (radio pulse) terms to
the non-oscillating unipolar (video pulse) term in the sum (4.1). Figure 3 shows the examples of possible radiated waveforms \( \xi \) calculated using the inversion integral. It can be seen that the waveform of the generated pulse strongly depends on the speed of the ionization front, its transverse size, the plasma frequency, and the initial dielectric permittivity of the slab. In the following sections of this section, we consider some limiting cases allowing one to obtain comprehensive analytical expressions for the frequencies and waveforms of...
radiated pulses. These expressions clearly show that the waveform and the modal composition of the radiated pulse can significantly depend on the parameters of the ionization front and the ionizable slab and reveal the nature of these relationships. In particular, in section 4.2, we consider the case of a sufficiently thin slab and obtain simple expressions describing the degeneracy situation mentioned previously.

Before that, let us expand the residues in (4.1) and obtain an expression for the waveform of the radiated pulse \( B_p(\xi) \). The magnetic field \( B_p \) at the slab boundary does not contain the static term, and the summation goes only over zeros of \( \Delta \),

\[
B_p = -\frac{D_0 \cos \theta}{\varepsilon_i} \sum_{k=1}^{\infty} \left[ \left( \frac{q_k}{\omega_p} \right)^2 + 1 \right] \exp \frac{q_k \xi}{P(q_k)}, \tag{4.7}
\]

where

\[
P(q) = \frac{\varepsilon_i \omega_i^2 \sin \theta}{\omega_n^2} \frac{a}{c} \left( \frac{q^2}{\omega_i^2} + 1 \right) \left( \frac{q^2}{\omega_p^2} + 1 \right) + \left( \frac{2}{\omega_n^2} - \frac{1}{\omega_p^2} \right) q^2 + 1.
\]

Here \( \omega_i^2 = \omega_p^2/(1 - \varepsilon_i^{-1}) \) is the squared frequency at which the dielectric permittivity of a collisionless plasma is unity and the plasma boundaries do not reflect waves; \( \omega_i^2 = \omega_p^2/(1 - \varepsilon_i^{-1} \cot^2 \theta) \) is the squared frequency at which the Brewster condition \( \varepsilon = \cot^2 \theta \) holds for collisionless plasma and the boundaries do not reflect waves with the grazing angles \( \theta \) in a vacuum and \( \pi/2 - \theta \) in plasma; \( \omega_i^2 = \omega_p^2/(1 - \varepsilon_i^{-1} \cos^2 \theta) \) is the squared frequency of the transverse wave in a collisionless plasma with phase velocity equal to the ionization front velocity. The frequencies \( \omega_0 \) and \( \omega_b \) can be purely real, purely imaginary, or infinite depending on the values of the dielectric permittivity \( \varepsilon_i \) and the angle \( \theta \). The frequency \( \omega_0 \) is always real and finite for the superluminal ionization front. At \( \varepsilon_i = 1 \), the expression for \( P(q) \) is simplified to

\[
P(q) = \cos^2 \theta + \sin^2 \theta \left( \frac{\omega_p a \sin \theta}{c} \frac{q}{\omega_p} + 1 \right) \left( \frac{q^2}{\omega_p^2} + 1 \right).
\]

Equation (4.7) is obtained for the nondegenerate case where all the poles are first-order ones, i.e.,

\[
P(q_k) \equiv c^2 q_0^{-4} q_k \left( \Delta(q) = q_k \right)/\partial q \neq 0 \quad \text{for} \quad k > 0.
\]

For the degenerate cases, the expression for \( B_p \) can be obtained from (4.7) as limiting cases.

In the case of collisions, \( \nu > 0 \), most of the poles are shifted closer to the imaginary axis of the complex plane relative to the collisionless case, and leaky waves are partly absorbed inside the slab. In the case of relatively rare collisions, collisional corrections \( \delta q_k^{(c)} \) to the poles can be found with the use of Taylor’s formula for the function \( \Delta(q, \nu) \) as follows:

\[
\delta q_k^{(c)} \approx -\nu \left. \frac{\partial \Delta/\partial \nu}{\partial \Delta/\partial q} \right|_{q=q_k} \approx -\frac{\nu}{2} \left[ \frac{P_0(q)}{P(q_k)} - \frac{\omega_n^2}{q_k^2} \left[ 1 - \frac{P_0(q_k)}{P(q_k)} \right] \right], \tag{4.8}
\]

where \( P_0(q) = P(q)|_{\nu=0} = \left[ 2/\omega_n^2 - 1/\omega_p^2 \right] q^2 + 1 \). As seen from (4.8), the correction linear in \( \nu \) has both nonzero real and imaginary parts, although in some limiting cases (for example, at \( \omega_0 \omega_p/c \rightarrow 0 \) or \( q_k \rightarrow \pm \omega_n \)), the imaginary part of the correction tends to zero. The absolute value of the correction real part can be considered the collisional decrement \( \delta q_k^{(c)} \) of a leaky wave.

The collisions also cause the previously static components of the magnetic field and the current density equations (4.4) and (4.5) to damp. These damped components are also present in the generated radiation; i.e., they present themselves as a leaky field. It possible to say that the collisions remove the infinite degeneracy for the eigenmode at zero frequency. As a result, an infinite number of additional leaky modes exist (and corresponding to them poles of the Laplace images). The complex frequencies of these modes are purely imaginary, and the corresponding poles are real and lie in the interval \((-\nu, 0)\). Point \( q = -\nu \) is for them an accumulation point.

4.2. Generated waveform for a thin slab

In this section, we investigate the limiting case of the ionization front moving inside a thin enough slab. In the quasistatic limit, at \( \omega_p a/c \rightarrow 0 \), the frequencies of all the leaky modes except for one tend to infinity (for \( \nu = 0 \)). The real part of the eigenfrequency \( \Omega_\nu \) of the remaining mode tends to the frequency of geometric resonance for the slab in the transverse electric field, i.e., to the plasma frequency, and the radiative decrement tends to zero. By approximating the function \( f(q, a) = (\Delta/s_p) \tanh s_p a \) with its second-order Taylor polynomial and equating it to zero, we find the corrections to the frequency and the decrement for a small but finite value of \( \omega_p a/c \):
\[ \Omega_i = \omega_p + \delta \Omega_i, \]
\[ \delta \Omega_i \approx \left( 1 - \frac{5}{4} \frac{\cos^2 \theta}{\varepsilon_i} \right) \alpha^2 \cos^2 \theta \omega_p, \tag{4.9} \]
\[ \gamma_i^{(0)} \approx \frac{\alpha \cos^2 \theta}{2\varepsilon_i} \omega_p, \tag{4.10} \]

where \( \alpha = \omega_p a/c \sin \theta \). Parameter \( \alpha \) can be considered the phase incursion that a wave at the plasma frequency acquires in a vacuum for a propagation distance \( a/\sin \theta \), i.e., during the travel through the slab of thickness \( a \) at angle \( \theta \) to its border. Corrections (4.9) and (4.10) can be used if \( \alpha \ll 1 \). This inequality is in fact a condition of applicability of the quasistatic approximation to the present problem.

Using (4.8), one can obtain the corresponding decrement \( \gamma_i^{(0)} \approx -\nu/2 \) associated with weak loss effects (at \( \nu \ll \omega_p \)). Then, using (4.7), we obtain the generated waveform
\[ B_b \approx -\frac{D_0 \alpha \cos \theta}{\varepsilon_i} \exp \left( -\frac{\alpha \cos^2 \theta}{2\varepsilon_i} \omega_p \xi - \frac{\nu \xi}{2} \right) \sin \Omega_i \xi. \tag{4.11} \]

The resultant solution is actually the known quasistatic weakly damped solution at a frequency close to the frequency of the geometric resonance. This is a poorly radiating solution with the emitted pulse having a small amplitude of about \( D_0 \alpha \varepsilon_i \cos \theta \ll D_0 /\varepsilon_i \).

Let us show now that there are qualitatively different solutions even in the case of a thin layer with \( \varepsilon_i^{1/2} \omega_p a/c < 1 \) when condition \( \alpha \ll 1 \) is violated. This can happen, for example, at small values of angle \( \theta \ll 1 \), or in the case \( \omega_p a/c \sim 1 \) for small values of dielectric permittivity \( \varepsilon_i \ll 1 \). We can obtain the solutions in the cases mentioned if we assume the argument of the hyperbolic cotangent in the expression for \( \Delta \) to be small near the poles: \( |s_\tau a| \ll 1 \) and \( \cosh s_\tau a \approx 1/s_\tau a \). Then the transcendental equation for the poles turns into a cubic one, \( Q(q) = 0 \), where
\[ Q(q) = \frac{\alpha q^2 (q + \nu)}{\omega_p \omega_n^2} + \frac{q(q + \nu)}{\omega_p^2} + \frac{\alpha q}{\omega_p} + 1. \tag{4.12} \]

The expression for the magnetic field image at the boundary becomes
\[ B_b = -\frac{D_0 \alpha \cos \theta}{\varepsilon_i \omega_p Q(q)}. \tag{4.13} \]

The general solution to the equation \( Q(q) = 0 \) is rather cumbersome. Therefore, we consider here only the case of dielectric permittivity close to unity or lower it when \( \omega_p^2 /\omega_n^2 = 1 - \varepsilon_i^{-1} \cos^2 \theta \ll 1/o, 1/\alpha^2 \). Later, in section 4.4, we also consider the limiting case of large \( \alpha \) for \( \varepsilon_i > 1 \). To satisfy strict inequality for poles \( |s_\tau a| \ll 1 \), we must assume \( \varepsilon_i^{1/2} \omega_p a/c \ll 1, |1 - \alpha o /\omega_p^2|/2 \).

If \( \omega_p^2 /\omega_n^2 \ll 1/\alpha, 1/\alpha^2 \), the cubic term in \( Q \) is small enough, and we ignore it. By doing this, we lose one root being sufficiently large in absolute value and not giving a significant contribution to the final result. The roots of the obtained quadratic equation are \( \omega_p [-\bar{a} \pm \sqrt{(\bar{a}^2 - 0.5)]} \), where \( \bar{a} = \alpha + \nu /\omega_p \). These two roots are complex conjugate to each other if \( \bar{a} < 2 \); they are the same and there is a degeneracy if \( \bar{a} = 2 \); they are real and different if \( \bar{a} > 2 \). We obtain the original
\[ B_b = -\frac{D_0 \alpha}{\omega_p} \exp \left( -\frac{\alpha \omega_p \xi}{2} \right) \cdot \begin{cases} \frac{\sin \left( \sqrt{1 - \bar{a}^2 /4} \omega_p \xi \right)}{\sqrt{1 - \bar{a}^2 /4}} & \text{for } \bar{a} < 2; \\ \frac{\sinh \left( \sqrt{\bar{a}^2/4 - 1} \omega_p \xi \right)}{\sqrt{\bar{a}^2/4 - 1}} & \text{for } \bar{a} = 2; \\ \frac{\sin \left( \sqrt{\bar{a}^2/4 - 1} \omega_p \xi \right)}{\sqrt{\bar{a}^2/4 - 1}} & \text{for } \bar{a} > 2. \end{cases} \tag{4.14} \]

This expression shows that the pulse waveform can change dramatically depending on the value of \( \alpha \), that is, depending on the angle \( \theta \) or the dimensionless slab half-thickness \( \omega_p a/c \). For small \( \alpha \), a weakly damped oscillating radio pulse is generated; for large \( \alpha \), a unipolar video pulse is all. This can be illustrated by the waveforms shown in figure 3(a). The video pulse with the shortest duration (about \( 2.45 /\omega_p \) at the field-maximum level) is achieved at \( \alpha = 2 \), i.e., at the degeneracy point. The amplitude of this shortest pulse is \( 2D_0/c \approx 0.74D_0 \) (here and only here, \( e \) denotes Euler’s number). Thus, even at \( \varepsilon_i^{1/2} \omega_p a/c < 1 \), well-radiating solutions exist; that is, the rather short pulses having an amplitude comparable to the external electric field in the slab can be generated, and the shortest pulses are generated when a mode degeneracy is present.
In general, the duration of the generated pulse can be characterized by the minimal (over all excited modes) radiative decrement. Numerical solution of the equation $\Delta = 0$ shows that the largest value of this minimal decrement at the given value $\varepsilon_i < 9/8$ and $\nu = 0$ is achieved near the multiple degeneracy points with the third- and fourth-order poles or pairs of complex conjugate second-order poles present. Although in general the third-order pole is at a boundary of the applicability domain of the equation $Q(q) = 0$, this equation can be used to estimate the position of the third-order pole $q$ and the parameter values at which we have a triple degeneracy. To do this, we equate $Q, \partial Q/\partial q$, and $\partial^2 Q/\partial q^2$ with zero and solve the obtained equation set assuming that $\varepsilon_i$ is given and there are no collisions, $\nu = 0$. For the point of triple degeneracy, we have

$$\cos^2 \theta \approx \frac{8\varepsilon_i}{9}, \quad \frac{\omega_p a}{c} \approx \sqrt{3 - \frac{8\varepsilon_i}{3}}, \quad q \approx -\sqrt{3} \omega_p.$$  

(4.15)

For comparison, the joint numerical solution to the equations $\Delta = 0$, $\partial \Delta / \partial q = 0$, and $\partial^2 \Delta / \partial q^2 = 0$ gives $\alpha_p a/c \approx 0.83$, $\nu \approx 0.42 \approx 24^\circ$ and $q \approx -2.03 \omega_p$ for the triple degeneracy point at $\varepsilon_i = 1$. For these parameter values, the radiated pulse has a duration at the field half-maximum level of about 1.69 $/\omega_p$ and an amplitude of about 0.82$D_h$. If $\varepsilon_i \geq 9/8$, there are no high-order poles. The value 9/8 in the preceding inequality is accurate; that is, at $\varepsilon_i \geq 9/8$, not only the equation $Q(q) = 0$ but also the equation $\Delta(q) = 0$ has no multiple zeros. This is due to the fact that the value of $s_p a$ at the third-order pole tends to zero at $\varepsilon_i \to 9/8 - 0$. Thus, the errors of the expression (4.15) also tend to zero. Similarly, the value of $s_p a$ for the third-order pole and the errors of the expression (4.15) tend to zero at $\varepsilon_i \to +0$. There are two values of the dielectric permittivity $\varepsilon_i$: $\varepsilon_i \approx 0.97$ and $\varepsilon_i \approx 0.27$, at which a quadruple degeneracy may exist. There is no triple degeneracy point at $\varepsilon_i$ between the mentioned values, but pairs of the second-order complex conjugate poles may exist. Such pairs also correspond to the well-radiating solutions.

4.3. Generated waveform for a thick slab

In this section, we examine the generated radiation for the case of a thick slab, i.e., for the case that is the opposite of the preceding case. For this purpose, we find the originals of the magnetic field $B_h^{(\infty)}(\xi) = \lim_{\varepsilon_i \to 0} B_h(\xi)$ at the boundary of the half-space, inside which the ionization front moves with superluminal velocity. For a slab of finite thickness at $\xi < 2(\varepsilon_i - \cos^2 \theta)^{1/2} a/c$, one of the plasma transverse boundaries is separated from the opposite boundary by a space-like interval. Thus, no signal can reach from one boundary to the opposite boundary, and $B_h[\xi \leq 2(\varepsilon_i - \cos^2 \theta)^{1/2} a/c] \equiv B_h^{(0)}$ due to the principle of causality. Therefore, the function $B_h^{(0)}$ describes the beginning of the radiated pulse for any value of slab thickness, and the thicker the slab, the later the beginning of the discrepancy between the generated waveform profile and $B_h^{(0)}$.

Note also that the causality principle makes it easy to find the field originals in the wedge-like region $0 \leq \xi \leq (\varepsilon_i - \cos^2 \theta)^{1/2}(a - \{|x|\})/c$ inside the slab. The interior points of this region are separated from both edges of the ionization front (i.e., lines $x = \pm a$, $\xi = 0$) by space-like intervals. Thus, in this region, the fields do not differ from the fields behind the one-dimensional (uniform) ionization front: $B_h = B_h$, $E_h = E_h \equiv \lim_{\varepsilon_i \to 0} E_h(x = 0)$, and $E_h = 0$. The Laplace transform of $B_h$ is given by (3.5), and the Laplace transform of $E_h$ can be obtained from $B_h$ with the use of the second equation from (3.2),

$$\tilde{E}_h = \left(1 - \frac{\cos^2 \theta}{\varepsilon_i}\right) \frac{D_0 q}{c^2 s^2 p}.$$  

The originals $B_h$ and $E_h$ can be found with the use of the inverse Laplace transform tables from [59],

$$E_h = \frac{D_h}{\varepsilon_i} \exp\left(-\frac{\nu \xi}{2}\right)\left(\cos \omega_h \xi + \frac{\nu}{2\omega_h} \sin \omega_h \xi\right),$$

$$B_h = \cos \theta \left(E_h - \frac{D_h}{\varepsilon_i}\right),$$  

where $\omega_h = (\omega_p^2 - \nu^2/4)^{1/2}$. This is the well-known solution for the fields in plasma behind the one-dimensional superluminal ionization front propagating in the external electric field. These fields are a superposition of the plane wave with a phase velocity equal to the velocity of the ionization front and the uniform constant magnetic field equal in modulus to the external electric field.

The expression for the magnetic field image $B_h^{(0)}$ at the boundary of plasma half-space can be obtained directly from (3.6) and (3.7) with $a \to \infty$. For definiteness, we choose the sign before the square root in $s_p$ in such a way that $\text{Re} s_p > 0$ in the convergence domain of the Laplace transforms. Thus, $\text{coth} s_p a \to 1$ and
\[ B_{\phi}^{(\infty)} = -\frac{D_{0}q \cos \theta}{c^{2}p_{s} + \varepsilon_{p} s_{p}} \left( \frac{\varepsilon_{p}}{\varepsilon_{l}} - 1 \right). \]

The last expression can be rewritten in the form

\[ B_{\phi}^{(\infty)} = \frac{D_{0} \cos \theta}{\varepsilon_{l} \omega_{h} \cos 2\theta} \left[ \phi \left( \frac{q}{\omega_{h}}, \frac{\sin \theta}{\sqrt{\varepsilon_{l} - \cos^{2} \theta}}, \frac{\nu}{\omega_{h}} \right) - \phi \left( \frac{q}{\omega_{h}}, \frac{\cos^{2} \theta}{\sin \theta \sqrt{\varepsilon_{l} - \cos^{2} \theta}}, \frac{\nu}{\omega_{h}} \right) \right], \]

where

\[ \phi(p, \gamma, \mu) = \frac{1}{(1 - \gamma)^{2}p(p + \mu) + 1} \left\{ (1 - \gamma^{2})(p + \mu) + \gamma \sqrt{p(p + \mu) + 1} \right\}. \]

Let \( \phi(\tau, \nu, \mu) \) be the inverse Laplace transform of the function \( \phi(p, \gamma, \mu) = \int_{0}^{\infty} \phi(\tau, \gamma, \mu) \exp(-pt) \, dt \) with respect to \( p \). Then the magnetic field original at the plasma boundary can be written as

\[ B_{\phi}^{(\infty)}(, , ) = \int_{0}^{\infty} \phi(, , ) \exp(-pt) \, dp. \]

The indeterminate form at \( \theta = \pi/4 \) can be opened using L’Hôpital’s rule,

\[ B_{\phi}^{(\infty)} = -\frac{D_{0} \cos \theta}{\varepsilon_{l} \omega_{h} - 1/2} \frac{\partial \phi}{\partial \gamma}\left( \frac{1}{\sqrt{2\varepsilon_{l}} - 1}, \frac{\nu}{\omega_{h}} \right), \]

where \( \partial \phi/\partial \gamma \) is the derivative of \( \phi \) with respect to the second argument. In the particular case \( \theta = \pi/4, \varepsilon_{l} = 1, \nu = 0 \), the magnetic field original can be found with the use of the inverse Laplace transform tables,

\[ B_{\phi}^{(\infty)} = -\sqrt{2} D_{0}\left[ \sqrt{2} J_{0}\left( \sqrt{2} \omega_{p} \phi_{l} \right) - J_{0}\left( \sqrt{2} \omega_{p} \phi_{l} \right) \right], \]

where \( J_{0} \) and \( f_{1} \) are the zero- and first-order Bessel functions, respectively. In general, to find the original \( \phi \), one can use Laplace transform properties (the convolution, differentiation, and translation theorems) and the inverse Laplace transform tables. The inversion method is similar to the one used in [44, 45] to solve the problem of wave reflection from an instantly created plasma half-space. The original can be expressed in terms of the convolution of trigonometric and cylindrical functions,

\[ \phi(\tau, \nu, \mu) = \exp \left[ -\frac{\mu \tau \nu}{2} \right] \left[ f(\tau) + \frac{\nu}{1 - \nu^{2}} \int_{0}^{\infty} \frac{\mu \tau}{2} J_{0}(\mu \tau) \ast f(\tau) \right], \]

\[ \phi(\nu, 0, \mu) = \exp \left[ -\frac{\mu \tau \nu}{2} \right] \left[ \frac{\partial}{\partial \nu} + \frac{\mu}{2} \int_{0}^{\infty} \frac{\mu \tau}{2} J_{0}(\mu \tau) \ast J_{0}(\tau) \right]. \]

Here \( J_{0} \) is the modified zero-order Bessel function,

\[ f(\tau) = \sqrt{1 - \nu^{2}} \left( \frac{\partial}{\partial \tau} + \frac{\mu}{2} \right) \sin \frac{\tau}{\sqrt{\nu^{2} - 1}}, \]

and the asterisk denotes the convolution, \((F \ast G)(\tau) = \int_{0}^{\infty} F(\tau') G(\tau - \tau') \, d\tau'\).

In the absence of collisions, \( \nu = 0 \) and \( \mu = 0 \), the expressions for \( \phi \) can be reduced as follows:

\[ \phi(\tau, \nu, 0) = \cos \frac{\tau}{\sqrt{1 - \nu^{2}}} + \frac{\nu}{\sqrt{1 - \nu^{2}}} \int_{0}^{\infty} J_{0}(\tau) \sin \frac{\tau - \tau'}{\sqrt{1 - \nu^{2}}} \, d\tau', \]

\[ \phi(\nu, 0, 0) = J_{0}(\tau). \]

In fact, \( \phi(\tau, \nu, 0) \) is the solution to an oscillator equation \((1 - \nu^{2}) \partial^{2} \phi/\partial \tau^{2} + \phi = \gamma J_{0}(\tau)\) with real or imaginary eigenfrequency and the exciting force proportional to the zero-order Bessel function. The initial conditions at \( \tau = 0 \) are \( \phi = 1 \) and \( \phi/\partial \tau = 0 \). For not excessively large values of \( \gamma \), the function \( \phi \) as the function of its first argument oscillates at a unit frequency and slowly decays as the argument increases (similar to the Bessel functions). At large \( \gamma \), slowly decaying oscillations take place on the exponentially falling background. For \( \gamma > 1 \), a good approximation is provided by the following empirical formula:
\[
\phi(\tau, \gamma > 1, 0) \approx q_b, \quad \left| \phi - q_b \right| < 0.1,
\]

\[
q_b = \exp \left( -\frac{\tau}{\sqrt{\gamma^2 - 1}} \right) + \frac{1}{\gamma} \int_0^\tau \exp \left( -\frac{\gamma \tau}{\sqrt{\gamma^2 - 1}} \right) d\tau.
\]

As is seen from (4.16), the main parameters which determine the pulse shape are the ratios sin \(\theta\)(\(\varepsilon_i - \cos^2 \theta\))^{1/2} = (1 - \omega_b^2 / \omega_0^2) and cos \(\theta\)/sin \(\theta\) (\(\varepsilon_i - \cos^2 \theta\))^{1/2} = (1 - \omega_b^2 / \omega_0^2)^2, and \(\omega_b\) determines the time scale of the radiated waveform (at \(\nu = 0\)). If both the mentioned ratios are not too large compared with unity, that is, \(\varepsilon_i, \theta\), and \(\varepsilon_i - \cos^2 \theta\) are not too small, the radiated pulse presents slow-decaying oscillations at frequency \(\omega_b\) (see curves 2 in figures 3(b) and (c)). If one of these ratios is large, the oscillations at frequency \(\omega_b\) take place on the exponentially decaying background proportional to \(\exp[-\omega_b |\xi|]\) or \(\exp[-\omega_b |\xi|]\) (curve 1 in figure 3(b)). Thus, one can speak about the generation of a video pulse whose front-edge duration is determined by frequency \(\omega_b\) and whose back-edge duration is determined by \(\omega_b\) and \(\omega_0\). If both ratios are large, the generated video pulse is proportional to \(\exp(-\omega_b |\xi|) - \exp(-\omega_0 |\xi|) / |\omega_0 - \omega_b|\). On the background of this pulse, weak oscillations at frequency \(\omega_b\) take place (curve 1 in figure 3(b)). Thus, for thick slabs as well as for thin ones, the generated waveforms can differ significantly depending on the angle \(\theta\) and the dielectric permittivity \(\varepsilon_i\).

Note an interesting feature of the obtained solutions: the substitution \(\theta \rightarrow \pi/2 - \theta, \varepsilon_i \rightarrow \varepsilon_i \tan^2 \theta\) swaps the values of frequencies \(\omega_\alpha\) and \(\omega_b\), whereas \(\omega_\alpha\) and \(\omega_b\) of the generated waveform do not change:

\[
B_\phi^\infty(\xi, \theta, \varepsilon_i, \omega) = \cot \theta \cdot B_\phi^\infty(\xi, \pi/2 - \theta, \varepsilon_i \tan^2 \theta, \omega).
\]

Another interesting feature can be seen from the asymptotics of the solution at \(\xi \rightarrow +\infty\). One can find these asymptotics by evaluating the inversion integral with the use of the standard method based on the stationary phase approximation. We get

\[
B_\phi^\infty \approx \frac{2D_0}{\sin 2\theta} \left( \frac{2\omega_b^2}{\pi \omega_0^2 \xi} \cos \left( \omega_0 \xi - \frac{\pi}{4} \right) \right)
\]

for \(\nu = 0\) and

\[
B_\phi^\infty \approx -D_0 \cot \theta \left( \frac{\nu}{\pi \omega_0^2 \xi} \right)
\]

for \(\nu > 0\). By comparing these two expressions, we find that the radiated magnetic field in the presence of collisions exceeds the one in their absence if \(\nu / \omega_0 > 2\varepsilon_i^2 \omega_p^2 / \omega_0^2 \cos^2 \theta\) and the time \(\xi\) is large enough (compare figures 3(b) and (d)). Therefore, even rare collisions may lead to an effective increase in the radiated field, accompanied by a significant change in the spectrum. This can be explained by the contribution from the quasi-static components originating from the static magnetic field (4.4) and the current density (4.5).

### 4.4. Generated waveform for a slightly superluminal ionization front

In the two preceding sections, we showed that the radiated field can be an oscillating weakly decaying radio pulse as well as a short video pulse. The latter situation is typical in the case of a slightly superluminal ionization front with speed close to the speed of light in a vacuum or in a non-ionized slab. We consider this limiting case in this section.

If \(\varepsilon_i < 1\), the speed of light in the slab is higher than that in a vacuum, and the limiting value of the angle \(\theta\) is equal to arccos(\(\varepsilon_i^{1/2}\)). For \(\theta \rightarrow \arccos(\varepsilon_i^{1/2}) + 0\) and \(\nu = 0\), the Laplace image (3.6) goes into

\[
B_\theta = -\frac{2D_0}{\sqrt{\varepsilon_i} \omega_p q^2 / \omega_p^2 + 2\beta q / \omega_p + 1},
\]

where \(\beta = \tanh(\varepsilon_i^{1/2} \omega_p a/c) / 2 (\varepsilon_i(1 - \varepsilon_i)^{1/2})\). The original \(B_\theta\) is described by an expression similar to (4.14),

\[
B_\theta \approx \frac{2D_0 \beta}{\sqrt{\varepsilon_i}} \exp\left( -\beta \xi |\omega_p| \right) \begin{cases} \sin \left( \sqrt{1 - \beta^2} \omega_p \xi \right) & \text{for } \beta < 1; \\ \omega_p \xi / \sqrt{1 - \beta^2} & \text{for } \beta = 1; \\ \sinh \left( \sqrt{\beta^2 - 1} \omega_p \xi \right) / \sqrt{\beta^2 - 1} & \text{for } \beta > 1. \end{cases}
\]
Thus, a radio pulse is generated at small values of $\beta$, $\beta \ll 1$, and a video pulse is produced at values of $\beta$ close to unity or greater. A double degeneracy comes if $\beta = 1$. Here, as well as in the preceding section, even rare collisions can significantly change the shape of generated pulses (mainly their back edge) if $\beta \gg 1$.

For $\epsilon_i > 1$, the speed of light in the slab is less than that in a vacuum, and the limiting angle is zero. When the ionization front speed tends to the speed of light in vacuum, $\theta \to +0$, one real pole tends to zero, and $s_{\alpha, \mu} \to \pi n$ for the other poles where $n$ is an integer. However, the respective amplitudes for the poles with $n \neq 0$ also tend to zero. Therefore, the main contribution to the solution at $\nu = 0$ is given just by three poles close to $0$ and $\pm i \omega_{\nu}$, respectively. For small finite values of $\theta$, when $\theta \ll \epsilon_i^{-1/2} \omega_{\nu} \alpha/c$, $(1 - \epsilon_i^{-1})^{1/2} \omega_{\nu} \rho/a$, these poles are approximately equal to $-\epsilon_i^{1/2} \theta \omega_{\nu} \coth(\epsilon_i^{1/2} \omega_{\nu} \rho/a)$, $\pm i \omega_{\nu} = \epsilon \theta (\epsilon_i - 1)$. This gives, for the original,

$$B_\nu \approx -D_1 \left[ \exp \left( -\sqrt{\epsilon_i} \theta \coth \frac{\epsilon_i^{1/2} \omega_{\nu} \rho/a}{c} \right) \right] - \exp \left( -\frac{\theta \epsilon_i \xi}{2a(\epsilon_i - 1)} \right) \cos \omega_{\nu} \xi.$$

The characteristic waveform depends on the ratio of radiative decrements of the two excited modes. When $(\epsilon_i^{1/2} \omega_{\nu} \rho/a) \coth(\epsilon_i^{1/2} \omega_{\nu} \rho/a) \ll 1/(\epsilon_i - 1)$, the decrement of the oscillating mode is large compared with the background decrement. In this case, the generated waveform is a video pulse which is a background for rapidly damped oscillations at frequency $\omega_{\nu}$. The front edge of this pulse has a characteristic timescale on the order of $1/\omega_{\nu}$, and the back edge has a timescale on the order of $\tanh(\epsilon_i^{1/2} \omega_{\nu} \rho/a) / \epsilon_i^{1/2} \theta \omega_{\nu}$. In the opposite case, $(\epsilon_i^{1/2} \omega_{\nu} \rho/a) \coth(\epsilon_i^{1/2} \omega_{\nu} \rho/a) \gg 1/(\epsilon_i - 1)$, the contribution of the oscillating mode remains perceptible for a long time, even after the slow background has already decayed. However, in this case, even rare collisions can change the relation between the mode decrements. This can be easily seen for $\epsilon_i^{1/2} \omega_{\nu} \rho/a \ll 1$. In this case, the position of the three poles is described by the equation $Q(q) = 0$, where $Q(q)$ is defined by (4.12). In the case we are interested in, $\alpha \gg 1$, the rare collisions almost do not change the damped decrement of the non-oscillating mode with the imaginary frequency, whereas the decrement of the oscillating mode is increased by $\epsilon_i/2$. The decrements of the two modes can be equal as early as $\nu / \omega_{\nu} \sim 2/\alpha \ll 1$, and for larger values of $\nu$, we may have the generation of video pulses similar to the ones described earlier. This situation is illustrated in figure 3(e).

The solutions obtained for the cases $\epsilon_i < 1$ and $\epsilon_i > 1$ do not work if $\epsilon_i$ is close to unity, $|\epsilon_i - 1|^{1/2} \ll \theta \ll 1$, $\omega_{\nu} \rho/a$. In this case, one of the poles is approximately equal to $-\theta \omega_{\nu} \coth(\omega_{\nu} \rho/a)$ and is close to zero, and all the other poles, in contrast, are sufficiently large and approximately equal to $-\omega_{\nu}/(\theta \sinh \rho_{2})$. Here $\rho_{2}$ are solutions to the equation $\rho = (\omega_{\nu} \rho/a) \cosh \rho$ such that $\Re \sinh \rho_{2} < 0$. This equation has two real positive roots and an infinite number of complex conjugate root pairs for $\omega_{\nu} \rho/a < (\theta^{2} - 1)^{1/2} \approx 0.66$ and only complex conjugate root pairs for $\omega_{\nu} \rho/a > (\theta^{2} - 1)^{1/2}$. Here $\rho \approx 1.20$ is the real positive root of the equation $\rho = \coth \rho$. At $\omega_{\nu} \rho/a \approx 0.66$ we have the second-order pole, which is approximately equal to $-(\theta^{2} - 1)^{1/2} \omega_{\nu} \rho/a \theta \approx -1.51 \omega_{\nu} / \theta$. The generated pulse is a video pulse whose background is occupied by weak and frequent oscillations. The amplitude of this pulse is close to $D_0$. The scale of the back edge of the pulse is determined by the pole $q_{s} \approx -\theta \omega_{\nu} \coth(\omega_{\nu} \rho/a)$. The front-edge scale is determined by the remaining poles, $-(\omega_{\nu}/\theta) \sinh \rho_{2}$.

### 4.5. Generated waveform for a strongly superluminal ionization front

In this section, we consider the limiting case that is the opposite of the preceding one. We assume here that the angle $\theta$ is close to $\pi/2$ and $\cos \theta \ll 1$, $\epsilon_i^{1/2}$, i.e., the ionization front speed is much greater than the speed of light in a vacuum and the speed of light in the non-ionized slab. In the case of the infinite ionization front speed, $\theta = \pi/2$, when the entire slab volume is ionized simultaneously, the solution is nonradiating oscillation at the plasma frequency. In this solution, the fields outside the slab do not change after the ionization; and inside the slab, the electric field is equal to $(D_0/\epsilon_i) \cos \omega_{\nu} t \hat{z}$, and the magnetic field is absent. In fact, for $\theta = \pi/2$, even weak transverse modes exist, and the equation $\Delta = 0$ has other solutions in addition to $q = \pm i \omega_{\nu}$. However, the electric field component directed perpendicular to the slab boundaries is zero in these modes. Therefore, the external electric field perpendicular to the slab boundaries cannot excite them. Thus, the amplitudes of all weak modes except for one tend to zero for $\theta \to \pi/2 - 0$. For the remaining mode, the real part of the eigenfrequency tends to $\omega_{\nu}$, and the radiative decrement tends to zero. The correction to the complex frequency of this mode at a finite nonzero value of the longitudinal (in the direction of the $z$ axis) wavenumber was found in [35] for $\epsilon_i = 1$. For an arbitrary value of $\epsilon_i$, the frequency and the decrement of radiation losses for this mode are described by the following expressions:
\[ \Omega_i = \omega_p + \delta \Omega_i, \quad \delta \Omega_i \approx \cos^2 \theta i \left( \frac{\omega_p a/c}{1 + \left( \omega_p a/c \right)^2} \right)^2 \omega_p, \]

\[ \gamma_i^{(r)} \approx \frac{\cos^2 \theta}{2\epsilon i} \frac{\omega_p a/c}{1 + \left( \omega_p a/c \right)^2} \omega_p. \]  

(4.18)

In the radiation field, the amplitudes of all the modes (including the mode with frequency \( \Omega_i \)) tend to zero if \( \theta \to \pi/2 - 0 \). Although the mode with frequency \( \Omega_1 \) provides the most significant contribution to the radiation field for \( \theta \) close to \( \pi/2 \), in general, the radiation field is not entirely determined by only this mode. This is easily seen if we find the respective contribution \( B_1 \) of this mode to the magnetic field \( B_1 \) at the slab boundary. In accordance with (4.7), this contribution is equal to

\[ B_1 \approx \frac{D_0 \cos \theta \omega_p a/c}{\epsilon_1 \left[ 1 + \left( \omega_p a/c \right)^2 \right]^2} \exp \left( -\gamma_i^{(r)} \xi \right) \left[ \sin \Omega_1 \xi - \frac{\omega_p a}{c} \cos \Omega_1 \xi \right]. \]  

(4.19)

It is easy to see that \( B_1 \) does not satisfy the initial condition for the magnetic field for not excessively small values of \( \omega_p a/c \) (when we cannot neglect the cosine in (4.19)). Nevertheless, all other modes have a significantly greater radiative decrement, and \( B_1 \) describes the radiated pulse rather well for sufficiently late times \( \xi \). Comparing the waveform of \( B_1 \) with those obtained in section 4.4, we can see that the change of the angle \( \theta \) can lead to a substantial reshaping of the radiated pulse waveform in a sufficiently wide range of parameters \( \epsilon_i \) and \( \omega_p a \).

Collisions here can significantly complicate the picture if the slab is not thin enough. The radiative decrement for the mode with frequency \( \Omega_1 \) is equal to \( \nu/2 \) in accordance with (4.8). Thus, if the slab is not very thick, the radiated waveform is described by the right-hand side of (4.19) multiplied by \( \exp(-\nu \xi/2) \) (see figure 3(f) and curve 2 in it). However, as we have seen in section 4.3, collisions lead not only to an increase in the damping decrements of oscillating modes but also to the emergence of an additional low-frequency radiation. This can significantly affect the radiated waveform for a slab sufficiently thick.

### 5. Radiation spectrum and energy

The Laplace transforms obtained allow us to get the expressions for a spectral density \( w_{rad} \) of the energy radiated from one of the slab sides (in the positive or negative direction of the x axis) per area unit of the slab:

\[ w_{rad}(\omega) = \frac{c}{4\pi^2} \text{Re} \left[ \mathcal{E}_x B_z^* \right]_{q=\pi} = \frac{c \sin \theta}{4\pi^2} |\mathbf{\beta}_0(q = i\omega)|^2. \]  

(5.1)

The total energy \( W_{rad} \) radiated from both sides of the slab per area unit can be obtained by integrating over the frequency: \( W_{rad} = 2 \int_0^\infty w_{rad}(\omega) \, d\omega \).

In the absence of internal losses, \( \nu = 0 \), the radiated energy can be calculated without resorting to the preceding integral with the use of Poynting’s theorem:

\[ \frac{\partial \mathbf{W}}{\partial t} + \mathbf{V} \mathbf{S} = \frac{\partial}{\partial \xi} \left( u - \frac{\cos \theta}{c} S_z \right) + \frac{\partial S_x}{\partial x} = 0, \]  

(5.2)

where \( u = (\epsilon_0 E^2 + B^2)/8\pi + 2\pi i/a_\nu \) is the volume density of the energy, including the electromagnetic energy and the kinetic energy of the ordered free-carrier motion, and \( \mathbf{S} = (c/4\pi) \mathbf{E} \times \mathbf{B} \) is the Poynting vector. We integrate (5.2) with respect to \( x \) and \( \xi \) and obtain

\[ W_{rad} = \int_0^\infty S_z|_{k=\omega} \, d\xi = \int_{-\infty}^\infty \left( \frac{\cos \theta}{c} S_z - u \right) \, dx \bigg|_{x=0}^{x=\infty}. \]  

(5.3)

Inside the slab, we have only the magnetic field (4.4) and the current density (4.5) for \( \xi \to +\infty \), which allows us to find the values of the Poynting vector and the energy density at \( \xi \to +\infty \). By substituting these values in (5.3), we find that \( W_{rad} = \eta W_0 \), where

\[ W_0 = \frac{D_0^2 a}{4\pi\epsilon_i} \]  

(5.4)
is the electrostatic energy (per unit of slab area) stored initially, and

\[ \eta = 1 - \frac{\cos^2 \theta}{e_i} \left[ 1 - \frac{\tanh \left( \sqrt{e_i} \alpha_p a/c \right)}{\sqrt{e_i} \alpha_p a/c} \right] \]  

(5.5)
is the radiated fraction of the initially stored energy.

A similar approach can be used to determine the mean square of the radiation frequency,

\[ \langle \omega^2 \rangle = 2 \int_{0}^{\infty} \omega^2 w_{rad} \, d\omega / W_{rad}. \]

Note here that the fields and the current density \( [E', B', j'] = \partial / \partial (\omega_p \xi) \) are the solution to this problem, similar to the original one but with the initial conditions

\[ E'|_{\xi=0} = 0, \quad B'|_{\xi=0} = 0, \quad j'|_{\xi=0} = \frac{\omega_p D_0}{4\pi} x \]

instead of the initial conditions in (2.3). On the one hand, the radiated energy \( W_{rad}^\prime \) in this problem is related to the average frequency square in the original problem:

\[ W_{rad}^\prime = 2 \omega_p \int_{0}^{\infty} \omega^2 w_{rad} \, d\omega = \frac{\langle \omega^2 \rangle}{\omega_p^2} W_{rad}, \]

which follows from the formula similar to (5.1) for the fields with primes. At the same time, we have the equality, similar to (5.5),

\[ W_{rad}^\prime = -\int_{-a}^{a} \left( \frac{\cos \theta}{c} S'_\xi - u' \right) dx \bigg|_{\xi=0}, \]

(5.6)

where \( S'_\xi \) and \( u' \) are the longitudinal component of the Poynting vector and the energy density in the problem for the fields and current density with primes. It is taken into account here that the fields and current density \( E', B', j' \) tend to zero when \( \xi \to +\infty \). By evaluating the integral in (5.6), we obtain \( W_{rad}^\prime = W_0 \) and \( \langle \omega^2 \rangle = \alpha_p^2 / \eta \). We can calculate the average value of the fourth power of the radiation frequency \( \langle \omega^4 \rangle = 2 \int_{0}^{\infty} \omega^4 w_{rad} \, d\omega / W_{rad} \) in quite the same way. The same procedure should be applied to the fields \( [E^\prime, B^\prime, j^\prime] = \partial / \partial (\alpha_p \xi)^2 \) \( [E, B, j] \), which at \( \nu = 0 \) satisfy the initial conditions

\[ E^\prime|_{\xi=0} = 0, \quad B^\prime|_{\xi=0} = 0, \quad j^\prime|_{\xi=0} = 0 \]

at \( |x| > a \) and

\[ E^\prime|_{\xi=0} = -\frac{D_0}{e_i - \cos^2 \theta} x, \quad B^\prime|_{\xi=0} = -\frac{D_0 \cos \theta}{e_i - \cos^2 \theta} x, \quad j = 0 \]

at \( |x| < a \). The radiated energy in this problem is equal to \( W_{rad}^\prime = W_0 / (1 - e^{-1} \cos^2 \theta) \) and

\[ \langle \omega^4 \rangle = \frac{\alpha_p^2 \omega_p^2}{\eta}. \]

Thus, the total radiated energy per unit of slab area is always positive and less than the stored energy, \( 0 < W_{rad} < W_0 \), even in the absence of any internal losses. In the absence of collisions, the energy stored at ionization is then spent not only for electromagnetic wave radiation but also for the generation of static longitudinal current density and magnetic field in plasma in accordance with (4.4) and (4.5). The energy fraction spent for the generation of the static current and magnetic field increases with the dimensionless half-thickness of the slab and decreases with the ionization front speed. The average frequency of the generated radiation increases with the increase of this fraction. The stored energy can be almost fully converted to the energy of the static current and magnetic field if \( e_i^{1/2} / \cos \theta \to 1 \) and the ionized slab is thick enough, \( e_i^{1/2} \alpha_p a / c \gg 1 \). The radiated energy fraction is \( \eta \approx 1 - e_i^{-1} \cos^2 \theta \ll 1 \) in this case, in accordance with (5.5). In the case of a sufficiently thin layer, \( e_i^{1/2} \alpha_p a / c \ll 1 \), on the other hand, only a small fraction of the energy is converted to the static current density and magnetic field, and almost all the energy is radiated as electromagnetic waves.

Note that (5.5) is applicable only if any internal losses in plasma, including collisions, are negligible. In other words, the radiated energy can be described by (5.5) only if all the characteristic times of internal losses are large compared with the characteristic time of radiation damping (i.e., duration of the generated electromagnetic pulse in the absence of internal losses). In particular, (5.5) gives a completely incorrect result in the limit \( \theta \to \pi/2 \to 0 \), which corresponds to the infinite increase in the ionization front speed. As is described in section 4.5, the case \( \theta = \pi/2 \) corresponds to the nonradiating solution with zero radiated energy. This clearly
contradicts (5.5). This contradiction occurs because the radiation time of the main excited mode increases without bound when \( \theta \to \pi/2 - 0 \). This can be seen from (4.18), which shows that the radiative decrement of the main mode decreases and tends to zero at \( \theta \to \pi/2 - 0 \). Thus, the radiated energy decreases and tends to zero with an infinite increase in the ionization front speed if internal losses such as collisions are taken into account. The same refers to the limit \( N \to 0 \), as can be seen from (4.10). In a similar way, one can consider the limits \( e_i \to \infty \) (with fixed plasma density \( N \) or fixed plasma frequency \( \omega_p \propto (N/e_i)^{1/2} \) and \( e_i \to +0 \), \( \theta \to \pi/2 - 0 \) (with fixed \( N \) and fixed ratio \( e_i^{1/2}/\cos \theta \propto e_i^{1/2}/V_0 \)). The latter limit corresponds to the limiting case of the slab placed between two metal planes, \( e_i \to +\infty \). (Here the substitution described at the end of section 2 and leading to \( e_i = 1 \) should be taken into account.) Note that all these limits are valid and correct for the obtained images of the fields and current density (equations (3.2)–(3.7)), but they lead to incorrect results if applied to (5.5).

The radiated energy \( W_{\text{rad}} \) is a decreasing function of the plasma density \( N \), as can be seen from (5.5). According to the preceding paragraph and formula (4.10), the collisions or other internal losses make the radiated energy a nonmonotonic function of the plasma density \( N \) (when all other parameters, including the collision frequency, are fixed). Therefore, there is an optimal plasma density value at which the radiated energy is at a maximum. This is illustrated in figure 4. Figure 4(a) shows the dependences of the radiated energy fraction on the slab half-thickness at various angles \( \theta \) with \( \alpha \omega/c \) fixed. The fixed value of \( \alpha \omega/c \) makes the figure present the dependence on plasma frequency with the collision frequency fixed. It is seen that the optimal value of the plasma density and the width of the optimum are smaller the smaller the angle \( \theta \). The radiation spectra at the optimum plasma density may strongly depend on the angle \( \theta \), as can be seen from figure 4(b). In accordance with the complex eigenfrequency dependencies discussed in sections 4.4 and 4.5, there is a high-\( Q \) leaky mode at angles close to \( \pi/2 \) which results in a pronounced spectral peak at the plasma frequency (line 3 in figure 4(b)) and there is no such mode at small angles (line 1 in figure 4(b)). Note that in figures 4 and 5 we give the radiation spectra in the near field, where the radiation is just a plane wave. Due to this, the spectral energy density can be nonzero at zero frequency. This near-field solution is applicable for the wavelengths which are shorter than the ionization front propagation path multiplied by \( \sin^2 \theta \).

It should also be noted here that collisions can lead to a decrease as well as to an increase in the total radiated energy. Due to the collisions, the energy spent on the excitation of the current density (4.5) and magnetic field (4.4) is partly dissipated in the plasma and partly radiated as electromagnetic waves into the surrounding space. The total radiated energy may increase due to these waves despite the collisional energy dissipation. The spectrum of this collision-induced radiation has a maximum at the zero frequency and width on the order of the collision frequency \( \nu \) and may essentially differ from the radiation spectrum in the absence of collisions. Due to this, the radiated energy dependence on the angle \( \theta \) may vary in a complex manner with the collisions taken into account. Without collisions, this dependence is a monotonically increasing function of the angle. The collisions make this function monotonically decreasing at \( \theta \to \pi/2 - 0 \) with zero value at \( \theta = \pi/2 \). On the other hand, when the ionization front speed is close to the speed of light in a vacuum or in the slab, the radiated energy may decrease as well as increase. In this case, the radiated energy calculated by integrating \( 2W_{\text{rad}}(\omega) \) over all frequencies can be either a monotonically decreasing or a nonmonotonic function of the angle. However, the generated radiation spectrum rearranges itself to lower frequencies when \( \theta \to 0 \), so \( W_{\text{rad}}(\omega) \to 0 \) for any finite value of the frequency \( \omega > 0 \). Therefore, the radiation energy in the frequency range \((\omega_{\text{min}}, \omega_{\text{max}})\) (where \( 0 < \omega_{\text{min}} < \omega_{\text{max}} \)) tends to zero when the ionization front speed tends to the speed of light in a vacuum. Thus, at \( e_i \geq 1 \), the energy radiated into a certain frequency interval is nonmonotonic, and there is an optimal value of the angle \( \theta \) at which this energy is at a maximum.
We illustrate this by calculating the energy radiated in the terahertz wavelength range (0.3–3 THz) in the ionization of a thin photoconductive slab. To do this, we integrate the energy spectral density \( w^2 \text{rad} \) only over the specified frequency range. Here we consider two cases. The first relates to an ionizable slab being part of a flat-layered solid-state structure, e.g., heterostructure, and \( \varepsilon = \varepsilon_i \). The second corresponds to a slab surrounded by medium with dielectric permittivity close to unity, e.g., air, and \( \varepsilon = 1, \varepsilon_i \gg 1 \). Figure 5 shows the terahertz radiation energy normalized to the stored electrostatic energy as a function of the angle at various slab thicknesses and the examples of the generated terahertz spectra. It is seen that there is an optimum value of the angle \( \theta \) at which the energy of terahertz radiation reaches a maximum in both cases. In the case of the layered solid-state structure, the shape of the generated terahertz radiation spectrum can vary significantly through tuning of the angle \( \theta \) around the optimum value (see figures 5(a) and (b)). This indicates the possibility of employing such schemes for sources of terahertz pulses with a tunable spectrum and waveform. In the case of the photoconductive slab surrounded by air, the shape of the radiated spectrum remains almost unchanged when the angle \( \theta \) is varied (see figures 5(c) and (d)). A significant change in the spectrum shape can occur here only at very small angles \( \theta \) when the terahertz energy is much less than its maximal value at the optimal angle. This may explain why such spectrum shape variations are usually not observed in experiments on the generation of terahertz radiation by photoconductor surfaces since in these experiments, the laser radiation is usually incident from air. It should be noted that the experimental dependences (see, for example, [28]) of terahertz yield on the angle \( \theta \) and the values of the optimum angle are usually consistent with those obtained here.

6. Conclusion

We have calculated the electromagnetic fields behind the superluminal ionization front propagating in a flat slab biased by a transverse external uniform static electric field. Using the Laplace transform, we have expressed the fields as a superposition of fast leaky waves phase-synchronized with the ionization front, i.e., waves with a ratio of the complex frequency and complex wavenumber equal to the velocity of the ionization front. These waves leak through the slab transverse boundaries and constitute an electromagnetic pulse which propagates at an angle to the direction of ionization front propagation. The angle value is determined by the ionization front speed.

For the first time, the dispersion properties of such leaky waves propagating along a flat plasma slab have been analyzed in detail, and the complex frequencies of these waves have been determined as functions of the slab thickness, plasma frequency, initial dielectric permittivity of the slab, and the ionization front speed. We have found the parameter values at which the frequency degeneracy, i.e., the coincidence of several (up to four) leaky wave eigenfrequencies, takes place. These parameters usually correspond to the generation of the shortest electromagnetic pulses. It has been shown that the radiated pulse waveform can vary significantly when a
parameter (for example, ionization front speed) is tuned continuously with passage through the frequency degeneracy. In this way, the radiated pulse can be tuned from a short unipolar video pulse to a long-lasting radio pulse. For thin layers, we have found a simple analytical expression \((4.14)\) which describes this rebuild of the waveform. A similar expression \((4.17)\) was obtained for ionization front speeds close to the speed of light in the slab. The case of the ionization front propagating in a half-space has also been examined in detail. Here we have obtained an exact analytical expression for the waveform of the pulse radiated through the half-space boundary in the term of the convolution of the trigonometric and cylindrical functions. This expression shows that, in this case, the radiated waveform also depends significantly on the ionization front speed and, for slightly superluminal fronts, differs essentially from the trivial oscillating-pulse waveform.

We have found an exact analytical expression for the radiated energy per area unit of the slab. Even in the absence of any internal losses, the radiated energy has turned out to be less than the electrostatic energy \((5.4)\) stored initially. This difference in energies is related to the generation of the longitudinal quasi-dc current and the consistent magnetic field inside the plasma slab. The generation of this current can be regarded as a new loss mechanism which should be taken into account when similar problems are considered.

We have obtained an analytical expression for the spectral power of the generated radiation and, by numerical integration of this expression, have calculated the amount of the radiated energy in the presence of collisions. The calculations show that there exists an optimal plasma density at which the radiated energy reaches its maximum (with all other parameters being held fixed). If the permittivity of the ionizable slab is higher than that of the environment, there is an optimum ionization front speed which corresponds to the maximum of the radiated energy in a certain frequency interval. The readjustment of the ionization front speed around this optimum value may cause a significant change in the generated radiation spectrum.

These results demonstrate the attractiveness of schemes similar to the one considered here for the creation of radiation sources (particularly in the terahertz frequency range) with a tunable spectrum and the waveform of a generated pulse. We hope that this study may also be of interest in the context of the general problems of the theory and application of leaky waves in planar plasma-like structures.

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