Quantum Mechanical Model of the Betatron

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Abstract. A quantum mechanical model of the betatron is proposed for two-dimensional electron gases, in which the betatron condition is satisfied globally under a special type of axially symmetric inhomogeneous magnetic field named beta-field. The eigenvalue problem for the electron in the beta-field is solved analytically. It is shown that, for electrons undergoing the cyclotron motion in a uniform magnetic field, the adiabatic increase of the beta-field accelerates the speed of the rotation without changing the orbital radius.

1. Introduction

In 1941, Kerst[1] succeeded in constructing an accelerator of electrons to the energy as high as 2.3MeV by using the principle of electromagnetic induction. This apparatus, now called a betatron, is operated based on the theory of stability orbit in an inhomogeneous time-dependent magnetic field. A bunch of electrons rotating in a vacuum chamber is accelerated by the induced electric field due to the increase of the magnetic flux linking the electron orbit, while the magnetic field at the orbit provides the central force (Lorentz force) that maintains the radius of the circular motion constant. For that purpose, the so-called 2:1 rule should be satisfied[1],

\[ \Phi(r, t) = 2\pi r^2 B(r, t), \]  

(1)

where \( \Phi(r, t) \) is the magnetic flux penetrating the orbit with radius \( r \), and \( B(r, t) \) is the magnetic field at the orbit. Thus, the magnetic flux within the orbit must be twice the value obtained if the flux density is uniform and is equal to the field at the orbit. The betatron has advantages that it can accelerate electrons avoiding the insulation breakdown of the direct voltage generators, and that the size of the apparatus is relatively small.

In the present work, I study the quantum mechanical properties of the betatron downsized to the mesoscopic scale. Such a small size of apparatus may be realized by the nano fabrication technique applied to the two-dimensional electron gases (2DEG), or quantum dots. It will be shown below that, for a special type of inhomogeneous time-dependent magnetic field, the electrons can be accelerated coherently in their circular motion without changing the orbital radius, just like the classical betatron. It will provide a new tool of controlling electronic systems, and also a new source of radiation, in mesoscopic scale.

2. Beta-field and stationary states

Let the coordinates of an electron confined in a plane be \( \vec{r} = (x, y, 0) \). We assume that a time-dependent axially symmetric magnetic field is applied perpendicularly to the plane,
\( \vec{B}(r, t) = B(r, t)\hat{e}_z \), where \( \hat{e}_z \) is the unit vector in z-direction and \( r = \sqrt{x^2 + y^2} \). In the cylindrical coordinate system \((r, \theta, z)\) defined as usual, the corresponding vector potential \( \vec{A} \) has only the azimuth angle component \( A_\theta(r, t) \) given by \( B(r, t) = r^{-1} \partial (r A_\theta(r, t)) / \partial r \).

The Hamiltonian is given by
\[
H(t) = -\frac{\hbar^2}{2m_e} \left[ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \left\{ -i \frac{\partial}{\partial \theta} + \frac{e r}{\hbar} A_\theta(r, t) \right\} \right]^2,
\]
in which \( m_e \) is the effective mass of the electron, and \( e (> 0) \) is the magnitude of the electric charge. Here and hereafter we suppress the argument \( z \). In this work, we neglect the interaction of the magnetic field with the spin of electrons. It will give rise to an additional spin-dependent force because of the inhomogeneity of the magnetic field like the Stern-Gerlach effect. Since the angular momentum \( L_\theta = -i \hbar \partial / \partial \theta \) is a constant of motion, the time-dependent Schrödinger equation \( i \hbar \partial \psi(\vec{r}, t) / \partial t = H(t) \psi(\vec{r}, t) \) is reduced to that for the radial function by the separation of the variable, \( \psi(\vec{r}, t) = \exp(i m \theta) R(r, t) \).

In considering the quantum betatron, we assume that the magnetic field \( \vec{B}(r, t) \) satisfies the betatron condition Eq.\((1)\) everywhere in the plane. This is natural since, in quantum mechanics, the wave function depends globally on the field profile. Furthermore, this is desirable from the experimental as well as practical point of view, since a substantial part of the electrons in the mesoscopic scale of 2DEG can be accelerated coherently without changing the orbital size. It can be easily shown that the profile of the magnetic field satisfying this requirement is given by
\[
B(r, t) = C(t) / r,
\]
where \( C(t)(> 0) \) is a function of time. The vector potential is given by \( A_\theta = C(t) \). We will call such a magnetic field a beta-field.

First we solve the eigenvalue problem for the time-independent beta-field, \( C(t) = C \). For the angular momentum quantum number \( m \), the Schrödinger equation for the radial wave function \( R(r) \) is given by
\[
\left\{ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \left( \frac{m}{r} + \beta \right)^2 + \epsilon \right\} R(r) = 0,
\]
where \( \beta = eC / \hbar \) and \( \epsilon = 2m_eE / \hbar^2 \) with \( E \) being the eigenenergy. If one neglects the second term, the problem becomes analogous to that in one-dimension with the effective potential \( V(r) = (m/r + \beta)^2 \). For \( m < 0 \), \( V(r) \) takes a minimum value at \( r_0 = |m|/\beta \), and tends to \( V(r \to \infty) = \beta^2 \), \( V(r \to 0) \to \infty \). Therefore, there are bound states with \( \epsilon \) satisfying \( 0 < \epsilon < \beta^2 \). The eigenstates with \( \epsilon \geq \beta^2 \) form a continuum of scattering states. For \( m > 0 \), the effective potential becomes repulsive, so that all eigenstates become scattering states with \( \epsilon \geq \beta^2 \).

For \( m < 0 \) and \( \epsilon \leq \beta^2 \), we transform Eq.\((4)\) into the form,
\[
\rho \frac{d^2}{d\rho^2} L + (2|m| + 1 - 2\rho) \frac{dL}{d\rho} - (2\lambda m + 2|m| + 1)L = 0,
\]
by the change of variables \( \rho = \alpha r \) with \( \alpha = \sqrt{\beta^2 - \epsilon} \), \( \lambda = \beta / \alpha \), and by setting \( R(r) = \rho^{[m]} \exp(-\rho)L(\rho) \). This is Laguerre’s equation. The quantization condition is derived by the requirement that \( L(\rho) \) is a polynomial as usual. Thus we find
\[
\lambda|m| - |m| - \frac{1}{2} = n, \quad n = 0, 1, 2, \ldots,
\]
and the eigenenergy as
\[
E_{n,m} = \frac{\hbar^2 \beta^2}{2m_e} \left( 1 - \frac{|m|^2}{(n + |m| + \frac{1}{2})^2} \right).
\]
The solution to Eq. (5) is given by the associated Laguerre polynomial \( L^{(2|m|)}_{n}(2\rho) \). The eigenenergies depend both on the principal quantum number \( n \) and the magnetic quantum number \( m \). For each \( m \), the eigenstates form an infinite ladder of eigenvalues from \( n = 0 \) to \( n \rightarrow \infty \).

The scattering states for the energy \( \epsilon \geq \beta^2 \) can be treated simultaneously both for \( m < 0 \) and \( m > 0 \). It can be easily shown that, by setting \( \rho = i\alpha r \) with \( \alpha = \sqrt{\epsilon - \beta^2} \), and \( R(r) = \rho^n \exp(-\rho)L(\rho) \) in this case, the solution is written by a hypergeometric function explicitly. The case of zero angular momentum requires a special attention. For \( m = 0 \), Eq. (4) reduces essentially to that for a free particle, with the change of the energy from \( \epsilon \) to \( \epsilon + \beta^2 \). The Schrödinger equation is solved trivially, and the solution is written by the Bessel function of order 0, as is well known.

3. Betatron

Feldman and Kahn [2] showed that a quantum mechanical description of the cyclotron motion under a uniform magnetic field \( B_0 \) is given by the two-dimensional coherent state representation. The wave function is characterized by two complex-valued parameters, \( \delta \) and \( \xi \). It is shown that the minimum uncertainty state with width of the quantum mechanical cyclotron radius \( L = \sqrt{\hbar/eB_0} \) undergoes a cyclotron motion around \( x = \Re \xi \), \( y = \Im \xi \) with radius \( |\delta| \) without changing the Gaussian wave form. The frequency of this cyclic motion is the cyclotron frequency \( \omega_c = eB_0/m_e \). The classical picture is recovered in the limit \( |\delta|/l \gg 1 \).

Suppose that the beta-field \( B(r, t) \) is switched on and increased adiabatically, superimposing the static uniform field \( B_0 \). The beta-field breaks the translational symmetry. We consider such an orbit encircling the origin with radius \( |\delta| \). The corresponding wave function is given by the superposition of the eigenstates of the angular momentum \( m \), with the maximum value of the weight taken at \( m \simeq |\delta|^2/2l^2[2] \). Under the coexistence of \( B_0 \) and \( B(r, t) \), the Hamiltonian is given by

\[
H(t) = -\frac{\hbar^2}{2m_e} \left\{ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \left( \frac{m}{r} + \frac{r}{2l^2} + \beta(t) \right)^2 \right\},
\]

in the subspace of the angular momentum \( m \). A crucial observation is that the effective potential

\[
V(r) = \left( \frac{m}{r} + \frac{r}{2l^2} + \beta(t) \right)^2
\]

takes the minimum value at \( r_m = \sqrt{2ml} \) \( (m > 0) \) independently of \( \beta(t) \). This is a unique feature of the beta-field. For \( m \gg 1 \), the lowest adiabatic eigenvalue of \( H(t) \) is approximately given by

\[
E^{(m)}_0(t) \simeq V(r_m) = \frac{\hbar^2}{2m_e} \left( \frac{\sqrt{2m}}{l} + \beta(t) \right)^2,
\]

where we neglect the zero point energy estimated by the curvature of \( V(r) \) at \( r = r_m \). For those states with \( |\delta|/l \gg 1 \), \( E^{(m)}_0(t) \) is approximated as \( E^{(m)}_0(t) \simeq m\hbar\tilde{\omega}_c(t) \) with \( \tilde{\omega}_c(t) = \omega_c(1 + l^2\beta(t)/|\delta|^2) \). Since the initial wave packet of the cyclotron motion is a coherent state given by the sum of the eigenstates of \( L_z \), the wave packet does not change the wave form nor the orbital radius, with only its frequency of circulation increasing as \( \beta(t) \) is increased adiabatically.

4. Discussion

It has been shown that the velocity of the cyclotron motion of the quantum mechanical electrons can be increased adiabatically, with its orbital radius kept nearly constant, by the induction of a magnetic field with a specially designed profile. This means that the Hamiltonian (2) has an
approximate adiabatic invariant \( \langle r \rangle \) for such type of magnetic field. At first sight, this seems paradoxical since the angular momentum is a strictly conserved quantity. It should be noted that the velocity of the electron is given by \( \vec{v} = (\vec{p} + e \vec{A})/m_e \) with \( \vec{p} \) being the momentum. Therefore, the kinetic part of the angular momentum \( \tilde{L}_z \) is given by

\[
\tilde{L}_z = (\vec{r} \times m_e \vec{v})_z = L_z + e r A_\theta.
\]

Thus, the increase of the angular momentum of the electromagnetic field is transferred to that of the kinetic part of the angular momentum.

\[\text{Figure 1. A possible design of the electromagnet that produces the beta-field.}\]

For actual implementation of the quantum betatron, a possible way of producing the beta-field may be to use an electromagnet with a cone-shaped core as a magnetic pole, with its tip of cone nearly touching the sample, as shown in Fig.1. An alternative and more interesting way will be to irradiate a circularly polarized laser pulse focused to the sample which is put on an appropriate magneto-optically active material. The inverse Faraday effect\[3\] will produce a beta-field with a designed spatio-temporal profile of the magnetic field.

Like classical betatrons, the accelerated electrons in the quantum betatron will emit radiations. It was shown\[4\] that in the nonrelativistic regime, the emission spectrum has a single peak at the cyclotron frequency. The accelerated frequency \( \tilde{\omega}_c \) is given by \( \tilde{\omega}_c = \omega_c F \) with the enhancement factor \( F \) given by \( F = (1 + B_\delta/B_0)^2 \), where \( B_\delta \) is the intensity of the beta-field at the orbit, \( B_\delta = C/|\delta| \). Therefore, \( \tilde{\omega}_c \) is higher for the electrons circulating closer to the origin. The quantum betatron will be a source of broad band radiation in mesoscopic scale.

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