ON THE STRUCTURE OF EINSTEIN WARPED PRODUCT SEMI-RIEmannIAN MANIFOLDS

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Abstract. In this paper we consider a class of Einstein warped product semi-Riemannian manifolds \( \tilde{M} = M^n \times_f N^m \) with \( n \geq 3 \) and \( m \geq 2 \). For \( \tilde{M} \) with compact base and Ricci-flat fiber, we prove that \( \tilde{M} \) is simply a Riemannian product space. Then, when the base \( M \) is conformal to a pseudo-Euclidean space which is invariant under the action of a \((n-1)\)-dimensional translation group, we classify all such spaces. Furthermore, we get new examples of complete Einstein warped products Riemannian manifolds.

1. Introduction

Einstein manifolds are related with many questions in geometry and physics, for instance: Riemannian functionals and their critical points, Yang-Mills theory, self-dual manifolds of dimension four, exact solutions for the Einstein equation field. Today we already have in our hands many examples of Einstein manifolds, even the Ricci-flat ones (see [4, 9, 11, 12]). However, finding new examples of Einstein metrics is not an easy task. A common tool to make new examples of Einstein spaces is to consider warped product metrics (see [11, 12]).

In [4], a question was made about Einstein warped products:

\[ \text{"Does there exist a compact Einstein warped product with nonconstant warping function?"} \] (1.1)

Inspired by the problem (1.1), several authors explored this subject in an attempt to get examples of such manifolds. Kim and Kim [10] considered a compact Riemannian Einstein warped product with nonpositive scalar curvature. They proved that a manifold like this is just a product manifold. Moreover, in [2, 6] they considered (1.1) without the compactness assumption. Barros, Batista and Ribeiro Jr [3] also studied (1.1) when the Einstein product manifold is complete and noncompact with nonpositive scalar curvature. It is worth to say that Case, Shu and Wei [5] proved that a shrinking quasi-Einstein metric has positive scalar curvature. Further, Sousa and Pina [12] were able to classify some structures of Einstein warped product on semi-Riemannian manifolds, they considered, for instance, the case in which the base and the fiber are Ricci-flat semi-Riemannian manifolds. Furthermore, they provided a classification for a noncompact Ricci-flat warped product semi-Riemannian manifold with 1-dimensional fiber, however the base is not necessarily a Ricci-flat manifold. More recently, Leandro and Pina [11] classified the static solutions for the vacuum Einstein equation field with cosmological constant not necessarily identically zero, when the base is invariant under the action of a translation group. In particular, they provided a necessarily condition for integrability of the system of differential equations given by the invariance of the base for the static metric.

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When the base of an Einstein warped product is a compact Riemannian manifold and the fiber is a Ricci-flat semi-Riemannian manifold, we get a partial answer for (1.1). Furthermore, when the base is not compact, we obtain new examples of Einstein warped products.

Now, we state our main results.

**Theorem 1.** Let \((\hat{M}^{n+m}, \hat{g}) = (M^n, g) \times_f (N^m, \bar{g})\), \(n \geq 3\) and \(m \geq 2\), be an Einstein warped product semi-Riemannian manifold (non Ricci-flat), where \(M\) is a compact Riemannian manifold and \(N\) is a Ricci-flat semi-Riemannian manifold. Then \(\hat{M}\) is a product manifold, i.e., \(f\) is trivial.

It is very natural to consider the next case (see Section 2).

**Theorem 2.** Let \((\hat{M}^{n+m}, \hat{g}) = (M^n, g) \times_f (N^m, \bar{g})\), \(n \geq 3\) and \(m \geq 2\), be an Einstein warped product semi-Riemannian manifold (i.e., \(\hat{\text{Ric}} = \lambda \bar{g}\); \(\lambda \neq 0\)), where \(M\) is a compact Riemannian manifold with scalar curvature \(R \leq \lambda(n-m)\), and \(N\) is a semi-Riemannian manifold. Then \(\hat{M}\) is a product manifold, i.e., \(f\) is trivial. Moreover, if the equality holds, then \(N\) is Ricci-flat.

Now, we consider that the base is a noncompact Riemannian manifold. The next result was inspired, mainly, by Theorem 2 and (1.1), and gives the relationship between Ricci tensor \(\hat{\text{Ric}}\) of the warped metric \(\hat{g}\) and the Ricci tensor \(\text{Ric}\) for the metric of the base \(g\).

**Theorem 3.** Let \((\hat{M}^{n+m}, \hat{g}) = (M^n, g) \times_f (N^m, \bar{g})\), \(n \geq 3\) and \(m \geq 2\), be an Einstein warped product semi-Riemannian manifold (i.e., \(\hat{\text{Ric}} = \lambda \bar{g}\)), where \(M\) is a noncompact Riemannian manifold with constant scalar curvature \(\lambda = \frac{R}{n-1}\), and \(N\) is a semi-Riemannian manifold. Then \(M\) is Ricci-flat if and only if the scalar curvature \(R\) is zero.

Considering a conformal structure for the base of an Einstein warped product semi-Riemannian manifold, we have the next results. Furthermore, the following theorem is very technical. We consider that the base for such Einstein warped product manifold is conformal to a pseudo-Euclidean space which is invariant under the action of a \((n-1)\)-dimensional translation group, and that the fiber is a Ricci-flat space. In order, for the reader to have a more intimate view of the next results, we recommend a previous reading of Section 2.

**Theorem 4.** Let \((\hat{M}^{n+m}, \hat{g}) = (\mathbb{R}^n, \bar{g}) \times_f (N^m, \bar{g})\), \(n \geq 3\) and \(m \geq 2\), be an warped product semi-Riemannian manifold such that \(N\) is a Ricci-flat semi-Riemannian manifold. Let \((\mathbb{R}^n, \bar{g})\) be a pseudo-Euclidean space with coordinates \(x = (x_1, \ldots, x_n)\) and \(g_{ij} = \delta_{ij} \varepsilon_i\), \(1 \leq i, j \leq n\), where \(\delta_{ij}\) is the delta Kronecker and \(\varepsilon_i = \pm 1\), with at least one \(\varepsilon_i = 1\).

Consider smooth functions \(\varphi(\xi)\) and \(f(\xi)\), where \(\xi = \sum_{k=1}^{n} \alpha_k x_k\), \(\alpha_k \in \mathbb{R}\), and \(\sum_{k=1}^{n} \varepsilon_k \alpha_k^2 = \kappa\).

Then \((\mathbb{R}^n, \bar{g}) \times_f (N^m, \bar{g})\), where \(\bar{g} = \frac{1}{\varphi} g\), is an Einstein warped product semi-Riemannian manifold (i.e., \(\hat{\text{Ric}} = \lambda \bar{g}\)) such that \(f\) and \(\varphi\) are given by:

\[
\begin{aligned}
(n-2) \varphi'' - m (G \varphi)' &= m G^2 \\
\varphi'' - (n-1) (\varphi')^2 + m G \varphi' &= \kappa \lambda \\
n G \varphi' - (G \varphi)' - m G^2 &= \kappa \lambda,
\end{aligned}
\]

and

\[
f = \Theta \exp \left( \int \frac{G}{\varphi} d\xi \right),
\]

where \(\Theta \in \mathbb{R} \setminus \{0\}\), \(G(\xi) = \pm \sqrt{\frac{\kappa (n-m) - \bar{R}}{m(m-1)}}\) and \(\kappa = \pm 1\). Here \(\bar{R}\) is the scalar curvature for \(\bar{g}\).
The next result is a consequence of Theorem 4.

**Theorem 5.** Let \((\tilde{M}^{n+m}, \tilde{g}) = (M^n, g) \times_f (N^m, \tilde{g})\), \(n \geq 3\) and \(m \geq 2\), be an Einstein warped product semi-Riemannian manifold, where \(\tilde{M}\) is conformal to a pseudo-Euclidean space invariant under the action of a \((n-1)\)-dimensional translation group with constant scalar curvature (possibly zero), and \(N\) is a Ricci-flat semi-Riemannian manifold. Then, \(\tilde{M}\) is either

1. a Ricci-flat semi-Riemannian manifold \((\mathbb{R}^n, g) \times_f (N^m, \tilde{g})\), such that \((\mathbb{R}^n, g)\) is the pseudo-Euclidean space with warped function \(f(\xi) = \Theta \exp(A\xi)\), where \(\Theta > 0\), \(A \neq 0\) are nonnull constants, or

2. conformal to \((\mathbb{R}^n, g) \times (N^m, \tilde{g})\), where \((\mathbb{R}^n, g)\) is the pseudo-Euclidean space. The conformal function \(\varphi\) is given by

\[
\varphi(\xi) = \frac{1}{(-G\xi + C)^2}; \quad \text{where} \quad G \neq 0, C \in \mathbb{R}.
\]

Moreover, the conformal function is defined for \(\xi \neq \frac{C}{G}\).

It is worth mentioning that the first item of Theorem 5 was not considered in [12].

From Theorem 5 we can construct examples of complete Einstein warped product Riemannian manifolds.

**Corollary 1.** Let \((N^m, \tilde{g})\) be a complete Ricci-flat Riemannian manifold and \(f(\xi) = \Theta \exp(A\xi)\), where \(\Theta > 0\) and \(A \neq 0\) are constants. Therefore, \((\mathbb{R}^n, g_{can}) \times_f (N^m, \tilde{g})\) is a complete Ricci-flat warped product Riemannian manifold.

**Corollary 2.** Let \((N^m, \tilde{g})\) be a complete Ricci-flat Riemannian manifold and \(f(x) = \frac{1}{x_n}\) with \(x_n > 0\). Therefore, \((\tilde{M}, \tilde{g}) = (\mathbb{R}^n, g_{can}) \times_f (N^m, \tilde{g})\) is a complete Riemannian Einstein warped product such that

\[
\hat{\text{Ric}} = -\frac{m+n-1}{n(n-1)}\tilde{g}.
\]

The paper is organized as follows. Section 2 is divided in two subsections, namely, General formulas and A conformal structure for the warped product with Ricci-flat fiber; where will be provided the preliminary results. Further, in Section 3 we will prove our main results.

2. Preliminary

Consider \((M^n, g)\) and \((N^m, \tilde{g})\), with \(n \geq 3\) and \(m \geq 2\), semi-Riemannian manifolds, and let \(f : M^n \to (0, +\infty)\) be a smooth function, the warped product \((\tilde{M}^{n+m}, \tilde{g}) = (M^n, g) \times_f (N^m, \tilde{g})\) is a product manifold \(M \times N\) with metric

\[
\hat{g} = g + f^2\tilde{g}.
\]

From Corollary 43 in [9], we have that (see also [10])

\[
(2.1) \quad \hat{\text{Ric}} = \lambda \hat{g} \iff \begin{cases} 
\text{Ric} - \frac{m}{f} \nabla^2 f = \lambda g \\
\tilde{\text{Ric}} = \mu \tilde{g} \\
f\Delta f + (m-1)|\nabla f|^2 + \lambda f^2 = \mu
\end{cases},
\]

where \(\lambda\) and \(\mu\) are constants. Which means that \(\tilde{M}\) is an Einstein warped product if and only if (2.1) is satisfied. Here \(\hat{\text{Ric}}, \text{Ric}\) and \(\tilde{\text{Ric}}\) are, respectively, the Ricci tensor for \(\hat{g}, \tilde{g}\) and \(g\). Moreover, \(\nabla^2 f, \Delta f\) and \(\nabla f\) are, respectively, the Hessian, The Laplacian and the gradient of \(f\) for \(g\).
2.1. General formulas. We derive some useful formulae from system \((2.1)\). Contracting the first equation of \((2.1)\) we get
\[
Rf^2 - mf \Delta f = nf^2 \lambda,
\]
where \(R\) is the scalar curvature for \(g\). From the third equation in \((2.1)\) we have
\[
mf \Delta f + m(m-1)|\nabla f|^2 + m\lambda f^2 = m\mu.
\]
Then, from \((2.2)\) and \((2.3)\) we obtain
\[
|\nabla f|^2 + \left[\frac{\lambda(m-n) + R}{m(m-1)}\right] f^2 = \frac{\mu}{(m-1)}.
\]
When the base is a Riemannian manifold and the fiber is a Ricci-flat semi-Riemannian manifold (i.e., \(\mu = 0\)), from \((2.4)\) we obtain
\[
|\nabla f|^2 + \left[\frac{\lambda(m-n) + R}{m(m-1)}\right] f^2 = 0.
\]
Then, either
\[
R \leq \lambda(n-m)
\]
or \(f\) is trivial, i.e., \(\widehat{M}\) is a product manifold.

2.2. A conformal structure for the Warped product with Ricci-flat fiber. In what follows, consider \((\mathbb{R}^n, g)\) and \((N^m, \tilde{g})\) semi-Riemannian manifolds, and let \(f : \mathbb{R}^n \rightarrow (0, +\infty)\) be a smooth function, the warped product \((\widehat{M}^{n+m}, \hat{g}) = (\mathbb{R}^n, g) \times_f (N^m, \tilde{g})\) is a product manifold \(\mathbb{R}^n \times N\) with metric
\[
\hat{g} = g + f^2 \tilde{g}.
\]
Let \((\mathbb{R}^n, g), n \geq 3\), be the standard pseudo-Euclidean space with metric \(g\) and coordinates \((x_1, \ldots, x_n)\) with \(g_{ij} = \delta_{ij} \varepsilon_i, 1 \leq i, j \leq n\), where \(\delta_{ij}\) is the delta Kronecker, \(\varepsilon_i = \pm 1\), with at least one \(\varepsilon_i = 1\). Consider \((\widehat{M}^{n+m}, \hat{g}) = (\mathbb{R}^n, g) \times_f (N^m, \tilde{g})\) a warped product, where \(\varphi : \mathbb{R}^n \rightarrow \mathbb{R} \setminus \{0\}\) is a smooth function such that \(\hat{g} = \frac{\varphi}{\varphi^2}\). Furthermore, we consider that \(\widehat{M}\) is an Einstein semi-Riemannian manifold, i.e.,
\[
\hat{\text{Ric}} = \lambda \hat{g},
\]
where \(\hat{\text{Ric}}\) is the Ricci tensor for the metric \(\hat{g}\) and \(\lambda \in \mathbb{R}\).

We use invariants for the group action (or subgroup) to reduce a partial differential equation into a system of ordinary differential equations \([8]\). To be more clear, we consider that \((\widehat{M}^{n+m}, \hat{g}) = (\mathbb{R}^n, g) \times_f (N^m, \tilde{g})\) is such that the base is invariant under the action of a \((n-1)\)-dimensional translation group \([11, 12, 13]\). More precisely, let \((\mathbb{R}^n, g)\) be the standard pseudo-euclidean space with metric \(g\) and coordinates \((x_1, \ldots, x_n)\), with \(g_{ij} = \delta_{ij} \varepsilon_i, 1 \leq i, j \leq n\), where \(\delta_{ij}\) is the delta Kronecker, \(\varepsilon_i = \pm 1\), with at least one \(\varepsilon_i = 1\). Let \(\xi = \sum_i \alpha_i x_i, \alpha_i \in \mathbb{R}\), be a basic invariant for a \((n-1)\)-dimensional translation group
\[
\alpha = \sum_i \alpha_i \frac{\partial}{\partial x_i}
\]
is a timelike, lightlike or spacelike vector, i.e., \(\sum_i \varepsilon_i \alpha_i^2 = -1, 0, \text{ or } 1\), respectively. Then we consider \(\varphi(\xi)\) and \(f(\xi)\) non-trivial differentiable functions such that
\[
\varphi_{x_i} = \varphi' \alpha_i \quad \text{and} \quad f_{x_i} = f' \alpha_i.
\]
Moreover, it is well known (see \([1, 11, 12]\)) that if \(\hat{g} = \frac{\varphi}{\varphi^2}g\), then the Ricci tensor \(\hat{\text{Ric}}\) for \(\hat{g}\) is given by
\[
\hat{\text{Ric}} = \frac{1}{\varphi^2} \{(n-2)\varphi \nabla^2 \varphi + [\varphi \Delta \varphi - (n-1)\nabla \varphi]^2 g\},
\]
where $\nabla^2 \varphi$, $\Delta \varphi$ and $\nabla \varphi$ are, respectively, the Hessian, the Laplacian and the gradient of $\varphi$ for the metric $g$. Hence, the scalar curvature of $\bar{g}$ is given by

$$\bar{R} = \sum_{k=1}^{n} \varepsilon_k \varphi^2 \left( \bar{Ric} \right)_{kk} = (n-1)(2\varphi \Delta \varphi - n|\nabla \varphi|^2)$$

(2.6)

$$= (n-1)|2\varphi \varphi'' - n\varphi|^2 \sum \varepsilon_i \alpha_i^2.$$  

In what follows, we denote $\kappa = \sum \varepsilon_i \alpha_i^2$.

When the fiber $N$ is a Ricci-flat semi-Riemannian manifold, we already know from Theorem 1.2 in [12] that $\varphi(\xi)$ and $f(\xi)$ satisfy the following system of differential equations

$$\begin{cases} 
(n-2)f\varphi'' - mf''\varphi - 2m\varphi' f' = 0; \\
2\varphi\varphi'' - (n-1)f(\varphi')^2 + m\varphi\varphi' f' = \kappa \lambda f; \\
(n-2)f\varphi\varphi' f' - (m-1)\varphi^2(f')^2 - f f''\varphi^2 = \kappa \lambda f^2. 
\end{cases}$$

(2.7)

Note that the case where $\kappa = 0$ was proved in [12]. Therefore, we only consider the case $\kappa = \pm 1$.

3. PROOF OF THE MAIN RESULTS

**Proof of Theorem 1** In fact, from the third equation of the system (2.1) we get that

$$\text{div} (f \nabla f) + (m-2)|\nabla f|^2 + \lambda f^2 = \mu.$$  

(3.1)

Moreover, if $N$ is Ricci-flat, from (3.1) we obtain

$$\text{div} (f \nabla f) + \lambda f^2 \leq \text{div} (f \nabla f) + (m-2)|\nabla f|^2 + \lambda f^2 = 0.$$  

(3.2)

Considering $M$ a compact Riemannian manifold, integrating (3.2) we have

$$\int_M \lambda f^2 dv = \int_M (\text{div} (f \nabla f) + \lambda f^2) dv \leq 0.$$  

(3.3)

Therefore, from (3.3) we can infer that

$$\lambda \int_M f^2 dv \leq 0.$$  

(3.4)

This implies that, either $\lambda \leq 0$ or $f$ is trivial. It is worth to point out that compact quasi-Einstein metrics on compact manifolds with $\lambda \leq 0$ are trivial (see Remark 6 in [10]).  

**Proof of Theorem 2** Let $p$ be a maximum point of $f$ on $M$. Therefore, $f(p) > 0$, $(\nabla f)(p) = 0$ and $(\Delta f)(p) \geq 0$. By hypothesis $R + \lambda(m-n) \leq 0$, then from (2.4) we get

$$|\nabla f|^2 \geq \frac{\mu}{m-1}.$$  

Whence, in $p \in M$ we obtain

$$0 = |\nabla f|^2(p) \geq \frac{\mu}{m-1}.$$  

Since $\mu$ is constant, we have that $\mu \leq 0$. Moreover, from the third equation in (2.1) we have

$$\lambda f^2(p) \leq (f \Delta f)(p) + (m-1)|\nabla f|^2(p) + \lambda f^2(p) = \mu \leq 0.$$
Implying that \( \lambda \leq 0 \). Then, from [10] the result follows.

Now, if \( R + \lambda (m - n) = 0 \) from [2.4] we have that

\[
|\nabla f|^2 = \frac{\mu}{m - 1}.
\]

Then, for \( p \in M \) we obtain

\[
0 = |\nabla f|^2(p) = \frac{\mu}{m - 1}.
\]

Therefore, since \( \mu \) is a constant we get that \( \mu = 0 \), i.e., \( N \) is Ricci-flat.

It is worth to say that if \( M \) is a compact Riemannian manifold and the scalar curvature \( R \) is constant, then \( f \) is trivial (see [5]).

**Proof of Theorem 3**

Considering \( \lambda = \frac{R}{n - 1} \) in equation (2.4) we obtain

(3.5)

\[
|\nabla f|^2 + \frac{R}{m(n - 1)} f^2 = \frac{\mu}{m - 1}.
\]

Then, taking the Laplacian we get

(3.6)

\[
\frac{1}{2} \Delta |\nabla f|^2 + \frac{R}{m(n - 1)} (|\nabla f|^2 + f \Delta f) = 0.
\]

Moreover, when we consider that \( \lambda = \frac{R}{n - 1} \) in (2.1), and contracting the first equation of the system we have that

(3.7)

\[-\Delta f = \frac{Rf}{m(n - 1)}.\]

From (3.7), (3.6) became

(3.8)

\[
\frac{1}{2} \Delta |\nabla f|^2 + \frac{R}{m(n - 1)} |\nabla f|^2 = \frac{R^2 f^2}{m^2(n - 1)^2}.
\]

The first equation of (2.1) and (3.5) allow us to infer that

\[
\frac{2f}{m} Ric(\nabla f) = \frac{2Rf}{m(n - 1)} \nabla f + 2 \nabla^2 f(\nabla f)
\]

\[
= \nabla \left( |\nabla f|^2 + \frac{Rf^2}{m(n - 1)} \right) = \nabla \left( \frac{\mu}{m - 1} \right) = 0.
\]

And since \( f > 0 \) we get

(3.9)

\[Ric(\nabla f, \nabla f) = 0.\]

Remember the Bochner formula

(3.10)

\[
\frac{1}{2} \Delta |\nabla f|^2 = |\nabla^2 f|^2 + Ric(\nabla f, \nabla f) + g(\nabla f, \nabla \Delta f).
\]

Whence, from (3.7), (3.9) and (3.10) we obtain

(3.11)

\[
\frac{1}{2} \Delta |\nabla f|^2 + \frac{R}{m(n - 1)} |\nabla f|^2 = |\nabla^2 f|^2.
\]

Substituting (3.8) in (3.11) we get

(3.12)

\[
|\nabla^2 f|^2 = \frac{R^2 f^2}{m^2(n - 1)^2}.
\]

From the first equation of (2.1), a straightforward computation give us

(3.13)

\[|Ric|^2 = \frac{m^2}{f^2} |\nabla^2 f|^2 + \frac{2mR\Delta f}{(n - 1)f} + \frac{nR^2}{(n - 1)^2}.\]
Finally, from (3.12), (3.11) and (3.13) we have that
\[ |Ric|^2 = \frac{R^2}{n-1}. \]
Then, we get the result. □

In what follows, we consider the conformal structure given in Section 2.2 to prove Theorem 4 and Theorem 5.

**Proof of Theorem 4** From definition,
\[ |\bar{\nabla} f|^2 = \sum_{i,j} \varphi^2 \varepsilon_i \delta_{ij} f_{x_i} f_{x_j} = \left( \sum_i \varepsilon_i \alpha_i^2 \right) \varphi^2 (f')^2 = \kappa \varphi^2 (f')^2, \]
where \( \bar{\nabla} f \) is the gradient of \( f \) for \( \bar{g} \), and \( \kappa \neq 0 \). Then, from (2.5) and (3.14) we have
\[ \kappa \varphi^2 (f')^2 + \left[ \frac{\lambda (m-n) + \bar{R}}{m(m-1)} \right] f^2 = \mu \]
(3.15)

Consider that \( N \) is a Ricci-flat semi-Riemannian manifold, i.e., \( \mu = 0 \), from (3.15) we get
\[ (n-1) \varphi'' - m \varphi' = \mu \]
(3.16)
where \( G(\xi) = \pm \sqrt{\varphi^2 (f')^2 - \mu} \). Which give us (1.3).

Now, from (3.16) we have
\[ \frac{f''}{f} = \frac{G(\bar{R})}{\varphi}, \]
(3.17)
Therefore, from (2.7), (3.16) and (3.17) we get (1.2).

**Proof of Theorem 5** Considering that \( \bar{R} \) is constant, from (1.2) we obtain
\[ \begin{cases} 
(n-2) \varphi'' - m \varphi' = mG^2 \\
\varphi'' - (n-1)(\varphi')^2 + mG \varphi' = \kappa \lambda \\
(n-1)G \varphi' - mG^2 = \kappa \lambda 
\end{cases} \]
(3.18)
The third equation in (3.18) give us that \( \varphi \) is an affine function. Moreover, since
\[ \varphi'(\xi) = \frac{\kappa \lambda + mG^2}{(n-1)G}, \]
(3.19)
we get \( \varphi'' = 0 \). Then, from the first and second equations in (3.18) we have, respectively,
\[ -m \varphi' = mG^2 \quad \text{and} \quad -(n-1)(\varphi')^2 + m \varphi' = \kappa \lambda. \]
This implies that
\[ (\varphi')^2 = \frac{\kappa \lambda + mG^2}{(n-1)}, \]
(3.20)
Then, from (3.19) and (3.20) we get
\[ (\varphi')^2 + G \varphi' = 0. \]
That is, \( \varphi' = 0 \) or \( \varphi' = -G \).
First consider that $\varphi' = 0$. From (2.6) and (3.18), it is easy to see that $\lambda = \bar{R} = 0$. Then, we get the first item of the theorem since, as mentioned, the case $\varphi' = 0$ was not considered in [12].

Now, we take $\varphi' = -G$. Integrating over $\xi$ we have
\begin{equation}
\varphi(\xi) = -G\xi + C; \quad \text{where} \quad G \neq 0, C \in \mathbb{R}.
\end{equation}
Then, from (3.19) we obtain
\begin{equation}
\kappa \lambda + mG^2 = -G.
\end{equation}
Since $G^2 = \frac{k[\lambda(n-m)-\bar{R}]}{m(m-1)}$, from (3.22) we obtain
\begin{equation}
\bar{R} = \frac{n(n-1)\lambda}{(m+n-1)}.
\end{equation}
Considering that $\lambda \neq 0$, we can see that $\bar{R}$ is a non-null constant. On the other hand, since $\varphi' = -G$, from (2.6) we get
\begin{equation}
\bar{R} = -n(n-1)\kappa G^2,
\end{equation}
where $G^2 = \frac{k[\lambda(n-m)-\bar{R}]}{m(m-1)}$. Observe that (3.23) and (3.24) are equivalent.

Furthermore, from (1.3) and (3.21) we get
\begin{equation}
f(\xi) = \frac{\Theta}{-G\xi + C}.
\end{equation}
Now the demonstration is complete. ♦

**Proof of Corollary 1**: It is a direct consequence of Theorem 5-(1).

**Proof of Corollary 2**: Remember that $\xi = \sum \alpha_i x_i$, where $\alpha_i \in \mathbb{R}$ (cf. Section 2.2).

Consider in Theorem 5-(2) that $\alpha_n = \frac{1}{G}$ and $\alpha_i = 0$ for all $i \neq n$. Moreover, taking $C = 0$ we get
\begin{equation}
f(\xi) = \frac{1}{x_n^2}.
\end{equation}
Moreover, taking $\mathbb{R}_{+}^n = \{ (x_1, \ldots, x_n) \in \mathbb{R}^n; x_n > 0 \}$. Then, $\left( \mathbb{R}_{+}^n, g_{can} = \delta_{ij} \frac{x_i}{x_n} \right) = (\mathbb{H}^n, g_{can})$ is the hyperbolic space. We pointed out that $\mathbb{H}^n$ with this metric has constant sectional curvature equal to $-1$. Then, from (3.23) we obtain $\lambda = -\frac{m+n-1}{n(n-1)}$, and the result follows. ♦

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