SOLVING A FRACTIONAL PROGRAMMING PROBLEM IN A COMMERCIAL BANK

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Abstract. We formulate a new optimization problem which arises in the Bank Asset and Liability Management (ALM). The problem is a fractional programming which belongs to a class of global optimization. Most of optimization problems in the Bank Asset and Liability Management are return maximization or risk minimization problems. For solving the fractional programming problem, we propose curvilinear multi-start algorithm which finds the best local solutions to the problem. Numerical results are given based on the balance sheets of 5 commercial banks of Mongolia.

1. Introduction. Bank asset and liability management are defined as the complex planning of all asset and liability positions on the bank’s balance sheet, which includes the purpose of managing interest rate risk, providing liquidity and enhancing value of the bank. Therefore, this purpose has led banks to determine their optimal balance among profitability, risk, liquidity and other uncertainties [11]. The optimal balance exists between the structure of a bank’s liability, equity and the composition of its assets.

Asset and liability management models have been studied in a number of works [11]-[15] and [4]-[10]. Among them portfolio optimization [15] plays an important role. According to this theory, bank managers utilize risk-averse utility function. However, this model omits matching an activity based on assets and liabilities. In order to achieve certain goals, such as maximization of the bank’s gross revenue, a linear programming problem was proposed in [5]. The main disadvantage of this approach is a deterministic simulation to observe portfolio behavior under various economic conditions [2]. Dynamic optimization problems have been examined in [19] and [16]. A dynamic programming in terms of asset problems has been considered in [6]. However, due to a small number of financial instruments that can be analyzed simultaneously, this approach has limited usage in practice.

A stochastic linear programming approach was proposed in [4]. This technique denotes each realization of the variables by a constraint and leads to large problems...
in realistic situations. A multi-objective linear programming model for commercial bank balance sheet management was developed in [9] considering profitability and solvency objectives subject to policy and managerial constraints.

The problem of management of assets and liabilities for banking was formulated as a multi-period stochastic linear programming in Cohen [4] and Eatman [9]. They show that the proposed model ALM [20]-[8] has an advantage in front of linear programming model. Another approach to model bank asset and liability management is a goal-programming model [12], [10] which has been solved by large-scale multi-criteria decision making problem [13]. Extended models of this approach determine the optimal balance among profit, risk, liquidity and other uncertainties by considering several goals, such as the maximization of returns, the minimization of risk, the maintenance of a desirable level of liquidity and solvency, the expansion of deposits and loans. Optimization of liquidity management by fuzzy programming was proposed in [17]. The stochastic programming approach has been discussed in [20].

A mixed approach for asset and liability management has been developed in [14]. Multistage stochastic integer programming with logical constraints on asset and liability management under uncertainty was developed in [8]. The work [22] deals with continuous-time asset and liability management under benchmark and mean-variance criteria in a jump diffusion market. The benchmark model was studied by stochastic dynamic programming.

It seems that there is a little attention paid to a fractional programming approach for bank asset and liability management. The aim of this paper is to introduce a fractional programming approach based on bank asset and liability management. The paper is organized as follows. In Section 2, we introduce the main ratio variables of Bank asset and liability management and their relationship. Section 3 is devoted to a fractional programming problem in ALM. A global search algorithm is described in Section 4. Computational results are provided in Section 5.

2. Bank asset and liability management model. The need of ALM to the bank industry has been highlighted due to the dramatic changes in relation with uses of funds and sources of funds. This has been accompanied by increased volatility of markets, diversification of bank product profiles, and intensified competition between banks on a global scale, all including the risk exposure of banks. Thus, banks increasingly need to match the maturities of the assets and liabilities, providing the objectives of profitability, liquidity, and risk. This has increased the significance of implementing structural variables of asset and liability management in banks. The work is to formulate this type of approach. The formulation is based on a balance sheet and 14 structural variables. 7 variables correspond to assets and others are liabilities.

In Table 1, we show these variables.

| Assets                        | Liabilities               |
|-------------------------------|---------------------------|
| \( A_1 \) : Cash and cash equivalents | \( L_1 \) : Current account |
| \( A_2 \) : Deposits to the Bank of Mongolia | \( L_2 \) : Time deposit  |
| \( A_3 \) : Deposits at other banks    | \( L_3 \) : Demand deposit |
| \( A_4 \) : Financial investments | \( L_4 \) : Placements by other banks |
| \( A_5 \) : Loans and advances    | \( L_5 \) : Other deposits |
| \( A_6 \) : Other financial assets | \( L_6 \) : Other liabilities |
| \( A_7 \) : Fixed assets        | \( E \) : Equity          |
| \( A \) : Total assets         | \( L + E = A \) : Total assets |
where $L$: total liabilities for $L = L_1 + L_2 + L_3 + L_4 + L_5 + L_6$, $D$: deposits without $L_4$, $L_5$ and $L_6$ for $D = L_1 + L_2 + L_3$. In most of the cases, asset and liability management committee (ALCO) shapes a bank’s balance sheet under an adequate risk management framework and regulatory compliance limits. Such limits include liquidity ratios, capital adequacy ratio and leverage ratio. The idea behind these limits is to provide sustainable structural variables on a balance sheet. Therefore, if ALCO approve, then the following inequalities hold:

\begin{align}
L_2 &\geq L_1 + L_3, \\
L_1 + L_2 &\geq L_4, \\
A_1 + A_2 + A_3 + A_4 &\geq L_4, \\
L_2 &\leq A_1 + A_2 + A_3 + A_4, \\
A_5 &\leq A_1 + A_2 + A_3 + 2A_4, \\
A_6 + A_7 &\leq E, \\
A_6 + A_7 &\leq A_4, \\
L_5 + L_6 &\geq E. \\
\end{align} 

(1)

In order to formulate our model, we consider the following ratio variables:

\begin{align}
v_1 &= \frac{A_1 + A_2 + A_3 + A_4}{L_1 + L_2 + L_3 + L_4 + L_5}, \\
v_2 &= \frac{A_5}{A}, \\
v_3 &= \frac{L_2}{L_1 + L_2 + L_3 + L_4 + L_5}, \\
v_4 &= \frac{A_1 + A_2 + A_3 + A_4}{A}, \\
v_5 &= \frac{L_4}{A}, \\
v_6 &= \frac{A_4 + A_5}{L_1 + L_2 + L_3}, \\
v_7 &= \frac{A_4}{A}, \\
v_8 &= \frac{L_1 + L_3}{L}, \\
v_9 &= \frac{L_4}{L}, \\
v_{10} &= \frac{E}{L},
\end{align} 

(2)

where $v_1$: liquidity ratio, $v_2$: loans to total assets ratio, $v_3$ and $v_5$: stability and instability ratio of deposit, respectively, $v_4$: liquidity assets to total assets ratio, $v_6$: placements by other banks to total assets ratio, $v_7$: earning assets to total assets ratio, $v_8$: financial investment to total assets ratio, $v_9$: placements by other banks to total liabilities ratio, $v_{10}$: equity to total liabilities ratio.

Taking into account (2), we can obtain the following inequalities:

\begin{align}
v_1 &\geq v_4, \\
v_6 &\geq v_2 + v_7, \\
v_5 &\geq v_9, \\
v_3 &\geq v_8, \\
v_9 &\geq v_5, \\
v_4 &\geq v_5, \\
v_3 &\leq v_4, \\
v_2 &\leq v_4 + v_7.
\end{align} 

(3)
Substituting (2) into (1), we have:

\[ v_2 + v_4 \geq \frac{v_5}{v_9}, \]
\[ v_2 + v_4 + v_7 \geq 1, \]
\[ v_3 + v_8 + v_9 + v_{10} \leq 1. \]

3. Fractional programming formulation. Based on the main ratio variables, we proved the following lemmas [1].

**Lemma 3.1.** For \( v_1, v_2, \ldots, v_9 \), we have

\[ v_{10} = \frac{v_1}{v_3v_4} \left( \frac{v_9}{v_5} \left( \frac{v_2 + v_7}{v_6} \right) - v_8 \right) - 1. \]

**Lemma 3.2.** For \( v_1, v_2, \ldots, v_9 \), we have the relationship

\[ v_1 = \frac{v_3v_4v_6v_9}{v_9(v_2 + v_7) - v_5v_6v_8}. \]

**Lemma 3.3.** For \( v_5, v_9 \), we have

\[ \frac{v_5}{v_9} = \frac{1}{1 + v_{10}}. \]

**Lemma 3.4.** If \( \frac{v_4}{v_1} \geq 0 \), then

\[ -\frac{v_4}{v_1} + \frac{1}{1 + v_{10}} \geq 0. \]

Now we consider the problem of maximizing equity to total liabilities ratio or a capital adequacy subject to constraints given by ALM variables. This problem is formulated as:

\[ \max f(v) = \frac{f_1(v)}{f_2(v)}, v \in R^9, \]

where,

\[ f_1(v) = v_1v_2v_9 + v_1v_7v_9 - v_1v_5v_6v_8 - v_3v_4v_5v_6, \]
\[ f_2(v) = v_3v_4v_5v_6, \]

subject to

\[ g_1 = v_1v_2v_9 + v_1v_7v_9 - v_1v_5v_6v_8 - v_3v_4v_5v_6 = 0, \]
\[ g_2 = \frac{v_5}{v_9} - \frac{v_3v_4v_5v_6}{v_1v_2v_9 + v_1v_7v_9 - v_1v_5v_6v_8} = 0, \]
\[ g_3 = -v_2 - v_4 + \frac{v_5}{v_9} \leq 0, \]
\[ g_4 = \frac{v_4}{v_1} - \frac{v_3v_4v_5v_6}{v_1v_2v_9 + v_1v_7v_9 - v_1v_5v_6v_8} \leq 0, \]
\[ g_5 = v_3 + v_8 + v_9 + \frac{v_1v_2v_9 + v_1v_7v_9 - v_1v_5v_6v_8 - v_3v_4v_5v_6}{v_3v_4v_5v_6} - 1 \leq 0, \]
\[ g_6 = -v_2 - v_4 - v_7 + 1 \leq 0. \]
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\[
\begin{pmatrix}
-1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4 \\
v_5 \\
v_6 \\
v_7 \\
v_8 \\
v_9
\end{pmatrix} \leq 0,
\]

(12)

where \( f \): a capital adequacy, \( g_1 \): liquidity ratio constraint, \( g_2 \): leverage, \( g_3 \): liquidity and loan ratio constraints, \( g_4 \): total deposits ratio constraints, \( g_5 \): total weighted deposits ratio constraint, and \( g_6 \): combined asset constraint; \( g_7 \): constraints for asset and liability management, \( v_i, \overline{v}_i \) are lower and upper bounds for variables of \( v_i \), \( \mu_i, \sigma_i \) are means and standard deviations for variable \( v_i \).

4. The curvilinear multi-start algorithm. In order to solve problem (9)-(13), we propose curvilinear multi-start algorithm [7]. The algorithm was originally developed for solving box-constrained optimization problems. We use the penalty function techniques [3] for finding local solutions. For each equality constraint \( g(v) = 0 \), we construct a simple penalty function \( \hat{g}(v) = g^2(v) \). For each inequality constraint \( g(v) \leq 0 \), we also construct the corresponding penalty function as follows:

\[
\hat{g}(v) = \begin{cases} 
0, & \text{if } g(v) \leq 0, \\
g^2(v), & \text{if } g(v) > 0.
\end{cases}
\]

Thus, we have the following box-constrained optimization problem:

\[
\hat{f}(v) = -f(v) + \frac{\gamma}{2} \left( \sum_{i=1}^{2} g_i^2(v) + \sum_{j=3}^{14} \hat{g}_j(v) \right) \to \min, \\
V = \{v \in \mathbb{R}^9 | \underline{v}_i \leq v_i \leq \overline{v}_i, \ i = 1, ..., 9\}.
\]

where \( \gamma \) is a penalty parameter, \( \underline{v} \) and \( \overline{v} \) - are lower and upper bounds. For original \( v \)-variables the constraint is the box \([0,1]\). Initial value of a penalty parameter \( \gamma \) is chosen not too large (something about 1000) and after finding some local minimums we increase it for searching another local minimum.

The proposed algorithm starts from some initial point \( v^1 \in V \). At each \( k \)-th iteration, the algorithm performs randomly “drop” of two auxiliary points \( \hat{v}^1 \) and \( \hat{v}^2 \) and generating a curve (parabola) which passes through all three points \( v^k, \hat{v}^1 \) and \( \hat{v}^2 \). Then we generate some random grid along this curve and try to found all convex triples inside the grid. For each founded triple, we perform refining the triple minimum value with using the golden section method. The best triple became a start point for the local optimization algorithm, the final point of which will be
Algorithm 1 The Curvilinear Multi-start Algorithm

**Input:** $v^1 \in V$ – initial (start) point; $K > 0$ – iterations count; $\delta > 0$; $N > 0$; $\varepsilon_\alpha > 0$ — algorithm parameters.

**Output:** Best local solution $v^*$ and $f^* = f(v^*)$

**for** $k \leftarrow 1, K$ **do**

1. generate stochastic point $\tilde{v}^1 \in V$
2. generate stochastic point $\tilde{v}^2 \in V$
3. generate stochastic $\alpha$-grid:
   
   $$-1 = \alpha_1 \leq \cdots \leq \alpha_i \leq -\delta \leq \delta \leq \alpha_{i+1} \leq \cdots \leq \alpha_N = 1$$

4. Let $\hat{v}(\alpha_i) = \text{Proj}_V (\alpha_i \left( (\tilde{v}^1 + \tilde{v}^2)/2 - v^k \right) + \alpha_i/2 (\tilde{v}^2 - \tilde{v}^1) + v^k)$

5. $f_k^* \leftarrow f_k$
6. $\alpha_k^* \leftarrow 0$
7. **for** $i \leftarrow 1, (N - 2)$ **do**
8. Convex triplet if $f(\hat{v}(\alpha_i)) > f(\hat{v}(\alpha_{i+1}))$ **and** $f(\hat{v}(\alpha_{i+1})) < f(\hat{v}(\alpha_{i+2}))$. Refining the value of minima using the Golden-Section search method with accuracy $\varepsilon_\alpha$.

9. $\alpha_k^* \leftarrow \text{GoldenSectionSearch}(f, \alpha_i, \alpha_{i+1}, \alpha_{i+2}, \varepsilon_\alpha)$
10. **if** $f(\hat{v}(\alpha_k^*)) < f_k^*$ **then**
11. $f_k^* \leftarrow f(\hat{v}(\alpha_k^*))$
12. $\alpha_k^* \leftarrow 0$
13. **end if**
14. **end for**
15. Start local optimization algorithm $v^{k+1} \leftarrow \text{LOptim}(\hat{v}(\alpha_k^*))$
16. **end for**
17. $v^* \leftarrow v^k$
18. $f^* \leftarrow f(v^k)$

a start point for the next iteration of the global method. Note that we can also use any one-dimensional optimization method instead golden section in Algorithm 1. We can use a global search algorithm for minimizing $f(\hat{v}(\alpha))$ with respect to $\alpha$ such as Piyavskii method [18] but it is computationally expensive for any triple $v^k, \tilde{v}^1, \tilde{v}^2$. Details are presented in Algorithms 1 and 2.

Algorithm 2 The Local Optimization Algorithm

**Input:** $v^1 \in V$ – initial (start) point; $\varepsilon_v > 0$ — accuracy parameter.

**Output:** Local minimum point $v^*$ and $f^* = f(v^*)$

1. Repeat
2. $d^k = v^k - \text{Proj}_V (v^k - \nabla f(v^k))$
   //Perform local relaxation step, for example, with using standard convex interval capture technique.
3. $v^{k+1} = \arg\min_{\alpha \geq 0} f(v^k + \alpha d^k)$
4. **until** $\|v^{k+1} - v^k\|_2 \leq \varepsilon_v$
A convergence of Algorithm 2 is given by the following theorem.

**Theorem 4.1.** [21] The sequence \( \{v^k, k = 1, 2, \ldots\} \) generated by Algorithm 2 converges to a local solution of problem (9)-(13).

5. **Computational results.** The proposed method was implemented in Matlab. Parameters of Algorithms 1 and Algorithm 2 were the following: \( \gamma = 1000, \delta = 0.01, \varepsilon_\alpha = 0.001, \varepsilon_v = 0.0001, k = 20 \). The main ratio variables used in this model were taken directly from the balance sheets of Top 5 commercial banks of Mongolia for 2017. These variables are given in Table 2.

Table 2. Initial values of \( v \) for banks

| Ratio | Khan   | TDB | XAC | State | Golomt | Mean | Stdev |
|-------|--------|-----|-----|-------|--------|------|-------|
| \( v_1 \) | 0.564  | 0.520 | 0.545 | 0.409  | 0.528  | 0.513 | 0.061 |
| \( v_2 \) | 0.457  | 0.403 | 0.442 | 0.599  | 0.445  | 0.469 | 0.075 |
| \( v_3 \) | 0.405  | 0.253 | 0.358 | 0.496  | 0.412  | 0.385 | 0.089 |
| \( v_4 \) | 0.477  | 0.452 | 0.483 | 0.362  | 0.472  | 0.449 | 0.050 |
| \( v_5 \) | 0.202  | 0.203 | 0.413 | 0.181  | 0.186  | 0.237 | 0.099 |
| \( v_6 \) | 0.998  | 1.648 | 1.522 | 0.988  | 0.859  | 1.203 | 0.356 |
| \( v_7 \) | 0.150  | 0.296 | 0.206 | 0.064  | 0.118  | 0.166 | 0.089 |
| \( v_8 \) | 0.300  | 0.232 | 0.116 | 0.251  | 0.311  | 0.242 | 0.078 |
| \( v_9 \) | 0.228  | 0.232 | 0.446 | 0.197  | 0.201  | 0.260 | 0.103 |
| \( v_{10} \) | 0.129  | 0.140 | 0.070 | 0.089  | 0.081  | 0.102 | 0.031 |

Source: Audited report of individual commercial bank

Lower and upper bounds are the following:

\[
0.483 \leq v_1 \leq 0.544, 0.432 \leq v_2 \leq 0.507, 0.340 \leq v_3 \leq 0.429, \\
0.424 \leq v_4 \leq 0.474, 0.188 \leq v_5 \leq 0.287, 1.025 \leq v_6 \leq 1.381, \\
0.122 \leq v_7 \leq 0.211, 0.203 \leq v_8 \leq 0.281, 0.209 \leq v_9 \leq 0.312.
\]

The best local solution to problem (9)-(13) found by the Algorithm 1 corresponding to the Bank Asset and Liability Management was \( f^* = 0.2068 \) and the relative ratio variables were: \( v_1^* = 0.5318, v_2^* = 0.4517, v_3^* = 0.3404, v_4^* = 0.4330, v_5^* = 0.1880, \\
v_6^* = 1.2785, v_7^* = 0.1428, v_8^* = 0.2266, \) and \( v_9^* = 0.2266 \). The number of local solutions found by Algorithm 2 was 3 and computational time was 1m:36sec.

6. **Conclusion.** The problem of maximizing a capital adequacy subject to the ratios of the bank indicators has been formulated as a fractional programming problem. The problem is a non-convex optimization problem and was solved numerically by curvilinear multi-start algorithm [7] and tested on the balance sheet of some Mongolian commercial banks.

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