The zeta function of a finite category and the series Euler characteristic

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Abstract
We prove that a certain conjecture holds true and the conjecture states a relationship between the zeta function of a finite category and the Euler characteristic of a finite category.

1 Introduction
In [NogA], the zeta function of a finite category was defined and one conjecture was proposed. The zeta function of a finite category $I$ is the formal power series defined by

$$\zeta_I(z) = \exp\left(\sum_{m=1}^{\infty} \frac{\#N_m(I)}{m} z^m\right)$$

where

$$N_m(I) = \{ (x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \ldots \xrightarrow{f_m} x_m) \text{ in } I \}.$$ 

The conjecture states a relationship between the zeta function of a finite category and the Euler characteristic of a finite category, called series Euler characteristic [BL08].

Conjecture 1.1. Suppose $I$ is a finite category which has series Euler characteristic. Then, we have

(C1) the zeta function of $I$ is a finite product of the following form

$$\zeta_I(z) = \prod \frac{1}{(1 - \alpha_i z)^{\beta_i}} \exp\left(\sum \frac{\gamma_j z^j}{j(1 - \delta_j z)^j}\right)$$

for some complex numbers $\alpha_i, \beta_i, \gamma_j, \delta_j$.

(C2) $\sum \beta_i$ is the number of objects of $I$.

(C3) each $\alpha_i$ is an eigen value of $A_I$. Hence, $\alpha_i$ is an algebraic integer.

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Key words and phrases: the zeta function of a finite category, the Euler characteristic of categories.

2010 Mathematics Subject Classification: 18G30
\[
\sum_{i} \frac{\beta_i}{\alpha_i} + \sum (-1)^j \frac{\gamma_j}{\delta_j + 1} = \chi_{\Sigma}(I).
\]

It was verified this conjecture holds true under certain additional conditions in [NogA] and [NogB].

In [NogA], it was verified the conjecture holds true in concrete cases, that is, when a finite category is a groupoid, an acyclic category and has two objects and so on. An acyclic category is a small category in which all endomorphisms and isomorphisms are identity morphisms. In [NogB], it was verified the conjecture holds true when a finite category has Möbius inversion. A finite category \( I \) has Möbius inversion if its adjacency matrix \( A_I \) has an inverse matrix where \( A_I \) is an \( N \times N \)-matrix whose \((i, j)\)-entry is the number of morphisms of \( I \) from \( x_i \) to \( x_j \) when the set of objects of \( I \) is

\[
\text{Ob}(I) = \{x_1, x_2, \ldots, x_N\}
\]

(see [Lei08] and [Lei]). In the sense of Leinster, this is called coarse Möbius inversion [Lei]. The class of finite categories which has coarse Möbius inversion is large and very important to consider the Euler characteristic of a finite category. Euler characteristic for categories is defined by various ways, the series Euler characteristic \( \chi \sum \) [BL08], the \( L^2 \)-Euler characteristic \( \chi^{(2)} \) [FLS11], the extended \( L^2 \)-Euler characteristic \( \chi^{(2)} \) [Nog], the Euler characteristic of an \( \mathbb{N} \)-filtered acyclic category \( \chi_{\text{fil}} \) [Nog11] and so on. If a finite category \( I \) has the coarse Möbius inversion, then \( I \) has Leinster’s Euler characteristic and series Euler characteristic and they coincide, \( \chi_{\text{fil}}(I) = \chi_{\Sigma}(I) \). A finite acyclic category \( A \) has the coarse Möbius inversion and all of the Euler characteristic above for \( A \) coincide.

In this paper, we prove the conjecture holds true without any additional conditions. The following is our main theorem.

**Main Theorem.** Suppose \( I \) has series Euler characteristic and

\[
\deg(|E - A_I z|) = N - r
\]

and

\[
\deg(\text{sum(adj}(E - A_I z))) = N - 1 - s
\]

and the polynomial \(|E - A_I z|\) is factored to the following form

\[
|E - A_I z| = d_{N-r}(z - \theta_1)^{e_1} \cdots (z - \theta_n)^{e_n}
\]

where each \( e_i \geq 1 \) and \( \theta_i \neq \theta_j \) if \( i \neq j \). Then the rational function

\[
\frac{\text{sum(adj}(E - A_I z)A_I)}{|E - A_I z|}
\]

has a partial fraction decomposition to the following form

\[
\frac{\text{sum(adj}(E - A_I z)A_I)}{|E - A_I z|} = \frac{1}{d_{N-r}} \sum_{k=1}^{n} \sum_{j=1}^{e_k} \frac{A_{k,j}}{(z - \theta_k)^j}
\]

for some complex numbers \( A_{k,j} \). Moreover,
1. Then the zeta function of $I$ is

$$
\zeta_I(z) = \prod_{k=1}^{n} \frac{1}{(1 - \frac{1}{\theta_k} z)^{A_{k,1} d_{N-r}}} \times \exp \left( \frac{1}{d_{N-r}} \sum_{k=1}^{n} \sum_{j=1}^{e_k-1} \frac{z^j}{j(1 - \frac{1}{\theta_k} z)} \left( \sum_{i=j}^{e_k-1} \frac{(i-1)(-1)^{i+1}}{(\frac{1}{\theta_k})^{i+j}} A_{k,i+1} \right) \right)
$$

2. \( \frac{A_{k,1}}{d_{N-r}} = N \)

3. Each $\frac{1}{\theta_k}$ is an eigen value of $A_I$. In particular, $\frac{1}{\theta_k}$ is an algebraic integer.

4. \( \sum_{k=1}^{n} \frac{A_{k,1}}{d_{N-r}} \theta_k \)

\[
+ \frac{1}{d_{N-r}} \sum_{k=1}^{n} \sum_{j=1}^{e_k-1} (-1)^{j-1} \left( \frac{1}{\theta_k} \right)^{i+j} A_{k,i+1} = \chi \Sigma(I).
\]

If we do not assume the condition that $I$ has series Euler characteristic, the part 1 is given by the following.

**Theorem 1.2 (Theorem 3.1).** Let $I$ be a finite category. Suppose the polynomial $|E - A_I z|$ is factored to the following form:

$$
|E - A_I z| = d_{N-r}(z - \theta_1)^{e_1} \ldots (z - \theta_n)^{e_n}
$$

where $1 \leq r \leq N - 1$ each $e_i \geq 1$ and $\theta_i \neq \theta_j$ if $i \neq j$. Suppose

$$
\text{sum(adj}(E - A_I z)A_I) = q(z)|E - A_I z| + r(z)
$$

where

$$
deg(r(z)) < deg |E - A_I z|
$$

and \( \frac{r(z)}{|E - A_I z|} \) has a partial fraction decomposition to the following form

$$
\frac{r(z)}{|E - A_I z|} = \frac{1}{d_{N-r}} \sum_{k=1}^{n} \sum_{j=1}^{e_k} \frac{A_{k,j}}{(z - \theta_k)^j}
$$

Then the zeta function of $I$ is

$$
\zeta_I(z) = \prod_{k=1}^{n} \frac{1}{(1 - \frac{1}{\theta_k} z)^{A_{k,1} d_{N-r}}} \times \exp \left( Q(z) + \frac{1}{d_{N-r}} \sum_{k=1}^{n} \sum_{j=1}^{e_k-1} \frac{z^j}{j(1 - \frac{1}{\theta_k} z)^j} \left( \sum_{i=j}^{e_k-1} \frac{(i-1)(-1)^{i+1}}{(\frac{1}{\theta_k})^{i+j}} A_{k,i+1} \right) \right)
$$

where $Q(z) = \int q(z)dz$ is a polynomial whose constant term is 0.
It is very important to study about behavior of singular points and zeros of a zeta function. By the following corollary, the problem is reduced to investigate properties of roots of $|E - A_I z|$.  

**Corollary 1.3.** Let $I$ be a finite category. A complex number $z$ is a singular point or zero of $\zeta_I$ if and only if $z$ is a root of $|E - A_I z|$. 

This paper is organized as follows.  
In section 2, we prove some lemmas for a proof of our main theorem.  
In section 3, we prove our main theorem.

## 2 Preparations for our main theorem

### 2.1 Notation

Throughout this paper, we will use the following notations.

1. We mean $I$ is a finite category which has $N$-objects.

2. The three polynomials $|E - A_I z|$, $\text{sum}(\text{adj}(E - A_I z))$ and $\text{sum}(\text{adj}(E - A_I z)A_I)$ which will be often used later are expressed by the following form

   $|E - A_I z| = d_0 + d_1 z + \cdots + d_N z^N,$

   $\text{sum}(\text{adj}(E - A_I z)) = k_0 + k_1 z + \cdots + k_{N-1} z^{N-1}$

   and

   $\text{sum}(\text{adj}(E - A_I z)A_I) = m_0 + m_1 z + \cdots + m_{N-1} z^{N-1}.$

By Lemma 2.2 of NogB, the degree of the third polynomial is less than or equal to $N - 1$. The coefficients $d_0, d_1$ and $d_N$ are $1, (-1)^N \text{tr}(A_I)$ and $(-1)^N |A_I|$, respectively. Hence, the degree of $|E - A_I z|$ is larger or equal to 1 if $I$ is not an empty category since $\text{tr}(A_I) \geq N$.

### 2.2 Some lemmas

In this subsection, we investigate the three polynomials above.

**Lemma 2.1.** The degree of $|E - A_I z|$ is $N - r$ if and only if $|A_I - Ez|$ can be divided by $z^r$, but can not be divided by $z^{r+1}$.

**Proof.** We have

$$|A_I - Ez| = (-1)^N (d_0 z^N + d_1 z^{N-1} + \cdots + d_{N-1} z + d_N).$$

Indeed, if we write

$$|A_I - Ez| = a_0 + a_1 z + \cdots + a_N z^N,$$

then we have

$$|E - A_I z| = (-1)^N z^N |A_I - E \frac{1}{z}||
$$

$$= (-1)^N z^N \left(a_0 + \frac{1}{z} + \cdots + \frac{1}{z^N} \right)$$

$$= (-1)^N (a_0 z^N + a_1 z^{N-1} + \cdots + a_N)$$

$$= d_0 + d_1 z + \cdots + d_N z^N.$$
Hence, we have $a_0 = (-1)^N d_N, a_1 = (-1)^N d_{N-1}, \ldots, a_N = (-1)^N d_0$.
Suppose $\deg(|E - A_I z|) = N - r$. Then, $d_N = d_{N-1} = \cdots = d_{N-r+1} = 0$, but $d_{N-r} \neq 0$. Hence, we have

$$|A - Ez| = (-1)^N d_0 z^N + \cdots + (-1)^N d_{N-r} z^r.$$ 

So $|A - Ez|$ can be divided by $z^r$, but can not be divided by $z^{r+1}$.

Conversely, if the polynomial $|A_I - Ez|$ can be divided by $z^r$, but can not be divided by $z^{r+1}$, then $d_N = d_{N-1} = \cdots = d_{N-r+1} = 0$ and $d_{N-r} \neq 0$. Hence, $\deg(|E - A_I z|) = N - r$.

**Lemma 2.2.** The degree of $\sum(\text{adj}(E - A_I z))$ is $N - 1 - s$ if and only if $\sum(\text{adj}(E - A_I z))$ can be divided by $z^s$, but can not be divided by $z^{s+1}$.

**Proof.** We have

$$\sum(\text{adj}(A_I - Ez)) = (-1)^{N-1} (k_0 z^{N-1} + k_1 z^{N-2} + \cdots + k_{N-1}).$$

Indeed, if we write

$$\sum(\text{adj}(A_I - Ez)) = b_0 + b_1 z + \cdots + b_{N-1} z^{N-1},$$

then we have

$$\begin{align*}
\sum(\text{adj}(E - A_I z)) &= (-z)^{N-1} \sum \left( \text{adj} \left( A_I - E \frac{1}{z} \right) \right) \\
&= (-z)^{N-1} \left( b_0 + b_1 \frac{1}{z} + \cdots + b_{N-1} \frac{1}{z^{N-1}} \right) \\
&= (-1)^{N-1} b_0 z^{N-1} + \cdots + (-1)^{N-1} b_{N-1} \\
&= k_0 + k_1 z + \cdots + k_{N-1} z^{N-1}.
\end{align*}$$

Hence, we have $b_0 = (-1)^{N-1} k_{N-1}, b_1 = (-1)^{N-1} k_{N-2}, \ldots, b_{N-1} = (-1)^{N-1} k_0$.

Suppose $\deg(\sum(\text{adj}(E - A_I z))) = N - 1 - s$. Then, $k_{N-1} = k_{N-2} = \cdots = k_{N-s} = 0$, but $k_{N-s-1} \neq 0$. Hence, we have

$$\sum(\text{adj}(E - A_I z)) = (-1)^{N-1} k_0 z^{N-1} + \cdots + (-1)^{N-1} k_{N-1-s} z^s.$$ 

So $\sum(\text{adj}(E - A_I z))$ can be divided by $z^s$, but can not be divided by $z^{s+1}$.

Conversely, if the polynomial $\sum(\text{adj}(E - A_I z))$ can be divided by $z^s$, but can not be divided by $z^{s+1}$, then $k_{N-1} = k_{N-2} = \cdots = k_{N-s} = 0$ and $k_{N-s} - 1 \neq 0$. Hence, $\deg(\sum(\text{adj}(E - A_I z))) = N - 1 - s$.

**Lemma 2.3.** Suppose the degree of $|E - A_I z|$ is $N - r$ and the degree of $\sum(\text{adj}(E - A_I z))$ is $N - 1 - s$. Then, $I$ has series Euler characteristic if and only if $s \geq r$. In this case, we have

$$\chi(I) = \begin{cases} 0 & \text{if } s > r \\ - \frac{k_{N-1-s}}{d_{N-r}} & \text{if } s = r. \end{cases}$$
Proof. The finite category $I$ has series Euler characteristic if and only if the rational function

$$\frac{\sum(\text{adj}(E - (A_I - E)t))}{|E - (A_I - E)t|}$$

can be substituted $-1$ to $t$ if and only if the rational function

$$\frac{\sum(\text{adj}(A_I - Ez))}{|A_I - Ez|}$$

can be substituted $0$ to $z$ (page 45 of [BL08]). Lemma 2.1 and Lemma 2.2 imply

$$\sum(\text{adj}(A_I - Ez)) \bigg|_{A_I - Ez} = z^s h(z) \bigg|_{A_I - Ez}$$

for some polynomials $g(z)$ and $h(z)$ of $\mathbb{Z}[z]$ such that $g(z)$ and $h(z)$ can not divided by $z$. Hence, the rational function

$$\frac{\sum(\text{adj}(A_I - Ez))}{|A_I - Ez|}$$

can be substituted $0$ to $z$ if and only if $s \geq r$. So the first claim is proved.

Suppose $I$ has series Euler characteristic. Then, we have $s \geq r$. If $s > r$, then it is clear $\chi_{\Sigma}(I) = 0$. If $s = r$, then we have

$$\frac{\sum(\text{adj}(A_I - Ez))}{|A_I - Ez|} = \frac{(-1)^N \sum (k_0 z^{N-1} + k_1 z^{N-2} + \cdots + k_{N-1-s} z^s)}{(-1)^N \sum (d_0 z^N + d_1 z^{N-1} + \cdots + d_{N-r} z^r)}$$

$$= \frac{k_0 z^{N-1-s} + \cdots + k_{N-1-s}}{d_0 z^{N-r} + \cdots + d_{N-r}}.$$

Hence, we obtain $\chi_{\Sigma}(I) = \frac{k_{N-1-s}}{d_{N-r}}$.

\[\Box\]

Lemma 2.4. If $I$ has series Euler characteristic, then we have

$$\deg \left( \sum(\text{adj}(E - A_I z) A_I) \right) = \deg(|E - A_I z|) - 1.$$

Proof. Lemma 2.2 of [NogB] implies

$$\sum(\text{adj}(E - A_I z) A_I) = \frac{1}{z} \left( \sum(\text{adj}(E - A_I z)) - N |E - A_I z| \right).$$

Note that the polynomial

$$\sum(\text{adj}(E - A_I z)) - N |E - A_I z|$$

has no constant term since $k_0 = N$ and $d_0 = 1$. Hence, we have

$$\deg \left( \sum(\text{adj}(E - A_I z) A_I) \right) = \deg \left( \sum(\text{adj}(E - A_I z)) - N |E - A_I z| \right) - 1.$$

Since $I$ has series Euler characteristic, Lemma 2.2 implies $s \geq r$. Hence, we have the inequality

$$\deg(\sum(\text{adj}(E - A_I z))) = N - 1 - s < N - r = \deg(|E - A_I z|).$$

So we obtain

$$\deg \left( \sum(\text{adj}(E - A_I z) A_I) \right) = \deg(|E - A_I z|) - 1.$$
Lemma 2.5. If \( I \) has series Euler characteristic and \( \deg(|E - A_I z|) = N - r \) and
\[
\deg(\sum(\text{adj}(E - A_I z))) = N - 1 - s,
\]
then for the polynomial
\[
\sum(\text{adj}(E - A_I z)A_I) = m_0 + m_1 z + \cdots + m_{N-1-r} z^{N-1-r},
\]
we have \( m_{N-1-r} = -Nd_{N-r} \) and
\[
m_{N-2-r} = \begin{cases} 
-Nd_{N-1-r} & \text{if } s > r \\
-Nd_{N-1-r} + k_{N-1-r} & \text{if } s = r.
\end{cases}
\]

Proof. Lemma 2.2 of \([\text{NogB}]\) implies
\[
\sum(\text{adj}(E - A_I z)A_I) = \sum(\text{adj}(E - A_I z)) = m_0 + m_1 z + \cdots + m_{N-1-r} z^{N-1-r}
\]
\[
= \frac{1}{z} \left( \sum(\text{adj}(E - A_I z)) - N |E - A_I z| \right)
\]
\[
= \frac{1}{z} \left( k_0 + k_1 z + \cdots + k_{N-1-s} z^{N-1-s} - N (d_0 + d_1 z + \cdots + d_{N-r} z^{N-r}) \right)
\]
\[
= (k_1 - Nd_1) + (k_2 - Nd_2) z + \cdots + (k_{N-1-s} - Nd_{N-1-s}) z^{N-2-s} - Nd_{N-s} z^{N-1-s} + \cdots - Nd_{N-r} z^{N-1-r}.
\]

Since \( I \) has series Euler characteristic, Lemma 2.4 implies \( s \geq r \). Hence,
\[
N - 1 - s < N - r,
\]
so that \( m_{N-1-r} = -Nd_{N-r} \).
If \( s > r \), then \( N - 1 - s < N - 1 - r \), so that \( m_{N-2-r} = -Nd_{N-1-r} \).
If \( s = r \), then \( m_{N-2-r} = -Nd_{N-1-r} + k_{N-1-r} \).

3 A proof of main theorem

Theorem 3.1. Let \( I \) be a finite category. Suppose the polynomial \( |E - A_I z| \) is factored to the following form:
\[
|E - A_I z| = d_{N-r}(z - \theta_1)^{e_1} \cdots (z - \theta_n)^{e_n}
\]
where \( 1 \leq r \leq N - 1 \) each \( e_i \geq 1 \) and \( \theta_i \neq \theta_j \) if \( i \neq j \). Suppose
\[
\sum(\text{adj}(E - A_I z)A_I) = q(z)|E - A_I z| + r(z)
\]
where
\[
\deg(r(z)) < \deg |E - A_I z|
\]
and \( \frac{r(z)}{|E - A_I z|} \) has a partial fraction decomposition to the following form

\[
\frac{r(z)}{|E - A_I z|} = \frac{1}{d_{N-r}} \sum_{k=1}^{n} \sum_{j=1}^{e_k} \frac{A_{k,j}}{(z - \theta_k)^j}
\]

Then the zeta function of \( I \) is

\[
\zeta_I(z) = \prod_{k=1}^{n} \frac{1}{(1 - \frac{1}{\theta_k} z)^{d_{N-r}}} \times \exp \left( Q(z) + \frac{1}{d_{N-r}} \sum_{k=1}^{n} \sum_{j=1}^{e_k-1} \frac{z^j}{j(1 - \frac{1}{\theta_k} z)^j} \left( \sum_{i=j}^{e_k-1} (-1)^{i-1} \left( \frac{1}{\theta_k} \right)^{i+j} A_{k,i+1} \right) \right)
\]

where \( Q(z) = \int q(z) dz \) is a polynomial whose constant term is 0.

Proof. Since \( \deg(r(z)) < \deg |E - A_I z| \), we can have a partial fraction decomposition of the following form

\[
\frac{r(z)}{|E - A_I z|} = \frac{1}{d_{N-r}} \sum_{k=1}^{n} \sum_{j=1}^{e_k} \frac{A_{k,j}}{(z - \theta_k)^j}
\]

for some complex numbers \( A_{k,j} \). Hence, we have

\[
\frac{\text{sum}(\text{adj}(E - A_I z)A_I)}{|E - A_I z|} = q(z) + \frac{r(z)}{|E - A_I z|}
\]

\[
= q(z) + \frac{1}{d_{N-r}} \sum_{k=1}^{n} \sum_{j=1}^{e_k} \frac{A_{k,j}}{(z - \theta_k)^j}
\]

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Proposition 2.1 of [NogB] implies
\[ \zeta_I(z) = \exp \left( \int q(z) dz + \int \frac{1}{d_{N-r}} \sum_{k=1}^{n} \sum_{j=1}^{e_k} A_{k,j} (z - \theta_k)^j dz \right) \]
\[ = \exp \left( \int q(z) dz + \frac{1}{d_{N-r}} \int \sum_{k=1}^{n} A_{k,1} (z - \theta_k) dz + \frac{1}{d_{N-r}} \sum_{k=1}^{n} \sum_{j=2}^{e_k} A_{k,j} (z - \theta_k)^j dz \right) \]
\[ = \exp \left( Q(z) + \frac{1}{d_{N-r}} \sum_{k=1}^{n} A_{k,1} \log(z - \theta_k) + \frac{1}{d_{N-r}} \sum_{k=1}^{n} \sum_{j=2}^{e_k} -A_{k,j} \frac{1}{(j-1)(z - \theta_k)^{j-1}} + C \right) \]
\[ = \prod_{k=1}^{n} \frac{1}{(z - \theta_k)^{\frac{A_{k,1}}{d_{N-r}}}} \times \]
\[ \exp \left( Q(z) + \frac{1}{d_{N-r}} \sum_{k=1}^{n} \sum_{j=1}^{e_k-1} -A_{k,j+1} \frac{1}{j} \frac{1}{(z - \theta_k)^j} \right) \exp C \]
\[ = \prod_{k=1}^{n} \frac{1}{(z - \theta_k)^{\frac{A_{k,1}}{d_{N-r}}}} \times \]
\[ \exp \left( Q(z) + \frac{-1}{d_{N-r}} \sum_{k=1}^{n} \sum_{j=1}^{e_k-1} A_{k,j+1} \frac{1}{j} \frac{1}{(z - \theta_k)^j} \right) C'' \]
where we did and will replace the constant term as \( C, C' \) and \( C'' \) . . . Lemma 2.7 of [NogB] implies
\[ \zeta_I(z) = \prod_{k=1}^{n} \frac{1}{(1 - \frac{1}{\theta_k} z)^{\frac{A_{k,1}}{d_{N-r}}}} \times \]
\[ \exp \left( Q(z) + \frac{-1}{d_{N-r}} \sum_{k=1}^{n} \sum_{j=1}^{e_k-1} A_{k,j+1} \frac{1}{j} \frac{1}{(z - \theta_k)^j} \right) C''' \]
Here, we use the boundary condition \( \zeta_I(0) = 1 \). This condition is directly implied by the definition of the zeta function. Hence, we obtain \( C''' = 1 \). By
exchanging $\sum_i$ and $\sum_j$, we have

$$\zeta_I(z) = \prod_{k=1}^{n} \frac{1}{(1 - \frac{1}{\theta_k} z)^{A_{k,1}}}$$

$$\times \exp \left( Q(z) + \frac{1}{d_{N-r}} \sum_{k=1}^{n} \sum_{j=1}^{e_{k}-1} \frac{z^j}{j(1 - \frac{1}{\theta_k} z)} \left( \sum_{i=j}^{e_{k}-1} \left( \frac{1}{\theta_k} \right)^{i+j} A_{k,i+1} \right) \right)$$

Hence, we obtain the result.

It is very important to study about behavior of singular points and zeros of a zeta function. By the following corollary, the problem is reduced to investigate properties of roots of $|E - A_I z|$.

**Corollary 3.2.** Let $I$ be a finite category. A complex number $z$ is a singular point or a zero of $\zeta_I$ if and only if $z$ is a root of $|E - A_I z|$.

**Proof.** Theorem 3.1 directly implies all of the singular points and zeros are roots of $|E - A_I z|$. Conversely, suppose $z$ is a root of $|E - A_I z|$ but $z$ is not a singular point and a zero. Then, $z = \theta_\ell$ for some $\ell$. The index $\frac{A_{k,1}}{d_{N-r}}$ must be 0. Namely, we have $A_{k,e_k} = 0$. For $j = e_\ell - 1$,

$$\sum_{i=e_\ell-1}^{e_\ell-1} -A_{e_\ell+1} \left( \frac{1}{\theta_\ell} \right)^{j-1} \left( \frac{1}{\theta_\ell} \right)^{-1}$$

must be 0 since $\zeta_I(z)$ is defined. Hence, we have $A_{k,e_k} = 0$. As this, we can show each $A_{k,j} = 0$ by the descent from $j = e_\ell - 1$. Hence, we have

$$\frac{r(z)}{|E - A_I z|} = \frac{1}{d_{N-r}} \sum_{k=1}^{n} \sum_{j=1}^{e_k} A_{k,j} \prod_{i=1}^{e_k} \frac{z - \theta_{i}}{(z - \theta_\ell)^j}$$

$$= \frac{1}{d_{N-r}} \sum_{k=1}^{n} \sum_{j=1}^{e_k} \frac{z - \theta_{\ell}}{(z - \theta_\ell)^j}$$

Hence, we obtain

$$|E - A_I z| = d_{N-r} (z - \theta_{1})^{e_1} \ldots (z - \theta_{n})^{e_n}$$

$$= d_{N-r} (z - \theta_{e_\ell - 1})^{e_1} \ldots (z - \theta_{e_\ell - 1})^{e_\ell - 1} (z - \theta_{e_\ell})^{e_\ell + 1} \ldots (z - \theta_{n})^{e_n}$$

The polynomial $|E - A_I z|$ has two different degrees since each $e_k \geq 1$. This contradiction implies $z = \theta_\ell$ is a singular point or a zero of $\zeta_I$.

**Theorem 3.3.** Suppose $I$ has series Euler characteristic and

$$\deg(|E - A_I z|) = N - r$$

and

$$\deg(\text{sum(adj}(E - A_I z))) = N - 1 - s$$
and the polynomial \(|E - A_I z|\) is factored to the following form

\[|E - A_I z| = d_{N-r}(z - \theta_1)^{e_1} \cdots (z - \theta_n)^{e_n}\]

where each \(e_i \geq 1\) and \(\theta_i \neq \theta_j\) if \(i \neq j\). Then the rational function

\[
\frac{\text{sum}(\text{adj}(E - A_I z)A_I)}{|E - A_I z|}
\]

has a partial fraction decomposition to the following form

\[
\frac{\text{sum}(\text{adj}(E - A_I z)A_I)}{|E - A_I z|} = \frac{1}{d_{N-r}} \sum_{k=1}^{n} \sum_{j=1}^{e_k} A_{k,j} (z - \theta_k)^j
\]

for some complex numbers \(A_{k,j}\). Moreover,

1. the zeta function of \(I\) is

\[
\zeta_I(z) = \prod_{k=1}^{n} \frac{1}{(1 - \frac{1}{\theta_k} z)^{\frac{A_{k,1}}{d_{N-r}}} - \frac{A_{k,1}}{d_{N-r}}}
\]

\[
\times \exp \left( \frac{1}{d_{N-r}} \sum_{k=1}^{n} \sum_{j=1}^{e_k-1} \frac{z^j}{j(1 - \frac{1}{\theta_k} z)} \left( \sum_{i=j}^{e_k-1} \binom{i-1}{j-1} (-1)^{i-1} \left( \frac{1}{\theta_k} \right)^{i+j} A_{k,i+1} \right) \right)
\]

2. the sum of all the indexes are the number of objects of \(I\), that is,

\[
\sum_{k=1}^{n} \frac{-A_{k,1}}{d_{N-r}} = N
\]

3. Each \(\frac{1}{\theta_k}\) is an eigen value of \(A_I\). In particular, \(\frac{1}{\theta_k}\) is an algebraic integer

4.

\[
\sum_{k=1}^{n} \frac{-A_{k,1}}{d_{N-r}} \frac{1}{\theta_k} + \frac{1}{d_{N-r}} \sum_{k=1}^{n} \sum_{j=1}^{e_k-1} (-1)^j \sum_{i=j}^{e_k-1} \binom{i-1}{j-1} (-1)^{i-1} \left( \frac{1}{\theta_k} \right)^{i+j} A_{k,i+1} = \chi \Sigma(I).
\]

We give a simple interpretation of the part \(\Box\). Put \(\alpha_k = \frac{1}{\theta_k}\), \(\beta_{k,0} = -\frac{A_{k,1}}{d_{N-r}}\) and

\[
\beta_{k,j} = \sum_{i=j}^{e_k-1} \binom{i-1}{j-1} (-1)^{i-1} \left( \frac{1}{\theta_k} \right)^{i+j} A_{k,i+1}.
\]

Then, the equation \(\Box\) is

\[
\sum_{k=1}^{n} \sum_{j=0}^{e_k-1} (-1)^j \frac{\beta_{k,j}}{\alpha_k} = \chi \Sigma(I).
\]

This theorem claims that this alternating sum is always a rational number and it is the series Euler characteristic \(\chi \Sigma(I)\) of \(I\).
Proof of Theorem 3.3. Lemma 2.4 implies
\[ \deg(\text{sum(adj}(E - A_I z)A_I)) < \deg(|E - A_I z|). \]

Hence, we have a partial fraction decomposition
\[ \frac{\text{sum(adj}(E - A_I z)A_I)}{|E - A_I z|} = \frac{1}{d_{N-r}} \sum_{k=1}^{n} \sum_{j=1}^{e_k} A_{k,j} (z - \theta_k)^j \]
for some complex numbers \( A_{k,j} \).

The part 1 is directly implied by Theorem 3.1 as \( Q(z) = 0 \).

Next we show the part 2. We observe the numerators of both sides
\[ \frac{\text{sum(adj}(E - A_I z)A_I)}{|E - A_I z|} = \frac{1}{d_{N-r}} \sum_{k=1}^{n} \sum_{j=1}^{e_k} A_{k,j} (z - \theta_k)^j. \]

For the right hand side, when it is transformed to the left hand side by a reduction to common denominator, the coefficient of \( z^{N-1-r} \) of the numerator is \( \sum_{k=1}^{n} A_{k,1} \). Lemma 2.5 implies \( \sum_{k=1}^{n} A_{k,1} = m_{N-1-r} = d_{N-r} \). Thus, we obtain
\[ \sum_{k=1}^{n} \frac{A_{k,1}}{d_{N-r}} = N. \]

We show the part 3. Since each \( \theta_k \) is a root of the polynomial \( |E - A_I z| \), we obtain
\[ |E - A_I \theta_k| = 0 \]
\[ (\theta_k)^N \left| \frac{1}{\theta_k} - A_I \right| = 0. \]
Hence, \( \frac{1}{\theta_k} \) is an eigen value of \( A_I \). Note that \( \theta_k \neq 0 \). Moreover, since \( |E\lambda - A_I| \) is a monic polynomial with coefficients in \( \mathbb{Z} \), \( \frac{1}{\theta_k} \) is an algebraic integer.
Finally, we show the part [4]. The equation (1) is

\[
(1) = \sum_{k=1}^{n} -\frac{A_{k,1}}{\theta_k} + \frac{1}{d_{N-r}} \sum_{k=1}^{n} \sum_{j=1}^{e_k-1} (-1)^j \frac{\sum_{i=j}^{e_k-1} (i-j)(-1)^{i-1} \left( \frac{1}{\theta_i} \right)^{i+j} A_{k,i+1}}{\left( \frac{1}{\theta_i} \right)^{j+1}}
\]

\[
= \sum_{k=1}^{n} \left( -\frac{\theta_k A_{k,1}}{d_{N-r}} \right) + \frac{1}{d_{N-r}} \sum_{j=1}^{e_k-1} \sum_{i=j}^{e_k-1} (-1)^j \left( -\frac{1}{\theta_k} \right)^{i-1} \left( \frac{i-1}{j-1} \right) A_{k,i+1}
\]

\[
= \sum_{k=1}^{n} \left( -\frac{\theta_k A_{k,1}}{d_{N-r}} \right) + \frac{1}{d_{N-r}} \sum_{i=1}^{e_k-1} (-1)^{i-1} A_{k,i+1} \left( \sum_{j=1}^{i} (-1)^j \left( \frac{i-1}{j-1} \right) \right)
\]

\[
= \frac{1}{d_{N-r}} \sum_{k=1}^{n} (-\theta_k A_{k,1} - A_{k,2}).
\]

So it is enough to show

\[
\frac{1}{d_{N-r}} \left( \sum_{k=1}^{n} -\theta_k A_{k,1} - A_{k,2} \right) = \chi \sum(I).
\]

(2)

By comparison of the numerators of both sides

\[
\frac{\text{sum}(\text{adj}(E - A_{1z}))}{|E - A_{1z}|} = \frac{1}{d_{N-r}} \sum_{k=1}^{n} \sum_{j=1}^{e_k} \frac{A_{k,j}}{(z - \theta_k)^j},
\]

we have

\[
m_{N-2-r} = \sum_{k=1}^{n} A_{k,2} - \sum_{k=1}^{n} A_{k,1} (\theta_1 e_1 + \cdots + \theta_k (e_k - 1) + \cdots \theta_n e_n).
\]

Hence, the left hand side of (2) is

\[
\frac{1}{d_{N-r}} \left( \sum_{k=1}^{n} -\theta_k A_{k,1} - A_{k,2} \right) = \frac{1}{d_{N-r}} \left( \sum_{k=1}^{n} -\theta_k A_{k,1} - m_{N-2-r} - A_{k,1} (\theta_1 e_1 + \theta_k (e_k - 1) + \cdots + \theta_n e_n) \right)
\]

\[
= \frac{1}{d_{N-r}} \left( - \left( \sum_{k=1}^{n} A_{k,1} \right) \left( \sum_{k=1}^{n} \theta_k e_k \right) - m_{N-2-r} \right).
\]

(3)
We have
\[ |E - A_I z| = d_0 + d_1 z + \cdots + d_{N-r} z^{N-r} = d_{N-r}(z - \theta_1)^{e_1} \cdots (z - \theta_n)^{e_n} = d_{N-r} z^{N-r} - \left( \sum_{k=1}^{n} \theta_k e_k \right) z^{N-1-r} + \cdots .\]

Hence, we obtain \(-d_{N-r} (\sum_{k=1}^{n} \theta_k e_k) = d_{N-1-r},\) so that
\[ -\sum_{k=1}^{n} \theta_k e_k = \frac{d_{N-1-r}}{d_{N-r}} .\]

We have already seen \(\sum_{k=1}^{n} A_{k,1} = -Nd_{N-r}.\) Therefore, the equation (3) is
\[ \frac{1}{d_{N-r}} \left( -\left( \sum_{k=1}^{n} A_{k,1} \right) \left( \sum_{k=1}^{n} \theta_k e_k \right) - m_{N-2-r} \right) = \frac{1}{d_{N-r}} \left( -Nd_{N-1-r} - m_{N-2-r} \right) .\] (4)

Here we have to consider two cases
\[ \chi_{\Sigma}(I) = \begin{cases} 0 & \text{if } s > r \\ -\frac{k_{N-1-r}}{d_{N-r}} & \text{if } s = r \end{cases} \]
(see Lemma 2.3).

If \(s > r,\) Lemma 2.3 implies \(m_{N-2-r} = -Nd_{N-1-r},\) so that the equation (4) is
\[ \frac{1}{d_{N-r}} \left( -Nd_{N-1-r} - m_{N-2-r} \right) = \frac{1}{d_{N-r}} \left( -Nd_{N-1-r} + Nd_{N-1-r} \right) = 0 \]
\[ = \chi_{\Sigma}(I). \]

If \(r = s,\) Lemma 2.3 implies \(m_{N-r-2} = k_{N-1-r} - Nd_{N-1-r}.\) Hence, the equation (4) is
\[ \frac{1}{d_{N-r}} \left( -Nd_{N-1-r} - m_{N-2-r} \right) = \frac{1}{d_{N-r}} \left( -Nd_{N-1-r} - k_{N-1-r} + Nd_{N-1-r} \right) \]
\[ = \frac{k_{N-1-r}}{d_{N-r}} \]
\[ = \chi_{\Sigma}(I). \]

Hence, we obtain the results. \(\square\)

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