RANDOM MATRIX ENSEMBLES
ASSOCIATED TO COMPACT SYMMETRIC SPACES

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Abstract. We introduce random matrix ensembles that correspond to the infinite families of irreducible Riemannian symmetric spaces of type I. In particular, we recover the Circular Orthogonal and Symplectic Ensembles of Dyson, and find other families of (unitary, orthogonal and symplectic) ensembles of Jacobi type. We discuss the universal and weakly universal features of the global and local correlations of the levels in the bulk and at the “hard” edge of the spectrum (i.e., at the “central points” ±1 on the unit circle). Previously known results are extended, and we find new simple formulas for the Bessel Kernels that describe the local correlations at a hard edge.

1. Introduction

Local correlations between eigenvalues of various ensembles of random unitary, orthogonal or symplectic matrices, in the limit when their size tends to infinity, are known to exhibit universal behavior in the bulk of the spectrum. Dyson’s “Threefold Way” [14] predicts that this behavior is to be expected universally in the bulk of the spectrum, depending only on the symmetry type of the ensemble (unitary, orthogonal or symplectic). Unfortunately, for general ensembles this conjecture remains open, though in the unitary case (modeled after the Gaussian Unitary Ensemble) the universality of the local correlations has been proven for some classes of families [9, 7, 3, 2]. In the orthogonal and symplectic cases the extension of results known for Gaussian ensembles is technically more complicated but some more recent work deals with families of such ensembles [20]. Most of the focus has been on non-compact (Gaussian and the like) matrix ensembles. In the present article we study families of compact (circular) ensembles including, in particular, Dyson’s circular ensembles: the COE, CUE and CSE [11]. First we fit Dyson’s ensembles into the framework

I wish to thank Prof. Peter Sarnak for his continued encouragement and guidance as my Ph. D. thesis advisor as well as Brian Conrey for making my stay at AIM possible. This research has been supported in part by the FRG grant DMS–00–74028 from the NSF.
of the theory of symmetric spaces, and then we proceed to associate
a matrix ensemble to every family of irreducible compact symmetric
space (all of these are known by the work of Cartan [4, 5]). The most
well-known of these are the families of classical orthogonal, unitary and
symplectic groups of matrices, for which questions about universality
have known answers [17]. These are the so-called compact symmetric
spaces of type II. Zirnbauer [30], on the other hand, has construc-
ted the “infinitesimal” versions of the other (type I) ensembles, namely
their tangent spaces at the identity element, which is enough to derive
their eigenvalue measures. We, however, construct the “global” ensem-
bles associated to the infinite families of compact symmetric spaces of
type I in a very explicit manner analogous to Dyson’s description of
his circular ensembles.

| Type   | G/K                                          | Parameters                        |
|--------|----------------------------------------------|-----------------------------------|
| A I (COE) | $U(R)/O(R)$                                  | $\beta = 1$ (not Jacobi)          |
| A II (CSE) | $U(2R)/USp(2R)$                             | $\beta = 4$ (not Jacobi)          |
| A III  | $U(2R + L)/U(R + L) \times U(R)$            | $\beta = 2$, $(a, b) = (L, 0)$   |
| BD I   | $O(2R + L)/O(R + L) \times O(R)$            | $\beta = 1$, $(a, b) = (\frac{L+1}{2}, \frac{L}{2})$ |
| D III  | $SO(4R)/U(2R)$                              | $\beta = 4$, $(a, b) = (0, 0)$   |
| C I    | $USp(2R)/U(R)$                              | $\beta = 1$, $(a, b) = (0, 0)$   |
| C II   | $USp(4R + 2L)/USp(2R + 2L) \times USp(2R)$ | $\beta = 4$, $(a, b) = (2L + 1, 1)$ |

Table 1. Parameters of the probability measure of the
eigenvalues for ensembles of type I.

Besides Dyson’s COE and CSE, the other compact matrix ensembles
of type I are Jacobi ensembles in the sense that their joint eigenvalue
measure is given by

$$d\nu(x_1, \ldots, x_R) \propto \prod_{1 \leq j < k \leq R} |x_j - x_k|^\beta \prod_{j=1}^R (1-x_j)^a(1+x_j)^b dx_j$$
on $[-1, 1]^R$

for some parameters $a, b > -1$ (depending on the ensemble, see ta-
ble[1]) and $\beta = 1, 2, 4$ (the “symmetry parameter”) in the orthogonal,
unitary and symplectic cases, respectively. Here, the “free” eigenval-
ues are $x_j \pm \sqrt{-1}y_j$ —excluding eigenvalues equal to $+1$ forced by
the symmetry built into the ensemble—. Also, $R$ stands for the rank of the
Matrix Ensembles Associated to Symmetric Spaces

3.

Corresponding symmetric space, and our interest is in the semiclassical limit of the eigenvalue statistics as $R \to \infty$ ($L \geq 0$ is a fixed parameter: different values of $L$ yield different ensembles.) The name “Jacobi ensembles” comes from the intimate connection between the measure and the classical Jacobi polynomials on the interval $[-1, 1]$.

Afterwards, we prove the universality of the local correlations for general unitary, orthogonal and symplectic Jacobi ensembles (previous results of Nagao and Forrester [23] are insufficient for our purposes). We rely on work of Adler et al [1]. At the “hard edges” $\pm 1$ of the interval, Dyson’s universality breaks down and we obtain simple formulas for the Bessel kernel in terms of which the hard edge correlations are expressed. In a nutshell, for Jacobi ensembles:

- Away from the “hard edge” $x = \pm 1$, the local correlations follow the universal law of the GOE ($\beta = 1$), GUE ($\beta = 2$) or GSE ($\beta = 4$). Namely, in terms of local parameters $\xi_j$ around a fixed $z_0 \in (-1, 1)$ so that $x_j = \cos(\alpha_o + (\pi/R)\xi_j)$ ($z_0 = \cos \alpha_o$), these local correlations are given by

$$L^{(n)}_\beta(z_0, \xi_1, \ldots, \xi_n) = \text{DET}(\hat{K}_\beta(\xi_j, \xi_k))_{n \times n},$$

where DET stands for either the usual ($\beta = 2$) or quaternion ($\beta = 1, 4$) determinant, and $K_\beta$ is the (scalar or quaternion) Sine kernel (cf., equations (74)–(78)).

- At the hard edge $z_0 = +1$, the local correlations depend on the parameter $a$ of the Jacobi ensemble as well as on $\beta$. In terms of local parameters $\xi_j > 0$ with $x_j = \cos((\pi/R)\xi_j)$ the same expression (2) holds except that the kernel $\hat{K}_\beta$ is to be replaced by a Bessel kernel $\hat{K}_\beta^{(a)}(\xi, \eta)$ given by equations (80)–(86). At the hard edge $z_0 = -1$ the result is obtained by replacing $a$ by $b$.

2. Dyson’s Circular Ensembles as Symmetric Spaces

For motivational purposes we start by reviewing the construction of the circular ensembles of Dyson and their probability measures of the eigenvalues in a manner in which the theory of Riemannian symmetric spaces is brought into play.

The Circular Unitary Ensemble (CUE) is the set $S = S(N)$ of all $N \times N$ unitary matrices $H$, endowed with the unique probability measure $d\mu(H)$ that is invariant under left (also right) multiplication by any unitary matrix. This requirement makes the measure invariant under unitary changes of bases, hence the ensemble’s name.
In the study of statistics of eigenvalues, the relevant probability measure is the one induced by $d\mu(H)$ on the torus $A = A(N) \subset S(N)$ consisting of unitary diagonal matrices

$$A = \{\text{diag}(\lambda_1 = e^{i\theta_1}, \ldots, \lambda_N = e^{i\theta_N})\},$$

where $\Theta = (\theta_1, \ldots, \theta_N) \in [0, 2\pi)^N$, say.

To be more precise, let us denote by $K = K(N)$ the unitary group of $N \times N$ matrices (its underlying set is just $S(N)$). Then we have a surjective mapping

$$K \times A \rightarrow S$$

$$ (k, a) \mapsto H = kak^{-1},$$

and correspondingly there exists a probability measure $d\nu(a)$ on $A$ such that, for any continuous function $f \in C(S)$,

$$\int_S f(H)d\mu(H) = \int_K \int_A f(kak^{-1})d\nu(a)d\text{Haar}(k),$$

where we denote by $d\text{Haar}(k)$ the unique translation-invariant probability measure on $K$ (so here $d\text{Haar} = d\mu$). This measure $d\nu(a)$ can be pulled back to some measure on the space $[0, 2\pi)^N$ of angles $\Theta$ which, abusing notation, we denote by $d\nu(\Theta)$. The measure $d\nu(\Lambda)$ (or $d\nu(\Theta)$) is the so-called probability measure of the eigenvalues (for the CUE).

We have

$$d\nu(\Theta) \propto |\text{Van}(e^{i\Theta})|^2 d\Theta \quad \text{on } [0, 2\pi)^N. \tag{6}$$

Here the symbol “$\propto$” stands for proportionality up to a constant (depending only on $N$), $d\Theta = d\theta_1 \ldots d\theta_N$ is the usual translation-invariant measure on the space of angles $\Theta$, $e^{i\Theta} = (e^{i\theta_1}, \ldots, e^{i\theta_N})$ and, for a vector $x = (x_1, \ldots, x_N)$, $\text{Van}(x)$ is the Vandermonde determinant

$$\text{Van}(x) = \det_{N \times N}(x_j^{k-1}) = \prod_{1 \leq j < k \leq N} (x_k - x_j). \tag{7}$$

The construction of the Circular Orthogonal Ensemble (COE) is as follows. One starts with the set $S = S(N)$ of $N \times N$ symmetric unitary matrices $H$. However, because $S(N)$ is not a group, the choice of the probability measure $d\mu(H)$ is not as obvious as it was for the CUE. Let $G = G(N)$ again be the group of $N \times N$ unitary matrices $g$, and $K = K(N) \subset G(N)$ be the group of orthogonal matrices. Let $\Omega(g) = (g^T)^{-1}$ be the involution of $G$ whose fixed-point set is $K$. Then we may identify

$$G/K \simeq S$$

$$G \ni g \mapsto H = g \Omega(g)^{-1} =: g^{1-\Omega}, \tag{8}$$
and by general principles the translation-invariant probability measures on $G$ and $K$ determine a unique $G$-invariant measure $d\mu(\bar{g}) = d\mu(H)$ on $G/K \simeq S$ which satisfies
\begin{equation}
\int_G f(g) d\text{Haar}(g) = \int_{G/K} \left( \int_K f(gk) d\text{Haar}(k) \right) d\mu(\bar{g}),
\end{equation}
where on the right-hand side $g$ stands for a choice of an element $g \in G$ such that $gK = \bar{g}$. The left translation-invariance of $d\text{Haar}(g)$ ensures that $d\mu(\bar{g})$ is invariant under left translations by elements of $K$, therefore the measure $d\mu(H)$ is invariant under orthogonal changes of bases, hence the ensemble’s name.

The probability measure of eigenvalues $d\nu(a) = d\nu(\Theta)$ is again that which satisfies (5) (with the same torus $A \subset S$ as for the CUE). It is known that \[ d\nu(\Theta) \propto |\text{Van}(e^{i\Theta})| d\Theta \text{ on } [0, 2\pi)^N. \]

The constructions of the Circular Symplectic Ensemble CSE and of its measure on eigenvalues $d\nu(\Theta)$ are very similar to the case of the COE. Here $S(N)$ consists of $2N \times 2N$ self-dual unitary matrices. Namely, letting
\begin{equation}
J = J_N = \begin{pmatrix} I_N & -I_N \\ I_N & \end{pmatrix},
\end{equation}
then a matrix $H$ is self-dual if it equals its dual $H^D := JH^TJ^T$. If we let $G = G(N)$ be the group of $2N \times 2N$ unitary matrices and $K = K(N)$ be the subgroup of symplectic matrices $k$ (they satisfy $kJk^T = J$) then $K$ is the fixed-point set of the involution $\Omega(g) = (g^D)^{-1}$. The identification \[ \Rightarrow \] continues to hold and \[ \Rightarrow \] again defines the probability measure $d\mu(H) = d\mu(\bar{g})$ of the ensemble. It is invariant under symplectic changes of bases.

The torus $A$ consists here of diagonal matrices:
\begin{equation}
A = \{ \text{diag}(e^{i\theta_1}, \ldots, e^{i\theta_N}, e^{i\theta_1}, \ldots, e^{i\theta_N}) \}
\end{equation}
with twice-repeated eigenvalues. Then the probability measure of the eigenvalues is characterized by \[ \Rightarrow \], and indeed
\begin{equation}
d\nu(\Theta) \propto |\text{Van}(e^{i\Theta})|^4 d\Theta \text{ on } [0, 2\pi)^N.
\end{equation}

Summing up, the measure on eigenvalues for the circular ensembles is given by
\begin{equation}
d\nu(\Theta) \propto |\text{Van}(e^{i\Theta})|^\beta d\Theta,
\end{equation}
where $\beta = 1, 2, 4$ in the orthogonal, unitary and symplectic cases, respectively.
Remark. It can be appreciated that the parameter $\beta$ determines the strength of the repulsion between nearby eigenvalues: this repulsion is stronger the larger $\beta$ is. Hence anything that measures the local interactions between eigenvalues is likely to depend on $\beta$. This is the case, in particular, of the “local correlations” between eigenvalues, cf. section 4.

Remark. The apparent dissimilarity in the construction of the measure $d\mu(H)$ in the case of the unitary vs. the orthogonal and symplectic ensembles is not essential. In fact, the unitary ensemble $S(N)$ is still a quotient $G(N)/K(N)$ where $G(N) = U(N) \times U(N)$ is the direct product of two copies of the unitary group, and $K(N)$ is the diagonal of $G(N)$ (isomorphic to the unitary group itself). If we identify $S(N)$ with the “anti-diagonal” $\{H = (g, g^{-1})\} \subset G(N)$ and take $\Omega(g, h) = (h, g)$ then the construction of the ensemble and of the measures $d\mu(H)$ and $d\nu(\Theta)$ follows through in essentially the same manner. We omit the details. The key observation is that the constructions above show that the circular ensembles are examples of Riemannian globally symmetric spaces.

3. Compact Symmetric Spaces as Matrix Ensembles

Any Riemannian globally symmetric space $X$ is locally isometric to a product of irreducible ones (the symbol “$\approx$” means “is locally isometric to”):

$$X \approx \prod_i X_i^{(c)} \times \prod_j X_j^{(nc)} \times E^\ell,$$

where the $X_i^{(c)}$ (resp., the $X_j^{(nc)}$) are irreducible symmetric spaces of compact (resp., non-compact) type, and $E^\ell = (E^1)^\ell$ is $\ell$-dimensional Euclidean space (a flat manifold). In the case of the circular ensembles, we have

$$\begin{align*}
\text{CUE} & = U(N) \approx SU(N) \times S^1 \\
\text{COE} & = U(N)/O(N) \approx (SU(N)/SO(N)) \times S^1 \\
\text{CSE} & = U(2N)/USp(2N) \approx (SU(2N)/USp(2N)) \times S^1
\end{align*}$$

where in each case the first factor is an irreducible symmetric space of the compact type and the other (Euclidean) factor is a circle $S^1 \approx E^1$ (we write $S^1$ rather than $E^1$ to emphasize that the spaces are compact). In the language of differential geometry, the probability measure of a circular ensemble is the one determined by the natural volume element of the manifold. Hence the natural question arises as to how to construct a random matrix ensemble corresponding to each (infinite)
family of irreducible symmetric spaces of compact type. The restriction to infinite families is due to the need to have a large parameter $N$ such that the number of eigenvalues grows with $N$, and then we are interested mainly in limiting statistics.

The presence of the Euclidean factor $S^1$ (which comes from the subset of scalar multiples of the identity matrix within the ensemble) is rather convenient and natural. If we were to define “irreducible” circular ensembles analogously to Dyson’s circular ensembles, except requiring that they consist of matrices with unit determinant, then the spaces so obtained would be irreducible symmetric spaces of the compact type (i.e., the factors $S^1$ would disappear from (16)). However, the measure on eigenvalues would no longer be translationally invariant (under transformations of the form $\Theta \mapsto \Theta + (t, \ldots, t)$). Namely, instead of the measure (14), we would obtain an asymmetric version given by the same formula but with $\Theta$ replaced by $\Theta = (\theta_1, \ldots, \theta_{N-1}, -\theta_1 - \cdots - \theta_{N-1})$ and with $d\theta_N$ omitted from the volume element $d\Theta$. As may be expected from such a loss of symmetry, a rigorous analysis of these “irreducible” ensembles would be more involved.

Since we are considering only compact symmetric spaces, it is possible to normalize the natural volume element to obtain a probability measure. This is not the case for symmetric spaces of non-compact type. To clarify the difference, we analyze the example of the classical Gaussian matrix ensembles, which also fit within the framework of the theory of symmetric spaces (the construction is analogous to that of the circular ensembles):

$$\text{GUE} \approx SL(N, \mathbb{C})/SU(N) \times E^1$$

$$\text{GOE} \approx SL(N, \mathbb{R})/SO(N) \times E^1$$

$$\text{GSE} \approx \text{SU}^*(2N)/\text{USp}(2N) \times E^1.$$ (17)

Finding the probability measure on eigenvalues also reduces to a factorization of measures $d\mu(H) = d\text{Haar}(k) d\nu(a)$ in the sense of (14), where $K$ is still the group of invariance (orthogonal, unitary, symplectic) of the ensemble’s measure, but where $A \simeq E^N$ is now a Euclidean space, which in the case of these ensembles consists of real diagonal matrices which can be parametrized by $N$-tuples $\Lambda = (\lambda_1, \ldots, \lambda_N)$ of real numbers. However, the measure $d\mu(H)$ is certainly not the one obtained from the Riemannian volume element $d\text{Haar}(g)$ of $G$ through (14) since the latter is not normalizable. A choice has to be made to make this measure into a finite one while preserving its left and right $K$-invariance. One possibility is provided by a “Gaussian” probability
measure on $G$ proportional to

$$e^{-\frac{\beta}{2} \text{tr} g^2} \text{dHaar}(g)$$

(the symmetry parameter $\beta = 1, 2, 4$ corresponds to the orthogonal, unitary and symplectic cases, respectively, just as in the case of the Orthogonal ensembles), which in turn yields the measure on eigenvalues:

$$d\nu(a) \propto e^{-\beta \sum \lambda_j^2 |\text{Van}(\Lambda)|} d\Lambda.$$ 

It can be rightfully argued that the choice of the Gaussian normalization for the measure on these matrix ensembles is rather arbitrary and motivated by analytical rather than conceptual considerations. The point we wish to state here is that making such a choice is unavoidable. For the compact spaces, however, no such choice needs to be made since their volume element already determines a unique probability measure. We will henceforth restrict our attention to compact ensembles for that reason.

The general definition of a Riemannian symmetric space of the compact type is as follows. We start with a compact semisimple Lie algebra $\mathfrak{g}$ (i.e., $\exp(\text{ad}(\mathfrak{g})) \subset \text{GL}(\mathfrak{g})$ is compact) having an involutive automorphism $\omega$. Then $\mathfrak{g}$ splits into the sum of the $(+1)$- and $(-1)$-eigenspaces of $\omega$ as

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$ 

(the subspace $\mathfrak{p} \subset \mathfrak{g}$ can be identified with the tangent space to $G/K$ at the identity coset $o = K/K$). $G/K$ is called a Riemannian symmetric space of the compact type if

1. $K \subset G$ are Lie groups ($G$ connected). Their Lie algebras are $\mathfrak{k}, \mathfrak{g}$, and
2. there is a (necessarily unique) involutive automorphism $\Omega$ of $G$ such that $(G^\Omega)_o \subset K \subset G^\Omega$, where $G^\Omega$ is the fixed-point set of $\Omega$ in $G$ (a Lie subgroup of $G$) and $(G^\Omega)_o$ is its identity component (then $d\Omega_o = \omega$).

The complete list of irreducible symmetric spaces (up to local isometry) is known by the classical work of Cartan. As we will explain later, it suffices to consider one matrix ensemble in each equivalence class of locally isometric symmetric spaces, because the measures on eigenvalues for locally isometric ensembles are the same.

The irreducible symmetric spaces of compact type are classified into spaces of “Type I” and “Type II”. Of these the latter are simplest to describe: they are the (connected) simple compact Lie groups $G$, provided with a bi-invariant (under both left and right translations)
Riemannian metric. Proving that such a $G$ is a bona fide symmetric space of the compact type as defined before involves expressing it as $(G \times G)/G$ in a manner analogous to what we did at the end of section 2 for the CUE.

\begin{table}[h]
\begin{tabular}{|c|c|}
\hline
Type & $G/K$ \\
\hline
A I & $SU(N)/SO(N)$ \\
A II & $SU(2N)/USp(2N)$ \\
A III & $SU(M + N)/S(U(M) \times U(N))$ \\
BD I & $SO(M + N)/SO(M) \times SO(N)$ & $\min(M, N)$ \\
D III & $SO(2N)/U(N)$ & $\lfloor N/2 \rfloor$ \\
C I & $USp(2N)/U(N)$ \\
C II & $USp(2M + 2N)/USp(2M) \times USp(2N)$ & $\min(M, N)$ \\
\hline
\end{tabular}
\caption{Table 2. The infinite families of symmetric spaces of type I.}
\end{table}

Up to local isometry, the infinite families of Type II spaces are those of orthogonal $SO(N)$, unitary $SU(N)$ and (compact) symplectic $USp(2N)$ groups. The random matrix theory of these spaces is well-known [17].

The Type I spaces, on the other hand, are those symmetric spaces $G/K$ of the compact type with $G$ simple. The bi-invariant Riemannian metric on $G$ determines that on the quotient $G/K$. Table 2 lists the infinite families of Type I spaces, up to local isometry.

Without loss of generality, we assume henceforth that $\min(M, N) = N$.

Choose a maximal abelian subalgebra $a$ of $g$ contained in $p$. Then the subgroup $A = \exp(a)$ is a torus that projects onto a totally flat submanifold $AK/K \subset G/K$ (a flat torus). This totally flat manifold is maximal, and its dimension is the rank $R$ of the symmetric space $G/K$. Thus, $R = \dim(AK/K) = \dim A = \dim a$.

Guided by the exposition in the previous section, it is reasonable to regard as ensembles the symmetric spaces $G/K$ of type I endowed with their normalized Riemannian volume elements $d\mu(\bar{g})$, which satisfy (9). However, the elements of these ensembles are not matrices but rather cosets $\bar{g} = gK \in G/K$.

**Theorem 1.** The infinite families of type I ensembles $G/K$ can be realized as matrix ensembles $S$. Indeed, maps $G/K$ bijectively onto a submanifold $S \subset G$, and $G$ is a classical group of matrices, hence $S$ is a space of matrices. Under this correspondence, $AK/K \subset G/K$ is mapped onto the torus $A$. The action of $K$ on $G/K$ by left translation corresponds to the conjugation $H \mapsto kHk^{-1}$ on matrices $H \in S$, and
any \( H \in S \) is conjugate to some \( a \in A \) under this action. Moreover, two matrices in \( A \) are conjugate under \( K \) if and only if they have the same eigenvalues.

The proof of the theorem is a long exercise in elementary linear algebra. We shall omit most of the details, which can be found in [10]. In what follows we describe the explicit matrix ensembles \( S \) which are the images of the imbedding [8].

In each case, we choose the involution \( \Omega \) of \( G \) so that its fixed-point set is exactly \( K \). The cases of A I (COE) and A II (CSE) have been discussed already. We introduce some notation (recall that \( J_N \) is defined by equation (11)):

\[
J'_N = \begin{pmatrix} I_N & I_N \\ I_N & -I_N \end{pmatrix}_{2N \times 2N},
\]

(21)

\[
J_{MN} = \begin{pmatrix} J_M & J_N \\ J_N & -J_M \end{pmatrix}_{(2M+2N) \times (2M+2N)},
\]

(22)

\[
J'_{MN} = \begin{pmatrix} J'_M & J'_N \\ J'_N & -J'_M \end{pmatrix}_{(2M+2N) \times (2M+2N)},
\]

(23)

\[
I'_{MN} = \begin{pmatrix} I_M & -I_N \\ -I_N & I_M \end{pmatrix}_{(M+N) \times (M+N)}.
\]

(24)

The canonical bilinear antisymmetric matrix \( J_n \) in the definition of the compact symplectic group \( USp(2n) \) will be taken to be [11] in the case of ensembles with one parameter \( N \) \( (n = N) \), and (22) in the case of ensembles with two parameters \( M, N \) \( (n = M + N) \).

A III. Take \( M \geq N \geq 1 \) and \( G(M, N) = U(M + N) \). Then \( K(M, N) = U(M) \times U(N) \) is the fixed-point set of the involution

\[
g \mapsto g^\Omega := I'gI',
\]

(25)

with \( I' = I'_{MN} \) as in (24).

The symmetric space \( U(M+N)/U(M) \times U(N) = SU(M+N)/SU(U(M) \times U(N)) \) is realized as the matrix ensemble

\[
S(M, N) := \{ H = GI' \text{ such that } G \in U(M + N) \}
\]

is Hermitian of signature \((M, N)\), under the identification [8]. A choice of the abelian torus \( A \) is given by

\[
A = \left\{ \begin{pmatrix} 1_{M-N} & \Re \Lambda_N & -\Im \Lambda_N \\ \Im \Lambda_N & \Re \Lambda_N \end{pmatrix} \right\}
\]

(27)
where $\Lambda_N = \text{diag}(\lambda_1, \ldots, \lambda_N)$ is an arbitrary diagonal unitary matrix. Besides the eigenvalue 1 with multiplicity $M - N$, the eigenvalues of the matrix in (27) come in $R = N$ pairs $\lambda_j, \lambda_j^{-1}, |\lambda_j| = 1$.

**BD I.** Let $M \geq N \geq 1$, $G(M, N) = O(M + N)$, and $K(M, N) = O(M) \times O(N)$ be the fixed-point set of the involution (23) with $I' = I_{MN}'$ as in (24). Then $G/K = O(M + N)/O(M) \times O(N) \approx SO(M + N)/SO(M) \times SO(N)$ (the last two spaces are locally isometric).

The symmetric space $O(M + N)/O(M) \times O(N)$ can be realized as the set of matrices
\[
S(M, N) := \{ H = gI' \text{ such that } g \in O(M + N) \\
\text{is symmetric of signature } (M, N) \},
\]
by means of (8). The torus $A$ is just as in (27) and we get the same description for the eigenvalues.

**D III.** Let $G(N) = SO(2N)$ and $K(N) = SO(2N) \cap Sp(2N, \mathbf{C}) \simeq U(N)$:
\[
U(N) \ni g \mapsto \begin{pmatrix} \Re g & -\Im g \\
\Im g & \Re g \end{pmatrix} \in K(N).
\]
Then $K(N)$ is the fixed-point set of the involution
\[
g \mapsto g^\Omega := J^T(g^{-1})^TJ = J^TgJ
\]
with $J = J_N$ as in (11). We can identify $G(N)/K(N)$ with the set
\[
S(N) := \{ H \in SO(2N) \text{ s. t. } HJ \text{ is "dexter" antisymmetric} \}
\]
using equation (8). We now explain what we mean by a dexter matrix. Say $G$ is a $2N \times 2N$ orthogonal antisymmetric matrix. Then an orthogonal change of basis puts it into the canonical form $J_N$. However, this may not be possible by means of a proper orthogonal change of basis (i.e., of determinant +1). Specifically, when $N$ is even, the two complex structures $\pm J_N$ are equivalent (under, say, the proper orthogonal change of basis $J'_N$ as in (21)), but when $N$ is odd they are not. We call $G$ dexter if, by a proper orthogonal change of basis, it can be taken into the canonical form $+J_N$. Thus, for $N$ even, all orthogonal antisymmetric matrices are dexter, whereas for $N$ odd, only half of them are (in this case, conjugation by $J'_N$ takes “dexter” matrices into “sinister” ones and vice-versa). Now, for $H \in S(N)$, $G := HJ$ is dexter antisymmetric, so our discussion above proves the surjectivity of the mapping.
The torus $A$ is

$$A = \begin{cases} 
\left( \begin{array}{cc}
\Re \Lambda_R & -\Im \Lambda_R \\
\Im \Lambda_R & \Re \Lambda_R 
\end{array} \right) & \text{for } N \text{ even;}
\left( \begin{array}{cc}
1 & -\Im \Lambda_R \\
\Im \Lambda_R & \Re \Lambda_R 
\end{array} \right),
\left( \begin{array}{cc}
\Re \Lambda_R & \Im \Lambda_R \\
-\Im \Lambda_R & \Re \Lambda_R 
\end{array} \right) & \text{for } N \text{ odd.}
\end{cases}$$

where $\Lambda_R = \text{diag}(\lambda_1, \ldots, \lambda_R)$ is a diagonal unitary matrix. Besides the double eigenvalue 1, which occurs for $N$ odd, the matrices in (32) have $R$ quadruples of eigenvalues $\lambda_j, \lambda_j, \lambda_j^{-1}, \lambda_j^{-1}$.

C I. Here $G(N) = USp(2N)$, and $K(N) \simeq U(N)$ is the fixed-point set of the involution (25) with $I' = I'_{NN}$ as in (24). Explicitly,

$$U(N) \ni g \mapsto \left( \begin{array}{c} g \\ (g^T)^{-1} \end{array} \right) \in K(N).$$

Identify $G(N)/K(N)$ with the set

$$S(N) := \{ H = GI' \text{ s.t. } G \in U(2N) \text{ is Hermitian and } JG = -\overline{GJ} \}$$

by means of (8). The torus $A$ is

$$A = \left( \begin{array}{cc}
\Re \Lambda_N & -\Im \Lambda_N \\
\Im \Lambda_N & \Re \Lambda_N 
\end{array} \right);$$

with $\Lambda_N$ a unitary diagonal matrix as before. The eigenvalues occur in pairs just as in the case of (24) with $M = N$.

C II. Let $M \geq N \geq 1$ and $G(M, N) = USp(2M + 2N)$. We take the complex structure $J = J_{MN}$ as in (22). Then $K(M, N) = USp(2M) \times USp(2N)$ consists exactly of those elements that also stabilize

$$I' = \left( \begin{array}{cc}
I'_{MN} & \\
& I'_{MN}
\end{array} \right),$$

with $I'_{MN}$ as in (24), so that $K(M, N)$ is the fixed-point set of the involution

$$g \mapsto g^\Omega := I'gI'.$$
We can realize the symmetric space \( G(M, N)/K(M, N) \) as the set of matrices

\[
S(M, N) := \{ H = GI' \text{ such that } G \in USp(2M + 2N) \text{ is Hermitian of signature } (M, N) \},
\]

where we mean the quaternionic signature as discussed below. We recall that any matrix \( G \in Sp(2n, \mathbb{C}) \) which is Hermitian \((G^T = G)\), has real eigenvalues and can be diagonalized with a symplectic matrix \( g \in Sp(2n, \mathbb{C}) \), that is,

\[
g^{-1}Gg = \begin{pmatrix} \Delta_n & \Delta_n^{-1} \\ \Delta_n^{-1} & -\Delta_n \end{pmatrix}
\]

for some real diagonal matrix \( \Delta_n \). The usual signature of \( G \) is of the form \((2a, 2b)\), so we call \((a, b)\) the quaternionic signature. The identification is, of course, given by (8). The torus \( A \) is

\[
A = \left\{ \begin{pmatrix} 1_{M-N} & \Re \Lambda_N - \Im \Lambda_N \\ \Im \Lambda_N & \Re \Lambda_N \end{pmatrix} \right\}
\]

with \( \Lambda_N \) unitary diagonal. Besides the eigenvalue 1 with multiplicity \( 2(M - N) \), the other eigenvalues occur in quadruples like those of the matrices in (32).

For each of the ensembles, the torus \( A \), which has dimension equal to the rank \( R \) of the symmetric space, is parametrized by diagonal unitary matrices

\[
\Lambda_R = \text{diag}(\lambda_1, \ldots, \lambda_R), \quad |\lambda_j| = 1.
\]

Abusing notation, we will also write \( \Lambda_R \) for the vector \((\lambda_1, \ldots, \lambda_R)\). The tangent space \( \mathfrak{a} \) to this torus at the identity is identified with the space of \( R \)-tuples \( i\Theta = (i\theta_1, \ldots, i\theta_R) \), \( \theta_j \in \mathbb{R} \). Recall that we identify \( \mathfrak{p} \) with the tangent space to \( G/K \) at the base-point \( o = K/K \). The exponential maps \( \text{Exp} \) of \( G/K \) and \( \exp \) of \( G \) are related by

\[
\text{Exp}(X) = \exp(X)K \in G/K
\]

for \( X \in \mathfrak{p} = T_o(G/K) \). For \( i\Theta \in \mathfrak{a} \), \( \exp(i\Theta) \) is given by the matrix on the right-hand side of equations (27), (32), (35) and (40), respectively, provided we choose \( \lambda_j = e^{i\theta_j} \) in (41).
Proposition 1 (KAK decomposition). Let $G/K$ be a symmetric space of the compact type and $A \subset G$ be as above. The mapping
\[
K \times A \times K \rightarrow G
\]
\[
(k_1, a, k_2) \mapsto k_1ak_2
\]
is a surjection.

The $KAK$ decomposition has an integral counterpart.

Proposition 2 (Weyl’s integration formula). There is a measure $d\bar{\nu}(a)$ on $A$ such that, for any $f \in C(G)$,
\[
\int_G f(g) dg = \int_K \int_K \int_A f(k_1ak_2)d\bar{\nu}(a)dk_2dk_1.
\]
(We have simplified our notation by dropping the name “Haar” of the respective invariant measures.) Denote by $\Xi^+$ the set of positive roots of the symmetric Lie algebra $(g, \omega)$, and by $m_\alpha$ the multiplicity of a positive root $\alpha \in \Xi^+$. Then
\[
d\bar{\nu}(a) \propto \prod_{\alpha \in \Xi^+} |\sin \alpha(\Theta)|^{m_\alpha} da = \Delta(\Theta)da,
\]
say, where $\Theta$ is chosen so $a = \exp(i\Theta)$.

(With the notation above, we write $i\Theta = \log(a)$. This $\Theta$ is well-defined modulo $2\pi$.)

Now recall that the (positive) roots of $(g, \omega)$ are certain non-zero real-valued linear functionals on $a$ (in fact one should speak about the roots which are positive with respect to a fixed Weyl chamber in $a$). The root systems of the irreducible orthogonal Lie algebras of compact type are well-known by Cartan’s work.

Proposition 3. The positive roots and multiplicities for the irreducible orthogonal Lie algebras of type I are as follows (let $L = M - N$ in the case of ensembles with two parameters).

- **A I.**

\[
\begin{array}{|c|c|}
\hline
\alpha & m_\alpha \\
\hline
\theta_k - \theta_j, & 1 \leq j < k \leq R \\
\hline
1 & 1 \\
\hline
\end{array}
\]

- **A II.**

\[
\begin{array}{|c|c|}
\hline
\alpha & m_\alpha \\
\hline
\theta_k - \theta_j, & 1 \leq j < k \leq R \\
\hline
4 & 1 \\
\hline
\end{array}
\]

- **A III.**

\[
\begin{array}{|c|c|}
\hline
\alpha & m_\alpha \\
\hline
\theta_k - \theta_j, & 1 \leq j < k \leq R \\
\hline
4 & 1 \\
\hline
\end{array}
\]

- **BD I.**
We are now ready to derive the measure on eigenvalues for ensembles of type I.

**Theorem 2.** The measure on eigenvalues for a symmetric space of type I is given by

\[
d
\nu(a) \propto \Delta(\Theta/2)da = \prod_{\alpha \in \Xi^+} \left| \sin \frac{1}{2} \alpha(\Theta) \right|^{m_\alpha} da, \quad i\Theta = \log a.
\]

We are now ready to derive the measure on eigenvalues for ensembles of type I.
Using Weyl’s integration formula, we deduce that, for any \( f \in C(S) \),

\[
\int_S f(H) d\mu(H) = \int_{G/K} f((gk)^{1-\Omega}) dk d\mu(g) \quad \text{(by (8))}
\]

\[
= \int_G f(g^{1-\Omega}) dg \quad \text{(since } k^{1-\Omega} = e) \]

\[
= \int_K \int_A \int_K f((k_1a_2)^{1-\Omega}) dk_2 d\nu(a) dk_1
\]

\[
= \int_K \int_A \int_K f((k_1a)^{1-\Omega}) dk_2 d\nu(a) dk_1
\]

\[
= \int_K \int_A \int_K f((ka)^{1-\Omega}) d\nu(a) dk
\]

\[
= \int_K \int_A f(ka^{2k^{-1}}) d\nu(a) dk. \quad \text{(since } a^{1-\Omega} = a^2) \]

This ought to be compared with (45), which defines the measure \( d\nu(\Lambda) \) on eigenvalues. A key property of the measure \( d\nu(a) \) defined by (45) is reflected in the fact that \( \Delta(\Theta) = \Delta(\Theta') \) if \( \Theta \equiv \Theta' \mod \pi \) (this follows in general from the fact that the roots take integral values on the “unit lattice” \( \exp^{-1}(e) \), and can be verified for ensembles of type I directly using proposition 3). From that observation, it follows that:

\[
\int_K \int_A f(ka^{2k^{-1}}) d\nu(a) dk \propto \int_K \int_{[0,2\pi]^R} f(k \exp(2i\Theta)k^{-1}) \Delta(\Theta) d\Theta dk
\]

\[
= 2^R \int_K \int_{[0,\pi]^R} f(k \exp(2i\Theta)k^{-1}) \Delta(\Theta) d\Theta dk
\]

\[
= \int_K \int_{[0,2\pi]^R} f(k \exp(i\Theta)k^{-1}) \Delta(\Theta/2) d\Theta dk
\]

\[
= \int_K \int_A f(kak^{-1}) \Delta(\Theta/2) da dk.
\]

When put together with (47), this proves (46).

Now we restrict attention to the most interesting case, that of “class functions” \( f \in C(K\setminus S) \), that is, those functions on \( S \) which depend only on the eigenvalues of the matrix, viz

\[
f(ka^{2k^{-1}}) = f(a).
\]

The tori \( A \) are parametrized by \( R \)-tuples \( (\lambda_j = e^{i\theta_j}) \). From the knowledge of the structure of the set of eigenvalues of the matrices

\[
f(kak^{-1}) = f(a).
\]
in these tori, we see that for all the ensembles of type I except for Dyson’s A I and A II, changing the sign of any $\theta_j$ does not change the set of eigenvalues since these always come in pairs $\{e^{\pm i\theta_j}\}$ (with single or double multiplicity), hence any class function $f \in C(K\backslash S)$ is determined by its values on $\exp([0, \pi]^R) \subset A$, and correspondingly

$$\int_S f(H) d\mu(H) = \int_A f(a) d\nu(a) \propto \int_{[-\pi, \pi]^R} f(\exp(i\Theta)) \Delta(\Theta/2) d\Theta$$

(50)

Hence, except in the cases of A I and A II, it is convenient to regard the measure on eigenvalues as one supported on $[0, \pi]^R$. Noting that the contribution of a pair of roots $\theta_k \pm \theta_j$ to $\Delta(\Theta/2)$ is

$$|\sin \left( \frac{\theta_k - \theta_j}{2} \right) \sin \left( \frac{\theta_k + \theta_j}{2} \right)| \propto |\cos \theta_k - \cos \theta_j|,$$

(51)

it is clear that for all the ensembles of type I, except for the COE and the CSE, the measure on eigenvalues is proportional to the measure

$$\prod_{1 \leq j < k \leq R} |\text{Van}(\cos \Theta)|^\beta \prod_{1 \leq j \leq R} |\sin \theta_j|^p |\sin(\theta_j/2)|^q d\Theta,$$

(52)

on $[0, \pi]^R$. (Here $\beta = 1, 2, 4$ according to the multiplicity $m_\alpha$ of the roots $\theta_k \pm \theta_j$.)

Because $|\sin \theta| = |1 - \cos \theta|^{1/2}|1 + \cos \theta|^{1/2}$ and $|\sin(\theta/2)| = 2^{-1/2}|1 - \cos \theta|^{1/2}$, the above is proportional to the measure

$$\prod_{1 \leq j < k \leq R} |\text{Van}(\cos \Theta)|^\beta \prod_{1 \leq j \leq R} |1 - \cos \theta_j|^p |1 + \cos \theta_j|^q d\Theta,$$

(53)

on $[0, \pi]^R$.

We make the change variables $\Theta \mapsto x = \cos \Theta$ to obtain

$$d\nu(x) \propto \prod_{1 \leq j < k \leq R} |\text{Van}(x)|^\beta \prod_{1 \leq j \leq R} |1 - x_j|^a |1 + x_j|^b dx, \quad \text{on } [-1, 1]^R,$$

(54)

where $a = p - 1/2$, $b = q - 1/2$, and $dx = dx_1 \ldots dx_R$. The weight function

$$w(x) = |1 - x|^a |1 + x|^b \quad \text{on } [-1, 1]$$

(55)

is that with respect to which the classical Jacobi orthogonal polynomials $P_n^{(a,b)}(x)$ are defined, so a matrix ensemble for which the probability measure of the eigenvalues is given by (54) is called a Jacobi ensemble (with parameters $(a, b)$). For $\beta = 1, 2, 4$ we call such an ensemble orthogonal, unitary or symplectic, respectively.
Recall that, for the COE and CSE, the probability measure of the eigenvalues is given by \((14)\). It coincides with that given by Weyl’s formula (proposition \(2\)) since

\[
|e^{i\theta_k} - e^{i\theta_j}| = 2 \left| \sin \left( \frac{\theta_k - \theta_j}{2} \right) \right|.
\]

For completeness, table 3 is the analogue of table 1 for (the infinite families of) symmetric spaces of type II (compact Lie groups). The CUE is a circular ensemble with \(\beta = 2\) and measure on eigenvalues \((14)\), whereas the orthogonal and symplectic groups are unitary Jacobi ensembles.

| Type   | \(S(N)\) | Parameters |
|--------|----------|------------|
| \(a_N\) (CUE) | \(U(N)\) | \(\beta = 2\) |
| \(b_N\) | \(SO(2N + 1)\) | \(\beta = 2, (a, b) = (\frac{1}{2}, -\frac{1}{2})\) |
| \(c_N\) | \(USp(2N)\) | \(\beta = 2, (a, b) = (\frac{1}{2}, \frac{1}{2})\) |
| \(d_N\) | \(SO(2N)\) | \(\beta = 2, (a, b) = (-\frac{1}{2}, \pm \frac{1}{2})\) |

Table 3. Parameters of the probability measure of the eigenvalues for ensembles of type II.

4. **Universality of Local Correlations**

In this section we analyze the limiting correlation functions for general Jacobi ensembles. As we have shown, with the exception of Dyson’s COE (A I) and CSE (A II), the ensembles of type I are special cases of (orthogonal, unitary or symplectic) Jacobi ensembles.

We consider the joint probability measure of the \(R\) levels (we speak about levels rather than eigenvalues since the natural variables to use are \(x_j = \Re \lambda_j\)) given in the general form

\[
d\nu(x_R) = P_R(x_R) \, dx_R,
\]

where \(x_R = (x_1, \ldots, x_R)\) is an \(R\)-tuple of levels. The \(n\)-level correlation function \(I_R^{(n)}(x_n)\) is defined by

\[
I_R^{(n)}(x_n) = \frac{R!}{(R - n)!} \int \cdots \int P_R(x_n, x_{n+1}, \ldots, x_R) \, dx_{n+1} \cdots dx_R.
\]

It is, loosely speaking, the probability that \(n\) of the levels, regardless of order, lie in infinitesimal neighborhoods of \(x_1, \ldots, x_n\) (but the total mass of the measure \(I_R^{(n)}(x_n)\) is now \(R!/(R - n)!\) and not 1).
The semi-classical limit $R \to \infty$ is of great interest. The so-called “universality conjecture” (which dates back to the work of Dyson [14]) states that the local correlations of the eigenvalues in the bulk of the spectrum tend to very specific limits that depend only on the symmetry parameter $\beta$. Special cases of the truth of this assertion are known. In particular, in the unitary case $\beta = 2$, the result is proven in certain generality [8, 7, 2, 3], but for $\beta = 1, 4$ it is known only for special ensembles such as the circular ensembles of Dyson [11, 12, 13] and, by work of Nagao and Forrester [23], for most Laguerre ensembles and Jacobi ensembles. However, the latter assumes that the parameters $a, b$ are strictly positive, hence it is not applicable to ensembles of type I (cf., table 1).

It is an extremely important fact that for general orthogonal, unitary and symplectic ensembles the correlation functions can be expressed as determinants (which discovery goes back, in the unitary case, to the work of Gaudin and Mehta [15, 19], and in the orthogonal and symplectic cases to Dyson’s study of his circular ensembles, and later extended by Chadha, Mahoux and Mehta [18, 6, 22] to the general case). In the case of unitary Jacobi ensembles there exists a scalar-valued kernel $K_{R^2}^{(a,b)}(x, y)$ defined in terms of the classical Jacobi orthogonal polynomials $P_n^{(a,b)}(x)$ (the projector kernel onto the span of the first $R$ Jacobi polynomials) satisfying [24]

$$I_{R^2}^{(n)}(x_n) = \det(K_{R^2}(x_j, x_k))_{j,k=1,...,n}.$$ 

In the case of the orthogonal (resp., symplectic) Jacobi ensembles, there exists a matrix-valued kernel [24] (alternatively, a “quaternion” kernel)

$$K_{R^2}^{(a,b)}(x, y) = \begin{pmatrix} S_{R^2}^{(a,b)}(x, y) & I_{R^2}^{(a,b)}(x, y) - \delta \epsilon(x - y) \\ D_{R^2}^{(a,b)}(x, y) & S_{R^2}^{(a,b)T}(x, y) \end{pmatrix},$$

where $\delta = 1$ (resp., $\delta = 0$—the $\epsilon$-term is absent in the symplectic case),

$$\epsilon(z) = \frac{1}{2} \text{sgn}(z) = \frac{1}{2} \frac{z}{|z|},$$

and the scalar kernel $S_{R^2}^{(a,b)}$ is defined in terms of the skew-orthogonal polynomials of the second (resp., first) kind depending on the weight [55].
and the other quantities are given by

$$I_{R\beta}^{(a,b)}(x,y) = - \int_x^y S_{\beta}^{(a,b)}(x,z) dz,$$

$$D_{R\beta}^{(a,b)}(x,y) = \partial_x S_{R\beta}^{(a,b)}(x,y),$$

$$S_{R\beta}^{(a,b)T}(x,y) = S_{R\beta}^{(a,b)}(y,x).$$

The matrix kernel (60) is self-dual in the sense that $K_{R\beta}^{(a,b)}(y,x) = K_{R\beta}^{(a,b)}(x,y)D$ (cf., section 2). The correlation functions themselves are given by

$$I_{R\beta}^{(n)}(x_n) = \sqrt{\det(K_{R\beta}(x_j,x_k))_{n\times n}}.$$

Indeed, if the matrix $(K_{R\beta}(x_j,x_k))_{n\times n}$ is interpreted as a quaternion self-dual matrix [20], then the right-hand side of (65) is its Dyson’s “quaternion determinant” $\text{qdet}$ [11, 12, 13], so (65) can be rewritten:

$$I_{R\beta}^{(n)}(x_n) = \text{qdet}(K_{R\beta}(x_j,x_k))_{n\times n}.$$

**Remark.** In what follows we will sometimes unify notation by writing $\text{DET}$ (all caps) to signify the usual determinant when $\beta = 2$ and the quaternion determinant when $\beta = 1, 4$. Thus, equations (59) and (66) will be written

$$I_{R\beta}^{(n)}(x_n) = \text{DET}(K_{R\beta}(x_j,x_k))_{n\times n}.$$

The first quantity of interest is the (global) level density. Indeed, since the first correlation function has total mass $R$, one might expect that the probability measure $R^{-1}I_{R}^{(1)}(x)dx$ on $[-1, 1]$ tend to a limiting measure as $R \to \infty$. We define the level density to be the corresponding probability density function:

$$\rho(x) = \lim_{R \to \infty} R^{-1}I_{R}^{(1)}(x).$$

Assuming $\rho(x)$ to be continuous, the bulk of the spectrum is the set $\{x : \rho(x) > 0\}$: points where the level density vanishes or blows up to infinity are excluded from the bulk of the spectrum.

**Theorem 3.** For the orthogonal, unitary or Jacobi ensembles associated to the weight function (55), the global level density is given by

$$\rho(x) = \frac{1}{\pi \sqrt{1-x^2}} \quad \text{on } (-1, 1).$$

The limit in (68) is attained uniformly on compact subsets of $(-1, 1)$. 


This theorem will be proved in the following section.

If we revert to the angular variable $\theta$ with $x = \cos \theta$, we see that

(70) \[ \rho(x) dx = \frac{d\theta}{\pi} = \varrho(\theta) d\theta \]

so the level density $\varrho(\theta) \equiv 1/\pi$ on $(0, \pi)$ is constant: the eigenvalues become equidistributed on the unit circle (with respect to its invariant measure), and uniformly so away from the central eigenvalues $\pm 1$, in the semiclassical limit $R \to \infty$. The bulk of the spectrum excludes the edges $\pm 1$.

The local $n$-level correlations are the “local” semi-classical limits of the $n$-level correlations $I_R^{(n)}$. When localizing near the neighborhood of a fixed level $z_o$ belonging to the bulk of the spectrum, these local correlations are universal in the sense that they depend neither on the specific ensemble nor on the choice of $z_o$ but only on the symmetry parameter $\beta$. In particular they coincide with the local correlations of the Gaussian Orthogonal ($\beta = 1$), Unitary ($\beta = 2$) or Symplectic ($\beta = 4$) ensemble, respectively. For Jacobi ensembles the bulk of the spectrum consists of the open interval $(-1, 1)$, whereas the local correlations near the “hard edges” $\pm 1$ (which correspond to the “central eigenvalues” $\pm 1$ on the unit circle) have a different behavior which is sensitive to the parameters $(a, b)$ of the ensemble.

**Remark.** As we shall see later, the level density vanishes to some order at, say, the hard edge $+1$ depending on the parameter $a$ (which is natural since $a$ determines the order to which the weight function (55) vanishes at $x_j = +1$). The local correlations fail to follow Dyson’s universal “threefold way”, but rather depend on this parameter. The same limiting behavior occurs at the hard edge 0 of Laguerre ensembles [23], so that, at least conjecturally, these “universal” laws—manifestly different from Dyson’s bulk regimes—describe the behavior of the local correlations at a hard edge for general orthogonal, unitary or symplectic ensembles.

We now fix a level $z_o \in [-1, 1]$. Given that the eigenvalue density is uniform, it is natural to change variables from $x$ to $\xi$ stretching the angles by a factor $R$, namely setting

(71) \[ x_j = \cos \left( \alpha_o + \frac{\pi}{R} \xi_j \right), \]

where $\alpha_o = \arccos z_o$ (note that the change of variables depends on $R$). The semiclassical limit of the correlation functions is obtained by letting $R$ tend to infinity. What the factor $\pi/R$ accomplishes is that,
on the bulk of the spectrum, the local level density (i.e., the local limit of the correlation function $I_{R\beta}^{(1)}$) will be $\bar{\rho}(\xi) \equiv 1$.

**Theorem 4.** For the orthogonal ($\beta = 1$), unitary ($\beta = 2$) and symplectic ($\beta = 4$) Jacobi ensembles associated to the weight function (55), the local correlations are as follows:

- **Bulk local correlations (independent of $\beta$ and of the choice of a fixed $z_0 = \cos \alpha_0 \in (-1, 1)$).**

  - **Local level density:**
    
    $\bar{\rho}(\xi) = \lim_{R \to \infty} (R\rho(x))^{-1} I_{R\beta}^{(1)}(x) \equiv 1, \quad \xi \in \mathbb{R}$.

  where $x$ depends on $\xi$ as in (71) and $\rho(x)$ is the global level density (69).

  - **Local correlations:**
    
    $L_{\beta}^{(n)}(z_0; \xi_n) = \lim_{R \to \infty} (R\rho(z_0))^{-n} I_{R\beta}^{(n)}(x_n) = \text{DET}(\bar{K}_\beta(\xi_j, \xi_k))_{n \times n}$.

where $x_n$ and $\xi_n$ are related by (71) (recall that DET stands for the usual or the quaternion determinant in the cases of $\beta = 2$ and $\beta = 1, 4$, respectively). In the case $\beta = 2$, $\bar{K}_2$ is the scalar Sine Kernel

$\bar{K}_2(\xi, \eta) = \begin{cases} 
\frac{\sin \pi(\xi - \eta)}{\pi(\xi - \eta)}; & \xi \neq \eta; \\
\bar{\rho}(\xi) = 1, & \xi = \eta.
\end{cases}$

In the case $\beta = 4$ the matrix Sine Kernel $\bar{K}_4$ is given by

$\bar{K}_4(\xi, \eta) = \begin{pmatrix} S_4(\xi, \eta) & I_4(\xi, \eta) \\
D_4(\xi, \eta) & S_4^T(\xi, \eta) \end{pmatrix},$

where

$\bar{S}_4(\xi, \eta) = \bar{K}_2(2\xi, 2\eta),
\bar{I}_4(\xi, \eta) = -\int_\xi^\eta \bar{S}_4(\xi, t)dt,
\bar{D}_4(\xi, \eta) = \partial_\xi \bar{S}_4(\xi, \eta),
\bar{S}_4^T(\xi, \eta) = S_4(\eta, \xi).$

In the case $\beta = 1$ the matrix Sine Kernel $\bar{K}_1$ is given by

$\bar{K}_1(\xi, \eta) = \begin{pmatrix} \bar{S}_1(\xi, \eta) & \bar{I}_1(\xi, \eta) - \epsilon(\xi - \eta) \\
\bar{D}_1(\xi, \eta) & \bar{S}_1^T(\xi, \eta) \end{pmatrix},$
where
\begin{align}
\bar{S}_1(\xi, \eta) &= \bar{K}_2(\xi, \eta), \\
\bar{I}_1(\xi, \eta) &= -\int_\xi^\eta \bar{S}_1(\xi, t) dt, \\
\bar{D}_1(\xi, \eta) &= \partial_\xi \bar{S}_1(\xi, \eta), \\
\bar{S}_1^T(\xi, \eta) &= \bar{S}_1(\eta, \xi).
\end{align}

- Hard edge \( z_0 = +1 \) (\( \alpha_0 = 0 \)).

- Central point level density. For \( \xi > 0 \):
\begin{align}
\lim_{R \to \infty} \left( \frac{R}{\pi} \right)^{-1} I^{(1)}_{R, 3}(x) &= \hat{\rho}_\beta(\xi) \tag{79}
\end{align}
(where \( x \) depends on \( \xi \) by (71)) is given by:
\begin{align}
\hat{\rho}_2^{(a)}(\xi) &= \frac{\pi}{2} (\pi \xi)^2 J_{a}(\pi \xi)^2 - J_{a-1}(\pi \xi) J_{a+1}(\pi \xi), \\
\hat{\rho}_1^{(a)}(\xi) &= \hat{\rho}_2^{(2a+1)}(\xi) + \frac{\pi}{2} J_{2a+1}(\pi \xi) \int_{\pi \xi}^{\infty} J_{2a+1}, \\
\hat{\rho}_4^{(a)}(\xi) &= \hat{\rho}_2^{(a)}(2\xi) - \frac{\pi}{2} J_{a-1}(2\pi \xi) \int_{0}^{2\pi \xi} J_{a+1}.
\end{align}

- Local correlations. For \( \xi_n > 0 \):
\begin{align}
L^{(n)}_{\beta}(+1; \xi_n) &= \lim_{R \to \infty} \left( \frac{R}{\pi} \right)^{-n} I^{(n)}_{R, 3}(x_n) = \text{DET}(\hat{K}_\beta(\xi_j, \xi_k))_{n \times n},
\end{align}
with \( x_n \) related to \( \xi_n \) by (71). The scalar “Bessel Kernel” \( \hat{K}_2 = \hat{K}_2^{(a)} \) is given by
\begin{align}
\hat{K}_2^{(a)}(\xi, \eta) &= \left\{ \begin{array}{ll}
\hat{\rho}_2^{(a)}(\xi) & \xi \neq \eta; \\
\frac{\xi \eta}{\pi} & \xi = \eta.
\end{array} \right.
\end{align}

For \( \beta = 1, 4 \) the matrix Bessel Kernels are given by the same expressions of (75)-(78), except that the bars are to be replaced by hats and \( \hat{S}_1 = \hat{S}_1^{(a)} \), \( \hat{S}_4 = \hat{S}_4^{(a)} \) are given by
\begin{align}
\hat{S}_1^{(a)}(\xi, \eta) &= \sqrt{\frac{\xi}{\eta}} \hat{K}_2^{(2a+1)}(\xi, \eta) + \frac{\pi}{2} J_{2a+1}(\pi \eta) \int_{\pi \xi}^{\infty} J_{2a+1}(t) dt, \\
\hat{S}_4^{(a)}(\xi, \eta) &= \sqrt{\frac{\xi}{\eta}} \hat{K}_2^{(a-1)}(2\xi, 2\eta) - \frac{\pi}{2} J_{a-1}(2\pi \eta) \int_{0}^{2\pi \xi} J_{a+1}(t) dt.
\end{align}

where the \( J_\nu \) are the Bessel functions of the first kind.
Remark. The local limits at the edge $z_0 = -1$ are given by the same formulae replacing the parameter $a$ by $b$.

Remark. The integral in (86) diverges for $-1 < a < 0$. However, in the next section we provide an alternative version of that equation which is well-defined for all $a > -1$.

Remark. In connection with the hard edge correlations for the classical orthogonal and symplectic groups (table 3), we remark that the unitary Bessel kernel (84), in the case $a = +1/2$ (resp., $a = -1/2$), coincides with the “odd” (resp., “even”) Sine Kernel [17]:

\[
\hat{K}_2^{(\pm 1/2)}(\xi, \eta) = \frac{\sin(\xi - \eta)}{\xi - \eta} \mp \frac{\sin(\xi + \eta)}{\xi + \eta}.
\]

5. Proofs

In this section we prove theorems 3 and 4. First we remark that the unitary case has been studied in the work of Nagao and Wadati [24, 25], but we reproduce the proofs here for completeness and also to show that the hypothesis $a > -1$ is, in a certain sense, unnecessary. Also we remark that Forrester and Nagao [23] have studied the hard edge correlations directly, using skew-orthogonal polynomial expressions for
the matrix kernels $K_{R1}, K_{R4}$, but their results apply only when the parameters $a, b$ are strictly positive, and in view of the application to symmetric spaces this restriction is unacceptable (see table II). Also, their somewhat more complicated formulas for the limiting quantities $\hat{S}_1, \hat{S}_4$ are given in terms of iterated integrals of Bessel functions. Here
we take advantage of the more recent work of Adler et al which provides simple “summation formulas” for the quantities $S_{R1}, S_{R4}$.

5.1. Some preliminary results and formulas. The various results we quote on Jacobi polynomials can be found in Szegő’s book [28] and in his article on asymptotic properties of Jacobi polynomials [27] (reproduced in his collected papers [29]). Stirling’s formula and the Bessel function identities can be found, for instance, in the tables of Gradshteyn and Ryzhik [16]. We denote by $P_{N}^{(A,B)}(x)$ the classical Jacobi polynomials defined by

\begin{equation}
(1 - x)^A(1 + x)^B P_N^{(A,B)}(x) = \frac{(-1)^N}{2^N N!} \left( \frac{d}{dx} \right)^N \left[ (1 - x)^{N+A}(1 + x)^{N+B} \right].
\end{equation}

When $A, B > -1$, these polynomials are orthogonal on $[-1, 1]$ with respect to the weight

\begin{equation}
w(x) = |1 - x|^A |1 + x|^B,
\end{equation}

but they are not normalized. However, the formula (88) is meaningful for arbitrary (real or complex) values of the parameters $A, B$, and defines a polynomial in $A, B, x$ of degree (at most) $N$ in $x$. In fact

\begin{equation}
P_N^{(A,B)}(x) = \sum_{k=0}^{N} \binom{A + N}{k} \binom{B + N}{N - k} \left( \frac{x - 1}{2} \right)^{N-k} \left( \frac{x + 1}{2} \right)^{k}.
\end{equation}

In particular

\begin{equation}
P_N^{(A,B)}(+1) = \binom{A + N}{N}.
\end{equation}

The derivative of a Jacobi polynomial is related to another Jacobi polynomial by the identity (the apostrophe denotes differentiation with respect to $x$)

\begin{equation}
P_N^{(A,B)'}(x) = \frac{1}{2} (N + A + B + 1) P_{N-1}^{(A+1,B+1)}(x).
\end{equation}

Proposition 4 (Darboux’s formula). (With an improved error term due to Szegő [27].) For arbitrary reals $A, B$,

\begin{equation}
P_N^{(A,B)}(\cos \theta) = (\pi N)^{-1/2} \left( \sin \frac{\theta}{2} \right)^{-A-1/2} \left( \cos \frac{\theta}{2} \right)^{-B-1/2} \cos(N' \theta + \gamma) + E,
\end{equation}

where

\begin{equation}
N' = N + \frac{A + B + 1}{2}, \quad \gamma = - \left( A + \frac{1}{2} \right) \frac{\pi}{2},
\end{equation}

and $E$ is a small error term.
for $0 < \theta < \pi$, where the error term $E$ satisfies

\begin{equation}
E = \theta^{-A-3/2}O(N^{-3/2}), \quad \text{uniformly for } c/N \leq \theta \leq \pi - \epsilon,
\end{equation}

for any positive constants $c, \epsilon$, and the constant implied by the $O$ symbol depends only on $c, \epsilon, A, B$.

**Proposition 5** (Hilb’s formula). (As generalized by Szegő to Jacobi polynomials [28].) For $A > -1$ and any real $B$:

\begin{equation}
\left( \sin \frac{\theta}{2} \right)^A \left( \cos \frac{\theta}{2} \right)^B P_N^{(A,B)}(\cos \theta) = N^{-A} \frac{\Gamma(N + A + 1)}{N!} \sqrt{\frac{\theta}{\sin \theta}} J_A(N'\theta) + E,
\end{equation}

where $N'$ has the same meaning as in (93) and the error term $E$ is given by

\begin{equation}
E = \begin{cases} 
\theta^{1/2}O(N^{-3/2}) & \text{if } c/N \leq \theta \leq \pi - \epsilon, \\
\theta^{A+2}O(N^A) & \text{if } 0 < \theta \leq c/N,
\end{cases}
\end{equation}

where $c, \epsilon$ are arbitrary but fixed positive constants, and the constants implied by the $O$ symbol depend on $A, B, c, \epsilon$ only.

The restriction to $A > -1$, however, is too strong for some purposes, and we will need the following formula, also due to Szegő [27] (reproduced in [29]):

\begin{equation}
P_N^{(A,B)}(\cos \theta) = \left( \sin \frac{\theta}{2} \right)^{-A} \left( \cos \frac{\theta}{2} \right)^{-B} \sqrt{\frac{\theta}{\sin \theta}} \left( 1 - \sqrt{\frac{\tan(\theta/2)}{2\theta}} \right) \times \right. \\
\times J_A(N'\theta) + R,
\end{equation}

with $N'$ as in (93). Here $A, B$ are arbitrary reals. The error term $R$ satisfies:

\begin{equation}
R = \begin{cases} 
\theta^{1/2}O(N^{-3/2}) & \text{if } c/N \leq \theta \leq \pi - \epsilon, \\
O(N^{A-2}) & \text{if } 0 < \theta \leq c/N,
\end{cases}
\end{equation}

where $c, \epsilon$ are fixed positive numbers, and the constants implied by the $O$ symbol depend only on $A, B, c, \epsilon$. It must be noted, however, that the error term $R$ of (98) does not depend on $\theta$ on the range $0 < \theta < c/N$, which makes this formula less useful than (95) with the error term (96) for $\theta$ in this range.

Recall Stirling’s asymptotic formula for the Gamma function:

\begin{equation}
\log \Gamma(x) = \left( x - \frac{1}{2} \right) \log x - x + \frac{1}{2} \log 2\pi + O(1/x), \quad \text{as } x \to \infty.
\end{equation}
The Bessel functions of the first kind are defined by the series
\[(100)\]
\[J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{2^{2k}k!\Gamma(\nu + k + 1)}, \quad z \in \mathbb{C}\setminus(-\infty, 0], \quad \nu \in \mathbb{R};\]

they satisfy, among many others, the relations:
\[(101)\]
\[J'_\nu(z) = J_{\nu-1}(z) - \frac{\nu}{z} J_\nu(z),\]
\[(102)\]
\[J'_\nu(z) = -J_{\nu+1}(z) + \frac{\nu}{z} J_\nu(z),\]
\[(103)\]
\[J'_\nu(z) = \frac{1}{2}[J_{\nu-1}(z) - J_{\nu+1}(z)],\]
\[(104)\]
\[J_{\nu+1}(z) = \frac{2\nu}{z} J_\nu(z) - J_{\nu-1}(z),\]
\[(105)\]
\[\frac{d}{dz}[z^\nu J_\nu(z)] = z^\nu J_{\nu-1}(z),\]
\[(106)\]
\[\frac{d}{dz}[z^{-\nu} J_\nu(z)] = -z^{-\nu} J_{\nu+1}(z).\]

We also have
\[(107)\]
\[\int J_\nu = 2 \sum_{k=0}^{\infty} J_{\nu+2k+1},\]
\[(108)\]
\[\int_0^{\infty} J_\nu(t)dt = 1 \quad \text{for} \ \nu > -1.\]

5.2. **Asymptotics of the Unitary Jacobi Kernel.** In this section we recall the proofs of some of the results of Nagao and Wadati [24], which will be needed later on in the analysis of the orthogonal and symplectic cases.

Using the Christoffel-Darboux summation formula [28], the scalar kernel \(K_{N2}^{(A,B)}\) can be written in the form
\[(109)\]
\[K_{N2}^{(A,B)}(x, y) = \frac{2^{-A-B}}{2N + A + B} \frac{\Gamma(N + 1)\Gamma(N + A + B + 1)}{\Gamma(N + A)\Gamma(N + B)} \times \sqrt{w(x)w(y)} \frac{P_N^{(A,B)}(x)P_{N-1}^{(A,B)}(y) - P_N^{(A,B)}(x)P_{N-1}^{(A,B)}(y)}{x - y},\]
for \( x \neq y \), and

\[
K^{(A,B)}_{N^2}(x, x) = \frac{2^{-A-B}}{2N + A + B} \frac{\Gamma(N + 1) \Gamma(N + A + B + 1)}{\Gamma(N + A) \Gamma(N + B)} \times w(x) [P^{(A,B)}_N(x) P^{(A,B)}_{N-1}(x) - P^{(A,B)}_{N-1}(x) P^{(A,B)}_N(x)].
\]

We observe that the kernel \( K_{N^2} \) given by (109) and (110) is well-defined for \( A, B > -c \) for any real constant \( c \) provided \( N \) is sufficiently large.

First consider the global level density

\[
\rho(x) = \lim_{N \to \infty} N^{-1} K(x, x).
\]

Using Darboux’s formula (93) together with the identity (92) in the expression (110) for the kernel, we find:

\[
K^{(a,b)}_{N^2}(x, x) = \frac{N}{\pi \sqrt{1 - x^2}} + O(1)
\]

where the implied constant depends only on \( \epsilon \) for \(-1 + \epsilon \leq x \leq 1 - \epsilon\).

Equation (112) proves (69) (in the unitary case).

A density function \( D = D(x_1, \ldots, x_n) \) defines a measure \( D dx_1 \ldots dx_n \).

Under a (monotonically increasing or decreasing) differentiable change of variables \( x_j = X(u_j) \), this density is transformed into the density

\[
D(u_1, \ldots, u_n) = \left( \prod_{j=1}^{n} |X'(u_j)| \right) D(X(u_1), \ldots, X(u_n)).
\]

If the density \( D \) is given as a determinant with a (scalar) kernel \( K(x, y) \), namely \( D = \det(K(x_j, x_k))_{n \times n} \), then the change of variables reflects itself in the kernel in the following fashion:

**Lemma 1.** After the (monotonic) differentiable change of variables \( u \to x = X(u) \), the correlation functions are given as the determinant defined using the kernel

\[
K(u, v) = \sqrt{|X'(u)X'(v)|} K(X(u), X(v)).
\]

This is clear since the introduction of the factor \( \sqrt{|X'(u)X'(v)|} \) results in multiplying the determinant (59) by \( \prod_{j=1}^{n} |X'(u_j)| \).

The localization at some \(-1 < z_o = \cos \alpha_o < 1\) given by the change of variables (71) leads us to consider the limit

\[
\bar{K}_2^{(a,b)}(\xi, \eta) = \lim_{N \to \infty} (N \sqrt{\rho(x)\rho(y)})^{-1} K^{(a,b)}_{N^2}(x, y)
\]

\[
= \lim_{N \to \infty} (N \rho(z_o))^{-1} K^{(a,b)}_{N^2}(x, y),
\]
with \( x, y \) related to \( \xi, \eta \) by (71), which from Darboux’s formula (93) can be easily seen to be the Sine Kernel (74), independently of the value of \( z_o \) (as long as \(-1 < z_o < 1\)), for any real \( a, b \), and the limit is attained uniformly on compacta.

For the localization at \( z_o = +1 (\alpha_o = 0) \) —localization at \( z_o = -1 \) is analogous provided \( a \) and \( b \) are interchanged—, we use the same change of variables (71) with \( \xi_n > 0 \). To compute the limit

\[
\hat{K}^{(a,b)}_2(\xi, \eta) = \lim_{N \to \infty} (N \sqrt{\rho(x)\rho(y)})^{-1} K^{(a,b)}_{N2}(x, y)
\]

we use Szegö’s formulas (95), (97), in conjunction with (109) and (110):

\[
\hat{K}^{(a)}_2(\xi, \eta) = \frac{\sqrt{\xi \eta}}{\xi^2 - \eta^2} \left[ \pi \xi J_a'(\pi \xi)J_a(\pi \eta) - J_a(\pi \xi)\pi \eta J_a'(\pi \eta) \right].
\]

Using the derivation formula (102) we rewrite this kernel in the form (84).

For the case \( \xi = \eta \) we start with the expression (110) and use the derivation formula (92) to find:

\[
\hat{\rho}^{(a)}_2(\xi) = \hat{K}^{(a)}_2(\xi, \xi)
\]

\[
= \frac{\pi}{2} \left[ J_a(\pi \xi)J_{a+1}(\pi \xi) + \pi \xi J'_{a+1}(\pi \xi)J_a(\pi \xi) - \pi \xi J_a(\pi \xi)J_{a+1}(\pi \xi) \right].
\]

Applying the derivation formula (101) and the recurrence formula (104) this can be rewritten in the form (80).

5.3. Asymptotics of the Orthogonal Jacobi Kernel. We start with some general remarks. If a density \( P = P(x_1, \ldots, x_n) \) is given as a quaternion determinant with a self-dual matrix kernel \( K(y, x) = K(x, y)^D \), namely \( P = \det(Q(x_j, x_k))^{1}_{n} \), then under a differentiable change of variables \( x_j = X(u_j) \) the density is still given as a quaternion determinant.

**Lemma 2.** After a (monotonic) differentiable change of variables \( u \to x = X(u) \), a density function

\[
P(x_1, \ldots, x_n) = \det(K(x_j, x_k))
\]

defined in terms of some self-dual matrix kernel \( \delta = 0, 1 \)

\[
K(x, y) = \begin{pmatrix}
S(x, y) & 1(x, y) - \delta\epsilon(x - y) \\
D(x, y) & S^T(x, y)
\end{pmatrix}
\]
with

\[(121) \quad I(x, y) = -\int_x^y S(x, z)dz,\]
\[(122) \quad D(x, y) = \partial_x S(x, y),\]
\[(123) \quad S^T(x, y) = S(y, x).\]

is transformed into the density

\[(124) \quad \mathcal{P}(u_1, \ldots, u_n) = q\det(K(u_j, u_k)),\]

where

\[(125) \quad K(u, v) = \begin{pmatrix} S(u, v) & \mathcal{I}(u, v) - \delta\epsilon(u - v) \\ D(u, v) & S^T(u, v) \end{pmatrix},\]
\[(126) \quad S(u, v) = S(X(u), X(v))|X'(v)| = \pm S(X(u), X(v))X'(v),\]
\[(127) \quad \mathcal{I}(u, v) = -\int_u^v S(u, w)dw,\]
\[(128) \quad D(u, v) = \partial_u S(u, v),\]
\[(129) \quad S^T(u, v) = S(v, u).\]

For the proof, we need first:

**Lemma 3.** Let \(H = H^D = J_nH^TJ_n^T\) be a \(2n \times 2n\) self-dual complex matrix. Let \(k_j, j = 1, 2, \ldots, n\) be arbitrary complex constants. Set \(K = \text{diag}(k_1, \ldots, k_n)\). Then the matrices

\[(130) \quad H_1 = \text{diag}(I, K)H \text{ diag}(K, I) \quad H_2 = \text{diag}(-I, K)H \text{ diag}(-K, I)\]

(where \(I = I_n\) is the \(n \times n\) identity matrix) are both self-dual, and

\[(131) \quad q\det(H_1) = \det(K) q\det(H) = q\det(H_2).\]

The verification that \(H_1\) and \(H_2\) are self-dual is trivial. On the other hand, since \((q\det X)^2 = \det X\) for any self-dual matrix \(X\), we have that

\[(132) \quad (q\det(H_1))^2 = (q\det(H_2))^2 = (\det(K))^2 \det(H) = (\det(K))^2 (q\det(H))^2.\]

Hence equation (131), which is an equality between polynomials in the entries of the matrices involved, must hold up to a sign. Setting \(K = I_n\) we see that the first equality in (131) holds, and setting \(K = -I_n\), so \(H_2 = -H\), the validity of the second equality in (131) is equivalent to the easy fact that \(q\det(-H) = (-1)^n q\det H = \det(-I_n) q\det H\).
Proceeding to the proof of lemma 2, we first observe that, after the change of variables $u \rightarrow x$, the density $P(x_1, \ldots, x_n)$ transforms into the density

$$P(u_1, \ldots, u_n) = P(X(u_1), \ldots, X(u_n))\prod_{j=1}^{n}|X'(u_j)|.$$  

We apply lemma 3 with $H = (K(X(u_j), X(u_k)))_{n \times n}$ and $k_j = |X'(u_j)|$ to conclude that (124) holds with either of the two kernels (we write $X(u,v)$ for $(X(u), X(v))$)

$$K_{\pm}(u,v) = \begin{pmatrix}
S(X(u,v))|X'(v)| & \pm (I - \delta \epsilon)(X(u,v)) \\
\pm D(X(u,v))|X'(u)||X'(v)| & S^T(X(u,v))|X'(u)|
\end{pmatrix}. $$

The plus and minus signs correspond to applying the first and second of the equalities in (131), respectively. If $x \rightarrow u$ preserves orientation, then we observe that $\epsilon(X(u) - X(v)) = \epsilon(u - v)$ and conclude by a simple application of the chain rule and a change of variables in the integral that the kernel $K_+$ coincides with $K$ from (125) for the choices (126)–(129). If $x \rightarrow u$ reverses orientation, we choose the minus signs, observe that $\epsilon(X(u) - X(v)) = -\epsilon(u - v)$ and proceed exactly as before to see that $K_-$ coincides with (125) in this case.

Lemma 2 explains the relations (78) between the entries of the limiting kernels $\bar{K}_\beta$ and also of $\hat{K}_\beta$ ($\beta = 1, 4$). The relations certainly hold when $R$ is finite after applying the change of variables (141) to the the matrix kernel $K_{R\beta}$ so as to obtain another kernel $K_{R\beta}$. They can be shown to continue to hold in the limit either by noting that the sequence of scalar kernels $\{S_{R\beta}(\xi, \eta)\}_{R=0}^{\infty}$ is a normal sequence of analytic functions (i.e., it converges uniformly on compacta), or by direct verification that each of the sequences $\{S_{R\beta}\}, \{I_{R\beta}\}, \{K_{R\beta}\}, \{S_{R\beta}^T\}$ converges to the correct limit as $R \rightarrow \infty$. In what follows we will only consider the limit of the quantity $S_{R\beta}$ which alone determines the matrix kernel $K_{R\beta}$.

Let $A = 2a + 1, B = 2b + 1$, where $a, b$ are the parameters of the orthogonal Jacobi ensemble. Assume also that $R$ is even. Observe that $A, B > -1$ if $a, b > -1$. The summation formula of Adler et al [1] expresses the orthogonal kernel $S_{R1}^{(a,b)}$ using the unitary kernel $K_{R_\beta}^{(A,B)}$ and another term. As we shall see, this other term is negligible in the localized limit (in the bulk of the spectrum), but it does contribute to the edge limit.
The summation formula for the quantity $S_{R_1}^{(a,b)}(x,y)$ of (60) is as follows [1]:

$$S_{R_1}^{(a,b)}(x,y) = \sqrt{\frac{1-x^2}{1-y^2}}K_{R-1,2}^{(A,B)}(x,y) + c_{R-2}\psi_{R-1}(y)\epsilon\psi_{R-2}(x).$$

Here $\epsilon$ denotes the integral operator (cf., eq. (61))

$$\epsilon f(x) = \int_{-1}^{1} \epsilon(x-y)f(y)dy,$$

and we have set

$$\psi_N(t) = \psi_N^{(A,B)}(t) = (1-t)^{(A-1)/2}(1+t)^{(B-1)/2}P_N^{(A,B)}(t)$$

and

$$c_N = 2^{-A-B-1}\frac{\Gamma(N+2)\Gamma(N+A+B+2)}{\Gamma(N+A+1)\Gamma(N+B+1)}.$$

The quantity $S_{R_1}^{(a,b)}$ determines the entries of the matrix kernel $K_{R_1}^{(a,b)}$ as per equations (62)–(64).

From Stirling’s formula (99), the asymptotic behavior of the coefficient $c_N$ is

$$c_N \sim 2^{-A-B-1}N^2, \quad \text{as } N \to \infty.$$

**Lemma 4.** For any real $A, B$: 

$$\lim_{N \to \infty} \psi_N^{(A,B)}(\cos \phi) = 0$$

for $0 < \phi < \pi$, uniformly on compacta.

This follows immediately from Darboux’s formula (93).

This lemma is, however, insufficient to understand the asymptotics of the function $\epsilon\psi_N$ as $N \to \infty$ since it says nothing about the behavior of $\psi_N$ near the edge. First we note:

**Lemma 5.** For $A > -1$ and $B$ arbitrary:

$$\lim_{N \to \infty} N^{-1}\psi_N^{(A,B)}(\cos(\phi/N)) = 2^{\frac{A+B}{2}}J_A(\phi),$$

$$\lim_{N \to \infty} \psi_N^{(A,B)}(\cos(\phi/N)) \sin(\phi/N) = 2^{\frac{A+B}{2}}J_A(\phi).$$

The limits hold uniformly on compact subsets of $(0, \infty)$.

These follow from Szegö’s formula (95).
Lemma 6. For $A, B$ real with $A > -1$ and any $0 < \theta < \pi$ we have:

\[
\lim_{N \to \infty} N \int_0^\theta \psi_N^{(A, B)}(\cos \phi) \sin \phi \, d\phi = 2^\frac{A+B}{2},
\]

\[
\lim_{N \to \infty} N \int_0^{\theta/N} \psi_N^{(A, B)}(\cos \phi) \sin \phi \, d\phi = 2^\frac{A+B}{2} \int_0^\theta J_A.
\]

These follow again from Szegő’s formula (95) and equation (108). When $-1 < A < 0$, the dependence on $\theta$ of the second of the error terms in (96) is critical to ensure that the contribution of this error term to the integral is negligible (in particular, this lemma cannot be proven using the alternate formula (98) unless $A > 0$.)

Corollary 1. For $-1 < A, B$ and $0 < \theta < \pi$:

\[
\lim_{N \to \infty} N (\varepsilon \psi_N^{(A, B)})(\cos \theta) = 0,
\]

\[
\lim_{N \to \infty} N (\varepsilon \psi_N^{(A, B)})(\cos(\theta/N)) = 2^{\frac{A+B}{2}} \left(1 - \int_0^\theta J_A\right) = 2^{\frac{A+B}{2}} \int_\theta^\infty J_A.
\]

This follows from the previous lemma applied to both $\psi_N^{(A, B)}$ and $\psi_N^{(B, A)}$. We also used (108) to obtain the last equality.

We localize at some $z_o = \cos \alpha_o \in (-1, 1)$ using the change of variable $x \to \xi$ of (71). The limit to consider is

\[
\bar{S}_{1}^{(a, b)}(\xi, \eta) = \lim_{R \to \infty} (N \rho(y))^{-1} \bar{S}_{R1}^{(a, b)}(x, y) = \lim_{R \to \infty} (N \rho(z_o))^{-1} \bar{S}_{R1}^{(a, b)}(x, y)
\]

By the lemmas above, the second term on the right-hand side of (135) is negligible in the limit. Also, the factor $\sqrt{\frac{1-x^2}{1-y^2}}$ is 1 in the limit. Thus, the limit (137) is equal to the limiting unitary kernel, namely the Sine Kernel, whence the expression (78).

As for the central point, let us now localize at $z = +1$. Using the summation formula (135), lemma 5 and corollary 1, we readily find:

\[
\hat{S}_1^o(\xi, \eta) = \sqrt{\frac{\xi}{\eta}} \hat{K}_2^{(2a+1)}(\xi, \eta) + \frac{\pi}{2} J_{2a+1}(\pi \eta) \left[1 - \int_0^{\pi \xi} J_{2a+1}(t) \, dt\right]
\]

\[
= \sqrt{\frac{\xi}{\eta}} \hat{K}_2^{(2a+1)}(\xi, \eta) + \frac{\pi}{2} J_{2a+1}(\pi \eta) \int_{\pi \xi}^\infty J_{2a+1}(t) \, dt.
\]
As we remarked already, the conditions $a > -1$ and $A > -1$ are equivalent since $A = 2a + 1$. Thus we have derived a weak universality law for the local correlations at the central points ±1 for any $a, b > -1$.

**Lemma 7.** Let $\kappa_\alpha(x, y) = x J_{\alpha+1/2}(x) J_{\alpha-1/2}(y) - J_{\alpha-1/2}(x) y J_{\alpha+1/2}(y)$. Then

\[
\sqrt{\frac{x}{y}} \kappa_{\alpha+1/2}(x, y) - \sqrt{\frac{y}{x}} \kappa_{\alpha+1/2}(x, y) = \mp \left( \frac{x^2 - y^2}{\sqrt{xy}} \right) J_{\alpha-1/2}(x) J_{\alpha-1/2}(y). \tag{149}
\]

(This equation stands for two different equations, one with the top signs and another with the bottom signs.)

We prove the equation with the choice of the top signs (the other case is analogous). Indeed, expanding the left-hand side we obtain:

\[
x^{3/2} J_{\alpha+1}(x) y^{-1/2} J_{\alpha}(y) - x^{1/2} J_{\alpha}(x) y^{1/2} J_{\alpha+1}(y)
- x^{1/2} J_{\alpha}(x) y^{1/2} J_{\alpha-1}(y) + x^{-1/2} J_{\alpha-1}(x) y^{3/2} J_{\alpha}(y).
\]

The central terms can be combined into $-2\alpha x^{1/2} J_{\alpha}(x) y^{-1/2} J_{\alpha}(y)$ using the identity \(104\) and expanded using this same identity into $-x^{3/2} J_{\alpha-1}(x) y^{-1/2} J_{\alpha}(y) - x^{3/2} J_{\alpha+1}(x) y^{-1/2} J_{\alpha}(y)$. Two terms cancel out, and the remaining two factor to give the right-hand side of \(149\).

We now have, using lemma \(14\)

\[
\sqrt{\frac{\xi}{\eta}} \tilde{K}_2^{(A)}(\xi, \eta) = \frac{\sqrt{\xi \eta}}{\xi^2 - \eta^2} \kappa_{A+1/2}(\pi \xi, \pi \eta)
= \frac{\xi}{\xi^2 - \eta^2} \kappa_{A-1/2}(\pi \xi, \pi \eta) + \pi J_{A}(\pi \xi) J_{A-1}(\pi \eta)
= \sqrt{\frac{\eta}{\xi}} \tilde{K}_2^{(A-1=2a)}(\xi, \eta) - \pi J_{A-1}(\pi \xi) J_{A}(\pi \eta),
\]

and similarly

\[
\sqrt{\frac{\xi}{\eta}} \tilde{K}_2^{(A)}(\xi, \eta) = \sqrt{\frac{\eta}{\xi}} \tilde{K}_2^{(A+1)}(\xi, \eta) + \pi J_{A+1}(\pi \xi) J_{A}(\pi \eta).
\]

From \(107\):

\[
\left( \int_{0}^{\pi \xi} J_{A} \right) \pm 2 J_{A+1}(\pi \xi) = \int_{0}^{\pi \xi} J_{A+2}.
\]

\[
\int_{0}^{\pi \xi} J_{A} \pm 2 J_{A+1}(\pi \xi) = \int_{0}^{\pi \xi} J_{A+2}.
\]
The last two equations provide alternative forms of the kernel $\hat{S}_1^{(a)}$, namely

\begin{align}
\hat{S}_1^{(a)}(\xi, \eta) &= \sqrt{\eta} \hat{K}_2^{(2a)}(\xi, \eta) + \frac{\pi}{2} J_{2a-1}(\pi \eta) \left[ 1 - \int_0^{\pi \xi} J_{2a-1}(t) dt \right] \\
\hat{S}_1^{(a)}(\xi, \eta) &= \sqrt{\eta} \hat{K}_2^{(2a+2)}(\xi, \eta) + \frac{\pi}{2} J_{2a+1}(\pi \eta) \left[ 1 - \int_0^{\pi \xi} J_{2a+3}(t) dt \right]
\end{align}

(154)

(155)

As before, the terms in brackets can be replaced by $\left[ \int_0^{\pi} \right]$. 

5.4. Asymptotics of the Symplectic Jacobi Kernel. Here we set $A = a - 1, B = b - 1$ where $a, b$ are the parameters of the symplectic Jacobi ensemble. Note that here $a, b > -1$ corresponds to $A, B > -2$. With $c_N$ as in (138) and $\psi_N = \psi^{(A,B)}_N$ as in (137), the summation formula in this case reads

\begin{align}
S_{R4}^{(a,b)}(x, y) &= \frac{1}{2} \sqrt{\frac{1 - x^2}{1 - y^2}} K_{2R,2}^{(A,B)}(x, y) - \frac{1}{2} c_{2R-1} \psi_{2R}(y) \delta \psi_{2R-1}(x),
\end{align}

(156)

where the operator $\delta$ acts by

\begin{align}
\delta f(x) &= \int_x^1 f(t) dt.
\end{align}

(157)

The formula (156) only holds verbatim when $a > 0$ (that is, $A, B > -1$), since the integral defining $\delta \psi^{(A,B)}_N$ is divergent for $A \leq -1$. However, we note that the skew orthogonal polynomials of the second kind are analytic functions of the parameters $a, b > -1$ (corresponding to $A, B > -2$), hence the kernel $K_{N4}$ is an analytic function on $a, b > -1$. Thus, we must find a suitable analytic continuation of (156) valid for $A, B > -2$. First we remark that, although the original kernel $K_{2R,2}^{(A,B)}$ of unitary Jacobi ensembles is defined for $A, B > -1$, equation (109) is well-defined and analytic for $A, B > -2$ if $R > 1$ (which we will
We write

$$\delta\psi_{N}^{(A,B)}(x) = \int_{x}^{1} (1 - t)^{(A-1)/2}(1 + t)^{(B-1)/2} P_{N}^{(A,B)}(t) dt$$



$$= \int_{x}^{1} (1 - t)^{(A-1)/2}(1 + t)^{(B-1)/2}(P_{N}^{(A,B)}(t) - P_{N}^{(A,B)}(1)) dt$$



$$+ P_{N}^{(A,B)}(1) \int_{x}^{1} (1 - t)^{(A-1)/2}(1 + t)^{(B-1)/2} dt.$$  

The first integral on the right-hand side is well-defined and analytic for $A > -2$. The term $P_{N}^{(A,B)}(1) = \binom{A+N}{N}$ (cf., equation (91)) vanishes for $A = -1$, which is sufficient to extend the second integral on the right-hand side to a well-defined analytic function on the range $A > -2$. It is easy to rewrite that integral as an incomplete Beta function and use well-known results to achieve the extension, but one can also proceed elementarily as follows. Integrating the second integral by parts we obtain, for $A > -1$:

$$\binom{A+N}{N} \int_{x}^{1} (1 - t)^{(A-1)/2}(1 + t)^{(B-1)/2} dt = \frac{2}{A+1} \binom{A+N}{N}(1 - x)^{(A+1)/2}(1 + x)^{(B-1)/2}$$



$$+ \frac{B-1}{A+1} \binom{A+N}{N} \int_{x}^{1} (1 - t)^{(A+1)/2}(1 + t)^{(B-3)/2} dt.$$  

Observe that

$$\frac{1}{A+1} \binom{A+N}{N} = \frac{1}{N} \binom{A+N}{N-1},$$

and the latter is an analytic function of all $A$. Then both terms on the right-hand side of (159) are analytic functions of $A > -2$ for $-1 < x \leq 1$, so this last equation provides the analytic extension of the integral (158) defining $\delta\psi_{N}(x)$, which is sensu stricti undefined for $A \leq -1$, to an analytic function on $A > -2$.

The rest of the reasoning is analogous to that in the orthogonal case. The only technical difficulty arises because the error term in Szegö’s formula does not depend on $\theta$ in the range $0 < \theta \leq c/N$, effectively making the reasoning of the previous section inapplicable when $-2 < A \leq -1$. This is to be expected since the summation formula only makes sense after being analytically continued. In what follows we prove that the various limits of the kernel do in fact depend
analytically on the parameter $A$, thus allowing the expressions obtained for $A > -1$ to be extended to $A > -2$.

Using Szego’s formula (97) (valid for all $A$), there is no problem to obtain this variant of lemma 6:

**Lemma 8.** For any $A, B, \theta$ real and $0 < \psi < \pi$ we have:

\[
\lim_{N \to \infty} N \int_{\theta/N}^{\phi} \psi_N^{(A,B)}(\cos \psi) \sin \psi d\psi = 2^{A+B} \int_{\theta}^{\infty} J_A.
\]

**Lemma 9.** Using equation (159), the expression

\[
N \int_{0}^{\theta/N} \psi_N^{(A,B)}(\cos \phi) \sin \phi d\phi
\]

can be analytically continued to a regular function on $A > -2$. As $N \to \infty$, this function tends to a limit which is also analytic for $A > -2$ and coincides with (144) for $A > -1$.

We change variables $\phi \to \phi/N$. As before, we split the integral to rewrite (162) in the form

\[
2^{(A+B)/2} \int_{0}^{\theta} \left( \sin \frac{\phi}{2N} \right)^A \left( \cos \frac{\phi}{2N} \right)^B \left[ P_N^{(A,B)} \left( \cos \frac{\phi}{N} \right) - P_N^{(A,B)}(1) \right] d\phi
\]

\[
+ 2^{(A+B)/2} P_N^{(A,B)}(1) \int_{0}^{\theta} \left( \sin \frac{\phi}{2N} \right)^A \left( \cos \frac{\phi}{2N} \right)^B P_N^{(A,B)}(1) d\phi
\]

The first of these terms is analytic for $A > -2$, the second one has an analytic continuation given by (159). It is easy to see that this second term has the asymptotic behavior:

\[
2^{(A+B)/2} \int_{0}^{\theta} \left( \sin \frac{\phi}{2N} \right)^A \left( \cos \frac{\phi}{2N} \right)^B P_N^{(A,B)}(1) d\phi
\]

\[
\sim 2^{\frac{A+B}{2}} \frac{1}{N} \left( \frac{A + N}{N - 1} \right) \left( \frac{\phi}{N} \right)^{A+1}
\]
as $N \to \infty$, and from Stirling’s formula (99), the binomial coefficient \( \binom{A+N}{N} = \frac{\Gamma(A+N+1)}{\Gamma(N+1)\Gamma(A+2)} = O(N^{A+1}) \), hence this second terms is asymptotically negligible. As for the first term in (163), we first write

\[
(165) \quad P_N^{(A,B)} \left( \cos \frac{\phi}{N} \right) - P_N^{(A,B)}(1) = -\frac{1}{N} \int_0^\phi P_N^{(A,B)' \left( \cos \frac{\psi}{N} \right) \sin \frac{\psi}{N} d\psi}
\]

where we have used the derivation formula (92). We can now use Szegő’s formula (95) to estimate $P_N^{(A+1,B+1)}$ since $A + 1 > -1$. The upshot is that the limit of (162) as $N \to \infty$ can be written as the following integral, which is an analytic function of $A > -2$:

\[
(166) \quad -2^{(A+B)/2} \int_0^\theta \int_0^\phi \phi^A \psi^{-A} J_{A+1}(\psi) d\psi d\phi.
\]

Using the Bessel function identity (106) we can simplify the above integral, for $A > -1$:

\[
(167) \quad 2^{(A+B)/2} \int_0^\theta J_A(\phi) d\phi,
\]

which is in agreement with lemma 6.

We note that the expression (167) can be easily continued to an analytic function of $A > -2$ without the need to rewrite it as the double integral (166). Namely, using (107) we have, for $A > -1,$

\[
(168) \quad \int_0^\theta J_A(\phi) d\phi = J_{A+1}(\theta) + \int_0^\theta J_{A+2}(\phi) d\phi.
\]

The expression on the right-hand side is analytic for $A > -2$ and provides the desired analytic continuation.

The global level density is derived identically to the previous section. The limiting kernel in the bulk of the spectrum is given by the sum of two terms: $\hat{S}_2^{(a)}(2\xi, 2\eta)$ and another term which is negligible in the limit. For the central point $z = +1$, the lemmas above yield the following expression for the limiting kernel:

\[
(169) \quad \hat{S}_4^{(a)}(\xi, \eta) = \sqrt{\frac{\xi}{\eta}} \hat{K}_2^{(A)}(2\xi, 2\eta) - \frac{\pi}{2} J_A(2\pi\eta) \int_0^{2\pi\xi} J_A(t) dt,
\]

where the last integral is to be understood in the sense of equation (168) for $A \leq -1$. Using equations (151) and (152) together
with (153) and the equation above, the kernel can be rewritten in either of the forms:

\[ \hat{S}_4^{(a)}(\xi, \eta) = \sqrt{\frac{\eta}{\xi}} K_2^{(a)}(2\xi, 2\eta) - \frac{\pi}{2} J_{a-1}(2\pi \eta) \int_0^{2\pi \xi} J_{a+1}(t) dt, \]

(170) \[ \hat{S}_4^{(a)}(\xi, \eta) = \sqrt{\frac{\eta}{\xi}} K_2^{(a-2)}(2\xi, 2\eta) - \frac{\pi}{2} J_{a-1}(2\pi \eta) \int_0^{2\pi \xi} J_{a-3}(t) dt. \]

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