Asymptotic of Non-Crossings probability of Additive Wiener Fields

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Abstract: Let $W_i = \{W_i(t_i), t_i \in \mathbb{R}_+ \}, i = 1, 2, \ldots, d$ be independent Wiener processes. $W = \{W(t), t \in \mathbb{R}^d_+ \}$ be the additive Wiener field define as the sum of $W_i$. For any trend $f$ in $\mathcal{H}$ (the reproducing kernel Hilbert Space of $W$), we derive upper and lower bounds for the boundary non-crossing probability

$$P_f = P\{ \sum_{i=1}^{d} W_i(t_i) + f(t) \leq u(t), t \in \mathbb{R}^d_+ \},$$

where $u : \mathbb{R}^d_+ \to \mathbb{R}_+$ is a measurable function. Furthermore, for large trend functions $\gamma f > 0$, we show that the asymptotically relation $\ln P_{\gamma f} \sim \ln P_{\gamma f}$ as $\gamma \to \infty$, where $\underline{f}$ is the projection of $f$ on some closed convex subset of $\mathcal{H}$.

Key words: Boundary non-crossing probability; reproducing kernel Hilbert space; additive Wiener field; asymptotics probability.

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1 Introduction

For $d$ be a positive integer, let $X_i = \{X_i(t), t \in \mathbb{R}_+ \}, i = 1, 2, \ldots, d$ be independent real-valued stochastic processes on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define the $d$-parameters real-valued additive field (additive process)

$$X(t) = X(t_1, t_2, \ldots, t_d) = \sum_{i=1}^{d} X_i(t_i), \quad t = (t_1, t_2, \ldots, t_d) \in \mathbb{R}^d_+.$$ 

The additive process which plays a key role in studying of the general multiparameter processes, multiparameter potential theory, fractal geometry, spectral asymptotic theory has been actively investigated recently. To have a glance of these results, we refer the reader to [1, 2, 3, 4, 5, 6, 7] and the references therein.

On the other hand, calculation of boundary non-crossing probabilities of Gaussian processes is a key topic both of theoretical and applied probability, see, e.g., [8, 9, 10, 11, 12, 13, 14]. Numerous applications concerned with the evaluation of boundary non-crossing probabilities relate to mathematical finance, risk theory, queuing theory, statistics, physics among many other fields. In the literature, most of contributions are only concentrate on the boundary non-crossing probabilities of Gaussian processes with one-parameter (e.g. Brownian motion, Brownian bridge and fractional Brownian motion), some important results of this field can see in [15, 16, 17, 18, 19, 20, 21, 22]. For multiparameter Gaussian processes, few cases are known about the boundary non-crossing
probabilities (see, e.g., [23, 24, 25]).

In this paper, we are concentrating on the calculation of boundary non-crossing probabilities of additive Wiener field $W$ which defined by

$$W(t) = W_1(t_1) + W_2(t_2) + \ldots + W_d(t_d), \quad t \in \mathbb{R}_+^d;$$

(1)

where $W_i = \{W_i(t), t \in \mathbb{R}_+\}, i = 1, 2, \ldots, d$ are independent Wiener processes define on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. It can be checked easily that $W$ is a Gaussian field with the convariance function given by

$$\mathbb{E}\{W(s)W(t)\} = \sum_{i=1}^{d} s_i \wedge t_i, \quad s = (s_1, s_2, \ldots, s_d), \quad t = (t_1, t_2, \ldots, t_d).$$

(2)

For two measurable functions $f, u : \mathbb{R}_+^d \to \mathbb{R}$ we shall investigate the upper and lower bounds for

$$P_f = \mathbb{P}\{W(t) + f(t) \leq u(t), \quad t \in \mathbb{R}_+^d\}$$

In the following, we consider $u$ a general measurable function and $f \neq 0$ to belong to the reproducing kernel Hilbert space (RKHS) of $W$ which is denote by $\mathcal{H}$. A precise description of $\mathcal{H}$ is given in section 2, where the inner product $(f, g)$ and the corresponding norm $\|f\|$ for $f, g \in \mathcal{H}$ are also defined.

As in [20], a direct application of Theorem 1' in [26] shows that for any $f \in \mathcal{H}$ we have

$$\left|P_f - P_0\right| \leq \frac{1}{\sqrt{2\pi}} \|f\|.$$  

(3)

Further, for any $g \in \mathcal{H}$ such that $g \geq f$, we obtain

$$\Phi(\alpha - \|g\|) \leq P_g \leq P_f \leq \Phi(\alpha + \|f\|),$$

(4)

where $\Phi$ is the distribution of an $N(0, 1)$ random variable and $\alpha = \Phi^{-1}(P_0)$ is a finite constant. When $f \leq 0$, then we can take always $g = 0$ above which make the lower bound of (4) useful if $\|f\|$ is large. When $\|f\|$ is small, the equation (3) provides a good bound for the approximation rate of $P_f$ by $P_0$. Since the explicit formulas for computing $p_f$ seem to be impossible, the asymptotic performance of the bounds for trend functions $\gamma f$ with $\gamma \to \infty$ and $\gamma \to 0$ are thus worthy of consideration. This paper we shall consider the former case, and we obtain the following:

If $f(t_0) > 0$ for some $t_0$ with non-negative components, then for any $g \geq f, g \in \mathcal{H}$ we have

$$\ln P_{\gamma f} \geq \ln \Phi(\alpha - \gamma f) \geq -(1 + o(1)) \frac{\gamma^2}{2} \|g\|^2, \quad \gamma \to \infty,$$

(5)

hence

$$\ln P_{\gamma f} \geq -(1 + o(1)) \frac{\gamma^2}{2} \|f\|^2, \quad \gamma \to \infty,$$

(6)

where $f$ (which is unique and exists) solves the following minimization problem

$$\min_{g, f \in \mathcal{H}, g \geq f} \|g\| = \|f\| > 0.$$  

(7)
In Section 2 we shall show that \( f \) is the projection of \( f \) on a closed convex set of \( \mathcal{H} \), and moreover we show that
\[
\ln P_{\gamma f} \sim \ln P_{\gamma f} \sim -\frac{\gamma^2}{2} \| f \|^2, \quad \gamma \to \infty.
\] (8)

The rest of this paper are organized as follows: In section 2 we briefly talk about the RKHS of additive Wiener field and construct the solution of the minimization problem (7). We present our main results in Section 3. The proofs of the results in this paper are shown in Section 4, and we conclude this paper by Appendix.

## 2 Preliminaries

This section reviews basic results of the reproducing kernel Hilbert space (RKHS), and we shall give a representation of the RKHS of additive Wiener field \( W \). We shall also construct \( V \) as a closed convex set of \( \mathcal{H} \), which finally enable us to prove that \( f \) in (7) is the projection of \( f \) on \( V \). The idea of constructing \( V \) comes from a similar result in one-parameter case (see e.g., [19, 14, 23, 18]).

In the following of this paper bold letters are reserved for vectors, so we shall write for instance \( t = (t_1, t_2, \ldots, t_d) \in \mathbb{R}^d_+ \) and \( \lambda_1 \) denote the Lebesgue measures on \( \mathbb{R}_+ \), whereas \( ds \) a mean integration with respect to this measure.

### 2.1 The RKHS of additive Wiener field

Recall that \( W_1 \) is an one-parameter Wiener process. It is well-known (see e.g., [27]) that the RKHS of the Wiener process \( W_1 \), denoted by \( \mathcal{H}_1 \), is characterized as follows
\[
\mathcal{H}_1 = \left\{ h : \mathbb{R}_+ \to \mathbb{R} \mid h(t) = \int_{[0,t]} h'(s)ds, \quad h' \in L_2(\mathbb{R}_+, \lambda_1) \right\},
\]
with the inner product \( \langle h, g \rangle_1 = \int_{\mathbb{R}_+} h'(s)g'(s)ds \) and the corresponding norm \( \| h \|_1^2 = \langle h, h \rangle \). The description of RKHS for \( W_i, i = 2, 3, \ldots, d \) are evidently the same. We now begin to construct the RKHS of additive Wiener field \( W \), for any
\[
\begin{align*}
    h_1(t) &= f_1(t_1) + f_2(t_2) + \ldots + f_d(t_d), \\
    h_2(t) &= g_1(t_1) + g_2(t_2) + \ldots + g_d(t_d),
\end{align*}
\]
where \( f_i(t_i), g_i(t_i) \in \mathcal{H}_1, i = 1, 2, \ldots, d \), define the inner product
\[
\langle h_1, h_2 \rangle = \sum_{i=1}^d \int_{\mathbb{R}_+} f_i'(s)g_i'(s)ds. \quad (9)
\]

**Remark 2.1.** From lemma 5.1 in Appendix we have the representation \( h(t) = h_1(t_1) + h_2(t_2) + \ldots + h_d(t_d) \) is unique, hence the above inner product is well defined.

Next, in view of lemma 5.2 in Appendix we have the following
Lemma 2.1. The RKHS for additive Wiener field $W$ is given by

$$\mathcal{H} = \left\{ h : \mathbb{R}^d_+ \to \mathbb{R} \mid h(t) = \sum_{i=1}^{d} h_i(t_i), \text{ where } h_i \in H_1, i = 1, 2, \ldots, d \right\}$$

(10)
equipped with the norm $\|h\|^2 = \langle h, h \rangle$.

For notational simplicity in the following we shall use the same notation $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ to present the inner product and norm respectively, on space $H_1$ and $\mathcal{H}$.

2.2 The solution of minimization problem

In this subsection, we begin to solve equation (7). For any $h \in H_1$, it has been shown (see [18]), that the smallest concave majorant of $h$ solves

$$\min_{g, f \in H_1, g \geq f} \|g\| = \|f\| > 0.$$ 

Moreover, as shown in [14] the smallest concave majorant of $h$, which we denote by $\underline{h}$, can be written analytically as the unique projection of $h$ on the closed convex set

$$V_1 = \{ h \in H_1 \mid h'(s) \text{ is a non-increasing function} \}$$
i.e., $\underline{h} = Pr_{V_1} h$. Here we write $Pr_A h$ for the projection of $h$ on some closed set $A$ also for other Hilbert spaces considered below. Further, if we define

$$\overline{V}_1 = \{ h \in H_1 \mid \langle h, f \rangle \leq 0 \text{ for any } f \in V_1 \}$$

be the polar cone of $V_1$. Then the following hold

Lemma 2.2. [24] With the above notation and definitions we have

(i) If $h \in V_1$, then $h \geq 0$.

(ii) If $h \in \overline{V}_1$, then $h \leq 0$.

(iii) We have $\langle Pr_{V_1} h, Pr_{\overline{V}_1} h \rangle = 0$ and further

$$h = Pr_{V_1} h + Pr_{\overline{V}_1} h.$$ 

(11)

(iv) If $h = h_1 + h_2$, $h_1 \in V_1$, $h_2 \in \overline{V}_1$ and $\langle h_1, h_2 \rangle = 0$, then $h_1 = Pr_{V_1} h$ and $h_2 = Pr_{\overline{V}_1} h$.

(v) The unique solution of the minimization problem $\min_{g \geq h, g \in H_1} \|g\|$ is $\underline{h} = Pr_{V_1} h$.

Since we are going to work with functions $f$ in $\mathcal{H}$ we need to consider the projection of such $f$ on a particular closed convex set. In the following we shall write $f = f_1 + f_2 + \ldots + f_d$ meaning that $f(t) = f_1(t_1) + f_2(t_2) + \ldots + f_d(t_d)$.
\[ f(t) = \sum_{i=1}^{d} f_i(t_i), \quad f_i(t_i) \in \mathcal{H}_1, \quad i = 1, 2, \ldots, d \]

where \( f_1, f_2, \ldots, f_d \in \mathcal{H}_1 \). Note in passing that this decomposition is unique for any \( f \in \mathcal{H} \). Define the closed convex set

\[ V_2 = \{ h = h_1 + h_2 + \ldots + h_d \in \mathcal{H} | h_1, h_2, \ldots, h_d \in V_1 \} \]

and let \( \overline{V}_2 \) be the polar cone of \( V_2 \) given by

\[ \overline{V}_2 = \{ h \in \mathcal{H} | \langle h, v \rangle \leq 0 \text{ for any } v \in V_2 \} \]

with inner product from (9). Analogous to Lemma 2.2 we have

Lemma 2.3. For any \( h = h_1 + h_2 + \ldots + h_d \in \mathcal{H} \), we have

(i) If \( h \in V_2 \), then \( h_i \geq 0, \quad i = 1, 2, \ldots, d \).

(ii) If \( h \in \overline{V}_2 \), then \( h_i \leq 0, \quad i = 1, 2, \ldots, d \).

(iii) We have \( \langle \text{Pr}_{V_2} h, \text{Pr}_{\overline{V}_2} h \rangle = 0 \) and further

\[ h = \text{Pr}_{V_2} h + \text{Pr}_{\overline{V}_2} h. \] (12)

(iv) If \( h = h_1 + h_2, \quad h_1 \in V_2, \quad h_2 \in \overline{V}_2 \) and \( \langle h_1, h_2 \rangle = 0 \), then \( h_1 = \text{Pr}_{V_2} h \) and \( h_2 = \text{Pr}_{\overline{V}_2} h \).

(v) The unique solution of the minimization problem \( \min_{g \geq h, g \in \mathcal{H}} \| g \| \) is

\[ h = \text{Pr}_{V_2} h = \text{Pr}_{V_1} h_1 + \text{Pr}_{V_1} h_2 + \ldots + \text{Pr}_{V_1} h_d. \] (13)

3 Main Result

Consider two measurable d-parameter functions \( f, u : \mathbb{R}_+^d \to \mathbb{R} \). Suppose that \( f(0) = 0 \) and \( f \in \mathcal{H} \). Hence we can write

\[ f(t) = \sum_{i=1}^{d} f_i(t_i), \quad f_i(t_i) \in \mathcal{H}_1, \quad i = 1, 2, \ldots, d \]

we also suppose \( f_i(0) = 0, \quad i = 1, 2, \ldots, d \) in the above decomposition. Recall their representations \( f_i(t_i) = \int_{[0,t_i]} f_i^*(s)ds, \quad f_i^* \in L_2(\mathbb{R}_+, \lambda_1), \quad i = 1, 2, \ldots, d \). We shall estimate the boundary non-crossing probability

\[ P_f = \mathbb{P}\{ W(t) + f(t) \leq u(t), \quad t \in \mathbb{R}_+^d \}. \]

In the following we set \( \underline{f}_i = \text{Pr}_{V_1} f_i, \quad i = 1, 2, \ldots, d \) and \( \underline{f} = \text{Pr}_{V_2} f \). We state next our main result:

Theorem 3.1. Let the following conditions hold:

\[ \lim_{t_i \to \infty} u(0, \ldots, t_i, 0, \ldots, 0)f_i'(t_i) = 0, \quad i = 1, 2, \ldots, d. \] (14)

Then we have

\[ P_f \leq P_{\underline{f} - \underline{f}} \exp \left( -\sum_{i=1}^{d} \int_{\mathbb{R}_+} u(0, \ldots, t_i, 0, \ldots, 0)df_i'(t_i) - \frac{1}{2} \| f \|^2 \right). \]
Remark 3.1. Note that \( f \) starts from zero therefore \( f \) can not be a constant unless \( f \equiv 0 \) but this case is trivial.

Remark 3.2. Conditions (14) of the theorem means that asymptotically the components of shifts and their derivatives are negligible in comparison with function \( u \).

Using Theorem 3.1, we can obtain an asymptotically property of \( P_{\gamma f} \), in fact, if \( u(t) \) is bounded above, then we have the following result

**Corollary 3.1.** If \( f \in \mathcal{H} \) is such that \( f(t_0) \) for some \( t_0 \), then

\[
\ln P_{\gamma f} \sim \ln \| f \|^2, \quad \gamma \to \infty.
\]

4 Proofs

**Proof of Lemma 2.2:** For \( h \in V_1 \), we have \( h' \) is non-increasing therefore \( h' \) is non-negative. Since \( h(0) = 0 \), thus \( h(u) \geq 0 \) for all \( u \). The proof of statements (ii) to (v) can see in [24], we do not repeat the proof here. □

**Proof of Lemma 2.3:** (i) If \( h \in V_2 \), from the definition of \( V_2 \), we obtain \( h_1, h_2, \ldots, h_d \in V_1 \). Thus \( h_i \geq 0, i = 1, 2, \ldots, d \) follow directly from (i) in Lemma 2.2

(ii) If \( h(t) = h_1(t_1) + h_2(t_2) + \ldots + h_d(t_d) \in \mathcal{V}_2 \), then \( h_i(t_i) \in \mathcal{H} \). For any \( f_i(t_i) \in V_1 \), let

\[ v(t) = f_i(t_i) \in V_2. \]

From the definition of \( \mathcal{V}_2 \), we obtain

\[ \langle h, v \rangle = \langle h_i, f_i \rangle \leq 0. \]

Therefore, \( h \in \mathcal{V}_1 \), and the results follow from (ii) in lemma 2.2.

The proof of statements (iii) and (iv) are similar to (iii) and (iv) in Lemma 2.2, and can obtain immediately from [14].

(v) For any \( h(t) \in \mathcal{H} \), let \( g(t) \in \mathcal{H} \) such that \( g \geq h \), we then have \( g_i \geq h_i, i = 1, 2, \ldots, d \), where

\[
\begin{align*}
  h &= h_1 + h_2 + \ldots + h_d, \\
  g &= g_1 + g_2 + \ldots + g_d.
\end{align*}
\]

The minimization problem

\[
\min_{g \geq h, g \in \mathcal{H}} \| g \| = \min_{g \geq h, g \in \mathcal{H}} (\| g_1 \| + \| g_2 \| + \ldots + \| g_d \|)
\]

\[
= \sum_{i=1}^{d} \min_{g_i \geq h_i, g_i \in \mathcal{H}_i} \| g_i \|
\]

\[
= \| h_1 \| + \| h_2 \| + \ldots + \| h_d \|.
\]

The equalizes above hold if and only if

\[
h = Pr_{\mathcal{V}_2} h = Pr_{\mathcal{V}_1} h_1 + Pr_{\mathcal{V}_1} h_2 + \ldots + Pr_{\mathcal{V}_1} h_d.
\]
Completing the proof. □

**Proof of Theorem 3.1:** Denote by \( P \) a probability measure that is defined via its Radon-Nikodym derivative

\[
\frac{dP}{dP} = \prod_{i=1}^{d} \exp \left( -\frac{1}{2} \| f_i \|^2 + \int_{\mathbb{R}^+} f_i'(t_i)dW^0_i(t_i) \right).
\]

According to Cameron-Martin-Girsanov theorem, \( W^0_i(t) = W_i(t) + \int_{[0,t]} f_i'(s)ds, \ i = 1, 2, \ldots, d \) are independent Wiener processes. Denote \( 1_u \{ X \} = 1 \{ X(t) \leq u(t), \ t \in \mathbb{R}^d \} \) and

\[
W^0(t) = W^0_1(t_1) + W^0_2(t_2) + \ldots + W^0_d(t_d).
\]

Note that \( \| f \|^2 = \| f_1 \|^2 + \| f_2 \|^2 + \ldots + \| f_d \|^2 \), hence using further (12) and (13) we obtain

\[
P_f
= \mathbb{E} \left\{ 1_u \left( \sum_{i=1}^{d} (W_i(t_i) + f_i(t_i)) \right) \right\}
= \mathbb{E}_P \left( \frac{dP}{dP} 1_u(W^0(t)) \right)
= \exp \left( -\frac{1}{2} \| f \|^2 \right) \mathbb{E} \left\{ \exp \left( \sum_{i=1}^{d} \int_{\mathbb{R}^+} f_i'(t_i)dW^0_i(t_i) \right) 1_u(W^0(t)) \right\}
= \exp \left( -\frac{1}{2} \| f \|^2 \right) \times \mathbb{E} \left\{ \prod_{i=1}^{d} \exp \left( -\frac{1}{2} \| P_{R_i} f_i \|^2 + \int_{\mathbb{R}^+} P_{R_i} f_i'(t_i)dW^0_i(t_i) \right) \times \exp \left( \sum_{i=1}^{d} \int_{\mathbb{R}^+} f_i'(t_i)dW^0_i(t_i) \right) 1_u(W^0(t)) \right\}.
\]

In order to re-write \( \int_{\mathbb{R}^+} f_i'(t_i)dW^0_i(t_1) \), we mention that in this integral \( dW^0_i(t_1) = d_1(W^0(t_1, 0, \ldots, 0)) \), therefore on the indicator \( 1_u \{ \sum_{i=1}^{d} W^0_i(t_i) \} = 1_u \{ W^0(t) \} \) under conditions of the theorem and using lemma 5.3 in the Appendix we have the relations

\[
\int_{\mathbb{R}^+} f_i'(t_1)dW^0_1(t_1) = \lim_{n \to \infty} \int_{[0,n]} f_i'(t_1)dW^0_1(t_1) = \lim_{n \to \infty} \left( f_i'(n)W^0(n, 0, \ldots, 0) + \int_{[0,n]} W^0(t_1, 0, \ldots, 0)d(-f_i')(t_1) \right), \tag{16}
\]

Similarly, for any \( i = 2, 3, \ldots, d \) we have

\[
\int_{\mathbb{R}^+} f_i'(t_i)dW^0_i(t_i) = \lim_{n \to \infty} \left( f_i'(n)W^0(0, \ldots, n, 0, \ldots, 0) + \int_{[0,n]} W^0(0, \ldots, t_i, 0, \ldots, 0)d(-f_i')(t_i) \right). \tag{17}
\]

Combining (16)–(17) and using conditions (14), we get that on the same indicator

\[
\sum_{i=1}^{d} \int_{\mathbb{R}^+} f_i'(t_i)dW^0_i(t_i) \leq \lim_{n \to \infty} \left( \sum_{i=1}^{d} f_i'(n)W^0(0, \ldots, n, 0, \ldots, 0) + \sum_{i=1}^{d} \int_{[0,n]} W^0(0, \ldots, t_i, 0, \ldots, 0)d(-f_i')(t_i) \right)
\leq -\sum_{i=1}^{d} \int_{\mathbb{R}^+} u(0, \ldots, t_i, 0, \ldots, 0)d[f_i'(t_i)]. \tag{18}
\]
On the other hand, we have
\[ P_{f} - f = \mathbb{E}\left\{ \prod_{i=1}^{d} \exp\left(-\frac{1}{2}\|f - f_i\|^{2} + \int_{\mathbb{R}^{+}} (f - f_i)'dW_{i}(t)\right)1_{u}(W^{0}(t)) \right\} \] (19)
\[ = \mathbb{E}\left\{ \prod_{i=1}^{d} \exp\left(-\frac{1}{2}\|Pr_{\tilde{V}_{i}}f_i\|^{2} + \int_{\mathbb{R}^{+}} Pr_{\tilde{V}_{i}}f_i'(t)dW_{i}(t)\right)1_{u}(W^{0}(t)) \right\}. \]

From (18) and (19), we conclude that
\[ P_{f} \leq P_{f} - f \exp\left(-\sum_{i=1}^{d} \int_{\mathbb{R}^{+}} u(0, \ldots, t_i, 0, \ldots, 0)df_i'(t_i) - \frac{1}{2}\|f\|^{2}\right). \]

\[ \square \]

**Proof of Corollary 3.1:** From (5) we obtain
\[ \ln P_{\gamma f} \geq -(1 + o(1)) \inf_{g \geq f} \frac{\gamma^{2}}{2}\|g\|^{2} = -(1 + o(1)) \frac{\gamma^{2}}{2}\|f\|^{2}, \quad \gamma \to \infty. \]

On the other hand, from theorem 3.1 we obtain
\[ P_{\gamma f} \leq P_{\gamma(f - f)} \exp(-(1 + o(1)) \frac{\gamma^{2}}{2}\|f\|^{2}). \]

Since \( f(t_0) > 0 \), then \( \lim_{\gamma \to \infty} P_{\gamma(f - f)} = \text{constant} > 0 \). Hence as \( \gamma \to \infty \),
\[ \ln P_{\gamma f} \leq -(1 + o(1)) \frac{\gamma^{2}}{2}\|f\|^{2}, \]
and the claim follows. \( \square \)

### 5 Appendix

**Lemma 5.1.** If the function \( h : \mathbb{R}_{+}^{d} \to \mathbb{R} \) admits the representation
\[ h(t) = h_1(t_1) + h_2(t_2) + \ldots + h_d(t_d), \] (20)
where \( h_i \in \mathcal{H}_1, i = 1, \ldots, d \), then the representation (20) is unique.

**Proof.** If the function \( h : \mathbb{R}_{+}^{d} \to \mathbb{R} \) admits the representation
\[ h(t) = \sum_{i=1}^{d} f_i(t_i) = \sum_{i=1}^{d} g_i(t_i), \] (21)
where \( f_i, g_i \in \mathcal{H}_1, i = 1, 2, \ldots, d \). For any \( i = 1, 2, \ldots, d \), we put \( t_j = 0 \) for \( j \neq i \), and note that \( f_j(0) = g_j(0) = 0 \), then we obtain \( f_i = g_i, i = 1, 2, \ldots, d \). Hence the representation (20) is unique. \( \square \)

Noting that the covariance function of \( W_i \) is \( s_i \wedge t_i \), and the covariance function of processes \( W(t) = W_1(t_1) + W_2(t_2) + \ldots + W_d(t_d) \) is given by
\[ R(s, t) := \mathbb{E}\{W(s)W(t)\} = \sum_{i=1}^{d} s_i \wedge t_i, \quad s = (s_1, s_2, \ldots, s_d), \quad t = (t_1, t_2, \ldots, t_d). \]
Next, we will identify the RKHS corresponding to a sum of \(d\) covariances. Suppose now \(R_i, i = 1, 2, \ldots, d\) are \(d\) covariances of Gaussian processes, the corresponding RKHS are \(\mathbb{K}_i, i = 1, 2, \ldots, d\). We suppose also \(\| \cdot \|\) the inner product of RKHS \(\mathbb{K}_i, i = 1, 2, \ldots, d\). The following is a well-known lemma and we refer the reader to [28] for its proof.

**Lemma 5.2.** The RKHS of Gaussian processes which with covariances \(R = R_1 + R_2 + \ldots + R_d\) is then given by the Hilbert space \(\mathbb{K}\) consists of all functions \(f((t)) = f_1(t_1) + f_2(t_2) + \ldots + f_d(t_d),\) with \(f_i(t_i) \in \mathbb{K}_i, i = 1, 2, \ldots, d,\) and the norm is given by

\[
\| f \| = \inf(\| f_1 \|_1 + \| f_2 \|_2 + \ldots + \| f_d \|_d),
\]

where the infimum taken for all the decomposition \(f(t) = f_1(t_1) + f_2(t_2) + \ldots + f_d(t_d),\) \(g(t) = g_1(t_1) + g_2(t_2) + \ldots + g_d(t_d)\) with \(f_i(t_i), g_i(t_i) \in \mathbb{K}_i, i = 1, 2, \ldots, d.\) Furthermore, if for any \(f \in \mathbb{K},\) the decomposition \(f(t) = f_1(t_1) + f_2(t_2) + \ldots + f_d(t_d)\) is unique, then the inner product of \(\mathbb{K}\) is

\[
\langle f, g \rangle = \langle f_1, g_1 \rangle + \langle f_2, g_2 \rangle + \ldots + \langle f_d, g_d \rangle.
\]

Also if we define the plus \(\oplus\) among \(\mathbb{K}_i, i = 1, 2, \ldots, d\) by \(\mathbb{K}_i \oplus \mathbb{K}_j := \{ f = f_i + f_j \mid f_i \in \mathbb{K}_i, f_j \in \mathbb{K}_j \},\) then we can rewritten \(\mathbb{K}\)

\[
\mathbb{K} = \mathbb{K}_1 \oplus \mathbb{K}_2 \oplus \ldots \oplus \mathbb{K}_d.
\]

Let \(W_1\) be a Wiener process, \(h : \mathbb{R}_+ \to \mathbb{R}\) be an integrable function, we can extend the integration of \(h\) w.r.t \(W_1\) on \(\mathbb{R}_+\) by the following sense

\[
\int_{\mathbb{R}_+} h(s)dW_1(s) = L_2 - \lim_{n \to \infty} \int_{[0,n]} h(s)dW_1(s)
\]

whenever this limit exists. Furthermore, for any \(h \in V_1,\) the derivative \(h' \in L^2(\mathbb{R}_+, \lambda_1)\) is non-increasing, therefore \(\int_{[0,n]} h'^2(s)ds \leq h'^2(0)n\) which implies that the integral \(\int_{[0,n]} h(s)dW_1(s)\) is correctly defined as Itô integral. We then can construct the integration-by-parts formula

**Lemma 5.3.** Let \(h \in V_1,\) and \(W_1\) be a Wiener process. Then for any \(T < \infty,\) we have the following:

\[
\int_{[0,T]} h(s)dW_1(s) = \int_{[0,T]} W_1(s)d(-h(s)) + h(T)W_1(T),
\]

where the integral in the right-hand side of (23) is a Riemann-Stieltjes integral.

**Proof.** From [29], for any partition \(\pi\) of interval \([0,T]\), we obtain that the integral \(\int_{[0,T]} h(s)dW_1(s)\) coincide with the limits in probability of integral sums

\[
\int_{[0,T]} h(s)dW_1(s) = L_2 - \lim_{|\pi| \to 0} \sum_{i=1}^{N} h(s_{i-1})(W_1(s_i) - W_1(s_{i-1}))
\]

\[
= L_2 - \lim_{|\pi| \to 0} \sum_{i=1}^{N} W_1(s_i)(h(s_{i-1}) - h(s_i)) + W_1(T)h(T)
\]

\[
= \int_{[0,T]} W_1(s)d(-h(s)) + W_1(T)h(T).
\]
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