GRADIENT BOUNDS FOR SOLUTIONS TO IRREGULAR PARABOLIC EQUATIONS WITH \((p, q)\)-GROWTH

Cristiana De Filippis

Abstract. We provide quantitative gradient bounds for solutions to certain parabolic equations with unbalanced polynomial growth and non-smooth coefficients.

Contents

1. Introduction \hspace{1cm} 1
2. Preliminaries \hspace{1cm} 3
2.1. Notation \hspace{1cm} 3
2.2. Main assumptions \hspace{1cm} 4
2.3. Auxiliary results \hspace{1cm} 5
3. Higher Sobolev regularity for non-degenerate systems \hspace{1cm} 7
4. Gradient bounds \hspace{1cm} 18
4.1. Uniform \(L^\infty\)-estimates \hspace{1cm} 18
4.2. Proof of Theorem 1 \hspace{1cm} 24
References \hspace{1cm} 26

1. INTRODUCTION

We focus on the Cauchy-Dirichlet problem

\[
\begin{aligned}
\partial_t u - \text{div} \ a(x, t, Du) &= 0 \quad \text{in} \ \Omega_T \\
u &= f \quad \text{on} \ \partial_{\text{par}} \Omega_T,
\end{aligned}
\]

with initial-boundary datum \(f: \mathbb{R}^{n+1} \to \mathbb{R}\) as in (2.5) below and nonlinear diffusive tensor \(a(\cdot)\) featuring \((p, q)\)-growth conditions as displayed in (2.2). The main novelties here are twofold: the map \(x \mapsto a(x, t, z)\) is only Sobolev-differentiable in the sense that

\[
|\partial_x a(x, t, z)| \leq \gamma(x, t) \left[\left((\mu^2 + |z|^2)^{\frac{p-1}{2}} + (\mu^2 + |z|^2)^{\frac{q-1}{2}}\right)^{\frac{1}{p}}\right],
\]

where \(\gamma\) possess a suitably high degree of integrability, cf. (2.4). Moreover, we can treat in a single shot both the degenerate case \(p \geq 2\) and the singular one \(p < 2\), allowing also for the case \(\mu = 0\). Precisely, we prove that

**Theorem 1.** If assumptions (2.1)-(2.5) are satisfied, Cauchy-Dirichlet problem (1.1) admits a solution \(u \in L^p(0,T;W^{1,p}(\Omega))\) such that

\[
Du \in L^\infty_{\text{loc}}(\Omega_T, \mathbb{R}^n), \quad V_{\mu,p}(Du) \in L^2_{\text{loc}}(0,T;W^{1,2}_{\text{loc}}(\Omega, \mathbb{R}^n))
\]

\begin{flushright}
2010 Mathematics Subject Classification. 35K20, 35K61, 35K92.

Key words and phrases. Regularity, parabolic equations, \((p, q)\)-growth.

Acknowledgements. This work is supported by the Engineering and Physical Sciences Research Council (EPSRC): CDT Grant Ref. EP/L015811/1.
\end{flushright}
and
\begin{equation}
(1.3) \quad u \in W^{1,2}_{\text{loc}}(0,T;L^2_{\text{loc}}(\Omega)) \quad \text{for all} \quad \epsilon \in \left(0, \frac{1}{2}\right).
\end{equation}

In particular, if \(Q_\epsilon \subseteq \Omega_T\) is any parabolic cylinder there holds that
\begin{equation}
(1.4) \quad \|H(Du)\|_{L^\infty(Q_\epsilon)} \leq \frac{c}{\epsilon^{\beta_1}} \left[1 + \left(\int_{Q_\epsilon} H(Du)^{\frac{2}{\beta_2}} \, dy\right)^{\frac{1}{\beta_2}}\right],
\end{equation}
with \(c \equiv c(\text{data})\) and \(\beta_1, \beta_2 \equiv \beta_1, \beta_2(n, p, q, d)\).

We refer to Sections 2.1-2.2 for a detailed description of the various quantities involved in the previous statement. Our analysis includes equations with double phase structure, such as
\begin{align*}
\partial_t u - \div (|Du|^{p-2}Du + b(x, t)|Du|^{q-2}Du) &= 0 \quad \text{in} \ \Omega_T \\
b &\in L^\infty(\Omega_T) \quad \text{with} \ \partial_x b \in L^d(\Omega_T);
\end{align*}
or equations with variable exponent:
\begin{align*}
\partial_t u - \div (|Du|^{p(x,t)-2}Du) &= 0 \quad \text{in} \ \Omega_T \\
p &\in L^\infty(\Omega_T) \quad \text{with} \ \partial_x p \in L^d(\Omega_T);
\end{align*}
and also anisotropic equations like
\begin{equation}
(1.5) \quad \left\{ \begin{array}{ll}
-\div a(x,Du) &= 0 &\text{in} \ \Omega \\
u &= f &\text{on} \ \partial\Omega
\end{array} \right.
\end{equation}
i.e., the elliptic counterpart of (1.1) started in [27–29] and, subsequently, has undergone an intensive development over the last years, see for instance [4–7, 10–13, 15, 19, 20, 25] and references therein. As suggested by the counterexamples contained in [19, 27], already in the elliptic setting the regularity of solution to (1.5) is strongly linked to the closeness of the exponents \((p, q)\) ruling the growth of the vector field \(a(\cdot)\). Precisely, it turns out that
\begin{equation}
(1.6) \quad 1 \leq \frac{q}{p} < 1 + \mathcal{M}(\text{problem's data}),
\end{equation}
where \(\mathcal{M}(\cdot)\) is in general a bounded function connecting the various informations given a priori about solutions. In this respect, we refer to [4] for an idea on the subtle yet quantifiable interplay between the regularity of solutions and the main parameters of the problem and to [5, 11, 12], where is shown that, as long as \(p\) and \(q\) stay close to each other, problems with \((p,q)\)-growth
can be interpreted as perturbations of problems having standard $p$-growth. In the parabolic setting, the regularity for solutions of (1.1) is very well understood when $a(\cdot)$ is modelled upon the parabolic $p$-laplacian, see e.g. [14,17,18,22,23] for an overview of the state of the art on this matter and [2,3], where more general structures are analyzed. Finally, the question of existence of regular solutions of (1.1) when the nonlinear tensor $a(\cdot)$ has unbalanced polynomial growth was treated in [8,9,31,32]. The theory exposed in these papers confirms that, as in the elliptic case, a restriction like (1.6) on the ratio $q/p$ suffices to prove existence of regular solutions to (1.1). Actually, the function $M(\cdot)$ is worsen for parabolic equations than for elliptic ones, due to the so-called phenomenon of caloric deficit, originated from the difference of scaling in space and time, see e.g. [8,32], in which $M(\cdot)$ is quantified as a function of $n$ and $p$. In our case, $M(\cdot)$ has to take into account also the integrability exponent of $\gamma$, therefore it depends on $n,p,d$ and, reversing the process of caloric deficit, it renders precisely the bound for elliptic equations with Sobolev-differentiable coefficients appearing in [11,12,26].

Organization of the paper. This paper is organized as follows: in Section 2 contains our notation, the list of the assumptions which will rule problem (1.1) and several by now classical tools in the framework of regularity theory for elliptic and parabolic PDE. Sections 3-4 are devoted to the proof of Proposition 3.1 and Theorem 1 respectively.

2. Preliminaries

In this section we display the notation adopted throughout the paper and list some well-known result which will be helpful in the various proofs presented.

2.1. Notation. In this paper, $\Omega_T := \Omega \times (0,T)$ is a space-time cylinder over an open, bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$ with $C^1$-boundary. If $\Omega \subseteq \Omega_0$ and $t_0 \in [0,T]$, by $\Omega_{t_0}$ we mean the subcylinder $\Omega \times (0,t_0) \subseteq \Omega_T$. Clearly, when $t_0 = 0$, $\Omega_0 \equiv \Omega$. We denote by $B_\rho(x_0) := \{x \in \mathbb{R}^n : |x-x_0| < \rho \}$ the $n$-dimensional open ball centered at $x_0 \in \mathbb{R}^n$ with radius $\rho > 0$. When working in the parabolic setting it is convenient to consider parabolic cylinders

$$Q_\rho(y_0) := B_\rho(x_0) \times (t_0 - \rho^2, t_0)$$

where $y_0 := (x_0, t_0) \in \mathbb{R}^{n+1}$, i.e., balls in the parabolic metric. With "$\rho" we shall always denote the couple $(x,t) \in \Omega_T$. Very often, when not otherwise stated, different balls (or cylinders) in the same context will share the same center. Given any differentiable map $G : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, with $\partial_t G(x, t, z)$ we mean the derivative of $G(\cdot)$ with respect to the $z$ variable, by $\partial_x G(x, t, z)$ the derivative in the time variable $t$ and by $\partial_x G(x, t, z)$ the derivative of $G$ with respect to the space variable $x$. We name "$c" a general constant larger than one. Different occurrences from line to line will be still denoted by $c$, while special occurrences will be denoted by $c_1, c_2, \tilde{c}$ and so on. Relevant dependencies on parameters will be emphasized using parentheses, i.e., $c_1 \equiv c_1(n, p)$ means that $c_1$ depends on $n, p$. For the sake of clarity, we shall adopt the shorthand notation

$${\text{data}} := (n, \nu, L, p, q, d, \|\gamma\|_{L^q(\Omega_T)}).$$

In most of the inequalities appearing in the proof of our results we will use the symbols "$\lesssim$" or "$\gtrsim$", meaning that the inequalities hold up to constants depending from some (or all) the parameters collected in data. We refer to Section 2.2 for more details on the quantities appearing in the expansion of data.
2.2. Main assumptions. When dealing with the Cauchy-Dirichlet problem (1.1), we assume that the nonlinear tensor $a: \Omega_T \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies:

\begin{equation}
\begin{cases}
  t \mapsto a(x,t,z) & \text{measurable for all } x \in \Omega, z \in \mathbb{R}^n \\
  x \mapsto a(x,t,z) & \text{differentiable for all } t \in (0,T), z \in \mathbb{R}^n \\
  z \mapsto a(x,t,z) & \in C(\mathbb{R}^n, \mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^n) \quad \text{for all } (x,t) \in \Omega_T
\end{cases}
\end{equation}

and

\begin{equation}
\begin{align*}
  &|a(x,t,z)| + (\mu^2 + |z|^2)^{\frac{p}{2}} |\partial_x a(x,t,z)| \leq L \left( [\mu^2 + |z|^2]^{\frac{p}{2}} + (\mu^2 + |z|^2)^{\frac{q-1}{2}} \right) \\
  &|\partial_x a(x,t,z)| \geq \nu (\mu^2 + |z|^2)^{\frac{p}{2}} |\xi|^2 \\
  &|\partial_x a(x,t,z)| \leq \nu (\mu^2 + |z|^2)^{\frac{p}{2}} + (\mu^2 + |z|^2)^{\frac{q-1}{2}},
\end{align*}
\end{equation}

which holds for all $(x,t) \in \Omega_T$ and $z, \xi \in \mathbb{R}^n$. In (2.2), $\mu \in [0,1]$ is any number, exponents $(p,q)$ are so that

\begin{equation}
q < p + 2 \left( \frac{1}{n+2} - \frac{p}{2d} \right) \quad \text{with} \quad p > \frac{2nd}{(n+2)(d-2)}
\end{equation}

and

\begin{equation}
\gamma \in L^d(\Omega_T) \quad \text{for some } d > \max \left\{ \frac{p}{2}, 1 \right\} (n+2).
\end{equation}

Finally, the function $f: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ satisfies

\begin{equation}
f \in C_\text{loc}(\mathbb{R}; L^2_\text{loc}(\mathbb{R}^n)) \cap L^p_\text{loc}(\mathbb{R}; W^{1,r}_\text{loc}(\mathbb{R}^n)), \quad \partial_t f \in L^p_\text{loc}(\mathbb{R}; W^{-1,p}_\text{loc}(\mathbb{R}^n)),
\end{equation}

where $r := p/(q-1)$. In this setting, we define a weak solution to (1.1) as follows.

**Definition 1.** A function $u \in f + L^p(0,T; W_0^{1,p}(\Omega))$ is a weak solution of problem (1.1) if and only if the identity

\begin{equation}
\int_{\Omega_T} [u \partial_t \varphi - a(x,t,Du) \cdot D\varphi] \ dy = 0
\end{equation}

holds true for all $\varphi \in C_\infty^0(\Omega_T)$ and, in addition, $u(\cdot,0) = f(\cdot,0)$ in the $L^2$-sense, i.e.:

\begin{equation}
\lim_{\delta \to 0} \frac{1}{\delta} \int_0^\delta \int_{\Omega_T} |u(x,s) - f(x,0)|^2 \ dx \ ds = 0.
\end{equation}

**Remark 2.1.** Let us compare the bound in (2.3) with the one in force in the elliptic setting, i.e.:

\begin{equation}
q < p + p \left( \frac{1}{n} - \frac{1}{d} \right),
\end{equation}

see [11, 12, 26]. The restriction imposed in (2.3) looks the right one: in fact, due to the different scaling in time, in (2.8) $n$ must be replaced by $n+2$. Moreover, the usual parabolic deficit coming from the growth of the diffusive part of the equation affects also $d$:

\begin{equation}
q < p + p \left( \frac{1}{n+2} - \left( \frac{d}{2} \cdot \frac{2}{p} \right)^{-1} \right) \cdot \frac{2}{p}.
\end{equation}

If we let $d \to \infty$ in (2.3) and reverse the transformation prescribed by the caloric deficit phenomenon, we obtain

\begin{equation}
q < p + \frac{p}{n},
\end{equation}

which is the same appearing in [19] when the space-depending coefficient is Lipschitz-continuous.
2.3. Auxiliary results. In this section we collect some well-known facts that will have an important role throughout the paper.

On Sobolev functions. Let \( w \in L^1(\Omega_T, \mathbb{R}^k) \), \( k \geq 1 \) be any function. If \( h \in \mathbb{R}^n \) is a vector, we denote by \( \tau_h : L^1(\Omega_T, \mathbb{R}^k) \to L^1(\Omega_{|h|} \times (0, T), \mathbb{R}^k) \) the standard finite difference operator in space, pointwise defined as
\[
\tau_h w(x) := w(x + h, t) - w(x, t) \quad \text{for a.e. } (x, t) \in \Omega_{|h|} \times (0, T),
\]
where \( \Omega_{|h|} := \{ x \in \Omega : \text{dist}(x, \partial \Omega) > |h| \} \) and by \( \Delta_h : L^1(\Omega_T, \mathbb{R}^k) \to L^1(\Omega_{|h|} \times (0, T), \mathbb{R}^k) \) the spacial difference quotient operator, i.e.:
\[
\Delta_h w(x, t) := \frac{w(x + h, t) - w(x, t)}{|h|} = |h|^{-1} (\tau_h w(x, t)).
\]
Moreover, if \( \tilde{h} \in \mathbb{R} \) is a number so that \( |h| < T \), we also recall the definition of finite difference operator in time \( \tilde{\tau}_h : L^1(\Omega_T) \to L^1(\Omega \times ([\tilde{h}], T - |\tilde{h}|)) \):
\[
\tilde{\tau}_h w(x, t) := w(x, t + h) - w(x, t).
\]

An important property of translation operators is their continuity in Lebesgue spaces.

Lemma 2.1. Let \( \varphi \in C_\infty^\infty(\Omega) \) be any map, \( h \in \mathbb{R}^n \) so that \( |h| \in \left(0, \frac{\text{dist}(\text{supp}(\varphi), \partial \Omega)}{2}\right) \) and \( w \in L^s_{\text{loc}}(\Omega_T, \mathbb{R}^k) \) with \( s \in [1, \infty) \) and \( k \in \mathbb{N} \). Then
\[
\|(w(\cdot + h, t) - w(\cdot, t))\varphi\|_{L^s(\Omega)} \to |h| \to 0 0.
\]

It is also useful to recall a basic property of difference quotient.

Lemma 2.2. Let \( w \in L^1_{\text{loc}}(\Omega_T) \) be any function. There holds that

- if \( w \in L^s_{\text{loc}}(0, T; W^{1,s}_{\text{loc}}(\Omega, \mathbb{R}^k)) \), \( s \in [1, \infty) \) and \( \tilde{\Omega} \subseteq \Omega \) is any open set, then
  \[
  \|\Delta_h w(t) - Dw(t)\|_{L^s(\tilde{\Omega})} \to |h| \to 0;
  \]

- if in addition \( s > 1 \) and \( \tilde{\Omega} \subseteq \Omega \) is any open set so that
  \[
  \sup_{|h| > 0} \int_0^T \int_{\tilde{\Omega}} |\Delta_h w(x, t)|^s \ dx \ dt < \infty,
  \]
  then \( Dw \in L^s(\tilde{\Omega} \times (0, T)) \) and \( \|\Delta_h w(\cdot, t) - Dw(\cdot, t)\|_{L^s(\tilde{\Omega})} \to |h| \to 0 \).

When dealing with parabolic PDE, solutions in general possess a modest degree of regularity in the time-variable, and, in particular, time derivatives exist only in the distributional sense. For this reason, we recall the definition and main properties of Steklov averages, see e.g. [14, Chapter 1].

Definition 2. Let \( w \in L^1(\Omega_T, \mathbb{R}^k) \), \( k \in \mathbb{N} \), be any function. For \( \delta \in (0, T) \), the Steklov averages of \( w \) are defined as
\[
w_\delta := \begin{cases}
\frac{1}{T} \int_0^{t+\delta} w(x, s) \ ds & t \in (0, T - \delta] \\
\frac{1}{\delta} \int_{t-\delta}^t w(x, s) \ ds & t > T - \delta
\end{cases}
\]
and
\[
w_\delta := \begin{cases}
\frac{1}{T} \int_0^t w(x, s) \ ds & t \in (\delta, T] \\
\frac{1}{\delta} \int_0^{t-\delta} w(x, s) \ ds & t < \delta.
\end{cases}
\]

Lemma 2.3. If \( w \in L^s_{\text{loc}}(\Omega_T) \), then \( w_\delta \to \delta \to 0 w \) in \( L^s_{\text{loc}}(\Omega_{T-\varepsilon}) \) for all \( \varepsilon \in (0, T) \). If \( w \in C(0, T; L^s(\Omega)) \), then as \( \delta \to 0 \), \( w_\delta(\cdot, t) \) converges to \( w(\cdot, t) \) for all \( t \in (0, T - \varepsilon) \) and all \( \varepsilon \in (0, T) \). A similar statement holds for \( w_\delta \) as well.

We also record the definition of fractional Sobolev spaces.
Definition 3. A function $w \in L^s(\Omega_T, \mathbb{R}^k)$ belongs to the fractional Sobolev space $W^{\alpha,\theta,s}(\Omega_T, \mathbb{R}^k)$, $\alpha, \theta \in (0,1)$, $k \in \mathbb{N}$ provided that

$$
\int_0^T \int_{\Omega_T} \frac{|w(x,t_1) - w(x,t_2)|^s}{|t_1 - t_2|^1 + \sigma \theta} \, dx \, dt + \int_0^T \int_{\Omega_T} \frac{|w(x,t_1) - w(x,t_2)|^s}{|t_1 - t_2|^1 + \sigma \theta} \, dx \, dt < \infty.
$$

The local variant of $W^{\alpha,\theta,s}(\Omega_T, \mathbb{R}^k)$ can be defined in the usual way.

The usual relation between Nikolski spaces and fractional Sobolev spaces holds in the parabolic setting as well.

Proposition 2.1. Let $w \in L^s(\Omega_T, \mathbb{R}^k)$, $(t_1, t_2) \in (0, T)$, $\Omega \subset \Omega$ be an open set, $h \in \mathbb{R}^n$ be any vector with $|h| < \frac{\text{dist}(\Omega, \partial \Omega)}{4}$ and $\bar{h} \in \mathbb{R}$ be a number so that $|\bar{h}| < \frac{\min|t_1, T-t_2|}{4}$. Assume that

$$
\int_{t_1}^{t_2} \int_{\Omega} |w(x,t+h) - w(x,t)| \, dx \, dt \leq c' |\bar{h}|^{s\theta} \quad \text{for some } \theta \in (0,1),
$$

where $c'$ is a positive, absolute constant. Then there exists a constant $c \equiv c(n, s, c', t_1, T-t_2) > 0$ such that

$$
\int_{t_1}^{t_2} \int_{\Omega} \frac{|w(x,t_1) - w(x,t_2)|^s}{|t_1 - t_2|^1 + \sigma \theta} \, dx \, dt \leq c < \infty \quad \text{for all } t \in (0, \theta).
$$

Suppose that

$$
\int_{t_1}^{t_2} \int_{\Omega} |w(x+h, t) - w(x, t)|^{\alpha} \, dx \, dt \leq c' |h|^{s\alpha} \quad \text{for some } \alpha \in (0,1),
$$

with $c'$ positive, absolute constant. Then,

$$
\int_{t_1}^{t_2} \int_{\Omega} \frac{|w(x,t_1) - w(x,t_2)|^s}{|x_1 - x_2|^{n+s\gamma}} \, dx \, dt \leq c < \infty \quad \text{for all } \gamma \in (0, \alpha),
$$

with $c \equiv c(n, s, c', \gamma, \text{dist}(\Omega, \partial \Omega))$.

We refer to [1,16,17,24] for more details on this matter. We close this part with a fundamental compactness criterion in parabolic Sobolev spaces, whose proof can be found in [30].

Lemma 2.4. Let $X \subset B \subset Y$ be three Banach spaces such that the immersion $X \hookrightarrow B$ is compact and $1 \leq a_1 \leq a_2 \leq \infty$ be numbers satisfying the balance condition $a_1 > a_2/(1 + \sigma a_2)$ for some $\sigma \in (0, 1)$. If the set $\mathcal{J}$ is bounded in $L^{a_2}(0, T; X) \cap W^{a_2, a_1}(0, T; Y)$, then $\mathcal{J}$ is compact in $L^{a_2}(0, T; B)$ and eventually in $C(0, T; B)$ when $a_2 = \infty$.

Tools for p-laplacian type problems. For a constant $\tilde{c} \in [0,1]$ and $z \in \mathbb{R}^n$ we introduce the auxiliary vector field

$$
V_{\tilde{c},s}(z) := (\tilde{c}^2 + |z|^2)^{\frac{s-2}{2}} z \quad s \in \{p, q\},
$$

which turns out to be very convenient in handling the monotonicity properties of certain operators.

Lemma 2.5. For any given $z_1, z_2 \in \mathbb{R}^n$, $z_1 \neq z_2$ there holds that

$$
|V_{\tilde{c},s}(z_1) - V_{\tilde{c},s}(z_2)|^2 \sim (\tilde{c}^2 + |z_1|^2 + |z_2|^2)^{\frac{s-2}{2}} |z_1 - z_2|^2,
$$

where the constants implicit in "~" depend only from $(n, s)$.

Another useful result is the following

Lemma 2.6. Let $s > -1$, $\tilde{c} \in [0,1]$ and $z_1, z_2 \in \mathbb{R}^n$ be so that $\tilde{c} + |z_1| + |z_2| > 0$. Then

$$
\int_0^1 \left[ (\tilde{c}^2 + |z_1 + \lambda(z_2 - z_1)|^2) \right]^{\frac{s}{2}} \, d\lambda \sim (\tilde{c}^2 + |z_1|^2 + |z_2|^2)^{\frac{s}{2}},
$$

with constants implicit in "~" depending only from $s$. 

Finally, the iteration lemma.

**Lemma 2.7.** Let \( Z : [n, R) \to [0, \infty) \) be a function which is bounded on every interval \([n, R)\) with \(R < R\). Let \( c \in (0, 1)\), \(a_1, a_2, \gamma_1, \gamma_2 \geq 0\) be numbers. If

\[
Z(\tau_1) \leq c Z(\tau_2) + \frac{a_1}{(\tau_2 - \tau_1)^{\gamma_1}} + \frac{a_2}{(\tau_2 - \tau_1)^{\gamma_2}} \quad \text{for all } \tau_1 < \tau_2 < R,
\]

then

\[
Z(\tau) \leq c \left[ \frac{a_1}{(R - \tau)^{\gamma_1}} + \frac{a_2}{(R - \tau)^{\gamma_2}} \right],
\]

holds with \( c \equiv c(\varepsilon, \gamma_1, \gamma_2) \).

3. **Higher Sobolev regularity for non-degenerate systems**

In this section we prove the existence of a suitably regular weak solution to Cauchy-Dirichlet problem

\[
\begin{cases}
\partial_t v - \text{div} \, \tilde{a}(x, t, Dv) = 0 & \text{in } \Omega_T \\
v = f & \text{on } \partial_{\text{par}} \Omega_T,
\end{cases}
\]

where \( f \) is as in (2.5) and the diffusive tensor \( \tilde{a} : \Omega_T \times \mathbb{R}^n \to \mathbb{R} \) satisfies

\[
\begin{align*}
t \mapsto \tilde{a}(x, t, z) & \quad \text{measurable for all } x \in \Omega, z \in \mathbb{R}^n \\
x \mapsto \tilde{a}(x, t, z) & \quad \text{differentiable for all } t \in (0, T), z \in \mathbb{R}^n \\
z \mapsto \tilde{a}(x, t, z) & \in C^1(\mathbb{R}^n, \mathbb{R}^n) \quad \text{for all } (x, t) \in \Omega_T
\end{align*}
\]

and

\[
\begin{align*}
|\tilde{a}(x, t, z)| + (\tilde{\mu}^2 + |z|^2)^{\frac{1}{2}} |\partial_z \tilde{a}(x, t, z)| & \leq \tilde{L} \left( (\tilde{\mu}^2 + |z|^2)^{\frac{p-1}{2}} + (\tilde{\mu}^2 + |z|^2)^{\frac{d-1}{2}} \right) \\
|\partial_z \tilde{a}(x, t, z) \xi : \xi| & \geq \nu (\tilde{\mu}^2 + |z|^2)^{\frac{p-1}{2}} |\xi|^2 \\
|\partial_x \tilde{a}(x, t, z)| & \leq \gamma(x, t) \left( (\tilde{\mu}^2 + |z|^2)^{\frac{p-1}{2}} + (\tilde{\mu}^2 + |z|^2)^{\frac{d-1}{2}} \right),
\end{align*}
\]

for all \((x, t) \in \Omega_T\) and \(z, \xi \in \mathbb{R}^n\). In (3.3), \((p, q)\) are linked by the relation in (2.3), \(\gamma\) is as in (2.4) and

\[
\tilde{\mu} > 0.
\]

Our first result is the following

**Proposition 3.1.** Let \( f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) be as in (2.5) and \( \tilde{a} : \Omega_T \times \mathbb{R}^n \to \mathbb{R} \) be a Carathéodory vector field satisfying (3.2), (3.3), (2.3), (2.4) and (3.4). Then there exists a weak solution \( v \in L^p(0, T; W^{1,p}(\Omega)) \) of Cauchy-Dirichlet problem (3.1) such that

\[
v \in L^{s}_{\text{loc}}(0, T; W^{1,s}_{\text{loc}}(\Omega)) \quad \text{for all } s \in \left[ 1, p + \frac{4}{n} \right]
\]

satisfying

\[
\partial_t v \in L^l_{\text{loc}}(\Omega_T) \quad \text{for some } l \equiv l(n, p, q, d) \in (1, \min\{2, p\})
\]

and

\[
Dv \in L^\infty_{\text{loc}}(0, T; L^2_{\text{loc}}(\Omega)) \quad \text{with } V_p(Dv) \in L^2_{\text{loc}}(0, T; W^{1,2}_{\text{loc}}(\Omega, \mathbb{R}^n)).
\]

For the sake of simplicity, we shall split the proof of Proposition 3.1 into eight steps.
Step 1: Approximating Cauchy-Dirichlet problems. For the ease of notation, we define numbers:

(3.8) \[ m := \frac{d}{d-2} > 1, \quad q := \max \left\{ \frac{p}{2}, 1 \right\}, \]

and, for \( j \in \mathbb{N} \), consider a usual family of non-negative mollifiers \( \{ \psi_j \} \) of \( \mathbb{R}^{n+1} \). We then regularize \( f \) via convolution against \( \{ \psi_j \} \), thus obtaining the sequence \( \{ f_j \} := \{ f * \psi_j \} \), set

(3.9) \[ \epsilon_j := \left( 1 + j + \| f_j \|_{L^{2m\tilde{q}}(\Omega_T)} \right)^{-1}, \quad \tilde{H}(z) := (\tilde{a}^2 + |z|^2), \]

correct the nonstandard growth of the diffusive tensor \( \tilde{a}(\cdot) \) as follows:

(3.10) \[ \tilde{a}_j(x, t, z) := \tilde{a}(x, t, z) + \epsilon_j \tilde{H}(z)^{\frac{-m-2}{2}} z \]

and consider solutions \( v_j \in L^{2m\tilde{q}}(0, T; W^{1,2m\tilde{q}}(\Omega)) \) of the following Cauchy-Dirichlet problem

(3.11) \[
\begin{cases}
\partial_t v_j - \text{div} \tilde{a}_j(x, t, Dv_j) = 0 & \text{in } \Omega_T \\
v_j = f_j & \text{on } \partial_{\text{part}} \Omega_T.
\end{cases}
\]

By (3.4), (2.2) and the definition in (3.10), it can be easily seen that (3.2) holds and

(3.12) \[
\begin{cases}
|\tilde{a}_j(x, t, z)| + \tilde{H}(z)|\partial_t \tilde{a}_j(x, t, z)| \leq c \left[ \tilde{H}(z)^{\frac{p-1}{2}} + \tilde{H}(z)^{\frac{2m-1}{2}} \right] + c \epsilon_j \tilde{H}(z)^{\frac{m-2}{2}} \\
|\partial_t \tilde{a}_j(x, t, z)| \leq c \gamma(x, t) \left[ \tilde{H}(z)^{\frac{p-1}{2}} + \tilde{H}(z)^{\frac{2m-1}{2}} \right],
\end{cases}
\]

for all \((x, t) \in \Omega_T, z, \xi \in \mathbb{R}^n\), with \( \gamma \) as in (2.4) and \( c \equiv c(u, \nu, L, p, q, d) \). We recall that the weak formulation associated to problem (3.1) reads as

(3.13) \[ \int_{\Omega_T} [v_j \partial_t \varphi - \tilde{a}(x, t, Dv_j) \cdot D\varphi] \, dy = 0 \quad \text{for all } \varphi \in C_c^\infty(\Omega_T) \]

and the attainment of the boundary datum \( f_j \) must be considered in the \( L^2 \)-sense as in Definition 1.

Step 2: Uniform energy bounds. Our main goal it to prove that the sequence \( \{ v_j \} \) is bounded, uniformly with respect to \( j \in \mathbb{N} \) in the space-time \( L^p \)-norm. Since this is quite a routine procedure, we will just sketch it and refer the reader to [8, 31], for more details. Modulo Steklov averages, we can test (3.13) against the difference \( v_j - f_j \) to get

(3.14) \[
\int_\Omega |v_j(x, t) - f_j(x, t)|^2 \, dx + \int_0^t \int_\Omega \tilde{a}_j(x, s, Dv_j) \cdot (Dv_j - Df_j) \, dx \, ds = -\int_0^t \langle v_j - f_j, \partial_t f_j \rangle_{W_0^{1,p}(\Omega)} \, ds \quad \text{for a.e. } t \in (0, T).
\]

By (3.12)₂, Hölder and Young inequalities, if \( p \geq 2 \) a straightforward computation renders that

\[
\int_0^t \int_\Omega |Dv_j|^p \, dx \, ds + \epsilon_j \int_0^t \int_\Omega |2m\tilde{q}| \, dx \, ds + \epsilon_j \int_0^t \int_\Omega |Df_j|^p \, dx \, ds + \epsilon_j \int_0^t \int_\Omega |2m\tilde{q}| \, dx \, ds,
\]

while if \( 1 < p < 2 \) there holds that

\[
\int_0^t \int_\Omega |Dv_j|^p \, dx \, ds + \epsilon_j \int_0^t \int_\Omega |2m\tilde{q}| \, dx \, ds.
\]
Here, we also used that \( q > \chi \) with bounded, piecewise continuous, non-negative first derivative and a ball with radius \( \frac{1}{\sigma} \), Section 3.1, we can test \((3.17)\), expanding \((3.16)\), Sobolev-Poincaré and Young inequalities we have

\[
\int_{0}^{t} \int_{\Omega} \left| \triangle f \right|^{p} \, dx \, ds + \varepsilon \int_{0}^{t} \int_{\Omega} \left| \nabla f \right|^{2m} \, dx \, ds + \int_{0}^{t} \int_{\Omega} \left| \nabla f \right|^{2m} \, dx \, ds + \int_{0}^{t} \int_{\Omega} \left| \nabla f \right|^{2m} \, dx \, ds
\]

\[
\lesssim \left[ \int_{0}^{t} \int_{\Omega} \left| \nabla f \right|^{p} \, dx \, ds + \varepsilon \int_{0}^{t} \int_{\Omega} \left| \nabla f \right|^{2} \, dx \, ds + \int_{0}^{t} \int_{\Omega} \left| \nabla f \right|^{2} \, dx \, ds \right] + \left[ \int_{0}^{t} \int_{\Omega} \left( 1 + \left| f \right|^{2} \right) \, dx \, ds \right]
\]

\[
\lesssim \left[ \int_{0}^{t} \int_{\Omega} \left| \nabla f \right|^{p} \, dx \, ds + \varepsilon \int_{0}^{t} \int_{\Omega} \left| \nabla f \right|^{2} \, dx \, ds + \int_{0}^{t} \int_{\Omega} \left| \nabla f \right|^{2} \, dx \, ds + \int_{0}^{t} \int_{\Omega} \left| \nabla f \right|^{2} \, dx \, ds \right] + 1
\]

\[
(3.15) \quad \lesssim \left[ \int_{0}^{t} \int_{\Omega} \left| \nabla f \right|^{p} \, dx \, ds + \varepsilon \int_{0}^{t} \int_{\Omega} \left| \nabla f \right|^{2} \, dx \, ds + \int_{0}^{t} \int_{\Omega} \left| \nabla f \right|^{2} \, dx \, ds \right] + 1.
\]

As stated at the end of Section 2.2, none of the constants implicit in "\( \lesssim " \) depends on \( t \in (0, T) \), therefore we can send \( t \to T \) on the right-hand side of \((3.15)\) to get

\[
\int_{0}^{t} \int_{\Omega} \left| \nabla f \right|^{p} \, dx \, ds + \varepsilon \int_{0}^{t} \int_{\Omega} \left| \nabla f \right|^{2} \, dx \, ds + \int_{0}^{t} \int_{\Omega} \left| \nabla f \right|^{2} \, dx \, ds + \int_{0}^{t} \int_{\Omega} \left| \nabla f \right|^{2} \, dx \, ds
\]

\[
(3.16) \quad \lesssim \left[ \int_{0}^{t} \int_{\Omega} \left| \nabla f \right|^{p} \, dx \, ds + \varepsilon \int_{0}^{t} \int_{\Omega} \left| \nabla f \right|^{2} \, dx \, ds + \int_{0}^{t} \int_{\Omega} \left| \nabla f \right|^{2} \, dx \, ds \right] + 1 =: C_f.
\]

Step 3: Caccioppoli inequality. Let \( h \in \mathbb{R}^{n} \setminus \{0\} \) any vector satisfying \( |h| \in (0, 1) \), \( B_{2\rho} \subset \Omega \) a ball with radius \( 0 < \rho \leq 1 \) and so that \( B_{2\rho} \subset \Omega \), \( g \in W^{1,\infty} (\mathbb{R}) \) a non-negative function with bounded, piecewise continuous, non-negative first derivative and \( \chi \in W^{1,\infty} ([0, T]) \) with \( \chi (0) = 0 \), \( \varphi \in C^{\infty} (B_{\rho}, [0, 1]) \) two cut-off functions. By the approximation procedure developed e.g. in [8, Section 3] or [32, Section 3.1], we can test \((3.13)\) against a suitably regularized version of the comparison map \( \frac{1}{2} \chi \Delta_h u_j g \left| \Delta_h u_j \right|^{2} \) and manipulate it to obtain, for a.e. \( \tau \in (0, \min \{T, 1\}) \),

\[
\frac{1}{2} \int_{B_{2\rho}} \frac{1}{2} \chi \left( \int_{0}^{t} \left| \Delta_h v_j \right|^{2} g(s) \, ds \right) \, dx + \frac{1}{2} \sum_{k=1}^{n} \int_{Q_{\tau}} \chi \left( \int_{0}^{t} \left| \Delta_h v_j \right|^{2} g(s) \, ds \right) \, dy
\]

\[
= - \frac{1}{2} \sum_{k=1}^{n} \int_{Q_{\tau}} \chi \left( \int_{0}^{t} \left| \Delta_h v_j \right|^{2} g(s) \, ds \right) \, dy,
\]

\[
(3.17) \quad \lesssim - \frac{1}{2} \sum_{k=1}^{n} \int_{Q_{\tau}} \chi \left( \int_{0}^{t} \left| \Delta_h v_j \right|^{2} g(s) \, ds \right) \, dy
\]
where we abbreviated \( Q_\tau := B_\tilde{x} \times (0, \tau) \). We also reduce further the size of \(|h|\): we ask that

\[
|h| \in \left(0, \frac{\text{dist}(\text{supp}(\chi), \partial B_{\tilde{x}})}{10000}\right).
\]

Using the mean value theorem, we rearrange \( \Delta_h a_j(x, t, Dv_j) \) in a more convenient way:

\[
\begin{align*}
\Delta_h \tilde{a}_j^k(x, t, Dv_j) &= |h|^{-1} \left[ \tilde{a}_k^l(x + h, t, Dv_j(x + h)) - \tilde{a}_k^l(x, t, Dv_j(x + h)) \right] \\
&\quad + |h|^{-1} \left[ H(Dv_j(x + h)) \frac{\partial H(Dv_j(x))}{\partial x} Dv_j(x + h) - \tilde{a}_k^l(x, t, Dv_j(x)) \right] \\
&\quad + |h|^{-1} \left[ \tilde{a}_k^l(x, t, Dv_j(x + h)) - \tilde{a}_k^l(x, t, Dv_j(x)) \right] \\
&= |h|^{-1} \sum_{l=1}^n \left[ \int_0^1 \partial_{x_l} \tilde{a}_k^l(x + \lambda h, t, Dv_j(x + h))h^l \, d\lambda \right] \\
&\quad + \sum_{l=1}^n \left[ \int_0^1 \partial_{x_l} \tilde{a}_k^l(x, t, Dv_j(x) + \lambda \tau_h Dv_j(x)) \, d\lambda \right] \Delta_h Dv_j.
\end{align*}
\]

Plugging this expansion in (3.17) we eventually get

\[
\begin{align*}
\frac{1}{2} \int_{B_{\tilde{x}}} &\varphi^2 \chi \left( \int_0^{||\Delta_h v_j||^2} g(s) \, ds \right) \, dx \\
&\quad + |h|^{-1} \sum_{k,l=1}^n \int_{Q_\tau} \varphi^2 \chi \left[ \int_0^1 \partial_{x_l} \tilde{a}_k^l(x + \lambda h, t, Dv_j(x + h))h^l \, d\lambda \right] D_h \left[ \Delta_h v_j g(||\Delta_h v_j||^2) \right] \, dy \\
&\quad + \sum_{k,l=1}^n \int_{Q_\tau} \varphi^2 \chi \left[ \int_0^1 \partial_{x_l} \tilde{a}_k^l(x, t, Dv_j(x) + \lambda \tau_h Dv_j(x)) \, d\lambda \right] \Delta_h Dv_j D_h \left[ \Delta_h v_j g(||\Delta_h v_j||^2) \right] \, dy \\
&= -2 |h|^{-1} \sum_{k,l=1}^n \int_{Q_\tau} \varphi \chi \left( \Delta_h v_j g(||\Delta_h v_j||^2) \right) \left[ \int_0^1 \partial_{x_l} \tilde{a}_k^l(x + \lambda h, t, Dv_j(x + h))h^l \, d\lambda \right] D_h \varphi \, dy \\
&\quad -2 \sum_{k,l=1}^n \int_{Q_\tau} \varphi \chi \left( \Delta_h v_j g(||\Delta_h v_j||^2) \right) \left[ \int_0^1 \partial_{x_l} \tilde{a}_k^l(x, t, Dv_j(x) + \lambda \tau_h v_j) \, d\lambda \right] \Delta_h Dv_j D_h \varphi \, dy \\
&\quad + \frac{1}{2} \int_{Q_\tau} \left( \int_0^{||\Delta_h v_j||^2} g(s) \, ds \right) \varphi^2 \partial_t \chi \, dy.
\end{align*}
\]

For reasons that will be clear in a few lines, we introduce the shorthands

\[
\mathcal{D}(h) := \left( \tilde{\mu}^2 + ||Dv_j(x + h)||^2 + ||Dv_j(x)||^2 \right) \quad \text{and} \quad \mathcal{G}(h) := \left( g(||\Delta_h v_j||^2) + ||\Delta_h v_j||^2 g'(||\Delta_h v_j||^2) \right),
\]

and notice that, by (3.4), \( \mathcal{D}(h) > \tilde{\mu}^2 > 0 \). Now we start estimating all the terms appearing in (3.19). For the sake of clarity, we split

\[
\begin{align*}
\mathcal{I} := |h|^{-1} \sum_{k=1}^n \int_{Q_\tau} \varphi^2 \chi \left[ \int_0^1 \partial_{x_l} \tilde{a}_k^l(x + \lambda h, t, Dv_j(x + h))h^l \, d\lambda \right] D_h \left[ \Delta_h v_j g(||\Delta_h v_j||^2) \right] \, dy \\
&\quad = |h|^{-1} \sum_{k=1}^n \int_{Q_\tau} \varphi^2 \chi \left[ \int_0^1 \partial_{x_l} \tilde{a}_k^l(x + \lambda h, t, Dv_j(x + h))h^l \, d\lambda \right] \Delta_h Dv_j D_h \left[ \Delta_h v_j g(||\Delta_h v_j||^2) \right] \, dy \\
&\quad + 2 |h|^{-1} \sum_{k=1}^n \int_{Q_\tau} \varphi^2 \chi \left[ \int_0^1 \partial_{x_l} \tilde{a}_k^l(x + \lambda h, t, Dv_j(x + h))h^l \, d\lambda \right] \left[ \Delta_h v_j||^2 g'(||\Delta_h v_j||^2) \right] \Delta_h Dv_j \, dy \\
&=: \mathcal{I}_1 + \mathcal{I}_2.
\end{align*}
\]
With (3.12)3, (2.4), Hölder and Young inequalities we bound
\[
|\text{(I)}_1| + |\text{(I)}_2| \leq c \int_{Q_r} \varphi^2 \chi \left( \int_0^1 \gamma(x + \lambda h, t) \, d\lambda \right) \left[ \mathcal{D}(h)^{\frac{m}{2}} + \mathcal{D}(h)^{\frac{m}{2} - \frac{q}{2}} \right] G(h) |\Delta_h \Psi_{v_j}| \, dy \\
\leq \sigma \int_{Q_r} \varphi^2 \chi \mathcal{G}(h) \mathcal{D}(h)^{\frac{m}{2}} |\Delta_h \Psi_{v_j}|^2 \, dy \\
+ \frac{c}{\sigma} \int_{Q_r} \varphi^2 \chi \left( \int_0^1 \gamma(x + \lambda h, t) \, d\lambda \right)^2 \left[ \mathcal{D}(h)^{\frac{m}{2}} + \mathcal{D}(h)^{\frac{m}{2} - \frac{q}{2}} \right] G(h) \, dy \\
\leq \sigma \int_{Q_r} \varphi^2 \chi \mathcal{G}(h) \mathcal{D}(h)^{\frac{m}{2}} |\Delta_h \Psi_{v_j}|^2 \, dy \\
+ \frac{c}{\sigma} \int_0^1 \|\gamma(\cdot, t)\|_{L^2(B_{2\rho})}^2 \left( \int_{B_{\rho}} \varphi^{2m} \chi^m \left[ \mathcal{D}(h)^{\frac{m}{2}} + \mathcal{D}(h)^{\frac{m}{2} - \frac{q}{2}} \right] G(h)^m \, dx \right)^{\frac{1}{m}} \, dt \\
\leq c \int_{Q_r} \varphi^2 \chi \mathcal{G}(h) \mathcal{D}(h)^{\frac{m}{2}} |\Delta_h \Psi_{v_j}|^2 \, dy + \frac{c}{\sigma} \left( \int_{Q_r} \varphi^{2m} \chi^m \left[ 1 + \mathcal{D}(h)^m \gamma^\frac{1}{m} \right] G(h)^m \, dy \right)^{\frac{1}{m}},
\]
for \( c \equiv c(\text{data}) \). Moreover, by (3.12)2, Lemmas 2.5 and 2.6, we obtain
\[
(\text{II}) := \sum_{k,l=1}^n \int_{Q_r} \varphi^2 \chi \left( \int_0^1 \partial_x \tilde{a}_k^l (x, t, Dv_j (x) + \lambda \tau_h Dv_j (x)) \, d\lambda \right) \Delta_h \Psi_{v_j} \Delta_k \Psi_{v_j} \left[ |\Delta_h \Psi_{v_j}|^2 \right] \, dy \\
= \sum_{k,l=1}^n \int_{Q_r} \varphi^2 \chi \left( \int_0^1 \partial_x \tilde{a}_k^l (x, t, Dv_j (x) + \lambda \tau_h Dv_j (x)) \, d\lambda \right) \Delta_h \Psi_{v_j} \Delta_k \Psi_{v_j} \left[ |\Delta_h \Psi_{v_j}|^2 \right] \, dy \\
+ 2 \sum_{k,l=1}^n \int_{Q_r} \varphi^2 \chi \left( \int_0^1 \partial_x \tilde{a}_k^l (x, t, Dv_j (x) + \lambda \tau_h Dv_j (x)) \, d\lambda \right) \Delta_h \Psi_{v_j} \Delta_k \Psi_{v_j} \left[ |\Delta_h \Psi_{v_j}|^2 \right] \, dy \\
\geq c|h|^{-2} \int_{Q_r} \varphi^2 \chi \mathcal{D}(h)^{\frac{2m}{2}} |\tau_h Dv_j|^2 G(h) \, dy + c|h|^{-2} \varepsilon_j \int_{Q_r} \varphi^2 \chi \mathcal{D}(h)^{\frac{2m}{2}} |\tau_h Dv_j|^2 G(h) \, dy \\
\geq c \int_{Q_r} \varphi^2 \chi \mathcal{G}(h) |\Delta_h \Psi_{\mu,p}(Dv_j)|^2 \, dy + c \varepsilon_j \int_{Q_r} \varphi^2 \chi \mathcal{G}(h) |\Delta_h \Psi_{\mu,p}(Dv_j)|^2 \, dy,
\]
with \( c \equiv c(a, \nu, p, q, d) \). With (3.12)1,3, Hölder and Young inequalities we finally obtain
\[
||\text{(III)}|| + ||\text{(IV)}|| := \\
- 2|h|^{-1} \sum_{k,l=1}^n \int_{Q_r} \varphi \chi \left( \Delta_h \Psi_{v_j} g(|\Delta_h \Psi_{v_j}|^2) \right) \left( \int_0^1 \partial_x \tilde{a}_k^l (x + h \lambda, t, Dv_j (x + j) h^l) \, d\lambda \right) \Delta_k \varphi \, dy \\
- 2 \sum_{k,l=1}^n \int_{Q_r} \varphi \chi \Delta_h \Psi_{v_j} g(|\Delta_h \Psi_{v_j}|^2) \left( \int_0^1 \partial_x \tilde{a}_k^l (x, t, Dv_j (x) + \lambda \tau_h Dv_j) \, d\lambda \right) \Delta_k \Psi_{v_j} \, dy \\
\leq \sigma \int_{Q_r} \varphi^2 \chi \mathcal{G}(h) \mathcal{D}(h)^{\frac{m}{2}} |\Delta_h \Psi_{v_j}|^2 \, dy + \sigma \varepsilon_j \int_{Q_r} \varphi^2 \chi \mathcal{G}(h) \mathcal{D}(h)^{\frac{m}{2} - \frac{q}{2}} |\Delta_h \Psi_{v_j}|^2 \, dy \\
+ \frac{c \varepsilon_j}{\sigma} \int_{Q_r} \chi |D \varphi|^2 |\Delta_h \Psi_{v_j}|^2 G(h)^{\frac{2m}{2}} \, dy + \int_{Q_r} \chi |D \varphi|^2 |\Delta_h \Psi_{v_j}|^2 G(h)^{\frac{2m}{2} - \frac{q}{2} - 1} \, dy \\
+ c \|\gamma\|_{L^q(\Omega_t)} \left( \int_{Q_r} \chi^{m} \varphi^{2m} g(|\Delta_h \Psi_{v_j}|^2)^m \left[ \mathcal{D}(h)^{\frac{m}{2}} + \mathcal{D}(h)^{\frac{m}{2} - \frac{q}{2}} \right] \, dy \right)^{\frac{1}{m}},
\]
where \( c \equiv c(\text{data}) \). Merging the content of all the previous displays and choosing \( \sigma > 0 \) sufficiently small, we end up with

\[
\frac{1}{2} \int_{B_\epsilon} \varphi^2 \chi \left( \int_0^{[\Delta x]_j} g(s) \, ds \right) \, dx + \int_{Q_{\epsilon}} \varphi^2 \chi \hat{G}(h) [\Delta h V_{\mu,p}(Dv_j)]^2 \, dy + \varepsilon_j \int_{Q_{\epsilon}} \varphi^2 \chi \hat{G}(h) [\Delta h V_{\mu,2m\sigma}(Dv_j)]^2 \, dy \\
\leq c \left( \int_{Q_{\epsilon}} \chi^m \varphi^{2m} \hat{G}(h)^m \left[ 1 + \mathcal{D}(h)^m(q-\frac{2}{3}) \right] \, dy \right)^{\frac{1}{m}} + c \int_{Q_{\epsilon}} \chi |D\varphi|^2 [\Delta h v_j]^2 g(|\Delta h v_j|^2) \, dy \]

\[+ c \varepsilon_j \int_{Q_{\epsilon}} \chi [\Delta h v_j]^2 g(|\Delta h v_j|^2) |D\varphi|^2 |\Delta h v_j|^2 \, dy \]

(3.20)

with \( c \equiv c(\text{data}) \). In (3.20), we also used that \( m > 1 \) and that, being \( p \leq q \) we have that \( \frac{m}{2} \leq \frac{q}{2} \leq q - \frac{2}{3} \). For \( z \in \mathbb{R}^n \), set \( \tilde{G}(z) \) := \( (g(|z|^2) + |z|^2 g'(|z|^2)) \). Now we recall (3.18) and that \( g(\cdot) \) is bounded with bounded, piecewise continuous, non-negative first derivative. Keeping (3.4) in mind, it is then easy to see that by Lemmas 2.1-2.2, we can use Fatou Lemma on the left-hand side of (3.20) and a well-known variant of the dominated convergence theorem on the right-hand side of (3.20) to end up with

\[
\frac{1}{2} \int_{B_\epsilon} \varphi^2 \chi \left( \int_0^{[Dv_j]_j} g(s) \, ds \right) \, dx + \int_{Q_{\epsilon}} \varphi^2 \chi \hat{G}(Dv_j) |DV_{\mu,p}(Dv_j)|^2 \, dy + \varepsilon_j \int_{Q_{\epsilon}} \varphi^2 \chi \hat{G}(Dv_j) |DV_{\mu,2m\sigma}(Dv_j)|^2 \, dy \\
\leq c \left( \int_{Q_{\epsilon}} \chi^m \left( |D\varphi|^{2m} + \varphi^{2m} \right) \tilde{G}(Dv_j)^m \left[ 1 + H(Dv_j)^m(q-\frac{2}{3}) \right] \, dy \right)^{\frac{1}{m}} + c \varepsilon_j \int_{Q_{\epsilon}} \chi |D\varphi|^2 g(|Dv_j|^2) \tilde{H}(Dv_j)^{m\sigma} \, dy \\
+ c \int_{Q_{\epsilon}} \left( \int_0^{[Dv_j]_j} g(s) \, ds \right) \varphi^2 \partial_\chi \, dy,
\]

(3.21)

with \( c \equiv c(\text{data}) \).

**Step 4: Higher weak differentiability and interpolation.** Our starting point is inequality (3.21) with the choice \( g \equiv 1 \), \( \frac{2}{3} \leq \tau_1 \leq \tau_2 \leq q \), \( \varphi \in C_0^\infty(B_\epsilon) \) so that

\[1_{B_{\tau_1}} \leq \varphi \leq 1_{B_{\tau_2}} \quad \text{and} \quad |D\varphi| \leq \frac{4}{\tau_2 - \tau_1}\]

and \( \chi \in W^{1,\infty}(\mathbb{R}, [0, 1]) \) with

\[\chi(t_0 - \tau_1^2) = 0, \quad \chi \equiv 1 \quad \text{on} \quad (t_0 - \tau_1^2, t_0), \quad 0 \leq \partial_\chi \leq \frac{4}{(\tau_2 - \tau_1)^2}.\]

Combining (3.21) with (3.16) we obtain

\[
\sup_{t_0 - \tau_2^2 < t < t_0} \int_{B_\epsilon} \varphi^2 \chi |Dv_j(x, t)|^2 \, dx + \int_{Q_\epsilon} \varphi^2 \chi |DV_{\mu,p}(Dv_j)|^2 \, dx
\]
+ \varepsilon_j \int_{Q_y} \varphi^2 \chi_{[V_{\tilde{\mu}, 2m\tilde{q}}]}(Dv_j)^2 \, dy \\
(3.22) \leq \frac{c}{(\tau_2 - \tau_1)^2} \left( \int_{Q_{\tau_2}} \left[ 1 + \tilde{H}(Dv_j)^{m\tilde{q}} \right] \, dy \right)^{\frac{1}{m\tilde{q}}} + \frac{cC_f}{(\tau_2 - \tau_1)^2},

with $c \equiv c(\text{data})$. Now we set

(3.23) \tilde{n} := \begin{cases} n & \text{if } n > 2 \\
\min \left\{ 2, \min \left\{ 2 \left( \frac{d}{d(q-p)+1} - 1 \right), \frac{2(d-2)}{d(q-p)+p} \right\} \right\} & \text{if } n = 2 \text{ and } \tilde{q} = q - \frac{p}{2} \\
\min \left\{ 2, \frac{2(d-2)}{2d-pd+2p} \right\} & \text{if } n = 2 \text{ and } \tilde{q} = 1
\end{cases}

and notice that, if $p \geq 2$

\[ \tilde{H}(z)^{\frac{\tilde{q}}{2}} \geq |V_{\tilde{\mu}, p}(z)|^2 \geq |z|^p \quad \text{for all } z \in \mathbb{R}^n, \]

or, if $1 < p < 2$

\[ \tilde{H}(z)^{\frac{\tilde{q}}{2}} \geq |V_{\tilde{\mu}, p}(z)|^2 \geq 2^{\frac{\tilde{q}}{2}}|z|^p \quad \text{for all } z \in \mathbb{R}^n \text{ with } |z| \geq \tilde{\mu}. \]

On a fixed time slice we use Hölder inequality and (3.22) to bound

\[
\int_{B_o} \varphi^{2(1 + \frac{\tilde{q}}{2})} |Dv_j|^{p+\frac{\tilde{q}}{2}} \, dx \leq \left( \int_{B_o} \varphi^{\frac{2d}{d(q-p)+1}} |Dv_j|^{\frac{d}{d(q-p)+1}} \, dx \right)^{\frac{d}{d(q-p)+1}} \left( \int_{B_o} \varphi^{2} |Dv_j|^2 \, dx \right)^{\frac{d}{d(q-p)+1}} \\
\leq c \left[ \left( \int_{B_o} \varphi^{\frac{2d}{d(q-p)+1}} \, dx \right)^{\frac{d}{d(q-p)+1}} + \left( \int_{B_o} \varphi^{\frac{2d}{d(q-p)+1} |V_{\tilde{\mu}, p}v_j|^{\frac{2d}{d(q-p)+1}} \, dx \right)^{\frac{d}{d(q-p)+1}} \right] \left( \int_{B_o} \varphi^{2} |Dv_j|^2 \, dx \right)^{\frac{d}{d(q-p)+1}} \\
\leq c \left[ \int_{B_o} |D\varphi|^2 \, dx + \int_{B_o} |D(\varphi V_{\tilde{\mu}, p}(Dv_j))|^2 \, dx \right] \left( \int_{B_o} \varphi^{2} |Dv_j|^2 \, dx \right)^{\frac{d}{d(q-p)+1}} \\
\leq c \left[ \int_{B_o} |D\varphi|^2 \, dx + \int_{B_o} \varphi^2 |DV_{\tilde{\mu}, p}(Dv_j)|^2 \, dx + \int_{B_o} |V_{\tilde{\mu}, p}(Dv_j)|^2 |D\varphi|^2 \, dx \right] \left( \int_{B_o} \varphi^{2} |Dv_j|^2 \, dx \right)^{\frac{d}{d(q-p)+1}}.
\]

We multiply both sides of the inequality in the previous display by $\chi^{1+\frac{\tilde{q}}{2}}$, integrate in time for $t \in (t_0 - \tau_2^2, t_0)$, take the supremum in the time variable of the last integral on the right-hand side, use (3.22) and eventually get

(3.24) \[
\int_{Q_{\tau_2}} |Dv_j|^{p+\frac{\tilde{q}}{2}} \, dy \leq \frac{c}{(\tau_2 - \tau_1)^{2(1+\frac{\tilde{q}}{2})}} \left( \int_{Q_{\tau_2}} \left[ 1 + \tilde{H}(Dv_j)^{m\tilde{q}} \right] \, dy \right)^{\frac{1}{m\tilde{q}}} + \frac{c}{(\tau_2 - \tau_1)^{2(1+\frac{\tilde{q}}{2})}}
\]

\[ \leq \frac{c}{(\tau_2 - \tau_1)^{2(1+\frac{\tilde{q}}{2})}} \left( \int_{Q_{\tau_2}} |Dv_j|^{2m\tilde{q}} \, dy \right)^{\frac{1}{2m\tilde{q}}} + \frac{c}{(\tau_2 - \tau_1)^{2(1+\frac{\tilde{q}}{2})}},\]

where $c \equiv c(\text{data}, C_f)$. We can rearrange (3.24) in the following way:

(3.25) \[
\|Dv_j\|_{L^{p+\frac{\tilde{q}}{2}}(B_1 \times (t_0 - \tau_1^2, t_0))} \leq \frac{c}{(\tau_2 - \tau_1)^{\frac{4}{n} + \frac{4}{n}}} \|Dv_j\|_{L^{2m\tilde{q}}(B_2 \times (t_0 - \tau_2^2, t_0))}^{\frac{2m\tilde{q}}{4\tilde{q} + 2}} + \frac{c}{(\tau_2 - \tau_1)^{\frac{4}{n} + \frac{4}{n}}}.
\]

Notice that, by (2.4) and (2.3), there holds that

(3.26) \[ p \leq q < 2m\tilde{q} < p + \frac{4}{n}, \]
so we can apply the interpolation inequality

\begin{equation}
\|Dv_j\|_{L^2(\Omega, \mathbb{R}^N)} \leq \|Dv_j\|^\theta_{L^p(\mathbb{R}^N)} \|Dv_j\|^{1-\theta}_{L^q(\mathbb{R}^N)} \|Dv_j\|^\theta_{L^{p+}\frac{2}{(p+1)}} \end{equation}

where \( \theta \in (0, 1) \) solves

\[
\frac{1}{2m\hat{q}} = \frac{1-\theta}{p} + \frac{\hat{q} \theta}{\hat{n}p + 4} \quad \Rightarrow \quad \theta = \frac{(\hat{n}p + 4)(2m\hat{q} - p)}{8m\hat{q}}.
\]

Plugging (3.27) into (3.25) we get

\begin{align*}
\|Dv_j\|_{L^{p+}\frac{2}{(p+1)}}(B_{r_2}(t_0-t_1^2, t_0)) & \leq \frac{c}{(\tau_2 - \tau_1)^{\frac{2(n+2)}{n+4}}} \|Dv_j\|^{\frac{2(n+2)}{2n+4}}_{L^p(\mathbb{R}^N)} \|Dv_j\|^{\frac{2(1-\theta)(n+2)}{4m}}_{L^{p+}\frac{2}{(p+1)}}(B_{r_2}(t_0-t_1^2, t_0)) \\
& \quad + \frac{c}{(\tau_2 - \tau_1)^{\frac{2(n+2)}{n+4}}},
\end{align*}

with \( c \equiv c(\text{data}, C_f) \). By (2.3) and (3.23) there holds that

\[
\frac{2\theta \hat{q}(\hat{n} + 2)}{\hat{n}p + 4} < 1,
\]

so we can apply Young inequality with conjugate exponents

\begin{equation}
\left(\frac{4m}{(2m\hat{q} - p)(\hat{n} + 2)}, \frac{4m}{4m - (2m\hat{q} - p)(\hat{n} + 2)}\right)
\end{equation}

to get

\begin{align*}
\|Dv_j\|_{L^{p+}\frac{2}{(p+1)}}(B_{r_2}(t_0-t_1^2, t_0)) & \leq \frac{1}{2m\hat{q}} \|Dv_j\|_{L^{p+}\frac{2}{(p+1)}}(B_{r_2}(t_0-t_1^2, t_0)) \\
& \quad + \frac{c(\text{data}, C_f)}{(\tau_2 - \tau_1)^{\theta}} \left[ 1 + \|Dv_j\|_{L^p(\mathbb{R}^N)}^{\beta} \right],
\end{align*}

where we set \( \hat{\theta} := \frac{8m(\hat{n} + 2)}{(\hat{n}p + 4)(4m - (2m\hat{q} - p)(\hat{n} + 2))} \) and \( \beta := \frac{8m(1-\theta)(\hat{n} + 2)}{4m - (2m\hat{q} - p)(\hat{n} + 2))((\hat{n}p + 4)} \). Now we are in position to apply Lemma 2.7 and (3.16) to the inequality in the previous display and conclude with

\begin{equation}
\|Dv_j\|_{L^{p+}\frac{2}{(p+1)}}(B_{r_2}(t_0-t_b^2, t_0)) \leq \frac{c}{\hat{\theta}} \left[ 1 + \|Dv_j\|_{L^p(\mathbb{R}^N)}^{\beta} \right]
\end{equation}

for \( c \equiv c(\text{data}) \). Finally, Hölder inequality and (3.31) in particular imply that

\begin{equation}
\|Dv_j\|_{L^q(\Omega, C^{s'}(\mathbb{R}^N))} \leq \frac{c(\text{data}, C_f, s)}{\hat{\theta}} \quad \text{for all } s \in \left[1, p + \frac{4}{\hat{n}}\right],
\end{equation}

thus (2.3) and (3.23) render that \( s = q \) and \( s = 2m\hat{q} \) are both admissible choices. In the previous two displays, we also expanded the expression of \( C_f \).

Step 5: Fractional differentiability in space. Let \( t_0 \in (0, T) \) be any number and \( \phi \in C_c^\infty(\mathbb{R}) \) and \( \chi \in W^{1,\infty}(\mathbb{R}, [0, 1]) \) be two cut-off functions satisfying

\begin{equation}
\chi(t_0 - \theta^2/4) = 0, \quad \chi = 1 \text{ on } (t_0 - \theta^2/16, t_0), \quad 0 \leq \partial_\theta \chi \leq \frac{4}{\theta^2}
\end{equation}

respectively. If \( p \geq 2 \), by Lemma 2.5 we have

\[
\int_{Q_{\theta/2}} \varphi^2 \chi |\Delta_h V_{\theta,p}(Dv_j)|^2 \, dy \sim |h|^{-2} \int_{Q_{\theta/2}} \varphi^2 \chi |D_h(Dv_j)|^2 \, dy.
\]
and a standard covering argument, we can conclude

\[ (3.35) \quad \geq |h|^{-2} \int_{Q_{e/2}} \varphi^2 |\tau_h Dv_j|^p \, dy, \]

while, for \( 1 < p < 2 \) we have that

\[ (3.36) \quad |h|^{-p} \int_{Q_{e/2}} \varphi^2 |\tau_j Dv_j|^p \, dy \leq \left( |h|^{-2} \int_{Q_{e/2}} \varphi^2 |\Delta_h V_{1,p}(Dv_j)|^2 \, dx \right)^{\frac{p}{2}} \left( \int_{Q_{e/2}} \varphi^2 |\Delta_h Dv_j|^p \, dy \right)^{\frac{2-p}{2}}. \]

Therefore, if \( p \geq 2 \), by (3.35), (3.20) with \( g \equiv 1 \), \( \varphi \) and \( \chi \) as in (3.33)-(3.34), (3.22) and (3.32) we obtain

\[ (3.37) \quad \lesssim \limsup_{|h| \to 0} g(h) \lesssim g^{-2} \left[ 1 + \left( \int_{Q_{e/2}} \tilde{H}(Dv_j)^{m\tilde{q}} \, dy \right)^{\frac{p}{m\tilde{q}}} \right] \lesssim g^{-\theta}, \]

while, for \( 1 < p < 2 \) we have, using also (3.16)

\[ (3.38) \quad \lesssim g^{-p} \left[ 1 + \left( \int_{Q_{e/2}} \tilde{H}(Dv_j)^{m\tilde{q}} \, dy \right)^{\frac{p}{m\tilde{q}}} \right] \lesssim g^{-\tilde{\theta}}, \]

In both, (3.37)-(3.38), \( \tilde{\theta} \equiv \tilde{\theta}(n,p,q,d) \) and the constants implicit in "\( \lesssim \)" depend on (data, \( C_f \)). Combining (3.37)-(3.38), Proposition 2.1 and a standard covering argument, we can conclude that

\[ (3.39) \quad Dv_j \in L^p_{\text{loc}}(0,T;W^{s/p}_{\text{loc}}(\Omega,\mathbb{R}^n)) \quad \text{for all} \quad \zeta \in \left( 0, \min \left\{ 1, \frac{2}{p} \right\} \right). \]

**Step 6: Fractional differentiability in time.** We aim to prove that

\[ (3.40) \quad \tilde{\alpha}_j(Dv_j) \in L^l_{\text{loc}}(0,T;W^{1,l}_{\text{loc}}(\Omega,\mathbb{R}^n)) \quad \text{for some} \quad l \equiv l(n,p,q,d) \in (1, \min\{2,p\}). \]

The forthcoming argument appears for instance in [18] for the \( p \)-laplace case with \( p \geq 2 \). Before going on, let us record some computations which will be helpful in a few lines. By the definition given in (3.8) it is clear that

\[ (3.41) \quad \max \left\{ \frac{p}{2}, q - \frac{p}{2}, m\tilde{q} \right\} = m\tilde{q}. \]

Moreover, by (3.26) we also have that there exists \( l \in (1,2) \) so that

\[ (3.42) \quad \max \{ s_1, s_2 \} < p + \frac{4}{n}, \]

where we set

\[ s_1 := \frac{2l(m\tilde{q} - 1)}{2-l} \quad \text{and} \quad s_2 := \frac{dl(q - 1)}{(d-l)}. \]
For \( \varphi, \chi \) as in (3.33)-(3.34) and \( h \) as in (3.18), we expand
\[
\int_{Q_{h/2}} \varphi^2 \chi \left[ \tau_n \tilde{a}_j(x, t, Dv_j) \right]^l \ dy \lesssim \int_{Q_{h/2}} \varphi^2 \chi \left[ \tilde{a}_j(x + h, t, Dv_j(x + h)) - \tilde{a}_j(x, t, Dv_j(x + h)) \right]^l \ dy
\]
\[
+ \int_{Q_{h/2}} \varphi^2 \chi \left[ \tilde{a}_j(x, t, Dv_j(x + h)) - \tilde{a}_j(x, t, Dv_j(x)) \right]^l \ dy =: (I) + (II)
\]
and estimate, via (3.12), (3.42) and Hölder inequality,
\[
(I) \lesssim |h|^l \int_{Q_{h/2}} \varphi^2 \chi \left( \int_0^1 \gamma(x + h \lambda, t) \ d\lambda \right)^l \left[ 1 + D(h)^{\frac{m-1}{2}} \right] \ dy
\]
\[
\lesssim |h|^l \| \gamma \|_{L^\infty(\Omega_T)} \left( \int_{Q_{h/2}} \left[ 1 + D(h)^{\frac{m-1}{2}} \right] \ dy \right)^{\frac{m-1}{2}}.
\]
Concerning term (II) we distinguish three cases: \( q \geq p \geq 2, q \geq 2 > p \) and \( 2 > q \geq p \). If \( q \geq p \geq 2 \), via (3.12), (3.41), (3.42), Hölder inequality, Lemmas 2.5 and 2.6 we get
\[
(II) \lesssim |h|^l \mu^{p-2} \left( \int_{Q_{h/2}} \varphi^2 \chi |\Delta_h Dv_j|^p \ dy \right)^{\frac{1}{p}}
\]
\[
+ |h|^l \left( \int_{Q_{h/2}} \varphi^2 \chi D(h)^{\frac{m(q-2)}{q-1}} \ dy \right)^{\frac{2q}{q-1}} \left( \int_{Q_{h/2}} \varphi^2 \chi \left| \Delta_h V_{\tilde{h}, \mu}(Dv_j) \right|^2 \ dy \right)^{\frac{q-1}{q}}
\]
\[
+ |h|^l \left( \int_{Q_{h/2}} \varphi^2 \chi D(h)^{\frac{m(q-2)}{q-1}} \ dy \right)^{\frac{2q}{q-1}} \left( \int_{Q_{h/2}} \varphi^2 \chi \left| \Delta_h V_{\tilde{h}, 2mq}(Dv_j) \right|^2 \ dy \right)^{\frac{q-1}{q}}.
\]
For \( q \geq 2 > p \), recalling (3.4) we obtain
\[
(II) \lesssim |h|^l \mu^{p-2} \left( \int_{Q_{h/2}} \varphi^2 \chi |\Delta_h Dv_j|^p \ dy \right)^{\frac{1}{p}}
\]
\[
+ |h|^l \left( \int_{Q_{h/2}} \varphi^2 \chi D(h)^{\frac{m(q-2)}{q-1}} \ dy \right)^{\frac{2q}{q-1}} \left( \int_{Q_{h/2}} \varphi^2 \chi \left| \Delta_h V_{\tilde{h}, \mu}(Dv_j) \right|^2 \ dy \right)^{\frac{q-1}{q}}
\]
\[
+ |h|^l \left( \int_{Q_{h/2}} \varphi^2 \chi D(h)^{\frac{m(q-2)}{q-1}} \ dy \right)^{\frac{2q}{q-1}} \left( \int_{Q_{h/2}} \varphi^2 \chi \left| \Delta_h V_{\tilde{h}, 2mq}(Dv_j) \right|^2 \ dy \right)^{\frac{q-1}{q}}.
\]
Finally, when \( 2 > q \geq p \) we use (3.4) to conclude that
\[
(II) \lesssim |h|^l \mu^{p-2} \left( \int_{Q_{h/2}} \varphi^2 \chi |\Delta_h Dv_j|^p \ dy \right)^{\frac{1}{p}}
\]
\[
+ \int_{Q_{h/2} \cap \{ \tilde{D}(h) \leq 1 \}} \varphi^2 \chi \left( D(h)^{\frac{m-1}{2}} D(h)^{\frac{m-1}{2}} |\tau_n Dv_j| \right)^l \ dy
\]
\[
+ \int_{Q_{h/2} \cap \{ \tilde{D}(h) > 1 \}} \left[ D(h)^{\frac{m-1}{2}} |\tau_n Dv_j| \right]^l \ dy.
\]
on the left-hand side of the chain of inequalities for (3.44) from (3.45).

Finally, applying Fataou’s lemma and Lemma 2.2 on the left-hand side of the chain of inequalities displayed above we obtain that

\[
\limsup_{|h| \to 0} \int_{Q_{x/4}} |D\bar{a}_j(x,t,Dv_j)|^l \, dy \lesssim \|\gamma\|_{L^p(\Omega_T)} \left( \int_{Q_{x/2}} 1 + |Dv_j|^{p_2} \, dy \right)^\frac{l(m-1)}{2s} \quad (3.43)
\]

with \( c \equiv c(\text{data}, C_f, \bar{\mu}) \) and \( \bar{\theta} \equiv \bar{\theta}(n, p, q, d) \). With (3.43) and a standard covering argument we deduce (3.40). Now, whenever we consider a subset of type \( \Omega \times (t_1, t_2) \subset \Omega_T \) with \( \Omega \subset \subset \Omega \) open, from (3.43) and (3.40) and a covering argument we have that

\[
\| \text{div} \bar{a}_j(\cdot, t, Dv_j) \|_{L^1(\Omega \times (t_1, t_2))} \leq c \| D\bar{a}_j(\cdot, t, Dv_j) \|_{L^1(\Omega \times (t_1, t_2))} \leq c,
\]

for \( c \equiv c(\text{data}, C_f, \mu, t_1, T - t_2, \text{dist}(\partial \Omega)) \). Finally, integrating by parts in (3.13) and using (3.44) we obtain that

\[
\partial_t v_j \in L^l_{\text{loc}}(\Omega_T) \quad \text{with} \quad l \equiv l(n, p, q, d) \in (1, \min\{p, 2\}).
\]

Step 6: Convergence. A standard covering argument combined with Proposition 2.1, (3.32), (3.37)-(3.39) and (3.44)-(3.45) respectively then implies that if \( \Omega \subset \subset \Omega \) is any open subset and \((t_1, t_2) \subset (0, T)\) is an interval, then

\[
\| Dv_j \|_{L^p(\Omega \times (t_1, t_2))} \leq c \quad \text{for all} \quad s \in \left[ 1, p + \frac{4}{n} \right];
\]

\[
\| v_j \|_{L^p(t_1, t_2; W^{1,1+\varsigma}(\Omega))} \leq c \quad \text{for all} \quad \varsigma \in \left( 0, \min\left\{ 1, \frac{2}{p} \right\} \right);
\]

\[
\| v_j \|_{W^{1,1+\varsigma}(t_1, t_2; L^p(\Omega))} \leq c \quad \text{for all} \quad \varsigma \in (0, 1),
\]

with \( c \equiv c(\text{data}, s, \varsigma, \varsigma, C_f, t_1, T - t_2, \text{dist}(\partial \Omega)) \). Estimates (3.47) and (3.48) render that

\( \{ v_j \} \) is uniformly bounded in \( W^{1,1+\varsigma}(0, T; L^p_{\text{loc}}(\Omega)) \cap L^p_{\text{loc}}(0, T; W^{1,1+\varsigma+\varsigma}_{\text{loc}}(\Omega)), \)
therefore we can first choose \( \ell \in \left( \frac{p-1}{p} \right) \) so that \( l > \frac{p}{1+p} \) and then apply Lemma 2.4 with \( a_1 = l, a_2 = p, \sigma = \ell, X = W^{1+p,+}_\text{loc}(\Omega), B = W^{1+p}_\text{loc}(\Omega) \) and \( Y = L^l_{\text{loc}}(\Omega) \) to conclude that

\[
(3.49) \quad \text{there exists a subsequence} \{ v_j \} \text{ strongly converging to} \ v \text{ in} \ L^l_{\text{loc}}(0,T; W^{1+p}_\text{loc}(\Omega)),
\]

where we also used that \( l < p \). Using (3.46) we also see that, again up to subsequences,

\[
(3.50) \quad Dv_j \to Dv \text{ in} \ L^s_{\text{loc}}(\Omega_T, \mathbb{R}^n) \quad \text{for all} \ s \in \left[1, p + \frac{4}{n}\right]
\]

which assures that

\[
(3.51) \quad \|Dv\|_{L^s((\tilde{\Omega} \times (t_1, t_2)))} \leq c(\text{data}, s, C_f, t_1, T - t_2, \text{dist}(\tilde{\Omega}, \partial \Omega)) \quad \text{and} \quad v|_{\partial \Omega \times \Omega_T} = f|_{\partial \Omega \times \Omega_T}.
\]

By (3.26), (3.49), (3.50), (3.51) and the interpolation inequality

\[
\|Dv_j - Dv\|_{L^s((\tilde{\Omega} \times (t_1, t_2)))} \leq \|Dv_j - Dv\|_{L^s((\tilde{\Omega} \times (t_1, t_2)))}^{1-\theta} \|Dv_j - Dv\|_{L^p(\tilde{\Omega} \times (t_1, t_2)))}^{\theta} \\
\leq c\|Dv_j - Dv\|_{L^p(\tilde{\Omega} \times (t_1, t_2)))}^{\theta}
\]

with \( c \equiv c(\text{data}, s, C_f, t_1, T - t_2, \text{dist}(\tilde{\Omega}, \partial \Omega)) \) and

\[
\frac{1}{s} = \frac{\tilde{n} \theta}{\tilde{n} p + 4} + \frac{1 - \theta}{l} \quad \implies \theta = \frac{(\tilde{n} p + 4)(s - l)}{s(\tilde{n} p + 4 - \tilde{n} l)},
\]

we can conclude that

\[
(3.52) \quad Dv_j \to Dv \text{ in} \ L^s_{\text{loc}}(0,T; L^s_{\text{loc}}(\Omega, \mathbb{R}^n)) \quad \text{for all} \ s \in \left[1, p + \frac{4}{n}\right].
\]

Once (3.52) is available, we can look back at (3.20) with \( g \equiv 1 \), send first \( j \to \infty \) and then \( |b| \to 0 \) and rearrange the right-hand side with the help of (3.31) to obtain (3.7). Moreover, using (3.52), (3.12) and the dominated convergence theorem, we can pass to the limit in (3.13) to conclude that \( v \) satisfies

\[
(3.53) \quad \int_{\Omega_T} \left[ v \partial_t \varphi - a(x, t, Dv) \cdot D\varphi \right] \, dy = 0 \quad \text{for all} \ \varphi \in C_c^\infty(\Omega_T).
\]

Once (3.5), (3.53) and (3.7) are available, we can repeat the same computations leading to (3.40)-(3.45) with \( \tilde{a}(\cdot), v, v_j \) to obtain (3.6).

**Step 8: The initial boundary condition.** With (3.53), the energy estimate (3.16) and the continuity of \( f \) in time prescribed by (2.5), we can proceed exactly as in [8, Section 6.5] to verify the requirements of Definition 1 (formulated for \( v \) and \( \tilde{a}(\cdot) \) of course).

### 4. Gradient bounds

This section is divided into two parts: in the first one we construct a sequence of maps satisfying suitable uniform estimates and in the second we prove that such a sequence converges to a weak solution of problem (1.1).

#### 4.1. Uniform \( L^\infty \)-estimates.

We consider again Cauchy-Dirichlet problem (1.1) with \( a(\cdot) \) described by (2.1)-(2.3) and \( f \) as in (2.5). To construct a suitable family of approximating problems, this time we only regularize the vector field \( a(\cdot) \) in the gradient variable by convolution against a sequence \( \{ \phi_j \} \) of mollifiers of \( \mathbb{R}^n \) with the following features:

\[
\phi \in C_c^\infty(B_1), \quad \|\phi\|_{L^1(\mathbb{R}^n)} = 1, \quad \phi_j(x) := j^n \phi(jx), \quad B_{3/4} \subset \text{supp}(\phi).
\]

This leads to the definition of the approximating vector field

\[
a_j(x, t, z) := \int_{B_1} a(x, t, z + j^{-1} z') \phi(z') \, dz',
\]
satisfying the structural conditions

\[
\begin{cases}
t \mapsto a_j(x, t, z) & \text{measurable for all } x \in \Omega, z \in \mathbb{R}^n \\
\quad x \mapsto a_j(x, t, z) & \text{differentiable for all } t \in (0, T), z \in \mathbb{R}^n \\
\quad z \mapsto a_j(x, t, z) \in C^1(\mathbb{R}^n, \mathbb{R}^n) & \text{for all } (x, t) \in \Omega_T
\end{cases}
\]  

(4.2)

and

\[
\begin{cases}
|a_j(x, t, z)| + H_j(z)\frac{1}{n} |\partial_z a_j(x, t, z)| \leq c \left[ H_j(z) \frac{n}{2} + H_j(z) \frac{n}{2} \right] \\
\partial_z a_j(x, t, z) \geq cH_j(z) \frac{1}{n} |x|^2 \\
|\partial_z a_j(x, t, z)| \leq \gamma(x, t) \left[ H_j(z) \frac{n}{2} + H_j(z) \frac{n}{2} \right],
\end{cases}
\]

(4.3)

for all \((x, t) \in \Omega_T, z, \xi \in \mathbb{R}^n, \gamma \text{ as in (2.4)}, \) with \(c \equiv c(n, \nu, L, p, q), \) see [12, Section 4.5] for more details on this matter. In (4.3),

\[
\mu_j := \mu + j^{-1} > 0 \quad \text{and} \quad H_j(z) := (\mu_j^2 + |z|^2).
\]

We then define problem

\[
\begin{cases}
\partial_t v - \text{div} a_j(x, t, Dv) = 0 & \text{in } \Omega_T \\
v = f & \text{on } \partial_{par} \Omega_T,
\end{cases}
\]

(4.4)

with \(f \) as in (2.5). By (4.2)-(4.3), we see that the assumptions of Proposition 3.1 are satisfied, thus problem (4.4) admits a solution \(u_j \in L^p(0, T; W^{1, p}(\Omega)) \) in the sense of Definition 1, satisfying (3.5), (3.6) and (3.7). In particular, (3.5) authorizes to test (2.6) against test functions defined as products of \(u_j \) with suitable cut-off functions, therefore, for such a solution, we can repeat almost the same computations leading to (3.21) (with \(\varepsilon_j \equiv 0, \) of course), for getting

\[
\begin{align*}
\frac{1}{2} \int_{B_{\varepsilon_i}} \varphi^2 \chi \left( \int_0^{Du_{\epsilon_j}} g(s) \, ds \right) \, dx \\
+ \int_{Q_r} \varphi^2 \chi \hat{G}(Du_j) |DV_{\mu_j, p}(Du_j)|^2 \, dy \\
\leq c \left( \int_{Q_r} \chi^m (|Du|^2 + \varphi^2) \hat{G}(Du_j)^m \left[ 1 + H_j(Du_j)^{m(v - \frac{1}{2})} \right] \, dy \right)^{\frac{1}{m}} \\
+ c \int_{Q_r} \left( \int_0^{Du_{\epsilon_j}} g(s) \, ds \right) \varphi^2 \partial_t \chi \, dy,
\end{align*}
\]

(4.5)

with \(c \equiv c(\text{data}), g \in W^{1, \infty}(\mathbb{R}) \) non-negative with bounded, non-negative, piecewise continuous first derivative, \(\varphi \in C_c^\infty(B_{\varepsilon_i}[0, 1]) \) and \(\chi \in W^{1, \infty}(0, T) \). The quantity \(\hat{G}(Du_j) \) is defined as in Step 3 of the proof of Proposition 3.1, clearly with \(u_j \) replacing \(v_j \). For \(i \in \mathbb{N}, \) we inductively define radii \(\varrho_i := \tau_i + (\tau_2 - \tau_1)2^{-i+1} \) with \(\frac{\varrho_i}{2} \leq \tau_1 < \tau_2 \leq \varrho_i \), select cut-off functions \(\varphi_i \in C_c^1(B_{\varrho_i}) \) so that

\[
\mathbb{I}_{B_{\varrho_{i+1}}} \leq \varphi_i \leq \mathbb{I}_{B_{\varrho_i}} \quad \text{and} \quad |D\varphi_i| \leq \frac{4}{\varrho_i - \varrho_{i+1}} = \frac{2^{i+2}}{(\tau_2 - \tau_1)},
\]

and \(\chi_i \in W^{1, \infty}_0((0, \varrho_i^2), [0, 1]) \) satisfying

\[
\chi_i(t_0 - \varrho_i^2) = 0, \quad \chi_i \equiv 1 \quad \text{on} \quad (t_0 - \varrho^2_{i+1}, t_0), \quad |\partial_t \chi_i| \leq \frac{4}{(\varrho_i - \varrho_{i+1})^2} \leq \frac{2^i}{(\tau_2 - \tau_1)^2}
\]

and numbers

\[
\kappa_1 \equiv 0, \quad \kappa_i := \frac{\Gamma}{m + \omega \kappa_{i-1}} \quad \text{for} \quad i \geq 2, \quad \alpha_i := m\tilde{q} + m\kappa_i,
\]

(4.6)
where we set
\[
\omega := \frac{1}{m} \left[ 1 + \frac{2}{n} \right] > 1 \quad \text{and} \quad \Gamma := \frac{p}{2} + \frac{2}{n} - m\tilde{q} > 0.
\]
In (4.5) we take \(\varphi \equiv \varphi_i, \chi \equiv \chi_i\) and, for \(M > 0\) set
\[
g(s) \equiv g_{i,M}(s) := \begin{cases} \mu_i^2 + s & \text{if } s \leq M \\ \mu_i^2 + M & \text{if } s > M, \end{cases}
\]
which is admissible by construction in (4.5). Clearly,
\[
g_{i,M}(s) \leq (\mu_i^2 + s)^{\kappa_i} \quad \text{for all } s \in [0, \infty).
\]
All in all, (4.5) becomes
\[
\frac{1}{2} \int_{B_{\varepsilon_i}} \varphi_i^2 \chi_i \left( \int_0^{|Du_j|} g_{i,M}(s) \, ds \right) \, dx
+ \int_{Q_{\varepsilon_i}} \varphi_i^2 \chi_i |D\tilde{g}_{i,M}(Du_j)| |DV_{\mu_i,p}(Du_j)|^2 \, dy
\leq c \left( \int_{Q_{\varepsilon_i}} \chi_i^m \left( |D\varphi_i|^2 \mu_i^2 + \varphi_i^2 \right)^m \tilde{g}_{i,M}(Du_j)^m \left[ 1 + H_j(Du_j)^m(q - \frac{2}{m}) \right] \, dy \right)^{\frac{1}{m}}
+ c \int_{Q_{\varepsilon_i}} \left( \int_0^{|Du_j|^2} g_{i,M}(s) \, ds \right) \varphi_i^2 \partial \chi_i \, dy,
\]
where we defined \(\tilde{g}_{i,M}\) in the obvious way: \(\tilde{g}_{i,M}(z) := (g_{i,M}(|z|) + |z|^2 g_{i,M}(|z|^2))\). As we only know that \(\{u_j\}\) satisfies (3.5)-(3.7), we have to proceed inductively. We shall prove that
\[
H_j(Du_j)^{\kappa_1} \in L^1(Q_{\varphi_i}) \Rightarrow H_j(Du_j)^{\kappa_{i+1}} \in L^1(Q_{\varphi_{i+1}}) \quad \text{for all } i \in \mathbb{N}.
\]

Basic step. Let us verify (4.10) for \(i = 1\). In this case, we immediately see that \(\tilde{g}_{1,M}(Du_j) \equiv 1\) and notice that, since the approximating sequence \(\{u_j\}\) we choose satisfies (3.5)-(3.7), all the computations made in Step 3 of Section 3 are legal without further corrections to the growth of the vector field defined in (4.1). Moreover, a quick inspection of estimates (3.21)-(3.22) points out the dependency of the constants from \(C_i\) is due only to the presence of the term multiplying \(\varepsilon_j\), which, in the present case is zero. Hence, (4.9) becomes (3.22) with \(\varepsilon_j \equiv 0, \varphi \equiv \varphi_1\) and \(\chi \equiv \chi_1\). Since \(\alpha_1 = m\tilde{q}\) and \(\alpha_2 = \frac{q}{2} + \frac{2}{m}\), we can easily deduce from (3.24) (with \(\tau_1 = \tau_2 = \tau_1\)) and no dependencies of the constants from \(C_i\) that \(H_j(Du_j)^{\kappa_2} \in L^1(Q_{\varphi_1})\).

Induction step. We assume now that
\[
H_j(Du_j)^{\kappa_i} \in L^1(Q_{\varphi_i})
\]
and expand into (4.9) the expression of \(\tilde{g}_{i,M}(Du_j)\) for getting, after a few standard manipulations:
\[
\frac{1}{2} \int_{B_{\varepsilon_i}} \varphi_i \chi_i \left( \int_0^{\min\{|Du_j|, M\}} (\mu_j^2 + s)^{\kappa_i} \, ds \right) \, dx
+ \int_{Q_{\varepsilon_i} \cap \{|Du_j|^2 \leq M\}} \varphi_i^2 \chi_i (\mu_j^2 + |Du_j|^2)^{\kappa_i} |DV_{\mu_j,p}(Du_j)|^2 \, dy
\leq c(1 + \kappa_i) \left( \int_{Q_{\varepsilon_i}} \chi_i^m \left( |D\varphi_i|^2 \mu_i^2 + \varphi_i^2 \right)^m \left[ 1 + H_j(Du_j)^m(q - \frac{2}{m}) \right] \, dy \right)^{\frac{1}{m}}
+ \frac{c}{1 + \kappa_i} \int_{Q_{\varepsilon_i}} \varphi_i^2 \partial \chi_i H_j(Du_j)^{1+\kappa_i} \, dy.
\]
\[
\leq c(1 + \kappa_i) \left( \int_{Q_{2\xi}} \left[ \chi^m \left( |D\varphi_i|^{2m} + \varphi_i^{2m} \right) + \varphi_i^{2m} |\partial_t \chi_i|^m \right] \left[ 1 + H_{j}(Du_j)^{\alpha_i} \right] \, dy \right)^{\frac{1}{\alpha_i}},
\]

for \( c \equiv c(\text{data}) \). For the inequality in the previous display we used in particular (4.8) and the definition of \( \varphi_i, \chi_i \). Now we can send \( M \to \infty \) in the previous display and apply Fatou Lemma on the left-hand side, the dominated convergence theorem, (4.6)3 and (11.1) on the right-hand side to conclude with

\[
\frac{1}{2} \int_{B_{\xi}} \varphi_i \chi_i H_j(Du_j)^{1+\kappa_i} \, dx + (1 + \kappa_i) \int_{Q_{2\xi}} \varphi_i^2 \chi_i H_j(Du_j)^{\kappa_i} |DV_{\mu, p}(Du_j)|^2 \, dy
\]

\[
\leq c(1 + \kappa_i)^2 \left( \int_{Q_{2\xi}} \left[ \chi^m \left( |D\varphi_i|^{2m} + \varphi_i^{2m} \right) + \varphi_i^{2m} |\partial_t \chi_i|^m \right] \left[ 1 + H_{j}(Du_j)^{\alpha_i} \right] \, dy \right)^{\frac{1}{\alpha_i}},
\]

where \( c \equiv c(\text{data}) \). Next, with (3.7) at hand, we compute

\[
|DH_{j}(Du_j)^{\frac{p-2\kappa_i}{2}}|^2 = \left( \frac{p + 2\kappa_i}{p} \right)^2 \frac{H_{j}(Du_j)^{\kappa_i}}{|DH_{j}(Du_j)^{\frac{p-2}{2}}|}^2 |Du_j|^2
\]

and

\[
|DV_{\mu, p}(Du_j)|^2 = \left( \frac{p - 2}{2} \right)^2 \frac{H_{j}(Du_j)^{\kappa_i}}{|Du_j|^2} |Du_j \cdot D^2 u_j|^2 + H_{j}(Du_j)^{\kappa_i} |D^2 u_j|^2 + (p - 2)H_{j}(Du_j)^{\kappa_i} |Du_j \cdot D^2 u_j|^2
\]

\[
\geq \min\{ 1, (p - 1) \} H_{j}(Du_j)^{\kappa_i} |D^2 u_j|^2,
\]

so, keeping in mind that

\[
|DH_{j}(Du_j)^{\frac{p-2\kappa_i}{2}}|^2 \leq \left( \frac{p}{2} \right)^2 \frac{H_{j}(Du_j)^{\kappa_i}}{4 \min\{ p - 1, 1 \}} |D^2 u_j|^2,
\]

we end up with

\[
(4.13) \quad |DH_{j}(Du_j)^{\frac{p-2\kappa_i}{2}}|^2 \leq \frac{(p + 2\kappa_i)^2}{4 \min\{ p - 1, 1 \}} H_{j}(Du_j)^{\kappa_i} |DV_{\mu, p}(Du_j)|^2.
\]

Plugging (4.13) into (4.12) we obtain, after routine calculation,

\[
\sup_{t_0 - (\eta, 2)} \int_{B_{\xi}} \varphi_i^2 \chi_i H_j(Du_j)^{1+\kappa_i} \, dx + \int_{Q_{2\xi}} \chi_i |D(\varphi_i[H_{j}(Du_j)^{\frac{p-2\kappa_i}{2}}] + 1)|^2 \, dy
\]

\[
\leq \sup_{t_0 - (\eta, 2)} \int_{B_{\xi}} \varphi_i \chi_i H_j(Du_j)^{1+\kappa_i} \, dx
\]

\[
+ c \int_{Q_{2\xi}} \chi_i \left[ \varphi_i^2 \left( \frac{p}{2} \right)^2 \frac{H_{j}(Du_j)^{\kappa_i}}{4 \min\{ p - 1, 1 \}} |D^2 u_j|^2 + |D\tilde{\phi}|^2 \left( H_{j}(Du_j)^{\kappa_i} \right)^2 + 1 \right] \, dy
\]

\[
(4.14) \quad \leq c(1 + \kappa_i)^4 \left( \int_{Q_{2\xi}} \left[ \chi^m \left( |D\varphi_i|^{2m} + \varphi_i^{2m} \right) + \varphi_i^{2m} |\partial_t \chi_i|^m \right] \left[ 1 + H_{j}(Du_j)^{\alpha_i} \right] \, dy \right)^{\frac{1}{\alpha_i}},
\]

with \( c \equiv c(\text{data}) \). For \( \tilde{n} \) as in (3.23), we define \( \tilde{\sigma} := 2(1 + \kappa_i)\tilde{n}^{-1} \). On a fixed time slice, we apply in sequence Hölder and Sobolev-Poincaré inequalities to get

\[
\int_{B_{\xi}} \varphi_i^{2\left(1 + \frac{\tilde{n}}{2}\right)} H_j(Du_j)^{\frac{p+2\kappa_i}{2} + \tilde{\sigma}} \, dx
\]

\[
\leq \left( \int_{B_{\xi}} \left[ \varphi_i^2 \left( H_j(Du_j)^{\frac{p+2\kappa_i}{2}} + 1 \right) \right]^\frac{\alpha_i}{2} \, dx \right)^{\frac{2}{\alpha_i}} \left( \int_{B_{\xi}} \varphi_i^{2\tilde{\sigma}} H_j(Du_j)^{\frac{\tilde{n}}{2} + \tilde{\sigma}} \, dx \right)^{\frac{1}{\tilde{n}}},
\]
(4.15) \[ \leq c \left( \int_{B_{r_i}} |\mathcal{D}[\varphi_i(H_j(Du_j)^{\frac{m+2\kappa}{2}} + 1)]|^2 \, dx \right) \left( \int_{B_{r_i}} \varphi_i^2 H_j(Du_j)^{\frac{m}{2}} \, dx \right)^{\frac{2}{m}} , \]

for \( c \equiv c(n,p,q,d) \). Now we multiply both sides of (4.15) by \( \dot{x}_i^\infty \), integrate with respect to \( t \in (t_0 - (r_i)\sigma)^2, t_0) \), take the supremum over \( t \in (t_0 - (r_i)\sigma)^2, t_0) \) on the right-hand side, use (4.14) and eventually obtain

(4.16) \[ \int_{Q_{n+1}} (\varphi_i^2 \chi_i)^{1+\frac{2}{m}} \left[ 1 + H_j(Du_j)^{\frac{m+2\kappa}{2} + \sigma_i} \right] \, dy \]

\[ \leq c(1 + \kappa_i)^4(1 + \frac{2}{m}) \left( \int_{Q_{n+1}} \left[ \chi_i^m \left( \varphi_i^{2m} + |D\varphi_i|^{2m} + \varphi_i^{2m} |\partial_\alpha \chi_i|^m \right) \left[ 1 + H_j(Du_j)^{\alpha_i} \right] \right] \, dy \right)^{\frac{2}{m}(1 + \frac{2}{m})} , \]

where \( c \equiv c(data) \). In the light of (4.6)- (4.7) we have

\[ \frac{p}{2} + \kappa_i + \sigma_i = \frac{p}{2} + \frac{2}{n} + m\omega\kappa_i = \left( \frac{p}{2} + \frac{2}{n} - m\tilde{q} \right) + m(\tilde{q} + \omega\kappa_i) \]

(4.17) \[ = m \left( \frac{\Gamma}{m} + \tilde{q} + \omega\kappa_i \right) = m(\tilde{q} + \kappa_{i+1}) = \alpha_{i+1} , \]

so, recalling also the definition of \( \chi_i, \varphi_i \) (4.16) becomes

\[ \int_{Q_{n+1}} H_j(Du_j)^{\alpha_{i+1}} \, dy \leq \frac{c(data,i)}{(\varrho_i - \varrho_{i+1})^2} \left( \int_{Q_{n+1}} \left[ 1 + H_j(Du_j)^{\alpha_i} \right] \, dy \right)^{\frac{2}{m}(1 + \frac{2}{m})} \]

and (4.11) is proved for all \( i \in \mathbb{N} \).

Now we know that the quantity appearing on the right-hand side of (4.16) is finite for all \( i \in \mathbb{N} \), we define

\[ A_i := \left( \int_{Q_{n+1}} \left[ 1 + H_j(Du_j)^{\alpha_i} \right] \, dz \right)^{\frac{2}{m}} . \]

From the definitions in (4.6), it is easy to see that whenever \( i \geq 2 \)

\[ \kappa_i = \frac{\Gamma}{m} \sum_{l=0}^{i-2} \omega^l \quad \text{and} \quad \alpha_i = m\tilde{q} + \Gamma \sum_{l=0}^{i-2} \omega^l , \]

so (4.7) yields that \( \alpha_i \to \infty \). In these terms, (4.16) can be rearranged as

(4.18) \[ A_i \leq \left[ \frac{c \omega_i^2(1 + \kappa_i)^2}{(\tau_2 - \tau_1)^2} \right]^{\frac{2\omega_i}{\alpha_{i+1}}} \left[ \frac{\omega_{i+1}}{\alpha_{i+1}} \right]^{\frac{2\omega_{i+1}}{\alpha_{i+1}}} A_i^{\frac{\omega_{i+1}}{\alpha_{i+1}}} , \]

for \( c \equiv c(data) \). Iterating (4.18) we obtain

(4.19) \[ A_{i+1} \leq \left( \frac{c}{\tau_2 - \tau_1} \right)^{\frac{4m}{\alpha_{i+1}}} \sum_{l=0}^{i} \omega^l \sum_{l=0}^{i-1} \left[ \frac{2^{3(i-l)}(1 + \kappa_{i-l})^2}{(\tau_2 - \tau_1)^2} \right]^{\frac{2\omega_{i-l+1}}{\alpha_{i-l+1}}} A_i^{\frac{\omega_{i-l+1}}{\alpha_{i-l+1}}} . \]

Let us study the asymptotics of the various constants appearing in (4.19). We have:

\[ \lim_{i \to \infty} \frac{4m}{\alpha_{i+1}} \sum_{l=0}^{i} \omega^l = \frac{4m\omega}{\Gamma} , \quad \lim_{i \to \infty} \frac{\omega_{i-l}}{\alpha_{i-l+1}} = \frac{m\tilde{q}(\omega - 1)}{\Gamma} . \]
and

\[
\lim_{i \to \infty} \prod_{l=0}^{i-1} \left[ 2^{8(i-l)}(1 + \kappa_{i-l})^2 \right]^{\frac{m\omega}{i+1}} \\
\leq \exp \left\{ \frac{4m(\omega - 1)}{\Gamma} \log \left( 4 \max \left\{ 2, \frac{\Gamma}{m(\omega - 1)} \right\} \sum_{l=1}^{[\frac{m\omega}{i+1}]+1} \omega^{-l} + 1 + e^{-1} \log \omega \right) \right\},
\]

where we also used that

\[
\sum_{l=1}^{\infty} \omega^{-l} \leq \sum_{l=1}^{[\frac{m\omega}{i+1}]+1} \omega^{-l} + 1 + e^{-1} \log \omega.
\]

As

\[
\int_{Q_{2i+1}} H_j(Du_j)^{\alpha_{i+1}} \leq A_{i+1}
\]

(4.20)

\[
\leq \left( \frac{c}{\tau_2 - \tau_1} \right)^{\frac{4m(\omega - 1)}{\Gamma} \sum_{l=1}^{i+1} \omega^{-l}} \prod_{l=0}^{i-1} \left[ 2^{4(i-l)}(1 + \kappa_{i-l})^2 \right]^{\frac{m\omega}{i+1}} A_{i+1},
\]

we can pass to the limit in (4.20) for concluding that

\[
\|H_j(Du_j)\|_{L^\infty(Q_{\tau_1})} \leq \left[ 1 + \|H_j(Du_j)\|_{L^\infty(Q_{\tau_2})} \right]^{\frac{m\tilde{q}}{\omega - 1}} \left( \int_{Q_{\tau_2}} \left[ 1 + H_j(Du_j)^{\tilde{q}} \right]^\frac{\omega - 1}{\omega - \tilde{q}} dy \right)^{\frac{1}{\omega - \tilde{q}}},
\]

(4.21)

\[
\leq \left( \frac{c}{\tau_2 - \tau_1} \right)^{\theta'} \left[ 1 + \|H_j(Du_j)\|_{L^\infty(Q_{\tau_2})} \right]^{\frac{m\tilde{q}}{\omega - 1}} \left( \int_{Q_{\tau_2}} \left[ 1 + H_j(Du_j)^{\tilde{q}} \right]^\frac{\omega - 1}{\omega - \tilde{q}} dy \right)^{\frac{1}{\omega - \tilde{q}}},
\]

with \( c \equiv c(\text{data}), \theta' \equiv \theta'(n, p, q, d) \) and \( \theta := \theta' + (n + 2)(\omega - 1)\Gamma^{-1} \). Recalling the definition given in (3.8) and the restriction imposed in (2.3), it is easy to see that

\[
\Gamma^{-1} \left( m\tilde{q} - \frac{p}{2} \right) (\omega - 1) < 1.
\]

(4.22)

In fact, verifying (4.22) is equivalent to check the validity of the following inequality

\[
\tilde{q} < \frac{p}{2m} + \frac{2}{\omega n m},
\]

which is satisfied by means of (2.3) and (3.23). So we can apply Young inequality with conjugate exponents \( (b_1, b_2) := \left( \frac{2\Gamma}{(2m\tilde{q} - p)(\omega - 1)\Gamma}, \frac{2\Gamma}{2\Gamma - (2m\tilde{q} - p)(\omega - 1)\Gamma} \right) \) in (4.21) to end up with

\[
\|H_j(Du_j)\|_{L^\infty(Q_{\tau_1})} \leq \frac{1}{2} \|H_j(Du_j)\|_{L^\infty(Q_{\tau_2})} + \left( \int_{Q_{\tau_2}} \left[ 1 + H_j(Du_j)^{\tilde{q}} \right]^\frac{\omega - 1}{\omega - \tilde{q}} dy \right)^\frac{1}{\omega - \tilde{q}}
\]

(4.23)

\[
\leq \frac{1}{2} \|H_j(Du_j)\|_{L^\infty(Q_{\tau_2})} + \left( \int_{Q_{\tau_2}} \left[ 1 + H_j(Du_j)^{\tilde{q}} \right]^\frac{\omega - 1}{\omega - \tilde{q}} dy \right)^\frac{1}{\omega - \tilde{q}}.
\]
with \( c \equiv c(\text{data}) \). Now we apply Lemma 2.7 to (4.23) to conclude that
\[
\|H_j(Du_j)\|_{L^\infty(Q_{4\theta})} \leq \frac{c}{\theta^\beta_1} \left[ 1 + \left( \int_{Q_{\theta}} H_j(Du_j)^{\frac{2}{\mu}} \, dy \right)^{\beta_2} \right],
\]
for \( c \equiv c(\text{data}) \), \( \beta_1 := \theta \beta_2 \) and \( \beta_2 := \frac{(\omega-1)\theta_2}{\theta} \).

4.2. Proof of Theorem 1. Let \( \{u_j\} \) be the sequence built in Section 4.1. As for each \( j \in \mathbb{N} \), \( u_j \) solves problem (4.4), which is driven by the nonlinear tensor \( a_j(\cdot) \) defined in (4.1), thus satisfying in particular (4.3), and has boundary datum \( f \) described by (2.5), we deduce that the uniform energy bound (3.16) holds true. Hence, combining (3.16) with (4.24) we obtain that
\[
\|H_j(Du_j)\|_{L^\infty(Q_{4\theta})} \leq \frac{c}{\theta^\beta_1} \left[ \|J_f\|_{L^\infty(Q_{4\theta})} + \|\partial_j f\|_{L^p(0,T;W^{1,p}(\Omega))} + 1 \right],
\]
with \( \beta := \beta_1 + (n + 2) \beta_2 \) and \( c \equiv c(\text{data}) \). Whenever \( (t_1, t_2) \in (0, T) \) and \( \tilde{\Omega} \subset \Omega \) is open, a standard covering argument and the content of the above display render that
\[
\|Du_j\|_{L^\infty(\tilde{\Omega} \times (t_1, t_2))} \leq c(\text{data}, C_f, \text{dist}(\tilde{\Omega}, \partial \Omega), t_1, T - t_2).
\]
Estimates (3.16) and (4.25) in turn imply that there exists a function \( u \in L^p(0, T; W^{1,p}(\Omega)) \) with gradient \( Du \in L^\infty(0,T;L^\infty(\Omega,\mathbb{R}^n)) \) so that
\[
\begin{cases}
  u_j \rightharpoonup u & \text{in } L^p(0,T;W^{1,p}(\Omega)), \\
  Du_j \rightharpoonup Du & \text{in } L^\infty(0,T;L^\infty(\Omega,\mathbb{R}^n)), \\
  u_j = f & \text{on } \partial_{\text{par}} \Omega.
\end{cases}
\]
In particular, by (4.25), (4.26) and weak*-lower semicontinuity we have
\[
\|Du\|_{L^\infty(\tilde{\Omega} \times (t_1, t_2))} \leq c(\text{data}, C_f, \text{dist}(\tilde{\Omega}, \partial \Omega), t_1, T - t_2).
\]
Such information is not sufficient to pass to the limit as \( j \to \infty \) in (3.13), therefore we shall prove that \( u_j \) admits some fractional derivative in space and in time which is controllable uniformly with respect to \( j \in \mathbb{N} \). Concerning the fractional derivative in space, we can use \textit{verbatim} the same argument leading to (3.37)-(3.39) to deduce that
\[
Du_j \in L^p_{\text{loc}}(0,T;W^{1+p,\alpha}_{\text{loc}}(\tilde{\Omega})) \quad \text{for all } \alpha \in \left( 0, \min \left\{ 1, \frac{2}{p} \right\} \right)
\]
with
\[
\|u_j\|_{L^p(t_1, t_2; W^{1+p,\alpha}(\tilde{\Omega}))} \leq c(\text{data}, \alpha, C_f, t_1, T - t_2, \text{dist}(\tilde{\Omega}, \partial \Omega)).
\]
On the other hand, we cannot borrow the corresponding estimates for the fractional derivative in time of the \( u_j \)'s developed in Step 6 of Section 3: the constant appearing on the right-hand side of (3.43) depends on \( \tilde{\mu}^{-1} \) and, since now \( \tilde{\mu} \equiv \mu_j \) it may blow up in the limit as \( j \to \infty \) if \( \mu = 0 \). Therefore we shall follow a different path, see [17, Section 9] for the case \( q = p = 2 \). Let
\[
0 < t_1 < \tilde{t}_1 < \tilde{t}_2 < t_2 < T \quad \text{and} \quad \tilde{h} > 0 \quad \text{be so that} \quad 0 < \tilde{h} < \frac{\min(\tilde{t}_1-t_1, t_2-\tilde{t}_2)}{1000}. \]
Using the forward Steklov average to reformulate (3.13) we obtain, for a.e. \( t \in (t_1, t_2) \),
\[
\int_{\Omega} \left[ \partial_t |u_j|_{\tilde{h}}\varphi + |a_j(x,t,Du_j)|_{\tilde{h}} \cdot D\varphi \right] \, dy = 0 \quad \text{for all } \varphi \in C^c_c(\Omega).
\]
Since \( \partial_t |u_j|_{\tilde{h}} = \tilde{h}^{-1} \tilde{\tau}_{\tilde{h}} u_j \), we can rearrange (4.29) as
\[
\int_{\Omega} \left[ \frac{\tilde{\tau}_{\tilde{h}}}{\tilde{h}} \varphi + |a_j(x,t,Du_j)|_{\tilde{h}} \cdot D\varphi \right] \, dy = 0.
\]
Modulo regularization, by (3.5), in the above display we can pick \( \varphi := \eta^2 \tilde{t}_h u_j \) with \( \eta \in C_c^\infty(\tilde{\Omega}) \) so that
\[
\|D\eta\|_{L^\infty(\tilde{\Omega})} \leq \frac{4}{\text{dist}(\tilde{\Omega}, \partial \Omega)},
\]
and integrate over the interval \((\hat{t}_1, \hat{t}_2)\) to get
\[
(4.30) \quad h^{-1} \int_{\hat{t}_1}^{\hat{t}_2} \int_{\Omega} |\tilde{t}_h u_j|^2 \eta^2 \, dx \, ds = - \int_{\hat{t}_1}^{\hat{t}_2} \int_{\tilde{\Omega}} \left[ a_j(x, t, Du_j) \right] \eta^2 \tilde{t}_h Du_j + 2 \tilde{t}_h u_j \eta D\eta \, dx \, ds.
\]
Recall that, for any function \( w \in L^1(\tilde{\Omega} \times (t_1, t_2)) \) there holds that
\[
\int_{\hat{t}_1}^{\hat{t}_2} \int_{\tilde{\Omega}} |\tilde{t}_h u_j| \, dx \, ds \leq \int_{\hat{t}_1}^{\hat{t}_2+h} \int_{\tilde{\Omega}} |w| \, dx \, ds \leq \int_{\hat{t}_1}^{\hat{t}_2} \int_{\tilde{\Omega}} |w| \, dx \, ds,
\]
therefore, by (4.31), Hölder and Young inequalities we estimate
\[
(4.31) \quad \frac{\tilde{h}^{-1}}{2} \int_{\hat{t}_1}^{\hat{t}_2} \int_{\tilde{\Omega}} \eta^2 |\tilde{t}_h u_j|^2 \, dx \, ds \leq \frac{\tilde{h}^{-1}}{2} \int_{\hat{t}_1}^{\hat{t}_2} \int_{\tilde{\Omega}} \eta^2 |\tilde{t}_h u_j|^2 \, dx \, ds + \tilde{h} \|D\eta\|_{L^\infty(\tilde{\Omega})}^2 \int_{\hat{t}_1}^{\hat{t}_2} \int_{\tilde{\Omega}} |a_j(x, t, Du_j)|^2 \, dx \, ds
\]
with \( c \equiv c(\text{data}, C_f, \text{dist}\tilde{\Omega}, \partial \Omega, t_1, T - t_2) \). Merging (4.30) and (4.31) we end up with
\[
\limsup_{\tilde{h} \to \infty} \left( \frac{\tilde{h}^{-1}}{2} \int_{\hat{t}_1}^{\hat{t}_2} \int_{\tilde{\Omega}} |\tilde{t}_h u_j|^2 \, dx \, ds \right) \leq c(\text{data}, C_f, \text{dist}\tilde{\Omega}, \partial \Omega, t_1, T - t_2),
\]
which, being \( \hat{t}_1, t_1, \hat{t}_2, t_2 \) arbitrary, and since we can repeat exactly the same procedure for the backward Steklov average of \( u_j \), we get
\[
u_j \in W^{1,2}_{\text{loc}}(0, T; L^2_{\text{loc}}(\Omega)) \quad \text{for all} \quad \iota \in \left( 0, \frac{1}{2} \right)
\]
and
\[
\|u_j\|_{W^{1,2}(0, T; L^2_{\text{loc}}(\tilde{\Omega})))} \leq c(\text{data}, \iota, C_f, \text{dist}\tilde{\Omega}, \partial \Omega, t_1, T - t_2).
\]
From (4.28) and (4.32) we deduce that
\[
\{u_j\} \text{ is bounded uniformly w.r.t. } j \in \mathbb{N} \text{ in } W^{1,2}_{\text{loc}}(0, T; L^2_{\text{loc}}(\Omega)) \cap L^p_{\text{loc}}(0, T; W^{1+\sigma,p}_{\text{loc}}(\Omega))
\]
for all \( \iota \in \left( 0, \frac{1}{2} \right) \), \( \sigma \in \left( 0, \min \left\{ 1, \frac{2}{p} \right\} \right) \), thus we can apply Lemma 2.4 with \( a_1 = p, a_2 = 2, \sigma = \iota, X = W^{1+\sigma,p}_{\text{loc}}(\Omega), B = W^{1,\min(2,p)}_{\text{loc}}(\Omega), Y = L^2_{\text{loc}}(\Omega) \), to obtain a (non-relabelled) subsequence \( \{u_j\} \) so that
\[
(4.33) \quad u_j \to u \quad \text{in } L^{\min(p,2)}_{\text{loc}}(0, T; W^{1,\min(p,2)}_{\text{loc}}(\Omega)).
\]
Combining (4.26), (4.33) and (4.27) we get
\[
(4.34) \quad Du_j \to Du \quad \text{in } L^p_{\text{loc}}(0, T; L^p(\Omega, \mathbb{R}^n)) \quad \text{for all } s \in (1, \infty),
\]
therefore we can pass to the limit in (3.13) to deduce that \( u \) satisfies (3.53). Moreover, repeating Step 8 of Section 3 we finally see that Definition 1 is satisfied, therefore \( u \) is a solution of problem (1.1) and, recalling also (4.27) we obtain (1.2). Once (1.2) is available, we can repeat the same procedure leading to (4.32) (with \( a(\cdot), u \) replacing \( a_j(\cdot), u_j \)) to obtain (1.3). Furthermore, by
(4.34), (4.27) and (3.16) we can pass to the limit for \( j \to \infty \) in (4.5) with \( q \equiv 1 \) and, after a standard covering argument, get (1.2)_2. Finally, combining (4.26)_2 and (4.34) with (4.24) we obtain (1.4). The proof of Theorem 1 is complete.

References

[1] B. Avellan, T. Kuusi, G. Mingione: Nonlinear Calderón-Zygmund theory in the limiting case. Arch. Ration. Mech. Anal. 227, nr. 2, 663-714, (2018).
[2] P. Baroni, Lorentz estimates for degenerate and singular evolutionary systems. J. Differential Equations 255, 2927-2951, (2013).
[3] P. Baroni, Riesz potential estimates for a general class of quasilinear equations. Calc. Var. & PDE 53(3-4), 12, pp. 803-846, (2015).
[4] P. Baroni, M. Colombo, G. Mingione, Regularity for general functionals with double phase. Calc. Var. & PDE 57:62, (2018).
[5] L. Beck, G. Mingione, Lipschitz bounds and non-uniform ellipticity. Comm. Pure Appl. Math., (2020). 
[6] P. Bella, M. Schaffner, Local boundedness and Harnack inequality for solutions of linear non-uniformly elliptic equations. Comm. Pure. Appl. Math., to appear.
[7] P. Bella, M. Schaffner, On the regularity of scalar integral functionals with \( (p,q) \)-growth. Anal. PDE, to appear.
[8] V. Bögelein, F. Duzaar, P. Marcellini, Parabolic Equations with \( p,q \)-growth. J. Math. Pures Appl. 100, 535-563, (2013).
[9] V. Bögelein, F. Duzaar, P. Marcellini, Parabolic Systems with \( p,q \)-Growth: A Variational Approach. Arch. Rational Mech. Anal. 210, 219-267, (2013).
[10] P. Bousquet, L. Brasco, \( C^1 \)-regularity of orthotropic \( p \)-harmonic functions in the plane. Anal. PDE 11, nr. 4, 813-846, (2018).
[11] C. De Filippis, G. Mingione, Lipschitz bounds and non-autonomous functionals. Preprint (2020).
[12] C. De Filippis, G. Mingione, On the Regularity of Minima of Non-autonomous Functionals. J. Geometric Analysis, (2019). https://doi.org/10.1007/s12220-019-00225-z
[13] C. De Filippis, J. Oh, Regularity for multi-phase variational problems. J. Diff. Equ. 267, 1631-1670, (2019).
[14] E. DiBenedetto, Degenerate Parabolic Equations. Universitext, Springer-Verlag New York, (1993).
[15] T. Di Marco, P. Marcellini, \( A \)-priori gradient bound for elliptic systems under either slow or fast growth conditions. Preprint (2019). https://arxiv.org/pdf/1910.04158.pdf
[16] E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhiker’s guide to the fractional Sobolev spaces. Bulletin des Sciences Mathématiques 136, 5, 521-573, (2012).
[17] F. Duzaar, G. Mingione, Second order parabolic systems, optimal regularity, and singular sets of solutions. Ann. I. H. Poincaré - AN 22, 705-751, (2005).
[18] F. Duzaar, G. Mingione, K. Steffen, Parabolic Systems with Polynomial Growth and Regularity. Memoirs AMS 1005, 214, (2011).
[19] L. Esposito, F. Leonetti, G. Mingione, Sharp regularity for functionals with \( (p,q) \)-growth. J. Diff. Equ. 204, 5-55, (2004).
[20] P. Hästö, J. Ok, Maximal regularity for local minimizers of non-autonomous functionals. Preprint, (2019). https://arxiv.org/pdf/1902.00261.pdf
[21] J. Hirsch, M. Schäffner, Growth conditions and regularity, an optimal local boundedness result. Preprint, (2019). https://arxiv.org/pdf/1911.12822.pdf
[22] T. Kuusi, G. Mingione, The Wolff gradient bound for degenerate parabolic equations. J. Eur. Math. Soc. (JEMS) 16, nr. 4, 835-892, (2014).
[23] T. Kuusi, G. Mingione, Gradient regularity for nonlinear parabolic equations. Ann. Sc. Norm. Super. Pisa Cl. Sci. 5, 12, nr. 4, 755-822, (2013).
[24] O. A. Ladyzhenskaya, V. A. Solonnikov, N. N. UralâĂŹtse va, Linear and quasi-linear equations of parabolic type. Transl. Math. Monographs AMS, 23, 1968.
[25] G. M. Lieberman, The natural generalization of the natural condition of Ladyzhenskaya and Ural’tseva for elliptic equation. Comm. PDE 16, 311-361, (1991).
[26] P. Marcellini, A variational approach to parabolic equations under general and \( p,q \)-growth conditions. Nonlinear Analysis, (2019). https://doi.org/10.1016/j.na.2019.02.010
[27] P. Marcellini, On the definition and the lower semicontinuity of certain quasiconvex integrals. Annales de l'I.H.P. Analyse non linéaire 3, nr. 5, 391-409, (1986).
[28] P. Marcellini, Regularity and Existence of Solutions of Elliptic Equations with \( p,q \)-Growth Conditions. J. Differential Equations 90, 1-30, (1991).
[29] P. Marcellini, Regularity of minimizers of integrals of the calculus of variations with non standard growth conditions. Arch. Rat. Mech. Anal. 105, 267-284, (1989).
[30] J. Simon, Compact sets in the space \( L^p(0,T;B) \). Ann. Math. Pura Appl. 146, (4), 65-96, 1987.
[31] T. Singer, Existence of weak solutions of parabolic systems with $p, q$-growth. *Manuscripta Math.* 151, 87-112, (2016).

[32] T. Singer, Parabolic Equations with $p, q$-growth: the subquadratic case. *Quart. J. Math.* 66, 707-742, (2015).

Cristiana De Filippis, Mathematical Institute, University of Oxford, Andrew Wiles Building, Radcliffe Observatory Quarter, Woodstock Road, Oxford, OX26GG, Oxford, United Kingdom

E-mail address: Cristina.DeFilippis@maths.ox.ac.uk