C-FIELD COSMOLOGY IN HIGHER DIMENSIONS

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Abstract

Hoyle and Narlikar's $C$-field cosmology is extended in the framework of higher dimensional spacetime and a class of exact solutions is obtained. Adjusting the arbitrary constants of integration one can show that our model is amenable to the desirable property of dimensional reduction so that the universe ends up in an effective 4D one. Further with matter creation from the $C$-field the mass density steadies with time and the usual bigbang singularity is avoided. An alternative mechanism is also suggested which seems to provide matter creation in the 4D spacetime although total matter in the 5D world remains conserved. Quintessence phenomenon and energy conditions are also discussed and it is found that in line with the physical requirements our model admits a solution with a decelerating phase in the early era followed by an accelerated expansion later. Moreover, as the contribution from the $C$-field is made negligible a class of our solutions reduces to the previously known higher dimensional models in the framework of Einstein's theory.

KEY WORDS: C-Field; Higher dimension; Singularity.

1 INTRODUCTION

For a long time the bigbang cosmology based on Einstein's field equations has come to be regarded as the only model that successfully explains the three important observations in astronomy, namely the phenomenon of the expanding universe, primordial nucleosynthesis and the observed isotropy of the cosmic microwave background radiation. However, the more sophisticated astronomical observations in the late eighties have revealed that the predictions of the Friedmann-Robertson-Walker type of models do not always exactly meet our expectations as was believed earlier \cite{1}. Some puzzling results regarding the redshifts from the extra galactic objects continue to evade theoretical explanation from the bigbang type of model, nor did the discovery of CMBR necessarily prove it to be a relic of the bigbang. In fact a nonrelic interpretation of the the CMBR is also plausible\cite{2}.

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So alternative theories are being proposed time to time – the most well-known being the Steady State theory by Bondi and Gold [3]. In this approach the universe does not have any singular beginning nor an end on the cosmic time scale. Moreover the statistical properties of the large scale features of the universe do not change. To account for the constancy of the mass density they have to envisage a very slow but continuous creation of matter going on in contrast to the one time infinite and explosive creation at $t = 0$ of the standard model. But it suffers from the serious disqualification that they do not give any physical justification in the form of any dynamical theory for the phenomenon of the continuous creation of matter, however insignificant. Thus the principle of conservation of matter is clearly sacrificed in this formalism. To overcome this difficulty in particular and a host others Hoyle and Narlikar [4] in the sixties adopted a field theoretic approach introducing a massless and chargeless scalar field $C$ in the Einstein-Hilbert action to account for the matter creation. In the $C$-field theory introduced by Hoyle and Narlikar there is also no bigbang type of singularity as in the earlier steady state theory of Bondi and Gold.

In the present work we have thought it worthwhile to extend the pioneering work of Hoyle and Narlikar in the framework of higher dimensional spacetime. Over the years the interests in higher dimensional theories have stemmed in their attempts to unify gravity with other forces in nature [5]. The recent spurt in activities is also due to its new field of application in brane cosmology[6]. Some of the important findings in the 5D $C$-field model of homogeneous dust universe in contrast with the 4D model of HN may be summarised as follows:

1. Unlike the 4D model of HN, in one case the dust density decreases continuously and finally vanishes. In this specific model the rate of fall of the dust density vanishes at both extremes $t = \pm \infty$.
2. One of our models demonstrates an explicit example of dimensional reduction, evolving into an effective 4D spacetime.
3. There is at least one situation where our higher dimensional $C$ field cosmology starts from the a singularity of infinite mass density and evolves with a deceleration followed by an acceleration at later stage reflecting the characteristic feature of the so-called 'quintessence' model. One must refer to the current ideas about an accelerated universe explained in terms of dark energy characterised by an equation of state $\omega = p/\rho$ being more negative than $-1/3$ after inclusion of other forms of matter.

Our paper is organised as follows:
In section 2 we have briefly introduced the original work of Hoyle and Narlikar. In section 3 an exact solution of HN theory in a 5D homogeneous spacetime is found. The section 4 deals with the dynamical behaviour of our model where one of the cases exhibits the desirable property of dimensional reduction such that with time the universe ends up as an effective 4D one. The section 5 is particularly interesting in the sense that we here present a case where the cosmology manifestly has a bigbang type of singularity. It emphasises the fact that mere inclusion of a $C$-field is no guarantee against the occurrence of singular epoch. The section further deals with the quintessence
phenomenon where interestingly one of the models yields a decelerating universe in the early phase and later the gravitationally self repulsive $C$ field causes the expansion to accelerate. This is in conformity with the present day observational status. The section 6 deals with the energy conditions in the framework of HN theory while in section 7 an alternative mechanism is naively suggested where the matter creation in the 3D space apparently occurs as a consequence of spontaneous compactification of the extra dimensions although the total matter in the higher dimensional sense is strictly conserved. We conclude in section 8.

2 HOYLE-NARLIKAR THEORY

As pointed out earlier Einstein’s field equations are modified in this formalism through the introduction of an external $C$ field such that (see ref. [11] for more details)

$$R_{ik} - \frac{1}{2}g_{ik}R = -8\pi \left( mT_{ik} + cT_{ik} \right)$$

(1)

where the first term in the r.h.s. refers to the matter tensor of Einstein’s theory and the second term is that due to the $C$-field given by

$$cT_{ik} = -f \left( C_i C_k - \frac{1}{2}g_{ik}C^\alpha C_\alpha \right)$$

(2)

where $f > 0$ and $C_i = \frac{dC^i}{dx^i}$.

Since $T^{00} < 0$ we see that the $C$ field has negative energy density producing repulsive gravitational field driving the expansion of the universe. Thus the energy conservation law reduces to

$$mT_{ik} = -cT_{ik} = fC_i C_k$$

(3)

i.e. matter creation through a nonzero l.h.s. is possible while conserving the overall energy and momentum. The last equation is identical with

$$m\frac{dx^i}{ds}g_{ik} - C_k = 0$$

(4)

which tells us that the 4-momentum of the created particle is compensated by 4-momentum of the $C$ field. Clearly to achieve this balance the $C$ field must have negative energy. Further the $C$ field satisfies the source equation $fC_i^i = j_i^i$ and $j^i = \rho\frac{dx^i}{ds} = \rho v^i$ where $\rho$ is the homogeneous mass density.

3 FIELD EQUATIONS AND ITS INTEGRALS

We here discuss a spatially flat 5D homogeneous cosmological model with the topology $M^1 \times R^3 \times S^1$ where $S^1$ is taken in the form of a circle such that

$$ds^2 = dt^2 - R^2(t) \left( dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2 \right) - A^2(t)dy^2$$

(5)
where \( R(t) \) is the scale factor for the 3D space and \( A(t) \), that for the extra dimension. The independent field equations for our metric (5) and energy momentum tensor (2) are

\[
3 \frac{\dot{R}^2}{R^2} + 3 \frac{\dot{R}}{R} \frac{\dot{A}}{A} = 8\pi \left( \rho - \frac{1}{2} f \dot{C}^2 \right) \tag{6}
\]

\[
2 \frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + 2 \frac{\dot{R}}{R} \frac{\dot{A}}{A} + \frac{\ddot{A}}{A} = 4\pi f \dot{C}^2 \tag{7}
\]

\[
3 \frac{\ddot{R}}{R} + 3 \frac{\dot{R}^2}{R^2} = 4\pi f \dot{C}^2 \tag{8}
\]

Following Hoyle and Narlikar we have also taken a zero pressure matter field in this work. From the Bianchi identity we further get

\[
\dot{\rho} + \left( 3 \frac{\dot{R}}{R} + \frac{\dot{A}}{A} \right) \rho = f \dot{C} \left[ \ddot{C} + \left( 3 \frac{\dot{R}}{R} + \frac{\dot{A}}{A} \right) \dot{C} \right] \tag{9}
\]

which when used in the source equation yields \( \dot{C} = \text{const.} = 1 \).

For economy of space we shall not give here the details of the intermediate mathematical analysis to solve the field equations and mention important steps only. We substitute \( \dot{R} = Y \) such that \( \ddot{R} = \frac{dY}{dt} = \frac{dy}{dR} \frac{dR}{dt} = YY'(R) \) where a prime overhead denotes differentiation with respect to \( R \) and now \( R \) becomes the new independent variable. The equation (8) now reduces to

\[
YY' + \frac{Y^2}{R} - \frac{p^2 R}{2} = 0 \tag{10}
\]

where \( p^2 = 8\pi f/3 \). It yields a first integral as

\[
Y^2 R^2 = k + \frac{p^2 R^4}{4} \tag{11}
\]

where \( k \) is an arbitrary constant of integration. From the last equation we finally get

\[
R^2 = R_0^2 \cosh pt \tag{12}
\]

when \( k < 0 \).

\[
R^2 = R_0^2 \sinh pt \tag{13}
\]

when \( k > 0 \). Moreover from the other two field equations (7)-(8) we obtain

\[
\ddot{A} + 2 \dot{A} \frac{\dot{R}}{R} - A \left( \frac{\ddot{R}}{R} + 2 \frac{\dot{R}^2}{R^2} \right) = 0 \tag{14}
\]

By inspection we get a particular solution of the above equation as \( A = R \).

Following the standard method of solving this type of second order linear differential equation we assume \( A(t) = R(t)u(t) \) such that the last equation reduces to

\[
R\ddot{u} + 4R\dot{u} = 0 \tag{15}
\]
yielding successively after two quadratures

\[ \dot{u} = \frac{\beta}{R^4} = \frac{\beta}{R_0^4 \cosh^2 pt} \]  

(16)

and

\[ u = \frac{\beta}{R_0^2} (\gamma + \tanh pt) \]  

(17)

where \( \gamma \) is an arbitrary constant of integration. So we finally get as general solution

\[ A = uR = \frac{a \cosh pt + b \sinh pt}{\sqrt{\cosh pt}} \]  

(18)

Using the above values of the metric coefficients we get the expression for the mass density as

\[ \rho = \frac{f}{2} \frac{a \cosh^2 pt + a \sinh^2 pt + 2b \cosh pt \sinh pt}{\cosh pt (a \cosh pt + b \sinh pt)} = f \left[ 1 - \frac{a}{2 \cosh pt (a \cosh pt + b \sinh pt)} \right] \]  

(19)

From the expression of the energy density we note that \( f > 0 \).

4 DYNAMICAL BEHAVIOUR

Depending on the nature of the arbitrary constants of integration several interesting possibilities of the evolution present itself.

Case I \((b = 0)\)

Our solutions reduce to

\[ R^2 = R_0^2 \cosh pt, \quad A^2 = a^2 \cosh pt, \quad \rho = \frac{f (1 + \tanh^2 pt)}{2}, \quad \dot{\rho} = \frac{f \sinh pt}{\cosh^3 pt} \]

(20)

When \( t \to -\infty \), \( R^2 = \frac{1}{2} R_0^2 e^{-pt} \to \infty \), \( A^2 = \frac{1}{2} a^2 e^{-pt} \to \infty \) whereas \( \rho \to f \) and \( \dot{\rho} \to 0 \).

So in the range \(-\infty < t \leq 0\), the density decreases but reaches a steady value and \( A^2 \) also reduces to a steady value \( a^2 \), which is an arbitrary constant quantity. As \( a \) may be chosen as small as possible there is dimensional reduction but no big bang type of singularity at any stage.

Moreover the 5D volume, \( R^3 A = R_0^3 a \cosh^2 pt \) tends to infinity as \( t \to \pm \infty \) and to \( R_0^3 a \) (a finite value) as \( t \to 0 \). So the 5D volume decreases to a minimum and then increases indefinitely. This model is rather puzzling in the sense that as the contraction is followed by an expansion the mass density finally reaches a steady value – even though both \( R \) and \( A \) approach infinity, which is consistent with the steady state theory where matter creation via the creation field maintains the balance. The model gives rise to a very interesting probability that the higher dimension gains significance in future which follows a dimensional reduction without any inconsistency in the present day scenario of the universe.

Case II \((a = 0)\)

The situation is distinctly different from what has been discussed in case I. Here

\[ R^2 = R_0^2 \cosh pt, \quad A^2 = b^2 \frac{\sinh^2 pt}{\cosh pt} \]  

and \( \rho = f \) (constant always). The range \(-\infty < t < 0\) is not valid here since the 5D volume \( R^3 A \) becomes negative. But in the range \( 0 \leq t < \infty \), the 5D volume
increases continuously and indefinitely starting from zero. But the mass density remains constant throughout and the behaviour mimics more a higher dimensional variant of the well known steady state theory of Bondi and Gold where the matter density remains unchanged due to the hypothesis of continuous creation as mentioned in the introduction. This also follows from the divergence relation (9), which on integration yields \( \rho = f + \frac{k}{R^3} \). Comparison with (19) gives \( k = \frac{faR^3}{2} \). For our case, \( a = 0 \) and hence \( k = 0 \), giving \( \rho = f \).

Case III \((b = -a)\)

In the context of higher dimensional theories this case is most interesting because with time the extra dimensional scale factor shrinks exhibiting the desirable feature of dimensional reduction. We presume that as the extra space reduces to the planckian size some stabilising mechanism (quantum gravity may be a possible candidate) would halt the shrinkage and the 5D cosmology becomes effectively four dimensional in nature.

Here \( R^2 = R_0^2 \cosh pt, A^2 = \frac{2a^2 e^{-2pt}}{e^{pt} + e^{-pt}} \), \( \rho = \frac{f}{2} (1 - \tanh pt) \) and \( \dot{\rho} = -fp/(2 \cosh^2 pt) \).

Moreover as \( t \to -\infty, R \to \infty, A \to \infty, \rho \to f, \dot{\rho} \to 0 \)

as \( t \to +\infty, R \to \infty, A \to 0, \rho \to 0, \dot{\rho} \to 0 \)

as \( t \to 0, R \to R_0, A \to a, \rho \to f/2, \dot{\rho} \to -\frac{fp}{2} \).

So the density decreases continuously in the range \(-\infty < t < +\infty\) starting from \( \rho = f \) to \( \rho \to 0 \) reaching steady state at two extremes \( \dot{\rho} \to 0 \). In this case to find the exact asymptotic nature we see that as \( t \to \infty, R^2 \sim R_0^2 e^{pt} \) and \( A^2 \sim 2a^2 e^{-3pt} \).

Hence dimensional reduction takes place and the 5D volume becomes \( \frac{R_0^3 a^2}{2} \), a finite magnitude. Thus there is no singularity throughout evolution.

5 SINGULAR SOLUTION IN HN THEORY

The creation field \( C \) was introduced by Hoyle and Narlikar to avoid any big bang type of singularity in cosmology sacrificing in the process the age old conservation principle. However we present here a set of solutions in \( C \) field theory in 5D spacetime with a big bang type singularity as follows:

\[
R^2 = R_0^2 \sinh pt \\
A = \frac{(a \sinh pt + b \cosh pt)}{\sqrt{\sinh pt}} \\
C = t \\
\rho = \frac{f}{2} \frac{a \sinh^2 pt + a \cosh^2 pt + 2b \cosh pt \sinh pt}{\sinh pt(a \sinh pt + b \sinh pt)}.
\]

This is obviously not a non singular model with everything blowing up at \( t = 0 \). However at \( t \to \infty \), the mass density becomes constant and \( \dot{\rho} \) vanishes, a characteristic of the steady state model. At this stage it may not be out of place to point out that this type of singular solutions in \( C \) fields cosmology is possibly unique in higher dimensional models only. One can not obtain solutions of similar kind in 4D spacetime. So in the context of higher dimensional spacetime the \( C \) field is not always effective in removing the perennial problem of singularity in cosmology.
The above model in the framework of creation field cosmology in 5D shows an extraordinary character in the behaviour of deceleration parameter. A little algebra shows that it is given by

$$q = -\frac{\dot{R}}{R^2} = \frac{1 - \sinh^2 pt}{1 + \sinh^2 pt}$$  \tag{25}

It is evident from (21) that the range $-\infty < t < 0$ is not valid for obvious reasons. However the scale factor is real in the range $0 \leq t < \infty$. Initially $R \to 0$ and the deceleration parameter is positive ($q \geq 0$). But at $t = t_1$ with $\sinh^2 pt_1 = 1$ we have $q = 0$, whereas subsequently for $t > t_1$, that is in the range $t_1 < t < \infty$ the deceleration parameter $q < 0$, which implies that the universe is accelerating consistent with the present day observations. This result is quite interesting vis a vis the present day attempts to construct accelerating models as in quintessence [10].

As the cosmology described in this section is somewhat similar to the big bang type it is tempting to compare the set (21)-(24) with the well known multidimensional homogeneous models in the absence of the $C$ field. Now as $f \to 0$, the contribution from the $C$ field becomes increasingly insignificant. Since $4\pi f = 3p^2/2$, the equations (21)-(24), as $f$ tends to zero reduce to $R \sim t^{1/2}$, $A \sim at^{1/2} + bt^{-1/2}$, $\rho \sim a^2 p^2 t^{2n} + bp^2$.

This resembles our solution [7] for a special case of homogeneous model when the inhomogeneity parameter is set to zero. If in addition we take $a = 0$, then $R \sim t^{1/2}$, $A \sim t^{-1/2}$ and $\rho = 0$, which is the well known soluton of Chodos and DetWeiler [8] for a 5D homogeneous empty universe. Relevant to point out that HN solution [4] are not amenable to similar reduction to FRW type of models when the creation field $C$ is switched off.

6 ENERGY CONDITIONS

As discussed in the previous section Hoyle and Narlikar had to invoke an extraneous scalar field with negative energy to get singularity free solutions. Following closely a recent work of Kolassis et al [9] we shall discuss, very briefly, the weak, dominant and the strong energy conditions in the context of $C$ field theory for our particular model. With the energy momentum tensor given in section 2 we write for a zero pressure model in 4D

$$T_{00} = (\rho - \frac{f}{2}), \quad T_{11} = T_{22} = T_{33} = -\frac{f}{2}$$  \tag{26}

in the locally Minkowskian frame. Obviously the roots of the matrix equation

$$[T_{ij} - r g_{ij}] = \text{diag} [(\rho - f/2 - r), (r - f/2), (r - f/2), (r - f/2)]$$  \tag{27}

give the eigenvalues $r$ of our energy momentum tensor as $r_0 = (\rho - f/2)$, $r_1 = r_2 = r_3 = f/2$. Skipping details (see ref. [9]) the energy conditions for our model may be briefly summed up as:

(a) Weak energy condition

$$r_0 \geq 0 \quad \text{i.e.,} \quad \rho \geq f/2 \quad \text{and} \quad (r_0 - r_i) \geq 0 \quad \text{i.e.,} \quad \rho \geq f.$$
(b) **Dominant energy condition**

\[ r_0 \geq 0 \quad \text{i.e., } \rho \geq f/2 \quad -r_0 \leq -r_i \leq r_0 \quad \text{i.e., } (-\rho + f/2) \leq -f/2 \leq (\rho - f/2). \]

Obviously \((\rho - f) \geq 0 \quad \text{i.e., } \rho > f.\)

(c) **Strong energy condition**

\[ (r_0 - \sum r_i) \geq 0. \]

It follows that \(\rho \geq 2f \quad (r_0 - r_i) \geq 0 \quad \text{implying } \rho \geq f.\)

So all the conditions clubbed together lead to \(\rho > 2f \) satisfying all the energy conditions.

A little algebra shows that for a \((n + 4)\) dimensional model the identical situation for strong energy condition leads to \((r_0 - \sum r_i) \geq 0 \quad \text{or } \rho \geq (2f + \frac{nf}{2})\). So apparently the extra dimensions put more stringent condition for a physically realistic energy momentum tensor.

### 7 AN ALTERNATIVE PROPOSAL

As pointed out in the section 2 that to avoid cosmological singularity the hypothesis of matter creation by Gold and Bondi is an adhoc assumption without any dynamical mechanism being offered to justify the process. To circumvent this difficulty HN introduced the so called *creation field* without much of any plausible physical justification. In both the cases the conservation principle is clearly violated. In the event that the spacetime has indeed extra dimension it is not suffient to have proved that spontaneous compactification occurs. Rather the cosmological consequences of the disappearance of the extra dimensions should also be studied. Here we briefly describe a scenario where matter may be created in the 3D space as a result of dimensional reduction of the extra space but matter conservation is still valid in the higher dimensional sense.

From the Bianchi identity it follows that in the absence of an external \(C\) field

\[ \rho R^3 A = \text{const} = M(0) \]  

(28)

where \(M(0)\) is the total mass in the 5D world which can not but conserve. But for a 4D observer the effective 4D matter will be given by \(\rho R^3 = M_4\) such that \(M_4(t) = M(0) A^{-1}\). Since for a physically realistic model the extra dimensions should shrink the above equation tells us although the overall \((4+1)\)D matter remains conserved there will be *matter leakage* from the internal space onto the effective 4D world. So it offers a natural mechanism for matter creation in 4D spacetime without the assumtion of an extraneous field.

### 8 DISCUSSION

We extend to higher dimension an earlier work of HN in *C*-field cosmology. Our work presents varied scenarios. Depending on the nature of the arbitrary constants the extra dimension either expand along with the 3D space or it shrinks with time with the cosmology ending up as an effective 4D one. We here chose a topology \(M^1 \times R^3 \times S^1\) but we believe that most of the findings may be extended if we take a large number of extra spatial dimensions. An important result of our investigation is the appearence of singular solution in the the *C* field theory. This has probably
no analogue in the 4D spacetime and it seemingly suggests that the mere presence of a $C$ field is no guarantee against singular solutions in cosmology. Moreover this set reduces to the well-known higher dimensional solutions in generalised Einstein’s equations when contribution from the $C$ field is made insignificant.

We have also put forward an alternative mechanism for matter creation in 3D space without invoking the existence of an extraneous creation field. This occurs as a consequence of dimensional reduction of the extra space although total matter is strictly conserved in the higher dimensional sense. However the idea is too premature to come to any definite conclusion in this regard and hence we are very brief on this point.

Recently Hoyle, Narlikar and Burbidge [11] modified their previous theory to Quasi steady state cosmology (QSSC) where the cosmology has an oscillatory phase superposed on a steadily expanding de Sitter type solution of field equation. As both QSSC and higher dimensional spacetime are both particularly relevant in the early phase of cosmic evolution an extension of the ideas of QSSC in the realm of multidimensional cosmology is urgently called for.

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