Diophantine equations coming from binomial near-collisions.

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1 Introduction

Given a positive integer \(d\) and a pair \((k, l)\) of unequal integers \(\geq 2\), we say that there exists a \((\text{binomial})\) \((k, l)\) near-collision with difference \(d\) if there exists a pair \((m, n)\) of integers with \(2 \leq k \leq n/2, 2 \leq l \leq m/2\), such that \(\binom{m}{l} - \binom{n}{k} = d\) and \(\binom{m}{l} \geq d^3\). In such a case, the quadruple \((n, k, m, l)\) is said to be a \((\text{binomial})\) near collision with difference \(d\).

Note that the above restrictions on \(k, l\) are very natural in view of the symmetries \(\binom{m}{l} = \binom{m}{m-l}\) and \(\binom{n}{k} = \binom{n}{n-k}\). The rather arbitrary condition \(\binom{m}{l} \geq d^3\) is just to ensure that the difference between the two binomial coefficients is quite small compared to the greater one. As explained in [1], it is probably more natural to replace the exponent 3 of \(d\) from the exponent 5.

If we consider \(k, l \geq 2\) and \(d \neq 0\) (not-necessarily positive) as given fixed integers with \(k \neq l\) we obtain the Diophantine equation

\[
\binom{m}{l} - \binom{n}{k} = d, 
\]

in the positive integer unknowns \(m, n\), without any restriction on the size of \(\binom{m}{l}\) compared to \(d\). In Section 2 we will solve (1) when \((k, l) = (3, 6)\) for various values of \(d\), and in Section 3 we will solve (1) with \((k, l) = (8, 2)\) and \(d = 1\). Our main results, Theorems 2.2.1, 2.3.1, 3.1.1 respectively imply Corollaries 2.2.2, 2.3.2, 3.1.2. As a consequence we have that \((k, l)\)-near collisions with difference 1 do not exist if \((k, l) \in \{(6, 3), (3, 6), (8, 2)\}\), establishing thus a conjecture stated in [1, Section 3.1].

We now sketch the method which we apply in Sections 2 and 3 for solving the equations mentioned above. For each equation we work as follows. We reduce its resolution to the problem of finding the points \((u, v)\) with integral coordinates on a certain elliptic curve \(C\) whose equation is not in Weierstrass form. We find a Weierstrass model \(E\) and an explicit birational transformation

\[
C \ni (u, v) \longrightarrow (x, y) = (\mathcal{X}(u, v), \mathcal{Y}(u, v)) \in E
\]

\[
C \ni (\mathcal{U}(x, y), \mathcal{V}(x, y)) = (u, v) \longleftarrow (x, y) \in E
\]

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between $C$ and $E$. This is accomplished by the \textsc{maple} implementation of van Hoeij’s algorithm \cite{5}. The typical point on $C$ is denoted by $P^C$ and the corresponding point on $E$ via the above birational transformation by $P^E$. We will also use the notation $(u(P), v(P))$ for the coordinates of the point $P$ viewed as a point on $C$, hence $(u(P), v(P)) = P^C$, and $(x(P), y(P))$ for the coordinates of the point $P$ viewed as a point on $E$, hence $(x(P), y(P)) = P^E$. Thus, if $P^C = (u, v) = (u(P), v(P))$ and $P^E = (x, y) = (x(P), y(P))$, then $x = \mathcal{X}(u, v)$, $y = \mathcal{Y}(u, v)$ and $u = \mathcal{U}(x, y)$, $v = \mathcal{V}(x, y)$.

Our problem is reduced to the following:

\begin{equation}
\text{To compute explicitly all points } P^E \in E(\mathbb{Q}) \text{ such that } P^C \in C(\mathbb{Z}).
\end{equation}

We deal with this problem as follows. Using the routine \textsc{MordellWeilBasis} of \textsc{magma} based on the work of many contributors, like J. Cremona, S. Donnelly, T. Fisher, M. Stoll, to mention a few of them, we compute a Mordell-Weil basis for $E(\mathbb{Q})$ and let $P^E_1, \ldots, P^E_r$ be generators of the free part of $E(\mathbb{Q})$. At this point we stress the fact that, in certain cases, especially when the rank of the elliptic curve is $\geq 5$, it is necessary to improve the Mordell-Weil basis computed by \textsc{magma}, in the sense explained in “Important computational issue” of Appendix \textsc{D}; we will need to do this in Sections \textsc{2} and \textsc{3}. Let $P^C = (u, v)$ denote the typical unknown point with integral coordinates. Its transformed point $P^E$ via the previously mentioned birational transformation is a point with rational coordinates, therefore $P^E = m_1 P^E_1 + \cdots + m_r P^E_r + T^E$, where $m_1, \ldots, m_r$ are unknown integers and $T^E$ denotes the typical torsion point (only finitely many and, actually, very few options for $T^E$ exist). To this we associate the linear form

\begin{equation}
L(P) = (m_0 + \frac{s}{t}) \omega_1 + m_1 l(P_1) + \cdots + m_r l(P_r) \{\pm l(P_0)\}.
\end{equation}

Some explanations have their place here. Firstly, $l$ denotes the map $l: E(\mathbb{R}) \rightarrow \mathbb{R}/\mathbb{Z}\omega_1$ closely related to the elliptic-logarithm function, which is defined and discussed in detail in Chapter 3 of \textsc{12}, especially, Theorem 3.5.2. Next, $\omega_1$ is the minimal positive real period of $E$, $m_0$ is an extra integer whose size depends explicitly on $M := \max_{1 \leq i \leq r} |m_i|$, and $s, t$ are relatively prime integers as follows: $t \geq 1$ divides the lcm of the orders of the non-zero torsion points of $E$ and $s$ is such that $-1/2 < s/t \leq 1/2$.\footnote{Note that, by a famous theorem of B. Mazur, $11 \neq t \leq 12$; see \textsc{6}, \textsc{7}, or \textsc{9} Theorem 7.5.}\footnote{Note that, by a famous theorem of B. Mazur, $11 \neq t \leq 12$; see \textsc{6}, \textsc{7}, or \textsc{9} Theorem 7.5.} Last, the indication $\{\}$ in the summand $l(P_0)$ means that this is present only in Section \textsc{2} where $P_0$ is a certain explicitly known point.

The \textit{Elliptic Logarithm Method} exploits the fact that $u, v$ are integers in order to find an upper bound for $|L(P)|$ in terms of $M$ (see \textsc{21}) and, on the other hand, applies a deep result of S. David \textsc{3} in order to obtain a lower bound for $|L(P)|$ in terms of $M$. Comparing the two bounds of $|L(P)|$ leads to a relation

\begin{equation}
\rho M^2 \leq \frac{c_1 c_{13}}{2\theta} (\log(\alpha M + \beta) + c_{14})(\log \log(\alpha M + \beta) + c_{15})^{r+3} + \gamma + \frac{c_{11}}{2\theta} \log \frac{c_9}{1 + \theta} + \frac{1}{2} c_{10},
\end{equation}

where all constants involved in it are explicit; see relation (9.8), Theorem 9.1.3 of \textsc{12}. It is clear that, if $M$ is larger than an explicit bound $B$, then the left-hand side is larger than the right-hand side and this contradiction certainly implies that $M \leq B$. Since $B$ is explicit, this allows us to compute all integer points $P^C = (u, v)$ as follows: For each $(m_1, \ldots, m_r)$ in the range $|m_i| \leq M$ ($i = 1, \ldots, r$) we compute each point $P^E = m_1 P^E_1 + \cdots + m_r P^E_r + T^E$ with $T^E$ a torsion point and
then we compute its transformed point $P^C$ via the previously mentioned birational transformation. If $P^C$ has integer coordinates, then we have gotten an integer point $P^C = (u, v)$.

In principle, this procedure allows to pick-up all integer points $(u, v)$ and, indeed, this is so if the bound $B$ is small, say around 30. But the bound which we obtain from (3) is huge and we must reduce it to a manageable size, which is accomplished with de Weger’s [13] technique, the basic tool of which is the *LLL-algorithm* of Lenstra-Lenstra-Lovász [4].

2 Equation (1) with $(k, l) = (3, 6)$

Replacing in (1) $d$ by $-d$ we obtain the equation

$$\left( \frac{n}{3} \right) = \left( \frac{m}{6} \right) + d,$$

which we study in this section. We have

$$\left( \frac{n}{3} \right) = \frac{n(n-1)(n-2)}{6} = \frac{1}{6} ((n-1)^3 - (n-1)) = \frac{1}{6} (u^3 - u),$$

where $u = n - 1$, and

$$\left( \frac{m}{6} \right) = \frac{m(m-1)(m-2)(m-3)(m-4)(m-5)}{6!}$$

$$= \left( \frac{(m - \frac{5}{2})^2 - \frac{25}{4}}{2} \right) \left( \frac{(m - \frac{5}{2})^2 - \frac{9}{4}}{2} \right) \left( \frac{(m - \frac{5}{2})^2 - \frac{1}{4}}{2} \right)$$

$$= \frac{6!}{6 \cdot 5 \cdot 3}$$

$$= \frac{(v - 3)(v - 1)v}{6 \cdot 5 \cdot 3},$$

where $v = \frac{1}{2} \left( \left( m - \frac{5}{2} \right)^2 - \frac{1}{4} \right) = (m - 2)(m - 3)/2$.

Thus, equation (4) implies

$$15(u^3 - u - 6d) = v^3 - 4v^2 + 3v,$$  \hspace{1cm} (5)

with $u, v$ related to $n$ and $m$ as above. We rewrite equation (5) as $g(u, v) = 0$, where

$$g(u, v) = 15u^3 - v^3 + 4v^2 - 90d - 15u - 3v.$$  \hspace{1cm} (6)
2.1 Equation \((6)\) when \(d = (N^3 - N)/6\)

Assuming that \(N\) is an explicitly known non-zero integer, we will show how the method of [12, Chapter 8] can be applied in order to compute—at least in principle—all integer solutions of \((6)\). A crucial fact is that certain parameters involved in the application of that method can be expressed uniformly in \(N\).

The curve \(C : g(u, v) = 0\), being a non-singular cubic, has genus one. Moreover, \((u, v) = (n, 1)\) is a rational point of \(C\), so that \(C\) is a model of an elliptic curve over \(\mathbb{Q}\). The MAPLE implementation of van Hoeij’s algorithm [5] gives the birational transformation between \(C\) and the Weierstrass model

\[
E : y^2 = x^3 - 1575x + 33750N^3 - 33750N - \frac{1366875}{4}N^6 + \frac{1366875}{2}N^4 - \frac{1366875}{4}N^2 + 52650. \tag{7}
\]

The birational transformation from \(C\) to \(E\) mentioned in page 4 is

\[
(u, v) \mapsto (x, y) = (\mathcal{X}(u, v), \mathcal{Y}(u, v))
\]

\[
(U(x, y), V(x, y)) = (u, v) \mapsto (x, y),
\]

where the functions \(\mathcal{X}\) and \(\mathcal{Y}\) are

\[
\mathcal{X}(u, v) = (-45N^3v + 45N^2uv + 120N^3 - 60N^2u - 60Nu^2 + 15Nv - 15uv + 3v^2 + 60u - 12v - 60N + 9) : (N - u)^2,
\]

\[
\mathcal{Y}(u, v) = -\frac{3}{2}(675N^6 - 675N^5u - 675N^4u^2 + 675N^3u^3 + 120N^3v^2 - 120N^2uv^2 - 675N^4 + 1125N^3u - 480N^3v - 225N^2u^2 + 390N^2uv - 225Nu^2v + 90Nu^2v + 420N^3 - 180N^2u - 180Nu^2 - 40Nu^2 - 60u^3 + 40uv^2 + 150N^2 - 300Nu + 190Nu + 150u^2 - 190uv + 6v^2 - 240N + 240u - 24v + 18) : (N - u)^3,
\]

and the functions \(U, V\) are given by

\[
U(x, y) = (-2x^3 - 120x^2 + 6xy - (-1525N^3 + 12375N - 540)x + (4050N^4 - 2700N^2 + 360N + 450)y - 1366875N^7 + 1366875N^5 + 135000N^4 - 455625N^3 - 67500N^2 + 58725N - 40500) : (-2x^3 - 360N^2x^2 - (9450N^2 + 4050)x + 2733750N^6 - 2733750N^4 + 297000N^3 + 911250N^2 - 243000N - 89100),
\]

\[
V(x, y) = (5467500N^6 - 7290000N^4 + 759375N^3 + 3037500N^2 - 577125N - 380700 - (91125N^5 - 60750N^3 + 8100N^2 + 10125N + 8100)x - 270N^2x + (5400N^3 - 1800N + 270)y - (-90N^2 + 30)xy) : (-2x^3 - 360N^2x^2 - (9450N^2 + 4050)x + 2733750N^6 - 2733750N^4 + 297000N^3 + 911250N^2 - 243000N - 89100).
\]
Let $\zeta$ denote the cubic root of 15; in our computations we view $\zeta$ as a real number. With the aid of MAPLE we find out that there is exactly one conjugacy class of Puiseux series $v(u)$ solving $g(u, v) = 0$. This unique class contains exactly three series and only the following one has real coefficients:

$$v_1(u) = \zeta u + 4/3 + \left(\frac{7}{135} \zeta^2 - \frac{1}{3} \zeta\right) u^{-1} + \left(-\frac{1}{3} \zeta n^3 + \frac{4}{243} \zeta + \frac{1}{3} \zeta n\right) u^{-2} + \left(\frac{7}{405} \zeta^2 - \frac{1}{9} \zeta\right) u^{-3}$$

$$+ \left(\frac{7}{405} \zeta^2 n^3 - \frac{7}{405} \zeta^2 n - \frac{2}{9} \zeta n^3 + \frac{2}{9} \zeta n + \frac{8}{729} \zeta - \frac{28}{32805} \zeta^2\right) u^{-4} + \ldots$$  

(8)

In the notation of Fact 8.2.1(a) in [12], $K = \mathbb{Q}(\zeta)$, $\mu_1 = -1$, $\nu_1 = 1$ and according to Fact 8.2.1(d) of [12], a constant $B_0$ can be explicitly computed with the property that, for $|u| > B_0$ the identity $g(u, v_1(u)) = 0$ holds. In our case it turns out from Appendix A that we can take $B_0 = |N| + 1$. Then, according to Lemma 8.3.1 in [12], for every integer solution $(u, v)$ of (6) with $|u| \geq |N| + 1$ we have $v = v_1(u)$. Thus in the notation of Proposition 8.3.2 in [12], $x(u) = \mathcal{X}(u, v_1(u))$ and, putting $u = t^{-\nu_1} = t^{-1}$ we write $x(u)$ as a series in $t$

$$x(t) = 45\zeta N^2 - 60N - 15\zeta + 3\zeta^2 + (45N^3\zeta + 6N\zeta^2 - 120N^2 - 15N\zeta - 4\zeta + 40) t$$

$$+ \left(45\zeta N^4 + \frac{34}{3} \zeta^2 N^2 - 120N^3 - 30\zeta N^2 - 8\zeta N - \frac{25}{9} \zeta^2 + 40N + 5\zeta + 3\right) t^2$$

$$+ \left(30\zeta N^5 + \frac{37}{3} \zeta^2 N^3 - 120N^4 - 10\zeta N^3 - \frac{304}{27} \zeta^2 N^2 - \frac{25}{9} \zeta^2 N + 40N^2 - \frac{44}{405} \zeta^2 + 6N + \frac{88}{81} \zeta\right) t^3$$

$$+ O(t^4)$$

(9)

Then the point $P_0^E$ that plays a crucial role in the resolution (see [12, Definition 8.3.3]) is

$$P_0^E = (45N^2\zeta + 3\zeta^2 - 60N - 15\zeta, 90 - 60\zeta^2 - 135\zeta N + \frac{675}{2} N - \frac{2025}{2} N^3 + 180\zeta^2 N^3).$$

(10)

Now we refer to the discussion of Section 11 whose notation etc we use. According to [12, Theorem 9.1.3], applied to “case of Theorem 8.7.2”, if $|u(P)| \geq \max\{B_2, B_3\}$, where $B_2$ and $B_3$ are explicit positive constants, then either $M \leq c_{12}$, where $c_{12}$ is an explicit constant, or the inequality (11) holds. As already mentioned in Section 11 all constants in (11) explicit. More specifically, as we show in Appendix 14

$$B_2 = 3|N|, \quad B_3 = |N| + 1, \quad \theta = 1, \quad c_9 = 0.17, \quad c_{10} = \log(200|N|^3), \quad c_{11} = 2,$$

while the remaining constants appearing in (11), namely, $\alpha, \beta, \gamma, r, \rho, c_{12}, c_{13}, c_{14}, c_{15}$ depend on the peculiarities of the elliptic curve $E$, like e.g. its rank and Mordell-Weil group which by no means can be expressed uniformly in terms of $N$. Thus, we have the following:

**Theorem 2.1.1.** If $|u(P)| \geq 3|N|$, then either $M \leq c_{12}$ or

$$\rho M^2 \leq c_{13}(\log(\alpha M + \beta) + c_{14})(\log \log(\alpha M + \beta) + c_{15})^{r^3} + \gamma + \log 0.085 + \frac{1}{2} \log(200|N|^3).$$
2.2 Equation (6) with \(d = -1\)

Since \((N^3 - N)/6 = -1\) for \(N = -2\), we can apply the general discussion of Section 2.1, the notation of which will be used throughout the present section. We have

\[
C : g(u, v) = 0, \quad \text{where} \quad g(u, v) = 15u^3 - v^3 + 4v^2 - 15u - 3v + 90
\]  

(11)

and

\[
E : y^2 = x^3 - 1575x - 12451725 =: f(x).
\]  

(12)

\(E(\mathbb{Q})\) has rank 5 (in the notation of Theorem 2.1.1 \(r = 5\)) and trivial torsion subgroup (in subsequent notation \(r_0 = 1\)). The free part of \(E(\mathbb{Q})\) is generated by the points

\[
P_E^1 = (35, 395), \quad P_E^2 = (615, 14805), \quad P_E^3 = (3055, 168805), \quad P_E^4 = (1350, 49455), \quad P_E^5 = \left(\frac{1185}{8}, -\frac{28935}{8}\right).
\]

Actually, the Mordell-Weil basis formed by the above five points is an improvement of the Mordell-Weil basis furnished by \textsc{magma}, in the sense of the “Important computational issue” of Appendix D.

The birational transformation between the models \(C\) and \(E\) is:

\[
\begin{align*}
\mathcal{X}(u, v) &= \frac{3(40u^2 + 55uv + v^2 - 60u + 106v - 277)}{(u + 2)^2} \\
\mathcal{Y}(u, v) &= \frac{3(2505u^3 + 90u^2v + 220uv^2 + 5595u^2 - 685uv + 437v^2 - 6360u - 1718v - 15069)}{(u + 2)^3},
\end{align*}
\]

and

\[
\begin{align*}
\mathcal{U}(x, y) &= \frac{2x^3 - 60x^2 + 3xy - 49455x + 26865y + 68298525}{-x^3 + 360x^2 - 20925x + 66442950} \\
\mathcal{V}(x, y) &= \frac{15(18x^2 + 11xy + 80325x - 1311y + 8004285)}{-x^3 + 360x^2 - 20925x + 66442950}.
\end{align*}
\]  

(13)

By (8) and the discussion immediately after it, for every real solution of \(g(u, v) = 0\) with \(|u| \geq 3\) it is true that \(v = v_1(u)\), where

\[
v_1(u) = \zeta u + \frac{4}{3} + \left(\frac{7}{135}\zeta^2 - \frac{1}{3}\zeta\right)u^{-1} + \frac{490}{243}\zeta u^{-2} + \left(\frac{7}{405}\zeta^2 - \frac{1}{9}\zeta\right)u^{-3} + \left(-\frac{686}{6561}\zeta^2 + \frac{980}{729}\zeta\right)u^{-4} + \ldots.
\]  

(14)

Also, by (10),

\[
P_0^E = (3\zeta^2 + 165\zeta + 120, 660\zeta^2 + 270\zeta + 7515),
\]
where $\zeta$ is the cubic root of 15.

Referring to the discussion of Section 11 we consider the linear form

$$L(P) = \left(m_0 + \frac{s}{t}\right)\omega_1 + m_1I(P_1) + m_2I(P_2) + m_3I(P_3) + m_4I(P_4) + m_5I(P_5) \pm I(P_0).$$

Since $f(X)$ has only one real root, namely $e_1 \approx 234.0452973361$, we have $E(\mathbb{R}) = E_0(\mathbb{R})$, therefore $I(P_1)$ coincides with the elliptic logarithm of $P_i^E$ for $i = 1, \ldots, 5$ (see Chapter 3 of [12], especially, Theorem 3.5.2). On the other hand, $P_0^E$ has irrational coordinates. As MAGMA does not possess a routine for calculating elliptic logarithms of non-rational points, we wrote our own routine in MAPLE for computing $l$-values of points with algebraic coordinates. Thus we compute

$$I(P_1) \approx -0.0771021779, \quad I(P_2) \approx -0.0404989783, \quad I(P_3) \approx -0.0180931954,$$

$$I(P_4) \approx -0.0272287725, \quad I(P_5) \approx 0.0607913520, \quad I(P_0) \approx 0.1159496335.$$

Note that the six points $P_i^E$, $i = 0, 1, \ldots, 5$ are $\mathbb{Z}$-linearly independent because their regulator is non-zero (see [8], Theorem 8.1). Therefore our linear form $L(P)$ falls under the scope of the second “bullet” in [12, page 99] and we have $r_0 = 1$, $s/t = s_0/t_0 = 0/1 = 0$, $d = 1$, $r = 5$, $n_i = m_i$ for $i = 1, \ldots, 4$, $n_5 = \pm 1$, $n_0 = m_0$, $k = r + 1 = 6$, $\eta = 1$ and $N = \max_{0 \leq i \leq 5} |n_i| \leq r_0 \max\{M, \frac{1}{2}rM + 1\} + \frac{1}{2}\eta r_\gamma = \frac{5}{2}M + \frac{3}{2}$, so that, in the relation (9.6) of [12] we can take

$$\alpha = 5/2, \beta = 3/2.$$

We compute the canonical heights of $P_1^E, P_2^E, P_3^E, P_4^E, P_5^E$ using MAGMA, and for the canonical height of $P_0^E$ we confine ourselves to the upper bound furnished by Lemma C.1. Thus we have

$$\hat{h}(P_1^E) \approx 2.2913414307, \quad \hat{h}(P_2^E) \approx 2.0649979264, \quad \hat{h}(P_3^E) \approx 3.3258621376,$$

$$\hat{h}(P_4^E) \approx 2.5707390271, \quad \hat{h}(P_5^E) \approx 2.6752327982, \quad \hat{h}(P_0^E) \leq 7.300572483.$$

The corresponding height-pairing matrix for the particular Mordell-Weil basis is

$$\mathcal{H} = \begin{pmatrix}
2.2913414307 & 1.0192652309 & 1.5359254535 & -1.2315544080 & -0.77710896815 \\
1.0192652309 & 2.0649979264 & 0.3597655203 & -0.4612024943 & 0.3804341218 \\
1.5359254535 & 0.3597655203 & 3.3258621376 & -1.9571170828 & -1.9878905154 \\
-1.2315544080 & -0.4612024943 & 1.9571170828 & 2.5707390271 & 1.3907956375 \\
-0.77710896815 & 0.3804341218 & -1.9878905154 & 0.3907956375 & 2.6752327982
\end{pmatrix}$$

with minimum eigenvalue

$$\rho \approx 0.772227489.$$

Next we apply [12, Proposition 2.6.3] in order to compute a positive constant $\gamma$ with the property that $\hat{h}(P^E) - \frac{1}{2}h(z(P)) \leq \gamma$ for every point $P^E = (x(P), y(P)) \in E(\mathbb{Q})$, where $h$ denotes Weil height; it turns out that

$$\gamma \approx 4.6451703657.$$

\footnote{For the definition of the canonical height we follow J.H. Silverman; as a consequence the values displayed here for the canonical heights are the halves of those computed by MAGMA and the least eigenvalue $\rho$ of the height-pairing matrix $\mathcal{H}$ below, is half that computed by MAGMA; cf. “Warning” at bottom of p. 106 in [12].}

\footnote{In the notation of [12, Proposition 2.6.3], as a curve $D$ we take the minimal model of $E$ which is $E$ itself.}
Finally, we have to specify the constants $c_{12}, c_{13}, c_{14}, c_{15}$ defined in [12, Theorem 9.1.2]. This is a rather straightforward task if one follows the detailed instructions of [12, “Preparatory to Theorem 9.1.2”] which can be carried out even with a pocket calculator, except for the computation of various canonical heights. Clearly, this is quite a boring job which, fortunately, can be carried out almost automatically with a MAPLE program. In this way we compute

$$c_{12} \approx 1.210103 \cdot 10^{27}, \quad c_{13} \approx 1.342820 \cdot 10^{281}, \quad c_{14} \approx 2.09861, \quad c_{15} \approx 25.03975. \quad (18)$$

Now, in view of Theorem 2.11 and (15), (16), (17), (18), we conclude that, if $|u(P)| \geq 6$, then either $M \leq c_{12}$ or

$$0.77222 \cdot M^2 \leq 1.34 \cdot 10^{281} \cdot (\log(2.5M + 1.5) + 2.0986) \cdot (\log(0.4342 \log(2.5M + 1.5)) + 25.03975)^5 + 5.4159.$$

But for all $M \geq 6.3 \cdot 10^{147}$, we check that the left-hand side is strictly larger than the right-hand side which implies that $M < 6.86 \cdot 10^{147}$, therefore

$$M \leq \max\{c_{12}, 6.86 \cdot 10^{147}\} = 6.86 \cdot 10^{147} \quad \text{provided that } |u(P)| \geq 6. \quad (19)$$

An easy straightforward computation shows that all integer points $P^C$ with $|u(P)| \leq 5$ (equivalently, all integer solutions $(u, v)$ of (11) with $|u| \leq 5$) are the following:

$$P^C = (-2, 0), (-2, 1), (-2, 3), (-1, 6), (0, 6), (1, 6). \quad (20)$$

In order to find explicitly all points $P^C$ with $|u(P)| \geq 6$ it is necessary to reduce the huge upper bound (19) to an upper bound of manageable size. This is accomplished in Appendix D where we show that $M \leq 27$. Therefore, we have to check which points

$$P^E = m_1P_1^E + m_2P_2^E + m_3P_3^E + m_4P_4^E + m_5P_5^E, \quad \text{with } \max_{1 \leq i \leq 5} |m_i| \leq 27,$$

have the property that $P^E = (x, y)$ maps via the transformation (13) to a point $P^C = (u, v) \in C$ with integer coordinates. We remark here that every point $P^C$ with $u(P)$ integer and $|u(P)| \geq 6$ is obtained in this way, but the converse is not necessarily true; i.e. if $\max_{1 \leq i \leq 5} |m_i| \leq 27$ and the above $P^E$ maps to $P^C$ with integer coordinates, it is not necessarily true that $|u(P)| \geq 6$.

If we were going to check all 5-tuples $(m_1, m_2, m_3, m_4, m_5)$ in the range $-27 \leq m_i \leq 27$ by “brute force” this would take more than 15 days of computation. Therefore, we apply a simple but very effective trick to speed up this final search. This trick, called in [10] inequality trick, is based in the observation that, for every 5-tuple $(m_1, m_2, m_3, m_4, m_5)$ corresponding to a point $P^E = m_1P_1^E + m_2P_2^E + m_3P_3^E + m_4P_4^E + m_5P_5^E$, the upper bound of $|L(P)|$ mentioned just above (3), more specifically,

$$|L(P)| \leq k_1 \exp(k_2 - k_4M^2) \quad (21)$$

must be satisfied for the six-tuple $(m_0, m_1, \ldots, m_5)$ where $m_0$ is the extra parameter appearing in (2) with $|m_0| \leq 27$. The heuristic observation is that the above inequality is very unlikely to be satisfied for points $P^E$, with at least one large coefficient $m_i$. The reason is that the elliptic
logarithms \( l(P) \) are more or less randomly distributed (at least there is no reason to assume otherwise) so that the linear \( L(P) \) is rarely very small. Checking whether the \( L(P) \) coming from a certain 6-tuple \((m_0, m_1, m_2, m_3, m_4, m_5)\) in the range \(-27 \leq m_i \leq 27\) satisfies the above displayed inequality requires real number computations which are considerably faster than those required for symbolically computing \( P_i^E = m_1 P_1^E + m_2 P_2^E + m_3 P_3^E + m_4 P_4^E + m_5 P_5^E \) and then checking whether the corresponding point \( P^C \) is integral. Actually, this reduces the computation to a few hours and furnishes us with the points figuring in Table 1.

**Important remark.** As mentioned in the “Important computational issue” at the end of Appendix D the online MAGMA calculator (V2.24-3) returns a different Mordell-Weil basis for the elliptic curve (12). The value of \( \rho \) corresponding to that basis is \( \rho \approx 0.410937 \). As a consequence, the initial upper bound for \( M \) (cf. (19)) is \( M < 8.63 \cdot 10^{147} \) and after four reduction steps, the final reduced upper bound is 34. Therefore the final check for all 6-tuples \((m_0, m_1, \ldots, m_5)\) in the range \(-34 \leq m_i \leq 34\) needs at least four times \((4 \approx (34/27)^6)\) more computation time; actually, it needs much more according to our experiments.

| \( m_1 \) | \( m_2 \) | \( m_3 \) | \( m_4 \) | \( m_5 \) | \( P^E = (x, y) \) | \( P^C = (u, v) \) |
|---|---|---|---|---|---|---|
| -1 | 0 | 0 | -1 | 1 | (27075, -4455045) | (-2, 1) |
| -1 | 0 | 0 | 0 | 0 | (235, -395) | (1, 6) |
| 0 | 0 | -1 | -1 | 0 | (495, -10395) | (-1, 6) |
| 0 | 0 | -1 | 0 | -1 | (555, 12555) | (-138, -339) |
| 0 | 0 | -1 | 0 | 0 | (3055, -168805) | (-2, 3) |
| 0 | 0 | 0 | 0 | 1 | (1185/4, -28935/8) | (0, 6) |

Table 1: All points \( P^E = \Sigma_i m_i P_i^E \) with \( P^C = (u, v) \in \mathbb{Z} \times \mathbb{Z} \).

Note that only the point \( P^C \) which corresponds to \((m_1, m_2, m_3, m_4, m_5) = (0, 0, -1, 0, -1)\) has \(|u(P)| \geq 6\). All other points \( P^C = (u, v) \), although they correspond to \((m_1, m_2, m_3, m_4, m_5)\) with \( \max_{1 \leq i \leq 5} |m_i| \leq 1 \), have \(|u| < 6\). These five points are of course contained in the already found list of points (20), which contains one more point, namely \((u, v) = (-2, 0)\), because this point cannot correspond via the (affine) birational transformation to a point \( P^E \); cf. page 6. We have thus proved the following:

**Theorem 2.2.1.** The integer solutions of the equation (11) are

\((u, v) = (-138, -339), (-2, 0), (-2, 1), (-2, 3), (-1, 6), (0, 6), (1, 6)\).

**Corollary 2.2.2.** No \((3, 6)\) near-collision with difference 1 exists.

**Proof.** Assume that \((n, 3, 6)\) is a near collision with difference 1. Then \(\binom{m}{n} - \binom{u}{3} = 1\), which is equation (1) with \(d = -1\). At the beginning of Section 2 we saw that if we put \(u = n - 1\) and \(v = (m - 2)(m - 3)/2\), then \((u, v)\) is an integer solution of the equation (5) with \(d = -1\), i.e. \((u, v)\) is an integral point on the curve (11). By the restrictions on the definition of collision, \(n \geq 6\), so \(u \geq 7\) and by Theorem 2.2.1 no solution \((u, v)\) to (11) exists with \(u \geq 7\).
2.3 Other cases with \( d = (N^3 - N)/6 \)

From the discussion at the beginning of Section 2.1 and (6) we will deal with the elliptic curve
\[ C : 15u^3 - v^3 + 4v^2 - 15u - 3v - 90d. \]

The birationally equivalent Weierstrass model \( E \) is, by (7),
\[ E : y^2 = x^3 - 1575x + a_6(N), \]
where
\[ a_6(N) = -\frac{1366875}{4}N^6 + \frac{1366875}{2}N^4 + 33750N^3 - \frac{1366875}{4}N^2 - 33750N + 52650. \]

Generally speaking, the method for computing all integer points on \( C \) is completely analogous to
the one we applied in Section 2.2. Moreover, for \( d = 1, 4, 10, 20 \) (corresponding to \( N = 2, 3, 4, 5 \)) the
final checking, after the reduction process (cf. the discussion just before the Table) is considerably
less time-consuming because the ranks of the elliptic curves are at most 4. Therefore, we think it is
easy to include all necessary information in Table 2. We remind the notation which is identical
to that of Section 2.2: \( r \) denotes rank; torsion subgroup is trivial for every \( N \geq 1 \), therefore,
“generators” in the table means always “generators of infinite rank”. The discriminant is negative
for every \( N \geq 1 \) and \( e_1 \) is the sole real root of the cubic polynomial in the right-hand side of the
defining equation of \( E \). Finally, \( \rho \) denotes the least eigenvalue of the (positive definite) regulator
matrix. All points \( P_i, i = 0, 1, 2, 3, 4 \) below refer to the model \( E \); for simplicity in the notation we
omit the superscript \( E \) from them.

Table 2: \( C : 15u^3 - v^3 + 4v^2 - 15u - 3v - 90d \) and \( E : y^2 = x^3 - 1575x + a_6(N) \)

| \( N \) | \( d \) | \( a_6(N) \) | \( r \) | Generators | \( \rho \) | \( e_1 \) |
|---|---|---|---|---|---|---|
| 2 | 1 | -12046725 | 2 | \( P_1 = (26745/4, -4373685/8) \), \( P_2 = (2995, 163855) \) | 1.8907445355 | 231.5297170832 |
| 3 | 4 | -195967350 | 2 | \( P_1 = (37845, 7362270) \), \( P_2 = (152325, -59450670) \) | 1.9685805562 | 581.7501698100 |
| 4 | 10 | -1228109850 | 3 | \( P_1 = (2530, 122320) \), \( P_2 = (3414, 196362) \), \( P_3 = (108705/49, 33758640/343) \) | 2.1464178968 | 1071.3824031820 |

continued on next page
N  d  \(a_6(N)\)  \(r\)  Generators  \(\rho\)  \(\epsilon_1\)
5  20  \(-4916647350\)  4  \(P_1 = (1232475,1368255420)\)
    \(P_2 = (2181,73854)\)
    \(P_3 = (136825,50611330)\)
    \(P_4 = (2235,79020)\)  1.5758474521  1700.7293293549

Remark. In the case \(N = 5\), the \(\rho\)-value corresponding to the set of generators computed by the online MAGMA calculator is 0.4945449338. For reasons explained in the remark after Table 4, we would like to have a set of generators with a \(\rho\)-value as large as possible. By applying unimodular transformations to the basis computed by MAGMA and computing the corresponding \(\rho\)'s we succeeded to compute the basis shown in Table 2.

| \(N\) | \(d\) | \(P_0\) | Upper bound of \(\hat{h}(P_0^E)\) |
|------|------|------|-------------------------------|
| 2    | 1    | \((3\zeta^2 + 165\zeta - 120, -660\zeta^2 + 270\zeta + 7335)\) | 7.6463097298 |
| 3    | 4    | \((3\zeta^2 + 390\zeta - 180, 1560\zeta^2 - 405\zeta - 26235)\) | 8.539616384 |
| 4    | 10   | \((3\zeta^2 + 705\zeta - 240, 2820\zeta^2 - 540\zeta - 63360)\) | 9.141125914 |
| 5    | 20   | \((3\zeta^2 + 1110\zeta - 300, 4440\zeta^2 - 675\zeta - 124785)\) | 13.29809473 |

Completely analogously to the case \(d = -1\) in Section 2.2 in order to obtain an upper bound of \(M\) we compute the parameters \(\alpha, \beta, c_{12}, c_{13}, c_{14}, c_{15}\), as well as the analogous to those just above relation (15), and apply Theorem 2.1.1 according to which, either \(M \leq c_{12}\) or (3) holds. Always \(k = r + 1\) and \(\alpha, \beta\) are “very small” integers explicitly calculable. We remind that the parameters \(c_{12}, c_{13}, c_{14}, c_{15}\) are defined in [12, Theorem 9.1.2] and calculated according to the instructions in the “Preparatory to Theorem 9.1.2” therein. The values of these parameters are shown in Table 4.

| \(n\) | \(d\) | \(c_{12}\) | \(c_{13}\) | \(c_{14}\) | \(c_{15}\) | \(\alpha\) | \(\beta\) | \(\gamma\) | \(k\) |
|------|------|----------|----------|----------|----------|--------|--------|--------|------|
| 2    | 1    | \(1.010 \cdot 10^{27}\) | \(1.162 \cdot 10^{116}\) | 2.098    | 24.097   | 1      | 3/2    | 4.6396592897 | 3    |
| 3    | 4    | \(2.074 \cdot 10^{30}\) | \(1.841 \cdot 10^{116}\) | 2.098    | 27.779   | 1      | 3/2    | 5.1045188249 | 3    |
The upper bounds \( B(M) \) of \( M \) and the respective reduced upper bounds which are obtained by a reduction process completely analogous to that of the case \( d = -1 \) (Appendix D) are shown in Table 5. Finally we pick all points \( P^E = \sum_i m_i P^E_i \) with \( |m_i| \) less that the reduced bound, such that their corresponding point \( P^C \) has integer coordinates, as discussed at the end of Section 1. Our results are shown in Table 6.

We have thus proved the following:

**Theorem 2.3.1.** For \( d \in \{1, 4, 10, 20\} \) all integer solutions of the equation \( (5) \) are those listed in the fourth column of Table 6.

**Corollary 2.3.2.** No \((6, 3)\) near-collision with difference 1 exists.

**Proof.** Assume that \((n, 6, m, 3)\) is a near collision with difference 1. Then \( \binom{m}{3} - \binom{n}{6} = 1 \) and, on interchanging \( m, n \), we are led to equation \( (4) \) with \( d = 1 \). According to Section 2 if in \( (4) \) we put \( u = n - 1 \) and \( v = (m - 2)(m - 3)/2 \), then \((u, v)\) is an integer solution of the equation \( (5) \) with \( d = 1 \). Moreover, by the restrictions on the definition of collision, \( n \geq 6 \), so \( u \geq 7 \). According to Theorem 2.3.1 for \( d = 1 \) there is no solution \((u, v)\) with \( u \geq 7 \), and this concludes the proof.
3 Equation \((1)\) with \((k, l) = (8, 2)\)

We write our equation as follows:

\[
\frac{(n^2 - 7n)(n^2 - 7n + 6)(n^2 - 7n + 10)(n^2 - 7n + 12)}{3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} + 2 = (m^2 - m).
\]

Putting

\[
u = \frac{1}{2}n^2 - \frac{7}{2}n + 6, \quad v = 210m - 105
\]

we are led to

\[
v^2 = 35u^4 - 350u^3 + 945u^2 - 630u + 315^2,
\]

hence, it suffices to explicitly solve equation (23). The most straightforward thing for doing this would be to turn to \textsc{magma}'s routine \texttt{IntegralQuarticPoints} which is based on [11] and was firstly developed in 1999 by Emmanuel Herrmann and further improved in the years 2006-2013 by Stephen Donnelly and other people of \textsc{magma} group. And indeed, we ran the above routine for (23), but after five days, \textsc{magma} gave up without results, with the message “Killed”. Consequently we must solve (23) “non-automatically”, following the method of [11], as exposed in [12] Chapter 6.

For the successful accomplishment of this, crucial role play:

1. Our Mordell-Weil basis which is an improvement of the one furnished by \textsc{magma}, as explained in the “Important remark” at the end of this section, and

2. The application of an \textit{inequality trick} completely analogous to that which we discuss a little before and after relation (21).

3.1 The equation \(v^2 = 35u^4 - 350u^3 + 945u^2 - 630u + 315^2\)

We will deal with the elliptic curve

\[
C : v^2 = Q(u) := 35u^4 - 350u^3 + 945u^2 - 630u + 315^2.
\]

We use the notation, results etc of [12] Chapter 6; thus we have \(a = 35, b = -350, c = 945, d = -630, e = 315\). By [12] Relation (6.3) the Weierstrass model which is birationally equivalent to the curve \(C\) is

\[
E : y^2 = f(x) := x^3 + Ax + B,
\]

where \(A = -13968675\) and \(B = 3410363250\), and the birational functions

\[
C \ni (u, v) \mapsto (X(u, v), Y(u, v)) = (x, y) \in E
\]

\[
E \ni (x, y) \mapsto (U(x, y), V(x, y)) = (u, v) \in C
\]
are

\[ X(u, v) = \frac{315(u^2 - 2u - 2v + 630)}{u^2} \]
\[ Y(u, v) = -\frac{630(175u^3 - 945u^2 - uv + 945u + 630v - 198450)}{u^3} \]  

(12 Relation (6.4)), and

\[ U(x, y) = -\frac{630(x + 109935 + y)}{x^2 - 630x - 13792275} \]
\[ V(x, y) = -315(x^4 + 630x^3 + 2x^2y - 529200x^2 + 439740xy + 22441718250x - 110933550y - 196956864680625) : (x^2 - 630x - 13792275)^2 \]

(12 Relations (6.5), (6.6)).

The roots \( e_1 > e_2 > e_3 \) of \( f(x) \) have approximate values

\[ e_1 \approx 3608.8322706141 > e_2 \approx 245.1990070867 > e_3 \approx -3854.0312777009. \]

A fundamental pair of periods for the Weierstrass \( \wp \) function associated to \( E \) is

\( \omega_1 \approx 0.043947022525096, \ \omega_2 \approx 0.042006613806929 \cdot i. \)

Now we refer to Section 1 the notation etc of which we adopt here.

The rank of \( E \) is 5 and the torsion subgroup \( E_{\text{tors}}(\mathbb{Q}) \) is trivial. The following points form a Mordell-Weil basis for \( E(\mathbb{Q}) \):

\[ P^E_1 = (-1799, 150724), \ P^E_2 = (105, -44100), \ P^E_3 = (-315, -88200), \]
\[ P^E_4 = (8985, 776700), \ P^E_5 = (3885, 88200). \]

We note that, for \( i = 1, 2, 3 \), the points \( P^E_i \) belong to \( E_1(\mathbb{R}) \), the bounded piece ("egg") of \( E(\mathbb{R}) \), therefore by "Conclusions and remarks" (1) in [12 page 51], \( l(P_i) \) is the elliptic logarithm of the point \( P^E_i + Q^E_2 \), where \( Q^E_2 = (e_2, 0) \). Now \( P^E_i + Q^E_2 \) belongs to the infinite piece \( E_0(\mathbb{R}) \) of \( E(\mathbb{R}) \) but its coordinates are non-rational, belonging to the cubic extension of \( \mathbb{Q}(e_2)/\mathbb{Q} \), therefore, for \( i = 1, 2, 3 \) we compute the elliptic logarithm of \( P^E_i + Q^E_2 \) using our maple routine (cf. page 17); thus we find

\[ \ell_1 := l(P_1) \approx -0.1233994082363, \ \ell_2 := l(P_2) \approx 0.318524714651, \ \ell_3 := l(P_3) \approx 0.635691508151. \]

The points \( P^E_4 \) and \( P^E_5 \) belong to \( E_0(\mathbb{R}) \), therefore their \( l \)-values are equal to their respective elliptic logarithms; thus we find

\[ \ell_4 := l(P_4) \approx -0.1074268089, \ \ell_5 := l(P_5) \approx -0.18720073188. \]

\[ ^4 \text{See the "Important remark at the end of this section.} \]
Next we need to calculate approximate values of the canonical heights:

\[ \hat{h}(P_E^1) \approx 2.7309763445, \quad \hat{h}(P_E^2) \approx 1.2722439353, \quad \hat{h}(P_E^3) \approx 1.0972517248, \]

\[ \hat{h}(P_E^4) \approx 2.5539836387, \quad \hat{h}(P_E^5) \approx 1.2394130665 \]

and the height-pairing matrix

\[
\mathcal{H} = \begin{pmatrix}
2.2913414307 & 1.0192652309 & 1.5359254535 & -1.2315944080 & -0.77710896815 \\
1.0192652309 & 2.0649979264 & 0.3597655203 & -0.4612024943 & 0.38043412180 \\
1.5359254535 & 0.3597655203 & 3.3258621376 & -1.9571170828 & -1.98789051540 \\
-1.2315944080 & -0.4612024943 & -1.9571170828 & 2.5707390271 & 1.39079563750 \\
-0.7771089681 & 0.3804341218 & -1.9878905154 & 0.3907956375 & 2.6753279820
\end{pmatrix}
\]

with minimum eigenvalue

\[ \rho \approx 0.5764009469. \]

We will need also to compute a positive number \( \gamma \) such that \( \hat{h}(P_E) - \frac{1}{2}h(x(P)) \leq \gamma \), where \( h \) denotes Weil height. This we do by applying Proposition 2.6.3 of [12]. In the notation of that proposition, as a curve \( D \) we take the minimal model of \( E \) which is \( E \) itself and, following the simple instructions therein we compute \( \gamma = 6.4974558131 \). Finally, in order to compute the necessary constants involved in [12, Theorem 9.1.2] which are necessary for the application of [12, Theorem 9.1.3], we replace the pair of fundamental periods \( \omega_1, \omega_2 \) for which \( \tau := \omega_1/\omega_2 \) does not belong to the fundamental region of the complex upper half-plane, by the pair \((\omega_2, -\omega_1)\); for this pair, \( \tilde{\tau} := \omega_1/\omega_2 \) satisfies \( |\tilde{\tau}| \geq 1, \Re \tilde{\tau} > 0 \) and \( |\Re \tilde{\tau}| < 1/2 \), hence belongs to the fundamental region.

In order to obtain a relation of the form of [12] we will apply Theorem 9.1.3 “Case of Theorem 6.8” of [12]. That theorem is applicable for points \( P^C = (u(P), v(P)) \) for which \( |u(P)| \) is sufficiently large. Table 6.1 in [12, Chapter 6] indicates a procedure for computing how large \( |u(P)| \) should be; actually, we must have \( |u(P)| \geq \max\{u^{**}, \bar{u}^{**}\} \) and \( u^{**}, \bar{u}^{**} \) are calculated as explained in that table. The existence of two columns in Table 6.1 of [12, Chapter 6] and its specialization to our case which is Table 7 below, is explained as follows: At this stage it is convenient, instead of searching for solutions of \( Q(u) = v^2 \) with \( v \geq 0 \) and \( u \) of whatever sign, to look for solutions of both equations \( Q(u) = v^2 \) and \( \bar{Q}(u) := Q(-u) = v^2 \) with \( u, v \geq 0 \). Thus, a “bar” over a constant refers to the second equation.

The constant \( \max\{c_7, \bar{c}_7\} (= 13 \text{ in our case}) \) is used in the application of Theorem 9.1.3 “Case of Theorem 6.8” of [12].

Table 7: Parameters and auxiliary functions for the solution of the quartic elliptic equation according to the table 6.1 in [12]

\[ \text{continued on next page} \]

\footnote{See footnote 2}
From Table 7 it follows that the conditions of [12, Theorem 6.8] which are necessary also for the application of [12, Theorem 9.1.3] are fulfilled for all points $P^C \in C(\mathbb{Z})$ with $v(P) > 0$ and $|u(P)| \ge 80$. A quick computer search shows that the only points in $P^C(\mathbb{Z})$ with $|u(P)| < 80$ are those points $(u, v)$ listed in Table 1 with $|u| < 80$.

From Table 7 it follows that, on applying Theorem 9.1.3 of [12] we must take $c_7 = 13$ and $L(P) = I(P) \pm I(P_0)$. We have already computed approximations of the coefficients $\omega_1$ and $\ell_i \ (i = 1, \ldots, 5)$ of the linear form $I(P)$, and using our Magma routine mentioned in page 7 we also compute $\ell_0 := I(P_0) \approx -0.179410143$.

Using the routine IsLinearlyIndependent of magma, we see that the points $P^E_i \ (i = 0, \ldots, 5)$ are $\mathbb{Z}$-linearly independent, so that we are in the situation described in the second “bullet”, page 99 in [12]. Therefore, the parameters in the linear form (9.2) of [12] are

$$k = r + 1 = 6, \quad d = 1, \quad r_0 = 1, \quad (n_1, n_2, n_3, n_4, n_5) = (m_1, m_2, m_3, m_4, m_5), \quad n_6 = \pm 1, \quad \ell_6 = \ell_0.$$  

In the notation of [12, relation (9.3)] we have $N_0 = \frac{5}{2}M + \frac{3}{2}$, hence $(\alpha, \beta) = (5/2, 3/2)$.

In order to compute various constants involved in the upper bound for $M$ furnished by Theorem 9.1.3 of [12], we also need to compute $\hat{h}(P^E_0)$. Since $P^E_0$ is not a rational point we confine ourselves to a reasonably good upper bound of its canonical height which we obtain from Proposition 2.6.4 in [12]. In the notation of that proposition we take as curve $D$ our curve $E$ and obtain the bound $\hat{h}(P^E_0) \le 14.72$.

We see that the degree of the number field generated by the coordinates of all points $P_i \ (i = 0, \ldots, 5)$ is 6, so that $D = 6$ in the notation of “Preparatory to Theorem 9.1.2” of [12]. Following the instructions in that “Preparatory” and Theorem 9.1.2 we compute

\begin{align*}
c12 &= 6.7621175190 \cdot 10^{30}, \quad c13 = 3.6856632904 \cdot 10^{286}, \\
& \quad c14 = 2.7917594692, \quad c15 = 28.9071122373 \\
& \quad c16 = 0.6761234039, \quad c17 = 1.831780823, \quad c18 = 1.
\end{align*}
By that theorem, which in our case is Theorem 2.1.1, we conclude: either \( M \leq c_{12} \), or \( B(M) > 0 \), where \( B(M) = c_{18}c_{13}(\log(\alpha M + \beta) + c_{14})(\log \log(\alpha M + \beta) + c_{15})^{k+2} + \gamma + c_{18} \log c_{16} + c_{17} - \rho \cdot M^2 \).

Note that all parameters of \( B(M) \) have already been computed and are displayed in this and the previous pages. Now it is straightforward to check that for \( M \geq 6.28 \cdot 10^{150} \) we have \( B(M) < 0 \), which implies that

\[
M \leq \max\{c_{12}, 6.28 \cdot 10^{150}\} = 6.28 \cdot 10^{150}.
\]

We cannot obtain an upper bound for \( M \) essentially better than the above using [12, Theorem 9.1.3]; indeed, we check that \( B(6.27 \cdot 10^{150}) > 0 \) which shows that a “little smaller” bound for \( M \) does not lead to a contradiction.

We are now in a situation completely similar to that after relation (19). There, we reduced the huge upper bound of \( M \) by working as explained in Appendix D. Here, we work similarly to obtain a small upper bound for \( M \). This time the reduction process is repeated three times to successively give the upper bounds 170, 30 and 28 for \( M \); the last upper bound cannot be further reduced. Next, we check which points \( P^E = m_1P_1^E + \cdots + m_5P_5^E \) in the range \( \max_{1 \leq i \leq m} |m_i| \leq 28 \) correspond to a point \( P^C \) with integral coordinates, using the inequality trick, as explained in the last paragraph above Table 1. The computation on a computer Intel i5-7200U @ 2.50GHz took a little more than 70 hours of computation and the results are comprised in Table 8.

**Theorem 3.1.1.** All integer solutions of the equation (23) are those listed in the seventh column of Table 8.

| \( m_1 \) | \( m_2 \) | \( m_3 \) | \( m_4 \) | \( m_5 \) | \( P^E = (x, y) \) | \( P^C = (u, v) \) |
|-----|-----|-----|-----|-----|-----|-----|
| 0   | 0   | 0   | 0   | 1   | (3885, 88200) | (111, -69615) |
| 1   | 1   | 1   | 1   | -1  | (-4427535/1369, 6153669900/50653) | (111, 69615) |
| 0   | 0   | 0   | 1   | -1  | (5355, 286650) | (-22, -3535) |
| 1   | 1   | 1   | 0   | -1  | (-465570/121, 18522000/1331) | (-22, 3535) |
| 0   | 0   | 1   | 0   | -1  | (-3570, 88200) | (-102, 64575) |
| 1   | 1   | 0   | 1   | 1   | (1228395/289, 709061850/4913) | (-102, -64575) |
| 0   | 1   | 1   | 0   | 0   | (-315, -88200) | (1, 315) |
| 1   | 1   | 0   | 1   | 0   | (396585, -249738300) | (1, -315) |
| 0   | 1   | -1  | 1   | -1  | (4110, 124200) | (-294, -520065) |
| 1   | 0   | 2   | 0   | 1   | (-170085/49, 34428150/343) | (-294, 520065) |
| 0   | 1   | 0   | 0   | 0   | (105, -44100) | (3, 315) |

continued on next page
Important remark. The online *magma* calculator (V2.24-3) returns the following Mordell-Weil basis for the elliptic curve (24):

\[
(19705/81, 3758300/729), \quad (14665/4, -307475/8), \quad (8985, -776700), \\
(693805, -577896200), \quad (28035, -4652550).
\]

The value of \(\rho\) corresponding to that basis is \(\rho \approx 0.1284705\). As a consequence, the initial upper bound for \(M\) is \(M < 1.34 \cdot 10^{151}\). This not essentially better than the above displayed upper bound for \(M\). However after four reduction steps–and here \(\rho\) plays its important role–the final reduced upper bound is 62 which cannot be further improved. Therefore, had we used the above Mordell-Weil basis, the final check for all 6-tuples \((m_0, m_1, \ldots, m_5)\) in the range \(-62 \leq m_i \leq 62\) would be at least \((62/28)^6\) times more expensive, which amounts to at least one year of computation time! We must also check the points \((x, y) \in E(\mathbb{Q})\) which are zeros of \(q(x) = x^2 - 630x - 13792275\) appearing in the denominator of \(U(x,y)\) and \(V(x,y)\). But the zeros of \(q(x)\) are irrational, so we do not have any new solutions.

Finally we come back to the collision equation \(m^2 = n^2 + 1\) from which we started. We have \(m = (v+105)/210\), hence \(105|v\), and \(2u = n^2 - 7n + 12\). The only solutions \((u, v)\) with \(v\) divisible by 105 are those listed in Table 9, where also the corresponding values of \((m, n)\) in \(\mathbb{N}^2\) are listed.
Table 9: Positive integer solutions of the collision equation \( (m) = (n) + 1 \)

| \((u, v)\) | \((m, n) \in \mathbb{N}^2\) |
|---|---|
| (1, 315) | (2, 5), (2, 2) |
| (3, 315) | (2, 6), (2, 1) |
| (36, 6615) | (32, 12) |
| (15, 945) | (2, 4), (2, 3) |
| (6, 315) | (2, 0), (2, 7) |
| (0, 315) | (2, 4), (2, 3) |

Note that no pair \((m, n)\) in the above table satisfies the condition \(m \geq 4\) and \(n \geq 16\), therefore we have proved the following:

**Corollary 3.1.2.** There is no \((8, 2)\) near-collision with difference 1.

**Appendix A  The constant \(B_0\) in Sections 2.1 and 2.2**

In [12] Fact 8.2.1 (d) \(B_0\) denotes the maximum modulus of the roots of the polynomial \(\text{Res}_v(g, \frac{\partial g}{\partial v}) \in \mathbb{Z}[u]\). Since the property that we actually need is the convergence of a certain power series in \(u\) for \(|u| > B_0\), we can take as \(B_0\) any number larger than this maximum modulus.

**Lemma A.1.** The maximum modulus of the roots of the polynomial \(\text{Res}_v(g, \frac{\partial g}{\partial v}) \in \mathbb{Z}[u]\) is \(< |n| + 1\). Therefore we can take \(B_0 = |N| + 1\).

**Proof.** (Based on an idea of E. Katsoprinakis, whom we thank.) We have

\[
\text{Res}_v(g, \frac{\partial g}{\partial v}) = u^6 - 2u^4 + a_1 u^3 + u^2 - a_1 u + a_0,
\]

where \(a_1 = -2N^3 + 2N + \frac{8}{81}\) and \(a_0 = N^6 - 2N^4 - \frac{8}{81}N^3 + N^2 + \frac{8}{81}N - \frac{4}{675}\).

\(\text{Res}_v(g, \frac{\partial g}{\partial v}) = 0 \iff (u^3 - u)^2 + a_1(u^3 - u) + a_0 = 0\). If \(u^3 - u = y\) then we solve the quadratic equation \(y^2 + a_1 y + a_0 = 0\) and we find \(y_1 = N^3 - N + \frac{4}{81} + \frac{14\sqrt{7}}{405}\) and \(y_2 = N^3 - N - \frac{4}{81} - \frac{14\sqrt{7}}{405}\).

We have to solve the cubic equation \(u^3 + pu + q = 0\), with \(p = -1\) and \(q = y_1\) or \(y_2\). We find that \(\frac{q^2}{4} + \frac{p^3}{27} > 0\) (for \(|n| \geq 2\)), so the cubic equation \(u^3 + pu + q = 0\) has one real root and two conjugates complex roots. So from Cardano’s method we have that the roots are

\[
u_1 = A + B
\]

\[
u_2 = -\frac{1}{2}(A + B) + \frac{i\sqrt{3}}{2}(A - B)
\]

\[
u_2 = -\frac{1}{2}(A + B) - \frac{i\sqrt{3}}{2}(A - B)
\]
where \( A = 3 \sqrt{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \) and \( B = 3 \sqrt{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \).

We have that \( A \cdot B = \frac{1}{3} \), so

\[
|u_2|^2 = u_1^2 - 1
\]

(27)

So the polynomial \( \text{Res}_{\nu}(g, \frac{\partial g}{\partial \nu}) \) has two real roots (the real roots are in the interval \((-|N| - 1, |N| + 1]\)) and two pair of conjugates complex roots. From (27) we have that, if \( \rho \) is root of polynomial then

\[
|\rho| < |N| + 1.
\]

Appendix B  The constants \( \theta, c_9, c_{10}, c_{11} \) in Section 2

In order to compute the constants \( \theta, c_9, c_{10}, c_{11} \) which are necessary for the resolution of equation (6) (see the paragraph before Theorem 2.1.1), we follow the detailed instructions of [12, Chapter 8], especially sections 8.5 and 8.6 therein. One needs first compute three positive constants \( B_1, B_2 \) and \( B_3 \) with the property that Theorem 2.1.1 holds for all points \( P \) with \( |u(P)| \geq \max\{B_2, B_3\} \), and the computation of \( B_2, B_3 \) requires the computation of the positive constant \( B_1 \) with the following property (cf. [12, Proposition 8.3.2]): \( B_1 \geq B_0 \) (for \( B_0 \) we refer to Appendix A), the functions \( x(u) := X(u, v_1(u)) \) and \( y(u) := Y(u, v_1(u)) \) are strictly monotonous in the interval \((B_1, +\infty)\) and \( y \) does not change sign in this interval. Therefore the values of the parameters that figure in the title of this appendix and are involved in (3) are computed under this restriction on \( u(P) \).

The following lemma is used in the computation of \( B_1 \); it is the correct version of Lemma 8.5.1 in [12].

**Lemma B.1.** Let \( F \in \mathbb{R}[X,Y] \) be a polynomial such that \( F(X,0) \neq 0 \) and let \( V : \mathbb{R} \to \mathbb{R} \) be a continuous function, such that \( F(u, V(u)) = 0 \) for \( |u| > U_0 \), where \( U_0 \) is a positive constant. Let \( \mathcal{R} \) be the set of all real roots of the polynomial \( F(X,0) \), and define \( U_{\min} = \min\{-U_0, \min \mathcal{R}\} \) and \( U_{\max} = \max\{U_0, \max \mathcal{R}\} \). Then the function \( V \) keeps a constant sign in the interval \((U_{\max}, +\infty)\) and so it does in the interval \((-\infty, U_{\min})\).

**Proof.** Contrary to the hypothesis, assume, for example, that \( V \) changes sign in the interval \((U_{\max}, +\infty)\). Then, by the continuity of \( V \), it follows that there exists a root, say \( u_0 \), of \( V \) and \( u_0 > U_{\max} \). Since \( U_{\max} \geq U_0 \), we have \( |u_0| = u_0 > U_{\max} \geq U_0 \), therefore, by hypothesis, \( F(u_0, V(u_0)) = 0 \), hence \( F(u_0, 0) = 0 \). This means that \( u_0 \in \mathcal{R} \), therefore \( u_0 \leq \max \mathcal{R} \). But, on the other hand, \( u_0 > U_{\max} \geq \max \mathcal{R} \) and we arrive at a contradiction.

Similarly we arrive at a contradiction if we assume that \( V \) changes sign in the interval \((-\infty, U_{\min})\).

For the computation of \( B_1 \) we will apply Lemma B.1. Based on this lemma and following the detailed instructions and notation of [12, Section 8.5] we compute a number of constants, namely, \( R, M_{1,max}, M_{1,min} \) and \( M_{2,max}, M_{2,min} \). This task requires several computational steps, which we
perform with the aid of MAPLE. Below we give just a rough description of the kind of computations
that we have to do; the notation is that of [12, Section 8.5].

First, it is easy to compute that we can take \( R = |N| \). For the computation of \( M_{1,max}, M_{1,min} \),
we need compute the polynomial \( H_1 \) in the variables \( u, v, Y \) satisfying \( H_1(u, v_1(u), Y(u, v_1(u))) = 0 \).
It is the sum of 16 monomials and deg\(_u H_1 = 3\), deg\(_v H_1 = 2\), deg\(_Y H_1 = 1\). Since \(|u| > B_0 = N + 1\), we
have \( g(u, v_1(u)) = 0 \). We also have \( H_1(u, v_1(u), Y(u, v_1(u))) = 0 \). This leads us to consider
the resultant \( R_1 \) with respect of the variable \( v \) of the polynomials \( g(u, v) \) and \( H_1(u, v, Y) \), so that
\( R_1(u, Y) = 0 \). It is the product of \(-(-u + N)^6\) with a polynomial in \( u \) and \( Y \). Since \(-u + N \neq 0\) for \(|u| > B_0 = N + 1\), we must have \( R_{10}(u, Y(u, v_1(u))) = 0 \).

The polynomial \( R_{10}(u, Y) \) is the sum of 94 monomials and deg\(_u R_{10} = 3\), deg\(_Y R_{10} = 3\). We apply
Lemma [B.1] with \( F = R_{10}, U_0 = B_0 = |N| + 1, V = Y \). The polynomial \( R_{10}(u, 0) \) is cubic with
exactly one real root in the interval \((-|N| - 1, |N| + 1)\) and by definition of \( M_{1,min}, M_{1,max} \), we
obtain \( M_{1,min} = -|N| - 1 \) \( M_{1,max} = |N| + 1 \).

For the computation of \( M_{2,max}, M_{2,min} \), we need compute the polynomial \( H_2 \) in the variables
\( u, v, X \) which satisfies \( H_2(u, v_1(u), X(u, v_1(u))) = 0 \). It is the sum of 10 monomials terms deg\(_u H_2 = 2\),
dege\(_v H_2 = 2\), deg\(_X H_2 = 1\). In analogy with what we did above, we consider the resultant \( R_2 \) with
respect of the variable \( v \) of the polynomials \( g(u, v) \) and \( H_2(u, v, X) \) which has the property that
\( R_2(u, X(v_1(u))) = 0 \). It is the product \(-(-u + N)^4\) with a certain polynomial \( R_{20}(u, X) \) which is
the sum of 36 monomials of degree 2 with respect of \( u \) and degree 3 with the respect to \( X \). Then,
necessarily, \( R_{20}(u, x(u)) = 0 \), where, for simplicity in the notation, we have put \( x = X(u, v_1(u)) \).

Differentiating this we obtain (this is equation (8.17) of [12])
\[
\frac{\partial R_{20}}{\partial u}(R_{20}(u, x(u))) + \frac{\partial R_{20}}{\partial x}(R_{20}(u, x(u))) \cdot x'(u) = 0
\]
with \( x'(u) \) meaning the derivative of \( x(u) \) with respect to \( u \). The left-hand side is a polynomial
\( H_3 \) in the variables \( u, X, X', \) linear in \( X' \), with the property \( H_3(u, x(u), x'(u)) = 0 \) identically.
This equation along with \( R_{20}(u, x(u)) = 0 \) suggest to consider the resultant, with respect of the
variable \( X \), of the polynomials \( H_3(u, X, X') \) and \( R_{20}(u, X) \). This we denote by \( R_3(u, X') \); it satisfies
\( R_3(u, x'(u)) = 0 \). According to our computations, \( R_3(u, X') \) is the product of an integer, times
\((-u + N)\), times the square of a linear polynomial in \( u \) (only) whose root belongs to the interval
\((-|N|, |N| + 1)\), times a polynomial \( R_{30}(u, X') \) which is a sum of 28 monomials and deg\(_u (R_{30}) = 9\)
and deg\(_X (R_{30}) = 3\). Since we assume that \(|u| > B_0 = |N| + 1\), then, necessarily, \( R_{30}(u, x'(u)) = 0 \)
and we apply Lemma [B.1] with \( F = R_{30}, V = x', U_0 = B_0 = |N| + 1 \). Now \( F(u, 0) \) is a cubic
polynomial with exactly one real root, approximately equal to \(-|N| - 1\). Therefore, in the notation
of the aforementioned Lemma, in the present situation we have \( U_{min} = -|N| - 1 \) and \( U_{max} =
|N| + 1 \). Consequently, by the definition of \( M_{2,min}, M_{2,max} \), we obtain \( M_{2,min} = -|N| - 1 \) and
\( M_{2,max} = |N| + 1 \).

According to Section 8.5 of [12], this implies that \( B_1 = |N| + 1 \).

We perform the computation of \( B_2, B_3 \) and the constants \( \theta, c_9, c_{10} \) following the detailed
instructions of [12, Section 8.6]. First we have to compute (symbolic computation) the rational
function \( G(u, v) \) defined explicitly in [12, Proposition 8.4.1]. Then, following Section 8.4 of [12],
we set \( g(u) = G(u, v_1(u)) \) (notice the difference between \( g \) and \( g \)). According to [12, Proposition 8.4.2],
there exist constants $B_2 \geq B_1$, $c_9 > 0$ and $\theta$ which satisfy

$$\left| \frac{g(u)}{g_v(u, v_1(u))} \right| \leq c_0|u|^{-1-\theta}$$

($g_v$ means derivative with respect to $v$). For the practical computation of these constants, a detailed example is discussed in [12, Section 8.6]. Here we follow an analogous method. We denote by $I = I(u)$ the rational function inside the absolute value in the left-hand side of the above displayed inequality. Clearing out the denominator in the last relation gives an explicit polynomial equation $H_4(u, v_1(u), I) = 0$. Our computations show that $H_4(u, v_1(u), I) = -(u + N)^6H_4(u, v_1(u), I)$, where $H_4(u, v, I)$ is a certain polynomial in $u, v, I$. Since $|u| > |N| + 1$, we must have $H_4(u, v_1(u), I) = 0$. But we also have $g(u, v_1(s)) = 0$, so that we can eliminate $v_1(u)$ from the last two equations to obtain a relation which, according to our computations, is the following: constant $\cdot (N - u)^6h(u)R_4(u, I) = 0$, where $h(u)$ is a quartic polynomial in $u$ which, as it is easily seen, has no real roots for $|u| \geq 2$. Then, necessarily, $R_4(u, I) = 0$ and, following the method and notation of Section 8.6 of [12], we write this equation as follows:

$$I^3 + q_2(u)I + q_3(u) = 0,$$

with

$$q_2(u) = \frac{28}{3q(u)}, \quad q_3(u) = \frac{8}{3q(u)},$$

where

$$q(u) = 2025u^6 - 4050u^4 + (-4050N^3 + 4050N + 200)u^3 + 2025u^2$$

$$+ (4050N^3 - 4050N - 200)u - 12 + 2025N^2 + 200N - 200N^3 - 4050N^4 + 2025N^6.$$

Now we work as follows. Consider $q_2(u)$. Its numerator has no real roots and those of the denominator belong to the interval $(-|N| - 1, |N| + 1)$. But we have already assumed that $|u| \geq |N| + 1$ so (for $|N| \geq 2$) we have $q_2(u) > 0$ and $q_3(u) > 0$. Consequently, in (28), $I < 0$. Setting $I = -J < 0$ we obtain the equation

$$J^3 + q_2(u)J - q_3(u) = 0$$

where now the strictly negative coefficient are $-q_3(u)$. By Cauchy’s rule\footnote{see “Cauchy’s rule” in Section 8.6 in [12]},

$$0 < J < \max\{q_3(u)^{1/3}\}.$$ 

If $|u| > 3|N|$ then $q(u) > 1800u^6$, implying $q_3(u) < \frac{28}{3 \cdot 1800}u^{-6}$. We obtain $0 < -I = J < 0.17|u|^{-2}$. Hence, in the notation of [12 Proposition 8.4.2],

$$B_2 = 3|N|, \quad \theta = 1 = \frac{1}{\nu_s}, \quad c_9 = 0.17.$$ 

Next we must compute constants $B_3$, $c_{10}$ and $c_{11}$ such that: If $g(u, v) = 0$ with $u$ an integer $> \max\{B_2, B_3\}$, and $x = \mathcal{X}(u, v)$, then $h(x) \leq c_{10} + c_{11} \log |u|$, where $h(x)$ denotes the absolute
logarithmic height of \( x \). For the practical computation of these constants we apply \cite{12} Proposition 8.7.1.

We write the relation \( g(u, v) = 0 \) in the form

\[
v^3 + a_1(u)v^2 + a_2(u)v + a_3(u) = 0
\]

where \( a_1(u) = -4, a_2(u) = 3 \) and \( a_3(u) = -15u^3 + 15u + 15N^3 - 15N \). Let \( B_3 \) be a constant larger than every root of every non-zero polynomial \( a_i \).

We easily check that we can take

\[
B_3 = |N| + 1.
\]

Thus, in the sequel we will assume that the point \( P^C = (u(P), v(P)) \) satisfies

\[
|u(P)| \geq 3|N|
\]

and, for simplicity in the notation we put \( (u, v) \).

Assume \( u \geq 3N \) and \( N \geq 2 \).

- If \( v \geq 0 \), then Cauchy’s rule implies

\[
0 \leq v \leq \max\{2|a_1(u)|, (2|a_3(u)|)^{\frac{1}{2}}\} = \max\{8, \sqrt[3]{2 |30u^3|^{\frac{2}{3}}}\} = 2\sqrt[3]{15} |u|.
\]

- If \( v < 0 \), we put \( v = -w \) with \( w > 0 \) so that \( g(u, v) = 0 \) is written as

\[
w^3 + b_1(u)w^2 + b_2(u)w + b_3(u) = 0,
\]

where \( b_1(u) = 4, b_2(u) = 3 \) and \( b_3(u) = 15u^3 - 15u - 15N^3 + 15N > 0 \), for \( u \geq 3N \). So the above polynomial has no real roots.

Therefore we conclude \( |v| \leq 2\sqrt[3]{15} |u| \).

Next assume \( u \leq -3N \) and \( N \leq -2 \).

Then, we consider \( f(u, v) = g(-u, v) \) instead of \( g(u, v) \). Working as above we obtain the bound \( |v| \leq 2\sqrt[3]{15} |u| \).

Thus, in general, for \( |u| \geq 3|N| \) \( (|N| \geq 2) \) we have \( |v| \leq 2\sqrt[3]{15} |u| \) and, consequently, the absolute value of the numerator of \( \mathcal{X}(u, v) \) is, easily, bounded by

\[
(180n^2 + 108)|u|^2 + (180n^3 + 60n^2 + 60n + 108)|u| + 120n^3 + 60n + 9 \leq 200n^3 |u|^2
\]

and, clearly, \( 200N^3 |u|^2 \) is an upper bound for the absolute value of denominator \( (N - u)^2 \) of \( \mathcal{X}(u, v) \). Thus

\[
h(x(P)) = \log |x(P)| = \\
= \log \max\{\text{numer}(\mathcal{X}(u(P), v(P))), \text{denom}(\mathcal{X}(u(P), v(P)))\} \\
\leq \log(200|N|^3 |u(P)|^2) = \log(200|N|^3) + 2 \log |u(P)|
\]

and consequently, \( c_{10} = \log(200|N|^3) \) and \( c_{11} = 2 \).

Summing up, our computations furnished us with the following values:

\[
B_2 = 3|N|, \quad \theta = 1, \quad c_9 = 0.17, \quad B_3 = |N| + 1, \quad c_{10} = \log(200|N|^3), \quad c_{11} = 2.
\]

(31)
Appendix C  The canonical height of $P^E_0$ in Section 2.2

In this appendix we compute an upper bound for the canonical height of the point $P_0$, by applying [12, Proposition 2.6.4].

Lemma C.1. For the elliptic curve $E : y^2 = x^3 - 1575x - 12451725$ and its point $P^E_0$, defined in Section 2.2, we have

$$
\hat{h}(P^E_0) \leq 7.647146073.
$$

Proof. According to Section 2.2, $x(P_0) = -15\zeta + 63\zeta^2$, where $\zeta = \sqrt[3]{15}$. The minimal polynomial of $x(P_0)$ is $x^3 - 360x^2 + 20925x - 66442950$, therefore, by [12, Proposition 2.4.2] we have $h(x(P_0)) = \frac{1}{3}\log(66442950)$.

The discriminant $\Delta$ and the $j$-invariant of $E$ are, respectively,

$$
\Delta = -66979386718470000, \quad j = \frac{592704009187844543}{9187844543}.
$$

Applying [12, Proposition 2.6.4], to the elliptic curve $E$ with $D = E$, we obtain $\hat{h}(P^E_0) \leq 7.300572483$.

Appendix D  Reduced upper bound of $M$ in Section 2.2

In this appendix we reduce the upper bound (19) of $M$ following the very explicit procedure described in the first four pages of [12, Chapter 10]. This is based on de Weger’s reduction process [13] which makes use of the LLL-algorithm [4] to problems of the following general type: Let $\lambda = n_0 + n_1\xi_1 + \cdots + n_k\xi_k$, where the $\xi_i$’s are explicitly known real numbers and $n_0, n_1, \ldots, n_k$ are unknown integers, such that $N = \max_{0 \leq i \leq k} |n_i| \leq B$ with $B$ an explicit “huge” positive number and $|\lambda| \leq \kappa_1 \exp(\kappa_2 - \kappa_3 N^2)$ with $\kappa_1, \kappa_2, \kappa_3$ explicit positive numbers. Exploit this to find a considerably smaller upper bound for $N$, which is of the size of $\log B$.

We keep the notations of [12, Chapter 10]. In our case $\lambda$ is the linear form $L(P)$ up to a multiplicative constant and the relation $|\lambda| \leq \kappa_1 \exp(\kappa_2 - \kappa_3 N^2)$ comes from [12, Theorem 6.8], which guarantees that, if $|u(P)| \geq 80$, then $|L(P)| \leq \alpha^{-1/2} \exp(0.5 \log(3c_7) + \gamma - \rho M^2)$. We have

$$
\lambda := \lambda(P) := \frac{d \ell_0}{\zeta_1} L(P) = n_0 + n_1 \xi_1 + n_2 \xi_2 + n_3 \xi_3 + n_4 \xi_4 + n_5 \xi_5
$$

where

$$
\xi_i = \frac{r_0 \ell_i}{\zeta_1} \quad (i = 1, \ldots, 4) \quad \text{and} \quad \xi_5 = \frac{r_0 \ell_0}{\zeta_1},
$$

where $\ell_i = l(P_i), \ i = 1, \ldots, 5$. In the notation of [12, Chapter 10] we have $k = 6, \ d = 1, \ r_0 = 1, \ \alpha = 5/2$ and $\beta = 3/2$ and $N = \frac{5}{2} M + \frac{3}{2}$. Therefore

$$
N \leq 2.6 M, \ \text{if} \ M \geq 15, \ \text{hence} \ M^2 \geq (2.6)^{-2} N^2,
$$
so, in the notation of [12, Chapter 10], \( \kappa_3 = (2.6)^{-2} = 0.1479 \) and, by [12 Relation (10.3)] \( \kappa_1 = 0.3458142306, \kappa_2 = 8.318175470, \) and \( \kappa_4 = 0.06077760153. \)

Choice of \( C \): According to (19), \( M \leq 6.86 \cdot 10^{147} \), therefore a first upper bound for \( N \) is \( B_1(N) := 6.86 \cdot 10^{147} \) and, according to [12 Relation (10.7)], the integer \( C \) must be somewhat larger than

\[
2^{k(k+1)/2} \left( k \frac{1}{2} \right)^{k+1} = 2^{21} \cdot 6.57 \cdot (6.86 \cdot 10^{147})^7 \approx 10^{1046}.
\]

We choose \( C = 10^{1050} \) and work with precision 1080 decimal digits. The linear form \( \lambda \) to which we apply the reduction process is

\[
\lambda = \frac{1}{\kappa_1} L(P) = n_0 + n_1 \left( \frac{\ell_1}{\xi_1} \right) + n_2 \left( \frac{\ell_2}{\xi_1} \right) + n_3 \left( \frac{\ell_3}{\xi_1} \right) + n_4 \left( \frac{\ell_4}{\xi_1} \right) + n_5 \left( \frac{\ell_5}{\xi_1} \right)
\]

\[
= n_0 + n_1 \xi_1 + n_2 \xi_2 + n_3 \xi_3 + n_4 \xi_4 + n_5 \xi_5
\]

\[
= n_0 + n_1(-48478...) + n_2(-254638...) + n_3(-11376...) + n_4(-17120...) + n_5(38222...).
\]

The lattice \( \Gamma \) which is generated by the columns of the matrix

\[
\mathcal{M}_\Gamma = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
[C\xi_1] & [C\xi_2] & [C\xi_3] & [C\xi_4] & [C\xi_5] & C
\end{bmatrix}
\]

is a sublattice of \( \mathbb{Z}^6 \), where

\[
[C\xi_1] = \underbrace{-4847823699 \ldots 3498111567}_{1080 \text{ digits}}, \quad [C\xi_2] = \underbrace{-2546386009 \ldots 1065528645}_{1080 \text{ digits}},
\]

\[
[C\xi_3] = \underbrace{-1137615345 \ldots 6305151167}_{1080 \text{ digits}}, \quad [C\xi_4] = \underbrace{-1712017639 \ldots 6800307853}_{1080 \text{ digits}},
\]

\[
[C\xi_5] = \underbrace{3822275389 \ldots 0614559892}_{1080 \text{ digits}}.
\]

All six integer coordinates of the first vector \( b_0 \) of the LLL-reduced basis have 180 digits and the length of \( b_0 \) is of the size of \( 7.85 \cdot 10^{179} \), satisfying thus the relation

\[
|b_0| > 2^{k/2} \left( k \frac{1}{2} \right) B_1(N)
\]

(cf. [12 Relation (10.6)]). It follows then by [12 Proposition 10.1.1] that

\[
\kappa_1 N^2 \leq \kappa_2 + \log(\kappa_1 C) - \log\left\{ \sqrt{2^{-k}|b_0|^2 - kB_1(N) - kB_1(N)} \right\}
\]

from which we obtain \( N \leq 186. \)
We set now \( B_1(N) = 186 \) and repeat the process, by choosing \( C = 10^{30} \). We obtain

\[
\text{new } b_0 = \begin{pmatrix}
-53853 \\
15304 \\
-25937 \\
245 \\
-36760 \\
12425
\end{pmatrix}.
\]

The new \( b_0 \) satisfies (33) hence, from (34) we obtain the new upper bound \( N \leq 33 \). Repeating the process we obtain the new upper bound \( N \leq 27 \), which cannot be further reduced.

**Important computational issue.** In (34) the parameter \( \kappa_4 \) is equal to an explicitly calculable multiple of \( \rho \), the least eigenvalue of the (positive definite) height-pairing matrix; this is detailed in the beginning of Chapter 10 of [12]. It is clear then that, the smaller \( \rho \) is, the larger is the upper bound for \( N \) which is obtained from (34). This shows that, as the reduction process goes on and \( C \) becomes smaller and smaller, the role of \( \rho \) becomes more and more important: The larger is \( \rho \) the smaller will be the reduced upper bound for \( N \). Therefore, it is important to compute a Mordell-Weil basis whose height-pairing matrix has its least eigenvalue as small as possible. We start from a Mordell-Weil basis furnished by MAGMA and then follow the algorithm of Stroeker & Tzanakis [10, Section 4], which we implemented in MAPLE. In our case, the online MAGMA calculator (V2.24–3) furnished us with the basis

\[
(235, 395), (750, -20205), (310, -4105), (495, 10395), (1075, 35045).
\]

and corresponding \( \rho \approx 0.410937 \). Using the above mentioned algorithm we obtained the Mordell-Weil basis that we use in Section 2.2. As explained in the “Important remark” of that section, just above Table 1 by using the improved basis a lot of computation time is gained.

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