Metaplectic Covariance of the Weyl-Wigner-Groenewold-Moyal
Quantization and Beyond

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Abstract

The metaplectic covariance for all forms of the Weyl-Wigner-Groenewold-Moyal quantization is established with different realizations of the inhomogeneous symplectic algebra. Beyond that, in its most general form $W_\infty$-covariance of this quantization scheme is investigated, and explicit expressions for the quantum-deformed Hamiltonian vector fields are presented. In a general basis the structure constants of the $W_\infty$-algebra are obtained and its subalgebras are analyzed.

I. INTRODUCTION

The Weyl-Wigner-Groenewold-Moyal (WWGM)-quantization scheme [1,2], like all the existing quantization methods [3,4], is an association process (in fact a collection of association processes) between the classical observables (c-number functions defined on a classical phase space) and quantum observables (operators acting in the corresponding Hilbert space $\mathcal{H}$). In a recent work [5], it has been shown that the WWGM-quantization has an infinity of covariances described by the recently found $W_\infty$-algebra [6]. In that study it was explicitly established that under the group actions of $W_\infty$-algebra generated by ordered products of operators realized in the tangent space of a classical phase space bases operators transform by similarity transformations which can be made unitary by taking suitable combinations of generators, or by choosing special rule of ordering. This seems to be great achievement in comparison with the well-known ”metaplectic covariance” (covariance with respect to affine canonical transformations) [4-9], of the WWGM-quantization. It is unfortunate to observe that even this small covariance property was established for only special forms of the WWGM-quantization.

In order to see how the metaplectic covariance of the WWGM-quantization emerges let us consider the Heisenberg-Weyl (HW) algebra: $[\hat{q},\hat{p}] = i\hbar \hat{I}$, where $\hbar, \hat{I}, \hat{q}$ and $\hat{p}$ are the Planck’s constant, the identity operator and the hermitian position and momentum operators, respectively. Here and henceforth operators and functions of operators acting in $\mathcal{H}$ are denoted by over letters. According to the Stone-von Neumann theorem [4], up to a central element generated by the identity operator $\hat{I}$, every irreducible representation of the
HW-group is unitarily equivalent to the Schrödinger representation given by the operators
\[ \hat{D}(\xi, \eta) = \exp i(\xi \hat{q} + \eta \hat{p}) \] which act irreducibly in \( \mathcal{H} \). \( U(1) \) being the center of the HW-group generated by \( \hat{I} \), the so-called displacement operators \( \hat{D}(\xi, \eta) \) are the representatives of the coset space \( HW/U(1) \) in the real \((\xi, \eta)\) parametrization of the HW-group space. The affine canonical covariance of the WWGM-quantization simply follows from the structure of automorphism group of the HW-group and from the Stone-von Neumann theorem. Because, in addition to inner automorphisms, the automorphism group of the HW-group contains the inhomogeneous symplectic group \( ISp(2) \) which is the semidirect product of the translation group and the symplectic group \( Sp(2) \) [4]. Thus, one can combine \( \hat{D}(\xi, \eta) \) with an element of \( ISp(2) \) to obtain another representation unitarily equivalent to the Schrödinger representation.

Like the metaplectic covariance, the \( W_\infty \)-covariance is also a direct result of the property of the operator bases which are \( s \)-parametrized \((s \in \mathbb{C})\) displacement operators and their Fourier transforms [10,11]

\[ \hat{D}(s) = e^{-ihs\xi/2} \hat{D}(\xi, \eta), \quad \hat{\Delta}_{qp}(s) = (h/2\pi) \int \int e^{-i(\xi q + \eta p)} \hat{D}(s) d\xi d\eta \] (1)

(All the integrals are from \(-\infty \) to \( \infty \)). Since they form complete operator bases, in the sense that any operator obeying certain conditions can be expanded in terms of them [10], they provide a unified approach to different quantization rules. The WWGM-quantization come into play by considering the parameters of the group space as the coordinate functions of a phase space. In this sense, the basis elements play a dual role; on the one hand they are operators parametrized by the coordinate functions of a phase space and acting in \( \mathcal{H} \), on the other hand they behave as operator-valued c-number functions defined on the same phase space. Each different basis is closely connected with the ordering of noncommuting \( \hat{q} \) and \( \hat{p} \) in the expansion of operators, and therefore, the so obtained symbols as well as the resulting phase spaces are different. The basis operators \( \hat{\Delta}_{qp} \) (for \( s = 0, \pm 1 \)) are known as the Grossmann-Royer displaced parity operators [12], and as the Kirkwood bases, respectively.

The first aim of this report is to extend and to sharpen the ideas introduced in reference [5] by emphasizing the metaplectic covariance of the WWGM-quantization in its as general form as possible and make manifest the algebraic foundation of this quantization scheme. Secondly, we analyse the structure of the \( W_\infty \)-algebra in a general basis. As far as we know the presentation of this important algebra in such a general framework does not exist in the literature. For the purposes of this report systems with only one degree of freedom and the corresponding phase spaces in real coordinates are considered. Generalizing the results of this report to systems with finite or denumerably infinite number of degrees of freedom and to phase spaces with complex coordinates are straightforward. We use the derivative-based approach developed in [5], which is different from the integral-based conventional one in that differential structures of the bases operators are given primary status.

The organization of the paper is as follows. Firstly, the metaplectic covariance of the WWGM-quantization will be explicitly established for all the operator bases of this quantization scheme, by giving all the possible realization of the \( isp(2) \)-algebra (Sec.III and IV). Secondly, two basic ingredients of the WWGM-quantization, the \( \star \) product and Moyal Brackets (MB) are recognized as the genuine properties of the bases operators (Sec.V). Finally, in a most general basis, explicit expressions for the quantum deformed Hamiltonian vector fields, the structure constants of the \( W_\infty \)-algebra and its subalgebra structures will be presented.
Two important properties of the operators $\hat{D}(s)$ are the factorization and the so-called displacement property

$$\hat{D}(s) = e^{i\hbar(1-s)q\eta/2}e^{i\xi\eta}e^{i\eta\hat{p}} = e^{-i\hbar(1+s)\xi\eta/2}e^{i\eta\hat{p}}e^{i\xi\hat{q}},$$

(2)

$$\hat{D}(s) \hat{f}(\hat{q}, \hat{p}) \hat{D}^{-1}(s) = \hat{f}(\hat{q} + \hbar \eta, \hat{p} - \hbar \xi).$$

(3)

We also have $\text{Tr}[\hat{D}(s)] = (2\pi/\hbar)\delta(\xi)\delta(\eta)$ and $\hat{D}^\dagger(s) = \hat{D}^{-1}(s)$, where $\text{Tr}$ stands for the trace, and $\dagger$ for hermitian conjugation and $\bar{s}$ denotes the complex conjugation of $s$. The corresponding relations for the $\Delta$ operators are $\hat{\Delta}_{qp}^{\dagger}(s) = \hat{\Delta}_{qp}(-\bar{s})$, and

$$\int \int \hat{\Delta}_{qp}(s) dq dp = \hbar, \quad \text{Tr}[\hat{\Delta}_{qp}(s)] = 1.$$  

(4)

For later use, we also note the relations

$$\xi \hat{D}(s) = \hbar^{-1}[\hat{p}, \hat{D}(s)] \quad ; \quad \eta \hat{D}(s) = -\hbar^{-1}[\hat{q}, \hat{D}(s)]$$

(5)

which easily follow from (3).

Now, a large class of associations and their inverse transformations can be defined as

$$\hat{F}(\hat{q}, \hat{p}) = \hbar^{-1} \int \int f^{(-s)}(q, p) \hat{\Delta}_{qp}(s) dq dp \quad ; \quad f^{(s)}(q, p) = \text{Tr}[\hat{F} \hat{\Delta}_{qp}(s)].$$

(6)

For special values $s = 1, 0, -1$ these are known, respectively, as the standart, the Wigner-Weyl, and the antistandard rules of associations $[1,13,14]$. There is one more special association defined by $\hat{F}(\hat{q}, \hat{p}) = (\hbar/2\pi) \int \int f(\xi, \eta) \hat{D}^{-1}(\xi, \eta)d\xi d\eta$, and $f(\xi, \eta) = \text{Tr}[\hat{F} \hat{D}(\xi, \eta)]$. This is known as the alternative Weyl association (or quantization), and the above mentioned Wigner quantization is, simply, the Fourier transform of it. Except for some particular values of $s$, which may give singularities $[10]$, these associations are norm preserving $1-1$ associations between the space of bounded operators and the space of square integrable functions. The quasi-probability distribution functions which enable us to carry out quantum mechanical calculations in a purely classical manner in the resulting phase space, are nothing more than the c-number functions associated to density operators.

By taking the derivatives of the various factorizations of $\hat{D}(s)$ we obtain

$$\partial_\xi \hat{D}(s) = (i/2)\hat{T}_{[\eta]}(s) \hat{D}(s) \quad ; \quad \partial_\eta \hat{D}(s) = (i/2)\hat{T}_{[\eta]}(s) \hat{D}(s)$$

(7)

where, and henceforth the notation $\partial_x \equiv \partial/\partial x$ will be used. In Eqs.(7), $\hat{L}_{\tilde{A}}$ and $\hat{R}_{\tilde{A}}$ being, respectively, multiplication from left and from right by $\tilde{A}$, we defined the Hilbert space operation $\hat{T}_{[\eta]}(s) = (1 + s)\hat{L}_{\tilde{A}} + (1 - s)\hat{R}_{\tilde{A}}$. Observing that for an arbitrary operator $\tilde{B}$
we can generalize Eqs. (7) as follows

$$\partial^{(m)} \partial^{(n)} \hat{D}(s) = (i/2)^{n+m} \hat{T}^m_{[\xi]} \hat{T}^m_{[\eta]} \hat{B}(s)$$

In fact by making use of (8) this equation can be rewritten in finitely many, differently looking but equivalent forms \[4\].

The s-ordered products \(\hat{i}^{(s)}_{nm} \equiv \{(\hat{q})^n (\hat{p})^m\}_s\) are, implicitly, defined in terms of the parametrized bases operators as follows \[11\]

$$\hat{i}^{(s)}_{nm} = (-i)^{n+m} \partial^{(m)} \partial^{(n)} \hat{D}(s)|_{\xi=\eta}. \quad (10)$$

Although, there are not any known physical applications apart from the three principle ones corresponding to \(s = 1, 0, -1\), embedding orderings in a continuum provides a natural context for viewing their differences and interrelationships in a continuous manner and enable us to carry out the related analyses in their most general forms. For these reasons, instead of implicit ones given by (10) we must have explicit expressions for the ordered products which can be easily obtained by making use of the differential structure of the basis operator derived above. Indeed, using (9) in (10) we get

$$\hat{i}^{(s)}_{nm} = (1/2)^{n+m} \hat{T}^m_{[\xi]} \hat{T}^m_{[\eta]} \hat{I} = (1/2)^{n+m} \hat{T}^m_{[\eta]} \hat{T}^m_{[\xi]} \hat{I}. \quad (11)$$

In view of Eq.(8), it is possible to write many equivalent forms of the above relations, but, for later use only two of them have been written. We note that ordering parameters \(s\) and \(-s\) in the last factors of the the above expressions do not contribute to the results since \(\hat{T}^m_{[\xi]} \hat{I} = 2^m A^m\). By making use of this observation and the binomial formula

$$\hat{T}^m_{[\xi]} = [(1 + s) \hat{L}_A + (1 - s) \hat{R}_A]^n = \sum_{j=0}^{n} \binom{n}{j} (1 + s)^j (1 - s)^{n-j} \hat{L}_A^j \hat{R}_A^{n-j} \quad (12)$$

we can rewrite expressions in (11) more explicitly as

$$\hat{i}^{(s)}_{nm} = 2^{-n} \sum_{j=0}^{n} \binom{n}{j} (1 + s)^j (1 - s)^{n-j} \hat{q}^j \hat{p}^m \hat{q}^{n-j}$$

$$= 2^{-m} \sum_{k=0}^{m} \binom{m}{k} (1 - s)^k (1 + s)^{m-k} \hat{p}^k \hat{q}^n \hat{p}^{m-k} \quad (13)$$

From these we have, for \(s = \pm 1\), \(\hat{i}^{(1)}_{nm} = \hat{L}_q \hat{R}_p \hat{I} = \hat{q}^n \hat{p}^m\); \(\hat{i}^{(-1)}_{nm} = \hat{L}_p \hat{R}_q \hat{I} = \hat{p}^m \hat{q}^n\) and for \(s = 0\), \(\hat{i}^{(0)}_{nm} = 2^{-n} \sum_{j=0}^{n} \binom{n}{j} \hat{q}^j \hat{p}^m \hat{q}^{n-j} = 2^{-m} \sum_{k=0}^{m} \binom{m}{k} \hat{p}^k \hat{q}^n \hat{p}^{m-k}\). While the first two of these expressions exhibit the standard and antistandard rule of orderings, respectively, that corresponding to \(s = 0\) are two well known expressions of the Weyl, or symmetricaly ordered products. In fact the usual expression known for the Weyl ordered form of \(\hat{i}^{(0)}_{nm}\) is a totally symmetrized form containing \(n\) factors of \(\hat{q}\) and \(m\) factors of \(\hat{p}\), normalized by dividing by the number of terms in the symmetrized expression. As a simple result of the approach followed here not only the above mentioned equivalences but the explicit expressions for many forms
of the \( s \)-ordered products and their equivalences, without using the usual commutation relations, naturally arise by noting only the relation (8).

Noting that \( \hat{T}[\hat{p}]_{(-s)} = \hat{T}[\hat{p}]_{(-st)} - (s - st)\text{ad}_{\hat{p}} \) and \( \text{ad}_{\hat{p}} \hat{q}^n \equiv [\hat{p}, \hat{q}^n] = -i\hbar \hat{q}^{n-1} \) from (11) we have

\[
\hat{t}^{(s)}_{nm} = \sum_{k=0}^{(n,m)} 2^{-k} b(k, n, m) [i\hbar (s - st)]^k \hat{t}^{(s)}_{n-k, m-k}
\]  

(14)

where \((n, m)\) denotes the smaller of the integers \(n\) and \(m\), and \(\binom{n}{k} = n!(n-k)!k!^{-1}\) being a binomial coefficient

\[
b(k, n, m) = \binom{n}{k} \binom{m}{k} k!
\]  

(15)

Alternatively, (14) can also be obtained by differentiating \( \hat{D}(s) = e^{-i\hbar (s-st)} \hat{\xi}/2 \hat{D}(s) \). This relation expresses an arbitrary \( s \)-ordered product in terms of a polynomial in \( s \)-ordered product, where \( s \) is also arbitrary. Note that \( i\hbar \) in Eq. (14) is the sign of the commutator of the corresponding operators there. Thus, the relation (11), or (13) can be used for any pair of the operators \( A, B \) of any algebra satisfying the commutation relation \([\hat{A}, \hat{B}] = i\lambda, \lambda \in \mathbb{C}\). These relations enable us to generalize the discussion in the case that one, or both of the integers \(n\) and \(m\) are negative and to determine the hermiticity property of a general \( s \)-ordered product. From (13) it easily follows that \([\hat{t}^{(s)}_{nm}]^\dagger = \hat{t}^{(s)}_{nm} \), that is, for general \(n, m\) integers, \(\hat{t}^{(s)}_{nm}\) are hermitian if and only if \(s = -s\). In particular, the Weyl ordered products \(\hat{t}^{(0)}_{nm}\) are hermitian. Since the result is independent from \(s\) when both or one of the integers \(n, m\) is zero, these special monomials are hermitian for any value of \(s\). For general \(s, \alpha \in \mathbb{C}\) one can find combinations such as \(\hat{\kappa}_{nm}(s) = \alpha \hat{t}^{(s)}_{nm} + \bar{\alpha} \hat{t}^{(-s)}_{nm}\) that are hermitian. In fact either of the relation given by (11) (or alternatively, by (13)) can be used as a definition of \( s \)-ordered product for any two operators, irrespective of the commutation relation between them. But, in such a case, connections among differently ordered forms, and more importantly the completeness of the ordered products will depend on the whole structure of the algebra they belong. Therefore, as peculiar properties of the algebras these issues must be separately investigated.

By defining the so called \( s \)-parametrized Bopp operators \(3,13\)

\[
Q_L(s) = -i\partial_{\xi} - s^\eta, \quad Q_R(s) = -i\partial_{\eta} + s^\xi,
\]

\[
P_L(s) = -i\partial_{\eta} + s^\xi, \quad P_R(s) = -i\partial_{\eta} - s^\xi
\]

(16)

where

\[
s^\mp = \frac{1}{2} \hbar (1 \mp s)
\]  

(17)

another way of writing the derivatives of the \( \hat{D}(s) \) such that quantities appearing at different sides belong to different spaces is achieved as

\[
Q^n_L(s) \hat{D}(s) = \hat{q}^n \hat{D}(s) \quad , \quad Q^n_R(s) \hat{D}(s) = \hat{D}(s) \hat{q}^n
\]

\[
P^n_L(s) \hat{D}(s) = \hat{p}^n \hat{D}(s) \quad , \quad P^n_R(s) \hat{D}(s) = \hat{D}(s) \hat{p}^n.
\]  

(18)

Being defined on the tangent space of the phase space, the \( s \)-parametrized Bopp operators obey the commutation relations
\[ [Q_L(s), P_L(s)] = -i\hbar = -[Q_R(s), P_R(s)] \]

All other commutators are zero. These relations show that the above defined Bopp operators form a concrete coordinate realization of a direct sum of two copies of the HW-algebra, and for real \( s \) they are hermitian on the Lebesque space defined on the phase space. The Bopp operators were defined only for the Wigner \((s=0)\) quantization \([13,15]\). Here we generalize for real \( s \) form a concrete coordinate realization of a direct sum of two copies of the HW-algebra, and these relations show that the above defined Bopp operators are hermitian on the Lebesque space defined on the phase space. The Bopp operators were defined only for the Wigner \((s=0)\) quantization \([13,15]\). Here we generalize for real \( s \).

Differential structures of the \( \Delta(s) \) bases are, formally, the Fourier transforms of that obtained for \( \hat{D}(s) \) bases. More simply, they can be derived from (1), (5) and (7) by elementary calculations as follows

\[
\hat{\partial}_q \hat{\Delta}_{qp}(s) = -i\hbar [\hat{q}, \hat{\Delta}_{qp}(s)], \quad \hat{\partial}_p \hat{\Delta}_{qp}(s) = \frac{i}{\hbar} [\hat{p}, \hat{\Delta}_{qp}(s)]
\]

\[
q \hat{\Delta}_{qp}(s) = \frac{1}{2} \hat{f}_{[\hat{q},(s) \hat{\Delta}_{qp}(s)]}, \quad p \hat{\Delta}_{qp}(s) = \frac{1}{2} \hat{f}_{[\hat{p},(-s) \hat{\Delta}_{qp}(s)]}
\]

As an application, making use of (8), these can be generalized such as \( q^n p^m \hat{\Delta}_{qp}(s) = 2^{-(n+m)} T_{[\hat{q},(s)]}^m T_{[\hat{p},(-s)]}^m \hat{\Delta}_{qp}(s) \). Now by taking the traces of both sides we have \( q^n p^m = Tr[\hat{f}_{[\hat{q},(s) \hat{\Delta}_{qp}(s)]}] \) which shows that the \( s \)-quantization of the monomial \( q^n p^m \) is \( \hat{f}_{[\hat{q},m]} \). For other applications which follow more easily by using the derivative-based approach than by using conventional integral-based one we refer to the work \([8]\).

From Eqs. (20) and (21) we have

\[
Q_{\Delta L}(s) \hat{\Delta}_{qp}(s) = \hat{q} \hat{\Delta}_{qp}(s), \quad Q_{\Delta R}(s) \hat{\Delta}_{qp}(s) = \hat{\Delta}_{qp}(s) \hat{q}
\]

\[
P_{\Delta L}(s) \hat{\Delta}_{qp}(s) = \hat{p} \hat{\Delta}_{qp}(s), \quad P_{\Delta R}(s) \hat{\Delta}_{qp}(s) = \hat{\Delta}_{qp}(s) \hat{p}
\]

where the Bopp operators for the \( \hat{\Delta}_{qp}(s) \) bases are found to be

\[
Q_{\Delta L}(s) = q - is^- \hat{\partial}_p, \quad Q_{\Delta R}(s) = q + i s^+ \hat{\partial}_p, \quad P_{\Delta L}(s) = p + i s^+ \hat{\partial}_q, \quad P_{\Delta R}(s) = p - i s^- \hat{\partial}_q
\]

\[
-[Q_{\Delta L}(s), P_{\Delta L}(s)] = i\hbar = [Q_{\Delta R}(s), P_{\Delta R}(s)].
\]

III. METAPLECTIC COVARIANCE OF THE WEYL BASIS

In this section, with the realization given by (24) below, we are going to work out the action of the inhomogeneous symplectic algebra \( isp(2) \) on the Weyl basis \( \hat{D}(s) \). \( isp(2) \) consists of two translations \( (N_1, N_2) \), two squeezes \( (B_1, B_2) \) and one rotation \( (L) \) generators. In the classical \((\xi, \eta)\) phase space a usual (hermitian) realization for them is given as follows \([7,8]\)

\[
N_1 = -i \hat{\partial}_\xi \quad , \quad B_1 = -i \left( \frac{\xi \hat{\partial}_\xi - \eta \hat{\partial}_\eta}{2} \right)
\]

\[
N_2 = -i \hat{\partial}_\eta \quad , \quad B_2 = i \left( \frac{\xi \hat{\partial}_\eta + \eta \hat{\partial}_\xi}{2} \right)
\]

\[
J = -i \left( \frac{\xi \hat{\partial}_\eta - \eta \hat{\partial}_\xi}{2} \right)
\]
They obey the commutation relations

\[
[N_1, N_2] = 0, \quad [B_1, B_2] = iJ, \quad [J, B_1] = -iB_2, \quad [J, B_2] = iB_1
\]

\[
- [N_1, B_1] = [N_2, J] = [N_2, B_2] = \frac{i}{2}N_1
\]

\[
- [N_1, J] = [N_2, B_1] = [N_1, B_2] = \frac{i}{2}N_2
\]

(25)

The translations \(N_1\) and \(N_2\) form the invariant subalgebra of the \(isp(2)\). The subalgebras generated by \((N_1, N_2, J)\) and \((J, B_1, B_2)\) are known as the Euclidean algebra \(e(2)\) and the homogeneous symplectic algebra \(sp(2)\), respectively. \((N_1, N_2, B_1)\) and \((N_1, N_2, B_2)\) are also solvable subalgebras of \(isp(2)\).

Making use of Eqs. (5) and (7) it is easy to verify the following actions

\[
N_1 \hat{D}(s) = \frac{1}{2} \hat{T}_{\|s\|} \hat{D}(s) , \quad N_2 \hat{D}(s) = \frac{1}{2} \hat{T}_{\|s\|} \hat{D}(s)
\]

(26)

\[
B_1 \hat{D}(s) = \frac{1}{4\hbar} \text{ad}_{(\hat{q}\hat{p} + \hat{p}\hat{q})} \hat{D}(s)
\]

(27)

\[
B_2 \hat{D}(s) = \frac{1}{4\hbar} [\text{ad}_{(\hat{q}^2 - \hat{p}^2)} + s \hat{T}_{[\hat{q}^2 - \hat{p}^2]}] - 2s \hat{M}_+ \hat{D}(s)
\]

(28)

\[
J \hat{D}(s) = \frac{1}{4\hbar} [\text{ad}_{(\hat{q}^2 - \hat{p}^2)} + s \hat{T}_{[\hat{q}^2 - \hat{p}^2]}] - 2s \hat{M}_- \hat{D}(s)
\]

(29)

where \(\hat{M}_\pm = \hat{L}_p \hat{R}_q \pm \hat{L}_q \hat{R}_p\). Eqs.(26) explicitly show that, with the realization given by (24), neither \(N_1\) nor \(N_2\) induce an infinitesimal transformation of \(\hat{D}(s)\) in \(\mathcal{H}\), but, as is apparent from (27) under the action of \(B_1\) an infinitesimal transformation generated by \(\hat{B}_1 = (4\hbar)^{-1}(\hat{q}\hat{p} + \hat{p}\hat{q})\) is induced for all values of \(s\). On the other hand, Eqs.(28) and (29) show that only for \(s = 0\), \(B_2\) and \(J\) induce the infinitesimal transformations. Thus, under the action of unitary representation of the the group \(Sp(2)\) generated by the realization of \(sp(2)\) given by (24) only the Weyl basis \(\hat{D} = \hat{D}(s = 0)\) transforms by unitary similarity transformations generated by

\[
\hat{B}_1 = \frac{1}{4\hbar}(\hat{q}\hat{p} + \hat{p}\hat{q}), \quad \hat{B}_2 = \frac{1}{4\hbar}(\hat{q}^2 - \hat{p}^2), \quad J = \frac{1}{4\hbar}(\hat{q}^2 + \hat{p}^2)
\]

(30)

which obey the commutation relations

\[
[\hat{J}, \hat{B}_1] = i\hat{B}_2, \quad [\hat{J}, \hat{B}_2] = -i\hat{B}_1, \quad [\hat{B}_1, \hat{B}_2] = -i\hat{J}
\]

(31)

Up to an overall minus sign, the quantum algebra generated by \((\hat{B}_1, \hat{B}_2, \hat{J})\) is the same as with \(sp(2)\). In fact, by the completeness of the \(\hat{D}\) basis, this quantum algebra is determined up to central element \(\hat{I}\). Therefore the induced algebra is, in fact, the central extension of the \(sp(2)\). This can be made manifest by considering the action of a general group element \(V(b_1, b_2, \theta) \in Sp(2)\)

\[
V(b_1, b_2, \theta) \hat{D} = \hat{U}(b_1, b_2, \theta) \hat{D} \hat{U}^+(b_1, b_2, \theta)
\]

(32)

where \(b_1, b_2, \theta \in \mathbb{R}\) being the group parameters of \(Sp(2)\), \(V(b_1, b_2, \theta) = \exp i(b_1 \hat{B}_1 + b_2 \hat{B}_2 + \theta \hat{J})\), and \(\hat{U}(b_1, b_2, \theta) = \exp i(b_1 \hat{B}_1 + b_2 \hat{B}_2 + \theta \hat{J})\). Eq.(32), which can be easily verified by
exponentiating the actions given by (26-29) for \( s = 0 \), explicitly shows that for a given element \( V \in Sp(2) \), \( \hat{U} \) is determined up to a phase factor, the choice of which can be made in one and only one way up to factors of \( \pm \), so that the group generated by \( \hat{U} \)'s provides a double valued representations of \( Sp(2) \).

If the algebra \((\hat{B}_1, \hat{B}_2, \hat{J})\) is enlarged by adding the generators

\[
\hat{N}_1 = \hat{q}, \quad \hat{N}_2 = \hat{p} \quad ; \quad [\hat{N}_1, \hat{N}_2] = i\hbar \hat{I}
\]

apart from an overall sign difference and the commutator given above they obey the same commutation relations given by (25). Therefore, the algebra generated by \( isp_c(2) \equiv (\hat{I}, \hat{N}_1, \hat{N}_2, \hat{B}_1, \hat{B}_2, \hat{J}) \) is the central extension of the \( isp(2) \). But, it is not induced under the action of \( isp(2) \) with the realization given by (24). At a glimpse to Eqs.(5) we see that infinitesimal transformations of \( \hat{D} \) generated by \( \hat{N}_1 \) and \( \hat{N}_2 \) correspond to multiplications by \( M_1 \equiv -\bar{\hbar} \eta \) and \( M_2 \equiv \bar{\hbar} \xi \). It is easy to check that, if \( N_1 \) and \( N_2 \) are replaced by \( M_1 \) and \( M_2 \), respectively, the commutation relations (25) remain unchanged. Therefore, the unitary transformations generated by \( isp_c(2) \) is induced under the action of \( isp(2) \) with the realization given by \((M_1, M_2, B_1, B_2, J)\).

In order to make the connection between two algebras more concrete we define operator-valued one column matrices by \( \hat{\chi}^t = (\hat{q}, \hat{p}, \hat{I}) \) and \( \hat{\chi}'^t = (\hat{q}', \hat{p}', \hat{I}) \), where the superscript \( t \) stands for transpose operation. Now it is straightforward to verify the following relations

\[
\hat{\chi}' = e^{i\theta \hat{J}} \hat{\chi} e^{-i\theta \hat{J}} = J(\theta) \hat{\chi}
\]

where \( k = 1, 2 \) and

\[
J(\theta) = \begin{pmatrix}
\cos \frac{\theta}{2} & \sin \frac{\theta}{2} & 0 \\
-\sin \frac{\theta}{2} & \cos \frac{\theta}{2} & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

\[
B_1(b_1) = \begin{pmatrix}
e^{b_1/2} & 0 & 0 \\
0 & e^{-b_1/2} & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

\[
B_2(b_2) = \begin{pmatrix}
\cosh \frac{b_2}{2} & -\sinh \frac{b_2}{2} & 0 \\
-\sinh \frac{b_2}{2} & \cosh \frac{b_2}{2} & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

\[
N_1(c_1) = \begin{pmatrix}1 & 0 & 0 \\
0 & 1 & -c_1 \hbar \\
0 & 0 & 1
\end{pmatrix}
\]

\[
N_2(c_2) = \begin{pmatrix}1 & 0 & c_2 \hbar \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
On the other hand, the action of the corresponding $ISp(2)$ in the $(\xi, \eta)$-phase space on the coordinate functions is as follows

\begin{align*}
e^{i\beta_1} \chi &= J(-\theta) \chi, \quad e^{i\beta_k} B_k \chi = B_k(b_k) \chi \\
e^{i\alpha_1 M_1} \chi &= e^{-i\alpha_1 h \eta} \chi, \quad e^{i\alpha_2 M_2} \chi = e^{i\alpha_2 h \xi} \chi
\end{align*}

where $\chi^t = (\xi, \eta, 1)$. Note that there is a sign difference between the corresponding rotation matrices and that the matrices generated by $\hat{N}_1, \hat{N}_2$ and by $M_1, M_2$ are completely different. Hence, the group generated by $isp_c$ is, in fact, not the central extension of $ISp(2)$, but, of the semidirect product of $U(1) \otimes U(1)$ and $Sp(2)$, where the $U(1)$ groups are generated by multiplication operators $M_1$ and $M_2$ which do not generate area preserving transformations.

Recalling that all these observations were valid only for the $\hat{D} \equiv \hat{D}(s = 0)$ basis we now ask the following question: Are there realizations of $isp(2)$ which induce well defined similarity transformations of the $\hat{D}(s)$ for all values of $s \in \mathbb{C}$? To answer this question we observe that with the realization given above the solvable subalgebra of $isp(2)$ generated by $(M_1, M_2, B_1)$ has this property. Thus, we have to search, only, for different realizations of the generators $B_2$ and $J$. A close inspection of the relations given by (5) and (7) shows that if $B_2$ and $J$ are replaced by

\begin{align*}
B_2(s) &= B_2 - \frac{1}{4} \hbar (\xi^2 + \eta^2), \\
J(s) &= J + \frac{1}{4} \hbar (\xi^2 - \eta^2)
\end{align*}

then, with these new realizations of the $isp(2)$ given by $(M_1, M_2, B_1, B_2(s), J(s))$, the same quantum algebra $isp_c(2)$ given above is induced which generate similarity transformation of $\hat{D}(s)$ for all values of $s \in \mathbb{C}$. We also note that, these $s$-parametrized realizations obey the same commutation relations given by (25) and they are hermitian for $s \in \mathbb{R}$.

### IV. METAPELECTIC COVARIANCE OF THE $\hat{\Delta}(s)$ BASES

Let us consider the following realization of the $isp(2)$ algebra

\begin{align*}
N_{\Delta 1} &= -i \hbar \partial_p, \quad B_{\Delta 1} = \frac{i}{2} (q \partial_q - p \partial_p) \\
N_{\Delta 2} &= i \hbar \partial_q, \quad B_{\Delta 2} = -\frac{i}{2} (q \partial_p + p \partial_q) \\
J_{\Delta} &= \frac{i}{2} (q \partial_p - p \partial_q)
\end{align*}

in the classical phase space $(q, p)$. This realization is obtained from (24) by the replacement $\hbar(\xi, \eta) \rightarrow (p, -q)$ in the corresponding letters. It satisfies the same commutation relations given by (25). By making use of Eqs.(20) and (21) one can easily verify the following actions on the $\hat{\Delta}_{qp}(s)$ bases

\begin{align*}
N_{\Delta 1} \hat{\Delta}_{qp}(s) &= [\hat{q}, \hat{\Delta}_{qp}(s)] \\
N_{\Delta 2} \hat{\Delta}_{qp}(s) &= [\hat{p}, \hat{\Delta}_{qp}(s)] \\
B_{\Delta 1} \hat{\Delta}_{qp}(s) &= \frac{1}{4 \hbar} [\hat{q} \hat{p} + \hat{p} \hat{q}, \hat{\Delta}_{qp}(s)] \\
B_{\Delta 2} \hat{\Delta}_{qp}(s) &= \frac{1}{4 \hbar} [a \hat{q}^2 - p^2 + s \tilde{T}_{[\hat{q}^2 - \hat{p}^2]}(s) - 2s \hat{M}_+] \hat{\Delta}_{qp}(s) \\
J_{\Delta} \hat{\Delta}_{qp}(s) &= \frac{1}{4 \hbar} [a \hat{q}^2 + \hat{p}^2 + s \tilde{T}_{[\hat{q}^2 - \hat{p}^2]}(s) - 2s \hat{M}_-] \hat{\Delta}_{qp}(s)
\end{align*}


which are formally equivalent to that given by (26-29), except the first two ones. Therefore, unlike the \( \hat{D}(s) \) bases, \( \hat{\Delta}_{qp}(s) \) bases have the well defined transformation property under the actions of the displacement operators \( (N_{\Delta 1}, N_{\Delta 2}) \). But, like \( \hat{D}(s) \), \( \hat{\Delta}_{qp}(s) \) transforms by similarity transformations only for the value \( s = 0 \). As a result, only the \( \hat{\Delta}_{qp}(s = 0) \) basis has the full metaplectic covariance property under the realization of the \( isp(2) \) given by (37), and the induced quantum algebra is the same as with that found in the preceding section (see the Table I).

Here again, from Eqs.(20) we observe that if \( B_{\Delta 2} \) and \( J_{\Delta} \) are replaced by

\[
B_{\Delta 2}(s) = B_{\Delta 2} + \frac{1}{4} \hbar (\partial_q^2 + \partial_p^2), \quad J_{\Delta}(s) = J_{\Delta} - \frac{1}{4} \hbar (\partial_q^2 - \partial_p^2)
\]

which act on \( \hat{\Delta}_{qp}(s) \) as

\[
B_{\Delta 2}(s)\hat{\Delta}_{qp}(s) = \frac{1}{4\hbar} [\hat{q}^2 - \hat{p}^2, \hat{\Delta}_{qp}(s)], \quad J_{\Delta}(s)\hat{\Delta}_{qp}(s) = \frac{1}{4\hbar} [\hat{q}^2 + \hat{p}^2, \hat{\Delta}_{qp}(s)]
\]

then complete metaplectic covariance of the \( \hat{\Delta}_{qp}(s) \) bases are established under the action of \( N_{\Delta 1}, N_{\Delta 2}, B_{\Delta 1}, B_{\Delta 2}(s), J_{\Delta}(s) \).

An important point that should be mentioned is that, although these \( s \)-dependent realizations of the \( isp(2) \)-algebra manifestly establish the complete metaplectic covariance of the \( \hat{D}(s) \) and \( \hat{\Delta}(s) \) bases for any values of \( s \in \mathbb{C} \), the actions of the generators \( B_2(s), J(s) \) on the coordinate functions do not seem to be expressible in a simple manner except for \( s = 0 \). On the other hand, while the actions of \( B_{\Delta 2}(s), J_{\Delta}(s) \) on the coordinate functions coincide with those of \( B_{\Delta 2}, J_{\Delta} \), their actions on an arbitrary function are quite different. These may be taken as points favouring the use of \( \hat{D} \) and \( \hat{\Delta}_{qp}(s = 0) \) bases.

**V. BEYOND THE METAPLECTIC COVARIANCE**

From Eqs.(13) and (18) one can easily verify that

\[
i_{nm}^{(r)} \hat{D}(s) = L_{nm}^{(-r)}(s) \hat{D}(s), \quad \hat{D}(s)i_{nm}^{(r)} = R_{nm}^{(r)}(s) \hat{D}(s)
\]

where \( X_{nm}^{(r)}(s) \equiv \{ Q_X(s)P_X^{m}(s) \}_{r}, X = L, R \). Similarly, by defining \( X_{\Delta nm}^{(r)}(s) \equiv \{ Q_{\Delta X}(s)P_{\Delta X}^{m}(s) \}_{r} \) from (13) and (22) we have

\[
i_{nm}^{(r)} \hat{\Delta}_{qp}(s) = L_{\Delta nm}^{(-r)}(s) \hat{\Delta}_{qp}(s), \quad \hat{\Delta}_{qp}(s)i_{nm}^{(r)} = R_{\Delta nm}^{(r)}(s) \hat{\Delta}_{qp}(s).
\]

That is, to the actions of the \( r \) ordered products on the basis operators in \( \mathcal{H} \) there corresponds the actions of the \( -r \) (or \( r \)) ordered products of the Bopp operators in the classical phase space. These equations immediately lead to

\[
T_{nm}^{(r)}(s) \hat{D}(s) = [i_{nm}^{(r)}, \hat{D}(s)], \quad \Gamma_{nm}^{(r)}(s) \hat{\Delta}_{qp}(s) = [i_{nm}^{(r)}, \hat{\Delta}_{qp}(s)]
\]

where

\[
T_{nm}^{(r)}(s) \equiv L_{nm}^{(-r)}(s) - R_{nm}^{(r)}(s)
\]

\[
\Gamma_{nm}^{(r)}(s) \equiv L_{\Delta nm}^{(-r)}(s) - R_{\Delta nm}^{(r)}(s)
\]

10
Eqs.(46) explicitly show the $W_\infty$ covariance of the $\hat{\mathcal{D}}(s)$ and $\hat{\Delta}_{qp}(s)$ bases. By multiplying both sides of these equations with an arbitrary bounded operator, and then taking trace of the resulting equations the $W_\infty$-covariance of the WWGM-quantization is easily seen at the algebra level. At the group level $W_\infty$ covariance of the bases operators are obtained by exponentiating the actions given by (46). The extensions of these observations to the case of complex coordinates and, by linearity, to arbitrary functions of operators which can be expanded to a series of ordered products are straightforward.

Here we have two $W_\infty$-algebra: the first one acting in $\mathcal{H}$ is generated by the ordered products $\hat{t}_nm^{(r)}; n, m \geq 0$, and the other one, build up by products of the Bopp operators explicitly realized in the tangent space of a phase space, is generated by $T_{nm}^{(r)}(s); n, m \geq 0$, or by $\Gamma_{nm}^{(r)}(s), n, m \geq 0$. Noting that the former algebra is indexed by one and the latter one by two complex order parameters, we have, in fact, a continuum of $W_\infty$-covariances for a given basis. By noting the relation

$$[\Gamma_{nm}^{(r)}(s), \Gamma_{kl}^{(r)}(s)] \hat{\Delta}_{qp}(s) = -[[\hat{t}_nm^{(r)}, \hat{t}_{kl}^{(r)}], \hat{\Delta}_{qp}(s)]$$

and the similar one for $\hat{\mathcal{D}}(s)$ basis which easily follows from (46), we see that quantum $W_\infty$ is, in fact, the central extension of the classical $W_\infty$.

From now on we shall specialize to the $\hat{\Delta}_{qp}(s)$ basis.

Taking care of the commutation relations given by (23) and the Eq.(14), we can write the following $-s$ and $s$ ordered product of the Bopp operators in terms of standart and antistandard ordered ones as follows

$$L_{\Delta nm}^{(-s)}(-s) = \sum_{k=0}^{(n,m)} b(k, n, m)(is^+)^k Q_{\Delta L}^{n-k}(-s)P_{\Delta L}^{m-k}(-s)$$

$$R_{\Delta nm}^{(s)}(-s) = \sum_{k=0}^{(n,m)} b(k, n, m)(is^+)^k P_{\Delta R}^{m-k}Q_{\Delta R}^{n-k}(-s)$$

By using the explicit expressions of the Bopp operators given by (23), and by formal Taylor expansion we have

$$L_{\Delta nm}^{(-s)}(-s)f(q, p) = \sum_{k=0}^{(n,m)} b(k, n, m)(is^+)^k q^{n-k}(p + is^+ \partial^L_q - is^- \partial^L_q)^{m-k}f(q, p - is^+ \partial^L_q)$$

$$= [ \sum_{k=0}^{(n,m)} b(k, n, m)(is^+)^k q^{n-k}p^{m-k} e^{is^+ \partial^L_q \partial^L_q - is^- \partial^R_q \partial^L_q - is^- \partial^L_q \partial^R_q} f(q, p)]$$

where $f$ is an arbitrary c-number function, and we used the convention that $\partial^L$ and $\partial^R$ are acting on the left(L) and on the right(R), respectively. Since

$$e^{is^+ \partial_q \partial_p (q^n p^m)} = \sum_{k=0}^{(n,m)} b(k, n, m)(is^+)^k q^{n-k}p^{m-k}$$

the expression in the square brackets of the last line of Eq.(51) is simply $q^n p^m$. Thus, this equation can be rewritten as follows

$$L_{\Delta nm}^{(-s)}(-s)f(q, p) = (q^n p^m) \star_{(-s)} f(q, p)$$

$$L_{\Delta nm}^{(-s)}(-s)f(q, p) = (q^n p^m) \star_{(-s)} f(q, p)$$

$$= \sum_{k=0}^{(n,m)} b(k, n, m)(is^+)^k q^{n-k}p^{m-k} e^{is^+ \partial^L_q \partial^L_q - is^- \partial^R_q \partial^L_q - is^- \partial^L_q \partial^R_q} f(q, p)$$

11
where s-star product \( \star_{(-s)} \) is defined to be
\[
\star_{(-s)} = \exp \frac{1}{2} i \hbar [(1 - s) \partial_p \partial_q^R - (1 + s) \partial_q \partial_p^R].
\] (54)

In a similar way leading to Eq.(53), we have
\[
R_{\Delta nm}^{(s)}(-s) f(q, p) = f(q, p) \star_{(-s)} (q^n p^m)
\] (55)

and by combining with (53)
\[
\Gamma_{nm}^{(s)}(-s) f(q, p) = \{q^n p^m, f(q, p)\}_{MB}^{(-s)}
\] (56)

where the s–Moyal Bracked \( \{ , \}_{MB}^{(-s)} \) is defined as follows
\[
\{ f_1(q, p), f_2(q, p)\}_{MB}^{(-s)} = f_1(q, p) \star_{(-s)} f_2(q, p) - f_2(q, p) \star_{(-s)} f_1(q, p).
\] (57)

These last relations give the different expressions for the star product and Moyal brackets, that appeared in the literature separately in a unified manner and generalize them for an arbitrary s-ordering.

Since the bases operators themselves can be thought of as c-number functions, similar to Eqs.(53) and (55) they satisfy the following relations
\[
L_{\Delta nm}^{(-s)}(-s) \hat{\Delta}_{qp}(-s) = (q^n p^m) \star_{(-s)} \hat{\Delta}_{qp}(-s) = \hat{i}_{nm}^{(s)} \hat{\Delta}_{qp}(-s)
\] (58)
\[
R_{\Delta nm}^{(s)}(-s) \hat{\Delta}_{qp}(-s) = \hat{\Delta}_{qp}(-s) \star_{(-s)} (q^n p^m) = \hat{\Delta}_{qp}(-s) \hat{i}_{nm}^{(s)}
\] (59)
\[
\Gamma_{nm}^{(s)}(-s) \hat{\Delta}_{qp}(-s) = \{q^n p^m, \hat{\Delta}_{qp}(-s)\}_{MB}^{(-s)} = [\hat{i}_{nm}^{(s)}, \hat{\Delta}_{qp}(-s)].
\] (60)

These relations reveal the important fact that in their most general forms the \( \star \) product and MB, which constitute two fundamental ingredients of the WWGM-quantization, emerge naturally under the actions of the ordered products on the basis operators prior to the quantization processes.

VI. \( W_{\infty} \) AND QUANTUM- DEFORMED HAMILTONIAN VECTOR FIELD IN A GENERAL BASIS

By multiplying the relation (60) by \( \hat{i}_{kl}^{(s)} \) and, then taking the trace of the both sides we have
\[
\Gamma_{nm}^{(s)}(-s)(q^k p^l) = \{q^n p^m, q^k p^l\}_{MB}^{(-s)}
\]
\[
= -Tr\{[\hat{i}_{nm}^{(s)}, \hat{i}_{kl}^{(s)}] \hat{\Delta}_{qp}(-s)\}
\] (61)

which shows that, up to an overall sign difference the structure constants of the quantum \( W_{\infty} \) in a general s ordered basis can be obtained from the s-MB of the c-number monomials \( q^n p^m \). Indeed, we find it much easier to use s-MB than to use the Lie brackets in investigating the algebraic structure of the \( W_{\infty} \) in a general basis. Converting a relation obtained through \( \{ , \}_{MB}^{(-s)} \) to a Lie brackets relation is achieved, in view of (61) or of (49), by replacing \( q^n p^m \) monomials by \( \hat{i}_{nm}^{(s)} \) and by noting the overall sign difference mentioned above.
In order to obtain the structure constants of the $W_\infty$ algebra in a general basis we begin by the relation

$$q^n p^m \ast_{(-s)} q^k p^l = \sum_{j=0}^{\infty} \frac{j^i}{j!} (s^- \partial_x \partial_y - s^+ \partial_z \partial_w) j^x m^n y^k u^l \big|_{y=q=z}$$

where the prime over the second summation indicates that the maximum value that $r$ may take is $r_{\text{max}} = (m, k)$ (i.e., the smaller of the integers $m$ and $k$) and

$$j_{\text{max}} = (n + r_{\text{max}}, l + r_{\text{max}}), \quad a_{nmkl, rj} = \frac{n!m!k!!}{(n + r - j)!(m - r)!(k - r)!(l + r - j)!}. \quad (63)$$

The restrictions imposed on summations also follows from the expression of $a_{nmkl, rj}$. In a similar way we have

$$q^n p^m \ast_{(-s)} q^k p^l = \sum_{j=0}^{j_{\text{max}}} \frac{j^i}{j!} \sum_{r=0}^{j_{\text{max}}} (j^r)(-s^-)^{j-r} (-s^+)^r a_{nmkl, rj} q^{n+k-j} p^{m+l-j} \quad (64)$$

Thus

$$\{q^n p^m, q^k p^l \}_{MB} = \sum_{j=0}^{j_{\text{max}}} \frac{j^i}{j!} \sum_{r=0}^{j_{\text{max}}} (j^r) f_{srj} a_{nmkl, rj} q^{n+k-j} p^{m+l-j} \quad (65)$$

where

$$f_{srj} = (s^-)^r (-s^+)^{j-r} - (s^-)^{j-r} (-s^+)^r \quad (66)$$

is the only factor depending on the chosen rule of ordering. In terms of the ordered products the corresponding commutation relation is

$$[\hat t^{(s)}_{kl}, \hat t^{(s)}_{nm}] = - \sum_{j=0}^{j_{\text{max}}} \frac{j^i}{j!} \sum_{r=0}^{j_{\text{max}}} (j^r) f_{srj} a_{nmkl, rj} \hat t^{(s)}_{n+k-l, m+j-l} \quad (67)$$

Here we observe that anti-MB of the monomials and anti-commutators of the ordered products are given by the same relations as (65) and (66), respectively, only provided that the ordering factor is replaced by

$$f_{srj}^+ = (s^-)^r (-s^+)^{j-r} + (s^-)^{j-r} (-s^+)^r$$

For the three known rules of ordering (66) is as follows

$$f_{srj} = \begin{cases} (-\hbar)^j (\delta_{r0} - \delta_{rj}) & \text{for } s = 1 \\ \frac{\hbar^2}{2} (1)^{j-r} [(-1)^j - 1] & \text{for } s = 0 \\ \hbar^j (\delta_{jr} - \delta_{r0}) & \text{for } s = -1 \end{cases}$$

If these expressions are used in (65) the resulting commutators
\[
\{ q^n p^m, q^k p^l \}_{MB}^{(-s)} = \left\{ \begin{array}{ll}
(\sum_{j=0}^{(n,l)} (ih)^j a_{mnkl,0j} - \sum_{j=0}^{(m,k)} (ih)^j a_{nmkl,jj}) q^{n+k-j} p^{m+l-j} & s = 1 \\
-2 \sum_{j=0}^{(n,l)} (ih/2)^{2j+1} \sum_{r=0}^{(2j+1)} (1)^r a_{nmkl,r2j+1} q^{n+k-2j-1} p^{m+l-2j-1} & s = 0 \\
\sum_{j=0}^{(n,l)} (ih)^j a_{mnkl,jj} - \sum_{j=0}^{(m,k)} (ih)^j a_{nmkl,0j} q^{n+k-j} p^{m+l-j} & s = -1
\end{array} \right.
\]

coincide with those appeared in literature \cite{17} \cite{19}. Note that there is only an overall sign difference between the structure constants that can be obtained from \( \{ t^{(s)}_{nm}, t^{(s)}_{kl} \} \) and that obtained above.

At the right hand side of (65) the term corresponding to \( j = 0; r = 0 \) is always zero since \( f_{s00} = 0 \). The first nonvanishing structure constant which corresponds to \( j = 1; r = 0 \) can be easily shown to be \( i\hbar (mk - nl) \). But, \( (mk - nl) \) are the structure constants that would be obtained from \( (ih)^{-1} \{ q^n p^m, q^k p^l \}_{MB}^{(-s)} \) in the limit as \( \hbar \to 0 \). Since in this limit \( (ih)^{-1} \{ , \}^{(-s)}_{MB} \) goes to the \( \{ , \}_{PB} \), this limiting algebra is the algebra formed by the monomials \( q^n p^m; n, m \geq 0 \) under the usual Poisson Brackets (PB)

\[
\{ q^n p^m, q^k p^l \}_{PB} = \partial_p (q^n p^m) \partial_q (q^k p^l) - \partial_q (q^n p^m) \partial_p (q^k p^l) = (mk - nl) q^{n+k-1} p^{m+l-1}
\]

This is the algebra of canonical diffeomorphisms of a phase space that is topologically equivalent to \( \mathbb{R}^2 \) and is known as \( w_\infty \), or, since the area element and symplectic form coincide in two dimensions, as the algebra of area preserving diffeomorphisms \( Diff_A \mathbb{R}^2 \) \cite{3}. The above mentioned \( W_\infty \) algebras are quantum (or, \( \hbar \)) deformation of this classical \( w_\infty \).

Equivalently, \( w_\infty \) can be considered as generated by the Hamiltonian vector fields (HVF) basis

\[
v_{nm} = \{ q^n p^m, \}_{PB} = q^{n-1} p^{m-1} (mq \partial_q - np \partial_p)
\]

which generate canonical transformations and close in the algebra

\[
[v_{nm}, v_{kl}] = v_{\{ q^n p^m, q^k p^l \}_{PB}}
\]

Thus, the pseudodifferential operators

\[
\Gamma^{(s)}_{nm}(-s) = \{ q^n p^m, \}_{MB}^{(-s)} = 2i q^n p^m e^{\frac{1}{2} \hbar s(\partial_p \hat{\partial}_q^R + \partial_q \hat{\partial}_p^R) \sin \left[ \frac{1}{2} \hbar (\partial_p \hat{\partial}_q^R - \partial_q \hat{\partial}_p^R) \right]}
\]

are the quantum deformation of the HVF. They close in the algebra

\[
[\Gamma^{(s)}_{nm}(-s), \Gamma^{(s)}_{kl}(-s)] = \Gamma^{(s)}_{\{ q^n p^m, q^k p^l \}_{MB}^{(-s)}}
\]

and generate the quantum counterpart of the classical canonical transformations.

Quantum-deformed HVF appeared in the recent literature only in some special basis \cite{20}. Here, by recognizing the fact that they are nothing more than the ordered products of parametrized Bopp operators we presented them in the most general basis. Moreover, as explicitly seen in (60) they naturally appear, prior to the WWGM-quantization, under the actions of the ordered products \( t^{(s)}_{nm} \) on the basis operators.
In the rest of this section we will discuss subalgebra structures of $W_\infty$-algebra in a general basis. Most of these subalgebras are known in the literature, but they were not presented in such a general framework as we are going to do here. As is investigated in Sec.III and IV, the finite dimensional subalgebra generated by the monomials $q^n p^m$; $n, m \leq 2$ is the central extension of $isp(2)$. Besides that $W_\infty$ has some infinite dimensional subalgebras structures to be discussed in a general basis.

Let us define the degree of a generator $q^n p^m$ as $\text{deg}(q^n p^m) = n - m$. Thus, as can be checked from (67), $W_\infty$ can be thought as the union of two disjoint infinite subalgebras $W_\infty^+$ and $W_\infty^-$ having positive and negative degrees, respectively, and of the infinite abelian subalgebra $W_\infty^0$ consisting the generators $H^n = q^n p^m$ of degree zero. Obviously, the set of the generators having the same degree is an infinite set. We use the same definition and the decomposition when $W_\infty$ is realized in terms of the ordered products and usual Lie brackets. In the former case commutativity of the generators $H^n$ can be easily verified by observing that the monomials and the $\ast(-s)$ product can be expressed in terms of a single variable $H \equiv qp$. In the latter case $[\hat{H}^{(s)}_n, \hat{H}^{(s)}_k] = 0$ simply follows from the fact that for any values of $n \geq 0$ and $s \in \mathbb{C}$ all the ordered products of the form $\hat{H}^{(s)}_n \equiv \hat{c}^{(n)}$ can be expressed in terms of single operator $\hat{\mathcal{H}} \equiv (\hat{qp} + \hat{pq})/2$. To prove this statement, let us first consider the standart ordered product $\hat{H}_n^{(1)} = \hat{q}^n \hat{p}^n$ which can be written as

$$\hat{H}_n^{(1)} = \hat{q}^{n-1}\hat{q}\hat{p}^{n-1} = \hat{q}^{n-1}\hat{p}^{n-1}[\hat{x} + \hat{c}(2n - 1)] = \hat{q}^{n-2}\hat{p}^{n-2}[\hat{x} + \hat{c}(2n - 3)][\hat{x} + \hat{c}(2n - 1)]$$

Hence, by induction, we have

$$\hat{H}_n^{(1)} = \prod_{j=1}^{n} [\hat{x} + \hat{c}(2j - 1)]$$

(73)

where $\hat{c} \equiv i\hbar \hat{I}/2$ and the relations $\hat{q}\hat{p} = \hat{x} + \hat{c}$; $[\hat{x}, \hat{p}^k] = 2\hat{c}\hat{p}^k$ are used. Since, in view of (14), any $s$-ordered product can be written in terms of the standart ordered forms the proof of the above statement is accomplished.

$W_\infty^+$ and $W_\infty^-$ contain the infinite abelian subalgebras generated by $\{q^n\}$ and $\{p^n\}$, respectively. Each of these subalgebras corresponds to an affine $U(1)$ Kac-Moody algebra, but they commute neither with each other nor with $W_\infty^0$

$$\{q^n, p^l\}_{MB}^{(-s)} = \sum_{j=1}^{(n,l)} \hat{q}^j b(j, n, l) \{(-s^+)^j - (s^-)^j\} q^{n-j} p^{l-j}$$

$$\{q^k, H^n\}_{MB}^{(-s)} = \sum_{j=1}^{(k,n)} \hat{q}^j b(j, k, n) \{(-s^+)^j - (s^-)^j\} q^{n+k-j} p^{n-j}$$

$$\{p^k, H^n\}_{MB}^{(-s)} = \sum_{j=1}^{(k,n)} \hat{q}^j b(j, k, n) \{(s^-)^j - (-s^+)^j\} q^{n-j} p^{n+k-j}.$$  

(74)

Apart from these abelian subalgebras, $W_\infty$ has two infinite, nonabelian subalgebras generated by $w_{n0} \equiv q^{n+1} p$ and $w_{n1} \equiv qp^{n+1}$ which obey
\[ \{ w_{n0}, w_{k0} \}_{MB}^{(-s)} = i\hbar (k - n) w_{n+k,0} \quad , \quad \{ w_{0n}, w_{0k} \}_{MB}^{(-s)} = i\hbar (n - k) w_{0,n+k} \] (75)

These are two noncommuting copies of the well-known centerless Virasoro algebra which is the underlying symmetry algebra of two dimensional conformal field theories (\cite{24}, and references therein). Here we note that the so-called negative modes corresponding to \( n \leq -2 \) are missing. In that case, remarkable enough, independence of the structure constants from \( s \) implies that they are the same for each basis when the algebras are expressed in terms of the ordered products and the Lie brackets, that is

\[ [\hat{w}_{n0}^{(s)}, \hat{w}_{k0}^{(s)}] = -i\hbar (k - n) \hat{w}_{n+k,0}^{(s)} \quad , \quad [\hat{w}_{0n}^{(s)}, \hat{w}_{0k}^{(s)}] = -i\hbar (n - k) \hat{w}_{0,n+k}^{(s)} \] (76)

where \( \hat{w}_{nm}^{(s)} \equiv \hat{q}_{n+1,m+1}^{(s)} \). Since \( [\hat{w}_{n0}^{(s)}, \hat{q}^k] = -i\hbar k \hat{q}^{n+k} \) and \( [\hat{w}_{0n}^{(s)}, \hat{p}^l] = i\hbar \hat{p}^{n+l} \) one can form two algebras by taking the semidirect product of the affine \( U(1) \) Kac-Moody algebras and appropriate Virasoro algebra.

All the abelian subalgebras of the \( W_{\infty} \) and the nonabelian ones having structure constants depending only on the first power of \( i\hbar \) are necessarily the subalgebras of the \( w_{\infty} \). This is a simple result of the fact that the structure constants of the \( W_{\infty} \) with respect to \( s-MB \) are power series in \( i\hbar \). If they vanish, so do the coefficients of all power of \( i\hbar \), and, in particular the coefficient of \( i\hbar \) which is the PB of the generators. But, the converse of the above statement is not true in general for it is an old known fact that the vanishing of the PB does not imply the vanishing of the MB. Thus, all the above mentioned subalgebras of the \( W_{\infty} \) are also the subalgebras of the \( w_{\infty} \) which remain unchanged under the deformation \( \{ , \}_{PB} \rightarrow \{ , \}_{MB}^{(-s)} \). As a final remark we note that, all the investigations given above are restricted to the positive powers of the generators. This is due to the fact that WWGM-association is valid for c-number functions accepting power series expansion. If we give up associating the c-number functions with Hilbert space operators, the structure constants of the full \( W_{\infty} \) for all values of \( n, m \in \mathbb{R} \) (more generally \( n, m \in \mathbb{C} \)) can be obtained via \( s-MB \) by simply changing the factorial in (63) by corresponding Gamma functions, and by removing the restrictions imposed on the summations.

**VII. CONCLUSION**

Any scheme of quantization is, in essence, an association between classical and quantum observables, hence, determination of their covariance properties is of the primary importance. By following the derivative based approach developed in \cite{5}, metaplectic covariance for all forms of the WWGM-quantization is established. It has been explicitly shown that while the underlying algebras are all isomorphic to the \( isp(2) \), the corresponding group structures are different for different quantization schemes. Beyond that WWGM-quantization has \( W_{\infty} \)-covariance which includes the metaplectic covariance as a small subset. (The Table I for \( n, m \leq 2 \) show the generators of the \( Isp(2) \) in the classical phase spaces and their central extensions in \( \mathcal{H} \)). Here we emphasize that like the metaplectic covariance this \( W_{\infty} \)-covariance is also a genuine property of the operator basis. We also show that the star product, the MB and the quantum deformed HVF in their most general forms naturally arise when basis operators are multiplied by the ordered products of the HW-algebra generators. These observations confirm and generalize the well-known fact that many essential properties of the
WWGM-quantization follows from the corresponding properties of the employed operator bases.

Besides its classical fields of application such as statistical mechanics, quantum optics, collision theory, classically chaotic nonlinear systems [4], and the theory of pseudodifferential operators [1], recently, the WWGM-quantization has been employed in the quantization of the Nambu mechanics [22], as well as in the construction of noncommutative geometry on a classical phase space [23]. On the other hand, \( W_\infty \) algebras are currently the subject of the active investigations in two dimensional gravity [1], conformal field theories [21,24], and, in connection with quantum Hall effect, in condensed matter physics (see [25], and references therein). Now, we see that, it also describes the complete covariance property the WWGM-quantization in its most general form. Thus, we expect that, the results of this study will shed more light on deeper understanding of these seemingly different active research fields and provide fruitful interactions among them.

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TABLE I. The generators of the quantum and classical $W_\infty$, corresponding to $n, m \leq 2$, for the $\hat{D}(s)$ and $\hat{\Delta}_{qp}(s)$ bases. The elements of the $isp(2)$–algebra and of its central extension discussed in sections III and IV are linear combinations of the generators given in the Table.

| $n,m$ | $\hat{t}_{nm}^{(0)}$ | $T_{nm}^{(0)}(s)$ | $\Gamma_{nm}^{(0)}(s)$ |
|-------|---------------------|-------------------|---------------------|
| 0,0   | $\hat{I}$           | 0                 | 0                   |
| 1,0   | $\hat{q}$           | $-\hbar \eta$    | $-i\hbar \partial_p$ |
| 0,1   | $\hat{p}$           | $\hbar \xi$      | $i\hbar \partial_q$ |
| 1,1   | $\frac{1}{2}(\hat{q}\hat{p} + \hat{p}\hat{q})$ | $-i\hbar(\xi \partial_\xi - \eta \partial_\eta)$ | $i\hbar(q \partial_q - p \partial_p)$ |
| 2,0   | $\hat{q}^2$         | $2i\hbar \eta \partial_\xi - s\hbar^2 \eta^2$ | $-2i\hbar q \partial_q + s\hbar^2 \partial_q^2$ |
| 0,2   | $\hat{p}^2$         | $-2i\hbar \xi \partial_\eta + s\hbar^2 \xi^2$ | $2i\hbar p \partial_q - s\hbar^2 \partial_q$ |