Generalized coordinate Bethe ansatz
for non diagonal boundaries

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Abstract

We compute the spectrum and the eigenstates of the open XXX model with non-diagonal (triangular) boundary matrices. Since the boundary matrices are not diagonal, the usual coordinate Bethe ansatz does not work anymore, and we use a generalization of it to solve the problem.
1 Introduction

The XXX model with periodic boundary conditions \[1\] is one of the most studied model in the realm of integrable systems. It was solved by Bethe \[2\], using what is now called the coordinate Bethe ansatz, and since then, many papers appeared on the subject. As for most of integrable models, the situation drastically changes when one considers the open case, i.e. when the boundary conditions are not periodic anymore. From the pioneer works of Cherednik \[3\] and Sklyanin \[4\], it is known that integrability of the model is preserved when the boundary conditions are coded by two independent matrices obeying the so-called reflection equation. Classification of such matrices amounts to classify integrable boundary conditions. However, although the models are known to be integrable, there is no general answer to get the Hamiltonian eigenstates for generic integrable boundaries (even when the periodic ones are known).

For the XXX model and its \(su(N)\) generalization, such classification of boundaries has been done in \[5–7\]. However, as mentioned, the models are still not solved for generic boundary matrices. The situation where the two matrices are diagonal is well-understood: the spectrum can be computed using analytical Bethe ansatz \[8–10\], the eigenfunctions can be obtained from coordinate \[11–13\] or algebraic \[4, 14–16\] Bethe ansatz, and correlation functions are well studied, see e.g. \[17–19\]. The case of simultaneously diagonalizable matrices is done in \[20\] (see also \[15\]).

The spectrum for some non diagonal cases, where the boundary parameters (entering the boundary matrices) obey relations, could be deduced from studies on XXZ models, as dealt in e.g. \[21, 22\]. Let us also mention \[23\] where numerical methods are used to get the spectrum and \[24\] where a deformation of Onsager algebra is studied to compute the spectrum for general boundary matrices. The first attempt to compute the spectrum and eigenfunctions for XXX model with boundary matrices that are not simultaneously diagonalizable can be found in \[25\]. There, the algebraic Bethe ansatz is used but one must restrict itself to one triangular boundary and one diagonal. Moreover, the \(su(2)\) invariance of the \(R\)-matrix is needed, so that the treatment is specific to the XXX model. The aim of this article is to tackle the same question but using a generalized coordinate Bethe ansatz (gCBA). Indeed, recently, in the case of XXZ model such problem has been successfully solved using such a method \[26, 27\]. It leads to a simpler presentation of the gCBA, that we present from the integrable models view point (while the presentation in \[26, 27\] was more ASEP like). We hope that this presentation permits to emphasize the novelties of the gCBA without the technical difficulties encountered in the ASEP model. The gCBA allows us to construct the eigenstates and compute the spectrum for one diagonal and one non-diagonal boundary matrix. The solution is equivalent to the \(q \to 1\) limit of the XXZ model \[21, 22, 27\] and the one found by algebraic Bethe ansatz \[25\]. The constraints on the boundary matrices for XXZ model found previously \[21, 22, 27\] being now replaced by the triangular form of the left boundary matrix and the diagonal form of the right one.

To be more specific, we will construct the eigenfunctions and compute the spectrum and

\[^1\text{Strictly speaking, references \[9, 14\] deal with the XXZ model. One has to perform a } q \to 1 \text{ limit to get the XXX model.}\]
Bethe equations for the following Hamiltonian:

\[ H = B^+_1 + H_{\text{bulk}} + B^-_L \]

where \( H_{\text{bulk}} = \sum_{\ell=1}^{L-1} h_{\ell,\ell+1} = \sum_{\ell=1}^{L-1} \left( P_{\ell,\ell+1} - I \right) \) \hspace{1cm} (1.1)

with the following boundary matrices

\[ B^+ = \begin{pmatrix} \alpha & \mu \\ 0 & \beta \end{pmatrix} \quad \text{and} \quad B^- = \begin{pmatrix} \gamma & 0 \\ 0 & \delta \end{pmatrix}. \] \hspace{1cm} (1.2)

Let us stress that, generically, both boundary matrices are not simultaneously diagonalizable (or even not diagonalizable at all), since they do not commute and that the total spin is not a good quantum number, since it does not commute with the Hamiltonian (because of the triangular boundary matrix). Of course, the method also applies when \( B^+ \leftrightarrow B^- \) or when one considers a lower triangular matrix instead of a upper triangular one. Moreover, one can also conjugate both \( B^+ \) and \( B^- \) by the same matrix: the spectrum is the same, and the eigenstates are constructed in an obvious way.

The plan of the paper is as follows. Notations are detailed in section 2. The construction of the eigenstates is done in section 3, and section 4 is devoted to the proof of the main property of the article. We conclude in section 5.

2 XXX model with boundaries

The model we consider has Hamiltonian (1.1). It acts on a spin chain with \( L \) sites, with a spin \( \frac{1}{2} \) (a \( \mathbb{C}^2 \) space) on each site of the chain. Hence, the Hilbert space is \( \mathcal{H} = (\mathbb{C}^2)^\otimes L \), and \( H \in \text{End}(\mathcal{H}) \).

We will use the auxiliary space notation, where indices indicate on which sites (of the chain) the operators act non trivially. For instance, for an operator acting on two sites, \( O \in \text{End}(\mathbb{C}^2 \otimes \mathbb{C}^2) \), \( O_{34} \) will denote the operator \( O \) acting on sites 3 and 4 of the chain, i.e.

\[ O_{34} = I \otimes I \otimes O \otimes (I)\otimes(L-4) \in \text{End}(\mathcal{H}), \] \hspace{1cm} (2.1)

where \( I \) denotes the \( 2 \times 2 \) identity matrix.

Hence, in (1.1), \( B^+_1 \) is the left boundary matrix \( B^+ \) acting on the first site 1:

\[ B^+_1 = B^+ \otimes \underbrace{I \otimes \ldots \otimes I}_{L-1}, \] \hspace{1cm} (2.2)

\( h_{\ell,\ell+1} \) are local Hamiltonians acting on sites \((\ell, \ell+1)\)

\[ h_{\ell,\ell+1} = \underbrace{I \otimes \ldots \otimes I \otimes h \otimes I \otimes \ldots \otimes I}_{\ell-1} \otimes \underbrace{I \otimes \ldots \otimes I \otimes \ldots \otimes I}_{L-\ell-1}, \] \hspace{1cm} (2.3)

and \( B^-_L \) is the right boundary matrix \( B^- \) acting on the last site \( L \).
$P_{\ell,\ell+1}$ is the permutation operator between site $\ell$ and site $\ell + 1$. As a matrix, the permutation $P$ takes the form

$$P = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}. \quad (2.4)$$

The Hamiltonian (1.1) describes the interaction of spins (up or down) among themselves, and with two boundaries describes by the matrices $B^\pm$. We are looking for its eigenstates and its spectrum.

Contrary to the periodic case, the open XXX Hamiltonian does not possess an $su(2)$ symmetry, so that the spin cannot be used as a quantum number. For non-diagonal boundary matrices, it does not commute with the $u(1)$ generator, so that the pseudo-excitation number is not a good quantum number either. We will come back on this point in the next section.

**Reference state**

The state

$$| \uparrow \ldots \uparrow \rangle \in (\mathbb{C}^2)^{\otimes L} \quad (2.5)$$

is an eigenstate:

$$H | \uparrow \ldots \uparrow \rangle = (\alpha + \gamma) | \uparrow \ldots \uparrow \rangle. \quad (2.6)$$

This state will be chosen as a reference state (the so-called pseudo-vacuum), and states with some spin down will be considered as pseudo-excitations above this reference state, see eq. (3.2). Let us stress that the reference state is \textit{not}, in general, the ground state, and that spins down are \textit{not} physical excitations. It is just a convenient way to parametrize all the states in $\mathcal{H}$.

On contrary, due to the left boundary, the state $| \downarrow \ldots \downarrow \rangle$ is \textit{not} an eigenstate.

**Hermiticity**

Note also that with this choice of boundaries, the Hamiltonian $H$ is not Hermitian anymore for generic values of the boundary parameters. However, we will see that the energy is the same as the one computed for diagonal boundary matrices, so that the energy is real when the parameters $\alpha$, $\beta$, $\gamma$ and $\delta$ are real.

Remark that $H$ can be pseudo-Hermitian:

$$H^\dagger = U H U^\dagger \quad \text{with} \quad U = \sigma^x \otimes \ldots \otimes \sigma^x \quad \text{when} \quad \alpha^* = \beta \quad ; \quad \delta^* = \gamma \quad ; \quad \mu^* = \mu, \quad (2.7)$$

where $*$ denotes complex conjugation. This case is still outside the range solved by usual coordinate Bethe ansatz (for $\mu \neq 0$).
**Integrability**

The model we consider is integrable. It can be built from the following transfer matrix

\[ t(u) = \text{Tr}_0 \left( K_0^+(u)T_{0,\langle 1,\ldots,L\rangle}(u)K_0^-(u)\hat{T}_{0,\langle 1,\ldots,L\rangle}(u) \right), \]

\[ T_{0,\langle 1,\ldots,L\rangle}(u) = R_{0L}(u)\ldots R_{01}(u) \quad \text{with} \quad R_{0\ell}(u) = P_{0\ell} + u, \]

\[ \hat{T}_{0,\langle 1,\ldots,L\rangle}(u) = R_{01}(u)\ldots R_{0L}(u), \]

\[ K^-(u) = I + u\tilde{B}^- \quad \text{with} \quad \tilde{B}^- = \begin{pmatrix} \gamma - \delta & 0 \\ 0 & \delta - \gamma \end{pmatrix}, \]

\[ K^+(u) = I + u\tilde{B}^+ \quad \text{with} \quad \tilde{B}^+ = \begin{pmatrix} \alpha - \beta & \mu \\ 0 & \beta - \alpha \end{pmatrix}, \]

where the \( K^-(u) \) matrix (resp. \( K^+(u) \) one) obey the reflection equation (resp. dual reflection one). Standard calculations \[4\] show that one recovers the Hamiltonian

\[ \left. \frac{1}{2} \frac{dt(u)}{du} \right|_{u=0} = H + \left( L - 1 - \frac{\alpha + \beta + \gamma + \delta}{2} \right) I. \]

3 **Generalized Coordinate Bethe Ansatz**

The fact that the left boundary \( B^+ \) is not diagonal anymore implies that the usual coordinate Bethe ansatz fails. Indeed, since \( B^+ \) is triangular, it can flip a spin down to spin up, which is interpreted as the annihilation of a pseudo-excitation (or equivalently as its transmission outside the system). Hence, one cannot consider an eigenfunction with a given (fixed) number of pseudo-excitations. This is also another way to see that the spin is not a ‘good’ quantum number, since the Hamiltonian changes it. However, since there is only annihilation, one can consider eigenfunctions having a fixed maximum number of pseudo-excitations. It leads us to the following ansatz:

\[ \Phi_n = \sum_{m=0}^{n} \sum_{x_{m+1}<\cdots<x_n} \sum_{g \in G_m} A_g^{(n,m)} e^{k_g^{(m)} x^{(m)}} |x_{m+1},\ldots,x_n\rangle, \]

where \( G_m \) is a full set of representatives of the coset \( BC_n/BC_m \) (\( G_0 = BC_n \), by convention) and \( BC_m \) is the \( B_m \) Weyl group, generated by transpositions \( \sigma_j, j = 1,\ldots,m-1 \) that exchange \( k_j \) and \( k_{j+1} \), and the reflection \( R_1 \) exchanging \( k_1 \) and \(-k_1 \). The vectors \( |x_{m+1},\ldots,x_n\rangle \) are given by

\[ |x_{m+1},\ldots,x_n\rangle = |\uparrow \cdots \uparrow \downarrow \cdots \uparrow \downarrow \cdots \uparrow \downarrow \cdots \uparrow \rangle \in (\mathbb{C}^2)^{\otimes L} \]

and we introduce the notation \( k_g^{(m)} \) for the following truncated vector

\[ k_g^{(m)} = (k_{g(m+1)},\ldots,k_{g(n)}). \]
For this definition to be consistent, the coefficients $A_{g}^{(n,m)}$ do not have to depend on the choice of the representative i.e.

$$A_{gh}^{(n,m)} = A_{g}^{(n,m)} \text{ for any } h \in BC_m.$$  \hfill (3.4)

The coefficients $A_{g}^{(n,m)}$ are complex numbers to be determined such that $\Phi_n$ is an eigenfunction of $H$ i.e. such that the following equation holds

$$H\Phi_n = E\Phi_n.$$ \hfill (3.5)

We project equation (3.5) on the different independent vectors to get constraints on the coefficients $A_{g}^{(n,m)}$.

Since $H$ is a sum of operators acting on (at most) two neighboring sites only, one has to single out the cases where the $x$’s obey the following constraints:

- all the $x_j$’s are far away one from each other ($1 + x_j < x_{j+1}$, $\forall j$) and are not on the boundary sites 1 and $L$. This case will be called generic\(^2\).
- $x_j + 1 = x_{j+1}$ for some $j$,
- $x_1 = 1$, or $x_m = L$.

As the eigenvalue problem is a linear problem, it is enough to treat the cases where at most one of the particular cases appears: more complicated cases just appear as superposition of ‘simple’ ones.

**Calculation of the energy: projection on $|x_1, \ldots, x_n\rangle$ for $(x_1, \ldots, x_n)$ generic**

As in the usual coordinate Bethe ansatz \[^2\], this projection provides the energy:

$$E = \alpha + \gamma + \sum_{j=1}^{n} \lambda(e^{ik_j}) \quad \text{where} \quad \lambda(u) = u + \frac{1}{u} - 2 = \frac{(u - 1)^2}{u}. \hfill (3.6)$$

Let us remark that, up to the boundary terms $\alpha$ and $\gamma$, the energy takes the same form as in the periodic case.

We want also to stress that the coefficient $\mu$ does not enter the energy, hence the spectrum for the model is the same as the spectrum of the model based on diagonal matrices\[^3\].

\[^2\]Here and below, unless explicitly specified, all the (sub)sets of $x_i$’s will be considered as generic.

\[^3\]To be precise, one should also check that the Bethe equations do not depend on $\mu$ either: one will see below that it is indeed the case.
Scattering matrix: projection on $|x_1, \ldots, x_j, x_{j+1} = 1 + x_j, x_{j+2}, \ldots, x_n\rangle$

As usual, this projection provides the scattering matrix between pseudo-excitations. It is given by a relation between $A^{(n,0)}_g$ and $A^{(n,0)}_{g\sigma_j}$ where $\sigma_j$ is the permutation of $j$ and $j+1$. Namely, we get

$$A^{(n,0)}_{g\sigma_j} = S(e^{ik_{gj}}, e^{ik_{g(j+1)}}) A^{(n,0)}_g,$$

with $S(u, v) = -\frac{a(u, v)}{a(v, u)}$ and $a(u, v) = i \frac{2v - uv - 1}{uv - 1}$.

The normalization chosen for the function $a(u, v)$ is for further simplifications. As expected, this relation is similar to the periodic case since the boundaries are not involved in this process.

First relation: projection on $|1, x_{m+1} \ldots, x_n\rangle$, $m \geq 1$

This relation is a new one. We get, for any $g \in G_m$,

$$\sum_{h \in H_m} A^{(n,m-1)}_{gh} e^{ik_{ghm}} \left( \beta - \alpha - 1 + e^{ik_{ghm}} - \sum_{j=1}^{m} \lambda(e^{ik_{gj}}) \right) = 0,$$

where $H_m = BC_m/BC_{m-1}$. To obtain this relation, we have used the following property (valid for any function $f$):

$$\sum_{g \in G_{m-1}} f(g) e^{ik_{g(m)}} e^{ik_{g(m)} x^{(m)}} = \sum_{g \in G_m} e^{ik_{g(m)}} x^{(m)} \sum_{h \in H_m} f(gh) e^{ik_{gh(m)}}.$$

Let us stress that equation (3.9) does not depend on the choice of representative of $G_m$.

Reflection coefficient for the left boundary

The reflection coefficient is computed using relation (3.9) at $m = 1$. In that case, $H_1 = BC_1$ is constituted of 2 elements only: the identity $id$ and the reflection $R_1$ that changes $k_1$ into $-k_1$. One gets

$$A^{(n,0)}_{gR_1} = R(e^{ik_{g1}}) A^{(n,0)}_g \quad \text{with} \quad R(u) = -u^2 \frac{1 - \frac{1}{u} + \beta - \alpha}{1 - u + \beta - \alpha} = \frac{r_+(1/u)}{r_+(u)},$$

where $r_+(u)$ is given in (3.15).

Remark that $R(u) R(\frac{1}{u}) = 1$: for the physical excitations, the left boundary is purely reflective (no loss of excitations).
Second relation: projection on $|x_{m+1} \ldots , x_n\rangle$, $m \geq 1$

This projection provides a relation between the coefficients from the level $m-1$ and $m$ since we must take into account that the left boundary can destroy one pseudo-excitation present on the site 1. We obtain the following constraint, for any $g \in G_m$,

$$\mu \sum_{h \in H_m} A^{(n,m-1)}_{gh} e^{ik_{gh(m)}} - \sum_{j=1}^{m} A^{(n,m)}_{g} \lambda(e^{ik_{gj}}) = 0 \quad (3.12)$$

Transmission coefficient for the left boundary

From (3.9) and (3.12), we may express all the $A^{(n,m)}_{g}$ ($m \geq 1$) in terms of $A^{(n,0)}_{g}$ thanks to the recursive relation

$$A^{(n,m)}_{g} = T^{(m)}(e^{ik_{g1}}, \ldots , e^{ik_{gm}}) A^{(n,m-1)}_{g} \quad \text{for} \quad 1 \leq m \leq n \quad (3.13)$$

with the following definitions:

$$T^{(m)}(u_1, \ldots , u_m) = \frac{\mu}{r_+(u_m)} \left( \prod_{j=1}^{m-1} a(u_m, u_j) a(u_j, 1/u_m) \right)^{-1} \quad (3.14)$$

$$r_+(u) = \lambda(u) \frac{1 - u + \beta - \alpha}{1 - u^2} = - \frac{(u - 1)(1 - u + \beta - \alpha)}{u(1 + u)} \quad (3.15)$$

Relation (3.13) can be interpreted as the transmission of one (among $m$) pseudo-excitation(s) through the left boundary: this pseudo-excitation has been destroyed (from the spin chain viewpoint).

Remark that $T^{(m)}$ is proportional to $\mu$, in accordance with the picture described at the beginning of the section: when $\mu = 0$, the boundary becomes diagonal, the usual coordinate Bethe ansatz works, and there is no relation between $A^{(n,m)}_{g}$ and $A^{(n,m-1)}_{g}$, each level $m$ providing an independent eigenfunction (with a fixed number of pseudo-excitations); when $\mu \neq 0$ all the levels $m$ are related, but we get an independent eigenfunction for each upper number $n$ of pseudo-excitations.

Iterative use of (3.13) leads to

$$A^{(n,m)}_{g} = T^{(m)}(e^{ik_{g1}}, \ldots , e^{ik_{gm}}) A^{(n,0)}_{g} \quad (3.16)$$

$$T^{(m)}(u_1, \ldots , u_m) = \frac{\mu^m}{r_+(u_m) r_+(u_{m-1}) \ldots r_+(u_1)} \left( \prod_{1 \leq j < \ell \leq m} a(u_\ell, u_j) a(u_j, 1/u_\ell) \right)^{-1} \quad (3.17)$$

The proof that (3.13-3.14) is a solution of both equations (3.9) and (3.12) is postponed to section 4 and relies on a residue computation. The integrability of the model plays a role at this place, since there are a priori too many constraints but not all of them are independent.
Note that relations (3.7), (3.11) and (3.17) show that all the coefficients $A_{g}^{(n,m)}$ can be expressed in term of $A_{id}^{(n,0)}$. It is important to notice that this computation is consistent since the obtained expression does not depend on the way we write $g$ in terms of the generators $\sigma_j$ and $R_1$.

Bethe equations: projection on $|x_1 \ldots, x_{n-1}, L\rangle$

This last constraint consists in the quantization of the pseudo-excitations moments since the system is in a finite volume. In the context of the coordinate Bethe ansatz, this quantization leads to the so-called Bethe equations, explicitly given by

$$\prod_{\ell=1}^{n} S(e^{ik_{\ell}}, e^{-ik_{j}}) S(e^{-ik_{j}}, e^{ik_{\ell}}) = e^{2iLk_{j}} \frac{r_{+}(e^{ik_{j}}) r_{-}(e^{-ik_{j}})}{r_{+}(e^{-ik_{j}}) r_{-}(e^{-ik_{j}})} \quad \text{for} \quad 1 \leq j \leq n \quad (3.18)$$

$$r_{-}(u) = \frac{u - 1}{u + 1} (1 - u + \delta - \gamma). \quad (3.19)$$

We remind that the scattering matrix $S(u, v)$ is given in (3.8), while the $r_{+}(u)$ function is defined in (3.15).

Let us remark that Bethe equations (3.18) do not depend on the parameter $\mu$. Therefore, as previously mentioned, the spectrum (i.e. the energy (3.6)) is also independent of this parameter. Therefore, the spectrum is similar to the one with diagonal boundaries ($\mu = 0$). In [25], similar results on the spectrum have been obtained via algebraic Bethe ansatz. In contrary, the eigenvectors depends on the parameter $\mu$ via relation (3.17) determining the coefficients entering in our ansatz. The eigenvectors computed thanks to the algebraic Bethe ansatz in [25] also depends on $\mu$ since the creation operators $\tilde{\mathcal{B}}(\lambda)$, used to construct the ansatz, depends on $\mu$. Unfortunately, a direct proof that the eigenvectors constructed via both methods are identical is a difficult task and beyond the scope of this paper.

Completeness of the ansatz

It is clear that two states $\Phi_{n_1}$ and $\Phi_{n_2}$ are independent when $n_1 \neq n_2$. Thus it remains to prove that the Bethe equations provide the right number of solutions, and that, at given $n$, these solutions are independent.

Moreover, it is believed that the coordinate Bethe ansatz for open XXX chain\footnote{To our knowledge it is only proven for closed spin chains, see e.g. [28][29].} is complete when $\mu = 0$. Since the Bethe equations do not depend on $\mu$, they provide the same number of solutions. Hence, from the conjecture at $\mu = 0$, we deduce that when $\mu \neq 0$, the set of solutions has the right dimension to get the complete set of eigenstates.
4 Proof of the transmission relation (3.13)

In this section, we prove that (3.13) implies (3.9) and (3.12).

We start with (3.12). First, we remark that a consequence of (3.13) is

\[ A^{(n,m)}_g = A^{(n,m)}_g \times \begin{cases} \frac{T^{(m)}(e^{ik_{g1}},...,e^{ik_{g(m-1)},e^{ik_{g(m+1)}}})}{T^{(m)}(e^{ik_{g1}},...,e^{ik_{g(m+1)}})} S(e^{ik_{gm}},e^{ik_{g(m+1)}}) & j = m, \\ S(e^{ik_{gj}},e^{ik_{g(j+1)}}) & j \geq m+1. \end{cases} \tag{4.1} \]

Then, using again (3.13) to express now \( A^{(n,m+1)}_g \) in terms of \( A^{(n,m)}_g \) and using (4.1) to express \( A^{(n,m)}_h \) in terms of \( A^{(n,m)}_g \), relation (3.12) becomes the functional relation

\[ \sum_{j=1}^{m+1} \left[ u_j r_+(u_j) \prod_{\ell=1}^{m+1} a(u_j, u_{\ell}) a(u_{\ell}, \frac{1}{u_j}) + \frac{1}{u_j} r_+ \left( \frac{1}{u_j} \right) \prod_{\ell=1}^{m+1} a\left( \frac{1}{u_j}, u_{\ell} \right) a(u_{\ell}, u_j) - \lambda(u_j) \right] = 0, \tag{4.2} \]

where \( u_j \) stands for \( \exp(ik_{gj}) \) and the functions are defined in (3.8) and (3.15).

To prove this last relation (4.2), let us introduce the following function

\[ F^{(m)}(u) = \frac{1-u+\beta-\alpha}{2(1-u)} \prod_{\ell=1}^{m} a(u, u_{\ell}) a(u_{\ell}, \frac{1}{u}). \tag{4.3} \]

Looking at the poles of \( F^{(m)}(u) \), one can compute the residues of this function:

\[ \text{Res}(F^{(m)}(u)) \bigg|_{u=u_j} = u_j r_+(u_j) \prod_{\ell=1, \ell \neq j}^{m} a(u_j, u_{\ell}) a(u_{\ell}, \frac{1}{u_j}), \tag{4.4} \]

\[ \text{Res}(F^{(m)}(u)) \bigg|_{u=1/u_j} = \frac{1}{u_j} r_+ \left( \frac{1}{u_j} \right) \prod_{\ell=1, \ell \neq j}^{m} a\left( \frac{1}{u_j}, u_{\ell} \right) a(u_{\ell}, \frac{1}{u_j}), \tag{4.5} \]

\[ \text{Res}(F^{(m)}(u)) \bigg|_{u=1} = - \frac{1}{2}(\beta-\alpha), \tag{4.6} \]

\[ \text{Res}(F^{(m)}(u)) \bigg|_{u=\infty} = \frac{1}{2}(\beta-\alpha) - \sum_{\ell=1}^{m} \lambda(u_{\ell}). \tag{4.7} \]

Then, (4.2) is equivalent to \( \sum_{\text{residue}} F^{(m)}(u) = 0 \) which proves (3.12).

From (3.9), we use the same procedure to obtain a new functional relation. After use of (4.2), this relation simplifies to

\[ \sum_{j=1}^{m+1} \left[ u_j^2 r_+(u_j) \prod_{\ell=1, \ell \neq j}^{m+1} a(u_j, u_{\ell}) a(u_{\ell}, \frac{1}{u_j}) + \frac{1}{u_j^2} r_+ \left( \frac{1}{u_j} \right) \prod_{\ell=1, \ell \neq j}^{m+1} a\left( \frac{1}{u_j}, u_{\ell} \right) a(u_{\ell}, u_j) \right] 
    + (\beta-\alpha-1 - \sum_{j=1}^{m} \lambda(u_j)) \left( \sum_{k=1}^{m} \lambda(u_k) \right) = 0. \tag{4.8} \]
Again, this relation is equivalent to a residue calculation, the function to consider being now 
\[ G^{(m)}(x) = x F^{(m)}(x) \].

5 Conclusion

We have computed the spectrum and the eigenfunctions of the open XXX model with one triangular boundary matrix using a generalized coordinate Bethe ansatz. The next step would be to compute scalar products of the eigenstates to get informations on the correlation functions of this model. As far as scalar products are concerned, we think that an approach à la Gaudin [11] should work. A determinant formula, similar to the Slavnov determinant [30] will be needed to compute the correlation functions in a simple way.

It remains to treat the case where the two boundary matrices are both triangular, and also the cases where one or two of the boundaries are general \(2 \times 2\) matrices. Work is in progress on these two cases. We expect to find constrains between the boundary parameters in the first case. We remind that the method has been applied successfully in the case of open XXZ model [27]. The second case needs a further generalization of the coordinate Bethe ansatz.

Finally, we believe also that the method presented here can be useful to find the spectrum for other integrable models with boundaries.

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