Abstract

Criteria for the existence of $T$-periodic solutions of nonautonomous parabolic equation $u_t = \Delta u + f(t, x, u), \quad x \in \mathbb{R}^N, \quad t > 0$ with asymptotically linear $f$ will be provided. It is expressed in terms of time average function $\hat{f}$ of the nonlinear term $f$ and the spectrum of the Laplace operator $\Delta$ on $\mathbb{R}^N$. One of them says that if the derivative $\hat{f}_\infty$ of $\hat{f}$ at infinity does not interact with the spectrum of $\Delta$, i.e. $\text{Ker}(\Delta + \hat{f}_\infty) = \{0\}$, then the parabolic equation admits a $T$-periodic solution. Another theorem is derived in the situation, where the linearization at 0 and infinity differ topologically, i.e. the total multiplicities of negative eigenvalues of the averaged linearizations at 0 and $\infty$ are different mod 2.

1 Introduction

We shall be concerned with time $T$-periodic solutions of the following parabolic problem
\begin{align*}
\begin{cases}
\frac{\partial u}{\partial t}(x, t) = \Delta u(x, t) + f(t, x, u(x, t)), \quad x \in \mathbb{R}^N, \quad t > 0, \\
u(\cdot, t) \in H^1(\mathbb{R}^N), \quad t \geq 0,
\end{cases}
\end{align*}
(1)

where $\Delta$ is the Laplace operator (with respect to $x$), $N \geq 3$ and a continuous function $f : [0, +\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is $T$-periodic in time:
\begin{equation}
f(t, x, u) = f(t + T, x, u) \quad \text{for all} \quad t \geq 0, \quad x \in \mathbb{R}^N, \quad u \in \mathbb{R}.
\end{equation}
(2)

The assumption that $N \geq 3$ is not essential, we make it only for the convenience and our results hold for $N = 1, 2$ after a proper restatement. Periodic problems for parabolic equations were widely studied by many authors by use of various methods. Some early results are due to Brezis and Nirenberg [5], Amman and Zehnder [2], Nkashama and Willem [17], Hirano [14, 15], Prüss [20], Hess [13], Shioji [23] and many others; see also [26] and the references therein. Most of these results treat the
where $\Omega$ is bounded and are based either on topological degree and coincidence index techniques in the spaces of functions depending both on $x$ and time $t$ or on the translation along trajectories operator to which fixed point theory is applied. In this paper we shall study the case $\Omega = \mathbb{R}^N$ by applying translation along trajectories approach together with fixed point index and Henry’s averaging (see [12]) as in [17] (for a general reference see also [3]). In this case the semigroup compactness arguments are no longer valid (since the Rellich-Kondrachov theorem on $\mathbb{R}^N$ does not hold and the semigroup of bounded linear operators generated by the linear heat equation $u_t = \Delta u$ on $\mathbb{R}^N$ is not compact). Therefore adequate topological fixed point theory for noncompact maps and the adaptation of proper averaging techniques is required.

We shall assume that $f \in C([0, +\infty) \times \mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ is such that, for all $t, s \in [0, +\infty), x \in \mathbb{R}^N, u, v \in \mathbb{R}$, one has

$$f(t, \cdot, 0) \in L^2(\mathbb{R}^N), \quad \|f(t, \cdot, 0)\|_{L^2} < M_0;$$

$$|f(t, x, u) - f(s, x, v)| \leq \left(\tilde{K}(x) + K(x)\|u\|\right)|t - s|^\theta + L(x, s)|u - v|;$$

$$(f(t, x, u) - f(t, x, v))(u - v) \leq -a|u - v|^2 + b(x)|u - v|^2$$

where $M_0 > 0$, $\theta \in (0, 1)$, $\tilde{K} \in L^2(\mathbb{R}^N)$, $K = K_0 + K_\infty$, $L = L_0 + L_\infty$ with $K_0, L_0(t, \cdot) \in L^p(\mathbb{R}^N), K_\infty, L_\infty \in L^\infty(\mathbb{R}^N)$, $\sup_{t \geq 0}(\|L_0(t, \cdot)\|_{L^p} + \|L_\infty(t, \cdot)\|_{L^\infty}) < +\infty, a > 0$ and $b \in L^p(\mathbb{R}^N)$ (see Remark 1.3 for examples of functions satisfying these assumptions).

We shall consider the averaged equation

$$\begin{cases}
\frac{\partial u}{\partial t}(x, t) = \Delta u(x, t) + \hat{f}(x, u(x, t)), & x \in \mathbb{R}^N, t > 0, \\
u(\cdot, t) \in H^1(\mathbb{R}^N), & t > 0,
\end{cases}$$

where the time average function $\hat{f} : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ of $f$ is given by

$$\hat{f}(x, u) := \frac{1}{T} \int_0^T f(t, x, u) \, dt.$$

Our main results are the following criteria for the existence of $T$-periodic solutions.

**Theorem 1.1.** Suppose that $f$ satisfies conditions (2), (3), (4), (5) and, for all $x \in \mathbb{R}^N$ and $t \geq 0$,

$$\lim_{|u| \to \infty} \frac{f(t, x, u)}{u} = \omega(t, x) := \omega_0(t, x) - \omega_\infty(t, x)$$

where $\omega_0(t, \cdot) \in L^p(\mathbb{R}^N)$, $N \leq p < +\infty, \omega_\infty(t, \cdot) \in L^\infty(\mathbb{R}^N)$ for all $t > 0$, and $\omega_\infty \geq \omega_\infty \geq 0$ for some real number $\tilde{\omega}_\infty$ and

$$\sup_{t \geq 0}(\|\omega_0(t, \cdot)\|_{L^p} + \|\omega_\infty(t, \cdot)\|_{L^\infty}) < +\infty. \quad \text{If}

\begin{cases}
\frac{\partial u}{\partial t}(x, t) = \lambda \Delta u(x, t) + \lambda \omega(t, x)u(x, t), & x \in \mathbb{R}^N, t > 0, \\
u(\cdot, t) \in H^1(\mathbb{R}^N), & t \geq 0,
\end{cases}$$

$$\lim_{|u| \to \infty} \frac{f(t, x, u)}{u} = \omega(t, x) := \omega_0(t, x) - \omega_\infty(t, x)$$

where $\omega_0(t, \cdot) \in L^p(\mathbb{R}^N)$, $N \leq p < +\infty, \omega_\infty(t, \cdot) \in L^\infty(\mathbb{R}^N)$ for all $t > 0$, and $\omega_\infty \geq \omega_\infty \geq 0$ for some real number $\tilde{\omega}_\infty$ and

$$\sup_{t \geq 0}(\|\omega_0(t, \cdot)\|_{L^p} + \|\omega_\infty(t, \cdot)\|_{L^\infty}) < +\infty. \quad \text{If}

\begin{cases}
\frac{\partial u}{\partial t}(x, t) = \lambda \Delta u(x, t) + \lambda \omega(t, x)u(x, t), & x \in \mathbb{R}^N, t > 0, \\
u(\cdot, t) \in H^1(\mathbb{R}^N), & t \geq 0,
\end{cases}$$
has no nonzero $T$-periodic solutions, for $\lambda \in (0, 1]$ and $\text{Ker}(\Delta + \hat{\omega}) = \{0\}$, where $\hat{\omega} : \mathbb{R}^N \to \mathbb{R}$ is the time average function of $\omega$, given by $\hat{\omega}(x) := \frac{1}{T} \int_0^T \omega(t,x) \, dt$, then the equation \((1)\) admits a $T$-periodic solution $u \in C([0, +\infty), H^1(\mathbb{R}^N)) \cap C([0, +\infty), H^2(\mathbb{R}^N)) \cap C^1([0, +\infty), L^2(\mathbb{R}^N))$.

Our second result applies in the case where there exists a trivial periodic solution $u \equiv 0$ and the previous theorem does not imply the existence of a nontrivial periodic solution.

**Theorem 1.2.** Suppose that all the assumptions of Theorem 1.1 are satisfied and additionally that, for all $x \in \mathbb{R}^N$ and $t \geq 0$, $f(t, x, 0) = 0$ and

$$\lim_{u \to 0} \frac{f(t, x, u)}{u} = \alpha(t, x) := \alpha_0(t, x) - \alpha_\infty(t, x)$$

(9)

where $\alpha_0(t, \cdot) \in L^p(\mathbb{R}^N)$, $N \leq p < +\infty$, $\alpha_\infty(t, \cdot) \in L^\infty(\mathbb{R}^N)$ for all $t > 0$, and $\alpha_\infty \geq \hat{\alpha}_\infty > 0$ for some real number $\hat{\alpha}_\infty > 0$ and $\sup_{t \geq 0} (||\alpha_0(t, \cdot)||_{L^p} + ||\alpha_\infty(t, \cdot)||_{L^\infty}) < +\infty$. If the equation

$$\begin{cases}
\frac{\partial u}{\partial t}(x, t) = \lambda \Delta u(x, t) + \lambda \alpha(t, x) u(x, t), 
\quad x \in \mathbb{R}^N, 
\quad t > 0,
\end{cases}$$

(10)

has no nonzero $T$-periodic solutions for $\lambda \in (0, 1]$ and $\text{Ker}(\Delta + \hat{\omega}) = \{0\}$ and $m_{-}(\infty) \not\equiv m_{-}(0) \mod 2$, where $m_{-}(0)$ and $m_{-}(\infty)$ are the total multiplicities of the negative eigenvalues of $-\Delta - \hat{\omega}$ and $-\Delta - \hat{\omega}$, respectively, then the equation \((1)\) admits a nontrivial $T$-periodic solution $u \in C([0, +\infty), H^1(\mathbb{R}^N)) \cap C([0, +\infty), H^2(\mathbb{R}^N)) \cap C^1([0, +\infty), L^2(\mathbb{R}^N))$.

**Remark 1.3.**

(a) Let us give an example of a class of functions satisfying \((2), \ (3), \ (4)\) and \((5)\). Consider $f : [0, +\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ given by

$$f(t, x, u) := U(t, x) + V(t, x) u + g(W(t, x) u)$$

where with $U, V, W \in C(\mathbb{R}^N \times [0, +\infty))$ such that $U(\cdot, t) \in L^2(\mathbb{R}^N)$ for all $t > 0$, $\sup_{t > 0} ||U(\cdot, t)||_{L^2} < +\infty$, $V = V_0 + V_\infty$, $W = W_0 + W_\infty$, $V_0, W_0 \in L^p(\mathbb{R}^N)$, $V_\infty, W_\infty \in L^\infty(\mathbb{R}^N)$ and, for all $t, s \geq 0$ and $x \in \mathbb{R}^N$, $|U(t, x) - U(s, x)| \leq L_U(x)|t - s|$, $|V(t, x) - V(s, x)| \leq L_V(x)|t - s|$ and $|W(t, x) - W(s, x)| \leq L_W(x)|t - s|$ with $L_U \in L^2(\mathbb{R}^N), L_V, L_W \in L^p(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$. Assume that $g : \mathbb{R} \to \mathbb{R}$ is a bounded Lipschitz function with a constant $L > 0$ such that $g(0) = 0$ and $g'(0)$ exists. Then the assumptions \((3)\) and \((4)\) are satisfied. If additionally there is $a > 0$ such that we have $V_\infty(x, x) + L|W_\infty(x, x)| \leq -a$ for all $t \geq 0$ and $x \in \mathbb{R}^N$, then \((3)\) holds. As a concrete example one may give $f(t, x, u) := -2au + \sin(au + bu(1 + |x|^{-s}) \cos t)$ where $a, b > 0$ and $s > N/p$. Moreover, in this particular case $\omega_\infty \equiv 2a$, $\omega_0 \equiv 0$, $\alpha_\infty \equiv a$, $\alpha_0(t, x) = b(1 + |x|^{-s}) \cos t$.

(b) The appearance of terms from the space $L^p(\mathbb{R}^N)$ with $N \leq p < +\infty$, is essential for our considerations. The function $\alpha_0$ (or $\omega_0$) assures that the negative part of the spectrum $\sigma(-\Delta - \hat{\omega})$ (or $\sigma(-\Delta - \hat{\omega})$) consists of a finite number of eigenvalues with finite dimensional eigenspaces (see Remark 6.2 for a more detailed
discussion). That makes the numbers \(m_-(0)\) and \(m_-(\infty)\) well-defined, i.e. the formulation of the above result is correct. Note that in Theorem 1.2 the appearance of the nontrivial term either \(\alpha_0\) or \(\omega_0\) belonging to \(L^p(\mathbb{R}^N)\) is necessary to satisfy the desired condition \(m_-(0) \neq m_-(\infty) \mod 2\). In consequence, the appearance of such a term determines the phase space for the equation, since we want the term \(\alpha_0(x,t)\) or \(f(t, \cdot, u(\cdot))\) to belong to \(L^2(\mathbb{R}^N)\). Therefore we need to consider solutions starting in \(H^1(\mathbb{R}^N)\) (instead of \(L^2(\mathbb{R}^N)\)).

(c) In this paper we focus on the case when there is no resonance both at 0 and at \(\infty\), i.e. when the nonexistence assumptions for (S) and (10) hold, respectively. From the technical point of view they enable us to use continuation along the parameter \(\lambda\) up to \(\lambda = 1\). For small parameter \(\lambda > 0\) the lack of \(T\)-periodic assumptions of (10) and (S) is implied by the conditions \(\ker(\Delta + \omega) = \{0\}\) and \(\ker(\Delta + \delta) = \{0\}\), respectively. The lack of nontrivial \(T\)-periodic solutions for the problem \(\frac{\partial u}{\partial t} = \omega_0(t, x) u + \alpha_0(t, x) u\), \(u(\cdot, t) \in H^1(\mathbb{R}^N)\), \(x \in \mathbb{R}^N\), \(t > 0\), is rather obvious if \(\alpha\) is independent of time (which is possible also when \(f\) depends on time). Moreover the nonexistence condition also holds in the general case if, for instance,

\[
\sup_{t \in [0, T]} \|\alpha_0(\cdot, t)\| < \frac{C(N)}{(N/2p)^N/2p \cdot C(N)^{N/2p^2}}
\]

where \(C(N) > 0\) is the constant in the Sobolev inequality \(\|u\|_{L^{2N/2}} \leq C(N) \|\nabla u\|_{L^2}\), \(u \in H^1(\mathbb{R}^N)\) (for details see Remark 5.6).

Following the tail estimates techniques of Wang [27], who studied attractors, and Prizzi [19], who studied stationary states and connecting orbits by use of Conley index, we develop a fixed point index setting applicable to parabolic equations on \(\mathbb{R}^N\). We shall show that the translation along trajectories operator \(\Phi_T : H^1(\mathbb{R}^N) \to H^1(\mathbb{R}^N)\) for (11) is ultimately compact, i.e. belongs to the class of maps for which the fixed point index \(\text{Ind}(\Phi_T, U)\), with respect to open subsets of \(H^1(\mathbb{R}^N)\), can be considered (see e.g. [1]). Clearly the nontriviality of that index will imply the existence of the fixed point of \(\Phi_T\) in \(U\), which is the starting point of the corresponding periodic solution. In order to determine the index \(\text{Ind}(\Phi_T, U)\), we use an averaging method, i.e. we embed the equation (11) into the family of problems

\[
\begin{align*}
\frac{\partial u}{\partial t}(x, t) &= \Delta u(x, t) + f \left( \frac{t}{\lambda}, x, u(x, t) \right), \quad x \in \mathbb{R}^N, \ t > 0, \ \lambda > 0, \\
u(t, t) &\in H^1(\mathbb{R}^N), \ t \geq 0.
\end{align*}
\]

(11)

According to Henry’s averaging principle the solutions of (11) converge to solution of (11) as \(\lambda \to 0^+\). Exploiting the tail estimate technique of Wang and Prizzi together with an extension of Henry’s averaging principle we prove that asymptotic assumptions on \(f\) imply a sort of a priori bounds conditions, i.e. that there are no \(\lambda T\)-periodic solution of (11), for \(\lambda \in (0, 1]\), with initial states of large \(H^1\) norm (in case of Theorem 1.1) and also of small \(H^1\) norm (in case of Theorem 1.2), i.e. initial states of \(\lambda T\)-periodic solutions are located outside some open bounded set \(U \subset H^1(\mathbb{R}^N)\). This enables us to use a sort of the averaging index formula stating that

\[
\text{Ind}(\Phi_T, U) = \lim_{\lambda \to 0^+} \text{Ind}(\tilde{\Phi}_T, U)
\]

(12)
where $\Phi_\ast$ is the translation along trajectories operator for (6). In computation of $\text{Ind}(\Phi_\ast, U)$, for small $t > 0$, the spectral properties of the operators $-\Delta + \hat{\alpha}$ and $-\Delta + \hat{\omega}$ are crucial. We strongly use the fact that their essential spectrum is contained in $(0, +\infty)$ and the rest consists of negative eigenvalues with finite dimensional eigenspaces and that numbers $m_- (\infty)$ and $m_- (0)$ are well-defined (i.e. finite).

The paper is organized as follows. In Section 2 we recall the concept of ultimately compact maps and fixed point index theory. In Section 3 we strengthen in a general setting of sectorial operators the initial condition continuity property and Henry’s averaging principle. Section 4 is devoted to the ultimate compactness property of the translation operator. In Section 5 we adapt the ideas of [7] to the case $\Omega = \mathbb{R}^N$, proving the averaging index formula (12) as well as verify a priori bounds conditions for $\lambda T$-periodic solutions of (11) with $\lambda \in (0, 1]$. Finally, in Section 6 the main results are proved.

2 Preliminaries

Notation. If $X$ is a normed space with the norm $\| \cdot \|$, then, for $x_0 \in X$ and $r > 0$, we put $B_X (x_0, r) := \{ x \in X \mid \| x - x_0 \| < r \}$. By $\partial U$ and $\overline{U}$ we denote the boundary and the closure of $U \subset X$. $\text{conv} V$ and $\text{conv}^X V$ stand for the convex hull and the closed (in $X$) convex hull of $V \subset X$, respectively. By $(\cdot, \cdot)_0$ is denoted the inner product in $X$.

Measure of noncompactness. If $X$ is a Banach space and $V \subset X$ is bounded, then by $\beta_X (V)$ we denote the infimum over all $r > 0$ such that $V$ can be covered with a finite number of open balls of radius $r$. Clearly $\beta_X (V)$ is finite and it is called the Hausdorff measure of noncompactness of the set $V$ in the space $X$. It is not hard to show that $\beta_X (V) = 0$ implies that $V$ is relatively compact in $X$. More properties of the measure of noncompactness can be found in [9] or [1].

Fixed point index. Below we recall basic definitions and facts from the fixed point index theory for ultimately compact maps. For details we refer to [1].

We say that a map $\Phi : D \to X$, defined on a subset $D$ of a Banach space $X$ is ultimately compact if $V \subset X$ is such that $\text{conv} \Phi (V \cap D) = V$, then $V$ is compact. We shall say that an ultimately compact map $\Phi : \overline{U} \to X$, defined on the closure of an open bounded set $U \subset X$, is called admissible if $\Phi (u) \neq u$ for all $u \in \partial U$. By an admissible homotopy between two admissible maps $\Phi_0, \Phi_1 : \overline{U} \to X$ we mean a continuous map $\Psi : \overline{U} \times [0, 1] \to X$ such that $\Psi (\cdot, 0) = \Phi_0$, $\Psi (\cdot, 1) = \Phi_1$, $\Psi (u, \mu) \neq u$ for all $u \in \partial U$ and $\mu \in [0, 1]$, and, for any $V \subset X$, if $\text{conv} \Psi ((V \cap \overline{U}) \times [0, 1]) = V$, then $V$ is relatively compact. $\Phi_0, \Phi_1$ are called homotopic then. A fixed point index for ultimately compact maps was constructed in [1] 1.6.3 and 3.5.6. Basic properties of the fixed point index are collected in the following

Proposition 2.1.

(i) (existence) If $\text{Ind}(\Phi, U) \neq 0$, then there exists $u \in U$ such that $\Phi (u) = u$.

(ii) (additivity) If $U_1, U_2 \subset U$ are open and $\Phi (u) \neq u$ for all $u \in \overline{U \setminus (U_1 \cup U_2)}$, then

$$\text{Ind}(\Phi, U) = \text{Ind}(\Phi, U_1) + \text{Ind}(\Phi, U_2).$$
following properties

\[ \text{Theorem 3.1.} \quad \text{Ind}(\Phi_0, U) = \text{Ind}(\Phi_1, U). \]

(iv) (normalization) Let \( u_0 \in X \) and \( \Phi_{u_0} : \overline{U} \to X \) be defined by \( \Phi_{u_0}(u) = u_0 \) for all \( u \in \overline{U} \). Then \( \text{Ind}(\Phi_{u_0}, U) \) is equal 0 if \( u_0 \notin U \) and 1 if \( u_0 \in U \).

**Remark 2.2.** If \( \Phi : \overline{U} \to X \) is a compact map then \( \text{Ind}(\Phi, U) \) is equal to the Leray-Schauder index \( \text{Ind}_{LS}(\Phi, U) \) (see e.g. [11]).

### 3 Remarks on abstract continuity and averaging principle

Let \( A : D(A) \to X \) be a sectorial operator such that for some \( \alpha > 0 \), \( A + aI \) has its spectrum in the half-plane \( \{ z \in \mathbb{C} \mid \text{Re} \, z > 0 \} \). Let \( X^\alpha \), \( 0 < \alpha < 1 \), be the fractional power space determined by \( A + aI \). It is well-known that there exists \( C_\alpha > 0 \) such that for all \( t > 0 \)

\[ \| e^{-tA}u \| \leq C_\alpha t^{-\alpha} e^{at} \| u \|_0, \]

where \( \{ e^{-tA} \}_{t \geq 0} \) is the semigroup generated by \( -A \). Consider the equation

\[ \begin{aligned}
\dot{u}(t) &= -Au(t) + F(t, u(t)), \quad t > 0, \\
u(0) &= \bar{u},
\end{aligned} \tag{13} \]

where \( \bar{u} \in X^\alpha \) and \( F : [0, +\infty) \times X^\alpha \to X \) is continuous, locally Lipschitz with respect to the second variable map with sublinear growth. We shall say that \( u : [0, +\infty) \to X^\alpha \) is a solution of above initial value problem if

\[ u \in C([0, +\infty), X^\alpha) \cap C((0, +\infty), D(A)) \cap C^1((0, +\infty), X) \]

and satisfies (13). By classical results (see [6] or [12]), the problem (13) admits a unique global solution \( u \in C([0, +\infty), X^\alpha) \cap C((0, +\infty), D(A)) \cap C^1((0, +\infty), X) \). Moreover, it is known that \( u \) being solution of (13) satisfies the following Duhamel formula

\[ u(t) = e^{-tA}u(0) + \int_0^t e^{-(t-s)A}F(s, u(s))ds, \quad t > 0. \]

**Theorem 3.1.** Assume that mappings \( F_n : [0, T) \times X^\alpha \to X \), \( n \geq 0 \), have the following properties

\[ \| F_n(t, u) \| \leq C(1 + \| u \|_0) \text{ for } t \in [0, T), \quad u \in X^\alpha, \quad n \geq 0, \]

for any \( (t, x) \in [0, T) \times X^\alpha \) there exists a neighborhood \( U \) of \( (t, x) \) in \([0, T) \times X^\alpha \) such that for all \( (t_1, u_1), (t_2, u_2) \in U \)

\[ \| F_n(t_1, u_1) - F_n(t_2, u_2) \| \leq L(|t_1 - t_2|^\theta + \| u_1 - u_2 \|_0) \]

for some \( L > 0 \) and \( \theta \in (0, 1) \) and, for each \( u \in X^\alpha \),

\[ \int_0^t F_n(s, u)ds \to \int_0^t F_0(s, u)ds \quad \text{in } X \text{ as } n \to +\infty \]

uniformly with respect to \( t \) from compact subsets of \([0, T)\). If \( u_n(0) \to u_0(0) \) in \( X \), then \( u_n(t) \to u_0(t) \) in \( X^\alpha \) uniformly with respect to \( t \) from compact subsets of \((0, T)\).
Remark 3.2. Recall that Henry’s result from [12] states that, under the above assumptions if \( u_n(0) \to u_0(0) \) in \( X^\alpha \), as \( n \to +\infty \), then \( u_n(t) \to u_0(t) \) in \( X^\alpha \) uniformly on compact subsets of \([0,T)\). Here, inspired by the proof of Proposition 2.3 of [19], we modify Henry’s proof.

In the proof we shall use the following lemma.

Lemma 3.3. ([12] Lemma 3.4.7) Under the assumptions of Theorem 3.1 for any continuous \( u : [0,T) \to X^\alpha \),
\[
\int_0^t e^{-(t-s)A}F_n(s,u(s)) \, ds \to \int_0^t e^{-(t-s)A}F_0(s,u(s)) \, ds \quad \text{in } X^\alpha \text{ as } n \to +\infty,
\]
uniformly with respect to \( t \) from compact subsets of \([0,T)\).

Proof of Theorem 3.1: By the Duhamel formula, for \( t \in (0,T) \),
\[
u_n(t) - u_0(t) = e^{-At}(u_n(0) - u_0(0)) + \int_0^t e^{-(t-s)A}(F_n(s,u_0(s)) - F_0(s,u_0(s))) \, ds
\]
\[+ \int_0^t e^{-(t-s)A}(F_n(s,u_n(s)) - F_n(s,u_0(s))) \, ds.
\]

This gives
\[
\|u_n(t) - u_0(t)\|_\alpha \leq \gamma_n(t) + C_\alpha L \int_0^t (t-s)^{-\alpha}\|u_n(s) - u_0(s)\|_\alpha \, ds
\]
with
\[
\gamma_n(t) := C_\alpha t^{-\alpha}\|u_n(0) - u_0(0)\|_0 + \left\| \int_0^t e^{-(t-s)A}(F_n(s,u_0(s)) - F_0(s,u_0(s))) \, ds \right\|_\alpha.
\]

By use of Lemma 7.1.1 of [12], we get
\[
\|u_n(t) - u_0(t)\|_\alpha \leq \gamma_n(t) + K \int_0^t (t-s)^{-\alpha}\gamma_n(s) \, ds
\]
for some constant \( K > 0 \). Now let us take an arbitrary \( \delta \in (0,T/2) \). Observe also that
\[
\int_0^t (t-s)^{-\alpha}\gamma_n(s) \, ds \leq \frac{2^\alpha}{\delta^\alpha} \int_0^{t-\delta/2} \gamma_n(s) \, ds + \int_{t-\delta/2}^t (t-s)^{-\alpha}\gamma_n(s) \, ds
\]
\[\leq \frac{2^\alpha}{\delta^\alpha} \int_0^T \gamma_n(s) \, ds + \frac{(\delta/2)^{1-\alpha}}{1-\alpha} \sup_{s \in [\delta/2,T-\delta]} \gamma_n(s).
\]

Since, in view of Lemma 3.3 \( \gamma_n(t) \to 0 \) uniformly with respect to \( t \) from compact subsets of \((0,T)\) and functions \( \gamma_n \), \( n \geq 1 \), are bounded by an integrable function (of the form \( t \mapsto C(t^{-\alpha} + t^{1-\alpha}) \) with some constant \( C > 0 \)), we infer, by the dominated convergence theorem, that \( \|u_n(t) - u_0(t)\|_\alpha \to 0 \) as \( n \to +\infty \) uniformly with respect to \( t \in [\delta, T - \delta] \).

\[\square\]
The above theorem allows us to strengthen Henry’s averaging principle. We assume that a continuous map \( F : [0, +\infty) \times X^\alpha \times [0, 1] \to X \) is such that there exists a continuous \( \tilde{F} : X^\alpha \times [0, 1] \to X \) and, for any \( t > 0 \) and \( \mu_0 \in [0, 1] \),

\[
\lim_{\tau \to +\infty, \mu \to \mu_0} \frac{1}{\tau} \int_0^\tau F(t, u, \mu) \, dt = \tilde{F}(u, \mu_0) \quad \text{in } X. 
\]

(14)

For \( \lambda > 0 \) and \( \mu \in [0, 1] \), consider

\[
\dot{u}(t) = -Au(t) + F(t/\lambda, u(t), \mu), \quad t > 0
\]

and its solution \( u \) with initial condition \( u(0) = \bar{u} \) for some \( \bar{u} \in X^\alpha \), denote by \( u(\cdot; \bar{u}, \mu, \lambda) \). By \( \hat{u}(\cdot; \bar{u}, \mu) \) we denote the solution of

\[
\dot{u}(t) = -Au(t) + \hat{F}(u(t), \mu), \quad t > 0, \quad u(0) = \bar{u} \in X^\alpha.
\]

**Theorem 3.4.** If \( \bar{u}_n \to \bar{u}_0 \) in \( X \), \( \mu_n \to \mu_0 \) in \([0, 1]\) and \( \lambda_n \to 0^+ \) as \( n \to +\infty \), and \( u(\cdot; \bar{u}_n, \lambda_n) \) and \( \hat{u}(\cdot; \bar{u}_0, \mu_0) \) are defined on \([0, T]\), then \( u(t; \bar{u}_n, \mu_n, \lambda_n) \to \hat{u}(t; \bar{u}_0, \mu_0) \) in \( X^\alpha \) uniformly with respect to \( t \) from compact subsets of \([0, T]\).

**Proof of Theorem 3.4.** Let \( F_n := F(\cdot; \lambda_n, \cdot, \mu_n) \) and \( F_0 := \hat{F}(\cdot, \mu_0) \). Observe that, using (14), we get, for any \( \bar{u} \in X^\alpha \) and \( t > 0 \),

\[
\int_0^t F_n(s, \bar{u}) \, ds = \lambda_n \int_0^{t/\lambda_n} F(\rho, \bar{u}, \mu_n) \, d\rho \to t\hat{F}(\bar{u}, \mu_0), \quad \text{in } X, \quad \text{as } n \to +\infty,
\]

which in view of Theorem 3.1 yields the assertion. \(\square\)

**Remark 3.5.**

(a) The above result is an improvement of the continuation theorem and the Henry averaging principle [12, Th. 3.4.9] to the case when initial values from \( X^\alpha \) converge in the topology of \( X \) (not \( X^\alpha \)). This will appear crucial when establishing the ultimate compactness property and verifying a priori estimates in the proofs of main results. We shall need to consider solutions in the the phase space \( X^{1/2} = H^1(\mathbb{R}^N) \) (cf. Remark 1.3), while the compactness of initial values is possible with respect to the \( L^2(\mathbb{R}^N) \) topology only.

(b) An averaging principle for parabolic equations on \( \mathbb{R}^N \) was also proved in [3] where time dependent coefficients of the elliptic operator were considered. Here we have provided a general abstract approach.

### 4 Translation operator for the parabolic equation

In order to transform \( (1) \) into an abstract evolution equation we define an operator \( A : D(A) \to X \) in the space \( X := L^2(\mathbb{R}^N) \) by

\[
Au := -\sum_{i,j=1}^N a_{ij} \frac{\partial^2 u}{\partial x_j \partial x_i}, \quad \text{for } u \in D(A) := H^2(\mathbb{R}^N),
\]
where \(a_{ij} \in \mathbb{R}, i, j = 1, \ldots, N\), are such that there exists \(\theta_0 > 0\) satisfying the following ellipticity condition
\[
\sum_{i,j=1}^{N} a_{ij} \xi_i \xi_j \geq \theta_0 |\xi|^2 \text{ for any } \xi \in \mathbb{R}^N
\]
and \(a_{ij} = a_{ji}\) for \(i, j = 1, \ldots, N\). It is well-known that \(A\) is a self-adjoint, positive and sectorial operator in \(L^2(\mathbb{R}^N)\). Define \(F : [0, +\infty) \times H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^2)\) by \(F(t, u)(x) := f(t, x, u(x))\), for a.e. \(x \in \mathbb{R}^N\).

**Lemma 4.1.** Under the above assumptions there are constants \(C_1, C_2 > 0\) depending only on \(M_0, K, \bar{K}, L, N\) and \(p\) such that, for all \(t_1, t_2 \geq 0\) and \(u_1, u_2 \in H^1(\mathbb{R}^N)\),
\[
\|F(t_1, u_1) - F(t_2, u_2)\|_{L^2} \leq C_1(1 + \|u_1\|_{H^1})|t_1 - t_2|^\theta + C_1\|u_1 - u_2\|_{H^1} \quad \text{and} \quad \|F(t, u)\|_{L^2} \leq C_2(1 + \|u\|_{H^1}) \text{ for any } t \geq 0 \text{ and } u \in H^1(\mathbb{R}^N).
\]

**Proof:** By use of (4), the Hölder and Sobolev inequalities, one finds a constant \(C_1 > 0\) such that, for any \(t_1, t_2 \geq 0\) and \(u_1, u_2 \in H^1(\mathbb{R}^N)\),
\[
\|F(t_1, u_1) - F(t_2, u_2)\|_{L^2} \leq \left(\|\bar{K}\|_{L^2} + C^{N/p} \|K_0\|_{L^p} \|u_1\|_{H^1} + \|K_{\infty}\|_{L^2} \|u_1\|_{L^2}\right)|t_1 - t_2|^\theta + C^{N/p} \|L_0(\cdot, t_2)\|_{L^p} \|u_1 - u_2\|_{H^1} + \|L_{\infty}(\cdot, t_2)\|_{L^\infty} \|u_1 - u_2\|_{L^2} \leq C_1(1 + \|u_1\|_{H^1})|t_1 - t_2|^\theta + C_1\|u_1 - u_2\|_{H^1}
\]
where \(C = C(N) > 0\) is the constant in the Sobolev inequality: \(\|u\|_{L^{2N/(N-2)}} \leq C\|u\|_{H^1}\) for all \(u \in H^1(\mathbb{R}^N)\).

Furthermore, by (4), one also has \(|f(t, x, u)| \leq |f(t, x, 0)| + L(x, t)|u|\) for \(t \geq 0\), \(x \in \mathbb{R}^N, u \in \mathbb{R}\). This gives the existence of \(C_2 > 0\) such that
\[
\|F(t, u)\|_{L^2} \leq \|F(t, 0)\|_{L^2} + C_2(1 + \|u\|_{H^1})
\]
for any \(t \geq 0\) and \(u \in H^1(\mathbb{R}^N)\). \(\square\)

Consider now the evolutionary problem
\[
\dot{u}(t) = -Au(t) + F(t, u(t)), \quad t \geq 0, \quad u(0) = \bar{u} \in H^1(\mathbb{R}^N). \tag{15}
\]

Due to Lemma 4.1 and standard results in theory of abstract evolution equations (see [12] or [6]) the problem (15) admits a unique global solution \(u \in C([0, +\infty), H^1(\mathbb{R}^N)) \cap C((0, +\infty), H^2(\mathbb{R}^N)) \cap C^1((0, +\infty), L^2(\mathbb{R}^N))\). We shall say that \(u : [0, T_0) \to H^1(\mathbb{R}^N), T_0 > 0\), is a solution \((H^1\text{-solution})\) of
\[
\left\{ \begin{array}{l}
\frac{\partial u}{\partial t}(x, t) = Au(x, t) + f(t, x, u(x, t)), \quad x \in \mathbb{R}^N, \ t \in (0, T_0), \\
u(x, 0) = \bar{u}(x), \quad x \in \mathbb{R}^N,
\end{array} \right.
\]
for some \(\bar{u} \in H^1(\mathbb{R}^N)\), where \(A = -\sum_{i,j=1}^{N} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}\) is an elliptic operator, if
\(u \in C([0, +\infty), H^1(\mathbb{R}^N)) \cap C((0, +\infty), H^2(\mathbb{R}^N)) \cap C^1((0, +\infty), L^2(\mathbb{R}^N))\) and (15) holds. In this sense we have global in time existence and uniqueness of solutions for the parabolic partial differential equation.

The continuity of solutions properties are collected below.
Proposition 4.2. [compare ([19] Prop. 2.3)] Assume that continuous functions $f_n : [0, +\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}, n \geq 0$, satisfy the assumptions (3) and (1) with common $L$ and that $f_n(t, x, u) \to f_0(t, x, u)$ for all $(t, x, u) \in [0, +\infty) \times \mathbb{R}^N \times \mathbb{R}$ and $f_n(t, \cdot, 0) \to f_0(t, \cdot, 0)$ in $L^2(\mathbb{R}^N)$ for all $t \geq 0$. Let $u : [0, T] \to H^1(\mathbb{R}^N)$ be a solution of (1) with $f := f_n$ such that, for some $R > 0$, $\|u(t)\|_{H^1} \leq R$, and $\|u_n(t)\|_{H^1} \leq R$ for all $t \in [0, T]$ and $n \geq 1$. Then $f_n(t, \cdot, u(\cdot)) \to f_0(t, \cdot, u(\cdot))$ in $L^2(\mathbb{R}^N)$ for any $u \in H^1(\mathbb{R}^N)$ and $t \geq 0$ and

(i) if $u_n(0) \to u(0)$ in $L^2(\mathbb{R}^N)$ as $n \to \infty$, then $u_n(t) \to u(t)$ in $H^1(\mathbb{R}^N)$ for $t$ from compact subsets of $(0, T]$.

(ii) if $u_n(0) \to u(0)$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$, then $u_n(t) \to u(t)$ in $H^1(\mathbb{R}^N)$ uniformly for $t \in [0, T]$.

Proof: Define $F_n : [0, +\infty) \times H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N), n \geq 0$, by $[F_n(t, u)](x) := f_n(t, x, u)$. Note that, in view of (1), for any $t \geq 0$ and $u \in H^1(\mathbb{R}^N)$ and a.e. $x \in \mathbb{R}^N$

$$|f_n(t, x, u(x)) - f_0(t, x, u(x))|^2 \leq 2|f_n(t, x, 0) - f_0(t, x, 0)|^2 + 4|L(x, t)u|^2.$$ 

Since, for any $t \geq 0$, $f_n(t, \cdot, 0) \to f_0(t, \cdot, 0)$ in $L^2(\mathbb{R}^N)$ as $n \to +\infty$, the right hand side can be estimated by an integrated function, which due to the Lebesgue dominated convergence theorem implies $F_n(t, u) \to F_0(t, u)$ in $L^2(\mathbb{R}^N)$ as $n \to +\infty$. Moreover, by use of Lemma 4.1 we may pass to the limit under the integral to get $\int_0^t F_n(s, u) \, ds \to \int_0^t F_0(s, u) \, ds$ in $L^2(\mathbb{R}^N)$ for any $t \geq 0$. This in view of Theorem 3.1 implies the assertion (i). The assertion (ii) comes from the standard continuity theorem from [12].

□

Lemma 4.3. Assume that a continuous function $f$ satisfies condition (3), (4) and (5). Suppose that $u : [0, T] \to H^1(\mathbb{R}^N)$ is a solution of (1) such that $\|u(t)\|_{H^1} \leq R$ for all $t \in [0, T]$. Then there exists a sequence $(\alpha_n)$ with $\alpha_n \to 0$ as $n \to \infty$ such that

$$\int_{\mathbb{R}^N \setminus B(0,n)} |u(t)|^2 \, dx \leq R^2 e^{-2at} + \alpha_n \quad \text{for all } t \in [0, T], n \geq 1,$$

where $\alpha_n$'s depend only on $M_0, L, \tilde{K}, a$ and $b$.

Proof: it goes along the lines of [19, Prop. 2.2]. The only difference is that here we have the modified dissipativity condition (5), i.e., (5) implies

$$f(t, x, u)u \leq -a|u|^2 + b(x)|u|^2 + f(t, x, 0)u$$

for $t \geq 0, x \in \mathbb{R}^N, u \in \mathbb{R}$, and one needs to modify the proof in a rather obvious way. □

Now suppose that $a_{ij} \in C([0, 1], \mathbb{R}), i, j = 1, \ldots, N$, are such that there exists $\theta_0 > 0$ satisfying the ellipticity condition $\sum_{i,j=1}^N a_{ij}(\mu)\xi_i\xi_j \geq \theta_0|\xi|^2$ for any $\xi \in \mathbb{R}^N$ and $\mu \in [0, 1]$. Let $A(\mu) : D(A(\mu)) \to L^2(\mathbb{R}^N), \mu \in [0, 1]$, be given by

$$A(\mu)u := -\sum_{i,j=1}^N a_{ij}(\mu) \frac{\partial^2 u}{\partial x_j \partial x_i}, \quad u \in D(A(\mu)) := H^2(\mathbb{R}^N).$$
Let \( h \in C([0, +\infty) \times \mathbb{R}^N \times \mathbb{R} \times [0, 1], \mathbb{R}) \) be such that, for all \( t, s \geq 0, x \in \mathbb{R}^N, u, v \in \mathbb{R} \) and \( \mu, \nu \in [0, 1] \),
\[
    h(t, \cdot, 0, \mu) \in L^2(\mathbb{R}^N), \quad \|h(t, \cdot, 0, \mu)\|_{L^2} \leq M_0,
\]
and
\[
    h(t, s, x, u, \mu) - h(s, x, v, \mu) \leq (\tilde{K}(x) + K(x)|u|)\|t-s\|^\theta + L(x, t)|u-v|,
\]
where \( M_0 \geq 0, \theta \in (0, 1), \tilde{K} \in L^2(\mathbb{R}^N), K = K_0 + K_\infty \) with \( K_0 \in L^p(\mathbb{R}^N), p \geq N, \)
\( K_\infty \in L^\infty(\mathbb{R}^N), L = L_0 + L_\infty \) with \( L_0, L_\infty \in L^p(\mathbb{R}^N), p \geq N, L_\infty(\cdot, t) \in L^\infty(\mathbb{R}^N) \)
for all \( t \geq 0 \), and \( \sup_{t \geq 0} (\|L_0(\cdot, t)\|_{L^p} + \|L_\infty(\cdot, t)\|_{L^\infty}) < +\infty \), \( \rho \in C([0, 1], \mathbb{R}), a > 0 \) and \( b \in L^p(\mathbb{R}^N) \).

Clearly, due to Lemma 4.1, we get the existence and uniqueness of solutions on \([0, +\infty)\). Denote by \( u(\cdot; \bar{u}, \mu) \) the solution of (20) satisfying the initial value condition \( u(0) = \bar{u} \).

The following tail estimates will be crucial in studying the compactness properties of the translation along trajectories operator of (20).

**Lemma 4.4.** Take any \( \bar{u}_1, \bar{u}_2 \in H^1(\mathbb{R}^N) \) and \( \mu_1, \mu_2 \in [0, 1] \) and suppose that there are solutions \( u(\cdot; \bar{u}_i, \mu_i) : [0, T] \to H^1(\mathbb{R}^N), i = 1, 2 \) of (20), for some fixed \( T > 0 \). If \( \|u(t; \bar{u}_1, \mu_1)\|_{H^1} \leq R \) and \( \|u(t; \bar{u}_2, \mu_2)\|_{H^1} \leq R \) for all \( t \in [0, T] \) and some fixed \( R > 0 \), then there exists a sequence \( (\alpha_n) \) with \( \alpha_n \to 0 \) as \( n \to \infty \) such that
\[
    \int_{\mathbb{R}^N \setminus B(0, n)} |u(t; \bar{u}_1, \mu_1) - u(t; \bar{u}_2, \mu_2)|^2 \, dx \leq e^{-2at}\|\bar{u}_1 - \bar{u}_2\|_{L^2}^2 + Q\eta(\mu_1, \mu_2) + \alpha_n,
\]
for all \( t \in [0, T] \) and \( n \geq 1 \), where \( \alpha_n \geq 0 \) and \( Q > 0 \) depend only on \( M_0, \tilde{K}, K, L, a, b \) and \( a_{ij} s \),
\[
    \eta(\mu_1, \mu_2) := \max \left\{ |\rho(\mu_1) - \rho(\mu_2)|, \max_{i, j = 1, \ldots, N} |a_{ij}(\mu_1) - a_{ij}(\mu_2)| \right\}.
\]

**Proof:** Let \( \phi : [0, +\infty) \to \mathbb{R} \) be a smooth function such that \( \phi(s) \in [0, 1] \) for \( s \in [0, +\infty), \phi|_{[0, 1]} \equiv 0 \) and \( \phi_{[2, +\infty]} \equiv 1 \) and let \( \phi_n : \mathbb{R}^N \to \mathbb{R} \) be defined by \( \phi_n(x) := \phi(|x|^2/n^2), x \in \mathbb{R}^N \). Put \( u_1 := u(\cdot; \bar{u}_1, \mu_1), u_2 := u(\cdot; \bar{u}_2, \mu_2) \) and \( v := u_1 - u_2 \).

Observe that
\[
    \frac{1}{2} \frac{d}{dt} (v(t), \phi_n v(t))_0 = \frac{1}{2} ((v(t), \phi_n \dot{v}(t))_0 + (v(t), \phi_n v(t))_0) = (\phi_n v(t), \dot{v}(t))_0 = I_1(t) + I_2(t) + I_3(t)
\]
where

\[ I_1(t) := (\phi_n v(t), -A^{(\mu_1)} u_1(t) + A^{(\mu_1)} u_2(t))_0, \]
\[ I_2(t) := (\phi_n v(t), -A^{(\mu_1)} u_2(t) + A^{(\mu_2)} u_2(t))_0, \]
\[ I_3(t) := (\phi_n v(t), H(t, u_1(t), \mu_1) - H(t, u_2(t), \mu_2))_0. \]

As for the first term we notice that

\[ I_1(t) = (\phi_n v(t), -A^{(\mu_1)} v(t))_0 \]
\[ = -\int_{\mathbb{R}^N} \sum_{i,j=1}^{N} a_{ij}(\mu_1) \frac{\partial}{\partial x_j} (\phi_n(x)v(t)) \frac{\partial}{\partial x_i} (v(t)) \, dx \]
\[ = -\int_{\mathbb{R}^N} \phi_n(x) \sum_{i,j=1}^{N} a_{ij}(\mu_1) \frac{\partial}{\partial x_j} (v(t)) \frac{\partial}{\partial x_i} (v(t)) \, dx \]
\[ - \frac{2}{n^2} \int_{\mathbb{R}^N} \sum_{i,j=1}^{N} \phi'(|x|^2/n^2)v(t)x_ja_{ij}(\mu_1) \frac{\partial}{\partial x_i} (v(t)) \, dx \]
\[ \leq \frac{2L\phi}{n^2} \int_{\{n \leq |x| \leq \sqrt{n}\}} \sum_{i,j=1}^{N} a_{ij}(\mu_1)|x||v(t)|||\nabla v(t)||_1 \, dx \]
\[ \leq \frac{2\sqrt{2}L\phi MN^2}{n} \|v(t)\|_{L^2} \|v(t)\|_{H^1}. \]

where \( L_{\phi} := \sup_{s \in [0, +\infty)} |\phi'(s)| < \infty \) (as \( \phi \) is smooth and \( \phi' \) nonzero on a bounded interval) and \( M := \max_{1 \leq i,j \leq N, \mu \in [0,1]} |a_{ij}(\mu)| \). Further, in a similar manner

\[ I_2(t) = -\int_{\mathbb{R}^N} \sum_{i,j=1}^{N} \frac{\partial}{\partial x_j} (\phi_n v(t)) (a_{ij}(\mu_1) - a_{ij}(\mu_2)) \frac{\partial}{\partial x_i} (u_2(t)) \, dx \]
\[ = -\int_{\mathbb{R}^N} \phi_n(x) \sum_{i,j=1}^{N} (a_{ij}(\mu_1) - a_{ij}(\mu_2)) \frac{\partial}{\partial x_j} (v(t)) \frac{\partial}{\partial x_i} (u_2(t)) \, dx \]
\[ - \frac{2}{n^2} \int_{\mathbb{R}^N} \sum_{i,j=1}^{N} \phi'(|x|^2/n^2)v(t)x_j(a_{ij}(\mu_1) - a_{ij}(\mu_2)) \frac{\partial}{\partial x_i} (u_2(t)) \, dx \]
\[ \leq \eta(\mu_1, \mu_2) \|v(t)\|_{H^1} \|u_2(t)\|_{H^1} + \frac{4\sqrt{2}L\phi \eta(\mu_1, \mu_2) N^2}{n} \|v(t)\|_{L^2} \|u_2(t)\|_{H^1}. \]
To estimate $I_3(t)$ we see that (19) implies
\[
I_3(t) = \int_{\mathbb{R}^N} \phi_n(x) (H(t, u_1(t), \mu_1) - H(t, u_2(t), \mu_2)) v(t) \, dx
\]
\[
\leq \int_{\mathbb{R}^N} \phi_n(x) (H(t, u_1(t), \mu_1) - H(t, u_2(t), \mu_1)) v(t) \, dx
\]
\[
+ \int_{\mathbb{R}^N} \phi_n(x) (H(t, u_2(t), \mu_1) - H(t, u_2(t), \mu_2)) v(t) \, dx
\]
\[
\leq -a \int_{\mathbb{R}^N} \phi_n(x) |v(t)|^2 \, dx + \int_{\mathbb{R}^N} \phi_n(x) b(x) |v(t)|^2 \, dx
\]
\[
+ \int_{\mathbb{R}^N} L(x, t) |\rho(\mu_1) - \rho(\mu_2)||u_2(t)||v(t)|| \, dx
\]
\[
\leq -a \int_{\mathbb{R}^N} \phi_n(x) |v(t)|^2 \, dx + C^{N/p} \|v(t)\|_{H^1}^2 \left( \int_{\{|x| \geq n\}} b(x)^p \, dx \right)^{1/p}
\]
\[
+ \eta(\mu_1, \mu_2) \left( C^{N/p} \|L_0(\cdot, t)\|_{L^p} \|u_2(t)\|_{H^1} + \|L_{\infty}(\cdot, t)\|_{L^\infty} \|u_2(t)\|_{L^2} \right) \|v(t)\|_{L^2},
\]
where $C$ is the constant of the Sobolev embedding $H^1(\mathbb{R}^N) \subset L^{2N/(N-2)}(\mathbb{R}^N)$. Hence we get, for any $n \geq 1$,
\[
\frac{d}{dt}(v(t), \phi_n v(t))_0 \leq -2a(v(t), \phi_n v(t))_0 + \tilde{C} \eta(\mu_1, \mu_2) + \alpha_n
\]
for some constant $\tilde{C} = \tilde{C}(p, N, L, R) > 0$. Multiplying by $e^{2at}$ and integrating over $[0, \tau]$ one obtains
\[
e^{2at}(v(\tau), \phi_n v(\tau))_0 - (v(0), \phi_n v(0))_0 \leq (2a)^{-1}(e^{2at} - 1) \left( \tilde{C} \eta(\mu_1, \mu_2) + \alpha_n \right),
\]
which gives
\[
(v(\tau), \phi_n v(\tau))_0 \leq e^{-2at} \|v(0)\|_{L^2}^2 + (2a)^{-1} \left( \tilde{C} \eta(\mu_1, \mu_2) + \alpha_n \right).
\]
And this finally implies the assertion as $\|\phi_n v(\tau)\|_{L^2}^2 \leq (v(\tau), \phi_n v(\tau))_0$. \qed

Let $\Psi_t : H^1(\mathbb{R}^N) \times [0, 1] \to H^1(\mathbb{R}^N)$, $t > 0$, be the translation operator for (20), i.e. $\Psi_t(\bar{u}, \mu) = u(t; \bar{u}, \mu)$ for $\bar{u} \in H^1(\mathbb{R}^N)$ and $\mu \in [0, 1]$.

**Proposition 4.5.** Suppose that (16), (18), (17) and (19) are satisfied.

(i) For any bounded $V \subset H^1(\mathbb{R}^N)$ and $t > 0$, $\beta_{L^2}(\Psi_t(V \times [0, 1])) \leq e^{-at} \beta_{L^2}(V)$;  

(ii) If a bounded $V \subset H^1(\mathbb{R}^N)$ is relatively compact as a subset of $L^2(\mathbb{R}^N)$, then $\Psi_t(V \times [0, 1])$ is relatively compact in $H^1(\mathbb{R}^N)$;  

(iii) If $V \subset \overline{\Psi_t(V \times [0, 1])}$ for some bounded $V \subset H^1(\mathbb{R}^N)$ and $t > 0$, then $V$ is relatively compact in $H^1(\mathbb{R}^N)$.

**Proof:** (i) Let $\phi_n$, $n \geq 1$ be functions from Lemma 4.4. Then, for each $n \geq 1$,
\[
\Psi_t(V \times [0, 1])) \subset \{u(t; \bar{u}, \mu) \mid \bar{u} \in V, \mu \in [0, 1] \} \subset W_n + R_n
\]
where $W_n := \{\chi_n u(t; \bar{u}, \mu) \mid \bar{u} \in V, \mu \in [0, 1]\}$ and $R_n := \{(1 - \chi_n) u(t; \bar{u}, \mu) \mid \bar{u} \in V, \mu \in [0, 1]\}$ where $\chi_n$ is the characteristic function of the ball $B(0, n)$. Note
Finally, by a passage to the limit with the relative compactness of $V^\mu(\bar{B}_h)$, where Kondrachov theorem, Now we need to estimate the measure of noncompactness of $R^\mu$ is such that

$$\Psi_t(V \times [0,1]) \leq \beta_{L^2}(R^\mu), \quad \text{for all } n \geq 1.$$ (21)

Now we need to estimate the measure of noncompactness of $R_n$ in $L^2(\mathbb{R}^N)$. To this end fix an arbitrary $\varepsilon > 0$. Choose a finite covering of $V$ consisting of balls $B_{L^2}(\bar{u}_k, r_k)$, $k = 1, \ldots, m_\varepsilon$, with $r_\varepsilon := \beta_{L^2}(V) + \varepsilon$ and such that $\bar{u}_k \in V$ for each $k = 1, \ldots, m_\varepsilon$ and cover $[0,1]$ with intervals $(\mu_l - \delta, \mu_l + \delta)$, $l = 1, \ldots, n_\delta$ where $\delta > 0$ is such that $\eta(\mu_1, \mu_2) < \varepsilon$ whenever $|\mu_1 - \mu_2| < \delta$. Put $\bar{u}_{k,l} := (1 - \chi_n)u(t; \bar{u}_k, \mu_l), \ k = 1, \ldots, m_\varepsilon, \ l = 1, \ldots, n_\varepsilon$.

Now take any $\tilde{u} \in R_n$. There are $\bar{u} \in V$ and $\mu \in [0,1]$ such that $\tilde{u} = (1 - \chi_n)u(t; \bar{u}, \mu)$. Clearly there exist $k_0 \in \{1, \ldots, m_\varepsilon\}$ and $l_0 \in \{1, \ldots, n_\delta\}$ such that $\|\tilde{u} - \bar{u}_{k_0}\| < r_\varepsilon$ and $|\mu - \mu_{l_0}| < \delta$. In view of Lemma 4.4

$$\|\tilde{u} - \bar{u}_{k_0,l_0}\|^2_{L^2} = \int_{\mathbb{R}^N \setminus B(0,n)} |u(t; \bar{u}, \mu) - u(t; \bar{u}_{k_0}, \mu_{l_0})|^2 \, dx$$

$$\leq e^{-2at}\|\bar{u} - \bar{u}_{k_0}\|^2_{L^2} + Q\eta(\mu, \mu_{l_0}) + \alpha_n$$

which means that $R_n$ is covered by the balls $B_{L^2}(\bar{u}_{k,l}, \sqrt{\varepsilon, n})$, $k = 1, \ldots, m_\varepsilon$, $l = 1, \ldots, n_\varepsilon$. This means that $\beta_{L^2}(R_n) \leq \sqrt{\varepsilon, n}$ for any $\varepsilon > 0$, and, in consequence, $\beta_{L^2}(R_n) \leq (e^{-2at}(\beta_{L^2}(V))^2 + \alpha_n)^{1/2}$. Using (21) we get

$$\beta_{L^2}(\Psi_t(V \times [0,1])) \leq (e^{-2at}(\beta_{L^2}(V))^2 + \alpha_n)^{1/2}, \quad \text{for } n \geq 1.$$ Finally, by a passage to the limit with $n \to \infty$ we obtain the required inequality as $\alpha_n \to 0^+$.

(ii) Take any $(\bar{u}_n)$ in $V$ and $(\mu_n)$ in $[0,1]$. We may assume that $\mu_n \to \mu_0$ for some $\mu_0 \in [0,1]$, as $n \to +\infty$. Since $(\bar{u}_n)$ is bounded, by the Banach-Alaoglu theorem, we may suppose that $(\bar{u}_n)$ converges weakly in $H^1(\mathbb{R}^N)$ to some $\bar{u} \in H^1(\mathbb{R}^N)$. By the relative compactness of $V$ in $L^2(\mathbb{R}^N)$ we may assume that $\bar{u}_n \to \bar{u}$ in $L^2(\mathbb{R}^N)$. Therefore, by use of Proposition 4.2, one has $\Psi_t(\bar{u}_n, \mu_n) \to \Psi_t(\bar{u}, \mu_0)$ in $H^1(\mathbb{R}^N)$, which ends the proof.

(iii) Observe that here, by use of (i), one gets

$$\beta_{L^2}(V) \leq \beta_{L^2}(\Psi_t(V \times [0,1])) \leq e^{-at}\beta_{L^2}(V).$$

This implies $\beta_{L^2}(V) = 0$, i.e. that $V$ is relatively compact in $L^2(\mathbb{R}^N)$. To see that $V$ is relatively compact in $H^1(\mathbb{R}^N)$ observe that, by (ii), $\Psi_t(V \times [0,1])$ is relatively compact in $H^1(\mathbb{R}^N)$.

5 Averaging index formula

Consider the following parameterized equation

$$\begin{cases}
\frac{\partial u}{\partial t}(x, t) = \Delta u(x, t) + h(t/\lambda, x; u(x, t), \mu), \quad t > 0, \ x \in \mathbb{R}^N, \\
u(t, t) \in H^1(\mathbb{R}^N), \quad t > 0,
\end{cases}$$

(22)

where $h$ is as in the previous section and $\lambda > 0$. Combining the compactness result with averaging principle we get the following result.
Lemma 5.1. Suppose $h$ satisfies conditions (16), (17), (18) and (19) and is $T$-periodic in the time variable ($T > 0$). If $(\bar{u}_n)$ is a bounded sequence in $H^1(\mathbb{R}^N)$, $(\mu_n)$ in $[0,1]$, $(\lambda_n)$ in $(0, +\infty)$ with $\lambda_n \to 0^+$ as $n \to +\infty$ and $u_n : [0, +\infty) \to H^1(\mathbb{R}^N)$ are solutions of (22) with $\lambda = \lambda_n$, $\mu = \mu_n$, such that $u_n(0) = u_n(\lambda_n T) = \bar{u}_n$, then there are a subsequence $(\bar{u}_{n_k})$ of $(\bar{u}_n)$ converging in $H^1(\mathbb{R}^N)$ to some $\bar{u}_0 \in H^2(\mathbb{R}^N)$ and a subsequence $(\mu_{n_k})$ of $(\mu_n)$ converging to some $\mu_0 \in [0,1]$, as $k \to +\infty$, such that $\bar{u}_0$ is a solution of

$$\Delta u(x) + \hat{h}(x,u(x),\mu_0) = 0, \quad x \in \mathbb{R}^N,$$

where $\hat{h} : \mathbb{R}^N \times \mathbb{R} \times [0,1] \to \mathbb{R}$, $\hat{h}(x,u,\mu) := \frac{1}{T} \int_0^T h(t,x,u,\mu) \, dt$, $(x,u,\mu) \in \mathbb{R}^N \times \mathbb{R} \times [0,1]$. Moreover, $u_{n_k}(t) \to \bar{u}_0$ in $H^1(\mathbb{R}^N)$, as $k \to +\infty$, uniformly with respect to $t$ from compact subsets of $(0, +\infty)$.

Proof: Recall that $u_n$ are solutions of $\dot{u} = -Au + H(t/\lambda_n, u, \mu_n)$ with $u_n(0) = u_n(\lambda_n T) = \bar{u}_n$, $n \geq 1$, where $A$ and $H$ are as in the previous section with $A = \Delta$. Clearly, by the sublinear growth, there exists $R > 0$ such that $\|u_n(t)\|_{H^1} \leq R$ for all $t > 0$ and $n \geq 1$. Fix $n \geq 1$ and take an arbitrary $M > 0$ and $k \in \mathbb{N}$ such that $k\lambda_n T > M$. In view of Lemma 4.3 for all $n \geq 1$,

$$\|(1 - \chi_n)u_n\|_{L^2}^2 = \|(1 - \chi_n)u_n(k\lambda_n T)\|_{L^2}^2 \leq R^2 e^{-2ak\lambda_n T} + \alpha_n \leq R^2 e^{-2aM} + \alpha_n,$$

where $\chi_n$ is the characteristic function of $B(0,n)$. Since $M > 0$ is arbitrary we see that $\|(1 - \chi_n)\bar{u}_n\|_{L^2} \leq \sqrt{\alpha_n}$. Since, due to the Rellich-Kondrachov, the sequence $(\chi_n \bar{u}_n)$ is relatively compact in $L^2(\mathbb{R}^N)$, we infer that $\{\bar{u}_n\}_{n \geq 1}$ is relatively compact in $L^2(\mathbb{R}^N)$. And since it is bounded in $H^1(\mathbb{R}^N)$ we get a subsequence $(\bar{u}_{n_k})$ such that $\bar{u}_{n_k} \to \bar{u}_0$ in $L^2(\mathbb{R}^N)$ for some $\bar{u}_0 \in H^1(\mathbb{R}^N)$. We may also assume that $\mu_{n_k} \to \mu_0$ for some $\mu_0 \in [0,1]$. Hence, in view of Theorem 3.1, $u_{n_k}(t) \to \hat{u}(t)$ uniformly for $t$ from compact subsets of $(0, +\infty)$ where $\hat{u} : [0, +\infty) \to H^1(\mathbb{R}^N)$ is a solution to

$$\dot{u} = -Au + \hat{H}(u,\mu_0), \quad t > 0,$$

with $\hat{H}(u,\mu) := \frac{1}{T} \int_0^T H(t,u,\mu) \, dt$ for $u \in H^1(\mathbb{R}^N)$, $\mu \in [0,1]$. Here note that, for each $u \in H^1(\mathbb{R}^N)$ and $\mu \in [0,1]$,

$$[\hat{H}(u,\mu)](x) = \hat{h}(x,u(x),\mu) \quad \text{for all a.a.} \quad x \in \mathbb{R}^N.$$

Finally, for any $t > 0$, we put $k_n := \lfloor t/\lambda_n T \rfloor$, $n \geq 1$, and see that

$$\bar{u}_n = u_n(0) = u_n(k_n\lambda_n T) \to \hat{u}(t) \quad \text{in} \quad H^1(\mathbb{R}^N), \quad \text{as} \quad n \to +\infty.$$

Hence $\hat{u}(t) = \hat{u}(0) = \bar{u}_0$ and $u_n \to \bar{u}_0$ in $H^1(\mathbb{R}^N)$. □

Remark 5.2. It clear that it follows from the proof of Lemma 5.1 that if $f_n : [0, +\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are as in Proposition 4.2 and satisfy (5) with common $a$ and $b$, then for any bounded sequence $(\bar{u}_n)$ in $H^1(\mathbb{R}^N)$, $(\lambda_n)$ in $(0, +\infty)$ with $\lambda_n \to 0^+$ as $n \to +\infty$ and $u_n : [0, +\infty) \to H^1(\mathbb{R}^N)$ being $\lambda_n T$-periodic solutions of

$$\begin{cases}
\frac{\partial u}{\partial t} = \Delta u + f_n(t/\lambda_n, x, u), \quad x \in \mathbb{R}^N, \quad t > 0,
\frac{\partial u}{\partial t} = u(x, \lambda_n T) = \bar{u}_n(x), \quad x \in \mathbb{R}^N,
\end{cases}$$

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there is a subsequence \((\tilde{u}_{n_k})\) of \((u_n)\) converging in \(H^1(\mathbb{R}^N)\) to some \(\tilde{u}_0 \in H^2(\mathbb{R}^N)\) being a solution of
\[
\Delta u(x) + \tilde{f}_0(x, u(x)) = 0 \quad \text{on } \mathbb{R}^N.
\]
Moreover, \(u_{n_k}(t) \rightarrow \tilde{u}_0\) in \(H^1(\mathbb{R}^N)\), as \(k \rightarrow +\infty\), uniformly with respect to \(t\) from compact subsets of \((0, +\infty)\). \(\square\)

Now consider the following problem
\[
\begin{align*}
\frac{\partial u}{\partial t}(x, t) &= \Delta u(x, t) + f\left(\frac{t}{\lambda}, x, u(x, t)\right), \quad t > 0, \quad x \in \mathbb{R}^N, \\
u(\cdot, t) &\in H^1(\mathbb{R}^N), \quad t > 0,
\end{align*}
\]
where \(f\) satisfies conditions (2), (3), (4) and (5). We intend to prove an averaging index formula that allows to express the fixed point index of translation along trajectories operator for (23) in terms of the averaged equation
\[
\begin{align*}
\frac{\partial u}{\partial t}(x, t) &= \Delta u(x, t) + \hat{f}(x, u(x, t)), \quad t > 0, \quad x \in \mathbb{R}^N, \\
u(\cdot, t) &\in H^1(\mathbb{R}^N), \quad t \geq 0,
\end{align*}
\]
where \(\hat{f}: \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}\) is defined by
\[
\hat{f}(x, u) := \frac{1}{T} \int_0^T f(t, x, u) \, dt, \quad x \in \mathbb{R}^N, \quad u \in \mathbb{R}.
\]

**Theorem 5.3.** Let \(U \subset H^1(\mathbb{R}^N)\) be an open bounded set and by \(\Phi^{(\lambda)}_t\) and \(\hat{\Phi}_t\), \(t > 0\), denote the translation along trajectories operators (by time \(t\)) for the equations (23) and (24), respectively. If the problem
\[
\begin{align*}
- \Delta u(x) &= \hat{f}(x, u(x)), \quad x \in \mathbb{R}^N, \\
u(x) &\in H^1(\mathbb{R}^N),
\end{align*}
\]
has no solution in \(\partial U\), then there exists \(\lambda_0 > 0\) such that, for all \(\lambda \in (0, \lambda_0]\), \(\Phi^{(\lambda)}_{\lambda T}(\bar{u}) \neq \bar{u}\), \(\hat{\Phi}_{\lambda T}(\bar{u}) \neq \bar{u}\) for all \(\bar{u} \in \partial U\), and
\[
\text{Ind}(\Phi^{(\lambda)}_{\lambda T}, U) = \text{Ind}(\hat{\Phi}_{\lambda T}, U).
\]
**Proof:** Define \(F: [0, +\infty) \times H^1(\mathbb{R}^N) \times [0, 1] \rightarrow L^2(\mathbb{R}^N)\) by
\[
[F(t, u, \mu)](x) := (1 - \mu)f(t, x, u(x)) + \mu \hat{f}(x, u(x)), \quad \text{for a.e. } x \in \mathbb{R}^N,
\]
and all \(t > 0, u \in H^1(\mathbb{R}^N)\). For a parameter \(\lambda > 0\) consider
\[
\dot{\bar{u}}(t) = -Au(t) + F(t/\lambda, u(t), \mu), \quad t \in [0, T],
\]
and the parameterized translation operator \(\Psi^{(\lambda)}_t: H^1(\mathbb{R}^N) \times [0, 1] \rightarrow H^1(\mathbb{R}^N)\) defined by
\[
\Psi^{(\lambda)}_t(\bar{u}, \mu) := u(t)
\]
where \( u : [0, T] \to H^1(\mathbb{R}^N) \) is the solution of (26) with \( u(0) = \bar{u} \). Observe that for \( \mu = 0 \), (26) becomes

\[
\dot{u}(t) = -Au(t) + F(t/\lambda, u(t)), \quad t \in [0, T],
\]

and we have \( \Phi_t^{(\lambda)} = \Psi_t^{(\lambda)}(\cdot, 0) \). In the same way for \( \mu = 1 \) the equation (26) becomes

\[
\dot{u}(t) = -Au(t) + \hat{F}(u(t)), \quad t \in [0, T]
\]

and one has \( \Phi_t = \Psi_t^{(\lambda)}(\cdot, 1) \) (it does not depend on \( \lambda \)).

We claim that there exists \( \lambda_0 > 0 \) such that, for all \( \lambda \in (0, \lambda_0] \),

\[
\Psi_{\lambda T}^{(\lambda)}(\bar{u}, \mu) \neq \bar{u} \quad \text{for all} \quad \bar{u} \in \partial U, \mu \in [0, 1]. \tag{27}
\]

Suppose the claim does not hold. Then there exist \( (\bar{u}_n) \) in \( \partial U, (\mu_n) \) in \([0, 1]\) and \( (\lambda_n) \) with \( \lambda_n \to 0^+ \) as \( n \to \infty \) such that

\[
\Psi_{\lambda_n T}^{(\lambda_n)}(\bar{u}_n, \mu_n) = \bar{u}_n \quad \text{for all} \quad n \geq 1.
\]

This means that for each \( n \geq 1 \) there is a \( \lambda_n T \)-periodic solution \( u_n : [0, +\infty) \to H^1(\mathbb{R}^N) \) of (26) with \( \lambda = \lambda_n, \mu = \mu_n \) and \( u_n(0) = \bar{u}_n \). By Lemma 5.1 we may assume that \( \bar{u}_n \to \bar{u}_0 \) in \( H^1(\mathbb{R}^N) \). Therefore \( \bar{u}_0 \in \partial U \cap D(A) \) and \( 0 = -A\bar{u}_0 + \hat{F}(\bar{u}_0) \), a contradiction with the assumption. This proves the existence of \( \lambda_0 > 0 \) such that, for all \( \lambda \in (0, \lambda_0] \), (27) holds.

Now, due to Proposition 4.5 (iii), for each \( \lambda \in (0, \lambda_0] \), \( \Psi_{\lambda T}^{(\lambda)} \) is an admissible homotopy in the sense of fixed point index theory for ultimately compact maps. Finally, by Proposition 2.1(iii), we get the desired equality of the indices. \( \square \)

As a consequence we get the following continuation principle.

**Corollary 5.4.** Suppose that an open bounded \( U \subset H^1(\mathbb{R}^N) \) is such that (25) has no solution in \( \partial U \), and for any \( \lambda \in (0, 1) \) the problem

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \lambda \Delta u + \lambda f(t, x, u), x \in \mathbb{R}^N, \ t > 0, \\
u(\cdot, t) &\in H^1(\mathbb{R}^N), \ t \geq 0 \\
u(x, 0) &= u(x, T), \ x \in \mathbb{R}^N,
\end{align*}
\tag{28}
\]

has no solution \( u : [0, +\infty) \to H^1(\mathbb{R}^N) \) with \( u(0) \in \partial U \). Then

\[
\text{Ind}(\Phi_T, U) = \lim_{t \to 0^+} \text{Ind}(\hat{\Phi}_t, U)
\]

where \( \Phi_T \) is the translation along trajectories operator for (11).

**Proof:** Let \( \lambda_0 > 0 \) be as in Theorem 5.3. Since there are no solutions to (28), we infer that

\[
\Phi_{\lambda T}^{(\lambda)}(\bar{u}) \neq \bar{u} \quad \text{for any} \quad \bar{u} \in \partial U, \lambda \in (0, 1).
\]

Now by Proposition 4.5 (iii) and the homotopy invariance of the index, for any \( \lambda \in (0, 1) \), we get \( \text{Ind}(\Phi_T, U) = \text{Ind}(\Phi_T^{(1)}, U) = \text{Ind}(\Phi_T^{(\lambda)}, U) = \text{Ind}(\Phi_{\lambda T}^{(\lambda)}, U) \), where
\( \Phi^{(\lambda)}_T \) is the translation along trajectories operator for the parabolic equation in (28) with the parameter \( \lambda \) and the last equality comes from a time rescaling argument saying that \( \Phi^{(\lambda)}_T = \Phi^{(\lambda)}_{\lambda T} \). Now an application of Theorem 5.3 completes the proof. \( \Box \)

The rest of the section is devoted to methods of verification the \textit{a priori} bounds conditions occurring in the above corollary and computation of fixed point index. We shall use a linearization approach.

**Proposition 5.5.** Suppose that \( f \) satisfies conditions (2), (3), (4), (5) and \( f(t, x, 0) = 0 \) for all \( x \in \mathbb{R}^N \) and \( t \geq 0 \).

(i) If (7) holds, \( \text{Ker} (\Delta + \hat{\omega}) = \{0\} \) and the linear equation

\[
\begin{align*}
\frac{\partial u}{\partial t}(x, t) &= \lambda \Delta u(x, t) + \lambda \omega(t, x)u(x, t), \quad x \in \mathbb{R}^N, \quad t > 0, \\
u(\cdot, t) &\in H^1(\mathbb{R}^N), \quad t \geq 0,
\end{align*}
\]

has no nonzero \( T \)-periodic solutions for \( \lambda \in (0, 1] \), then there exists \( R > 0 \) such that, for any \( \lambda \in (0, 1] \) the problem (28) has no \( T \)-periodic solutions \( u : [0, +\infty) \to H^1(\mathbb{R}^N) \) with \( \|u(0)\|_{H^1} \geq R \).

(ii) If (9) holds, \( \text{Ker} (\Delta + \hat{\alpha}) = \{0\} \) and the linear equation

\[
\begin{align*}
\frac{\partial u}{\partial t}(x, t) &= \lambda \Delta u(x, t) + \lambda \alpha(t, x)u(x, t), \quad x \in \mathbb{R}^N, \quad t > 0, \\
u(\cdot, t) &\in H^1(\mathbb{R}^N), \quad t \geq 0,
\end{align*}
\]

has no nonzero \( T \)-periodic solutions, then there exists \( r > 0 \) such that, for any \( \lambda \in (0, 1] \) the problem (28) has no \( T \)-periodic solutions \( u : [0, +\infty) \to H^1(\mathbb{R}^N) \) with \( 0 < \|u(0)\|_{H^1} \leq r \).

**Proof:** (i) Suppose to the contrary, i.e. that for any \( n \geq 1 \) there exist \( \lambda_n \in (0, 1) \) and a time \( T \)-periodic solution \( u_n : [0, +\infty) \to H^1(\mathbb{R}^N) \) of

\[
\frac{\partial u}{\partial t} = \lambda_n \Delta u + \lambda_n f(t, x, u), \quad x \in \mathbb{R}^N, \quad t > 0
\]

with \( \|u_n(0)\|_{H^1} \to +\infty \). This means that \( v_n \) given by \( z_n(t) := u_n(t/\lambda_n) \) is a solution of

\[
\frac{\partial z}{\partial t} = \Delta z + f(t/\lambda_n, x, z), \quad x \in \mathbb{R}^N, \quad t > 0.
\]

It is also clear that \( v_n := \rho_n z_n, \quad \rho_n := (1 + \|u_n(0)\|_{H^1})^{-1} \), is a solution of

\[
\frac{\partial v}{\partial t} = \Delta v + \rho_n f(t/\lambda_n, x, \rho_n^{-1}v), \quad x \in \mathbb{R}^N, \quad t > 0.
\]

It is clear that \( \rho_n \to 0^+ \) and we may suppose that \( \lambda_n \to \lambda_0 \), as \( n \to +\infty \) for some \( \lambda_0 \in [0, 1] \). Take an arbitrary \( M > 0 \) and observe that Lemma 5.3 gives, for integers \( k \geq 1 \) such that \( k\lambda_n T > M \),

\[
\|(1 - \chi_n)v_n(0)\|_{L^2}^2 = \|(1 - \chi_n)v_n(k\lambda_n T)\|_{L^2}^2 \leq R_0^2 e^{-2a k \lambda_n T} + \alpha_n \leq R_0^2 e^{-2aM} + \alpha_n
\]
with $\alpha_n \to 0^+$ as $n \to +\infty$ and $R_0 > 0$ such that $\|v_n(t)\|_{H^1} \leq R_0$ for all $t \geq 0$ and $n \geq 1$. Since $M > 0$ is arbitrary we see that $\|(1 - \chi_n)v_n(0)\|_{L^2} \leq \sqrt{\alpha_n}$ for $n \geq 1$. Due to the Rellich-Kondrachov $\{\chi_n u_n(0)\}_{n \geq 1}$ is relatively compact in $L^2(\mathbb{R}^N)$.  Therefore $\{v_n(0)\}_{n \geq 1}$ is relatively compact in $L^2(\mathbb{R}^N)$. As a bounded sequence in $H^1(\mathbb{R}^N)$ it contains a subsequence convergent in $L^2(\mathbb{R}^N)$ to some $\tilde{v}_0 \in H^1(\mathbb{R}^N)$. Therefore we may assume that $v_n(0) \to \tilde{v}_0$ in $L^2(\mathbb{R}^N)$.

First consider the case when $\lambda_0 \in (0, 1]$ and define $g_n : [0, +\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, $g_n(t, x, v) := \rho_n f(t/\lambda_n, x, \rho_n^{-1} v)$, $n \geq 1$, $t \geq 0$, $x \in \mathbb{R}^N$, $v \in \mathbb{R}$. Further note that (7) and (2) yield

$$\lim_{n \to +\infty} g_n(t, x, v) = \omega(t/\lambda_0, x)v \quad \text{for all } t \geq 0, x \in \mathbb{R}^N, v \in \mathbb{R},$$

and $\|g_n(t, \cdot, 0)\|_{L^2} = \rho_n \|f(t/\lambda_n, \cdot, 0)\|_{L^2} \leq \rho_n \sup_{\tau \geq 0} \|f(\tau, \cdot, 0)\|_{L^2} \to 0$, as $n \to +\infty$.

Since the assumptions of Proposition 4.2 are satisfied we infer that $v_n(t) \to v_0(t)$ in $H^1(\mathbb{R}^N)$ uniformly with respect to $t$ from compact subsets of $(0, +\infty)$, where $v_0 : [0, +\infty) \to H^1(\mathbb{R}^N)$ is a solution of

$$\frac{\partial v}{\partial t} = \Delta v + \omega(t/\lambda_0, x)v, \quad v(\cdot, 0) = \tilde{v}_0$$

Since $\|\tilde{v}_0\|_{H^1} \neq 0$, by rescaling time we get a nontrivial $T$-periodic solution of (29) with $\lambda = \lambda_0$, a contradiction proving the desired assertion.

Now consider the situation when $\lambda_0 = 0$. Define $h : [0, +\infty) \times \mathbb{R}^N \times \mathbb{R} \times [0, 1] \to \mathbb{R}$ by

$$h(t, x, v, \mu) := \begin{cases} \mu f(t, x, \mu^{-1} v), & t \geq 0, x \in \mathbb{R}^N, v \in \mathbb{R}, \mu \in (0, 1] \\ \omega(t, x)v, & t \geq 0, x \in \mathbb{R}^N, v \in \mathbb{R}, \mu = 0. \end{cases}$$

Define $H : [0, +\infty) \times H^1(\mathbb{R}^N) \times [0, 1] \to L^2(\mathbb{R}^N)$ by $[H(t, u, \mu)](x) := h(t, x, u, \mu)$. By the assumptions $H$ is well-defined, has sublinear growth and is locally Hölder continuous in time. Moreover, $h$ is continuous on $[0, +\infty) \times \mathbb{R}^N \times \mathbb{R} \times (0, 1]$ and observe that, for any $(t, x, v) \in [0, +\infty) \times \mathbb{R}^N \times \mathbb{R}$,

$$\lim_{\mu \to 0^+} h(t, x, v, \mu) = \lim_{\mu \to 0^+} \mu f(t, x, \mu^{-1} v) = \omega(t, x)v,$$

and

$$\|h(t, \cdot, 0, \mu)\|_{L^2} = \mu \|f(t, \cdot, 0)\|_{L^2} \leq \mu \sup_{\tau \geq 0} \|f(\tau, \cdot, 0)\|_{L^2} \to 0 \quad \text{as } \mu \to 0.$$

Hence, due to Proposition 4.1 $H$ is continuous. Now using Lemma 5.1 we get $v_n(t) \to \tilde{v}_0$ uniformly on $[0, +\infty)$ and that

$$0 = \Delta \tilde{v}_0(x) + \bar{\alpha}(x)\tilde{v}_0(x), \quad x \in \mathbb{R}^N,$$

which contradicts the assumption and completes the proof.

The proof of (ii) is analogical to the proof of (i). Suppose that assertion does not hold. Then there exist $\lambda_n \in (0, 1)$ and a $T$-periodic solution $u_n : [0, +\infty) \to H^1(\mathbb{R}^N)$ of

$$\frac{\partial u}{\partial t} = \lambda_n \Delta u + \lambda_n f(t, x, u), \quad x \in \mathbb{R}^N, \quad t > 0$$

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Let us remark that the nonexistence of solutions for (29) or (30) may be also verified
Remark 5.6. 
proof.
Note that
\[ \frac{\partial v}{\partial t} = \Delta v + \rho_n^{-1} f(t/\lambda_n, x, \rho_n v), \quad x \in \mathbb{R}^N, \quad t > 0. \]
We may assume that \( \lambda_n \to \lambda_0 \) as \( n \to +\infty \) for some \( \lambda_0 \in [0, 1] \). Applying the same arguments as in the first part of the proof, one can deduce that \( (v_n(0)) \) converges (up to a subsequence) in \( L^2(\mathbb{R}^N) \) to some \( \bar{v}_0 \in H^1(\mathbb{R}^N) \).

Suppose now that \( \lambda_0 \in (0, 1) \). Let \( g_n : [0, +\infty) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R} \) be given by
\[ g_n(t, x, v) := \rho_n^{-1} f(t/\lambda_n, x, \rho_n v), \quad n \geq 1, \quad t \geq 0, \quad x \in \mathbb{R}^N, \quad v \in \mathbb{R}. \]
By (31) we infer that
\[ \lim_{n \to +\infty} g_n(t, x, v) = \alpha(t/\lambda_0, x)v \quad \text{for all} \quad t \geq 0, \quad x \in \mathbb{R}^N, \quad v \in \mathbb{R}. \]
and observe that
\[ g_n(t, x, 0) = 0, \quad \text{for} \quad n \geq 1. \]
This allows us to use again Proposition 4.2 and one can deduce that \( v_n(t) \to v_0(t) \) in \( H^1(\mathbb{R}^N) \) uniformly with respect to \( t \) from compact subsets of \( (0, +\infty) \), where \( v_0 : [0, +\infty) \to H^1(\mathbb{R}^N) \) is the solution of
\[ \frac{\partial v}{\partial t} = \Delta v + \alpha(t/\lambda_0, x)v, \quad v(\cdot, 0) = \bar{v}_0, \]
i.e after time rescaling we obtain the existence of nontrivial \( T \)-periodic solution of (30), a contradiction. In case \( \lambda_0 = 0 \) we define \( h : [0, +\infty) \times \mathbb{R}^N \times \mathbb{R} \times [0, 1] \to \mathbb{R} \) by
\[ h(t, x, v, \mu) := \begin{cases} 
\mu^{-1} f(t, x, \mu v), & t \geq 0, \quad x \in \mathbb{R}^N, \quad v \in \mathbb{R}, \quad \mu \in (0, 1] \\
\alpha(t, x)v, & t \geq 0, \quad x \in \mathbb{R}^N, \quad v \in \mathbb{R}, \quad \mu = 0.
\end{cases} \]
Note that
\[ \lim_{\mu \to 0^+} h(t, x, v, \mu) = \lim_{\mu \to 0^+} \mu^{-1} f(t, x, \mu v) = \alpha(t, x)v, \]
and
\[ h(t, x, 0, \mu) = 0 \quad \text{for} \quad t \geq 0, \quad x \in \mathbb{R}^N, \quad \mu \in [0, 1]. \]
Finally, exactly in the same manner as in the proof of (i), we conclude that \( v_n(t) \to \bar{v}_0 \) uniformly on \( [0, +\infty) \), \( \bar{v}_0 \neq 0 \) and \( \bar{v}_0 \in \text{Ker}(\Delta + \bar{\alpha}) \), a contradiction. This ends the proof.

Remark 5.6.
Let us remark that the nonexistence of solutions for (29) or (30) may be also verified if \( \alpha \) or \( \omega \) are time dependent. Assume that
\[ \sup_{t \in [0, T]} \| \alpha_0(\cdot, t) \|_{L^p} < \frac{\bar{\alpha}}{\alpha_{\infty}^{1-N/2p}} \left( \frac{N}{2p} \right)^{N/2p} C(N)^{N/p} \]  
(31)
where \( C(N) > 0 \) is the constant in the Sobolev inequality \( \| u \|_{L^{2N/N-2}} \leq C(N) \| \nabla u \|_{L^2}, u \in H^1(\mathbb{R}^N) \). Suppose that \( u \) is a nonzero \( T \)-periodic solution of (29). Then, for all \( t > 0, \)
\[ \frac{d}{dt} \frac{1}{2\lambda} \| u(t) \|_{L^2}^2 = - \int_{\mathbb{R}^N} |\nabla u(t)|^2 dx - \int_{\mathbb{R}^N} \alpha_\infty(t, x)|u(t)|^2 dx + \int_{\mathbb{R}^N} \alpha_0(t, x)|u(t)|^2 dx. \]
This yields, by use of the Sobolev and Hölder inequalities,

\[ \int_0^T (\| \nabla u(t) \|^2 + \alpha_\infty \| u(t) \|^2) \, dt \leq \int_0^T \| \alpha_0(t, \cdot) \|_{L^p} \| u(t) \|_{L^{N/p}}^{N/p} \, dt \]

\[ \leq C(N)^{N/p} \int_0^T \| \alpha_0(\cdot, t) \|_{L^p} \| \nabla u(t) \|_{L^{N/p}}^{N/p} \| u(t) \|_{L^{2-N/p}}^{2-N/p} \, dt. \]

In the Young inequality \( ab \leq \frac{a^r}{r} + \frac{b^s}{s} \) where \( a, b \geq 0, \epsilon > 0 \) and \( r \in (1, +\infty) \) such that \( \frac{1}{r} + \frac{1}{s} = 1 \), put \( a := \| \alpha_0(\cdot, t) \|_{L^p} \| \nabla u(t) \|_{L^{N/p}}, b := \| u(t) \|_{L^{2-N/p}} \) and \( r := 2p/N \) to obtain

\[ \| \alpha_0(\cdot, t) \|_{L^p} \| \nabla u(t) \|_{L^{N/p}} \| u(t) \|_{L^{2-N/p}} \leq \frac{N/2p}{\epsilon} \| \alpha_0(\cdot, t) \|_{L^p}^{2p} \| \nabla u(t) \|_{L^{N/p}}^{2p} + \left(1 - \frac{N}{2p}\right) \epsilon^{2p/N} \| u(t) \|_{L^{2-N/p}}^2 \]

for any \( \epsilon > 0 \) and fixed \( t \in [0, T] \). If we take \( \epsilon = \epsilon(t) \) so that \( \frac{N/2p}{\epsilon(t)^{2p/N}} \| \alpha_0(\cdot, t) \|_{L^p}^{2p} = 1 \), i.e. \( \epsilon(t) = (N/2p)^{N/2p} C(N)^{N/2p^2} \| \alpha(\cdot, t) \|_{L^p} \) and apply (31), then

\[ \tilde{\alpha}_\infty \int_0^T \| u(t) \|_{L^2}^2 \, dt \leq C(N)^{N/p} \left(1 - \frac{N}{2p}\right) \int_0^T \epsilon(t)^{2p} \| u(t) \|_{L^2}^2 \, dt \]

\[ \leq (N/2p)^N \| \alpha_0(\cdot, t) \|_{L^p}^{2p} \| u(t) \|_{L^{2-N}} \int_0^T \| \alpha_0(\cdot, t) \|_{L^p}^{2p} \| u(t) \|_{L^{2-N}} \, dt \]

\[ \leq (N/2p)^N \| \alpha_0(\cdot, t) \|_{L^p}^{2p} \sup_{t \in [0, T]} \| \alpha_0(\cdot, t) \|_{L^p}^{2p} \int_0^T \| u(t) \|_{L^2}^2 \, dt \]

\[ < \tilde{\alpha}_\infty \int_0^T \| u(t) \|_{L^2}^2 \, dt, \]

a contradiction proving that (29) has no nontrivial \( T \)-periodic solutions.

6 Proofs of Theorems 1.1 and 1.2

We start with a linearization method for computing the fixed point index of the translation operator in the autonomous case.

Proposition 6.1. Assume that \( f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R} \) satisfies conditions (3), (4) and (5) (in their time-independent versions) and let \( \Phi_t \) be the translation along trajectories for the autonomous equation

\[
\left\{ \begin{array}{l}
\frac{\partial u}{\partial t}(x, t) = \Delta u(x, t) + f(x, u(x, t)), \ x \in \mathbb{R}^N, \ t > 0, \\
u(\cdot, t) \in H^1(\mathbb{R}^N), \ t \geq 0.
\end{array} \right.
\]

(i) If (7) holds and \( \text{Ker}(\Delta + \omega) = \{0\} \), then there exists \( R_0 > 0 \) such that \(-\Delta u(x) = f(x, u(x)), \ x \in \mathbb{R}^N, \) has no solutions \( u \in H^1(\mathbb{R}^N) \) with \( \| u \|_{H^1} \geq R_0 \) and there exists \( \bar{t} > 0 \) such that, for all \( t \in (0, \bar{t}], \ \Phi_t(u) \neq \bar{u} \) for all \( \bar{u} \in H^1(\mathbb{R}^N) \setminus B_{H^1}(0, R_0) \) and, for all and all \( t \in (0, \bar{t}] \) and \( R \geq R_0, \)

\[ \text{Ind}(\Phi_t, B_{H^1}(0, R)) = (-1)^{m_{\infty}} \]

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where $m_{-}(\infty)$ is the total multiplicity of the negative eigenvalues of $-\Delta + \omega$.

(ii) If (11) holds and $\text{Ker}(\Delta + \alpha) = \{0\}$, then there exists $r_0 > 0$ such that $-\Delta u(x) = f(x, u(x))$, $x \in \mathbb{R}^N$, has no solutions with $0 < ||u||_{H^1} \leq r_0$ and there exists $t > 0$ such that, for all $t \in (0, \bar{t}]$, $\Phi_t(\bar{u}) \neq \bar{u}$ for all $\bar{u} \in B_{H^1}(0, r_0) \setminus \{0\}$ and, for each $t \in (0, \bar{t}]$,

$$\text{Ind}(\Phi_t, B_{H^1}(0, r_0)) = (-1)^{m_{-}(0)}$$

where $m_{-}(0)$ is the total multiplicity of the negative eigenvalues of $-\Delta - \alpha$.

Remark 6.2. Recall the known arguments on the spectrum of $-\Delta - \omega_0 + \omega_{\infty}$. To this end, define $B_0 : D(B_0) \to L^2(\mathbb{R}^N)$ with $D(B_0) := H^1(\mathbb{R}^N)$ by $B_0 u := \omega_0 u$ and $B_\infty : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ by $B_\infty u := \omega_{\infty} u$. By [18], $A_\infty := A - B_0 + B_\infty$ is a $C_0$ semigroup generator and its spectrum $\sigma(A_\infty)$ is contained in an interval $(-c, +\infty)$ with some $c > 0$. It is clear that $\sigma(A + B_\infty) \subset [\omega_{\infty}, +\infty)$. Since $B_0(A + B_\infty)^{-1} : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is a compact linear operator – see [19, Lem. 3.1], by use of the Weyl theorem on essential spectra, we obtain $\sigma_{\text{ess}}(A_\infty) = \sigma_{\text{ess}}(A + B_\infty) \subset \sigma(A + B_\infty) \subset [\omega_{\infty}, +\infty)$ (see e.g. [21]). Hence, by general characterizations of essential spectrum, we see that $\sigma(A_\infty) \cap (-\infty, 0)$ consists of isolated eigenvalues with finite dimensional eigenspaces (see [21]).

Proof of Proposition 6.1. (i) We start with an observation that there exists $R_0 > 0$ such that the problem

$$0 = \Delta u + (1 - \mu)f(x, u) + \mu \omega(x)u, \ x \in \mathbb{R}^N, \ (32)$$

has no weak solutions in $H^1(\mathbb{R}^N) \setminus B_{H^1}(0, R_0)$. To see this, suppose to the contrary that there exist a sequence $(\mu_n)$ in $[0, 1]$ and solutions $\bar{u}_n$, $n \geq 1$, of (32) with $\mu = \mu_n$ such that $||\bar{u}_n||_{H^1} \to +\infty$ as $n \to +\infty$. Put $\rho_n := (1 + ||\bar{u}_n||_{H^1})^{-1}$ and observe that $\bar{v}_n := \rho_n \bar{u}_n$ are solutions of

$$0 = \Delta v + (1 - \mu_n)\rho_n f(x, \rho_n^{-1}v) + \mu_n \omega(x)v, \ x \in \mathbb{R}^N.$$

Clearly

$$\rho_n f(x, \rho_n^{-1}v) \to \omega(x)v \text{ as } n \to +\infty \text{ for all } t \geq 0 \text{ and a.a. } x \in \mathbb{R}^N.$$

Hence, by use of Remark 5.2, we see that $(\bar{u}_n)$ contains a sequence convergent to some $\bar{u}_0 \in H^1(\mathbb{R}^N)$ being a weak nonzero solution of $0 = \Delta u + \omega(x)u$, $x \in \mathbb{R}^N$, a contradiction proving that (32) has no solutions outside some ball $B_{H^1}(0, R_0)$.

Now consider the equation

$$\frac{\partial u}{\partial t} = \Delta u + (1 - \mu)f(x, u) + \mu \omega(x)u, \ x \in \mathbb{R}^N, \ t > 0, \ (33)$$

where $\mu \in [0, 1]$ is a parameter. Let $\Psi_t : H^1(\mathbb{R}^N) \times [0, 1] \to H^1(\mathbb{R}^N), \ t > 0$, be the parameterized translation along trajectories operator for the above equation. In view of Theorem 5.3, there exists $t > 0$ such that

$$\Psi_t(\bar{u}, \mu) \neq \bar{u} \text{ for all } t \in (0, \bar{t}], \bar{u} \in \partial B_{H^1}(0, R_0).$$

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By Proposition 4.5 (iii), the homotopy $\Psi$, is admissible in the sense of the fixed point theory for ultimately compact maps (see Section 2). Therefore using the homotopy invariance one has, for $t \in (0, \tilde{t})$,

$$\text{Ind}(\Phi_t, B_{H^1}(0, R_0)) = \text{Ind}(e^{-tA_\infty}, B_{H^1}(0, R_0))$$

(34)

where $A_\infty := A + B_0 - B_\infty$.

It is left to determine the fixed point index of $e^{-tA_\infty}$. We note that the set $\sigma(A_\infty) \cap (-\infty, 0)$ is bounded and closed. Hence, in view of the spectral theorem (see [25]) there are closed subspaces $X_\sigma \subset L^2(\mathbb{R}^N)$ such that $X_- \oplus X_+ = L^2(\mathbb{R}^N)$, $A_\infty(X_-) \subset X_-$, $A_\infty(D(A)) \subset X_+$, $\sigma(A_\infty|_{X_-}) = \sigma(A_\infty) \cap (-\infty, 0)$, $\sigma(A_\infty|_{X_+}) = \sigma(A_\infty) \cap (0, +\infty)$. Define $\Theta_t : H^1(\mathbb{R}^N) \times [0, 1) \to H^1(\mathbb{R}^N)$ by

$$\Theta_t(\bar{u}, \mu) := (1 - \mu)e^{-tA_\infty}\bar{u} + \mu e^{-tA_\infty}P_-\bar{u},$$

where $P_- : H^1(\mathbb{R}^N) \to H^1(\mathbb{R}^N)$ is the restriction of the projection onto $X_- \cap H^1(\mathbb{R}^N)$ in $L^2(\mathbb{R}^N)$. Since $\dim X_- < +\infty$ we infer that $P_-$ is continuous. It also claim that $\Theta_t$ is ultimately compact. To see this take a bounded set $V \subset H^1(\mathbb{R}^N)$ such that $V = \text{conv}^{H^1}\Theta_t(V \times [0, 1])$. This means that $V \subset \text{conv}^{H^1} e^{-tA_\infty}(V \cup P_- V)$. Since $V \cup P_- V$ is bounded, Proposition 4.5 (ii) implies that $V$ is relatively compact in $H^1(\mathbb{R}^N)$, which proves the ultimate compactness of $\Theta_t$. Since $\ker(I - \Theta_t(\cdot, \mu)) = \{0\}$ for $\mu \in [0, 1]$, by the homotopy invariance and the restriction property of the Leray-Schauder fixed point index, one gets

$$\text{Ind}(e^{-tA_\infty}, B_{H^1}(0, R_0)) = \text{Ind}_{\text{LS}}(e^{-tA_\infty}P_-, B_{H^1}(0, R_0)) = \text{Ind}_{\text{LS}}(e^{-t(A_\infty|_{X_-})}, B_{H^1}(0, R_0) \cap X_-) = (-1)^{m_-}(\infty).$$

The latter equality comes from the fact that $\sigma(A_\infty|_{X_-}) \subset (-\infty, 0)$ consists of isolated eigenvalues of finite dimensional eigenspaces. This ends the proof of (i) together with [24].

(ii) First we shall prove the existence of $r_0 > 0$ such that the problem

$$0 = \Delta u + (1 - \mu)f(x, u) + \mu\alpha(x)u, \quad x \in \mathbb{R}^N,$$

(35)

has no solutions in $B_{H^1}(0, r_0) \setminus \{0\}$. Suppose to the contrary that there exist a sequence $(\mu_n)$ in $[0, 1]$ and solutions $\bar{u}_n : [0, +\infty) \to H^1(\mathbb{R}^N)$, $n \geq 1$, of (35) with $\mu = \mu_n$ such that $\|\bar{u}_n\|_{H^1} \to 0^+$ as $n \to +\infty$ and $\|\bar{u}_n\|_{H^1} \neq 0$, $n \geq 1$. Put $\rho_n := \|\bar{u}_n\|_{H^1}$. Then $\bar{v}_n := \frac{\bar{u}_n}{\rho_n}$ are solutions of

$$0 = \Delta v + (1 - \mu_n)\rho_n^{-1} f(x, \rho_n v) + \mu_n\alpha(x)v, \quad x \in \mathbb{R}^N.$$

Observe that

$$\rho_n^{-1} f(x, \rho_n v) \to \alpha(x)v \quad \text{as} \quad n \to \infty \text{ for a.a. } x \in \mathbb{R}^N.$$

Using again Remark 5.2 one can see that $(\bar{u}_n)$ (up to a subsequence) converges to some nonzero solution of $0 = \Delta u + \alpha(x)u$, $x \in \mathbb{R}^N$, a contradiction. Summing up, there is $r_0 > 0$ such that (35) has no solutions $u \in H^1(\mathbb{R}^N)$ with $0 < \|u\|_{H^1} \leq r_0$. 

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The rest of the proof runs as before: by \( \Psi_t : H^1(\mathbb{R}^N) \times [0,1] \to H^1(\mathbb{R}^N) \), \( t > 0 \) we denote the translation along trajectories operator for the equation

\[
\frac{\partial u}{\partial t} = \Delta u + (1-\mu)f(x,u) + \mu\alpha(x)u, \ x \in \mathbb{R}^N, \ t > 0, \ \mu \in [0,1],
\]  

and, by applying Theorem 5.3 we obtain the existence of \( \bar{t} > 0 \) such that

\[
\Psi_t(\bar{u}, \mu) \neq \bar{u} \quad \text{for all} \quad t \in (0, \bar{t}], \ \bar{u} \in \partial B_{H^1}(0, r_0).
\]

Next Proposition 4.5 (iii) ensures the admissibility of \( \Psi_t \) and by homotopy invariance, for \( t \in (0, \bar{t}] \), we have

\[
\text{Ind}(\Phi_t, B_{H^1}(0, r_0)) = \text{Ind}(e^{-tA_0}, B_{H^1}(0, r_0)) \tag{37}
\]

where \( A_0 := A + C_0 - C_\infty \) and operators \( C_i : D(C_i) \to L^2(\mathbb{R}^N) \) with \( D(C_i) = H^1(\mathbb{R}^N) \) are given by \( C_i u := \alpha_i u, i \in \{0, \infty\} \). Now one can easily determine fixed point index of \( e^{-tA_0} \) by arguing as in part (i) (with \( A_\infty \) replaced by \( A_0 \) and \( B_{H^1}(0, R_0) \) replaced by \( B_{H^1}(0, r_0) \)) and, as a consequence, obtain that

\[
\text{Ind}(e^{-tA_0}, B_{H^1}(0, r_0)) = \text{Ind}_{LS}(e^{-tA_0} P_-, B_{H^1}(0, r_0)) = \text{Ind}_{LS}(e^{-tA_0|X_-}, B_{H^1}(0, r_0) \cap X_-) = (-1)^{m-(0)}.
\]

This completes the proof. \( \square \)

Now we are ready to conclude and provide proofs of our main results.

**Proof of Theorem 1.1** Let \( \Phi_t, t > 0 \), be the translation operator for \( \hat{f} \). It is clear that

\[
\lim_{|u| \to +\infty} \frac{\hat{f}(x,u)}{u} = \hat{\omega}(x), \ \text{for any} \ x \in \mathbb{R}^N.
\]

Hence, by applying Proposition 6.1 (i) we obtain \( R_0 > 0 \) such that

\[
\Delta u(x) + \hat{f}(x, u(x)) = 0, \ x \in \mathbb{R}^N,
\]

has no solutions in the set \( H^1(\mathbb{R}^N) \setminus B_{H^1}(0, R_0) \) and there exists \( t_0 > 0 \) such that, for \( t \in (0, t_0] \),

\[
\text{Ind}(\hat{\Phi}_t, B_{H^1}(0, R_0)) = (-1)^{m-(\infty)}. \tag{38}
\]

Due to Proposition 5.3 and the assumption, increasing \( R_0 \) if necessary, we can assume that (29) has no \( T \)-periodic solutions starting from \( H^1(\mathbb{R}^N) \setminus B_{H^1}(0, R_0) \). Taking \( U := B_{H^1}(0, R_0) \) and applying Corollary 5.4 we get

\[
\text{Ind}(\Phi_T, B_{H^1}(0, R_0)) = \lim_{t \to 0^+} \text{Ind}(\hat{\Phi}_t, B_{H^1}(0, R_0)),
\]

which along with (38) yields \( \text{Ind}(\Phi_T, B_{H^1}(0, R_0)) = (-1)^{m-(\infty)} \). This and the existence property of the fixed point index imply that there exists \( \bar{u} \in B_{H^1}(0, R_0) \) such that \( \Phi_T(\bar{u}) = \bar{u} \), i.e. there exists a \( T \)-periodic solution of (1). \( \square \)
Proof of Theorem 1.2: First use Proposition 6.1 to get $R_0, r_0 > 0$ such that

$$
\lim_{t \to 0^+} \text{Ind}(\hat{\Phi}_t, B_{H^1}(0, R)) = (-1)^{m-\infty} \text{ if } R \geq R_0 \tag{39}
$$
and

$$
\lim_{t \to 0^+} \text{Ind}(\hat{\Phi}_t, B_{H^1}(0, r)) = (-1)^{m-0} \text{ if } 0 < r \leq r_0. \tag{40}
$$

Now, due to Proposition 5.5 there exist $R \geq R_0$ and $r \in (0, r_0]$ such that, for any $\lambda \in (0, 1], (28)$ has no solutions with $u(0) \in B_{H^1}(0, r) \cup (H^1(\mathbb{R}^N) \setminus B_{H^1}(0, R))$. Next we put $U := B_{H^1}(0, R) \setminus B_{H^1}(0, r)$ and apply Corollary 5.4 to get

$$
\text{Ind}(\Phi_T, U) = \lim_{t \to 0^+} \text{Ind}(\hat{\Phi}_t, U).
$$

This together with (39) and (40), by use of the additivity property of the fixed point index, yields

$$
\text{Ind}(\Phi_T, U) = \lim_{t \to 0^+} \text{Ind}(\hat{\Phi}_t, U) - \lim_{t \to 0^+} \text{Ind}(\hat{\Phi}_t, B_{H^1}(0, R)) = (-1)^{m-\infty} - (-1)^{m-0} \neq 0,
$$

which gives the existence of the fixed point of $\Phi_T$ in $U$. \qed

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