The Regularity of Conformal Target Harmonic Maps

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Abstract: We establish that any weakly conformal $W^{1,2}$ map from a Riemann surface $S$ into a closed oriented sub-manifold $N^n$ of an euclidian space $\mathbb{R}^m$ realizes, for almost every sub-domain, a stationary varifold if and only if it is a smooth conformal harmonic map from $S$ into $N^n$.

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I Introduction

In [5] the author developed a viscosity method in order to produce closed minimal 2 dimensional surfaces into any arbitrary closed oriented sub-manifolds $N^n$ of any euclidian spaces $\mathbb{R}^m$ by min-max type arguments. The method consists in adding to the area of an immersion $\Phi$ of a surface $\Sigma$ into $N^n$ a more coercive term such as the $L^2$ norm of the second fundamental form preceded by a small parameter $\sigma^2$:

$$A^\sigma(\Phi) := \text{Area}(\Phi) + \sigma^2 \int_\Sigma \left[ 1 + \|\vec{I}_\Phi\|^2 \right]^p \, dvol_{g_\Phi}$$

where $\vec{I}_\Phi$ is the second fundamental form of the immersion $\Phi$ and $dvol_{g_\Phi}$ is the volume form associated to the induced metric. For $p > 1$ and $\sigma > 0$ one proves that the Lagrangians $A^\sigma$ are Palais-Smale in some ad-hoc Finsler bundle of immersions complete for the Palais distance. By applying the now classical Palais-Smale deformation theory in infinite dimensional space one can then produce critical points $\Phi_\sigma$ to $A^\sigma$. It is proved in [4] that, for a sequence of parameters $\sigma_k \to 0$, the sequence of integer rectifiable varifolds associated to the immersion of $\Sigma$ by $\Phi_{\sigma_k}$ does not necessarily converge to a stationary integer rectifiable varifold. However, by applying Struwe’s monotonicity trick one can always select a sequence $\sigma_j \to 0$ such that the following additional “entropy estimate” holds

$$\sigma_j \int_\Sigma \left[ 1 + \|\vec{I}_{\Phi_{\sigma_j}}\|^2 \right]^p \, dvol_{g_{\Phi_{\sigma_j}}} = o \left( \frac{1}{\sigma_j \log(\sigma_j^{-1})} \right)$$

Assuming this additional estimate, the main achievement of [5] is to prove that the immersion of $\Sigma$ by $\Phi_{\sigma_j}$ varifold converges to a stationary integer rectifiable varifold given by the image of a smooth Riemann surface $S$ by a weakly conformal $W^{1,2}$ map $\Phi$ into $N^n$ equipped by an integer multiplicity. The main result of the paper is to prove that, when this multiplicity is constant, such a map is smooth and satisfies the harmonic map equation. To state our main result we need two definitions.

Definition I.1. A property is said to hold for almost every smooth domain in $\Sigma$, if for any smooth domain $\Omega$ and any smooth function $f$ such that $f^{-1}(0) = \partial \Omega$ and $\nabla f \neq 0$ on $\partial \Omega$ then for almost every $t$ close enough to zero and regular value for $f$ the property holds for the domain contained in $\Omega$ or containing $\Omega$ and bounded by $f^{-1}\{t\}$. $\Box$

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1even modulo extraction of subsequences and in a weak sense such as the varifold distance topology.
Precisely we define the notion of target harmonic map as follows.

**Definition I.2.** Let \((\Sigma, h)\) be a smooth closed Riemann surface equipped with a metric compatible with the complex structure. A map \(\Phi \in W^{1,2}(\Sigma, N^n)\) is target harmonic if for almost every smooth domain \(\Omega \subset \Sigma\) and any smooth function \(F\) supported in the complement of an open neighborhood of \(\Phi(\partial \Omega)\) we have

\[
\int_{\Omega} \left\langle d(F(\Phi)), d\Phi \right\rangle_h - F(\Phi) A(\Phi)(d\Phi, d\Phi)_h \, \text{dvol}_h = 0 \quad (I.1)
\]

where \(h\) is any metric compatible with the chosen conformal structure on \(S\) and where \(A(q)(X, Y)\) denotes the second fundamental form of \(N^n\) at the point \(q\) and acting on the pair of vectors \((X, Y)\) and by an abuse of notation we write

\[
A(\Phi)(d\Phi, d\Phi)_h := \sum_{i,j=1}^n h_{ij} A(\Phi)(\partial_{x_i} \Phi, \partial_{x_j} \Phi).
\]

Observe that the main difference with the general definition of being harmonic is that, for target harmonic one restricts (I.1) to test functions \(F\) supported in the target while for the definition of harmonic, one requires (I.1) to hold for any \(W^{1,2}\) test function defined on the domain. Therefore, being harmonic implies to be target harmonic and the proof of the reverse is the goal of the present work. For a weakly conformal \(W^{1,2}\) map into \(N^n\) the condition for \(\Phi\) to be target harmonic is equivalent to saying that the mapping of \(\Sigma\) in \(N^n\) defines a stationary integer rectifiable varifold (see proposition A.1). Our main result in the present paper is the following.

**Theorem I.1.** Any weakly conformal target harmonic map into an arbitrary closed sub-manifold \(N^n\) of \(\mathbb{R}^m\) in two dimension is smooth and satisfy the harmonic map equation.

It is not known if, without the conformality assumption, a target harmonic map is an harmonic map in the classical sense. Following the main lines of the proof below one can prove that this is indeed the case in one dimension.

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**II The partial regularity.**

Since the problem is local we shall work in a chart and hence consider maps in \(W^{1,2}(D^2, N^n)\) exclusively. Indeed, being target harmonic in \(S\) implies to be target harmonic in any sub-domain of \(S\). Hence if one proves that target harmonic on a disc implies harmonic on that disc in the classical sense we deduce that target harmonic on \(S\) implies harmonic on \(S\). Therefore we can reduce to the case \(S = D^2\).

Let \(\Phi\) be a \(W^{1,2}\) map from the disc \(D^2\) into \(N^n\) satisfying the weak conformality condition

\[
|\partial_{x_1} \Phi|^2 = |\partial_{x_2} \Phi|^2 \quad \text{and} \quad \partial_{x_1} \Phi \cdot \partial_{x_2} \Phi = 0 \quad \text{a.e. on } D^2 \quad (II.2)
\]

\(^2\)Observe that for almost every domain \(\Omega\) the restriction of \(\Phi\) to \(\partial \Omega\) is Hölder continuous and \(\Phi(\partial \Omega)\) is then closed.

\(^3\)One observe that the condition (I.1) is conformally invariant and hence independent of the choice of the metric \(h\) within the conformal class given by \(\Sigma\).
For any $A > 1$ we introduce

$$G^A := \begin{cases} 
  x \text{ is a Lebesgue point for } \bar{\Phi} \text{ and } \nabla \bar{\Phi} & \\
  x \text{ is a point of } L^2 \text{ app. differentiability} & \\
  A^{-1} < |\nabla \bar{\Phi}(x)| < A & 
\end{cases}$$

We are going to prove the following lemma

**Lemma II.1.** Let $\bar{\Phi}$ be a target harmonic map on $D^2$. Under the previous notations we have that

$$G := \cup_{A > 1} G^A$$

is an open subset of $D^2$, $\bar{\Phi}$ is smooth and harmonic into $N^\alpha$ in the strong sense on $G$. Moreover

$$\mathcal{H}^2 \left( \bar{\Phi}(D^2 \setminus G) \right) = \frac{1}{2} \int_{D^2 \setminus G} |\nabla \bar{\Phi}|^2 \, dy^2 = 0 \quad \text{.}$$  \hspace{1cm} (II.3) \hspace{1cm} \Box

**Proof of lemma [II.1]**. Let $A > 0$. For such an $x \in G^A$ we shall denote

$$e^{\lambda(x)} := |\partial_{x_1} \bar{\Phi}(x)| = |\partial_{x_2} \bar{\Phi}(x)| \quad \text{and} \quad \bar{e}_i(x) := e^{-\lambda(x)} \partial_{x_i} \bar{\Phi}(x) \quad .$$

We have in particular

$$\frac{1}{|B_r(x)|} \int_{B_r(x)} |\nabla \bar{\Phi}(y)|^2 \, dy^2 > A^2 + o(1)$$

and hence

$$\lim_{r \to 0} \frac{\int_{B_r(x)} |\nabla \bar{\Phi}(y) - \nabla \bar{\Phi}(x)|^2 \, dy^2}{\int_{B_r(x)} |\nabla \bar{\Phi}(y)|^2 \, dy^2} = 0 \quad .$$  \hspace{1cm} (II.4)

And

$$\lim_{r \to 0} \frac{\int_{B_r(x)} r^{-2} |\bar{\Phi}(y) - \bar{\Phi}(x) - \partial_{x_i} \bar{\Phi}(x)(y_1 - x_1) - \partial_{x_2} \bar{\Phi}(x)(y_2 - x_2)|^2 \, dy^2}{\int_{B_r(x)} |\nabla \bar{\Phi}(y)|^2 \, dy^2} = 0 \quad (II.5)$$

For any $\varepsilon > 0$ there exists $r_0 > 0$ such that for any $r < r_0$

$$\frac{\int_{B_r(x)} |\nabla \bar{\Phi}(y) - \nabla \bar{\Phi}(x)|^2 + r^{-2} |\bar{\Phi}(y) - \bar{\Phi}(x) - \partial_{x_1} \bar{\Phi}(x)(y_1 - x_1) - \partial_{x_2} \bar{\Phi}(x)(y_2 - x_2)|^2 \, dy^2}{\int_{B_r(x)} |\nabla \bar{\Phi}(y)|^2 \, dy^2} < \varepsilon^2$$

Using *Fubini theorem* together with the *mean value Theorem*, for any such $r$ there exists $\rho_{r,x} \in [r/2, r]$ such that

$$\frac{r}{2} \int_{\partial B_{\rho_{r,x}}(x)} |\nabla \bar{\Phi}(y) - \nabla \bar{\Phi}(x)|^2 + r^{-2} |\bar{\Phi}(y) - \bar{\Phi}(x) - \partial_{x_1} \bar{\Phi}(x)(y_1 - x_1) - \partial_{x_2} \bar{\Phi}(x)(y_2 - x_2)|^2 \, dl_{\partial B_{\rho_{r,x}}(x)}$$

$$< \varepsilon^2 \int_{B_r(x)} |\nabla \bar{\Phi}(y)|^2 \, dy^2 = \pi r^2 \varepsilon^2 e^{2\lambda(x)}(1 + o_\varepsilon(1)) \quad (II.6)$$
We have moreover
\[ \Gamma_{\rho} \]
Denote by \( \Gamma_{\rho} \)
Hence combining (II.7) and (II.11) we obtain
\[ \| \Phi_{\rho, \xi, 0} - \Phi_{\rho, \xi, 0} - e^{\lambda(\xi)} \rho_{\rho, \xi} [\cos \theta - 1] e_{1} + e^{\lambda(\xi)} \rho_{\rho, \xi} \sin \theta e_{2} \|_{L^{\infty}(0, 2\pi)} \]
\[
\leq \int_{0}^{2\pi} \left| \frac{\partial \Phi_{\rho, \xi, \emptyset}}{\partial \theta}(y) + \partial_{x_{1}} \Phi_{\rho, \xi, \emptyset}(x) \cos \theta \right| \, d\theta \]
\[
= \int_{\partial B_{\rho, \xi, \emptyset}(x)} \left| \frac{1}{\rho_{\rho, \xi}} \partial_{\rho, \xi} \Phi_{\rho, \xi, \emptyset}(y) + \partial_{x_{1}} \Phi_{\rho, \xi, \emptyset}(x) \cos \theta \right| \, d\theta_{\partial B_{\rho, \xi, \emptyset}} \]
\[
\leq \int_{\partial B_{\rho, \xi, \emptyset}(x)} \left| \nabla \Phi_{\rho, \xi}(y) - \nabla \Phi_{\rho, \xi}(x) \right| \, d\theta_{\partial B_{\rho, \xi, \emptyset}} < 2\pi \rho_{\rho, \xi} e^{\lambda(\xi)}(1 + o_r(1)) \]
Denote \( \tilde{L}_{\rho}(y) := \Phi_{\rho, \xi}(x) + \partial_{x_{1}} \Phi_{\rho, \xi}(x)(y_{1} - x_{1}) + \partial_{x_{2}} \Phi_{\rho, \xi}(x)(y_{2} - x_{2}) \). Considering (II.6), we have also chosen \( \rho_{\rho, \xi} \) in such a way that
\[
\left\| \Phi_{\rho, \xi, \emptyset}(y) - \tilde{L}_{\rho}(y) \right\|_{L^{\infty}(0, 2\pi)} \leq 2 \rho_{\rho, \xi}^{2} e^{\lambda(\xi)}(1 + o_r(1)) \]
We have
\[
\int_{\partial B_{\rho, \xi, \emptyset}(x)} \Phi_{\rho, \xi, \emptyset}(y) \, d\theta_{\partial B_{\rho, \xi, \emptyset}} = \frac{1}{2\pi} \int_{0}^{2\pi} \Phi_{\rho, \xi, \emptyset}(y) \, d\theta \]
where $\tilde{R}_m$ is the generalized curvature of the varifold given by $\tilde{\Phi}$ in $\mathbb{R}^m$. The generalized curvature is by definition given by

$$\int \tilde{H}_m \cdot F(q) \, d\mathcal{H}^2 = - \int \Sigma \, d(F(\tilde{\Phi})) \cdot d\tilde{\Phi} \, dvol_{g_\mathcal{E}}$$

Using the stationarity in $N^n$ we have (A.2) and then we have

$$\tilde{H}_m := - A(\tilde{\Phi}) (d\tilde{\Phi}, d\tilde{\Phi})_{g_\mathcal{E}}$$

Hence, since $|d\tilde{\Phi}|_{g_\mathcal{E}} = 1$ we deduce that $|\tilde{H}_m| \leq \|A\|_{L^\infty(N^n)}$. Inserting this bound in (II.14) we obtain

$$\rho_{r,x}^{-2} \mathcal{H}^2 \left( \Sigma_{r,x} \cap B_{e^{\lambda(x)}}(\tilde{\Phi}(x)) \right) - \pi e^{2\lambda(x)} \geq - C \rho_{r,x}^{-1} \mathcal{H}^2 \left( \Sigma_{r,x} \cap B_{e^{\lambda(x)}}(\tilde{\Phi}(x)) \right)$$

Thus

$$\rho_{r,x}^{-2} \mathcal{H}^2 \left( \Sigma_{r,x} \cap B_{e^{\lambda(x)}}(\tilde{\Phi}(x)) \right) \geq \pi e^{2\lambda(x)} (1 - C \rho_{r,x})$$

Thus having chosen $r_0 < \varepsilon^2$, we have

$$\mathcal{H}^2 \left( \Sigma_{r,x} \cap B_{e^{\lambda(x)}}(\tilde{\Phi}(x)) \right) \geq \pi e^{2\lambda(x)} \rho_{r,x}^2 (1 - C \varepsilon^2)$$

In the mean time, due to (II.4) we have

$$\mathcal{H}^2 \left( \Sigma_{r,x} \cap B_{e^{\lambda(x)}}(\tilde{\Phi}(x)) \right) \leq \frac{1}{2} \int_{B_{r}(x)} |\nabla \tilde{\Phi}(y)|^2 \, dy \leq \pi e^{2\lambda(x)} \rho_{r,x}^2 \left( 1 + C \varepsilon^2 \right)$$

The *tilt excess* of $\Sigma_{r,x}$ in the sense of Allard [1] is given by

$$E \left( \Sigma_{r,x}, e_1 \wedge e_2, \tilde{\Phi}(x), \rho_{r,x} \right) = \frac{1}{\rho_{r,x}^2} \int_{B_{r}(x) \cap \mathcal{G}} |e^{-2\lambda(y)} \partial_{x_1} \tilde{\Phi}(y) \wedge \partial_{x_2} \tilde{\Phi}(y) - e_1 \wedge e_2|^2 e^{2\lambda(y)} \, dy$$

Since we have the pointwise bound

$$2 |e^{\lambda(y)} - e^{\lambda(x)}| = ||\nabla \tilde{\Phi}(x) - \nabla \tilde{\Phi}(y)|| \leq |\nabla \tilde{\Phi}(x) - \nabla \tilde{\Phi}(y)|$$

hence we deduce from (II.6) that

$$\int_{B_{r}(x) \cap \mathcal{G}} |e^{\lambda(y)} - e^{\lambda(x)}|^2 \, dy \leq \pi e^{2\lambda(x)} \rho_{r,x}^2$$

Hence, denoting

$$E_\varepsilon := \{ y \in B_{r}(x) \cap \mathcal{G} : |\lambda(y) - \lambda(x)| > \sqrt{\varepsilon} \}$$

we have

$$|E_\varepsilon| \leq \varepsilon |B_{r}(x)|$$

Observe

$$\int_{E_\varepsilon} |\nabla \tilde{\Phi}(y)|^2 \, dy \leq 2 \int_{B_{r}(x) \cap \mathcal{G}} |\nabla \tilde{\Phi}(y) - \nabla \tilde{\Phi}(x)|^2 \, dy + 2 \int_{E_\varepsilon} |\nabla \tilde{\Phi}(x)|^2 \, dy$$

$$\leq 4 \pi \varepsilon e^{2\lambda(x)} \rho_{r,x}^2 e^{2\lambda(x)} \leq 3 \pi \varepsilon \rho_{r,x}^2 e^{2\lambda(x)}$$

4We are restricting to the integration on $B_{r}(x) \cap \mathcal{G}$ because one requires $\lambda$ to be defined as a measurable function.
Observe that we have the pointwise bound

\[|e^{-2\lambda(y)}\partial_{x}, \Phi(y) \wedge \partial_{x} \Phi(y) - \vec{e}_{1} \wedge \vec{e}_{2}|^2 e^{2\lambda(y)} = |\vec{e}_{1}(y) \wedge \partial_{x} \Phi - \vec{e}_{1}(x) \wedge \vec{e}_{2}(x) e^{\lambda(y)}|^2\]

\[\leq 2 |\vec{e}_{1}(x) - \vec{e}_{1}(y)|^2 e^{2\lambda(y)} + 2 |\vec{e}_{2}(x) - \vec{e}_{2}(y)|^2 e^{2\lambda(y)} \leq 4 |e^{\lambda(x)} - e^{\lambda(y)}|^2 + 2 |\nabla \Phi(x) - \nabla \Phi(y)|^2\]

\[\leq 4 |\nabla \Phi(x) - \nabla \Phi(y)|^2\]

where we have also used (II.19). We have using one more time (II.6) together with (II.21) and the previous pointwise inequality

\[E \left( \Sigma_{\rho, x}, \vec{e}_{1} \wedge \vec{e}_{2}, \Phi(x), \rho_{r, x} \right) \leq \frac{4}{\rho_{r, x}^2} \int_{E_{r}} |\nabla \Phi(y)|^2 \, dy^2\]

\[+ \frac{1}{\rho_{r, x}^2} \int_{B_{r, x}(x) \cap \Phi \setminus E_{r}} |e^{-2\lambda(y)} \partial_{x} \Phi(y) \wedge \partial_{x} \Phi(y) - \vec{e}_{1} \wedge \vec{e}_{2}|^2 e^{2\lambda(y)} \, dy^2\]

\[\leq 12 \pi \varepsilon A^2 + \frac{1}{\rho_{r, x}^2} \int_{B_{r, x}(x)} |\nabla \Phi(y)|^2 \, dy^2 \leq 14 \pi \varepsilon A^2\]  \hspace{1cm} (II.22)

where we recall that \(x\) has been chosen in such a way that \(2 e^{2\lambda(x)} \leq A^2\). Combining (II.18) and (II.22) we can apply Allard main regularity result and precisely we obtain for \(\varepsilon\) small enough the existence of \(\gamma < 1\) such that \(S_{r, x} := \Sigma_{\rho, x} \cap B_{r, x}^{m} (\Phi(x))\) is a smooth minimal sub-manifold. Moreover, because of the upper semi-continuity of the density function for a stationarity varifold we can assume that all points \(\vec{q} \in S_{r, x}\) have multiplicity 1 (See for instance the presentation of the absence of hole in \([2]\) section 7). In other words we have that

\[\int_{B_{r, x}^{m} (\Phi(x))} dH^{2} \Sigma_{\rho, x} = \text{Area}(S_{r, x}) = \int_{\Phi^{-1}(S_{r, x})} dvol_{g_{k}} = \frac{1}{2} \int_{\Phi^{-1}(S_{r, x})} |\nabla \Phi|^{2} \, dx^2\]  \hspace{1cm} (II.23)

and, in the area formula, the counting function

\[\mathcal{H}^{0}(\Phi^{-1}(S_{r, x}) \cap \Phi^{-1}\{\vec{q}\}) = 1 \quad \text{for} \quad \mathcal{H}^{2} = \text{a.e.} \quad \vec{q} \in S_{r, x}\]  \hspace{1cm} (II.24)

We claim that for \(s\) small enough \(\Phi(B_{s}(x)) \subset S_{r, x}\). Assuming there is point \(\vec{p} \notin B_{r, x}^{m} (\Phi(x))\) but \(\vec{p} \in \Phi(B_{s}(x))\). As \(s\) goes to zero we have in one hand, using (II.6),

\[\int_{B_{s}(x)} |\nabla \Phi|^{2} \, dy^2 \leq 2 \pi s^2 (1 + \varepsilon) e^{2\lambda(x)}\]  \hspace{1cm} (II.25)

but the stationarity of \(\Sigma_{\rho, x} := \Phi(B_{s, r}(x)) \cap \Gamma_{r, x}^{\varepsilon} \cap \Pi_{\rho, x}^{\varepsilon}\) together with the monotonicity formula would imply

\[\pi(\gamma r - s)^2 e^{2\lambda(x)} (1 - o_{r}(1)) \leq \pi(\text{dist}(\vec{p}, \Gamma_{r, x}^{\varepsilon}) (1 - o_{r}(1)) \leq \mathcal{H}^{2}(\Sigma_{\rho, x}) \leq \frac{1}{2} \int_{B_{s}(x)} |\nabla \Phi|^{2} \, dy^2\]

which contradicts (II.25) for \(s\) very small compared to \(r\). Hence, for \(s\) small enough

\[\Phi(B_{s}(x)) \subset S_{r, x} = \Sigma_{\rho, x} \cap B_{s}^{m} (\Phi(x))\]

\(^{5}\)Observe that based on similar arguments one could prove directly that \(\Phi\) is continuous on the whole \(\Sigma\) without using Allard’s result but this is not needed.
Let $\Psi$ be a smooth conformal diffeomorphism from $\Sigma_{\rho,z} \cap B_{m}^{\infty}(\Phi(x))$ into the disc $D^{2}$. Denote by $e^{2\mu} \ dz^{2} = \Psi^{*}g_{\Sigma}$ and consider $f := \Psi \circ \Phi$. This is in particular a $W^{1,2}$ weakly conformal map from $B_{s}(x)$ into $\mathbb{C}$. Consider any map $g \in W^{1,2}(B_{\rho},(x))$ such that $g = f$ on $\partial B_{\rho}(x)$. The union of $f$ and $g$ realizes a $W^{1,2} \cap L^{\infty}$ map from $S^{2}$ into $\mathbb{C}$ that we denote $\tilde{f}$. The cycle $\tilde{f}[S^{2}]$ has then an algebraic covering number equal to 0 $\mathcal{H}^{2}$-almost everywhere in $\mathbb{C}$ that is to say\footnote{Indeed due to the $W^{1,2}$ nature of $f$ and since we are in 2 dimension we have that $d$ and $\ast$ commute (this is clearly not the case in higher dimension) and we have in particular $f^{*}dz_{1} \wedge dz_{2} = d(f_{1}df_{2})$}
\[
\int_{S^{2}} \tilde{f}^{*}dz_{1} \wedge dz_{2} = 0.
\]

Away from the compact 1 rectifiable set $f(\partial B_{\rho}(x))$, because of $\text{Lemma III.2}$, the image $f(B_{\rho}(x))$ has covering number $+1$, $-1$, or 0. Hence, $g$ must have an odd covering number almost everywhere in $\mathbb{C} \setminus f(\partial B_{\rho}(x))$ whenever the covering number of $f$ is non zero. This homological fact implies
\[
\frac{1}{2} \int_{B_{\rho}(x)} e^{2\mu} |\nabla f|^{2} \ dx^{2} = \int_{B_{\rho}(x)} e^{2\mu} |\partial_{x_{1}}f \times \partial_{x_{2}}f| \ dx_{1} \wedge dx_{2} \\
\leq \int_{B_{\rho}(x)} e^{2\mu} |\partial_{x_{1}}g \times \partial_{x_{2}}g| \ dx_{1} \wedge dx_{2} \leq \frac{1}{2} \int_{B_{\rho}(x)} e^{2\mu} |\nabla g|^{2} \ dx^{2}
\]

Hence by the Dirichlet principle $f$ coincides with it’s harmonic extension\footnote{The harmonic extension is unique for r small enough since one is taking value in a convex geodesic ball of $(D^{2}, e^{2\mu} dz^{2})$.} in $(D^{2}, e^{2\mu} dz^{2})$. The map $\Psi \circ \Phi$ is then smooth (holomorphic or anti-holomorphic) on $B_{\rho}(x))$. This implies Lemma $\text{III.2}$ \hfill $\Box$

### III The Harmonicity of Conformal Target Harmonic Maps

#### III.1 The integration by part formula.

The goal of the present subsection is to prove the following integration by parts formula.

**Lemma III.2.** Let $F(\vec{p}) = (F_{ij}(\vec{p}))_{1 \leq i \leq m, 1 \leq j \leq k}$ be a $C^{1}$ map on $N^{n}$ taking values into $m \times k$ real matrices. Then, for any smooth compactly supported function $\phi$ in $D^{2}$ and for almost every regular value $t > 0$ of $\phi$ one has
\[
\sum_{i=1}^{m} \int_{\Omega_{t}} F(\vec{\Phi})_{ij} \frac{\partial \vec{\Phi}_{i}}{\partial \nu} \ dx + \int_{\Omega_{t}} \nabla F(\vec{\Phi})_{ij} \nabla \vec{\Phi}_{i} \ dy^{2} + \int_{\Omega_{t}} F(\vec{\Phi})_{ij} A_{i}(\vec{\Phi})(d\vec{\Phi}, d\vec{\Phi}) \ dy^{2} = 0 \tag{III.26}
\]
where $\Omega_{t} = \phi^{-1}((t, +\infty))$ and $\nu$ is the exterior unit normal to the level set $\nu := \nabla \phi/|\nabla \phi|$. \hfill $\Box$

In order to prove lemma $\text{III.2}$ we will first establish some intermediate results.

Let $x \in D^{2}$ and choose $r$ such that
\[
\frac{1}{2} \int_{r-\varepsilon}^{r+\varepsilon} ds \int_{d\partial B_{r}(x)} |\nabla \vec{\Phi}|^{2} \ dx_{1} \wedge dx_{2} = \int_{d\partial B_{r}(x)} |\nabla \vec{\Phi}|^{2} \ dx_{1} \wedge dx_{2} < +\infty \tag{III.27}
\]

Hence the restriction of $\vec{\Phi}$ to $\partial B_{r}(x)$ is $W^{1,2}$ and the continuous image of $\partial B_{r}(x)$ by $\vec{\Phi}$, $\Gamma_{r,x} := \vec{\Phi}(\partial B_{r}(x))$ has finite length and is rectifiable and compact.

We denote $B := D^{2} \setminus G$. Because of the previous lemma $\text{III.1}$ we have that $B$ is closed. Hence
\[
B = \bigcap_{\varepsilon > 0} B_{\varepsilon} \quad \text{where} \quad B_{\varepsilon} := \{x \in D^{2} : \text{dist}(x, B) \leq \varepsilon\}
\]
Since the integral of $|\nabla \tilde{\Phi}|^2$ over $\mathcal{B}$ is zero, we clearly have
\[
\lim_{\varepsilon \to 0} \int_{\mathcal{B}_\varepsilon} |\nabla \tilde{\Phi}|^2(y) \, dy = 0 \tag{III.28}
\]
We denote also
\[
\mathcal{G}_\varepsilon := D^2 \setminus \mathcal{B}_\varepsilon
\]
Finally we denote by $\mathfrak{B}_\varepsilon$ the rectifiable image of $\mathcal{B}_\varepsilon$ by $\tilde{\Phi}$ that is the image by the approximate continuous representative of $\tilde{\Phi}$ of the intersection of $\mathcal{B}_\varepsilon$ with the points of approximate differentiability of $\tilde{\Phi}$. We claim the following.

**Lemma III.3.** Under the previous notations we have
\[
\lim_{\varepsilon \to 0} \int_{D^2 \cap \tilde{\Phi}^{-1}(\mathfrak{B}_\varepsilon)} |\nabla \tilde{\Phi}|^2(y) \, dy = 0 \tag{III.29}
\]
**Proof of lemma III.3.** Identity (III.28) implies that
\[
\lim_{\varepsilon \to 0} \mathcal{H}^2(\mathfrak{B}_\varepsilon) = 0 \tag{III.30}
\]
Hence for any $\delta > 0$ there exists $\varepsilon_\delta > 0$ and for any $\varepsilon < \varepsilon_\delta$ there exists a covering of $\mathfrak{B}_\varepsilon$ by balls $(B_{\varepsilon}^{m_s}(\tilde{p}_i))_{i \in \Lambda}$ such that
\[
\sum_{i \in \Lambda} r_i^2 < \delta
\]
Using the monotonicity formula for stationary varifolds we obtain
\[
\int_{D^2 \cap \tilde{\Phi}^{-1}(\mathfrak{B}_\varepsilon)} |\nabla \tilde{\Phi}|^2(y) \, dy \leq \sum_{i \in I} \int_{\tilde{\Phi}^{-1}(B_{\varepsilon}^{m_s}(\tilde{p}_i))} |\nabla \tilde{\Phi}|^2(y) \, dy \leq C \sum_{i \in \Lambda} r_i^2 < C \delta
\]
which implies the lemma. $\square$

We shall prove the following.

**Lemma III.4.** Under the previous notations we have
\[
\lim_{s \to 0} \int_{\mathcal{G}_\varepsilon} \left[ 1 + |\nabla \tilde{\Phi}|^2(y) \right] \frac{\mathcal{H}^2(\mathfrak{B}_\varepsilon \cap B_{\varepsilon}^{m_s}(\tilde{\Phi}(y)))}{\pi s^2} - \theta^2_\varepsilon(\tilde{\Phi}(y)) \, dy^2 = 0 \tag{III.31}
\]
where
\[
\theta^2_\varepsilon(\tilde{p}) := \lim_{s \to 0} \frac{\mathcal{H}^2(\mathfrak{B}_\varepsilon \cap B_{\varepsilon}^{m_s}(\tilde{p}))}{\pi s^2}
\]
exists $\mathcal{H}^2$ almost everywhere since $\mathfrak{B}_\varepsilon \subset \cup_{\eta > 0} \tilde{\Phi}(\mathcal{G}_\eta)$ is 2-rectifiable. Moreover since $\tilde{\Phi}$ is a smooth immersion on $\mathcal{G}_\varepsilon$, $\theta^2_\varepsilon(\tilde{\Phi}(y))$ is a well defined measurable function on $\mathcal{G}_\varepsilon$.

**Proof of lemma III.4.** Since $|\nabla \tilde{\Phi}|$ is uniformly bounded on $\mathcal{G}_\varepsilon$ we have
\[
\int_{\mathcal{G}_\varepsilon} \left[ 1 + |\nabla \tilde{\Phi}|^2(y) \right] \frac{\mathcal{H}^2(\mathfrak{B}_\varepsilon \cap B_{\varepsilon}^{m_s}(\tilde{\Phi}(y)))}{\pi s^2} - \theta^2_\varepsilon(\tilde{\Phi}(y)) \, dy^2 \\
\leq \left[ 1 + \|\nabla \tilde{\Phi}\|_{L^2(\mathcal{G}_\varepsilon)} \right] \int_{\mathcal{G}_\varepsilon} \frac{\mathcal{H}^2(\mathfrak{B}_\varepsilon \cap B_{\varepsilon}^{m_s}(\tilde{\Phi}(y)))}{\pi s^2} - \theta^2_\varepsilon(\tilde{\Phi}(y)) \, dy^2
\]
\[8\]
Since $\Phi$ is a smooth immersion on $G_\varepsilon$, $\theta^2_\varepsilon(\Phi(y))$ is a well defined measurable function on $G_\varepsilon$ and we have for almost $y$

$$\lim_{s \to 0} \frac{\mathcal{H}^2 \left( \mathcal{B}_\varepsilon \cap B_s^m(\Phi(y)) \right)}{\pi s^2} = \theta^2_\varepsilon(\Phi(y))$$

The monotonicity formula for the stationary varifold given by the image of $D^2$ by $\Phi$ gives

$$\sup_{s > 0; \ y \in D^2} \frac{\mathcal{H}^2 \left( \mathcal{B}_\varepsilon \cap B_s^m(\Phi(y)) \right)}{\pi s^2} \leq \sup_{s > 0; \ y \in D^2} \frac{\mathcal{H}^2 \left( \Phi(D^2) \cap B_s^m(\Phi(y)) \right)}{\pi s^2} \leq C$$

for some $C$. The lemma follows by a direct application of dominated convergence. □

**Proof of lemma III.2** In order to simplify the presentation we shall restrict to $\Omega$ to be balls. Because of (III.29), for every $x \in D^2$ and almost every $r > 0$ we have

$$\lim_{t \to 0} \int_{\mathcal{G} \cap \partial B_r(x)} |\nabla\Phi|(y) \, d\mathcal{H}_N = 0$$

(III.32)

We choose $r$ such that (III.32) holds true and such that also

$$\lim_{t \to 0} \int_{\mathcal{G} \cap \partial B_r(x)} \left[ 1 + |\nabla\Phi|^2(y) \right] \left( \frac{\mathcal{H}^2 \left( \mathcal{B}_\varepsilon \cap B_s^m(\Phi(y)) \right)}{\pi s^2} - \theta^2_\varepsilon(\Phi(y)) \right) d\mathcal{H}_N = 0 \ .$$

(III.33)

Hence in particular $\mathcal{H}^1(\Phi(B_\varepsilon \cap \partial B_r(x)))$ is converging to zero as $\varepsilon$ goes to zero. Since $\Phi$ is continuous on $\partial B_r(x)$ and since $B_\varepsilon$ is a closed set we have that $\Phi(B_\varepsilon \cap \partial B_r(x))$ is a compact subset of $N^m$. Because of the previous, for any $\delta > 0$, there exists $\varepsilon_\delta > 0$ such that for any $\varepsilon < \varepsilon_\delta$, we can include $\Phi(B_\varepsilon \cap \partial B_r(x))$ in finitely many balls $(B_{r_i}^m(\tilde{p}_i))_{i \in \Lambda}$ such that

$$\sum_{i \in \Lambda} r_i < \delta \ .$$

(III.34)

We also choose $\varepsilon_\delta > 0$ such that for any $\varepsilon < \varepsilon_\delta$

$$\int_{\mathcal{G} \cap \partial B_r(x)} |\nabla\Phi|(y) \, d\mathcal{H}_N < \delta \ .$$

(III.35)

Let $\chi$ be a cut-off function on $\mathbb{R}_+$ such that $\chi \equiv 1$ on $[2, +\infty)$ and $\chi \equiv 0$ on $[0, 1]$. We introduce

$$\xi_\varepsilon(\Phi(y)) := \prod_{i \in \Lambda} \chi \left( \frac{|\Phi(y) - \tilde{p}_i|}{r_i} \right)$$

Observe that $\xi_\varepsilon(\Phi(y))$ is zero on $B_\varepsilon \cap \partial B_r(x)$. For any $s > 0$ we also introduce

$$\eta_s(\Phi(y)) := \chi \left( \frac{\text{dist}(\Phi(y), \Gamma_{r,x})}{s} \right)$$

where we recall that $\Gamma_{r,x} := \Phi(\partial B_r(x))$. Using the assumption III.1 we have for any $F$ as in the statement of the lemma

$$\sum_{i=1}^m - \int_{B_r(x)} \nabla(\xi_\varepsilon(\Phi) \eta_s(\Phi) F(\Phi)_{ij}) \nabla\Phi_i \, dy + \int_{B_r(x)} \xi_\varepsilon(\Phi) \eta_s(\Phi) F(\Phi)_{ij} A_i(\Phi) (d\Phi, d\Phi) \, dy = 0$$

(III.36)
We have
\[
\left| \int_{B_r(x)} \nabla (\xi_c(\Phi)) \eta_s(\Phi) F(\Phi)_{ij} \nabla \Phi_i \, dy \right| \leq C \|F\| \sum_{i \in \Lambda} \frac{1}{r_l} \int_{B_r(x)} |x' - \frac{\Phi(y) - x}{r_l}| |\nabla \Phi|^2(y) \, dy
\]
\[
\leq C \|F\| \sum_{i \in \Lambda} r_l^{-1} \int_{\partial B_r(x)} |\nabla \Phi|^2(y) \, dy \leq C \|F\| \sum_{i \in \Lambda} r_l \leq C \|F\| \delta
\]
where we observe that the bound is independent of $s$. We now write
\[
\int_{B_r(x)} \xi_c(\Phi) \nabla (\eta_s(\Phi)) F(\Phi)_{ij} \nabla \Phi_i \, dy = \int_{B_r(x) \cap B_s} \cdots + \int_{B_r(x) \cap G_e} \cdots \quad (\text{III.38})
\]
We have
\[
\left| \int_{B_r(x) \cap B_s} \xi_c(\Phi) \nabla (\eta_s(\Phi)) F(\Phi)_{ij} \nabla \Phi_i \, dy \right| \leq \frac{C \|F\|}{s} \int_{B_r(x) \cap B_s} 1_{\text{dist}(\Phi(y), \Gamma_r,s) < s} |\nabla \Phi|^2(y) \, dy
\]
(\text{III.39})

Where $\Gamma_r,s$ is the smooth immersed curve $\Phi(\partial B_r(x)) \cap G_e$ and $1_{\text{dist}(\Phi(y), \Gamma_r,s) < s}$ is the characteristic function of the set of $y$ such that $\Phi(y)$ is at the distance at most $s$ to $\Gamma_r,s$. The fact that we can restrict to $\Gamma_r,s$ instead of $\Gamma_r,s$ is due to the fact that we are cutting off $\Gamma_r,s \setminus \Gamma_r,s$ by multiplying by $\xi_c(\Phi)$. Observe that since the curve $\Gamma_r,s$ is a smooth immersion of the open subset of $\partial B_r(x)$ given by $\partial B_r(x) \cap G_e$ we have for $s$ small enough
\[
1_{\text{dist}(\Phi, \Gamma_r,s) < s} \leq \frac{1}{s} \int_{\Gamma_r,s} 1_{\text{dist}(\Phi, \Gamma_r,s) < 2s} \, d\mathcal{H}^1(y)
\]
(\text{III.40})

Inserting this inequality in (\text{III.39}) gives
\[
\left| \int_{B_r(x) \cap B_s} \xi_c(\Phi) \nabla (\eta_s(\Phi)) F(\Phi)_{ij} \nabla \Phi_i \, dy \right| \leq \frac{C \|F\|}{s^2} \int_{\partial B_r(x) \cap G_e} |\nabla \Phi|(y) \, d\partial B_r \int_{B_r(x) \cap B_s} 1_{\text{dist}(\Phi(y), \Phi(z)) < 2s} |\nabla \Phi|^2(y) \, dy
\]
(\text{III.41})

Using (\text{III.33}) we then obtain
\[
\limsup_{s \to 0} \left| \int_{B_r(x) \cap B_s} \xi_c(\Phi) \nabla (\eta_s(\Phi)) F(\Phi)_{ij} \nabla \Phi_i \, dy \right| \leq C \|F\| \int_{\partial B_r(x) \cap G_e} |\nabla \Phi|(z) \, d\partial B_r(z)
\]
(\text{III.42})
and using the uniform bound on the density which itself comes from the monotonicity formula

$$\limsup_{s \to 0} \left| \int_{B_r(x) \cap B_s} \xi_s(\Phi) \nabla (\eta_s(\Phi)) F(\Phi)_{ij} \nabla \Phi_i \ dy \right|$$ (III.43)

$$\leq C \| F \|_\infty \int_{\partial B_r(x) \cap \mathcal{G}_s \cap \partial \mathcal{G}_-^{-1}(\mathcal{A}_s)} |\nabla \Phi|^2 (z) \ d\sigma_{\partial B_r}(z) \leq C \| F \|_\infty \delta$$

where we have used (III.55). Observe that since we have cut-off \( \Gamma_{r,x} \setminus \Gamma_{r,x}^c \) by multiplying by \( \xi_s(\Phi) \) we have for \( s \) small enough

$$\lim_{s \to 0} \int_{B_r(x) \cap \mathcal{G}_s} \xi_s(\Phi) \nabla (\eta_s(\Phi)) F(\Phi)_{ij} \nabla \Phi_i \ dy^2$$ (III.44)

$$\lim_{s \to 0} \int_{B_r(x) \cap \mathcal{G}_s} \xi_s(\Phi) \nabla \left( \eta \left( \frac{\text{dist}(\Phi(y), \Gamma_{r,x}^c)}{s} \right) \right) F(\Phi)_{ij} \nabla \Phi_i \ dy^2$$

Using the coarea formula this gives

$$\lim_{s \to 0} \sum_{i=1}^m \int_{B_r(x) \cap \mathcal{G}_s} \xi_s(\Phi) \nabla \left( \eta \left( \frac{\text{dist}(\Phi(y), \Gamma_{r,x}^c)}{s} \right) \right) F(\Phi)_{ij} \nabla \Phi_i \ dy^2$$ (III.45)

$$= \lim_{s \to 0} \frac{1}{s} \int_0^{2s} \chi'(\frac{\sigma}{s}) \ ds \int_{\text{dist}(\Phi(\gamma), \Gamma_{r,x}^c) = \sigma} \xi_s(\Phi) F_j(\Phi) \cdot \nu \ d\mathcal{H}^1 \mathcal{S}_{r,x}^c$$

where \( \mathcal{S}_{r,x}^c \) is the immersed sub-manifold \( \Phi(B_r(x)) \cap \mathcal{G}_s \) and \( \nu \) is the unit exterior vector in this sub-manifold orthogonal to the level set \( \text{dist}(\overline{\Phi}, \Gamma_{r,x}^c) = \sigma \) and \( F_j(\Phi) \cdot \nu = \sum_{i=1}^m F_{ij}(\Phi) \nu_i \). Since \( \Phi \) is an immersion in a neighborhood of \( \partial B_r \cap \mathcal{G}_s \), for \( s \) small enough and being a regular value of \( f_{r,x}(\Phi) := \text{dist}(\overline{\Phi}, \Gamma_{r,x}^c) \) the level set is made of the following union

$$\left( f_{r,x}^c \right)^{-1}(\sigma) = \gamma_{r,x}^c(\sigma) \cup_{\alpha \in A_s} \partial \omega_{\alpha}(\sigma)$$

where \( \gamma_{r,x}^c(\sigma) \) is a smooth curve converging to \( \Gamma_{r,x}^c \) and \( \omega_{\alpha}(\sigma) \) are subdomains of \( B_r(x) \) included in \( \text{dist}(\Phi(y), \Gamma_{r,x}^c)^{-1}(\{0, \sigma\}) \). The Taylor expansion of \( \Phi \) with respect to each point \( y \in \partial B_r(x) \) gives

$$\lim_{\sigma \to 0} \int_{\gamma_{r,x}^c(\sigma)} \xi_s(\Phi) F_j(\Phi) \cdot \nu \ d\mathcal{H}^1 \mathcal{S}_{r,x}^c = \int_{\partial B_r(x) \cap \mathcal{G}_s} \xi_s(\Phi) F_j(\Phi) \frac{\partial \Phi_i}{\partial r} \ d\sigma_{\partial B_r}$$ (III.46)

Hence

$$\lim_{s \to 0} \frac{1}{s} \int_0^{2s} \chi'(\frac{\sigma}{s}) \ ds \int_{\gamma_{r,x}^c(\sigma)} \xi_s(\Phi) F_j(\Phi) \cdot \nu \ d\mathcal{H}^1 \mathcal{S}_{r,x}^c$$ (III.47)

$$= \int_{\partial B_r(x) \cap \mathcal{G}_s} \xi_s(\Phi) \sum_{i=1}^m F_{ij}(\Phi) \frac{\partial \Phi_i}{\partial r} \ d\sigma_{\partial B_r}$$

For the other contributions we have

$$\lim_{s \to 0} \frac{1}{s} \int_0^{2s} \chi'(\frac{\sigma}{s}) \ ds \sum_{\alpha \in A_s} \int_{\partial \omega_{\alpha}(\sigma)} \xi_s(\Phi) F_j(\Phi) \cdot \nu \ d\mathcal{H}^1 \mathcal{S}_{r,x}^c$$ (III.48)

$$\lim_{s \to 0} \frac{1}{s} \int_0^{2s} \chi'(\frac{\sigma}{s}) \ ds \sum_{\alpha \in A_s} \int_{\omega_{\alpha}(\sigma)} \text{div}_{\gamma_{r,x}^c} (\xi_s(\Phi) F_j(\Phi)) \ d\mathcal{H}^2 \mathcal{S}_{r,x}^c$$
Since $\omega_\sigma(\sigma)$ is included in a $\sigma$ neighborhood of the smooth curve $\Gamma_{r,x}^\varepsilon$, using the monotonicity formula, for $\sigma$ small enough, covering such a neighborhood by $\simeq \sigma^{-1} \mathcal{H}^1(\Gamma_{r,x}^\varepsilon)$ balls of radius $2\sigma$ we have the following bound

$$\sum_{\alpha \in A_x} \mathcal{H}^2(\omega_\alpha(\sigma)) \leq C \sigma \mathcal{H}^1(\Gamma_{r,x}^\varepsilon) \sup_{t,\overline{\nu}} \frac{\mathcal{H}^2(\Sigma_{r,x}^\varepsilon \cap B_t^m(\overline{\nu}))}{t^2} \leq C\varepsilon \sigma$$  (III.49)

Hence we have

$$\lim_{s \to 0} \frac{1}{s} \int_0^{2s} x' \left( \frac{a}{s} \right) \, da \sum_{\alpha \in A_x} \int_{\partial \omega_\alpha(\sigma)} \xi_\varepsilon(\overline{\nu}) F_j(\overline{\nu}) \cdot \nu \, d\mathcal{H}^1 \Sigma_{r,x}^\varepsilon \leq \lim_{s \to 0} \frac{C\varepsilon}{s} \int_0^{2s} \|\nabla \Sigma_{r,x}^\varepsilon (\xi_\varepsilon(\overline{\nu}) F_j(\overline{\nu})) \|_{L^\infty(\Sigma_{r,x}^\varepsilon)} \sigma \, da \leq C\varepsilon \sigma$$  (III.50)

Combining (III.44) ... (III.50) we obtain

$$\lim_{s \to 0} \int_{B_r(x) \cap \varnothing_x} \xi_\varepsilon(\overline{\nu}) \nabla(\eta_\varepsilon(\overline{\nu})) F(\overline{\nu})_{ij} \nabla \overline{\nu}_i \, dy^2 = \int_{\partial B_r(x) \cap \varnothing_x} \xi_\varepsilon(\overline{\nu}) \sum_{i=1}^m F_{ij}(\overline{\nu}) \frac{\partial \overline{\nu}_i}{\partial r} \, dl_{\partial B_r}$$  (III.51)

Collecting (III.35), (III.37), (III.43) and (III.51) we obtain

$$\left| \sum_{i=1}^m \int_{\partial B_r(x)} F(\overline{\nu})_{ij} \frac{\partial \overline{\nu}_i}{\partial r} \, dl_{\partial B_r} - \int_{B_r(x)} \nabla(F(\overline{\nu})_{ij}) \nabla \overline{\nu}_i \, dy^2 + \int_{B_r(x)} F(\overline{\nu})_{ij} A_i(\overline{\nu})(d\overline{\nu},d\overline{\nu}) \right| < C\|F\|_{L^\infty} \delta$$  (III.52)

This holds for any $\delta > 0$ and hence lemma [1.12] is proved.

### III.2 Proof of theorem [1.1]

Let $\varphi \in C_0^\infty(D^2)$, be a non negative function. The co-area formula gives

$$\int_{D^2} \Delta \varphi \overline{\Phi} \, dx^2 = - \int_{D^2} \nabla \varphi \cdot \nabla \overline{\Phi} = \int_0^{+\infty} dt \int_{\varphi^{-1}(t)} \partial_t \varphi \, dl_{\varphi^{-1}(t)}$$  (III.53)

Using lemma [1.2] for $F_{ij} := \delta_{ij}$, we obtain

$$\int_{D^2} \Delta \varphi \overline{\Phi} \, dx^2 = - \int_0^{+\infty} dt \int_{\varphi^{-1}(t,+\infty)} A(\overline{\Phi})(d\overline{\nu},d\overline{\nu}) \, dy^2$$  (III.54)

This implies that $\overline{\Phi}$ satisfies weakly the harmonic map equation into $\mathbb{N}^n$

$$-\Delta \overline{\Phi} = A(\overline{\Phi})(d\overline{\nu},d\overline{\nu}) \quad \text{in } \mathcal{D}'(D^2)$$

and using Hélein’s regularity result [3], we prove theorem [1.1].
A Appendix

Proposition A.1. Let \( N^n \) be a \( C^2 \) sub-manifold of the euclidian space \( \mathbb{R}^m \). Let \((\Sigma, h)\) be a compact Riemann surface (equipped with a metric compatible with the complex structure) possibly with boundary. Let \( \Phi \) be a map in \( W^{1,2}(\Sigma, N^n) \). Assume \( \Phi \) is weakly conformal and continuous on \( \partial \Sigma \). The integer rectifiable varifold associated to \((\Phi, \Sigma)\) is stationary in \( N^n \setminus \Phi(\partial \Sigma) \) if and only if

\[
\forall \ F \in C^\infty_0(N^n \setminus \Phi(\partial \Sigma), \mathbb{R}^m) \quad \int_\Sigma \left[ \left( d(F(\Phi)), d\Phi \right)_h - F(\Phi) A(\Phi)(d\Phi, d\Phi)_h \right] \ dvol_h = 0
\]

where \( A(\Phi)(d\Phi, d\Phi)_h \) denotes the second fundamental form of \( N^n \) at the point \( \Phi \) and acting on the pair of vectors \((X, Y)\) and by an abuse of notation we write

\[
A(\Phi)(d\Phi, d\Phi)_h := \sum_{i,j=1}^2 h_{ij} A(\Phi)(\partial_x_i \Phi, \partial_x_j \Phi) .
\]

Proof of proposition [A.1]. For any \( \Phi \in N^n \) one denotes by \( P_T(\Phi) \) the symmetric matrix giving the orthogonal projection onto \( T_\Phi N^n \). The integer rectifiable varifold given by \((\Phi, \Sigma)\) is by definition the following Radon measure on \( G_2(T\mathbb{R}^m) \) the Grassman bundle of un-oriented 2-planes over \( \mathbb{R}^m \) given by

\[
\forall \ (S, q) \in G_2(T\mathbb{R}^m) \quad A(\Phi)(d\Phi, d\Phi)_h := \sum_{i,j=1}^2 h_{ij} A(\Phi)(\partial_x_i \Phi, \partial_x_j \Phi) .
\]

By definition (see [1]), the varifold \( v_\Phi \) is stationary in \( N^n \) if

\[
\forall \ F \in C^\infty_0(N^n \setminus \Phi(\partial \Sigma), \mathbb{R}^m) \quad \int_\Sigma \nabla S(P_T F)(\Phi) \ dvol_\Phi(S, q) = 0 \tag{A.1}
\]

In local conformal coordinates at a point where \( |\partial_x_i \Phi| = |\partial_x_j \Phi| = e^{\lambda} \), introducing the orthonormal basis of \( S := \Phi_* T_\Sigma \) given by \( e_i := e^{-\lambda} \partial_x_i \Phi \), one has by definition

\[
\nabla v_{\Phi_* T_\Sigma}(P_T F)(\Phi) := \sum_{i=1}^2 \partial_{x_i} (P_T F)(\Phi) \cdot e_i = \sum_{i=1}^m \sum_{k=1}^m e_i^k \partial_{x_k} (P_T F)(\Phi) \cdot e_i
\]

where \( e_i := \sum_{k=1}^m e_i^k \partial_{x_k} \). Hence we have

\[
\nabla v_{\Phi_* T_\Sigma}(P_T F)(\Phi) = e^{-2\lambda} \sum_{i=1}^2 \sum_{k=1}^m e_i^k \partial_{x_k} (P_T F)(\Phi) \cdot \partial_{x_i} \Phi
\]

where we used respectively that \( P_T(\Phi) \nabla \Phi = \nabla \Phi \) and that \( A(\Phi)(\nabla \Phi, \nabla \Phi) = -\partial_x P_T(\Phi) \cdot \nabla \Phi \). Multiplying by \( dvol_\Phi = e^{2\lambda} dx_1 \wedge dx_2 \) we obtain at almost every point \( x \) where \( \nabla \Phi(x) \neq 0 \)

\[
\nabla v_{\Phi_* T_\Sigma}(P_T F)(\Phi) \ dvol_\Phi = \left[ \left( d(F(\Phi)), d\Phi \right)_h - F(\Phi) A(\Phi)(d\Phi, d\Phi)_h \right] \ dvol_\Phi \tag{A.2}
\]

This concludes the proof of the lemma. \( \square \)
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