Thermal buckling of ballasted tangent track

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Abstract
Euler type analysis usually used for compression members in structural engineering does not work for railroad track. Euler type analytical formulas for horizontal and vertical buckling endorsed in a recent literature is reviewed to demonstrate its weakness. Using definition of moment and curvature as well as principle of equilibrium, the author suggests formula for horizontal buckling load of railroad track and demonstrates validation in context with currently accepted values, published results, and past field tests. The buckling load from suggested formula agrees with the recent buckling load formula based on total energy theorem. A formula is suggested to study the effect of misalignment on critical temperature differential or critical load. A vertical buckling load formula is derived from horizontal buckling load formula. A buckling process is narrated through step by step computation. Formulas are suggested to compute the effect of track misalignment on critical buckling load and threshold radius of a vertical curve.

Keywords
Ballast resistance, critical load, critical temperature differential, critical alignment defect, Euler buckling load, stress neutral temperature, thermal buckling, thermal load, wave length of buckled track, amplitude of buckled track

Introduction
In terms of analysis, buckling in a railroad track is a quite complex phenomenon due to the following:

- The “end” conditions of the rails are neither perfectly fixed nor perfectly pinned nor free, but provide varying rotational and lateral resistance depending on numerous factors,
- Resistance to lateral displacement is provided by the lateral stiffness of rails, track frame, and ballast resistance,
- Railroad track invariably contains initial curvature and imperfections,
- Axial loading through the centroid of the track would seldom occur etc.

Literature review of track buckling field tests is done to present the track buckling load, amplitude of deflection etc. The information from these field tests are subsequently used to validate assumption used, findings made, and formula derived by the author.

Kerr detailed the shortcomings of field tests and analytical formula. Kerr stated that the field test spans were short. The author established that field tests spans are not short to determine buckling load only, rather they are short to capture the whole buckling process. The author suggests two field test span – one to determine only buckling load and the other, to capture both buckling load and modal shape of buckling.
Since the early 1930s till 1978, many track stability analyses were published but no reliable analyses are available to compute the buckling load or buckling temperature differential of a railroad track. The linear differential equation used to describe lateral response of a buckled track is based on the plane section hypothesis whereas in an actual track this assumption is not satisfied, because the ties and fasteners are not rigid. Euler-type analysis is applied on compression members; if the length between two supports increases buckling load decreases. In case of track, lateral displacement is resisted by ballast resistance and the buckling load remains same whatever be the track length. The length of a test track should be sum of length of buckled track and breathing length at both ends of buckled track. The actual buckling load is supposed to be more than the buckling load in the field if the test track is too short. Thus, a minimum length is required for a test track; buckling load would be same for all test spans beyond the minimum length as buckling in railroad track is essentially a local phenomenon. A sufficiently long test track as suggested by Kerr would exhibit real life modal shapes in addition to buckling load. In 1932, A. Bloch showed that the analysis needed for thermal track buckling is different than the Euler-type analysis usually utilized in structural mechanics.

A formula based on total energy theorem from recent literature is found to offer acceptable value of critical buckling load. Two analytical formulas for horizontal buckling from recent literature are reviewed – Euler type one is found to be unacceptable and the other based on energy theorem is found to be acceptable. In the course of review, the equivalent lateral moment of inertia of a track frame is suggested to be 10 (10%) more than the lateral moment of inertia of two rails of a track. One Euler type analytical formula for vertical buckling is reviewed and found to be unacceptable too.

Using definition of moment and curvature as well as principle of equilibrium, the author suggests formulas for horizontal and vertical buckling load and corresponding critical temperature differential. The formulas are applied and the results are validated against current formula from Esveld, field test data, currently accepted value, and published values of critical buckling load and temperature differential. A vertical buckling load formula is derived from the horizontal buckling load formula.

The buckling process from critical to fully buckled state is described step by step with related calculations. Finally, threshold radius of a vertical railroad curve is suggested to avoid vertical buckling.

### Review of track buckling tests

The possibility of buckling of a pointless track due to constrained thermal expansion was discussed, as early as, 1902 by A Harmann. This problem did not get attention of railroad research engineers until about 30 years later. From 1932 field tests were conducted to investigate track buckling on a test track segment. Some test cases are summarized from literature.

| Test case | Researchers, year | Test track length | Rail, tie, and track type | Load introduced by | Buckling mode | Track condition | Number of half waves | Buckled length | Amplitude of deflection | Critical/Buckling load, $P_c$ | Critical temperature differential, $\Delta T_c$ |
|-----------|-------------------|-------------------|--------------------------|-------------------|---------------|------------------|---------------------|----------------|------------------------|---------------------------|-------------------------------|
| A.        | O. Ammann and C.v. Gruenewaltt; 1932 | 30 m | Steel tie, K-type track | Hydraulic jack | Buckled in vertical plane. | No information | One, 30 m | 30 m | 800 mm | 200 tons (approximately) | 49°C assuming $E = 2.1 \times 10^6$ kg/cm$^2$, $\alpha = 1.18 \times 10^{-5}$/°C, $A = 2 \times 82.5 = 165$ cm$^2$ |
| B.        | O. Ammann and C.v. Gruenewaltt; 1932 | 60 m | Wood tie, K-type track | Hydraulic jack | Buckled in horizontal plane. | No information | Two, each 19 m | 38 m ($=2 \times 19 m$) | 400 mm | 200 tons (approximately) | 49°C as calculated under A |

(continued)
| C. Researchers, year, reference         | J. Nemcsek; 1933 |
|----------------------------------------|------------------|
| Test track length                      | 60 m             |
| Tie and track type                     | Wood tie         |
| Load introduced by                     | Hydraulic jack   |
| Buckling mode                          | Except for one test, all seven test tracks buckled in horizontal plane. However, in a number of tests, a small lift-off was observed before the test track buckled sideways. |
| Track condition                        | Made up of worn rails with sizeable geometric imperfections. |
| Number of half waves                   | Two              |
| Buckled length                         | No information   |
| Amplitude of deflection                | No information   |
| $P_c$                                  | 180 tons         |
| $\Delta T_c$                           | $44^\circ C$ ($=49 \times 180/200; \text{cf. case A}$) |

| D. Researchers, year                   | F. Rabb, Technical University of Karlsruhe, Germany, 1934 |
|----------------------------------------|-----------------------------------------------------------|
| Test track length                      | 46.17 m                                                   |
| Tie and track type                     | Wood tie, K-type track                                     |
| Load introduced by                     | Heating rails by electric current                          |
| Buckling mode                          | Buckled in horizontal plane.                               |
| Track condition                        | No information                                            |
| Number of half waves                   | Three                                                     |
| Buckled length                         | No information                                            |
| Amplitude of deflection                | No information                                            |
| $P_c$                                  | 173 tons                                                  |
| $\Delta T_c$                           | $42.5^\circ C$ ($=49 \times 173/200; \text{cf. case A}$) |

| E. Researchers, year                   | F. Birmann and F. Rabb; Technical University of Karlsruhe, Germany, 1960 |
|----------------------------------------|--------------------------------------------------------------------------|
| Test track length                      | 46.17 m                                                   |
| Tie and track type                     | Wood tie, K-type track                                     |
| Load introduced by                     | Electric current                          |
| Buckling mode                          | The rail-tie structure did not lift off prior to horizontal buckling, but rather it lifted off simultaneously and gradually with increasing lateral displacements. |
| Track condition                        | Track has geometric imperfections.                             |
| Number of half waves                   | Two, three or four, each 5 m                                    |
| Buckled length                         | 10, 15, and 20 m (computed)                                    |
| Amplitude of deflection                | 250 mm (max)                                               |
| $P_c$                                  | 173 tons                                                  |
| $\Delta T_c$                           | $42.5^\circ C$ ($=49 \times 173/200; \text{cf. case A}$) |

| F. Researchers, year                   | British Transport Commission, 1961                           |
|----------------------------------------|-----------------------------------------------------------|
| Test track length                      | 36.6 m                                                   |
| Rail, tie and track type               | Wood and reinforced concrete tie                           |
| Load introduced by                     | Parabolic reflector fitted with electric elements          |
| Buckling mode                          | The observed buckling mode consists of 1, 2, 3, or 4 half-waves. No information was given regarding vertical lift-off prior to and during buckling. |
| Track condition                        | No information                                            |
| Number of half waves                   | One, two, three or four                                    |
| Buckled length                         | No information                                            |
| Amplitude of deflection                | 250 mm (max)                                               |
| $P_c$                                  | 173 tons                                                  |
| $\Delta T_c$                           | $42.5^\circ C$ ($=49 \times 173/200; \text{cf. case A}$) |

| G. Researchers, year                   | E. M. Bromberg, Central Railroad Research Institute, USSR, 1966 |
|----------------------------------------|--------------------------------------------------------------------------|
| Test track length                      | 100 m                                                   |
| Rail, tie and track type               | P 65, Wood and reinforced concrete tie                           |
| Load introduced by                     | Heating by electric current                          |
| Buckling mode                          | The observed buckling mode consists of 3, 4 or 5 half-waves. No information was given regarding vertical lift-off prior to and during buckling. |

(continued)
The aforementioned seven field tests yield a critical temperature differential of 42.5°C to 50°C corresponding to a track buckling load of 173 to 200 tons; the lower buckling load was observed on test tracks with geometric imperfection. It appears that tie type does not influence buckling load. Geometric imperfections reduce the buckling load by 14%. The test track length varies between 30 to 100 m with no significant drop of buckling load with increased length. This implies the buckling is essentially a local phenomenon and a test track of 30 m length is adequate to capture buckling load only.

The critical buckling load is calculated using the following formulas and data:

\[
\Delta T_c = \sqrt{\frac{8.7F_{QVW}I^*}{\alpha^2 A^2 E f^*}} \quad (1)
\]

\[
F_0 = \alpha \Delta T_{crit} E A \quad (2)
\]

\[
L = \frac{3\pi}{\sqrt{\frac{2EI}{F_0}}} \quad (3)
\]

Data:
Rail: UIC 60
Tie: Concrete B70w tie, tensioned

\[
\begin{aligned}
I^* &= 2200 \text{ cm}^4 \\
A &= 2 \times 76.9 = 153.8 \text{ cm}^2 \\
F_{QVW} &= 100 \text{ N/cm} \\
E &= 21,000,000 \text{ N/cm}^2 \\
\alpha &= 0.000012^o/\text{C} \\
f^* &= 2-2.5 \text{ cm}
\end{aligned}
\]

A uniform temperature rise of 50°C induces in the track an axial compression force of 204 tons which may be sufficient to buckle the track. The critical temperature differential, \(\Delta T_c\), and critical buckling load, \(F_0\), are 115.7°C and 4484 kN (450 tons) as per equations (1) and (2). The \(\Delta T_c\) of 115.7°C is more than double of the field tests values of 42.5°C to 50°C and currently accepted value of 50°C. The critical buckling load of 4484 kN (450 tons) is more than double of the field tests values of 173 to 200 tons and currently accepted value of 200 tons. Interestingly Lichtberger labeled 50°C temperature differential as a critical value for buckling elsewhere in his book. Some published values of critical temperature differential are presented in Table 1.

**Table 1.** \(\Delta T_c\) compared to vertical and horizontal buckling with a tie spacing of 60 cm (American).2,5

| Rail weight (kg/m) | Comparable to rail profile | Mode of buckling vertical/horizontal | Critical increase in temperature [°C] |
|-------------------|---------------------------|-------------------------------------|-------------------------------------|
| 47                | 115 RE                    | vertical/horizontal                 | 46/49                               |
| 59                | UIC 60                    | vertical/horizontal                 | 48/43                               |
| 68                | 136 RE                    | vertical/horizontal                 | 50/42                               |

AREMA allows a thermal stress of 124 N/mm² (18,000 psi) which corresponds to a temperature differential of 50°C. AREMA needs to review its allowable thermal stress. The equation (1) would suggest \(\Delta T_c = 80^o/°C\) if the lateral resistance, \(F_{QVW}\), is reduced by 50%; this implies that a track would not buckle at above 100°C (assuming a stress neutral temperature of 20°C) even if all ballast above tie-ballast interface is removed. Thus, if the value by the equation (1) is correct then there would have been no requirement of hot weather patrolling, no speed restriction on a tamp.
track without dynamic track stabilizer, no speed restriction on a newly laid track built without dynamic stabilizer, and no further research to enhancement tie resistance since 1969 etc.

The author intends to modify two aforementioned input data to check if the formula can still be used:

The equivalent moment of inertia of the track grid, \( I' \) with UIC 90 rail and concrete B70w tie is 2200 cm\(^4\). The moment of inertia of two UIC 90 rails, \( 2I \) is 1026 cm\(^4\). This implies that a track grid increases the lateral stiffness by 114%; the huge increase in lateral stiffness by a track grid is not supported by the following facts:

The longitudinal resistance of a track equals to the lateral resistance of the track; this implies that lateral bending stiffness has no significant contribution toward lateral resistance of a track and almost entire resistance comes from the ballast. In fact, the longitudinal load developed by the combination of thermal stress in a continuously welded rail and by traffic is restrained by mass internal friction of ballast. Dogneton showed influence of 23 factors on the lateral resistance of the track wherein he mentioned tie spacing has little or no influence. Zarembski also compiled a list of 23 parameters which influence lateral displacement resistance, tie spacing is not in the list. Track grid is not supposed to influence ballast resistance significantly when tie spacing does not. Torsional rigidity of the track grid has minor influence on lateral resistance. The stiffness of rail-tie structure does not play a very important role in the lateral rigidity of the entire track. The track frame contributes 5% to 10% toward the lateral resistance of the track. It has been estimated that ballast resistance increased by 10% is sufficient to compensate the.increase in lateral stiffness by only 10%. Thus, the equivalent inertia of a track frame with UIC 60 rail is estimated to be 1130 cm\(^4\). Using this value in the equation (1), the \( \Delta T_C \) would be 83°C; still the value is unacceptable. In fact, no sensible value of equivalent inertia of a track would offer a \( \Delta T_C \) of 50°C.

The critical misalignment value, \( f' \) given is in a range of 2 to 2.5 cm. If 2.5 cm is used, the \( \Delta T_C \) would be 74°C \((= 83 \times (2.290^2))\); still the value is high and unacceptable. This is to be noted that at present, there is no suitable theoretical model to compute the critical misalignment.

The buckling length formula (ref: equation (3)) offers a buckling length of 13.53 m which seems to be acceptable. The value of buckling length cannot be correct if the buckling load is faulty. The equivalent moment of inertia of the track grid is multiplied by two in the equation (3) which is not understood. With modified input data, the buckling length would be 6.85 m \((= 13.53/1130/(2 \times 2200))\) which is too short to accept.

Thus, even with the modified input data, the above analytical equation (1) does not work.

On the basis of total energy theorem, the critical load is given by Esveld:

\[
P_{\text{crit}} = 2 \sqrt{\frac{\tau_0 EI}{\pi f_0}}
\]  

The critical length is given by

\[
L^2_{\text{crit}} = 2 \sqrt{\frac{\pi^5 f_0 EI}{\tau_0}}
\]

in which

\( EI \) bending stiffness,  
\( f_0 \) critical misalignment,  
\( \tau_0 \) displacement resistance.

Based on following parameters:

\( \tau_0 = 10 \text{ kN/m}, EI = 8000 \text{ kNm}^2, \) and \( f_0 = 0.025 \text{ m} \)

the critical load is calculated to be 2018 kN which agrees with the past field test values.

The wave length is calculated to be 12.51 m.

**Derivation of a new formula**

A centerline of an initially tangent track which has been bent by the thermal load, \( P \) is shown in Figure 1.

The bending moment at any point of the deflected centerline is given by

\[
M = Py
\]  
\[
2EI \frac{d^2 y}{dx^2} = Py
\]  
\[
\frac{d^2 y}{dx^2} = \frac{Py}{2EI}
\]

The curvature \( k(x) \) for an arc of the form \( y = f(x) \) is given by

![Figure 1. Center line of a track under compressive load.](image)
\[ k(x) = \frac{\frac{d^2y}{dx^2}}{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{1/2}} \]  

(9)

Neglecting \( dy/dx \) being small compared to unity,

\[ k(x) = \frac{d^2y}{dx^2} \]  

(10)

The radius of curvature of an arc at a point is defined as the reciprocal of the absolute value of curvature at that point\(^{10} \); that is

\[ R = \frac{1}{k(x)} \]  

(11)

From equations (10) and (11),

\[ \frac{1}{R} = \frac{d^2y}{dx^2} \]  

(12)

The curvature is approximated by the above relation in the Klingel theory on lateral movement of a wheelset on tangent track.\(^2 \) The same relation is used to derive the formula of vertical curve in railway engineering. From equations (8) and (12),

\[ \frac{1}{R} = \frac{Py}{2EI} \]  

(13)

Equating thermal load per unit length of track with the lateral resistance of track, it may be written,

\[ \frac{P}{R} = F_{QVW} \]  

(14)

\[ \frac{1}{R} = \frac{F_{QVW}}{P} \]  

(15)

From equations (13) and (15),

\[ \frac{Py}{2EI} = \frac{F_{QVW}}{P} \]  

(16)

\[ y = \frac{2F_{QVW}EI}{P^2} \]  

(17)

\[ y_c = \frac{2F_{QVW}EI}{P_c^2} \]  

(18)

Changing the unit of critical load, \( P_c \) from N to kN, unit of lateral resistance, \( F_{QVW} \) from N/mm to N/cm, the equation (18) is written as

\[ y_c = \frac{2F_{QVW}EI}{10^3 P_c^2} \]  

(19)

Considering 10% contribution toward lateral rigidity by track frame as explained earlier, the equation (19) is modified as

\[ y_c = \frac{2F_{QVW}EI}{10^3 P_c^2} \]  

(19)

Figure 2. Lateral displacement resistance diagram.

\[ y_c = \frac{2.2F_{QVW}EI}{10^3 P_c^2} \]  

(20)

\[ P_c = \sqrt{\frac{2.2F_{QVW}EI}{10^3 y_c}} \]  

(21)

The equation (21) shows a correlation between four parameters and critical load which is exactly same as the correlation shown in the equation (4) based on energy theorem from Esveld. The derivation of equation (21) is much simpler than that of equation (4).

\[ P_c = 2a\Delta T \cdot E \cdot A \]  

(22)

The force value arising when ties are displaced by 2 mm is usually taken as lateral displacement resistance. Where there is greater displacement the tie begins to slide as static friction changes to sliding friction as shown in Figure 2.\(^3 \)

Thus, the tie would start sliding in a plastic zone at a displacement beyond 2 mm. In the aforementioned calculation example, a critical misalignment of 2 cm is assumed to compute the \( \Delta T_c \). There is no suitable theoretical model to compute the critical misalignment.\(^9 \)

The assumption of critical misalignment of 2 to 2.5 cm is questionable for at least three reasons: (1) A tie resistance is determined at a displacement of 2 mm; the track displacement enters into plastic zone after 2 mm displacement. Thus, the critical misalignment is expected to be close to, not far off 2 mm. (2) As per European (EN) specification, the admissible limit of misalignment is 2 mm on a 10 m chord; this limit would have been more than 2 mm if the critical value is 25 mm. (3) As per equations (1), (4), and (21),

\[ \Delta T_c \propto \frac{1}{\sqrt{y_c}} \]  

(23)

With a track misalignment of \( y_{mis} \) the reduced temperature differential is given by
The equation (25) can be used to study the effect of track misalignment on critical temperature differential and critical buckling load. The $D_{TC}$ and $P_c$ is reduced by 13.4% for $ymis = 2\text{ mm}$ and $yc = 6\text{ mm}$. This is supported by the field tests where the critical load is reduced to 173 to 180 tons from 200 tons, that is, 10% to 13.5% due to geometric imperfections (ref: cases under B, C, E). On the other hand, the $D_{TC}$ and $F_0$ is reduced by 3.8% for $ymis = 2\text{ mm}$ and $yc = 25\text{ mm}$, which is not sensible. Moreover, for $ymis = 3.36\text{ mm}$ and $yc = 6\text{ mm}$, $\Delta T_{C} = 40^\circ\text{C}$, the track is supposed to buckle at 60°C assuming SNT = 20°C and $\Delta T_{C} = 50^\circ\text{C}$. Thus, a misalignment of 3.36 mm is risky as the rail temperature in North America might hit 60°C. On the other hand, for $ymis = 14\text{ mm}$ and $yc = 25\text{ mm}$, $\Delta T_{C} = 40^\circ\text{C}$, the track is supposed to buckle at 60°C. Thus, a misalignment of 14 mm is risky which implies a misalignment below 14 mm may be admissible which is unacceptable. The aforementioned analysis by the equation (25) proves a critical misalignment of 25 mm is unacceptable.

The Buckling analysis normally gives only the upper bound of load that can be applied to the system before encountering loss of stability. A lateral displacement resistance of 90 and 150 N/cm are assumed for well ballasted and fully stabilized wood and concrete tie track respectively. Still to solve the equation (21) a value of critical misalignment, $yc$ is to be assumed. As ballast resistance plays significant role in buckling issue, hence it is chosen as basis to select the critical misalignment value. At present, there is no suitable theoretical model to compute the critical misalignment. A critical misalignment of 6 mm is assumed for a ballast resistance of 90 N/cm (wood tie track) and 9 mm for 150 N/cm (concrete tie track).

Incorporating $F_{QVW} = 100\text{ N/cm}$, $E = 210,000\text{ N/mm}^2$, and $I = 5,130,000\text{ mm}^4$ (UIC 60 rail), $yc = 6\text{ mm}$ in the equation (20) the critical load or upper bound of buckling load comes out to be 1987 kN (cf. equation (4) from Esvelt gives 2018 kN).

The critical load and critical temperature differential are computed in Table 2.

A published value from Table 1 shows 49°C (cf. 49°C on first row in Table 2) as critical temperature differential.
differential for horizontal buckling with wood tie and 47 kg/m rail.\(^3\)\(^,\)\(^1\)\(^1\) No value is given for concrete tie. The Table 1 shows 42°C (cf. 48°C on third row in Table 2) as critical temperature differential with wood tie and 68 kg/m rail. The Table 1 shows 46°C (cf. 50°C on last row in Table 2) as critical temperature differential with concrete tie and 68 kg/m rail. Thus, the values computed in Table 2 are comparable with the published values in Table 1. 

The assumption of critical misalignment of 6 mm for displacement resistance of 90 N/cm and 9 mm for 150 N/cm render acceptable value of critical load and critical temperature differential. The modal shape of buckling of a tangent track is usually assumed as a sine wave shown in Figure 3.

\[
y = y_c \sin \left( \frac{2\pi x}{\lambda_c} \right) \tag{26}
\]

\[
\lambda_c = \frac{2\pi x_c}{\sin^{-1} \left( \frac{x}{r} \right)} \tag{27}
\]

Assuming first tie at the point of deflection, the second tie would be at a distance of S. The second tie should be displaced by at least 2 mm or a little more. Incorporating \(x = S = 0.75\) m and \(y/y_c = 2/6\) in the equation (25), the wavelength, \(\lambda_c = 13.8\) m for 115 RE rail track. The wavelength of the traditional buckling is up to 15 m with amplitudes of up to 1 m.\(^3\) The equation (20) gives a buckling load of 1987 kN and the equation (4) from Esveld gives a buckling load of 2018 kN.

A buckling process for the above buckled track of 13.8 m wave length is depicted below:

A stress neutral temperature, SNT of 20°C is assumed. The track is expected to buckle at 70°C (= 50 + 20°C). The arc length of the rail is calculated to be 13,800.04 mm by integration; the extra 0.04 mm is breathed from two ends of the buckled track. Assuming \(F_{QW} = 90\) N/cm for wood tie track, a breathing length, \(L_{B1}\) of 2.6 m is required to breath 0.02 mm from each end of buckled track and the corresponding drop in temperature, \(\Delta T_1\) would be 1.3°C. The rail temperature will drop to 68.7°C (= 70 – 1.3); the affected track length in critical state, \(L_c = 19\) m (= 2.6 + 13.8 + 2.6 m). If the temperature increases to a value below 68.7°C, the track is not supposed to distort. The track starts to move at 68.7°C. The minimum length of test track shall be 19 m to determine the buckling load only. A solution based on true expression for curvature enables the deflection to be determined and shows that small increases in load above the critical load produces significantly large deflection.\(^12\) With a slight increase in buckling load, the track is exploded to full buckling state releasing more energy. This is why buckling load is assumed to be constant throughout the buckling process.\(^2\) Assume the rail temperature drops to SNT that is, 20°C; the temperature differential, \(\Delta T_2\) would be 48.7°C (= 68.7 – 20°C) and the corresponding breathing length, \(L_{B2} = 97\) m. The breathing lengths contribute 55.72 mm (= 2 × 27.86 mm) of rail into the buckled track; all parameters – amplitude, wave length, and affected track length would grow further. The equation (25) may be applied to compute the amplitude of the full buckled track; in practice \(\Delta T_2^*\) would not be zero as the amplitude of a buckled track has a finite value. Assuming \(\Delta T_2^* = 4.5 – 5^\circ C\), the amplitude comes out to be 594 to 735 mm, respectively. The affected track length after buckling, \(L_f\) would be 213 m (= 97 + 19 + 97 m). Thus, a test track length of 213 m would capture both buckling load and modal shape of buckling. Kerr mentioned that the test track lengths (ref: case A–E) are inadequate.\(^1\) In aforementioned field test cases the maximum amplitude observed is 400 mm due to inadequate test lengths. As the breathing lengths breathe at the ends only, the final wave length is not expected to change significantly. The critical wave length might be disturbed by the sudden buckling movement of the track. Thus, the final wave length, \(\lambda_f\) may be estimated by \((\lambda_c + L_c)/2\) that is, 16.4 (= 13.8 + 19)/2. Similar calculations are done for three other cases and results are shown in Table 3.

Some author assumes critical wave length to remain same throughout entire process of buckling.\(^2\) From Table 3 the wave length of a concrete tie track is about 23 to 24 m and that of wood tie track, 16 to 17 m. The chord length to measure longitudinal alignment may be half of wave length, that is, 8 to 12 m, currently 10 m chord is used. A test track length of 30 m (ref: \(L_c\) in Table 3) is quiet adequate to capture buckling load only. The track lengths in aforementioned field tests are 30, 36.6, 46.17, 60, and 100 m; hence the buckling load from field tests are acceptable. The minimum test track length shall be 250 m (ref: \(L_f\) in Table 3) for wood tie track and 220 m (ref: \(L_f\) in Table 3) for concrete tie track to capture modal shapes of a buckled track. The test track lengths in aforementioned field test are not adequate to capture modal shapes, hence the descriptions of modal shape, for example, amplitude, wave length from aforementioned field tests are unacceptable.

If a track is buckled then at least a track length equal to full affected length, \(L_f\) should be distressed to put it at original design SNT.

**Vertical buckling**

A relation between critical vertical misalignment and critical vertical buckling load may be expressed from the equation (16) simply replacing lateral resistance term, \(F_{QW}\) by the weight of track, \(w\) (N/cm) and moment of inertia, \(I_y\) term by \(I_x\) as under:

\[
y_c = \frac{2wEh_{x-y}}{10^5P_{c}^2} \tag{28}
\]
The product of two terms, that is, $y_c p_c^2$ is an indicator of track buckling strength. Thus, the ratio of vertical buckling to lateral buckling strength, $r$ is given by

$$r = \frac{w}{F_{Q\ell W}} \frac{I_{x-x}}{I_{y-y}} = 6.25 \frac{w}{F_{Q\ell W}}$$ (29)

The critical temperature differential for vertical buckling is given by

$$\Delta T_c^V = 6.25 \frac{w}{F_{Q\ell W}} \Delta T_c$$ (30)

The vertical buckling of a track can occur simultaneously with the horizontal buckling if the following condition is met:

$$F_{Q\ell W} = 6.25w$$ (31)

The vertical buckling of a track can occur prior to horizontal buckling if the following condition is met:

$$F_{Q\ell W} > 6.25w$$ (32)

A well ballasted and fully stabilized concrete tie track exhibits a lateral displacement resistance of 150 N/cm. The weight of track with UIC 60 rail and B70 concrete tie@60 cm, $w$ is 62.6 N/cm. The $\Delta T_c$ for horizontal buckling is calculated to be 51°C for horizontal buckling of a track with UIC 60 rail and B70 concrete tie. Thus $\Delta T_c^V$ for vertical buckling will be 2.6 (=6.25 × 62.6/150) times the $\Delta T_c$, that is, 133°C (=2.6 × 51°C). The critical misalignment for vertical buckling would be 0.15 times (=1/2.62) the critical misalignment of horizontal buckling, that is, 0.9 to 1.35 mm (=0.15 × 6 – 0.15 × 9) if the track resistance exceeds 391 N/cm (=6.25 × 62.6). But the lateral resistance of the track is only 150 N/cm. Hence, lateral buckling will take place (cf. 51°C, 133°C); vertical buckling is an absurd event with concrete tie track.

A well ballasted and fully stabilized wood tie track exhibits a lateral displacement resistance of 80 N/cm. The weight of track with UIC 90 rail and wood tie@60 cm, $w$ is 28.1 N/cm. The $\Delta T_c$ for horizontal buckling of a track with UIC 60 rail and wood tie is assumed to be 45°C. Thus $\Delta T_c^V$ for vertical buckling will be 2.2 times the critical temperature for horizontal buckling, that is, 99°C. The critical misalignment for vertical buckling would be 0.2 times (=1/2.22) the critical misalignment of horizontal buckling that is, 1.2 to 1.8 mm (=0.2 × 6 – 0.2 × 9) if the track resistance exceeds 175 N/cm (=6.25 × 28.1). But the lateral displacement resistance is only 80 N/cm. Hence, lateral buckling will take place (cf. 45°C, 99°C), vertical buckling is an absurd event with wood tie track.

Under the influence of the lift-off wave, the bottom friction is reduced by 20% to 40%. Typical track buckling below train in summertime at high rail temperature is caused in the area of the lift-off wave. There is always a segment of unloaded track in front of wheel or in between two trucks due to negative deflection or rail lifting. The distance between first and second point of zero deflection from a wheel load is $\pi L$ (=7$\pi L/4$ – 3$\pi L/4$), approximately 3 m, which is the maximum length of unloaded lifted track. The negative deflection is $e^{-\pi}$ that is, 4.3% of the deflection under the wheel; the negative deflection computation does not consider weight of the track. If the deflection under the wheel is 6 mm, the negative deflection would be 0.26 mm. The dead weight of modern concrete tie track compensates the negative deflection. The dead weight of wood tie track would reduce the negative deflection. Thus the misalignment of lift-off wave is zero for concrete tie and less than critical misalignment of wood tie track. Moreover, the length of the lifted track is very small. Thus, lift-off wave under wheel load is not a concern for buckling of the track.

**Literature review on vertical buckling**

There is an analytical formula with a calculation example for computing critical temperature differential for vertical buckling in literature as follows:

$$\Delta T_c = \frac{4}{\alpha A} \sqrt{\frac{L G_{crit}}{E f^*}}$$ (33)

$$F_0 = \alpha \Delta T_c E A$$ (34)

$$L = 2\pi \sqrt{\frac{2E I}{F_0}}$$ (35)

$$G_{crit} = \frac{E A}{16 I} (\alpha A \Delta T_{crit})^2$$ (36)

Data:
- **Rail**: UIC 60
- **Tie**: Concrete B70w tie, tensioned
- $I$ = 6110 cm$^4$
- $A$ = 2 × 76.9 = 153.8 cm$^2$
- $E$ = 21,000,000 N/cm$^2$
- $\alpha$ = 0.000012°C$^{-1}$
- $f^*$ = 1.5 cm
- $G$ = 62.6 N/cm

The formula offers a critical temperature differential of 238.8°C for vertical buckling for a track with B70w concrete tie and UIC 60 rail. The published value is 70°C with concrete tie and 59 kg/m rail. Thus, the computed value is simply unacceptable in context with the published value, test value from field (ref: case A), and aforementioned discussion. Although a formula for critical track weight is given (ref: equation (36), but the calculation uses the actual track weight of 62.6 N/cm. Moreover, an assumed value of critical misalignment of
15 mm seems to be very high to offset the self-weight of track in light of aforementioned discussion.

The equation (1) yields a critical temperature differential of 115.7°C for horizontal buckling for the same track; the ratio of vertical to horizontal buckling load, \( r \) is 2.06 \((= 238.8/115.7)\). The ratio of two values seem acceptable although the individual values are unacceptable. The author computes the ratio between vertical and horizontal buckling load to be 2.6. In fact the ratio can be more than 2.6 as the formulation is based on track weight only which is a simplification; the restraining influence of the ballast around the tie is not considered.

### Threshold radius of a vertical curve

The threshold radius of a vertical (crest) curve may be expressed by equating track weight and critical buckling load on a curve. Adjusting units, the threshold radius is given by

\[
R_T = \frac{2.6200}{37} \approx 320 \text{ m} \quad (1)
\]

For a buckling load of 200 kN, the threshold radius of a vertical (crest) curve on a concrete tie track would be 320 m \((= 10 \times 200/62.6)\). For a buckling load of 200 kN, the threshold radius of a vertical (crest) curve on a wood tie track would be 1779 m, \((= 10 \times 200/28.1)\) say 1800 m. If the radius is below the threshold value, thermal load would initiate lifting of the track and tie resistance will be dropped to 50% causing horizontal buckling at a temperature differential lower than the critical temperature differential. A vertical curve radius of 2000 m must be considered as the absolute minimum.\(^2\) This exercise validates earlier work on vertical buckling indirectly.

### Conclusion

An analytical formula is suggested to compute the critical load and critical temperature differential for horizontal thermal buckling of a tangent track. The critical temperature differential for horizontal and vertical buckling is related by track weight and lateral ballast resistance; from this relation critical temperature differential for vertical buckling is determined. A track might start to shift at 2°C below the critical temperature for (horizontal) buckling. A track is likely to go under vertical mode of buckling under high compressive load if the lateral displacement resistance of a track exceeds 6.25 times the self-weight of a track. A test track length of 30 m is quiet adequate to capture buckling load only. A calculation process is shown to narrate buckling process and to determine the wave length, affected track length, and amplitude of a buckled track. A formula to assess the effect of track misalignment on critical buckling load is given. A formula to determine threshold radius of a vertical curve is suggested; the lighter the track the higher the threshold radius.

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Appendix I

Notation

The following symbols are used in this paper:

- $A$: sectional area of a rail (mm$^2$),
- $E$: Young’s modulus (210,000 N/mm$^2$),
- $F_0$: critical rail pressure force (N),
- $F_{QVW}$: lateral displacement resistance (N/cm),
- $f^*$: critical track defect (2–2.5 cm),
- $I$: moment of inertia of a single rail (cm$^4$, mm$^4$),
- $I^*$: moment of inertia of the track grid (cm$^4$),
- $L$: characteristic length (m),
- $L_C$: affected track length at critical position (m),
- $L_F$: affected track length after buckling (m),
- $P_c$: critical (buckling) load (kN),
- $R$: curve radius (m),
- $R^*_F$: threshold radius (m),
- $r$: ratio between critical temperatures, $\Delta T_C^r/\Delta T_C$,
- $S$: spacing of tie (m),
- $SNT$: stress neutral temperature (°C),
- $w$: track weight (N/cm),
- $\alpha$: coefficient of expansion (0.0000118/°C),
- $\Delta T_C$: critical temperature differential for horizontal buckling (°C),
- $\Delta T_C^r$: reduced critical temperature differential due to track misalignment (°C),
- $\Delta T_C^v$: critical temperature differential for vertical buckling (°C),
- $\lambda_C$: critical wave length (m) as in Figure 3, wave length after buckling (m).