Asymptotic Properties of Primal-Dual Algorithm for Distributed Stochastic Optimization Over Random Networks

Jinlong Lei, Han-Fu Chen, and Hai-Tao Fang

Abstract

This paper studies a distributed stochastic optimization problem over random networks with imperfect communications subject to a global constraint, which is the intersection of local constraint sets assigned to agents. The global cost function is the sum of local cost functions, each of which is the expectation of a random cost function. By incorporating the augmented Lagrange technique with the projection method, a stochastic approximation based distributed primal-dual algorithm is proposed to solve the problem. Each agent updates its estimate by using the local observations and the information derived from neighbors. For the constrained problem, the estimates are first shown to be bounded almost surely (a.s.), and then are proved to converge to the optimal solution set a.s. Furthermore, the asymptotic normality and efficiency of the algorithm are addressed for the unconstrained case. The results demonstrate the influence of random networks, communication noises, and gradient errors on the performance of the algorithm. Finally, numerical simulations demonstrate the theoretic results.

Index Terms

Distributed stochastic optimization, random networks, primal-dual algorithm, stochastic approximation, asymptotic normality, asymptotic efficiency.

I. INTRODUCTION

For recent years, extensive efforts have been paid to the distributed estimation and optimization problems motivated by their wide applications in sensor networks [1], [2], cognitive networks [3], multi robots [4], as well as in distributed learning [5]. This paper studies a distributed optimization problem, where \( n \) agents connected in a network collectively minimize a convex cost function \( \sum_{i=1}^{n} f_{i}(x) \) subject to a convex set constraint \( \bigcap_{i=1}^{n} \Omega_{i} \). The local cost function of agent \( i \) takes the form \( f_{i}(x) = E[h_{i}(x, \vartheta_{i})] \), where \( \vartheta_{i} \) is a random variable. In such a problem, the cost function \( f_{i}(x) \) is difficult to calculate, but samples of the cost function \( h_{i}(x, \vartheta_{i}) \) may serve as estimates for its expectation. It is assumed that the local constraint set \( \Omega_{i} \) of agent \( i \) is closed and convex, and the communication relationship among the agents is described by a random network. Besides, communications are imperfect since there are noises in the channels through which agents exchange information.

There exist many papers considering the related problems. A unconstrained cooperative optimization problem is investigated in [7] and [8] over the deterministic and the random switching networks, respectively. A distributed stochastic subgradient projection algorithm is proposed in [9] to solve a constrained optimization problem, where all agents are subject to a common convex constraint set, and subgradients of local cost functions are corrupted by stochastic errors. Effects of stochastic subgradient
errors on the convergence of the algorithm over deterministic switching networks are investigated. A distributed asynchronous algorithm with two diminishing step sizes is proposed in [10] to solve the distributed constrained stochastic optimization problem. The estimates are shown to converge to a random point in the optimal set a.s., when constraint sets are compact, the global constraint set has a nonempty interior set, and cost functions are non-smooth but with bounded subgradients. A distributed algorithm based on dual subgradient averaging is proposed in [11], where it is shown how do the network size and the spectral gap of the network influence convergence rates. Consensus-based distributed primal-dual subgradient methods are given in [12], [13], where [13] solves a problem with the cost function being the sum of local cost functions and with global convex inequality constraints known to all agents, while [12] solves a problem with a coupled global cost function and with inequality constraints. A primal-dual algorithm with constant step size is proposed and its convergence is shown in [14] for the deterministic unconstrained optimization problem over undirected connected graph with perfect communications. Besides, performance of the continuous time primal-dual algorithms is also investigated in [15], [16]. Generally speaking, the above mentioned algorithms can be divided into three categories: [7]–[10] belong to the primal domain algorithms, [11] belongs to the dual domain algorithm, while [12]–[16] belong to the primal-dual domain algorithms.

In this paper, we propose a stochastic approximation based distributed primal-dual algorithm to solve the distributed constrained stochastic optimization problem. Since it is equivalent to a convex optimization problem with a linear equality constraint and a convex set constraint, by incorporating the augmented Lagrange technique with the projection method, a distributed primal-dual algorithm is derived. The algorithm is distributed as in an iteration each agent updates its estimate using the noisy observations for gradients of the local functions and the noisy observations for both primal and dual variables of the neighboring agents.

Contributions of the paper are as follows. 1) Stability and convergence of the algorithm are proved for the constrained problem. The communication graphs are assumed to be independent identically distributed (i.i.d) with the mean graph being undirected and connected. Communication noises and gradient errors are assumed to be martingale difference sequences (mds). Convex sets are required to have smooth boundaries with the global constraint having at least one relative interior, and gradient functions are required to be Lipschitz continuous. Then with diminishing step-size, the estimates are shown to be bounded a.s. by using the convergence theorem for martingales, and to converge to the optimal solution set a.s. by use of the results for constrained stochastic approximation [27]. Compared with [10], here gradients are only required to be Lipschitz continuous without boundedness assumption, constraint sets are not assumed to be compact, and the global constraint is only required to have at least one relative interior point. Different from [12]–[14], the stochastic optimization problem is investigated over random networks with imperfect communications. 2) Asymptotic properties are considered for the unconstrained problem. Through dimensionality reduction, asymptotic normality and efficiency of the algorithm are established. In comparison with [11], we have shown the exact influence of random networks, imperfect communications, gradient errors and the structure of cost functions on the rate of convergence.

The organization of the paper is as follows. In Section II, some preliminary information about graph
theory and convex analysis is provided and the problem is formulated. In Section III, the basic assumptions are introduced and a stochastic approximation based distributed primal-dual algorithm is designed. Convergence for the constrained problem is established in Section IV while asymptotic properties for the unconstrained problem are given in Section V. Numerical simulations are demonstrated in Section VI with some concluding remarks given in Section VII.

II. PRELIMINARIES AND PROBLEM STATEMENT

We first introduce some preliminary results about graph theory, convex functions and convex sets, then formulate the distributed optimization problem.

A. Graph Theory

Consider a network of $n$ agents. The communication relationship among agents is described by a digraph $G = (V, E, A)$, where $V = \{1, \cdots, n\}$ is the node set with node $i$ representing agent $i$; $E \subset V \times V$ is the edge set, and $(j, i) \in E$ if and only if agent $i$ can get information from agent $j$; $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ is the adjacency matrix of $G$, where $a_{ij} > 0$ if $(j, i) \in E$, and $a_{ij} = 0$, otherwise. Here, we assume the self-edge $(i, i)$ is not allowed, i.e., $a_{ii} = 0 \ \forall i \in V$. The Laplacian matrix of graph $G$ is defined as $L = \Delta G - A$ with $\Delta G = diag\{\sum_{j=1}^{n} a_{1j}, \cdots, \sum_{j=1}^{n} a_{nj}\}$, where and hereafter $diag\{D_1, \cdots, D_n\}$ denotes the block diagonal matrix with main diagonal blocks being square matrices $D_i$, $i = 1, \cdots, n$, and with the off-diagonal blocks being zero matrices.

For a bidirectional graph $G$, $(i, j) \in E_{\overline{G}}$ if and only if $(j, i) \in E_{\overline{G}}$. The graph $G$ is undirected if $A$ is symmetric. The undirected graph $G$ is connected if for any pair $i, j \in V$, there exists a sequence of nodes $i_1, \cdots, i_p \in V$ such that $(i, i_1) \in E_{\overline{G}}$, $(i_1, i_2) \in E_{\overline{G}}, \cdots, (i_p, j) \in E_{\overline{G}}$. For matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ with $a_{ij} \geq 0 \ \forall i, j = 1, \cdots, n$, denote by $G_A = (V, E_A, A_G)$ the digraph generated by $A$, where $V = \{1, \cdots, n\}$, $A_G = A$, and $(j, i) \in E_A$ if $a_{ij} > 0$.

The following lemma presents some properties of the Laplacian matrix $L$ corresponding to an undirected graph $G$.

Lemma 2.1: [29] The Laplacian matrix $L$ of an undirected graph $G$ has the following properties:

i) $L$ is symmetric and positive semi-definite;

ii) $L$ has a simple zero eigenvalue and the corresponding eigenvector is 1, and all the other eigenvalues are positive if and only if $G$ is connected, where 1 denotes the vector with all entries equal to 1.

B. Gradient, Projection Operator and Normal Cone

For a given function $f : \mathbb{R}^m \to (-\infty, \infty]$, denote its domain as $\text{dom}(f) \triangleq \{x \in \mathbb{R}^m : f(x) < \infty\}$. Let $f(\cdot)$ be a convex function, and let $x \in \text{dom}(f)$. For a smooth (differentiable) function $f(\cdot)$, denote by $\nabla f(x)$ and by $\nabla^2 f(x)$ the gradient and Hessian of $f(\cdot)$ at point $x$, respectively. Then

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) \ \forall y \in \text{dom}(f), \quad (1)$$

where $x^T$ denotes the transpose of $x$. 

DRAFT
For a nonempty closed convex set $\Omega \subset \mathbb{R}^m$ and a point $x \in \mathbb{R}^m$, we call the point in $\Omega$ that is closest to $x$ the projection of $x$ on $\Omega$ and denote it by $P_\Omega(x)$. $P_\Omega(x)$ contains only one element for any $x \in \mathbb{R}^m$, and it satisfies the following non-expansive property [17, Theorem 2.13]

$$\| P_\Omega(x) - P_\Omega(y) \| \leq \| x - y \| \quad \forall x, y \in \mathbb{R}^m.$$ (2)

Consider a convex closed set $\Omega \subset \mathbb{R}^m$ and a point $x \in \Omega$. Define the normal cone to $\Omega$ at $x$ as $N_\Omega(x) \triangleq \{ v \in \mathbb{R}^m : \langle v, y - x \rangle \leq 0 \quad \forall y \in \Omega \}$. It is shown that [17, Lemma 2.38]

$$N_\Omega(x) = \{ v \in \mathbb{R}^m : P_\Omega(x + v) = x \} \quad \forall x \in \Omega.$$ (3)

A set $C$ is affine if it contains the lines that pass through any pairs of points $x, y \in C$ with $x \neq y$. Let $\Omega \subset \mathbb{R}^m$ be a nonempty convex set. We say that $x \in \mathbb{R}^m$ is a relative interior point of $\Omega$ if $x \in \Omega$ and there exists an open sphere $S$ centered at $x$ such that $S \cap \text{aff}(\Omega) \subset \Omega$, where $\text{aff}(\Omega)$ is the intersection of all affine sets containing $\Omega$. A pair of vectors $x^* \in \Omega$ and $z^* \in \Psi$ is called a saddle point of the function $\Phi(x, z)$ in $\Omega \times \Psi$ if

$$\Phi(x^*, z) \leq \Phi(x^*, z^*) \leq \Phi(x, z^*) \quad \forall x \in \Omega, \quad \forall z \in \Psi.$$ 

These definitions can be found in [19].

C. Problem Statement

Consider a network of $n$ agents. The objective of the network is to solve the following constrained optimization problem

$$\text{minimize} \quad f(x) = \sum_{i=1}^{n} f_i(x),$$

subject to $x \in \Omega_o = \bigcap_{i=1}^{n} \Omega_i,$

where $f_i(x) : \mathbb{R}^m \to \mathbb{R}$ is the local cost function of agent $i$, and $\Omega_i \subset \mathbb{R}^m$ is the local constraint set of agent $i$. Assume that $f_i(\cdot)$ is a smooth convex function on $\Omega_i$, and $\Omega_i$ is a closed convex set only known to agent $i$. Assume there exists at least one finite solution $x^*$ to the problem (4). For the problem (4), denote by $f^* = \min_{x \in \Omega_o} f(x)$ the optimal value, and by $\Omega^*_o = \{ x \in \Omega_o : f(x) = f^* \}$ the optimal solution set.

Further, assume that for each $i \in \mathcal{V}$, the values of $f_i(\cdot)$ and $\nabla f_i(\cdot)$ are observed with noises. For example, $f_i(x) = E[h_i(x, \vartheta_i)]$, where $h_i : \mathbb{R}^m \times \Theta_i \to \mathbb{R}$ with $\vartheta_i$ being a random variable defined on $\Theta_i$, and the expectation $E[\cdot]$ is taken with respect to $\vartheta_i$. In this case, one may only observe $\nabla h_i(x_i, \vartheta_i)$ for some given samples of $\vartheta_i$, while the exact gradient $\nabla f_i(x_i)$ is difficult to calculate.

Let the communication relationship among agents at time $k$ be described by a directed graph $\mathcal{G}_k = \{ \mathcal{V}, \mathcal{E}_{\mathcal{G}_k}, \mathcal{A}_{\mathcal{G}_k} \}$, where $\mathcal{V} = \{1, \ldots, n\}$ is the node set, $\mathcal{E}_{\mathcal{G}_k}$ is the edge set, and $\mathcal{A}_{\mathcal{G}_k} = [a_{ij,k}]_{i,j=1}^{n}$ is the adjacency matrix. Denote by $\mathcal{L}_k = [l_{ij,k}]_{i,j=1}^{n}$ the Laplacian matrix of digraph $\mathcal{G}_k$. Denote by $\mathcal{N}_i,k = \{ j \in \mathcal{V} : (j, i) \in \mathcal{E}_{\mathcal{G}_k} \}$ the neighbors of agent $i$ at time $k$. Besides, neighboring agents exchange information through channels which may contain noises. The noises may be introduced by quantization errors [22], [23], or actively introduced to achieve differential privacy [24].

DRAFT
III. PRIMAL-DUAL ALGORITHM

We now propose a distributed primal-dual algorithm to solve the distributed stochastic optimization problem, and list some conditions and preliminary lemmas to be used in the sequel.

A. Algorithm Design

Denote by \( x_{i,k} \in \mathbb{R}^m \) the estimate for the optimal solution to problem (4) given by agent \( i \) at time \( k \), and by \( \lambda_{i,k} \in \mathbb{R}^m \) the auxiliary variable of agent \( i \). Hereafter, we call \( x_{i,k} \) and \( \lambda_{i,k} \) the primal and dual variables for agent \( i \) at time \( k \). Agents exchange information in the following way: if \( (j,i) \in \mathcal{E}_k \), then agent \( i \) gets the noisy observations \( \{ x_{ij,k}, \lambda_{ij,k} \} \) of \( \{ x_{j,k}, \lambda_{j,k} \} \) given as follows:

\[
\begin{align*}
    x_{ij,k} &= x_{j,k} + \omega_{ij,k} \quad \text{if} \ (j,i) \in \mathcal{E}_k, \\
    \lambda_{ij,k} &= \lambda_{j,k} + \zeta_{ij,k} \quad \text{if} \ (j,i) \in \mathcal{E}_k,
\end{align*}
\]

where \( \omega_{ij,k} \) and \( \zeta_{ij,k} \) denote the communication noises.

The sequences \( \{ x_{i,k} \} \) and \( \{ \lambda_{i,k} \} \) are updated as follows:

\[
\begin{align*}
    x_{i,k+1} &= P_{\Omega_k}(x_{i,k} - \gamma_k g_{i,k} - \gamma_k \sum_{j \in \mathcal{N}_{i,k}} a_{ij,k}(\lambda_{i,k} - \lambda_{ij,k}) - \gamma_k \sum_{j \in \mathcal{N}_{i,k}} a_{ij,k}(x_{i,k} - x_{ij,k})), \\
    \lambda_{i,k+1} &= \lambda_{i,k} + \gamma_k \sum_{j \in \mathcal{N}_{i,k}} a_{ij,k}(x_{i,k} - x_{ij,k}),
\end{align*}
\]

where \( \gamma_k \) is the step size and \( g_{i,k} \) denotes the noisy observation of \( \nabla f_i(x_{i,k}) \):

\[
g_{i,k} = \nabla f_i(x_{i,k}) + v_{i,k},
\]

where \( v_{i,k} \) is the observation noise. Note that the algorithm (6) is distributed as in an iteration each agent updates its local estimates only using the local gradient observations and the noisy observations for primal and dual variables of its neighbors.

Set \( X_k \triangleq \text{col}\{x_{1,k}, \ldots, x_{n,k}\} \), \( \Lambda_k \triangleq \text{col}\{\lambda_{1,k}, \ldots, \lambda_{n,k}\} \), and \( \nabla \bar{f}(X_k) \triangleq \text{col}\{\nabla f_1(x_{1,k}), \ldots, \nabla f_n(x_{n,k})\} \), where by \( \text{col}\{x_1, \ldots, x_n\} \) we mean \( (x_1^T, \ldots, x_n^T)^T \). Define \( v_k \triangleq \text{col}\{v_{1,k}, \ldots, v_{n,k}\} \), \( \omega_k \triangleq \text{col}\{\omega_{1,k}, \ldots, \omega_{n,k}\} \) with \( \omega_{i,k} \triangleq \sum_{j=1}^n a_{ij,k}\omega_{ij,k} \) and \( \zeta_k \triangleq \text{col}\{\zeta_{1,k}, \ldots, \zeta_{n,k}\} \) with \( \zeta_{i,k} \triangleq \sum_{j=1}^n a_{ij,k}\zeta_{ij,k} \). Then the algorithm (6) can be rewritten in the compact form as follows:

\[
\begin{align*}
    X_{k+1} &= P_{\Omega}(X_k - \gamma_k \nabla \bar{f}(X_k) - \gamma_k (\mathcal{L}_k \otimes \mathbf{I}_m)(\Lambda_k + X_k) + \gamma_k (\zeta_k + \omega_k - v_k)), \\
    \Lambda_{k+1} &= \Lambda_k + \gamma_k (\mathcal{L}_k \otimes \mathbf{I}_m)X_k - \gamma_k \omega_k,
\end{align*}
\]

where \( \Omega = \prod_{i=1}^n \Omega_i \) denotes the Cartesian product, the symbol \( \otimes \) denotes the Kronecker product, and \( \mathbf{I}_m \) denotes the identity matrix of size \( m \).

B. Assumptions

We impose the following assumptions on the constraint sets and on the cost functions.

Assumption 1:

a) \( \Omega_o \) has at least one relative interior point.
b) There exists a constant $L_f > 0$ such that for any $i \in \mathcal{V}$
\[ \| \nabla f_i(x) - \nabla f_i(y) \| \leq L_f \| x - y \| \quad \forall x, y \in \Omega_i. \quad (9) \]

c) For any $i \in \mathcal{V}$, the set $\Omega_i$ is determined by $p_i$ inequalities:
\[ \Omega_i = \{ x \in \mathbb{R}^m : q_{ij}(x) \leq 0, \forall j = 1, \cdots, p_i \}, \]
where $q_{ij}()$, $j = 1, \cdots, p_i$ are continuously differentiable real-valued functions on $\mathbb{R}^m$. Moreover, $\{\nabla q_{ij}(x), j \in A_i(x)\}$ are linearly independent, where $A_i(x) = \{ j : q_{ij}(x) = 0 \}$.

**Remark 3.1:** The existence of the relative interior point will be used to guarantee that the primal and dual problems defined in Section III C have the same optimal solution. The globally Lipschitz condition is used to guarantee the boundedness of the estimates. Assumption 1-c indicates that all local constraint sets have smooth boundaries. In fact, Assumption 1-c corresponds to A4.3.2 in [27] but without compactness requirement.

The following conditions are imposed on the communication graphs and on the adjacency matrices.

**Assumption 2:** (Mean graph is connected and undirected)
a) $(\mathcal{A}_{\bar{G}})_k \geq 0$ is an i.i.d sequence with expectation denoted by $\bar{A} = E[\mathcal{A}_{\bar{G}}]$.
b) The graph $\mathcal{G}_{\bar{A}}$ generated by $\bar{A}$ is undirected and connected.
c) There exists a constant $\eta > 0$ such that
\[ E[a_{ij,k}^2] = \sigma_{ij}^2 \leq \eta^2 \quad \forall i, j \in \mathcal{V}. \]
d) $\mathcal{L}_k$ is independent of $\mathcal{F}_{k-1}$, where
\[ \mathcal{F}_k = \sigma \{ X_0, \Lambda_0, \omega_{ij,t}, \zeta_{ij,t}, v_{i,t}, \mathcal{L}_t, 0 \leq t \leq k, 1 \leq i, j \leq n \}. \quad (10) \]

**Remark 3.2:** Note that Assumption 2 does not require the random graph at any instance be undirected or strongly connected. It only requires the mean graph be undirected and connected. The gossip-based communication protocol [20] and the broadcast-based communication [21] both satisfy Assumption 2 when the underlying graph is bidirectional and strongly connected.

Set
\[ \mathcal{F}'_k = \sigma \{ \mathcal{L}_{k+1}, \mathcal{F}_k \}. \quad (11) \]

Note that the adjacency matrix $\mathcal{A}_{\bar{G}}$ is uniquely defined by $\mathcal{L}_k$ with $a_{ij,k} = -l_{ij,k} \forall i \neq j$ and $a_{ii,k} = 0$. Thus, the covariance of $\mathcal{L}_k$ is finite by Assumption 2-c, $\mathcal{A}_{\bar{G}}$ is independent of $\mathcal{F}_{k-1}$ by Assumption 2-d, and $\mathcal{L}_k$ is adapted to $\mathcal{F}'_{k-1}$ by its definition (11).

The following conditions are imposed on the communication noises and gradient errors.

**Assumption 3:**
a) For any $i, j \in \mathcal{V}$, $\{\omega_{ij,k}, \mathcal{F}'_k\}$ is an mds with
\[ E[\omega_{ij,k}|\mathcal{F}'_{k-1}] = 0, \quad E[||\omega_{ij,k}\|^{2}|\mathcal{F}'_{k-1}] \leq \mu^2, \]
and
\[ E[\omega_{ij,k}\omega_{ij,k}^T|\mathcal{F}'_{k-1}] \triangleq R_{\omega,ij}. \quad (12) \]
b) For any $i,j \in \mathcal{V}$, \{\(\zeta_{ij,k}, \mathcal{F}_k^j\)\} is an mds with
\[
E[\zeta_{ij,k} | \mathcal{F}_{k-1}^j] = 0, \quad E[\|\zeta_{ij,k}\|^2 | \mathcal{F}_{k-1}^j] \leq \mu^2,
\]
and
\[
E[\zeta_{ij,k} \zeta_{ij,k}^T | \mathcal{F}_{k-1}^j] \triangleq R_{\zeta,ij}.
\] (13)

For any $i \in \mathcal{V}$, \{\(v_{i,k}, \mathcal{F}_k^j\)\} is an mds with
\[
E[v_{i,k} | \mathcal{F}_{k-1}^j] = 0, \quad E[\|v_{i,k}\|^2 | \mathcal{F}_{k-1}^j] \leq c_v (1 + \|x_{i,k}\|^2),
\] (14)
\[
\lim_{k \to \infty} E[v_{i,k} v_{i,k}^T | \mathcal{F}_{k-1}^j] \triangleq R_{v,i}.
\] (15)

In Section IV, (12), (13) and (15) are not needed. The simplified version of Assumption 3 with (12), (13) and (15) removed will be called Assumption 4.

**Assumption 4:**

a) Assumption 3-a with (12) removed.

b) Assumption 3-b with (13) removed.

c) Assumption 3-c with (15) removed.

**Remark 3.3:** The communication noises introduced by the probabilistic quantization [22], [23] is shown to be an i.i.d sequence with bounded second moments, and hence satisfy Assumption 4-a and 4-b. Assumption 3-c holds true in many cases, for example, in the quadratic distributed stochastic optimization problem (101) discussed in Section VI.

By Assumption 2-a, \{\(\mathcal{L}_k\)\}_{k \geq 0} is an i.i.d sequence. Set \(\bar{\mathcal{L}} \triangleq E[\mathcal{L}_k]\). Then \(\bar{\mathcal{L}}\) is the Laplacian matrix of the undirected connected graph \(G_{\bar{A}}\). Define
\[
e_{1,k} \triangleq ((\bar{\mathcal{L}} - \mathcal{L}_k) \otimes I_m) (\Lambda_k + X_k),
\] (16)
\[
e_{2,k} \triangleq \zeta_k + \omega_k - v_k,
\] (17)
\[
e_{3,k} \triangleq ((\mathcal{L}_k - \bar{\mathcal{L}}) \otimes I_m) X_k - \omega_k.
\] (18)

Then (8) can be rewritten as:
\[
X_{k+1} = P_{\Omega} \left( X_k - \gamma_k \nabla \tilde{f}(X_k) - \gamma_k (\bar{\mathcal{L}} \otimes I_m)(\Lambda_k + X_k) + \gamma_k (e_{1,k} + e_{2,k}) \right),
\] (19)
\[
\Lambda_{k+1} = \Lambda_k + \gamma_k (\bar{\mathcal{L}} \otimes I_m) X_k + \gamma_k e_{3,k}.
\]

We impose the following condition on the step size \{\(\gamma_k\)\}.

**Assumption 5:**

\[
\gamma_k > 0, \quad \sum_{k=1}^{\infty} \gamma_k = \infty, \text{ and } \sum_{k=1}^{\infty} \gamma_k^2 < \infty.
\]
C. Preliminary Lemmas

We now give some preliminary results about the formulated distributed optimization problem.

**Lemma 3.4:** The problem (4) is equivalent to the following constrained optimization problem

\[
\begin{align*}
\text{minimize} & \quad \tilde{f}(X) \triangleq \sum_{i=1}^{n} f_i(x_i), \\
\text{subject to} & \quad (\bar{L} \otimes I_m)X = 0, \quad X \in \Omega,
\end{align*}
\]

(20)

where \(X = \text{col}\{x_1, \cdots, x_n\}\).

The result can be easily derived since \((\bar{L} \otimes I_m)X = 0\) if and only if \(x_i = x_j \forall i, j \in V\).

Define \(\Phi(X, \Lambda) \triangleq \tilde{f}(X) + \Lambda^T(\bar{L} \otimes I_m)X\) as the Lagrange function, where \(\Lambda \in \mathbb{R}^{mn}\) is the Lagrange multiplier. Then the problem (20) can be rewritten as \(\inf_{X \in \Omega} \sup_{\Lambda \in \mathbb{R}^{mn}} \Phi(X, \Lambda)\), while the dual problem is defined as follows \(\sup_{\Lambda \in \mathbb{R}^{mn}} \inf_{X \in \Omega} \Phi(X, \Lambda)\).

**Lemma 3.5:** Assume Assumption 1-a and Assumption 2-b hold. Then \(\Phi(X, \Lambda)\) has at least one saddle point in \(\Omega \times \mathbb{R}^{mn}\). A pair \((X^*, \Lambda^*)\) is the primal-dual solution to the problems (20) and (21) if and only if \((X^*, \Lambda^*)\) is a saddle point of \(\Phi(X, \Lambda)\) on \(\Omega \times \mathbb{R}^{mn}\).

**Proof:** Assumption 1-a implies that there exists a relative interior \(\bar{X}\) of set \(\Omega\) such that \((L \otimes I_m)\bar{X} = 0\). Since \(f^*\) is finite, by [19, Proposition 5.3.3] we know that \(\inf_{X \in \Omega} \sup_{\Lambda \in \mathbb{R}^{mn}} \Phi(X, \Lambda) = \sup_{\Lambda \in \mathbb{R}^{mn}} \inf_{X \in \Omega} \Phi(X, \Lambda)\), and there exists at least one dual optimal solution.

Since the minimax equality (22) holds, by [19, Proposition 3.4.1] \(X^*\) is the primal optimal solution and \(\Lambda^*\) is the dual optimal solution if and only if \((X^*, \Lambda^*)\) is a saddle point of \(\Phi(X, \Lambda)\) on \(\Omega \times \mathbb{R}^{mn}\). Since there exists at least one primal and dual optimal solution pair, we conclude that \(\Phi(X, \Lambda)\) has at least one saddle point in \(\Omega \times \mathbb{R}^{mn}\). This completes the proof.  

IV. CONVERGENCE THEOREMS

In this section, we analyze stability and convergence of the algorithm (6). For notational simplicity, we assume \(m = 1\) in this section. This does not influence the convergence analysis for the general case \(m \geq 1\).

A. Stability Analysis

**Theorem 4.1:** (Stability) Let \(\{x_{i,k}\}\) and \(\{\lambda_{i,k}\}\) be produced by the algorithm (6) with any initial values \(x_{i,0}, \lambda_{i,0}\). Let Assumptions 1-a, 1-b, 2, 4 and 5 hold. Then \(\|X_k - X^*\|^2 + \|\Lambda_k - \Lambda^*\|^2\) converges a.s., where \((X^*, \Lambda^*)\) is a saddle point of \(\Phi(X, \Lambda)\) in \(\Omega \times \mathbb{R}^{mn}\).

This theorem establishes that the sequences \(\{X_k\}\) and \(\{\Lambda_k\}\) are bounded a.s., and the distance between the pair \((X_k, \Lambda_k)\) and the saddle point \((X^*, \Lambda^*)\) converges a.s. Before proving the theorem, we first give some preliminary lemmas. The following lemma establishes properties of noise sequences \(\{e_{1,k}\}\), \(\{e_{2,k}\}\) and \(\{e_{3,k}\}\) defined in (16), (17), and (18), respectively.
Lemma 4.2: Let Assumptions 2\(a\), 2\(c\), 2\(d\), and 4 hold. Then the following assertions take place a.s.

\[ E[e_{1,k} | F_{k-1}] = 0, \quad E[\|e_{1,k}\|^2 | F_{k-1}] \leq C_{01} \|\Lambda_k + X_k\|^2, \]  
\[ E[e_{2,k} | F_{k-1}] = 0, \quad E[\|e_{2,k}\|^2 | F_{k-1}] = C_{02} + 3c_n \|X_k\|^2, \]  
\[ E[e_{3,k} | F_{k-1}] = 0, \quad E[\|e_{3,k}\|^2 | F_{k-1}] \leq C_{01} \|X_k\|^2 + C_{03}, \]  

where \( C_{01} = E[\|\mathcal{L}_k - \bar{L}\|^2], \ C_{02} = 3c_n n + 6n^3 \mu^2 \eta^2, \) and \( C_{03} = n^3 \mu^2 \eta^2. \)

**Proof:** By Assumption 2\(d\) we have

\[ E[\bar{L} - L_k | F_{k-1}] = \bar{L} - E[L_k] = 0. \]  

Since \( X_k \) and \( \Lambda_k \) are adapted to \( F_{k-1} \) by (6) (10), from (16) (26) it follows that

\[ E[e_{1,k} | F_{k-1}] = E[\bar{L} - L_k | F_{k-1}] (\Lambda_k + X_k) = 0, \]  

and

\[ E[\|e_{1,k}\|^2 | F_{k-1}] \leq E[\|\bar{L} - L_k\|^2 | F_{k-1}] \cdot \|\Lambda_k + X_k\|^2 \]  
\[ = E[\|L_k - \bar{L}\|^2] \cdot \|\Lambda_k + X_k\|^2. \]

Therefore, (23) holds.

Since \( a_{i,j,k} \) is adapted to \( F_{k-1}^{\prime} \) by (11), from Assumption 4\(a\) it follows that for any \( i \in \mathcal{V} \)

\[ E[\omega_{i,k} | F_{k-1}^{\prime}] = \sum_{j=1}^{n} a_{i,j,k} E[\omega_{i,j,k} | F_{k-1}^{\prime}] = 0. \]  

(27)

Similarly, by Assumption 4\(b\) it is shown that

\[ E[\zeta_{i,k} | F_{k-1}^{\prime}] = 0 \quad \forall i \in \mathcal{V}. \]  

(28)

Then from (27) (28) and Assumption 4\(c\), by (17) we derive

\[ E[e_{2,k} | F_{k-1}^{\prime}] = E[\omega_{k} | F_{k-1}^{\prime}] + E[\zeta_{k} | F_{k-1}^{\prime}] + E[v_{k} | F_{k-1}^{\prime}] = 0. \]  

(29)

Since \( F_{k-1} \subset F_{k-1}^{\prime} \), by (27) (29) we see

\[ E[\omega_{k} | F_{k-1}] = E[E[\omega_{k} | F_{k-1}^{\prime}] | F_{k-1}] = 0 \quad \text{a.s.}, \]  

(30)

\[ E[e_{2,k} | F_{k-1}] = E[E[e_{2,k} | F_{k-1}^{\prime}] | F_{k-1}] = 0 \quad \text{a.s.}. \]

Since \( a_{i,j,k} \) is adapted to \( F_{k-1}^{\prime} \) by (11), from Assumption 4\(a\) it follows that

\[ E[\|a_{i,j,k} \omega_{i,j,k}\|^2 | F_{k-1}^{\prime}] \leq a_{i,j,k}^2 E[\|\omega_{i,j,k}\|^2 | F_{k-1}^{\prime}] \leq a_{i,j,k}^2 \mu^2. \]

Since \( a_{i,j,k} \) is independent of \( F_{k-1} \) by Assumption 2\(d\), from \( F_{k-1} \subset F_{k-1}^{\prime} \) by Assumption 2\(c\) we obtain

\[ E[\|a_{i,j,k} \omega_{i,j,k}\|^2 | F_{k-1}] = E[E[\|a_{i,j,k} \omega_{i,j,k}\|^2 | F_{k-1}^{\prime}] | F_{k-1}] \]  
\[ \leq \mu^2 E[a_{i,j,k}^2] \leq \mu^2 \eta^2 \quad \forall i \in \mathcal{V} \quad \text{a.s.} \]
Then by the conditional Minkowski inequality \( \left( E[\| \sum_{i=1}^{k} X_i \|^2 | \mathcal{F} ] \right)^{\frac{1}{2}} \leq \sum_{i=1}^{k} \left( E[\| X_i \|^2 | \mathcal{F} ] \right)^{\frac{1}{2}} \), and by \( \omega_{i,k} = \sum_{j=1}^{n} a_{ij,k} \omega_{ij,k} \) we derive

\[
\left( E[\| \omega_{i,k} \|^2 | \mathcal{F}_{k-1} ] \right)^{\frac{1}{2}} \leq \sum_{j=1}^{n} \left( E[\| a_{ij,k} \omega_{ij,k} \|^2 | \mathcal{F}_{k-1} ] \right)^{\frac{1}{2}} \leq n \mu \eta \ \forall i \in \mathcal{V} \text{ a.s.}
\]

Similarly, by Assumption F b we derive

\[
\left( E[\| \zeta_{i,k} \|^2 | \mathcal{F}_{k-1} ] \right)^{\frac{1}{2}} \leq n \mu \eta \ \forall i \in \mathcal{V} \text{ a.s.}
\]

Then by the definitions of \( \omega_k \) and \( \zeta_k \) we conclude that

\[
E[\| \omega_k \|^2 | \mathcal{F}_{k-1} ] = \sum_{i=1}^{n} E[\| \omega_{i,k} \|^2 | \mathcal{F}_{k-1} ] \leq n^3 \mu^2 \eta^2 \text{ a.s.,}
\]

\[
E[\| \zeta_k \|^2 | \mathcal{F}_{k-1} ] = \sum_{i=1}^{n} E[\| \zeta_{i,k} \|^2 | \mathcal{F}_{k-1} ] \leq n^3 \mu^2 \eta^2 \text{ a.s.}
\]

By (14) we have

\[
E[\| v_k \|^2 | \mathcal{F}_{k-1} ] = \sum_{i=1}^{n} E[\| v_{i,k} \|^2 | \mathcal{F}_{k-1} ] \leq c_v (n + \| X_k \|^2).
\]

Then by noticing that \( \mathcal{F}_{k-1} \subset \mathcal{F}_{k-1}' \) and \( X_k \) is adapted to \( \mathcal{F}_{k-1} \) we have

\[
E[\| v_k \|^2 | \mathcal{F}_{k-1} ] = E\left[ E[\| v_k \|^2 | \mathcal{F}'_{k-1} ] | \mathcal{F}_{k-1} \right] \leq c_v (n + \| X_k \|^2) \text{ a.s.}
\]

Thus, by (17) from (31) (32) we obtain

\[
E[\| e_{2,k} \|^2 | \mathcal{F}_{k-1} ] = 3 \left( E[\| \omega_k \|^2 | \mathcal{F}_{k-1} ] + E[\| \zeta_k \|^2 | \mathcal{F}_{k-1} ] + E[\| v_k \|^2 | \mathcal{F}_{k-1} ] \right)
\]

\[
\leq 6n^3 \mu^2 \eta^2 + 3c_v (n + \| X_k \|^2) \text{ a.s.}
\]

Hence (24) holds.

We now consider properties of the noise sequence \( \{ e_{3,k} \} \) defined in (18). Since \( X_k \) is adapted to \( \mathcal{F}_{k-1} \), by (26) (30) we have

\[
E[\omega_{3,k} | \mathcal{F}_{k-1} ] = E[\tilde{\mathcal{L}} - \mathcal{L}_k | \mathcal{F}_{k-1} ] X_k - E[\omega_k | \mathcal{F}_{k-1} ] = 0 \text{ a.s.}
\]

Since \( X_k, \mathcal{L}_k \) are adapted to \( \mathcal{F}'_{k-1} \) and \( \mathcal{F}_{k-1} \subset \mathcal{F}'_{k-1} \), by (27) we derive

\[
E[\omega_{3,k}^T (\mathcal{L}_k - \tilde{\mathcal{L}}) X_k | \mathcal{F}_{k-1} ] = E\left[ E[\omega_{3,k}^T | \mathcal{F}'_{k-1} ] (\mathcal{L}_k - \tilde{\mathcal{L}}) X_k | \mathcal{F}_{k-1} \right] = 0 \text{ a.s.}
\]

Hence by (31) and Assumption 2 d we conclude that

\[
E[\| e_{3,k} \|^2 | \mathcal{F}_{k-1} ] = E[\| (\mathcal{L}_k - \tilde{\mathcal{L}}) X_k \|^2 | \mathcal{F}_{k-1} ] + E[\| \omega_k \|^2 | \mathcal{F}_{k-1} ] + 2E[\omega_k^T (\mathcal{L}_k - \tilde{\mathcal{L}}) X_k | \mathcal{F}_{k-1} ]
\]

\[
\leq E[\| \mathcal{L}_k - \tilde{\mathcal{L}} \|^2 ] \| X_k \|^2 + n^3 \mu^2 \eta^2 \text{ a.s.}
\]

Therefore, (25) holds.
Lemma 4.3: Let Assumptions \[a, b, c, d \text{ and } 4\] hold. Then for any \( X \in \Omega \) and \( \Lambda \in \mathbb{R}^{mn} \)

\[
E[\|X_{k+1} - X\|^2 | \mathcal{F}_{k-1}] \leq \|X_k - X\|^2 + \gamma_k^2 \|\nabla \tilde{f}(X_k) + \tilde{L}(\Lambda_k + X_k)\|^2 \\
+ 2\gamma_k (\Phi(X, \Lambda_k) - \Phi(X_k, \Lambda_k)) - 2\gamma_k (X_k - X)^T \tilde{L} X_k \\
+ C_{01} \gamma_k^2 \|\Lambda(k) + X_k\|^2 + 3c_v \gamma_k^2 \|X_k\|^2 + C_{02} \gamma_k^2 \text{ a.s.,}
\]

and

\[
E[\|\Lambda_{k+1} - \Lambda\|^2 | \mathcal{F}_{k-1}] \leq \|\Lambda_k - \Lambda\|^2 + \gamma_k^2 \|\tilde{L} X_k\|^2 + C_{03} \gamma_k^2 \\
+ 2\gamma_k (\Phi(X, \Lambda_k) - \Phi(X_k, \Lambda)) + C_{01} \gamma_k^2 \|X_k\|^2 \text{ a.s.}
\]

Proof: By using the non-expansive property \((2)\) of the projection operator, from \((19)\) we obtain

\[
\|X_{k+1} - X\|^2 \leq \|X_k - \gamma_k \nabla \tilde{f}(X_k) - \gamma_k \tilde{L}(\Lambda_k + X_k) - X + \gamma_k (e_{1,k} + e_{2,k})\|^2 \\
\leq I_0(k) + \gamma_k^2 I_1(k) + 2\gamma_k I_2(k) \quad \forall X \in \Omega,
\]

where \( I_0(k) = \|X_k - \gamma_k \nabla \tilde{f}(X_k) - \gamma_k \tilde{L}(\Lambda_k + X_k) - X\|^2 \), \( I_1(k) = \|e_{1,k} + e_{2,k}\|^2 \), \( I_2(k) = (e_{1,k} + e_{2,k})^T (X_k - \gamma_k \nabla \tilde{f}(X_k) - \gamma_k \tilde{L}(\Lambda_k + X_k) - X) \).

Since \( e_{1,k} \) is adapted to \( \mathcal{F}_{k-1} \) by \((11)\) \((16)\), by \( \mathcal{F}_k \subset \mathcal{F}_k' \) and \((29)\) we see that

\[
E[e_{1,k}^T e_{2,k} | \mathcal{F}_{k-1}] = E[E[e_{1,k}^T e_{2,k} | \mathcal{F}_{k-1}'] | \mathcal{F}_{k-1}] = E[E[e_{1,k}^T e_{2,k} | \mathcal{F}_{k-1}'] | \mathcal{F}_{k-1}] = 0 \text{ a.s.}
\]

Thus, from here by \((23)\) \((24)\) we derive

\[
E[I_1(k) | \mathcal{F}_{k-1}] = E[\|e_{1,k}\|^2 | \mathcal{F}_{k-1}] + E[\|e_{2,k}\|^2 | \mathcal{F}_{k-1}] + 2E[e_{1,k}^T e_{2,k} | \mathcal{F}_{k-1}] \\
\leq C_{01} \|\Lambda_k + X_k\|^2 + C_{02} + 3c_v \|X_k\|^2 \text{ a.s.}
\]

Since \( X_k, \Lambda_k \) are adapted to \( \mathcal{F}_{k-1} \), by \((23)\) \((24)\) we derive

\[
E[I_2(k) | \mathcal{F}_{k-1}] = E[e_{1,k} + e_{1,k} | \mathcal{F}_{k-1}]^T (X_k - \gamma_k \nabla \tilde{f}(X_k) - \gamma_k \tilde{L}(\Lambda_k + X_k) - X) = 0 \text{ a.s.}
\]

Since \( I_0(k) \) is adapted to \( \mathcal{F}_{k-1} \), combining \((37)\), \((39)\), \((40)\) we obtain

\[
E[\|X_{k+1} - X\|^2 | \mathcal{F}_{k-1}] \leq I_0(k) + C_{01} \gamma_k^2 \|\Lambda_k + X_k\|^2 + 3c_v \gamma_k^2 \|X_k\|^2 + C_{02} \gamma_k^2 \text{ a.s.}
\]

Note that

\[
I_0(k) = \|X_k - \gamma_k \nabla \tilde{f}(X_k) - \gamma_k \tilde{L}(\Lambda_k + X_k) - X\|^2 \\
\leq \|X_k - X\|^2 + \gamma_k^2 \|\nabla \tilde{f}(X_k) + \tilde{L}(\Lambda_k + X_k)\|^2 \\
- 2\gamma_k (X_k - X)^T (\nabla \tilde{f}(X_k) + \tilde{L}(\Lambda_k + X_k))
\]

Since \( \Phi(X, \Lambda_k) \) is convex in \( X \in \Omega \), by \((1)\) we derive

\[
\Phi(X, \Lambda_k) \geq \Phi(X_k, \Lambda_k) + (X - X_k)^T (\nabla \tilde{f}(X_k) + \tilde{L} \Lambda_k),
\]

and hence

\[
-(X_k - X)^T (\nabla \tilde{f}(X_k) + \tilde{L} \Lambda_k) \leq \Phi(X, \Lambda_k) - \Phi(X_k, \Lambda_k).
\]
Then by (42) we conclude that
\[
I_0(k) \leq \|X_k - X\|^2 + \gamma_k^2 \|\nabla \bar{f}(X_k) + \bar{L}(\Lambda_k + X_k)\|^2 \\
+ 2\gamma_k (\Phi(X, \Lambda_k) - \Phi(X, \Lambda_k)) - 2\gamma_k (X_k - X)^T \bar{L}X_k,
\]
which incorporating with (41) yields (35).

For any \( \Lambda \in \mathbb{R}^m \)
\[
\|\Lambda_{k+1} - \Lambda\|^2 = \|\Lambda_k + \gamma_k \bar{L}X_k - \Lambda + \gamma_k e_{3,k}\|^2 \\
= I_3(k) + \gamma_k^2 \|e_{3,k}\|^2 + 2\gamma_k e_{3,k}^T (\Lambda_k + \gamma_k \bar{L}X_k - \Lambda),
\] (43)
where \( I_3(k) = \|\Lambda_k + \gamma_k \bar{L}X_k - \Lambda\|^2 \).

Since \( X_k \) and \( \Lambda_k \) are adapted to \( F_{k-1} \), from (25) we see
\[
E[e_{3,k}^T (\Lambda_k + \gamma_k \bar{L}X_k - \Lambda)|F_{k-1}] = E[e_{3,k}^T |F_{k-1}] (\Lambda_k + \gamma_k \bar{L}X_k - \Lambda) = 0 \text{ a.s.}
\]
Noticing that \( I_3(k) \) is adapted to \( F_{k-1} \), from here by (25) (43) we obtain
\[
E[\|\Lambda_{k+1} - \Lambda\|^2|F_{k-1}] \leq I_3(k) + C_01\gamma_k^2 \|X_k\|^2 + C_03\gamma_k^2 \text{ a.s.}
\] (44)

By the definition of \( \Phi(X, \Lambda) \), we derive
\[
\Phi(X_k, \Lambda_k) = \Phi(X_k, \Lambda) + (\Lambda_k - \Lambda)^T \bar{L}X_k
\]
and hence
\[
I_3(k) = \|\Lambda_k - \Lambda\|^2 + \|\gamma_k \bar{L}X_k\|^2 + 2\gamma_k (\Lambda_k - \Lambda)^T \bar{L}X_k \\
= \|\Lambda_k - \Lambda\|^2 + \|\gamma_k \bar{L}X_k\|^2 + 2\gamma_k (\Phi(X, \Lambda_k) - \Phi(X, \Lambda)),
\]
which incorporating with (44) yields (35).

**Proof of Theorem 4.1.** Summing up both sides of (35) and (36), and by replacing \((X, \Lambda)\) with \((X^*, \Lambda^*)\) we obtain
\[
E[\|X_{k+1} - X^*\|^2|F_{k-1}] + E[\|\Lambda_{k+1} - \Lambda^*\|^2|F_{k-1}] \\
\leq \|X_k - X^*\|^2 + \|\Lambda_k - \Lambda^*\|^2 + \gamma_k^2 \|\nabla \bar{f}(X_k) + \bar{L}(\Lambda_k + X_k)\|^2 \\
+ 2\gamma_k (\Phi(X^*, \Lambda_k) - \Phi(X_k, \Lambda^*)) - 2\gamma_k (X_k - X^*)^T \bar{L}X_k \\
+ C_01\gamma_k^2 \|\Lambda_k + X_k\|^2 + \gamma_k^2 \|\bar{L}X_k\|^2 + \gamma_k^2 (C_01 + 3C_0) \|X_k\|^2 + (C_02 + C_03)\gamma_k^2 \text{ a.s.}
\] (45)

Since \((X^*, \Lambda^*)\) is a saddle point for \( \Phi(X, \Lambda) \), by Lemma 3.5 \( X^* \) is the optimal solution to the problem (20). Then from Lemma 3.4 it follows that
\[
\bar{L}X^* = 0, \text{ and } \bar{L}X_k = \bar{L}(X_k - X^*),
\] (46)
and hence
\[
\nabla \bar{f}(X_k) + \bar{L}(\Lambda_k + X_k) = \nabla \bar{f}(X_k) - \nabla \bar{f}(X^*) + \bar{L}(\Lambda_k - \Lambda^*) + \bar{L}(X_k - X^*) + \bar{L}\Lambda^* + \nabla \bar{f}(X^*).
\]
Then by (9) we obtain
\[
\|\nabla f(X_k) + \tilde{L}(\Lambda_k + X_k)\|^2 \\
\leq 4(\|\nabla f(X_k) - \nabla f(X^*)\|^2 + \|\tilde{L}(\Lambda_k - \Lambda^*)\|^2 + \|\tilde{L}(X_k - X^*)\|^2 + \|\tilde{L}\Lambda^* + \nabla f(X^*)\|^2)
\] (47)
\[
\leq 4c_1\|\Lambda_k - \Lambda^*\|^2 + (4c_1 + 4L_f^2)\|X_k - X^*\|^2 + c_2,
\]
where \(c_1 = \|\tilde{L}\|^2\), \(c_2 = 4\|\tilde{L}\Lambda^* + \nabla f(X^*)\|^2\). From (46) we derive
\[
\|\tilde{L}X(k)\|^2 \leq c_1\|X_k - X^*\|^2.
\] (48)

Note that \(\|\Lambda_k + X_k\|^2 \leq 3(\|\Lambda_k - \Lambda^*\|^2 + \|X_k - X^*\|^2 + \|\Lambda^* + X^*\|^2)\) and \(\|X_k\|^2 \leq 2(\|X_k - X^*\|^2 + \|X^*\|^2)\).

Then by (45), (47) and (48) we derive
\[
E[\|X_{k+1} - X^*\|^2|\mathcal{F}_{k-1}] + E[\|\Lambda_{k+1} - \Lambda^*\|^2|\mathcal{F}_{k-1}] \\
\leq (1 + (5c_1 + 5C_{10} + 4L_f^2 + 6c_v)\gamma_k^2)\|X_k - X^*\|^2 \\
+ (1 + (4c_1 + 3C_{01})\gamma_k^2)\|\Lambda_k - \Lambda^*\|^2 + \gamma_k^2(\|X_k - X^*\|^2 + 2\gamma_k(\Phi(X^*, \Lambda_k) - \Phi(X_k, \Lambda^*)) - 2\gamma_k(X_k - X^*)^T\tilde{L}X_k)
\] (49)

Since \(\tilde{L}\) is the Laplacian matrix of some connected undirected graph by Assumption 2b, from (46) and Lemma 2.1 it follows that
\[
(X_k - X^*)^T\tilde{L}X_k = (X_k - X^*)^T\tilde{L}(X_k - X^*) \geq 0.
\] (50)

Noticing \(X_k \in \Omega\), by definition of the saddle point we see
\[
\Phi(X^*, \Lambda_k) \leq \Phi(X_k, \Lambda^*) \leq \Phi(X_k, \Lambda^*) \quad \forall k \geq 0.
\]

Then by setting \(V_k = \|X_k - X^*\|^2 + \|\Lambda_k - \Lambda^*\|^2\), from (49), (50) we derive
\[
E[V_{k+1}|\mathcal{F}_{k-1}] \leq (1 + C_{11}\gamma_k^2)V_k + C_{12}\gamma_k^2 \quad a.s.,
\]
where \(C_{11} = 5c_1 + 5C_{10} + 4L_f^2 + 6c_v\), and \(C_{12} = c_2 + C_{10} + 3C_{01}\|\Lambda^* + X^*\|^2 + 2(3c_v + C_{01})\|X^*\|^2\).

Consequently, by Assumption 5 and Lemma A.1 in Appendix we conclude that \(\|X_k - X^*\|^2 + \|\Lambda_k - \Lambda^*\|^2\) converges a.s.

\[\square\]

\section*{B. Consensus and Consistency}

The following theorem shows that the estimates given by all agents reach a consensus belonging to the optimal solution set of problem (4).

\textit{Theorem 4.4:} Let \(\{x_i,k\}\) and \(\{\lambda_i,k\}\) be produced by the algorithm (6) with any initial values \(x_{i,0}\), \(\lambda_{i,0}\). Let Assumptions 1, 2, 4, and 5 hold. Then

i) (Consensus) \(\lim_{k \to \infty} (x_{i,k} - x_{j,k}) = 0 \quad \forall i, j \in \mathcal{V} \quad a.s.\)

ii) (Consistency) \(\lim_{k \to \infty} d(x_{i,k}, \Omega^*_\infty) = 0 \quad \forall i \in \mathcal{V} \quad a.s.,\) (51)
Thus, \( E \) where in the last inequality we have used we conclude that \( \theta \)

\( B_2 \) holds. By the definition of \( \Omega \) to some limit set of the following projected ODE in Appendix to prove the theorem. Thus, we have to verify \( B_1-B_4 \).

we can rewrite (19) in the form of algorithm (A.1) with \( Y \)

we can rewrite (19) in the form of algorithm (A.1) with \( Y_k = g(\theta_k) + e_k \). We intend to use Lemma A.2 in Appendix to prove the theorem. Thus, we have to verify \( B_1-B_4 \).

Since \( X_k, \, \Lambda_k \) are bounded a.s., from (23), (24), (25) we conclude that

\[
E[\|e_k\|^2] \leq 2E[\|e_{1,k}\|^2] + 2E[\|e_{2,k}\|^2] + E[\|e_{3,k}\|^2] < \infty.
\]

Thus, \( E[\|Y_k\|^2] = 2E[\|g(\theta_k)\|^2] + 2E[\|e_k\|^2] < \infty \), and hence \( B_1 \) holds. From (23) (24) (25) it follows that \( B_2 \) holds. By the definition of \( g(\theta) \), from Assumption 1-b it is seen that \( B_3 \) holds. By Theorem 4.1 we conclude that \( \theta_k \) is bounded a.s., and hence \( B_4 \) holds.

In summary, we have validated \( B_1-B_4 \). Then by Lemma A.2 we conclude that \( (X_k, \Lambda_k) \) converge a.s. to some limit set of the following projected ODE in \( \Omega \times \mathbb{R}^{mn} \):

\[
\dot{X}(t) = -\nabla \bar{f}(X(t)) - \bar{L}(X(t) + \Lambda(t)) - Z(t), \quad Z(t) \in N_{\Omega}(X(t)),
\]

\[
\dot{\Lambda}(t) = \bar{L}X(t),
\]

where \( Z(\cdot) \) is the projection or constraint term, the minimum force needed to keep \( X(\cdot) \) in \( \Phi \).

Define \( V(X, \Lambda) = \|X - X^*\|^2 + \|\Lambda - \Lambda^*\|^2 \). By (53) we derive

\[
\dot{V}(X, \Lambda) = (X - X^*)^T \dot{X} + (\Lambda - \Lambda^*)^T \dot{\Lambda}
\]

\[
= -(X - X^*)^T \bar{f}(X) - (X - X^*)^T \bar{L}(X + \Lambda) - (X - X^*)^T Z + (\Lambda - \Lambda^*)^T \bar{L}X.
\]

Since \( Z(t) \in N_{\Omega}(X(t)) \), by the definition of normal cone we derive \( (X - X^*)^T Z \geq 0 \). Since \( \bar{L} \) is symmetric, by (46) we derive

\[
\dot{V}(X, \Lambda) \leq -(X - X^*)^T \bar{f}(X) - X^T \bar{L}X - X^T \bar{L} \Lambda + \Lambda^T \bar{L}X - (\Lambda^*)^T \bar{L}(X - X^*)
\]

\[
\leq -(X - X^*)^T (\bar{f}(X) + \bar{L} \Lambda^*) - X^T \bar{L}X
\]

\[
\leq \Phi(X^*, \Lambda^*) - \Phi(X, \Lambda^*) - X^T \bar{L}X,
\]

where in the last inequality we have used \( \Phi(X^*, \Lambda^*) \geq \Phi(X, \Lambda^*) + (X^* - X)^T (\nabla \bar{f}(X) + \bar{L} \Lambda^*) \) since \( \Phi(X, \Lambda^*) \) is convex with respect to \( X \). Noting that \( \bar{L} \) is positive semi-definite, by the definition of saddle point we derive

\[
\dot{V}(X, \Lambda) \leq 0.
\]

By the LaSalle invariant theorem [30], the trajectories produced by (53) converge to the largest invariant set contained in the set \( S = \{(X, \Lambda) \in \Omega \times \mathbb{R}^{mn} : \dot{V}(X, \Lambda) = 0 \} \). By (54) it is clear that \( S = \{(X, \Lambda) \in \Omega \times \mathbb{R}^{mn} : X^T \bar{L}X = 0, \, \Phi(X^*, \Lambda^*) - \Phi(X, \Lambda^*) = 0 \} \).
If $X^T \bar{L}X = 0$, then by noticing that $\bar{L}$ is the Laplacian matrix of an undirected connected graph, from Lemma 2.1 we have $X = 1 \otimes x$ for some $x \in \mathbb{R}^m$. Since $(X^*, \Lambda^*)$ is a saddle point of $\Phi(X, \Lambda)$, $X^*$ is an optimal solution to the problem (20) by Lemma 3.5. Thus, $\Phi(X^*, \Lambda^*) = \tilde{f}(X^*) + (\Lambda^*)^T \bar{L} X^* = \bar{f}(X^*) = f^*$. If $\Phi(X, \Lambda^*) - \Phi(X^*, \Lambda^*) = 0$, then from $X = 1 \otimes x$ and $X \in \Omega$ we conclude that $f(x) = f^*$, $x \in \Omega_0$. Thus, $x$ is also an optimal solution to problem (4), and hence $S = \{(X, \Lambda) : X = 1 \otimes x, x \in \Omega^*_0, \Lambda \in \mathbb{R}^{mn}\}$. Therefore, $(X_k, \Lambda_k)$ converges to the largest invariant set in set $S$. Consequently, the estimates given by all agents finally reach consensus, and hence (51) holds.

Furthermore, if $\Omega^*_0 = \{x^*\}$, then by (51) we derive (52). The proof is completed.

V. ASYMPTOTIC PROPERTIES

In this section, we establish asymptotic properties of the distributed primal-dual algorithm (6) when there is no constraint, i.e., $\Omega_i = \mathbb{R}^m \ \forall i \in \mathcal{V}$.

A. Dimensionality Reduction

We now introduce a linear transformation to the algorithm (19). Note that $\bar{L}$ is the Laplacian matrix of an undirected connected graph by Assumption 2-b. Then by Lemma 2.1 $\bar{L}$ has a simple zero eigenvalue while all other eigenvalues are positive. Thus, there exists an orthogonal matrix $V = (V_1 V_2)$, where $V_2 = \frac{1}{\sqrt{n}}$ and each column of $V_1 \in \mathbb{R}^{n \times (n-1)}$ is an eigenvector corresponding to some positive eigenvalue of $\bar{L}$, such that

$$ V^T \bar{L} V = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} $$

(55)

where $S = \text{diag}\{\kappa_2, \ldots, \kappa_n\} \in \mathbb{R}^{(n-1) \times (n-1)}$ with $\kappa_i, i = 2, \ldots, n$ being positive eigenvalues of $\bar{L}$.

By multiplying both sides of (55) from left with $\mathcal{V}$, it follows that

$$ \mathcal{L} \mathcal{V} = (\mathcal{V}_1 \mathcal{V}_2) \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} = (\mathcal{V}_1 S \ 0). $$

(56)

Similarly, by multiplying both sides of (55) from right with $\mathcal{V}^T$, we obtain

$$ \mathcal{V}^T \bar{L} = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{V}_1^T \\ \mathcal{V}_2^T \end{pmatrix} = \begin{pmatrix} S \mathcal{V}_1^T \\ 0 \end{pmatrix}. $$

(57)

Let $(X^*, \Lambda^*)$ be the primal-dual solution pair of the problems (20) and (21) when $\Omega_i = \mathbb{R}^m \ \forall i \in \mathcal{V}$. Then by Lemma 3.5 $(X^*, \Lambda^*)$ satisfies

$$ \Phi(X^*, \Lambda) \leq \Phi(X^*, \Lambda^*) \leq \Phi(X, \Lambda^*) \ \forall X, \Lambda \in \mathbb{R}^{mn}, $$

and hence

$$ \nabla \tilde{f}(X^*) + (\bar{L} \otimes I_m) \Lambda^* = 0, \ \ (\bar{L} \otimes I_m) X^* = 0. $$

(58)

The first equality in (58) is the optimality condition for $\min_X \tilde{f}(X) + X^T (\bar{L} \otimes I_m) \Lambda^*$, where the minimum is attained at $X^*$. 
Therefore, from (19) (58) and $\Omega = \mathbb{R}^{mn}$ it follows that
\[
X_{k+1} - X^* = X_k - X^* - \gamma_k \left( \nabla \tilde{f}(X_k) - \nabla \tilde{f}(X^*) \right)
- \gamma_k (\tilde{L} \otimes I_m) (\Lambda_k - \Lambda^* + X_k - X^*) + \gamma_k (e_{1,k} + e_{2,k}),
\]
(59)
\[
\Lambda_{k+1} - \Lambda^* = \Lambda_k - \Lambda^* + \gamma_k (\tilde{L} \otimes I_m) (X_k - X^*) + \gamma_k e_{3,k}.
\]
(60)

Define
\[
\tilde{\Lambda}_{1,k} \triangleq (V_1^T \otimes I_m)(\Lambda_k - \Lambda^*), \quad \tilde{\Lambda}_{2,k} \triangleq (V_2^T \otimes I_m)(\Lambda_k - \Lambda^*).
\]

Then by multiplying both sides of (60) with $V^T \otimes I_m$ from left, by the rule of Kronecker product
\[
(A \otimes B)(C \otimes D) = (A \otimes C)(B \otimes D)
\]
(61)
and by (57) we obtain
\[
\left( \begin{array}{c}
\tilde{\Lambda}_{1,k+1} \\
\tilde{\Lambda}_{2,k+1}
\end{array} \right) = \left( \begin{array}{c}
\tilde{\Lambda}_{1,k} \\
\tilde{\Lambda}_{2,k}
\end{array} \right) + \gamma_k \left( \begin{array}{c}
SV_1^T \otimes I_m \\
0
\end{array} \right) \tilde{X}_k + \gamma_k \left( \begin{array}{c}
V_1^T \otimes I_m \\
V_2^T \otimes I_m
\end{array} \right) e_{3,k}.
\]
Hence
\[
\tilde{\Lambda}_{1,k+1} = \tilde{\Lambda}_{1,k} + \gamma_k (SV_1^T \otimes I_m) \tilde{X}_k + \gamma_k (V_1^T \otimes I_m) e_{3,k}.
\]
(62)

Since $VV^T = I_m$, by (56) and by (61) we derive
\[
(\tilde{L} \otimes I_m)(\Lambda_k - \Lambda^*) = (\tilde{L} \otimes I_m)(V \otimes I_m)(V^T \otimes I_m)(\Lambda_k - \Lambda^*)
= (V_1 S \otimes I_m \ 0) \left( \begin{array}{c}
\tilde{\Lambda}_{1,k} \\
\tilde{\Lambda}_{2,k}
\end{array} \right) = (V_1 S \otimes I_m) \tilde{\Lambda}_{1,k}.
\]

Then by setting $\tilde{X}_k \triangleq X_k - X^*$, from (59) we derive
\[
\tilde{X}_{k+1} = \tilde{X}_k - \gamma_k (\nabla \tilde{f}(\tilde{X}_k + X^*) - \nabla \tilde{f}(X^*))
- \gamma_k (\tilde{L} \otimes I_m) \tilde{X}_k - \gamma_k (V_1 S \otimes I_m) \tilde{\Lambda}_{1,k} + \gamma_k (e_{1,k} + e_{2,k}).
\]
(63)

B. Asymptotic Normality and Efficiency

To investigate the asymptotic properties of the algorithm (62)(63), we need the following conditions.

**Assumption 6:** $\gamma_k = \frac{1}{k^\nu}$ with $\nu \in (\frac{2}{3}, 1)$.

**Assumption 7:**

a) $f(\cdot)$ is strictly convex and the unique optimal solution is $x^*$.

b) The Hessian matrix of $f_i(\cdot)$ at point $x^*$ is $H_i$, and $\sum_{i=1}^n H_i$ is positive definite.

c) There exists a constant $c > 0$ such that $\|\nabla f_i(x) - \nabla f_i(x^*) - H_i(x - x^*)\| \leq c\|x - x^*\|^2 \forall i \in \mathcal{V}$.

**Remark 5.1:** By Assumption 7b, the Hessian matrix $\nabla^2 f(x^*)$ is positive definite. If in addition, for any $i \in \mathcal{V}$, the Hessian matrix function $\nabla^2 f_i(\cdot)$ is globally Lipschitz, then by [28, Lemma 1.2.4] we derive
\[
\|\nabla f_i(y) - \nabla f_i(x) - \nabla^2 f_i(x)(y - x)\| \leq \frac{M}{2} \|y - x\|^2,
\]
where $M > 0$ is a constant. Hence Assumption 7c holds.

**Assumption 8:**
a) For any \( i \neq j \in \mathcal{V} \), \( v_{i,k} \) and \( v_{j,k} \) are conditionally independent given \( \mathcal{F}'_{k-1} \).

b) For any \((i_1, j_1) \neq (i_2, j_2)\) with \(i_1, i_2, j_1, j_2 \in \mathcal{V}\), \( \omega_{i_1,j_1,k} \) and \( \omega_{i_2,j_2,k} \) are conditionally independent given \( \mathcal{F}'_{k-1} \).

c) For any \( i, j \in \mathcal{V} \), \( v_{i,k} \), \( \omega_{ij,k} \), and \( \zeta_{ij,k} \) are conditionally independent given \( \mathcal{F}'_{k-1} \).

d) For any \( i \in \mathcal{V} \), \( v_{i,k} \) and \( \mathcal{L}_k \) are conditionally independent given \( \mathcal{F}_{k-1} \).

Define

\[
R_{\omega,i} \triangleq \sum_{j=1}^{n} \sigma_{ij} R_{\omega,ij}, \quad R_{\zeta,i} \triangleq \sum_{j=1}^{n} \sigma_{ij} R_{\zeta,ij}, \quad R_v \triangleq \text{diag}\{R_{v,1}, \ldots, R_{v,n}\},
\]

\[
R_\omega \triangleq \text{diag}\{R_{\omega,1}, \ldots, R_{\omega,n}\}, \quad R_\zeta \triangleq \text{diag}\{R_{\zeta,1}, \ldots, R_{\zeta,n}\},
\]

\[
S_1 \triangleq E\left[ (\mathcal{L}_k - \tilde{\mathcal{L}}) \mathcal{V}_1 \mathcal{V}_1^T \otimes I_m \right] \nabla \bar{f}(X^*) \nabla \bar{f}(X^*)^T (\mathcal{V}_1 \mathcal{V}_1^{-1} \mathcal{V}_1^T (\mathcal{L}_k - \tilde{\mathcal{L}})^T \otimes I_m)]
\]

\[
\mathcal{H} = \text{diag}\{H_1, \ldots, H_n\}, \quad S_2 = R_v + R_\omega + R_\zeta.
\]

**Theorem 5.2**: (Asymptotic Normality) Set \( \Omega_i = \mathbb{R}_m \forall i \in \mathcal{V} \). Let Assumptions 1-b, 2, 3, 4, 7 and 8 hold. Then \( \theta_k = \text{col}\{\bar{X}_k, \bar{A}_{1,k}\} \) is asymptotically normal:

\[
\theta_k/\sqrt{\gamma_k} \xrightarrow{d} N(0, \Sigma),
\]

where \( \Sigma = \int_0^\infty e^{Ft} \Sigma_1 e^{F^T t} dt \),

\[
F \triangleq - \begin{pmatrix} (\tilde{\mathcal{L}} \otimes I_m) + \mathcal{H} & \mathcal{V}_1 \mathcal{S} \otimes I_m \\ -\mathcal{S} \mathcal{V}_1^T \otimes I_m & 0 \end{pmatrix},
\]

and

\[
\Sigma_1 = \begin{pmatrix} S_1 + S_2 & -R_\omega (\mathcal{V}_1 \otimes I_m) \\ -(\mathcal{V}_1^T \otimes I_m) R_\omega & (\mathcal{V}_1^T \otimes I_m) R_\omega (\mathcal{V}_1 \otimes I_m) \end{pmatrix}.
\]

**Theorem 5.3**: Set \( \Omega_i = \mathbb{R}_m \forall i \in \mathcal{V} \). Let Assumptions 1-b, 2, 3, 4, 7 and 8 hold. Define \( \bar{\theta}_n = \frac{1}{n} \sum_{k=1}^{n} \theta_k \). Then \( \{\bar{\theta}_k\} \) is asymptotically efficient:

\[
\sqrt{k} \bar{\theta}_k \xrightarrow{d} N(0, F^{-1} \Sigma_1 (F^{-1})^T).
\]

C. **Proof of Theorems 5.2 and 5.3**

Before proving the results, we give some lemmas to be used in the proof of Theorem 5.2.

**Lemma 5.4**: [6, Lemma 2] Let a block matrix \( F \) have the following form

\[
F = - \begin{pmatrix} X & Y^T \\ -Y & 0 \end{pmatrix},
\]

and let \( X \in \mathbb{R}^{p \times p} \) be positive definite and \( Y \in \mathbb{R}^{p \times q} \) be of full row rank. Then the matrix \( F \) is Hurwitz.

**Lemma 5.5**: Let Assumption 2-b and Assumption 7-b hold. Then \( F \) defined by (65) is Hurwitz.

**Proof**: Since \( H_i \forall i \in \mathcal{V} \) are Hessian matrices of convex functions, \( \mathcal{H} \) is semi-positive definite. The matrix \( \tilde{\mathcal{L}} \) is semi-positive definite since it is a Laplacian matrix of an undirected graph. Therefore, a nonzero vector \( x \in \mathbb{R}^{mn} \) satisfies \( x^T (\tilde{\mathcal{L}} \otimes I_m + \mathcal{H}) x = 0 \) if and only if

\[
x^T (\tilde{\mathcal{L}} \otimes I_m) x = 0, \quad x^T \mathcal{H} x = 0.
\]

(66)
Since \( \bar{L} \) is the Laplacian matrix of an undirected connected graph, by Lemma 2.1 a nonzero vector \( x \in \mathbb{R}^{mn} \) satisfies \( x^T(\bar{L} \otimes I_m)x = 0 \) if and only if \( x = 1 \otimes u \forall u \neq 0 \in \mathbb{R}^m \). However, by Assumption 7b
\[
(1 \otimes u)^T H(1 \otimes u) = u^T(\sum_{i=1}^n H_i)u > 0 \quad \forall u \neq 0.
\]

Therefore, the two equalities in (66) do not hold simultaneously. Thus, \((\bar{L} \otimes I_m) + H\) is positive definite. Note that \(SV_1^T\) is of full row rank. Then, by Lemma 5.4 we see that \( F \) defined by (65) is Hurwitz. ■

**Proof of Theorem 5.2:** Set
\[
\theta \triangleq \begin{pmatrix} \bar{X} \\ \bar{\Lambda} \end{pmatrix}, \quad e_k \triangleq \begin{pmatrix} e_{1,k} + e_{2,k} \\ (V_1^T \otimes I_m)e_{3,k} \end{pmatrix},
\]
\[
g(\theta) \triangleq -\begin{pmatrix} g_1(\theta) \\ g_2(\theta) \end{pmatrix} = -\begin{pmatrix} \nabla \bar{f}(\bar{X} + X^*) - \nabla \tilde{f}(X^*) + (\bar{L} \otimes I_m)\bar{X} + (V_1 S \otimes I_m)\bar{\Lambda} \\ -(SV_1^T \otimes I_m)\bar{X} \end{pmatrix}.
\]

Then we can rewrite (62) (63) as
\[
\theta_{k+1} = \theta_k + \gamma_k Y_k,
\]
where \( Y_k = g(\theta_k) + e_k \).

We want to apply Lemma 4.4 i). For this, we have to validate conditions C0-C3.

**Step 1:** We first show C0. By Assumption 1b, from [28, Theorem 2.1.5] it follows that
\[
\langle x - y, \nabla f_i(x) - \nabla f_i(y) \rangle \geq \frac{1}{L_f} \parallel \nabla f_i(x) - \nabla f_i(y) \parallel^2 \quad \forall x, y \in \mathbb{R}^m.
\]

Set \( V_1(\theta) \triangleq \frac{1}{2}(\parallel \bar{X} \parallel^2 + \parallel \bar{\Lambda} \parallel^2) \). Then by (67) (68) we obtain
\[
\nabla V_1(\theta)^T g(\theta)
\]
\[
= -\bar{X}^T(\nabla \bar{f}(\bar{X} + X^*) - \nabla \tilde{f}(X^*) + (\bar{L} \otimes I_m)\bar{X} + (V_1 S \otimes I_m)\bar{\Lambda}) - \bar{\Lambda}^T(SV_1^T \otimes I_m)\bar{X}
\]
\[
= -\bar{X}^T(\bar{L} \otimes I_m)\bar{X} - \alpha \tilde{\Lambda}^T(\nabla \bar{f}(\bar{X} + X^*) - \nabla \tilde{f}(X^*))
\]
\[
\leq -\bar{X}^T(\bar{L} \otimes I_m)\bar{X} - \frac{1}{L_f} \parallel \nabla \bar{f}(\bar{X} + X^*) - \nabla \tilde{f}(X^*) \parallel^2
\]
\[
\leq -\bar{X}^T(\bar{L} \otimes I_m)\bar{X}.
\]

Set \( V_2(\theta) \triangleq \tilde{f}(\bar{X} + X^*) - \bar{f}(X^*) - \bar{X}^T \nabla \bar{f}(X^*) + \frac{1}{2}(\bar{X}^T(\bar{L} \otimes I_m)\bar{X}) + \bar{X}^T(V_1 S \otimes I_m)\bar{\Lambda} \). Then
\[
\nabla V_2(\theta)^T g(\theta) = -\parallel g_1(\theta) \parallel^2 + \parallel (SV_1^T \otimes I_m)\bar{X} \parallel^2.
\]

By (67) we have
\[
(V^T \bar{L} \otimes I_m)\bar{X} = col\{(SV_1^T \otimes I_m)\bar{X}, 0\}.
\]

Then by \( \forall V^T = I_n \) and the properties of Kronecker products (61) we derive
\[
\parallel (SV_1^T \otimes I_m)\bar{X} \parallel^2 = \parallel (V^T \bar{L} \otimes I_m)\bar{X} \parallel^2 = \parallel \bar{X}^T(\bar{L}^2 \otimes I_m)\bar{X} \parallel^2.
\]

Hence by (70) we derive
\[
\nabla V_2(\theta)^T g(\theta) = -\parallel g_1(\theta) \parallel^2 + \bar{X}^T(\bar{L}^2 \otimes I_m)\bar{X}.
\]
Set $V(\theta) \triangleq V_1(\theta) + \alpha V_2(\theta)$ with $0 < \alpha < \frac{1}{\kappa^*}$, where $\kappa^* = \max_{i=2,\ldots,n} \kappa_i$. Then by (69) and (71) we derive
\[
\nabla V(\theta)^T g(\theta) = -\bar{X}^T((\bar{\mathcal{L}} - \alpha \bar{\mathcal{L}}^2) \otimes \mathbf{I}_m) \bar{X} - \alpha \|g_1(\theta)\|^2.
\] (72)

Since $V^T \mathcal{L} V = \text{diag}\{0, \kappa_2, \ldots, \kappa_n\}$, we have $\nabla^T \bar{\mathcal{L}}^2 V = \text{diag}\{0, \kappa_2^2, \ldots, \kappa_n^2\}$. Then all possible distinct eigenvalues of $\bar{\mathcal{L}} - \alpha \bar{\mathcal{L}}^2$ are 0, and $\kappa_i - \alpha \kappa_i^2$, $i = 2, \ldots, n$. By $0 < \alpha \leq \frac{1}{\kappa_n}$ we derive $\alpha \kappa_i \leq 1 \ \forall i = 1, \ldots, n$, and hence $\kappa_i - \alpha \kappa_i^2 = \kappa_i(1 - \alpha \kappa_i) \geq 0 \ \forall i = 1, \ldots, n$. Thus for any $\alpha$ with $0 < \alpha < \frac{1}{\kappa^*}$, the matrix $\bar{\mathcal{L}} - \alpha \bar{\mathcal{L}}^2$ is positive semi-definite. Then by (72) we have
\[
\nabla V(\theta)^T g(\theta) \leq 0 \ \forall \theta \in \mathbb{R}^{(2n-1)m}.
\] (73)
The equality holds if and only if $\bar{X}^T((\bar{\mathcal{L}} - \alpha \bar{\mathcal{L}}^2) \otimes \mathbf{I}_m) \bar{X} = 0$, $g_1(\theta) = 0$.

Since the matrix $\bar{\mathcal{L}} - \alpha \bar{\mathcal{L}}^2$ is positive semi-definite, the equality $\bar{X}^T((\bar{\mathcal{L}} - \alpha \bar{\mathcal{L}}^2) \otimes \mathbf{I}_m) \bar{X} = 0$ implies that $\bar{X} = \mathbf{1} \otimes \bar{x}$. Then by multiplying both sides of
\[
\nabla f(\bar{X} + x^*) - \nabla f(x^*) + (\bar{\mathcal{L}} \otimes \mathbf{I}_m) \bar{X} + (\mathcal{V}_1 S \otimes \mathbf{I}_m) \tilde{\Lambda}_1 = 0
\] (74)
from left with $1^T \otimes \mathbf{I}_m$, from $1^T \mathcal{V}_1 = 0$ and $1^T \bar{\mathcal{L}} = 0$ by (61) it follows that
\[
\nabla f(x^* + \bar{x}) - \nabla f(x^*) = 0.
\]

Since $f(\cdot)$ is strictly convex with $x^*$ being the unique optimal solution, $\nabla f(x^* + \bar{x}) = \nabla f(x^*) = 0$, and hence, $\bar{x} = 0$. Then from (74) we see $(\mathcal{V}_1 S \otimes \mathbf{I}_m) \tilde{\Lambda}_1 = 0$, and hence $(\mathcal{V}_1^T \mathcal{V}_1 S \otimes \mathbf{I}_m) \tilde{\Lambda}_1 = 0$. By noticing that $\mathcal{V}_1^T \mathcal{V}_1 = \mathbf{I}_{n-1}$ and $S$ is a diagonal matrix with positive diagonal entries, we obtain $\tilde{\Lambda}_1 = 0$. Consequently, $\nabla V(\theta)^T g(\theta) = 0$ only if $\theta = 0$. Therefore, by (73) we derive C0.

**Step 2:** We now verify C1. We use Lemma A.3 to prove $\lim_{k \to \infty} \theta_k = 0$ a.s.

Note that $X_k, \Lambda_k$ are bounded with probability one by Theorem 4.1. Then by the definition of $\theta_k$ we know that C1’ holds. From Lemma 4.2 and Assumption 6 by the convergence theorem for mds [26]
\[
\sum_{k=1}^{\infty} \gamma_k e_k < \infty \ a.s.,
\]
and hence C2’ holds. By the definition of $g(\theta)$ it is seen that C3’ holds. Since it has already been proven in Step 1 that C0 holds, by Lemma A.3 we obtain C1.

**Step 3:** We now verify C2. Define $\varepsilon_k = e_k I_{\|\theta_k\| \leq \varepsilon}, \nu_k = e_k I_{\|\theta_k\| > \varepsilon}$, where $\varepsilon > 0$ is a constant.

By noting that $\lim_{k \to \infty} \theta_k = 0$ a.s., there exists $k_0$ possibly depending on samples such that
\[
\|\theta_k\| \leq \varepsilon \ \forall k \geq k_0 \ a.s.
\] (75)

Thus, $\nu_k = 0 \ \forall k \geq k_0$ a.s., and hence (A.3) holds.

Since $\theta_k$ is adapted to $\mathcal{F}_{k-1}$, from (23), (24), (25) and by $\varepsilon_k$ defined in (67) we derive
\[
E[\varepsilon_k I_{\|\theta_k\| \leq \varepsilon}] = E[\varepsilon_k I_{\|\theta_k\| \leq \varepsilon}] I_{\|\theta_k\| \leq \varepsilon} = 0 \ a.s.
\] (76)

Since $(\mathcal{L}_k \otimes \mathbf{I}_m)X^* = 0$, by (16) we derive
\[
e_{1,k} = ((\bar{\mathcal{L}} - \mathcal{L}_k) \otimes \mathbf{I}_m) (\Lambda_k - \Lambda^*) + ((\bar{\mathcal{L}} - \mathcal{L}_k) \otimes \mathbf{I}_m) \bar{X}_k + ((\bar{\mathcal{L}} - \mathcal{L}_k) \otimes \mathbf{I}_m) \Lambda^*.
\]
By noticing that \( V_1^T V_1^T = I_n - \frac{11^T}{n} \) and \( \hat{\mathcal{L}}_1 = \mathcal{L}_k \mathcal{L}_1 = 0 \), we derive
\[
\hat{\mathcal{L}} - \mathcal{L}_k = (\hat{\mathcal{L}} - \mathcal{L}_k) V_1 V_1^T.
\]
(77)

Then by \( \tilde{A}_{1,k} = (V_1^T \otimes I_m)(V_k - V^*) \) we see that
\[
(V_1^T \otimes I_m)(\hat{\mathcal{L}}_k - \mathcal{L}_k)(V_k - V^*) = (\hat{\mathcal{L}} - \mathcal{L}_k) V_1 \otimes I_m \tilde{A}_{1,k}.
\]

(78)

Then by multiplying both sides of the first equality in (58) with \( V_1^T \otimes I_m \) from left, and by (61) (57) we obtain
\[
-(V_1^T \otimes I_m) \nabla \tilde{f}(X^*) = (V_1^T \otimes I_m)(\hat{\mathcal{L}} - \mathcal{L}_k)(V_k - V^*) = (\hat{\mathcal{L}} - \mathcal{L}_k) V_1 \otimes I_m \tilde{A}_{1,k}.
\]

(79)

Thus, \( \tilde{A}_{1,k} = (V_1^T \otimes I_m)(\hat{\mathcal{L}}_k - \mathcal{L}_k)(V_k - V^*) \) where
\[
\mathcal{L}_k \mathcal{L}_k^{-1} = \Lambda_k \mathcal{L}_k \Lambda_k^{-1} = \Lambda_k \mathcal{L}_k \Lambda_k^{-1} = \Lambda_k \mathcal{L}_k \Lambda_k^{-1} = \Lambda_k \mathcal{L}_k \Lambda_k^{-1} = \Lambda_k \mathcal{L}_k \Lambda_k^{-1} = \Lambda_k \mathcal{L}_k \Lambda_k^{-1} = \Lambda_k \mathcal{L}_k \Lambda_k^{-1}
\]
consequently, by (76) and (83) we know that (A.4) holds.

Note that \( X_k \) and \( \Lambda_k \) are adapted to \( \mathcal{F}_{k-1} \), and that \( \mathcal{L}_k \) is independent of \( \mathcal{F}_{k-1} \) by Assumption 2 d. Then by (79) we derive
\[
E[\|e_{1,k}\|^2|\mathcal{F}_{k-1}] \leq 3C_{01}(\|\tilde{X}_k\|^2 + \|\Lambda_{1,k}\|^2 + C_{04}) \quad a.s.,
\]
(80)

where \( C_{04} = \|V_1 S^{-1} V_1^T \otimes I_m \nabla \tilde{f}(X^*)\|^2 \). Since \( X_k = X^* + \tilde{X}_k \), by (24) (25) we derive
\[
E[\|e_{2,k}\|^2|\mathcal{F}_{k-1}] \leq C_{02} + 6c_v(\|X^*\|^2 + \|\tilde{X}_k\|^2) \quad a.s.,
\]
(81)

\[
E[\|e_{3,k}\|^2|\mathcal{F}_{k-1}] \leq 2C_{01}(\|X^*\|^2 + \|\tilde{X}_k\|^2) + C_{03} \quad a.s.
\]
(82)

Since \( \theta_k \) is adapted to \( \mathcal{F}_{k-1} \), by (80) (81) (82) we know that there exists a constant \( K > 0 \) such that
\[
E[\|e_{k}\|^2|\mathcal{F}_{k-1}] = E[\|e_{k}\|^2|\mathcal{F}_{k-1}]I_{\|\theta_k\| \leq \epsilon} \leq K \quad \forall k \geq 1 \quad a.s.
\]
(83)

Consequently, by (76) and (83) we know that (A.4) holds.

By the Chebyshev’s inequality from (83) we have
\[
\mathbb{P}(\|e_k\| > a) \leq \frac{E[\|e_k\|^2]}{a^2} \leq \frac{K}{a^2} \quad \forall k \geq 1.
\]

Then by the Schwarz inequality from (83) we derive
\[
E[\|e_kI_{\|e_k\| > a}\|] \leq (E[\|e_k\|^2])^{\frac{1}{2}}(E[I_{\|e_k\| > a}])^{\frac{1}{2}} \leq \sqrt{K} \sqrt{\mathbb{P}(\|e_k\| > a)} \leq \frac{K}{a} \quad \forall k \geq 1.
\]

Therefore, \( \lim_{a \to \infty} \sup_k E[\|e_kI_{\|e_k\| > a}\|] = 0 \), and hence (A.6) holds.
Note that

\[ e_k^T e_k = \left( e_{1,k} + e_{2,k} \right) \left( e_{1,k}^T + e_{2,k}^T \right) \left( e_{3,k}^T \right) \] (84)

and that \( \lim_{k \to \infty} \bar{X}_k = 0 \) a.s., \( \lim_{k \to \infty} \bar{A}_{1,k} = 0 \) a.s., and \( \bar{X}_k, \bar{A}_{1,k} \) are adapted to \( \mathcal{F}_{k-1} \). Then by Assumption 2 d, from (75) (79) and the definition of \( S_1 \) given in (64) we derive

\[ E[e_{1,k}^T e_{1,k} I[\|\theta_i\| \leq \epsilon] | \mathcal{F}_{k-1}] \xrightarrow[k \to \infty]{\text{a.s.}} S_1 \text{ a.s.} \] (85)

By Assumptions 3 a and 8 b we derive

\[ E[\omega_{1,i,j,k}^T \omega_{1,j,k}^T | \mathcal{F}_{k-1}^r] = E[\omega_{1,i,j,k}^T | \mathcal{F}_{k-1}^r] E[\omega_{1,j,k}^T | \mathcal{F}_{k-1}^r] = 0 \text{ } \forall (i_1, j_1) \neq (i_2, j_2). \] (86)

Thus, noticing that \( a_{ij,k} \forall i, j \in \mathcal{V} \) are adapted to \( \mathcal{F}_{k-1}^r \) we obtain

\[ E[\omega_{1,i,j,k}^T \omega_{1,j,k}^T | \mathcal{F}_{k-1}^r] = \sum_{j_1, j_2=1}^{n} a_{i_1,j_1,k} a_{i_2,j_2,k} E[\omega_{1,i_1,j_1,k}^T \omega_{1,i_2,j_2,k}^T | \mathcal{F}_{k-1}^r] = 0 \text{ } \forall i_1 \neq i_2. \] (87)

By (86) and Assumption 3 a we obtain

\[ E[\omega_{i,k}^T \omega_{i,k}^T | \mathcal{F}_{k-1}^r] = \sum_{j_1, j_2=1}^{n} a_{i_1,j_1,k} a_{i_2,j_2,k} E[\omega_{i,j_1,k}^T \omega_{i,j_2,k}^T | \mathcal{F}_{k-1}^r] = \sum_{j=1}^{n} a_{i,j,k}^2 E[\omega_{i,j,k}^T \omega_{i,j,k}^T | \mathcal{F}_{k-1}^r] = \sum_{j=1}^{n} a_{i,j,k}^2 R_{\omega,ij}. \]

Then by Assumptions 2 c and 2 d, from \( \mathcal{F}_k \subset \mathcal{F}_{k-1}^r \) it follows that

\[ E[\omega_{i,k}^T \omega_{i,k}^T | \mathcal{F}_{k-1}] = E[\omega_{i,k}^T \omega_{i,k}^T | \mathcal{F}_{k-1}^r] = E[\sum_{j=1}^{n} a_{i,j,k}^2 R_{\omega,ij} | \mathcal{F}_{k-1}] = \sum_{j=1}^{n} E[a_{i,j,k}^2 R_{\omega,ij} | \mathcal{F}_{k-1}] = R_{\omega,i} \text{ a.s.} \] (88)

Similarly,

\[ E[\zeta_{i,k} \zeta_{i,k}^T | \mathcal{F}_{k-1}] = 0 \text{ } \forall i_1 \neq i_2 \text{ a.s.}, \text{ } E[\zeta_{i,k} \zeta_{i,k}^T | \mathcal{F}_{k-1}] = R_{\zeta,i} \text{ a.s.} \]

From here, by (87) and (88) we conclude that

\[ E[\omega_{k}^T \omega_{k}^T | \mathcal{F}_{k-1}] = \text{diag} \{ R_{\omega,1}, \cdots, R_{\omega,N} \} = R_{\omega} \text{ a.s.}, \] \( E[\zeta_{k} \zeta_{k}^T | \mathcal{F}_{k-1}] = \text{diag} \{ R_{\zeta,1}, \cdots, R_{\zeta,N} \} = R_{\zeta} \text{ a.s.} \) (89)

By Assumptions 3 and 8 c, similar to (89) we can show that

\[ E[\omega_{k} \zeta_{k}^T | \mathcal{F}_{k-1}] = 0 \text{ a.s., } E[\omega_{k} v_{k}^T | \mathcal{F}_{k-1}] = 0 \text{ a.s., } E[\zeta_{k} v_{k}^T | \mathcal{F}_{k-1}] = 0 \text{ a.s.} \] (90)
From Assumptions 3c and 8a it follows that

$$E[v_{i,k}^T | \mathcal{F}_{k-1}'] = E[v_{i,k} | \mathcal{F}_{k-1}] E[v_{j,k}^T | \mathcal{F}_{k-1}] = 0 \quad \forall i \neq j,$$

and hence by Assumption 3c we obtain

$$E[v_k v_k^T | \mathcal{F}_{k-1}'] = \text{diag} \{ E[v_1 v_1^T | \mathcal{F}_{k-1}], \ldots, E[v_{n,k} v_{n,k}^T | \mathcal{F}_{k-1}] \} \xrightarrow{k \to \infty} \text{diag} \{ R_v, \ldots, R_{v,n} \} = R_v \ a.s.$$

Noting that $v_k$ and $\mathcal{L}_k$ are conditionally independent given $\mathcal{F}_{k-1}$ by Assumption 8d, by [26, Corollary 7.3.2] we have

$$E[v_k v_k^T | \mathcal{F}_{k-1}'] = E[v_k v_k^T | \mathcal{F}_{k-1}, \mathcal{L}_k] = E[v_k v_k^T | \mathcal{F}_{k-1}] \xrightarrow{k \to \infty} R_v \ a.s.$$

For $e_{2,k}$ defined by (17), by (89) and (90) we derive

$$E[e_{2,k} e_{2,k}^T | \mathcal{F}_{k-1}'] = E[v_k v_k^T + \omega_k \omega_k^T + \zeta_k \zeta_k^T | \mathcal{F}_{k-1}] \xrightarrow{k \to \infty} R_v + R_{\omega} + R_{\zeta} = S_2 \ a.s.$$

Thus by noticing that $\theta_k$ is adapted to $\mathcal{F}_{k-1}'$, from (75) we derive

$$E[e_{2,k} e_{2,k}^T I[\|\theta_k\| \leq \epsilon] | \mathcal{F}_{k-1}] = E[e_{2,k} e_{2,k}^T I[\|\theta_k\| \leq \epsilon] | \mathcal{F}_{k-1}] \xrightarrow{k \to \infty} S_2 \ a.s. \quad (91)$$

Since $\mathcal{F}_k \subset \mathcal{F}'_k$ and $e_{1,k}, \theta_k$ are adapted to $\mathcal{F}'_{k-1}$, by (29) we obtain

$$E[e_{1,k} e_{2,k}^T I[\|\theta_k\| \leq \epsilon] | \mathcal{F}_{k-1}] = E[E[e_{1,k} e_{2,k}^T I[\|\theta_k\| \leq \epsilon] | \mathcal{F}_{k-1}'] | \mathcal{F}_{k-1}]$$

$$= E[e_{1,k} I[\|\theta_k\| \leq \epsilon] E[e_{2,k}^T | \mathcal{F}_{k-1}'] | \mathcal{F}_{k-1}] = 0 \ a.s.,$$

which incorporating with (85) (91) yields

$$E[(e_{1,k} + e_{2,k})(e_{1,k} + e_{2,k})^T I[\|\theta_k\| \leq \epsilon] | \mathcal{F}_{k-1}] \xrightarrow{k \to \infty} S_1 + S_2 \ a.s. \quad (92)$$

By (27) we see that

$$E[\omega_k | \mathcal{F}_{k-1}'] = 0. \quad (93)$$

Hence, noticing that $\mathcal{F}_k \subset \mathcal{F}'_k$ and that $e_{1,k} I[\|\theta_k\| \leq \epsilon]$ is adapted to $\mathcal{F}_{k-1}'$, we obtain

$$E[e_{1,k} I[\|\theta_k\| \leq \epsilon] \omega_k^T | \mathcal{F}_{k-1}] = E[E[e_{1,k} I[\|\theta_k\| \leq \epsilon] \omega_k^T | \mathcal{F}_{k-1}' | \mathcal{F}_{k-1}'] | \mathcal{F}_{k-1}]$$

$$= E[e_{1,k} I[\|\theta_k\| \leq \epsilon] E[\omega_k | \mathcal{F}_{k-1}'] | \mathcal{F}_{k-1}] = 0 \ a.s. \quad (94)$$

Note that

$$(\mathcal{L}_k - \bar{\mathcal{L}}) \otimes I_m X_k = (\mathcal{L}_k - \bar{\mathcal{L}}) \otimes I_m \bar{X}_k, \quad (95)$$

and that $\bar{X}_k$ and $\bar{\Lambda}_{1,k}$ are adapted to $\mathcal{F}_{k-1}$. Then from (18) (79) (94), by Assumption 2c and 2d we derive

$$E[e_{1,k} e_{3,k}^T I[\|\theta_k\| \leq \epsilon] | \mathcal{F}_{k-1}] = E[e_{1,k} ((\mathcal{L}_k - \bar{\mathcal{L}}) \otimes I_m \bar{X}_k)^T I[\|\theta_k\| \leq \epsilon] | \mathcal{F}_{k-1}] \xrightarrow{k \to \infty} 0 \ a.s., \quad (96)$$

where the limit takes place because $\lim_{k \to \infty} \bar{X}_k = 0 \ a.s.$ and $\lim_{k \to \infty} \bar{\Lambda}_{1,k} = 0 \ a.s.$
Since \( \mathcal{L}_k, X_k \) are adapted to \( \mathcal{F}'_{k-1} \), by (18)(29) we derive

\[
E[e_{2,k} e_{3,k}^T | \mathcal{F}'_{k-1}] = E[e_{2,k} ((\mathcal{L}_k - \bar{\mathcal{L}}) \otimes I_m X_k) | \mathcal{F}'_{k-1}] - E[e_{2,k} \omega_k^T | \mathcal{F}'_{k-1}]
\]

\[
= E[e_{2,k} e_{3,k} | \mathcal{F}'_{k-1}] ((\mathcal{L}_k - \bar{\mathcal{L}}) \otimes I_m X_k) T - E[e_{2,k} \omega_k^T | \mathcal{F}'_{k-1}] = -E[e_{2,k} \omega_k^T | \mathcal{F}'_{k-1}].
\]

Then by \( \mathcal{F}_k \subset \mathcal{F}'_k \) we conclude that

\[
E[e_{2,k} e_{3,k}^T | \mathcal{F}_{k-1}] = E \left[ E[e_{2,k} e_{3,k}^T | \mathcal{F}'_{k-1}] | \mathcal{F}_{k-1} \right]
\]

\[
= -E \left[ E[e_{2,k} \omega_k^T | \mathcal{F}'_{k-1}] | \mathcal{F}_{k-1} \right] = -E[e_{2,k} \omega_k^T | \mathcal{F}_{k-1}] \text{ a.s.}
\]

Noticing \( e_{2,k} \) defined by (17), by (89) and (90) we derive

\[
E[e_{2,k} e_{3,k}^T | \mathcal{F}_{k-1}] = -E[\omega_k^T | \mathcal{F}_{k-1}] = -R_\omega \text{ a.s.}
\]

Since \( \theta_k \) is adapted to \( \mathcal{F}_{k-1} \), by (75) we obtain

\[
\lim_{k \to \infty} E[e_{2,k} e_{3,k}^T I[\|\theta_k\| \leq \epsilon] | \mathcal{F}_{k-1}] = \lim_{k \to \infty} E[e_{2,k} e_{3,k}^T | \mathcal{F}_{k-1}] I[\|\theta_k\| \leq \epsilon] = -R_\omega \lim_{k \to \infty} I[\|\theta_k\| \leq \epsilon] = -R_\omega \text{ a.s.,}
\]

which incorporating with (96) yields

\[
E[(e_{1,k} + e_{2,k}) e_{3,k}^T (\mathcal{L}_1 \otimes I_m) I[\|\theta_k\| \leq \epsilon] | \mathcal{F}_{k-1}] \xrightarrow{k \to \infty} -R_\omega (\mathcal{L}_1 \otimes I_m) \text{ a.s. (97)}
\]

Since \( \mathcal{L}_k \) and \( \bar{X}_k \) are adapted to \( \mathcal{F}'_{k-1} \), by (93) and \( \mathcal{F}_k \subset \mathcal{F}'_k \) we obtain

\[
E[\omega_k ((\mathcal{L}_k - \bar{\mathcal{L}}) \otimes I_m \bar{X}_k) T | \mathcal{F}_{k-1}]
\]

\[
= E \left[ E[\omega_k ((\mathcal{L}_k - \bar{\mathcal{L}}) \otimes I_m \bar{X}_k) T | \mathcal{F}'_{k-1}] | \mathcal{F}_{k-1} \right]
\]

\[
= -E \left[ E[\omega_k^T | \mathcal{F}'_{k-1}] | \mathcal{F}_{k-1} \right] = 0 \text{ a.s. (98)}
\]

By definition of \( e_{3,k} \) and (95) we see

\[
e_{3,k} e_{3,k}^T = -((\mathcal{L}_k - \bar{\mathcal{L}}) \otimes I_m \bar{X}_k \omega_k^T - \omega_k ((\mathcal{L}_k - \bar{\mathcal{L}}) \otimes I_m \bar{X}_k) T
\]

\[
+ (\mathcal{L}_k^2 \otimes I_m) \bar{X}_k \omega_k - (\mathcal{L}_k^2 \otimes I_m) \bar{X}_k T)
\]

\[
\text{Since } \lim_{k \to \infty} \bar{X}_k = 0 \text{ a.s., and } \bar{X}_k \text{ is adapted to } \mathcal{F}_{k-1}, \text{ by Assumption } 2c \text{ and } 2d \text{ we obtain}
\]

\[
\|E[((\mathcal{L}_k - \bar{\mathcal{L}}) \otimes I_m \bar{X}_k) ((\mathcal{L}_k - \bar{\mathcal{L}}) \otimes I_m) | \mathcal{F}_{k-1}]\| \leq E[\|\mathcal{L}_k - \bar{\mathcal{L}}\|^2] \|\bar{X}_k\|^2 \xrightarrow{k \to \infty} 0 \text{ a.s. (99)}
\]

Then by (89), (98), (99) we obtain

\[
E[e_{3,k} e_{3,k} T I[\|\theta_k\| \leq \epsilon] | \mathcal{F}_{k-1}] \xrightarrow{k \to \infty} R_\omega \text{ a.s.,}
\]

which incorporating with (84), (92) and (97) yields

\[
E[e_{k} e_{k} T I[\|\theta_k\| \leq \epsilon] | \mathcal{F}_{k-1}] \xrightarrow{k \to \infty} \Sigma_1 \text{ a.s. (100)}
\]

Hence by the definition of \( \varepsilon_k \) we obtain

\[
E[\varepsilon e_{k} T | \mathcal{F}_{k-1}] \xrightarrow{k \to \infty} \Sigma_1 \text{ a.s. (100)}
\]
By (83) we derive
\[ E \left[ \sup_k E[\|\varepsilon_k\|_2^2 |\mathcal{F}_{k-1}] \right] \leq K. \]

Then by the Lebesgue dominated convergence theorem [26, Corollary 4.2.3] and by (100) we have
\[ \lim_{k \to \infty} E[E[\|\varepsilon_k\|_2^2 |\mathcal{F}_{k-1}]] = \lim_{k \to \infty} E[\|\varepsilon_k\|_2^2 |\mathcal{F}_{k-1}]] = \Sigma_1. \]

Thus, \( k \to \infty \) \( E[E[\|\varepsilon_k\|_2^2 |\mathcal{F}_{k-1}]] \) = \( \Sigma_1 \), and hence (A.5) holds. So, C2 has been verified.

Step 4: It remains to check C3. By (67) and (65) we derive
\[
g(\theta) - F\theta = \left( (\bar{L} \otimes I_m + \mathcal{H}) \bar{X} + \mathcal{V}_1 S \otimes I_m \tilde{\Lambda}_1 \right) - \left( \nabla \tilde{f}(\bar{X} + X^*) - \nabla \tilde{f}(X^*) + (\bar{L} \otimes I_m) \bar{X} + (\mathcal{V}_1 S \otimes I_m) \tilde{\Lambda}_1 \right)
- \left( \nabla \tilde{f}(\bar{X} + X^*) - \nabla \tilde{f}(X^*) - \mathcal{H} \bar{X} \right) = 0.
\]

Then by Assumption [7]c we obtain
\[ \|g(\theta) - F\theta\|^2 \leq c\|\bar{X}\|^2 \leq c\|\theta\|^2, \]
and hence C3 holds.

In summary, we have verified C0-C3. Then by Lemma [A.4]i) the assertion of the theorem follows. ■

**Proof of Theorem 5.3.** Since it is shown in the proof of Theorem 5.2 that C0-C3 hold, by Lemma [A.4]ii) we immediately derive the assertion. ■

**VI. NUMERICAL EXAMPLES**

In this section, we do simulations for the distributed parameter estimation problem considered in [6]. We aim at estimating the unknown \( m \)-dimensional vector \( x^* \) based on the data gathered by \( n \) spatially distributed sensors in the network. Each agent \( i = 1, \cdots, n \) at time \( k \) has access to its real scalar measurement \( d_{i,k} \) given by the following linear time-varying model
\[
d_{i,k} = u_{i,k} x^* + \nu_{i,k},
\]
where \( u_{i,k} \in \mathbb{R}^{1 \times m} \) is the regression vector accessible to agent \( i \), and \( \nu_{i,k} \) is the local observation noise of agent \( i \).

Assume that \( \{u_{i,k}\} \) and \( \{
u_{i,k}\} \) are mutually independent iid Gaussian sequences with distributions \( N(0, R_{u,i}) \) and \( N(0, \sigma^2_{i,\nu}) \), respectively. Besides, we allow some covariance matrices nonpositive definite, but require \( \sum_{i=1}^n R_{u,i} \) be positive definite. This parameter estimation problem is modeled as solving the following distributed stochastic optimization problem
\[
\min_x f(x) = \sum_{i=1}^n f_i(x) \overset{\Delta}{=} E[\|d_{i,k} - u_{i,k} x\|^2]. \tag{101}
\]
Fig. 1: Estimates $x_{1,k}$, histograms and limit distributions for $(x_{1,k} - x^*)/\sqrt{\gamma_k}$ at $k = 1000$

So, $f_i(x) = (x - x^*)^T R_{u,i} (x - x^*) + \sigma_{i,\nu}^2$ and $\nabla f_i(x) = R_{u,i} (x - x^*)$. Therefore, $x^*$ is the unique optimal solution to (101) when $\sum_{i=1}^n R_{u,i}$ is positive definite.

Let $m = 3$. Set $x^* = (1, 2, 3)$, and

$$R_{u,1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad R_{u,2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_{u,3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma_{i,\nu} = \sqrt{0.1} \forall i \in \mathcal{V}.$$

Set $n = 3$ with the underlying undirected graph being fully connected. At any time $k \geq 0$, with equal probability $\frac{1}{3}$ for each edge, we randomly choose one edge from the graph. Set $a_{ij,k} = a_{ji,k} = 1$ when the edge between $i$ and $j$ is chosen. For any $i, j \in \mathcal{V}$, let the communication noises $\{\omega_{ij,k}\}$ and $\{\zeta_{ij,k}\}$ be mutually independent iid Gaussian sequences $N(0, 0.1 I_3)$.

Set $\gamma_k = \frac{1}{k^{0.75}}$. By using $u_{i,k}$ and $d_{i,k}$ observed at time $k$, the noisy observation of the gradient $\nabla f_i(x_{i,k})$ is constructed as $g_{i,k} = u_{i,k}^T u_{i,k} x_{i,k} - d_{i,k} u_{i,k}^T x_{i,k}$. Let $\{x_{i,k}\}$ and $\{\lambda_{i,k}\}$ be produced by the algorithm (6) with initial values $x_{i,0} = 0$, $\lambda_{i,0} = 0$. Since $v_{i,k} = (u_{i,k}^T u_{i,k} - R_{u,i})(x_{i,k} - x^*) - \nu_{i,k} u_{i,k}^T$, it is seen that $\{v_{i,k}\}$ satisfies Assumption 4-c with $c_v = \max\{E[||u_{i,k}^T u_{i,k} - R_{u,i}||^2], \sigma_{i,\nu}^2 ||R_{u,i}||\}$. Then $\lim_{k \to \infty} x_{i,k} = x^*$ by Theorem 4.4, and hence $\lim_{k \to \infty} E[v_{i,k}^T v_{i,k} | F_{k-1}] = \sigma_{i,\nu}^2 R_{u,i}$. As a result, the gradient observation noise for the distributed parameter estimation problem satisfies Assumption 3-c.

The algorithm (6) is calculated for 1000 independent samples with $k \leq 1000$. For $i = 1, 2, 3$, the
Fig. 2: Estimates $x_{2,k}$, histograms and limit distributions for $(x_{2,k} - x^*)/\sqrt{\gamma_k}$ at $k = 1000$

estimates $x_{i,k}$ and the histograms for each component of $(x_{i,k} - x^*)/\sqrt{\gamma_k}$ at time 1000 are shown in Figs. 1, 2, and 3, respectively. We use the normal distribution to fit the 1000 independent samples for each component of $(x_{i,k} - x^*)/\sqrt{\gamma_k}$, $i = 1, 2, 3$ with $k = 1000$. It is shown that the data are fitted with the normal distribution by the Kolmogrov-Smirnov test with the significance level $\alpha = 0.05$. Fig. 1-a demonstrates estimates given by agent 1 for components of $x^* = (1, 2, 3)$, where the real lines denote true values, while the dashed lines are their estimates. The estimation errors $(x_{i,k} - x^*)/\sqrt{\gamma_k}$ are presented in Figs. 1-b, 1-c, 1-d, where the histograms are given by errors of 1000 samples at time $k = 1000$, which are fitted by Gaussian densities. Figs. 2 and 3 are for agents 2 and 3, respectively.

VII. CONCLUSIONS

In this work, a stochastic approximation based distributed primal-dual algorithm is proposed to solve the distributed constrained stochastic optimization problem over random networks with imperfect communications. The local estimates derived at all agents all shown to a.s. reach a consensus belonging to the optimal solution set. Besides, we established conditions for the unconstrained problem, under which the asymptotic normality and asymptotic efficiency of the proposed algorithm are established. The influence on the convergence rate of the conditional covariance matrices of communication noises and gradient errors, properties of the cost function like gradients and Hessian matrices at the optimal point, as well as the random graphs and its mean graph is demonstrated in the paper.
**Fig. 3:** Estimates $x_{3,k}$, histograms and limit distributions for $(x_{3,k} - x^*)/\sqrt{\gamma_k}$ at $k = 1000$

### APPENDIX A

**SOME RESULTS ON STOCHASTIC APPROXIMATION**

To ease reading, some results on non-negative super-martingales [25] and some information from stochastic approximation [18] [27] are cited below.

**Lemma A.1:**[25] Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots$ be a sequence of sub-$\sigma$-algebras of $\mathcal{F}$. Let $\{d_k\}$ and $\{w_k\}$ be nonnegative $\mathcal{F}_k$-measurable random variables such that

$$E[d_{k+1} | \mathcal{F}_k] \leq (1 + \alpha_k)d_k + w_k,$$

where $\alpha_k \geq 0$ and $\sum_{k=1}^{\infty} \alpha_k < \infty$. If $\sum_{k=1}^{\infty} w_k < \infty$ a.s., then $\{d_k\}$ converges a.s.

We now introduce the convergence results for the constrained stochastic approximation algorithm [27].

Consider the following recursion

$$\theta_{k+1} = P_{\Phi}(\theta_k + \gamma_k Y_k), \quad (A.1)$$

where $\Phi \in \mathbb{R}^m$ is a convex constraint set. We list the conditions to be used.

**B1:** $\sup_k E[\|Y_k\|^2] < \infty$ a.s.

**B2:** There is a function $g(\cdot)$ such that

$$E_k[Y_k] = E[Y_k | \theta_0, Y_i, i < k] = g(\theta_k) \text{ a.s.}$$
B3: $g(\cdot)$ is continuous.

B4: $\theta_k$ is bounded a.s.

Lemma A.2: [27, Theorem 5.2.3] Let $\{\theta_k\}$ be generated by (A.1). Assume that the convex set $\Phi$ satisfies the same condition as Assumption 1c imposed on $\Omega_i$. Let B1-B4, and Assumption 5 hold. Then $\theta_k$ converges a.s. to the limit set of the following projected ODE [27] in $\Phi$:

$$\dot{\theta} = g(\theta) + z, \quad z(t) \in -N_\Phi(\theta(t)),$$

where $z(\cdot)$ is the projection or constraint term, the minimum force needed to keep $\theta(\cdot)$ in $\Phi$.

We introduce asymptotic properties of the sequence $\{\theta_k\}$ generated by the following recursion:

$$\theta_{k+1} = \theta_k + \gamma_k g(\theta_k) + \gamma_k e_k. \quad (A.2)$$

We need the following conditions.

C0 There exists a continuously differentiable function $v(\cdot)$ such that

$$g(x)^T \nabla v(x) < 0 \quad \forall x \neq 0.$$  

C1’ $\theta_k$ is bounded a.s.

C1 $\lim_{k \to \infty} \theta_k = 0$ a.s.

C2’ $\sum_{k=1}^{\infty} \gamma_k e_k < \infty$ a.s.

C2 The noise sequence $\{e_k\}$ can be decomposed into two parts $e_k = \varepsilon_k + \nu_k$ such that

$$\nu_k = o(\sqrt{\gamma_k}) \quad a.s., \quad (A.3)$$

and $\{\varepsilon_k, \mathcal{F}_k\}$ is an mds satisfying conditions:

$$E[\varepsilon_k | \mathcal{F}_{k-1}] = 0, \quad \sup_k E[\|\varepsilon_k\|^2 | \mathcal{F}_{k-1}] \leq \sigma \quad \text{with } \sigma \text{ being a constant}, \quad (A.4)$$

$$\lim_{k \to \infty} E[\varepsilon_k \varepsilon_k^T | \mathcal{F}_{k-1}] = \lim_{k \to \infty} E[\varepsilon_k \varepsilon_k^T] = S_0 \quad a.s., \quad (A.5)$$

$$\lim_{a \to \infty} \sup_k E[\|\varepsilon_k\|^2 I_{\|\varepsilon_k\| > a}] = 0. \quad (A.6)$$

C3’ $g(\cdot)$ is measurable and locally bounded.

C3 $g(\cdot)$ is measurable and locally bounded. As $\theta \to 0$,

$$\|g(\theta) - F\theta\| \leq c\|\theta\|^2,$$

where $c > 0$ and $F$ is stable.

Lemma A.3: [18, Theorem 2.2.1] Let $\{\theta_k\}$ by generated by (A.2) with an arbitrary initial value $\theta_0$. Let Assumption 6 and C0, C1’, C2’, and C3’ hold. Then

$$\lim_{k \to \infty} \theta_k = 0 \quad a.s.$$  

Lemma A.4: Let $\{\theta_k\}$ generated by (A.2). Let Assumption 6 and C0, C1, C2, and C3 hold. Then

i) $\frac{1}{\sqrt{\gamma_k}} \theta_k$ is asymptotically normal:

$$\frac{1}{\sqrt{\gamma_k}} \theta_k \xrightarrow{d} N(0, S).$$  

DRAFT
where $S = \int_0^\infty e^{Ft} S_0 e^{F^T t} dt$;

ii) $\tilde{\theta}_k$ is asymptotically efficient:

$$\sqrt{k}\tilde{\theta}_k \xrightarrow{k \to \infty} N(0, S),$$

where $S = F^{-1} S^0 (F^{-1})^T$, and $\tilde{\theta}_k = \frac{1}{k} \sum_{p=1}^k \theta_p$.

Remark A.5: Lemma A.4 i) is [18, Theorem 3.3.2] for the case: $r = 0, \beta = 1, \alpha = 0, x^0 = 0$. Since the noise sequence $\{e_k\}$ satisfies C2, by [18, Remarks 3.4.1 and 3.4.2] it is seen that A3.4.3 in [18] holds. Then by [18, Theorem 3.4.2] with $\beta = 1, x^0 = 0$ the assertion of Lemma A.4 ii) follows.

REFERENCES

[1] Y. Shi, J. Zhang, B. O’Donoghue, and K. B. Letaief, “Large-scale convex optimization for dense wireless cooperative networks,” IEEE Trans. Signal Process., vol. 63, no. 18, pp. 4729–4743, 2015.

[2] A. Bertrand, and M. Moonen “Consensus-based distributed total least squares estimation in Ad Hoc wireless sensor networks,” IEEE Trans. Signal Process., vol. 59, no. 5, pp. 2320–2330, 2011.

[3] G. Mateos, J. A. Bazerque, and G. B. Giannakis, “Distributed sparse linear regression,” IEEE Trans. Signal Process., vol. 58, no. 11, pp. 5262–5276, Nov. 2010.

[4] B. Johansson, A. Speranzon, M. Johansson, and K. H. Johansson, “On decentralized negotiation of optimal consensus,” Automatic, vol. 44, no. 4, pp. 1175–1179, 2008.

[5] S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein, “Distributed optimization and statistical learning via the alternating direction method of multipliers,” Found. Trends Machine Learning, vol. 3, no. 1, pp. 1–122, 2011.

[6] Z. J. Towfic, ans A. H. Sayed, “Stability and performance limits of adaptive primal-dual networks,” IEEE Trans. Signal Process., vol. 63, no. 11, pp. 2888–2903, 2015.

[7] A. Nedić, and A. Ozdaglar, “Distributed subgradient methods for multi-agent optimization,” IEEE Trans. Autom. Control, vol. 54, no. 1, pp. 48-61, 2009.

[8] I. Lobel, and A. Ozdaglar, “Distributed Subgradient Methods for Convex Optimization Over Random Networks,” IEEE Trans. Autom. Control, vol. 56, no. 6, pp. 1291-1306, 2011.

[9] S. S. Ram, A. Nedić, and V.V. Veeravalli, “Distributed stochastic subgradient projection algorithms for convex optimization,” J Optim Theory Appl, vol. 147, pp. 516-545, 2010.

[10] K. Srivastava, and A. Nedić, “Distributed asynchronous constrained stochastic optimization,” IEEE Journal of Selected Topics in Signal Processing, vol. 5, no. 4, pp. 772-790, 2011.

[11] J. C. Duchi, A. Agarwal, and M. J. Wainwright, “Dual Averaging for Distributed Optimization: Convergence Analysis and Network Scaling” IEEE Trans. Autom. Control, vol. 57, no. 3, pp. 592-606, 2012.

[12] T. H. Chang, A. Nedić, and A. Scaglione, “Distributed constrained optimization by consensus-based primal-dual perturbation method,” IEEE Trans. Autom. Control, vol. 59, no. 6, pp. 1524-1538, 2014.

[13] M. Zhu, and S. Martinez, “On distributed convex optimization under inequality and equality constraints” IEEE Trans. Autom. Control, vol. 57, no. 1, pp. 151-164, 2012.

[14] W. Shi, Q. Ling, G. Wu, and W. Yin, “EXTRA: An exact first-order algorithm for decentralized consensus optimization,” SIAM Journal on Optimization, vol. 25, no. 2, pp. 944-966, 2015.

[15] Q. Liu, and J. Wang, “A second-order multi-agent network for bounded constrained distributed optimization,” IEEE Trans. Autom. Control, vol. 60, no. 12, pp. 3310–3315, 2015.

[16] D. Feijer, and F. Paganini, “Stability of primal-dual gradient dynamics and applications to network optimization,” Automatic, pp. 1974–1981, vol. 46, 2010.

[17] A. Ruszczynski, Nonlinear Optimization, Princeton University Press, new Jersey, 2006.

[18] H. F. Chen, Stochastic approximation and its applications, Dordrecht, The Netherlands: Kluwer, 2002.

[19] D. P. Bertsekas, Convex Optimization Theory, Athena Scientific and Tsinghua University Press, 2010.

[20] S. Boyd, A. Ghosh, B. Prabhakar, and D. Shah, “Randomized Gossip Algorithms”, IEEE/ACM Trans. netw., vol. 14, no. 6, pp. 2508-2530, 2006.
[21] T. Aysal, M. Yildiz, A. Sarwate, and A. Scaglione, “Broadcast Gossip Algorithms for Consensus,” IEEE Trans. Signal Process., vol. 57, no. 7, pp. 2748-2761, 2009.

[22] S. Zhu, Y. C. Soh, and L. Xie “Distributed Parameter Estimation With Quantized Communication via Running Average,” IEEE Trans. Signal Process., vol. 63, no. 17, pp. 4634–4646, 2015.

[23] T. C. Aysal, M. J. Cotaes, and M. G. Rabbat, “Distributed Average Consensus With Dithered Quantization,” IEEE Trans. Signal Process., vol. 56, no. 10, pp. 4905–4917, 2008.

[24] S. Han, U. Topcu, G. J. Pappas, Differentially Private Distributed Constrained Optimization, arXiv: 1411.4105, http://arxiv.org/abs/1411.4105

[25] H. Robbins and D. Siegmund, A convergence theorem for non negative almost supermartingales and some applications, Optimizing Methods in Statistics, 1971:233-257.

[26] Y. S. Chow, and H. Teicher, Probability Theory. Springer, 1997.

[27] H. J. Kushner, and G. Yin, Stochastic Approximation Algorithms and Applications. Springer-Verlag, new York, 1997.

[28] Y. Nesterov, Introductory Lectures on Convex Programming Volume I: Basic Course, 2008.

[29] C. D. Godsil and G. Royle, Algebraic Graph Theory. New York: Springer-Verlag, 2001.

[30] H. K. Khalil, Nonlinear Systems. Prentice Hall, 2001.