HARMONIC NUMBERS AT HALF INTEGER AND BINOMIAL SQUARED SUMS

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Abstract. Half integer values of harmonic numbers and reciprocal binomial squared coefficients sums are investigated in this paper. Closed form representations and integral expressions are developed for the infinite series.

1. Introduction

In this paper we will develop new families of identities of the form

\[
\int_0^1 \frac{1}{1-x} \left\{ 3F_2 \left[ \begin{array}{c} 1, 1, 1 \\ 2 + k, 2 + k \end{array} \right] - 1 \right\} - 3F_2 \left[ \begin{array}{c} 1, 1, 1 \\ 2 + k, 2 + k \end{array} \right] - x \right\} \, dx
\]

in terms of closed form representations of alternating Euler sums of the form

\[
M(k, p) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n - \frac{1}{2} \left( \frac{n}{k} \right)^2}{n^p \binom{n+k}{k}}
\]

for \( p = 0, 1 \) and \( 2 \) and for \( k \in \mathbb{N} := \{1, 2, 3, \cdots\} \) is the set of positive integers, and \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \). \( 3F_2 \left[ \begin{array}{c} \cdot \cdot \cdot \\ \cdot \cdot \cdot \end{array} \right] \) is the classical generalized hypergeometric function and \( H_n = H_n^{(1)} \) is the harmonic number of order

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one. It is known that the harmonic number $H_n$ has the usual definition

$$H_n = \sum_{r=1}^{n} \frac{1}{r} = \sum_{j=1}^{\infty} \frac{n}{j(n)} = \int_{0}^{1} \frac{1-x^n}{1-x} \, dx \quad (H_0 := 0)$$

for $n \in \mathbb{N}$. Choi [4] recently, in terms of log-sine functions gave the definition

$$H_n = -4n \int_{0}^{\frac{\pi}{2}} \ln(sinx) \sin x \cos x^{2n-1} \, dx \quad (3)$$

An unusual, but intriguing representation has recently been given by Ciaurri et. al. [6] as follows:

$$H_n = \pi \int_{0}^{1} \left( x - \frac{1}{2} \right) \frac{\cos \left( \frac{(4n+1)\pi x}{2} \right) - \cos \left( \frac{\pi x}{2} \right)}{\sin \left( \frac{\pi x}{2} \right)} \, dx.$$ 

Let $\mathbb{R}$ and $\mathbb{C}$ denote, respectively, the sets of real and complex numbers. We define harmonic numbers at half integer values as $H_{n-\frac{1}{2}}$, which may be expressed in terms of the digamma (or Psi) function $\psi(z)$, $z \in \mathbb{R}$ and the Euler-Mascheroni constant, $\gamma$ as $H_{n-\frac{1}{2}} = \gamma + \psi \left( n + \frac{1}{2} \right)$. The digamma function is defined by

$$\psi(z) := \frac{d}{dz} \{ \log \Gamma(z) \} = \frac{\Gamma'(z)}{\Gamma(z)} \quad \text{or} \quad \log \Gamma(z) = \int_{1}^{z} \psi(t) \, dt.$$ 

A generalized binomial coefficient $\binom{\lambda}{\mu}$ ($\lambda, \mu \in \mathbb{C}$) is defined, in terms of the familiar (Euler’s) gamma function, by

$$\binom{\lambda}{\mu} := \frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)\Gamma(\lambda-\mu+1)}, \quad (\lambda, \mu \in \mathbb{C}),$$

which, in the special case when $\mu = n$, $n \in \mathbb{N}_0$, yields

$$\binom{\lambda}{0} := 1 \quad \text{and} \quad \binom{\lambda}{n} := \frac{\lambda(\lambda-1)\cdots(\lambda-n+1)}{n!} = \frac{(-1)^n (-\lambda)_n}{n!} \quad (n \in \mathbb{N}),$$

where $(\lambda)_\nu$ ($\lambda, \nu \in \mathbb{C}$) is the Pochhammer symbol defined, also in terms of the gamma function, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \ \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N}; \ \lambda \in \mathbb{C}) \end{cases}$$
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it being understood conventionally that \( (0)_0 := 1 \) and assumed that the \( \Gamma \)-quotient exists. A generalized harmonic number \( H_n^{(m)} \) of order \( m \) is defined, for positive integers \( n \) and \( m \), as follows:

\[
H_n^{(m)} := \sum_{r=1}^{n} \frac{1}{r^m}, \quad (m, n \in \mathbb{N}) \quad \text{and} \quad H_0^{(m)} := 0 \quad (m \in \mathbb{N})
\]

and

\[
\psi^{(n)}(z) := \frac{d^n}{dz^n} \{ \psi(z) \} = \frac{d^{n+1}}{dz^{n+1}} \{ \log \Gamma(z) \} \quad (n \in \mathbb{N}_0).
\]

In the case of non integer values of the argument \( z = \frac{r}{q} \), we may write the generalized harmonic numbers, \( H_z^{(\alpha+1)} \), in terms of polygamma functions

\[
H_z^{(\alpha+1)} = \zeta(\alpha + 1) + \left( -1 \right)^{\alpha} \frac{\psi(\alpha)}{\alpha!} \psi\left( \frac{r}{q} + 1 \right), \quad r \neq -1, -2, -3, \ldots
\]

where \( \zeta(z) \) is the zeta function. We also define

\[
H_{\frac{r}{q}} = H^{(1)}_{\frac{r}{q}} = \gamma + \psi\left( \frac{r}{q} + 1 \right).
\]

The evaluation of the polygamma function \( \psi^{(\alpha)}\left( \frac{r}{q} \right) \) at rational values of the argument can be explicitly done via a formula as given by Köblig [8], or Choi and Cvijovic [2] in terms of the polylogarithmic or other special functions. Some specific values are given as follows:

\[
H^{(1)}_{3} = \frac{8}{3} - 2 \ln 2, \quad H^{(3)}_{2} = 8 - 6\zeta(3), \quad H^{(4)}_{2} = 16 - 14\zeta(4),
\]

many others are listed in the books [14], [20] and [21]. While there are many results for sums of harmonic numbers with positive terms, see for example, [1], [3], [5], [9], [7], [10], [11], [12], [13], [15], [18], [19], [22], [23], [24] and references therein. There are no results, that the author is aware of, for sums of the type (1).

The following lemma will be useful in the development of the main theorems.

**Lemma 1.1.** Let \( r \in \mathbb{N} \). Then for \( p \in \mathbb{N} \),

\[
\sum_{j=1}^{r} \frac{(-1)^j}{j^p} = \frac{1}{2^p} \left( H^{(p)}_{\frac{r}{2}} + H^{(p)}_{\frac{r+1}{2}} \right) - H^{(p)}_{2\left(\frac{r+1}{2}\right)} - 1.
\]

For \( p = 1 \),
\[ \sum_{j=1}^{r} \frac{(-1)^j}{j} = H_{\lfloor \frac{j}{2} \rfloor} - H_r \]

where \([x]\) is the integer part of \(x\). We also have the known results, for \(0 < t \leq 1\),

\[ \ln^2(1 + t) = 2 \sum_{n=1}^{\infty} \frac{(-t)^{n+1} H_n}{n + 1} \]

and when \(t = 1\),

\[ \ln^2 2 = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n}{n + 1} = \zeta(2) - \frac{\pi^2}{6} \left( \frac{1}{2} \right) . \]

\[ t \ln (1 + t) = \sum_{n=1}^{\infty} \frac{(-t)^{n+1}}{n} , \text{ hence} \]

\[ \ln 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \sum_{n=1}^{\infty} \frac{1}{n2^n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{H_n}{2^n} . \]

\[ U(0) : = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_{n-\frac{1}{2}}}{n} = \frac{3}{4} \zeta(2) - 2 \ln^2 2, \]

\[ V(0) : = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_{n-\frac{3}{2}}}{n^2} = 2\pi G - \ln 2 \zeta(2) - \frac{7}{2} \zeta(3) . \]

where \(G := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \approx .91596 \) is Catalan’s constant.

*Proof.* The proof of (4) is given in the paper [16].
Firstly (5) and (6) are standard known results. Next from the definition (3),

\[
U (0) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_{n-\frac{1}{2}}}{n} \\
= -2 \int_{0}^{\frac{\pi}{2}} \ln (\sin x) \sin x \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (2n - 1) (\cos x)^{2n-2}}{n} \, dx \\
= -2 \int_{0}^{\frac{\pi}{2}} \ln (\sin x) \sin x (2 + (1 + \sec^2 x) \ln (1 + \cos^2 x)) \, dx \\
= \frac{3}{4} \zeta (2) - 2 \ln^2 2.
\]

Similarly

\[
V (0) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_{n-\frac{1}{2}}}{n^2} \\
= -2 \int_{0}^{\frac{\pi}{2}} \ln (\sin x) \sin x \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (2n - 1) (\cos x)^{2n-2}}{n^2} \, dx \\
= -2 \int_{0}^{\frac{\pi}{2}} \ln (\sin x) \sin x \left( \text{Li}_2 (\cos^2 x) + 2 \ln (1 + \cos^2 x) \right) \, dx \\
= 2 \pi G - \ln 2 \zeta (2) - \frac{7}{2} \zeta (3),
\]

\[\square\]

**Lemma 1.2.** Let \( r \in \mathbb{N} \), then we have the recurrence relation

\[
U (r) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_{n-\frac{1}{2}}}{n + r} = -U (r - 1) + \frac{\pi}{2r-1} + 2 \ln 2 \left( \frac{(-1)^r}{2r-1} - \frac{1}{r} \right) \\
+ \frac{2 (-1)^r}{2r-1} \left( H_{\left\lfloor \frac{r-1}{2} \right\rfloor} - H_{r-1} \right),
\]
with solution

\[ U (r) = (-1)^r U (0) + (-1)^r \left( 2H_r - 2H_{\left[ \frac{r}{2} \right]} + H_{r-\frac{1}{2}} + 2 \ln 2 \right) \ln 2 \]

\[ + (-1)^r \pi \sum_{j=1}^{r} \frac{(-1)^j}{2j-1} + 2 (-1)^r \sum_{j=1}^{r} \frac{H_{\left[ \frac{j}{2} \right]} - H_{j-1}}{2j-1} \]

(10)

and \( U (0) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n - \frac{1}{2}}{n} = \frac{3}{4} \zeta (2) - 2 \ln^2 2 \).

Also

\[ V (r) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n - \frac{1}{2}}{(n + r)^2} = -V (r - 1) + \frac{2 \pi}{(2r - 1)^2} + 2 \ln 2 \left( \frac{2 (-1)^r}{(2r - 1)^2} - \frac{1}{r^2} \right) \]

\[ + \frac{4 (-1)^r}{(2r - 1)^2} \left( H_{\left[ \frac{r}{2} \right]} - H_{r-1} \right) + \frac{(-1)^r}{2(2r - 1)} \left( H^{(2)}_{\left[ \frac{r}{2} \right]} - H^{(2)}_{\left[ \frac{r}{2} \right] - \frac{1}{2}} \right), \]

(11)

with solution

\[ V (r) = (-1)^r V (0) + 2 (-1)^r \pi \sum_{j=1}^{r} \frac{(-1)^j}{(2j-1)^2} + 4 (-1)^r \sum_{j=1}^{r} \frac{H_{\left[ \frac{j}{2} \right]} - H_{j-1}}{(2j - 1)^2} \]

\[ + (-1)^r \left( 3 \zeta (2) + H^{(2)}_{r-\frac{1}{2}} - \frac{1}{2} \left( H^{(2)}_{\left[ \frac{r}{2} \right]} - H^{(2)}_{\left[ \frac{r}{2} \right] - \frac{1}{2}} \right) \right) \ln 2 \]

\[ + \frac{(-1)^r}{2} \sum_{j=1}^{r} \frac{1}{2j-1} \left( H^{(2)}_{\left[ \frac{j}{2} \right]} - H^{(2)}_{\left[ \frac{j}{2} \right] - \frac{1}{2}} \right). \]

(12)

and \( V (0) = 2 \pi G - \ln 2 \zeta (2) - \frac{7}{2} \zeta (3) \).
Proof. The proof of (9) and (10) are detailed in the paper [17]. Next, by a change of counter, we consider (11) and write

\[
V(r) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_{n-\frac{1}{2}}}{(n+r)^2} = \sum_{n=2}^{\infty} \frac{(-1)^n H_{n-\frac{1}{2}}}{(n+r-1)^2}
\]

\[
= \sum_{n=2}^{\infty} \frac{(-1)^n}{(n+r-1)^2} \left( H_{n-\frac{1}{2}} - \frac{2}{2n-1} \right)
\]

\[
= -\sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_{n-\frac{1}{2}}}{(n+r-1)^2} + \sum_{n=1}^{\infty} \frac{2 (-1)^{n+1}}{(2n-1) (n+r-1)^2} + \frac{H_{\frac{1}{2}}}{r^2} - \frac{2}{r^2}
\]

\[
= -V(r-1) + \frac{1}{r^2} \left( H_{\frac{1}{2}} - 2 \right) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2r-1)^2} \left( \frac{8}{2n-1} - \frac{4}{n+r-1} \right)
\]

\[+ \sum_{n=1}^{\infty} \frac{2 (-1)^{n+1}}{(2r-1)(n+r-1)^2},\]

\[
= -V(r-1) - \frac{2 \ln 2}{r^2} + \frac{2\pi}{(2r-1)^2} - 4 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2r-1)^2 (n+r-1)}
\]

\[-2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2r-1)(n+r-1)^2}.
\]

From lemma 1.1 and using the known results

\[
V(r) = -V(r-1) - \frac{2 \ln 2}{r^2} + \frac{2\pi}{(2r-1)^2}
\]

\[+ \frac{4 (-1)^r}{(2r-1)^2} \left( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} - \sum_{n=1}^{r-1} \frac{(-1)^{n+1}}{n} \right)
\]

\[+ \frac{2 (-1)^r}{2r-1} \left( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} - \sum_{n=1}^{r-1} \frac{(-1)^{n+1}}{n^2} \right)
\]

which, for \( r \geq 1 \), results in the recurrence relation (11). By the subsequent reduction of the \( V(r) \), \( V(r-1) \), \( V(r-2) \), ..., \( V(1) \) terms in (16), we arrive at the identity (12).

It is of some interest to note that \( V(r) \) may be expanded in a slightly different way so that it gives rise to another unexpected harmonic series identity. This is pursued in the next lemma.
Lemma 1.3. For \( r \in \mathbb{N} \), we have the identity

\[
A(r) = \sum_{n=1}^{\infty} \frac{(4n + 2r - 1) H_{2n-\frac{1}{2}}}{(2n + r)^2 (2n + r - 1)^2} = V(r) + \frac{2}{(2r - 1)^2} H_{r-\frac{1}{2}} + \frac{1}{2(2r - 1)} H_{r-\frac{1}{2}}^{(2)} - \frac{1}{2(2r - 1)} \zeta(2) + 2 \frac{2}{(2r - 1)^2} \left(3 \ln 2 - \frac{\pi}{2}\right).
\]

(13)

For \( r = 0 \)

\[
A(0) = \sum_{n=1}^{\infty} \frac{(4n - 1) H_{2n-\frac{1}{2}}}{(2n)^2 (2n - 1)^2} = 2\pi G + \frac{3}{2} \zeta(2) + 2 \ln 2 - \pi - \ln 2 \zeta(2) - \frac{7}{2} \zeta(3)
\]

\[
= \int_{0}^{1} \frac{1}{1 - x} \left(\frac{1}{2} \zeta(2) - \frac{1}{4} x^2 \Phi \left(x^2, 2, \frac{1}{2}\right) + \frac{Li_2 \left(x^2\right)}{4\sqrt{x}}\right) \, dx.
\]

Proof. By expansion

\[
V(r) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_{n-\frac{1}{2}}}{(n + r)^2} = \sum_{n=1}^{\infty} \left(\frac{H_{2n-\frac{3}{2}}}{(2n + r - 1)^2} - \frac{H_{2n-\frac{3}{2}}}{(2n + r)^2}\right),
\]

\[
= \sum_{n=1}^{\infty} \left(\frac{(4n + 2r - 1) H_{2n-\frac{3}{2}}}{(2n + r - 1)^2 (2n + r)^2} - \frac{2}{(4n - 1) (2n + r - 1)^2}\right),
\]

and by rearrangement

\[
A(r) = \sum_{n=1}^{\infty} \frac{(4n + 2r - 1) H_{2n-\frac{1}{2}}}{(2n + r)^2 (2n + r - 1)^2} = V(r) + \sum_{n=1}^{\infty} \frac{2}{(4n - 1) (2n + r - 1)^2}
\]

\[
= V(r) + \frac{2}{(2r - 1)^2} H_{r-\frac{1}{2}} + \frac{H_{r-\frac{1}{2}}^{(2)}}{2(2r - 1)} - \frac{\zeta(2)}{2(2r - 1)} + \frac{2}{(2r - 1)^2} \left(3 \ln 2 - \frac{\pi}{2}\right),
\]

where \( V(r) \) is given by (10).
From (2) we can write, for \( r = 0 \)

\[
\sum_{n=1}^{\infty} \frac{(4n-1)H_{2n-\frac{1}{2}}}{(2n)^2(2n-1)^2} = \int_0^1 \frac{1}{1-x} \sum_{n=1}^{\infty} \frac{(4n-1)\left(1-x^{2n-\frac{1}{2}}\right)}{(2n(2n-1))^2} \, dx
\]

\[
= \int_0^1 \frac{1}{1-x} \left(\frac{1}{2} \zeta(2) - \frac{1}{4}x^2 \Phi \left(x^2, 2, \frac{1}{2}\right) + \frac{L_i_2(x^2)}{4\sqrt{x}}\right) \, dx
\]

\[
= 2\pi G + \frac{3}{2}\zeta(2) + 2\ln 2 - \pi - 2\zeta(2) - \frac{7}{2}\zeta(3)
\]

\[
= V(0) + \frac{3}{2}\zeta(2) - \pi + 2\ln 2.
\]

Here \( \Phi(z, t, a) = \sum_{m=0}^{\infty} \frac{z^m}{(m+a)^t} \) is the Lerch transcendent defined for \( |z| < 1 \) and \( \Re(a) > 0 \) and satisfies the recurrence

\[ \Phi(z, t, a) = z \Phi(z, t, a + 1) + a^{-t}. \]

The Lerch transcendent generalizes the Hurwitz zeta function at \( z = 1 \),

\[ \Phi(1, t, a) = \sum_{m=0}^{\infty} \frac{1}{(m+a)^t} \]

and the Polylogarithm, or de Jonquières’s function, when \( a = 1 \),

\[ Li_t(z) := \sum_{m=1}^{\infty} \frac{z^m}{mt^t}, \quad t \in \mathbb{C} \text{ when } |z| < 1; \quad \Re(t) > 1 \text{ when } |z| = 1. \]

Moreover

\[
\int_0^1 \frac{Li_t(px)}{x} dx = \begin{cases} 
\zeta(1+t), & \text{for } p = 1 \\
(2^{-t} - 1)\zeta(1+t), & \text{for } p = -1
\end{cases}
\]

\[ \Box \]

**Example 1.4.**

\[
V(5) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} H_{n-\frac{1}{2}}}{(n+5)^2} = \frac{4790057}{22381400} + \frac{182738\pi}{99225} - 2\pi G - \frac{563}{315}\zeta(2)
\]

\[
- \frac{5090387\ln 2}{793800} + \ln 2\zeta(2) + \frac{7}{2}\zeta(3).
\]
The next few theorems relate the main results of this investigation, namely the closed form and integral representation of (1).

2. Closed form and integral identities

We now prove the following theorems.

**Theorem 2.1.** Let \( k \in \mathbb{N} \), then from (1) with \( p = 0 \) we have:

\[
M (k, 0) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_{2n-\frac{1}{2}}}{((2n+4)(2n+5))^2} = \frac{7}{2} \zeta (3) + \ln 2 \zeta (2) - 2\pi G - \frac{5031587 \ln 2}{793800} + \frac{73926639}{44762800} + \frac{181513\pi}{99225} - \frac{129}{70} \zeta (2).
\]

Theorem 2.2. Let \( k \in \mathbb{N} \), then from (1) with \( p = 0 \) we have:

\[
\sum_{n=1}^{\infty} \frac{(4n+9) H_{2n-\frac{1}{2}}}{((2n+4)(2n+5))^2} = \frac{7}{2} \zeta (3) + \ln 2 \zeta (2) - 2\pi G
\]

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\[
M (k, 0) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_{n-\frac{1}{2}}}{((n+k) \choose k)^2}
\]

\[
= \sum_{r=1}^{k} \left( r \left( \frac{k}{r} \right) \right)^2 (2 (H_{r-1} - H_{k-r}) U (r) + V (r))
\]

where \( U (r) \) is given by (10) and \( V (r) \) is given by (12).

**Proof.** Consider the expansion

\[
M (k, 0) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_{n-\frac{1}{2}}}{((n+k) \choose k)^2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (k!)^2}{((n+1)k)^2} H_{n-\frac{1}{2}}
\]

\[
= \sum_{n=1}^{\infty} (k!)^2 H_{n-\frac{1}{2}} \sum_{r=1}^{k} \frac{\Lambda_r}{n+r} + \frac{\Omega_r}{(n+r)^2}
\]

where

\[
\Lambda_r = \lim_{n \to -r} \left\{ \frac{(n+r)^2}{k \prod_{r=1}^{k} (n+r)^2} \right\} = \left( \frac{r}{k!} \binom{k}{r} \right)^2
\]
and
\[
\Omega_r = \lim_{n \to -r} \frac{d}{dn} \left\{ \frac{(n+r)^2}{\prod_{r=1}^{k} (n+r)^2} \right\} = 2 \left( \frac{r}{k!} \binom{k}{r} \right)^2 (H_{r-1} - H_{k-r}).
\]

Hence
\[
M(k, 0) = \sum_{r=1}^{k} \left( \frac{k}{r} \right)^2 \sum_{n=1}^{\infty} (-1)^{n+1} H_n^{-\frac{1}{2}} \left( \frac{\Lambda_r}{n+r} + \frac{\Omega_r}{(n+r)^2} \right)
\]
\[
= \sum_{r=1}^{k} \left( \frac{k}{r} \right)^2 (2(H_{r-1} - H_{k-r})U(r) + V(r)).
\]

The other two cases of \(M(k, 1), M(k, 2)\) can be evaluated in a similar fashion. We list the results in the next corollary.

**Corollary 2.2.** Under the assumptions of Theorem 2.1, we have,
\[
M(k, 1) = \sum_{n=1}^{\infty} \frac{(-1)^n H_n^{-\frac{1}{2}}}{n} \left( \frac{n+k}{k} \right)^2 = \frac{3}{4} \zeta(2) - 2 \ln^2 2
\]
\[- \sum_{r=1}^{k} \left( \frac{k}{r} \right)^2 (1 + 2(H_{r-1} - H_{k-r}))U(r) + rV(r))
\]
with \(M(0, 1) = \frac{3}{4} \zeta(2) - 2 \ln^2 2\). Also
\[
M(k, 2) = \sum_{n=1}^{\infty} \frac{(-1)^n H_n^{-\frac{1}{2}}}{n^2} \left( \frac{n+k}{k} \right)^2 = 2\pi G - \ln 2 \zeta(2) - \frac{7}{2} \zeta(3)
\]
\[-2 \left( \frac{3}{4} \zeta(2) - 2 \ln^2 2 \right) H_k + \sum_{r=1}^{k} \left( \frac{k}{r} \right)^2 (2(H_r - H_{k-r})U(r) + V(r))
\]
with \(M(0, 2) = 2\pi G - \ln 2 \zeta(2) - \frac{7}{2} \zeta(3)\).

**Proof.** The proof follows directly from theorem 2.1 and using the same technique.
It is possible to represent the alternating harmonic number sums (14), (15) and (16) in terms of an integral, this is developed in the next theorem.

**Theorem 2.3.** Let \( k \in \mathbb{N} \), then we have:

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n + k)^2} = \int_0^1 \frac{1}{1-x} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n + k)^2} \frac{(1 - x^{-\frac{1}{2}})}{dx} = \sum_{r=1}^{k} \left( r \left( \frac{k}{r} \right) \right)^2 \left( 2 (H_{r-1} - H_{k-r}) U (r) + V (r) \right),
\]

where \( U (r) \) is given by (10) and \( V (r) \) is given by (12).

**Proof.** From (2) we can write

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n + k)^2} = \int_0^1 \frac{1}{1-x} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n + k)^2} \frac{(1 - x^{-\frac{1}{2}})}{dx} = \sum_{r=1}^{k} \left( r \left( \frac{k}{r} \right) \right)^2 \left( 2 (H_{r-1} - H_{k-r}) U (r) + V (r) \right),
\]

therefore

\[
\int_0^1 \frac{1}{1-x} \left( 3F_2 \left[ \begin{array}{c} 1,2,2 \\ 2+k,2+k \end{array} \right] -1 \right) \left( -\sqrt{x} \right) 3F_2 \left[ \begin{array}{c} 1,2,2 \\ 2+k,2+k \end{array} \right] -x \right) dx = (1 + k)^2 \sum_{r=1}^{k} \left( r \left( \frac{k}{r} \right) \right)^2 \left( 2 (H_{r-1} - H_{k-r}) U (r) + V (r) \right),
\]

hence (17) and (18) follows.

Similar integral representations can be evaluated for \( M (k,1) \) and \( M (k,2) \), the results are recorded in the next theorem.
Theorem 2.4. Let the conditions of theorem 2.3 hold, then we have:

\[
\frac{1}{(1+k)^2} \int_0^1 \frac{1}{1-x} \left( 3F_2 \begin{bmatrix} 1,1,2 & 2+k,2+k \end{bmatrix} - 1 \right) \left( -\sqrt{x} 3F_2 \begin{bmatrix} 1,1,1 & 2+k,2+k \end{bmatrix} - x \right) dx
\]

= \frac{3}{4} \zeta(2) - 2 \ln^2 2

- \sum_{r=1}^{k} \binom{k}{r}^2 ((1 + 2 (H_{r-1} - H_{k-r})) U(r) + r V(r))

= M(k, 1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_{n - \frac{1}{2}}}{n \left( \frac{n+k}{k} \right)}

Also for \(M(k, 2)\):

\[
\frac{1}{(1+k)^2} \int_0^1 \frac{1}{1-x} \left( 3F_2 \begin{bmatrix} 1,1,1 & 2+k,2+k \end{bmatrix} - 1 \right) \left( -\sqrt{x} 3F_2 \begin{bmatrix} 1,1,1 & 2+k,2+k \end{bmatrix} - x \right) dx
\]

= \frac{2\pi G - \ln 2 \zeta(2) - \frac{7}{2} \zeta(3) - 2 \left( \frac{3}{4} \zeta(2) - 2 \ln^2 2 \right) H_k}{n^2 \left( \frac{n+k}{k} \right)}

+ \sum_{r=1}^{k} \binom{k}{r}^2 (2 (H_r - H_{k-r}) U(r) + V(r))

= M(k, 2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_{n - \frac{1}{2}}}{n^2 \left( \frac{n+k}{k} \right)^2}.
Similarly for $V(r)$, for (12), we have the integral representation
\[\int_0^1 \frac{1}{1-x} \left( \frac{1}{4} \left( H_2^{(2)} - H_{2,-1}^{(2)} \right) \right) - x^{1/2} \Phi(-x,2,1+r) \, dx = V(r).\]

**Proof.** The proof follows the same pattern as that employed in theorem 2.3. \qed

**Example 2.5.** Some illustrative examples follow:

\[
\frac{1}{36} \int_0^1 \frac{1}{1-x} \left( 3F_2 \left[ \begin{array}{c} 1,1,2 \\ 7,7 \end{array} \right| -1 \right) - \sqrt{x} \, 3F_2 \left[ \begin{array}{c} 1,2,1 \\ 7,7 \end{array} \right| -x \right) \, dx
\]
\[= -\frac{53380703}{476280} - \frac{2418002}{19845} \pi + 60\pi G + \frac{16548097}{31752} \ln 2 + \frac{15490}{63} \zeta(2)
\]
\[+ 150 \ln 2 \zeta(2) + 525 \zeta(3).\]

Also from
\[\int_0^1 \frac{1}{1-x} \left( \frac{1}{4} \left( H_2^{(2)} - H_{2,-1}^{(2)} \right) \right) - x^{1/2} \Phi(-x,2,1+2) \, dx = V(2)\]
we have
\[
\int_0^1 \frac{1}{1-x} \left( \frac{1}{2} \zeta(2) - \frac{3}{4} + \frac{1}{x \frac{3}{2}} - \frac{1}{4x \frac{5}{2}} + \frac{Li_2(-x)}{x \frac{5}{2}} \right) \, dx = 2\pi G
\]
\[-\ln 2 \zeta(2) - \frac{7}{2} \zeta(3) + \frac{4}{3} \zeta(2) + \frac{107}{18} \ln 2 - \frac{16}{9} \pi - \frac{10}{9}.\]

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