On a mass functional for initial data in \(4+1\) dimensional spacetime

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Abstract

We consider a broad class of asymptotically flat, maximal initial data sets satisfying the vacuum constraint equations, admitting two commuting rotational symmetries. We construct a mass functional for ‘\(t − \phi\)’ symmetric data which evaluates to the ADM mass. We then show that \(\mathbb{R} \times U(1)^2\)-invariant solutions of the vacuum Einstein equations are critical points of this functional amongst this class of data. We demonstrate positivity of this functional for a class of rod structures which include the Myers-Perry initial data. The construction is a natural extension of Dain’s mass functional to \(D = 5\), although several new features arise.

1 Introduction

Dain has established the remarkable inequality \(m \geq \sqrt{|J|}\), for complete asymptotically flat, axisymmetric maximal initial data \((\Sigma, h, K)\) of the vacuum Einstein equations in four dimensions [1–4]. Here \(m\) is the ADM mass of the Riemannian manifold and \(J\) is the conserved angular momenta, defined from the existence of the \(U(1)\) isometry. This result was subsequently strengthened to a more general class of metrics, multiple asymptotic ends and weaker asymptotic fall-off conditions [5]. Dain’s equality is saturated if and only if \((\Sigma, h, K)\) is that of constant-Boyer-Lindquist time hypersurface of the extreme Kerr black hole.

An important step in the proof of this inequality was the construction of a well-defined mass functional \(M\), defined for \(t − \phi\) symmetric, asymptotically flat maximal initial data. \(M = M(v, Y)\) depends on two scalar functions \(v\) and \(Y\) which can be
shown to fully specify the initial data set. The proof shows that \( m = \mathcal{M}(v, Y) \) for \( t - \phi \) symmetric maximal initial data, and that \( m \geq \mathcal{M} \) for arbitrary axisymmetric maximal data. \( \mathcal{M}(v, Y) \) can be shown to be positive-definite and the unique minimizer is extreme Kerr, completing the elegant argument.

It is natural to expect an analogous inequality would hold in \( D = 5 \) dimensions, under suitable restrictions on the initial data. The situation is particularly interesting as there are potentially two candidates for minimizers: extreme Myers-Perry black holes with \( S^3 \) horizon topology \cite{6}, and extreme black rings with \( S^1 \times S^2 \) horizon topology. The masses of these solutions satisfy

\[
M^3 = \frac{27\pi}{32}(|J_1| + |J_2|)^2 \quad \text{(Myers-Perry)} \tag{1}
\]
\[
M^3 = \frac{27\pi}{32}|J_1|(|J_1| + |J_2|) \quad \text{(black ring)} \tag{2}
\]

where \( J_i \) are conserved angular momenta computed in terms of Komar integrals. Of course it is not manifestly clear how an expression which is derived from the ADM mass (i.e. evaluated at spatial infinity) would capture information on the topology of the horizon - indeed, at the level of the initial data, the horizon is a minimal surface in the interior. It is worth noting that another, related class of geometric inequalities relating the area of marginally outer trapped surfaces to the angular momenta (and charge) have also been established in three spatial dimensions \cite{7, 8}. Once again the geometries which uniquely saturate the bound were the horizon geometries corresponding to the extreme Kerr geometry. Recently, Hollands has derived an area-angular momenta inequality in general dimension \( D \), for spaces admitting a \( U(1)^{D-3} \) action as isometries \cite{9}. In this case, the inequality depends on the topology of the marginally outermost trapped surface.

As a first step towards establishing a mass-angular momenta inequality in five dimensions, in this work we investigate a generalization of Dain’s mass functional \( M(v, Y) \) to \( D > 4 \) for maximal spatial slices of five-dimensional vacuum spacetimes with \( U(1)^2 \) isometry. Note that most of the local analysis works equally well for \( D \)-dimensional spacetimes with \( U(1)^{D-3} \) isometry. However, such spacetimes could only be asymptotically flat for \( D = 5 \) (there might be a useful extension to spaces that are asymptotically Kaluza-Klein). Hence we will focus on this case, although it will be sometimes convenient to leave \( D \) as a free parameter.

Our first goal is to construct a positive-definite functional which evaluates to the mass for a broad class of asymptotically flat, maximal initial data with ‘\( t - \phi \)’ symmetry \((i = 1 \ldots D - 3)\). Such data can be thought of as data that is ‘stationary at a moment in time’. In particular, it allows us to specify the extrinsic curvature in terms of \( D - 3 \) twist potentials, using the construction of transverse traceless tensors given in \cite{10}. The functional is defined over functions on the orbit space \( \mathcal{B} = \Sigma/U(1)^{D-3} \) and depends on a matrix \( \lambda'_{ij} \) specifying the metric on surfaces of transitivity of the \( U(1)^{D-3} \) action (often called the ‘Gram’ matrix) and \( D - 3 \) twist potentials. Setting \( D = 5 \), this amounts to five independent functions.
Carter has established a variational formulation of the stationary, axisymmetric vacuum Einstein equations [11]. Our main result is to demonstrate the mass functional we have defined, when integrated over an appropriate domain, is the same as Carter’s functional up to (divergent) boundary terms. Therefore \( \mathbb{R} \times U(1)^2 \)-invariant vacuum solutions arise as critical points of the mass amongst all asymptotically flat, \( t - \phi^i \) symmetric initial data (see Bardeen’s result [12] for the 3+1-dimensional case). In this sense our proposed functional is an extension of Dain’s functional \( \mathcal{M}(v, Y) \), which also has this property. However, there are a number of important differences. As we will elaborate, our functional contains boundary terms which encode the ‘rod structure’ of the initial data. In particular, the initial data may contain 2-cycles (‘bubbles’) which also contribute to the mass. The rod structure plays an important role in the black hole uniqueness theorem [13,14] in five-dimensional vacuum gravity. In the case of stationary black holes containing additional 2-cycles, the usual laws of black hole mechanics have been shown to be modified [15]. More recently, an explicit example of a black hole space-time containing an 2 cycle in the exterior region was constructed in supergravity [16].

This paper is organized as follows. In Section 2 we introduce a broad class of \( U(1)^2 \)-invariant maximal initial data \((\Sigma, h_{ab}K_{ab})\) for the vacuum Einstein equations and discuss the particular case of \( t - \phi^i \)-symmetric data, which allows us to construct a general class of transverse-traceless tensors in the geometry. The resulting data is parameterized by functions on the orbit space and we discuss in detail their various boundary and asymptotic conditions that we must impose. In Section 3 we introduce a functional defined for asymptotically flat initial data which evaluates to the ADM mass and discuss some of its properties. Section 4 investigates the relationship of this functional to Carter’s variational formulation for stationary, axisymmetric vacuum solutions. We conclude with an argument that demonstrates positivity of this functional for a particular class of rod structure which includes Myers-Perry black hole initial data.

2 Initial data with rotational isometries

2.1 Conformal Data

Consider a stationary vacuum solution of the Einstein equations with \( U(1)^{D-3} \) isometry group. It is a well known result [13,17] that Forebenius’ theorem and the vacuum equations imply orthogonal transitivity of the \( \mathbb{R} \times U(1)^{D-3} \) action, and the metric can be written in the form

\[
g = G_{\alpha\beta}d\zeta^\alpha d\zeta^\beta + g_{AB}dx^A dx^B
\]

where \( d/d\zeta^\alpha \) generate the isometry group \((\alpha, \beta = 0, 1, 2)\) and \( x^A \) are coordinates on the two-dimensional surfaces orthogonal to the surfaces of transitivity. We may write this explicitly as

\[
g = -Hdt^2 + \frac{\chi^{ij}}{H^{1/N}}(d\phi^i - w^i dt)(d\phi^j - w^j dt) + e^{2\nu}(d\rho^2 + dz^2)
\]
where $N = D - 1$ and $\rho^2 = \det \chi'$ is harmonic on the spacetime orbit space $\mathcal{B} \equiv M/U(1)^{D-3}$ and $dz$ is the harmonic conjugate of $d\rho$. Further details on the functions $\chi_{ij}$ and one-forms $\omega^i$, and an analysis of $\mathcal{B}$, are given in [13,14]. Note that constant time slices in this spacetime have induced metric $h = H^{-1/N} \tilde{h}$ where

$$
\tilde{h} = \chi'_{ij} d\phi^i d\phi^j + e^{2U}(d\rho^2 + dz^2)
$$

where $e^{2U} = e^{2\nu} H^{1/N}$. If we consider five-dimensional asymptotically flat black hole solutions, it is known that in an appropriate coordinate system, the spacetime metric takes the form [13]

$$
d\sigma^2 = -\left(1 - \frac{\mu}{R^2} + \mathcal{O}(R^{-2})\right) dt^2 + \left(\frac{2\mu a_1 (R^2 + a_1^2)}{R^4} \sin^2 \theta + \mathcal{O}(R^{-3})\right) dt d\phi_1 \\
+ \left(\frac{2\mu a_2 (R^2 + a_2^2)}{R^4} \cos^2 \theta + \mathcal{O}(R^{-3})\right) dt d\phi_2 + \left(1 + \frac{\mu}{2R^2} + \mathcal{O}(R^{-3})\right) \\
\times \left[\frac{R^2 + a_1^2 \cos^2 \theta + a_2^2 \sin^2 \theta}{(R^2 + a_1^2)(R^2 + a_2^2)} R^2 dR^2 + (R^2 + a_1^2 \cos^2 \theta + a_2^2 \sin^2 \theta) d\theta^2 \\
+ (R^2 + a_1^2) \sin^2 \theta d\phi_1 + (R^2 + a_2^2) \cos^2 \theta d\phi_2\right]
$$

where $R \to \infty$ corresponds to spatial infinity and $(\mu, a_i)$ are parameters related to the mass $M$ and angular momenta $J_i$ of the black hole respectively. It is clear that $t =$constant slices in the above asymptotic geometry can be written $h = \Phi^2 \tilde{h}$, where $\tilde{h}$ has vanishing ADM mass.

We now focus attention on a general class of vacuum initial data. It is important to note that the results discussed in this work apply to spacetimes which will evolve from this data. In particular, the evolution need not be stationary, and so the results apply to dynamical spacetimes. We will consider solutions of the vacuum constraint equations in $N$ space dimensions (Latin indices $a, b = 1, \ldots, N$)

$$
R_h - K^{ab} K_{ab} + (\text{Tr}_h K)^2 = 0 \\
\nabla^b (K_{ab} - \text{Tr}_h K h_{ab}) = 0
$$

where $(\Sigma, h_{ab}, K_{ab})$ refer to an asymptotically flat Riemannian manifold $(\Sigma, h_{ab})$ with second fundamental form $K_{ab}$ in spacetime. This initial data set is assumed to be maximal $(\text{Tr}_h K = 0)$ and invariant under a $U(1)^{N-2}$ isometry group generated by commuting Killing vector fields $m_i$, that is

$$
\mathcal{L}_{m_i} h_{ab} = 0, \quad \mathcal{L}_{m_i} K_{ab} = 0
$$

Motivated by the above asymptotic geometry of black hole slices, we will focus on the case $N = 4$ and $h_{ab}$ is conformal to a $U(1)^2$ invariant metric $\tilde{h}$ of the form (5) with vanishing ADM mass. This encompasses a broad class of initial data (we have checked
this explicitly for the initial data for Myers-Perry black holes and the extreme doubly spinning black ring). For generic initial data, of course, one need not have orthogonal transitivity of the $U(1)^{N-2}$ action, and, indeed, $\rho = \sqrt{\det \lambda}$ need not be harmonic and so the two-dimensional metric will not take the conformally flat form above (i.e. $h_{\rho z} \neq 0$). We expect, however, that these restrictions can be removed (see e.g. [18]).

By Froebenius’ theorem the two-dimensional subspace of the tangent space at each point which are spanned by vectors orthogonal to $m_1$ and $m_2$ are integrable (tangent to two-dimensional surfaces) if and only if

$$\nabla_{[a}\eta_{b]} = l_{[a}\eta_{b]} + s_{[a}\gamma_{b]}$$
$$\nabla_{[a}\gamma_{b]} = p_{[a}\eta_{b]} + q_{[a}\gamma_{b]}$$

where we have set $\eta^a \equiv m^a_1$, $\gamma^a \equiv m^a_2$ and $\eta = \eta_a \eta^a$ and $\gamma = \gamma_a \gamma^a$, and $l_a, s_a, p_a, q_a$ are arbitrary 1-forms. It is straightforward to verify that these imply the following identities:

$$(\det \lambda') \nabla_a \eta_b = \gamma \eta_{[b} \nabla_a] \eta - 2L \gamma^c \eta_{[b} \nabla_a] \eta_c - L \gamma_{[b} \nabla_a] \eta + 2\eta \gamma^c \gamma_{[b} \nabla_a] \eta_c$$
$$(\det \lambda') \nabla_a \gamma_b = -L \eta^c_{[b} \nabla_a] \gamma + 2\gamma^c \eta_{[b} \nabla_a] \gamma_c + \eta \gamma_{[b} \nabla_a] \gamma - 2L \gamma^c \gamma_{[b} \nabla_a] \gamma_c$$

where $L = \eta_a \gamma^a$. We will use these shortly.

The Ricci tensor for the metric (5) is straightforward to compute [17]. We will divide the indices $a,b$ along $A,B = 1 \ldots 2$ and $i,j = 1 \ldots N-2$. Our main interest is the scalar curvature. Since $\lambda_{ij} R_{ij} = 0$ we have

$$\tilde{R} = g^{AB} R_{AB} = e^{-2U} \left[ -2 \nabla^2 U + \frac{1}{\rho^2} - \frac{1}{4} \lambda^{ik} \nabla_A \lambda^j_{kj} \lambda^{jm} \nabla_B \lambda^l_{lm} \delta^{AB} \right]$$

where $\nabla$ refers to the flat partial derivative operator $\partial_A$. We will also denote by $\cdot$ the scalar product with respect to the flat metric $\delta_{AB}$. The last term can also be written in the compact form

$$-\frac{1}{4} \text{Tr} \left[ (\lambda^{-1} d \lambda')^2 \right]$$

If $N = 3$, the final two terms in (12) cancel, and $\tilde{R}$ takes a particularly simple form. This fact is crucial to establish positive-definiteness of the mass functional for three-dimensional initial data.

We consider solutions $(\Sigma, h_{ab}, K_{ab})$ of (7) expressed by the conformal rescaling

$$h_{ab} = \Phi^2 \tilde{h}_{ab}, \quad K_{ab} = \Phi^{-2} \tilde{K}_{ab}$$

in terms of which the constraint equations for maximal slices become (note $\text{Tr} \tilde{h} \tilde{K} = 0$)

$$\Delta_{\tilde{h}} \Phi - \frac{1}{6} R_{\tilde{h}} \Phi + \frac{1}{6} \tilde{K}_{ab} \tilde{K}^{ab} \Phi^{-5} = 0.$$ 
$$\nabla_b \tilde{K}^{ab} = 0.$$
The Lichnerowicz equation \((15)\) is a second-order non-linear elliptic PDE for the conformal factor \(\Phi\) on a fixed Riemannian manifold \((\Sigma, \tilde{h}_{ab})\) with a given symmetric, traceless, divergenceless rank two tensor field \(\tilde{K}_{ab}\). The existence and uniqueness of solutions of \((15)\) is guaranteed by the results of [19] (see Section VIII) under suitable regularity conditions, i.e. \(\tilde{K}_{ab}\tilde{K}^{ab}\) belongs to a particular weighted Sobolev space and \((\Sigma, \tilde{h})\) is in the positive Yamabe class. Clearly, the latter condition is true, because \(\tilde{h}\) is conformal to \(h\), which must have positive scalar curvature. The former condition is an additional condition we impose on the data.

We now focus attention on a class of axisymmetric initial data sets \((\Sigma, \tilde{h}_{ab}, \tilde{K}_{ab})\) for which the extrinsic curvature can be specified completely from scalar potentials. The construction of transverse, traceless tensors under these conditions is given in [10] and we will only briefly review it here. Let \(\phi^i\) be angular coordinates adapted to the commuting Killing fields \(m_i\). Following [20] we define an initial data set to be \(t-\phi^i\)-symmetric if (1) \(\partial/\partial\phi^i\) generate a \(U(1)^2\) isometry and (2) \(\phi^i \rightarrow -\phi^i\) is a diffeomorphism that preserves \(\tilde{h}\) but reverses the sign of \(\tilde{K}_{ab}\). Condition (1) is obviously trivially satisfied by construction. In terms of the Weyl coordinate system used above, condition (2) implies \(K_{ij} = K_{AB} = 0\) (initial data with this property arise naturally within the context of slices of stationary, axisymmetric black holes for \(N = 3, 4\)). Note that \(t-\phi^i\) symmetry implies that the \(U(1)^2\) action is orthogonally transitive, i.e. the identities (11) hold.

As a consequence of this symmetry, \(K_{ab}\) is automatically traceless. Using the divergenceless condition and the property \(\Sigma\) is simply connected [10], we can express \(\tilde{K}_{ab}\) in a compact form. Define two scalar potentials \(Y^i\) and one-forms

\[
S^i = \frac{1}{2 \det \lambda'} i_{m_1} i_{m_2} \star dY^i
\]

Note \(d \star S^i = 0\). Then an \(arbitrary\) divergenceless \(t-\phi^i\)-symmetric extrinsic curvature can be expressed as [10]

\[
\tilde{K}_{ab} = \frac{2}{\det \lambda'} \left[ \left( \lambda'^{12} S^1(a m_{1b}) - \lambda'^{12} S^2(a m_{1b}) \right) + \left( \lambda'^{11} S^2(a m_{2b}) - \lambda'^{12} S^1(a m_{2b}) \right) \right].
\]  \(18\)

The vanishing of the trace of \((18)\) is obvious since \(S^i\) and the \(m_i\) are orthogonal. The divergenceless condition is more involved and requires the use of the identities (11). Hence for \(t-\phi^i\) symmetric initial data, the extrinsic curvature is completely characterized by the scalar potentials \(Y^i\) as well as the metric functions \(\lambda'_{ij}\). One can show [10] that these potentials are simply the pull-backs of the spacetime twist potentials defined in the usual way, i.e. \(dY^i = \star_5 (m_1 \wedge m_2 \wedge dm_i)\).

Further, it is useful to note that the full contraction of this tensor is

\[
\tilde{K}_{ab} \tilde{K}^{ab} = e^{-2U} \frac{\Tr (\lambda'^{-1} dY \cdot dY')}{2 \det \lambda'}
\]

where for simplicity we use the notation \(dY = (dY^1, dY^2)^t\) to define a column vector. If we consider the extrinsic curvature \(\tilde{K}_{ab}\) of a \(U(1)^2\)-invariant, non \(t-\phi^i\)-symmetric
initial data set, one can show that

$$K_{ab}K^{ab} = \tilde{K}_{ab}\tilde{K}^{ab} + K_{AB}K_{CD}g^{AC}g^{BD} + K_{ij}K_{kl}\lambda'_{ik}\lambda'_{jl} \geq \tilde{K}_{ab}\tilde{K}^{ab} \quad (20)$$

where $g_{AB} = e^{2U}\delta_{AB}$. In particular there is equality if and only if $t - \phi^i$-symmetry holds.

In summary, we are considering the class of $U(1)^2$-invariant maximal initial data sets $(h_{ab}, K_{ab})$ of the form (14) satisfying (8). The conformal metric (5) has vanishing ADM mass and is specified by the functions $U$ and $\lambda'_{ij}$. Finally, if we impose in addition that the initial data be $t - \phi^i$ symmetric, then the extrinsic curvature is fully characterized by specifying in addition to the other data, two twist potentials $Y^i$.

### 2.2 Geometry of $\Sigma$

To conclude this section we discuss general properties of the manifold $(\Sigma, h)$ and its $U(1)^2$ action. We assume $\Sigma$ is complete and simply connected and it may have several asymptotic ends, each asymptotically flat or asymptotically cylindrical. As a simple example, the maximal initial data slices of Schwarzschild have this property; the topology of the slice is $\mathbb{R} \times S^3$ which has two asymptotic flat ends. Asymptotically cylindrical ends arise in the context of initial data for extreme black holes.

Holland, Hollands and Ishibashi [21] have shown that for asymptotically flat spacetime metrics with $\mathbb{R} \times U(1)$ isometry we must have

$$\Sigma \cong (\mathbb{R}^4 \# p(S^2 \times S^2) \# p' \mathbb{C}P^2) \setminus B \quad (21)$$

where $B$ is the black hole region and $p, p'$ are non-negative integers. For spherical topology, this is equivalent to removing a point from the manifold. However, if the black hole horizon is $S^1 \times S^2$, then $B = S^1 \times S^3$, where $S^n$ is an $n$-ball [22]. We will assume our spatial slices to have a similar form, as explained below.

We consider $\Sigma$ which possess an arbitrary number of asymptotic ends and an arbitrary number of 2-cycles (we assume $p' = 0$ above). The invariance of the functions under the $U(1)^2$ isometry implies we need only consider them as functions on the two-dimensional orbit space $B \equiv \Sigma/U(1)^2$. As in the spacetime case, $B$ is a non-compact, simply connected manifold with 1d boundary $\partial B \cong \mathbb{R}$ and corners [23]. Since $\Sigma$ is asymptotically flat, $B$ has an asymptotic end corresponding to this spatial infinity. In the interior of $B$, the matrix $\lambda'_{ij}$ has rank 2, whereas on $\partial B$ and the corners it has rank 1 and rank 0 respectively. $\partial B$ consists of finite spatial intervals (‘rods’), and two semi-infinite intervals [13]. These represent co-dimension 2 surfaces upon which an integer linear combination of the rotational Killing fields vanish, and the two rotation axes which extend to spatial infinity, respectively. As we discuss in more detail below, the finite-length rods correspond to 2-cycles in the spacetime. Thus in the absence of any additional asymptotic ends this situation represents initial data on $\mathbb{R}^4$ with ‘bubbles’, and is qualitatively different to the situation in $\mathbb{R}^3$, where topological censorship rules out the existence of 2-cycles.

If we are considering initial data for spacetimes containing black holes, then $\Sigma$ will have regions removed (i.e. $\Sigma \cong \mathbb{R}^4 - pt$ for a spatial slice of Myers-Perry that does not
intersect the event horizon). Hence the orbit space will have, in addition to a boundary consisting of intervals and an asymptotic boundary, additional points removed from it. By axisymmetry, these points must lie on the axis $\rho = 0$. Of course, these removed points represent entire asymptotic regions that are an infinite proper distance from other points in $\Sigma$. Note that a similar situation in the Lorentzian setting occurs when analyzing the rod structure of extreme black holes, when the rod corresponding to timelike Killing field becoming null shrinks to zero size \cite{14}. In summary, the boundary $\rho = 0$ of the orbit space for more general initial data will consist of two semi-infinite rods, possibly finite rods, and points removed between the rods. We illustrate this in Figure 1, which shows the orbit space of a black hole spacetime and its associated standard constant time maximal slice for the Myers-Perry black hole \cite{6} and a black ring \cite{24}.

We now discuss the behaviour of the functions appearing in the class of conformal metrics (5) on the boundary and asymptotic regions of the orbit space. These will be required to analyze properties of the mass functional to be defined in the next section.

2.2.1 Asymptotic behaviour

First of all, note that $\mathbb{R}^4$ in the $\rho, z$ chart given in (5) is

\[
\delta_4 = \frac{d\rho^2 + dz^2}{2\sqrt{\rho^2 + z^2}} + (\sqrt{\rho^2 + z^2} - z)d\phi^2 + (\sqrt{\rho^2 + z^2} + z)d\psi^2
\]

(22)

where $\rho \in \mathbb{R}^+ \cup \{0\}$ and $z \in \mathbb{R}$. This can be put in a more familiar chart by setting

\[
\rho = \frac{r^2}{2\sqrt{1-x^2}}, \quad z = \frac{r^2}{2}x
\]

(23)

and noting that $r^2 = 2\sqrt{\rho^2 + z^2}$, the metric is

\[
\delta_4 = dr^2 + \frac{r^2dx^2}{4(1-x^2)} + \frac{r^2}{2}((1-x)d\phi^2 + (1+x)d\psi^2)
\]

(24)
where \( r \geq 0 \) and \( -1 \leq x \leq 1 \) and \( \phi, \psi \) have period \( 2\pi \). Hence our asymptotically flat metrics must approach \( \delta_4 \) with appropriate fall-off conditions. Note that asymptotic infinity corresponds to \( r \to \infty \) so that \( \rho, z \to \infty \) with \( z(\rho^2 + z^2)^{-1/2} \) fixed and the axes of rotation \( x = \pm 1 \) lie on the axis \( \rho = 0 \) with finite \( z \) [14]. In particular, the boundary of the orbit space for \((\mathbb{R}^4, \delta_4)\) consists of the semi-infinite rods \( I_- : -\infty < z < 0 \) and \( I_+ : 0 < z < \infty \) where \( \partial_{\psi} \) and \( \partial_{\phi} \) vanish respectively.

Let us now consider our class of asymptotically flat conformal metrics \( \tilde{h} \). We will consider \( U \) and \( \lambda_{ij}' \) as functions on the orbit space \( \mathcal{B} \). First of all, asymptotic flatness implies \( e^{-2U} \to 2\sqrt{\rho^2 + z^2} \). Since we assume the conformal metric has zero ADM mass, it is convenient to decompose \( U \) as

\[
U = V - \frac{1}{2} \log \left( 2\sqrt{\rho^2 + z^2} \right) \tag{25}
\]

where \( V = O(r^{-2}) \), that is

\[
V = \frac{\tilde{V}(x)}{r^2} + o(r^{-2}) \quad r \to \infty \tag{26}
\]

and \( \tilde{V} \) satisfies the condition that the integral given in (A.3) vanishes. As shown in Appendix A this is equivalent to the requirement that \( \tilde{h} \) has vanishing ADM mass. Next, we take the fall-off conditions of the Killing metric \( \lambda_{ij}' \) to be

\[
\lambda_{11}' = \frac{r^2}{2}(1-x)[1+f(x)+o(r^{-2})], \quad \lambda_{22}' = \frac{r^2}{2}(1+x)[1+g(x)+o(r^{-2})], \quad \lambda_{12}' = (1-x^2)o(r^{-2}) \tag{27}
\]

with \( f(x) + g(x) = 0 \) because \( \det \lambda' = \rho^2 \) where \( \rho \) is given by (23). We also assume the following fall off at infinity \( r \to \infty \)

\[
Y^1 = y_1 - \frac{J_1(x+1)^2}{\pi} + O(r^{-2}) \quad Y^2 = y_2 - \frac{J_2(3-x)(x+1)}{\pi} + O(r^{-2}) \tag{28}
\]

where \( J_i \) are angular momenta and \( y_i \) are constants [14]. Therefore we have

\[
\tilde{K}_{ab} = o(1/r^3). \tag{29}
\]

Finally, we have assumed

\[
\Phi - 1 = O(1/r^2) \quad \Phi,_{r} = O(1/r^3) \tag{30}
\]

which is sufficient to ensure finite ADM mass of \((\Sigma, h)\).

### 2.2.2 Boundary conditions on the axis

The boundary of the orbit space \( \partial \mathcal{B} \) lies on the \( z \)-axis \( \rho = 0 \). We know by [17] that the eigenspace for the eigenvalue zero of the matrix \( \lambda_{ij}' \) for a given \( z \) is one-dimensional, except for isolated values of \( z \). These isolated points are denoted \( a_1, \ldots, a_n \) and we
can divide the axis into subintervals \((-\infty, a_1), (a_1, a_2), \ldots, (a_n, \infty)\). On each interval a particular integer linear combination of the \(m_i\) vanishes. The semi-infinite rods \(I_\pm\) at the ends correspond to the axes of rotation of the asymptotic \(\mathbb{R}^4\) region. Without loss of generality, we can choose \(m_1, m_2\) to vanish on \(I_+\) and \(I_-\) respectively.

The finite-length rods, on the other hand, correspond to 2-cycle s in \(\Sigma\). Suppose that on a particular rod \(I_i\) we have

\[
\lambda'_{ij} v^j = 0, \quad v = v^i \frac{\partial}{\partial \phi^i}, \quad v^i \in \mathbb{Z}
\]

By an \(SL(2,\mathbb{Z})\) change of basis \((m_1, m_2) \rightarrow (m'_1, m'_2)\) of the Killing fields, one can always choose \(v = m'_1\) and another basis vector \(w = m'_2\) such that \(w\) is non-vanishing on \(I_i\) except at its two endpoints \(a_i\) and \(a_{i+1}\). Hence the interval \(I_i\) is topologically an \(S^2\) submanifold of \(\Sigma\).

The functions \(V\) and \(\lambda'_{ij}\) must satisfy regularity requirements as \(\rho \rightarrow 0\). We will review these briefly here. Let \(\psi\) be a coordinate such that \(\partial/\partial \psi = v\). Then in order to ensure the absence of conical singularities of the metric we impose that

\[
\Delta \psi = 2\pi \lim_{\rho \rightarrow 0} \sqrt{\frac{\rho^2 e^{2\psi}}{\lambda'_{ij} v^i v^j}} = 2\pi \lim_{\rho \rightarrow 0} \sqrt{\frac{\rho^2 e^{2\psi}}{2\sqrt{\rho^2 + z^2 \lambda'_{ij} v^i v^j}}} = 2\pi \quad z \in (a_i, a_{i+1})
\]

and hence

\[
V = \frac{1}{2} \log \left(2\sqrt{\rho^2 + z^2 \lambda'_{ij} v^i v^j}/\rho^2\right) = \frac{1}{2} \log V_i \quad \text{where} \quad z \in (a_i, a_{i+1}) \quad \text{and} \quad \rho \rightarrow 0
\]

where \(V_i = V_i(z)\) is a bounded function.

Now consider the behaviour of \(V\) on the semi-infinite rods \(I_\pm\) defined above. The metric must take the asymptotic form of the flat metric. Then on either axis, we have, as \(z \rightarrow \pm\infty\),

\[
\lambda'_{ij} v^i v^j = \frac{1}{2|z|} \rho^2 + O(\rho^4)
\]
where \( v \) is taken to be \( m_1 \) or \( m_2 \). Thus we see that \( V_\pm \to 1 \) as \( z \to \pm \infty \), where \( V_\pm \) refers to the function \( V_i \) evaluated on \( I_\pm \).

Moreover, consider the rod \( I_i \) (either finite or semi-infinite). Near \( I_i \), in the adapted basis \((m'_1, m'_2)\) discussed above we have the following behaviour for \( \lambda'_{ij} \):

\[
\lambda'_{ij} = \begin{pmatrix}
O(\rho^2) & O(\rho^3) \\
O(\rho^2) & O(1)
\end{pmatrix} \quad \rho \to 0
\]  

(34)

In addition, we will require that near \( I_i \), the twist potentials in this basis behave as

\[
Y^1 = C_1 + O(\rho^4) \quad Y^2 = C_2 + O(\rho^2) \quad \rho \to 0
\]

(35)

where \( C_1 \) and \( C_2 \) are constants. The fall-off of \( Y^1 \) is more restrictive than simply requiring \( dY^1 = O(\rho^2) \) along a rod where \( v = m'_1 \) vanishes, but this condition is satisfied by the twist potentials of the Myers-Perry and black ring solutions. The condition \( (35) \) will be needed to show certain terms in the mass functional are finite as \( \rho \to 0 \).

Finally, on the axis \( \rho = 0 \), apart from isolated points corresponding to asymptotic ends, we require

\[
\Phi = O(1) \, .
\]

(36)

2.2.3 Behaviour near asymptotic ends

As discussed above, \( \Sigma \) may have additional asymptotic ends. Note that to have non-trivial angular momenta, \( \Sigma \) must have non trivial topology. More precisely, if \( S \) represents the sphere at infinity, then since \( d \star dm_i = 0 \) by virtue of the vacuum spacetime equations, it follows that the Komar angular momenta \( J_i \) vanishes unless \( S \) is not the boundary of some compact domain contained in \( \Sigma \). Such a situation arises when isolated points are removed from \( \Sigma \), yielding additional asymptotic ends. By the \( U(1)^2 \) symmetry assumption, these lie on a point on the axis \( \rho = 0 \). For example, in the case that the initial data arises from a stationary black hole, the location of the removed point corresponds to the location of the event horizon. We will allow for both asymptotically flat ends and cylindrical ends, which the latter arise in the context of initial data for extreme black holes [10]. We impose singular boundary conditions on the conformal factor \( \Phi \):

\[
\Phi = O(r_i^{-2}) \quad \partial_{r_i} \Phi = O(r_i^{-3}) \quad \text{asymptotically flat} \]

(37)

\[
\Phi = O(r_i^{-1}) \quad \partial_{r_i} \Phi = O(r_i^{-2}) \quad \text{cylindrical end} \]

(38)

where \( r_i \) represents the distance to the asymptotic end. We assume that the conformal metric \( \tilde{h} \) approaches the flat metric on \( \mathbb{R}^4 \) in the former case, whereas in the latter case, the conformal metric \( \tilde{h} \) will be asymptotically cylindrical, so \( \tilde{h} = \Omega^2 (dr_i^2 + r_i^2 \gamma) \) where \( \gamma \) is a metric on a compact manifold (the example of extreme Myers-Perry is discussed in detail in Appendix B). This assumption is most easily illustrated by considering the example of a maximal constant-\( t \) slice of the five-dimensional Schwarzschild geometry,

\[
h = \left( 1 + \mu \left[ \frac{r}{2r^2} \right] \right)^2 \delta_4
\]

(39)
where \( \tilde{h} = \delta_{ij} \) is the flat metric on \( \mathbb{R}^4 \) given in (22) and \( r = 2\sqrt{\rho^2 + z^2} \). One easily sees that the point \( r = 0 \) corresponds to another asymptotically flat end and the conformal factor \( \Phi \) has the singular behaviour (37). In this simple case, this asymptotic region corresponds to the corner point \((\rho, z) = (0, 0)\) on the axis of the orbit space where both Killing fields vanish.

For the initial data of the Myers-Perry black hole, the conformal factor \( \Phi \) diverges in the same way and the removed point is again located at a corner \([13]\) of the orbit space (the point \( \bar{a}_E \) in Figure 1 (c)). However, for the black ring, the removed point is not at a corner, but instead lies on the rod corresponding to the Killing field which vanishes on the \( S^2 \) of the horizon. One can again verify that \( \Phi \) has the above singular behaviour at this point, and is in fact regular at the point \( \bar{a}_2 \) (see Figure 1 (d)). Finally, we impose the same condition (26) on \( V \) with \( r \) replaced by \( r_i \) and \( \tilde{V} \to \tilde{V}_i \).

3 A mass functional for initial data

We now follow the approach of Dain \([25]\) to construct a proposal for a mass functional \( \mathcal{M} \). This functional depends on the functions \((\lambda_{ij}', Y^i)\) and should evaluate to the mass for the class of initial data we considered in the previous section when \( t - \phi^i \) symmetry holds.

Our starting point is the ADM mass for the asymptotically flat, complete Riemannian manifold \((\Sigma, h_{ab})\):

\[
M_{ADM} = \frac{1}{16\pi} \lim_{r \to \infty} \int_{S^3_r} (\partial_a h_{ac} - \partial_c h_{aa}) n^c \, ds_h
\]

where \( S^3_r \) refers to a three-sphere of coordinate radius \( r \) with volume element \( ds_h \) in the Euclidean chart outside a large compact region and \( n \) is the unit normal. If we evaluate the ADM mass in terms of the conformally scaled initial data and by the assumptions in the previous section, one has

\[
M_{ADM} = -\frac{3}{8\pi} \lim_{r \to \infty} \int_{S^3_r} n^c \tilde{\nabla}_c \Phi \, ds_h
\]

where \( \tilde{\nabla} \) refers to the covariant derivative with respect to \( \tilde{h}_{ab} \). Using (11) and the fact that \( \Phi \to 1 \) as \( r \to \infty \), define

\[
m = -\frac{3}{8\pi} \lim_{r \to \infty} \int_{S^3_r} \frac{\tilde{\nabla}_c \Phi}{\Phi} n^c \, ds_h
\]

\[
= -\frac{3}{8\pi} \int_{\Sigma} \tilde{\nabla}^c \frac{\tilde{\nabla}_c \Phi}{\Phi} \, ds_{\tilde{h}} + \frac{3}{8\pi} \lim_{r_i \to 0} \int_{S^3_{i}} \frac{\tilde{\nabla}_c \Phi}{\Phi} n^c \, ds_{\tilde{h}}
\]

\[
= \frac{3}{8\pi} \int_{\Sigma} \left( -\frac{\tilde{R}}{6} + \frac{\tilde{K}_{ab} \tilde{K}^{ab}}{6\Phi^6} + \frac{\tilde{\nabla}^c \Phi \tilde{\nabla}_c \Phi}{\Phi^2} \right) \, ds_{\tilde{h}} + \frac{3}{8\pi} \lim_{r_i \to 0} \int_{S^3_{i}} \frac{\tilde{\nabla}_c \Phi}{\Phi} n^c \, ds_{\tilde{h}}
\]
where in passing from the first line to the second line, we have used the divergence theorem. Provided the behaviour of $\Phi$ at the asymptotic ends is given by (37) the last term in $m$ is zero.

This form of expressing the mass as a bulk integral is important for defining the functional $\mathcal{M}$. For our class of initial data, we can reduce the integral to one over the orbit space $\mathcal{B}$. Note that $\det \tilde{h} = e^{4U} \rho^2$. Performing the trivial integrals over the angles gives

$$m = \frac{3\pi}{2} \int_{\mathcal{B}} \left(-\frac{\mathcal{R} e^{2U}}{6} + \frac{\mathcal{K}_{ab} \mathcal{K}^{ab} e^{2U}}{6\Phi^6} + \frac{\partial_A \Phi \partial^A \Phi}{\Phi^2}\right) \rho d\rho dz$$

(43)

Equivalently, in terms of the flat metric on $\mathbb{R}^3$ in cylindrical coordinates

$$\delta = d\rho^2 + dz^2 + \rho^2 d\varphi^2$$

(44)

where $\varphi$ is an auxiliary angular coordinate with period $2\pi$, we can write

$$m = \frac{3}{4} \int_{\mathbb{R}^3} \left(-\frac{\mathcal{R} e^{2U}}{6} + \frac{\mathcal{K}_{ab} \mathcal{K}^{ab} e^{2U}}{6\Phi^6} + \frac{\partial_A \Phi \partial^A \Phi}{\Phi^2}\right) d\mu_0$$

and $d\mu_0$ is the volume element of $\delta$.

We are now in the position to define our mass functional for arbitrary $t-\phi^i$ symmetric initial data. Recall for this data, the extrinsic curvature is specified in terms of twist potentials $Y^i$ as given by (18) and its square is given by the contraction (19). Using the expression for the scalar curvature $\mathcal{R}$ (12) we have

$$m = \frac{1}{8} \int_{\mathbb{R}^3} \left(2\Delta_2 U - \frac{1}{\rho^2} + \frac{1}{4} \text{Tr} \left[ (\lambda'^{-1} d\lambda')^2 \right] + e^{-6v} \frac{\text{Tr} (\lambda'^{-1} dY dY^t)}{2 \det \lambda'} + 6 (dv)^2 \right) d\mu_0$$

(45)

where $v = \log \Phi$. As an integral over $\mathbb{R}^3$, this expression appears similar to the analogous formula for $m$ when $N = 3$ first given in [25]. However, in $N = 4$ there are a number of key qualitative differences.

Firstly, consider the terms

$$-\frac{1}{\rho^2} + \frac{1}{4} \text{Tr} \left[ (\lambda'^{-1} d\lambda')^2 \right] .$$

(46)

In $N = 3$, it is easily seen that the above expression vanishes identically. This is no longer the case in $N = 4$. We note for later use the identity:

$$-\frac{1}{\rho^2} + \frac{1}{4} \text{Tr} \left[ (\lambda'^{-1} d\lambda')^2 \right] = -\frac{1}{4} \left(\text{Tr} (\lambda'^{-1} d\lambda')\right)^2 + \frac{1}{4} \text{Tr} \left[ (\lambda'^{-1} d\lambda')^2 \right]$$

$$= -\frac{1}{2} \frac{\det d\lambda'}{\det \lambda'} \quad \text{for} \ 2 \times 2 \ \text{matrices}$$

(47)

where we are using the notation $\det d\lambda' = \frac{1}{2} \epsilon^{ijk} \epsilon^{ijkl} d\lambda'_{ij} \cdot d\lambda'_{kl}$. 

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A second important difference is that, unlike in \( N = 3 \), the integral over \( \Delta_2 U \) does not vanish. Indeed, in terms of the rod structure formalism, the class of three-dimensional initial data studied in [25] has \( \partial \mathcal{B} \) consisting of a single rod (the rotation axis of the generator of the \( U(1) \) symmetry) with points removed corresponding to asymptotic ends. As we shall now explain, this is sufficient, along with appropriate fall-off conditions, to prove that the first term in \( m \) does not contribute either. However, we shall see the situation is more complicated in our present case. Note that

\[
\Delta_2 U = \frac{\partial^2 U}{\partial \rho^2} + \frac{\partial^2 U}{\partial z^2} \tag{48}
\]

Using our expression for \( U \), we have

\[
\int_{\mathcal{B}} \Delta_2 U \rho \, d\rho d z = \int_{\mathcal{B}} \left( \Delta_2 V - \frac{1}{2} \Delta_2 \log \left( 2\sqrt{\rho^2 + z^2} \right) \right) \rho \, d\rho d z \]

\[
= \int_{\mathcal{B}} \Delta_2 V \rho \, d\rho d z = \int_{\mathcal{B}} d\alpha \tag{49}
\]

where we have defined the one-form

\[
\alpha \equiv (\rho V_\rho - V) \, dz - \rho V_z \, d\rho \tag{50}
\]

Recall the boundary of the orbit space consists of the asymptotic region \( \mathcal{B}_\infty \equiv \{ z, \rho \to \infty, z(\rho^2 + z^2)^{-1/2} \text{ finite} \} \), i.e. \( r \to \infty \), and the axis \( \rho = 0 \) denoted by \( \partial \mathcal{B} \). Using the asymptotic condition [26], we find to leading order \( \alpha = -\tilde{V}(x)dx \) as \( r \to \infty \). Hence by Stokes’ theorem we have

\[
\int_{\mathcal{B}} d\alpha = \int_{\partial \mathcal{B} \cup \mathcal{B}_\infty} \alpha = \int_{\partial \mathcal{B}} \alpha = \int_{I_- \cup I_1 \cup \cdots \cup I_+} \alpha \\
= \int_{I_- \cup I_1 \cup \cdots \cup I_+} (\rho V_\rho - V) \big|_{\rho=0} \, dz - \int_{I_- \cup I_1 \cup \cdots \cup I_+} V \big|_{\rho=0} \, d\rho \\
= -\frac{1}{2} \sum_{\text{rods}} \int_{I_i} \log V_i \, dz
\]

Note that the integral over \( \mathcal{B}_\infty \) vanishes as a consequence of the condition \( \tilde{h} \) has vanishing ADM mass (A.3).

The above considerations lead us to define the following mass functional for \( t - \phi^i \)-symmetric, maximal asymptotically flat vacuum data:

\[
\mathcal{M} \equiv \frac{\pi}{4} \int_{\mathcal{B}} \left( -\frac{1}{\rho^2} + \frac{1}{4} \text{Tr} \left[ (\lambda'^{-1} d\lambda')^2 \right] + e^{-6v} \frac{\text{Tr} \left( (\lambda'^{-1} dY dY^* \right)}{2 \det \lambda'} + 6 (dv)^2 \right) \rho \, d\rho \, d z \\
- \frac{\pi}{4} \sum_{\text{rods}} \int_{I_i} \log V_i \, d z \tag{51}
\]

where \( \mathcal{M} = \mathcal{M}(\lambda'_{ij}, Y^i, v) \). Note that if we consider maximal, \( U(1)^2 \)-invariant data without \( t - \phi^i \) symmetry, we have \( m \geq \mathcal{M} \) as a consequence of [20].

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The mass functional depends on the matrix \( \lambda'_{ij} \), the twist potentials \( Y^i \), and the conformal factor \( v \). It also depends upon the boundary values of the \( \lambda'_{ij} \) along finite rods on the axis, via the functions \( V_i \). Define \( \mathcal{A} \equiv \{ (\lambda', Y, v) : \mathcal{M}(\lambda', Y, v) \text{ is bounded} \} \), then \( \mathcal{M} \) will be well-defined on \( \mathcal{A} \). Of course, not all elements belonging to \( \mathcal{A} \) will represent the mass of some initial data set. There are three functions in \( \lambda'_{ij} \) with a constraint \( \det \lambda' = \rho^2 \) so there are two independent functions in \( \lambda'_{ij} \), two independent potentials \( Y^i \) and conformal factor \( \Phi \) (or \( v = \log \Phi \)). We have seen that all axisymmetric and \( t - \phi^i \) symmetric data can be generated by six functions \( (U, \lambda', Y, v) \). These functions are coupled by the Lichnerowicz equation (15) which can be rewritten as

\[
\Delta_3 v + \frac{1}{3} \Delta_2 U - \frac{1}{6\rho^2} + \frac{1}{24} \Tr \left( (\lambda'^{-1} d\lambda')^2 \right) = e^{-6v} \frac{\Tr (\lambda'^{-1} dY dY^t)}{12 \det \lambda'}
\]

(52)

where \( \Delta_3 \) is three dimensional Laplace operator with respect to metric \( \delta \). Now for given \( (\lambda'_{ij}, Y, v) \), we have a linear two dimensional Poisson equation for \( U \). Then by equation (25) we have a linear two dimensional Poisson equation for \( V \):

\[
\Delta_2 V = F(v, \lambda', Y)
\]

(53)

with boundary (26) at infinity and we have \( V = O(1) \) at \( \rho = 0 \). Now let \( \mathcal{A}_1 \) be a solution of equation (53) with appropriate fall-off conditions. Then \( \mathcal{M}(\lambda'_{ij}, Y^i, v) \) will give us the mass of an initial data set only if the given data is selected from \( \mathcal{A}_1 \subset \mathcal{A} \).

Finally, using the asymptotic and boundary conditions on the orbit space functions, we now show that \( \mathcal{M} \) is finite. By asymptotic condition (27) the behaviour of the first two terms of \( \mathcal{M} \) near infinity is

\[
- \frac{1}{\rho^2} + \frac{1}{4} \Tr \left( (\lambda'^{-1} d\lambda')^2 \right) = \mathcal{O}(r^{-8}) \quad \text{as} \quad r \to \infty
\]

(54)

Thus it is bounded at infinity. Near the axis, we must analyze the behaviour of these terms near each rod. One can check

\[
- \frac{1}{\rho^2} + \frac{1}{4} \Tr \left( (\lambda'^{-1} d\lambda')^2 \right) = \mathcal{O}(1) \quad \text{as} \quad \rho \to 0
\]

(55)

The third term in the mass functional by equations (27) and (28) has following behaviour at infinity

\[
e^{-6v} \frac{\Tr (\lambda'^{-1} dY dY^t)}{2 \det \lambda'} = \mathcal{O}(r^{-10}) \quad \text{as} \quad r \to \infty
\]

(56)

and near the axis one has, using (34) and (35),

\[
e^{-6v} \frac{\Tr (\lambda'^{-1} dY dY^t)}{2 \det \lambda'} = \mathcal{O}(1) \quad \text{as} \quad \rho \to 0
\]

(57)

Finally, if one uses the conditions near additional asymptotically flat ends, one can ensure that \( \mathcal{M} \) is finite, assuming continuity of the functions in the interior of \( \mathcal{B} \).
4 Stationary, biaxisymmetric data

Let us return to vacuum solutions with $\mathbb{R} \times U(1)^2$ isometry group. As discussed above, the metric takes the canonical form

$$ g = -H dt^2 + \frac{\lambda^\prime_{ij}}{H^{1/2}} (d\phi^i - w^i dt) (d\phi^j - w^j dt) + e^{2\nu} (d\rho^2 + dz^2) \quad (58) $$

where $\rho^2 = \det \lambda^\prime$ is harmonic on the orbit space. Remarkably, the vacuum field equations for this spacetime can be derived from the critical points of the following functional, as first discussed by Carter for $D = 4$ in [11] (see [13] for general dimension):

$$ \mathcal{M}' = \frac{\pi}{16} \int_{\tilde{\mathcal{B}}} \text{Tr} \left( \mathcal{V}^{-1} d\mathcal{V} \right)^2 \rho d\rho dz \quad (59) $$

where $\tilde{\mathcal{B}}$ is the orbit space of spacetime, $\mathcal{V}$ is the $3 \times 3$ unimodular matrix

$$ \mathcal{V} = \begin{pmatrix} \frac{1}{\det \lambda} & -\frac{\nu_i}{\det \lambda} \\ -\frac{\nu_j}{\det \lambda} & \lambda_{ij} + \frac{\nu_i \nu_j}{\det \lambda} \end{pmatrix} \quad (60) $$

where

$$ \lambda_{ij} = \frac{\lambda^\prime_{ij}}{H^{1/2}} \quad (61) $$

and $\nu$ are the spacetime twist potentials. Note that it follows that $H = \rho^2 (\det \lambda)^{-1}$. That is, the Euler-Lagrange equations for $\mathcal{M}'$ are precisely those for the vacuum field equations for the above form of the metric. Once $\Phi$ is determined, the remaining functions $H$ and conformal factor $\nu$ are determined by quadrature.

Expanding out the Lagrangian gives

$$ \text{Tr} \left[ (\mathcal{V}^{-1} d\mathcal{V})^2 \right] = \left( \frac{d \det \lambda}{\det \lambda} \right)^2 + \text{Tr} \left[ (\lambda^{-1} d\lambda)^2 \right] + 2 \frac{\text{Tr} (\lambda^{-1} d\nu d\nu^t)}{\det \lambda} \quad (62) $$

We wish to express the action in terms of $\lambda^\prime_{ij}$. Since

$$ d\lambda = \frac{1}{2} \left( \frac{\det \lambda}{\rho^2} \right)^{-\frac{1}{2}} \left( \frac{d \det \lambda}{\rho^2} - 2 \frac{\det \lambda d\rho}{\rho^3} \right) \lambda' + \left( \frac{\det \lambda}{\rho^2} \right)^{\frac{1}{2}} d\lambda', \quad (63) $$

a calculation yields

$$ \text{Tr} \left[ (\lambda^{-1} d\lambda)^2 \right] = \frac{1}{2} \left( \frac{d \det \lambda}{\det \lambda} \right)^2 - 2 \left( \frac{d\rho \cdot d\rho}{\rho^2} \right) + \text{Tr} \left[ (\lambda^{-1} d\lambda')^2 \right] . $$

Note $d\rho \cdot d\rho = 1$. 

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Consider a constant-time spatial slice of the stationary, axisymmetric metric (58). The metric can be placed in our general form for our initial data provided

\[ \Phi^2 = e^{2v} = \frac{1}{H^{1/2}} \left[ \frac{\det \lambda}{\det \lambda'} \right]^{1/2} \]  

(64)

which implies

\[ v = \frac{1}{4} \log(\det \lambda) - \frac{\log \rho}{2}. \]  

(65)

We then deduce

\[ d v \cdot dv = \left( \frac{d \det \lambda}{4 \det \lambda} - \frac{d \rho}{2 \rho} \right)^2 = \frac{1}{16} \left( \frac{d \det \lambda}{\det \lambda} \right)^2 - \frac{1}{4} d (\log \rho) \cdot d \log \left( \frac{\rho}{\det \lambda} \right). \]  

(66)

Using equations (61) and (65) one can replace the independent variables \( v \) and \( \lambda'_{ij} \) by \( \det \lambda \) and \( \lambda_{ij} \) in the mass functional. Then \( \det \lambda, \lambda_{ij}, \) and \( Y^i \) are taken to be independent, and we have

\[ \mathcal{M}_{\tilde{\mathcal{B}}} = \mathcal{M} + \frac{\pi}{4} \int_{\mathcal{B}} \left[ \frac{d \rho \cdot d \rho}{\rho} - \frac{3}{2} \frac{d \rho \cdot d \det \lambda}{\det \lambda} \right] d\rho \wedge dz + \frac{\pi}{2} \int_{\partial \tilde{\mathcal{B}}} \alpha \]

\[ = \mathcal{M}' + \frac{\pi}{4} \int_{\mathcal{B}} \left[ \log \left( \frac{\rho}{(\det \lambda)^{3/2}} \right) \right] d\rho \wedge dz + \frac{\pi}{2} \int_{\partial \tilde{\mathcal{B}}} \alpha \]

\[ = \mathcal{M}' + \frac{\pi}{4} \int_{\partial \tilde{\mathcal{B}} \cup \mathcal{B}_\infty} \left[ 2\alpha + \log \left( \frac{\rho}{(\det \lambda)^{3/2}} \right) \right] dz \]  

(67)

where \( \alpha \) is the one-form defined in (50). Note that we have taken the domain of integration in \( \mathcal{M} \) to be over \( \tilde{\mathcal{B}} \) when demonstrating this equivalence. This is an important point, because \( \partial \tilde{\mathcal{B}} \) will, in general, contain additional finite timelike rods on the axis \( \rho = 0 \) corresponding to Killing horizons (i.e. where a timelike Killing vector field becomes null) which are not present on \( \partial \mathcal{B} \). The domain of integration of \( \mathcal{M}' \) covers only the exterior region to the black hole, with an inner boundary representing the horizon. In contrast, our mass functional is naturally defined over \( \mathcal{B} \) and covers a complete manifold with no inner boundary, and in particular may have additional asymptotic regions. In general, \( \mathcal{M}_{\tilde{\mathcal{B}}} \) will be singular because it may diverge on the horizon rod, whereas, \( \mathcal{M} \) is finite. In the special case of extremal horizons, however, the orbit spaces coincide, because the timelike horizon rod shrinks to a point and corresponds to an asymptotically cylindrical region [14].

In summary, we have shown that over an appropriate domain, \( \mathcal{M} \) equals Carter’s functional, up to a divergent boundary term. Equivalently, we have proved that if one considers the change of variables \( (v, \lambda'_{ij}) \rightarrow \lambda_{ij} \) given by (61) and (65), then \( \mathcal{M} \) is precisely the same as \( \mathcal{M}' \) up to a boundary term. It follows they have the same Euler-Lagrange equations, provided we consider variations which are fixed on \( \partial \tilde{\mathcal{B}} \). Hence the critical points of Carter’s functional, i.e. the stationary, axisymmetric vacuum solutions, are also critical points of the mass functional.
It is interesting to directly compute the Euler-Lagrange equations of $\mathcal{M}$. The details are tedious and we simply summarize the result here. First we vary the mass functional with respect to functions on the orbit space $\bar{\lambda}'$, $\bar{v}$ and $\bar{Y}$, which have compact supported on the interior of $\mathcal{B}$ (and in particular vanish on $\partial \mathcal{B}$ and $\mathcal{B}_\infty$). Therefore, by [10] the angular momenta will be preserved. We define

$$E(\epsilon) = \mathcal{M}(v + \epsilon \bar{v}, \lambda' + \epsilon \bar{\lambda}', Y + \epsilon \bar{Y})$$

Then we have

$$\delta E(0) = \frac{\pi}{4} \int_\mathcal{B} \left[ 12dv \cdot d\bar{v} - \frac{1}{2\rho^2} (d\bar{\lambda}_{11}' \cdot d\lambda'_{22} + d\lambda'_{11} \cdot d\bar{\lambda}_{22}' - 2d\bar{\lambda}_{12}' \cdot d\lambda'_{12} ) ight]$$

$$+ e^{-6v} \left( (\det \bar{\lambda}') \frac{\text{Tr}(\lambda^{-1}dYdY^t)}{2\rho^2} + \frac{\text{Tr}(\lambda^{-1}dYdY^t)}{\rho^2} - 3\bar{v} \frac{\text{Tr}(\lambda^{-1}dYdY^t)}{\rho^2} \right) \rho d\rho dz$$

Performing the appropriate integration by parts and imposing $\delta E(0) = 0$ yields

$$4\Delta_3 v + e^{-6v} \frac{\text{Tr}(\lambda^{-1}dYdY^t)}{\rho^2} = 0,$$

$$d \left( \frac{\lambda'}{\rho^2} \right) + e^{-6v} \frac{dY \cdot dY^t}{\rho^2} = 0, \quad d \left( \frac{e^{-6v}}{\rho^2} \lambda^{-1}dY \right) = 0$$

where $\Delta_3$ is the Laplacian with respect to the metric (44). Consider the critical points of $\mathcal{M}$ that are extreme, stationary, axisymmetric vacuum solutions. We can use the above to show that the mass functional is positive definite for these data. We have $h = e^{2v} \bar{h}$ and thus

$$\tilde{R} = Re^{-2v} + 6\Delta_3 v + 6e^{-2U}(dv)^2$$

Since $\Delta_3 = e^{-2U} \Delta_3$ on $U(1)^2$-invariant functions, using (17), (19) and (70) yields

$$\tilde{R} = e^{-2U} \left( -2e^{-6v} \frac{\text{Tr}(\lambda^{-1}dYdY^t)}{2\rho^2} + 6(dv)^2 \right)$$

Substitution into the expression (43) gives

$$\mathcal{M}_{cp} = \frac{3\pi}{4} \int_\mathcal{B} e^{-6v} \frac{\text{Tr}(\lambda^{-1}dYdY^t)}{2 \det \lambda'} \rho d\rho dz$$

where $\mathcal{M}_{cp}$ is the restriction of $\mathcal{M}$ to these critical points. Clearly this is positive definite.

5 Positivity of $\mathcal{M}$

In this section we investigate the positivity of $\mathcal{M}$. Positivity is a desirable property as it plays a key role in applications to geometric inequalities for three-dimensional initial
data [1–3] and investigating the linear stability of extreme black holes [28]. Gibbons and Holzegel [29] have generalized Brill’s proof of positive mass for a restricted class of four-dimensional initial data with \( U(1)^2 \) isometry, by expressing the mass in a manifestly positive definite way. We will show that for a particular set of initial data, \( M \) can be expressed in a form such that the arguments of [29] can be adapted to demonstrate positivity. It is important to note that our boundary conditions are weaker than the ones used in [29]. A proof of positivity for arbitrary rod data remains to be found. In the following, we will consider asymptotically flat data with a single additional asymptotic end.

Introduce the coordinates \((r, x)\) given by the transformation (23). This is equivalent to introducing a map from \( B \cong \mathbb{R} \times \mathbb{R}^+ \backslash \{a_E\} \) to the infinite strip \( B \cong \mathbb{R} \times [-1, 1] \) [14]. This map will divide the axis \( \rho = 0 \) into two disconnected axes \( I^\pm = \{r > 0, x = \pm 1\} \) and another end, \( a_E = \{r = 0, |x| \leq 1\} \). Note that the rod structure is contained on \( I^\pm \), see Figure 3.

Consider the mass functional (45) and (51). The inner products are taken with respect to \( \delta^2 = d\rho^2 + dz = r^2(dr^2 + \frac{r^2 dx^2}{4(1-x^2)}) = r^2\delta'^2 \). We will rewrite the functional with respect to \( \delta'^2 \) and as an integral over the infinite strip. Thus we have

\[
\mathcal{M} = \frac{1}{32} \int_B \left( -\frac{\det d\lambda'}{2 \det \lambda'} + e^{-6v} \frac{\text{Tr} (\lambda'^{-1}dY \cdot dY^t)}{2 \det \lambda'} + 6 (dv)^2 \right) r^3 dr dx + \frac{1}{4} \int_{\partial B \cup B_\infty} \alpha \quad (75)
\]

where all scalar products of one-forms are taken with respect to \( \delta'^2 \). Note that \((\rho, z)\) and \((x, r)\) have positive orientation. The boundary is \( \partial B \cup B_\infty = I_E + I_\infty + I^+ + I^- \) where \( I_E \equiv \{r = 0, -1 \leq x \leq 1\} \), \( I_\infty \equiv \{r = \infty, -1 \leq x \leq 1\} \). In terms of the \((r, x)\) chart, we have

\[
\alpha = - (r(1-x^2)V_x + rxV) \, dr + \left( \frac{r^3}{4} V_r - \frac{r^2}{2} V \right) \, dx \quad (76)
\]

The stability argument uses positivity of the second variation of the mass functional about extreme Kerr. This energy is related to the recent construction of Hollands-Wald [26] of a canonical energy, which has recently been used to demonstrate the existence of instabilities of (near)-extreme black holes [27].

Figure 3: The rod point \( a_E = \{\rho, z = 0\} \) is another end. (a) and (b) illustrates the map from the \( z + i\rho \) complex plane to the \( y + ix \) complex plane where \( y = \log r \).
Consider the integral (75). We are given $n$ rod points $a_i$. Subdivide the infinite strip into $n$ rectangular columns $A_i$ with

$$A_i = \{-1 \leq x \leq 1, b_i < r < b_{i+1}\} , \quad i = 0 \ldots n - 1$$

where $b_i$ correspond to the location of the rod points $a_i$ after ordering along the $y = \log r$ axis (see Figure 4). For convenience, we have chosen $b_1 < b_2 < \ldots < b_{n-1}$ to correspond to the asymptotic end $a_E$ and $b_n$ to correspond to the asymptotically flat end $r \to \infty$. We then express (75) as

$$\mathcal{M} = \sum_{i=0}^{n-1} \int_{A_i} \mathcal{M}_i$$

where $\mathcal{M}_i$ is the restriction of $\mathcal{M}$ to $A_i$.

Fix a region $A_i$. Then one of the following two possibilities must occur: (a) distinct Killing fields $v^{(i)}_i$ and $w^{(i)}_i$ vanish on $A_i \cap I^+$ and $A_i \cap I^-$ respectively (in this case $A_i$ is topologically $S^3 \times \mathbb{R}$), or (b) the same Killing field $v^{(i)}_i = v^{(i)}_m m_i$ vanishes on both of the disjoint sub intervals $A_i \cap I^\pm$ (in this case $A_i$ is topologically $S^2 \times D$ where $D$ is a non-contractible disc). We can demonstrate positivity for case (a). In this case without loss
of generality we can select the following parameterization of the 3 independent functions contained in $\lambda'_{ij}$ and $v$:

$$
\begin{align*}
\lambda'_{11} &= \frac{r^2(1-x)}{2\sqrt{1-W^2}} e^{v_1-v_2} \\
\lambda'_{12} &= \frac{r^2\sqrt{1-x^2}W}{2\sqrt{1-W^2}} \\
v &= V_1 + V_2 + \log \sqrt{1-W^2}
\end{align*}
$$

where without loss of generality we have chosen $v_{(i)} = \partial_{\phi_1}$ and $w_{(i)} = \partial_{\phi_2}$. $V_1$, $V_2$ and $W$ are $C^1$ functions whose boundary conditions on the axis are induced from those of $\lambda'_{ij}$ and $v$ (26) and (27). In particular, we have $\det \lambda' = \rho^2$ and to remove conical singularities on $\mathcal{I}^\pm$ we require:

$$2V - V_1 + V_2 = 0 \quad \text{on} \quad \mathcal{I}^+, \quad 2V - V_2 + V_1 = 0 \quad \text{on} \quad \mathcal{I}^-, \quad W = 0 \quad \text{on} \quad \mathcal{I}^\pm$$

Note that since $\lambda'_{ij}$ and $v$ are continuous across the boundary of $A_i$, this will impose boundary conditions on the parameterization functions in adjacent subregions. Secondly, we rewrite the second and fourth terms of $\mathcal{M}$ as functions of $V_1$, $V_2$, and $W$, yielding:

$$
\frac{\det d\lambda'}{2 \det \lambda'} = \frac{-1}{2(1-W^2)} \left[ (dV_1 - dV_2)^2 - \frac{8}{r^2} \partial_x(V_1 - V_2) + (dW)^2 + \frac{W^2(dW)^2}{1-W^2} + \frac{4W^2}{r^2(1-x^2)} \right]
$$

and

$$6(dv)^2 = \frac{3}{2}(dV_1 + dV_2)^2 - \frac{3}{2} \frac{W^2(dW)^2}{1-W^2} - \frac{3W}{1-W^2}(dV_1 \cdot dW + dV_2 \cdot dW)
$$

Therefore, we have

$$
\mathcal{M}_i = \frac{1}{32} \int_{A_i} \left( R e^{2v+2U} + (dV_1 + dV_2)^2 + (dV_1)^2 + (dV_2)^2 \right)
$$

$$+ \frac{W^2}{2(1-W^2)} \left[ (dV_1 - dV_2)^2 - \frac{6}{W}(dV_1 \cdot dW + dV_2 \cdot dW) \right]
$$

$$+ \frac{W^2}{r^2(1-W^2)} \left[ 4\partial_x V_2 - 4\partial_x V_1 + \frac{2}{1-x^2} \right] + \frac{(dW)^2}{2(1-W^2)} + \frac{2W^2(dW)^2}{(1-W^2)^2} \right) r^3 dr dx
$$

$$+ \frac{1}{8} \int_{b_i}^{b_{i+1}} r((V_1 - V_2)|_{x=1} - (V_1 - V_2)|_{x=1}) dr + \frac{1}{4} \int_{b_i}^{b_{i+1}} r(V|_{x=1} + V|_{x=1}) dr
$$

$$= \frac{1}{32} \int_{A_i} \left( R e^{2v+2U} + (dV_1 + dV_2)^2 + (dV_1)^2 + (dV_2)^2 \right)
$$

$$+ \frac{W^2}{2(1-W^2)} \left[ (dV_1 - dV_2)^2 - \frac{6}{W}(dV_1 \cdot dW + dV_2 \cdot dW) \right]
$$

$$+ \frac{W^2}{r^2(1-W^2)} \left[ 4\partial_x V_2 - 4\partial_x V_1 + \frac{2}{1-x^2} \right] + \frac{(dW)^2}{2(1-W^2)} + \frac{2W^2(dW)^2}{(1-W^2)^2} \right) r^3 dr dx
$$

$$= \frac{1}{32} \int_{A_i} \left( R e^{2v+2U} + (dV_1 + dV_2)^2 + (dV_1)^2 + (dV_2)^2 \right)
$$

$$+ \frac{W^2}{2(1-W^2)} \left[ (dV_1 - dV_2)^2 - \frac{6}{W}(dV_1 \cdot dW + dV_2 \cdot dW) \right]
$$

$$+ \frac{W^2}{r^2(1-W^2)} \left[ 4\partial_x V_2 - 4\partial_x V_1 + \frac{2}{1-x^2} \right] + \frac{(dW)^2}{2(1-W^2)} + \frac{2W^2(dW)^2}{(1-W^2)^2} \right) r^3 dr dx
$$

$$21$$
Consider the first equality. The first term follows from the constraint equation for maximal slices, (19), and (14). The remaining bulk terms follow from (82) and (83) while the first boundary term comes from (82) and the second from (77). The second equality is obtained by noting the boundary contributions cancel by regularity on the axes (81). The remaining terms can be shown to be positive by a straightforward application of the arguments given in section 4.3 of [29]. Therefore, $M_i \geq 0$.

From this result it follows that provided all subregions $A_i$ fall into class (a) then $M$ is positive-definite. In particular, for the rod structure of Myers-Perry initial data, there is only one region $A_0$ of class (a) and hence for any data with the same rod structure, $M \geq 0$. One might expect a similar argument to hold for class (b). This case of course includes initial data for black rings (the same Killing vector field vanishes on either side of the asymptotic end). By choosing a general parametrization for the various functions in this region, one finds that the boundary term has an indefinite sign. However our strategy is merely sufficient to demonstrate positivity, and we expect positivity will hold for general rod structure. Interestingly, for the initial data for extreme black rings, the expression (74) shows $M \geq 0$.

6 Discussion

We have constructed a mass functional $M$ valid for a broad class of asymptotically flat $t-\phi^i$-symmetric maximal initial data for the vacuum Einstein equations in five dimensions. $M$ can be considered an extension of a similar functional defined for three-dimensional initial data sets [25]. We can check this mass functional is finite and evaluates to the ADM mass provided certain boundary and asymptotic conditions are met. These conditions encompass a large class of initial data, and in particular we have checked this explicitly for the usual maximal constant-time slices for the Myers-Perry black hole (see Appendix B) and the extreme vacuum black ring solution. Moreover, we proved that $\mathbb{R} \times U(1)^2$-invariant solutions of the vacuum Einstein equations are critical points of this functional amongst this class of data. Finally, we have shown explicitly that the mass functional is positive for a particular class of rod structures as explained in detail above, although it remains to show this for an arbitrary rod data. This property is relevant to investigate geometric inequalities for five-dimensional vacuum solutions. An starting towards this goal is to show a local mass-angular momenta inequality along the lines of [11]. This problem is currently under investigation.

7 Acknowledgments

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A Mass of conformal metric

Assume that we have an asymptotically flat initial data set \((\Sigma, h_{ab}, K_{ab})\) of Einstein’s equation. The ADM mass of this data is given by formula (40). But by a rescaling similar to (14) we have

\[
M_{ADM} = -\frac{3}{8\pi} \lim_{r \to \infty} \int_{S^3} \tilde{n} \tilde{\nabla} c \Phi \, ds + \tilde{M}_{ADM}
\]  

(A.1)

where \(\tilde{M}_{ADM}\) is the ADM mass of \(\tilde{h}\). Now as in Section 5 we can introduce a chart with coordinates \((r, x)\) such that the asymptotically flat conformal metric takes the form

\[
\tilde{h} = e^{2V} \left( dr^2 + \frac{r^2}{4(1-x^2)} dx^2 \right) + f_2 \frac{r^2}{2} (1-x) d\phi^2 + f_3 \frac{r^2}{2} (1+x) d\psi^2 + f_4 r^2 (1-x^2) d\phi d\psi
\]

(A.2)

with the fall-off conditions \(e^{2V} - 1, f_2 - 1,\) and \(f_3 - 1 = O(r^{-2})\) and \(f_4 = o(r^{-2})\) as \(r \to \infty\). Then the ADM mass of the conformal metric is

\[
\tilde{M}_{ADM} = -\frac{1}{16\pi} \lim_{r \to \infty} \int_{S^3} \left( r^2 \partial_r [ r(f_2 + f_3 - 2)] + r^5 \partial_r \left( \frac{e^{2V} - 1}{r^2} \right) \right) \, d\Omega_3
\]

\[
= \frac{1}{16\pi} \int_{S^3} (f(x) + g(x)) \, d\Omega_3 + \frac{1}{2\pi} \int_{S^3} \tilde{V}(x) \, d\Omega_3
\]

\[
= \frac{\pi}{2} \int_{-\infty}^{\infty} \tilde{V}(x) \, dx
\]

(A.3)

The first equality is the definition of ADM mass applied to (A.2). The second equality uses the expansion of \(\lambda'_{ij}\) and \(V\) at infinity (27), (26). Therefore, we can see the ADM mass of the conformal metric is zero if and only if the right hand side of (A.3) vanishes. It is trivially satisfied if \(V = o(r^{-2})\). In general however, one may wish to consider weaker fall-off conditions on \(V\) that still lead to vanishing ADM mass of the conformal metric. In particular, we have checked explicitly for the general Myers-Perry black hole and for the extreme doubly spinning black ring that the right hand side of (26) vanishes, although \(\tilde{V}(x) \neq 0\) in these cases.

B Myers-Perry initial data

Here we consider the Myers-Perry solution with coordinates \((t, \tilde{r}, \theta, \phi_1, \phi_2)\) [30]. The \(\phi_i\) have period \(2\pi\). Then we have following metric functions

\[
\omega^1 = \frac{\mu a \lambda_{22} \sin^2 \theta - \mu b \lambda_{12} \cos^2 \theta}{\Sigma \det \lambda}
\]

\[
\omega^2 = \frac{\mu b \lambda_{11} \cos^2 \theta - \mu a \lambda_{12} \sin^2 \theta}{\Sigma \det \lambda}
\]

(B.1)

\[
\lambda_{11} = \frac{a^2 \mu}{\Sigma} \sin^4 \theta + (\tilde{r}^2 + a^2) \sin^2 \theta
\]

\[
\lambda_{12} = \frac{ab \mu}{\Sigma} \sin^2 \theta \cos^2 \theta
\]

(B.2)

\[
\lambda_{22} = \frac{b^2 \mu}{\Sigma} \cos^4 \theta + (\tilde{r}^2 + b^2) \cos^2 \theta
\]

(B.3)
\[ \Sigma = \tilde{r}^2 + b^2 \sin^2 \theta + a^2 \cos^2 \theta, \quad (B.4) \]
\[ \Delta(\tilde{r}) = (\tilde{r}^2 + a^2) (\tilde{r}^2 + b^2) - \mu \tilde{r}^2. \quad (B.5) \]

The metric on a constant time slice will be
\[ h = \frac{\Sigma}{\Delta(\tilde{r})} dr^2 + \Sigma d\theta^2 + \lambda_{ij} d\phi^i d\phi^j \quad (B.6) \]

This metric is singular at two roots \( \tilde{r}_\pm \) of \( \Delta(\tilde{r}) \) which correspond to spacetime inner and outer horizons. One can define a quasi-isotropic coordinate as
\[ \tilde{r}^2 = r^2 + \frac{1}{2} \left( \mu - a^2 - b^2 \right) + \frac{\mu (\mu - 2a^2 - 2b^2) + (a^2 - b^2)^2}{16r^2} \quad (B.7) \]

Note the outer horizon at \( \tilde{r}_+ \) is shifted to \( r = 0 \) and the slice metric will be
\[ h = \frac{\Sigma}{\tilde{r}^2} \left( dr^2 + r^2 d\theta^2 \right) + \lambda_{ij} d\phi^i d\phi^j \quad (B.8) \]

where \( 0 < r < \infty, \ 0 < \theta < \pi/2, \) and \( 0 < \phi_1, \phi_2 < 2\pi. \) The point \( r = 0 \) is another asymptotic infinity (see figure 1) and one can show this with computing the distance to \( r = 0 \) along a curve of constant \( (\theta, \phi_1, \phi_2) \) from \( r = r_0, \) i.e.
\[ \text{Distance} = \int_r^{r_0} \frac{\sqrt{\Sigma}}{r} dr \to \infty \quad \text{as} \quad r \to 0 \quad (B.9) \]

In the extreme limit \( \mu = (a + b)^2 \) the quasi-isotropic radius simplifies to
\[ \tilde{r}^2 = r^2 + ab \quad (B.10) \]

The conformal metric \( \tilde{h} \) can be determined by the relations
\[ \Phi^2 = \frac{\sqrt{\det \lambda}}{\rho}, \quad e^{2U} = \frac{\rho \Sigma}{r^4 \sqrt{\det \lambda}}, \quad \lambda'_{ij} = \Phi^{-2} \lambda_{ij} \quad (B.11) \]

where \( \rho = \frac{1}{2} r^2 \sin 2\theta \) and \( z = \frac{1}{2} r^2 \cos 2\theta. \) The potentials in the general case are cumbersome, but in the extreme case simplify to
\[ Y^1 = \frac{a(a^2 - b^2)(r^2 + ab + b^2) \cos^2 \theta - r^2 a(2a^2 + 2ab + r^2)}{(a - b)^2} + \frac{a(r^2 + ab + a^2)(r^2 + ab + b^2)}{\Sigma(a - b)^2} \]
\[ Y^2 = \frac{br^2((a + b)^2 + r^2) - b(a^2 - b^2)(r^2 + ab + a^2) \cos^2 \theta - b(r^2 + ab + a^2)(r^2 + ab + b^2)^2}{(a - b)^2} \quad (B.12) \]
The expansion at infinity is

\[ Y^1 = \frac{a^3(a+b)^2}{(a-b)^2} - \frac{4J_1}{\pi} \cos^2 \theta(2 - \cos^2 \theta) + \mathcal{O}(r^{-2}) \] (B.13)

\[ Y^2 = -\frac{ab^2(a+b)^2}{(a-b)^2} - \frac{4J_2}{\pi} \cos^4 \theta + \mathcal{O}(r^{-2}) \] (B.14)

The asymptotic behaviour of the conformal factor at infinity is given by

\[ \Phi = 1 + \frac{\mu}{4r^2} + \mathcal{O}(r^{-4}) \quad r \to \infty \] (B.15)

The region \( r \to 0 \) corresponds to another asymptotic region. In the non-extreme case, we have

\[ \Phi = \frac{\sqrt{\mu - (a+b)^2}(\mu - (a-b)^2)^2}{4r^2} + \mathcal{O}(1), \quad \Phi_r = \mathcal{O}(r^{-3}) \quad r \to 0 \] (B.16)

and it is easy to verify that \( \tilde{h} \) approaches the flat metric on \( \mathbb{R}^4 \). Hence this region is an asymptotically flat end. In the extreme case, however, one can check that

\[ \Phi = \frac{(ab(a+b)^3)^{1/4}}{(a \cos^2 \theta + b \sin^2 \theta)r} + O(r) \quad r \to 0 \] (B.17)

By examining the behaviour of the metric \( h \), one can see that the asymptotic region \( r \to 0 \) is a cylindrical end. In fact, explicit computation of \( U \) and \( \lambda_i' \) shows that the conformal metric \( \tilde{h} \) approaches the metric of a cone over an \( S^3 \) equipped with an inhomogeneous metric,

\[ \tilde{h} = \Omega^2 \left( dr^2 + r^2 \gamma \right) \] (B.18)

where \( \Omega = \Omega(\theta) \neq 0 \) and \( \gamma \) is conformal to the inhomogeneous metric on cross-sections of the horizon of the extreme Myers-Perry black hole.

The conformal factor in either case satisfies conditions of (30), (41), and we have at the asymptotic ends

\[ \frac{3}{8\pi} \lim_{r \to 0} \int_{S_r} \frac{\nabla_c \Phi}{\Phi} n^c ds_{\tilde{h}} = 0 \] (B.19)

where \( ds_{\tilde{h}} = r^3 \sin \theta \cos \theta + \mathcal{O}(r^6) \). Now, one can expand the function \( V \) at infinity and at the origin. As we discussed before we only consider behaviour of \( V \) near \( \rho = 0 \). We find

\[ V = \frac{(a^2 - b^2) \cos 2\theta}{4r^2} + \mathcal{O}(r^{-4}) \quad r \to \infty \] (B.20)

\[ V_+ = \frac{2z + a^2 + ab}{\sqrt{4z^2 + 3a^2b^2 + 2a^2z + b^4 + 2b^2z + 4abz + a^4b + 3ab^2}} \quad z \in I_+ \] (B.21)

\[ V_- = \frac{-2z + b^2 + ab}{\sqrt{a^4 + 3a^2b + 3a^2b^2 - 2a^2z + ab^3 - 4abz - 2b^2z + 4z^2}} \quad z \in I_- \] (B.22)
Thus $V$ satisfies condition (26) and (32). In particular, we read off $\bar{V} = \frac{1}{4}(a^2 - b^2)x$ and hence from (A.3) we see $\bar{h}$ (see (B.11)) has vanishing ADM mass. In addition, when $z \to \pm \infty$ we have $V_\pm \to 1$ and $V_\pm$ are bounded continuous functions on rods $I_\pm$. Therefore, they are integrable. Let us consider boundedness of other terms in the mass functional (51). We will consider explicitly the non-extreme case so the end is asymptotically flat. First we have the following expansion for $v$ at origin and infinity

$$(dv)^2 = -\frac{\mu}{2r^5} + O(r^{-7}) \quad r \to \infty$$

$$(dv)^2 = -\frac{2}{r^3} + O(r^{-1}) \quad r \to 0$$  (B.23)

since the volume element is $\rho d\rho dz = r^5 \sin \theta \cos \theta \, dr d\theta$, $(dv)^2$ is bounded at origin and infinity. Now we consider term which related to scalar curvature in mass functional (51).

We use identity (47) and we have

$$\det \frac{d\lambda'}{d\lambda} = O(r^{-8}) \quad r \to \infty$$

$$\det \frac{d\lambda'}{d\lambda} = O(1) \quad r \to 0$$  (B.24)

This is clearly bounded. One can check numerically over a range of $(a, b)$ that $\det d\lambda' < 0$ everywhere. The only term remaining is related to the full contraction of extrinsic curvature and we have

$$\frac{\text{Tr} (\lambda'^{-1} dY dY')}{2 \det \lambda'} = O(r^{-10}) \quad r \to \infty$$

$$e^{-6v} \frac{\text{Tr} (\lambda'^{-1} dY dY')}{2 \det \lambda'} = O(r^2) \quad r \to 0$$  (B.25)

Therefore, non-extreme Myers-Perry lies in the domain on which the mass functional (51) is defined. By similar steps the same result holds for the extreme case.

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