Soft confinement in a 3-d spin system

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Abstract We consider a 1+3 dimensional spin system. The spin-wave (magnon) field is described by the O(3) non-linear sigma model with a symmetry-breaking potential. This interacts with a slow spin SU(2) doublet Schrödinger fermion. The interaction is described by a generalized nonperturbative Yukawa coupling, and the self-consistency condition is solved with the aid of a non-relativistic Gribov equation. When the Yukawa coupling is sufficiently strong, the solution exhibits supercriticality and soft confinement, in a way that is quite analogous to Gribov’s light-quark confinement theory.

The solution corresponds to a new type of spin polaron, whose condensation may lead to exotic superconductivity.

1 Introduction

1.1 Motivation

The motivation for our work is twofold.

First, we have the colour confinement problem, to which one suggested solution is Gribov’s light-quark confinement theory [1,2,3,4]. The discovery of an analogy or application for it in another system will be beneficial for deepening our understanding about light-quark confinement theory and confinement in general.

Second, the behaviour of fermionic spin in a polarized background is often described by spin polarons [5], which are known to occur in some limiting strongly correlated cases. It would be interesting to find some other instances of spin physics in which fermionic spin is strongly affected by the polarized background.

1.2 General remarks

When discussing the colour confinement problem, one traditional approach is to look primarily at the gluodynamics at large scales. As a representative example, we have Wilson’s work [6] which is based on lattice quantization. In this picture, in simplest terms, coloured objects are confined because the long-distance effective potential is unbounded from above. When a quark–antiquark pair is pulled apart, for example, it requires infinite energy to separate them by an infinite distance.

However, in the real world, there are light quarks. When two quarks are pulled apart, they will fragment into hadrons. This being the case, we arrive at an alternative line of thought, that large long-distance interaction is not necessarily the true essence of confinement as it relates to our world. What would then be the condition that governs confinement?

This problem is dealt with by Gribov’s light-quark confinement theory [1,2,3,4], according to which confinement occurs when moderately strong coupling binds together light quarks supercritically. The supercriticality changes the vacuum structure, and the response of the modified vacuum is described by the contribution of Goldstone pions.

A somewhat unobvious outcome of this picture is that although moderately strong gluonic interaction is responsible for changing the vacuum structure, it is the interaction of pions that confines the quarks. One intuitive explanation for this is that when a quark–antiquark pair is pulled apart, they fragment into more mesons such as pions, with no reference to dynamics...
that grows large at long distances. Hadrons are fragile objects.

Let us call the first picture ‘hard confinement’ and the second picture ‘soft confinement’ [4] (note, however, that the first picture is called ‘soft’ in ref. [3]). We would now like to look for their counterparts in spin systems.

An analogy for hard confinement is in the so-called ‘string polarons’ [7], which occurs in two-dimensional antiferromagnetic systems with small spin exchange energy $J$ at low carrier (electron or hole) concentration. Here, moving a single carrier over the antiferromagnetic background disturbs the spin. The amount of disturbance is proportional to the distance $\ell$ moved by the carrier, and therefore there is an effective potential $V(\ell) \propto \ell$ which traps the carrier, or confines it into spin-singlet pairs.

Our question is whether there is any such analogy for soft confinement. Pions being Goldstone bosons, and magnons which are the quanta of spin wave being Goldstone bosons, we are led to consider magnon-fermion systems. The most obvious candidate is then three-dimensional spin-wave system with linear dispersion relation, which is coupled to a spin-doublet carrier through a Yukawa-like interaction.

We shall study this system, and shall show that soft confinement does indeed occur when the interaction is strong. This solution corresponds to a new type of spin polaron state, in which fermionic spin is confined not by large spatial long-distance interaction but by the temporal decay of the false vacuum. More phenomenologically speaking, a fermionic spin that is excited will decay by emitting a magnon, and will not have time to behave like an itinerant spin.

Our work utilizes Gribov equations [1,2,3,4], but they are used here for somewhat different set of reasons than in the QCD case.

We shall formulate the problem in Sec. 2. We shall write down the Gribov equations in Sec. 3 and solve them in Sec. 4. We analyze the solutions corresponding to both weak and strong coupling regions in Sec. 5. The nature of the supercritical solutions is discussed in Sec. 6, together with some phenomenological remarks and a comparison with QCD. The conclusions are stated at the end.

**2 Formulation of the problem**

The spin-wave system is described by the following Lagrangian density:

$$L_\Phi = \frac{\hbar^2}{2} \left[ \left( \frac{\partial \Phi}{\partial t} \right)^2 - u^2 (\nabla \Phi)^2 \right] - V(\Phi).$$

$u$, which adopts the role of $c$ or the speed of light, is the spin-wave velocity. $V(\Phi)$ is a potential such as $-\mu/2 |\Phi|^2 + \gamma/4 |\Phi|^4$, whose minimum is found at $|\Phi| = \nu_h$, $\nu_h$ is non-zero, and this implies finite magnetization. $\Phi$ may then be parametrized as

$$\Phi = (\phi_1, \phi_2, \nu_h + h).$$

The magnon Green’s function may then be written as

$$D_\pm(\omega, k) = \frac{1}{\omega^2 - u^2 k^2 + i0}.$$ (3)

$\pm$ refers to the $S_z = \pm 1$ magnon modes, i.e. $\phi_x \pm i \phi_y$ when magnetization is along the $z$ axis. In principle, there will also be the amplitude mode which we may denote

$$D_0(\omega, k) = \frac{1}{\omega^2 - u^2 k^2 - M_h^2 u^4 + i0}.$$ (4)

but we shall not consider the contributions of this mode in this study. That is, $M_h u^2$ will be assumed to be large.

The fermion $\Psi$ is assumed to move at a velocity $\sim \nu_F$ that is much smaller than $u$, so that the momentum-dependent terms may be neglected. The Lagrangian is given by

$$L_\Psi = \Psi^\dagger \left[ \frac{ih}{\hbar} \frac{d}{dt} - \mu_F + f^{-1} \Delta_{\text{ex}} \sigma \cdot \Phi \right] \Psi.$$ (5)

$T$ and $\Delta_{\text{ex}}$ (generalized exchange energy) are some functions of the time derivative. $\Psi$ is an SU(2) doublet. $f = 2 \nu_h$ is the form factor of the magnon, and has the dimension [energy x volume]^{-1/2}. $\mu_F$ is the Fermi energy. When $\Phi$ is as parametrized by eqn. (2), the $S_z = \pm 1/2$ spin states $\psi_{\pm}$ are given by

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}.$$ (6)

The $\psi_-$ state is more energetically favourable.

Equation (5) generalizes the weak-coupling expression

$$L_\Psi = \Psi^\dagger \left[ \frac{ih}{\hbar} \frac{d}{dt} - \mu_F + f^{-1} \Delta_0 \sigma \cdot \Phi \right] \Psi.$$ (7)

The exchange energy $\Delta_0$ is constant. In this case, the Green’s function is given by

$$G_\pm^{\text{weak}}(\omega) = \frac{1}{\omega - \mu_F \pm \Delta_0/2}.$$ (8)

We see that $\Delta_0 = G_{-1}(\omega) - G_{+1}(\omega)$. We then generalize this result as

$$\Delta_{\text{ex}}(\omega) = G_{-1}(\omega) - G_{+1}(\omega).$$ (9)
Note that in the weak-coupling expression of eqn. (7), the interaction term may be rotated away by the transformation

\[ \Psi(x,t) \rightarrow \exp \left( \int \frac{i}{\hbar} f^{-1} \Delta_0 \sigma \cdot \Phi(x,t) dt \right) \Psi(x,t). \]  

(10)

This is not the case when the interaction is strong. Our writing eqn. (9) is based on the conservation of spin current. Let us consider a current of the form shown in fig. 1. Note that \( S_z \) is conserved.

\[ + \rightarrow \Gamma_{\mu} \rightarrow - \mu \]

Fig. 1 Off-diagonal spin current. There is a spin \( S_z \) conservation law at the vertex. Fermions carry \( \pm \frac{1}{2} \) \( S_z \) charge, whereas the magnon has \( S_z = \pm \frac{1}{2} \).

Whatever is the form of \( \Gamma_{\mu} \), its contraction with the incoming 4-momentum \( q_{\mu} \) will be zero in the soft limit \( q_{\mu} \rightarrow 0 \). In this limit, the violation of the Ward identity will be proportional to \( G_{-1}^{-1} - G_{+1}^{-1} \). This must be cancelled by the magnon contribution, and therefore the coupling is of the form of eqn. (5) together with eqn. (9).

Let us now consider the one-loop self-energy diagram, shown in fig. 2.

\[ \phi_+(q-k) \]

\[ \psi_+(q) \downarrow \psi_-(k) \rightarrow \psi_+(q) \]

Fig. 2 The self-energy diagram for \( \psi_+ \).

We adopt the four-vector notation with the metric diag(1, \(-1\), \(-1\), \(-1\)) when \( u = c = 1 \).

At the dressed one-loop level, and corresponding to fig. 2 the self-energy is given by

\[ \Sigma_+(q) = \int \frac{d^4k}{(2\pi)^4} i f^{-2} \Delta_{\text{ex}}(k,q) G_-(k) D_{\pm}(q-k). \]  

(11)

\( \Sigma_+ \) refers to the correction to \( G_{-1}^{-1} \). \( \Sigma_- \) is given by

\[ \Sigma_-(q) = \int \frac{d^4k}{(2\pi)^4} i f^{-2} \Delta_{\text{ex}}(k,q) G_+(k) D_{\pm}(q-k). \]  

(12)

This set of equations needs special emphasis. Suppose that, instead of the above formulation, we had started from the weak coupling formulation of eqn. (7) and then defined the full theory based on some perturbative expansion. In that case, if we define \( \Delta_{\text{ex}} \) perturbatively as the bare vertex plus corrections, then starting at the three-loop level, there will apparently be double counting in eqns. (11,12), as shown in fig. 3. One would think that each of fig. 3a, b corrects one or the other of \( \Delta_{\text{ex}} \) whereas the two diagrams are in fact equivalent.

\( \Sigma_+ \) refers to the correction to \( G_{-1}^{-1} \). \( \Sigma_- \) is given by

\[ \left( \begin{array}{c} \psi_+(q) \\ \psi_-(k) \\ \psi_+(q) \end{array} \right) \]

Fig. 3 The two perturbative three-loop self-energy diagrams a and b are equivalent.

However, this is not quite true. Diagrams of the form shown in fig. 3 cannot be part of the renormalization of \( \Delta_{\text{ex}} \), if \( \Delta_{\text{ex}} \) is defined by eqn. (9). For instance, the leading self-energy correction contribution starts at the one-loop order as can be seen in fig. 2, whereas the leading vertex-correction contribution starts at the two-loop order, as can be seen in fig. 4.

\[ \left( \begin{array}{c} \psi_+(q) \\ \psi_+(q) \end{array} \right) \]

Fig. 4 The leading perturbative vertex correction contribution.

Therefore \( \Delta_{\text{ex}} \) in eqns. (11,12) cannot be interpreted as the bare vertex plus perturbative corrections of the form shown in fig. 3. Equations (11,12) must instead be solved self-consistently. We shall do so using the Gribov equation formalism [1,2,3,4].

3 Gribov equations

Following refs. [1,2,3,4], we now apply

\[ \frac{\partial^2}{\partial q_0} - \frac{1}{w} \frac{\partial^2}{\partial q^2} \equiv \frac{\partial^2}{\partial q^0} \]  

(13)

to eqn. (11).

Let us assume that the variations of \( \Delta_{\text{ex}}(k,q) \) with respect to \( k-q \) are small, so that we need only consider
the effect of applying $\partial^2$ on $D_\pm (q-k)$. This is correct in the weak coupling case, but becomes an approximation in the strong coupling case, as we find that $\Delta_{\infty}$ tends to fall at large energies as $1/E$. Even so, one expects that the structure of $G_\pm$ is determined mostly by soft-exchange contributions, $k-q \ll k$, in which case $\Delta_{\infty}(k,q) \approx \Delta_{\infty}(k)$ is a good approximation. Accepting this, the integral sign then disappears because of the following identity:

$$\frac{\partial^2}{(q_0 - k_0)^2 - w^2(q-k)^2 + i0} = \frac{4\pi^2 i}{u^3}\delta^{(4)}(q-k).$$

(14)

We then obtain

$$\frac{\partial^2}{\partial q_0^2} G_+^0(q) = \frac{(f^{-1} \Delta_{\infty}(q))^2}{4\pi^2h^3u^3} G_-(q),$$

(15)

and the same for $\Sigma$ when $G_-$ is replaced by $G_+$ on the right-hand side.

Our approximation is that $\psi$ moves much more slowly than $\phi$ does. We may therefore take the large $u$ limit of eqn. (15), in which case only the energy derivative remains:

$$\frac{\partial^2}{\partial q_0^2} G_+^0(q) = \frac{(f^{-1} \Delta_{\infty}(q))^2}{4\pi^2h^3u^3} G_-(q).$$

(16)

We have replaced $\Sigma_+$ with $-G_+^{-1}$ on the right-hand side because the double energy derivative of the bare inverse propagator vanishes for Schrödinger fields.

In order to simplify the equations, let us introduce a new variable $\omega_f = 2\pi f(hu)^{1/2}$, which has the dimension of energy. We obtain

$$\begin{align*}
(G_+^0)' &= -\omega_f^{-2}(G_+^{-1} - G_+^{-1})' G_-, \\
(G_-^{-1})' &= -\omega_f^{-2}(G_-^{-1} - G_+^{-1})' G_+.
\end{align*}$$

(17)

Prime refers to the energy derivative $\partial/\partial q_0$.

### 4 Solution to Gribov equations

Let us proceed to solve eqns. (17). We first consider the expression

$$G_+(G_+^{-1})' - G_-(G_-^{-1})' = 0,$$

(18)

which follows from eqns. (17). We then use the identity $x^{-1}x'' \equiv (\ln x)' + ((\ln x)')^2$:

$$\ln(G_+^{-1}/G_-^{-1})' + ((\ln G_+^{-1})')^2 - ((\ln G_-^{-1})')^2 = 0.$$  

(19)

Let us define $z = G_+^{-1}/G_-^{-1}$:

$$(\ln z)'' + (\ln z)'(\ln(G_+^{-1}G_-^{-1}))' = 0.$$  

(20)

Integrating this expression once yields

$$(\ln z)' = c_0 G_+ G_-.$$  

(21)

c_0 is a constant of integration with the dimension of energy. It is easy to see that $c_0 = \Delta_0$ in the weak-coupling limit. Let us therefore denote it as such. We then obtain

$$G_+^2 = \Delta_0^{-1} z', \quad G_-^2 = \Delta_0^{-1} z'/z^2.$$  

(22)

Let us make use of eqn. (22) to eliminate the energy derivatives in eqns. (17). This gives us, almost trivially,

$$G_+^2 \frac{d^2 G_+}{dz^2} G_- = (\omega_f \Delta_0)^{-2}(z + z^{-1} - 2).$$  

(23)

The equation for $G_+$ is obtained by substituting $G_+$ for $G_-$ and $z^{-1}$ for $z$.

Let us proceed to solve eqn. (23) numerically. The definition of $z$ is such that the divergences of $G_\pm$ map to $z \to \infty$ and $z \to 0$. We would like to replace it with some variable where both singularities appear at a finite value.

In this regard, we notice that the left-hand side of eqn. (23) has a conformal symmetry with respect to transformations of $z$. This symmetry becomes more manifest when we return to eqns. (17) and now eliminate $G_-$ and $G_+$ using eqn. (22). After some elementary algebra, we obtain

$$\frac{z''}{2z} = \frac{3}{4} \left( \frac{z''}{z'} \right)^2 = \omega_f^{-2}(z + z^{-1} - 2).$$  

(24)

We then see that the left-hand side of this equation is invariant under the Möbius transformation:

$$z \to w = \frac{az + b}{cz + d}.$$  

(25)

Let us adopt

$$w = \frac{1 + z}{1 - z}, \quad z = \frac{1 - w}{1 + w}.$$  

(26)

The singularities now map to $w = \pm 1$. We also define

$$y = \left( \frac{\omega_f \sqrt{3}}{4} \frac{dw}{dE} \right)^{1/2} = \left( \frac{\omega_f \Delta_0 \sqrt{3}}{2 \Delta_{\infty}^2} \right)^{1/2}.$$  

(27)

In terms of these variables, eqn. (24) reduces to

$$y^3 \frac{d^2 y}{dw^2} = -\frac{3}{4(1 - w^2)}.$$  

(28)

This may be derived also by applying eqn. (26) directly to eqns. (23) and (22).

By the symmetry between $G_\pm$, we have one boundary condition, namely $dy/dw = 0$ at $w = 0$. The other boundary condition is $y = y_0$ at $w = 0$, where $y_0$ is a parameter. Since $y$ is proportional to $\sqrt{\Delta_0/\Delta_{\infty}}$, large $y_0$ corresponds to weak coupling and small $y_0$ to strong coupling.
We solved eqn. (28) numerically using the above boundary conditions. The result is shown in fig. 5. The numbers were obtained using the classical Runge–Kutta method with the step size of 0.0005, and were found to be stable against modifications of the step size.

![Graph](image)

Fig. 5 Numerical results for $y$ versus $|w|$, for five representative values of $y_0 = y(0)$.

There is a critical value $y_0^{\text{crit}}$ of $y_0$, where $y$ vanishes when $w = \pm 1$, with the limiting behaviour $y \to (1 - w^2)^{1/4}$. The behaviour of the solution changes above and below $y_0^{\text{crit}}$. We have found numerically that $y_0^{\text{crit}} = 1.061$ to three decimal places.

### 5 Analysis of the solutions

We would now like to discuss the nature of the solutions both below and above $y_0^{\text{crit}}$.

Let us define the scaled energy $x$ as

$$x = \frac{4}{\sqrt{3}} \frac{E - E_0}{\omega_f},$$  

with the boundary condition $w = 0$ at $x = 0$. By eqn. (27), we obtain

$$\int dx = \int \frac{dw}{y^2(w)}.$$  

We can use this equation to convert the $y$ versus $z$ results into relations involving scaled energy. As an example, we show $z$ as a function of $x$ in fig. 6 which is calculated numerically using the data of fig. 5 and using Simpson’s rule for integration.

Let us now discuss the three regions of $y_0$.

#### 5.1 Subcritical case ($y_0 > y_0^{\text{crit}}$)

When $y_0$ is large, eqn. (28) is solved by the following approximate solution when $|w|$ is not large:

$$y \approx y_0 - \frac{3}{8y_0^2} [(1 + w) \ln(1 + w) + (1 - w) \ln(1 - w)].$$  

When $|w|$ is large, the limiting behaviour is given by $y \to A - B |w|$, with $A \approx y_0$ and $B \propto y_0^3$. If $y_0$ is large, $B$ is small. Since $y = \text{const}$ corresponds to constant $\Delta_{\text{ex}}$, this behaviour corresponds to the weak coupling limit, as expected. The system is as described by eqn. (4).

It can be seen that the singularity condition $w = \pm 1$ is satisfied at finite and real $x$. Explicitly, $w = \pm 1$ at near $x = \pm 1/y_0^2$. The Green’s functions have approximate poles on the real axis. Their behaviour is as shown in fig. 4. The two Green’s functions are separated by a constant exchange energy.

Above the poles, $w$ grows linearly with $x$ at first, and saturates at $|w| = A/B$. This means, by the definition of $w$ and $z$, that the two Green’s functions are renormalized asymmetrically. The asymmetry is proportional to $B/A$ and is therefore small when $y_0$ is large.

#### 5.2 Supercritical case ($y_0 < y_0^{\text{crit}}$)

When $y_0$ is small, eqn. (28) is solved by the following approximate solution when $w^2 \ll 1$:

$$y \approx \left(y_0 - \frac{3w^2}{4y_0^2}\right)^{1/2}. $$  

![Graph](image)

Fig. 6 Numerical results for $z$ versus $x$, for five representative values of $y_0 = y(0)$.
functions are then given by

$$y$$

So long as

The Green’s functions in the strong-coupling limit as

Fig. 8

This solution corresponds to the approximation $1 - w^2 \approx 1$ in eqn. (28).

Equation (30) now yields

$$w = \frac{2}{\sqrt{3} y_0^2} \tanh \left( \frac{\sqrt{3}}{2} x \right).$$

(33)

So long as $y_0^2 < \sqrt{3}/2$, the solution has no singularities $w = \pm 1$ for real $x$. That is, $\psi$ is confined.

Let us calculate $G_{\pm}$. Equation (33) implies

$$z = -\frac{\cosh \sqrt{3}/2 (x + \delta_0)}{\cosh \sqrt{3}/2 (x - \delta_0)},$$

(34)

where $\delta_0 = (2/\sqrt{3}) \tanh^{-1} (2y_0^2/\sqrt{3}) \approx 4y_0^2/3$. The Green’s functions are then given by

$$G_{\pm}^{-1} = \pm \frac{\sqrt{3}}{2} (x \pm \delta_0),$$

(35)

up to an irrelevant normalization. This result is shown in fig. 8. The Green’s functions decay rapidly away from $x \approx 0$. The $\psi$ field only exists in a narrow region of energy.

Singularities now appear off the real axis. The form of eqn. (35) implies a cyclic series of singularities, of which the ones nearest to the real axis are at

$$x = \delta_0 \pm \frac{i\pi}{\sqrt{3}}, \quad -\delta_0 \pm \frac{i\pi}{\sqrt{3}}.$$  

(36)

Hence the positions of the singularities, and the decay rate of the false vacuum [2], are determined primarily by $\omega f$ by eqn. (29). Note that the behaviour of the Green’s functions very close to the singularities are modified because the approximation $w^2 \ll 1$ fails.

Concerning the behaviour near $w = \pm 1$, roughly the same considerations as the weak-coupling case apply, and the singularities in the strong coupling limit are asymptotically of the form of simple poles. One difference is in that, as can be seen from eqn. (32), singularities $w = \pm 1$ occur for pure imaginary $y$. This implies that, since $y^2 = dw/dx$, the sign of $x$ that is, the relative sign between $G_{\pm}$, is reversed.

Our findings are consistent with the discussion of ref. [2], in that in a confining theory, no real singularities appear, and complex singularities represent the decay of the false vacuum into a vacuum with vacant negative energy states and occupied positive energy states. These states emerge because $\psi_+\psi_-$ pairs are bound together into supercritical bound states.

However, the appearance of the cyclic series of singularities requires explanation. This is due to the cyclic series of states that occur in the critical case, to be discussed in the following.

5.3 Critical case ($y_0 = y_0^{\text{crit}}$)

Last of all, let us discuss the critical case $y_0 = y_0^{\text{crit}}$.

The approximate solution is now given by $y = (1 - w^2)^{1/4}$. This leads to

$$w = \sin x, \quad z = \frac{\sin x - 1}{\sin x + 1} = -\tan^2 \left( \frac{x - \pi/2}{2} \right).$$  

(37)

This implies that, by eqn. (22),

$$G_+ = (\cos x)^{1/2} (\sin x + 1)^{-1},$$  

(38)

$$G_- = (\cos x)^{1/2} (\sin x - 1)^{-1},$$  

(39)

up to an irrelevant overall normalization. This behaviour is shown in fig. 9. Note that we have plotted Green’s functions squared here unlike in the previous two figures.

We see that the solution is cyclic. Near the singularities, e.g., $x = \pi/2 - \epsilon$, these functions behave as

$$G_- \rightarrow \frac{\sqrt{\epsilon}}{2}, \quad G_+ \rightarrow -2e^{-\epsilon/2}.$$  

(40)

This is clearly a rather exotic behaviour.
Green’s functions squared (critical $g_0$)

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig9.png}
\caption{Approximate Green’s functions squared in the critical case as a function of $x$. The normalization of axes is arbitrary.}
\end{figure}

Let us recall the Levinson theorem \cite{8,9} which states that the number of states in between energies $E_A$ and $E_B$ is given by the difference in the phase shifts $\delta$:

$$N = \frac{1}{\pi} \left[ \delta (E_B) - \delta (E_A) \right]. \quad (41)$$

In this sense, we can say that there are $-1/2 \psi_-$ states and $+3/2 \psi_+ \text{ states at } x = \pi/2$. Similarly, there are $-1/2 \psi_+ \text{ states and } +3/2 \psi_- \text{ states at } x = 3\pi/2$. This is if we adopt the convention of moving singularities above the real axis for $E > \mu_F$ and below the real axis for $E < \mu_F$. But this convention cannot be right when it produces negative number of states.

The negative-number states need to be occupied or vacated, and become occupied positive-energy states and vacant negative-energy states, and therefore their cuts are located on the other side of the real axis, as shown in fig. 10.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig10.png}
\caption{Moving away real-axis singularities, for $G_-$ in the critical case. $x_F$ corresponds to the Fermi energy.}
\end{figure}

Let us now say that we excite a $\psi_+$ state at $x = 0$. This $\psi_+$ can emit a critical $\psi_+ \psi_- \text{ pair without any cost of energy, and change into } \psi_+^\dagger \text{ at } x = 0$. Thus there is a mixing between $\psi_+$ and $\psi_+^\dagger$, and therefore the presence of a $\psi_-$ hole (i.e., occupied positive energy state) at the same energy as $\psi_+$ is explained.

Our convention has been that $\psi_+$ states are at a higher energy than $\psi_-$ states. In this case, the energy difference between the states in terms of $x$ is $\pi$. As a result of having both $\psi_+$ and $\psi_-$ states at the same energy, it follows that there are states both $\pi$ above and $\pi$ below $x = 0$. The same applies to states at $x = \pi$ and so, continuing ad infinitum, we come to the conclusion that there must indeed be a tower of states, though this seems strange and exotic.

6 Discussion

6.1 Nature of the critical pairs

From the structure of the self-energy diagram, and of the solutions, the complex singularities can be seen to correspond to $\psi_+ \psi_- \text{ (super-)critical bound states. These pairs are strange objects: they are not Cooper pairs which are formed out of electrons that are near the Fermi surface.}

At least mathematically, the critical solution to Gribov equations implies the following:

1. A real $\psi_+$ state is accompanied by a $\psi_+^\dagger$ hole state, because of the emergence of $\psi_+ \psi_-$ critical states.
2. $\psi_+ \psi_- \text{ states are formed by pairing a } \psi_+ \text{ state with the excitation } (\psi_+^\dagger)^\dagger \text{ of the accompanying } \psi_+^\dagger \text{ state.}

This is an unpleasant chicken-or-egg situation. A more intuitive statement is that pairs are formed by the interaction of, let us say, a real $\psi_+$ with a virtual $\psi_-$.

This should be classified as a spin polaron state \cite{5}. However, unlike the string polaron \cite{7} where confinement is due to interaction which grows at long distances, we have a dynamics which is described by the temporal decay of the false vacuum, i.e., soft confinement.

Physically, when fermionic spin is excited, it will decay by emitting a magnon, and will not behave like an itinerant spin.

Whether or not the condensation of supercritical pairs leads to superconductivity depends on whether these pairs are delocalized (and hence superconducting) or localized (and hence insulating). The equations suggest delocalized pairs, but presumably there are both possibilities, depending on the separation between charge carriers, for instance, which information is omitted in our framework. Since the supercritical pairs are not Cooper pairs, the superconductivity, even if it is realized, would be of a very different type to conventional ones.
6.2 Comparison with real spin systems

One important issue concerns whether our system, which is equipped with an $\omega \propto |k|$ spin-wave coupled to slow fermions, describes some real spin system.

On the outset, this appears unlikely, because a linear dispersion relation usually occurs in antiferromagnetic systems whereas the Yukawa coupling is appropriate to ferromagnetic systems.

One possibility concerns the case of doped antiferromagnetic insulators. In this case, the carriers will tend to move by hopping and so their velocity will typically be much slower than the spin-wave velocity. Furthermore, if the system has a sufficiently metallic character, one may expect intuitively that the Yukawa interaction becomes a reasonable description.

This suggests that somewhere in the metal–insulator transition phase diagram, there may exist a region in which the $\psi_+\psi_-$ condensate arises, corresponding to exotic insulator or superconductor.

This has not been observed in three spatial dimensions where, to our knowledge, there are no reported instances of antiferromagnetic metal to antiferromagnetic insulator transition to start with.

On the other hand, in two spatial dimensions, where our results are not directly applicable, high-$T_c$ superconductivity does occur in cuprate \cite{10,11} and iron pnictide \cite{12} systems. Further work is desirable.

The tower of states that occurs in our case at critical coupling apparently has no counterpart in QCD, even though it is suggestive of meson trajectories.

7 Conclusion

We have written down a coupled system of Gribov equations for slow fermions $\psi$ interacting with fast $\omega \propto |k|$ magnons $\phi$ in three spatial dimensions.

The solution exhibits qualitatively different behaviour depending on the strength of interaction. When the interaction is sufficiently strong, supercritical pairs of $\psi$ condense, much like the corresponding situation in Gribov’s light-quark confinement scenario \cite{11,12,13}. This corresponds to a new spin polaron state and may lead to exotic superconductivity.

A particularly exotic behaviour is the presence of an infinite tower of (real or complex) states. This behaviour arises because a state at a certain energy level is necessarily accompanied by a spin-flipped state either above or below it. The unusual occupation of states then requires the presence of this unusual tower of states.

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