BILINKAGE IN CODIMENSION 3 AND CANONICAL SURFACES OF DEGREE 18 IN $\mathbb{P}^5$

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Abstract. We study the behavior of the bilinkage process in codimension 3. In particular, we construct a smooth canonically embedded and linearly normal surface of general type of degree 18 in $\mathbb{P}^5$; this is probably the highest degree such surface may have. Next, we apply our construction to find a geometric description of Tonoli Calabi–Yau threefolds in $\mathbb{P}^6$.

1. Introduction

Let $S$ be a minimal surface of general type defined over the field of complex numbers. Then, by the inequality of Noether and Bogomolov-Miyaoka-Yau, we have

$$2\chi(O_S) - 6 \leq K_S^2 \leq 9\chi(O_S).$$

On the other hand, if we assume that the canonical system of $S$ gives a birational map then, by the Castelnuovo inequality, we deduce $3\chi(O_S) - 10 \leq K_S^2$. Note that we know from [Bom73] that $5K_S$ always gives a birational morphism for surfaces of general type. In this context, it is a natural problem (cf. [Ash91], [Cat97]) to construct surfaces of general type with birational canonical map in the region $3\chi(O_S) - 10 \leq K_S^2 \leq 9\chi(O_S)$. Many works are related to this problem [Ash91], [IVdV74], [Som84], however, the part with $\chi(O_S) \leq 7$ seems out of reach with those methods. The general surface of general type with $\chi(O_S) = 7$ and $h^1(O_S) = 0$ should admit a birational canonical map to $\mathbb{P}^5$. The image of such a map is a subcanonical surface of codimension 3 in $\mathbb{P}^5$.

On the other hand, it was proven in [Wal96] that submanifolds of codimension 3 in projective spaces $X \subset \mathbb{P}^N$ with $N - 3$ not divisible by 4 that are subcanonical are Pfaffian i.e. their ideal sheaf admits a Pfaffian resolution of the form

$$0 \to O_{\mathbb{P}^N}(-2s-t) \to E^*(-s-t) \xrightarrow{\varphi} E(-s) \xrightarrow{\psi} I_X \to 0$$

where $E$ is a vector bundle of odd rank and $s, t \in \mathbb{Z}$. The study of codimension 3 manifolds is reduced in this way to the study of the Hartshorne-Rao modules of the submanifolds. However, complicated algebraic problems appear when we want to classify such modules (see [KKb]). Catanese [Cat97] applied the Pfaffian construction in order to construct canonically embedded surfaces in $\mathbb{P}^5$ and found constructions of surfaces with $K_S^2 \leq 17$. Later, in his thesis [Ton04], Tonoli constructed Calabi–Yau threefolds in $\mathbb{P}^6$ but found only examples of degree $\leq 17$. Since the Pfaffian construction becomes more and more complicated when the degree increases, it is natural to address the problem of deciding whether there are any canonical surfaces of degree $\geq 18$ in $\mathbb{P}^5$. Our main result is the following:

**Theorem 1.1.** There exists a surface of general type with $K^2 = 18$, $p_g = 6$, $q = 0$ whose canonical map is an isomorphism onto its image.

We describe the construction of such surfaces in Section 3 concluding with Theorem 3.5. We expect, by [KKb], that this is the highest degree of such canonically embedded surface in $\mathbb{P}^5$. In fact, in Theorem 3.5, we find an explicit description of a 20-dimensional sub-family of the at least
36 dimensional family of degree 18 canonical surfaces in $\mathbb{P}^5$. Having our existence result, it is a natural problem (see [KKb]) to find a Pfaffian resolution for a general canonical surface of degree 18 in $\mathbb{P}^5$.

We begin by analyzing how the Pfaffian resolution changes after the bilinkage. Then the idea of the construction is to take a special central projection to $\mathbb{P}^5$ of the image $V_9$ of the third Veronese embedding of $\mathbb{P}^2$ in $\mathbb{P}^9$ and perform a bilinkage. More precisely, we find a special $\mathbb{P}^3 \subset \mathbb{P}^9$ such that the image $D_9 \subset \mathbb{P}^5$ of $V_9 \subset \mathbb{P}^9$ by the projection centered in this $\mathbb{P}^3$ is smooth and contained in the complete intersection of two cubics. Note here that it is contained in a single cubic for a generic projection (cf. [Kap, Rem. 5.5]). Then we perform a bilinkage of $D_9$ through the intersection of these two cubics obtaining a special smooth canonically embedded general type surface of degree 18 (cf. [KKb]). In Proposition 3.8 we show that all the known examples of canonical surfaces in $\mathbb{P}^5$ from [Cat97] can be obtained by such bilinkage construction. We believe that our construction can be applied in more general situations and describe a geometric law having a big impact to submanifolds of codimension 3.

Since the method of bilinkage worked better than the Pfaffian construction in the case of surfaces, we applied it in Section 4 to study Tonoli Calabi–Yau threefolds in $\mathbb{P}^6$. Those threefolds were constructed before in [Ton04] (cf. [ST02], [KKb]) by using the Pfaffian resolution (1.1). Our first result, Proposition 4.2, says that starting from del Pezzo threefolds of degree $d \leq 7$ in $\mathbb{P}^6$ we obtain families of Tonoli Calabi–Yau threefolds of degree $d + 9$ by performing the bilinkage construction through the intersection of two cubics (cf. [HR00]). In the remaining degree 17, there are three families of Calabi–Yau threefolds that we call after Tonoli of type $k = 8$, $k = 9$, and $k = 11$. The corresponding degree 8 del Pezzo threefold is the double Veronese embedding of $\mathbb{P}^3$ projected to $\mathbb{P}^6$. Analogously as before, we can find a special center of projection such that the image of the del Pezzo threefold of degree 8 is smooth and contained in a three dimensional space of cubics. Note that it is contained in no cubics for a general projection. So we can perform a bilinkage and it’s result is a natural degenerations of the degree 17 Tonoli family of type $k = 9$. Note, that the examples of type $k = 8$ an $k = 11$ cannot be constructed by bilinkage. This shows that the construction that we propose in [KKb, Thm. 1.2], by operations on vector bundle from the Pfaffian resolution is, in this context, a strict generalization of the one using bilinkages. We close Section 4 by the construction of a singular degree 18 threefold in $\mathbb{P}^6$ birational to a Calabi–Yau threefold.

Finally, in Section 5 we discuss relations between constructions by bilinkage and by unprojection; finding that the former are more general in our situation. This confirms the general Reid philosophy about the relation between these constructions. As a result, we analyze an example of non-Gorenstein unprojection that should be of independent interest.

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2. **Preliminary**

We shall apply the following construction to relate a given del Pezzo surface $F \subset \mathbb{P}^5$ (resp. del Pezzo threefold) with a surface of general type $X \subset \mathbb{P}^5$ (resp. Calabi–Yau threefold).

**Construction 2.1.** Write

$$\mathbb{P}^n \ni F \Rightarrow X' \sim X \subset \mathbb{P}^n,$$
where \( F \Rightarrow X' \) means that \( F \) and \( X' \) are bilinked and \( X' \leadsto X \) means that \( X' \) is a degeneration of \( X \) i.e. that there is a proper flat family over a disc such that \( X' \) is its special element and \( X \) a general one. Then we say that \( X \) is constructed from \( F \) by the bilinkage construction.

Before we study the possible applications of this construction, let us consider bilinkages of Pfaffian varieties in general.

2.1. Bilinkages of Pfaffians. Let us make some useful remarks on the construction of bilinkages between Pfaffian varieties by relating vector bundles defining them. More precisely, we aim at proving that under some assumptions if two bundles of odd rank \( E \) and \( F \) differ by a sum of line bundles then the Pfaffian varieties associated to general sections of their twisted wedge squares are in the same complete intersection biliaison class.

Let \( X \subset \mathbb{P}^N \) be a Pfaffian variety defined by a section \( \varphi \in H^0(\wedge^2 E \otimes \mathcal{O}_{\mathbb{P}^N}(t)) \) for some vector bundle \( E \) of rank \( 2r + 1 \) for some \( r \in \mathbb{N} \) and \( t \in \mathbb{Z} \). Denote \( s = c_1(E) + rt \).

The map \( \varphi \) in the Pfaffian resolution \((1.1)\) is identified with the section
\[
\varphi \in H^0(\mathbb{P}^N, \bigwedge^2 E \otimes \mathcal{O}_{\mathbb{P}^N}(t)).
\]
and \( \psi \) is the map
\[
E(-s) \to \mathcal{I}_X = \text{Im}(\psi) \subset \bigwedge^{2r+1} E \otimes \mathcal{O}_{\mathbb{P}^N}(rt - s) = \mathcal{O}_{\mathbb{P}^N}
\]
defined as the wedge product with the \( r \)-th divided power of \( \varphi \):
\[
\frac{1}{r!}(\varphi \wedge \varphi \wedge \cdots \wedge \varphi) \in H^0(\mathbb{P}^N, \bigwedge^{2r} E \otimes \mathcal{O}_{\mathbb{P}^N}(rt)).
\]

**Assumption 2.2.** Assume that \( E \) satisfies \( H^1(E^*(l)) = 0 \) for \( l \in \mathbb{Z} \).

Observe that Assumption 2.2 is satisfied when \( E \) is obtained as the kernel of a surjective map between decomposable bundles.

Under Assumption 2.2 on \( E \) we claim that every hypersurface of degree \( d \) containing \( X \) is defined as a Pfaffian hypersurface given by a section of the bundle \( \bigwedge^2(E \oplus \mathcal{O}_{\mathbb{P}^N}(d - s - t)) \otimes \mathcal{O}_{\mathbb{P}^N}(t) \) of even rank \( 2r + 2 \). Indeed, we can split the Pfaffian sequence into two short exact sequences:
\[
0 \to \mathcal{O}_{\mathbb{P}^N}(-2s - t) \to E^*(d - s - t) \to F \to 0,
\]
\[
0 \to F \to E(-s) \xrightarrow{\psi} \mathcal{I}_X \to 0,
\]
for some sheaf \( F \). Taking the cohomology of the second, we obtain an exact sequence
\[
H^0(E(d - s)) \to H^0(\mathcal{I}_X(d)) \to H^1(F(d)).
\]
On the other hand by the first exact sequence we have:
\[
H^1(E^*(d - s - t)) \to H^1(F(d)) \to H^2(\mathcal{O}_{\mathbb{P}^N}(d - 2s - t)),
\]
and it follows by the assumption on \( E \) and the fact that \( N \geq 3 \) that \( H^1(F(d)) = 0 \). We hence have a surjection
\[
H^0(E(d - s)) \to H^0(\mathcal{I}_X(d))
\]
induced by \( \psi \). Thus every hypersurface of degree \( d \) in the ideal of \( X \) is identified with a section
\[
s \wedge \varphi^{(r)} \in H^0(\mathbb{P}^N, \bigwedge^{2r+1} E \otimes \mathcal{O}_{\mathbb{P}^N}(d - s + rt)),
\]
for some section \( s \in H^0(E(d - s)) \). It is now enough to observe that
\[
\mathcal{O}_{\mathbb{P}^N}(d) = \bigwedge^{2r+1} E \otimes \mathcal{O}_{\mathbb{P}^N}(d - s + rt) = \bigwedge^{2r+2} (E \otimes \mathcal{O}_{\mathbb{P}^N}(d - s - t)) \otimes \mathcal{O}_{\mathbb{P}^N}((r + 1)t)
\]
and the section $s \land \varphi^{(r)}$ corresponds to the Pfaffian of the section

$$(\varphi, s) \in H^0(\wedge^2 (E \oplus \mathcal{O}_{\mathbb{P}^r}(d - s - t)) \otimes \mathcal{O}_{\mathbb{P}^r}(t)) = H^0(\wedge^2 E \otimes \mathcal{O}_{\mathbb{P}^r}(t)) \oplus H^0(E \otimes \mathcal{O}_{\mathbb{P}^r}(d - s))$$

under the above identification.

**Lemma 2.3.** Let $X$, $E$, $r$, $s$, $\varphi$ be as above and let $E$ satisfy Assumption 2.2. Assume that $X$ is contained in two hypersurfaces $H_{d_1}$ and $H_{d_2}$ of degree $d_1$ and $d_2$ respectively. Let $s_i$ be the section of $H^0(E(d_i - s))$ corresponding to $H_{d_i}$ for $i = 1, 2$. Then $H_{d_1} \cap H_{d_2}$ is a codimension 2 complete intersection if and only if the section $(\varphi, s_1, s_2, l)$ defines a rational function on $T$ of degree 3 Pfaffian variety for general $l \in H^0(\mathcal{O}_{\mathbb{P}^6}(d_1 + d_2 - 2s - t))$. Moreover, if $Y$ is a Pfaffian variety defined by a section $(\varphi, s_1, s_2, l)$ then $Y$ is bilinked to $X$ via the intersection of the two hypersurfaces $H_{d_1}$ and $H_{d_2}$.

**Proof.** Assume that $T = H_{d_1} \cap H_{d_2}$ is a codimension 2 complete intersection. Let us choose $l \in H^0(\mathcal{O}_{\mathbb{P}^6}(d_1 + d_2 - 2s - t))$ such that $T \cap \{1 = 0\}$ is of codimension 3. We can now easily check that at $\lambda = 0$ the degeneracy loci of these sections $(\varphi, s_1, s_2, l)$ degenerate to a subvariety of $X \cup (Y \cap \{l = 0\})$, hence the general element of the family has codimension 3 as expected. By simple base change we get that the map associated to $(\varphi, s_1, s_2, t)$ degenerates along a codimension 3 Pfaffian variety for general $\lambda$.

For the converse let us assume that a section $(\varphi, s_1, s_2, l)$ defines a codimension 3 variety. Let us consider two sections $g_1$ and $g_2$

$$g_1 = s_1 \land s_2 \land \varphi \land \cdots \land \varphi \in H^0(\bigwedge^{2r+3}_{r-1} (E \oplus \mathcal{O}_{\mathbb{P}^6}(d_1 - s - t) \oplus \mathcal{O}_{\mathbb{P}^6}(d_2 - s - t))((r + 1)t))$$

and

$$g_2 = l \otimes (t \land \varphi \land \cdots \land \varphi) \in H^0(\bigwedge^{2r+3}_{r} (E \oplus \mathcal{O}_{\mathbb{P}^6}(d_1 - s - t) \oplus \mathcal{O}_{\mathbb{P}^6}(d_2 - s - t))((r + 1)t)).$$

It is a simple exercise in linear algebra to check on fibers of this bundle that these two sections are proportional on $H_{d_1} \cap H_{d_2}$ i.e. when $s_i \land \varphi \land \cdots \land \varphi$ vanish for $i = 1, 2$. The ratio between these two sections defines a rational function $g$ on $H_{d_1} \cap H_{d_2}$. Observe that the function $g + 1$ vanishes only along the Pfaffian variety defined by $(\varphi, s_1, s_2, l)$ which is of codimension 3 by assumption. It follows that $H_{d_1} \cap H_{d_2}$ is of codimension 2.

For the proof of the last statement of the lemma, let $Y$ be a Pfaffian variety defined by the section of $(\varphi, s_1, s_2, l)$. In particular, $Y$ is of codimension 3 and $T = H_{d_1} \cap H_{d_2}$ is a complete intersection of two hypersurfaces. Now, $Y$ defines a generalized divisor in the sense of [Har94]. We claim that $Y$ is linearly equivalent as a generalized divisors to $X + H$, where $H$ is the restriction of the hyperplane section on $T$. Indeed, the rational function $g - 1$ is a rational function on $H_{d_1} \cap H_{d_2}$ which defines $Y - X - H$. By definition of biliaison, it follows that $Y$ is related to $X$ by a biliaison of height 1. Finally, by [Har94] Prop 4.4], this means that $Y$ is bilinked to $X$. \qed
3. Degree 9 del Pezzo surface and degree 18 canonically embedded surfaces

The analogy discussed in [KKb] suggests that one might try to construct a canonically embedded surface of general type of degree 18 in \( \mathbb{P}^5 \) if we find an appropriate description of a del Pezzo surface of degree 9 in \( \mathbb{P}^5 \). In this section we collect information on such del Pezzo surface and next present a construction of canonically embedded surfaces of general type of degree 18.

3.1. Del Pezzo surface of degree 9. Recall that a del Pezzo surface of degree 9 is just \( \mathbb{P}^2 \) and its anti-canonical embedding is the image \( V_9 \) of the triple Veronese embedding. We shall denote by \( D_9^A \subset \mathbb{P}^5 = : \mathbb{P}(W) \) the surface obtained as the image of the projection of the image of this embedding from a general 3-dimensional linear subspace \( \Lambda \subset \mathbb{P}^9 \). Let \( D = D_9^A \) for some general 3-dimensional linear subspace \( \Lambda \subset \mathbb{P}^9 \). Our aim is to understand the module \( M := \oplus_{k=0}^{\infty} H^1(I_D(k+2)) \), the shifted Hartshorne–Rao module of \( D \). From [KKb, lem 3.1], we know that the Hilbert function of the Hartshorne–Rao module of \( D \) has values \((0, 4, 7, 0, \ldots) \) starting from grade 0. We can thus write \( M = M_{-1} \oplus M_0 = H^1(I_D(1)) \oplus H^1(I_D(2)). \)

By working out a random example in Macaulay2, we prove that \( M \) is generated in degree \(-1\). Moreover, the minimal resolution of \( M \) is the following:

\[
0 \leftarrow M \leftarrow 4R(1) \leftarrow 17R \oplus 18R(-1) \oplus 4R(-2) \oplus 29R(-2) \oplus 80R(-3) \leftarrow 81R(-4) \leftarrow 38R(-5) \leftarrow 7R(-6) \leftarrow 0
\]

Trying to extend theorem [KKb] Thm 1.2 to the case of del Pezzo surfaces of degree 9, we should look for Pfaffian varieties associated to bundles in \( \text{Ext}^1(2\mathcal{O}_{\mathbb{P}^5}, S_{yz}M) \). However, for modules corresponding to general del Pezzo surfaces \( D_9^A \) it is hard to find any such bundle for which a Pfaffian variety would exist. In particular, even the bundle \( S_{yz}M \oplus 2\mathcal{O}_{\mathbb{P}^5} \) has no twisted self skew map defining a Pfaffian variety. For this reason, instead of trying to find a special element in \( \text{Ext}^1(2\mathcal{O}_{\mathbb{P}^5}, S_{yz}M) \) for \( M \) being the shifted Hartshorne–Rao module of a general del Pezzo surface of degree 9 in \( \mathbb{P}^5 \), we look for special smooth projected del Pezzo surface of degree 9 for which the Betti table of the resolution of its Hartshorne–Rao module has different shape.

**Proposition 3.1.** There exists a 3-dimensional linear subspace \( \Lambda \subset \mathbb{P}^9 \) such that the projected del Pezzo surface \( D_9^A \subset \mathbb{P}^5 \) is smooth and is contained in a complete intersection \( Y \) of two smooth cubic hypersurfaces and such that the singular locus of \( Y \) consists of 60 isolated singularities.

**Proof.** Let \( V_9 \) be the del Pezzo surface of degree 9 obtained as the image of the following Veronese embedding

\[
\mathbb{P}^2 \ni (x : y : z) \mapsto (x^3 : y^3 : z^3 : 3x^2y : 3xy^2 : 3xz^2 : 3yz^2 : 3y^2z : 6xyz) \in \mathbb{P}^9.
\]

Let us denote the corresponding coordinates in \( \mathbb{P}^9 \) by \((a_0, \ldots, a_9)\).

Recall that the ideal of the del Pezzo surface of degree 9 in \( \mathbb{P}^9 \) is defined by the \( 2 \times 2 \) minors of the Catalecticant matrix.

\[
A = \begin{bmatrix}
3a_0 & a_4 & a_6 & 2a_3 & 2a_5 & a_9 \\
a_3 & 3a_1 & a_8 & 2a_4 & a_9 & 2a_7 \\
a_5 & a_7 & 3a_2 & a_9 & 2a_6 & 2a_8
\end{bmatrix}.
\]

It is also known that the \( 3 \times 3 \) minors of \( A \) define the secant locus of the del Pezzo surface. Moreover, the secant locus is of codimension 4 in \( \mathbb{P}^9 \).

By the probabilistic method, one can easily construct in positive characteristic an example of a three-dimensional projective space \( \Lambda_0 \) disjoint from the secant locus such that the projection is contained in a pencil of cubics. An example of a matrix defining a projection from such a \( \Lambda_0 \) in
characteristic 17 is the following:

\[
N_0 = \begin{bmatrix}
0 & 0 & -1 & -2 & 0 & 0 \\
1 & 0 & -2 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & -1 \\
1 & 0 & 0 & -1 & 0 & 0 \\
1 & 1 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 \\
-1 & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 \\
\end{bmatrix}.
\]

We check using Macaulay2 that the projection of \( V_9 \) from \( \Lambda_0 \) is contained in a complete intersection \( Y \) of two smooth cubics \( C_1 \) and \( C_2 \) such that \( Y \) has 60 distinct singular points.

We shall show that \( \Lambda_0 \) lifts to characteristic 0 to some \( \Lambda \) such that the projection of \( V_9 \) from \( \Lambda \) is contained into two cubics and such that these two cubics specialize to \( C_1 \) and \( C_2 \). First observe that a projection from \( \Lambda \) corresponds to a linear map \( \mathbb{C}^{10} \to \mathbb{C}^6 \) hence a \( 10 \times 6 \) matrix \( N \) with complex entries. This projection composed with the Veronese embedding is a map

\[
\varphi : \mathbb{P}^2 \to \mathbb{P}^6
\]
defined by a base point free linear system of cubics, described by \( N \). If we further compose \( \varphi \) with the triple Veronese embedding \( \psi : \mathbb{P}^6 \to \mathbb{P}^{55} \), then the dimension of the space of cubics containing \( \Pi_\Lambda(V_9) \) is equal to the codimension of the span of the image \( \psi \circ \varphi(\mathbb{P}^2) \) in \( \mathbb{P}^{55} \). On the other hand, it is clear that \( \psi \circ \varphi \) factors through a 9-tuple Veronese embedding of \( \mathbb{P}^2 \to \mathbb{P}^{54} \) and a linear map \( L_N : \mathbb{P}^{54} \to \mathbb{P}^{55} \). It follows that the image of the projection \( \Pi_\Lambda(V_9) \) is always contained in a cubic hypersurface. Moreover, the projection is contained in a two dimensional space if and only if \( L_N \) has non-maximal rank. Observe that, following the above description, \( L_N \) can be written explicitly as a \( 55 \times 56 \) matrix depending on the entries of \( N \). If we now consider a 60 dimensional vector space \( V \) parametrizing matrices \( N \) then we obtain a \( 55 \times 56 \) matrix \( L \) with entries being cubic polynomials on \( V \). Denote by \( \Gamma \) the degeneracy locus of \( L \). It is a subvariety of \( V \) of codimension \( \leq 2 \). We can now proceed to explicit lifting of the constructed case over \( F_{17} \).

Consider any lift \( N_0' \) of \( N_0 \) to \( \mathbb{Z} \) and a random line \( l \) in \( V \) passing through \( N_0 \). More precisely, we choose \( l \) by choosing a parametrization \( \mathbb{C} \ni \lambda \mapsto N_0' + \lambda N_1 \in l \) with random \( N_1 \), for example

\[
\mathcal{N} := \begin{bmatrix}
0 & \lambda & -2\lambda - 1 & -2 & 0 & 0 \\
1 & 2\lambda & -2 & -\lambda & 0 & 0 \\
0 & 0 & -1 & 0 & 2\lambda & -1 \\
\lambda + 1 & 2\lambda & 0 & -1 & 0 & 0 \\
1 & 1 & 2 & -\lambda & 0 & 0 \\
0 & -\lambda & 0 & -2\lambda & 2 & 2\lambda \\
-2\lambda - 1 & 2\lambda & 1 & 1 & 1 & 1 \\
2\lambda + 1 & -\lambda & 0 & 0 & 0 & 0 \\
\lambda + 1 & -2\lambda & -\lambda & -2\lambda & 1 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 \\
\end{bmatrix}.
\]

We can now easily compute in Macaulay2 the Smith normal form \( L_{SNF} \) of the matrix \( L_\mathcal{N} \) restricted to the line \( l \) (i.e. over \( \mathbb{Q}[\lambda] \)). It is a matrix with entries being polynomials in \( \lambda \). More precisely, in our specific case, it has one entry being a polynomial \( p \) of degree 150 with integer entries whereas the remaining diagonal entries are 1. By definition, \( L_\mathcal{N} \) drops rank if and only \( L_{SNF} \) drops rank. The degeneracy locus of \( L_{SNF} \) is clearly defined by the vanishing of \( p \). We also
check that the reduction mod 17 of this polynomial is also of degree 150. It follows by the Valuative Criterion for Properness that there exists a number field $K$ and a prime $p$ in its ring of integers $O_K$ with $O_K/p \cong \mathbb{F}_{17}$ such that $N_0$ is the specialization of an $O_{K,p}$ valued point of $\Gamma$. In our case, this can be done explicitly. Indeed, the polynomial of degree 150 decomposes into two irreducible (over $\mathbb{Q}[\lambda]$) polynomials $P_1$ and $P_2$ with integer coefficients and of degrees 60 and 90 respectively. We check that $P_1$ reduced mod $17$ has a root in $\lambda = 0$. We can hence consider the number field $K = \mathbb{Z}[\lambda]/(P_1)$ and the prime ideal generated by $(17, \lambda)$ in $O_K$. It is clear that a projection $\Pi$ defined by the matrix $L$ with $\lambda$ being any root of the polynomial $P_1$ maps $V_9$ to a variety contained in two cubics.

Observe now that the two cubics containing $\Pi(V_9)$ are also computed with parameter $\lambda$ while computing the Smith normal form. More precisely, from the algorithm, we obtain matrices $S_1$, $S_2$ invertible over $\mathbb{Q}[\lambda]$ such that $S_1LNS_2 = L_{SNF}$. In particular, the appropriate columns of $S_2$ corresponding to vanishing columns of $L_{SNF}$ define cubics containing the image $\Pi(V_9)$. We check easily that the ideal generated by these two cubics specializes via our specialization map to the ideal generated by the two cubics computed over $\mathbb{F}_{17}$. Moreover, since the kernel of the projection over $\mathbb{F}_{17}$ is disjoint from the secant locus, this is also the case for the lifted projection. It follows that there exists a lift of the projection found over $\mathbb{F}_{17}$ such that the image of $V_9$ is smooth and contained in a complete intersection $Y$ of two smooth cubics and such that $Y$ has singularities not worse than 60 isolated singular points. 

**Remark 3.2.** Note that the computation of $c_2(I_{D_9}/I_{D_9}^2(3)) = 60$ gives us the expected number of singular points. We indeed check using Macaulay 2 that the two cubics containing the projection of $V_9$ constructed are smooth and intersect in a variety having 60 isolated singularities. However, the singularities of $Y$ are not nodes but isolated triple points with tangent cone being the cone over the projection of the cubic scroll to $\mathbb{P}^3$.

**Remark 3.3.** One can describe the set of matrices defining projections of $V_9$ that are contained in two cubics in terms of maximal minors of $L$ in $V$, hence, a variety of expected codimension 2. On the other hand, from the proof of Proposition 3.1 for a random choice of line $l$, the degeneracy locus restricted to this line had two components over $\mathbb{Q}[\lambda]$. One component of degree 60 is passing through our lift and the other of degree 90 corresponds to the locus of $\Lambda$ which intersect the secant variety of $V_9$ (it is a simple exercise to prove that for any such $\Lambda$ the projection is indeed contained in two cubics). This suggests that the locus of projections satisfying the assertion of Proposition 3.1 is an open subset of a hypersurface of degree 60 in $\mathbb{P}(V)$. Additional evidence follows from the Jacobi formula for the derivative of the determinant. Indeed, using this formula we can compute the tangent space to the variety $\Gamma$ in any constructed point even if writing down the equations of $\Gamma$ is out of reach for our computer. In our case, we get a codimension 1 tangent space. It is an interesting problem to find a geometric interpretation as in [HIR00] for the centers of projection contained in $\Gamma$.

Another interesting problem is the geometric description of the cubics containing the projected variety. For instance, for a generic choice of the center of projection $\Lambda$, the surface $D_9 \subset \mathbb{P}^3$ is contained in a unique cubic singular along a non degenerated curve of degree 6. From Lazarsfeld [Laz09] such cubic has to be determinantal.

**Remark 3.4.** When the center of projection $\Lambda$ intersects the secant variety of the surface $V_9$ in one point then $D_9^\Lambda$ is also contained in a pencil of cubics. However, in this case, we have $h^1(I_{D_9^\Lambda}) = 3$. Moreover, if $\Lambda$ meets the secant variety in three (resp. four) points then $D_9^\Lambda$ is contained in a four (resp. five) dimensional space of cubics. And if the intersection is a line then there is seven dimensional space of cubics in the ideal.
3.2. **Surfaces of general type of degree 18 in \( \mathbb{P}^5 \).** Let us consider a special \( \Lambda \) such that \( D_9^\Lambda \subset \mathbb{P}^5 \) (see Proposition 3.1) is contained in a complete intersection threefold of degree 9 with 60 isolated singularities as above.

**Theorem 3.5.** The surface \( D_9^\Lambda \) can be bilinked through the complete intersection of two cubics to a smooth surface of general type \( S_0 \) of degree 18.

**Proof.** Let us denote by \( H \) the class of the hyperplane on \( \mathbb{P}^5 \). Denote by \( F_k \) the residual surface to \( D_9 \) through a complete intersection of type \( (3,3,k) \) defined as the intersection of the base locus of the system of cubics containing \( D_9^\Lambda \) with a general hypersurface of degree \( k \) for some \( k \geq 4 \). From the exact sequence

\[
0 \to \mathcal{I}_{D_9 \cup F_k} \to \mathcal{I}_{F_k} \to \omega_{D_9}(-k) \to 0,
\]

using the fact that \( D_9 \cup F_k \) is aCM and \( \omega_{D_9} = \mathcal{O}_{D_9}(-1) \), we infer \( h^0(\mathcal{I}_{D_9 \cup F_k}(k+1)) + 1 = h^0(\mathcal{I}_{F_k}(k+1)) \). The surface \( F_k \) is thus linked through two cubics and a surface of degree \( k+1 \) to a surface \( S_0 \). Let \( Y \) be the complete intersection of our cubics. Using Macaulay 2 in characteristic 17, we proved that the singular locus of \( Y \) is a smooth zero dimensional scheme of degree 60. Moreover, in characteristic 17 this scheme is also the intersection scheme of \( D_9 \) and \( S_0 \). It follows that also in characteristic 0 the surfaces \( D_9 \) and \( S_0 \) intersect in isolated points in a transversal way i.e. their tangent spaces intersect transversely in each point of intersection. Since \( S_0 \) and \( D_9 \) are contained in a smooth cubic it follows that \( S_0 \) is smooth at each point of the intersection. To prove that \( S_0 \) is smooth everywhere we use Bertini theorem outside the singular locus of \( Y \).

Indeed, observe first that by [Har94, prop 4.4] the variety \( S_0 \) is an almost Cartier generalized divisor linearly equivalent to the almost Cartier generalized divisor \( D_9 + H \). It follows that on \( Y \setminus \text{Sing}(Y) \) the surface \( S_0 \) is a general element of the the system \( |D_9 + H| \) which is base point free on \( Y \setminus \text{Sing}(Y) \) because it is base point free outside \( D_9 \) and because \( S_0 \) does not meet \( D_9 \) in \( Y \setminus \text{Sing}(Y) \). Furthermore, by adjunction formula, since \( S_0 \) is a smooth divisor, Cartier in codimension 2, we have \( K_{S_0} = (K_Y + D + H)|_{S_0} = H_{S_0} \). This means that \( S_0 \) is a surface of general type embedded canonically in \( \mathbb{P}^5 \). It is clearly linearly normal since by basic properties of liaison its Hartshorne-Rao module is the Hartshorne-Rao module of the del Pezzo surface with gradation lifted by 1. Finally, the degree of \( S_0 \) is 18 by construction. \( \square \)

**Remark 3.6.** Observe that if \( E \) is the bundle defining the del Pezzo surface \( D_9 \) through the Pfaffian construction, then \( S_0 \) is defined by the bundle \( E \oplus 2 \mathcal{O}_{\mathbb{P}^5} \). Indeed, let us first point out that since we are dealing with almost Cartier divisors we can perform all computations on \( Y \setminus \text{Sing}(Y) \) and then extend the result to \( Y \). In particular, by Lemma 2.3 we have a 6 dimensional subsystem of the linear system \( |D_9 + H| \) consisting of varieties obtained as sections of \( E \oplus 2 \mathcal{O}_{\mathbb{P}^5} \). To prove that this gives the complete linear system \( |D_9 + H| \) we make a simple dimension count basing on the exact sequence:

\[
0 \to \mathcal{O}_Y(H) \to \mathcal{O}_Y(D + H) \to \mathcal{O}_D(D + H) \to 0,
\]

the fact that the singularities of \( Y \) are normal of codimension 3, and the equalities \( \mathcal{O}_D(D+H) = \mathcal{O}_D \), \( h^1(Y, \mathcal{O}_Y(H)) = 0 \). We obtain that \( h^0(Y, \mathcal{O}_Y(D + H)) = h^0(Y, \mathcal{O}_Y(H)) + h^0(D, \mathcal{O}_D) = 7 \), hence the system \( |D_9 + H| \) is of dimension 6.

**Remark 3.7.** Observe that the dimension of the family of surfaces obtained in Theorem 3.5 is at most 20. Indeed, since we know that the general choice of center of projection does not lead to a variety contained in the complete intersection of two cubics, it follows that the dimension of the space of \( \Lambda \)’s up to linear automorphisms is at most \( \dim G(4,10) - 1 - 8 = 15 \). We find also that for a given \( D_9 \subset \mathbb{P}^5 \) contained in the intersection of the cubics there is a 5-dimensional family \( (= h^0(D + H) - 1) \) of bilinked surfaces. This together gives a space of dimension at most 20. Moreover, by Remark 3.3 and the Betti table of a constructed surface, we expect this dimension to be exactly 20.
We know that the dimension of the Kuranishi space $K$ of a surface of general type $S$ is not smaller than $h^1(T_S) - h^2(T_S)$. On the other hand, we find by Noether’s formula $c_2(S) = 66$. So, by Riemann–Roch theorem applied to $T_S$, we infer $h^1(T_S) - h^2(T_S) = 36 + h^0(T_S) \geq 36$. Since $h^1(T_{P^5}|S) = 0$, we have also $H^0(N_{S|P^5}) \to H^1(T_S)$. Thus there should be an at least 36-dimensional family of canonically embedded surfaces of general type of degree 18 in $P^5$ with special element $S_0$. The general element of this family should have a simpler Hartshorne-Rao module. It is a natural problem to construct such general Hartshorne–Rao module.

### 3.3. Bilinkages of surfaces of general type

Let us now study how the construction \[\text{2.1}\] works in the case of canonical surfaces in $P^5$ constructed in \[\text{[Cat97]}\].

**Proposition 3.8.** Any smooth linearly normal canonical surface $S \subset P^5$ of degree $d_S \leq 17$ satisfying the maximal rank assumption is obtained by Construction \[\text{2.1}\] from a del Pezzo surface $D$ of degree $d_D = d_S - 9$ with bilinkage performed in a complete intersection of two cubics.

**Proof.** We compare the description, by \[\text{[Cat97]}\], of a canonical surface $S_d$ of degree $12 \leq d + 9 \leq 17$ in $P^5$ satisfying the maximal rank assumption, with the description of a projected del Pezzo surface $D_d$ of degree $d$ contained in \[\text{[KKb]}\]. Since all surfaces $S_d$ satisfying the assumptions above are deformation equivalent, it is enough to obtain one such surface in each degree using Construction \[\text{2.1}\] as in the assertion. We observe that for $d \leq 6$ the bundle $E_d$ constructed by Catanese is related to the corresponding bundle $E_{d'}$ from \[\text{[KKb]}\] by $E_d \oplus 2O_{P^5} = F_d$. Since for $d \leq 7$ the del Pezzo surface $D_d$ is contained in a complete intersection of two cubic hypersurfaces, by Lemma \[\text{2.3}\] the del Pezzo surface $D_d$ is bilinked to a Gorenstein surface of general type defined by the bundle $E_d \oplus 2O_{P^5}$ through the Pfaffian construction. Let us denote the general such surface by $\tilde{S}_d$. Since, for $d \leq 6$, we have $E_d \oplus 2O_{P^5} = F_d$, it follows that $\tilde{S}_d$ is a smooth canonical surface in $P^5$ of degree $d + 9$.

For $d = 7$ we have $E_7 \oplus 2O_{P^5}$ and $F_7$ both appear as kernels of some surjective maps $13O_{P^5} \to 2O_{P^5}$. Moreover $F_7$ is the kernel of a generic such map. It is now enough to take a one-parameter family parametrized by $\lambda \in \mathbb{C}$ of above maps such that for $\lambda \neq 0$ the kernel is isomorphic to $F_7$ whereas for $\lambda = 0$ the kernel is $E_7 \oplus 2O_{P^5}$. It follows that there is a bundle $E$ on $P^6 \times \{0\}$ whose restriction to $P^6 \times \{\lambda\}$ for $\lambda \neq 0$ is isomorphic to $E$. Moreover, since $h^0({\mathcal{L}}^2((E_7 \oplus 2O_{P^5})) = h^0({\mathcal{L}}^2 F_7))$, we infer that each section of $\mathcal{L}(E_7 \oplus 2O_{P^5})(1)$ is extendable to a section of $\mathcal{L}(E)(1)$. It follows by \[\text{[KKb]}\] Lem. \[\text{2.3}\] that $\tilde{S}_7$ is a degeneration of a family of canonical surfaces of degree 16 satisfying the maximal rank assumption, since all such surfaces are deformation equivalent the assertion follows in the case $d = 7$.

For $d = 8$ the situation is similar. More precisely, let $D_8$ be a del Pezzo surface of degree 8 (either of type $D_8^1$ or $D_8^2$) defined by some bundle $E_8$. Then $D$ is contained in a variety being the complete intersection of two cubics and $E_8 \oplus 2O_{P^5}$ is the kernel of some special surjective map $16O_{P^5} \to 3O_{P^5}$. The bundle $F_8$ being the kernel of a generic such map defines by \[\text{[Cat97]}\] a canonical surface of general type satisfying the maximal rank assumption. We then construct $E$ in the same way as above and conclude by \[\text{[KKb]}\] Lem. \[\text{2.3}\] and the equality $h^0({\mathcal{L}}^2((E_8 \oplus 2O_{P^5})(1)) = h^0({\mathcal{L}}^2 F_8(1)).$ \[\square\]

### 4. Del Pezzo threefolds and Tonoli Calabi–Yau threefolds

In order to obtain a Calabi–Yau threefold by construction \[\text{2.1}\] we consider a del Pezzo threefold $F$ embedded in $P^6$ by a subsystem of the half-anti-canonical class. The first examples of such threefolds are the del Pezzo threefolds of degree $d \leq 5$ embedded by a complete linear system. Moreover, for a threefold in $Y \subset P^n$ with $n \geq 7$ which is not contained in a hyperplane, there exists a smooth projection of $Y$ into $P^6$ if and only if $Y$ is defective. It follows that if we want to consider only smooth $F \subset P^6$ in the case $d \geq 6$ we are restricted to the consideration of del Pezzo threefolds defective in their half-anti-canonical embedding. There are only a few examples of such;
Lemma 4.1. A smooth del Pezzo threefold $F$ of degree $\deg(F) \geq 6$ in its half-anticanonical embedding is defective if and only if one of the following holds:

1. $\deg(F) = 6$ and $F$ is hyperplane section of the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$;
2. $\deg(F) = 7$ and $F$ is the projection of the image $T$ of the double Veronese embedding of $\mathbb{P}^3$ into $\mathbb{P}^9$ from a linear space intersecting the secant of $T$ in a point lying on $T$;
3. $\deg(F) = 8$ and $F$ is the projection of the image $T$ of the double Veronese embedding of $\mathbb{P}^3$ into $\mathbb{P}^9$ from a linear space (possibly empty) disjoint from the secant of $T$;

Proof. The theorem follows by the comparison of the classical classification of defective threefolds due to Scorza (see [CC01] for a modern approach) and the classification of del Pezzo threefolds due to Iskovskikh (see [IP99]).

We aim at proving the following.

Proposition 4.2. For $d \leq 7$, every smooth del Pezzo threefold $T_d$ of degree $d$ in $\mathbb{P}^6$ is related by construction [2.7] to a smooth Calabi–Yau threefold of degree $d + 9$ in $\mathbb{P}^6$.

Proof for $d \leq 7$. We have observed that the list above is the complete list of smooth del Pezzo threefolds in $\mathbb{P}^6$. Moreover, by [KKb, Rem. 3.6], del Pezzo threefolds are defined through the Pfaffian construction by the bundles $E_d'$ therein. Since for $d \leq 7$ the del Pezzo threefold $T_d$ is contained in a complete intersection of two cubic hypersurfaces, by Lemma [2.3] the del Pezzo threefold of degree $d$ is bilinked to a Gorenstein Calabi–Yau threefold (a Gorenstein threefold with $\omega_X = 0$ and $h^1(X, \mathcal{O}_X) = h^2(X, \mathcal{O}_X) = 0$) $\tilde{X}_d$ defined by the bundle $E_d' \oplus 2\mathcal{O}_{\mathbb{P}^6}$ through the Pfaffian construction. Now for $d \leq 6$ we have $E_d' \oplus 2\mathcal{O}_{\mathbb{P}^6} = F_d$ and hence $\tilde{X}_d$ is a smooth Tonoli Calabi–Yau threefold of degree $d + 9$. For $d \leq 7$ it is enough to observe that there is a bundle $\mathcal{E}$ on $\mathbb{P}^6 \times \mathbb{C}$ whose restriction to $\mathbb{P}^6 \times \{0\}$ is the bundle $E_d' \oplus 2\mathcal{O}_{\mathbb{P}^6}$ and the restriction to any fiber $\mathbb{P}^6 \times \{\lambda\}$ for $\lambda \neq 0$ is isomorphic to $F_7$. We also compute using Macaulay 2 that the dimension of the space of sections $\wedge^2(E_d' \oplus 2\mathcal{O}_{\mathbb{P}^6})(1)$ is equal to the dimension of the space of sections of $\wedge^2 F_7(1)$. We infer that $\wedge^2(E_d' \oplus 2\mathcal{O}_{\mathbb{P}^6})(1)$ is extendable to a section of $\wedge^2 \mathcal{E}(1)$. It follows by [KKb, Lem. 2.3], that $\tilde{X}_d$ is a degeneration of a family of Tonoli Calabi–Yau threefolds.

Let, now, $F$ be a del Pezzo threefold of degree 8 in $\mathbb{P}^6$ i.e. $F$ is the projection to $\mathbb{P}^6$ of the image $T$ of the double Veronese embedding of $\mathbb{P}^3$ into $\mathbb{P}^9$ from a plane disjoint from the secant of $T$. Using the methods from [HK08], we deduce that the ideal of $F \subset \mathbb{P}^6$ is generated by 45 quartics and does not contain any cubic.

In order to perform a bilinkage, we can find however a special center of projection also disjoint from the secant variety, such that the image $F$ is contained in a 3-dimensional system of cubics.

Proposition 4.3. There exists a center of projection such that $F \subset \mathbb{P}^6$ can be bilinked to a Gorenstein Calabi–Yau threefold (not necessarily normal) $X'$ of degree 17. Moreover, one can choose the bilinkage in such a way that $X'$ admits a smoothing to a family of Tonoli Calabi–Yau threefolds of degree 17 with $k = 9$.

Proof. Recall that the $2 \times 2$ minors of the following matrix

\[
A = \begin{bmatrix}
 a & x & y & z \\
 x & b & t & u \\
 y & t & c & v \\
 z & u & v & w \\
\end{bmatrix}
\]

define the second Veronese embedding of $\mathbb{P}^3$ in $\mathbb{P}^9(a, b, c, x, y, z, t, u, v, w)$. Let us consider a special $\Lambda = \mathbb{P}^2$ being the center of projection defined by the following equations:
Although it is hard to check by hand, it is straightforward to check using Macaulay 2 that the image of the projection $F \subset \mathbb{P}^6$ is contained in three independent cubics. Then the residual to $F$ of the intersection of these cubics is a threefold $G$ of degree 19 that is contained in a quartic that does not contain $F$ (cf. [Kap] Lemma 3.1). The residual to $G$ in the intersection of the cubics with the quartic containing $G$ is a threefold $X'$ of degree 17 (we say that $X'$ is bilinked with $F$ through two cubics with height 1).

As in Proposition 3.8 and Proposition 17.2 by Lemma 2.3 we infer that there is a Pfaffian variety $X$ associated to the vector bundle $E \oplus 2\mathcal{O}_{\mathbb{P}^6}$ which is a Gorenstein Calabi–Yau threefold of degree 17.

For the second part, note that, by the general properties of bilinkage, the threefold $X$ has the same Hartshorne–Rao module as $F$ but shifted by one. It follows by [KKb] proof of Thm 1.2] that this Hartshorne–Rao module is determined by some special $\mathbb{P}^{13} \subset \mathbb{P}^2 \times \mathbb{P}^6$ containing a linear space $\mathcal{P}$ spanned by the graph of some double Veronese embedding (composed with a linear embedding) of $\mathbb{P}^2$ to $\mathbb{P}^6$. Observe moreover that, by [KKb] Thm 1.1] the Hartshorne–Rao module of a Tonoli Calabi–Yau threefold of degree 17 with $k = 9$ corresponds to a $\mathbb{P}^{15} \subset (\mathbb{P}^2 \times \mathbb{P}^6)$ containing such a linear space $\mathcal{P}$. We now claim that it follows that the bundle $E \oplus 2\mathcal{O}_{\mathbb{P}^6}$ appears as a flat deformation of a family of bundles associated to such Calabi–Yau threefolds. Indeed, the bundle $E \oplus 2\mathcal{O}_{\mathbb{P}^6}$ is obtained as the kernel of a map $16\mathcal{O}_{\mathbb{P}^6} \rightarrow 3\mathcal{O}_{\mathbb{P}^6}(1)$ whose columns span the $\mathbb{P}^{13}$ whereas the chosen Tonoli Calabi–Yau threefolds appear as a Pfaffian variety associated to a bundle obtained as the kernel of a similar map but with columns spanning a $\mathbb{P}^{15}$ containing our $\mathbb{P}^{13}$. It is easy to see that by degenerating two columns of the map to zero (for example by multiplying them by the parameter $\lambda$) one obtain the desired flat deformation.

Observe now that there exists a subspace $V$ of dimension 9 of the space of sections $\bigwedge^2(E \oplus 2\mathcal{O})(1)$ consisting of sections which admit extensions to our deformation family. By [KKb] Lem. 2.3 the varieties given by these sections admit smoothings to Tonoli Calabi–Yau threefolds of degree 17 with $k = 9$.

**Remark 4.4.** Observe that, in the proof of Proposition 13.3, not every section of $\bigwedge^2(E \oplus 2\mathcal{O}_{\mathbb{P}^6})(1)$ extends to the deformation family. It follows that taking a general section of $\bigwedge^2(E \oplus 2\mathcal{O}_{\mathbb{P}^6})(1)$ in the proof of Proposition 13.3 one obtains a Gorenstein Calabi–Yau threefold representing a different component of the Hilbert scheme of Calabi–Yau threefolds consisting possibly of only singular threefolds.

5. Unprojections

Recall that *unprojection* is the inverse process to projection (see [PR04] for a general discussion). We discuss in this section the relations between the constructions by unprojection and by bilinkage in the context of submanifolds of codimension 3.

For the construction of Calabi–Yau threefolds using bilinkages with del Pezzo threefolds we are not restricted to starting with smooth Fano threefolds. A natural choice for singular del Pezzo threefolds are cones over del Pezzo surfaces. These are always contained in many cubics and a bilinkage can be performed. This construction is a way to directly relate the del Pezzo surface...
with the Calabi–Yau threefolds constructed. When the cone is Gorenstein it is related to so-called Kustin–Miller unprojections. This construction was studied in \cite{KK10, Kap11, Kap}. In particular, a straightforward generalization of \cite[Prop.4.1]{Kap11} (cf. \cite{BP12}) shows that the unprojection of a codimension 3 variety defined by Pfaffians of a decomposable bundle $E$ on $\mathbb{P}^n$ in a codimension 2 complete intersection can be seen as as some special Pfaffian variety associated to the bundle $E' \oplus \mathcal{O}_{\mathbb{P}^n+1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^n+1}(a_2)$ with $E'$ denoting the trivial extension of the decomposable bundle $E$ to $\mathbb{P}^{n+1}$ and $a_1$ and $a_2$ are appropriate numbers depending on the degrees of the generators of the complete intersection and the degrees in the decomposition of $E$.

In the case of a complete intersection of two cubics containing a projectively Gorenstein del Pezzo surface in $\mathbb{P}^5$, the result of the unprojection is a special Pfaffian variety associated to $F = E' \oplus 2\mathcal{O}_{\mathbb{P}^6}$. More precisely it is given as the degeneracy locus of a skew-symmetric map $\rho : F^* \to F \otimes \mathcal{O}_{\mathbb{P}^6}(1)$ corresponding to a section of the form

\[(\varphi, c_1, c_2, x_6) \in H^0(\bigwedge^2 F \otimes \mathcal{O}_{\mathbb{P}^6}(1)) = H^0(\bigwedge^2 E' \otimes \mathcal{O}_{\mathbb{P}^6}(1)) \oplus 2H^0(E'(1)) \oplus H^0(\mathcal{O}_{\mathbb{P}^6}(1)),\]

where $\varphi$ defines the cone over the del Pezzo surface, $c_1, c_2$ are sections which correspond via the Pfaffian resolution to two cubics containing the del Pezzo surface, and $x_6$ is the new variable of $\mathbb{P}^6$.

The following follows

**Corollary 5.1.** Every Tonoli Calabi–Yau threefold of degree $d \leq 14$ is a smoothing of a Gorenstein Calabi–Yau threefold obtained as the unprojection of a del Pezzo surface of degree $d \leq 5$ in a complete intersection of two cubics.

In the case $d \geq 6$ a standard Kustin–Miller unprojection cannot be performed because the del Pezzo surface is not projectively Gorenstein. This case is the first case in which the cone over the del Pezzo surface is not Gorenstein in its vertex and, as such, it cannot be written in terms of the Pfaffian construction applied to a vector bundle. We can however proceed somehow ignoring this fact and proposing a non-Gorenstein unprojection instead of the standard construction due to Kustin and Miller. More precisely by a non-Gorenstein unprojection we mean that having a variety $D \subset Y \subset \mathbb{P}^N$ with $D$ not projectively Gorenstein we construct a variety $X \subset \mathbb{P}^{N+1}$ singular in some point $p$ such that the projection of $X$ from $p$ is $Y$ and the exceptional locus is $D$.

**Proposition 5.2.** A Tonoli Calabi–Yau threefold of degree 15 can be obtained as a smoothing of a singular variety obtained as a non-Gorenstein unprojection of a del Pezzo surface of degree $d = 6$ in a complete intersection of two cubics.

**Proof.** Observe that, although the cone over the del Pezzo surface $D_6$ is not Gorenstein, we have its description in terms of some similar Pfaffian construction applied to the sheaf $E'$: the sheaf trivially extending $E$ to $\mathbb{P}^6$. In this case the special Pfaffian variety associated to the sheaf $F = E' \oplus 2\mathcal{O}_{\mathbb{P}^6}$ obtained by copying the unprojection procedure above in the context of sheaves is a non-Gorenstein variety $X$. We shall prove that it admits a smoothing to a Calabi–Yau threefold of degree 15. More precisely, we proceed in the following way. We start with a del Pezzo surface $D_6$. It is obtained as a Pfaffian variety associated to the bundle $E = \Omega^1_{\mathbb{P}^5}(1) \oplus 2\mathcal{O}_{\mathbb{P}^5}$ i.e. defined as the degeneracy locus of a general skew-symmetric map $\phi : E^*(-1) \to E$. We consider two cubics in the ideal of the del Pezzo surface. From the Pfaffian sequence they correspond to two sections of $E(1)$ giving a map $\psi : 2\mathcal{O}_{\mathbb{P}^5}(-1) \to E$. We can now extend the bundle $E$ to a sheaf $E'$ on $\mathbb{P}^6$ defined as the kernel of the map $8\mathcal{O}_{\mathbb{P}^6} \to \mathcal{O}_{\mathbb{P}^6}(1)$ given by the matrix $[x_0, \ldots, x_5, 0, 0]$. Then we consider the skew symmetric map $\rho : (E' \oplus 2\mathcal{O}_{\mathbb{P}^6})^*(-1) \to E' \oplus 2\mathcal{O}_{\mathbb{P}^6}$ defined by $\phi$, $\psi$ and multiplication by the new variable $x_6$. The degeneracy locus of $\rho$ is a codimension 3 variety $X'$ which is singular in the point $(x_0, \ldots, x_6) = (0, \ldots, 0, 1)$, the tangent cone being the cone over the projected del Pezzo surface $D_6$. The latter singularity is not Gorenstein. And hence our variety cannot be described as a Pfaffian variety associated to a vector bundle (we have its description as a kind of Pfaffian variety.
associated to the sheaf $E'$. It is however straightforward to check that the projection form the point $(0 \ldots 0, 1) \in \mathbb{P}^6$ maps $X'$ to the complete intersection of the two cubics containing the del Pezzo surface, and the exceptional locus is $D_6$.

Having the description of $X'$ in terms of Pfaffians (associated to a sheaf), we perform a similar reasoning as in [KKa, Prop. 7.2] and prove that $X'$ although it is not Gorenstein and not normal it can nonetheless be smoothed to a Tonoli Calabi–Yau threefold of degree 15. More precisely, following [Cat97] we can consider $\rho$ as a $10 \times 10$ skew symmetric matrix $A$ of linear forms satisfying the vanishing

$$[x_0 \ldots x_5, 0, 0, 0, 0] \cdot A$$

and its degeneracy locus is given by $8 \times 8$ Pfaffians of $A$. Observe that by the shape of unprojection and the assumption on $A$ we can write $A$ in the following form

$$
\begin{pmatrix}
B & a_0 & D^T \\
\vdots & \vdots & \vdots \\
-a_0 \ldots -a_5 & 0 & a_7 a_8 a_9 \\
D & -a_7 & K \\
-a_8 & -a_9 & K
\end{pmatrix},
$$

where the variable $x_6$ appears only in the matrix $K$ (more precisely in a $2 \times 2$ skew-symmetric submatrix of $K$). Since $[a_0 \ldots a_5]$ satisfies a Koszul relation there exists a skew symmetric $5 \times 5$ matrix $B'$ with complex entries such that

$$
\begin{pmatrix}
a_0 \\
\vdots \\
a_5
\end{pmatrix} = B' \cdot \begin{pmatrix}
x_0 \\
\vdots \\
x_5
\end{pmatrix}.
$$

Moreover since $a_i$ do not depend on $x_6$ there is clearly a unique $3 \times 6$ matrix $D'$ with complex entries such that

$$
\begin{pmatrix}
a_7 \\
a_8 \\
a_9
\end{pmatrix} = D' \cdot \begin{pmatrix}
x_0 \\
\vdots \\
x_5
\end{pmatrix}.
$$

Consider now the family of skew symmetric matrices:

$$A_\lambda = \begin{pmatrix}
B + \lambda x_6 B' & a_0 & D^T + (\lambda x_6 D')^T \\
\vdots & \vdots & \vdots \\
-a_0 \ldots -a_5 & 0 & a_7 a_8 a_9 \\
D + \lambda x_6 D' & -a_7 & K \\
-a_8 & -a_9 & K
\end{pmatrix},
$$

parametrized by $\lambda \in \mathbb{C}$. Observe that in this case $[x_0 \ldots x_5, \lambda x_6, 0, 0, 0] \cdot A_\lambda = 0$. Hence the matrices $A_\lambda$ induce sections of $\bigwedge^2 E_\lambda(1)$, with $E_\lambda$ isomorphic to $\Omega^1_{\mathbb{P}^6}(1) \oplus 3 \mathcal{O}_{\mathbb{P}^6}$ and the ideals generated by their $8 \times 8$ Pfaffians correspond to Pfaffian varieties associated to $E_\lambda$. To finish the proof it is enough to observe that the considered family is flat around $\lambda = 0$.  

**Remark 5.3.** Observe that we used the special form of the section defining the unprojected variety. In particular the construction could not be performed if we were unable to find a matrix $A$ with all of the $a_i$ for $i \in \{7, 8, 9\}$ independent of $x_6$. This suggests that the Hilbert scheme of Calabi–Yau
threefolds of degree 15 has at least two components: one giving the Tonoli family of degree 15; the other parametrizing a family of non Gorenstein threefolds probably birational to the degree 15 threefolds in $\mathbb{P}(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,2)$ constructed in [Kap]. In such case, the unprojected threefolds $X'$ above would correspond to some points in the intersection of these two components.

One can try to extend the construction from the case of $d = 6$ to higher degree del Pezzo surfaces. For instance, similarly as in the proof of Proposition 4.3 the Hartshorne–Rao modules of the cones over $D_1^3$ and $D_2^2$ are degenerations of Hartshorne–Rao modules associated to Tonoli Calabi–Yau threefolds of degree 17 and $k = 9, 11$ respectively. The sheafified first syzygy modules of their Hartshorne–Rao modules are not vector bundle but more general sheaves. However, one can still hope that, as in the case of degree $d = 6$, these non-Gorenstein threefolds admit smoothing to Calabi–Yau threefolds. Proceeding further, we compute the dimension of the space of sections of the twisted second wedge power corresponding to the unprojection and in each case we obtain a bigger space than the space of sections of second wedge power of the bundle defining the appropriate families of Tonoli Calabi–Yau threefolds. Thus we again (cf. Remarks 4.4, 5.3) obtain distinct components of the Hilbert scheme of Calabi–Yau threefolds of degree 17 in $\mathbb{P}^6$. The smoothing might possibly be performed only for very special unprojections. It is also not clear whether the varieties representing the general points of any of these components are smooth Calabi–Yau threefold.

5.1. **Calabi–Yau threefolds of degree 18 via unprojection.** Using the method of unprojection, we can also construct a non-Gorenstein projective threefold with one singular point with singularity locally isomorphic to the cone over a projected del Pezzo surface of degree 9. More precisely, let us start with a del Pezzo surface $D_9^A$ from Proposition 3.1. It is contained in a complete intersection $Y$ of two cubic hypersurfaces. Let $E$ be the vector bundle on $\mathbb{P}^5$ defining $D_9^A$. Consider the non-Gorenstein unprojection of $D_9^A$ in $Y$, i.e., a threefold $X$ defined as the degeneracy locus of a special skew-symmetric map between the sheaf $E' \oplus 2\mathcal{O}_{\mathbb{P}^6}$ and its twisted dual, as in the case of degree $d = 6$. Here, $E'$ is the sheaf on $\mathbb{P}^6$ obtained as the trivial extension of $E$. In this case, $X$ is a threefold with one singular point such that the projection form this point is $Y$ and the exceptional locus is $D_9^A$. Moreover, $X$ has degree 18 and is birational to a Calabi–Yau threefold. Unfortunately $X$ has no smoothing.

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