Going beyond modelling and inversion of first-order reflection data - Generalization of Kirchhoff reflectivity and consequences.

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Abstract

We emphasize in a novel way the connections and differences between Kirchhoff and Born modelling. We show how they lead to similar expressions, and derive a general expression for the conversion of a velocity model perturbation into a reflectivity through the “generalized reflectivity” concept. This allows the modelling of first-order effects within the Kirchhoff scheme that go beyond first-order reflections (for example first-order diffractions). Also, we clarify some aspects related to Kirchhoff modelling, in the context of possibly non-smooth propagating media (introducing travel-time branches or multipathing) and the linearity approximation on reflectors.

The scheme we develop offers opportunities within the framework of Kirchhoff inversion, i.e. for the interpretation of seismic-migrated images: generalized reflectivity gives a basis to interpret more information than that associated with first-order reflections (for instance the amplitudes of first-order diffractors). Also the scheme offers opportunities for full-waveform inversion (FWI) approaches that include a reflectivity, showing how to rigorously convert the reflectivity into a velocity perturbation.

1. Introduction.

The aim of seismic imaging \cite{1} is to characterize the geological structures of the subsurface from the analysis of seismic waves \cite{2,3}. A central component of seismic imaging is the scale separation, i.e. the separation of a smooth background velocity containing the long wavelength components of the true subsurface velocity model from the short wavelength components \cite{1,4}. This separation is justified regarding the physical behavior of these two components with respect to band limited data (typically 3 to 80 Hz) \cite{1,2,3}: the background velocity exhibits a strongly non-linear behavior with respect to the data, affecting the kinematics of the seismic events, while the short wavelength components have a much more linear behavior, affecting mostly the amplitudes of the events. As a consequence, recovery of the background velocity and of the short wavelength components is usually done sequentially \cite{2,5}. The first step is to compute the background velocity, typically by non-linear tomographic methods \cite{5}. The second step is to compute the short wavelength components through a linear inversion process, considering first-order scattered events (reflections and diffractions) \cite{4,5,10}, called seismic migration or imaging. There are two ways in seismic migration of linearly representing the short wavelength components of the velocity model:

- Using the Born approximation \cite{11,12,13,14}, based on a velocity model perturbation.
- Using the Kirchhoff approximation \cite{1,14}, where the short wavelength components are represented through a reflectivity distribution, i.e. a volumetric distribution of reflection coefficients. Bleistein’s groundbreaking work \cite{10,14} fundamentally establishes the reflectivity and shows how it can, at a later stage, be converted into material properties of the subsurface through an additional inversion process \cite{10,16}.

Full waveform inversion (FWI) \cite{4,17} is another approach for characterizing the subsurface velocity. Its ultimate aim is to invert band-limited seismic data non-linearly for the full range of wavelength components of the velocity model. In common FWI applications, a local optimization scheme is used (each iteration being related to a linearization, i.e. the Born approximation) \cite{4}, so that an initial velocity model that is sufficiently good kinematically is needed to avoid local minima. Then, the non-linearity is sufficiently weak and FWI can invert for long wavelength components. A reflectivity can be introduced within FWI to model first-order reflections but it must be converted into a velocity perturbation at each iteration for the velocity update \cite{1,18}.
Those two representations of the short wavelength components of the velocity model, i.e. model perturbation (Born) versus reflectivity (Kirchhoff), are commonly used. They are both based on a linearization, but each offers some specifics. For instance, Born approximation allows for modelling of first-order reflections on weak discontinuities and first-order diffractions, whereas Kirchhoff approximation allows for modelling of first-order reflections on strong discontinuities and postcritical reflections. The connections and differences between the two have been studied from different points of view, see e.g. [14, 15, 20, 21], but we believe some formal aspects still need to be detailed. In this paper, we propose to emphasize those connections and differences in a refreshing way.

First, we recall the chain of approximations leading to Kirchhoff and Born modelling equations and point out, from a fundamental point of view, the strengths and weaknesses of those schemes. We seize the opportunity to clarify some aspects related to Kirchhoff modelling, concerning possibly non-smooth propagating media (introducing travel-time branches or multipathing) and the linearity approximation on reflectors.

Then, we detail how Kirchhoff and Born modelling can lead to very similar expressions and how we can derive a general expression for the conversion from velocity model perturbation into reflectivity through a “generalized reflectivity” concept. This offers opportunities for FWI approaches based on reflectivity, showing how to rigorously convert reflectivity into velocity perturbation. This also allows us to model within a Kirchhoff scheme first-order effects that go beyond first-order reflections (like first-order diffractions).

Finally we show how this offers opportunities into the framework of Kirchhoff inversion, i.e. for the analysis of seismic images: generalized reflectivity gives a basis to interpret more information than that associated only with reflectors, in particular the amplitudes of imaged diffractors.

2. Kirchhoff modelling and inversion.

2.1. Exact modelling for one interface.

In the following, \( t \) represents time, \( \mathbf{r} = (x, y, z) \), position in the subsurface, \( \mathbf{r}_s \), position of an impulsive source of signature \( s(t) \) and \( \mathbf{r}_r \), the receiver positions. Our time-direction Fourier transform convention is \( A(\omega) = \int_{-\infty}^{+\infty} dt e^{-i\omega t} a(t) \). We use capital letters for the Fourier transform result.

Seismic waves are frequently modelled assuming a constant density acoustic approximation, i.e. using the scalar wave equation where the subsurface model is parameterized by the velocity. The subsurface wavefield \( p(\mathbf{r}_s, \mathbf{r}, t) \) generated by a point source at \( \mathbf{r}_s \) then obeys

\[
\begin{align*}
\text{For } \mathbf{r}_s \in \mathbb{R}^3, & \forall \mathbf{r} \in \mathbb{R}^3: \\
& \left[ \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta \right] p(\mathbf{r}, \mathbf{r}, t) = \delta(\mathbf{r} - \mathbf{r}_s)s(t) \\
p(\mathbf{r}_s, \mathbf{r}, t) &= 0 \quad \text{and} \quad \frac{\partial}{\partial t} p(\mathbf{r}_s, \mathbf{r}, t) = 0 \quad \text{for} \quad t \leq 0. \quad (1)
\end{align*}
\]

c is the velocity of the subsurface. \( p \) is assumed to satisfy proper boundary conditions (free surface and Sommerfeld radiation condition).

Now, we recall the chain of approximations leading to the Kirchhoff modelling equation [14]. We first consider an arbitrary (possibly “virtual”) interface surface in the subsurface that is infinitely spread (or “extended”). \( S_k \) denotes the positions of the interface surface. The Green function \( g(\mathbf{r}_s, \mathbf{r}, t) \) of the subsurface satisfies eq. \( (1) \) with \( s(t) = \delta(t) \). It is decomposed above \( S_k \) into:

- An “incident” field \( g_{\text{inc}} \) that is generated by the source and does not interact with \( S_k \) and the medium below \( S_k \); in other terms it satisfies eq. \( (1) \) above \( S_k \) with \( s(t) = \delta(t) \) and (or “absorbing”) boundary conditions on \( S_k \).
- A field \( g_{\text{ref}} = g - g_{\text{inc}} \) that represents what remains, i.e. events generated on \( S_k \) and below that “come back” into the medium above \( S_k \). They are described through a boundary condition on \( S_k \).

In the following “above \( S_k \)” means in the “incident” medium by slight abuse of language, see Fig. \( \square \).

Some manipulations using the representation theorem \( 2 \) \( 14 \) allow us to demonstrate

\[
G_{\text{ref}}(\mathbf{r}_s, \mathbf{r}, \omega) = \int_{S_k} d\mathbf{r}_k \left( \nabla G_{\text{ref}}(\mathbf{r}_s, \mathbf{r}, \omega) \cdot \n(\mathbf{r}) \right) G_{\text{inc}}(\mathbf{r}_s, \mathbf{r}, \omega) - G_{\text{ref}}(\mathbf{r}_s, \mathbf{r}, \omega) \nabla G_{\text{inc}}(\mathbf{r}_s, \mathbf{r}, \omega) \cdot \n(\mathbf{r}), \quad (2)
\]

where \( \n(\mathbf{r} \in S_k) \) denotes the unit vector normal to \( S_k \) that points “downward”, see Fig. \( \square \). Here we have made no approximation. \( G_{\text{inc}} \) is known from eq. \( (1) \) solved in the medium above \( S_k \) (with \( s(t) = \delta(t) \)) and \( G_{\text{ref}} \) is obtained by solving eq. \( (2) \). The latter is not an easy task because the integral depends on the \( G_{\text{ref}} \) values. The Kirchhoff approximation allows us to ease this task by finding approximate \( G_{\text{ref}} \) values for those that enter into the integral.
2.2. Kirchhoff modelling considering one reflector.

Reflectors are defined by discontinuities in the subsurface model $c$ that generate reflections. Reflections are defined within 0-order geometrical optics (0-g.o.) or high-frequency approximation \[22, 23\] by the events that satisfy the Snell-Descartes law, which imposes a particular direction to a reflected ray according to the direction of the corresponding incident ray \[14, 22, 23\]. (Contrariwise diffraction events do not satisfy the Snell-Descartes law: diffracted rays radiate in all directions.)

We consider a subsurface composed of sufficiently separated reflectors (in a sense that will be clarified later), with a sufficiently smooth velocity between reflectors from the 0-g.o. point of view. Let us choose for $S_k$ an interface that follows one of the reflectors. We use the 0-g.o. approximation \[22, 23\] for the Green function $G_{inc}$, that makes it possible to separate $G_{inc}$ into the contributions $G_{inc}^{(j)}$ related to each of the travel-time branches (or ray paths) that reach $S_k$ from $r_s$:

$$\forall r \in S_k : \quad G_{inc}(r_s, r, \omega) \approx \sum_{j \geq 1} G_{inc}^{(j)}(r_s, r, \omega)$$

where $j$ denotes the travel-time branch numbers. $N(r_s)$ denotes the number of direct travel-time branches (i.e. non-reflected, or refracted due to velocity inhomogeneities) and $j \leq N(r_s)$ refers to these arrivals. $j > N(r_s)$ refers to travel-time branches reflected (once or multiple times) within the medium above $S_k$ but not on $S_k$ (remembering that $G_{inc}$ denotes the field that does not interact with $S_k$ and the medium below). The wavefield on $S_k$ related to a travel-time branch is parameterized by an amplitude $A^{(j)}$ and travel-time $T^{(j)}(r_s, r \in S_k)$ defines the direction of the $j^{th}$ travel-time branch “ray” on the reflector. Fig. 1 gives an illustration, with $j > N(r_s)$ for the source travel-time branch.

![Figure 1: Source and receiver travel-time branches in a configuration where they are related by a reflection from above on a given reflector $k$.](image)

The essence of the Kirchhoff approximation is to assume the following relationship between the Green functions along the interface $S_k$:

$$\forall r \in S_k : \quad G_{ref}^{(i)}(r_s, r, \omega) \approx R^{(i)}(r_s, r)G_{inc}^{(i)}(r_s, r, \omega)$$

$$\nabla G_{ref}^{(i)}(r_s, r, \omega).n(r) \approx -R^{(i)}(r_s, r)\nabla G_{inc}^{(i)}(r_s, r, \omega).n(r),$$

Fig. 1 gives an illustration, with $j > N(r_s)$ for the source travel-time branch.

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1 When we use geometrical optics, the Green functions implicitly become “analytical signals”, obtained from the real signal by equating to 0 the negative frequencies, to be consistent with possibly complex reflection coefficients \[22, 23\]. In other words, all equations that imply the 0-g.o. approximation \[43\] are defined for $\omega \geq 0$ only. The real signal is recovered by “symmetrizing” the negative frequencies.
where \( R^{(i)} \) is a possibly complex function and \( G_{\text{ref}}^{(i)} \) represents the contribution to \( G_{\text{ref}} \) of an interaction of the \( i^{th} \) "source" travel-time branch with the reflector. The "incidence angle" \( \theta_{\text{inc}}^{(i)} \) is defined by the acute angle at position \( r \in S_k \) between \( \nabla_r T^{(i)} \) and the normal \( n \) to the reflector, the normal being well defined for smooth \( S_k \) only. The angle \( \theta_{\text{inc}}^{(i)} \) is then uniquely defined for each travel-time branch. An analytical expression for \( R^{(i)} \), called the reflection coefficient, is obtained within 0-g.o. in the case of a reflector

\[
R^{(i)}(r_s, r) = R(r, \theta_{\text{inc}}^{(i)}(r_s, r)) = \lim_{\epsilon \to 0} \frac{c_+ \left( r \right) \cos(\theta_{\text{inc}}^{(i)}(r_s, r)) - \sqrt{c_+^2(r) - \sin^2(\theta_{\text{inc}}^{(i)}(r_s, r))c_s^2(r)}}{c_+ \left( r \right) \cos(\theta_{\text{inc}}^{(i)}(r_s, r)) + \sqrt{c_+^2(r) - \sin^2(\theta_{\text{inc}}^{(i)}(r_s, r))c_s^2(r)}}
\]

\[
c_+(r) = \lim_{\epsilon \to 0^+} c(r + \epsilon n(r)), \quad c_-(r) = \lim_{\epsilon \to 0^+} c(r - \epsilon n(r)).
\]

A reflector is also called a "smooth interface"\[14\] because its surface must have sufficiently small curvature and be associated with a reflection coefficient that varies slowly enough along the interface. Note that using eqs. \[2\] and \[3\] as boundary conditions for \( G_{\text{ref}}^{(i)} \) implies that we consider single events reflected from above on \( S_k \) and that (for now) we neglect events generated below \( S_k \).

Inserting previous results in eq. \[2\], using the 0-g.o. approximation \[3\] for \( G_{\text{inc}}^{(i)} \) and keeping only the high-frequency leading terms gives \[14\]

\[
G_{\text{ref}}(r_s, r_r, \omega) = \sum_{i,j \geq 0} G_{\text{ref}}^{(i,j)}(r_s, r_r, \omega)
\]

\[
G_{\text{ref}}^{(i,j)}(r_s, r_r, \omega) = i \omega S(\omega) \int_{S_k} dr \frac{R(r, \theta_{\text{inc}}^{(j)}(r_s, r)) n(r).\nabla (T^{(i)}(r_s, r) + T^{(j)}(r_r, r))}{A^{(i)}(r_s, r).A^{(j)}(r_r, r)e^{-i\omega(T^{(i)}(r_s, r) + T^{(j)}(r_r, r))}}
\]

where \( G_{\text{ref}}^{(i,j)} \) contains one reflection event on \( S_k \). It represents the contribution of the \( i^{th} \) "source" travel-time branch, coupled with a single reflection from above on reflector \( k \) to the \( j^{th} \) "receiver" travel-time branch. Again, events generated beyond \( S_k \) are neglected for now.

Eq. \[6\] is not symmetric (reciprocal) under the exchange of \( r_s \) and \( r_r \) because of the angle \( \theta_{\text{inc}}^{(j)}(r_s, r) \) that depends on \( r_s \) and \( i \) (and implicitly contains knowledge of the geological dip at every position on the reflector), but does not depend on \( r_r \) and \( j \). This is conceptually annoying because Green functions should satisfy symmetry under the exchange of \( r_s \) and \( r_r \)\[14\]. To recover this symmetry we use the property that the phase of eq. \[6\] is stationary when the Snell-Descartes law for reflections is satisfied\[14\], i.e. for "specular" source and receiver ray pairs. Then, using the stationary phase approximation principle described for instance in Ref. \[14\], valid for sufficiently high frequencies, we can make the following replacements in eq. \[6\]\[14\]\[10\]:

\[
\forall r \in S_k : \quad \theta_{\text{inc}}^{(i)}(r_s, r) \Leftrightarrow \theta^{(i)}(r_s, r, r_r) \quad R(r, \theta_{\text{inc}}^{(j)}(r_s, r)) \Leftrightarrow R(r, \theta^{(j)}(r_s, r, r_r)) \quad n(r).\nabla (T^{(i)}(r_s, r) + T^{(j)}(r_r, r)) \Leftrightarrow |\nabla (T^{(i)}(r_s, r) + T^{(j)}(r_r, r))| = 2 \cos \left( \theta^{(i)}(r_s, r, r_r) \right) / c(r),
\]

where \( \theta^{(i)}(r_s, r, r_r) \) is half the angle at reflector positions between the \( i^{th} \) source travel-time branch ray and the \( j^{th} \) receiver travel-time branch ray (possibly non specular ray pairs), see Fig \[\text{II}\]. By inserting those results in eq. \[6\], factorizing the 0-g.o.-Green functions, and introducing the source signature to deal with wavefields, we obtain

\[
P_{\text{ref}}(r_s, r_r, \omega) = \sum_{i,j \geq 0} P_{\text{ref}}^{(i,j)}(r_s, r_r, \omega)
\]

\[
P_{\text{ref}}^{(i,j)}(r_s, r_r, \omega) = \int_{S_k} dr R(r, \theta^{(j)}(r_s, r, r_r)) \frac{2 \cos \left( \theta^{(i)}(r_s, r, r_r) \right)}{c(r)} L_{\text{inc}}^{(i,j)}(r_s, r_r, r, \omega)
\]

\[
L_{\text{inc}}^{(i,j)}(r_s, r_r, r, \omega) = i \omega S(\omega) G_{\text{inc}}^{(i,j)}(r_s, r, \omega) G_{\text{inc}}^{(j)}(r, r_r, \omega),
\]

where \( P_{\text{ref}} = S(\omega) G_{\text{ref}}(r_s, r_r, \omega) \) denotes the total wavefield reflected on \( S_k \) and measured at the earth’s surface. Each \( P_{\text{ref}}^{(i,j)}(r_s, r_r, \omega) = S(\omega) G_{\text{ref}}^{(i,j)}(r_s, r_r, \omega) \) is related to a single reflection event on \( S_k \) for the corresponding source and receiver travel-time branches reaching the reflector. This is the so-called Kirchhoff modelling equation for one reflector \( S_k \).
The approximations we performed share the same high-frequency approximation basis and are thus valid for sufficiently large frequencies. The range of validities of those approximations are discussed in Ref. [14]. Still within geometrical optics, several extensions of those approximations exist [22, 23]. Here we keep the simplest one because it does not affect the conclusions of this article.

2.3. Kirchhoff modelling considering many reflectors.

2.3.1. Linearity approximation on reflectors.

Until now we have considered events occurring on a single reflector in Kirchhoff modelling equation (8). Now suppose the subsurface reflectors are in a configuration where they are separable almost everywhere, i.e. a not too dense configuration in a sense that will be clarified later. Each reflector is identified by \(k \in \mathbb{N}^{*}\). The idea behind the linearity approximation on reflectors is to consider a Kirchhoff modelling equation like eq. (8) for each reflector and to sum them, in order to account for the contributions of all reflectors.

We add subscript \(k\) in eq. (8) to make explicit that it concerns reflector \(k\): obviously to \(P_{\text{ref},k}^{(ij)}\) and \(P_{\text{inc},k}^{(ij)}\), but also to \(\theta_{k}^{(ij)}\) and \(G_{\text{inc},k}^{(ij)}\) because they describe results of propagation in the medium above \(S_{k}\) excluding \(S_{k}\), thus different propagations result when different reflectors \(k\) are considered. We obtain

\[
\begin{align*}
P_{\text{ref}}(r_{s}, r_{r}, \omega) &= \sum_{k \geq 1} \sum_{i,j \geq 1} P_{\text{ref},k}^{(ij)}(r_{s}, r_{r}, \omega) \\
P_{\text{ref},k}^{(ij)}(r_{s}, r_{r}, \omega) &= \int_{S_{k}} dr \frac{R(\theta_{k}, r_{s}, r_{r})}{c(r)} \cos\frac{\theta_{k}^{(ij)}(r_{s}, r, r_{r})}{e(r)} L_{\text{inc},k}^{(ij)}(r_{s}, r_{r}, \omega) \\
L_{\text{inc},k}^{(ij)}(r_{s}, r_{r}, \omega) &= i \omega S(\omega) G_{\text{inc},k}^{(ij)}(r_{s}, r_{r}, \omega) G_{\text{inc},k}^{(ij)}(r_{r}, \omega).
\end{align*}
\]

Again, in \(P_{\text{ref},k}^{(ij)}\), only single reflections from above on reflector \(k\) are considered through the integral, even if multipathing (direct arrivals and multiple reflections above reflector \(k\)) is considered through the travel-time branches. Note that crossing reflectors can naturally be considered in eq. (8) including a \(k\)-dependency to the normal \(\mathbf{n}(r)\), leading to an explicitly \(k\)-dependent reflection coefficient through eq. (5).

We discuss why the contributions of each reflector can be described separately and then summed. The Born approximation (which will be detailed in §5) tells us that first-order scattering effects (such as first-order, or single, reflections and diffractions) can be modelled linearly regarding the wavefield if the velocity perturbation is not too strong. Applied to Kirchhoff modelling (based on the reflection coefficient and not on velocity perturbations), this linearity implies that each single reflection event recorded at the earth’s surface \(P_{\text{ref},k}^{(ij)}\) can be associated with one Kirchhoff modelling equation of the form (8) if the reflection coefficients are not too large. The total reflected wavefield is obtained by “summing” the \(P_{\text{ref},k}^{(ij)}\) (here over the reflectors and the travel-time branches). Although single reflections on reflector \(k\) are considered in each \(P_{\text{ref},k}^{(ij)}\), reflection events can still be “strong”.

The result obtained here, eq. (9), represents the traditional Kirchhoff modelling scheme with multiple travel-time branches and different Green functions for different reflectors. It makes it possible to model the reflected events part of the subsurface wavefield measured at the earth’s surface. We now describe the further approximations that lead to the traditional Kirchhoff modelling scheme.

2.3.2. Traditional linearity approximation on reflectors and use of smooth velocities.

In the \(P_{\text{ref},k}^{(ij)}\) of eq. (9) we allow possible multiple reflections on reflectors above \(S_{k}\) during propagations from source or to receivers (but, rigorously, not on \(S_{k}\) even if this restriction can be overcome) through the travel-time branches, and single reflections from above on \(S_{k}\) through the integral. This allows us to keep the problem linear in terms of the reflection coefficients. An extension to multiple reflections from above and below on \(S_{k}\) has been proposed, see e.g. [18]. Then the problem becomes non-linear in terms of the reflection coefficients.

On the other hand, the traditional linearity approximation on reflectors [14] proposes a further simplification: it considers only the \(P_{\text{ref},k}^{(ij)}\) contributions to \(P_{\text{ref}}\) related to direct (or refracted) source and receiver travel-time branches, i.e. \(i \in [1, N(r_{s})]\) and \(j \in [1, N(r_{r})]\) using the notation of (2.2). Consequently, single (first-order) reflections on \(S_{k}\) only are modelled and \(G_{\text{inc},k}^{(ij)}\) and \(\theta_{k}^{(ij)}\) can be considered as independent of reflectors \(k\) (the presence of reflectors above a subsurface position does not condition directly the number of direct waves reaching the subsurface position); we denote them by \(G_{\text{inc}}^{(ij)}\) and \(\theta^{(ij)}\).
We have

\[ P_{\text{ref}}(r_s, r_r, \omega) \approx \sum_{k \geq 1} \sum_{i=1}^{N(r_s)} \sum_{j=1}^{N(r_r)} P_{\text{ref},k}^{(ij)}(r_s, r_r, \omega) \]

\[ P_{\text{ref},k}^{(ij)}(r_s, r_r, \omega) = \int_{S_k} dr \delta g_k(r_s, r_r, \omega) \frac{2\cos(\theta^{(ij)}(r_s, r_r))}{c(r)} L_{\text{inc}}^{(ij)}(r_s, r_r, r, \omega) \]

\[ L_{\text{inc}}^{(ij)}(r_s, r_r, \omega) = i\omega S(\omega) \rho_{\text{inc}}^{(ij)}(r_s, r_r, \omega) \rho_{\text{inc}}^{(ij)}(r_s, r_r, r, \omega). \]  

(10)

\( G_{\text{inc}} \) should be computed with the true subsurface velocity \( c \) considering only a direct travel-time branch. This is not always easy as \( c \) can contain discontinuities. For instance within a 0-g.o. propagation this would imply resolving boundary conditions through each discontinuity in \( c \). Within a wave propagation this would imply “muting” all reflections or the use of one-way propagators \([1, 18]\). To avoid the need for this, it is common practice to introduce in the modelling a smooth velocity \( c_{\text{inc}} \) that best reproduces travel-times and amplitudes of a wavefield generated at the earth’s surface and measured at the reflector positions. A smooth velocity \( c_{\text{inc}} \) that meets as well as possible those criteria can be defined through tomography \([24, 25]\).

However, if a strong reflector (i.e. a large velocity contrast such as a salt dome) is present in the true subsurface, i.e. in \( c \), the use of a unique smooth velocity \( c_{\text{inc}} \) will not be able to reproduce good amplitudes at positions below the reflector. A solution might involve considering two different smooth velocities: \( c_{\text{inc}}^{\text{above}} \) for propagations related to events occurring above the large contrast and \( c_{\text{inc}}^{\text{below}} \) for propagations related to events occurring below the large contrast, i.e. for propagations from the surface through this large contrast. This would permit more freedom in the modelling to better reproduce the phase and amplitude below the reflector. This remains in the spirit of the general form of the linearity approximation on reflectors presented in \([2, 3, 11]\) where the Green functions \( G_{\text{inc,k}}^{\text{inc}} \) can be different for different reflectors.

### 2.3.3. Reflectivity distribution, Kirchhoff inversion and interpretation.

We convert the surface integral in eq. (10) into a volume integral to introduce the reflectivity \( \hat{R} \), i.e. a volumetric distribution of the reflectivity coefficients. We use the “singular function of the reflector’s surface”, i.e. the Dirac delta distribution \( \delta_{S_k}(r) \) that spikes on \( S_k \)

For any volume \( V' \) that contains \( S_k \)

\[ \int_{V'} \int_{S_k} \delta_{S_k}(r) \]

where \( A'(r \in V') \) is any well-behaved extension of function \( A \) defined only for \( r \in S_k \) in the whole volume \( V' \). If \( g_k(r) = 0 \) is an equation that defines the position of the surface \( S_k \), its singular function is defined by \([14]\)

\[ \delta_{S_k}(r) = |\nabla g_k(r)|[\delta g_k(r)]. \]  

(12)

We then obtain (choosing \( V' \) to be the whole space under the earth’s surface located at \( z = 0 \)) \([14]\)

\[ \int_{S_k} \int_{V'} \delta_{S_k}(r) \]

\[ \hat{R}(r, \theta^{(ij)}(r_s, r_r)) = \sum_{k \geq 1} \int_{V'} \int_{S_k} \delta_{S_k}(r) \]

\[ \frac{2\cos(\theta^{(ij)}(r_s, r_r))}{c_{\text{inc}}(r)} \delta_{S_k}(r). \]  

(13)

\( R \) is extended in the whole volume through eq. (13) where \( n \) is continuously extended in-between reflector positions (then \( R \) is different from 0 only along reflectors).

Eq. (13) represents the Kirchhoff volumetric modelling equation \([14]\). It is based on the reflectivity \( \hat{R} \), that represents a volumetric distribution that “points” on reflectors through \( \delta_{S_k} \), and contains information on the reflectivity coefficients through \( R \).

The reflectivity concept becomes interesting in the context of Kirchhoff inversion. Suppose we recorded seismic data \( P \) at the earth’s surface, pre-processed to retain only first-order reflections (especially multiple reflections having been filtered out). Suppose we also produced a smooth subsurface model \( c_{\text{inc}} \) that allows computing \( L_{\text{inc}}^{(ij)} \). One can then invert the linear equation (13) to recover the reflectivity \( \hat{R} \).
Let us consider only given travel-time branches, i.e. given \( i \) and \( j \) values (for instance the ones related to the shortest travel-times). Using eq. (7), valid for reflections, we return to \( \theta_{\text{inc}}^{(ij)} \leftrightarrow \phi^{(ij)} \) for the practical purpose of removing the \( r_s \) dependency of the reflectivity \( \hat{R}(r, \theta_{\text{inc}}^{(ij)}(r_s, r)) \approx \hat{R}(r, \phi^{(ij)}(r_s, r, r_s)) \) and allow inversions per full shot. One then obtains the following linear inversion for each shot (i.e. each \( r_s \))

\[
\forall r_s : \quad \hat{R}_{\text{inv}}(r_s, r) = \arg \min_{\hat{R}(r, r_s)} \int d\omega \int dr \hat{R}(r_s, r, \omega) - \int_{z \geq 0} dr \hat{R}(r_s, r) \delta F_{\text{inc}}^{(ij)}(r_s, r, r, r, \omega)^2. \quad (14)
\]

This is called least-squares Kirchhoff inversion by shots \([1, 4, 10]\); the result \( \hat{R}_{\text{inv}}(r_s, r) \) of the inversion obviously gives an estimate for the reflectivity \( \hat{R}(r, \theta_{\text{inc}}^{(ij)}(r_s, r)) \), that is band-limited (limited \( \omega \) range) and aperture-limited (limited \( r_s \) range) in practice. Of course enough frequency and receiver aperture ranges are needed so that the inversion gives an unambiguous result, depending amongst others on the number of samples that describe the reflectivity.

By making certain assumptions, amongst others one of real reflection coefficients, Bleistein \([10]\) demonstrated that \( 2 \)

\[
\hat{R}_{\text{inv}}(r_s, r) \approx \sum_{k \geq 1} R(r, \theta_{\text{inc}}^{(ij)}(r_s, r)) \left( \frac{2 \cos(\theta_{\text{inc}}^{(ij)}(r_s, r))}{c_{\text{inc}}(r)} \right)^2 \delta_{bl}(g_k(r))
\]

\[
\delta_{bl}(g_k(r)) = \frac{1}{\pi} \Re \int d\omega e^{i\omega g_k(r)} F(\omega),
\]

where \( F \) represents the residual (band-limited) wavelet present in the data (after pre-processing). It maps in the reflectivity through eq. (15). We see that the band-limited Dirac \( \delta_{bl} \) peaks where \( g_k(r) = 0 \), i.e. on the reflector positions \( S_k \). This is why the inverted reflectivity \( \hat{R}_{\text{inv}} \) is also called an image of the reflectors of the subsurface, or seismic image \([4, 10, 1]\). We can demonstrate that \( F \) maps in the image in the direction perpendicular to the reflectors, see Appendix A. Fig. 2 presents an example of a seismic image obtained from Sigsbee (band-limited and aperture limited) synthetic data. Reflectors are visible and may be picked.

Eq. (15) allows us to deduce (using \( \forall r \in S_k : g_k(r) = 0 \) \([10]\))

\[
\forall r \in S_k : \quad \hat{R}_{\text{inv}}(r_s, r) \approx \alpha R(r, \theta_{\text{inc}}^{(ij)}(r_s, r)) \left( \frac{2 \cos(\theta_{\text{inc}}^{(ij)}(r_s, r))}{c_{\text{inc}}(r)} \right)^2
\]

\[
\alpha = \frac{1}{\pi} \Re \int d\omega F(\omega).
\]

Suppose we could pick the amplitude variations along the amplitude peaks of continuous events in the image, i.e. along reflector positions \( S_k \) and compute the incident angles \( \theta_{\text{inc}} \) using rays \([22, 23]\) or wavefield decomposition techniques and picked reflector dips. Then, using the definition of the reflection coefficient, eq. (6), we can invert eq. (15) for \( c \) around reflector positions. This common method of interpretation of the seismic image is called “amplitude versus angle” (AVA) analysis \([16, 10]\).

Now we clarify what we previously meant by reflectors in a not-too-dense configuration or separable almost everywhere. From the Kirchhoff inversion point of view this means reflectors separated almost everywhere from each other by more than the source or receiver wavefield wavelengths at the dominant frequency.

2.4. Open questions.

We have recalled the approximations underlying Kirchhoff modelling and inversion. In particular we clarified some aspects related to Kirchhoff modelling, concerning possibly non-smooth propagating media introducing travel-time branches and concerning the linearity approximation on reflectors. Some questions still remain open:

- In the case of a very dense configuration of reflectors where it is no longer possible to separate each reflector almost everywhere, can a reflectivity still be defined?

---

2 A key point is that Bleistein showed that by virtue of stationary phase considerations we can locally consider \(|\nabla g_k(r)| \leftrightarrow 2 \cos(\theta_{\text{inc}}^{(ij)}(r_s, r))/c_{\text{inc}}(r)\). This implies a proper choice for \( g_k \) that has the dimension of a time. The reflectivity \( R \) then has the dimension of a time divided by a squared distance.
• Is it possible to model effects that go beyond first-order reflections (like first-order diffractions) within the Kirchhoff scheme, i.e. through a reflectivity?

• In the Kirchhoff inversion context, is it possible to interpret more information in the image than only that associated with reflectors (for instance the amplitudes of diffraectors)?

We explore these questions in the following sections.

3. Generalization of the Kirchhoff reflectivity.

3.1. Born modelling approximation.

We recall the steps leading to Born modelling approximation \[14\]. We consider eq. \(11\) and decompose the subsurface velocity \(c\) into

\[
\frac{1}{c^2(r)} = \frac{1}{c_0^2(r)} + \delta l(r),
\]

where \(c_0\) is called a “reference” medium velocity, and \(\delta l\) is the squared slowness related to a “perturbation” of the reference medium. We decompose the subsurface wavefield into

\[
P(r_s, r, \omega) = P_0(r_s, r, \omega) + P_{\delta l}(r_s, r, \omega),
\]

where the reference medium wavefield \(P_0\) satisfies

\[
\left[-\frac{\omega^2}{c_0^2(r)} - \Delta\right]p_0(r_s, r, t) = \delta(r - r_s)s(t)
\]

\[
p_0(r_s, r, t) = 0 \quad \text{and} \quad \frac{\partial}{\partial t} p_0(r_s, r, t) = 0 \quad \text{for} \quad t \leq 0,
\]

and \(P_{\delta l}\) is the wavefield related to the perturbation \(\delta l\). \(P_{\delta l}\) can be decomposed into a linear contribution and a non-linear contribution \[14\]

\[
P_{\delta l}(r_s, r_r, \omega) = P_L(r_s, r_r, \omega) + P_{NL}[P_{\delta l}](r_s, r_r, \omega),
\]

where \(P_L\) and \(P_{NL}\) represent the linear and non-linear contributions, respectively.
where (considering the earth’s surface at \(z = 0\))

\[
P_L(r_s, r_r, \omega) = -(i\omega)^2 S(\omega) \int_{z \geq 0} dr \, \delta(l) G_0(r_s, r, \omega) G_0(r, r_r, \omega) \tag{21}
\]

\[
P_{NL}[P_{NL}](r_s, r_r, \omega) = -(i\omega)^2 \int_{z \geq 0} dr \, \delta(l) P_{NL}(r_s, r, \omega) G_0(r, r_r, \omega),
\]

where \(g_0\) denotes the causal Green function in the reference medium, i.e. satisfying eq. (19) with \(s(t) = \delta(t)\). Until now eqs. (20) and (21) do not involve any approximation. Any reference medium can be chosen, in particular ones with non-smooth \(c_0\), thus also any “strength” for the perturbation \(\delta l\).

The Born modelling approximation deals only with the linear contribution \(P_L\) for the modelling of the wavefield perturbation

\[
P_{NL}(r_s, r, \omega) \approx P_L(r_s, r, \omega),
\]

which represents a good approximation if

\[
[P_L(r_s, r, \omega)] \gg [P_{NL}[P_{NL}](r_s, r, \omega)] \iff \int_{z \geq 0} dr \frac{1}{c_0^2(r)} \gg \left| \int_{z \geq 0} dr \, \delta(l) \right| \text{ and } \frac{1}{c_0^2(r)} \gg |\delta(l)|. \tag{22}
\]

This implies that \(c_0\) remains close to \(c\) on average, i.e. that \(c_0\) reproduces the travel-times of first-order scattered events.

3.2. Reformulation of Born modelling and generalized reflectivity.

We reformulate the Born modelling equation in a way that allows a direct comparison with the Kirchhoff modelling equation (13). This comparison is a fundamental result of the article. Other demonstrations that share a similar spirit can be found in Refs. [19, 20]. Here we propose a different demonstration that involves travel-time branches, and provide more detail on properties and consequences.

Firstly, to achieve our goal, we must constrain the perturbation \(\delta l\) to describe at least all reflectors of the true subsurface model \(c\); more generally, we constrain it to describe all discontinuities of the subsurface. As a consequence \(c_0\) will be smooth. Secondly, as the Kirchhoff approximation involves 0-g.o. approximation as discussed in (2.2) we introduce 0-g.o. for the propagation of \(G_0\). As \(c_0\) is smooth, we consider only direct (refracted) travel-time branches

\[
G_0(r_s, r, \omega) \approx \sum_{j=1}^{N(r_s)} G_0^{(j)}(r_s, r, \omega)
\]

\[
G_0^{(j)}(r_s, r, \omega) = A^{(j)}(r_s, r) e^{-i\omega T^{(j)}(r_s, r)}. \tag{23}
\]

We use in this section similar notations as in (2.2).

For other choices of \(c_0\) and \(\delta l\) where \(c_0\) is non-smooth, travel-time branches with reflections would have to be considered in \(G_0\) in the spirit of eq. (4). For instance if \(\delta l\) describes only one reflector \((k)\) present in \(c\), \(c_0\) would contain the discontinuities related to all the other reflectors. Then, considering eq. (3) within the same steps as the following demonstration would lead to a result comparable to eq. (9) but where single reflections on reflector \(k\) from above and below would be allowed. We do not detail this branch here. In the following we wish to be comparable to eq. (13) and thus consider a single travel-time branch related to a smooth \(c_0\).

We naturally constrain \(c_0\) so that it reproduces the travel-times of the first-order events, or in other words, that it minimizes the travel-time corrections present in \(P_{NL}\). A velocity that meets this criterion as much as possible can be defined through tomography (24, 25). There is thus a close link between \(c_0\) and the smooth velocity \(c_{inc}\) introduced in Kirchhoff modelling in (2.3.2) and (2.3.3). So, in the following we consider

\[
c_0(r) \leftrightarrow c_{inc}(r) \quad \text{and} \quad G_0^{(j)}(r_s, r, \omega) \leftrightarrow G_{inc}^{(j)}(r_s, r, \omega). \tag{24}
\]

We next denote by \(r_{sr}(r) = (x_{sr}(r), y_{sr}(r), z_{sr}(r))\) any set of curvilinear coordinates obtained by transformation of the Cartesian coordinates \(r = (x, y, z)\); the superscript “sr” denotes that the curvilinear coordinates can be different for different \(r_s\) and \(r_r\), positions. The transformation must be well-defined, i.e. its Jacobian must be non-null at every position \(r\). To that aim, the curvilinear abscissa must not cross. We choose a transformation \(r_{sr}^{(j)}(r) = (x, y, z_{sr}^{(j)}(r))\), where \(z_{sr}^{(j)}(r)\) is a curvilinear coordinate in the average direction of the a direct ray that links \(r_s\) to \(r\), and of a direct ray that links \(r_r\) to \(r\), see Fig 3.
The unit vector in the direction of the curvilinear abscissa $z_{sr}^{(i)}(r)$ is (using standard rules of 0.g.o., see Fig. 3)

$$e_{sr}^{(i)}(r) = \frac{\nabla (T^{(i)}(r_s, r) + T^{(j)}(r_r, r))}{|\nabla (T^{(i)}(r_s, r) + T^{(j)}(r_r, r))|} = \frac{\partial}{\partial z_{sr}^{(i)}(r)} e_{sr}^{(i)}(r).$$

$$\frac{\partial}{\partial z_{sr}^{(i)}(r)} (T^{(i)}(r_s, r) + T^{(j)}(r_r, r)) = |\nabla (T^{(i)}(r_s, r) + T^{(j)}(r_r, r))| = \frac{2 \cos (\theta^{(i)}(r_s, r, r_r))}{c_{inc}(r)}. \quad (25)$$

The Jacobian of the transformation is invertible because we have a unique incidence angle $\theta^{(i)}$ at each position. Using eq. (23), we can show that for sufficiently large frequencies (i.e. for the high-frequency leading term) we have

$$\forall r \text{ such that } \theta^{(i)}(r_s, r, r_r) \neq \pi/2 :$$

$$G_{inc}^{(i)}(r_s, r, r_r, \omega)G_{inc}^{(j)}(r, r_r, \omega) \approx -\frac{1}{i \omega \theta_{sr}^{(i)}(r)} \left\{ \frac{c_{inc}(r)}{2 \cos (\theta^{(i)}(r_s, r, r_r))} G_{inc}^{(i)}(r_s, r, \omega)G_{inc}^{(j)}(r, r_r, \omega) \right\}. \quad (26)$$

The area where $\theta^{(i)}(r_s, r, r_r) = \pi/2$ corresponds to the area where one ray can be directly traced between $r_s$ and $r_r$, i.e. to the “diving waves” area. To avoid singularities, we constrain the Green functions $G_{inc}^{(i)}$ to contain no diving-wave travel-time branches. In areas where only diving waves occur in the subsurface, the Green functions $G_{inc}^{(i)}$ are thus null. This does not reduce the generality of our considerations as the diving waves are described by the $P_0$ term in eq. (15) while the $P_L$ term in eq. (26) describes first-order scattered events (first-order reflections and diffractions). We also constrain without loss of generality $\delta l$ to be null at the earth’s surface for practical purposes. We then can consider

$$\delta l(r) \frac{\partial}{\partial z_{sr}^{(i)}(r)} \left\{ \frac{c_{inc}(r)}{2 \cos (\theta^{(i)}(r_s, r, r_r))} G_{inc}^{(i)}(r_s, r, \omega)G_{inc}^{(j)}(r, r_r, \omega) \right\} \text{ to be well defined everywhere.}$$

Inserting eq. (26) in eq. (24) and using integration by parts gives

$$P_L(r_s, r_r, \omega) \approx i \omega S(\omega) \sum_{i=1}^{N(r_s)} \sum_{j=1}^{N(r_r)} \int_{z \geq 0} \delta l(r) \frac{\partial}{\partial z_{sr}^{(i)}(r)} \left\{ \frac{c_{inc}(r)}{2 \cos (\theta^{(i)}(r_s, r, r_r))} G_{inc}^{(i)}(r_s, r, \omega)G_{inc}^{(j)}(r, r_r, \omega) \right\}$$

$$\approx i \omega S(\omega) \sum_{i=1}^{N(r_s)} \sum_{j=1}^{N(r_r)} \int_{z \geq 0} \delta l(r) \frac{\partial}{\partial z_{sr}^{(i)}(r)} \left\{ \frac{c_{inc}(r)}{2 \cos (\theta^{(i)}(r_s, r, r_r))} G_{inc}^{(i)}(r_s, r, \omega)G_{inc}^{(j)}(r, r_r, \omega) \right\}$$

$$+ i \omega S(\omega) \sum_{i=1}^{N(r_s)} \sum_{j=1}^{N(r_r)} \int_{z \geq 0} \delta l(r) \frac{\partial}{\partial z_{sr}^{(i)}(r)} \left\{ \frac{c_{inc}(r)}{2 \cos (\theta^{(i)}(r_s, r, r_r))} G_{inc}^{(i)}(r_s, r, \omega)G_{inc}^{(j)}(r, r_r, \omega) \right\}.$$
the Kirchhoff modelling equation. In our case where \( \delta l \) contains discontinuities, \( \frac{\partial \delta l}{\partial \gamma_{sr}}(r) \) in eq. (27) is obviously defined from the distributional derivative point of view.

We rewrite eq. (27) as

\[
P_L(r_s, r_r, \omega) \approx \sum_{i=1}^{N(r_s)} \sum_{j=1}^{N(r_r)} \int_{z \geq 0} d\mathbf{r} \left\{ -\frac{\partial \delta l(r)}{\partial z_{sr}}(r) \frac{c_{inc}(r)}{2 \cos \left( \theta^{(ij)}(r_s, r_r) \right)} \right\} G^{(i)}_{inc}(r_s, r_r, \omega) G^{(j)}_{inc}(r_r, r_r, \omega)
\]

Details leading to the penultimate relationship are given in this footnote 3. This starts to look like the Kirchhoff modelling equation. In our case where \( \delta l \) contains discontinuities, \( \frac{\partial \delta l}{\partial \gamma_{sr}}(r) \) in eq. (27) is obviously defined from the distributional derivative point of view.

We rewrite eq. (27) as

\[
\begin{align*}
P_L(r_s, r_r, \omega) & \approx \sum_{i=1}^{N(r_s)} \sum_{j=1}^{N(r_r)} \int_{z \geq 0} d\mathbf{r} \hat{R}_{gen}(r, \theta^{(ij)}(r_s, r_r)) L^{(ij)}_0(r_s, r_r, \omega) \\
& - \frac{\frac{\partial \delta l}{\partial \gamma_{sr}}(r)}{c_{inc}(r)} \frac{c_{inc}(r)}{2 \cos \left( \theta^{(ij)}(r_s, r_r) \right)} e_{sr}^{(ij)}(r, \nabla \delta c(r)).
\end{align*}
\]

This main result consists of a reformulation of the Born approximation using 0-g.o. [19, 20]. It looks like the Kirchhoff modelling equation (13), where \( \hat{R}_{gen} \) is the counterpart of the Kirchhoff reflectivity distribution. We call it generalized reflectivity.

We now have to understand precisely the differences between \( \hat{R}_{gen} \) and Kirchhoff reflectivity \( \hat{R} \). Note that the Kirchhoff approximation describes first-order reflections, whereas the Born approximation may contain more: it can describe any first-order events present in \( P_L \) like first-order diffractions. We call \( \hat{R}_{gen} \) “generalized reflectivity” even if it can describe more than reflections.

3.3. Link between Born generalized reflectivity and Kirchhoff reflectivity.

Kirchhoff modelling considers only reflectors in the subsurface. We have first verify if Born generalized reflectivity \( \hat{R}_{gen} \), eq. (28), reduces to the Kirchhoff reflectivity \( \hat{R} \), eq. (13), when the perturbation contains only reflectors.

We introduce the velocity perturbation \( \delta c \) defined by

\[
c = c_{inc} + \delta c.
\]

We have (using eqs. (17) and (24)) \( \delta l = 1/(c_{inc} + \delta c)^2 - 1/c_{inc}^2 \). Because \( \delta c \) must be sufficiently small, we can perform a 1\textsuperscript{st}-order Taylor development and obtain

\[
\delta l(r) \approx -\frac{2 \delta c(r)}{c_{inc}^3(r)} \nabla \delta c(r).
\]

As \( c_{inc} \) is smooth and \( \delta c \) contains all the rapid velocity variations of the subsurface, we can consider

\[
\nabla \delta l(r) \approx -\frac{2 \delta c(r)}{c_{inc}^3(r)} \nabla \delta c(r).
\]

Inserting this result in eq. (28) we obtain \( \hat{R}_{gen} \) as a function of the velocity perturbation

\[
(c_{inc} \text{ smooth}) \quad \hat{R}_{gen}(r, \theta^{(ij)}(r_s, r_r)) = \frac{1}{\cos \left( \theta^{(ij)}(r_s, r_r) \right)} \frac{e_{sr}^{(ij)}(r)}{c_{inc}(r)} \nabla \delta c(r).
\]
Let us study what happens to the Born generalized reflectivity for a subsurface, i.e. a perturbation \( \delta c \), composed only of reflectors. This can be modelled by

\[
\delta c(r) = \sum_{k \geq 1} a_k(r) \left[ H(g_k(r)) - 0.5 \right],
\]

where \( H \) is the Heaviside function, and \( a_k(r) \) a smooth (continuously differentiable) function with compact support that has the dimension of velocity, and simply “adjusts” the Heaviside jumps. When \( r \) is on reflector \( k \), \( a_k(r) \) equals the velocity jump \( \Delta c(r) \) across the reflector (we use notation of eq. \( \text{(33)} \) where \( n \) is the normal to the reflectors continuously extended between reflectors)

\[
\forall r \in S_k : \quad a_k(r) = \Delta c(r) = c_+(r) - c_-(r).
\]

Evaluating the reflection coefficient, eq. \( \text{(33)} \), when \( \Delta c \) is small we obtain the linearized reflection coefficient \( R_{\text{lin}} \) (to 1st-order in \( \delta c \))

\[
R(r, \theta^{(i)}(r_s, r, r_r)) \propto \frac{\Delta c(r)}{2c_{\text{inc}}(r) \cos^2 \left( \theta^{(i)}(r, r_s, r_r) \right)}.
\]

We wish to compute the generalized reflectivity \( \hat{R}_{\text{gen}} \) corresponding to the velocity perturbation \( \delta c \). We have

\[
e^{\text{(i)}}_{sr}(r) \nabla \delta c(r) - \sum_{k \geq 1} a^{\text{(i)}}_{sr}(r) \nabla c_k(r) \left[ H(g_k(r)) - 0.5 \right] = \sum_{k \geq 1} \Delta c(r) e^{\text{(i)}}_{sr}(r) \nabla g_k(r) \delta(g_k(r))
\]

When inserted into the modelling integral, eqs. \( \text{(33)} \), the contribution of the \( \sum_{k \geq 1} a^{\text{(i)}}_{sr}(r) \nabla c_k(r) \left[ H(g_k(r)) - 0.5 \right] \) term is negligible compared to the contribution of the \( \sum_{k \geq 1} \Delta c(r) e^{\text{(i)}}_{sr}(r) \nabla g_k(r) \delta(g_k(r)) \) term. Indeed, the second term leads to a sum of surface integrals (because of \( \delta(g_k(r)) \)) for which stationary phases exist for reflections whereas the first term (where \( a_k \) is smooth) leads to a sum of volume integrals for which stationary phases exist for diving waves only. As the Green functions \( G_{\text{inc}}^{(i)} \) entering into our Born modelling equation were constrained to contain no diving wave, the impact of the second term can be neglected in our case. The dominant contribution of \( e^{\text{(i)}}_{sr}(r) \nabla \delta c(r) \) is thus given by (using eq. \( \text{(33)} \))

\[
e^{\text{(i)}}_{sr}(r) \nabla \delta c(r) \rightarrow \sum_{k \geq 1} \Delta c(r) e^{\text{(i)}}_{sr}(r) \nabla g_k(r) \delta(g_k(r))
\]

Inserting this result in eq. \( \text{(33)} \) gives

\[
\hat{R}_{\text{gen}}(r, \theta^{(i)}(r_s, r, r_r)) = \sum_{k \geq 1} R_{\text{lin}}(r, \theta^{(i)}(r_s, r, r_r)) \frac{2 \cos \left( \theta^{(i)}(r_s, r, r_r) \right) e^{\text{(i)}}_{sr}(r) \nabla g_k(r) \delta(g_k(r))}{c_{\text{inc}}(r)}.
\]

We can consider, again from stationary phase reasoning \( \text{[14, 10, 19]} \)

\[
e^{\text{(i)}}_{sr}(r) \nabla g_k(r) \leftrightarrow |\nabla g_k(r)|.
\]

Finally, using eq. \( \text{(32)} \), we obtain

\[
\hat{R}_{\text{gen}}(r, \theta^{(i)}(r_s, r, r_r)) = \sum_{k \geq 1} R_{\text{lin}}(r, \theta^{(i)}(r_s, r, r_r)) \frac{2 \cos \left( \theta^{(i)}(r_s, r, r_r) \right) \delta S_k(r)}{c_{\text{inc}}(r)}.
\]

\(^4\) Each \( a_k \) has a compact support around reflector \( k \) positions. The “second term” has the same compact support. Nevertheless this leads to a volume integral.
Gathering previous results we have

\[\tilde{R}_{gen}(\mathbf{r}, \phi^{(i)}(\mathbf{s}, \mathbf{r}, \mathbf{r}')) = \frac{1}{\cos(\phi^{(i)}(\mathbf{r}, \mathbf{r}')) c_{\text{inc}}(\mathbf{r})} \mathbf{e}^{(i)}(\mathbf{r}) \cdot \nabla \delta c(\mathbf{r}) \]

\[\rightarrow \text{describes first-order effects related to any kind of small perturbations } \delta c \]

(first-order reflections, diffractions)

if reflectors only

\[\sum_{k \geq 1} R_{lin}(\mathbf{r}, \phi^{(i)}(\mathbf{s}, \mathbf{r}, \mathbf{r}')) \frac{2 \cos(\phi^{(i)}(\mathbf{r}, \mathbf{r}'))}{c_{\text{inc}}(\mathbf{r})} \delta S_k(\mathbf{r}) \text{ with } \phi^{(i)}(\mathbf{s}, \mathbf{r}, \mathbf{r}') \leftrightarrow \phi^{(i)}(\mathbf{r}, \mathbf{s}, \mathbf{r}') \]

\[\rightarrow \text{describes first-order reflections related to sufficiently weak velocity discontinuities.} \]

For comparison let us remember the content of Kirchhoff reflectivity, eq. (13)

\[\tilde{R}(\mathbf{r}, \phi^{(i)}(\mathbf{s}, \mathbf{r}, \mathbf{r}')) = \sum_{k \geq 1} R(\mathbf{r}, \phi^{(i)}(\mathbf{s}, \mathbf{r}, \mathbf{r}')) \frac{2 \cos(\phi^{(i)}(\mathbf{r}, \mathbf{r}'))}{c_{\text{inc}}(\mathbf{r})} \delta S_k(\mathbf{r}) \text{ with } \phi^{(i)}(\mathbf{s}, \mathbf{r}, \mathbf{r}') \leftrightarrow \phi^{(i)}(\mathbf{r}, \mathbf{s}, \mathbf{r}') \]

\[\rightarrow \text{describes first-order (possibly postcritical) reflections related to (possibly large) velocity discontinuities.} \]

Let us discuss what we have learned until now. From the propagation point of view, Born has been constrained here to be very close to Kirchhoff’s \(c_0\) (we considered them as identical). According to considerations of previous sections, Kirchhoff has a slight advantage over Born because Kirchhoff modelling may be more “effective” (different modelings for events above and below strong reflectors...).

From the reflectivity point of view, we notice main differences if we compare the Born generalized reflectivity (35) to Kirchhoff reflectivity (36):

1. For reflections:
   - Born is based on a linearized version \(R_{lin}\) of the 0-g.o. reflection coefficient \(R\), whereas Kirchhoff is based on the full 0-g.o. reflection coefficient \(R\) defined by eq. (5). This represents an advantage for Kirchhoff in describing the following:
     - (a) reflections due to large velocity contrasts,
     - (b) large incidence angle reflections,
     - (c) complex reflections.
   - These effects affect the phase of the wavefield, and are thus contained in the non-linear term \(P_{NL}\) of eq. (20). Because the Born approximation does not account for it, these effects cannot be included in Born generalized reflectivity. As those effects that are non-linear with respect to the wavefield perturbation are still linear with respect to the full 0-g.o. reflection coefficient, they are accounted for by Kirchhoff.

2. For very dense reflector configurations:
   - The generalized reflectivity remains well defined in the limit of a configuration where it is no longer possible to separate each reflector almost everywhere.

3. For other first-order events (like first-order diffractions):
   - Born can describe them whereas traditional Kirchhoff cannot. The price to pay is that the reflectivity must then depend explicitly on the receiver positions \(\mathbf{r}_r\) through \(\phi^{(i)}(\mathbf{s}, \mathbf{r}, \mathbf{r}_r)\) (obvious for diffractions because they radiate in every direction).

According to the first point, Kirchhoff contains more than Born. According to the second and third points, Born contains more than Kirchhoff. The considerations of this article allow us to gather the strengths of both schemes in a unique scheme: one may use Born’s generalized reflectivity \(\tilde{R}_{gen}\) together with the full reflection coefficient \(R\) instead of the linearized reflection coefficient \(R_{lin}\) to model the reflections (only).

We mention that the Kirchhoff inversion procedure introduced in previous sections has more flexibility than the direct inversion of the Born modelling equation. Indeed, the latter attempts to directly recover a
material property of the subsurface $\delta l$ (or $\delta c$), which is independent of the source position. The least-squares inversion of Born modelling equation (21) is a procedure that combines the data from all sources ($P$ represents data recorded at the earth’s surface, pre-processed to retain only first-order events)

$$
\min_{\delta l(r)} \int dr_s \int d\omega \int dr_r |P(r_s, r_r, \omega) + (i\omega)^2 S(\omega) \int_{\omega \geq 0} dr \delta l(r) G_0(r_s, r, \omega) G_0(r, r_r, \omega)|^2.
$$

(37)

Kirchhoff inversion attempts to recover a reflectivity (or a seismic image) that is not a material property of the subsurface and depends on the source position. Thus Kirchhoff least-squares inversion, eq. (14), is done for each shot (or offset if the data are reorganized) independently. As a second step, the material properties of the subsurface are recovered by inverting the obtained reflectivity. This step has already been presented in §2.3.3 to interpret amplitudes of the reflectors present in the seismic image. In the following we show how the generalized reflectivity concept allows us to interpret events other than reflectors in the seismic image.

4. Utility of the generalized reflectivity for interpretation.

4.0.1. Conversion of the generalized reflectivity into a velocity perturbation.

Suppose we recorded data $P$ at the earth’s surface and pre-processed to retain only first-order events, with non-linear (or higher-order) events filtered out (in particular multiple reflections). Suppose we also produced a smooth subsurface model $c_{inc}$ that allows us to compute $C^{(i)}_{inc}$ and $\theta^{(i)}$ for chosen direct travel-time branches. Traditional Kirchhoff inversion, eq. (14), inverts for a reflectivity $\hat{R}_{inv}(r_s, r)$ that does not depend on the receiver positions $r_r$. [10]

As our generalized reflectivity considerations fit into a Kirchhoff modelling framework, eq. (38) can directly be used to interpret the result of a Kirchhoff inversion (14). Or course the most exhaustive scheme to invert for the generalized reflectivity should depend on the receiver positions because $\theta^{(i)}(r_s, r, r_r)$ is included in the definition of the generalized reflectivity. This would not cause any formal difficulties but would lead to an inversion scheme that is different from the commonly implemented scheme that does not depend on the receiver positions, eq. (14), with additional ill-posed problems. Here we want our considerations to be applicable to the commonly implemented scheme. The obtained $\hat{R}_{inv}$ then represents an average of $\hat{R}_{gen}$, eq. (38), over the receiver positions. If $S_{rec}$ denotes the area over which the receiver are spread, we have

$$
\hat{R}_{inv}(r_s, r) \approx \frac{1}{S_{rec}} \int_{S_{rec}} dr_r \hat{R}_{gen}(r, \theta^{(i)}(r_s, r, r_r))
\approx \mathbf{a}(r_s, r) \nabla \delta c(r)
$$

where $\mathbf{a}(r_s, r) = \frac{1}{c_{inc}(r)} \frac{1}{S_{rec}} \int_{S_{rec}} dr_r \frac{1}{\cos \theta^{(i)}(r_r, r_s, r_r)/2} \mathbf{e}^{(i)}_{sr}(r)$. [38]

Inverting eq. (38) for each shot would allow us to convert the seismic image (i.e. the reflectivity $\hat{R}_{inv}$ computed by traditional Kirchhoff inversion schemes) into a velocity perturbation $\delta c$ and vice-versa. More generally one could also invert eq. (38) for each source-receiver configuration. This offers opportunities for further interpretation of seismic images and also for FWI approaches that include a reflectivity, showing how to rigorously convert the reflectivity into a velocity perturbation.

4.0.2. Example of imaged spike diffractor amplitudes.

We discussed in §2.3.3 the traditional way to interpret seismic images, considering only the amplitudes of reflectors. But seismic data and images also contain events that are not related to reflections, for instance those related to diffractions. It has been kinematically understood why migration collapses first-order diffractions in the seismic image [1, 2], and the considerations of this article explain this from a fundamental point of view. But to our knowledge, it was not clear how to interpret the corresponding amplitudes. Our generalized reflectivity considerations open some doors: eq. (38) can be used to interpret events related to first-order diffractions, i.e. recover the perturbation $\delta c$ that generates them.

Fig. 2 presents an example of a seismic image. The stack of the reflectivities inverted for each shot is shown, i.e. $\int dr_s \hat{R}_{inv}(r_s, r)$. The image contains many reflectors but also some visible diffractions due to spikes in the true subsurface model (for example in the area surrounded by orange). Eq. (38) explicitly states the relationship between the imaged diffractions and the corresponding velocity perturbation; as we deal with a band-limited inversion and stack over shots, we have

$$
\int dr_s \hat{R}_{inv}(r_s, r) \approx \mathbf{a}_{stack}(r) \nabla \delta c_{bd}(r)
$$

14
\[
\mathbf{a}_{\text{stack}}(\mathbf{r}) = \frac{1}{c_{\text{inc}}(\mathbf{r})} \frac{1}{S_{\text{rec}}} \int_{S_{\text{inc}}} d\mathbf{r}_s \int d\mathbf{r}_s \frac{1}{\cos(\theta^{(ij)}(\mathbf{r}_s, \mathbf{r}, \mathbf{r}_r))} \mathbf{e}_s^{(ij)}(\mathbf{r}),
\]

where \(\delta \mathbf{c}_{\text{bl}}\) is the band-limited version of the velocity perturbation (because in practice we deal with band-limited Kirchhoff inversion). The direction of the vector \(\mathbf{a}_{\text{stack}}\) depends on the acquisition configuration and the propagation in the subsurface (through the "illumination direction" \(\mathbf{e}_s^{(ij)}\) and \(\theta^{(ij)}\)). As already mentioned in \([2,3]\) this behavior is different from reflectors, for which a residual wavelet maps in the direction perpendicular to the reflectors whatever the acquisition. For diffractors, the wavelet will map in the \(\mathbf{a}_{\text{stack}}\) direction in the image stacked over shots, this direction depending on the acquisition configuration.

From eq. \(39\), one has to integrate the image over the curvilinear abscissa \(a_{\text{stack}}\) defined by \(\mathbf{a}_{\text{stack}}\) to recover the band-limited velocity perturbation \(\delta \mathbf{c}_{\text{bl}}\)

\[
\delta \mathbf{c}_{\text{bl}}(\mathbf{r}) = \int_0^{a_{\text{stack}}(\mathbf{r}')} da_{\text{stack}}(\mathbf{r}') \int d\mathbf{r}_s \hat{R}_{\text{inv}}(\mathbf{r}_s, \mathbf{r}') + \text{constant.}
\]

\(\mathbf{a}_{\text{stack}}\) is computable through eq. \(39\) through knowledge of the propagation directions of the source and receiver wavefields. The constant can be computed by imposing that \(\delta \mathbf{c}_{\text{bl}}(\mathbf{r})\) is null at the earth's surface. This illustrates that the interpretation of more information than that associated with reflectors is possible by our considerations.

In the particular case of Fig. 2, the sources and receivers are evenly distributed over the earth’s surface. As \(\mathbf{a}_{\text{stack}}\) is related to an average over all shots and receivers of the \(\mathbf{e}_s^{(ij)}\) vectors, it will in this case more or less tend to point in the \(z\) direction and \(\int d\mathbf{r}_s \hat{R}_{\text{inv}}(\mathbf{r}_s, \mathbf{r})\) will approximately tend to be a scaled version of \(\partial \delta \mathbf{c}_{\text{bl}}(\mathbf{r})/\partial z\). The band-limited velocity perturbation corresponding to the diffractors of Fig. 2 is a band-limited spike centered on position \(\mathbf{r}_d\): \(\delta \mathbf{c}_{\text{bl}}(\mathbf{r}) \propto \delta \mathbf{c}_{\text{bl}}(\mathbf{r} - \mathbf{r}_d);\) thus \(\int d\mathbf{r}_s \hat{R}_{\text{inv}}(\mathbf{r}_s, \mathbf{r})\) must be a scaled version of \(\partial \delta \mathbf{c}_{\text{bl}}(\mathbf{r} - \mathbf{r}_d)/\partial z\) at diffractor positions. This is consistent with what we observe in Fig. 2 an oscillating wavelet maps in the \(z\)-direction at diffractor positions due to the derivative \(\partial / \partial z\).

5. Conclusions.

We recalled the chain of approximations leading to Kirchhoff and Born modelling equations. They both contain a linearization but each offers some specifics. The Kirchhoff approximation allows, for example, the modelling of first-order reflections on strong discontinuities and postcritical reflections. Born approximation makes it possible to model reflections on weak discontinuities only, but also first-order effects that go beyond first-order reflections (like first-order diffractions). We showed how Kirchhoff and Born modelling lead to very similar expressions and derived a general expression for the conversion from velocity model perturbation to reflectivity (and conversely) through the generalized reflectivity concept.

The scheme we have developed may offer some applications:

- It makes it possible to model within the Kirchhoff scheme, first-order effects that go beyond first-order reflections (like first-order diffractions).
- It offers opportunities in the framework of Kirchhoff inversion, i.e. for the interpretation of seismic-migrated images: the generalized reflectivity gives a basis to interpret more information than the amplitudes associated to first-order reflections, for instance the amplitudes of first-order diffractors. Also, it would allow us to go beyond AVA analysis, inverting for the whole seismic image amplitude information (not only amplitude information at peaks) to recover the related velocity model perturbation.
- Finally it offers opportunities on FWI approaches that include a reflectivity, showing how to rigorously convert the reflectivity into a velocity perturbation.

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Appendix A. Time-depth equivalency.

Time-depth equivalency is often mentioned in seismic. It, among other things, implies that the residual wavelet \( f \) present in Kirchhoff-inverted reflectivity maps in the direction perpendicular to the reflectors. In other terms \( \delta_\theta(g_k(r)) \) in eq. (15) represents a band-limited delta distribution that “peaks” in the direction perpendicular to the interface \( k \). This is what we demonstrate now.

We perform a 1-st order Taylor expansion of \( g_k(r) \) around the orthogonal projection of \( r \) on interface \( k \), denoted by \( r_k(r) \). We obtain

\[
g_k(r) = g_k(r_k(r)) + \nabla_{r_k} g_k(r_k(r)).(r - r_k(r)) + o(|r - r_k(r)|^2)
\]

\[
= \frac{\nabla_{r_k} g_k(r_k(r))}{|\nabla_{r_k} g_k(r_k(r))|} (r - r_k(r)) |\nabla_{r_k} g_k(r_k(r))| + o(|r - r_k(r)|^2)
\]

\[
= n(r_k(r)).(r - r_k(r)) |\nabla_{r_k} g_k(r_k(r))| + o(|r - r_k(r)|^2),
\]

(A.1)

where \( n(r_k(r)) = \frac{\nabla_{r_k} g_k(r_k(r))}{|\nabla_{r_k} g_k(r_k(r))|} \) is the unit vector normal to the interface \( k \) at position \( r_k(r) \) that points “downward”. The remainder term is negligible within 0-g.o., i.e. when the interface curvatures (the 2-nd order derivatives of \( g_k \)) is sufficiently small. Using the result (A.1) together with eq. (15), we have

\[
\delta_\theta(g_k(r)) \approx \frac{1}{|\nabla_{r_k} g_k(r_k(r))|} \delta_\theta(n(r_k(r)).(r - r_k(r))).
\]

(A.2)

We see that the smearing of \( \delta_\theta(g_k(r)) \) (due to the residual wavelet \( F \)) occurs in the direction \( n(r_k(r)) \) perpendicular to the reflector.

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