SLICE-TORUS LINK INVARIANTS, COMBINATORIAL INVARIANTS, AND
POSITIVITY CONDITIONS.

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ABSTRACT. We prove some necessary conditions for a link to be: concordant to a quasi-positive link,
quasi-positive, positive, or the closure of a positive braid. The main applications of our results are: a
characterisation of positive links with unlinking number 1 and 2, and a combinatorial criterion to test if
a positive link is the closure of a positive braid.

1. INTRODUCTION

In this note we study slice-torus and combinatorial invariants of multi-component links – instead
of knots – in $S^3$ satisfying various positivity conditions.

The aim of this note is dual; first, we provide a necessary condition for a link to be concordant
to a quasi-positive link, and compare it to other known obstructions. Second, we investigate certain
combinatorial invariants of quasi-positive, positive and positive-braid links, providing obstructions
for a link to fall in one of the aforementioned classes. As an application we give a characterisation of
positive links whose unlinking number is either one or two.

Notations and conventions. In what follows all manifolds and sub-manifolds are smooth and ori-
ented, unless otherwise stated. Boundaries are endowed with the induced orientation. Two links are
equivalent if ambient isotopic via an orientation-preserving ambient isotopy. In this paper no distinc-
tion shall be made between a link and its equivalence class.

The letter $L$ (possibly with subscripts or superscripts) is reserved to links, while $K$ (possibly with
subscripts or superscripts) is used for knots. The letter $\ell$ (possibly with subscripts or superscripts) is
used to denote the number of connected components of a link $L$ (with corresponding subscripts or
superscripts) – e.g. $\ell'$ denotes the number of components of the link $L'$.

The symbol $\cup$ denotes the usual set-theoretic union, while the symbol $\bigsqcup$ is reserved for the split
(or distant) union. By split union of links (or diagrams) we mean the union of links (or diagrams)
embedded into pairwise disjoint disks. In particular, with the notation $L = K_1 \cup \ldots \cup K_\ell$ we denote
a link with $\ell$ components with knot-type $K_1, \ldots, K_\ell$, which are possibly linked. On the other hand,
the notation $L = K_1 \bigsqcup \ldots \bigsqcup K_\ell$ indicates that the link $L$ has $\ell$ components which are, up to ambient
isotopy, contained in pairwise disjoint closed disks inside $S^3$ – and therefore unlinked.

1.1. Quasi-positivity and concordance. Since there are various notions of concordance between
multi-components links, we start by recalling the one we are interested in.

Definition 1.1. Let $L_0$ and $L_1$ be two links. Then, $L_0$ and $L_1$ are concordant if there exists a collection
of annuli $A_1, \ldots, A_\ell$, properly and disjointly embedded in $S^3 \times [0, 1]$, such that: both $A_i \cap S^3 \times \{0\}$
and $A_i \cap S^3 \times \{1\}$ are non-empty for each $i$, and

$$(A_1 \cup \ldots \cup A_\ell) \cap S^3 \times \{0\} = L_0, \quad \text{and} \quad (A_1 \cup \ldots \cup A_\ell) \cap S^3 \times \{1\} = rL_1^\ast,$$

where $rL$ denotes the reverse of a link $L$, and $L^\ast$ denotes the mirror of $L$. 

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The slice-torus link invariants are $\mathbb{R}$-valued concordance invariants of links, which vanish on the unlinks [8]. Prominent examples of slice-torus link invariants are the Ozsváth-Szabó-Rasmussen $\tau$-invariant [6, 21, 23], and (a re-scaling of) the Rasmussen $s$-invariant [3, 22]. It is known that the slice-torus link invariants provide an obstruction to be concordant to a quasi-positive link; indeed, it follows from [8, Theorem 3.2] that if $L$ is concordant to a quasi-positive link, then
\begin{equation}
2\nu(L) - \ell = -\chi_4(L),
\end{equation}
where $\chi_4$ is the slice Euler characteristic, and $\nu$ is any slice-torus invariant.

**Remark.** By [7, Proposition 1.6] (cf. [15, Proposition 3]) if $L$ is concordant to a quasi-positive link, then Equation (1.1) holds also by replacing $\nu$ with the Hom-Wu invariant $\nu^+$ [7, 15]. However, again by [7, Proposition 1.6], if Equation (1.1) holds, then also $2\nu^+(L) - \ell = -\chi_4$.

In this note we provide an obstruction for a link to be concordant to a quasi-positive link based on the slice-torus link invariants and linking numbers.

**Theorem 1.2.** Let $L$ be a link, and let $\nu$ be a slice-torus link invariant. If $L$ is concordant to a quasi-positive link, then
\begin{equation}
\nu(L) - \sum_{i=1}^k \nu(L_i) \leq \sum_{i < j} \ell_k(L_i, L_j),
\end{equation}
for each partition of $L$ into disjoint (non-empty) sub-links $L_1, ..., L_k$.

We point out that Theorem 1.2 poses constraints on the linking matrix of a quasi-positive link in terms of its sub-links. This is slightly surprising for two reasons; first, all linking matrices can be realised by quasi-positive links. Second, all links arise as sub-links of a quasi-positive link by adding a single unknotted component. The former fact – originally proved by Rudolph [24] – is ultimately a consequence of the fact that all links are topologically concordant to a quasi-positive link [5]. A proof of the second fact is included in the appendix; despite it being well-known to experts, we were unable to find any reference to it in the literature.

The obstruction provided Theorem 1.2 is independent from the obstruction provided by Equation (1.1). As an example we present an infinite family of 2-bridge links.

**Theorem 1.3.** Denote by $L_k$ the two-bridge link in Figure 1. For all $k \in \mathbb{N}$, and any slice-torus link invariant $\nu$, we have
\begin{equation}
2\nu(L_k) - 2 = -\chi_4(L_k) = 2k - 2.
\end{equation}
However, $L_k$ is not concordant to any quasi-positive link for all $k \geq 1$. In particular, the Whitehead link $L_1$ is not concordant to any quasi-positive link.

**Proof.** One can compute $\nu(L_k) = k$ using [8, Theorem 1.3] (cf. Proposition 2.4) and the diagram $D_k$ in Figure 1. Moreover, a quick inspection of $D_k$ shows that $L_k$ can be unlinked with $k$ crossing changes involving a single component (cf. [9, Proposition 2.7]), therefore $-\chi_4(L_k) \leq 2k - 2$. The equality follows by [8, Proposition 2.11], which implies that $2\nu(L) - \ell \leq -\chi_4(L)$. Since $\ell_k(L_k) = 0$, and since $L_k$ has trivial components, the statement follows directly from Theorem 1.2.

We conclude the first part of the paper by studying the maximal self-linking number in a concordance class $sl_c$; this is a well-defined integer concordance invariant which can be used to obstruct the concordance to quasi-positive links. We compute $sl_c$ for closures of pure braids, and characterise the closures of alternating pure braids which are concordant to a quasi-positive link (cf. Propositions 3.4 and 3.5 respectively).
Remark. In [10, Proposition 1.6] the maximal self-linking number $sl_{\text{max}}$ is used to show that $L_k$ is not quasi-positive for $k \geq 1$. A similar reasoning, where $sl_{\text{max}}$ is replaced by $sl_c$, can be used to (re-)prove Theorem 1.3—cf. Proposition 3.1 and Example 3.2.

1.2. Obstructions to positivity conditions. In the second part of this paper we investigate necessary conditions for a link to be quasi-positive, positive, or the closure of a positive braid. Our first result is the following necessary condition for certain links to be quasi-positive.

**Theorem 1.4.** Let $L$ be a quasi-positive link. If there exists a partition of $L$ into disjoint (non-empty) sub-links $L_1, \ldots, L_k$ such that

\[
\nu(L) - \sum_{i=1}^{k} \nu(L_i) = \sum_{i<j} \ell_k(L_i, L_j),
\]

then

\[
2\nu(L_i) - \ell_i = sl_{\text{max}}(L_i), \quad \text{for all } i \in \{1, \ldots, k\}.
\]

The second class of links we are concerned with is the class of positive links. A **positive link** is a link which admits a diagram with only positive crossings (see Figure 4). It has been observed by Nakamura [20] and Rudolph [26] that positive links are quasi-positive. In this paper we make use of combinatorial invariants of links to establish necessary conditions for a link to be positive. First, we need to establish some terminology. A **Gordian path** is a sequence of links $L^{(0)}, L', \ldots, L^{(k)}$ such that, for each $i \in \{1, \ldots, k\}$, a diagram for $L^{(i)}$ is obtained from a diagram of $L^{(i-1)}$ by changing a single crossing. A crossing change is a **mixed crossing change** if it involves two distinct components of the link, otherwise is a **self crossing change**. We call a Gordian path **strong** if it features only mixed crossing changes.

**Definition 1.5.** The **splitting number** $sp(L)$ (resp. **strong splitting number** $\tilde{sp}(L)$) of a link $L$ is the minimal length of a Gordian path (resp. strong Gordian path) from $L$ to a split union of knots. Such a Gordian path is called **splitting sequence** (resp. strong splitting sequence).

**Definition 1.6.** The **unlinking number** $u(L)$ of a link $L$ is the minimal length of a Gordian path from $L$ to an unlink. Such a Gordian path is called **unlinking sequence**. The unlinking number of a knot is also called **unknotting number**.

Since the necessary terminology and notation have been established, we can state our result.

**Theorem 1.7.** If $L = K_1 \cup \cdots \cup K_\ell$ is a positive link, then we have

\[
\ell_k(L) = sp(L) = \tilde{sp}(L) = u(L) - \sum_{i=0}^{\ell} u(K_i),
\]

where $\ell_k(L)$ is the total linking number, that is the sum of $\ell_k(K_i, K_j)$, for all $i < j$.

Nakamura in [20, Theorem 5.1] proved that if $K$ is positive knot, then $u(K) = 1$ if, and only if, $K$ is a (non-trivial) twist knot. As an application of Theorem 1.7 we extend Nakamura’s result to a characterisation all positive links with unlinking number one or two.
Proposition 1.8. Let $L$ be a positive link. Then, $u(L) = 1$ if, and only if, $L$ is the split union of a (possibly empty) unlink with either the positive Hopf link, or a (non-trivial) twist knot.

Proposition 1.9. Let $L$ be a positive link. Then, $u(L) = 2$ if, and only if, $L$ is the split union of a (possibly empty) unlink and one of the following

a. a positive link with total linking number 2 and unknotted components;

b. the connected sum of a positive Hopf link and a positive twist knot;

c. the split union of a positive Hopf link and a positive twist knot;

d. the split union of two non-trivial positive twist knots;

e. a positive knot with unknotting number 2.

Furthermore, each of the above families contains infinitely-many links.

Finally, we focus on links which are obtained as the closure of positive braids, which is to say positive-braid links. In this case we can strengthen the statement of Theorem 1.7.

Theorem 1.10. If $L = K_1 \cup \cdots \cup K_\ell$ is a positive-braid link, then

$$\nu(L) - \sum_{i=1}^\ell \nu(K_i) = \ell k(L) = sp(L) = \tilde{sp}(L),$$

for any slice-torus link invariant $\nu$. Furthermore, given any partition of $L$ into sub-links, say $L_1, ..., L_k$, we have

$$\nu(L) - \sum_{i=1}^k \nu(L_i) = u(L) - \sum_{i=0}^k u(L_i) = \sum_{1 \leq i < j \leq k} \ell k(L_i, L_j).$$

As consequences of Theorem 1.10 we obtain two results; the first result is a formula for the unlinking number of positive links with braid positive components (Proposition 4.6). The second result is the following combinatorial criterion to test whether a positive link is also positive-braid link.

Proposition 1.11. Let $D$ be a positive diagram for $L = K_1 \cup \cdots \cup K_\ell$. Denote by $D_1, ..., D_\ell$ the diagrams for $K_1, ..., K_\ell$ obtained by deleting all but the corresponding component from $D$. If $L$ is a positive-braid link, then

$$o(D) = \sum_i o(D_i)$$

where $o(D')$ denotes the number of Seifert circles of $D'$.

Example 1.12. In Figure 7 is depicted a positive diagram $D$ of a link $L = K_1 \cup K_2$. It is easily verified that $o(D) = 6 \neq 4 = o(D_1) + o(D_2)$, where $D_1$ and $D_2$ are defined as in Proposition 1.11. It follows that $L$ cannot be the closure of a positive braid.

1.3. Outline of the paper. In Section 2 we review some background material. In Section 3 we prove Theorems 1.2 and 1.4. Finally, in Section 4 we concern ourselves with positive and positive-braid links. The appendix is dedicated to the proof of the fact that all links arise as sub-links of a quasi-positive link, which we were unable to find in the literature.

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2. BACKGROUND MATERIAL AND NOTATION

The main scope of this section is to fix the notation and review some results needed in the proofs of the main theorems. The section is organised as follows; we start with some facts concerning braids. Then, we pass to the definition of self-linking number, and the description of some of its properties. Afterward we present some basic results on slice-torus link invariants. We conclude with some basic facts on unlinking and splitting numbers.

2.1. Braids. The braid group on \( n \)-strands, here denoted by \( B_n \), is the group generated by \( \sigma_1, \ldots, \sigma_{n-1} \) (called Artin generators), and subject to the relations

\[
\sigma_i \sigma_j = \sigma_j \sigma_i, \quad \text{if } |i - j| > 1, \quad \text{and} \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad \forall i, j.
\]

Alternatively, one can define the braid group diagrammatically as illustrated in Figure 2. A braid is a word in the Artin generators and their inverses. In the rest of the paper no difference shall be made between the algebraic and the diagrammatic interpretations of the braid group.

By joining the start and end points of each strand, as illustrated in Figure 2 one obtains a link diagram from a braid. The link represented by such a diagram is called the closure of the given braid. A classical theorem of Alexander states that all links can be obtained as closed braids – see \[2\].

**Definition 2.1.** A braid is quasi-positive if it is the product of conjugates of Artin generators, and it is positive if it is the product of Artin generators. A link is called quasi-positive (resp. positive-braid) if it is the closure of a quasi-positive (resp. positive) braid.

![Diagram of braid and its closure](image)

**Figure 2.** In the order from left to right: the Artin generator \( \sigma_i \) and its inverse \( \sigma_i^{-1} \), the product of two braids \( \beta \) and \( \beta' \), and the closure of a braid \( \beta'' \).

Recall that a braid is alternating if moving along each strand overpasses and underpasses alternate. A braid (or a link) is called non-split if its closure (resp. the link itself) is not the split union of two links. The following lemma of is easy to prove, and thus left as an exercise to the reader.

**Lemma 2.2.** A braid is non-split and alternating if, and only if, the following properties are satisfied:

(i) all \( \sigma_i \) appear (possibly with negative exponents),

(ii) each \( \sigma_i \) appears with exponents of a fixed sign (that is all positive or all negative),

(iii) \( \sigma_i \) and \( \sigma_{i+1} \) always appear with exponents of opposite signs.

Let \( \beta \) be a braid representing a link \( L \) – that is, the closure of \( \beta \) is \( L \) – and let \( L' \) be a sub-link \( L \). By deleting from \( \beta \) all strands which belong to \( L \setminus L' \) in \( \beta \)'s closure, one obtains a braid \( \beta' \) whose closure is \( L' \). We shall refer to \( \beta' \) as the sub-braid of \( \beta \) associated to \( L' \). Note that \( \beta' \) does not belong to the same braid group as \( \beta \) – cf. Figure 3.
2.2. The self-linking number. To each braid is attached an integer called self-linking number. Concretely, the self-linking number of a braid $\beta$ is defined as
\[ \text{sl}(\beta) = w(\beta) - n \]
where $w(\beta)$ denotes the writhe, or algebraic crossing count, of $\beta$. The self-linking number of a braid is not an invariant of its closure. However, the maximal self-linking number among all braids representing a given link, denoted by $\text{sl}_{\text{max}}$, is well-defined integer-valued link invariant – this fact follows from the celebrated Bennequin inequality [4].

Lemma 2.3. Let $L$ be a link partitioned into disjoint sub-links $L_1,\ldots,L_k$. Given a braid $\beta$ representing $L$, denote by $\beta_i$ the sub-braid associated to $L_i$. Then, we have
\[ \text{sl}(\beta) - \sum_{i=1}^{k} \text{sl}(\beta_i) = 2 \sum_{1 \leq i < j \leq k} \ell_k(L_i,L_j) \]

Proof. Since the number of strands of the $\beta_i$ adds up to the number of strands in $\beta$, the right hand side of Equation (2.1) is the signed count of all crossings in $\beta$ which do not belong to any $\beta_i$. Which is twice the total linking number between the $L_j$s, as claimed. \hfill \square

Lemma 2.3 shall play a pivotal role in the rest of the paper.

2.3. Slice-torus link invariants. Recall from the introduction that a slice-torus link invariant is an $\mathbb{R}$-valued link concordance invariant which vanishes for the unlinks. We refer the reader to [8] for the precise definition. We focus on some specific properties of slice-torus link invariants, and in particular on the so-called combinatorial bound. Before stating the combinatorial bound we need some preliminary definitions.

Let $D$ be an oriented link diagram. The oriented resolution of $D$ is the set of oriented circles obtained by replacing each crossing in $D$ with its oriented smoothing – see Figure 4. The circles appearing in the oriented resolution are called Seifert circles and their number is denoted by $o(D)$. Denote by $s_+(D)$ (resp. $s_-(D)$) the number of connected components of the graph obtained by smoothing all negative (resp. positive) crossings, and flattening all positive (resp. negative) crossings. Denote by $w(D)$ the algebraic crossing count (or writhe) of $D$. Finally, denote by $\ell_s(D)$ the number of connected components of the graph obtained by flattening all crossings in $D$, regardless of the sign – in particular, it follows that $\ell_s(D) \leq s_{\pm}(D)$. 

![Figure 3](image_url)
The above-defined (diagram-dependent) combinatorial quantities can be used to estimate the value of all slice-torus link invariants. More precisely, we have the following combinatorial bounds.

**Proposition 2.4** ([8, Theorem 1.3]). Let \( \nu \) be a slice-torus link invariant. For each oriented diagram \( D \) representing the \( \ell \)-component link \( L \), we have

\[
\frac{w(D) - o(D) + 2s_+(L) + \ell - 2s_s(D)}{2} \leq \nu(L) \leq \frac{w(D) + o(D) - 2s_- - (L) - \ell + 2s_s(D)}{2}
\]

An immediate consequence is the following refinement of the slice-Bennequin inequality.

**Corollary 2.5.** For each braid \( \beta \) representing the \( \ell \)-component link \( L \), we have

\[
sl(\beta) \leq 2\nu(L) - \ell.
\]

**Proof.** If we use the closure of \( \beta \in B_n \) as a diagram, then the number of strands \( n \) equals the number of Seifert circles. The inequality follows from Proposition 2.4 and from the fact that \( \ell_s \leq s_+ \). \( \Box \)

A diagram for which the upper and lower bounds in Proposition 2.4 coincide is called homogeneous. An homogeneous link is a link which admits an homogeneous diagram. Examples of homogeneous links are alternating links, and positive links. Homogeneous links where originally introduced by Cromwell [12]. Our characterisation follows from a result of Abe [1, Theorem 3.4] – which is stated for knots, but it is easily seen to hold for non-split links – together with the additivity of slice-torus link invariants under split union.

### 2.4. Unlinking and splitting numbers.

In this subsection we collected two lemmata concerning unlinking and splitting numbers. We start with the following basic fact.

**Lemma 2.6** ([9, Lemma 2.1]). Let \( L \) be a link. If there exists a splitting sequence with \( m \) mixed crossing changes, then

\[
|\ell_k|(L) \leq m,
\]

where \( |\ell_k| \) denotes the absolute linking number, that is the sum of the absolute value of the linking numbers between the components of \( L \) – not to be confused with \( |\ell_k|(L) \).

If the absolute linking number equals the strong splitting number, one can say something about the unlinking number, and splitting number \( sp \).

**Lemma 2.7.** Let \( L = K_1 \cup \cdots \cup K_\ell \) be a link. If \( sp(L) = |\ell_k|(L) \), then

(i) \( sp(L) = |\ell_k|(L) \);

(ii) \( u(L) - \sum_{i=1}^\ell u(K_i) = |\ell_k|(L) \).

**Proof.** The equality in (i) has been proved in [9, Remark 2.3]. Consider a minimal unlinking sequence for \( L \) with \( m \) mixed crossing changes and \( s \) self crossing changes. Since an unlinking sequence is also a splitting sequence, Lemma 2.6 implies \( |\ell_k|(L) \leq m \). Note that mixed crossing change do not change
the isotopy class of the components. Thus, the sum of the unknotting numbers of the $K_i$ is less than, or equal to, $s$. It follows that

$$|\ell_k|(L) + \sum_i u(K_i) \leq m + s = u(L).$$

The opposite inequality, and hence the equality, follows from a simple observation; a particular unlinking sequence is given by a minimal strong splitting sequence followed by a minimal unknotting sequence for each component. □

3. CONCORDANCE TO QUASI-POSITIVE LINKS

In this section we prove our obstructions to the existence of a concordance between a given link and quasi-positive links, that is Theorem 1.2. Then, we also prove an obstruction to quasi-positivity, namely Theorem 1.4. We conclude the section with some remarks concerning the concordance self-linking number – that is the maximal value of the self-linking number a concordance class. The concordance self-linking number $sl_c$ is extremely difficult to compute in general. In this section we compute $sl_c$, for the closure of pure braids, as a consequence we obtain a necessary condition for a pure braid to be concordant to a quasi-positive link. Finally, we characterise the closures of alternating pure braids which are concordant to a quasi-positive link.

3.1. Obstacation to quasi-positivity and concordance to quasi-positive links. This short subsection is dedicated to the proofs of Theorems 1.2 and 1.4. We start with Theorem 1.2.

Proof of Theorem 1.2. Let $L = L_1 \cup \cdots \cup L_k$ be a quasi-positive link, and let $\beta$ be a quasi-positive braid representing $L$. Denote by $\beta_i \in B_{n_i}$ the sub-braid of $\beta$ associated to $L_i$, and by $\ell_i$ the number of components of $L_i$. From [8, Theorem 3.2] it follows that

$$2\nu(L) - \ell = sl(\beta).$$

Therefore, we have

$$2 \left( \nu(L) - \sum_{i+1}^k \nu(L_i) \right) = sl(\beta) - \sum_{i=1}^k (2\nu(L_i) - \ell_i) \leq sl(\beta) - \sum_{i=1}^k sl(\beta_i),$$

where the last inequality follows from Corollary 2.5. The statement now follows from Lemma 2.3, and from the concordance invariance of both $\nu$ and the linking matrix. □

We already gave an example of Theorem 1.2, thus we go straight to the proof of Theorem 1.4.

Proof of Theorem 1.4. Let $L = L_1 \cup \cdots \cup L_k$ be a quasi-positive link partitioned into sub-links. Consider a quasi-positive braid representing $L$, say $\beta$, and denote by $\beta_i$ the sub-braid representing $L_i$. Fix a slice-torus link invariant $\nu$, and assume that

$$\nu(L) - \sum_{i=1}^k \nu(L_i) = \sum_{i<j} \ell_k(L_i, L_j).$$

It follows from (1.1) and Lemma 2.3 that

$$\sum_{i=1}^k (2\nu(L_i) - \ell_i) = \sum_{i=1}^k sl(\beta_i).$$

Since, by Corollary 2.5, we have $sl(\beta_i) \leq (2\nu(L_i) - \ell_i)$ the equality follows for each $i$. □
3.2. **Concordance self-linking number.** The maximal self-linking number \( sl_{\text{max}} \) is an integer link invariant which is useful to detect quasi-positivity. In fact, \([25]\) Lemma in Section 3) states that if a link \( L \) is quasi-positive, then \( sl_{\text{max}}(L) = -\chi_4(L) \). Unfortunately, the maximal self-linking number is not a concordance invariant; for instance, the knot \( 6_1 \) is concordant to the unknot \( U \), however \( sl_{\text{max}}(6_1) = -5 \), \( sl_{\text{max}}(6_1^*) = -3 \), and \( sl_{\text{max}}(U) = -1 \), see \([11]\). Nonetheless, Rudolph’s slice-Bennequin inequality \([25]\) implies that \( sl_{\text{max}} \) is bounded from above in each concordance class. Thus, the **concordance self-linking number** defined as follows

\[
sl_c(L) = \max \{ sl_{\text{max}}(L') \mid L' \text{ is concordant to } L \},
\]

and \( sl_c \) is a well-defined integer-valued concordance invariant. Furthermore, it follows that if an \( \ell \)-component link \( L \) is concordant to a quasi-positive link, then

\[
(3.1) \quad sl_c(L) = 2\nu(L) - \ell = -\chi_4(L),
\]

for each slice-torus link invariant \( \nu \). Hence, the concordance self-linking number can be used to obstruct the concordance to quasi-positive links.

The concordance self-linking number is extremely difficult to compute. Now we prove a property of \( sl_c \) which can be used to compute it for pure braid closures.

**Proposition 3.1.** Let \( L \) be a link partitioned into disjoint sub-links \( L_1,...,L_k \). Then, we have

\[
sl_c(L) \leq \sum_{i=1}^{k} sl_c(L_i) + 2 \sum_{1 \leq i < j \leq k} \ell k(L_i,L_j), \quad \text{where } \ast \in \{\text{max, c}\}.
\]

**Proof.** First we prove the case \( \ast = \text{max} \). Let \( \beta \) be a braid representing \( L \) and realising the maximal self-linking number. Denote by \( \beta_i \) the sub-braid of \( \beta \) associated to \( L_i \). By Lemma \([2.3]\) we have

\[
sl_{\text{max}}(L) = sl(\beta) = \sum_{i=1}^{k} sl(\beta_i) + 2 \sum_{1 \leq i < j \leq k} \ell k(L_i,L_j) \leq \sum_{i=1}^{k} sl_{\text{max}}(L_i) + 2 \sum_{1 \leq i < j \leq k} \ell k(L_i,L_j).
\]

Now, we can prove the case \( \ast = c \). Since \( sl_{\text{max}}(L_i) \leq sl_c(L_i) \), we have

\[
(3.2) \quad sl_{\text{max}}(L) \leq \sum_{i=1}^{k} sl_c(L_i) + 2 \sum_{1 \leq i < j \leq k} \ell k(L_i,L_j).
\]

Notice that any strong concordance establishes a bijection between the components of two links, and thus a bijection between sub-links. Therefore, for any link \( L' \) concordant to \( L \), we can find a partition into sub-links \( L'_1,...,L'_k \subset L' \) such that \( L_i \) is concordant to \( L'_i \), and \( \ell k(L'_i,L'_j) = \ell k(L_i,L_j) \) – since the linking matrix is a concordance invariant. Hence, applying Equation \((3.2)\) we get

\[
sl_{\text{max}}(L') \leq \sum_{i=1}^{k} sl_c(L'_i) + 2 \sum_{1 \leq i < j \leq k} \ell k(L'_i,L'_j) = \sum_{i=1}^{k} sl_c(L_i) + 2 \sum_{1 \leq i < j \leq k} \ell k(L_i,L_j).
\]

The statement follows by taking \( L' \) concordant to \( L \) and such that \( sl_{\text{max}}(L') = sl_c(L) \). \( \square \)

**Example 3.2.** Consider the link \( L_k \) with \( k \geq 1 \). By Proposition \([3.1]\) we have \( sl_c(L_k) \leq -2 \). By Theorem \([12]\) we have \( -\chi_4(L_k) = 2k - 2 \), thus \( L_k \) cannot be concordant to a quasi-positive link.

**Corollary 3.3.** If \( L \) has unknotted components, then \( sl_c(L) \leq 2\ell k(L) - \ell \).

We can now compute the concordance self-linking number of pure braid closures.
Proposition 3.4. If \( L = K_1 \cup ... \cup K_\ell \) is concordant to the closure of a pure braid, then
\[
\text{sl}_c(L) = 2\ell k(L) - \ell.
\]
In particular, if \( L \) is also concordant to a quasi-positive link, then \( \nu(L) = \ell k(L) \).

Proof. A pure braid \( \beta \) has as many strands as components in its closure. Furthermore, in \( \beta \) there are only crossings between strands belonging distinct components. It follows immediately that \( 2\ell k(L) - \ell = \text{sl}(\beta) \leq \text{sl}_c(L) \). The first part of the statement now follows from Corollary 3.3. The second part of the statement follows from Equation (3.1). □

As an application of Proposition 3.4, it is possible to identify all the closures of alternating pure braids which are concordant to a quasi-positive link.

Proposition 3.5. Let \( L \) be the closure of an alternating pure braid. Then, \( L \) is concordant to a quasi-positive link if, and only if, \( L \) is the split union of unknots and positive \((2, 2n)\)-torus links.

Proof. By [13, Corollary 7.6.4] we may assume, without loss of generality, that \( L \) is non-split.

From Lemma 2.6 follows that the diagram \( D_\beta \) obtained as the closure of a non-split alternating braid \( \beta \in B_n \) is such that \( s_+(D_\beta) \in \{ \lfloor n/2 \rfloor, \lceil n/2 \rceil \} \). The combinatorial bound in Proposition 2.4 is sharp for \( D_\beta \), therefore
\[
\text{sl}(\beta) + 2(s_+(D_\beta) - 1) = 2\nu(L) - \ell.
\]

If \( L \) is concordant to a quasi-positive link, then \( \text{sl}_c(L) = 2\nu(L) - \ell \). Moreover, Proposition 3.4 implies that pure braids realise the maximal self linking number. Thus, if \( \beta \) is an alternating pure braid, and if its closure is concordant to a quasi-positive link, then \( s_+(D_\beta) - 1 = 0 \).

Since \( s_+(D_\beta) \in \{ \lfloor n/2 \rfloor, \lceil n/2 \rceil \} \), either \( n = 1 \) and \( L \) is the unknot, or \( n = 2 \) and \( L \) is a \((2, n)\)-torus link. In the latter case, \( L \) is concordant to a quasi-positive link if, and only if, \( L \) is positive (otherwise it would contradict (1.1)). □

Corollary 3.6. The Borromean rings are not concordant to any quasi-positive link.

Proof. The Borromean rings are the closure of the non-split alternating pure 3-braid \((\sigma_1 \sigma_2^{-1})^3\). □

4. Positive and Positive-Braid Links

In this final section we prove the results concerning positive and positive-braid links.

4.1. Positive links. We call a link \emph{simply-linked} if it admits a diagram such that the number crossings between distinct components equals twice the absolute linking number. Positive and negative links are simply-linked. More in general, every link which has a diagram such that the crossings between every two fixed components have the same sign is simply-linked (and \emph{vice-versa}). The splitting numbers of simply-linked links are equal to the absolute linking number, more precisely we have the following.

Proposition 4.1. Let \( L \) be simply-linked. Then, we have the following equalities
\[
\bar{\text{sp}}(L) = \text{sp}(L) = |\ell k|(L) = u(L) - \sum_{i=1}^{\ell} u(K_i)
\]

Proof. By Lemma 2.6 and Lemma 2.7 it is sufficient to prove the inequality
\[
\bar{\text{sp}}(L) \leq |\ell k|(L).
\]

The result follows by noticing that all diagrams can be reduced to the diagram of a split union of knots by changing at most half of the crossings between different components. The latter fact is easy to prove, and hence left to the reader. □
Proof of Theorem 1.7. For all positive links we have $|\ell_L| = \ell_L$. Since positive links are simply-linked, the result follows directly from Proposition 4.1.

Proposition 4.2. If $L$ is a positive link with $\ell_L = 1$ and two unknotted components, then $L$ is the positive Hopf link.

Proof. Let $D$ be a positive diagram representing $L$. Since $\ell_L = 1$ there are only two crossings in $D$ which involve both components. It follows that $D$ must be of the form illustrated in Figure 5 and therefore represents the connected sum of the Hopf link, a knot $K_1$ on one components, and a knot $K_2$ on the other component. Because the components of $L$ are unknotted, the knots $K_1$ and $K_2$ must be trivial, and the statement follows.

![Figure 5](image.png)

Figure 5. The depiction of two positive diagrams of a link with two components, and linking number one. Each box indicate a (possibly trivial) $(1,1)$-tangle.

Proof of Proposition 1.8. Let $D$ be a positive diagram representing a link $L = K_1 \cup \ldots \cup K_\ell$, and assume $L$ to have unlinking number one. By Theorem 1.7 we have that

\begin{equation}
\ell_L + \sum_i u(K_i) = u(L) = 1.
\end{equation}

Since the unknotting number is non-negative, one obtains that $\ell_L \leq 1$. Positive links have non-negative linking numbers, therefore $\ell_L \in \{0,1\}$. If $\ell_L = 0$, then $\tilde{sp}(L) = 0$ and $\sum_i u(K_i) = u(L) = 1$. It follows that $L$ is the split union of knots, and precisely one among the $K_i$ has unknotting number one. Therefore, $L$ is the split union of an unlink and a positive knot whose unknotting number is one – which is a twist knot by Nakamura’s result.

If $\ell_L = 1$, then (4.2) implies that all components have unknotting number 0, and therefore are unknotted. Moreover, since positive diagrams are simply-linked, $D$ has precisely two crossing between distinct components. It follows that $L$ is the split union of an unlink, and a positive link with two unknotted components and linking number 1. Hence, by Proposition 4.2, $L$ is the split union of an unlink and a positive Hopf link.

The proof of Proposition 1.9 proceeds similarly to the proof of Proposition 1.8.

Proof of Proposition 1.9. Let $D$ be a positive diagram representing a link $L = K_1 \cup \ldots \cup K_\ell$, and assume $L$ to have unlinking number two. Theorem 1.7 implies

\begin{equation}
\ell_L + \sum_i u(K_i) = u(L) = 2,
\end{equation}

and thus $\ell_L \in \{0,1,2\}$.

If $\ell_L = 2$ then all components must have unknotting number 0, and therefore be unknotted.
If \( \ell_k(L) = 1 \) then \( D \) must be the union of some knot diagrams and a diagram of the form shown in Figure 5. Only one among the components of \( L \) has unknotting number one: thus, either \( L \) is the split union of an unlink, a twist knot, and an Hopf link, or \( L \) is the split union of an unlink and the connected sum of an Hopf link and a twist knot.

Finally, if \( \ell_k(L) = 0 \) then \( L \) is the split union of knots by Theorem 1.7. In this case, the sum of the unknotting numbers of the components of \( L \) is 2. Hence, either a single \( K_i \) has unknotting number two, or there exists \( K_i \) and \( K_j \) with \( i \neq j \) which are twist knots.

What is left is to show that there are infinitely many knot for each case. Since it is known that there are infinitely many twist knots, it is sufficient to show that there are infinitely many

(i) positive links with linking number two and trivial components;

(ii) positive knots with unknotting number 2.

In Figure 6 is represented a link with unkotted components and linking number 2. Since the diagram illustrated is alternating and reduced, it has a minimal number of crossings \([14, 19, 27]\). Thus, Figure 6 represents distinct links for distinct values of \( k \in \mathbb{N} \), which proves that family (i) is infinite.

Now, we shall argue that \( u(K \# K') = 2 \) when \( K \) and \( K' \) are positive twist knots. Then, family (ii) must be infinite since there are infinitely-many twist knots. The unlinking number is sub-additive, thus \( u(K \# K') \leq u(K) + u(K') = 2 \). The sum of two positive knots is a positive knot. Hence \( u(K \# K') = 1 \) if, and only if, \( K \# K' \) is a twist knot by \([20\text{ Theorem 5.1}]\). However, twist knots are prime, and therefore it must be \( u(K \# K') \geq 2 \).

\[\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{figure6.png}
\caption{An alternating positive link \( L \) with unkotted components and \( \ell_k(L) = 2 \). The box indicates the presence of \( k \) negative full twists.}
\end{figure}\]

4.2. **Positive-braid links.** In this final subsection we prove Theorem 1.10 and its consequences.

*Proof of Theorem 1.10.* Let \( \beta \) be a positive braid representing \( L = L_1 \cup \ldots \cup L_k \). Denote by \( \beta_i \) the braid associated to the sub-link \( L_i \). Since both \( \beta \) and the \( \beta_i \)'s are (quasi-)positive braids, \( 2\nu(L) - \ell = sl(\beta) \) and \( 2\nu(L_i) - \ell_i = sl(\beta_i) \). Therefore, by Lemma 2.3 we obtain

\[
\sum_{i=1}^{k} \nu(L_i) = \sum_{1 \leq i < j \leq k} \ell_k(L_i, L_j).
\]

Comparing (4.4) with Theorem 1.7 we have

\[
\nu(L) - \sum_{i=1}^{\ell} \nu(K_i) = u(L) - \sum_{i=1}^{\ell} u(K_i).
\]
where $K_1, \ldots, K_\ell$ denote the components of $L$. Each $L_i$ is the union of some components of $L$, say $K_{r_1,i} \cup \cdots \cup K_{r_\ell,i}$. Since each $L_i$ is a positive-braid link, we have

$$u(L) - \sum_{i=1}^{\ell} u(L_i) = u(L) - \sum_{i=1}^{\ell} u(K_i) - \sum_{i=1}^{\ell} \left( u(L_i) - \sum_{j=1}^{\ell_i} u(K_{r_j,i}) \right) = \nu(L) - \sum_{i=1}^{\ell} \nu(L_i),$$

where we applied (4.5) to both $L$ and the $L_i$. 

Example 4.3. In Figure 7 is depicted a positive diagram of a link $L = U \cup T$, where $U$ is an unknot (in red) and $T$ a positive trefoil (in blue). By Proposition 2.4 we have $\nu(L) = 4$, $\nu(U) = 0$, and $\nu(T) = 1$. It is known that $u(U) = 0$ and $u(T) = 1$, thus $u(L) = 5$ by Theorem 1.7. Hence $L$ is not a positive-braid link by Theorem 1.10. In particular, $L$ is an example of a fibred positive non-algebraic link with algebraic components.

Kawamura computed the unlinking number of positive-braid links; that is, he proved the following.

**Proposition 4.4** ([16, Theorem 1.3]). Let $L$ be a positive-braid link, and $D$ a diagram obtained from closing a positive braid representing $L$. Then,

$$2u(L) - \ell = x(D) - o(D) + \ell,$$

where $x$ denotes the number of crossings.

Comparing Proposition 4.4 with [8, Theorem 1.3] (cf. Proposition 2.4) one obtains that

(4.6)  

$$u(L) = \nu(L)$$

for each positive-braid link $L$, and each slice-torus link invariant $\nu$. This observation, together with Theorem 1.7 implies Theorem 1.10. Moreover, since $u = \nu$ for positive-braid links, it follows from [8, Theorem 1.3] that the unlinking number of a positive-braid link can be computed using the same formula from any positive diagram, and not only from the closures positive braids.

**Remark 4.5.** A quick check on Knotinfo [17] shows that there are 16 positive prime knots with less than 12 crossings which can be shown not to be positive-braid knots thanks to Equation (4.6).
Proof of Proposition 4.6. Let $D$ be a positive diagram of a positive-braid link $L = K_1 \cup \ldots \cup K_t$. Denote by $D_i$ the diagram obtained from $D$ by deleting the strands belonging to all the components of $L$ but $K_i$. Since the components of a positive-braid link are positive-braid knots, what we observed above implies $2u(L) - \ell = x(D) - o(D) + \ell$, and $2u(K_i) - 1 = x(D_i) - o(D_i) + 1$, for each $i$. By Theorem 1.7 we have

$$u(L) - \sum_i u(K_i) = \ell k(L).$$

On the other hand, a simple observation akin to the proof of Lemma 2.3 shows that in this case

$$x(D) - \sum_i x(D_i) = w(D) - \sum_i w(D_i) = 2\ell k(L).$$

Thus we obtain

$$x(D) - \sum_i x(D_i) = 2\ell k(L) = 2(u(L) - \sum_i u(K_i)) = x(D) - o(D) - \sum_i (x(D_i) - o(D_i)).$$

It follows that $o(D) - \sum_i o(D_i) = 0$, which concludes the proof.

Finally, combining Kawamura’s result and Theorem 1.7 one obtains a formula for the unlinking number for positive links with positive-braid components.

**Proposition 4.6.** Let $L = K_1 \cup \ldots \cup K_t$ be a positive link. Denote by $D_i$ the diagram obtained by deleting all the components but $K_i$ from $D$. If $K_1, \ldots, K_t$ are positive braid knots, then

$$u(L) = \frac{x(D) - \sum_i o(D_i) + \ell}{2},$$

where $x$ denotes the number of crossings.

**Proof.** Since $L$ is positive, it follows from Theorem 1.7 that

$$2u(L) = 2 \sum_i u(D_i) + 2\ell k(L) = \sum_i 2u(D_i) + (x(D) - \sum_i x(D_i)).$$

By Proposition 4.4 we have that $2u(K_i) - 1 = x(D_i) - o(D_i) + 1$ for each $i$. Therefore

$$2u(L) = \sum_i (x(D_i) - o(D_i) + 1) + (x(D) - \sum_i x(D_i)) = x(D) - \sum_i o(D_i) + \ell.$$
and that the rest of the components. Therefore, the braid associated to $L$ is unknotted component $U$.

Finally, we have to show that the link $L'$ can be obtained removing all the $w_r$ from $\beta'$ thus re-obtaining $\beta$.

\begin{proof}
Let $\beta = \sigma_{i_1}^{\varepsilon_1} \cdot \ldots \cdot \sigma_{i_k}^{\varepsilon_k} \in B_n$, be a braid representative for $L$, where $\varepsilon_1, \ldots, \varepsilon_k \in \{\pm 1\}$. Our aim is to define a new braid $\beta' \in B_{n+1}$ which is quasi-positive, and whose closure contains $L$ as a sub-link. First, define the following braid

$$w_r = \begin{cases} 
\sigma_{n+1} \cdot \ldots \cdot \sigma_{i_r} \cdot \sigma_{i_r} \cdot \ldots \cdot \sigma_{n+1} & \text{if } \varepsilon_r = -1 \\
1 & \text{otherwise}
\end{cases}, \text{ for each } r \in \{1, \ldots, k\}$$

Notice that the permutation associated to each $w_r$ is trivial. Now, define

$$\beta' = (\sigma_{i_1}^{\varepsilon_1} w_1) \cdot \ldots \cdot (\sigma_{i_k}^{\varepsilon_k} w_k) \in B_{n+1}.$$ 

Clearly, the permutation associated to $\beta'$ fixes $n + 1$, and behaves as the permutation associated to $\beta$ on $\{1, \ldots, n\}$. This fact implies that the closure of $\beta'$ has an unknotted component $U$ which starts and ends on the $(n + 1)$-th strand. The braid $\beta'$ is quasi-positive since it is a product of quasi-positive braids. Indeed, we have

$$\sigma_{i_r}^{\varepsilon_r} w_r = \begin{cases} 
\sigma_{i_r}^{-1}(\sigma_{n+1} \cdot \ldots \cdot \sigma_{i_r}) \sigma_{i_r} (\sigma_{i_r} \cdot \ldots \cdot \sigma_{n+1}) & \text{if } \varepsilon_r = -1 \\
\sigma_{i_r} & \text{otherwise}
\end{cases}.$$ 

Finally, we have to show that the link $L$ is a sub-link of the closure $L'$ of $\beta'$. We remark that the unknotted component $U$ has no self-crossings in $\beta'$, and the $w_r$ are precisely the crossings between $U$ and that the rest of the components. Therefore, the braid associated to $L \setminus U$ is obtained removing all the $w_r$ from $\beta'$ thus re-obtaining $\beta$.
\end{proof}