A Cosmological Model with Dark Spinor Source

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Abstract

In this paper, we discuss the system of Friedman-Robertson-Walker (FRW) metric coupling with massive nonlinear dark spinors in detail, where the thermodynamic movement of spinors is also taken into account. The results show that, the nonlinear potential of the spinor field can provide a tiny negative pressure, which resists the Universe to become singular. The solution is oscillating in time and closed in space, which approximately takes the following form

\[ g_{\mu\nu} = \tilde{R}^2 (1 - \delta \cos t)^2 \text{diag}(1, -1, -\sin^2 r, -\sin^2 r \sin^2 \theta), \]

with \( \tilde{R} = (1 \sim 2) \times 10^{12} \) light year, and \( \delta = 0.96 \sim 0.99 \). The present time is about \( t \sim 18^\circ \).

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1 Introduction

Combined with the fluid model, the general relativity usually gives the cosmological models with singularity. After a series of singularity theorems, which is based on the conditions such as positive energy, closed trapped surface, have been proved by Penrose, Hawking and so on[1, 2, 3, 4], it seems that the singularity has been quite naturally accepted as an objective existence. However the intrinsic singularities are after all abnormal and inconceivable. The fluid model is just a simplified description of matter, so one is unnecessary to take it too serious if the intrinsic essences of the spacetime and matter are involved.

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The global solutions of Einstein’s equation are always interesting[5]-[11]. The Lemaitre’s Phoenix picture, a regular and oscillating cosmological model is reconsidered in [12, 13], it can give a more natural explanation for some famous cosmological puzzles such as the flatness and horizon problems. However this model is obtained by hypothesis rather than by solution of Einstein equation. Recently the coupling system of spinor field with space-time has been studied by several authors. The principal motivation of the works was to find out the regular solutions of the corresponding field equations[14]-[17] or explain the accelerating expansion of the present Universe[18]-[21]. These works indeed obtained singularity-free solutions, and showed that the nonlinear spinor field results in a rapid expansion of the Universe.

In what follows, we also investigate the cosmological model with nonlinear spinor fields, which is derived from a general framework of field theory[22]-[25]. Different from the treatment of [15]-[21], in which the authors introduced only one spinor field independent of spatial coordinates, and suffered from the energy momentum tensor incompatible with the high symmetry of the Friedman-Robertson-Walker(FRW) metric. In our point of view, one spinor field just describes one particle and the high symmetry is a result in mean sense. Then all problems can be easily solved, and the result is quite natural and reasonable.

2 The Basic Equations

A large number of empirical data shows that our Universe is highly homogeneous and isotropic. So the Universe can be described by FRW metric in mean sense, the corresponding line element is given by

\[ ds^2 = d\tau^2 - a^2(\tau) \left( \frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2 \theta d\varphi^2) \right), \]  

where \( K = 1, 0 \) and \(-1\) correspond to the closed, flat and open Universe respectively. However this form of FRW metric is not convenient for analysis, and the solution is difficult to expressed by elementary functions. so in this paper we adopt the conformal coordinate system[13]. The corresponding metric becomes

\[ g_{\mu\nu} = a^2(t)\text{diag}(1, -1, -f^2(r), -f^2(r) \sin^2 \theta), \]  

where \( d\tau = a(t)dt \),

\[ f = \begin{cases} 
\sin r & \text{if } K = 1, \\
r & \text{if } K = 0, \\
\sinh r & \text{if } K = -1.
\end{cases} \] (2.3)

The Lagrangian of the gravitational field is generally given by[26]

\[ \mathcal{L}_g = \frac{1}{16\pi G} (R - 2\Lambda), \] (2.4)
where $R$ is the scalar curvature, $\Lambda$ the cosmological factor. For the metric (2.2), we have the scalar curvature and Einstein tensor $G_{\mu \nu}$ as follows

\begin{align*}
R &= 6 \frac{a''}{a^3} + Ka, \\
G_{00} &= -3\left( \frac{a'^2}{a^2} + K \right), \\
G_{11} &= 2 \frac{a''}{a} - \frac{a'^2}{a^2} + K, \\
G_{22} &= G_{11} f^2, \quad G_{33} = G_{11} f^2 \sin^2 \theta,
\end{align*}

(2.5)-(2.8)

where $a' = \frac{d}{dt} a$, $f' = \frac{d}{df} f$.

Now we construct the Lagrangian of matter. Denote the Pauli matrices by

\begin{equation}
\bar{\sigma} = (\sigma^k) = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.
\end{equation}

(2.9)

Define $4 \times 4$ Hermitian matrices by

\begin{align*}
\alpha^\mu &= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \quad \begin{pmatrix} 0 & \bar{\sigma} \\ \bar{\sigma} & 0 \end{pmatrix}, \\
\beta &= \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},
\end{align*}

(2.10)

Instead of Dirac matrices $\gamma^\mu$, we adopt the above Hermitian matrices (2.10) for the convenience of calculation. Obviously, the most part of matter should be described by spinors. For some reasons, the following Lagrangian is the most natural candidate for matter fields, especially for dark matter. In flat spacetime the Lagrangian is given by

\begin{equation}
\mathcal{L} = \sum_k \phi_k^+(\alpha^\mu i \partial_\mu - \mu_k \gamma) \phi_k + F(\bar{\alpha}_k^\mu, \bar{\beta}_k, \bar{\gamma}_k),
\end{equation}

(2.11)

where we assign $\phi_k$ to the $k$-th dark spinor $S_k$, $\mu_k > 0$ is constant mass, which takes one value for the same kind particles,

\begin{equation}
\bar{\alpha}_k^\mu = \phi_k^+ \alpha^\mu \phi_k, \quad \bar{\beta}_k = \phi_k^+ \beta \phi_k, \quad \bar{\gamma}_k = \phi_k^+ \gamma \phi_k,
\end{equation}

(2.12)

$F$ is the nonlinear coupling term, which is small in value ($|F| \ll \mu_k$), but decisive for the structure of matter[24], [27]-[30] and the Universe.

Considering that all the particle-like solutions of the nonlinear spinor equation have a mean diameter less than $10^2$ Compton wave lengths, and all $|\phi_k|$ decay exponentially with respect to the distance from the center[24, 29, 30], so we can neglect the nonlinear interaction terms among spinors, but keep the self coupling terms only. By the Pauli-Fierz identities $\bar{\alpha}_k^\mu \alpha_k = \bar{\beta}_k^2 + \bar{\gamma}_k^2[31]$ and the stability of the fields, $F$ should take the following form

\begin{equation}
F = \sum_k V_k, \quad V_k \equiv V(\bar{\gamma}_k),
\end{equation}

(2.13)
where the self-coupling potential $V$ is a differentiable, even and concave function satisfying
\[ V(0) = 0, \quad V'(x)x - V(x) > 0. \] (2.14)

The covariant form of spinor equation in curved spacetime have studied by many authors[32]-[40]. The generalized form of (2.11) with diagonal metric is given by
\[ \mathcal{L}_m = \sum_k \left( \Re \left( \phi^+_k \bar{g}^\mu i \partial_\mu \phi_k \right) - \mu_k \bar{\gamma}_k + V_k \right), \] (2.15)
in where
\[ \bar{g}^\mu = \begin{pmatrix} \alpha^0/a, & \alpha^1/a, & \alpha^2/af, & \alpha^3/af \sin \theta \end{pmatrix}. \] (2.16)
The variation of (2.15) with respect to $\phi^+_k$ gives dynamic equation for $S_k[39, 40]$\[ \bar{g}^\mu i \nabla_\mu \phi_k = (\mu_k - V_k) \gamma \phi_k, \quad (\forall k) \] (2.17)
where $\nabla_\mu = (\partial_\mu + \Upsilon_\mu)$, $\Upsilon_\mu$ is the spinor connection
\[ \Upsilon_\mu = \begin{pmatrix} 3a'/2a, & f'/f, & 1/2 \cot \theta, & 0 \end{pmatrix}. \] (2.18)

Coupling (2.4) with (2.15), we get the total Lagrangian of the Universe with dark spinors
\[ \mathcal{L} = \frac{1}{16\pi G} (R - 2\Lambda) + \sum_k \left( \Re \left( \phi^+_k \bar{g}^\mu i \partial_\mu \phi_k \right) - \mu_k \bar{\gamma}_k + V_k \right). \] (2.19)
Generally the variation (2.19) with respect to $g_{\mu\nu}$ gives the Einstein equation\[ G^{\mu\nu} + \Lambda g^{\mu\nu} + 8\pi G T^{\mu\nu} = 0. \] (2.20)
In the spacetime with orthogonal coordinates, the energy momentum tensor $T^{\mu\nu}$ is relatively simple[38, 40]
\[ T^{\mu\nu} = \sum_k \left( \frac{1}{2} \Re \left( \phi^+_k (\bar{g}^\mu i \partial_\nu + \bar{g}^\nu i \partial_\mu) \phi_k \right) - \mathcal{L}_m \bar{g}^{\mu\nu} \right) \]
\[ = \sum_k \left( \frac{1}{2} \Re \left( \phi^+_k (\bar{g}^\mu i \nabla_\nu + \bar{g}^\nu i \nabla_\mu) \phi_k \right) + (V_k' \bar{\gamma}_k - V_k) \bar{g}^{\mu\nu} \right). \] (2.21)
In (2.21) we substituted the dynamic equation (2.17) into $\mathcal{L}_m$.

For the spinor at particle state[25, 41], we have the following classical approximation
\[ \phi^+_k \bar{g}^\mu \phi_k \rightarrow u_k^\mu \sqrt{1 - v_k^2} \delta(\vec{x} - \vec{X}_k), \] (2.22)
\[ i \nabla_\mu \phi_k \rightarrow p_k^\mu \phi_k = m_k u_k^\mu \phi_k \] (2.23)
where $\vec{X}_k$ is the central coordinate of $S_k$, $u_k^\mu$ is its 4-vector speed. Then the classical approximation of (2.21) provides the energy momentum tensor of the ideal gas with a tiny negative pressure term
\[ T^{\mu\nu} = \sum_k (m_k u_k^\mu u_k^\nu + W_k g^{\mu\nu}) \sqrt{1 - v_k^2} \delta^3(\vec{x} - \vec{X}_k), \] (2.24)
where \( W_k = \int_{R^3} (V'_k \tilde{\gamma}_k - V_k) d^3\overline{x} > 0 \) is the proper energy of \( S_k \) contributed by \( V_k \), which acts as negative pressure. The details of the classical approximation and local Lorentz transformation see [41, 42]. The numerical results show that \( 0 < W_k \ll m_k \), but it keep a spinor to be an independent particle[24, 29, 30]. In what follows we see how this wizardly tiny term resists the Universe from singularity.

3 Solution to the Equation

The complete Einstein equation (2.20) is an overdetermined system under the assumption of FRW metric, because the FRW metric only holds in mean sense. For an overdetermined system we should appeal to the variation principle. The action corresponding to (2.19) is given by

\[
I = \int dt \int_\Omega L a^4 d\Omega, \tag{3.1}
\]

where \( d\Omega = f^2 \sin \theta d\theta d\phi \) is the angular volume independent of \( t \). Variation (3.1) with respect to \( a \), we get Euler equation as follows

\[
\frac{3a'' + 3Ka - 2\Lambda a^3}{4\pi G} + \sum_{X_k \in \Omega} \left( 3 \Re \left( \phi_k^\dagger \gamma^\mu i \nabla_\mu \phi_k \right) - 4 \mu_k \tilde{\gamma}_k + 4V_k \right) a^3 d\Omega = 0, \tag{3.2}
\]

where \( \Omega \) is any angular volume large enough to contain sufficient particles, such that the statistical principle is valid. Substituting (2.17) into (3.2) we get

\[
a'' + Ka - \frac{2}{3} \Lambda a^3 = \frac{4\pi G}{3\Omega} \sum_{X_k \in \Omega} (\mu_k \tilde{\gamma}_k + 3V'_k \tilde{\gamma}_k - 4V_k) a^3 d\Omega. \tag{3.3}
\]

Define the proper mass of \( S_k \) by

\[
m_k = \int_{R^3} (\mu_k \tilde{\gamma}_k + 3V'_k \tilde{\gamma}_k - 4V_k) d^3\overline{x}_k, \tag{3.4}
\]

where \( \overline{x}_k \) is the local Cartesian coordinates of the central coordinate system of \( S_k \). Obviously \( m_k \) is a constant independent of \( t \). Substituting it into (3.3) and making local Lorentz transformation[41], we get

\[
a'' + Ka - \frac{2}{3} \Lambda a^3 = \frac{4\pi G}{3\Omega} \sum_{X_k \in \Omega} m_k \sqrt{1 - v_k^2}. \tag{3.5}
\]

The drifting speed \( v_k \) depends on \( a(t) \), solving the geodesic of \( S_k \) we have[43]

\[
v_k = \frac{b_k}{\sqrt{a^2 + b_k^2}}, \quad \text{or} \quad \sqrt{1 - v_k^2} = \frac{a}{\sqrt{a^2 + b_k^2}}, \tag{3.6}
\]

where \( b_k \) is a constant determined by initial speed. Substituting (3.6) into (3.5), we get

\[
a'' + Ka - \frac{2}{3} \Lambda a^3 = \frac{4\pi G}{3\Omega} \sum_{X_k \in \Omega} \frac{m_k a}{\sqrt{a^2 + b_k^2}}. \tag{3.7}
\]
Multiply (3.7) by $a'$ and integrate it, again by (3.6) we have
\[ a'^2 + Ka'^2 - \frac{1}{3} \Lambda a'^4 = \frac{8\pi G}{3\Omega} \sum_{x_k \in \Omega} \frac{m_k a}{\sqrt{1 - v_k^2}} + C_0. \tag{3.8} \]

Now we examine the meaning of $C_0$. By (3.8), (2.5)-(2.8) and (2.21) we have
\[ C_0 = a'^2 + Ka'^2 - \frac{1}{3} \Lambda a'^4 - \frac{8\pi G}{3\Omega} \sum_{x_k \in \Omega} \frac{m_k a}{\sqrt{1 - v_k^2}} \approx \frac{8\pi G}{3} \left( a'^2 T_0^0 - \frac{1}{\Omega} \sum_{x_k \in \Omega} \frac{m_k a}{\sqrt{1 - v_k^2}} \right). \tag{3.9} \]

The equivalent pressure for stationary $S_k$ is given by\[ P_k \equiv \frac{1}{3} (T_0^0 - T_\mu^\mu) = \frac{1}{3} \left( \phi_k^+ \alpha_0^0 \bar{\alpha} \phi_k - \mu_k \bar{\gamma}_k - 2V_k' \bar{\gamma}_k + 3V_k \right), \tag{3.10} \]
where $\bar{x}_k$ stands for the central coordinate system of $S_k$. The numerical results show that for ground state
\[ \int_{R^3} P_k d^3 \bar{x}_k < 0. \tag{3.11} \]

In statistical sense, we have
\[
M_k = \int_{R^3} \Re \left( \phi_k^+ \phi_0^0 \nabla_0 \phi_k \right) a^3 d\Omega \\
= \int_{R^3} \frac{1}{\sqrt{1 - v_k^2}} \left( (1 + \frac{1}{3} v_k^2) \phi_k^+ \phi_0^0 \nabla_0 \phi_k - \frac{1}{v_k^2} \phi_k^+ \alpha_0^0 \bar{\alpha} \phi_k \right) d^3 \bar{x}_k \\
= \int_{R^3} \frac{1}{\sqrt{1 - v_k^2}} \left( (1 + \frac{1}{3} v_k^2) 3P_k + \mu_k \bar{\gamma}_k + 2V_k' \bar{\gamma}_k - 3V_k \right) - \frac{1}{3} v_k^2 (\mu_k - V_k) d^3 \bar{x}_k \\
= \int_{R^3} \frac{1}{\sqrt{1 - v_k^2}} \left[ (3 + v_k^2) P_k + \mu_k \bar{\gamma}_k + 2V_k' \bar{\gamma}_k - 3V_k + v_k^2 (V_k' \bar{\gamma}_k - V_k) \right] d^3 \bar{x}_k,
\]
where we also made local Lorentz transformation for the integrals of $S_k$. Then we get
\[
T_0^0 = \frac{1}{a' \Omega} \int_{R^3} \sum_{x_k \in \Omega} \left( \Re \left( \phi_k^+ \phi_0^0 \nabla_0 \phi_k \right) + V_k' \bar{\gamma}_k - V_k \right) a^3 d\Omega \\
= \frac{1}{a' \Omega} \sum_{x_k \in \Omega} \left( M_k + \int_{R^3} (V_k' \bar{\gamma}_k - V_k) \sqrt{1 - v_k^2} d^3 \bar{x}_k \right) \\
= \frac{1}{a' \Omega} \sum_{x_k \in \Omega} \int_{R^3} \frac{1}{\sqrt{1 - v_k^2}} \left[ (3 + v_k^2) P_k + \mu_k \bar{\gamma}_k + 3V_k' \bar{\gamma}_k - 4V_k \right] d^3 \bar{x}_k. \tag{3.12}
\]

Substituting (3.4) and (3.12) into (3.9) we get
\[ C_0 \approx \frac{8\pi Ga}{3\Omega} \sum_{x_k \in \Omega} \frac{3 + v_k^2}{\sqrt{1 - v_k^2}} \int_{R^3} P_k d^3 \bar{x}_k < 0. \tag{3.13} \]

So $C_0$ is a negative quantity related to the nonlinear potential $V$. Of course (3.13) can not be used as definition in (3.8), because $C_0^2$ and $T_0^0$ are “overdetermined functions” with respect to FRW metric.

The kinetic energy of $S_k$ is calculated by
\[ K_k = \frac{m_k^2}{\sqrt{1 - v_k^2}} - m_k. \tag{3.14} \]
Although the spinors are fermions, but if the mean distance among them is much larger than their mean radiuses, then they are all at particle states and the effects of exclusion principle actually vanish. So the kinetic energy of such spinors should satisfy the classical distribution, rather than Fermi-Dirac statistics. Considering the case of relativity, we have the following Maxwell-like distribution

\[ f(K) dK = \sqrt{\frac{4K}{\pi kT^3}} \exp \left(-\frac{K}{kT}\right) dK. \] (3.15)

Substituting (3.14) and (3.15) into (3.8), we get

\[ a'^2 + Ka^2 - \frac{1}{3} \Lambda a^4 = \frac{8\pi Ga}{3\Omega} \sum_{x_k \in \Omega} \int_0^\infty (K + m_k) f(K) dK + C_0 \]
\[ = \frac{8\pi Ga}{3\Omega} \sum_{x_k \in \Omega} \left( \frac{3}{2} kT + m_k \right) + C_0. \] (3.16)

In [43] we derived the relation \( T(a) \) as follows

\[ \frac{1}{2} kT = \frac{\bar{m}b^2}{5a(a + \sqrt{a^2 + b^2})}, \] (3.17)

where \( \bar{m} \) is mean mass of all particles. \( b \) is a constant determined by the initial data

\[ \frac{b}{a_0} = \sqrt{\frac{5kT_0}{\bar{m}}} \left( 1 + \frac{5kT_0}{4\bar{m}} \right) \ll 1, \quad (1eV \sim 10^4 K). \] (3.18)

Substituting (3.17) into (3.16) we finally get

\[ a'^2 + Ka^2 - \frac{1}{3} \Lambda a^4 = 2\bar{R} \left( \frac{3b^2}{5(a + \sqrt{a^2 + b^2})} + a \right) + C_0, \] (3.19)

where

\[ \bar{R} = \frac{4\pi G}{3\Omega} \sum_{x_k \in \Omega} m_k \] (3.20)

is the radius scale of the Universe, which is independent of \( a \).

In [44], the influences of the constants in (3.19) on the solution were analyzed in detail. The main results are that: (R1). \( b \ll a_0 \) mainly influences the behavior as \( a \to 0 \), it has little influence on the present Universe. (R2). \( \Lambda \) mainly influences the behavior as \( a \geq \bar{R} \), it seems to be superfluous term without any effective use, so \( \Lambda = 0 \) is the most natural choice. (R3). Eq.(3.19) has singularity-free solution as long as \( C_0 < -\frac{6}{5} \bar{R}b \).

According to these results, we set \( \Lambda = b = 0 \) for the following computation, the detailed computation will be given elsewhere. Then (3.19) becomes

\[ a'^2 + Ka^2 = 2\bar{R}a + C_0. \] (3.21)
Solving (3.21), we get the singularity-free solutions as follows

\[
a = \begin{cases} 
\bar{R}(1 - \delta \cos t), & (C_0 = -\bar{R}^2(1 - \delta^2), \ 0 < \delta < 1), \text{ if } f = \sin r, \\
\bar{R}(\delta + \frac{1}{2}t^2), & (C_0 = -2\bar{R}^2\delta, \ \delta > 0), \text{ if } f = r, \\
\bar{R}(\delta \cosh t - 1), & (C_0 = -\bar{R}^2(\delta^2 - 1), \ \delta > 1), \text{ if } f = \sinh r.
\end{cases}
\] (3.22)

Calculating the energy-momentum tensor \(T_{\mu\nu}\) by \(8\pi G T_{\mu\nu} = -G_{\mu\nu}\), we have

\[
T_{00} = \frac{3(1 - 2\delta \cos t + \delta^2)}{8\pi G(1 - \delta \cos t)^2}, \quad T_{11} = \frac{\delta^2 - 1}{8\pi G(1 - \delta \cos t)^2}, \text{ if } f = \sin r, \tag{3.23}
\]

\[
T_{00} = \frac{3\delta^2}{8\pi G(\delta + \frac{1}{2}t^2)^2}, \quad T_{11} = \frac{-2}{8\pi G(\delta + \frac{1}{2}t^2)^2}, \text{ if } f = r, \tag{3.24}
\]

\[
T_{00} = \frac{3(2\delta \cosh t - 1 - \delta^2)}{8\pi G(1 - \delta \cosh t)^2}, \quad T_{11} = \frac{1 - \delta^2}{8\pi G(1 - \delta \cosh t)^2}, \text{ if } f = \sinh r. \tag{3.25}
\]

From the above equations we learn that, only for the closed case the energy \(T_{00}\) is always positive, so only this solution is meaningful in physics. Moreover we find \(T_{11} < 0\) for all cases. \(T_{11} < 0\) is inconceivable for perfect fluid model, but from (2.21) and (2.24), we learn that it is normal for nonlinear spinor field.

### 4 Sketch of the Universe

In this section we associate the closed case in (3.22) with some observational data and give a sketch of the Universe. The detailed analysis will be given elsewhere. Define the cosmological time by

\[
\tau = \int_0^t a(t)dt = \bar{R}(t - \delta \sin t). \tag{4.1}
\]

Then the age of the Universe \(T\) reads

\[
T = \bar{R}(t_a - \delta \sin t_a), \tag{4.2}
\]

where \(t_a\) stands for the age in angular coordinate system. The Hubble’s parameter is defined by

\[
H = \frac{da}{adt} = \frac{a'}{a^2} = \frac{\delta \sin t_a}{\bar{R}(1 - \delta \cos t_a)^2}. \tag{4.3}
\]

The mass density \(\rho\) and the pressure \(P\) in usual sense are defined by

\[
\rho = T^0_0 = \frac{3}{8\pi GR^2} \cdot \frac{1 - 2\delta \cos t_a + \delta^2}{(1 - \delta \cos t_a)^4}. \tag{4.4}
\]

\[
P = -T^1_1 = \frac{-1}{8\pi GR^2} \cdot \frac{1 - \delta^2}{(1 - \delta \cos t_a)^4}. \tag{4.5}
\]

Denote the critical density \(\rho_c = \frac{3}{8\pi G}H^2 \sim 8 \times 10^{-30}\text{g/cm}^3\), we have

\[
\Omega_{\text{tot}} = \frac{\rho}{\rho_c} = 1 + \frac{a^2}{a'^2} = \frac{1 - 2\delta \cos t_a + \delta^2}{(\delta \sin t_a)^2}. \tag{4.6}
\]
The equation of state is defined by
\[ w = \frac{P}{\rho} = -\frac{1 - \delta^2}{3(1 - 2\delta \cos t_a + \delta^2)}. \] (4.7)

By (4.7) we find \( w(t_a) < 0 \) is an increasing function for \( 0 \leq t_a < \pi \), and
\[ w \to -\frac{1 + \delta}{3(1 - \delta)}, \quad \text{as} \quad t_a \to 0. \]

So \( w \) of spinors is a function crossing the value \(-1\).

There are several authoritative empirical data[45]-[53]. We take the data in [51] as basic parameters for computation.
\[ T = 14.2 \text{(Gyr)}, \quad \Omega_{\text{tot}} = 1.02 \sim 1.08, \quad \omega = -1 \sim -0.2. \] (4.8)

By computation we get
\[ t_a = 12^\circ \sim 22^\circ, \quad \delta = 0.96 \sim 0.99, \quad \bar{R} = (1.0 \sim 2.0) \times 10^{12}\text{yr}. \] (4.9)

By \( t_a \) we learn that the Universe is very young. The radius of the present Universe is about \( \pi \bar{R}(1 - \delta \cos(t_a)) \sim 300\text{Gyr} \). The period for a photon to travel a cycle in the Universe is just the life period of the Universe herself, so we can never observe the ghost of a galaxy at any time. However, by the functional digraphes of \( H(t) \), \( \Omega_{\text{tot}} \) and \( w(t) \), we find they are rapidly varying functions of \( t \) of the past, then consequently, the observational data strongly depend on the distances of the galaxies. So the parameters provided here could only be used as rough scales, the more values should be obtained dynamically under the aid and constraint of specific cosmological models.

From (3.22) we learn that, instead of the initial singularity there is a cradle of rebirth \( t = 0 \). At this time, the drifting speed of the particles or equivalently the cosmic temperature takes the maximum, and the volume of the Universe takes the minimum. This situation will result in violent collision among galaxies and particles. Such collisions will recover the vitality of the dead stars and combined particles. Just like the rebirth of the Phoenix, the Universe gets her rebirth via the destructive high temperature.

Of course the above analysis are based on the idealized model of particles. What is important for this paper is that, it reveals the solutions of the Einstein’s equation are sensitively related to the properties of the source. The interactional mechanism between the Universe and fundamental particles has been widely investigated by the group of the Center for Cosmoparticle Physics[54]-[66], and their methods and results will be important for further study on detailed interaction.

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