Lepton Mixing Predictions from $\Delta(6n^2)$ Family Symmetry

Stephen F. King, Thomas Neder and Alexander J. Stuart

School of Physics and Astronomy,
University of Southampton,
Southampton, SO17 1BJ, U.K.

Abstract

We obtain predictions of lepton mixing parameters for direct models based on $\Delta(6n^2)$ family symmetry groups for arbitrarily large $n$ in which the full Klein symmetry is identified as a subgroup of the family symmetry. After reviewing and developing the group theory associated with $\Delta(6n^2)$, we find many new candidates for large $n$ able to yield reactor angle predictions within $3\sigma$ of recent global fits. We show that such $\Delta(6n^2)$ models with Majorana neutrinos predict trimaximal mixing with reactor angle $\theta_{13}$ fixed up to a discrete choice, an oscillation phase of either zero or $\pi$ and the atmospheric angle sum rules $\theta_{23} = 45^\circ \mp \theta_{13}/\sqrt{2}$, respectively, which are consistent with recent global fits and will be tested in the near future.
INTRODUCTION

The measurement of a rather large reactor mixing angle by the Daya Bay [1], RENO [2], and Double Chooz [3] collaborations adds further complexity to an already difficult puzzle of flavour. Perhaps the best way to address this dilemma is to utilise the methods developed in the era of an unmeasured reactor angle and introduce an additional discrete family symmetry, $G_f$, under which all fields transform. This family symmetry will then be spontaneously broken in order to generate the observed fermionic masses and mixings [4]. However before even considering the construction of a model, it may be insightful to know some of the possible candidate symmetries for $G_f$. Herein lies the goal of this work, shedding light on a particular class of candidates for $G_f$, i.e. the $\Delta(6n^2)$ groups.

In the following text, we demand that the discrete group $G_f$ be a subgroup of the continuous group $SU(3)$ (or $U(3)$) because its fundamental representation is 3-dimensional. We further restrict ourselves to working with the $\Delta(6n^2) \cong (Z_n \times Z_n) \rtimes S_3$ subgroups of $SU(3)$ due to the past and current popularity of $S_4 \cong \Delta(24)$ ($n = 2$) in flavour model building (see [5] and references contained therein) as well as recent publications demonstrating that $\Delta(96)$ ($n = 4$) [6], $\Delta(150)$ ($n = 5$) [7, 8], $\Delta(600)$ ($n = 10$) [8, 9] and $\Delta(1536)$ ($n = 16$) [9] generate phenomenologically viable predictions for the lepton mixing angles. We further limit ourselves to working only with the $\Delta(6n^2)$ groups where $n$ is even because these are the only $\Delta(6n^2)$ groups which can contain a complete Klein subgroup, i.e. all four Klein subgroup elements with generators denoted by $S, U$, where the invariance of the neutrino mass matrix under the Klein symmetry group is sufficient to completely fix the neutrino mass matrix for Majorana neutrinos (for a review see e.g. [5]).

Thus with the preliminary assumptions and goals of this work put forth, we proceed by introducing the framework in which we will work. Afterwards, a brief review of the representations of $\Delta(6n^2)$ will be given. Finally, the details of our method elucidated and the results presented.

FROM $G_f$ TO LEPTON MIXING

As previously mentioned, to address the puzzling issue of flavour, we will introduce a discrete family symmetry which will be spontaneously broken to different subgroups in the
charged lepton and neutrino sectors, thereby generating the observed lepton masses and mixings. In such a direct model of flavour, the family group is broken to some abelian subgroup $Z_m^T$ (m an integer) in the charged lepton sector and to the $Z_2^S \times Z_2^U$ Klein Symmetry Group in the neutrino sector. The superscripts denote that $S$, $T$ and $U$ are the generators of their corresponding $Z_m$ group in the diagonal charged lepton basis. Hence, the $Z_2^S \times Z_2^U$ transformations on $\nu_L$ and the $Z_m^T$ transformations on $e_{L,R}$ leave the Lagrangian invariant. This implies that

$$\left[ S, M^\nu \right] = \left[ U, M^\nu \right] = 0 \text{ and } \left[ T, M^e \right] = 0, \quad (1)$$

where $M^\nu$ and $M^e$ represent the mass matrices multiplied by their Hermitian conjugates. Since $S$ and $U$ commute with $M^\nu$ they are diagonalised by the same matrix $V^\nu$. Similarly $T$ and $M^e$ are diagonalised by the same matrix $V^e$. Since $M^\nu$ and $M^e$ relate to the left-handed fields, the PMNS matrix is then given by

$$V = V^e V^\nu. \quad (2)$$

To obtain the matrices $V^\nu$ and $V^e$, and hence the PMNS matrix, we only need to identify the generators $S$, $U$ and $T$ and diagonalise them. In practice, this amounts to finding the eigenvectors of $S$, $U$ and $T$ which form the columns of $V^\nu$ and $V^e$. This is straightforward for $T$ since the eigenvalues are non-degenerate due to the fact that $T$ must be an element of $G_f$ of order 3 or greater. However for the $S$ and $U$ generators the situation is slightly different because they are $3 \times 3$ matrices of order 2. Thus, each eigenvalue of $S$ or $U$ can only be $\pm 1$. Without loss of generality, we choose $\det(S) = \det(U) = +1$, so that each generator has two $-1$ eigenvalues, rendering the corresponding eigenvectors non-unique. Since the three matrices $S$, $U$ and $SU$ each have one (unique) $+1$ eigenvalue this allows for the calculation of three unique eigenvectors (one for each non-trivial Klein group generator), each providing an $i$th column of the matrix $V^\nu$:

$$G_i V^\nu_i = +V^\nu_i, \text{ for } G_i \in \{S, U, SU\}. \quad (3)$$

In this way all three columns of $V^\nu$ can be obtained.

The remarkable method outlined in this section enables the calculation of the lepton mixing matrix by only considering the family group’s representation matrices. However, this certainly requires explicit representation matrices for the $\Delta(6n^2)$ group’s representations. We construct these in the next section.
THE GROUP THEORY OF $\Delta(6n^2)$

The $\Delta(6n^2)$ groups are finite non-Abelian subgroups of $SU(3)$ ($U(3)$) of order $6n^2$. They are isomorphic to the semidirect product

$$\Delta(6n^2) \cong (Z_n \times Z_n) \rtimes S_3.$$  \hfill (4)

The Klein group $Z_2^S \times Z_2^U$ (in direct models) can either originate purely from the $Z_n \times Z_n$ or it will involve the $S_3$ generators as well, both possibilities requiring even $n$. We may re-express Eq. (4) in a more illuminating form by taking advantage of the structure of $S_3$:

$$\Delta(6n^2) \cong (Z_n^c \times Z_n^d) \rtimes (Z_3^a \rtimes Z_2^b).$$  \hfill (5)

Notice that in Eq. (5), $(Z_n^c \times Z_n^d)$ forms a normal, abelian subgroup of $\Delta(6n^2)$, generated by the elements $c$ and $d$, and $(Z_3^a \rtimes Z_2^b)$ is nothing more than $S_3$ rewritten in terms of its generators $a$ and $b$. From Eq. (5) follows that the relevant presentation of $\Delta(6n^2)$ is [10]:

$$a^3 = b^2 = (ab)^2 = c^n = d^n = 1, \quad cd = dc,$$

$$aca^{-1} = c^{-1}d^{-1}, \quad ada^{-1} = c,$$

$$bcb^{-1} = d^{-1}, \quad bdb^{-1} = c^{-1}. \hfill (6)$$

Another advantage of the presentation in Eqs. (4)-(5) is that every group element can be written as

$$g = a^\alpha b^\beta c^\gamma d^\delta,$$  \hfill (7)

with $\alpha = 0,1,2$, $\beta = 0,1$ and $\gamma,\delta = 0,\ldots,n-1$, making the computation of all group elements for a certain representation/basis computationally simple. All that needs to be known next is the explicit forms of generators.

In order to find the explicit forms for the generators, we restrict ourselves to 3-dimensional irreducible representations of $\Delta(6n^2)$. Then, it can be shown that $\Delta(6n^2)$ has $2(n-1)$ 3-
dimensional irreducible representations denoted by $3^l_k$ and explicitly generated by (8):

$$a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad b = (-1)^{k+1} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$c = \begin{pmatrix} \eta^l & 0 & 0 \\ 0 & \eta^{-l} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \eta^l & 0 \\ 0 & 0 & \eta^{-l} \end{pmatrix},$$

where $\eta = e^{2\pi i/n}$; $k = 1, 2$; and $l = 1, \ldots, n-1$.

We further restrict ourselves to working with only faithful irreducible representations of $\Delta(6n^2)$. Thus, we exclude all representations in Eq. (8) where $l$ divides $n$, as they are unfaithful. Of the remaining representations, $3^l_k$ and $3^{l'}_k$ are complex conjugates of each other if $l + l' = n$. Therefore, they will provide complex conjugated mixing matrices. The remaining representations provide the same sets of mixing matrices because the generators $a$ and $b$ are the same for all $l$ and

$$c(3^l_k) = c(3^1_k)^l \text{ and } d(3^l_k) = d(3^1_k)^l.$$  (9)

Then, by considering Eq. (7) and Eq. (9) it is clear that each power of the $c$ and $d$ generators will appear in every 3-dimensional irreducible representation. For these reasons, it suffices if one only considers $S$, $T$, and $U$ as representation matrices from $3^1_k$. Notice that $k = 2$ has been chosen because in this case the determinant of the elements of order 2 is +1.

Having reduced the possible cases needed for consideration, the next step is to calculate all Klein subgroups of $\Delta(6n^2)$. This is accomplished by first calculating all order two elements. From the generators and rules given in Eq. (6) it follows that all order 2 elements in $\Delta(6n^2)$ are given by:

$$c^{n/2}, d^{n/2}, c^{n/2}d^{n/2}, bcd^\epsilon, abc^\gamma, \text{ and } a^2bd^\delta,$$  (10)

where $\epsilon, \gamma, \delta = 0, \ldots, n-1$.

The order 2 elements found in Eq. (10) serve as a starting point for calculating Klein Symmetry groups of $\Delta(6n^2)$. Using Eq. (6) and Eq. (10) reveals the Klein subgroups of
$\Delta(6n^2)$ for even $n$ to be:

$$\{1, c^{n/2}, d^{n/2}, c^{n/2}d^{n/2}\},$$  \hspace{1cm} (11)

$$\{1, c^{n/2}, abc\gamma', abc\gamma'+n/2\},$$  \hspace{1cm} (12)

$$\{1, d^{n/2}, a^2bd\delta', a^2bd\delta'+n/2\},$$  \hspace{1cm} (13)

$$\{1, c^{n/2}d^{n/2}, bc\epsilon' d', bc\epsilon'−n/2d\epsilon'−n/2\},$$  \hspace{1cm} (14)

where $\gamma', \delta', \epsilon' = 1, \ldots, n/2$. Notice that Eq. (11) corresponds to the Klein symmetry originating completely from $Z_n \times Z_n$ whereas Eqs. (12)-(14) involve also $S_3$. In the basis of Eq. (8), one of the Klein generators (taken to be $S$) is diagonal for all cases, while in the case of Eq. (11) both Klein generators $S, U$ are diagonal.\[13\]

As previously discussed, the $T$ generator which controls the charged lepton sector must be at least of order 3. As shown in the Appendix, only the minimal order 3 case is phenomenologically viable and so we only consider this possibility. In $\Delta(6n^2)$ groups where 3 does not divide $n$, all elements of order 3 are expressible as \[10\]:

$$ac\gamma d\delta, a^2c\gamma d\delta$$  \hspace{1cm} (15)

where $\delta, \gamma = 0 \ldots n − 1$ \[14\]. Without loss of generality we may choose to order three generator to be

$$T = a,$$  \hspace{1cm} (16)

since $a$ and $a^2$ only differ by a permutation of rows and columns and in the basis of Eq. (8), it can be seen that multiplication by $c\gamma d\delta$ only yields phases which may be absorbed into the charged lepton fields.

Notice that the $T$ of Eq. (16) can be diagonalised by the matrix,

$$V^e = \frac{1}{\sqrt{3}} \begin{pmatrix}
\omega^2 & \omega & 1 \\
\omega & \omega^2 & 1 \\
1 & 1 & 1
\end{pmatrix},$$  \hspace{1cm} (17)

where $\omega = e^{2\pi i/3}$. The ordering of the columns and rows in the above $V^e$ determines the ordering of the eigenvalues in $T$:

$$T \rightarrow V^e V^a V^e = \begin{pmatrix}
\omega^2 & 0 & 0 \\
0 & \omega & 0 \\
0 & 0 & 1
\end{pmatrix}.$$  \hspace{1cm} (18)
For example, changing the order of the eigenvalues of $T$ by applying $a^{\alpha}$ to $T$ by $a^{\alpha\dagger}Ta^{\alpha}$ ($\alpha = 1, 2$) changes $V_e$ to $a^{\alpha}V_e$ which just permutes the rows of $V$ in Eq. (2).

RESULTS

Using the results of the previous section one can compute the columns of the lepton mixing matrix which correspond to each possible Klein subgroup of a certain $\Delta(6n^2)$ group where $n$ is even and we assume $T = a$. The steps for this procedure are summarised as follows.

We shall generate all Klein group elements in Eqs. (11)-(14) in the explicit $3_1^2$ representation matrices given in Eq. (8), then transform each Klein group’s element to the basis where $T$ is diagonal via $V_e$, c.f. Eq. (18). Here, the eigenvectors with $+1$ eigenvalue correspond to the columns of possible mixing matrices as in Eq. (3). Since the ordering of the columns and rows of the mixing matrix calculated this way is arbitrary, without loss of generality we take the smallest absolute value from each mixing matrix and assign this as $V_{13}$ with its corresponding column being the third column eigenvector of $V$. This completed procedure is unique up to interchanging the second and third rows of $V$, corresponding to two predictions for the atmospheric angle.

Implementing the preceding procedure for calculating the mixing matrix resulting from the Klein group in Eq. (11) with $T = a$ yields the old trimaximal mixing matrix [11] given by the $V_e$ in Eq. (17) up to permutation of its rows and columns. Clearly, this is not a phenomenologically viable mixing matrix, so we discard this possibility.

We do not have to consider all the Klein groups in Eqs. (12)-(14) since they all result in identical PMNS matrices up to permutations of rows and columns. This is because the Klein group elements in Eq. (13) and Eq. (14) are related to $G_i$ in Eq. (12) by $a^2G_i a$ and $aG_i a^2$ respectively, where $a$ and $a^2$ from Eq. (8) interchange rows and columns.

Thus, it is sufficient to consider the Klein subgroup given in Eq. (12), where the element $c^{n/2}$ becomes the “traditional” $S$ generator in the basis in which $T$ is diagonal,

$$S \rightarrow V_e^\dagger c^{n/2}V_e = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$$ (19)
This predicts one trimaximal middle column (TM), i.e. $(1, 1, 1)^T / \sqrt{3}$, in lepton mixing [13]. The other elements of the same Klein subgroup also provide columns of $V$ which is then up to the order of rows and columns given by

$$ V = \begin{pmatrix} \frac{1}{\sqrt{6N_+}}(c + \sqrt{3}s + 1) & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6N_-}}(c + \sqrt{3}s - 1) \\ \frac{1}{\sqrt{N_+}} \frac{N_+}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6N_-}} N_+ \\ \frac{1}{\sqrt{6N_+}}(2c - 1) & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6N_-}}(2c + 1) \end{pmatrix}, \quad (20) $$

where $c = \cos \vartheta$ and $s = \sin \vartheta$, with $\vartheta = 2\pi \gamma'/n$ and

$$ N_\pm = \sqrt{2 \pm \sqrt{3}s \mp c}. \quad (21) $$

Since $\gamma' = 1, \ldots, n/2$ we obtain discrete predictions for the mixing angles corresponding to $\vartheta = 2\pi/n, \ldots, \pi$. In general we cannot predict the order of the rows and columns, so we pick the entry with the smallest absolute value and assign it to be $|V_{13}|$. For different values of $\vartheta$, different elements of Eq. (20) play the role of $V_{13}$. After $V_{13}$ has been fixed, the second and third row can be interchanged, leading to two different predictions for the atmospheric angle, corresponding to $\delta_{CP} = 0$ and $\delta_{CP} = \pi$, leading to the testable sum rules, $\theta_{23} = 45^\circ \mp \theta_{13}/\sqrt{2}$, respectively [13]. Note that Klein subgroups do not predict Majorana phases.

FIG. 1 shows all possible predictions for $|V_{13}|$ corresponding to the different Klein subgroups for each $\Delta(6n^2)$ of even $n$ one obtains using the method previously discussed. As $n$ increases the number of possible values of $|V_{13}|$ predicted by $\Delta(6n^2)$ also increases according to the above discussion.

CONCLUSIONS

In this paper we have obtained predictions of lepton mixing parameters for direct models based on $\Delta(6n^2)$ family symmetry groups for arbitrarily large $n$ in which the full Klein symmetry is identified as a subgroup of the family symmetry. After reviewing and developing the group theory associated with $\Delta(6n^2)$, we confirmed some known results of the recent numerical searches and found many new possible mixing patterns for large $n$ able to yield lepton mixing angle predictions within $3\sigma$ of recent global fits. Previously, $\Delta(6n^2)$ had only been analysed within particular scans up to a much lower order than we considered. All the
FIG. 1: The possible values that $|V_{13}|$ can take in $\Delta(6n^2)$ family symmetry groups with even $n$. Examples include $|V_{13}| = 0.211, 0.170, 0.160, 0.154$ for $n = 4, 10, 16, 22$, respectively. The lines denote the present approximate $3\sigma$ range of $|V_{13}|$.

examples predict exact TM$_2$ mixing with oscillation phase zero or $\pi$ corresponding to two possible predictions for the atmospheric angle but differ in the prediction of $|V_{13}|$ as shown in FIG. [1]

For large $n$, it is clear that the predictions for $|V_{13}|$ densely fill the allowed range. Nevertheless, our general method of analysing $\Delta(6n^2)$ family symmetry groups is of interest since it represents for the first time a model independent treatment of an infinite class of theories. The general predictions for the considered class of theories based on $\Delta(6n^2)$ are Majorana neutrinos, trimaximal lepton mixing with reactor angle fixed up to a discrete choice, an oscillation phase of either zero or $\pi$ and sum rules $\theta_{23} = 45^\circ \pm \theta_{13}/\sqrt{2}$, respectively, which are consistent with the recent global fits and will be tested in the near future.

ACKNOWLEDGEMENTS

The authors acknowledge partial support from the European Union FP7 ITN-INVISIBLES (Marie Curie Actions, PITN- GA-2011- 289442). SFK and AJS acknowledge
support from the STFC Consolidated ST/J000396/1 grant. SFK acknowledges support from EU ITN UNILHC PITN-GA-2009-237920.

**APPENDIX**

In this Appendix we show that $T$ generators of order greater than 3 are not viable. We begin by considering the order of $T$ to be even. Then, $T^m = 1$ with $m = 2q$ where $q$ is an integer. We first note that diagonal $T$ candidates in the basis of Eq. (8) will not lead to acceptable mixing. After removing unphysical phases, all non-diagonal $T$ candidates of even order $m = 2q$ can be written without loss of generality as,

$$ T = be^{\xi n/q}, \xi = 1, \ldots, q - 1. $$  \hspace{1cm} (22)

The matrices of Eq. (22) are diagonalised by

$$ V^e = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & e^{-i\pi\xi/q} & e^{-i\pi\xi/q} \\ \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}. $$  \hspace{1cm} (23)

Applying the above matrix to $c^{n/2}$ results in:

$$ U \rightarrow V^e c^{n/2} V^e = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. $$  \hspace{1cm} (24)

The unique eigenvector of this generator is given by $(0, 1, 1)/\sqrt{2}$. Picking the smallest element of the mixing matrix as $V_{13}$ gives $V_{13} = 0$. For $n = 2$ this results in a completely bimaximal mixing matrix [12].

If the order of $T$ is not even but can be divided by 3, application of a unitary transformation $R = c^\pi d^\rho$ can remove all phases implying only $T = a$ remains, yielding the previously discussed predictions for $T = a$.

Continuing the systematic consideration of candidate $T$ generators leads us to consider the case of a $T$ generator in which the order is odd, not divisible by 3 but larger than 3. A $\Delta(6n^2)$ group can only contain such an element if $m$ divides $n$. Then, for this case the possible $T$ generators are given by

$$ T = c^{mn/m} d^{mn/m} $$  \hspace{1cm} (25)
where $\mu, \rho = 0, \ldots, m - 1$ and $\mu, \rho$ are not simultaneously zero. These yield no phenomenologically viable predictions. Therefore, only candidate $T$ generators from $Z_3$ subgroups of $\Delta(6n^2)$ are phenomenologically viable.

* Electronic address: S.F.King@soton.ac.uk
† Electronic address: T.Neder@soton.ac.uk
‡ Electronic address: A.Stuart@soton.ac.uk

[1] F. P. An et al. [DAYA-BAY Collaboration], Phys. Rev. Lett. 108 (2012) 171803 [arXiv:1203.1669];
[2] J. K. Ahn et al. [RENO Collaboration], Phys. Rev. Lett. 108 (2012) 191802 [arXiv:1204.0626].
[3] Y. Abe et al. [Double Chooz Collaboration], arXiv:1301.2948 [hep-ex].
[4] J. Beringer et al. [Particle Data Group Collaboration], Phys. Rev. D 86 (2012) 010001.
[5] S. F. King and C. Luhn, Rep. Prog. Phys. 76 (2013) 056201 [arXiv:1301.1340 [hep-ph]].
[6] R. d. A. Toorop, F. Feruglio and C. Hagedorn, Phys. Lett. B 703 (2011) 447 [arXiv:1107.3486 [hep-ph]]; R. de Adelhart Toorop, F. Feruglio and C. Hagedorn, Nucl. Phys. B 858 (2012) 437 [arXiv:1112.1340 [hep-ph]]; G. -J. Ding, Nucl. Phys. B 862 (2012) 1 [arXiv:1201.3279 [hep-ph]]; S. F. King, C. Luhn and A. J. Stuart, Nucl. Phys. B 867 (2013) 203 [arXiv:1207.5741 [hep-ph]].
[7] C. S. Lam, Phys. Rev. D 87 (2013) 013001 [arXiv:1208.5527 [hep-ph]].
[8] C. S. Lam, arXiv:1301.1736 [hep-ph].
[9] M. Holthausen, K. S. Lim and M. Lindner, Phys. Lett. B 721 (2013) 61 [arXiv:1212.2411 [hep-ph]].
[10] J. A. Escobar and C. Luhn, J. Math. Phys. 50 (2009) 013524 [arXiv:0809.0639 [hep-th]].
[11] P. F. Harrison, D. H. Perkins and W. G. Scott, Phys. Lett. B 349 (1995) 137; P. F. Harrison, D. H. Perkins and W. G. Scott, Phys. Lett. B 458 (1999) 79 [hep-ph/9904297].
[12] V. D. Barger, S. Pakvasa, T. J. Weiler and K. Whisnant, Phys. Lett. B 437 (1998) 107 [hep-ph/9806387]; G. Altarelli, F. Feruglio and L. Merlo, JHEP 0905 (2009) 020 [arXiv:0903.1940 [hep-ph]].
[13] As an example of the Klein subgroups in Eqs. (11)-(14), in $\Delta(96)(n=4)$, it was found that for the bi-trimaximal mixing example $S = d^2$ and $U = a^2bd^3$, implying that these generators
are contained in the Klein subgroups defined in Eq. (13).

[14] When \( n \) is divisible by 3, there exist more order three elements given by \( c^{n/3}, c^{2n/3}, d^{n/3}, d^{2n/3}, c^{n/3}d^{n/3}, c^{2n/3}d^{n/3}, c^{n/3}d^{2n/3}, c^{2n/3}d^{2n/3} \). In the basis of Eq. (8), these are diagonal matrices of phases. Since \( S \) is also diagonal in this basis, this would result in phenomenologically unacceptable predictions for leptonic mixing.

[15] Note that a Klein symmetry corresponding to \( V \) with a fixed column of \( 1/\sqrt{6}(2, -1, -1)^T \) (TM\(_1\) mixing) cannot be identified as a subgroup of \( \Delta(6n^2) \).