Beyond Standard Models and Grand Unifications:

Anomalies, Topological Terms and

Dynamical Constraints via Cobordisms

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Abstract

We classify and characterize fully all invertible anomalies and all allowed topological terms related to various Standard Models (SM), Grand Unified Theories (GUT), and Beyond Standard Model (BSM) physics. By all anomalies, we mean the inclusion of (1) perturbative/local anomalies captured by perturbative Feynman diagram loop calculations, classified by $\mathbb{Z}$ free classes, and (2) non-perturbative/global anomalies, classified by finite group $\mathbb{Z}_N$ torsion classes. Our theory built from [arXiv:1812.11967] fuses the math tools of Adams spectral sequence, Thom-Madsen-Tillmann spectra, and Freed-Hopkins theorem. For example, we compute bordism groups $\Omega_d^G$ and their invertible topological field theory invariants, which characterize $d d$ topological terms and $(d - 1) d$ anomalies, protected by the following symmetry group $G$: Spin $\times \frac{SU(3) \times SU(2) \times U(1)}{\mathbb{Z}_2^q}$ for SM with $\Gamma_q := \mathbb{Z}_q$ as $q = 1, 2, 3, 6$; Spin $\times Spin(n)$ or Spin $\times SO(n)$ for SO(10) or SO(18) GUT as $n = 10, 18$; Spin $\times SU(n)$ for Georgi-Glashow SU(5) GUT as $n = 5$; Spin $\times \frac{SU(4) \times SU(2) \times SU(2)}{\mathbb{Z}_2^q}$ for Pati-Salam GUT as $q = 1, 2$; and others. Our approach offers an alternative view of all anomaly matching conditions built from the lower-energy (B)SM or GUT, in contrast to high-energy Quantum Gravity or String Theory Landscape v.s. Swampland program, as bottom-up/top-down complements. Symmetries and anomalies provide constraints of kinematics, we further suggest constraints of quantum gauge dynamics, and new predictions of possible extended defects/excitations plus hidden BSM non-perturbative topological sectors.

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1 Introduction

1.1 Physics Guide

The world where we reside, to our present knowledge, can be described by quantum theory, gravity theory, and the underlying long-range entanglement. Quantum field theory (QFT), specifically gauge field theory, under the name of Gauge Principle following Maxwell, Hilbert, Weyl \cite{1}, Pauli, and others, forms a cornerstone of the fundamental physics. Yang-Mills (YM) gauge theory \cite{2}, generalizing the U(1) abelian gauge group to a non-abelian Lie group, has been proven theoretically and experimentally essential to describe the Standard Model (SM) physics \cite{3-5}.

The SM of particle physics is a gauge theory encoding three of the four known fundamental forces or interactions (the electromagnetic, weak, and strong forces, but without gravity) in the universe. The SM also classifies all experimentally known elementary particles: fermions including three generations of quarks and leptons, while bosons including the electromagnetic force mediator photon γ, the strong force mediator gluon g, the weak force mediator W± and Z° gauge bosons, and Higgs particle; while the graviton has not yet been detected and is not in SM. Physics experiments have confirmed that at a higher energy of SM, the electromagnetic and weak forces are unified into an electroweak interaction sector. Grand Unifications and Grand Unified Theories (GUT) predict that at further higher energy, the strong and the electroweak interactions will be unified into an electroweak-nuclear GUT interaction sector. The GUT interaction is characterized by one larger gauge group and its force carrier mediator gauge bosons with a single unified coupling constant.\(^1\) Examples of GUT that we will encounter in this

\(^1\)Unifying gravity with the GUT interaction gives rise to a Theory of Everything (TOE). However, in our present work, the gravity only plays the role of the background probed fields instead of dynamical gravity. As we will classify and characterize, the background probed gravity also gives new constraints, such as in the gravitational anomaly or the mixed gauge-gravitational anomaly. We however will comment the implications for dynamical gravity such as in Quantum Gravity in Sec. 7.
article includes Georgi-Glashow SU(5) GUT [6], Fritzsch-Minkowski SO(10) GUT [7] and Pati-Salam model [8] and others.

In our present work, we aim to classify and characterize fully all (invertible) anomalies and all allowed topological terms associated with various Standard Models (SM), Grand Unified Theories (GUT), and Beyond Standard Model physics (BSM) in 4d. Then we will suggest the dynamical constraints on SM, GUT and BSM via non-perturbative statements based on anomalies and topological terms.

By “anomalies” of a theory in physics terminology, physicists may mean one of the following:

(1): Classical global symmetry is violated in a quantum theory, such that the classical global symmetry fails to be a quantum global symmetry, e.g. Adler-Bell-Jackiw anomaly [9,10].

(2): Quantum global symmetry is well-defined kinematically. However, there is an obstruction known as “‘t Hooft anomaly [11],” to gauge the global symmetry, detectable via coupling the charge operator (i.e., symmetry generators or symmetry defects, which measures the global symmetry charge of charged objects) to background fields. Specifically, we may detect an obstruction to even weakly gauge the symmetry or couple the symmetry to a non-dynamical background probed field (sometimes as background gauge field/connection). “‘t Hooft anomaly [11],” is sometimes regarded as a “background gauged anomaly” in condensed matter. Namely, the path integral or partition function $Z$ does not sum over background gauge fields. We only fix a background gauge field and the $Z$ only depends on the background gauge connection as a classical field or as a classical coupling constant.

(3): Quantum global symmetry is well-defined kinematically. However, once we promote the global symmetry to a dynamical local gauge symmetry of the dynamical gauge theory, then the gauge theory becomes ill-defined. Some people call this as a “dynamical gauge anomaly” prohibiting a quantum theory to be well-defined. Namely, the path integral after summing over dynamical gauge fields becomes ill-defined. Therefore, the anomaly-free or anomaly-matching conditions are crucial to avoid the sickness and ill-defineness of quantum gauge theory.

In fact, it is obvious to observe that the anomalies from (3) are descendants of anomalies from (2). 

(α). Anomalies from (3) can be related to anomalies from (2) via the gauging principle.

(β). Anomalies from (2) can be related to anomalies from (3) via the ungauging principle.

Thus our key idea is that if we know the gauge group of a gauge theory (e.g., SM, GUT or BSM), we may identify its ungauged global symmetry group as an internal symmetry group, say $G_{\text{internal}}$ via ungauging.4

To start, we should rewrite the global symmetries of an ungauging theory into the form of

$$G \equiv \left( \frac{G_{\text{spacetime}} \ltimes G_{\text{internal}}}{N_{\text{shared}}} \right),$$

\[1.1\]
where the $G_{\text{spacetime}}$ is the spacetime symmetry, the $G_{\text{internal}}$ is the internal symmetry,\(^5\) the $\kappa$ is a semi-direct product from a “twisted” extension,\(^6\) and the $N_{\text{shared}}$ is the shared common normal subgroup symmetry between $G_{\text{spacetime}}$ and $G_{\text{internal}}$.

In the later sections of our work, we write down the ungauged global symmetry groups $G$ of SMs, GUTs and BSMs. Then we should determine, classify and characterize all of their associated (invertible) anomalies and topological terms. Moreover, based on the descendant relations between the anomalies from (2) and (3), and the gauging/ungauging principles relate $(\alpha)$ and $(\beta)$, we thus also determine both:

(A). For the ungauged SM, GUT and BSM theories:
Invertible ’t Hooft anomalies and background probed topological terms associated to a global symmetry group $G$. This is related to a relation $(\beta)$.

(B). For the gauged SM, GUT and BSM theories:
Dynamical gauge anomalies and dynamical topological terms associated to a gauge group $G$, descent via the relation $(\alpha)$.

But by far, there are some pertinent questions that the readers may wonder in order to decide to keep reading our work (or decide to quit reading). We should confront, comfort, and answer the readers immediately:

[I]. **What do we mean by all (invertible) anomalies and all topological terms?** (See a disclaimer in Footnote 7.)

By “all (invertible) anomalies,” we mean the inclusion of:

(i). **Perturbative/local anomalies** captured by perturbative Feynman diagram loop calculations, classified by the integer group $\mathbb{Z}$ classes, or the so-called free classes in mathematics. Some selective examples from QFT or gravity include:
   (1): Perturbative fermionic anomalies from chiral fermions with U(1) symmetry, originated from Adler-Bell-Jackiw (ABJ) anomalies [9,10] with $\mathbb{Z}$ classes.
   (2): Perturbative bosonic anomalies from bosonic systems with U(1) symmetry with $\mathbb{Z}$ classes.
   (3): Perturbative gravitational anomalies [13].

(ii). **Non-perturbative/global anomalies**, classified by a product of finite groups such as $\mathbb{Z}_N$, or the so-called torsion classes in mathematics. Some selective examples from QFT or gravity include:
   (1): An SU(2) anomaly of Witten in 4d or in 5d [14] with a $\mathbb{Z}_2$ class, which is a gauge anomaly.
   (2): A new SU(2) anomaly in 4d or in 5d [15] with another $\mathbb{Z}_2$ class, which is a mixed gauge-gravity anomaly.
   (3): Some higher ’t Hooft anomalies for a pure 4d SU(2) YM theory with a second-Chern-class topological term [16–18] (or the so-called SU(2)\(_{\theta=\pi}\) YM): The higher anomaly involves a discrete 0-form time-reversal symmetry and a 1-form center $\mathbb{Z}_2$-symmetry. The first anomaly is discovered in [16]; later the anomaly is refined via a mathematical well-defined 5d bordism invariant as its topological term, with additional new anomaly for four siblings of YM [17,18].
   (4): Global gravitational anomalies [19].

By “all topological terms,” we mean the anomaly-inflow relation [20]: The $(d-1)d$ anomalies can be systematically captured by a one higher dimensional $dd$ topological invariants or $dd$ topological terms.

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\(^5\) Later we denote the probed background spacetime $M$ connection over the spacetime tangent bundle $TM$, e.g. as $w_j(TM)$ where $w_j$ is $j$-th Stiefel-Whitney (SW) class. We also denote the probed background internal-symmetry/gauge connection over the principal bundle $E$, e.g. as $w_j(E) = w_j(V_{\text{internal}})$ where $w_j$ is also $j$-th SW class.

\(^6\) The “twisted” extension is due to the symmetry extension from $G_{\text{internal}}$ by $G_{\text{spacetime}}$, for a trivial extension $\kappa$ becomes a direct product $\rtimes$. 

5
Recently Ref. [21] gives a non-perturbative description of anomaly inflow including both local and global anomalies based on Dai-Freed theorem [22] and the Atiyah-Patodi-Singer η-Invariant [23]. Therefore, by determining all \((d-1)d\) anomalies, we also determine all \(dd\) topological terms, and vice versa.\(^7\)

[II]. What tools are we using to classify and characterize all anomalies and all topological terms?

The short answer is based on the Freed-Hopkin’s theorem [28] and our prior work [29,30].\(^8\) The long answer is follows. Based on the Freed-Hopkin’s theorem [28] and an extended generalization that we propose [29,30], there exists a one-to-one correspondence between “the invertible topological quantum field theories (iTQFTs) with symmetry (including higher symmetries or generalized global symmetries [12])” and “a cobordism group.” In condensed matter physics, this means that there is a relation from iTQFT to “the symmetric invertible topological order (iTO, see a review [32])” with symmetry (including higher symmetries) or symmetry-protected topological state (SPTs, see [32–34]) that can be regularized on a lattice in its own dimensions.”

More precisely, it is a one-to-one correspondence (isomorphism “\(\cong\)”) between the following two well-defined “mathematical objects” (which turn out to be abelian groups):

\[
\begin{align*}
\{ & \text{Deformation classes of the reflection positive invertible } d \text{-dimensional extended } \\
& \text{topological field theories (iTQFT) with symmetry group } G_{\text{spacetime}} \ltimes G_{\text{internal}} \\
& \text{over } N_{\text{shared}} \} \\
\cong & \left[ MT\left( \frac{G_{\text{spacetime}} \ltimes G_{\text{internal}}}{N_{\text{shared}}} \right), \Sigma^{d+1} IZ \right]_{\text{tors}}. \quad (1.3)
\end{align*}
\]

We shall explain the notation above: \(MTG\) is the Madsen-Tillmann-Thom spectrum [35] of the group \(G\), \(\Sigma\) is the suspension, \(IZ\) is the Anderson dual spectrum, and \(\text{tors}\) means the torsion group by taking only

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\(^7\)There is however a disclaimer and some caveats:

(a) By all anomalies and all topological terms, their classifications and characterizations depend on the category of manifolds that can detect them. The categories of manifolds can be: TOP (topological manifolds), PL (piecewise linear manifolds), or DIFF (differentiable thus equivalently smooth manifolds), etc. These categories are different, and they are related by

\[
\text{TOP } \supseteq \text{PL } \supseteq \text{DIFF.} \quad (1.2)
\]

Since the SM, GUT and BSM are given by continuum QFT data, in this work, we only focus on the DIFF manifolds and their associated all possible anomalies and topological terms. However, if we refine the data of QFT later in the future to include PL or TOP data from PL or TOP manifolds, we may also need to refine the corresponding SM, GUT and BSM. Thus, we will have a new set of so-called all anomalies and all topological terms. The tools we use in either case would be a certain version of cobordism theory suitable for a specific category of manifolds. See more in [24].

(b) By anomalies and topological terms for some SM, GUT and BSM theories in our work, we either mean

(A). Invertible ‘t Hooft anomalies and background probed topological terms for the ungauged SM, GUT and BSM, or

(B). Dynamical gauge anomalies and dynamical topological terms for the gauged SM, GUT and BSM.

But after gauging \(G\)-symmetry of (A), for the gauged SM, GUT and BSM in (B), there could be additional new higher ‘t Hooft anomalies associated to the higher symmetries (depending on the group representations of the matter fields) whose charged objects are dynamical extend objects (e.g. 1-lines, 2-surfaces, etc.). In this present work, we do not discuss these additionally gained new higher ‘t Hooft anomalies after dynamically gauging, but will leave them for future work [24]. Examples of such higher ‘t Hooft anomalies can be found in [16–18,25,26] and References therein.

(c) Another possible loop hole is that we do only focus on invertible anomalies captured by invertible topological quantum field theories (iTQFTs), we do not study non-invertible anomalies (e.g. [27]). Different experts and different research fields may regard and define anomalies in different ways. After all, Laozi (600 B.C.) in Dao De Jing had long ago educated us that “The Way that can be told of is not an eternal way; The names that can be named are not eternal names. It was from the Nameless that Heaven and Earth sprang. The named is but the mother that rears the ten thousand creatures, each after its kind.”

---

\(^8\)See also a precursor of works for Ref. [29,30] by one of the present authors: Ref. [31].
the finite group sector. The right hand side is the torsion subgroup of homotopy classes of maps from a Madsen-Tillmann-Thom spectrum (MTG) to a suspension shift (Σ^{d+1}) of the Anderson dual to the sphere spectrum (IZ).

In other words, we classify the deformation classes of symmetric iTQFTs and also symmetric invertible topological orders (iTOs), via this particular cobordism group defined as follows

\[ \Omega^d_G \equiv \Omega^d_{(G_{\text{spacetime}} \ltimes G_{\text{internal}})} \]  
\[ \equiv \text{TP}_d(G) \equiv [MTG, \Sigma^{d+1}IZ]. \]  

(1.4)

by classifying the cobordant relations of smooth, differentiable and triangulable manifolds with a stable G-structure, via associating them to the homotopy groups of T hom-Madsen-Tillmann spectra [35,36], given by a theorem in Ref. [28]. Ref. [28] introduced TP which means the abbreviation of “Topological Phases” classifying the above symmetric iTQFT, where our notations follow [28] and [29]. (For an introduction of the mathematical background and mathematical notations explained for physicists, the readers can consult the Appendix A of [31] or [29].)

Now let us pause for a moment to trace back some recent history of relating these anomalies/topological terms to a cobordism theory. The d dimensional ’t Hooft anomaly of ordinary 0-form global symmetries is known to be captured by a (d + 1) dimensional iTQFT. In the condensed matter literatures, these (d + 1)d iTQFTs describe Symmetry-Protected Topological states (SPTs) or symmetric invertible topological orders (iTO)\(^9\) [32–34]. SPTs and symmetric iTO are interacting systems (interacting systems of bosons and fermions at the lattice scale UV with a local Hilbert space) beyond the free fermion or K theory classification [37] for the (non-interacting or so-called free) topological insulators/superconductors [38,39]. The relations between the SPTs and the response probe field theory partition functions have been systematically studied, selectively, in [31,40–43] (and References therein),\(^10\) and climaxed to the evidence of cobordism classification of SPTs [45,46]. Recently, the iTQFTs and SPTs are found to be systematically classified by a powerful cobordism theory of Freed-Hopkins [28], following the earlier framework of Thom-Madsen-Tillmann spectra [35,36].

A new ingredient in our work [29,30] is a generalization of the calculations and the cobordism theory of Freed-Hopkins [28] involving higher symmetries: Instead of the ordinary group G or ordinary classifying space BG, we consider a generalized cobordism theory studying spacetime manifolds endorsed with \(G_{\text{spacetime}}\) structure, with an additional higher group G (i.e., generalized as principal-G bundles) and higher classifying spaces BG.\(^11\)

[III]. What do we mean by classifications and characterizations?

- By classification, we mean that given certain physics theories or phenomena (here, higher-iTQFT and higher quantum anomalies), given a spacetime dimensions (here \(d + 1d\) for higher-iTQFT or \(dd\) for higher quantum anomalies), and their spacetime \(G_{\text{spacetime}}\)-structure and the internal higher global symmetry \(G_{\text{internal}}\), we compute how many classes (a number to count them) there are? Also, we aim to determine the mathematical structures of classes (i.e. here group structure as for (co)bordism groups: would the classes be a finite group \(Z_N\) or an infinite group \(Z\) or others, etc.).

\(^9\)We abbreviate both Symmetry-Protected Topological state and Symmetry-Protected Topological states as SPTs. We also abbreviate both Symmetry-Enriched Topologically ordered state and Symmetry-Enriched Topologically ordered states as SETs.

\(^10\)For example, the interacting versions of 10 Cartan symmetry classes of fermionic superconductors/insulators classifications (e.g. [44] in 4d or 3+1D) from condensed matter can be captured precisely by bordism invariants as invertible TQFTs [31].

\(^11\)Although most of our results in this article focus on the ordinary symmetry group, our framework does allow us to consider possible higher symmetries and higher anomalies [29,30].
• By *characterization*, we mean that we formulate their mathematical invariants (here, we mean the bordism invariants) to fully describe or capture their mathematical essences and physics properties. Hopefully, one may further compute their physical observables from mathematical invariants.

Since some of readers are still with us reading this sentence (after we answer the three questions [I], [II] and [III]), we believe that these readers decide to be interested in understanding our results.

Here we concern theories of 4d SMs, GUTs and BSMs and their anomalies and topological terms. Their 4d ’t Hooft anomalies captured by 5d iTQFTs. These 5d iTQFTs or bordism invariants are defined on the $d$-manifolds ($d = 5$). In fact, in our work, we present all $(d - 1)d$ ’t Hooft anomalies captured by $dd$ iTQFTs, for $d = 1, 2, 3, 4, 5$, associated with various SMs, GUTs and BSM (ungauged) symmetries. The manifold generators for the bordism groups are actually the closed $d$-manifolds. We should clarify that although there are ’t Hooft anomalies for $(d - 1)d$ QFTs (so $G_{\text{internal}}$ may not be gauge-able on the boundary), the SPTs/topological invariants defined in the closed $dd$ actually have $G_{\text{internal}}$ always gauge-able in that $dd$. This is related to the fact that the *bulk* $dd$ SPTs in condensed matter physics has an onsite local internal $G_{\text{internal}}$-symmetry (or on-$n$-simplex-symmetry as a generalization for higher-SPTs), thus this $G_{\text{internal}}$ must be gauge-able. By gauging the topological terms, this idea has been used to study the vacua of YM gauge theories coupling to dynamically gauged SPT terms (like the orbifold techniques in string theory, but here we generalize this thinking to any dimension), for example, in [31] and references therein. There are other uses and interpretations of our cobordism theory data that we will explain in Sec. 7.

We should emphasize that several recent pursuits are also along the fusions between the non-perturbative physics of SMs, GUTs, and BSMs via a cobordism theory:

• Garcia-Etxebarria-Montero [47] studies global anomalies of some SMs and GUTs model via a Dai-Freed theorem and Atiyah-Hirzebruch spectral sequence (AHSS) [22].

• Wang-Wen [48], independently, studies the non-perturbative definitions (e.g. on a lattice) and the global anomalies of SO(10) GUTs or SO(18) GUTs via $\Omega_5^{\text{Spin}(d) \times \text{Spin}(n)}$ with $n = 10, 18$ and SU(5) GUTs via $\Omega_5^{\text{Spin}(5)}(\text{BSU}(5))$. They ask what are all allowed thus all possible anomalies for a fermionic theory with $\text{Spin}(d) \times \text{Spin}(10)$-symmetry. Under the interaction effects, the answer turns out to be a $Z_2$ class (or a mod 2 class) global anomaly captured by the 5d iTQFT:

$$e^{i\pi \int_{M^5} w_2(TM) w_3(TM)}, \quad (1.5)$$

where $w_j(TM)$ is the $j$-th-Stiefel-Whitney class for the tangent bundle of 5D spacetime $M^5$. We note that on a $M^5$, we have a $\text{Spin}(d=5) \times \text{Spin}(n)$ connection — a mixed gravity-gauge connection, rather than the pure gravitational $\text{Spin}(d = 5)$ connection, such that $w_2(TM) = w_2(V_{\text{SO}(n)}) \equiv w_2(\text{SO}(n))$ and $w_3(TM) = w_3(V_{\text{SO}(n)}) \equiv w_3(\text{SO}(n))$, where $w_j(V_{\text{SO}(n)}) := w_j(\text{SO}(n))$ is the $j$-th-Stiefel-Whitney class for the associated vector bundle of an $\text{SO}(n)$ gauge bundle. The $M^5$ can be

12Before dynamically gauging $\text{Spin}(10)$, SO(10)-GUT is one kind of such theory: an $\text{Spin}(10)$-chiral gauge theory with fermions in the half-integer (iso)spin-representation. So we may call this ungauged theory as SO(10)-GUT chiral fermion theory.
a non-spin manifold. This is the same global anomaly known as “a new SU(2) anomaly” studied in Ref. [15]. But Ref. [48] and [15] show explicitly, since

$$\text{Spin}(10) \supset \text{Spin}(3) = \text{SU}(2),$$

if the SO(10) GUT chiral fermion theory is free from “the new SU(2) anomaly [15]” (which indeed is true), then the SO(10) GUT chiral fermion theory contains no anomaly at all. Thus this SO(10) GUT is all anomaly-free [15, 48].” This leads to a possible non-perturbative construction of SO(10) GUTs on the lattice proposed in [49, 50], rooted in the idea of gapping the mirror chiral fermions of Eichten-Preskill [51]. However, we will not pursue this idea of [48] nor [49, 50] (such as the lattice regularization) further in this work, but leave this for a future exploration [52].

- Wan-Wang [29, 30] attempts to classify all invertible local or global anomalies and the invertible higher-anomalies, based on a generalized cobordism group classification of invertible TQFTs and invertible higher-TQFTs in one higher dimensions via Adams spectral sequence. Ref. [29] computes the cobordism classification relevant for perturbative anomalies of chiral fermions (e.g. originated from Adler-Bell-Jackiw [9, 10]) or chiral bosons with U(1) symmetry in any even spacetime dimensions; they also compute the cobordism classification for the non-perturbative global anomalies such as Witten anomaly [14] and the new SU(2) anomaly [15] in 4d and 5d. Ref. [29] also obtains the cobordism classification relevant for higher ’t Hooft anomalies for a pure 4d SU(N) YM theory with a second-Chern-class $\theta = \pi$ topological term [16–18].

- McNamara-Vafa [58] studies the cobordism classes and the constraints on the Quantum Gravity or String Theory Landscape v.s. Swampland. QFT must satisfy some consistent criteria in order to be part of a consistent theory of Quantum Gravity. Those QFT not obeying those criteria are quoted to reside in Swampland.

- Davighi-Gripaios-Lohitsiri [59] studies also the global anomalies in various SMs and BSMs, based on Atiyah-Hirzebruch spectral sequence.

- Freed-Hopkins [60] studies a global anomaly cancellation involving the time-reversal symmetry relevant for the 11-dimensional M theory.

The outline of our article is the following. We consider the following models/theories, their co/bordism groups, TP groups, topological terms and anomalies, written in terms of iTQFTs.

1. Standard Models (SM) in Sec. 2 and Sec. 3:

   • Sec. 2: $\text{Spin} \times \frac{\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)}{\Gamma_q}$ model for $\Gamma_q := \mathbb{Z}_q$ from $\Omega_{d}^{\text{Spin}}(B^{\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)})$ with $q = 1, 2, 3, 6$. The Lie algebra of SM is known to be $\text{su}(3) \times \text{su}(2) \times \text{u}(1)$, but it is known that the global structure of gauge group allows a quotient group of

   $\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)$  \hspace{1cm} \text{(1.6)}

   as

   $\frac{\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)}{\Gamma_q}$, \hspace{1cm} \text{(1.7)}

   which is well-explained, for example, in [61].

13 In addition to Ref. [48–50], related ideas about interactions gapping the mirror chiral fermions also occur in 1+1d in Ref. [53–55] based on the anomaly cancellation constraints, or in 3+1d based on arguments about the topological defects in Ref. [56, 57].

14 More generally, the global structure of gauge group as $G$ or $G/\Gamma$ makes real physical differences in observables [62].
• Sec. 3: Spin $\times_{\mathbb{Z}_2} \mathbb{Z}_4 \times \frac{\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)}{\Gamma_q}$ model from $\Omega^\text{Spin}_d(\frac{\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)}{\Gamma_q})$ with $q = 1, 2, 3, 6$. This is an interesting group suggested by \cite{47, 63}.

2. Grand Unified Theories (GUT) in Sec. 4, Sec. 5 and Sec. 6:

• Sec. 4: Pati-Salam model $\text{Spin} \times \frac{\text{SU}(4) \times (\text{SU}(2) \times \text{SU}(2))}{\mathbb{Z}_2^F}$ with $q = 1, 2$. Here the $\mathbb{Z}_2^F$ is the well-known fermion parity symmetry, which acts on any fermionic operator $\Psi$ by
  \[ \mathbb{Z}_2^F : \Psi \to -\Psi, \]
  such that the operation is $(-1)^{N_F}$ where $N_F$ is the fermion number.

• Sec. 5: SO(10) and SO(18) GUT from either $\Omega^\text{Spin}_d(\frac{\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)}{\Gamma_q})$ and $\Omega^\text{Spin}_d(\text{BSO}(n))$.

• Sec. 6: SU(5) GUT from either $\Omega^\text{Spin}_d(\text{BSU}(5))$ or more generally $\Omega^\text{Spin}_d(\text{BSU}(n))$.

We provide an overview how this data can be used to dynamically constrain SMs, GUTs and BSMs in Conclusions in Sec. 7. After all these physics stories and inputs, and especially thanks to the readers are still staying in tune with us, now let us reward the readers by introducing some mathematics preliminary in Sec. 1.2.

1.2 Mathematics Preliminary

Adams spectral sequence is a mathematical tool to compute the homotopy groups of spectra \cite{64}. In particular, the homotopy group of the Madsen-Tillmann spectrum $MTG$ \cite{35} is the bordism group $\Omega^G_d$. We use Adams spectral sequence to compute several bordism groups related to Standard Models (SM), Grand Unified Theories (GUT) and beyond. We also compute the group $\text{TP}_d(G)$ classifying the topological phases (i.e., the topological terms in QFT or the topological phases of quantum matter) based on the computation of bordism groups and a short exact sequence.

We aim to compute the bordism group $\Omega^G_d$ and the group $\text{TP}_d(G)$ for several groups $G$, our computation is based on Adams spectral sequence, see \cite{28, 29, 31} for primers.

The Adams spectral sequence shows:

\[ \text{Ext}^{s,t}_{A_p}(H^*(Y, \mathbb{Z}_p), \mathbb{Z}_p) \Rightarrow \pi_{t-s}(Y)^\wedge_{p}, \]

where the $\text{Ext}$ denotes the extension functor, $A_p$ is the mod $p$ Steenrod algebra, and $Y$ is any spectrum. In particular, the mod 2 Steenrod algebra $A_2$ is generated by Steenrod squares $Sq^i$. The $H^*(Y, \mathbb{Z}_p)$ is an $A_p$-module whose internal degree $t$ is given by the $*$. The $\pi_{t-s}(Y)^\wedge_{p}$ is the $p$-completion of the $(t-s)$-th homotopy group of the spectrum $Y$. We note that, for any finitely generated abelian group $G$, then $G^\wedge_p = \lim_{n \to \infty} G/p^n G$ is the $p$-completion of $G$; if $G$ is the infinite cyclic group $\mathbb{Z}$, then the $\mathbb{Z}^\wedge_p$ is the ring of $p$-adic integers. Here the $G$ is meant to be substituted by a homotopy group $\pi_{t-s}(Y)$ in (1.9). Here are some explanations and inputs:

(1) Here the double-arrow \(\Rightarrow\) means "convergent to." The $E_2$ page are groups $\text{Ext}^{s,t}_{A_p}$ with double indices $(s, t)$, we reindex the bidegree by $(t - s, s)$. There are differentials $d_2$ in $E_2$ page which are arrows from
In Adams spectral sequence, we consider $\text{Ext}^{s,t}_{MT}$. The Madsen-Tillmann spectrum $\text{MT}$ is the Thom spectrum $\text{Thom}(B\text{G})$. We can compute the $E_2$ page, $E_3$ page and so on. Finally, $E_r$ page equals $E_{r+1}$ page (there are no differentials) for $r \geq N$, we call this $E_N$ page as the $E_\infty$ page, we can read the result $\pi_d$ at $d = t - s$. See further details discussed in Ref. [29]'s Sec. 2.3.

(2). In Adams spectral sequence, we consider $\text{Ext}^{s,t}_R(L, Z_p)$. Here we have the ring or the algebra $R = A_p$ or $A_2(1)$ for $p = 2$, and $L$ is an $R$-module. The $A_2(1)$ is the subalgebra of $A_2$ generated by the Steenrod squares $\text{Sq}^1$ and $\text{Sq}^2$. The index $s$ refers to the degree of resolution, and the index $t$ is the internal degree of the $R$-module $L$. Ext groups are defined by firstly taking a projective $R$-resolution $P_*$ of $L$, then secondly computing the (co)homology group of the (co)chain complex $\text{Hom}(P_*, Z_p)$. Here a projective $R$-resolution $P_*$ is an exact sequence of $R$-modules $\cdots \rightarrow P_s \rightarrow P_{s-1} \rightarrow \cdots \rightarrow P_0 \rightarrow L$ where $P_s$ is projective for $s \geq 0$.

For $Y = MTG$, where $MTG$ is the Madsen-Tillmann spectrum of a group $G$, the Adams spectral sequence shows:

$$\text{Ext}^{s,t}_{A_p}(H^*(MTG, Z_p), Z_p) \Rightarrow \pi_{t-s}(MTG) = (\Omega_{d=t-s}^G)_{p}.$$  (1.10)

The last equality is by the generalized Pontryagin-Thom isomorphism, we have an equality between the $d$-th bordism group of $G$ given by $\Omega_d^G$ and the $d$-th homotopy group of $MTG$ given by $\pi_d(MTG)$, namely

$$\Omega_d^G = \pi_d(MTG).$$  (1.11)

The $T$ in $MTG$ means that the $G$-structures are on tangent bundles instead of normal bundles. For Spin, the Madsen-Tillmann spectrum $MT\text{Spin} = MS\text{pin}$ is equivalent to the Thom spectrum.

We also compute the cobordism group of topological phases (TP) defined in [28] as

$$\text{TP}_d(G).$$  (1.12)

The $\text{TP}_d(G)$ classifies deformation classes of reflection positive invertible $d$-dimensional extended topological field theories with symmetry group $G_d$. The $\text{TP}_d(G)$ and the bordism group $\Omega_d^G$ are related by a short exact sequence

$$0 \rightarrow \text{Ext}^1(\Omega_d^G, Z) \rightarrow \text{TP}_d(G) \rightarrow \text{Hom}(\Omega_{d+1}^G, Z) \rightarrow 0.$$  (1.13)

We can compute the $E_2$ page of $A_2(1)$-module based on Lemma 11 of [29]. More precisely, in order to compute $\text{Ext}^{s,t}_{A_2(1)}(L_2, Z_2)$, we find a short exact sequence of $A_2(1)$-modules

$$0 \rightarrow L_1 \rightarrow L_2 \rightarrow L_3 \rightarrow 0,$$  (1.14)

then we apply Lemma 11 of [29] to compute $\text{Ext}^{s,t}_{A_2(1)}(L_2, Z_2)$ by the given data of $\text{Ext}^{s,t}_{A_2(1)}(L_1, Z_2)$ and $\text{Ext}^{s,t}_{A_2(1)}(L_3, Z_2)$. Our strategy is choosing $L_1$ to be the direct sum of suspensions of $L_2$ on which $\text{Sq}^1$ and $\text{Sq}^2$ act trivially, then we take $L_3$ to be the quotient of $L_2$ by $L_1$. We can use this procedure again and again until $\text{Ext}^{s,t}_{A_2(1)}(L_3, Z_2)$ is determined.

If $G = \text{Spin} \times G'$, then $BG = B\text{Spin} \times B\text{G}'$. By definition, the Madsen-Tillmann spectrum $MTG = \text{Thom}(BG, V)$ where $V$ is the induced virtual bundle of dimension 0 by the map $BG \rightarrow BO$. By the properties of Thom space (see further details discussed in Ref. [29]'s Sec. 1.3), we have

$$MTG = MS\text{pin} \wedge (BG')_+.$$  (1.15)
The ∧ is the smash product.\textsuperscript{15} The \((BG')_+\) is the disjoint union of the classifying space \(BG'\) and a point.\textsuperscript{16}

Below we will use the (1.10) and (1.11) to compute the \(d\)-th bordism group of \(G\) given by \(\Omega_G^d\). Then we will use (1.13) to compute the \(d\)-th cobordism group of topological phases of \(G\) given by \(\text{TP}_d(G)\).

By [65], the Thom spectrum \(M\text{Spin}\) splits as \(ko \cup \Sigma^8 ko \cup \cdots\) where \(ko\) is the \((-1)\)-connected cover of the real K-theory spectrum, so \(\pi_d(M\text{Spin} \vee X) = \pi_d(ko \vee X)\) for \(d < 8\) and any spectrum \(X\). The ∧ is the smash product and the ∨ is the wedge sum of spectra.

By the theorem of change of rings/Frobenius reciprocity, we have

\[
\text{Ext}_{\mathcal{A}_2}^{s,t}(\mathcal{A}_2 \otimes_{\mathcal{A}_2(1)} H^*(X, \mathbb{Z}_2), \mathbb{Z}_2) = \text{Ext}_{\mathcal{A}_2(1)}^{s,t}(H^*(X, \mathbb{Z}_2), \mathbb{Z}_2).\tag{1.17}
\]

If \(MTG = M\text{Spin} \vee X\) where \(X\) is any spectrum, then \(\pi_d(MTG) = \pi_d(ko \vee X)\) for \(d < 8\). So for the dimension \(d = t - s < 8\), we have

\[
\text{Ext}_{\mathcal{A}_2(1)}^{s,t}(H^*(X, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow (\Omega_{t-s}^G)^\wedge.\tag{1.18}
\]

The \(H^*(X, \mathbb{Z}_2)\) is an \(\mathcal{A}_2(1)\)-module whose internal degree \(t\) is given by the ∗.

Throughout the article, we will use \(\text{CS}_{2n-1}^V\) to denote the Chern-Simons \(2n - 1\)-form for the Chern class (if \(V\) is a complex vector bundle) or the Pontryagin class (if \(V\) is a real vector bundle). Note that \(p_i(V) = (-1)^i e_{2i}(V \otimes \mathbb{C})\). The relation between the Chern-Simons form and the Chern class is

\[
c_n(V) = d\text{CS}_{2n-1}^V\tag{1.19}
\]

where \(d\) is the exterior differential and \(c_n(V)\) is regarded as a closed differential form in de Rham cohomology.

There is also another kind of Chern-Simons form for Euler class [66], we denote it by \(\text{CS}_{2n-1,e}^V\), it satisfies

\[
e_{2n}(V) = d\text{CS}_{2n-1,e}^V.\tag{1.20}
\]

The relations between Pontryagin class, Euler class and Stiefel-Whitney class are

\[
p_i(V) = w_{2i}(V)^2 \mod 2\tag{1.21}
\]
and
\[ e_{2n}(V) = w_{2n}(V) \mod 2 \] (1.22)
where \( 2n = \dim V \).

By the Hirzebruch signature theorem, the relation between the signature and the first Pontryagin class of a 4-manifold \( M \) is
\[ \sigma = \frac{1}{3} \int_M p_1(TM). \] (1.23)

We use the standard notation \( w_i \) for the Stiefel-Whitney class, \( c_i \) for the Chern class, \( p_i \) for the Pontryagin class, \( e_n \) for the Euler class. Note that the Euler class only appears in the total dimension of the vector bundle. We use the notation \( w_i(G) \), \( c_i(G) \), \( p_i(G) \) and \( e_n(G) \) to denote the characteristic classes of the associated vector bundle of the principal \( G \) bundle (normally denoted as \( w_i(V_G) \), \( c_i(V_G) \), \( p_i(V_G) \) and \( e_n(V_G) \)).

## 2 Standard Models

Now we consider the co/bordism classes relevant for Standard Model (SM) physics [3–5].

### 2.1 Spin \( \times \) SU(3) \( \times \) SU(2) \( \times \) U(1) model

We consider \( G = \text{Spin} \times \text{SU}(3) \times \text{SU}(2) \times \text{U}(1) \), the Madsen-Tillmann spectrum \( MTG \) of the group \( G \)
is
\[ MTG = M\text{Spin} \wedge (B(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)))_+. \] (2.1)

The \( (B(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)))_+ \) is the disjoint union of the classifying space \( B(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)) \) and a point, see footnote 16.

For the dimension \( d = t - s < 8 \), since there is no odd torsion,\(^{17}\) for \( MTG = M\text{Spin} \wedge (B(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))))_+ \), for the dimension \( d = t - s < 8 \), by (1.18), we have the Adams spectral sequence
\[ \text{Ext}^{s,t}_{\mathcal{A}_2(1)}(H^*(B(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)), \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\text{Spin} \times \text{SU}(3) \times \text{SU}(2) \times \text{U}(1)}. \] (2.2)

We have
\[ H^*(B\text{SU}(n), \mathbb{Z}_2) = \mathbb{Z}_2[c_2, \ldots, c_n] \] (2.3)
and
\[ H^*(B\text{U}(n), \mathbb{Z}_2) = \mathbb{Z}_2[c_1, \ldots, c_n]. \] (2.4)

\(^{17}\)By computation using the mod \( p \) Adams spectral sequence for an odd prime \( p \), we find there is no odd torsion.
We also have the Wu formula\textsuperscript{18}

\[ Sq^{2j}(c_i) = \sum_{k=0}^{j} \binom{i - k - 1}{j - k} c_{i+j-k} c_k \text{ for } 0 \leq j \leq i. \] (2.6)

By Künneth formula, we have

\[ H^*(B(SU(3) \times SU(2) \times U(1)), \mathbb{Z}_2) = \mathbb{Z}_2[c_2, c_3] \otimes \mathbb{Z}_2[c_2'] \otimes \mathbb{Z}_2[c_1'']. \] (2.7)

The \( \mathcal{A}_2(1) \)-module structure of \( H^*(B(SU(3) \times SU(2) \times U(1)), \mathbb{Z}_2) \) below degree 6 and the \( E_2 \) page are shown in Figure 1, 2. Here we have used the correspondence between \( \mathcal{A}_2(1) \)-module structure and the \( E_2 \) page shown in Figure 31 and 33.

Figure 1: The \( \mathcal{A}_2(1) \)-module structure of \( H^*(B(SU(3) \times SU(2) \times U(1)), \mathbb{Z}_2) \) below degree 6.

\textsuperscript{18}There is another Wu formula which will be used later:

\[ Sq^i(x_{d-i}) = u_i x_{d-i} \] (2.5)

on \( d \)-manifold \( M \) where \( x_{d-i} \in H^{d-i}(M, \mathbb{Z}_2) \) and \( u_i \) is the Wu class.
In Adams chart, the horizontal axis labels the integer degree $d = t - s$ and the vertical axis labels the integer degree $s$. The differential $d_{r,t}^{s,t}: E_{r,t}^{s,t} \to E_{r+1,t}^{s+r,t+r-1}$ is an arrow starting at the bidegree $(t - s, s)$ with direction $(-1, r)$. $E_{r+1}^{s,t} := \frac{\text{Ker} d_{r,t}^{s,t}}{\text{Im} d_{r,t}^{s,t}}$ for $r \geq 2$. There exists $N$ such that $E_{N+k} = E_N$ stabilized for $k > 0$, we denote the stabilized page $E_\infty := E_N$.

To read the result from the Adams chart in Figure 2, we look at the stabilized $E_\infty$ page, one dot indicates a finite group $\mathbb{Z}_p$, a vertical finite line segment connecting $n$ dots indicates a finite group $\mathbb{Z}_{p^n}$. But when $n = \infty$, the infinite line connecting infinite dots indicates a $\mathbb{Z}$. Here $p$ is given by the mod $p$ Adams spectral sequence in (1.10). Here in Figure 2, $p = 2$, we can read from the Adams chart $\Omega_{Spin}^{SU(3) \times SU(2) \times U(1)} = \mathbb{Z}$ (an infinite line), $\Omega_{Spin}^{SU(3) \times SU(2) \times U(1)} = \mathbb{Z}_2$ (a dot), $\Omega_{Spin}^{SU(3) \times SU(2) \times U(1)} = \mathbb{Z} \times \mathbb{Z}_2$ (an infinite line and a dot), $\Omega_{Spin}^{SU(3) \times SU(2) \times U(1)} = 0$ (nothing), $\Omega_{Spin}^{SU(3) \times SU(2) \times U(1)} = \mathbb{Z}^4$ (four infinite lines), $\Omega_{Spin}^{SU(3) \times SU(2) \times U(1)} = \mathbb{Z}_2$ (a dot), $\Omega_{Spin}^{SU(3) \times SU(2) \times U(1)} = \mathbb{Z}^5 \times \mathbb{Z}_2$ (five infinite lines and a dot). Thus we obtain the bordism group $\Omega_d^{Spin \times SU(3) \times SU(2) \times U(1)}$ shown in Table 1.
Bordism group

\begin{tabular}{c|c|c}
\hline
$d$ & $\Omega_d^{\text{Spin}\times\text{SU}(3)\times\text{SU}(2)\times\text{U}(1)}$ & generators \\
\hline
0 & $\mathbb{Z}$ &  \\
1 & $\mathbb{Z}_2$ & $\tilde{\eta}$ \\
2 & $\mathbb{Z} \times \mathbb{Z}_2$ & $c_1(\text{U}(1)), \text{Arf}$ \\
3 & 0 &  \\
4 & $\mathbb{Z}^4$ & $\frac{\sigma}{16}, \frac{c_1(\text{U}(1))^2}{2}, c_2(\text{SU}(2)), c_2(\text{SU}(3))$ \\
5 & $\mathbb{Z}_2$ & $c_2(\text{SU}(2))\tilde{\eta}$ \\
6 & $\mathbb{Z}^5 \times \mathbb{Z}_2$ & $\frac{c_1(\text{U}(1))(\sigma - F \cdot F)}{8}, c_1(\text{U}(1))^3, c_1(\text{U}(1))c_2(\text{SU}(2))$, $c_1(\text{U}(1))c_2(\text{SU}(3)), \frac{c_3(\text{SU}(3))}{2}, c_2(\text{SU}(2))\text{Arf}$ \\
\hline
\end{tabular}

Table 1: Bordism group. $\tilde{\eta}$ is a mod 2 index of 1d Dirac operator. Arf is a 2d Arf invariant. $\sigma$ is the signature of manifold. F is the characteristic 2-surface [67]. $F \cdot F$ is the product of two F. Note that $c_1(\text{U}(1))^2 = Sq^2c_1(\text{U}(1)) = (w_2 + w_1^2)c_1(\text{U}(1)) = 0 \mod 2$ on Spin 4-manifolds, $c_3(\text{SU}(3)) = Sq^2c_2(\text{SU}(3)) = (w_2 + w_1^2)c_2(\text{SU}(3)) = 0 \mod 2$ on Spin 6-manifolds.

By (1.13), we obtain the cobordism group $\text{TP}_d(\text{Spin} \times \text{SU}(3) \times \text{SU}(2) \times \text{U}(1))$ shown in Table 2.

\begin{tabular}{c|c|c}
\hline
$d$ & $\text{TP}_d(\text{Spin} \times \text{SU}(3) \times \text{SU}(2) \times \text{U}(1))$ & generators \\
\hline
0 & 0 &  \\
1 & $\mathbb{Z} \times \mathbb{Z}_2$ & $\text{CS}_1^{\text{U}(1)}, \tilde{\eta}$ \\
2 & $\mathbb{Z}_2$ & $\text{Arf}$ \\
3 & $\mathbb{Z}^4$ & $\frac{\text{CS}_1^{\text{U}(1)}}{8}, \frac{1}{2}c_1(\text{U}(1)), c_3(\text{SU}(2)), c_3(\text{SU}(3))$ \\
4 & 0 &  \\
5 & $\mathbb{Z}^5 \times \mathbb{Z}_2$ & $\mu(\text{PD}(c_1(\text{U}(1))))$, $\frac{1}{2}\text{CS}_1^{\text{U}(1)}c_1(\text{U}(1))^2, c_1(\text{U}(1))c_2(\text{SU}(2))$, $c_1(\text{U}(1))c_2(\text{SU}(3)), \frac{1}{7}\text{CS}_5^{\text{SU}(3)}, c_2(\text{SU}(2))\text{Arf}$ \\
\hline
\end{tabular}

Table 2: Topological phase classification ($\equiv \text{TP}$) as a cobordism group, following Table 1. $\tilde{\eta}$ is a mod 2 index of 1d Dirac operator. Arf is a 2d Arf invariant. $\sigma$ is the characteristic 2-surface [67]. $F \cdot F$ is the product of two F. The PD is the Poincaré dual. The $TM$ is the spacetime tangent bundle. The $\mu$ is the 3d Rokhlin invariant. If $\partial M^4 = M^3$, then $\mu(M^3) = ((\sigma - FF)/8)(M^4)$, thus $\mu(\text{PD}(c_1(\text{U}(1))))$ is related to $\frac{c_1(\text{U}(1))(\sigma - F \cdot F)}{8}$ in Table 1.

2.2 Spin $\times \frac{\text{SU}(3)\times\text{SU}(2)\times\text{U}(1)}{\mathbb{Z}_2}$ model

We consider $G = \text{Spin} \times \frac{\text{SU}(3)\times\text{SU}(2)\times\text{U}(1)}{\mathbb{Z}_2} = \text{Spin} \times \text{SU}(3) \times \text{U}(2)$, the Madsen-Tillmann spectrum $MTG$ of the group $G$ is

$$MTG = M\text{Spin} \wedge (B(\text{SU}(3) \times \text{U}(2)))_+.$$ (2.8)

The $(B(\text{SU}(3) \times \text{U}(2)))_+$ is the disjoint union of the classifying space $B(\text{SU}(3) \times \text{U}(2))$ and a point, see footnote 16.
For the dimension \( d = t - s < 8 \), since there is no odd torsion (see footnote 17), by (1.18), we have the Adams spectral sequence

\[
\text{Ext}^{s,t}_{\mathcal{A}_2(1)}(H^*(B(SU(3) \times U(2)), \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega^{\text{Spin} \times SU(3) \times U(2)}_{t-s}.
\] (2.9)

By Künneth formula, we have

\[
H^*(B(SU(3) \times U(2)), \mathbb{Z}_2) = \mathbb{Z}_2[c_2, c_3] \otimes \mathbb{Z}_2[c'_1, c'_2].
\] (2.10)

The \( \mathcal{A}_2(1) \)-module structure of \( H^*(B(SU(3) \times U(2)), \mathbb{Z}_2) \) below degree 6 and the \( E_2 \) page are shown in Figure 3, 4. Here we have used the correspondence between \( \mathcal{A}_2(1) \)-module structure and the \( E_2 \) page shown in Figure 31 and 33.

![Diagram](image)

Figure 3: The \( \mathcal{A}_2(1) \)-module structure of \( H^*(B(SU(3) \times U(2)), \mathbb{Z}_2) \) below degree 6.
Figure 4: $\Omega_{\text{Spin} \times SU(3) \times U(2)}$.

| $d$ | $\Omega_d^{\text{Spin} \times SU(3) \times U(2)}$ | generators                  |
|-----|---------------------------------|----------------------------|
| 0   | $\mathbb{Z}$                    |                            |
| 1   | $\mathbb{Z}_2$                  | $\tilde{\eta}$             |
| 2   | $\mathbb{Z} \times \mathbb{Z}_2$| $c_1(U(2))$, Arf           |
| 3   |                                 |                            |
| 4   | $\mathbb{Z}^4$                  | $\frac{\sigma}{16}$, $c_1(U(2))^2$, $c_2(SU(3))$, $c_2(U(2))$ |
| 5   |                                 |                            |
| 6   | $\mathbb{Z}^5$                  | $\frac{c_1(U(2))\sigma - F \cdot F}{8}$, $c_3(U(2))$, $c_1(U(2))c_2(U(2))$, $c_1(U(2))^3$, $c_1(U(2))c_2(SU(3))$, $c_2(SU(3))^2$ |

Table 3: Bordism group. $\tilde{\eta}$ is a mod 2 index of 1d Dirac operator. Arf is a 2d Arf invariant. $\sigma$ is the signature of manifold. $F$ is the characteristic 2-surface [67]. $F \cdot F$ is the product of two $F$. Note that $c_1(U(2))^2 = \text{Sq}^2 c_1(U(2)) = (w_2 + w_1^2)c_1(U(2)) = 0$ mod 2 on Spin 4-manifolds, $c_3(SU(3)) = \text{Sq}^2 c_2(SU(3)) = (w_2 + w_1^2)c_2(SU(3)) = 0$ mod 2 on Spin 6-manifolds.
Cobordism group

| d | $\text{TP}_d(\text{Spin} \times \frac{\text{SU}(3)\times\text{SU}(2)\times\text{U}(1)}{\mathbb{Z}_2})$ | generators |
|---|---|---|
| 0 | 0 |  |
| 1 | $\mathbb{Z} \times \mathbb{Z}_2$ | $\text{CS}_1^{\text{U}(2)}, \tilde{\eta}$ |
| 2 | $\mathbb{Z}_2$ | $\text{Arf}$ |
| 3 | $\mathbb{Z}^4$ | $\frac{\text{CS}_1^{\text{SU}(3)}}{\mathbb{R}}, \frac{1}{7}\text{CS}_1^{\text{U}(2)} c_1(\text{U}(2)), \text{CS}_3^{\text{U}(2)} c_2(\text{U}(2)), \text{CS}_1^{\text{SU}(3)} c_1(\text{U}(2))^2, \text{CS}_1^{\text{U}(2)} c_2(\text{SU}(3))$ |
| 4 | 0 |  |
| 5 | $\mathbb{Z}^5$ | $\mu(\text{PD}(c_1(\text{U}(2)))), \frac{1}{7}\text{CS}_5^{\text{SU}(3)}, \frac{1}{7}\text{CS}_1^{\text{U}(2)} c_2(\text{U}(2)), \text{CS}_1^{\text{U}(2)} c_1(\text{U}(2))^2, \text{CS}_1^{\text{U}(2)} c_2(\text{SU}(3))$ |

Table 4: Topological phase classification ($\equiv \text{TP}$) as a cobordism group, following Table 3. $\tilde{\eta}$ is a mod 2 index of 1d Dirac operator. Arf is a 2d Arf invariant. F is the characteristic 2-surface [67]. $F \cdot F$ is the product of two F. The PD is the Poincaré dual. The $TM$ is the spacetime tangent bundle. The $\mu$ is the 3d Rokhlin invariant. If $\partial M^4 = M^3$, then $\mu(M^3) = ((\sigma - FF)/8)(M^4)$, thus $\mu(\text{PD}(c_1(\text{U}(2))))$ is related to $\frac{1}{8}(c_1(\text{U}(2))(\sigma - F \cdot F))$ in Table 3.

2.3 Spin $\times \frac{\text{SU}(3)\times\text{SU}(2)\times\text{U}(1)}{\mathbb{Z}_3}$ model

We consider $G = \text{Spin} \times \frac{\text{SU}(3)\times\text{SU}(2)\times\text{U}(1)}{\mathbb{Z}_3} = \text{Spin} \times \text{U}(3) \times \text{SU}(2)$, the Madsen-Tillmann spectrum $MTG$ of the group $G$ is

$$MTG = M\text{Spin} \wedge (\text{B}(\text{U}(3) \times \text{SU}(2)))_+. \quad (2.11)$$

The $(\text{B}(\text{U}(3) \times \text{SU}(2)))_+$ is the disjoint union of the classifying space $\text{B}(\text{U}(3) \times \text{SU}(2))$ and a point, see footnote 16.

The localization of $M\text{Spin}$ at the prime 3 is the wedge sum of suspensions of the Brown-Peterson spectrum $BP$ (here $M\text{Spin}_{(3)} = BP \vee \Sigma^5 BP \vee \cdots$) and $H^*(BP, \mathbb{Z}_3) = A_3/(\beta_{(3,3)})$ where $(\beta_{(3,3)})$ is the two-sided ideal generated by $\beta_{(3,3)}$, and $\beta_{(3,3)}$ is the Bockstein homomorphism associated to the extension $\mathbb{Z}_3 \to \mathbb{Z}_9 \to \mathbb{Z}_3$. Note that

$$\cdots \to \Sigma^2 A_3 \oplus \Sigma^6 A_3 \oplus \cdots \to \Sigma A_3 \oplus \Sigma^5 A_3 \oplus \cdots \to A_3 \to A_3/(\beta_{(3,3)}) \quad (2.12)$$

is an $A_3$-resolution of $A_3/(\beta_{(3,3)})$ (denoted by $P_\bullet$) where the differentials $d_1$ are induced by $\beta_{(3,3)}$.

The Adams chart of $\text{Ext}^*_{A_3}(H^*(M\text{Spin}, \mathbb{Z}_3), \mathbb{Z}_3)$ is shown in Figure 5. $P_\bullet \otimes H^*(\text{B}(\text{U}(3) \times \text{SU}(2)), \mathbb{Z}_3)$ is a projective $A_3$-resolution of $H^*(BP, \mathbb{Z}_3) \otimes H^*(\text{B}(\text{U}(3) \times \text{SU}(2)), \mathbb{Z}_3)$ (since $P_\bullet$ is actually a free $A_3$-resolution), the differentials $d_1$ are induced by $\beta_{(3,3)}$. 

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We have the Adams spectral sequence
\[ \text{Ext}_{A_3}^{s,t}(H^*(M\text{Spin}, \mathbb{Z}_3) \otimes H^*(B(U(3) \times SU(2)), \mathbb{Z}_3), \mathbb{Z}_3) \Rightarrow (\Omega^{\text{Spin} \times U(3) \times SU(2)}_{t-s})^\wedge. \]  
\[ (2.13) \]

By Künneth formula, we have
\[ H^*(B(U(3) \times SU(2)), \mathbb{Z}_3) = \mathbb{Z}_3[c_1, c_2, c_3] \otimes \mathbb{Z}_3[e_2']. \]  
\[ (2.14) \]

The Adams chart of \( \text{Ext}_{A_3}^{s,t}(H^*(M\text{Spin}, \mathbb{Z}_3) \otimes H^*(B(U(3) \times SU(2)), \mathbb{Z}_3), \mathbb{Z}_3) \) is shown in Figure 6. There is no differential since the arrow of the differential \( d_r \) is of bidegree \((-1, r)\), while all lines are of interval 2 at degree \( t-s \).

So there is actually no 3-torsion in \( \Omega^{\text{Spin} \times U(3) \times SU(2)}_d \).
For the dimension $d = t - s < 8$, since there is no odd torsion (see footnote 17), by (1.18), we have the Adams spectral sequence

$$\text{Ext}^{s,t}_{\mathcal{A}_3}(H^*(M\text{Spin}, \mathbb{Z}_3) \otimes H^*(B(U(3) \times SU(2)), \mathbb{Z}_3), \mathbb{Z}_3).$$

(2.15) By Künneth formula, we have

$$H^*(B(U(3) \times SU(2)), \mathbb{Z}_2) = \mathbb{Z}_2[c_1, c_2, c_3] \otimes \mathbb{Z}_2[c'_2].$$

(2.16) The $\mathcal{A}_2(1)$-module structure of $H^*(B(U(3) \times SU(2)), \mathbb{Z}_2)$ below degree 6 and the $E_2$ page are shown in Figure 7, 8. Here we have used the correspondence between $\mathcal{A}_2(1)$-module structure and the $E_2$ page shown in Figure 31 and 33.
Figure 7: The $A_2(1)$-module structure of $H^*(B(U(3) \times SU(2)), \mathbb{Z}_2)$ below degree 6.

Figure 8: $\Omega_{\text{Spin} \times U(3) \times SU(2)}$
Bordism group

| $d$ | $\Omega^*_d \text{Spin} \times \text{SU(3) \times SU(2) \times U(1)} \mathbb{Z}_6$ | generators |
|-----|-------------------------------------------------|-------------|
| 0   | $\hat{Z}$                                        |             |
| 1   | $\hat{Z} \times \hat{Z}$                        | $c_1(\text{U}(3))$, $\text{Arf}$ |
| 2   | $\mathbb{Z}^4$                                    | $c_2(\text{SU(2)})\hat{\eta}$ |
| 3   | 0                                               |             |
| 4   | $\mathbb{Z}^5 \times \hat{Z}_2$                  | $c_1(\text{U}(3))^2$, $c_2(\text{U}(3))$, $c_2(\text{SU(2)})$, $c_2(\text{SU(2)})\hat{\eta}$ |

Table 5: Bordism group. $\hat{\eta}$ is a mod 2 index of 1d Dirac operator. Arf is a 2d Arf invariant. $\sigma$ is the signature of manifold. F is the characteristic 2-surface [67]. $F \cdot F$ is the product of two F. Note that $c_1(\text{U}(3))^2 = \text{Sq}^2 c_1(\text{U}(3)) = (w_2 + w_1^2) c_1(\text{U}(3)) = 0 \mod 2$ on Spin 4-manifolds, $c_1(\text{U}(3)) c_2(\text{U}(3)) + c_3(\text{U}(3)) = \text{Sq}^2 c_2(\text{U}(3)) = (w_2 + w_1^2) c_2(\text{U}(3)) = 0 \mod 2$ on Spin 6-manifolds.

| $d$ | $\text{TP}_d(\text{Spin} \times \text{SU(3) \times SU(2) \times U(1)}) \mathbb{Z}_6$ | generators |
|-----|---------------------------------------------------------------------------------|-------------|
| 0   | 0                                                                               |             |
| 1   | $\hat{Z} \times \hat{Z}_2$                                                      | $\text{CS}_1(\text{U}(3)), \hat{\eta}$ |
| 2   | $\hat{Z}_2$                                                                     | $\text{Arf}$ |
| 3   | $\mathbb{Z}^4$                                                                   | $\text{CS}_1(\text{U}(3)), \text{CS}_3(\text{U}(3)), \text{CS}_3(\text{SU}(2))$ |
| 4   | 0                                                                               |             |
| 5   | $\mathbb{Z}^5 \times \hat{Z}_2$                                                  | $\mu(\text{PD}(c_1(\text{U}(3)))), \text{CS}_1(\text{U}(3))^2$, $\text{CS}_1(\text{U}(3))^2$, $\text{CS}_1(\text{U}(3))^2$, $\text{CS}_3(\text{U}(3))^2$, $\text{CS}_3(\text{SU}(2))$, $\text{CS}_3(\text{SU}(2))\hat{\eta}$ |

Table 6: Topological phase classification ($\equiv \text{TP}$) as a cobordism group, following Table 5. $\hat{\eta}$ is a mod 2 index of 1d Dirac operator. Arf is a 2d Arf invariant. $\sigma$ is the characteristic 2-surface [67]. $F \cdot F$ is the product of two F. The PD is the Poincaré dual. The $TM$ is the spacetime tangent bundle. The $\mu$ is the 3d Rokhlin invariant. If $\partial M = M^3$, then $\mu(\text{PD}(c_1(\text{U}(3)))) = (\wedge_3 F)^2/8$, thus $\mu(\text{PD}(c_1(\text{U}(3))))$ is related to $c_1(\text{U}(3))^2$ in Table 5.

2.4 $\text{Spin} \times \frac{\text{SU(3) \times SU(2) \times U(1)}}{\mathbb{Z}_6}$ model

We consider $G = \text{Spin} \times \frac{\text{SU(3) \times SU(2) \times U(1)}}{\mathbb{Z}_6} = \text{Spin} \times S(\text{U}(3) \times \text{U}(2))$. This SM group is particularly interesting because:

$$\text{SO}(10) \supset \text{SU}(5) \supset \frac{\text{SU(3) \times SU(2) \times U(1)}}{\mathbb{Z}_6}. \quad (2.17)$$

Thus the $\text{SO}(10)$ GUT and $\text{SO}(5)$ GUT can be Higgs down to $\frac{\text{SU(3) \times SU(2) \times U(1)}}{\mathbb{Z}_6}$ SM. We have

$$\frac{(\text{SU(3) \times SU(2) \times U(1)})}{\mathbb{Z}_6} = \frac{(\text{SU(3) \times U(2)})}{\mathbb{Z}_3} = \frac{(\text{SU(2) \times U(3)})}{\mathbb{Z}_2}. \quad (2.18)$$
Let the hypercharge be:

\[
Y = \text{diag}(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{2}, \frac{1}{2}).
\]  

(2.19)

The group \( \frac{\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)}{\mathbb{Z}_6} \) is just the subgroup of SU(5) commuting with the group generated by \( Y \).

The subgroup is \( \text{S(U}(3) \times \text{U}(2)) = (A, B) \in \text{U}(3) \times \text{U}(2) \mid \det A \cdot \det B = 1 \) and \( \text{(SU}(3) \times \text{SU}(2) \times \text{U}(1)) / \mathbb{Z}_6 = \text{S(U}(3) \times \text{U}(2)) \subset \text{SU}(5) \).

Since

\[
H^*(\text{B}(\frac{\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)}{\mathbb{Z}_6}), \mathbb{Z}_3) = H^*(\text{B}(\frac{\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)}{\mathbb{Z}_3}), \mathbb{Z}_3),
\]  

(2.20)

similarly as the discussion in Sec. 2.3, there is no 3-torsion in \( \Omega^\text{Spin\times S(U}(3) \times \text{U}(2))_{d} \).

The Madsen-Tillmann spectrum \( MTG \) of the group \( G \) is

\[
MTG = \text{MSpin} \wedge (\text{B}(\text{SU}(3) \times \text{U}(2))))^+.
\]  

(2.21)

The \( (\text{B}(\text{SU}(3) \times \text{U}(2))))^+ \) is the disjoint union of the classifying space \( \text{B}(\text{SU}(3) \times \text{U}(2)) \) and a point, see footnote 16.

For the dimension \( d = t - s < 8 \), since there is no odd torsion (see footnote 17), by (1.18), we have the Adams spectral sequence

\[
\text{Ext}^s_{A_2(1)}(H^*(\text{B}(\text{SU}(3) \times \text{U}(2))), \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega^\text{Spin\times S(U}(3) \times \text{U}(2))_{t-s}.
\]  

(2.22)

We have the following commutative diagram with exact columns

\[
\begin{array}{ccc}
\text{S(U}(3) \times \text{U}(2)) & \rightarrow & \text{SU}(5) \\
\downarrow & & \downarrow \\
\text{U}(3) \times \text{U}(2) & \rightarrow & \text{U}(5) \\
\downarrow \text{det} & & \downarrow \text{det} \\
\text{U}(1) & & \text{U}(1).
\end{array}
\]  

(2.23)

So by Kunneth formula, we have

\[
H^*(\text{B}(\text{SU}(3) \times \text{U}(2))), \mathbb{Z}_2) = \mathbb{Z}_2[c_1, c_2, c_3] \otimes \mathbb{Z}_2[c'_1, c'_2]/(c_1 = c'_1).
\]  

(2.24)

The \( A_2(1) \)-module structure of \( H^*(\text{B}(\text{SU}(3) \times \text{U}(2))), \mathbb{Z}_2) \) below degree 6 and the \( E_2 \) page are shown in Figure 9, 10. Here we have used the correspondence between \( A_2(1) \)-module structure and the \( E_2 \) page shown in Figure 31 and 33.
Figure 9: The $A_2(1)$-module structure of $H^*(B(S(U(3) \times U(2)), \mathbb{Z}_2)$ below degree 6.

![Diagram](image)

Figure 10: $\Omega^\text{Spin\times S(U(3)\times U(2))}_\ast$.
Table 7: Bordism group. £η is a mod 2 index of 1d Dirac operator. Arf is a 2d Arf invariant. σ is the signature of manifold. F is the characteristic 2-surface \([\sigma - F \cdot F]\). F \cdot F is the product of two F. Here \(c_1(\text{U}(3))\) is identified with \(c_1(\text{U}(2))\). Note that \(c_1(\text{U}(3))^2 = Sq^2 c_1(\text{U}(3)) = (w_2 + w_1^2) c_1(\text{U}(3)) = 0 \) mod 2 on Spin 4-manifolds, \(c_1(\text{U}(3)) c_2(\text{U}(3)) + c_3(\text{U}(3)) = Sq^2 c_2(\text{U}(3)) = (w_2 + w_1^2) c_2(\text{U}(3)) = 0 \) mod 2 on Spin 6-manifolds.

Table 8: Topological phase classification (≡ TP) as a cobordism group, following Table 7. £η is a mod 2 index of 1d Dirac operator. Arf is a 2d Arf invariant. F is the characteristic 2-surface \([\sigma - F \cdot F]\). F \cdot F is the product of two F. The PD is the Poincaré dual. The T M is the spacetime tangent bundle. The μ is the 3d Rokhlin invariant. If \(\partial M^4 = M^3\), then \(\mu(M^3) = ((\sigma - F F)/8)(M^4)\), thus \(\mu(\text{PD}(c_1(\text{U}(3))))\) is related to \(c_1(\text{U}(3))(\sigma - F F)/8\) in Table 7.

2.5 Comparison

- Our approach [29, 30] is based on Adams spectral sequence (ASS), which includes the more refined data, containing both module and group structure, thus with the benefits of having less differentials. In addition, we can conveniently read and extract the iTQFT (namely, co/bordism invariants) from the Adams chart [29, 30].

- Ref. [47, 59] is based on Atiyah-Hirzebruch spectral sequence (AHSS), which includes only the group structure, but with the disadvantage of having more differentials and some undetermined extensions.

Specifically, in Ref. [59], using Atiyah-Hirzebruch spectral sequence, the authors compute the cobordism groups \(\Omega^d_{\text{Spin}}(B(SU(3) \times SU(2) \times U(1)))\) for \(0 \leq d \leq 5\) and \(\Gamma = 1, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_6\), but their re-
sult for the $\Gamma = \mathbb{Z}_6$ case is not completely determined. Ref. [59] determines the cobordism groups $\Omega^\text{Spin}_2(B(\frac{SU(3) \times SU(2) \times U(1)}{\mathbb{Z}_6}))$ in 2d and 4d as some undetermined extensions. Namely, they obtain that $\Omega^\text{Spin}_2(B(\frac{SU(3) \times SU(2) \times U(1)}{\mathbb{Z}_6})) = e(\mathbb{Z}_3, \mathbb{Z} \times \mathbb{Z}_2)$, and $\Omega^\text{Spin}_4(B(\frac{SU(3) \times SU(2) \times U(1)}{\mathbb{Z}_6})) = e(\mathbb{Z}_3, e(\mathbb{Z}_3, \mathbb{Z}_4))$ where $e(Q, N)$ is the group extension of $Q$ by $N$. So the group $e(Q, N)$ fits into the short exact sequence $0 \to N \to e(Q, N) \to Q \to 0$ but it may not be uniquely determined.

In contrast, our results are more refined and can uniquely determine the extension in this case. Our result from Adams spectral sequence demonstrates that each step of extensions is nontrivial, while the trivial extension yields 3-torsion. Using Adams spectral sequence, we find that there is no 3-torsion for the $\Gamma = \mathbb{Z}_6$ case. So we also provide the solutions to the extension problems in Ref. [59], given by the nontrivial extension $\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_3$. We obtained the precise answer

$$\Omega^\text{Spin}_2(B(\frac{SU(3) \times SU(2) \times U(1)}{\mathbb{Z}_6})) = \mathbb{Z} \times \mathbb{Z}_2$$

and

$$\Omega^\text{Spin}_4(B(\frac{SU(3) \times SU(2) \times U(1)}{\mathbb{Z}_6})) = \mathbb{Z}_4.$$  

### 3 Standard Models with additional discrete symmetries

#### 3.1 $\text{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4 \times SU(3) \times SU(2) \times U(1)$ model

Inspired by the models of [47,63], below we consider the co/bordism classes relevant for Standard Models with additional discrete symmetries.\(^{19}\)

We consider $G = \text{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4 \times SU(3) \times SU(2) \times U(1)$.

We have a homotopy pullback square

$$\begin{array}{ccc}
B(\text{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4) & \longrightarrow & \mathbb{BZ}_2 \\
\downarrow & & \downarrow \alpha^2 \\
\mathbb{BSO} & \longrightarrow & \mathbb{B}^2\mathbb{Z}_2 \end{array}$$

\hspace{1cm}(3.1)

where $\alpha$ is the generator of $H^1(\mathbb{BZ}_2, \mathbb{Z}_2)$.

By [69], since there is a homotopy pullback square

$$\begin{array}{ccc}
B(\text{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4) & \longrightarrow & * \times \mathbb{BZ}_2 \\
\downarrow & & \downarrow \\
\mathbb{BO} & \longrightarrow & K(\mathbb{Z}_2, 1) \times K(\mathbb{Z}_2, 2) \end{array}$$

\hspace{1cm}(3.2)

\(^{19}\)JW is grateful to Miguel Montero [47] for informing his unpublished note [63]. To make comparison, our approach is based on the Adams spectral sequence, while [47, 63] uses Atiyah-Hirzebruch spectral sequence (AHSS). Two approaches between ours [29, 30] and Garcia-Etxebarria-Montero [47, 63] are rather different. We will not make any attempts to interpret the data of these models and leave physics implications for future work, see [63, 68].
which is equivalent to the homotopy pullback square

$$\begin{array}{ccc}
B(\text{Spin} \times_{Z_2} Z_4) & \longrightarrow & B\text{Spin} \\
\downarrow & & \downarrow \\
BSO \times BZ_2 & \longrightarrow & BSO,
\end{array}$$

(3.3)

$MT(\text{Spin} \times_{Z_2} Z_4) = M\text{Spin} \wedge (BZ_2)^{2\xi} = M\text{Spin} \wedge \Sigma^{-2}\mathbb{R}P_2^\infty$ where $2\xi : BZ_2 \to BSO$ is twice the sign representation, the final identification is by [70].

The Madsen-Tillmann spectrum $MTG$ of the group $G$ is

$$MTG = M\text{Spin} \wedge (BZ_2)^{2\xi} \wedge (B(SU(3) \times SU(2) \times U(1)))_+$$

$$= M\text{Spin} \wedge \Sigma^{-2}\mathbb{R}P_2^\infty \wedge (B(SU(3) \times SU(2) \times U(1)))_+.$$  (3.4)

The $(B(SU(3) \times SU(2) \times U(1)))_+$ is the disjoint union of the classifying space $B(SU(3) \times SU(2) \times U(1))$ and a point, see footnote 16.

For the dimension $d = t - s < 8$, since there is no odd torsion (see footnote 17), by (1.18), we have the Adams spectral sequence

$$\text{Ext}_{A_2(1)}^{s,t}(H^{*+2}(\mathbb{RP}_2^\infty, Z_2) \otimes H^*(B(SU(3) \times SU(2) \times U(1)), Z_2), Z_2) \Rightarrow \Omega^{Spin \times_{Z_2} Z_4 \times SU(3) \times SU(2) \times U(1)}_{t-s}.$$  (3.5)

The $A_2(1)$-module structure of $H^{*+2}(\mathbb{RP}_2^\infty, Z_2)$ is shown in Figure 11.

Figure 11: The $A_2(1)$-module structure of $H^{*+2}(\mathbb{RP}_2^\infty, Z_2)$.

The $A_2(1)$-module structure of $H^{*+2}(\mathbb{RP}_2^\infty, Z_2) \otimes H^{*+2}(C\eta, Z_2)$ is shown in Figure 12. Here $\eta : S^3 \to S^2$ is the Hopf fibration, the mapping cone is $C\eta = S^2 \cup e^1 = \mathbb{C}P^2$. The $A_2(1)$-module structure of $H^{*+2}(C\eta, Z_2)$ has two elements in degree 0 and 2 attached by a $\text{Sq}^2$.

\footnote{Beware that we use $\eta$ to denote the $\eta$ invariant (e.g. the APS $\eta$ invariant), while we use the up-greek font eta $\eta : S^3 \to S^2$ to denote the Hopf fibration.}
Based on Figure 1 and 12, we obtain the \( A_2(1) \)-module structure of \( H^{*+2}(\mathbb{R}\mathbb{P}_2^\infty, \mathbb{Z}_2) \otimes H^{*+2}(\mathbb{C}_\eta, \mathbb{Z}_2) \) below degree 6, as shown in Figure 13.

The \( E_2 \) page is shown in Figure 14. Here we have used the correspondence between \( A_2(1) \)-module structure and the \( E_2 \) page shown in Figure 33, 34 and 35.
Figure 14: $\Omega_s^{\text{Spin} \times \mathbb{Z}_2 \times \text{SU}(3) \times \text{SU}(2) \times \text{U}(1)}$.

| $d$ | $\Omega_d^{\text{Spin} \times \mathbb{Z}_2 \times \text{SU}(3) \times \text{SU}(2) \times \text{U}(1)}$ | generators |
|-----|-------------------------------------------------|-------------|
| 0   | $\mathbb{Z}$                                   |             |
| 1   | $\mathbb{Z}_4$                                 | $\eta'$     |
| 2   | $\mathbb{Z}$                                   | $c_1(\text{U}(1))$ | |
| 3   | $\mathbb{Z}_2$                                 | $ac_1(\text{U}(1))$ | |
| 4   | $\mathbb{Z}_4$                                 | $a^2c_1(\text{U}(1)) + c_1(\text{U}(1))^2$, $c_2(\text{SU}(3))$, $c_2(\text{SU}(2))$, $\sigma_{-\text{FF}}$ | |
| 5   | $\mathbb{Z}_2 \times \mathbb{Z}_4^2 \times \mathbb{Z}_4$ | $ac_2(\text{SU}(3))$, $c_2(\text{SU}(2))\eta'$, $c_1(\text{U}(1))^2\eta'$, $\eta(\text{PD}(a))$ | |
| 6   | $\mathbb{Z}_5$                                 | $c_1(\text{U}(1))^3$, $c_2(\text{SU}(3))$, $c_1(\text{U}(1))c_2(\text{SU}(3))$, $c_1(\text{U}(1))c_2(\text{SU}(2))$ | |

Table 9: Bordism group. $\eta'$ is a $\mathbb{Z}_4$ valued 1d eta invariant. $\sigma$ is the signature of manifold. $\eta$ is the 4d eta invariant. F is the characteristic 2-surface [67]. $\text{F} \cdot \text{F}$ is the product of two F. Note that $c_1(\text{U}(1))^2 = \text{Sq}^2c_1(\text{U}(1)) = (w_2 + w_1^2)c_1(\text{U}(1)) = a^2c_1(\text{U}(1))$ mod 2. $c_3(\text{SU}(3)) = \text{Sq}^2c_2(\text{SU}(3)) = (w_2 + w_1^2)c_2(\text{SU}(3)) = a^2c_2(\text{SU}(3))$ mod 2. On 5-manifolds with Spin $\times \mathbb{Z}_2 \mathbb{Z}_4$ structure, there is a Pin$^+$ structure on PD($a$) and there is an isomorphism $\Omega_5^{\text{Spin} \times \mathbb{Z}_2 \mathbb{Z}_4} \xrightarrow{\text{PD}(a)} \Omega_4^{\text{Pin}^+}$ [71], so the bordism invariant of $\Omega_5^{\text{Spin} \times \mathbb{Z}_2 \mathbb{Z}_4} = \mathbb{Z}_16$ is $\eta(\text{PD}(a))$. |
Cobordism group

| d (Spin × Z₂ Z₄ × SU(3) × SU(2) × U(1)) | generators |
|-----------------------------------------|-------------|
| 0                                      | 0           |
| 1                                      | Z × Z₄      |
|                                          | CS₁SU₁(1), η' |
| 2                                      | 0           |
| 3                                      | Z⁴ × Z₂      |
|                                          | a² CS₁SU₁(1), c₁(U(1))CS₁SU₁(1)², CS₃SU₁(3), CS₃SU₁(2), μ, ac₁(U(1)) |
| 4                                      | 0           |
| 5                                      | Z⁵ × Z₂ × Z₄² × Z₁₆ |
|                                          | μ(PD(c₁(U(1))))², a² CSSU₁(3)CSU₁(2), c₁(U(1))CSSU₁(3), c₁(U(1))CSSU₁(2), ac₂(SU₁(3)), c₂(SU₁(2))η', c₁(U(1))^2 η', η(PD(a)) |

Table 10: Topological phase classification (≡ TP) as a cobordism group, following Table 9. η' is a Z₄ valued 1d eta invariant. η is the 4d eta invariant. F is the characteristic 2-surface [67]. F · F is the product of two F. The PD is the Poincaré dual. The TM is the spacetime tangent bundle. The μ is the 3d Rokhlin invariant. If ∂M⁴ = M³, then μ(M³) = ((σ − FF)/8)(M⁴), thus μ(PD(c₁(U(1)))) is related to c₁(U(1))(σ − FF)/8 in Table 9.

3.2 Spin × Z₂ Z₄ × SU(3) × SU(2) × U(1) model

We consider G = Spin × Z₂ Z₄ × SU(3) × SU(2) × U(1) = Spin × Z₂ Z₄ × SU(3) × U(2), the Madsen-Tillmann spectrum MTG of the group G is

\[
MTG = MSU₂ × (BSU₂)^{2Z} × (BSU₃ × U(2))^+,
\]

The (BSU₃ × U(2))^+ is the disjoint union of the classifying space B(SU₃ × U(2)) and a point, see footnote 16.

For the dimension d = t − s < 8, since there is no odd torsion (see footnote 17), by (1.18), we have the Adams spectral sequence

\[
\text{Ext}_{A₂(1)}^{s,t}(H^{*+2}(RP₂^{∞}, Z₂) ⊗ H^*(B(SU₃ × U(2)), Z₂), Z₂) \Rightarrow O_{t−s}^{Spin×Z₂Z₄×SU₃×U(2)}. \tag{3.7}
\]

Based on Figure 3 and 12, we obtain the A₂(1)-module structure of H^{*+2}(RP₂^{∞}, Z₂) ⊗ H^*(B(SU₃ × U(2)), Z₂) below degree 6, as shown in Figure 15.
Figure 15: The $\mathcal{A}_2(1)$-module structure of $H^{*+2}(\mathbb{RP}_2^\infty, \mathbb{Z}_2) \otimes H^*(B(SU(3) \times U(2)), \mathbb{Z}_2)$ below degree 6.

The $E_2$ page is shown in Figure 16. Here we have used the correspondence between $\mathcal{A}_2(1)$-module structure and the $E_2$ page shown in Figure 33, 34 and 35.

Figure 16: $\Omega^c_{\text{Spin} \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times SU(3) \times U(2)}$.
Table 11: Bordism group. \( \eta' \) is a \( \mathbb{Z}_4 \) valued 1d eta invariant. \( \eta \) is the 4d eta invariant. \( \sigma \) is the signature of manifold. \( F \) is the characteristic 2-surface [67]. \( F \cdot F \) is the product of two \( F \). Note that 
\[c_1(U(2))^2 = \text{Sq}^2 c_1(U(2)) = (w_2 + w_1^2) c_1(U(2)) = a^2 c_1(U(2)) \mod 2,\]
\[c_3(SU(3)) = \text{Sq}^2 c_2(SU(3)) = (w_2 + w_1^2) c_2(SU(3)) = a^2 c_2(SU(3)) \mod 2.\]
\[c_1(U(2)) c_2(U(2)) = \text{Sq}^2 c_2(U(2)) = (w_2 + w_1^2) c_2(U(2)) = a^2 c_2(U(2)) \mod 2.\]

Table 12: Topological phase classification (\( \equiv \text{TP} \)) as a cobordism group, following Table 11. \( \eta' \) is a \( \mathbb{Z}_4 \) valued 1d eta invariant. \( \eta \) is the 4d eta invariant. \( F \) is the characteristic 2-surface [67]. \( F \cdot F \) is the product of two \( F \). The PD is the Poincaré dual. The \( T \) is the spacetime tangent bundle. The \( \mu \) is the 3d Rokhlin invariant. If \( \partial M^4 = M^3 \), then \( \mu(M^3) = ((\sigma - FF)/8)(M^4) \), thus \( \mu(\text{PD}(c_1(U(2)))) \) is related to \( c_1(U(2))(\sigma - FF) \) in Table 11.

### 3.3 \( \text{Spin} \times \mathbb{Z}_2 \mathbb{Z}_4 \times \frac{\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)}{\mathbb{Z}_3} \) model

We consider \( G = \text{Spin} \times \mathbb{Z}_2 \mathbb{Z}_4 \times \frac{\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)}{\mathbb{Z}_3} = \text{Spin} \times \mathbb{Z}_2 \mathbb{Z}_4 \times \text{U}(3) \times \text{SU}(2) \), the Madsen-Tillmann spectrum \( MTG \) of the group \( G \) is

\[
MTG = \text{MSpin} \wedge (B\mathbb{Z}_2)^{2\mathbb{Z}} \wedge (B(\text{U}(3) \times \text{SU}(2)))_+, \\
= \text{MSpin} \wedge \Sigma^{-2} \mathbb{RP}^{\infty}_{\mathbb{Z}_2} \wedge (B(\text{U}(3) \times \text{SU}(2)))_+. \tag{3.8}
\]

The \((B(\text{U}(3) \times \text{SU}(2)))_+\) is the disjoint union of the classifying space \( B(\text{U}(3) \times \text{SU}(2)) \) and a point, see footnote 16.
Since
\[ H^\ast(M\text{Spin} \wedge (B\mathbb{Z}_2)^{2\xi}, \mathbb{Z}_3) = H^\ast(M\text{Spin}, \mathbb{Z}_3), \] (3.9)
similarly as the discussion in Sec. 2.3, there is no 3-torsion in \( \Omega_d^{\text{Spin} \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times U(3) \times SU(2)}. \)

For the dimension \( d = t - s < 8 \), since there is no odd torsion (see footnote 17), by (1.18), we have the Adams spectral sequence
\[ \text{Ext}^{s,t}_{A_2(1)}(H^{s+2}(\mathbb{RP}_2^\infty, \mathbb{Z}_2) \otimes H^\ast(B(U(3) \times SU(2)), \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\text{Spin} \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times U(3) \times SU(2)}. \] (3.10)

Based on Figure 7 and 12, we obtain the \( A_2(1) \)-module structure of \( H^{s+2}(\mathbb{RP}_2^\infty, \mathbb{Z}_2) \otimes H^\ast(B(U(3) \times SU(2)), \mathbb{Z}_2) \) below degree 6, as shown in Figure 17.

```
Figure 17: The \( A_2(1) \)-module structure of \( H^{s+2}(\mathbb{RP}_2^\infty, \mathbb{Z}_2) \otimes H^\ast(B(U(3) \times SU(2)), \mathbb{Z}_2) \) below degree 6.
```

The \( E_2 \) page is shown in Figure 18. Here we have used the correspondence between \( A_2(1) \)-module structure and the \( E_2 \) page shown in Figure 33, 34 and 35.
Figure 18: $\Omega^{{\text{Spin}} \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \text{SU}(3) \times \text{SU}(2)}$. 

| $d$ | $\Omega^{{\text{Spin}} \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \text{SU}(3) \times \text{SU}(2)}$ | generators |
|-----|-----------------------------------------------------------------|-------------|
| 0   | $\mathbb{Z}$                                                   |             |
| 1   | $\mathbb{Z}_4$                                                 | $\eta'$     |
| 2   | $\mathbb{Z}$                                                   | $c_1(U(3))$ |
| 3   | $\mathbb{Z}_2$                                                 | $ac_1(U(3))$|
| 4   | $\mathbb{Z}^4$                                                 | $a^2c_1(U(3)) + c_1(U(3))^2, c_2(U(3)), c_2(SU(2)), \eta - \eta'$, $\eta(\text{PD}(a))$ |
| 5   | $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_{16}$     | $ac_2(U(3)), c_3(SU(2))\eta', c_1(U(3))^2\eta', \eta(\text{PD}(a))$ |
| 6   | $\mathbb{Z}^5$                                                 | $c_1(U(3))\eta - \eta', a^2c_1(U(3)) + c_1(U(3))c_2(U(3)) = c_1(U(3))^3, c_3(U(3)), c_1(U(3))c_2(U(3)), c_2(U(3))$ |

Table 13: Bordism group. $\eta'$ is a $\mathbb{Z}_4$ valued 1d eta invariant. $\eta$ is the 4d eta invariant. $\sigma$ is the signature of manifold. F is the characteristic 2-surface [67]. $F \cdot F$ is the product of two F. Note that $c_1(U(3))^2 = \text{Sq}^2 c_1(U(3)) = c_2(U(3)) = \text{mod } 2$. $c_1(U(3))c_2(U(3)) + c_3(U(3)) = \text{Sq}^2 c_2(U(3)) = (w_2 + w_1^2)c_2(U(3)) = \text{mod } 2$. $c_1(U(3))^2 = \text{Sq}^2 c_1(U(3)) = \text{mod } 2$. $c_1(U(3))^3 = \text{mod } 2$. $c_2(U(3)) = \text{mod } 2$. $c_3(U(3)) = \text{mod } 2$. $\eta(\text{PD}(a)) = \text{mod } 2$. $\eta - \eta' = \text{mod } 2$.
We consider $\times$ MT $G_{\equiv U(2))}^3$ Rokhlin invariant. If $\partial M \equiv Spin$ is the 4d eta invariant. $F$ is the characteristic 2-surface $[\Sigma, 2]$ since there is no odd torsion (see footnote 6). $\sigma \in Spin(U(3)) = H^*(B(S(U(3)), U(2))), the Madsen-Tillmann spectrum $MTG$ of the group $G$ is

$$MTG = MSpin \wedge (BZ_2)^{2\xi} \wedge (B(S(U(3) \times U(2))))_+$$

$$= MSpin \wedge \Sigma^{-2}\mathbb{RP}_\infty^\infty \wedge (B(S(U(3) \times U(2))))_+.$$  \hspace{1cm} (3.11)

The $(B(S(U(3) \times U(2))))_+$ is the disjoint union of the classifying space $B(S(U(3) \times U(2)))$ and a point, see footnote 16.

Since

$$H^*(MSpin \wedge (BZ_2)^{2\xi}, Z_3) = H^*(MSpin, Z_3), \hspace{1cm} (3.12)$$

and

$$H^*(B(\frac{SU(3) \times SU(2) \times U(1)}{Z_6}), Z_3) = H^*(B(\frac{SU(3) \times SU(2) \times U(1)}{Z_6}), Z_3), \hspace{1cm} (3.13)$$

similarly as the discussion in Sec. 2.3, there is no 3-torsion in $\Omega^Z_{d \in Spin \times Z_4 \times Spin(S(U(3) \times U(2)))}$.

For the dimension $d = t - s < 8$, since there is no odd torsion (see footnote 17), by (1.18), we have the Adams spectral sequence

$$\text{Ext}^{*,*}_{A_2(1)}(H^{*,*}(\mathbb{RP}_2^\infty, Z_2) \otimes H^*(B(S(U(3) \times U(2))), Z_2)) \Rightarrow \Omega^Z_{d \in Spin \times Z_4 \times Spin(S(U(3) \times U(2)))}. \hspace{1cm} (3.14)$$

Based on Figure 9 and 12, we obtain the $A_2(1)$-module structure of $H^{*,*}(\mathbb{RP}_2^\infty, Z_2) \otimes H^*(B(S(U(3) \times U(2))), Z_2)$ below degree 6, as shown in Figure 19.
Figure 19: The $A_2(1)$-module structure of $H^{**2}(\mathbb{R}P^{\infty}_2, \mathbb{Z}_2) \otimes H^*(B(S(U(3) \times U(2))), \mathbb{Z}_2)$ below degree 6.

The $E_2$ page is shown in Figure 20. Here we have used the correspondence between $A_2(1)$-module structure and the $E_2$ page shown in Figure 33, 34 and 35.

Figure 20: $\Omega_{\text{Spin} \times Z_2 Z_4 \times S(U(3) \times U(2))}$. 

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Bordism group

| $d$ | $\Omega^d_Z$ | generators |
|-----|-------------|------------|
| 0   | $Z$         |            |
| 1   | $Z_4$       | $\eta'$   |
| 2   | $\mathbb{Z}$ | $c_1(U(3))$ |
| 3   | $Z_2$       | $ac_1(U(3))$ |
| 4   | $\mathbb{Z}^4$ | $a^2c_1(U(3)) + c_1(U(3))^2\eta'$, $c_2(U(3))$, $c_2(U(2))$, $\sigma - FF$ |
| 5   | $\mathbb{Z}_2^3 \times Z_4 \times Z_{16}$ | $ac_2(U(3))$, $ac_2(U(2))$, $c_1(U(3))^2 \eta'$, $\eta(PD(a))$ |
| 6   | $\mathbb{Z}^5$ | $c_1(U(3))^2 \eta'$, $a^2c_2(U(3)) + c_1(U(3))c_2(U(3)) + c_3(U(3))$, $a^2c_2(U(2)) + c_1(U(2))c_2(U(2))$, $c_1(U(3))^3$, $c_3(U(3))$ |

Table 15: Bordism group. $\eta'$ is a $\mathbb{Z}_4$ valued 1d eta invariant. $\eta$ is the 4d eta invariant. $\sigma$ is the signature of manifold. $F$ is the characteristic 2-surface [67]. $F \cdot F$ is the product of two $F$. Here $c_1(U(3))$ is identified with $c_1(U(2))$. Note that $c_1(U(3))^2 = Sq^2 c_1(U(3)) = (w_2 + w_1^2)c_1(U(3)) = a^2 c_1(U(3))$ mod 2. $c_1(U(3))c_2(U(3)) + c_3(U(3)) = Sq^2 c_2(U(3)) = (w_2 + w_1^2)c_2(U(3)) = a^2 c_2(U(3))$ mod 2. $c_1(U(2))c_2(U(2)) = Sq^2 c_2(U(2)) = (w_2 + w_1^2)c_2(U(2)) = a^2 c_2(U(2))$ mod 2.

Cobordism group

| $d$ | $\text{TP}_d$ | generators |
|-----|-------------|------------|
| 0   | $0$         |            |
| 1   | $Z \times Z_4$ | $CS_1^{U(3)}, \eta'$ |
| 2   | $Z^4 \times Z_2$ | $a^2 CS_1^{U(3)} + c_1(U(3))CS_1^{U(3)}$, $CS_3^{U(3)}, CS_3^{U(2)}, \mu, ac_1(U(3))$ |
| 3   | $Z^5 \times Z_2^2 \times Z_4 \times Z_{16}$ | $\mu(PD(c_1(U(3)))), a^2 CS_1^{U(3)} + c_1(U(3))CS_1^{U(3)} + CS_3^{U(3)}$, $a^2 CS_3^{U(2)} + c_1(U(3))CS_3^{U(2)}$, $c_1(U(3))^2 CS_1^{U(3)}$, $c_3(U(3))^2$, $ac_2(U(3)), ac_2(U(2)), c_1(U(3))^2 \eta', \eta(PD(a))$ |

Table 16: Topological phase classification ($\equiv \text{TP}$) as a cobordism group, following Table 15. $\eta'$ is a $\mathbb{Z}_4$ valued 1d eta invariant. $\eta$ is the 4d eta invariant. $F$ is the product of two $F$. The PD is the Poincaré dual. The $TM$ is the spacetime tangent bundle. The $\mu$ is the 3d Rokhlin invariant. If $\partial M^4 = M^3$, then $\mu(M^3) = ((\sigma - FF)/8)(M^4)$, thus $\mu(PD(c_1(U(3))))$ is related to $c_1(U(3))(\sigma - FF)$ in Table 15.

4 Pati-Salam models

Now we consider the co/bordism classes relevant for Pati-Salam GUT models [8]. There are actually two different cases for modding out different discrete normal subgroups.
4.1 Pati-Salam model

We consider $G = \text{Spin} \times SU(4) \times SU(2) \times SU(2)$. 

Note that $SU(4) \times \mathbb{Z}_2 = SO(6)$, $SU(2) \times SU(2) \times \mathbb{Z}_2 = SO(4)$. 

We have a homotopy pullback square

$$
\begin{array}{ccc}
BG & \rightarrow & B(SO(6) \times SO(4)) \\
\downarrow & & \downarrow \quad w_2' + w_2'' \\
BSO & \rightarrow & B^2 \mathbb{Z}_2.
\end{array}
$$

(4.1)

Here $w_2' = w_2(SO(6))$, $w_2'' = w_2(SO(4))$. 

$BG$ sits in a fibration sequence

$$
BG \rightarrow BSO \times BSO(6) \times BSO(4) \xrightarrow{w_2(TM) + w_2(V_1) + w_2(V_2)} B^2 \mathbb{Z}_2.
$$

(4.2)

Map the three-fold product to itself by the matrix

$$
\begin{pmatrix}
1 & -V_1 + 6 & -V_2 + 4 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
$$

(4.3)

we find that $BG$ is homotopy equivalent to $B\text{Spin} \times BSO(6) \times BSO(4)$. 

So we can identify $BG \rightarrow BO$ with $B\text{Spin} \times BSO(6) \times BSO(4) \rightarrow BO$: $(W, V_1, V_2) \mapsto -W - (V_1 + V_2 - 10)$. 

Hence the Madsen-Tillmann spectrum $MTG$ of the group $G$ is

$$
MTG = M\text{Spin} \wedge \Sigma^{-6}M\text{SO}(6) \wedge \Sigma^{-4}M\text{SO}(4).
$$

(4.4)

For the dimension $d = t - s < 8$, since there is no odd torsion (see footnote 17), by (1.18), we have the Adams spectral sequence

$$
\text{Ext}^{s,t}_{\mathcal{A}_2(1)}(H^{+6}(M\text{SO}(6), \mathbb{Z}_2) \otimes H^{+4}(M\text{SO}(4), \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{M\text{Spin} \times SU(4) \times SU(2) \times SU(2) \times \mathbb{Z}_2}. 
$$

(4.5)

$$
H^{+6}(M\text{SO}(6), \mathbb{Z}_2) = \mathbb{Z}_2[w_2', w_3', w_4', w_5', w_6'][U], \quad H^{+4}(M\text{SO}(4), \mathbb{Z}_2) = \mathbb{Z}_2[w_2'', w_3'', w_4''][V].
$$

Here $U$ and $V$ are Thom classes with $\text{Sq}^1U = 0$, $\text{Sq}^2U = w_2'U$, $\text{Sq}^1V = 0$ and $\text{Sq}^2V = w_2''V$. 

The $\mathcal{A}_2(1)$-module structure of $H^{+6}(M\text{SO}(6), \mathbb{Z}_2) \otimes H^{+4}(M\text{SO}(4), \mathbb{Z}_2)$ and the $E_2$ page are shown in Figure 21, 22. Here we have used the correspondence between $\mathcal{A}_2(1)$-module structure and the $E_2$ page shown in Figure 32, 36, 37, 39 and 42.
Figure 21: The $A_2(1)$-module structure of $H^{*+6}(MSO(6), \mathbb{Z}_2) \otimes H^{*+4}(MSO(4), \mathbb{Z}_2)$ below degree 6.
Figure 22: $\Omega_{\text{Spin} \times SU(4) \times (SU(2) \times SU(2)) \mathbb{Z}_2}$.

Table 17: Bordism group. Here $e_i$ is the Euler class, $p_i$ is the Pontryagin class. Here $w'_i = w_i(\text{SO}(6))$, $w''_i = w_i(\text{SO}(4))$, $p'_1 = p_1(\text{SO}(6))$, $p''_1 = p_1(\text{SO}(4))$, $e'_i = e_i(\text{SO}(6))$, $e''_i = e_i(\text{SO}(4))$. Here $w_2 = w_2'' + w_2'$, $w_3 = w_3'' + w_3'$. Since $w_2 + w_2' = 0$ on 2-manifold, we have $w_2 = w_2' + w_2'' = w_2^i = 0$ on oriented 2-manifold. Since $\text{Sq}^2(w'_2) = w'_3 = (w_2 + w_2')w'_2 = (w_2'' + w_2')w'_2$ on oriented 6-manifold by Wu formula, we have $w'_2w'_2 = w'_3 + w'_2$. Since $w_2^2 = \text{Sq}^2(w'_2) = (w_2 + w_2')w'_2 = (w_2'' + w_2')w'_2$ on oriented 4-manifold by Wu formula, we have $w'_2w'_2 = 0$ on oriented 4-manifold. On a 4-manifold, the oriented bundle of rank 6 splits as the direct sum of an oriented bundle of rank 4 and a trivial plane bundle, the Euler class $e'_4$ is the Euler class of the subbundle of rank 4.
Cobordism group

| d  | TP_d((-Spin x SU(4) x (SU(2) x SU(2)))/Z_2^2) | generators |
|-----|---------------------------------------------|------------|
| 0   | 0                                           |            |
| 1   | 0                                           |            |
| 2   | Z_2                                         | w/z''     |
| 3   | Z^4                                         | CS_{SO(6)}, CS_{SO(4)}^{3}, CS_{SO(6)}^{3}, CS_{SO(4)}^{3} |
| 4   | 0                                           |            |
| 5   | Z x Z_2                                     | CS_{SO(6)}^{3}, w/z'' |

Table 18: Topological phase classification (≡ TP) as a cobordism group, following Table 17.

4.2 Spin x SU(4) x (SU(2) x SU(2)) Pati-Salam model

We consider $G = \frac{Spin \times SU(4) \times (SU(2) \times SU(2))}{Z_2}$, let $G' = \frac{SU(4) \times (SU(2) \times SU(2))}{Z_2}$, then we have a fibration

$$B G' \xrightarrow{w/z''} BSO(6) \times BSO(4) \xrightarrow{w/z'' + w/z''} B^2Z_2.$$ (4.6)

Here $w/z'' = w_2(SO(6)), w/z'' = w_2(SO(4))$.

We have a homotopy pullback square

$$\begin{array}{ccc}
BG & \xrightarrow{w/z'' = w/z''} & BSO \\
\downarrow & & \downarrow \\
BSO & \xrightarrow{w/z''} & B^2Z_2.
\end{array}$$ (4.7)

There is a homotopy equivalence $f : BSO \times BG' \xrightarrow{\sim} BSO \times BG'$ by $(V, W) \mapsto (V - W + 10, W)$, and there is obviously also an inverse map. Note that the pullback $f^*(w_2) = w_2(V - W) = w_2(V) + w_1(V)w_1(W) + w_2(W) = w_2 + w_2'$. Then we have the following homotopy pullback

$$\begin{array}{ccc}
BG & \xrightarrow{\sim} & BSpin \times BG' \\
\downarrow & & \downarrow \\
BSO \times BG' & \xrightarrow{w/z'' + w/z''} & B^2Z_2
\end{array}$$ (4.8)

This implies that the two classifying spaces $BG \sim BSpin \times BG'$ are homotopy equivalent.

By definition, the Madsen-Tillmann spectrum $MTG$ of the group $G$ is $MTG = Thom(BG; -V)$, where $V$ is the induced virtual bundle (of dimension 0) by the map $BG \to BO$. 

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We can identify $BG \to BO$ with $B\text{Spin} \times BG' \xrightarrow{V-W+10} BSO \to BO$.

So the spectrum $MTG$ is homotopy equivalent to Thom($B\text{Spin} \times BG'; -(V-W+10)$), which is

$$MTG = M\text{Spin} \wedge \Sigma^{-10} MG'.$$  \hspace{1cm} (4.9)

For the dimension $d = t - s < 8$, since there is no odd torsion (see footnote 17), by (1.18), we have the Adams spectral sequence

$$\text{Ext}^{s,t}_{A_2(1)}(H^{*+10}(MG', \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_4^{t-s}. \quad (4.10)$$

There is a fibration

$$\xymatrix{B\mathbb{Z}_2 \ar[r] \ar[d] & BG' \ar[d] \cr BSO(6) \times BSO(4).} \quad (4.11)$$

So we have the Serre spectral sequence

$$H^p(BSO(6) \times BSO(4), H^q(B\mathbb{Z}_2, \mathbb{Z}_2)) \Rightarrow H^{p+q}(BG', \mathbb{Z}_2). \quad (4.12)$$

Let $a$ be the generator of $H^1(B\mathbb{Z}_2, \mathbb{Z}_2)$, then we have the differentials $d_2(a) = w'_2 + w''_2$, $d_3(a^2) = Sq^1d_2(a) = w'_2 + w''_3$, $d_4(a^4) = Sq^2d_3(a^2) = w'_2w'_3 + w'_5 + w''_2w'_3$ and so on. So in $H^*(BG', \mathbb{Z}_2)$, we have $w'_2 = w''_2$, $w'_3 = w''_3$ and $w'_5 = 0$.

So below degree 6, we have

$$H^{*+10}(MG', \mathbb{Z}_2) = ((\mathbb{Z}_2[w'_2, w'_3, w'_4, w'_5, w'_6] \otimes \mathbb{Z}_2[w''_2, w''_3, w''_4, w''_5])/(w'_2 = w''_2, w'_3 = w''_3, w'_5 = 0))U \quad (4.13)$$

where $U$ is the Thom class with $Sq^1U = 0$, $Sq^2U = w'_2U = w''_2U$.

The $A_2(1)$-module structure of $H^{*+10}(MG', \mathbb{Z}_2)$ below degree 6 and the $E_2$ page are shown in Figure 23, 24. Here we have used the correspondence between $A_2(1)$-module structure and the $E_2$ page shown in Figure 31, 32, 33, 38, 40 and 41.
Figure 23: The $A_2(1)$-module structure of $H^{*+10}(MG', \mathbb{Z}_2)$ below degree 6.

Figure 24: $\Omega_{s, 2^2}^{Spin \times SU(4) \times (SU(2) \times SU(2))}$. 

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Table 19: Bordism group. Here $e_i$ is the Euler class, $p_i$ is the Pontryagin class. Here $w_i' = w_i(SO(6))$, $w_i'' = w_i(SO(4))$, $p_i' = p_i(SO(6)) = p_i(SO(4))$, $e_i' = e_i(SO(6))$, $e_i'' = e_i(SO(4))$. Here $w_2 = w_2' = w_2''$, $w_3 = w_3' = w_3''$. $\eta$ is a mod 2 index of 1d Dirac operator. Arf is a 2d Arf invariant. Since $Sq^2w'_4 = w_2w'_4 + w_6' = (w_2 + w_2^2)w_4' = w_2w_4'$ on oriented 6-manifold by Wu formula, we have $e_6' = w_6' = 0 \mod 2$. On a 4-manifold, the oriented bundle of rank 6 splits as the direct sum of an oriented bundle of rank 4 and a trivial plane bundle, the Euler class $e_4'$ is the Euler class of the subbundle of rank 4. There are 3 undetermined bordism invariants which we mark them as $?_1$, $?_2$, and $?_3$. The $?_1$ in 4d is $\mathbb{Z}$ valued and we read from the Adams chart that there is an extension: $\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}$ where the middle $\mathbb{Z}$ is $?_1$, the first $\mathbb{Z}$ is $\frac{\sigma_{PD}(w_2)}{16}$, and the last $\mathbb{Z}$ is $w_3\eta$. The $?_2$ in 5d is $\mathbb{Z}_2$ valued and we read from the Adams chart that it looks like $w_3\text{Arf}$. The $?_3$ in 6d is $\mathbb{Z}_2$ valued and we read from the Adams chart that it looks like $\frac{\sigma_{PD}(w_2)}{16}$ mod 2.

Table 20: Topological phase classification ($\equiv TP$) as a cobordism group, following Table 19. The $?_4$ in 3d is related to the $?_1$ in 4d in Table 19. The $?_2$ in 5d is explained in Table 19.

5 SO(10), SO(18) and SO(n) Grand Unifications

Now we consider the co/bordism classes relevant for Fritzsch-Minkowski SO(10) GUT \cite{7}. There are actually two cases, depending on whether the gauged SO(10) GUT allows gauge-invariant fermions, or whether the gauged SO(10) GUT only allows gauge-invariant bosons.
5.1 Spin × SO(n) for n ≥ 7: Spin × SO(10) and Spin × SO(18)

Here we consider $G = \text{Spin} \times \text{SO}(n)$ for $n \geq 7$, especially for SO(10) and SO(18). The Madsen-Tillmann spectrum $MTG$ of the group $G$ is

$$MTG = M\text{Spin} \wedge (\text{BSO}(n))_+.$$ (5.1)

The $(\text{BSO}(n))_+$ is the disjoint union of the classifying space $\text{BSO}(n)$ and a point, see footnote 16.

For the dimension $d = t - s < 8$, since there is no odd torsion (see footnote 17), by (1.18), we have the Adams spectral sequence

$$\text{Ext}^{s,t}_{A_2(1)}(H^*(\text{BSO}(n), \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega^{\text{Spin} \times \text{SO}(n)}_{t-s}. $$ (5.2)

We have

$$H^*(\text{BSO}(n), \mathbb{Z}_2) = \mathbb{Z}_2[w'_2, w'_3, \ldots, w'_n].$$ (5.3)

We also have the Wu formula

$$\text{Sq}^j w'_i = \sum_{k=0}^{j} \binom{i-k-1}{j-k} w'_{i+j-k}w'_k \text{ for } 0 \leq j \leq i.$$ (5.4)

For $n \geq 7$, the $A_2(1)$-module structure of $H^*(\text{BSO}(n), \mathbb{Z}_2)$ below degree 6 is shown in Figure 25.

![Figure 25: The $A_2(1)$-module structure of $H^*(\text{BSO}(n), \mathbb{Z}_2)$ below degree 6 for $n \geq 7.$](image)

The $E_2$ page is shown in Figure 26. Here we have used the correspondence between $A_2(1)$-module structure and the $E_2$ page shown in Figure 31, 32 and 38.
Figure 26: $\Omega_{s}^{\text{Spin} \times \text{SO}(n)}$ for $n \geq 7$.

| Bordism group |
|---------------|
| $d$ | $\Omega_{d}^{	ext{Spin} \times \text{SO}(n)}$ for $n \geq 7$ | generators |
| 0  | $\mathbb{Z}$ | |
| 1  | $\mathbb{Z}_2$ | $\tilde{\eta}$ |
| 2  | $\mathbb{Z}_2^2$ | Arf, $w'_2$ |
| 3  | 0 | |
| 4  | $\mathbb{Z}_2^2 \times \mathbb{Z}_2$ | $\sigma$, $\frac{\epsilon_2}{2}$, $w'_4$ |
| 5  | 0 | |
| 6  | $\mathbb{Z}_2^5$ | $w'_4$, $w'_6$ |

Table 21: Bordism group. Here $w'_i = w_i(\text{SO}(10))$, $p'_1 = p_1(\text{SO}(10))$. $\tilde{\eta}$ is a mod 2 index of 1d Dirac operator. Arf is a 2d Arf invariant. $\sigma$ is the signature of manifold.

| Cobordism group |
|----------------|
| $d$ | $\text{TP}_d(\text{Spin} \times \text{SO}(n))$ for $n \geq 7$ | generators |
| 0  | 0 | |
| 1  | $\mathbb{Z}_2$ | $\tilde{\eta}$ |
| 2  | $\mathbb{Z}_2^2$ | Arf, $w'_2$ |
| 3  | $\mathbb{Z}_2^2$ | $\frac{1}{8} \text{CS}_3^{TM}$, $\frac{1}{2} \text{CS}_3^{\text{SO}(n)}$ |
| 4  | $\mathbb{Z}_2$ | $w'_4$ |
| 5  | 0 | |

Table 22: Topological phase classification ($\equiv \text{TP}$) as a cobordism group, following Table 21. $\tilde{\eta}$ is a mod 2 index of 1d Dirac operator. Arf is a 2d Arf invariant. The $TM$ is the spacetime tangent bundle.
Let $G = \frac{\text{Spin} \times \text{Spin}(n)}{\mathbb{Z}_2}$, we have a homotopy pullback square

\[
\begin{array}{ccc}
BG & \rightarrow & \text{BSO}(n) \\
\downarrow & & \downarrow w_2' \\
\text{BSO} & \rightarrow & B^2\mathbb{Z}_2.
\end{array}
\]

Here $w_2' = w_2(\text{SO}(n))$.

There is a homotopy equivalence $f : \text{BSO} \times \text{BSO}(n) \xrightarrow{\sim} \text{BSO} \times \text{BSO}(n)$ by $(V, W) \mapsto (V - W + n, W)$, and there is obviously also an inverse map. Note that the pullback $f^*(w_2) = w_2(V - W) = w_2(V) + w_1(V)w_1(W) + w_2(W) = w_2 + w_2'$. Then we have the following homotopy pullback

\[
\begin{array}{ccc}
BG & \overset{\sim}{\rightarrow} & \text{BSpin} \times \text{BSO}(n) \\
\downarrow & & \downarrow w_2 + 0 \\
\text{BSO} \times \text{BSO}(n) & \xrightarrow{f} & \text{BSO} \times \text{BSO}(n) \\
(V, W) \mapsto V & \xrightarrow{w_2 + w_2'} & (V, W) \mapsto V + W - n \\
\downarrow & & \downarrow \\
\text{BSO} & \rightarrow & B^2\mathbb{Z}_2.
\end{array}
\]

This implies that the two classifying spaces $BG \sim \text{BSpin} \times \text{BSO}(n)$ are homotopy equivalent.

By definition, the Madsen-Tillmann spectrum $MTG$ of the group $G$ is $MTG = \text{Thom}(BG; -V)$, where $V$ is the induced virtual bundle (of dimension 0) by the map $BG \rightarrow \text{BO}$.

We can identify $BG \rightarrow \text{BO}$ with $\text{BSpin} \times \text{BSO}(n) \xrightarrow{V - V_n + n} \text{BSO} \rightarrow \text{BO}$.

So the spectrum $MTG$ is homotopy equivalent to $\text{Thom}(\text{BSpin} \times \text{BSO}(n); -(V - V_n + n))$, which is $M\text{Spin} \wedge \Sigma^{-n}M\text{SO}(n)$.

We consider $G = \frac{\text{Spin} \times \text{Spin}(10)}{\mathbb{Z}_2}$, the Madsen-Tillmann spectrum $MTG$ of the group $G$ is

\[
MTG = M\text{Spin} \wedge \Sigma^{-10}M\text{SO}(10).
\]

We have $w_2 = w_2'$, namely $w_2(TM) = w_2(\text{SO}(10))$.

For the dimension $d = t - s < 8$, since there is no odd torsion (see footnote 17), by (1.18), we have the Adams spectral sequence

\[
\text{Ext}^{s,t}_{A_2(1)}(H^{*+10}(\text{MSO}(10), \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s} \frac{\text{Spin} \times \text{Spin}(10)}{\mathbb{Z}_2}.
\]

We have

\[
H^{*+10}(\text{MSO}(10), \mathbb{Z}_2) = \mathbb{Z}_2[w_2', w_3', \ldots, w_{10}'] U
\]
where $U$ is the Thom class with $\text{Sq}^1 U = 0$, $\text{Sq}^2 U = w'_2 U$.

The $A_2(1)$-module structure of $H^{*+10}(\text{MSO}(10), \mathbb{Z}_2)$ below degree 6 and the $E_2$ page are shown in Figure 27, 28. Here we have used the correspondence between $A_2(1)$-module structure and the $E_2$ page shown in Figure 32, 39, and 42.

Figure 27: The $A_2(1)$-module structure of $H^{*+10}(\text{MSO}(10), \mathbb{Z}_2)$ below degree 6. Here $U$ is a Thom class.

Figure 28: $\Omega_* \mathbb{Z}_2^{\infty}$. 

}\begin{figure}[h] \centering \includegraphics[width=\textwidth]{figure27.png} \caption{The $A_2(1)$-module structure of $H^{*+10}(\text{MSO}(10), \mathbb{Z}_2)$ below degree 6. Here $U$ is a Thom class.} \end{figure}
Table 23: Bordism group. Here $e_i$ is the Euler class, $p_i$ is the Pontryagin class. Here $w'_i = w_i(SO(10))$, $p'_1 = p_1(SO(10))$, $e'_i = e_i(SO(10))$. On a 4-manifold, the oriented bundle of rank 10 splits as the direct sum of an oriented bundle of rank 4 and a trivial bundle of rank 6, the Euler class $e'_4$ is the Euler class of the subbundle of rank 4. Same result for $\Omega^{\text{Spin} \times \text{Spin}(n)}_d$ with $n \geq 7$ and $0 \leq d \leq 6$.

Table 24: Topological phase classification ($\equiv \text{TP}$) as a cobordism group, following Table 23. Same result for $\text{TP}_d^{\text{Spin} \times \text{Spin}(n)}$ with $n \geq 7$ and $0 \leq d \leq 5$.

6 SU(5) and SU(n) Grand Unifications: Spin × SU(n): Spin × SU(5)

Now we consider the co/bordism classes relevant for Georgi-Glashow SU(5) GUT [6].

We consider $G = \text{Spin} \times \text{SU}(5)$, the Madsen-Tillmann spectrum $MTG$ of the group $G$ is

$$MTG = M\text{Spin} \land (\text{BSU}(5))_+. \quad (6.1)$$

The $(\text{BSU}(5))_+$ is the disjoint union of the classifying space $\text{BSU}(5)$ and a point, see footnote 16.

For the dimension $d = t - s < 8$, since there is no odd torsion (see footnote 17), by (1.18), we have the Adams spectral sequence

$$\text{Ext}_{A_2(1)}^{s,t}(H^*(\text{BSU}(5), \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\text{Spin} \times \text{SU}(5)}. \quad (6.2)$$
The $A_2(1)$-module structure of $H^*(BSU(5), \mathbb{Z}_2)$ below degree 6 and the $E_2$ page are shown in Figure 29, 30. Here we have used the correspondence between $A_2(1)$-module structure and the $E_2$ page shown in Figure 31 and 33.

![Diagram](image)

Figure 29: The $A_2(1)$-module structure of $H^*(BSU(5), \mathbb{Z}_2)$ below degree 6.

![Diagram](image)

Figure 30: $\Omega^*_{Spin \times SU(5)}$.

| Bordism group | $\Omega^*_{Spin \times SU(5)}$ generators |
|---------------|------------------------------------------|
| $d$           | $\mathbb{Z}$                             |
| 0             | $\mathbb{Z}$                             |
| 1             | $\mathbb{Z}_2$                           |
| 2             | $\mathbb{Z}_2$                           |
| 3             | 0                                        |
| 4             | $\mathbb{Z}^2$                           |
| 5             | 0                                        |
| 6             | $\mathbb{Z}$                             |

Table 25: Bordism group. $\tilde{\eta}$ is a mod 2 index of 1d Dirac operator. Arf is a 2d Arf invariant. $\sigma$ is the signature of manifold. Note that $c_3 = \text{Sq}^2 c_2 = (w_2 + w_1^2) c_2 = 0 \mod 2$ on Spin 6-manifolds.


Actually $\Omega_{d}^{\text{Spin} \times SU(n)} = \Omega_{d}^{\text{Spin} \times SU(n+1)}$ for $n \geq 3$ and $0 \leq d \leq 6$.

| d  | $\text{TP}_d(\text{Spin} \times SU(5))$ | generators |
|-----|---------------------------------|------------|
| 0   | 0                               |            |
| 1   | $\mathbb{Z}_2$                  | $\tilde{\eta}$ |
| 2   | $\mathbb{Z}_2$                  | Arf        |
| 3   | $\mathbb{Z}^2$                  | $\frac{1}{8}\text{CS}_{3}^{TM}$, $\text{CS}_{3}^{SU(5)}$ |
| 4   | 0                               |            |
| 5   | $\mathbb{Z}$                    | $\frac{1}{2}\text{CS}_{5}^{SU(5)}$ |

Table 26: Topological phase classification ($\equiv \text{TP}$) as a cobordism group, following Table 25. $\tilde{\eta}$ is a mod 2 index of 1d Dirac operator. Arf is a 2d Arf invariant. The $TM$ is the spacetime tangent bundle.

7 Conclusions, and Explorations on Non-Perturbative and Topological Sectors of BSM

7.1 Summary

- In Sec. 2, for $\Gamma_q := \mathbb{Z}_q$, such that here $\Gamma_q = 1, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_6$, we compute the bordism groups $\Omega_d^{\text{Spin} \times SU(3) \times SU(2) \times U(1)}_{\Gamma_q}$ and the bordism invariants for $0 \leq d \leq 6$. We also determine the group $\text{TP}_d(\text{Spin} \times SU(3) \times SU(2) \times U(1))_{\Gamma_q}$ and the topological terms for $0 \leq d \leq 5$.

  We find that there is only 2-torsion in these bordism groups, and the bordism groups $\Omega_d^{\text{Spin} \times SU(3) \times SU(2) \times U(1)}_{\mathbb{Z}_3}$ and $\Omega_d^{\text{Spin} \times SU(3) \times SU(2) \times U(1)}_{\mathbb{Z}_2}$ are isomorphic, while $\Omega_d^{\text{Spin} \times SU(3) \times SU(2) \times U(1)}_{\mathbb{Z}_6}$ are isomorphic.

  We use the 3d Rokhlin invariant and Chern-Simons forms to express the topological terms.

- In Sec. 3, for $\Gamma_q = 1, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_6$, we compute the bordism groups $\Omega_d^{\text{Spin} \times \mathbb{Z}_2 \times SU(4) \times (SU(2) \times SU(2)) \times U(1)}_{\mathbb{Z}_q}$ and the bordism invariants for $0 \leq d \leq 6$. We also determine the group $\text{TP}_d(\text{Spin} \times \mathbb{Z}_2 \times SU(4) \times (SU(2) \times SU(2)) \times U(1))_{\mathbb{Z}_q}$ and the topological terms for $0 \leq d \leq 5$.

  We find that there is only 2-torsion in these bordism groups, and the bordism groups $\Omega_d^{\text{Spin} \times \mathbb{Z}_2 \times SU(4) \times (SU(2) \times SU(2)) \times U(1)}_{\mathbb{Z}_3}$ and $\Omega_d^{\text{Spin} \times \mathbb{Z}_2 \times SU(4) \times (SU(2) \times SU(2)) \times U(1)}_{\mathbb{Z}_6}$ are isomorphic, while $\Omega_d^{\text{Spin} \times \mathbb{Z}_2 \times SU(4) \times (SU(2) \times SU(2)) \times U(1)}_{\mathbb{Z}_2}$ are isomorphic.

  We also use the 3d Rokhlin invariant and Chern-Simons forms to express the topological terms. Compared to Sec. 2, there are new bordism invariants. For example, in 1d, $\tilde{\eta}$ becomes $\eta'$ which is $\mathbb{Z}_4$ valued; in 4d, $\frac{\sigma_{16}}{8}$ becomes $\frac{\sigma-FF}{8}$ where F is a characteristic surface [67]; in 5d, there is a new $\eta(\text{PD}(a))$ since there is an induced Pin$^+$ structure on PD(a) where $\eta$ is the 4d eta invariant, PD is the Poincaré dual.

- In Sec. 4, we compute the bordism groups $\Omega_d^{\text{Spin} \times SU(4) \times (SU(2) \times SU(2))}$ for $0 \leq d \leq 5$.
We also determine the group \( \text{TP}_d(\frac{\text{Spin} \times \text{SU}(4) \times (\text{SU}(2) \times \text{SU}(2))}{\mathbb{Z}_2^d}) \) and \( \text{TP}_d(\frac{\text{Spin} \times \text{SU}(4) \times (\text{SU}(2) \times \text{SU}(2))}{\mathbb{Z}_2^d}) \) for \( 0 \leq d \leq 5 \).

We find that Euler class appears in the bordism invariants and the Chern-Simons form for Euler class appears in the topological terms.

- In Sec. 5, we compute the bordism groups \( \Omega_d^{\text{Spin} \times \text{SO}(n)} \) and \( \Omega_d^{\frac{\text{Spin} \times \text{Spin}(n)}{\mathbb{Z}_2^d}} \) for \( n \geq 7 \) and \( 0 \leq d \leq 6 \). We also determine the group \( \text{TP}_d(\frac{\text{Spin} \times \text{SO}(n)}{\mathbb{Z}_2^d}) \) and \( \text{TP}_d(\frac{\text{Spin} \times \text{Spin}(n)}{\mathbb{Z}_2^d}) \) for \( n \geq 7 \) and \( 0 \leq d \leq 5 \).

We find that for \( n \geq 7 \), \( \Omega_d^{\text{Spin} \times \text{SO}(n)} = \Omega_d^{\text{Spin} \times \text{SO}(n+1)} \) and \( \Omega_d^{\frac{\text{Spin} \times \text{Spin}(n)}{\mathbb{Z}_2^d}} = \Omega_d^{\frac{\text{Spin} \times \text{Spin}(n+1)}{\mathbb{Z}_2^d}} \) for \( 0 \leq d \leq 6 \).

We also find that Euler class appears in the bordism invariants of \( \Omega_d^{\text{Spin} \times \text{Spin}(n)} \) and the Chern-Simons form for Euler class appears in the topological terms of \( \text{TP}_d(\frac{\text{Spin} \times \text{Spin}(n)}{\mathbb{Z}_2^d}) \).

- In Sec. 6, we compute the bordism groups \( \Omega_d^{\text{Spin} \times \text{SU}(5)} \) for \( 0 \leq d \leq 6 \). We also determine the group \( \text{TP}_d(\frac{\text{Spin} \times \text{SU}(5)}{\mathbb{Z}_2^d}) \) for \( 0 \leq d \leq 5 \).

We find that for \( n \geq 3, \Omega_d^{\text{Spin} \times \text{SU}(n)} = \Omega_d^{\text{Spin} \times \text{SU}(n+1)} \) for \( 0 \leq d \leq 6 \).

## 7.2 Constraints on Quantum Dynamics

We have discussed and summarized potential anomalies and topological terms in SM, GUT and BSM. There are actually two versions of anomalies we are speaking of: One is the Anomaly (2) for the ungauged SM, GUT and BSM with the \( G \) is simply a global symmetry. Another is the Anomaly (3) for the gauged SM, GUT and BSM with the \( G \) internal dynamically gauged inside \( G \). In more details,

I. For ungauged SM, GUT and BSM, we can use the ’t Hooft anomaly (Anomaly (2)) to the gappability of these models’ matter field sectors. For ungauged SM, GUT and BSM, we can simply have matter field contents (e.g. fermions: quarks and leptons) without dynamical gauge fields. In fact, Ref. [48–50] use the all anomaly free conditions to support that a non-perturbative definition (lattice regularization) of SO(10) GUT is doable, by checking. By anomaly free, there exists non-perturbative interactions for gapping the mirror world chiral fermions [52]. This mirror-fermion gapping can help to get rid of the mirror world chiral fermion doublers, surpassing the Niels-Ninomiya fermion doubling theorem (which is only true for the free non-interacting systems). The fact that all anomaly free gapless theories can be deformed to a fully gapped trivial vacuum is also consistent with the concept of Seiberg’s deformation class [72]. More details can be found in [52].

II. For gauged SM, GUT and BSM, we can use the dynamical gauge anomaly matching conditions (Anomaly (3)) to rule out inconsistent theories. Importantly, depending on the matter contents and their representations in \( G_{\text{internal}} \), we may gain or loss some global symmetries. For example, for fermions in the adjoint representation, we can have a 1-form center symmetry for the gauge theory. Thus, we should beware potentially additional new higher ’t Hooft anomalies (see [29,30]) for gauged SM, GUT and BSM can help us to constrain quantum dynamics (also more discussions below).

We use the path integral and the action to understand the basic kinematics and the global symmetry of the QFTs. We can apply the spacetime geometric topology properties to constrain QFTs, such as
doing the spacetime surgery for QFTs [73–76]. We can also determine the anomalies of QFTs at UV. However, given the potentially complete anomalies, we can constrain the IR dynamics by UV-IR anomaly matching. The consequence of anomaly matching implies that the IR theories with ‘t Hooft anomalies in $G$-symmetry must be matched by at least one of the following dynamics scenarios:

1. Symmetry-breaking:
   • (say, discrete or continuous $G$-symmetry breaking).

2. Symmetry-preserving:
   • Degenerate ground states (like the “Lieb-Schultz-Mattis theorem [77, 78]”),
   • Gapless, e.g., conformal field theory (CFT),
   • Symmetry-preserving TQFT: Intrinsic topological orders.

3. Symmetry-extension [79]: Symmetry-extension is a rather exotic possibility, which does not occur naturally without fine-tuning or artificial designed, explained in [79]. However, the symmetry-extension approach is a useful intermediate stepstone, to construct another earlier scenario: symmetry-preserving TQFT, via gauging the extended-symmetry [79].

In more details, suppose there are mixed anomalies between the ordinary 0-form global symmetries and higher-form symmetries (say 1-form symmetries) in the certain gauged SM, GUT, and BSM as in Model II, we have further refinement of possibilities:

(i) Ordinary 0-form symmetry broken (spontaneously or explicitly).

(ii) 1-form center discrete $\mathbb{Z}_{N,[1]}^e$ symmetry broken (spontaneously or explicitly) as deconfinement:

   (2-i) 1-form $\mathbb{Z}_{N,[1]}^e$-symmetry breaking and deconfined TQFTs, i.e., topological order in condensed matter terminology.

   (2-ii) 1-form $\mathbb{Z}_{N,[1]}^e$-symmetry breaking and deconfined gapless theories (e.g. CFTs).

(iii) 1-form center discrete $\mathbb{Z}_{N,[1]}^e$ symmetry unbroken as confinement:

   (3-i) 1-form symmetry-extended invertible TQFT: The exotic scenario of symmetry-extended invertible TQFT is systematically studied in Ref. [79]. This idea is generalized to higher symmetries for higher-symmetry-extended invertible TQFT in Ref. [26].

   (3-ii) 1-form symmetry-preserving TQFT: In fact, under certain criteria, there is a no go theorem to match some specific anomalies by this scenario. Hinted by the obstruction of constructing the symmetry-extended TQFTs [18, 26], Cordova-Ohmori [80] proved a no go theorem for TQFTs preserving both the 1-form symmetry and time-reversal symmetry while saturating the 4d SU(N) YM anomaly with $\theta = \pi$.

(iv) Full symmetry-preserving gapless theory (e.g. CFTs).

In summary, based on the kinematics, global symmetries (ordinary or higher symmetries), anomalies and spacetime topology constraints, we may suggest new quantum dynamical constraints.
7.3 Anomaly Cancellation or Anomaly-Matched Hidden Sectors for BSM

The previous section also has phenomenology implications of the gauged SM, GUT, and BSM as in Model II.

- Suppose we prove that the theory is fully 't Hooft anomaly free (for global symmetries, not the dynamical gauge anomalies), then there is an application for gapping the mirror world chiral fermions (see Ref. [48–50, 52] and References therein).
- Suppose we prove that the theory has a certain 't Hooft anomaly (for global symmetries, not the dynamical gauge anomalies), then we discover either of the following (see more details in):

1. Anomaly-Matched Topological Field Theories: Anomaly-Matched Hidden Topological Sectors. The gapped topological sector has its low energy physics described by a certain TQFT. But the gapped topological sector may have a finite energy gap $\Delta_E$ (instead of infinite large gap $\Delta_E \to \infty$) which has experimental measurable consequences.

2. Anomaly-Matched Hidden Gapless Sectors: This may be more surprising for phenomenology grounds with additional massless or gapless beyond SM around the energy scale $\Delta_E$ with anyonic excitations of particles, strings or branes (e.g. see [75, 76, 81] and References therein).

3. Anomaly-Matched Symmetry-Breaking Sectors: Global symmetries can be spontaneously broken.

4. All Anomaly-Free conditions are matched: This suggests the relation between the gapping criteria and defining fermion/gauge theories. In particular, the challenge of defining chiral fermion/gauge theories can be overcome by gapping the mirror world chiral fermions [48–50, 52].

5. Swapland Implications and Defects: In the recent work of McNamara-Vafa [58], by combing the ideas of Quantum Gravity/String Landscape and Swampland (e.g. Ref. [82, 83], see a review [84]) together with no internal global symmetry for quantum gravity [85–88], Ref. [58] argues that all the cobordism classes in the full Quantum Gravity (QG) must be vanished as in the trivial class. Effectively, the effective cobordism classes of QG named $\Omega_k^{QG}$ must be vanished. This gives several powerful constraints for the cobordism class data in any dimension. If we find any anomalies or topological terms in lower dimensions, say $k = 1, 2, 3, 4, 5$ in our case, this suggests that in the full QG, their cobordism classes must be cancelled by some other new objects/excitations/extended operators. Two particular interesting possibilities are [58]:

(1). Symmetry is broken: Mathematically, this suggests a map

$$\Omega_k^{QG} \to \Omega_k^{QG+defects},$$

which means the cancellation of cobordism classes via new additional symmetry defects. Namely, the kernel in $\Omega_k^{QG}$ of this map is mapped to a trivial class in $\Omega_k^{QG+defects}$.

(2). Symmetry is gauged: Mathematically, this suggests a map

$$\Omega_k^{QG+gauge\ sectors} \to \Omega_k^{QG},$$

which means the cancellation of cobordism classes via additional new gauge sectors. Namely, the cokernel in $\Omega_k^{QG}$ of this map is mapped reversely to a trivial class in $\Omega_k^{QG+gauge\ sectors}$.

We leave the further detailed explorations of the above predictions in upcoming works [24, 30, 68].
8 Acknowledgements

The authors are listed in the alphabetical order by the standard convention. During the completion of this manuscript, we become aware that a recent work has obtained related results [59] on global anomalies of SMs and GUTs; also Ref. [89] studies various topological terms of 4d gauge theories analogous to [62] but on generic non-spin manifolds. JW is grateful to Miguel Montero [47] for informing his unpublished note [63], and thanks colleagues for warmly encouragements [68]. Major tools of this work were built in Fall 2018 in a prior work Ref. [29], and a major part of calculations of this work were done in Fall 2018 - Summer 2019, when ZW was at USTC and when JW was at Institute for Advanced Study. Part of this work is also presented by JW in the workshop Lattice for Beyond the Standard Model physics 2019, on May 2-3, 2019 at Syracuse University [90]. JW appreciates the support and feedback from the workshop. JW thanks the authors of Ref. [59] for the Sec. 2.5. ZW acknowledges previous supports from NSFC grants 11431010 and 11571329. ZW is supported by the Shuimu Tsinghua Scholar Program. JW was supported by NSF Grant PHY-1606531. This work is also supported by NSF Grant DMS-1607871 “Analysis, Geometry and Mathematical Physics” and Center for Mathematical Sciences and Applications at Harvard University.

A The correspondence between $A_2(1)$-module structure and the $E_2$ page

In this Appendix, we list the correspondence between $A_2(1)$-module $L$ and its $E_2$ page used in our computation before.

![Figure 31: The $A_2(1)$-module $L = \mathbb{Z}_2$ and its $E_2$ page.](image)

\footnote{The topological terms of 4d gauge theories in Ref. [89] are related to the bordism invariants that we enumerate in our present work, and also those bordism invariants enumerated in Ref. [18].}

\footnote{Although the methods of ours [29,30] and theirs [47,59] are rather different: ours is via Adams spectral sequence, while theirs is via Atiyah-Hirzebruch spectral sequence. In addition, we can conveniently read the iTQFT (namely, co/bordism invariants) from the Adams chart [29,30], while iTQFT cannot be extracted easily from Ref. [47,59].}
Figure 32: The $A_2(1)$-module $L = A_2(1)$ and its $E_2$ page.

Figure 33: The $A_2(1)$-module $L = H^{*+2}(C\eta, \mathbb{Z}_2)$ and its $E_2$ page.

Figure 34: The $A_2(1)$-module $L = H^{*+2}(\mathbb{R}P_2^\infty, \mathbb{Z}_2)$ and its $E_2$ page.
Figure 35: The $A_2(1)$-module $L_1$ which appears in Figure 12 and its $E_2$ page.

Figure 36: The $A_2(1)$-module $L_2$ which appears in Figure 21 and its $E_2$ page.

Figure 37: The $A_2(1)$-module $L_3$ which appears in Figure 21 and its $E_2$ page.
Figure 38: The $A_2(1)$-module $L_4$ which appears in Figure 23 and 25 and its $E_2$ page.

Figure 39: The $A_2(1)$-module $L_5$ which appears in Figure 21 and 27 and its $E_2$ page.

Figure 40: The $A_2(1)$-module $L_6$ which appears in Figure 23 and its $E_2$ page.
Figure 41: The $A_2(1)$-module $L_7$ which appears in Figure 23 and its $E_2$ page.

Figure 42: The $A_2(1)$-module $L_8$ which appears in Figure 21 and 27 and its $E_2$ page.

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