ON REFINED BRUHAT DECOMPOSITIONS AND ENDOMORPHISM ALGEBRAS OF GELFAND-GRAEV REPRESENTATIONS

ALESSANDRO PAOLINI AND IULIAN I. SIMION

Abstract. Let \( G \) be a finite reductive group defined over \( \mathbb{F}_q \), with \( q \) a power of a prime \( p \). Motivated by a problem recently posed by C. Curtis, we first develop an algorithm to express each element of \( G \) into a canonical form in terms of a refinement of a Bruhat decomposition, and we then use the output of the algorithm to explicitly determine the structure constants of the endomorphism algebra of a Gelfand-Graev representation of \( G \) when \( G = \text{PGL}_3(q) \) for an arbitrary prime \( p \), and when \( G = \text{SO}_5(q) \) for \( p \) odd.

Let \( G \) be the fixed point subgroup \( \overline{G}^F \) of a reductive algebraic group \( \overline{G} \) under a Frobenius endomorphism \( F \). A special role in the representation theory of finite reductive groups is played by Gelfand-Graev representations. These are certain representations induced from a linear character of a maximal unipotent subgroup of \( G \). If the center of the ambient algebraic group \( \overline{G} \) is connected, then a Gelfand-Graev representation \( \Gamma \) is unique up to conjugacy and the irreducible components of \( \Gamma \) were described in [DL76]. In general, these representations were parametrized and decomposed in [DM91, Section 14] and [DLM92, Section 3].

Denote by \( \mathcal{H} \) the Hecke algebra (that is, the \( G \)-endomorphism algebra) of the module affording \( \Gamma \). Since \( \Gamma \) is multiplicity free [Car85, Theorem 8.1.3], the algebra \( \mathcal{H} \) is abelian. C. Curtis parametrized in [Cur93] the irreducible representations of \( \mathcal{H} \) by pairs \((T, \theta)\) where \( T = \overline{T}^F \) for an \( F \)-stable maximal torus \( \overline{T} \) of \( \overline{G} \) and where \( \theta \) is an irreducible character of \( T \). Each irreducible representation \( f_{T,\theta} \) of the algebra \( \mathcal{H} \) is shown to have the following factorization [Cur93, Theorem 4.2],

\[
    f_{T,\theta} = \hat{\theta} \circ f_T,
\]

where \( f_T : \mathcal{H} \to \mathbb{Q}_\ell T \) is a homomorphism of algebras and \( \hat{\theta} \) is the linear extension of \( \theta \) to the group algebra \( \mathbb{Q}_\ell T \). The homomorphism \( f_T \) is independent of \( \theta \). Curtis’ homomorphism \( f_T \) was considered in [BK08] in the context of \( \ell \)-modular Gelfand-Graev representations with \( \ell \neq p \). Here it is shown that if \( \ell \) does not divide the order of the Weyl group of \( G \), then the behavior of such endomorphism algebras is generic along all prime powers \( q \).

In order to obtain a constructive description of each \( f_T \), one needs to consider structure constants of \( \mathcal{H} \) with respect to some basis. Building on work of Kawanaka [Kaw77] and Deodhar [Deo85], Curtis described in [Cur15] an algorithm for obtaining structure constants for \( \mathcal{H} \) with respect to a standard basis parametrized by certain elements in \( N_G(T) \). In general, the determination of the structure
constants requires a deep understanding of the following intersections of shifted $U$-double cosets,

$$UxU \cap zUy^{-1}U, \quad \text{for} \quad U \in \text{Syl}_p(G) \quad \text{and} \quad x, y, z \in N_G(T).$$

A weaker form of these intersections are the *refined Bruhat cells* which are obtained by replacing $U$ with $B = N_G(U)$, i.e. $BxB \cap zBy^{-1}B$. Here Curtis also raises the problem of explicitly obtaining formulas for the structure constants, which can be used to give a combinatorial proof of the existence of the homomorphisms $f_{T,\theta}$, see [Cur15, Section 4].

The interest in the above cell intersections and the problem raised by Curtis on an explicit determination of structure constants of $\mathcal{H}$ are the motivation lying at the core of our work. The work is divided in two parts. Firstly, we determine an algorithm to decompose explicitly the intersection of Bruhat cells in terms of left $U$-coset representatives when $G$ is of small rank. Secondly, we use the output of this algorithm to determine the structure constants of $\mathcal{H}$ in the case of adjoint types $A_2$ and $B_2$ and some generation properties in these algebras. The knowledge of the structure constants is essential for the determination of the $f_{T,\theta}$, see [Cha76, Section 4]. As the computations are involved and the methods need to be slightly adapted when moving from the simply connected case to the adjoint case, we postpone the problem of determining the $f_{T,\theta}$’s to subsequent work.

We discuss now the methods and results of each of the two parts of the work. Let us focus on the algorithm described in the first part. The intersections of our interest have the form $UxU \cap zUy^{-1}U$ with $x, y$ and $z$ of the form $\tilde{w}t_w$, with $\tilde{w}$ a certain lift in $N_G(T)$ of an element $w$ of the Weyl group $W$ of $G$, and $t_w \in T$. Further, $x$ comes with a fixed reduced expression in $W$. From results of Curtis [Cur88, Cur09, Cur15] which rely on work of Deodhar [Deo85], we can decompose the intersection into disjoint sets indexed by *distinguished subexpressions* of the fixed reduced expression of $x$. Building on results of [Sim16], and exploiting the theory of the invariants for distinguished subexpressions, we first determine the intersections for certain $t_w$ and then we extend this result to all elements of $N_G(T)$. This decomposition method resembles the principle of “collection from the left”, see for example [Sim14, §9.4]. This algorithm has been implemented in Python and is available in the GitHub repository [Chevalley]. It has been successfully run for groups of small rank, in particular in types $A_2$, $B_2$ and $G_2$ exhaustively. An outline of the algorithm is given in Section 2.

We now move on to the investigation of the endomorphism algebras of Gelfand-Graev representations and their structure constants. We first mention some well-known results. The case $G = SL_2$ is worked out in detail in [Cur93]. In [Cha76], the structure constants for $\mathcal{H}$ in the case $G = GL_3$ are calculated and extensively used for the determination of the homomorphisms $f_{T,\theta}$. However, the knowledge of such structure constants does not give directly the ones for PGL$_3(q)$. In fact, the endomorphism algebra with respect to PGL$_3(q)$ is contained in the one with respect to GL$_3(q)$, but *not* a subquotient of it. In other words, a standard basis in the case of PGL$_3(q)$ is not obtained from the one in the case of GL$_3(q)$ in a natural way, and vice versa.

The focus of the second part of this work is on the description of the structure constants of endomorphism algebras of Gelfand-Graev representations for non-twisted groups of rank 2 with connected center, by means of the determination of the $U$-coset representatives obtained in the first part which parametrize a standard basis for $\mathcal{H}$ (described in [CR81, Proposition 11.30], see also §1.3). The choice of considering the *adjoint* version of the simple algebraic group having $G$ as fixed-point subgroup (see §1.1) is motivated by the following two facts. Firstly, there is a unique Gelfand-Graev character, see
for instance [Car85, §8.1]. Secondly, it is easy to parametrize a maximally split torus which allows a more compact description of the standard basis.

Without loss of generality, we may assume that a root datum for $G$ is chosen such that the character $\psi$ corresponding to a Gelfand-Graev representation restricts to the same character $\phi$ on each of the simple root groups. We state our main result.

**Theorem.** Let $q$ be a power of a prime $p$, and let $G = \text{PGL}_3(q)$ for any $p$ or $G = \text{SO}_5(q)$ for $p \geq 3$.

Let $\psi$ be the Gelfand-Graev character of $G$. The structure constants of the endomorphism algebra of $\psi$ with respect to its standard basis are given in Table 4.1 and Table 4.3 respectively. In particular, the elements $e_1(a), e_2(b)$ and $e_3$ for $a, b \in \mathbb{F}_q^\times$ generate the endomorphism algebra of $\psi$ when $G = \text{PGL}_3(q)$.

The methods employed to obtain this result are as follows. The formula given in [Cur15, Section 4] (see §1.3) gives a way to compute the structure constants of $H$ in terms of shifted $U$-double coset intersections, which have been determined in the first part of our work for groups of small rank. The formula can then be written as a sum over certain indeterminates in $\mathbb{F}_q$ and $\mathbb{F}_q^\times$ of the character $\phi$ evaluated on a rational polynomial function in such indeterminates. The remaining task is then to solve some equations over $\mathbb{F}_q$ and to use these solutions, whose number may vary with $q$, to express each sum coming from the formula in [Cur15, Section 4] in a form which is as closed as possible. We can complete this step for groups of type $A_2$ and $B_2$.

In some cases, the structure constants involve certain generalizations of Gauss and Kloosterman sums (see [IK04, Section 11]), and it seems not to be possible to further simplify those calculations. We provide the calculations explicitly in the most involved cases. Details for the remaining computations can be found on [PS]. Although we were able to parametrize the standard basis of $H$ and determine the sums giving the structure constants when $G = G_2(q)$, finding a satisfactory way of expressing these constants is still open. Here the major hurdle is the investigation of equations of degree 5 or higher, whose generic sets of solutions are not easy to control.

We finish by presenting further directions for future work. An important open problem mentioned before is to describe explicitly the algebra homomorphisms $f_{T, \theta} : H \rightarrow \mathbb{C}$. The methods are likely to involve the determination of the structure constants of $H$ and a minimal set of algebra generators, see [Cha76, Section 4–6]. The standard basis elements corresponding to the Weyl group elements 1, $s_1s_2$ and $s_2s_1$ generate $H$ in the cases of $\text{GL}_3(q)$ and $\text{PGL}_3(q)$. However, as explained in Section 4, the techniques for the computations in type $A_2$ do not generalize to type $B_2$. We formulate the following question, an answer to which would generalize [Cha76, Theorem 2.1] to other types.

**Question.** Can the algebra $H$ be always generated by elements of a standard basis which correspond to Weyl group elements of co-length less than or equal to 1?

Lastly, we mention the problem of finding a compact way to express the structure constants of $H$, when the prime $p$ is large enough, for untwisted groups of type $G_2$ or rank higher than 2 and for twisted groups.

The structure of the work is as follows. In Section 1 we describe the setup for groups, Gelfand-Graev characters and the standard basis of $H$, where structure constants of $H$ are expressed as sums over intersections of shifted $U$-double cosets. In Section 2 we show how these intersections can be obtained explicitly by means of the above mentioned algorithm. In Section 3 we discuss the sums which intervene in our description of the structure constants and we provide explicit computations.
in the most involved case in type $B_2$. We collect in Section 4 the structure constants in types $A_2$ and $B_2$.

Acknowledgement: The authors deeply thank G. Malle for his precious comments and feedback on an earlier version of the paper. Part of the work was developed during a research visit of the first author hosted at, and supported by, the Babeș-Bolyai University, and of the second author hosted at the Technische Universität Kaiserslautern and supported by the SFB–TRR 195. The authors would like to thank both institutions for the kind hospitality.

1. Preliminaries

1.1. The group $G$. We consider a simple algebraic group $G$ of adjoint type and $G = \check{G}^F$, the fixed point subgroup under a Frobenius endomorphism $F$ which is not twisted. If $K$ is the subgroup of $G$ generated by unipotent elements, then with few exceptions, $K$ is simple [GLS98, Thm 2.2.7]. By [GLS98, 2.5.8(a)] $G$ is an extension of $K$ by diagonal automorphisms. Moreover, since $G$ is adjoint, we have $G = K\check{T}^F$ for a split maximal torus $\check{T}$ of $G$ and $T = \check{T}^F$ induces the full group of diagonal automorphisms of $K$.

The group $G$, viewed as extension of the finite simple group $K$ by all diagonal automorphisms, can be described in the Chevalley group setting as follows. We regard $K$ as an adjoint Chevalley group [Car89, Theorem 11.1.2] and consider the diagonal automorphisms described there. The group $\check{T}^F$ corresponds to $\check{H}$ in [Car89, §7.1].

Throughout, $T = \check{T}^F \subseteq B = \check{B}^F$ will denote a pair of a split maximal torus and Borel subgroup corresponding to fixed choices of the analogue $F$-stable subgroups $\check{T} \subseteq B$ in $G$. The unipotent radical of $B$ is $\check{U}$ so $U$ is the Sylow $p$-subgroup of $B$. We fix the root system $\Phi$ with respect to $\check{T}$. The simple roots $\Delta$ and the positive roots $\Phi^+$ are chosen with respect to $\check{B}$. The root subgroups of $G$ are of the form $U_\alpha = \check{U}_\alpha^F$ with $\check{U}_\alpha \subseteq \check{B}$ a root subgroup of $G$. The subgroup of $U$ generated by root groups $U_\alpha$ corresponding to positive non-simple roots is called $U^*$.

Further, for $a, b \in \mathbb{F}_q$ and $d \in \mathbb{F}_q^*$ we denote by $u_\alpha(a)$, $n_\alpha(b)$ the elements in $G$ which in [Car89] are denoted by $x_\alpha(a)$, $n_\alpha(b)$ respectively and by $t_\alpha(d)$ the cocharacters corresponding to the coroots (see [Car85, p.76] or [GLS98, Remark 2.5.11(c)]). In addition, $n_\alpha$ stands for $n_\alpha(1)$. To simplify notation, we write $i$ instead of $\alpha_i$ whenever $\alpha_i$ is an index, e.g. $u_{\alpha_1}(a) = u_1(a)$ and $t_{\alpha_b}(c) = t_b(c)$.

The root groups $U_\alpha$ are the images of $u_\alpha$ and the split maximal torus $T$ is generated by the elements of the form $t_\alpha(c)$ . A set of $T$-coset representatives in the Weyl group $W = N_G(T)/T$ can be selected from words in the elements $n_\alpha \in G$ with $\alpha \in \Delta$; these choices will be made in the sequel. The set of these fixed representatives is $\check{W}$ and for $w \in \check{W}$ we denote by $w$ the corresponding representative. The simple reflections of $W$ are $\{s_\alpha = n_\alpha T : \alpha \in \Delta\}$, the length of a reduced word of $w \in W$ in terms of these simple reflections is denoted by $\ell(w)$ and the lifts of $s_\alpha$ are $\check{s}_\alpha = n_\alpha \in \check{W}$. In the sequel, we sometimes use the letter $\ell$ to denote a positive integer other than the length function; the meaning of $\ell$ is in any case determined by the local use.

Using $t_\alpha$ to parametrize $T$ we have for every $\alpha, \beta \in \Phi^+, \lambda \in \mathbb{F}_q^\times$ and $\mu \in \mathbb{F}_q$ (see [Car85, p.76]),

\begin{equation}
\ell_\alpha(\lambda)u_\beta(\mu) = \ell_\alpha(\lambda)u_\beta(\mu)\ell_\alpha(\lambda)^{-1} = u_\beta(\lambda^{\alpha_\beta} \mu).
\end{equation}

The ground field of $\check{G}$ is of characteristic $p > 0$ and the Frobenius endomorphism $F$ is such that $|U_\alpha^F| = q$ for some power $q$ of $p$, i.e. $K \subseteq \check{G}^F$ is a Chevalley group over the field $\mathbb{F}_q$. 
1.2. Gelfand-Graev characters. Following [Cur15], let $\psi$ be a linear representation of $U$ and let $e = |U|^{-1} \sum_{u \in U} \psi(u^{-1})u \in CG$ be the idempotent such that the induced representation $\psi^G$ is afforded by the module $CGe$. The Hecke algebra corresponding to $\psi$ is (as in [CR81, §11.D])

$$H = eCGe = \text{End}_{CG} CGe.$$  

Further, $\psi^G$ is a Gelfand-Graev character if $\psi(U\alpha) \neq 1$ for simple roots $\alpha$ and $\psi(U\alpha) = 1$ for all other roots $\alpha$ [Car85, §8.1]. By [Car85, Theorem 8.1.3], the algebra $H$ is proved to be abelian due to an isometry of Gelfand-Graev characters which has order two.

The character $\psi$ has $U^*$ in its kernel. Since $U/U^* \cong \prod_{\alpha \in \Delta} U\alpha \cong k|\Delta|$, for any group isomorphism $u_\alpha : k \rightarrow U\alpha$ we obtain characters $\phi_\alpha = (\psi|_{U\alpha}) \circ u_\alpha : k \rightarrow C$. Denoting by $f : F_q \rightarrow F_p$ the field trace map, it then follows from [Car85, Proposition 8.1.2] that we may assume

$$\phi_\alpha = \phi \quad \forall \alpha \in \Delta, \quad \text{where} \quad \phi(x) = e^{\frac{2\pi i \text{Tr}(x)}{p}}.$$  

When determining the character values of $\psi$ and $\phi$, we will frequently use delta-notation for the description of the obtained structure constants,

$$\delta_P = \begin{cases} 1 & \text{if } P \text{ is true}, \\ 0 & \text{if } P \text{ is false}, \end{cases} \quad \text{and} \quad \delta_{a,b} = \delta_{a=b} = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{if } a \neq b, \end{cases}$$  

for example $\delta_{1\in\{2,3\}} = 0$.

1.3. Standard basis and structure constants. The set of $U$-double coset representatives of $G$ is $\{\dot{w}t : w \in W, t \in T\}$. The standard basis $B$ of the algebra $H$ (with respect to $T \subseteq B$) is described in [CR81, Proposition 11.30], it is

$$B = \left\{ e_n = q^{\ell(w)}ene, \quad \text{where} \quad n = \dot{w}t, \ w \in W, \ t \in T, \ \text{such that} \quad n\psi = \psi \text{ on } U \cap nU \right\}.$$  

For $e_\ell, e_m, e_n \in B$ the corresponding structure constant is

$$[e_\ell e_m : e_n] = \sum_{u\ell u' = nvm^{-1} \in UU \cap nUm^{-1}U} \psi((uu')^{-1}v).$$

An algorithm for obtaining these constants was provided by C. Curtis (see [Cur15]). A variation of this algorithm, with different choices for $U$-coset representatives was described in [Sim16]. We use this latter version (see Section 2) to obtain the elements

$$u\ell u_1 = nvm^{-1} \in U\ell U \cap nUm^{-1}U$$

explicitly in both their forms ($u\ell u_1$ and $nvm^{-1}$) for $G$ of type $A_2$ and $B_2$.

Some of the more involved structure constants obtained here are expressed in terms of Gauss sums, Kloosterman sums and generalizations thereof described in Section 3. Similar calculations are obtained for $G_2$; however, finding concise formulae for the structure constants which are meaningful in terms of character values of $\phi$ and symmetries of $H$ remains open in this case.
1.4. Types $A_2$ and $B_2$. We assume throughout that $\Phi$ is of type $A_2$ or $B_2$, with positive roots $\Phi^+$ ordered as follows,

$$A_2 : \alpha_1, \alpha_2, \alpha_1 + \alpha_2, \quad \text{and} \quad B_2 : \alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2.$$ 

The negative roots in $\Phi$ are ordered by $-\alpha_i = e_{i+1} \epsilon |_{+1} i$, e.g. $\alpha_6 = -\alpha_1 - \alpha_2$ in type $A_2$ and $\alpha_6 = -\alpha_2$ in type $B_2$. The structure constants for the group $G$ are as in [Car89, §5.2]. According to the choice of extra-special pairs, the non-trivial commutator relations among root elements are

$$u_1(x_1)u_2(x_2) = u_1(x_1)u_2(x_2)u_1(x_1)^{-1} = u_2(x_2)u_2(x_1x_2)$$

in type $A_2$, and

$$u_1(x_1)u_2(x_2) = u_2(x_2)u_3(x_1x_2)u_4(x_1^2x_2) \quad \text{and} \quad u_1(x_1)u_3(x_3) = u_3(x_3)u_4(2x_1x_3)$$

in type $B_2$.

1.4.1. For the description of $B$, let $e_n \in B$ with $n = \dot{w}t$, $w \in W$ and $t \in T$. By [Ste67, (1) in proof of Thm. 49], if $e_n \neq 0$ then $w = w_0 w_\pi$ where $w_0$ is the longest element of $W$ and $w_\pi$ is the longest element of the parabolic subgroup $W_\pi \leq W$ corresponding to the subset $\pi$ of simple roots. Notice that $\dot{w}$ and $\dot{w}_0 \dot{w}_\pi$ might differ.

In rank 2, there are 4 standard parabolic subgroups of $W$. Their longest elements are $1, s_1, s_2, s_1s_2s_1$ for $A_2$ and $s_1s_2s_1s_2$ for $B_2$, so

$$w_0w_\pi = \begin{cases} s_1s_2s_1, & s_1s_2, & s_2s_1, & 1 \text{ for } A_2, \\ s_1s_2s_1s_2, & s_2s_1s_2, & s_1s_2s_1, & 1 \text{ for } B_2. \end{cases}$$

We number the lifts in $N_G(T)$ of the above elements in the following way,

$$\dot{w}_0 = n_1n_2n_1, \quad \dot{w}_1 = n_1n_2, \quad \dot{w}_2 = n_2n_1, \quad \dot{w}_3 = 1 \text{ for } A_2,$$

$$\dot{w}_0 = n_1n_2n_1n_2, \quad \dot{w}_1 = n_2n_1n_2, \quad \dot{w}_2 = n_1n_2n_1, \quad \dot{w}_3 = 1 \text{ for } B_2.$$

1.4.2. Recall that the character $n^\psi$ is defined such that $n^\psi(u) = \psi(u^n)$ for all $u \in U$. Thus in order to obtain the standard basis (3), we need the condition

$$\psi(u) = \psi(u^n) \quad \text{for} \quad u \in U \cap nU$$

(6) to be satisfied. Hence

$$B = \left\{ e_n = q^{(w)}e_n \right\}, \quad \text{where} \quad n = \dot{w}t, \quad w = w_0w_\pi \in W, \quad t \in T, \quad \psi(u) = \psi(u^n) \text{ for } u \in U \cap nU.$$

Using commutator relations, in particular (1), we can impose the condition $\psi(u) = \psi(u^n)$ for $u \in U \cap nU$ from (6) in our particular cases. Since $U \cap U^{\dot{w}_0} = 1$, the condition for $w_0$ is empty, so $\dot{w}_0t \in B$ for all $t \in T$. For the element $n = n_1(a,b) = \dot{w}_1t_1(a)t_2(b)$, we have

$$U \cap nU = \begin{cases} U_2 & \text{for } A_2, \\ U_1 & \text{for } B_2 \end{cases} \quad \text{and} \quad \begin{cases} u_2(x)^{n_1(a,b)} = u_1(a^{-1}x) & \text{for } A_2, \\ u_1(x)^{n_1(a,b)} = u_1(a^{-1}x) & \text{for } B_2. \end{cases}$$

With (2), we have that an element of the form $n_1(a,b)$ is in $B$ if and only if for all $x \in F_q$ we have

$$\psi(u_2(x)) = \psi(u_1(a^{-1}x)) \quad \text{and} \quad \psi(u_1(x)) = \psi(u_1(a^{-1}x)) \quad \iff \phi(x) = \phi(a^{-1}x) \iff a = 1 \text{ for } A_2 \text{ and } B_2.$$

(7)
Now consider the element $n_2(a, b) = \tilde{w}_2 t_1(a) t_2(b)$. We have

$$U \cap nU = \begin{cases} U_1 & \text{for } A_2, \\ U_2 & \text{for } B_2 \end{cases}$$

and

$$\begin{align*}
& u_1(x)^{n_2(a, b)} = u_2(b^{-1}x) \quad \text{for } A_2, \\
& u_2(x)^{n_2(a, b)} = u_2(b^{-1}x) \quad \text{for } B_2.
\end{align*}$$

Hence, as previously argued, we have $n_2(a, b) \in \mathcal{B}$ if and only if $b = 1$.

For $\dot{w}_3 = 1$ we consider $n = n_3(a, b) = t_1(a) t_2(b)$, so $U \cap nU = U$ and $n_3(a, b) \in \mathcal{B}$ if and only if

$$(8) \quad \psi(u_1(x_1) u_2(x_2)) = \psi(u_1(x_1 a^{-1}) u_2(x_2 b^{-1})) \iff \phi(x_1 + x_2) = \phi(a^{-1} x_1 + b^{-1} x_2) \iff a = b = 1$$

in all cases. These conditions determine $\mathcal{B}$, namely, the elements in $\mathcal{B}$ are parametrized by the elements

$$(9) \quad n_0(a, b) := \tilde{w}_0 t_1(a) t_2(b), \quad n_1(c) := \tilde{w}_1 t_2(c), \quad n_2(d) := \tilde{w}_2 t_1(d), \quad n_3 := \tilde{w}_3 = 1$$

with $a, b, c, d \in \mathbb{F}_q^\times$. The corresponding basis elements of $\mathcal{H}$ are

$$\mathcal{B} = \{ e_0(a, b) = e n_0(a, b) e, \ e_1(c) = e n_1(c) e, \ e_2(d) = e n_1(d) e, \ e_3 = e n_3 e = e, \ a, b, c, d \in \mathbb{F}_q^\times \}.$$  

We sometimes omit the parameters $a, b, c, d$ and write $e_i$ and $n_i$ for these elements.

## 2. $U$-Double Cosets

2.1. Once the basis $\mathcal{B}$ of $\mathcal{H}$ is determined, we obtain the structure constants by (4). In order to do so, we need to determine all left $U$-coset representatives in the intersections (5):

$$U x U \cap z U y^{-1} U \quad \text{for } x, y, z \in \mathcal{B}.$$  

From results of C. Curtis [Cur88, Cur09, Cur15] building on results of V. Deodhar [Deo85], the above intersection decomposes into disjoint sets parametrized by distinguished subexpressions of the fixed reduced expression for $x$,

$$U x U \cap z U y^{-1} U = \bigsqcup_{j \in J(x,y,z)} D_j U \quad \text{for } x, y, z \in N_G(T).$$

The sets $D_j$ of left $U$-coset representatives are described inductively. For concretely determining these intersections in both the $U x U$-form and $z U y^{-1} U$-form which are required for our purposes, one needs to translate the inductive descriptions into the two forms. By making use of the $D_j$ described in [Sim16], we illustrate how the algorithm works in general. Such a procedure has been implemented in Python [Chevalley].

The problem is divided in two steps. Since $x, y$ and $z$ are of the form $w t_w$ for $w \in W$ and $t_w \in T$, the intersections can be obtained by

I. determining first the intersections for certain $t_w$, and
II. deducing then the intersections for arbitrary $t_w$.

We show in §2.4 how the algorithm works in the case of $Un_0 U \cap n_0 Un_0 U$ for $G$ of type $B_2$.  

2.2. Part I. The set \( W \) of reduced expressions in terms of simple reflections was chosen in \( \S 1.4.1 \). Fix \( x, y, z \in W \) and consider the fixed reduced expression \( s_{i_n} \ldots s_{i_1} \) of \( x \), which we may identify with the tuple \( i = [i_n, \ldots, i_1] \). A subexpression \( j \in \{0, i_n\} \times \cdots \times \{0, i_1\} \) of \( i \) has the following sets attached to it,

\[
\begin{align*}
A_j &= \{ m : \ell(s_{j_m} \tau_{m-1}(j)y) < \ell(\tau_{m-1}(j)y) \}, \\
B_j &= \{ m : \ell(s_{j_m} \tau_{m-1}(j)y) = \ell(\tau_{m-1}(j)y) \} = \{ m : j_m = 0 \}, \\
C_j &= \{ m : \ell(s_{j_m} \tau_{m-1}(j)y) > \ell(\tau_{m-1}(j)y) \},
\end{align*}
\]

where \( s_0 = 1 \) and \( \tau_m \) is the truncation map \( j \mapsto \tau_m(j) = s_{j_m} \ldots s_{j_1} \). Using these sets we define, for a tuple \( \mu \in \mathbb{F}_q^\ell(x) \),

\[
D_j(\mu) := \prod_{m=1}^{n} \left[ u_m(\mu_m)s_m \right]^{\delta_m \in A_j} \left[ u_{-m}(\mu_m)\hat{s}_m \right]^{\delta_m \in B_j} \left[ \hat{s}_m \right]^{\delta_m \in C_j}
\]

where the product is from right to left, with \( \mu_m \in \mathbb{F}_q \) if \( m \in A_j \), \( \mu_m \in \mathbb{F}_q^* \) if \( m \in B_j \), and \( \mu = 1 \) if \( m \in C_j \) and where \( \hat{s}_\alpha = n_\alpha \) and \( u_{-m} = u_{-\alpha_m} \). Here the exponents are the delta-functions defined in \( \S 1.2 \). In this case, since \( m \) is in exactly one of the sets \( A_j, B_j \) or \( C_j \), we take the term corresponding to the set in which \( m \) lies.

We also define

\[
D'_j(\mu) := \prod_{m=1}^{n} \left[ u_m(\mu_m)s_m \right]^{\delta_m \in A_j} \left[ u_{-m}(\mu_m)^{-1}\hat{s}_m(-\mu_m)^{-1}u_m(\mu_m^{-1}) \right]^{\delta_m \in B_j} \left[ \hat{s}_m \right]^{\delta_m \in C_j}
\]

where the product is again from right to left and where \( \mu_m \) is as before. We say that \( m \) is of type A, B or C according to the set \( A_j, B_j \) or \( C_j \) it lies in. It is a small calculation in type \( A_1 \) (using for example the relations in \([\text{Car}89]\)) to show that the \( m \)'s of type B are equal in the two expressions so we have

\[
D_j(\mu) = D'_j(\mu).
\]

It is also easy to notice that \( D'_j(\mu) \) parametrizes distinct left \( U \)-coset representatives in \( UxU \) (exactly \( q^{|A_j|}(q-1)^{|B_j|} \) such representatives for the allowed values of \( \mu \)). Moreover, using commutator relations in \([\text{Car}89]\), one can calculate the \( UxU \)-form of the element \( D'_j(\mu) \), that is,

\[
D'_j(\mu) = U'_j(\mu)xt_{\mu}V_{\mu}(\mu),
\]

where \( t_{\mu} \in T \) appears due to the B-type positions in \( j \). Now, if \( j \) is a distinguished subexpression of \( i \), using \([\text{Sim}16, \text{Lemma 3.8}]\) one can calculate the \( zUy^{-1}U \)-form of the element \( D_j(\mu) \), namely

\[
D_j(\mu) = zU_j(\mu)y^{-1}t_0V_j(\mu)
\]

where \( 1 \neq t_0 \in T \) may appear in the expression due to non-reduced products of elements in \( W \).

It is in this part of the algorithm that the property of \( j \) being a distinguished subexpression is used in an essential way. Namely this property allows one to move the root group elements corresponding to negative roots on the left-hand side of

\[
z^{-1}D_j(\mu) = U_j(\mu)y^{-1}t_0V_j(\mu)
\]

to the left, such that they end up in \( U_j(\mu) \subseteq U \), (for details see \([\text{Sim}16, \S 3.3]\)). In this way we obtain

\[
D_j(\mu)U \subseteq UxU \cap zUy^{-1}t_0U.
\]
Since the elements $D_j(\mu)$ are distinct left $U$-coset representatives for different $j \in J(x,y,z)$ and different parameters $\mu$, it follows from [Cur09, Lemma 2.3] that we obtain all left $U$-cosets in $U x t_\mu U \cap z U y^{-1} t_0 U$.

2.3. **Part II.** We keep the assumptions and notation form Part I and consider the intersection

\begin{equation}
U x t_x U \cap z t_z U (y t_y)^{-1} U
\end{equation}

for $t_x, t_y, t_z \in T$. Since $U_j'(\mu) x t_\mu V_j'(\mu) = D_j(\mu) = z U_j(\mu) y^{-1} t_0 V_j(\mu)$ we have

\[ U_j'(\mu) x t_\mu t_0 [V_j'(\mu)V_j(\mu)^{-1}] t_0 = D_j(\mu) = z U_j(\mu) y^{-1} \]

hence

\[ z t_z U_j(\mu)(y t_y)^{-1} = z t_z U_j(\mu)t_y^{-1} y^{-1} = (t_z)^z^{-1} U_j'(\mu) x t_\mu t_0 [V_j'(\mu)V_j(\mu)^{-1}] t_0 (t_y^{-1}) y^{-1} \]

which equals

\[ (U_j'(\mu))^{(t_z)^{z^{-1}}} \cdot x \cdot (t_z)^{z^{-1}} x t_\mu t_0 (t_y^{-1}) y^{-1} \cdot [V_j'(\mu)V_j(\mu)^{-1}] t_0 (t_y^{-1}) y^{-1}. \]

It follows that, for fixed $x, y, z, t_x, t_y, t_z$ the intersection

\[ U x t_x U \cap z t_z U (y t_y)^{-1} U \]

is non-empty if and only if

\begin{equation}
(t_z)^{z^{-1}} x t_\mu t_0 (t_y^{-1}) y^{-1},
\end{equation}

in which case the elements

\[ \left\{ z t_z U_j(\mu)(y t_y)^{-1} = [U_j'(\mu)]^{(t_z)^{z^{-1}}} x t_x [V_j'(\mu)V_j(\mu)^{-1}] t_0 (t_y^{-1}) y^{-1} : \mu \in F_q^\ell(x), j \in J(x,y,z) \right\}, \]

where each entry $\mu_m$ is restricted to the conditions remarked at the beginning of §2.2, form a complete set of left $U$-coset representatives for $U x t_x U \cap z t_z U (y t_y)^{-1} U$.

The elements in (10) can thus be determined with the following steps:

**Input:** Fixed reduced expressions for Weyl group elements. Elements $x, y, z \in W$ and $t_x, t_y, t_z \in T$.

**Result:** A set of left $U$-coset representatives in $U x t_x U \cap z t_z U (y t_y)^{-1} U$.

result ← ∅;

\begin{algorithm}
begin
  \textbf{foreach} distinguished expression $j \in J(x,y,z)$ \textbf{do}
  consider the products $D_j(\mu) = D_j'(\mu)$;
  bring $D_j(\mu)$ in the form $x U_j(\mu) y^{-1} t_0 V_j(\mu)$;
  bring $D_j'(\mu)$ in the form $U_j'(\mu) x t_\mu V_j'(\mu)$;
  \textbf{if} $t_x = (t_z)^{z^{-1}} x t_\mu t_0 (t_y^{-1}) y^{-1}$ \textbf{then}
  \textbf{add} $\dot{z} t_z U_j(\mu)(\dot{y} t_y)^{-1} = [U_j'(\mu)]^{(t_z)^{z^{-1}}} \dot{z} t_x [V_j'(\mu)V_j(\mu)^{-1}] t_0 (t_y^{-1}) y^{-1}$ to result;

  return result;
\end{algorithm}
In our calculations however, \( t_x, t_y \) and \( t_z \) do not range over \( T \), namely they are restricted to the toral elements intervening in the standard basis as in (9).

2.4. \([e_0 e_0, e_0]\) in type \( B_2 \). In order to determine the constants \([e_0(a_1, b_1)e_0(a_2, b_2) : e_0(a_3, b_3)]\) in type \( B_2 \), we first need to determine the intersection in (5), namely

\[ n_0(a_3, b_3)U n_0(a_2, b_2)^{-1}U \cap U n_0(a_1, b_1)U. \]

The reduced expression for \( w_0 \) is \( s_1 s_2 s_1 s_2 \) and the distinguished subexpressions in \( J(w_0, w_0, w_0) \) are

\[ [0, 0, 0, 0] \text{ of type BBBB, } [0, 2, 0, 2] \text{ of type BCBA and } [1, 0, 1, 0] \text{ of type CBAB.} \]

2.4.1. For \( j = [0, 2, 0, 2] \) we have \( D_j = u_5(x_1)n_2(1)u_5(x_3)u_2(x_4)n_2(1) \) so

\[
 z^{-1}D_j = \frac{u_1(-x_1)u_2(x_4)u_3(x_3) n_1(-1)n_2(-1)n_1(-1)n_2(-1) t_1(-1)}{u'_j} y^{-1} t_0.
\]

We also have \( D'_j = u_1(x_1^{-1})n_1(-x_1^{-1})u_1(x_1^{-1})n_2(1)u_3(x_3^{-1})n_1(-x_3^{-1})n_1(x_3^{-1})u_2(x_4)n_2(1) = u_1(x_1^{-1})u_2 \left( \frac{x_1(x_1x_4 - 2x_3)}{x_3^2} \right) u_3(x_3^{-1})n_1(1)n_2(1)n_1(1)n_2(1) t_1(x_1^2)t_2 \left( \frac{x_2}{x_1} \right) u_4 \left( \frac{x_3 - x_1x_4}{x_1x_3} \right) u_3(x_3^{-1})u_4 \left( \frac{x_4}{x_3^2} \right)
\]

which gives the first part of the algorithm. Condition (11) on the elements in \( T \) is

\[
t_x = t_1(a_1)t_2(b_1) = t_1(-a_2a_3x_1^2)t_2 \left( b_2b_3x_3^2 \right), \quad \text{where } \ t_y = t_1(a_2)t_2(b_2) \quad \text{and} \quad t_z = t_1(a_3)t_2(b_3).
\]

Hence, for this distinguished expression the elements in \( n_0(a_3, b_3)U n_0(a_2, b_2)^{-1} \cap U n_0(a_1, b_1)U \) are

\[
n_0(a_3, b_3)u_1(-x_1)u_2(x_4) \cdots n_0(a_2, b_2)^{-1} = u_1 \left( \frac{1}{a_3x_1} \right) u_2 \left( \frac{x_1(x_1x_4 - 2x_3)}{b_3x_3^2} \right) \cdots n_0(a, b)u_1 \left( \frac{x_1x_4 - x_3}{a_2x_1x_3} \right) \cdots
\]

where \( \mu = (x_1, 1, x_3, x_4) \in \mathbb{F}_q^\times \times \{1\} \times \mathbb{F}_q^\times \times \mathbb{F}_q \) and where the dots indicate terms in \( U_3 \) and \( U_4 \).

For our purposes, we are only interested in \( U_1 \) and \( U_2 \), so, once \( D_j = D'_j \) has been brought in the \( zUy^{-1}U \)-form and in the \( UxU \)-form, we may ignore the terms in \( U_3 \) and \( U_4 \), in that they lie in \( \ker \psi \) and do not contribute to the sum (4) with distinct summands.

2.4.2. For \( j = [1, 0, 1, 0] \) we have \( D_j = n_1(1)u_6(x_2)u_1(x_3)n_1(1)u_6(x_4) \) so

\[
 z^{-1}D_j = \frac{u_1(x_3)u_2(-x_4)u_4(-x_2)n_1(-1)n_2(-1)n_1(-1)n_2(-1)}{u'_j} y^{-1} t_0.
\]

Here \( t_0 = 1 \). We also have \( D'_j = n_1(1)u_2(x_2^{-1})n_2(-x_2^{-1})u_2(x_2^{-1})u_1(x_3)n_1(1)u_2(x_4^{-1})n_2(-x_4^{-1})u_2(x_4^{-1}) = u_2 \left( \frac{1}{x_4} + \frac{x_3^2}{x_2^2} \right) u_3 \left( 1 + \frac{x_3}{x_2} \right)u_4 \left( \frac{x_2}{x_4} \right) u_1 \left( \frac{x_3x_4}{x_2} \right) u_2 \left( \frac{x_3^{-1}}{x_4} \right) u_4 \left( x_3^{-1} \right)u_4 \left( x_2^{-1} \right).
\]
The condition (11) on elements in $T$ is

$$t_x = t_1(a_1)t_2(b_2) = t_1\left(\frac{a_2a_3x_2}{x_4}\right) t_2\left(\frac{b_2b_3x_4}{x_4}\right), \text{ where } t_y = t_1(a_2)t_2(b_2) \text{ and } t_z = t_1(a_3)t_2(b_3).$$

For this distinguished expression the elements in $n_0(a_3, b_3)U n_0(a_2, b_2)^{-1}U \cap U n_0(a_1, b_1)$ are

$$n_0(a_3, b_3)u_1(x_3)u_2(-x_4) \cdots n_0(a_2, b_2)^{-1}u_2\left(\frac{x_2+x_3^2}{b_3^2x_2x_4}\right) \cdots = u_1\left(\frac{x_3x_4}{a_2x_2}\right) u_2\left(\frac{1}{b_2x_4}\right) \cdots n_0(a, b)$$

where $\mu = (1, x_2, x_3, x_4) \in \{1\} \times \mathbb{F}_q^\times \times \mathbb{F}_q \times \mathbb{F}_q^\times$ and where the dots indicate terms in $U_3$ and $U_4$.

2.4.3. For $j = [0, 0, 0, 0]$ we have $D_j = u_5(x_1)u_6(x_2)u_5(x_3)u_6(x_4) = u_{-1}(x_1)u_{-2}(x_2)u_{-1}(x_3)u_{-2}(x_4)$, so

$$z^{-1}D_j = \frac{u_1(-x_1-x_3)u_2(-x_2-x_4)u_3(-x_2x_3)u_4(-x_3x_4^3)}{U_j} n_1(-1)n_2(-1)n_1(-1)n_2(-1).$$

Moreover,

$$D_j' = u_1(x_1^{-1})u_1(x_1^{-1})u_2(x_2^{-1})u_2(x_2^{-1})u_3(x_3^{-1})u_3(x_3^{-1})u_4(x_4^{-1})u_4(x_4^{-1})$$

and calculating the Bruhat form, one finds that

$$D_j' = u_1(x_1^{-1}) u_2\left(\frac{x_2^2+x_1+x_3+x_1^2}{x_2^2x_4^3}\right) u_3\left(-\frac{x_1+x_3}{x_1x_2x_3}\right) u_4\left(\frac{1}{x_1x_2}\right) n_0(1)n_2(1)n_1(1)n_2(1).$$

For the second part of the algorithm we have $t_x = t_1(a_1)t_2(b_1)$, $t_y = t_1(a_2)t_2(b_2)$, $t_z = t_1(a_3)t_2(b_3)$ and condition (11) on elements in $T$ is

$$t_1(a_1)t_2(b_1) = t_x = (t_x)^{-1}t_\mu t_0(t_y)^{-1} = t_1\left(\frac{a_2a_3x_1^2x_2}{x_4}\right) t_2\left(\frac{b_2b_3x_2^2x_4}{x_4}\right).$$

In fact, here $z^{-1}x = 1$ and $t_y^{-1} = t_y^{-1}$. Continuing, the elements in $n_0(a_3, b_3)U n_0(a_2, b_2)^{-1}U \cap U n_0(a, b)$ are

$$n_0(a_3, b_3)u_1(-x_1-x_3)u_2(-x_2-x_4) \cdots n_0(a_2, b_2)^{-1} = u_1\left(\frac{1}{a_3x_1}\right) u_2\left(\frac{x_1x_2+x_4(x_1+x_3)}{b_3x_2x_4}\right) \cdots n_0(a_1, b_1) u_1\left(\frac{x_1x_2+x_4(x_1+x_3)}{a_2x_1x_2x_3}\right) u_2\left(\frac{1}{b_2x_4}\right) \cdots$$

$$\left[U_j(\mu)\right]^{t_\mu^{-1}} \left[U_j(\mu)\right]^{t_0(t_y)^{-1}} = \left[U_j(\mu)\right]^{t_\mu^{-1}} \left[U_j(\mu)\right]^{t_0(t_y)^{-1}}$$

where $\mu = (x_1, x_2, x_3, x_4) \in (\mathbb{F}_q^\times)^4$, where the dots indicate terms in the root groups $U_3$ and $U_4$ and where $t_x$, $t_y$ and $t_z$ satisfy the relation previously deduced.
3. Calculating structure constants

We describe in this section the methods employed to obtain the values of the structure constants of endomorphism algebras of Gelfand-Graev representations in types $A_2$ and $B_2$. Such methods involve manipulating some well-known sums over $\mathbb{F}_q$ which generalize the so-called *Gauss sums* over finite fields and involve values of the character $\phi$ fixed in (2). These sums, in particular quadratic Gauss sums and Kloosterman sums, frequently occur in the description of the structure constants in Tables 4.1 and 4.3 and allow a more compact way of collecting the structure constants.

3.1. We first remark some properties of roots in finite fields. Let $F$ be a finite field and involve values of the character $\phi$ fixed in (2). For (i) and (ii), we refer to [Cha76, Lemma 1.2]. We now show (iii). Denote by $\zeta_1, \ldots, \zeta_r$ the $r$ distinct $r$-th roots of $d$ in $\mathbb{F}_q$. Then it is easy to see that $d \neq (a/c)^r$ then $a - c\zeta_i \neq 0$ for every $i = 1, \ldots, r$, while if $d = (a/c)^r$ then $a - c\zeta_i = 0$ for exactly one index $i \in \{1, \ldots, r\}$. The claim now follows by applying (i).

3.2. The *standard quadratic Gauss sum* has the form

$$ (12) \quad G := \sum_{x \in \mathbb{F}_q} \phi(x^2). $$

We first assume that $p$ is an odd prime. Let $a \in \mathbb{F}_q^\times$. Then it is easy to see that

$$ \sum_{x \in \mathbb{F}_q} \phi(ax^2) = (2\delta_{a \in \mathbb{F}_q^\times} - 1)G, $$

for example by applying (i) in §3.1. For $q = p$, we have that this sum evaluates to $\sqrt{p}$ if $p$ is congruent to 1 modulo 4, and to $i\sqrt{p}$ if $p$ is congruent to -1 modulo 4. For generic odd $q$, we refer to [IK04, Chapters 3 and 11]; in particular, an explicit value of this sum is in general not known.

It is then straightforward to obtain the following generalization,

$$ \sum_{x \in \mathbb{F}_q} \phi(Ax^2 + Bx + C) = (2\delta_{a \in \mathbb{F}_q^\times} - 1)G\phi \left( C - \frac{B^2}{4A} \right). $$

Quadratic gauss sums and their generalizations yield different sum values if $p = 2$. Namely in this case we have that $G$ as defined in (12) evaluates to 0, as the map $x \mapsto x^2$ is an automorphism of $\mathbb{F}_q$. 


when \( q = 2^f \). Moreover, we recall that in this case we have that \( \phi(x^2 + x) = 1 \) for every \( x \in \mathbb{F}_q \). As a consequence, if we let \( A, B, C \in \mathbb{F}_q^\times \), then we have

\[
\sum_{x \in \mathbb{F}_q} \phi(Ax^2 + Bx + C) = \begin{cases} 
q\phi(C), & \text{if } A = B^2 \\
0, & \text{otherwise}.
\end{cases}
\]

3.3. Let \( \ell \mid q - 1 \). We define the following sum, for \( B, a, b, a', b' \in \mathbb{F}_q \),

\[
\tilde{S}_\ell(B, a, b, a', b') := \sum_{\zeta \in \mathbb{F}_q^\times \mid \zeta^\ell = B} \phi \left( a' \zeta^2 + a\zeta + \frac{b}{\zeta} + \frac{b'}{\zeta^2} \right).
\]

A special case of the sum \( \tilde{S}_\ell(B, a, b, a', b') \) repeatedly appears in Tables 3.4.1.4 and 3.4.3, namely

\[
S_\ell(B, a, b) := \tilde{S}_\ell(B, a, b, 0, 0) = \sum_{\zeta \in \mathbb{F}_q^\times \mid \zeta^\ell = B} \phi \left( a\zeta + \frac{b}{\zeta} \right).
\]

We call \( \tilde{S}_\ell(B, a, b, a', b') \) and \( S_\ell(B, a, b) \) generalized Kloosterman sums, as the value of \(-S_{q-1}(1, a, b)\) is equal to the value of the Kloosterman sum in [IK04, §11.5] when the maps \( \psi \) and \( \varphi \) appearing there are multiplications by \( a \) and \( b \) respectively. Notice that \( \tilde{S}_\ell(1, a, b, a', b') = \tilde{S}_\ell(1, b, a, b', a') \), thus \( S_\ell(1, a, b) = S_\ell(1, b, a) \). Although \(|S_\ell(1, a, b)|\) can be sharply bounded, no simpler formula for the sum \( S_\ell(B, a, b) \) exists, see [IK04, §1.4].

3.4. \([e_0 e_0 : e_0] \text{ in type } B_2\). We are now in a position to continue the calculation of the structure constants with the use of (4). We provide here in full details the computations giving the structure constants \([e_0(a_1, b_1)e_0(a_2, b_2) : e_0(a_3, b_3)] \) in type \( B_2 \). This is the most complicated case to study from Tables 3.4.1 and 3.4.3. The computational details for the other structure constants in such tables are collected in [PS] and can be obtained more easily by using the same methods.

Since there are three distinguished subexpressions \( j \), the sum describing these structure constants splits into three parts which we treat separately.

3.4.1. \( j = [0, 2, 0, 2] \). By §2.4.1 we have that

\[
a_1 = -a_2 a_3 x_1^2 \quad \text{and} \quad b_1 = \frac{b_2 b_3 x_3^2}{x_1^2}.
\]

The subsum in (4) restricted to the elements in §2.4.1 is

\[
\sum \psi \left( u_1 \left( -x_1 - \frac{1}{a_3 x_1} - \frac{x_4}{a_2 x_3} + \frac{1}{a_2 x_1} \right) u_2 \left( x_4 - \frac{x_1^2 x_4}{b_3 x_3^2} + \frac{2x_1}{b_3 x_3} \right) \right)
\]

with summation over \((x_1, x_2, x_3, x_4) \in \mathbb{F}_q^\times \times \{1\} \times \mathbb{F}_q^\times \times \mathbb{F}_q\). This equals

\[
\sum \phi \left( -x_1 - \frac{1}{a_3 x_1} - \frac{x_4}{a_2 x_3} + \frac{1}{a_2 x_1} + x_4 - \frac{x_1^2 x_4}{b_3 x_3^2} + \frac{2x_1}{b_3 x_3} \right).
\]
If \(-\frac{a_1}{a_2a_3} \not\in \mathbb{F}^\times_{q,2}\) or \(\frac{b_1}{b_2b_3} \not\in \mathbb{F}^\times_{q,2}\), then the equations (13) do not have a common solution, and the sum is zero. We assume \(-\frac{a_1}{a_2a_3}, \frac{b_1}{b_2b_3} \in \mathbb{F}^\times_{q,2}\). Then we have that
\[
(x_1, x_3) \in \left\{ (\zeta_1, \zeta_1\zeta_2) \mid \zeta_1^2 = \frac{a_1}{a_2a_3}, \zeta_2^2 = \frac{b_1}{b_2b_3} \right\}.
\]
Hence we can write the sum as
\[
\sum_{\zeta_1^2 = \frac{a_1}{a_2a_3}} \sum_{\zeta_2^2 = \frac{b_1}{b_2b_3}} \sum_{x_4 \in k} \phi(x_4(1 - \frac{b_2}{b_1} + \frac{a_3b_2b_3}{a_1b_1}\zeta_1 - \zeta_1 + \frac{a_2}{a_1}\zeta_1 - \frac{a_3}{a_1}\zeta_1 + \frac{2b_2}{b_1}\zeta_2)),
\]
and by (iii) in §3.1 this can be written as
\[
\sum_{\zeta_1^2 = \frac{a_1}{a_2a_3}} \sum_{\zeta_2^2 = \frac{b_1}{b_2b_3}} \sum_{x_4 \in k} q^{\delta_{b_2}} a_3b_2b_3\zeta_1\zeta_2 \phi\left(-\zeta_1 + \frac{a_2}{a_1}\zeta_1 - \frac{a_3}{a_1}\zeta_1 + \frac{2b_2}{b_1}\zeta_2\right).
\]

3.4.2. \(j = [1, 0, 1, 0]\). By §2.4.2 we have that
\[
a_1 = \frac{a_2a_3x_2}{x_4} \quad \text{and} \quad b_1 = b_2b_3x_4^2.
\]
The subsum in (4) restricted to the elements in §2.4.2 is
\[
\sum \psi \left( u_1 \left( x_3 - \frac{x_3x_4}{a_2x_2} \right) \right) u_2 \left( -x_4 - \frac{1}{b_3x_2} - \frac{x_3^2}{b_3x_2} - \frac{1}{b_2x_4} \right)
\]
with summation over \((x_1, x_2, x_3, x_4) \in \{1\} \times \mathbb{F}^\times_q \times \mathbb{F}^\times_q \times \mathbb{F}^\times_q\).

If \(\frac{b_1}{b_2b_3} \not\in \mathbb{F}^\times_{q,2}\), then we have no possible value for \(x_4\), and the sum is zero. We then assume \(\frac{b_1}{b_2b_3} \in \mathbb{F}^\times_{q,2}\).

In this case, we have
\[
(x_2, x_4) \in \left\{ \left( \frac{a_1}{a_2a_3}\zeta, \zeta \right) \mid \zeta^2 = \frac{b_1}{b_2b_3} \right\},
\]
and we can write the above sum as
\[
\sum_{\zeta^2 = \frac{b_1}{b_2b_3}} \sum_{x_3 \in k} \phi(x_3 - \frac{x_3\zeta a_2a_3}{a_2} - \zeta - \frac{1}{b_3\zeta} - \frac{x_3^2 a_2a_3}{b_3} - \frac{1}{b_2\zeta}) = \sum_{\zeta^2 = \frac{b_1}{b_2b_3}} \sum_{x_3 \in k} \phi\left(A(\zeta)x_3^2 + Bx_3 + C(\zeta)\right),
\]
where \(A(\zeta) = \frac{a_2a_3}{a_1b_3\zeta}, B = 1 - \frac{a_2}{a_1}\) and \(C(\zeta) = \zeta + \frac{1}{b_2\zeta} + \frac{1}{b_3\zeta}\). The above sum can be written by §3.2 as
\[
G \sum_{\zeta^2 = \frac{b_1}{b_2b_3}} (2\delta_{A(\zeta) \in \mathbb{F}^\times_{q,2}} - 1) \phi\left(C(\zeta) - \frac{B^2}{4A(\zeta)}\right).
\]
3.4.3. \( j = [0, 0, 0, 0] \). In this case, the subsum in (4) restricted to the elements in \( \Phi_{4.4.3} \) is
\[
\sum \psi(u_1(-x_1 - x_3 - \frac{1}{a_3 x_1} - \frac{1}{a_2 x_3} - \frac{x_1}{a_3 x_1} - \frac{x_4}{a_2 x_3})u_2(-x_2 - x_4 - \frac{x_2^2}{b_3 x_3 x_4} - \frac{x_4^2}{b_2 x_3 x_4} - \frac{2x_1}{b_3 x_3 x_4} - \frac{1}{b_2 x_4}))
\]
\[
= \sum \phi(-x_1 - x_3 - \frac{1}{a_3 x_1} - \frac{1}{a_2 x_3} - \frac{x_1}{a_3 x_1} - \frac{x_4}{a_2 x_3})-x_2 - x_4 - \frac{x_2^2}{b_3 x_3 x_4} - \frac{x_4^2}{b_2 x_3 x_4} - \frac{2x_1}{b_3 x_3 x_4} - \frac{1}{b_2 x_4}),
\]
and by \( \Phi_{4.4.3} \) we have \( a_1 = a_2 a_3 x_1^2 x_2 / x_4 \) and \( b_1 = b_2 b_3 x_3^2 / x_4 \). The first equation yields \( x_4 = \frac{a_2 a_3 x_1^2 x_2}{a_1} \).

Substituting this into the second equation yields \( (x_1 x_2 x_3)^2 = (\frac{a_1}{a_2 a_3})^2 \frac{b_1}{b_2 b_3} \).

If \( \frac{b_1}{b_2 b_3} \notin \mathbb{F}_{q,2}^\times \), then the equalities are not both satisfied at the same time, hence the sum is zero.
Let us then assume \( \frac{b_1}{b_2 b_3} \notin \mathbb{F}_{q,2}^\times \). Let us put \( A := \frac{a_1}{a_2 a_3} \) and \( B := \frac{b_1}{b_2 b_3} \). Then we have that
\[
(x_1, x_2, x_3, x_4) \in \left\{ (t, u, \frac{A \zeta}{t u}, \frac{t^2 u}{A}) \mid t, u \in \mathbb{k}^\times, \zeta^2 = B \right\}.
\]
The above sum can now be written as
\[
\sum_{\zeta^2 = B, t, u \in \mathbb{k}^\times} \phi(-t - \frac{A \zeta}{t u} - \frac{1}{a_3 t} - \frac{t u}{a_2 A} - \frac{t^3 u}{a_2 A^2 \zeta} - \frac{t}{a_2 A} - \frac{t^2 u}{A} - \frac{t^4 u}{b_3 A} - \frac{b_3 A^2 B}{b_3 A} - \frac{2}{b_3 A} - \frac{1}{b_2 t^2 A}),
\]
which can be put in the following more compact form,
\[
\sum_{t \in \mathbb{k}^\times} \phi \left( -\frac{2t^2}{b_3 A} - \frac{t a_2 A + 1}{a_2 A} - \frac{1}{a_3 t} \right) S_{q-1} \left( 1, -\frac{t^4}{b_3 A^2 \zeta^2} - \frac{t^3}{a_2 A^2 \zeta} - \frac{t^2 b_3 A^2 \zeta^2}{b_3 A} - \frac{t}{a_2 A \zeta} - \frac{1}{b_3} - \frac{A \zeta}{t} - \frac{A}{b_2 t^2} \right).
\]
This sum, which we cannot reduce further, is the most complicated sum appearing in Table 4.3. All other sums in Tables 4.1 and 4.3 can be expressed in a compact form, namely by at most four summands which only involve \( \phi \) and the generalized Kloosterman sums \( S_{q-1}(B, a, b) \) and \( \tilde{S}_{q-1}(B, a, b, a', b') \).

4. Structure constants in types \( A_2 \) and \( B_2 \)

In this section we summarize our findings. In their action on the left, the elements of the standard basis of \( \mathcal{H} \) are given in the sequel. These elements are \( e_0 = e_0(a, b), e_1 = e_1(c), e_2 = e_2(d) \) and \( e_3 = 1 \) (as in \( \Phi_{4.4.2} \)). It is clear that \( [e_3] = [e_3 e_j : e_i]_{j=0,3} \) is just the identity matrix.

We denote by \( S^k_{ij} = [e_i e_j : e_k] \) the structure constants. Clearly \( S^k_{ij} \) will depend on certain parameters: we use \( a_1, b_1, c_1, d_1 \in \mathbb{k}^\times \) for the index \( i \), we use \( a_2, b_2, \ldots \) for the index \( j \) and \( a_3, b_3, \ldots \) for the index \( k \). For example \( S^1_{12} = S^1_{12}(a_1, b_1, c_3, d_2) \). Since \( \mathcal{H} \) is abelian, \( S^k_{ij} = S^k_{ji} \) and we mark this redundant information with \( * \).

The structure constants in types \( A_2 \) and \( B_2 \) are given in the second column of Tables 4.1 and 4.3 respectively. The third columns contain the distinguished subexpressions which parametrize the decompositions of \( U n_i U \cap n_k U n_j^{-1} U \) as disjoint unions of left \( U \)-cosets (as in \( \Phi_{2.1} \)). The last column gives the types of the distinguished subexpressions (as defined in \( \Phi_{2.2} \)). We recall that all details for the computations in Tables 4.1 and 4.3, most of which have been automated via the algorithm in Section 2, can be found in [PS].

The structure constants which are easiest to obtain, namely those of the form \( q^{\ell(w)} \), are the ones corresponding to the \( S^3_{ij} \) with \( w_i^{-1} = w_j \). For such elements \( w_i \) there is only one distinguished
subexpression, namely the entire fixed reduced expression of \( w_i \). Moreover, all its entries are of type A. For example, consider \( S_{21}^3 \) which is a sum over left \( U \)-coset representatives in \( Un_2 U \cap n_3 U n_1^{-1} U \). The decomposition as in §2.1 has only one element, namely \( J(w_2, w_1, w_3) = \{ [2, 1] \} \) corresponding to the fixed expression \( s_2 s_1 \) of \( w_2 \). The type of this expression is AA. Moreover, it is easy to see from (4) that all the summands are \( \psi(1) = \phi(0) = 1 \).

We also recall that the 0’s in the subexpressions mean that the corresponding simple reflection is omitted, e.g. \([1, 2, 0] = s_1 s_2\) and \([1, 0, 1] = s_1 s_1\) are subexpressions of \([1, 2, 1] = s_1 s_2 s_1\).

4.1. Type \( A_2 \). In type \( A_2 \) there is an obvious extra symmetry to the structure constants obtained by interchanging the two simple roots, i.e. interchanging the indices 1 and 2 in \( S_{ij}^k \).

\[
[c_0] = \begin{bmatrix}
S_{00}^0 & S_{01}^0 & S_{02}^0 & \delta_{a_1,a_3}\delta_{b_1,b_3} \\
S_{10}^0 & S_{11}^0 & S_{12}^0 & 0 \\
S_{20}^0 & S_{21}^0 & S_{22}^0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

\[
[c_1] = \begin{bmatrix}
* & S_{11}^0 & S_{12}^0 & 0 \\
0 & * & 0 & \delta_{c_1,c_3} \\
0 & S_{21}^0 & * & 0 \\
0 & 0 & q^2\delta_{c_1,-c_2} & 0
\end{bmatrix},
\quad
[c_2] = \begin{bmatrix}
* & * & S_{22}^0 & 0 \\
* & 0 & S_{22}^0 & 0 \\
0 & q^2\delta_{c_2,-c_3} & 0 & 0
\end{bmatrix}.
\]

In the row corresponding to \( S_{00}^0 \) in Table 4.1, we put

\[
\sigma_1 = -1 - \frac{b_3}{b_1} + \zeta \left( -\frac{a_2 b_2 b_3}{a_1 b_1} - \frac{a_2 b_3}{b_1} \right) + \frac{1}{\zeta} \left( -\frac{1}{a_2} - \frac{1}{a_3} \right),
\]

\[
\sigma_2 = -\frac{1}{b_3} - \zeta - \frac{b_1}{a_2 b_2 b_3} \zeta.
\]

4.2. Generation in type \( A_2 \). We prove that \( e_3, e_2(d) \) and \( e_1(c) \) generate \( A_2 \). By looking at the constant \( S_{02}^0 \) as in our table, we get

\[
e_1(c_1) e_2(d_2) - q^2\delta_{c_1,-d_2} c_3 = \sum_{a_3,b_3 \in F_q^*} \delta_{a_3 d_1,b_3} \phi(a_3/b_2) e_0(a_3,b_3) = \sum_{t \in F_q^*} \phi(t) e_0(d_2 t,c_1 t).
\]

In order to obtain just one \( e_0(c,d) \) on the right hand side, we exploit the properties (i)–(iii) of \( \phi \) in §3.1. We evaluate the above at \( c_1 z^{-1} \) and \( d_2 z^{-1} \) respectively, and we obtain

\[
e_1(c_1 z^{-1}) e_2(d_2 z^{-1}) - q^2\delta_{c_1 z^{-1},-d_2 z^{-1}} c_3 = \sum_{t \in F_q^*} \phi(t) e_0(d_2 t z^{-1}, c_1 t z^{-1}).
\]

Notice that \( \delta_{c_1 z^{-1},-d_2 z^{-1}} = \delta_{c_1,-d_2} \).

Let us consider a function \( f : F_q \to \mathbb{C} \). By multiplying the above equation by \( f(z) \) and then summing over \( z \), we obtain

\[
\sum_z f(z) (e_1(c_1 z^{-1}) e_2(d_2 z^{-1}) - q^2\delta_{c_1,-d_2} c_3) = \sum_{t} \sum_z f(z) \phi(t) e_0(d_2 t z^{-1}, c_1 t z^{-1}).
\]

We can now bring the term in \( e_0 \) outside one sum by the change of variable \( s = t/z \), namely

\[
\sum_{s} \sum_z f(z) \phi(sz) e_0(d_2 s,c_1 s) = \sum_{s} e_0(d_2 s,c_1 s) \sum_z f(z) \phi(sz).
\]
In particular we have obtained more explicitly, for each $x, y, s \in F_q^\times$, satisfying the above property does exist. Namely if $f(z) = \phi(z) - 1$ then
\[
\sum_z (\phi(z) - 1) \phi(sz) = \sum_z \phi((s + 1)z) - \sum_z \phi(sz) = (q - 1)\delta_{s,-1} - \delta_{s\neq -1} + 1 = q\delta_{s,-1}.
\]

In particular we have obtained more explicitly, for each $x, y \in F_q^\times$,
\[
eq 0(x, y) = q^{-1} \sum_{z \in F_q^\times} (\phi(z) - 1)e_1(-yz^{-1})e_2(-xz^{-1}) + q^2\delta_{x,-y}e_3.
\]
4.3. Type $B_2$. We assume here that $p \neq 2$.

\[
[e_0] = \begin{bmatrix}
S_{00}^0 & S_{01}^0 & 0 & \delta_{a_1,a_3}\delta_{b_1, b_3} \\
S_{01}^0 & S_{12}^0 & 0 & 0 \\
S_{00}^1 & S_{12}^1 & 0 & 0 \\
q^4\delta_{a_1,a_2}\delta_{b_1, b_2} & 0 & 0 & 0
\end{bmatrix}
\]

\[
[e_1] = \begin{bmatrix}
* & S_{01}^0 & 0 & 0 \\
* & 0 & S_{12}^1 & \delta_{c_1, c_3} \\
* & S_{11}^1 & S_{12}^2 & 0 \\
0 & q^3\delta_{c_1, c_2} & 0 & 0
\end{bmatrix},
\]

\[
[e_2] = \begin{bmatrix}
* & * & S_{00}^0 & 0 \\
* & * & S_{11}^1 & 0 \\
* & * & 0 & \delta_{d_1, d_3} \\
0 & 0 & q^3\delta_{d_1, d_2} & 0
\end{bmatrix}
\]

| $S_{00}^0$ | $q\sum_{c_2^2=-\frac{b_1}{2\zeta}}\delta_{c_1,a_2}\delta_{b_1, b_2} S_{q-1}(1,-1-\frac{a_2}{a_2}-\frac{a_3}{a_3}) = 0 = \frac{a_2b_1}{a_2b_1} - \frac{a_3c_1}{a_3c_1} - \frac{a_1c_2}{a_1c_2}$ | [1,0,0,0] | ABBB |
| $S_{00}^1$ | $q\sum_{c_2^2=\frac{b_1}{2\zeta}}\delta_{c_1,a_2}\delta_{b_1, b_2} S_{q-1}(1,1,1,1) = 0 = \frac{a_2b_1}{a_2b_1} - \frac{a_3c_1}{a_3c_1} - \frac{a_1c_2}{a_1c_2}$ | [2,0,2,0] | ACBA |
| $S_{02}^0$ | $q\sum_{c_2^2=-\frac{b_1}{2\zeta}}\delta_{c_1,a_2}\delta_{b_1, b_2} S_{q-1}(1,\frac{a_1}{a_1}, \frac{a_2}{a_2}, \frac{a_3}{a_3}) = 0 = \frac{a_2b_1}{a_2b_1} - \frac{a_3c_1}{a_3c_1} - \frac{a_1c_2}{a_1c_2}$ | [0,2,1,0] | BACB |
| $S_{02}^1$ | $q\sum_{c_2^2=\frac{b_1}{2\zeta}}\delta_{c_1,a_2}\delta_{b_1, b_2} S_{q-1}(1,\frac{a_1}{a_1}, \frac{a_2}{a_2}, \frac{a_3}{a_3}) = 0 = \frac{a_2b_1}{a_2b_1} - \frac{a_3c_1}{a_3c_1} - \frac{a_1c_2}{a_1c_2}$ | [0,2,0,2] | BABA |

**Table 2.** Structure constants of $H$ when $G = SO_5(q)$ and $p$ is odd.
In the rows corresponding to $S_{02}^0$ and $S_{00}^2$ in Table 4.3, we put
\[
\tau_1 = a_3 d_2 \left( \frac{2}{a_1 b_3} \zeta + \frac{1}{a_1 b_1} + \frac{1}{a_1 b_1} \right), \quad \tau_2 = -1 - \zeta - \frac{a_3}{a_1} - \frac{a_3}{a_1}, \quad \tau_3 = -\frac{1}{a_3}, \quad \tau_4 = \frac{a_1}{a_3 d_2},
\]
\[
\tau'_1 = a_2 d_3 \left( \frac{2}{a_1} - \frac{b_2}{a_1 b_1} - \frac{1}{a_1} \right), \quad \tau'_2 = d_3 \left( \frac{1}{a_1} - \frac{1}{a_1} \right), \quad \tau'_3 = \frac{a_1}{a_2 d_3} - \frac{1}{d_3}, \quad \tau'_4 = -\frac{a_1}{a_2 b_2 d_3}.
\]

4.4. **Generation in type $B_2$.** The problem of determining whether each of the $e_0(a, b)$ is generated by elements of the form $e_1(c)$, $e_2(d)$ and $e_3$ seems much harder in this case. Here we again have
\[
(14) \quad e_1(c_1)e_2(d_2) + K = \sum_{a_3, b_3} S_{12}^0 e_0(a_3, b_3)
\]
for some element $K$ generated by $e_1(c)$, $e_2(d)$ and $e_3$. Notice that if we let $(\mu) = \mathbb{F}_q^\times$ then we can write
\[
\sum_{a_3, b_3} S_{12}^0 e_0(a_3, b_3) = \sum_{t} \sum_{i=1}^{q-1} \left( \phi(t + \frac{1}{c_1 \mu^{2i} t} + \mu^i + \frac{1}{d_2 \mu^{2i} t}) + \phi(t + \frac{1}{c_1 \mu^{2i} t} - \mu^i - \frac{1}{d_2 \mu^{2i} t}) \right) e_0(d_2 t, c_1 \mu^{2i})
\]
\[
= \sum_{t} \sum_{u} \left( \phi(t + \frac{1}{c_1 t u^2} + u + \frac{1}{d_2 t u}) \right) e_0(d_2 t, c_1 u^2)
\]
since $\mu^{(q-1)/2} = -1$.

We try to apply the same method as in §4.2. Let us consider a function $f : \mathbb{F}_q^2 \to \mathbb{C}$. By replacing $c_1$ and $d_2$ in (14) with $c_1 z^{-2}$ and $d_2 w^{-1}$ respectively, multiplying by $f(z, w)$, and summing over $z$ and $w$, we get
\[
\sum_{z, w} f(z, w) e_1(c_1 z^{-2}) e_2(d_2 w^{-1}) + K' = \sum_{x, y} e_0(d_2 x, c_1 y^2) \sum_{z, w} f(z, w) \phi \left( zy + wx + \frac{1}{d_2 xyz} + \frac{1}{c_1 xyz^2} \right),
\]
The problem of finding a suitable choice of $f$ which allows to express the second sum on the right hand side as a product of suitable delta functions, which would leave just one term $e_0(x, y)$, remains an open question.

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Technische Universität Kaiserslautern, Fachbereich Mathematik, Postfach 3049, 67653 Kaiserslautern, 
E-mail address: paolini@mathematik.uni-kl.de

Babeș-Bolyai University, Department of Mathematics, Mathematica 9, 400157, Ploiești 23-25, Cluj-Napoca, 
E-mail address: simion@math.ubbcluj.ro