Non-minimal coupling, boundary terms and renormalization of the Einstein-Hilbert action and black hole entropy

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Abstract

A consistent variational procedure applied to the gravitational action requires according to Gibbons and Hawking a certain balance between the volume and boundary parts of the action. We consider the problem of preserving this balance in the quantum effective action for the matter non-minimally coupled to metric. It is shown that one has to add a special boundary term to the matter action analogous to the Gibbons-Hawking one. This boundary term modifies the one-loop quantum corrections to give a correct balance for the effective action as well. This means that the boundary UV divergences do not require independent renormalization and are automatically renormalized simultaneously with their volume part. This result is derived for arbitrary non-minimally coupled matter. The example of 2D Maxwell field is considered in much detail. The relevance of the results obtained to the problem of the renormalization of the black hole entropy is discussed.

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1 Introduction

It was realised by Gibbons and Hawking [1] that a well-defined variational procedure for the Einstein-Hilbert (EH) action

\[ W_{EH} = -\frac{1}{16\pi G} \int_M R \]  

(1.1)
on manifold \( M \) with boundary \( \partial M \) requires fixing the metric but not its normal derivative on the boundary. Therefore, they added to the volume action (1.1) the boundary term the role of which is to compensate variations of the normal derivatives of the metric on \( \partial M \) that come from the variation of (1.1) after integrating by parts. The resulting action takes the form

\[ W_{EH} = -\frac{1}{16\pi G} \left( \int_M R + 2 \int_{\partial M} k \right) \]  

(1.2)

where \( k = g^{\mu\nu} k_{\mu\nu} \) \( (k_{\mu\nu} = \frac{1}{2}(\nabla_\mu n_\nu + \nabla_\nu n_\mu)) \) is the extrinsic curvature of the boundary, \( n^\mu \) is outward pointing normal vector to \( \partial M \). The variation procedure being applied to (1.2) is now consistent.

On the other hand, it is well known that the term like (1.1) is typically induced by quantum corrections. And it was even proposed to treat the EH as being completely induced by quantum matter fields interacting with background classical gravitational field [2]. However, it is natural to ask whether the boundary term in EH (1.2) can also be induced in such a way that the “correct” balance between the volume and boundary parts of (1.2) is preserved. The related but somewhat more general question is that does the quantum effective action, obtained by integrating out quantum matter fields and being functional of background metric, have automatically such a form to get the consistent variational procedure?

The technique of the heat kernel expansion [3]

\[ W_{eff} = \frac{1}{2} \ln \det(-\Delta) = -\frac{1}{2} \int_c^\infty \frac{ds}{s} Tr K_M(s), \]

\[ K_M(s) = e^{s\Delta} = \frac{1}{(4\pi s)^{\frac{d}{2}}} \sum_n a_n s^n, \quad s \to 0 \]  

(1.3)
gives us the useful tool to investigate the problem. The EH like term in the quantum effective action \( W_{eff} \) appears in the first coefficient \( a_1 \) of the expansion for the matter field operator \( \Delta \). For example, for the scalar field minimally coupled to the gravitational field

\[ W_{sc} = \frac{1}{2} \int_M \phi(-\Delta_0)\phi, \]  

(1.4)

where \( \Delta_0 = \nabla_\alpha \nabla^\alpha \), we get the promising result that

\[ a_1(\Delta_0) = \frac{1}{6} \left( \int_M R + 2 \int_{\partial M} k \right). \]
However, already the non-minimally coupled scalar field

\[ W_{sc}^\xi = \frac{1}{2} \int_M \phi (-\Delta_0^\xi) \phi , \]  

where \( \Delta_0^\xi = \Delta_0 - \xi R \), and vector field described by the action

\[ W_{vec} = \frac{1}{2} \int_M A^\mu (-\Delta_1)_{\mu \nu} A^\nu , \]  

where \( (\Delta_1)_{\mu \nu} = g_{\mu \nu} \nabla^\alpha \nabla_\alpha - R_{\mu \nu} \) is the Beltrami-Laplace operator for one-forms, break this idyll:

\[ a_1(\Delta_0^\xi) = \left( \frac{1}{6} - \xi \right) \int_M R + \frac{1}{3} \int_{\partial M} k ; \]
\[ a_1(\Delta_1) = \left( \frac{d}{6} - 1 \right) \int_M R + \frac{d}{3} \int_{\partial M} k , \]  

where \( d = \text{dim} M \).

The reason for this obviously lies in the non-minimality of the coupling with metric in (1.3), (1.6). Namely, the curvature tensors enter directly the matter action (1.3)-(1.6). Therefore, if we calculate the variation of the action with respect to background metric we observe the same problem as for (1.1): there are some non-zero variations of the normal derivatives of the metric on the boundary. We need to add some boundary term like in (1.2) in order to kill these variations. More generally, before demanding the quantum effective action to have the consistent metric variation we need to start with the classical matter action satisfying this requirement. The metric variation of the action gives us the stress-energy tensor \( T_{\mu \nu} \) of the matter. So, the modification of the actions (1.3), (1.4) by appropriate boundary term would give us a well-defined \( T_{\mu \nu} \) without any peculiarities on the boundary. It is easy to find such a modification for the scalar case (1.5):

\[ W_{\text{mat}} = \frac{1}{2} \int_M \phi (-\Delta_0^\xi) \phi + \xi \int_{\partial M} k \phi^2 . \]  

It should be noted that the boundary term in (1.8) appears even in the flat space \( (R = 0) \) when \( \partial M \) has a non-zero extrinsic curvature. Similarly, it is well-known that \( T_{\mu \nu} \) for matter described by (1.8) is modified by \( \xi \)-dependent terms even in flat spacetime.

Now, if we start with the action (1.8) and quantize \( \phi \) we could expect that the resulting effective action possesses the needed property of a balance between the volume and boundary terms like in (1.2). Indeed, the boundary terms like that in (1.8) modify the heat kernel expansion in the way to get the correct balance. This is the aim of this paper to demonstrate how this happens in the general case of fields of arbitrary spin.

The action (1.2) is a particular form of a more general expression\[1\]

\[ W = \int_M U^{\mu \nu \alpha \beta} R_{\mu \nu \alpha \beta} - 4 \int_{\partial M} U^{\mu \nu \alpha \beta} n_\mu n_\beta k_{\nu \alpha} , \]  

\[1\]It should be noted that adding to (1.9) quadratic term \( U_{\mu \nu \alpha \beta}^2 \) we would obtain the first order form of the higher-derivative \( R^2 \)-theory of gravity.
where $U^{\mu\nu\alpha\beta}$ is arbitrary tensor not containing derivatives of the metric. Below we prove that (1.9) has a well-defined metric variation. What we are really going to demonstrate is that the part of the effective action which is of the first order in curvature always takes the form (1.9) for some tensor $U^{\mu\nu\alpha\beta}$, the concrete form of which depends on the concrete non-minimal matter. For known types of matter $U^{\mu\nu\alpha\beta}$ is some combination of the metric $g_{\mu\nu}$ but not of its derivatives. Then (1.9) certainly reproduces the form (1.2).

The coefficient $a_1$ of the heat kernel expansion (1.3) typically represents the power $1/\epsilon^{d-2}$ ultraviolet divergence of the effective action. The fact that the structure of the divergent term repeats the form (1.2) is significant. This means that we do not need a special renormalization for removing the UV divergences on the boundary as it could happen for the theories (1.5)-(1.6). Instead, all the divergences, volume and at the boundary, are removed by only the renormalization of the gravitational constant $G$ in (1.2).

One of the important applications of this result is the calculation and renormalization of black hole entropy[4, 5, 6, 7, 12, 13, 14]. For matter non-minimally coupled to gravity this calculation becomes non-trivial. This was pointed in [12] in the context of the conical approach to black hole entropy. The reason for this is that the Riemann tensors behave as distributions at the conical singularity. This fact gives rise to appearance of contact interaction of the matter that is concentrated at the singularity (that is horizon in the black hole case). As a result of this interaction, one obtains the modification of the heat kernel expansion on the conical space by terms defined on the singular subspace. Taking them into account was shown to be important for the black hole entropy and allowed in [12] to extend the statement of simultaneous renormalization of the entropy and effective action to the case of non-minimally coupled matter. In the present paper we put this problem in another way and demonstrate that the renormalization statement is closely related with the ultraviolet renormalization of the volume and surface part of the gravitational effective action. In this sight on the problem we are mostly close to the work of Larsen and Wilczek [13] where it is argued that the universality of one renormalization for both the gravitational constant $G$ and the entropy of a black hole follows from the low-energy theorem – the generic local structure of the low-energy effective action. However, the work [13] does not reveal the concrete mechanism of this phenomenon for the general case of non-minimal coupling, not to say about its extension beyond the black hole entropy framework (see discussion in Sect.6). The purpose of this paper is to establish this mechanism and its general validity at the one-loop order.

2 The boundary term

We start with the action describing matter field of arbitrary spin which is non-minimally coupled to gravity:

$$W_{\text{mat}} = \frac{1}{2} \int_M \phi^A ( - \Box_{AB} ) \phi^B,$$  \hspace{1cm} (2.1)
with operator

\[ \Box_{AB} = \eta_{AB} \nabla_\alpha \nabla^\alpha - U_{\mu \nu \alpha \beta}^{\mu \nu \alpha \beta}, \tag{2.2} \]

where \( \{ \phi^A \}, \ A = 1, ..., D \) is a section of the matter bundle over manifold \( M \) with invariant metric \( \eta_{AB} \) and covariant derivative \( \nabla_\alpha \). The tensor \( U_{\mu \nu \alpha \beta}^{\mu \nu \alpha \beta} \) has symmetries of the Riemann tensor \( R_{\mu \nu \alpha \beta} \) with respect to upper indexes. In principle it can be arbitrary tensor not containing metric derivatives. However, for known operators it is an combination of the metric \( g_{\mu \nu} \) and \( \eta_{AB} \).

In order to find the boundary term to be added to (2.1), let us consider a small vicinity of the boundary \( \partial M \). There the manifold \( M \) can be represented as direct product \( M = \partial M \otimes I \). Let the parameter \( t \) label the hypersurfaces of the foliation, the boundary \( \partial M \) being one of them. The outward pointing normal to the hypersurfaces is \( n_\alpha = -N \nabla_\alpha t \) where \( N^2 = (\nabla t)^2 \) defines the lapse function \( N \). The hypersurface metric is \( h_{\mu \nu} = g_{\mu \nu} - n_\mu n_\nu \). Then the Gauss-Codazzi equation implies (see [8]):

\[ R_{\mu \nu \alpha \beta} = 4 n_\mu [L_n k_\nu]_\alpha [n_\beta] + \ldots \tag{2.3} \]

where \( L_n \) is the Lie derivative along \( n^\mu \). Since \( k_{\mu \nu} = \frac{1}{2} L_n h_{\mu \nu} \) the first term at r.h.s. of (2.3) is of the second order with respect to normal derivatives \( L_n L_n h_{\mu \nu} \). The \( \ldots \) terms in (2.3) are of lower order in the normal derivative.

Thus, under variation of the Riemann tensor in the expression

\[ \int_M U_{\mu \nu \alpha \beta}^{\mu \nu \alpha \beta} R_{\mu \nu \alpha \beta} \]

only the first term in (2.3) produces the second normal derivative of the metric variation \( L_n L_n \delta h_{\mu \nu} \) that after integration by parts gives the variation on the boundary

\[ 2 \int_{\partial M} U_{\mu \nu \alpha \beta}^{\mu \nu \alpha \beta} n_\mu n_\beta L_n \delta h_{\nu \alpha} \cdot \]

This variation can be canceled if we add the boundary term as follows

\[ \int_M U_{\mu \nu \alpha \beta}^{\mu \nu \alpha \beta} R_{\mu \nu \alpha \beta} - 4 \int_{\partial M} U_{\mu \nu \alpha \beta}^{\mu \nu \alpha \beta} n_\mu n_\beta k_{\nu \alpha} \cdot \]

This is exactly the form announced in (1.9).

Applying this result to (2.1)-(2.2) we obtain the action with the boundary term

\[ W_{\text{mat}} = \frac{1}{2} \int_M \phi^A (-\Box_{AB}) \phi^B - 2 \int_{\partial M} \phi^A \phi^B U_{AB}^{\mu \nu \alpha \beta} n_\mu n_\beta k_{\nu \alpha} \cdot \tag{2.4} \]

which is our starting point for the quantization and derivation of the heat kernel expansion.
3 The heat kernel expansion

Considering the path integral over fields $\phi^A$ the dynamics of which is described by the action (2.4) we get

$$Z = \int [\mathcal{D}\phi] e^{-W_{mat}} = \int [\mathcal{D}\phi] e^{\int_{\partial M} \phi^A \phi^B V_{AB} - \frac{1}{2} \int_M \phi^A (-\Box_{AB}) \phi^B}$$

(3.1)

where we denote $V_{AB} = 2 U_{\mu \nu \alpha \beta} n_\mu n_\beta k_{\nu \alpha}$. Taking the first term in (3.1) perturbatively we can formally expand it in powers of $V_{AB}$. In the leading order we get

$$Z = \bar{Z} \left(1 + \langle \int_{\partial M} \phi^A \phi^B V_{AB} \rangle_{\bar{Z}} + \langle O(V_{AB}^2) \rangle_{\bar{Z}}\right),$$

(3.2)

where the average $\langle \rangle_{\bar{Z}}$ is taken with respect to measure defined by functional integral

$$\bar{Z} = \int [\mathcal{D}\phi] e^{-\frac{1}{2} \int_M \phi^A (-\Box_{AB}) \phi^B}$$

(3.3)

without boundary term. Equivalently, we can write for the effective action $W_{eff} = -\ln Z$ ($\bar{W}_{eff} = -\ln \bar{Z}$):

$$W_{eff} = \bar{W}_{eff} + \langle \int_{\partial M} \phi^A \phi^B V_{AB} + O(V_{AB}^2) \rangle_{\bar{Z}}.$$

(3.4)

In the case when the boundary term is not included we have a standard heat kernel expansion:

$$\bar{W}_{eff} = \frac{1}{2} \ln \det(-\Box_{AB}) = -\frac{1}{2} \int_s^\infty \frac{ds}{s} Tr \bar{K}_M(s),$$

$$\langle \phi^A(x) \phi^B(x') \rangle_{\bar{Z}} = \int_2^\infty ds \bar{K}^{AB}_M(x, x', s),$$

$$\bar{K}^{AB}_M(s) = e^{s\Box_{AB}} = \frac{1}{(4\pi s)^{\frac{D}{2}}} \sum_n a_n^{AB} s^n, \quad s \to 0$$

(3.5)

where $n$ in the sum runs $0, 1/2, 1, 3/2, \ldots$. For manifold with boundary one typically imposes some boundary condition on the quantum field $\phi^A$: $\phi^A|_{\partial M} = 0$ for the Dirichlet condition and $\mathcal{L}_n \phi^A|_{\partial M} = 0$ for the Neumann one. Correspondingly, this condition is imposed on the heat kernel $\bar{K}_M(x, x', s)$ when one of the points $x$ or $x'$ lies on the boundary. Therefore, one could naively expect that the second term in (3.4) is zero for the Dirichlet condition. However, this does not happen since the limit of the coincident points is considered, which is rather peculiar (see the derivation based on the method of images in [9]). In particular, we have

$$\tilde{a}_{0,AB}(x, x) = \eta_{AB}$$

(3.6)

even if $x$ lies at $\partial M$, that is consequence of the other condition

$$\bar{K}_M(x, x', 0) = \delta(x, x')$$
imposed at $s = 0$. The first few coefficients of the expansion for the operator (2.2) can be found from known results (see for example [9, 10]). In particular, the trace of $\bar{a}_{1AB}(x, x')$ is given by

$$\bar{a}_{1}(\Box) = \int_{M} \left( \frac{D}{6} R - \eta^{AB} U_{AB}^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta} \right) + \frac{D}{3} \int_{\partial M} k. \quad (3.7)$$

Note, that (3.6) and (3.7) are the same for both Dirichlet and Neumann conditions.

Inserting (3.5) into (3.4) we obtain

$$W_{eff} = -\frac{1}{2} \int_{\epsilon}^{\infty} ds \frac{d}{s} TrK_{M}(s),$$

$$TrK_{M}(s) = Tr\tilde{K}_{M}(s) + 2s Tr_{\partial M} \left( V\tilde{a}_{n-1}(x, s) \right) + O(V^{2}), \quad (3.8)$$

where the $x$-integration in $Tr_{\partial M}$ is taken over only the boundary $\partial M$.

From (3.8) we have the following expression for the heat kernel $K_{M}(s)$:

$$TrK_{M}(s) = \frac{1}{(4\pi s)^{d/2}} \sum_{n} a_{n}s^{n},$$

$$a_{n} = \int_{M} \bar{a}_{n}(x, s) + 2 \int_{\partial M} Tr(V\bar{a}_{n-1}(x, s)) + O(V^{2}).$$

(3.9)

Note that typically every new power of $V$ in the perturbation theory for the heat kernel brings at least one extra power of $s$ (see the derivation in [3]), whence $O(V^{2})$ term here contributes to $a_{n}$ starting with $a_{2}$ and does not affect the calculation of $a_{1}$. Thus, for the $a_{1}$ coefficient we obtain, taking into account (3.6) and (3.7) that

$$a_{1}(\Box) = \int_{M} \left( \frac{D}{6} R + 2 \int_{\partial M} k \right) - \left( \int_{M} \eta^{AB} U_{AB}^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta} - 4 \int_{\partial M} \eta^{AB} U_{AB}^{\mu\nu\alpha\beta} \eta_{\mu\nu} n_{\alpha} k_{\nu\alpha} \right) \quad (3.10)$$

The Eq. (3.10) is our main result. It shows that the linear in the curvature term of the effective action for non-minimal matter fields indeed repeats the EH form (1.2) (or, more generally, the form (1.9)), if we include the boundary term as in (2.4). Though there exists possibility to consider arbitrary tensor $\eta^{AB} U_{AB}^{\mu\nu\alpha\beta}$, not necessarily related to metric, for known types of matter it is the combination of the metric tensor. Then the second term in (3.10) repeats with some overall coefficient the form of the first term. In the particular cases of non-minimal matter considered in the Introduction, we have $D = 1$ for the scalar (1.5) and $\eta_{AB} \equiv g_{\mu\nu}$, $D = d \equiv dimM$ for vector (1.6) and in the both cases (up to factor $\xi$ for scalar ) we get that $\eta^{AB} U_{AB}^{\mu\nu\alpha\beta} = \frac{1}{2}(g^{\mu\alpha} g^{\nu\beta} - g^{\mu\beta} g^{\nu\alpha})$. Then, the corresponding coefficients $a_{1}$ calculated according to (3.10) read

$$a_{1}(\Delta_{0}^{\xi}) = \left( \frac{1}{6} - \xi \right) \left( \int_{M} R + 2 \int_{\partial M} k \right) \quad (3.11)$$

$$a_{1}(\Delta_{1}) = \left( \frac{d}{6} - 1 \right) \left( \int_{M} R + 2 \int_{\partial M} k \right). \quad (3.12)$$

Eq. (3.9) allows calculate other coefficients of the heat kernel expansion for both integer and half-integer $n$. We are not doing this here.
The functional integral (3.1) can be written as an average of the boundary operator
\[ \langle \hat{V}[\partial M] \rangle = \langle e^{i \int_{\partial M} A_{\mu} d \sigma^\mu} V_{\alpha \beta} \rangle \]. It is worth noting that this operator is similar to other objects that appeared earlier in field theory models: Wilson loop \( \langle P e^{i \int_{C} A_{\mu} d \sigma^\mu} \rangle \) in the theory of non-Abelian gauge fields and vertex operator \( V = e^{i p \cdot X(z)} \) describing the contact interaction in string theory. Following this analogy, we may interpret our boundary operator as describing some (contact) interaction at the boundary.

4 2D Maxwell field

As a simple application of the results obtained let us consider Maxwell field on two-dimensional manifold with boundary. We define the partition function for the Maxwell field, including the contribution of ghosts, as follows
\[ Z = [\det' \Delta_1]^{-1/2} \det' \Delta_0 \] (4.1)
where \( \det' \) is calculated only on non-zero modes of operators; \( \Delta_k = (d \delta + \delta d)_{(k)} \) is the Beltrami-Laplace operator acting on \( k \)-forms.

In two dimensions we have a remarkable property for closed manifold that the set of non-zero eigenvalues of \( \Delta_1 \) is given by a union non-zero eigenvalues of \( \Delta_0 \) and \( \Delta_2 \). This is simply a consequence of the cohomological algebra of the operators \( \Delta_k, \delta_k \) and \( d_k \) (see for example [14]). Moreover, due to Hodge duality in two dimensions the eigenvalues of operators \( \Delta_0 \) and \( \Delta_2 \) are the same. Hence we have that \( \det' \Delta_1 = \det' \Delta_0 \det' \Delta_2 = (\det' \Delta_0)^2 \) and, therefore, (4.1) is trivial, \( Z = 1 \). In particular, there is not any ultraviolet divergence in \( \ln Z \). The cohomological arguments can be applied to the open manifold as well and one could expect the same result.

To proceed with the heat kernel, it is worth noting that for an arbitrary elliptic operator \( A \) the formula (1.3) is modified due to zero modes
\[ \ln \det' A = - \int_{\varepsilon^2}^{\infty} \frac{dt}{t} \text{Tr} [e^{-t A} - P(A)] \] (4.2)
where \( P(A) \) is a projector onto the subspace of zero modes of \( A \); \( \text{Tr} P(A) = N(A) \) — the number of zero modes of \( A \). In \( d \) dimensions the zero modes contribute to the coefficient \( a_{d/2} \) in the heat kernel expansion.

For the divergent part of (1.3) in two dimensions we have
\[ (\ln Z)_{\text{div}} = \left[ \frac{1}{2\pi} \left[ \frac{1}{2} a_1(\Delta_1) - a_1(\Delta_0) \right] + [2N_0 - N_1] \right] \ln \frac{L}{\varepsilon} \] (4.3)
where \( N_k = N(\Delta_k) \). For a closed 2D manifold the combination of numbers \( (2N_0 - N_1) \) in (4.3) is the Euler number \( \chi(M) = \frac{1}{4\pi} (\int_M R + 2 \int_{\partial M} k) \) of 2D manifold. Using a

\[ ^2 \text{The general definition of the Euler number in two dimensions reads: } \chi(M) = N_0 - N_1 + N_2. \] However, for a 2D closed manifold we have by Hodge duality that \( N_2 = N_0 \) and hence \( \chi(M) = 2N_0 - N_1 \).
standard (1.7) or modified (3.12) expression for the coefficients (for a closed manifold they are the same) we obtain that (1.3) indeed vanishes.

For an open 2D manifold the situation is more complicated. First of all, we have to impose some boundary conditions on \( p \)-forms eigen vectors in the partition function (1.1). The possible choice is to put Dirichlet condition for zero-forms and the generalized Dirichlet condition for the one-form \( A = A_\mu dx^\mu; \) \( A^\mu \epsilon_{\mu\nu} n^\nu|_{\partial M} = 0, \) where \( n^\mu \) is a unit vector normal to the boundary. For a disk with polar coordinates \((r, \phi)\) the later conditions means \( A_\phi = 0 \) on the boundary. The important observation now is that the combination \( (2N_0 - N_1) \) is no longer equal to the Euler number of the manifold. Indeed, for boundary conditions as above for a disk we have \( N_0 = N_1 = 0 \) while the Euler number of a disk is \( \chi = 1. \) But the combination \( (2N_0 - N_1) \) is still a metric independent quantity and its metric variation vanishes. Then, using the standard coefficients (1.7) in (4.3) for open manifold we obtain the disbalance between volume \( \int_M R \) and boundary \( \int_{\partial M} k \) parts. The balance certainly restores if we take into account the boundary operator according to (3.1) when calculating the quantity \( \ln \text{det}'(-\Delta_1) \) in (4.1). This is easily checked by inserting the expression (3.12) for \( a_1(\Delta_1) \) in (4.3) instead of the standard one. We then obtain for an open manifold

\[
(\ln Z)_{\text{div}} = [-\chi(M) + (2N_0 - N_1)] \ln \frac{L}{\epsilon}.
\]

(4.4)

The metric variation of (4.4) is well-defined. However, it does not take exactly the form (1.2) due to the contribution of zero modes. Note, that the form of the heat kernel coefficients is universal being independent of the dimension of the manifold. Otherwise, the contribution of zero-modes in (1.2) is essentially dependent on spacetime dimensionality. Therefore, the structure of (4.4) is very special and appears only in two dimensions. The UV divergence (4.4) is similar to that of obtained by Kabat [14] though in his calculation he neglected the role of terms on the boundary \( \partial M \) and contribution of zero modes.

5 Renormalization of black hole entropy

The calculation of the heat kernel in Sect.3 is very similar to the calculation of the (divergent) quantum correction to the black hole entropy in [12]. This fact is certainly not occasional. In order to demonstrate that our results are relevant to the black hole entropy we note that it is a boundary term in (1.2), or in a more general expression (1.9), that is responsible for the entropy. There are different ways to show this, we follow ref. [15].

Consider the Euclidean black hole instanton with metric

\[
ds^2 = g(\rho)d\tau^2 + d\rho^2 + r^2(\rho)d\Omega^2,
\]

(5.1)

where \( d\Omega^2 = \gamma_{ab}(\theta)d\theta^a d\theta^b \) is metric of \( (d - 2) \)-sphere of unity radius; the period of \( \tau \) in (5.1) is chosen to remove the singularity at the horizon surface \( \Sigma \) defined by the equation
\[ g(\rho_{\Sigma}) = 0. \] We take the coordinate \( \rho \) such that \( \rho_{\Sigma} = 0 \), then \( g(\rho) = \rho^2 / \beta_H^2 + O(\rho^4) \) and \( r^2(\rho) = r_{\Sigma}^2 + O(\rho^2) \). Take a small ball \( B_\delta \) of radius \( \delta \) \( (0 \leq \rho \leq \delta) \) surrounding the surface \( \Sigma \). Vector normal to \( \partial B_\delta \) has only one non-zero component \( n_\rho = 1 \). Then the components of the extrinsic curvature of \( \partial B_\delta \) read\( k_{\mu\nu} = \frac{1}{2} \partial_{\mu} g_{\nu \rho} \). When \( B_\delta \) shrinks \( (\delta \to 0) \) the boundary \( \partial B_\delta \) goes to \( \Sigma \).

Take the gravitational action in the form (1.9) on \( B_\delta \)
\[
W[B_\delta] = \int_{B_\delta} U^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta} - 4 \int_{\partial B_\delta} U^{\mu\nu\alpha\beta} n_\mu n_\nu n_\alpha n_\beta. \tag{5.2}
\]
The volume term in (5.2) then vanishes when \( \delta \to 0 \) while the boundary term gives the integral over the surface \( \Sigma \)
\[
W[B_{\delta \to 0}] = -4\pi \int_{\Sigma} U^{\mu\nu\alpha\beta} n_\mu n_\nu n_\alpha n_\beta, \tag{5.3}
\]
where \( \{n^i\} \) is a pair of vectors normal to \( \Sigma \) (the only non-zero components are \( n^i_\rho = 1 \), \( n^i_\tau = \frac{\rho}{\beta_H^2} \)). In fact, the boundary term in (5.2) produces combination of the type \( \sqrt{\gamma} \rho (U^{\mu\nu\rho\rho} k_{\tau\tau} + U^{\mu\nu\rho\rho} k_{ab}) \), where for small \( \rho \) we have for the extrinsic curvature: \( k_{\tau\tau} = \frac{\rho}{\beta_H^2} + O(\rho^3) \); \( k_{ab} = \frac{1}{2} \gamma_{ab} \partial_{\rho} \rho^2 \). Only component \( U^{\rho\tau\rho\rho} \) is assumed to be divergent (as \( \frac{1}{\rho^4} \)) in the center of the polar coordinates \( (\rho, \tau) \) while components \( U^{\rho\mu\rho\rho} \) lie in the orthogonal space and remain finite for \( \rho = 0 \). Therefore, taking limit \( \rho = \delta \to 0 \) we obtain the result (5.3). The expression (5.3) coincides exactly (with minus) the black hole entropy calculated by a number of other methods (see recent paper [16] where the different methods are compared). So, the entropy of a black hole (at least for the theories linear in curvature) can be treated as a gravitational action
\[
S_{BH} = -W[B_{\delta \to 0}] \tag{5.4}
\]
defined for the infinitesimal ball \( B_{\delta \to 0} \) surrounding the horizon surface \( \Sigma \). Note, that above calculation is essentially off-shell that makes it similar to the calculation within the conical method of [15, 5]. Other remark is that this calculation can be also applied to a higher-derivative theory of gravity if the later preliminary re-expressed in the first order form (see [16]).

One can see now that there is a strong correlation between the volume and boundary terms in the classical gravitational action (1.2) (or (1.9)) due to the necessity of a consistent variational procedure. This correlation is preserved, as we have shown, in the divergent part of the quantum effective action. Therefore, we need to renormalize only the gravitational constant \( G \) to remove both the volume and boundary parts of the UV divergences. The boundary divergent term of the effective action (see (3.8)-(3.10)) by the same line of reasoning as in (5.3)-(5.4) gives the divergence of the entropy \( S_q \)
\[
S_q = \frac{1}{\epsilon^{d-2}} \frac{1}{(d-2)} \frac{1}{(4\pi)^{\frac{d-1}{2}}} \left( \frac{D}{6} \int_{\Sigma} 1 + \int_{\Sigma} \eta^{AB} U_{AB}^{\mu\nu\alpha\beta} n_\mu n_\nu n_\alpha n_\beta \right) \tag{5.5}
\]
\(^3\)We do not consider here the logarithmic (\( \ln \epsilon \)) divergence (for \( d > 2 \)) of the entropy originating from \( R^2 \)-terms in the effective action [1].
due to the non-minimally coupled matter of the general type (2.4), which is a quantum addition to its classical counterpart (5.3)-(5.4). Since the entropy is related to the boundary term in the action, we obtain from our consideration a simple proof of the statement that the black hole entropy is automatically renormalized by the same procedure as the effective action. This is just the consequence of (5.4) and of the balance between the volume and boundary parts established above for the quantum effective action.

For the scalar (1.5) and one-form (1.6) matter we get the quantum entropy correspondingly as follows:

\[ S_{\text{sc}}^{q} = \frac{1}{\epsilon^{d-2}} \frac{1}{(d-2)} \left( \frac{1}{6} - \xi \right) \int \Sigma \]

\[ S_{\text{vec}}^{q} = \frac{1}{\epsilon^{d-2}} \frac{1}{(d-2)} \left( \frac{d}{6} - 1 \right) \int \Sigma \]

The result (5.6) coincides with that previously obtained in [12].

The results of Section 4 can be used to study the renormalization of the Maxwell field in two dimensions. The use of boundary conditions considered in Sect.4 allows one to neglect the contribution of the zero modes in (4.4). Then the Maxwell field in two dimensions gives rise to a constant (UV-divergent) contribution to the entropy which is renormalized in the same manner as 2D (effective) gravitational coupling. The non-zero result for the entropy of Maxwell fields seems puzzling (see [14]) in view of the absence of their propagating degrees of freedom in two dimensions. Its statistical entropy, therefore, is expected to be zero [14]. A possible resolution of this puzzle is that our method of calculation yields the thermodynamical entropy which may differ from the statistical one by the constant independent of spacetime geometry. The same happens with 2D topological gravity described by the action

\[ W_{\text{top}} = \int_{M} R + 2 \int_{\partial M} k. \]

This model has a constant contribution to entropy, while \( W_{\text{top}} \) does not describe any dynamical degrees of freedom. In view of this, the fact that the Maxwell field entropy is just a (UV-divergent) constant, independent on the black hole geometry, can be considered as a manifestation of trivial nature of this field in two dimensions.

To make the correspondence with the conical method considered in [4, 5, 12] note that the effect of the conical singularity is concentrated in the infinitesimal region near the singular surface participating in the construction of (5.4). On the other hand, we do not concern here the higher curvature terms in the effective action. It is not quite clear how to generalize the considerations of this paper to include such terms.

Another problem to be mentioned is whether the same balance is true for other boundary conditions on the metric. Indeed, instead of fixing the metric on the boundary (Dirichlet problem) we could fix its normal derivative that changes the boundary term in the gravitational action. These questions in more detail will be considered elsewhere.
6 Conclusion

The main question addressed in this paper is whether the Einstein-Hilbert action can be generated by quantum matter exactly in the form suggested by Gibbons and Hawking that possesses the consistent variation with respect to metric subject to Dirichlet conditions for the metric coefficients of the boundary. We argue that in order to get this one has to start with a matter action the metric variation of which on manifold with a boundary is well defined. It is shown that the action of matter non-minimally coupled with metric requires some special boundary term analogous to the Gibbons-Hawking one. We derive this term for arbitrary non-minimal matter. Then in the effective action of the quantized matter the term of the first order in the curvature is generated in the correct form. In particular, this means that the corresponding boundary UV divergences do not require an independent renormalization and are automatically renormalized simultaneously with their volume part linear in the curvature. We relate this fact to the problem of the renormalization of a black hole entropy and arrive at the same result as [12]. The similar conclusion was done by the authors of [13]. We should emphasize, however, the essential difference in methods and generality of results of the present paper from those of [13].

For the renormalization of the gravitational coupling constant and black hole entropy the authors of [13] used two different methods adjusted correspondingly to two different definitions of entropy: the Gibbons-Hawking thermodynamic entropy and the so-called geometric one (defined by differentiating the effective action with respect to the deficit angle on the conical manifold). The first method is a standard local Schwinger-DeWitt expansion and proper time regularization on the manifold with smooth metric and sufficiently small curvature. The second one uses the trick of decomposing the spacetime in the vicinity of a conical singularity into a product of two spaces (two-dimensional and transversal, angular, one) with a subsequent use of a powerful two-dimensional machinery. Generically, the conical singularity technique for non-minimally coupled matter was developed in [12]. The first method allows the authors of [13] to establish the validity of the main result only for a minimally coupled scalar field, which boils down to the statement of eq.(1.5) in our Introduction, because they do not use the correct form of the boundary action like in (1.10) and (2.4) and do not include its contribution in the expansion (3.2) and in the final result (3.10). Without this it is impossible to reach a correct result for generic nonminimally coupled fields. So, as a remedy for a general case the second method is used in [13] reproducing the results of [13] for the particular case of $2 + (D - 2)$ decomposition near the horizon. The authors emphasize a conceptual inequivalence of these two methods and domains of their applicability, but recognize a miraculous coincidence of their results in case of a minimal scalar field. In our derivation we don’t use the conical singularity method that has a disadvantage of manipulating with singular operators. To arrive at conclusions of a common renormalization of the volume Einstein-Hilbert term and the relevant surface term for arbitrary boundary (not necessarily related to conical singularities) we have to use in all entirety a regular technique of the present paper.

The last comment concerns the use of low-energy theorem arguments in [13] for
a better foundation of the main result. Certainly it seems tempting to declare that the correctness of the variational problem for a local low-energy effective action should guarantee the same renormalization of its bulk and surface terms. In reality, however, such arguments have only heuristic nature and should be supported by verifying the quantitative mechanism of this phenomenon. There are examples, maybe in the different context, when quantum corrections can potentially violate the asymptotically flat boundary conditions given at the classical level, and nontrivial intrinsic cancelation of dangerous terms should be checked to maintain one and the same boundary value problem in classical and quantum domains [1]. Another example is the same problem as posed in the present paper but with Neumann boundary conditions for metric. It was used for constructing the microcanonical gravitational ensemble [15], when instead of the 3-metric of the boundary the quasilocal energy and momentum are being fixed there. At present, it is far from obvious for us if and how the same conclusions will hold for this problem, although the low-energy theorem argument would seem to be equally applicable.

The range of open issues related to this work can be further continued. It is not clear whether this can be generalized to terms of higher powers in curvature in the effective action. It seems reasonable that starting with a matter action having a well defined metric variation we would obtain under quantization the effective action in the right form to have the same property. However, the general proof of this, even in the case of a minimal matter [14], is still absent. Moreover, the formulation of the consistent variational problem for the full effective action on spacetimes with boundaries is not yet clear due to its essentially nonlocal nature.

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