Boundary Kondo impurities in the generalized supersymmetric $t - J$ model

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Abstract

We study the generalized supersymmetric $t - J$ model with Kondo impurities in the boundaries. We first construct the higher spin operator K-matrix for the XXZ Heisenberg chain. Setting the boundary parameter to be a special value, we find a higher spin reflecting K-matrix for the supersymmetric $t - J$ model. By using the Quantum Inverse Scattering Method, we obtain the eigenvalue and the corresponding Bethe ansatz equations.

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1 Introduction

There has been extensive interests in the investigation of low-dimensional correlated electron systems with impurities. Recently, using renormalization group techniques, Kane and Fisher [1] studied the transport properties of a 1D interacting electron gas in the presence of a potential barrier. They showed that a single potential scatter may dramatically influence the physics in the presence of repulsive $e^{-e}$ interactions. The system behaves like a Tomonaga-Luttinger liquid rather than a Fermi liquid. Some different techniques were also applied to study similar systems [2, 3]. The Kondo impurities in a Tomonaga-Luttinger liquid have been investigated in great detail [4, 5, 6, 7].

Attempts to study the effects due to the presence of impurities in 1D quantum chains in the framework of integrable models have a long successful history [8]-[14]. Andrei and Johannesson [9] studied an arbitrary spin $S$ embeded in a spin-1/2 Heisenberg chain. This method was generalized to other cases. Recently the supersymmetric $t-J$ model with impurities has attracted considerable interests. The Hamiltonian of the $t-J$ model includes the near-neighbour hopping ($t$) and antiferromagnetic exchange ($J$) [15, 16].

$$H = \sum_{j=1}^{L} \left\{ -t \mathcal{P} \sum_{\sigma = \pm 1} (c_{j+1,\sigma}^\dagger c_{j,\sigma} + c_{j,\sigma}^\dagger c_{j+1,\sigma}) \mathcal{P} + J (S_j S_{j+1} - \frac{1}{4} n_j n_{j+1}) \right\}. \quad (1)$$

It is known that this model is supersymmetric and integrable for $J = \pm 2t$ [7, 8]. The supersymmetric $t-J$ model was also studied in Refs. [9, 10, 11, 12], for a review, see Ref. [13] and the references therein. Essler and Korepin et al [22] showed that the one-dimensional Hamiltonian can be obtained from the transfer matrix of the two-dimensional supersymmetric exactly solvable lattice model [21, 24].

By use of the Quantum Inverse Scattering Method (QISM) [25], the supersymmetric $t-J$ model with higher spin impurity was first investigated in the periodic boundary conditions [14]. Recently, the supersymmetric $t-J$ model with impurities have been studied extensively in both periodic and reflecting (open) boundary conditions [26, 27, 28, 29].

The open boundary condition was studied extensively in the last decade. There have been several methods to study the the problem of open boundary condition [30, 31]. In the end of 80’s, Sklyanin [32] proposed a systematic approach to handle the open boundary condition problem in the framework of the QISM. Besides the Yang-Baxter equation [33], the reflection equation proposed by Cherednik [34] also plays a key role in proving the commutativity of the trasfer matrix. We know that the Hamiltonian of the model is usually written as the logarithmic derivative of a transfer matrix at zero spectral parameter. The boundary terms in the Hamiltonina are determined by the reflecting $K$-matrix which is a solution to the reflection equation. In the usual boundary problem, the $K$-matrix is a c-number matrix. The operator $K$-matrices which determine the Kondo impurities in the Hamiltonian are studied recently for several models [35], including the supersymmetric $t-J$ model [26, 27].

The Hamiltonian (1) of the supersymmetric $t-J$ model can be obtained from the transfer matrix constructed by the rational $R$-matrix. We can also use the trigonometric $R$-matrix to formulate the transfer matrix. The corresponding Hamiltonina is a generalization of the original supersymmetric $t-J$ model [36]. This Hamiltonian satisfies a symmetry of the quantum group $SU_q(2|1)$. In this paper, we shall study the generalized supersymmetric $t-J$ model with higher spin boundary impurities. The operator $K$-matrix is first constructed for the XXZ Heisenberg spin chain with higher spin impurities. We then find a higher spin operator $K$-matrix for the supersymmetric $t-J$ model. Using the graded algebraic
Bethe ansatz method, we obtain the eigenvalue of the transfer matrix and the Bethe ansatz equations.

The paper is organized as follows: We introduce the model in section 2. In section 3, we study the XXZ spin chain with higher spin Kondo impurities and present the higher spin reflecting matrices for the generalized supersymmetric $t-J$ model. In section 4, using the nested algebraic Bethe ansatz method, we obtain the eigenvalues of the transfer matrix for the generalized supersymmetric $t-J$ model. Section 5 includes a brief summary and discussions.

2 The Model

We first review the generalized supersymmetric $t-J$ model. For convenience, we choose similar notations as those in [22] and our previous paper [37]. The Hamiltonian of the generalized supersymmetric model takes the following form:

$$
H = \sum_{j=1}^{N} \sum_{\sigma=\pm} [c_{j,\sigma}^\dagger (1-n_{j,-\sigma})c_{j+1,\sigma} (1-n_{j+1,-\sigma}) + c_{j+1,\sigma}^\dagger (1-n_{j+1,-\sigma})c_{j,\sigma} (1-n_{j,-\sigma})] - 2\sum_{j=1}^{N} \frac{1}{2} (S^\dagger_j S_{j+1} + S_j S^\dagger_{j+1}) + \cos(\eta)S_j^z S_{j+1}^z - \frac{\cos(\eta)}{4} n_j n_{j+1} + isin(\eta) \sum_{j=1}^{N} [S_j^z n_{j+1} - S_{j+1}^z n_j].
$$

(2)

When the anisotropic parameter $\eta = 0$, this Hamiltonian reduces to an equivalent form of the original Hamiltonian (1). The operators $c_{j,\sigma}$ and $c_{j,\sigma}^\dagger$ mean the annihilation and creation operators of electron with spin $\sigma$ on a lattice site $j$, and we assume the total number of lattice sites is $N$. $\sigma = \pm$ represent spin down and up, respectively. These operators are canonical Fermi operators satisfying anticommutation relations

$$\{c_{j,\sigma}^\dagger, c_{j,\tau}\} = \delta_{ij} \delta_{\sigma\tau}.
$$

(3)

We denote by $n_{j,\sigma} = c_{j,\sigma}^\dagger c_{j,\sigma}$ the number operator for the electron on a site $j$ with spin $\sigma$, and by $n_j = \sum_{\sigma=\pm} n_{j,\sigma}$ the number operator for the electron on a site $j$. The Fock vacuum state $|0\rangle$ is defined as $c_{j,\sigma}|0\rangle = 0$. Due to the exclusion of double occupancy, there are altogether three possible electronic states at a given lattice site $j$

$$|0\rangle, \quad |\uparrow\rangle = c_{j,\uparrow}^\dagger |0\rangle, \quad |\downarrow\rangle = c_{j,\downarrow}^\dagger |0\rangle.
$$

(4)

$S^z_j, S_j, S_j^\dagger$ are spin operators satisfying $su(2)$ algebra and can be expressed as:

$$S_j = c_{j,\uparrow}^\dagger c_{j,\downarrow}, \quad S_j^\dagger = c_{j,\downarrow}^\dagger c_{j,\uparrow}, \quad S_j^z = \frac{1}{2} (n_{j,1} - n_{j,-1}).
$$

(5)

The above Hamiltonian can be obtained from the logarithmic derivative of the transfer matrix at zero spectral parameter. In the framework of QISM, the transfer matrix is constructed by the trigonometric R-matrix of the Perk-Schultz model [38]. The non-zero entries of the R-matrix are given by

$$
\tilde{R}(\lambda)_{aa}^{aa} = \sin(\eta + \epsilon_a \lambda), \quad \tilde{R}(\lambda)_{ab}^{ab} = (-1)^{\epsilon_a \epsilon_b} \sin(\lambda), \quad \tilde{R}(\lambda)_{ba}^{ab} = e^{isign(a-b)\lambda} \sin(\eta), \quad a \neq b,
$$

(6)
where $\epsilon_a$ is the Grassman parity, $\epsilon_a = 0$ for boson and $\epsilon_a = 1$ for fermion, and

$$\text{sign}(a - b) = \begin{cases} 1, & \text{if } a > b \\ -1, & \text{if } a < b. \end{cases}$$  \hspace{1cm} (7)$$

This R-matrix of the Perk-Schultz model satisfies the usual Yang-Baxter equation:

$$R_{12}(\lambda - \mu)R_{13}(\lambda)R_{23}(\mu) = R_{23}(\mu)R_{13}(\lambda)R_{12}(\lambda - \mu)$$  \hspace{1cm} (8)

In this paper, we shall concentrate our discussion only to the Fermionic, Fermionic and Bosonic case (FFB), that means $\epsilon_1 = \epsilon_2 = 1, \epsilon_3 = 0$. And we shall use the graded formulae to study this model. For supersymmetric $t-J$ model, the spin of the electrons and the charge 'hole' degrees of freedom play a very similar role forming a graded superalgebra with two fermions and one boson. The holes obey boson commutation relations, while the spinons are fermions\cite{[23]}. The graded approach has an advantage of making clear distinction between bosonic and fermionic degrees of freedom \cite{[3]}. Introducing a diagonal matrix $\Pi_{ac}^{bd} = (-)^{a_c} \delta_{ab} \delta_{cd}$, we change the original R-matrix to the following form

$$R(\lambda) = \Pi \tilde{R}(\lambda).$$  \hspace{1cm} (9)

From the non-zero elements of the R-matrix $R_{ac}^{bd}$, we see that $\epsilon_a + \epsilon_b + \epsilon_c + \epsilon_d = 0$. One can show that the R-matrix satisfies the graded Yang-Baxter equation

$$R(\lambda - \mu)^{b_1 b_2}_{a_1 a_2} R(\lambda)^{c_1 c_2}_{b_1 b_2} R(\mu)^{c_3 c_4}_{b_1 b_2} (-)^{\epsilon_{b_1} + \epsilon_{c_1} + \epsilon_{b_2}} = R(\mu)^{b_2 b_3}_{a_2 a_3} R(\lambda)^{a_1 b_1}_{c_1 c_2} R(\lambda - \mu)^{c_3 c_4}_{b_1 b_2} (-)^{\epsilon_{a_1} + \epsilon_{b_1} + \epsilon_{b_2}}.$$  \hspace{1cm} (10)

Explicitly the R-matrix is written as

$$R(\lambda) = \begin{pmatrix} a(\lambda) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b(\lambda) & 0 & -c_-(\lambda) & 0 & 0 & 0 & 0 \\ 0 & 0 & b(\lambda) & 0 & 0 & 0 & c_-(\lambda) & 0 \\ 0 & -c_+(\lambda) & 0 & b(\lambda) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a(\lambda) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b(\lambda) & 0 & c_-(\lambda) & 0 \\ 0 & 0 & c_+(\lambda) & 0 & 0 & b(\lambda) & 0 & 0 \\ 0 & 0 & 0 & 0 & c_+(\lambda) & 0 & b(\lambda) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & w(\lambda) & 0 \end{pmatrix},$$  \hspace{1cm} (11)

where

$$a(\lambda) = \sin(\lambda - \eta), \quad w(\lambda) = \sin(\lambda + \eta), \quad b(\lambda) = \sin(\lambda), \quad c_\pm(\lambda) = e^{\pm i \lambda} \sin(\eta).$$  \hspace{1cm} (12)

In the framework of the QISM, we can construct the $L$ operator from the R-matrix as:

$$L_n(\lambda) = \begin{pmatrix} b(\lambda) - (b(\lambda) - a(\lambda))e^n_{11} & -c_-(\lambda)e^n_{21} & c_-(\lambda)e^n_{31} \\ -c_+(\lambda)e^n_{12} & b(\lambda) - (b(\lambda) - a(\lambda))e^n_{22} & c_-(\lambda)e^n_{32} \\ c_+(\lambda)e^n_{13} & e^n_{23} & b(\lambda) - (b(\lambda) - a(\lambda))e^n_{33} \end{pmatrix}. \hspace{1cm} (13)$$

Here $e^n_{ab}$ acts on the $n$-th quantum space. Thus we have the (graded) Yang-Baxter relation

$$R_{12}(\lambda - \mu)L_1(\lambda)L_2(\mu) = L_2(\mu)L_1(\lambda)R_{12}(\lambda - \mu).$$  \hspace{1cm} (14)
In this sense we say that the model is integrable. Expanding the transfer matrix in the powers of $\tau$

As a consequence of the Yang-Baxter relation (18) and the unitarity property of the R-matrix, we can

For periodic boundary condition, the transfer matrix $T_N(\lambda)$ is defined as a matrix product over the $N$ operators on all sites of the lattice,

where the subscript $a$ represents the auxiliary space, and the tensor product is in the graded sense. Explicitly we write

By repeatedly using the Yang-Baxter relation (14), one can prove easily that the monodromy matrix also satisfies the Yang-Baxter relation

For periodic boundary condition, the transfer matrix $\tau_{\text{peri}}(\lambda)$ of this model is defined as the supertrace of the monodromy matrix in the auxiliary space

As a consequence of the Yang-Baxter relation (18) and the unitarity property of the R-matrix, we can prove that the transfer matrix commutes with each other for different spectral parameters,

In this sense we say that the model is integrable. Expanding the transfer matrix in the powers of $\lambda$, we can find conserved quantities. The first non-trivial conserved quantity is the Hamiltonian (1),

where $P_{ij}$ is the graded permutation operator expressed as $P_{ac}^{bd} = \delta_{ad}\delta_{bc}(-1)^{a+c}$.

In this paper, we consider the reflecting boundary condition case. In addition to the Yang-Baxter equation, a reflection equation should be used in proving the commutativity of the transfer matrix with boundaries. The reflection equation takes the form [14]

For the graded case, the reflection equation remains the same as the above form. We only need to change the usual tensor product to the graded tensor product. We write it explicitly as

$$R(\lambda - \mu)^{b_1b_2}_{a_1a_2} K(\lambda)^{c_1}_{b_1} R(\lambda + \mu)^{d_1d_2}_{c_2c_1} K(\mu)^{a_2a_1}_{d_2d_1} (-)^{e_{a_1} + e_{c_1}} r_{b_1} r_{b_2}$$

$$= K(\mu)^{b_2}_{a_2} R(\lambda + \mu)^{b_1c_2}_{a_1b_2} K(\lambda)^{c_1}_{b_1} R(\lambda - \mu)^{d_2d_1}_{c_2c_1} (-)^{e_{a_1} + e_{c_1}} r_{b_2}. \quad (23)$$
Instead of the monodromy matrix $T(\lambda)$ for periodic boundary conditions, we consider the double-row monodromy matrix
\[
T(\lambda) = T(\lambda)K(\lambda)T^{-1}(-\lambda)
\] (24)
for the reflecting boundary conditions. Using the Yang-Baxter relation, and considering the boundary K-matrix which satisfies the reflection equation, one can prove that the double-row monodromy matrix $T(\lambda)$ also satisfies the reflection equation
\[
R(\lambda - \mu)_{a_1a_2}^{b_1b_2} T(\lambda)_{b_1}^{c_1} R(\lambda + \mu)_{c_2}^{d_2} T(\mu)_{c_2}^{d_1} (-)^{e_1+e_2} = T(\mu)_{a_1}^{a_2} R(\lambda + \mu)_{b_1}^{b_2} T(\lambda)_{b_1}^{c_1} R(\lambda - \mu)_{c_2}^{d_2} (-)^{e_1+e_2}.
\] (25)

Next, we study the properties of the R-matrix. We define the super-transposition $st$ as
\[
(A^{st})_{ij} = A_{ji}(-1)^{( \epsilon_i + 1) \epsilon_j}.
\] (26)
For FFB grading used in this paper, $\epsilon_1 = \epsilon_2 = 1$, $\epsilon_3 = 0$, we can rewrite the above relation explicitly as
\[
\begin{pmatrix}
A_{11} & A_{12} & B_1 \\
A_{21} & A_{22} & B_2 \\
C_1 & C_2 & D
\end{pmatrix}^{st} =
\begin{pmatrix}
A_{11} & A_{21} & C_1 \\
A_{12} & A_{22} & C_2 \\
-B_1 & -B_2 & D
\end{pmatrix}.
\] (27)
We also define the inverse of the super-transposition $\bar{st}$ as $\{A^{st}\}^{\bar{st}} = A$.

One can prove directly that the R-matrix (11) satisfy the following unitarity and cross-unitarity relations:
\[
R_{12}(\lambda)R_{21}(-\lambda) = \rho(\lambda) \cdot id., \quad \rho(\lambda) = \sin(\eta + \lambda)\sin(\eta - \lambda),
\] (28)
\[
R^{st_{12}}(\eta - \lambda)M_1 R^{st_{1}}_{21}(\lambda)M^{-1}_1 = \tilde{\rho}(\lambda) \cdot id., \quad \tilde{\rho}(\lambda) = \sin(\lambda)\sin(\eta - \lambda).
\] (29)
Here the matrix $M = \text{diag.}(e^{2i\eta},1,1)$ is determined by the R-matrix. The cross-unitarity relation can also be written as the following form
\[
\{M_1^{-1}R^{st_{1},st_{2}}_{12}(\eta - \lambda)M_1\}^{st_2} R^{st_1}_{21}(\lambda) = \tilde{\rho}(\lambda),
\] (30)
\[
R^{st_{12}}_{12}(\lambda) \{M_1 R^{st_{1},st_{2}}_{21}(\eta - \lambda)M^{-1}_1\}^{st_2} = \tilde{\rho}(\lambda).
\] (31)
In order to construct the commuting transfer matrix with boundaries, besides the reflection equation, we need the dual reflection equation. In general, the dual reflection equation which depends on the unitarity and cross-unitarity relations of the R-matrix takes different forms for different models. For the models considered in this paper, we can write the dual reflection equation in the following form:
\[
R^{st_{1},st_{2}}_{21}(\mu - \lambda)K_{1}^{+st_{1}}(\lambda)M^{-1}_1 R^{st_{1},st_{2}}_{12}(\eta - \lambda - \mu)M_1 K_{2}^{+st_{2}}(\mu) = K_{2}^{+st_{2}}(\mu)M_1 R^{st_{1},st_{2}}_{21}(\eta - \lambda - \mu)M^{-1}_1 K_{1}^{+st_{1}}(\lambda)R^{st_{1},st_{2}}_{12}(\mu - \lambda).
\] (32)
Then the transfer matrix with boundaries is defined as
\[
t(\lambda) = strK^+(\lambda)T(\lambda).
\] (33)
The commutativity of \( t(\lambda) \) can be proved by using unitarity and cross-unitarity relations, reflection equation and the dual reflection equation. The detailed proof of the commuting transfer matrix with boundaries for super (graded) case can be found, for instance, in Ref. [40, 41, 42] etc. With a normalization \( K(0) = \text{id.} \), the Hamiltonian can be obtained as

\[
H = \frac{1}{2} \sin(\eta) \frac{d \ln t(\lambda)}{d \lambda} \bigg|_{\lambda=0} + \sum_{j=1}^{N-1} P_{j,j+1} L'_{j,j+1}(0) + \frac{1}{2} \sin(\eta) K_1'(0) + \frac{str_a K_a^+(0) P_{N_a} L'_N(0)}{str_a K_a(0)}. \tag{34}
\]

3 Higher spin solution to the reflection equation for supersymmetric \( t - J \) model

In order to find the higher spin solution to the reflection equation for the generalized supersymmetric \( t - J \) model, we first construct the higher spin reflecting matrix for the XXZ Heisenberg chain.

3.1 XXZ Heisenberg chain with higher spin boundary impurities

The higher spin R-matrix can be constructed by using the fusion procedure [43]. The Hamiltonian of the XXZ Heisenberg chain is written as

\[
H_{XXX} = \sum_{j=1}^{N} [\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+ + \frac{1}{2} \cos(\eta) \sigma_j^z \sigma_{j+1}^z]. \tag{35}
\]

Here \( \sigma^\pm = 1/2(\sigma^x \pm \sigma^y) \) and \( \sigma^x, \sigma^y \) and \( \sigma^z \) are Pauli matrices. The R-matrix is known to be the standard six-vertex model,

\[
r_{12}(\lambda) = \begin{pmatrix}
\sin(\lambda + \eta) & 0 & 0 & 0 \\
0 & \sin(\lambda) & \sin(\eta) & 0 \\
0 & \sin(\eta) & \sin(\lambda) & 0 \\
0 & 0 & 0 & \sin(\lambda + \eta)
\end{pmatrix}. \tag{36}
\]

In the framework of QISM, the L-operator constructed by the r-matrix is written as:

\[
L_{ak}(\lambda) = \begin{pmatrix}
\sin(\lambda + \frac{1}{2} \eta + \frac{1}{2} \eta \sigma_k^z) & \sin(\eta) \sigma_k^- \\
\sin(\eta) \sigma_k^+ & \sin(\lambda + \frac{1}{2} \eta - \frac{1}{2} \eta \sigma_k^z)
\end{pmatrix}, \tag{37}
\]

where \( a \) represents auxiliary space. As usual, we can construct the row-to-row monodromy matrix \( T_a(\lambda) = L_{aN}(\lambda) \cdots L_{a1}(\lambda) \), and we have the Yang-Baxter relation

\[
r_{12}(\lambda - \mu) T_1(\lambda) T_2(\mu) = T_2(\mu) T_1(\lambda) r_{12}(\lambda - \mu), \tag{38}
\]

where the tensor product is a non-graded one.

Next, we consider the higher spin operators. Let the higher spin L operator take the form [43, 44]

\[
\mathcal{L}(\lambda) = \begin{pmatrix}
\sin(\lambda + S^\eta) & \sin(\eta) S^- \\
\sin(\eta) S^+ & \sin(\lambda - S^\eta)
\end{pmatrix}, \tag{39}
\]
where $S^\sigma$, $S$ and $S^\dagger$ are spin-$s$ operators satisfying the following commutation relations,

$$[S^\sigma, S^\pm] = \pm S^\pm, \quad [S^+, S^-] = \frac{\sin(2S^\sigma \eta)}{\sin(\eta)}, \quad (40)$$

We also have the following relations for spin-$s$ operators:

$$\sin(S^\sigma \eta)\sin(\eta + S^\sigma \eta) + \sin^2(\eta)S^-S^+ = \sin^2(\eta)S^+S^- + \sin(S^\sigma \eta)\sin(S^\sigma \eta - \eta) = \sin(s\eta)\sin(s\eta + \eta) \quad (41)$$

A more general relations can be written as

$$\sin(\lambda + S^\sigma \eta)\sin(\eta + S^\sigma \eta - \lambda) + \sin^2(\eta)S^-S^+ = \sin^2(\eta)S^+S^- + \sin(\lambda - S^\sigma \eta)\sin(-\lambda - S^\sigma \eta + \eta) = \sin(\lambda + s\eta)\sin(s\eta + \eta - \lambda). \quad (42)$$

One can prove that the higher spin $L$ operator also satisfy the Yang-Baxter relation

$$r_{12}(\lambda - \mu)L_1(\lambda)L_2(\mu) = L_2(\mu)L_1(\lambda)r_{12}(\lambda - \mu). \quad (43)$$

Now, let us consider the reflecting boundary condition. We can find a c-number solution to the reflection equation $K_c(\lambda) = \text{diag.}(\sin(\xi + \lambda), \sin(\xi - \lambda))$, where $\xi$ is an arbitrary parameter. This is a general c-number diagonal solution to the reflection equation. In particular, if $\xi \to -i\infty$, we find $K(\lambda) = \text{diag.}(e^{2i\lambda}, 1)$ is a solution to the reflection equation.

It is interesting to find a higher spin operator $K$-matrix. We can construct the operator $K$-matrix by $K_{XXZ}(\lambda) = L(\lambda + c)K_c(\lambda)L^{-1}(-\lambda + c)$, one can find easily that $K(\lambda)$ is an operator reflecting matrix satisfying the reflection equation. Explicitly, the higher spin reflecting $K$ has the form $K_{XXZ}(\lambda) = \left( \begin{array}{cc} K(\lambda)_1^1 & K(\lambda)_1^2 \\ K(\lambda)_2^1 & K(\lambda)_2^2 \end{array} \right)$ with

$$K(\lambda)_1^1 = \sin(\lambda - \xi)\sin(\lambda + c + s\eta)\sin(\lambda + c - \eta - s\eta) + \sin(2\lambda)\sin(\lambda + c + S^\sigma \eta)\sin(\xi - c + \eta + S^\sigma \eta)$$

$$K(\lambda)_2^2 = -\sin(\xi + \lambda)\sin(\lambda + c + s\eta)\sin(\lambda + c - \eta - s\eta) + \sin(2\lambda)\sin(\lambda + c - S^\sigma \eta)\sin(\xi + c - \eta + S^\sigma \eta),$$

$$K(\lambda)_1^2 = \sin(\eta)\sin(2\lambda)\sin(\xi + c + S^\sigma \eta)S^-,$$

$$K(\lambda)_2^1 = \sin(\eta)\sin(2\lambda)\sin(\xi - c + S^\sigma \eta)S^+ \quad \text{(44)}.$$

By use of the cross-unitarity relation of the r-matrix, the operator reflecting matrix to the dual reflection equation can also be found. The eigenvalues of the transfer matrix can be obtained by applying the algebraic Bethe ansatz method. These results will be presented in a separate paper [45].

### 3.2 Higher spin reflecting matrix for the supersymmetric $t - J$ model

We know that the generalized supersymmetric $t - J$ model has a $SU_q(2)$ symmetry. We suppose that the operator $K$-matrix takes the following form:

$$K(\lambda) = \left( \begin{array}{ccc} A(\lambda) & B(\lambda) & 0 \\ B(\lambda) & C(\lambda) & 0 \\ 0 & 0 & 1 \end{array} \right) \quad \text{(45)}$$
Inserting this matrix into the reflection equation (23), we can find the following non-trivial relations:

\[
\hat{r}(\lambda - \mu)_{a_1a_2}^{b_1b_2}K(\lambda)_{b_1}^{c_1} \hat{r}(\lambda + \mu)_{b_2c_1}^{c_2d_1}K(\mu)_{c_2}^{d_2} = K(\mu)_{a_2}^{b_2} \hat{r}(\lambda + \mu)_{a_1b_2}^{b_1c_2}K(\lambda)_{b_2}^{c_1} \hat{r}(\lambda - \mu)_{c_1}^{d_1},
\]

and

\[
K(\lambda)_{a_1}^{b_1}K(\mu)_{b_1}^{d_1} = K(\mu)_{a_1}^{b_1}K(\lambda)_{b_1}^{d_1},
\]

where all indices take values 1,2, and we have introduced

\[
\eta = \xi + \Lambda
\]

Next, let us consider the higher spin reflecting matrix to the dual reflection equation (32). We find the higher spin reflecting matrix as

\[
A(\lambda) = f(\lambda)e^{-2i\lambda}K(\lambda)^{1/2},
B(\lambda) = f(\lambda)e^{-i\lambda}K(\lambda)^{1/2},
C(\lambda) = f(\lambda)e^{-i\lambda}K(\lambda)^{1/2},
D(\lambda) = f(\lambda)K(\lambda)^{1/2},
\]

and after some tedious calculations, we find that if we take \( \xi \to -i\infty \), and \( f(\lambda) = -1/e^{2i\lambda}\sin(\lambda - c - \eta - sn)\sin(\lambda - c + sn) \), all relations obtained from the reflection equation can be satisfied. So, we finally find the higher spin reflecting matrix as

\[
A(\lambda) = g(\lambda) \left( e^{-4i\lambda} \sin(\lambda + c - sn) \sin(\lambda + c + \eta + sn) - \sin(2\lambda) \sin(u + c + S^\eta) e^{-i(3\lambda + c + \eta + S^\eta)} \right),
B(\lambda) = g(\lambda) \sin(\eta) \sin(2\lambda) e^{-i(2\lambda + c + S^\eta)} S^-,
C(\lambda) = g(\lambda) \sin(\eta) \sin(2\lambda) e^{-i(2\lambda + c + S^\eta)} S^+,
D(\lambda) = g(\lambda) \left( \sin(\lambda + c - sn) \sin(\lambda + c + \eta + sn) - \sin(2\lambda) \sin(\lambda + c + S^\eta) e^{-i(\lambda - c - \eta + S^\eta)} \right),
\]

where \( g(\lambda) = 1/\sin(\lambda - c - \eta - sn) \sin(\lambda - c + sn) \).

Next, let us consider the higher spin reflecting matrix to the dual reflection equation (32). We suppose \( K^+ \) has the similar form as \( K \). By direct calculation, we can find \( R_{12}^{st,1} = I_1 R_{21}(\lambda) I_1 \) with \( I = \text{diag}(-1, -1, 1) \). For the form (13), we have \( IK(\lambda)I = K(\lambda) \). Then with the help of property \( [M_1M_2, R(\lambda)] = 0 \), we can write the dual reflection equation as

\[
R_{12}^{st,2}(\mu - \lambda)K_1^{+st,1}(\lambda)M_1^{-1}R_{21}(\eta - \lambda - \mu)K_2^{+st,2}(\mu)M_2^{-1} = K_2^{+st,2}(\mu)M_2^{-1}R_{12}(\eta - \lambda - \mu)K_1^{+st,1}(\lambda)M_1^{-1}R_{21}(\mu - \lambda).
\]

We see that there is an isomorphism between \( K \) and \( K^+ \):

\[
K(\lambda) \mapsto K^+({\lambda}) = K(\frac{\eta}{2} - \lambda)M.
\]

Given a solution to the reflection equation (23), we can also find a solution to the dual reflection equation (32). Remark that in the sense of the transfer matrix, the reflection equation and the dual reflection
equation are independent of each other. We can write the higher spin reflecting matrix to the dual reflection equation as

\[ K^+(\lambda) = \begin{pmatrix} A^+(\lambda) & B^+(\lambda) & 0 \\ B^+(\lambda) & C^+(\lambda) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \] (53)

with

\[ A^+(\lambda) = g^+(\lambda) [e^{4i\lambda} \sin(\lambda + \tilde{c} - \eta + \tilde{s}\eta)\sin(\lambda + \tilde{c} - 2\eta - \tilde{s}\eta)] - \sin(2\lambda - \eta)\sin(u + \tilde{c} - \eta - \tilde{s}\eta)e^{i(3\lambda + \tilde{c} - \eta + \tilde{s}\eta)}], \]

\[ B^+(\lambda) = -g^+(\lambda)\sin(\eta)\sin(2\lambda - \eta)e^{i(2\lambda - \tilde{c} + \tilde{s}\eta)}S^-, \]

\[ C^+(\lambda) = -g^+(\lambda)\sin(\eta)\sin(2\lambda - \eta)e^{i(2\lambda - \tilde{c} + \tilde{s}\eta)}S^+, \]

\[ D^+(\lambda) = g^+(\lambda)[\sin(\lambda + \tilde{c} - \eta + \tilde{s}\eta)\sin(\lambda + \tilde{c} - 2\eta - \tilde{s}\eta)] - \sin(2\lambda - \eta)\sin(\lambda + \tilde{c} - \eta + \tilde{s}\eta)e^{i(\lambda - \tilde{c} + \tilde{s}\eta)}], \] (54)

where \( g^+(\lambda) = 1/\sin(\lambda - \tilde{c} + \eta + \tilde{s}\eta)\sin(\lambda - \tilde{c} - \tilde{s}\eta). \)

Thus we find the higher spin reflecting matrices for the generalized supersymmetric \( t - J \) model. We should remark that these higher spin reflecting matrices are the kind of 'singular' matrices. It can not be constructed directly by the Sklyanin's 'dressing' procedure. In the rational limit, it reduces to the result obtained in [27]. The rational higher spin K-matrix has been analyzed in detail by the projecting method [29]. Our result should also be obtained by the projecting method.

4 Algebraic Bethe ansatz method for the generalized supersymmetric \( t - J \) model with higher spin impurities

4.1 First level algebraic Bethe ansatz

We denote the double-row monodromy matrix as

\[ T(\lambda) = \begin{pmatrix} A_{11}(\lambda) & A_{12}(\lambda) & B_1(\lambda) \\ A_{21}(\lambda) & A_{22}(\lambda) & B_2(\lambda) \\ C_1(\lambda) & C_2(\lambda) & D(\lambda) \end{pmatrix}. \] (55)

For later discussions, we introduce the following transformations

\[ A_{ab}(\lambda) = \tilde{A}_{ab}(\lambda) + \delta_{ab} \frac{e^{-2i\lambda\sin(\eta)}}{\sin(2\lambda + \eta)}D(\lambda). \] (56)

As mentioned in section 2, the double-row monodromy matrix satisfies the reflection equation [25], we have the following commutation relations:

\[ C_{d_1}(\lambda)C_{d_2}(\mu) = \hat{C}_{d_1}(\lambda - \mu)C_{d_2}(\mu)C_{d_1}(\lambda), \] (57)

\[ D(\lambda)C_{d}(\mu) = \frac{\sin(\lambda + \mu)\sin(\lambda - \mu - \eta)}{\sin(\lambda + \mu + \eta)\sin(\lambda - \mu)}C_{d}(\mu)D(\lambda) + \frac{\sin(2\mu)\sin(\eta)e^{i(\lambda - \mu)}}{\sin(\lambda - \mu)\sin(2\mu + \eta)}C_{d}(\lambda)D(\mu) - \frac{\sin(\eta)e^{i(\lambda + \mu)}}{\sin(\lambda + \mu + \eta)}C_{d}(\lambda)\tilde{A}_{bd}(\mu), \] (58)
\[ \hat{A}_{a,d_1}(\lambda)C_{d_2}(\mu) = \frac{\delta_{d_2,1} \delta_{d_1,2}}{\sin(\lambda + \mu + \eta)\sin(\lambda - \mu)} \hat{C}_{c_1}(\mu) \hat{A}_{c_1,b_1}(\lambda) \]

\[ + \frac{\sin(\eta)e^{-i(\lambda - \mu)}}{\sin(\lambda - \mu)\sin(2\lambda + \eta)} \hat{r}_{12}(\lambda + \eta) \hat{C}_{b_1}(\lambda) \hat{A}_{b_2,d_2}(\mu) \]

\[ - \frac{\sin(2\mu)\sin(\eta)e^{-i(\lambda + \mu)}}{\sin(\lambda + \mu + \eta)\sin(2\mu + \eta)} \hat{r}_{12}(\lambda + \eta) \hat{C}_{b_2}(\lambda) \hat{D}(\mu). \] (59)

Here the indices take values 1,2, and the matrix \( \hat{r} \) is defined in [10].

We define a reference state in the n-th quantum space as \(|0 >_{n} = (0,0,1)^{t}\), and reference states for the boundary operators as \( S^{-}|0 >_{r}, = 0, S^{*}|0 >_{r} = -\hat{s}|0 >_{r}, S^{+}|0 >_{r} \neq 0, \) and \( S^{-}|0 >_{\tilde{r}} = 0, S^{*}|0 >_{\tilde{r}} = -\tilde{s}|0 >_{\tilde{r}}, S^{+}|0 >_{\tilde{r}} \neq 0. \) The vacuum state is then defined as \(|0 > = |0 >_{\tilde{r}} \otimes_{k=1}^{N} |0 >_{k} \otimes |0 >_{r} \). Acting the double-row monodromy matrix on this vacuum state, we have

\[ B_{a}(\lambda)|0 > = 0, \]

\[ C_{a}(\lambda)|0 > \neq 0, \]

\[ D(\lambda)|0 > = sin^{2N}(\lambda + \eta)|0 >, \]

\[ \hat{A}_{ab}(\lambda)|0 > = sin^{2N}(\lambda)[K(\lambda)^{b}_{a} - \delta_{ab}\frac{sin(\eta)e^{-2i\lambda}}{sin(2\lambda + \eta)}]|0 > = W_{ab}(\lambda)sin^{2N}(\lambda)|0 >, \] (60)

where

\[ W_{12}(\lambda) = 0, \quad W_{21}(\lambda) = C(\lambda), \]

\[ W_{11}(\lambda) = g(\lambda)\frac{e^{i\eta}sin(2\lambda)}{sin(2\lambda + \eta)}[e^{-i(4\lambda + 2\eta)}sin(\lambda + \lambda + 2\eta + \eta)] - sin(2\lambda + \eta)sin(\lambda + \lambda + \eta + \eta)e^{-i(3\lambda + \lambda + 3\eta - \eta)}], \]

\[ W_{22}(\lambda) = -e^{-2i\lambda}\frac{sin(2\lambda)sin(\lambda + \lambda + \eta - \eta)}{sin(2\lambda + \eta)sin(\lambda + \lambda + \eta)}. \] (61)

The transfer matrix ([33]) can be written as

\[ t(\lambda) = -K^{+}(\lambda)^{b}_{a}\hat{A}_{ab}(\lambda) + D(\lambda) \]

\[ = -K^{+}(\lambda)^{b}_{a}\hat{A}_{ab}(\lambda) + \left(1 - \frac{sin(\eta)e^{-2i\lambda}}{sin(2\lambda + \eta)}[A^{+}(\lambda) + D^{+}(\lambda)]\right)D(\lambda). \] (62)

Acting this transfer matrix on the ansatz of the eigenvector

\[ C_{d_{1}}(\mu_{1})C_{d_{2}}(\mu_{2}) \cdots C_{d_{n}}(\mu_{n})|0 > \sim F^{d_{1} \cdots d_{n}}, \] (63)

where \( F^{d_{1} \cdots d_{n}} \) is a function of the spectral parameters \( \mu_{j} \), we have

\[ t(\lambda)C_{d_{1}}(\mu_{1})C_{d_{2}}(\mu_{2}) \cdots C_{d_{n}}(\mu_{n})|0 > \sim F^{d_{1} \cdots d_{n}} \]

\[ \sim \frac{sin(2\lambda - \eta)sin(\lambda - \tilde{c} + \eta - \tilde{s} \eta)sin(\lambda - \tilde{c} + 2\eta + \tilde{s} \eta)}{sin(2\lambda + \eta)sin(\lambda - \tilde{c} + \eta + \tilde{s} \eta)sin(\lambda - \tilde{c} - \tilde{s} \eta)} \]

\[ \times sin^{2N}(\lambda + \eta)\prod_{i=1}^{n} \frac{sin(\lambda + \mu_{i})sin(\lambda - \mu_{i} - \eta)}{sin(\lambda + \mu_{i} + \eta)sin(\lambda - \mu_{i})}C_{d_{1}}(\mu_{1}) \cdots C_{d_{n}}(\mu_{n})|0 > \sim F^{d_{1} \cdots d_{n}} \]
with the grading \( \epsilon \)
conditions corresponding to the anisotropic case
\( u.t. \)
where
\[ \text{defined, with the help of the relation (59), as} \]
\[ K \]
where
\[ \text{interpreted as an operator matrix with higher spin. Explicitly, with the help of (61, 62), we have} \]
\[ \text{With} \]
\[ M \]
By use of the cross-unitarity relation \( \hat{r} \)
\[ \text{relation is just the dual reflection equation which we need.} \]
\[ \text{We find that this nested transfer matrix can be regarded as a transfer matrix with reflecting boundary conditions corresponding to the anisotropic case} \]
\[ t^{(1)}(\lambda) = \text{str} K^{(1)+}(\lambda')T^{(1)}(\lambda', \{ \mu'_i \})K^{(1)}(\lambda')T^{(1)-1}(-\lambda', \{ \mu'_i \}) \] (66)
with the grading \( \epsilon_1 = \epsilon_2 = 1 \). Here, we denote \( \lambda' = \lambda + \frac{\pi}{2}, \mu' = \mu + \frac{\pi}{2} \). The reflecting matrix can also be interpreted as an operator matrix with higher spin. Explicitly, with the help of (61, 62), we have
\[ K^{(1)}(\lambda') = e^{i\eta \frac{\sin(2\lambda' - \eta)}{\sin(2\lambda')}} \begin{pmatrix} A(\lambda', c') & B(\lambda', c') \\ C(\lambda', c') & D(\lambda', c') \end{pmatrix}, \]
\[ K^{(1)+}(\lambda') = \begin{pmatrix} A^+(\lambda' - \frac{\pi}{2}) & B^+(\lambda' - \frac{\pi}{2}) \\ C^+(\lambda' - \frac{\pi}{2}) & D^+(\lambda' - \frac{\pi}{2}) \end{pmatrix}, \] (67)
where \( c' = c + \frac{\pi}{2} \). Note that the solution of the reflection equation can be changed by a gauge transformation. In order to prove that the above defined nested transfer matrix is still a transfer matrix with higher spin reflecting matrix, we should prove that \( K^{(1)}(\lambda') \) and \( K^{(1)+}(\lambda') \) satisfy the reduced reflection equation and its corresponding dual reflection equation. Indeed, it can be shown that the following reflection equation holds,
\[ \hat{r}_{12}(\lambda' - \mu')K^{(1)}_{12}(\lambda')\hat{r}_{21}(\lambda' + \mu')K^{(2)}_{12}(\mu') = K^{(2)}_{12}(\mu')\hat{r}_{12}(\lambda' + \mu')K^{(1)}_{12}(\lambda')\hat{r}_{21}(\lambda' - \mu'). \] (68)
With \( M^{(1)} = \text{diag}(e^{2i\eta}, 1) \), and the isomorphism (62), we find that \( K^{(1)+} \) satisfies the following relation
\[ \hat{r}_{12}(\lambda' - \mu')K^{(1)+}_{12}(\lambda')\hat{r}_{21}(\lambda' + \mu')K^{(2)}_{12}(\lambda' - \mu') \]
\[ = K^{(2)+}_{12}(\lambda') M^{(1)}_{12}^{-1} \hat{r}_{12}(\lambda' + \mu')K^{(1)+}_{12}(\lambda') \]
\[ \cdot \hat{r}_{21}(\lambda' - \mu')K^{(2)}_{12}(\lambda') M^{(1)}_{12}^{-1} \hat{r}_{21}(\lambda' - \mu'). \] (69)
By use of the cross-unitarity relation \( \hat{r}_{12}^{21}(2\eta - \lambda)M_{12}^{(1)}\hat{r}_{21}(\lambda)M_{12}^{(1)-1} = \sin(\lambda)\sin(2\eta - \lambda) \cdot \text{id.} \), the above relation is just the dual reflection equation which we need.

The row-to-row monodromy matrix \( T^{(1)}(\lambda', \{ \mu'_i \}) \) (corresponding to the periodic boundary condition) and its inverse are defined as
\[ T^{(1)}_{da_n}(\lambda', \{ \mu'_i \})_{e_1 \cdots e_n} = \hat{r}(\lambda' + \mu'_1)_{a_1}^{e_1} \hat{r}(\lambda' + \mu'_2)_{a_2}^{e_2} \cdots \hat{r}(\lambda' + \mu'_n)_{a_n}^{e_n} \]
\[ T^{(1)-1}_{b_n a}(\lambda', \{ \mu'_i \})_{e_n \cdots e_1} = \hat{r}_{21}(\lambda' - \mu'_n)_{b_n e_n}^{\cdots e_1} \cdots \hat{r}_{21}(\lambda' - \mu'_1)_{b_1 e_1}^{d_1}. \] (70)
We show that a problem to find the eigenvalue of the original transfer matrix \( t(\lambda) \) reduces to a problem to find the eigenvalue of the nested transfer matrix \( t^{(1)}(\lambda) \). The nested transfer matrix is still a boundary case with higher spin reflecting matrix.
In order to ensure the assumed eigenvector is indeed the eigenvector of the transfer matrix, \( \mu_1, \cdots, \mu_n \) should satisfy the following Bethe ansatz equations,

\[
\frac{\sin(2\mu_j - \eta)\sin(\mu_j - \bar{c} + \eta - \bar{s} \eta)\sin(\mu_j - \bar{c} + 2\eta + \bar{s} \eta)}{\sin(2\mu_j + \eta)\sin(\mu_j - \bar{c} + \eta + \bar{s} \eta)\sin(\mu_j - \bar{c} - \bar{s} \eta)} \sin^{2N}(\mu_j + \eta) \\
\times \prod_{i=1}^{n} \sin(\mu_j + \mu_i)\sin(\mu_j - \mu_i - \eta) = -\sin^{2N}(\mu_j)\Lambda^{(1)}(\mu_j), \quad j = 1, 2, \cdots, n. \tag{72}
\]

Here we have used the notation \( \Lambda^{(1)}(\lambda) \) to denote the eigenvalue of the nested transfer matrix \( t^{(1)}(\lambda) \).

### 4.2 Bethe ansatz for the six-vertex model with higher spin reflecting matrices

We repeat almost the same procedure as that of the first level algebraic Bethe ansatz method. We only write down some results without the detailed calculations here. We have

\[
e^{i\eta} \frac{\sin(2\lambda' - \eta)}{\sin(2\lambda')} D(\lambda', \epsilon') |0 \rangle >_r = -e^{-i2\lambda} \frac{\sin(2\lambda)\sin(\lambda + \epsilon - \eta + s\eta)}{\sin(2\lambda + \eta)\sin(\lambda - \epsilon + s\eta)} |0 \rangle \equiv U_2 |0 \rangle, \tag{73}
\]

\[
e^{i\eta} \frac{\sin(2\lambda' - \eta)}{\sin(2\lambda')} [A(\lambda', \epsilon') + D(\lambda', \epsilon') \frac{\sin(\eta)e^{-i2\lambda'}}{\sin(2\lambda' - \eta)}] |0 \rangle >_r, \tag{74}
\]

and

\[
A^+(\lambda' - \frac{\eta}{2}) |0 \rangle >_r = -e^{i(2\lambda + \eta)} \frac{\sin(\lambda + \bar{c} - \eta + \bar{s} \eta)}{\sin(\lambda - \bar{c} + \eta + \bar{s} \eta)} |0 \rangle \equiv U_1^+ |0 \rangle >_r, \tag{76}
\]

\[
[D^+(\lambda' - \frac{\eta}{2}) - A^+(\lambda' - \frac{\eta}{2}) \frac{\sin(\eta)e^{-i2\lambda'}}{\sin(2\lambda' - \eta)}] |0 \rangle >_r, \tag{77}
\]

We write the nested double-row monodromy matrix as

\[
\mathcal{T}^{(1)}(\lambda, \{\mu_i\}) = \begin{pmatrix} A^{(1)}(\lambda) & B^{(1)}(\lambda) \\ C^{(1)}(\lambda) & D^{(1)}(\lambda) \end{pmatrix}. \tag{78}
\]

From the results obtained above, we know that this double-row monodromy matrix also satisfies the reflection equation \([B]\). Considering the transformation

\[
A^{(1)}(\lambda) = \tilde{A}^{(1)}(\lambda) - \frac{\sin(\eta)e^{-2i\lambda}}{\sin(2\lambda - \eta)} D^{(1)}(\lambda), \tag{79}
\]

we can find the following commutation relations which are useful for the algebraic Bethe ansatz method,

\[
D^{(1)}(\lambda)C^{(1)}(\mu) = \frac{\sin(\lambda - \mu + \eta)\sin(\lambda + \mu)}{\sin(\lambda - \mu)\sin(\lambda + \mu - \eta)} C^{(1)}(\mu)D^{(1)}(\lambda) - \frac{\sin(2\mu)\sin(\eta)e^{i(\lambda - \mu)}}{\sin(\lambda - \mu)\sin(2\mu - \eta)} C^{(1)}(\lambda)D^{(1)}(\mu) + \frac{\sin(\eta)e^{i(\lambda + \mu)}}{\sin(\lambda + \mu - \eta)} C^{(1)}(\lambda)\tilde{A}^{(1)}(\mu). \tag{80}
\]
The eigenvalues of the transfer matrix for the generalized supersymmetric $t-J$ model are given as follows:

$$\hat{A}^{(1)}(\lambda)C^{(1)}(\mu) = \frac{\sin(\lambda - \mu - \eta)\sin(\lambda + \mu - 2\eta)}{\sin(\lambda - \mu)\sin(\lambda + \mu - \eta)}C^{(1)}(\mu)\hat{A}^{(1)}(\lambda)$$

$$+ \frac{\sin(\eta)\sin(2\lambda - 2\eta)e^{-i(\lambda - \mu)}}{\sin(\lambda - \mu)\sin(2\lambda - \eta)}C^{(1)}(\lambda)\hat{A}^{(1)}(\mu)$$

$$- \frac{\sin(2\mu)\sin(2\lambda - 2\eta)\sin(\eta)e^{-i(\lambda + \mu)}}{\sin(\lambda + \mu - \eta)\sin(2\lambda - \eta)\sin(2\mu - \eta)}C^{(1)}(\lambda)D^{(1)}(\mu),$$

where $\mu$ satisfies the corresponding Bethe ansatz equations. In what follows, we give a summary of our main result.

4.3 Result

The eigenvalues of the transfer matrix for the generalized supersymmetric $t-J$ model are given as follows:

$$\Lambda^{(1)}(\lambda') = -U_1^+U_1\prod_{i=1}^{n}[\sin(\lambda' + \mu_i)\sin(\lambda' - \mu_i)]\prod_{i=1}^{m}\left\{\frac{\sin(\lambda' - \mu_i^{(1)} + \eta)\sin(\lambda' + \mu_i^{(1)} - \eta)}{\sin(\lambda' - \mu_i^{(1)})\sin(\lambda' + \mu_i^{(1)} - \eta)}\right\},$$

where $\mu_1^{(1)}, \ldots, \mu_m^{(1)}$ should satisfy the corresponding Bethe ansatz equations. In what follows, we give a summary of our main result.
where $\mu_1, \cdots, \mu_n$ and $\mu^{(1)}_1, \cdots, \mu^{(1)}_n$ should satisfy the Bethe ansatz equations

$$
\frac{\sin(\mu^{(1)}_j + c + \eta + s\eta)\sin(\mu^{(1)}_j - c - \eta + s\eta)\sin(\mu^{(1)}_j + \tilde{c} - \eta + \tilde{s}\eta)\sin(\mu^{(1)}_j - \tilde{c} - \tilde{s}\eta)}{\sin(\mu^{(1)}_j - c - \eta - s\eta)\sin(\mu^{(1)}_j + c + \eta - s\eta)\sin(\mu^{(1)}_j + \tilde{c} - \eta - \tilde{s}\eta)\sin(\mu^{(1)}_j - \tilde{c} + \eta - \tilde{s}\eta)} = \prod_{i=1}^{n} \frac{\sin(\mu^{(1)}_j + \mu_i)\sin(\mu^{(1)}_j - \mu_i - \eta)}{\sin(\mu^{(1)}_j + \mu_i + \eta)\sin(\mu^{(1)}_j - \mu_i)_{i=1, j} \prod_{i=1, j} \frac{\sin(\mu^{(1)}_j - \mu^{(1)}_i + \eta)\sin(\mu^{(1)}_j + \mu^{(1)}_i + \eta)}{\sin(\mu^{(1)}_j - \mu^{(1)}_i - \eta)\sin(\mu^{(1)}_j + \mu^{(1)}_i - \eta)},
$$

and

$$
\frac{\sin(\mu_j + \tilde{c} - \eta - \tilde{s}\eta)\sin(\lambda + c + \eta - s\eta)}{\sin(\mu_j - \tilde{c} + 2\eta + \tilde{s}\eta)\sin(\lambda - c + s\eta)} = \frac{\sin^{2n}(\mu_j + \eta)}{\sin^{2n}(\mu_j)} \prod_{i=1}^{m} \frac{\sin(\mu_j - \mu^{(1)}_i + \eta)\sin(\mu_j + \mu^{(1)}_i + \eta)}{\sin(\mu_j - \mu^{(1)}_i - \eta)\sin(\mu_j + \mu^{(1)}_i - \eta)},
$$

\[ j = 1, \cdots, m, \tag{86} \]

\[ j = 1, \cdots, n. \tag{87} \]

5 Summary

In this paper, we have studied the generalized supersymmetric $t - J$ model with Kondo impurities in the boundaries. Using the higher spin $L$ operator of XXZ Heisenberg chain and the general diagonal solution to the reflection equation for six vertex model, we find a higher spin reflecting matrix for the generalized supersymmetric $t - J$ model. Applying the graded algebraic Bethe ansatz method, we obtain the eigenvalues of the transfer matrix for the $t - J$ model with higher spin boundaries.

It is interesting to solve this problem in other background gradings, for example, FB or BFF. The higher spin reflecting matrix should be constructed from the BF or FB six vertex models. The analysis of ground state properties, low-lying excitations and thermodynamic Bethe ansatz is always worth performing.

One can find that the $SU_q(2)$ higher spin reflecting matrix also satisfy the reflection equation of $SU_q(N)$ model. The eigenvalues of $SU_q(N)$ model with $SU_q(2)$ higher spin boundary impurities can be obtained by using the nested algebraic Bethe ansatz method. Actually, the $SU_q(2)$ higher spin boundary impurities could be embedded into $SU_q(M|N)$ spin chains with $M \geq 2$ or $N \geq 2$.

After we put our article to the cond-mat e-print archive, X.Y.Ge and H.Q.Zhou inform us that they solve the same problem independently [10].

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