Charged Renyi entropies for free scalar fields

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Abstract
I first calculate the charged spherical Rényi entropy by a numerical method that does not require knowledge of any eigenvalue degeneracies, and applies to all odd dimensions.

An image method is used to relate the full sphere values to those for an integer covering, \( n \). It is shown to be equivalent to a ‘transformation’ property of the zeta-function.

The \( n \to \infty \) limit is explicitly constructed analytically and a relation deduced between the limits of corner coefficients and the effective action (free energy) which generalises, for free fields, a result of Bueno, Myers and Witczak-Krempa and Elvang and Hadjiantonis to any dimension.

Finally, the known polynomial expressions for the Rényi entropy on even spheres at zero chemical potential are re–derived in a different form and a simple formula for the conformal anomaly given purely in terms of central factorials is obtained.

Keywords: Rényi entropy, charged spherical, free fields

(Some figures may appear in colour only in the online journal)

1. Introduction and summary

Belin et al [19] introduced an extension of Rényi entropy to include a chemical potential enabling the entanglement in different charge sectors of the field theory to be investigated.

In a recent work I discussed one aspect of this charged spherical Rényi entropy viz. the conformal weights of twist operators. I now turn to the Rényi entropies themselves.

Although the object of the many papers around this topic are results that apply to any conformal field theory, the case of free fields in special geometries is often used as a valuable exemplar of techniques and putative theorems. Explicit expressions for the entropy for...
a spherical entangling surface were obtained in [19] using several methods one of which
involved conformal mappings to a branched sphere and the associated spectral problem. Only
the case of the three-sphere was analysed and the resulting free energy emerged a sum of
derivatives of Hurwitz $\zeta$-functions. Moving to higher dimensions would appear to be organi-
sationally awkward. In the present work I wish to revisit this situation using an approach
valid for all dimensions. I employ a different coordinate system and am able to avoid the
messy calculation of the degeneracies that arises in [19]. Also, instead of an $n$-fold covered
sphere, I employ, initially, an orbifolded sphere, $S^d/\mathbb{Z}_q$ used earlier many times. A further,
descriptive difference, is that I do not use the language of ‘Wilson loops’ but simply use a
quasi-periodicity interpreted as the effect of an Aharonov–Bohm flux.

The entropy is determined from the effective action, or logdet, of the relevant propagating
operator. Technically one needs the derivative of the corresponding $\zeta$-function, $\zeta(s)$, at $s = 0$
which requires the eigenvalues. The uncharged eigenproblem is textbook and has been given
many times. In the next section I detail the charged extension, in simplified form, before
outlining its application to the entropy, spending most time on the derivation of the effective
action. For its evaluation I have been content to use a purely numerical approach, via a known
quadrature, the results of which are shown in graphs. The charged Rényi entropy derived
therethrough is exhibited in section 3. The remaining sections are distinct and more analytical.

Section 4 analyses the entropy in the limit $n \to \infty$, $n$ being the covering integer, $n = 1/q$.
The answer, in odd dimensions, is a sum of Dirichlet $\eta$-functions. The relation to corner
coefficients and conformal dimensions of twist operators is also explored making contact with
earlier work.

Finally, in section 5, the Rényi entropy on even spheres (for zero chemical potential) is
reconsidered and the exact expression derived in a neater form agreeing, in results, to an ear-
lier formula of Casini and Huerta, [13]. Also a compact, easily calculated formula is obtained
for the conformal anomaly.

In appendix A, an image relation, derived elsewhere, is outlined and an explicit illustrative
example calculated. In general, it allows quantities on the $n$-fold covered sphere to be obtained
from those on the single sphere but with non-zero chemical potential. The image expression
is shown, in the spherical situation, to be a consequence of a ‘transformation’ property of the
$\zeta$-function.

Appendix B contains a different formulation of the effective action in terms of Barnes’ mul-
tiple $\Gamma$-function, $\Gamma_d$, which, for the full sphere in odd dimensions, reduces to an integral involv-
ing elementary functions and gives a more explicit dependence on the chemical potential.

There is also a very brief description of another method involving $\Gamma_d$ that leads to an in-
tegral over Gauss’ $\psi$ function and holds for all dimensions.

2. Spectrum on the orbifolded sphere with flux

I deal with the $q$-orbifolded $d$-sphere, $S^d/\mathbb{Z}_q$, which, in order to allow for a chemical potential,
has an Aharonov–Bohm flux running between the south and north poles, in either direction
equivalently.

The orbifolded sphere is made up of $2q$ segments of apex angle $\pi/q$. I refer to these as
lunes. The combination of two adjacent lunes I call a doubled lune. These tile the sphere under
an SO(2) action, neighbouring (single) lunes being related by a reflection.

It is convenient to set $\beta = \pi/q$ and generalise to a lune of arbitrary angle. The tiling prop-
erty is then geometrically lost.

The $d$-lune can be defined inductively by giving its metric in the nested form,
which is iterated down to the 1-lune of metric \( d\theta^2_i \) with \( 0 \leq \theta_i \leq \pi \). The angle \( \theta_1 \) is referred to as the polar angle and conventionally written \( \phi \). The angle of the lune (there are two of them) is \( \beta \). The conical deformation of the sphere can be traced to the deformation of the \( \phi \) circle. This disappears when \( \beta = 2\pi \).

The boundary of the lune comprises two pieces corresponding to \( \phi = 0 \) and \( \phi = \beta \). The metric (1) shows immediately that these are unit \( (d-1) \)-hemispheres because their polar angle, \( \theta_2 \), runs only from 0 to \( \pi \). Conditions, typically Dirichlet and Neumann, can be applied at the boundary. The boundary parts intersect, with a constant dihedral angle of \( \beta \), in a \( (d-2) \)-sphere, of unit radius, which constitutes a set of points fixed under \( O(2) \) rotations parametrised by \( \phi \).

It can be seen that the 2-lune submanifold, with coordinates \( \theta_2 \) and \( \phi \), has a wedge singularity at its north and south poles. These poles are at \( \theta_2 = 0 \) and \( \theta_2 = \pi \) and are the 0-hemispheres of a 0-sphere. In the \( d \)-lune, the submanifolds, \( \theta_2 = 0 \) and \( \theta_2 = \pi \), are the \( (d-2) \)-hemispheres of the \( (d-2) \)-sphere of fixed points just mentioned and constitute the entangling surface.

In the absence of the flux, the mode problem is more or less standard. I consider the propagating operator and eigenvalue equation,

\[
\mathcal{O} \psi = -\Delta_2 + c^2 \\
\mathcal{O} \psi = \lambda \psi, \\
-\Delta_2 \psi = \lambda \psi, \\
\Lambda = \lambda + c^2,
\]

where \( \Delta_2 \) is the Laplacian on the sphere with eigenvalues, \(-\lambda\), determined by boundary and periodicity conditions on the double lune, \( c \) is a constant.

The separation of variables follows the nesting structure in (1) and is a classic calculation. The hyperspherical harmonics are labeled by the set of \( d \) separation constants \( l_i \), \( i = 1, \ldots, d \). The first one, \( l_1 \), is associated with the \( \phi \) eigenvalue equation, and I leave it unspecified for the moment. The remaining constants, \( l_i \), which can be taken as non-negative, are associated with the intervening spheres, \( S_i \). They satisfy,

\[
l_i = l_{i-1} + n_i, \quad n_j = 0, 1, \ldots, \infty, \quad i = 2, \ldots, d,
\]

with the last one determining the eigenvalue \( \lambda \) by,

\[
\lambda = l_d(l_d + d - 1) = \left( l_d + \frac{d-1}{2} \right)^2 - \frac{(d-1)^2}{4}.
\]

Iteration of (2) gives

\[
l_d = l_1 + n_2 + \ldots + n_d.
\]

In this approach the degeneracy of a particular eigenvalue is given by coincidences as the free labels vary. If \( l_1 \) is an integer then the distribution of \( d \) integers is involved, otherwise it is that of \( d-1 \) integers. It will turn out unnecessary to find the degeneracies explicitly, although it is a standard situation.

The calculation of \( \lambda \) devolves upon the constant \( l_1 \) coming from the eigenproblem on the deformed \( \phi \) circle with a flux running through, an old question see [21]. I approach this via pseudo–periodic eigenfunctions which satisfy the defining equation, for a complex \( \psi \),

\[^1\text{For the full sphere it was first given by George Green who, for some reason, worked in } d \text{ dimensions at a time when the notion of higher dimensional geometry had yet to be formulated.}\]
with the periodicity condition on the double lune,
\[ \psi(\phi + 2\beta) = e^{2\pi i \delta} \psi(\phi). \] (5)

Then the eigenfunction is,
\[ \psi_l(\phi) \propto e^{i l_1 \phi}, \]
with
\[ l_1 = \frac{\pi}{\beta} (m_1 + \delta), \quad m_1 = -\infty, \ldots, 0, \ldots, \infty, \] (6)
and all quantities are periodic in \( \delta \) with period 1. Furthermore all real physical quantities are unchanged under the reversal \( \delta \rightarrow -\delta \). As remarked in [11], for complex fields the Green function, as a typical quantity, is \( G_{23,3} + G_{23,3,-\delta} \) where \( G_{23,3} \) is computed from fields obeying (5). These two conditions require that the periodising unit cell be symmetrical about zero, \(-1/2 < \delta \leq 1/2\). My basic approach is to calculate the physical quantities in a unit cell and simply extend them by periodicity.

Generally, one might expect the physical quantity to have singularities at the cell boundary, and its iterates. The present calculation shows that this not so, agreeing with [19]. The unit cell could then be chosen \( 0 < \delta < 1 \).

I turn now to the eigenvalues and note that \( l_1 \) can be negative but because the Legendre equation for the next spherical function involves \( l_1 \) it is sufficient to replace \( l_1 \) by \(|l_1|\) so that the expression for \( l_1 \), (4), takes the form,
\[ l_1 = q |m_1 \pm \delta| + \mathbf{l} \cdot \mathbf{n}, \]
where \( q = \pi \beta \) and \( \mathbf{l} \) is a \( d - 1 \)-dimensional set with \( \mathbf{l} = (1,1,\ldots,1) \) and where \( \mathbf{n} \) is a set of integers whose dimension is that of the vector it is multiplying.

Using (3), the eigenvalue \( \Lambda \) is,
\[ \Lambda(\pm \delta) = \left( \frac{d-1}{2} + q |m_1 \pm \delta| + \mathbf{l} \cdot \mathbf{n} \right)^2 - \frac{(d-1)^2}{4} = c^2. \]

For conformal invariance in \( d \) dimensions,
\[ \Lambda(\pm \delta) = \left( \frac{d-1}{2} + q |m_1 \pm \delta| + \mathbf{l} \cdot \mathbf{n} \right)^2 - \frac{1}{4}, \]
and the conformal \( \zeta \)-function for the operator \( \mathcal{O} \) is then \( (\alpha = 1/2) \), after some rearrangement and assuming that \( |\delta| \leq 1 \),
\[ \zeta(s, \mu, q|\omega) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{\mathbf{n}=0}^{\infty} \left[ \left( \frac{d-1}{2} + q |m_1 \pm \delta| + \mathbf{l} \cdot \mathbf{n} \right)^2 - \alpha^2 \right]^{-s} \]
\[ = 2 \sum_{\mathbf{n}=0}^{\infty} \left[ \left( \frac{d-1}{2} + |\mu| + \mathbf{\omega} \cdot \mathbf{n} \right)^2 - \alpha^2 \right]^{-s} \]
\[ + 2 \sum_{\mathbf{n}=0}^{\infty} \left[ \left( \frac{d-1}{2} + q - |\mu| + \mathbf{\omega} \cdot \mathbf{n} \right)^2 - \alpha^2 \right]^{-s}, \] (7)
where $\omega$ is the set of $d$ integers, $(q, 1)$, but can be considered, formally, as real non-negative and referred to as the parameters or degrees.

I have introduced the chemical potential $\mu = q\delta$. In particular, for the orbifolded sphere, $S^d/\mathbb{Z}_q$ is integral, otherwise not. The range of $\mu$ is extended by periodicity.

When $\mu = 0$ the first sum corresponds to the Neumann lune and the second to the Dirichlet one.

The factor of 2 in (7) is a consequence of the charged nature of the field and so the effective action (sometimes called the free energy), is,

$$A_d(q, \mu) = -\frac{1}{2} \zeta'(0, \mu, q\omega).$$

Another notation is $A_d(q, \mu) \equiv A_d(1/n, \mu)$.

A means of calculating the derivative $\zeta'(0)$ is presented in [26] which I will employ later but first I give a numerical method used, e.g. in [20] and valid only in odd dimensions.

Since the references and technicalities of the method are detailed in [20] I feel I can just give the answer which is, after minor manipulation,

$$A_d(q, \mu) = -\frac{1}{2^{d-2}} \int_0^\infty \frac{dx}{x^{d-2}} \text{Re} \frac{\cosh(q - 2\mu)\tau/2 \cosh\tau/2}{\tau \sinh q\tau/2 \sinh\tau/2}$$

for $\tau = x + iy$ with $y < 2\pi/q$. The integral converges in the relevant range of $\mu$. I just compute it numerically. More formal expressions for the effective action are given in appendix B.

In order to compare with results in [19], figures 1 and 3 show the ratio, $A_d(q, 0)/A_d(0)$, of effective actions plotted against chemical potential in three dimensions. Figure 2 has the results for $d = 5$.

### 3. Charged Rényi entropy

The definition of the Rényi entropy is,

$$S_d(\mu) = \frac{n A_d(1, \mu) - A_d(1/n, \mu)}{1 - n}, \quad n = 1/q,\quad (9)$$

where $A_d(q, \mu)$ is the effective action on the $d$-dimensional space–time deformed by a conical singularity of angle $2\pi/q$ computed in the previous sections.

Figures 4 and 5 show the variation of the Rényi entropy against $n$, considered as a continuous variable for various chemical potentials.

### 4. The $n \to \infty$ limit

For large $n$, the entropy tends to a constant,

$$\lim_{n \to \infty} S_d(\mu) = \lim_{q \to 0} q A_d(q, \mu) - A_d(1, \mu), \quad n = 1/q,\quad (10)$$

The last term is the full sphere value and could be calculated by the other methods described in appendix B. There is an explicit integral for the first term and I consider it further, in its own right, by a contour method which leads to a simple closed form.

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2 If $q = 1/n (n \in \mathbb{N})$, $\delta$ is the flux through the $n$-fold covering so that $\delta/n$ is the flux through a single sheet i.e. the chemical potential as defined here, and in [19].
Figure 1. Ratio of effective actions for $d = 3$.

Figure 2. Ratio of effective actions for $d = 5$.

Figure 3. Ratio of effective actions for $d = 3$. 
Figure 4. Renyi entropy $d = 3$.

Figure 5. Renyi entropy $d = 5$.

Figure 6. $q \rightarrow 0$ effective action.
I use $q$ in preference to $n$. The first thing to note is that, because of periodicity ($\propto q$), the limit is independent of $\mu^3$ (which is here defined to remain finite as $q \to 0$). All values $\mu$ are equivalent to $\mu = 0$ giving, as the limit of (8),

$$
q \lim_{q \to 0} A_\rho(q, \mu) = -\frac{1}{2^{d-1}} \int_{-\infty+iy}^{\infty+iy} \frac{dz}{z^3} \cosh \frac{z}{z^2} \sinh^{d-1} z
$$

$$
= \frac{1}{2^{d-2}(d-2)} \int_{-\infty+iy}^{\infty+iy} \frac{dr}{r^3} \frac{1}{z} \sinh^{d-2} z
$$

$$
\equiv A^x_\rho.
$$

(11)

It is possible to carry the calculation forward for any (odd) dimension. For this purpose, as in earlier works, I employ the expansion,

$$
cosech^{2r+1} z = \frac{(-1)^r}{(2r)!} \sum_{\rho=0}^{r} (-1)^\rho G^r_\rho \frac{d^{2\rho}}{dz^{2\rho}} \cosech z,
$$

(12)

for odd powers in terms of even derivatives and then integrate by parts. The integer coefficients, $G^r_\rho$, are known. This results in,

$$
A^x_\rho = \frac{2(-1)^r}{2^{d+1}(2r+1)!} \sum_{l=1}^{\infty} \sum_{\rho=0}^{r} \frac{(2+2\rho)!}{2!} \frac{G^r_\rho}{\pi^{2\rho+2}} \sum_{l=1}^{\infty} (-1)^l \frac{1}{l^{2\rho+3}}
$$

$$
= -\frac{(-1)^r}{2^{d+1}(2r+1)!} \sum_{\rho=0}^{r} (2+2\rho)! \frac{G^r_\rho \eta(2\rho + 3)}{\pi^{2\rho+2}},
$$

(13)

where $d = 2r + 3$.

The numbers, $G^r_\rho$, have been tabulated. Some are below,

| \rho | 0 | 1 | 2 | 3 | 4 |
|-----|---|---|---|---|---|
| $G^r_0$ | 1 | 1 | 9 | 225 | 11,025 |
| $G^r_1$ | 1 | 10 | 259 | 12,916 | 1 |
| $G^r_2$ | 1 | 35 | 1,974 | 2 |
| $G^r_3$ | 1 | 84 | 3 |
| $G^r_4$ | 1 | 4 |

giving for example,

$$
A^x_3 = \frac{\eta(3)}{\pi^2} \approx 0.091 345 371 175 179
$$

$$
A^x_5 = \frac{-\eta(3)}{24\pi^2} - \frac{\eta(5)}{2\pi^4} \approx -0.008 795 939 288 7171
$$

$$
A^x_7 = \frac{3\eta(3)}{640\pi^2} + \frac{\eta(5)}{16\pi^4} + \frac{3\eta(7)}{16\pi^6} \approx 0.001 245 502 543 8325,
$$

(14)

the first of which is given by Klebanov et al [25].

Purely numerically, the integral form (11) is more convenient and, although the integral is suppose to hold only for $d$ integral and odd, in figure 6 the limiting value is plotted against a (continuous) dimension. This interpolates the values, (14).

For zero chemical potential, Bueno et al [17], have discussed the relation between a corner coefficient, $\sigma^d_n$, and the effective action, $A_\rho(1/n, 0)$, in the $n \to \infty$ limit. $\sigma^d_n$ is related to the

<sup>3</sup> A hint of this behaviour can be seen in figures 1 and 2. The ripples become smaller and closer, the mean tending to one.
conformal dimension, $h_n^\alpha$, of twist operators and so I investigate the connection between $h^\infty$ and $A_\alpha^\infty$, for odd dimensions. In fact this relation holds, rather trivially, for any (Euclidean) chemical potential because, in the limit, they are all equivalent to zero as a consequence of periodicity.

In using a conformal transformation to a conical space, an expression for $h_n^\alpha$ was derived from the energy density. As shown there, this depends on the integral, (I have already set the chemical potential to zero),

$$ W_d(q) = \sin \pi q \int_0^\infty \frac{d\tau}{\cosh^d \tau/2} \frac{1}{\cosh q\tau - \cos q\pi}. $$

(15)

by, for conformal scalars,

$$ h_n^\alpha = -\frac{2\Gamma(d/2)}{2\pi^{d/2}} \left( \frac{d}{d-1} W_d(q) - \frac{d-2}{d-1} W_{d-2} \right). $$

(16)

Taking the $n \to \infty$ limit of (15),

$$ \lim_{q \to 0} W_d(q) = \frac{2}{q} \int_0^\infty \frac{d\tau}{\cosh^d \tau/2} \frac{\pi}{\tau^2 + \pi^2} $$

$$ \equiv \frac{1}{q} W^{\infty}_d. $$

(17)

I likewise define $h^{\infty}_n = \frac{1}{n} \lim_{n \to \infty} h^\alpha_n$.

The integral in (17) was encountered, and computed, in [20] in connection with the effective action on a full sphere (i.e. $q = 1$). It takes the form of a sum of $\eta$-functions plus a $\log 2$ term with coefficients involving the $G_\rho^\alpha$. For completeness, I briefly recall the results.

In terms of the notation in [20],

$$ W^{\infty}_d = 2J(d). $$

To calculate (17), this time one uses the recursion,

$$ \sech^{2r+1} x = \frac{1}{(2r)!} \sum_{\rho=0}^r (-1)^\rho \frac{G_\rho^\alpha}{\pi^{2\rho}} \frac{d^{2\rho}}{dx^{2\rho}} \sech x, $$

(18)

and $J(d)$ is found to be (now $d = 2r + 1$),

$$ J(2r+1) = \sum_{\rho=1}^r \frac{G_\rho^\alpha (2\rho)!}{(2r)!} \frac{\eta(2\rho+1)}{\pi^{2\rho}} + \frac{1}{(2r)!} G_0^\alpha \log 2. $$

On taking the conformal combination, $J(2r+1) - \frac{2r-1}{2r} J(2r-1)$, one finds, firstly, that the $\log 2$ terms cancel in view of the recursion relation

$$ G_\rho^\alpha = (2r-1)^2 G_{\rho-1}^{\alpha-1} + G_{\rho-1}^{\alpha-1}, $$

(19)

and the rest of the series equals,

$$ \sum_{\rho=1}^r \frac{G_\rho^\alpha (2\rho)!}{(2r)!} \frac{\eta(2\rho+1)}{\pi^{2\rho}} = \frac{2r-1}{2r} \sum_{\rho=1}^{r-1} \frac{G_{\rho-1}^{\alpha-1} (2\rho)!}{(2r-2)!} \frac{\eta(2\rho+1)}{\pi^{2\rho}}. $$

Extend the second sum to $\rho = r$ which is possible because $G_\rho^\alpha$ is zero if $\rho > r$. Then one finds for this combination,
\[
\sum_{\rho=1}^{r}(2\rho)! \left[ \frac{G^\rho_r}{(2\rho-1)!} - (2\rho - 1) G^{\rho-1}_r \right] \frac{\eta(2\rho + 1)}{\pi^{2\rho}} \frac{1}{2r(2r-2)!} \\
= \sum_{\sigma=0}^{r-1} G^{\sigma}_{\sigma}(2\sigma + 2)! \frac{\eta(2\sigma + 3)}{(2\sigma)!} \frac{1}{\pi^{2\sigma}}. 
\]

A comparison with (13) shows agreement, to a factor, if one remembers that in (13), \( d = 2r + 3 \). To allow for this, shift \( r \) by 1, then

\[
A^\infty_d = \frac{(-1)^r}{2^{2r-1}(2r-1)!} \sum_{\rho=0}^{r-1} (2 + 2\rho)! \frac{G^{\rho-1}_r \eta(2\rho + 3)}{\pi^{2\rho+2}}. 
\]

and, from (16),

\[
h^\infty_d = -\frac{4\Gamma(r + 1/2)}{2^{2r+1}} \frac{1}{\pi^{r+1/2}} \frac{2^{2r-1}}{(2\sigma)!} \sum_{\sigma=0}^{r-1} G^{\sigma^{-1}_{\sigma}}(2\sigma + 2)! \frac{\eta(2\sigma + 3)}{\pi^{2\sigma}}. 
\]

So I find the relation,

\[
h^\infty_d = (-1)^r (2r - 1)! \frac{4\Gamma(r + 1/2)}{2^{2r+1} \pi^{r+1/2}} A^\infty_d \\
= (-1)^r (2r + 1/2) \frac{\Gamma(r + 1/2)}{2\pi^{r+1/2}} A^\infty_d, 
\]

and, in particular,

\[
h^\infty_3 = \frac{1}{4\pi} A^\infty_3. 
\]

According to [17] the limit, \( \sigma^\infty_3 \), of the corner coefficient, \( \sigma^a_3 \), is related to \( h^\infty_3 \) by \( \sigma^\infty_3 = h^\infty_3 / \pi \) (in three dimensions), and so (23) implies that

\[
\sigma^\infty_3 = \frac{1}{4\pi^2} A^\infty_3. 
\]

I have thus confirmed, in a rather detailed fashion, the results of [17] who prove this relation for any 3d conformal field theory using the mapping to the hyperbolic cylinder. The result was found earlier by Elvang and Hadjiantonis, [12], for free bosons and fermions, from calculated expressions for the \( \sigma^a_3 \).

In [18] a conjecture is made regarding the relation between the corner coefficient and the conformal weight in any dimension. Assuming this relation, a generalisation of (24) can be found, for odd dimensions \( d = 2r + 1 \),

\[
\sigma^\infty_d = (-1)^r \frac{2^{r-1}}{\pi^2} \frac{(r - 1)!}{(2r - 1)!} \frac{\Gamma^2}{A^\infty_d}. 
\]

As mentioned above, the integral \( J(d) \) in (17) occurs in the effective action on the full sphere. I give the expression which is, [20],

\[
A_{2r+1}(1, 0) = \frac{(-1)^r}{2^{2r-1}} (J(2r + 1) - J(2r - 1)), 
\]

\(^4\) A similar conclusion holds for even \( d \) using the explicit polynomial expression for \( h^a_d \). The details might be presented at a later time.
which shows that the the limiting form of the Rényi entropy, (10), can be written just in terms of the basic integral, $J(d)$.

5. Rényi entropies on even spheres revisited

The Rényi entropy, or, rather, just its universal log coefficient (at zero chemical potential) was computed on even spheres, dimension by dimension, some time ago by Casini and Huerta, [14], using their mapping to the hyperbolic cylinder (see also [25]). In [20] the same quantity was found from the heat-kernel via known expressions for the conformal anomaly for any (even) dimension. All this for scalar fields, to which I am always restricting myself. The answer was given in terms of generalised Bernoulli polynomials, which, although easily obtained, are, perhaps, not conveniently organised for some purposes. In this section I present an alternative arrangement which offers a simplification and extension of known results.

Rather than extracting the log coefficient using a heat-kernel expansion, of one sort or another, I employ the method described in [7] which involves off-shell thermodynamic arguments. The logarithm arises from the divergent integration up to the de Sitter horizon, as opposed to a regularised odd hyperbolic volume or a generic UV cut off.

The Rényi entropy is determined, (9), by the effective action, $\mathcal{A}$. The thermodynamic free-energy, $F$, is obtained from $\mathcal{A}$ by removing the (imaginary) time circle factor,

$$
A(q) = −\beta F = −\int_{0}^{\beta} d\beta E(\beta), \quad q = 2\pi l\beta;
$$

(26)

where I have introduced the internal energy, $E(\beta)$. The entropy is then just (9) for $\mu = 0$,

$$
S_q = \frac{A_\mu(1) - qA_\mu(q)}{q - 1}.
$$

(27)

$E(\beta)$ is found from the integral of the energy density $\langle T_{00}\rangle_{\text{dS}}$ on de Sitter, which is given by conformal transformation from that on a conical flat space, derived somewhat earlier. The general structure is,

$$
\langle T_{00}\rangle_{\text{dS}} = \frac{1}{(4\pi)^{d/2}} P(d, q) \frac{(1 + Z^2)^d}{2^d Z^d}.
$$

(28)

$P$ is a polynomial in $q$ determined by the field dynamics. The last factor is a geometric one giving the dependence on the radial–like coordinate, $Z$, of the static de Sitter metric,

$$
d s^2 = \frac{4a^2}{(1 + Z^2)^2} \left[ Z^2 d(t/a)^2 - dZ^2 \right] - a^2 \left[ \frac{1 - Z^2}{1 + Z^2} \right] d s^2_{(d-2)\text{-sphere}}.
$$

(29)

The total energy, $E$, is obtained by integrating (28) over the spatial section of (29) up to an infinitesimal distance, $\epsilon$, from the horizon, $Z = 0$. The expansion in $\epsilon$ then allows the coefficient of log $\epsilon$ to be extracted.

If the effective action is constructed via the thermodynamic relation, (26), with the energy coming from (28), the log coefficient in $A(q)$ is found to be, after algebra,

$$
\text{logcoeff}(q) = (-1)^{d/2} \frac{1}{2^{d-2} \Gamma(d/2)} \int_{q}^{\infty} dq \frac{P(d, q)}{q^2}.
$$

(30)
and the coefficient in the entropy follows immediately from (27) if the integral can be found. For this one needs the polynomial, $P(d,q)$, which follows by conformal transformation from the energy density on flat conical space (equivalent to Rindler space) and gives,

$$
\langle T_{00}\rangle_{ds} = \frac{(d-1)\Gamma(d/2)}{(4\pi)^{d/2}} \frac{q}{\pi} \left( W_d(q) - \frac{d-2}{d-1} W_{d-2}(q) \right) \frac{(1+Z^2)^d}{2^d Z^d}.
$$

(31)

Hence the polynomial is (I set $d = 2g$),

$$
P(2g, q) = (2g-1)\Gamma(g) \frac{q}{\pi} \left( W_{2g}(q) - \frac{2(g-1)}{2g-1} W_{2g-2}(q) \right).
$$

(32)

So far the analysis is the same as in [7]. There, the quantities, $W_d$, were given as generalised Bernoulli polynomials and the integration in (30) performed dimension by dimension to give, it turns out, the conformal anomaly, in the case of the entanglement entropy. The difference here is that they are computed via a trigonometric sum, which has an image interpretation, derived by Jeffery, [16], see [15], which allows the integration to be done explicitly for all $d$. Jeffery’s expression is,

$$
\frac{q}{\pi} W_{2g}(q) = \frac{2^{2g-1}}{\Gamma(2g)} \sum_{i=0}^{g-1} \frac{\Gamma(2i+2)}{2^{2i}} A^g_i \left( 1 - q^{2i+2} \right) \frac{\zeta(2i+2)}{\pi^{2i+2}},
$$

(33)

where the $A^g_i$ are constants related to central factorial numbers. They vanish if $i \geq g$ and were early tabulated (See also [10], and below).\(^5\)

To get the scalar energy density, according to (31), a conformal combination is required,

$$
W_{2g}(q) - \frac{2g-2}{2g-1} W_{2g-2}(q) = \frac{\pi}{q} \frac{2^{2g-1}}{\Gamma(2g)} \sum_{i=0}^{g-1} \frac{\Gamma(2i+2)}{2^{2i}} B^g_i \left( 1 - q^{2i+2} \right) \frac{\zeta(2i+2)}{\pi^{2i+2}},
$$

(34)

using $A^{g-1}_{g-1} = 0$ and where the $B^g_i$ are the easily found constants,

$$
B^g_i = A^g_i - (g-1)^2 (A^g_{i-1} - \delta_{i,g-1}) = A^{g-1}_{i-1} + (g-1)^2 \delta_{i,g-1},
$$

upon using the recursion, e.g. [22],

$$
A^{k+1}_{k+1} = A^k_k + k^2 A^k_{k+1}.
$$

(35)

The definition of $B$ is such that $B^g_0 = A^{g-1}_{g-1}$, except $B^g_{g-1} = 1$.

Then,

$$
P(2g, q) = \frac{(2g-1)\Gamma(g)}{\Gamma(2g)} \frac{1}{\pi} \sum_{i=1}^{g-1} \frac{2^{2g-2i-1}\Gamma(2i+2)}{2^{2i-1}\Gamma(2i+2)} B^g_i \left( 1 - q^{2i+2} \right) \frac{\zeta(2i+2)}{\pi^{2i+2}},
$$

employing $A^g_{g-1} = 1$ and $B^g_0 = 0$.

There is no term proportional to $q^2$. This is a consequence of conformal coupling, [11], i.e. of the specific combination in (32).

Then the integral in (30) is,

\(^5\)The validity of (33) for any real $q$ depends on Carlson’s theorem. There are a few more details.
\[ \int_{q}^{\infty} dq \frac{P(2g, q)}{q^2} = -\frac{(2g - 1) \Gamma(g)}{\Gamma(2g) q} \sum_{i=1}^{g-2} 2^{2g-2i} \Gamma(2i + 1) B_i^g \left( 1 + \frac{q^{2i+2}}{2i + 1} \right) \zeta(2i + 2, \pi^{2i+2}). \]

and, from (30),

\[ \logcoeff A(q) = \frac{(-1)^g}{q(2g - 2)!} \sum_{i=1}^{g-1} \Gamma(2i + 2) B_i^g \left( 1 + \frac{q^{2i+2}}{2i + 1} \right) \zeta(2i + 2, \pi^{2i+2}), \quad (36) \]

with the particular on shell value,

\[ \frac{(-1)^g}{(2g - 2)!} \sum_{i=1}^{g-1} (2i) ! (i + 1) B_i^g \frac{\zeta(2i + 2)}{2^{2i+2}\pi^{2i+2}}. \quad (37) \]

The log coefficient in the Rényi entropy, (27), can then be assembled,

\[ \logcoeff S_q = \frac{(-1)^g}{(2g - 2)!} \sum_{i=1}^{g-1} (2i) ! (i + 1) B_i^g \frac{\zeta(2i + 2)}{2^{2i+2}\pi^{2i+2}}. \quad (38) \]

Casini and Huerta, [14], obtain an expression of similar form, but the constants are different in appearance as they come from the heat–kernel on hyperbolic space. A general formula is not given, although the expression is calculated explicitly for \( d = 2, 4 \) and 6 dimensions.

The only unknowns in (38) are the constants, \( A_i^g \). As mentioned, these are tabulated or can easily be found from the recursion, (35), so that (38) can be quickly programmed and I find agreement with [13]. For 8 and 10 dimensions I obtain, very quickly,

\[ \frac{(n + 1)(79 n^6 + 79 n^4 + 23 n^2 + 3)}{907 200 n^7}, \]

and

\[ \frac{(n + 1)(1759 n^8 + 1759 n^6 + 571 n^4 + 109 n^2 + 10)}{119 750 400 n^9}. \]

One asset of the form, (38), is that it facilitates the limit \( q \rightarrow 1 \) which gives for the log coefficient of the entanglement entropy, \( S_1 \),

\[ \logcoeff S_1 = \frac{(-1)^g}{(2g - 2)!} \sum_{i=1}^{g-1} (2i) ! (i + 1) B_i^g \frac{\zeta(2i + 2)}{2^{2i+2}\pi^{2i+2}}. \quad (39) \]

The same as (37).

The heat-kernel approach (or \( \zeta \)-function) gives an expression for \( \logcoeff S_q \) in terms of conformal anomalies, which reduces to just the standard round anomaly in the case of \( q = 1 \). (39) is therefore expected to equal this, as evaluation confirms.

Direct derivations of the anomaly via the \( \zeta \)-function and expansion of the degeneracy, [6], also produce a sum over Riemann \( \zeta \)-functions but it would need some manipulation to show formal agreement with (39).

An even simpler form can be obtained by rewriting the generalised Bernoulli form, [27], of the anomaly, \( C \), to give (see [23, 24]) for complex fields,

\[
\logcoeff S_1 = \frac{(-1)^g}{(2g - 2)!} \sum_{i=1}^{g-1} (2i) ! (i + 1) B_i^g \frac{\zeta(2i + 2)}{2^{2i+2}\pi^{2i+2}}. \quad (39)
\]

The same as (37).

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An even simpler form can be obtained by rewriting the generalised Bernoulli form, [27], of the anomaly, \( C \), to give (see [23, 24]) for complex fields,
\[ C_{2g} = \frac{4(-1)^g}{(2g)!} \int_0^1 dt \prod_{i=0}^{g-1} (i^2 - t^2), \]

which can easily be evaluated dimension by dimension. However, use of the coefficients, \( A_i^g \), allows\(^6\) the integral to be done, rather trivially, for all \( d \), yielding the compact expression,

\[ C_{2g} = \frac{4(-1)^g}{(2g)!} \sum_{j=1}^{g} (-1)^j \int_0^1 dt \, t^{2j} A_{j-1}^g \]
\[ = \frac{4(-1)^g}{(2g)!} \sum_{j=1}^{g} (-1)^j \frac{A_{j-1}^g}{2j + 1}. \]

(40)

which allows hand calculation, for smallish dimensions, more readily than (39).

The equality of (39) and (40) entails the sum rule involving central factorials and Bernoulli numbers,

\[ \sum_{i=1}^{g} (-1)^i \frac{A_{i-1}^g}{2i + 1} = \frac{g(2g - 1)}{2} \sum_{i=1}^{g-1} \frac{1}{2i + 1} B_i^g B_{2i+2}. \]

6. Comments

The calculational methods used in this work have the significant advantage that the degeneracies, which can become quite involved, do not appear and therefore need not be expanded. The calculation of [19] uses the explicit forms of these degeneracies leading to a traditional sum of derivatives of Hurwitz \( \zeta \)-functions, which have still to be numerically evaluated\(^7\).

Another feature of the present technique is that it allows the covering, \( n \), to be taken continuous. It also allows an explicit treatment of the infinite \( n \) limit.

Appendix A. Images

An image formula given in [11], and further discussed in, has the particular case, when expressed in terms of the integer covering effective action (free energy), \( A_n(\mu) \),

\[ A_n(\mu) = \sum_{i=0}^{n-1} A_i^1(\mu \pm \frac{s}{n}), \quad n \in \mathbb{N}, \]

which can be used to check the numerical work or to obtain the left-hand side from quantities on the ordinary sphere (with a flux). It could be termed a replica relation. The equivalence of the \( \pm \) signs is a consequence of periodicity.

It is interesting to spell this out explicitly, the easiest case being dimension three (the only odd one considered in [19]). For this once, I use the explicit degeneracies.

Things simplify on the full sphere, \( q = 1 = n \). I define, just for algebraic convenience, the \( \zeta \)-function occurring in (7) in three dimensions,

\(^6\) Relatedly, the coefficients occur when expanding the degeneracy. The connection with the direct \( \zeta \)-function and hyperbolic cylinder approaches could then be made.

\(^7\) The degeneracies are coded into the expression (8) and would emerge, in one form or another, on developing the integral using a contour approach into \( \zeta \)-functions.
\[
\zeta(s, a) = \frac{1}{(a + 1 \cdot m)^2 - \frac{1}{4}} = \frac{1}{2} \sum_{m=0}^{\infty} \left(\frac{m + 2}{(a + m)^2 - \frac{1}{4}}\right)^2.
\]

(A.2)

In odd dimensions there is no multiplicative anomaly, and, factorising the eigenvalue, we reach the auxiliary, or surrogate, \(\zeta\)-function,

\[
\zeta_q(s, w) = \frac{1}{2} \sum_{m=0}^{\infty} \frac{(m + 1)(m + 2)}{(w + m)^2},
\]

where \(w = a \pm 1/2\) and \(a\) takes the two values, according to (7), \(a = 1 + |\mu|\), and \(a = 2 - |\mu|\).

Hence, to spell things out, defining \(\mu = \frac{-1}{2}\), \(w\) has the four values

\[
w^{(p)} = p \pm \mu, \quad p = 1, 2,
\]

showing the symmetry under \(\mu \rightarrow -\mu\) so that one need calculate only for the range \(0 < \mu < 1/2\).

As is usual in these circumstances the degeneracy is expanded,

\[(n + 1)(n + 2) = (w + n)^2 + e(w + n) + f.
\]

Hence, \(e = 3 - 2w\) and \(f = 2 - 3w + w^2\).

The auxiliary \(\zeta\)-function, (A.3) then becomes

\[
\frac{1}{2}(\zeta_q(s - 2, w) + e \zeta_q(s - 1, w) + f \zeta_q(s, w))
\]

in terms of the Hurwitz \(\zeta\)-function, as in many earlier works.

The theorem, [26], is that the derivative at \(s = 0\) of (A.2) is

\[
\zeta'(0, a) = \frac{1}{2}(\zeta_q(-2, w) + e(w) \zeta_q(-1, w) + f(w) \zeta_q(0, w)).
\]

(A.5)

This is the same calculation as in [19], only the mode structure and metric are different\(^{10}\).

An easy case is \(\mu = 1/2\) for which the coefficients have the values,

\[
w = 2, \quad 1, \quad 2, \quad 1
\]

\[
e = -1, \quad 1, \quad -1, \quad 1
\]

\[
f = 0, \quad 0, \quad 0, \quad 0,
\]

and the corresponding total auxiliary \(\zeta\)-function, (A.5), is,

\[
\zeta_q(s - 2, 2) - \zeta_q(s - 1, 2) + \zeta_q(s - 2, 1) + \zeta_q(s - 1, 1)
\]

\[
= \zeta_q(s - 2) - \zeta_q(s - 1) + \zeta_q(s - 2) + \zeta_q(s - 1)
\]

\[
= 2\zeta_q(s - 2)
\]

where \(\zeta_q(s)\) is the Riemann \(\zeta\)-function.

Hence for (7)

\[
\mathcal{A}_q(1/2) = \zeta'(0, 1/2) = 4\zeta'(-2) = \frac{\zeta(3)}{2\pi^2}.
\]
The values of $A_n(0)$ are old and standard. For example $A_0(0) = \log \frac{2}{4} - \frac{\zeta(3)}{8\pi^2}$ and $A_2(0) = \log \frac{2}{4} + \frac{\zeta(3)}{8\pi^2}$. Therefore,

$$A_2(0) = A_0(0) + A_1(1/2),$$

which is the simplest example of the image relation, (A.1). It can also be derived from the integral (8) using (hyperbolic) trigonometric relations. Further remarks can be found in section 4.

To work from the right-hand side of (A.1) to the left, the values of $A_2(w)$ are required and so I return to the expression (A.5) which is treated numerically. The total $\zeta$-function derivative is

$$\zeta'(0, \mu) = \frac{1}{2} \sum_{p, \pm} (\chi_{\mu}(\pm 2, w_{\mu}^{(\pm)}) + e(w_{\mu}^{(\pm)}) \chi_{\mu}^{-1}(1, w_{\mu}^{(\pm)}) + f(w_{\mu}^{(\pm)}) \zeta_{\mu}^{(0)}(0, w_{\mu}^{(\pm)})) \quad (A.6)$$

For the final term the standard result,

$$\zeta'(0, w) = \log(\Gamma(w)) \sqrt{2\pi}.$$

can be used. The other two derivatives can also be related to Gamma-type functions but it is probably best to treat them numerically, for example from asymptotic expansions (e.g. [5]). Of course if just a number is wanted, the integral form, (8), is far easier, but alternative methods are handy checks and the two methods do agree. This is the end of this specific example of images.

Although the image relation, (A.1), is fairly general, in the present spherical case it follows as a result of a ‘transformation’ property of the $\zeta$-function, (7), which is, written symbolically,

$$\zeta(s, \mu, q|\omega) = \sum_{i=0}^{n-1} \zeta(s, \mu + \sum_{i=0}^{n} \omega_i q|\omega) \quad (A.7)$$

$\omega/n$ stands for the set of $d$-degrees $(\omega_1/n_1, \ldots, \omega_d/n_d)$ and

$$\sum t \omega_i \equiv \frac{t_1 \omega_1}{n_1} + \ldots + \frac{t_d \omega_d}{n_d}. \quad (A.8)$$

I give the expression in generality as it might be useful for later investigations. For present application, $\omega = 1$ and all the $n_i$ are equal to 1, except $n_1 = n$, the covering integer, so that all the $t_i$ are zero except $t_1 \equiv t$. There is, therefore, only one term in the sum in (A.8).

The image relation, (A.7), is derived by Barnes, [1], for the $\zeta$-function

$$\zeta_{\mu}(s, a|\omega) = \sum_{m=0}^{\infty} \frac{1}{(a + m \omega)^s}, \quad \text{Re } s > d \quad (A.9)$$

by the simple expedient of introducing residue classes mod $n$. This can be applied to (7) and the sum in (A.7) is over these classes.

**Appendix B. Formal expressions. The Barnes Gamma function**

In [26] the auxiliary $\zeta$-function method was derived and applied to general $\zeta$-functions of the type (7). Factorising the denominators in (7) leads to a sum of Barnes $\zeta$-functions (A.9), with four values for the parameter $a$,

$$a_p^{(\pm)} = \frac{d + q}{2} \pm p \pm p$$

$$= b \pm p - p, \quad p = 0, 1,$$
so that $a^{(\pm)}_i = d^{(\pm)}_0 - 1$.

The auxiliary $\zeta$-function is then,

\[ \sum_{p=0,1} \zeta(s, a^{(\pm)}_p | \omega), \]

where the $d$-degrees are $\omega = (q, 1)$, and the derivative of the original $\zeta$-function, (7), reads,

\[ \frac{1}{2} \zeta'(0, q | \omega) = \sum_{p=0,1} \zeta'(0, a^{(\pm)}_p | \omega) \]

\[ = \log \frac{1}{\rho^2(\omega)} \prod_{p=0,1} \Gamma(d^{(\pm)}_p | \omega) \]

\[ = \log \prod_{p=0,1} \Gamma_d(a^{(\pm)}_0 | \omega) \Gamma_{d+1}(a^{(\pm)}_0 | \omega) \]

\[ = \log \frac{\Gamma_{d+1}(a^{(\pm)}_0 - 1 | \omega) \Gamma_{d+1}(a^{(\pm)}_0 + 1 | \omega)}{\Gamma_{d+1}(a^{(\pm)}_0 - 1 | \omega) \Gamma_{d+1}(a^{(\pm)}_0 + 1 | \omega)}. \]

In general, this presents no advantage for actual computation. However for the full sphere, $(q = 1)$, it can be taken further to elementary functions.

Dropping the dependence on $\omega = 1$,

\[ \frac{1}{2} \zeta'(0, \mu, 1) = \log \frac{\Gamma_{d+1}(b + \pi - 1)}{\Gamma_{d+1}(-b + \pi + 1)} - \log \frac{\Gamma_{d+1}(b + \pi + 1)}{\Gamma_{d+1}(-b + \pi - 1)}. \]

which can be evaluated in terms of standard functions if $d$ is odd. The details are given in [24] and [8] with the result,

\[ \frac{1}{2} \zeta'(0, \mu, 1) = \log \text{Sin}_{d+1}(b + \pi + 1) - \log \text{Sin}_{d+1}(b - \pi - 1) \]

\[ = -\frac{1}{d!} \int_{b-\pi-1}^{b+\pi+1} dz B^{d+1}_d(z) \pi \cot \pi z. \]

$\text{Sin}_d$ is Kurokawa’s generalised sine function and the Bernoulli polynomial has the product form,

\[ B^{d+1}_d(z) = (x-1)(x-2) \cdots (x-d). \]

To recapitulate, (B.3) gives another, more analytical, formula for the effective action (on a full sphere with flux) again showing the explicit dependence on $\mu$. For example, the $\mu$ derivative could be found analytically as a function of $\mu$ \(^{11}\).

Equation (B.3) provides another means of numerically computing the effective action on the full sphere and the values agree with the previous methods. Unfortunately, it again applies only for odd dimensions.

Although these relations appertain just to the full sphere, they can be extended by images to integer coverings.

Another Gamma function form, that holds for all dimensions, can be found detailed in [24] (with references) and makes use of Barnes’ theory of the multiple $\psi$-function, $\psi^{(d)}(z)$ which is the $p$th logarithmic derivative of the multiple $\Gamma$-function, $\Gamma_d$. Actually, the only one required is $\psi^{(d)}(z) \equiv \psi_d(z)$ since by integration, trivially, the desired ratio in (B.2) is,

\[ ^{11} \text{Alternatively, this could be derived directly from the Barnes } \zeta\text{-functions and then integrated to give (B.3).} \]
\[
\log \frac{\Gamma_d(z_2)}{\Gamma_d(z_1)} = \int_{z_1}^{z_2} \psi_d(z) \\
= \frac{(-1)^{d-1}}{(d-1)!} \int_{z_1}^{z_2} dz \left( B_{d-1}^{(0)}(z) \psi(z) + Q_d(z) \right)
\]

where \( \psi(z) \) is the ordinary \( \psi \)-function, if \( \omega = 1 \), and \( Q \) is an easily obtained polynomial. This equation results from recursion relations.

Since \( \psi \) is numerically available, the effective action can be calculated this way for even, as well as for odd, dimensions. I do not pursue this at this time.

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