Prediction of random variables by excursion metric projections

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Abstract

We use the concept of excursions for the prediction of random variables without any moment existence assumptions. To do so, an excursion metric on the space of random variables is defined which appears to be a kind of a weighted $L^1$-distance. Using equivalent forms of this metric and the specific choice of excursion levels, we formulate the prediction problem as a minimization of a certain target functional which involves the excursion metric. Existence of the solution and weak consistency of the predictor are discussed. An application to the extrapolation of stationary heavy-tailed random functions illustrates the use of the aforementioned theory. Numerical experiments with the prediction of Gaussian, $\alpha$-stable and further heavy–tailed time series round up the paper.

Keywords: extrapolation, (linear) prediction, forecasting, excursion, level set, Gini metric, stationary random field, $\alpha$–stable random function, heavy tails, time series, statistical learning.

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1 Introduction

Let $Y : \Omega \to \mathbb{R}$ be a square integrable random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{G} \subset \mathcal{F}$ be a sub–$\sigma$–algebra generated by a family of random variables $\{Z_j\}$ which are observable. The classical $L^2$–theory of prediction of random variables states that the best unbiased predictor of $Y$ with respect to $\mathcal{G}$ is given by the conditional expectation $E(Y|\mathcal{G})$ which is an orthogonal $L^2$–projection of $Y$ onto the space of $\mathcal{G}$–measurable square integrable random variables. But as far as $Y$ has no finite moments, no unified widely accepted prediction theory exists, to the best of our knowledge. Our paper is an attempt to create such theory which also applies to the finite variance case. Its main idea is the following. Let $u \in \mathbb{R}$ be an excursion level chosen according to a finite measure $m(\cdot)$ on $\mathbb{R}$. For any two random variables $Y_1, Y_2 : \Omega \to \mathbb{R}$ introduce the quantity

$$E_m(Y_1, Y_2) := E \int_{\mathbb{R}} \mathbb{1}(\{Y_1 > u\} \triangle \{Y_2 > u\}) \, m(du) = \int_{\mathbb{R}} \mathbb{P}(\{Y_1 > u\} \triangle \{Y_2 > u\}) \, m(du),$$

which is (by Fubini’s theorem) a $m$-weighted average probability of symmetric difference of excursions of $Y_1$ and $Y_2$ over $u \in \mathbb{R}$. Then, we say that $Z$ is a prediction of a random variable $Y$ onto the $\sigma$–algebra $\mathcal{G}$ introduced above if

$$Z = \text{argmin}_{Y_0} E_m(Y, Y_0),$$

whenever this minimum (taken over all $\mathcal{G}$-measurable random variables $Y_0$) exists and is unique. Sometimes it is also plausible to add more constraints to the geometry of our projection space saying that, additionally to $\mathcal{G}$-measurability, $Y_0 \overset{d}{=} Y$ (here, $d$ means the equality in law) or that $Y_0$ is a linear combination of $\{Z_j\}$. Apparently, the above solution $Z$, its existence and uniqueness may heavily depend on the choice of measure $m(\cdot)$. A natural candidate for this would be the distribution of $Y$ as we explain it later. As it is shown in

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Definition 1. Let $E_m$ be a metric on the space of random variables whenever the distribution function of $m(\cdot)$ is strictly increasing.

The intuition behind the use of the new metric is the following. Assume that a stationary heavy–tailed time series $\{Y_t, t \in \mathbb{R}\}$ is observed at locations $t_1, \ldots, t_n$ in a compact window $W \subset \mathbb{R}$. As proposed in [4], the linear predictor $\hat{Y}_t = \frac{m}{n} \sum_{j=1}^{n} \lambda_j Y_t, t \notin \{t_1, \ldots, t_n\}$, is a minimizer of the functional

$$
\int_{\mathbb{R}} E [v_1 (A_Y(u) \Delta A^{\hat{\varphi}}(u))] m(du) = \int_{W} E_m(Y_t, \hat{Y}_t) \, dt
$$

with respect to the choice of weights $\lambda_1, \ldots, \lambda_n$ subject to the constraint $\hat{Y}_t \sim Y_t$. The above equality holds by Fubini's theorem, whereas the left hand side term is the mean length of the symmetric difference of excursion sets $A_Y(u) := \{t \in W : Y_t > u\}$ and $A^{\hat{\varphi}}(u) := \{t \in W : \hat{Y}_t > u\}$ averaged over the levels $u \in \mathbb{R}$ picked up according to the measure $m(\cdot)$. The term $E [v_1 (A_Y(u) \Delta A^{\hat{\varphi}}(u))]$ is called error-in-measure and quantifies the prediction error measured by the symmetric difference of excursions. Here and in what follows, $\Delta \cdot := \max(\cdot - \cdot)$ is the Lebesgue measure on $\mathbb{R}$. In view of the said above, $E_m$ will be named the excursion metric. In the previous literature, the minimization of probability metrics was used mainly in context of optimal mass transportation and parameter inference, see e.g. [13].

The paper is organised as follows: in Section 2 the properties of $E_m$ are studied. It is shown that $E_m$ coincides with the so–called separation (pseudo) metric [16] whenever $m$ is a probability measure. The maximal value attained by $E_m$ with respect to the choice of measure $m$ as well as implications of the choice $m = P_Y$ are given in Section 3. Restricted to the space of random variables $Y$ with the same absolutely continuous distribution $F$, the metric $E_F$ turns to be distribution–free depending only on bivariate copulas. We call this metric (in analogy to Gini coefficient from econometrics [8] [18] a Gini metric. It properties are investigated within the same section. The excursion metric with $m = P_Y$ is applied to the prediction of random variable $Y$ in Section 4. There, we give three possible forms of minimization problems leading to such forecasting. Existence of the solution and consistency of the predictor are discussed in Section 5. A special case of extrapolating heavy–tailed time series is considered in Section 6. Numerical examples predicting Gaussian, $\alpha$–stable and autoregressive heavy–tailed stationary time series follow in Section 7.

2 Excursion metric and its properties

Let $L^0(\Omega, \mathcal{F}, \mathbb{P})$ be the set of all real–valued random variables define on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

We introduce the excursion (pseudo)metric mentioned in Section 1 in a slightly different (but equivalent) form:

**Definition 1.** Let $m$ be a finite non-negative measure on $\mathbb{R}$, and $Y_1, Y_2 \in L^0(\Omega, \mathcal{F}, \mathbb{P})$. The excursion metric $E_m$ is given by

$$
E_m(Y_1, Y_2) := \int_{\mathbb{R}} (\mathbb{P}(Y_1 > u) + \mathbb{P}(Y_2 > u) - 2\mathbb{P}(Y_1 > u, Y_2 > u)) m(du). \tag{1}
$$

In order to understand when functional $E_m$ is indeed a metric on the space $L^0(\Omega, \mathcal{F}, \mathbb{P})$, we present several equivalent forms of (1).

In the sequel, we will need the distribution function of measure $m$ given by

$$
F_U(x) := \int_{-\infty}^{x} m(du), x \in \mathbb{R}, \quad F_U(x–) = \lim_{y \to x–} F_U(y),
$$

and the notation

$$
\Delta Y_1, Y_2(u) := \mathbb{P}(Y_1 > u) + \mathbb{P}(Y_2 > u) - 2\mathbb{P}(Y_1 > u, Y_2 > u).
$$

It clearly holds $E_m(Y_1, Y_2) = \int_{\mathbb{R}} \Delta Y_1, Y_2(u) m(du)$. Denote by $Y_1 \lor Y_2$ ($Y_1 \land Y_2$) the maximum (minimum, resp.) of the random variables $Y_1$ and $Y_2$. 

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Remark 1. Let $F_1$ and $F_2$ be the distribution functions of $Y_1$ and $Y_2$, respectively, and $C$ be the copula of $(Y_1, Y_2)$. Then one writes
\[
\Delta_{Y_1,Y_2}(u) = \mathbb{P}(Y_1 \vee Y_2 > u) - \mathbb{P}(Y_1 \wedge Y_2 > u) = \mathbb{P}(Y_1 \wedge Y_2 \leq u) - \mathbb{P}(Y_1 \vee Y_2 \leq u) = F_1(u) + F_2(u) - 2F_1(u)F_2(u),
\]
where the last relation follows from Sklar’s theorem \[^{15}\]. Moreover, it follows
\[
\Delta_{Y_1,Y_2}(u) = \frac{1}{2} \mathbb{P}(Y_1 > u) + \frac{1}{2} \mathbb{P}(Y_2 > u) - \mathbb{P}(Y_1 > u, Y_2 > u) + \frac{1}{2} \mathbb{P}(Y_1 \leq u) + \frac{1}{2} \mathbb{P}(Y_2 \leq u) - \mathbb{P}(Y_1 \leq u, Y_2 \leq u)
\]
Thus, it follows the relation $0 \leq E_m(Y_1, Y_2) \leq m(\mathbb{R})$.

Without loss of generality, we may thus divide both sides of the last inequality by $m(\mathbb{R})$ and consider $m$ to be a probability measure which yields $0 \leq E_m(Y_1, Y_2) \leq 1$. Let $U$ be a random variable with probability law $m$ which is independent of $Y_1, Y_2$. It can be interpreted as a random excursion level which we choose to build the metric $E_m$.

Lemma 2.1. Let $m$ be a probability measure on $\mathbb{R}$ with c.d.f. $F_U(x) = \int_{-\infty}^{x} m(dy)$, $x \in \mathbb{R}$. Then
\[
E_m(Y_1, Y_2) = E|F_U(Y_2) - F_U(Y_1)|.
\]
Proof. We have from relation \[^{2}\] that
\[
E_m(Y_1, Y_2) = \int_{\mathbb{R}} \mathbb{E}\left( 1\{u < Y_1 \vee Y_2\} - 1\{u < Y_1 \wedge Y_2\} \right) m(du)
\]
\[
= \mathbb{E}\left( F_U(Y_1 \vee Y_2) - F_U(Y_1 \wedge Y_2) \right) = \mathbb{E}|F_U(Y_2) - F_U(Y_1)|. \]

Equation \[^{4}\] can be interpreted as a probability that $U$ separates $Y_1$ and $Y_2$:
\[
E_m(Y_1, Y_2) = \mathbb{P}(Y_1 \wedge Y_2 \leq U < Y_1 \vee Y_2).
\]

Seen this way, $E_m$ coincides with the separation (pseudo)metric introduced by M. Taylor \[^{10}\]. The first part of the following corollary is also contained in \[^{10}\] Remark 1:

Corollary 2.2. If $F_U$ is continuous then it holds $F_U(x-) = F_U(x), x \in \mathbb{R}$, and
\[
E_m(Y_1, Y_2) = \mathbb{E}|F_U(Y_2) - F_U(Y_1)|.
\]
If, in addition, $m$ is absolutely continuous with density $\psi$ then
\[
E_m(Y_1, Y_2) = \mathbb{E} \left[ \int_{1}^{Y_2} \psi(u) du \right] = \frac{1}{2} \mathbb{E} \left[ |Y_1 - Y_2| \int_{-1}^{1} \psi \left( s \frac{|Y_1 - Y_2|}{2} + \frac{Y_1 + Y_2}{2} \right) ds \right].
\]

The next theorem was also proven (under different assumptions) in \[^{10}\] Theorem 2:

Theorem 2.3. Let $X_S$ be the space of random variables with support $S \subseteq \mathbb{R}$. If $F_U$ is strictly increasing on $S$, then $E_m$ is a metric on $X_S \times X_S$.

Proof. The symmetry of $E_m$ is trivial. The triangle inequality follows for arbitrary $Y_1, Y_2, Y_3 \in X_S$ from
\[
E_m(Y_1, Y_2) = \mathbb{E}|F_U(Y_2) - F_U(Y_1)|
\]
\[
\leq \mathbb{E}|F_U(Y_2) - F_U(Y_3)| + \mathbb{E}|F_U(Y_3) - F_U(Y_1)| = E_m(Y_1, Y_3) + E_m(Y_3, Y_2).
\]
Let for some $Y_1, Y_2 \in X_S : \mathbb{E}|F_U(Y_2) - F_U(Y_1)| = 0$, then $F_U(Y_1) = F_U(Y_2)$ a.s. Since $F_U$ is strictly increasing on $S$, $\mathbb{P}(Y_1 = Y_2) = 1$. Thus, $E_m$ is a metric. \[\square\]

It can be easily shown that $E_m$ given in \[^{5}\] (with a specific choice of $U : (0,1) \to \mathbb{R}$ being a homeomorphism) coincides with the metric $d_{h,p}$ for $p = 1$, $h = U$ from \[^{17}\] which metrizes the weak convergence in the space of distribution functions.\[\square\]
3 Optimal choice of a weighting measure \( m \)

In this section, we assume that random variables \( Y_1 \) and \( Y_2 \) are absolutely continuously distributed with support \( S \subseteq \mathbb{R} \), cumulative distribution functions (c.d.f.’s) \( F_1, F_2 \) and copula \( C(\cdot, \cdot) \). Let \( D \) be the space of all probability measures on \( \mathbb{R} \).

**Theorem 3.1.** The maximum

\[
\max_{m \in D} E_m(Y_1, Y_2)
\]

is attained at a measure \( m \) with c.d.f. \( F^*_1(x) = 1\{u^* \leq x\} \), where

\[
u^* = \arg \max_{u \in \mathbb{R}} (F_1(u) + F_2(u) - 2C(F_1(u), F_2(u))).
\]

If additionally \( Y_1 \overset{d}{=} Y_2 \) then

\[
\max_{m \in D} E_m(Y_1, Y_2) = 2 \max_{x \in [0,1]} (x - C(x, x)),
\]

whereas \( u^* = F_1^{-1}(x^*) \) and \( x^* = \arg \max_{x \in [0,1]} (x - C(x, x)) \).

**Proof.** Recall from (3) that \( \Delta_{Y_1,Y_2}(u) = F_1(u) + F_2(u) - 2C(F_1(u), F_2(u)) \in [0,1] \). Therefore, there exists, not necessarily unique,

\[
u^* = \arg \max_{u \in \mathbb{R}} [F_1(u) + F_2(u) - 2C(F_1(u), F_2(u))].
\]

Then

\[
E_m(Y_1, Y_2) = \int_{\mathbb{R}} \Delta_{Y_1,Y_2}(u) m(du) \leq F_1(u^*) + F_2(u^*) - 2C(F_1(u^*), F_2(u^*)) = \int_{\mathbb{R}} \Delta_{Y_1,Y_2}(u) \delta_{u^*}(du).
\]

If \( Y_1 \overset{d}{=} Y_2 \) then \( F_1 = F_2 \) and hence

\[
2 \max_{u \in \mathbb{R}} [F_1(u) - C(F_1(u), F_1(u))] = 2 \max_{x \in [0,1]} [x - C(x, x)],
\]

since \( F_1 \) is non-decreasing.

**Corollary 3.2.** Let the random vector \((Y_1, Y_2)\) have a density function \( p : \mathbb{R}^2 \to \mathbb{R}_+ \) with \( Y_1 \overset{d}{=} Y_2 \). Then \( u^* \) satisfies the equation

\[
\int_{u^*}^{+\infty} p(y, u^*)dy = \int_{-\infty}^{u^*} p(u^*, y)dy.
\]

If \( p \) is additionally unimodal and symmetric around its mode \((\mu, \mu)\) then \( u^* = \mu, x^* = 0.5 \), and

\[
\max_{m \in D} E_m(Y_1, Y_2) = 1 - 2C(0.5, 0.5).
\]

**Proof.** It holds

\[
\max_{x \in [0,1]} (x - C(x, x)) = \max_{u \in \mathbb{R}} (F(u) - C(F(u), F(u)))
\]

\[
= \max_{u \in \mathbb{R}} \left( \int_{\mathbb{R}} \int_{-\infty}^{u} p(y_1, y_2)dy_1dy_2 - \int_{-\infty}^{u} \int_{-\infty}^{u} p(y_1, y_2)dy_1dy_2 \right)
\]

\[
= \max_{u \in \mathbb{R}} \int_{u}^{+\infty} \int_{-\infty}^{u} p(y_1, y_2)dy_1dy_2.
\]

The maximum is reached on an extremal point \( u \) such that

\[
\frac{d}{du} \int_{u}^{+\infty} \int_{-\infty}^{u} p(y_1, y_2)dy_1dy_2 = 0.
\]
or
\[ \int_{u}^{+\infty} p(y, u)dy = \int_{-\infty}^{u} p(u, y)dy. \]

Now let \( p \) be unimodal and symmetric around its mode \((\mu, \mu)\), then
\[ \int_{\mu}^{+\infty} p(y, \mu)dy = \int_{-\infty}^{\mu} p(y, \mu)dy = \frac{1}{2} \int_{\mathbb{R}} p(y, \mu)dy, \]
\[ \int_{-\infty}^{\mu} p(\mu, y)dy = \int_{\mu}^{+\infty} p(\mu, y)dy = \frac{1}{2} \int_{\mathbb{R}} p(\mu, y)dy. \]

Thus, the maximum is reached at \( u^* = \mu \).

**Remark 2.** Although the maximum of \( E_m \) is reached on \( m = \delta_{u^*} \), \( E_{\delta_{u^*}} \) is not a metric but just a pseudo metric.

Which choice of \( m \) is preferable to keep \( E_m \) a metric which is relatively easy to compute, infer and interpret? We know that \( F_U \) should be strictly increasing on support \( S \). If we take \( F_U = F_1 \) then it follows from the proof of Lemma 2.1 that
\[ E_{F_1}(Y_1, Y_2) = EF_1(Y_1 \vee Y_2) - EF_1(Y_1 \wedge Y_2) \]
\[ = 2EF_1(Y_1 \vee Y_2) - EF_1(Y_1) - EF_1(Y_2) = 2EF_1(Y_1 \vee Y_2) - EF_1(Y_2) - \frac{1}{2}, \quad (7) \]
since it holds \( F_1(Y_1) \sim U(0, 1) \) with expected value \( 1/2 \). \( E_{F_1} \) is a metric on the space of all random variables with absolutely continuous distributions on support \( S \).

If additionally \( Y_1 \overset{d}{=} Y_2 \), we get
\[ E_{F_1}(Y_1, Y_2) = 2EF_1(Y_1 \vee Y_2) - 1 = 2EF_1(Y_1) \vee Y_2(1) - 1 \]
from equation (7) since \( F_1 \) is strictly increasing. It follows from [6, p. 68] that
\[ C(x, x) = P(F_1(Y_1) \vee Y_1(2) \leq x), \quad x \in [0, 1]. \]

Using relation (3) one writes after the substitution \( x = F_1(u) \) that
\[ E_{F_1}(Y_1, Y_2) = 2 \int_{\mathbb{R}} [F_1(u) - C(F_1(u), F_1(u))] dF_1(u) = 2 \int_{0}^{1} [x - C(x, x)] dx = 1 - 2 \int_{0}^{1} C(x, x) dx. \quad (8) \]
Since the term \( 2 \int_{0}^{1} [x - C(x, x)] dx \) is equal to the Gini coefficient of the Lorenz curve
\[ \{(x, C(x, x)), \ x \in [0, 1]\} \]
in case of a convex \( \{C(x, x), \ x \in [0, 1]\} \), we come to the following definition:

**Definition 2.** Let \( L_{F_1} \) be a space of random variables with absolutely continuous c.d.f. \( F_1 \). The metric \( G = E_{F_1} \) given by
\[ G(Y_1, Y_2) = 1 - 2 \int_{0}^{1} C(x, x) dx, \quad Y_1, Y_2 \in L_{F_1}, \]
where \( C \) is the copula of \((Y_1, Y_2)\), is called a Gini metric on \( L_{F_1} \).

By definition, the Gini metric is distribution-free: it takes only the dependence structure between \( Y_1 \) and \( Y_2 \) into account, but not the marginal distribution of \( Y_1, Y_2 \).

**Remark 3.** In case \( Y_1 \overset{d}{=} Y_2 \), the maximum value \( [6] \) equals the \( L^\infty([0, 1]) \)-distance between the diagonals of \( C \) and of the upper Fréchet–Hoeffding bound \( M_2(x, y) = \min\{x, y\}, \ x, y \in [0, 1] \), cf. e.g. [6, Theorem 1.7.3]. Namely,
\[ \sup_{m \in \mathcal{D}} E_m(Y_1, Y_2) = 2\|M_2(x, x) - C(x, x)\|_{L^\infty([0, 1])}. \quad (9) \]
Similarly, relation [8] yields \( G(Y_1, Y_2) = 2\|M_2(x, x) - C(x, x)\|_{L^1([0, 1])} \). In other words, the Gini metric \( G \) measures the \( L^1 \)-deviation of the diagonal of the copula \( C \) of \((Y_1, Y_2)\) to the diagonal of the comonotonicity copula \( M_2 \).
Lemma 3.3. It holds $0 \leq G(Y_1, Y_2) \leq 1/2$, $Y_1, Y_2 \in L_{F_1}$. The upper bound is attained whenever $Y_2 = f(Y_1)$ a.s. for some decreasing function $f$ such that $F_1(x) = 1 - F_1(f^{-1}(x))$, $x \in \mathbb{R}$.

Proof. Using the Fréchet–Hoeffding bounds [6, Theorem 1.7.3]

$$W_2(x, y) \leq C(x, y) \leq M_2(x, y), \quad x, y \in [0, 1]$$

with $W_2(x, y) = \max\{0, x + y - 1\}$ being a copula of linearly dependent random variables, one can easily calculate

$$1/4 \leq \int_0^1 C(x, x) \, dx \leq 1/2.$$  

Thus, relation [8] yields the bounds $0 \leq G(Y_1, Y_2) \leq 1/2$, whereas the upper bound is attained by [6, Theorem 2.5.13 (d)] whenever there exists a decreasing function $f$ such that $Y_2 = f(Y_1)$ a.s. Since we assume $Y_1 \overset{d}{=} Y_2$ here, it can be only the case if

$$F_1(x) = \mathbb{P}(Y_1 \leq x) = \mathbb{P}(f(Y_2) \leq x) = \mathbb{P}(Y_2 \geq f^{-1}(x)) = 1 - \mathbb{P}(Y_2 < f^{-1}(x)) = 1 - F_1(f^{-1}(x))$$

for any real $x$. In the latter relation, we used the absolute continuity of the distribution of $Y_2$. \hfill \Box

Example 3.4. The upper bound $1/2$ is attained by Gini metric in Lemma 3.3 if $Y_1 + Y_2 = \mu$ a.s. for some $\mu \in \mathbb{R}$ and the distribution of $Y_1$ is symmetric about $\mu$: $F_1(x) = 1 - F(\mu - x), x \in \mathbb{R}$. To see this, just take $f(x) = \mu - x$ in Lemma 3.3.

Example 3.5. The Gini distance between stochastically independent random variables $Y_1, Y_2 \in L_{F_1}$ is equal to $1/3$, since in this case $C(x, y) = xy$, $x, y \in [0, 1]$, and thus $G(Y_1, Y_2) = 1 - 2 \int_0^1 x^2 \, dx = 1/3$.

To summarize, the choice $m = \mathbb{P}_{Y_1}$ for an absolutely continuous law $\mathbb{P}_{Y_1}$ seems natural to make $E_m$ a metric. Gini metric $G$ is easy to calculate and distribution-free (depending only on the copula $C$ of $(Y_1, Y_2)$, cf. relation (3)) when $\mathbb{P}_{Y_1} = \mathbb{P}_{Y_2}$. Informally speaking, the excursion levels are chosen here according to the same law as $\mathbb{P}_{Y_1} = \mathbb{P}_{Y_2}$ which makes the corresponding excursion sets non-empty (with positive probability) and representative for $Y_1, Y_2$.

4 Prediction of random variables

In this section, we discuss the prediction of a value of random variable $X$ with continuous distribution function $F_X$ based on the set $\mathcal{X}_n := (X_1, \ldots, X_n)$ of realizations of $X$ via the excursion metric metric $E_{F_X}$. Namely, we propose a predictor $\hat{X}_\lambda := g(\lambda, \mathcal{X}_n)$, where $\lambda = (\lambda_1, \ldots, \lambda_d) \in \Lambda \subset \mathbb{R}^d$ is deterministic and $g : \mathbb{R}^n \times \Lambda \to \mathbb{R}$, $d, n \in \mathbb{N}$, is a continuous measurable function such that the excursion metric is minimal:

$$E_{F_X}(X, \hat{X}_\lambda) \to \min_{\lambda \in \Lambda}. \quad \text{(10)}$$

Here, the set of admissible parameters $\Lambda$ as well as the analytic form of $g(\lambda, \mathcal{X}_n)$ depend on the law $\mathbb{P}_X$. For instance, the choice $g(\lambda, \mathcal{X}_n) = \sum_{j=1}^n \lambda_j X_j$ makes sense for infinitely divisible laws of $X$, whereas $g(\lambda, \mathcal{X}_n) = \max_{j=1, \ldots, n} \lambda_j X_j$ might be a better choice for max-stable $X$. In both cases, we assume $d = n$. The set $\Lambda$ may incorporate additional constraints onto $\mathbb{R}_\hat{X}$, for instance, $\hat{X}_\lambda \overset{d}{=} X$. Since $g$ and $F_X$ are continuous, the constraint $\hat{X}_\lambda \overset{d}{=} X$ is equivalent to $F_X(\hat{X}_\lambda) \overset{d}{=} F_X(X) \overset{d}{=} U$, where $U \sim U(0, 1)$. Under additional assumptions onto the joint probability law of $(X, \mathcal{X}_n)$ and onto $g$, the set $\Lambda_g := \{ \lambda \in \mathbb{R}^d : F_X(g(\lambda, \mathcal{X}_n)) \overset{d}{=} U \}$ is a manifold in $\mathbb{R}^d$. Unfortunately, the analytic form of $\Lambda_g$ can be found only in specific cases when the pre-knowledge of the distribution of $(X, \mathcal{X}_n)$ is available, such as in the Gaussian or $\alpha$-stable case, cf. [4].

Should our prediction be law-preserving (i.e., $\hat{X}_\lambda \overset{d}{=} X$), it holds $E_{F_X} = G$, and the optimization problem $G(X, \hat{X}_\lambda) \to \min_{\lambda \in \Lambda}$ with $\Lambda = \Lambda_g$ rewrites using (8) as

$$\int_0^1 C_{X, \hat{X}_\lambda}(x, x) \, dx \to \max_{\lambda \in \Lambda_g}. \quad \text{(11)}$$
where \( C_{X,\hat{X}_\lambda} \) is the copula of \((X, \hat{X}_\lambda)\). In order to avoid a tricky statistic assessment of copulas, we use however the following forms of prediction which are motivated by (7) and require only expectations to be inferred into:

**Definition 3.** The excursion predictor \( \hat{X}_\lambda \) is given by \( \hat{X}_\lambda = g(\hat{\lambda}, X_n) \), where

\[
\hat{\lambda} := \arg\min_{\lambda \in \Lambda} \left[ 2EF_X(X \lor \hat{X}_\lambda) - EF_X(\hat{X}_\lambda) \right] \tag{12}
\]

in general case, or

\[
\hat{\lambda} := \arg\min_{\lambda \in \Lambda} \left\{ EF_X(X \lor \hat{X}_\lambda) \right\} \tag{13}
\]

in case of law-preserving prediction \( \hat{X}_\lambda \overset{d}{=} X \).

Let us consider the law-preserving case in more detail. If the analytic form of \( \Lambda \) is given explicitly but \( \Lambda_g \) is hardly available, we modify the minimization functional in (13) by adding a term which penalizes a difference between the law of \( Y_1 = F_X(\hat{X}_\lambda) \) and \( Y_2 \sim U(0, 1) \):

\[
\hat{\lambda} := \arg\min_{\lambda \in \Lambda} \left\{ 2EF_X(X \lor \hat{X}_\lambda) - EF_X(\hat{X}_\lambda) + \gamma \rho^2(F_{Y_1}, F_{Y_2}) \right\}, \tag{14}
\]

where \( \gamma > 0 \) is a penalty weight and \( \rho \) is an arbitrary (but handy) metric on the space of continuous distribution functions of random variables. For simplicity reasons, we use the 2-Wasserstein distance

\[
\rho(F_1, F_2) = \left( \int_0^1 [F_1^{-1}(x) - F_2^{-1}(x)]^2 dx \right)^{1/2}
\]

between two c.d.f.'s \( F_1 \) and \( F_2 \) with quantile functions \( F_1^{-1} \) and \( F_2^{-1} \), respectively.

In the case \( Y_2 \sim U(0, 1) \) we have \( F_{Y_2}^{-1}(x) = x, \ x \in [0, 1] \). Hence, the squared 2-Wasserstein distance equals

\[
\rho^2(F_{Y_1}, F_{Y_2}) = \int_0^1 x^2 dx + \int_0^1 y^2 dF_{Y_1}(y) - \int_0^1 y dF_{Y_2}(y) = \frac{1}{3} + EY_1^2 - E[Y_1 \lor Y], \tag{15}
\]

where \( Y \) is an independent copy of \( Y_1 \). The latter relation holds since \( P(Y_1 \lor Y \leq y) = F_{Y_1}^2(y), \ y \in \mathbb{R} \). Due to \( Y_1 = F_X(\hat{X}_\lambda) \in [0, 1] \) a.s., it holds \( E[Y_1 \lor Y] \geq EY_1^2 \). Thus, the minimization problem (14) rewrites in an equivalent form:

\[
\hat{\lambda} := \arg\min_{\lambda \in \Lambda} \left\{ 2EF_X(X \lor \hat{X}_\lambda) - EF_X(\hat{X}_\lambda) + \gamma \left[ EF_X^2(\hat{X}_\lambda) - EF_X(\hat{X}_\lambda) \right] \right\}, \tag{16}
\]

where \( Y \) is an independent copy of \( F_X(\hat{X}_\lambda) \). Compared with formulation (13), the new prediction method (16) does not require an explicit knowledge of \( \Lambda_g \), but it realizes the constraint \( X_\lambda \overset{d}{=} X \) only in approximation form: \( \rho(F_X, F_{\hat{X}_\lambda}) \leq \varepsilon \) for some small \( \varepsilon > 0 \).

Sometimes it is advantageous to use the integration by parts in (15) and write

\[
\rho^2(F_{Y_1}, F_{Y_2}) = \frac{1}{3} + \int_0^1 F_{Y_1}(y) [F_{Y_1}(y) - 2y] \ dy \tag{17}
\]

which allows for an equivalent reformulation

\[
\hat{\lambda} := \arg\min_{\lambda \in \Lambda} \left\{ 2EF_X(X \lor \hat{X}_\lambda) - EF_X(\hat{X}_\lambda) + \gamma \int_0^1 F_{Y_1}(y) [F_{Y_1}(y) - 2y] \ dy \right\} \tag{18}
\]

of the problem (16).

To summarize, prediction approach (12) will be used for unconstrained prediction of a random variable \( X \) with an absolutely continuous c.d.f. \( F_X \) based on its realizations \( X_n = (X_1, \ldots, X_n) \). For the law-preserving prediction, approaches (13), (16) or (18) will be used depending on whether the restrained parameter set \( \Lambda_g \) is given explicitly or not.
5 Existence of a solution

A solution to the above optimization problems exists on compact parametric sets due to the continuity of the corresponding target functionals:

**Theorem 5.1.** Let the joint distribution of the random vector \((X, X_n)\) be absolutely continuous with respect to the Lebesgue measure on \(\mathbb{R}^{n+1}\). Introduce the following assumptions:

(i) \(\Lambda\) (or \(\Lambda_g\), respectively) is a compact in \(\mathbb{R}^n\).

(ii) The copula diagonal \(C_{X, X_n}(x, x)\) of \((X, X_n)\) is continuous on \(\Lambda\) (or \(\Lambda_g\), respectively) uniformly w.r.t. \(x \in [0, 1]\).

(iii) For each \(\lambda \in \Lambda\), \(\hat{X}_{\lambda}\) has an absolutely continuous distribution with density \(p_{\hat{X}_{\lambda}}\) such that the map \(p_{\hat{X}_{\lambda}} : \Lambda \to L^1(\mathbb{R})\) is continuous on \(\Lambda\) w.r.t. the \(L^1\)-norm.

If the conditions (i)-(ii) hold then there exists a solution to the problems (12). If the conditions (i)-(iii) hold then there exists a solution to the problems (12) and (18).

**Proof.** To show the existence of a solution, it is sufficient to assume (i) and show that the target functional \(\Phi(\lambda)\) to be maximized or minimized is continuous on \(\Lambda_g\) or \(\Lambda\).

In case of the problem (13) we have

\[
\Phi(\lambda) = \int_0^1 C_{X, \hat{X}_{\lambda}}(x, x) \, dx, \quad \lambda \in \Lambda_g
\]

with regard to (11). Condition (ii) ensures the continuity of \(\Phi\) on \(\Lambda_g\) in view of the corresponding theorem for the continuity of integrals with parameters.

For the problem (12), we have

\[
\Phi(\lambda) = 2 - 2 \int_0^1 C_{X, \hat{X}_{\lambda}}(x, x) \, dx - EF_X(\hat{X}_{\lambda}), \quad \lambda \in \Lambda
\]

together with

\[
EF_X(\hat{X}_{\lambda}) = \int_0^1 P(F_X(\hat{X}_{\lambda}) > y) \, dy = 1 - \int_0^1 F_{\hat{X}_{\lambda}}(F_X^{-1}(y)) \, dy,
\]

where \(F_{\hat{X}_{\lambda}}\) is the c.d.f. of the predictor \(\hat{X}_{\lambda}\). The latter relation holds since the c.d.f. \(F_X\) is strictly increasing on \(S = \text{supp}(X)\). Due to the absolute continuity of the distribution of \(\hat{X}_{\lambda}\), the function \(F_{\hat{X}_{\lambda}}(F_X^{-1}(y))\) is continuous on \([0, 1]\) for each \(\lambda \in \Lambda\). Moreover, for any sequence \(\{\lambda_k\} \subset \Lambda\) with \(\lambda_k \to \lambda_0 \in \Lambda\) as \(k \to \infty\) we have

\[
\sup_{x \in S} \left| F_{\hat{X}_{\lambda_k}}(x) - F_{\hat{X}_{\lambda_0}}(x) \right| = \sup_{x \in \mathbb{R}} \left| \int_{-\infty}^{x} \left[ p_{\hat{X}_{\lambda_k}}(y) - p_{\hat{X}_{\lambda_0}}(y) \right] \, dy \right| \leq \int_{\mathbb{R}} \left| p_{\hat{X}_{\lambda_k}}(y) - p_{\hat{X}_{\lambda_0}}(y) \right| \, dy \to 0
\]

as \(k \to \infty\) by assumption (iii) which means the continuity of \(F_{\hat{X}_{\lambda}}(F_X^{-1}(y))\) on \(\Lambda\) uniformly w.r.t. \(y \in [0, 1]\).

The application of the theorem on the continuity of integrals with parameters finishes the proof.

In the problem (18), the target functional rewrites

\[
\Phi(\lambda) = 2 - 2 \int_0^1 C_{X, \hat{X}_{\lambda}}(x, x) \, dx - EF_X(\hat{X}_{\lambda}) + \gamma \int_0^1 F_{Y_1}(y) \left[ 2y - F_{Y_1}(y) \right] \, dy, \quad \lambda \in \Lambda.
\]

Similarly to the previous case, it is not difficult to show that the integrand \(F_{Y_1}(y) \left[ 2y - F_{Y_1}(y) \right]\) is continuous on \(\Lambda\) uniformly w.r.t. \(y \in [0, 1]\) provided that condition (iii) holds true.

**Remark 4.** Condition (iii) of Theorem 5.1 means that for any sequence \(\{\lambda_k\} \subset \Lambda\) with \(\lambda_k \to \lambda_0 \in \Lambda\) as \(k \to \infty\)

\[
d_{TV}(\hat{X}_{\lambda_k}, \hat{X}_{\lambda_0}) = \frac{1}{2} \int_{\mathbb{R}} \left| p_{\hat{X}_{\lambda_k}}(y) - p_{\hat{X}_{\lambda_0}}(y) \right| \, dy = \frac{1}{2} \left\| p_{\hat{X}_{\lambda_k}} - p_{\hat{X}_{\lambda_0}} \right\|_1 \to 0, \quad k \to \infty,
\]

where \(d_{TV}\) is the total variation distance and \(\left\| \cdot \right\|_1\) is the norm in \(L^1(\mathbb{R})\). It implies that \(\hat{X}_{\lambda_k} \to \hat{X}_{\lambda_0}\) in total variation as \(k \to \infty\).
Let us give some examples of the compacts $\Lambda_g \subset \mathbb{R}^n$. In what follows, the random vector $(X, \mathcal{X}_n)$ will have a joint $\alpha$–stable distribution for some $\alpha \in (0, 2]$, hence it is natural to consider the linear predictor $\hat{X}_\lambda = \sum_{j=1}^{n} \lambda_j X_j$.

**Example 5.2.**

1. If $(X, \mathcal{X}_n)$ is a Gaussian random vector with marginal distribution $N(\mu, \sigma^2)$ and $\Sigma$ is the covariance matrix of $\mathcal{X}_n$ then the manifold $\Lambda_g$ is an ellipsoid of dimension $n - 1$ given by

$$\Lambda_g = \left\{ \lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^n : \lambda^\top \Sigma \lambda = \sigma^2, \sum_{j=1}^{n} \lambda_j = 1 \right\}.$$

2. If $(X, \mathcal{X}_n)$ is a subgaussian random vector with stability index $\alpha \in (0, 2)$ and i.i.d. standard Gaussian components then it follows from [14] p. 80-81 that $\Lambda_g$ is a unit sphere $S^{n-1}$ in $\mathbb{R}^n$ given by

$$\Lambda_g = \left\{ \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n : \int_{S^{n-1}} |\langle \lambda, s \rangle|^\alpha \Gamma(ds) = 1 \right\}.$$

The dominated convergence theorem helps showing that $\Lambda_g$ is a closed set. For $\alpha \in [1, 2)$, rewrite the constraint in $\Lambda_g$ as

$$|\lambda| = h_K^{-1}(u), \quad u = |\lambda| \in S^{n-1},$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$ and

$$h_K(u) = \left( \int_{S^{n-1}} |\langle \lambda, s \rangle|^\alpha \Gamma(ds) \right)^{1/\alpha}$$

is the support function of a convex set $K = \Gamma^{1/\alpha}(S^{n-1})E_\alpha[-\eta, \eta]$ named $L_\alpha$-zonoid. Here $\eta$ is a random vector on $S^{n-1}$ distributed according to $\Gamma(\cdot)/\Gamma(S^{n-1})$ and $E_\alpha[-\eta, \eta]$ is the Firey $\alpha$–expectation of the random segment $[-\eta, \eta]$, cf. e.g. [12]. If $K$ is full-dimensional (which is e.g. the case if $\Gamma$ has a density w.r.t. to the surface area measure on $S^{n-1}$ which is bounded away from zero everywhere on $S^{n-1}$) we have $\inf_{u \in S^{n-1}} h_K(u) > 0$ and thus $\Lambda_g$ is bounded, hence a compact.

**Example 5.3.** Show that conditions (i)-(iii) of Theorem 5.1 are satisfied if $(X, \mathcal{X}_n)$ is a Gaussian random vector with marginal distribution $N(0, 1)$. For any $\lambda \in \Lambda_g$, it holds $\hat{X}_\lambda \sim N(0, 1)$. Condition (i) was shown in Example 5.2. Let us check condition (ii). In view of [14], the copula diagonal writes

$$C_{X, \hat{X}_\lambda}(x, x) = x^2 + \frac{1}{2\pi} \int_{0}^{\sin^{-1}(\rho_\lambda)} \exp \left( -\left( \varphi^{-1}(x) \right)^2 \frac{1 - \sin(\theta)}{\cos^2(\theta)} \right) d\theta,$$

where $\varphi^{-1}(x)$ is the quantile function of $N(0, 1)$ and $\rho_\lambda = \text{Corr} \left( X, \hat{X}_\lambda \right) = \sum_{j=1}^{n} \lambda_j \text{Cov}(X, X_j)$. Since the exponential function under the integral is nonnegative and bounded from above by one, we get

$$\left| C_{X, \hat{X}_{\lambda_1}}(x, x) - C_{X, \hat{X}_{\lambda_2}}(x, x) \right| \leq \frac{1}{2\pi} \left| \sin^{-1}(\rho_{\lambda_1}) - \sin^{-1}(\rho_{\lambda_2}) \right|, \quad \lambda_1, \lambda_2 \in \Lambda_g$$

uniformly on $x \in [0, 1]$ which shows the uniform continuity of $C_{X, \hat{X}_\lambda}(x, x)$ on $\Lambda_g$. To show (iii), let the covariance matrix $\Sigma$ of $\mathcal{X}_n$ be positive definite. Then $\hat{X}_\lambda \sim N(0, \lambda^\top \Sigma \lambda)$ where $\lambda^\top \Sigma \lambda > 0$ for all $\lambda \neq 0$, hence it has a Gaussian density which is continuous on $\mathbb{R}^n \setminus \{0\}$ in the $L^1$–norm. To see this, just use the multivariate mean value theorem for this density with respect to $\lambda$. 


The next result shows that it is sufficient to consider bounded spaces Λ or Λₐ in minimization problems \((12), (13), \text{ and } (16)\). If these spaces are additionally closed, the existence of a solution is guaranteed by Theorem 5.1. For instance, so is often the choice

\[
\Lambda = \{ \lambda \in \mathbb{R}^n : |\lambda| \leq M \} \text{ or } \Lambda = \{ \lambda \in \mathbb{R}^n_+ : |\lambda| \leq M \}
\]

for a suitable \(M > 0\). Introduce the notation

\[
\Phi_1(\lambda) := \mathbb{E}F_X(X \vee \tilde{X}_\lambda), \quad \Phi_2(\lambda) := 2\mathbb{E}F_X(X \vee \tilde{X}_\lambda) - \mathbb{E}F_X(\tilde{X}_\lambda), \quad \Phi_3(\lambda) := \Phi_2(\lambda) + \gamma \left[ \mathbb{E}F_X(\tilde{X}_\lambda) - \mathbb{E}[F_X(\tilde{X}_\lambda) \vee Y] \right]
\]

for the target functionals in minimization problems \((13), (12), (16)\) or \((18)\), respectively.

**Proposition 5.4.** Assume that there exists \(\lambda_0 \in \Lambda_0 \) or \(\Lambda\) such that \(\Phi_j(\lambda_0) < 1, \ j = 1, 2, 3\). Let \(\tilde{X}_{\lambda_k} \xrightarrow{P} +\infty\) as \(k \to \infty\) for any sequence \(\{\lambda_k\} \subset \Lambda\) such that \(|\lambda_k| \to +\infty\). Then there exists \(M > 0\) such that

\[
\min_{\lambda \in \Lambda_0} \Phi_1(\lambda) = \min_{\lambda \in \Lambda_0 : |\lambda| \leq M} \Phi_1(\lambda), \quad \min_{\lambda \in \Lambda} \Phi_j(\lambda) = \min_{\lambda \in \Lambda : |\lambda| \leq M} \Phi_j(\lambda), \ j = 2, 3.
\]

**Proof.** Sequences of random variables \(\{F_X(\tilde{X}_{\lambda_k})\}, \{F_X^2(\tilde{X}_{\lambda_k})\}, \{F_X(X \vee \tilde{X}_{\lambda_k})\}, \{F_X(\tilde{X}_{\lambda_k}) \vee Y\}\) are uniformly integrable since they are a.s. bounded by zero and one. Hence, their expectations tend to one as \(|\lambda_k| \to +\infty\) by properties of a c.d.f. Then it holds

\[
\Phi_j(\lambda_k) \to 1, \ |\lambda_k| \to +\infty, \ j = 1, 2, 3.
\]

Take \(M > 0\) such that \(\Phi_j(\lambda_k) > \Phi_j(\lambda_0)\) for all \(k\) such that \(|\lambda_k| > M, j = 1, 2, 3\). The assertion is proven. \(\square\)

**Example 5.5.** Assume that there exists a \(\lambda_0 \in \Lambda_0 \) or \(\Lambda\) such that

1. \(\tilde{X}_{\lambda_0} = X\) a.s. This is the case for some prediction functions \(g\) if \(X = X_{j_0}\) a.s., \(j_0 \in \{1, \ldots, n\}\). Then it can be easily shown that \(\Phi_1(\lambda_0) = \Phi_2(\lambda_0) = 1/2, \ \Phi_3(\lambda_0) = 1/2 - \gamma/3 < 1\) for all \(\gamma > 0\).

2. \(\tilde{X}_{\lambda_0}\) and \(X\) are stochastically independent. This can be the case if \(X, X_n\) form an \(m\)-dependent sequence with \(m < n\). Then it can be easily shown that \(\Phi_1(\lambda_0) = 2/3\) (cf. Example 3.5),

\[
\Phi_2(\lambda_0) = 1 + \int_0^1 F_{Y_1}(x)(1 - 2x)dx < 1
\]

with \(Y_1 = F_X(\tilde{X}_{\lambda_0})\), because \(\int_0^1 F_{Y_1}(x)(1 - 2x)dx = \int_0^1 (F_{Y_1}(x) - F_{Y_1}(1/2))(1 - 2x)dx < 0\), and

\[
\Phi_3(\lambda_0) \leq \Phi_2(\lambda_0) < 1.
\]

**Example 5.6.** Condition \(\tilde{X}_{\lambda_k} \xrightarrow{P} +\infty\) as \(k \to \infty\) for any sequence \(\{\lambda_k\} \subset \Lambda\) such that \(\lambda_k = (\lambda_k(1), \ldots, \lambda_k(n))\), \(|\lambda_k| \to +\infty\) is satisfied for \(\Lambda = \mathbb{R}_+^n\), \(\tilde{X}_{\lambda_k} = \sum_{j=1}^n \lambda_k(j)X_j\) or \(\tilde{X}_{\lambda_k} = \max_{j=1,\ldots,n} \lambda_k(j)X_j\) a.s. nonnegative random variables \(X_j, j = 1, \ldots, n\).

The question of uniqueness of a solution \(\lambda\) to problems \((12), (13)\) and \((16)\) cannot be resolved in such generality. As illustrated in the Gaussian case \(\mathbb{1}\), it will require further specification of the dependence structure of observations \(X_j\) within the set \(X_0\), of the statistic \(g\) and parameter set \(\Lambda\). This will be done for some specific classes of random variables in forthcoming research.
6 Excursion-based extrapolation of stationary heavy–tailed random fields

In this section, we will use the above prediction approach to extrapolate a real–valued strictly stationary ergodic random field $X = \{X(t), t \in \mathbb{R}^d\}$ with absolutely continuous (but possibly heavy–tailed) marginal distribution $F_{\theta_0}$. We assume that $F_{\theta_0}$ belongs to an appropriate parametric family of possible marginal distributions $\{F_{\theta}, \theta \in \Theta\}$, $\Theta \subseteq \mathbb{R}^k$. This ansatz can be useful, in particular, for heavy–tailed time series forecasting in insurance/finance (d = 1, compare Section 7) or in image analysis (d = 2, 3) for upsampling of low-resolution 2D and 3D gray scale images (cf. the so-called super–resolution problem [2, 9, 7, 5]).

Denote by $\mathbb{Z}_h = (h_1 \mathbb{Z}) \times \cdots \times (h_d \mathbb{Z})$ the d–dimensional grid with mesh sizes $h = (h_1, \ldots, h_d) \in (0, +\infty)^d$. Let $X$ be potentially observed at points $T_0 := W_o \cap \mathbb{Z}_h$, where $W_o \subset \mathbb{R}^d$ is a compact. The observed values form a sample $X_{T_0} := \{X(t_j), t_j \in T_0\}$.

Let us predict the value $X(t)$ at a location $t \in \mathbb{Z}_h$, $t \notin W_o$ from the knowledge of the so–called forecast sample $T_f := \{t_1, \ldots, t_n\} \neq \emptyset$, $T_f \subset \mathbb{Z}_h$. The predictor $\hat{X}_\lambda = g(\lambda, X(T_f))$ with $\lambda \in \Lambda \subseteq \mathbb{R}^n$ and $X(T_f) := (X(t_1), \ldots, X(t_n))^\top$ requires weights $\lambda$ to be a solution of minimization problems (19), (16) or (18), i.e.,

$$\hat{\lambda} = \arg\min_{\lambda \in \Lambda} \left\{ 2E \left[ F_{\theta_0}(X(t)) \lor F_{\theta_0}(\hat{X}_\lambda) \right] - 2EF_{\theta_0}(\hat{X}_\lambda) \right\},$$ (19)

$$\hat{\lambda} = \arg\min_{\lambda \in \Lambda} \left\{ 2E \left[ F_{\theta_0}(X(t)) \lor F_{\theta_0}(\hat{X}_\lambda) \right] - 2EF_{\theta_0}(\hat{X}_\lambda) + \gamma \left[ EF_{\theta_0}^2(\hat{X}_\lambda) - 2EF_{\theta_0}(\hat{X}_\lambda) \lor Y \right] \right\},$$ or (20)

$$\hat{\lambda} = \arg\min_{\lambda \in \Lambda} \left\{ 2E \left[ F_{\theta_0}(X(t)) \lor F_{\theta_0}(\hat{X}_\lambda) \right] - 2EF_{\theta_0}(\hat{X}_\lambda) + \gamma \int_0^1 F_{\theta_0}(\hat{X}_\lambda)(y) \left[ F_{\theta_0}(\hat{X}_\lambda)(y) - 2y \right] dy \right\},$$ (21)

respectively, due to strict monotonicity of $F_{\theta_0}$, where $Y$ is an independent copy of $F_{\theta_0}(\hat{X}_\lambda)$ and $F_{\theta_0}(\hat{X}_\lambda)$ is the c.d.f. of the random variable $F_{\theta_0}(\hat{X}_\lambda)$. As already mentioned in Section 4, we use

$$\hat{X}_\lambda = \lambda^\top X(T_f)$$

for infinitely divisible $X$ and

$$\hat{X}_\lambda = \max_{j=1, \ldots, n} \lambda(j) X(t_j)$$

with $\lambda = (\lambda(1), \ldots, \lambda(n))^\top$ for max–stable $X$.

The excursion predictors [13], [19] – [21] are consistent under very mild assumptions.

**Theorem 6.1.** Let the random field $X = \{X(t), t \in \mathbb{R}^d\}$ be stochastically continuous. Assume that there exists $\hat{\lambda}_k \in \Lambda_k$ such that $g(\hat{\lambda}_k, X(T_f)) = X(t_k)$ a.s. for any $k = 1, \ldots, n$ and $\min_{j=1, \ldots, n} \|t_j - t_k\|_2 \to 0$ as $n \to \infty$. Then $\hat{X}_\lambda(t) \overset{P}{\to} X(t)$ as $n \to \infty$, where $\hat{X}_\lambda$ is an excursion predictor [13], [19], [20] or [21].

**Proof.** For the method [19], let $\hat{X}_\lambda = g(\hat{\lambda}, X(T_f))$, where $\hat{\lambda} := \arg\min_{\lambda \in \Lambda} \left\{ 2E_{F_X}(X \lor \hat{X}_\lambda) - 2E_{F_X}(\hat{X}_\lambda) \right\}$. Denote by $\tilde{\lambda}_n \in \Lambda_k \subset \Lambda$ such that $g(\tilde{\lambda}_n, X(T_f)) = X(t_n)$, where $t_n = \arg\min_{j=1, \ldots, n} \|t_j - t\|_2$. Then excursion metric writes

$$E_{F_X}(\hat{X}_\lambda(t), X(t)) = E|F_X(X(t)) - F_X(\hat{X}_\lambda(t))| = \min_{\lambda \in \Lambda} \left\{ 2E_{F_X}(X(t) \lor \hat{X}_\lambda(t)) - 2E_{F_X}(\hat{X}_\lambda(t)) \right\} - \frac{1}{2} \leq 2E_{F_X}(X(t) \lor \hat{X}_{\tilde{\lambda}_n}(t)) - 2E_{F_X}(\hat{X}_{\tilde{\lambda}_n}(t)) - \frac{1}{2} = 2E_{F_X}(X(t) \lor X(t_n)) - 2E_{F_X}(X(t_n)) - \frac{1}{2} = E|F_X(X(t)) - F_X(X(t_n))|.$$ 

Sequence $\{F_X(X(t)) - F_X(X(t_n))\}_{n \geq 1}$ is obviously uniformly integrable and $F_X(X(t_n)) \overset{P}{\to} F_X(X(t))$ as $t_n \to t$. Therefore, $F_X(\hat{X}_\lambda(t)) \overset{P}{\to} F_X(X(t))$ in $L^1$–sense and, consequently, $F_X(\hat{X}_\lambda(t)) \overset{P}{\to} F_X(X(t))$. Due to the continuous mapping theorem, $\hat{X}_\lambda(t) \overset{P}{\to} X(t)$ as $n \to \infty$. 

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The excursion predictor given by (13) is consistent as well by similar arguments:

\[
E_{F_X}(\tilde{X}_\lambda(t), X(t)) = \mathbb{E}[F_X(X(t)) - F_X(\tilde{X}_\lambda(t))] = \min_{\lambda \in \Lambda} \left[ 2E_{F_X}(X(t) \vee \tilde{X}_\lambda(t)) - 1 \right] 
\]

\[
\leq 2E_{F_X}(X(t) \vee \tilde{X}_{\lambda_n}(t)) - 1 = 2E_{F_X}(X(t) \vee \tilde{X}_{\lambda_n}(t)) - 1 = \mathbb{E}[F_X(X(t)) - F_X(\tilde{X}_{\lambda_n}(t))] \to 0
\]
as \(n \to \infty\), where \(\lambda_n \in \Lambda_g\) such that \(g(\lambda_n, X(T_f)) = X(t)\) a.s.

As for the excursion predictor (23), choose a \(\gamma > 0\) and write for \(Y_1 = F_X(\tilde{X}_{\lambda}(t)), Y_2 \sim U(0, 1)\) that

\[
E_{F_X}(\tilde{X}_{\lambda}(t), X(t)) + \gamma \rho(F_{Y_1}, F_{Y_2}) = 2E_{F_X}(X(t) \vee \tilde{X}_{\lambda}(t)) - E_{F_X}(\tilde{X}_{\lambda}(t)) 
\]

\[
+ \gamma \left[ E_{F_{\hat{X}}}(\tilde{X}_{\lambda}(t)) - E(F_X(\tilde{X}_{\lambda}(t)) \vee Y) \right] + \frac{\gamma}{3} - \frac{1}{2} 
\]

\[
= \min_{\lambda \in \Lambda} \left[ 2E_{F_X}(X(t) \vee \tilde{X}_{\lambda}(t)) - E_{F_X}(\tilde{X}_{\lambda}(t)) 
\right] 
\]

\[
+ \gamma \left( \frac{1}{3} + E_{F_{\hat{X}}}(\tilde{X}_{\lambda_n}(t)) - E(F_X(\tilde{X}_{\lambda_n}(t)) \vee Y) \right) 
\]

\[
= 2E_{F_X}(X(t) \vee X(\tilde{X}_{\lambda_n})) - E_{F_X}(X(\tilde{X}_{\lambda_n})) - 1/2 
\]

\[
+ \gamma \left( \frac{1}{3} + E_{F_{\hat{X}}}(X(\tilde{X}_{\lambda_n})) - E(F_X(X(\tilde{X}_{\lambda_n})) \vee Y) \right) 
\]

\[
= \mathbb{E}[F_X(X(t)) - F_X(X(\tilde{X}_{\lambda_n}))] \to 0
\]
as \(n \to \infty\), since \(X(\tilde{X}_{\lambda_n}) \stackrel{d}{=} X(t)\).

If the parameter \(\theta_0\) is unknown, we assess it by a statistic \(\hat{\theta}\) in order to find \(F_{\hat{\theta}}\), which is a plug-in estimator of \(F_{\theta_0}\). By ergodicity of \(X\), we substitute expectations in (19)-(21) by the corresponding empirical moments. The prediction problems above get the form

\[
\Phi_k(\lambda) := \sum_{j=1}^N Q_j^{(k)}(\lambda) \to \min_{\lambda \in \Lambda} \ k = 2, 3, 4,
\]

where

\[
Q_j^{(2)}(\lambda) := 2F_{\hat{\theta}}(X(t + h_j)) \vee F_{\hat{\theta}}(g(\lambda, X(T_f + h_j))) - F_{\hat{\theta}}(g(\lambda, X(T_f + h_j)))
\]

for unconstrained prediction (19),

\[
Q_j^{(3)}(\lambda) := Q_j^{(2)}(\lambda) + \gamma \left[ F_{\hat{\theta}}^2(g(\lambda, X(T_f + h_j))) - F_{\hat{\theta}}^2(g(\lambda, X(T_f + h_j))) \vee Y_j \right]
\]

for the (approximatively) law-preserving prediction (20), and

\[
Q_j^{(4)}(\lambda) := Q_j^{(2)}(\lambda) + \gamma F^2_{\hat{\theta}}(g(\lambda, X(T_f + h_j))) 
\]

\[- \frac{\gamma}{N} \left[ F_{\hat{\theta}}(g(\lambda, X(T_f + h_j))) + 2 \sum_{i=1}^{j-1} F_{\hat{\theta}}(g(\lambda, X(T_f + h_i))) \vee F_{\hat{\theta}}(g(\lambda, X(T_f + h_j))) \right]
\]

for the law-preserving prediction variant (21), where the convention \(\sum_{i=1}^{0} = 0\) is used. Here \(T_f + h_j, j = 1, \ldots, N\) with \(\{h_1, \ldots, h_N\} := \{s \in \mathbb{Z}_h : s + T_f \cup \{t\} \subset T_0\}\) are the so-called learning samples, and \(Y_j\) are independent copies of \(F_{\hat{\theta}}(g(\lambda, X(T_f + h_j)))\). In practice, the sample \(\{Y_1, \ldots, Y_N\}\) can be obtained for each \(\lambda\) by bootstrap, i.e. resampling of

\[
\{F_{\hat{\theta}}(g(\lambda, X(T_f + h_1))), \ldots, F_{\hat{\theta}}(g(\lambda, X(T_f + h_N)))\}.
\]
Remark 5. The extrapolation methods \([19]-[21]\) with \(\theta_0\) replaced by \(\theta_t\) can be also used for the extrapolation of non–stationary random fields \(X\) if their marginal distributions \(F_{\lambda_0}(x) = P(X(t) \leq x)\) are known in advance. In this case, the empirical moments \((23)-(26)\) in the problem \((22)\) have to be rewritten with \(F_{\lambda_0}\) in lieu of \(F_\theta\).

In order to find the minimum of \(\bar{\Phi}_k(\lambda)\), we use a subgradient descent. Assume that \(F_\theta\) has a density \(p_\theta\). Since the marginal distribution of \(X\) is absolutely continuous and \(X(t + h_j) \not\in X(T_f + h_j)\) for all \(j\) a.s. if the joint probability density of \((X(t + h_j), X(T_f + h_j))\) exists, it is reasonable to assume that

\[
P(X(t + h_j) = g(\lambda, X(T_f + h_j))) = 0,
\]

\[
P(Y_j = F_\theta(g(\lambda, X(T_f + h_j)))) = 0, \quad j = 1, \ldots, N,
\]

\[
P(F_\theta(g(\lambda, X(T_f + h_i))) = F_\theta(g(\lambda, X(T_f + h_j)))) = 0, \quad i, j = 1, \ldots, N.
\]

Then the subgradients \(\nabla^* Q_j^{(k)}(\lambda)\), \(k = 2, 4\), write with probability one as

\[
\nabla^* Q_j^{(2)}(\lambda) = \left[ 2 \mathbb{1}\{X(t + h_j) < g(\lambda, X(T_f + h_j))\} - 1 \right] p_\theta(g(\lambda, X(T_f + h_j))) \nabla^* g(\lambda, X(T_f + h_j)),
\]

\[
\nabla^* Q_j^{(4)}(\lambda) = \nabla^* Q_j^{(2)}(\lambda) + \gamma \left[ 2 F_\theta(g(\lambda, X(T_f + h_j))) - \frac{1}{N} - \frac{2}{N} \sum_{i=1}^{j-1} \mathbb{1}\{F_\theta(g(\lambda, X(T_f + h_i)) < F_\theta(g(\lambda, X(T_f + h_j)))\} \right.
\]

\[
- \frac{2}{N} \sum_{i=1}^{j-1} \mathbb{1}\{F_\theta(g(\lambda, X(T_f + h_i)) > F_\theta(g(\lambda, X(T_f + h_j)))\} \bigg] p_\theta(g(\lambda, X(T_f + h_i))) \nabla^* g(\lambda, X(T_f + h_i)),
\]

respectively. The subgradient \(\nabla^* Q_j^{(3)}(\lambda)\) can be written as

\[
\nabla^* Q_j^{(3)}(\lambda) = \nabla^* Q_j^{(2)}(\lambda) + \gamma \left[ 2 F_\theta(g(\lambda, X(T_f + h_j))) - \mathbb{1}\{g(\lambda, Y(T_f + h_j)) < g(\lambda, X(T_f + h_j))\} \right.
\]

\[
\times p_\theta(g(\lambda, X(T_f + h_j))) \nabla^* g(\lambda, X(T_f + h_j))
\]

\[
- \gamma \mathbb{1}\{g(\lambda, Y(T_f + h_j)) \geq g(\lambda, X(T_f + h_j))\} p_\theta(g(\lambda, Y(T_f + h_j))) \nabla^* g(\lambda, Y(T_f + h_j)),
\]

where \(Y\) is an independent copy of \(X\).

Here is e.g. \(\nabla^* g(\lambda, X(T_f + h_j)) = X(T_f + h_j)\) for infinitely divisible \(X\) and

\[
\nabla^* g(\lambda, X(T_f + h_j)) = \left( X(t_i + h_j) \mathbb{1}\left\{ \lambda^{(j)} X(t_i + h_j) = \max_{k=1,\ldots,n} \lambda^{(k)} X(t_k + h_j) \right\}, \quad i = 1, \ldots, n \right)
\]

for max–stable \(X\).

Remark 6. The speed of convergence of \(\frac{1}{N} \sum_{j=1}^{N} Q_j^{(k)}(\lambda)\) to its expectation \(\Phi_k(\lambda)\) as \(N \to \infty\) in the ergodic theorem for correlated data \(Q_j^{(k)}(\lambda)\) highly depends on their correlation rate. Hence, large values of \(N\) \((N \approx 1000)\) are recommended for practical use. In the case of random processes with infinite variance, the speed of convergence can be determined via their \(\beta\)–mixing properties, see e.g \([19]\).

Now use the classical (batch) subgradient descent \([10]\) with e.g. \(\lambda_0 \in \Lambda\),

\[
\lambda_{l+1} = \Pi_\Lambda \left[ \lambda_l - \frac{\eta_l}{N} \sum_{j=1}^{N} \nabla^* Q_j(\lambda_l) \right], \quad l \in \mathbb{N},
\]

(27)

where \(\Pi_\Lambda[\cdot]\) is the metric projection onto \(\Lambda\) and \(\eta_l > 0\) is a step length factor which has to be tuned numerically. The iterations stop at some \(l^*\) whenever \(|\lambda_{l^*+1} - \lambda_{l^*}| < \delta\) for some small threshold value \(\delta > 0\)
yielding \( \hat{\lambda} = \lambda_{t-} \). In order to avoid costly computations at each step, a stochastic (or online) subgradient descent \( \Pi \) may be performed instead. Here, at each step \( l \in \mathbb{N} \), we do

\[
\lambda_{l+1} = \Pi_\Lambda \left[ \lambda_l - \eta_l \nabla^* Q_j (\lambda_l) \right],
\]

where \( j \) is chosen at random uniformly from \( \{1, \ldots, N\} \). For the sequence \( \{\eta_l\} \), we may require

\[
\sum_{l=1}^{\infty} \eta_l = \infty, \quad \sum_{l=1}^{\infty} \eta_l^2 < \infty,
\]

for instance, \( \eta_l = l^{-1} \). In addition, the Polyak-Ruppert averaging can be used after a burn-in period of length \( l_0 \): the resulting weight vector

\[
\hat{\lambda} = \frac{1}{l - l_0} \sum_{l=l_0}^{l-1} \lambda_l.
\]

Alternatively, we may set \( \hat{\lambda} \) to be equal to the value of \( \lambda_l \) with the smallest target functional \( \Phi (\lambda_l) \).

Remark 7. In general, we assume the weight space \( \Lambda \) to be a convex cone within \( \mathbb{R}^n \). A more accurate choice of \( \Lambda \) should reflect the constraints onto the support of the distribution of \( X(t) \). For instance, if \( X(t) \geq 0 \) a.s. we may take \( \Lambda = \mathbb{R}_{\geq 0}^n \). However, for practical reasons of avoiding back projection \( \Pi_\Lambda \) onto \( \Lambda \) at each iteration step, it is better to modify the predictor \( \hat{X}_\Lambda \) and make \( g(\lambda, \cdot) \) be dependent on \( \lambda^\top = (\lambda^2(1), \ldots, \lambda^2(n))^\top \) instead of \( \lambda = (\lambda(1), \ldots, \lambda(n))^\top \). Doing so, the formulas for the subgradient \( \nabla^* g(\lambda, X(T_j + h_j)) \) have to be modified accordingly.

Remark 8. The advantage of optimization formulation (25) in comparison with (24) is that a bootstrap step (generating instances \( Y_j \) and thus increasing the variance of the forecast) is not needed for the evaluation of the subgradient \( \nabla^* Q_j (\lambda) \). However, there is a fee to pay: a more slow calculation of \( \nabla^* Q_j (\lambda) \) due to the sum inside.

It is worth mentioning that many existing optimization routines (e.g. those built in R or Mathlab) can be used to minimize the functional (22) in lieu of (27) or (28). They sometimes work more accurately but are rather slow, cf. Table 1. Under several additional assumptions, the a.s. convergence of the stochastic gradient descent method (28) can be shown; however, this would blow up the length of this paper and thus will be the matter of future papers.

7 Numerical examples

In this section, we test our prediction methods on simulated data. Although our approach works for random fields on \( W \subset \mathbb{R}^d \), we take \( d = 1 \) in order to simplify computations and the representation of results.

A random process \( X \) is observed at points \( T_0 = W_o \cap \mathbb{Z}_{h_1} \cup T_f \), where \( h_1 = 0.02, W_o = [0, 30 - h_1] \), and \( T_f \) is the forecast sample. We take \( T_f = \{30.0, 30.1, \ldots, 30.9\} \) for the extrapolation and \( T_f = \{30.0, 30.5, 31.0, \ldots, 34.5\} \) for the interpolation problems. In both cases \( n = |T_f| = 10 \).

We predict the values \( X(t) \) at locations \( t \in \mathbb{Z}_{h_2} \cap [30, 35], t \notin T_0 \) via predictor \( \hat{X}_\Lambda = \lambda \top X \), where \( \lambda \in \mathbb{R}^n \) and \( X = (X(t_1), \ldots, X(t_n)) \top \), \( t_j \in T_f, j = 1, \ldots, n \).

We solve the arising minimization problems by the stochastic subgradient descent method from Section 6.7. Our preliminary numerical studies show that \( \gamma = 5 \) is a good value for the constrained optimization. The minimization sequence \( \hat{\Phi} (\lambda) \) obtained by the classical (batch) subgradient descent very often stacks in some local minima. A stochastic (or online) subgradient descent has much better performance in a sense that \( \Phi_k (\lambda) \) reaches lower levels. After a series of numerical experiments, we can recommend the use of the sequence \( \eta_l = 10(10 + l)^{-\beta} \) with \( \beta = 0.7 \) in (28) with \( l \leq 300 \). Moreover, the value of \( \beta \) has a two-sided effect. Decreasing \( \beta \), the volatility of \( \lambda_l \) increases, which produces more possibilities of gaining a global minimum. But then the sequence of \( \lambda_l \) converges slower, and the number of computational steps increases as well.

One can also use the result of the stochastic subgradient descent method as an initial value for other optimization routines. This combines the advantages of two procedures, but increases the runtimes.
The choice of an initial value $\lambda_0$ is crucial for the good convergence of (28). Based on our experience, we provide the following practical recommendations. First, produce a finite number of "candidates" $\lambda_{0,j}$ for $\lambda_0$. Then the initial value $\lambda_0$ is chosen as $\arg \min_j \bar{f}_k(\lambda_{0,j})$. One possible set of such candidates may be $\lambda_{0,j} = (0, \ldots, 1, \ldots, 0)$, $j = 1, \ldots, n$ or the value obtained from the optimization problem for the neighbour point $t$. Another one may consist of a fixed number of $\lambda$’s generated randomly on $[0, 1]^n$ such that $||\lambda||_1 = 1$.

We choose three models of stationary infinitely divisible random processes: Gaussian stochastic process, moving average with $\alpha$-stable marginals, and an autoregressive model with Student $t$-distributed innovations.

As Gaussian random processes are well studied and their behaviour is determined by the covariance function, they allow us to compare the performance of our prediction method via excursions and some popular procedures, cf. kriging.

In the $\alpha$-stable moving average case, we model the dependence within $X$ by a deterministic kernel function and determine the marginal distribution of $X$ via the choice of a random integrator measure. If $\alpha \in (0, 2)$, the variance of $X(t)$ is infinite, and the $L^2$-forecasting techniques are not applicable.

We simulate also an autoregressive model in order to study two effects: the accuracy of the solutions of the minimization problems, and the method’s performance without knowing the marginal distribution.

The R code for our prediction methods can be found in [11]. The marginal distributions functions are taken as $F_\theta$ from the corresponding parametric family, and parameters’ estimates $\hat{\theta}$ are obtained from the one sample trajectories $X(t)$, $t \in W_o \cap \mathbb{Z}_{h_1}$.

We also do not solve the minimization problems for the times points from the forecast sample $T_f$. Naturally, we put $\hat{X}(t_k) = X(t_k)$, $t_k \in T_f$. We see from the further plots that the predicted trajectories are continuous functions and $\hat{X}(t) \approx X(t_k)$ if $t$ is close to $t_k$. Therefore, we can avoid computations for $t \in T_f$.

### 7.1 Gaussian random processes

In the case of a Gaussian random process $X$, the exact solution $\hat{\lambda}_c(t)$ of minimization problem (13) is given in [4] Theorem 3.5 by

$$\hat{\lambda}_c(t) = \sqrt{\text{Var}X(t)} \cdot \frac{\Sigma^{-1}c_t}{\sqrt{c_t^\top \Sigma^{-1}c_t}},$$

where $\Sigma$ is the covariance matrix of $X$ and $c_t = (\text{Cov}(X(t), X(t_1)), \ldots, \text{Cov}(X(t), X(t_n)))$. For numerical illustration, we take $X$ with standard normal marginal distribution and covariance function $C(t) = e^{-|t|^2/2}$, $t \in \mathbb{R}$.

For each $T_f$, we find numerical solutions $\hat{\lambda}_n(t)$ of (19) and $\hat{\lambda}_c(t)$ of (20), $t \in \mathbb{Z}_{h_1} \cap [30, 35] \setminus T_f$. We compare the corresponding predicted trajectories $\hat{X}_n$ and $\hat{X}_c$ with $\hat{X}_e$ obtained via [3] Theorem 3.5 and simple kriging $\hat{X}_{sk}$, see Figures 1 (interpolation) and 3 (extrapolation).

**Remark 9.** The extrapolated trajectory $\hat{X}_c$ becomes constant shortly after the last point of observation $t_n$. Mathematically, $\hat{\lambda}_c(t) \rightarrow (0, \ldots, 0, 1)^\top$ as $t \gg t_n$. Indeed, $\Sigma$ does not depend on $t$ and $c_t = (e^{-(t_n-t)/2}, \ldots, 1)^\top = c_n$ for $C(t) = e^{-|t|^2/2}$. Thus,

$$\Sigma \hat{\lambda}_c(t) = \sqrt{\text{Var}X(t)} \cdot \frac{c_t}{C(t-t_n)} \left( \frac{c_t^\top \Sigma^{-1}c_t}{C(t-t_n)} \right)^{-1/2} = \sqrt{\text{Var}X(0)c_n (c_n^\top \Sigma^{-1}c_n)^{-1/2}},$$

and, consequently, $\hat{\lambda}_c(t) = (0, \ldots, 0, 1)^\top$.

One can observe that the trajectories of $\hat{X}_n$ and $\hat{X}_{sk}$ are relatively close, which may indicate that the solution of unconstrained minimization problem (19) approximates the minimizers obtained by the simple kriging method.
Figure 4 also shows that the trajectory \( \hat{X}_c \) is not so close to \( \hat{X}_c \). This effect has two sources. First, there is no exact constraint of the equality of marginal distributions of the predictor and the random process. Second, the minimization functional is approximated by its sample mean. Hence, one should increase the size \( N \) of the learning sample and the weight \( \gamma \) in order to obtain a closer match between \( \hat{\lambda}(t) \) and \( \hat{\lambda}(t) \).

While the prediction weights \( \hat{\lambda}_u \) and \( \hat{\lambda}_c \) are computed based on one learning sample, the quality of prediction is evaluated on 1000 independently simulated trajectories of \( X \) on \([30, 35] \cap \mathbb{Z} \). We compute the corresponding sample values of excursion metric \( E_{F_{X_0}}(X(t), \hat{X}(t)) \) presented in Figures 2 and 3. The similarity between the marginal distributions of \( X(t) \) and \( \hat{X}(t) \) is measured by Wasserstein distance \( \rho(F(X(t), F(\hat{X}(t))) \), whose values can be found in Figures 8 and 11.

7.2 \( \alpha \)-stable moving averages

We consider the moving average process \( X_\alpha = \{X_\alpha(t), t \in \mathbb{Z}_{h_1} \} \) given by \( X_\alpha(t) = \sum_{m \in \mathbb{Z}} m(t/h_1 - x)\xi_{\alpha}(x) \) where \( \xi_{\alpha}(x) \) are independent \( S_\alpha(1, \beta, 0) \)-random variables and \( m : \mathbb{Z} \to \mathbb{R}_+ \) is a kernel function such that \( \|m\|_\alpha := (\sum_{x \in \mathbb{Z}} m^\alpha(x))^{1/\alpha} < \infty \). \( X_\alpha \) is stationary with marginal distribution \( S_\alpha(||m||_\alpha, \beta, 0) \).

Two cases of heaviness of the tails are chosen: Cauchy distribution (\( \alpha = 1, \beta = 0 \)) and Lévy distribution (\( \alpha = 0.5, \beta = 1 \)). The kernel function is given by

\[
m(x) = \begin{cases} 
  e^{-0.02x} & (1 - e^{-0.02}) (1 - e^{-5.02})^{-1} \mathbb{1}(x \in \mathbb{Z} \cap [0, 251]) , & \alpha = 1, \\
  e^{-0.02x} & (1 - e^{-0.01})^2 (1 - e^{-2.51})^{-2} \mathbb{1}(x \in \mathbb{Z} \cap [0, 251]) , & \alpha = 0.5. 
\end{cases}
\]

It holds \( \|m\|_\alpha = 1 \) and thus \( X_\alpha(t) \sim S_\alpha(1, \beta, 0) \). We choose \( S_{0.5}(1, 1, 0) \) and \( S_1(1, 0, 0) \) because there are simple analytical formulas for their c.d.f.’s:

\[
F_{X_{0.5}}(y) = \frac{1}{\sqrt{2\pi}} y^{-3/2} e^{-\frac{y^2}{2}} \mathbb{1}(y > 0), \quad F_{X_1}(y) = \frac{1}{2} + \frac{1}{\pi} \arctan(y), \quad y \in \mathbb{R}. 
\] (29)

Moreover, \( \mathbb{E}[X_{\alpha}(t)] = +\infty \) and \( \mathbb{E}X_{\alpha}^2(t) = +\infty \) in both cases.

For each \( T_f, X_{0.5}, \) and \( X_1 \), we find numerical solutions \( \hat{\lambda}_u(t) \), and \( \hat{\lambda}_c(t) \), \( t \in \mathbb{Z}_{h_1} \cap [30, 35] \setminus T_f \) of minimization problems (19) and (20), which leads to predicted trajectories \( \hat{X}_{0.5}, \hat{X}_{1,u}, \) and \( \hat{X}_{0.5,c}, \hat{X}_{1,c} \), respectively, see Figures 9, 10, and 13.

One can observe that the predicted trajectory \( \hat{X}_{1,c} \) in Figure 7 look like a step-wise functions and the extrapolated trajectories in Figure 11 are quite volatile. In order to understand the source of these phenomena, we apply two different methods of numerical solution. For \( \alpha = 1 \), we use stochastic subgradient descent method (21) and, for \( \alpha = 0.5 \), we apply Remark 4 and existing optimization routine in \( \mathbb{R} \), avoiding the modification of subgradients.

The predicted trajectories in Figures 13 and 16 seem to fit the real trajectory better. Therefore, the existing minimization procedures can be more stable and accurate than the stochastic subgradient descent algorithm, which is nonetheless compensated by much larger runtimes.

We repeat the simulation and prediction procedure 1000 times and compute the corresponding values of excursion metrics \( E_{F_{X_0}}(X_\alpha(t), \hat{X}_{\alpha,\lambda}(t)) \), see Figures 8, 11, 14, and 17. The corresponding Wasserstein distances \( \rho(F(X_\alpha(t), F(\hat{X}_{\alpha,\lambda}(t))) \) are given in Figures 9, 12, 15, and 18.

In Figures 5, 11, and 17, the excursion metrics tend asymptotically to value 1/3 as the distance to the last observed point increases. This alludes to Example 3.5 and shows that predictor \( \hat{X}_{\alpha,\lambda}(t) \) and true random variable \( X_\alpha(t) \) become asymptotically independent (for large \( t \)).

7.3 Autoregressive stationary process

A random process which can be nicely predicted via a linear forecast is an autoregressive time series \( AR(p) \) given by

\[
X(t) = \varphi_1 X(t - ph) + \varphi_2 X(t - ph + h) + \ldots + \varphi_p X(t - h) + \xi_t,
\]

where \( \xi_t, t \in \mathbb{Z}_h \) are independent random variables and \( \varphi_k, k = 1, \ldots, p \) are regression coefficients.

We examine our method on a simulated trajectory of \( X(t), t \in T_0 \cup T_f \) with \( T_f = \{30.0, 30.1, 30.2\}, h = 0.02 \) and \( n = p = 3 \). We take \( \varphi_1 = 0.1, \varphi_2 = 0.25, \varphi_3 = 0.5 \) in order to get a stationary \( AR(3) \) process,
Figure 1: Interpolation of a Gaussian random process $X$. True trajectory $X(t)$ (black), predicted trajectories $\hat{X}_u$ (red, solid), $\hat{X}_c$ (blue, solid), $\hat{X}_{sk}$ (red, dashed), $\hat{X}_e$ (blue, dashed).

Figure 2: Interpolation of a Gaussian random process $X$. Excursion metric for predictors $\hat{X}_u$ (red, solid), $\hat{X}_c$ (blue, solid), $\hat{X}_{sk}$ (red, dashed), $\hat{X}_e$ (blue, dashed).

Figure 3: Interpolation of a Gaussian random process $X$. Wasserstein distance between $F(X)$ and predictors $F(\hat{X}_u)$ (red, solid), $F(\hat{X}_c)$ (blue, solid), $F(\hat{X}_{sk})$ (red, dashed), $F(\hat{X}_e)$ (blue, dashed).
Figure 4: Extrapolation of a Gaussian random process $X$. True trajectory $X(t)$ (black), predicted trajectories $X_u$ (red, solid), $X_c$ (blue, solid), $X_{sk}$ (red, dashed), $X_e$ (blue, dashed).

Figure 5: Extrapolation of a Gaussian random process $X$. Excursion metric for predictors $X_u$ (red, solid), $X_c$ (blue, solid), $X_{sk}$ (red, dashed), $X_e$ (blue, dashed).

Figure 6: Extrapolation of a Gaussian random process $X$. Wasserstein distance between $F(X)$ and predictors $F(X_u)$ (red, solid), $F(X_c)$ (blue, solid), $F(X_{sk})$ (red, dashed), $F(X_e)$ (blue, dashed).
Figure 7: Interpolation of a moving average $X_1$ with Cauchy distributed marginals: True trajectory $X_1(t)$ (black), predicted trajectories $\hat{X}_{1,u}$ (red) and $\hat{X}_{1,c}$ (blue).

Figure 8: Interpolation of a moving average $X_1$ with Cauchy distributed marginals: Excursion metric for predictors $\hat{X}_{1,u}$ (red) and $\hat{X}_{1,c}$ (blue).

Figure 9: Interpolation of a moving average $X_1$ with Cauchy distributed marginals: Wasserstein distance between $F(X_1)$ and predictors $F(\hat{X}_{1,u})$ (red), $F(\hat{X}_{1,c})$ (blue).
Figure 10: Extrapolation of a moving average $X_1$ with Cauchy distributed marginals: True trajectory $X_1(t)$ (black), predicted trajectories $\hat{X}_{1,u}$ (red) and $\hat{X}_{1,c}$ (blue).

Figure 11: Extrapolation of a moving average $X_1$ with Cauchy distributed marginals: Excursion metric for predictors $\hat{X}_{1,u}$ (red) and $\hat{X}_{1,c}$ (blue).

Figure 12: Extrapolation of a moving average $X_1$ with Cauchy distributed marginals: Wasserstein distance between $F(X_1)$ and predictors $F(\hat{X}_{1,u})$ (red), $F(\hat{X}_{1,c})$ (blue).
Figure 13: Interpolation of a moving average $X_{0.5}$ with Lévy distributed marginals: True trajectory $X_{0.5}(t)$ (black), predicted trajectories $\hat{X}_{0.5,u}$ (red) and $\hat{X}_{0.5,c}$ (blue).

Figure 14: Interpolation of a moving average $X_{0.5}$ with Lévy distributed marginals: Excursion metric for predictors $\hat{X}_{0.5,u}$ (red) and $\hat{X}_{0.5,c}$ (blue).

Figure 15: Interpolation of a moving average $X_{0.5}$ with Lévy distributed marginals: Wasserstein distance between $F(X_{0.5})$ and predictors $F(\hat{X}_{0.5,u})$ (red), $F(\hat{X}_{0.5,c})$ (blue).
Figure 16: Extrapolation of a moving average $X_{0.5}$ with Lévy distributed marginals: True trajectory $X_{0.5}(t)$ (black), predicted trajectories $\hat{X}_{0.5,u}$ (red) and $\hat{X}_{0.5,c}$ (blue).

Figure 17: Extrapolation of a moving average $X_{0.5}$ with Lévy distributed marginals: Excursion metric for predictors $\hat{X}_{0.5,u}$ (red) and $\hat{X}_{0.5,c}$ (blue).

Figure 18: Extrapolation of a moving average $X_{0.5}$ with Lévy distributed marginals: Wasserstein distance between $F(X_{0.5})$ and predictors $F(\hat{X}_{0.5,u})$ (red), $F(\hat{X}_{0.5,c})$ (blue).
and set $\xi$ being standard Student $t$-distributed with $\nu = 0.8$ degrees of freedom which is infinitely divisible with $\mathbb{E}[\xi] = \infty$. Indeed, it is not hard to verify that the roots of the equation $\sum_{j=1}^{p} \varphi_t z^2 = 1$, $z \in \mathbb{C}$ lie outside the unit circle, and the tail probabilities of $\xi$ are regularly varying, cf. conditions in [3].

Remark 10. In this case, we do not know the exact marginal distribution of $X$. However, we can use the excursion predictor $\hat{X}_\lambda$ from Definition 3 with c.d.f. function $F$ having the same support as $X(t)$. If $\lambda$ is defined via (12) or (19), one can show that $\hat{X}_\lambda$ is a consistent estimator of $X(t)$ as well. In the case of constrained minimization (20)–(21), the distribution of $F(X(t))$ is not uniform any more, and the relation $\mathbb{E}F^2(\hat{X}_\lambda) - \mathbb{E}[F(\hat{X}_\lambda) \lor Y]$ does not correspond to the squared 2-Wasserstein distance $\rho^2(\mathbb{L}aw[F(\hat{X}_\lambda)], \mathbb{L}aw[F(X(t))])$. By triangle inequality, it holds

$$\rho(\mathbb{L}aw(\hat{X}_\lambda)), \mathbb{L}aw[F(X(t))]) \leq \rho(\mathbb{L}aw[F(\hat{X}_\lambda)), U[0, 1]) + \rho(U[0, 1], \mathbb{L}aw[F(X(t))]),$$

and hence the minimization of $\rho(\mathbb{L}aw[F(\hat{X}_\lambda)], U[0, 1])$ leads to the approximative minimization of $\rho(\mathbb{L}aw(\hat{X}_\lambda)), \mathbb{L}aw[F(X(t))])$, if $\rho(U[0, 1], \mathbb{L}aw[F(X(t))])$ is small enough. So, we expect our predictor estimator to be robust regarding the choice of function $F$.

Therefore, we use the excursion metric $E_F$ with c.d.f. $F = F_\theta$ being close to the true marginal distribution of $X$ and having the same support. We choose $F_\theta$, $\theta = (\mu, \sigma, \nu)$ from the parametric family of Student $t$-distributions $ST(\mu, \sigma, \nu)$. Based on a simulated trajectory of $X$, we find that $F_\theta$ with $\theta = (0, 10, 0.7)$ best approximates the marginals of $X$.

In this example, we do not extrapolate the trajectories of $X$ on a wide interval, but we study the performance of the minimization algorithm. That is why the prediction coefficients $\lambda(t) = (\lambda_1(t), \lambda_2(t), \lambda_3(t))$ are computed at points $t \in \{30.3, 30.4, 30.5, 30.6\}$. We do this for minimization problems [19] and [20] via stochastic subgradient descent and via standard minimization methods implemented in R language resulting in $\lambda_k^\theta$ and $\lambda_k^\gamma$, $k = u, c$, respectively. The results are given in Tables 2 (unconstrained) and 3 (constrained).

One can observe that the minimal values of $\Phi_2$ and $\Phi_3$ increase slightly when the prediction point $t$ moves away from the forecast sample $T_f$. We see also that the standard implemented minimization methods reach the lower minimal values of $\Phi_2$ and $\Phi_3$. However, the stochastic subgradient descent method is much faster and the differences $\|\lambda^\theta(t) - \lambda^\gamma(t)\|_2$ and $\|\lambda^\gamma(t) - \lambda^\gamma(t)\|_2$ are small for the prediction points which are close to the forecast sample.

The values of $\lambda^\theta(t) = (0.10490, 0.24573, 0.49832)$ and $\lambda^\gamma(t) = (0.12154, 0.23066, 0.48210)$ at point $t = 30.3$ are very close to regression coefficients $\varphi$. Therefore, one can use the reliable prediction method via excursion metric $E_{FX}$ in the case when the marginal distribution is not known a-priori and has to be statistically assessed.

| $t$         | $\lambda^\theta(t)$ | $\Phi_2(\lambda^\theta(t))$ | $\lambda^\gamma(t)$ | $\Phi_2(\lambda^\gamma(t))$ | $\|\lambda^\theta(t) - \lambda^\gamma(t)\|_2$ |
|------------|----------------------|-------------------------------|----------------------|-------------------------------|---------------------------------|
| 30.3       | (0.10490, 0.24573, 0.49832) | 0.04423                       | (0.10832, 0.24299, 0.49852) | 0.04423                       | 0.00438                         |
| 30.4       | (0.05293, 0.22722, 0.45865)  | 0.06358                       | (0.05914, 0.22262, 0.49288) | 0.06358                       | 0.00827                         |
| 30.5       | (0.04532, 0.19385, 0.46395)  | 0.08082                       | (0.06593, 0.18834, 0.44610) | 0.08076                       | 0.02782                         |
| 30.6       | (0.15467, 0.19230, 0.28920)  | 0.09702                       | (0.05452, 0.16589, 0.41035) | 0.09579                       | 0.13938                         |
| $\varphi$  | (0.10000, 0.25000, 0.50000)  | (0.10000, 0.25000, 0.50000)  | (0.10000, 0.25000, 0.50000)  |                               |                                 |

Table 2: AR(3) stationary process. Extrapolation coefficients $\lambda^\theta$ and $\lambda^\gamma$ for unconstrained minimization.

| $t$         | $\lambda^\theta(t)$ | $\Phi_3(\lambda^\theta(t))$ | $\lambda^\gamma(t)$ | $\Phi_3(\lambda^\gamma(t))$ | $\|\lambda^\theta(t) - \lambda^\gamma(t)\|_2$ |
|------------|----------------------|-------------------------------|----------------------|-------------------------------|---------------------------------|
| 30.3       | (0.12154, 0.23066, 0.48210) | 0.04784                       | (0.12081, 0.24737, 0.47881) | 0.04770                       | 0.01765                         |
| 30.4       | (0.05638, 0.23474, 0.50164)  | 0.05212                       | (0.07559, 0.22932, 0.48422) | 0.05207                       | 0.02650                         |
| 30.5       | (0.33077, 0.26291, 0.39731)  | 0.08588                       | (0.10881, 0.18561, 0.43390) | 0.07623                       | 0.24629                         |
| 30.6       | (-0.04932, 0.21927, 0.53541) | 0.08633                       | (0.08251, 0.17675, 0.42702) | 0.08513                       | 0.17588                         |
| $\varphi$  | (0.10000, 0.25000, 0.50000)  | (0.10000, 0.25000, 0.50000)  | (0.10000, 0.25000, 0.50000)  |                               |                                 |

Table 3: AR(3) stationary process. Extrapolation coefficients $\lambda^\theta$ and $\lambda^\gamma$ for constrained minimization.
8 Summary

We introduced the new predictors for random variables, processes and fields with possibly infinite moments via the minimization of a functional based on excursion sets. We explored several advantages of our excursion predictors using theoretical results and computational studies. Namely, they are computationally fast, consistent for stochastically continuous random fields, and work for random fields without finite moments. The research presented in this paper is introductory and covers only the first important properties of our methods. These results reveal a great potential for many real world applications. A further theoretical investigation of our methods including the uniqueness of solutions, accuracy of prediction as well as the improvement of computation routines will be the topic of our next papers.

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