Self-field $QED_{1+1}$ with massless matter fields: Two-body problem

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We consider two-body problem in the self-field $(1+1)$-dimensional quantum electrodynamics on the circle. We present two different formulations of the problem which correspond to two different types of variational principles and prove that both formulations lead to the same spectrum of the two-body Hamiltonian with massless matter fields. We give the exact and complete solution of the relativistic two-body equation in the massless case.

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1 INTRODUCTION

The relativistic two-body problem attracts attention in both quantum mechanics of point particles and field theory [1]-[4]. To formulate the problem we usually assume that two or many body systems are described by a composite field. This is general in all formulations. What brings a difference is variational principle.

We can rewrite the action of the two-body system entirely in terms of the composite field and then require the action to be stationary with respect to the variations of this field only. This leads to a single two-body equation [3]. However, if we first vary the action with respect to the individual fields, then we come to a pair of coupled equations on the composite field [3, 4]. These two different types of variational principles produce therefore different types of two-body equations.

In the present paper, we aim (i) to compare the two formulations for the (1 + 1)-dimensional quantum electrodynamics (QED) known as the Schwinger model (SM) [7] and (ii) to solve exactly the two-body problem for this model in the massless case. QED in lower dimensions is interesting as a simpler model for discussion of many body aspects of particle physics, for example, spontaneous positron production by supercritical potentials [8]. Moreover, under certain conditions such lower dimensions may be physically realizable in condensed matter and statistical systems [9]. There is a discussion of many-body problems in (1 + 1)-dimensions, however under instantaneous phenomenological, e.g., δ-functional potentials [10].

We use the self-field version of QED [11] which is a first-quantized theory, so both matter and electromagnetic fields are not quantized. The electromagnetic field has no separate local degrees of freedom and can be eliminated between the coupled Maxwell-Dirac equations, but then we must include nonlinear self-field terms. We consider two matter Dirac fields coupled to a $U(1)$ gauge or electromagnetic field and work on the circle where the electromagnetic field has a global physical degree of freedom.

Our paper is organized as follows. In Sec. 2, for our two-body system we present two alternative formulations based on the variational principles mentioned above and derive the corresponding two-body equations in configuration space. We give the Hamiltonian form of these equations in both cases. The single two-body equation formulation is one-time formulation. In contrast the formulation with the composite field governed by two coupled Dirac equations has two time coordinates and includes the relative energy. In Sec. 3, we find the eigenfunctions and the spectrum of the two-body Hamiltonian in the single two-body equation formulation with massless matter fields. In [12] we solved this problem with the self-potentials neglected. Now we treat the case of the massless matter fields completely. We take into account in the two-body equation the self-potentials responsible for the radiative corrections and solve it exactly, i.e., get the exact and complete solution of the two-body problem. In Sec. 4, we consider the eigenvalue problem for the two-body Hamiltonian in the pair of Dirac equations formulation as well. We prove that both formulations lead to the same spectrum and are therefore equivalent to each other. Sec. 5 contains our conclusions.

2 TWO-BODY SYSTEM

The action of the system is

$$W[\psi, A] = \int_{-\infty}^{\infty} dt \int_0^L dx \sum_{k=1}^2 \{ \bar{\psi}_k \gamma^\mu (i \partial_\mu - e_k A_\mu) \psi_k - m_k \bar{\psi}_k \psi_k \} - \frac{1}{4} F_{\mu \nu} F^{\mu \nu},$$  (2.1)

where $(\mu, \nu = 0, 1)$, $\gamma^0 = -i\sigma_2$, $\gamma^0 \gamma^1 = \gamma^5 = \sigma_3$, $\sigma_i$ $(i = 1, 3)$ are Pauli matrices. The fields $\psi_k$ are 2-component Dirac spinors, and $\bar{\psi}_k = \psi_k^* \gamma^0$. 
We suppose that space is a circle of length $L$, $0 \leq x < L$, and impose the following boundary conditions for the fields

$$ A_\mu(L, t) = A_\mu(0, t), $$
$$ \psi_k(L, t) = e^{i2\pi \kappa_k} \psi_k(0, t), \quad k = 1, 2, $$

$\kappa_1, \kappa_2$ being arbitrary numbers. The charges $e_1, e_2$ are not arbitrary on the circle, one of the charges must be a multiple of another \cite{13}.

We work in the Coulomb gauge

$$ A_1(x, t) = b(t), $$

where

$$ b(t) \equiv \frac{1}{L} \int_0^L dx A_1(x, t) $$

is the electromagnetic field global degree of freedom.

The electromagnetic field equations deduces from the action (2.1) are

$$ \partial_\nu F^{\nu \mu} = J^\mu, \quad (2.2) $$

where the total matter current

$$ J^\mu = \sum_{k=1}^2 e_k \bar{\psi}_k \gamma^\mu \psi_k $$

is conserved, $\partial_\mu J^\mu = 0$.

If we solve the electromagnetic field equations, express $A_\mu$ in terms of $J^\mu$ and insert the expressions obtained into (2.1), then we get the action written in terms of the matter fields

$$ W[\psi, A] = \int_{-\infty}^{\infty} dt \int_0^L dx \sum_{k=1}^2 \bar{\psi}_k (\gamma^\mu i \partial_\mu - m_k) \psi_k + \frac{1}{2} \int_{-\infty}^{\infty} dt \int_0^L dx \int_0^L dy J^0(x, t) D(x, y|L) J^0(y, t) $$

$$ - \frac{1}{2} \int_{-\infty}^{\infty} dt \int_0^L dx J^1(x, t) b(t). \quad (2.3) $$

The last term represents the interaction of the matter currents with the global electromagnetic field degree of freedom $b$, while the middle term is a sum of current-current interactions containing both the mutual and self-interaction terms.

The Green’s function in (2.3) is

$$ D(x, y|L) \equiv \frac{1}{2} |x - y| + \frac{xy}{L}. $$

### 2.1 First formulation: single two-body equation

Following the relativistic configuration space formalism \cite{5}, we define the composite field

$$ \Phi(x_1, t|x_2, t) \equiv \psi_1(x_1, t) \otimes \psi_2(x_2, t), $$

which is 4-component spinor field. The configuration space $(x_1, x_2)$ is a torus with the circle length $(0 \leq x_1 < L$, $0 \leq x_2 < L)$.
We can rewrite our action (2.3) entirely in terms of the composite field $\Phi$. In order to do this we multiply the kinetic energy terms with the normalization factors

$$\int_0^L dx \psi_k^*(x,t) \psi_k(x,t) = 1, \quad k = 1 \text{ or } 2. \quad (2.4)$$

We have to do this twice on the self-interaction terms. The resultant action is

$$W[\Phi, A] = \int_{-\infty}^{\infty} dt \int_0^L dx_1 \int_0^L dx_2 \Phi(x_1, t|x_2, t)\{(\gamma^\mu p_{(1),\mu} - m_1) \otimes \gamma^0 + \gamma^0 \otimes (\gamma^\mu p_{(2),\mu} - m_2)
+ \frac{1}{2}(\gamma^0 \otimes \gamma^0)(e_1 \phi^{(1)} + e_2 \phi^{(2)}) - \frac{1}{2} e_1 b(\gamma^1 \otimes \gamma^0) - \frac{1}{2} e_2 b(\gamma^0 \otimes \gamma^1)
+ e_1 e_2(\gamma^0 \otimes \gamma^0)D(x_1, x_2|L)} \Phi(x_1, t|x_2, t), \quad (2.5)$$

where

$$p_{(i),\mu} \equiv i \frac{\partial}{\partial x_i^\mu},$$

and

$$\phi^{(1)}(x_1, t) = e_1 \int_0^L dy \int_0^L dz D(x_1, z|L)(\Phi(z, t|y, t)(\gamma^0 \otimes \gamma^0)\Phi(z, t|y, t),
$$

$$\phi^{(2)}(x_2, t) = e_2 \int_0^L dy \int_0^L dz D(x_2, y|L)(\Phi(z, t|y, t)(\gamma^0 \otimes \gamma^0)\Phi(z, t|y, t),$$

the self-potentials $\phi^{(k)}$ being non-linear integral expressions. The spin matrices are written here in the form of tensor products $\otimes$, the first factor always referring to the spin space of particle 1, the second to particle 2.

Let us note that the last term in (2.3) can also be put into the self-potentials $\phi^{(k)}$, one half for each particle; the total potentials then take the form

$$\phi^{(1)} \rightarrow \phi^{(1)},
\phi^{(2)} \rightarrow \phi^{(2)},$$

where

$$\phi^{(1)}(x, t) \equiv \int_0^L dy \int_0^L dz (e_1 D(x, z|L) + e_2 D(x, y|L))\Phi(z, t|y, t)(\gamma^0 \otimes \gamma^0)\Phi(z, t|y, t). \quad (2.6)$$

We must now specify a variational principle for the matter fields. We could vary the action with respect to individual fields $\psi_1$ and $\psi_2$ separately. This results in non-linear coupled equations for these fields (see below). Instead, we require the action (2.5) to be stationary with respect to the total composite field only. This is a weaker condition which leads to the following two-body equation

$$\{((\gamma^\mu p_{(1),\mu} - m_1) \otimes \gamma^0 + \gamma^0 \otimes (\gamma^\mu p_{(2),\mu} - m_2) + (\gamma^0 \otimes \gamma^0)(e_1 \phi^{(1)} + e_2 \phi^{(2)})
- \frac{1}{2} e_1 b(\gamma^1 \otimes \gamma^0) - \frac{1}{2} e_2 b(\gamma^0 \otimes \gamma^1) + e_1 e_2(\gamma^0 \otimes \gamma^0)D(x_1, x_2|L)} \Phi(x_1, t|x_2, t) = 0. \quad (2.7)$$

If we define the generalized (kinetic) momenta as

$$\pi_{(i),\mu} \equiv p_{(i),\mu} + e_i A_{(i),\mu}^{\text{self}}$$
with

\[ A_{(1),0}^{\text{self}} = \phi_{(1)}, \quad A_{(2),0}^{\text{self}} = \phi_{(2)}^{\text{self}}, \]

\[ A_{(1),1}^{\text{self}} = A_{(2),1}^{\text{self}} = -\frac{1}{2}b, \]

then the two-body equation takes the compact form

\[
\{(\gamma^\mu \pi_{(1),\mu} - m_1) \otimes \gamma^0 + \gamma^0 (\gamma^\mu \pi_{(2),\mu} - m_2) + e_1 e_2 (\gamma^0 \otimes \gamma^0) D(x_1, x_2|L)\} \Phi(x_1, t|x_2, t) = 0. \tag{2.8}
\]

In the center of mass and relative coordinates

\[ \Pi = \pi_{(1)} + \pi_{(2)}, \quad \pi = \pi_{(1)} - \pi_{(2)}, \]

\[ P = p_{(1)} + p_{(2)}, \quad p = p_{(1)} - p_{(2)}, \]

\[ x_+ = x_1 + x_2, \quad x_- = x_1 - x_2, \]

the configuration space \((x_-, x_+)\) is again a torus, but with the circle length \(2L (-L \leq x_- < L, 0 \leq x_+ < 2L)\), while the function \(D(x_1, x_2|L)\) becomes a sum of center of mass and relative parts depending only on \(x_-\) and \(x_+\), respectively,

\[
D(x_1, x_2|L) = D_-(x_-|L) + D_+(x_+|L),
\]

\[
D_-(x_-|L) = \frac{1}{2} |x_-| - \frac{1}{4L} x_-^2,
\]

\[
D_+(x_+|L) = \frac{1}{4L} x_+^2.
\]

Eq. (2.8), without the self-field terms, becomes

\[
\{\Gamma^\mu p_\mu + k^\mu p_\mu + e_1 e_2 (\gamma^0 \otimes \gamma^0) D - m_1 (I \otimes \gamma^0) - m_2 (\gamma^0 \otimes I)\} \Phi(x_-, t|x_+, t) = 0, \tag{2.9}
\]

where we have introduced

\[
\Gamma^\mu \equiv \frac{1}{2} (\gamma^\mu \otimes \gamma^0 + \gamma^0 \otimes \gamma^\mu),
\]

\[
k^\mu \equiv \frac{1}{2} (\gamma^\mu \otimes \gamma^0 - \gamma^0 \otimes \gamma^\mu),
\]

and \(I\) is identity matrix. Since \(k^0\) vanishes, the zero component of \(p_\mu\), i.e., the relative energy \(p_0\) drops out of the two-body equation automatically. Thus we have only one time variable conjugate to the center of mass energy \(P_0\), one degree of freedom for the center of mass momentum \(P^1\) and one degree of freedom for the relative momentum \(p^1\). By multiplying (2.9) by \(\Gamma^{-1}_0\) we obtain the Hamiltonian form of the two-body equation

\[
P_0 \Phi = \{\alpha_+ P^1 + \alpha_- p^1 - e_1 e_2 D + \beta_1 m_1 + \beta_2 m_2\} \Phi, \tag{2.10}
\]

with

\[
\alpha_\pm \equiv \frac{1}{2} (\alpha_1 \pm \alpha_2), \quad \alpha_1 \equiv \gamma^5 \otimes I, \quad \alpha_2 \equiv I \otimes \gamma^5,
\]

\[
\beta_1 \equiv \gamma^0 \otimes I, \quad \beta_2 \equiv I \otimes \gamma^0,
\]
and the relative and center of mass terms in the Hamiltonian $P_0$ being additive,

$$P_0 = H_{\text{c.m.}} + H_{\text{rel}},$$

$$H_{\text{c.m.}} \equiv \alpha_+ P^1 - e_1 e_2 D_+,$$

$$H_{\text{rel}} \equiv \alpha_- p^1 - e_1 e_2 D_- + \beta_1 m_1 + \beta_2 m_2.$$ 

Eq. (2.10) has the form of a generalized Dirac equation, now a 4-component wave equation. The self-potentials break in general the above mentioned additivity of the center of mass and relative parts of $P_0$.

2.2 Second formulation: pair of Dirac equations

Let us use now a different variational principle and vary the action (2.3) with respect to each field $\psi_k$ separately. In this way we come to a pair of coupled nonlinear equations

$$(\gamma^\mu i \partial_\mu - m_1) \psi_1(x,t) - \frac{1}{2} e_1 b(t) \gamma^1 \psi_1(x,t) + e_1 \int_0^L dyD(x,y|L)J^0(y,t)\gamma^0 \psi_1(x,t) = 0,$$  \hspace{1cm} (2.12a)

$$(\gamma^\mu i \partial_\mu - m_2) \psi_2(x,t) - \frac{1}{2} e_2 b(t) \gamma^1 \psi_2(x,t) + e_2 \int_0^L dyD(x,y|L)J^0(y,t)\gamma^0 \psi_2(x,t) = 0.$$  \hspace{1cm} (2.12b)

To describe our two-body system we define the composite field

$$\Phi(x_1, t_1 | x_2, t_2) = \psi_1(x_1, t_1) \otimes \psi_2(x_2, t_2)$$

composed of the individual matter fields at different times. Multiplying Eq.(2.12a) taken at $(x,t) = (x_1, t_1)$ by $\gamma^0 \psi_2(x_2, t_2)$ and Eq.(2.12b) taken at $(x,t) = (x_2, t_2)$ by $\gamma^0 \psi_1(x_1, t_1)$ as well as the nonlinear self-field terms in both equations by the normalization factors leads to

$$G_1 \Phi(x_1, t_1 | x_2, t_2) \equiv \{ (\gamma^\mu p_{(1),\mu} - m_1) \otimes \gamma^0 - \frac{1}{2} e_1 b(t_1) (\gamma^1 \otimes \gamma^0)$$

$$+ (\gamma^0 \otimes \gamma^0) e_1 \phi^{\text{self}}(1) \} \Phi(x_1, t_1 | x_2, t_2) = 0,$$  \hspace{1cm} (2.13a)
\[ G_2 \Phi(x_1, t_1 | x_2, t_2) \equiv \{ \gamma^0 \otimes (\gamma^\mu p_{(2), \mu} - m_2) - \frac{1}{2} e_2 b(t_2)(\gamma^0 \otimes \gamma^1) \]

\[ + (\gamma^0 \otimes \gamma^0) e_2 \phi^{\text{self}}(2) \} \Phi(x_1, t_1 | x_2, t_2) = 0, \quad (2.13b) \]

where the self-potential is

\[ \phi^{\text{self}}(x|t_1, t_2) \equiv \int_0^L dy \int_0^L dz D(x, y|L) \{ e_1 \overline{\Phi}(y, t_1 | z, t_2)(\gamma^0 \otimes \gamma^0) \Phi(y, t_1 | z, t_2)

\[ + e_2 \overline{\Phi}(z, t_2 | y, t_1)(\gamma^0 \otimes \gamma^0) \Phi(z, t_2 | y, t_1) \}, \]

and

\[ \phi^{\text{self}}(1) \equiv \phi^{\text{self}}(x_1 | t_1, t_2), \quad \phi^{\text{self}}(2) \equiv \phi^{\text{self}}(x_2 | t_2, t_1), \]

i.e., we have a pair of Dirac equations on \( \Phi \) instead of a single one in the first formulation. For \( t_1 = t_2 \equiv t \), \( \phi^{\text{self}}(x|t, t) \) coincides with the self-potential \( \phi^{\text{self}}(x|t) \) given by (2.6).

The compatibility condition for the two equations is

\[ [G_1, G_2] \Phi = 0. \quad (2.14) \]

It can be checked that this condition reduces to

\[ e_1 \frac{\partial \phi^{\text{self}}(1)}{\partial t_2} = e_2 \frac{\partial \phi^{\text{self}}(2)}{\partial t_1}, \quad (2.15) \]

i.e. requires a specific time dependence of the self-potentials.

Taking the sum and the difference of Eqs.(2.13a-b) we get

\[ \{ (\gamma^\mu p_{(1), \mu} - m_1) \otimes \gamma^0 + \gamma^0 \otimes (\gamma^\mu p_{(2), \mu} - m_2) + (\gamma^0 \otimes \gamma^0)(e_1 \phi^{\text{self}}(1) + e_2 \phi^{\text{self}}(2)) \]

\[ - \frac{1}{2} e_1 b(t_1)(\gamma^1 \otimes \gamma^0) - \frac{1}{2} e_2 b(t_2)(\gamma^0 \otimes \gamma^1) \} \Phi(x_1, t_1 | x_2, t_2) = 0, \quad (2.16a) \]

and

\[ \{ (\gamma^\mu p_{(1), \mu} - m_1) \otimes \gamma^0 - \gamma^0 \otimes (\gamma^\mu p_{(2), \mu} - m_2) + (\gamma^0 \otimes \gamma^0)(e_1 \phi^{\text{self}}(1) - e_2 \phi^{\text{self}}(2)) \]

\[ - \frac{1}{2} e_1 b(t_1)(\gamma^1 \otimes \gamma^0) + \frac{1}{2} e_2 b(t_2)(\gamma^0 \otimes \gamma^1) \} \Phi(x_1, t_1 | x_2, t_2) = 0. \quad (2.16b) \]

The first equation is in fact the two-body equation derived earlier with the Coulomb potential included into the self-potentials, the only difference being in the number of time variables, while (2.16b) is a new equation on \( \Phi \).

To make clear the nature of the new equation, we use again the center of mass and relative coordinates. Acting along similar lines as above, we obtain the Hamiltonian form of the equations on \( \Phi \):

\[ p_0 \Phi = \{ \alpha_+ \Pi^1 + \alpha_- \pi^1 - e_1 \phi^{\text{self}}(1) - e_2 \phi^{\text{self}}(2) + \beta_1 m_1 + \beta_2 m_2 \} \Phi, \quad (2.17a) \]

\[ p_0 \Phi = \{ \alpha_+ \pi^1 + \alpha_- \Pi^1 - e_1 \phi^{\text{self}}(1) + e_2 \phi^{\text{self}}(2) + \beta_1 m_1 - \beta_2 m_2 \} \Phi. \quad (2.17b) \]

In addition to the two-body equation we have therefore an equation which includes the relative energy \( p_0 \). While the center of mass energy plays the role of the "Hamiltonian" of the two-body system, the relative energy (or its conjugate variable, the relative time) is an unphysical variable and
must be eliminated to avoid possible unphysical effects, for example, relative energy excitations in
the spectrum.

In the spectrum problem we can simply put
\[ p_0 \Phi = 0, \quad (2.18) \]
i.e., assume that \( \Phi \) does not depend on the relative time \( \tau = t_1 - t_2 \). We could also start from
the beginning with the field \( \Phi \) composed of the individual matter fields taken at the same time
\( t_1 = t_2 = t \). Then the compatibility condition of the two Dirac equations would be
\[ \frac{\partial}{\partial t} (e_1 \phi^{self}(x_1, t) - e_2 \phi^{self}(x_2, t)) = 0. \quad (2.19) \]
We shall continue our discussion of the pair of Dirac equations formulation in Sec. 4.

### 3 MASSLESS CASE

There are three types of interactions in the first quantized two-body Hamiltonian \( P_0 \), namely, interaction described by the self-potentials , interaction between the matter fields and global electro-
magnetic field degree of freedom and the Coulomb interaction . All these interactions influence the spectrum, the self-potentials being responsible for radiative processes.

Let us find the eigenfunctions and the spectrum of \( P_0 \) in the single two-body equation formulation.
The consideration below is at fixed time \( t = 0 \). The equation for the eigenfunctions is
\[ (\alpha_+ \Pi^1 + \alpha_- \pi^1 + \beta_1 m_1 + \beta_2 m_2) \Phi = (E + V) \Phi, \quad (3.1) \]
where
\[ \Pi^1 = 2i \frac{\partial}{\partial x_+} - \frac{1}{2} (e_1 + e_2) b, \]
\[ \pi^1 = 2i \frac{\partial}{\partial x_-} - \frac{1}{2} (e_1 - e_2) b, \]
and
\[ V(x_-, x_+) = \phi + e_1 \phi^{self}_1 + e_2 \phi^{self}_2. \]

If we denote the components of \( \Phi \) as
\[ \Phi^{11} \equiv \eta_1, \quad \Phi^{12} \equiv \eta_2, \]
\[ \Phi^{21} \equiv \eta_3, \quad \Phi^{22} \equiv \eta_4, \]
then (3.1) reduces to the system of four equations
\[ 2i \frac{\partial}{\partial x_+} \eta_1 - (V + E + \frac{1}{2} (e_1 + e_2) b) \eta_1 = -m_1 \eta_3 - m_2 \eta_2, \]
\[ 2i \frac{\partial}{\partial x_+} \eta_4 + (V + E - \frac{1}{2} (e_1 + e_2) b) \eta_4 = m_1 \eta_2 + m_2 \eta_3, \quad (3.2) \]
\[ 2i \frac{\partial}{\partial x_-} \eta_2 - (V + E + \frac{1}{2} (e_1 - e_2) b) \eta_2 = -m_1 \eta_4 - m_2 \eta_1, \]
\[ 2i \frac{\partial}{\partial x_-} \eta_3 + (V + E - \frac{1}{2}(e_1 - e_2)b) \eta_3 = m_1 \eta_1 + m_2 \eta_4. \]  

(3.3)

The global electromagnetic field degree of freedom shows itself in all four equations. For \( e_1 = -e_2 \), \( b \) drops out of the first pair of the equations, and for \( e_1 = e_2 \) of the second one.

We see from these equations that

\[ \eta_1^*(E, -e_1, -e_2) = \eta_4(E, e_1, e_2), \]  

(3.4a)

\[ \eta_2^*(E, -e_1, -e_2) = \eta_3(E, e_1, e_2), \]  

(3.4b)

so only half of all solutions correspond to physical particles.

The conditions (3.4a-b) are modified in the case of the massless matter fields and vanishing total potential \( V \),

\[ \eta_1(-E, e_1, e_2) = \eta_4(E, e_1, e_2), \]

\[ \eta_2(-E, e_1, e_2) = \eta_3(E, e_1, e_2), \]

i.e., the negative energy solutions of \( \eta_1 \) and \( \eta_2 \) coincide correspondingly with the positive energy solutions of \( \eta_4 \) and \( \eta_3 \). Again only half of all solutions correspond to physical particles.

The boundary and normalization conditions for \( \eta_i \) (\( i = 1, 4 \)) deduced from the ones for the individual matter fields are

\[ \eta_i(L\|L) = \exp\{i2\pi\kappa_i^{(1)}\} \eta_i(0\|0), \]

\[ \eta_i(-L\|L) = \exp\{i2\pi\kappa_i^{(2)}\} \eta_i(0\|0), \]

\[ \eta_i(0\|2L) = \exp\{i2\pi(\kappa_1^{(i)} + \kappa_2^{(i)})\} \eta_i(0\|0), \]

and

\[ \int_{-L}^{L} dx_- \int_{0}^{2L} dx_+ \eta_i^*(x_-|x_+) \eta_i(x_-|x_+) = 1, \]

respectively (no summation over \( i \)).

For the massless matter fields, \( m_1 = m_2 = 0 \) and \( \eta_i \) decouple from each other in Eqs.(3.2) - (3.3) which are therefore simplified. In what follows we consider in detail the two-body problem for the massless matter fields.

### 3.1 The case \( \phi_{{\text{self}}}{(1)} = \phi_{{\text{self}}}{(2)} = 0 \)

In [12] we put the self-potentials \( \phi_{{\text{self}}}(k) \) equal to zero and solved Eqs.(3.2)-(3.3) for the massless matter fields only in the presence of the Coulomb interaction and \( b \) treated as an external field. Here we want to give the same solution but without the additional assumption \( A_0(0, t) = 0 \) used earlier. For this reason the expressions for the eigenfunctions and the spectrums given below are slightly different from the ones in [12].

The solution is

\[ \eta_i^c, n = \frac{1}{2L} \exp\{-i\frac{e_1 e_2}{2} I_1(x_-, x_+) - i\frac{1}{2}(E_{1,n}^c + \frac{1}{2}(e_1 + e_2)b)x_+\}, \]

(3.5)
\[ \eta_{2,n}^c = \frac{1}{2L} \exp\left\{ -\frac{i}{2} e_2 I_2(x_-, x_+) - \frac{i}{2} (E_{2,n}^c + \frac{1}{2} (e_1 - e_2)b) x_- \right\} , \] 

(3.6)

where

\[ I_1(x_-, x_+) \equiv \frac{1}{2} x_+ D_+(x_+|L) + x_+ D_-(x_-|L) - \frac{1}{24L} x_+^3, \]
\[ I_2(x_-, x_+) \equiv \frac{1}{2} x_- D_-(x_-|L) + x_+ D_+(x_+|L) + \frac{1}{24L} x_-^3. \]

The eigenvalues \( E_{1,n}^c, E_{2,n}^c \) are determined by the boundary conditions. From the boundary condition connecting the values of \( \eta_1 \) at the points \((x_- = 0, x_+ = 0)\) and \((x_- = 0, x_+ = 2L)\) we get

\[ E_{1,n}^c = -\frac{1}{2L} \int_0^{2L} dz V(0,z) + \frac{2\pi n}{L} - \frac{1}{2} (e_1 + e_2)b, \quad n \in \mathbb{Z}, \]

while the boundary conditions connecting the values of \( \eta_2 \) at \((x_- = 0, x_+ = 0)\) and \((x_- = \pm L, x_+ = L)\) give

\[ E_{2,n}^c = -\frac{1}{2L} \int_{-L}^{L} dz V(z,L) + \frac{2\pi n}{L} - \frac{1}{2} (e_1 - e_2)b, \quad n \in \mathbb{Z}. \]

For \( V = \phi \) (with the assumption \( A_0(0,t) = 0 \) both parts of the Coulomb potential \( \phi_+ \) and \( \phi_- \) would be asymmetric \cite{12}), we easily evaluate the integrals, so the spectrums become

\[ E_{1,n}^c = -\frac{1}{3} e_1 e_2 L + \frac{2\pi}{L} n - \frac{1}{2} (e_1 + e_2)b, \]
\[ E_{2,n}^c = -\frac{5}{12} e_1 e_2 L + \frac{2\pi}{L} n - \frac{1}{2} (e_1 - e_2)b. \]

The eigenfunctions \( \eta_{3,n}^c, \eta_{4,n}^c \) are obtained from Eqs.\((3.5)-(3.6)\) by making use of the relations \((3.4a-b)\), the corresponding spectrums being

\[ E_{3,n}^c = -\frac{5}{12} e_1 e_2 L + \frac{2\pi}{L} n + \frac{1}{2} (e_1 - e_2)b, \]
\[ E_{4,n}^c = -\frac{1}{3} e_1 e_2 L + \frac{2\pi}{L} n + \frac{1}{2} (e_1 + e_2)b, \quad n \in \mathbb{Z}. \]

The superscript “c” indicates that the eigenfunctions \( \eta_{i,n}^c \) and the eigenvalues \( E_{i,n}^c \) represent the solution of our two-body problem in the presence of the Coulomb interaction, but without the self-potentials.

The boundary conditions fix also the phases \( \kappa_{1,n}^{(i)}, \kappa_{2,n}^{(i)} \),

\[ \kappa_{1,n}^{(1)} = \kappa_{1,n}^{(4)} = -\kappa_{2,n}^{(1)} = -\kappa_{2,n}^{(4)} = \frac{n}{2}, \]
\[ \kappa_{1,n}^{(2)} = \kappa_{1,n}^{(3)} = \kappa_{2,n}^{(2)} = \kappa_{2,n}^{(3)} = \frac{n}{2}. \]

3.2 The case \( \phi_{(1)}^{\text{self}} \neq 0, \phi_{(2)}^{\text{self}} \neq 0 \)

Let us now solve Eqs.\((3.2)-(3.3)\) in the presence of the self-potentials. In the self-field approach to quantum electrodynamics in four dimensions the self-field effects are calculated by an iteration
procedure. To lowest order of iteration we take the fields to be given by the solutions without the self-energy terms, and the energies to be shifted by a small amount:

\[ \eta_{i,n} = \eta_{i,n}^c, \]

\[ E_n = E_n^c + \Delta E_n. \]

But in two-dimensional quantum electrodynamics for some problems we need not apply the iteration procedure, because these problems can be solved exactly. We show below that it is the case for the two-body problem in the massless 2-dim QED.

For the vanishing matter masses, the general structure of the solutions to the Eqs. (3.2)-(3.3) with the self-potentials is the same as of the corresponding solutions in the case when the self-field effects are neglected. All the solutions are exponents. So if these solutions are normalized, they fulfil the conditions

\[ \eta^*_i \eta_i = \frac{1}{4L^2}, \quad i = 1, 2, 3, 4. \]

The bilinear combinations \( \Phi(\gamma^0 \otimes \gamma^0) \Phi \) which enter the expressions for the self-potentials can be therefore easily evaluated as

\[ \Phi(\gamma^0 \otimes \gamma^0) \Phi = \sum_{i=1}^{4} \eta^*_i \eta_i = \frac{1}{L^2}. \]

The self-potentials take the exact and closed form

\[ \phi^{\text{self}}_{(1)} = \frac{e_1}{2L} \left( x^2 + \frac{L^2}{2} \right), \]

\[ \phi^{\text{self}}_{(2)} = \frac{e_2}{2L} \left( x^2 + \frac{L^2}{2} \right). \]

In terms of the center of mass and relative coordinates the self-field part of the total potential \( V \) becomes

\[ e_1 \phi^{\text{self}}_{(1)} + e_2 \phi^{\text{self}}_{(2)} = \frac{1}{8L} (e_1^2 + e_2^2) (x_+^2 + x_-^2) + \frac{1}{4L} (e_1^2 - e_2^2) x_+ x_- + \frac{L}{4} (e_1^2 + e_2^2), \]

i.e., only for \( e_1 = \pm e_2 \) the self-potentials do not destroy the additivity of the center of mass and relative Hamiltonians.

With the self-potentials, Eqs. (3.2)-(3.3) are solved by the eigenfunctions

\[ \eta_{1,n} = \frac{1}{2L} \exp \{- \frac{i}{2} e_1 e_2 I_1(x_-, x_+) - \frac{i}{2} (E_{1,n} + \frac{1}{2} (e_1 + e_2)b) x_+ \}
\]

\[ \quad - \frac{i}{2} e_1^2 J^{(1)}_1(x_-, x_+) - \frac{i}{2} e_2^2 J^{(2)}_1(x_-, x_+) \}, \quad (3.7) \]

\[ \eta_{2,n} = \frac{1}{2L} \exp \{- \frac{i}{2} e_1 e_2 I_2(x_-, x_+|a) - \frac{i}{2} (E_{2,n} + \frac{1}{2} (e_1 - e_2)b)(x_- - aL) \}
\]

\[ \quad - \frac{i}{2} e_1^2 J^{(1)}_2(x_-, x_+|a) - \frac{i}{2} e_2^2 J^{(2)}_2(x_-, x_+|a) \}, \quad (3.8) \]

where

\[ I_2(x_-, x_+|a) = \frac{1}{2} x_- D_-(x_-|L) + (x_- - aL) D_+(x_+|L) + \frac{1}{24L} x_-^3 + I_2(0,0|a), \]
and
\[ I_2(0, 0|a) \equiv \begin{cases} \frac{L^2}{24} a^2 (2a + 3) & \text{for } a < 0, \\ \frac{L^2}{24} a^2 (2a - 9) & \text{for } a > 0, \end{cases} \]

while
\[ J_1^{(1)}(x_-, x_+) \equiv \frac{1}{24L} ((x_+ + x_-)^3 - x_-^3) + \frac{L}{4} x_+, \]
\[ J_1^{(2)}(x_-, x_+) \equiv \frac{1}{24L} ((x_+ - x_-)^3 + x_-^3) + \frac{L}{4} x_+, \]
\[ J_2^{(1)}(x_-, x_+|a) \equiv \frac{1}{24L} ((x_+ + x_-)^3 - (x_+ + aL)^3) + \frac{L}{4} (x_- - aL), \]
\[ J_2^{(2)}(x_-, x_+|a) \equiv \frac{1}{24L} (- (x_+ - x_-)^3 + (x_+ - aL)^3) + \frac{L}{4} (x_- - aL). \]

The constant \( a \) depends on the charges \( e_1, e_2 \), namely,
\[ a = (e_2 - e_1)/(e_2 + e_1) \text{ for } e_1 \neq \pm e_2 \text{ and } a = 0 \text{ for } e_1 = \pm e_2. \]

In the Coulomb case when the self-potentials are not taken into account, \( a \) vanishes, and
\[ I_2(x_-, x_+|0) = I_2(x_-, x_+), \]

the functions \( \eta_{1,n}, \eta_{2,n} \) reducing to the Coulomb eigenfunctions \( \eta_{1,n}^c, \eta_{2,n}^c \).

The eigenvalues acquire a shift,
\[ E_{1,n} = E_{1,n}^c + \Delta E_1, \quad (3.9a) \]
\[ E_{2,n} = E_{2,n}^c + \Delta E_2, \quad (3.9b) \]

which is nothing else than the self-energy
\[ \Delta E_1 \equiv - \frac{1}{2L} \int_0^{2L} dz (e_1 \phi_{(1)}^{\text{self}}(0, z) + e_2 \phi_{(2)}^{\text{self}}(0, z)), \]
\[ \Delta E_2 \equiv - \frac{1}{2L} \int_{-L}^{L} dz (e_1 \phi_{(1)}^{\text{self}}(z, L) + e_2 \phi_{(2)}^{\text{self}}(z, L)). \]

The shift is the same for both spectrums
\[ \Delta E_1 = \Delta E_2 \equiv \Delta E = - \frac{5}{12} e_1^2 L - \frac{5}{12} e_2^2 L. \]

The eigenfunctions \( \eta_{3,n}, \eta_{4,n} \) are related to \( \eta_{1,n}, \eta_{2,n} \) by Eqs. (3.4a-b), the corresponding spectrums being shifted by the same amount \( \Delta E \).

The self-potentials contribute also to the boundary conditions phases
\[ \kappa_{1,n}^{(1)} = - \kappa_{2,n}^{(1)} = \frac{n}{2} + \kappa_1^{\text{self}}, \]
\[ \kappa_{1,n}^{(2)} = \kappa_{2,n}^{(2)} = \frac{n}{2} + \kappa_2^{\text{self}}, \]
\[ \kappa_{1,n}^{(3)} = - \kappa_{2,n}^{(3)} = \frac{n}{2} - \kappa_2^{\text{self}}, \]
\[ \kappa_{1,n}^{(4)} = - \kappa_{2,n}^{(4)} = \frac{n}{2} - \kappa_1^{\text{self}}. \]
where
\[
\kappa_{1\text{self}}^1 = \frac{(e_2^2 - e_1^2)L^2}{32\pi},
\]
\[
\kappa_{2\text{self}}^2 = \frac{(e_1 + e_2)^2L^2}{32\pi}a(2 - a^2).
\]

The additional phases \(\kappa_{1\text{self}}, \kappa_{2\text{self}}\) vanish in the case \(e_1 = \pm e_2\).

With the Coulomb and self-interaction shifts, the spectrums for \(e_1 = -e_2 \equiv e\), for instance, become
\[
E_{1,n} = -\frac{1}{2}e^2L + \frac{2\pi}{L}n,
\]
\[
E_{2,n} = -\frac{5}{12}e^2L + \frac{2\pi}{L}n - eb, \quad n \in \mathbb{Z}.
\]

Eqs.(3.7)-(3.9) represent the complete and exact solution of the two-body problem for the massless matter fields.

\section{4 EQUIVALENCE}

In the pair of Dirac equations formulation, the Coulomb potential is included into the self-field terms. With the assumptions that the composite matter field does not depend on the relative time and \(t_1 = t_2\), the total self-potential coincides with the corresponding one in the single two-body equation formulation and can be evaluated exactly as
\[
\phi^{\text{self}}(x) = \frac{e_1 + e_2}{2L}(x^2 + \frac{L^2}{2}).
\]
It is time-independent and satisfies the compatibility condition (2.19).

The eigenvalue problem for the two-body Hamiltonian reduces to the system of two equations for each component of the composite field. For \(\eta_1\), we have
\[
(\Pi ^1 - V - E)\eta_1 = 0, \tag{4.1}
\]
\[
(\pi ^1 - U)\eta_1 = 0, \tag{4.2}
\]
where the last equation means the vanishing of the relative energy, and
\[
V(x_-, x_+) = e_1\phi^{\text{self}}(\frac{x_+ + x_-}{2}) + e_2\phi^{\text{self}}(\frac{x_+ - x_-}{2}),
\]
\[
U(x_-, x_+) \equiv e_1\phi^{\text{self}}(\frac{x_+ + x_-}{2}) - e_2\phi^{\text{self}}(\frac{x_+ - x_-}{2}).
\]
The potentials \(V\) and \(U\) fulfil the relations
\[
\frac{\partial V}{\partial x_-} = \frac{\partial U}{\partial x_+}, \tag{4.3a}
\]
\[
\frac{\partial V}{\partial x_+} = \frac{\partial U}{\partial x_-}. \tag{4.3b}
\]
the first one being the compatibility condition for Eqs. (4.1) and (4.2).

The general solution of (4.1) is

\[ \eta_1(x_-, x_+) = \chi_1(x_-) \exp \left\{ -\frac{i}{2} \int_{0}^{x_+} dz V(x_-, z) - \frac{i}{2} (E + \frac{1}{2} (e_1 + e_2) b)x_+ \right\}, \tag{4.4} \]

where \( \chi_1 \) is a function depending only on the relative coordinate \( x_- \).

Substituting this solution into (4.2) and using the relations (4.3a-b), we get the equation for \( \chi_1 \),

\[ (\pi^1 - U(x_-, 0)) \chi_1 = 0, \tag{4.5} \]

which is solved by

\[ \chi_1(x_-) = \exp \left\{ -\frac{i}{2} \int_{0}^{x_-} dz U(z, 0) - \frac{1}{4} (e_1 - e_2) bx_- \right\}. \tag{4.8} \]

Although the solution (4.4)-(4.5) includes both potentials \( V \) and \( U \), only the potential \( V \) contributes to the eigenvalue spectrum. Indeed, \( \chi_1(0) = 1 \) at the boundary points \((x_- = 0, x_+ = 0)\) and \((x_- = 0, x_+ = 2L)\). Since just the boundary condition connecting the values of \( \eta_1 \) at these points determines the spectrum, the potential \( U \) drops out of this boundary condition, and for the spectrum we get the same expression as in the single two-body equation formulation.

For the second component \( \eta_2 \), the system of equations for the eigenfunctions is

\[ (\pi^1 + V - E) \eta_2 = 0, \tag{4.6a} \]
\[ (\Pi^1 + U) \eta_2 = 0. \tag{4.6b} \]

The solution is given by

\[ \eta_2(x_-, x_+) = \chi_2(x_+) \exp \left\{ \frac{i}{2} \int_{0}^{x_-} dz V(z, x_+) - \frac{i}{2} (E + \frac{1}{2} (e_1 + e_2) b)(x_- - aL) \right\}, \tag{4.7} \]

where

\[ \chi_2(x_+) = \exp \left\{ \frac{i}{2} \int_{0}^{x_-} dz U(0, z) - \frac{i}{4} (e_1 + e_2) bx_+ \right\}. \tag{4.8} \]

From the boundary conditions relating the points \((x_- = 0, x_+ = 0)\) and \((x_- = \pm L, x_+ = L)\) and with \( \kappa_1^{(2)} = \kappa_2^{(2)} \) we get the following condition determining the spectrum

\[ \eta_2(-L|L) = \eta_2(L|L). \]

In both parts of this condition we have the function \( \chi_2(x_+) \) taken at the same center of mass coordinate \( x_+ = L \), so the potential \( U \) drops out again.

Thus, in both formulations the eigenvalue spectrums coincide. This proves that the two formulations are equivalent to each other in the spectrum problem.

### 5 DISCUSSION

1. For \((1+1)\)-dimensional self-field QED, we have presented two different formulations of the two-body problem in accordance with two different types of variational principles. These two formulations are closely related but not identical. In the first formulation we vary the action with respect to the composite matter field and get a single two-body equation. In the second formulation we require
the action be stationary with respect to the individual fields. This condition is stronger and leads to a pair of equations for the composite field. In addition to the two-body equation we have an equation including the relative energy. While the single two-body equation formulation is one-time formulation without any restrictions on the self-potentials, the second formulation has in general two time coordinates. Only for a special time dependence of the self-potentials one of the time coordinates, i.e., the relative time can be eliminated.

We have shown that for the massless matter fields the eigenvalue spectrums of the two-body Hamiltonian in both formulations coincide. The two formulations are therefore equivalent in the spectrum problem. Nevertheless, the second formulation with two compatible equations on the same composite field provides more complete information about the eigenfunctions. The single two-body equation does not fix the eigenfunctions uniquely. We can multiply the components $\eta_1$ and $\eta_4$ by an arbitrary function depending only on the relative coordinate $x_-$ and the components $\eta_2$ and $\eta_3$ by one depending only on $x_+$.  

2. We have proved that the relativistic two-body problem in the massless two-dimensional quantum electrodynamics is exactly soluble. In the single two-body equation formulation, we have solved the covariant two-body equation with both mutual and self-interactions and found the eigenfunctions and the spectrum of the two-body Hamiltonian.

For the massive matter fields, the eigenvalue problem for the two-body Hamiltonian becomes essentially more complicated. In this case, $\eta$'s are not decoupled in the system of equations (3.2)-(3.3). If we try to decouple them, then we arrive at a set of second-order differential equations which can be solved only in some approximation. We can take the masses $m_1, m_2$ as small parameters and consider the mass contribution to the two-body Hamiltonian eigenfunctions and eigenvalues as small corrections to the corresponding eigenfunctions and eigenvalues for the vanishing masses. The discussion of the massive case will be given elsewhere.

There is an essential difference in the Coulomb and self-interaction shifts in the spectrums. The Coulomb interaction shifts the discrete energy spectrums by a value which is different for $E_{1,n}$ and $E_{2,n}$ ($E_{4,n}$ and $E_{3,n}$), the difference being equal to $\frac{1}{12}e_1e_2L$, while the self-interaction shift is the same for all four spectrums.

If we call formally the first and second components of the two-component fields $\psi_k$ as "up" and "down" components, then the two-body system states described by $\eta_{i,n}$ ($i = 1, 4$) can be interpreted correspondingly as "up-up", "up-down", "down-up" and "down-down" states. The Coulomb interaction shift is therefore the same for the "up-up" and "down-down" states and takes a different value for the "up-down" and "down-up" states. Thus we can recognize the effects of spin-spin interactions in the first quantized theory.

For arbitrary values of $e_1, e_2$, all the spectrums $E_{i,n}$ depend on the global electromagnetic field degree of freedom $b$. The global degree of freedom contribution to the spectrums is specific to models defined on the circle. For models on the line, the electromagnetic field has neither local nor global physical degrees of freedom and so can be eliminated completely from the two-body Hamiltonian.

For $e_1 = -e_2$, only the spectrums $E_{2,n}$ and $E_{3,n}$ corresponding to the "up-down" and "down-up" states, and for $e_1 = e_2$ only $E_{1,n}$ and $E_{4,n}$ corresponding to the "up-up" and "down-down" states depend on $b$.

3. The standard SM with a single matter field of charge $e$ is equivalent to the theory of a free scalar field with mass $e^2/\pi$ [7]. In our work, we have looked at the SM from a different point of view. We have constructed the mass spectrum for the model with two matter fields. The spectrum obtained does not contain the boson of the SM. This result is not surprising. It is well known from the second quantized version of $(1 + 1)$-dimensional QED that only on light front the SM boson can be represented as a bound state of two fields, fermion and antifermion [14, 15]. The study of the
self-field SM on light front was given in [10].

In (3 + 1)-dimensions the formulations presented above give us a possibility to make calculations for real two-body systems. In the self-field approach, the single two-body equation formulation was used in [5] to calculate the energy spectrum for positronium and muonium. In the framework of the constraint approach, the pair of Dirac equations formulation was applied to the phenomenological calculation of the $q - ar{q}$ meson bound state spectrum as well as to the study of the dynamics of quarkonium systems [3, 4]. Both formulations produce results which agree with experiment. However, to clarify the difference between the two formulations in (3 + 1)-dimensions an analytical work along the lines given in the present paper for (1 + 1)-dimensions is needed.

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