Antitrace maps and light transmission coefficients for a
generalized Fibonacci multilayers

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Abstract

By using antitrace map method, we investigate the light transmission for a
generalized Fibonacci multilayers. Analytical results are obtained for trans-
mission coefficients in some special cases. We find that the transmission coef-
ficients possess two-cycle property or six-cycle property. The cycle properties
of the trace and antitrace are also obtained.
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The transmission of light through the multilayers arranged by the Fibonacci [1–3], non-Fibonacci sequence [4,5], Thue-Morse sequence [6], and the generalized Thue-Morse sequence [7] was studied in the literature. Schwartz [8] suggested a possibility of quasiperiodic multilayers as optical switches and memories. Huang et al [9] and Yang et al [10] have found an interesting switch-like property in the light transmission through Fibonacci-class sequences.

On the other hand, the trace-map technique, first introduced in 1983 [11], has proven to be a powerful tool to investigate the properties of various aperiodic systems. However, as pointed by Dulea et al [4], we must know the so-called “antitrace map” when we evaluate the light transmission coefficients through aperiodic sequences. Here, the so-called “antitrace” of a \( 2 \times 2 \) matrix

\[
A = \begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix}
\] (1)

is defined as \( y = A_{21} - A_{12} \). Recently we have given a detailed study on the antitrace maps for various aperiodic systems and shown that the antitrace maps exist for arbitrary substitution sequences [12].

In this paper, we use the antitrace map method to evaluate the transmission coefficients through a generalized Fibonacci sequence. Its substitution rule is [13]

\[
b \rightarrow a, \quad a \rightarrow a^n b.
\] (2)

Now we consider that the light transmit vertically through the generalized Fibonacci multilayer which is sandwiched by two media of type \( a \). The corresponding transfer matrices \( A_l \) are written as [1]

\[
A_1 = P_{ab}P_bP_{ba},
\]

\[
A_2 = P_a,
\]

\[
A_{l+1} = A_l^n A_{l-1},
\] (3)

where \( P_{ab}(P_{ba}) \) stands for the propagation matrix from layer \( a(b) \) to \( b(a) \) and \( P_a(P_b) \) is the propagation matrix through single layer \( a(b) \). They are given by [1]
\[ P_{ab} = P_{ba}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & n_a/n_b \end{pmatrix}, \]
\[ P_{a(b)} = \begin{pmatrix} \cos \delta_{a(b)} & -\sin \delta_{a(b)} \\ \sin \delta_{a(b)} & \cos \delta_{a(b)} \end{pmatrix}, \]

where \( \delta_{a(b)} = k n_{a(b)} d_{a(b)} \), \( n_{a(b)} \) is the refraction index of media \( a(b) \), \( d_{a(b)} \) are the thickness of layers, and \( k \) is the wave number in vacuum. The quantity \( \delta_{a(b)} \) is the phase difference between the ends of a layer.

We remark here that the trace and antitrace map of the generalized Fibonacci sequence is identical to that of the Fibonacci-class sequence \[10\] since they have the same recursion relations for the transfer matrix. However the initial two transfer matrices \( A_1 \) and \( A_2 \) are different.

The transmission coefficient is given by \[1\]
\[ T_l = \frac{4}{|A_l|^2 + 2}, \]
where \( |A_l|^2 \) is the sum of squares of the four elements of \( A_l \). Since the transfer matrix is unimodular, we can express the transmission coefficient in the following form
\[ T_l = \frac{4}{x_l^2 + y_l^2}, \]
where \( x_l \) and \( y_l \) denote the trace and antitrace of the transfer matrix \( A_l \), respectively.

From Eq. \[3\], we see that the transmission coefficient is completely determined by the trace and antitrace, i.e., a complete description of the light transmission through general aperiodic multilayers requires both the trace and antitrace map \[4\].

In the following discussion we need to know the \( n \)th power of a unimodular 2×2 matrix \( A \), which can be written as \[14\] \[17\]
\[ A^n = U_n(x_A)A - U_{n-1}(x_A)I, \]
where \( I \) is the unit matrix and
\[ U_n(x_A) = \frac{\lambda_+^n - \lambda_-^n}{\lambda_+ - \lambda_-}, \]
\[ \lambda_{\pm} = x_A \pm \sqrt{x_A^2 - 4}, \]
Here $x_A$ and $\lambda_{\pm}$ denote the trace and the two eigenvalues of $A$, respectively.

Using Eq. (7), we can write the recursion relation of the transfer matrix (3) as

$$A_{l+1} = U_n(x_l)A_lA_{l-1} - U_{n-1}(x_l)A_{l-1}. \quad (9)$$

From the above equation, the trace map is easily obtained as

$$x_{l+1} = U_n(x_l)v_l - U_{n-1}(x_l)x_{l-1},$$
$$v_{l+1} = U_{n+1}(x_l)v_l - U_n(x_l)x_{l-1}, \quad (10)$$

where $v_l = \text{tr}(A_lA_{l-1})$ is a subsidiary quantity.

In order to study antitrace maps we need the following identity for two unimodular transfer matrices $A$ and $B$ [4]

$$y_{AB} = x_By_A + x_Ay_B - y_{BA}, \quad (11)$$

where $y_A$ denotes the antitrace.

Using the above equation and Eq. (9), we obtain the antitrace map as

$$y_{l+1} = U_n(x_l)\bar{w}_l - U_{n-1}(x_l)y_{l-1},$$
$$\bar{w}_{l+1} = x_{l+1}y_l + U_{n-1}(x_l)\bar{w}_l - U_{n-2}(x_l)y_{l-1}. \quad (12)$$

Here $\bar{w} = y_{A_lA_{l-1}}$ is also subsidiary. The trace and antitrace map are completely determined by Eqs. (10) and (12). The forms of trace and antitrace maps are different from that in Refs. [4] and [10]. These forms are easy to be obtained and are convenient for application. If we know the initial conditions, the transmission coefficients can be determined from the trace and antitrace map. Note that the coefficients in Eq. (12) are dependent on the traces of the transfer matrices. We must know the trace map when we make use of the antitrace map.

We choose appropriate thickness of the layers $d_a$ and $d_b$ to make $n_ad_a = n_bd_b$. Then we have $\delta_a = \delta_b = \delta$. For $\delta = (k + 1/2)\pi$, the propagation matrices $P_{a(b)}$ become

$$P_a = P_b = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (13)$$
Then from Eqs. (3), (4), and (13) the initial conditions for the trace and antitrace map are obtained as
\begin{align*}
x_1 &= 0, \\
x_2 &= 0, \\
v_2 &= -(R + R^{-1}), \\
y_1 &= R + R^{-1}, \\
y_2 &= 2, \\
\bar{w}_2 &= 0,
\end{align*}
(14)
where \( R = n_a/n_b \). The initial conditions depend on a single parameter \( R \).

Now we classify the generalized Fibonacci sequence into two classes, i.e., the even family with \( n = 2m \) and the odd family with \( n = 2m + 1 \), \( m=0,1,2,\ldots \).

First we consider the even family. From the trace map, the initial conditions and the following property of the function \( U_n(x) \)
\begin{align*}
U_{2m}(0) &= 0, \quad U_{2m+1}(0) = (-1)^m,
\end{align*}
(15)
we see that the trace \( x_l = 0 \) for any \( l \). Therefore the transmission coefficient is only dependent on the antitrace \( y_l \). From the antitrace map equation and the initial conditions, we obtain the antitrace \( y_l \) and the transmission coefficients. The result is shown in Table I.

| \( l \) | \( y_l \) (odd \( m \)) | \( y_l \) (even \( m \)) | \( T_l \) |
|-------|----------------|----------------|-----|
| 1     | \( R + R^{-1} \) | \( R + R^{-1} \) | \( \frac{4}{R^2 + R^{-2} + 2} \) |
| 2     | 2               | 2              | 1   |
| 3     | -(\( R + R^{-1} \)) | \( R + R^{-1} \) | \( \frac{4}{R^2 + R^{-2} + 2} \) |
| 4     | -2              | 2              | 1   |

We see the antitrace has four-cycle property (odd \( m \)) or two-cycle property (even \( m \)). The transmission coefficients have two-cycle property. For even \( l \), the transmission coefficient
is 1, while for odd \( l \), the transmission coefficient is \( 4/(R^2 + R^{-2} + 2) \). Actually from the first line of Eq. (12) and Eq. (15), we obtain

\[
y_{l+1} = (-1)^m y_{l-1}.
\]  

(16)

This leads to Table I immediately.

Next we consider the odd family. From the trace map, antitrace map, and the initial conditions, we obtain Table II.

Table II. The antitrace and transmission coefficients for the odd family. The upper sign refers to even, and the lower sign refers to odd values of \( m \).

| \( l \) | \( x_l \) | \( y_l \) | \( T_l \) |
|------|------|------|------|
| 1 | 0 | \( R + R^{-1} \) | \( \frac{4}{R^2 + R^{-2} + 2} \) |
| 2 | 0 | 2 | 1 |
| 3 | \( \mp (R + R^{-1}) \) | 0 | \( \frac{4}{R^2 + R^{-2} + 2} \) |
| 4 | 0 | \( \mp (R^n + R^{-n}) \) | \( \frac{4}{R^{2n} + R^{-2n} + 2} \) |
| 5 | 0 | \( \pm (R^{n-1} + R^{-n+1}) \) | \( \frac{4}{R^{2n-2} + R^{-2n+2} + 2} \) |
| 6 | \( \pm (R + R^{-1}) \) | 0 | \( \frac{4}{R^2 + R^{-2} + 2} \) |

The trace \( x_l \), antitrace \( y_l \) and the transmission coefficients all have six-cycle property. From Table II, we know that the trace and antitrace is not completely same for odd and even \( m \). They have a sign difference. The trace and antitrace are zero alternatively. We also see that the transmission coefficient is 1 for \( l = 6k + 2(k = 0, 1, 2, \ldots) \), i.e., the light is transparent in this case.

In conclusion, we have studied the light transmission for the generalized Fibonacci sequences by using the antitrace method. The analytical results for the transmission coefficient are obtained in some special cases. The transmission coefficients have two-cycle property for even \( n \) and six-cycle property for odd \( n \). The analysis in this paper shows that it is very convenient to use the antitrace map method to evaluate the transmission coefficients.
This method can be used not only in the generalized Fibonacci sequences, but also in other substitution sequences.

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