HELMHOLTZ AND DISPERSIVE EQUATIONS WITH VARIABLE COEFFICIENTS ON EXTERIOR DOMAINS

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Abstract. We prove smoothing estimates in Morrey-Campanato spaces for a Helmholtz equation

\[ -Lu + zu = f, \quad -Lu := \nabla^b(a(x)\nabla^b u) - c(x)u, \quad \nabla^b := \nabla + ib(x) \]

with fully variable coefficients, of limited regularity, defined on the exterior of a starshaped compact obstacle in \( \mathbb{R}^n, n \geq 3 \), with Dirichlet boundary conditions. The principal part of the operator is a long range perturbation of a constant coefficient operator, while the lower order terms have an almost critical decay. We give explicit conditions on the size of the perturbation which prevent trapping.

As an application, we prove smoothing estimates for the Schrödinger flow \( e^{itL} \) and the wave flow \( e^{it\sqrt{L}} \) with variable coefficients on exterior domains and Dirichlet boundary conditions.

1. Introduction

The Helmholtz equation

\[ \Delta u + zu = f(x) \]  

(1.1)

where \( z \in \mathbb{C} \) and \( x \) varies on \( \mathbb{R}^n \) or on the exterior \( \Omega = \mathbb{R}^n \setminus K \) of an obstacle \( K \), is used to model standing waves in many different applications in physics and engineering. When \( z \not\in \mathbb{R} \), (1.1) can be written as the resolvent equation \( u = R_0(z)f \) for \( R_0(z) = (-\Delta - z)^{-1} \), and the interesting problem is to prove uniform estimates for \( z \) approaching the spectrum of the operator. In addition, smoothing estimates for (1.1) have become an important tool to prove smoothing and Strichartz estimates for many dispersive equations including wave, Schrödinger, Klein-Gordon and Dirac equations.

The theory of (1.1) is classical, while in recent years generalizations of (1.1) with lower order terms have attracted some attention. In [38], [39] the Helmholtz equation on \( \mathbb{R}^n \) with a variable refraction index

\[ \Delta u - n(x)u + zu = f(x) \]

is studied; first order perturbations were examined in [26] on the whole space, and on exterior domains in [3].

Resolvent estimates for elliptic operators of various type have been studied intensely, initially in the framework of spectral theory and scattering, more recently for use in proving decay estimates for evolution equations. The two strains of research have independent origin in the seminal papers [28] (see also [30]), [1] and in Morawetz’ work on wave equations [35], [36]; Fourier transform with respect to the time variable acts as a bridge between the two points of view. Resolvent estimates for \(-\Delta + V(x)\) were proved in [6], [7], and more recently for singular potentials in
Magnetic potentials \((i \nabla - A(x))^2 + V(x)\) were studied in [21] (small \(A\)) and [25], [15] (large \(A\)).

The case of Laplace-Beltrami operators on some classes of manifolds has also attracted intense attention. Among the many contributions, we mention [9], [50], [10], [16], [51], [8] for high frequency resolvent estimates on asymptotically flat or conical manifolds, possibly with boundary. Global smoothing estimates for all frequencies in the case of manifolds which are flat outside a compact region were obtained in [44], and quantitative bounds for more general manifolds were proved in [43].

A key point in the previous results is that the principal part of the operator, and the obstacle, must be nontrapping, meaning that the Hamiltonian flow of the operator leaves compact sets in a finite time. Actually, weaker resolvent estimates are still true in the trapped case, but the bounds may grow exponentially in the frequency, making them unsuitable for applications to dispersive equations. The importance of this condition in the context of smoothing estimates was noted at an early stage, see [18], [24], [10].

In this paper we prove smoothing estimates on a starshaped exterior domain for a Helmholtz equation with fully variable coefficients, and for the corresponding Schrödinger and wave equations. In summary, under our assumptions, the principal part is a (small) long range perturbation of a constant coefficient operator, while the lower order terms have an almost critical decay at infinity. Our method of proof, based on an adaptation of the multiplier method, permits to give explicit quantitative nontrapping conditions. By this we mean that the assumptions can be explicitly checked in concrete examples, in contrast with the usual formulation of nontrapping in terms of the bicharacteristic flow.

We consider a Helmholtz equation

\[
\nabla^b \cdot (a(x) \nabla^b v) - c(x)v + (\lambda + i\epsilon)v = f, \quad \lambda \in \mathbb{R}, \quad \epsilon > 0
\]

on an exterior domain \(\Omega = \mathbb{R}^n \setminus \omega\), with \(\omega\) bounded and possibly empty; coefficients are real valued, \(a(x) = [a_{jk}(x)]_{j,k=1}^n\) and \(\nabla^b\) denotes the covariant derivatives

\[
\nabla^b = (\partial^b_1, \ldots, \partial^b_n), \quad \partial^b_j = \partial_j + ib_j(x).
\]

The conditions on the coefficients are the following:

**Assumptions on** \(a(x)\). The symmetric, real valued matrix \(a(x) = [a_{jk}(x)]_{j,k=1}^n\) satisfies

\[
NI \geq a(x) \geq \nu I, \quad N \geq \nu > 0, \quad (1.3)
\]

and denoting with \(|a(x)|\) the operator norm of the matrix \(a(x)\) and writing \(|a'| = \sum_{|\alpha|=1} |\partial^\alpha a(x)|, |a''| = \sum_{|\alpha|=2} |\partial^\alpha a(x)|\) and \(|a'''| = \sum_{|\alpha|=3} |\partial^\alpha a(x)|\), we have

\[
|a'(x)| + |x||a''(x)| + |x|^2|a'''(x)| \leq C_a(x)^{-1-\delta}, \quad \delta \in (0,1). \quad (1.4)
\]

**Assumptions on** \(b(x)\). The coefficients \(b(x) = (b_1, \ldots, b_n)\) are real valued and the matrix \(db(x) := [\partial_j b_k - \partial_k b_{j,l=1}^n]_{j,k=1}^n\) satisfies

\[
|db(x)| \leq \frac{C_{b}}{|x|^2+|x|^2+|x|^2} \quad (1.5)
\]

**Assumptions on** \(c(x)\). The potential \(c(x)\) is real valued and satisfies

\[
-\frac{C_c^2}{|x|^2+|x|^2+|x|^2} \leq c(x) \leq \frac{C_c^2}{|x|^2}. \quad (1.6)
\]

Moreover we assume that \(c(x)\) is repulsive with respect to the metric \(a(x)\), meaning that

\[
a(x)x \cdot \nabla c \leq \frac{C_c}{|x|^2(x)+|x|^2} \quad (1.7)
\]
ASSUMPTIONS ON THE DOMAIN. The domain $\Omega \subseteq \mathbb{R}^n$ is an exterior domain, i.e. the complement of a compact and possibly empty set. We assume that $\partial \Omega$ is $C^1$ and $a(x)$-starshaped, meaning that at all points of $\partial \Omega$ the exterior normal $\vec{v}$ to $\partial \Omega$ satisfies
\[ a(x)\vec{x} \cdot \vec{v} \leq 0. \tag{1.8} \]
When $a = I$, (1.8) reduces to the condition that $\omega$ is starshaped with respect to the origin.

Remark 1.1 (Selfadjointness). By the previous assumptions, the operator $L = -A^b + c$ is symmetric and satisfies the inequality
\[ (Lv, v)_{L^2(\Omega)} \geq \nu \|\nabla v\|_{L^2} - C^2 \|(|x|^{2+\delta} + |x|^{2-\delta})^{-1/2}v\|_{L^2(\Omega)} \quad \forall v \in C_c(\Omega). \]
By the magnetic Hardy inequality (3.15), this implies that
\[ (Lv, v)_{L^2(\Omega)} \geq -C\|v\|_{L^2(\Omega)}. \]
Thus the operator $L$ has a selfadjoint Friedrichs extension on $L^2(\Omega)$, which is sufficient for our purposes. Actually, it is possible to prove that $L$ is essentially selfadjoint and hence the selfadjoint extension is unique: this follows by Chernoff’s result [17] since the wave equation $ut_{tt} + Lu = 0$ has finite speed of propagation.

Our first result is a homogeneous smoothing resolvent estimate, expressed in terms of the Morrey-Campanato type norms
\[ \|v\|_X^2 := \sup_{R > 0} \frac{1}{R^n} \int_{\mathbb{R}^n \cap \{|x| = R\}} |v|^2 dx, \quad \|v\|_Y^2 := \sup_{R > 0} \frac{1}{R} \int_{\Omega \cap \{|x| \leq R\}} |v|^2 dx, \]
while the $\hat{Y}^*$ norm is predual to the $\hat{Y}$ norm (see an explicit characterization in Section 3). Note that the result is only partially satisfactory in the case $n = 3$ which will be considered in detail below (see Remark 1.5).

Theorem 1.1. Let $n \geq 3$. Consider the Helmholtz equation (1.2) with Dirichlet boundary conditions on an exterior domain $\Omega$ as in (1.8). Assume that (1.3), (1.4), (1.5), (1.6) and (1.7) hold. Finally, assume the ratio $N/\nu$ satisfies
\[ \frac{N}{\nu} \leq \sqrt{\frac{n^2 + 2n + 15}{n(n+2)}} \quad \text{for } 3 \leq n \leq 46, \quad \frac{N}{\nu} < \frac{3n-1}{n+3} \quad \text{for } n \geq 47 \tag{1.9} \]
and the constants $C_a, C_- , C_c, C_b$ are small enough that
\[ C_a(N + C_a) \leq \frac{K\delta}{2m}, \quad C_b \leq \frac{K\delta}{5N^2}, \quad C_- \leq \frac{K\delta}{18(N+2)}, \quad C_c \leq K\delta, \tag{1.10} \]
where $K = \min \left\{ 1, \frac{\nu^2}{8}, \frac{(a+3)\nu^2}{18n+3} - \frac{3n-1}{n+3} \right\}$. Then the solution to (1.2) satisfies
\[ \|v\|_X^2 + \|\nabla v\|_Y^2 \leq M_0 \|f\|_Y^2, \tag{1.11} \]
and
\[ |\lambda||v|_Y^2 \leq 2n^2(\nu + 1)^2M_0 \|f\|_Y^2, \quad |\nu||v|_Y^2 \leq 9(\nu + 1)M_0 \|f\|_Y^2. \tag{1.12} \]
where $M_0 = 64n^2K^{-2}(\nu + 1)^2(C_+ + \nu + 1)^2$.

Remark 1.2. The use of multiplier methods has several advantages versus the phase space approach, besides simplicity. Indeed, one can prove sharp estimates in terms of Morrey-Campanato norms which are stronger than the usual weighted $L^2$ norms; in addition, one obtains quantitative bounds which would be impossible to prove when using e.g. Fredholm theory. The technique used here was introduced in [38], and then improved in [4], [5], [23], [3], and the main novelty of the present paper is the adaptation of the method to a general elliptic operator with variable coefficients. In spirit, this paper is close to [43] where a version of the multiplier method for manifolds is developed.
However, in our opinion, the best feature of the method is the possibility to obtain explicit (although non sharp) criteria to check if an operator is nontrapping. In addition, tracking the constants with precision allows to see their qualitative dependence on the parameters of the problem. As a simple example, consider the diagonal case
\[ \nabla \cdot (\alpha(x)\nabla u) + \frac{C}{|x|^2} u + zu = f \quad \text{on} \quad \mathbb{R}^d, \]
where \( C \geq 0 \) and \( \alpha(x) : \mathbb{R}^d \to \mathbb{R} \) is a scalar function satisfying for some \( \delta \in (0,1) \)
\[ 1 \leq \alpha(x) \leq \sqrt{\frac{12}{\delta^2}}, \quad |\alpha'| + |x| |\alpha''| + |x^2| |\alpha'''| \leq \frac{\delta}{\sqrt{\delta^2}(x)^{1-\delta}}. \]
Under these conditions, all the assumptions of the Theorem are satisfied, and the smoothing estimates (1.11), (1.12) are true.

**Remark 1.3.** For high energies \( \lambda >> 1 \) the estimates are valid for the more general class of potentials with Coulomb decay \( |c(x)| \leq \frac{C}{|x|^d} \). Indeed, if we perturb the equation with a term \( c_0(x)v \):
\[ A^b v - c(x)v + (\lambda + i\epsilon)v = f - c_0(x)v =: \tilde{f}(x) \]
and we apply the previous result, we obtain
\[ \|\nabla v\|_Y + \sqrt{|\lambda| + |\epsilon|}\|v\|_Y + \|v\|_X - C\|c_0(x)v\|_Y \leq C\|f\|_Y. \]
Since
\[ \|c_0v\|_Y \lesssim \|\lambda\|^\frac{1}{2}\|v\|_Y, \]
we see that we can absorb the negative term in the term \( |\lambda|^\frac{1}{2}\|v\|_Y \), provided
\[ \lambda \gtrsim \|\lambda\|\|c_0\|_{L^\infty}. \]
Note that a similar argument allows to absorb small first order perturbations of critical decay in the term \( \|\nabla v\|_Y \).

**Remark 1.4.** For the proof of the Theorem, instead of the second condition in (1.9), it is sufficient to assume the following weaker pointwise inequality:
\[ 2|\alpha(x)|^2_{HS} + \pi^2 - \pi(x)\hat{a}(x) + 15\hat{a}^2(x) - 12|\alpha(x)\hat{x}|^2 \geq 0 \quad \forall x \in \Omega. \quad (1.13) \]
where \( |\alpha(x)|_{HS} \) is the Hilbert-Schmidt norm of the matrix \( \alpha(x) \), \( \pi(x) \) its trace, \( \hat{a}(x) = a(x)\hat{x} \cdot \hat{x} \) and \( \hat{x} = x/|x| \). Since
\[ 2|\alpha|_{HS} + \pi^2 - 6\pi\hat{a} + 15\hat{a}^2 - 12|\alpha\hat{x}|^2 \geq (2n + n^2 + 15)\nu^2 - 6(n + 2)N^2 \]
we see that the second condition in (1.9) implies (1.13).

**Remark 1.5.** In the case \( n = 3 \), assumption (1.9) forces \( \nu = N \) so that \( a(x) \) must be a diagonal operator, and this is of course too restrictive for our purposes. There is not much to gain if we revert to the weaker assumption (1.13): for instance, if \( a(x) = \text{diag}[1,1,1+\epsilon] \) and we choose \( \hat{x} = (0,0,1) \), the quantity in (1.13) is equal to \(-8\epsilon\), thus generic small perturbations of \( I \) are ruled out also under the weaker assumption.

For this reason we complement Theorem 1.1 with an additional result in which \( a(x) \) is allowed to be any small perturbation of identity. The drawback is that we obtain a slightly weaker *nonhomogeneous* estimate, which is expressed in terms of the norms
\[ \|v\|_X^2 := \sup_{R > 0} \frac{1}{(4\pi)^{3/2}} \int_{\Omega \cap \{|x|=R\}} |v|^2 dS, \quad \|v\|_Y^2 := \sup_{R > 1} \frac{1}{\pi} \int_{\Omega \cap \{|x|\leq R\}} |v|^2 dx, \]
and the \( Y^* \) norm, predual to the \( Y \) norm. Then we have:
Theorem 1.2. Let \( n = 3 \). Consider the Helmholtz equation (1.2) with Dirichlet boundary conditions on an exterior domain \( \Omega \) as in (1.8). Assume that (1.3), (1.4) hold and that \( \text{db}(x) \) satisfies
\[
|\text{db}(x)| \leq \frac{C_2}{|x|^{1+\delta}}
\]
for some \( \delta \in (0, 1) \), while the electric potential \( c(x) \) satisfies
\[
-\frac{C_2^2}{(x)^{2+\sigma}} \leq c(x) \leq \frac{C_2^2}{|x|^{2+\sigma}}
\]
and
\[
a(x) x \cdot \nabla c \leq \frac{C_2}{|x|^{3+\sigma}}.
\]
Finally, assume the principal part is close to identity in the following sense:
\[
|a(x) - I| \leq C_1(x)^{-\delta}, \quad C_1 < 1/100
\]
and the constants \( C_a, C_- , C_t, C_c, C_b \) are small enough that
\[
C_a \leq \frac{4}{3000}, \quad C_t \leq \frac{5}{3000}, \quad C_b \leq \frac{4}{3000}, \quad C_- \leq \frac{4}{3000}, \quad C_c \leq \frac{4}{3000}.
\]
Then the solution to (1.2) satisfies the estimates
\[
\|v\|_X^2 + \|\nabla^b v\|_Y^2 \leq 10^9 (C_+^2 + 1) ||f||_Y^2,
\]
and
\[
|\lambda||v||_X^2 \leq 10^10 (C_+^2 + 1)^2 ||f||_Y^2, \quad |\epsilon||v||_Y^2 \leq 10^10 (C_+^2 + 1) ||f||_Y^2.
\]

We conclude the Introduction with some implications of the previous estimates for equations of Schrödinger and wave type connected to the operator \( A^b - c \).

1.1. Applications. The estimates in Theorems 1.1 and 1.2 have several applications. Natural consequences are a limiting absorption principle for the operator \( L = -A^b + c \), the absence of embedded or zero eigenvalues and resonances, and the existence and uniqueness of solutions for the Helmholtz equation \((-L + \lambda)v = f\) under a Sommerfeld radiation condition at infinity. We shall study these and related questions in a forthcoming paper; here we will focus on the applications to dispersive evolution equations connected to the operator \( L = -A^b + c \).

Namely, we consider the Schrödinger equation
\[
iu_t - A^b u + c(x) u = F(t, x), \quad u(0, x) = u_0(x)
\]
and the wave equation
\[
u_t - A^b u + c(x) u = F(t, x), \quad u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x)
\]
associated with \( L \). Smoothing and decay properties of solutions are best expressed in terms of the corresponding Schrödinger flow \( e^{itL} \) and wave flow \( e^{it\sqrt{-L}} \). Kato’s smoothing theory [29] (see also [30]) provides a direct link between resolvent estimates and smoothing estimates for the Schrödinger flow. Actually it is possible to prove that estimates for the full resolvent, \textit{supersmoothing} estimates in the terminology of Kato–Yajima, are equivalent to \textit{nonhomogeneous} estimates (like (1.26), (1.27) in the following Corollary), while the \textit{homogeneous} estimates of the form (1.25) are equivalent to weaker estimates for the imaginary part of the resolvent, which are properly called \textit{smoothing} estimates. A detailed exposition of the theory, together with an extension which allows to include wave and Klein-Gordon equations at the same level of generality, can be found in [19]. An additional consequence of the resolvent estimates is the pointwise decay in time of local norms of the solution, which can be interpreted as a stronger form of the RAGE theorem of [46] for these flows.

In order to simplify the exposition, we shall make two additional hypotheses on the operator \( L \) (but see also Remarks 1.6 and 1.8):
Positivity of \(L\). We assume that \(L\) is a positive operator, i.e., \((Lu,v)_{L^2(Ω)} ≥ 0\) for all \(v\) in the domain of \(L\). Note that in view of the magnetic Hardy inequality (3.15) proved below and the previous assumptions on the coefficients, we have
\[
(Lu,u)_{L^2(Ω)} ≥ ν∥∇^b u∥^2_{L^2(Ω)} - C^2 ∥|x|^{-1} u∥^2_{L^2(Ω)} ≥ \frac{1}{4}ν - C^2 1 104\end{equation}
thus in order to have \(L ≥ 0\) (and actually coercive) it is sufficient to assume that
\[
C_− < \frac{n−2}{2} \sqrt{ν}.
\]

Weighted elliptic estimate. We assume that the operator \(L\) satisfies the weighted \(L^2\) estimates
\[
∥⟨x⟩−\frac{1}{2}−σ L^\frac{1}{2} v∥_{L^2(Ω)} ≤ ∥⟨x⟩−\frac{1}{2}−σ'(−Δ)^\frac{1}{2} e∥_{L^2(Ω)}
\]
for \(σ > σ' > 0\) close to 0. Note that the right hand side in (1.24) is equivalent to \(∥⟨x⟩−\frac{1}{2}−σ'∇v∥_{L^2(Ω)}\) since \(⟨x⟩−\frac{1}{2}−σ\) is an \(A_2\) weight. This assumption is not too restrictive: indeed, this kind of elliptic estimates is known in several cases (see e.g. [43] for elliptic operators with smooth coefficients on \(\mathbb{R}^n\)). More generally, the estimate follows directly from, or can be proved by the techniques of [13] for any selfadjoint operator \(L\) whose heat kernel \(e^{L}\) satisfies an upper gaussian estimate; this covers the case of elliptic operators on exterior domains with bounded coefficients [37] and magnetic Schrödinger operators with singular coefficients on \(\mathbb{R}^n\) [31].

Then we can prove the following results:

**Corollary 1.3.** Assume the selfadjoint operator \(L = −\mathcal{A}^b + c\) with Dirichlet b.c. on \(L^2(Ω)\) and the exterior domain \(Ω\) satisfy the assumptions of Theorem 1.1. Assume in addition (1.23) and (1.24). Then we have the estimates
\[
∥⟨x⟩^{−1}−e^{itL} f∥_{L^2(Ω)} + ∥⟨x⟩^{−\frac{1}{2}}−L^\frac{1}{2} e^{itL} f∥_{L^2(Ω)} ≤ ∥f∥_{L^2(Ω)},
\]
\[
∥∫_0^t⟨x⟩^{−1}−e^{i(t−s)L} Fds∥_{L^2(Ω)} ≤ ∥⟨x⟩^{1+} F∥_{L^2(Ω)},
\]
\[
∥∫_0^t⟨x⟩^{−\frac{1}{2}}−L^\frac{1}{2} e^{i(t−s)L} Fds∥_{L^2(Ω)} ≤ ∥⟨x⟩^{\frac{1}{2}+} F∥_{L^2(Ω)}.
\]
Moreover, for every \(f \in L^2(Ω)\), we have the RAGE type property
\[
∥⟨x⟩^{−\frac{1}{2}}−e^{itL} f∥_{L^2(Ω)} → 0 \text{ as } t → ±∞.
\]

**Remark 1.6.** Note that we can also apply the Morawetz multiplier method directly to equation (1.21), instead of using Kato’s theory. This approach does not require assumptions (1.23), (1.24) and gives an estimate of the form
\[
∥∇ e^{itL} f∥_{Y L^2} + ∥e^{itL} f∥_{XL^2} ≤ ∥f∥_{X^{1+}},
\]
but it does not seem easy to transfer the \(1/2\) derivative from the right to the left hand side. Note also that the norms at the l.h.s. are of reversed type, with an inner integration in \(t\). However, this method can be used to prove interaction Morawetz estimates, which will be the topic of a forthcoming paper.

**Remark 1.7.** Assume, in addition to the above, that the operator \(L\) coincides with the Laplacian \(−Δ\) outside a bounded set, and moreover that the local in time (non endpoint) Strichartz estimates hold for solutions of equation (1.21) which are compactly supported in space. Then, by a well known procedure due to Burq [11], it is possible to deduce from the smoothing estimate (1.25) the full set of global in time (non endpoint) Strichartz estimates for the same equation.

The same remark applies to the wave equation (1.22) which we consider next.
Moreover, for every available also in the case of fully variable coefficients, both local and global in time wave and Dirac equations with electric and magnetic potentials; some results are mentioned at least the papers \[ \text{[48, 12, 33, 42, 34].} \]

\section{Basic identities}

In this section we show how to adapt the method of Morawetz multipliers to a general elliptic operator with variable coefficients. Using implicit summation over repeated indices, we can write the principal part in the form

\[ A^b v := \nabla^b \cdot (a(x) \nabla^b v) = \partial_j (a_{jk}(x) \partial_k v). \]

We also write

\[ Av := \nabla \cdot (a(x) \nabla v) = \partial_j (a_{jk}(x) \partial_k v) \]

and we recall the notations

\[ \hat{x} = \frac{x}{|x|}, \quad a(w, z) = a_{jk}(x) w_k z_j. \]

The appropriate multiplier for the operator \( A^b \) is the quantity

\[ [A^b, \psi] = (Av) \overline{\psi} + 2a(\nabla \psi, \nabla^b v). \]

(2.1)

The following identities are based on the multiplier (2.1) and the simpler multiplier \( \phi \overline{\psi} \).

\section{Helmholtz Equation}

\begin{corollary}
Assume \( L = -A^b + c \) and \( \Omega \) are as in the previous corollary. Then we have the estimates

\[ \|\langle x \rangle^{-\frac{1}{2}} e^{it\sqrt{\mathcal{T}}} f\|_{L^2 L^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)}, \]

(1.29)

\[ \|\langle x \rangle^{-1} e^{it\sqrt{\mathcal{T}}} f\|_{L^2 L^2(\Omega)} \lesssim \|f\|_{H^\frac{1}{2}(\Omega)}, \]

(1.30)

\[ \| \int_0^t \langle x \rangle^{-\frac{1}{2}} e^{i(t-s)\sqrt{\mathcal{T}}} F ds \|_{L^2 L^2(\Omega)} \lesssim \|\langle x \rangle^{\frac{1}{2}} F\|_{L^2 L^2(\Omega)}, \]

(1.31)

\[ \| \int_0^t \langle x \rangle^{-1} e^{i(t-s)\sqrt{\mathcal{T}}} F ds \|_{L^2 L^2(\Omega)} \lesssim \|\langle x \rangle^{\frac{1}{2}} L^\frac{1}{2} F\|_{L^2 L^2(\Omega)}, \]

(1.32)

Moreover, for every \( f \in L^2(\Omega) \), we have the RAGE type property

\[ \|\langle x \rangle^{-\frac{1}{2}} e^{it\sqrt{\mathcal{T}}} f\|_{L^2(\Omega)} \to 0 \quad \text{as} \quad t \to \pm \infty. \]

(1.33)

\end{corollary}

Remark 1.8. As for the Schrödinger equation, a direct application of the multiplier method to the wave equation (1.22) gives an estimate of the form

\[ \|\nabla e^{itL} f\|_{Y L^2} + \|e^{it\sqrt{\mathcal{T}}} f\|_{X L^2} \lesssim \|f\|_{H^1}. \]

Remark 1.9. There is a large literature on smoothing properties for Schrödinger, wave and Dirac equations with electric and magnetic potentials; some results are available also in the case of fully variable coefficients, both local and global in time (see e.g. [52, 40, 18, 27, 41], [46, 3, 32, 43, 47, 39, 20, 14] and the references therein). On the other hand, the case of exterior domains is less studied: we mention at least the papers [48, 12, 33, 42, 34].
where
\[ \alpha_{\ell m} = 2a_{jm}\partial_j(a_{\ell k}\partial_k\psi) - a_{jk}\partial_k\psi\partial_ja_{\ell m}, \]
and
\[ \partial_jP_j = A^b v \cdot \phi \overline{\psi} + a(\nabla^b v, \nabla^b v)\phi - \frac{1}{2}A\phi|v|^2 + i3a(\nabla^b v, v\nabla\phi). \quad (2.4) \]

**Proof.** As mentioned above, both identities can be deduced by multiplying the quantity \( A^b v - V(x)v \) by (2.1) or \( \phi\overline{\psi} \) respectively. The computations are long but elementary, and once the identities (2.3) and (2.4) are known, it is straightforward to check their validity. We omit the details. \( \Box \)

Applying the two identities of the previous Proposition to a solution of the Helmholtz equation with \( V = c(x) - \lambda - ie \), then adding them, after a few elementary manipulations we obtain:

**Theorem 2.2.** Assume \( v \in H^2_{loc}(\Omega) \) satisfies
\[ A^b v - (c(x) - \lambda - ie)v = f(x) \quad (2.5) \]
for \( a, b, c \) real valued, \( a(x) \) symmetric, \( \lambda, e \in \mathbb{R} \). Then the following identities hold for any real weights \( \phi(x), \psi(x) \):
\[ \Re\partial_j\{Q_j + P_j\} = -\frac{1}{2}A(\nabla^b v, \nabla^b v)v^2 - [a(\nabla^b v, \nabla^c) - c\phi + \lambda\phi]|v|^2 \]
\[ + \Re[\alpha_{\ell m}\partial_\ell\nabla^b v - a(\nabla^b v, \nabla^b v)\phi] + 2\Re[\partial_\ell\alpha_{\ell m}\partial_m\nabla^b v]
\]
\[ + \Re[\partial_\ell\alpha_{\ell m}\psi]\overline{\nabla}\overline{f} + 2a(\nabla^b v, \nabla^b v)f \quad (2.6) \]
and
\[ \partial_jP_j = a(\nabla^b v, \nabla^b v)\phi + (c - \lambda - ie)|v|^2\phi + f\overline{\nabla}\overline{\psi} - \frac{1}{2}A\phi|v|^2 + i3a(\nabla^b v, v\nabla\phi), \quad (2.7) \]
where
\[ P_j = a_{jk}\partial_k\phi \nabla v - \frac{1}{2}\partial_{jk}\partial_k\phi|v|^2 \]
\[ Q_j = a_{jk}\partial_k\phi \nabla v - \frac{1}{2}a_{jk}(\partial_k\phi)|v|^2 - a_{jk}\partial_k\psi \left[(c - \lambda)|v|^2 + a(\nabla^b v, \nabla^b v)\right] \]
and
\[ \alpha_{\ell m} = 2a_{jm}\partial_j(a_{\ell k}\partial_k\psi) - a_{jk}\partial_k\psi\partial_ja_{\ell m}. \]

3. **Morrey-Campanato type norms and their properties**

In this section we prove some relations between the Morrey-Campanato type norms \( \tilde{X}, \tilde{Y}, X, Y \) and usual weighted \( L^2 \) norms. If \( \Omega \) is an open subset of \( \mathbb{R}^n \), \( n \geq 2 \), we write
\[ \Omega_{=R} = \Omega \cap \{x: |x| = R\}, \quad \Omega_{<R} = \Omega \cap \{x: |x| < R\}, \quad \Omega_{>R} = \Omega \cap \{x: |x| > R\}, \]
\[ \Omega_{R_1 \leq |x| \leq R_2} = \Omega \cap \{x: R_1 \leq |x| \leq R_2\}. \]
The homogeneous norm \( \tilde{X} \) and the corresponding predual norm \( \tilde{X}^* \) of a function \( v: \Omega \to \mathbb{C} \) are defined as
\[ \|v\|_{\tilde{X}}^2 := \sup_{R > 0} \frac{1}{2\pi} \int_{\Omega_{=R}} |v|^2 dS, \quad \|v\|_{\tilde{X}^*} := \int_0^{+\infty} r \left( \int_{\Omega_{=r}} |v|^2 dS \right)^{\frac{1}{2}} dr, \]
where \( dS \) is the surface measure on \( \Omega_{=R} \). The corresponding nonhomogeneous versions, denoted by \( X, X^* \), are
\[ \|v\|_X^2 := \sup_{R > 0} \frac{1}{2\pi} \int_{\Omega_{=R}} |v|^2 dS, \quad \|v\|_{X^*} := \int_0^{+\infty} (r) \left( \int_{\Omega_{=r}} |v|^2 dS \right)^{\frac{1}{2}} dr \quad (3.1) \]
where \( \langle R \rangle = \sqrt{1 + R^2} \). We shall also need proper Morrey-Campanato spaces, both in the homogeneous version \( \dot{Y} \) and in the non homogeneous version \( Y \); their norms are defined as

\[
\|v\|_{Y^*}^2 := \sup_{R > 0} \frac{1}{R} \int_{\Omega \subseteq R} |v|^2 \, dx, \quad \|v\|_{\dot{Y}^*}^2 := \sup_{R > 0} \frac{1}{R^2} \int_{\Omega \subseteq R} |v|^2 \, dx.
\]

(3.2)

The following equivalence is easy to prove:

\[
\|v\|_{Y^*}^2 \leq \sup_{R \geq 1} \frac{1}{R} \int_{\Omega \subseteq R} |v|^2 \leq \sqrt{2} \|v\|_{Y^*}^2.
\]

(3.3)

A concrete characterization of the predual norms \( \dot{Y}^* \), \( Y^* \) is less immediate. The simplest approach is to introduce an equivalent dyadic norm

\[
\|v\|_{Y^*} := \sup_{j \in \mathbb{Z}} 2^{-j/2} \|v\|_{L^2(\Omega_j, \mathbb{R}^d)}
\]

which satisfies, as it is readily seen,

\[
2^{-1} \|v\|_{Y^*} \leq \|v\|_{Y} \leq 2 \|v\|_{Y^*}
\]

and similarly

\[
\|v\|_{\dot{Y}^*} := \|v\|_{L^2(\Omega_j, \mathbb{R}^d)} + \sup_{j \geq 1} 2^{-j/2} \|v\|_{L^2(\Omega_{j-1}, \mathbb{R}^d)}
\]

which satisfies

\[
3^{-1} \|v\|_{\dot{Y}^*} \leq \|v\|_{Y} \leq 3 \|v\|_{\dot{Y}^*}
\]

We then obtain the following characterizations:

\[
\|v\|_{Y^*} \sim \sum_{j \in \mathbb{Z}} 2^{j/2} \|v\|_{L^2(\Omega_j, \mathbb{R}^d)}, \quad \|v\|_{\dot{Y}^*} \sim \|v\|_{L^2(\Omega_j, \mathbb{R}^d)} + \sum_{j \geq 1} 2^{j/2} \|v\|_{L^2(\Omega_{j-1}, \mathbb{R}^d)}.
\]

The following Lemmas contain several estimates to be used in the rest of the paper.

**Lemma 3.1.** For any \( v \in C_c^\infty(\mathbb{R}^n) \),

\[
\|x\|_{Y^*} \leq \|v\|_{X^*}, \quad \|x\|_{\dot{Y}^*} \leq \|v\|_{X}.
\]

(3.4)

**Proof.** Both inequalities are immediate in polar coordinates: for the first we have

\[
\sup_{R > 0} \frac{1}{R} \int_{\Omega \subseteq R} \frac{|v|^2}{|x|^2} \leq \sup_{R > 0} \frac{1}{R} \int_0^R \rho \, d\rho \, \frac{1}{\rho} \int_{\Omega \subseteq \rho} |v|^2 \, dS \, d\rho \leq \|v\|_{X^*} \sup_{R > 0} \frac{1}{R} \int_0^R \rho \, d\rho.
\]

and the second one is similar.

\( \square \)

**Lemma 3.2.** For any \( 0 < \delta < 1 \) and \( v \in C_c^\infty(\mathbb{R}^n) \),

\[
\int_{\Omega} \frac{|v|^2}{|x|^2 \delta} \leq 2\delta^{-1} \|v\|_{X^*}^2,
\]

(3.5)

\[
\int_{\Omega \subseteq 1} \frac{|v|^2}{|x|^2 \delta} \leq \int_{\Omega \subseteq 1} \frac{|v|^2}{|x|^2 \delta} \leq 2\delta^{-1} \|v\|_{X^*}^2,
\]

(3.6)

\[
\int_{\Omega \subseteq \rho} \frac{|v|^2}{|x|^2 \delta} \leq 8\delta^{-1} \|v\|_{Y} \leq 8\delta^{-1} \|v\|_{Y^*}^2.
\]

(3.7)

**Proof.** To prove (3.5), (3.6) we write

\[
\int_{\Omega} \frac{|v|^2}{|x|^2 \delta} \leq \|v\|_{X}^2 \int_0^\infty \langle \rho \rangle^{-1-\delta} \, d\rho, \quad \int_{\Omega \subseteq 1} \frac{|v|^2}{|x|^2 \delta} \leq \|v\|_{X}^2 \int_1^\infty \frac{\rho^2}{\rho^{1+\delta}} \, d\rho
\]

and notice that \( \langle \rho \rangle \geq (1 + \rho)/\sqrt{2} \), and that \( \langle \rho \rangle^2 \leq 2\rho^2 \) for \( \rho \geq 1 \). To prove (3.7) we split \( \Omega = \Omega_{\leq 1} \cup \Omega_{\geq 1} \); we have immediately for the first piece

\[
\int_{\Omega_{\leq 1}} \frac{|v|^2}{|x|^2 \delta} \leq \int_{\Omega_{\leq 1}} |v|^2 \leq \sqrt{2} \|v\|_{Y^*}^2.
\]

For the remaining piece we use a dyadic decomposition \( \Omega_{\geq 1} = \bigcup_{j \geq 0} \Omega_{2^j, \leq |x| \leq 2^{j+1}} \), we notice that

\[
\int_{\Omega_{2^j, \leq |x| \leq 2^{j+1}}} \frac{|v|^2}{|x|^2 \delta} \leq \frac{2^j}{(2^j + 1)^2} \int_{\Omega_{2^j+1}} |v|^2 \leq \|v\|_{Y} \frac{(2^j+1)^2}{(2^j+1)^2}.
\]
and we sum over \( j \geq 0 \). Since
\[
\sum_{j \geq 0} \frac{j^{2+\delta}}{(2j+1)^{2\delta+1}} \leq 2^{3/2} \sum_{j \geq 0} 2^{-j\delta} \leq \frac{2^{3/2+\delta}}{2\delta-1} \leq 2^{5/2}\delta^{-1}
\]
we obtain (3.7) with a constant \( \sqrt{2} + 2^{5/2} \leq 8 \).

□

**Lemma 3.3.** For any \( v \in C^\infty_c(\mathbb{R}^n) \),
\[
\sup_{R>1} \frac{1}{R} \int_{|x| \leq 2R} \frac{|vw|}{|x|^2} dx \leq 3\|v\|_X^2,
\]
(3.8)
\[
\sup_{R>0} \frac{1}{R} \int_{R^{-1} \leq |x| \leq 2R} \frac{|vw|}{|x|^2} dx \leq \frac{3}{2}\|v\|_X^2,
\]
(3.9)
\[
\sup_{R>0} \int_{R^{-1} \leq |x| \leq 2R} \frac{|v|^2}{|x|^2} dx \leq \frac{3}{2}\|v\|_X^2,
\]
(3.10)

**Proof.** Trivial.

□

**Lemma 3.4.** For any \( R > 0 \), \( 0 < \delta < 1 \) and \( v, w \in C^\infty_c(\mathbb{R}^n) \),
\[
\int_{|x| \leq 2R} \frac{|vw|}{|x|^2} dx \leq 3R^2\|v\|_X\|w\|_Y,
\]
(3.11)
\[
\int_{|x| \leq 2R} \frac{|vw|}{|x|^2} dx \leq 3(R)^2\|v\|_X\|w\|_Y,
\]
(3.12)
\[
\int_{|x| \leq 2R} \frac{|vw|}{|x|^2} dx \leq 3(R)^2\|v\|_X\|w\|_Y,
\]
(3.13)
\[
\int_{|x| \leq 2R} \frac{|vw|}{|x|^2} dx \leq 12\delta^{-1}\|v\|_X\|w\|_Y.
\]
(3.14)

**Proof.** The first inequalities in (3.11) and (3.12) are trivial; the second ones follow by duality and by the estimates \( \|1_{\Omega_{\leq n}}v\|_Y \leq R\|v\|_X \) and \( \|1_{\Omega_{\leq n}}v\|_Y \leq (R)\|v\|_X \), where \( 1_K \) denotes the characteristic function of a set \( K \). To prove (3.13), we use (3.11) to write
\[
\int_{|x| \leq 1} \frac{|vw|}{|x|^2} dx \leq \sum_{j \geq 1} 2^{j(2-\delta)} \int_{2^{-j}\leq |x| \leq 2^{-j+1}} |vw| \leq \|v\|_X\|w\|_Y \sum_{j < 1} 2^{j(2-\delta)} \cdot 3 \cdot 2^{-2j}
\]
which gives
\[
\int_{|x| \leq 1} \frac{|vw|}{|x|^2} dx \leq \frac{3}{2\delta-1}\|v\|_X\|w\|_Y
\]
and similarly
\[
\int_{|x| \leq 1} \frac{|vw|}{|x|^2} dx \leq \sum_{j \geq 0} 2^{-j(2+\delta)} \int_{2^{j+1}\leq |x| \leq 2^{j+1}} |vw| \leq \frac{3^{2j+1}}{2\delta-1}\|v\|_X\|w\|_Y
\]
so that, summing up,
\[
\int_{|x| \leq 1} \frac{|vw|}{|x|^2} dx \leq \frac{3^{2j+1}}{2\delta-1}\|v\|_X\|w\|_Y \leq \frac{3}{2}\|v\|_X\|w\|_Y.
\]

For the nonhomogeneous estimate (3.14) we write, using (3.12),
\[
\int_{|x| \leq 1} \frac{|vw|}{|x|^2} dx \leq \sum_{j \geq 0} 2^{-j(2+\delta)} \int_{|x| \leq 2^{j+1}} |vw| \leq 3 \sum_{j \geq 0} 2^{-j(2+\delta)} \langle 2^j \rangle^2 \cdot \|v\|_X\|w\|_Y
\]
and notice that
\[
\sum_{j \geq 0} 2^{-j(2+\delta)} \langle 2^j \rangle^2 \leq 2 \sum_{j \geq 0} 2^{-j\delta} = \frac{2^{j+1}}{2\delta-1} \leq \frac{4}{\delta}.
\]

□

In the following Lemma we prove some magnetic Hardy type inequalities, which require \( n \geq 3 \), expressed in terms of the nonhomogeneous \( X, Y \) norms:
Lemma 3.5. Let $n \geq 3$ and assume $b(x) = (b_1(x), \ldots, b_n(x))$ is continuous up to the boundary of $\Omega$ with values in $\mathbb{R}^n$. For any $0 < \delta < 1$ and $v \in C_c^\infty(\Omega)$, we have:

\[
\|\frac{|x|^{-1}}{n}v\|_{L^2(\Omega)} \leq \frac{1}{n}\|\nabla v\|_{L^2(\Omega)},
\]

\[
\|\frac{|x|^{-1}}{n}v\|^2_{L^2(\Omega)} \leq 6\|\nabla v\|^2_{L^2(\Omega)} + 3\|v\|^2_{L^2(\Omega)},
\]

\[
f_{\Omega_{\leq 1}} |\nabla v| |x|^{-1} dx + \int_{\Omega_{\geq 1}} |\nabla v| |x|^{-1} dx \leq 9 \delta^{-1} (\|\nabla v\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)}),
\]

\[
\|v\|_{L^2(\Omega)} \leq 4 \sup_{R > 1} R^{-2} \int_{\Omega_{\leq R}} |v|^2 dS + 13\|\nabla v\|^2_{L^2(\Omega)},
\]

\[
f_{\Omega_{\geq 1}} |\nabla v| |x|^{-1} dx \leq 8 \delta^{-1} \|v\|^2_{L^2(\Omega)} + 9 \int_{\Omega_{\geq 1}} |\nabla v|^2 dx.
\]

Proof. We start from the identity

\[
\nabla \cdot \left( \frac{|x|^{-2}}{n} |v|^2 \right) = \frac{n-2}{|x|^2} |v|^2 + \frac{2}{|x|^2} \nabla(\nabla \cdot (v \cdot \nabla v)).
\]

Integrating over $\Omega_{\leq R}$ and noticing that $u|_{\partial \Omega} = 0$, we get

\[
(n-2) \int_{\Omega_{\leq R}} |v|^3 |x|^{-2} dx = \frac{n-2}{|x|^2} \int_{\Omega_{\leq R}} |v|^2 dS - \frac{2}{|x|^2} \int_{\Omega_{\leq R}} |\nabla v|^2 dx.
\]

Estimating the last term with Cauchy-Schwartz

\[-\int_{\Omega_{\geq 1}} \frac{2}{|x|^2} |\nabla(\nabla \cdot (v \cdot \nabla v))| dx \leq \frac{n-2}{|x|^2} \int_{\Omega_{\leq R}} |v|^2 dS + \frac{2}{n-2} \int_{\Omega_{\leq R}} |\nabla v|^2 dx\]

we obtain

\[
\frac{1}{|x|^2} \int_{\Omega_{\leq R}} |v|^3 |x|^{-2} dx \leq \frac{n-2}{|x|^2} \int_{\Omega_{\leq R}} |v|^2 dS + \frac{4}{(n-2)^2} \int_{\Omega_{\leq R}} |\nabla v|^2 dx.
\]

Letting $R \to \infty$ in (3.21) we obtain (3.15). On the other hand, for $0 < R < 1$ this gives

\[
\frac{1}{|x|^2} \int_{\Omega_{\leq R}} |v|^3 |x|^{-2} dx \leq 2 \int_{\Omega_{\leq 1}} |v|^2 dS + 4 \int_{\Omega_{\leq 1}} |\nabla v|^2 dx \leq 2 \sqrt{2} \|v\|^2_{L^2} + 4 \sqrt{2} \|\nabla v\|^2_{L^2},
\]

while for $R > 1$ it gives

\[
\frac{1}{|x|^2} \int_{\Omega_{\leq R}} |v|^3 |x|^{-2} dx \leq \frac{2}{|x|^2} \int_{\Omega_{\leq 1}} |v|^2 dS + \frac{4}{|x|^2} \int_{\Omega_{\leq 1}} |\nabla v|^2 dx \leq 2 \sqrt{2} \|v\|^2_{L^2} + 4 \|\nabla v\|^2_{L^2}.
\]

The last two inequalities together imply (3.16). To prove (3.17), we split $\Omega = \Omega_{\leq 1} \cup \Omega_{\geq 1}$ and we remark that

\[
\int_{\Omega_{\leq 1}} |v|^3 |x|^{-2} dx \leq \frac{1}{\sqrt{2}} \int_{\Omega_{\leq 1}} |v|^2 dS + \frac{1}{2 \sqrt{2}} \int_{\Omega_{\leq 1}} |\nabla v|^2 dx \leq \frac{3}{2} \|v\|^2_{L^2} + \sqrt{2} \|\nabla v\|^2_{L^2}
\]

using (3.22), while using (3.14)

\[
\int_{\Omega_{\geq 1}} |v|^3 |x|^{-2} dx \leq 6 \delta^{-1} \|\nabla v\|^2_{L^2} + 6 \delta^{-1} \|v\|^2_{L^2}.
\]

Summing up, we obtain (3.17).

On the other hand, from (3.20) we can deduce also, for all $0 < R < 1$,

\[
\int_{\Omega_{\leq R}} |v|^2 dS \leq (n-2) \int_{\Omega_{\leq 1}} |v|^2 |x|^{-2} dx + \int_{\Omega_{\geq 1}} |v|^2 |x|^{-2} dx + \int_{\Omega_{\leq R}} |\nabla v|^2 dx
\]

\[
\leq (n-1) \int_{\Omega_{\leq 1}} |v|^2 |x|^{-2} dx + \int_{\Omega_{\geq 1}} |\nabla v|^2 dx.
\]

Together with (3.21) with $R = 1$, this gives, for all $0 < R < 1$,

\[
\int_{\Omega_{\leq R}} |v|^2 dS \leq \frac{2(n-1)}{(n-2)^2} \int_{\Omega_{\leq 1}} |v|^2 dS + \left( \frac{4(n-1)}{(n-2)^2} + 1 \right) \int_{\Omega_{\geq 1}} |\nabla v|^2 dx
\]

and recalling that $n \geq 3$ we have proved for all $0 < R < 1$

\[
\int_{\Omega_{\leq R}} |v|^2 dS \leq 4 \int_{\Omega_{\leq 1}} |v|^2 dS + 9 \int_{\Omega_{\geq 1}} |\nabla v|^2 dx.
\]

from which (3.18) and (3.19) follow easily. \(\square\)
By a density argument, it is clear that the estimates in Lemmas 3.1–3.5 are valid not only for smooth functions but also for functions belonging to the domain of the operator $D(-A^k + c)$; in particular, Lemmas 3.5 hold in view of the Dirichlet boundary conditions in the definition of the operator.

4. Proof of the Theorems

The proof consists in integrating the identity (2.6) on $\Omega$ and estimating all the terms. Since the arguments for both Theorems 1.1 and 1.2 largely overlap, we shall proceed with both proofs in parallel. The proof is divided into several steps.

4.1. Notations. Recall that we are using implicit summation over repeated indices and the notations

$$\hat{x} = \frac{x}{|x|} = (\hat{x}_1, \ldots, \hat{x}_n), \quad \hat{x}_j = \frac{\langle \hat{x}, e_j \rangle}{|x|},$$

$$\hat{a}(x) = a_{tm}(x)\hat{x}_t \hat{x}_m, \quad \hat{\pi}(x) = \text{trace} a(x) = a_{mm}(x).$$

Since $a(x)$ is positive definite, we have

$$0 \leq \hat{a} = a\hat{x} \cdot \hat{x} \leq |a\hat{x}| \leq \hat{\pi}.$$

We denote with a semicolon the partial derivatives:

$$a_{jk;\ell} := \partial_\ell a_{jk}, \quad a_{jk;tm} := \partial_t \partial_m a_{jk}, \quad a_{jk;tmp} := \partial_t \partial_m \partial_p a_{jk}.$$

Notice the formulas

$$\partial_k (\hat{x}_\ell) = |x|^{-1} [\delta_{k\ell} - \hat{x}_k \hat{x}_\ell],$$

$$\partial_k (\hat{x}_t \hat{x}_m) = |x|^{-1} [\delta_{kt} \hat{x}_m + \delta_{km} \hat{x}_t - 2\hat{x}_k \hat{x}_t \hat{x}_m],$$

$$\partial_j \partial_k (\hat{x}_t \hat{x}_m) = \frac{1}{|x|^2} [\delta_{kt} \delta_{jm} + \delta_{km} \delta_{jt} + 2\hat{x}_j \hat{x}_k \hat{x}_t \hat{x}_m]$$

$$- 2\delta_{kt} \hat{x}_j \hat{x}_m - 2\delta_{km} \hat{x}_t \hat{x}_\ell - 2\delta_{jk} \hat{x}_t \hat{x}_m - 2\delta_{jm} \hat{x}_k \hat{x}_\ell$$

which imply

$$a_{jk} a_{tm} \partial_j (\hat{x}_t \hat{x}_m) = 2|x|^{-1} [a\hat{x}^2 - \hat{a}^2],$$

and

$$a_{jk} a_{tm} \partial_j \partial_k (\hat{x}_t \hat{x}_m) = \frac{2}{|x|^2} [a_{tm} a_{tm} - 4(a\hat{x}^2 - \hat{a}^2) - \hat{\pi}].$$

By the previous identities, for any radial function $\psi(x) = \psi(|x|)$ we can write

$$A\psi(x) = \partial_j (a_{tm} \hat{x}_m \psi') = \hat{a} \psi'' + \hat{\pi} - \hat{a} \psi' + a_{tm} \hat{x}_m \psi'$$

(4.1)

where $\psi'$ denotes the derivative of $\psi(r)$ with respect to the radial variable.

4.2. Estimate of the $\epsilon$ term. We begin with the $\epsilon$-term in (2.6)

$$I_\epsilon := 2\epsilon \Im [a(v\nabla\psi, \nabla^k v)].$$

We need a few auxiliary estimates. Choosing $\phi = 1$ in (2.7) and taking the real part we get

$$a(\nabla^k v, \nabla^k v) = (\lambda - c)|v|^2 - \Re (f \overline{\pi}) + \Re \partial_j \{\overline{\pi} a_{jk} \partial_k^j v\}$$

and assuming that

$$c(x) \geq -\sigma(x)^2$$

for some nonnegative function $\sigma(x)$ which will be precised in the following (accordin to assumptions (1.6)–(1.15)), we obtain, with $\lambda_+ = \max\{\lambda, 0\},$

$$a(\nabla^k v, \nabla^k v) \leq (\lambda_+ + \sigma^2)|v|^2 - \Re (f \overline{\pi}) + \Re \partial_j \{\overline{\pi} a_{jk} \partial_k^j v\}. \quad (4.2)$$

On the other hand taking the imaginary part of (2.7) with $\phi = 1$ gives

$$\epsilon |v|^2 = \Im (f \overline{\pi}) - \Im \partial_j \{\overline{\pi} a_{jk} \partial_k^j v\}$$
which can be written
\[ |\alpha|v^2 = 3((\text{sgn} \, \epsilon) f) - \Im \partial_j \{ (\text{sgn} \, \epsilon) \mathbf{v}_a \partial^a_j \mathbf{v} \}. \] (4.3)

Now consider \( I_\epsilon \) for an arbitrary function \( \psi \) with bounded derivatives; by Cauchy-Schwartz
\[ |I_\epsilon| \leq |\epsilon| a(v \nabla \psi, \nabla \psi)(\lambda_+ + \sigma^2 + |\epsilon|)^{1/2} + |\epsilon| a(\nabla^b v, \nabla^b v)(\lambda_+ + \sigma^2 + |\epsilon|)^{-1/2}. \]

Since by \( a(x) \leq N I \) we have
\[ a(v \nabla \psi, v \nabla \psi) \leq N \|\nabla \psi\|_{L^\infty}^2 v^2, \]
using (4.2) we obtain
\[ |I_\epsilon| \leq N \|\nabla \psi\|_{L^\infty}^2 (\lambda_+ + \sigma^2 + |\epsilon|)^{1/2} |\epsilon| v^2 + \frac{|\epsilon|}{(\lambda_+ + \sigma^2 + |\epsilon|)^{1/2}} \Im (\partial_j \{ \mathbf{v}_a \partial^a_j \mathbf{v} \} - f) \]
and hence
\[ |I_\epsilon| \leq (N \|\nabla \psi\|_{L^\infty}^2 + 1)(\lambda_+ + \sigma^2 + |\epsilon|)^{1/2} |\epsilon| v^2 + \frac{|\epsilon|}{(\lambda_+ + \sigma^2 + |\epsilon|)^{1/2}} \Im (\partial_j \{ \mathbf{v}_a \partial^a_j \mathbf{v} \} - f). \]

Using now (4.3) we get
\[ |I_\epsilon| \leq (N \|\nabla \psi\|_{L^\infty}^2 + 1)(\lambda_+ + \sigma^2 + |\epsilon|)^{1/2} \Im ((\text{sgn} \, \epsilon) f) - \partial_j \{ (\text{sgn} \, \epsilon) \mathbf{v}_a \partial^a_j \mathbf{v} \} - \frac{|\epsilon|}{(\lambda_+ + \sigma^2 + |\epsilon|)^{1/2}} \Im (\partial_j \{ \mathbf{v}_a \partial^a_j \mathbf{v} \} - f) \]
which implies
\[ |I_\epsilon| \leq (N \|\nabla \psi\|_{L^\infty}^2 + 2)(\lambda_+ + \sigma^2 + |\epsilon|)^{1/2} |\epsilon| \|f\|_\infty + \frac{|\epsilon|}{(\lambda_+ + \sigma^2 + |\epsilon|)^{1/2}} \Im (\partial_j \{ \mathbf{v}_a \partial^a_j \mathbf{v} \} - f) \]

Then we notice that for any vector valued function \( Z = Z(x) \)
\[(\lambda_+ + \sigma^2 + |\epsilon|)^{1/2} \nabla \cdot Z = \nabla \cdot \{ (\lambda_+ + \sigma^2 + |\epsilon|)^{1/2} Z \} - \frac{\sigma \nabla \sigma \cdot Z}{(\lambda_+ + \sigma^2 + |\epsilon|)^{1/2}} \]
\[ \leq \nabla \cdot \{ (\lambda_+ + \sigma^2 + |\epsilon|)^{1/2} Z \} + \|\nabla \sigma \cdot Z\| \]
and similarly
\[ \frac{|\epsilon| |\nabla \cdot Z|}{(\lambda_+ + \sigma^2 + |\epsilon|)^{1/2}} \leq \nabla \cdot \{ (\lambda_+ + \sigma^2 + |\epsilon|)^{1/2} Z \} + \|\nabla \cdot Z\| \]

Applying these estimates to the previous inequality we get
\[ |I_\epsilon| \leq (N \|\nabla \psi\|_{L^\infty}^2 + 2) \left[ (\lambda_+ + \sigma^2 + |\epsilon|)^{1/2} \|f\|_\infty + |a(\nabla^b v, \nabla^b \sigma)| \right] + \partial_j G_j \] (4.4)
where
\[ G_j = \frac{|\epsilon| |\Im (\mathbf{v}_a \partial^a_j \mathbf{v})|}{(\lambda_+ + \sigma^2 + |\epsilon|)^{1/2}} - (N \|\nabla \psi\|_{L^\infty}^2 + 1)(\lambda_+ + \sigma^2 + |\epsilon|)^{1/2} \text{sgn} \, \epsilon) \Im (\mathbf{v}_a \partial^a_j \mathbf{v}). \]

We then integrate (4.4) over the set \( \Omega \cap \{|x| \leq R\} \). The boundary terms \( G_j \) vanish at \( \partial \Omega \) in view of the Dirichlet conditions; on the other hand, at the remaining part of the boundary \( \Omega \cap \{|x| = R\} \) we get the quantity
\[ \int_{\Omega \cap \{|x| = R\}} \nu_j G_j dS \]
where \( \nu_j = (\nu_1, \ldots, \nu_n) \) is the exterior normal and \( dS \) is the surface measure on the sphere \( \{|x| = R\} \). Since \( v \in H^1(\Omega) \), and the coefficients and \( \sigma(x) \) are bounded for \( x \) large, we get
\[ \lim \inf_{R \to +\infty} \int_{\Omega \cap \{|x| = R\}} \nu_j G_j dS = 0 \]
and hence the boundary term vanishes after integration over \( \Omega \):
\[ \int_{\Omega} (\partial_j G_j) dx = 0. \]
Thus integration of (4.4) over $\Omega$ gives
$$\int_{\Omega} |f| \, dx \leq (N\|\nabla \varphi\|_{L^\infty}^2 + 2) \int_{\Omega} [(\lambda_+ + \sigma^2 + |v|)^{1/2} |f| + |a(\nabla^b c, v \nabla \sigma)|] \, dx. \quad (4.5)$$
In order to control the RHS of (4.5) we need a few more estimates, which are different for the homogeneous and the nonhomogeneous cases. We begin with the homogeneous estimate, under assumption (1.6). Thus we can take
$$\sigma(x) = \frac{C_2}{|x|^2 + \sigma_2 |x|^2} \leq C_- |x|^{-1}$$
so that
$$\int \sigma |f| \varphi |x|^{-1} \leq C_- \|f\|_{L^\infty} \|\nabla \varphi\| \leq C_- \|f\|_{L^\infty} \|\nabla \varphi\|$$
(see (3.4)). Second, we take the imaginary part of (2.7)
$$\partial_j \Im P_j = -c|v|^2 \phi + \Im (f \varphi) + \Im a(\nabla^b v, v \nabla \phi),$$
and choose $\phi$ as follows:
$$\phi(x) = 1 \text{ if } |x| \leq R, \quad \phi(x) = 2 - \frac{|x|}{R} \text{ if } R \leq |x| \leq 2R, \quad \phi(x) = 0 \text{ if } |x| \geq R. \quad (4.7)$$
We note also that the integral of $\partial_j P_j$ over $\Omega$ vanishes as above by the Dirichlet boundary conditions. This gives easily, using (3.11),
$$|c| \int_{|x| \leq R} |v|^2 \leq \int_{|x| \leq R} |f| \varphi + \frac{N}{R} \int_{R^2 \leq |x| \leq 2R} |v| |\nabla^b v| \leq 2R \|f\|_{L^\infty} \|\varphi\| + 3NR \|v\| \|\nabla^b v\|_{L^\infty}. \quad (4.8)$$
Dividing by $R$ and taking the sup for $R > 0$ gives
$$|c| \|v\|_N^2 \leq 3(1 + N) \left[ \|f\|_{L^\infty} \|\nabla^b v\|_{L^\infty} \right] \|v\|_X. \quad (4.9)$$
As a consequence, we can write
$$|c|^{1/2} \int_{\Omega} |f| \varphi \leq \|f\|_{L^\infty} \left[ 3(1 + N) \right]^{1/2} \left[ \|f\|_{L^\infty} \|\nabla^b v\|_{L^\infty} \right]^{1/2} \|v\|_{X}^{1/2}$$
and this implies
$$|c|^{1/2} \int_{\Omega} |f| \varphi \leq (N + 1)^{1/2} \left[ \|f\|_{L^\infty} \|\nabla^b v\|_{L^\infty} \right] \|v\|_{X}. \quad (4.10)$$
If we take instead the real part of (2.7), split $c = c_+ - c_-$ into positive and negative part, and write $\lambda_\pm = \max\{\pm \lambda, 0\}$, we have
$$(c_- + \lambda_+)|v|^2 \phi = -\partial_j \Re P_j + a(\nabla^b v, \nabla v) \phi + (c_+ + \lambda_-)|v|^2 \phi + \Re (f \varphi) \phi - \frac{1}{2} A \phi |v|^2.$$ We choose the same function $\phi$ as in (4.7). When $\lambda = \lambda_+ \geq 0$, by assumptions (1.3), (1.4) we can write
$$-\frac{1}{2} A \phi = \frac{\partial_\lambda \Im a + \partial_\lambda}{R^2} \delta_{|x|=2R} \frac{1R_{|x| \leq |2R-\delta_{|x|=R}|}}{2} \leq \frac{N^2 + C_a}{R^2} \frac{1R_{|x| \leq 2R}}{2} + \frac{N}{R^2} \delta_{|x|=2R}.$$ Integrating over $\Omega$, the boundary term disappears as usual, and dividing by $R$ and taking the sup over $R > 0$ we get
$$\lambda_+ \|v\|_N^2 \leq 2N \|\nabla^b v\|_Y^2 + 2|c_+| \|v\|_N^2 + 4(4N(n+1) + n^2 C_a + 1) \|v\|_X^2 + \|f\|_{L^\infty}^2,$$
(see (3.11), (3.9)). Using the second half of assumption (1.6) and (3.4), this gives
$$\lambda_+ \|v\|_N^2 \leq 2N \|\nabla^b v\|_Y^2 + 4(4C_a^2 + N(n+1) + n^2 C_a + 1) \|v\|_X^2 + \|f\|_{L^\infty}^2,$$
and hence
$$\sqrt{\lambda_-} \int_{\Omega} |f| \varphi \leq \|f\|_{L^\infty} \left[ 2N \|\nabla^b v\|_Y^2 + 4(4C_a^2 + N(n+1) + n^2 C_a + 1) \|v\|_X^2 + \|f\|_{L^\infty}^2 \right]^{1/2}. \quad (4.13)$$
On the other hand, when $\lambda \leq 0$ i.e. $\lambda = -\lambda_-$, we rewrite (2.7) in the form
\[
(c_+ + \lambda_-)|v|^2 \phi = \partial_t \mathfrak{R} P_j - a(\nabla^b v, \nabla^b v) + (c_- + \lambda_+)|v|^2 \phi - \mathfrak{R}(f \phi \sigma) \phi + \frac{1}{2} A \phi |v|^2
\]
and with the same choice of $\phi$ as above we have (since $\overline{\sigma} \geq \hat{\sigma}$)
\[
\frac{1}{2} A \phi = -\frac{\pi a}{|x|^{\alpha/2}} \mathfrak{R} P_j 1_{R \leq |x| \leq 2R} + \frac{1}{R^2} (\delta_{|x|=R} - \delta_{|x|=2R}) \leq \frac{\pi a C}{R} 1_{R \leq |x| \leq 2R} + \frac{\pi}{R^2} \delta_{|x|=R}
\]
which implies ($\lambda_+ = 0$)
\[
\lambda_-|v|^2 1_{|x| \leq R} \leq \partial_t \mathfrak{R} P_j + (c_-|v|^2 + |f \overline{\sigma}|) 1_{|x| \leq 2R} + \frac{\pi a C}{R} |v|^2 1_{R \leq |x| \leq 2R} + \frac{\pi}{R^2} |v|^2 \delta_{|x|=R}.
\]
Proceeding exactly as above but using the fact that $c_-(x) \leq C_2^2 |x|^{-2}$ by (1.6), we conclude that
\[
\lambda_- \|v\|^2 \leq 2(C_2^2 + N + n^2 C_\alpha + 1)\|v\|^2_X + \|f\|^2_{Y^*}.
\]
For the remaining term in (4.5), we note that
\[
\sigma(x) = \frac{C_-}{|x|^{\alpha/2} \sqrt{|v(x)|}} \implies |\nabla \sigma| \leq 2C_- \begin{cases} |x|^{-2+\delta/2} & \text{if } |x| \leq 1, \\ |x|^{-2-\delta/2} & \text{if } |x| \geq 1. \end{cases}
\]
and by (13.3) we get
\[
\int_{\Omega} |a(\nabla^b v, \nabla \sigma)| \leq 36\delta^{-1} N C_- \|\nabla^b v\|_{Y^*} \|v\|_{\tilde{X}}.
\]
We assume now that
\[
\|\nabla v\|_{L^\infty} \leq 1
\]
the explicit choice of the weight will be done in the next step) so that, summing up, we can estimate (4.5) via (4.6), (4.10), (14.13), (14.16) to obtain
\[
(N + 2)^{-1} \int_{\Omega} |l| \leq 4(C_2^2 + C_\alpha^2 + \frac{a}{2} N n + n^2 C_\alpha + 1)^{1/2} \|f\|_{Y^*} \|v\|_{X} + \frac{a}{2} \sqrt{N + 1} \|f\|_{Y^*} \|\nabla^b v\|_{Y^*} + \|f\|^2_{Y^*} + 36\delta^{-1} N C_- \|\nabla^b v\|_{Y^*} \|v\|_{\tilde{X}}
\]
whence one gets, for any $0 < \zeta \leq 1$, the final homogeneous estimate of $I_5$
\[
\int_{\Omega} |l| \leq \zeta + 18\delta^{-1} N (N + 2) C_- \|\nabla^b v\|^2_{Y^*} + \|v\|^2_{\tilde{X}} + 8(N + 2)^2 (C_2^2 + C_\alpha^2 + N n + n^2 C_\alpha + 1) \zeta^{-1} \|f\|^2_{Y^*},
\]
under assumption (1.6).

We show now how to get a nonhomogeneous estimate for $I_5$ under the stronger assumption (1.15), thus we can take now
\[
\sigma(x) = C_- \langle x \rangle^{-1-\delta/2}.
\]
Starting again from (4.5), we have, since $\sigma(x) \leq C_- |x|^{-1}$,
\[
\int |\nabla \sigma| \leq C_- \|f\|_{Y^*} \|v\|_{X} |v|^{-1} \|v\|_{Y} \leq 3C_- \|f\|_{Y^*} \|v\|_{X} \|v\|^2_{Y^*} + \|\nabla^b v\|^2_{Y^*})^{1/2}
\]
by (3.16). Then we consider again (4.8) where we estimate as follows
\[
|\varphi| \|v\|^2 \leq \langle 2R \rangle \|v\|_{X} \|f\|_{Y^*} + \frac{a}{2} \cdot 3(R^2) \|v\|_{X} \|\nabla^b v\|_{Y^*}
\]
by (3.12). Dividing by $R$, taking the sup over $R > 1$ and recalling (3.3) we get
\[
|\varphi| \|v\|^2 \leq \sqrt{\frac{5}{3}} \|v\|_{X} \|f\|_{Y^*} + 6N \|v\|_{X} \|\nabla^b v\|_{Y^*}
\]
Thus
\[
|\varphi| \|v\|^2 \leq 2(N + 1)^{1/2} \|v\|^2_{Y^*} + \|f\|_{Y^*} \|\nabla^b v\|_{Y^*} + \|f\|_{Y^*} \|v\|_{X}.
\]
and this implies
\[
|\varphi| \|v\|^2 \leq 2(N + 1)^{1/2} \|v\|^2_{Y^*} + \|f\|_{Y^*} \|\nabla^b v\|_{Y^*} + \|f\|_{Y^*} \|v\|_{X}
\]
and
\[
|\varphi| \|v\|^2 \leq 2(N + 1)^{1/2} \|v\|^2_{Y^*} + \|f\|_{Y^*} \|\nabla^b v\|_{Y^*} + \|f\|_{Y^*} \|v\|_{X}
\]
Next, in the case $\lambda = \lambda_+ \geq 0$ we integrate (4.11) over $\Omega$ and we take the sup over $R > 1$ to get
\[
\lambda_+ \|v\|_2^2 \leq 2\sqrt{2}N\|\nabla^b v\|_2^2 + 2\sqrt{2}\|c_+^{1/2} v\|_2^2 + (3n(N + nC_a) + 2N + 2)\|v\|_X^2 + \|f\|_Y^2.
\]
where we used (3.12) and (3.8). Since $c_+(x) \leq C_+ |x|^{-2}$, by (3.16) we obtain
\[
\lambda_+ \|v\|_2^2 \leq 3(N + 6C_a^2)\|\nabla^b v\|_2^2 + 3(N(n + 1) + n^2 C_a + 3C_a^2)\|v\|_X^2 + \|f\|_Y^2.
\]
and this implies
\[
\sqrt{\lambda_+} \int_\Omega |f|^p dx \leq \|f\|_Y \left(3(N + 6C_a^2)\|\nabla^b v\|_2^2 + 3(N(n + 1) + n^2 C_a + 3C_a^2)\|v\|_X^2 + \|f\|_Y^2 \right)^{1/2}.
\]
(4.23)

On the other hand, when $\lambda = -\lambda_- \leq 0$, we integrate (4.14) over $\Omega$, divide by $R$ and take the sup over $R > 1$ to obtain, using (3.12), (3.8),
\[
\lambda_- \|v\|_2^2 \leq 2\sqrt{2}\|c_+^{1/2} v\|_2^2 + (2(N + 1) + 3n^2 C_a)\|v\|_X^2 + \|f\|_Y^2.
\]
and since $c_-(x) \leq C_- |x|^{-2}$, by (3.16) we get
\[
\lambda_- \|v\|_2^2 \leq 18C_a^2 \|\nabla^b v\|_2^2 + 3(N + n^2 C_a + 6C_a^2 + 1)\|v\|_X^2 + \|f\|_Y^2.
\]
(4.24)

To estimate last term in (4.5) we notice that $|\nabla \sigma| \leq 2C_- |x|^{-2-2/2}$ and hence
\[
\int_\Omega |a(\nabla^b v, v \nabla \sigma)| \leq 2NC_- \int_\Omega \frac{|\nabla^b v|}{|x|^{2+2/2}} dx \leq 18\delta^{-1}NC_- (\|\nabla^b v\|_2^2 + \|v\|_X^2) \tag{4.25}
\]
by (3.17). Summing up we get in a few steps (using again (4.17))
\[
(N + 2)^{-1} \int_\Omega |\epsilon| \leq 9(C_+^2 + 4C_a^2 + Nn + n^2 C_a + 1)^{1/2}\|f\|_Y \|v\|_X + \|\nabla^b v\|_Y
\]
\[
+ 3(N + 1)\|f\|_Y^2 + 18\delta^{-1} NC_- (\|\nabla^b v\|_2^2 + \|v\|_X^2)
\]
and hence, for every $0 < \zeta \leq 1$, we obtain
\[
\int_\Omega |\epsilon| \leq (\zeta + 18\delta^{-1} N(N + 2C_-)) (\|\nabla^b v\|_2^2 + \|v\|_X^2)
\]
\[
+ 44(N + 2)^2(C_+^2 + 4C_a^2 + Nn + n^2 C_a + 1)\zeta^{-1}\|f\|_Y^2. \tag{4.26}
\]
under assumption (1.15).

4.3. **Choice of the weight $\psi$.** Our choice of the weight function $\psi$ in (2.6) is inspired by [23], [3] (see also [22], [26]). Define
\[
\psi_0(r) = \int_r^\infty \psi'(s) ds \tag{4.27}
\]
where
\[
\psi'_r(r) = \begin{cases} \frac{a-1}{2n}, & r \leq 1 \\ \frac{1}{2} - \frac{1}{2a \nu + 1}, & r > 1. \end{cases}
\]
Then $\psi$ is the radial function, depending on a scaling parameter $R > 0$,
\[
\psi(|x|) \equiv \psi_R(|x|) := R \psi_1 \left( \frac{|x|}{R} \right).
\]
Here and in the following, with a slight abuse, we shall use the same letter $\psi$ to denote a function $\psi(r)$ defined for $r \in \mathbb{R}^+$ and the radial function $\psi(x) = \psi(|x|)$ defined on $\mathbb{R}^n$. We compute the first radial derivatives $\psi^{(j)}(r) = (\frac{1}{|x|} \cdot \nabla^j)^\psi(x)$ for $|x| > 0$:
\[
\psi'(x) = \begin{cases} \frac{a-1}{2n} \cdot \frac{|x|}{R^{a-1}}, & |x| \leq R \\ \frac{1}{2} - \frac{1}{2a \nu + 1}, & |x| > R. \end{cases}
\]
(4.28)
which can be equivalently written as
\[
\psi'(x) = \frac{|x|}{2a \nu} \left[ n \frac{R}{|x|} - \left( \frac{R}{|x|} \right)^n \right]
\]
and implies in particular that the assumption (4.17) used in the previous step is satisfied, and actually
\[ 0 \leq \psi' \leq \frac{1}{2}. \] (4.29)
Then we have
\[
\psi''(x) = -\frac{n-1}{2a} \frac{R^{n-1}}{|x|^n} = \frac{n-1}{2a} \left\{ \begin{array}{ll}
\frac{1}{R} & |x| \leq R \\
\frac{1}{|x|^n} & |x| > R,
\end{array} \right.
\] (4.30)
\[
\psi'''(x) = -\frac{n-1}{2} \frac{R^{n-1}}{|x|^{n+1}} 1_{|x| \geq R}
\] (4.31)
\[
\psi^{IV}(x) = \frac{n^2-1}{2} \frac{R^{n-1}}{|x|^{n+1}} 1_{|x| \geq R} - \frac{n-1}{2a} \frac{1}{R^n} \delta_{|x|=R}.
\] (4.32)
Note in particular that
\[
\psi'' - \frac{\psi'}{|x|} = \begin{cases}
0 & |x| \leq R \\
-\frac{1}{2|x|} \left( 1 - \frac{R^{n-1}}{|x|^{n}} \right) & |x| > R.
\end{cases}
\] (4.33)
Moreover, we choose \( \phi = 0 \) and we see that (see (4.1))
\[
A \psi + \phi = A \psi = \hat{a} \psi'' + \frac{\hat{a} - a}{|x|} \psi' + a_{\ell m, \ell} \hat{x}_m \psi'.
\] (4.34)
is continuous and piecewise Lipschitz.

4.4. Estimate of the terms in \(|v|^2\). Since \( \phi \equiv 0 \), these terms reduce to
\[
I_{|v|^2} = -\frac{1}{2} A^2 \psi|v|^2 - a(\nabla \psi, \nabla c)|v|^2.
\] (4.35)
First of all we need to compute the quantity \( A(A \psi + \phi) \equiv A^2 \psi \). Using the identity
\[
A(ux) = (Au)v + u(Av) + 2a(\nabla u, \nabla v)
\]
we can write
\[
A^2 \psi = I + II + III + A(a_{\ell m, \ell} \hat{x}_m \psi')
\]
where
\[
I = \hat{a} \cdot A \psi'' + (\hat{a} - \hat{\alpha}) A \left( \frac{\psi'}{|x|} \right),
\]
\[
II = \hat{a} \cdot \psi'' + A(\hat{a} - \hat{\alpha}) \frac{\psi'}{|x|}
\]
and
\[
III = 2a(\nabla \hat{a}, \nabla \psi'') + 2a(\nabla (\hat{a} - \hat{\alpha}), \nabla \frac{\psi'}{|x|}).
\]
We separate the terms which do not contain derivatives of the coefficients \( a_{jk} \) from the others. We have \( I = I_1 + I_2 \) with
\[
I_1 = \hat{a}^2 \psi^{IV} + \hat{a}(\hat{a} - \hat{\alpha}) \psi'' + \frac{(\hat{a} - \hat{\alpha})(\hat{a} - \hat{\alpha})}{|x|^2} \left( \psi'' - \frac{\psi'}{|x|} \right),
\]
\[
I_2 = \hat{a} a_{\ell m, \ell} \hat{x}_m \psi'' + (\hat{a} - \hat{\alpha}) a_{\ell m, \ell} \hat{x}_m \left( \frac{\psi''}{|x|} - \frac{\psi'}{|x|^2} \right)
\]
where
\[
\psi''' = -\frac{n-1}{2} \frac{R^{n-1}}{|x|^{n+1}} 1_{|x| \geq R},
\] (4.36)
\[
\psi^{IV} = \frac{n^2-1}{2} \frac{R^{n-1}}{|x|^{n+1}} 1_{|x| \geq R} + \frac{n-1}{2a} \frac{1}{R^n} \delta_{|x|=R}.
\] (4.37)
In a similar way, \( II = II_1 + II_2 \) with
\[
II_1 = \frac{2}{|x|^2} a_{\ell m} a_{\ell m} - 4(|a\hat{x}|^2 - \hat{\alpha}^2) \left( \psi'' - \frac{\psi'}{|x|} \right),
\]
\[
II_2 = \partial_j (a_{jk} a_{\ell m, k} \hat{x}_m) + \partial_j (a_{jk} a_{\ell m}) \partial_k (\hat{x}_m \hat{x}_m) \left( \psi'' - \frac{\psi'}{|x|} \right) + (A\hat{a}) \frac{\psi'}{|x|}
\]
and \( III = III_1 + III_2 \) with
\[
III_1 = \frac{2}{|x|^2} |a\hat{x}|^2 - \hat{\alpha}^2 \left( \psi'' - \frac{\psi'}{|x|} + \frac{\psi'}{|x|^2} \right),
\]
\[ III_2 = 2a_{jk}a_{\ell m;k}\hat{x}_j\hat{x}_m. \]

Collecting all the terms we get

\[ A^2\psi = S(x) + R(x) \quad (4.38) \]

where

\[ S(x) = 2\hat{\nu}^4 + 2\hat{\alpha}^2(\hat{\alpha}^2 - \hat{\alpha}^2) + \left( \frac{2}{|x|^2} [a_{\ell m}a_{\ell m} - \hat{\alpha} - 4(|\hat{x}|^2 - \hat{\alpha}^2)] \right) \left( \frac{\psi'''}{|x|^2} \right) + \]

\[ + \left[ \psi'''' - \frac{\psi'''}{|x|^2} \right] + 2a_{\ell m}a_{\ell m}a_{\ell m}a_{\ell m} \hat{x}_j \left( \psi''' \right) + 2a(\nabla \hat{\alpha}, \nabla \psi') + \]

\[ R(x) = a_{\ell m}a_{\ell m}a_{\ell m} + (\hat{\alpha} - \hat{\alpha})a_{\ell m}a_{\ell m} \hat{x}_j \left( \psi''' \right) + \]

\[ + [\psi'''] + A_{\ell m}a_{\ell m}a_{\ell m}a_{\ell m} \hat{x}_j \left( \psi''' \right) + 2a(\nabla \hat{\alpha}, \nabla \psi'). \]

The remainder \( R(x) \) can be estimated as follows: recalling that, by definition of \( \psi' \),

\[ |\psi' \leq \frac{n-1}{2n(C_n + C_n)}, \quad |\psi'' \leq \frac{n-1}{2n(C_n + C_n)}, \quad |\psi''' \leq \frac{n-1}{2n(C_n + C_n)} \]

and the assumptions (1.4), then after a long but elementary computation we find \((n \geq 3)\)

\[ |R(x)| \leq \frac{12nC_n(N + C_n)}{|x|^{n+3}(R \vee |x|)}. \quad (4.39) \]

We focus now on the main term \( S(x) \), which can be written

\[ S(x) = 2\hat{\nu}^4 + [2\alpha^2 - 6\alpha^2 + 4|\hat{x}|^2] \frac{\psi'''}{|x|^2} + \]

\[ + [2a_{\ell m}a_{\ell m} + \alpha^2 - 6\alpha^2 + 15\alpha^2 - 12|\hat{x}|^2] \left( \frac{\psi'''}{|x|^2} \right). \quad (4.40) \]

With our choice of the weight \( \psi \) we have in the region \(|x| \leq R\)

\[ S(x) = -\frac{n+1}{2n}\alpha^2 \hat{\alpha}^2 \delta_{|x|=R} \quad (4.41) \]

while in the region \(|x| > R\)

\[ S(x) = (n - 1) \left[ \frac{\alpha^2}{|x|^2} - \frac{1}{2n} - 2(n - 1)|\hat{x}|^2 - \hat{\alpha}^2 \right] \left( \frac{\psi'''}{|x|^2} \right) - \]

\[ - [2a_{\ell m}a_{\ell m} + \alpha^2 - 6\alpha^2 + 15\alpha^2 - 12|\hat{x}|^2] \left( 1 - \frac{R}{|x|} \right)^{n-1} \frac{1}{|x|^2}. \quad (4.42) \]

Note that \( a_{\ell m}a_{\ell m} \) is the square of the Hilbert-Schmidt norm of the matrix \( a(x) \).

To proceed, we handle the cases of Theorems 1.1 and 1.2 separately. For Theorem 1.1, we deduce from assumption (1.3)

\[ nN \geq \pi \geq \nu, \quad N \geq |\hat{x}| \geq \tilde{\alpha} \geq \nu, \quad a_{\ell m}a_{\ell m} \geq n\nu^2, \]

so that

\[ S(x) \leq -\frac{n-1}{2n}\alpha^2 \hat{\alpha}^2 \delta_{|x|=R} \quad (4.43) \]

while, recalling also assumption (1.13), we obtain

\[ S(x) \leq (n - 1) \left[ \frac{n+3}{2n} - n - \nu \right] \alpha^2 \delta_{|x|=R} \quad (4.44) \]

On the other hand, for Theorem 1.2 with \( n = 3 \), writing \( a(x) = I + q(x) \) i.e. \( a_{\ell m} = a_{\ell m} - \delta_{\ell m} \) we have, with the usual notations \( \hat{q} = q_{\ell m}\hat{x}_m \) and \( \hat{\nu} = \hat{q}_{\ell t} \),

\[ a_{\ell m}a_{\ell m} = \delta_{\ell m} + 2\delta_{\ell m}q_{\ell m} + q_{\ell m}q_{\ell m} = 3 + 2\hat{q} + q_{\ell m}q_{\ell m} \]
and also
\[ \hat{a} = 1 + \hat{q}, \quad \overline{a} = 3 + \overline{q}, \quad |a\hat{x}|^2 = 1 + 2\hat{q} + |q\hat{x}|^2. \]

Note that \( |q| = |a(x) - I| \leq C_I(x)^{-\delta} < 1 \) by assumption (1.17), which implies
\[ |\overline{q}| \leq 3C_I(x)^{-\delta}, \quad |\hat{q}| \leq C_I(x)^{-\delta}, \quad |q\hat{x}| \leq C_I(x)^{-\delta} \]
so that
\[ 2a_{\ell m}a_{\ell m} + \overline{a} - 6\overline{m} \hat{a} + 15\hat{a} - 12|a\hat{x}|^2 = 47 - 12\hat{q} + 2a_{\ell m}a_{\ell m} + \overline{a} - 6\overline{m} \hat{a} + 15\hat{a} - 12|a\hat{x}|^2 \]
\[ \geq 47 - 12\hat{q} - 6\overline{q} - 12|q\hat{x}|^2 \geq 46C_I(x)^{-\delta}. \]

We have also \( 1 - C_I \leq \hat{a} \leq 1 + C_I \) so that \((n = 3)\)
\[-\frac{n-1}{2}\hat{a}^2 \leq -(1 - C_I)^2, \quad \left( \frac{n+3}{2}\hat{a} - \overline{a} \right) \hat{a} \leq 6C_I(1 + C_I) < 12C_I \]
Thus under the assumptions of Theorem 1.2 we obtain the estimates
\[ S(x) \leq -(1 - C_I)^{-\frac{1}{2}} |x| = R \quad \text{for } |x| \leq R \]
and
\[ S(x) \leq 24C_I \left[ \frac{|x|^2}{|x|^2} + \frac{1}{|x|^2(x)^2} \right] \quad \text{for } |x| > R. \]

We are now ready to estimate the integral
\[ -\int_{\Omega} A(A\psi + \phi)|v|^2\,dx \equiv -\int_{\Omega} A^2\psi|v|^2\,dx = I + II \]
where
\[ I = -\int_{\Omega} S(x)|v|^2\,dx, \quad II = -\int_{\Omega} R(x)|v|^2\,dx. \]

In Theorem 1.1 by (4.39) and (3.5) we have immediately for any \( R > 0 \)
\[ II \geq -24n\delta^{-1}C_a(N + C_a)\|v\|^2_X. \]

In Theorem 1.2 we use a different estimate, which is valid for \( R > 1 \) only: by (4.39) and (3.19) we have
\[ R > 1 \quad \Rightarrow \quad II \geq -324\delta^{-1}C_a(N + C_a) \left[ \|v\|^2_X + \|\nabla^b\psi\|^2_{L^2(\Omega_{\delta})} \right]. \]

Concerning the \( S(x) \) term, in Theorem 1.1 we have by (4.43), (4.44)
\[ I \geq \frac{n-1}{2}\nu\frac{1}{2} \int_{\Omega_{\delta}} \hat{a}|v|^2\,dS - \left[ \frac{n+3}{2}N - \nu \right] (n - 1) \int_{\Omega_{\delta}} \hat{a}\left( \frac{n-1}{2} \right) \]
for all \( R > 0 \). If \( \frac{n+3}{2}N \geq \nu \), we can use (3.10) to get
\[ I \geq \frac{n-1}{2}\nu\frac{1}{2} \int_{\Omega_{\delta}} \hat{a}|v|^2\,dS - \left[ \frac{n+3}{2}N - \nu \right] \left\| \hat{a}^{1/2}v \right\|^2_X \]
and taking the sup over \( R > 0 \) we conclude
\[ \sup_{R > 0} I \geq K_0\|\hat{a}^{1/2}v\|^2_X \]
provided we define \( K_0 \) as
\[ K_0 := \frac{n-1}{2} - \frac{n+3}{2}\frac{N}{\nu} + n > 0; \]

notice that the condition \( K_0 > 0 \) is equivalent to assumption (1.9). On the other hand, if \( \frac{n+3}{2}N \leq \nu \) (which implies \( K_0 \leq \frac{n+1}{2} \)) from (4.49) we have directly
\[ I \geq \frac{n-1}{2}\nu\frac{1}{2} \int_{\Omega_{\delta}} \hat{a}|v|^2\,dS \geq K_0\frac{1}{2} \int_{\Omega_{\delta}} \hat{a}|v|^2\,dS \]
and taking the sup over \( R > 0 \) we obtain again (4.50).

In Theorem 1.2, using (3.10) and (3.6) in (4.45), (4.46), we have for all \( R > 1 \)
\[ R > 1 \quad \Rightarrow \quad I \geq (1 - C_I)^{-\frac{1}{2}} \int_{\Omega_{\delta}} |v|^2\,dS - 72C_I\delta^{-1}\|v\|^2_X. \]

It remains to consider the second term in (4.35); in Theorem 1.1 we have
\[ -a(\nabla\psi, \nabla c)|v|^2 = -a(\hat{a}, \nabla c)\psi' |v|^2 \geq -\frac{C_a}{|x|^2(x)^2} \psi' |v|^2 \]
thanks to assumption (1.7). Since \( 0 < \psi' < 1/2 \), by estimate (3.5) we obtain
\[
- \int_{\Omega} a(\nabla \psi, \nabla c) |v|^2 \geq - C_\delta \delta^{-1} \|v\|_X^2
\] (4.54)
and taking into account (4.50), (4.47) and the inequality \( \alpha \geq \nu \), we obtain
\[
\sup_{R > 0} \int_{\Omega} I_{|v|^2} dx \geq \left[ K_\nu \nu^2 - (24\pi C_a (N + C_a) + C_\delta \delta^{-1}) \right] \|v\|_X^2.
\] (4.55)
In Theorem 1.2 we use the stronger assumption (1.16) and (3.19), to obtain
\[
- \int_{\Omega} a(\nabla \psi, \nabla c) |v|^2 \geq - 8\delta^{-1} C_a \|v\|_X^2 - 9 C_c \|\nabla^b v\|_{L^2(\Omega \leq 1)}^2.
\] (4.56)
Putting together (4.48), (4.52) and (4.56) we obtain
\[
\int_{\Omega} I_{|v|^2} dx \geq (1 - C_I) \frac{1}{\Omega_{\Omega R}} \int_{\Omega_{\leq R}} |v|^2 dS - \frac{72 C_I \delta^{-1} + 8 \delta^{-1} C_c}{\Omega_{\Omega R}} \|v\|_X^2 - 9 C_c \|\nabla^b v\|_{L^2(\Omega \leq 1)}^2
\]
whence, noticing that \( \|u\|_{L^2(\Omega \geq 1)} \leq \sqrt{2} \|u\|_Y \), we have for all \( R > 1 \)
\[
\int_{\Omega} I_{|v|^2} dx \geq (1 - C_I) \left[ \frac{1}{\Omega_{\Omega R}} \int_{\Omega_{\leq R}} |v|^2 dS - 8\delta^{-1} C_c + 9 C_I + 41 C_a (N + C_a) \right] \|v\|_X^2
\]
\[
- 13 \delta^{-1} C_c + 36 C_a (N + C_a) \|\nabla^b v\|_{L^2(\Omega \geq 1)}^2.
\] (4.57)

4.5. Estimate of the terms in \( \|\nabla^b v\|^2 \). We consider now the terms in (2.6) which are quadratic in \( \nabla^b v \): since \( \phi = 0 \) they reduce to
\[
I_{|v|^2} = \alpha_{\ell m} \cdot \Re(\partial^b_v \partial^b_{m} v)
\]
with
\[
\alpha_{\ell m} = 2 a_{\ell m} \partial_j (a_{k} \delta_k \psi) - a_{j \ell k} a_{\ell m, k} \partial_k \psi.
\]
We split the coefficient as
\[
\alpha_{\ell m} = s_{\ell m}(x) + r_{\ell m}(x)
\]
where the remainder \( r_{\ell m} \) gathers the terms containing derivatives of the \( a_{jk} \). Since the weight \( \psi \) is radial we have
\[
s_{\ell m}(x) = 2 a_{\ell m} a_{k} \partial_j \partial_k \psi = 2 a_{\ell m} a_{j k} \frac{\psi'}{|x|}
\]
while
\[
r_{\ell m}(x) = [2 a_{\ell m} a_{k, j} - a_{j k} a_{\ell m, j}] \partial_k \psi.
\]
We estimate directly
\[
|s_{\ell m}(x)\Re(\partial^b_v \partial^b_{m} v)| \leq 3 |a| |a'||\|\nabla^b v\|^2
\]
and by assumption (1.4) we obtain
\[
|r_{\ell m}(x)\Re(\partial^b_v \partial^b_{m} v)| \leq 3 N C_a (x^{-1 - \delta}) \|\nabla^b v\|^2.
\]
Then integration on \( \Omega \) gives, using (3.7),
\[
\int_{\Omega} |s_{\ell m}(x)\Re(\partial^b_v \partial^b_{m} v)| dx \leq 24 N C_a \|\nabla^b v\|_{L^2}^2.
\] (4.58)
Concerning \( s_{\ell m} \), in the region \( |x| > R \) we have
\[
s_{\ell m}(x) = [a_{j m} a_{j} - a_{j m} a_{\ell k} \partial_j \partial_k \partial_{x m}] \frac{\psi'}{|x|} + \frac{\delta^{\Delta-1}}{|x|^m} a_{j m} a_{\ell k} \partial_j \partial_k \partial_{x m} - a_{j m} a_{j} \frac{\delta^{\Delta-1}}{|x|^m}
\]
so that, in the sense of positivity of matrices,
\[
s_{\ell m}(x) \geq [a_{j m} a_{j} - a_{j m} a_{\ell k} \partial_j \partial_k \partial_{x m}] \frac{\psi'}{|x|} \geq 0 \quad \text{for } |x| > R
\]
(indeed, one has \( a_{j m} a_{j} \geq a_{j m} a_{\ell k} \partial_j \partial_k \partial_{x m} \) as matrices); on the other hand, in the region \( |x| \leq R \) we have
\[
s_{\ell m}(x) = a_{j m} a_{j} \frac{\psi'}{|x|} \quad \text{for } |x| \leq R.
\]
Thus, by the assumption \( a(x) \geq \nu I \), one has for all \( x \)
\[
s_{\text{en}}(x) \Re (\partial^2_t \psi_0^m \psi_m) \geq \frac{a - 1}{m} \nu^2 1_{|x| \leq R(x)} |\nabla^b v|^2.
\]
Integrating on \( \Omega \) and recalling (4.58) we obtain
\[
\int_{\Omega} I_v |\nabla v|^2 \, dx \geq \frac{a - 1}{m} \nu^2 \int_{|x| \leq R} |\nabla^b v|^2 \, dx - 24 NC_\alpha \delta^{-1} \|\nabla^b v\|^2_Y.
\]

4.6. Estimate of the magnetic terms. Consider the term
\[
I_b := 23 |a_j \partial^b \psi (\partial_t b - \partial_t b_j) \alpha_{en} \partial_m \psi | \equiv 23 \left[ (\partial_t \cdot \partial^b) \cdot (a \nabla^b v) \nabla^b \psi \right]
\]
where the identity holds for any radial \( \psi \), while \( \partial^b \) is the matrix
\[
db = [\partial_j b_t - \partial_t b_j]_{t=1}.
\]
Since \( 0 \leq \psi' \leq 1/2, |a(x)| \leq N \), we have
\[
|I_b(x)| \leq 2 N^2 |\partial_t b(x)| \cdot |\nabla^b v| |\psi| \leq N^2 |\partial_t b(x)| \cdot |\nabla^b v| |v|
\]
and by assumption (1.5) we get
\[
\int_{\Omega} |I_b| \leq N^2 C_b \int_{\Omega} \frac{|\nabla^b v| |v|}{|x|^\alpha + |x|^\beta} \leq 5 \delta^{-1} N^2 C_b (\|\nabla^b v\|^2_Y + \|v\|^2_X)
\]
using (3.13). Under the stronger assumption (1.14) we have instead, using (3.17)
\[
\int_{\Omega} |I_b| \leq N^2 C_b \int_{\Omega} \frac{|\nabla^b v| |v|}{|x|^\alpha + |x|^\beta} \leq 9 \delta^{-1} N^2 C_b (\|\nabla^b v\|^2_Y + \|v\|^2_X).
\]

4.7. Estimate of the terms containing \( f \). Consider now the terms
\[
I_f := \Re (A \psi + \phi) \nabla f + 2 \Re (\partial^b \psi, \nabla^b v) f
\]
with \( \phi \equiv 0 \). We have easily
\[
\psi'', \frac{\psi''}{|x|^\alpha} \leq \frac{1}{2(\alpha R^2)} \quad \implies \quad 0 \leq \hat{\psi}'' + \frac{\hat{\psi}}{|x|^\alpha} \psi' \leq \frac{N_n}{2(\alpha R^2)} \leq \frac{\nu}{2(\alpha R^2)}
\]
and recalling (4.34) we get
\[
|A \psi| \leq \frac{N_n}{2(\alpha R^2)} + \frac{1}{2} |a| \leq \frac{N_n}{2(\alpha R^2)} + \frac{C_\alpha}{2(\alpha R^2)}
\]
by \( \psi' \leq 1/2 \) and assumption (1.4). Thus by (3.4) we can write, for all \( R > 0 \),
\[
\int_{\Omega} |(A \psi) \nabla f| \leq \frac{1}{2} (N_n + C_\alpha) \|f\|_{Y^\cdot} \|x|^{-1} |v| \leq \frac{1}{2} (N_n + C_\alpha) \|f\|_{Y^\cdot} \|v\|_{X^\cdot}
\]
while, using instead the second estimate in (3.4), we can write for all \( R > 1 \)
\[
\int_{\Omega} |(A \psi) \nabla f| \leq (N_n + C_\alpha) \|f\|_{Y^\cdot} \|x|^{-1} |v| \leq (N_n + C_\alpha) \|f\|_{Y^\cdot} \|v\|_{X^\cdot}.
\]
For the second term in \( I_f \) we have simply
\[
|2 a (\nabla \psi, \nabla^b v) f| \leq N |\nabla^b v||f|
\]
and summing up, for any \( 0 < \zeta_1 \leq 1 \), we obtain
\[
R > 0 \quad \implies \quad \int_{\Omega} |I_f| \leq (N_n + C_\alpha)^2 \zeta_1^{-1} \|f\|_{X^\cdot}^2 + \zeta_1 \|v\|_{X^\cdot}^2 + \|\nabla^b v\|_{Y^\cdot}^2.
\]
and also
\[
R > 1 \quad \implies \quad \int_{\Omega} |I_f| \leq (N_n + C_\alpha)^2 \zeta_1^{-1} \|f\|_{X^\cdot}^2 + \zeta_1 \|v\|_{X^\cdot}^2 + \|\nabla^b v\|_{Y^\cdot}^2.
\]
4.8. Estimate of the boundary terms. It remains to consider the term in divergence form $\partial_j R Q_j$ appearing in identity (2.6), where

$$Q_j = a_{jk} \nabla^b v \cdot [A^b, \psi] \nabla + \frac{1}{2} a_{jk} \partial_k A \psi \big| v \big|^2 - a_{jk} \partial_k \psi \left[ (c - \lambda) |v|^2 + a(\nabla b v, \nabla b v) \right].$$

Note that

$$[A^b, \psi] \nabla = (A \psi) \nabla + 2a(\nabla \psi, \nabla b v).$$

As before we integrate over the set $\Omega \cap \{|x| \leq R\}$ and then let $R \to +\infty$. The integral over $|x| = R$ tends to zero, and we are left to consider the integral over $\partial \Omega$. After canceling several terms due to the Dirichlet boundary condition, and noticing that $\nabla b v = \nabla v + ib v = \nabla v$ on $\partial \Omega$, we are left with

$$\int_{\Omega} \partial_j R Q_j = \Re \int_{\Omega} \big| 2a(\nabla v, \hat{v}) \cdot a(\hat{v}, \nabla v) - a(\nabla v, \nabla v) \cdot a(\hat{v}, \hat{v}) \big| dS$$

where $\hat{v}$ is the exterior unit normal to $\partial \Omega$. Dirichlet boundary conditions imply that $\nabla v$ is normal to $\partial \Omega$ so that

$$\nabla v = (\hat{v} \cdot \nabla v) \hat{v}$$

and hence

$$a(\nabla v, \hat{v}) = (\hat{v} \cdot \nabla v) a(\hat{v}, \hat{v}), \quad a(x, \nabla v) = (\hat{v} \cdot \nabla v) a(\hat{v}, \hat{v}), \quad a(\nabla v, \nabla v) = \big| \nabla v \big|^2 a(\hat{v}, \hat{v})$$

and

$$\int_{\Omega} \partial_j R Q_j = \int_{\Omega} \big| \nabla v \big|^2 a(\hat{v}, \hat{v}) dS.$$ 

In particular when the obstacle $\mathbb{R}^n \setminus \Omega$ satisfies (1.8) we have

$$\int_{\Omega} \partial_j R Q_j \leq 0.$$  

(4.66)

4.9. Conclusion of the proof. We now integrate the identity (2.6) on $\Omega$; by (4.66) we obtain

$$\int_{\Omega} (|v|^2 + |\nabla v|^2) \leq \int_{\Omega} (|L| + |b| + |f|).$$  

(4.67)

4.9.1. Proof of Theorem 1.1. We use estimates (4.18), (4.55), (4.60), (4.61), (4.63), and take the sup over $R > 0$. Note in particular that by (4.60) we have

$$\sup_{R > 0} \int_{\Omega} |v|^2 d\nu \geq \left[ \frac{\alpha - 1}{\beta} \nu^2 - 24nNC_\delta^{-1} \right] \| \nabla b v \|^2_{\gamma}.$$ 

Thus we obtain

$$M_1 \| v \|^2_{\chi} + M_2 \| \nabla b v \|^2_{\gamma} \leq M_3 \| f \|^2_{\gamma},$$

where

$$M_1 = K_0 \nu^2 - 24nC_\delta^{-1} \frac{N + C_\delta^{-1}}{\delta^{-1}} - 18C_\delta^{-1} N(N + 2)C_\delta^{-1} - 5\delta^{-1} N^2 C_b - \zeta - \zeta_1$$

$$M_2 = \frac{\alpha - 1}{\beta} \nu^2 - 24nC_\delta^{-1} \frac{N + C_\delta^{-1}}{\delta^{-1}} - 18C_\delta^{-1} N(N + 2)C_\delta^{-1} - 5\delta^{-1} N^2 C_b - \zeta - \zeta_1$$

$$M_3 = 8(N + 2)^2 (C_\delta^{-1} + C_\delta + 2N + n^2 C_a + 1)(\zeta^{-1} + (N + C_a^2)^2 \zeta^{-1} - 1).$$

Recall that

$$K_0 := \frac{\alpha - 1}{\beta} - \frac{\alpha^2}{\beta^2} \frac{N}{\beta^2} > 0 \implies N \leq 3\nu$$

while $\zeta, \zeta_1 \in (0, 1]$ are arbitrary. We define

$$K := \min \left\{ 1, \frac{\nu^2}{\beta^2}, \frac{\alpha^2}{\beta^2} K_0 \right\},$$

we impose the conditions

$$24nC_\delta(N + C_\delta^{-1}) \delta^{-1}, \quad C_\delta \delta^{-1}, \quad 18C_\delta^{-1} N(N + 2)C_\delta^{-1}, \quad 5\delta^{-1} N^2 C_b$$

are $\leq K$ and we choose $\zeta = \zeta_1 = K$. Then we get

$$\| v \|^2_{\chi} + \| \nabla b v \|^2_{\gamma} \leq K^{-1} M_3 \| f \|^2_{\gamma}.$$ 

It is easily checked that with our choices $C_\delta \leq \nu, C_a \leq \nu, N \leq 3\nu$ so that

$$M_3 \leq 64n^2 K^{-1} (\nu + 1)^2 (C_\delta + \nu + 1)^2$$
and finally

\[ \|v\|^2_X + \|\nabla^b v\|^2_Y \leq 64n^2K^{-2}(\nu + 1)^2(C_+ + \nu + 1)^2\|f\|^2_Y \quad (4.68) \]

while the previous conditions on the coefficients are implied by assumptions (1.10).

On the other hand, estimates (4.12), (4.15) with the previous choices give

\[ \lambda_+\|v\|^2_Y \leq 6\nu\|\nabla^b v\|^2_Y + 4(C_+^2 + 3\nu\nu + 1)\|v\|^2_X \leq \lambda_+\|v\|^2_Y, \]

\[ \lambda_-\|v\|^2_Y \leq 2(\nu^2 + 3\nu + n^2\nu + 1)\|v\|^2_X + \|f\|^2_Y, \]

while estimate (4.9) gives

\[ |\epsilon|\|v\|^2_Y \leq 3(3\nu + 1)\left[ \|v\|^2_X + \|\nabla^b v\|^2_Y + \|f\|^2_Y \right]. \]

Summing up, and using (4.68), we obtain (1.12).

4.9.2. Proof of Theorem 1.2. We use estimates (4.26), (4.57), (4.60), (4.62), (4.64) and take the sup over \( R > 1 \). In particular we have, by (4.60),

\[ \sup_{R > 1} \int_{\Omega} I_N|v|^2 dx \geq \left[ \frac{24n^2\nu^2 - 24NC_\delta^{-1}}{n} \right] \|\nabla^b v\|^2_Y. \]

Note also that we can take \( \nu = 1 - C_I \) and \( N = 1 + C_I \) by assumption (1.17), while \( n = 3 \). We obtain

\[ (1 - C_I)\sup_{R > 1} \frac{1}{R^2} \int_{\Omega^\ast R} |v|^2 dS = m_1\|v\|^2_X + M_2\|\nabla^b v\|^2_Y \leq M_3\|f\|^2_Y, \]

where

\[ m_1 = 8\delta^{-1}[41C\nu(N + C_\delta) + 9C_\delta + C_\delta] + 12\delta^{-1}N(N + 2)C_\delta + 9\delta^{-1}N^2C_\delta + \zeta + \zeta_1 \]

\[ M_2 = 4[1 - C_I]^2 - 24NC_\delta^{-1} [36C\nu(N + C_\delta) + C_\delta] - 12\delta^{-1}N(N + 2)C_\delta - 9\delta^{-1}N^2C_\delta - \zeta - \zeta_1 \]

\[ M_3 = 44(C_\delta^2 + 4C_\delta^2 + 3N + 9C_\delta + 1)(N + 2)^22\zeta^{-1} + (3N + C_\delta)^22\zeta^{-1}. \]

We impose that \( C_I \leq 1/100 \) so that we can take \( N = 101/100, \nu = 99/100 \). If we choose \( \zeta = \zeta_1 = 1/200 \) and impose that the quantities

\[ 12\delta^{-1} - 36C\nu(N + C_\delta), \quad 135\delta^{-1}C_I, \quad 135\delta^{-1}C_\delta, \quad 18\delta^{-1}N(N + 2)C_\delta, \quad 9\delta^{-1}N^2C_\delta \]

are \( \leq 1/100 \), we see that \( 24NC_\delta^{-1} \leq 6 \cdot 10^{-5}, 8\delta^{-1} \cdot 41C\nu(N + C_\delta) \leq 8 \cdot 10^{-3}, C_\delta \leq 3 \cdot 10^{-5}, C_\delta^2 \leq 2 \cdot 10^{-4}, 8\delta^{-1}C_\delta \leq 7 \cdot 10^{-3} \). With these choices, we obtain easily

\[ \sup_{R > 1} \frac{1}{R^2} \int_{\Omega^\ast R} |v|^2 dS - \frac{46}{1000}\|v\|^2_X + \frac{60}{1000}\|\nabla^b v\|^2_Y \leq 38000(C_\delta^2 + 1)\|f\|^2_Y. \]

and using (3.18) we arrive at

\[ \|v\|^2_X + \|\nabla^b v\|^2_Y \leq 5 \cdot 10^8\|f\|^2_Y. \quad (4.69) \]

It is straightforward to check that the previous conditions are (generously) implied by the smallness assumptions (1.18).

We recall also estimates (4.22) and (4.24), which with the previous choice of constants imply

\[ \lambda_+\|v\|^2_Y \leq 18(C_\delta^2 + 1)(\|\nabla^b v\|^2_Y + \|v\|^2_X) + \|f\|^2_Y, \]

\[ \lambda_-\|v\|^2_Y \leq 10^{-4}\|\nabla^b v\|^2_Y + 7\|v\|^2_X + \|f\|^2_Y, \]

while estimate (4.20) implies now

\[ |\epsilon|\|v\|^2_Y \leq 5\|v\|^2_X + 5\|\nabla^b v\|^2_Y + \|f\|^2_Y. \]

Taking into account (4.69), we finally obtain (1.20).
5. Proof of the Corollaries

The proof of Corollary 1.3 is standard. Denote by $R(z) = (L - z)^{-1}$ the resolvent operator of $L$, then estimates (1.11) and (1.19) imply
\[
\| \langle x \rangle^{-\frac{1}{2}} R(z) v \|_{L^2(\Omega)} + \| \langle x \rangle^{-\frac{1}{2}} \nabla R(z) v \|_{L^2(\Omega)} \lesssim \| \langle x \rangle^{\frac{1}{2}} v \|_{L^2(\Omega)}
\]
where we used estimates (3.5), (3.6) and (3.7) from Section 3. From the first term by complex interpolation we obtain
\[
\| \langle x \rangle^{\frac{1}{2}} R(z) (\langle x \rangle^{-\frac{1}{2}} v) \|_{L^2(\Omega)} \lesssim \| v \|_{L^2(\Omega)}
\]
which means that the operator $\langle x \rangle^{\frac{1}{2}}$ is $L$-supersmooth; on the other term, from the second term and the elliptic estimate (1.24) we have
\[
\| \langle x \rangle^{-\frac{1}{2}} L^\frac{1}{2} R(z) L^\frac{1}{2} v \|_{L^2(\Omega)} = \| \langle x \rangle^{-\frac{1}{2}} L^\frac{1}{2} R(z) v \|_{L^2(\Omega)} \lesssim \| \langle x \rangle^{-\frac{1}{2}} \nabla R(z) v \|_{L^2(\Omega)} \lesssim \| \langle x \rangle^{\frac{1}{2}} v \|_{L^2(\Omega)}
\]
and this implies that $\langle x \rangle^{-\frac{1}{2}} L^\frac{1}{2}$ is also $L$-supersmooth. From the standard Kato theory (see the details in [19]) we obtain (1.25), (1.26), (1.27).

Also the proof of (1.28) is standard: we can write for any $f \in D(L)$ and any test function $g$ (scalar product and norm of $L^2(\Omega)$)
\[
(e^{itL} f, g) = (f_t^{t+1} \partial_x [(s-t)e^{isL} f] ds, g) = i \int_t^{t+1} ((s-t)Le^{isL} f, g) ds - \int_t^{t+1} (e^{isL} f, g) ds
\]
which implies by Cauchy–Schwarz
\[
\| (e^{itL} f, g) \| \leq \| \langle x \rangle^{\frac{1}{2}} \| \cdot \| e^{itL} f \|_2 \| + \| \langle x \rangle^{-\frac{1}{2}} e^{itL} f \|_2^2 \| ds \|^{\frac{1}{2}}
\]
Using (1.25) we obtain
\[
\int_0^\infty \| \langle x \rangle^{-\frac{1}{2}} e^{itL} f \|_2^2 ds \leq \| L^\frac{1}{2} f \|^2
\]
and this implies that for every $\epsilon > 0$ we can find $T_\epsilon$ such that
\[
\int_0^\infty \| \langle x \rangle^{-\frac{1}{2}} e^{itL} f \|_2^2 ds \leq \epsilon \quad \text{for} \quad t > T_\epsilon.
\]
The second term can be handled in a similar way. Thus we have proved that for any $f \in D(L)$, any test function $g$ and any $\epsilon > 0$
\[
\| (e^{itL} f, g) \| \leq \| \langle x \rangle^{\frac{1}{2}} \| \cdot \epsilon \quad \text{for} \quad t > T_\epsilon + 1
\]
which implies
\[
\| \langle x \rangle^{-\frac{1}{2}} e^{itL} f \| \leq \epsilon \quad \text{for} \quad t > T_\epsilon + 1.
\]
By density (note that $L$ has no eigenvalues) we obtain (1.28).

We recall if $L$ is a selfadjoint operator on a Hilbert space $\mathcal{H}$ and $A : \mathcal{H} \to \mathcal{H}_1$ is a closed operator to a second Hilbert space $\mathcal{H}_1$, $A$ is called $L$-supersmooth if the following estimate holds for all $z \in \mathbb{C} \setminus \mathbb{R}$
\[
\| A(z - L)^{-1} A^* v \|_{\mathcal{H}_1} \leq C \| v \|_{\mathcal{H}}
\]
with a constant uniform in $z$. From this estimate one obtains the following estimates for the Schrödinger flow associated to $L$:
\[
\| A e^{-itL} v \|_{L^2(\Omega)} \leq C \| v \|_{\mathcal{H}}, \quad \| \int_0^t A e^{i(t-s)\xi} A^* h(s) ds \|_{L^2(\Omega)} \leq C \| h \|_{L^2(\Omega)}.
\]
The proof of Corollary 1.4 is immediate using the following result which is a special case of Corollary 2.5 in [19]:

**Proposition 5.1.** Let $L \geq 0$ be a selfadjoint operator on the Hilbert space $\mathcal{H}$ and let $P$ be the orthogonal projection onto $\ker(L)^\perp$. Assume $A$ and $AL^{-\frac{1}{2}} P$ are closed operators with dense domain from $\mathcal{H}$ to a second Hilbert space $\mathcal{H}_1$. If $A$ is $L$-supersmooth then $AL^{-\frac{1}{2}} P$ is $\sqrt{L}$-supersmooth.
Applying this result we obtain
\[ \langle x \rangle^{1+} \text{ is } L\text{-supersmooth} \implies \langle x \rangle^{1+}L^{-\frac{1}{4}} \text{ is } \sqrt{L}\text{-supersmooth} \]
and
\[ \langle x \rangle^{-\frac{1}{2}} \text{ is } L\text{-supersmooth} \implies \langle x \rangle^{-\frac{1}{2}} \text{ is } \sqrt{L}\text{-supersmooth} \]
and Corollary 1.4 follows directly. The RAGE-type property is proved similarly to the previous one.

References

[1] Shmuel Agmon. Spectral properties of Schrödinger operators and scattering theory. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 2(2):151–218, 1975.
[2] Serge Alinhac. On the Morawetz–Keel-Smith-Sogge inequality for the wave equation on a curved background. *Publ. Res. Inst. Math. Sci.*, 42(3):705–720, 2006.
[3] Juan Antonio Barceló, Luca Fanelli, Alberto Ruiz, and Maricruz Vilela. A priori estimates for the Helmholtz equation with electromagnetic potentials in exterior domains. *Proc. Roy. Soc. Edinburgh Sect. A*, 143(1):1–19, 2013.
[4] Juan Antonio Barceló, Alberto Ruiz, and Luis Vega. Some dispersive estimates for Schrödinger equations with repulsive potentials. *J. Funct. Anal.*, 236(1):1–24, 2006.
[5] Juan Antonio Barceló, Alberto Ruiz, Luis Vega, and M. C. Vilela. Weak dispersive estimates for Schrödinger equations with long range potentials. *Comm. Partial Differential Equations*, 34(1-3):74–105, 2009.
[6] Matania Ben-Artzi. Global estimates for the Schrödinger equation. *J. Funct. Anal.*, 107(2):362–368, 1992.
[7] Matania Ben-Artzi and Sergiu Klainerman. Decay and regularity for the Schrödinger equation. *J. Anal. Math.*, 58:25–37, 1992. Festschrift on the occasion of the 70th birthday of Shmuel Agmon.
[8] Nicolas Burq. Décroissance de l’énergie locale de l’équation des ondes pour le problème extérieur et absence de résonance au voisinage du réel. *Acta Math.*, 180(1):1–29, 1998.
[9] Nicolas Burq. Semi-classical estimates for the resolvent in nontrapping geometries. *Int. Math. Res. Not.*, (5):221–241, 2002.
[10] Nicolas Burq. Global Strichartz estimates for nontrapping geometries: about an article by H. F. Smith and C. D. Sogge. *Comm. Partial Differential Equations*, 28(9-10):1675–1683, 2003.
[11] Nicolas Burq, Patrick Gérard, and Nikolay Tzvetkov. On nonlinear Schrödinger equations in exterior domains. *Comm. Partial Differential Equations*, 28(9-10):1675–1683, 2003.
[12] Paul R. Chernoff. Essential self-adjointness of powers of generators of hyperbolic equations. *J. Functional Analysis*, 12:401–414, 1973.
[13] Walter Craig, Thomas Kappeler, and Walter Strauss. Microlocal dispersive smoothing for the Schrödinger equation. *Comm. Pure Appl. Math.*, 48(8):769–860, 1995.
[14] Pierre D’Ancona. Kato smoothing and strichartz estimates for wave and Dirac equations with a magnetic potential. *J. Differential Equations*, 246(12):4552–4567, 2009.
[15] Pierre D’Ancona and Reinhard Racke. Evolution equations on non-flat waveguides. *Arch. Ration. Mech. Anal.*, 206(1):81–110, 2012.
[24] Shin-ichi Doi. Smoothing effects for Schrödinger evolution equation and global behavior of geodesic flow. *Math. Ann.*, 318(2):355–389, 2000.
[25] M. Burak Erdoğan, Michael Goldberg, and Wilhelm Schlag. Strichartz and smoothing estimates for Schrödinger operators with almost critical magnetic potentials in three and higher dimensions. *Forum Mathematum*, 21:687–722, 2009.
[26] Luca Fanelli. Non-trapping magnetic fields and Morrey-Campanato estimates for Schrödinger operators. *J. Math. Anal. Appl.*, 357(1):1–14, 2009.
[27] Lev Kapitanski and Yuri Safarov. Dispersive smoothing for Schrödinger equations. *Math. Res. Lett.*, 3(1):77–91, 1996.
[28] Tosio Kato. Wave operators and unitary equivalence. *Pacific J. Math.*, 15:171–180, 1965.
[29] Tosio Kato. Wave operators and similarity for some non-selfadjoint operators. *Math. Ann.*, 162:258–279, 1965/1966.
[30] Tosio Kato and Kenji Yajima. Some examples of smooth operators and the associated smoothing effect. *Rev. Math. Phys.*, 1(4):481–496, 1989.
[31] Kazuhiro Kurata. An estimate on the heat kernel of magnetic Schrödinger operators and uniformly elliptic operators with non-negative potentials. *J. London Math. Soc. (2)*, 62(3):885–903, 2000.
[32] Jeremy Marzuola, Jason Metcalfe, and Daniel Tataru. Strichartz estimates and local smoothing estimates for asymptotically flat Schrödinger equations. *J. Funct. Anal.*, 255(6):1497–1553, 2008.
[33] Jason Metcalfe and Christopher D. Sogge. Global existence for Dirichlet-wave equations with quadratic nonlinearities in high dimensions. *Math. Ann.*, 336(2):391–420, 2006.
[34] Jason Metcalfe and Daniel Tataru. Decay estimates for variable coefficient wave equations in exterior domains. In *Advances in phase space analysis of partial differential equations*, volume 78 of *Progr. Nonlinear Differential Equations Appl.*, pages 201–216. Birkhäuser Boston Inc., Boston, MA, 2009.
[35] Cathleen S. Morawetz. Time decay for the nonlinear Klein-Gordon equations. *Proc. Roy. Soc. Ser. A*, 306:291–296, 1968.
[36] Cathleen S. Morawetz. Decay of solutions of the exterior problem for the wave equation. *Comm. Pure and Applied Math.*, 28:229–264, 1975.
[37] El Maati Ouhabaz. *Analysis of heat equations on domains*, volume 31 of *London Mathematical Society Monographs Series*. Princeton University Press, Princeton, NJ, 2005.
[38] Benoit Perthame and Luis Vega. Energy concentration and Sommerfeld condition for Helmholtz equation with variable index at infinity. *Geom. Funct. Anal.*, 17(5):1685–1707, 2008.
[39] Masa ıoshi Tsutsumi. Global solutions of nonlinear Schrödinger equations with variable coefficients in exterior domains of a three-dimensional space. *Differentsialnye Uravneniya*, 29(3):523–536, 1993.
[40] András Vasy and Jared Wunsch. Morawetz estimates for the wave equation at low frequency. *Math. Ann.*, 355(4):1221–1254, 2013.
[41] András Vasy and Maciej Zworski. Semiclassical estimates in asymptotically Euclidean scattering. *Comm. Math. Phys.*, 212(1):205–217, 2000.
[51] Georgi Vodev. Local energy decay of solutions to the wave equation for nontrapping metrics. 
*Mat. Contemp.*, 26:129–133, 2004.

[52] Kenji Yajima. On smoothing property of Schrödinger propagators. In *Functional-analytic methods for partial differential equations (Tokyo, 1989)*, volume 1450 of *Lecture Notes in Math.*, pages 20–35. Springer, Berlin, 1990.

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