The choice of the Bondi radial coordinate and the interpretation of the news function in axisymmetric spacetimes

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Abstract.
In the Bondi formulation of the axisymmetric vacuum Einstein equations, we argue that the “surface area” coordinate condition determining the “radial” coordinate is part of the initial data and can be chosen in a way that gives information about the physical problem whose solution is sought. We suggest a coordinate choice that follows from interpreting the radial coordinate, near infinity, as the (inverse of the) Newtonian potential. In this way, physical quantities that specify the problem (mass moments) enter the equations from the very beginning and play the role of “source” terms. A natural identification of the news function in terms of these “source” terms is suggested, leading to an expression for the radiated energy that formally differs from the standard quadrupole formula. We consider ways to reconcile this conclusion with the classical result.

1. Introduction
The evolution of a dynamical gravitational system in General Relativity is determined in terms of appropriate initial data given on an “initial” hypersurface [1]. These data must describe the initial state of the gravitational field (the geometry of the initial hypersurface and the positions, velocities, etc., of the gravitating masses generating the field). As the very definition of the problem (distances, velocities) depends on the unknown metric, it is clear that physical initial data can only be given approximately. Moreover, as the theory is invariant under coordinate transformations, the form of the data depends on the coordinates: particular “pieces” of the data can be “gauged away” by a choice of coordinates.

The required initial data depend on the nature of the initial hypersurface. For a space-like hypersurface, \( t = 0 \), one must give Cauchy data [2]: the metric on the hypersurface and its first time derivative (second fundamental form). The main difficulty in this formulation is that the initial data are not free: they must satisfy four non-linear constraint equations. Moreover, even if the data become flat outside a sphere containing the source, there is no guarantee that the evolved spacetime will remain asymptotically flat in future null directions and will contain only outgoing radiation, as required for an isolated system (global existence). These are formidable problems that, despite the extensive literature on the subject, have not yet been answered convincingly in sufficient generality [3].

When the initial hypersurface is null, the initial data are free [4]: given the (degenerate) metric on this null hypersurface, and additional boundary conditions on a timelike or null hypersurface.
intersecting the initial null hypersurface, then the field equations determine the remaining metric functions on this hypersurface, as well as the future evolution of the full metric. Moreover, it is straightforward to impose conditions which guarantee that the solution is asymptotically flat and contains no incoming radiation. However, the characteristic formulation is limited in that it is valid only asymptotically: the spacetime cannot be expected to be foliated globally by a set of null hypersurfaces \( u = \text{const} \), as these hypersurfaces are expected to bend and fold-up on themselves near the source (focusing of null geodesics in a strong gravitational field). Thus, the most interesting region of the spacetime – where the sources are located – which would determine the physical interpretation of the parameters in the solution, cannot be described in this formulation.

The Cauchy-Characteristic-Matching (CCM) [5] approach tries to combine the best features of the two formulations by matching, across a time-like world tube, an asymptotically flat solution containing no incoming radiation, obtained using the characteristic formulation outside the world tube, to a solution of the Cauchy problem for a particular distribution of sources in the interior of the tube. In this way, the arbitrary functions of integration appearing in the exterior solution are interpreted in terms of the properties of the source in the interior, while, at the same time, the interior solution is constrained to have proper behavior at infinity.

In a seminal 1962 paper [6], Bondi, van der Burg and Metzner used the characteristic formulation to obtain the general, exact¹ asymptotically flat solution (in the form of a formal series expansion, valid near null infinity) of the equations of General Relativity describing the evolution of an axisymmetric distribution of matter. The solution contains an arbitrary function of two variables and its interpretation depends on identifying this free function of integration with the properties of the source. In the original paper, this identification was made by appealing to the static form of the solution in a mathematically non-rigorous way that Bondi calls “distinctly crude”. In this paper, which generalizes recent results in [7], we argue that a physical interpretation of the solution, that contains no mathematically unjustified steps, can be obtained if we make a different choice for the “radial” coordinate: one that depends on the properties of the source.

2. Generalized Bondi coordinates

We will restrict our considerations to the case of axial symmetry. We will use the symbols \( u, \xi, \eta, \varphi \) for the four coordinates.² \( u \) is a null coordinate labeling the family of null hypersurfaces which contains the initial hypersurface \( u = u_0 \), while \( \eta, \varphi \) are angular coordinates that are constant on the null geodesics in the \( u = \text{const} \) hypersurfaces. Then, assuming also that the axial Killing vector trajectories (parametrized by \( \varphi \)) are hypersurface orthogonal, the metric tensor in these coordinates satisfies

\[
g^{uu} = g^{u\eta} = g^{u\varphi} = g^{\xi\varphi} = g^{\eta\varphi} = 0. \tag{1}
\]

One more condition must be imposed to define the coordinate \( \xi \), the “radial” coordinate parametrizing the null geodesics. Bondi imposes the condition that the determinant of the metric on the 2-surfaces \( u = \text{const}, \xi = \text{const} \) equal \( -\xi^4 \), so that they have the spherical area \( 4\pi \xi^2 \), while Penrose [8], Newman-Unti [9] impose the condition \( g^{u\xi} = 1 \) so that \( \xi \) is an affine parameter along the null geodesics. We will refrain, for the time being, from making any choice and write the metric satisfying (1) as

\[
ds^2 = \frac{V}{B} du^2 + \frac{2}{B} du d\xi - \frac{K^4}{R^4} (d\eta - U du)^2 - R^2 d\varphi^2, \tag{2}
\]

¹ “general” = contains an arbitrary function of two variables; “exact” = to all non-linear orders.
² \( \xi \) and \( \eta \) will be required to behave as spherical coordinates \( r \) and \( \cos \theta \) near infinity.
where \( B, V, K, R, U \) are arbitrary functions of \( u, \xi, \eta \). The metric retains its form under a change of the radial coordinate \( \xi \to \Xi(u, \xi, \eta) \), the functions \( B, V, U \) being redefined appropriately. The boundary conditions for asymptotic flatness, as this generalized Bondi radial coordinate \( \xi \to \infty \), are
\[
K \to \xi, \quad R \to \xi \sqrt{(1 - \eta^2)}, \quad B \to 1, \quad \xi U \to 0, \quad V \to 1,
\]
while the condition for the absence of incoming radiation [9, 10] is that the Weyl tensor component \( \Psi_0 \) in a null NP frame\(^3\) with \( l = du \) falls off as \( \Psi_0 \approx O(1/\xi)^5 \).

The degenerate metric on the null hypersurfaces \( u = \text{const} \) is
\[
d\sigma^2 = \frac{K^4}{R^2} du^2 + R^2 d\varphi^2,
\]
and is independent of the \( \xi \) coordinate condition. We observe that the function \( K \) (as well as \( R \)), entering the metric on the initial \( u = \text{const} \) hypersurface is part of the initial data. And an appropriate choice of the coordinate \( \xi \) would be one that would determine \( K \) as a function of \( u, \xi, \eta \) in terms of physical quantities describing the source. Such a choice is made in Section 4, after we verify in the next Section that Bondi’s hierarchical integration of the equations is not affected by the choice of \( K \).

3. Series solution of the vacuum Einstein equations
We assume that the functions \( K, R \) have formal series expansions in powers of \( \xi \), which are valid near \( \xi \to \infty \). In view of the boundary conditions (3), we write
\[
K \approx \xi + \frac{k_1(u, \eta)}{\xi} + \frac{k_2(u, \eta)}{\xi^2} + \frac{k_3(u, \eta)}{\xi^3} + O(1/\xi)^4, \quad \text{(5)}
\]
\[
R \approx \xi \sqrt{(1 - \eta^2)} \left( 1 + \frac{c_1(u, \eta)}{\xi} + \frac{c_2(u, \eta)}{\xi^2} + \frac{c_3(u, \eta)}{\xi^3} + O(1/\xi)^4 \right). \quad \text{(6)}
\]
The term with \( k_0(u, \eta) \) is missing as the field equations imply that \( k_0 \) must vanish for \( V \) to have the proper limit at infinity. Treating the \( k_n \) as known functions of \( (u, \eta) \), we can carry out the well-known hierarchical series of \( \xi \) integrations\(^4\) to obtain the other metric functions. To calculate tensor components we use a canonical Newman-Penrose null frame\(^3\), in which \( ds^2 = l \otimes n + n \otimes l - m \otimes \overline{m} - \overline{m} \otimes m \), where:
\[
l = \frac{1}{B} du, \quad n = d\xi + \frac{V}{2} du, \quad m = \frac{-1}{\sqrt{2}} \left( \frac{K^2}{R} \left( d\eta - U du \right) + i R d\varphi \right),
\]
\[
\Delta = B \left( \frac{\partial}{\partial u} - \frac{V}{2} \frac{\partial}{\partial \xi} + U \frac{\partial}{\partial \eta} \right), \quad D = \frac{\partial}{\partial \xi}, \quad \delta = \frac{1}{\sqrt{2}} \left( \frac{R}{K^2} \frac{\partial}{\partial \eta} + i \frac{\partial}{R} \frac{\partial}{\partial \varphi} \right). \quad \text{(7)}
\]
Thus the vanishing of the Ricci component \( \Phi_{00} \) determines the function \( B \) via the equation
\[
\frac{B\xi}{B} = - \frac{K\xi}{K} + 2 \frac{R\xi}{R} - \frac{K\xi\xi}{K\xi} - \frac{K R \xi^2}{R^2 K \xi}, \quad \text{(8)}
\]
\(^3\) We use the standard NP notation [8]: the complex null-tetrad basis \( \{ l, n, m, \overline{m} \} \) is normalized to \( l \cdot n = -m \cdot \overline{m} = 1 \). We will restrict the symbols \( \{ l, n, m, \overline{m} \} \) to denote the basis co-vectors (differential forms) while the vectors (differential operators) will be denoted by the standard symbols \( \{ D, \Delta, \delta, \overline{\delta} \} \).
\(^4\) Remarkably, the hierarchical structure of the “main” equations, allowing the functions \( B, U, V \) to be determined successively, is not destroyed by non-linear couplings through the non-zero \( u, \eta \) derivatives of \( K \).
whose solution, satisfying the boundary condition (3), is

\[ B \simeq 1 + \frac{2 k_1(u, \eta) + c_1(u, \eta)^2}{2 \xi^2} + \ldots \]  

(9)

We then compute the series expansion of the Weyl tensor component \( \Psi_0 \), obtaining

\[ \Psi_0 \simeq - \frac{2 k_1(u, \eta) + c_1(u, \eta)^2 - 2 c_2(u, \eta)}{\xi^4} + \ldots \]  

(10)

The condition for absence of incoming radiation \( \Psi_0 \simeq O(\frac{1}{\xi^5}) \), thus, implies that

\[ c_2(u, \eta) = k_1(u, \eta) + \frac{1}{2} c_1(u, \eta)^2. \]  

(11)

Next, the Ricci component \( \Phi_{0,1} \) can be written

\[ \left( \frac{B K^6 U_\xi}{R^2} \right) \frac{u}{\xi} = K^2 \left( \frac{B \xi B_\eta}{B^2} - \frac{B \xi \eta}{B} + 2 \frac{R \xi R_\eta}{R^2} + 2 \frac{R \xi \eta}{R} \right) + 2 K K \xi \left( \frac{B \eta}{B} - \frac{R_\eta}{R} \right), \]  

(12)

so that, with the series expansions of \( K, R, B \) known, \( U \) can be obtained. We find

\[ U \simeq \frac{(1 - \eta^2) c_1, u(y, \eta)}{\xi^2} + \frac{(1 - \eta^2) [4 c_1(u, \eta)c_1, u(y, \eta) - x_0(u, \eta)] - 8 \eta c_1(u, \eta)^2}{3 \xi^3} + \ldots \]  

(13)

where \( x_0(u, \eta) \) is an arbitrary function of integration. The second such function, to be added to \( U \), must be set equal to zero for asymptotic flatness.

Proceeding in the same way, equation \( \Phi_{1,1} + 3 \Lambda = 0 \) can be solved for \( ( V K K \xi) \xi \) in terms of known quantities, and, with \( V \) known, equation \( \Phi_{0,2} = 0 \) gives \( (K R_u/R_\xi). \) Denoting by \( y_0(u, \eta), z_0(u, \eta) \) the functions of integration, \( V \) and \( R_u \) are given by

\[ V \simeq 1 + \frac{y_0(u, \eta)}{\xi} + \ldots \]  

(14)

\[ R_u \simeq \sqrt{(1 - \eta^2)} \left[ z_0(u, \eta) + \frac{c_1(u, \eta) z_0(u, \eta) + k_1, u(u, \eta)}{\xi} \right] + \ldots \]  

(15)

Finally the requirement that this agrees with the \( u \) derivative of (6) determines the \( u \) derivatives of the coefficients \( c_n(u, \eta) \) (except for \( c_2(u, \eta) \) which is given by (11)):

\[ c_{1, u} = z_0, \quad c_{3, u} = C3(c_1, x_0, y_0, k_1, k_2), \quad c_{4, u} = C4(c_1, c_3, x_0, y_0, k_1, k_2, k_3). \]  

(16)

This completes the integration of the so-called “main” equations. Of the remaining equations, \( \Phi_{1,1} = 0 \) is satisfied identically, while the vanishing of \( \Phi_{1,2}, \Phi_{2,2} \) impose the following two conditions – conservation laws – on the three functions of integration \( x_0(u, \eta), y_0(u, \eta) \) and \( z_0(u, \eta) = c_{1, u}(u, \eta) \):

\[ x_0, u = -y_0, \eta + c_1, u, \eta - 3 c_1, u c_1, \eta - k_{1, u}, \eta, \]  

\[ y_0, u = 2 c_{1, u}^2 - 2 c_{1, u} + (1 - \eta^2) c_{1, u}, \eta - 4 \eta c_{1, u}, \eta + 2 k_{1, u}, \eta. \]  

(17)

We observe that the conservation laws (17) and the evolution equations (16) also form a hierarchical system: given \( c_1 \) (and assuming the \( k_n \) known), (17) determine successively \( y_0, x_0 \) and then (16) determine successively \( c_3, c_4, \ldots, c_n \). Thus the entire solution is determined by the free “source” functions \( k_n(u, \eta) \) (which are to be specified by imposing a coordinate condition on \( \xi \)) and the single arbitrary function of integration \( z_0(u, \eta) \) (or \( c_1(u, \eta) \)).
4. Physical interpretation of the solution and the choice of the coordinate $\xi$

To interpret the solution obtained in the previous section we rewrite the functions of integration $y_0$, $c_1$ as follows

\begin{align}
y_0(u, \eta) &= -2 M(u, \eta), \\
c_1(u, \eta) &= (1 - \eta^2) q(u, \eta),
\end{align}

so that $M(u, \eta)$ is Bondi’s “mass aspect” (see (14)) and $q(u, \eta)$ is finite on the axis. The second of (17) then becomes

\begin{align}
-M_{,u} &= (1 - \eta^2)^2 q_{,u \eta \eta} - 4 \eta q_{,u u} - 2 q_{,u}(1 - 3 \eta^2). \\
&= \xi_{,u}^2(1 - \eta^2)^2 + (1 - \eta^2) \frac{1}{2} q_{,u \eta \eta} - 4 \eta q_{,u u} + k_{1, u u} - 2 q_{,u}(1 - 3 \eta^2).
\end{align}

For this equation to describe the rate of energy loss due to gravitational radiation, its rhs must satisfy two requirements:

- it must be positive definite;
- its angular dependence must be $\sim (1 - \eta^2)^2$ as appropriate for a spin-2 field.

These requirements can be satisfied provided that $k_1 = f(u) (1 - 3 \eta^2)$ for some function $f(u)$. Because then we can choose the free function $q = f/2$ and obtain $-M = (f/2) (1 - \eta^2)^2$.

Now $k_1$ (as well as the other $k_n$ in the series expansion of $K$) can be determined by giving a physical interpretation to the 2-surfaces $u = \text{const}, \xi = \text{const}$, whose determinant is $K^4$. To satisfy the boundary condition $K \to \xi$ as $\xi \to \infty$, these surfaces must become spheres near infinity. And as the leading order correction to the flat metric near infinity is the Newtonian potential, we can give a physical meaning to the coordinate $\xi$ by identifying these surfaces with the surfaces of constant Newtonian potential in Euclidean 3-space. Thus, for the third coordinate condition determining $\xi$, we will require that $K^4$ will be given by the determinant of the metric on the surfaces of constant Newtonian potential (per unit mass) in Euclidean 3-space in orthogonal coordinates $\xi, \eta, \varphi$ which behave as $r, \cos \theta, \varphi$ at infinity.

4.1. The form of $K$

To obtain the form of $K$ following from this identification, we proceed as follows:

Let $r, x(= \cos \theta), \varphi$ be spherical coordinates in $E^3$. Then, define the functions $\xi(r, x), \eta(r, x)$ by the equations

\begin{align}
\frac{1}{\xi} &= \frac{1}{r} + \frac{M_1 x}{r^2} + \frac{M_2 (3 x^2 - 1)}{2 r^3} + \ldots, \\
\nabla \xi \cdot \nabla \eta &= \xi_r \eta_r + \frac{1 - x^2}{r^2} \xi_x \eta_x = 0,
\end{align}

where $M_l$ (assumed to depend on $u$) denotes the $2^l$-pole moment per unit mass, so that $\frac{1}{\xi}$ is the Newtonian potential per unit mass. Solving (22) for $\eta$ we find

\begin{equation}
\eta = x - (1 - x^2)(\frac{M_1}{r} + \frac{3 x M_2}{2 r^2} + \ldots).
\end{equation}

Using now $\xi, \eta$ as coordinates, the Euclidean 3-metric becomes

\begin{equation}
ds^2 = \frac{d\xi^2}{|\nabla \xi|^2} + \frac{d\eta^2}{|\nabla \eta|^2} + r^2(1 - x^2)d\varphi^2,
\end{equation}
so that the surfaces of constant $\xi$ have determinant $r^2(1-x^2)/|\nabla \eta|^2 = K^4$ (see (4)). Expressing this as a function of $\xi$, $\eta$ by inverting the coordinate transformation (21), (23) to obtain $r(\xi, \eta)$, $x(\xi, \eta)$ for large $\xi$, we finally arrive at (the $P_1(\eta)$ are Legendre polynomials)

$$K = \xi + (M_1^2 - M_2)P_2(\eta) \frac{1}{2\xi} - (M_3 - 3M_1M_2 + 2M_1^3)P_3(\eta) + \ldots$$

(25)

We note that this form of $K$ has no $k_0$ term (even when $M_1 = 0$), as required by asymptotic flatness, and that $k_1 = f(u)(1 - 3\eta^2)$ as required for satisfying the spin-2 positive energy radiation condition. We also note that the appearance of the Legendre polynomials, whose integral over the sphere vanishes, implies that the area of the radiation condition. We also note that the appearance of the Legendre polynomials, whose integral over the sphere vanishes, implies that the area of the $\xi = \text{const}$ surfaces equals $4\pi \xi^2$ to a very good approximation, so that the coordinate $\xi$ is also a very good “luminosity distance”.

We will now set $M_1 = 0$ (choice of origin) and write $M_2 = Q/M_0$ ($Q$=quadrupole moment, $M_0$=total mass). Then, choosing $c_1 = \dot{Q}/(8M_0)(1 - \eta^2)$, the radiated energy becomes

$$-\dot{\mathcal{M}} = \left(\frac{\dot{Q}}{8M_0}\right)^2(1 - \eta^2)^2,$$

(26)

which can readily be interpreted as the rate of energy loss due to gravitational radiation. This formula is an exact result of the equations with our choice of radial coordinate and of $c_1$; the coefficient $\dot{Q}/(8M_0)$, however, seems to disagree with the standard quadrupole formula $\dot{\mathcal{M}} = \dot{Q}/2$. In the next section we discuss several reasons that might explain this discrepancy.

5. Discussion

In comparing (26) with the standard quadrupole formula, one must keep in mind the following points:

- The quadrupole formula is an exact result of the linearized equations.
- The quadrupole formula incorporates the effects of the equations of motion.
- The symbol $Q$ and the “dots” in the two formulas have different meanings.

We will now briefly discuss how each of these points can be used to reconcile the two expressions.

5.1. The effect of non-linear terms

One can obtain a relation between $c_3$ and $c_1$ by differentiating the expression for $c_3$ in (16) and using equations (17) to eliminate the $u$-derivatives of $M$ and $x_0$. If we also put all the $k_n$ equal to zero (Bondi’s choice of radial coordinate), and define $Q_B(u)$ and $q_B(u)$ by writing

$$c_3 = \frac{1}{2}Q_B(1 - \eta^2), \quad c_1 = q_B(1 - \eta^2),$$

(27)

we find the relation

$$\ddot{Q}_B = 2\ddot{q}_B + M\ddot{q}_B + \text{terms non-linear in } q_B.$$  

(28)

Since $\dot{M} \sim \ddot{q}_B$, we must set $M = M_c = \text{const}$ when we drop the non-linear $q_B$ terms. After one integration, the resulting equation, $\ddot{Q}_B = 2\dot{q}_B + M_c\ddot{q}_B$, gives the standard quadrupole formula when $q_B \gg M_c\ddot{q}_B$, but agrees with (26) when $q_B \ll M_c\ddot{q}_B$ (taking $M_c = 8M_0$). Thus, when non-linear terms are taken into account in Bondi’s derivation of the radiated energy, the resulting quadrupole formula changes in a way that approaches (26). This result has been noted before [11, 12].

5 Calculating $K$ to several orders higher, we find that the first correction to the spherical area appears at $O(\frac{1}{\xi})^4$.  

6
5.2. The effect of the equations of motion

The third time derivative in the standard quadrupole formula is the result of using the equations of motion to convert the integral over the space components of the energy-momentum tensor of the source, determining the linear solution, to an integral over the leading component $T_{00}$. As the equations of motion hold on the particle trajectories, we cannot formulate or use them in the asymptotic solution. We can, however, ask the following question: what equation of motion would make the radiation formula (26) agree with the classical result?

Using Bondi’s value for $c_1$, we find that the two formulae would agree if $Q\dot{} = -Q/(4M_0)$, implying

$$Q = Q_0 \exp(-u - u_0/4M_0).$$

(29)

To the extent that this behavior for the quadrupole moment is reasonable (the diminishing quadrupole moment implies that gravity is, in the very least, attractive!), one can argue that, conversely, the “correct” equation of motion applied to (26) could bring it closer to the classical result.

An interesting special case of (29) is that of two point particles separated by a distance $a(u)$, for which $Q \sim a(u)^2$, so that (29) implies that $a(u) = a_0 \exp(-u - u_0/8(m_1 + m_2))$. In this case, all multiple moments and, therefore, all coefficients $k_n$ are proportional to powers of $a(u)$. Then all conservation laws (17) and evolution equations (16) can be integrated in closed form as the $u$-integrations involve sums of exponentials only. Finally, choosing the arbitrary functions (of $\eta$) of integration to vanish except for the final mass, $M_f = 8(m_1 + m_2)$, the resulting series solution (which can be continued to any desired order) has the properties

- it is an exact (to all nonlinear orders), closed form (all series coefficients are known explicitly) solution of the Einstein equations, including the evolution equations;
- the three parameters defining the problem, $m_1$, $m_2$, $a_0$, can have arbitrary relative values\(^6\);
- it reduces to the Schwarzschild solution when $u \to \infty$, as all $c_n$ and $k_n$ vanish with $a(u)$.

This solution is interesting in its own right and can be considered as representing qualitatively the late-time behavior of the head-on collision of two black holes.

One might object that the exponential decay of the distance parameter $a(u)$, implying that $\dot{a}(u) \to 0$ as $u \to \infty$, makes this solution unphysical. However, interpreting $\dot{a}(u)$ as relative “velocity” is unjustified as the meaning of symbols cannot be easily extrapolated to the vicinity of the source – which brings us to the last point of this discussion.

5.3. The meaning of symbols

We have, so far, intentionally avoided giving a precise definition for the symbols used to denote physical quantities. We will now point out the differences in the meaning of the same symbols in (26) and in the standard quadrupole formula.

First, the quadrupole moment $Q$ in (26) is defined at infinity in terms of the multipole expansion of the Newtonian potential and is axiomatically allowed to depend on $u$ (the same is true of Bondi’s $Q_B$), while $Q$ in the classical formula is defined in terms of a retarded-time integral over the spatial mass distribution whose dependence on time is determined by the equations of motion. Secondly, the classical formula involves time derivatives, while (26) contains derivatives with respect to the null coordinate $u$. To compare the two one must express the time function near the source (satisfying the harmonic coordinate condition to first order in the deviation of the metric from Minkowski space) in terms of our coordinates: $t(u, \xi, \eta)$. This would require

\(^6\) Except for an upper limit to the ratio $a_0/(m_1 + m_2)$ implied by the total radiated energy/mass.
obtaining a solution valid near the source and matching it across a timelike tube $\xi = \text{constant}$ to the asymptotic solution obtained here. The effect of using such a time coordinate to interpret physical quantities can be seen in the special case discussed previously, where the equation of motion that makes (26) agree with the quadrupole formula implies that two particles approach each other with exponentially vanishing “velocity” $\dot{a}(u)$. If, near the source, the time function has a characteristic [13, 14] logarithmic dependence on the radial coordinate of the form

$$
t = u + \xi - 2a(u) + 8 (m_1 + m_2) \log \left( \frac{\xi}{a_0} \right),
$$

then, on the curve $\xi = a(u) = a_0 \exp(-\frac{u-u_0}{8(m_1+m_2)})$, the parameter $a$ satisfies

$$
\frac{da}{dt} = -1.
$$

This, admittedly ad hoc, example shows how a different slicing of spacetime into “space” and “time” can drastically change physical conclusions and that the exponential-decay solution cannot be easily dismissed as unphysical.

6. Summary and conclusions

This work is an attempt to interpret the general, exact, asymptotically flat, series solution of the characteristic initial value problem, which is valid only near null infinity, in terms of the physical properties of a dynamical source of gravitating matter and the radiation it emits. The main point made is that the free function $K(u, \xi, \eta)$, which is part of the initial data in the characteristic formulation of the initial value problem, should not be gauged away but should be used to define the source of the gravitational field. And it is pointed out that this function must have the asymptotic behavior

$$
K \simeq \xi + \frac{f(u)(1-3\eta^2)}{\xi} + O\left(\frac{1}{\xi}\right)^2
$$

in order to allow a choice of the free function $c_1$ that will make the energy conservation equation (20) take a form that is appropriate for the emission of gravitational waves. A particular, necessarily approximate, physical definition of the coordinate $\xi$ near infinity is suggested that leads to such a $K$ with $f(u) \sim Q/M_0$, $Q$ being a formal, asymptotically defined “quadrupole moment” and $M_0$ a total “mass”. And it is pointed out that a better physical interpretation of these symbols requires extending the asymptotic solution to the near zone.

Defining the dynamical system via the choice of coordinates avoids the serious approximations that Bondi is forced to make in generalizing the static solution to a dynamic one (see the discussion in section VII of [7]). But there is clearly a problem of uniqueness in this approach: a different physical definition of the $u = \text{const}$, $\xi = \text{const}$ surfaces will, in general, lead to a different expression for $f(u)$. For example, identifying these surfaces with the surfaces on which the magnitude of the Newtonian acceleration is constant, leads to an extra factor of $3/2$ in the expression for $f(u)$. The way out of this dilemma is to require only that $k_1 = f(u)(1-3\eta^2)$ and $c_1 = \dot{f}/2(1-\eta^2)$, without restricting the other $k_n$ or attaching any meaning to $f(u)$. Then, matching this asymptotic solution to a sufficiently general interior solution across the time-like tube $\xi = \text{constant}$, as in the CCM [5] approach, will give a physical meaning both to the function $f(u)$ and to the coordinate $\xi$.

Finally, it should be stressed that our results follow from a change of coordinates only; and they could have been obtained using Bondi’s choice of radial coordinate, provided his mass aspect $M_B$ was taken equal to

$$
M_B = M + \dot{f}(1-3\eta^2).
$$
Then \( M \) would satisfy the simple radiation law (26) and, to eliminate the extra term in Bondi’s mass aspect (which distorts the angular distribution of the emitted radiation), one would have to make the change of coordinates
\[
\xi_B = \xi + f(u) \left( 1 - 3 \eta^2 \right) / \xi + O\left( \frac{1}{\xi}^2 \right).
\]
In fact, in deriving his radiation formula, Bondi obtains such a first-order (in \( \dot{f} \sim c_1 \)) correction to the mass aspect without discussing its physical meaning. Thus, if we were to adopt Bondi’s “distinctly crude” derivation, we would conclude that \( \dot{f} = -\ddot{Q} \), in agreement with the quadrupole formula. Note that the total Bondi mass is invariant under the redefinition (33), as the extra term vanishes when integrated over angles. More detailed discussion of these matters can be found in [7].

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