THE L-EQUIVALENT COUNTERPART OF THE M-III EQUATION

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Abstract

The connection between differential geometry of curves and the integrable (2+1)-dimensional spin system (the M-III equation) is established. Using the proposed geometrical formalism, the L-equivalent counterpart of the M-III equation is found.
1 Introduction

Consider the Myrzakulov III (M-III) equation [1]

\[ S_t = (S \wedge S_y + uS)_x + 2l(cl + d)S_y - 4cvS_x + S \wedge W \quad (1a) \]

\[ u_x = -S \cdot (S_x \wedge S_y) \quad (1b) \]

\[ v_x = \frac{1}{4(2cl + d)^2} (S_x^2)_y \quad (1c) \]

\[ W_x = JS_y \quad (1d) \]
which was introduced in [1]. Here \( S = (S_1, S_2, S_3) \) is the spin vector, \( S^2 = \beta = \pm 1 \), \( u \) and \( v \) are scalar functions, \( c, d, l \) - constants. The M-III equation (1) contains several interesting integrable particular cases:

i) \( c = 0, d = 1 \), yields the isotropic Myrzakulov I (M-I) equation

\[
S_t = (S \wedge S_y + uS)_x
\]

\[
u_x = -S \cdot (S_x \wedge S_y).
\]  

(2a)

(2b)

ii) \( d = 0 \) yields the Myrzakulov II (M-II) equation

\[
S_t = (S \wedge S_y + uS)_x + 2\lambda^2 S_y - 4cvS_x
\]

\[
u_x = -S \cdot (S_x \wedge S_y)
\]

\[

v_x = \frac{1}{16c^2\lambda^2}(S^2)_y.
\]  

(3a)

(3b)

(3c)

iii) \( c = 0, d = 1 \), \( J = \text{diag}(0,0,\triangle) \), yields the M-I equation with one-ion anisotropy

\[
S_t = (S \wedge S_y + uS)_x + 2l(cl + d)S_y + S \wedge W
\]

\[
u_x = -S \cdot (S_x \wedge S_y)
\]

\[
W_x = JS_y.
\]  

(4a)

(4b)

(4c)

iv) The isotropic M-III equation as \( J = \text{diag}(0,0,0) \)

\[
S_t = (S \wedge S_y + uS)_x + 2l(cl + d)S_y - 4cvS_x
\]

\[
u_x = -S \cdot (S_x \wedge S_y)
\]

\[

v_x = \frac{1}{4(2cl + d)^2}(S^2)_y
\]  

(5a)

(5b)

(5c)

and so on [1]. All of these equations are integrable. For instance, the Lax representation of the isotropic M-III equation (5) has the form [1]

\[
\phi_x = U' \phi
\]

\[
\phi_t = -2(c\lambda^2 + d\lambda)\Phi_y + V' \phi
\]  

(6a)

(6b)

with

\[
U' = [ic(\lambda^2 - l^2) + id(\lambda - l)]S + \frac{c(\lambda - l)}{2c\lambda + d}SS_x,
\]

\[
V' = [2c(\lambda^2 - l^2) + 2d(\lambda - l)]B + \lambda^2 F_2 + \lambda F_1 + F_0,
\]  

(7a)

(7b)

where

\[
F_2 = -4ic^2VS,
\]

\[
F_1 = -4icdVS - \frac{4c^2V}{2cl + d}VSS_x - \frac{ic}{2cl + d}S(SS_x)_y - [SS_x, B],
\]

\[
F_0 = -lF_1 - l^2F_2, B = \frac{1}{4}([S, S_y] + 2isu), S = \vec{S} \cdot \vec{\sigma}.
\]
In [1] we proposed a new class integrable and nonintegrable spin systems. And we also suggested the geometrical formalism to establish the connection between differential geometry of curves and surfaces and nonlinear evolution equations (NEE), including soliton equations (the A-, B-, C-, D-approaches) (see, also, the refs. [4, 5]). In this paper, using the D-approach we will establish the connection between the differential geometry of curves and the isotropic M-III equation (5). Also we find the L-equivalent (Lakshmanan equivalent [1, 2]) counterpart of this equation.

2 On the (2+1)-dimensional curve and soliton equations.

2.1 The M-LXVII equation

Using D-approach, in [1] we have established the connection between differential geometry of curves and well known soliton equations in 2+1 dimensions. It is remarkable that this approach simultaneously permit determine the L-equivalent counterpart of the under consideration spin systems. Here we will consider a (2+1)-dimensional curves which are given by the M-LXVII equation. The M-LXVII equation reads as [1]

\begin{equation}
\begin{pmatrix}
e_1 \\
e_2 \\
e_3 
\end{pmatrix}_x = \begin{pmatrix} 0 & k & -\gamma \\
-\beta k & 0 & \tau \\
\beta \gamma & -\tau & 0
\end{pmatrix} \begin{pmatrix} e_1 \\
e_2 \\
e_3 
\end{pmatrix}
\tag{8a}
\end{equation}

\begin{equation}
\begin{pmatrix} e_1 \\
e_2 \\
e_3 
\end{pmatrix}_t = C \begin{pmatrix} e_1 \\
e_2 \\
e_3 
\end{pmatrix} + D \begin{pmatrix} e_1 \\
e_2 \\
e_3 
\end{pmatrix}
\tag{8b}
\end{equation}

where we assume that \( k = k(\lambda, x, y, t) \), \( \tau = \tau(\lambda, x, y, t) \), \( \gamma = \gamma(\lambda, x, y, t) \), \( C = C(\lambda, x, y, t) \), \( D = D(\lambda, x, y, t) \) and \( \lambda \) is some complex parameter, \( C, D \) are some matrices.

And also let us we suppose that

\begin{equation}
k = \sum_{j=1}^{n} k_j \lambda^j, \quad \gamma = \sum_{j=1}^{n} \gamma_j \lambda^j, \quad \tau = \sum_{j=1}^{n} \tau_j \lambda^j \tag{9a}
\end{equation}

\begin{align}
C &= \sum_{j=1}^{m} C_j \lambda^j, \quad D = \sum_{j=1}^{m} D_j \lambda^j, \quad e_1^2 = \beta = \pm 1, \quad e_2^2 = e_3^2 = 1. \tag{9b}
\end{align}

Here \( k_j = k_j(x, y, t) \), \( \tau_j = \tau_j(x, y, t) \), \( \gamma_j = \gamma_j(x, y, t) \), \( C_j = C_j(x, y, t) \), \( D_j = D_j(x, y, t) \).

2.2 The M-LXVII equation associated with the M-III equation

In this subsection we require that the vector \( e_1 \equiv S \) satisfies the M-III equation (5). Then we must put

\begin{equation}
k_0 = -d(q + p), \quad k_1 = -2c(q + p), \quad k_j = 0, \quad j \geq 2 \tag{10a}
\end{equation}
\[ \gamma_0 = -d(q - p), \quad \gamma_1 = -2c(q - p), \quad \gamma_j = 0, \quad j \geq 2 \]  
\[ \tau_0 = 0, \quad \tau_1 = -2d, \quad \tau_2 = -2c, \quad \tau_j = 0, \quad j \geq 3 \]  
\[ C_0 = 0, \quad C_1 = 2d, \quad C_2 = 2c, \quad C_j = 0, \quad j \geq 3 \]  
\[ D = \lambda^2 D_2 + \lambda D_1 + D_0, \quad D_j = 0 \quad j \geq 3. \]

Here
\[ D = \lambda^2 D_2 + \lambda D_1 + D_0 = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\beta \omega_3 & 0 & \omega_1 \\ \beta \omega_2 & -\omega_1 & 0 \end{pmatrix} \]  

\[ q, p \] are some function. So in our case the M-LXVII equation takes the form [1]
\[ \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}_x = \begin{pmatrix} 0 & -(2c\lambda + d)(q + p) & i(2c\lambda + d)(q - p) \\ \beta(2c\lambda + d)(q + p) & 0 & -2(c\lambda^2 + d\lambda) \\ -\beta i(2c\lambda + d)(q - p) & 2(c\lambda^2 + d\lambda) & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \]  
\[ \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}_t = 2(c\lambda^2 + d\lambda) \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} + (\lambda^2 D_2 + \lambda D_1 + D_0) \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \]

where the explicit forms of \( D_j \) are given in [1]. In terms of matrix this equation we can write in the form
\[ \hat{e}_{1x} = -(2c\lambda + d)(q + p)\hat{e}_2 + i(2c\lambda + d)(q - p)\hat{e}_3 \]  
\[ \hat{e}_{2x} = \beta(2c\lambda + d)(q + p)\hat{e}_1 - 2(c\lambda^2 + d\lambda)\hat{e}_3 \]  
\[ \hat{e}_{3x} = -\beta i(2c\lambda + d)(q - p)\hat{e}_1 + 2(c\lambda^2 + d\lambda)\hat{e}_2 \]  
\[ \hat{e}_{1t} = 2(c\lambda^2 + d\lambda)\hat{e}_{1y} + \omega_3\hat{e}_2 - \omega_2\hat{e}_3 \]  
\[ \hat{e}_{2t} = 2(c\lambda^2 + d\lambda)\hat{e}_{2y} - \beta \omega_3 \hat{e}_1 + \omega_1 \hat{e}_3 \]  
\[ \hat{e}_{3t} = 2(c\lambda^2 + d\lambda)\hat{e}_{3y} + \beta \omega_2 \hat{e}_1 - \omega_1 \hat{e}_2 \]

where
\[ \hat{e}_1 = g^{-1}\sigma_3 g, \quad \hat{e}_2 = g^{-1}\sigma_2 g, \quad \hat{e}_3 = g^{-1}\sigma_1 g. \]

Here \( \sigma_j \) are Pauli matrices
\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

So we have
\[ \sigma_1 \sigma_2 = i\sigma_3 = -\sigma_2 \sigma_1, \quad \sigma_1 \sigma_3 = -i\sigma_2 = -\sigma_3 \sigma_1, \quad \sigma_3 \sigma_2 = -i\sigma_1 = -\sigma_2 \sigma_3 \]  
and
\[ \sigma_j^2 = I = \text{diag}(1,1). \]

Equations (13) we can rewrite in the form
\[ [\sigma_3, U] = -(2c\lambda + d)(q + p)\sigma_2 + i(2c\lambda + d)(q - p)\sigma_1 \]
\[ [\sigma_2, U] = \beta(2c\lambda + d)(q + p)\sigma_3 - 2(c\lambda^2 + d\lambda)\sigma_1 \] (17b)
\[ [\sigma_1, U] = -\beta i(2c\lambda + d)(q - p)\sigma_3 + 2(c\lambda^2 + d\lambda)\sigma_2 \] (17c)
\[ [\sigma_3, V] = \omega_3\sigma_2 - \omega_2\sigma_1 \] (17d)
\[ [\sigma_2, V] = -\beta\omega_3\sigma_3 + \omega_1\sigma_1 \] (17e)
\[ [\sigma_1, V] = \beta\omega_2\sigma_3 - \omega_1\sigma_2. \] (17f)

Here
\[ U = g_xg^{-1}, \quad V = g_tg^{-1} - 2(c\lambda^2 + d\lambda)g_xg^{-1}. \] (18)

Hence we get
\[ U = i[(c\lambda^2 + d\lambda)\sigma_3 + (2c\lambda + d)Q] \] (19a)
\[ V = \lambda^2B_2 + \lambda B_1 + B_0 \] (19b)

with
\[ B_2 = -4ic^2v\sigma_3, \quad B_1 = -4icdv\sigma_3 + 2cQ_y\sigma_3 - 8ic^2vQ, \quad B_0 = \frac{d}{2c}B_1 - \frac{d^2}{4c^2}B_2 \] (20)

and
\[ Q = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}, \quad p = \beta\bar{q}. \] (21)

Thus the matrix-function \( g \) satisfies the equations
\[ g_x = Ug \] (22a)
\[ g_t = 2(c\lambda^2 + d\lambda)g_y + Vg \] (22b)

The compatibility condition of these equations gives the following M-III equation
\[ iq_t = q_{xy} - 4ic(vq)_x + 2d^2vq = 0 \] (23a)
\[ -ip_t = p_{xy} + 4ic(vp)_x + 2d^2vp = 0 \] (23b)
\[ v_x = (pq)_y \] (23c)

So we have identified the curve, given by the M-LXVII equation (12) with the M-III equation (5). On the other hand, the compatibility condition of equations (22) is equivalent to the equation (23). So that we have also established the connection between the curve (the M-LXVII equation) and the equation (23). It means that the M-III equation (5) and the equation (23) are L-equivalent (Lakshmanan equivalent) to each other.
2.3 Particular cases

2.3.1 The M-LXVII equation and the M-I equation

For the M-I equation \((c = 0, d = 1)\) the associated M-LXVII equation has the form

\[
\begin{pmatrix}
    e_1 \\
    e_2 \\
    e_3
\end{pmatrix}_x =
\begin{pmatrix}
    0 & -(q + p) & i(q - p) \\
    \beta(q + p) & 0 & -2\lambda \\
    -\beta i(q - p) & 2\lambda & 0
\end{pmatrix}
\begin{pmatrix}
    e_1 \\
    e_2 \\
    e_3
\end{pmatrix}
\] (24a)

where the explicit form of \(D_0\) given in [1]. After some algebra we obtain the following L-equivalent of the isotropic M-I equation (2)

\[
iq_t = q_{xy} + 2vq
\] (26a)

\[
-ip_t = p_{xy} + 2vp
\] (26b)

\[
v_x = (pq)_y
\] (26c)

It is the Zakharov equation (ZE) [6].

2.3.2 The M-LXVII equation and the M-II equation

Now we put \(d = 0\). Then we get the following version of the M-LXVII equation

\[
\begin{pmatrix}
    e_1 \\
    e_2 \\
    e_3
\end{pmatrix}_x =
\begin{pmatrix}
    0 & -2c\lambda(q + p) & 2c\lambda(q - p) \\
    2\beta c\lambda(q + p) & 0 & -2c\lambda^2 \\
    -2\beta c\lambda(q - p) & 2c\lambda^2 & 0
\end{pmatrix}
\begin{pmatrix}
    e_1 \\
    e_2 \\
    e_3
\end{pmatrix}
\] (27a)

where the explicit forms of \(D_j\) are given in [1]. This M-LXVII equation is associated with the M-II equation (3). Proceeding as above we get the following L-equivalent of the M-II equation

\[
iq_t = q_{xy} - 4ic(vq)_x
\] (28a)

\[
-ip_t = p_{xy} + 4ic(vp)_x
\] (28b)

\[
v_x = (pq)_y
\] (28c)

which is the Strachan equation [7].
3 The M-LXI equation and the M-III equation

3.1 The M-LXI equation

Now, we consider the (2+1)-dimensional curve which is given by the M-LXI equation

\[
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3
\end{pmatrix}_x = A \begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3
\end{pmatrix}
\]

(29a)

\[
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3
\end{pmatrix}_y = B \begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3
\end{pmatrix}
\]

(29b)

\[
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3
\end{pmatrix}_t = C \begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3
\end{pmatrix}
\]

(29c)

where

\[
A = \begin{pmatrix}
  0 & k & 0 \\
  -\beta k & 0 & \tau \\
  0 & -\tau & 0
\end{pmatrix}
\]

(30a)

\[
B = \begin{pmatrix}
  0 & m_3 & -m_2 \\
  -\beta m_3 & 0 & m_1 \\
  \beta m_2 & -m_1 & 0
\end{pmatrix}
\]

(30b)

\[
C = \begin{pmatrix}
  0 & \omega_3 & -\omega_2 \\
  -\beta \omega_3 & 0 & \omega_1 \\
  \beta \omega_2 & -\omega_1 & 0
\end{pmatrix}
\]

(30c)

3.2 The M-LXII equation

From (29), we obtain the following M-LXII equation [1]

\[
A_y - B_x + [A, B] = 0
\]

(31a)

\[
A_t - C_x + [A, C] = 0
\]

(31b)

\[
B_t - C_y + [B, C] = 0
\]

(31c)

Equation (31a) gives

\[
k_y - m_{3x} - \tau m_2 = 0
\]

(32a)

\[
-m_{2x} + \tau m_3 - km_1 = 0
\]

(32b)

\[
\tau_y - m_{1x} + \beta km_2 = 0.
\]

(32c)

The M-LXII equation (31), we can rewrite in form

\[
k_y - m_{3x} = \beta e_3 \cdot (e_{3x} \wedge e_{3y})
\]

(33a)

\[
-m_{2x} = \beta e_2 \cdot (e_{2x} \wedge e_{2y})
\]

(33b)
\[ \tau_y - m_{1x} = e_1 \cdot (e_{1x} \wedge e_{1y}) \]  

Also from (31) we get

\[ k_1 - \omega_{3x} = \tau \omega_2 = \beta e_3 \cdot (e_{3x} \wedge e_{3t}) \]  
\[ -\omega_{2x} = -\tau \omega_3 + k_1 \omega_1 = \beta e_2 \cdot (e_{2x} \wedge e_{2y}) \]  
\[ \tau_t - \omega_{1x} = -\beta k_2 \omega_2 = e_1 \cdot (e_{1x} \wedge e_{1t}) \]  

and

\[ m_{1t} - \omega_{1y} = -\beta(m_3 \omega_2 - m_2 \omega_3) = e_1 \cdot (e_{1y} \wedge e_{1t}) \]  
\[ m_{2t} - \omega_{2y} = -m_1 \omega_3 + m_3 \omega_1 = \beta e_2 \cdot (e_{2y} \wedge e_{2t}) \]  
\[ m_{3t} - \omega_{3y} = -m_2 \omega_1 + m_1 \omega_2 = \beta e_3 \cdot (e_{3y} \wedge e_{3t}) \]  

### 3.3 On the topological invariants

It is interesting to note that the M-LXII equations allows the following integrals of motion [1]

\[ K_1 = \int \int k m_{2x} dxdy, \quad K_2 = \int \int \tau m_{2x} dxdy, \quad K_3 = \int \int (\tau m_3 - k m_1) dxdy \]  

or

\[ K_1 = \int \int e_1 (e_{1x} \wedge e_{1y}) dxdy \]  
\[ K_2 = \int \int e_2 (e_{2x} \wedge e_{2y}) dxdy \]  
\[ K_3 = \int \int e_3 (e_{3x} \wedge e_{3y}) dxdy. \]  

So we have the following three topological invariants

\[ Q_1 = \frac{1}{4\pi} \int \int e_1 \cdot (e_{1x} \wedge e_{1y}) dxdy \]  
\[ Q_2 = \frac{1}{4\pi} \int \int e_2 \cdot (e_{2x} \wedge e_{2y}) dxdy \]  
\[ Q_3 = \frac{1}{4\pi} \int \int e_3 \cdot (e_{3x} \wedge e_{3y}) dxdy \]  

### 3.4 The M-III equation

Now we wish show how connected the M-LXI (29), M-LXII (31) and M-III (5) equations. Let us we identify S with e_1, i.e.

\[ e_1 = S. \]  

Then we have

\[ m_1 = u + \partial_x^{-1} \tau_y \]  
\[ m_2 = \frac{1}{k} u_x \]
\[ m_3 = \partial_{x}^{-1}(ky - \tau m_2) = \partial_{x}^{-1} - \frac{\tau u_x}{k} \] (40c)

and

\[ \omega_1 = \frac{1}{k}[-\omega_2 + \tau \omega_3] \] (41a)
\[ \omega_2 = -m_3x - \tau m_2 + 2l(cl + d)m_2 \] (41b)
\[ \omega_3 = m_2x - \tau m_3 - uk + 2l(cl + d)m_3 - 4cvk. \] (41c)

3.5 The \text{L}-equivalent of the \text{M-III} equation

We now introduce the function \( q \) according to the following expression

\[ q = \frac{k}{2(2cl + d)} \exp\{2l(cl + d)x - \partial_{x}^{-1}\tau\}. \] (42)

It is not difficult to verify that the function \( q \) satisfy the following nonlinear Schrodinger-type equation

\[ iq_t = q_{xy} - 4ic(vq)_x + 2d^2vq = 0 \] (43a)
\[ -ip_t = p_{xy} + 4ic(vp)_x + 2d^2vp = 0 \] (43b)
\[ v_x = (pq)_y. \] (43c)

It coincide with the \( \text{M-III}_q \) equation (23). So we have proved that the equations (23) and (5) are L-equivalent to each other. As well known that these equations are G-equivalent to each other [3]. Equation (43) contains two reductions: the Zakharov equation as \( c = 0 \) [6] and the Strachan equation as \( d = 0 \) [7].

4 The mM-LXI equation and the M-III equation

4.1 The mM-LXI equation

One of the significant model of the (2+1)-dimensional curves is the modified M-LXI (mM-LXI) equation. It reads as [1]

\[
\begin{pmatrix}
e_1 \\
e_2 \\
e_3
\end{pmatrix} = A_m \begin{pmatrix}
e_1 \\
e_2 \\
e_3
\end{pmatrix}
\] (44a)
\[
\begin{pmatrix}
e_1 \\
e_2 \\
e_3
\end{pmatrix} = B_m \begin{pmatrix}
e_1 \\
e_2 \\
e_3
\end{pmatrix}
\] (44b)
\[
\begin{pmatrix}
e_1 \\
e_2 \\
e_3
\end{pmatrix} = C_m \begin{pmatrix}
e_1 \\
e_2 \\
e_3
\end{pmatrix}
\] (44c)

where

\[
A_m = \begin{pmatrix}
0 & k & -\sigma \\
-\beta k & 0 & \tau \\
\beta \sigma & -\tau & 0
\end{pmatrix}
\] (45a)
\[
B_m = \begin{pmatrix} 0 & m_3 & -m_2 \\ -\beta m_3 & 0 & m_1 \\ \beta m_2 & -m_1 & 0 \end{pmatrix} \quad (45b)
\]
\[
C_m = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\beta \omega_3 & 0 & \omega_1 \\ \beta \omega_2 & -\omega_1 & 0 \end{pmatrix} . \quad (45c)
\]

4.2 The mM-LXII equation

The compatibility condition of the mM-LXI equations (44) gives the modified M-LXII (mM-LXII) equation \[1\]

\[
A_{my} - B_{mx} + [A_m, B_m] = 0 \quad (46a)
\]
\[
A_{mt} - C_{mx} + [A_m, C_m] = 0 \quad (46b)
\]
\[
B_{mt} - C_{my} + [B_m, C_m] = 0 . \quad (46c)
\]

From (46a) we get

\[
k_y - m_3 x + \sigma m_1 - \tau m_2 = 0 \quad (47a)
\]
\[
\sigma_y - m_2 x + \tau m_3 - k m_1 = 0 \quad (47b)
\]
\[
\tau_y - m_1 x + \beta (k m_2 - \sigma m_3) = 0 . \quad (47c)
\]

The mM-LXII equation (47a), we can rewrite in form

\[
k_y - m_3 x = \beta e_3 \cdot (e_{3x} \wedge e_{3y}) \quad (48a)
\]
\[
\sigma_y - m_2 x = \beta e_2 \cdot (e_{2x} \wedge e_{2y}) \quad (48b)
\]
\[
\tau_y - m_1 x = e_1 \cdot (e_{1x} \wedge e_{1y}) . \quad (48c)
\]

Also from (14) we get

\[
k_t - \omega_3 x = \sigma_1 - \tau_2 = +\beta e_3 (e_{3x} \wedge e_{3t}) \quad (49a)
\]
\[
\tau_t - \omega_1 x = \beta (k \omega_2 - \sigma \omega_3) = -e_1 \cdot (e_{1x} \wedge e_{1t}) \quad (49b)
\]
\[
\sigma_t - w_2 x = \tau w_3 - k w_1 = \beta e_2 \cdot (e_{2x} \wedge e_{2t}) \quad (49c)
\]

and

\[
m_{1t} - \omega_1 y = -\beta (m_3 \omega_2 - m_2 \omega_3) = e_1 \cdot (e_{1y} \wedge e_{1t}) \quad (50a)
\]
\[
m_{2t} - \omega_2 y = -m_1 \omega_3 + m_3 \omega_1 = \beta e_2 \cdot (e_{2y} \wedge e_{2t}) \quad (50b)
\]
\[
m_{3t} - \omega_3 y = -m_2 \omega_1 + m_1 \omega_2 = \beta e_3 \cdot (e_{3y} \wedge e_{3t}) . \quad (50c)
\]
4.3 On the topological invariants

It is interesting to note that the M-LXII equations allows the following integrals of motion

\[ K_1 = \int \int (\kappa m_2 + \sigma m_3) dxdy, \quad K_2 = \int \int (-\tau m_3 + k m_1) dxdy, \quad K_3 = \int \int (\tau m_2 - \sigma m_1) dxdy \]  

\[ (51a) \]

or

\[ K_1 = \int \int e_1 (e_{1x} \wedge e_{1y}) dxdy \]  

\[ (51b) \]

\[ K_2 = \int \int e_2 (e_{2x} \wedge e_{2y}) dxdy \]  

\[ (51c) \]

\[ K_3 = \int \int e_3 (e_{3x} \wedge e_{3y}) dxdy \]  

\[ (51d) \]

So we have the following three topological invariants

\[ Q_1 = \frac{1}{4\pi} \int \int e_1 \cdot (e_{1x} \wedge e_{1y}) dxdy \]  

\[ (52a) \]

\[ Q_2 = \frac{1}{4\pi} \int \int e_2 \cdot (e_{2x} \wedge e_{2y}) dxdy \]  

\[ (52b) \]

\[ Q_3 = \frac{1}{4\pi} \int \int e_3 \cdot (e_{3x} \wedge e_{3y}) dxdy \]  

\[ (52c) \]

4.4 The Lax representation of the mM-LXI equation

To find the Lax representation (LR) of the mM-LXI equation (44), we rewrite it in the following matrix form

\[ \dot{e}_{1x} = k \dot{e}_2 - \sigma \dot{e}_3 \]  

\[ (53a) \]

\[ \dot{e}_{2x} = -\beta \dot{e}_1 + \tau \dot{e}_3 \]  

\[ (53b) \]

\[ \dot{e}_{3x} = \beta \sigma \dot{e}_1 - \tau \dot{e}_2 \]  

\[ (53c) \]

\[ \dot{e}_{1y} = m_3 \dot{e}_2 - m_2 \dot{e}_3 \]  

\[ (54a) \]

\[ \dot{e}_{2y} = -\beta m_3 \dot{e}_1 + m_1 \dot{e}_3 \]  

\[ (54b) \]

\[ \dot{e}_{3y} = \beta m_2 \dot{e}_1 - m_1 \dot{e}_2 \]  

\[ (54c) \]

\[ \dot{e}_{1t} = \omega_3 \dot{e}_2 - \omega_2 \dot{e}_3 \]  

\[ (55a) \]

\[ \dot{e}_{2t} = -\beta \omega_3 \dot{e}_1 + \omega_1 \dot{e}_3 \]  

\[ (55b) \]

\[ \dot{e}_{3t} = \beta \omega_2 \dot{e}_1 - \omega_1 \dot{e}_2 \]  

\[ (55c) \]

where \( \dot{e}_j \) are given by (14). Equations (53)-(55) we can rewrite in the form (below we put \( \beta = 1 \))

\[ [\sigma_3, U] = k \sigma_2 - \sigma \sigma_1 \]  

\[ (56a) \]

\[ [\sigma_2, U] = -\beta k \sigma_3 + \tau \sigma_1 \]  

\[ (56b) \]

\[ [\sigma_1, U] = \beta \sigma \sigma_3 - \tau \sigma_2 \]  

\[ (56c) \]
\[ [\sigma_3, V] = m_3\sigma_2 - m_2\sigma_1 \]  
\[ [\sigma_2, V] = -\beta m_3\sigma_3 - m_1\sigma_1 \]  
\[ [\sigma_1, V] = \beta m_2\sigma_3 - m_1\sigma_2 \]  
\[ [\sigma_3, W] = \omega_3\sigma_2 - \omega_2\sigma_1 \]  
\[ [\sigma_2, W] = -\beta\omega_3\sigma_3 + \omega_1\sigma_1 \]  
\[ [\sigma_1, W] = \beta\omega_2\sigma_3 - \omega_1\sigma_2 \]  

where
\[ U = g_x g^{-1}, \quad V = g_y g^{-1}, \quad W = g_t g^{-1}. \]  

Hence we get
\[ U = \frac{1}{2i} \begin{pmatrix} \tau & k - i\sigma \\ \beta(k + i\sigma) & -\tau \end{pmatrix} \]  
\[ V = \frac{1}{2i} \begin{pmatrix} m_1 & m_3 - im_2 \\ \beta(m_3 + im_2) & -m_1 \end{pmatrix} \]  
\[ W = \frac{1}{2i} \begin{pmatrix} \omega_1 & w_3 - iw_2 \\ \beta(w_3 + iw_1) & -w_1 \end{pmatrix}. \]  

Thus the matrix-function \( g \) satisfies the equations
\[ g_x = U g, \quad g_y = V g, \quad g_t = W g. \]  

This equation is the LR of the mM-LXI equation. Apropos as \( \sigma = 0 \) the equation (61) is the LR of the M-LXI equation (29). From the compatibility condition of the equations (61) we get the new form of the mM-LXII equation (46)
\[ U_y - V_x + [U, V] = 0 \]  
\[ U_t - W_x + [U, W] = 0 \]  
\[ V_t - W_y + [V, W] = 0 \]  

4.5 The M-III equation

In this subsection, we want establish the connection between the mM-LXI (44), mM-LXII (46) and M-III (5) equations. To this purpose, as above, we identify \( S \) with \( e_1 \), i.e. works the identity (39). Then the identifying variables for the M-III equation (5) are given by
\[ m_1 = u + \partial_x^{-1}\tau_y \]  
\[ m_2 = \frac{1}{k}(u_x + \sigma m_3) \]  
\[ m_3 = \partial_x^{-1}(k_y + \sigma m_1 - \tau m_2) \]  

and
\[ \omega_1 = \frac{1}{k}[\sigma_t - \omega_{2x} + \tau \omega_3] \]  
\[ \omega_2 = -m_{3x} - \tau m_2 + u\sigma + 2l(cl + d)m_2 - 4cv\sigma \]  
\[ \omega_3 = m_{2x} - \tau m_3 - uk + 2l(cl + d)m_3 - 4cvk \]
4.6 The L-equivalent of the M-III equation

Return to the function $q$. Let this function has the form

$$q = \frac{k^2 + \sigma^2}{2(2c_1 + d)} \exp i \{2l(c_1 + d)x - \partial_x^{-1} \tau\}$$

(65)

Then $q$ again satisfies is the equation (23). So, we have again shown that the M-III equation (5) and the equation (23) are L-equivalent to each other.

It is remarkable that this result is consistent with the other result namely that these equations are G-equivalent (gauge equivalent) to each other [3].

5 Geometry of curves and Bilinear representation of the M-III equation

In this section we establish self-coordination of the our geometrical formalism that presented above with the other powerful tool of soliton theory - the Hirota’s bilinear method. We demonstrate our idea in example the M-III equation (5).

For the curve we take the mM-LXI and mM-LXII equations. Usually, for the spin vector $S = (S_1, S_2, S_3)$ takes the following transformation

$$S^+ = S_1 + iS_2 = \frac{2\bar{f}g}{\Lambda}, \quad S_3 = \frac{\bar{f}f - \bar{g}g}{\Lambda}, \quad \Lambda = \bar{f}f + \bar{g}g.$$  

(66)

Also in this section, we assume the (39) is holds. In [4] was shown that for the mM-LXI equation (44) is correct the following representation

$$e_1^+ = \frac{2f\bar{g}}{\Lambda}, \quad e_{13} = \frac{\bar{f}f - \bar{g}g}{\Lambda}$$

(67a)

$$e_2^+ = i\frac{\bar{f}^2 + \bar{g}^2}{\Lambda}, \quad e_{23} = i\frac{f\bar{g} - \bar{f}\bar{g}}{\Lambda}$$

(67b)

$$e_3^+ = \frac{\bar{f}^2 - \bar{g}^2}{\Lambda}, \quad e_{33} = -\frac{f\bar{g} + \bar{f}\bar{g}}{\Lambda}$$

(67c)

and

$$k = -i\frac{D_x(g \circ f - \bar{g} \circ \bar{f})}{\Lambda}, \quad \sigma = -\frac{D_x(g \circ f + \bar{g} \circ \bar{f})}{\Lambda}, \quad \tau = -i\frac{D_x(\bar{f} \circ f + g \circ g)}{\Lambda}$$

(68a)

$$m_1 = -i\frac{D_y(\bar{f} \circ f + \bar{g} \circ \bar{g})}{\Lambda}, \quad m_2 = -\frac{D_y(g \circ f + \bar{g} \circ \bar{f})}{\Lambda} \quad m_3 = -i\frac{D_y(g \circ f - \bar{g} \circ \bar{f})}{\Lambda}$$

(68b)

Here $e_j = (e_{j1}, e_{j2}, e_{j3})$,  

$$e_j^\pm = e_{j1} \pm ie_{j2}.$$  

Now we take  

$$\tau = 0 \quad m_1 = u.$$  

(69)

Then hence and from (68) we have

$$D_x(\bar{f} \circ f + \bar{g} \circ g) = 0$$

(70a)

$$u = -i\frac{D_y(\bar{f} \circ f + \bar{g} \circ g)}{\Lambda}$$

(70b)

So for the M-III equation (5) we have the bilinear representation (66) and (70).

This representation allows construct the bilinear form of the M-III equation which is left in future.
6 The M-III equation as the integrable particular case of the M-0 equation

Consider the (2+1)-dimensional M-0 equation [1]
\[ S_t = a_{12}e_2 + a_{13}e_3, \quad S_x = b_{12}e_2 + b_{13}e_3, \quad S_y = c_{12}e_2 + c_{13}e_3 \]  
(71)
where
\[ \triangle = b_{12}c_{13} - b_{13}c_{12}. \]  
(72)

All known spin systems (integrable and nonintegrable) in 2+1 dimensions are the particular reductions of the M-0 equation (71). In particular, the M-III equation (5) is the integrable reduction of equation (71). In this case, we have
\[ a_{12} = \omega_3, \quad a_{13} = -\omega_2, \quad b_{12} = k, \quad b_{13} = -\sigma, \quad c_{12} = m_3, \quad c_{13} = -m_2. \]  
(73)

Sometimes we use the following form of the M-0 equation [1]
\[ S_t = d_2S_x + d_3S_y \]  
(74)
with
\[ d_2 = \frac{a_{12}c_{13} - a_{13}c_{12}}{\triangle}, \quad d_3 = \frac{a_{12}b_{13} - a_{13}b_{12}}{\triangle}. \]  
(75)

7 The equation for \( \lambda \)

Let us consider the M-LXVII equation in the form (12) for the case \( q = p = 0 \). We have
\[ \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2(c\lambda^2 + d\lambda) \\ 0 & 2(c\lambda^2 + d\lambda) & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \]  
(76a)
\[ \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}_t = 2(c\lambda^2 + d\lambda) \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}_y \]  
(76b)

Hence we get
\[ \lambda_t = 2(c\lambda^2 + d\lambda)y \]  
(77)
So for the M-III equation (5) the spectral parameter satisfies the equation (77), i.e. in this case we have a nonisospectral problem. From (77) we obtain
1) for the M-I equation (2)
\[ \lambda_t = 2\lambda y. \]  
(78)
2) for the M-II equation (3)
\[ \lambda_t = 2c\lambda^2 y. \]  
(79)

Now consider the general form of such equations
\[ \lambda_t = k\lambda^n y, \quad k = \text{const}. \]  
(80)
This equation has the following solution

\[ \lambda = \left( \frac{y + c}{a - kt} \right)^{\frac{1}{n}} \quad (81) \]

where \( a(c) \) is real (complex) constant.

8 Conclusion

In this paper, we have used a geometrical approach pioneered by Lakshmanan to analyze the connection between differential geometry of curves and spin systems to establish such connection with the integrable (2+1)-dimensional spin system—called M-III equation. Simultaneously our approach permit construct the corresponding L-equivalent of the given spin system and Lax representation of it. Some other consequences of this geometrical formalism are presented.

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