Another tetraquark structure in the $\pi^+\chi_{c1}$ invariant mass distribution

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Abstract

In this article, we assume that there exists a scalar hidden charm tetraquark state in the $\pi^+\chi_{c1}$ invariant mass distribution, and study its mass using the QCD sum rules. The numerical result $M_Z = (4.36 \pm 0.18)$ GeV is consistent with the mass of the $Z(4250)$. The $Z(4250)$ may be a tetraquark state, other possibilities, such as a hadro-charmonium resonance and a $D_1^+\bar{D}^0 + D^+\bar{D}_1^0$ molecular state are not excluded.

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1 Introduction

Recently the Belle collaboration reported the first observation of two resonance-like structures in the $\pi^+\chi_{c1}$ invariant mass distribution near 4.1 GeV in the exclusive $\bar{B}^0 \to K^-\pi^+\chi_{c1}$ decays [1]. The Breit-Wigner masses and the widths are about $M_1 = 4051 \pm 14^{+20}_{-14}$ MeV, $\Gamma_1 = 82^{+21}_{-17-22}$ MeV, $M_2 = 4248^{+44}_{-29-35}$ MeV and $\Gamma_2 = 177^{+54}_{-39-61}$ MeV (thereafter we will denote them as $Z(4050)$ and $Z(4250)$ respectively). The significance of each of the $\pi^+\chi_{c1}$ structures exceeds 5σ, including the effects of systematics from various fit models. Their quark contents must be some special combinations of $c\bar{c}u\bar{d}$, just like the $Z(4430)$, they cannot be the conventional mesons [2].

The $Z$ (denote the $Z(4050)$ and $Z(4250)$) may be loosely deuteron-like bound states (molecules) of the charm mesons or compact nucleon-like bound states of the diquark-antiquark pair. The spins of the Z are not determined yet, they can be scalar or vector states.

In the meson-exchange model, the $Z(4050)$ is probably a loosely molecular state $D^*\bar{D}^*$ with $J^P = 0^+$ [3] and the $Z(4250)$ is unlikely an S-wave $D_1D$ or $D_0D^*$ molecular state [4], while the $SU(3)$ chiral quark model indicates that the $Z(4050)$ is unlikely an S-wave $D^*\bar{D}^*$ molecular state [5]. In Ref. [6, 7], the authors study the mesons $Z(4050)$ and $Z(4250)$ as the $D^*\bar{D}^*$ and $D^+_1\bar{D}^0 + D^+\bar{D}_1^0$ molecular states with $J^P = 0^+$ and $J^P = 1^-$ respectively using the QCD sum rules, and draw the conclusion that the $D^*\bar{D}^*$ state is probably a virtual state which is not related with the $Z(4050)$ and the $Z(4250)$ is a possible $D^+_1\bar{D}^0 + D^+\bar{D}_1^0$ molecular state. In a relativistic quark model, the $Z(4250)$ can be tentatively interpreted as the charged $P$-wave $1^-$ tetraquark state $SS$ or as the $P$-wave $0^+$ tetraquark state $(SA \pm \bar{SA})/\sqrt{2}$ [8], where the $S$ and $A$ denote the scalar and axial vector diquarks respectively.

The colored objects (diquarks) in a confining potential can result in a copious spectrum, there maybe exist a series of orbital angular momentum excitations; while the colorless objects (mesons) bound by a short range potential (through meson-exchange) should have

\[ M_Z = (4.36 \pm 0.18) \text{ GeV} \]

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a very limited spectrum. In the heavy quark limit, the $c$ quark can be viewed as a static well potential, and binds the light quark $q$ to form a diquark in the color antitriplet channel. We take the diquarks as the basic constituents following Jaffe and Wilczek [9] [10]. The heavy tetraquark system could be described by a double-well potential with two light quarks $q'\bar{q}$ lying in the two wells respectively.

In Refs. [11, 12, 13, 14, 15], Maiani et al take the diquarks as the basic constituents, examine the rich spectrum of the diquark-antidiquark states from the constituent diquark masses and the spin-spin interactions, and try to accommodate some of the newly observed charmonium-like resonances not fitting a pure $c\bar{c}$ assignment. The predictions depend heavily on the assumption that the light scalar mesons $a_0(980)$ and $f_0(980)$ are tetraquark states, the basic parameters (constituent diquark masses) are estimated thereafter.

In Ref. [16], we assume that the hidden charm mesons $Z(4050)$ and $Z(4250)$ are vector tetraquark states, and study their masses using the QCD sum rules. The numerical results indicate that the masses of the vector hidden charm tetraquark states are about $M_Z = (5.12 \pm 0.15) \text{ GeV}$ or $M_Z = (5.16 \pm 0.16) \text{ GeV}$, which are inconsistent with the experimental data and also much larger than the predictions of the constituent diquark model [12, 13, 14, 15].

The diquarks have five Dirac tensor structures, scalar $C\gamma_5$, pseudoscalar $C$, vector $C\gamma_\mu\gamma_5$, axial vector $C\gamma_\mu$ and tensor $C\sigma_{\mu\nu}$. The structures $C\gamma_\mu$ and $C\sigma_{\mu\nu}$ are symmetric, the structures $C\gamma_5$, $C$ and $C\gamma_\mu\gamma_5$ are antisymmetric. The attractive interactions of one-gluon exchange favor formation of the diquarks in color antitriplet $3_c$, flavor antitriplet $3_f$ and spin singlet $1_s$ [17, 18]. The scalar hidden charm tetraquark states may have smaller masses than the corresponding vector states.

The mass is a fundamental parameter in describing a hadron, in order to identify the $Z(4050)$ and $Z(4250)$ as tetraquark states, we must prove that the masses of the corresponding tetraquark states lie in the region $(4.1 - 4.3) \text{ GeV}$. Furthermore, whether or not there exist such hidden tetraquark configurations is of great importance itself, because it provides a new opportunity for a deeper understanding of the low energy QCD.

In this article, we assume that there exists a scalar hidden charm tetraquark state in the $\pi^+\chi_{c1}$ invariant mass distribution, and construct the $C\gamma_5 - C\gamma_5$ type current $J_1(x)$ and $C - C$ type current $J_2(x)$ (and their superposition $J(x)$) to interpolate it,

$$J_1(x) = \epsilon^{ijk}\epsilon^{ijn}u_j^T(x)c_k(x)\bar{c}_m(x)\gamma_5C\bar{d}_n(x),$$
$$J_2(x) = \epsilon^{ijk}\epsilon^{ijn}u_j^T(x)c_k(x)\bar{c}_m(x)Cd_n^T(x),$$
$$J(x) = \cos\theta J_1(x) + \sin\theta J_2(x),$$

where the $i, j, \cdots, n$ are color indexes; then study its mass using the QCD sum rules [19, 20]. The hidden charm mesons $X(3872)$, $Y(4260)$, $Y(4350)$, $Y(4660)$, $Z(4430)$ have also been studied with the QCD sum rules as the tetraquark or molecular states [21, 22, 23, 24].

In the QCD sum rules, the operator product expansion is used to expand the time-ordered currents into a series of quark and gluon condensates which parameterize the long distance properties of the QCD vacuum. Based on the quark-hadron duality, we can obtain copious information about the hadronic parameters at the phenomenological side [19, 20].

The article is arranged as follows: we derive the QCD sum rules for the mass of the
In the following, we write down the two-point correlation function \( \Pi(p) \) in the QCD sum rules,

\[
\Pi(p) = i \int d^4 x e^{ip \cdot x} \langle 0 | T \left\{ J(x) J(0) \right\} | 0 \rangle ,
\]

we choose the scalar current \( J(x) \) to interpolate the tetraquark state \( Z \).

We can insert a complete set of intermediate hadronic states with the same quantum numbers as the current operator \( J(x) \) into the correlation function \( \Pi(p) \) to obtain the hadronic representation \[19, 20\]. After isolating the ground state contribution from the pole term of the \( Z \), we get the following result,

\[
\Pi(p) = \frac{\lambda_Z^2}{M_Z^2 - p^2} + \cdots ,
\]

where the pole residue (or coupling) \( \lambda_Z \) is defined by

\[
\lambda_Z = \langle 0 | J(0) | Z(p) \rangle .
\]

In the following, we briefly outline the operator product expansion for the correlation function \( \Pi(p) \) in perturbative QCD. The calculations are performed at the large space-like momentum region \( p^2 \ll 0 \). We write down the “full” propagators \( S_{ij}(x) \) and \( C_{ij}(x) \) of a massive quark in the presence of the vacuum condensates firstly \[20\],

\[
S_{ij}(x) = \frac{i \delta_{ij}}{2\pi^2 x^2} - \frac{\delta_{ij} m_q}{4\pi^2 x^2} - \frac{\delta_{ij}}{12} \langle \bar{q}q \rangle + \frac{i \delta_{ij}}{48} m_q \langle \bar{q}q \rangle \frac{\sigma}{x} - \frac{\delta_{ij} x^2}{192} \langle \bar{q}g_s \sigma Gq \rangle
\]

\[
+ \frac{i \delta_{ij} x^2}{1152} m_q \langle \bar{q} g_s \sigma Gq \rangle \frac{\sigma}{x} - \frac{i}{32\pi^2 x^2} G^{ij}_{\mu \nu}(\bar{x} \sigma^{\mu \nu} + \sigma^{\mu \nu} \bar{x}) + \cdots ,
\]

\[
C_{ij}(x) = \frac{i}{(2\pi)^4} \int d^4 k e^{-ik \cdot x} \left\{ \frac{\delta_{ij}}{k - m_c} - \frac{g_s G_{ij}^{\alpha \beta} \sigma_{\alpha \beta} (k + m_c) + (k + m_c) \sigma_{\alpha \beta}}{4 (k^2 - m_c^2)^2} \right. \\
\left. + \frac{\alpha_s G G}{3 \pi} \delta_{ij} m_c \frac{k^2 + m_c k}{(k^2 - m_c^2)^4} + \cdots \right\} ,
\]

where \( \langle \bar{q} g_s \sigma Gq \rangle = \langle \bar{q} g_s \sigma G \rangle \) and \( \langle \frac{\alpha_s G G}{\pi} \rangle = \langle \frac{\alpha_s G_{\alpha \beta} G^{\alpha \beta}}{\pi} \rangle \), then contract the quark fields in the correlation function \( \Pi(p) \) with Wick theorem, and obtain the result:

\[
\Pi(p) = \frac{i e^{ijk} e^{imn} e^{i'h'k'}}{\epsilon} \int d^4 x e^{ip \cdot x} \left\{ \cos^2 \theta Tr \left[ \gamma_5 C_{kk'}(x) \gamma_5 C_S^{T}_{jj'}(x) C \right] \right. \\
\left. + Tr \left[ \gamma_5 C_{m'n'}(x) \gamma_5 C_{S_{m'n'}}^{T}(x) C \right] \right. \\
\left. + sin^2 \theta Tr \left[ C_{kk'}(x) C S_{jj'}^{T}(x) C \right] \right. \\
\left. + Tr \left[ C_{m'n'}(x) C S_{m'n'}^{T}(x) C \right] \right\} .
\]
Substitute the full $u$, $d$ and $c$ quark propagators into the correlation function $\Pi(p)$ and complete the integral in the coordinate space, then integrate over the variables in the momentum space, we can obtain the correlation function $\Pi(p)$ at the level of the quark-gluon degrees of freedom.

We carry out the operator product expansion to the vacuum condensates adding up to dimension-10 and take the assumption of vacuum saturation for the high dimension vacuum condensates, they are always factorized to lower condensates with vacuum saturation in the QCD sum rules, factorization works well in large $N_c$ limit. In calculation, we observe that the contributions from the gluon condensate are suppressed by large denominators and would not play any significant roles [27, 28, 29, 30, 31]. Furthermore, we neglect the terms proportional to the $m_u$ and $m_d$ as their contributions are of minor importance.

Once analytical results are obtained, then we can take the quark-hadron duality and perform Borel transform with respect to the variable $P^2 = −p^2$, finally we obtain the following sum rule:

$$\lambda^2 Z e^{-\frac{M^2}{M^2}} = \int_{4m^2_c}^{s_0} ds \rho(s) e^{-\frac{s}{M^2}},$$

(10)
\[ \rho(s) = \frac{1}{512\pi^6} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{1-\alpha} d\beta \alpha \beta (1 - \alpha - \beta)^3 (s - \bar{m}_c^2)^2 (7s^2 - 6s\bar{m}_c^2 + \bar{m}_c^4) + t \frac{m_c}{16\pi^4} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{1-\alpha} d\beta (1 - \alpha - \beta) (\alpha + \beta) (s - \bar{m}_c^2) (\bar{m}_c^2 - 2s) + t \frac{m_c^2 \langle \bar{q} \bar{q} Gq \rangle}{64\pi^4} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{1-\alpha} d\beta (\alpha + \beta) (3s - 2\bar{m}_c^2) + \frac{m_c^2 \langle \bar{q} \bar{q} Gq \rangle^2}{12\pi^2} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{1-\alpha} d\beta \left[ 1 + \frac{\bar{m}_c^2}{M^2} \right] \delta \left( s - \bar{m}_c^2 \right) + \frac{m_c^2 \langle \bar{q} \bar{q} Gq \rangle^2}{192\pi^2 M^6} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha s^2 \delta \left( s - \bar{m}_c^2 \right) + \frac{1}{512\pi^4} \langle \alpha_s G \rangle \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{1-\alpha} d\beta (\alpha + \beta) (1 - \alpha - \beta)^2 (10s^2 - 12s\bar{m}_c^2 + 3\bar{m}_c^4) + \frac{m_c^2 \langle \bar{q} \bar{q} Gq \rangle}{384\pi^4} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{1-\alpha} d\beta (\alpha + \beta) (\alpha^3 + \beta^3) (1 - \alpha - \beta)^3 \left[ 1 + \frac{\bar{m}_c^2}{M^2} \right] \delta \left( s - \bar{m}_c^2 \right) + \frac{m_c^2 \langle \bar{q} \bar{q} Gq \rangle}{288\pi^2} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{1-\alpha} d\beta \left[ 1 + \frac{(\alpha^3 + \beta^3)(1 - \alpha - \beta)}{\alpha^2 \beta^2} \right] \left[ 1 + \frac{\bar{m}_c^2}{M^2} \right] \delta \left( s - \bar{m}_c^2 \right) - \frac{1}{96\pi^4} \langle \alpha_s G \rangle \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{1-\alpha} d\beta \left[ 1 + \frac{(\alpha^3 + \beta^3)(1 - \alpha - \beta)}{\alpha^2 \beta^2} \right] [2 + s \delta \left( s - \bar{m}_c^2 \right) ] - \frac{1}{1152\pi^2 M^4} \langle \alpha_s G \rangle \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{1-\alpha} d\beta (\alpha + \beta) (\alpha^3 + \beta^3) \frac{s \delta \left( s - \bar{m}_c^2 \right) }{\alpha^2 \beta^2} + \frac{1}{384\pi^2} \langle \alpha_s G \rangle \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{1-\alpha} d\beta \left[ 1 + \frac{(\alpha^3 + \beta^3)(1 - \alpha - \beta)}{\alpha^2 \beta^2} \right] \left[ 1 + \frac{\bar{m}_c^2}{M^2} \right] \delta \left( s - \bar{m}_c^2 \right) + \frac{m_c^2 \langle \bar{q} \bar{q} Gq \rangle}{1728 M^2} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{1-\alpha} d\beta \left[ 1 + \frac{1}{\alpha^2 + (1 - \alpha)^2} \right] \delta \left( s - \bar{m}_c^2 \right) - \frac{m_c^2 \langle \bar{q} \bar{q} Gq \rangle}{216 M^4} \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{1-\alpha} d\beta \left[ 1 + \frac{1}{\alpha^3 + (1 - \alpha)^3} \right] \delta \left( s - \bar{m}_c^2 \right) , \]  

(11)

where \( \alpha_{\max} = \frac{1 + \sqrt{1 - 4m_c^2}}{2} \), \( \alpha_{\min} = \frac{1 - \sqrt{1 - 4m_c^2}}{2} \), \( \beta_{\min} = \frac{\alpha m_c^2}{\alpha - m_c^2} \), \( \bar{m}_c^2 = \frac{(\alpha + \beta)m_c^2}{\alpha \beta} \), \( \bar{m}_c^2 = \frac{m_c^2}{\alpha (1 - \alpha)} \),

\( t = \cos 2\theta \in [-1, 1] \).

Differentiating the Eq.(10) with respect to \( \frac{1}{M^2} \), then eliminate the pole residue \( \lambda_Z \), we
can obtain a sum rule for the mass of the $Z$,

$$M_Z^2 = \frac{\int_{s_0}^{s_{\text{max}}} ds \int dx \rho(s) e^{-x^2}}{\int_{s_0}^{s_{1/2}} ds \rho(s) e^{-x^2}},$$  \hspace{0.5cm} \text{(12)}$$

3 Numerical results and discussions

The input parameters are taken to be the standard values $\langle \bar{q}q \rangle = -(0.24 \pm 0.01 \text{GeV})^3$, $\langle \bar{q}g_{\sigma}Gq \rangle = m_0^2 \langle \bar{q}q \rangle$, $m_0^2 = (0.8 \pm 0.2) \text{GeV}^2$, $\langle \bar{q}q_{GG} \rangle = (0.33 \text{GeV})^4$, and $m_c = (1.35 \pm 0.10) \text{GeV}$ at the energy scale about $\mu = 1 \text{GeV}$ \cite{11, 20, 25}.

The Belle collaboration observed the resonance-like structures $Z(4050)$ and $Z(4250)$ in the $\pi^+\chi_{c1}$ invariant mass distribution near 4.1 GeV in the exclusive $B^0 \to K^-\pi^+\chi_{c1}$ decays \cite{1}. If they are scalar tetraquark states, the central value of the threshold parameter can be taken as $s_0 = (4.248 + 0.5)^2 \text{GeV}^2 \approx 23 \text{GeV}^2$, where we tentatively choose the energy gap between the ground states and the first radial excited states to be 0.5 GeV.

The present experimental knowledge about the phenomenological hadronic spectral densities of the tetraquark states is rather vague, whether or not there exist tetraquark states is not confirmed with confidence, and no knowledge about the high resonances; we can borrow some ideas from the baryon spectra \cite{26}.

For the octet baryons with the quantum numbers $I(J^P) = \frac{1}{2}(\frac{1}{2}^+)$, the mass of the proton (the ground state) is $M_p = 938 \text{MeV}$, and the mass of the first radial excited state $N(1440)$ (the Roper resonance) is $M_{1440} = (1420 - 1470) \text{MeV} \approx 1440 \text{MeV}$ \cite{26}. For the decuplet baryons with the quantum numbers $I(J^P) = \frac{3}{2}(\frac{3}{2}^+)$, the mass of the $\Delta(1232)$ (the ground state) is $M_{1232} = (1231 - 1233) \text{MeV} \approx 1232 \text{MeV}$, and the mass of the first radial excited state $\Delta(1600)$ is $M_{1600} = (1550 - 1700) \text{MeV} \approx 1600 \text{MeV}$ \cite{20}. The energy gap between the ground states and the first radial excited states can be tentatively taken as 0.5 GeV for the light flavor baryons.

In Ref.\cite{14}, Maiani et al assume the $Z(4430)$ is the $2S$ $1^- \pi^+$ hidden charm tetraquark state to take into account the decay mode $Z(4430) \to \psi(2S) + \pi^+$, as the $1S$ $1^- \pi^+$ hidden charm tetraquark states lie at the interval $(3750 - 3880) \text{MeV}$ and have the decay mode $X^+(1S) \to \psi(1S) + \pi^+$ or $\eta_c(1S) + \rho^+$ in the constituent diquark model \cite{22}. The energy gap between the ground state and the first radial excited state is estimated to be $M_{\psi(2S)} - M_{\psi(1S)} \approx 0.59 \text{GeV}$ for the heavy tetraquark states.

We take it for granted that the energy gap between the ground states and the first radial excited states is about 0.5 GeV, and use this value as a guide to determine the threshold parameter $s_0$ with the QCD sum rules.

We explore whether or not there exist scalar tetraquark states which consist of a scalar (pseudoscalar) diquark-antidiquark pair at the energy interval $(4.1 - 4.3) \text{GeV}$, and choose the larger value $s_0 = (4.248 + 0.5)^2 \text{GeV}^2 \approx 23 \text{GeV}^2$ rather than the smaller value $s_0 = (4.051 + 0.5)^2 \text{GeV}^2 \approx 22 \text{GeV}^2$ to take into account all possible contributions from the ground states.

In the conventional QCD sum rules \cite{19, 20}, there are two criteria (pole dominance and convergence of the operator product expansion) for choosing the Borel parameter $M^2$ and threshold parameter $s_0$.

\footnote{For the tetraquark states consist of light flavors, if the perturbative terms have the main contribution}
Figure 1: The contributions from different terms with variation of the parameter $t$. The $A$ and $B$ correspond to the threshold parameters $s_0 = 23 \text{GeV}^2$ and $25 \text{GeV}^2$ respectively. The notations $\alpha$, $\beta$, $\gamma$, $\lambda$ and $\tau$ correspond to the perturbative term, $\langle \bar{q}q \rangle + \langle \bar{q}g_s \sigma Gq \rangle$ term, $\langle \frac{2G}{\pi} \rangle + \langle \frac{2G}{\pi} \rangle \left[ \langle \bar{q}q \rangle + \langle \bar{q}g_s \sigma Gq \rangle + \langle \bar{q}q \rangle^2 \right]$ term, $\langle \bar{q}q \rangle^2 + \langle \bar{q}q \rangle \langle \bar{q}g_s \sigma Gq \rangle + \langle \bar{q}g_s \sigma Gq \rangle^2$ term and perturbative $\langle \bar{q}q \rangle + \langle \bar{q}g_s Gq \rangle$ term, respectively. Here we take $M^2 = 3 \text{GeV}^2$ and the central values of other input parameters.

Figure 2: The pole residue (or coupling) with variation of the parameter $t$. The notation $\alpha$, $\beta$ and $\gamma$ correspond to the threshold parameters $s_0 = 23 \text{GeV}^2$, $24 \text{GeV}^2$ and $25 \text{GeV}^2$, respectively. Here we take $M^2 = 3 \text{GeV}^2$ and the central values of other input parameters.
The contributions from different terms with variation of the parameter $t$ in the operator product expansion are shown in Fig.1. From the figure, we can see that the contributions from the term $\langle \bar{q}q \rangle + \langle \bar{q}g_s \sigma Gq \rangle$ are negative at the interval $-1 \leq t < 0$, which cancel out the contribution from the perturbative term greatly. The net contributions from the perturbative term $\langle \bar{q}q \rangle + \langle \bar{q}g_s \sigma Gq \rangle$ increase with variation of the $t$, and reach the largest value at $t = 1$. The contributions from the gluon condensates $\langle \frac{\alpha_s G G}{\pi} \rangle + \langle \frac{\alpha_s G G}{\pi} \rangle \left[ \langle \bar{q}q \rangle + \langle \bar{q}g_s \sigma Gq \rangle + \langle \bar{q}q \rangle^2 \right]$ are very small and decrease with the parameter $t$ monotonously; the contributions from the high dimension condensates $\langle \bar{q}q \rangle \langle \bar{q}g_s \sigma Gq \rangle + \langle \bar{q}g_s \sigma Gq \rangle^2$ also decrease with the parameter $t$ monotonously. In other words, the operator product expansion converges more quickly for larger $t$ at the interval $t \in [-1, 1]$, we can choose the value $t = 1$.

On the other hand, the coupling of the interpolating current $J(x)$ to the tetraquark state becomes stronger with larger $t$, see Fig.2. It is reasonable to take the interpolating current with the strongest coupling to the tetraquark state.

In Figs.3-4, we plot the contributions from different terms in the operator product expansion. The contribution from the term $\langle \frac{\alpha_s G G}{\pi} \rangle$ is tiny and can be safely neglected. The contributions from the terms involving the gluon condensates are less than 8% even at very small Borel parameter $M^2$, the gluon condensate plays a minor important role. The

\begin{equation}
\left[ \langle \bar{q}q \rangle + \langle \bar{q}g_s \sigma Gq \rangle + \langle \bar{q}q \rangle^2 \right]
\end{equation}

\begin{equation}
\text{(in the conventional QCD sum rules, the perturbative terms always have the main contribution), we can approximate the spectral density with the perturbative term (where the $A$ are some numerical coefficients) [32].}
\end{equation}

\begin{equation}
B \Pi t = A \int_0^\infty s^4 e^{-M^4 s} ds = AM^{10} \int_0^\infty t^4 e^{-t^4} dt,
\end{equation}

then take the pole dominance condition,

\begin{equation}
\frac{\int_0^{t_0} t^4 e^{-t^4} dt}{\int_0^\infty t^4 e^{-t^4} dt} \geq 50%,
\end{equation}

and obtain the approximated relation,

\begin{equation}
t_0 = \frac{s_0}{M^2} \geq 4.7.
\end{equation}

The superpositions of different interpolating currents can only change the contributions from different terms in the operator product expansion, and improve convergence, they cannot change the leading behavior of the spectral density $\rho(s) \propto s^4$ of the perturbative term [32].

This relation is difficult to satisfy for the light flavor tetraquark states [27, 28, 29, 30, 31], if we take the Borel parameter has the typical value $M^2 = 1$ GeV$^2$, $s_0 \geq 4.7$ GeV$^2$, the threshold parameter is too large for the light tetraquark state candidates $f_0(980)$, $a_0(980)$, etc.

The hidden charm (or bottom) tetraquark states and open bottom tetraquark states may satisfy the relation, as they always have larger Borel parameter $M^2$ and threshold parameter $s_0$ [21, 22, 23, 24]. Their spectral densities have the form $\rho(s) = C_1 s^4 + C_2 s^3 + C_3 s^2 + \ldots$, where the $C_i$ are coefficients, and exhibit the same leading behavior $\rho(s) \propto s^4$ as the light flavor tetraquark states. If we take $M^2 = 1$ GeV$^2$, $s_0 \geq 4.7$ GeV$^2$, the threshold parameter $s_0$ is too low for the hidden charm or open bottom tetraquark states, there is a large room for choosing larger threshold parameter to take into account the ground state contribution. We draw the conclusion that the hidden charm (bottom) tetraquark states and open bottom tetraquark states have possibility to satisfy the pole dominance condition.

In this article, the vacuum condensate of the highest dimension $\langle \bar{q}g_s \sigma Gq \rangle^2$ serve as a criterion for choosing the Borel parameter $M^2$. At the value $M^2_{\text{min}} \geq 2.2$ GeV$^2$, its contribution is less than 10% (see Fig.3), we expect the operator product expansion is convergent. The relation in Eq.(15) indicates $s_0 \geq 10.5$ GeV$^2$, if we take a large Borel parameter $M^2 \geq 2M^2_{\text{min}}$, then $s_0 \geq 21$ GeV$^2$, our phenomenological estimation $s_0 \sim 23$ GeV$^2$ is reasonable.
vacuum condensate of the highest dimension $\langle \bar{q}q_\sigma Gq \rangle^2$ serve as a criterion for choosing the Borel parameter $M^2$. At the value $M^2_{\text{min}} \geq 2.2 \text{ GeV}^2$, its contribution is less than 10%, we expect the operator product expansion is convergent.

The contributions from the vacuum condensates of high dimension $\langle \bar{q}q \rangle^2 + \langle \bar{q}q \rangle \langle \bar{q}q_\sigma Gq \rangle$ vary with the threshold parameter $s_0$ remarkably and serve as a criterion for choosing the threshold parameter $s_0$. At the value $s_0 \geq 23 \text{ GeV}^2$, their contributions are less than (or equal) 10% (see Fig.3-A), we expect the operator product expansion is convergent. The contributions from the vacuum condensates $\langle \bar{q}q \rangle^2 + \langle \bar{q}q \rangle \langle \bar{q}q_\sigma Gq \rangle + \langle \bar{q}q_\sigma Gq \rangle^2$ are less than 13.5% at the values $M^2 \geq 2.2 \text{ GeV}^2$ and $s_0 \geq 23 \text{ GeV}^2$. The contributions from the vacuum condensates $\langle \bar{q}q \rangle^2 + \langle \bar{q}q \rangle \langle \bar{q}q_\sigma Gq \rangle + \langle \bar{q}q_\sigma Gq \rangle^2 + (\frac{\alpha_{GG}}{\pi}) + (\frac{\alpha_{GG}}{\pi}) [\langle \bar{q}q \rangle + \langle \bar{q}q_\sigma Gq \rangle + \langle \bar{q}q \rangle^2]$ are less than 18%, the main contributions come from the perturbative term $+ \langle \bar{q}q \rangle + \langle \bar{q}q_\sigma Gq \rangle$, see Fig.4. The operator product expansion is convergent at the values $M^2_{\text{min}} \geq 2.2 \text{ GeV}^2$ and $s_0 \geq 23 \text{ GeV}^2$.

In Fig.5, we plot the contribution from the pole term with variation of the threshold parameter $s_0$. For the central values of the input parameters (except for $t = 1$), the contribution from the pole term is larger than 50% at the values $M^2_{\text{max}} \leq 3.2 \text{ GeV}^2$ and $s_0 \geq 23 \text{ GeV}^2$.

In this article, the threshold parameter and the Borel parameter are taken as $s_0 = (24 \pm 1) \text{ GeV}^2$ and $M^2 = (2.2 - 3.2) \text{ GeV}^2$ respectively, the contribution from the pole term is about $(51 - 88\%)$, the two criteria of the QCD sum rules are full filled [19, 20]. We can take smaller Borel parameter and threshold parameter to satisfy the two criteria of the QCD sum rules marginally, however, the Borel window is rather small, $M^2_{\text{max}} - M^2_{\text{min}} < 1 \text{ GeV}^2$.

Taking into account all uncertainties of the input parameters, finally we obtain the values of the mass and pole reside of the $Z$, which are shown in Figs.4-5. From the figures, we can see that at the value $M^2 \leq 2.6 \text{ GeV}^2$, the mass and the pole reside change remarkably with variation of the Borel parameter, we take the value $M^2 = (2.6 - 3.2) \text{ GeV}^2$, and obtain

\[
M_Z = (4.36 \pm 0.18) \text{ GeV},
\]

\[
\lambda_Z = (3.38 \pm 0.65) \times 10^{-2} \text{ GeV}^5. \tag{16}
\]

The meson $Z(4250)$ may be a scalar tetraquark state, other possibilities, such as a hadro-charmonium resonances and a $D_1^{\pm} D^0 + D^+ D_1^0$ molecular states are not excluded.

The $Z(4250)$ lie about $(0.5 - 0.6) \text{ GeV}$ above the $\pi^+ \chi_{c1}$ threshold, if it is a tetraquark state, the decay $Z \rightarrow \pi^+ \chi_{c1}$ can take place with the OZI super-allowed "fall-apart" mechanism, which can take into account the large total width naturally; on the other hand, if it is a $D_1^{\pm} D^0 + D^+ D_1^0$ molecular state, the decay can occur through the final-state re-scattering effects, $Z \rightarrow D_1^{\pm} D^0 + D^+ D_1^0 \rightarrow \pi^+ \chi_{c1}$, and the corresponding width may be narrow, we have to search for other decay channels to accommodate the large total width.

The typical decay mode $Z \rightarrow D^+ D^0$ is kinematically allowed, we can determine the spins of the $Z(4250)$ with the angular distribution of the final state $D^+ D^0$. If the decay $Z \rightarrow D^+ D^0$ is not observed (or the width is rather narrow), the $Z(4250)$ may be a hadro-charmonium resonance (bound state of a relatively compact charmonium ($\chi_{c1}$) inside a light hadron ($\pi^+$) having a larger spatial size) [33]. The decay $Z \rightarrow \pi^+ \chi_{c1}$ occurs with the "fall-apart" mechanism and the width is large; while the decay $Z \rightarrow D^+ D^0$ takes place through the final-state re-scattering effects ($Z \rightarrow \pi^+ \chi_{c1} \rightarrow D^+ D^0$) and the width may be narrow.
Figure 3: The contributions from the vacuum condensates with variation of the Borel parameter $M^2$. The $A$, $B$, $C$, $D$, $E$ and $F$ correspond to the contributions from the $(\bar{q}q)^2 + \langle \bar{q}q \rangle \langle \bar{q}q \rangle \sigma_{Gq}$ term, $(\bar{q}q)\sigma_{Gq})^2$ term, $(\bar{q}q)^2 + \langle \bar{q}q \rangle \langle \bar{q}q \rangle \sigma_{Gq}$ + $(\bar{q}q)\sigma_{Gq})^2$ term, $(\alpha_{GG})^2$ term, $(\alpha_{GG})^2 + (\alpha_{GG}) \langle \bar{q}q \rangle + (\bar{q}q) \sigma_{Gq})^2$ term and $(\bar{q}q)^2 + \langle \bar{q}q \rangle \langle \bar{q}q \rangle \sigma_{Gq}$ + $(\bar{q}q)\sigma_{Gq})^2 + (\alpha_{GG})^2 + (\alpha_{GG}) \langle \bar{q}q \rangle + (\bar{q}q) \sigma_{Gq})^2$ term, respectively. The notations $\alpha$, $\beta$, $\gamma$, $\lambda$, $\tau$ and $\xi$ correspond to the threshold parameters $s_0 = 20 \text{ GeV}^2$, $21 \text{ GeV}^2$, $22 \text{ GeV}^2$, $23 \text{ GeV}^2$, $24 \text{ GeV}^2$ and $25 \text{ GeV}^2$, respectively. Here we take $t = 1$ and the central values of other input parameters.
Figure 4: The contributions from different terms with variation of the Borel parameter $M^2$. The $A$ and $B$ correspond to the threshold parameters $s_0 = 23\,\text{GeV}^2$ and $25\,\text{GeV}^2$ respectively. The notations $\alpha$, $\beta$ and $\gamma$ correspond to the perturbative term, $\langle \bar{q}q \rangle + \langle \bar{q}g_s \sigma Gq \rangle$ term and $\langle \bar{q}q \rangle + \langle \bar{q}g_s \sigma Gq \rangle^2 + \langle \frac{\alpha_s G G}{\pi} \rangle + \langle \frac{\alpha_s G G}{\pi} \rangle \left[ \langle \bar{q}q \rangle + \langle \bar{q}g_s \sigma Gq \rangle + \langle \bar{q}q \rangle^2 \right]$ term, respectively. Here we take $t = 1$ and the central values of other input parameters.

Figure 5: The contribution from the pole term with variation of the Borel parameter $M^2$. The notations $\alpha$, $\beta$, $\gamma$, $\lambda$, $\tau$ and $\xi$ correspond to the threshold parameters $s_0 = 20\,\text{GeV}^2$, $21\,\text{GeV}^2$, $22\,\text{GeV}^2$, $23\,\text{GeV}^2$, $24\,\text{GeV}^2$ and $25\,\text{GeV}^2$, respectively.
Figure 6: The mass $M_Z$ with variation of the Borel parameter $M^2$.

Figure 7: The pole residue $\lambda_Z$ with variation of the Borel parameter $M^2$. 
4 Conclusion

In this article, we assume that there exists a scalar hidden charm tetraquark state in the $\pi^+\chi_{c1}$ invariant mass distribution, and study its mass using the QCD sum rules. The numerical result indicates that the mass is about $M_Z = (4.36 \pm 0.18)$ GeV, which is consistent with the experimental data. The hidden charm meson $Z(4250)$ may be a tetraquark state. Other possibilities, such as a hadro-charmonium resonance and a $D^+_s\bar{D}^0 + D^+\bar{D}^0$ molecular state are not excluded; more experimental data are still needed to identify it.

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