Symplectic quantization, inequivalent quantum theories, and Heisenberg’s principle of uncertainty

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We analyze the quantum dynamics of the non-relativistic two-dimensional isotropic harmonic oscillator in Heisenberg’s picture. Such a system is taken as toy model to analyze some of the various quantum theories that can be built from the application of Dirac’s quantization rule to the various symplectic structures recently reported for this classical system. It is pointed out that these quantum theories are inequivalent in the sense that the mean values for the operators (observables) associated with the same physical classical observable do not agree with each other. The inequivalence does not arise from ambiguities in the ordering of operators but from the fact of having several symplectic structures defined with respect to the same set of coordinates. It is also shown that the uncertainty relations between the fundamental observables depend on the particular quantum theory chosen. It is important to emphasize that these (somewhat paradoxical) results emerge from the combination of two paradigms: Dirac’s quantization rule and the usual Copenhagen interpretation of quantum mechanics.

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I. INTRODUCTION

The usual textbook treatment of the Hamiltonian formulation of dynamical systems consists in writing the equations of motion

\[ \dot{q}^i = f^i(q, p), \quad \dot{p}_i = g_i(q, p), \tag{1} \]

for autonomous systems in the form

\[ \dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}, \quad i, j = 1, 2, \ldots, n, \tag{2} \]

where \( H \) is “the Hamiltonian of the system,” the variables \((q^i, p_i)\) are canonically conjugate to each other in the sense that

\[ \{q^i, q^j\} = 0, \quad \{q^i, p_j\} = \delta^i_j, \quad \{p_i, p_j\} = 0, \tag{3} \]

with \{,\} the Poisson bracket defined by

\[ \{f, g\} = \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i}, \tag{4} \]

where from now on Einstein’s sum convention over the contracted indices is understood.

If the classical system admits the Hamiltonian formulation previously mentioned, then the standard recipe to go from its classical to its quantum dynamics from the canonical point of view consists in finding an irreducible representation for the fundamental operators which satisfy the Heisenberg-Born-Jordan relations or simply canonical commutation relations

\[ [\hat{q}^i, \hat{q}^j] = 0, \quad [\hat{q}^i, \hat{p}_j] = i\hbar\delta^i_j, \quad [\hat{p}_i, \hat{p}_j] = 0, \tag{5} \]

which are the quantum version of Eqs. (5). In Eq. (6), the “hat” over each symbol indicates the operator corresponding to the variable under consideration and the square bracket \([,]\) indicates the commutator of operators. For most of dynamical systems with a finite number of degrees of freedom the specific representation of these operators does not matter, on account of the Stone-von Neumann theorem [ nevertheless, an exception where the theorem does not apply is the system of a “particle in a box”]. It is important to emphasize that the standard procedure to go from the classical to the quantum realm, known as canonical quantization [6], is not completely free of ambiguities. Among them one has the choice of the measures on the several Hilbert spaces involved and sometimes some ambiguities in the ordering of the product of operators. Even though these ambiguities are important, they are not relevant for the present discussion and they are mentioned just for completeness in the description of canonical quantization.

Coming back to the classical dynamics and before mentioning the ideas developed in this paper, it is convenient to remind the reader that the equations of motion can be put in the form

\[ \dot{x}^\mu = \omega^{\mu\nu} \frac{\partial H}{\partial p^\nu}, \quad \mu, \nu = 1, 2, \ldots, 2n, \tag{6} \]
with \((x^\mu) = (q^1, q^2, ..., q^n, p_1, p_2, ..., p_n)\) and
\[
(\omega^{\mu\nu}) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},
\tag{7}
\]
where 0 and \(I\) are \(n \times n\) matrices \([1]\). Also, Eq. \((11)\) acquires the form
\[
\{f, g\} = \frac{\partial f}{\partial x^{\mu}} \omega^{\mu\nu} \frac{\partial g}{\partial x^{\nu}},
\tag{8}
\]
from which Eq. \((3)\) can be rewritten as
\[
\{x^{\mu}, x^{\nu}\} = \omega^{\mu\nu}.
\tag{9}
\]
Thus, from this perspective, the coordinates \((x^{\mu})\) locally label the points of the phase space \(\Gamma\) of the system where dynamics takes place, the Hamiltonian \(H\) is a real function defined on \(\Gamma\), and the definition of the Poisson bracket is equivalent to the introduction of a symplectic structure \(\omega = \frac{1}{2} \omega_{\mu\nu} dx^{\mu} \wedge dx^{\nu}\) on the phase space, where the matrix \((\omega_{\mu\nu})\) is the inverse matrix of \((\omega^{\mu\nu})\). The 2-form \(\omega\) is non-degenerate, i.e., \(\omega_{\mu\nu} v^{\nu} = 0\) implies \(v^{\mu} = 0\) which means that there exists the inverse matrix \((\omega^{\mu\nu})\). Also, \(\omega\) is closed, i.e., \(\partial_{\nu} \omega_{\mu\gamma} + \partial_{\gamma} \omega_{\nu\mu} + \partial_{\mu} \omega_{\nu\gamma} = 0\) which is equivalent to the fact that the Poisson bracket satisfies the Jacobi identity \([1, 2]\).

Therefore, it is clear that the symplectic geometry is the geometric structure underlying the Hamiltonian formulation of mechanics \([1, 2]\). Moreover, Eqs. \((10)\) are covariant in the sense that they maintain their form if the canonical coordinates are replaced by a completely arbitrary set of coordinates in terms of which \((\omega^{\mu\nu})\) need not be given by Eq. \((7)\). It should be remarked that even in the standard formulation of Lagrangian or Hamiltonian mechanics one always has the possibility of using completely arbitrary coordinates in the configuration or in the phase space; the usual procedure consists in finding first the expression for the Lagrangian or the Hamiltonian function making use of an inertial reference frame and then make the desired coordinate transformation.

In a similar way, one can retain the original coordinates \((q^i, p_i)\) and still write the original equations of motion \([11]\) in the Hamiltonian form \((1)\), but now employing alternative symplectic structures \(\omega^{\mu\nu}(x)\), distinct to that given in Eq. \((4)\), and by taking as Hamiltonian any real function on \(\Gamma\) which is a constant of motion for the system. This means that the writing of the equations of motion of a dynamical system in Hamiltonian form is not unique [see Sect. \([11]\)]. It is pretty obvious that any description of the dynamics for a given classical system from the symplectic point of view is mathematically and physically acceptable.

However, it is a priori far from being obvious whether or not the various quantum theories emerging from the combination of Dirac’s quantization rule
\[
[\hat{f}, \hat{g}] = i\hbar \{f, g\},
\tag{10}
\]
with alternative symplectic structures are mathematically and physically equivalent to each other in the generic case. Again, in Eq. \((10)\), the “hat” over each symbol indicates the operator corresponding to the classical variable under consideration. Therefore, \(\{f, g\}\) is the operator corresponding to \(\{f, g\}\). In particular, the combination of these two ingredients gives rise to the following questions: what are the consequences in the quantum theory of choosing alternative symplectic structures on the phase space of the theory when the pairs \((q^i, p_i)\) are not necessarily canonical ones from the very beginning? Is it possible in such cases to build a mathematically “consistent” quantum theory? If the answer is in the affirmative, does it make sense physically?

In this paper, we are going to try to answer these kinds of questions.

At first sight it might appear that this way of approaching quantum mechanics is the one of geometric quantization \([7]\). Nevertheless, there, people frequently choose a symplectic structure in such a way that \((q^i, p_i)\) are canonical coordinates to start with the quantization programme.

Here, as we mentioned, we are not interested in keeping \((q^i, p_i)\) as canonical coordinates but exactly the other way around, we want to analyze the quantum theories that emerge from Dirac’s quantization rule \([10]\) when alternative symplectic structures (defined with respect to the same set of coordinates of phase space) are taken into account. To investigate this point, the quantum dynamics of the two-dimensional isotropic harmonic oscillator is analyzed in this paper. In particular, it is shown that several quantum theories can consistently be built from alternative symplectic structures associated with the same classical system and that the corresponding quantum theories are not equivalent in the sense that the expectation values for the operators (observables) associated to the same physical entity do not agree with each other in all these quantum theories. Moreover, it is shown that Heisenberg’s uncertainty principle in the usual way that it is normally stated does not hold. In our opinion, these results are just a reflection of the fact that the notions involved in quantum mechanics are not expressed in a “covariant” way but they are tied to the case when \((q^i, p_i)\) are canonical coordinates as we discuss in Sects. \([11, 14]\) and \([15]\).

II. FREEDOM IN THE SYMPLECTIC
DESCRIPTION OF CLASSICAL DYNAMICS

Before going into the quantum theory, it is convenient to review the classical dynamics of the non-relativistic two-dimensional isotropic harmonic oscillator which will be used as toy model to study the consequences on the quantum theory of choosing symplectic structures alternative to the usual one. The dynamics of this system is given by the equations of motion
\[
\dot{x} = \frac{p_x}{m}, \quad \dot{y} = \frac{p_y}{m},
\]
\[
\dot{p}_x = -m\omega^2 x, \quad \dot{p}_y = -m\omega^2 y,
\tag{11}
\]
where the dot “.” stands for the time derivative with respect to the Newtonian time \( t \), \( m \) is the mass and \( \omega \) the angular frequency. The solution to the equations of motion \( 11 \) is

\[
\begin{align*}
x &= x_0 \cos \omega t + \frac{p_{x0}}{m} \sin \omega t, \\
p_x &= -m \omega x_0 \sin \omega t + p_{x0} \cos \omega t, \\
y &= y_0 \cos \omega t + \frac{p_{y0}}{m} \sin \omega t, \\
p_y &= -m \omega y_0 \sin \omega t + p_{y0} \cos \omega t,
\end{align*}
\]

where \( x_0 = x(t = 0) \), \( y_0 = y(t = 0) \), \( p_{x0} = p_x(t = 0) \), and \( p_{y0} = p_y(t = 0) \) are the initial data (at \( t = 0 \)) of the dynamical variables for the system.

**Alternative viewpoints of symplectic dynamics.** As explained in Refs. 8, 9, it is not mandatory to interpret \((x, p_x)\) and \((y, p_y)\) as if they were per se canonical coordinates, and many other choices of the pair \((\omega, H)\) where \( \omega \) is a symplectic structure and \( H \) is a Hamiltonian are allowed. The following four pairs were introduced in Ref. 8:

i) the equations of motion \( 11 \) can be put in a Hamiltonian form \( 14 \) by taking \((x^\mu) = (x^1, x^2, x^3, x^4) = (x, y, p_x, p_y)\),

\[
(\omega^{\mu\nu}) = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

and

\[
H = S_0 := \frac{1}{2} \left( \frac{(p_x)^2}{m} + m \omega^2 x^2 + \frac{(p_y)^2}{m} + m \omega^2 y^2 \right),
\]

or, equivalently, the non-vanishing Poisson brackets are

\[
\{x, p_x\}_0 = 1, \quad \{y, p_y\}_0 = 1,
\]

which is the same as

\[
\omega_0 = dp_x \wedge dx + dp_y \wedge dy.
\]

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\[
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0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

and

\[
H = S_1 := \frac{p_x p_y}{m} + m \omega^2 xy,
\]

or, equivalently, the non-vanishing Poisson brackets are

\[
\{x, p_y\}_1 = 1, \quad \{y, p_x\}_1 = 1,
\]

which is the same as

\[
\omega_1 = dp_y \wedge dx + dp_x \wedge dy
\]

[cf. Eqs. 15 and 16].

ii) the equations \( 11 \) can also be obtained from \((x^\mu) = (x^1, x^2, x^3, x^4) = (x, y, p_x, p_y)\),

\[
(\omega^{\mu\nu}) = \begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix},
\]

and

\[
H = S_2 := \left( \frac{(p_x)^2}{2m} - \frac{(p_y)^2}{2m} \right) + \frac{1}{2} m \omega^2 (y^2 - x^2),
\]

or, equivalently, the non-vanishing Poisson brackets are

\[
\{x, p_x\}_2 = -1, \quad \{y, p_y\}_2 = 1,
\]

which is the same as

\[
\omega_2 = -dp_x \wedge dx + dp_y \wedge dy
\]

[cf. Eqs. 15 and 16].

iii) similarly, the equations of motion \( 11 \) can be gotten from \((x^\mu) = (x^1, x^2, x^3, x^4) = (x, y, p_x, p_y)\),

\[
(\omega^{\mu\nu}) = \begin{pmatrix}
0 & -\frac{1}{m \omega} & 0 & 0 \\
\frac{1}{m \omega} & 0 & 0 & 0 \\
0 & 0 & 0 & -m \omega \\
0 & 0 & m \omega & 0
\end{pmatrix},
\]

and

\[
H = S_3 := \omega (xp_y - yp_x),
\]

or, equivalently, the non-vanishing Poisson brackets are

\[
\{x, y\}_3 = -\frac{1}{m \omega}, \quad \{p_x, p_y\}_3 = -m \omega,
\]

which is the same as

\[
\omega_3 = m \omega dx \wedge dy + \frac{1}{m \omega} dp_x \wedge dp_y
\]

[cf. Eqs. 15 and 16].

Some remarks follow: 0 it is important to recall that the alternative Hamiltonians are just constants of motion, that do not correspond nor they are required to correspond to the energy of the system, as it can be easily verified combining each Hamiltonian function with the Poisson bracket proposed in each case or by using the equations of motion \( 11 \) directly. Moreover, the energy is
conserved in each case because it is a constant of motion, 1) the equations of motion do not uniquely determine a single pair \((\omega, H)\) formed by a symplectic structure \(\omega\) and a Hamiltonian \(H\). In the present case, the triple \((\Gamma = \mathbb{R}^4, \omega = \omega_\mu, H = S_\mu)\), \(\mu = 0, 1, 2, 3\) give rise by means of Eq. (10) to the same equations of motion (11). Therefore, phrases involving an absolute connotation like “the Hamiltonian of the dynamical system” are not correct because there is not a single Hamiltonian for a dynamical system, rather, there are many of them \([10, 11]\). In addition, to state that \(x\) and \(p_x\) (same for \(y\) and \(p_y\)) “do not commute classically” is, on account of the previously displayed symplectic structures, also not correct because the commutation or not (in the Poisson bracket sense) is not something intrinsic to the variables \(x\) and \(p_x\) but it depends on the symplectic structure chosen \(^2\), 2) the fact of having several symplectic structures \(\omega_\mu\) should not be interpreted as a reflection of Darboux’s theorem \(^2\), which applies once a symplectic 2-form has been defined on a manifold of even dimension. Here, there is no such a fixed symplectic structure from the very beginning, rather, one is defining four alternative symplectic structures from the very beginning, 3) it must be emphasized that even if the symplectic structure \(\omega\) were fixed to be \(\omega_0\), there would still be an ambiguity in the definition of the Hamiltonian \(H\), a constant \(a\) might be added to \(H\) to get a new Hamiltonian \(H + a\). The converse is also true: if the Hamiltonian \(H\) were fixed to be \(\omega_0\), there would still be several ways of choosing the symplectic structure \(\omega\) in such a way that these choices, via Eq. (6), reproduce the equations of motion \((11)\) [see Refs. 8, 9 for more details], 4) note also that the difference among the several symplectic structures is not a change of coordinates, the coordinates \((x^\mu) = (x^1, x^2, x^3, x^4) = (x, y, p_x, p_y)\) that label the points of the phase space \(\Gamma = \mathbb{R}^4\) are the same in all cases, what changes is the choice of the pair \((\omega, H)\) formed by a symplectic structure \(\omega\) and a Hamiltonian \(H\), 5) note that the Hamiltonian \(S_0\) is bounded from below while the Hamiltonians \(S_i\), \(i = 1, 2, 3\) are not, 6) the symplectic structure of the case iii) implies classically a non-commutativity between the coordinates \((x, y)\) and between the momenta \((p_x, p_y)\).

By using \((12)\) it is possible to compute the corresponding symplectic structures on the physical phase space \(\Gamma_{\text{phys}}\) whose points are labelled by the coordinates \((x_0, y_0, p_{x0}, p_{y0})\). One gets

\[
\begin{align*}
\Omega_0 &= dp_{x0} \wedge dx_0 + dp_{y0} \wedge dy_0, \\
\Omega_1 &= dp_{y0} \wedge dx_0 + dp_{x0} \wedge dy_0, \\
\Omega_2 &= -dp_{x0} \wedge dx_0 + dp_{y0} \wedge dy_0, \\
\Omega_3 &= m\omega dx_0 \wedge dy_0 + \frac{1}{m\omega} dp_{x0} \wedge dp_{y0},
\end{align*}
\]

respectively. Obviously

\[
\Omega_\mu = (\phi_t)^* \omega_\mu, \quad \mu = 0, 1, 2, 3,
\]

with \(\phi_t : \Gamma_{\text{phys}} \to \Gamma\) given by Eq. \((12)\), i.e.,

\[
(\Gamma_{\text{phys}}, \Omega_\mu) \xrightarrow{\phi_t} (\Gamma, \omega_\mu), \quad \mu = 0, 1, 2, 3.
\]

Thus, even when the evolution in \(t\) is a “canonical transformation” there exists to our disposal the freedom to choose the symplectic structure in the target (in \(\Gamma\)) with the corresponding symplectic structure on the source (in \(\Gamma_{\text{phys}}\)) with respect to which the “abstract transformation” given in \((12)\) becomes canonical [see Eq. \((31)\)].

In summary, there are many ways of making the description of classical dynamics from the symplectic viewpoint, we have just listed four of them, and all of these choices are mathematically and physically allowed. The reader interested in the description of the non-relativistic two-dimensional harmonic oscillator (as well as of any other dynamical system with first class constraints only) from the parameterized point of view (which is also covariant in the sense that the Newtonian time \(t\) is treated on the same footing as the other configuration variables) can see Ref. [12], in particular if he/she wants to understand the consequences on the constraints formalism of the fact of having various symplectic structures (with respect to the same set of coordinates) on the extended, on the constraints surface, and on the reduced phase spaces associated with the same dynamical system.

### III. INEQUIVALENT QUANTUM THEORIES

In the previous section, several forms of describing the classical dynamics of the system \((11)\) from a symplectic point of view were displayed. Now the idea is to explore, in the framework of symplectic quantization\(^4\), the quantum theories that emerge from each of these symplectic

\(^1\) A real function \(f\) defined on the phase space \(\Gamma\) is a constant of motion if and only if \(df/dt = 0\). Therefore, to check if a function \(f\) is a constant of motion one needs to use the equation of motion only without having to choose a particular Hamiltonian \(H\) and its corresponding symplectic structure \(\omega\). Of course, if one makes a choice of the pair \((H, \omega)\), then one can also use this knowledge to check it.

\(^2\) Note, that the alternative symplectic matrices \((\omega^{\mu\nu})\) of the cases i), ii), and iii) are not obtained from the matrix \((\omega^{\mu\nu})\) of the case 0) by making the matrix product of the later by a matrix \(K\), i.e., to write the equations of motion in a Hamiltonian form it is not required that the alternative symplectic matrices \((\omega^{\mu\nu})\) are obtained by making the matrix product of \(K\) by another matrix \(K\).

\(^3\) Moreover, to state that “dynamical variables referring to different degrees of freedom do always commute” is not correct. The consequences of this in the quantum theory and its relationship with Heisenberg’s principle of uncertainty will be discussed later in this paper.

\(^4\) We think that it is more appropriate to use the term symplectic quantization instead of canonical quantization when, as in the present cases, \((x, p_x)\) and \((y, p_y)\) are not always canonical pairs.
structures under consideration. It will be shown that, in the context of the so-called Copenhagen interpretation, the quantum theories are not equivalent in the sense that the mean values of the operators (observables) associated to the same physical classical entity do not agree.

The description of the quantum dynamics for the system will be given in the Heisenberg picture. Thus, the quantum mechanical relations analogous to those given in Eq. (12) are given by

\[
\begin{align*}
\hat{x}(t) &= x_0 \cos \omega t + \frac{\hat{p}_0}{m\omega} \sin \omega t, \\
\hat{p}_x(t) &= -m\omega x_0 \sin \omega t + \hat{p}_x_0 \cos \omega t, \\
\hat{y}(t) &= \hat{y}_0 \cos \omega t + \frac{\hat{p}_0}{m\omega} \sin \omega t, \\
\hat{p}_y(t) &= -m\omega y_0 \sin \omega t + \frac{\hat{p}_0}{i\omega} \cos \omega t,
\end{align*}
\]  

(32)

So far, in the right-hand side of Eq. (32) the operators \(\hat{x}_0, \hat{y}_0, \hat{p}_x, \) and \(\hat{p}_0\) are “abstract”, i.e., the concrete commutation relations satisfied by them have not, at this stage, been specified. Moreover, the specification of the algebraic relations satisfied by them gives rise precisely to distinct quantum theories. Let \(\hat{O}(t = 0)\) be any of the fundamental operators \(\hat{x}_0, \hat{y}_0, \hat{p}_x, \) or \(\hat{p}_y\). The corresponding quantum theories emerging from each of the symplectic structures are the following:

0) The quantum theory emerging from the triple \((\Gamma = R^4, \omega = \omega_0, H = S_0)\) is defined by a representation of the algebra

\[
[\hat{x}_0, \hat{p}_0] = i\hbar, \quad [\hat{y}_0, \hat{p}_0] = i\hbar,
\]  

(33)

associated with Eq. (15). Let the Hilbert space be the space of square-integrable functions in \(R^2, \mathcal{F} = L^2(R^2, d\mu = dx dy)\), then

\[
\hat{x}_0 = x, \quad \hat{p}_x_0 = \frac{\hbar}{i} \frac{\partial}{\partial x},
\]

\[
\hat{y}_0 = y, \quad \hat{p}_0 = \frac{\hbar}{i} \frac{\partial}{\partial y},
\]  

(34)

is a Schrödinger or coordinate representation of the fundamental operators which satisfies (33). In addition, the relationship between the coordinate and momentum basis can be obtained from

\[
\begin{align*}
\langle x_0, y_0 | x, y \rangle &= x | x, y \rangle, \\
\langle y_0, x_0 | x, y \rangle &= y | x, y \rangle, \\
\langle x_0, y_0 | px, py \rangle &= px | px, py \rangle, \\
\langle y_0, x_0 | px, py \rangle &= py | px, py \rangle,
\end{align*}
\]  

(35)

and Eq. (34) from which, after normalization,

\[
\langle x, y | px, py \rangle = \frac{1}{2\pi\hbar} e^{i(px_0+py_0)/\hbar}.
\]  

(36)

Moreover, the representation of the operators given in Eq. (34) is unitarily equivalent to

\[
\begin{align*}
\hat{x}(t) &= x_0 \cos \omega t + \frac{\hat{p}_0}{m\omega} \sin \omega t \frac{\partial}{\partial x}, \\
\hat{p}_x(t) &= -m\omega x_0 \sin \omega t + \frac{\hbar}{m\omega} \cos \omega t \frac{\partial}{\partial x}, \\
\hat{y}(t) &= \hat{y}_0 \cos \omega t + \frac{\hbar}{i\omega} \sin \omega t \frac{\partial}{\partial y}, \\
\hat{p}_y(t) &= -m\omega y_0 \sin \omega t \frac{\hbar}{i\omega} \cos \omega t \frac{\partial}{\partial x},
\end{align*}
\]  

(37)

obtained by using

\[
\hat{O}(t) = e^{i\hat{S}_0 t/\hbar} \hat{O}(0) e^{-i\hat{S}_0 t/\hbar},
\]  

(38)

and Eqs. (39) and (41), i.e., equation (39) is the concrete version of Eq. (32) after the use of Eq. (41).

1) Similarly, the quantum theory built from the triple \((\Gamma = R^4, \omega = \omega_1, H = S_1)\) is defined by a representation of the algebra

\[
[\hat{x}_0, \hat{p}_0] = i\hbar, \quad [\hat{y}_0, \hat{p}_x] = i\hbar,
\]  

(39)

in agreement with Eq. (19). Let the Hilbert space be \(\mathcal{F} = L^2(R^2, d\mu = dx dy)\) then

\[
\begin{align*}
\hat{x}_0 &= x, \quad \hat{p}_x_0 = \frac{\hbar}{i} \frac{\partial}{\partial x}, \\
\hat{y}_0 &= y, \quad \hat{p}_y_0 = \frac{\hbar}{i} \frac{\partial}{\partial y},
\end{align*}
\]  

(40)

is a Schrödinger representation of the fundamental operators which satisfies (39) [cf. Eq. (41)]. Again, the relationship between the coordinate and momentum basis can be obtained from

\[
\begin{align*}
\langle x_0, y_0 | x, y \rangle &= x | x, y \rangle, \\
\langle y_0, x_0 | x, y \rangle &= y | x, y \rangle, \\
\langle x_0, y_0 | px, py \rangle &= px | px, py \rangle, \\
\langle y_0, x_0 | px, py \rangle &= py | px, py \rangle,
\end{align*}
\]  

(41)

and Eq. (40) from which, after normalization,

\[
\langle x, y | px, py \rangle = \frac{1}{2\pi\hbar} e^{i(px_0+py_0)/\hbar}.
\]  

(42)

[cf. Eq. (40)]. Note that we have, in this case, something that might be called a “crossed Fourier transform” in the sense that a packet sharpened in the \(x\) direction spreads out in the \(p_y\) direction (same for \(y\) and \(p_x\)). Moreover, by the Stone-von Neumann theorem the operators given in Eq. (40) are unitarily equivalent to

\[
\begin{align*}
\hat{x}(t) &= x \cos \omega t + \frac{\hbar}{m\omega} \sin \omega t \frac{\partial}{\partial x}, \\
\hat{p}_x(t) &= -m\omega x \sin \omega t + \frac{\hbar}{m\omega} \cos \omega t \frac{\partial}{\partial x}, \\
\hat{y}(t) &= y \cos \omega t + \frac{\hbar}{i\omega} \sin \omega t \frac{\partial}{\partial y}, \\
\hat{p}_y(t) &= -m\omega y \sin \omega t \frac{\hbar}{i\omega} \cos \omega t \frac{\partial}{\partial x},
\end{align*}
\]  

(43)
obtained by using
\[ \hat{O}(t) = e^{i\hat{S}_t\hat{t}/\hbar} \hat{O}(t = 0) e^{-i\hat{S}_t\hat{t}/\hbar}, \]
and Eqs. (39) and (40), i.e., equation (43) is the concrete version of Eq. (32) in the present quantum theory.

2) In the case of the triple \((\Gamma = \mathbb{R}^3, \omega = \omega_2, H = S_2)\) the quantum theory is defined by a representation of the algebra
\[ [\hat{\tilde{x}}_0, p_{x0}] = -i\hbar, \quad [\hat{\tilde{y}}_0, \hat{\tilde{p}}_{y0}] = i\hbar, \]
in agreement with Eq. (23). Let the Hilbert space be \(\mathcal{H} = L^2(\mathbb{R}^2, d\mu = dx dy)\) then
\[ \hat{\tilde{x}}_0 = x, \quad \hat{\tilde{p}}_{x0} = -i\frac{\hbar}{m_0} \frac{\partial}{\partial x}, \]
\[ \hat{\tilde{y}}_0 = y, \quad \hat{\tilde{p}}_{y0} = i\frac{\hbar}{m_0} \frac{\partial}{\partial y}, \]
is a Schrödinger representation of the fundamental operators which satisfies (45) [cf. Eq. (34)]. Once again, the relationship between the coordinate and momentum basis can be obtained from
\[ \langle x, y | p_x, p_y \rangle = \langle x | p_x \rangle \langle y | p_y \rangle = \frac{1}{2\pi\hbar} e^{i(-xp_x + yp_y)/\hbar}, \] [cf. Eq. (26)]. As expected, on account of the Stone-von Neumann theorem, the operators given in Eq. (26) are unitarily equivalent to
\[ \hat{t}(t) = x \cos \omega t - \frac{\hbar}{m_0} \omega \sin \omega t \frac{\partial}{\partial x}, \]
\[ \hat{\tilde{p}}_x(t) = -m_0 \omega x \sin \omega t - \frac{\hbar}{i} \cos \omega t \frac{\partial}{\partial x}, \]
\[ \hat{\tilde{y}}(t) = y \cos \omega t + \frac{\hbar}{m_0} \omega \sin \omega t \frac{\partial}{\partial y}, \]
\[ \hat{\tilde{p}}_y(t) = -m_0 y \sin \omega t + \frac{\hbar}{i} \cos \omega t \frac{\partial}{\partial y}, \]
with the unitary transformation given by
\[ \hat{O}(t) = e^{i\hat{S}_t\hat{t}/\hbar} \hat{O}(t = 0) e^{-i\hat{S}_t\hat{t}/\hbar}, \]
and taking into account Eqs. (44) and (45), i.e., equation (46) is the concrete version of Eq. (32) in the present case.

3) finally, the quantum theory associated with the triple \((\Gamma = \mathbb{R}^4, \omega = \omega_3, H = S_3)\) is built from a representation of the algebra
\[ [\hat{\tilde{x}}_0, \hat{\tilde{y}}_0] = -i\hbar \frac{m_0}{\omega_3}, \quad [\hat{\tilde{p}}_{x0}, \hat{\tilde{p}}_{y0}] = -i\hbar m_0 \omega, \]
in agreement with Eq. (27). However, this case is a little bit different from the cases (0), 1), and 2). There, independently of the case, the operators \(\hat{\tilde{p}}_{x0}\) and \(\hat{\tilde{p}}_{y0}\) commute. This is also the case of the operators \(\hat{\tilde{x}}_0\) and \(\hat{\tilde{y}}_0\). This fact was used to build a “coordinate” and “momentum” basis and their interconnection was displayed in all cases. But now, \(\hat{\tilde{p}}_{x0}\) and \(\hat{\tilde{p}}_{y0}\) do not commute anymore (same for the operators \(\hat{x}_0\) and \(\hat{y}_0\)). Thus, in the present case, it is not possible to build a common basis for these operators as before. Nevertheless, it makes sense to talk about a Schrödinger or “coordinate representation” for the operators involved. By this, we mean
\[ \hat{\tilde{x}}_0 = x, \quad \hat{\tilde{p}}_{x0} = m_0 \omega y, \]
\[ \hat{\tilde{y}}_0 = \frac{i\hbar}{m_0} \frac{\partial}{\partial x}, \quad \hat{\tilde{p}}_{y0} = \frac{i\hbar}{m_0} \frac{\partial}{\partial y}, \]
which satisfies Eq. (51). This representation for the operators is, by means of the Stone-von Neumann theorem, unitarily equivalent to
\[ \hat{\tilde{x}}(t) = x \cos \omega t + y \sin \omega t, \]
\[ \hat{\tilde{p}}_x(t) = -m_0 \omega x \sin \omega t + m_0 \omega y \cos \omega t, \]
\[ \hat{\tilde{y}}(t) = \frac{i\hbar}{m_0} \omega \cos \omega t \frac{\partial}{\partial x} + \frac{i\hbar}{m_0} \omega \sin \omega t \frac{\partial}{\partial y}, \]
\[ \hat{\tilde{p}}_y(t) = -i\hbar \omega \sin \omega t \frac{\partial}{\partial x} + i\hbar \omega \cos \omega t \frac{\partial}{\partial y}, \]
via
\[ \hat{O}(t) = e^{i\hat{S}_t\hat{t}/\hbar} \hat{O}(t = 0) e^{-i\hat{S}_t\hat{t}/\hbar}, \]
and Eqs. (54) and (55), i.e., equation (56) is the concrete version of Eq. (32) in this case.

Inequivalence of the quantum theories. So far, four mathematically consistent quantum theories have been obtained by using Dirac’s quantization rule, which is a cornerstone of quantum mechanics. In each of these theories, evolution in \(t\) is a unitary transformation. Now, according to Heisenberg’s picture of quantum mechanics if the system is left (prepared) on by means of a certain experimental arrangement in the state |\(\Psi\rangle\) (which might be even a wave packet) at \(t = 0\) then
\[ \langle \Psi | \hat{O}(t) | \Psi \rangle \]
yields the expected (central) value in the distribution of the corresponding physical quantity associated with the observable \(\hat{O}(t)\) if that quantity were to be measured at time \(t\). At first sight it might appear that the numerical value of the expectation value \(\langle \Psi | \hat{O}(t) | \Psi \rangle\) for certain (and fixed) observable \(\hat{O}(t)\) is the same in all the four quantum theories under consideration, after all Eq. (52) which is required to compute \(\langle \Psi | \hat{O}(t) | \Psi \rangle\) has, apparently, the same functional form for all of these theories. However, this is not so for the simple reason that in each of the quantum theories described above the fundamental operators \(\hat{x}_0, \hat{y}_0, \hat{p}_{x0}, \) and \(\hat{p}_{y0}\) act very differently on the state |\(\Psi\rangle\) in which
the system was prepared on because such operators have quite distinct representations on account of the specific algebraic relations they must satisfy in each theory. For instance

\[
\begin{align*}
x \cos \omega t - \frac{i\hbar}{m\omega} \sin \omega t \frac{\partial}{\partial x}, \\
x \cos \omega t - \frac{i\hbar}{m\omega} \sin \omega t \frac{\partial}{\partial y}, \\
x \cos \omega t + \frac{i\hbar}{m\omega} \sin \omega t \frac{\partial}{\partial x}, \\
x \cos \omega t + y \sin \omega t,
\end{align*}
\]

(56)

are the corresponding operators associated to the observable \( \hat{x}(t) \) in the quantum theories 0), 1), 2), and 3); respectively. Therefore, the various quantum theories are inequivalent in the sense that the expectation value of the fundamental operators \( \hat{O}(t) \) computed by using one quantum theory is not the same expectation value than the one obtained with any other of the quantum theories analyzed above when the system is prepared in the state \( |\Psi\rangle \) (same for all theories). Note that this inequivalence between the various quantum theories does not arise from an ambiguity in the order of the operators as usually happens when there are several quantum theories associated to a single classical theory. The origin of the inequivalence comes from: 1) the various quantum theories emerging from the implementation of Dirac’s quantization rule to the several symplectic structures chosen to make the classical description and 2) keeping the interpretation that the state in which the system is prepared on by the experimental arrangement has the same functional form in all the quantum theories.

From the previous discussion it is clear that, at this stage, theoretical consistency in the construction of the quantum theories does not provide a unique way of relating theoretical predictions with experimental outcomes. Either:

a) nature prefers just one of the various quantum theories in the sense that only one of these quantum theories matches the experimental data. Even if this were the case, there would still be something missing in the theoretical formalism whose knowledge might allow us to pick up a particular quantum theory and discard the remaining ones solely on theoretical grounds, i.e., we would need to specify that hypothetic rule that would allow us to single out the “right” quantum theory and also to uncover the fundamental cause of this, or

b) all the quantum theories are mathematically and physically viable. From this perspective, one would be assuming that there should exist a (yet unknown) covariant quantization scheme without the need of restricting ourselves to the use of a particular symplectic structure as starting point to build the quantum theory. Nevertheless, due to the fact that the expected values computed in each theory are numerically distinct, this would mean that there should exist a (yet unknown) criterion whose knowledge and its implementation would lead to the same theoretical predictions (which will match the experimental data) no matter which symplectic structure were chosen from the very beginning.

**IV. HEISENBERG’S UNCERTAINTY PRINCIPLE AND MEASURING PROCESS**

The consequences of having various quantum theories built from the implementation of Dirac’s quantization rule to the various symplectic structures for the fundamental variables \( x, y, p_x, \) and \( p_y \) are much more stronger when the several uncertainty relations coming from such quantization schemes are analyzed. To appreciate this, it is convenient to remind the reader that the physical meaning of the observables \( \hat{x}(t), \hat{y}(t), \hat{p}_x(t), \) and \( \hat{p}_y \) is the same in spite of the specific representation the operators acquire in each one of the theories under study.

The non-trivial products of quantum uncertainties in the measurement of \( \hat{x}(t), \hat{y}(t), \hat{p}_x(t), \) and \( \hat{p}_y \) are, in each theory, given by:

\[
\begin{align*}
\Delta x \Delta p_x & \geq \frac{\hbar}{2}, \\
\Delta y \Delta p_y & \geq \frac{\hbar}{2}, \\
\Delta x \Delta p_y & \geq \frac{\hbar}{2}, \\
\Delta y \Delta p_x & \geq \frac{\hbar}{2}.
\end{align*}
\]

(57) 1)

(58) 2)

(59) 3)

\[
\Delta x \Delta y \geq \frac{\hbar}{2m\omega}, \\
\Delta p_x \Delta p_y \geq \frac{\hbar m\omega}{2}.
\]

(60)

Once again, from a) and b) of Sect. III either nature prefers a single quantum theory or the sets of product of uncertainties given in Eqs. 57-60 are just a reflection of the fact the standard uncertainty relation is not expressed in a covariant way.

Moreover, from Eq. 60 one has, just to list an example

\[
[\hat{x}(t), \hat{x}(t')] = \frac{1}{m\omega} \sin \omega(t' - t) [\hat{x}_0, \hat{p}_x],
\]

(61)

which means, according to the standard interpretation of quantum mechanics, that the variable \( \hat{x}(t) \) can be monitored without affecting its evolution in the framework of theories 1) and 3) but not in the quantum theories 0) and 2) [see page 380 of Ref. 13].
V. DISCUSSION

In the symplectic viewpoint of dynamics it is possible to make the description of a dynamical system without having the necessity of restricting ourselves to the case where \((q_i, p_i)\) are canonical pairs. The various quantum theories built from the application of Dirac's quantization rule to these alternative symplectic structures yields inequivalent quantum theories in the sense that the expectation values for observables representing the same physical quantity are different. However, we think that it should be possible to build a quantization scheme which matches experimental outcomes no matter if the \((q'_i, p'_i)\) are or not canonical. After all, nature should not care which type of symplectic structure one uses to describe it. Experimental results should be independent of each particular choice of symplectic structure. Thus, our philosophical position is closer to the point b) of Sect. Finally, the consequences of choosing different pairs \((\omega, H)\) formed by a symplectic structure \(\omega\) and a Hamiltonian \(H\) in field theory (where expansion on harmonic oscillators is frequently done) as well as on classical and quantum statistical mechanics are not explored, but deserve to be done. Also, the possibility of choosing alternative symplectic theories in realistic theories such as general relativity or string theories and the consequences of this fact on their quantum theories should be investigated.

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