Lower Bounds for the Bandwidth Problem

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Abstract

The Bandwidth Problem seeks for a simultaneous permutation of the rows and columns of the adjacency matrix of a graph such that all nonzero entries are as close as possible to the main diagonal. This work focuses on investigating novel approaches to obtain lower bounds for the bandwidth problem. In particular, we use vertex partitions to bound the bandwidth of a graph. Our approach contains prior approaches for bounding the bandwidth as special cases. By varying sizes of partitions, we achieve a trade-off between quality of bounds and efficiency of computing them. To compute lower bounds, we derive a Semidefinite Programming relaxation. We evaluate the performance of our approach on several data sets, including real-world instances.

Keywords. Bandwidth Problem, Graph Partition, Semidefinite Programming.

1 Introduction

The Bandwidth Problem (BP) is the problem of labeling the vertices of a given undirected graph with distinct integers such that the maximum difference between the labels of adjacent vertices is minimal. It originated in the 1950s from sparse matrix computations, and received much attention since Harary’s \cite{Harary58} description of the problem and Harper’s paper \cite{Harper67} on the bandwidth of the \(n\)-cube (see also \cite{Cvetkovic80, Godsil85}). Berger-Wolf and Reingold \cite{BergerWolf05} showed that the problem of designing a code to minimize distortion in multi-channel transmission can be formulated as the Bandwidth Problem for generalized Hamming graphs. The BP belongs to a class of combinatorial optimization problems known as graph layout problems. The Cyclic Bandwidth \cite{Cvetkovic80, Cvetkovic88}, Cutwidth \cite{Cvetkovic80, Cvetkovic88}, Antibandwidth \cite{Chen96} and Linear Arrangement Problem \cite{Cvetkovic80, Cvetkovic88} also belong to this class of problems. The Bandwidth Problem arises in many different engineering applications related to efficient storage and processing. It also plays a role in designing parallel computation networks, VLSI layouts, and constraint satisfaction problems, see e.g., \cite{Cvetkovic80, Cvetkovic88, Cvetkovic88} and the references therein.

Determining the bandwidth is NP-hard \cite{Cvetkovic80} and even approximating the bandwidth within a given factor is known to be NP-hard \cite{Cvetkovic80}. Moreover, the BP is known to be NP-hard even on trees with maximum degree three \cite{Cvetkovic80} and on caterpillars with hair length three \cite{Cvetkovic80}. On the other hand, the Bandwidth Problem has been solved for a few families of graphs having special properties. Among these are the path, the complete graph, the complete bipartite graph \cite{Cvetkovic80}, the
hypercube graph [18], the grid graph [8], the complete $k$-level $t$-ary tree [36], the triangular graph [22], and the triangulated triangle [20]. Blum et al. [3] and Dunagan and Vempala [13] propose an $O(\log^3 n \sqrt{\log \log n})$ approximation algorithm for the bandwidth, where $n$ is the number of vertices.

Several lower and upper bounding approaches for the bandwidth of a graph are considered in the literature. Cuthill and McKee [9] proposed a heuristic to relabel the vertices of the graph so as to reduce the bandwidth after relabeling. It is widely used in practice, see for instance [38]. MATLAB offers the command `symrcm` as an implementation of this heuristic. For graphs with symmetry there exists an improved reverse Cuthill-McKee algorithm, see [40]. However, it is much more difficult to obtain lower bounds on the bandwidth. The following two approaches have been proposed in the literature.

**Lower bounds based on 3-partitions**  Juvan and Mohar [23] consider 3-partitions of the vertices into partition blocks $S_1, S_2, S_3$ of (fixed) sizes $m_1, m_2$ and $m_3$. If all such partitions have edges joining $S_1$ and $S_3$, then clearly the bandwidth must be bigger than $m_2$. Juvan and Mohar introduce eigenvalue-based lower bounds on the bandwidth which were refined by Helmberg et al. [19] leading to the following bound based on eigenvalues of the Laplacian $L$ of the graph

$$bdw > \frac{n\lambda_2(L)}{\lambda_3(L)},$$

see also the subsequent section. The same lower bound was derived by Haemers [15] by exploiting interlacing of Laplacian eigenvalues. Povh and Rendl [32] showed that this eigenvalue bound can also be obtained by solving a Semidefinite Programming (SDP) relaxation for a special Minimum Cut (MC) problem. They further tightened the SDP relaxation and consequently obtained a stronger lower bound for the Bandwidth Problem. Rendl and Sotirov [33] showed how to further tighten the SDP relaxation from [32].

**Bounds based on permutations** A labeling of the vertices of a graph corresponds to a simultaneous permutation of the rows and columns of the adjacency matrix. This may be expressed by pre- and post-multiplication with a permutation matrix, leading to quadratic assignment formulations of the bandwidth. De Klerk et al. [11] proposed two lower bounds based on SDP relaxations of the resulting Quadratic Assignment Problem (QAP). The numerical results in [11] show that both their bounds dominate the bound of Blum et al. [3], and that in most of the cases their bounds are stronger than the bound by Povh and Rendl [32].

In [30], the authors derived an SDP relaxation of the minimum cut problem by strengthening the well-known SDP relaxation for the QAP. They derive strong bounds for the bandwidth of highly symmetric graphs with up to 216 vertices by exploiting symmetry. For general graphs, their approach is rather restricted. Above mentioned bounds are either unsatisfyingly weak, or computing them is challenging already for small (general) graphs, i.e., graphs of about 30 vertices.

**Our contribution** We introduce a general $k$-partition model to get lower bounds on the bandwidth. It contains (with $k = 3$) the 3-partition model from Juvan and Mohar [23] and (with $k = n$) the permutation-based formulation of the problem, see Section 3 below. The $k$-partition problem is still NP-complete. Therefore, we introduce tractable relaxations based on SDP. In Section 4 such a relaxation based on the “matrix-lifting” idea is introduced. It leads to an SDP in matrices of order $n \cdot k$. It is known that the feasible region of such a relaxation always has a nullspace of
dimension $n + k - 1$. We identify an $n$-dimensional part of this nullspace, which can be eliminated using a simple combinatorial argument. Finally, in Section 5 we show that the new partition model leads to improved lower bounds for the bandwidth, even in case of small values of $k$, like $k \leq 6$. Moreover, we provide strong bounds for graphs with up to 128 vertices in a reasonable time frame.

**Notation** The space of $n \times n$ symmetric matrices is denoted by $S_n$ and the space of $n \times n$ symmetric positive semidefinite matrices by $S_n^+$. For two matrices $X, Y \in \mathbb{R}^{n \times n}$, $X \geq Y$, means $x_{ij} \geq y_{ij}$, for all $i, j$. The set of $n \times n$ permutation matrices is denoted by $\Pi_n$. Further, for a matrix $A$ the corresponding transposed matrix is denoted by $A^T$ while $A^\perp$ denotes the orthogonal complement. We use $I_n$ to denote the identity matrix of order $n$, and $e_n^i$ to denote the $i$-th standard basis vector of length $n$. Similarly, $J_n$ and $e_n$ denote the $n \times n$ all-ones matrix and all-ones $n$-vector, respectively.

The trace operator is denoted by $\text{trace}$, and $\langle \cdot, \cdot \rangle$ denotes the trace inner product. The Hadamard product of two matrices $A$ and $B$ of the same size is denoted by $A \odot B$ and defined as $(A \odot B)_{ij} = a_{ij} \cdot b_{ij}$ for all $i, j$. The $\text{diag}$ operator maps an $n \times n$ matrix to the $n$-vector given by its diagonal, while the $\text{vec}$ operator stacks the columns of a matrix in a vector. We denote by $\text{Diag}$ the adjoint operator of $\text{diag}$.

## 2 The Bandwidth Problem

We now formally introduce the Bandwidth Problem as a Quadratic Assignment Problem with special data matrices $A$ and $B_{r,n}$.

Let $G = (V, E)$ be an undirected simple graph with $|V| = n$ vertices and edge set $E$. A bijection $\phi : V \to \{1, \ldots, n\}$ is called a labeling of the vertices of $G$. The bandwidth of a graph $G$ with respect to the labeling $\phi$ is defined as follows

$$\text{bdw}(\phi, G) := \max_{(i,j) \in E} |\phi(i) - \phi(j)|.$$ 

The bandwidth of a graph $G$ is defined as the minimum of $\text{bdw}(\phi, G)$ over all labelings $\phi$, i.e.,

$$\text{bdw}(G) := \min \{ \text{bdw}(\phi, G) : \phi \text{ labeling of } G \}.$$ 

Equivalently, one can consider the adjacency matrix $A$ of the graph $G$. The bandwidth of $A$ amounts to a simultaneous permutation of the rows and columns of the adjacency matrix such that the largest distance of a nonzero entry from the main diagonal is as small as possible. The bandwidth of an adjacency matrix $A$ is defined as:

$$\text{bdw}(A) := \text{bdw}(G).$$ 

Therefore, from now on we assume that a graph $G$ is given through its adjacency matrix $A$. Since in terms of matrices the BP seeks for a simultaneous permutation of the rows and columns of $A$ such that all nonzero entries are as close as possible to the main diagonal, a “natural” problem formulation is as follows.

Let $r$ be an integer such that $1 \leq r \leq n - 2$, and $B_{r,n} = (b_{ij})$ be the symmetric matrix of order $n$ defined as follows

$$b_{ij} := \begin{cases} 
1, & \text{for } |i - j| > r, \\
0, & \text{otherwise}.
\end{cases}$$
Then, the following holds:

$$\min_{Q \in \Pi_n} \langle Q^T A Q, B_{r,n} \rangle = \begin{cases} 0, & \text{if } \text{bdw}(A) \leq r, \\ >0, & \text{if } \text{bdw}(A) > r. \end{cases} \quad (1)$$

The minimization problem has the form of a QAP, which might be even harder to solve than actually computing \( \text{bdw}(A) \). The idea of formulating the Bandwidth Problem as a QAP was suggested by Helmberg et al. \[19\]. De Klerk et al. \[11\] considered two SDP-based bounds for the Bandwidth Problem that are obtained from the SDP relaxations for the QAP introduced in \[42\] and \[10\]. The results show that it is hard to obtain bounds for graphs with 32 vertices, even though the symmetry in the graphs under consideration was exploited.

Since it is very difficult to solve QAPs in practice for sizes larger than 30 vertices other approaches are needed for deriving bounds for the bandwidth of graphs.

### 3 Partition Approach

We show how to use vertex partitions in order to obtain lower bounds for the bandwidth of a graph. For \(3 \leq k \leq n\) let \(m \in \mathbb{N}^k\) be given with \(m_i \geq 1 (i = 1, \ldots, k), \sum_{i=1}^k m_i = n\). We consider partitions of the vertex set \(V\) into \(k\) subsets \(\{S_1, \ldots, S_k\}\) such that \(|S_j| = m_j, j = 1, \ldots, k\). These are in one-to-one correspondence with \(n \times k\) partition matrices:

$$\mathcal{P}_m := \{X \in \{0,1\}^{n \times k} : X e_k = e_n, \ X^T e_n = m\}, \quad (2)$$

where for the partition \((S_1, \ldots, S_k)\) we set \(x_{ij} = 1\) whenever \(i \in S_j, i = 1, \ldots, n\). Since any vertex \(i \in V\) is assigned to precisely one of the blocks \(S_j\) we can define the map \(p : V = \{1, \ldots, n\} \mapsto \{1, \ldots, k\}\) given by

$$p(i) = j \Leftrightarrow x_{ij} = 1 \Leftrightarrow i \in S_j,$$

which identifies the partition block containing vertex \(i\). Thus, given the partition matrix \(X \in \mathcal{P}_m\) we get \(S_j = \{i \in V : p(i) = j\}\) for all \(1 \leq j \leq k\). For \(1 \leq r \leq k - 2\) let \(B_{r,k} = (b_{ij})\) be the 0–1 matrix of order \(k\) with

$$b_{ij} = \begin{cases} 1, & |i - j| > r, \\ 0, & |i - j| \leq r. \end{cases} \quad (3)$$

Suppose that \(i \in S_u, j \in S_v\), i.e., \(p(i) = u, p(j) = v\). Then for \(X \in \mathcal{P}_m\) the following holds:

$$(X B_{r,k} X^T)_{ij} = e_k^u X B_{r,k} e_k^v = \begin{cases} 1, & |u - v| > r, \\ 0, & |u - v| \leq r. \end{cases}$$

Therefore we get

$$\frac{1}{2} \langle A, X B_{r,k} X^T \rangle = \sum_{i,j \in V, i < j} a_{ij} (X B_{r,k} X^T)_{ij} = \sum_{[i,j] \in E} (X B_{r,k} X^T)_{ij} = \sum_{[i,j] \in E, |p(i) - p(j)| > r} 1.$$

Hence, this term counts the number of edges with endpoints in partition blocks of distance greater than \(r\) under the map \(p\).
\textbf{Basic Partition}  It will be convenient to introduce the special partition matrix $X$ corresponding to the \textit{basic partition} $\bar{p}$ which assigns the first $m_1$ vertices to $S_1$ the next $m_2$ vertices to $S_2$ and so on. Thus, the $n \times k$ matrix $X$ is characterized by columns of consecutive blocks of ones of appropriate lengths. Therefore the $n \times n$ matrix
\[ B := XB_{r,k}X^T \]
is a block matrix with blocks of sizes $m_i \times m_j$. The nonzero blocks of this matrix correspond to all-ones matrices of size $m_i \times m_j$ whenever the entry $(B_{r,k})_{ij} = 1$, see also Figure I below. Thus, for a given $n \times n$ adjacency matrix $A$ the term $\frac{1}{2}\langle A, XB_{r,k}X^T \rangle$ counts the number of edges joining vertices in partition blocks of distance greater than $r$.

\textbf{General Partition}  In general, any partition matrix $X \in \mathcal{P}_m$ can be obtained from the basic partition matrix $X$ by row-permutations that are defined by a permutation matrix $P \in \Pi_n$. Thus
\[ \mathcal{P}_m = \{ P\overline{X} : P \in \Pi_n \}, \]
where $\overline{X}$ is the basic partition matrix. The following transformation is obtained by replacing $X$ by $P\overline{X}$:
\[ \frac{1}{2}\langle A, XB_{r,k}X^T \rangle = \frac{1}{2}\langle A, P\overline{X}B_{r,k}\overline{X}^TP^T \rangle = \frac{1}{2}\langle P^TAP, XB_{r,k}X^T \rangle. \]
This shows that the permutation $P \in \Pi_n$ can be applied either to the adjacency matrix $A$ or to the matrix $XB_{r,k}X^T$.

The following example serves as an illustration of this property.

\textbf{Example 1.}  We consider a $15 \times 15$ matrix and the partitioning $m = (3, 3, 3, 3)^T$. Moreover, we choose $r = 2$. If $\langle A, XB_{r,k}X^T \rangle > 0$, then there must be an edge with endpoints in blocks of distance larger than $r = 2$. Such edges could either join vertices in $S_1$ and $S_4$, or in $S_1$ and $S_5$, or in $S_2$ and $S_5$, which require to “jump” over \{${S_2, S_3}$\} or \{${S_3, S_4}$\} at least. We illustrate this in Figure I.

The following theorem forms the basis for our lower bounds on the bandwidth.

\textbf{Theorem 1.}  Let $A$ be an $n \times n$ adjacency matrix, and let $3 \leq k \leq n$ and $m \in \mathbb{N}^k$ be given with \[ \sum_{i=1}^k m_i = n. \]
Let $1 \leq r \leq k - 2$. If
\[ \min_{P \in \Pi_n} \frac{1}{2}\langle P^TAP, XB_{r,k}X^T \rangle > 0, \]
then
\[ \text{bw}(A) \geq \min\{m_2 + \cdots + m_{r+1}, m_3 + \cdots + m_{r+2}, \ldots, m_{k-r} + \cdots + m_{k-1}\}. \]

\textit{Proof.}  If $\langle P^TAP, XB_{r,k}X^T \rangle > 0$, then some nonzero entry of $P^TAP$ is multiplied with a nonzero entry of $XB_{r,k}X^T$. The nonzeros of this matrix closest to the main diagonal are in the positions
\[ (m_1, m_1 + \cdots + m_{r+1} + 1), \ldots, (m_1 + \cdots + m_{k-r-1}, m_1 + \cdots + m_{k-1} + 1). \]
As an illustration, these positions are marked with bullets in Figure I below. The distances of these positions to the main diagonal are given by
\[ m_2 + \cdots + m_{r+1}, \ldots, m_{k-r} + \cdots + m_{k-1}. \]
Therefore $\text{bw}(A)$ must be larger than the smallest of these numbers. \( \square \)
In case that the above minimum is zero, we have to consider the zeros of $XB_{r,k}X^T$ with largest possible distance to the main diagonal. These are marked with crosses in Figure 1.

**Theorem 2.** Let $A$ be an $n \times n$ adjacency matrix, and let $3 \leq k \leq n$ and $m \in \mathbb{N}^k$ be given with $\sum_{i=1}^{k} m_i = n$. Let $1 \leq r \leq k - 1$. If

$$\min_{X \in P_m} \frac{1}{2} \langle A, XB_{r,k}X^T \rangle = 0,$$  

then

$$\text{bdw}(A) < \max \{ m_1 + m_2 + \cdots + m_{r+1}, m_2 + m_3 + \cdots + m_{r+2}, \ldots, m_{k-r} + \cdots + m_k \}.$$  

The proof is similar to Theorem 1 and is therefore omitted. In Figure 1, we illustrate the lower and upper bounds given by Theorems 1 and 2, respectively.

![Figure 1: $A \in S_{15}$, $m = (3,3,3,3,3)^T$, and $r = 2$. The crosses (bullets) indicate possible positions of the non-zero entries in terms of lower (upper) bounds.](image)

The following **Minimal Partition Problem** (minPart):

$$\text{minPart}(m, r) := \min_{X \in P_m} \frac{1}{2} \langle A, XB_{r,k}X^T \rangle$$  

serves as the basis to derive lower bounds on the bandwidth of $A$. From a practical point of view, we are interested in selections of $m$ where the minimum in Theorem 1 is attained in each term. Some particular cases are summarized in the following corollaries.

**Corollary 3.** Let $A$ be an $n \times n$ adjacency matrix of $G$, and let $3 \leq k \leq n$ and $m \in \mathbb{N}^k$ be given with $\sum_{i=1}^{k} m_i = n$. Let $r = 1$. Further, suppose that $m_2 = \cdots = m_{k-1}$.

If there exists $X \in P_m$ such that $\langle A, XB_{r,k}X^T \rangle > 0$, then $\text{bdw}(A) > m_2$.

**Corollary 4.** Let $A$ be an $n \times n$ adjacency matrix of $G$, and let $r = 2$ and $m \in \mathbb{N}^k$ be given with $\sum_{i=1}^{k} m_i = n$. Further, suppose $m = (m_1, m_2, m_3, m_2, m_3, \ldots, m_k)^T$.

If there exists $X \in P_m$ such that $\langle A, XB_{r,k}X^T \rangle > 0$, then $\text{bdw}(A) > m_2 + m_3$.

By cyclically repeating the sizes, we can insure that the minimum in Theorem 1 is attained in each term simultaneously as above also for values $r > 2$.  


3.1 Relation to Prior Work

We present below two important special cases of our new modelling approach and their relation to prior work.

The case $k = 3$ Given $k = 3$ the only allowable choice for $r$ is $r = 1$ and therefore the only nonzero elements in $B_{1,3}$ are $b_{1,3} = b_{3,1} = 1$. Hence for $m = (m_1, m_2, m_3)^T$ Theorem \[\text{[1]}\] states that if there exists $X \in \mathcal{P}_m$ such that $\langle A, XB_{1,3}X^T \rangle > 0$, then $\text{bdw}(A) > m_2$. This observation is used in \[\text{[19]}\] to derive lower bounds on $\text{bdw}(A)$, and is further refined in \[\text{[32, 40]}\].

The case $k = n$ Another notable case occurs for $k = n$, which implies that $m_1 = \cdots = m_n = 1$. Hence, for any $r \in \{1, \ldots, n-2\}$ it follows from Theorem \[\text{[1]}\] that $\text{bdw}(A) > r$, if there exists a partition matrix $X \in \mathcal{P}_m$ such that $\langle A, XB_{r,n}X^T \rangle > 0$. However, in this case the basic partition matrix becomes the identity matrix of rank $n$, i.e., $X = I_n$. Thus, $X$ becomes a permutation matrix $Q \in \Pi_n$ and we recover the statement

$$\min_{Q \in \Pi_n} \langle Q^T A Q, B_{r,n} \rangle > 0 \Rightarrow \text{bdw}(A) > r,$$

from \[\text{[1]}\]. This approach is used e.g., in \[\text{[11]}\] to derive lower bounds on $\text{bdw}(A)$.

In Summary, we have shown that once the minPart problem has a positive value for given $B_{r,k}$ and $m$, we get a nontrivial lower bound on the bandwidth from Theorem \[\text{[1]}\]. The minPart problem is itself NP-complete, so our strategy is to consider tractable lower bounds for the minPart problem. If some lower bound turns out to be positive for given $r$ and $m$, then clearly minPart has a positive value, and our bounding argument can be applied. In the following section we consider relaxations of minPart, based on semidefinite optimization.

4 SDP models

In this section, we derive several Semidefinite Programming relaxations for the Minimal Partition Problem. Our first two SDP relaxations are obtained by matrix lifting and therefore have matrix variables of order $\mathcal{O}(n \cdot k)$, while the third relaxation has $k$ matrix variables of order $n$.

4.1 SDP model in $S_{n \cdot k+1}$

In this section, we derive an SDP relaxation whose matrix variable is of order $n \cdot k + 1$.

Let $X \in \mathcal{P}_m$ be a partition matrix, see \[\text{[2]}\]. Let $x_1, \ldots, x_k$ be the columns of $X$, i.e., $X = [x_1 \ \cdots \ x_k]$, and $x := \text{vec}(X) \in \mathbb{R}^{n \cdot k}$. Now, the constraint $Y = xx^T$ may be weakened to $Y - xx^T \succeq 0$ which is well-known to be equivalent to the following convex constraint

$$Z := \begin{bmatrix} Y & x \\ x^T & 1 \end{bmatrix} \succeq 0.$$
Further, we use the following block notation for $Z \in \mathcal{S}_{n,k+1}$:

$$
Z = \begin{bmatrix}
X_1 & X_{12} & \ldots & X_{1k} & x_1 \\
X_{12} & X_2 & \ldots & X_{2k} & x_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
X_{1k} & X_{2k} & \ldots & X_k & x_k \\
x_1^T & x_2^T & \ldots & x_k^T & 1
\end{bmatrix},
$$

where $X_i$ corresponds to $x_ix_i^T$, $i = 1, \ldots, k$, and $X_{ij}$ to $x_ix_j^T$, $i \neq j$, $i, j = 1, \ldots, k$.

For any $X_i$, $i = 1, \ldots, k$, we have $\text{diag}(X_i) = \text{diag}(x_ix_i^T) = x_i$ and thus $\text{trace}(X_i) = x_i^Te_n = m_i$. For all $i = 1, \ldots, k$ we have:

$$
\langle J_n, X_i \rangle = \text{trace}(e_ne_n^Tx_ix_i^T) = \text{trace}((x_i^Te_n)^2) = m_i^2.
$$

Similarly, we have

$$
\langle J_n, X_{ij} + X_{ij}^T \rangle = \text{trace}(J_nX_{ij} + J_nX_{ij}^T) = 2 \cdot \text{trace}(e_ne_n^Tx_ix_i^T) = 2m_im_j, \quad \forall i, j.
$$

From orthogonality of vectors $x_i$, $i = 1, \ldots, k$, it follows $\text{diag}(X_{ij}) = 0$.

Let us describe the matrix (3) as the sum of symmetric matrices having only two non-zero entries, i.e., $B_{r,k} = \sum_{|u-v|>r} (e_u^ve_v^T + e_v^ue_u^T)$. Hence, we derive

$$
XB_{r,k}X^T = \sum_{|u-v|>r} \left( Xe_u^ve_v^TX^T + Xe_v^ue_u^TX^T \right) = \sum_{|u-v|>r} (x_u^vx_v^T + x_v^ux_u^T) = \sum_{|u-v|>r} (X_{uv} + X_{vu}).
$$

Therefore, we can rewrite the Minimal Partition Problem, see (4), as:

$$
\min_{X \in \mathcal{P}_m} \frac{1}{2} \langle A, XB_{r,k}X^T \rangle = \min_{X \in \mathcal{P}_m} \frac{1}{2} \langle A, \sum_{|u-v|>r} X_{uv} + X_{vu} \rangle = \min_{X \in \mathcal{P}_m} \sum_{|u-v|>r} \langle A, X_{uv} \rangle.
$$

Finally, we collect all above mentioned constraints and propose the following model for the
Minimal Partition Problem based on the matrix lifting approach.

\[
\begin{aligned}
\text{min} & \quad \sum_{|u-v|>r} \langle A, X_{uv} \rangle, \\
\text{s.t.} & \quad \text{diag}(X_i) = x_i, \quad i = 1, \ldots, k, \\
& \quad \text{diag}(X_{ij}) = 0, \quad i \neq j, \quad i, j = 1, \ldots k, \\
& \quad \text{trace}(X_i) = m_i, \quad i = 1, \ldots, k, \\
& \quad \langle J_n, X_i \rangle = m_i^2, \quad i = 1, \ldots, k, \\
& \quad \langle J_n, X_{ij} + X_{ij}^T \rangle = 2m_im_j, \quad i \neq j, \quad i, j = 1, \ldots, k, \\
\end{aligned}
\]

Here \( Z \in S^{+}_{kn+1} \). The feasible region of the above SDP relaxation equals the feasible region of the SDP relaxation for the graph partition problem derived by Wolkowicz and Zhao [41]. In order to further improve the relaxation, one can add nonnegativity constraints.

Below, we analyze the feasible region of the model (5).

**Lemma 5.** Let \( Z \) satisfy (5b), (5c), (5d), (5e), and (5g). Then

\[
\begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
-m_1 \\
\vdots \\
-m_2 \\
\vdots \\
-m_k
\end{bmatrix} \quad \begin{bmatrix}
e_n \\
e_n \\
\vdots \\
e_n \\
e_n \\
-e_n^T
\end{bmatrix} \quad \begin{bmatrix}I_n \\
I_n \\
\vdots \\
I_n \\
-e_n^T
\end{bmatrix}
\]

spans the nullspace of \( Z \).

For a proof we refer the reader to [33, Lemma 10 and Section 5.2] as well as to [41]. We observe in particular that this result holds independent of (5f).

Note that the vectors from Lemma 5 correspond to a \((n \cdot k + 1) \times (n + k)\) matrix. As the sum of the first \( k \) columns is equal to the sum of the last \( n \) columns, the nullspace of \( Z \) has dimension \( n + k - 1 \).

**Lemma 6.** Let \( Z \) satisfy (5b), (5c), (5d), (5e), and (5g). Then

\[
\begin{aligned}
X_1 + X_{12} + \cdots + X_{1k} &= x_1 e_n^T \\
\vdots \\
X_{k1} + X_{k1} + \cdots + X_{k} &= x_k e_n^T \\
x_1 + x_2 + \cdots + x_k &= e_n
\end{aligned}
\]
Again, we refer the reader to [33, Section 5.2, and [41] for a formal proof. As a consequence of the previous lemma, the block 
\[X_{k1} \ X_{k2} \ \ldots \ X_{k,k-1} \ X_k \ x_{k}\] is determined by \(X_1, \ldots, X_{k-1}, X_{ij}, (i \neq j, i, j = 1, \ldots, k-1),\) and \(x_1, \ldots, x_{k-1}.\) Hence, matrix \(Z\) can be reduced by one block of rows and their corresponding columns without loss of information. This leads us to the reduced SDP model presented in the following section.

One can also derive the Slater feasible version of the SDP relaxation (5) by exploiting a basis of the orthogonal complement to the nullspace of \(Z\) given in Lemma 5. For details see e.g., [33, 42]. The Slater feasible version may be efficiently solved by using the Alternating Direction Method of Multipliers (ADMM) as described in [30]. The ADMM is a first-order method for convex problems that decomposes an optimization problem into subproblems that may be easier to solve.

4.2 Reduced SDP Model in \(S_{n \cdot (k-1)+1}\)

In this section, we provide an SDP relaxation that is equivalent to the one from the previous subsection, but contains less variables. In particular, based on Lemma 6, we propose the following SDP relaxation for the Minimal Partition Problem.

\[
\begin{align*}
\min \quad & \sum_{|u-v|>r} \langle A, X_{uv} \rangle, \\
\text{s.t.} \quad & \text{diag} (X_i) = x_i, \quad i = 1, \ldots, k-1, \\
& \text{diag} (X_{ij}) = 0, \quad i \neq j, i, j = 1, \ldots, k-1, \\
& \text{trace} (X_i) = m_i, \quad i = 1, \ldots, k-1, \\
& \langle J_n, X_i \rangle = m_i^2, \quad i = 1, \ldots, k-1, \\
& \langle J_n, X_{ij} + X_{ij}^T \rangle = 2m_i m_j, \quad i \neq j, i, j = 1, \ldots, k-1, \\
& \tilde{Z} = \begin{bmatrix} X_1 & X_{1,2} & \ldots & X_{1,k-1} & x_1 \\ X_{1,2}^T & X_2 & \ldots & X_{2,k-1} & x_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ X_{1,k-1}^T & X_{2,k-1}^T & \ldots & X_{k-1} & x_{k-1} \\ x_1^T & x_2^T & \ldots & x_{k-1}^T & 1 \end{bmatrix} \succeq 0. 
\end{align*}
\]

Here \(\tilde{Z} \in S_{n \cdot (k-1)+1}^+\). Note that the nullspace of the reduced matrix \(\tilde{Z}\) has rank \(k-1).\) We show below that the SDP relaxation (6) is equivalent to (5). The number of equations in this SDP is still \(O(n \cdot k)\), but we saved about \(n\) equations as compared to the original model.

Additional sign constraints
\[
X_{uv} \geq 0, \quad |u-v|>r
\]
insure that the lower bound from this model is always nonnegative.

**Lemma 7.** From (6b) - (6g) follow (5b) - (5g).

**Proof.** Step 1: From Lemma 6 directly follows that, given \(\tilde{Z}\), the “missing” entries of \(Z\) can be
expressed by:

\[ x_k = e_n - x_1 - \cdots - x_{k-1} \geq 0, \]

\[ X_{ik} = x_i e_n^T - X_i - \sum_{j=1\atop j \neq i}^{k-1} X_{ij}, \quad i = 1, \ldots, k-1, \]

\[ X_k = x_k e_n^T - \sum_{j=1}^{k-1} X_{kj}. \]

Nonnegativity of \( x_k \) follows from (6c) and (6g).

**Step 2:**

**Constraint (5g)** From [33, Section 5], we know that under (6b) – (6g) it holds that

\[ \tilde{Z} \succeq 0 \land Z = WUW^T \Rightarrow Z \succeq 0, \]

where

\[ W := \begin{bmatrix} e_n & 0_n & \cdots & 0_n & I_n \\ 0_n & e_n & \cdots & 0_n & I_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_n & 0_n & \cdots & e_n & I_n \\ -m_1 & -m_2 & \cdots & -m_k & -e_n \end{bmatrix}. \]

Hence, it holds (5g).

**Constraint (5b)** In addition to (6b), \( \text{diag}(X_k) = x_k \) must hold. In particular, from **Step 1** it follows

\[ \text{diag}(X_k) = \text{diag}(x_k e_n^T - \sum_{j=1}^{k-1} X_{kj}) = x_k - \sum_{j=1}^{k-1} \text{diag}(X_{kj}) = x_k. \]

**Constraint (5c)** In addition to (6c), \( \text{diag}(X_{ik}) = 0, \ i = 1, \ldots, k-1, \) must hold. Again, by using **Step 1** we have:

\[ \text{diag}(X_{ik}) = \text{diag}(x_i e_n^T - X_i - \sum_{j=1\atop j \neq i}^{k-1} X_{ij}) = 0. \]

**Constraint (5d)** From (6b) and **Step 1** we have (5b).

Thus, from \( \text{diag}(X_k) = x_k \) it follows \( \text{trace}(X_k) = m_k. \)

**Constraint (5e)** From \( \langle J_n, x_k e_n^T \rangle = m_k \cdot n \) and \( \langle J_n, X_{kj} \rangle = \langle e_n e_n^T, x_k x_j^T \rangle = m_j \cdot m_k \), we have

\[ \langle J_n, X_k \rangle = \langle J_n, x_k e_n^T - \sum_{j=1}^{k-1} X_{kj} \rangle = m_k(n - \sum_{j=1}^{k-1} m_j) = m_k^2. \]
In addition to (6), \( \langle J_n, X_{ik} + X_{ik}^T \rangle = 2m_i m_k, \ i = 1, \ldots, k - 1 \), must hold.

\[
\langle J_n, X_{ik} + X_{ik}^T \rangle = 2 \cdot \langle J_n, X_{ik} \rangle = 2 \cdot \left[ \langle J_n, x_ie_n^T \rangle - \langle J_n, X_i \rangle - \sum_{j=1 \atop i \neq j}^{k-1} \langle J_n, X_{ij} \rangle \right]
\]

\[= 2m_i m_k.\]

Note that the inverse to the one in the lemma follows directly. To make the SDP relaxation (6) with additional nonnegativity constraints equivalent to SDP relaxation (5) with additional nonnegativity constraints, we need to add nonnegativity constraints to the “missing” blocks \([X_{k1} \ X_{k2} \ \cdots \ X_{kk-1} \ X_k \ \ x_k]\) in (5). In particular, we have the following proposition.

**Proposition 8.** The SDP relaxation (5) with additional constraints \( Z \geq 0 \) is equivalent to the SDP relaxation (6) with additional constraints \( \tilde{Z} \geq 0 \) and

\[
1 - \sum_{r=1}^{k-1} (X_r)_{i,i} - \sum_{r=1}^{k-1} (X_r)_{j,j} + \sum_{r=1}^{k-1} \sum_{p=1}^{k-1} (X_{rp})_{i,j} \geq 0, \ i > j,
\]

\[
(X_r)_{i,i} - \sum_{l=1}^{k-1} (X_{lr})_{i,j} \geq 0, \ i \neq j, \ r \in \{1, \ldots, k-1\},
\]

where \( i, j = 1, \ldots, n. \)

In Section 5, we demonstrate the strength of our SDP relaxation.

### 5 Computational Experiments

#### 5.1 Solving the SDP relaxation

The partition-based lower bounds for the bandwidth problem lead to semidefinite programs with one matrix of dimension \( n \cdot (k - 1) + 1 \), see (9). The resulting relaxations can be solved using standard SDP packages such as SDPT3 only for limited values of \( n \) and \( k \). We also consider nonnegativity constraints which add another \( O(n^2 k^2) \) potentially violated sign constraints to our relaxation. Interior-point based methods for such a scenario turn out to be too slow. Hence, we propose to use the ADMM method, which works well for SDPs with simple sign constraints. To use the ADMM, we use the Slater feasible version of the SDP relaxation (5) as described in the previous section. The resulting SDP relaxation has a matrix variable of order \( (k - 1) \cdot (n - 1) + 1 \), see e.g., [41]. Then, we proceed in the same manner as described in [21, 30].

#### 5.2 Strength of the partition bounds

As a first experiment we investigate the quality of the SDP relaxations (5) and (6) to assess

\[\text{minPart}(m, r) > 0\]

for given \( m \) and \( r \). We recall that \( \text{minPart}(m, r) \) denotes the number of edges in the minimal partition specified by \( m \) and \( r \), see [41]. We are primarily interested in parameter settings for \( m \) and \( r \) where \( \text{minPart}(m, r) > 0 \) but small. For such values of \( m \) and \( r \) it is a nontrivial task to prove positive lower bounds for minPart using our SDP models.
5.2.1 Test problems

We investigate the practical performance of our lower bounds on the following classes of graphs.

Torus graphs For given integer \( k \) the torus graph \( T_k \) has \( k^2 \) vertices which we label by \((i, j)\) for \( i, j \in \{1, \ldots, k\} \). We introduce “vertical” edges of the form \([\,(i, j), \,(i + 1, j)\)] for \( 1 \leq i \leq k - 1 \) and \([\,(1, j), \,(k, j)\)]\}. Altogether there are \( k^2 \) such edges. In a similar way we add “horizontal” edges of the form \([\,(i, j), \,(i, j + 1)\)] for \( j < k \) together with \([\,(i, 1), \,(i, k)\)]\}. This graph therefore has \( n := k^2 \) vertices and \( 2n \) edges. These graphs are interesting for the following reason. They are extremely sparse (\( n \) vertices and \( 2n \) edges), but their bandwidth is quite large. Namely, it is known that \( \text{bdw}(T_k) = 2k - 1 \), see e.g., [1, 26].

Torus graphs plus Hamiltonian path Here we start out with the torus graph \( T_k \), choose a labeling of its vertices yielding a bandwidth of size \( 2k \), and add the Hamiltonian path from the first to the last vertex in this labeling. The resulting graph is denoted by \( TH_k \). It is still sparse having roughly \( 3|V(TH_k)| \) edges and bandwidth again at most \( 2k \).

Hypercubes The Hamming graph \( H(d, q) \) is the Cartesian product of \( d \) copies of the complete graph \( K_q \). The Hamming graph \( H(d, 2) \) is also known as the hypercube (graph) \( Q_d \). Thus, the hypercube graph \( Q_d \) has \( 2^d \) vertices. The bandwidth of the hypercube graph was determined by Harper [18] and is given by the following expression:

\[
\text{bdw}(Q_d) = \sum_{i=0}^{d-1} \left( i \cdot \left\lfloor \frac{i}{2} \right\rfloor \right).
\]

We use the hypercube graphs \( Q_d \) to test the quality of our partition bounds.

5.2.2 Computations

Torus graphs In the tables to follow we always provide the following information. The first block of data contains the vector \( m \) of cardinalities for the partition blocks. We consider partitions into \( k \in \{4, 5, 6\} \) blocks. We set \( r = 1 \) and ask that \( m_2 = m_3 = \cdots = m_{k-1} \).

The sizes \( m_1 \) and \( m_k \) are chosen such that \( \sum_{i=1}^{k} m_i = n \) and \( |m_1 - m_k| \leq 1 \). Next we provide upper and lower bounds for the Minimal Partition Problem. The upper bound (ub) is obtained by running a standard Simulated Annealing heuristic [4] to find a good partition. The lower bound (lb) is obtained from the SDP relaxation (5) with all nonnegativity constraints included. Our main interest lies in values of \( m \), where the obtained lower bound is nontrivial, i.e., \( lb > 0 \). We give an illustration of the obtained solutions in Figure 2.

First, we consider Table 1, which contains computational results for the Torus graph \( T_7 \). Initially, we consider 4 blocks with \( m_2 = m_3 = 8 \) leading to a lower bound \( lb > 1.23 \). Hence, Corollary 3 allows us to conclude that \( \text{bdw}(T_7) > 8 \). We next try \( m_2 = m_3 = 9 \) where we only obtain the trivial lower bound of 0. Therefore, we get no further restriction on \( \text{bdw}(T_7) \) from 4-partitions. The 5-partition with \( m_2 = m_3 = m_4 = 9 \) however yields a positive lower bound and therefore \( \text{bdw}(T_7) > 9 \). Also, 6-partitions, given in the last block of the Table 1, do not lead to a further tightening of \( \text{bdw}(T_7) \).
Figure 2: Illustration of the $TH_6$ graph. On the left, we show the unpermuted graph, in the center, the permuted graph is shown, on the right, the obtained solution of the minPart problem with $m = (9, 9, 9)^T$ is shown. The value of minPart is 3, the corresponding entries are indicated by stars.

$$T_7 \quad (n = 49)$$

| $m_1$ | $m_2$ | $m_3$ | $m_4$ | $m_5$ | $m_6$ | ub | lb |
|-------|-------|-------|-------|-------|-------|-----|----|
| 16    | 8     | 8     | 17    |       |       | 6   | 1.23|
| 15    | 9     | 9     | 16    |       |       | 5   | -  |
| 11    | 9     | 9     | 9     | 11    |       | 6   | 0.68|
| 9     | 10    | 10    | 10    | 10    |       | 5   | -  |
| 6     | 9     | 9     | 9     | 9     | 7     | 6   | 0.56|
| 4     | 10    | 10    | 10    | 5     |       | 4   | -  |

Table 1: Torus graph $T_7$.

The results for the Torus graphs $T_8, T_9,$ and $T_{10}$ are summarized in Table 2. We proceed as before and consider partitions with $k \in \{4, 5, 6\}$. We can prove a lower bound of 11 for $bdw(T_8)$ and $bdw(T_9)$. It turns out that proving positive lower bounds for our partition problems gets increasingly difficult as either $n$ or $k$ increases. For $T_{10}$, the use of a 6-partition allows us to prove a lower bound of 14.

As a second experiment, we consider the graphs $TH_7, \ldots, TH_{10}$ consisting of the union of the Torus graph and a Hamiltonian path such that $bdw(TH_k) \leq 2k$ is insured. The results are summarized in Table 3. Compared to the Torus graphs we get slightly stronger lower bounds even though these graphs are still quite sparse, with $|E(TH_k)| < 3|V(TH_k)|$. Again, we see increasing gaps between lower and upper bounds as the number of nodes of the graph increases.

We summarize the bandwidth information for all variations of the Torus graphs in Table 4. Our partitioning approach provides nontrivial lower bounds on all instances.

Now, let us provide some information on computation time. To compute 4-partitions for graphs with 49 vertices we need about 20 seconds, for 5-partitions about 30 seconds, and for 6-partitions about 90 seconds. On the other hand, to compute a 4-partition (6-partition) on a graph with 100 vertices, our ADMM code needs about 200 seconds (700 seconds). Clearly, computation times
Table 2: Torus graphs $T_8$, $T_9$, $T_{10}$.

| $T_8$  $(n = 64)$ |   |   |   |   |   |   |
|---|---|---|---|---|---|---|
| $m_1$ | $m_2$ | $m_3$ | $m_4$ | $m_5$ | $m_6$ | $ub$ | $lb$ |
| 23 | 9 | 9 | 23 | 7 | 1.01 |
| 22 | 10 | 10 | 22 | 6 | - |
| 17 | 10 | 10 | 10 | 17 | 7 | 0.84 |
| 15 | 11 | 11 | 11 | 16 | 7 | - |
| 12 | 10 | 10 | 10 | 10 | 12 | 8 | 0.99 |
| 10 | 11 | 11 | 11 | 11 | 10 | 6 | - |

| $T_9$  $(n = 81)$ |   |   |   |   |   |   |
|---|---|---|---|---|---|---|
| $m_1$ | $m_2$ | $m_3$ | $m_4$ | $m_5$ | $m_6$ | $ub$ | $lb$ |
| 31 | 9 | 9 | 32 | 9 | 1.53 |
| 30 | 10 | 10 | 31 | 8 | - |
| 25 | 10 | 10 | 10 | 26 | 10 | 1.63 |
| 24 | 11 | 11 | 11 | 24 | 9 | - |
| 20 | 10 | 10 | 10 | 10 | 21 | 9 | 1.91 |
| 18 | 11 | 11 | 11 | 11 | 19 | 9 | - |

| $T_{10}$  $(n = 100)$ |   |   |   |   |   |   |
|---|---|---|---|---|---|---|
| $m_1$ | $m_2$ | $m_3$ | $m_4$ | $m_5$ | $m_6$ | $ub$ | $lb$ |
| 41 | 9 | 9 | 41 | 11 | 1.62 |
| 40 | 10 | 10 | 40 | 10 | - |
| 32 | 12 | 12 | 12 | 32 | 10 | 0.68 |
| 30 | 13 | 13 | 13 | 31 | 9 | - |
| 24 | 13 | 13 | 13 | 13 | 24 | 10 | 0.52 |
| 22 | 14 | 14 | 14 | 14 | 22 | 10 | - |

increase with respect to increasing partition sizes and number of vertices of the graphs. However, we obtain bounds in reasonable time for all tested graphs. All experiments were performed on a Windows 7 64-bit machine equipped with an Intel Core i5-5300U (2×2300 MHz) and 12 GB RAM using MATLAB 2016b.

**Hypercubes** Results for the hypercubes $Q_5$, $Q_6$, and $Q_7$ are summarized in Table 5. The table reads similar to the previous tables. To show a lower bound of 10 for $\text{bdw}(Q_5)$, our ADMM needs only 4 seconds. For comparison purposes we computed a lower bound for $Q_5$ and the case $k = 32$. Thus, we solved the QAP relaxation for that instance and obtained 11 as the lower bound of the BP.

For the hypercube $Q_6$ the 4-partition with $m_2 = m_3 = 17$ and $r = 1$ yields a positive lower bound, and therefore $\text{bdw}(Q_6) \geq 18$. We also compute the 6-partition with $m = (15, 9, 8, 9, 8, 15)^T$ and $r = 2$, and obtain a positive lower bound, which leads again to the conclusion that $\text{bdw}(Q_6) \geq 18$. Finally, we prove a lower bound of 33 for the hypercube $Q_7$.  


Table 3: Torus graphs plus Hamiltonian paths $TH_7$, $TH_8$, $TH_9$, $TH_{10}$.

5.3 Bandwidth of Matrices from Applications

In this section, we evaluate the performance of our approach on matrices that are given by real-world applications. We collected symmetric matrices, having 48 to 115 vertices. These are taken from the HB, Pothen, and Pajek groups of the SuiteSparse Matrix Collection [37]. We also selected matrices from the Newman collection available on the NIST Matrix Market [29].

Considering the Bandwidth Problem, only the structural properties of the matrices are of interest. Therefore, for a matrix $A$, we set $\text{diag}(A) = 0$. Moreover, we set all nonzero entries equal
to one.

In our computational evaluation, we select the partitioning $m$ such that $m_2 = \ldots = m_{k-1}$, $m_1 = \lceil \frac{n-d}{2} \rceil$, and $m_k = \lceil \frac{n-d}{2} \rceil$ where $d = \sum_{i=2}^{k-1} m_i$. We set $r = 1$, except when applying the 6-partition to *adjnoun* and *football* where we had to set $r = 2$. In the later case, we apply Corollary 4.

We summarize the results in Table 6. We provide the number of nodes (column labeled $n$) and the number of edges (column labeled $|E(G)|$). The column labeled $(bdw \leq)$ provides an upper bound on the bandwidth which we found by running a Simulated Annealing heuristic. We did not find any bandwidth information on these data in the literature. We also determined the density relative to the bandwidth, i.e., proportion of edges within the bandwidth, in the column labeled (bdw-dens). Finally, and most interestingly, we provide lower bounds based on $k$-partitions for $k \in \{3, 4, 5, 6\}$. The results in the column for $k = 3$ reflect the previous state-of-the-art using 3-partitions. The remaining columns show the improvement of the lower bound using partitions into $k \in \{4, 5, 6\}$ blocks. The lower bound is substantially improved in all cases. These results clearly indicate that our general partition approach yields a significant improvement over the 3-partition

| $k$ | $n$ | $T_k$ | $TH_k$ | $bdw \geq$ | $bdw \leq$ |
|-----|-----|------|--------|-------------|-------------|
| 7   | 49  | 10   | 12     | 14          |
| 8   | 64  | 11   | 13     | 16          |
| 9   | 81  | 11   | 14     | 18          |
| 10  | 100 | 14   | 15     | 20          |

Table 4: Summary of bounds for the bandwidth.

| $m_1$ | $m_2$ | $m_3$ | $m_4$ | $m_5$ | $m_6$ | $ub$ | $lb$ | $bdw \geq$ |
|-------|-------|-------|-------|-------|-------|------|------|------------|
| 6     | 10    | 10    | 6     | -     | -     | 0    | -    | 10         |
| 7     | 9     | 9     | 9     | 7     | -     | 4    | 0.99 | 10         |

Hypercube $Q_5$ \hspace{1cm} $n = 32$, \hspace{0.5cm} $bdw = 13$

| $m_1$ | $m_2$ | $m_3$ | $m_4$ | $m_5$ | $m_6$ | $ub$ | $lb$ | $bdw \geq$ | $r$ |
|-------|-------|-------|-------|-------|-------|------|------|-------------|-----|
| 15    | 17    | 17    | 15    | -     | -     | 10   | 1.18 | 18          | 1   |
| 15    | 9     | 8     | 9     | 8     | 15    | 14   | 1.18 | 18          | 2   |
| 14    | 9     | 9     | 9     | 9     | 14    | 9    | -    | -           |     |

Hypercube $Q_6$ \hspace{1cm} $n = 64$, \hspace{0.5cm} $bdw = 23$

| $m_1$ | $m_2$ | $m_3$ | $m_4$ | $m_5$ | $m_6$ | $ub$ | $lb$ | $bdw \geq$ |
|-------|-------|-------|-------|-------|-------|------|------|-------------|
| 33    | 31    | 31    | 33    | -     | -     | 19   | -    | -           |
| 34    | 30    | 30    | 34    |       |       | 31   | 3.11 | 31          |
| 16    | 32    | 32    | 32    | 16    |       | 18   | 0.93 | 33          |

Hypercube $Q_7$ \hspace{1cm} $n = 128$, \hspace{0.5cm} $bdw = 43$

Table 5: Hypercubes.
bounds from \[19, 23, 32, 33\].

| Name     | n   | $|E(G)|$ | bdw $\leq$ | bdw-dens | bdw $\geq$ |
|----------|-----|-------|------------|-----------|------------|
|          |     |       |            |           | partitioning |
|          |     |       | 3 | 4 | 5 | 6 |
| DWT59    | 59  | 104   | 6 | 0.381 | 3 | 4 | 4 | 5 |
| DWT87    | 87  | 227   | 10| 0.278 | 5 | 6 | 7 | 8 |
| NOS4     | 100 | 247   | 10| 0.261 | 6 | 7 | 7 | 8 |
| ASH85    | 85  | 219   | 9 | 0.304 | 4 | 6 | 7 | 7 |
| CAN61    | 61  | 248   | 13| 0.353 | 5 | 9 | 9 | 11|
| CAN73    | 73  | 152   | 16| 0.147 | 7 | 11| 14| 14|
| CAN96    | 96  | 336   | 13| 0.290 | 7 | 10| 11| 12|
| GD97-b   | 47  | 132   | 15| 0.226 | 5 | 11| 12| 11|
| mesh1e1  | 48  | 129   | 11| 0.279 | 6 | 9 | 10| 10|
| sphere2  | 66  | 192   | 13| 0.250 | 7 | 9 | 11| 12|
| dolphins | 62  | 159   | 13| 0.222 | 7 | 9 | 11| 11|
| lesmis   | 77  | 254   | 20| 0.191 | 5 | 11| 16| 17|
| polbooks | 105 | 441   | 20| 0.233 | 9 | 11| 14| 17|
| adjnoun  | 112 | 425   | 39| 0.119 | 23| 32| 31| 32|
| football | 115 | 613   | 37| 0.173 | 28| 33| 33| 33|

Table 6: Graphs from the literature.

5.4 Discussion

Based on our computational experiments we reach the following conclusions.

- The partitioning approach leads to acceptable lower bounds for the Bandwidth Problem. Our results indicate that the bounds get weaker as the number of nodes increases. This should come as no surprise in view of the hardness results known for the Bandwidth Problem.

- Our approach offers some flexibility in choosing the number $k$ of partition blocks to estimate the bandwidth. A larger $k$ would result in tighter bounds at higher computational cost.

- Further tightening of the semidefinite models is possible by adding additional constraints, e.g., triangle inequalities. This results in SDPs which require a refined computational setup.

- We could prove significantly better lower bounds for the Bandwidth Problem compared to the previous state-of-the-art of using 3-partitions.

6 Summary and Conclusion

We have shown that the partition approach provides a versatile tool to obtain lower bounds for the bandwidth of a graph. The choice of the model parameters $k$, $m$, and $r$ are highly problem dependent. However, our experiments indicate that even with a small number of partition blocks
we are able to derive nontrivial lower bounds on the bandwidth, even for very sparse graphs. Further research is necessary to explore this approach for larger graphs.

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