NOTES ON LOCAL COHOMOLOGY AND DUALITY

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Abstract. We provide a formula (see Theorem 1.5) for the Matlis dual of the injective hull of $R/p$ where $p$ is a one dimensional prime ideal in a local complete Gorenstein domain $(R, m)$. This is related to results of Enochs and Xu (see [4] and [3]). We prove a certain 'dual' version of the Hartshorne-Lichtenbaum vanishing (see Theorem 2.2). There is a generalization of local duality to cohomologically complete intersection ideals $I$ in the sense that for $I = m$ we get back the classical Local Duality Theorem. We determine the exact class of modules to which a characterization of cohomologically complete intersection from [6] generalizes naturally (see Theorem 4.4).

In this paper we prove a Matlis dual version of Hartshorne-Lichtenbaum Vanishing Theorem and generalize the Local Duality Theorem. The latter generalization is done for ideals which are cohomologically complete intersections, a notion which was introduced and studied in [6]. The generalization is such that local duality becomes the special case when the ideal $I$ is the maximal ideal $m$ of the given local ring $(R, m)$. We often use formal local cohomology, a notion which was introduced and studied by the second author in [13]. Formal local cohomology is related to Matlis duals of local cohomology modules (see [5, Sect. 7.1 and 7.2] and Corollary 3.4).

We start in Section 1 with the study of the Matlis duals of local cohomology modules $H^n_{I^{-1}}(R)$, where $n = \dim R$. The latter is also the formal local cohomology module $\varprojlim H^1_m(R/I^n)$ provided $R$ is a Gorenstein ring. We describe this module as the cokernel of a certain canonical map. As a consequence we derive a formula (see Theorem 1.5) for the Matlis dual of $E_R(R/p)$, where $p \in \text{Spec } R$ is a 1-dimensional prime ideal. In some sense this is related to results by Enochs and Xu (see [4] and [3]).

In Section 3 we generalize the Local Duality (see Theorem 3.1). The canonical module in the classical version is replaced by the dual of $H^c_I(R)$ where $I$ is a cohomologically complete intersection ideal of grade $c$ (the case $I = m$ specializes to the classical local duality). See also [11, Theorem 6.4.1]. It is a little bit surprising that the $d$-th formal local cohomology occurs as the duality module for the duality of cohomologically complete intersections in a Gorenstein ring (see Corollary 3.4).

In Theorem 4.4 we generalize the main result [6, Theorem 3.2]. This provides a characterization of the property of 'cohomologically complete intersection' given for ideals to finitely generated modules. Finally, in Section 5 we fill a gap in our proof of [6, Lemma 1.2]. To this end we use a result on inverse limits as it was shown by the second author (see [9]). Some of the results of Section 4 are obtained independently by W. Mahmood (see [9]).

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1. ON FORMAL LOCAL COHOMOLOGY

Let \((R, m)\) be a local ring, let \(I \subset R\) be an ideal. In the following let \(\hat{R}^I\) denote the \(I\)-adic completion of \(R\). Let \(0 = \cap_{i=1}^r q_i\) denote a minimal primary decomposition of the zero ideal. Then we denote by \(\dim R/(p_i + I) > 0\), where \(\Rad q_i = p_i, i = 1, \ldots, r\).

For the definition and basic properties of \(\varprojlim H^*_m(R/I^\alpha)\), the so-called formal local cohomology, we refer to \([13]\). We denote the functor of global transform by \(T(\cdot) = \varprojlim \Hom_R(m^\alpha, \cdot)\), in order to distinguish it from Matlis duality
\[
D(M) = \Hom_R(M, E_R(R/m)),
\]
where \(E_R(R/m)\) is a fixed \(R\)-injective hull of \(k := R/m\).

**Lemma 1.1.** Let \(I \subset R\) denote an arbitrary ideal. Then there is a short exact sequence
\[
0 \to \hat{R}^I/u(I\hat{R}^I) \to \varprojlim T(R/I^\alpha) \to \varprojlim H^1_m(R/I^\alpha) \to 0.
\]

**Proof.** For each \(\alpha \in \mathbb{N}\) there is the following canonical exact sequence
\[
0 \to H^0_m(R/I^\alpha) \to R/I^\alpha \to T(R/I^\alpha) \to H^1_m(R/I^\alpha) \to 0.
\]
It splits up into two short exact sequences
\[
0 \to H^0_m(R/I^\alpha) \to R/I^\alpha : \langle m \rangle \to 0 \text{ and } 0 \to R/I^\alpha : \langle m \rangle \to T(R/I^\alpha) \to H^1_m(R/I^\alpha) \to 0.
\]

Now the inverse systems at the left side of both of the exact sequences satisfy the Mittag-Leffler condition. That is, by passing to the inverse limits it provides two short exact sequences. Putting them together there is an exact sequence
\[
0 \to \varprojlim H^0_m(R/I^\alpha) \to \hat{R}^I \to \varprojlim T(R/I^\alpha) \to \varprojlim H^1_m(R/I^\alpha) \to 0.
\]
Now it follows that \(\varprojlim H^0_m(R/I^\alpha) = u(I\hat{R}^I)\), see \([13\text{ Lemma 4.1}]\). This finally proves the statement. \(\square\)

Of a particular interest in the above Corollary is the case where \(I \subset R\) is an ideal such that \(\dim R/I = 1\).

**Corollary 1.2.** Suppose that \(\dim R/I = 1\). Then there is a short exact sequence
\[
0 \to \hat{R}^I/u(I\hat{R}^I) \to \oplus_{i=1}^s \hat{R}_{p_i} \to \varprojlim H^1_m(R/I^\alpha) \to 0,
\]
where \(p_i, i = 1, \ldots, s\), denote those prime ideals \(p\) of \(\Ass R/I\) such that \(\dim R/p = 1\).

**Proof.** Since \(\dim R/I = 1\) there is an element \(x \in m\) that is a parameter for \(R/I^\alpha\) for all \(\alpha \in \mathbb{N}\). Then there is an isomorphism \(T(R/I^\alpha) \simeq R_x/I^\alpha R_x\) for all \(\alpha \in \mathbb{N}\).

Now let \(S = \cap_{i=1}^s (R \setminus p_i)\). Since \(x \in S\) there is a natural isomorphism (by the local-global-principle)
\[
R_x/I^\alpha R_x \simeq R_S/I^\alpha R_S\text{ for all }\alpha \in \mathbb{N}.
\]
Then \(R_S\) is a semi local ring. The Chinese Remainder Theorem provides isomorphisms
\[
R_S/I^\alpha R_S \simeq \oplus_{i=1}^s R_{p_i}/I^\alpha R_{p_i}\text{ for all }\alpha \in \mathbb{N}.
\]
Now \( \text{Rad} IR_{p_i} = p_i R_{p_i}, i = 1, \ldots, s \). So by passing to the inverse limit we get the isomorphism

\[
\lim \left( R/I^\alpha \right) \simeq \bigoplus_{i=1}^s \overline{R}_{p_i}.
\]

Therefore \( \square \) finishes the proof of the statement.

**Remark 1.3.** In the case that \( R/I \) is one dimensional it follows that

\[
H^m_n(R/I^\alpha) \simeq H^1_x(R/I^\alpha) \simeq H^1_x(R) \otimes R/I^\alpha \simeq (R_x/R) \otimes R/I^\alpha
\]

for all \( \alpha \in \mathbb{N} \), where \( x \in m \) denotes a parameter of \( R/I \). This finally implies that

\[
\lim \left( H^1_n(R/I^\alpha) \right) \simeq \widetilde{R_x}/R.
\]

**Corollary 1.4.** Suppose that \( I \subset R \) is a one dimensional ideal in a local Gorenstein ring \((R, m)\) with \( n = \dim R \). Then there is a short exact sequence

\[
0 \rightarrow \hat{R}^l / u(I \hat{R}^l) \rightarrow \bigoplus_{i=1}^s \overline{R}_{p_i} \rightarrow \text{Hom}_R(H^1_0(R, E) \rightarrow 0,
\]

where \( p_i, i = 1, \ldots, s \), denote those prime ideals \( p \) of \( \text{Ass} R/I \) such that \( \dim R/p = 1 \).

*Proof.* This is clear because of \( \lim H^1_n(R/I^\alpha) \simeq \text{Hom}_R(H^1_0(R, E) \) as it is a consequence of the Local Duality Theorem for Gorenstein rings (the Hom-functor in the first place transforms a direct limit into an inverse limit).

In particular the Matlis dual of \( H^1_0(R) \) is exactly the cokernel of the canonical map \( \hat{R}^l \rightarrow \bigoplus_{i=1}^s \overline{R}_{p_i} \). This generalizes \([5, \text{Lemma 3.2.1}] \) (see also \([7, \text{Lemma 1.5}] \)).

If we assume in addition that \( I = p \) is a one dimensional prime ideal and that \( R \) is a complete domain, then by \([6, \text{Theorem 3.2}] \) the fact \( H^1_p(R) = 0 \) (as follows by the Hartshorne-Lichtenbaum Vanishing Theorem) is equivalent to: The minimal injective resolution of \( H^{n-1}_p(R) \) looks as follows:

\[
0 \rightarrow H^{n-1}_p(R) \rightarrow E_R(R/p) \rightarrow E_R(R/m) \rightarrow 0.
\]

On the other hand we have (see Corollary 1.4) a short exact sequence

\[
0 \rightarrow R \rightarrow \hat{R}_p \rightarrow D(H^1_n(R)) \rightarrow 0.
\]

Note that the natural map \( R \rightarrow \hat{R}_p \) is injective because \( R \) is a complete domain. Therefore, \( u(p) = u(p \hat{R}^p) = 0 \). The comparison of the two exact sequences has the following consequence:

Applying the functor \( D \) to the first exact sequence it induces a natural homomorphism

\[
R = D(E_R(R/m)) \rightarrow D(E_R(R/p)).
\]

Since the latter is an \( R_p \)-module, we get a map \( R_p \rightarrow D(E_R(R/p)) \) and therefore a family of homomorphisms

\[
R_p/p^\alpha R_p \rightarrow D(E_R(R/p))/p^\alpha D(E_R(R/p))
\]

for any integer \( \alpha \in \mathbb{N} \). But now we have the isomorphisms

\[
D(E_R(R/p)) = \lim \underset{\alpha}{\text{Hom}} R(p^\alpha, E_R(R/p)), E_R(R/m)) = \lim \underset{\alpha}{\text{D}(E_R(R/p))/p^\alpha D(E_R(R/p)).}
\]
Therefore the above inverse systems induce a homomorphism $f : \hat{R}_p \to D(E_R(R/p))$. Clearly the natural homomorphism $R = D(E_R(R/m)) \to D(E_R(R/p))$ factors through $f$. So the above two short exact sequences induce a commutative diagram

$$
\begin{array}{ccccc}
0 & \to & R & \to & \hat{R}_p & \to & D(H_{p}^{n-1}(R)) & \to & 0 \\
& & | & & | & & | & & \\
0 & \to & D(E_R(R/m)) = R & \to & D(E_R(R/p)) & \to & D(H_{p}^{n-1}(R)) & \to & 0
\end{array}
$$

All maps in this commutative diagram are canonical and it is easy to see that the vertical homomorphism on the right is the identity. Therefore $f$ is an isomorphism too. We conclude with the following result:

**Theorem 1.5.** Let $p$ be a prime ideal of height $n - 1$ in an $n$-dimensional local, complete Gorenstein domain $(R, m)$. Then the Matlis dual of $E_R(R/p)$ is $\hat{R}_p$.

This is related to results from Enochs and Xu: $D(E_R(R/p))$ is flat and cotorsion by [4, Lemma 2.3] (see also [3, Theorem 1.5]), therefore (see [4, Theorem]), it is isomorphic to a direct product of modules $T_q$ (over $q \in \text{Spec } R$) where each $T_q$ is the completion of a free module over $R_q$. It was also proved in [4] that in this direct product the ranks of these free modules are uniquely determined. By [3, Theorem 2.2] these ranks are

$$\pi_0(q, D(E_R(R/p))) = \dim_{k(q)} k(q) \otimes_{R_q} \text{Hom}_R(R_q, D(E_R(R/p)))$$

(all higher $\pi_i$ for $i > 0$ vanish since $D(E_R(R/q))$ is flat: Its minimal flat resolution is trivial). For each $q$ different from $p$ the latter rank is zero: In case $p \neq m$ this follows from

$$\text{Hom}_R(R_q, D(E_R(R/p))) = D(R_q \otimes_R E_R(R/p)) = 0,$$

and in case $q = m$ we have

$$(R/m) \otimes_R D(E_R(R/p)) = D(\text{Hom}_R(R/m, E_R(R/p))) = 0.$$

Therefore, the use of those results from [4, 3] leads us to

$$D(E_R(R/p)) = T_p,$$

where $T_p$ is the completion of a free $R_p$-module.

Our Theorem 1.5 gives the more precise information that the rank of this free module is exactly 1, i.e. $T_p \cong \hat{R}_p$.

**Question 1.6.** Is it possible to deduce the fact that this rank is 1 directly from [3, Theorem 2.2], i.e., without using our theorem 1.5?

2. A Remark on the Hartshorne-Lichtenbaum Vanishing Theorem

In this Section there is a comment on the Hartshorne-Lichtenbaum Vanishing Theorem in view to the previous investigations. Let $I$ denote an ideal in a local Noetherian ring $(R, m)$ with $\dim R = n$. As above let $\hat{R}^I$ denote the $I$-adic completion of $R$. Let $0 = \cap_{i=1}^r q_i$ denote a minimal primary decomposition of the zero ideal. Then we denote by $v(I)$ the intersection of those $q_i, i = 1, \ldots, r$, such that $\dim R/(q_i + I) > 0$ and $\dim R/q_i = n$. 
Recall that $v(I) = u(I)$ (for the ideal $u(I)$ as introduced at the beginning of Section 1) if $R$ is equi-dimensional.

The following result provides a variant of the Hartshorne Lichtenbaum Vanishing Theorem.

**Theorem 2.1.** ([11, Theorem 2.20]) Let $I \subset R$ denote an ideal and $n = \dim R$. Then

$$H^n_I(R) \cong \text{Hom}_R(v(I\hat{R}), E_R(R/m)).$$

That is $H^n_I(R)$ is an Artinian $R$-module and $H^n_I(R) = 0$ if and only if $\dim \hat{R}/(I\hat{R} + p) > 0$ for all $p \in \text{Ass} \ R$ with $\dim \hat{R}/p = n$.

For an ideal $I$ of a Noetherian ring $R$ let $\text{Ass} I$ denote the ultimate constant (see [1]) value of $\text{Ass} R/I^\alpha$ for $\alpha \gg 0$. We define the multiplicatively closed set $S = \cap_{p \in \text{Ass} I \setminus \{m\}} R\setminus p$. Then there is an exact sequence

$$0 \to I^\alpha : (m)/I^\alpha \to R/I^\alpha \to R_S/I^\alpha R_S \quad \text{for all } \alpha \gg 0.$$

Since the modules on the left are of finite length the corresponding inverse system satisfies the Mittag-Leffler condition. By passing to the inverse limit it induces an exact sequence

$$0 \to u(I\hat{R}^I) \to \hat{R}^I \to \hat{R}_S^I$$

(see [13, Lemma 4.1]). Now let $R$ denote a complete equidimensional local ring. Then the natural homomorphism $R \to \hat{R}_S^{-I}$ is injective if and only if $H^n_I(R) = 0$. This follows since $v(I) = u(I) = 0$ if and only if $H^n_I(R) = 0$ under the additional assumption on $R$.

If in addition $\dim R/I = 1$ we have as above that $R_S/I^\alpha R_S \cong \bigoplus_{i=1}^s R_{p_i}/I^\alpha R_{p_i}$. Therefore, if $I$ is a one dimensional ideal in an equidimensional complete local ring $(R, m)$. Then the natural homomorphism $R \to \bigoplus_{i=1}^s \hat{R}_{p_i}$ is injective if and only if $H^n_I(R) = 0$ (the $p_i$ are defined as above). In case $R$ is in addition a domain then our map $R \to \bigoplus_{i=1}^s \hat{R}_{p_i}$ is clearly injective and, therefore, $H^n_I(R) = 0$.

In the following we shall continue with this series of ideas in the case of $(R, m)$ a Gorenstein ring. To this end we put $V(I)_1 = \{p \in V(I) | \dim R/p = 1\}$.

**Theorem 2.2.** Let $(R, m)$ denote a Gorenstein ring with $n = \dim R$. Let $I \subset R$ denote an ideal. Then $D(H^n_I(R))$ is isomorphic to the kernel of the natural map $\hat{R} \to \prod_{p \in V(I)_1} \hat{R}_{p}$. In particular, this homomorphism is injective if and only if $H^n_I(R) = 0$.

**Proof.** By applying the section functor $\Gamma_I(\cdot)$ to the minimal injective resolution of the Gorenstein ring $R$ it provides an exact sequence

$$\bigoplus_{p \in V(I)_1} E_R(R/p) \to E_R(R/m) \to H^n_I(R) \to 0.$$

Now apply the Matlis duality functor $D(\cdot)$ to the sequence. It provides the exact sequence

$$0 \to D(H^n_I(R)) \to \hat{R} \to \prod_{p \in V(I)_1} D(E_R(R/p)).$$

By virtue of Theorem [15, 1.5] it follows that $D(E_R(R/p)) \cong \hat{R}_{p}$. Therefore $D(H^n_I(R))$ is isomorphic to the kernel of the natural map $\hat{R} \to \prod_{p \in V(I)_1} \hat{R}_{p}$. Matlis duality provides the claim. \qed
If in addition \( \dim R/I = 1 \), then \( V(I)_1 \) is a finite set. Therefore the direct product in Theorem 2.2 is in fact a direct sum. Hence, the result in Theorem 2.2 is a generalization of the considerations above for the case \( \dim R/I = 1 \).

Moreover, in a certain sense Theorem 2.2 is a dual version to [2, Proposition] shown by Call and Sharp.

3. ON A DUALITY FOR COHOMOLGICALLY COMPLETE INTERSECTIONS

As above let \((R, \mathfrak{m})\) denote a local Noetherian ring. An ideal \( I \subset R \) is called a cohomologically complete intersection whenever \( H^i_I(R) = 0 \) for all \( i \neq c \) for some \( c \) (see [3] for the definition and a characterization). If \( I \) is a cohomologically complete intersection, then in the paper of Zargar and Zakeri (see [15]) the ring \( R \) is called Cohen-Macaulay with respect to \( I \).

The main aim of the present section is to prove a generalized local duality for a cohomologically complete intersection \( I \). A corresponding result was already obtained by W. Mahmood (see [9]) resp. by the second author in [5, Theorem 6.4.1] by different means.

**Theorem 3.1.** Let \( I \subset R \) denote a cohomologically complete intersection with \( c = \text{grade } I \). Let \( X \) denote an arbitrary \( R \)-module. Then there are the following functorial isomorphisms

(a) \( \text{Tor}^R_{c-i}(X, H^i_I(R)) \cong H^i_I(X) \) and

(b) \( \text{Ext}^{c+i}_{R}(X, \text{Hom}_R(H^i_I(R), E_R(k))) \cong \text{Hom}_R(H^i_I(X), E_R(k)) \)

for all \( i \in \mathbb{Z} \).

**Proof.** First of all choose \( \underline{x} = x_1, \ldots, x_r \) a system of elements of \( R \) such that \( \text{Rad} \underline{x} R = \text{Rad} I \). Then we consider the Čech complex \( \check{C}_{\underline{x}}^* \). This is a bounded complex of flat \( R \)-modules with \( H^i(\check{C}_{\underline{x}}) = 0 \) for all \( i \neq c \) and \( H^c(\check{C}_{\underline{x}}) \cong H^c_I(R) \). Moreover, \( H^i(X \otimes_R \check{C}_{\underline{x}}) \cong H^i_I(X) \) (see e.g. [12]). In order to compute the cohomology \( \text{Tor}_i(\check{C}_{\underline{x}}, X) \) there is the following spectral sequence

\[
E^{i,j}_2 = \text{Tor}^R_{c-i}(H^j_I(R), X) \Rightarrow E^{i+j}_\infty = \text{Tor}^R_{-i-j}(\check{C}_{\underline{x}}, X)
\]

Since \( I \) is a cohomologically complete intersection we get a degeneration to the following isomorphisms

\[
\text{Tor}^R_{c-i}(H^j_I(R), X) \cong \text{Tor}^R_{-i-j}(\check{C}_{\underline{x}}, X) \cong H^j_I(X)
\]

for all \( i \in \mathbb{Z} \). This proves the isomorphisms of the statement in (a).

For the proof of (b) note that

\[
\text{Hom}_R(\text{Tor}^R_{c-i}(H^j_I(R), X), E_R(k)) \cong \text{Ext}^c_{R}(X, \text{Hom}_R(H^j_I(R), E_R(k)))
\]

as follows by adjunction since \( E_R(k) \) is an injective \( R \)-module. \( \square \)

The Matlis dual \( \text{Hom}_R(H^j_I(R), E_R(k)) = D(H^j_I(R)) \) plays a central rôle in the above generalized duality. It allows to express the Matlis dual of \( H^j_I(X) \) in terms of an Ext module.

**Definition 3.2.** Let \( I \subset R \) denote a cohomologically complete intersection with \( c = \text{grade } I \). Then we call \( D_I(R) = \text{Hom}_R(H^j_I(R), E_R(k)) = D(H^j_I(R)) \) the duality module of \( I \).
In general the structure of $D_I(R)$ is difficult to determine. In the following we want to discuss a few particular cases of cohomologically complete intersections and their duality module. To this end let $K(R)$ denote the canonical module of $R$, provided it exists.

**Corollary 3.3.** Let $(R, \mathfrak{m})$ denote a local ring such that $\mathfrak{m}$ is a cohomologically complete intersection. There are natural isomorphisms

$$H_{\mathfrak{m}}^{n-i}(M) \cong \text{Hom}_R(\text{Ext}_R^i(M, K(R)), E_R(k)), n = \text{dim } R,$$

for a finitely generated $R$-module $M$ and any $i \in \mathbb{Z}$. Note that $R$ is Cohen-Macaulay ring and $D_{\mathfrak{m}}(R) \cong K(R)$.

**Proof.** In case $\mathfrak{m}$ is a cohomologically complete intersection, then depth $R = \text{grade } \mathfrak{m} = \text{dim } R$ since $H_{\mathfrak{m}}^i(R)$ is the only non-vanishing local cohomology module. This follows by the non-vanishing of $H_{\mathfrak{m}}^i(R)$ for $i = \text{depth } R$ and $i = \text{dim } R$ (see e.g. [11]). Therefore $R$ is a Cohen-Macaulay ring.

Moreover, $\text{Hom}_R(H_{\mathfrak{m}}^d(R), E_R(k)) \cong \text{Hom}_R(H_{\mathfrak{m}}^d(\hat{R}), (E_R(k)))$ since $E_R(k)$ is an Artinian $R$-module. Then $\hat{R}$ admits a canonical module and $K(\hat{R}) \cong \text{Hom}_R(H_{\mathfrak{m}}^d(\hat{R}), (E_R(K)))$ (see also [11] for more details).

It is known (and easy to see) that $H_{\mathfrak{m}}^i(M)$ is an Artinian $R$-module for any $i$ and a finitely generated $R$-module $M$. Then the isomorphisms follow by Theorem 3.1 (b) by the aid of Matlis duality. □

In the particular case of a Gorenstein ring it follows that $K(\hat{R}) \cong \hat{R}$. So Corollary 3.3 provides the classically known Local Duality Theorem for a Gorenstein ring. In the following we shall consider the case of an arbitrary cohomologically complete intersection $I$ in a Gorenstein ring.

**Corollary 3.4.** Let $I \subset R$ denote a cohomologically complete intersection in a Gorenstein ring $(R, \mathfrak{m})$. Then there is the isomorphism $D_I(R) \cong \lim_{\leftarrow} H_{\mathfrak{m}}^d(R/I^\alpha)$, where $d = \text{dim } R/I$. That is, there are natural isomorphisms

$$\text{Hom}_R(H_{I}^{d-i}(X), E_R(k)) \cong \text{Ext}_R^i(X, \lim_{\leftarrow} H_{\mathfrak{m}}^d(R/I^\alpha))$$

for any $R$-module $X$ and all $i \in \mathbb{Z}$.

**Proof.** By the definition of local cohomology there are the following isomorphisms

$$H_{I}^j(R) \cong \lim_{\rightarrow} \text{Ext}_R^j(R/I^\alpha, R)$$

for all $i \in \mathbb{Z}$. By the duality we get the isomorphisms

$$\text{Hom}_R(H_{I}^j(R), E_R(k)) \cong \text{Hom}_R(\lim_{\rightarrow} \text{Ext}_R^j(R/I^\alpha, R), E_R(k)) \cong \lim_{\leftarrow} H_{\mathfrak{m}}^d(R/I^\alpha).$$

Note that the Hom-functor in the first place transforms a direct limit into an inverse limit. □

Note that $\lim_{\leftarrow} H_{\mathfrak{m}}^d(R/I^\alpha)$ was studied in [13] under the name formal local cohomology. See also [13] for more details. It is a little bit surprising that the $d$-th formal local cohomology occurs as the duality module for the duality of cohomologically complete intersections in a Gorenstein ring.

Now we consider the particular case of a one dimensional cohomologically complete intersection in a Gorenstein ring.
Corollary 3.5. Let $I \subset R$ denote a one dimensional cohomologically complete intersection in a Gorenstein ring $R$ with $n = \dim R$. Let $x \in \mathfrak{m}$ be a parameter of $R/I$ and let $X$ denote an arbitrary $R$-module. Then for all $i \in \mathbb{Z}$ there are natural isomorphisms

$$\text{Hom}_R(H^c_i(X), E_R(k)) \cong \text{Ext}^i_R(X, D),$$

where $D$ denotes the cokernel of the natural homomorphism $\hat{R}^l \to \hat{R}^l_x$.

Proof. The proof is an obvious consequence of Corollary 3.4 by the aid of the results from section 1. \hfill \square

Another interpretation of the duality module $D$ in Corollary 3.5 can be done as the cokernel of the natural map $\hat{R}^l \to \oplus_{i=0}^s \hat{R}_m$ as done in Corollary 1.2.

4. Cohomologically complete intersections: A generalization to modules

In this section let $I$ be an ideal of a local ring $(R, \mathfrak{m})$. Let $M$ denote a finitely generated $R$-module. Let $E_R(M)$ denote a minimal injective resolution of the $R$-module $M$. The cohomology of the complex $\Gamma(I)(E_R(M))$ is by definition the local cohomology $H^i_I(M)$, $i \in \mathbb{N}$. Suppose that $c = \text{grade}(I, M)$. Then $\Gamma(I)(E^*_R(M)) = 0$ for all $i < c$. Therefore $H^i_I(M) = \text{Ker}(\Gamma(I)(E^*_R(M)) \to \Gamma(I)(E^{c+1}_R(M))$ and there is an embedding $H^i_I(M)[-c] \to \Gamma(I)(E^*_R(M))$ of complexes.

Definition 4.1. The cokernel of the embedding $H^i_I(M)[-c] \to \Gamma(I)(E^*_R(M))$ is defined by $C^i_M(I)$, the truncation complex of $M$ with respect to $I$. So there is a short exact sequence

$$0 \to H^i_I(M)[-c] \to \Gamma(I)(E^*_R(M)) \to C^i_M(I) \to 0$$

of complexes of $R$-modules. In particular $H^i(C^i_M(I)) = 0$ for all $i \leq c$ and $H^i(C^i_M(I)) \cong H^i_I(M)$ for all $i > c$.

Note that the definition of the truncation complex was used in the case of $M = R$ a Gorenstein ring in [6]. This construction is used in order to obtain certain natural homomorphisms.

Lemma 4.2. Let $M$ denote a finitely generated $R$-module with $c = \text{grade}(I, M)$. Then there are natural homomorphisms

$$H^c_m(H^i_I(M)) \to H^i_m(M)$$

for all $i \in \mathbb{N}$. These are isomorphisms for all $i \in \mathbb{Z}$ if and only if $H^c_m(C^i_M(I)) = 0$ for all $i \in \mathbb{Z}$.

Proof. Take the short exact sequence of the truncation complex (cf. 4.1) and apply the derived functor $\Gamma_m(\cdot)$. In the derived category this provides a short exact sequence of complexes

$$0 \to \Gamma_m(H^c_I(M)[-c] \to \Gamma_m(\Gamma_I(E^*_R(M))) \to \Gamma_m(C^i_M(I)) \to 0.$$

Since $\Gamma_I(E^*_R(M))$ is a complex of injective $R$-modules we might use $\Gamma_m(\Gamma_I(E^*_R(M)))$ as a representative of $\Gamma_m(\Gamma_I(E^*_R(M)))$. But now there is an equality for the composite of section functors $\Gamma_m(\Gamma_I(\cdot)) = \Gamma_m(\cdot)$. Therefore $\Gamma_m(E^*_R(M))$ is a representative of $\Gamma_m(\Gamma_I(E^*_R(M)))$ in the derived category.
First of all it provides the natural homomorphisms of the statement. Then the long exact cohomology sequence provides that these maps are isomorphisms if and only if $H^i_m(C_M(I)) = 0$ for all $i \in \mathbb{Z}$.

\textbf{Definition 4.3.} The finitely generated $R$-module $M$ is called cohomologically complete intersection with respect to $I$ in case there is an integer $c \in \mathbb{N}$ such that $H^i(I) = 0$ for all $i \neq c$. Clearly $c = \text{grade}(I, M)$.

This notion extends those of a cohomologically complete intersection $I \subset R$ in a Gorenstein ring $R$ as it was studied in [6].

It is our intention now to generalize part of [6, Theorem 5.1] to the situation of a module $M$ and an ideal $I \subset R$ satisfying the requirements of Definition 4.3. See also [9] for similar results.

\textbf{Theorem 4.4.} Let $(R, m)$ be a local ring, let $M$ be a finitely generated $R$-module, $I$ an ideal of $R$. Let $c := \text{grade}(I, M)$. Then the following conditions are equivalent:

(i) $H^i_I(M) = 0$ for all $i \neq c$.
(ii) The natural map

$$H^i_p\gamma_p(H^c_I M_p) \rightarrow H^i_p\gamma_p(M_p)$$

is an isomorphism for all $p \in V(I) \cap \text{Supp} M$ and all $i \in \mathbb{Z}$.

\textbf{Proof.} We begin with the proof of the implication (i) $\Rightarrow$ (ii). By the assumption it follows easily that $c = \text{grade}(IR_p, M_p)$ for all $p \in V(I) \cap \text{Supp} M$. That is we might reduce the proof to the case of the maximal ideal. By the assumption in (i) it follows (see Definition 4.1) that $C_M(I)$ is an exact bounded complex. Therefore $H^i_m(C_M(I)) = 0$ for all $i \in \mathbb{Z}$. So the claim follows by virtue of Lemma 1.2.

For the proof of (ii) $\Rightarrow$ (i) we proceed by an induction on $\dim V(I) \cap \text{Supp} M =: t$. In the case of $t = 0$, i.e. $V(I) \cap \text{Supp} M = \{m\}$, it follows that $R\Gamma_m(C_M(I)) \cong C_M(I)$. Then the claim is true by virtue of Lemma 1.2. Now suppose that $t > 0$ and by induction hypothesis the statement holds for all smaller dimensions. Then it follows that $\text{Supp} H^i_I(M) \subseteq \{m\}$ for all $i \neq c$. By the definition of $C_M(I)$ we get that $\text{Supp} H^i(I) \subseteq \{m\}$. Therefore it follows that $H^i_m(C_M(I)) \cong H^i(C_M(I))$ for all $i \in \mathbb{Z}$. By the assumption in (i) for $p = m$ it implies (see Lemma 1.2) that

$$H^i_m(C_M(I)) \cong H^i(C_M(I)) = H^i_I(M) = 0$$

for all $i \neq c$. This completes the proof.

We remark that Theorem 4.4 works without the hypothesis ”$R$ is Gorenstein”. In the paper [6] the authors considered only the case of a Gorenstein ring.

5. A note on direct and inverse limits

In the proof of [6, Lemma 1.2(a)] it is claimed that $\text{Ext}$ of a direct limit in the first variable is the projective limit of the corresponding $\text{Ext}$’s. In general, this is not true: E.g. it is well-known and not very difficult to see that $\text{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z})$ is non-zero (it is actually uncountable), while $\mathbb{Q}$ can be written as a direct limit of copies of $\mathbb{Z}$’s and each copy of $\text{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})$ is of course zero. We explain how this problem can be overcome (literally all results from [6] are valid – apart from lemma 1.2 (a) ).
The general result is the following:

**Theorem 5.1.** ([14] Lemma 2.6) Let \( \{M_i\} \) be a direct system of \( R \)-modules. Let \( N \) denote an arbitrary \( R \)-module. Then there is a short exact sequence

\[
0 \to \lim_{\leftarrow} \text{Ext}^{i-1}_R(M_i, N) \to \text{Ext}^i_R\left(\lim_{\leftarrow} M_i, N\right) \to \lim_{\leftarrow} \text{Ext}^i_R(M_i, N) \to 0
\]

for all \( i \in \mathbb{Z} \). In particular, \( \text{Hom}_R\left(\lim_{\leftarrow} M_i, N\right) \cong \lim_{\leftarrow} \text{Hom}_R(M_i, N). \)

The previous Lemma 5.1 gives the corrected version of [6, Lemma 1.2 (a)]. In the following we shall explain how to derive the other results of [6, Lemma 1.2].

**Lemma 5.2.** Let \((R, m, k)\) be an \( n \)-dimensional local Gorenstein ring. For each \( R \)-module \( X \) there are canonical isomorphisms

\[
\text{Ext}^{n-i}_R(X, \hat{R}) \cong \text{Hom}_R(H^i_m(X), E)
\]

for all \( i \in \mathbb{Z} \). Here \( \hat{R} \) denotes the completion of \( R \).

**Proof.** The proof follows immediately from Corollary 3.4 or from [5, Theorem 6.4.1]. We put \( I := m \). Recall that \( D_I(R) = \hat{R}. \)

Having established this general version of Local Duality, it is easy to produce the statement of [6] lemma 1.2 (b)].

**Corollary 5.3.** Let \( I \) be a proper ideal of height \( c \) in a \( n \)-dimensional local Gorenstein ring \((R, m)\). There are canonical isomorphisms

\[
\text{Ext}^{n-i}_R(H^i_m(R), \hat{R}) \cong \text{Hom}_R(H^i_m(H^j_m(R)), E) \cong \lim_{\leftarrow} \text{Ext}^{n-i}_R(\text{Ext}^j_R(R/I^\alpha, R), \hat{R})
\]

for all \( i, j \in \mathbb{Z}. \)

**Proof.** The first of the isomorphisms is a consequence of Lemma 5.2 applied to \( H^1_m(R) \). Lemma 5.2 applied to \( \text{Ext}^j_R(R/I^\alpha, R) \) provides a family of isomorphisms

\[
\text{Hom}_R(H^i_m(\text{Ext}^j_R(R/I^\alpha, R), E) \cong \text{Ext}^{n-i}_R(\text{Ext}^j_R(R/I^\alpha, R), \hat{R}), \quad \text{for all } \alpha \in \mathbb{N},
\]

which are compatible with the inverse systems induced by the natural surjections. So, it induces an isomorphism

\[
\lim_{\leftarrow} \text{Hom}_R(H^i_m(\text{Ext}^j_R(R/I^\alpha, R), E) \cong \lim_{\leftarrow} \text{Ext}^{n-i}_R(\text{Ext}^j_R(R/I^\alpha, R), \hat{R}),
\]

for all \( i \) and \( j \). Since the inverse limit commutes with the direct limit under \( \text{Hom} \) in the first place (see Theorem 5.1) it induces an isomorphism

\[
\lim_{\leftarrow} \text{Hom}_R(H^i_m(\text{Ext}^j_R(R/I^\alpha, R), E) \cong \text{Hom}_R(\lim_{\rightarrow} \text{Ext}^j_R(R/I^\alpha, R), E).
\]

This finally completes the proof since \( H^1_m(R) \cong \lim_{\rightarrow} \text{Ext}^j_R(R/I^\alpha, R) \) and because local cohomology commutes with direct limits. \( \square \)

With these results in mind the proof of [6, Lemma 1.2 (c)] follows the same line of arguments as in the original paper. In the proof of [6, Corollary 2.9] there is a reference to [6, Lemma 1.2(b)]: However 2.9 can be easily deduced from the minimal injective resolution \( 0 \to H^i_m(R) \to \Gamma_I(E^\ast) \to \Gamma_I(E^{\ast+1}) \to \ldots \) (where \( 0 \to R \to E^\ast \) is a minimal injective resolution of \( R \)) of \( H^i_m(R) \); note that we know what indecomposable injective modules occur in the complex \( E^\ast \) since \( R \) is Gorenstein.
In the proof of (iii) \(\iff\) (iv) of [6, Theorem 3.1] there is a reference to [6, lemma 1.2(b)]; But this equivalence (iii) \(\iff\) (iv) follows from Lemma 5.2.

**Remark 5.4.** Combining the statements in Theorem 5.1 with those of Corollary 5.3 it follows that

\[
\lim_{\leftarrow} \Ext^i_R(\Ext^j_R(R/I\alpha, R), \hat{R}) = 0
\]

for all \(i, j \in \mathbb{Z}\).

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