A dual algorithm for stochastic control problems: Applications to Uncertain Volatility Models and CVA

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April 24, 2015

Abstract

We derive an algorithm in the spirit of Rogers [18] and Davis, Burstein [4] that leads to upper bounds for stochastic control problems. Our bounds complement lower biased estimates recently obtained in Guyon, Henry-Labordère [10]. We evaluate our estimates in two numerical examples motivated from mathematical finance.

1 Introduction

Solving stochastic control problems, for example by approximating the Hamilton-Jacobi-Bellman equation, is an important problem in applied mathematics. Classical PDE methods are effective tools for solving such equations in low dimensional settings, but quickly become computationally intractable as the dimension of the problem increases: a phenomenon commonly referred to as "the curse of dimensionality". Probabilistic methods on the other hand such as Monte-Carlo simulation are less sensitive to the dimension of the problem. It was demonstrated in Pardoux & Peng [17] and Cheridito, Soner, Touzi & Victoir [3] that first and second backward stochastic differential equations (in short BSDE) can provide stochastic representations that may be regarded as a non-linear generalisation of the classical Feynman-Kac formula for semi-linear and fully non-linear second order parabolic PDEs.

The numerical implementation of such a BSDE based scheme associated to a stochastic control problem was first proposed in Bouchard & Touzi [2], also independently in Zhang

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Further generalization was provided in Fahim, Touzi & Warin [8] and in Guyon & Henry-Labordère [10]. The algorithm in [10] requires evaluating high-dimensional conditional expectations, which are typically computed using parametric regression techniques. Solving the BSDE yields a sub-optimal estimation of the stochastic control. Performing an additional, independent (forward) Monte-Carlo simulation using this sub-optimal control, one obtains a biased estimation: a lower bound for the value of the underlying stochastic control problem. Choosing the right basis for the regression step is in practice a difficult task, particularly in high-dimensional settings. In fact, a similar situation arises for the familiar Longstaff-Schwarz algorithm, which also requires the computation of conditional expectations with parametric regressions and produces a low-biased estimate.

As the algorithm in [10] provides a biased estimate, i.e. a lower bound it is of limited use in practice, unless it can be combined with a dual method that leads to a corresponding upper bound.

Such a dual expression was obtained by Rogers [18], building on earlier work by Davis and Burstein [4]. While the work of Rogers is in the discrete time setting, it applies to a general class of Markov processes. Previous work by Davis and Burstein [4] linking deterministic and stochastic control using flow decomposition techniques (see also Diehl, Friz, Gassiat [5] for a rough path approach to this problem) is restricted to the control of a diffusion in its drift term. In the present paper we are also concerned with the control of diffusion processes, but allow the control to act on both the drift and the volatility term in the diffusion equation. The basic idea underlying the dual algorithm in all these works is to replace the stochastic control by a pathwise deterministic family of control problems that are not necessarily adapted. The resulting "gain" of information is compensated by introducing a penalisation analogous to a Lagrange multiplier. In contrast to [4], [5] we never consider continuous pathwise, i.e. deterministic, optimal control problems. Instead, we rely on a discretisation result for the HJB equation due to Krylov [13] and recover the solution of the stochastic control problem as the limit of deterministic control problems over a finite set of discretised controls.

Our paper is structured as follows. In Section 2 we introduce the stochastic control problem and derive the dual bounds in the Markovian setting for European type payoffs. The key analytic ingredient used in our estimates is Lemma 2.2, a discretisation result for the HJB equation due to Krylov [13]. In Section 3.1 we generalise our estimates to a non-Markovian setting, i.e. where the payoff has a path dependence. Finally, in Section 3.2 we consider a setting suitable for pricing American style options in a Markov setting. We evaluate the quality of the upper bounds obtained in two numerical examples. First, we consider the pricing of a variety of options in the uncertain volatility model. Based on our earlier estimates we transform the stochastic optimisation problem into a family of suitably discretised deterministic optimisations, which we can in turn approximate for example using local optimisation algorithms. Second, we consider the a problem arising in credit valuation adjustment. In this example, the deterministic optimisation can particularly efficiently be solved by deriving a recursive ODE solution to the corresponding Hamilton-Jacobi equations. Our algorithm
complements the lower bounds derived in [10] by effectively re-using some of the quantities already computed when obtaining the lower bounds (cf. Remark 2.7).

2 Duality result for European options

2.1 Notations

We begin by introducing some basic notations. For any $k \in \mathbb{N}$ let

$$\Omega^k := \{\omega : \omega \in C([0, T], \mathbb{R}^k), \omega_0 = 0\}.$$ 

Let $d, m \in \mathbb{N}$ and $T > 0$. Define $\Omega := \Omega^d$, $\Theta := [0, T] \times \Omega$ and let $B$ denote the canonical process on $\Omega^m$ with $F = \{F_t\}_{0 \leq t \leq T}$ the filtration generated by $B$. Finally, denote by $P_0$ the Wiener measure.

For $h > 0$, consider a finite partition $\{t^i_h\}_i$ of $[0, T]$ with mesh less than $h$, i.e. such that $t^i_{h+1} - t^i_h \leq h$ for all $i$. For some $M > 0$, let $A$ be a compact subset of $O_M := \{x \in \mathbb{R}^k : |x| \leq M\}$, for some $k \in \mathbb{N}$, and $N^h$ be a finite $h$-net of $A$, i.e. for all $a, b \in N^h \subset A$, we have $|a - b| \leq h$. We define sets:

- $A := \{\varphi : \Theta \to \mathbb{R}^k : \varphi$ is $\mathbb{F}$-adapted, and takes values in $A\}$;
- $A_h := \{\varphi \in A : \varphi$ is constant on $[t^i_h, t^i_{h+1})$ for $i$, and takes values in $N^h\}$;
- $\mathcal{U} := \{\varphi : \Theta \to \mathbb{R}^d : \varphi$ is bounded and adapted\};
- $\mathcal{D}_h := \{f : [0, T] \to \mathbb{R}^k : f$ is constant on $[t^i_h, t^i_{h+1})$ for $i$, and takes values in $N^h\}$.

For the following it is important to note that $\mathcal{D}_h$ is a finite set of piecewise constant functions.

2.2 The Markovian case

We consider stochastic control problems of the form:

$$u_0 = \sup_{\alpha \in A} \mathbb{E}^0_0 \left[ \int_0^T e^{-\int_0^t r(s, \alpha_s, X^\alpha_s)ds} f(t, \alpha_t, X^\alpha_t)dt + e^{-\int_0^T r(s, \alpha_s, X^\alpha_s)ds} g(X^\alpha_T) \right], \quad (2.1)$$

where $X^\alpha$ is a $d$-dimensional controlled diffusion defined by

$$X^\alpha := \int_0^t \mu(t, \alpha_t, X^\alpha_t)dt + \int_0^t \sigma(t, \alpha_t, X^\alpha_t)dB_t,$$

and the functions $\mu$, $\sigma$, $f$, $r$ satisfy the following assumption.

**Assumption 2.1** The functions $\mu, \sigma, f, r$ defined on $\mathbb{R}^+ \times A \times \mathbb{R}^d$ takes values in $\mathbb{R}^d, \mathbb{R}^{d \times m}, \mathbb{R}, \mathbb{R}$ respectively. Assume that $\mu, \sigma, f, r$ are uniformly bounded Hölder continuous in $t$, continuous in $\alpha$ and Lipschitz in $x$, uniformly in $(\alpha, x)$. Also assume that $g : \mathbb{R}^d \to \mathbb{R}$ is continuous.
Our main result is a duality in the spirit of [4] that allows us to replace the stochastic control problem by a family of suitably discretised deterministic control problems. The key analytic ingredient in our estimate is the following lemma which is a direct consequence of Theorem 2.3 in Krylov [13].

Define the function
\[ u_0^h := \sup_{\alpha \in A_h} E_0^h \left[ \int_0^T e^{-\int_0^t r(s, \alpha_s, X_{s}^\alpha)ds} f(t, \alpha_t, X_t^\alpha)dt + e^{-\int_0^T r(s, \alpha_s, X_{s}^\alpha)ds} g(X_T^\alpha) \right]. \]

Lemma 2.2 Suppose Assumption 2.1 holds and \( g \) is bounded. We have for any family of partition of \([0, T]\) with mesh tending to zero that
\[ u_0 = \lim_{h \to 0} u_0^h. \] (2.2)

Remark 2.3 Theorem 2.3 in [13] also gives a rate of convergence for the discretisation in Lemma 2.2. There exists a constant \( C > 0 \) such that
\[ |u_0 - u_0^h| \leq Ch^{\frac{1}{3}} \]
for all \( 0 < h \leq 1 \).

For the following statement, we introduce:
\[ v^h := \inf_{\varphi \in \mathcal{U}} E_0^h \left[ \sup_{a \in \mathcal{D}_h} \left\{ e^{-\int_0^T r(s, a_s, X_{s}^a)ds} g(X_T^a) + \int_0^T e^{-\int_0^t r(s, a_s, X_{s}^a)ds} f(t, a_t, X_t^a)dt \right. \right. \]
\[ - \left. \left. \int_0^T e^{-\int_0^t r(s, a_s, X_{s}^a)ds} \varphi(t, a_t, X_t^a) \sigma(t, a_t, X_t^a)dB_t \right]\right\}. \] (2.3)

Remark 2.4 It is noteworthy that stochastic integrals are defined in \( L^2 \)-space, so it is in general meaningless to take the pathwise supremum of a family of stochastic integrals. However, as we mentioned before, the set \( \mathcal{D}_h \) is of finite elements. So there is a unique random variable in \( L^2 \) equal to the maximum value of the finite number of stochastic integrals, \( F_0 \)-a.s.

The following theorem allows to recover the stochastic optimal control problem as a limit of discretised deterministic control problems.

Theorem 2.5 Suppose Assumption 2.1 holds and \( g \) is bounded. Then we have
\[ u_0 = \lim_{h \to 0} v^h. \]

Proof We first prove that \( u_0 \leq \lim_{h \to 0} v^h \). Recall \( u_0^h \) defined in (2.2). For all \( \varphi \in \mathcal{U} \), the process \( \int_0^T e^{-\int_0^t r(s, a_s, X_{s}^a)ds} \varphi(t, a_t, X_t^a) \sigma(t, a_t, X_t^a)dB_t \) is a martingale and we have
\[ u_0^h = \sup_{\alpha \in A_h} E_0^h \left[ e^{-\int_0^T r(s, a_s, X_{s}^a)ds} g(X_T^a) + \int_0^T e^{-\int_0^t r(s, a_s, X_{s}^a)ds} f(t, a_t, X_t^a)dt \right. \]
\[ \left. - \int_0^T e^{-\int_0^t r(s, a_s, X_{s}^a)ds} \varphi(t, a_t, X_t^a) \sigma(t, a_t, X_t^a)dB_t \right] \]
\[ \leq E_0^h \left[ \sup_{a \in \mathcal{D}_h} \left\{ e^{-\int_0^T r(s, a_s, X_{s}^a)ds} g(X_T^a) + \int_0^T e^{-\int_0^t r(s, a_s, X_{s}^a)ds} f(t, a_t, X_t^a)dt \right. \right. \]
\[ \left. \left. - \int_0^T e^{-\int_0^t r(s, a_s, X_{s}^a)ds} \varphi(t, a_t, X_t^a) \sigma(t, a_t, X_t^a)dB_t \right\}. \]
The desired result follows.

To show \( u_0 \geq \lim_{h \to 0} u^h \) we construct an explicit minimiser \( \varphi^* \). First note that under Assumption 2.1 it is easy to verify that \( u_t \) defined as

\[
\begin{align*}
  u_t(x) := \sup_{a \in A} \mathbb{E}^0 \left[ \int_0^T e^{-\int_0^T r(t, a_t, X^a_t)dt} f(s, a_s, X^a_s)ds + e^{-\int_0^T r(s, a_s, X^a_s)ds} g(X^a_T) | X^a_t = x \right],
\end{align*}
\]

is a viscosity solution to the Dirichlet problem of the HJB equation:

\[
- \partial_t u - \sup_{a \in A} L^a u = 0, \quad u_T = g,
\]

where \( L^a u := \mu(t, a, x) \cdot \partial_a u + \frac{1}{2} \text{Tr}(\sigma(t, a, x) \sigma^T(t, a, x) \partial_x^2 u) - \sigma(t, a, x) \partial_x u + f(t, a, x) \).

We next define the mollification \( u^{(\varepsilon)} := u \ast K^{(\varepsilon)} \) of \( u \), where \( K \) is a smooth function with compact support in \((-1,0) \times O_1 (O_1 \text{ the unit ball in } \mathbb{R}^d) \), and \( K^{(\varepsilon)}(x) := \varepsilon^{-d-2} K(t/\varepsilon^2 , x/\varepsilon) \).

Clearly, \( u^{(\varepsilon)} \in C^\infty_0 \) and \( u^{(\varepsilon)} \) converges uniformly to \( u \). Further, by a convexity argument as in Krylov [13] proof of Theorem 2.1, we obtain that \( u^{(\varepsilon)} \) is a classical supersolution to the HJB equation (2.4).

Consequently for all \( a \in A \), we have

\[
\begin{align*}
  I^{(\varepsilon)}_t := e^{-\int_0^t r(s, a_s, X^a_s)ds} g(X^a_t) + \int_0^t e^{-\int_0^s r(s, a_s, X^a_s)ds} f(t, a_t, X^a_t)dt \\
  - e^{-\int_0^t r(s, a_s, X^a_s)ds} u^{(\varepsilon)}(X^a_t) + u^{(\varepsilon)}_0 + \int_0^t e^{-\int_0^s r(s, a_s, X^a_s)ds} L^a u^{(\varepsilon)}(t, X^a_t)dt \\
  \leq e^{-\int_0^t r(s, a_s, X^a_s)ds} (g(X^a_t) - u^{(\varepsilon)}(X^a_t)) + u^{(\varepsilon)}_0.
\end{align*}
\]

By Assumption 2.1 it is clear that \( I^{(\varepsilon)}_t \) is uniformly bounded from above. It is easy to verify that the function \( u \) is continuous and therefore uniformly continuous on \( S_L := [0, T] \times \{ |x| \leq L \} \) for any \( L > 0 \) and \( u^{(\varepsilon)} \) converges uniformly to \( u \) on \( S_L \).

Letting

\[
\rho_L(\varepsilon) := \max_{|x| \leq L} |g(x) - u^{(\varepsilon)}_T(x)|
\]

we note that \( \lim_{\varepsilon \to 0} \rho_L(\varepsilon) = 0 \). Therefore,

\[
w^{(\varepsilon)} := \mathbb{E}^0 \left[ \sup_{a \in A} I^{(\varepsilon)}_t \right] = \mathbb{E}^0 \left[ \sup_{a \in A} I^{(\varepsilon)}_T \mid |X^a_T| \leq L \right] + \mathbb{E}^0 \left[ \sup_{a \in A} I^{(\varepsilon)}_T \mid |X^a_T| > L \right] \\
\leq C \rho_L(\varepsilon) + u^{(\varepsilon)}_0 + CP \left[ \sup_{a \in A} |X^a_T| > L \right],
\]

where \( C \) is a constant independent of \( L \) and \( \varepsilon \). Letting \( \varepsilon \) tend to zero we deduce that

\[
\lim_{\varepsilon \to 0} w^{(\varepsilon)} \leq u_0 + CP \left[ \sup_{a \in A} |X^a_T| > L \right]
\]

for any \( L > 0 \). Further, since \( \mathbb{P}_0 \left[ \sup_{a \in A} |X^a_T| > L \right] \to 0 \) as \( L \to \infty \), we conclude that

\[
\lim_{\varepsilon \to 0} w^{(\varepsilon)} \leq u_0.
\]

It follows from the Itô formula that

\[
\begin{align*}
  e^{-\int_0^T r(s, a_s, X^a_s)ds} u^{(\varepsilon)}_t(X^a_T) - u^{(\varepsilon)}_0 = \int_0^T e^{-\int_0^s r(r, a_r, X^a_r)ds} L^a u^{(\varepsilon)}(t, X^a_t)dt \\
  + \int_0^T e^{-\int_0^s r(s, a_s, X^a_s)ds} \partial_x u^{(\varepsilon)}(X^a_t)^T \sigma(t, a_t, X^a_t)dB_t, \quad \text{for all } a \in A_h, \ \mathbb{P}_0\text{-a.s.}
\end{align*}
\]
and, therefore,
\[
    w^h = \mathbb{E}^0 \left[ \sup_{a \in \mathcal{A}} \left\{ e^{-\int_0^T r(s,a_s,X_s^a)ds} g(X_T^a) + \int_0^T e^{-\int_0^s r(s,a_s,X_s^a)ds} f(t,a_t,X_t^a)dt \right. \right.
    \\
    \left. \left. \quad - \int_0^T e^{-\int_0^s r(s,a_s,X_s^a)ds} \partial_x u_t^c(X_t^a)^T \sigma(t,a_t,X_t^a)dB_t \right\} \right] \geq v^h. \tag{2.6}
\]

Combining (2.5) and (2.6), we conclude that \( v^h \leq u_0 \) for all \( 1 \geq h > 0 \).

The boundedness assumption on \( g \) may be relaxed by means of a simple cut off argument:

**Corollary 2.6** Assume that \( g \) is of polynomial growth. Let \( M > 0 \), \( g^M \) a continuous compactly supported function that agrees with \( g \) on \( O_M \subseteq \mathbb{R}^d \) and satisfies \( |g^M| \leq |g| \). Let \( v^{h,M} \) denote the approximations defined in (2.3), with respect to \( g^M \) in place of \( g \). Then we have
\[
    \lim_{M \to 0} |u_0 - \lim_{h \to 0} v^{h,M}| = 0.
\]

**Proof** Define \( u_0^M \) as in (2.1) by using the approximation \( g^M \). By Theorem 2.5 we know that \( u_0^M = \lim_{h \to 0} v^{h,M} \). Further, we have
\[
    |u_0 - u_0^M| \leq C \sup_{a \in \mathcal{A}} \mathbb{E}^0 \left[ g(X_T^a) - g^M(X_T^a) \right]
    \leq C \left. \sup_{a \in \mathcal{A}} \mathbb{E}^0 \left[ |X_T^a|^p + 1; |X_T^a| \geq M \right] \right.
    \leq \frac{C'}{M}.
\]

The last estimate is due the Chebyshev inequality and the moment estimate in Krylov [14] (Lemma 2 on page 78). The proof is completed.

We conclude the section with two remarks, both relevant to the numerical simulation of the approximation derived in Theorem 2.5.

**Remark 2.7** To approximate \( v^h \) in our numerical examples we will as in the proof of Theorem 2.5 use fixed functions \( \varphi^h \) for the minimisation. The calculations in (2.5) and (2.6) make it clear that the natural choice for these minimisers are the (numerical approximations) of the function \( \partial_x u_t \). Note that these approximations are readily available from the numerical scheme [11] that is used to compute the complementary lower bounds.

**Remark 2.8** In the proof of Theorem 2.5 we showed that \( u_0^h \leq v^h \leq u_0 \). It therefore follows from Remark 2.3 that there exists a constant \( C > 0 \) such that
\[
    |u_0 - v^h| \leq C h^{1/2}
\]
for all \( 0 < h \leq 1 \land T \).
3 Some extensions

3.1 The non-Markovian case

In our first extension we consider stochastic control problems of the form

$$u_0 = \sup_{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{P}_0} \left[ g(X^\alpha_{T_{\Lambda^\varepsilon}}) \right],$$

where $X^\alpha$ is a $d$-dimensional diffusion defined by $X^\alpha : \equiv \int_0^\cdot \mu(t, \alpha_t) dt + \int_0^\cdot \sigma(t, \alpha_t) dB_t$. Note that in this setting $\mu$ and $\sigma$ only depend on $\alpha$ and $t$, but the payoff function $g$ is path dependent.

Remark 3.1 The arguments in this subsection are based on the "path-freezing" approach developed in Ekren, Touzi and Zhang [6]. In order to be able to apply their approach we have restricted the class of diffusions $X^\alpha$ we consider compared to the Markovian control problem.

Writing $\mathbb{P}_\alpha : = \mathbb{P}_0 \circ (X^\alpha)^{-1}$, we have

$$u_0 = \sup_{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{P}_\alpha} \left[ g(B_{T_{\Lambda^\varepsilon}}) \right].$$

Throughout this subsection we will impose the following regularity assumptions.

Assumption 3.2 The functions $\mu, \sigma : \mathbb{R}^+ \times A \to E$ ($E$ is the respective metric space) and $g : \Omega^d \to \mathbb{R}$ are uniformly bounded such that

(i) $\mu, \sigma$ are Hölder continuous in $t$, continuous in $\alpha$;
(ii) $g$ is uniformly continuous.

Example 3.3 Arguing as in Corollary 2.6 we may also consider unbounded payoffs. Hence, possible path-dependent payoffs that fit our framework include e.g. the maximum $\max_{s \in [0,T]} \omega_s$ and Asian options $\int_0^T \omega_s ds$.

Let $\Lambda^\varepsilon : = \{ t_0 = 0, t_1, t_2, \cdots, t_n = T \}$ be a partition of $[0,T]$ with mesh bounded above by $\varepsilon$. For $k \leq n$ and $\pi_k = (x_1 = 0, x_2, \cdots, x_k) \in \mathbb{R}^{d \times k}$, denote by $\Gamma^{\varepsilon,k}_{\Lambda^\varepsilon} (\pi_k)$ the path generated by the linear interpolation of the points $\{(t_i, x_i)\}_{0 \leq i \leq k}$. Where no confusion arises with regards to the underlying partition we will in the following drop the superscript $\Lambda^\varepsilon$ and write $\Gamma^{\varepsilon,k}_k (\pi_k)$ in place of $\Gamma^{\varepsilon,k}_{\Lambda^\varepsilon} (\pi_k)$, but it must be emphasised that the entire analysis in this subsection is carried out with a fixed but arbitrary partition $\Lambda^\varepsilon$ in mind. Define the interpolation approximation of $g$ by

$$g^\varepsilon (\pi_n) : = g \left( \Gamma^{\varepsilon,k}_n (\pi_n) \right)$$

and define an approximation of the value function by letting

$$\theta^\varepsilon_0 : = \sup_{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{P}_\alpha} \left[ g^\varepsilon (B_{\pi_n}) \right].$$

The following lemma justifies the use of linear interpolation for approximating dependent payoff.
Lemma 3.4 Under Assumption 3.1, we have
\[ \lim_{\varepsilon \to 0} \theta^\varepsilon_0 = u_0. \]

Proof Recall that \( g \) is uniformly continuous. Let \( \rho \) be a modulus of continuity of \( g \). If necessary, we may choose \( \rho \) to be concave. Further, we define
\[ w_B(\varepsilon, T) := \sup_{s \leq t \leq T \mid t - s \leq \varepsilon} |B_s - B_t|. \]
Clearly, we have
\[ \left| \sup_{a \in A} \mathbb{E}^\pi_a \left[ g^\varepsilon ((B_{t_0})_{0 \leq t \leq \varepsilon}) \right] \right| \leq \sup_{a \in A} \mathbb{E}^\pi_a \left[ \rho(w_B(\varepsilon, T)) \right] \leq \rho \left( \sup_{a \in A} \mathbb{E}^\pi_a [w_B(\varepsilon, T)] \right). \]
It follows from Theorem 1 in Fisher and Nappo [9] that \( \mathbb{E}^\pi_a [w_B(\varepsilon, T)] \) converges to 0 uniformly in \( \alpha \), as \( \varepsilon \to 0. \)

We next define the controlled diffusion with time-shifted coefficients by setting
\[ X^{\alpha,t} := \int_0^t \mu(t + r, \alpha_r) dr + \int_0^t \sigma(t + r, \alpha_r) dB_r, \quad s \in [0, T - t], \ P_0\text{-a.s.}, \]
and the corresponding law:
\[ \mathbb{P}^\alpha := \mathbb{P}_0 \circ (X^{\alpha,t})^{-1}. \]
Further, for \( 1 \leq k \leq n - 2 \) let \( \eta_k := t_{k+1} - t_k \), and define recursively a family of stochastic control problems:
\[ \theta^\varepsilon(\pi_{n-1}; t, x) := \sup_{a \in A} \mathbb{E}^{\pi_{n-1}, t, x} \left[ g^\varepsilon (\pi_{n-1, t, x}^a + x + B_{\eta_{n-1} - t}) \right], \ t \in [0, \eta_{n-1}], \ x \in \mathbb{R}^d \]
\[ \theta^\varepsilon(\pi_k; t, x) := \sup_{a \in A} \mathbb{E}^{\pi_{k}, t, x} \left[ \theta^\varepsilon (\pi_k, x_k + x + B_{\eta_k - t}) \right], \ t \in [0, \eta_k], \ x \in \mathbb{R}^d. \]
Clearly, \( \theta^\varepsilon(0, 0, 0) = \theta^\varepsilon_0. \)

Lemma 3.5 Fix \( \varepsilon > 0. \) The function \( \theta^\varepsilon(\pi; t, x) \) is Borel-measurable in all the arguments and uniformly continuous in \( (t, x) \) uniformly in \( \pi. \)

Proof It follows from the uniform continuity of \( g \) and the fact that interpolation with respect to a partition \( \Lambda \varepsilon \) is a Lipschitz function (in this case from \( \mathbb{R}^{n \times d} \) into the continuous functions), that \( g^\varepsilon \) is also uniformly continuous. Denote by \( \rho^\varepsilon \) a modulus of continuity of \( g^\varepsilon \), chosen to be increasing and concave if necessary. For any \( \pi_{n-1}, \pi'_{n-1} \in \mathbb{R}^{(n-1) \times d} \), given \( t \in [0, \eta_{n-1}], \ x, x' \in \mathbb{R}^d \), we have
\[ \left| \theta^\varepsilon(\pi_{n-1}; t, x) - \theta^\varepsilon(\pi'_{n-1}; t, x') \right| \leq \sup_{a \in A} \mathbb{E}^{\pi_{n-1}, t, x} \left[ \left| g^\varepsilon (\pi_{n-1, t, x}^a + x + B_{\eta_{n-1} - t}) - g^\varepsilon (\pi'_{n-1, t, x'}^a + x' + B_{\eta_{n-1} - t}) \right| \right] \leq \rho^\varepsilon (|\pi_{n-1} - \pi'_{n-1}|, |x - x'|). \]
Similarly, for any \( k < n - 1 \) and \( \pi_k, \pi'_k \in \mathbb{R}^{k \times d} \), given \( t \in [0, \eta_k], \ x, x' \in \mathbb{R}^d \), we have
\[ \left| \theta^\varepsilon(\pi_k; t, x) - \theta^\varepsilon(\pi'_k; t, x') \right| \leq \sup_{a \in A} \mathbb{E}^{\pi_k, t, x} \left[ \left| \theta^\varepsilon (\pi_k, x_k + x + B_{\eta_k - t}) - \theta^\varepsilon (\pi'_k, x_k + x' + B_{\eta_k - t}) \right| \right] \leq \rho^\varepsilon (|\pi_k - \pi'_k|, |x - x'|). \]
(3.2)
For $0 \leq t^0 < t^1 \leq \eta_k$, it follows from the dynamic programming principle that
\[
\theta^\varepsilon(\pi_k; t^0, x) = \sup_{\alpha \in A} E^{\delta_{t^0}} \left[ \theta^\varepsilon(\pi_k; t^1, x + B_{t^1 - t^0}) \right]
\] (3.3)
and (3.3) and (3.2) we deduce that
\[
|\theta^\varepsilon(\pi_k; t^0, x) - \theta^\varepsilon(\pi_k; t^1, x)| \leq \sup_{\alpha \in A} E^{\delta_{t^0}} \left[ |\theta^\varepsilon(\pi_k; t^1, x + B_{t^1 - t^0}) - \theta^\varepsilon(\pi_k; t^1, x)| \right]
\]
\[
\leq \sup_{\alpha \in A} E^{\delta_{t^0}} \left[ \rho^\varepsilon(|B_{t^1 - t^0}|) \right]
\]
\[
\leq \rho^\varepsilon \left( \sup_{\alpha \in A} E^{\delta_{t^0}} \left[ \sup_{|\cdot| \leq C} |B_{t^1 - t^0}| \right] \right).
\] (3.4)
For any $\hat{\mu}$ and $\hat{\sigma}$ satisfying Assumption 3.1, define the controlled diffusion and the corresponding law:
\[
X^t_{\hat{\mu}, \hat{\sigma}} = \int_0^t \hat{\mu} \, ds + \int_0^t \hat{\sigma} dB_s, \quad \mathbb{P}^{\hat{\mu}, \hat{\sigma}} := \mathbb{P}_0 \circ (X^\hat{\mu}, \hat{\sigma})^{-1}.
\]
Note that the bound
\[
\sup_{\alpha \in A} E^{\delta_{t^0}} \left[ |B_{t^1 - t^0}| \right] \leq \sup_{|\cdot| \leq C} E^{\rho^\varepsilon} \left[ |B_{t^1 - t^0}| \right]
\] (3.5)
do not depend on $\pi_k$ and $t_0$. It follows from (3.4) that
\[
|\theta^\varepsilon(\pi_k; t^0, x) - \theta^\varepsilon(\pi_k; t^1, x)| \leq \rho^\varepsilon \left( \sup_{|\cdot| \leq C} E^{\rho^\varepsilon} \left[ |B_{t^1 - t^0}| \right] \right).
\] (3.6)
Hence, combining (3.2) and (3.3) we conclude that $\theta^\varepsilon(\pi_k; t, x)$ is uniformly continuous in $(t, x)$ uniformly in $\pi_k$.

The functions $\theta^\varepsilon(\pi_k; \cdot, \cdot)$ are defined as the value functions of stochastic control problems, and one can easily check that they are viscosity solutions to the corresponding Hamilton-Jacobi-Bellman equations. For $j = 1, \ldots, n - 1$, we define a family of PDEs by letting
\[
L_t^j \theta = 0, \quad \text{on } [0, \eta_j) \times \mathbb{R}^d,
\]
where $L_t^j \theta := -\partial_t \theta - \sup_{\alpha \in A} \left\{ \mu(t_j + \cdot, \alpha) \cdot \partial_\alpha \theta + \frac{1}{2} \text{Tr}((\sigma(t_j + \cdot, \alpha) \sigma^T(t_j + \cdot, \alpha)) \partial^2 \theta) \right\}$.
\] (3.7)
The following proposition links the stochastic control problems with the PDE and applies, analogous to the Markovian case, a mollification argument.

**Proposition 3.6** There exists a function $u^{(c)} : (\pi, t, x) \mapsto \mathbb{R}$ such that $u^{(c)}(0, 0, 0) = \theta_0 + \varepsilon$ and for all $\pi_k \in \Pi_\mathcal{E}$, $u^{(c)}(\pi_k; \cdot, \cdot)$ is a classical supersolution to the PDE (3.7) with $j = k$ and the boundary condition:
\[
u^{(c)}(\pi_k; \eta_k, x) = u^{(c)}((\pi_k, x); 0, 0), \quad \text{if } k < n - 1;
\]
\[
u^{(c)}(\pi_k; \eta_k, x) \geq g^{\varepsilon}((\pi_k, x)), \quad \text{if } k = n - 1.
\]

**Proof** Define $\theta^\varepsilon(\pi_k; \cdot, \cdot) := \theta^\varepsilon(\pi_k; \cdot, \cdot) \ast K^\delta$ for all $\pi_k \in \mathbb{R}^{k \times d}$, $k \leq n$, where $K$ is a smooth function with compact support in $(-1, 0) \times O_1$ ($O_1$ is the unit ball in $\mathbb{R}^d$), and $K^\delta(t, x) := \delta^{-d-2}K(t/\delta^2, x/\delta)$. By Lemma 3.5 $\theta^\varepsilon(\pi_k, \cdot, \cdot)$ converges uniformly to $\theta^\varepsilon(\pi_k; \cdot, \cdot)$ uniformly in $\pi_k$, as $\delta \to 0$. Take $\delta$ small enough so that $||\theta^\varepsilon - \theta^\varepsilon|| \leq \frac{\varepsilon}{\mathcal{E}}$. As in the Markovian case (compare the proof of Theorem 2.5) using a convexity argument analogous Krylov [13], we can
prove that \( \theta^{\varepsilon, \delta}(\pi_k; \cdot, \cdot) \) is a classical supersolution for (3.7). Note that \( \theta^{\varepsilon, \delta}(\pi_k; \cdot, \cdot) + c \) is still a supersolution for any constant \( c \). So there exists a smooth function \( v^\varepsilon(0; \cdot, \cdot) \) on \([0, t_1] \times \mathbb{R}^d\) such that

\[
v^\varepsilon(0; 0, 0) = \theta^\varepsilon(0; 0, 0) + \frac{\varepsilon}{n}, \quad v^\varepsilon(0; \cdot, \cdot) \geq \theta^\varepsilon(0; \cdot, \cdot)
\]

and smooth functions \( v^\varepsilon(\pi_k; \cdot, \cdot) \) on \([0, \eta_k] \times \mathbb{R}^d\) for \( 1 \leq k \leq n - 1 \) such that

\[
v^\varepsilon(\pi_k; 0, 0) = v^\varepsilon(\pi_{k-1}; \eta_{k-1}, x_{k} - x_{k-1}) + \frac{\varepsilon}{n}, \quad v^\varepsilon(\pi_k; \cdot, \cdot) \geq \theta^\varepsilon(\pi_k; \cdot, \cdot).
\]

Finally, we define for \( \pi_k \in \mathbb{R}^{k \times d} \) and \((t, x) \in [0, \eta_k] \times \mathbb{R}^d\)

\[
u^\varepsilon(\pi_k; t, x) := v^\varepsilon(\pi_k; t, x) + \frac{n - k + 1}{n} \varepsilon.
\]

It is now straightforward to check that \( u^\varepsilon \) satisfies the requirements.

The discrete framework we just developed may be linked to pathspace by means of linear interpolation along the partition \( \Lambda_n \). Recall that \( \Theta \) was defined to be \([0, T] \times \Omega\).

**Corollary 3.7** Define \( \tilde{u}^\varepsilon : \Theta \rightarrow \mathbb{R} \) by

\[
\tilde{u}^\varepsilon(t, \omega) := u^\varepsilon((\omega_t)_{0 \leq i \leq k}; t - t_k, \omega_t - \omega_{t_k}), \quad \text{for } t \in [t_k, t_{k+1}).
\]

There exist adapted processes \( \lambda(t, \alpha), \mu(t, \alpha), \varphi_t(x), \eta(t, x) \) such that for all \( \alpha \in \mathcal{A} \)

\[
\tilde{u}^\varepsilon(T, X^\alpha) = u_0^\varepsilon + \int_0^T \left( \lambda + \mu(t, \alpha) \varphi + \frac{1}{2} \text{Tr}(\sigma \sigma^T(t, \alpha) \eta)(t, X^\alpha) \right) dt + \int_0^T \varphi_t(X^\alpha)^T \sigma(t, \alpha) dB_t,
\]

\( \mathbb{P}_0 \)-a.s., and

\[
(\lambda + \mu(t, \alpha) \varphi + \frac{1}{2} \text{Tr}(\sigma \sigma^T(t, \alpha) \eta)(t, \omega) \leq 0, \quad \text{for all } \alpha \in \mathcal{A}, (t, \omega) \in \Theta.
\]

**Proof** The result follows by applying Itô’s formula on each interval \([t_k, t_{k+1})\) and using the supersolution property of \( u^\varepsilon \) in Proposition 3.6.

Finally, we prove an approximation analogous to Theorem 2.5 in our non-Markovian setting.

**Theorem 3.8** Suppose Assumption 3.3 holds. Then we have

\[
u_0 = \lim_{h \rightarrow 0} v^h, \quad \text{where } v^h := \inf_{\varphi \in \mathcal{D}^h} \mathbb{E}^{\mathbb{P}_0} \left[ \sup_{a \in \mathcal{A}_h} \left\{ g(X^\alpha_{T \land}) - \int_0^T \varphi_t^T(X^\alpha) \sigma(t, a_t) dB_t \right\} \right].
\]

**Proof** Arguing as in the proof of Theorem 2.5 one can easily deduce using the Ito formula that \( u_0 \leq \lim_{h \rightarrow 0} v^h \).

Consider the function \( \tilde{u}^\varepsilon \) and let \( \varphi \) be the process defined in Corollary 3.7. We have

\[
v_h \leq \mathbb{E}^{\mathbb{P}_0} \left[ \sup_{a \in \mathcal{A}_h} \left\{ g(X^\alpha_{T \land}) - \int_0^T \varphi_t^T(X^\alpha) \sigma(t, a_t) dB_t \right\} \right] \\
\leq \mathbb{E}^{\mathbb{P}_0} \left[ \sup_{a \in \mathcal{A}_h} \left\{ g(X^\alpha_{T \land}) - \tilde{u}^\varepsilon(X^\alpha) + u_0^\varepsilon \right\} \right] \\
\leq \mathbb{E}^{\mathbb{P}_0} \left[ \sup_{a \in \mathcal{A}_h} \left\{ g(X^\alpha_{T \land}) - g^\varepsilon((X^\alpha_t)_{0 \leq i \leq n}) \right\} \right] + \theta_0 + \varepsilon.
\]
For the last inequality, we use the fact that $u_0^{(ε)} = u^{(ε)}(0; 0, 0) = θ_0^ε + ε$. Note that there are only finite elements in the set $D_h$. Therefore, by Lemma 3.4

$$\lim_{ε \to 0} \left( \mathbb{E}^{P_0} \left[ \sup_{a \in D_h} \left\{ g(X^a_{T \wedge}) - g^ε((X^a_{t_i})_{0 \leq t_i \leq n}) \right\} \right] + θ_0^ε + ε \right) \leq \lim_{ε \to 0} \left( \sum_{a \in D_h} \mathbb{E}^{P_0} \left[ \left| g(X^a_{T \wedge}) - g^ε((X^a_{t_i})_{0 \leq t_i \leq n}) \right| \right] + θ_0^ε + ε \right)$$

$$\leq u_0.$$ We conclude that $v^h \leq u_0$ for all $h \in (0, 1 \wedge T)$.

3.2 Example of a duality result for an American option

In this subsection we give an indication how our approach may be extended to American options. To this end we consider a toy model, in which the $d$-dimensional controlled diffusion $X^a$ takes the particular form $X^a := \int_0^T \alpha^0_t dt + \int_0^T \alpha^1_t dB_t$ and carry out the analysis in this elementary setting. The stochastic control problem is now

$$u_0 = \sup_{\alpha \in A, \tau \in T} \mathbb{E}^{P_0}[g(X^a_{\tau})],$$

where $T_T$ is the set of all stopping times smaller than $T$. Throughout this subsection we will make the following assumption:

**Assumption 3.9** Suppose $g : \mathbb{R}^d \to \mathbb{R}$ to be bounded and uniformly continuous.

For $\alpha \in A$ define probability measures $P_\alpha := P_0 \circ (X^a)^{-1}$, let $P := \{P_\alpha : \alpha \in A\}$ and define the nonlinear expectation $\mathbb{E}[: \mathbb{E}^P[\cdot]] := \sup_{P \in P} \mathbb{E}^P[\cdot]$. It will be convenient to use the shorthand $\alpha^1 \cdot B$ for the stochastic integral $\int_0^T \alpha^1_s dB_s$. We have

$$u_0 = \sup_{\tau \in T_T} \mathbb{E}^P[g(B_{\tau})].$$

Further, we define the dynamic version of the control problem:

$$u_t(x) := \sup_{\tau \in T_{T-t}} \mathbb{E}^P[g(x + B_{\tau})], \quad \text{for } (t, x) \in [-1, T] \times \mathbb{R}^d.$$

The following lemma shows that the function $u$ satisfies a dynamic programming principle (see for example Lemma 4.1 of [7] for a proof).

**Lemma 3.10** The value function $u$ is continuous in both arguments, and we have

$$u_{t_1}(x) = \sup_{\tau \in T_{T-t_1}} \mathbb{E}^P[g(x + B_{\tau})1_{\tau < t_2} + u_{t_2}1_{\tau \geq t_2}].$$

In particular, $u$ is a $P$-supermartingale for all $P \in P$.

Next we apply the familiar mollification technique already employed in Section 2.2. Define $u^{(ε)} := u \ast K^{(ε)}$.

**Lemma 3.11** $\{u^{(ε)}(t, B_t)\}_t$ is a $P$-supermartingale for all $P \in P$, and $u^{(ε)} \geq g^{(ε)} := g \ast K^{(ε)}$. 
Proof  For any $s \leq t \leq T$ and $x \in \mathbb{R}$, we have by Lemma 4.10

$$
\mathcal{E}[u^{(e)}(t, x + B_{t-s})] = \mathcal{E}\left[ \int u(t-r, x-y + B_{t-s})K^{(e)}(r,y)dydr \right] \\
\leq \int \mathcal{E}[u(t-r, x-y + B_{t-s})]K^{(e)}(r,y)dydr \\
\leq \int u(s-r, x-y)K^{(e)}(r,y)dydr = u^{(e)}(s, x).
$$

This implies that for all $P \in \mathcal{P}$ we have

$$
\mathbb{E}^P[u^{(e)}(t, x + B_{t-s})] \leq u^{(e)}(s, x).
$$

Therefore, $\{u^{(e)}(t, B_t)\}_t$ is a $\mathbb{P}$-supermartingale for all $\mathbb{P} \in \mathcal{P}$. On the other hand, it is clear from the definition of $u$ that $u \geq g$ and, hence, $u^{(e)} \geq g^{(e)}$.

Again, the stochastic control problem can be discretised.

**Lemma 3.12** Under Assumption 3.9, it holds

$$
u_0 = \lim_{h \to 0} u_0^h, \quad \text{where } u_0^h := \sup_{\alpha \in \mathcal{A}_h, \tau \in T_h} \mathbb{E}^P_0 \left[ g(X^\alpha_\tau) \right]. \tag{3.8}
$$

**Proof** We only prove the case $\alpha^0 = 0$ and $\alpha = \alpha^1 \in \mathbb{R}$, i.e. $X^\alpha = (\alpha \cdot B)$. The general case follows by a straightforward generalisation of the same arguments. Note that it is sufficient to show that $u_0 \leq \lim_{h \to 0} u_0^h$. Fix $\varepsilon > 0$. There exists $\alpha^\varepsilon \in \mathcal{A}$ such that

$$
u_0 < \sup_{\tau \in T_h} \mathbb{E}^P_0 \left[ g((\alpha^\varepsilon \cdot B)_{\tau}) \right] + \varepsilon. \tag{3.9}
$$

For any $h$ sufficiently small define a process $\hat{\alpha}^h$ by letting

$$
\hat{\alpha}^h_t := \sum_{i} \frac{1}{h} \int_{t_i}^{t_{i+1}} \mathbb{E}^P_0 (\alpha^\varepsilon_s) ds 1_{[t_i, t_{i+1})}(t).
$$

Clearly, $\hat{\alpha}^h$ is piecewise constant on each interval $[t_i, t_{i+1})$. Further, define $\tilde{\alpha}^h := h \left[ \frac{\hat{\alpha}^h}{h} \right]$ and note that we have $\tilde{\alpha}^h \in \mathcal{A}_h$. A standard argument using the martingale convergence theorem yields

$$
\lim_{h \to 0} \mathbb{E}^P_0 \int_0^T (\alpha^\varepsilon_s - \tilde{\alpha}^h_s)^2 ds = 0 \tag{3.10}
$$

and, hence,

$$
\lim_{h \to 0} \mathbb{E}^P_0 \int_0^T (\alpha^\varepsilon_s - \hat{\alpha}^h_s)^2 ds = 0.
$$

With $\rho$ an increasing and concave modulus of continuity of $g$ we have

$$
\sup_{\tau \in T_h} \mathbb{E}^P_0 \left[ g((\alpha^\varepsilon \cdot B)_{\tau}) \right] \leq \sup_{\tau \in T_h} \mathbb{E}^P_0 \left[ g((\tilde{\alpha}^h \cdot B)_{\tau}) \right] \leq \mathbb{E}^P_0 \left[ \rho(\|\alpha^\varepsilon - \tilde{\alpha}^h\|) \right] \leq \rho \left( \mathbb{E}^P_0 \left[ \int_0^T (\alpha^\varepsilon_s - \hat{\alpha}^h_s)^2 ds \right] \right) \tag{3.11}
$$
Combining (3.9), (3.11) we have

\[ u_0 < \sup_{r \in T} \mathbb{E}^0 \left[ g \left( (\alpha^h \cdot B)_r \right) \right] + \rho \left( \mathbb{E}^0 \left[ \int_0^T (\alpha^h_s - \bar{\alpha}^h_s)^2 ds \right] \right) + \varepsilon \]

\[ \leq u_0^h + \rho \left( \mathbb{E}^0 \left[ \int_0^T (\alpha^h_s - \bar{\alpha}^h_s)^2 ds \right] \right) + \varepsilon. \]

Letting \( h \to 0 \) we deduce

\[ u_0 \leq \lim_{h \to 0} u_0^h + \varepsilon. \]

for all \( \varepsilon > 0. \)

We conclude the section by proving the analogous approximation result for American options.

**Theorem 3.13** Suppose Assumption 3.9 holds. Then we have

\[ u_0 = \lim_{h \to 0} v^h, \text{ where } v^h := \inf_{\varphi \in \mathcal{U}} \mathbb{E}^0 \left[ \sup_{t \in [0,T]} \left\{ g(X^a_t) - \int_0^T \varphi^T(X^a_t) \alpha_s dB_s \right\} \right]. \]

**Proof** We first prove that the left hand side is smaller. Recall \( u_0^h \) defined in (3.8). For all \( \varphi \in \mathcal{U} \), the process \( \int_0^t \varphi(X^a)^T \alpha_s dB_s \) is a martingale, and we have

\[ u_0^h \leq \sup_{t \in [0,T]} \mathbb{E}^0 \left[ g(X^a_t) - \int_0^t \varphi(X^a_t)^T \alpha_s dB_s \right] \]

\[ \leq \mathbb{E}^0 \left[ \sup_{t \in [0,T]} \left\{ g(X^a_t) - \int_0^t \varphi(X^a_t)^T \alpha_s dB_s \right\} \right], \text{ for all } \varphi \in \mathcal{U}. \]

The desired result follows by Lemma 3.12. For the converse note that since \( u^{(c)} \in C^{1,2} \) and \( u^{(c)}(t, B_t) \) is a \( \mathbb{P} \)-supermartingale for all \( \mathbb{P} \in \mathcal{P} \), we have

\[ \partial_s u^{(c)} + \sup_{a \in A} \left\{ a^0 \partial_s u^{(c)} + \frac{1}{2} \text{Tr} (a_s^1 a_s^1 T \partial_s^2 u^{(c)}) \right\} \leq 0. \]

Hence, for all \( h > 0 \)

\[ u_h \leq \sup_{t \in [0,T]} \left\{ g(X^a_t) - \int_0^t \partial_s u^{(c)}(X^a_s)^T \alpha_s dB_s \right\} \]

\[ \leq \mathbb{E}^0 \left[ \sup_{t \in [0,T]} \left\{ g(X^a_t) - u^{(c)}(X^a_t) + u_0^{(c)} \right. \right. \]

\[ + \int_0^t \left( \partial_s u^{(c)}(X^a_s) + \partial_s u^{(c)}(X^a_s) \alpha_s \right) + \frac{1}{2} \text{Tr} (a_s^1 a_s^1 T \partial_s^2 u^{(c)}(X^a_s)) ds \right\} \]

\[ \leq \mathbb{E}^0 \left[ \sup_{t \in [0,T]} \left\{ g(X^a_t) - g^{(c)}(X^a_t) \right\} + u_0^{(c)} \right], \]

where we have used Ito's formula and the inequality \( u^{(c)} \geq g^{(c)} \) proved in Lemma 3.11. It is straightforward to check that

\[ \lim_{\varepsilon \to 0} \left[ \mathbb{E}^0 \left[ \sup_{t \in [0,T]} \left\{ g(X^a_t) - g^{(c)}(X^a_t) \right\} + u_0^{(c)} \right] \right] = u_0. \]
4 Examples:

4.1 Uncertain volatility model

As a first example, we consider an uncertain volatility model (UVM), first considered in [1] and [16]. We will consider a range of options with payoff \( F_T \) at a maturity \( T \). Let \( D \subseteq \mathbb{R}^d \times \mathbb{R}^d \times d \) be a compact domain such that for all \( \xi := (\sigma^i, \rho^{ij})_{1 \leq i,j \leq d} \in D \) the matrix

\[
\left( \rho^{ij} \sigma^i \sigma^j \right)_{1 \leq i,j \leq d}
\]

is positive semi-definite, \( \rho^{ij} = \rho^{ji} \in [-1, 1] \) and \( \rho^{ii} = 1 \). If \( d = 2 \) an example of such a domain is obtained by setting

\[
D = \left( \prod_{i=1}^2 \left[ \sigma_i, \overline{\sigma} \right] \right) \times \left\{ \left( \frac{1}{\rho}, \rho \right) : \rho \in [\underline{\rho}, \overline{\rho}] \right\},
\]

where \( 0 \leq \sigma_i \leq \overline{\sigma} \) and \( -1 \leq \rho \leq \overline{\rho} \leq 1 \). An adapted process \( (\sigma, \rho) = (\sigma_t, \rho_t)_{0 \leq t \leq T} \) is in the set of admissible volatility processes \( \Xi_D \) if it takes values in \( D \).

In the UVM the stock prices follow the dynamics

\[
dX_i^t = \sigma_i X_i^t dW_i^t, \quad d\langle W_i, W_j \rangle^t = \rho^{ij} dt, \quad 1 \leq i < j \leq d,
\]

where \( \sigma, \rho \in \Xi_D \) is the unknown volatility process and correlation. The time-\( t \) value of the option in the UVM, interpreted as a super-replication price under uncertain volatilities, is given by

\[
u_t = \sup_{(\sigma, \rho) \in \Xi_D} \mathbb{E}^Q [F_T | \mathcal{F}_t].
\]

For European style payoffs \( F_T = g(X_T) \), the value \( \nu(t, x) \) is then the unique viscosity solution (under suitable growth condition on \( g \)) of the nonlinear PDE:

\[
\partial_t \nu(t, x) + H(x, D^2 \nu(t, x)) = 0, \quad \nu(T, x) = g(x)
\]

with the Hamiltonian

\[
H(X, \Gamma) = \frac{1}{2} \max_{(\sigma^i, \rho^{ij})_{1 \leq i,j \leq d}} \sum_{i,j=1}^d \rho^{ij} \sigma^i \sigma^j X_i^t \Gamma^{ij}.
\]

Denote by \( S^d \) the space of symmetric \( d \times d \) matrices. The 2-BSDE associated to this PDE has driver

\[
f(x, \Gamma) = \frac{1}{2} \max_{(\sigma^i, \rho^{ij})_{1 \leq i,j \leq d} \in D} \sum_{i,j=1}^d \rho^{ij} \sigma^i \sigma^j x^i x^j
\]

and dynamics

\[
\begin{align*}
\hat{X}_i^t &= \hat{\sigma}^i \hat{X}_i^t dW_i^t, \quad dW_i^t dW_j^t = \rho^{ij} dt, \quad 1 \leq i < j \leq d, \\
\hat{Y}_i^t &= -f \left( \hat{X}_i^t, \Gamma_t \right) dt + \sum_{i=1}^d Z_i^t \circ \hat{\sigma}^i \hat{X}_i^t dW_i^t, \\
\hat{Z}_i^t &= A_i^t dt + \sum_{j=1}^d \Gamma_i^{ij} \hat{\sigma}^j \hat{X}_i^t dW_j^t, \\
Y_T &= g \left( \hat{X}_T \right).
\end{align*}
\]
Note that $\tilde{X}_t$ follows a log normal diffusion with some admissible variance $\tilde{\sigma}$ (we are free to choose e.g. $\tilde{\sigma} = (\sigma + 2)/2$) and some constant correlation $\tilde{\rho}$. Let $(Y_t, Z_t, \Gamma_t, A_t)$ be the quadruple of adapted processes taking values in \( \mathbb{R}, \mathbb{R}^d, S^d \) and \( \mathbb{R}^d \) respectively, solving the 2-BSDE (in the sense of [3]). For details regarding existence and uniqueness of this 2-BSDE solution we refer to [10, p.52f] and the references therein.

We recall a numerical scheme based on this second-order backward stochastic differential equation that has been proposed in [8] and [10]. For the dual bounds derived in this paper we next determine the numerical approximation for \( \nabla_x u \), which may be computed using the sub optimal control, we obtain a lower bound $u_0^{LS} \leq u_0$. So far the primal algorithm developed in [10]. For the dual bounds derived in this paper we next determine the numerical approximation for $\nabla_x u$, which will serve as the minimiser $\varphi^*$, which may be computed using the relation

$$\tilde{\sigma}^j \hat{X}_0^j (\varphi^*_i) = \mathbb{E}^p_{i-1}[Y_i U_i]$$

and letting

$$\varphi^*_i = \sum_{i=1}^N \varphi^*_{i-1}(\hat{X}_{i-1}(\hat{X}_i - \hat{X}_{i-1})$$

Below, we denote $Y_0 = u_0^{BSDE}$. Using our candidate $\varphi^*$ in the minimisation, we get an upper bound

$$u_0^{LS} \leq u_0 \leq u_0^{dual} \equiv \lim_{N \to \infty} \mathbb{E} \left[ \sup_{(\sigma, \rho) \in D^N} \{g(\hat{X}_N) - \sum_{i=1}^N \varphi^*_{i-1}(\hat{X}_{i-1})(\hat{X}_i - \hat{X}_{i-1})\} \right]$$

where

$$\hat{X}_i = \hat{X}_0 e^{-(\sigma^2 + \sigma^2 W^k_i)}$$

4.1.1 The algorithm

The algorithm can be summarized by the following four steps:

1. Simulate $N_1$ replications of $X$ with a lognormal diffusion (we choose $\tilde{\sigma} = (\sigma + 2)/2$).
2. Apply the backward algorithm using a regression approximation. Basis coefficients for the Delta at each discretization time are stored. Compute $Y_0 = u_0^{\text{BSDE}}$.

3. Simulate $N_2$ independent replication of $X$ using the sub-optimal controls. Give a low-biased estimate $u_0^{\text{LS}}$.

4. Simulate independent increment $\Delta W_t$ and optimize $g(X_{tN}) = \sum_{i=1}^N \varphi^*_t(X_{t_{i-1}}) (X_{t_i} - X_{t_{i-1}})$ over $(\sigma)$. In our numerical experiments, as the payoff may be non-smooth, we have used a direct search polytope algorithm. Then average.

4.1.2 Numerical experiments

In our experiments, we take $T = 1$ year and for each asset $\alpha$, $X_0^\alpha = 100$, $\xi^\alpha = 0.1$, $\sigma^\alpha = 0.2$ and we use the constant mid-volatility $\ddot{\sigma}^\alpha = 0.15$ to generate the first $N_1$ replication of $X$.

For the second independent Monte-Carlo using our sub-optimal control, we take $N_{\text{LS}} = 2^{15}$ replications of $X$ and a time step $\Delta_{\text{LS}} = 1/400$. In the backward and dual algorithms, we pick $N_1 = 2^{15}$ and choose the $\Delta = (1/2, 1/4, 1/8, 1/12)$ that give the higher $u_0^{\text{LS}}$ and the lower $u_0^{\text{dual}}$. The conditional expectations at $t_i$ are computed using parametric regressions. The regression basis consists in some polynomial basis with the Black-Scholes price/delta/gamma with mid-volatilities. The exact price is obtained by solving the (one or two-dimensional) HJB equation with a finite-difference scheme.

1. 90 – 110 call spread $(X_T - 90)^+ - (X_T - 110)^+$, basis= 5-order polynomial:
$$u_0^{\text{LS}} = 11.07 < u_0^{\text{PDE}} = 11.20 < u_0^{\text{dual}} = 11.70, \quad u_0^{\text{BSDE}} = 10.30$$

2. Digital option $\1_{X_T \geq 100}$, basis= 5-order polynomial:
$$u_0^{\text{LS}} = 62.75 < u_0^{\text{PDE}} = 63.33 < u_0^{\text{dual}} = 66.54, \quad u_0^{\text{BSDE}} = 52.03$$

3. Outperformer option $(X_T^2 - X_T^1)^+$ with 2 uncorrelated assets,
$$u_0^{\text{LS}} = 11.15 < u_0^{\text{PDE}} = 11.25 < u_0^{\text{dual}} = 11.84, \quad u_0^{\text{BSDE}} = 11.48$$

4. Outperformer option with 2 correlated assets $\rho = -0.5$
$$u_0^{\text{LS}} = 13.66 < u_0^{\text{PDE}} = 13.75 < u_0^{\text{dual}} = 14.05, \quad u_0^{\text{BSDE}} = 14.14$$

5. Outperformer spread option $(X_T^2 - 0.9X_T^1)^+ - (X_T^2 - 1.1X_T^1)^+$ with 2 correlated assets $\rho = -0.5$,
$$u_0^{\text{LS}} = 11.11 < u_0^{\text{PDE}} = 11.41 < u_0^{\text{dual}} = 12.35, \quad u_0^{\text{BSDE}} = 9.94$$

Note that in examples 3.-5. the regression basis we used consisted of
$$\{1, X^1, X^2, (X^1)^2, (X^2)^2, X^1 X^2\}.$$

The dual bounds we have derived complement the lower bounds derived in [10]. They allow us to access the quality of the regressors used in computing the conditional expectations.
4.2 Credit valuation adjustment

Our second example arises in credit valuation adjustment. We will show that for this particular example, we can solve the deterministic optimisation problems arising in the dual algorithm efficiently by recursively solving ODEs. More specifically, we consider the stochastic control problem

\[ u^{HJB}(t, X_t) = \sup_{\lambda_t \in [0, c]} \mathbb{E}[e^{-\int_t^T \lambda_s ds} g(X_T)], \quad \text{where } dX_t = \sigma(t, X_t) dW_t, \]

for which the HJB equation reads

\[ \partial_t u^{HJB} + \frac{1}{2} \sigma^2(t, x) \partial^2_{xx} u^{HJB} + c(u^{HJB})^- = 0. \quad (4.2) \]

4.2.1 CVA interpretation

The nonlinear PDE (4.2) corresponds to the pricing equation in the case of unilateral counterparty value adjustment (see [11] for more details). We have one counterparty, denoted by C, that may default and another, B, that cannot. We assume that B is allowed to trade dynamically in the underlying \( X \) - that is described by a local martingale \( dX_t = \sigma(t, X_t) dW_t \) under a risk-neutral measure Q. The default of C is modeled by a Poisson jump process with a constant intensity \( c \). We denote by \( u \) the value of B’s long position in a single derivative contracted by C, given that C has not defaulted so far. For simplicity, we assume zero rate. The no-arbitrage condition gives that \( u(t, X_t) \) is a \( Q \)-martingale, characterized by

\[ \partial_t u + \frac{1}{2} \sigma^2(t, x) \partial^2_{xx} u + c (\tilde{u} - u) = 0. \]

where \( \tilde{u} \) is the derivative value just after the counterparty has defaulted. At the default event, in the case of zero recovery, \( \tilde{u} \) is given by

\[ \tilde{u} = (-u)^+ \]

Indeed, if the value of \( u \) is positive, meaning that \( u \) should be paid by the counterparty, nothing will be received by B after the default. If the value of \( u \) is negative, meaning that \( u \) should be received by the counterparty, B will pay \( u \) in the case of default of C. Finally, we obtain the relation

\[ u(t, x) = e^{-c(T-t)} u^{HJB}(t, x). \]

4.2.2 Dual Bound

We are interested in deriving an efficient upper bound for \( u^{HJB}(0, X_0) \). Writing \( \Lambda_t = \int_0^T \lambda_s ds \) and letting \( \mathcal{D}_u := \left\{ \int_0^T \lambda_s ds : \lambda \in \mathcal{D}_\lambda \right\} \) our dual expression is

\[ u^{HJB}(0, X_0) = \lim_{k \to \infty} \inf_{\varphi \in \mathcal{U}} \mathbb{E} \left[ \sup_{\Lambda_t \in \mathcal{D}_u} \left\{ e^{-\Lambda_T} g(X_T) - \int_0^T e^{-\Lambda_s} \varphi(t, X_t) dX_t \right\} \right] \leq \lim_{k \to \infty} \mathbb{E} \left[ \sup_{\Lambda_t \in \mathcal{D}_u} \left\{ e^{-\Lambda_T} g(X_T) - \int_0^T e^{-\Lambda_s} \varphi^*(t, X_t) dX_t \right\} \right]. \]
where $\varphi^*$ is a fixed strategy. Rewriting the integral in Stratonovich form we have

$$
\int_0^T e^{-\lambda_t} \varphi^*(t, X_t) dX_t \\
= \int_0^T e^{-\lambda_t} \varphi^*(t, X_t) \circ dX_t - \frac{1}{2} \int_0^T e^{-\lambda_t} \left( \frac{\partial}{\partial x} \varphi^* \right) (t, X_t) \sigma^2(t, X_t) dt
$$

Therefore, using the classical Zakai approximation of the Stratonovich integral, it follows that

$$
E \left[ \sup_{\lambda \in \mathcal{D}_k} \{ e^{-\lambda_T} g(X_T^n) - \int_0^T e^{-\lambda_t} \varphi^*(t, X_t) dX_t \} \right] \\
= \lim_{n \to \infty} E \sup_{\lambda \in \mathcal{D}_k} \{ e^{-\lambda_T} g(X_T^n) \\
- \int_0^T e^{-\lambda_t} \varphi^*(t, X_t^n) \circ dX_t^n + \frac{1}{2} \int_0^T e^{-\lambda_t} \left( \frac{\partial}{\partial x} \varphi^* \right) (t, X_t^n) \sigma^2(t, X_t^n) dt \}
$$

$$
\leq \lim_{n \to \infty} E \left[ \sup_{\lambda \in \mathcal{D}_k} \{ e^{-\lambda_T} g(X_T^n) \\
- \int_0^T e^{-\lambda_t} \varphi^*(t, X_t^n) \sigma(t, X_t^n) \dot{W}_t^n - \frac{1}{2} \left( \frac{\partial}{\partial x} \varphi^* \right) (t, X_t^n) \sigma^2(t, X_t^n) dt \} \right],
$$

where $\mathcal{D}$ denotes the set of all absolutely continuous controls. For almost every $\omega$ we may consider for all $n$ the following deterministic optimisation problem. Set

$$
g_{\omega, n} = g(X_T^n(\omega)), \quad \alpha_{\omega, n}(t) = -\varphi^*(t, X_t^n(\omega)) \sigma(t, X_t^n(\omega)) \dot{W}_t^n(\omega),
$$

$$
\beta_{\omega, n}(t) = \frac{1}{2} \left( \frac{\partial}{\partial x} \varphi^* \right) (t, X_t^n(\omega)) \sigma^2(t, X_t^n(\omega)),
$$

and consider the function:

$$
u^H_{\omega, n}(t) = \sup_{\lambda \in [0, c]} \left\{ e^{-\lambda_T + \lambda_t} g_{\omega, n} + \int_0^T e^{-\lambda_s + \lambda_t} \left( \alpha_{\omega, n}(s) + \beta_{\omega, n}(s) \right) ds \right\}.
$$

Note that $u^H$ is the solution of the (path-wise) Hamilton-Jacobi equation:

$$(u^H_{\omega, n})'(t) + c \left( -u^H_{\omega, n}(t) \right) + \alpha_{\omega, n}(t) + \beta_{\omega, n}(t) = 0, \quad u^H_{\omega, n}(T) = g_{\omega, n}.
$$

The ODE for $u^H_{\omega, n}$ can be solved analytically. Fix a $t^0 \in [0, T]$, and let

$$
t^* = \sup \left\{ s < t^0 : u^H_{\omega, n}(t^0) u^H_{\omega, n}(s) < 0 \right\} \vee 0.
$$

For all $t \in [t^*, t_0]$ we get the following recurrence equation:

$$
u^H_{\omega, n}(t) = \begin{cases} 
- \int_t^{t^0} e^{-c(s-t)} \left( \alpha_{\omega, n}(s) + \beta_{\omega, n}(s) \right) ds + u^H_{\omega, n}(t^0)c^*(t^0-t), \quad u^H_{\omega, n}(t^0) < 0 \\
- \int_t^0 \left( \alpha_{\omega, n}(s) + \beta_{\omega, n}(s) \right) ds + u^H_{\omega, n}(t^0), \quad u^H_{\omega, n}(t^0) > 0
\end{cases}
$$

$$
u^H_{\omega, n}(T) = g_{\omega, n}.
Finally, we observe that,

\[ u_{HJB}^{0}(0, X_0) \leq \lim_{n \to \infty} \mathbb{E}[u_{\omega,n}^{HJ}(0)]. \]

We illustrate the quality of our bounds by the following numerical example.

**Remark 4.1** This example falls into the framework of [4], [5]. By virtue of their (continuous) pathwise analysis the upper bounds derived above could in the limit be replaced with equalities. Only the error introduced by the choice of \( \varphi^* \) remains.

**Numerical example**

We take \( \sigma(t, x) = 1, T = 1 \) year, \( X_0 = 0 \). \( g(x) = x \). We use two choices: \( \varphi^*(t, x) = e^{-c(T-t)} \) (which corresponds to \( \partial_x u_{HJB}^{0} \) at the first-order near \( c = 0 \)) and \( \varphi^*(t, x) = 0 \). We have computed \( \mathbb{E}[u_{\omega,n}^{HJ}(0)] \) as a function of the time discretization (see Table 1 and 2). The exact value has been computed using a one-dimensional PDE solver (see column PDE). We have used different values of \( c \) corresponding to a probability of default at \( T \) equal to \( (1 - e^{-cT}) \).

The approximation has two separate sources of error. First, there is the suboptimal choice of the minimiser \( \varphi^* \) for the discretised optimisation implying an upper bias. The second error arises from the discretisation of the deterministic optimisation problems, which in this example underestimates the true value of the optimisation. The choice \( \varphi^* = e^{-c(T-t)} \) in our example - as expected - close to optimal, so for small values of \( n \) in the discretisation of the deterministic optimisation problems the optimisation error dominates converging at a rate \( n^{-1/2} \) to the upper bound. The case \( \varphi^* = 0 \) demonstrates the effect of the gain of information, when the stochastic optimisation problem is replaced by the deterministic problems without a Lagrange multiplier to compensate.
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