ON PARATOPOLOGICAL GROUPS

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Abstract. In this paper, we firstly construct a Hausdorff non-submetrizable paratopological group \( G \) in which every point is a \( G_\delta \)-set, which gives a negative answer to Arhangel’ski”ı and Tkachenko’s question [Topological Groups and Related Structures, Atlantis Press and World Sci., 2008]. We prove that each first-countable Abelian paratopological group is submetrizable. Moreover, we discuss developable paratopological groups and construct a non-metrizable, Moore paratopological group. Further, we prove that a regular, countable, locally \( k_\omega \)-paratopological group is a discrete topological group or contains a closed copy of \( S_\omega \). Finally, we discuss some properties on non-H-closed paratopological groups, and show that Sorgenfrey line is not H-closed, which gives a negative answer to Arhangel’ski”ı and Tkachenko’s question [Topological Groups and Related Structures, Atlantis Press and World Sci., 2008]. Some questions are posed.

1. Introduction

A semitopological group \( G \) is a group \( G \) with a topology such that the product map of \( G \times G \) into \( G \) is separately continuous. A paratopological group \( G \) is a group \( G \) with a topology such that the product map of \( G \times G \) into \( G \) is jointly continuous. If \( G \) is a paratopological group and the inverse operation of \( G \) is continuous, then \( G \) is called a topological group. However, there exists a paratopological group which is not a topological group; Sorgenfrey line (16, Example 1.2.2) is such an example. Paratopological groups were discussed and many results have been obtained [7, 8, 6, 14, 22, 23, 24, 19].

Proposition 1.1. [29] For a group with topology \( (G, \tau) \) the following conditions are equivalent:

(1) \( G \) is a paratopological group;
(2) The following Pontrjagin conditions for basis \( B = \mathcal{B}_\tau \) of the neutral element \( e \) of \( G \) are satisfied.

(a) \( (\forall U, V \in B)(\exists W \in B): W \subset U \cap V; \)
(b) \( (\forall U \in B)(\exists V \in B): V^2 \subset U; \)
(c) \( (\forall U \in B)(\forall x \in U)(\exists V \in B): Vx \subset U; \)
(d) \( (\forall U \in B)(\forall x \in G)(\exists V \in B): xVx^{-1} \subset U; \)

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The paratopological group $G$ is Hausdorff if and only if
\[(e) \cap \{UU^{-1} : U \in \mathcal{B}\} = \{e\};\]
The paratopological group $G$ is a topological group if and only if
\[(f) \left( \forall U \in \mathcal{B} \right) \left( \exists V \in \mathcal{B} \right): V^{-1} \subset U.\]

In this paper, we mainly discuss the following questions.

**Question 1.2.** Open problem 3.3.1] Suppose that $G$ is a Hausdorff (regular) paratopological group in which every point is a $G_\delta$-set. Is $G$ submetrizable?

**Question 1.3.** Open problem 5.7.2] Let $G$ be a regular first-countable $\omega$-narrow paratopological group. Is $G$ submetrizable?

**Question 1.4.** Problem 20] Is every regular first countable (Abelian) paratopological group submetrizable?

**Question 1.5.** Problem 22] Is it true that every regular first countable (Abelian) paratopological group $G$ has a zero-set diagonal?\footnote{We say that a space $X$ has a zero-set diagonal if the diagonal in $X \times X$ is a zero-set of some continuous real-valued function on $X \times X$.}

**Question 1.6.** Problem 21] Is every regular first countable (Abelian) paratopological group Dieudonné complete?

**Question 1.7.** Open problem 3.4.3] Let $G$ be a regular $\omega$-narrow first-countable paratopological group. Does there exist a continuous isomorphism of $G$ onto a regular (Hausdorff) second-countable paratopological group?

**Question 1.8.** Is a regular symmetrizable paratopological group metrizable?

**Question 1.9.** Open problem 5.7.5] Is every paratopological group, which is Moore space, metrizable?

**Question 1.10.** Open problem 3.6.5] Must the Sorgenfrey line $S$ be closed in every Hausdorff paratopological group containing it as a paratopological subgroup?

We shall give negative answers to Questions 1.2, 1.8, 1.10, and 4.2, and give a partial answer to Question 1.3. Moreover, we shall also give affirmative answers to Questions 1.4, 1.5, 1.7 and 1.6 when the group $G$ is Abelian.

2. Preliminaries

**Definition 2.1.** Let $\mathcal{P} = \bigcup_{x \in X} \mathcal{P}_x$ be a cover of a space $X$ such that for each $x \in X$, (a) if $U, V \in \mathcal{P}_x$, then $W \subset U \cap V$ for some $W \in \mathcal{P}_x$; (b) the family $\mathcal{P}_x$ is a network of $x$ in $X$, i.e., $x \in \bigcap \mathcal{P}_x$, and if $x \in U$ with $U$ open in $X$, then $P \subset U$ for some $P \in \mathcal{P}_x$.

The family $\mathcal{P}$ is called a weak base for $X$ \footnote{We say that a space $X$ has a zero-set diagonal if the diagonal in $X \times X$ is a zero-set of some continuous real-valued function on $X \times X$.} if, for every $A \subset X$, the set $A$ is open in $X$ whenever for each $x \in A$ there exists $P \in \mathcal{P}_x$ such that $P \subset A$. The space $X$ is weakly first-countable if $\mathcal{P}_x$ is countable for each $x \in X$.

**Definition 2.2.** (1) A space $X$ is called an $S_\omega$-space if $X$ is obtained by identifying all the limit points from a topological sum of countably many convergent sequences.

We say that a space $X$ has a zero-set diagonal if the diagonal in $X \times X$ is a zero-set of some continuous real-valued function on $X \times X$.\footnote{We say that a space $X$ has a zero-set diagonal if the diagonal in $X \times X$ is a zero-set of some continuous real-valued function on $X \times X$.}
A space $X$ is called an $S_2$-space (Arens’ space) if $X = \{ \infty \} \cup \{ x_n : n \in \mathbb{N} \} \cup \{ x_n(m) : m, n \in \mathbb{N} \}$ and the topology is defined as follows: Each $x_n(m)$ is isolated; a basic neighborhood of $x_n$ is $\{ x_n \} \cup \{ x_n(m) : m > k, \text{for some } k \in \mathbb{N} \}$; a basic neighborhood of $\infty$ is $\{ \infty \} \cup \bigcup \{ V_n : n > k \text{ for some } k \in \mathbb{N} \}$, where $V_n$ is a neighborhood of $x_n$.

**Definition 2.3.** Let $X$ be a space and $\{ P_n \}$ a sequence of collections of open subsets of $X$.

1. $\{ P_n \}$ is called a quasi-development [10] for $X$ if for every $x \in U$ with $U$ open in $X$, there exists an $n \in \mathbb{N}$ such that $x \in \text{st}(x, P_n) \subset U$.
2. $\{ P_n \}$ is called a development [30] for $X$ if $\{ \text{st}(x, P_n) \}$ is a neighborhood base at $x$ in $X$ for each point $x \in X$.
3. $X$ is called quasi-developable (resp. developable), if $X$ has a quasi-development (resp. development).
4. $X$ is called Moore, if $X$ is regular and developable.

A subset $B$ of a paratopological group $G$ is called $\omega$-narrow in $G$ if, for each neighborhood $U$ of the neutral element of $G$, there is a countable subset $F$ of $G$ such that $B \subset F \cup UF$.

A space $X$ is called a submetrizable space if it can be mapped onto a metric space by a continuous one-to-one map. A space $X$ is called a subquasimetrizable space if it can be mapped onto a quasimetric space by a one-to-one map.

All spaces are $T_0$ unless stated otherwise. The notations $\mathbb{R}, \mathbb{Q}, \mathbb{P}, \mathbb{N}, \mathbb{Z}$ are real numbers, rational numbers, irrational numbers, natural numbers and integers respectively. The letter $e$ denotes the neutral element of a group. Readers may refer to [8, 16, 17] for notations and terminology not explicitly given here.

3. **Submetrizability of first-countable paratopological groups**

In this section, we firstly give a negative answer to Question 1.2, then an answer to Question 1.3. We also give affirmative answers to Questions 1.4, 1.5, 1.7 and 1.6 when $G$ is Abelian.

**Proposition 3.1.** [27] The following conditions are equivalent for an arbitrary space $X$.

1. The space $X$ is submetrizable.
2. The free paratopological group $F_p(X)$ is submetrizable.
3. The free Abelian paratopological group $A_p(X)$ is submetrizable.

**Proposition 3.2.** [27] The following conditions are equivalent for an arbitrary space $X$.

1. The space $X$ is subquasimetrizable.
2. The free paratopological group $F_p(X)$ is subquasimetrizable.
3. The free Abelian paratopological group $A_p(X)$ is subquasimetrizable.

**Example 3.3.** There exist a Hausdorff paratopological group $G$ in which every point is a $G_\delta$-set, and $G$ is not submetrizable.

**Proof.** Let $X$ be the lexicographically ordered set $X = (\mathbb{R} \times \{0\}) \cup (\mathbb{P} \times \mathbb{Z})$. Then $X$ is a non-metrizable linearly ordered topological space without $G_\delta$-diagonal ([11 Example 2.4]), hence $X$ is not submetrizable. However, $X$ is quasi-developable [25]. It is well known that quasi-developability in a generalized ordered space is
equivalent to the existence of a \( \sigma \)-disjoint base, or of a \( \sigma \)-point-finite base [12, Theorem 4.2]. Hence \( X \) has \( \sigma \)-point finite base. Therefore, \( X \) is quasi-metrizable since a space with a \( \sigma \)-point finite base is quasi-metrizable [17, Page 489]. Let \( G \) be the free Abelian paratopological group \( A_p(X) \) over \( X \). Since a totally order space endowed the order topology is Tychonoff, then \( X \) is Tychonoff. It follows from [27, Proposition 3.8] that \( G \) is Hausdorff. By Propositions 3.1 and 3.2, \( G \) is subquasimetrizable and non-submetrizable. Since \( G \) is subquasimetrizable, every singleton of \( G \) is a \( G_\delta \)-set. \( \square \)

Next we partially answer Question 1.3.

The \textit{weak extent} [9] of a space \( X \), denoted by \( \omega e(X) \), is the least cardinal number \( \kappa \) such that for every open cover \( \mathcal{U} \) of \( X \) there is a subset \( A \) of \( X \) of cardinality no greater than \( \kappa \) such that \( \text{st}(A; \mathcal{U}) = X \), where \( \text{st}(A; \mathcal{U}) = \bigcup \{ U : U \in \mathcal{U}, U \cap A \neq \emptyset \} \). If \( X \) is separable, then \( \omega e(X) = \omega \).

**Theorem 3.4.** [9] If \( X^2 \) has countable weak extent and a regular \( G_\delta \)-diagonal, then \( X \) condenses onto a second countable Hausdorff space.

**Theorem 3.5.** [19] Each \( \omega \)-narrow first-countable paratopological group is separable.

**Theorem 3.6.** If \( G \) is a regular \( \omega \)-narrow first-countable paratopological group, then \( G \) condenses onto a second countable Hausdorff space.

**Proof.** It is straightforward to prove that the product of two \( \omega \)-narrow paratopological groups is an \( \omega \)-narrow paratopological group. Then \( G^2 \) is an \( \omega \)-narrow first-countable paratopological group, and hence \( G^2 \) is separable by Theorem 3.5. Then \( G^2 \) has countable weak extent. Moreover, it follows from [22] that \( G \) has a regular \( G_\delta \)-diagonal. Therefore, \( G \) condenses onto a second countable Hausdorff space by Theorem 3.4. \( \square \)

**Corollary 3.7.** Let \( (G, \tau) \) be a regular \( \omega \)-narrow first-countable paratopological group. There exists a continuous isomorphism of \( G \) onto a Hausdorff second-countable space.

A paratopological group \( G \) with a base at the neutral element \( \mathcal{B} \) is a **SIN-group** (Small Invariant Neighborhoods), if for each \( U \in \mathcal{B} \) there exists a \( V \in \mathcal{B} \) such that \( xVx^{-1} \subset U \) for each \( x \in G \).

**Theorem 3.8.** If \( (G, \tau) \) is a Hausdorff SIN first-countable paratopological group, then \( G \) is submetrizable.

**Proof.** Let \( \{ U_n : n \in \mathbb{N} \} \) be a countable local base of \( (G, \tau) \) at the neutral element \( e \), where \( U_{n+1} \subset U_n \) for each \( n \in \mathbb{N} \).

For \( x \in G \), let \( \mathcal{B}_x = \{ xu_nu_n^{-1} : n \in \mathbb{N} \} \). Then \( \{ \mathcal{B}_x \}_{x \in G} \) has the following properties.

(BP1) For every \( x \in G \), \( \mathcal{B}_x \neq \emptyset \) and for every \( U \in \mathcal{B}_x \), \( x \in U \).

(BP2) If \( x \in U \in \mathcal{B}_y \), then there exists a \( V \in \mathcal{B}_x \) such that \( V \subset U \).

In fact, if \( x \in U = yu_1u_1^{-1} \in \mathcal{B}_y \), then \( x = yu_1u_1^{-1} \) for some \( u_1, u_2 \in U \). Pick \( U_j, U_k \in \{ U_n : n \in \mathbb{N} \} \) such that \( U_k \subset U_j, u_1U_k \subset U_j, u_2U_k \subset U_j, u_2^{-1}U_j, u_2 \subset U_k \).

\(^2\)A space \( X \) is said to have a **regular \( G_\delta \)-diagonal** if the diagonal \( \Delta = \{(x, x) : x \in X \} \) can be represented as the intersection of the closures of a countable family of open neighborhoods of \( \Delta \) in \( X \times X \).
Then \( xU_jU_j^{-1} = yu_1u_2^{-1}U_jU_j^{-1} \subseteq yu_1U_ku_2^{-1}U_j^{-1} \subseteq yU_i(U_ju_2)^{-1} \subseteq yU_iU_i^{-1} \subseteq yU_iU_i^{-1} = U. \)

(BP3) For any \( V_1, V_2 \in \mathcal{B}_x \), there exists a \( V \in \mathcal{B}_x \) such that \( V \subseteq V_1 \cap V_2. \)

Let \( \tau^* \) be the topology generated by the neighborhood system \( \{ \mathcal{B}_x \}_{x \in G} \). Obviously, the topology of \( (G, \tau^*) \) is coarser than \( (G, \tau) \) and it is first-countable. We prove that \( (G, \tau^*) \) is a Hausdorff topological group.

It is easy to see that (a), (d) and (f) in Proposition 3.11 are satisfied. (BP2) implies (c). We check conditions (b) and (d).

Fix \( n \in \mathbb{N} \). Then there is \( k > n \) such that \( U_k^2 \subseteq U_n \) since \( (G, \tau) \) is a paratopological group. \( G \) is a SIN-group, there exists a continuous isomorphism of \( G \) onto a Tychonoff second-countable topological group. Hence there exists a continuous isomorphism of \( G \) onto a Tychonoff second-countable topological group. \( \square \)

It is well known that all submetrizable spaces have a zero-set diagonal. Therefore, Theorem 3.11 gives a partial answer to Question 1.15

**Corollary 3.9.** If \( (G, \tau) \) is a Hausdorff Abelian first-countable paratopological group, then \( G \) is submetrizable.

Indeed, we have the following more stronger result.

**Theorem 3.10.** If \( (G, \tau) \) is a Hausdorff Abelian paratopological group with a countable \( \pi \)-character, then \( G \) is submetrizable.

**Proof.** Let \( \mathcal{B} = \{ U_\alpha : \alpha < \kappa \} \) be a local base at the neutral element \( e \). It follows from the proof of Theorem 3.11 that the family \( \{ U_\alpha U_\alpha^{-1} : \alpha < \kappa \} \) is a local base at \( e \) in the Tychonoff topological group \( (G, \tau^*) \).

Let \( \mathcal{C} = \{ V_n : n \in \mathbb{N} \} \) be a local \( \pi \)-base at \( e \). Put \( \mathcal{F} = \{ V_nV_n^{-1} : n \in \mathbb{N} \} \). Then \( \mathcal{F}' = \{ \text{int}(V_nV_n^{-1}) : n \in \mathbb{N} \} \) is a local base at \( e \) in \( \tau^* \).

Indeed, for each \( n \in \mathbb{N} \) and fix a point \( x \in V_n \), then \( x^{-1}V_n \) is an open neighborhood at \( e \) in \( \tau \), and hence there exists an \( U_\alpha \in \mathcal{B} \) such that \( U_\alpha \subseteq x^{-1}V_n \). Thus \( U_\alpha U_\alpha^{-1} \subseteq x^{-1}V_nV_n^{-1} \). On the other hand, fix \( \alpha < \kappa \), there is \( n \in \mathbb{N} \) such that \( V_n \subseteq U_\alpha \). Therefore, \( \mathcal{F}' = \{ \text{int}(V_nV_n^{-1}) : n \in \mathbb{N} \} \) is a local base at \( e \) in \( \tau^* \).

Since first-countable topological group metrizable, we have \( (G, \tau^*) \) is metrizable. Therefore, \( G \) is submetrizable. \( \square \)

Since every submetrizable space is (hereditarily) Dieudonné complete, Corollary 3.9 give a partial answer to Question 1.16

**Theorem 3.11.** Let \( (G, \tau) \) be a Hausdorff separable SIN first-countable paratopological group. There exists a continuous isomorphism of \( G \) onto a Tychonoff second-countable topological group.

**Proof.** By the proof of Theorem 3.11 we know that \( (G, \tau^*) \) is metrizable. Since \( G \) is separable and \( \tau^* \subseteq \tau \), \( (G, \tau^*) \) is separable, and hence \( (G, \tau^*) \) is a second-countable topological group. Hence there exists a continuous isomorphism of \( G \) onto a Tychonoff second-countable topological group. \( \square \)
By Theorem 3.5, we have the following corollary, which gives a partial answer to Question 1.7.

**Corollary 3.12.** Let \((G, \tau)\) be a Hausdorff \(\omega\)-narrow first-countable SIN paratopological group. There exists a continuous isomorphism of \(G\) onto a Tychonoff second-countable topological group.

The following two theorems give another answers to Questions 1.4 and 1.6.

**Theorem 3.13.** If \((G, \tau)\) is a Hausdorff saturated\(^3\) first-countable paratopological group, then \(G\) is submetrizable.

*Proof.* Suppose that \(\{U_n : n \in \mathbb{N}\}\) is a countable local base at \(e\). Let

\[\sigma = \{U \subset G : \text{There exists an } n \in \mathbb{N} \text{ such that } xU_n U_n^{-1} \subset U \text{ for each } x \in U\}.\]

Since \(G\) is saturated, it follows from \([13, \text{Theorem 3.2}]\) that \((G, \sigma)\) is a topological group. Obvious, \((G, \sigma)\) is \(T_1\) since \((G, \tau)\) is a Hausdorff, and hence \((G, \sigma)\) is regular. Then \((G, \sigma)\) is first-countable, and thus it is metrizable. Therefore, \((G, \tau)\) is submetrizable.

**Theorem 3.14.** If \((G, \tau)\) is a Hausdorff feebly compact\(^4\) first-countable paratopological group, then \(G\) is submetrizable.

*Proof.* Suppose that \(\{U_n : n \in \mathbb{N}\}\) is a local base at \(e\). Then the family \(\{\text{int}U_n : n \in \mathbb{N}\}\) is a local base at \(e\) for a regular paratopological group topology \(\sigma\) on \(G\). Obviously, \((G, \sigma)\) is feebly compact. Since \((G, \sigma)\) is first-countable, \((G, \sigma)\) has a regular \(G_\delta\)-diagonal \([22]\). Then \((G, \sigma)\) is metrizable since a regular feebly compact space with a regular \(G_\delta\)-diagonal is metrizable. Therefore, \((G, \tau)\) is submetrizable.

Next, we gives an another answer to Question 1.2.

**Theorem 3.15.** Let \((G, \tau)\) be a Hausdorff SIN paratopological group. If \(G\) is locally countable, then \(G\) is submetrizable.

*Proof.* Since \(G\) is locally countable, there exists an open neighborhood \(U\) of \(e\) such that \(U\) is a countable set. Then \(UU^{-1}\) is also a countable set. Let \(UU^{-1} \setminus \{e\} = \{x_n : n \in \mathbb{N}\}\). Since \(G\) is a paratopological group, we can find a family of countably many neighborhoods \(\{V_n : n \in \omega\}\) of \(e\) satisfying the following conditions:

(i) \(V_0 = U\); 
(ii) For each \(n \in \omega\), then \(V_{n+1} \subset V_n\); 
(iii) For each \(n \in \mathbb{N}\), then \(x_n \notin V_n V_n^{-1}\) (This is possible since \(G\) is Hausdorff.) 
(iv) For each \(n \in \omega\), then \(xV_{n+1} x^{-1} \subset V_n\) for each \(x \in G\).

Since \(G\) is a SIN group, the topology \(\sigma\) generated by the neighborhood basis \(\{V_n : n \in \mathbb{N}\}\) is a paratopological group. Clearly, \((G, \sigma)\) is coarser than \(\tau\). Since \(\bigcap_{n \in \mathbb{N}} V_n V_n^{-1} = \{e\}\) by (iii), \(\sigma\) is Hausdorff. Therefore, \((G, \sigma)\) is a Hausdorff SIN first-countable paratopological group, and it follows from Theorem 3.8 that \(G\) is submetrizable.

**Corollary 3.16.** Let \((G, \tau)\) be a Hausdorff Abelian paratopological group. If \(G\) is locally countable, then \(G\) is submetrizable.

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\(^3\)A paratopological group \(G\) is saturated if for any neighborhood \(U\) of \(e\) the set \(U^{-1}\) has nonempty interior in \(G\).

\(^4\)A space \(X\) is called feebly compact if each locally finite family of open subsets in \(X\) is finite.
4. Developable paratopological groups

Recall that a topological space is *symmetrizable* if its topology is generated by a symmetric, that is, by a distance function satisfying all the usual restrictions on a metric, except for the triangle inequality \([1]\).

Now, we give a negative answer\(^5\) to Question 1.8 by modifying \([23]\, Example 2.1\).

**Example 4.1.** There exists a separable, Moore paratopological group \(G\) such that \(G\) is not metrizable.

**Proof.** Let \(G = \mathbb{R} \times \mathbb{Q}\) be the group with the usual addition. Then we define a topology on \(G\) by giving a local base at the neutral element \((0,0)\) in \(G\). For each \(n \in \mathbb{N}\), let

\[
U_n(0,0) = \{(0,0)\} \cup \{(x, y) : y \geq nx, y < \frac{1}{n}, y \in \mathbb{Q}, x \geq 0\}.
\]

Let \(\sigma\) be the topology generated by the local base \(\{U_n : n \in \mathbb{N}\}\) at the neutral element \((0,0)\). It is easy to see that \((G, \sigma)\) is a semitopological group. Now, we prove that is a paratopological group. Since \(G\) is Abelian, it only need to prove that for each \(n \in \mathbb{N}\) there exists an \(m \in \mathbb{N}\) such that \(U_n^2 \subseteq U_m\). Indeed, fix an \(n \in \mathbb{N}\), Then we have \(U_n^2 \subseteq U_n\). For two points \((x_i, y_i) \in U_{4n}(i = 1, 2)\), where \(x_i \geq 0, y_i < \frac{1}{4n}, y_i \geq 4nx_i, y_i \in \mathbb{Q}(i = 1, 2)\). Let

\[
(x, y) = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2).
\]

Obviously, we have

\[
x = x_1 + x_2 \geq 0, y = y_1 + y_2 < \frac{1}{4n} + \frac{1}{4n} = \frac{1}{2n} < \frac{1}{n},
\]

and

\[
y = y_1 + y_2 \geq 4nx_1 + 4nx_2 = 4nx \geq nx.
\]

Then \((x, y) \in U_n\), and hence \(U_n^2 \subseteq U_n\). Moreover, it is easy to see that \(G\) is regular, separable and first-countable space.

For each \(q \in \mathbb{Q}\), it is easy to see that the family \(\{(x, q) : x \in \mathbb{R}\}\) is uncountable discrete and closed, hence \(G\) is a \(\sigma\)-space, and thus it is a \(\beta\)-space\(^6\). \(G\) is a Moore space by \([23]\, Corollary 2.1\). Hence \(G\) is semi-metrizable \([17]\). Therefore \(G\) is symmetrizable since a space is semi-metrizable if only if it is first-countable and symmetrizable \([17]\). However, \(G\) is not metrizable since \(G\) is separable and contains an uncountable discrete closed subset. \(\square\)

**Question 4.2.** Is every quasi-developable paratopological (semitopological) group a \(\beta\)-space?

Next, we give a partial answer to Question 4.2.

**Lemma 4.3.** \([4]\, Lemma 1.2\) Suppose that \(G\) is a paratopological group and not a topological group. Then there exists an open neighborhood \(U\) of the neutral element \(e\) of \(G\) such that \(U \cap U^{-1}\) is nowhere dense in \(G\), that is, the interior of the closure of \(U \cap U^{-1}\) is empty.

\(^5\)Li, Mou and Wang \([19]\) also obtained a non-metrizable Moore paratopological group.

\(^6\)Let \((X, \tau)\) be a topological space. A function \(g : \omega \times X \rightarrow \tau\) satisfies that \(x \in g(n, x)\) for each \(x \in X, n \in \omega\). A space \(X\) is a \(\beta\)-space \([17]\) if there is a function \(g : \omega \times X \rightarrow \tau\) such that if \(x \in g(n, x_n)\) for each \(n \in \omega\), then the sequence \(\{x_n\}\) has a cluster point in \(X\).
Theorem 4.4. A regular Baire quasi-developable paratopological group $G$ is a metrizable topological group.

Proof. Claim: Let $U$ be an arbitrary open neighborhood of $e$. Then $U^{-1}$ is also a neighborhood of $e$.

Suppose that $\{\mathcal{U}_n : n \in \mathbb{N}\}$ is a family of open subsets of $G$ such that for each $x \in G$ and $x \in V$ with $V$ open in $G$, there exists an $n \in \mathbb{N}$ such that $x \in \text{st}(x, \mathcal{U}_n) \subset V$. For each $n$, put $A_n = \{x \in G : \text{st}(x, \mathcal{U}_n) \subset x \cdot U\}$.

It is easy to see that $G = \bigcup_0 (A_n : n \in \mathbb{N})$. Since $G$ is Baire, there exists $n_0 \in \mathbb{N}$ such that $\text{int} A_{n_0} \neq \emptyset$. Therefore, there exist a point $x_0 \in G$ and $n_1 \in \mathbb{N}$ such that $\text{st}(x_0, \mathcal{U}_{n_1}) \subset A_{n_0}$. Let $\mathcal{V} = \{U_1 \cap U_2 : U_1 \in \mathcal{U}_{n_0}, U_2 \in \mathcal{U}_{n_1}\}$. Put $W = \text{st}(x_0, \mathcal{V})$.

For each $W \cap A_{n_0}$, it is easy to see

$$x_0 \in \text{st}(y, \mathcal{V}) \subset \text{st}(y, \mathcal{U}_{n_0}) \subset y \cdot U,$$

hence $y^{-1}x_0 \in U$, so $x_0^{-1}y \in U^{-1}$, hence $x_0^{-1} \cdot (W \cap A_{n_0}) \subset U^{-1}$. Moreover, since $W \subset A_{n_0}$, then $W \subset W \cap A_{n_0}$. Therefore, we have

$$e \in x_0^{-1}W \subset x_0^{-1} \cdot W \cap A_{n_0} \subset x_0^{-1} \cdot (W \cap A_{n_0}) \subset U^{-1}.$$

Since $x_0^{-1}W$ is an open neighborhood of $e$, the set $U^{-1}$ is also a neighborhood of $e$.

If $G$ is developable, then the answer is affirmative. Indeed, it was proved that each Baire Moore semitopological group $G$ is a metrizable topological group [14].

Finally, we pose some questions about developable paratopological groups.

Question 4.5. Is each regular Baire quasi-developable semitopological group $G$ a paratopological group?

If $G$ is developable, then the answer is affirmative. Indeed, it was proved that each Baire Moore semitopological group $G$ is a metrizable topological group [14].

Question 4.6. Is every developable or Moore paratopological group submetrizable?

Question 4.7. Is every normal Moore paratopological group submetrizable?

Question 4.8. Is every paratopological group with a base of countable order\footnote{Recall that a space is Baire if the intersection of a sequence of open and dense subsets is dense.} developable?

Question 4.9. Is every paratopological group with a base of countable order\footnote{A space $X$ is said to have a base of countable order (BCO) \cite{17} if there is a sequence $\{B_n\}$ of base for $X$ such that whenever $x \in b_n \in B_n$ and $(b_n)$ is decreasing (by set inclusion), then $\{b_n : n \in \mathbb{N}\}$ is a base at $x$.} developable?
5. Fréchet-Urysohn paratopological groups

First, we need the following Lemma.

**Lemma 5.1.** [8 Theorem 4.7.5] Every weakly first-countable Hausdorff paratopological group is first-countable.

Arhangel’skií proved that if a topological group $G$ is an image of a separable metrizable space under a pseudo-open map, then $G$ is metrizable [4]. We have the following.

**Theorem 5.2.** Let $G$ be an uncountable paratopological group. Suppose that $G$ is a pseudo-open image of a separable metric space, then $G$ is a separable and metrizable.

**Proof.** We introduce a new product operation in the topological space $G$ by the formula: $a \times b = ba$, for $a, b \in G$ and denote the space with this operation by $H$. Put $T = \{(g, g^{-1}) \in G \times H, g \in G\}$. $|T| > \omega$ since $G$ is uncountable. By [7, Proposition 2.9], $H$ is a paratopological group and $T$ is closed in the space $G \times H$ and is a topological group.

Since $G$ is a pseudo-open image of a separable metric space, then the space $G$ is a Fréchet-Urysohn space with a countable k-network. $G \times H$ has a countable network. Hence $T$ has a countable network. By the proof of [17, Theorem 4.9], $T$ is a one-to-one continuous image of a separable metric space $M$. Let $D$ be a countable dense subset of $M$, there is a sequence $L \subset D$ converging to some point in $M \setminus D$ since $M$ is uncountable. Therefore, there is a non-trivial sequence $\{(g_n, g_n^{-1}) : n \in \mathbb{N}\}$ converging to $(e, e)$ (note that $T$ is homogeneous), and hence there exists a sequence $C_0 = \{g_n : n \in \mathbb{N}\} \subset G$ converging to $e$ and its inverse $C_1 = \{g_n^{-1} : n \in \mathbb{N}\}$ also converges to $e$. $G$ contains no closed copy of $S_2$ since $G$ is Fréchet-Urysohn. By [22, Theorem 2.4], $G$ contains no closed copy of $S_\omega$. Since $G$ is a sequential space with a point-countable k-network and contains no closed copy of $S_\omega$, then $G$ is weakly first-countable [20], and hence $G$ is first-countable by Lemma 5.1. Therefore $G$ is separable and metrizable [17, Theorem 11.4(ii)]. □

A quotient image of a topological sum of countably many compact spaces is called a $k_\omega$-space. Every countable $k_\omega$-space is a sequential $\aleph_0$-space, and a product of two $k_\omega$-spaces is itself a $k_\omega$-space, see [26].

**Theorem 5.3.** Let $G$ be a regular countable, locally $k_\omega$, paratopological group. Then $G$ is a discrete topological group or contains a closed copy of $S_\omega$.

**Proof.** Suppose that $G$ is an $\alpha_4$-space. Then since $G$ is locally $k_\omega$, $G$ is sequential and $\aleph_0$, and thus $G$ is weakly first-countable [21]. Then $G$ is first-countable by Lemma 5.1 and hence $G$ is a separable metrizable space since $G$ is countable. If $G$ is not discrete, $G$ has no isolated points. Then $G$ is homeomorphic to the rational number set $\mathbb{Q}$ since a separable metrizable space is homeomorphic to the rational number set $\mathbb{Q}$ provided that it is infinite, countable and without any isolated points [18]. However, $\mathbb{Q}$ is not a locally $k_\omega$-space, which is a contradiction. Then $G$ is discrete, and $G$ is a topological group.

If $G$ is not an $\alpha_4$-space, and thus $G$ contains a copy of $S_\omega$. Since every point of $G$ is a $G_\delta$-set, it follows from [20, Corollary 3.4] that $G$ contains a closed copy of $S_\omega$. □

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9A map $f : X \to Y$ is pseudo-open if for each $y \in Y$ and every open set $U$ containing $f^{-1}(y) \subset U$ one has $y \in \text{int}(f(U))$.  

By Theorem 5.3, it is easy to obtain the following corollary.

**Corollary 5.4.** Let \( G \) be a regular, countable, non-discrete, Fréchet-Urysohn\(^{10}\) paratopological group. If \( G \) is \( k_{\omega} \), then \( G \) contains a closed copy of \( S_{\omega} \) and no closed copy of \( S_{2} \).

Since the closed image of a locally compact, separable metric space is a Fréchet-Urysohn and \( k_{\omega} \)-space, we have the following corollary.

**Corollary 5.5.** Let \( G \) be a countable non-discrete paratopological group. If \( G \) is a closed image of a locally compact, separable metric space, then \( G \) contains a closed copy of \( S_{\omega} \), and hence \( G \) is not metrizable.

The condition “locally \( k_{\omega} \)-space” is essential in Theorem 5.3, and we can not omit it.

**Example 5.6.** There exists a regular non-discrete countable second-countable paratopological group \( G \) such that \( G \) contains no closed copy of \( S_{\omega} \) and \( G \) is not a topological group.

**Proof.** Let \( G = \mathbb{Q} \) be the rational number and endow with the subspace topology of the Sorgenfrey line. Then \( \mathbb{Q} \) is a second-countable paratopological group and non-discrete. Obviously, \( G \) contains no closed copy of \( S_{\omega} \) and \( G \) is not a topological group. \qed

### 6. Non-H-closed Paratopological Groups

A paratopological group is H-closed if it is closed in every Hausdorff paratopological group containing it as a subgroup.

Let \( U \) be a neighborhood of \( e \) in a paratopological group \( G \). We say that a subset \( A \subset G \) is \( U \)-unbounded if \( A \nsubseteq KU \) for every finite subset \( K \subset G \).

Now, we give a negative answer to Question 1.10.

**Lemma 6.1.** Let \( G \) be an abelian paratopological group of the infinite exponent. If there exists a neighborhood \( U \) of the neutral element such that a group \( nG \) is \( U U^{-1} \)-unbounded for every \( n \in \mathbb{N} \), then the paratopological group \( G \) is not H-closed.

**Theorem 6.2.** The Sorgenfrey line \((\mathbb{R}, \tau)\) is not H-closed.

**Proof.** Obvious, \( \mathbb{R} \) is an abelian paratopological group of the infinite exponent. Let \( U = [0, 1) \). Then \( U U^{-1} = (-1, 1) \) For each \( n \in \mathbb{N} \), it is easy to see that \( n\mathbb{R} \nsubseteq K(-1, 1) \) for every finite subset \( K \subset \mathbb{R} \). Indeed, for each finite subset \( K \) of \( \mathbb{R} \), since \( K \) is finite set, there exists an \( n \in \mathbb{N} \) such that \( |x| \leq n \) for each \( x \in K \), and then \( K \subset (-1, 1) \subset (-n, n) \). Therefore, we have \( n\mathbb{R} \nsubseteq K(-1, 1) \). By Lemma 6.1, \( \mathbb{R} \) is not H-closed. \qed

However, we have the following theorem.

**Theorem 6.3.** Let \((\mathbb{R}, \tau)\) be the Sorgenfrey line. Then the quotient group \((\mathbb{R}/\mathbb{Z}, \xi)\) is H-closed, where \( \xi \) is the quotient topology.

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\(^{10}\)A space \( X \) is said to be Fréchet-Urysohn if, for each \( x \in \overline{A} \subset X \), there exists a sequence \( \{x_n\} \) such that \( \{x_n\} \) converges to \( x \) and \( \{x_n : n \in \mathbb{N}\} \subset A \).
Proof. Let \((\mathbb{R}/\mathbb{Z}, \sigma)\) be the finest group topology such that \(\sigma \subset \xi\). Then \((\mathbb{R}/\mathbb{Z}, \sigma)\) is compact. Therefore, \((\mathbb{R}/\mathbb{Z}, \sigma)\) is H-closed. Hence \((\mathbb{R}/\mathbb{Z}, \xi)\) is H-closed by [28, Proposition 10].

Let \(G\) be an abelian non-periodic paratopological group. We say that \(G\) is strongly unbounded if there exists non-periodic element \(x_0\) and open neighborhood \(U\) of \(e\) such that \(\langle x_0 \rangle \cap U U^{-1} = \{e\}\) and \(\langle x_0 \rangle\) is closed in \(G\), where \(\langle x_0 \rangle\) is a subgroup generated by \(x_0\). Obviously, every strongly unbounded paratopological group is not H-closed.

Next, we discuss some non-H-closed paratopological groups.

Given any elements \(a_0, a_1, \cdots, a_n\) of an abelian group \(G\) put

\[
X(a_0, a_1, \cdots, a_n) = \{a_0^{x_0} a_1^{x_1} \cdots a_n^{x_n} : 0 \leq x_i \leq n, 0 \leq i \leq n\}.
\]

**Theorem 6.4.** Let \((G, \tau)\) be a Hausdorff strongly unbounded paratopological group. Then there exists a Tychonoff paratopological group topology \(\tau\) on \(G \times \mathbb{Z}\) satisfies the following conditions:

1. There exists a Hausdorff paratopological group topology \(\sigma \subset \tau\) on \(G \times \mathbb{Z}\) such that \(\sigma_{|G \times \{0\}} = \tau\) and \((G \times \mathbb{Z}, \sigma)\) contains closed copies of \(S_2\) and \(S_\omega\);
2. \((G \times \mathbb{Z}, \gamma)\) is a strongly zero-dimensional, paracompact \(\sigma\)-space;
3. The remainder \(b(G \times \mathbb{Z}, \gamma) \setminus (G \times \mathbb{Z}, \gamma)\) of every Hausdorff compactification \(b(G \times \mathbb{Z}, \gamma)\) is pseudocompact.

**Proof.** Since \(G\) is strongly unbounded paratopological group, there exist a non-periodic element \(x_0\) of \(G\) and open neighborhood \(U\) of \(e\) such that \(\langle x_0 \rangle \cap U U^{-1} = \{e\}\) and \(\langle x_0 \rangle\) is closed in \(G\). Obvious, \(\langle x_0 \rangle\) is an abelian group. Then the mapping \(f : (nx_0, m) \mapsto (n, m)\) is naturally isomorphic from \(\langle x_0 \rangle \times \mathbb{Z}\) onto a group \(\mathbb{Z} \times \mathbb{Z}\). We may assume that \(\mathbb{Z} \times \mathbb{Z} \subset G \times \mathbb{Z}\). Now we define a zero dimensional paratopological group topology on \(G \times \mathbb{Z}\). Obvious, we can define a positively natural number sequence \(\{a_n\}\) satisfies the following conditions:

1. \(a_n > n\);
2. \(a_n > 2a\) for each \(a \in X(a_1, \cdots, a_{n-1})\).

Define a base \(\mathcal{B}\) at the neutral element of paratopological group topology \(\sigma\) on the group \(G \times \mathbb{Z}\) as follows. Put \(A_n^+ = \{(e, 0)\} \cup \{a_k : k > n\}\). For every strictly increasing sequence \(\{n_k\}\) put \(A[n_k] = \bigcup_{i \in \mathbb{N}} A_{n_i}^+\cdots A_{n_k}^+\). Put \(\mathcal{B}_n = \{A[n_k]\}\). Then \(\gamma\) is a zero dimensional paratopological group topology on \(G \times \mathbb{Z}\).

By the proof of [28, Lemma 3], we can define a topology \(\sigma\) on \(G \times \mathbb{Z}\) such that \(\sigma_{|G \times \{0\}} = \tau\) and \(\sigma_{|\mathbb{Z} \times \mathbb{Z}} = \gamma_{|\mathbb{Z} \times \mathbb{Z}}\). Let \(\gamma_{|\mathbb{Z} \times \mathbb{Z}} = \xi\). Since \((\mathbb{Z} \times \mathbb{Z}, \xi)\) is zero dimensional and countable, the space \((\mathbb{Z} \times \mathbb{Z}, \xi)\) is Tychonoff.

**Claim 1:** \((\mathbb{Z} \times \mathbb{Z}, \xi)\) is a closed in \((G \times \mathbb{Z}, \sigma)\).

Since \(\langle 1 \rangle \cap U U^{-1} = \{e\}\) and \(\langle 1 \rangle\) is closed in \(G\), it is easy to see that \((\mathbb{Z} \times \mathbb{Z}, \xi)\) is closed in \((G \times \mathbb{Z}, \sigma)\).

**Claim 2:** \((\mathbb{Z} \times \mathbb{Z}, \xi)\) contains a closed copy of \(S_\omega\).

For each \(n \in \mathbb{N}\), let \(\beta_n = \{(na_{n+k}, n)\}_{k \in \mathbb{N}}\). Obvious, each \(\beta_n\) converges to \((0,0)\) as \(k \to \infty\).

Let \(X = \{(na_{n+k}, n) : k, n \in \mathbb{N}\} \cup \{(0,0)\}\). It is easy to see that \(X\) is a closed copy of \(S_\omega\).

**Claim 3:** \((\mathbb{Z} \times \mathbb{Z}, \xi)\) contains a closed copy of \(S_2\).
Let $\alpha_0 = \{(a_k, 1)\}_{k \in \mathbb{N}}$. For each $n \in \mathbb{N}$, let $\alpha_n = \{(a_n + (n-1)a_k, n)\}_{k \in \mathbb{N}}$. Obviously, $\alpha_0$ converges to $(0, 0)$ as $k \to \infty$ and each $\alpha_n$ converges to $(a_n, 1)$ as $k \to \infty$.

Let $X = \{a_n + (n-1)a_k, n \in \mathbb{N}\} \cup \alpha_0 \cup \{(0, 0)\}$. It is easy to see that $X$ is a closed copy of $S_2$.

$(2)$ Let $i : \mathbb{Z} \to \mathbb{Z}$ be the identity map. Since $\mathbb{Q}$ is a divisible group, the map $i$ can be extended to a homomorphism $\phi : G \to \mathbb{Q}$. Put $|x| = |\phi(x)|$ for every element $x \in G$. Then $|\cdot|$ is a seminorm on the group $G$ such that for all $x, y \in G$ holds $|x + y| \leq |x| + |y|$. For each $n \in \mathbb{N}$, let $\mathcal{B}_n = \{(x, y) : \phi(x) \leq n, y \in \mathbb{Z}\}$. Then it is easy to see that each $\mathcal{B}_n$ is a discrete family of closed subsets. Since it follows from Theorem $3.15$ that the space $(G \times \mathbb{Z}, \gamma)$ is submetrizable, there exists a metrizable topology $\mathcal{F}$ such that $\mathcal{F} \subset \gamma$. Obvious, $\bigcup_{n \in \mathbb{N}} \mathcal{B}_n$ is a network for $(G \times \mathbb{Z}, \mathcal{F})$, and it follows from $[3]$ Theorem $7.6.6$ that $(G \times \mathbb{Z}, \gamma)$ is a paracompact $\sigma$-space. Moreover, each subspace $F_n = \bigcup_{s \in \mathbb{N}} \mathcal{B}_s$ is strongly zero-dimensional and $G \times \mathbb{Z} = \bigcup_{n \in \mathbb{N}} F_n$, and hence $(G \times \mathbb{Z}, \gamma)$ is strongly zero-dimensional by $[13]$ Theorem $2.2.7$.

$(3)$ **Claim 4:** $(G \times \mathbb{Z}, \gamma)$ has no strong $\pi$-base at any compact subset of $G \times \mathbb{Z}$.

Indeed, suppose there exists a compact subset $K \subset G \times \mathbb{Z}$ such that $K$ has a strong $\pi$-base $\varphi$. It is easy to see that the set $\{n : K \cap (G \times \{n\}) \neq \emptyset, n \in \mathbb{Z}\}$ is finite. Moreover, without loss of generalization, we may assume that strong $\pi$-base $\varphi$ at $K$ is countable and each element of $\varphi$ is an open neighborhood of some point in $G \times \mathbb{Z}$. Let $A = \{(b_i, n(i)) : i \in \mathbb{N}, n(i) \in \mathbb{Z}\}$, and let $\varphi = \{(b_i, n(i)) + A[n_i] : i \in \mathbb{N}\}$.

Case 1: The set $\{n(i) : i \in \mathbb{N}\}$ is infinite.

Then it is easy to see that $A$ contains a discrete closed subset $B$ such that $B \cap K = \emptyset$. Let $U$ be an open neighborhood at $K$. Then $U \setminus B$ is also an open neighborhood at $K$, and each $(b_i, n(i)) + A[n_i] \subset U \setminus B$, which is a contradiction.

Case 2: The set $\{n(i) : i \in \mathbb{N}\}$ is finite.

Let $n_0 = \max\{n(i) : i \in \mathbb{N}\} \cup \{n : K \cap (G \times \{n\}) \neq \emptyset, n \in \mathbb{Z}\} \cup \{0\} + 1$. For each $i + n_0 \in \mathbb{N}$, we can choose a $(c_i, i + n_0) \in (b_i, n(i)) + A[n_i]$. Then $\{(c_i, i + n_0) : i \in \mathbb{N}\}$ is closed and discrete subset in $(G, \gamma)$ and $\{(c_i, i + n_0) : i \in \mathbb{N}\} \cap K = \emptyset$. Let $U$ be an open neighborhood at $K$. Then $U \setminus \{(c_i, i + n_0) : i \in \mathbb{N}\}$ is also an open neighborhood at $K$, and each $(b_i, n(i)) + A[n_i] \subset U \setminus \{(c_i, i + n_0) : i \in \mathbb{N}\}$, which is a contradiction.

It follows from Claim 4 and $[6]$ Corollary 4.3 that $b(G \times \mathbb{Z}) \setminus (G \times \mathbb{Z})$ is pseudocompact.

**Remark** (1) Sorgenfrey line $(\mathbb{R}, \tau)$ is a Hausdorff strongly unbounded paratopological group, and hence we can define a Hausdorff paratopological group topology $\sigma$ on $G \times \mathbb{Z}$ such that $\sigma_{G \times \{0\}} = \tau$ and $(G \times \mathbb{Z}, \sigma)$ contains closed copies of $S_2$ and $S_\omega$.

$(2)$ Let $\mathbb{Q}$ be the rationals with the subspace topology of usual topology $\mathbb{R}$. Then $\mathbb{Q}$ is a topological group. It easily check that $\mathbb{Q}$ is a strongly zero-dimensional, nowhere locally compact, paracompact $\sigma$-space. In additional, $\mathbb{Q}$ has a strong $\pi$-base at each point since it is first-countable. It follows from $[3]$ Lemma $2.1$ that the remainder of any Hausdorff compactification of $\mathbb{Q}$ is not pseudocompact.

A paratopological group $G$ is said to have the property $(**)$, if there exists a sequence $\{x_n : n \in \mathbb{N}\}$ of $G$ such that $x_n \to e$ and $x_n^{-1} \to e$.

---

$^{[1]}$A strong $\pi$-base of a space $X$ at a subset $F$ of $X$ is an infinite family $\gamma$ of non-empty open subsets of $X$ such that every open neighborhood of $F$ contains all but finitely many elements of $\gamma$. Clearly, a strong $\pi$-base can be always assumed to be countable.
In [22], C. Liu proved the following theorem.

**Theorem 6.5.** Let $G$ be a paratopological group having the property (**). Then $G$ has a (closed) copy of $S_2$ if it has a (closed) copy of $S_\omega$.

It is natural to ask the following.

**Question 6.6.** Can we omit the property (**) in Theorem 6.5.

By Lemma 6.1, the space $G$ in Theorem 6.4 is not H-closed. Moreover, it is easy to see that the paratopological group topology on $\mathbb{Z} \times \mathbb{Z}$ in Theorem 6.4 does not have the the property (**). Then we have the following questions.

**Question 6.7.** Let $(G, \tau)$ be a H-closed paratopological group. Does there exist a Hausdorff paratopological group topology $\sigma$ on $G \times \mathbb{Z}$ such that $\sigma|G \times \{0\} = \tau$ and $(G \times \mathbb{Z}, \sigma)$ contains closed copies of $S_2$ and $S_\omega$?

**Question 6.8.** Let $G$ be a not H-closed paratopological group. Is it true that $G$ has a (closed) copy of $S_2$ if it has a (closed) copy of $S_\omega$?

**Question 6.9.** Let $G$ be a H-closed paratopological group. Is it true that $G$ has a (closed) copy of $S_2$ if it has a (closed) copy of $S_\omega$?

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