Quark masses and mixing angles in heterotic orbifold models

Pyungwon Ko\textsuperscript{1}, Tatsuo Kobayashi\textsuperscript{2} and Jae-hyeon Park\textsuperscript{3}

\textsuperscript{1,3}Department of Physics, KAIST, Daejon 305-701, Korea
\textsuperscript{2}Department of Physics, Kyoto University, Kyoto 606-8502, Japan

Abstract

We study systematically the possibility for realizing realistic values of quark mass ratios $m_c/m_t$ and $m_s/m_b$ and the mixing angle $V_{cb}$ by using only renormalizable Yukawa couplings derived from heterotic orbifold models. We assume one pair of up and down sector Higgs fields. We find many realistic Yukawa matrices.

\textsuperscript{1}E-mail address: pko@muon.kaist.ac.kr
\textsuperscript{2}E-mail address: kobayash@gauge.scphys.kyoto-u.ac.jp
\textsuperscript{3}E-mail address: jhpark@muon.kaist.ac.kr
1 Introduction

What is the origin of fermion masses and mixing angles is one of important issues in particle physics. They are determined by Yukawa couplings within the framework of the standard model as well as its extension. In a sense, $O(1)$ of Yukawa couplings seem natural. From this viewpoint, how to derive hierarchically suppressed Yukawa couplings is a key-point in understanding the hierarchy of fermion masses and mixing angles.

Superstring theory is a promising candidate for unified theory. Thus, it must predict fermion masses and mixing angles. Actually, Yukawa couplings have been studied in several types of 4D string models, that is, selection rules have been investigated and $O(1)$ of Yukawa couplings have been calculated explicitly in many 4D string models. Among them, heterotic orbifold models [1] as well as intersecting D-brane models are interesting, because they lead to suppressed Yukawa couplings depending on moduli [2, 3, 4]. Calculations of such moduli-dependent Yukawa couplings are possible in orbifold models [2, 3, 5, 6], since string theory can be solved on orbifolds. Calculations of Yukawa couplings in intersecting D-brane models are similar to those in heterotic orbifold models [7, 8, 9]. Furthermore, the selection rule due to space group invariance in orbifold models seems unique [2, 10, 11], e.g. compared with $Z_N$ discrete symmetries. It allows non-trivially off-diagonal couplings. Hence, orbifold models have a possibility for leading to realistic mixing angles as well as fermion masses. On the other hand, the selection rule for allowed couplings in intersecting D-brane models is model-dependent. As much as we are aware, there are no intersecting D-brane models with realistic values of mixing angles at tree-level with the minimal number of Higgs fields. Therefore, it is important to study systematically the possibility for leading to realistic fermion masses and mixing angles in heterotic orbifold models.

Indeed, a similar study has been done in Ref. [12]. However, the analysis in Ref. [12] concentrated rather only to the second and third diagonal elements of Yukawa matrices, that is, the mass ratios between the second and third families. The other entries were assumed to be generated through higher dimensional operators like the Froggatt-Nielsen mechanism. Indeed, it is plausible that non-renormalizable couplings play a role for suppressed entries in Yukawa matrices. However, in realistic patterns of Yukawa matrices, the $(2,3)$ and $(3,2)$ entries are the same as or larger than the $(2,2)$ entries. Thus, if we assume that the $(2,2)$ entries are originated from stringy renormalizable couplings, it is natural to expect the $(2,3)$ and $(3,2)$ entries are also obtained as stringy renormalizable couplings. In this paper we study the possibility for predicting a realistic mixing angle as well as mass ratios. We concentrate ourselves mainly to $(2 \times 2)$ sub-matrices of the second and third quark families. We study systematically the possibility for obtaining realistic values of $V_{cb}$ and mass ratios $m_c/m_t$ and $m_s/m_b$. Then, we will show examples to lead to them. To our knowledge, our result is the first examples, which show explicitly the possibility for predicting realistic values of
mixing angles by use of only renormalizable couplings in string models, when we consider a pair of up and down Higgs fields, although already there are proposals to introduce more Higgs fields to lead to realistic Yukawa matrices [13].

2 Orbifold models and selection rule

2.1 Fixed points and twisted sectors

Here we give a brief review on the structure of fixed points on orbifolds and corresponding twisted states. (See for their details Ref. [10, 11].) An orbifold is defined as a division of a torus by a discrete rotation, i.e., a twist \( \theta \). For example, the 2D \( \mathbb{Z}_3 \) orbifold is obtained by dividing \( \mathbb{R}^2 \) by the \( SU(3) \) root lattice and its automorphism \( \theta \), that is, the Coxeter element of \( SU(3) \) algebra, which transforms the \( SU(3) \) simple roots \( e_1 \) and \( e_2 \),

\[
\theta e_1 \rightarrow e_2, \quad \theta e_2 \rightarrow -e_1 - e_2.
\]  

(1)

Thus, the twist \( \theta \) is the \( \mathbb{Z}_3 \) rotation. Similarly, the 2D \( \mathbb{Z}_6 \) orbifold is obtained as a division of \( \mathbb{R}^2 \) by the \( G_2 \) lattice and the \( G_2 \) Coxeter element, which transforms the \( G_2 \) simple roots \( e_1 \) and \( e_2 \),

\[
\theta e_1 \rightarrow -e_1 - e_2, \quad \theta e_2 \rightarrow 3e_1 + 2e_2,
\]  

(2)

that is, the \( \mathbb{Z}_6 \) twist.

The 6D \( \mathbb{Z}_3 \) orbifold is a direct product of three 2D \( \mathbb{Z}_3 \) orbifolds. Similarly, the so-called 6D \( \mathbb{Z}_6 \text{-I} \) orbifold is a direct product of two 2D \( \mathbb{Z}_6 \) orbifolds and one 2D \( \mathbb{Z}_3 \) orbifold, that is, eigenvalues of its twist \( \theta \) are obtained as \( \theta = \text{diag}(e^{2\pi i/6}, e^{2\pi i/6}, e^{-2\pi i/3}) \). Other 6D orbifolds are also defined in the same way as a division of \( \mathbb{R}^6 \) by a Lie lattice \( \Lambda \) and its Coxeter element \( \theta \).

On an orbifold, there are two types of closed strings. One type is untwisted strings, which are closed before orbifold twisting, and the other type is twisted strings. The latter plays a important role here and has the following twisted boundary condition,

\[
X^i(\sigma = 2\pi) = (\theta^k X)^i(\sigma = 0) + ne^i,
\]  

(3)

where \( e^i \) is the lattice vector defining \( \Lambda \) and \( n \) is an integer. Thus, ground states of \( \theta^k \) twisted sectors are assigned with fixed points \( f \) on the orbifold, which are defined as

\[
f^i = (\theta^k f)^i + ne^i.
\]  

(4)

\footnote{See Ref. [10, 11] for details of Lie lattices and their Coxeter elements, which realize \( Z_N \) orbifold twists.}
This fixed point \( f \) is presented by the corresponding space group element \((\theta^k, ne^i)\). For example, the 2D \( Z_3 \) orbifold has three fixed points under \( \theta \), and those are obtained as

\[
\begin{align*}
   g_{Z_3,1}^{(0)} &= (0, 0), \\
   g_{Z_3,1}^{(1)} &= (2/3, 1/3), \\
   g_{Z_3,1}^{(2)}(1/3, 2/3),
\end{align*}
\]

in the \( SU(3) \) simple root basis, up to \((1 - \theta)\Lambda_{SU(3)}\). Furthermore, those are denoted as

\[
g_{Z_3,1}^{(n)} : (\theta, ne^1),
\]

where \( n = 0, 1, 2 \). The corresponding twisted ground states are obtained as \(|g_{Z_3,1}^{(n)}\rangle\) with \( n = 0, 1, 2 \). Twenty seven fixed points on the 6D \( Z_3 \) orbifold are obtained as direct products of three fixed points on the 2D orbifolds as

\[
g_{Z_3,1}^{(n)} \otimes g_{Z_3,1}^{(n')} \otimes g_{Z_3,1}^{(n'')},
\]

where \( n, n', n'' = 0, 1, 2 \), and moreover the corresponding twisted ground states are denoted as

\[
|g_{Z_3,1}^{(n)}\rangle \otimes |g_{Z_3,1}^{(n')}\rangle \otimes |g_{Z_3,1}^{(n'')}\rangle.
\]

\( \theta^2 \)-twisted states are CPT conjugates of \( \theta \)-twisted states.

Similarly, fixed points on the 2D \( Z_6 \) orbifold are obtained. There is only one fixed point under \( \theta \) on the 2D \( Z_6 \) orbifold, that is,

\[
g_{Z_6,1}^{(0)} = (0, 0),
\]

in the \( G_2 \) simple root basis. Furthermore, there are three fixed points under \( \theta^2 \),

\[
\begin{align*}
   g_{Z_6,2}^{(0)} &= (0, 0), \\
   g_{Z_6,2}^{(1)} &= (0, 1/3), \\
   g_{Z_6,2}^{(2)} &= (0, 2/3).
\end{align*}
\]

Recall that these fixed points are defined up to the \( G_2 \) lattice. \(^5\) The corresponding three twisted ground states are denoted as \(|g_{Z_6,2}^{(i)}\rangle\). However, note that all of three points \( g_{Z_6,2}^{(i)} \) are not fixed under \( \theta \). While \( g_{Z_6,2}^{(0)} \) is still a fixed point of \( \theta \), the two fixed points \( g_{Z_6,2}^{(1)} \) and \( g_{Z_6,2}^{(2)} \) are transformed to each other by \( \theta \). Physical states consist of \( \theta \)-eigenstates. Thus, we take linear combinations of states corresponding to \( g_{Z_6,2}^{(1)} \) and \( g_{Z_6,2}^{(2)} \) as \([14, 10]\)

\[
|g_{Z_6,2}^{(1)}; \pm 1\rangle \equiv \frac{1}{\sqrt{2}} \left(|g_{Z_6,2}^{(1)}\rangle \pm |g_{Z_6,2}^{(2)}\rangle\right),
\]

with the eigenvalues \( \gamma = \pm 1 \), while the state \(|g_{Z_6,2}^{(0)}\rangle\) corresponding to the fixed point \( g_{Z_6,2}^{(0)} \) is a \( \theta \)-eigenstate.

\(^5\)For example, for the fixed point \( g_{Z_6,2}^{(2)} = (0, 2/3) \), it is often useful to use the following point, \( g_{Z_6,2}^{(2)} = (1, 2/3) \), in order to calculate Yukawa couplings, because this point is closer to the origin than \((0, 2/3)\) and we have exactly \( \theta(0, 1/3) = (1, 2/3) \).
In addition, there are four fixed points under \( \theta^3 \),

\[
\begin{align*}
g_{Z_6,3}^{(0)} &= (0, 0), & g_{Z_6,3}^{(1)} &= (0, 1/2), \\
g_{Z_6,3}^{(2)} &= (1/2, 0), & g_{Z_6,3}^{(3)} &= (1/2, 1/2).
\end{align*}
\] (12)

Recall again that these fixed points are defined up to the \( G_2 \) lattice. \(^6\) Not all of four points are fixed under \( \theta \). The \( \theta \)-eigenstates for each \( G_2 \) part are obtained as

\[
|g_{Z_6,3}^{(0)}; \gamma\rangle = \frac{1}{\sqrt{3}} \left( |g_{Z_6,3}^{(1)}; \gamma\rangle + \gamma |g_{Z_6,3}^{(2)}; \gamma\rangle + \gamma^2 |g_{Z_6,3}^{(3)}; \gamma\rangle \right),
\] (13)

where \( \gamma = 1, \omega, \omega^2 \) with \( \omega = e^{2\pi i/3} \).

Fixed points and twisted ground states for the 6D \( Z_6 \)-I orbifold are obtained as direct products of two 2D \( Z_6 \) orbifolds and one 2D \( Z_3 \) orbifold. The \( \theta \) twisted sector has the following fixed points,

\[
g_{Z_6,1}^{(0)} \otimes g_{Z_6,1}^{(0)} \otimes g_{Z_3,1}^{(i)},
\] (14)

for \( i = 0, 1, 2 \), and the corresponding ground states are denoted as

\[
|g_{Z_6,1}^{(0)} \otimes g_{Z_6,1}^{(0)} \otimes g_{Z_3,1}^{(i)}\rangle.
\] (15)

The \( \theta^2 \) twisted sector has the following fixed points,

\[
g_{Z_6,2}^{(i)} \otimes g_{Z_6,2}^{(i')} \otimes g_{Z_3,2}^{(j)},
\] (16)

for \( i, i', j = 0, 1, 2 \). The \( \theta \)-eigenstates are obtained as

\[
\begin{align*}
|g_{Z_6,2}^{(0)} \otimes g_{Z_6,2}^{(0)} \otimes g_{Z_3,2}^{(j)}\rangle, \\
|g_{Z_6,2}^{(0)} \otimes g_{Z_6,2}^{(i)} \otimes g_{Z_3,2}^{(j)}\rangle, \\
|g_{Z_6,2}^{(i)} \otimes g_{Z_6,2}^{(0)} \otimes g_{Z_3,2}^{(j)}\rangle, \\
|g_{Z_6,2}^{(i)} \otimes g_{Z_6,2}^{(i')} \otimes g_{Z_3,2}^{(j)}\rangle, \\
|g_{Z_6,2}^{(i)} \otimes g_{Z_6,2}^{(i')} \otimes g_{Z_3,2}^{(j')}\rangle.
\end{align*}
\] (17)

for \( \gamma, \gamma' = \pm 1 \). The \( \theta^3 \) twisted sector has the following fixed points,

\[
g_{Z_6,3}^{(m)} \otimes g_{Z_6,3}^{(m')}.
\] (18)

for \( m, m' = 0, 1, 2, 3 \). The \( \theta \)-eigenstates are obtained as

\[
\begin{align*}
|g_{Z_6,3}^{(0)} \otimes g_{Z_6,3}^{(0)}\rangle, \\
|g_{Z_6,3}^{(0)} \otimes g_{Z_6,3}^{(1)}\rangle, \\
|g_{Z_6,3}^{(0)} \otimes g_{Z_6,3}^{(1)} \otimes g_{Z_3,3}^{(j)}\rangle, \\
|g_{Z_6,3}^{(1)} \otimes g_{Z_6,3}^{(1)} \otimes g_{Z_3,3}^{(j)}\rangle, \\
|g_{Z_6,3}^{(1)} \otimes g_{Z_6,3}^{(1)} \otimes g_{Z_3,3}^{(j')}\rangle.
\end{align*}
\] (19)

\(^6\)For Yukawa calculations, the following fixed points are often useful, \( g_{Z_6,3}^{(0)} = (0, 0), g_{Z_6,3}^{(1)} = (1, 1/2), g_{Z_6,3}^{(2)} = (1/2, 0), g_{Z_6,3}^{(3)} = (1/2, 1/2) \).
where $\gamma, \gamma' = 1, \omega, \omega^2$.

Similarly, we can study fixed points and the structure of ground states for other $Z_N$ orbifolds, and those are explicitly shown in Ref. [10, 11]. In what follows, the $\theta^k$ twisted sector of $Z_N$ orbifold models is denoted as $\hat{T}_k$.

### 2.2 Selection rule

Here we give a brief review on the selection rule for Yukawa couplings in orbifold models. (See for their details Ref. [10, 11].) The fixed point $f$ is denoted by its space group element, $(\theta^k, (1 - \theta^k)f)$, as said in the previous subsection. Thus, the three states corresponding to three fixed points $(\theta^k_i, (1 - \theta^k_i)f_i)$ for $i = 1, 2, 3$ can couple if the product of their space group elements $\prod_i(\theta^k_i, (1 - \theta^k_i)f_i)$ is equivalent to identity. Here, note that the fixed point $(\theta^k, (1 - \theta^k)f)$ is equivalent to $(\theta^k, (1 - \theta^k)(f + \Lambda))$, that is, they belong to the same conjugacy class. Thus, the Yukawa coupling among three states is allowed if

$$\prod_i(\theta^k_i, (1 - \theta^k_i)f_i) = (1, \sum_i(1 - \theta^k_i)\Lambda).$$

(20)

This is called the space group selection rule [2]. That includes the point group selection rule, which requires a product of twists, $\prod_i \theta^k_i$, to be identity.

For example, let us consider the $\hat{T}_1\hat{T}_1\hat{T}_1$ coupling in the $Z_3$ orbifold models. The $\hat{T}_1$ sector of the 2D orbifold has three fixed points, $g_{Z3,1}^{(i)} (i = 0, 1, 2)$, and three states corresponding fixed points $g_{Z3,1}^{(i_1)}$, $g_{Z3,1}^{(i_2)}$ and $g_{Z3,1}^{(i_3)}$ can couple when the following equation is satisfied,

$$i_1 + i_2 + i_3 = 0 \pmod{3}.$$  

(21)

Hence, couplings on the 2D $Z_3$ orbifold are diagonal, that is, when we choose two states, the other state, which is allowed to couple with them, is fixed uniquely. Since the $\hat{T}_1$ sector on the 6D $Z_3$ orbifold is obtained as a direct product of those on the 2D $Z_3$, all of Yukawa couplings on the 6D $Z_3$ orbifold are diagonal. Therefore, we always obtain the following form of Yukawa matrix,

$$Y = \begin{pmatrix} Y_{11} & 0 & 0 \\ 0 & Y_{22} & 0 \\ 0 & 0 & Y_{33} \end{pmatrix},$$

(22)

when we consider only one pair of up and down Higgs fields. Thus, in this type of models we can not derive non-vanishing mixing angles. All of $Z_N$ orbifold models with $N = 1$ 4D supersymmetry are classified as $Z_3$, $Z_4$, $Z_6$-I, $Z_6$-II, $Z_7$, $Z_8$-I, $Z_{8'}$-II, $Z_{12}$-I and $Z_{12'}$-II orbifolds [1, 15, 16]. The situation in $Z_7$ orbifold models is the same as one in $Z_3$ orbifold models, and one can not derive non-vanishing mixing angles only with renormalizable couplings and the minimal number of up and down Higgs fields.
However, the situation is different for non-prime order orbifold models, and off-diagonal couplings are allowed. For example, let us consider $Z_6$-I orbifold models. The 6D $Z_6$-I orbifold consists of $G_2 \times G_2 \times SU(3)$ part. The $SU(3)$ part is the same as the $Z_3$ orbifold, and only diagonal couplings are allowed. Thus, this part is irrelevant to us, and we concentrate to the $G_2 \times G_2$ part. Among Yukawa couplings of twisted sectors, the point group selection rule and $H$-momentum conservation [17] allow only the following couplings [14],

$$\hat{T}_1\hat{T}_2\hat{T}_3, \quad \hat{T}_2\hat{T}_2\hat{T}_2. \quad (23)$$

For the $\hat{T}_1\hat{T}_2\hat{T}_3$ couplings, the space group selection rule requires only that the product of $\theta$-eigenvalues among coupling states must be equal to identity, that is, $\prod \gamma = 1$. Thus, the twisted states relevant to allowed $\hat{T}_1\hat{T}_2\hat{T}_3$ couplings are the single $\hat{T}_1$ state,

$$|g_{Z_6,1}^{(0)}\rangle \otimes |g_{Z_6,1}^{(0)}\rangle, \quad (24)$$

and the five $\hat{T}_2$ states,

$$\begin{align*}
\hat{T}_2^{(1)} & \equiv |g_{Z_6,2}^{(0)}\rangle \otimes |g_{Z_6,2}^{(0)}\rangle, \\
\hat{T}_2^{(2)} & \equiv |g_{Z_6,2}^{(0)}\rangle \otimes |g_{Z_6,2}^{(1)}; +1\rangle, \\
\hat{T}_2^{(3)} & \equiv |g_{Z_6,2}^{(1)}; +1\rangle \otimes |g_{Z_6,2}^{(0)}\rangle, \\
\hat{T}_2^{(4,\gamma)} & \equiv |g_{Z_6,2}^{(1)}; \gamma\rangle \otimes |g_{Z_6,2}^{(1)}; \gamma^{-1}\rangle, \quad (25)
\end{align*}$$

where $\gamma = \pm 1$, and six $\hat{T}_3$ states,

$$\begin{align*}
\hat{T}_3^{(1)} & \equiv |g_{Z_6,3}^{(0)}\rangle \otimes |g_{Z_6,3}^{(0)}\rangle, \\
\hat{T}_3^{(2)} & \equiv |g_{Z_6,3}^{(0)}\rangle \otimes |g_{Z_6,3}^{(1)}; +1\rangle, \\
\hat{T}_3^{(3)} & \equiv |g_{Z_6,3}^{(1)}; +1\rangle \otimes |g_{Z_6,3}^{(0)}\rangle, \\
\hat{T}_3^{(4,\gamma)} & \equiv |g_{Z_6,3}^{(1)}; \gamma\rangle \otimes |g_{Z_6,3}^{(1)}; \gamma^{-1}\rangle, \quad (26)
\end{align*}$$

where $\gamma = 1, \omega, \omega^2$. Hence, in the case that fermions are assigned with $\hat{T}_2$ and $\hat{T}_3$ sectors and the Higgs is assigned with $\hat{T}_1$, one can obtain non-trivial Yukawa matrices for three flavors, whose determinant does not vanish and diagonalizing matrix is not identity.

On the other hand, the space group selection for the $\hat{T}_2\hat{T}_2\hat{T}_2$ couplings is exactly the same as one of $\hat{T}_1\hat{T}_1\hat{T}_1$ coupling in $Z_3$ orbifold models, when we consider the basis of twisted states corresponding directly to fixed points. However, in the $Z_6$-I orbifold models, we take linear combinations as Eq. (11), and such linear combinations can lead to non-trivial mixing.

We give a comment on the third plane, again. In order to allow Yukawa couplings through $\hat{T}_1\hat{T}_2\hat{T}_3$ couplings, we have to assign the same fixed point on the $SU(3)$ plane for $\hat{T}_1$ and $\hat{T}_2$, while the $SU(3)$ part is the fixed torus for $\hat{T}_3$. In
In this case, we just have $O(1)$ contribute, which are universal to different flavors. That implies that also for $\hat{T}_2 \hat{T}_2 \hat{T}_2$ couplings the contribution from the third part is universal, because we have to assign all of three families of left-handed quarks to the same fixed point on the third plane. That leads to an overall suppression factor or an overall factor of $O(1)$. Anyway, that does not contribute to the mixing angles or ratios of fermion masses. We neglect this part, and concentrate to the $G_2 \times G_2$ part for the $\hat{T}_2 \hat{T}_2 \hat{T}_2$ couplings, too. Actually, in the following section we assume all of relevant modes correspond to the same fixed point on the $SU(3)$ plane.

Similarly, we can study Yukawa couplings for other non-prime order $Z_N$ orbifold models. In general, they allow off-diagonal couplings. The numbers of twisted states relevant to allowed off-diagonal couplings in $Z_4$, $Z_6$-II, $Z_8$-II $Z_{12}$-II orbifold models are smaller than one in $Z_6$-I orbifold models. In the next subsection, we will mention our reason why we do not study $Z_8$-I or $Z_{12}$-I orbifold models. Thus, here we concentrate ourselves to analysis on Yukawa matrices in $Z_6$-I models. Yukawa matrices in other orbifold models will be studied systematically elsewhere.

### 2.3 Yukawa couplings

The strength of Yukawa couplings has been calculated by use of 2D conformal field theory. It depends on locations of fixed points. The Yukawa coupling strength of the $\hat{T}_1 \hat{T}_2 \hat{T}_3$ coupling in $Z_6$-I orbifold models is obtained for the $G_2 \times G_2$ part as

$$Y = \sum_{f_{23}=f_2-f_3+\Lambda} \exp[-\frac{\sqrt{3}}{4\pi} f_{23}^T M f_{23}], \quad (27)$$

up to an overall normalization factor, where

$$M = \begin{pmatrix}
R_1^2 & -\frac{3}{2} R_1^2 & 0 & 0 \\
-\frac{3}{2} R_1^2 & 3R_1^2 & 0 & 0 \\
0 & 0 & R_2^2 & -\frac{3}{2} R_2^2 \\
0 & 0 & -\frac{3}{2} R_2^2 & 3R_2^2
\end{pmatrix}, \quad (28)$$

in the $G_2 \times G_2$ root basis. Here, $f_2$ and $f_3$ denote fixed points of $\hat{T}_2$ and $\hat{T}_3$ sectors, respectively, and $R_i$ corresponds to the radius of the $i$-th torus, which can be written as a real part of the $i$-th Kähler moduli $T_i$ up to a constant factor, and the imaginary part of $T_i$ can lead to CP phases. However, we will concentrate ourselves to the $(2 \times 2)$ sub-matrices. On the other hand, the full $(3 \times 3)$ matrices are necessary to study physical CP phase. Thus, we do not consider imaginary

\footnote{See Ref. [18] for the proper normalization of the moduli and Yukawa couplings such that the transformation $T_i \rightarrow T_i + i$ is a symmetry, that is, we have the relation $Re(T_i) = \sqrt{3} R_i^2/(16\pi^2)$.}
parts of the Kähler moduli. The states with fixed points in the same conjugacy class contribute to the Yukawa coupling. Thus, we take summation of those contributions in eq. (27). However, the states corresponding to the nearest fixed points \((f_2, f_3)\) contribute dominantly to the Yukawa coupling for a large value of \(R_i\). Hence, we calculate Yukawa couplings for the nearest fixed points \((f_2, f_3)\).

Similarly, the strength of \(^dT_2 \tilde{T}_2 \tilde{T}_2\) Yukawa couplings is obtained as

\[
Y = \sum_{f_{23}=f_2-f_3+\Lambda} \exp\left(-\frac{\sqrt{3}}{16\pi} f_{23}^T M f_{23}\right),
\]

where \(M\) is the same matrix as Eq.(28). Here, \(f_2\) and \(f_3\) denote two of three fixed points in \(\tilde{T}_2\) sectors. Recall that when we choose two states, the other state, which is allowed to couple, is uniquely fixed in the basis of states corresponding directly to fixed points.

Similarly, we can calculate Yukawa couplings in generic orbifold models, and those for all of \(Z_N\) orbifold models are shown in Ref. [11]. Off-diagonal couplings are originated from the \(SO(9)\) lattice part for the \(Z_8\)-I orbifold and the \(F_4\) lattice part for the \(Z_{12}\)-II orbifold. Thus, in these models only one parameter, say \(R_2^1\), corresponding to the volume of the \(SO(9)\) torus or the \(F_4\) torus is relevant to mass ratios and mixing angles, while in \(Z_6\)-I models two parameters \(R_1^2\) and \(R_2^2\) are relevant. That is the reason why we concentrate ourselves to \(Z_6\)-I orbifold models here, but we will study other \(Z_N\) orbifold models elsewhere.

\(^8\)Yukawa couplings also depend on other moduli, e.g. continuous Wilson line moduli, but here we consider only the dependence of \(R_i\).

\(^9\)To be explicit, the \(\tilde{T}_1 \tilde{T}_2 \tilde{T}_3\) couplings have the factor \(4Re(T_1)Re(T_2)(2Re(T_3))^{1/2}\) due to the normalization of the Kähler metric, and the \(\tilde{T}_2 \tilde{T}_2 \tilde{T}_2\) couplings have the factor \(8Re(T_1)Re(T_2)Re(T_3)\) due to the normalization of the Kähler metric. Furthermore, Yukawa couplings \(\hat{Y}_{ijk}\) in global supersymmetric models are related with Yukawa couplings \(Y_{ijk}^{(SUGRA)}\) in supergravity as \(\hat{Y}_{ijk} = Y_{ijk}^{(SUGRA)} e^{(K)^{1/2}}\) up to Kähler metric [19]. Here, \(K\) is the Kähler potential and obtained as \(K = -\sum_i \ln(T_i + \bar{T}_i) + \cdots\), where ellipsis denotes the contributions due to other fields with large vacuum expectation values. Hence, the Kähler potential \(K\) itself is also relevant to the overall magnitude of Yukawa matrices, but irrelevant to mass ratios and mixing angles.
Table 1: Input values used for fitting the moduli. The current quark masses in the \( \overline{\text{MS}} \) scheme at \( m_W \) scale are from Ref. [20], and the CKM matrix elements from Ref. [21]. (See also Ref. [22].) We supposed that \( V_{ub} \) were real in the analysis.

### 3 Quark masses and mixing angles

Here we study systematically the possibilities for leading to realistic quark masses and the mixing angle for the second and third families in \( Z_6 \)-I orbifold models by use of the structure of fixed points and the strength of Yukawa couplings explained in the previous section. We assume the minimal number of up and down Higgs fields. Concerned about the other entries of Yukawa matrices, we assume that Yukawa entries relevant to the first family may be originated from higher dimensional operators.

The experimental values are listed in Table 1, and the ratios of running masses at the weak scale are displayed in the last row of Table 3. The Yukawa couplings (27), (29) are obtained at the Planck scale. Precise values at low energy depend on renormalization group flows, that is, the matter content of models. Thus, we do not try to derive exact values, but we try to fit their orders. We concentrate to the mass ratios, \( m_c/m_t \) and \( m_s/m_b \), and the mixing angle \( V_{cb} \), but pay less attention to the overall magnitude like \( Y_t \) and \( Y_b \), because moduli values other than \( R_1 \) and \( R_2 \), in general, contribute to the overall magnitude as said in the previous section and we have ambiguity.

\( Z_6 \)-I orbifold models have two types of Yukawa couplings, \( \hat{T}_1 \hat{T}_2 \hat{T}_3 \) and \( \hat{T}_2 \hat{T}_2 \hat{T}_2 \). The \( \hat{T}_1 \) sector has a single relevant state for the \( G_2 \times G_2 \) part, that is, there is no variety for families. That implies that we have to assign matter fields with \( \hat{T}_2 \) and \( \hat{T}_3 \). Hence, we have five classes of assignments, which are shown in Table 2. Here, \( Q \), \( u \) and \( d \) denote the left-handed quarks, the up and down sector of right-handed quarks, respectively. In Assignments 1 and 2, both the up and down sectors of Yukawa couplings are originated from \( \hat{T}_1 \hat{T}_2 \hat{T}_3 \) couplings. On the other hand, in Assignment 3, the up sector Yukawa couplings are originated from \( \hat{T}_1 \hat{T}_2 \hat{T}_3 \) couplings, and the down sector Yukawa couplings are originated from \( \hat{T}_2 \hat{T}_2 \hat{T}_2 \) couplings. Oppositely, in Assignment 4, the up sector Yukawa couplings are originated from \( \hat{T}_2 \hat{T}_2 \hat{T}_2 \) couplings, and the down sector Yukawa couplings are originated from \( \hat{T}_1 \hat{T}_2 \hat{T}_3 \) couplings. In Assignment 5, both the up and down sectors of Yukawa couplings are originated from \( \hat{T}_2 \hat{T}_2 \hat{T}_2 \) couplings.

Here we concentrate ourselves to the \( \hat{T}_2 \) and \( \hat{T}_3 \) states with \( \gamma = 1 \). The relevant number of \( \hat{T}_2 \) states is 4 as Eq.(25). Thus, the possible number that two flavors
are assigned with $\hat{T}_2$ states leading to different magnitudes of Yukawa couplings is equal to 6. Similarly, the possible number that two flavors are assigned with $\hat{T}_3$ states leading to different magnitudes of Yukawa couplings is also equal to 6. Hence, in Assignment 1 there are $6^3 = 216$ possibilities, and Assignment 2 also has 216 possibilities.

For Assignment 3, there are further four possibilities to assign $H_d$ with $\hat{T}_2$ states. Totally, there are $6^3 \times 4 = 864$. Similarly, Assignment 4 has 864 possibilities. Moreover, Assignment 5 has $6^3 \times 4^2 = 3456$.

We investigate systematically all of these possibilities, varying two independent parameters $R_1$ and $R_2$. We find many configurations leading to realistic values of $m_c/m_t$, $m_s/m_b$ and $V_{cb}$, which are consistent with experimental values up to $O(1)$ factor. In particular, the numbers of realistic examples in Assignments 1 and 2 are larger than those in other assignments. Here we show one of the best fitting examples in each class of Assignment. Table 3 shows examples leading to realistic values of $m_c/m_t$, $m_s/m_b$ and $V_{cb}$. The first column shows the class of Assignments. The second column shows assignments of quarks and Higgs fields with twisted states. The third and fourth columns show the values of $R_1^2$ and $R_2^2$ corresponding to the best fit with the experimental values. The last three columns show predicted values of $m_c/m_t$, $m_s/m_b$ and $V_{cb}$. We have also studied the cases including $\hat{T}_2$ states with $\gamma \neq 1$, but we have not obtained more realistic results than those shown in Table 3. We do not need to consider $\hat{T}_3$ states with $\gamma \neq 1$, since they lead to the same strength of $\hat{T}_1\hat{T}_2\hat{T}_3$ type Yukawa coupling as $\gamma = 1$.

The second row in Table 3 shows an example in the class of Assignment 1. Its explicit form of up and down Yukawa matrices are obtained as

$$Y_u = \begin{pmatrix} 0.0416 & 0.718 \\ 0.0557 & 0.848 \end{pmatrix}, \quad Y_d = \begin{pmatrix} 0.0313 & 0.0416 \\ 0.0370 & 0.0557 \end{pmatrix}. \quad (30)$$

We have neglected a common factor of $O(1)$. In this example, the down sector corresponds to a democratic form, while the up sector is hierarchical. We have the

| Class       | $Q$ | $u_1$ | $u_2$ | $u_3$ | $H_u$ | $H_d$ |
|-------------|-----|-------|-------|-------|-------|-------|
| Assignment 1| $T_2$| $T_3$ | $T_3$ | $T_1$ | $T_1$ |       |
| Assignment 2| $T_3$| $T_2$ | $T_2$ | $T_1$ | $T_1$ |       |
| Assignment 3| $T_2$| $T_3$ | $T_2$ | $T_1$ | $T_2$ |       |
| Assignment 4| $T_2$| $T_2$ | $T_3$ | $T_2$ | $T_1$ |       |
| Assignment 5| $T_2$| $T_2$ | $T_2$ | $T_2$ | $T_2$ |       |

Table 2: Four classes of Assignments

\[^{10}\text{In practice, we have removed trivial possibilities, e.g. matrices with vanishing determinant and matrices with degenerate eigenvalues, before numerical study.}\]
ratio $Y_t/Y_b = 13$. In the class of Assignment 1, there are many examples leading to similarly realistic results for $m_c/m_t$, $m_s/m_b$ and $V_{cb}$, with both $Y_t/Y_b = O(1)$ and $Y_t/Y_b = O(10)$, including the case with different overall magnitude.

The third row in Table 3 shows an example in the class of Assignment 2. The explicit Yukawa matrices are obtained as

$$
Y_u = \begin{pmatrix}
0.0281 & 0.439 \\
0.0371 & 0.665
\end{pmatrix}, \quad Y_d = \begin{pmatrix}
0.0199 & 0.0281 \\
0.0302 & 0.0371
\end{pmatrix}.
$$

(31)

This form is similar to Eq. (30) and leads to the ratio $Y_t/Y_b = 14$. In the class of Assignment 2, we have many examples leading to similarly realistic values of $m_c/m_t$, $m_s/m_b$ and $V_{cb}$, with both $Y_t/Y_b = O(1)$ and $Y_t/Y_b = O(10)$, including the case with different overall magnitude.

The fourth row in Table 3 shows an example in the class of Assignment 3. The explicit Yukawa matrices are obtained as

$$
Y_u = \begin{pmatrix}
0.0000636 & 2.61 \times 10^{-7} \\
0.0000344 & 0.0168
\end{pmatrix}, \quad Y_d = \begin{pmatrix}
0 & 0.0251 \\
0.225 & 0.500
\end{pmatrix}.
$$

(32)

This leads to realistic values for $m_c/m_t$, $m_s/m_b$ and $V_{cb}$, but the ratio $Y_t/Y_b$ is small. If we change the configuration on the $SU(3)$ part for the down sector such that $Q$, $d$ and $H_d$ correspond to different fixed points on the $SU(3)$ part, we can obtain a small value of $Y_t$. However, this example leads to too much suppressed top Yukawa coupling. Similarly, in Assignment 3, most of examples leading to realistic values of $m_c/m_t$, $m_s/m_b$ and $V_{cb}$ predict the top Yukawa coupling, which is smaller than $O(1)$. Thus, these examples may lead to smaller top mass than the experimental value. One solution for this problem is to enhance the overall magnitude of Yukawa couplings by choosing suitable values of moduli, which contribute only the overall size of Yukawa couplings (through the Kähler potential), but not ratios or mixing angles.

The fifth row in Table 3 shows an example in the class of Assignment 4. The explicit form of Yukawa matrices in this example are obtained as

$$
Y_u = \begin{pmatrix}
0 & 0.00569 \\
0.0179 & 0.159
\end{pmatrix}, \quad Y_d = \begin{pmatrix}
0.000133 & 3.83 \times 10^{-7} \\
2.88 \times 10^{-12} & 0.00379
\end{pmatrix}.
$$

(33)

This leads to the realistic mass ratios and mixing angle, but the small overall magnitude similarly to the example in Assignment 3. We need some enhancement of the overall magnitude.

The sixth row in Table 3 shows an example in the class of Assignment 5. The explicit form of Yukawa matrices in this example are obtained as

$$
Y_u = \begin{pmatrix}
0 & 0.0309 \\
0.0309 & 0.500
\end{pmatrix}, \quad Y_d = \begin{pmatrix}
0.00132 & 0 \\
0.0214 & 0.0309
\end{pmatrix}.
$$

(34)
This leads to the ratio $Y_t/Y_b = 13$. In Assignment 5, we also have examples leading to similarly realistic results for $m_c/m_t$, $m_s/m_b$ and $V_{cb}$, and the ratio $Y_t/Y_b = O(1)$.

As results, we have found many examples of assignments leading to realistic values of the mass ratios $m_c/m_t$ and $m_s/m_b$ and the mixing angle $V_{cb}$. It is quite non-trivial to derive reasonable values of three observables $m_c/m_t$, $m_s/m_b$ and $V_{cb}$ by only two independent parameters $R_1$ and $R_2$ in models with renormalizable couplings, which can be derived from string models.

So far, we have considered the Yukawa couplings which are induced only for the nearest fixed points. Such approximation is reliable in our results, because our realistic examples are obtained for large values of $R_1$. Actually, we have examined Yukawa coupling contributions due to quite far fixed points. Then we have obtained almost same results.

We can extend the above analysis to the full $(3 \times 3)$ Yukawa matrices. For example, we assign three families of $Q$, $u$, $d$ as follows,

$$
Q : \hat{T}_3^{(1)}, \hat{T}_3^{(2)}, \hat{T}_3^{(4)},
$$

$$
u : \hat{T}_2^{(1)}, \hat{T}_2^{(2)}, \hat{T}_2^{(3)},
$$

$$
d : \hat{T}_2^{(2)}, \hat{T}_2^{(3)}, \hat{T}_2^{(4)},
$$

and both of Higgs fields with $\hat{T}_1$. Then, this example leads to the following mass ratios and mixing angles,

$$
\frac{m_u}{m_t} = 3.7 \times 10^{-9}, \quad \frac{m_c}{m_t} = 3.8 \times 10^{-3},
$$

$$
\frac{m_d}{m_b} = 1.7 \times 10^{-3}, \quad \frac{m_s}{m_b} = 9.9 \times 10^{-3},
$$

$$
V_{us} = 0.22, \quad V_{cb} = 4.6 \times 10^{-9}, \quad V_{ub} = 4.8 \times 10^{-9},
$$

when we take $R_1^2 = 37.1$ and $R_2^2 = 572$. In this example, all of mass ratios except $m_u/m_t$ are reasonable values, and the Cibibbo angle $V_{us}$ is predicted as a realistic value by only two parameters, $R_1$ and $R_2$, but the other mixing angles are suppressed too much, although it is non-trivial to fix seven observables by
only two parameters. Similarly, we have investigated all of possibilities, but it seems difficult to fit all of quark masses and mixing angles to be consistent with the experimental values by only two parameters \( R_1 \) and \( R_2 \). Some extension is necessary in \( Z_6 \)-I orbifold models in order to derive all of quark masses and mixing angles. It may be reasonable that higher dimensional operators can contribute to small couplings \(^{11}\) like Yukawa matrix entries relevant to the first family, as we assumed in the analysis for \((2 \times 2)\) sub-matrices. It is also plausible that loop-effects have non-trivial contributions for some entries. Alternatively, it would be helpful to introduce more than one pair of \( H_u \) and \( H_d \).

### 4 Conclusion

We have systematically studied the possibility for leading to realistic values of \( m_c/m_t \), \( m_s/m_b \) and \( V_{cb} \) in \( Z_6 \)-I orbifold models. We have found realistic examples of Yukawa matrices. In particular, the classes of Assignments 1 and 2 have many realistic Yukawa matrices. Our result is the first examples to show the possibility for deriving the realistic mixing angle by renormalizable couplings in string models with one pair of \( H_u \) and \( H_d \).

To realize our results, the moduli \( R_1 \) and \( R_2 \) must be stabilized at proper values. How to stabilize these moduli is an important issue to study further.

One can extend our analysis to other non-prime order \( Z_N \) orbifold models. Similarly we can discuss \( Z_N \times Z_M \) orbifold models, and extensions to non-supersymmetric orbifold models might also be interesting. Such systematical study will be done elsewhere.

Another important extension is to study the lepton sector. The situation would change for realizing large mixing angles. It is interesting to investigate systematically whether one can obtain realistic lepton masses and mixing angles by renormalizable couplings derived from string models. Such systematical analysis will also be done elsewhere.

### Acknowledgment

The authors would like to thank Oleg Lebedev for useful discussions. One of the authors (T. K.) would like to thank hospitality of KAIST, where a part of this work was studied. T. K. is supported in part by the Grant-in-Aid for Scientific Research (#16028211) and the Grant-in-Aid for the 21st Century COE “The Center for Diversity and Universality in Physics” from the Ministry of Education, Culture, Sports, Science and Technology of Japan. PK and JP are supported in part by KOSEF Sundo Grant R02-2003-000-10085-0, KRF grant KRF-2002-070-

\(^{11}\)See for related subjects e.g. Ref. [23].
C00022, BK21 Haeksim program and KOSEF SRC program through CHEP at Kyungpook National University.

References

[1] L. J. Dixon, J. A. Harvey, C. Vafa and E. Witten, Nucl. Phys. B 261, 678 (1985); Nucl. Phys. B 274, 285 (1986).

[2] L. J. Dixon, D. Friedan, E. J. Martinec and S. H. Shenker, Nucl. Phys. B 282, 13 (1987).

[3] S. Hamidi and C. Vafa, Nucl. Phys. B 279, 465 (1987).

[4] L. E. Ibanez, Phys. Lett. B 181, 269 (1986).

[5] T. T. Burwick, R. K. Kaiser and H. F. Muller, Nucl. Phys. B 355, 689 (1991); J. Erler, D. Jungnickel, M. Spalinski and S. Stieberger, Nucl. Phys. B 397, 379 (1993).

[6] T. Kobayashi and O. Lebedev, Phys. Lett. B 566, 164 (2003); Phys. Lett. B 565, 193 (2003).

[7] D. Cremades, L. E. Ibanez and F. Marchesano, JHEP 0307, 038 (2003).

[8] M. Cvetic and I. Papadimitriou, Phys. Rev. D 68, 046001 (2003).

[9] S. A. Abel and A. W. Owen, Nucl. Phys. B 663, 197 (2003).

[10] T. Kobayashi and N. Ohtsubo, Int. J. Mod. Phys. A 9, 87 (1994).

[11] J. A. Casas, F. Gomez and C. Munoz, Int. J. Mod. Phys. A 8, 455 (1993).

[12] J. A. Casas, F. Gomez and C. Munoz, Phys. Lett. B 292, 42 (1992).

[13] See e.g., S. A. Abel and C. Munoz, JHEP 0302, 010 (2003).

[14] T. Kobayashi and N. Ohtsubo, Phys. Lett. B 245, 441 (1990).

[15] L. E. Ibanez, J. Mas, H. P. Nilles and F. Quevedo, Nucl. Phys. B 301, 157 (1988).

[16] Y. Katsuki, Y. Kawamura, T. Kobayashi, N. Ohtsubo, Y. Ono and K. Tanikawa, Nucl. Phys. B 341, 611 (1990).

[17] D. Friedan, E. J. Martinec and S. H. Shenker, Nucl. Phys. B 271, 93 (1986).

[18] O. Lebedev, Phys. Lett. B 521, 71 (2001).
[19] S. K. Soni and H. A. Weldon, Phys. Lett. B 126, 215 (1983); V. S. Kaplunovsky and J. Louis, Phys. Lett. B 306, 269 (1993).

[20] F. Caravaglios, P. Roudeau and A. Stocchi, Nucl. Phys. B 633, 193 (2002).

[21] K. Hagiwara et al. [Particle Data Group Collaboration], Phys. Rev. D 66, 010001 (2002).

[22] See, e.g., H. Fusaoka and Y. Koide, Phys. Rev. D 57, 3986 (1998).

[23] A. E. Faraggi and E. Halyo, Nucl. Phys. B 416, 63 (1994); T. Kobayashi, Phys. Lett. B 358, 253 (1995); T. Kobayashi and Z. z. Xing, Mod. Phys. Lett. A 12, 561 (1997); Int. J. Mod. Phys. A 13, 2201 (1998); J. Giedt, Nucl. Phys. B 595, 3 (2001) [Erratum-ibid. B 632, 397 (2002)].