Emission of Two Hard Photons in Large–Angle Bhabha Scattering

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Abstract

A closed expression for the differential cross section of the large–angle Bhabha $e^+e^-$ scattering which explicitly takes into account the leading and next–to–leading contributions due to the emission of two hard photons is presented. Both collinear and semi–collinear kinematical regions are considered. The results are illustrated by numerical calculations.

Key words: Bhabha scattering, double bremsstrahlung, large angles, high energy

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1 Introduction

The large–angle Bhabha process is well suited for the determination of the luminosity $\mathcal{L}$ at $e^+e^-$ colliders of the intermediate energy range $\sqrt{s} = 2\varepsilon \sim 1\text{GeV}$ [1,2]. Small scattering angle kinematics of Bhabha scattering is used for high–energy colliders such as LEP I [3]. As far as 0.1% accuracy is desirable in the determination of $\mathcal{L}$, the corresponding requirement

$$\left| \frac{\delta \sigma}{\sigma} \right| \leq 10^{-3}$$

(1)

on the Bhabha cross section theoretical description appears. The quantity $\Delta \sigma$ is an unknown uncertainty in the cross section due to higher order radiative corrections. A great attention was paid to this process during the last decades (see review [4] and references therein). The Born cross section with weak interactions taken into account and the first order QED radiative corrections to it were studied in detail [5]. Both contributions, the one enhanced by the large logarithmic multiplier $L = \ln(s/m^2)$ (where $s = (p_+ + p_-)^2 = 4\varepsilon^2$ is the total center–of–mass (CM) energy squared, $m$ is the electron mass), and the one without $L$ are to be kept in the limits (1): $\alpha L/\pi, \alpha/\pi$. As for the corrections in the second order of the perturbation theory, they are necessary in the leading and next–to–leading approximations and take the following orders, respectively:

$$\left( \frac{\alpha}{\pi} \right)^2 L^2, \quad \left( \frac{\alpha}{\pi} \right) L.$$  

(2)

The total two–loop $(\sim (\alpha/\pi)^2)$ correction could be constructed from: 1) the two–loop corrections arising from the emission of two virtual photons; 2) the one–loop corrections to a single real (soft and hard) photon emission; 3) the ones arising from the emission of two real photons; 4) the virtual and real $e^+e^-$ pair production [6]. As for the corrections in the third order of perturbation theory, only the leading ones proportional to $(\alpha L/\pi)^3$ are to be taken into account.

In this paper we consider the emission of two real hard photons:

$$e^+(p_+) + e^-(p_-) \rightarrow e^+(q_+) + e^-(q_-) + \gamma(k_1) + \gamma(k_2).$$

(3)

The relevant contribution to the experimental cross section has the following form

$$\sigma_{\text{exp}} = \int d\sigma \Theta_+ \Theta_-,$$

(4)

where $\Theta_+$ and $\Theta_-$ are the experimental restrictions providing the simultaneous detection of both the scattered electron and positron. First, this means that their energy fractions
should be larger than a certain (small) quantity $\varepsilon_{th}/\varepsilon$, $\varepsilon_{th}$ is the energy threshold of the detectors. The second condition restricts their angles with respect to the beam axes. They should be larger than a certain finite value $\psi_0$ ($\psi_0 \sim 35^\circ$ in the experimental conditions accepted in [1]):

$$\pi - \psi_0 > \theta_-, \quad \theta_+ > \psi_0, \quad \theta_\pm = \frac{q_0 p_-}{m^2},$$

(5)

where $\theta_\pm$ are the polar angles of the scattered leptons with respect to the beam axes ($p_-$). We accept the condition on the energy threshold of the charged–particle registration $q_0^0 > \varepsilon_{th}$. Both photons are assumed to be hard. Their minimal energy

$$\omega_{\text{min}} = \Delta \varepsilon, \quad \Delta \ll 1,$$

(6)

could be considered as the threshold of the photon registration.

The main ($\sim (\alpha L/\pi)^2$) contribution to the total cross section (5) comes from the collinear region: when both the emitted photons move within narrow cones along the charged particle momenta (they may go along the same particle). So we will distinguish 16 kinematical regions:

$$\vec{a}^k_1 \text{ and } \vec{a}^k_2 < \theta_0, \quad \vec{b}^k_1 \text{ and } \vec{b}^k_2 < \theta_0,$$

$$m/\varepsilon \ll \theta_0 \ll 1, \quad a \neq b, \quad a, b = p_-, p_+, q_-, q_+ .$$

(7)

The matrix element module square summed over spin states in the regions (7) is of the form of the Born matrix element multiplied by the so–called collinear factors. The contribution to the cross section of each region has also the form of $2 \to 2$ Bhabha cross sections in the Born approximation multiplied by factors of the form

$$d\sigma_i^{\text{coll}} = d\sigma_{\text{0i}} \left[ a_i(x_j, y_j) \ln^2 \left( \frac{\varepsilon^2 \theta_0^2}{m^2} \right) + b_i(x_j, y_j) \ln \left( \frac{\varepsilon^2 \theta_0^2}{m^2} \right) \right],$$

(8)

where $x_j = \omega_j/\varepsilon$, $y_1 = q_0^-/\varepsilon$, $y_2 = q_0^+/\varepsilon$ are the energy fractions of the photons and of the scattered electron and positron. The dependence on the auxiliary parameter $\theta_0$ will be cancelled in the sum of the contributions of the collinear and semi–collinear regions. The last region corresponds to the kinematics, when only one photon is emitted inside the narrow cone $\theta_1 < \theta_0$ along one of the charged particle momenta. And the second photon is emitted outside any cone of that sort along charged particles ($\theta_2 > \theta_0$):

$$d\sigma_i^{\text{sc}} = \frac{\alpha}{\pi} \ln \left( \frac{4\varepsilon^2}{m^2} \right) d\sigma_{\text{0i}}^\gamma(k_2),$$

(9)

where $d\sigma_{\text{0i}}^\gamma$ has the known form of the single hard bremsstrahlung cross section in the Born approximation [7].
Below we show explicitly that the result of the integration over the single hard photon emission in eq. (9) in the kinematical region $\theta_2 > \theta_0$ ($\theta_2$ is the emission angle of the second hard photon with respect to the direction of one of the four charged particles) has the following form

$$\int d\sigma_0^\gamma(k_2) = -2 \ln\left(\frac{\theta_0^2}{4}\right) a_i(x, y) d\sigma_0^i + d\tilde{\sigma}^i. \quad (10)$$

The collinear factors in the double bremsstrahlung process were first considered in papers of the CALCUL collaboration [8]. Unfortunately they have a rather complicated form which is less convenient for further analytical integration in comparison with the expressions given below. The method of calculation of the collinear factors may be considered as a generalization of the quasi–real electron method [9] to the case of multiple bremsstrahlung. Another generalization is required for the calculations of the cross section of process $e^+e^- \rightarrow 2e^+2e^-$ [6].

It is interesting that the collinear factors for the kinematical region of the two hard photon emission along the projectile and the scattered electron are found the same as for the electron–proton scattering process considered by one of us (N.P.M.) in paper [10].

There are 40 Feynman diagrams of the tree type which describe the double bremsstrahlung process in $e^+e^-$ collisions. The differential cross section in terms of helicity amplitudes was computed about ten years ago [8,11]. It has a very complicated form. We note that the contribution from the kinematical region in which the angles (in the CM system) between any two final particles are large compared with $m/\varepsilon$ is of the order

$$\frac{\alpha^2 r_0^2 m^2}{\pi^2 \varepsilon^2} \sim 10^{-36} \text{cm}^2, \quad (11)$$

($r_0$ is the classical electron radius). So, the corresponding events will possess poor statistics at the colliders with the luminosity $L \sim 10^{31} - 10^{32} \text{cm}^{-2}\text{s}^{-1}$. More probable are the cases of double bremsstrahlung imitating the processes $e^+e^- \rightarrow e^+e^-$ or $e^+e^- \rightarrow e^+e^-\gamma$, which corresponds to the emission of one or two photons along charged–particle momenta.

2 Kinematics in the collinear region

It is convenient to introduce, in the collinear region, new variables and transform the phase volume of the final state in the following way (from now on we will work in the CM system):

$$\int d\Gamma = \int \frac{d^3q_- d^3q_+ d^3k_1 d^3k_2}{16q_0^0 q_+^0 \omega_1 \omega_2 (2\pi)^8} \delta^{(4)}(\eta_1 p_- + \eta_2 p_+ - \lambda_1 q_- - \lambda_2 q_+).$$
Table 1

|  | \(p_−p_−\) | \(q_−q_−\) | \(p_+p_+\) | \(q_+q_+\) | \(p_−p_+\) | \(q_−q_+\) | \(p_−q_+\) | \(p_+q_+\) | \(p_−q_−\) | \(p_+q_−\) |
|---|---|---|---|---|---|---|---|---|---|---|
| \(\eta_1\) | \(y\) | 1 | 1 | 1 | 1−\(x_1\) | 1 | 1−\(x_1\) | 1 | 1−\(x_1\) | 1 |
| \(\eta_2\) | 1 | 1 | \(y\) | 1 | 1−\(x_2\) | 1 | 1 | 1−\(x_1\) | 1−\(x_1\) | 1 |
| \(\lambda_1\) | 1 | \(\frac{1}{y}\) | 1 | 1 | \(\frac{1}{1−x_1}\) | 1 | \(\frac{1}{1−x_1}\) | 1 | \(\frac{1}{1−x_1}\) | 1 |
| \(\lambda_2\) | 1 | 1 | 1 | \(\frac{1}{y}\) | 1 | \(\frac{1}{1−x_2}\) | 1 | \(\frac{1}{1−x_1}\) | 1 | \(\frac{1}{1−x_1}\) | 1 |

\[
\frac{m^4\pi^2}{4(2\pi)^6} \int_{\Delta}^1 \frac{dx_1}{\Delta} \int_{\Delta}^1 \frac{dx_2}{\Delta} x_1 x_2 \int_0^{2\pi} d\phi \int_0^{z_0} dz_1 \int_0^{z_0} dz_2 \int d\Gamma_q, \tag{12}
\]

\[
\int d\Gamma_q = \int \frac{d^3q_+ d^3q_+}{4q_0^0 q^0_1 (2\pi)^2} \delta^{(4)}(\eta_1 p_− + \eta_2 p_+ - \lambda_1 q_− - \lambda_2 q_+),
\]

\[
z_{1,2} = \left(\frac{\theta_i \varepsilon}{m}\right)^2, \quad \phi = k_{1\perp} k_2\perp, \quad x_i = \frac{\omega_i}{\varepsilon}, \quad z_0 = \left(\frac{\theta_0 \varepsilon}{m}\right)^2 \gg 1, \quad \Delta = \frac{\omega_{\text{min}}}{\varepsilon},
\]

where \(\theta_i\) (\(i = 1, 2\)) is the polar angle of the \(i\)–photon emission with respect to the momentum of the charged particle that emits the photon; \(\eta_\pm\) and \(\lambda_\pm\) depend on the specific emission kinematics, they are given in Table 1.

The columns of the Table correspond to a certain choice of the kinematics in the following way: \(p_−p_−\) means the emission of both the photons along the projectile electron, \(p_+q_−\) means that the first of the photons goes along the projectile positron; the second, along the scattered electron, and so on. The contributions from 6 remaining kinematical regions (when the photons in the last 6 columns are interchanged) could be found by the simple substitution \(x_1 \leftrightarrow x_2\). We will use the momentum conservation law

\[
\eta_1 p_− + \eta_2 p_+ = \lambda_1 q_− + \lambda_2 q_+, \tag{13}
\]

and the following relations coming from it:

\[
\eta_1 + \eta_2 = \lambda_1 y_1 + \lambda_2 y_2, \quad \lambda_1 y_1 \sin \theta_− = \lambda_2 y_2 \sin \theta_+, \quad y_{1,2} = \frac{q_{1,2}^0}{\varepsilon},
\]

\[
\lambda_2 y_2 = \frac{\eta_1^2 + \eta_2^2 + (\eta_2^2 - \eta_1^2)c}{\eta_1 + \eta_2 + (\eta_2 - \eta_1)c}. \tag{14}
\]

Each of 16 contributions to the cross section of process (3) can be expressed in terms of the corresponding Born–like cross section multiplied by its collinear factor.
Expressions (17) agree with the results of paper [8] except for a simpler form of $\mathcal{K}(q_{-q_+})$.

The sum over $(\eta, \lambda)$ means the sum over 16 collinear kinematical regions. The corresponding $(\eta, \lambda)$ could be found in Table 1. The quantities $\mathcal{K}_i(\eta, \lambda)$ are as follows:

\[
\mathcal{K}(p_{-p_-}) = \frac{2}{y} \mathcal{A}(A_1, A_2, A, x_1, x_2, y), \quad \mathcal{K}(q_{-q_-}) = 2y \mathcal{A}(B_1, B_2, B, \frac{-x_1}{y}, \frac{-x_2}{y}, \frac{1}{y}),
\]

\[
\mathcal{K}(p_{+p_+}) = \frac{2}{y} \mathcal{A}(C_1, C_2, C, x_1, x_2, y), \quad \mathcal{K}(q_{+q_+}) = 2y \mathcal{A}(D_1, D_2, D, \frac{-x_1}{y}, \frac{-x_2}{y}, \frac{1}{y}),
\]

\[
\mathcal{A}(A_1, A_2, A, x_1, x_2) = -\frac{yA_2}{A^2A_1} - \frac{yA_1}{A^2A_2} + \frac{1 + y^2}{x_1x_2A_1A_2} + \frac{r_1^3 + yr_2}{AA_1x_2}
+ \frac{r_2^3 + yr_1}{AA_2x_1} + \frac{2m^2(y^2 + r_1^2)}{AA_2^2} + \frac{2m^2(y^2 + r_2^2)}{AA_1^2},
\]

\[
\mathcal{K}(p_{-p_+}) = 2K_1K_2, \quad \mathcal{K}(p_{-q_-}) = -2K_1K_3, \quad \mathcal{K}(p_{+p_+}) = -2K_4K_5,
\]

\[
\mathcal{K}(q_{-q_+}) = 2K_6K_7, \quad \mathcal{K}(q_{-q_-}) = -2K_1K_5, \quad \mathcal{K}(p_{+q_+}) = -2K_4K_3, \quad \mathcal{K}(q_{+q_+}) = -2K_4K_3.
\]

\[
K_1 = \frac{g_1}{A_1x_1r_1} + \frac{2m^2}{A_1^2}, \quad K_2 = \frac{g_2}{C_2x_2r_2} + \frac{2m^2}{C_2^2}, \quad K_3 = \frac{g_4}{D_2x_2t_2} - \frac{2m^2}{D_2^2},
\]

\[
K_4 = \frac{g_1}{C_1x_1r_1} + \frac{2m^2}{C_1^2}, \quad K_5 = \frac{g_3}{B_2x_2t_1} - \frac{2m^2}{B_2^2}, \quad K_6 = \frac{g_1}{B_1x_1} - \frac{2m^2}{B_1^2},
\]

\[
K_7 = \frac{g_2}{D_2x_2} - \frac{2m^2}{D_2^2}, \quad r_1 = 1 - x_1, \quad r_2 = 1 - x_2,
\]

\[
g_1 = 1 + r_1^2, \quad g_2 = 1 + r_2^2, \quad g_3 = y_1^2 + t_1^2, \quad g_4 = y_2^2 + t_2^2,
\]

\[
t_1 = y_1 + x_2, \quad t_2 = y_2 + x_2, \quad y = 1 - x_1 - x_2,
\]

$y_1, y_2$ are the energy fractions of the scattered electron and positron defined in eq. (14).

Expressions (17) agree with the results of paper [8] except for a simpler form of $\mathcal{K}(q_{-q_+})$. 

5
As for eq. (16) it has an evident advantage in comparison to the corresponding formulae given in paper [8]. Let us note that the remaining factors \( K(p, q) \) could be obtained from the ones given in eq. (17) using relations of the following type:

\[
K(p_-, q_+)(x_1, x_2, A_1, B_2) = K(q_-, p_+)(x_2, x_1, A_2, B_1).
\]

(18)

Note also that terms of the kind \( m^4/(B_1^2 C_1^2) \) do not give logarithmically enhanced contributions, and we will omit them below. The denominators of the propagators entering into eqs. (16), (17) are:

\[
A_i = (p_+ - k_i)^2 - m^2, \quad A = (p_+ - k_1 - k_2)^2 - m^2,
\]

\[
B_i = (q_+ + k_i)^2 - m^2, \quad B = (q_+ + k_1 + k_2)^2 - m^2,
\]

\[
C_i = (k_i - p_+)^2 - m^2, \quad C = (k_1 + k_2 - p_+)^2 - m^2,
\]

\[
D_i = (q_+ + k_i)^2 - m^2, \quad D = (q_+ + k_1 + k_2)^2 - m^2.
\]

(19)

For further integration it is useful to rewrite the denominators in terms of the photon energy fractions \( x_{1,2} \) and their emission angles. In the case of the emission of both the photons along \( p_- \) we would have

\[
\frac{A}{m^2} = -x_1(1 + z_1) - x_2(1 + z_2) + x_1 x_2 (z_1 + z_2) + 2 x_1 x_2 \sqrt{z_1 z_2} \cos \phi,
\]

\[
\frac{A_i}{m^2} = -x_i(1 + z_i),
\]

(20)

where \( z_i = (\epsilon \theta_i/m)^2 \), \( \phi \) is the azimuthal angle between the planes containing the space vector pairs \((p_-, k_1)\) and \((p_-, k_2)\). In the same way one can obtain in the case \( k_1, k_2 || q_- \):

\[
\frac{B}{m^2} = \frac{x_1}{y_1} (1 + y_1^2 z_1) + \frac{x_2}{y_1} (1 + y_1^2 z_2) + x_1 x_2 (z_1 + z_2) + 2 x_1 x_2 \sqrt{z_1 z_2} \cos \phi,
\]

\[
\frac{B_i}{m^2} = \frac{x_i}{y_1} (1 + y_1^2 z_i).
\]

(21)

Then we perform the elementary azimuthal angle integration and the integration over \( z_1, z_2 \) within the logarithmical accuracy using the procedure suggested in paper [10]:

\[
\bar{\sigma} = m^4 \int_0^{z_0} dz_1 \int_0^{z_0} dz_2 \int_0^{2\pi} \frac{d\phi}{2\pi} a.
\]

(22)

The list of the relevant integrals is given in Appendix A. In this way one obtains the differential cross section in the collinear region:
\[
\frac{d\sigma_{\text{coll}}}{d\Omega} = \frac{\alpha^4 L}{4\pi^2 s} \frac{d^3 q_+ d^3 q_- dx_1 dx_2}{x_1 x_2} (1 + \mathcal{P}_{1,2}) \left\{ \frac{1}{y_1^2} \left[ \frac{1}{2} (L + 2l) g_1 g_5 \right. \right.
+ (y^2 + r_1^2) \ln \frac{x_2 r_1^2}{x_1 y} + x_1 x_2 (y - x_1 x_2) - 2 r_1 g_5 \left[ B_{p-p_- p_-} + B_{p_+ p_- \delta p_+ p_-} \right]
+ \frac{1}{y_1^2} \left[ \frac{1}{2} (L + 2l + 4 \ln y) g_1 g_5 \right.
+ (y^2 + r_1^2) \ln \frac{x_1 r_1^2}{x_2 y} + x_1 x_2 (y - x_1 x_2) - 2 r_1 g_1 \left. \right]
\times [B_{q_- q_+ \delta q_+ q_-} + B_{q_+ q_- \delta q_+ q_-}] + B_{p-p_- p_-} \left[ (L + 2l) \frac{g_1 g_2}{r_1 r_2} - \frac{2 g_1}{r_1} - \frac{2 g_2}{r_2} \right]
\left. + B_{q_+ q_- \delta q_+ q_-} \left[ (L + 2l + 2 \ln (r_1 r_2)) \frac{g_1 g_2}{r_1 r_2} - \frac{2 g_1}{r_1} - \frac{2 g_2}{r_2} \right] \right.
+ \left. \left[ B_{p_- p_+ \delta p_+ q_-} + B_{p_+ p_- \delta p_+ q_-} \right] \left[ (L + 2l + 2 \ln y_1) \frac{g_1 g_3}{r_1 y_1 t_1} - \frac{2 g_1}{r_1} - \frac{2 g_3}{y_1 t_1} \right] \right.
\left. + \left[ B_{p_+ q_- \delta p_+ q_-} + B_{p_- q_+ \delta p_+ q_-} \right] \left[ (L + 2l + 2 \ln y_2) \frac{g_1 g_4}{r_1 y_2 t_2} - \frac{2 g_1}{r_1} - \frac{2 g_4}{y_2 t_2} \right] \right\}. \tag{23}
\]

We used the symbol \( \mathcal{P}_{1,2} \) for the interchange operator \( \mathcal{P}_{1,2} f(x_1, x_2) = f(x_2, x_1) \). We used the notation (see also eq. (17)):

\[
l = \ln \left( \frac{\beta_0^2}{4} \right), \quad g_5 = y^2 + r_1^2, \tag{24}\]  

where \( \theta_0 \) is the collinear parameter. Delta–function \( \delta_{p,q} \) corresponds to the specific conservation law of the kinematical situation defined by the pair \( p, q \) (see Table 1): \( \delta_{p,q} = \delta^{(4)}(\eta_2 p_+ + \eta_1 p_- - \lambda_1 q_- - \lambda_2 g_+) \). Besides, we imply that the first photon is emitted along the momentum \( p \); and the second, along the momentum \( q \) \( (p, q = p_-, p_+, q_-, q_+) \). These \( \delta \)–functions could be taken into account in the integration as is made in the expression for \( d\Omega_{\eta, \lambda} \) (see eq. (15)). Finally, we define

\[
B_{p,q} = \left( \frac{\eta_2 s}{\lambda_1 t} + \frac{\lambda_1 t}{\eta_2 s} + 1 \right)^2, \quad t = (p_- - q_-)^2. \tag{25}\]  

3 Contribution of the semi–collinear region

We will suggest for definiteness that the photon with momentum \( k_2 \) moves inside a narrow cone along the momentum direction of one of the charged particles, while the other photon moves in any direction outside that cone along any charged particle. This choice allows us to omit the statistical factor \( 1/2! \). The quasireal electron method [9] may be used to obtain the cross section:

\[
d\sigma_{\text{sc}} = \frac{\alpha^4}{32 s \pi^4} \frac{d^3 q_- d^3 q_+ d^3 k_1}{q_-^0 q_+^0 k_1^0} V \frac{d^3 k_2}{k_2^0} \left\{ \mathcal{K}_{p_-} \delta_{p_-} R_{p_-} \right\}
\]

7
where the function $R$ is the known accompanying radiation factor; $V$ we omitted the terms of the kind $m^2/(p\cdot k_2)^2$ in eq. (26) because their contribution does not contain the large logarithm $L$. The quantities entering into eq. (26) are given by:

$$
V = \frac{s}{k_1p_+ \cdot k_1p_-} + \frac{s'}{k_1q_+ \cdot k_1q_-} - \frac{t'}{k_1p_+ \cdot k_1q_+} - \frac{t}{k_1p_- \cdot k_1q_-} + \frac{u'}{k_1p_+ \cdot k_1q_-} + \frac{u}{k_1q_+ \cdot k_1p_-}.
$$

$(27)$

$V$ is the known accompanying radiation factor; $K_i$ are the single photon emission collinear factors:

$$
K_{p_-} = K_{p_+} = \frac{g_2}{x_2r_2}, \quad K_{q_-} = \frac{y_2^2 + (y_1 + x_2)^2}{x_2(y_1 + x_2)}, \quad K_{q_+} = \frac{y_2^2 + (y_2 + x_2)^2}{x_2(y_2 + x_2)}.
$$

$(28)$

Quantities $R_i$ read:

$$
R_{p_-} = R[sp_2, tr_2, ur_2, s', t', u'], \quad R_{p_+} = R[sp_2, t, u, s', t'r_2, u'r_2],
$$

$$
R_{q_-} = R[s, t_1, y_1, u, s', t_1', u', t_1'], \quad R_{q_+} = R[s, t, u, t_2, y_2, s', t_2', t_2, u'],
$$

$(29)$

where the function $R$ has the form [12]:

$$
R[s, t, u, s', t', u'] = \frac{1}{s's'tt'}[ss'(s^2 + s'^2) + tt'(t^2 + t'^2) + uu'(u^2 + u'^2),
$$

$$
s = (p_+ + p_-)^2, \quad s' = (q_+ + q_-)^2, \quad t = (p_+ - p_-)^2,
$$

$$
t' = (p_+ - q_+)^2, \quad u = (p_+ - q_-)^2, \quad u' = (p_+ - q_-)^2.
$$

$(30)$

Finally, we define

$$
\delta_{p_-} = \delta^{(4)}(p_- r_2 + p_+ - q_+ - q_- - k_1),
$$

$$
\delta_{p_+} = \delta^{(4)}(p_- + p_+ r_2 - q_+ - q_- - k_1),
$$

$$
\delta_{q_-} = \delta^{(4)}(p_- + p_+ - q_+ - q_- \frac{y_1 + x_2}{y_1} - k_1),
$$

$$
\delta_{q_+} = \delta^{(4)}(p_- + p_+ - q_+ \frac{y_2 + x_2}{y_2} - q_- - k_1).
$$

$(31)$

Performing the integration over angular variables of the collinear photon we obtain
\[d\sigma^{\text{sc}} = \frac{\alpha^4 L}{16 \pi s^3} \frac{d^3q_- d^3q_+ d^3k_1}{q_-^0 q_+^0 k_1^0} \frac{dx_2 V}{2} \left\{ \mathcal{K}_{p_-} [R_{p_-} \delta_{p_-} + R_{p_+} \delta_{p_+}] \\
+ \frac{1}{y_2} \mathcal{K}_{q_+} R_{q_+} \delta_{q_+} + \frac{1}{y_1} \mathcal{K}_{q_-} R_{q_-} \delta_{q_-} \right\}. \] (32)

To see that the sum of cross sections (23) and (32)

\[d\sigma^{\gamma\gamma} = d\sigma^{\text{coll}} + \int dO_1 \left( d\sigma^{\text{sc}} \right) \] (33)

does not depend on the auxiliary parameter \(\theta_0\). We verify that terms \(L \cdot l\) from eq. (23) cancel out with the terms

\[L \frac{k_1^0 q_i^0}{2\pi} \int \frac{dO_1}{k_1 q_i} \approx -L \cdot l, \] (34)

which arise from 16 regions in the semi–collinear kinematics.

4 Numerical results and discussion

We separated the contribution of the collinear and semi–collinear regions using the auxiliary parameter \(\theta_0\). By direct numerical integration according to the presented formulae we had convinced ourselves that the total result is independent on the choice of \(\theta_0\).

It is convenient to compare the cross section of double hard photon emission with the Born cross section

\[\sigma^{\text{Born}} = \frac{\alpha^2 \pi}{2s} \int_{-\cos \psi_0}^{\cos \psi_0} \left( \frac{3 + c^2}{1 - c} \right)^2 dc. \] (35)

For illustrations we integrated over some typical experimental angular acceptance and chose the following values of the parameters:

\[\psi_0 = \pi/4, \quad \sqrt{s} = 0.9 \text{ GeV}, \quad \Delta_1 = 0.4, \quad \Delta = 0.05, \quad \theta_0 = 0.05, \]
\[L = 15.0, \quad l = -7.38, \] (36)

where \(\Delta_1\) defines the energy threshold for the registration of the final electron and positron: \(q_i^0 > \varepsilon_{\text{th}} = \varepsilon \Delta_1\). Note that restrictions on \(\theta_0\) (7) and (12) \(z_0 = \exp\{L + l\} \gg 1\) are fulfilled.
For the chosen parameters we get

\[
\sigma^{\text{Born}} = 1.2 \text{ mkb}, \quad \frac{\sigma^{\text{coll}}}{\sigma^{\text{Born}}} \cdot 100\% = -0.25\%, \\
\frac{\sigma^{\text{sc}}}{\sigma^{\text{Born}}} \cdot 100\% = 0.81\% , \quad \delta\sigma^{\text{tot}} = \frac{\sigma^{\text{sc}} + \sigma^{\text{coll}}}{\sigma^{\text{Born}}} \cdot 100\% = 0.56\% .
\]  

(37)

The phenomenon of negative contribution to the cross section from the collinear kinematics is an artifact of our approach. Namely, we systematically omitted positive terms without large logarithms, among them we dropped terms proportional to \( l^2 \). The cancellation of \( l^2 \) terms can be seen only after adding the contribution of the non–collinear kinematics (when both photons are emitted outside narrow cones along charged–particle momenta). The non–collinear kinematics does not provide any large logarithm \( L \).

Both quantities \( \sigma^{\text{coll}} \) and \( \sigma^{\text{sc}} \) depend on auxiliary parameter \( \theta_0 \). We eliminated by hands from eq. (23) the terms proportional to \( l \) and obtained the following quantity:

\[
\frac{\sigma^{\text{bare}}^{\text{coll}}}{\sigma^{\text{Born}}} \cdot 100\% = 1.43\% .
\]  

(38)

This quantity corresponds to an approximation for the correction under consideration in which one considers only the collinear regions and takes into account only terms proportional to \( L^2 \) and \( L \) (all terms dependent on \( \theta_0 \) are to be omitted). Having in mind the cancellation of \( \theta_0 \)–dependence in the sum of the collinear and semi–collinear contributions, we may subtract from the value of the semi–collinear contribution the part which is associated with \( l \):

\[
\sigma^{\text{bare}}^{\text{sc}} = \sigma^{\text{sc}} + (\sigma^{\text{coll}} - \sigma^{\text{bare}}^{\text{coll}}), \quad \frac{\sigma^{\text{bare}}^{\text{sc}}}{\sigma^{\text{Born}}} \cdot 100\% = -0.87\% .
\]

Looking at bare quantities one can get an idea of relative impact of two considered regions. We see that at the precision level of 0.1% the next–to–leading contributions of semi–collinear regions are important.

In figure 1 we illustrated the dependence on parameter \( \Delta \) of the bare collinear contribution for different fixed values of \( \Delta_1 \). Large growing in the region of small \( \Delta \) corresponds to an infrared singularity, which will be cancelled after adding contributions of virtual and soft photon emission.

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Appendix A

We present here the list of integrals (see eqs. (19 – 22)):

\[
\begin{align*}
\frac{A_2}{A_1} & = \frac{L_0}{x_1 x_2 r_1^2} \left[ \frac{1}{2} L_0 + \ln \frac{x_2 r_1^2}{x_1 y} - 1 + \frac{x_1 x_2}{y} \right], \\
\frac{1}{AA_1} & = \frac{L_0}{x_1 x_2 r_1} \left[ \frac{1}{2} L_0 + \ln \frac{x_2 r_1^2}{x_1 y} \right], \\
\frac{1}{A_1 A_2} & = \frac{L_0^2}{x_1 x_2}, \\
\frac{1}{A_1 B_2} & = -\frac{L_0}{y_1 x_1 x_2} (L_0 + 2 \ln y_1), \\
L_0 & = \ln z_0 \equiv L + l, \quad l = \ln \left( \frac{\theta_0^2}{4} \right), \quad L = \ln \left( \frac{4 \varepsilon^2}{m^2} \right).
\end{align*}
\]

The remaining integrals could be obtained by simple substitutions defined in eqs. (19 – 22).

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Fig. 1. The ratio $\frac{\sigma_{\text{bare}}}{\sigma_{\text{Born}}}$ in percent as functions of $\Delta$ for $\psi_0 = \pi/4$ and different values of $\Delta_1$. 