THE DIFFERENTIAL ANALYTIC INDEX IN
SIMONS-SULLIVAN DIFFERENTIAL K-THEORY

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Dedicated to my father Kar-Ming Ho

Abstract. We define the Simons-Sullivan differential analytic index by
translating the Freed-Lott differential analytic index via explicit ring iso-
morphisms between Freed-Lott differential K-theory and Simons-Sullivan
differential K-theory. We prove the differential Grothendieck-Riemann-
Roch theorem in Simons-Sullivan differential K-theory using a theorem
of Bismut.

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1. Introduction

As explained in [3, 4, 7, 10], the physics motivation for differential K-
theory is a quantum field theory whose Lagrangian has differential form
field strength. This leads to a generalized cohomology theory with a map
to ordinary cohomology that implements charge quantization. In [7] Freed
argued that there should be a similar extension of topological K-theory. We
refer to [8, §1.4] for a historical discussion. The mathematical motivation for
differential K-theory can be traced to Cheeger-Simons differential characters
[6], the unique differential extension of ordinary cohomology [14], and to the
work of Karoubi [11]. It is thus natural to look for differential extensions
of generalized cohomology theories, for example topological K-theory. The

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differential extension of topological $K$-theory is now known as differential $K$-theory. Roughly speaking, differential $K$-theory is topological $K$-theory combined with differential form data in a complicated way, just as differential characters combine ordinary cohomology with differential form data. Various definitions of differential $K$-theory have been proposed, notably by Bunke-Schick [3], Freed-Lott [8], Hopkins-Singer [10] and Simons-Sullivan [15]. Axioms for differential extensions of generalized cohomology theories are given in [4], where it is shown that two differential extensions of a fixed generalized cohomology theory satisfying certain conditions are uniquely isomorphic. In particular the four models of differential $K$-theory mentioned above are isomorphic by this abstract result. For more details and an introduction to differential $K$-theory, see [5].

The Atiyah-Singer family index theorem can be formulated as the equality of the analytic and topological pushforward maps $\text{ind}^{\text{an}} = \text{ind}^{\text{top}} : K(X) \to K(B)$. Applying the Chern character, we get the Grothendieck-Riemann-Roch theorem, the commutativity of the following diagram

$$
\begin{array}{c}
K(X) \xrightarrow{\text{ch}} H^{\text{even}}(X; \mathbb{Q}) \\
\downarrow \text{ind}^{\text{an}} \\
K(B) \xrightarrow{\text{ch}} H^{\text{even}}(B; \mathbb{Q})
\end{array}
$$

Analogous theorems hold in differential $K$-theory. Bunke-Schick prove the differential Grothendieck-Riemann-Roch theorem (dGRR) [3, Theorem 6.19], i.e., for a proper submersion $\pi : X \to B$ of even relative dimension, the following diagram is commutative:

$$
\begin{array}{c}
\widehat{K}_{\text{BS}}(X) \xrightarrow{\widehat{\text{ch}}_{\text{BS}}} \hat{H}^{\text{even}}(X; \mathbb{R}/\mathbb{Q}) \\
\downarrow \text{ind}^{\text{an}}_{\text{BS}} \\
\widehat{K}_{\text{BS}}(B) \xrightarrow{\widehat{\text{ch}}_{\text{BS}}} \hat{H}^{\text{even}}(B; \mathbb{R}/\mathbb{Q})
\end{array}
$$

where $\hat{H}(X; \mathbb{R}/\mathbb{Q})$ is the ring of differential characters [6], $\widehat{\text{ch}}_{\text{BS}}$ is the differential Chern character [3, §6.2], $\text{ind}^{\text{an}}_{\text{BS}}$ is the Bunke-Schick differential analytic index [3, §3] and $\int_{X/B} \widehat{\text{Tod}}(\nabla^T^V X)^* \text{d}z$ is a modified pushforward of differential characters [3, §6.4]. The notation is explained more fully in later sections. Freed-Lott prove the differential family index theorem [8, Theorem 7.32] $\text{ind}^{\text{an}}_{\text{FL}} = \text{ind}^{\text{top}}_{\text{FL}} : \widehat{K}_{\text{FL}}(X) \to \widehat{K}_{\text{FL}}(B)$, where $\text{ind}^{\text{an}}_{\text{FL}}$ and $\text{ind}^{\text{top}}_{\text{FL}}$ are the Freed-Lott differential analytic index [8, Definition 3.11] and the differential topological index [8, Definition 5.33]. Applying the differential Chern character $\widehat{\text{ch}}_{\text{FL}}$ yields the dGRR [8, Corollary 8.23]. Since $\text{ind}^{\text{an}}_{\text{BS}} = \text{ind}^{\text{an}}_{\text{FL}}$ [3, Corollary 5.5], the two dGGR theorems are essentially the same. See [3] for
a short proof of the dGRR.

To this point, the differential index theorem formulated in Simons-Sullivan differential $K$-theory has not appeared. The purpose of this paper is to fill this gap by both defining the differential analytic index and proving the dGRR in Simons-Sullivan differential $K$-theory.

The first main result of this paper (Theorem 1) is the construction of explicit ring isomorphisms between Simons-Sullivan differential $K$-theory and Freed-Lott differential $K$-theory. While these theories must be isomorphic by [4, Theorem 3.10], the explicit isomorphisms have not been appeared in literature as far as we know. Moreover, it follows from Corollary [1] that the flat parts of these differential $K$-theories are also isomorphic via the restriction of the explicit ring isomorphisms in Theorem [1]. This result is a more explicit version of [4, Theorem 5.5] in this case. The advantage of these explicit ring isomorphisms is that we see which elements in these differential $K$-groups correspond to each other.

The second main result of this paper is the dGRR in Simons-Sullivan differential $K$-theory. We first define the Simons-Sullivan differential analytic index by translating the Freed-Lott analytic index via the explicit isomorphisms in Theorem 1. To be precise, we study the special case where the family of kernels $\ker(D^E)$ forms a superbundle. The general case follows from a standard perturbation argument as in [8, §7]. The Simons-Sullivan differential analytic index of an element $\mathcal{E} = [E, h^E, [\nabla^E]] \in \hat{K}_{SS}(X)$, in the special case, is given by

$$
\text{ind}^{\text{an}}_{SS}(\mathcal{E}) = [\ker(D^E), h^{\ker(D^E)}, [\nabla^{\ker(D^E)}]] + [V, h^V, [\nabla^V]] - [\dim(V), h, [d]],
$$

where $[V, h^V, [\nabla^V]] := \tilde{CS}^{-1} (\tilde{\eta}(\mathcal{E}))$, and all the terms will be explained below. The general case of $\text{ind}^{\text{an}}_{SS}(\mathcal{E})$ is given by a similar formula. This formula is considerably more complicated than the Freed-Lott differential analytic index. This indicates that Simons-Sullivan differential $K$-theory is not the easiest setting for differential index theory, although the Simons-Sullivan construction of the differential $K$-group is perhaps the simplest among all the existing ones. We then prove the dGRR (Theorem 2) in the special case, i.e., the commutativity of the following diagram

$$
\begin{array}{ccc}
\hat{K}_{SS}(X) & \xrightarrow{\hat{\chi}_{SS}} & \hat{H}^{\text{even}}(X; \mathbb{R}/\mathbb{Q}) \\
\text{ind}^{\text{an}}_{SS} & & \downarrow \int_{X/B} \text{Todd}(\varphi^V_X) \ast (\cdot) \\
\hat{K}_{SS}(B) & \xrightarrow{\hat{\chi}_{SS}} & \hat{K}_{SS}(B)
\end{array}
$$

in Simons-Sullivan differential $K$-theory, using a theorem of Bismut [1, Theorem 1.15]. The general case of the dGRR follows by a similar argument,
since [1] Theorem 1.15] can be extended to the general case.

In principle all the theorems and proofs can be transported from Freed-Lott differential $K$-theory to Simons-Sullivan differential $K$-theory by the explicit isomorphisms given by Theorem 1. However, with [1] Theorem 1.15] the proof of the dGRR is easier.

The paper is organized as follows: the next two sections contain all the necessary background material. Section 2 reviews Simons-Sullivan differential $K$-theory. Section 3 reviews Freed-Lott differential $K$-theory and the construction of the Freed-Lott differential analytic index. The main results of the paper are proved in Section 5, including the explicit isomorphisms between Simons-Sullivan differential $K$-theory and Freed-Lott differential $K$-theory, the formula for the differential analytic index in Simons-Sullivan differential $K$-theory and the dGRR.

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2. Simons-Sullivan differential $K$-theory

In this section we review Simons-Sullivan differential $K$-theory [15]. For our purpose, we use the Hermitian version of structured bundles instead of the complex version. Consider a triple $(V, h, \nabla)$, where $V \to X$ is a Hermitian vector bundle over a compact manifold $X$ with a Hermitian metric $h$ and a unitary connection $\nabla$. Recall that the Chern character form $\text{ch}(\nabla) \in \Omega^{\text{even}}(X; \mathbb{R})$ and the Chern-Simons transgression form $\text{cs}(\nabla^t) \in \Omega^{\text{odd}}(X; \mathbb{R})$ of two connections $\nabla^0, \nabla^1$ on $V \to X$ joined by a smooth curve $\nabla^t$ of connections are related by the equality

$$d\text{cs}(\nabla^t) = \text{ch}(\nabla^1) - \text{ch}(\nabla^0).$$

Define

$$\text{CS}(\nabla^0, \nabla^1) := \text{cs}(\nabla^t) \mod \text{Im}(d : \Omega^{\text{even}}(X) \to \Omega^{\text{odd}}(X)),$$

where $\nabla^t$ is a smooth curve joining the connection $\nabla^1$ and $\nabla^0$. Since $\text{cs}(\nabla^t)$ only depends on the curve joining the connections up to an exact form [15, Proposition 1.6], $\text{CS}(\nabla^0, \nabla^1)$ is well defined.

It follows from (1) that $d\text{CS}(\nabla^0, \nabla^1) = \text{ch}(\nabla^1) - \text{ch}(\nabla^0)$. There are other sign convention, for example see [8]. We will use the convention $d\text{CS}(\nabla^0, \nabla^1) = \text{ch}(\nabla^1) - \text{ch}(\nabla^0)$ in this paper.

For two connections $\nabla^0, \nabla^1$ on $V \to X$, we set $\nabla^0 \sim \nabla^1$ if and only if $\text{CS}(\nabla^0, \nabla^1) = 0. \sim$ is an equivalence relation.

The triple $\mathcal{V} = (V, h, [\nabla])$ is called a (Hermitian) structured bundle. Two
structured bundles $V = (V, h^V, [\nabla V])$ and $W = (W, h^W, [\nabla W])$ are isomorphic if there exists a vector bundle isomorphism $\sigma : V \to W$ such that $\sigma^* h^W = h^V$ and $\sigma^* ([\nabla W]) = [\nabla V]$. Denote by $\text{Struct}(X)$ the set of all isomorphism classes of structured bundles. Direct sum and tensor product of structured bundles are well-defined \cite{15}, so $\text{Struct}(X)$ is an abelian semi-ring.

The Simons-Sullivan differential $K$-group is defined to be

$$\widehat{K}_{SS}(X) = K(\text{Struct}(X)).$$

Thus, Simons-Sullivan differential $K$-theory is a $K$-theory of vector bundles with connections.

To be precise, $[V_1] - [W_1] = [V_2] - [W_2]$ in $\widehat{K}_{SS}(X)$ if and only if there exists a structured bundle $(G, h^G, [\nabla G]) \in \text{Struct}(X)$ such that $V_1 \oplus W_2 \oplus G \cong W_1 \oplus V_2 \oplus G$ and $\text{CS}(\nabla V_1 \oplus \nabla W_2 \oplus \nabla G, \nabla V_2 \oplus \nabla W_1 \oplus \nabla G) = 0$.

Define

$$\text{Struct}_{ST}(X) = \{V \in \text{Struct}(X) | V \text{ is stably trivial}\}$$

$$\text{Struct}_{SF}(X) = \{V \in \text{Struct}(X) | V \oplus \mathcal{F} \cong \mathcal{H}\}$$

where $\mathcal{F} \to X$ and $\mathcal{H} \to X$ are flat structured bundles. Elements in $\text{Struct}_{SF}(X)$ are said to be stably flat. Let $U := \varprojlim U(n)$. Denote by $\theta \in \Omega^1(U, u)$ the canonical left invariant $u$-valued form on $U$. Define

$$b_j = \frac{1}{(j-1)!} \left( \frac{1}{2\pi i} \right)^j \int_0^1 (t^2 - t)^{j-1} dt, j \in \mathbb{N}$$

$$\Theta = \sum_{j=1}^{2j-1} b_j \text{tr}(\theta \wedge \cdots \wedge \theta) \in \Omega^{\text{odd}}(U)$$

Then define

$$\Omega_U(X) = \{g^*(\Theta) + d\beta | g : X \to U, \beta \in \Omega^{\text{even}}(X)\}$$

$$\Omega_{BU}^*(X) = \{\omega \in \Omega_{\bullet=0}^*(X) | [\omega] \in \text{Im} \left( \text{ch} : K^{-(\bullet \mod 2)}(X) \to H^\bullet(X; \mathbb{Q}) \right)\}.$$ 

where $\bullet \in \{\text{even, odd}\}$. The so-called Venice lemma in \cite{15} shows that the map $\widehat{\text{CS}} : \frac{\text{Struct}_{ST}(X)}{\text{Struct}_{SF}(X)} \to \frac{\Omega^{\text{odd}}(X)}{\Omega_U(X)}$ defined by

$$\widehat{\text{CS}}(V) := \text{CS}(\nabla V \oplus \nabla F, \nabla H) \mod \frac{\Omega_U(X)}{\Omega^{\text{odd}}_{\text{exact}}(X)}$$

is an isomorphism, where $F \to X$ and $H \to X$ are trivial bundles over $X$ such that $H \cong V \oplus F$ and $\nabla F, \nabla H$ are flat connections on $F, H$, respectively.

\footnote{This definition differs from the one in \cite{15} Proposition 2.4 by a sign.}
Also, the homomorphism
\[ \Gamma : \text{Struct}_{ST}(X) \to \hat{K}_{SS}(X) \]
defined by \( \Gamma(V) = [V] - [\text{dim}(V)] \) is injective, for \( \text{dim}(V) \) the trivial structured bundle of rank \( V \) with the trivial metric and connection. Thus the homomorphism
\[ i : \Omega_{\text{odd}}(X) \to \hat{K}_{SS}(X) \]
defined by \( i(\phi) = \Gamma \circ \hat{CS}_{-1}(\phi) \) is injective. If we pick \( V \in \hat{CS}_{-1}(\phi) \), then \( V \) is a stably trivial structured bundle and
\[ d\phi = d\text{CS}(\nabla^V \oplus \nabla^F, \nabla^H) = \text{ch}(\nabla^V) - \text{rank}(V) \mod \Omega_{\text{odd}}(X) \]
is independent of the choice of \( V \).

In the following hexagon the diagonal and the off-diagonal sequences are exact, and every square and triangle commutes:

In [15] the homomorphism \( \text{ch}_{\hat{K}_{SS}} : \hat{K}_{SS}(X) \to \Omega_{\text{even}}^{BU}(X) \) is just denoted by \( \text{ch} \), which is a well defined lift of the Chern character form of a connection on a vector bundle to elements in \( \hat{K}_{SS}(X) \). We use the notation \( \text{ch}_{\hat{K}_{SS}} \) in order to keep track of the Chern character in different usage.

**Remark 1.** We show that \( \Omega_U(X) = \Omega_{\text{odd}}^{BU}(X) \), and we will use this identification throughout this paper. This is implicitly stated in [13] Diagram 1. We include the easy proof here for completeness. Let \( d \) be the trivial connection on the trivial bundle \( X \times \mathbb{C}^N \to X \) for some \( N \in \mathbb{N} \). By the proof of [15] Lemma 2.3, the connection \( d + g^*(\theta) \) on \( X \times \mathbb{C}^N \to X \), where \( g : X \to U \) is an arbitrary but fixed smooth map, has trivial holonomy. Following the proof of [15] Lemma 2.3, we have \( g^*(\Theta) = \text{CS}(d, d + g^*(\theta)) = \text{CS}(d, d + g^{-1}dg) =: \text{ch}^{\text{odd}}([g]), \) so \( \Omega_U(X)\Omega_{\text{odd}}^{BU}(X) \).
3. **Freed-Lott differential $K$-theory**

In this section we review Freed-Lott differential $K$-theory \[8\]. If

\[
0 \longrightarrow E_1 \xrightarrow{i} E_2 \xrightarrow{j} E_3 \longrightarrow 0
\]

(3)

is a split short exact sequence of complex vector bundles with connections $\nabla_i$ on $E_i \to X$, for $i = 1, 2, 3$, we define the relative Chern-Simons transgression form $CS(\nabla_1, \nabla_2, \nabla_3) \in \frac{\Omega^{\text{odd}}(X)}{\text{Im}(d)}$ by

\[
CS(\nabla_1, \nabla_2, \nabla_3) := CS((i \oplus s)^*\nabla_2, \nabla_1 \oplus \nabla_3),
\]

noting that $i \oplus s: E_1 \oplus E_3 \to E_2$ is a vector bundle isomorphism.

The Freed-Lott differential $K$-group $\hat{K}_{FL}(X)$ is defined to be the abelian group with the following generators and relation: a generator of $\hat{K}_{FL}(X)$ is a quadruple $E = (E, h, \nabla, \phi)$, where $(E, h, \nabla)$ is as before and $\phi \in \frac{\Omega^{\text{odd}}(X)}{\text{Im}(d)}$.

The only relation is $E_2 = E_1 + E_3$ if and only if there exists a short exact sequence of Hermitian vector bundles (3) and

\[
\phi_2 = \phi_1 + \phi_3 - CS(\nabla_1, \nabla_2, \nabla_3).
\]

For $E_1, E_2 \in \hat{K}_{FL}(X)$, the addition

\[
E_1 + E_2 := (E_1 \oplus E_2, h^{E_1} \oplus h^{E_2}, \nabla^{E_1} \oplus \nabla^{E_2}, \phi_1 + \phi_2)
\]

is well defined. Note that $E_1 = E_2$ if and only if there exists $(F, h^F, \nabla^F, \phi^F) \in \hat{K}_{FL}(X)$ such that

1. $E_1 \oplus F \cong E_2 \oplus F$, and
2. $\phi_1 - \phi_2 = CS(\nabla^{E_2} \oplus \nabla^F, \nabla^{E_1} \oplus \nabla^F)$.

The Freed-Lott differential Chern character $\hat{\text{ch}}_{FL}: \hat{K}_{FL}(X) \to \hat{H}^{\text{even}}(X; \mathbb{R}/\mathbb{Q})$ is defined by

\[
\hat{\text{ch}}_{FL}(E) = \hat{\text{ch}}(E, \nabla) + i_2(\phi),
\]

where $\hat{H}^{\text{even}}(X; \mathbb{R}/\mathbb{Q})$ is the $\mathbb{R}/\mathbb{Q}$ Cheeger-Simons differential characters \[6\], $E = (E, h, \nabla, \phi) \in \hat{K}_{FL}(X)$, $\hat{\text{ch}}(E, \nabla)$ is the differential Chern character defined in \[6\] §4, and $i_2: \frac{\Omega^{\text{odd}}(X)}{\Omega_2^{\text{odd}}(X)} \to \hat{H}^{\text{even}}(X; \mathbb{R}/\mathbb{Q})$ is an injective homomorphism defined by $i_2(\omega)(z) := \int_z \omega \mod \mathbb{Q}$ for $z \in Z_{\text{even}}(X)$ \[6\] Theorem 1.1.
3.1. The Freed-Lott differential analytic index. In this subsection we review the construction of the Freed-Lott differential analytic index. Consider the following diagram:

In this diagram, $\pi : X \to B$ is a proper submersion with closed fibers of even relative dimension and $T^V X \to X$ is the vertical tangent bundle, which is assumed to have a metric $g^{TV}$. $T^H X \to X$ is a horizontal distribution, $g^{TB}$ is a Riemannian metric on $B$, the metric on $TX \to X$ is defined by $g^{TX} := g^{TV} \oplus \pi^*g^{TB}$, $\nabla^{TX}$ is the corresponding Levi-Civita connection, and $\nabla^{TV} := P \circ \nabla^{TX} \circ P$ is a connection on $T^V X \to X$, where $P : TX \to T^V X$ is the orthogonal projection. $T^V X \to X$ is assumed to have a Spin$^c$ structure. Denote by $S^V X \to X$ the Spin$^c$-bundle associated to the characteristic Hermitian line bundle $L^V \to X$ with a unitary connection. The connections on $T^V X \to X$ and $L^V X \to X$ induce a connection $\hat{\nabla}^{TV}$ on $S^V X \to X$. Define an even form $\text{Todd}(\hat{\nabla}^{TV}) \in \Omega^{\text{even}}(X)$ by

$$\text{Todd}(\hat{\nabla}^{TV}) = \tilde{A}(\nabla^{TV}) \wedge \mathbb{L}^{1/2}(\nabla^{LV}).$$

The modified pushforward of forms $\pi_* : \Omega^{\text{odd}}(X) \to \Omega^{\text{odd}}(B)$ is defined by

$$\pi_*(\phi) = \int_{X/B} \text{Todd}(\hat{\nabla}^{TV}) \wedge \phi.$$

The Freed-Lott differential analytic index $\text{ind}^{an} : \hat{K}_{FL}(X) \to \hat{K}_{FL}(B)$ [8, Definition 3.11] is defined by

$$\text{ind}^{an}(E) = (\ker(D^E), h^{\ker(D^E)}, \nabla^{\ker(D^E)}, \pi_*(\phi) + \tilde{\eta}(E)),$$

where $E = (E, h, \nabla, \phi) \in \hat{K}_{FL}(X)$, $\tilde{\eta}(E)$ is the Bismut-Cheeger eta form characterized, up to exact form, by

$$d\tilde{\eta}(E) = \int_{X/B} \text{Todd}(\hat{\nabla}^{TV}) \wedge \text{ch}(\nabla) - \text{ch}(\nabla^{\ker(D^E)}),$$

$D^E$ is the family of Dirac operators on $S^V X \otimes E$, and $\ker(D^E)$ is assumed to form a superbundle over $B$.

4. Main results

4.1. Explicit isomorphisms between $\hat{K}_{FL}$ and $\hat{K}_{SS}$. In this subsection we construct explicit isomorphisms between the Simons-Sullivan differential $K$-group and the Freed-Lott differential $K$-group.
Theorem 1. Let $X$ be a compact manifold. The maps
\[ f : \hat{K}_{SS}(X) \to \hat{K}_{FL}(X), \quad g : \hat{K}_{FL}(X) \to \hat{K}_{SS}(X) \]
defined by
\[ f([E, h^E, [\nabla^E]]) - [F, h^F, [\nabla^F]] = (E, h^E, \nabla^E, 0) - (F, h^F, \nabla^F, 0), \]
\[ g(E, h^E, \nabla^E, \phi) = [E, h^E, [\nabla^E]] + [V, h^V, [\nabla^V]] - [\dim(V), h, [d]], \]
where $\mathcal{V} = (V, h^V, [\nabla^V]) \in \widehat{\mathcal{CS}}^{-1}(\phi)$, are well defined ring isomorphisms, with $f^{-1} = g$. Moreover, $f$ is natural and unique [41 Theorem 3.10], and is compatible with the structure maps $i$, $j$, $\delta$ and $\text{ch}_{\hat{K}_{SS}}$ in [2].

Proof. First we show that the maps $f$ and $g$ are well defined. For the map $f$, if $[E_1, h^{E_1}, [\nabla^{E_1}]] - [F_1, h^{F_1}, [\nabla^{F_1}]] = [E_2, h^{E_2}, [\nabla^{E_2}]] - [F_2, h^{F_2}, [\nabla^{F_2}]]$ in $\hat{K}_{SS}(X)$, then
\[ (E_1, h^{E_1}, \nabla^{E_1}, 0) - (F_1, h^{F_1}, \nabla^{F_1}, 0) = (E_2, h^{E_2}, \nabla^{E_2}, 0) - (F_2, h^{F_2}, \nabla^{F_2}, 0), \]
since there exists $(G, h^G, [\nabla^G]) \in \text{Struct}(X)$ such that $E_1 \oplus F_2 \oplus G \cong F_1 \oplus E_2 \oplus G$ and
\[ 0 = \text{CS}(\nabla^{E_1} \oplus \nabla^{F_2} \oplus \nabla^G, \nabla^{F_1} \oplus \nabla^{E_2} \oplus \nabla^G) = \text{CS}(\nabla^{E_1} \oplus \nabla^{F_2} \oplus \nabla^G, \nabla^{F_1} \oplus \nabla^{E_2}). \]
It follows that the map $f$ is well defined.

For the map $g$, if $(E, h^E, \nabla^E, \phi) = (F, h^F, \nabla^F, \omega)$ in $\hat{K}_{FL}(X)$, then there exists $(G, h^G, [\nabla^G], \phi^G) \in \hat{K}_{FL}(X)$ such that $E \oplus G \cong F \oplus G$ and $\phi - \omega = \text{CS}(\nabla^F \oplus \nabla^G, \nabla^E \oplus \nabla^G)$. We want
\[ [E, h^E, [\nabla^E]] + [V, h^V, [\nabla^V]] - [\dim(V), h, [d]] = [F, h^F, [\nabla^F]] + [W, h^W, [\nabla^W]] - [\dim(W), h, [d]], \]
where $\widehat{\text{CS}}(V) = \phi$ and $\widehat{\text{CS}}(W) = \omega$. We need to show that there exists $(G', h^{G'}, [\nabla^{G'}]) \in \text{Struct}(X)$ such that
\[ (E, h^E, [\nabla^E]) + (V, h^V, [\nabla^V]) + (\dim(W), h, [d]) + (G', h^{G'}, [\nabla^{G'}]) = (F, h^F, [\nabla^F]) + (W, h^W, [\nabla^W]) + (\dim(V), h, [d]) + (G', h^{G'}, [\nabla^{G'}]), \tag{4} \]
and $\text{CS}(\nabla^E \oplus \nabla^V \oplus dW \oplus \nabla^G, \nabla^F \oplus \nabla^W \oplus dV \oplus \nabla^{G'}) = 0$. (4) is equivalent to
\[ (E \oplus V \oplus \dim(W) \oplus G', h^E \oplus h^V \oplus h \oplus h^{G'}, [\nabla^E \oplus \nabla^V \oplus d \oplus \nabla^{G'}]) = (F \oplus W \oplus \dim(V) \oplus G', h^F \oplus h^W \oplus h \oplus h^{G'}, [\nabla^F \oplus \nabla^W \oplus d \oplus \nabla^{G'}]). \]
Since $V$ and $W$ are stably trivial, there exist trivial bundles $V'$ and $W'$ with connections $\nabla^{V'}$ and $\nabla^{W'}$ such that
\[ H^V := \dim(V) \oplus V' = V \oplus V', \quad H^W := \dim(W) \oplus W' = W \oplus W', \]
and
\[ \phi = \text{CS}(\nabla^V \oplus \nabla^{V'}, \nabla^{H^V}), \quad \omega = \text{CS}(\nabla^W \oplus \nabla^{W'}, \nabla^{H^W}). \]
By taking $G' = G \oplus V' \oplus W'$, we have
\begin{align*}
(E \oplus V \oplus \dim(W)) \oplus (G \oplus V' \oplus W') \\
\cong (E \oplus G) \oplus (V \oplus V') \oplus (\dim(W) \oplus W') \\
\cong (F \oplus G) \oplus (\dim(V) \oplus V') \oplus (W \oplus W') \\
\cong (F \oplus W \oplus \dim(V)) \oplus (G \oplus V' \oplus W')
\end{align*}
\tag{5}
and for $d^V, d^W$ the trivial connections on $\dim(V), \dim(V)$, respectively,
\begin{align*}
\CS(\nabla_E \oplus \nabla^V \oplus d^W \oplus \nabla^G \oplus \nabla^{V'} \oplus \nabla^W, \nabla^F \oplus \nabla^W \oplus d^V \oplus \nabla^G \oplus \nabla^{V'} \oplus \nabla^W) \\
= \CS(\nabla^E \oplus \nabla^G, \nabla^F \oplus \nabla^G) + \CS(\nabla^V \oplus \nabla^{V'}, d^V \oplus \nabla^{V'}) + \CS(d^W \oplus \nabla^W, \nabla^W \oplus \nabla^{W'}) \\
= -\phi + \omega + \CS(\nabla^V \oplus \nabla^{V'}, \nabla^H) + \CS(\nabla^{H^W}, \nabla^W \oplus \nabla^W) \\
= -\phi + \omega + \phi - \omega \\
= 0
\end{align*}
\tag{6}

(5) and (6) imply (4), so the map $g : \hat{K}_{FL}(X) \to \hat{K}_{SS}(X)$ is well defined.

We now show that $f$ and $g$ are inverses. Note that
\begin{align*}
(g \circ f)([E, h^E, [\nabla^E]] - [F, h^F, [\nabla^F]]) &= g((E, h^E, \nabla^E, 0) - (F, h^F, \nabla^F, 0)) \\
&= [E, h^E, [\nabla^E]] - [F, h^F, [\nabla^F]]
\end{align*}
as $\CS^{-1}(0) = 0 \in \Struct_{ST}(X)$. For the other direction, we consider
\begin{align*}
(f \circ g)(E, h^E, \nabla^E, \phi) &= (E, h^E, \nabla^E, 0) + (V, h^V, \nabla^V, 0) - (\dim(V), h, d, 0),
\end{align*}
where $\CS(V) = \phi$ for $V := (V, h^V, [\nabla^V]) \in \Struct_{ST}(X)$. It suffices to show
$(E, h^E, \nabla^E, \phi) + (\dim(V), h, d^V, 0) = (E, h^E, \nabla^E, 0) + (V, h^V, \nabla^V, 0)$,
which is equivalent to
\begin{align*}
(E \oplus \dim(V), h^E \oplus h, \nabla^E \oplus d^V, \phi) = (E \oplus V, h^E \oplus h^V, \nabla^E \oplus \nabla^V, 0). \tag{7}
\end{align*}
To see this, since $V = (V, h^V, [\nabla^V])$ is stably trivial, there exist trivial structured bundles $\mathcal{F} = (F, h, [d^F])$ and $\mathcal{H} = (H, h, [d^H])$ such that $V \oplus \mathcal{F} \cong H$ and $\phi = \CS(d^H, \nabla^V \oplus d^F)$. Thus $E \oplus \dim(V) \oplus \dim(F) \cong E \oplus \dim(H) \cong E \oplus \dim(F)$, and
\begin{align*}
\CS(\nabla^E \oplus \nabla^V, \nabla^E \oplus d^V) &= \CS(\nabla^E \oplus \nabla^V \oplus d^F, \nabla^E \oplus d^V \oplus d^F) \\
&= \CS(\nabla^E \oplus \nabla^V \oplus d^F, \nabla^E \oplus d^H) = \phi.
\end{align*}
This proves (7).

$f$ is obviously a natural ring homomorphism. Since $g = f^{-1}$, $g$ is also a ring homomorphism. \hfill \square

The following corollary follows from the compatibility of $f$ and $\ch_{\hat{K}_{SS}}$. 
Corollary 1. Let $X$ be a compact manifold. The following diagram is commutative.

$$
\begin{array}{c}
0 \longrightarrow K_{SS}^{-1}(X; \mathbb{R}/\mathbb{Z}) \xrightarrow{j} \widehat{K}_{SS}(X) \xrightarrow{\text{ch}} \Omega_{BU}(X) \longrightarrow 0 \\
\downarrow f \downarrow \quad \downarrow \quad \downarrow = \\
0 \longrightarrow K_{L}^{-1}(X; \mathbb{R}/\mathbb{Z}) \xrightarrow{j'} \widehat{K}_{FL}(X) \xrightarrow{\omega} \Omega_{BU}(X) \longrightarrow 0
\end{array}
$$

where $\bar{f}$ is the restriction of $f$ to $K_{SS}^{-1}(X; \mathbb{R}/\mathbb{Z})$. Here $\omega : \widehat{K}_{FL}(X) \to \Omega_{BU}(X)$ is defined by $\omega(E, h^E, \nabla^E, \phi) = \text{ch}((\nabla^E) + d\phi$.

Note that the horizontal sequences are exact by [8, 15].

4.2. The differential analytic index in $\widehat{K}_{SS}$. In this subsection we give the formula for the differential analytic index in Simons-Sullivan differential $K$-theory.

Let $\pi : X \to B$ be a proper submersion of even relative dimension and its fibers are assumed to be Spin$^c$. The differential analytic index in Simons-Sullivan differential $K$-theory is given by forcing the following diagram to be commutative:

$$
\begin{array}{c}
\widehat{K}_{SS}(X) \xrightarrow{f} \widehat{K}_{FL}(X) \\
\downarrow \text{ind}_{SS}^{an} \quad \downarrow \text{ind}_{FL}^{an} \\
\widehat{K}_{SS}(B) \xleftarrow{g} \widehat{K}_{FL}(B)
\end{array}
$$

Let $E := [E, h^E, [\nabla]] \in \widehat{K}_{SS}(X)$. Since

$$(g \circ \text{ind}_{FL}^{an} \circ f)(E) = [\ker(D^E), h^{\ker(D^E)}, [\nabla^{\ker(D^E)}]]$$

$$+ [V, h^V, [\nabla^V]] - [\dim(V), h, [d]],$$

where $V := (V, h^V, [\nabla^V]) \in \text{Struct}_{ST}(B) / \text{Struct}_{SF}(B)$ is uniquely determined by the condition $\widehat{\text{CS}}(V) = \widehat{\eta}(E) \mod \Omega^{\text{odd}}_{U}(B)$, it follows that the differential analytic index in the Simons-Sullivan differential $K$-theory $\text{ind}_{SS}^{an} : \widehat{K}_{SS}(X) \to \widehat{K}_{SS}(B)$ is given by

$$\text{ind}_{SS}^{an}(E) = [\ker(D^E), h^{\ker(D^E)}, [\nabla^{\ker(D^E)}]] + [V, h^V, [\nabla^V]] - [\dim(V), h, [d]],$$

(8)

where $\ker(D^E)$ is assumed to form a superbundle over $B$. Although $V := \widehat{\text{CS}}^{-1}(\widehat{\eta}(E))$ is uniquely determined up to a stably flat structured bundle, its class $[V] \in \widehat{K}_{SS}(B)$ is unique since the differential $K$-theory class of a stably flat structured bundle is zero. Moreover, since $\text{ind}_{FL}^{an}$ is well defined (see [9, Proposition 1] for a proof which does not use the differential family index theorem), it follows that $\text{ind}_{SS}^{an}$ is well defined too.
If one defines the Simons-Sullivan differential analytic index $\text{ind}_{SS}^{an}(E)$ without considering the other differential analytic indices, a natural candidate would be, say, in the special case when $\text{ker}(D^E) \to B$ is a superbundle,

$$\text{ind}_{SS}^{an}(E) = [\text{ker}(D^E), h^{\text{ker}(D^E)}, [\nabla^{\text{ker}(D^E)}]].$$

This definition coincides with $[8]$ if and only if $V \in \text{Struct}_{SF}(B)$, which is equivalent to saying that $\tilde{\eta}(E) \in \Omega_{T}(B) = \Omega_{\text{odd}}^{BU}(B)$. However, this is not true since

$$d\tilde{\eta}(E) = \int_{X/B} \text{Todd}(\hat{\nabla}^{TV,X}) \wedge \text{ch}(\nabla) - \text{ch}(\nabla^{\text{ker}(D^E)}),$$

which shows that $\tilde{\eta}(E)$ is not closed in general.

**Lemma 1.** Let $E = [E, h, [\nabla]] \in \widehat{K}_{SS}(X)$. Then

$$\text{ch}_{\widehat{K}_{SS}}(\text{ind}_{SS}^{an}(E)) = \text{ch}(\nabla^{\text{ker}(D^E)}) + d\tilde{\eta}(E).$$

It follows from Lemma 1 and the local family index theorem that

$$\text{ch}_{\widehat{K}_{SS}}(\text{ind}_{SS}^{an}(E)) = \text{ch}(\nabla^{\text{ker}(D^E)}) + d\tilde{\eta}(E)$$

$$= \int_{X/B} \text{Todd}(\hat{\nabla}^{TV,X}) \wedge \text{ch}(\nabla^{E})$$

$$= \pi_{*}(\text{ch}_{\widehat{K}_{SS}}(E)).$$

We define the Simons-Sullivan differential Chern character $\hat{\text{ch}}_{SS} : \widehat{K}_{SS}(X) \to \overline{H}^{\text{even}}(X; \mathbb{R}/\mathbb{Q})$ by

$$\hat{\text{ch}}_{SS}(E) := \hat{\text{ch}}(E, \nabla),$$

where $E = [E, h, [\nabla]]$.

It is instructive to note that the following diagram commute,

$$\begin{array}{ccc}
\widehat{K}_{SS}(X) & \xrightarrow{i} & \Omega_{\text{odd}}^{BU}(X) \\
\Omega_{\text{odd}}^{BU}(X) & \xrightarrow{f} & \widehat{K}_{FL}(X) \\
\end{array}$$

where $f : \widehat{K}_{SS}(X) \to \widehat{K}_{FL}(X)$ is the isomorphism given by Theorem 1.
4.3. **The differential Grothendieck-Riemann-Roch theorem.** In this subsection we prove the dGRR in Simons-Sullivan differential $K$-theory. To be precise, we first prove the special case that the family of kernels $\text{ker}(D^E)$ forms a superbundle by a theorem of Bismut reviewed below. The general case follows from the standard perturbation argument as in [8, §7].

We now recall Bismut’s theorem. For the geometric construction of the analytic index given in §4.2, with the fibers assumed to be Spin, and $\text{ker}(D^E) \to B$ assumed to form a superbundle, we have

\[
\hat{\chi}(\text{ker}(D^E), \nabla^{\text{ker}(D^E)}) + i_2(\overline{\eta}) = \int_{X/B} \hat{A}(T^V X, \nabla^{T^V X}) \ast \hat{\chi}(E, \nabla^E) \tag{9}
\]

[1, Theorem 1.15], where $\int_{X/B}$ is the pushforward of differential characters for proper submersion [8, §8.3], $\ast$ is the multiplication of differential characters [6, §1], and $\hat{A}(T^V Z, \nabla^{T^V X}) \in \hat{H}^{\text{even}}(X; \mathbb{R}/\mathbb{Q})$ is the differential character associated to the $\hat{A}$-class (see [6, §2]). If the fibers are assumed to be Spin$^c$, [9] has the obvious modification, and in our notation becomes

\[
\hat{\chi}(\text{ker}(D^E), \nabla^{\text{ker}(D^E)}) + i_2(\overline{\eta}) = \int_{X/B} \hat{\text{Todd}}(T^V X, \nabla^{T^V X}) \ast \hat{\chi}(E, \nabla^E), \tag{10}
\]

for $\hat{\text{Todd}}(T^V X, \nabla^{T^V X}) \in \hat{H}^{\text{even}}(X; \mathbb{R}/\mathbb{Q})$ the differential character associated to the Todd class (see [9, §2]). We will write $\hat{\text{Todd}}(T^V X, \nabla^{T^V X})$ as $\hat{\text{Todd}}(\nabla^{T^V X})$ in the sequel. Note that (9) and (10) extend to the general case where $\text{ker}(D^E) \to B$ does not form a bundle [11, p. 23].

**Theorem 2 (Differential Grothendieck-Riemann-Roch theorem).** Let $\pi : X \to B$ be a proper submersion with closed Spin$^c$-fibers of even dimension. The following diagram is commutative:

\[
\begin{array}{ccc}
\hat{K}_{SS}(X) & \xrightarrow{\hat{\chi}_{SS}} & \hat{H}^{\text{even}}(X; \mathbb{R}/\mathbb{Q}) \\
\text{ind}_{SS} \downarrow & & \downarrow \int_{X/B} \hat{\text{Todd}}(\nabla^{T^V X}) \ast (\cdot) \\
\hat{K}_{SS}(B) & \xrightarrow{\hat{\chi}_{SS}} & \hat{H}^{\text{even}}(B; \mathbb{R}/\mathbb{Q})
\end{array}
\]

i.e., if $\mathcal{E} = [E, h, [\nabla^E]] \in \hat{K}_{SS}(X)$, then

\[
\hat{\chi}_{SS}(\text{ind}_{SS} (\mathcal{E})) = \int_{X/B} \hat{\text{Todd}}(\nabla^{T^V X}) \ast \hat{\chi}_{SS}(\mathcal{E}).
\]
Proof.
\[
\widehat{ch}_{SS}(\text{ind}^{an}_{SS}(\mathcal{E})) = \widehat{ch}_{SS}([\ker(D^E), h^{\ker(D^E)}, [\nabla^{\ker(D^E)}]] + [V, h^V, [\nabla^V]] - [\dim(V), h, [d]])
\]
\[
= \widehat{ch}(\ker(D^E), \nabla^{\ker(D^E)}) + i_2(\bar{\eta}(\mathcal{E}))
\]
\[
= \int_{X/B} \text{Todd}(\hat{T}^{VX}) \ast \widehat{ch}(E, \nabla^E)
\]
\[
= \int_{X/B} \text{Todd}(\hat{T}^{VX}) \ast \widehat{ch}_{SS}(\mathcal{E})
\]
where the second equality follows from Proposition ?? and the third equality follows from [10]. \hfill \Box

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