BOUNDS AND REGULARITY OF SOLUTIONS OF PLANAR DIV-CURL PROBLEMS

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Abstract. New 2-norm bounds are described for the least energy solutions of planar div-curl boundary value problems on bounded regions in space. Prescribed flux, tangential or mixed flux and tangential boundary conditions are treated. A harmonic decomposition of planar fields is used to separate the solutions due to source terms from harmonic components that are determined by boundary data. Some regularity results are described.

1. Introduction

This paper describes some properties of solutions of boundary value problems for div-curl systems on bounded regions \( \Omega \subset \mathbb{R}^2 \). This is a degenerate elliptic system of two equations in two unknowns where the existence and uniqueness conditions for solutions depend on both properties of the data and also the topology of the region and the boundary conditions. Questions about well-posedness and uniqueness of solutions were studied using variational methods in Alexander and Auchmuty [1]. Here primary attention is devoted to finding energy (2-norm) bounds on solutions, their dependence on boundary conditions and associated regularity results.

In particular some different decompositions of the fields will be used to obtain different and better energy inequalities and regularity results. Prescribed flux prescribed tangential and mixed flux and tangential boundary conditions will be studied. The analysis is based on a "harmonic decomposition", which differs from the Hodge-Weyl decompositions used in [1]. In this decomposition, the potentials associated with sources are solutions of zero-Dirichlet problems for Poisson’s equation. Then the harmonic component is determined as solutions of homogeneous equations with nontrivial Neumann boundary data.

The energy (2-norm) bounds obtained here hold for the least energy solutions when the boundary value problem is underdetermined. To simplify the regularity statements, and also some estimates, only the simplest div-curl system is treated. A number of other authors have regard this system is a prototype for degenerate elliptic systems, while K.O. Friedrichs [16] has called an inequality that bounds the the energy of a field by norms of its divergence and curl, the main inequality of vector analysis. This system has been used to model many
different situations in both fluid mechanics and electromagnetic field theories. Many of the results described here may be generalized to the case of general elliptic coefficient matrix $E(x)$, as used in $[11]$ using standard assumptions and techniques.

The bounds described here are quite different to those in papers such as that of Krizek and Neittaanmaki $[19]$ or the Schauder estimates of Bolik and von Wahl $[9]$ for example. Since a major interest is the dependence of solutions on boundary data, this analysis is also quite different to the work of Brezis and Bourgain $[11]$ who studied these problems in $\mathbb{R}^N$ or with periodic boundary conditions. A sophisticated analysis of the prescribed flux and tangential boundary problems has been given by Mitrea $[20]$. She used layer potentials and Besov spaces to study these problems on bounded regions with Lipschitz boundaries. There also has been considerable work on the numerical analysis and simulation of solutions of these problems by a large number of authors. See Bramble and Pasciak $[10]$ for example, or Monk $[21]$ for an overview.

In section 3, some results about the regularity of orthogonal projections of $L^2$–vector fields are described. It is shown that, when a field is smooth on an open subset of $\Omega$, so are its potentials. (The function that is often called a stream function will be called a potential here.) Also the class of harmonic vector fields that can be represented by conjugate harmonic functions is characterized.

In section 4, some orthogonal decompositions of the space $L^2(\Omega; \mathbb{R}^2)$ the standard inner product are described. Readers can look at page 314 of Dautray and Lions $[12]$ to see the possible orthogonal representations available. While that diagram is for 3-dimensional regions, essentially the same analysis holds for planar regions. The only simplification for 2-dimensional problems is that the dimensions of the spaces of special harmonic fields must be equal in 2-d while they can vary in 3-d. Theorem 4.2 here is the crucial theorem about the existence and regularity of representations of irrotational or solenoidal vector fields in $L^p(\Omega; \mathbb{R}^2)$ using potentials. It should be commented that the non-uniqueness of the Helmholtz decomposition of vector fields has been a source of many problems in applications as many of the commonly used splittings are not described by projections. As a corollary a proof that harmonic vector fields must be $C^\infty$ on $\Omega$ is proved. This generalizes Weyl’s lemma for harmonic functions.

Section 5 describes some properties of scalar Laplacian boundary value problems are collected for later use. Some properties of the solution of zero-Dirichlet boundary value problem for Poisson’s equation that are not readily accessible are described. One consequence is that a harmonic decomposition that applies to $L^1$ and $L^1_{loc}$ fields is obtained. A different Hilbert space $H_0(\Delta, \Omega)$ is introduced so that results may be given when the boundary $\partial \Omega$ is not necessarily $C^1$. Explicit formulae, and estimates, for solutions of Neumann, and other, harmonic boundary value problems in terms of Steklov eigenvalues and eigenfunctions follow from the author’s work in citeAuH and are needed to obtain results about the dependence of solutions on boundary data.

Sections 6 and 7 describe results about the least norm solutions of the prescribed flux and prescribed tangent div-curl boundary value problem respectively when the necessary compatibility conditions hold. When each of the data is $L^2$, solution estimates depending
on the principal Dirichlet and Steklov eigenvalues $\lambda_1$ and $\delta_1$ of the Laplacian on $\Omega$. These estimates are sharp. When $\Omega$ is not simply connected these solutions are not unique and the extra information required for well-posedness were studied in \[1\].

When mixed tangential and normal boundary conditions are imposed on $\partial \Omega$, no compatibility conditions on the data are required for the existence of solutions. Under natural assumptions on the data, it is shown how the solutions may be represented using two potentials and estimates of these solutions are found in terms of some different eigenvalues associated with the Laplacian on $\Omega$.

This paper aims to provide a self-contained description of some basic results about these problems in a manner that can be used by numerical analysts and others interested in the approximation, and properties, of solutions. Thus some of the results here are variants of results known, often in much greater generality, to researchers in linear elliptic boundary value problems.

2. Definitions and Notation.

In this paper, standard definitions as given in Evans text [14] or Attouch, Butazzo and Michaille [2] will generally be used. - specialized to $\mathbb{R}^2$ since this paper only treats planar problems. Cartesian coordinates $x = (x_1, x_2)$ will be used and Euclidean norms and inner products are denoted by $|\cdot|$ and $x \cdot y$. A region is a non-empty, connected, open subset of $\mathbb{R}^2$. Its closure is denoted $\overline{\Omega}$ and its boundary is $\partial \Omega := \overline{\Omega} \setminus \Omega$. Often the position vector $x$ is omitted in formulae for functions and fields and equality should be interpreted as holding a.e. with respect to 2-dimensional Lebesgue measure $d^2x = dx_1 dx_2$ on $\Omega$.

A number of the results here depend on the differential topology of the region $\Omega$. A curve in the plane is said to be a simple Lipschitz loop if it is a closed, non-self-intersecting curve with at least two distinct points and a uniformly Lipschitz parametrization. Such loops will be compact and have finite, nonzero, length. Arc-length will be denoted $s(\cdot)$ and our standard assumption is

**Condition B1.** $\Omega$ is a bounded region in $\mathbb{R}^2$ with boundary $\partial \Omega$ the union of a finite number of disjoint simple Lipschitz loops $\{\Gamma_j : 0 \leq j \leq J\}$.

Here $\Gamma_0$ will always be the exterior loop and the other $\Gamma_j$ will enclose holes in the region $\Omega$. The interior region to the loop $\Gamma_0$ defined by the Jordan curve theorem will be denoted $\Omega_0$. When $J = 0$, $\Omega$ is said to be simply connected and then $\Omega_0 = \Omega$.

The outward unit normal to a region at a point on the boundary is denoted $\nu(z) = (\nu_1(z), \nu_2(z))$. Then $\tau(z) := (-\nu_2(z), \nu_1(z))$ is the positively oriented unit tangent vector at a point $z \in \partial \Omega$. $\nu, \tau$ are defined s.a.e. on $\partial \Omega$ when (B1) holds.

In this paper, all functions are assumed to be at least $L^1_{\text{loc}}$ and derivatives will be taken in a weak sense. The spaces $W^{1,p}(\Omega), W^{1,p}_0(\Omega)$ are defined as usual for $p \in [1, \infty]$ with standard norms denoted by $\|\cdot\|_{1,p}$. When $p = 2$ the spaces will also be denoted $H^1(\Omega), H^1_0(\Omega)$. 
When $\Omega$ is bounded, the trace of Lipschitz continuous functions on $\overline{\Omega}$ restricted to $\partial \Omega$ is again Lipschitz continuous. The extension of this linear mapping is a continuous linear mapping of $W^{1,p}(\Omega)$ to $L^p(\partial \Omega, ds)$ for all $p \in [1, \infty]$ when (B1) holds. See [13], Section 4.2 for details. From Morrey’s theorem, $\gamma$ maps $W^{1,p}(\Omega)$ into $C^\alpha(\partial \Omega)$ when $p > 2$ and $\alpha = 1 - 2/p$. Di Benedetto [13] proposition 18.1 shows that when $\varphi \in H^1(\Omega)$ then $\gamma(\varphi) \in L^q(\partial \Omega, ds)$ for all $q \in [1, \infty)$. Also if $\varphi \in W^{1,p}(\Omega)$ with $p \in [1, 2]$ then $\gamma(\varphi) \in L^q(\partial \Omega, ds)$ for all $q \in [1, p_T]$ with $p_T = p/(2 - p)$ under stronger regularity conditions on the boundary.

The region $\Omega$ is said to satisfy a compact trace theorem provided the trace mapping $\gamma : H^1(\Omega) \to L^2(\partial \Omega, ds)$ is compact. Theorem 1.5.1.10 of Grisvard [18] proves an inequality that implies the compact trace theorem when $\partial \Omega$ satisfies (B1).

We will generally use the following equivalent inner product on $H^1(\Omega)$

$$
[\varphi, \psi]_\partial := \int_\Omega \nabla \varphi \cdot \nabla \psi \, d^2x + \int_{\partial \Omega} \gamma(\varphi) \gamma(\psi) \, ds
$$

(2.1)

The associated norm is denoted $\|\varphi\|_\partial$. The proof that this norm is equivalent to the usual $(1, 2)$–norm on $H^1(\Omega)$ when (B1) holds is Corollary 6.2 of [4] and also is part of theorem 21A of [22]. The inner product on $H^1_0(\Omega)$ is the restriction of this inner product.

When $\Omega$ satisfies (B1), then the Gauss-Green theorem holds in the forms

$$
\int_\Omega D_j \varphi(x) \, d^2x = \int_{\partial \Omega} \gamma(\varphi)(z) \nu_j(z) \, ds(z) \quad \text{for all } \varphi \in W^{1,1}(\Omega) \quad \text{and} \quad (2.2)
$$

$$
\int_\Omega \varphi(x) D_j \psi(x) \, dx = \int_{\partial \Omega} \gamma(\varphi) \gamma(\psi) \nu_j \, ds - \int_\Omega \psi(x) D_j \varphi(x) \, dx \quad \text{for each } j \quad (2.3)
$$

and all $\varphi, \psi$ in $W^{1,p}(\Omega)$ with $p \geq 4/3$. Often the trace operator will be implicit in boundary integrals.

When $\varphi \in W^{1,1}(\Omega)$ is weakly differentiable, then the gradient and Curl of $\varphi$ are the vector fields

$$
\nabla \varphi(x) := (D_1 \varphi(x), D_2 \varphi(x)) \quad \text{and} \quad \nabla^\perp \varphi(x) := (D_2 \varphi(x), -D_1 \varphi(x)).
$$

(2.4)

Here $D_j \varphi$ or $\varphi, j$ denotes the weak $j$-th derivative.

A function $\rho \in L^1_{loc}$ is defined to be the Laplacian of $\varphi$ provided one has

$$
\int_\Omega \varphi \, \Delta v \, d^2x = \int_\Omega \rho \, v \, d^2x \quad \text{for all } \quad v \in C^2_c(\Omega)
$$

A function $\varphi \in W^{1,1}(\Omega)$ is said to be harmonic on $\Omega$ provided

$$
\int_\Omega \nabla \varphi \cdot \nabla \chi \, d^2x = 0 \quad \text{for all } \chi \in C^2_c(\Omega).
$$

(2.5)

The subspace of all harmonic functions in $H^1(\Omega)$ will be denoted $\mathcal{H}(\Omega)$ and it is straightforward to observe that $H^1(\Omega) = H^1_0(\Omega) \oplus \mathcal{H}(\Omega)$ and that $\mathcal{H}(\Omega)$ is isomorphic to the trace space $H^{1/2}(\partial \Omega)$. Later use will be made of the analysis in [5] where this is described and $\partial$–orthogonal bases of the space $\mathcal{H}(\Omega)$ are found that involve the Steklov eigenfunctions of the Laplacian on $\Omega$. 

3. Projections and Potentials in \( L^2(\Omega; \mathbb{R}^2) \).

Here we will first describe the representation of planar vector fields by scalar potentials \( \varphi, \psi \) in the form
\[
\mathbf{v}(x) = \nabla \psi(x) - \nabla \varphi(x) \quad \text{on} \quad \Omega.
\]
(3.1)

Often \( \psi \) is called a stream function and a Cartesian frame on \( \Omega \) is used. This representation is generally called a Helmholtz decomposition and many different choices of \( \varphi, \psi \) have been used by scientists and engineers for different boundary value problems. The choice of signs in (3.1) is commonly used in applications and also introduces some mathematical consistency.

In this paper, decompositions of the form (3.1) that are defined by projections and also have orthogonality properties will be analyzed in some detail.

For \( p \in [1, \infty] \), \( L^p(\Omega; \mathbb{R}^2) \) is the space of planar vector fields \( \mathbf{v}(x) = (v_1(x), v_2(x)) \) on \( \Omega \) whose components are \( L^p \)-functions on \( \Omega \). Let \( L^2(\Omega; \mathbb{R}^2) \) is the real Hilbert space of \( L^2 \)-vector fields on \( \Omega \) with the inner product
\[
\langle \mathbf{v}, \mathbf{w} \rangle := \int_{\Omega} \mathbf{v} \cdot \mathbf{w} \, d^2x.
\]
(3.2)

Throughout this paper if fields or subspaces are said to be orthogonal, without any further adjectives, this inner product is implied.

Define \( G(\Omega), G_0(\Omega), \text{Curl}(\Omega), \text{Curl}_0(\Omega) \) be the subspaces of gradients and Curls with potentials \( \varphi \) in \( H^1(\Omega), H^1_0(\Omega) \) respectively. These subspaces will first be shown to be closed subspaces of \( L^2(\Omega; \mathbb{R}^2) \) and then some properties of the associated orthogonal projections are obtained. This will be done using a variational characterization of projections based on Riesz’ projection theorem as described in section 3 of Auchmuty [3].

First consider the approximation of two-dimensional vector fields by gradient fields. This becomes a problem of minimizing the functional \( \mathcal{E}_v \) defined by
\[
\mathcal{E}_v(\varphi) := \int_{\Omega} \left[ |\nabla \varphi|^2 + 2 \mathbf{v} \cdot \nabla \varphi \right] \, d^2x
\]
on \( H^1(\Omega), H^1_0(\Omega) \) respectively. This functional differs from \( \| \mathbf{v} + \nabla \varphi \|^2 \) by \( \| \mathbf{v} \|^2 \) so minimizing this is equivalent to finding the best approximation of \( \mathbf{v} \) by gradients in \( L^2(\Omega; \mathbb{R}^2) \).

As is standard, the space \( H^1(\Omega) \) is replaced by the space \( H^1_m(\Omega) \) of all potentials with mean value \( \bar{\varphi} = 0 \). The inner product on both \( H^1_m(\Omega) \) and \( H^1_0(\Omega) \) is taken as \( \langle \varphi, \chi \rangle \mathbf{v} := \langle \nabla \varphi, \nabla \chi \rangle \).

The results about these variational principles may be summarized as follows.

**Theorem 3.1.** Assume \( \Omega \) obeys (B1) and \( v \in L^2(\Omega; \mathbb{R}^2) \). Then there is a unique \( \varphi_v \in H^1_m(\Omega) \) that minimizes \( \mathcal{E}_v \) on \( H^1_m(\Omega) \) and it satisfies
\[
\int_{\Omega} (\nabla \varphi + \mathbf{v}) \cdot \nabla \chi \, d^2x = 0 \quad \text{for all} \quad \chi \in H^1(\Omega).
\]
(3.4)

Moreover (i) \( \| \nabla \varphi_v \| \leq \| \mathbf{v} \| \) and (ii) if \( \mathbf{v} = - \nabla \psi \) then there is a constant \( c \) such that \( \varphi_v + \psi \equiv c \) on \( \Omega \).
Proof. When (B1) holds there is a \( \lambda_m > 0 \) such that
\[
\int_{\Omega} |\nabla \varphi|^2 \, d^2x \geq \lambda_m \int_{\Omega} \varphi^2 \, d^2x \quad \text{for all } \varphi \in H^1_m(\Omega). \tag{3.5}
\]
Hence \( \mathcal{E}_v \) is continuous, strictly convex and coercive on \( H^1_m(\Omega) \) so there is a unique minimizer of \( \mathcal{E}_v \). This functional is Gateaux differentiable and the minimization condition is \( (3.4) \).

Choose \( \chi = \nabla \varphi_v \) in \( (3.4) \) then (i) follows from Cauchy-Schwarz. (ii) follows as each function in \( H^1(\Omega) \) has a unique decomposition of the form \( \varphi = \varphi_m + c \) where \( \varphi_m \in H^1_m(\Omega) \) and \( c = \overline{\varphi} \). \( \square \)

Define \( P_G : L^2(\Omega; \mathbb{R}^2) \to L^2(\Omega; \mathbb{R}^2) \) by \( P_G \mathbf{v} := -\nabla \varphi_v \). This result implies that \( G(\Omega) \) is a closed subspace of \( L^2(\Omega; \mathbb{R}^2) \) from corollary 3.3 of citeAu1 and (ii) shows that \( P_G \) is the projection of \( L^2(\Omega; \mathbb{R}^2) \) onto \( G(\Omega) \). The extremality condition \( (3.4) \) implies that \( \varphi_v \) is a weak solution of the Neumann problem
\[
-\Delta \varphi = \text{div } \mathbf{v} \quad \text{on } \Omega \quad \text{and} \quad D_v \varphi = -\mathbf{v} \cdot \nu \quad \text{on } \partial \Omega. \tag{3.6}
\]

The orthogonal complement of this projection is \( Q_G := I - P_G \) and is the 2d version of the Leray projection of fluid mechanics. Equation \( (3.4) \) says that \( Q_G \) and \( P_G \) are orthogonal projections.

**Theorem 3.2.** Assume \( \Omega \) obeys (B1) and \( \mathbf{v} \in L^2(\Omega; \mathbb{R}^2) \). Then there is a unique \( \varphi_{v0} \in H^1_0(\Omega) \) that minimizes \( \mathcal{E}_v \) on \( H^1_0(\Omega) \) and it satisfies
\[
\int_{\Omega} (\nabla \varphi + \mathbf{v}) \cdot \nabla \chi \, d^2x = 0 \quad \text{for all } \chi \in H^1_0(\Omega) \tag{3.7}
\]
and \( \| \nabla \varphi_{v0} \| \leq \| \mathbf{v} \| \).

Proof. When (B1) holds there is a \( \lambda_1 > 0 \) such that
\[
\int_{\Omega} |\nabla \varphi|^2 \, d^2x \geq \lambda_1 \int_{\Omega} \varphi^2 \, d^2x \quad \text{for all } \varphi \in H^1_0(\Omega). \tag{3.8}
\]
Hence \( \mathcal{E}_v \) is continuous, strictly convex and coercive on \( H^1_0(\Omega) \) so there is a unique minimizer of \( \mathcal{E}_v \). This functional is Gateaux differentiable and the minimization condition is \( (3.7) \).

Choose \( \chi = \nabla \varphi_v \) in \( (3.7) \) then the last part follows from Cauchy-Schwarz. \( \square \)

The extremality condition \( (3.4) \) says that \( \varphi_{v0} \in H^1_0(\Omega) \) is a weak solution of the Dirichlet problem for
\[
-\Delta \varphi = \text{div } \mathbf{v} \quad \text{on } \Omega \tag{3.9}
\]

Define \( P_{G_0} : L^2(\Omega; \mathbb{R}^2) \to L^2(\Omega; \mathbb{R}^2) \) by \( P_{G_0} \mathbf{v} := -\nabla \varphi_{v0} \). This result implies that \( G_0(\Omega) \) is a closed subspace of \( L^2(\Omega; \mathbb{R}^2) \) from corollary 3.3 of \cite{Au} and (ii) shows that \( P_{G_0} \) is the projection of \( L^2(\Omega; \mathbb{R}^2) \) onto \( G_0(\Omega) \). The complementary projection \( Q_{G_0} := I - P_{G_0} \) is the projection onto the null space of the divergence operator - see theorem \textbf{4.2} of the next section.
These characterizations of these projections allow the proof that they preserve interior regularity. A vector field $v$ is said to be $H^m$ on an open subset $\mathcal{O}$ provided each component $v_j$ is of class $H^m$ on $\mathcal{O}$. Specifically the following holds.

**Theorem 3.3.** Suppose $\Omega$ obeys (B1) and $\mathcal{O}$ is open with $\overline{\mathcal{O}} \subset \Omega$. If $v \in H^m(\mathcal{O})$ with $m \geq 1$, then so are $P_G v$, $P_{G0} v$, $Q_G v$, and $Q_{G0} v$.

**Proof.** When $v \in H^m$ on $\mathcal{O}$, then $\text{div} \, v$ is $H^{m-1}$ and thus $\varphi_v$ is $H^{m+1}$ from standard elliptic regularity results for solutions of (3.6) as in Evans, [14] chapter 6 or elsewhere. Hence the gradient is $H^m$ so the results hold for $P_G v$ and $Q_G v$. The result for $P_{G0} v$, $Q_{G0} v$ is proved in the same way since the potentials now are solutions of (3.9).

Analogous analyses hold for projections onto spaces of Curls. First note that

$$\| v - \nabla^\perp \psi \|^2 - \| v \|^2 = \int_\Omega \left[ |\nabla \psi|^2 - 2 v \wedge \nabla \psi \right] \, d^2 x$$

where $\wedge$ denotes the 2d vector product. Consider the variational problems of minimizing the functional

$$\mathcal{C}_v(\varphi) := \int_\Omega \left[ |\nabla \varphi|^2 - 2 v \wedge \nabla \varphi \right] \, d^2 x \quad (3.10)$$

on $H^1(\Omega)$, $H^1_0(\Omega)$ respectively. Results about these variational principles may be summarized as follows.

**Theorem 3.4.** Assume $\Omega$ obeys (B1) and $v \in L^2(\Omega; \mathbb{R}^2)$. Then there is a unique $\psi_v \in H^1_m(\Omega)$ that minimizes $\mathcal{C}_v$ on $H^1_m(\Omega)$ and it satisfies

$$\int_\Omega (\nabla^\perp \psi - v) \cdot \nabla^\perp \chi \, d^2 x = 0 \quad \text{for all } \chi \in H^1(\Omega). \quad (3.11)$$

Then (i) $\| \nabla^\perp \psi_v \| \leq \| v \|$, (ii) if $v = \nabla^\perp \psi$ then there is a constant $c$ such that $\psi_v - \psi \equiv c$ on $\Omega$ and (iii) if $\mathcal{O}$ is an open subset of $\Omega$ with $\overline{\mathcal{O}} \subset \Omega$ and $v \in H^m(\mathcal{O})$, then $\nabla^\perp \psi_v \in H^m(\mathcal{O})$.

**Proof.** This proof just involves appropriate modifications to those of theorems 3.1 and 3.3.

The extremality condition (3.11) says that $\psi_v$ is a weak solution of the Neumann problem

$$- \Delta \psi = \text{curl} \, v \quad \text{on } \Omega \text{ and } D \psi = - v \cdot \tau \quad \text{on } \partial \Omega. \quad (3.12)$$

Define $P_C : L^2(\Omega; \mathbb{R}^2) \to L^2(\Omega; \mathbb{R}^2)$ by $P_C \, v := \nabla^\perp \psi_v$. The theorem implies that $\text{Curl}(\Omega)$ is a closed subspace of $L^2(\Omega; \mathbb{R}^2)$ from corollary 3.3 of citeAu1 and (b) shows that $P_C$ is the projection of $L^2(\Omega; \mathbb{R}^2)$ onto $\text{Curl}(\Omega)$. The orthogonal complement of this projection is $Q_C := I - P_C$ and (3.11) says that $Q_C$ and $P_C$ are orthogonal projections.

Similarly, the problem of minimizing the functional $\mathcal{C}_v$ on $H^1_0(\Omega)$ has solutions that satisfy the following. The proof is similar to that of theorems 3.2 and 3.3.
Theorem 3.5. Assume $\Omega$ obeys (B1) and $v \in L^2(\Omega; \mathbb{R}^2)$. Then there is a unique $\psi_{v0} \in H^1_0(\Omega)$ that minimizes $C_v$ on $H^1_0(\Omega)$ and it satisfies
\[
\int_{\Omega} (\nabla^\perp \psi - v) \cdot \nabla^\perp \chi \, d^2x = 0 \quad \text{for all } \chi \in H^1_0(\Omega). \tag{3.13}
\]
Thus (i) $\|\nabla^\perp \psi_{v0}\| \leq \|v\|$, (ii) if $v = \nabla^\perp \psi$ with $\psi \in H^1_0(\Omega)$ then $\psi_{v0} = \psi$ and (iii) if $\mathcal{O}$ is an open subset of $\Omega$ with $\mathcal{O} \subset \Omega$ and $v \in H^m(\mathcal{O})$, then $\nabla^\perp \psi_{v0} \in H^m(\mathcal{O})$.

Define $P_{C_0} : L^2(\Omega; \mathbb{R}^2) \to L^2(\Omega; \mathbb{R}^2)$ by $P_{C_0} v := \nabla^\perp \psi_{v0}$. This result implies that $\text{Curl}_0(\Omega)$ is a closed subspace of $L^2(\Omega; \mathbb{R}^2)$ from corollary 3.3 of [3] and (iii) shows that $P_{C_0}$ is the projection of $L^2(\Omega; \mathbb{R}^2)$ onto $\text{Curl}_0(\Omega)$. The extremality condition (3.13) says that $\psi_{v0} \in H^1_0(\Omega)$ is a weak solution of the Dirichlet problem for
\[
-\Delta \varphi = \text{curl } v \quad \text{on } \Omega. \tag{3.14}
\]

The complementary projection $Q_{C_0} := I - P_{C_0}$ is the projection onto the null space of the curl operator as described in the next section. These results may be combined to yield the following result that has been central in the study of 2-dimensional perfect fluids and much classical study of vector fields.

Theorem 3.6. Assume $\Omega$ satisfies (B1) and $v \in G(\Omega) \cap \text{Curl}(\Omega)$ then the potentials $\varphi_v, \psi_v$ are conjugate harmonic functions on $\Omega$ and $v$ is $C^\infty$ on $\Omega$.

Proof. The assumption is that there are functions $\varphi_v, \psi_v$ such that $v = \nabla \varphi_v = \nabla^\perp \psi_v$ on $\Omega$. These are the Cauchy-Riemann equations. Then (2.5) follows for each of $\varphi_v, \psi_v$ upon using Gauss Green and the commutativity of weak derivatives. The potentials are harmonic functions so they are $C^\infty$ and thus $v$ is.

\[ \square \]

4. Div, Curl and Orthogonality of Planar Vector Fields

A basic question for these fields is how the projections onto the spaces of gradients and Curls defined above are related to the vectorial operators div and curl on Sobolev-type spaces of vector fields? In this section such spaces are described and some orthogonality results obtained.

The curl of a vector field $v \in L^1_{\text{loc}}(\Omega; \mathbb{R}^2)$ is a function $\omega \in L^1_{\text{loc}}(\Omega)$, (or possibly a distribution) that satisfies
\[
\int_{\Omega} \nabla^\perp \psi \cdot v \, d^2x = \int_{\Omega} \omega \psi \, d^2x \quad \text{for all } \psi \in C^1_c(\Omega). \tag{4.1}
\]
In this case we write $\text{curl } v := \omega$. Similarly the divergence of $v \in L^1_{\text{loc}}(\Omega; \mathbb{R}^2)$ is defined to be the function $\rho \in L^1_{\text{loc}}(\Omega)$ provided $\rho$ satisfies
\[
\int_{\Omega} \nabla \varphi \cdot v \, d^2x = -\int_{\Omega} \rho \varphi \, d^2x \quad \text{for all } \varphi \in C^1_c(\Omega). \tag{4.2}
\]
When this holds we write $\text{div}\, \mathbf{v} := \rho$.

In the following only fields whose curl and div are locally integrable functions on $\Omega$ will be studied. Note these definitions do not require that the individual components of the derivative matrix are finite, or even defined. When the components of a planar vector field $\mathbf{v}$ are in $W^{1,1}(\Omega)$, then the derivative of the field is the matrix valued function $D\mathbf{v}(x) := (v_{j,k})$ whose entries are $L^1$ functions on $\Omega$. It is straightforward to verify that then

$$\text{div}\, \mathbf{v} = v_{1,1} + v_{2,2} \quad \text{and} \quad \text{curl}\, \mathbf{v} = v_{2,1} - v_{1,2}.$$  \hspace{1cm} (4.3)

A field $\mathbf{v} \in L^1_{\text{loc}}(\Omega; \mathbb{R}^2)$ is irrotational, or solenoidal, respectively provided

$$\int_{\Omega} \nabla^\perp \phi \cdot \mathbf{v} \, d^2x = 0 \quad \text{or} \quad \int_{\Omega} \nabla \phi \cdot \mathbf{v} \, d^2x = 0 \quad \text{for all} \ \phi \in C^1_c(\Omega). \hspace{1cm} (4.4)$$

A field $\mathbf{v} \in L^1_{\text{loc}}(\Omega; \mathbb{R}^2)$ is harmonic if it is both irrotational and solenoidal on $\Omega$. Let $H^0(\Omega, \mathbb{R}^2)$ be the closed subspace of all harmonic vector fields in $L^2(\Omega; \mathbb{R}^2)$. Observe that the space $H_{GC}(\Omega) := G(\Omega) \cap \text{Curl}(\Omega)$ of vector fields that are both gradients and curls of $H^1-$ functions is a space of harmonic fields.

Let $N(\text{curl}), N(\text{div})$ be the subspaces of irrotational, solenoidal vector fields in $L^2(\Omega; \mathbb{R}^2)$ respectively. Note that fields in $G(\Omega)$ are in $N(\text{curl})$ and fields in $\text{Curl}(\Omega)$ are in $N(\text{div})$ from the commutativity of weak differentiation. A first orthogonal decomposition result is the following CGH decomposition - which is independent of the differential topology of $\Omega$.

**Theorem 4.1.** Assume $\Omega$ satisfies (B1), then

(a) $L^2(\Omega; \mathbb{R}^2) = \text{Curl}_0(\Omega) \oplus N(\text{curl}) = G_0(\Omega) \oplus N(\text{div})$, and

(b) $L^2(\Omega; \mathbb{R}^2) = \text{Curl}_0(\Omega) \oplus G_0(\Omega) \oplus H^0(\Omega, \mathbb{R}^2)$. \hspace{1cm} (4.6)

**Proof.** (a) follows from the definition of $N(\text{curl})$ and $N(\text{div})$ since $C^1(\Omega)$ is dense in $H^1_0(\Omega)$. Then (b) follows as $\mathbf{v} \in H^0(\Omega, \mathbb{R}^2)$ iff it is orthogonal to both $\text{Curl}_0(\Omega)$ and $G_0(\Omega)$. \hspace{1cm} $\square$

This theorem implies that the projections $Q_{G0}, Q_{C0}$ of the preceding section are the projections onto the subspaces $N(\text{div}), N(\text{curl})$ respectively since $P_{G0}, P_{C0}$ are the projections onto their orthogonal complements.

The splitting of (4.6) will be called the *harmonic decomposition* and will be the primary representation used from now on in this paper. It is different to the usual Hodge-Weyl decompositions where the zero boundary conditions are imposed on only one of the potentials. The following common physical sign convention will be used.

$$\mathbf{v} = \nabla^\perp \psi - \nabla \varphi + \mathbf{h} \quad \text{with} \quad \psi, \varphi \in H^1_0(\Omega) \quad \text{and} \quad \mathbf{h} \in H^0(\Omega, \mathbb{R}^2). \hspace{1cm} (4.7)$$

Let $\mathcal{O} := I_1 \times I_2$ be an open rectangle in $\mathbb{R}^2$. Poincaré’s lemma provides explicit formulae for the potentials $\varphi_p, \psi_p$ of irrotational and solenoidal $C^1-$fields on $\Omega$. Namely given a point $P \in \mathcal{O}$ and a piecewise $C^1-$ curve $\Gamma_x$ joining $P$ to $x = (x_1, x_2) \in \mathcal{O}$, then

$$\varphi_p(x) := \int_{\Gamma_x} v_1 \, dx_1 + v_2 \, dx_2 \quad \text{and} \quad \psi_p(x) := \int_{\Gamma_x} v_1 \, dx_2 - v_2 \, dx_1 \hspace{1cm} (4.8)$$
are well-defined $C^1-$ functions on $\Omega$. When $v$ is irrotational, $\nabla \varphi_p = v$ and when $v$ is solenoidal then $\nabla \perp \psi_p = v$ on $\mathcal{O}$. See Dautray and Lions [12], Chapter IX, section 1, lemma 3 for a proof in the case where $\Omega$ is a block in $\mathbb{R}^3$. The proof there is easily modified for this 2 dimensional case.

The line integrals in (4.8) are not well-defined when the field $v$ is only $L^p$ on $\Omega$. Nevertheless, potentials may be proved to exist using a density argument. The following result is known for such fields with $p = 2$ and $\mathcal{O} \subset \mathbb{R}^3$; see Girault - Raviart [17] or Monk, [21] theorem 3.37.

**Theorem 4.2.** Assume $\Omega$ satisfies (B1) and $p \in [1, \infty)$. If $v \in L^p(\Omega; \mathbb{R}^2)$ is irrotational then there is a $\varphi \in W^{1,p}(\Omega)$ such that $v = \nabla \varphi$ on $\Omega$. If $v \in L^p(\Omega; \mathbb{R}^2)$ is solenoidal then there is a $\psi \in W^{1,p}(\Omega)$ such that $v = \nabla \perp \psi$ on $\Omega$.

**Proof.** First assume that $\Omega$ is convex, then Poincaré’s lemma implies this result holds when $v$ is $C^1$ on $\Omega$. To prove this holds for any $L^2$ field introduce a $C^1-$ mollifier $\Phi$ and consider fields on the open convex neighborhood $\Omega_1$ of points within distance 1 of $\Omega$. The sequence of $C^1-$ fields $v^{(m)}$ defined by convolution $v^{(m)} := \Phi_m \ast v$ converges to the zero extension of $v$ to $\Omega_1$ in $L^2(\Omega)$. Each of these $v^{(m)} = \nabla \perp \psi^{(m)}$ on $\Omega_1$ from Poincaré’s lemma. Normalize the $\psi^{(m)}$ to have mean value zero. Since these fields are a Cauchy sequence in $L^2(\Omega)$, the $\psi^{(m)}$ are Cauchy in $H_{m}^1(\Omega_1)$, so they converge to a limit $\tilde{\psi}$. Taking limits, $\nabla \perp \tilde{\psi}$ is the zero extension of $v$ to $\Omega_1$, and the result follows by considering the restriction to $\Omega$. A similar proof works for the second part, using the second part of the classical Poincaré lemma.

When $\Omega$ is not convex, then choose $\Omega_1$ in the above proof be the neighborhood of distance 1 from the convex hull of $\Omega$. Then the same arguments yield the statement of the theorem. \hfill \Box

Note that the preceding proof extends to 3-dimensional vector fields and regions, with the usual modifications, as the construction of Poincaré’s lemma is valid there - and the other ingredients are independent of dimension. A corollary is the following vector-valued version of Weyl’s lemma, that also extends to 3-d vector fields.

**Corollary 4.3.** Assume $\Omega$ satisfies (B1) and $p \in [1, \infty)$. If $v \in L^p(\Omega; \mathbb{R}^2)$ is a harmonic vector field then it is $C^\infty$ on $\Omega$.

**Proof.** Since $v$ is irrotational, there is a $\varphi \in W^{1,p}(\Omega)$ such that $v = \nabla \varphi$ on $\Omega$. As $v$ also is solenoidal, $\varphi$ is a weak solution of Laplace’s equation. Thus, from Weyl’s lemma, $\varphi$ is $C^\infty$ on $\Omega$, and thus $v$ is also. \hfill \Box

It should be noted that the above results do not require any topological conditions on the region $\Omega$. The theorem implies that $G(\Omega)^\perp \subset \text{Curl}(\Omega)$ and $\text{Curl}(\Omega)^\perp \subset G(\Omega)$ for any region $\Omega$ satisfying (B1) - and it is well-known that these are strict inclusions when $\Omega$ is not simply connected.
5. DIV-CURL AND LAPLACIAN BOUNDARY VALUE PROBLEMS

The div-curl boundary value problem is to find a vector field $v$ defined on a bounded region $\Omega \subset \mathbb{R}^2$ that satisfies

$$\text{div } v(x) = \rho(x) \quad \text{and} \quad \text{curl } v(x) = \omega(x) \quad \text{for } x \in \Omega$$  \hspace{1cm} (5.1)

subject to prescribed boundary conditions on $\partial \Omega$.

Generally either the normal component $v \cdot \nu$, or the tangential component $v \cdot \tau$, of the field at the boundary are prescribed in applications. When the normal component is prescribed everywhere on the boundary we have a normal Div-Curl boundary value problem that will be analyzed in the next section. Problems where the tangential component is prescribed everywhere are called the tangential Div-Curl boundary value problem and are studied in section 7. When normal components are prescribed on part of the boundary and tangential components on the complementary subset, it will be called a mixed Div-Curl boundary value problem.

To obtain bounds and regularity results for these problems, some results about Laplacian boundary value problems on regions obeying (B1) are required. Although this is a standard example of a second order elliptic boundary problem, the author has not found many of these results in the literature - so they are proved here for completeness. Stronger regularity results are well-known when the boundary $\partial \Omega$ is $C^k$ with $k \geq 1$ or solutions are sought in various Schauder spaces. For many physical and numerical problems, however, it is desirable to have results on Lipschitz regions that allow for “corners.”

The solutions of these boundary value problems will be found by introducing appropriate potentials $\varphi_0, \psi_0$ so that (b) of theorem 4.1 can be used. They will be solutions of Poisson’s equation with zero Dirichlet boundary data and are characterized by variational principles.

Given $\rho \in L^p(\Omega)$, consider the problem of minimizing the functional $\mathcal{D}$ defined by

$$\mathcal{D}(\varphi) := \int_{\Omega} \left[ |\nabla \varphi|^2 - 2\rho \varphi \right] d^2x$$  \hspace{1cm} (5.2)

on $H^1_0(\Omega)$. The essential results about this classical problem may be summarized as follows

**Theorem 5.1.** Suppose (B1) holds, $p \in (1, \infty]$ and $\rho \in L^p(\Omega)$. Then there is a unique minimizer $\varphi_0 := \mathcal{G}_D \rho$ of $\mathcal{D}$ on $H^1_0(\Omega)$ and it satisfies

$$\int_{\Omega} [\nabla \varphi \cdot \nabla \chi - \rho \chi] = 0 \quad \text{for all } \chi \in H^1_0(\Omega).$$  \hspace{1cm} (5.3)

The linear operator $\mathcal{G}_D$ is a 1-1 and compact map of $L^p(\Omega)$ into $H^1_0(\Omega)$.

**Proof.** When $\Omega$ is bounded then the imbedding $i : H^1_0(\Omega) \rightarrow L^q(\Omega)$ is continuous for all $q \in [1, \infty)$ from the Sobolev imbedding theorem. Thus the linear term in $\mathcal{D}$ is weakly continuous. The existence of a unique minimizer then holds as $\lambda_1 > 0$ in (3.8), so $\mathcal{D}$ is continuous, convex and coercive on $H^1_0(\Omega)$. 

\( D \) is \( G \)-differentiable on \( H_0^1(\Omega) \) and the extremality condition for a minimizer is (5.3). When the minimizer is denoted \( G_D\rho \) it follows that \( G_D \) is a linear mapping that satisfies
\[
\|\nabla \varphi\|_2 \leq C_{p'} \|\rho\|_p
\] (5.4)
with \( p' \) conjugate to \( p \). Here \( C_q \) is the imbedding constant for \( H_0^1(\Omega) \) into \( L^q(\Omega) \). Thus \( G_D \) is continuous. It is 1-1 from the maximum principle for harmonic functions.

To prove \( G_D \) is compact let \( \{\rho_m : m \geq 1\} \) be a weakly convergent sequence in \( L^p(\Omega) \) with \( p > 1 \). The imbedding of \( L^p(\Omega) \) into \( H^{-1}(\Omega) \) is compact for \( p \in (1, \infty) \) by duality to the Kondratchev theorem, so the sequence \( \{\rho_m\} \) is strongly convergent in \( H^{-1}(\Omega) \). A standard result is that \( G_D \) is a continuous linear map of \( H^{-1}(\Omega) \) to \( H_0^1(\Omega) \) so it is a compact linear mapping of \( L^p(\Omega) \) to \( H_0^1(\Omega) \) when \( p > 1 \) by composition. \( \Box \)

It is worth noting that this result enables proofs of many of the results about the approximation of solutions of (5.3) by eigenfunction expansions in terms of the eigenfunctions of the zero-Dirichlet Laplacian eigenproblem. It is well-known that orthonormal bases of \( H_0^1(\Omega) \) of such eigenfunctions can be constructed. Since \( G_D \) is compact, finite rank approximations using these eigenfunctions will converge to the solution \( G_D\rho \) for all \( \rho \) in these \( L^p(\Omega) \) and this solution has the standard spectral representation arising from the spectral theorem for compact self-adjoint maps on \( L^2(\Omega) \).

This result enables a generalization of the harmonic decomposition of \( L^2(\Omega) \) to fields as follows.

**Theorem 5.2.** (Harmonic Decomposition) Suppose (B1) holds, \( v \in L^1_{\text{loc}}(\Omega; \mathbb{R}^2), \) (or \( L^1(\Omega; \mathbb{R}^2) \)), with \( \text{div} \, v, \, \text{curl} \, v \in L^p(\Omega) \) for some \( p > 1 \). Then there are \( \varphi_0, \psi_0 \in H_0^1(\Omega) \) and a harmonic field \( h \in L^1_{\text{loc}}(\Omega; \mathbb{R}^2), \) (or \( L^1(\Omega; \mathbb{R}^2) \)) such that
\[
v = \nabla^\perp \psi - \nabla \varphi + h \quad \text{on} \quad \Omega.
\] (5.5)

**Proof.** Let \( \rho := \text{div} \, v, \omega := \text{curl} \, v \) and \( \varphi_0, \psi_0 \) are the associated solutions of (5.3). They exist and are in \( H_0^1(\Omega) \) from the theorem. Then \( h := v - \nabla^\perp \psi + \nabla \varphi \) is a harmonic field that is in \( L^1_{\text{loc}}(\Omega; \mathbb{R}^2), \) (or \( L^1(\Omega; \mathbb{R}^2) \)), respectively when \( v \) is. \( \Box \)

Note that the three components in this decomposition are \( L^2 \)-orthogonal. Also potentials \( \varphi_0, \psi_0 \) and the harmonic field \( h \) here will have better regularity when stronger conditions are imposed on the field, its divergence or curl using standard results from regularity theory.

For div-curl problems we seek potentials in the subspace \( H_0(\Delta, \Omega) \) be the subspace of \( H_0^1(\Omega) \) of all functions whose Laplacians are in \( L^2(\Omega) \). This is a real Hilbert space with respect to the inner product
\[
\langle \varphi, \chi \rangle_\Delta := \int_\Omega \left[ \Delta \varphi \, \Delta \chi + \nabla \varphi \cdot \nabla \chi \right] \, d^2x
\] (5.6)

When \( \Omega \) satisfies (B1) and \( \partial \Omega \) is \( C^1 \), then it is well known that \( H_0(\Delta, \Omega) = H_0^1(\Omega) \cap H^2(\Omega) \). See Evans chapter 8 for example. Under weaker conditions on \( \partial \Omega \) such as our (B1),
this need not hold - such issues have been studied by Grisvard [18], Jerison, Koenig and others.

Obviously the problem (5.3) has a unique solution \( \varphi = GD\rho \in H_0(\Delta, \Omega) \) if and only if \( \rho \in L^2(\Omega) \). In this case the following holds.

**Theorem 5.3.** Assume that (B1) holds and \( \lambda_1 \) is the constant in (3.8). Then the operator is \( GD \) is a homeomorphism of \( L^2(\Omega) \) and \( H_0(\Delta, \Omega) \) with \( \varphi_0 = GD\rho \) satisfying

\[
\|\varphi_0\|_2 \leq \frac{1}{\lambda_1} \|\rho\|_2, \quad \|\nabla \varphi_0\|_2 \leq \frac{1}{\sqrt{\lambda_1}} \|\rho\|_2 \quad \text{and} \quad \|D_\nu \varphi_0\|_{2,\partial\Omega} \leq C_0 \|\rho\|_2.
\]

(5.7)

where \( C_0 > 0 \) depends only on \( \Omega \).

**Proof.** The first two inequalities here follow from the spectral representation of \( GD \) in terms of the Dirichlet eigenfunctions of the Laplacian on \( \Omega \). The inequality for \( D_\nu \varphi_0 \) is theorem 3.2 in [8]. \( \square \)

Note also that if \( \mathcal{O} \) is an open subset of \( \Omega \) with \( \overline{\mathcal{O}} \subset \Omega \) and \( \rho \) is \( H^m \) on \( \mathcal{O} \), then the solution \( \varphi_0 \) of (5.3) will be of class \( H^{m+1} \) on \( \mathcal{O} \) from the usual interior regularity analysis. When \( \rho, \omega \in L^p(\Omega) \), let \( \varphi_0 = GD\rho, \psi_0 := GD\omega \) be solutions of (5.3) and consider

\[
h(x) := v - \nabla^\perp \psi_0(x) + \nabla \varphi_0(x).
\]

(5.8)

Substituting in (5.1), one sees that \( h \) will be a harmonic field with

\[
h \cdot \nu = v \cdot \nu + D_\nu \varphi_0 \quad \text{and} \quad h \cdot \tau = v \cdot \tau + D_\nu \psi_0 \quad \text{on} \ \partial\Omega.
\]

(5.9)

That is, the solvability of this div-curl system is decomposed into zero-Dirichlet boundary value problems involving the source terms and a boundary value problem for a harmonic field. So the following sections will concentrate on issues regarding different types of boundary value problems for harmonic vector fields.

Some related results about the Neumann problem for the Laplacian will also be needed later. The harmonic components of solutions of our problems involve potentials \( \chi \in H^1(\Omega) \) that satisfy

\[
\int_{\Omega} \nabla \chi \cdot \nabla \xi \, dx - \int_{\partial\Omega} \eta \xi \, ds = 0 \quad \text{for all} \ \xi \in H^1(\Omega).
\]

(5.10)

This is the weak form of Laplace’s equation subject to \( D_\nu \chi = \eta \) on \( \partial\Omega \).

To study the existence of solutions of this problem, consider the problem of minimizing the functional \( \mathcal{N} : H^1(\Omega) \to \mathbb{R} \) defined by

\[
\mathcal{N}(\chi) := \int_{\Omega} |\nabla \chi|^2 \, dx - 2 \int_{\partial\Omega} \eta \chi \, ds
\]

(5.11)

with \( \eta \in L^2(\partial\Omega, ds) \). A necessary condition for the existence of a solution of this problem is that

\[
\int_{\partial\Omega} \eta \, ds = 0.
\]

(5.12)
To prove the existence of a solution of this problem we need the fact that there is a \( \delta_1 > 0 \) such that
\[
\int_\Omega |\nabla \varphi|^2 d^2x \geq \delta_1 \int_{\partial \Omega} \varphi^2 ds \quad \text{for all} \quad \varphi \in H^1(\Omega) \text{ that satisfy (5.12).} \tag{5.13}
\]
This \( \delta_1 \) is the first nonzero harmonic Steklov eigenvalue for the region \( \Omega \).

When (B1) and (5.12) hold then a standard variational argument says that there is a unique minimizer \( \chi = B \eta \) of \( N \) on \( H^1_{m}(\Omega) \) that satisfies (5.10).

This solution operator \( B \) may be regarded as an integral operator that maps functions from \( L^2(\partial \Omega, ds) \) to \( H(\Omega) \subset H^1(\Omega) \). In particular it has a nice expression in terms of the harmonic Steklov eigenfunctions of \( \Omega \). See Auchmuty [4] and [6] for a discussion of the Steklov eigenproblem for the Laplacian on bounded regions and [8] for further results about \( H_0(\Delta, \Omega) \).

A function \( s_j \in H(\Omega) \) is a Steklov eigenfunction for the Laplacian on \( \Omega \) provided it is a nontrivial solution of the system
\[
\int_\Omega \nabla s_j \cdot \nabla \xi d^2x = \delta \int_{\partial \Omega} s_j \xi ds \quad \text{for all} \quad \xi \in H^1(\Omega). \tag{5.14}
\]
Here \( \delta \in \mathbb{R} \) is the associated Steklov eigenvalue. Let \( \Lambda := \{ \delta_j; j \geq 0 \} \) be the set of Steklov eigenvalues repeated according to multiplicity and with \( \delta_j \) an increasing sequence. The first eigenvalue is \( \delta_0 = 0 \) and the corresponding eigenfunctions are constants on \( \overline{\Omega} \). It is a simple eigenvalue and the next eigenvalue is \( \delta_1 > 0 \) of (5.13). Normalize an associated set of Steklov eigenfunctions \( S := \{ s_j : j \geq 0 \} \) to be \( L^2 \)-orthonormal on \( \partial \Omega \). Then
\[
\int_\Omega \nabla s_j \cdot \nabla s_k d^2x = \delta_j \quad \text{when} \quad j = k \quad \text{and} \quad 0 \quad \text{otherwise.} \tag{5.15}
\]

Theorem 4.1 of [5] says that this sequence can be chosen to be an orthonormal basis of \( L^2(\partial \Omega, ds) \) with the usual inner product. If \( \eta \in L^2(\partial \Omega, ds) \) satisfies the compatibility condition (5.12) then it has the representation
\[
\eta(z) = \sum_{j=1}^{\infty} \hat{\eta}_j s_j(z) \quad \text{on} \quad \partial \Omega \quad \text{with} \quad \hat{\eta}_j = \int_{\partial \Omega} \eta s_j ds. \tag{5.16}
\]

For \( M \geq 1 \), consider the boundary integral operators \( B_M : L^2(\partial \Omega, ds) \to H(\Omega) \) defined by
\[
B_M \eta(x) := \int_{\partial \Omega} B_M(x, z) \eta(z) ds \quad \text{with} \quad B_M(x, y) := \sum_{j=1}^{M} \delta_j^{-1} s_j(x) s_j(z). \tag{5.17}
\]
These are finite rank operators and the following SVD type representation theorem holds for the operator \( B \).

**Theorem 5.4.** Assume (B1) holds, \( \Lambda \) is the set of harmonic Steklov eigenvalues on \( \Omega \) repeated according to multiplicity and \( S \) is a \( \partial \)-orthogonal set of harmonic Steklov eigenfunctions and a orthonormal basis of \( L^2(\partial \Omega, ds) \). When \( \eta \in L^2(\partial \Omega, ds) \), the unique solution
\[ \mathcal{B} \eta \text{ of (5.10) in } H^1_0(\Omega) \text{ is} \]
\[ \chi(x) = \mathcal{B} \eta(x) = \lim_{M \to \infty} B_M \eta(x). \quad (5.18) \]

\( \mathcal{B} \) is a continuous linear transformation of \( L^2(\partial \Omega, ds) \) to \( \mathcal{H}(\Omega) \) with \( \| \nabla \chi \|_2 \leq \delta_1^{-1} \| \eta \|_{2, \partial \Omega} \) and \( \chi \) is \( C^\infty \) on \( \Omega \).

**Proof.** The first part of this theorem is proved in [5] where it is shown that \( \mathcal{S} \) is an orthonormal basis of \( L^2(\partial \Omega, ds) \). Thus (5.16) holds. When \( \chi \) is a solution of equation (5.10), then \( \chi(x) = \sum_{j=1}^{\infty} \hat{\chi}_j s_j(x) \) on \( \Omega \) as \( \mathcal{S} \) is a maximal orthogonal set in \( \mathcal{H}(\Omega) \). Take \( \xi = s_j \) in (5.10) then the coefficients \( \hat{\chi}_j = \hat{\eta}_j / \delta_j \) for \( j \geq 1 \), and (5.18) holds in the \( \partial \)–norm of \( \mathcal{H}(\Omega) \). The function \( \chi \) is \( C^\infty \) as it is harmonic and the bound on \( \| \nabla \chi \|_2 \) follows from the orthogonality of \( \mathcal{S} \).

It is worth noting that the solution \( \chi \) of this problem is \( C^\infty \) on \( \Omega \). It will be \( H^1 \) when the Neumann data is in \( H^{-1/2}(\partial \Omega) \) and more generally will be in the space \( \mathcal{H}^s(\Omega) \) defined as in [6] when \( \eta \in H^{s-3/2}(\partial \Omega) \).

### 6. The normal Div-curl Boundary Value Problem

The normal div-curl boundary value problem is to find a field \( \mathbf{v} \in L^2(\Omega; \mathbb{R}^2) \) that solves (5.1) subject to

\[ \mathbf{v}(z) \cdot \nu(z) = \eta_\nu(z) \quad \text{on } \partial \Omega. \quad (6.1) \]

with \( \eta_\nu \in L^2(\partial \Omega, ds) \). From the divergence theorem, a necessary condition for (5.1) - (6.1) to have a solution is the compatibility condition

\[ \int_{\Omega} \rho \, d^2x = \int_{\partial \Omega} \eta(\nu) \, ds. \quad (6.2) \]

When the solution has the form (4.7), then the potentials are solutions of (5.3) with \( \psi_0 = G_D \omega, \phi_0 = G_D \rho \). Since \( \psi_0 \equiv 0 \) on \( \partial \Omega \), the harmonic component satisfies

\[ \mathbf{h}(z) \cdot \nu(z) = \eta_\nu(z) + D_\nu \phi_0(z) \quad \text{for } z \in \partial \Omega. \quad (6.3) \]

Consider the problem of finding a gradient field that solves this problem. If \( h = \nabla \chi \) is a solution of this problem, then \( \chi \) is a harmonic function that satisfies (5.10) with \( \eta(z) \) given by the right hand side of (6.3). From theorem 5.3, this problem has a solution given by (5.18) and the following result holds.

**Theorem 6.1.** Assume (B1) holds, \( \rho, \omega \in L^2(\Omega), \eta_\nu \in L^2(\partial \Omega, ds) \) and (6.2) holds. Let \( \phi_0 = G_D \rho, \psi_0 = G_D \omega \). Then there is a unique \( \chi \in H^1_0(\Omega) \) such that \( \mathbf{h}(x) = \nabla \chi(x) \) is a harmonic field satisfying (5.10) with \( \eta \) from (6.3). The field \( \mathbf{v} = \nabla^\perp \psi_0 - \nabla \phi_0 + \nabla \chi \) is a solution of (5.1) - (6.1) with

\[ \| \mathbf{v} \|_2 \leq \frac{1}{\sqrt{\lambda_1}} \left[ \| \rho \|_2 + \| \omega \|_2 \right] + \frac{1}{\sqrt{\delta_1}} \left[ \| \eta_\nu \|_{2, \partial \Omega} + C_0 \| \rho \|_2 \right]. \quad (6.4) \]
Proof. Given $\rho, \omega \in L^2(\Omega)$, theorem 5.3 yields the first two terms in the inequality (6.4). Note that (6.2) implies the compatibility condition (5.12), so there is a unique $\chi \in H^1_0(\Omega)$ that is harmonic on $\Omega$ and satisfies the boundary condition (6.3) from Theorem 5.4. The three fields in this representation of $v$ are $L^2$-orthogonal so it only remains to bound $\|\nabla \chi\|_2$. This bound now follows from the last parts of theorems 5.3 and 5.4. □

It is worth noting that the constants in this inequality are best possible in that there are choices of $\rho, \omega$ and $\eta$ for which the right hand side equals the 2-norm of a solution of the problem. If $\rho, \omega \in L^p(\Omega)$ for some $p > 1$, then (5.2) implies that

$$\|v - \nabla \chi\|_2 \leq C_p \left[ \|\rho\|_p + \|\omega\|_p \right].$$

(6.5)

Corollary 6.2. Suppose (B1) holds, curl $v$, div $v \in L^2(\Omega)$, $v \cdot \nu \in L^2(\partial \Omega, ds)$, then $v \in H^1_0(\Omega) \cap H(\text{curl}, \Omega)$ and there is a $C > 0$ such that

$$\|v\|_2^2 \leq C \left[ \|\text{curl} \ v\|_2^2 + \|\text{div} \ v\|_2^2 + \|v \cdot \nu\|_{2,\partial \Omega}^2 \right].$$

(6.5)

Proof. The inequality (6.4) implies (6.5) for an appropriate choice of $C$. Since (6.5) holds, $v$ is in both $H^1_0(\partial \Omega) \cap H(\text{curl}, \Omega)$ provided the region satisfies (B1). □

Thus the solutions of this problem may be written as series expansions involving the Dirichlet and Steklov eigenfunctions of the Laplacian as $G_D$ and $B$ from (5.15) have representations with respect to these bases of $H^1_0(\Omega)$ and $H(\Omega)$ respectively.

When $\Omega$ is not simply connected this boundary value problem has further solutions. These was studied in [1] where the well-posed problem was shown to require the circulations around each handle be further specified for uniqueness. The above solution is the least energy (2-norm) solution of the problem.

7. The tangential Div-curl Boundary Value Problem

The tangential div-curl boundary value problem is to find a vector field $v \in L^2(\Omega; \mathbb{R}^2)$ that satisfies (5.1) subject to

$$v(z) \cdot \tau(z) = \eta_r(z) \quad \text{on} \quad \partial \Omega.$$
with \( \eta_r \in L^2(\partial \Omega, ds) \). A necessary condition, from the divergence theorem, for this problem to have a solution is that

\[
\int_{\Omega} \omega \, d^2x = \int_{\partial \Omega} \eta_r(z) \, ds. \tag{7.2}
\]

When the solution has the form (4.7), then the potentials \( \phi_0, \psi_0 \) are solutions of (5.3) given by

\[
\psi_0 = G_D \omega, \phi_0 = G_D \rho. \tag{7.2}
\]

Then the fact that \( \phi \equiv 0 \) on \( \partial \Omega \) implies that the harmonic component satisfies

\[
- h(z) \cdot \tau(z) = \eta_r(z) + D_\nu \psi_0(z) \quad \text{on} \quad \partial \Omega. \tag{7.3}
\]

Suppose that this harmonic field is given by \( h = -\nabla^\perp \chi \). Then \( \chi \) is a harmonic function that satisfies (5.10) with \( \eta(z) \) given by the right hand side of (7.3). From theorem 5.2 this problem has a solution of the form (5.18) and the following result holds.

**Theorem 7.1.** Assume (B1) holds, \( \rho, \omega \in L^2(\Omega) \), \( \eta_r \in L^2(\partial \Omega, ds) \) and (7.2) holds. Let \( \varphi_0 = G_D \rho, \psi_0 = G_D \omega \). Then there is a unique \( \chi \in H^1_m(\Omega) \) such that \( h(x) = -\nabla^\perp \chi(x) \) is a harmonic field satisfying (7.3) on \( \partial \Omega \). The field \( v = \nabla^\perp \psi_0 - \nabla \varphi_0 - \nabla^\perp \chi \) is a solution of (5.1) - (7.1) with

\[
\|v\|_2 \leq \frac{1}{\sqrt{\lambda_1}} \left[ \|\rho\|_2 + \|\omega\|_2 \right] + \frac{1}{\sqrt{\delta_1}} \left[ \|\eta_r\|_{2, \partial \Omega} + C_0 \|\omega\|_2 \right]. \tag{7.4}
\]

**Proof.** This proof is essentially the same as that of theorem 6.1. The compatibility condition (7.2) implies that compatibility condition for the solvability of (5.10) with \( \eta \) given by the right hand side of (7.3) holds. The estimates now follow as in theorem 6.1. \( \square \)

In a similar manner to corollary 6.2 of the last section one has

**Corollary 7.2.** Suppose (B1) holds and curl \( v \), div \( v \) \( \in L^2(\Omega) \), \( v \cdot \tau \in L^2(\partial \Omega, ds) \), then \( v \in H(\text{div}, \Omega) \cap H_\rho(\text{curl}, \Omega) \) and there is a \( C > 0 \) such that

\[
\|v\|_2 \leq C \left[ \|\text{curl} \ v\|_2 + \|\text{div} \ v\|_2 + \|v \cdot \tau\|_{2, \partial \Omega} \right]. \tag{7.5}
\]

When the region \( \Omega \) has holes, (that is its boundary has more than one connected component, then the solution of this boundary value problem is non-unique. There are non-zero harmonic vector fields associated with potential differences between different components of the boundary. This was studied in [1] where the well-posed problem was described and the solution described in the above theorem - is the least energy (2-norm) solution of the problem.

8. Mixed boundary conditions

In electromagnetic field theory, problems where given flux conditions are prescribed on part of the boundary and tangential boundary data is prescribed on the complementary part need to be solved. The well-posedness and uniqueness of solutions of these problems was studied in sections 12 - 15 of Alexander and Auchmuty [1]. Here our primary interest is
in obtaining 2-norm bounds on solutions in terms of the data. The constant in the relevant estimate will be the value of a natural optimization problem that is related to an eigenvalue in the case where the data is \( L^2 \).

The analysis of such problem differs considerably from that for the normal and tangential boundary value problems. First no compatibility conditions on the data are required for \( L^2 \)-solvability. In addition the two potentials are each found directly by solving similar variational problems on appropriate closed subspaces of \( H^1(\Omega) \) that is determined by the topology of the boundary data. The aim here is to obtain bounds on the solutions of these problems, in particular of their energy, in terms of norms of the data.

The mixed div-curl boundary value problem is to find vector fields \( v \in L^2(\Omega; \mathbb{R}^2) \) that satisfy

\[
\begin{align*}
\mathbf{v}(z) \cdot \tau(z) &= \eta_{\tau}(z) \quad \text{on } \Gamma_{\tau} \quad \text{and} \\
\mathbf{v}(z) \cdot \nu(z) &= \eta_{\nu}(z) \quad \text{on } \Gamma_{\nu}.
\end{align*}
\]

Here \( \Gamma_{\tau}, \Gamma_{\nu} \) are nonempty open subsets of \( \partial \Omega \) whose union is dense in \( \partial \Omega \). Our analysis will use the following requirement on these sets.

**Condition B2.** \( \Gamma \) is an nonempty open subset of \( \partial \Omega \) with a finite number of disjoint components \( \{ \gamma_1, \ldots, \gamma_L \} \) and there is a finite distance \( d_0 \) such that \( d(\gamma_j, \gamma_k) \geq d_0 \) when \( j \neq k \).

When \( \Gamma \) satisfies (B2), define \( H^1_{\Gamma_0}(\Omega) \) to be the subspace of \( H^1(\Omega) \) of functions whose traces are zero on the set \( \Gamma \subset \partial \Omega \). This is a closed subspace of \( H^1(\Omega) \) from lemma 12.1 of [1]. When (B1) and (B2) hold then \( H^1_{\Gamma_0}(\Omega) \) is a real Hilbert space with the \( \partial \)-inner product

\[
\langle \varphi, \psi \rangle_{\partial, \Gamma} := \int_\Omega \nabla \varphi \cdot \nabla \psi \, d^2x + \int_{\tilde{\Gamma}} \varphi \psi \, ds
\]

where \( \tilde{\Gamma} \) is the complement of \( \Gamma \) in \( \partial \Omega \).

**Lemma 8.1.** Suppose \( \Omega, \partial \Omega, \Gamma \) satisfy (B1) and (B2). Then

(i) \( \langle \varphi, \psi \rangle_1 := \langle \nabla \varphi, \nabla \psi \rangle \) is an equivalent inner product on \( H^1_{\Gamma_0}(\Omega) \) to the \( \partial \)-inner product.

(ii) For \( q \in (1, \infty) \), there is an \( M_q(\Gamma) \) such that

\[
\| \varphi \|_{q, \partial \Omega}^q + \| \varphi \|_{q, \partial \Omega}^q \leq M_q(\Gamma) \| \nabla \varphi \|_2^q \quad \text{for all } \varphi \in H^1_{\Gamma_0}(\Omega). \tag{8.3}
\]

(iii) if \( \Gamma_1 \supset \Gamma \), then \( M_q(\Gamma_1) \leq M_q(\Gamma) \).

**Proof.** (i) Let \( \lambda_1(\Gamma) \) be the least eigenvalue of the Laplacian on \( H^1_{\Gamma_0}(\Omega) \) so that

\[
\int_\Omega |\nabla \varphi|^2 \, d^2x \geq \lambda_1(\Gamma) \int_\Omega \varphi^2 \, d^2x \quad \text{for all } \varphi \in H^1_{\Gamma_0}(\Omega). \tag{8.4}
\]

This exists and is positive when (B2) holds as \( \sigma(\Gamma) > 0 \); see proposition 13.2 in [1] for a proof. Since the \( \partial \)-norm and the standard norm on \( H^1(\Omega) \) are equivalent, there is a \( C > 0 \) such that

\[
\| \varphi \|^2_{\partial} \leq C \| \varphi \|^2_{L^2} \leq C (1 + \lambda_1(\Gamma)^{-1}) \| \nabla \varphi \|_2^2
\]

Thus the norm from (i) is equivalent to the \( \partial \)-norm.
Consider the functional $G_q(\varphi) := \|\varphi\|_q^q + \|\varphi\|_{q,\partial\Omega}^q$ on $H^1_{10}(\Omega)$. This functional is convex and weakly continuous as the imbedding of $H^1(\Omega)$ into $L^q(\Omega)$ and $L^q(\partial\Omega, ds)$ are compact for any $q \geq 1$ when (B1) holds. Let $B_1$ be the unit ball in $H^1_{10}(\Omega)$ with respect to the inner product of (i). Define $M_q(\Gamma) := \sup_{\varphi \in B_1} G_q(\varphi)$. This sup is finite and (8.3) follows upon scaling.

(iii) When $\Gamma_1 \supset \Gamma$, then $H^1_{10}(\Omega) \subset H^1_{00}(\Omega)$, so the associated unit ball is smaller and thus $M_q(\Gamma_1) \leq M_q(\Gamma)$.

It appears that the value of $M_q(\Gamma)$ increases to $\infty$ as $\sigma(\Gamma)$ decreases to zero. It would be of interest to estimate or quantify this dependence. When $q = 2$ the constant $M_2(\Gamma)$ is related to the least eigenvalue of an eigenvalue problem for the Laplacian where the eigenvalue appears in both the equation and the boundary condition. Suppose that $\lambda_1(\Omega, \Gamma)$ is the least eigenvalue of

$$-\Delta u = \lambda u \quad \text{on} \quad \Omega \quad \text{with} \quad u = 0 \quad \text{on} \quad \Gamma, \quad D_\nu u = \lambda u \quad \text{on} \quad \tilde{\Gamma}, \quad (8.5)$$

then $M_2(\Gamma) = \lambda_1(\Omega, \Gamma)^{-1}$.

The conditions required here are that $\Gamma_{\tau}, \Gamma_\nu$ are proper subsets of $\partial\Omega$ satisfying the following.

**Condition B3.** $\Gamma_\nu$ and $\Gamma_\tau$ are disjoint, satisfy (B2) and have union dense in $\partial\Omega$.

Let $G_{\Gamma_\nu}(\Omega), \text{Curl}_{\Gamma_\nu}(\Omega)$ be the spaces of gradients of functions in $H^1_{10}(\Omega)$ and curls of functions in $H^1_{00}(\Omega)$ respectively. These spaces are $L^2 -$ orthogonal as fields in $L^2(\Omega; \mathbb{R}^2)$. The vector field $v := \nabla \perp \psi - \nabla \varphi$ will satisfy the boundary condition (8.1) in a weak sense provided $\varphi \in H^1_{10}(\Omega)$ and $D_\nu \varphi + \eta_\nu = 0$ on $\Gamma_\nu$ and $\psi \in H^1_{10}(\Omega)$ with $D_\nu \psi + \eta_\tau = 0$ on $\Gamma_\tau$.

As described in [1] there are variational principles for the potentials in this representation and the field $v$ will be a solution of (5.1) - (8.1) provided $\varphi \in H^1_{10}(\Omega)$ is a solution of

$$\int_\Omega [\nabla \varphi \cdot \nabla x - \rho \chi] \, dx + \int_{\Gamma_\nu} \eta_\nu \chi \, ds = 0 \quad \text{for all} \quad \chi \in H^1_{10}(\Omega). \quad (8.6)$$

Similarly $\psi \in H^1_{10}(\Omega)$ is a solution of

$$\int_\Omega [\nabla \psi \cdot \nabla x - \omega \chi] \, dx + \int_{\Gamma_\tau} \eta_\tau \chi \, ds = 0 \quad \text{for all} \quad \chi \in H^1_{10}(\Omega). \quad (8.7)$$

Note that these equations are of the same type; they differ only in that $\Gamma_{\tau}, \Gamma_\nu$ are interchanged from one to the other. They can be written as a problem of finding $\varphi \in H^1_{10}(\Omega)$ satisfying

$$\int_\Omega \nabla \varphi \cdot \nabla \chi \, dx = \mathcal{F}(\chi) \quad \text{for all} \quad \chi \in H^1_{10}(\Omega). \quad (8.8)$$

Here $\mathcal{F}(\chi)$ is the linear functional defined by $\mathcal{F}(\chi) = \int_\Omega \rho \chi \, dx - \int_{\partial\Omega} \eta_\chi \, ds$. For notational convenience the functions $\eta_\nu, \eta_\tau$ are extended to all of $\partial\Omega$ by zero.

The general result about this problem may be described as follows.
Theorem 8.2. Assume that $\Omega, \Gamma$ satisfy (B1)-(B2) with $\rho \in L^q(\Omega), \eta \in L^q(\partial \Omega, ds), q > 1$. Then there is a unique solution $\tilde{\varphi} \in H^1_{\Gamma_0}(\Omega)$ of (8.8) and it satisfies
\[
\| \nabla \tilde{\varphi} \|^2_2 \leq M_q'(\Gamma)^q \left[ \| \rho \|^q + \| \eta \|^q_{q, \Gamma} \right]^{q-1}
\] (8.9)

Proof. When $\tilde{\varphi}$ is a solution of (8.8) and $|F(\chi)| \leq C \| \nabla \chi \|_2$ for all $\chi \in H^1_{\Gamma_0}(\Omega)$, then $\| \nabla \tilde{\varphi} \|_2 \leq C$. So the result just requires an appropriate estimate of $F(\chi)$. Two applications of Holder’s inequality to the definition of $F$ yield that
\[
|F(\chi)| \leq \left[ \| \rho \|^q + \| \eta \|^q_{q, \partial \Omega} \right]^{1/q} \left[ \| \chi \|^q_{q'} + \| \chi \|^q_{q', \partial \Omega} \right] \left[ \| \rho \|^q + \| \eta \|^q_{q, \partial \Omega} \right]^{1/q} 
\]
for all $\chi \in H^1_{\Gamma_0}(\Omega)$. Then (8.2) yields
\[
|F(\chi)| \leq \left[ \| \rho \|^q + \| \eta \|^q_{q, \partial \Omega} \right]^{1/q} M_q'(\Gamma)^{1/q'} \| \nabla \chi \|_2.
\]
This inequality yields (8.9).

Corollary 8.3. Assume that $\Omega, \Gamma_\nu, \Gamma_\tau$ satisfy (B1)-(B3) with $\rho, \omega \in L^q(\Omega), \eta_\nu, \eta_\tau \in L^q(\partial \Omega, ds)$ and $q > 1$. Then there is a solution $\tilde{v} = \nabla \tilde{\psi} - \nabla \tilde{\varphi}$ of (5.1) - (8.1) with
\[
\| \tilde{v} \|^2_2 \leq C_q(\Gamma_\tau) \left[ \| \rho \|^q + \| \eta_\nu \|^q_{q, \Gamma_\nu} \right]^{2/q} + C_q(\Gamma_\nu) \left[ \| \omega \|^2 + \| \eta_\tau \|^2_{q, \Gamma_\tau} \right]^{2/q} \]
(8.10)

If $v$ is any solution of this mixed div-curl system, then $\| v \|_2 \geq \| \tilde{v} \|_2$.

Proof. Let $\tilde{\varphi}, \tilde{\psi}$ be the solutions of (8.6) - (8.7) respectively. Then their orthogonality implies that $\| \tilde{v} \|^2_2 = \| \nabla \tilde{\varphi} \|^2_2 + \| \nabla \tilde{\psi} \|^2_2$. Theorem 8.2 implies that there is a constant such that
\[
\| \nabla \tilde{\varphi} \|^2_2 \leq C_q(\Gamma_\tau) \left[ \| \rho \|^q + \| \eta_\nu \|^q_{q, \Gamma_\nu} \right]^{2/q}.
\]
Similarly the other mixed boundary value problem has solution $\tilde{\psi}$ with
\[
\| \nabla \tilde{\psi} \|^2_2 \leq C_q(\Gamma_\nu) \left[ \| \omega \|^q + \| \eta_\tau \|^q_{q, \Gamma_\tau} \right]^{2/q}.
\]
Adding these two expressions leads to the inequality of (8.10).

In general there is an affine subspace of solutions of (5.1) - (8.1) as described in section 14 of [1]. To find a well-posed problem certain linear functionals of the solutions must be further specified and the energy of the solution depends on these extra imposed conditions.

Since these are problems of interest to researchers in a variety of different areas, some relevant references to the literature may have been omitted from the following bibliography. The author would appreciated being informed about further papers that treat these topics analytically.
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