PLAYING COOPERATIVELY WITH POSSIBLY TREACHEROUS PARTNER

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Abstract. We investigate an alternative concept of Nash equilibrium, m-equilibrium, which slightly resembles Harsanyi-Selten risk dominant equilibrium although it is a different notion. M-equilibria provide nontrivial solutions of normal form games as shown by comparison of the Prisoner's Dilemma with the Traveler's Dilemma. They are also resistant on the deep iterated elimination of dominated strategies.

1. Introduction

The games of interest here are two person general sum games in normal form. Concerning notation and definitions the reader is asked to consult the next Section.

First one has to provide additional information about the game. The main assumption, usually made implicitly, is that the players can communicate with each other. This remedies the coordination problem quoted below (see [HaMa] for complexity issues).

Example 1. [Coordination] Let $S_1 = S_2 = \{1, 2, 3\}$,

$$G = \begin{bmatrix}
[2, 2] & [0, 0] & [0, 0] \\
[0, 0] & [1, 1] & [0, 0] \\
[0, 0] & [0, 0] & [2, 2]
\end{bmatrix}. $$

Among three equilibria $(1, 1)$, $(2, 2)$ and $(3, 3)$ only two are pleasant (as Pareto dominant), namely $(1, 1)$ and $(3, 3)$. If players cannot communicate, then they have to use randomization (e.g., coin flipping). The Bernoulli scheme would let them synchronize choices with small probability of failure (cf. Theorem 11.3 in [AlGa, chap.11.4]).

Thus we see that some amount of communication (say pre-play/cheap talk) and a sort of cooperation cannot be dismissed even in the case of such competitive/“selfish” notion like the Nash equilibrium (e.g., [An2, MiMo, Ro]). The classic stag hunt game exploits
another issue of miscoordination – the Wald criterion of worst possible scenario. In the vein of a stag hunt’s strategic security consider

**Example 2.** Let $S_1 = S_2 = \{1, 2\}$,

\[
G = \begin{bmatrix}
[4,4] & [1,4] \\
[4,1] & [3,3]
\end{bmatrix}.
\]

The pairs $(1, 1)$ and $(2, 2)$ are Nash equilibria with $(1, 1)$ being Pareto dominant. Nevertheless one cannot guarantee that previously agreed among players equilibrium $(1, 1)$ would be realized in practice. If the player is confident in fair play of his partner, he might switch strategy without any loss of income. The pair of strategies $(2, 2)$ is threat-safe although yields smaller payoffs than $(1, 1)$.

Players make their final decisions independently of others. Therefore communication provides only weak cooperation (\cite{HiKo}). No one can force fair play, even if it is profitable for all (free rider’s problem).

Largely discussed traveler’s dilemma (seen sometimes as the extension of prisoner’s dilemma) underlines anomalous behavior in widely accepted procedure called an iterative elimination of dominated strategies (\cite{Ba, BaBeSt, CpGo, Gi, HiPs}). Unlike the original formulation our assumes communication between players.

**Example 3.** [Traveler’s dilemma] Let $S_1 = S_2 = \{2, 3, \ldots, 100\}$, $P_1(x, y) = P_2(y, x) = \min(x, y) + 2 \cdot \text{sign}(y - x)$ for $x \in S_1$, $y \in S_2$, $G = (S_1, S_2; P_1, P_2)$. Then $(2, 2)$ is the only Nash equilibrium of $G$. It arises through the elimination of dominated strategies, although most strategy pairs Pareto dominate it.

Observe that the players could choose a pair of strategies which yields much higher payoffs than $(2, 2)$. Moreover, the player can still play very profitably after his partner betrayed and switched strategy to get higher payoffs. If more than 4% partners play “moderately” (at least 54), then we can expect higher gain from playing “dummy” 100 than from playing “wise” 2. If more than 10% partners play “high” (at least 90), then playing 100 we can expect over 400% income of that which we could earn playing 2.

Our point here is that one should calculate secure gains incorporating possible threats from his partner. This allows sometimes for much higher payoffs than those arising in the Nash equilibrium. That was the main motivation for introducing $m$-equilibria as we do in Section \ref{sec:m-equilibria}.

We silently assume that the games under consideration are not repeated and one-shot; see also the discussion around mixed equilibria in Examples \ref{ex:repeated-games} and \ref{ex:one-shot-games}. We understand the payoff to be NTU (nontransferable utility); that some “transfers” are still possible ensures us Example \ref{ex:transfer} in Section \ref{sec:transfer} and informal discussion of fair choice among equilibria in Section \ref{sec:fair-choice}.

For standard notions and theorems of game theory we refer to the textbook \cite{Wa} (comp. \cite{Da, Au}). Throughout the paper the language of multivalued (or set-valued) analysis shall be utilized in several places (consult \cite{HuPa, Be, Au}).
2. Notation and definitions

Let $\Gamma = (\Sigma_1, \Sigma_2; P_1, P_2 : \Sigma_1 \times \Sigma_2 \rightarrow \mathbb{R})$ be a two person game (in normal form). $\Sigma_i$ is the set of strategies and $P_i$ is the payoff function of the $i$-th player.

An accent will be put further on the case of a finite game $G = (S_1, S_2; W_1, W_2 : S_1 \times S_2 \rightarrow \mathbb{R})$, i.e., the game with the finite strategy sets $S_1, S_2$, and its mixed extension $\Delta(G) = (\Delta(S_1), \Delta(S_2); EW_1, EW_2 : \Delta(S_1) \times \Delta(S_2) \rightarrow \mathbb{R})$. $\Delta(S)$ stands for the standard simplex of probabilistic measures (mixed strategies) spanned on the finite set $S$ of (pure) strategies; $S \subset \Delta(S)$ due to the identification via Dirac measures: $S \ni x \mapsto \delta_x \in \Delta(S)$. The expected payoffs are given by

$$EW_i(\rho_1, \rho_2) = \sum_{(x,y) \in S_1 \times S_2} \rho_1(x) \cdot W_i(x, y) \cdot \rho_2(y)$$

for $\rho_i \in \Delta(S_i)$, $i = 1, 2$. Finite games shall appear in the examples as bimatrixes $[W_1(x, y), W_2(x, y)]_{(x,y) \in S_1 \times S_2}$, $S_1 \ni x$ - the number of the row, $S_2 \ni y$ - the number of the column.

It is customary to employ the convention $((\sigma_i, \sigma_{-i})) = (\sigma_1, \sigma_2) \in \Sigma_1 \times \Sigma_2$, $\Sigma_{-i} = \Sigma_{3-i}$ and $P_{-i} = P_{3-i}$ which emphasizes the role of the $i$-th player ($i = 1, 2$); the same applies to $((x, y)) \in S_1 \times S_2$, $((\pi, y)) \in \Delta(S_1) \times \Delta(S_2)$, $W_{-i}$ etc.

For technical reasons we shall tacitly assume payoff functions $P_i$ to be bounded from below in the co-player’s variable, i.e., $\inf_{\sigma_{-i} \in \Sigma_{-i}} P_i((\sigma_i, \sigma_{-i})) > -\infty$ for $\sigma_i \in \Sigma_i$, $i = 1, 2$. The continuity of $P_i$ is not assumed and, in the case of finite strategy sets $S_i$, of no use.

The game $\Gamma$ is said to be strictly competitive, if

$$\forall (\sigma_1, \sigma_2) : (\sigma'_1, \sigma'_2) \in \Sigma_1 \times \Sigma_2 \quad P_1(\sigma'_1, \sigma'_2) > P_1(\sigma_1, \sigma_2) \Leftrightarrow P_2(\sigma'_1, \sigma'_2) < P_2(\sigma_1, \sigma_2).$$

Let us note that strictly competitive games are essentially zero-sum games (see [AdDaPa]).

We call $\Gamma$ quantitatively symmetric, if $P_1(\sigma_1, \sigma_2) = P_2(\sigma_2, \sigma_1)$ for $\sigma_1, \sigma_2 \in \Sigma_1 = \Sigma_2$.

**Proposition 1.** If $G$ is a finite strictly competitive game, resp. quantitatively symmetric game, then its mixed extension $\Delta(G)$ is of the same character.

**Definition 1.** A pair of strategies $(\sigma_1^*, \sigma_2^*) \in \Sigma_1 \times \Sigma_2$ is

- Pareto optimum, $PO(\Gamma)$, if
  $$\neg \exists (\sigma_1, \sigma_2) : (\sigma_1, \sigma_2) \in \Sigma_1 \times \Sigma_2 \forall i = 1, 2 \quad P_i(\sigma_1, \sigma_2) > P_i(\sigma_1^*, \sigma_2^*),$$

- strong Pareto optimum, $SPO(\Gamma)$, if
  $$\neg \exists (\sigma_1, \sigma_2) : (\sigma_1, \sigma_2) \in \Sigma_1 \times \Sigma_2 \quad [\forall i = 1, 2 \quad P_i(\sigma_1, \sigma_2) \geq P_i(\sigma_1^*, \sigma_2^*) \land \exists i = 1, 2 \quad P_i(\sigma_1, \sigma_2) > P_i(\sigma_1^*, \sigma_2^*)],$$

- Wald solution, $W(\Gamma)$, if
  $$\forall i = 1, 2 \quad \sigma_i^* \in \arg \max_{\sigma_i \in \Sigma_i} \min_{\sigma_{-i} \in \Sigma_{-i}} P_i((\sigma_i, \sigma_{-i})),$$

- Nash equilibrium, $NE(\Gamma)$, if
  $$\forall \sigma_1 \in \Sigma_1, \sigma_2 \in \Sigma_2 \forall i = 1, 2 \quad P_i((\sigma_i, \sigma_{-i})) \leq P_i((\sigma_i^*, \sigma_{-i}^*)), \quad P_i((\sigma_i^*, \sigma_{-i}^*)) \leq P_i((\sigma_i, \sigma_{-i})).$$
strict Nash equilibrium, \( SNE(\Gamma) \), if
\[
\forall \sigma_i \in \Sigma_i \setminus \{\sigma_i^*\}, \, \sigma_j \in \Sigma_j \setminus \{\sigma_j^*\} \forall_{i=1,2} \ P_i((\sigma_i, \sigma_{-i}^*)) < P_i((\sigma_i^*, \sigma_{-i}^*)),
\]
semi-strict Nash equilibrium, \( SSNE(\Gamma) \), if it is Nash equilibrium and
\[
\forall \sigma_i \in \Sigma_i, \sigma_j \in \Sigma_j \forall_{i=1,2} \ P_i((\sigma_i, \sigma_{-i}^*)) = P_i((\sigma_i^*, \sigma_{-i}^*)) \Rightarrow P_{-i}((\sigma_i, \sigma_{-i}^*)) = P_{-i}((\sigma_i^*, \sigma_{-i}^*)) \]
weakly semi-strict Nash equilibrium, \( WSSNE(\Gamma) \), if it is Nash equilibrium and
\[
\forall \sigma_i \in \Sigma_i, \sigma_j \in \Sigma_j \forall_{i=1,2} \ P_i((\sigma_i, \sigma_{-i}^*)) = P_i((\sigma_i^*, \sigma_{-i}^*)) \Rightarrow P_{-i}((\sigma_i, \sigma_{-i}^*)) \geq P_{-i}((\sigma_i^*, \sigma_{-i}^*)) \]
coupled in wealth improvement, \( CWI(\Gamma) \), if
\[
\forall \sigma_i \in \Sigma_i, \sigma_j \in \Sigma_j \forall_{i=1,2} \ P_i((\sigma_i, \sigma_{-i}^*)) \geq P_i((\sigma_i^*, \sigma_{-i}^*)) \Rightarrow P_{-i}((\sigma_i, \sigma_{-i}^*)) \geq P_{-i}((\sigma_i^*, \sigma_{-i}^*)) \]

Remark that \( WSSNE(\Gamma) = NE(\Gamma) \cap CWI(\Gamma) \). The last three concepts (SSNE, WSSNE, and CWI) are provided by ourselves and their role shall be clear in view of further investigations. Loosely speaking (weakly) semi-strict equilibria retain most important features of strict equilibria, especially those associated with strategic uncertainty as shown in Example 2. Remark also that the (weakly) semi-strict equilibrium is a concept different from the weakly strict equilibrium introduced in \([\text{BoCaGJMN}]\) and quasi strict equilibrium in \([\text{No}]\). Finally the reader should be warned that the Wald solution becomes the maximin solution if additionally the “minimax identity” holds
\[
\forall_{i=1,2} \ P_i(\sigma_i^*, \sigma_2^*) = \max_{\sigma_i \in \Sigma_i} \min_{\sigma_{-i} \in \Sigma_{-i}} P_i((\sigma_i, \sigma_{-i})).
\]

We have the following relations

\[
SNE(\Gamma) \subsetneq SSNE(\Gamma) \subsetneq WSSNE(\Gamma) \subsetneq NE(\Gamma).
\]

The inclusions are immediate from the definitions. That they are strict illustrates:

**Example 4.** [3-4-5 game] Let \( S_1 = S_2 = \{1, 2, 3, 4\} \),
\[
G = \begin{bmatrix}
[3,3] & [0,0] & [0,0] & [0,0] \\
[0,0] & [4,4] & [0,0] & [4,4] \\
[0,0] & [0,0] & [3,3] & [5,3] \\
[0,0] & [4,4] & [3,5] & [5,5]
\end{bmatrix}.
\]

Then \((4, 4) \in NE(G) \setminus WSSNE(G), (3, 3) \in WSSNE(G) \setminus SSNE(G), (2, 2) \in SSNE(G) \setminus SNE(G), (1, 1) \in SNE(G)\).

**Proposition 2.** The equilibria of a finite game \( G \) and its mixed extension \( \Delta(G) \) are related as follows:

1. \( NE(G) \subset NE(\Delta(G)) \),
2. \( SNE(G) \subset SNE(\Delta(G)) \),
3. \( SSNE(G) \subset SSNE(\Delta(G)) \),
4. \( WSSNE(G) \subset WSSNE(\Delta(G)) \).
Proof. We only check the last invariance, the rest can be performed analogously. Let 
\((x^*, y^*) \in WSSNE(G)\) and suppose that some deviation from \(x^* \in S_i\) to a mixed strategy 
\(\pi \in \Delta(S_i)\) still gives equally good gain for the \(i\)-th player i.e. \(EW_i((x^*, y^*)) = EW_i((\pi, y^*))\). 
First note that since \((x^*, y^*)\) is a Nash equilibrium of \(G\), then \(W_i((x, y^*)) \leq W_i((x^*, y^*))\) 
for all \(x \in supp \pi\). Now if \(W_i((\pi, y^*)) < W_i((x^*, y^*))\) for some \(\pi \in supp \pi\), then 
\[EW_i((\pi, y^*)) = \pi(\pi) \cdot W_i((\pi, y^*)) + \sum_{\pi \neq x \in supp \pi} \pi(x) \cdot W_i((x, y^*)) \]
\[< \left( \pi(\pi) + \sum_{\pi \neq x \in supp \pi} \pi(x) \right) \cdot W_i((x^*, y^*)) = EW_i((x^*, y^*)) \]
Therefore in fact we have \(W_i((x, y^*)) = W_i((x^*, y^*))\) for \(x \in supp \pi\). Recalling that 
\((x^*, y^*) \in WSSNE(G)\) we obtain that \(W_{-i}(x^*, y^*) \geq W_{-i}(x^*, y^*)\) for \(x \in supp \pi\). Finally 
\[EW_{-i}(x^*, y^*) = \sum_{x \in supp \pi} \pi(x) \cdot W_{-i}(x^*, y^*) \]
\[\geq \left( \sum_{x} \pi(x) \right) \cdot W_{-i}(x^*, y^*) = EW_{-i}(x^*, y^*) \]
\( \qed \)

Theorem 1. In the strictly competitive game \(\Gamma\) all equilibria are semi-strict, \(NE(\Gamma) = SSNE(\Gamma)\).

Proof. Let \((\sigma^*_1, \sigma^*_2) \in NE(\Gamma)\). Competitiveness implies that equal payoffs of one player 
correspond to equal payoffs of the other one. Therefore 
\[P_i((\sigma_i, \sigma_{-i}^*)) = P_i((\sigma_i^*, \sigma_{-i}^*)) \Rightarrow P_{-i}((\sigma_i, \sigma_{-i}^*)) = P_{-i}((\sigma_i^*, \sigma_{-i}^*)), \quad i = 1, 2, \]
so \((\sigma^*_1, \sigma^*_2) \in SSNE(\Gamma)\). \( \qed \)

Recall that the Nash equilibria of the strictly competitive game are precisely maximin solutions.

3. LOWER PAYOFF

In any game \(\Gamma = (\Sigma_1, \Sigma_2; P_1, P_2 : \Sigma_1 \times \Sigma_2 \to \mathbb{R})\) predicted possible behavior of players 
may be described via the response map \(R_i : \Sigma_1 \times \Sigma_2 \to 2^{\Sigma_i}, \ i = 1, 2, \) The best response 
map \(BR_i : \Sigma_1 \times \Sigma_2 \to 2^{\Sigma_i}\) defined by 
\[BR_i((\sigma_i, \sigma_{-i})) = \left\{ \tilde{\sigma}_i \in \Sigma_i : P_i((\tilde{\sigma}_i, \sigma_{-i})) = \max_{\sigma_i' \in \Sigma_i} P_i((\sigma_i', \sigma_{-i})) \right\} \]
for \(i = 1, 2, ((\sigma_i, \sigma_{-i}) \in \Sigma_1 \times \Sigma_2, \) is a usual choice, provided that the payoffs are (upper semi-) 
continuous and the strategy spaces are compact. We would like to investigate other
option: not worse response. The not worse response map \( NW R_i : \Sigma_1 \times \Sigma_2 \rightarrow 2^{\Sigma_i} \) is defined by
\[
NW R_i((\sigma_i, \sigma_{-i})) = \{ \tilde{\sigma}_i \in \Sigma_i : P_i((\tilde{\sigma}_i, \sigma_{-i})) \geq P_i((\sigma_i, \sigma_{-i})) \}
\]
for \( i = 1, 2 \), \((\sigma_i, \sigma_{-i}) \in \Sigma_1 \times \Sigma_2 \).

Indeed any player who wants to maximize his payoff would not change his strategy into a new one, if it leads to lower payoff. Therefore any response map \( R_i \) should obey the following restrictions
\[
BR_i((\sigma_i, \sigma_{-i})) \subset R_i((\sigma_i, \sigma_{-i})) \subset NW R_i((\sigma_i, \sigma_{-i})).
\]

Nevertheless one is not forced to play the best response. We should only incorporate in our calculations the possible answers of the co-player to estimate sure gain. This is reflected by the lower payoff function \( P_i^\# : \Sigma_1 \times \Sigma_2 \rightarrow \mathbb{R} \),
\[
P_i^\#((\sigma_i, \sigma_{-i})) = \inf \{ P_i((\sigma_i, \tilde{\sigma}_{-i})) : P_{-i}((\tilde{\sigma}_{-i}, \sigma_i)) \geq P_{-i}((\sigma_{-i}, \sigma_i)) \}.
\]

In the case of our choice \( R_i = NW R_i \):
\[
P_i^\#((\sigma_i, \sigma_{-i})) = \inf \{ P_i((\sigma_i, \tilde{\sigma}_{-i})) : P_{-i}((\tilde{\sigma}_{-i}, \sigma_i)) \geq P_{-i}((\sigma_{-i}, \sigma_i)) \}.
\]

Let us note a simple but useful property

**Lemma 1.** There holds estimation \( P_i^\# \leq P_i, \ i = 1, 2 \). Moreover, \( P_i^\#(\sigma_1, \sigma_2) = P_i(\sigma_1, \sigma_2) \) for \( i = 1, 2 \) if and only if \( (\sigma_1, \sigma_2) \in CW I(\Gamma) \).

**Theorem 2.** Let \( \Sigma_1, \Sigma_2 \) be compact spaces. If \( P_1, P_2 : \Sigma_1 \times \Sigma_2 \rightarrow \mathbb{R} \) are continuous, then \( P_1^\#, P_2^\# : \Sigma_1 \times \Sigma_2 \rightarrow \mathbb{R} \) are lower semicontinuous.

**Proof.** By continuity of \( P_i \) the map \( NW R_i : \Sigma_1 \times \Sigma_2 \rightarrow 2^{\Sigma_i} \) has closed graph for \( i = 1, 2 \). Compactness of the domain \( \Sigma_1 \times \Sigma_2 \) implies that the map
\[
\Sigma_1 \times \Sigma_2 \ni ((\sigma_i, \sigma_{-i})) \mapsto ((\sigma_i, NW R_{3-i}((\sigma_{-i}, \sigma_i)))) \subset \Sigma_1 \times \Sigma_2
\]
is upper semicontinuous with compact values. Composing it with continuous \( P_i \) and Hausdorff nonexpansive min : \( 2^\mathbb{R} \rightarrow \mathbb{R} \) yields lower semicontinuity of \( P_i^\# \).

**Proposition 3.** If \( \Gamma = (\Sigma_1, \Sigma_2; P_1, P_2 : \Sigma_1 \times \Sigma_2 \rightarrow \mathbb{R}) \) is the strictly competitive game, then for \( \sigma_i \in \Sigma_i, \sigma_{-i} \in \Sigma_{-i} \)
\[
P_i^\#((\sigma_i, \sigma_{-i})) = \inf_{\sigma'_{-i} \in \Sigma_{-i}} P_i((\sigma_i, \sigma'_{-i})).
\]

**Proof.** Let \( i = 1, 2, \sigma_i, \sigma'_i \in \Sigma_i, \sigma_{-i}, \sigma'_{-i} \in \Sigma_{-i} \). Observe that
\[
P_{-i}(\sigma'_1, \sigma'_2) \geq P_{-i}(\sigma_1, \sigma_2) \Leftrightarrow P_i(\sigma'_1, \sigma'_2) \leq P_i(\sigma_1, \sigma_2).
\]
Hence
\[
P_i^\#((\sigma_i, \sigma_{-i})) = \inf \{ P_i((\sigma_i, \sigma_{-i})) : P_{-i}((\sigma_i, \sigma'_{-i})) \geq P_{-i}((\sigma_i, \sigma_{-i})) \} = \inf \{ P_i((\sigma_i, \sigma_{-i})) : P_i((\sigma_i, \sigma'_{-i})) \leq P_i((\sigma_i, \sigma_{-i})) \} = \inf_{\sigma'_{-i} \in \Sigma_{-i}} P_i((\sigma_i, \sigma'_{-i})).
\]
Roughly speaking in the competitive game the lower payoff of the player depends only upon his own strategy.

4. M-equilibrium

We associate with $\Gamma = (\Sigma_1, \Sigma_2; P_1, P_2 : \Sigma_1 \times \Sigma_2 \to \mathbb{R})$ the flat-game $\Gamma^\flat = (\Sigma_1, \Sigma_2; P_1^\flat, P_2^\flat : \Sigma_1 \times \Sigma_2 \to \mathbb{R})$.

**Definition 2.** An $m$-equilibrium of the game $\Gamma$ is the Nash equilibrium of $\Gamma^\flat$, $\text{ME}(\Gamma) = \text{NE}(\Gamma^\flat)$.

The Nash and $m$-equilibria are not related in a straightforward way.

**Example 5.** Let $S_1 = S_2 = \{1, 2, 3\}$,

$$G = \begin{bmatrix} 1, 4 & 0, 0 & 4, 4 \\ 0, 0 & 3, 3 & 5, 3 \\ 4, 4 & 3, 5 & 5, 3 \end{bmatrix}.$$  

Then $(2, 2) \in \text{ME}(G) \cap \text{WSSNE}(G)$, $(2, 3) \in \text{ME}(G) \cap (\text{NE}(G) \setminus \text{WSSNE}(G))$, $(3, 1) \in \text{ME}(G) \setminus \text{NE}(G)$, $(3, 3) \in \text{NE}(G) \setminus \text{ME}(G)$ with strongly Pareto optimal last pair. ♦

Any decision rule which uses lower payoff estimations (stick to an $m$-equilibrium in our case) is resistant on iterated elimination of dominated strategies. Formally

**Proposition 4.** Let the $i$-th player, $i \in \{1, 2\}$, expect at least $v_i = P_i^\flat((\sigma_i, \sigma_{-i}))$ from playing $((\sigma_i, \sigma_{-i}))$ in $\Gamma$. Suppose that the partner of $i$ changes his strategy $\sigma_{-i}$ into $\sigma'_{-i}$ according to possibly higher payoff $P_{-i}((\sigma'_{-i}, \sigma_i)) \geq P_{-i}((\sigma_{-i}, \sigma_i))$. Then the $i$-th player is still satisfied, because $P_i((\sigma_i, \sigma'_{-i})) \geq v_i$.

This obvious assertion (restatement of the definition of the lower payoff) explains why the player does not need to reject his dominated strategies.

To understand possible quirks of being content with warranted lower payoff one should consider two classic zero-sum games.

**Example 6.** [Hide a coin] Let $S_1 = S_2 = \{1, 2\}$,

$$G = \begin{bmatrix} [-10, 10] & [15, -15] \\ [15, -15] & [-20, 20] \end{bmatrix}.$$  

Then $\text{NE}(G) = \emptyset$, although $\text{ME}(G) = \text{NE}(G^b) = \{1\} \times S_2$,

$$G^b = \begin{bmatrix} [-10, -15] & [-10, -15] \\ [-20, -15] & [-20, -15] \end{bmatrix}.$$
The first player cannot ensure payoff higher than $-10$, the second player cannot ensure payoff higher than $-15$.

Example 7. [Matching pennies] Let $S_1 = S_2 = \{1, 2\}$,
\[
G = \begin{bmatrix}
[1, -1] & [1, -1] \\
[1, -1] & [1, -1]
\end{bmatrix}.
\]
Then $NE(G) = \emptyset$, although $ME(G) = NE(G^\flat) = S_1 \times S_2$,
\[
G^\flat = \begin{bmatrix}
[-1, -1] & [-1, -1] \\
[-1, -1] & [-1, -1]
\end{bmatrix}.
\]
None of the players can ensure payoff higher than $-1$.

On the other hand $NE(\Delta(G)) = \{(1/2, 1/2, 1/2, 1/2)\}$ and this constitutes the main argument for mixed strategies if we view strategic interaction as did von Neumann: trying to outmanoeuvre other participants.

The key to resolve inconsistency of $m$-equilibrium with the hiding player’s choice policy behind a mixed strategy is to recognize that the $m$-equilibrium concentrates on the question what can be warranted in one-shot game rather than what can be gambled during repeated play. This seems paradoxical, but one also needs to take into account injurious though rational player as during analysis in Example 2 (comp. comment from p.9 before Theorem 3).

We investigate further properties of flat-games and $m$-equilibria.

**Proposition 5.** Lower value of a game does not change when substitute payoffs with lower payoffs:
\[
\sup_{\sigma_i \in \Sigma_i} \inf_{\sigma_{-i} \in \Sigma_{-i}} P_i((\sigma_i, \sigma_{-i})) = \sup_{\sigma_i \in \Sigma_i} \inf_{\sigma_{-i} \in \Sigma_{-i}} P_i^\flat((\sigma_i, \sigma_{-i})).
\]

**Proof.** Fix $\sigma_i \in \Sigma_i$, $\sigma_{-i} \in \Sigma_{-i}$, $i = 1, 2$. By the definition of the lower payoff and Lemma 1
\[
\inf_{\sigma_{-i} \in \Sigma_{-i}} P_i((\sigma_i, \sigma_{-i})) \leq P_i^\flat((\sigma_i, \sigma_{-i})) \leq P_i((\sigma_i, \sigma_{-i})).
\]
Thus
\[
(1) \quad \inf_{\sigma_{-i} \in \Sigma_{-i}} P_i((\sigma_i, \sigma_{-i})) = \inf_{\sigma_{-i} \in \Sigma_{-i}} P_i^\flat((\sigma_i, \sigma_{-i})).
\]

$M$-equilibrium is a strategic concept – it depends upon mutual preferences of players.

**Proposition 6.** Let $\Gamma = (\Sigma_1, \Sigma_2; P_1, P_2 : \Sigma_1 \times \Sigma_2 \to \mathbb{R})$ be a game and $\varphi_i : \mathbb{R} \to \mathbb{R}$ be strictly increasing right continuous functions, $i = 1, 2$. Then the game $\tilde{\Gamma} = (\Sigma_1, \Sigma_2; \tilde{P}_1, \tilde{P}_2 : \Sigma_1 \times \Sigma_2 \to \mathbb{R})$ transformed from $\Gamma$ via the formula $\tilde{P}_i = \varphi_i \circ P_i$ admits the same $m$-equilibria as the original game $\Gamma$; symbolically $ME(\tilde{\Gamma}) = ME(\Gamma)$. 

Proof. Observe that \( \varphi_i(\inf Z) = \inf \varphi_i(Z) \) for \( Z \subset \mathbb{R}, i = 1, 2 \). Then a direct calculation shows that \( \tilde{P}_i^\circ = \varphi_i \circ P_i^\circ \) whence the conclusion follows. \( \Box \)

We postpone a more technically subtle discussion of the above property to the Appendix. Weakly semi-strict Nash equilibria are those equilibria which survive “flattenization” of the game. Due to a one-sided exploitation of player’s trust the other player might change his strategy without loss of his payoff just to lower the payoff of his partner, which explains why not all Nash equilibria are \( m \)-equilibria (cf. Example 2). Despite possible complications illustrated by Example 3 a positive criterion for a Nash equilibrium to be an \( m \)-equilibrium provides

**Theorem 3.** The equilibria of the game \( \Gamma \) and its flat \( \Gamma^\circ \) are related as follows:

1. \( WSSNE(\Gamma) \subset WSSNE(\Gamma^\circ) \),
2. \( SSNE(\Gamma) \subset SSNE(\Gamma^\circ) \),
3. \( SNE(\Gamma) \subset SNE(\Gamma^\circ) \).

**Proof.** We only check the first inclusion, the rest being analogous.

Let \( (\sigma_i^1, \sigma_i^2) \in WSSNE(\Gamma), i = 1, 2 \). By Lemma 1 for \( \sigma_i \),

\[
(2) \quad P_i^\circ((\sigma_i, \sigma_i^*) \leq P_i((\sigma_i, \sigma_i^*)) \leq P_i((\sigma_i^*, \sigma_i^*)) = P_i^\circ((\sigma_i^*, \sigma_i^*)),
\]

which shows \( \sigma_i^1, \sigma_i^2 \in NE(\Gamma^\circ) \).

Suppose that \( P_i^\circ((\sigma_i, \sigma_i^*) = P_i^\circ((\sigma_i^*, \sigma_i^*)) \). Then from (2) \( P_i((\sigma_i, \sigma_i^*)) = P_i((\sigma_i^*, \sigma_i^*)) \).

Observe that

\[
P_i^\circ((\sigma_i, \sigma_i^*)) = \inf \left\{ P_i((\tilde{\sigma}_i, \sigma_i^*)) : P_i((\tilde{\sigma}_i, \sigma_i^*)) \geq P_i((\sigma_i, \sigma_i^*)) \right\}
\]

where the last equality assures Lemma 1. \( \Box \)

**Proposition 7.** If \( \Gamma \) is quantitatively symmetric, then \( \Gamma^\circ \) too.

Neither zero-sum, nor strict competitiveness of the game is preserved under “flattenization” procedure as show Examples 6 and 7.

5. Motivating examples

We bring to the readers attention three classic games where the \( m \)-equilibrium turns out to be a nontrivial concept.

**Example 8.** [Traveler’s dilemma – continuation] Let \( G \) be as in Example 3. We have \( NE(G) = SNE(G) = \{(2, 2)\} \) and \( P_i^\circ(x, y) = P_i^\circ(y, x) = \min(x, y) - 4 + 2 \cdot \text{sign}(x - y) \) for \( x, y \in \{2, 3, \ldots, 100\} \). Hence \( G^\circ \) admits two equilibria, so that \( ME(G) = \{(2, 2), (100, 100)\} \).
The outcome (100, 100) was often proposed by people (cf. Sciam) as a reasonable Pareto optimal solution regardless of a possible treacherous behavior of the co-player. (Interestingly, $G^{o}$ possesses three equilibria, which shows that $NE(G^{o}) \neq NE(G)$ in general.)

**Example 9.** [Cournot duopoly; BiChKoSz, Wa] Let $\Gamma = (\Sigma_1, \Sigma_2; P_1, P_2 : \Sigma_1 \times \Sigma_2 \to \mathbb{R})$, $S_1 = S_2 = [0, L]$, $L > 0$, $P_1(x, y) = P_2(y, x) = x \cdot (L - (x + y))$ for $x, y \in [0, L]$. Then $NE(\Gamma) = SNE(\Gamma) = \left(\frac{L}{3}, \frac{L}{3}\right)$.

Using elementary methods (cf. Wa) we find that $P^{0}_1(x, y) = x \cdot \min(y, L - (x + y))$ and

$$ME(\Gamma) = \{(x, L - 2x) : 0 \leq x \leq L/3\} \cup \left\{ \left( x, \frac{L - x}{2} \right) : L/3 \leq x \leq L \right\}.$$  

Note that from the cartel point of view, a Pareto dominating TU-solution $(L/4, L/4)$ would be superior. Unfortunately this “natural” solution is not an m-equilibrium. Nevertheless the joint payoff $P_1 + P_2$ is maximized at two boundary m-equilibria: $(0, L)$ and $(L, 0)$. This suggests that under Cournot duopoly pricing it is profitable for firms to choose an active monopolist and the other firm rest with no production. Switching the role of monopolist between firms could become a strategy (in repeated game) for hidden transfer of utility despite the payoff in game was assumed to be NTU.

**Example 10.** [Puu duopoly; BiChKoSz, Puu] Let $\Gamma = (\Sigma_1, \Sigma_2; P_1, P_2 : \Sigma_1 \times \Sigma_2 \to \mathbb{R})$, $S_1 = S_2 = [0, L]$, $L > 1$,

$$P_1(x, y) = P_2(y, x) = \left(\frac{L}{x + y^2} - 1\right) \cdot x$$

for $x, y \in [0, L]$ with convention that $P_i(0, 0) = 0, i = 1, 2$. Then $NE(\Gamma) = \left(\frac{L}{4}, \frac{L}{4}\right)$.

By elementary (though a bit cumbersome) calculations

$$P^{0}_1(x, y) = \left(\frac{L}{x + y^2} - 1\right) \cdot x$$

for $(x, y) \neq (0, 0)$, where $y^2 = \max\left(y, \left(\frac{L}{x + y} - 1\right) \cdot x\right)$. Hence $P^{0}_1(x, y) = P^{0}_2(y, x) = \min(y, P_1(x, y))$ for all $x, y \in [0, L]$.

Puu duopoly enjoys a rich set of m-equilibria. Denote by $L_\ast \approx 3.0796$ the unique positive root of the polynomial $1 + 4L + 6L^2 + 4L^3 + L^4 - L^5$ and put

$$E = \begin{cases} 
\{(\sqrt{L}, \sqrt{L} \cdot (\sqrt{L} - 1)), (\sqrt{L} \cdot (\sqrt{L} - 1), \sqrt{L})\}, & \text{when } L = L_\ast, \\
\emptyset, & \text{otherwise.}
\end{cases}$$

Further, denote

$$N = \begin{cases} 
\{(L/4, L/4)\}, & \text{when } L > 16, \\
\emptyset, & \text{otherwise.}
\end{cases}$$

We have

$$ME(\Gamma) = \bigcup_{x \in [0, \sqrt{L}]} \{x\} \times [0, \sqrt{L} - x] \cup E \cup N.$$
The extraordinary pair of m-equilibria at $L = L^*$ is an unexpected phenomenon. (It seems to be a noneconomic artifact bond to the formal model). That Nash equilibria of $\Gamma$ need not be m-equilibria unless $L$ is sufficiently large, is an effect of weakness of Nash equilibrium: when taking into account the security of payoff, a treacherous partner can switch precomitted (during cheap talk) strategy to a strategy indifferent for him but harmful for his co-player. Formally, $(L/4, L/4)$ is not a (weakly semi-) strict Nash equilibrium for small $L$.

Finally observe that for $L < 4$ the m-equilibrium $(\sqrt{L}/2, \sqrt{L}/2)$ Pareto dominates the Nash equilibrium $(L/4, L/4)$. One can stipulate that such m-equilibria might “explain” cartels in a game theoretic way without a recourse to exterior (outside game) constructs.

There is no doubt that the traveler’s dilemma was the driving force of our research. Note that m-equilibria do not bring anything new to the (in)famous prisoner’s dilemma (PD). This confirms that the traveler’s dilemma is not merely an extension of PD – it is something qualitatively different. On the other hand simultaneous simplicity and nontriviality of PD shows that having a good solution concept does not negate the reason to perform the play at all: knowing consequences is not freeing us from making decisions.

6. Existence of m-equilibrium

We know from Theorem 3 that the class of games which possess at least one m-equilibrium is quite large. Unfortunately we do not know whether m-equilibria exist under suitably mild assumptions in general. Nevertheless competitive games admit pure m-equilibria.

**Theorem 4.** If $\Gamma = (\Sigma_1, \Sigma_2; P_1, P_2 : \Sigma_1 \times \Sigma_2 \to \mathbb{R})$ is the strictly competitive game with compact metrizable strategy spaces $\Sigma_i$ and continuous payoffs $P_i$, $i = 1, 2$, then $ME(\Gamma) \neq \emptyset$. Namely, Wald solutions rest in m-equilibrium.

**Proof.** Observe that for $\sigma_i, \sigma'_i \in \Sigma_i$, $i = 1, 2$

$$d_H(P_i((\sigma_i, \Sigma_{-i})), P_i((\sigma'_i, \Sigma_{-i}))) \leq \sup_{\sigma_{-i} \in \Sigma_{-i}} |P_i((\sigma_i, \sigma_{-i})) - P_i((\sigma'_i, \sigma_{-i}))|,$$

where $d_H$ stands for the Hausdorff distance in $2^\mathbb{R}$. Since $P_i$ are uniformly continuous (as continuous on the compactum), we know that $\Psi_i : \Sigma_i \to 2^\mathbb{R}$, $\Psi_i(\sigma_i) = P_i((\sigma_i, \Sigma_{-i})$) for $\sigma_i \in \Sigma_i$, are Hausdorff continuous with compact values. The Hausdorff nonexpansiveness of $\min : 2^\mathbb{R} \to \mathbb{R}$ yields then the continuity of the map

$$\Sigma_i \ni \sigma_i \mapsto \min_{\sigma_{-i} \in \Sigma_{-i}} P_i((\sigma_i, \sigma_{-i})) = \Psi_i(\sigma_i).$$

This shows that we can define

$$\sigma^*_i \in \arg\max_{\sigma_i \in \Sigma_i} \min_{\sigma_{-i} \in \Sigma_{-i}} P_i((\sigma_i, \sigma_{-i}))$$

for $i = 1, 2$. So $(\sigma^*_1, \sigma^*_2) \in W(\Gamma) \neq \emptyset$. 
Further by Proposition 3

\[
\max_{\sigma_i \in \Sigma_i} \min_{\sigma_{-i} \in \Sigma_{-i}} P_i((\sigma_i, \sigma_{-i})) = \max_{\sigma_i \in \Sigma_i} P_i^\#((\sigma_i, \sigma_{-i})) = \max_{\sigma_i \in \Sigma_i} P_i^\#((\sigma_i, \sigma_i^*)) = \max_{\sigma_i \in \Sigma_i} P_i^\#((\sigma_i^*, \sigma_i)) \geq P_i^\#((\sigma_i^*, \sigma_{-i}^*))
\]

for any \(\sigma_i \in \Sigma_i\), \(\sigma_{-i} \in \Sigma_{-i}\). Therefore \(W(\Gamma) \subset NE(\Gamma^\#) = ME(\Gamma)\).

Unfortunately the results established so far in the literature (cf. [Ba, Re, MoSc]) which are concerned with the existence of equilibria in games with discontinuous payoff functions do not seem to be applicable for the kind of problems considered here.

7. Mixed strategies and risk

The reason to calculate lower payoffs is establishing sure gains. Therefore one might question the use of expected payoffs to evaluate gains. We single out this phenomenon in the case of zero sum game.

Example 11. [extended matching pennies] Let \(S_1 = S_2 = \{1, 2, 3\}\),

\[
G = \begin{bmatrix}
-1,1 & 1,1 & 0,0 \\
1,1 & -1,1 & 0,0 \\
0,0 & 0,0 & 0,0
\end{bmatrix}
\]

Then \(NE(\Delta(G)) = \{(\delta_3, \delta_3), (\frac{1}{2}\delta_1 + \frac{1}{2}\delta_2, \frac{1}{2}\delta_1 + \frac{1}{2}\delta_2), (\frac{1}{2}\delta_1 + \frac{1}{2}\delta_2, \delta_3), (\delta_3, \frac{1}{2}\delta_1 + \frac{1}{2}\delta_2)\}\). Although all equilibria yield the same expected payoff, they differ significantly from the point of view of the risk. Namely the variance in payoff is nonzero unless both players use pure strategies (standard property of random variables). This has consequence for risk averse players usually not considered in the classic von Neumann’s minimax theory.

Let \(S = \{x_1, x_2, x_3, x_4, \ldots\}\) be the set of prizes with the associated utility function \(U : S \rightarrow \mathbb{R}\), such that \(U(x_1) < U(x_3) < U(x_2)\). Risk neutral players calculate gain for the lottery \((S, \pi), \pi \in \Delta(S)\), via the expected utility

\[
EU(\pi) = \sum_{x \in S} \pi(x) \cdot U(x).
\]

Hence they are indifferent in the choice between two lotteries \((\{x_1, x_2\}, \rho)\) and \(\{x_3\}\) as long as \(EU(\rho) = U(x_3)\).

However loss averse players would rather calculate the minimal gain

\[
E^{\min}U(\pi) = \min_{x \in \text{supp} \; \pi} U(x)
\]

for the lottery \((S, \pi)\). Then \(E^{\min}U(\rho) < U(x_3)\) and \(\{x_3\}\) is preferred over \((\{x_1, x_2\}, \rho)\) whenever \(x_1 \in \text{supp} \; \rho\). Take into account another lottery \((\{x_1, x_2\}, \rho')\) such that \(x_1 \in \text{supp} \; \rho'\). Then \(E^{\min}U(\rho') = E^{\min}U(\rho)\), so \(\rho\) and \(\rho'\) seem equally good. Still loss averse players might evaluate which of the given two lotteries with the same minimal gain has higher expected gain (as secondary criterion for preferences), \(EU(\rho)\) or \(EU(\rho')\)?
Concerning mixed strategies one should also be aware that the probability distribution might be also interpreted deterministically as a set of weights describing “fair” allocation of welfare/payoff induced by the choice of strategies. We discuss related questions in the next Section.

8. Equilibrium selection

The concept of m-equilibrium takes loss aversion and correlated decision into serious consideration. It demands communication and sure gains to be estimated. However it is not correlated equilibrium of Aumann. It also accounts for losses on the more basic level than the Harsanyi-Selten risk dominance selection criterion. Nevertheless m-equilibria (being Nash equilibria of the game with flattened payoffs) suffer the same curse of nonuniqueness (of payoff) as other notions of solution designed for non zero sum games.

To avoid complicated matters of the formal definitions of communication (pre-play) we simply say that the players can communicate to establish the final decision in a game \( \Gamma = (\Sigma_1, \Sigma_2; P_1, P_2 : \Sigma_1 \times \Sigma_2 \to \mathbb{R}) \), provided there exists a “communication channel” \( C : \Sigma_1 \times \Sigma_2 \to ME(\Gamma) \), where \( C \) is a random variable distributed on the set of m-equilibria according to probability \( \alpha \in \Delta[ME(\Gamma)] \). Vector \( \alpha \) will be interpreted further also as the set of weights of welfare allocation among m-equilibria.

Although communication restores Pareto-efficient equilibrium selection in Example 1 and the stag hunt game, we will face classical coordination dilemma posed by the battle of the sexes game. In presence of equal power and credibility players, the coordination dilemma is often resolved via fair allocation rule: “once for me, once for you”. We believe that cooperative social choice among various equilibria is the appropriate answer to equilibrium selection in both, one-shot and repeated games. Together with a social rule providing the allocation vector \( \alpha \in \Delta[ME(\Gamma)] \), some stochastic tie-breaking rules are indispensable. (During repeated play the variance of payoff outcomes arises as another problem. Alternate choice of equilibria minimizes this variance).

**Example 12.** [Battle of the sexes] Let \( S_1 = S_2 = \{1, 2\} \),

\[
G = \begin{bmatrix}
[3,2] & [0,0] \\
[0,0] & [2,3]
\end{bmatrix}.
\]

Then \( NE(G) = ME(G) = \{(1,1), (2,2)\} \); \( \alpha = \frac{1}{2} \cdot \delta_{(1,1)} + \frac{1}{2} \cdot \delta_{(2,2)} \). The only way to get rid of the question “who’s equilibrium played first” is to apply randomization device according to distribution given by \( \alpha \). This is an instance of Szaniawski’s probabilistically equal choice principle ([Li]). In one-shot games stochastic mechanism for choosing the player who selects preferred equilibrium to be played seems very reasonable also according to Laplace’s criterion of insufficient reason.

Once players receive the recommended equilibrium after pre-play phase, they form their beliefs and strategic properties of m-equilibrium warrant the appropriate payoff levels regardless of whether one of the players tries to exploit this information. Roughly speaking,
9. Final comments

The following problems are very important for the discussion of the relevance of the concept of m-equilibrium:

(1) Do (pure strategy) m-equilibria always exist under reasonable assumptions about payoff functions?
(2) What other than traveler’s dilemma games admit “intuitively superior” m-equilibria impossible within standardly interpreted Nash framework?
(3) How to cope with risk and welfare allocation? Does there exist any clear risk dominance criterion? (Cf. [Hs]).

To prove a general existence theorem on m-equilibria definitely one cannot use continuity of lower payoffs, but some assumptions about payoffs and strategy sets are indispensable.

**Example 13.** Let $\Gamma = (\Sigma_1, \Sigma_2; P_1, P_2 : \Sigma_1 \times \Sigma_2 \to \mathbb{R})$, $x, y \in \Sigma_1 = \Sigma_2 = [0, \infty)$ and $L \geq 2C > 0$. We define $P_1(x, y) = P_2(y, x) = \frac{Lx}{x+y} - \frac{C}{x}$, when $x > 0$ and $P_1(0, y) = 0$ otherwise. We assume here (unlike [Puu, BiChKoSz]) that the total cost of production decreases $C/x \searrow 0$ as the production of the firm increases $x \nearrow \infty$. One can think about this opportunity as the effect of scale (globalization). It turns out that under our assumption of diminishing cost, the Puu duopoly behaves qualitatively in a similar fashion to that observed in the competition of “giants”: each player has an incentive to grow production for overtaking the market; in practice we expect the mirroring behavior of firms, because it warrants maximal payoff share (according to TU value).

Interestingly $P^1_1(x, y) = P^2_2(y, x) = -\frac{C}{x}$, which reflects an old truth that in business one might bear the cost of production without any profit (“fall of a giant”). Consequently $ME(\Gamma) = \{(0, 0)\}$. If $\Sigma_1 = \Sigma_2 = (0, \infty)$, then $\Gamma$ has no m-equilibrium. A reasonable workaround could be then to allow for an epsilon-m-equilibrium (produce as little as possible).

A sky-rocketing competition in the above Example tells us that the dynamic view of games is necessary when the game is played more than once.

We do not consider multiplayer games in this article because we believe that only good understanding of two player games can give rise for reasonable extensions of static duel games to the situation of multiple interacting agents. We are aware of specific “phase transitions” and emergent effects when passing from the case of two players to the case of multiple players.

The adaptation of the notion of m-equilibrium for multiplayer games should be done carefully. Let $\Gamma = (\Sigma_1, \ldots, \Sigma_N; P_1, \ldots, P_N : \Sigma_1 \times \ldots \times \Sigma_N \to \mathbb{R})$ be a game with $N$ players. Since the player can only be sure of his own declaration and the communicated decisions of others might be changed, the following definition of the lower payoff seems to be suitable.
in this sort of situation:

$$P_i^\sigma(\sigma_1, \ldots, \sigma_N) = \inf \{ P_i(\varsigma^J) : \exists J \subset \{1, \ldots, N\} \forall j \in J \ P_j(\varsigma^J) \geq P_j(\sigma_1, \ldots, \sigma_N) \}$$

for $(\sigma_1, \ldots, \sigma_N)$, $\varsigma^J \in \Sigma_1 \times \ldots \times \Sigma_N$, where $\varsigma^J_i = \sigma_i$ when $i \notin J$, i.e., $\varsigma^J$ may differ from $(\sigma_1, \ldots, \sigma_N)$ for players $i$ contributing to a virtual coalition $J$.

Some other ideas aiming to resolve the dominated strategies conundrum in traveler’s dilemma were reported in [HlP]. Another concept of solution suitable for traveler’s dilemma (and accounting for the lack of communication unlike in the present article) is Hofstadter’s superrationality which can be argued within bayesian framework, e.g., [Ms]. However our intention was to dispose off as much probability as possible.

The cryptic term “m-equilibrium” was thought off by the author in accordance with the notion of meta-stable equilibrium appearing among others in chemistry and physics; that is an extraordinary equilibrium (or higher state) possible only under very specific conditions.

**APPENDIX: ISOMORPHISM OF GAMES**

We say that a function $\varphi : \mathbb{R} \supset Z \to \mathbb{R}$ is

- **strictly inf-increasing**, if for nonempty $U_1, U_2 \subset Z$

  $$\inf U_1 < \inf U_2 \Rightarrow \inf \varphi(U_1) < \inf \varphi(U_2),$$

- **inf-continuous**, if for every nonempty $U \subset Z$ such that $\inf U \in Z$ holds $\varphi(\inf U) = \inf \varphi(U)$,

- **right continuous**, if for every $z_0 \in Z$ and every (w.l.o.g. decreasing) sequence $z_n \in Z$, $z_0 \leq z_n \to z_0$ holds $\varphi(z_n) \to \varphi(z_0)$.

**Proposition 8.** Let $\varphi : \mathbb{R} \supset Z \to \mathbb{R}$.

1. If $\varphi$ is strictly inf-increasing, then it is strictly increasing.
2. If $\varphi$ is (not necessarily strictly) increasing, then it is inf-continuous if and only if it is right continuous.
3. If $Z = \mathbb{R}$ and $\varphi$ is strictly increasing inf-continuous, then it is strictly inf-increasing.

We warn that infima are taken in the whole $\mathbb{R}$, not in the ordered subset $Z \subset \mathbb{R}$. That the notion of strictly inf-increasing function is essentially stronger than strictly increasing function illustrates.

**Example 14.** Let $Z = \{0\} \cup (1, \infty) \subset \mathbb{R}$, $\varphi : Z \to \mathbb{R}$, $\varphi(z) = \max\{z - 1, 0\}$ for $z \in Z$. Although $\varphi$ is strictly increasing inf-continuous function it is not strictly inf-increasing. ♦

**Proposition 9.** Let $\Gamma = (\Sigma_1, \Sigma_2; P_1, P_2 : \Sigma_1 \times \Sigma_2 \to \mathbb{R})$ be a game and $\varphi_i : P_i(\Sigma_1 \times \Sigma_2) \to \mathbb{R}$ be order-preserving maps, $i = 1, 2$, i.e.

$$\forall u, v \in P_i(\Sigma_1 \times \Sigma_2) \ u < v \Rightarrow \varphi_i(u) < \varphi_i(v).$$
Then the game $\tilde{\Gamma} = (\Sigma_1, \Sigma_2; \tilde{P}_1, \tilde{P}_2 : \Sigma_1 \times \Sigma_2 \to \mathbb{R})$ transformed from $\Gamma$ via the formula $\tilde{P}_i = \varphi_i \circ P_i$ admits the same m-equilibria as the original game $\Gamma$; symbolically $ME(\tilde{\Gamma}) = ME(\Gamma)$.

**Proof.** Direct calculation shows that $\tilde{P}_i^{\flat} = \varphi_i \circ P_i^{\flat}$ whence the conclusion follows. [Needed $\varphi(\inf Z) = \inf \varphi(Z)$].

An analysis of “equivalent” prisoner’s dilemmas shows that isomorphic games may have nonequivalent risk structure. Therefore an appropriate concept of isomorphism of normal form games is no less controversial than the choice of satisfactory definition of the solution of a game or the equilibrium selection problem.

**Acknowledgement**

The author’s research was ignited in 2008 by Sławomir Plaskacz (differential inclusions, Hamilton-Jacobi equation in control and optimization, differential games). We had a lot of vigorous discussions.

A criticism of the earlier concepts proposed by the author (Nash-von Neumann co-operative solution, correlated Pareto optimum, retaliatory safe optimum) led to the concept of m-equilibrium. I would like to thank all participants of the three seminars where, around 2009, I referred those unsatisfactory concepts: Seminar of the Chair of Nonlinear Mathematical Analysis and Topology at the Nicolaus Copernicus University (Wojciech Krysiewski’s group), Seminar of the Game and Decision Theory Group at the Polish Academy of Sciences (Andrzej Wieczorek’s group) and Seminar of the Chair of Mathematical Economics at the Poznan University of Economics (Emil Panek’s group).

Almost all of this work has been done by the author during 2009-2010 in the Faculty of Mathematics and Computer Science at the Nicolaus Copernicus University (Toruń, Poland).

It would be really hard to grasp the current state of the research in game theory, if not *books.google* and various free resources provided by the experts in the subject.

**References**

[HuPa] S. Hu, N.S. Papageorgiou, *Handbook of Multivalued Analysis. Vol. I*, Kluwer, Dordrecht 1997.

[Be] G. Beer, *Topologies on closed and closed convex sets*, Kluwer, Dordrecht 1993.

[RoWe] R.T. Rockafellar, R.J-B. Wets, *Variational Analysis*, Springer 1997

[Au] J-P. Aubin, *Optima and Equilibria*, Springer, Berlin 1998.

[AuHa] R. Aumann, S. Hart (eds.), *Handbook of Game Theory with Economic Applications. Vol. 3*, North-Holland, Amsterdam 2004

[Da] E. van Damme, *Strategic equilibrium*, in: R. Aumann, S. Hart (eds.), *Handbook of Game Theory with Economic Applications. Vol. 3*, North-Holland, Amsterdam 2004, 1521–1596.

[Wa] J. Watson, *Strategy: an introduction to game theory*, W.W. Norton 2002.

[HiKo] J. Hillas, E. Kohlberg, *Foundations of strategic equilibrium*, in: R. Aumann, S. Hart (eds.), *Handbook of Game Theory with Economic Applications. Vol. 3*, North-Holland, Amsterdam 2004, 1597–1663.
[Gi] H. Gintis, The bounds of reason: game theory and the unification of the behavioral sciences, Princeton University Press, Princeton 2009.

[HHVa] S. Hargreaves-Heap, Y. Varoufakis, Game theory: A critical introduction, Routledge, London 1995.

[AlGa] S. Alpern, Sh. Gal, The Theory of Search Games and Rendezvous, Springer 2003.

[Li] G. Lissowski, Principles of Fair Distribution of Goods (in Polish), Scholar 2008.

[BiChKoSz] G.-I. Bischi, C. Chiarella, M. Kopel, F. Szidarovszky, Nonlinear Oligopolies: Stability and Bifurcations, Springer, Berlin 2010.

[Puu] T. Puu, Oligopoly: Old Ends – New Means, Springer, Berlin 2011.

[MoSc] J. Morgan, V. Scalzo, Pseudocontinuous functions and existence of Nash equilibria, J. Math. Econom. 43 (2007), 174–183.

[Re] P.J. Reny, On the existence of pure and mixed strategy Nash equilibria in discontinuous games, Econometrica 67 (1999), 1026–1056.

[Ba] A. Bagh, Variational convergence: Approximation and existence of equilibria in discontinuous games, J. Econom. Theory 145 (2010), 1244–1268.

[BoCaGMN] P.E.M. Borm, R. Cao, I. García-Jurado, L. Méndez-Naya, Weakly strict equilibria in finite normal form games, OR Spektrum 17 (1995), no. 4, 235–238.

[No] H. Norde, Bimatrix games have quasi-strict equilibria, Math. Program. 85 (1999), no. 1, Ser. A, 35–49.

[Ba] K. Basu, The traveler’s dilemma: Paradoxes of rationality in game theory, American Economic Review, Vol. 84 (1994), No. 2, 391–395.

[BaBeSt] K. Basu, L. Becchetti, L. Stanca, Experiments with the Traveler’s Dilemma: Welfare, Strategic Choice and Implicit Collusion, Soc. Choice Welf., forthcoming.

[CpCbGo2] C.M. Capra, S. Cabrera, R. Gómez, The Effects of Common Advice on One-shot Traveler’s Dilemma Games: Explaining Behavior through an Introspective Model with Errors, Economic Working Papers at Centro de Estudios Andalucés E2003/17, Centro de Estudios Andalucés.

[CbCpGo] S. Cabrera, C.M. Capra, R. Gómez, Behavior in one-shot traveler’s dilemma games: model and experiments with advice, Span. Econ. Rev. 9 (2007), 129–152.

[Au2] R.J. Aumann, Nash equilibria are not self-enforcing, in Economic Decision-Making: Games, Econometrics and Optimisation: Contributions in Honor of Jacques H. Drèze, eds. J.J. Gabszewicz, J.F. Richard, L.A. Wolsey, North-Holland 1990, pp.201–206.

[Ro] E.E. Rosinger, The Nash-Equilibrium requires strong cooperation, arXiv:math/0507013v2 (2005), 1–19.

[HaMa] S. Hart, Y. Mansour, How long to equilibrium? The communication complexity of uncoupled equilibrium procedures, Games Econom. Behav. 69 (2010), 107–126.

[MiMo] J.H. Miller, S. Moser, Communication and coordination, Complexity 9 (2004), no. 5, 31–40.

[GsHo] O. Gossner, J. Hörner, When is the lowest equilibrium payoff in a repeated game equal to the minmax payoff?, J. Econom. Theory 145 (2010), 63–84.

[Vo] M. Voorneveld, Preparation, Games Econom. Behav. 48 (2004), 403–414.

[DuNoRiTi] M. Dufwenberg, H. Norde, H. Reijnierse, S.Tijs, The consistency principle for set-valued solutions and a new direction for normative game theory, Math Meth Oper Res 54 (2001) 119–131.

[Ry] M. Ryan, The maximin criterion and randomized behavior reconsidered, Economics Working Papers, University of Auckland, June 1999.

[Co] A.M. Colman, Reasoning about strategic interaction: Solution concepts in game theory, in Psychology of reasoning: Theoretical and historical perspectives, eds. K. Manktelow, M. C. Chung, London: Psychology Press 2004, pp.287–308.

[AdDaPa] I. Adler, C. Daskalakis, C.H. Papadimitriou, A note on strictly competitive games, in Internet and Network Economics, ed. S. Leonardi, Springer 2009, pp.471–474.
[Hs] J. C. Harsanyi, A new theory of equilibrium selection for games with complete information, Games Econom. Behav. 8 (1995), 91–122.

[Ms] J. Masel, A Bayesian model of quasi-magical thinking can explain observed cooperation in the public good game, J. Economic Behavior Organization 64 (2007), no. 2, 216–231.

[HIPs] J. Y. Halpern, R. Pass, Iterated regret minimization: A new solution concept, Games Econom. Behav. 74 (2012), 184–207.

[SciAm] Scientific American webpage comments.

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