Laws of large numbers and nearest neighbor distances

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Dedicated to Sreenivasa Rao Jammalamadaka to mark his 65th year

Abstract

We consider the sum of power weighted nearest neighbor distances in a sample of size $n$ from a multivariate density $f$ of possibly unbounded support. We give various criteria guaranteeing that this sum satisfies a law of large numbers for large $n$, correcting some inaccuracies in the literature on the way. Motivation comes partly from the problem of consistent estimation of certain entropies of $f$.

1 Introduction

Nearest-neighbor statistics on multidimensional data are of long-standing and continuing interest, because of their uses, for example, in density estimation and goodness-of-fit testing \[3, 11, 21\], and entropy estimation \[2, 4, 9, 10\]. They form a multivariate analog to the one-dimensional spacings statistics in which the work of S. R. Jammalamadaka, the dedicatee of this paper, has featured prominently. For example, \[16\] uses

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nearest neighbor balls to generalize the maximum spacings method to high dimensions and to establish consistency in estimation questions.

In the present note we revisit, extend and correct some of the laws of large numbers concerned with sums of power-weighted nearest-neighbor distances that have appeared in recent papers, notably Penrose and Yukich [15], Wade [19], Leonenko et al. [10].

Fix $d \in \mathbb{N}$ and $j \in \mathbb{N}$. Given a finite $\mathcal{X} \subset \mathbb{R}^d$, and given a point $x \in \mathcal{X}$, let $\text{card}(\mathcal{X})$ denote the number of elements of $\mathcal{X}$, and let $D(x, \mathcal{X}) := D_j(x, \mathcal{X})$ denote the Euclidean distance from $x$ to its $j$th nearest neighbor in the point set $\mathcal{X} \setminus \{x\}$, if $\text{card}(\mathcal{X}) > j$; set $D(x, \mathcal{X}) := 0$ if $\text{card}(\mathcal{X}) \leq j$. Let $f$ be a probability density function on $\mathbb{R}^d$, and let $(X_i)_{i \in \mathbb{N}}$ be a sequence of independent random $d$-vectors with common density $f$. For $n \in \mathbb{N}$, let $\mathcal{X}_n := \{X_1, \ldots, X_n\}$. Let $\alpha \in \mathbb{R}$ and set

$$S_{n,\alpha} := \sum_{x \in \mathcal{X}_n} (n^{1/d}D(x, \mathcal{X}_n))^\alpha = \sum_{i=1}^n (n^{1/d}D(X_i, \mathcal{X}_n))^\alpha.$$  

Certain transformations of the $S_{n,\alpha}$ have been proposed [9, 10] as estimators for certain entropies of the density $f$ which are defined in terms of the integrals

$$I_\rho(f) := \int_{\mathbb{R}^d} f(y)^\rho dy \quad (\rho > 0).$$

For $\rho > 0$ with $\rho \neq 1$, the Tsallis $\rho$-entropy (or Havrda and Charvát $\rho$-entropy [7]) of the density $f$ is defined by $H_\rho(f) := (1 - I_\rho(f))/(1 - \rho)$, while the Rényi entropy [17] of $f$ is defined by $H^*_\rho(f) := \log I_\rho(f)/(1 - \rho)$.

Rényi and Tsallis entropies figure in various scientific disciplines, being used in dimension estimation and the study of nonlinear Fokker-Planck equations, fractal random walks, parameter estimation in semi-parametric modeling, and data compression (see [4] and [10] for further details and references).

A problem of interest is to estimate the Rényi and Tsallis entropies, or equivalently, the integrals $I_\rho(f)$, given only the sample $\{X_i\}_{i=1}^n$ and their pairwise distances. Let $\omega_d := \pi^{d/2}/\Gamma(1 + d/2)$ denote the volume of the unit radius Euclidean ball in $d$ dimensions, and set $\gamma(d, j) := \omega_d^{-\alpha/d} \left( \frac{\Gamma(j + \alpha/d)}{\Gamma(j)} \right)$. This note provides sufficient conditions on the density $f$ establishing that $\gamma(d, j)^{-1}n^{-1}S_{n,\alpha}$ converges to $I_{1-\alpha/d}(f)$ in $L^1$, or in $L^2$.  

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In other words, since $L^1$ convergence implies convergence of means, we provide sufficient conditions on $f$ guaranteeing that $\gamma(d,j)^{-1}n^{-1}S_{n,\alpha}$ is an asymptotically unbiased and consistent estimator of $I_{1-\alpha/d}(f)$.

2 Results

Two of our results can be stated without further ado.

Theorem 2.1 Let $\alpha > 0$. Suppose the support of $f$ is a finite union of convex bounded sets with nonempty interior, and $f$ is bounded away from zero and infinity on its support. Then as $n \to \infty$ we have $L^2$ and almost sure convergence

$$n^{-1}S_{n,\alpha} \to \omega_d^{-\alpha/d} \left( \frac{\Gamma(j + \alpha/d)}{\Gamma(j)} \right) I_{1-\alpha/d}(f).$$  \hspace{1cm} (2.1)

Theorem 2.2 Let $q = 1$ or $q = 2$. Let $\alpha \in (-d/q, 0)$ and suppose $f$ is bounded. Then (2.1) holds with $L^q$ convergence.

For the interesting case when $\alpha > 0$ and $f$ has unbounded support, our results require further notation. Let $|\cdot|$ denote the Euclidean norm on $\mathbb{R}^d$. For $r > 0$, define the integral

$$M_r(f) := \mathbb{E}[|X_1|^r] = \int_{\mathbb{R}^d} |x|^r f(x) dx,$$

and define the critical moment $r_c(f) \in [0, \infty]$, by

$$r_c(f) := \sup\{r \geq 0 : M_r(f) < \infty\}.$$

If $r < s$ and $M_s(f) < \infty$, then $M_r(f) < \infty$. Hence $M_r(f) < \infty$ for $r < r_c(f)$ and $M_r(f) = \infty$ for $r > r_c(f)$.

For $k \in \mathbb{N}$, let $A_k$ denote the annular shell centered around the origin of $\mathbb{R}^d$ with inner radius $2^k$ and outer radius $2^{k+1}$, and let $A_0$ be the ball centered at the origin with radius 2. For Borel measurable $A \subset \mathbb{R}^d$, set $F(A) := P[X_1 \in A] = \int_A f(x) dx$.

We can now state the rest of our results.
Theorem 2.3 Let $q = 1$ or $q = 2$. Let $\alpha \in (0, d/q)$. Suppose $I_{1-\alpha/d}(f) < \infty$, and $r_c(f) > q\alpha/(d - q\alpha)$. Then (2.1) holds with $L^q$ convergence.

We shall deduce from Theorem 2.3 that when $f(x)$ decays as a power of $|x|$, the condition $I_{1-\alpha/d}(f) < \infty$ is sufficient for $L^1$ convergence:

Corollary 2.1 Suppose there exists $\beta > d$ such that $f(x) = \Theta(|x|^{-\beta})$ as $|x| \to \infty$, i.e. such that for some finite positive $C$ we have

$$C^{-1}|x|^{-\beta} < f(x) < C|x|^{-\beta}, \quad \forall x \in \mathbb{R}^d, \ |x| \geq C. \quad (2.2)$$

Suppose also that $I_{1-\alpha/d}(f) < \infty$ for some $\alpha \in (0, d)$. Then (2.1) holds with $L^1$ convergence.

Our final result shows that in general, the condition $I_{1-\alpha/d}(f) < \infty$ is not sufficient alone for $L^1$ convergence, or even for convergence of expectations. It can also be viewed as a partial converse to Theorem 2.3 showing, under the additional regularity condition (2.3), that when $q = 1$ the condition $r_c(f) > q\alpha/(d - q\alpha)$ is close to being sharp.

Theorem 2.4 Let $0 < \alpha < d$. Then (i) if $r_c(f) < \alpha d/(d - \alpha)$, and also for some $k_0 \in \mathbb{N}$ we have

$$0 < \inf_{k \geq k_0} \frac{F(A_k)}{F(A_{k-1})} \leq \sup_{k \geq k_0} \frac{F(A_k)}{F(A_{k-1})} < \infty, \quad (2.3)$$

then $\limsup_{n \to \infty} \mathbb{E}[n^{-1}S_{n,\alpha}] = \infty$;

(ii) for $0 < r < \alpha d/(d - \alpha)$ there exists a bounded continuous density function $f$ on $\mathbb{R}^d$ satisfying (2.3), such that $I_{1-\alpha/d}(f) < \infty$, but with $r_c(f) = r$ so that $\limsup_{n \to \infty} \mathbb{E}[n^{-1}S_{n,\alpha}] = \infty$ by part (i).

The value of the limit in (2.1) was already known (see Lemma 3.1). The contribution of the present paper is concerned with the conditions under which the convergence (2.1) holds; in what follows we compare our conditions with the existing ones in the literature.
and also comment on related limit results. For conditions under which $n^{-1/2}(S_{n,\alpha} - \mathbb{E} S_{n,\alpha})$ is asymptotically Gaussian, we refer to [14, 11, 12].

Remarks.

(i) Theorem 2.1 The condition in Theorem 2.1 is a slight relaxation of condition C1 of the $L^2$ convergence results in [15] or [19], which assume a polyhedral support set. When the support of $f$ is the unit cube, Theorem 2.2 of [8] gives an alternative proof of almost sure convergence in (2.1) (we remark that Theorem 2.2 of [8] contains an extraneous $\mathbb{E}$ in the left-hand side). The convergence of means implied by Theorem 2.1 was previously obtained, under some extra differentiability conditions on $f$, in [5].

(ii) Theorem 2.2 The $L^1$ convergence of Theorem 2.2 improves upon Theorem 3.1 of [10], which establishes mean convergence; the $L^2$ convergence of Theorem 2.2 is contained in Theorem 3.2 of [10] and we include this for completeness.

(iii) Theorem 2.3 The condition in Theorem 2.3 corrects the condition of the corresponding result given [15], where for $L^1$ convergence it is stated that we need $r_c(f) > d/(d - \alpha)$; in fact we need instead the condition $r_c(f) > \alpha d/(d - \alpha)$. In the proof of Theorem 2.3 below, we shall indicate the errors in the proof in [15] giving rise to this discrepancy. This correction also applies to condition C2 in Theorem 2 of [19], the proof of which relies on the result stated in [15].

(iv) Theorem 2.4 The condition (2.3) holds, for example, if $f(x)$ is a regularly varying function of $|x|$. Given (2.3) and given $I_{1-\alpha/d} < \infty$, Theorem 2.4 shows that the condition $r_c(f) \geq \alpha d/(d - \alpha)$ is necessary for $L^1$ convergence of $n^{-1}S_{n,\alpha}$, while Theorem 2.3 says that $r_c(f) > \alpha d/(d - \alpha)$ is sufficient. It would be of interest to try to find more refined necessary and sufficient conditions when $r_c(f) = \alpha d/(d - \alpha)$.

(v) General $\phi$. For $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ put $S_{n,\phi} := \sum_{x \in X_n} \phi(n^{1/d}D(x, X_n))$. If $\phi$ has polynomial growth of order $\alpha$, that is if there is a constant $\alpha \in (0, \infty)$ such that $\phi(x) \leq C(1 + x^\alpha)$ for all $x \in \mathbb{R}^+$, then straightforward modifications of the proofs show that under the conditions of Theorem 2.1 or Theorem 2.3 we have the corresponding $L^q$ convergence

$$n^{-1}S_{n,\phi} \to \int_{\mathbb{R}^d} \mathbb{E} [\phi(D(0, \mathcal{P} f(x)))] f(x)dx,$$
where for all $\tau > 0$, $\mathcal{P}_\tau$ is a homogeneous Poisson point process in $\mathbb{R}^d$ having constant intensity $\tau$, and $D(0, \mathcal{P}_\tau)$ is the distance between the origin of $\mathbb{R}^d$ and its $j$th nearest neighbor in $\mathcal{P}_\tau$.

(vi) Minimal spanning trees. Given a finite $\mathcal{X} \subset \mathbb{R}^d$ and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$, let

$$L_\phi(\mathcal{X}) := \sum_{e \in \text{MST}(\mathcal{X})} \phi(|e|),$$

where $\text{MST}(\mathcal{X})$ denotes the edges in the graph of the minimal spanning tree on $\mathcal{X}$. Thus $L_\phi(\mathcal{X})$ is the sum of the $\phi$-weighted edge lengths in the minimal spanning tree on $\mathcal{X}$. Let $q = 1$ or 2. If $\phi$ has polynomial growth of order $\alpha$, with $\alpha \in (0, d/q)$, if $I_{1-\alpha/d}(f) < \infty$, and if $r_c(f) > q ad/(d - qa)$ then, as may be seen by following the proof of Theorem 2.3 of [15], the proof of Theorem 2.3(iii) of [15] in fact shows that as $n \to \infty$ we have

$$L_\phi(\mathcal{X}_n) \to \frac{1}{2} \int_{\mathbb{R}^d} \mathbb{E} \left[ \sum_{e \in \text{MST}(0, \mathcal{P}_f(x))} \phi(|e|) \right] f(x) dx,$$

where the convergence is in $L^q$, and where $\text{MST}(0, \mathcal{P}_f(x))$ denotes the edges in the minimal spanning tree graph on $0 \cup \mathcal{P}_f(x)$ incident to $0$, the origin of $\mathbb{R}^d$. When $q = 2$, this is new whereas for $q = 1$ and $\alpha \in (0, 1)$, this improves upon Theorem 2.3(iii) of [15], which requires $r_c(f) > \max(ad/(d - \alpha), d/(d - \alpha))$.

(vii) Non-existence of density. If the $\{X_i\}_{i=1}^n$ fail to have a density, then normalization of $S_{n, \alpha}$ may involve exotic functions of $n$, including log periodic normalizations, as is the case when the $\{X_i\}_{i=1}^n$ have a Cantor distribution on $[0, 1]$; see [18].

(viii) Comparison with [10]. The convergence of expectations corresponding to (2.1) is given as the main conclusion in Theorem 3.1 of [10]. In the case $1 - \alpha/d < 1$ of that result, it is claimed that this convergence of expectations holds without any extra conditions besides finiteness of $I_{1-\alpha/d}$. Theorem 2.4 here disproves this assertion; the argument in [10] requires that convergence in distribution implies convergence of $r$th moments, which is not in general true. On the other hand, Corollary 2.1 shows that if we assume $f(x)$ decays as some power of $|x|$ then finiteness of $I_{1-\alpha/d}$ is indeed a sufficient condition for convergence in $L^1$, and hence also convergence of expectations.
3 Proofs

This section provides the proofs of the results stated in the preceding section. We denote by \( c, C, C', \) and \( C'' \) various strictly positive finite constants whose values may change from line to line. The proofs of Theorems 2.1, 2.2 and 2.3 use the following result.

**Lemma 3.1** Let \( q \in \{1, 2\} \) and \( \alpha \in \mathbb{R} \). Suppose for some \( p > q \) that \( \mathbb{E} \left[ (n^{1/d} D(X_1, X_n))^{\alpha p} \right] \) is a bounded function of \( n \). Then (2.1) holds with \( L^q \) convergence.

*Proof.* Since \( D \) is a stabilizing functional on homogeneous Poisson point processes \([15]\), we can apply Theorem 2.2 of \([15]\) or Theorem 2.1 of \([15]\) to get \( L^q \) convergence of \( n^{-1} S_{n, \alpha} \) to a limit which is expressed as an integrated expectation in \([15]\) (see eqn (2.15) of \([15]\)). It was shown in \([19]\) that this limit is equal to the right hand side of (2.1) (and this is also consistent with the limiting constant in \([5]\)). \(\Box\)

*Proof of Theorem 2.1.* Recall that we assume the support of \( f \), namely \( \text{supp}(f) := \{x \in \mathbb{R}^d : f(x) > 0\} \), is a finite union of bounded convex sets with nonempty interior, here denoted \( B_1, \ldots, B_m \). Set \( \lambda := \sup \{|x - y| : x \in \text{supp}(f), y \in \text{supp}(f)\} \), the diameter of the support of \( f \). By assumption, \( \lambda < \infty \). Also we assert that there is a constant \( c > 0 \) such that for \( r \in (0, \lambda] \),

\[
F(B_r(x)) \geq cr^d, \quad \forall x \in \text{supp}(f), \tag{3.1}
\]

To see this, take \( \delta_1 > 0 \) such that for \( 1 \leq i \leq m \) there is a ball \( B_i^- \) of radius \( \delta_1 \) contained in \( B_i \). There is a constant \( \delta_2 > 0 \) such that for \( 1 \leq i \leq m \), if \( x \in B_i \), and \( r \leq \delta_1 \), then the intersection of the ball of radius \( r \) centered at \( x \) with the convex hull of the union of \( B_i^- \) and \( x \) has volume at least \( \delta_2 r^d \). This region is contained in \( B_i \) and (3.1) follows for \( r \in (0, \delta_1] \). But then (with a different choice of \( c \)) (3.1) follows for \( r \leq \lambda \). Hence, for \( 0 < t \leq \lambda n^{1/d} \) and with \( B(x; r) \) denoting the Euclidean ball of radius \( r \) centered at
Moreover this probability is clearly zero for $t > \lambda n^{1/d}$. Hence, for $\alpha > 0$ and $p > 2$,

$$
\mathbb{E}[(n^{1/d}D(X_1, \mathcal{X}_n)^{\alpha p}]] = \int_0^\infty P[n^{1/d}D(X_1, \mathcal{X}_n) > u^{1/(\alpha p)}]du \\
\leq C \int_0^\infty \exp(-(c/2)u^{d/(\alpha p)})du
$$

which is finite and does not depend on $n$. Therefore we can apply Lemma 3.1 to get the $L^2$ convergence (2.1).

For almost sure convergence, we apply Theorem 2.2 of [13], where here the test function considered in that result (and denoted $f$ there, not to be confused with the notation $f$ as used here) is the identity function. It is well known (see [3], or Lemma 8.4 of [20]) that there is a constant $C := C(d)$ such that for any finite $\mathcal{X} \subset \mathbb{R}^d$, any point $x \in \mathcal{X}$ is the $j$th nearest neighbor of at most $C$ other points of $\mathcal{X}$. Therefore adding one point to a set $\mathcal{X}$ within the bounded region $\text{supp}(f)$ changes the sum of the power-weighted $j$th nearest neighbor distances by at most a constant. Therefore (2.9) of [13] holds here (with $\beta = 1$ and $p' = 5$ say), and the almost sure convergence follows by Theorem 2.2 of [13].

**Proof of Theorem 2.2** The proof depends on the following lemma. Recall that $(X_i)_{i \geq 1}$ are i.i.d. with density $f$. Given $X_1$, let $V_n$ denote the volume of the $d$-dimensional ball centered at $n^{1/d}X_1$ whose radius equals the distance to the $j$th nearest point in $n^{1/d}(\mathcal{X}_n \setminus X_1)$, where for $r > 0$ and $\mathcal{X} \subset \mathbb{R}^d$ we write $r\mathcal{X}$ for $\{rx : x \in \mathcal{X}\}$. For
all $x \in \mathbb{R}^d$, for all $n = 2, 3, \ldots$ and for all $v \in (0, \infty)$ let

$$F_{n,x}(v) := P[V_n \leq v | X_1 = x]. \quad (3.2)$$

**Lemma 3.2** If $f$ is bounded and $\delta \in (0, 1)$, then

$$\sup_n \mathbb{E} V_n^{-\delta} = \sup_n \int_{\mathbb{R}^d} \int_0^{\infty} v^{-\delta} dF_{n,x}(v) f(x) dx < \infty.$$  

Proof of Lemma 3.2. Since $\int_0^{\infty} v^{-p} dF(v) = p \int_0^{\infty} v^{-p-1} F(v) dv$ for any $p \in (0, 1)$ whenever both integrals exist (see e.g. Lemma 1 on p. 150 of [6]), we have for all $x \in \mathbb{R}^d$

$$\int_0^{\infty} v^{-\delta} dF_{n,x}(v) = \delta \int_0^{\infty} v^{-\delta-1} F_{n,x}(v) dv$$

$$\leq \int_0^{1} v^{-\delta-1} F_{n,x}(v) dv + \delta \int_1^{\infty} v^{-\delta-1} dv = \int_0^{1} v^{-\delta-1} F_{n,x}(v) dv + 1.$$  

With $\tilde{B}_v(x)$ denoting the ball of volume $v$ around $x$, for all $v \in (0, 1)$ we have

$$F_{n,x}(v) = P[V_n \leq v | X_1 = x] = 1 - P[\text{card}(n^{1/d} X_{n-1} \cap \tilde{B}_v(n^{1/d} x)) < j]$$

$$\leq 1 - P[\text{card}(n^{1/d} X_{n-1} \cap \tilde{B}_v(n^{1/d} x)) = 0]$$

$$= 1 - \left(1 - \int_{\tilde{B}_{v/n}(x)} f(z) dz\right)^{n-1}. \quad (3.3)$$

Since $f$ is assumed bounded we have

$$F_{n,x}(v) \leq 1 - \exp (-(n-1) \log(1 - \|f\|_{\infty} v/n)).$$

When $n$ is large enough, for all $v \in (0, 1)$ we have $(n-1) \log(1 - \|f\|_{\infty} v/n) \geq -2\|f\|_{\infty} v$, and so for all $x \in \mathbb{R}^d$

$$F_{n,x}(v) \leq 1 - \exp(-2\|f\|_{\infty} v) \leq 2\|f\|_{\infty} v.$$  

Hence for all $n$ large enough and all $x$ we have $\int_0^{1} v^{-\delta-1} F_{n,x}(v) dv \leq 2\|f\|_{\infty} \int_0^{1} v^{-\delta-1} v dv$, demonstrating Lemma 3.2. \qed
Now to prove Theorem 2.2, we choose \( p > q \) such that \(-1 < \alpha p/d < 0\) and invoke Lemma 3.2 to conclude \( \sup_n \mathbb{E} [V_n^{\alpha p/d}] < \infty \). We now apply Lemma 3.1 to complete the proof of \( L^q \) convergence.

The proof of Theorem 2.3 uses the following lemma. Recall from Section 2 the definition of the regions \( A_k, k \geq 0 \).

**Lemma 3.3** Let \( 0 < s < d \). If \( r_c(f) > sd/(d-s) \), then \( \sum_{k=1}^{\infty} 2^{ks}(F(A_k))^{(d-s)/d} < \infty \).

**Proof.** We modify some of the arguments on page 85 of [20]. For all \( \varepsilon > 0 \), by Hölder’s inequality we have

\[
\sum_k 2^{ks}(F[A_k])^{(d-s)/d} = \sum_k 2^{-\varepsilon ks}(F[A_k])^{(d-s)/d} 2^{(1+\varepsilon)ks} \leq \left( \sum_k (2^{-\varepsilon ks})^{d/s} \right)^{s/d} \left( \sum_k F[A_k] (2^{(1+\varepsilon)ks})^{d/(d-s)} \right)^{(d-s)/d} \leq C(\varepsilon, s) \left( \sum_k \int_{A_k} |x|^{1+s/(d-s)} f(x) dx \right)^{(d-s)/d}
\]

which, for \( \varepsilon \) small enough, is finite by hypothesis. \( \square \)

**Proof of Theorem 2.3.** We follow the proof in [15], but correct it in some places and give more details in others. We aim to use Lemma 3.1. Since we assume \( 0 < \alpha < d/q \), we can take \( p > q \) with \( \alpha p < d \). Clearly

\[
\mathbb{E} [(n^{1/d} D(X_1, \mathcal{X}_n))^{\alpha p}] = n^{\alpha p/d-1} \mathbb{E} \left[ \sum_{i=1}^{n} D(X_i, \mathcal{X}_n)^{\alpha p} \right] = n^{\alpha p/d-1} \mathbb{E} [L^{\alpha p}(\mathcal{X}_n)], \quad (3.4)
\]

where for any finite point set \( \mathcal{X} \subset \mathbb{R}^d \), and any \( b > 0 \), we write \( L^b(\mathcal{X}) \) for \( \sum_{x \in \mathcal{X}} D(x, \mathcal{X})^b \) (and set \( L^b(\emptyset) := 0 \)). Note that for some finite constant \( C = C(d, j) \) the functional \( \mathcal{X} \mapsto L^b(\mathcal{X}) \) satisfies the simple subadditivity relation

\[
L^b(\mathcal{X} \cup \mathcal{Y}) \leq L^b(\mathcal{X}) + L^b(\mathcal{Y}) + Ct^b \quad (3.5)
\]

for all \( t > 0 \) and all finite \( \mathcal{X} \) and \( \mathcal{Y} \) contained in \([0, t]^d\) (cf. (2.2) of [20]).
As in (7.21) of [20] or (2.21) of [15] we have that

\[ L^\alpha p(X_n) \leq \left( \sum_{k=0}^{\infty} L^\alpha p(X_n \cap A_k) \right) + C(p) \max_{1 \leq i \leq n} |X_i|^{\alpha p}. \]  

(3.6)

In the last sentence of the proof of Theorem 2.4 of [15] it is asserted that the last term in (3.6) is not needed, based on a further assertion that one can take \( C = 0 \) in (3.5) here, but these assertions are incorrect. For example, if \( \text{card}(\mathcal{Y}) \leq j \) then \( L^b(\mathcal{Y}) = 0 \) but \( L^b(\mathcal{X} \cup \mathcal{Y}) \) could be strictly greater than \( L^b(\mathcal{X}) \). Similarly, if \( \text{card}(\mathcal{X}_n \cap A_k) \leq j \) then the term in (3.6) from that \( k \) is zero but the corresponding contribution to the left side of (3.6) is non-zero.

Combining (3.6) with (3.4) yields

\[ \mathbb{E}\left[(n^{-\alpha/d}D(X_1, \mathcal{X}_n))^{\alpha p}\right] \leq n^{(\alpha p-d)/d} \mathbb{E} \left[ \sum_k L^\alpha p(X_n \cap A_k) \right] + C(p) \mathbb{E} \left[ n^{(\alpha p-d)/d} \max_{i} |X_i|^{\alpha p} \right]. \]  

(3.7)

By Jensen’s inequality and the growth bounds \( L^\alpha p(\mathcal{X}) \leq C(\text{diam}\mathcal{X})^{\alpha p}(\text{card}(\mathcal{X}))^{(d-\alpha p)/d} \) (see Lemma 3.3 of [20]), we can bound the first term in the right hand side of (3.7) by

\[ C \sum_k 2^{k \alpha p} (F[A_k])^{(d-\alpha p)/d}. \]  

(3.8)

Recall that we are assuming \( 0 < \alpha < d/q \) and also \( r_c(f) > qd\alpha/(d - q\alpha) \) (the last assumption did not feature in [15], but in fact we do need it). Let \( p > q \) be chosen so that \( r_c(f) > d\alpha p/(d - \alpha p) \) as well as \( \alpha p < d \). Setting \( s = \alpha p \) in Lemma 3.3, we get that the expression (3.8) is finite. Thus the first term in the right hand side of (3.7) is bounded by a constant independent of \( n \).

The second term in the right hand side of (3.7) is bounded by

\[ C(p) \left( \int_0^1 P \left[ \max_{1 \leq i \leq n} |X_i|^{\alpha p} \geq tn^{(d-\alpha p)/d} \right] dt + \int_1^\infty P \left[ \max_{1 \leq i \leq n} |X_i|^{\alpha p} \geq tn^{(d-\alpha p)/d} \right] dt \right) \leq C(p) \left( 1 + n \int_1^\infty P[|X_1|^{\alpha p d/(d-\alpha p)} \geq t^{d/(d-\alpha p)}n]dt \right). \]

By Markov’s inequality together with the assumption \( r_c(f) > d\alpha p/(d - \alpha p) \), this last integral is bounded by a constant independent of \( n \).
Therefore the expression (3.7) is bounded independently of $n$, so we can apply Lemma 3.1 to get the $L^q$ convergence in (2.1).

**Proof of Corollary 2.1.** Suppose for some $\beta > d$ that $f(x) = \Theta(|x|^{-\beta})$ as $|x| \to \infty$. Then it is easily verified that given $\alpha \in (0, d)$, the condition $I_{1-\alpha/d}(f) < \infty$ implies that $-\beta(1-\alpha/d) + d < 0$ and hence $\beta > d^2(d-\alpha)^{-1}$. Moreover, it is also easily checked that $r_c(f) = \beta - d$ so that if $\beta > d^2(d-\alpha)^{-1}$ then $r_c(f) > d\alpha/(d-\alpha)$.

Therefore, if $I_{1-\alpha/d}(f) < \infty$ we can apply the case $q=1$ of Theorem 2.3 to get (2.1) with $L^1$ convergence.

The proof of Theorem 2.3 shows that

$$
\mathbb{E} \left[ (n^{1/d}D(X_1, X_n))^\varepsilon \right] < \infty,
$$

(3.9)

if $\varepsilon > 0$ is such that $\varepsilon d/(d-\varepsilon) < r_c(f)$. The proof of Theorem 2.4 given below, shows that the condition $\varepsilon d/(d-\varepsilon) < r_c(f)$ cannot be dropped in general.

**Proof of Theorem 2.4.** Let $0 < \alpha < d$. Suppose that $r_c(f) < \alpha d/(d-\alpha)$, and (2.3) holds for some $k_0 \in \mathbb{N}$. Choose $r, s$ such that $r_c(f) < r < s < \alpha d/(d-\alpha)$. Then $M_r(f) = \infty$, so $\sum_k 2^k F(A_k) = \infty$ and therefore there is an infinite subsequence $\mathcal{K}$ of $\mathbb{N}$ such that

$$
2^{sk} F(A_k) \geq 1, \quad k \in \mathcal{K}.
$$

(3.10)

Indeed, if no such $\mathcal{K}$ existed, then for all but finitely many $k$ we would have $2^{rk} F(A_k) \leq 2^{(r-s)k}$ which is summable in $k$.

Given $k \in \mathbb{N}$, and set $n(k) = \lfloor (F(A_k))^{-1} \rfloor$, the smallest integer not less than $(F(A_k))^{-1}$. Let $E_k$ be the event that $X_1 \in A_k$ but $X_i \notin A_{k-1} \cup A_k \cup A_{k+1}$ for $2 \leq i \leq n(k)$. Then by the condition (2.3), there is a strictly positive constant $c$, independent of $k$, such that for $k \geq k_0$ we have

$$
P[E_k] = F(A_k)(1 - F(A_{k-1} \cup A_k \cup A_{k+1}))^{n(k)-1} \geq c F(A_k).
$$
If $E_k$ occurs then $D(X_1, X_{n(k)}) \geq 2^{k-1}$, so for $n = n(k)$ we have (for a different constant $c$) that
\[
\mathbb{E} [n^{-1} S_{n, \alpha}] = \mathbb{E} [(n^{1/d} D(X_1, X_{n}))^\alpha] \geq n^{\alpha/d} \mathbb{E} [D(X_1, X_{n})^\alpha 1_{E_k}] \\
\geq cn^{\alpha/d} F(A_k) 2^{k\alpha} \geq c(F(A_k))^{1-\alpha/d} 2^{k\alpha}.
\]
By (3.10), for $k \in K$ this lower bound is at least a constant times $2^{k(\alpha - s(d-\alpha)/d)}$, and therefore tends to infinity as $k \to \infty$ through the sequence $K$, concluding the proof of part (i).

For part (ii), for each $k \geq 2$ choose, in an arbitrary way, a unit radius ball $B_k$ that is contained in $A_k$. Given $r \in (0, \alpha d/(d - \alpha))$, consider the density function $f$ with $f(x) = C 2^{-rk}$ for $x \in B_k, k \geq 2$, and with $f(x) = 0$ for $x \in \mathbb{R}^d \setminus \bigcup_{k=2}^{\infty} B_k$; here the normalizing constant $C$ is chosen to make $f$ a probability density function. This gives $F(A_k) = C \omega d 2^{-rk}$ for each $k \geq 2$; it is easy to see that this $f$ has $r_c(f) = r$, and that (2.3) holds with $k_0 = 3$. Also, for any $\rho > 0$ we have $I_\rho(f) = \omega d C^\rho \sum_{k \geq 2} 2^{-rk\rho}$ which is finite, so in particular $I_{1-\alpha/d} < \infty$. This choice of $f$ is bounded but not continuous, but can easily be modified to a continuous density with the same properties, for example by modifying $f$ in an annulus near the boundary of each ball $B_k$ so as to make it continuous, and then adjusting the normalizing constant $C$ accordingly.

\[\square\]

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