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To cite this version:
Yves Colin de Verdière, David Vicente. Large-time asymptotics of the wave fronts length I The Euclidean disk. 2020. hal-03009034v2

HAL Id: hal-03009034
https://hal.science/hal-03009034v2
Preprint submitted on 27 Nov 2020

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Large-time asymptotics of the wave fronts length I
The Euclidean disk

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November 27, 2020

In the paper [Vi-20], the second author proves that the length $|S_t|$ of the wave front $S_t$ at time $t$ of a wave propagating in an Euclidean disk $\mathbb{D}$ of radius 1, starting from a source $q$, admits a linear asymptotics as $t \to +\infty$: $|S_t| = \lambda(q) t + o(t)$ with $\lambda(q) = 2 \arcsin a$ and $a = d(0, q)$. We will give a more direct proof and compute the oscillating corrections to this linear asymptotics. The proof is based on the “stationary phase” approximation.

1 Wave fronts

Let us consider a 2D-Riemannian compact manifold $(X, g)$ possibly with a smooth convex boundary. We denote by $g^*: T^*X \to \mathbb{R}$ the half of the dual metric which is the Hamiltonian of the geodesic flow.

We denote by $\pi_X$ the canonical projection of $T^*X$ onto $X$ and $\phi_t : T^*X \to T^*X$, $t \in \mathbb{R}$ the Hamiltonian flow of $g^*$ which is the geodesic flow. If $X$ has a non empty boundary, we define $\phi_t$ using the law of reflection. Let $q \in X$ be given. For any $t > 0$, we define the wave front $S_t$ at time $t$ as the set of
points of $X$ of the form $\pi_X(\phi_t(\Sigma^q))$ where $\Sigma^q := \{(q, \xi) \in T^*X|g^*(q, \xi) = 1\}$. The set $S_t$ could also be defined as the image by the exponential map at $q$ of the circle $\Sigma_t$ of radius $t$ in the tangent space $T_qX$.

Let us define the length of $S_t$ and denote it by $|S_t|$. The wave front $S_t$ is a curve parametrized by a circle: $S_t := \exp_q(\Sigma_t)$. This allows to define its length using the Riemannian metric. Note that $S_t$ can admit some singular points. The length of the corresponding part vanishes and the remaining part is an immersed co-oriented curve with only transversal self-intersections.

In this article, we focus on the case where $X$ is the unit disk in $\mathbb{R}^2$ and $g$ is the Euclidean metric. In this context, we will prove that the following expansion holds:

$$|S_t| = 2\alpha_0 t + t \sum_{n=0}^{\infty} J_n^{\text{approx}}(t) + O(1)$$

as $t \to +\infty$, with

$$J_n^{\text{approx}}(t) = \frac{-8\sqrt{2}}{\pi^2(2n+1)^{5/2}} \cos((2n+1)\pi a) \cos\left(\pi \left((2n+1)t + \frac{1}{4}\right)\right)$$

where $a$ is the distance from the point $q$ to the center of the disk.

The case of closed surfaces with integrable geodesic flows will be the subject of [CV-20].

2 Numerics

In this section, we will compare the expansion given by (1) with the numerical calculations. We introduce a (small) time step $\delta_t > 0$, a (large) number of points $n$ which compose the wave front, two vectors $M$ and $V$ in $(\mathbb{R}^2)^n$ such that, for any $k \in [1, n]$, $X_k \in \mathbb{R}^2$ represents the position and $V_k \in \mathbb{R}^2$ the speed of the $k$th point of the wave front at a given time. We fix $a \in ]0, 1[$ such that $(a, 0)$ are the coordinates of the source $q$. Thus, we introduce the
following iterative scheme

**initialization:**
\[ M \leftarrow ((a,0), \ldots, (a,0)) \in (\mathbb{R}^2)^n, \]
for any \( k \in [1, n], V_k \leftarrow (\cos \left( \frac{2k\pi}{n} \right), \sin \left( \frac{2k\pi}{n} \right)) , \]

**iterative step:**
\[ \tilde{M} \leftarrow M + \delta_t V, \]
for any \( k \in [1, n], \)
\[ \begin{align*}
  \text{if } \tilde{M}_k \in \mathbb{D} & \text{ then } M_k \leftarrow \tilde{M}_k, \\
  \text{else } & \text{ compute } \delta_t^k \text{ s.t. } \| M_k + \delta_t^k V_k \| = 1 \text{ and } \delta_t^k \geq 0, \\
  & M_k \leftarrow M_k + \delta_t^k V_k, \\
  & V_k \leftarrow V_k - 2 \left( V_k \| M_k \| \right) \left( \frac{M_k}{\| M_k \|} \right), \\
  & M_k \leftarrow M_k + (\delta_t - \delta_t^k) V_k.
\end{align*} \]

The iterative loop consists in the computation of a linear motion outside the boundary and at the boundary one applies the familiar law *the angle of incidence equals the angle of reflection*. After \( p \) iterations, \( M \) represents the points of the wave front (see Figure 1 and Videos 1).

![Wave Front for \( a = 0.5 \) and \( t \in \{0.5, 10, 20, 50\} \)](image)

Figure 1: Wave Front for \( a = 0.5 \) and \( t \in \{0.5, 10, 20, 50\} \)

First, we can observe that \( |S_t| \) admits a linear asymptotic as \( t \) grows to \(+\infty\). Then, the oscillations are of period 2 with a phase independent of \( a \) (see Figure 2). One may remark the following points.

1. For \( a = 0 \), the family of curves \( (S_t)_t \) are concentric circles and \( |S_t| \) is of period 2.

2. For \( a = 0.5 \), the terms \( J_k^{\text{approx}}(t) \) vanish for any \( t \) and, in this case, this expansion is not able to capture the oscillating part of \( t \mapsto |S_t| \).

\[ \text{https://www.youtube.com/channel/UCMTvpxuhYwbYBYDErS1UI0EA/} \]
3. The terms $|J_k^{\text{approx}}(t)|$ are bounded by $Ck^{-5/2}t^{-1/2}$, where $C$ is a constant. For $t$ fixed, this ensures the (fast) convergence of the serie $\sum_k J_k^{\text{approx}}(t)$ and then the amplitude of $t \mapsto t \sum_{k \in \mathbb{N}} J_k^{\text{approx}}(t)$ is of order $t^{1/2}$ (see Figure 2).

3 A short proof of the Arcsinus formula

In the paper [Vi-20], the author was able to prove by elementary calculations the

\begin{align*}
\text{Theorem 3.1} & \quad \text{If } X \text{ is the unit disk, } |S_t| = \lambda(q)t + o(t) \text{ as } t \to +\infty \text{ with radius 1 with } \lambda(q) = 2 \arcsin a \text{ where } a \text{ is the distance from } q \text{ to the center of the disk.}
\end{align*}

We will reprove it using tools which will be extended to integrable geodesic flows in a forthcoming paper. For this, we will prove an integral formula:

\begin{align*}
\text{Theorem 3.2} & \quad \text{Let } \psi \text{ be the function periodic of period 1 whose restriction to } [0, 1] \text{ is given by } \psi(\theta) = |2\theta - 1|. \text{ We have}
\end{align*}

\begin{align*}
|S_t| = t \Sigma_{\pm} \int_{I_{a,q}} \psi \left( \theta_{2,q}(\xi) - \frac{t}{2 \sin \xi} \right) d\xi + O(1) \quad (2)
\end{align*}
as \( t \rightarrow +\infty \), where \( I_{\alpha_0} := [\pi/2 - \alpha_0, \pi/2 + \alpha_0] \), \( \alpha_0 = \arcsin a \) and

\[
\theta_{2\pm}^\pm(\xi) = \frac{1}{2} \pm \frac{\sqrt{a^2 - \cos^2 \xi}}{2 \sin \xi}.
\]

This integral can also be written as an integral over \( T \):

\[
|S_t| = t \int_T \psi \left( \frac{1}{2} - \frac{a \cos \alpha + t}{2 \sqrt{1 - a^2 \sin^2 \alpha}} \right) \frac{a \cos \alpha}{\sqrt{1 - a^2 \sin^2 \alpha}} d\alpha + O(1) \quad (3)
\]

as \( t \rightarrow +\infty \).

Let us show how Theorem 3.1 follows from Theorem 3.2. We consider an integral

\[
I(t) = \int_{I_{\alpha_0}} \psi \left( \theta(\xi) - \frac{t}{2 \sin \xi} \right) d\xi
\]
with $\theta$ smooth. We first approximate uniformly $\psi$ by a sequence of trigonometric polynomials $\psi_N(u) = \sum_{|n| \leq N} a_n \exp(2i\pi nu)$ with $a_0 = \int_0^1 \psi(\theta) d\theta = \frac{1}{2}$.

This way we get

$$I_N(t) = 2\alpha_0 + \sum_{|n| \leq N, \ n \neq 0} a_n \int_{I_{\alpha_0}} e^{2i\pi n\theta(\xi)} e^{-2i\pi nt/\sin \xi} d\xi$$

It follows from the stationary phase approximations that all these integrals tend to 0 as $t \to \infty$, Theorem 3.1 follows.

**Proof of Theorem 3.2** –

We will first parametrize the dynamics using angle coordinates on tori. Let us denote by $m(s) = (\cos s, \sin s)$ on the circle and by $\vec{u}_s$ the vector $\vec{0} \to \vec{m}(s)$. Let us introduce a set of coordinates. In what follows, we parametrize the 2D-submanifold of the phase space consisting of oriented chords joining a point $m(s)$ to $m(s + 2\xi)$ with speed 1 by $\xi \in ]0, \pi[$. Changing the orientation of the chords moves $\xi$ into $\pi - \xi$. For $\xi \in ]0, \pi[ \ and \ r \in [0, 2 \sin \xi]$, we define $F_\xi(s, r) = m(s) + r\vec{u}_s + \xi + \pi/2$. This describes the chord $C_\xi$ between $m(s)$ and $m(s + 2\xi)$. The function $F_\xi$ is extended as a function on $\mathbb{R}^2$ periodic with respect to the lattice $L_\xi$ spanned by the vectors $(2\pi, 0)$ and $(2\xi, -2\sin \xi)$.

The function $F_\xi$ is continuous, but only piecewise smooth. The pull-back under $F_\xi$ on $\mathbb{R}^2$ of the billiard dynamics is generated by the vector $\partial_r$.

The coordinates $(s, r)$ range over a torus $\mathbb{R}^2/L_\xi$. In order to continue the computation, we need to fix the lattice $Z^2$. For that we introduce the linear map $M_\xi : \mathbb{R}_{\theta_1, \theta_2}^2 \to \mathbb{R}_{s, r}^2$, sending the canonical basis of $Z^2$ onto the previous basis of $L_\xi$. The dynamics on the torus $\mathbb{R}^2/L_\xi$ is the image of $\partial_r$ under $M_\xi^{-1}$; let us denote it by $V$.

$$V = \frac{1}{2\pi \sin \xi} (\xi \partial_{\theta_1} - \pi \partial_{\theta_2})$$

Then, we need to compute the Euclidean norm of $F'_\xi(M_\xi(\partial_r V))$. We have

$$\partial_\xi V = \frac{-\cos \xi}{2\pi \sin^2 \xi} (\xi \partial_{\theta_1} - \pi \partial_{\theta_2}) + \frac{1}{2\pi \sin \xi} \partial_{\theta_1}$$

Hence

$$M_\xi(\partial_\xi V) = \frac{1}{\sin \xi} (-\cos \xi \partial_r + \partial_s)$$

Then

$$F'_\xi(\partial_r) = \vec{u}_{s+\xi+\pi/2}, \ F'_\xi(\partial_s) = \vec{u}_{s+\pi/2} - r\vec{u}_{s+\xi}$$
This gives
\[
\|\partial_\xi V\| = \frac{|r - \sin \xi|}{\sin \xi}
\]
As could have been anticipated, this length vanishes on the caustic! We now take the pull back of \(\|\partial_\xi V\|\) under \(M_\xi\) and get \(|2\theta_2 - 1|\).

Let us parametrize the chords starting from \(q\) by the angle \(\alpha \in \mathbb{T}\) defined by \(\alpha := \langle q, C_\xi \rangle\). We get \(\cos \xi = a \sin \alpha\). Hence \(\xi\) is the smooth function \(\xi(\alpha) = \arccos(a \sin \alpha)\). The length \(|S_t|\) is given by
\[
|S_t| = \int_\mathbb{T} \| \frac{d}{d\alpha}(\phi_t(\tilde{u}_\alpha)) \| d\alpha
\]
where \(\phi_t\) is the geodesic flow. Let us denote by \(\theta(\alpha)\) the coordinates of \(q\) in \(\mathbb{T}_2^\theta\). We get, using the parametrization of the flow on the tori \(\mathbb{T}_\theta\),
\[
|S_t| = \int_\mathbb{T} \| (F'_\xi \circ M_\xi)_{\theta(\alpha)+tV(\alpha)}(\theta'(\alpha)+tV'(\alpha)) \| d\alpha,
\]
\[
= t \int_\mathbb{T} \| (F'_\xi \circ M_\xi)_{\theta(\alpha)+tV(\alpha)}(V'(\alpha)) \| d\alpha + O(1)
\]
as \(t \to +\infty\). We rewrite the integral in terms of \(\xi\), using \(\cos \xi = a \sin \alpha\) and \(\theta_2(\xi) = \frac{1}{2} \pm \sqrt{\frac{a^2 - \cos^2 \xi}{2 \sin \xi}}\) with + if \(\alpha \in [\pi/2, 3\pi/2]\) and − otherwise. From this follows the result.

4 Local asymptotics of the length

In this section, we describe the asymptotics of the length of the intersection of the wave front with a smooth domain \(K\) included in the disk \(\mathbb{D}\). We have

**Theorem 4.1** We have
\[
l(S_t \cap K) \sim \frac{2t}{\pi} \int_K \Psi(\sqrt{x^2 + y^2}) |dxdy|
\]
as \(t \to +\infty\), where
\[
\Psi(r) = \frac{\min(r, a)}{\sqrt{1 - \min(r, a)^2}}
\]
Note that the function $\Psi$ is continuous, vanishes at $r = 0$ and is constant for $a \leq r \leq 1$. This implies that the density of the wave front is smaller near the center of the disk.

**Proof.** Let $\phi \in C(\mathbb{D}, \mathbb{R}^+)$, we want to calculate the asymptotics of the length $|S_{t,\phi}|$ of $S_t$ computed in the metric $\phi^2$Eucl. Following the proof of Theorem 3.1 we get $|S_{t,\phi}|/t \to \lambda(q, \phi)$ as $t \to +\infty$, with

$$\lambda(q, \phi) = 2 \int_{I_{\alpha_0}} \int_{T^2} |2\theta_2 - 1|/\phi \circ G(\theta, \xi)d\xi d\theta$$

with $G(\theta, \xi) = F_\xi \circ M_\xi(\theta)$. We will first make the change of variable $(\theta, \xi) \to (s, r, \xi)$ whose Jacobian is $4\pi \sin \xi$. This gives

$$\lambda(q, \phi) = 1/2\pi \int_{I_{\alpha_0}} \int_{\mathbb{R}^2/L_\xi} \left| \frac{r - \sin \xi}{\sin^2 \xi} \right| \phi \circ F_\xi(s, r)d\xi ds dr$$

Finally, we pass from $(s, r)$ to $(x, y)$. We have $|dxdy| = |r - \sin \xi|/ds dr$. The domain of integration is $\rho = \sqrt{x^2 + y^2} \geq \cos \xi$ which is covered twice by the torus $\mathbb{R}^2/L_\xi$, we get hence

$$\lambda(q, \phi) = 1/\pi \int_{I_{\alpha_0}} \int_{\cos \xi \leq \rho} \frac{1}{\sin^2 \xi} \phi(x, y)d\xi dxdy$$

An elementary calculus gives then

$$\lambda(q, \phi) = 2/\pi \int_{\mathbb{D}} \Psi(\rho)\phi(x, y)dxdy$$

The result follows then by approximating the characteristic function of $K$ by continuous functions. $\square$

## 5 Oscillations of the length

The numerical computations of the second author in [Vi-20] show clearly some regular oscillations of the length $|S_t|$ around the linear asymptotics. These oscillations are given in the

**Theorem 5.1** The following expansion holds:

$$|S_t| = 2\alpha_0 t + t \sum_{n=0}^{\infty} J_n^{\text{approx}}(t) + O(1)$$
as $t \to +\infty$, with

$$J_n^{\text{approx}}(t) = \frac{-8\sqrt{2}}{\pi^2(2n+1)^{5/2}} \cos ((2n+1)\pi a) \cos \left( \pi \left( (2n+1)t + \frac{1}{4} \right) \right)$$

The oscillations have an amplitude of the order of $\sqrt{t}$, are periodic of period 2. If $a = \frac{1}{2}$, we get $|S_t| = \pi t/3 + O(1)$ as $t \to +\infty$.

We start from the formula given by Equation (2):

$$|S_t| = t \Sigma_{\pm} \int_{I_{\alpha_0}} \psi \left( \theta_{2}^{\pm,q}(\xi) - \frac{t}{2 \sin \xi} \right) d\xi + O(1)$$

as $t \to +\infty$, where $I_{\alpha_0} := [\pi/2 - \alpha_0, \pi/2 + \alpha_0]$ and $\psi$ restricted to $[0,1]$ is given by $\psi(\theta) = |2\theta - 1|$ and $\psi$ is periodic of period 1. We have

$$\theta_{2}^{\pm,q}(\xi) = \frac{1}{2} \pm \frac{\sqrt{a^2 - \cos^2 \xi}}{2 \sin \xi}$$

The idea is to start with the Fourier expansion of $\psi$ and then to apply the stationary phase asymptotics.

We have

$$\psi(\theta) = \frac{1}{2} + \sum_{n \in \mathbb{Z}} \frac{2}{\pi^2(2n+1)^2} e^{2(2n+1)i\pi \theta}$$

We need to evaluate the integrals

$$I_n(t) = \frac{-4}{((2n+1)\pi)^2} \int_{I_{\alpha_0}} \cos \left( (2n+1)\pi \frac{\sqrt{a^2 - \cos^2 \xi}}{\sin \xi} \right) e^{-i\pi(2n+1)^{\frac{1}{2}}t} d\xi$$

and then we have

$$|S_t| = 2\alpha_0 t + t \sum_{n \in \mathbb{Z}} I_n(t) + O(1)$$

as $t \to +\infty$. Note first that the function $\cos \left( (2n+1)\pi \frac{\sqrt{a^2 - \cos^2 \xi}}{\sin \xi} \right)$ is smooth on $I_{\alpha_0}$ with a non vanishing derivative at the boundaries. The non vanishing contributions come from the critical point $\xi = \pi/2$ and the boundaries of $I_{\alpha_0}$. The boundary contributions are $O(1/t)$. They contribute to the $O(1)$ remainder. The contribution of the critical point can be calculated using the
formula (4). We get an asymptotic for $J_n = I_n + I_{n-1}$, $n = 0, 1, \cdots$ given by

$$J_n(t) \sim J_n^{\text{approx}} = \frac{-8\sqrt{2}}{\pi^2(2n+1)^{5/2}\sqrt{t}} \cos((2n+1)\pi a) \cos\left(\pi \left((2n+1)t + \frac{1}{4}\right)\right)$$

The previous calculation is only formal. We need to control the remainder terms in a uniform way with respect to $n$. Let us rewrite the integral $I_n$ as combination of integrals of the form

$$\int e^{-i\pi(2n+1)t} \left(\frac{1}{\sin\xi} - \frac{1}{t} \frac{\sqrt{a^2 - \cos^2 \xi}}{\sin\xi}\right) d\xi$$

and apply the stationary phase with the phase functions depending on $t$: $\Phi_t(\xi) = \frac{1}{\sin\xi} - \frac{1}{t} \frac{\sqrt{a^2 - \cos^2 \xi}}{\sin\xi}$. This phase function is non degenerate and converges in $C^\infty$ topology to $\frac{1}{\sin\xi}$ as $t \to \infty$. Hence the remainder is $O((nt)^{-3/2})$ as $t \to +\infty$, uniformly with respect to $n$.

### A Stationary phase

For this section, we refer the reader to [GS-77], chap. 1.

We want to evaluate the asymptotics as $t \to +\infty$ of integrals of the form

$$I(t) := \int_T e^{itS(x)}a(x)dx$$

where $S$ is a real valued smooth function. We assume that the critical points of $S$, ie the zeroes of $S'$, are non degenerate, ie $S''(x) \neq 0$. We will first assume that $a \in C_0^\infty(\mathbb{R})$ with only one critical point $x = 0$ in the support of $a$. Then $I(t)$ admits a full asymptotic expansion given by

$$I(t) = \frac{\sqrt{2\pi}e^{i\pi/4}}{|tS''(0)|^{1/2}} e^{itS(0)} (a(0) + O(t)) \quad (4)$$

as $t \to +\infty$, with $\varepsilon = \pm 1$ depending on the sign of $S''(0)$. We will need some uniform estimates in the remainder term. This is provided by the following

**Proposition A.1** Let us consider the integrals

$$I(t; S, a) := \int_T e^{itS(x)}a(x)dx$$
Let $S_0$ be a smooth real valued Morse function and $a_0$ be a smooth function. Let $S_\lambda$ and $a_\lambda$ be smoothly dependent of a real parameter $\lambda$. Then, for $\lambda$ small enough,

$$I(t; S_\lambda, a_\lambda) := I_{\text{asymp}}(t, \lambda) + O(t^{-3/2})$$

as $t \to +\infty$, where $O$ is uniform and $I_{\text{asymp}}(t, \lambda)$ is the sum of terms given by the formula (4) for all critical points of $S_\lambda$.

If $\lambda$ is small enough, $S_\lambda$ is still a Morse function. We localize the integrals near the critical points and apply the Morse Lemma with parameters. We are then reduced locally to the case where $S_\lambda(x) = \pm x^2$. We apply then any proof of the stationary phase approximation.

It will also be useful to consider the case of an integral on a closed interval $[c, d]$ with $c < d$.

$$I(t) := \int_c^d e^{itS(x)} a(x) dx$$

Assuming that $S'$ does not vanish on the support of $a$ and that $a$ is $C^1$, we have

$$I(t) = \frac{1}{it} \left( a(d)e^{itS(d)} - a(c)e^{itS(c)} + O(t) \right)$$

as $t \to +\infty$.

Note that in both asymptotic formulae, the remainders “$O(t)$” are uniform if $S'$ (resp. $a'$) is close to $S$ (resp. close to $a$) in $C^2$ topology.

The previous asymptotics extend to higher dimensional integrals.

**References**

[CV-20] Yves Colin de Verdière. *Large time asymptotics of the wave fronts length II: surfaces with an integrable Hamiltonian.* In preparation (2020?).

[GS-77] Victor Guillemin & Shlomo Sternberg. *Geometric asymptotics.* AMS (1977).

[Ta-05] Serge Tabachnikov. *Geometry and Billiards.* AMS (2005).

[Vi-20] David Vicente. *Une goutte d’eau dans un bol.* Quadrature 117:13–22, 45 (2020).