Noncommutative spacetime realized in $AdS_{n+1}$ space: Nonlocal field theory out of noncommutative spacetime

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In $\kappa$-Minkowski spacetime, the coordinates are Lie algebraic elements such that time and space coordinates do not commute, whereas space coordinates commute with each other. The noncommutativity is proportional to a Planck-length-scale constant $\kappa^{-1}$, which is a universal constant other than the velocity of light, under the $\kappa$-Poincaré transformation. In this sense, the spacetime has a structure called “doubly special relativity.” Such a noncommutative structure is known to be realized by $SO(1, 4)$ generators in 4-dimensional de Sitter space. In this paper, we try to construct a noncommutative spacetime having a commutative $n$-dimensional Minkowski spacetime based on $AdS_{n+1}$ space with $SO(2, n)$ symmetry. We also study an invariant wave equation corresponding to the first Casimir invariant of this symmetry as a nonlocal field equation expected to yield finite loop amplitudes.

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1. Introduction

The $\kappa$-Minkowski spacetime is a noncommutative spacetime characterized by an algebraic structure with a constant $\kappa$ other than the light velocity; in this sense, the framework of $\kappa$-Minkowski spacetime is called “doubly special relativity” (DSR) [1–3]. $\kappa$ is a Planck-energy-scale constant, which is usually said to be a trace of quantum gravity through the combination $\kappa \sim \sqrt{G/\Lambda}$ (or $\sqrt{\hbar/G}$) in some limit of $G, \Lambda \to \infty$ [4]. Here, $G$ and $\Lambda$ are respectively the gravitational constant and the cosmological constant.

Associated with this dimensional constant, the coordinates of the $\kappa$-Minkowski spacetime form the Lie algebra characterized by

\[ [\hat{x}_i, \hat{x}_j] = 0, \]
\[ [\hat{x}_0, \hat{x}_i] = -i\kappa^{-1} \hat{x}_i, \]

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where \( i \) runs over \((1, 2, 3)\). The characteristic relations (1) and (2) spoil the symmetry under the Lorentz boost in the Minkowski spacetime, although they are symmetric under space rotation. Nevertheless, this framework is symmetric under the \( \kappa \)-Poincaré transformation, which reduces to the Lorentz transformation according as \( \kappa \to \infty \).

Historically, another Lie algebra type of noncommutative spacetime was first discussed in 1947 by H. S. Snyder [5, 6] in the context of nonlocal field theory extended in spacetime with the fundamental length \( l \sim \kappa^{-1} \). Although the algebraic structure of Snyder’s spacetime is slightly different from that of the \( \kappa \)-Minkowski spacetime, the symmetry under the Lorentz boost is broken in this spacetime too. In both types of noncommutative spacetime, the dispersion relation of particles embedded in this spacetime becomes highly nonlinear due to \( \kappa \neq 0 \). As a result of those nonlinear dispersion relations, the wave equations of particles possess nonlocal structure by regarding those equations as effective in the usual commutative spacetime.

On the other hand, it is well known that H. Yukawa proposed a nonlocal field theory [7–10], the bilocale field theory, in the same period as Snyder’s noncommutative spacetime theory appeared. Yukawa’s proposal is motivated by a unified description of elementary particles and to get divergence-free field theories by introducing a fundamental length in spacetime. After a long history following this line of thought, Yukawa arrived at the field theory of elementary domains [11, 12], which obey a difference equation instead of a differential equation. The equation of a domain keeps the Lorentz invariance, but it is not consistent with the causality, since the field equation allows time-like extension of fields.

Although the field theories based on \( \kappa \)-Minkowski spacetime and domains like nonlocal field theories are standing on different bases of thought, they look as if they have a close connection to each other. The purpose of this paper is, thus, to study the relationship between a domain type of field theory and a nonlocal field theory based on a \( \kappa \)-Minkowski-like spacetime, which is modified so as to be symmetric under the Lorentz boost.

In the next section, we formulate the \( \kappa \)-Minkowski spacetime from the viewpoint regarding noncommutative coordinates as \( SO(1, 4) \) generators in \( dS_4 \) space, the 4-dimensional de Sitter space. As an extension of this formulation, in Sect. 3, we discuss a modified noncommutative spacetime realized in \( AdS_{n+1} \) space so that the framework is symmetric under the Lorentz transformation. In Sect. 4, we discuss a wave equation of a nonlocal field characterized by the first Casimir invariant in this space, and a detailed analysis of on-mass-shell particles is given. In Sect. 5, discussions on \( \phi^3 \) types of interaction of such a field are presented. Therein, we show that some loop diagrams become finite due to the nonunitary structure of such a field in the energy scale of \( \kappa \).

Section 6 is the discussion and summary. In Appendices A and B, the mathematical background of Sects. 2 and 3 is summarized.

2. \( \kappa \)-Minkowski spacetime based on \( dS_4 \)

A simple way to construct the Lie algebraic coordinates (1) and (2) is to start from \( dS_4 \) with coordinates \( y = (y^A) = (y^0, y^i, y^4) \) \((i = 1, 2, 3)\) characterized by

\[
y^2 = g_{AB} y^A y^B = (y^0)^2 - (y^1)^2 - (y^2)^2 - (y^3)^2 - (y^4)^2 = -\kappa^2. \tag{3}
\]

In terms of these coordinates, the generators of the \( SO(1, 4) \) isometric group can be written as

\[
M_{AB} = i\kappa^{-1}(y_A \partial_B - y_B \partial_A), \quad \partial_A = \frac{\partial}{\partial y^A}, \tag{4}
\]

from which one can verify that

\[
[M_{AB}, M_{CD}] = i\kappa^{-1}(g_{BC} M_{AD} + g_{AD} M_{BC} - g_{AC} M_{BD} - g_{BD} M_{AC}). \tag{5}
\]
It is obvious that there are ten independent components of these generators; that is, the generators of space rotation \( \{ M_{ij} \} \), the Lorentz boost \( \{ M_{i0} \} \), and remaining four generators \( \{ M_{i4} \} \), \((\mu) = (0, i))\). The noncommutative coordinates \( \hat{x}^0 \) and \( \hat{x}^i \) are constructed out of those remaining generators by
\[
\hat{x}_0 = M_{04},
\]
\[
\hat{x}_i = M_{i0} + M_{i4}.
\]

Here, the inverse sign \( g_{00} = -g_{44} \) in the metric is essential to realize the commutation relations (1) and (2).\(^1\) For the latter purpose, it is convenient to introduce light-cone variables between \((y^0, y^4)\); that is, we put \((y^A) = (y^i, y^+, y^-) \), \((A = i, \pm)\), where \( y^\pm = y^0 \pm y^4 \). Then we can write the invariant length in \( dS_4 \) as
\[
y^2 = \bar{g}_{AB} y^A y^B = -(y^i)^2 + y^+ y^- \quad \text{with the metric}
\]
\[
(\bar{g}_{AB}) = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & \frac{1}{2} \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

In this basis, the noncommutative coordinates can be expressed as
\[
\hat{x}_i = 2i\kappa^{-1}(y_i \partial_+ - y_+ \partial_i) = 2M_{i+},
\]
\[
\hat{x}_0 = 2i\kappa^{-1}(y_- \partial_+ - y_+ \partial_-) = 2M_{-+}.
\]

In order to find the invariant wave equation in the \( \kappa \)-Minkowski spacetime, let us consider the unitary transformation \( U(\omega) = e^{\frac{i}{2}\kappa \omega^{AB} M_{AB}} \), which causes a finite \( SO(1, 4) \) transformation in \( dS_4 \) in such a way that \( y^A = U(\omega) y^A U^\dagger(\omega) = (\kappa^{-1}\omega)^A y^B \) with \( \omega = (\omega^A_B) \). In particular, for the contracted transformation defined by \( \omega^{BC} = a^B b^C - b^B a^C \) with \((b^i, b^-, b^+) = (0, 2, 0)\), the exponent of \( U(\omega) \) reduces to \( \frac{1}{2}\kappa \omega^{BC} M_{BC} = a^- x^0 - a^i \hat{x}^i \); namely, \( U(\omega) \) becomes an exponential function in the \( \kappa \)-Minkowski spacetime in this contracted case. Furthermore, \( a^i \) and \( a^- \) are related to four-momenta conjugate to \{\( \hat{x}^\mu \)\} in some sense, which will be discussed below.

Now, \( SO(1, 4) \) transformation of a c-number vector \( u = (u^A) \), \((u^2 = -\kappa^2)\) in \( dS_4 \) is defined by \( U(\omega)(u \cdot y) U(\omega)^\dagger = ((\kappa^{-1}\omega)^A B)^u y_B = u(\omega) \cdot y \), so that \( u(\omega)^2 = -\kappa^2 \) holds. If we choose, for sake of simplicity, \( u = (u^0, u^i, u^4) = (0, 0, 0, 0, \kappa) \), then \( u(\omega)^A = \kappa (\kappa^{-1}\omega)^A x^0 \) becomes a nonlinear realization of a vector in \( dS_4 \) in terms of \((a^i, a^-)\). Here, a little calculation leads to the explicit form of \( e^{-\kappa^{-1}\omega} \) such that (Appendix A)
\[
e^{-\kappa^{-1}\omega} = 1 + \frac{1}{(\kappa a^-)^2} \left\{ \cosh(\kappa^{-1} a^-) - 1 \right\} \omega^2 - \frac{1}{a^-} \sinh(\kappa^{-1} a^-) \omega.
\]

\(^1\) By definition, \( \hat{x}^i \) is a three-vector under the rotation by \{\( M_{ij} \)\}; but it is not any three-vector under the Lorentz boost by \{\( M_{i0} \)\}. To make clear the meaning of \( \hat{x}^\mu \), let us put \( y_4 = +\sqrt{\gamma_0} y^i + \kappa^2 \) after calculating the commutation relations \([\hat{x}^\mu, y_4]\); then we obtain
\[
[\hat{x}_0, y_0] = -i\kappa^{-1} y_4 \rightarrow -i, \; (\kappa \rightarrow \infty),
\]
\[
[\hat{x}_i, y_j] = i\kappa^{-1} g_{ij} (-y_0 - y_4) \rightarrow -ig_{ij}, \; (\kappa \rightarrow \infty).
\]

Thus, \( \hat{x}^\mu \) tends to the \( p \)-representation of coordinates in flat Minkowski spacetime in the limit \( \kappa \rightarrow \infty \). In other words, \{\( y^\mu \)\} is the momentum space in flat Minkowski spacetime.
By taking \((\omega_{04}, \omega_{44}) = (-a^-, -a_i, 0)\) and \((\omega_{04}^2, \omega_{14}^2, \omega_{44}^2) = (-a^2, -a^- a_i, -(a^-)^2 + a^2)\) into account, we thus arrive at the expression
\[
\tilde{u}_i(\omega) = (1 - e^{-\kappa_0 - a^-}) \frac{a_i}{a^-} \tag{12}
\]
\[
\tilde{u}_0(\omega) = -\left\{ \cosh(\kappa_0 - a^-) - 1 \right\} \left( \frac{a}{a^-} \right)^2 + \sinh(\kappa_0 - a^-), \tag{13}
\]
\[
\tilde{u}_4(\omega) = -\cosh(\kappa_0 - a^-) + \left\{ \cosh(\kappa_0 - a^-) - 1 \right\} \left( \frac{a}{a^-} \right)^2, \tag{14}
\]
where we have written \(\tilde{u}_A(\omega) = \kappa_0 - a_0 U_0(\omega)\); henceforth, we use the same notation \(\tilde{f} = \kappa_0 - f\) for any \(f\). Further, we note that the vector \((\tilde{u}^A)\) is usually introduced in relation to the bicovariant differentials of \(U(\omega)\), since \(\tilde{u}(\omega)^A = (e^{-\kappa_0 \omega}) A_4\) satisfies (Appendix A)
\[
dU(\omega) = i \kappa \left\{ dx_0 \tilde{u}_\mu^\mu + dx_A(\tilde{u}_4 - 1) \right\} U(\omega). \tag{15}
\]

The next task is to identify \(U(\omega)\) with an ordered exponential function \(e^{-ik^0k^0 + ik^i \xi^i}\) in some way; the typical cases are
\[
\hat{e}(k) = e^{-i(k^0 \xi^0) + ik^i \xi^i} = e^{-i(k_0^0 \xi^0) + ik^i \xi^i} = e^{i(k_0^0 \xi^0) - ik^i \xi^i}, \tag{16}
\]
\[
\hat{e}_R(k) = e^{i(k_0^0 \xi^0) - ik^i \xi^i} = e^{-i(k_0^0 \xi^0) - ik^i \xi^i}, \tag{17}
\]
\[
\hat{e}_L(k) = e^{-i(k_0^0 \xi^0) + ik^i \xi^i} = e^{i(k_0^0 \xi^0) - ik^i \xi^i}, \tag{18}
\]
\[
\hat{e}_S(k) = e^{i(k_0^0 \xi^0) - ik^i \xi^i} = e^{-i(k_0^0 \xi^0 - k^i \xi^i)}, \tag{19}
\]
where
\[
k_R = (k_0^0, k_0^i) = (k^0, \frac{\kappa_0}{1 - e^{-k_0^0} k^i}), \tag{20}
\]
\[
k_L = (k_0^0, k_0^i) = (k^0, \frac{\kappa_0}{e^{k_0^0} - 1} k^i), \tag{21}
\]
\[
k_S = (k_0^0, k_0^i) = (k^0, \frac{\kappa_0}{e^{k_0^0/2} - e^{-k_0^0/2}} k^i). \tag{22}
\]

For example, if we put \(U(\omega) = \hat{e}_R(k)\), the right ordering of the exponential function, we get the well-known expression of the vector in \(dS_4\) such that\(^2\)
\[
\tilde{u}^i_k(k) = -e^{k_0^0 \kappa_0}, \tag{23}
\]
\(^2\)Similarly, the substitutions \((a^-, a^i) \rightarrow -k_L/S\) give the other expressions of vectors in \(dS_4\):
\[
(a^-, a^i) \rightarrow -k_L, \quad \tilde{u}^i_L(k) = -e^{k_0^0 \kappa_0}, \tag{24}
\]
\[
(a^-, a^i) \rightarrow -k_S, \quad \tilde{u}^i_S(k) = -e^{\frac{1}{2} k_0^0 \kappa_0}, \tag{25}
\]
\[
\tilde{u}^0_L(k) = 2e^{k_0^0} k^0 - \sinh(k^0), \quad \tilde{u}^0_S(k) = -2 e^{k_0^0} k^0 - \sinh(k^0), \tag{26}
\]
\[
\tilde{u}^4_L(k) = \cosh(k^0) - \frac{1}{2} e^{-k_0^0} k^2, \quad \tilde{u}^4_S(k) = \cosh(k^0) - \frac{1}{2} e^{k_0^0} k^2. \tag{27}
\]
\[ \tilde{u}^0(k)_R = -\frac{1}{2} e^{\tilde{k}_0} \tilde{k}^2 - \sinh(\tilde{k}_0), \]
\[ \tilde{u}^4(k)_R = \cosh(\tilde{k}_0) - \frac{1}{2} e^{\tilde{k}_0} \tilde{k}^2. \]

Therefore, if we put \( P^A(k) = -\kappa \tilde{u}^A(k)_R \), then Eq. (15) can be read as
\[ d\hat{e}_R(k) = -i \left\{ dx_\mu P^\mu(k) + dx_4(P^4(k) - \kappa) \right\} \hat{e}_R(k), \]
where \( P(k)^A \) and \( A = \mu, 4 \) are five-momenta satisfying \( P(k)^A P(k)_A = -\kappa^2 \).

Finally, we discuss the \( SO(1, 3) \) transformation realized in \( \{ \tilde{u}(k)^\mu \} \) through the transformation of \( \{ k^\mu \} \); then, their resultant form should be
\[ \hat{L}_{\mu\nu} \tilde{u}_\rho = (g_{\mu\rho} \tilde{u}_\nu - g_{\nu\rho} \tilde{u}_\mu), \quad \hat{L}_{\mu\nu} \tilde{u}_4 = 0, \]
so that the constraint \( \tilde{A}^A \tilde{a}_A = -1 \) holds. The \( \hat{L}_{\mu\nu} \)s are actions on \( k^\mu \) causing nonlinear transformations in general. The space rotation is simple, since \( \tilde{u}(k)^0 \) and \( \tilde{u}(k)^i \) transform respectively as a scalar and a vector under the rotation of \( \{ k^i \} \); then, the action of \( \hat{L}_{ij} \) on \( \tilde{k}_\mu \) becomes, as usual,
\[ \hat{L}_{ij} \tilde{k}_0 = 0, \quad \hat{L}_{ij} \tilde{k}_l = -(\delta_{il} \tilde{k}_j - \delta_{jl} \tilde{k}_i). \]

The Lorentz boost is somewhat difficult, and is given by (Appendix B)
\[ \hat{L}_{i0} \tilde{k}_0 = -\tilde{k}_i, \]
\[ \hat{L}_{i0} \tilde{k}_j = -\left\{ \delta_{ij} \left( \frac{\tilde{k}^2}{2} + \frac{1 - e^{-2\tilde{k}_0}}{2} \right) - \tilde{k}_j \tilde{k}_i \right\}. \]

The closed algebra (28)–(30) consisting of \( \{ \hat{L}_{ij}, \hat{L}_{0i}, k_\mu \} \) is known as the \( \kappa \)-Poincaré algebra \([1–3]\), in which, by definition, \( C_1 = P_4(k) \) and \( C_2 = P(k)^\mu P(k)_\mu \) form respectively the first and the second Casimir invariants. The invariant wave equation under the \( \kappa \)-Poincaré transformations is, thus, \( P_4(k) \Psi = 0 \) or \( P(k)^\mu P(k)_\mu \Psi = 0 \). We may read those equations as nonlocal field equations in the Minkowski spacetime by substitution \( k_\mu \rightarrow i \frac{\partial}{\partial \gamma^\mu} \). Then, those equations describe nonlocal fields with time-like extension, which spoils the symmetry under the Lorentz boost. It is, however, likely that if we start with a higher-dimensional spacetime with a time-like extra dimension, then we can realize the noncommutativity between the usual spacetime and the extra dimension, so that the Lorentz covariance is maintained.

### 3. Space noncommutatively realized in an \( AdS_{n+1} \) spacetime

We are interested in the \( (n + 1) \)-dimensional noncommutative spacetime with the coordinates \( (\hat{x}^\mu, \hat{x}^n) \) characterized by
\[ [\hat{x}_\mu, \hat{x}_\nu] = 0, \]
\[ [\hat{x}_n, \hat{x}_\mu] = i\kappa^{-1} \hat{x}_\mu, \]
where \( \hat{\mu} = (\mu, i) \) runs over \( \{ \mu \} = (0, 1, 2, 3) \) and \( (i) = (4, 5, \ldots, n - 1) \). The metric, here, is assumed to be \( g_{\hat{\mu}\hat{\nu}} = \text{diag}(+, -, -, \ldots, -) \). One can realize the closed algebra (31) and (32) by the combination of generators of the isometry group of \( AdS_{n+1} \) with coordinates \( (y^A) = (y^\mu, y^n, y^{n+1}) \) defined by
\[ g_{AB} y^A y^B = \eta_{\mu\nu} y^\mu y^\nu - (y^n)^2 + (y^{n+1})^2 = \kappa^2. \]

In terms of those coordinates, the generators of the isometry group, \( SO(2, n) \), can be written as \( M_{AB} = i\kappa^{-1} (y_A \frac{\partial}{\partial y^B} - y_B \frac{\partial}{\partial y^A}) \), \( (A, B = \hat{\mu}, n, n + 1) \), to which the same type of algebra as (5)
holds. The light-cone variables in this case are defined by $y^\pm = y^{n+1} \pm y^n$, by which the invariant length (33) for $y = (y^\mu, y^+, y^-)$ can be written as $g_{\hat{A}B} y^\mu y^\nu = g_{\hat{A}\hat{B}} y^\mu y^\nu + y^+ y^-$ with the metric

$$
(g_{\hat{A}B}) = \begin{pmatrix}
1 & 0 & \ldots & 0 & 0 & 0 \\
0 & -1 & \ldots & 0 & 0 & 0 \\
0 & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & \vdots & \ldots & -1 & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 & \frac{1}{2} \\
0 & 0 & \ldots & 0 & \frac{1}{2} & 0 \\
\end{pmatrix}.
$$

(34)

Then, it is easy to verify that the combination

$$
\hat{x}_\hat{\mu} = M_{\hat{\mu}, n} + M_{\hat{\mu}, n+1} = 2M_{\hat{\mu}+},
$$

(35)

$$
\hat{x}_n = M_{\hat{\mu}, n+1} = -2M_{\hat{\mu}+},
$$

(36)

satisfy Eqs. (31) and (32).

As in the previous section, we can again construct the vector in $AdS_{n+1}$ space using the contracted $SO(2, n)$ transformation $U(\omega) = e^{\frac{\omega}{2} BC M_{BC}}(\omega)^{BC} = a^B b^C = b^B a^C$ associated with the light-like vector $b = (b^1, \ldots, b^{n-1}, b^n, b^{n+1}) = (0, \ldots, 0, 1, 1)$. Then one can verify that $\frac{1}{2} \omega^B M_{BC} = a^\hat{B} \hat{x}_{\hat{\mu}} - a^- \hat{x}_n$, and the finite transformation defined by $y^A(\omega) = U(\omega) y^A U^\dagger(\omega) = (e^{\omega^-} \omega)^A B y^B$ can be obtained as

$$
e^{\frac{\omega^-}{2}} = 1 + \frac{1}{(a^-)^2} \left[ \cosh(\kappa^- a^-) - 1 \right] \omega^2 + \frac{1}{a^-} \sinh(\kappa^- a^-) \omega.
$$

(37)

Then, each component of the vector $\tilde{u}_A = \kappa^- (e^{-\tilde{\omega}})_{A, N+1}$ in $AdS_{n+1}$ becomes

$$
\tilde{u}_\hat{\mu}(\omega) = (e^{-\tilde{\omega}})_{\hat{\mu}, n+1} = \{ \cosh(\tilde{\omega}) - 1 \} \left( \frac{\tilde{a}_\hat{\mu}}{a^-} - \sinh(\tilde{\omega}) \right) \left( \frac{\tilde{a}_\hat{\mu}}{a^-} \right),
$$

(38)

$$
\tilde{u}_n(\omega) = (e^{-\tilde{\omega}})_{n, n+1} = \{ \cosh(\tilde{\omega}) - 1 \} \frac{\tilde{a}_\hat{\mu} \tilde{a}_\hat{\mu}}{(a^-)^2} - \sinh(\tilde{\omega}),
$$

(39)

$$
\tilde{u}_{n+1}(\omega) = (e^{-\tilde{\omega}})_{n+1, n+1} = \cosh(\tilde{\omega}) - \{ \cosh(\tilde{\omega}) - 1 \} \frac{\tilde{a}_\hat{\mu} \tilde{a}_\hat{\mu}}{(a^-)^2},
$$

(40)

to which $\tilde{u}^A \tilde{u}_A = 1$ is obviously satisfied. It is also straightforward to rewrite those components in terms of the wave numbers associated with an ordered exponential function. In what follows, for reasons of symmetry, we consider the case of symmetric ordering such as

$$
\tilde{e}_S(k) = e^{i k^n \hat{\xi}^n } e^{-i k^\mu \hat{\xi}_\mu} e^{i k^\mu \hat{\xi}_\mu} = e^{-i k^\mu \hat{\xi}_\mu + i k^n \hat{\xi}^n},
$$

(41)

which leads to

$$
k_S = (k^\mu_S, k^n_S) = \left( \frac{\tilde{k}^n}{e^{k^n/2} - e^{-k^n/2}} k^\mu, k^n \right).
$$

(42)

Then, the substitution $(a^\hat{\mu}, a^-) = (-k^\mu_S, k^n_S)$ gives rise to the expressions

$$
\tilde{u}^\hat{\mu}(k) = e^{-i \tilde{\xi}^\mu \tilde{k}^\mu},
$$

(43)

$$
\tilde{u}^n(k) = -\frac{1}{2} \tilde{k}^\mu \tilde{k}_\mu + \sinh(\tilde{k}^n),
$$

(44)

$$
\tilde{u}^{n+1}(k) = \cosh(\tilde{k}^n) - \frac{1}{2} \tilde{k}^\mu \tilde{k}_\mu.
$$

(45)
In this case, one can again define a nonlinear \((\tilde{k}^{\mu}, k^{n})\) transformation which causes the linear \(SO(2, n)\) transformation of \((\tilde{u}^{A})\). In particular, from expressions (43), (44), and (45), the \(SO(1, n - 1)\) transformation, the \((n - 1)\)-dimensional Lorentz transformations such as

\[
\mathcal{L}_{\tilde{\mu}\tilde{\nu}}\tilde{u}_{\tilde{\rho}} = \tilde{u}_{\tilde{\mu}}\tilde{g}_{\tilde{\nu}\tilde{\rho}} - \tilde{u}_{\tilde{\nu}}\tilde{g}_{\tilde{\mu}\tilde{\rho}},
\]

(46)

\[
\mathcal{L}_{\tilde{\mu}\tilde{\nu}}\tilde{u}_{n} = \mathcal{L}_{\tilde{\mu}\tilde{\nu}}\tilde{u}_{n+1} = 0,
\]

(47)

are equivalent to

\[
\mathcal{L}_{\tilde{\mu}\tilde{\nu}}\tilde{k}_{\tilde{\rho}} = \tilde{k}_{\tilde{\mu}}\tilde{g}_{\tilde{\nu}\tilde{\rho}} - \tilde{k}_{\tilde{\nu}}\tilde{g}_{\tilde{\mu}\tilde{\rho}}.
\]

(48)

\[
\mathcal{L}_{\tilde{\mu}\tilde{\nu}}\tilde{k}_{n} = 0.
\]

(49)

The Lorentz boost which causes the mixing between a new time component \(\tilde{u}^{n+1}\) and the spacetime components \(\tilde{u}^{\tilde{\mu}}\) can again be represented as a nonlinear transformation among \(\{\tilde{k}^{\mu}\}\); the resultant form is

\[
\mathcal{L}_{\tilde{\mu},n+1}\tilde{k}_{\tilde{\nu}} = e^{-\frac{\tilde{k}_{\tilde{\lambda}}}{2}}\left[-\frac{1}{2}\tilde{k}_{\tilde{\mu}}\tilde{k}_{\tilde{\lambda}} + \tilde{g}_{\tilde{\mu}\tilde{\nu}}\left\{\frac{1}{2}\tilde{k}_{\tilde{\nu}}^{2} - \cosh(\tilde{k}_{n})\right\}\right],
\]

(50)

\[
\mathcal{L}_{\tilde{\mu},n+1}\tilde{k}_{n} = e^{-\frac{\tilde{k}_{\tilde{\lambda}}}{2}}\tilde{k}_{\tilde{\mu}}.
\]

(51)

As in the previous section, \(P_{A} = \kappa \tilde{u}_{A}\) is a momentum vector in \(AdS_{n+1}\) space, and under the transformations from (48) to (51), \(C_{1} = \tilde{u}_{n}(k)\) and \(C_{2} = \tilde{u}^{\tilde{\mu}}(k)\tilde{u}_{\tilde{\mu}}(k) + (P_{n+1}(k))^{2}\) are the first and the second Casimir invariants, respectively.

4. Non-local field in the background of noncommutative spacetime

Let us now consider the wave equation for a scalar field, which is invariant under the \(SO(1, n - 1)\) transformation in \(\{\tilde{u}_{A}\}\) space. It is obvious that linear combinations of the first and the second Casimir invariants are candidates, which tend to the Klein–Gordon equation in the limit \(\kappa \to \infty\). In what follows, we consider a wave equation with the first Casimir invariant only because of its simple structure; that is, we put

\[
(-2\tilde{u}_{n} - \tilde{m}^{2}) \Phi = \left[\tilde{k}^{\tilde{\mu}}\tilde{k}_{\tilde{\mu}} - 2\sinh(\tilde{k}^{n}) - \tilde{m}^{2}\right] \Phi = 0
\]

(52)
as the free field equation. Here, \(m = \kappa \tilde{m}\) is a \(\kappa\)-dependent mass-dimension parameter that is introduced to adjust the lowest mass for this field.

At this stage, the dimensional parameter in the theory other than the additional \(m_{0}\) is \(\kappa\) only, which characterizes the spacetime in the Planck-scale physics. We now modify the above field equation by introducing a new energy scale \(\mu(\kappa)\) according to the following two steps: In the first, we note that \(k^{n}\) is nothing but the \(a^{-}\) in \(\omega^{AB} = a^{[A}b^{B]}\), which defines the vector \(\tilde{u}_{A} = (a^{-}\tilde{w})_{A,n+1} \in AdS_{n+1}\). Since \(b\) is a fixed vector in \(AdS_{n+1}\) with \(b^{+}\) component only, \(a^{A}\) may be a vector in \(AdS_{n+1}\) with a free \(a^{+}\) component. Then, by shifting \(a^{+} \to a^{+} + \kappa^{2}/a^{-}\), we can put \(a^{A}\) at a projective boundary of \(AdS_{n+1}\) such as \(a^{A}a_{A} = 0\), on which \(SO(1, n + 1)\) acts as a conformal transformation. Secondly, we break this conformal symmetry by introducing a scale parameter \(\mu\) lower than \(\kappa\) in such a way that

\[
a^{+} = \mu \left\{\frac{\tilde{k}^{n}}{2 \sinh(\tilde{k}^{n}/2)}\right\}^{2} \sim \mu.
\]

(53)
Since this equation gives rise to \( \tilde{k}^n = \frac{1}{\mu k} k^\mu k^\mu = -\frac{1}{\mu} \tilde{k}^\mu \tilde{k}_\mu \), the field equation (52) is modified so that
\[
\left[ \tilde{k}^\mu \tilde{k}_\mu + 2 \sinh(\kappa_\mu k^\mu k^\mu) \right] \Phi = 0,
\]
where \( \kappa_\mu = \frac{k}{\mu} \). Here, the scale parameter \( \mu \) is introduced by hand without any principle; however, it may be natural to read \( \kappa_\mu \lesssim 10^3 \sim 10^5 \), the order of unification.

The above equation is invariant under the \( n \)-dimensional Lorentz transformation of \( \{ k^\mu \} \), and tends to the Klein–Gordon equation \( \{(1 + 2\kappa_\mu)k^\mu k^\mu - m^2\} \Psi = 0 \) for \( |k^\mu k^\mu| \ll \mu k \). It is, thus, convenient to deal with the free field equation by adjusting the scale so that
\[
K(k) \Psi = W_k \left\{ \tilde{k}^\mu \tilde{k}_\mu + 2 \sinh(\kappa_\mu \tilde{k}^\mu \tilde{k}_\mu) \right\} \Psi = 0, \tag{55}
\]
\[
\sim (\tilde{k}^\mu \tilde{k}_\mu - m_0^2) \Psi = 0 \text{ (for } |k^\mu k^\mu| \ll \mu k), \tag{56}
\]
where \( k^\mu = (k^\mu) = \kappa(\tilde{k}^\mu) = \kappa \tilde{k}^\mu \), \( W_k = k^2(1 + 2\kappa_\mu)^{-1} \), and \( m_0^2 = W_k m^2 \simeq \frac{1}{2\kappa_\mu} m^2 \). Then \( m_0 \) becomes a very small mass parameter when we read \( m_\kappa \) as an ordinary low-energy mass parameter.

Substituting \( i\partial_\mu \) for \( k_\mu \) in \( K(k) \), the free field equation (55) becomes nothing but a nonlocal one in \( \{ x^\mu \} \) space, which is no longer a noncommutative space. In a practical model, further, the space of extra dimensions \( (x^i) = (x^4, x^5, \ldots, x^{n-1}) \) must be compact. For example, if we require the \( U(1) \) cyclicity \( x^i = x^i + 2\pi r_0 \), then \( k^i \) takes the spectrum \( k^i = \frac{l_i}{r_0}, \) \( (l_i = 0, \pm 1, \ldots) \), which we assume, henceforth, for sake of simplicity in addition to \( r_0 \approx \kappa^{-1} \).

The solutions of \( K(k) = 0 \), then, appear at the intersections of \( y = -2 \sinh(x) \) and \( y = \kappa_\mu^{-1} x - \tilde{m} \). From Fig. 1, it is obvious that the intersection gives rise to a real \( k^2 > 0 \); i.e., a real-time-like five-momentum \( \tilde{k} \). Other than such a time-like solution, there are complex solutions of \( k^2 \); all the solutions are approximately expressed as \( \tilde{k}^2 \simeq \kappa_\mu^{-1} \left\{ \frac{(-1)^n}{2} \tilde{m}^2 + i\pi n \right\}, \) \( n = 0, \pm 1, \pm 2, \ldots; \)

\[\text{Fig. 1. The intersections of sinh curves and the straight line represent the solutions of } \kappa_\mu^{-1} x \pm 2 \sinh(x) - \tilde{m}^2 = 0.\]
Fig. 2. The solutions associated with the intersection at \( x > 0 \) and \( x < 0 \) in Fig. 1 correspond to even \( n \) and odd \( n \) poles, respectively.

Therefore, the mass square \( M^2 = k^\mu k_\mu = \tilde{k}^2 + k^i k^i \) of particles in four-dimensional spacetime takes the spectra

\[
M^2_{n,l} \simeq r_0^{-2} l^2 + \frac{(-1)^n}{2} m^2 + i(\mu \kappa)n, \quad (n = 0, \pm 1, \ldots; |n| \lesssim 2\kappa \mu), \quad (57)
\]

where \( l = (l_4, \ldots, l_{n-1}) \).

In the right-hand side of Eq. (57), the first term is the order of \( \kappa^2 \) except the ground state \( l = 0 \).

The third term adds an imaginary component to \( M^2_{n,l} \), which may spoil the unitarity in the energy of the order of \( \sqrt{\mu \kappa} \). In other words, the present effective theory will go beyond the limits of validity at an energy scale larger than \( \sqrt{\mu \kappa} \), where the spacetime gets back to a noncumulative one. We finally note that the \( K^{-1}(k) \), the propagator of the free field, has simple poles at

\[
z_n = \left\{ \frac{(-1)^n}{2} \tilde{m}^2 + i\pi n \right\}, \quad (n = 0, \pm 1, \pm 2, \ldots) \quad (58)
\]

as a function of \( z = \kappa_\mu \tilde{k}^2 \). Then, one can verify that

\[
R_n \simeq \left[ W_\kappa \left\{ \kappa^{-1}_\mu + \frac{(-1)^n 2}{\cosh\left(\frac{\tilde{m}^2}{2}\right)} \right\} \right]^{-1} \simeq (-1)^n (\mu \kappa)^{-1} \quad (59)
\]

are residues of \( K^{-1}(k) \) at \( z = z_n \) characterized by \( K^{-1}\big|_{z \simeq z_n} \simeq R_n(z - z_n)^{-1} \). Those poles are expected to play an effective role in internal lines of loop diagrams, though those poles are negligible in low-energy physics.

5. An attempt at interacting fields

In the usual \( \kappa \)-Minkowski spacetime, it is not easy to formulate the interaction of fields because of its noncommutative structure among \( \hat{x}_0 \) and \( \hat{x}^i, \quad (i = 1, 2, 3) \) [14–19]. In our approach, discussed in the previous section, the resultant spacetime is a commutative one, although its fields obey a nonlocal field equation. Nevertheless, local interactions of such fields are not excluded in principle; we now

\[\text{\footnotesize in terms of } z = \xi + i\eta, \text{ the equation } \kappa^{-1}_\mu z + 2 \sinh(z) - \tilde{m}^2 = 0 \text{ is decomposed into simultaneous equations } \kappa^{-1}_\mu \xi + 2 \cos(\eta) \sinh(\xi) - \tilde{m} = 0 \text{ and } \kappa^{-1}_\mu \eta + 2 \sin(\eta) \cosh(\xi) = 0. \text{ The latter leads to } \sin(\eta) = -(2\kappa_\mu \cosh(\xi))^{-1} \eta \simeq 0; \text{ and so, we obtain } y \simeq \pi n, \quad (n = 0, \pm 1, \ldots; |n| \lesssim 2\kappa_\mu). \text{ Substituting these values for the former, the equation for } \xi \text{ becomes } \kappa^{-1}_\mu \xi + 2(-1)^n \sinh(\xi) - \tilde{m} = 0. \text{ Figure 1 shows that the solutions for } \xi \text{ exist near } \xi = 0 \text{ only; and so approximating } \sinh(\xi) \simeq \xi, \text{ we obtain } \xi \simeq \frac{(-1)^n \pi}{2} \tilde{m}^2.\]
attempt to study a $\phi^3$ type of field interaction, which is characterized by the free equation (55) and the following action:

$$S[\phi] = \int d^{n+1}x \left(-\frac{1}{2} \phi K(i\partial)\phi + \frac{g}{3!}\phi^3\right). \quad (60)$$

For simplicity, we confine our attention to the case of $n = 5$ with a compact fifth dimension such as $x^5 \equiv x^4 + 2\pi r_0$; and so, the wave number vector in (55) has the form $(\tilde{k}^\mu) = (k^\mu, r_0^{-1}l) = \kappa \{\tilde{k}^\mu, (\kappa r_0)^{-1}l\}$. (l = 0, ±1, . . ).

Now, the sinh term in the free propagator $K^{-1}(k)$ plays the role of ultraviolet convergent for both time-like and space-like regions of $k^\mu$ in Feynman diagrams. To see this situation in detail, let us study the propagator up to the order of one-loop corrections consisting of connected diagrams described by Fig. 3, for which we have the expression

$$i \langle T(\phi_x\phi_y) \rangle_0 \simeq \langle x|K^{-1}(i\partial) - \frac{ig^2}{2}K^{-1}(i\partial) \left(K^{-1} * K^{-1}(i\partial) + K^{-1}(0)I\right)K^{-1}(i\partial)|y \rangle$$

$$\simeq \frac{1}{2\pi r_0} \sum_l \int \frac{d^4p}{(2\pi)^4} \frac{\varepsilon^{-ip^\mu(x-y)\tilde{\mu}}}{K(p) - (\Sigma(p) + m_0^{-2}I)}, \quad (61)$$

where $K^{-1} * K^{-1}$ is the convolution of $K^{-1}$. By this convolution, the self-energy term $\Sigma(p)$, the Fourier transform of $\langle x| - \frac{ig^2}{2}K^{-1} * K^{-1}(i\partial)|y \rangle$, can be expressed as

$$\Sigma(p) = -\frac{ig^2}{2} \frac{1}{2\pi r_0} \sum_l \int \frac{d^4k}{(2\pi)^4} K^{-1}(k)K^{-1}(p+k). \quad (62)$$

Further, $I = \langle x|K^{-1}(i\partial)|x \rangle$ is the tadpole term of Fig. 4, for which we have the expression

$$I = -\frac{ig^2}{2} \frac{1}{2\pi r_0} \sum_l \int \frac{d^4k}{(2\pi)^4} K^{-1}(k). \quad (63)$$

We first evaluate the tadpole term (63) in detail, since its structure is rather simple. For this purpose, it is not possible to apply a simple Wick rotation with respect to $k^0$, since $K^{-1}(k)$ has poles on the complex $k^0$ plane. However, remembering that $K(k)$ is a function of $\kappa$,$\tilde{\kappa}^2 = (\mu\kappa)^{-1}(k^2 - r_0^{-2}l^2)$,
we can write (63) in the following form:

\[ I = -\frac{ig^2}{2} \frac{1}{2\pi r_0} \int \frac{d^4k}{(2\pi)^4} \int \frac{d\lambda}{2\pi} \int \frac{dz}{K[z]} e^{i\lambda (z-k_\mu^2)} \]

\[ \simeq -\frac{ig^2}{2} \frac{1}{2\pi r_0 (2\pi)^4} \int \frac{d\lambda}{2\pi} \left\{ -\left( \frac{\mu \kappa}{\lambda} \right)^4 \right\} \frac{1}{2} \int \frac{dz}{K[z]} e^{i\lambda z}, \]  

(64)

where \( K[z] = K(k) \mid _{z=k_\mu^2} \). We have also approximated the summation with respect to \( l \) to the leading \( (l = 0) \) term only,\(^5\) since \( l \neq 0 \) damps (63) by the factor \( e^{-2\kappa_\mu^2} \).

The second term in the right-hand side of the above equation is the logarithmically divergent one; that is, we put \( I \) in (63), we again discuss the case \( l = (63) \), the case \( l = 0 \) to handle this term.

The next task is to evaluate the \( z \) integral; that is, the tadpole term \( I \) can be verified that \( I \) is approximated as usual, we obtain

\[ I \simeq -\frac{ig^2}{2} \frac{1}{2\pi r_0 (2\pi)^4} \int \frac{d\lambda}{2\pi} \frac{\mu \kappa}{\lambda} \frac{1}{2} \int \frac{dz}{K[z]} e^{i\lambda z}, \]

(64)

where \( \Theta(\lambda) \equiv \left( e^{i\lambda z} - e^{-i\lambda z} + i\pi \right) \frac{1 - e^{-2\pi N|\lambda|}}{e^{2\pi \lambda} - 1}. \)

The parameter \( N (\sim 2\kappa_\mu) \) plays a role in excluding the region \( \lambda \lesssim N^{-1} \). Indeed, if we simply approximate \( \tilde{m} = 0 \) in (66), we can verify that

\[ \Theta(\lambda) \simeq \begin{cases} -\theta(-\lambda) & |\lambda| \gtrsim N^{-1} \\ -\frac{\pi}{N|\lambda|} & |\lambda| < N^{-1} \end{cases}, \]

(67)

where \( \theta(x) \) is the step function defined so that \( \theta(x) = 0 \) or 1 according as \( x < 0 \) or \( x > 0 \). Unfortunately, however, since the integrand of the \( \lambda \) integration in (64) has the form \( \lambda^{-2} \Theta(\lambda) \), a logarithmic divergence still remains in \( I \); that is, the tadpole term \( I \) can be evaluated as

\[ I \simeq -\frac{ig^2}{2} \frac{1}{2\pi r_0 (2\pi)^4} \int \int \frac{d\lambda}{2\pi} \frac{\mu \kappa}{\lambda} \frac{1}{2} \int \frac{dz}{K[z]} e^{i\lambda z}. \]

(68)

The second term in the right-hand side of the above equation is the logarithmically divergent one; we therefore need a renormalization with a cut-off to handle this term.

Next, let us study the self-energy term defined in (62) according to the same line of approach to \( I \). For the same reason as in (63), we again discuss the case \( l = 0 \) in both the external and internal lines in the above integral; that is, we put \( p = (p, 0) \) and \( \hat{k} = (k, 0) \). Then, we obtain the expression

\[ \Sigma(p, 0) \simeq -ig^2 \frac{1}{2\pi r_0} \int \frac{d^4k}{(2\pi)^4} \int \frac{dz_1}{K[z_1]} \int \frac{dz_2}{K[z_2]} \int \frac{d\lambda_1}{2\pi} e^{i\lambda_1(z_1-k_\mu^2)} \int \frac{d\lambda_2}{2\pi} e^{i\lambda_2(z_2-k_\mu^2)} \]

\[ = -ig^2 \frac{1}{2\pi r_0} \int \frac{1}{2\pi} \int \frac{d\lambda_1}{2\pi} \int \frac{d\lambda_2}{2\pi} \Theta(\lambda_1) \Theta(\lambda_2) \frac{1}{\lambda_1 + \lambda_2} \left[ \pi (\mu \kappa) \right] \frac{1}{(\lambda_1 + \lambda_2)} \left( [\pi (\mu \kappa)] \right]^2 e^{i\lambda_1 z_1/2} e^{i\lambda_2 z_2} \kappa_\mu^2 \]

(69)

Here, since \( \Theta(\lambda) = 0 \) for \( \lambda > 0 \), we can insert \( \int_0^\infty d\tau \delta(\tau + \lambda_1 + \lambda_2) = 1 \) into the above integral. Then, carrying out the integration with respect to \( \lambda_2 \) after the scaling \( \lambda_i = \tau \lambda_i \), we arrive at

\[ 3 \simeq 1 \quad \text{for} \quad \lambda \gtrsim 2\kappa_\mu^{-1}. \]

\(^5\) Strictly speaking, the sum with respect to \( l \) gives rise to Jacobi’s theta function \( \sum e^{-2\kappa_\mu^2 l^2} = \vartheta_3(z, q) \) with \( z = 1 \) and \( q = e^{-2\kappa} \), which can be evaluated as \( \vartheta_3 \sim 3 \) for \( \lambda \gtrsim 2\kappa_\mu^{-1} \).
the expression
\[ \Sigma(p, 0) \simeq \frac{g^2}{2} \frac{1}{2\pi r_0 (2\pi)^4} \int_0^\infty \frac{d\tau}{\tau} \int d\lambda_1 \Theta (\tau \lambda_1) \Theta (-\tau (\lambda_1 + 1)) e^{i\lambda_1 (\lambda_1 + 1) \tau \kappa_\mu \hat{p}_\mu} \]

\[ \simeq \frac{g}{2} \frac{1}{2\pi r_0 (2\pi)^4} \frac{1}{\sqrt{\kappa_\mu}} \int_0^\infty \frac{d\tau}{\tau^2} e^{-\frac{i\pi}{4} \kappa_\mu \hat{p}_\mu^2} \int dx e^{i\hat{p}_\mu x^\mu D_\tau(x)}, \tag{70} \]

where \( x = \sqrt{\tau \kappa_\mu (\lambda_1 + \frac{1}{2})} \) and

\[ D_\tau(x) = \Theta \left( \tau \left[ \frac{x}{\sqrt{\tau \kappa_\mu}} - \frac{1}{2} \right] \right) \Theta \left( -\tau \left[ \frac{x}{\sqrt{\tau \kappa_\mu}} + \frac{1}{2} \right] \right). \tag{71} \]

One can find that \( D_\tau(x) \) equals 1 for most of the region \(|x| < \frac{\sqrt{\tau \kappa_\mu}}{2} \) and vanishes for \(|x| > \frac{\sqrt{\tau \kappa_\mu}}{2} \); that is, the interval of the integration with respect to \( x \) is \(-\frac{\sqrt{\tau \kappa_\mu}}{2} < x < \frac{\sqrt{\tau \kappa_\mu}}{2} \). Strictly speaking, near both limits of integration, we have to modify the edges of \( D_\tau(x) \) so as to approach 0 continuously, reflecting the behavior of \( \Theta(\lambda) \) near \( \lambda = 0 \). The condition for both ends of the interval of \( x \) integration close to 0 should be \( \sqrt{\kappa_\mu \tau} \lesssim \kappa_\mu^{-1} \); that is, \( \tau \lesssim \kappa_\mu^{-3} \). Under those conditions, we can put \( e^{i\hat{p}_\mu x^\mu D_\tau(x)} \simeq (\pi N)^2 \tau^2 \left\{ \left( \frac{1}{\kappa_\mu \tau} - \frac{i\pi^2}{4} \right) x^2 - \frac{1}{4} \right\} \) to the order of \( x^2 \). On the other side, we may extend the interval of \( x \) integration \(-\frac{\sqrt{\kappa_\mu \tau}}{2}, \frac{\sqrt{\kappa_\mu \tau}}{2} \) up to the order of \( \tau \). Therefore, we can roughly evaluate the \( x \) integration so that

\[ \int dx e^{i\hat{p}_\mu x^\mu D_\tau(x)} \simeq \theta(\kappa_\mu^{-3} - \tau) \frac{\pi^2}{3} \left\{ \left( \kappa_\mu \tau \right)^{\frac{5}{2}} - i\hat{p}_\mu^2 (\kappa_\mu \tau)^2 \right\} + \theta(\tau - \kappa_\mu^{-3}) \sqrt{-\frac{\pi}{4\hat{p}_\mu^2}}. \tag{72} \]

Substituting this expression for (70), it follows that

\[ \Sigma(p, 0) \simeq \frac{g^2}{2} \frac{1}{2\pi r_0 (2\pi)^4} \left[ c_0 + c_1 \hat{p}_\mu^2 + \frac{i}{2} \sqrt{\pi \kappa_\mu \Gamma} \left( -\frac{1}{2} \hat{p}_\mu^2 \frac{5}{4\hat{p}_\mu^2} \right) \right], \tag{73} \]

where \( c_0 = \frac{\pi^2}{6} \kappa_\mu^{-4} \), \( c_1 = -\frac{i\pi^2}{36} \kappa_\mu^{-6} \), and \( \Gamma(-\frac{1}{2}, a) \) is the incomplete gamma function with the lower limit of integration \( a(=\frac{1}{4} (\hat{p}/\kappa_\mu)^{-2}) \), which is almost 0 in an energy scale lower than the Planck one. Thus, the third term in the right-hand side of equation (73) is also a constant; and those constants in the self-energy term are able to absorb into \( W, \kappa, \) and \( \hat{m}^2 \) in \( K \). In other words, as for the one-loop self-energy term, the renormalization can be carried out with finite renormalization constants.

**6. Summary and discussion**

In this paper, we have studied the \( \kappa \)-Minkowski spacetime from two points of view. One is the construction of the noncommutative spacetime coordinates based on \( SO(1, 4) \) generators in \( dS_4 \) spacetime and its modification to \( AdS_{n+1} \) background spacetime, which allows commutative four-dimensional spacetime. Another is a nonlocal field theory based on such a modified \( \kappa \)-Minkowski spacetime.

As for the former, in Sect. 2, we could show that the noncommutative coordinates \((\hat{x}_0, \hat{x}_i)\) in four-dimensional \( \kappa \)-Minkowski spacetime are nothing but generators of transformations between light-cone coordinate \( y^+ \) and others \((y^-, y^i)\) in \( dS_4 \). The plane wave in the \( \kappa \)-Minkowski spacetime, then, has the meaning of a finite \( SO(1, 4) \) transformation. From this definition of the plane wave, the five-momentum \( P_A \) in \( dS_4 \) associated with the bi-covariant differential of the plane wave is naturally understood as a resultant vector obtained by a finite transformation of \( e_4 = (0, 0, 0, 0, 1) \).
The invariant wave equations in the $\kappa$-Minkowski spacetime are defined in terms of the first or the second Casimir invariants in the background $SO(1, 4)$ symmetry. Our intention is that such a wave equation defines a nonlocal field theory having a similarity to Yukawa’s domain theory, though the wave equation spoils four-dimensional Lorentz invariance. To secure the Lorentz invariance, in Sect. 3, we studied a noncommutative spacetime associated with an $AdS_{n+1}$ type of background spacetime. In such a spacetime, there appears another time-like coordinate $y_{n+1}$ in addition to $y_0$, from which one can construct a $\kappa$-Minkowski-like spacetime characterized by the noncommutativity $[\hat{x}_n, \hat{x}_\mu] = i\kappa^{-1}\hat{x}_\mu$ and $[\hat{x}_\mu, \hat{x}_\nu] = 0$. In Sect. 4, we put the wave equation in this spacetime by using the first $SO(2, n + 1)$ Casimir invariant. Then, the wave equation is not invariant under the transformations between $\hat{x}_\mu$ and $\hat{x}_n$ but is invariant under the Lorentz transformations among $\{\hat{x}_\mu\}$. Further, by introducing a new scale parameter $\mu$ at the projective boundary of $AdS_{n+1}$, the wave equation is reduced to a nonlocal field equation in commutative $\{x_\mu\}$ spacetime, which is invariant under the Lorentz transformation.

In the resultant spacetime, we need not worry about the noncommutativity of spacetime variables. Then, in Sect. 5, we discussed a local interaction of fields, which obeys nonlocal field equations characterized by a free field equation including an infinite higher-derivative term such as $\sinh[(\kappa \mu)^{-1}\partial^2]$. There, we tried to evaluate one-loop diagrams by assuming a $\phi^3$ type of local interaction for those fields. At first, it is expected to get finite results for those diagrams, since the sinh term in the propagator plays the role of strong damping factor in the both space-like and time-like regions of momentum squared, $k^2 = \hat{k}^\mu k_\mu$. However, the situation is not so simple, because the propagator contains complex poles of $k^2$, which may spoil the unitarity of the interactions at the Planck energy scale. The contribution of those poles, fortunately, again produces a damping factor to internal lines of loop diagrams: the more the number of internal lines increase, the more the damping effect grows. These effects are not trivial, and one can expect to get convergent results by the same mechanism in higher loop diagrams too.

We also note that the second scale parameter $\mu$ characterizing the resultant spacetime is introduced by hand without enough guiding principles. The wave equation, the first Casimir invariant, then, becomes a nonlocal field equation that resembles Yukawa’s domain at some points. One of the purposes of domain theory is to improve the divergent problem in local field theories. Therefore, the investigation of the meaning of $\mu$ in more detail will be an interesting future problem.

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Appendix A: Bi-covariant differential of $U(\omega)$

Here we derive the bi-covariant differential of $U(\omega) = e^{i\Omega}$, $(\Omega = \frac{1}{2}\omega^{AB}M_{AB})$, equation (15), which yields the relation between an ordered plane wave $\hat{e}(k)$ and the corresponding momentum $P^A(k)$. The key is to notice that

$$\Omega = a^\mu \hat{x}_\mu - a^n \hat{x}_n = -e^4 \cdot \omega \cdot \hat{x} \quad (A1)$$

for $(\omega)^{AB} = \omega^{AB} = a^A b^B - b^A a^B$, where $(e^4)_A = \delta^4_A$ and $\hat{x} = (\hat{x}^0, \hat{x}^i, \hat{x}^A)$. Then, the differential operator for $\Omega$ should be defined by $[\Omega, d] = -d\Omega = -e_4 \cdot \omega \cdot dx = -e_4 \kappa \cdot \tilde{\omega} \cdot dx$, where $dx^A = d\hat{x}^A$ are commutative quantities satisfying

$$[M_{AB}, dx^C] = i\kappa^{-1}(g_{BC}dx^A - g_{AC}dx_B), \quad (A2)$$
which leads to the form of n-times commutator
\[
[\Omega, [\Omega, \cdots [\Omega, -d\Omega], \cdots]]] = (-i)^{n-1}\kappa e_4 \cdot \vec{\omega}^{n+1} \cdot dx.
\] (A3)

Then, it is straightforward to verify that
\[
dU(\omega) = e^{i\Omega} (e^{-i\Omega} d e^{i\Omega}) = e^{i\Omega} \sum_{n=0}^{\infty} (-i)^n [\Omega, [\Omega, \cdots [\Omega, d], \cdots]]] = -i dU(\omega).
\]
The result is a nonlinear realization\(^6\) of \(dS_4\) vector \(\hat{u}\) out of \(\omega(\hat{a})\).

As for the bi-covariant differential of the ordered plane wave associated with \(AdS_{n+1}\), we can follow the same way as the one in \(dS_4\). In this case,
\[
\Omega = \frac{1}{2} \omega^{AB} M_{AB} = a^\mu \hat{\chi}_\mu - a^- \cdot \hat{x}_n = -e_{n+1} \cdot \omega \cdot \hat{x},
\] (A5)
from which we have \(-d\Omega = e_{n+1} \cdot \omega \cdot dx\) and the n-times commutator \([\Omega, \cdots [\Omega, -d\Omega], \cdots]\] = \((-i)^n e_{n+1} \cdot \omega \cdot dx\). Then the counterpart of (A4) in the present case becomes
\[
dU(\omega) = e^{i\Omega} i e_{n+1} \cdot (e^{-i\omega} - 1) \cdot dx = i \kappa dx \cdot (1 - e^{-i\omega}) \cdot e_{n+1} e^{i\Omega}
\] (A6)

\[^6\]The nonlinear realization of a \(dS_4\) vector in a different basis is discussed in Ref. [20].
that is, that
\[ \mathcal{L}_{10} \tilde{k}_0 = \tilde{k}_i. \]  
(B5)

Substituting this result for (B1), it can be derived that
\[ \mathcal{L}_{10} \tilde{k}_j = \left\{ \delta_{ij} \left( \frac{\tilde{k}^2}{2} + \frac{1 - e^{-2\tilde{k}_0}}{2} \right) - \tilde{k}_i \tilde{k}_j \right\}. \]  
(B6)

In parallel to the above, the boosts \( \mathcal{L}_{\mu, n+1} \tilde{u}_A = \tilde{u}_\mu g_{n+1} A - \tilde{u}_{n+1} g_{\mu, \tilde{v}} \) for \( SO(2, n) \) vector \( \{u^A(\omega)\} \) can also be rewritten in terms of \( (\tilde{k}_\mu, \tilde{k}_n) \) associated with the symmetric ordering. First, for the component \( A = \tilde{v} \), we obtain the following as the counterpart of Eq. (B2):
\[ \mathcal{L}_{\tilde{\mu}, n+1} \tilde{u}_n = \frac{1}{2} \left( \mathcal{L}_{\tilde{\mu}, n+1} \tilde{k}_n \right) e^{\frac{1}{2}\tilde{k}_0} \tilde{k}_n + e^{\frac{1}{2}\tilde{k}_0} \left( \mathcal{L}_{\tilde{\mu}, n+1} \tilde{k}_n \right) = g_{\mu \tilde{v}} \left\{ \frac{1}{2} \tilde{k}^2 - \cosh(\tilde{k}_n) \right\}, \]  
(B7)

from which follows
\[ \frac{1}{2} \left( \mathcal{L}_{\tilde{\mu}, n+1} \tilde{k}_n \right) \tilde{k}_n + \left( \mathcal{L}_{\tilde{\mu}, n+1} \tilde{k}_n \right) = \tilde{k}_n e^{-\frac{1}{2}\tilde{k}_n} \left\{ \frac{1}{2} \tilde{k}^2 - \cosh(\tilde{k}_n) \right\}. \]  
(B8)

Secondly, \( \mathcal{L}_{\tilde{\mu}, n+1} \tilde{u}_{n+1} = \tilde{u}_\mu \) can be read as
\[ \mathcal{L}_{\tilde{\mu}, n+1} \tilde{u}_{n+1} = - \left( \mathcal{L}_{\tilde{\mu}, n+1} \tilde{k}_n \right) \tilde{k}_n + \left( \mathcal{L}_{\tilde{\mu}, n+1} \tilde{k}_n \right) + \sinh(\tilde{k}_n) = e^{\frac{1}{2}\tilde{k}_n} \tilde{k}_n. \]  
(B9)

Addition of (B8) and (B9) yields
\[ \mathcal{L}_{\tilde{\mu}, n+1} \tilde{k}_n = \tilde{k}_n e^{-\frac{1}{2}\tilde{k}_n}. \]  
(B10)

Substituting this result for (B7), we arrive at
\[ \mathcal{L}_{\tilde{\mu}, n+1} \tilde{k}_n = -e^{-\frac{1}{2}\tilde{k}_n} \tilde{k}_n + \sinh(\tilde{k}_n) = e^{\frac{1}{2}\tilde{k}_n} \tilde{k}_n. \]  
(B11)

The above results are nothing but (50) and (51).

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