Majority and plurality problems

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Abstract. Given a set of $n$ balls each colored with a color, a ball is said to be majority, $k$-majority, plurality if its color class has size larger than half of the number of balls, has size at least $k$, has size larger than any other color class; respectively. We address the problem of finding the minimum number of queries (a comparison of a pair of balls if they have the same color or not) that is needed to decide whether a majority, $k$-majority or plurality ball exists and if so then show one such ball. We consider both adaptive and non-adaptive strategies and in certain cases, we also address weighted versions of the problems.

1 Introduction

Two very much investigated problems in combinatorial search theory are the so-called majority and plurality problems. In this context, we are given $n$ balls in an urn, each colored with one color. A majority ball is one such that its color class has size strictly larger than $n/2$. A plurality ball is one such that its color class is strictly larger than any other color class. The aim is either to decide whether there exists a majority/plurality ball or even to show one (if there exists one). Note that if the number of colors is two, then the majority and the plurality problems coincide. Although there are other models (e.g. [6]), in the original settings a query is a pair of balls and the answer to the query tells us whether the two balls have the same color or not. Throughout the paper we consider queries of this sort.

We distinguish two types of algorithms for each problem we consider. An algorithm is adaptive if the $i$th query might depend on the answers received for the first $i - 1$ queries. A non-adaptive algorithm is simply a set of queries that should be answered at the same time. Clearly, any non-adaptive algorithm can be viewed as an adaptive one and therefore for any kind of combinatorial search problem, the minimum number of queries required in an adaptive algorithm is not more than the minimum number of queries required in a non-adaptive algorithm.

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The first results concerning plurality and majority problems are due to Fisher and Salzberg [7] and Saks and Werman [8]. In [7] it is proved that if the number of possible colors is unknown, then the minimum number of queries in an adaptive search for a majority ball is \( \lceil 3n/2 \rceil - 2 \), while [8] contains the result that if the number of colors is two, then the minimum number of queries needed to find a majority ball is \( n - b(n) \), where \( b(n) \) is the number of 1’s in the binary representation of \( n \). The latter result was later reproved in a simpler way by Alonso, Reingold, and Schott [3] and Wiener [10].

The adaptive version of the plurality problem was first considered by Aigner, De Marco, and Montangero in [2], where they showed that for any fixed positive integer \( c \), if the number of possible colors is at most \( c \), then the minimum number of queries needed in an adaptive search for a plurality ball is of linear order, and the constants depend on \( c \). Non-adaptive and other versions of the plurality problem were considered in [1].

Non-adaptive strategies were also studied by Chung, Graham, Mao, and Yao [4, 5]. They showed a linear upper bound for the majority problem in case the existence of a majority color is assumed. They mention a quadratic lower bound without this extra assumption. We precisely determine the minimum number of queries needed. They also obtain lower and upper bounds on the plurality problem in the non-adaptive case. We improve those bounds and find the correct asymptotics of the minimum number of queries.

1.1 Preliminaries and notation and main results

To state our results we introduce some notations. \( M_c(n) \) denotes the minimum number of queries that is needed to determine if there exists a majority color and if so, then to show one ball of that color and \( P_c(n) \) denotes the minimum number of queries that is needed to determine if there exists a plurality color and if so, then to show one ball of that color. In both cases the subscript \( c \) stands for the number of possible colors. The corresponding non-adaptive parameters are denoted by \( M^*_c(n) \) and \( P^*_c(n) \). A ball is said to be \( k \)-majority if its color class contains at least \( k \) balls. \( M_c(n, k) \) denotes the minimum number of queries that is needed to determine if there exists a \( k \)-majority color and if so, then to show one ball of that color and \( M^*_c(n, k) \) denotes the parameter of the non-adaptive variant.

We also consider weighted problems. Let \( S = \{w(1), \ldots, w(n)\} \) be a multiset of positive numbers, where \( w(i) \) is considered to be the weight of the \( i \)th ball. For all weighted problems considered in the paper, we