Combining Semilattices and Semimodules

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Abstract. We describe the canonical weak distributive law \( \delta: SP \to PS \) of the powerset monad \( P \) over the \( S \)-left-semimodule monad \( S \), for a class of semirings \( S \). We show that the composition of \( P \) with \( S \) by means of such \( \delta \) yields almost the monad of convex subsets previously introduced by Jacobs: the only difference consists in the absence in Jacobs’s monad of the empty convex set. We provide a handy characterisation of the canonical weak lifting of \( P \) to \( EM(S) \) as well as an algebraic theory for the resulting composed monad. Finally, we restrict the composed monad to finitely generated convex subsets and we show that it is presented by an algebraic theory combining semimodules and semilattices with bottom, which are the algebras for the finite powerset monad \( P_f \).

Keywords: algebraic theories · monads · weak distributive laws.

1 Introduction

Monads play a fundamental role in different areas of computer science since they embody notions of computations [31], like nondeterminism, side effects and exceptions. Consider for instance automata theory: deterministic automata can be conveniently regarded as certain kind of coalgebras on \( Set \) [32], nondeterministic automata as the same kind of coalgebras but on \( EM(P_f) \) [34], and weighted automata on \( EM(S) \) [36]. Here, \( P_f \) is the finite powerset monad, modeling nondeterministic computations, while \( S \) is the monad of semimodules over a semiring \( S \), modelling various sorts of quantitative aspects when varying the underlying semiring \( S \). It is worth mentioning two facts: first, rather than taking coalgebras over \( EM(T) \), the category of algebras for the monad \( T \), one can also consider coalgebras over \( \mathbb{K}(T) \), the Kleisli category induced by \( T \) [19]; second, these two approaches based on monads have lead not only to a deeper understanding of the subject, but also to effective proof techniques [6,7,13], algorithms [1,8,21,35,38] and logics [18,20,26].

Since compositionality is often the key to master complex structures, computer scientists devoted quite some efforts to compose monads [39] or the equivalent notion of algebraic theories [23]. Indeed, the standard approach of composing monads by means of distributive laws [3] turned out to be somehow unsatisfactory. On the one hand, distributive laws do not exist in many relevant cases:

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see [27, 40] for some no-go theorems; on the other hand, proving their existence is error-prone: see [27] for a list of results that were mistakenly assuming the existence of a distributive law of the powerset monad over itself.

Nevertheless, some sort of weakening of the notion of distributive law—e.g., distributive laws of functors over monads [25]—proved to be ubiquitous in computer science: they are GSOS specifications [37], they are sound inductive up-to techniques [7] and complete abstract domains [5]. In this paper we will exploit weak distributive laws in the sense of [14] that have been recently shown successful in composing the monads for nondeterminism and probability [16].

The goal of this paper is to somehow combine the monads $P_f$ and $S$ mentioned above. Our interest in $S$ relies on the wide expressiveness provided by the possibility of varying $S$: for instance by taking $S$ to be the Boolean semiring, one obtains the monad $P_f$; by fixing $S$ to be the field of reals, coalgebras over $EM(S)$ turn out be linear dynamical systems [33].

We proceed as follows. Rather than composing $P_f$, we found it convenient to compose the full, not necessarily finite, powerset monad $P$ with $S$. In this way we can reuse several results in [11] that provide necessary and sufficient conditions on the semiring $S$ for the existence of a canonical weak [14] distributive law $\delta: SP \rightarrow PS$. Our first contribution (Theorem 21) consists in showing that such $\delta$ has a convenient alternative characterisation, whenever the underlying semiring is a positive semifield, a condition that is met, e.g., by the semirings of Booleans and non-negative reals.

Such characterisation allows us to give a handy definition of the canonical weak lifting of $P$ over $EM(S)$ (Theorem 24) and to observe that such lifting is almost the same as the monad $C: EM(S) \rightarrow EM(S)$ defined by Jacobs in [24] (Remark 25): the only difference is the absence in $C$ of the empty subset. Such difference becomes crucial when considering the composed monads, named $CM: Set \rightarrow Set$ in [24] and $P_cS: Set \rightarrow Set$ in this paper: the latter maps a set $X$ into the set of convex subsets of $SX$, while the former additionally requires the subsets to be non-empty. It turns out that while $Kl(CM)$ is not CPPO-enriched, a necessary condition for the coalgebraic framework in [19], $Kl(P_cS)$ indeed is (Theorem 30).

Composing monads by means of weak distributive laws is rewarding in many respects: here we exploit the fact that algebras for the composed monad $P_cS$ coincide with $\delta$-algebras, namely algebras for both $P$ and $S$ satisfying a certain pentagonal law. One can extract from this law some distributivity axioms that, together with the axioms for semimodules (algebras for the monad $S$) and those for complete semilattices (algebras for the monad $P$), provide an algebraic theory presenting the monad $P_cS$ (Theorem 32).

We conclude by coming back to the finite powerset monad $P_f$. By replacing, in the above theory, complete semilattices with semilattices with bottom (algebras for the monad $P_f$) one obtains a theory presenting the monad $P_{fc}S$ of finitely generated convex subsets (Theorem 35), which is formally defined as a restriction of the canonical $P_cS$. The theory, displayed in Table 1, consists of the
Table 1. The sets of axioms $E_{SL}$ for semilattices (left), $E_{LSM}$ for $S$-semimodules (right) and $E_{D'}$ for their distributivity (bottom).

|                | $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$ | $(x + y) + z = x + (y + z)$ | $(\lambda + \mu) \cdot x = \lambda \cdot x + \mu \cdot x$ |
|----------------|-----------------------------------------------|-----------------------------------------------|---------------------------------------------------------|
|                | $x \sqcup y = y \sqcup x$                     | $x + y = y + x$                              | $0_S \cdot x = 0$                                       |
|                | $x \sqcup \bot = x$                           | $x + 0 = x$                                  | $(\lambda \mu) \cdot x = \lambda \cdot (\mu \cdot x)$ |
|                | $x \sqcup x = x$                              | $\lambda \cdot (x \sqcup y) = (\lambda \cdot x) \sqcup (\lambda \cdot y)$ | $\lambda \cdot 0 = 0$                                  |

theory presenting the monad $\mathcal{P}_f$ and the theory presenting the monad $S$ with four distributivity axioms.

Notation. We assume the reader to be familiar with monads and their maps. Given a monad $(M, \eta^M, \mu^M)$ on $\mathcal{C}$, $\text{EM}(M)$ and $\text{Kl}(M)$ denote, respectively, the Eilenberg-Moore category and the Kleisli category of $M$. The latter is defined as the category whose objects are the same as $\mathcal{C}$ and a morphism $f : X \to Y$ in $\text{Kl}(M)$ is a morphism $f : X \to M(Y)$ in $\mathcal{C}$. We write $U^M : \text{EM}(M) \to \mathcal{C}$ and $U_M : \text{Kl}(M) \to \mathcal{C}$ for the canonical forgetful functors, and $F^M : \mathcal{C} \to \text{EM}(M)$, $F_M : \mathcal{C} \to \text{Kl}(M)$ for their respective left adjoints. Recall, in particular, that $F^M(X) = (X, \mu^X)$ and, for $f : X \to Y$, $F^M(f) = M(f)$. Given $n$ a natural number, we denote by $\mathbb{n}$ the set $\{1, \ldots, n\}$.

2 (Weak) Distributive laws

Given two monads $S$ and $T$ on a category $\mathcal{C}$, is there a way to compose them to form a new monad $ST$ on $\mathcal{C}$? This question was answered by Beck [3] and his theory of distributive laws, which are natural transformations $\delta : TS \to ST$ satisfying four axioms and that provide a canonical way to endow the composite functor $ST$ with a monad structure. We begin by recalling the classic definition. In the following, let $(T, \eta^T, \mu^T)$ and $(S, \eta^S, \mu^S)$ be two monads on a category $\mathcal{C}$.

Definition 1. A distributive law of the monad $S$ over the monad $T$ is a natural transformation $\delta : TS \to ST$ such that the following diagrams commute.

$$
\begin{array}{ccc}
TTS & \xrightarrow{\delta} & STS \\
\downarrow_{\mu^S} & & \downarrow_{\mu^S T} \\
TS & \xrightarrow{\delta} & ST
\end{array}
\quad
\begin{array}{ccc}
TTT & \xrightarrow{\delta T} & TST \\
\downarrow_{\mu^T} & & \downarrow_{S \mu^T} \\
TT & \xrightarrow{\delta} & ST
\end{array}
\quad
\begin{array}{ccc}
TSS & \xrightarrow{\delta S} & STS \\
\downarrow_{\eta^S} & & \downarrow_{\eta^S T} \\
TS & \xrightarrow{\delta} & ST
\end{array}
\quad
\begin{array}{ccc}
TTS & \xrightarrow{\delta} & TST \\
\downarrow_{\eta^T} & & \downarrow_{S \eta^T} \\
TS & \xrightarrow{\delta} & ST
\end{array}
$$
One important result of Beck’s theory is the bijective correspondence between
distributive laws, liftings to Eilenberg-Moore algebras and extensions to Kleisli
categories, in the following sense.

**Definition 2.** A lifting of the monad $S$ to $\text{EM}(T)$ is a monad $(\tilde{S}, \eta^{\tilde{S}}, \mu^{\tilde{S}})$ where

$$
\begin{array}{ccc}
\text{EM}(T) & \overset{\tilde{S}}{\longrightarrow} & \text{EM}(T) \\
\downarrow F^T & & \downarrow F^T \\
C & \overset{S}{\longrightarrow} & C
\end{array}
$$

commutes, $U^T \eta^{\tilde{S}} = \eta^S U^T$, $U^T \mu^{\tilde{S}} = \mu^S U^T$.

An extension of the monad $T$ to $\text{Kl}(S)$ is a monad $(\tilde{T}, \eta^{\tilde{T}}, \mu^{\tilde{T}})$ such that

$$
\begin{array}{ccc}
C & \overset{T}{\longrightarrow} & C \\
\downarrow F_S & & \downarrow F_S \\
\text{Kl}(S) & \overset{\tilde{T}}{\longrightarrow} & \text{Kl}(S)
\end{array}
$$

commutes, $\eta^T F_S = F_S \eta^T$, $\mu^T F_S = F_S \mu^T$.

Böhm [10] and Street [36] have studied various weaker notions of distributive
law; here we shall use the one that consists in dropping the axiom involving $\eta^T$
in Definition 1 following the approach of Garner [14].

**Definition 3.** A weak distributive law of $S$ over $T$ is a natural transformation $\delta: TS \to ST$ such that the diagrams in (1) regarding $\mu^S$, $\mu^T$ and $\eta^S$ commute.

There are suitable weaker notions of liftings and extensions which also bijectively
correspond to weak distributive laws as proved in [10,14].

**Definition 4.** A weak lifting of $S$ to $\text{EM}(T)$ consists of a monad $(\tilde{S}, \eta^{\tilde{S}}, \mu^{\tilde{S}})$ on $\text{EM}(T)$ and two natural transformations

$$
U^T \tilde{S} \overset{\iota}{\longrightarrow} SU^T \overset{\pi}{\longrightarrow} U^T \tilde{S}
$$

such that $\pi \iota = \text{id}_{U^T \tilde{S}}$ and such that the following diagrams commute:

$$
\begin{array}{ccc}
U^T \tilde{S} & \overset{\iota}{\longrightarrow} & SU^T \\
\downarrow U^T \mu^S & & \downarrow U^T \mu^S \\
U^T \tilde{S} & \overset{\iota}{\longrightarrow} & SU^T
\end{array} \quad \text{(2)}
\begin{array}{ccc}
U^T \tilde{S} & \overset{\iota}{\longrightarrow} & SU^T \\
\downarrow U^T \eta^{\tilde{S}} & & \downarrow U^T \eta^{\tilde{S}} \\
U^T \tilde{S} & \overset{\iota}{\longrightarrow} & SU^T
\end{array}
\begin{array}{ccc}
SSU^T & \overset{\iota}{\longrightarrow} & SU^T \\
\downarrow \mu^S U^T & & \downarrow \mu^S U^T \\
SSU^T & \overset{\iota}{\longrightarrow} & SU^T
\end{array} \quad \text{(3)}
\begin{array}{ccc}
\mu^S U^T & \overset{\pi}{\longrightarrow} & U^T \tilde{S} \\
\downarrow U^T \eta^{\tilde{S}} & & \downarrow U^T \eta^{\tilde{S}} \\
\mu^S U^T & \overset{\pi}{\longrightarrow} & U^T \tilde{S}
\end{array}
$$

A weak extension of $T$ to $\text{Kl}(S)$ is a functor $\tilde{T}: \text{Kl}(S) \to \text{Kl}(S)$ together with a natural transformation $\tilde{\mu}^T: \tilde{T} \tilde{T} \to \tilde{T}$ such that $F_S T = \tilde{T} F_S$ and $\tilde{\mu}^T F_S = F_S \mu^T$.

**Theorem 5 (3,10,14).** There is a bijective correspondence between (weak) distributive laws $TS \to ST$, (weak) liftings of $S$ to $\text{EM}(T)$ and (weak) extensions of $T$ to $\text{Kl}(S)$. 
3 The Powerset and Semimodule Monads

The Monad $\mathcal{P}$. Let us now consider, as $S$, the powerset monad $(\mathcal{P}, \eta^\mathcal{P}, \mu^\mathcal{P})$, where $\eta^\mathcal{P}_X(x) = \{x\}$ and $\mu^\mathcal{P}_X(U) = \bigcup_{U \subseteq U} U$. Its algebras are precisely the complete semilattices and we have that $\mathcal{Kl}(\mathcal{P})$ is isomorphic to the category $\mathcal{Rel}$ of sets and relations. Hence, giving a distributive law $T\mathcal{P} \to \mathcal{P}T$ is the same as giving an extension of $T$ to $\mathcal{Rel}$: for this to happen the notion of weak cartesian functor and natural transformation is crucial.

**Definition 6.** A functor $T : \text{Set} \to \text{Set}$ is said to be weakly cartesian if and only if it preserves weak pullbacks. A natural transformation $\varphi : F \to G$ is said to be weakly cartesian if and only if its naturality squares are weak pullbacks.

Kurz and Velebil [28] proved, using an original argument of Barr [2], that an endofunctor $T$ on $\text{Set}$ has at most one extension to $\mathcal{Rel}$ and this happens precisely when it is weakly cartesian; similarly a natural transformation $\varphi : F \to G$, with $F$ and $G$ weakly cartesian, has at most one extension $\tilde{\varphi} : \tilde{F} \to \tilde{G}$, precisely when it is weakly cartesian. The following result is therefore immediate.

**Proposition 7 ([14, Corollary 16]).** For any monad $(T, \eta^T, \mu^T)$ on $\text{Set}$:

1. There exists a unique distributive law of $\mathcal{P}$ over $T$ if and only if $T$, $\eta^T$ and $\mu^T$ are weakly Cartesian.
2. There exists a unique weak distributive law of $\mathcal{P}$ over $T$ if and only if $T$ and $\mu^T$ are weakly Cartesian.

The Monad $\mathcal{S}$. Recall that a semiring is a tuple $(S, +, \cdot, 0, 1)$ such that $(S, +, 0)$ is a commutative monoid, $(S, \cdot, 1)$ is a monoid, $\cdot$ distributes over $+$ and $0$ is an annihilating element for $\cdot$. In other words, a semiring is a ring where not every element has an additive inverse. Natural numbers $\mathbb{N}$ with the usual operations of addition and multiplication form a semiring. Similarly, integers, rationals and reals form semirings. Also the booleans $\text{Bool} = \{0, 1\}$ with $\lor$ and $\land$ acting as $+$ and $\cdot$, respectively, form a semiring.

Every semiring $S$ generates a semimodule monad $\mathcal{S}$ on $\text{Set}$ as follows. Given a set $X$, $\mathcal{S}(X) = \{\varphi : X \to S \mid \text{supp } \varphi \text{ finite}\}$, where $\text{supp } \varphi = \{x \in X \mid \varphi(x) \neq 0\}$. For $f : X \to Y$, define for all $\varphi \in \mathcal{S}(X)$

$$S(f)(\varphi) = \left(y \mapsto \sum_{x \in f^{-1}(y)} \varphi(x)\right) : Y \to S.$$ 

This makes $\mathcal{S}$ a functor. The unit $\eta^S_X : X \to \mathcal{S}(X)$ is given by $\eta^S_X(x) = \Delta_x$, where $\Delta_x$ is the Dirac function centred in $x$, while the multiplication $\mu^S_X : \mathcal{S}^2(X) \to \mathcal{S}(X)$ is defined for all $\Psi \in \mathcal{S}^2(X)$ as

$$\mu^S_X(\Psi) = \left(x \mapsto \sum_{\varphi \in \text{supp } \Psi} \Psi(\varphi) \cdot \varphi(x)\right) : X \to S.$$
Table 2. Definition of some properties of a semiring $S$. Here $a, b, c, d \in S$.

| Positive | $a + b = 0 \implies a = 0 = b$ |
|----------|----------------------------------|
| Semifield | $a \neq 0 \implies \exists! x, a \cdot x = x \cdot a = 1$ |
| Refinable | $a + b + c + d \implies \exists x, y, z, t, x + y = a, z + t = b, x + z = c, y + t = d$ |
| (A)      | $a + b = 1 \implies a = 0$ or $b = 0$ |
| (B)      | $a \cdot b = 0 \implies a = 0$ or $b = 0$ |
| (C)      | $a + c = b + c \implies a = b$ |
| (D)      | $\forall a, b, \exists x, a + x = b$ or $b + x = a$ |
| (E)      | $a + b = c \cdot d \implies \exists t: \{(x, y) \in S^2 \mid x + y = d\} \to S$ such that $\sum_{x+y=d} t(x, y) x = a, \sum_{x+y=d} t(x, y) y = b, \sum_{x+y=d} t(x, y) = c$. |

An algebra for $S$ is precisely a left-$S$-semimodule, namely a set $X$ equipped with a binary operation $+$, an element $0$ and a unary operation $\lambda$ for each $\lambda \in S$, satisfying the equations in Table 1. Indeed, if $X$ carries a semimodule structure then one can define a map $a: SX \to X$ as, for $\varphi \in SX$,

$$a(\varphi) = \sum_{x \in X} \varphi(x) \cdot x$$

where the above sum is finite because so is $\text{supp} \varphi$. Vice versa, if $(X, a)$ is an $S$-algebra, then the corresponding left-semimodule structure on $X$ is obtained by defining for all $\lambda \in S$ and $x, y \in X$

$$x +^\alpha y = a(x \mapsto 1, y \mapsto 1), \quad 0^\alpha = a(\varepsilon), \quad \lambda \cdot^\alpha x = a(x \mapsto \lambda).$$

Above and in the remainder of the paper, we write the list $(x_1 \mapsto s_1, \ldots, x_n \mapsto s_n)$ for the only function $\varphi: X \to S$ with support $\{x_1, \ldots, x_n\}$ mapping $x_i$ to $s_i$ and we write the empty list $\varepsilon$ for the function constant to $0$. For instance, for $a = \mu^S_X: SSX \to SX$, the left-semimodule structure is defined for all $\varphi_1, \varphi_2 \in SX$ and $x \in X$ as

$$(\varphi_1 + \mu^S \varphi_2)(x) = \varphi_1(x) + \varphi_2(x), \quad 0^{\mu^S}(x) = 0, \quad (\lambda \cdot^\alpha \varphi_1)(x) = \lambda \cdot \varphi_1(x).$$

Proposition 7 tells us exactly when a (weak) distributive law of the form $TP \to PT$ exists for an arbitrary monad $T$ on $\text{Set}$. Take then $T = S$: when are the functor $S$ and the natural transformations $\eta^S$ and $\mu^S$ weakly cartesian? The answer has been given in [11] (see also [17]), where a complete characterisation in purely algebraic properties for $S$ is provided. In Table 2 we recall such properties.

Theorem 8 ([11]). Let $S$ be a semiring.

1. The functor $S$ is weakly cartesian if and only if $S$ is positive and refinable.
2. $\eta^S$ is weakly cartesian if and only if $S$ enjoys (A) in Table 2.
3. If $S$ is weakly cartesian, then $\mu^S$ is weakly cartesian if and only if $S$ enjoys (B) and (E) in Table 2.
Remark 9. In [11, Proposition 9.1] it is proved that if \( S \) enjoys (C) and (D), then \( S \) is refinable; if \( S \) is a positive semifield, then it enjoys (B) and (E). In the next Proposition we prove that if \( S \) is a positive semifield then it is also refinable, hence \( S \) and \( \mu^S \) are weakly cartesian.

Proposition 10. If \( S \) is a positive semifield, then it is refinable.

Proof. Let \( a, b, c \) and \( d \) in \( S \) be such that \( a + b = c + d \). If \( a + b = 0 \), then take \( x = y = z = t = 0 \), otherwise take \( x = ac c + d, y = ad c + d, z = bc c + d, t = bd c + d \).

Then \( x + y = a, z + t = b, x + z = c, y + t = d \). \( \square \)

Example 11. It is known that, for \( S = \mathbb{N} \), a distributive law \( \delta : \mathcal{SP} \rightarrow \mathcal{PS} \) exists. Indeed one can check that all conditions of Theorem 8 are satisfied, therefore we can apply Proposition 7.1. In this case, the monad \( S X \) is naturally isomorphic to the commutative monoid monad, which given a set \( X \) returns the collection of all multisets of elements of \( X \). The law \( \delta \) is well known (see e.g. [14,22]): given a multiset \( \langle A_1, \ldots, A_n \rangle \) of subsets of \( X \) in \( \mathcal{SP} X \), where the \( A_i \)'s need not be distinct, it returns the set of multisets \( \{ \langle a_1, \ldots, a_n \rangle \mid a_i \in A_i \} \).

Convex Subsets of Left-semimodules. Theorem 8 together with Proposition 7.1 tell us that whenever the element 1 of \( S \) can be decomposed as a non-trivial sum there is no distributive law \( \delta : \mathcal{SP} \rightarrow \mathcal{PS} \). Semirings with this property abound, for example \( \mathbb{Q}, \mathbb{R}, \mathbb{R}^+ \) with the usual operations of sum an multiplication, as well as \( \mathbb{B} \) (since \( 1 \lor 1 = 1 \)). Such semirings are precisely those for which the notion of convex subset of their left-semimodules is non-trivial. For the existence of a weak distributive law, however, this condition on 1_S is not required: convexity will indeed play a crucial role in the definition of the weak distributive law.

Definition 12. Let \( S \) be a semiring, \( X \) an \( S \)-left-semimodule and \( A \subseteq X \). The convex closure of \( A \) is the set

\[
\overline{A} = \left\{ \sum_{i=1}^{n} \lambda_i \cdot a_i \mid n \in \mathbb{N}, \ a_i \in A, \ \sum_{i=1}^{n} \lambda_i = 1 \right\} \subseteq X.
\]

The set \( A \) is said to be convex if and only if \( A = \overline{A} \).

Recalling that the category of \( S \)-left-semimodules is isomorphic to \( \mathbb{EM}(S) \), we can use (4) to translate Definition 12 of convex subset of a semimodule into the following notion of convex subset of a \( S \)-algebra \( a; SX \rightarrow X \).

Definition 13. Let \( S \) be a semiring, \((X,a) \in \mathbb{EM}(S), A \subseteq X \). The convex closure of \( A \) in \((X,a)\) is the set

\[
\overline{A} = \left\{ a(\varphi) \mid \varphi \in SX, \ \text{supp} \varphi \subseteq A, \ \sum_{x \in X} \varphi(x) = 1 \right\}.
\]
A is said to be convex in \((X, a)\) if and only if \(A = \overline{A}\). We denote by \(\mathcal{P}_c X\) the set of convex subsets of \(X\) with respect to \(a\).

Remark 14. Observe that \(\emptyset\) is convex, because \(\overline{\emptyset} = \emptyset\), since there is no \(\varphi \in \mathcal{S}X\) with empty support such that \(\sum_{x \in X} \varphi(x) = 1\).

Example 15. Suppose \(S\) is such that \(\eta^S\) is weakly cartesian (equivalently (A) holds: \(x + y = 1 \implies x = 0\) or \(y = 0\)), for example \(S = \mathbb{N}\), and let \((X, a) \in \mathcal{EM}(S)\). A \(\varphi \in \mathcal{S}X\) such that \(\sum_{x \in X} \varphi(x) = 1\) and \(\text{supp} \, \varphi \subseteq A\) is a function that assigns 1 to \(\text{exactly one}\) element of \(A\) and 0 to all the other elements of \(X\). These functions are precisely all the \(\Delta_x\) for those elements \(x \in A\). Since \(a: \mathcal{S}X \to X\) is a structure map for an \(S\)-algebra, it maps the function \(\Delta_x\) into \(x\). Therefore \(\overline{A^a} = \{a(\Delta_x) \mid x \in A\} = \{x \mid x \in A\} = A\). Thus all \(A \in \mathcal{PS}X\) are convex.

Example 16. When \(S = \mathbb{B}ool\), we have that \(\mathcal{S}\) is naturally isomorphic to \(\mathcal{P}_f\), the finite powerset monad, whose algebras are idempotent commutative monoids or equivalently semilattices with a bottom element. So, for \((X, a) \in \mathcal{EM}(S)\), a \(\varphi \in \mathcal{S}X\) such that \(\sum_{x \in X} \varphi(x) = 1\) and \(\text{supp} \, \varphi \subseteq A\) is any finitely supported function from \(X\) to \(\mathbb{B}ool\) that assigns 1 to at least one element of \(A\). Intuitively, such a \(\varphi\) selects a non-empty finite subset of \(A\), then \(a(\varphi)\) takes the join of all the selected elements. Thus, \(\overline{A^a}\) adds to \(A\) all the possible joins of non-empty finite subsets of \(A\): \(A\) is convex if and only if it is closed under binary joins.

4 The Weak Distributive Law \(\delta: \mathcal{SP} \to \mathcal{PS}\)

Weak extensions of \(\mathcal{S}\) to \(\mathcal{Kl}(\mathcal{P}) = \mathbb{Rel}\) only consist of extensions of the functor \(\mathcal{S}\) and of the multiplication \(\mu^S\), for which necessary and sufficient conditions are listed in Theorem 3. Hence for semirings \(S\) satisfying those criteria a weak distributive law \(\delta: \mathcal{SP} \to \mathcal{PS}\) does exist, and it is unique because there is only one extension of the functor \(\mathcal{S}\) to \(\mathbb{Rel}\).

Theorem 17. Let \(S\) be a positive, refinable semiring satisfying (B) and (E) in Table 3. Then there exists a unique weak distributive law \(\delta: \mathcal{SP} \to \mathcal{PS}\) defined for all sets \(X\) and \(\Phi \in \mathcal{SP}X\) as:

\[
\delta_X(\Phi) = \left\{ \varphi \in \mathcal{S}X \mid \exists \psi \in \mathcal{S}(\exists X) \left( \begin{array}{l}
\forall A \in \mathcal{PX}, \Phi(A) = \sum_{x \in A} \psi(A, x) \\
\forall x \in X, \varphi(x) = \sum_{A \ni x} \psi(A, x)
\end{array} \right) \right\}
\]

where \(\exists X\) is the set \(\{(A, x) \in \mathcal{PX} \times X \mid x \in A\}\).

The above \(\delta\), which is obtained by following the standard recipe of Proposition 7 (see the proof in Appendix 3), is illustrated by the following example.

Example 18. Take \(S = \mathbb{R}^+\) with the usual operations of sum and multiplication. Consider \(X = \{x, y, z, a, b\}\), \(A_1 = \{x, y\}\), \(A_2 = \{y, z\}\) and \(A_3 = \{a, b\}\). Let \(\Phi \in \mathcal{S}(\mathcal{PX})\) be defined as:

\[
\Phi = (A_1 \mapsto 5, A_2 \mapsto 9, A_3 \mapsto 13)
\]
and $\Phi(A) = 0$ for all other sets $A \subseteq X$, so $\text{supp} \Phi = \{A_1, A_2, A_3\}$. In order to find an element $\varphi \in \delta_X(\Phi)$, we can first take a $\psi \in S(\exists X)$ satisfying condition (a) in (6) and then compute the $\varphi \in SX$ using condition (b).

Among the $\psi \in S(\exists X)$, consider for instance the following:

$$\psi = \begin{pmatrix} (A_1, x) \mapsto 2 \\ (A_2, y) \mapsto 4 \\ (A_3, a) \mapsto 6 \end{pmatrix}.$$ 

Since $\Phi(A_1) = \psi(A_1, x) + \psi(A_1, y), \Phi(A_2) = \psi(A_2, y) + \psi(A_2, z)$ and $\Phi(A_3) = \psi(A_3, a) + \psi(A_3, b)$, we have that $\psi$ satisfies condition (a) in (6). Condition (b) forces $\varphi$ to be the following:

$$\varphi = (x \mapsto 2, \ y \mapsto 3 + 4, \ z \mapsto 5, \ a \mapsto 6, \ b \mapsto 7).$$

**Remark 19.** If $S$ enjoys (A) in Table 2 then the transformation $\delta$ given in (6) is actually a distributive law, and for $S = \mathbb{N}$ we recover the well-known $\delta$ of Example 11 Example 18 can be repeated with $c$ where the set $A$ elements of $\Phi$ and $\psi$ among those $\psi$'s, there are some special, minimal ones as it were, that choose for each $A$ in $\text{supp} \Phi$ exactly one element of $A$, and assign to it $\Phi(A)$. The induced $\varphi$ in $\delta_X(\Phi)$ can be described as $\sum_{A \in u^{-1}(x)} \Phi(A)$ (equivalently $S(u)(\Phi')$) where $u: \text{supp} \Phi \to X$ is a function selecting an element of $A$ for each $A \in \text{supp} \Phi$ (that is $u(A) \in A$). We denote the set of such $\varphi$'s by $c(\Phi)$.

$$c(\Phi) = \{S(u)(\Phi') \mid u: \text{supp} \Phi \to X \text{ such that } \forall A \in \text{supp} \Phi, u(A) \in A\} \quad (7)$$

**Example 20.** Take $X, A_1$ and $A_2$ as in Example 18 but a different, smaller, $\Phi \in S(\mathcal{P}X)$ defined as $\Phi = (A_1 \mapsto 1, \ A_2 \mapsto 2)$. There are only four functions $u: \text{supp} \Phi \to X$ such that $u(A) \in A$ and thus only four functions $\varphi$ in $c(\Phi)$:

$$\begin{array}{ll}
\varphi_1 = (x \mapsto 1, \ y \mapsto 2) \\
\varphi_2 = (x \mapsto 1, \ z \mapsto 2) \\
\varphi_3 = (y \mapsto 3) \\
\varphi_4 = (y \mapsto 1, \ z \mapsto 2)
\end{array}$$

Observe that the function $\varphi = (x \mapsto 1, \ y \mapsto 1, \ z \mapsto 1)$ belongs to $\delta_X(\Phi)$ but not to $c(\Phi)$. Nevertheless $\varphi$ can be retrieved as the convex combination $\frac{1}{3} \varphi_1 + \frac{1}{3} \varphi_2$.\footnote{More precisely, we should write $S(u)(\Phi')$ where $\Phi'$ is the restriction of $\Phi$ to $\text{supp} \Phi$.}
Our key result states that every \( \varphi \in \delta_X(\Phi) \) can be written as a convex combination (performed in the \( S \)-algebra \((SX, \mu_X)\)) of functions in \( c(\Phi) \), at least when \( S \) is a positive semifield, which by Remark 9 and Proposition 10 satisfies all the conditions that make (6) a weak distributive law. The proof is laborious and can be found in Appendix 8; we only remark that divisions in \( S \) play a crucial role in it.

**Theorem 21.** Let \( S \) be a positive semifield. Then for all sets \( X \) and \( \Phi \in S\mathcal{P}X \)

\[
\delta_X(\Phi) = \left\{ \mu_X^S(\Psi) \mid \Psi \in S^2X, \sum_{\varphi \in \delta_X} \Psi(\varphi) = 1, \text{supp} \Psi \subseteq c(\Phi) \right\} = \overline{c(\Phi)}^{S_X}. \tag{8}
\]

**Remark 22.** If we drop the hypothesis of semifield and only have the minimal assumptions of Theorem 17, then (8) does not hold any more: \( S = \mathbb{N} \) is a counterexample. Indeed, in this case every subset of \( SX \) is convex with respect to \( \mu_X^S \) (see Example 15), therefore we would have \( \delta_X(\Phi) = c(\Phi) \), which is false: the function \( \varphi \) of Example 18 is an example of an element in \( \delta_X(\Phi) \setminus c(\Phi) \).

**Remark 23.** When \( S = \text{Bool} \) (which is a positive semifield), the monad \( S \) coincides with the monad \( \mathcal{P}f \). The function \( c(\cdot) \) in (7) can then be described as

\[
c(\mathcal{A}) = \{ \mathcal{P}_f(u)(\mathcal{A}) \mid u : \mathcal{A} \rightarrow X \text{ such that } \forall \mathcal{A} \in \mathcal{A}. u(\mathcal{A}) \in \mathcal{A} \}
\]

for all \( \mathcal{A} \in \mathcal{P}_f\mathcal{P}X \). It is worth remarking that this is the transformation \( \chi \) appearing in Example 9 of [26] (which is in turn equivalent to the one in Example 2.4.7 of [30]). This transformation was erroneously supposed to be a distributive law, as it fails to be natural (see [27]). However, by taking its convex closure, as displayed in [8], one can turn it into a weak distributive law.

### 5 The Weak Lifting of \( \mathcal{P} \) to \( \mathcal{E}\mathcal{M}(S) \)

By exploiting the characterisation of the weak distributive law \( \delta \) (Theorem 21), we can now describe the weak lifting of \( \mathcal{P} \) to \( \mathcal{E}\mathcal{M}(S) \) generated by \( \delta \).

Recall from Definition 13 that \( \mathcal{P}_aX \) is the set of convex subsets of \( X \) with respect to the \( S \)-algebra \( a : SX \rightarrow X \). The functions \( \iota_{(X,a)} : \mathcal{P}_aX \rightarrow \mathcal{P}X \) and \( \pi_{(X,a)} : \mathcal{P}X \rightarrow \mathcal{P}_aX \) are defined for all \( A \in \mathcal{P}_aX \) and \( B \in \mathcal{P}X \) as

\[
\iota_{(X,a)}(A) = A \quad \text{and} \quad \pi_{(X,a)}(B) = \overline{B}^a,
\tag{9}
\]

that is \( \iota_{(X,a)} \) is just the obvious set inclusion and \( \pi_{(X,a)} \) performs the convex closure in \( a \). The function \( \alpha_a : S\mathcal{P}_aX \rightarrow \mathcal{P}_aX \) is defined for all \( \Phi \in S\mathcal{P}_aX \) as

\[
\alpha_a(\Phi) = \{ a(\varphi) \mid \varphi \in c(\Phi) \}. \tag{10}
\]

To be completely formal, above we should have written \( c(S(\iota)(\Phi)) \) in place of \( c(\Phi) \), but it is immediate to see that the two sets coincide. Proving that \( \alpha_a : S\mathcal{P}_aX \rightarrow \mathcal{P}_aX \) is well defined (namely, \( \alpha_a(\Phi) \) is a convex set) and forms an
\(\mathcal{S}\)-algebra requires some ingenuity and will be shown later in Section \[24\]. The assignment \((X, a) \mapsto (\mathcal{P}^a_c X, \alpha_a)\) gives rise to a functor \(\widehat{\mathcal{P}}: \text{EM}(\mathcal{S}) \rightarrow \text{EM}(\mathcal{S})\) defined on morphisms \(f: (X, a) \rightarrow (X', a')\) as

\[
\widehat{\mathcal{P}}(f)(A) = \mathcal{P}f(A) \tag{11}
\]

for all \(A \in \mathcal{P}^a_c X\). For all \((X, a)\) in \(\text{EM}(\mathcal{S})\), \(\eta_{(X,a)}^\mathcal{P}: (X, a) \rightarrow \widehat{\mathcal{P}}(X, a)\) and \(\mu_{(X,a)}^\mathcal{P}: \widehat{\mathcal{P}}\widehat{\mathcal{P}}(X, a) \rightarrow \widehat{\mathcal{P}}(X, a)\) are defined for \(x \in X\) and \(A \in \mathcal{P}^a_c (\mathcal{P}^a_c X)\) as

\[
\eta_{(X,a)}^\mathcal{P}(x) = \{x\} \quad \text{and} \quad \mu_{(X,a)}^\mathcal{P}(A) = \bigcup_{A \in A} A. \tag{12}
\]

**Theorem 24.** Let \(\mathcal{S}\) be a positive semifield. Then the canonical weak lifting of the powerset monad \(\mathcal{P}\) to \(\text{EM}(\mathcal{S})\), determined by \([3]\), consists of the monad \((\widehat{\mathcal{P}}, \eta^\mathcal{P}, \mu^\mathcal{P})\) on \(\text{EM}(\mathcal{S})\) defined as in \([10], [11], [12]\) and the natural transformations \(\iota: U^S\widehat{\mathcal{P}} \rightarrow \mathcal{P}U^S \) and \(\pi: \mathcal{P}U^S \rightarrow U^S\mathcal{P}\) defined as in \([9]\).

It is worth spelling out the left-semimodule structure on \(\mathcal{P}^a_c X\) corresponding to the \(\mathcal{S}\)-algebra \(\alpha_a: \mathcal{SP}^a_c X \rightarrow \mathcal{P}^a_c X\). Let us start with \(\lambda^{a\alpha} A\) for some \(A \in \mathcal{P}^a_c X\). By \([6]\), \(\lambda^{a\alpha} A = \alpha_a(\Phi)\) where \(\Phi = (A \mapsto \lambda)\). By \([10]\), \(\alpha_a(\Phi) = \{a(\varphi) \mid \varphi \in \mathcal{C}(\Phi)\}\). Following the definition of \(\mathcal{C}(\Phi)\) given in \([7]\), one has to consider functions \(u: \text{supp} \Phi \rightarrow X\) such that \(u(B) \in B\) for all \(B \in \text{supp} \Phi\): if \(\lambda \neq 0\), then \(\text{supp} \Phi = \{A\}\) and thus, for each \(x \in A\), there is exactly one function \(u_x: \text{supp} \Phi \rightarrow X\) mapping \(A\) into \(x\). It is immediate to see that \(S(u_x)(\Phi)\) is exactly the function \((x \mapsto \lambda)\) and thus \(a(S(u_x)(\Phi))\) is, by \([5]\), \(\lambda^{a\alpha} x\). Now if \(\lambda = 0\), then \(\text{supp} \Phi = \emptyset\), so there is exactly one function \(u: \text{supp} \Phi \rightarrow X\) and \(S(u)(\Phi)\) is the function mapping all \(x \in X\) into 0 and thus, by \([15]\), \(a(S(u)(\Phi)) = 0^a\).

Summarising,

\[
\lambda^{a\alpha} A = \begin{cases} 
\{\lambda^{a\alpha} x \mid x \in A\} & \text{if } \lambda \neq 0 \\
\{0^a\} & \text{if } \lambda = 0
\end{cases} \tag{13}
\]

Following similar lines of thoughts, one can check that

\[
A +^{a\alpha} B = \{x +^a y \mid x \in A, \ y \in B\} \quad \text{and} \quad 0^{a\alpha} = \{0^a\}. \tag{14}
\]

**Remark 25.** By comparing \([14]\) and \([15]\) with \([4]\) and \([5]\) in \([24]\), it is immediate to see that our monad \(\widehat{\mathcal{P}}\) coincides with a slight variation of Jacobs's convex powerset monad \(\mathcal{C}\), the only difference being that we do allow for \(\emptyset\) to be in \(\mathcal{P}^a X\). Jacobs insisted on the necessity of \(\mathcal{C}(X)\) to be the set of non-empty convex subsets of \(X\), because otherwise he was not able to define a semimodule structure on \(\mathcal{C}(X)\) such that \(0 \cdot \emptyset = \{0^a\}\). However, we do manage to do so, since by \([13]\), \(0 \cdot A = 0^a\) for all \(A\) and in particular for \(A = \emptyset\). At first sight, this may look like an ad-hoc solution, but this is not the case: it is intrinsic in the definition of the unique weak lifting of \(\mathcal{P}\) to \(\text{EM}(\mathcal{S})\), as stated by Theorem \[24\] and shown next.
5.1 Proof of Theorem 24

By Theorem 5, the weak distributive law (6) corresponds to a weak lifting \( \tilde{P} \) of \( P \) to \( EM(S) \), which we are going to show coincides with the data of (9)-(12). The image along \( \tilde{P} \) of a \( S \)-algebra \( (X, a) \) will be a set \( Y \) together with a structure map \( \alpha_a \) that makes it a \( S \)-algebra in turn. Garner [14, Proposition 13] gives us the recipe to build \( Y \) and \( \alpha_a \) appropriately. \( Y \) is obtained by splitting the following idempotent in \( Set \):

\[
e(x,a) = P(X) \xrightarrow{\eta_{P(X)}} S(PX) \xrightarrow{S(X)} P(SX) \xrightarrow{P(X)} P(X)
\]

as a composite \( e(x,a) = \iota_{(X,a)} \circ \pi_{(X,a)} \), where \( \pi_{(X,a)} \) is the corestriction of \( e_{(X,a)} \) to its image and \( \iota_{(X,a)} \) is the set-inclusion of the image of \( e_{(X,a)} \) into \( P(X) \). In other words, \( Y \) is the set of fixed points of \( e_{(X,a)} \). \( \alpha_a \) is obtained as the composite

\[
\alpha_a = SY \xrightarrow{S(\iota_{(X,a)})} SPX \xrightarrow{S(\pi_{(X,a)})} P(SX) \xrightarrow{P(X)} P(X) \xrightarrow{\pi_{(X,a)}} Y.
\]

Let us try to calculate \( \alpha_a \): given \( \Phi : \mathcal{P}^a_c X \to S \) with finite support, we have that \( S(\iota_{(X,a)}(\Phi)) \) is just the extension of \( \Phi \) to \( PX \) which assigns 0 to each non-convex subset of \( X \). If we write \( \iota \) instead of \( \iota_{(X,a)} \) for short, we have

\[
\alpha_a(\Phi) = P(a(S(\iota)(\Phi)))^a.
\]

Next, we can use the following technical result, whose proof is in Appendix 9.
Proposition 27. Let \((X, a)\) be a \(S\)-algebra. If \(A\) is a convex subset of \((SX, \mu_X^S)\), then \(Pa(A)\) is convex in \((X, a)\).

Since \(\delta_X(\Phi')\) is the convex closure of \(c(\Phi')\) in \((SX, \mu_X^S)\) for every \(\Phi' \in SPX\), by Proposition 27 we can avoid to perform the \(a\)-convex closure in (16). Therefore

\[
\alpha_a(\Phi) = Pa(\delta_X(S(\iota)(\Phi))) = Pa(c(S(\iota)(\Phi))) = Pa(\delta_X(\Phi)).
\]

In the next Proposition we show that also the \(\mu_X^S\)-convex closure is superfluous, due to the fact that \(\Phi \in SP^a \subseteq \) and not simply \(SPX\), thus obtaining (10).

Proposition 28. Let \(S\) be a positive semifield, \((X, a)\) a \(S\)-algebra, \(\Phi \in SP^a X\). Then \(Pa(\delta_X(S(\iota)(\Phi))) = Pa(c(S(\iota)(\Phi)))\).

Proof. In this proof we shall simply write \(\Phi\) instead of the more verbose \(S(\iota)(\Phi)\).

We want to prove that

\[
Pa(\delta_X(\Phi)) = \left\{ a(\psi) \mid \psi \in SX, \exists u: \text{supp} \Phi \rightarrow X. u(A) \in A, \forall x \in X. \psi(x) = \sum_{\alpha \in \text{supp} \Phi} \Phi(A) \right\}
\]

where we have, by Theorem 21 that

\[
Pa(\delta_X(\Phi)) = \{a(\mu_X^S(\psi)) \mid \Psi \in S^2X, \sum_{\varphi \in SX} \Psi(\varphi) = 1, \supp \Psi \subseteq c(\Phi)\}.
\]

First of all, \(\emptyset\) is not a \(S\)-algebra, because there is no map \(S(\emptyset) \rightarrow \emptyset\) given that \(S(\emptyset) = \{\emptyset\} \rightarrow S\), hence \(X \neq \emptyset\). Next, if \(\Phi = \varepsilon: PX \rightarrow S\), namely the function constant to 0, then \(c(\Phi) = \{\varepsilon: X \rightarrow S\}\) therefore one can easily see that the left-hand side of (17) is equal to \(a(\varepsilon: X \rightarrow S)\). For the same reason, the right-hand side is also equal to \(a(\varepsilon: X \rightarrow S)\). Moreover, if \(\Phi(\emptyset) \neq 0\), then there is no \(u: \text{supp} \Phi \rightarrow X\) such that \(u(\emptyset) \in \emptyset\), so \(c(\Phi) = \emptyset\) and so is the left-hand side of (17); for the same reason, also the right-hand side is empty.

Suppose then, for the rest of the proof, that \(\Phi \neq 0\) and that \(\Phi(\emptyset) = 0\).

For the right-to-left inclusion in (17): given \(\psi \in c(\Phi)\), consider \(\Psi = \eta_X^{S}(\psi) = \Delta_\psi \in S^2X\). Then \(\Psi\) clearly satisfies all the required properties and \(\mu_X^S(\Psi) = \psi\).

The left-to-right inclusion is more laborious. Let \(\Psi \in S^2X\) be such that

\[
\sum_{\chi \in SX} \Psi(\chi) = 1 \quad \text{and such that } \supp \Psi \subseteq c(\Phi), \text{ that is, for all } \varphi \in \supp \Psi \text{ there is } u\varphi: \text{supp} \Phi \rightarrow X \text{ such that } u\varphi(A) \in A \text{ for all } A \in \text{supp} \Phi \text{ and } \varphi = S(u\varphi)(\Phi). \]

We have to show that \(a(\mu(\Psi)) = a(\psi)\) for some \(\psi \in SX\) of the form

\[
\sum_{A \in \text{supp} \Phi} \Phi(A).u(A) \text{ for some choice function } u: \text{supp} \Phi \rightarrow X.
\]

Notice that the given \(\Psi\) is a convex linear combination of functions \(\varphi\)'s in \(SX\) like the one we have to produce: the trick will be to exploit the fact that each \(A \in \text{supp} \Phi\) is convex.

Here we shall only give a sketch of the proof; the detailed version can be found in Appendix 9. Suppose \(\text{supp} \Phi = \{A_1, \ldots, A_n\}\) and \(\text{supp} \Psi = \{\varphi^1, \ldots, \varphi^m\}\). Call
$u^\iota$ the choice function that generates $\wp^\iota$. Then $\Psi$ is of this form:

$$
\Psi = \left( \begin{array}{c}
u^1(A_1) \mapsto \Phi(A_1) \\
\vdots \\
u^1(A_n) \mapsto \Phi(A_n)
\end{array} \right) \mapsto \Psi(\wp^1), \ldots, 
\left( \begin{array}{c}
u^m(A_1) \mapsto \Phi(A_1) \\
\vdots \\
u^m(A_n) \mapsto \Phi(A_n)
\end{array} \right) \mapsto \Psi(\wp^m)
$$

Define the following element of $S^2X$:

$$
\Psi' = \left( \begin{array}{c}
u^1(A_1) \mapsto \Psi(\wp^1) \\
\vdots \\
u^m(A_1) \mapsto \Psi(\wp^m)
\end{array} \right) \mapsto \Phi(A_1), \ldots, 
\left( \begin{array}{c}
u^1(A_1) \mapsto \Psi(\wp^1) \\
\vdots \\
u^m(A_1) \mapsto \Psi(\wp^m)
\end{array} \right) \mapsto \Phi(A_n)
$$

Observe that $\nu^1(A_i), \ldots, \nu^m(A_i) \in A_i$ by definition, and $A_i$ is convex by assumption: since $\sum_{i=1}^n \Psi(\wp^i) = 1$, we have that $\alpha(\chi^i) \in A_i$. Set then $u(A_i) = a(\chi^i)$ and define $\psi = S(a)(\wp^i)$; we have $\psi \in c(\wp)$ with $U$ as the generating choice function. It is not difficult to see that $\mu^S(\psi) = \mu^S(\wp^i)$, therefore we have

$$a(\psi) = a(S(a)(\wp^i)) = a(\mu^S(\wp^i)) = a(\mu^S(\psi))$$

as desired. \hfill \Box

The rest of the proof of Theorem 24, concerning the action of $\tilde{P}$ on morphisms and the unit and multiplication of the monad $\tilde{P}$, consists in following the recipe provided by Garner [14]; the details can be found in Appendix 9.

6 The Composite Monad: an Algebraic Presentation

We can now compose the two monads $\mathcal{P}$ and $\mathcal{S}$ by considering the monad arising from the composition of the following two adjunctions:

$$
\begin{array}{ccc}
\text{Set} & \xrightarrow{\perp} & \text{EM}(\mathcal{S}) \\
\xleftarrow{\perp} & \text{EM}(\tilde{\mathcal{P}}) & \xrightarrow{\perp}
\end{array}
$$

Direct calculations show that the resulting endofunctor on Set, which we call $\mathcal{P}_c\mathcal{S}$, maps a set $X$ and a function $f: X \to Y$ into, respectively,

$$
\mathcal{P}_c\mathcal{S}X = \mathcal{P}_c^{\tilde{\mathcal{P}}}(\mathcal{S}X) \quad \text{and} \quad \mathcal{P}_c\mathcal{S}(f)(A) = \{\mathcal{S}(f)(\Phi) \mid \Phi \in A\} \quad (18)
$$

for all $A \in \mathcal{P}_c\mathcal{S}X$. For all sets $X$, $\eta^\mathcal{P}_c\mathcal{S}_X: X \to \mathcal{P}_c\mathcal{S}X$ and $\mu^\mathcal{P}_c\mathcal{S}_X: \mathcal{P}_c\mathcal{S}\mathcal{P}_c\mathcal{S}X \to \mathcal{P}_c\mathcal{S}X$ are defined as

$$
\eta^\mathcal{P}_c\mathcal{S}_X(x) = \{\Delta x\} \quad \text{and} \quad \mu^\mathcal{P}_c\mathcal{S}_X(\mathcal{O}) = \bigcup_{\Omega \in \mathcal{O}} \alpha^\mathcal{P}_c\mathcal{S}_X(\Omega) \quad (19)
$$

for all $x \in X$ and $\mathcal{O} \in \mathcal{P}_c\mathcal{S}\mathcal{P}_c\mathcal{S}X$. 

Theorem 29. Let \( S \) be a positive semifield. Then the canonical weak distributive law \( \delta : SP \to PS \) given in Theorem 29 induces a monad \( P,S \) on \( S \) with endofunctor, unit and multiplication defined as in \( \delta \) and \( \delta \).

Recall from Remark 25 that the monad \( C : EM(S) \to EM(S) \) from 24 coincides with our lifting \( \hat{P} \) modulo the absence of the empty set. The same happens for the composite monad, which is named \( CM \) in 24. The absence of \( \emptyset \) in \( CM \) turns out to be rather problematic for Jacobs. Indeed, in order to use the standard framework of coalgebraic trace semantics [19], one would need the Kleisli category \( Kl(CM) \) to be enriched over \( CPPO \), the category of \( \omega \)-complete partial orders with bottom and continuous functions. \( Kl(CM) \) is not \( CPPO \)-enriched since there is no bottom element in \( CM(X) \). Instead, in \( P,SX \) the bottom is exactly the empty set; moreover, \( Kl(P,S) \) enjoys the properties required by [19].

Theorem 30. The category \( Kl(P,S) \) is enriched over \( CPPO \) and satisfies the left-strictness condition: for all \( f : X \to P,SY \) and \( Z \) set, \( \bot_{Y,Z} \circ f = \bot_{X,Z} \).

It is immediate that every homset in \( Kl(P,S) \) carries a complete partial order. Showing that composition of arrows in \( Kl(P,S) \) preserves joins (of \( \omega \)-chains) requires more work: the proof, shown in Appendix 10, crucially relies on the algebraic theory presenting the monad \( P,S \), illustrated next.

An Algebraic Presentation. Recall that an algebraic theory is a pair \( T = (\Sigma,E) \) where \( \Sigma \) is a signature, whose elements are called operations, to each of which is assigned a cardinal number called its arity, while \( E \) is a class of formal equations between \( \Sigma \)-terms. An algebra for the theory \( T \) is a set \( A \) together with, for each operation \( o \) of arity \( \kappa \) in \( \Sigma \), a function \( o_A : A^\kappa \to A \) satisfying the equations of \( E \). A homomorphism of algebras is a function \( f : A \to B \) respecting the operations of \( \Sigma \) in their realisations in \( A \) and \( B \). Algebras and homomorphisms of an algebraic theory \( T \) form a category \( Alg(T) \).

Definition 31. Let \( M \) be a monad on \( S \), and \( T \) an algebraic theory. We say that \( T \) presents \( M \) if and only if \( EM(M) \) and \( Alg(T) \) are isomorphic.

Left \( S \)-semimodules are algebras for the theory \( LSM = (\Sigma_{LSM},E_{LSM}) \) where \( \Sigma_{LSM} = \{+,-,0\} \cup \lambda \cdot \lambda | \lambda \in S \) and \( E_{LSM} \) is the set of axioms in Table 1. As already mentioned in Section 3 left \( S \)-semimodules are exactly \( S \)-algebras and morphisms of \( S \)-semimodules coincide with those of \( S \)-algebras. Thus, the theory \( LSM \) presents the monad \( S \).

Similarly, semilattices are algebras for the theory \( SL = (\Sigma_{SL},E_{SL}) \) where \( \Sigma_{SL} = \{\bot,\cup\} \) and \( E_{SL} \) is the set of axioms in Table 1. It is well known that semilattices are algebras for the finite powerset monad. Actually, this monad is presented by \( SL \). In order to present the full powerset monad \( P \) we need to take joins of arbitrary arity. A complete semilattice is a set \( X \) equipped with joins \( \bigsqcup_{x \in A} x \) for all—not necessarily finite—\( A \subseteq X \). Formally the (infinitary) theory of complete semilattices is given as \( CSL = (\Sigma_{CSL},E_{CSL}) \) where \( \Sigma_{CSL} = \{\bigsqcup_i | \)
Table 3. The sets of axioms $E_{\text{CSL}}$ for complete semilattices: the second axiom generalises the usual idempotency and commutativity properties of finitary $\sqcup$, while the third one generalises associativity and neutrality of $\sqcap$.

\[
\begin{align*}
\bigcup_{i \in \{0\}} x_i &= x_0 \\
\bigcup_{j \in J} x_j &= \bigcup_{i \in I} x_{f(i)} \text{ for all } f : I \to J \text{ surjective} \\
\bigcup_{i \in I} x_i &= \bigcup_{j \in J} \bigcup_{k \in f^{-1}(j)} x_i \text{ for all } f : I \to J
\end{align*}
\]

$I$ set} and $E_{\text{CSL}}$ is the set of axioms displayed in Table 3 (for a detailed treatment of infinitary algebraic theories see, for example, [29]).

We can now illustrate the theory $(\Sigma, E)$ presenting the composed monad $P \text{cS}$; the operations in $\Sigma$ are exactly those of complete semilattices and $S$-semimodules, while the axioms are those of complete semilattices and $S$-semi modules together with the set $E_D$ of distributivity axioms illustrated below.

\[
\lambda \cdot \bigcup_{i \in I} x_i = \bigcup_{i \in I} \lambda \cdot x_i \text{ for } \lambda \neq 0, \quad \bigcup_{i \in I} x_i + \bigcup_{j \in J} y_j = \bigcup_{(i,j) \in I \times J} x_i + y_j \tag{20}
\]

In short, $\Sigma = \Sigma_{\text{CSL}} \cup \Sigma_{\text{LSM}}$ and $E = E_{\text{CSL}} \cup E_{\text{LSM}} \cup E_D$.

Theorem 32. The monad $P \text{cS}$ is presented by the algebraic theory $(\Sigma, E)$.

The presentation crucially relies on the fact that $P \text{cS}$ is obtained by composing $P$ and $S$ via $\delta$. Indeed, we know from general results in [10,14] that $P \text{cS}$-algebras are in one to one correspondence with $\delta$-algebras [3], namely triples $(X,a,b)$ such that $a : SX \to X$ is a $S$-algebra, $b : PX \to X$ is a $P$-algebra and the following diagram commutes.

\[
\begin{array}{ccc}
SPX & \xrightarrow{\delta_X} & PSX \\
\downarrow{s_b} & & \downarrow{p_a} \\
SX & & PX \\
\downarrow{a} & \searrow{b} & \searrow{b} \\
X & & 
\end{array}
\tag{21}
\]

The $S$-algebra $a$ corresponds to a $S$-semimodule $(X, +, 0, \cdot)$, the $P$-algebra $b$ to a complete lattice $(X, \sqcup)$ and the commutativity of diagram (21) expresses exactly the distributivity axioms in (20). The proof is given in Appendix 10.

Example 33. Let $S$ be $\mathbb{R}^+$ and let $[a, b]$ with $a, b \in \mathbb{R}^+$ denote the set $\{x \in \mathbb{R}^+ \mid a \leq x \leq b\}$ and $[a, \infty)$ the set $\{x \in \mathbb{R}^+ \mid a \leq x\}$. For $1 = \{x\}$, $P \text{cS}(1) = \{\emptyset\} \cup \{[a, b] \mid a, b \in \mathbb{R}^+\} \cup \{(a, +\infty) \mid a \in \mathbb{R}^+\}$. The $P \text{cS}$-algebra $\mu_{1}^{P \text{cS}} : P \text{cSPcS}1 \to$
The \( \mathbb{R}^+ \)-semimodule is as expected, e.g., \([a_1, b_1] + [a_2, b_2] = [a_1 + a_2, b_1 + b_2] \).

**Finite Joins and Finitely Generated Convex Sets.** We now consider the algebraic theory \((\Sigma', E')\) obtained by restricting \((\Sigma, E)\) to finitary joins. More precisely, we fix

\[
\Sigma' = \Sigma_{\mathcal{SL}} \cup \Sigma_{\mathcal{LSM}} \quad E' = E_{\mathcal{SL}} \cup E_{\mathcal{LSM}} \cup E_{\mathcal{D}}
\]

where \((\Sigma_{\mathcal{SL}}, E_{\mathcal{SL}})\) is the algebraic theory for semilattices, \((\Sigma_{\mathcal{LSM}}, E_{\mathcal{LSM}})\) is the one for \(S\)-semimodules, and \(E_{\mathcal{D}}\) is the set of distributivity axioms illustrated in Table\(\text{II}\). Thanks to the characterisation provided by Theorem\(\text{32}\), we easily obtain a function translating \(\Sigma'\)-terms into convex subsets (the proof is in Appendix\(\text{[10]}\)).

**Proposition 34.** Let \(T_{\Sigma', E'}(X)\) be the set of \(\Sigma'\)-terms with variables in \(X\) quotiented by \(E'\). Let \([\cdot]_X : T_{\Sigma', E'}(X) \to \mathcal{P}_S(X)\) be the function defined as

\[
\begin{align*}
[x] & = \{ A_x \} \text{ for } x \in X, \\
[0] & = \{ 0^{\mathcal{LS}} \}, \\
[\bot] & = \emptyset, \\
[\lambda \cdot t] & = \begin{cases} 
\{ \lambda, a^{\mathcal{LS}} f \mid f \in [t] \} & \text{if } \lambda \neq 0 \\
\{ 0^{\mathcal{LS}} \} & \text{otherwise}
\end{cases}, \\
[t_1 + t_2] & = \{ f_1 + a^{\mathcal{LS}} f_2 \mid f_1 \in [t_1], \ f_2 \in [t_2] \} \\
[t_1 \sqcup t_2] & = \{ t_1 \} \sqcup \{ t_2 \}^{\mathcal{LS}}
\end{align*}
\]

Let \([\cdot] : T_{\Sigma', E'} \to \mathcal{P}_S\) be the family \(\{ [\cdot]_X \}_{X \in \mathbb{Set}}\). Then \([\cdot] : T_{\Sigma', E'} \to \mathcal{P}_S\) is a map of monads and, moreover, each \([\cdot]_X : T_{\Sigma', E'}(X) \to \mathcal{P}_S(X)\) is injective.

We say that a set \(A \in \mathcal{P}_S(X)\) is finitely generated if there exists a finite set \(B \subseteq \mathcal{S}(X)\) such that \(\mathfrak{S} = A\). We write \(\mathcal{P}_{f, S}(X)\) for the set of all \(A \in \mathcal{P}_S(X)\) that are finitely generated. The assignment \(X \mapsto \mathcal{P}_{f, S}(X)\) gives rise to a monad \(\mathcal{P}_{f, S} : \mathbb{Set} \to \mathbb{Set}\) where the action on functions, the unit and the multiplication are defined as for \(\mathcal{P}_S\). The reader can find a proof of this, as well as of the following Theorem, in Appendix\(\text{[10]}\).

**Theorem 35.** The monads \(T_{\Sigma', E'}\) and \(\mathcal{P}_{f, S}\) are isomorphic. Therefore \((\Sigma', E')\) is a presentation for the monad \(\mathcal{P}_{f, S}\).

**Example 36.** Recall \(\mathcal{P}_S(1)\) for \(S = \mathbb{R}^+\) from Example\(\text{33}\). By restricting to the finitely generated convex sets, one obtains \(\mathcal{P}_{f, S}(1) = \{ \emptyset \} \cup \{ [a, b] \mid a, b \in \mathbb{R}^+ \}\), that is the sets of the form \([a, \infty)\) are not finitely generated. Table\(\text{4}\) illustrates the isomorphism \([\cdot] : T_{\Sigma', E'}(1) \to \mathcal{P}_S(1)\). It is worth observing that every closed interval \([a, b]\) is denoted by a term in \(T_{\Sigma', E'}(1)\) for \(1 = \{ x \}\): indeed,

\[\text{for the sake of brevity, we are ignoring the case where some } A_i = \emptyset.\]
Table 4. The inductive definition of the function \([\cdot]_1: T_{SP}(1) \to P_cS(1)\) for \(1 = \{x\}\).

\[ [\lambda \cdot t] = \begin{cases} [\lambda \cdot a, \lambda \cdot b] & \text{if } \lambda \neq 0, [t] = [a, b] \\ \emptyset & \text{if } \lambda \neq 0, [t] = \emptyset \\ [0, 0] & \text{otherwise} \end{cases} \]

\[ [x] = [1, 1] \]
\[ [0] = [0, 0] \]
\[ [\perp] = \emptyset \]

\[ [t_1 + t_2] = \begin{cases} [a_1 + a_2, b_1 + b_2] & \text{if } [t_1] = [a_1, b_1] \\ \emptyset & \text{otherwise} \end{cases} \]

\[ [t_1 \perp t_2] = \begin{cases} [\min a_1, \max b_1] & \text{if } [t_1] = [a_1, b_1] \\ [a_1, b_1] & \text{if } [t_1] = [a_1, b_1], [t_2] = \emptyset \\ [a_2, b_2] & \text{if } [t_2] = [a_2, b_2], [t_1] = \emptyset \\ \emptyset & \text{otherwise} \end{cases} \]

\[(a \cdot x) \sqcup (b \cdot x) = [a, b].\] For \(2 = \{x, y\}\), \(P_{fcS}(2)\) is the set containing all convex polygons; for instance the term \((r_1 \cdot x + s_1 \cdot y) \sqcup (r_2 \cdot x + s_2 \cdot y) \sqcup (r_3 \cdot x + s_3 \cdot y)\) denote a triangle with vertexes \((r_1, s_1), (r_2, s_2), (r_3, s_3)\). For \(n = \{x_0, \ldots, x_{n-1}\}\), it is easy to see that \(P_{fcS}(n)\) contains all convex \(n\)-polytopes.

7 Conclusions: Related and Future Work

Our work was inspired by [16] where Goy and Petrisan compose the monads of powerset and probability distributions by means of a weak distributive law in the sense of Garner [14]. Our results also heavily rely on the work of Clementino et al. [11] that illustrates necessary and sufficient conditions on a semiring \(S\) for the existence of a weak distributive law \(\delta: SP \to PS\). However, to the best of our knowledge, the alternative characterisation of \(\delta\) provided by Theorem 21 was never shown.

Such characterisation is essential for giving a handy description of the lifting \(\hat{P}: EM(S) \to EM(S)\) (Theorem 24) as well as to observe the strong relationships with the work of Jacobs (Remark 25) and the one of Klin and Rot (Remark 23). The weak distributive law \(\delta\) also plays a key role in providing the algebraic theories presenting the composed monad \(P_cS\) (Theorem 24) and its finitary restriction \(P_{fcS}\) (Theorem 35). These two theories resemble those appearing in, respectively, [16] and [9] where the monad of probability distributions plays the role of the monad \(S\) in our work.

Theorem 30 allows to reuse the framework of coalgebraic trace semantics [19] for modelling over \(\mathbb{K}l(P_cS)\) systems with both nondeterminism and quantitative features. The alternative framework based on coalgebras over \(EM(P_cS)\) directly leads to nondeterministic weighted automata. A proper comparison with those in [12] is left as future work. Thanks to the abstract results in [7], language equivalence for such coalgebras could be checked by means of coinductive up-to techniques. It is worth remarking that, since \(\delta\) is a weak distributive law,
then thanks to the work in \cite{15}, up-to techniques are also sound for “convex-bisimilarity” (in coalgebraic terms, behavioural equivalence for the lifted functor $\tilde{\mathcal{P}} : \mathcal{EM}(\mathcal{S}) \to \mathcal{EM}(\mathcal{S})$).

We conclude by recalling that we have two main examples of positive semifields: $\mathbb{B}ool$ and $\mathbb{R}^+$. Booleans could lead to a coalgebraic modal logic and trace semantics for alternating automata in the style of \cite{20}. For $\mathbb{R}^+$, we hope that exploiting the ideas in \cite{33} our monad could shed some lights on the behaviour of linear dynamical systems featuring some sort of nondeterminism.

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8 Appendix to Section 4

Proof of Theorem 17. We calculate $\delta$ by first analysing the canonical extension $\mathcal{S}$ of $S$ to $\mathbb{Rel}$ and then by following the proof of Theorem 5.
The formula to extend the functor $S$ to $\mathbb{R}el$ is the following: on objects it behaves like $S$; if $R \subseteq A \times B$ is a relation and $\pi_A : R \to A$ and $\pi_B : R \to B$ are the two projections, we have that

$$\tilde{S}(R) = \Gamma(S(\pi_B)) \circ \Gamma(S(\pi_A))^{-1}$$

(composition performed in $\mathbb{R}el$) where $\Gamma$ computes the graph of a function. Now, for all $\psi : R \to S$

$$S(\pi_A)(\psi) = \left( a \mapsto \sum_{r \in \pi_A^{-1}(a)} \psi(r) \right) = \left( a \mapsto \sum_{b \in B \atop a R b} \psi(a,b) \right)$$

and similarly for $S(\pi_B)(\psi)$. So,

$$\tilde{S}(R) = \left\{ \left( a \mapsto \sum_{b \in B \atop a R b} \psi(a,b), (b \mapsto \sum_{a \in A \atop a R b} \psi(a,b)) \right) \mid \psi : R \to S, \supp \psi \text{ finite} \right\}.$$

From this we obtain a weak distributive law as follows. Recall the identity-on-objects isomorphism of categories $F : \mathbb{K}l(\mathcal{P}) \to \mathbb{R}el$ where for $f : X \to \mathcal{P}(Y)$ in $\mathbb{S}et$, $F(f) = \{(x,y) \mid y \in f(x)\} \subseteq X \times Y$ and for $R \subseteq X \times Y$, $F^{-1}(R) = \{x \mapsto \{y \in Y \mid x R y\}\} : X \to \mathcal{P}(Y)$. Consider $id_{\mathcal{P}X} : \mathcal{P}X \to \mathcal{P}X$ in $\mathbb{S}et$. This is a Kleisli map $id_{\mathcal{P}X} : \mathcal{P}X \to X$ in $\mathbb{K}l(\mathcal{P})$, which corresponds to the relation $\exists X : \mathcal{P}X \to X = \{(A,x) \mid x \in A\}$. Then we have

$$\tilde{S}(\exists X) = \left\{ \left( (A \mapsto \sum_{x \in A} \psi(A,x)), (x \mapsto \sum_{A \exists x} \psi(A,x)) \right) \mid \psi \in S(\exists X) \right\}.$$

Remember that $\tilde{S}$ and $S$ coincide on objects. This relation, seen back as a Kleisli map $S\mathcal{P}X \to S\mathcal{X}$, gives us the $X$-th component of the desired weak distributive law, which is indeed (6). \qed

The Proof of Theorem 21. We need a lemma that generalises the distributivity property in an arbitrary semiring. The statement of the following Lemma makes sense only for commutative semirings, but it can be adapted for arbitrary semirings by restricting it to sets $L$ only of the form $\underline{n} = \{1, \ldots, n\}$.

Lemma 37. For all $n \in \mathbb{N}$, $L \subseteq \mathbb{N}$ such that $|L| = n$, for all $(s_k)_{k \in L} \in \mathbb{N}^n$, for all $(\lambda_j^k)_{j \in L}$ family of elements of $S$:

$$\sum_{u \in \prod_{k \in L} s_k} \prod_{k \in L} \lambda_{u_k}^k = \prod_{k \in L} \sum_{j=1}^{s_k} \lambda_j^k. \quad (22)$$

Proof. By induction on $n$. If $n = 0$, both sides of (22) are 0.
Let now \( n \geq 0 \), suppose that the statement of the Lemma holds for \( n \), let us consider a \( L \subset \mathbb{N} \) such that \( |L| = n + 1 \), \((s_k)_{k \in L} \in \mathbb{N}^{n+1} \), \((\lambda^k_j)_{j \in L} \). Without loss of generality we can assume that \( L = n + 1 \). Then we have:

\[
\sum_{w \in \prod_{k=1}^{n+1}} \prod_{k=1}^{n+1} \lambda^k w_k = \sum_{j=1}^{n+1} \left( \sum_{w \in \prod_{k=1}^{n}} \prod_{k=1}^{n} \lambda^k w_k \right) \cdot \lambda^j_{n+1} \\
= \sum_{j=1}^{n+1} \left( \sum_{w \in \prod_{k=1}^{n}} \prod_{k=1}^{n} \lambda^k w_k \right) \cdot \lambda^j_{n+1} \\
= \left( \prod_{k=1}^{n} \sum_{u=1}^{s_k} \lambda^k u \right) \cdot \left( \sum_{j=1}^{n+1} \lambda^j_{n+1} \right) \\
= \prod_{k=1}^{n+1} \sum_{j=1}^{s_k} \lambda^j_k.
\]

\( \square \)

**Theorem 38.** If \( S \) is a positive, refinable semifield, then for all \( X \) set and \( \Phi \in \mathcal{SP} X \), \( \delta_X(\Phi) = \overline{c(\Phi)}^{\overline{S}^X} \).

**Proof.** We shall discuss first some preliminary cases involving the empty set, excluding each time all the cases previously covered. Recall that

\[
\delta_X(\Phi) = \left\{ \varphi \in \mathcal{S}(X) \mid \exists \psi \in \mathcal{S}(\emptyset), \begin{cases} \forall A \in \mathcal{P} X, \Phi(A) = \sum_{x \in A} \psi(A, x) \\
\forall x \in X, \varphi(x) = \sum_{A \ni x} \psi(A, x) \end{cases} \right\}
\]

and write, within the scope of this proof, \( \delta_X'(\Phi) = \overline{c(\Phi)}^{\overline{S}^X} \), that is:

\[
\delta_X'(\Phi) = \left\{ \mu_X^S(\Psi) \mid \Psi \in \mathcal{S}^2 X, \sum_{\chi \in \mathcal{S} X} \Psi(\chi) = 1, \text{ supp } \Psi \subseteq c(\Phi) \right\}
\]

where

\[
c(\Phi) = \{ \chi \in \mathcal{S} X \mid \exists u \in \prod_{A \in \text{ supp } \Phi} A, \forall x \in X, \chi(x) = \sum_{A \ni x} \Phi(A) \}
\]

(this is of course an equivalent formulation of \((\square)\)).
Case 1: \(\Phi(\emptyset) \neq 0\). We have that \(\delta_X(\Phi) = \emptyset\) because there is no \(\psi \in \mathcal{S}(\emptyset)\) such that \(\Phi(\emptyset) = \sum_{x \in \emptyset} \psi(A, x) = 0\). At the same time, also \(\delta'_X(\Phi) = \emptyset\), because \(c(\Phi) = \emptyset\) since \(\prod_{A \in \text{supp } \Phi} A = \emptyset\) hence there is no \(\Psi \in \mathcal{S}^2 X\) with empty support that can satisfy \(\sum_{\chi \in \mathcal{S}_X} \Psi(\chi) = 1\).

Case 2: \(X = \emptyset\). (We also assume that \(\Phi(\emptyset) = 0\) from now on.) We have that \(\mathcal{S} \mathcal{P} \emptyset = \{\Omega: \{\emptyset\} \to S\}\), therefore \(\Phi = 0: \{\emptyset\} \to S\). Moreover, \(\exists \subseteq \mathcal{P}(\emptyset) \times \emptyset = \emptyset\) and \(\mathcal{S}(\emptyset) = \{0: \emptyset \to S\}\) is the singleton of the empty map, so

\[
\delta_\emptyset(\Phi) = \{\varphi \in \mathcal{S}(\emptyset) \mid \exists \psi \in \mathcal{S}(\emptyset), \forall A \subseteq \emptyset, \Phi(A) = \sum_{x \in \emptyset} \psi(A, x)\}
\]

\[
= \{\varphi \in \mathcal{S}(\emptyset) \mid \Phi(\emptyset) = 0\}
\]

\[
= \{0: \emptyset \to S\}
\]

because, by assumption, \(\Phi(\emptyset) = 0\). On the other hand, we have that, since \(\text{supp } \Phi = \emptyset\),

\[
c(\Phi) = \{\chi \in \mathcal{S}(\emptyset) \mid \exists u \in \prod_{A \in \emptyset} A\} = \mathcal{S}(\emptyset)
\]

because the zero-ary product is a choice of a terminal object of \(\text{Set}\), a singleton. So,

\[
\delta'_\emptyset(\Phi) = \{\mu(\Psi) \mid \Psi \in \mathcal{S}^2 \emptyset, \sum_{\chi \in \emptyset} \Psi(\chi) = 1\}
\]

\[
= \{\mu(\mathcal{S}(\emptyset) \ni \emptyset \to 1)\} = \{0: \emptyset \to S\}.
\]

Case 3: \(\Phi = 0: \mathcal{P} X \to S\). (We also assume that \(X \neq \emptyset\) from now on.) We have that the only \(\psi \in \mathcal{S}(\emptyset)\) such that for all \(\sum_{x \in A} \psi(A, x) = 0\) for all \(A \subseteq X\) is the null function, therefore \(\delta_X(0: \mathcal{P} X \to S) = \{0: X \to S\}\). On the other hand, we have that \(\text{supp } \Phi = \emptyset\), so

\[
c(\Phi) = \{\chi \in \mathcal{S} X \mid \exists u \in \prod_{A \in \emptyset} A. \forall x \in X, \sum_{A \in \emptyset} \Phi(A)\} = \{0: X \to S\}.
\]

It follows then that

\[
\delta_X(0: \mathcal{P} S \to S) = \{\mu(\Psi) \mid \Psi \in \mathcal{S}^2 X, \sum_{\chi \in \mathcal{S} X} \Psi(\chi) = 1, \text{supp } \Psi \subseteq \{0: X \to S\}\}
\]

\[
= \{\mu(\mathcal{S} X \ni 0 \to 1)\} \subseteq \{0: X \to S\}.
\]

We have now discussed all the preliminary cases. For the rest of the proof, we shall assume that \(X \neq \emptyset\), \(\Phi \neq 0: \mathcal{P} X \to S\) and \(\Phi(\emptyset) = 0\).

We first prove \(\delta_X(\Phi) \subseteq \delta'_X(\Phi)\). To this end, let \(\varphi \in \mathcal{S} X\) and \(\psi \in \mathcal{S}(\emptyset)\) such that \(\Phi(A) = \sum_{x \in A} \psi(A, x)\) for all \(A \in \mathcal{P} X\) and \(\varphi(x) = \sum_{A \ni x} \psi(A, x)\) for all \(x \in X\). Observe that:
– for all \((A, x) \in \text{supp } \psi\) we have that \(A \in \text{supp } \Phi\) and \(x \in \text{supp } \varphi \cap A\),
– for all \(A \in \text{supp } \Phi\) there is \(x \in \text{supp } \varphi \cap A\) such that \((A, x) \in \text{supp } \psi\),
– for all \(x \in \text{supp } \varphi\) there exists \(A \in \text{supp } \Phi\) such that \(A \ni x\) and \((A, x) \in \text{supp } \psi\).

Hence the first elements of pairs in \(\text{supp } \psi\) range over all and only elements of \(\text{supp } \Phi\), and the second entries of pairs in \(\text{supp } \psi\) range over all and only elements of \(\text{supp } \varphi\). In other words, say \(\text{supp } \Phi = \{A_1, \ldots, A_n\}\): then we have

\[
\text{supp } \psi = \left\{(A_1, x_1^1), \ldots, (A_1, x_{s_1}^1), \ldots, (A_n, x_1^n), \ldots, (A_n, x_{s_n}^n)\right\}
\]

where \(\bigcup_{i=1}^n \{x_1^i, \ldots, x_{s_i}^i\} = \text{supp } \varphi\). Notice that for all \(i \in \mathbb{N}, u, v \in s_i\) if \(u \neq v\) then \(x_u^i \neq x_v^i\), but we may have \(x_u^i = x_v^j\) if \(i \neq j\), because the \(A_i\)'s are distinct but not disjoint. We can then write:

\[
\begin{align*}
(\text{I}) & \quad \Phi(A_i) = \sum_{j=1}^{s_i} \psi(A_i, x_j^i) \text{ for all } i \in \mathbb{N} \\
(\text{II}) & \quad \varphi(x) = \sum_{i \in \mathbb{N}, 3j \in s_i, x=x_j^i} \psi(A_i, x) \text{ for all } x \in X.
\end{align*}
\]

We want to find a convex linear combination \(\Psi\) of elements of \(\mathcal{S}X\) of the form \(\sum_{i=1}^n \Phi(A_i) a_i\) for some \(a_i \in A_i\) such that \(\mu(\Psi) = \varphi\). Now, for every \(i \in \mathbb{N}\), we have many candidates for \(a_i\), namely \(x_1^i, \ldots, x_{s_i}^i\). Given that every \(\chi \in \text{supp } \Psi\) can only involve one \(x_j^i\) for every \(i\), we shall have as many \(\chi\)'s as the number of ways to pick one element of \(\{x_1^i, \ldots, x_{s_i}^i\}\) for each \(i\): let then \(w \in \prod_{i=1}^n s_i\) (it is a tuple of indexes \(w_i\) which we are going to use as \(j\)'s). Define \(\chi_w\) as

\[
\chi_w(x_1^i) = \Phi(A_1), \ldots, \chi_w(x_{s_i}^i) = \Phi(A_n)
\]

with the understanding that if \(x_{w_i}^i = x_{w_j}^j\) then \(\chi_w(x_{w_i}^i) = \Phi(A_i) + \Phi(A_j)\). Written more precisely, we define for all \(x \in X\)

\[
\chi_w(x) = \sum_{i \in \mathbb{N}, x=x_{w_i}^i} \Phi(A_i).
\]

We now define \(\Psi \in \mathcal{S}^2X\) with \(\text{supp } \Psi = \{\chi_w \mid w \in \prod_{i=1}^n s_i\}\) that assigns to each \(\chi_w\) a number such that \(\sum_{w} \Psi(\chi_w) = 1\). We shall use the previous Lemma to show that by defining

\[
\Psi(\chi_w) = \frac{\prod_{i=1}^n \psi(A_i, x_{w_i}^i)}{\prod_{i=1}^n \Phi(A_i)}
\]

for all \(w\), or more precisely by defining

\[
\Psi(\chi) = \sum_{w \in \prod_{i=1}^n s_i} \frac{\prod_{i=1}^n \psi(A_i, x_{w_i}^i)}{\prod_{i=1}^n \Phi(A_i)}
\]
for all \( \chi \in \mathcal{S}X \) we indeed have that \( \Psi \) satisfies the conditions of \( \delta'_{\chi}(\Phi) \) and \( \varphi = \mu(\Psi) \). First we prove that \( \Psi \) is in fact a convex linear combination:

\[
\sum_{\chi \in \text{supp } \Psi} \Psi(\chi) = \frac{1}{\prod_{i=1}^{n} \Phi(A_i)} \sum_{w \in \prod_{i=1}^{s_k} \psi(A_i, x^{i}_{w_{i}})} \prod_{i=1}^{n} \Phi(A_i) \prod_{i=1}^{n} \sum_{j}^{s_k} \psi(A_i, x^i_j)
\]

Lemma 37

\[
= \frac{1}{\prod_{i=1}^{n} \Phi(A_i)} \prod_{i=1}^{n} \sum_{j}^{s_k} \psi(A_i, x^i_j)
\]

because of (I)

\[
= 1.
\]

Next, notice that for every \( w \) the vector \( u \in \prod_{i=1}^{n} A_i \) required by the definition of \( \delta'_{\chi}(\Phi) \) is exactly \( (x^{i}_{w_{i}}, \ldots, x^{n}_{w_{n}}) \). Finally, we compute \( \mu(\Psi)(x) \) for an arbitrary \( x \in X \). The equations marked with (*) will be explained later.

\[
\mu^S_{\chi}(\Psi)(x) = \sum_{\chi \in \text{supp } \Psi} \Psi(\chi) \cdot \chi(x)
\]

\[
= \sum_{w \in \prod_{i=1}^{n} s_k} \left[ \frac{\prod_{i=1}^{n} \psi(A_i, x^i_{w_{i}})}{\prod_{i=1}^{n} \Phi(A_i)} \sum_{i \in \mathbb{N}} \Phi(A_i) \right] \quad (*)
\]

\[
= \sum_{i \in \mathbb{N}} \sum_{w \in \prod_{i=1}^{n} s_k} \left[ \frac{\prod_{k=1}^{n} \psi(A_k, x^k_{w_k})}{\prod_{k=1}^{n} \Phi(A_k)} \Phi(A_i) \right]
\]

\[
= \sum_{i \in \mathbb{N}} \sum_{\exists j \in \mathbb{N}, x = x^i_j} \left[ \Phi(A_i) \prod_{k=1}^{n} \Phi(A_k) \prod_{k \neq i}^{n} \psi(A_k, x^k_{w_k}) \right] \quad (**) \]

Now, use Lemma 37 with \( L = \{1, \ldots, i-1, i+1, \ldots, n\} \) and \( \lambda^k_j = \psi(A_k, x^k_{j}) \) in the following chain of equations:

\[
\sum_{w \in \prod_{i=1}^{n} s_k} \prod_{k \neq i}^{n} \psi(A_k, x^k_{w_k}) = \sum_{w' \in \prod_{k \in L} s_k} \prod_{k \in L} \psi(A_k, x^k_{w_k})
\]
\[
\mu^S_X(\Phi) = \sum_{i \in \mathbb{N}} \Phi(A_i) \cdot \frac{\psi(A_i, x)}{\prod_{k=1}^{n_i} \Phi(A_k)} \prod_{k \neq i} \Phi(A_k)
\]

Therefore, we obtain

\[
\mu^S_X(\Phi) = \sum_{i \in \mathbb{N}} \Phi(A_i) \cdot \frac{\psi(A_i, x)}{\prod_{k=1}^{n_i} \Phi(A_k)} \prod_{k \neq i} \Phi(A_k)
\]

It remains to explain equations \(*_1\) and \(*_2\) above. The latter is simply due to the fact that if \(i\) is such that there is no \(j\) for which \(x = x'_j\), then there is not a \(w\) such that \(x = x'_{w_i}\) either, and vice versa. The former is more delicate. It may be the case that \(\chi_w = \chi_w'\) for \(w \neq w'\). So, let us write

\[
supp \Psi = \{\chi_{w^1}, \ldots, \chi_{w^m}\}
\]

where the \(\chi_{w^i}\) are now all distinct. Then we have

\[
\sum_{\chi \in supp \Psi} \Psi(\chi) \cdot \chi(x) = \sum_{i=1}^{m} \sum_{w} \frac{\prod_{i=1}^{n_i} \psi(A_i, x_{w_i}^i)}{\prod_{i=1}^{n_i} \Phi(A_i)} \chi_{w^i}(x)
\]

which is the right-hand side of \((*_1)\).
Proof of Lemma 26. First of all, if \( A = \emptyset \), then the left-hand side is empty, because there is no \( \psi \in S(\emptyset) \) such that \( 1 = \Delta_\emptyset(\emptyset) = \sum_{x \in \emptyset} \psi(B, x) = 0 \), and so

\[
\sum_{x \in A} \psi(A, x) = \sum_{i=1}^{n} \sum_{\substack{j \in m_j \colon x = u_i^j}} \psi(\chi_j)\Phi(A_i) = \sum_{i=1}^{n} \sum_{\substack{j \in m_j \colon x = u_i^j}} \psi(\chi_j)\Phi(A_i) = \Phi(A) \sum_{j=1}^{m} \psi(\chi_j)
\]

where equation \(*_3\) holds because for all \( j \in m \) the addend \( \psi(\chi_j)\Phi(A_i) \) appears on the left-hand side exactly once, when in the first sum we are using, as \( x \), precisely \( u_i^j \in A_i \). In other words, given \( j \), there is a unique \( x \in A_i \) such that \( x = u_i^j \), so \( \psi(\chi_j)\Phi(A_i) \) appears in the left-hand side of \(*_3\) and it does so only once. Therefore \( \mu^S_X(\Psi) \in \delta_X(\Phi) \), and the proof is complete. \( \square \)

9 Appendix to Section 5

We have \( supp \psi = \{(A_i, u_i^j) \mid i \in \mathbb{n}, j \in \mathbb{m} \} \) is finite and so \( \psi \in S(\emptyset) \). We now verify the two conditions required in the definition of \( \delta_X(\Phi) \). For all \( x \in X \):

\[
\sum_{A \ni x} \psi(A, x) = \sum_{i=1}^{n} \psi(A_i, x) = \sum_{i=1}^{n} \sum_{j \in m_i \colon x = u_i^j} \psi(\chi_j)\Phi(A_i) = \sum_{i=1}^{n} \sum_{j \in m_i \colon x = u_i^j} \psi(\chi_j)\Phi(A_i)
\]

while for all \( A \subseteq X \), if \( A \notin supp \Phi \), then \( \sum_{x \in A} (\psi(A, x)) = 0 = \Phi(A) \) by definition and, for all \( i \in \mathbb{n} \):

\[
\sum_{x \in A_i} \psi(A_i, x) = \sum_{x \in A_i} \sum_{j \in m_j \colon x = u_i^j} \psi(\chi_j)\Phi(A_i) \equiv \sum_{j=1}^{m} \psi(\chi_j)\Phi(A_i) = \Phi(A_i) \sum_{j=1}^{m} \psi(\chi_j)
\]

The other inclusion, \( \delta_X(\Phi) \subseteq \delta_X(\Phi) \), is easier. Let \( \Psi \in S^2 X \) be such that \( \sum_{\chi \in \mathcal{S} X} \Psi(\chi) = 1 \) and

\[
supp \Psi \subseteq \{ \chi \in \mathcal{S} X \mid \exists u \in \prod_{A \in supp \Phi} A, \forall x \in X, \chi(x) = \sum_{A \in supp \Phi} \Phi(A) \}.\]

Write again \( supp \Phi = \{A_1, \ldots, A_n\} \) and let \( supp \Psi = \{\chi_1, \ldots, \chi_m\} \). For all \( j \in m \), let \( w^j \in \prod_{i=1}^{n} A_i \) be such that \( \chi_j(x) = \sum_{i \in \mathbb{n}} \Phi(A_i) \). Then we have

\[
\mu^S_X(\Psi)(x) = \sum_{j=1}^{m} \Psi(\chi_j)(x) = \sum_{j=1}^{m} \Psi(\chi_j) \sum_{\substack{i \in \mathbb{n} \colon x = u_i^j}} \Phi(A_i).
\]

Define then

\[
\psi(B, x) = \begin{cases} \sum_{\substack{j \in \mathbb{m} \colon x = u_i^j}} \Psi(\chi_j)\Phi(A_i) & \exists(\forall) i \in \mathbb{n}, B = A_i \\ 0 & \text{otherwise} \end{cases}
\]

We verify the two conditions required in the definition of \( \delta_X(\Phi) \). For all \( x \in X \):

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is the right-hand side, as the only function whose support is empty is the null function, which cannot satisfy $\sum_{x \in X} \varphi(x) = 1$.

Suppose then that $A \neq \emptyset$. For the left-to-right inclusion: observe first of all that $\text{supp} \, \varphi \neq \emptyset$, because otherwise we would have that $\psi(B, x) = 0$ for all $x \in X$ and for all $B \ni x$ due to the fact that $S$ is a positive semiring. This would then lead to the contradiction $1 = \Delta_A(A) = \sum_{x \in A} \psi(A, x) = 0$. Let then $x \in \text{supp} \, \varphi$: then $\varphi(x) \neq 0$, hence there exists $B \ni x$ such that $\psi(B, x) \neq 0$. It is necessarily the case, however, that $B = A$, because if $B \neq A$, then $0 = \Delta_A(B) = \sum_{y \in B} \psi(B, y) \geq \psi(B, x) \neq 0$, which is a contradiction. Thus $\text{supp} \, \varphi \subseteq A$. Moreover,

$$\sum_{x \in X} \varphi(x) = \sum_{x \in A} \varphi(x) = \sum_{x \in A} \sum_{B \ni x} \psi(B, x) = \sum_{x \in A} \psi(A, x) = \Delta_A(A) = 1$$

where equation * holds because for all $B \neq A$ and for all $y \in B$ we have $\psi(B, y) = 0$, hence the only addend of $\sum_{B \ni x} \psi(B, x)$ which is possibly not-null is the "$A$-th one": $\psi(A, x)$.

Vice versa, let $\varphi : X \to S$ such that $\text{supp} \, \varphi$ is finite and contained in $A$ and satisfying $\sum_{x \in X} \varphi(x) = 1$. Then the map

$$\exists \xrightarrow{\psi} \mathbb{R}^+$$

$$(B, x) \longmapsto \begin{cases} \varphi(x) & B = A \\ 0 & \text{otherwise} \end{cases}$$

has finite support, as it is in bijective correspondence with $\text{supp} \, \varphi$; for all $B \in \mathcal{P}X$, if $B \neq A$ then $\sum_{x \in B} \psi(B, x) = 0$, otherwise

$$\sum_{B \ni x} \psi(B, x) = \sum_{x \in A} \varphi(x) = \sum_{x \in X} \varphi(x) = 1;$$

and finally, for all $x \in X$, $\sum_{B \ni x} \psi(B, x) = \psi(A, x) = \varphi(x).$  

\[ \square \]

**The proof of Proposition 27.** We actually state and prove a more precise result, that implies Proposition 27.

**Proposition 39.** Let $(X, a)$ be a $S$-algebra and let $A \subseteq SX$. Then

$$\mathcal{P}a\left(\mathcal{A}^S_X\right) = \mathcal{P}a(A)^S.$$

**Proof.** We have

$$\mathcal{P}a\left(\mathcal{A}^S_X\right) = \{ a(\mu_X^S(\Psi)) \mid \Psi \in S^2X, \sum_{\psi \in SX} \Psi(\psi) = 1, \text{supp} \Psi \subseteq A\}$$
while

$$\overline{\text{Pa}(A)} = \{a(\varphi) \mid \varphi \in SX, \sum_{x \in X} \varphi(x) = 1, \text{supp } \varphi \subseteq \{a(\psi) \mid \psi \in A\}\}.$$ 

For the left-to-right inclusion: let $$\Psi \in S^2 X$$ be such that $$\sum_{\psi \in SX} \Psi(\psi) = 1$$ and with $$\text{supp } \Psi \subseteq A$$. Define $$\varphi = S(a)(\Psi)$$. Then $$x \in \text{supp } \varphi$$ if and only if there exists $$\psi \in \text{supp } \Psi$$ such that $$x = a(\psi)$$. Hence, if $$x \in \text{supp } \varphi$$, then $$x = a(\psi)$$ for some $$\psi \in A$$. Moreover,

$$\sum_{x \in X} \varphi(x) = \sum_{x \in X} \sum_{\psi \in \text{supp } \Psi} \Psi(\psi) = \sum_{\psi \in SX} \Psi(\psi) = 1.$$ 

Since $$a(\mu_X^S(\Psi)) = a(S(a)(\Psi))$$ by the properties of $$(X, a)$$ as $$S$$-algebra, and $$S(a)(\Psi) = \varphi$$ by definition, we conclude.

Vice versa, given $$\varphi \in SX$$ such that $$\sum_{x \in X} \varphi(x) = 1$$ and with $$\text{supp } \varphi = \{a(\psi_1), \ldots, a(\psi_n)\}$$ where $$\psi_1, \ldots, \psi_n \in A$$, define

$$\Psi = \begin{pmatrix}
\psi_1 & \varphi(a(g_1)) \\
\vdots & \vdots \\
\psi_n & \psi(a(f_n))
\end{pmatrix}$$

Then $$\sum_{\psi \in SX} \Psi(\psi) = 1$$ and

$$a(\mu_X^S(\Psi)) = a(S(a)(\Psi)) = a(\varphi)$$

because $$S(a)(\Psi) = \varphi$$. \hfill \Box

**Details of the Proof of Proposition 28.** In the same notations set up in the proof of Theorem 28, let $$A \in \text{supp } \Phi$$. Define for all $$x \in X$$:

$$\chi_A(x) = \sum_{\varphi \in \text{supp } \Psi_{x = u^\varphi(A)}} \Psi(\varphi).$$

Then $$\text{supp } \chi_A = \{u^\varphi(A) \mid \varphi \in \text{supp } \Psi\}$$ is finite, hence $$\chi_A \in SX$$. Let now

$$u(A) = a(\chi_A).$$

We have that

$$\sum_{x \in X} \chi_A(x) = \sum_{x \in X} \sum_{\varphi \in \text{supp } \Psi_{x = u^\varphi(A)}} \Psi(\varphi) = \sum_{\varphi \in \text{supp } \Psi} \Psi(\varphi) = 1,$$
hence \( u(A) \in A \) because \( A \) is convex. Define, for all \( x \in X \):

\[
\psi(x) = \sum_{A \in \text{supp } \varphi, x = u(\chi_A)} \Phi(A).
\]

Then \( \text{supp} \psi = \{ u(A) \mid A \in \text{supp } \Phi \} \) so \( \psi \in \mathcal{S}X \). To conclude, we have to prove that \( a(\mu_X^S(\Psi)) = a(\psi) \). To that end, let for all \( \chi \in \mathcal{S}X \)

\[
\Psi'(\chi) = \sum_{A \in \text{supp } \varphi, \chi_A = \chi} \Phi(A).
\]

Then \( \text{supp} \Psi' = \{ \chi_A \mid A \in \text{supp } \Phi \} \) is finite. Then we have for all \( x \in X \) that

\[
\mu_X^S(\Psi')(x) = \sum_{\chi \in \text{supp } \Psi'} \Psi'(\chi) \cdot \chi(x)
= \sum_{A \in \text{supp } \Phi} \Phi(A) \cdot \sum_{\varphi \in \text{supp } \Psi, x = u^a(A)} \Psi(\varphi)
= \sum_{\varphi \in \mathcal{S}X} \Psi(\varphi) \cdot \sum_{A \in \text{supp } \Phi, x = u^a(A)} \Phi(A)
= \sum_{\varphi \in \mathcal{S}X} \Psi(\varphi) \cdot \varphi(x)
= \mu_X^S(\Psi)(x)
\]

where equation \(*\) is explained in a similar way than \((*)_1\) in the proof of Theorem 38. We also have that

\[
\mathcal{S}(a)(\Psi')(x) = \sum_{\chi \in a^{-1}(x)} \Psi'(\chi) = \sum_{\chi \in \mathcal{S}X} \sum_{A \in \text{supp } \Phi, \chi_A = \chi} \Phi(A) = \sum_{A \in \text{supp } \Phi, a(\chi_A) = x} \Phi(A) = \psi(x).
\]

Therefore,

\[
a(\psi) = a(\mathcal{S}(a)(\Psi')) = a(\mu_X^S(\Psi')) = a(\mu_X^S(\Psi)).
\]

**Continuation of the Proof of Theorem 24.** We continue the proof of Theorem 24 from p. 14 by analysing the action of \( \hat{\mathcal{P}} \) on morphisms and the unit and the multiplication of \( \hat{\mathcal{P}} \).

First of all, technically for \( f: (X, a) \rightarrow (X', a') \) in \( E \mathcal{M}(\mathcal{S}) \), \( \hat{\mathcal{P}}(f) \) is defined as

\[
\hat{\mathcal{P}}(f)(A) = \overline{\mathcal{P}(f)(A)}^\prime \quad \text{for all } A \in \mathcal{P}^0 X,
\]

however it is not difficult to see that \( \mathcal{P}(f)(A) \) is convex in \( (X', a') \) using the fact that \( f \) is a morphism of \( \mathcal{S} \)-algebras.
The unit of the monad $\tilde{\mathcal{P}}$ is given, for every $(X, a)$ object of $\mathbb{EM}(S)$, as the unique morphism $\eta^\tilde{\mathcal{P}}_{(X, a)} : (X, a) \to \tilde{\mathcal{P}}(X, a)$ that makes the two triangles of (2) and (3) commute, which are in our case:

By definition of $\iota$ and $\eta^\mathcal{P}$, the only arrow that makes the left triangle above commutative is necessarily

and this makes also the right triangle commutative, since $\{x\} = \{x\}$. Similarly, the multiplication $\mu^\tilde{\mathcal{P}}$ is defined, for every $S$-algebra $(X, a)$, as the unique morphism making the two rectangles of (2) and (3) commute. These are in our case:

By definition of $\iota$ and $\mu^\mathcal{P}$, the only arrow that makes the left rectangle above commutative is necessarily

for all $(X, a) \in \mathbb{EM}(S)$. The reader may wonder why such morphism is well defined, namely why $\mu^\tilde{\mathcal{P}}_{(X, a)}(A)$ is in $\mathcal{P}^a(X)$. This is the case because the abstract results guarantee existence and uniqueness of a such morphism. The interested reader may find next a more concrete proof of this and also the fact that this $\mu^\tilde{\mathcal{P}}$ makes the right rectangle commutative.

**Proposition 40.** Let $(X, a)$ be a $S$-algebra and $A \in \mathcal{P}^a(X)$. Then $\bigcup_{A \in \mathcal{A}} A$ is convex in $(X, a)$.

**Proof.** We want to prove the following equality:

$$\bigcup_{A \in \mathcal{A}} A = \{a(\varphi) \mid \varphi \in SX, \sum_{x \in X} \varphi(x) = 1, \text{supp} \varphi \subseteq \bigcup_{A \in \mathcal{A}} A\}.$$
The left-to-right inclusion is trivial: for all $A \in \mathcal{A}$ and $x \in A$, $x = a(\Delta_x)$. Vice versa, let $\varphi \in SX$ be such that $\sum_{x \in X} \varphi(x) = 1$ and $\text{supp} \varphi \subseteq \bigcup_{A \in \mathcal{A}} A$. We want to find an element $A \in \mathcal{A}$ such that $a(\varphi) \in A$. Suppose

$$\text{supp} \varphi = \{x_1, \ldots, x_n\}.$$  

Then for all $i \in \mathbb{N}$ there exists $A_i \in \mathcal{A}$ such that $x_i \in A_i$. Notice that if $i \neq j$ then it is not necessarily the case that $A_i \neq A_j$. Define $\Phi \in S(P_X)$ as

$$\Phi(B) = \sum_{\begin{subarray}{c} i \in \mathbb{N} \\ B = A_i \end{subarray}} \varphi(x_i)$$

for all $B \in P_X$. Then $\text{supp} \Phi = \{A_i \mid i \in \mathbb{N}\}$ is finite and

$$\sum_{B \in P_X} \Phi(B) = \sum_{B \in P_X} \sum_{\begin{subarray}{c} i \in \mathbb{N} \\ B = A_i \end{subarray}} \varphi(x_i) = \sum_{i \in \mathbb{N}} \varphi(x_i) = 1.$$

Since $\mathcal{A}$ is convex in $(P_X, \alpha_a)$, we have that $\alpha_a(\Phi) \in \mathcal{A}$. This is going to be our desired $A$: we shall prove that $a(\varphi) \in \alpha_a(\Phi)$.

We have $\alpha_a(\Phi) = Pa(\delta_X(S(i)(\Phi)))$, hence, if we prove that $\varphi \in \delta_X(S(i)(\Phi))$, we have finished. To this end, recall the characterisation of $\delta_X$ using elements of $S(\exists)$, for $\exists \subseteq PX \times X$:

$$\delta_X(S(i)(\Phi)) = \left\{ a(\chi) \mid \chi \in SX. \exists \psi \in S(\exists), \begin{cases} \forall B \subseteq X. S(i)(\Phi)(B) = \sum_{x \in B} \psi(B, x) \\ \forall x \in X. \chi(x) = \sum_{B \ni x} \psi(B, x) \end{cases} \right\}.$$  

Define, for all $(B, x) \in PX \times X$ such that $B \ni x$:

$$\psi(B, x) = \sum_{\begin{subarray}{c} i \in \mathbb{N} \\ (B, x) = (A_i, x_i) \end{subarray}} \varphi(x_i).$$

Then $\text{supp} \psi = \{(A_i, x_i) \mid i \in \mathbb{N}\}$ is finite, for all $B \subseteq X$

$$\sum_{x \in B} \psi(B, x) = \sum_{x \in B} \sum_{\begin{subarray}{c} i \in \mathbb{N} \\ (B, x) = (A_i, x_i) \end{subarray}} \varphi(x_i) = \sum_{\begin{subarray}{c} i \in \mathbb{N} \\ B = A_i \end{subarray}} \varphi(x_i) = S(i)(\Phi)(B)$$

and, for all $x \in X$,

$$\sum_{B \ni x} \psi(B, x) = \sum_{B \ni x} \sum_{\begin{subarray}{c} i \in \mathbb{N} \\ (B, x) = (A_i, x_i) \end{subarray}} \varphi(x_i) = \begin{cases} 0 & \forall i \in \mathbb{N}, x \neq x_i \\ \varphi(x_i) & \exists(\exists) i \in \mathbb{N}, x = x_i \end{cases} = \varphi(x)$$

where if there is $i$ such that $x = x_i$, then such $i$ is unique due to the fact that the $x_j$'s are distinct. This proves that $\varphi \in \delta_X(S(i)(\Phi))$. □
Proposition 41. The following diagram commutes.

\[
\begin{array}{ccc}
P & \xrightarrow{\mu_X} & P(P_{c}^\alpha X) \\
\downarrow{\mu_X} & & \downarrow{\mu_X} \\
\mathcal{P}X & \xrightarrow{(-)^a} & P_{c}^\alpha X \\
\end{array}
\]

Proof. In this proof we shall simply write \(\overline{A}\) for \(\overline{A}\) for any \(A \subseteq X\), because there is no risk of confusion. We have to show that

\[
\{ a(\varphi) \mid \varphi \in \mathcal{S}X, \sum_{x \in X} \varphi(x) = 1, \text{ supp } \varphi \subseteq \bigcup_{U \in \mathcal{U}} U \} = \bigcup_{\phi \in \mathcal{S}P_{c}^\alpha X} \{ \alpha_\alpha(\Phi) \mid \sum_{C \in \mathcal{P}_{c}^\alpha X} \Phi(C) = 1, \text{ supp } \Phi \subseteq \bigcup_{U \in \mathcal{U}} U \}
\]

For the left-to-right inclusion: let \(\varphi \in \mathcal{S}X\) such that \(\sum_{x \in X} \varphi(x) = 1\), and suppose \(\text{ supp } \varphi = \{x_1, \ldots, x_n\}\). We have that for all \(i \in \mathfrak{n}\) there exists \(A_i \in \mathcal{U}\) such that \(x_i \in A_i\). While the \(x_i\)'s are distinct, the \(A_i\)'s need not be. Define, for all \(B \in \mathcal{P}_{c}^\alpha X\),

\[
\Phi(B) = \sum_{i \in \mathfrak{n}, B = A_i} \varphi(x_i).
\]

Then \(\text{ supp } \Phi = \{\overline{A_1}, \ldots, \overline{A_\mathfrak{n}}\}\) hence \(\Phi \in \mathcal{S}P_{c}^\alpha X\). One can then prove that \(a(\varphi) \in \alpha_\alpha(\Phi)\) in the same way as showed in the proof of Proposition 40.

Vice versa, an element of the right-hand side is of the form \(a(\psi)\) for some \(\Phi \in \mathcal{S}P_{c}^\alpha X\) such that \(\sum_{B \in \mathcal{P}_{c}^\alpha X} \Phi(B) = 1\), \(\text{ supp } \Phi = \{\overline{A_1}, \ldots, \overline{A_\mathfrak{n}}\}\) with \(A_i \in \mathcal{U}\) for all \(i \in \mathfrak{n}\) and for some \(\psi \in \mathcal{S}X\) and \(u \in \prod_{i=1}^{\mathfrak{n}} \overline{A_i}\) such that

\[
\psi(x) = \sum_{i \in \mathfrak{n}, x = u_i} \Phi(\overline{A_i}).
\]

We want to prove that \(a(\psi) = a(\varphi)\) for an appropriate \(\varphi \in \mathcal{S}X\) such that \(\sum_{x \in X} \varphi(x) = 1\) and \(\text{ supp } \varphi \subseteq \bigcup_{U \in \mathcal{U}} U\).

Since \(u_i \in \overline{A_i}\), we have that \(u_i = a(\varphi_i)\) for some \(\varphi_i \in \mathcal{S}X\) such that \(\sum_{x \in X} \varphi_i(x) = 1\) and \(\text{ supp } \varphi_i \subseteq A_i\). These \(\varphi_i\)'s are not necessarily distinct. Let then, for all \(\varphi \in \mathcal{S}X\),

\[
\Psi(\varphi) = \sum_{\varphi_i \subseteq \varphi} \Phi(\overline{A_i}).
\]

Then \(\text{ supp } \Psi = \{f_1, \ldots, f_\mathfrak{n}\}\) so \(\Psi \in \mathcal{S}^2 X\) and it is easy to prove, by means of direct calculations, that \(\psi = S(a)(\Psi), \sum_{x \in X} \mu_X^S(\Psi) = 1\) and that \(\text{ supp } \mu_X^S(\Psi) \subseteq \bigcup_{U \in \mathcal{U}} U\). Then we have that \(a(\psi) = a(S(a)(\Psi)) = \alpha_\alpha(\mu_X^S(\Psi)), \) and \(\mu_X^S(\Psi)\) is the \(\varphi\) we were looking for. \(\square\)
10 Appendix for Section 6

The Presentation of $\mathcal{P}_S\mathcal{S}$. For $\delta: SP \to PS$ given in (3), we have that a $\delta$-algebra is a set $X$ together with a $\mathcal{S}$-algebra structure $a: SX \to X$ and a $\mathcal{P}$-algebra-structure $b: PX \to X$ such that pentagon (21) commutes. A morphism of $\delta$-algebras is a morphism of $\mathcal{P}$- and $\mathcal{S}$-algebra simultaneously by definition. Hence, by defining $+^a$, $\lambda^a$ and $0^a$ as in (5), we have that $X$ is a $\mathcal{S}$-left-semimodule; moreover, if we define for each set $I$

$$\bigsqcup_{i \in I} x_i = b(\{x_i \mid i \in I\})$$

(23)

then $X$ is also a complete semilattice. This means that $X$ satisfies all the equations in $E_{CSL}$ and $E_{LSM}$; moreover, every morphism of $\delta$-algebras preserves all the operations in $\Sigma$. The commutativity of pentagon (21) will instead ensure that the equations in $E_D$ listed in (20) are satisfied, as we prove in the following.

For $A \subseteq X$, we shall sometimes write $\bigsqcup b_i^A$ for $\bigsqcup b_i \{x \in A \mid i \in I\}$.

**Proposition 42.** Let $(X, a: SX \to X, b: PX \to X)$ be a $\delta$-algebra. For each $A \subseteq X$,

$$\bigsqcup_{x \in A} x = \bigsqcup_{x \in A} x.$$

**Proof.** Let $\Phi = \eta_{\mathcal{P}X}(A) = \Delta_A \in \mathcal{S}PX$. Then

$$a(\mathcal{S}b(\Phi)) = a(\mathcal{S}b(\Delta_A)) = a(\Delta_{\bigsqcup A}) = \bigsqcup A$$

while

$$\delta_X(\Delta_A) = \{\varphi \in SX \mid \text{supp } \varphi \subseteq A, \sum_{x \in X} \varphi(x) = 1\}$$

because of Lemma [26], hence

$$\mathcal{P}a(\delta_X(\Delta_A)) = \bigsqcup A.$$  

By the commutativity of diagram (21), we have that $\bigsqcup A = \bigsqcup A^\alpha$.  

We can then prove that every $\delta$-algebra satisfies the equations in $E_D$.

**Theorem 43.** Let $(X, a: SX \to X, b: PX \to X)$ be a $\delta$-algebra. Then for all $A, B \subseteq X$ and for all $\lambda \in S \setminus \{0_S\}$:

$$\lambda \cdot \bigsqcup_{x \in A} x = \bigsqcup_{x \in A} \lambda \cdot x, \quad \bigsqcup_{x \in A} x +^a \bigsqcup_{y \in B} y = \bigsqcup_{(x,y) \in A \times B} x +^a y.$$
Proof. Let $\Phi = (A \mapsto \lambda) \in \mathcal{SP}_X$. We prove that the images along the two legs of pentagon \ref{21} of $\Phi$ coincide with the two sides of the first equation in the statement. It holds that

$$a(S(b)(\Phi)) = a(x \mapsto \sum_{U \in b^{-1}(x)} \Phi(U))$$

$$= a(b(A) \mapsto \lambda)$$

$$= a(\bigsqcup_{x \in A} x \mapsto \lambda) \quad \text{Definition of } \bigsqcup_{x \in A} \ref{23}$$

$$= \lambda \cdot a(\bigsqcup_{x \in A} x) \quad \text{Definition of } \lambda \cdot \ref{5}$$

On the other hand we have that

$$b(P_{\mathcal{A}}(\delta_X(\Phi))) = \bigsqcup_{x \in A} b(P_{\mathcal{A}}(\epsilon(\Phi)^{\mathcal{A}}_X)) \quad \text{Theorem } \ref{21}$$

$$= \bigsqcup_{x \in A} b(P_{\mathcal{A}}(\epsilon(\Phi))) \quad \text{Proposition } \ref{39}$$

$$= \bigsqcup_{x \in A} P_{\mathcal{A}}(\epsilon(\Phi)) \quad \text{Proposition } \ref{12}$$

Following the discussion after Theorem \ref{24} $\epsilon(\Phi) = \{ (x \mapsto \lambda) \mid x \in A \}$ if $\lambda \neq 0$. Therefore, if $\lambda \neq 0$, then $P_{\mathcal{A}}(\epsilon(\Phi)) = \{a(x \mapsto \lambda) \mid x \in A\}$ which by \ref{5} is exactly $\{\lambda \cdot a \mid x \in A\}$.

Therefore

$$b(P_{\mathcal{A}}(\delta_X(\Phi))) = \bigsqcup_{x \in A} \lambda \cdot a.$$ 

We then conclude by using the commutativity of diagram \ref{21}. Using the function $\Phi' = (A \mapsto 1, B \mapsto 1) \in \mathcal{SP}_X$ and a similar argument, one shows the second equation as well. 

Vice versa, we want to prove that every algebra for the theory $(\Sigma, E)$ is also a $\delta$-algebra. To this end, let $(X, +, \lambda \cdot, 0_X, \bigsqcup)$ be a $(\Sigma, E)$-algebra. Then $(X, +, \lambda \cdot, 0_X)$ is a $S$-left-semimodule, $(X, \bigsqcup)$ is a complete sup-semilattice. By defining

$$\mathcal{S}X \overset{a}{\longrightarrow} X \quad \sum_{x \in \text{supp} f} f(x) \cdot x \quad \mathcal{P}X \overset{b}{\longrightarrow} X \quad A \longmapsto \bigsqcup_{x \in A} x$$

we have that $(X, a) \in \text{EM}(\mathcal{S})$ and $(X, b) \in \text{EM}(\mathcal{P})$. We now have to check that pentagon \ref{21} commutes. Given $A \subseteq X$, we define its convex closure $\overline{A}$ in the usual way:

$$\overline{A} = \{ \sum_{i=1}^n p_i \cdot a_i \mid n \in \mathbb{N}, (p_i)_{i \in \mathbb{N}} \in S_1^n, (a_i)_{i \in \mathbb{N}} \in A^n \}$$

$X, a) \in \text{EM}(\mathcal{S})$ and $(X, b) \in \text{EM}(\mathcal{P})$. We now have to check that pentagon \ref{21} commutes. Given $A \subseteq X$, we define its convex closure $\overline{A}$ in the usual way:

$$\overline{A} = \{ \sum_{i=1}^n p_i \cdot a_i \mid n \in \mathbb{N}, (p_i)_{i \in \mathbb{N}} \in S_1^n, (a_i)_{i \in \mathbb{N}} \in A^n \}$$
Theorem 45. proved the following theorem.

\[ \sum_{A \in \text{supp } \Phi} \Phi(A) \cdot \bigcup A = \bigcup \left\{ \sum_{A \in \text{supp } \Phi} \Phi(A) \cdot u(A) \mid u : \text{supp } \Phi \to X, \forall A. u(A) \in A \right\} \]

where the left-hand side is \( a(S(b)(\Phi)) \) and the right-hand side is \( b(P(a)(\delta_X(\Phi))) \).

In the following Lemma we prove that if \( X \) is a \( (\Sigma, E) \)-algebra and \( A \subseteq X \), then \( \bigcup A = \bigcup \bigcup A \). Using this fact and the distributivity property (20), we will have shown the commutativity of pentagon (21).

**Lemma 44.** Let \( (X, +, \lambda \cdot, 0_X, \bigcup_I) \) be a \( (\Sigma, E) \)-algebra. Then for all \( A \subseteq X \)

\[ A = \bigcup A. \]

**Proof.** Let \( n \in \mathbb{N} \). Define \( S^n_1 = \{ (p_1, \ldots, p_n) \in S^n \mid \sum_{i=1}^n p_i = 1 \} \). Let \( (p_i)_{i \in \mathbb{N}} \in S^n_1 \). Then

\[ \bigcup A = 1 \cdot \bigcup A = \bigcup_{i=1}^n (p_i) \cdot \bigcup A = \bigcup_{i=1}^n \left\{ \sum_{i=1}^n | p_i \cdot a_i | (a_i)_{i \in \mathbb{N}} \in A^n \right\} \]

because of the properties of \( S \)-semimodule and the distributivity (20). Let \( \leq \) be the partial order determined by the complete semilattice structure of \( X \). We have that

\[ \forall n \in \mathbb{N}, \forall (p_i) \in S^n_1, \forall (a_i) \in A^n. \sum_{i=1}^n p_i \cdot a_i \leq \bigcup \left\{ \sum_{i=1}^n p_i \cdot b_i | (b_i)_{i \in \mathbb{N}} \in A^n \right\} = \bigcup A \]

hence

\[ \bigcup A = \bigcup \left\{ \sum_{i=1}^n p_i \cdot a_i | n \in \mathbb{N}, (p_i) \in S^n_1, (a_i)_{i \in \mathbb{N}} \in A^n \right\} \leq \bigcup A \]

while the other inequality is trivial because \( A \subseteq \left\{ \sum_{i=1}^n p_i \cdot a_i \mid n \in \mathbb{N}, (p_i) \in S^n_1, (a_i)_{i \in \mathbb{N}} \in A^n \right\} \).

Finally, again a morphism of \( (\Sigma, E) \)-algebras is a morphism respecting all the operations of \( \Sigma \), which means of \( \Sigma_{LSM} \) (thus is a morphism of \( \text{EM}(S) \)) and of \( \Sigma_{CSL} \) (thus is a morphism of \( \text{EM}(P) \)) at the same time. We have therefore proved the following theorem.

**Theorem 45.** The category \( \text{EM}(\delta) \) of \( \delta \)-algebras is isomorphic to the category \( \text{Alg}(\Sigma, E) \) of \( (\Sigma, E) \)-algebras.

Since \( \text{EM}(\delta) \) is canonically isomorphic to \( \text{EM}(P, S) \) (10.14), we have proved Theorem 32.
The Kleisli Category of $\mathcal{P}_c\mathcal{S}$. In this section we aim to prove, using the algebraic presentation of the monad $\mathcal{P}_c\mathcal{S}$, that its Kleisli category satisfies the conditions required to perform a coalgebraic trace semantics, as stated in [19].

$(\mathcal{P}_c\mathcal{S}X, \mu^\mathcal{P}_c\mathcal{S})$ is a $\mathcal{P}_c\mathcal{S}$-algebra, hence $\mathcal{P}_c\mathcal{S}X$ has a structure of semimodule and complete lattice where, via the canonical isomorphism $\text{EM}(\mathcal{P}_c\mathcal{S}) \rightarrow \text{EM}(\delta)$, we have

$$A + B = \{ \varphi + \psi \mid \varphi \in A, \psi \in B \}$$

$$\lambda \cdot A = \{ \lambda \cdot \varphi \mid \varphi \in A \}$$

$$\bigcup_{i \in I} A_i = \bigcup_{i \in I} A_i$$

for all $A, B \in \mathcal{P}_c\mathcal{S}X$ and $\lambda \in S$, $\lambda \neq 0$.

Consider now the Kleisli category $\mathcal{Kl}(\mathcal{P}_c\mathcal{S})$. Each homset $\mathcal{Kl}(\mathcal{P}_c\mathcal{S})(X, Y)$ of functions $f: X \rightarrow \mathcal{P}_c\mathcal{S}Y$ is partially ordered point-wise and inherits the structure of complete semilattice from $\mathcal{P}_c\mathcal{S}Y$, in particular its bottom element $\perp_{X,Y}$ is the constant function mapping $x$ to $\emptyset$ for all $x \in X$. Given a function $g: Y \rightarrow \mathcal{P}_c\mathcal{S}Z$, its Kleisli extension $g^\#: \mathcal{P}_c\mathcal{S}Y \rightarrow \mathcal{P}_c\mathcal{S}Z$ is given by $\mu^\mathcal{P}_c\mathcal{S} \circ \mathcal{P}_cSg$, which for all $A \in \mathcal{P}_c\mathcal{S}Y$ computes:

$$g^\#(A) = \bigcup_{\varphi \in A} \sum_{y \in \text{supp} \varphi} \varphi(y) \cdot g(y)$$

$$= \bigcup_{\varphi \in A} \left\{ \sum_{y \in \text{supp} \varphi} \varphi(y) \cdot \psi_y \mid \forall y \in \text{supp} \varphi. \psi_y \in g(y) \right\}.$$

Composition of a function $f: X \rightarrow \mathcal{P}_c\mathcal{S}Y$ and $g: Y \rightarrow \mathcal{P}_c\mathcal{S}Z$ is therefore given as

$$(g \circ f)(x) = g^\#(f(x)) = \bigcup_{\varphi \in f(x)} \sum_{y \in \text{supp} \varphi} \varphi(y) \cdot g(y) \in \mathcal{P}_c\mathcal{S}Z.$$

**Theorem 46.** The category $\mathcal{Kl}(\mathcal{P}_c\mathcal{S})$ is enriched over the category of directed-complete partial orders and satisfies the left-strictness condition:

$$\perp_{Y,Z} \circ f = \perp_{X,Z}$$

for all $f: X \rightarrow \mathcal{P}_c\mathcal{S}Y$ and $Z$ set.

**Proof.** Let $f: X \rightarrow \mathcal{P}_c\mathcal{S}Y$ and $\{g_i \mid i \in I\}$ a directed subset of $\mathcal{Kl}(\mathcal{P}_c\mathcal{S})(Y, Z)$. Then:

$$\left( \bigcup_{i \in I} g_i \circ f \right)(x) = \bigcup_{\varphi \in f(x)} \sum_{y \in \text{supp} \varphi} \varphi(y) \cdot \left( \bigcup_{i \in I} g_i \right)(y)$$

$$= \bigcup_{\varphi \in f(x)} \sum_{y \in \text{supp} \varphi} \varphi(y) \cdot \bigcup_{i \in I} (g_i)(y)$$

$$= \bigcup_{\varphi \in f(x)} \sum_{y \in \text{supp} \varphi} \bigcup_{i \in I} \varphi(y) \cdot g_i(y).$$
Combining Semilattices and Semimodules

\[ \bigcup_{\varphi \in f(x)} \left( \sum_{y \in \text{supp } \varphi} \varphi(y) \cdot g_i(y) \right) \]

\[ \bigcup_{\varphi \in f(x)} \left( \sum_{y \in \text{supp } \varphi} \varphi(y) \cdot \psi_{i,y} \right) \]

\[ \bigcup_{i \in I} \left( \sum_{\varphi \in f(x) \atop y \in \text{supp } \varphi} \varphi(y) \cdot g_i(y) \right) \]

\[ \bigcup_{i \in I} \left( g_i \circ f \right)(x) \]

Equation (\(\ast\)) holds because of the distributivity of addition over joins (20), while (\(\dagger\)) holds because the family \(\{g_i \mid i \in I\}\) is directed.

Given, instead, an arbitrary subset \(\{f_i \mid i \in I\}\) of \(Kl(P_c S)(X,Y)\) and a \(g : Y \to P_c S Z\), we have

\[ \left( g \circ \left( \bigsqcup_{i \in I} f_i \right) \right)(x) = g^Y \left( \bigsqcup_{i \in I} f_i(x) \right) \]

\[ \overset{\dagger}{=} \bigsqcup_{i \in I} g^Y(f_i(x)) \]

\[ = \bigsqcup_{i \in I} (g \circ f)(x) \]

where equation (\(\dagger\)) holds because \(g^Y\) is a morphism of \(P_c S\)-algebras (as it is given by the universal property of the free algebra \(P_c S Y\)), hence it preserves arbitrary suprema. Finally,

\[ (\perp_Y, z \circ f)(x) = \bigsqcup_{\varphi \in f(x) \atop y \in \text{supp } \varphi} \varphi(y) \cdot \emptyset = \bigsqcup_{\varphi \in f(x) \atop y \in \text{supp } \varphi} \emptyset = \bigsqcup_{\varphi \in f(x)} \emptyset = \emptyset \]

(notice that \(\varphi(y) \cdot \emptyset = \emptyset\) because \(\varphi(y) \neq 0\) when \(y \in \text{supp } \varphi\)).

The Term Monad for \((\Sigma, E)\). The algebraic theory \((\Sigma, E)\) described in Theorem 32 determines a monad on \(\text{Set} T_{\Sigma,E}\) where, for any set \(X\), \(T_{\Sigma,E}(X)\) is the set of all \(\Sigma\)-terms with variables in \(X\) quotiented by the equations in \(E\). Recall that a \(\Sigma\)-term with variables in \(X\) is defined inductively as:

- every variable \(x\) is a \(\Sigma\)-term,
- if \(o\) is an operation in \(\Sigma\) with arity \(\kappa\) and \(t_1, \ldots, t_\kappa\) are \(\Sigma\)-terms, then \(o(t_1, \ldots, t_\kappa)\) is a \(\Sigma\)-term.

If \(f : X \to Y\) is a function, \(T_{\Sigma,E}(f) : T_{\Sigma,E}(X) \to T_{\Sigma,E}(Y)\) sends a term \(t\) in \(t[f(x)/x]\), where every variable \(x\) is substituted by its image \(f(x)\). The unit \(\eta^T\)
is simply defined as $\eta_X^T(x) = x$, while the multiplication is defined by induction as:

$$
T_{\Sigma,E}(T_{\Sigma,E}(X)) \xrightarrow{\mu_X^T} T_{\Sigma,E}(X)
$$

$$
t \in T_{\Sigma,E} \xrightarrow{\cdot} t
$$

$$
o_\kappa(t_1, \ldots, t_\kappa) \xrightarrow{\cdot} o_\kappa(\mu_X^T(t_1), \ldots, \mu_X^T(t_\kappa))
$$

This construction is standard for finitary algebraic theories, where every operation in $\Sigma$ has finite arity. The fact that it makes sense also for our case, where we have an operation for every cardinal, is ensured by the fact that our $(\Sigma, E)$ is tractable in the sense of [29, Definition 1.5.44], because we proved in Theorem 32 that it presents a monad on $\mathcal{S}$, namely $\mathcal{P}_c\mathcal{S}$. Tractability ensures that the class of $\Sigma$-terms, once quotiented by $E$, is forced to be a set. Notice, moreover, that if we represent a $\Sigma$-term, as usual, as a tree whose nodes are operation symbols in $\Sigma$, which have each as many branches as their arity, and whose leaves are variables, then we end up with a tree with infinite branches but finite height. This helps in giving an intuition of why we can define functions $\varphi: T_{\Sigma,E}(X) \to Y$ by induction on the complexity of terms.

Now, the category of Eilenberg-Moore algebras for the monad $T_{\Sigma,E}$ is, in fact, isomorphic to $\text{Alg}(\Sigma, E)$, hence also to $\text{EM}(\mathcal{P}_c\mathcal{S})$, via a functor $F$ such that

$$
\text{EM}(\mathcal{P}_c\mathcal{S}) \xrightarrow{F} \text{EM}(T_{\Sigma,E})
$$

$$
\text{Set} \xrightarrow{id_{\text{Set}}} \text{Set}
$$

$$
\text{Set} \xrightarrow{U} \text{Set}
$$

$$
\text{Set} \xrightarrow{id} \text{Set}
$$

This generates an isomorphism of monads $\varphi: T_{\Sigma,E} \to \mathcal{P}_c\mathcal{S}$ where, for all $X$ set, $\varphi_X = F(\mu_X^{\mathcal{P}_c\mathcal{S}}) \circ T_{\Sigma,E}(\eta^{\mathcal{P}_c\mathcal{S}})$, thanks to the following general result:

**Theorem 47.** Let $(S, \eta^S, \mu^S)$ and $(T, \eta^T, \mu^T)$ be monads on a category $\mathcal{C}$. Suppose $\text{EM}(S)$ and $\text{EM}(T)$ are isomorphic via a functor $F: \text{EM}(S) \to \text{EM}(T)$ such that $U^TF = U^S$. Then $T$ and $S$ are isomorphic as monads, that is, there is a natural isomorphism $\varphi: T \to S$ such that

$$
\text{id}_\mathcal{C} \xrightarrow{\eta^T} T \xleftarrow{\mu^T} T^2
$$

$$
S \xrightarrow{S^2} \xleftarrow{S^2} \varphi \circ \varphi
$$

commutes, where $\varphi \circ \varphi = \varphi S \circ \varphi T = S \varphi \circ \varphi T$. Specifically, $\varphi_X$ is given as the unique morphism of $T$-algebras (hence, a morphism in $\mathcal{C}$) granted by the
universal property of \((TX, \mu^X_T)\) as the free \(T\)-algebra on \(X\), induced by \(\eta^S_X\):

\[
\begin{array}{c}
TX \xrightarrow{T(\varphi_X)} TSX \\
\downarrow \mu^X_T \downarrow \exists \varphi_X \\
TX \xrightarrow{\eta^X_T} SX
\end{array}
\]

\[\varphi_X = F(\mu^S_X) \circ T(\eta^S_X).\]

Direct calculations show that our \(\varphi_X : T_{\Sigma,E}(X) \to \mathcal{P}_cS(X) = \mathcal{P}_{c^\mu}SX\) acts as follows:

\[
\begin{align*}
\varphi_X(x) &= \{\Delta_x\} \quad \text{for} \ x \in X \\
\varphi_X(0) &= \{0 : X \to S\} \\
\varphi_X(t_1 + t_2) &= \varphi_X(t_1) + \varphi_X(t_2) \\
\varphi_X(\lambda \cdot t) &= \lambda \cdot \varphi_X(t) \\
\varphi_X \left( \bigsqcup_{i \in I} \{t_i\} \right) &= \bigcup_{i \in I} \varphi_X(t_i)
\end{align*}
\]

where \(\varphi_X(t_1) + \varphi_X(t_2)\) is the result of adding up in the \(S\)-algebra \((\mathcal{P}_{c^\mu}SX, \alpha_{\mu^S_X})\) (see Theorem 28) seen as a \(S\)-semimodule the convex subsets \(\varphi_X(t_1)\) and \(\varphi_X(t_2)\) of \(SX\). Similarly \(\lambda \cdot \varphi_X(t)\) is the scalar-product in \(\mathcal{P}_{c^\mu}SX\). Hence:

\[
\begin{align*}
\varphi_X(t_1 + t_2) &= \left\{ \mu^S_X \left( \begin{array}{c} f_1 \\
\downarrow \mu^S_X \\
f_2 \\
\downarrow \mu^S_X \end{array} \right) \mid f_1 \in \varphi_X(t_1), f_2 \in \varphi_X(t_2) \right\} \\
\lambda \cdot \varphi_X(t) &= \{\lambda \cdot f \mid f \in \varphi_X(t)\}.
\end{align*}
\]

It is clear, then, that if we restrict the action of \(\varphi_X\) to all those terms involving only finite suprema, we obtain exactly the function \([\ [\cdot] \]_X\) of Proposition 34. Since \(\varphi_X\) is a bijection, its restriction \([\ [\cdot] \]_X\) is injective. This proves Proposition 34.

The Restriction of \(\mathcal{P}_cS\) to \(\mathcal{P}_{fc}S\). The aim of this section is to prove that if we restrict the image of the functor \(\mathcal{P}_cS\) on a set \(X\) to those convex subsets \(A \subseteq SX\) that are finitely generated, then all the remaining structure of the monad \(\mathcal{P}_cS\) still works with no adaptation. This means that we have to prove that:

- for \(f : X \to Y\), \(\mathcal{P}_cS(f) : \mathcal{P}_cS(X) \to \mathcal{P}_cS(Y)\) restricts and corestricts to \(\mathcal{P}_{fc}S(X) \to \mathcal{P}_{fc}S(Y)\),
- \(\eta^X_{\mathcal{P}_{fc}S}\) can be corestricted to \(\mathcal{P}_{fc}S(X)\) (trivial),
- \(\mu^X_{\mathcal{P}_{fc}S}\) restricts and corestricts to \(\mathcal{P}_{fc}SP_{fc}SX \to \mathcal{P}_{fc}SX\).
We do so in the next few results. From now on, if $B \subseteq SX$, we shall simply write $\overline{B}$ in lieu of $\overline{S^X}$.

**Proposition 48.** Let $f : X \to Y$, $A \subseteq SX$ such that $A = \overline{B}$ for some finite $B \subseteq A$. Then

$$\{Sf(\varphi) \mid \varphi \in A\} = \{Sf(\psi) \mid \psi \in \overline{B}\}.$$  

**Proof.** For the left to right inclusion: let $\varphi \in A = \overline{B}$. Then there exists $\Psi \in S^2X$ such that $\sum_{x \in SX} \Psi(x) = 1$, $supp \Psi \subseteq B$, $\varphi = \mu_X^S(\Psi)$. We have to prove that there is a $\Psi' \in S^2Y$ such that $\sum_{y \in SY} \Psi'(y) = 1$, $supp \Psi' \subseteq \{Sf(\psi) \mid \psi \in B\}$, $\mu_Y^S(\Psi') = \varphi$.

Now, because of the naturality of $\mu^S$, we have

$$Sf(\varphi) = Sf(\mu_X^S(\Psi)) = \mu_Y^S(S^2f(\Psi)).$$

One can easily see that $S^2f(\Psi)$ works for our desired $\Psi'$.

Vice versa, let $\Psi' \in S^2Y$ be such that $\sum_{y \in SY} \Psi'(y) = 1$ and $supp \Psi' \subseteq \{Sf(\psi) \mid \psi \in B\}$. We have to show that there is $\varphi \in A$ such that $\mu_Y^S(\Psi') = Sf(\varphi)$.

We have that $\Psi'$ is of the form

$$\begin{pmatrix}
Sf(\psi_1) & \rightarrow & \Psi'(Sf(\psi_1)) \\
& \vdots & \\
Sf(\psi_n) & \rightarrow & \Psi'(Sf(\psi_n))
\end{pmatrix}$$

Then that means that $\Psi' = S(Sf)(\Psi)$ where $\Psi$ is defined as

$$\Psi = \begin{pmatrix}
\psi_1 & \rightarrow & \Psi(Sf(\psi_1)) \\
& \vdots & \\
\psi_n & \rightarrow & \Psi(Sf(\psi_n))
\end{pmatrix} \in S^2X$$

Then, again by naturality of $\mu^S$, we have

$$\mu_Y^S(\Psi') = \mu_Y^S(S^2f(\Psi)) = Sf(\mu_X^S(\Psi))$$

and $\varphi = \mu_X^S(\Psi)$ is indeed in $A$ because $A = \overline{B}$. $\square$

This tells us that $P_{fc}S$ is an endofunctor on $Set$. Next, $\eta_{X}^{P_{fc}S}(x) = \{\Delta_x\}$ and $\{\Delta_x\}$ is obviously finitely generated, therefore $\eta^{P_{fc}S}$ corestricts to $P_{fc}S$. How about $\mu^{P_{fc}S}$?

Recall that $\mu^{P_{fc}S} : P_{fc}S P_{fc}SX \to P_{fc}SX$ is defined, for every $\omega$ convex subset of $S(P_{fc}(SX))$, as

$$\mu^{P_{fc}S}(\omega) = \bigcup_{\Omega \in \omega} \{\mu_X^S(F) \mid F \in c(\Omega)\}$$
where

\[ c(\Omega) = \{ F \in S^2X \mid \forall A \in \text{supp} \Omega. \exists u_A \in A. \forall \varphi \in SX. F(\varphi) = \sum_{A \in \text{supp} \Phi} \Omega(A) \} \]

We aim to prove that \( \bigcup_{\Omega \in \mathcal{A}} \{ \mu_X^S(F) \mid F \in c(\Omega) \} \) is, in fact, finitely generated in the hypothesis that \( \mathcal{A} \in \mathcal{P}_{f,c}SP_{f,c}SX \). We will achieve this in three steps.

- **Step 1**: let \( \mathcal{B} \) be a finite subset of \( \mathcal{A} \) such that \( \mathcal{A} = \mathcal{B} \). Then we prove that

\[
\bigcup_{\Omega \in \mathcal{A}} \{ \mu_X^S(F) \mid F \in c(\Omega) \} = \bigcup_{\Theta \in \mathcal{B}} \{ \mu_X^S(G) \mid G \in c(\Theta) \}.
\]

showing therefore that we can reduce ourselves to a finite union.

- **Step 2**: we prove that each \( \{ \mu_X^S(G) \mid G \in c(\Theta) \} \) as of Step 1 is convex and finitely generated.

- **Step 3**: we prove that the convex closure of a finite union of convex and finitely generated sets is in turn finitely generated.

The next three Lemmas will perform each step.

**Lemma 49.** Let \( \mathcal{A} \in \mathcal{P}_{f,c}SP_{f,c}SX \) and let \( \mathcal{B} \) be a finite subset of \( \mathcal{A} \) such that \( \mathcal{A} = \mathcal{B} \). Then

\[
\bigcup_{\Omega \in \mathcal{A}} \{ \mu_X^S(F) \mid F \in c(\Omega) \} = \bigcup_{\Theta \in \mathcal{B}} \{ \mu_X^S(G) \mid G \in c(\Theta) \}.
\]

**Proof.** Let \( \Omega \in \mathcal{A} = \mathcal{B} \). We have that

\[
\Omega = \mu_X^S_{\mathcal{P}_{f,c}SX}
\begin{pmatrix}
\Theta_1 & \longrightarrow & \sigma_1 \\
\vdots & & \vdots \\
\Theta_t & \longrightarrow & \sigma_t
\end{pmatrix}
\]

where \( \Theta_i \in \mathcal{B} \) and \( \sum_{i=1}^t \sigma_i = 1 \). Notice that \( \text{supp} \Omega = \bigcup_{i=1}^t \text{supp} \Theta_i \). Now, if \( \text{supp} \Omega = \{ A_1, \ldots, A_n \} \) say, we have that any \( F \in c(\Omega) \) is of the form

\[
F = \begin{pmatrix}
\varphi_1 & \longrightarrow & \Omega(A_1) = \sum_{i=1}^t \sigma_i \cdot \Theta_i(A_1) \\
\vdots & & \vdots \\
\varphi_n & \longrightarrow & \Omega(A_n) = \sum_{i=1}^t \sigma_i \cdot \Theta_i(A_n)
\end{pmatrix}
\]

where each \( \varphi_i \in A_i \subseteq SX \). This leads us to define, for each \( l \in \mathcal{A} \), a function \( G_l \in S^2X \) as:

\[
G_l = \begin{pmatrix}
\varphi_1 & \longrightarrow & \Theta_l(A_1) \\
\vdots & & \vdots \\
\varphi_n & \longrightarrow & \Theta_l(A_n)
\end{pmatrix}
\]
Notice that, in fact, $G_l \in \mathcal{c}(\Theta_l)$. Indeed, fixed $l$, we have that $\text{supp} \Theta_l \subseteq \text{supp} \Omega$, so it can be the case that $\Theta_l(A_i) = 0$ for some $i \in \mathcal{A}$. In which case we have $G_l(\varphi_i) = 0$. Nonetheless, for each $A_i$ in $\text{supp} \Theta_l$, the function $G_l$ chooses one of its elements and associates to it $\Theta_l(A_i)$.

Define then

$$H = \begin{pmatrix}
\mu_X^S(G_1) & \sigma_1 \\
\vdots & \vdots \\
\mu_X^S(G_t) & \sigma_t
\end{pmatrix}$$

Then clearly $\sum_{\chi \in \mathcal{S}X} H(\chi) = 1$, and

$$\text{supp} \chi \subseteq \bigcup_{t=1}^t \{ \mu_X^S(G) \mid G \in \mathcal{c}(\Theta_l) \} \subseteq \bigcup_{\Theta \in \mathcal{B}} \{ \mu_X^S(G) \mid G \in \mathcal{c}(\Theta) \}.$$ 

It is a matter of direct calculation to show that $\mu_X^S(F) = \mu_X^S(H)$. This proves the left-to-right inclusion. The vice versa is immediate to see, recalling that we know that the left-hand side is convex. $\Box$

Applying the following Lemma for the $\mathcal{S}$-algebra $(\mathcal{S}X, \mu_X^S)$, we obtain Step 2.

**Lemma 50.** Let $(X, a)$ in $\mathbb{EM}(\mathcal{S})$. Then for all $\Phi \in \mathcal{SP}_{df}^a X$ we have that

$$\{ a(\varphi) \mid \varphi \in \mathcal{c}(\Phi) \} = \overline{\{ a(\psi) \mid \psi \in \mathcal{c}(\Phi') \}}$$

where, if $\text{supp} \Phi = \{A_1, \ldots, A_n\}$ say, and for all $i \ A_i = \overline{B_i}$ with $B_i \subseteq A_i$ finite, then

$$\Phi' = (B_1 \mapsto \Phi(A_1), \ldots, B_n \mapsto \Phi(A_n)).$$

**Proof.** Let $\varphi \in \mathcal{c}(\Phi)$. Then we can write $\varphi$ as

$$\varphi = (u_1 \mapsto \Phi(A_1), \ldots, u_n \mapsto \Phi(A_n))$$

where $u_i \in A_i$ for all $i$. Notice that it is possible for $u_i = u_j$ for $i \neq j$: in that case, the notation above implicitly says that $u_i \mapsto \Phi(A_i) = \Phi(A_j)$. Now, since $A_i = \overline{B_i}$, we have that $u_i = a(\chi_i)$ for some $\chi_i \in \mathcal{S}X$ such that $\sum \chi_i(x) = 1$ and $\text{supp} \chi_i \subseteq B_i$. So:

$$\varphi = (a(\chi_1) \mapsto \Phi(A_1), \ldots, a(\chi_n) \mapsto \Phi(A_n))$$

$$= \mathcal{S}(a)(\chi_1 \mapsto \Phi(A_1), \ldots, \chi_n \mapsto \Phi(A_n))$$
Call \( \Psi = (\chi_1 \mapsto \Phi(A_1), \ldots, \chi_n \mapsto \Phi(A_n)) \). Then we just said that \( \varphi = S(a)(\Psi) \).

We can write \( \Psi \) more explicitly, by listing down the action of each \( \chi_i \):

\[
\Psi = \begin{pmatrix}
\chi_1 = \begin{pmatrix}
x_1^1 \mapsto \lambda_1^1 \\
\vdots \\
x_{s_1}^1 \mapsto \lambda_{s_1}^1
\end{pmatrix} & \mapsto & \Phi(A_1) \\
\vdots & \vdots & \vdots \\
\chi_n = \begin{pmatrix}
x_n^1 \mapsto \lambda_n^1 \\
\vdots \\
x_{s_n}^n \mapsto \lambda_{s_n}^n
\end{pmatrix} & \mapsto & \Phi(A_n)
\end{pmatrix}
\]

where, for each \( k = 1, \ldots, n \), \( \sum_{j=1}^{s_k} \lambda_j^k = 1 \) and for all \( j = 1, \ldots, s_k \) we have \( x_j^k \in B_k \).

Now, define for all \( w \in \prod_{k=1}^n s_k \), a \( n \)-tuple whose \( j \)-th entry is a number between 1 and \( s_j \), the function

\[
\psi_w = (x_1^1 \mapsto \Phi(A_1), \ldots, x_n^w \mapsto \Phi(A_n)), \quad \text{i.e. } \psi_w(x) = \sum_{i \in \mathbb{N}}^\infty \Phi(A_i).
\]

Define also

\[
\Psi' = (\psi_w \mapsto \prod_{k=1}^n \lambda^k_{w_k})_{w \in \prod_{k=1}^n s_k} \quad \text{i.e. } \Psi'(\psi) = \sum_{w \in \prod_{k=1}^n s_k} \prod_{k=1}^n \lambda^k_{w_k}.
\]

Notice that, by Lemma 37, we have that

\[
\sum_{w \in \prod_{k=1}^n s_k} \prod_{k=1}^n \lambda^k_{w_k} = \prod_{k=1}^n s_k \sum_{j=1}^{s_k} \lambda_j^k = \prod_{k=1}^n 1 = 1.
\]

This immediately implies that \( \sum_{\psi \in S^X} \Psi'(\psi) = 1 \). Next, we show that \( \mu_{X}^S(\Psi') = \mu_{X}^S(\Psi) \).

\[
\mu_{X}^S(\Psi')(x) = \sum_{\psi \in S^X} \Psi'(\psi) \cdot \psi(x)
= \sum_{w} \prod_{k=1}^n \lambda^k_{w_k} \cdot \sum_{i \in \mathbb{N}}^\infty \Phi(A_i)
= \sum_{w} \sum_{i \in \mathbb{N}}^\infty \left[ \prod_{k=1}^n \lambda^k_{w_k} \cdot \Phi(A_i) \right]
\]
Lemma 51. Let \( \Phi \) be a \( S \)-function. Then

\[
\mu^S_X(\Phi)(x) = \sum_{i=1}^{\infty} \sum_{w} \left[ \Phi(A_i) \cdot \chi_i(x) \cdot \prod_{k \neq i} \lambda_w^k \right]_{\chi_i(x) = \chi_i(x)} = \sum_{i=1}^{n} \Phi(A_i) \cdot \chi_i(x) \cdot \prod_{k \neq i} \lambda_w^k = \mu^S_X(\Psi)(x).
\]

Hence:

\[
a(\varphi) = a(S(\varphi)) = a(\mu^S_X(\Psi)) = a(\mu^S_X(\varphi)) = a(S(\varphi))
\]

where \( a(S(\varphi)) \) indeed belongs to \( \{a(\psi) \mid \psi \in \mathcal{C}(\Phi)\} \) because

\[
supp(S(a)(\varphi')) = \{a(\psi) \mid w \in \prod_{k=1}^{n} s_k \subseteq \{a(\psi) \mid \psi \in \mathcal{C}(\Phi')\}
\]

and the sum of all its images is \( \sum_{w} \prod_{k=1}^{n} \lambda_w^k = 1. \) \( \square \)

Notice that if \( \Phi' \in SPX \) is such that \( supp \Phi' \subseteq \mathcal{P}_f(X) \), then \( \mathcal{C}(\Phi') \) is finite. Finally, the next Lemma, when \( A \) and \( B \) are finite, proves Step 3.

**Lemma 51.** Let \((X, a) \in E\mathcal{M}(S), A, B \subseteq X.\) Then \( \overline{A \cup B} = \overline{A} \cup \overline{B}.\)

**Proof.** We have:

\[
\overline{A \cup B} = \{a(\varphi) \mid \varphi \in SX, \sum_x \varphi(x) = 1, supp \varphi \subseteq \overline{A \cup B}\}
\]

\[
\overline{A} \cup \overline{B} = \{a(\psi) \mid \psi \in SX, \sum_x \psi(x) = 1, supp \psi \subseteq A or supp \psi \subseteq B\}
\]

\[
\overline{A \cup B} = \{a(\chi) \mid \chi \in SX, \sum_x \chi(x) = 1, supp \chi \subseteq A \cup B\}.
\]

Let then \( x \in \overline{A \cup B}. \) Then

\[
x = a(\varphi) = a\left(\begin{array}{c}
a(\psi_1) \\
\vdots \\
a(\psi_n)
\end{array} \right) \rightarrow \varphi(a(\psi_1)) = a(S(a)(\Phi)) = a(\mu^S_X(\Phi))
\]

where \( \Phi = (\psi_1 \mapsto \varphi(a(\psi_1)), \ldots, \psi_n \mapsto \varphi(a(\psi_n))). \) Calling \( \chi = \mu^S_X(\Phi), \) one can easily check that \( \sum_x \chi(x) = 1 \) and that \( supp \chi \subseteq A \cup B, \) hence \( \overline{A} \cup \overline{B} \subseteq \overline{A \cup B}. \) The other inclusion is obvious, given that \( A \cup B \subseteq \overline{A} \cup \overline{B}. \) \( \square \)
**Proof of Theorem 35.** We first observe that the function
\[ \lfloor \cdot \rfloor_X : T_{\Sigma', E'}(X) \to \mathcal{P}_c S(X) \]
factors as
\[ T_{\Sigma', E'}(X) \xrightarrow{\lfloor \cdot \rfloor_X} \mathcal{P}_f S(X) \xrightarrow{\iota_X} \mathcal{P}_c S(X) \]
where \( \iota_X : \mathcal{P}_f S(X) \to \mathcal{P}_c S(X) \) is the obvious set-inclusion. This can be easily checked by induction on \( T_{\Sigma', E'}(X) \).

Observe that, since \( \lfloor \cdot \rfloor_X \) is injective by Proposition 34, then also \( \lfloor \cdot \rfloor'_X \) is injective. We conclude by showing that it is also surjective.

Let \( A \in \mathcal{P}_f S(X) \). Since \( A \) is finitely generated there exists a finite set \( B \subseteq S(X) \) such that \( B = A \). If \( A = \emptyset \), then \( \lfloor \emptyset \rfloor'_X = A \). If \( B = \{ \varphi_1, \ldots, \varphi_n \} \) with \( \varphi_i \in S(X) \) then, for all \( i \), we take the term
\[ t_i = \varphi_i(x_1) \cdot x_1 + \cdots + \varphi_i(x_m) \cdot x_m \]
where \( \{ x_1, \ldots, x_m \} \) is the support of \( \varphi_i \). It is easy to check that \( \lfloor t_i \rfloor'_X = \{ \varphi_i \} \).

Then by the inductive definition of \( \lfloor \cdot \rfloor'_X \), one can easily verify that \( \lfloor t_1 \sqcup \cdots \sqcup t_n \rfloor'_X = \{ \varphi_1, \ldots, \varphi_n \} = A \).

\[ \square \]

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