Robust Safe Control for Uncertain Dynamic Models

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Abstract—Model mismatches prevail in real-world applications. Hence it is important to design robust safe control algorithms for systems with uncertain dynamic models. The major challenge is that uncertainty results in difficulty in finding a feasible safe control in real-time. Existing methods usually simplify the problem such as restricting uncertainty type, ignoring control limits, or forgoing feasibility guarantees. In this work, we overcome these issues by proposing a robust safe control framework for bounded state-dependent uncertainties. We first guarantee the feasibility of safe control for uncertain dynamics by learning a control-limits-aware, uncertainty-robust safety index. Then we show that robust safe control can be formulated as convex problems (Convex Semi-Infinite Programming or Second-Order Cone Programming) and propose corresponding optimal solvers that can run in real-time. In addition, we analyze when and how safety can be preserved under unmodeled uncertainties. Experiment results show that our method successfully finds robust safe control in real-time for different uncertainties and is much less conservative than a strong baseline algorithm.

I. INTRODUCTION

Safety is critical in robotic systems. Safe control, as the last defense of the system, ensures real-time safety by keeping the system state in a safe set (known as forward invariance) [1]. Energy-function based methods were proposed to realize safe control [2]. With the energy function (also called safety index, barrier function, and Lyapunov functions), the safe control problem is converted to an online quadratic programming (QP). QP-based safe control has been widely studied for deterministic dynamic models [2].

However, dynamic models are approximations of the real-world systems [3]. There can always be uncertainties. Therefore, it is important to enabling safe control for uncertain dynamic models (UDM). But generalizing QP-based safe control to UDM faces many challenges.

The first challenge is how to design a safety index that guarantees the existence of real-time safe control for all time (also known as persistent feasibility). It is difficult to design such a safety index because there are infinitely many states, and the control has to be safe for all these states under all possible models and within the control limits. And such a safety index even may not exist due to the nature of the dynamics or uncertainties. Existing works usually make simplifications, such as only consider additive uncertainty [4], assume finite control [3], ignore control limit [5] and only provide empirical guarantees [6].

On the other hand, even if we have a persistent-feasible safety index, how to optimally solve the QP-based safe control for UDM in real time remains unclear. There can be infinitely many constraints in such a QP because each possible dynamic model leads to one constraint. To address this problem, many existing methods find the safe control using the upper bound of the residual time derivative of the barrier function induced by uncertainty [7], [8], [9]. However, sometimes it is unclear how to choose the bound. And the resulting safe control can be conservative since the bound is often over-approximated and may be state-independent or control-independent. There are other works that find robust safe control under simplified scenarios, such as only considering parametric uncertainty [10] or ignoring control limits [5]. Besides, the modeled uncertainty may not be accurate. It is unclear what happens when the actual uncertainty is greater than we expected.

We propose a general robust safe control framework to address these challenges. First, we propose a sound but not complete method UR-SIS (uncertainty-robust safety index synthesis) to synthesize a control-limit aware and uncertainty-robust safety index where the resulting safe control constraint is guaranteed to be persistently feasible. In case such a safety index does not exist or we can not find one, UR-SIS tries to maximize the number of feasible states. Second, we show that the QP-based safe control for bounded uncertainties can be converted to a convex semi-infinite programming (CSIP). And we propose a real-time optimal solver based on the cutting-plane method. We also show that, if the uncertainty can be bounded by hyper-ellipsoids, the CSIP can be further reduced into a Second Order Cone Programming (SOCP), which has only one constraint and can be solved more efficiently. Compared to existing methods, our method considers a concrete and tight uncertainty bound as well as control limits, therefore is less conservative. Lastly, we provide a theoretical analysis of what happens when unmodeled uncertainty exists. In particular, we present the condition on the unmodeled uncertainty that the system state can be kept in a larger set (than the safe set) with the learned UR-SI. Combining all the parts, we are able to ensure robustness and non-conservativeness of the safe control, and forward invariance of the safe set for bounded dynamic model uncertainties and even unmodeled uncertainties.

The remainder of the paper is organized as follows: Section II formulates the problem and introduces notations. Section III presents the proposed robust safe control framework in detail. Section IV evaluates the proposed methods.
on two examples: SCARA and Segway.

II. Formulation

We consider the following nonlinear control-affine system with uncertain dynamic models (UDM)\(^1\):

\[
\dot{x} = f(x) + g(x)u, \quad f(x) \in \Sigma_f(x), \quad g(x) \in \Sigma_g(x),
\]

where state \(x \in X \subseteq \mathbb{R}^n\) and control \(u \in U \subseteq \mathbb{R}^m\). The terms \(f(x) \in \mathbb{R}^n\) and \(g(x) \in \mathbb{R}^{n \times m}\) are random matrices (elements \(f(x), g(x)_{i,j}\) are random variables) which are bounded by \(\Sigma_f(x) \subseteq \mathbb{R}^n\) and \(\Sigma_g(x) \subseteq \mathbb{R}^{n \times m}\) respectively. \(\Sigma_f(x)\) and \(\Sigma_g(x)\) are measurable functions on the state space which are state-dependent. For simplicity, in the following discussion, we may write \(\forall f(x), g(x)\) to represent the condition \(\forall f(x) \in \Sigma_f(x), g(x) \in \Sigma_g(x)\).

We consider the safety specification as a requirement that the system state should be constrained in a closed and connected set \(X_S \subseteq X\), which we call the safe set. We assume \(X_S\) is the zero-sublevel set of a safety index \(\phi_0 : X \mapsto \mathbb{R}\) given by the user. That is \(X_S = \{x \mid \phi_0(x) \leq 0, x \in X\}\). Constraining states inside \(X_S\) can be expressed as a forward invariance problem: when \(\phi_0(x(t_0)) \leq 0\), ensure \(\phi_0(x(t)) \leq 0, \forall t > t_0\). Forward invariance can be guaranteed with minimal invasion by QP-based safe set algorithms.

Algorithm 1 (QP-based safe set algorithms). Given a reference control \(u_{\text{ref}}\) and a safety index \(\phi\), QP-based safe set algorithms find safe control \(u\) by:

\[
\min_u \|u - u_{\text{ref}}\|^2 \quad \text{s.t.} \quad \dot{\phi}(x,u) \leq -\gamma(\phi(x)), \quad (2)
\]

where \(\gamma\) is a piecewise smooth function and \(\gamma(x) > 0\) when \(\phi(x) > 0\). \(\gamma\) can be non-continuous and designed differently. A special case of \(\gamma\) is an extended class \(K\) function on \(\phi(x)\), corresponding to the control barrier function (CBF) method [2]. Safe set algorithms require \(\phi\) to be persistent feasible: \(\forall x, \exists u\) such that \(\dot{\phi}(x,u) \leq -\gamma(x)\). However, a user-defined safety index \(\phi_0\) may not be naturally persistently feasible. It is often necessary to design a \(\phi\) based on \(\phi_0\) such that we can constrain the state in \(S : \{\phi \leq 0\} \subseteq X_S\) [1].

Extending algorithm 1 to UDM requires the safety constraint holds for all possible models. We define the extended problem as uncertainty-robust QP (UR-QP).

**Definition 1** (UR-QP).

\[
\min_u \|u - u_{\text{ref}}\|^2 \quad \text{s.t.} \quad \dot{\phi}(x,u) \leq -\gamma(\phi(x)), \quad \forall f(x), g(x). \quad (3)
\]

And we say \(\phi\) is an uncertainty-robust safety index (UR-SI) if it ensures persistent feasibility for all possible models.

**Definition 2** (UR-SI). A safety index \(\phi\) is a UR-SI if there exists a piecewise smooth, strictly increasing function \(\gamma\), and \(\gamma(0) = 0\), such that the following feasibility condition holds:

\[
\forall x, \exists u, \quad \dot{\phi}(x,u) \leq -\gamma(\phi(x)), \quad \forall f(x), g(x) \quad (4)
\]

\(^1\)Although our method assumes control affine dynamics, it is applicable to non-control affine systems, since we can always have a control affine form through dynamics extension [11].

To solve the UR-QP eq. (3) or to verify eq. (4), we are interested in the robust safe control set \(U_r(x)\):

\[
U_r(x) := \{u \mid \dot{\phi}(x,u) \leq -\gamma(\phi(x)), \quad \forall f(x), g(x)\}. \quad (5)
\]

With \(U_r(x)\), eq. (3) can be converted into the following form

\[
\min_{u \in U \cap U_r(x)} \|u - u_{\text{ref}}\|^2. \quad (6)
\]

Prior work [10] has shown that \(U_r(x)\) can be defined with the Lie derivatives [1]. Safe set algorithms require eq. (4) to hold for all \(x\), which involves infinitely many states. And, how to deal with unmodeled uncertainty remains unknown.

Then the safety constraint can be transformed as follows:

\[
\forall f,g, \quad \dot{\phi}(x,u) \leq -\gamma(\phi) \iff \max_{f,g} \dot{\phi}(x,u) \leq -\gamma(\phi) \iff \max_{L_f \phi, L_g \phi} L_f \phi + L_g \phi \cdot u \leq -\gamma(\phi) \iff \max_{L_g \phi \in V_g} L_g \phi \cdot u \leq -\gamma(\phi) - \max_{L_f \phi \in V_f} L_f \phi =: c
\]

Then \(U_r\) can be defined equivalently by

\[
U_r = \{u \mid \max_{v \in V_g} v^T u \leq c\} = \{u \mid v^T u \leq c, \forall v \in V_g\}. \quad (10)
\]

Although \(U_r\) is defined by linear constraints, the lack of a closed form of \(U_r\) makes it difficult to solve eq. (6). Even deciding if \(U_r\) is empty is difficult. Prior work [10] considers a simplified scenario and proposes a minimax formulation. But the minimax is generally intractable. In this work, we study how to solve this problem precisely and efficiently.

Besides, it is challenging to synthesize a UR-SI. Because a UR-SI requires eq. (4) to hold for all \(x\), which involves infinitely many states. And, how to deal with unmodeled uncertainty remains unknown.

III. Method

To address these challenges, we first introduce a method to synthesize a UR-SI with finite states, which addresses the infinitely many state problem. Then we present two convex formulations of the UR-QP and corresponding realtime solvers, which give optimal solutions of eq. (6) for polytope or hyper-ellipsoid bounded uncertainties. In the end, we analyze how the forward invariance set changes in presence of unmodeled large uncertainty.

A. Uncertainty-robust safety index synthesis (UR-SIS)

The purpose of safety index synthesis is to design a UR-SI \(\phi\) based on \(\phi_0\) such that the closed-loop trajectory is constrained in the safe set \(X_S\) even under uncertainty.

To synthesize a UR-SI, we extend our previous work on synthesizing safety index for deterministic learned dynamics [12]. We proved that the feasibility of the whole state space can be ensured by only verifying finite sampled states. In this work, we generalize the proof to UDM and UR-SI and adjust the sampling density based on the uncertainty. Then
we will be able to use an evolutionary algorithm to find a valid UR-SI with the sampled states.

We first prove that $\phi$ is a UR-SI if it has feasible solutions on a sampled state set $B$ based on the following assumption:

**Assumption 1.** $\gamma$, $\phi$ and $\nabla\phi$ are Lipschitz continuous functions with Lipschitz constants $k_{\phi}$, $k_{\psi}$ and $k_{\nabla\phi}$ respectively. The set change $\Delta(S, x', x'')$ depends on the state, which is deterministic. Therefore, all the parameterization also applies to UDM because $B_0$ are bounded by $M_u$ and $M_{\bar{x}}$ respectively.

**Lemma 1.** Suppose 1) we sample a state subset $B \subset X$ such that $\forall x \in X$, $\min_{x' \in X} \|x - x'\| \leq \delta$, where $\delta$ is an arbitrary constant representing the sampling density; 2) $\forall x' \in B$, there exists a safe control $u$, s.t. $\phi(x', u) \leq -\gamma(\phi(x')) - \epsilon$, $\forall f(x'), g(x')$, where $\epsilon = k_{\phi}(k_{\Sigma_f} + k_{\Sigma_g}M_u)\delta + k_{\nabla\phi}M_{\bar{x}} + k_{\phi}\delta$. Then $\phi$ satisfies eq. (4) and therefore is a UR-SI.

**Proof.** According to condition 1), $\forall x \in X$, $\exists x' \in B$ such that $\|x - x'\| \leq \delta$. And based on assumption 1, we have $\|f(x) - f(x')\| \leq k_{\Sigma_f}\delta$, and $\|g(x) - g(x')\| \leq k_{\Sigma_g}\delta$. According to condition 2), for this $x'$, we can find $u$ s.t. $\phi(x', u) \leq -\gamma(\phi(x')) - \epsilon$. Next we show $x$ and $u$ satisfy eq. (4) by triangle inequality and Lipschitz condition:

$$
\phi(x, u) = \phi(x, u) - \phi(x', u) + \phi(x', u) \\
\leq \|\nabla\phi(x')\| x + \|\nabla\phi(x')\| x' + \phi(x', u)
$$

We define a polytope $\Sigma_f$ and $\Sigma_g$. We assume the uncertainties are bounded by polytopes $\Sigma_f$ and $\Sigma_g$. That is, $\exists A_f, b_f, A_g, b_g$ such that

$$
\Sigma_f = \{ f | A_f f < b_f \}, \quad \Sigma_g = \{ g | A_g g < b_g \}.
$$

**Corollary.** When $\Sigma_f, \Sigma_g$ are polytopes, $V_f$ and $V_g$ are also polytopes because $\nabla\phi$ is a linear transformation.

We also consider a polytope $\Sigma_f$ to ease the computation of $c$ in the RHS of eq. (9). In this case, $c$ can be solved by linear optimization because $V_f$ is a linear convex set, as done in [13]. Next, we propose a quick way to check the existence of $U_r$ and an efficient method to solve eq. (6).

**Lemma 2.** When $c < 0$, if $0 \in \text{int} V_g$, then $U_r = \emptyset$.

**Proof.** We prove this by contradiction. $0 \in \text{int} V_g \implies \exists v > 0$, s.t. $v \in V_g$ if $\|v\| < r$. Given a $v$ that $\|v\| < r$, suppose $U_r \neq \emptyset$, then $\exists u \in U_r$ such that $v^T u < c < 0$. But $-v \in V_g$ (because $\|v\| < r$) and $-v^T u > 0$, therefore $u \notin U_r$. Contradicts with the assumption.

**Algorithm 2** (Polytope RSSA). Our key insight is that the boundary of $U_r$ is mostly defined by $H$; the vertices of $V_g$. We define $U_r := \{u | u^T v \leq c, \forall v \in H \subset V_g\}$. It is easy to see that $U_r \subset \hat{U}_r$, as shown in fig. 1. To solve the UR-QP eq. (6), we first minimize $\|u - u_{ref}\|$ in $U \cap \hat{U}_r$, which is a QP with finite linear constraints. Then, if $u \notin \hat{U}_r$, we add $[B^u]$, where $B^u := \{x' | \min_u \phi(x', u) \leq -\gamma(\phi(x')) - \epsilon, \forall f(x'), g(x') \in B\}$ is the feasible sampled state set. To decide if a state $x'$ belongs to $B^u$, we use an evolutionary algorithm to find a safe control solver to be proposed next. The algorithm stops when the feasible rate $r$ converges. For $r = 1$, then we find a UR-SI $\phi$. This method is sound but not complete. We will investigate complete methods in the future.
To further simplify the constraint of max violation: 
\[ v^* = \arg \max_{v \in V_g} v \cdot u \] to 
\[ \hat{U}_r \] and repeat the process.

**Lemma 3.** Algorithm 2 converges to the optimal solution.

**Proof.** We point out that eq. (6) is a special case of convex semi-infinite programming (CSIP), and Polytope RSSA is a variant of the cutting-plane method [14]. It has been proved that the method always converges to an optimal solution in [14]. □

**C. Robust safe control for ellipsoid bounded uncertainties**

One of the shortcomings of Polytope RSSA is that constructing polytopes may be time-consuming. Therefore, we propose a more efficient algorithm that considers hyper-ellipsoid uncertainty bounds, which is particularly useful when we consider chance constraints for Gaussian uncertainties.

**Assumption 3** (Ellipsoid bounds). \( \Sigma_f, \Sigma_g, \Sigma_f, \Sigma_g \) are hyper-ellipsoids, that is \( \exists \mu_f \in \mathbb{R}^n, Q_f \in \mathbb{R}^{n \times n}, Q_f > 0, d_f \in \mathbb{R}^n \), and \( \exists \mu_g \in \mathbb{R}^m, Q_g \in \mathbb{R}^{m \times m}, Q_g > 0, d_g \in \mathbb{R}^m, \) s.t.

\[
\Sigma_f := \{ f | f - \mu_f \|^2 Q_f^{-1} < d_f \}, \\
\Sigma_g := \{ g | g - \mu_g \|^2 Q_g^{-1} < d_g \}.
\]

**Remark.** If \( f \sim N(\mu_f, Q_f) \), \( g \sim N(\mu_g, Q_g) \), \( d_f = \chi^2_n(\alpha) \) and \( d_g = \chi^2_m(\alpha) \), where \( \alpha \in \mathbb{R}^+ \) and \( \chi^2 \) is the chi-square function. Then the above hyper-ellipsoids correspond to the \( 1 - \alpha \) confidence bounds.

**Corollary.** \( L_g \phi, L_f \phi \) are also bounded by hyper-ellipsoids. Let \( \mu_v = \nabla \phi \cdot \mu_g, Q_v = \nabla \phi^T Q_g \nabla \phi \). Then

\[
V_g := \{ v | (v - \mu_v)^T Q_v^{-1} (v - \mu_v) \leq d_g \}.
\]

Similarly, we can define \( V_f \) and compute \( c \) as in eq. (9).

With \( V_f \) and \( c \), the lemma below gives us an equivalent form of \( U_r := \{ u | v^T u \leq c, \forall v \in V_f \} \) using the hyper-ellipsoid bounds, which is defined by only one constraint.

**Lemma 4.** Under assumption 3, \( \exists L, s.t. LL^T = d_g Q_g \). And \( U_r \) has the second-order-cone form:

\[
U_r = \{ u | ||L^T u|| \leq -\mu_u^T u + c \}.
\]

**Proof.** For simplicity, let \( Q := d_g Q_g \). Then, the hyper-ellipsoid can be described as

\[
\Sigma_v := \{ v | (v - \mu_v)^T Q_v^{-1} (v - \mu_v) \leq 1 \}
\]

Because \( Q > 0 \), there exists an invertible matrix \( L \), s.t. \( Q = LL^T \). Define \( v' = L^{-1} (v - \mu_v) \), then \( V_g \) could be transformed to \( V_g' = \{ v' | v'^T v' \leq 1 \} \). Substitute \( v \) with \( v' \) in \( U_r \), we could get

\[
U_r = \{ u | (L^T v' + \mu_v)^T u \leq c, \forall v' \in V_g' \}
\]

To further simplify \( U_r \), we first introduce an auxiliary variable \( u' \) s.t. \( u = (L^T)^{-1} u' \). Then \( U_r \) is

\[
U_r = \{ u = (L^T)^{-1} u' | u'^T u' + (L^{-1} \mu_v)^T u' \leq c, \forall v' \in V_g' \}
\]

Notice that \( \max_{v' \in [1]} u'^T u' = ||u'||. \) Therefore,

\[
U_r = \{ u = (L^T)^{-1} u' | ||u'|| + (L^{-1} \mu_v)^T u' \leq c \} \quad (25)
\]

\[
= \{ u | ||L^T u|| \leq -\mu_u^T u + c \} \quad (26)
\]

In this way, we converted the UR-QP eq. (3) to a Second Order Cone Programming (SOCP), which can be solved efficiently by Semi-Definite Programming (SDP) solvers.

**D. Forward invariance under unmodeled uncertainty**

We have discussed how to design a robust safety index and how to find robust safe control with known uncertainty ranges. However, in practice, sometimes we can not acquire an accurate uncertainty range in advance. Further analysis for unmodeled uncertainties is desired. We present the condition on the unmodeled uncertainties such that the forward invariance is still preserved (in a larger set).

**Definition 5** (Residual dynamics). We define the unmodeled uncertainty as unknown residual dynamics \( \hat{f}(x) \) and \( \hat{g}(x) \):

\[
\dot{x} = f(x) + g(x)u + \hat{f}(x) + \hat{g}(x)u,
\]

\[
f(x) \in \Sigma_f(x), g(x) \in \Sigma_g(x)
\]

**Lemma 5.** A UR-SI \( \phi \) ensures the forward invariance of the set \( \hat{S} = \{ x | \phi(x) \leq \gamma^{-1} (k_0 M_{\tilde{z}}) \} \) with unmodeled uncertainties shown in eq. (27), where \( M_{\tilde{z}} := \max_{x,u} \| \hat{f}(x) + \hat{g}(x)u \| \) represents the maximum norm of the residual dynamics:

\[
\dot{\phi} = \nabla \phi \cdot \hat{f}(x) + \hat{g}(x)u \leq \gamma \phi(x)
\]

**Proof.** It suffices to show that applying \( \phi \) on the nominal dynamics eq. (1) ensures \( \phi \leq 0 \) on \( \partial \hat{S} := \{ x | \phi(x) = \gamma^{-1} (k_0 M_{\tilde{z}}) \} \) for the true dynamics eq. (27).

\( \phi \) can be decomposed as \( \phi = \phi_{nom} + \phi_{res} \), where \( \phi_{nom} := \nabla \phi[f(x) + g(x)u], \phi_{res} := \nabla \phi(x) \hat{f}(x) + \hat{g}(x)u] \). Because \( \forall x, \exists u, \phi_{nom}(x) \leq -\gamma \phi(x) \), \( \forall f(x), \forall g(x) \) and

\[
\phi_{res}(x) = \nabla \phi(x) \hat{f}(x) + \hat{g}(x)u \leq \gamma \phi(x)
\]

\[
\leq k_0 \| \hat{f}(x) + \hat{g}(x)u \| \leq k_0 M_{\tilde{z}}.
\]

We have \( \forall x \in \partial \hat{S}, \exists u \) such that \( \forall f(x), \forall g(x) \)

\[
\phi(x) = \phi_{nom}(x) + \phi_{res}(x) \leq -\gamma \phi(x) + k_0 M_{\tilde{z}}
\]

\[
= -\gamma (k_0 M_{\tilde{z}}) + k_0 M_{\tilde{z}} = 0
\]

Therefore, \( \hat{S} \) is forward invariant with the UR-SI \( \phi \). □

**IV. EXPERIMENT**

Our method applies to arbitrary dynamic model uncertainties as long as they can be bounded by state-dependent polytopes or ellipsoids, including but not limited to Gaussian Process, neural network dynamic models, and parameterized models. For simplicity, we show the effectiveness of our method on two parameterized models with non-Gaussian and Gaussian uncertainties.
α search ranges for the parameters are tuned safety index φ in the whole state space. Fig. 3 compares φ set check how many sampled states have feasible control. We k resid

m residual time derivative of φ certainty as residual dynamics, and uses a constant to bound the uncertainty. Specifically, we assume m = [x^α, y^α]^T and p_{ee} = [x^h, y^h]^T. Then the safety constraint φ_0 = -Δx = x_{ee} - x_{wall} < 0 should always be satisfied. The detailed dynamic model can be found in section V-A.

We assume the payload m_2 (the mass of the second arm) is uncertain. Specifically, we assume m_2 ~ N(1,0,0.3^2) * I_{(0,1.1,9)}. But after transformation, the dynamic model uncertainty is not Gaussian (the transformation involves multiplicative inverse and matrix inverse).

1) Feasibility: We first show that the safety index learning ensures feasibility. To learn a UR-SI, we use the parameterized form proposed by [1]: \( \phi = \max \{ \phi_0, -x^α_{wall} + x^α_{ee} + k_v x_{ee} + \beta \} \), where \( \alpha, k_v, \beta \) are learnable parameters. The search ranges for the parameters are: \( \alpha \in (0.1,5.0), k_v \in (0.1,5.0), \beta \in (0.001,1.0) \). For each parameter set, we check how many sampled states have feasible control. We set x_{wall} = 1.5, and uniformly sample 250000 states in the whole state space. Fig. 3 compares φ_0, a manually tuned safety index φ_h (\( \alpha = 1.0, k_v = 0.2, \beta = 0.0 \)), and a synthesized UR-SI \( \phi \) with the Polytope RSSA solver. (\( \alpha = 0.57, k_v = 2.15, \beta = 0.072 \)). UR-SI achieves a 0 infeasible rate.

2) Robust safe control solver: We compare the performance of our solvers Polytope RSSA and Ellipsoid RSSA, with a popular method: constant bounded robust safe control (Constant RSSA) [7, [8], [9], which considers the uncertainty as residual dynamics, and uses a constant to bound the residual time derivative of φ: \( \phi_{res} \) caused by uncertainties.

A. SCARA

We first test our method on a SCARA platform. We define the safety specification as not colliding with the vertical wall. Define the position and velocity of SCARA’s end-effector as \( p_{ee} = [x_{ee}, y_{ee}]^T \) and \( \dot{p}_{ee} = [\dot{x}_{ee}, \dot{y}_{ee}]^T \). Then the safety constraint \( \phi_0 = -\Delta x = x_{ee} - x_{wall} < 0 \) should always be satisfied. The detailed dynamic model can be found in section V-A.

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We construct the uncertainty bounds for both Ellipsoid RSSA and Polytope RSSA by sampling methods. We first get 50 dynamic model samples by sampling m_2 from the distribution and execute the nonlinear dynamics transformation. Then we either fit a Gaussian distribution for Ellipsoid RSSA or construct a polytope for Polytope RSSA.

For a fair comparison, we learned a UR-SI for each solver, denoted by \( \phi_{poly}, \phi_{elp} \) and \( \phi_{cst} \). With a true yet unknown m_2 = 0.5, we plot trajectories of \( \Delta x \) of all the solvers with the three UR-SIs under two different initial conditions: 1. when \( \phi_0 < 0 \) and \( \phi < 0 \), fig. 4 (a) shows that all the methods ensures safety, but as for the conservativeness, Polytope RSSA < Ellipsoid RSSA < Constant RSSA. 2. when \( \phi_0 < 0 \) but \( \phi > 0 \), fig. 4 (b) shows that Ellipsoid RSSA violates the safety constraint; To explain it in detail, we visualize the uncertainty bounds and corresponding safe
control sets during the execution in fig. 5. It shows that the Gaussian distribution does not capture the uncertainty correctly. Leading to incorrect safe control sets and therefore safety violations. This also demonstrates that existing methods relying on Gaussian assumptions may not be reliable [3], [5]. Note that the ellipsoid does not have to be constructed from Gaussian, and Ellipsoid RSSA also guarantees the safety if the ellipsoid bound is correctly constructed.

3) Unmodeled uncertainties: We study the forward invariance set change under unmodeled uncertainties. We test several $m_2$ that are out of the modeled uncertainty bound $[0.1, 1.9]$ in safety index learning and record $\phi_{\text{max}}$: the largest $\phi$ of many sampled trajectories, as an indicator of the forward invariant set boundary. Table I shows that the forward invariant set grows incrementally with the uncertainty level, and $\phi_{\text{max}}$ is always below the theoretical bound predicted in lemma 5.

| $m_2$  | 0.005 | 0.08 | 1.0 | 2.0 | 3.0 |
|-------|-------|------|-----|-----|-----|
| $\phi_{\text{max}}$ | 32.09 | 0.41 | -0.0013 | 0.92 | 1.51 |
| Theoretical bound | 256.4 | 3.51 | 0.0 | 1.07 | 8.75 |

* TABLE I*

**FORWARD INVARIANCE WITH UNMODELLED UNCERTAINTIES.**

B. Segway

To highlight the optimality and efficiency of Ellipsoid RSSA in case of Gaussian uncertainty. We also test our method on a realistic Segway model. We consider a tracking task with a safety specification on the tilt angle: $\phi_0 = |\phi| - 0.1$. We consider a safety index $\phi = \max\{|\varphi\alpha - 0.1\alpha + |\varphi\alpha + k_r\text{sign}(\varphi)\dot{\varphi} + \beta\}$, where $\alpha, k_r, \beta$ are learnable parameters. The nominal controller is designed to maintain $\dot{\theta}$ at $1\text{m/s}$. The detailed dynamics is shown in section V-B.

We assume the motor torque constant $K_m$ is a truncated Gaussian distribution: $K_m \sim N(2.524, 0.3^2) \cdot I_{[1.624, 3.424]}$. After transformation, the dynamic model uncertainty is a state-dependent Gaussian distribution.

1) Robust safe control solvers: We can efficiently compute the correct ellipsoid bounds for Ellipsoid RSSA because the dynamic model follows Gaussian distributions. But for Polytope RSSA, we still rely on the sampling method to construct the bound. As shown in fig. 6 (a), they both ensure safety and are non-conservative. But the computation time of Ellipsoid RSSA does not grow with the number of samples as shown in fig. 6 (b), which is a significant advantage when the dimensions are high and require more samples. The constant RSSA is overly conservative because $\dot{\phi}_{\text{res}}$ is too large.

V. CONCLUSION

In this work, we proposed a general framework of robust safe control for uncertain dynamic models. Our method applies to arbitrary bounded uncertainties and solves in real time. The future work includes generalizing the method to the case that $f$ and $g$ have correlations and high-dimensional applications that require a large number of samples to learn the safety index and construct bounds.

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APPENDIX

A. SCARA

Define joint positions of the robot arm as \( \theta = [\theta_1, \theta_2]^T \) and joint velocities as \( \dot{\theta} = [\dot{\theta}_1, \dot{\theta}_2]^T \). Here we assume all states \( x = [\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2]^T \) and control inputs \( u \) are bounded:

\[ X : [-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}] \times [-2, 2] \times [-2, 2] \]  
\[ U : [-20, 20] \times [-20, 20] \]

The control-affine dynamic model for SCARA is as follows:

\[
\frac{d}{dt} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 \\ -M(\theta)^{-1} H(\theta, \dot{\theta}) \end{bmatrix} + \begin{bmatrix} 0 \\ M(\theta)^{-1} B \end{bmatrix} u
\]

(32)

where \( M(\theta) \) is the mass matrix and \( H(\theta, \dot{\theta}) \) is the Coriolis matrix:

\[
M(\theta_1, \theta_2) = \begin{bmatrix} 2A + 2B + 2C \cos(\theta_2) & 2B + C \cos(\theta_2) \\ 2B + C \cos(\theta_2) & 2B \end{bmatrix}
\]

(33)

\[
H(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = \begin{bmatrix} -C \sin(\theta_2) \cdot (2\dot{\theta}_1 + \dot{\theta}_2) \dot{\theta}_2 \\ C \sin(\theta_2) \cdot \dot{\theta}_2^2 \end{bmatrix}
\]

(34)

And \( A, B, C \) only depend on the robot arm’s masses and lengths:

\[
\begin{align*}
A &= \frac{1}{2}m_1 l_1^2 + \frac{1}{2}m_2 l_1^2 \\
B &= \frac{1}{2}m_2 l_2^2 \\
C &= \frac{1}{2}m_2 l_1 l_2
\end{align*}
\]

(35)

B. Segway

Given wheel’s position \( p \) and frame’s tilt angle \( \varphi \), we define \( q = [p, \varphi]^T \) and \( \dot{q} = [\dot{p}, \dot{\varphi}]^T \). SegWay’s dynamic model can be written as

\[
\frac{d}{dt} \begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \dot{\varphi} \\ -M(q)^{-1} H(q, \dot{q}) \end{bmatrix} + \begin{bmatrix} 0 \\ M(q)^{-1} B \end{bmatrix} u
\]

(36)

where \( M(q), H(q, \dot{q}), B \) and \( b_t \) are defined as follows:

\[
M(q) = \begin{bmatrix} m_0 + mL \cos(\varphi) \\ mL \cos(\varphi) \end{bmatrix}
\]

(37)

\[
H(q, \dot{q}) = \begin{bmatrix} -mL \sin(\varphi) \dot{\varphi}^2 + \frac{b_t}{R} (\dot{p} - R \dot{\varphi}) \\ -mgL \sin(\varphi) - b_t (\dot{p} - R \dot{\varphi}) \end{bmatrix}
\]

(38)

\[
B = \begin{bmatrix} \frac{K_m}{R} \\ \frac{K_m}{m} \end{bmatrix}
\]

(39)

\[
b_t = \frac{K_m K_b}{R}
\]

(40)