CATEGORICAL DIMENSION OF BIRATIONAL AUTOMORPHISMS AND FILTRATIONS OF CREMONA GROUPS

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Abstract. Using a filtration on the Grothendieck ring of triangulated categories, we define the categorical dimension of a birational map between smooth projective varieties. We show that birational automorphisms of bounded categorical dimension form subgroups, which provide a nontrivial filtration of the Cremona group. We discuss some geometrical aspect and some explicit example. In the case of threefolds, we can moreover recover the genus of a birational automorphism, and the filtration defined by Frumkin.

1. Introduction

In the last decades, derived categories of coherent sheaves and their semiorthogonal decompositions attracted a growing amount of interest, thanks, among other features, to the conjectural interplay with birationality questions. Since the seminal work of Bondal and Orlov [13], examples and ideas based on the motivic behavior of semiorthogonal decompositions have lead to formulate some natural question about the possibility to define a birational invariant, or, at least, to obtain necessary conditions for rationality of a given variety from semiorthogonal decompositions of its derived category. We refer to [22] and [4] for recent reports on motivations, open questions, conjectures, technical problems, and comparison to other theories.

In an effort to understand the above questioning, and inspired by the classical notion of representability of Chow groups, the definition of categorical representability of a smooth projective variety $X$ was given in [7]. Roughly speaking, such an $X$ is categorically representable in dimension $m$ if the derived category $D^b(X)$ admits a semiorthogonal decomposition whose components can be realized in semiorthogonal decompositions of varieties of dimension at most $m$. Based on blow-up formulas and Hironaka resolution of singularities, the upshot of this definition is to understand whether being representable in codimension 2 is a necessary condition for rationality.

A fundamental invariant one can consider to study the birational geometry of a given variety $X$ is the group of birational automorphisms $\text{Bir}(X)$. The definitions and the results presented here came out of an attempt to understand the interplay between semiorthogonal decompositions and the group $\text{Bir}(X)$. The guiding idea is the possibility to define, basing upon weak factorization and categorical representability, the notion of categorical dimension of a given birational map $\phi : X \rightarrow Y$, see Definition 3.1. This is, to the best of the author’s knowledge, a first, though very little, attempt to obtain informations on $\text{Bir}(X)$ using semiorthogonal decompositions, while their interplay with other commutative birational invariants is already been treated (see [4] for a recent report) and sometimes even well understood (see [2] for the Hodge theory of cubic fourfolds, or [8] for intermediate Jacobians).

Let us resume the circle of ideas leading to Definition 3.1. Given a weak factorization of $\phi$, one has to consider the centers of all the blow-ups involved, and calculate the maximal value of categorical representability of such centers. One would like to define the categorical dimension of $\phi$ to be the minimum of such values on all the possible weak factorizations of $\phi$. However, one of the main technical (potential) obstructions to have this value well-behaved is the lack of Jordan-Hölder property for semiorthogonal decompositions, see [21] or [11]. In particular, if we used such a definition, we would not be able to compare the dimensions of $\phi$ and $\phi^{-1}$, a central result in order to have a group filtration on $\text{Bir}(X)$.

In order to obtain a well-behaved definition, we need to work in the Grothendieck rings $PT(k)$ and $K_0(\text{Var}(k))$ of dg categories and, respectively, varieties over $k$. Roughly speaking, in these
rings, semiorthogonal decompositions and, respectively, blow-ups turn into sums. Moreover, categorical representability induces a ring filtration $PT_d(k)$ on $PT(k)$. Bondal, Larsen and Lunts \cite{12} defined a \textit{motivic measure} (i.e. a ring homomorphism) $\mu : K_0(\text{Var}(k)) \rightarrow PT(k)$ by sending $[X]$ to the class of $D^b(X)$ for any smooth projective variety $X$. We define then the \textit{motivic categorical dimension} of $X$ to be the smallest $d$ such that $\mu([X])$ lies in $PT_d(k)$. This value is bounded above by the categorical representability of $X$, but it is not known if they always coincide. Given a birational map $\phi : X \dashrightarrow Y$, replacing categorical representability by motivic categorical dimension, we define the \textit{categorical dimension} of $\phi$ as we sketched above: running through all possible weak factorizations, we take the minimum value of the maximal motivic categorical dimension of the centers involved in the factorization. In the case where $X = Y$ this gives a filtration of $\text{Bir}(X)$.

\textbf{Theorem 1.} Let $X$ be smooth and projective and $\text{Bir}_d(X) \subset \text{Bir}(X)$ to be the subset of birational maps of categorical dimension at most $d$. Then $\text{Bir}_d(X)$ is a subgroup, which coincides with the whole group if $d \geqslant \dim (X) - 2$.

A slightly more general version of Theorem 1 will be stated and proved as Theorem 3.3.

Let us illustrate some geometric applications of Theorem 1 in the most interesting cases, that is $X = \mathbb{P}^n$. First of all, using Hochschild homology, we can prove that we actually defined a proper filtration.

\textbf{Theorem 2.} Let $n \geqslant 3$, and assume $k \subset \mathbb{C}$ is algebraically closed. Then $\text{Bir}_d(\mathbb{P}^n) \neq \text{Bir}_{d+1}(\mathbb{P}^n)$ for all $d < n - 2$.

Theorem 2 will be stated and proved as Theorem 3.3. The assumption on the base field is required in the proof, since we use results from Hodge theory and comparisons with Hochschild homology.

Given a birational map, being able to provide a weak factorization should give interesting informations on the birational map itself. For example, using more refined motivic arguments, we can see that if a birational map $\phi : X \dashrightarrow X$ admits a weak factorization with rational centers, then $\phi$ belongs to $\text{Bir}_{n-4}(X)$, see Corollary 3.2. Finally, if such a $\phi$ admits a weak factorization whose centers are all (abstractly isomorphic to) toric varieties, one can use a result of Kawamata \cite{15} to show that $\phi$ belongs to $\text{Bir}_0(X)$. It follow that the subgroup of $\text{Bir}(\mathbb{P}^n)$ generated by the standard Cremona transformation and automorphisms is a subgroup of $\text{Bir}_0(\mathbb{P}^n)$. Notice that $\text{Bir}_0(\mathbb{P}^n) \subset \text{Bir}_{n-4}(\mathbb{P}^n)$ is strict as soon as $n \geqslant 5$, and that there exist rational varieties of dimension at least three with positive motivic categorical dimension. If one were able to construct a birational map $\phi$ contracting rational varieties and such that $\text{cdim}(\phi) > 0$, then we would have that the group generated by maps contracting rational varieties is strictly bigger than the one generated by the standard Cremona transformation, a result proven in \cite{10} in the case where $n$ is odd. However, the proof of Theorem 2 heavily relies on the construction of birational maps contracting non-rational loci. See Section 4 for more details.

In \cite{16}, Frumkin defined a filtration of the group $\text{Bir}(X)$ for a uniruled complex threefold $X$. This is done by defining the \textit{genus} of a birational map $\phi$ to be the maximum of the genera among the centers of the blow-ups in regular resolutions of $\phi^{-1}$. As shown by Lamy \cite{23}, this is exactly the same as the maximum of the genera of irreducible divisors contracted by $\phi$. Using the theory of noncommutative motives and the fact, proved in \cite{3}, that one can reconstruct intermediate Jacobians and their polarizations via semiorthogonal decompositions, we can extend further the definition of genus to birational maps $\phi : X \dashrightarrow X$ in the case where both $X$ and all the centers of a weak factorization of $\phi$ have, roughly speaking, well-behaved principally polarized intermediate Jacobians. This recovers and extends Frumkin’s definition, see Proposition 4.3.

We conclude by recalling that Dmitrov, Haiden, Katzarkov and Konstevich have defined in \cite{15} the notion of entropy for an endofunctor of a triangulated category. On the other hand, the entropy of a birational map is a well-known, very interesting object of study. As asked by the above authors, it would be very interesting to understand if and how these two notions
can be related to each other [15, §4.3]. Here, noncommutative methods are used to produce a filtration of the group Bir(X). It is not clear to the author, whether knowing the categorical dimension of a birational map could give any information on its topological entropy.

We work exclusively over a field k of characteristic zero to ensure the existence of weak factorizations [1].

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2. Noncommutative tools

In this section we aim to give a short introduction to the noncommutative tools that are actively involved in the next: the filtration in the Grothendieck ring of triangulated categories, Hochschild homology and Jacobians of noncommutative motives. We assume the reader to be familiar with notions such as semiorthogonal decompositions and noncommutative motives; references can be find in [4] and [29] respectively.

2.1. Categorical representability. Using semiorthogonal decompositions, one can define a notion of categorical representability for triangulated categories. In the case of smooth projective varieties, this is inspired by the classical notions of representability of cycles, see [7].

Definition 2.1. A triangulated category A is representable in dimension m if it admits a semiorthogonal decomposition

\[ A = \langle A_1, \ldots, A_r \rangle, \]

and for each \( i = 1, \ldots, r \) there exists a smooth projective k-variety \( Y_i \) with \( \dim Y_i \leq m \), such that \( A_i \) is equivalent to an admissible subcategory of \( D^b(Y_i) \).

We use the following notation

\[ \text{rdim}(A) := \min \{ m \in \mathbb{N} \mid A \text{ is representable in dimension } m \}, \]

whenever such finite \( m \) exists.

Definition 2.2. Let \( X \) be a smooth projective k-variety. We say that \( X \) is categorically representable in dimension \( m \) (or equivalently in codimension \( \dim (X) - m \)) if \( D^b(X) \) is representable in dimension \( m \).

We will use the following notations:

\[ \text{rdim}(X) := \text{rdim}(D^b(X)), \quad \text{rcodim}(X) := \dim (X) - \text{rdim}(D^b(X)), \]

and notice that they are both nonnegative integer numbers.

Let \( A \) be representable in dimension \( m \) via a semiorthogonal decomposition \( A = \langle A_1, \ldots, A_r \rangle \), and let \( A = \langle B_1, \ldots, B_s \rangle \) be another semiorthogonal decomposition. Notice that the Jordan–Hölder property for semiorthogonal decompositions does not hold in general (see [11] or [21]), so that there is no known relation, in general, between the triangulated categories \( A_i \)'s and \( B_j \)'s. It follows that one does not know in general whether the \( B_j \)'s are also representable in dimension \( m \). As we will see later in this chapter, we can anyhow compare some invariant and obtain bounds for the representability of the \( B_j \)'s.

2.2. The Grothendieck ring of pretriangulated dg categories and its filtration. We sketch Bondal-Larsen-Lunts construction of the Grothendieck ring of pretriangulated dg categories [12]. Consider the free \( \mathbb{Z} \)-module generated by equivalence classes of pretriangulated dg categories, set \( I(A) \) for the class of such a category \( A \), and introduce the following relation:

\[ I(A) = I(B) + I(C) \quad \text{if} \quad A = \langle B, C \rangle. \]

We denote the quotient group by \( PT(k) \) (see [12, §5.1]).
Elements of $PT(k)$ will be also denoted by a lower case letter. For a smooth projective variety $X$, notice that there is a unique structure of pretriangulated dg category on $D^b(X)$ (see [20]). We will often use the notation $I(X)$ for the class $I(D^b(X))$ in $PT(k)$. In general, given any element $a$ in $PT(k)$, it is not known whether there exists a pretriangulated dg category $A$ such that $a = I(A)$. But if $B$ and $C$ are pretriangulated dg categories, and $a = I(B) + I(C)$, then $a = I(B \oplus C)$.

In the additive commutative group $PT(k)$, define the following associative product:

$$I(A) \bullet I(B) = I(A \otimes B).$$

**Proposition 2.3** ([12], Cor. 5.7). The group $PT(k)$ endowed with the product $\bullet$ is a commutative associative ring with unit $1 = I(D^b(Spec(k)))$.

The notion of categorical representability induces a ring filtration on $PT(k)$, as follows.

**Definition 2.4.** Let $d$ be a nonnegative integer. The subset $PT_d(k) \subset PT(k)$ is the set of $\mathbb{Z}$-linear combinations of classes of pretriangulated dg categories $A$ such that there exist pretriangulated dg categories $A_1, \ldots, A_r$ and $B$ with $\text{rdim}(B) \leq d$ such that $I(B) = I(A) + \sum_{i=1}^r I(A_i)$.

Equivalently, one can see $PT_d(k)$ as the additive subgroup generated by the smallest saturated monoid of $PT(k)$ containing classes of pretriangulated dg categories of categorical dimension at most $d$.

We notice that this definition is slightly different from the one introduced in [4, §8.1].

**Proposition 2.5.** The subsets $PT_i(k)$ give a filtration on the ring $PT(k)$. More precisely, suppose that $a$ is in $PT_i(k)$ and $b$ is in $PT_j(k)$. Then

$$a + b \text{ is in } PT_{\max(i,j)}(k),$$

$$a \bullet b \text{ is in } PT_{i+j}(k).$$

In particular, $PT_i(k)$ is an additive subgroup for any $i$.

**Proof.** First of all, by definition, $PT_i(k) \subset PT_{i+1}(k)$ for any integer $i \geq 0$. Now, let $A$ and $A'$ be pretriangulated dg categories such that $I(A)$ is in $PT_i(k)$ and $I(A')$ is in $PT_j(k)$. By definition, there exist $B$ and $B'$ such that $I(A)$ is a summand of $I(B)$ and $I(A')$ is a summand of $I(B')$, and $\text{rdim}(B) = i$, while $\text{rdim}(B') = j$. Then $I(A) + I(A')$ is a summand of $I(B' \oplus B) = I(B) + I(B')$ and $\text{rdim}(B \oplus B') \leq \max(i,j)$.

A similar argument works for the product. Indeed, with the same notations as above, it is enough to consider the case where $B$ (resp. $B'$) is an admissible subcategory of $D^b(X)$ (resp. $D^b(X')$), for some smooth projective variety $X$ (resp. $X'$) of dimension $i$ (resp. $j$). $B \oplus B'$ is admissible in $D^b(X \times X')$ (see, e.g., [19]), so that $\text{rdim}(B \oplus B') \leq i + j$ by an easy dimension calculation. Using the ring structure of $PT(k)$, one sees moreover that $I(A) \bullet I(A')$ is a summand of $I(B) \bullet I(B') = I(B \otimes B')$. \hfill \square

**Definition 2.6.** Let $A$ be a pretriangulated dg category. The **motivic categorical dimension** of $A$ is the smallest integer $d$ such that $I(A)$ belongs to $PT_d(k)$. We denote this value (which is either a nonnegative number or infinity), by $\text{mcd}(A)$. If $X$ is a smooth and projective scheme, then we also set $\text{mcd}(X) := \text{mcd}(D^b(X))$.

**Lemma 2.7.** For any pretriangulated dg category $A$, we have:

$$\text{mcd}(A) \leq \text{rdim}(A).$$

For any smooth and projective variety $X$, we have:

$$\text{mcd}(X) \leq \text{rdim}(X) \leq \text{dim}(X).$$

**Proof.** The first inequality follows by definition of the subset $PT_d(k)$: if $\text{rdim}(A) = d$, then clearly $I(A)$ belongs to $PT_d(k)$, hence $\text{mcd}(A) \leq d$. Now, if $X$ is smooth and projective, the first inequality is nothing but the previous one with $A = D^b(X)$. The second inequality follows by taking the trivial semiorthogonal decomposition $D^b(X) = \langle D^b(X) \rangle$. \hfill \square
Remark 2.8. We notice that strict inequality $\mathrm{mcd}(A) < \mathrm{rdim}(A)$ can hold over any field $k$, as Bondal-Kuznetsov’s counterexample [21] shows: there exists a category without exceptional objects which can be realized as a subcategory of a category generated by exceptional objects. A natural question is then to find conditions for which the equality $\mathrm{mcd}(A) = \mathrm{rdim}(A)$ holds. As we will see in Corollary 2.8, there exist smooth and projective manifolds $X$ with $\mathrm{mcd}(X) = \mathrm{rdim}(X) = \dim(X)$. For example, this holds if $X$ is Calabi-Yau (see Example 2.13 as well).

Remark 2.9. Notice that, if $X$ is a smooth projective $k$-variety, then $I(X) \neq 0$ in $PT(k)$. This can be shown, for example, using that Hochschild homology (see below) is nontrivial. Hence, if $X$ and $Y$ are smooth and projective $k$-varieties, and $m_i$ and $n$ nonnegative integers, then $mI(X) + nI(Y) = 0$ if and only if $m = n = 0$. Indeed, the former is the class of the scheme $(\mathbb{P}^m \times X) \sqcup (\mathbb{P}^n \times Y)$.

Consider now the Grothendieck ring $K_0(\text{Var}(k))$ of $k$-varieties whose unit $1 = [\text{Spec}(k)]$ is the class of the point, and recall that a motivic measure is a ring homomorphism $\mu : K_0(\text{Var}(k)) \to R$ to some ring $R$. Using weak factorization, the ring $K_0(\text{Var}(k))$ can be seen as the $\mathbb{Z}$-module generated by isomorphism classes of smooth proper varieties, where we set $[\emptyset] = 0$ and with the relation $[X] - [Z] = [Y] - [E]$ whenever $Y \to X$ is the blow-up along the smooth center $Z$ with exceptional divisor $E$, see [9]. The class of the affine line in $K_0(\text{Var}(k))$ is denoted by $L$.

Bondal, Larsen, and Lunts [12] show that, in this case, the assignment
\[
(3) \quad \mu : K_0(\text{Var}(k)) \to PT(k), \quad [X] \mapsto I(D^b(X))
\]
defines a motivic measure. Recall the semiorthogonal decomposition $D^b(\mathbb{P}^1) = \langle \mathcal{O}, \mathcal{O}(1) \rangle$, where both components are equivalent to $D^b(\text{Spec}(k))$. Since $[\mathbb{P}^1] = L + 1$ in $K_0(\text{Var}(k))$, we deduce that $\mu(L) = 1$. Moreover, $\mu([\mathbb{P}^m]) = m + 1$ for any positive integer $m$.

2.3. Hochschild homology. An additive invariant is a functor from the category of pretriangulated dg categories $\text{dgCat}(k)$ to some additive category that sends semiorthogonal decompositions into direct sums. Many of such invariants are the noncommutative interpretation of well-known cohomology theories. We present here one of them, Hochschild homology, which can be thought of as a noncommutative interpretation of the (vertically graded) Hodge structure on Betti cohomology. Hochschild homology will turn out to be a very useful tool in our proofs. However, there are several more additive invariants that one can consider, see [29] §2 for a detailed account.

Let $A$ be a dg algebra, and consider $A$ as a bimodule over itself (or, better, as an $A \otimes A^{\text{op}}$-bimodule) in a canonical way. Then define
\[
\text{HH}_*(A) := A \otimes_{A \otimes A^{\text{op}}} A.
\]
Such complex is well-defined up to Morita equivalence. More generally (see [29] §2), let $\text{dgCat}_{\text{geo}}(k)$ be the category of pretriangulated dg categories that can be realized as admissible subcategories of derived categories of smooth projective $k$-varieties. We have, for any integer $n$, functors
\[
\text{HH}_* : \text{dcCat}_{\text{geo}}(k) \longrightarrow D^b(k),
\]
\[
\text{HH}_n : \text{dcCat}_{\text{geo}}(k) \longrightarrow \text{Vect}(k)
\]
where $\text{HH}_n(A)$ denotes the $n$-th cohomology of the complex $\text{HH}_*(A)$. If $A = D^b(X)$, we will use the shorthand $\text{HH}_n(X) := \text{HH}_n(D^b(X))$. Notice that $\text{HH}_n$ and $\text{HH}_*$ are additive invariants, see, e.g., [29] §2.2.8. We hence obtain the following proposition either via noncommutative motives, as done by Tabuada [29], or in an explicit geometric way, as done by Kuznetsov [20].

**Proposition 2.10.** Let $X$ be smooth and projective, and $A \subset D^b(X)$ an admissible subcategory. If $A = \langle A_1, \ldots, A_r \rangle$ is a semiorthogonal decomposition of $A$, we have:
\[
\text{HH}_n(A) \simeq \bigoplus_{i=1}^r \text{HH}_n(A_i), \quad \text{HH}_n(A) \simeq \bigoplus_{i=1}^r \text{HH}_n(A_i)
\]
for every integer $n$.

**Proposition 2.10** is a very useful tool to bound the motivic categorical dimension of a smooth projective variety.
Proposition 2.11. Let $A$ be a pretriangulated dg category. If $\text{rep}(A) = m$, then $HH_i(A) = 0$ for $|i| > m$.

Proof. Using Proposition 2.10, it is enough to consider the case of $A$ admissible in $D^b(X)$ for some smooth and projective $X$ of dimension $m$. Then we have that $HH_i(X) = 0$ for $|i| > m$, and the proof follows.

If $k \subset \mathbb{C}$ is an algebraically closed subfield, and $X$ a smooth projective variety over $k$, Weibel [32] has shown that:

$$HH_n(X) \simeq \bigoplus_{p-q=n} H^{p,q}(X),$$

where $H^{p,q}(X) = H^q(X, \Omega^p_X)$. This gives another criterion to bound the motivic categorical dimension of a smooth projective variety.

Corollary 2.12. If $X$ is a smooth projective variety over an algebraically closed $k \subset \mathbb{C}$, set $m := \max\{i \mid \text{there exist } p, q \text{ such that } p - q = i \text{ and } H^{p,q}(X) \neq 0\}$.

Then $\text{rdim}(X) \geq \text{mcd}(X) \geq m$.

In particular, if $X$ has dimension $n$ and $H^{n,0}(X) \neq 0$, then $\text{rdim}(X) = \text{mcd}(X) = n$.

Proof. The first inequality has already been proved in Lemma 2.7. To obtain the second inequality, it is enough to apply Proposition 2.11 to the category $D^b(X)$ and use the formula (4) to see that we cannot have $\text{mcd}(X) < m$. The last chain of equalities easily follows from $\text{rdim}(X) \leq n$.

Example 2.13. Let $X \subset \mathbb{P}^{n+1}$ be a smooth projective hypersurface of degree $n + 2$. Then $\text{mcd}(X) = n$. Indeed, we have that $H^{n,0}(X)$ is one-dimensional.

The above criteria and example will provide a very useful tool in the proof of Theorem 2. However, the converse to the statement in Proposition 2.11 is not true as soon as $\text{rep}(A) \geq 2$. The following result shows that categorical representability captures indeed much finer invariants than Hochschild homology.

Proposition 2.14. Let $S$ be a smooth complex projective surface with $p_g = q = 0$, and such that $K_0(S)$ has nontrivial torsion. Then $HH_i(S) = 0$ for $i \neq 0$, and $\text{rdim}(S) = 2$.

Proof. The proof can be retraced along [41, §6], but let us quickly recall it. First, $HH_i(S) = 0$ since $H^{p,q}(S) = 0$ as soon as $p \neq q$, by assumption.

On the other hand, $\text{rdim}(S) \leq 2$ by Lemma 2.7. Now, for any smooth projective complex variety $X$, the vanishing $\text{rdim}(X) = 0$ is equivalent to the existence of a full exceptional collection, and this implies that $K_0(X)$ is free of finite rank. Then $\text{rdim}(S) \geq 1$. If $\text{rdim}(S) = 1$, then there is a curve $C$ of positive genus and a fully faithful functor $D^b(C) \to D^b(S)$. But this would give $0 \neq H^{1,0}(C) \subset HH_1(C) \subset HH_1(S)$, which contradicts the assumption $q = 0$, and the proof is concluded.

Example 2.15. There are examples of surfaces, in any nonnegative Kodaira dimension, satisfying the assumptions of Proposition 2.14, as for example Enriques surfaces and classical Godeaux surfaces. For a more exhaustive list, see [41, Ex. 6.1.14].

2.4. Jacobians of noncommutative Chow motives. Let $k \subset \mathbb{C}$ be algebraically closed. Recall from André [3, §4] the construction of the category $\text{Chow}(k)_Q$ of Chow motives and of the monoidal functor $M(-)_Q : \text{SmProj}(k)^{op} \to \text{Chow}(k)_Q$,

where $\text{SmProj}(k)$ is the category of smooth projective $k$-schemes. As proved in [3, Proposition 4.2.5.1], de Rham cohomology factors through Chow motives, so that every morphism of Chow motives induces a morphism in de Rham cohomology. For $X$ an irreducible $k$-scheme of
dimension $d$, we can consider the $\mathbb{Q}$-vector spaces $NH^{2i+1}_{dR}(X), 0 \leq i \leq d - 1$, defined by the formula

\begin{equation}
\sum_{C} \sum_{\gamma \in \text{Hom}(M(C)_{\mathbb{Q}}, M(X)_{\mathbb{Q}}^{(i)})} \text{Im}\left(H^1_{dR}(C) \xrightarrow{H^1_{dR}(\gamma)} H^{2i+1}_{dR}(X)\right),
\end{equation}

where $C$ is a curve and $\gamma$ a morphism from $M(C)_{\mathbb{Q}}$ to $M(X)_{\mathbb{Q}}^{(i)}$. Roughly speaking, the $NH^{2i+1}_{dR}(X)$ are the odd pieces of de Rham cohomology that are generated by curves. Restricting the classical intersection bilinear pairings on de Rham cohomology (see [3, §3.3]) to these pieces gives pairings

\begin{equation}
\langle -, - \rangle: NH^{2i-2}_{dR}(X) \times NH^{2i+1}_{dR}(X) \rightarrow k, \quad 0 \leq i \leq d - 1.
\end{equation}

We consider now the category $\text{NNum}(k)_{\mathbb{Q}}$ of noncommutative numerical motives with rational coefficients, see [29, §4.6, 4.7] for an account. As proved by Marcolli and Tabuada [27], the category $\text{NNum}(k)_{\mathbb{Q}}$ is semi-simple. The Jacobian functor

\[ J(-): \text{NChow}(k)_{\mathbb{Q}} \rightarrow \text{NNum}(k)_{\mathbb{Q}} \rightarrow \text{Ab}(k)_{\mathbb{Q}} \]

with values in the category of abelian varieties up to isogeny was constructed by Marcolli and Tabuada [28] as follows:

(i) the category of abelian varieties up to isogeny $\text{Ab}(k)_{\mathbb{Q}}$ can be identified with an abelian semi-simple full subcategory of $\text{NNum}(k)_{\mathbb{Q}}$, via natural functors

\[ \text{Ab}(k)_{\mathbb{Q}} \rightarrow \text{Num}(k)_{\mathbb{Q}} \rightarrow \text{NNum}(k)_{\mathbb{Q}} \]

through the category of (commutative) numerical motives;

(ii) any noncommutative numerical motive $N$ admits a unique finite direct sum decomposition $S_1 \oplus \cdots \oplus S_r$ into simple objects by semisimplicity of noncommutative numerical motives;

(iii) one defines $J(N)$ as the smallest piece of the noncommutative numerical motive $N \cong S_1 \oplus \cdots \oplus S_r$ which contains the simple objects belonging to the subcategory $\text{Ab}(k)_{\mathbb{Q}}$.

On the other hand, recall the construction of the intermediate Jacobians

\[ J^i(X) := \frac{F^{i+1}H^{2i+1}(X, \mathbb{C})}{H^{2i+1}(X, \mathbb{Z})}, \]

for $0 \leq i \leq \dim(X) - 1$, where $F^*$ is the Hodge filtration on Betti cohomology. The intermediate Jacobians are complex tori, not (necessarily) algebraic. Nevertheless, they contain an algebraic torus $J^i_\mathbb{A}(X) \subseteq J^i(X)$ defined by the image of the Abel-Jacobi map

\begin{equation}
A^i: A^{i+1}(X)_{\mathbb{Z}} \rightarrow J^i(X) \quad 0 \leq i \leq \dim(X) - 1, \end{equation}

where $A^{i+1}(X)_{\mathbb{Z}}$ denotes the group of algebraically trivial cycles of codimension $i + 1$; see Voisin [31, §12] for further details. The algebraic intermediate Jacobian $J^i_\mathbb{A}(X)$ is an Abelian variety, well-defined up to isogeny.

As proved in [28, Theorem 1.7], whenever the above pairings (6) are non-degenerate for all $i$, one has an isomorphism $J(D^b(X)) \cong \prod_{i=0}^{\dim(X) - 1} J^i_\mathbb{A}(X)$ in $\text{Ab}(k)_{\mathbb{Q}}$. As explained in loc. cit., (6) is always non-degenerate for $i = 0$ and $i = d - 1$. Moreover, if Grothendieck’s standard conjecture of Lefschetz type is true for $X$, then (6) is non-degenerate for all $i$; see Vial [31, Lemma 2.1].

By its motivic construction, it follows that if $\mathcal{A} = \langle \mathcal{A}_1, \ldots, \mathcal{A}_r \rangle$ is a semiorthogonal decomposition, then $J(\mathcal{A}) = J(\mathcal{A}_1) \oplus \cdots \oplus J(\mathcal{A}_r)$. It follows that, if $\mathcal{A}$ is generated by exceptional objects, then $J(\mathcal{A}) = 0$. We can state the following corollary.

**Corollary 2.16.** Let $\mathcal{A}$ be a pretriangulated dg category with $J(\mathcal{A}) \neq 0$. Then $\text{mcd}(\mathcal{A}) > 0$. In particular, if $C$ is a curve of positive genus, then $\text{mcd}(C) = \text{rdim}(C) = 1$.

Notice that Corollary 2.16 can be seen as a particular case of Proposition 2.11 as one can see by the construction of the Jacobians and the relation between Hochschild homology and Hodge cohomology. We want to stress it nevertheless because of its high geometrical significance.
3. Categorical dimension of birational maps

3.1. Definition and main properties. Let $X$ and $Y$ be smooth projective varieties. Given a birational map $\phi : X \to Y$, a weak factorization of $\phi$ of type $(b_1, c_1, \ldots, b_r, c_r)$ is a diagram of the form:

$$
\begin{array}{ccccccc}
X_0 = X & \xrightarrow{b_1} & Y_1 & \xleftarrow{c_1} & X_1 & \xrightarrow{b_2} & Y_2 & \xleftarrow{c_2} & \cdots & \xrightarrow{b_r} & Y_r & \xleftarrow{c_r} & X_r = Y,
\end{array}
$$

where $b_i$ and $c_i$ are compositions of finite numbers of blow-ups along smooth centers and $X_i$ and $Y_i$ are smooth and projective. We also denote by $\{B_{i,j}\}$ the loci blown-up by the $b_i$’s and by $\{C_{i,j}\}$ the loci blown-up by the $c_i$’s. Notice that, even if we don’t stress it to keep clear notations, the ranges for the indices $j$ depend on $i$. Recall that we assumed $k$ to have characteristic zero, so that any birational map has a weak factorization $[1]$.

**Definition 3.1.** Let $\phi : X \to Y$ be a birational map. The categorical dimension of a weak factorization of $\phi$ of type $(b_1, c_1, \ldots, b_r, c_r)$ is the minimal integer $d$ such that $\text{mcd}(C_{i,j}) \leq d$ for all $C_{i,j}$ blown-up by the $c_i$’s. The categorical dimension of $\phi$ is the integer

$$
c\dim(\phi) := \min\{d \mid \text{there is a weak factorization of } \phi \text{ of categorical dimension } d\}
$$

**Example 3.2.** Let $\sigma : X \to Y$ be the blow-up of a smooth subscheme $Z$ of $Y$ such that $\text{mcd}(Z) = d$. Then $c\dim(\sigma) \leq d$ and $c\dim(\sigma^{-1}) = 0$. This is easily obtained from the diagram:

$$
\begin{array}{ccc}
X & \xrightarrow{id} & Y.
\end{array}
$$

Using Bittner’s presentation of the Grothendieck group $K_0(\text{Var}(k))$ and Bondal-Larsen-Lunts motivic measure, we can prove our main result.

**Theorem 3.3.** Let $\phi : X \to Y$ be a birational map and assume $I(X) = I(Y)$ in $PT(k)$. This is the case if, for example, $[X] = [Y]$ in $K_0(\text{Var}(k))$. Then $c\dim(\phi^{-1}) = c\dim(\phi)$.

In particular, setting $\text{Bir}_d(X) \subset \text{Bir}(X)$ to be the subset of birational maps $X \to X$ of categorical dimension at most $d$, we have that $\text{Bir}_d(X)$ is a subgroup, which coincides with the whole group if $d \geq \dim(X) - 2$.

**Proof.** Let $\phi$ have categorical dimension $d$. By definition, there is a weak factorization of $\phi$ of type $(b_1, c_1, \ldots, b_r, c_r)$ with centers $C_{i,j}$ such that $I(C_{i,j})$ is in $PT_d(k)$ for all $i$ and $j$.

On the other hand, $\phi^{-1}$ clearly admits a weak factorization of type $(c_r, b_r, \ldots, c_1, b_1)$. Let us denote by $\alpha_j$ (resp. $\beta_j$) the codimensions of $B_{i,j}$ (resp. $C_{i,j}$) in their ambient variety. Using Bittner’s presentation of $K_0(\text{Var}(k))$, we obtain that

$$
[X] + \mathbb{L} \sum_{i=1}^{r} \sum_{j=1}^{s_i} [B_{i,j}]^{[\mathbb{P}^{\alpha_j-2}]} = [Y] + \mathbb{L} \sum_{i=1}^{r} \sum_{j=1}^{t_i} [C_{i,j}]^{[\mathbb{P}^{\beta_j-2}]},
$$

in the Grothendieck group $K_0(\text{Var}(k))$ (see, e.g., [24]). Now we apply the motivic measure $\mu$ to the formula (8). Using the fact that $\mu(\mathbb{L}) = 1$ and $\mu(\mathbb{P}^m) = m + 1$, we obtain the following formula in the ring $PT(k)$:

$$
I(X) + \sum_{i=1}^{r} \sum_{j=1}^{s_i} (\alpha_j - 1)I(B_{i,j}) = I(Y) + \sum_{i=1}^{r} \sum_{j=1}^{t_i} (\beta_j - 1)I(C_{i,j}).
$$

We can cancel out $I(X)$ and $I(Y)$ in (9) by our assumption. This gives

$$
\sum_{i=1}^{r} \sum_{j=1}^{s_i} (\alpha_j - 1)I(B_{i,j}) = \sum_{i=1}^{r} \sum_{j=1}^{t_i} (\beta_j - 1)I(C_{i,j}).
$$
The right hand side of (11) belongs to $PT_d(k)$ by assumption, so does the left hand side. Notice that all the coefficients in both sides of (11) are strictly positive. It follows by Remark 2.4 and by the definition of $PT_d(k)$, that all of the $I(B_{i,j})$ also belong to $PT_d(k)$.

Now, it is easy to see that if $\phi$, $\psi$ are both in Bir$_d(X)$, then their compositions $\phi \circ \psi$ and $\psi \circ \phi$ also are in Bir$_d(X)$. Indeed, a weak factorization of the composition is easily written from the weak factorizations of $\phi$ and $\psi$. This shows that Bir$_d(X)$ is a subgroup. The last statement is evident, since any $\phi$ in Bir($X$) has categorical dimension bounded above by dim ($X$) – 2. 

**Remark 3.4.** Notice that the proof of Theorem 3.3 works, more generally, whenever one has $M \subset PT(k)$ a saturated monoid. In such a case, one can define Bir$_M(X)$ to be the set of birational maps admitting a weak factorization of type $(b_1, c_1, \ldots, b_r, c_r)$ such that all the $I(C_{i,j})$ belong to $M$. Then one has that Bir$_M(X)$ is a subgroup of Bir($X$). However, we stick here to the definition of categorical dimension, since semiorthogonal decompositions give all the information on any additive invariant, and, hence, on most of the interesting informations about $X$ that one can retrieve from $I(X)$.

### 3.2. Cremona transformations of given categorical dimension.

Here we consider $k \subset \mathbb{C}$ to be an algebraically closed field. The results treated in the above sections allow us to explain how to construct birational maps with given categorical dimension. This enables us to prove Theorem 2.

**Theorem 3.5.** We have Bir$_d(\mathbb{P}^n) \neq$ Bir$_{d+1}(\mathbb{P}^n)$ for all $d < n - 2$.

**Proof.** First of all, we prove that we Bir$_{n-3}(\mathbb{P}^n) \neq$ Bir$_{n-2}(\mathbb{P}^n)$. To this end, it is enough to construct a birational map of $\mathbb{P}^n$ of categorical dimension exactly $n - 2$. Let us consider homogeneous coordinates $[x_0 : \ldots : x_n]$, and a general homogeneous polynomial $f$ of degree $n$ in the variables $(x_1, \ldots, x_n)$. Then $f$ defines a Calabi–Yau smooth hypersurface $X$ in the projective hyperplane $x_0 = 0$. As remarked in Example 2.13 we have $\text{mcd}(X) = \dim (X) = n - 2$. Fix any homogeneous polynomial $g$ of degree $n - 2$ in the variables $(x_1, \ldots, x_n)$ such that $\text{pgcd}(f, g) = 1$, and define a birational involution $\phi_{n-2}$ of $\mathbb{P}^n$ on the open subset $x_0 \neq 0$ by the affine formula $(x_1, \ldots, x_n) \mapsto (x_1 \frac{g}{f}, \ldots, x_n \frac{g}{f})$. The fact that this gives a (birational) involution of $\mathbb{P}^n$ can be checked on the affine space $x_0 \neq 0$: iterating the formula twice, and using the fact that $\deg(g) - \deg(f) = -2$, we have that $\phi_{n-2}^2$ writes:

$$(x_1, \ldots, x_n) \mapsto \left( x_1 \frac{g}{f} : \cdots : \frac{g}{f} \right)^{-2} = (x_1, \ldots, x_n),$$

that is, $\phi_{n-2}^2$ equals the identity on the open affine subset $x_0 \neq 0$ of $\mathbb{P}^n$. It is not difficult to see, via the homogeneous expression

$$\phi_{n-2}(x_0 : x_1 : \ldots : x_n) = [f : x_0 x_1 g : \ldots : x_0 x_n g]$$

that the $(n - 2)$-dimensional Calabi-Yau variety $X$ is contained in the base locus of $\phi_{n-2}$, since the map is not defined when $x_0 = 0$ and $f = 0$ simultaneously.

Let $\sigma : Y \to \mathbb{P}^n$ be a regular resolution of $\phi_{n-2}$. This means that $\sigma$ is a finite sequence of blow-ups with smooth centers, and that there is a birational morphism $\tau : Y \to \mathbb{P}^n$ such that $\phi_{n-2} = \sigma \circ \tau^{-1}$. Then $\sigma$ has to blow-up (a strict transform of) $X$, and we let $E$ be the exceptional divisor of such blow-up. In particular, $E$ is birational to $X \times \mathbb{P}^1$. On the other hand, since $\phi_{n-2}$ contracts $X$, then $\tau$ contracts $E$. It follows that, for any weak factorization of $\phi$, say $(b_1, c_1, \ldots, b_r, c_r)$ is its type, there is an $i$ such that one of the exceptional divisors of $c_i$, call it $F$, is birational to $E$. If $C_{i,j}$ is the center of such a blow-up, then $F$ is birational to $C_{i,j} \times \mathbb{P}^1$, for some $l > 0$.

The above argument gives a birational map $X \times \mathbb{P}^1 \dashrightarrow C_{i,j} \times \mathbb{P}^1$. Note that $\kappa(X) = 0$, since $X$ is Calabi-Yau. A result of Liu and Sebag [23, Thm. 2] implies that $l = 1$ and $C_{i,j}$ is birational to $X$. In particular, since the numbers $h^{1,0}$ are birational invariants, we obtain that $h^{n-2,0}(C_{i,j}) \neq 0$. It follows by Corollary 2.3 that $\text{mcd}(C_{i,j}) = n - 2$, and hence that $\text{cdim}(\phi_{n-2}) \geq n - 2$. The equality follows.
Now we proceed to construct a birational map \( \psi \) of \( \mathbb{P}^n \) of categorical dimension exactly \( d + 1 \). Consider the standard birational map \( \sigma : \mathbb{P}^n \to \mathbb{P}^{d+3} \times \mathbb{P}^{n-d-3} \), and let \( \psi := \sigma^{-1} \circ (\phi_{d+1}, \text{id}) \circ \sigma \), where \( \phi_{d+1} \) is constructed as the above \( \phi_{d-1} \) via general homogeneous coprime polynomials \( f \) and \( g \) of degree \( d + 3 \) and \( d + 1 \) respectively in \( d + 3 \) variables.

First of all, since \( \sigma \) is obtained by resolving a linear projection, it clearly admits a weak factorization whose centers are all (proper transforms of) linear subspaces, which have then motivic categorical dimension 0. On the other hand, we have seen that any weak factorization of \( \phi_{d+1} \) has categorical dimension \( d + 1 \). It follows that we can construct a weak factorization of \( \psi \) of categorical dimension \( d + 1 \), so that \( \text{cdim} (\psi) \leq d + 1 \).

Let \( X \) be the zero locus of \( f \) in the corresponding hyperplane. In particular, \( X \) is a \((d + 1)\)-dimensional Calabi-Yau hypersurface in a hyperplane \( \mathbb{P}^{d+2} \subset \mathbb{P}^{d+3} \), and we have \( h^{d+1,0} (X) \neq 0 \). By the same argument above, since \( X \) is contracted by \( \psi \), in any weak factorization there has to be an exceptional divisor \( F \), blow up of some center \( C_{i,j} \), which is birational to \( X \times \mathbb{P}^{n-d-2} \).

In particular, there is some \( l \) such that \( X \times \mathbb{P}^{n-d-2} \) is birational to \( C_{i,j} \times \mathbb{P}^l \). As above, we can appeal to the result of Liu and Sebag [25, Thm. 2], but this only allows us to show that either \( l = n - d - 2 \) and \( C_{i,j} \) is birational to \( X \), or that \( l < n - d - 2 \) and \( \kappa (C_{i,j}) < 0 \).

In the first case, we are done, as above. In the second case, we can use the Künneth formula to show that \( h^{d+1,0} (X \times \mathbb{P}^{n-d-2}) = h^{d+1,0} (X) \neq 0 \), and that \( h^{d+1,0} (C_{i,j} \times \mathbb{P}^l) = h^{d+1,0} (C_{i,j}) \).

Now, recall that the numbers \( h^{i,j} \) are birational invariants. This implies that \( h^{d+1,0} (C_{i,j}) \neq 0 \), and then that \( \text{mcd} (C_{i,j}) \geq d + 1 \) by Corollary [23]. As a consequence, we have that \( \text{cdim} (\psi) \geq d + 1 \), as required.

The proof of Theorem [5.3] makes an extensive use of Hochschild homology. One could define the Hochschild dimension of a pretriangulated category \( A \) to be the maximal integer \( i \) such that \( HH_i (A) \neq 0 \), and the Hochschild dimension of a birational map by replacing the motivic categorical dimension with the Hochschild dimension in Definition [4.1]. The above proof would prove that this notion also would give a proper filtration of the Cremona group.

However, the notions of categorical representability and dimension are much richer, since the noncommutative motive is the universal additive invariant (see [29, §2]) and we could use any other additive invariant to produce subgroups of the group of birational maps of a smooth projective variety. Any such invariant will be controlled, as much as the Hochschild homology, by the categorical dimension, or, more generally, by the derived categories of the centers of the weak factorization.

Finally, as we noticed in Proposition [2.14] the notion of Hochschild dimension would be weaker than the notion of categorical dimension, and it would be maybe interesting to study some natural problem.

Example 3.6. Suppose that \( \phi : \mathbb{P}^n \to \mathbb{P}^n \) is a birational map contracting a closed subset (birational to) \( \mathbb{P}^d \times S \) with \( S \) an Enriques surface. Then \( \text{cdim} (\phi) \geq 2 \): this can be seen following the same argument in the proof of Theorem [5.5] and the fact that, for surfaces, stable birationality coincides with birationality. On the other hand, assume moreover that any weak factorization has centers \( C_{i,j} \) such that \( HH_l (C_{i,j}) = 0 \) for \( l = 0 \). This is, a priori, possible, since \( HH_l (S) = 0 \). The birational map \( \phi \) has then Hochschild dimension 0 and categorical dimension at least 2.

Question 3.7. It is not known to the author whether a map \( \phi \) such the one considered in Example [4.6] exists, it would be interesting to construct it. On the other hand, it is a formal fact that, for any \( d \leq n - 2 \), one could have \( \text{Bir}_d (\mathbb{P}^n) \) filtered by subgroups of maps of Hochschild dimension \( i \), with \( 0 \leq i \leq d \). It would be interesting to explore such a finer filtration.

4. The genus of a birational map

4.1. Weak factorizations and Jacobians. Let \( k \subset \mathbb{C} \) be algebraically closed. For all varieties we consider in this section, we assume that the above pairings [10] are non-degenerate for all \( i \).

That is, we assume that for a birational map \( X \to Y \) with a weak factorization of type \( (b_1, c_1, \ldots, b_r, c_r) \), the pairings are non-degenerate for \( X, Y \) and for all the \( B_{i,j} \) and \( C_{i,j} \). Recall
that, under this assumption, the noncommutative Jacobians are isogenous to the product of the algebraic intermediate Jacobians.

**Proposition 4.1.** Suppose \( \phi : X \to Y \) as above, and assume that \( \dim\left( \prod_{i=0}^{d-1} J^i_a(X) \right) = \dim\left( \prod_{i=0}^{d-1} J^i_a(Y) \right) \). Then:

\[
\sum_{i=1}^{r} \sum_{j=1}^{s_i} (\alpha_j - 1) \dim \left( \prod_{l=1}^{r} J^l_a(B_{i,j}) \right) = \sum_{i=1}^{r} \sum_{j=1}^{t_i} (\beta_j - 1) \dim \left( \prod_{l=1}^{s_i} J^l_a(C_{i,j}) \right).
\]

Proof. It is enough to prove the statement for \( r = 1 \), that is for a diagram:

\[
\begin{array}{ccc}
W & & Y \\\n\downarrow b & & \downarrow c \\\nX & \leftarrow & Y,
\end{array}
\]

where \( b \) blows-up smooth centers \( \{B_i\}_{i=1}^r \) of codimension \( \alpha_i \) and \( c \) blows-up smooth centers \( \{C_j\}_{j=1}^t \) of codimension \( \beta_j \). Applying blow-up formula we obtain two semiorthogonal decompositions of \( D^b(W) \) inducing two decompositions of its noncommutative motive, and hence of its noncommutative Jacobian

\[
J(W) = J(X) \oplus \bigoplus_{i=1}^{s} J(B_i)^{\oplus \alpha_i - 1} = J(Y) \oplus \bigoplus_{i=1}^{t} J(C_i)^{\oplus \beta_i - 1},
\]

as an Abelian varieties up to isogeny, as explained in \([24]\). Calculating \( \dim(J(W)) \), the formula follows by \([28, \text{Theorem 1.7}]\) and by simplifying \( \dim(\prod_{i=0}^{d-1} J^i_a(X)) \) and \( \dim(\prod_{i=0}^{d-1} J^i_a(Y)) \) on each side.

**Corollary 4.2.** Let \( \phi : X \to Y \) be as above, denote by \( n \) the dimension of \( X \), and assume that \( B_{i,j} \) and \( C_{i,j} \) have dimension at most one. This is the case in particular if \( n = 3 \). Then the above formula \((\text{11})\) yields:

\[
\sum_{i=1}^{r} \sum_{j=1}^{s_i} g(B_{i,j}) = \sum_{i=1}^{r} \sum_{j=1}^{t_i} g(C_{i,j}).
\]

Proof. Notice that, under our assumptions, \( B_{i,j} \) and \( C_{i,j} \) have only one nontrivial Jacobian if and only if they are curves of positive genus, in which case the dimensions of the Jacobians are \( g(B_{i,j}) \) and \( g(C_{i,j}) \) respectively. Moreover, in \((\text{11})\), all the codimensions \( \alpha_j \) and \( \beta_j \) are equal to \( n - 2 \), so that this factor can be simplified.

**Remark 4.3.** Notice that Corollary \([12]\) apply more generally, without simplification of the codimensional coefficients, to the cases where the algebraic Jacobians of the \( B_{i,j} \) and of the \( C_{i,j} \) are split by Jacobians of curves, replacing \( g(B_{i,j}) \) and \( g(C_{i,j}) \) by the genera of the curves splitting the respective Jacobians.

### 4.2. A generalization for the genus of birational automorphisms of rationally connected threefolds.

In the case where \( X \) admits a unique nontrivial intermediate Jacobian \( J(X) \) which is principally polarized by an incidence polarization, we say that \( X \) is verepresentable. All smooth projective curves, many Fano threefolds, all projective spaces, quadric hypersurfaces and intersections of two quadrics, or of three even-dimensional quadrics are verepresentable. We refer to \([8]\) for more details. In these cases, the Jacobian \( J(X) \) contains the information about the principal polarization of \( J(X) \), as shown in \([8]\).

**Theorem 4.4.** Let \( X \) and \( Y \) be verepresentable varieties, such that the pairings \([6]\) are nondegenerate for both \( X \) and \( Y \), and suppose that

\[
D^b(X) = \langle A, B \rangle \quad D^b(Y) = \langle A', C \rangle.
\]

Assuming that \( J(B) = 0 \), and that \( A \cong A' \) as pretriangulated dg categories, there is an injective morphism of principally polarized abelian varieties \( \tau : J(X) \to J(Y) \), that is \( J(Y) = J(X) \oplus A \) for some abelian variety \( A \). Moreover, if \( J(C) = 0 \) as well, \( \tau \) is an isomorphism.
**Definition 4.5.** Let $X$ and $Y$ be vererpresentable $2n+1$-folds. A birational map $\phi: X \rightarrow Y$ is well-polarized if there is a weak factorization of type $(b_1, c_1, \ldots, b_r, c_r)$ such that the centers $C_{i,j}$ are vererrepresentable and all the varieties appearing in the weak factorization are vererrepresentable.

We call the Abelian type (with respect to the given weak factorization) of a well-polarized birational morphism the collection $J(C_{i,j})$ of all nontrivial algebraic Jacobians of the centers $C_{i,j}$. If $S_A$ is a set of indecomposable principally polarized Abelian varieties such that all of the $J(C_{i,j})$ are split by elements of $S_A$, we say that the Abelian type of $\phi$ is split by $S_A$. If, moreover, $S_A$ only contains Jacobian of curves, we say that the Abelian type is Jacobian.

The next Proposition uses Theorem 4.14 to produce subgroups of Bir($X$). Notice that similar arguments were used by Clemens and Griffiths [14] to show that any rational complex threefold has its intermediate Jacobian split by Jacobian of curves as a principally polarized abelian variety (i.e. any birational map between vererrepresentable threefolds is well polarized of Abelian Jacobian type). A categorical rephrasing of this criterion can be found in [6] (see also [7] and [4]).

**Proposition 4.6.** Let $X$ be a vererrepresentable $(2n+1)$-fold. Given a set of indecomposable principally polarized Abelian varieties $S_A$, the set of well-polarized birational automorphisms $\phi$ of Abelian type split by $S_A$ is a subgroup of Bir($X$).

**Proof.** If $\phi$ is well-polarized and its Abelian type is split by $S_A$, then there is a weak factorization of $\phi$ of type $(b_1, c_1, \ldots, b_r, c_r)$ such that all the varieties involved are vererrepresentable and the intermediate Jacobians of the $C_{i,j}$ are direct sums of elements of $S_A$. Considering the weak factorization of type $(c_r, b_r, \ldots, c_1, b_1)$ of $\phi^{-1}$ we have that the intermediate Jacobians of the $B_{i,j}$ are also split into Abelian varieties belonging to $S_A$: this is just an iterated application of Theorem 4.14 together with the fact that the category of principally polarized Abelian varieties is semisimple.

Finally, let $\phi$ and $\psi$ be well-polarized with Abelian type is split by $S_A$. Then also $\phi \circ \psi$ is well polarized since it admits a weak factorization which is just given by juxtaposition of the weak factorizations of $\phi$ and $\psi$, and it is clear that it has Abelian type split by $S_A$. \hfill \Box

**Question 4.7.** Construct examples of well-polarized birational maps whose Abelian type is not split by the set $\{0\}$ for varieties of dimension $\geq 5$, for example for $\mathbb{P}^5$.

**Definition 4.8.** If $\phi$ is well-polarized by a weak factorization of type $(b_1, c_1, \ldots, b_r, c_r)$, the maximum $\max\{\dim(J(C_{i,j}))\}$ over all the centers $C_{i,j}$ is called the genus of the given weak factorization. The genus $g(\phi)$ of the birational map $\phi$ is the smallest genus of any possible well-polarized weak factorization.

It is clear that if $n = 1$ and $X$ and $Y$ are vererrepresentable, then all $\phi$ are well-polarized, and have Jacobian Abelian type. Moreover, for any such a $\phi: X \rightarrow Y$, Frumkin [10] defined a notion of genus of $\phi$, which we denote by $g_F(\phi)$, to be the maximum of the genera among the centers of the blow-ups in regular resolutions of $\phi^{-1}$. Recall that by regular resolution of $\phi^{-1}$, we mean a blow-up $\sigma: Y' \rightarrow Y$ along smooth centers such that there is a birational morphism $\rho: Y' \rightarrow X$ and the composition of these two maps is $\phi^{-1}$. On the other hand, Lamy defines the genus of a birational map to be the maximal genus of a curve $C$ such that there exist a divisor $D$ in $X$ contracted by $\phi$, birational to $C \times \mathbb{P}^1$, and shows that his definition coincides with Frumkin’s one [29].

**Proposition 4.9.** Let $X$ and $Y$ be vererrepresentable threefolds and $\phi: X \rightarrow Y$ a birational map. Then $\phi$ is well-polarized of Jacobian Abelian type and its genus coincide with the genus defined by Frumkin.

**Proof.** First of all, it is easy to see that any weak factorization of $\phi$ is a composition of blow-ups along curves or points. It follows that $\phi$ is well polarized of Jacobian Abelian type.
Consider a regular resolution \( X \xrightarrow{\rho} Y' \xrightarrow{\sigma} Y \) of \( \phi^{-1} \), and a regular resolution \( Y \xrightarrow{\mu} X' \xrightarrow{\nu} X \) of \( \phi \), and the commutative diagram:

\[
\begin{array}{ccc}
X' & \xrightarrow{\psi} & Y' \\
\sigma' \downarrow & & \downarrow \sigma \\
X & \xrightarrow{\phi} & Y
\end{array}
\]

In particular, thanks to [10, Prop. 2.2] (see also [23, Prop. 7]), the map \( \psi \) has genus zero, hence any regular resolution of \( \psi \) and of \( \psi^{-1} \) blows-up only points and rational curves. It follows, that there exists a weak factorization of \( \psi \), say of type \((b_1, c_1, \ldots, b_r, c_r)\), whose centers are all rational curves or points. Then, the above diagram gives a weak factorization \((\sigma', b_1, c_1, \ldots, b_r, c_r, \sigma)\) of \( \phi \), where only \( \sigma \) and \( \sigma' \) can blow-up curves of positive genus. Then \( g(\phi) \leq g_F(\phi) \).

On the other hand, if \( \phi \) contracts a divisor birational to \( C \times \mathbb{P}^1 \) with \( g(C) > 0 \), then we have, as in the proof of Theorem 3.3, that any weak factorization of \( \phi \) must blow up a curve of genus \( g(C) \). Using Lamy’s equivalent definition, we have then that \( g_C(\phi) \leq g(\phi) \), and we obtain the proof. \( \square \)

5. Toric versus rational centers

5.1. The noncommutative motivic rational defect and \( \text{Bir}_{n-4}(X) \). Let \( Y \) be a rational \( n \)-dimensional variety. As remarked in [17], weak factorization gives

\[
[Y] = [\mathbb{P}^n] + LM_Y
\]

in \( K_0(\text{Var}(k)) \), where \( M_Y \) is a \( \mathbb{Z} \)-linear combination of classes of varieties of dimension bounded above by \( n-2 \). Galkin and Shinder define then \( ([Y] - [\mathbb{P}^n])/L \in K_0(\text{Var}(k))[L^{-1}] \) as the rational defect of \( Y \) [17]. Applying the motivic measure \( \mu \) defined in [3], and recalling that \( \text{mcd}(\mathbb{P}^n) = 0 \), we obtain the following statement (see [3, Prop. 8.1.2]).

**Proposition 5.1.** If \( Y \) is rational smooth, and projective, then \( \text{mcd}(Y) \leq n - 2 \).

We call the class of \( I(Y) \) in \( PT(k)/PT_{n-2}(k) \) the noncommutative motivic rational defect of the variety \( Y \).

**Corollary 5.2.** Let \( \phi : X \dashrightarrow X \) be a birational map. Assume there is a weak factorization of type \((b_1, c_1, \ldots, b_r, c_r)\) such that all the \( C_{i,j} \) are rational, then:

- if \( \dim(X) = 3 \), this is equivalent to having \( \text{cdim}(\phi) = 0 \),
- otherwise, this implies \( \text{cdim}(\phi) \leq n - 4 \).

In particular, if \( R_n \subset \text{Bir}(X) \) is the group generated by birational maps contracting only rational varieties, then \( R_n \subset \text{Bir}_{n-4}(X) \).

**Proof.** By proposition 5.1 a rational variety \( C \) of dimension \( m \) has \( \text{mcd}(C) \leq m - 2 \). If follows then, that in the above assumptions, \( \text{mcd}(C_{i,j}) \leq \dim(C_{i,j}) - 2 \leq n - 4 \) for any \( i, j \). The second statement is then proved, and also the fact that \( R_n \subset \text{Bir}_{n-4}(X) \), since one can find a weak factorization blowing up only rational varieties (see [11]).

The stronger statement in the case \( n = 3 \) is just a rephrasing of the fact that having rational centers is equivalent to having genus zero. \( \square \)

In the case of fourfolds, we consider centers of dimension at most two. A conjecture attributed to Orlov (see, e.g., [2, Conj. 6.2.1]) asks whether a complex surface is rational if and only if it has a full exceptional collection. We can formulate therefore the following Conjecture.

**Conjecture 5.3.** Suppose that \( \dim(X) \leq 4 \), and let \( \phi : X \dashrightarrow X \) be a birational automorphism. Then \( \text{cdim}(\phi) = 0 \) if and only if there exists a weak factorization of type \((b_1, c_1, \ldots, b_r, c_r)\) such that all the \( C_{i,j} \) are rational.

We notice that Hochschild homology is certainly not fine enough to study the above Conjecture, as Proposition 2.14 shows.
On the other hand, it is easy to see that there are rational threefolds $X$ such that $\text{mcd}(X) = 1$. For example, any blow-up of $\mathbb{P}^3$ along a smooth curve of positive genus, or the complete intersection of two quadrics. Suppose then $\phi$ is a birational map of $\mathbb{P}^n$ contracting some rational variety $Z$ with $\text{mcd}(Z) > 0$. One would be tempted to deduce that $\text{cdim}(\phi) > 0$, but the arguments used in the proof of Theorem 3.5 are deeply based on birational geometry. We formulate then the following question.

**Question 5.4.** Let $n \geq 5$. Does there exist $\phi$ is Bir$(\mathbb{P}^n)$, such that $\phi$ contracts only rational varieties and $\text{cdim}(\phi) > 0$?

5.2. Toric loci, and the standard Cremona transformation. Let $X$ be a smooth projective variety. We say that a birational automorphism $\phi : X \dasharrow X$ has toric loci if it admits a weak factorization of type $(b_1, c_1, \ldots, b_r, c_r)$ where all the $C_{i,j}$ are toric. By this, me mean that $C_{i,j}$ is isomorphic, as a smooth projective variety, to a toric variety. For example, a birational map $\phi : X \dasharrow X$ admitting a weak factorization such that all the $C_{i,j}$ are projective spaces has toric loci. Notice however that $\phi$ having toric loci does not imply that $X$ itself is toric, nor that any of the blow-ups is a toric map. Hence, this is a weaker notion then asking that the weak factorization is itself made of toric blow-ups of toric varieties. By Kawamata [18], the derived category of any toric variety is generated by exceptional objects. We therefore have the following easy remark.

**Proposition 5.5.** If $\phi : X \dasharrow X$ has toric loci, then $\text{cdim}(\phi) = 0$.

Notice that we do not know whether $\phi$ having toric loci implies that $\phi^{-1}$ has toric loci. It is not difficult indeed to have varieties generated by exceptional objects which are not toric, for example any del Pezzo surface of degree smaller than 5, or, more generally, any non-toric blow-up along points of a variety generated by exceptional objects.

A very well known example of a birational map with toric loci is given by the standard Cremona transformation $\sigma_n : \mathbb{P}^n \dasharrow \mathbb{P}^n$ which admits a factorization:

$$
\begin{array}{ccc}
\mathbb{P}^n & \xrightarrow{\sigma_n} & \mathbb{P}^n \\
\uparrow & & \downarrow \\
X & \xrightarrow{c} & \mathbb{P}^n \\
\downarrow & & \uparrow \\
& b & \mathbb{P}^n
\end{array}
$$

as follows: there are $n + 1$ points in general position on $\mathbb{P}^n$ naturally associated to $\sigma_n$. The map $c$ is the composition of the blow-ups of these points, followed by the blow-ups of the strict transforms of all the lines through two of them, then by the blow-ups of the strict transforms of planes through three of them and so on, the last step being the blow-ups of the strict transforms of the $(n-2)$-dimensional linear subspaces spanned by all the possible choices of $n-1$ points. Notice that $X$ is also known as the Losev–Manin space, a compactification of the moduli space of $(n+3)$-pointed curves of genus zero. It follows that all the standard Cremona transformations have toric loci, and are indeed toric. We remark that here we could have used that projective spaces have motivic categorical dimension 0, without appealing to toric varieties. We have the following Corollary of Proposition 5.5.

**Corollary 5.6.** For any $n$, we have

$$
G_n := \langle \text{PGL}_{n+1}(k), \sigma_n \rangle \subset \text{Bir}_0(\mathbb{P}^n).
$$

We finally notice, as a possible application of Corollary 5.6 that a positive answer to Question 5.4 would imply that the group generated by maps contracting rational varieties is strictly bigger than $G_n$, a result which was proved by Blanc and Hedén [10] if $n \geq 3$ is odd.

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