Four-dimensional Yang–Mills theory, gauge invariant mass and fluctuating three-branes

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Abstract

We are interested in a gauge invariant coupling between four-dimensional Yang–Mills field and a three-brane that can fluctuate into higher dimensions. For this we interpret the Yang–Mills theory as a higher dimensional bulk gravity theory with dynamics that is governed by the Einstein action, and with a metric tensor constructed from the gauge field in a manner that displays the original gauge symmetry as an isometry. The brane moves in this higher dimensional spacetime under the influence of its bulk gravity, with dynamics determined by the Nambu action. This introduces the desired interaction between the brane and the gauge field in a way that preserves the original gauge invariance as an isometry of the induced metric. After a prudent change of variables the result can be interpreted as a gauge invariant and massive vector field that propagates in the original spacetime $\mathbb{R}^4$. The presence of the brane becomes entirely invisible, except for the mass.

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1. Introduction

The Higgs mechanism is the way to introduce a mass for a gauge vector. But the original version with a fundamental Higgs field as employed in the standard Weinberg–Salam model [1] has issues with the stability of its mass under radiative corrections. This has led to the introduction of various alternatives such as models with a composite Higgs, supersymmetries and more recently theories with higher dimensions and the little(st) Higgs that emerges as a pseudo-Goldstone boson [2].

Here we describe a novel variant to equip a Yang–Mills theory with a gauge invariant mass. For this we consider a gauge invariant coupling between the four-dimensional Yang–Mills field...
and a three-brane, by a geometric reformulation of the Yang–Mills theory as a gravity theory with the Einstein action in a higher dimensional spacetime. This spacetime emerges when we replace the matrix-valued Lie algebra generators by Killing vector fields that act on an internal Riemannian manifold with an isometry group that coincides with the original gauge group. The standard $D = 4$ flat spacetime Yang–Mills action is obtained from the higher dimensional gravity action when we average over the internal manifold. This computation of the averages over the internal manifold replaces the evaluation of the matrix traces over Lie algebra generators in the conventional formulation. We then proceed to introduce a three-brane in this higher dimensional spacetime. Asymptotically the brane stretches into $\mathbb{R}^4$ but it can locally fluctuate into the internal manifold where it moves under the influence of the bulk gravity and with dynamics determined by the Nambu action. The gravitational interaction of the brane leads to an effective interaction between the original Yang–Mills gauge field and brane fluctuations. When viewed from the point of view of the original flat four-dimensional spacetime, this can be interpreted in terms of a massive vector field that resides in $\mathbb{R}^4$. Much like in the conventional Higgs effect where a gauge field combines with a Higgs boson, the gauge field now entirely eats up the higher dimensional brane fluctuations and becomes massive so that at the end, there is nothing else left in the theory that reveals the presence of a brane except the vector boson mass. The internal Riemannian manifold can be chosen to be any manifold with an isometry group that coincides with the original gauge group, and different choices give rise to different kinds of mass terms. Examples include the group manifold itself and its co-adjoint orbits.

2. Gauge group as a manifold

The $SU(N)$ Yang–Mills action in $\mathbb{R}^4$ is

$$S_{\text{YM}} = -\frac{1}{2e^2} \int d^4x \text{Tr}\{F_{\mu\nu}F_{\mu\nu}\} = \frac{1}{4e^2} \int d^4xF_a^{\mu\nu}F_a^{\mu\nu},$$

(1)

with

$$F_a^{\mu\nu}(A) = \partial_\mu A_a^{\nu} - \partial_\nu A_a^{\mu} + f^{abc}A_b^{\mu}A_c^{\nu}.$$ 

(2)

The trace is over anti-Hermitean matrices $T^a$ that represent the Lie algebra of the gauge group $SU(N)$, normalized so that

$$\text{Tr}\{T^a T^b\} = -\frac{1}{2} \delta^{ab}.$$ 

Here we are interested in the interaction between the Yang–Mills field and a three-brane. Such an interaction can either be difficult to introduce or lacks a proper interpretation in the strictly four-dimensional realm of (1). For this we replace the matrices $T^a$ by the Killing vector fields $K^a$ that act on an internal Einstein manifold with an isometry group $SU(N)$. This procedure is quite common in the context of Kaluza–Klein theories [3] and nonlinear $\sigma$-models [4], but is rarely used in the conventional Yang–Mills theory. Instead of the matrix trace we now have an integral over the internal manifold, and we shall assume that this integral generically gives us

$$\text{Tr}\{T^a T^b\} \rightarrow \mu^{\dim[V_{\text{int}}]} \int \sqrt{g} d\vartheta g_{mn}(\vartheta) K^{an}(\vartheta) K^{bm}(\vartheta) = \mu^{\dim[V_{\text{int}}]} \frac{\dim[V_{\text{int}}]}{\dim[SU(N)]} V_{\text{int}} \delta^{ab}.$$ 

(3)

Here $\vartheta^m$ are the local coordinates, $V_{\text{int}}$ is the volume of the internal manifold and $\mu$ is a mass scale. $K^{an}(\vartheta)$ are the components of Killing vectors

$$K^a(\vartheta) = K^{an} \frac{\partial}{\partial \vartheta^m}$$
that satisfy the Lie algebra of the gauge group:

$$[K^a, K^b] = f^{abc} K^c.$$  

(4)

The metric is invariant:

$$\mathcal{L}^a g_{mn} = g_{ma} \partial_n K^a + g_{na} \partial_m K^a + K^a \partial_k g_{mn} = 0.$$  

(5)

Furthermore, when

$$g_{mn} = K^a_m K^b_n \delta_{ab}$$  

(6)

we get from (4) the Maurer–Cartan equation

$$\partial_m K^a_n - \partial_n K^a_m = f^{abc} K^b_n K^c_m.$$  

In order to describe the interaction between the gauge field and the brane, instead of the Yang–Mills action (1) it is more convenient to take as the starting point the metric tensor

$$ds^2 = g_{\alpha \beta} dy^\alpha dy^\beta = (dx^\mu)^2 + g_{mn}(\vartheta) \{d\vartheta^m + K^m(\vartheta) A^a(\mathbf{x}) dx^a\} \{d\vartheta^n + K^n(\vartheta) A^b(\mathbf{x}) dx^b\}.$$  

(7)

This metric tensor is akin to the one that is widely employed in Kaluza–Klein theories [3] except that we have not included the dilaton fields. The reason for this choice is that here our goal is very different. Our goal is not to study a higher dimensional gravity theory per se but a geometric reformulation of the standard four-dimensional Yang–Mills theory with the help of a higher dimensional gravity theory. And we wish to interpret all of our results solely from the perspective of the ordinary $\mathbb{R}^4$ Yang–Mills theory. This dictates us to choose the metric tensor (7). Note that we have selected $g_{mn}$ and $K^m$ to depend only on the internal coordinates $\vartheta^m$, and the gauge field $A^a_\mu$ depends only on the coordinates $x^\mu$ of $\mathbb{R}^4$.

Relation (5) implies that metric (7) remains intact under the following (general) diffeomorphism:

$$\vartheta^m \to \vartheta^m - K^m(\vartheta) \epsilon^a(\mathbf{x}),$$  

(8)

provided

$$A^a_\mu(\mathbf{x}) \to A^a_\mu(\mathbf{x}) + \partial_\mu \epsilon^a(\mathbf{x}) + f^{abc} A^b_\mu(\mathbf{x}) \epsilon^c(\mathbf{x}).$$  

(9)

This coincides with the familiar transformation law of a gauge field under infinitesimal gauge transformations. To ensure that this diffeomorphism is an invariance in our reformulated Yang–Mills theory, we choose the Einstein action in the higher dimensional spacetime with a cosmological constant, and we evaluate it on metric (7). This yields

$$S_E = \frac{1}{\kappa} \int\sqrt{g} \, d^4 x \, d\vartheta \, [R - 2\Lambda] = \frac{1}{\kappa} \int\sqrt{g} \, d^4 x \, d\vartheta \left\{ -\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} K^m K^b_m + R_{int} - 2\Lambda \right\}.$$  

(10)

Here $R_{int}$ is the scalar curvature of the internal manifold. For an Einstein manifold $R_{int}$ is a constant, and we select the cosmological constant to cancel it.

Note that even though the rank of metric (6) may be smaller than the dimension of the gauge group, when we average over the internal manifold we may still obtain result (3). In the following we shall tacitly assume this to be the case, and if we use relation (3) we are left with

$$S_E = \mu^{\dim[V_{int}]} \frac{\dim[V_{int}]}{\dim[SU(N)]} \frac{V_{int}}{4\kappa} \int d^4 x \, F^a_{\mu\nu} F^a_{\mu\nu}.$$  

(11)

This coincides with the original Yang–Mills action (1) with

$$e^2 = \frac{\kappa}{V_{int}} \mu^{\dim[V_{int}]} \frac{\dim[SU(N)]}{\dim[V_{int}]}.$$
3. Mass form three-brane

We introduce a three-brane $B$ that is asymptotically stretched into the spacetime $\mathbb{R}^4$, but is allowed to locally fluctuate into the internal manifold $[5, 6]$. This brane is described by

$$\vartheta^m = \chi^m(x), \quad (12)$$

and we couple the brane to the Yang–Mills field by defining the brane dynamics to be determined by the Nambu action

$$S_{\text{Nambu}} = T \int d^4x \sqrt{G^{\text{ind}}}. \quad (13)$$

Here $G^{\text{ind}}$ is the determinant of the induced metric on the brane and $T$ is the brane tension. We ensure that (13) is finite by assuming that the brane fluctuations are contractible and have a compact support so that at large enough distances the world-sheet of the brane coincides with our spacetime $\mathbb{R}^4$:

$$\chi^m(x) = 0 \quad \text{as} \quad |x| > R. \quad (14)$$

Here $R$ is some (finite) distance scale. This condition states that for distances that are larger than $R$ the brane world-sheet merges with the original spacetime $\mathbb{R}^4$, which is chosen so that it coincides with $\vartheta^m = 0$ in the ambient space.

The induced brane metric is the pullback of the bulk metric (7) to the world-sheet surface (12), obtained by using the vielbein components

$$E_\alpha^\mu = \delta_\alpha^\mu + \partial_\mu \chi^m \delta_\alpha^m. \quad (16)$$

Explicitly the result is

$$G^{\text{ind}}_{\mu\nu} = E_\mu^\alpha E_\nu^\beta g_{\alpha\beta} = \delta_{\mu\nu} + \left( \partial_\mu \chi^m + K^{am} A^a_\mu \right) g_{mn} \left( \partial_\nu \chi^n + K^{bn} A^n_\nu \right).$$

The determinant can be evaluated using Sylvester’s theorem:

$$G^{\text{ind}} = 1 + \delta_{\mu\nu} \left( \partial_\mu \chi^m + K^{am} A^a_\mu \right) g_{mn} \left( \partial_\nu \chi^n + K^{bn} A^n_\nu \right)$$

and to the leading order in the brane fluctuation the Nambu action is

$$S_{\text{Nambu}} = T \int d^4x + \frac{T}{2} \int d^4x \delta^{\mu\nu} J^m_{\mu}(X) J^n_{\nu} + \cdots, \quad (15)$$

where we have defined

$$J^m_{\mu}(X) = \partial_\mu \chi^m + K^{am} A^a_\mu. \quad (16)$$

We remove the first term in (15) by re-adjusting the cosmological constant in (10). The second term is a mass term

$$S_{\text{mass}} = \frac{T}{2} \int d^4x g_{mn}(X) J^m_{\mu} J^n_{\nu}. \quad (17)$$

Due to the presence of the metric $g_{mn}(X)$ the mass apparently depends on the brane position but we shall soon find out that this is not the case, at least when the metric tensor admits the vielbein decomposition (6). But if the rank of the metric tensor is smaller than the dimension of the gauge group, the number of massive components $J^m_{\mu}$ is smaller than the number of gauge fields, and in that case the massive combinations in general may depend on the brane position.

We first verify that the Nambu action with our induced metric preserves the $SU(N)$ isometries (8) and (9) of the metric tensor (7), corresponding to the original gauge symmetry.
For this we establish the gauge invariance of the current $J^m$. We consider a diffeomorphism $(8)$ of the internal manifold, generated by the Killing vector $K^m$. We get

$$\delta e(\delta (X^m) = -K^{am}(X)\partial_e e^a(x),$$

(18)

while

$$\delta e(K^{am} A^a) = \delta e(K^{am}) \cdot A^a + K^{am} \cdot \delta e(A^a)$$

$$\Rightarrow (e \cdot e^b K^{am}) A^a + K^{am} \left( \partial_e e^a + f^{abc} A^b_{\mu} e^c \right) = K^{am} \partial_e e^a.$$

Consequently, (16) and in particular the Nambu action is gauge invariant, i.e. we conclude that the $SU(N)$ isometry of (7) is preserved by the coupling between the gauge field and the brane.

Consider next the quantity

$$B^a_{\mu} = K^a_{\mu} \partial_{\mu} X^m.$$

(19)

We compute

$$\partial_{\mu} B^a_{\nu} - \partial_{\nu} B^a_{\mu} = \partial_{\mu} \left( \partial_{\nu} X^m K^a_m \right) - \partial_{\nu} \left( \partial_{\mu} X^m K^a_m \right) = \left( \partial_{\mu} K^a_m - \partial_{\nu} K^a_m \right) \partial_{\nu} X^m \partial_{\mu} X^n$$

$$= -f^{abc} K^b_m \partial_{\nu} X^m \partial_{\mu} X^n = -f^{abc} B^b_{\mu} B^c_{\nu}.$$

Consequently (19) obeys the Maurer–Cartan equation, i.e. it is a pure gauge. In particular, we can write

$$B^a_{\mu} = K^a_{\mu} \partial_{\mu} X^m.$$

(20)

where $T^a$ are the matrices in a defining representation of the gauge group $SU(N)$ and $U$ is an element of the gauge group. We introduce the vielbein basis

$$\hat{e}^a = e^a T^i = U T^a U^{-1}.$$

(21)

Next we introduce the composite vector field:

$$J^i_{\mu} T^i = (A^a_{\mu} + B^a_{\mu}) e^a T^i = U \left( \partial_{\mu} + A_{\mu} \right) U^{-1}.$$

(22)

This vector is diffeomorphism a.k.a. gauge invariant under (8) and (9). When we resolve (22) for $A^a_{\mu}$ and substitute the result in (2) we get

$$F_{\mu \nu}^a(A) = \left( \partial_{\mu} J^i_{\nu} - \partial_{\nu} J^i_{\mu} + f_{ijk} J^j_{\mu} J^k_{\nu} \right) e^a_i \equiv F_{\mu \nu}^i (J) e^a_i.$$

Furthermore, when we assume that the metric tensor has the vielbein decomposition (6) we can also write the Nambu (mass) term entirely in terms of (22). Combining the Nambu action with the Yang–Mills action we then get the following manifestly diffeomorphism a.k.a. gauge invariant action:

$$S_{YM} + S_{Nambu} = \int d^4 x \left\{ \frac{1}{4e^2} F_{\mu \nu}^a (J) F_{\mu \nu}^i (J) + T \sqrt{1 + J_{\mu}^i J_{\nu}^i} \right\}$$

$$= \int d^4 x \left\{ \frac{1}{4e^2} F_{\mu \nu}^i (J) F_{\mu \nu}^i (J) + \frac{T}{2} J_{\mu}^i J_{\nu}^i + \cdots \right\}.$$

(23)

Intrinsically this action describes the interaction between the Yang–Mills field $A^a_{\mu}$ with the three-brane that fluctuates into the internal manifold. But remarkably, when we write it in terms of the variable $J^i_{\mu}$ it depends only on this variable and all reference to higher dimensions and in particular to the fluctuating brane has disappeared. The action (23) has a direct interpretation in terms of a massive vector field with $SU(N)$ invariant dynamics that takes place in the original spacetime $\mathbb{R}^4$. We can also interpret this so that the gauge field has ‘eaten up’ the brane fluctuations and the result is the massive vector field $J^i_{\mu}$, furthermore with translationally invariant dynamics in the original flat spacetime $\mathbb{R}^4$ since all dependence on the brane position
has also disappeared. The only thing that reveals the presence of the brane in our final theory is the presence of the mass term in $R^4$.

More generally, we can show that the Nambu action is both $SU(N)$ isometric and independent of the brane position whenever the Killing vectors act transitively on the internal manifold, and the dimension of the internal manifold does not exceed the number of the Killing vectors. This follows directly from the previous construction. We have verified that the vector field (16) is gauge invariant, i.e. its Lie derivative along the flow (8) and (9) vanishes. Consequently we can locally introduce a diffeomorphism generated by the Killing vectors that brings the brane coordinates to a constant value, for example

$$X^m(x) = 0.$$  

Consequently we can write the mass term patchwise as

$$S_{\text{mass}} = \frac{T}{2} \int d^4x g_{mn}(0) J^m_\mu J^n_\nu,$$

which establishes the independence on the brane position.

4. Anomalies and monopoles

Since the variable $J^\mu_\mu$ in (22) is gauge invariant, any Lorentz invariant action constructed from it is also gauge invariant. But in order to motivate the introduction of natural candidates we re-introduce the brane variable and re-write the mass contribution to the Nambu action in the following standard (Skyrme) form of a gauged nonlinear $\sigma$-model:

$$S_{\text{mass}} = -\frac{T}{4} \int d^4x \text{Tr}[U(\partial_\mu + A_\mu)U^{-1} \cdot U(\partial_\mu + A_\mu)U^{-1}] = \frac{T}{2} \int d^4x g_{mn}(X) \nabla_\mu X^m \nabla_\mu X^n.$$

This allows us to better relate our construction to the known results [4, 7, 8]. The covariant derivative is defined by

$$\nabla_\mu X^m = \partial_\mu X^m + K^{am}_{\mu} A^a_\mu. \tag{24}$$

This $\sigma$-model version proposes us to consider additional terms that have a natural $\sigma$-model interpretation. A general class of such terms is obtained by starting from the integral [4]

$$\int_B K_{mnpq}(\theta) d\theta^m d\theta^n d\theta^p d\theta^q. \tag{25}$$

Here the integral extends over the entire four-dimensional world-sheet of the fluctuating three-brane. When we pull back (25) into $R^4$ and replace derivatives with covariant derivatives we obtain a diffeomorphism invariant a.k.a. gauge invariant action in $R^4$ under (8) and (9):

$$S_K = \frac{1}{4!} \int d^4x \epsilon^{mnpqr} K_{mnpq}(X) \nabla_\mu X^m \nabla_\nu X^n \nabla_\rho X^p \nabla_\sigma X^q, \tag{26}$$

provided

$$L^m K = 0.$$  

Furthermore, since the brane fluctuations are local we can interpret its world-sheet to be the boundary of a contractible five-dimensional disk $D_5$ in the internal manifold. Due to the boundary condition that at large distances the brane coincides with $R^4$ the disk also includes the point $\theta^m = 0$, which corresponds to the brane position of the original spacetime $R^4$. We then use Stokes’ theorem to convert integral (25) into an integral over the entire disk $D_5$. The result is an integral of the form

$$\int_{D_5} H_{mnpqr}(\theta) d\theta^m d\theta^n d\theta^p d\theta^q d\theta^r, \tag{27}$$

6
where $H_{mnpqr}(\bar{\vartheta})$ are the components of the five-form:

$$H_{mnpqr}(\bar{\vartheta}) \, d\vartheta^m \, d\vartheta^n \, d\vartheta^p \, d\vartheta^q \, d\vartheta^r = d(K_{mnpq}(\vartheta) \, d\vartheta^m \, d\vartheta^n \, d\vartheta^p \, d\vartheta^q).$$

However, if we allow the five-form $H$ in (27) to be closed but not exact the ensuing four-form $K$ can only be introduced locally. In that case the extension of (27) into a diffeomorphism invariant quantity cannot be constructed simply by minimal substitution. An example is the following Wess–Zumino functional [4, 7, 8]:

$$S_{WZ} = -\frac{i}{2\pi^2} \cdot \frac{5!}{2} \int_{D_5} \epsilon^{\alpha\beta\gamma\delta\eta} \operatorname{Tr}(B_\alpha B_\beta B_\gamma B_\delta B_\eta).$$

This corresponds to the closed five-form

$$H_{mnpqr}(\vartheta) = K_a K_b K_c K_d K_e \cdot \operatorname{Tr}[T^a T^b T^c T^d T^e].$$

Its diffeomorphism invariant extension is

$$S_{WZ} = -\frac{i}{4!} \cdot \frac{1}{2\pi^2} \int d^4x \epsilon^{\mu\nu\rho\sigma} \left[ d_{abc} A_\mu^a B_\nu^b B_\rho^c + C_{abcd} A_\mu^a B_\nu^b A_\rho^c B_\sigma^d ight] + K_{mnpq} \partial_\mu X^m \partial_\nu X^n \partial_\rho X^p \partial_\sigma X^q,$$

where

$$\frac{i}{2} d_{abc} = \operatorname{Tr}[T^a \{ T^b, T^c \}] \quad \text{and} \quad C_{abcd} = \operatorname{Tr}[T^a T^b T^c T^d].$$

This is invariant under (18) only if [8]

$$\frac{\delta S_{WZ}}{\delta e^a(x)} = \frac{i}{24\pi^2} \epsilon^{\mu\nu\rho\sigma} \partial_\mu \left( d_{abc} A_\nu^b \partial_\rho A_\sigma^c + C_{abcd} A_\nu^b A_\rho^c A_\sigma^d \right) = 0. \quad (28)$$

When (28) is non-vanishing we have the familiar non-Abelian anomaly equation [8], due to a single Weyl fermion in interaction with the gauge field. We can interpret this in alternative ways. The presence of a gauge anomaly in a Yang–Mills theory with Weyl fermions leads to a breaking of diffeomorphism invariance in spacetime fluctuations away from $\mathbb{R}^4$. Alternatively, a non-Abelian gauge anomaly that arises from Weyl fermions can be removed by allowing for appropriate three-brane fluctuations that cancel those that emerge from the Weyl fermions. In this manner our approach provides a very natural interpretation and setting for the consistent quantization of anomalous gauge theories [8].

We also note that in the present approach we entirely avoid the conventional introduction of an ad hoc five-dimensional disk [4, 7, 8]. This disk now has a natural geometric interpretation in the context of our higher dimensional ambient space, and its boundary is the fluctuating three-brane.

We return to (21). We choose $H^a$ to be the Cartan subset of the $SU(N)$ generators $T^a$ and we introduce the ensuing subset $m^\alpha_i$ of the vielbeins $e^a_i$ in (21) [9]:

$$\hat{m}^\alpha = m^\alpha_i T^i = iH^a H^{-1}. \quad (29)$$

It is straightforward to verify that

$$[\hat{m}^\alpha, \hat{m}^\beta] = 0 \quad \text{and} \quad [\hat{m}^\alpha, \hat{m}^\beta] = d^{\alpha\beta\gamma} \hat{m}^\gamma \quad \text{and} \quad \operatorname{Tr}(\hat{m}^\alpha \partial_\mu \hat{m}^\beta) = 0.$$
Using (20) we can also show that
\[ d\hat{m}^\alpha = [\hat{m}^\alpha, B]. \]
We introduce the following closed two-forms:
\[ \Omega_H^\alpha = \text{Tr}(H^\alpha [U^{-1} dU, U^{-1} dU]) = f_{ijk} m^\alpha_i \partial_\mu m^\beta_j \partial_\nu m^\gamma_k \, dx^\mu \wedge dx^\nu. \] (30)
These are the symplectic two-forms on the orbit \( SU(N)/U(1)^{N-1} \). Recall that according to the Borel–Weil theorem each of the linear combinations
\[ \sum \alpha \frac{n_\alpha}{\Omega_1^\alpha} H, \]
where \( n_\alpha \in \mathbb{Z} \), corresponds to an irreducible representation of \( SU(N) \). We can show that
\[ B_\mu \, dx^\mu = U \partial_\mu U^{-1} \, dx^\mu = C_H \cdot \hat{m} + [d\hat{m}, \hat{m}], \] (31)
where
\[ \partial_\mu C_H^\alpha - \partial_\nu C_H^\alpha = \Omega_H^\mu \nu. \] (32)
This reveals a relation between the \( SU(N) \) magnetic monopoles in the original spacetime \( \mathbb{R}^4 \), representations of \( SU(N) \) and the non-triviality of the topological structure of the three-brane \( B \).

5. \( SU(2) \) as an example

As an explicit example we consider the case of \( SU(2) \). For the internal manifold we first take \( SU(2) \sim S^3 \). We use the following explicit Euler angle parametrization:
\[ U = -i \begin{pmatrix} \sin \frac{\theta}{2} e^{i \phi} & -\cos \frac{\theta}{2} e^{-i \phi} \\ -\cos \frac{\theta}{2} e^{i \phi} & -\sin \frac{\theta}{2} e^{-i \phi} \end{pmatrix}, \] (33)
where \( 0 \leq \theta \leq \pi \) and \( 0 \leq \phi \leq 2\pi \) are the local coordinates on \( S^3 \). The natural metric \( g_{mn} \) on \( S^3 \) is the bi-invariant Killing two-form:
\[ ds^2 = 2 \text{Tr}(dU dU^{-1}) = g_{mn} \, d\sigma^m \, d\sigma^n = (d\theta)^2 + \sin^2 \frac{\theta}{2} (d\phi_+)^2 + \cos^2 \frac{\theta}{2} (d\phi_-)^2. \] (34)
We write the Maurer–Cartan one-form as follows:
\[ B_\mu = U \, dU^{-1} = B_m^a \, d\sigma^m = \frac{1}{2i} \tau^a, \] (35)
where \( \tau^a \) are the Pauli matrices. We relate the components \( B_m^a \) to the Dreibeins for metric (34):
\[ g_{mn} = \delta_{ab} B_m^a B_n^b. \] (36)
The one-forms \( K^a = B_m^a \, d\sigma^m \) are subject to the \( SU(2) \) Maurer–Cartan equation:
\[ dB^a = -\frac{1}{4} \epsilon^{abc} B^b \wedge B^c. \] (37)
Explicitly, we write
\[ B^1 = n^1 \, d\psi_+ - e_2^1 \, d\theta \] (38)
\[ B^2 = n^2 \, d\psi_+ - e_2^2 \, d\theta \] (39)
\[ B^3 = n^3 \, d\psi_+ - d\psi_- \]. (40)
where we have defined
\[ \psi_\pm = \frac{1}{2}(\phi_+ \pm \phi_-) \]
and we have introduced the right-handed unit triplet
\[ \vec{e}_1 = \begin{pmatrix} \cos \psi_- \cos \theta \\ \sin \psi_- \cos \theta \\ -\sin \theta \end{pmatrix} \quad \text{and} \quad \vec{e}_2 = \begin{pmatrix} -\sin \psi_- \\ \cos \psi_- \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{n} = \begin{pmatrix} \cos \psi_- \sin \theta \\ \sin \psi_- \sin \theta \\ \cos \theta \end{pmatrix}. \]

There are three invariant Killing vector fields
\[ K^a = (K^a)_m \frac{\partial}{\partial \vartheta^m} \quad (m = 1, 2, 3) \]
that can be identified as the canonical duals of the one-forms \( B^a \). With (38)–(40) this gives us the explicit realization:
\[ K^1 = \{ \sin \psi_- \partial_\theta + \cos \psi_- \cot \theta \partial_\psi_+ \} + \frac{\cos \psi_-}{\sin \theta} \partial_\psi_- \]
\[ K^2 = \{ -\cos \psi_- \partial_\theta + \sin \psi_- \cot \theta \partial_\psi_+ \} + \frac{\sin \psi_-}{\sin \theta} \partial_\psi_- \]
\[ K^3 = -\partial_\psi_- \]
and the commutators of the Killing vectors determine a representation of the \( SU(2) \) Lie algebra:
\[ [K^a, K^b] = -\epsilon^{abc} K^c. \]

Using (31) and (41) we write
\[ B_\mu = \mathcal{C}_\mu \hat{n} + [\hat{n}, \partial_\mu \hat{n}], \]
where
\[ \mathcal{C}_\mu = \vec{e}^+ \cdot \partial_\mu \vec{e}^- \]
with
\[ \vec{e}^\pm = \frac{1}{2} e^{-\psi} (\vec{e}_1 \pm i \vec{e}_2). \]

Explicitly, in terms of the angular variables in (41)
\[ \mathcal{C}_\mu = -\frac{1}{2} (\cos \theta \partial_\mu \psi_- + \partial_\mu \psi_+) \]
and for (32) we get
\[ \partial_\mu \mathcal{C}_\nu - \partial_\nu \mathcal{C}_\mu = \vec{n} \cdot \partial_\mu \vec{n} \times \partial_\nu \vec{n} + \Sigma_{\mu\nu}, \]
where
\[ \Sigma_{\mu\nu} = -\frac{1}{2} [\partial_\mu, \partial_\nu] \psi_+ \]
is the familiar Dirac string tensor. Indeed, in (47) we identify the familiar structure of point-like Dirac monopoles. We note that this structure is also intimately related to the presence of knot-like configurations in the space \( \mathbb{R}^3 \) [10], and these knots are the natural candidates for describing the (glueball) spectrum of the Yang–Mills theory.

Instead of \( S^3 \), we can also take the internal manifold to be the co-adjoint orbit \( SU(2)/U(1) \sim \mathbb{S}^2 \). The Killing vectors are now
\[ K^1 = -\sin \phi \partial_\theta - \cot \theta \sin \phi \partial_\phi \]
The rank of the metric tensor on \( S^2 \) is two, but for integral (3) we get
\[
\int_{S^2} \sin \theta \, d\theta \, d\phi \, g_{mn} K^{am} K^{bn} = 4\pi \cdot \frac{2}{3} \delta^{ab},
\]
and consequently we obtain the Yang–Mills action from (10) and (11). But (17) now gives a non-vanishing mass to only two of the vector fields \( J^m \mu \), and the massless combination is
\[
J_\mu = \sin \phi(x) \sin \theta(x) A^1_\mu + \cos \phi(x) \sin \theta(x) A^2_\mu + \cos \theta(x) A^3_\mu,
\]
where \( \phi(x) \) and \( \theta(x) \) are the spherical coordinates of the brane position. We note that by properly implementing the diffeomorphisms (8) and (9) we can locally transport the brane position e.g. to the north pole \( \theta = 0 \) so that the massless mode becomes \( A^3_\mu \). This corresponds to selecting a unitary gauge in the original Yang–Mills theory.

6. Conclusions

In conclusion, we have considered a gauge invariant coupling between a four-dimensional \( SU(N) \) gauge field and a three-brane. For this we have geometrically re-interpreted the four-dimensional Yang–Mills theory, reformulating it as a gravity theory in a higher dimensional spacetime by replacing the Lie algebra generators with Killing vector fields. This enables us to employ the Nambu action of the three-brane to introduce an interaction between the Yang–Mills field and the three-brane that fluctuates in the ensuing higher dimensional spacetime. We have shown that the Nambu action for the brane leads to a vector field mass. The final theory describes the interactive dynamics of massive and massless vector fields in the original flat Euclidean four-space, with the mass constituent depending on the choice of the internal manifold. In particular, all other reference to the three-brane besides the presence of a vector mass term in \( \mathbb{R}^4 \) becomes entirely removed. We have also investigated topologically non-trivial brane structures, and established their connection to magnetic monopoles in \( \mathbb{R}^4 \). Our results suggest that a gauge invariant mass in Yang–Mills theory could well have its origin in higher dimensions. Or, at least our construction appears to give a novel and good motivation for introducing certain familiar Skyrme-like effective fields to describe both the mass gap and the gauge anomaly in a continuum Yang–Mills theory. From the four-dimensional point of view these theories are non-renormalizable. However, with the present higher dimensional gravity/membrane formulation, maybe there is a completion into a consistent theory. But we leave the inspection of quantum effects in our proposal as a future challenge.

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