Gradient of the Objective Function for an Anisotropic Centroidal Voronoi Tessellation (CVT) - A revised, detailed derivation

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Abstract
In their recent article (2010), Levy and Liu introduced a generalization of Centroidal Voronoi Tessellation (CVT) - namely the $L_p$-CVT - that allows the computation of an anisotropic CVT over a sound mathematical framework. In this article a new objective function is defined, and both this function and its gradient are derived in closed-form for surfaces and volumes. This method opens a wide range of possibilities, also described in the paper, such as quad-dominant surface remeshing, hex-dominant volume meshing or fully-automated capturing of sharp features. However, in the same paper, the derivations of the gradient and of the new objective function are only partially expanded, in the appendices, and some relevant requisites on the anisotropy field are left implicit. In order to better harness the possibilities described there, in this work the entire derivation process is made explicit. In the authors’ opinion, this also helps understanding the working conditions of the method and its possible applications.

keywords: Centroidal Voronoi tessellation, anisotropic meshing, surface reconstruction, topology preservation, computational geometry and object modeling

Introduction
In their recent article (2010), Levy and Liu introduce a generalization of Centroidal Voronoi Tessellation - namely the $L_p$-CVT - that “minimizes a higher-order moment of the coordinates on the Voronoi cells”. Levy and Liu take as reference the standard CVT objective function, and extend its behavior injecting the $L_p$ norm and an anisotropy term, in the form of a matrix obtained from the anisotropy field, in the function itself. This method opens a wide range of possible applications, also described in the article, such as quad-dominant surface remeshing, hex-dominant volume meshing or fully-automated capturing of sharp features. In particular, in the authors’ opinion, this method could also increase the resistance to noise in surface reconstruction and remeshing, which is relevant in the application of methods such as the one described in Piastra (2013).

In the work by Levy and Liu (2010), however, the derivation of the gradient, as well as the definition of the new objective function, are only partially described, and some conditions on the anisotropy field are left implicit. Application of the same method under different conditions, e.g. a specific anisotropy field, or for different purposes, involves a complete comprehension of the mathematical frame on which the method is based. This work is intended to analyze thoroughly the derivation both of the objective function and of its gradient, in order to understand the functioning of the method and the conditions of applicability.

For sake of clarity, the notation used here is slightly different from the one in the original work, due to of the different structure of this paper.

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1 The Energy Function $F_{L_p}$

1.1 The objective function of $L_p$-Centroidal Voronoi Tessellation

Given a set $W$ of $k$ vectors $w_1, \ldots, w_k \in \mathbb{R}^n$, the set of all points in $\mathbb{R}^n$ for which a particular $w_i$ is the nearest vector is called the Voronoi Region $\Omega_i$ of this vector, defined as:

$$\Omega_i := \{ x \in \mathbb{R}^n | i = \arg \min_j ||x - w_j|| \}.$$  (1)

The partition of $\mathbb{R}^n$, or of a manifold $\Omega \subseteq \mathbb{R}^n$, formed by all the Voronoi regions is called Voronoi or Dirichlet Tessellation, and each vector $w_i$ is referred to as Voronoi landmark or generator.

A Centroidal Voronoi Tessellation (Du, Faber, and Gunzburger, 1999) is a Voronoi tessellation whose generating points are the centroids (centers of mass) of the corresponding Voronoi regions, and it minimizes an energy function $F_{CVT}(W)$, the expected quantization error, defined as:

$$F_{CVT}(W) = \int_{\Omega} ||x - w_{i(x)}||^2 \, dx,$$  (2)

where $i(x)$ is the index $i$ of the Voronoi landmark nearest to $x$.

When the partitioned manifold is a set $\Omega \subset \mathbb{R}^n$, some of the Voronoi regions will not be completely contained in $\Omega$, thus leading to the definition of a restricted Voronoi cell as $\Omega_i \cap \Omega$. We can then decompose the integral in (2) as the sum of the integrals calculated on each restricted Voronoi region:

$$F_{CVT}(W) = \sum_{i} \int_{\Omega_i \cap \Omega} ||x - w_i||^2 \, dx,$$  (3)

$L_p$-Centroidal Voronoi Tessellation (Levy and Liu, 2010) enables the meshing of a manifold $\Omega$ to be controlled by a given anisotropy field. It is defined as the minimizer of the $L_p$-objective function $F_{L_p}$, obtained by injecting an anisotropy term, i.e. the anisotropy matrix $M_x$, and the $L_p$-norm into the standard CVT energy (3):

$$F_{L_p}(W) = \sum_{i} \int_{\Omega_i \cap \Omega} ||M_x(x - w_i)||_p^2 \, dx,$$  (4)

here $|| \cdot ||_p$ denotes the $L_p$-norm ($||V||_p = \sqrt[p]{x^p + y^p + z^p}$) and $||V||_p = \langle x^p + y^p + z^p \rangle$. For even values of $p$, the $L_p$-norm becomes $||V||_p^p = x^p + y^p + z^p$. We will examine the case in which the boundary of $\Omega$ is a piecewise linear complex (PLC).

In Levy and Liu (2010) the anisotropy field is characterized by a symmetric tensor field $G_x$. The spectral theorem states that:

**Theorem 1.1** (Spectral Theorem). If a $n \times n$ matrix $A$ is symmetric, then there is a basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$ whose elements are $n$ eigenvectors of $A$.

This means that we can decompose $G_x$ as:

$$G_x = QAQ^t,$$  (5)

where $Q$ is a matrix whose columns are the eigenvectors $q_i$ of $G$ and $A$ is a diagonal matrix whose diagonal entries are the relative eigenvalues $\lambda_i$.

In Levy and Liu (2010) it is implicitely assumed that $\Lambda$ is positive definite, so that it is possible to define $\Sigma$ and $M_x$ as:

$$\Sigma \Sigma^t = \Lambda, \quad \sigma_{ii} = \sqrt{\lambda_i}; \quad M_x := \Sigma^t Q^t = (Q \Sigma)^t.$$
and we can rewrite (5) as:

\[ G_x = M_x^T M_x. \]  

(6)

If we write explicitly \( M_x \) we can see that:

\[
M_x = \Sigma^t Q^t = \begin{bmatrix}
\sigma_1 & 0 & 0 \\
0 & \sigma_2 & 0 \\
0 & 0 & \sigma_3
\end{bmatrix}
\begin{bmatrix}
q_{1x} & q_{1y} & q_{1z} \\
q_{2x} & q_{2y} & q_{2z} \\
q_{3x} & q_{3y} & q_{3z}
\end{bmatrix} = \begin{bmatrix}
\sigma_1 q_{1x} & \sigma_1 q_{1y} & \sigma_1 q_{1z} \\
\sigma_2 q_{2x} & \sigma_2 q_{2y} & \sigma_2 q_{2z} \\
\sigma_3 q_{3x} & \sigma_3 q_{3y} & \sigma_3 q_{3z}
\end{bmatrix} = \begin{bmatrix}
[\sigma_1 q_{1x}]^t \\
[\sigma_2 q_{2x}]^t \\
[\sigma_3 q_{3x}]^t
\end{bmatrix}.
\]

(7)

In order to understand the meaning of the matrix \( M_x \) we will examine the quadratic form of \( G_x(v) : \mathbb{R}^n \rightarrow \mathbb{R} \) applied to a generic vector \( v \):

\[ G_x(v) = v^t G_x v = \sum_i \sum_j \sigma_i \sigma_j q_i q_j v_i v_j = \|M_x v\|^2. \]

(8)

If we define the vector \( v_i := x - w_i \), i.e. the distance between a generic point \( x \in S \) and its nearest Voronoi landmark inside the corresponding Voronoi region \( \Omega_i \), then \( G_x(v) \) can be interpreted as the squared error and we can calculate the expected quantization error induced by the Voronoi tessellation, considering the anisotropy \( G_x \), as:

\[ \sum_i \int_{\Omega_i \cap \Omega} \|M_x (x - w_i)\|^2 \, dx, \]

which is just \( F_{L_2} \), i.e. the \( L_2 \)-based version of (4).

1.2 The integration \( F_{L_p}^T \) over an integration simplex \( T \)

**Theorem 1.2.** Given that each closed Voronoi region in three dimension is a convex polyhedron \((\text{Okabe, 2000})\) and that the surface \( S \) is a piecewise linear complex, we can divide each region \( \Omega_i \cap \Omega \) into tetrahedrons, i.e. three-dimensional simplices, and further decompose the integral in (4). Assuming (a) that each tetrahedron is formed by a Voronoi landmark \( w_i \) and three vertices \( C_1, C_2, C_3 \), (b) that the determinant of the transformation matrix \( M_T \) is 1, i.e. the transformation doesn’t change the volume or energy and (c) that the index \( p \) of the \( L_p \)-norm is an even number, the integration \( F_{L_p}^T \) of the quantization error over each integration simplex \( T = T(w_i, C_1, C_2, C_3) \), is given by (see \( \text{Levy and Link, 2014} \)) and appendix A, equation (30):

\[
F_{L_p}^T = \int_T \|M_T(x - w_i)\|_p^p \, dx
= \frac{|T|}{a^n} \sum_{\alpha+\beta+\gamma=p} U_1^{\alpha} U_2^{\beta} U_3^{\gamma},
\]

(9)

where:

\[
\begin{align*}
U_j &= M_T(C_j - w_i) \\
V_1 \ast V_2 &= [x_1 x_2, y_1 y_2, z_1 z_2]^t \\
V^{\ast a} &= V \ast \ldots \ast V \ (a \ times) \\
\nabla &= x + y + z
\end{align*}
\]

**Proof.** The general rule for integration by substitution, in the multi-variable case, says that:

\[
\int_U f(\varphi(x)) |\det(J_{\varphi})(x)| \, dx = \int_{\varphi(U)} f(u) \, du,
\]

(10)

with the continuously differentiable substitution function \( \varphi(x) = u \).

If we define:

\[
\varphi(\cdot) = M_T(\cdot - w_i); \quad f(\cdot) = \|\cdot\|_p^p; \quad U = T(w_i, C_1, C_2, C_3),
\]

The subscripts should indicate even that this vertices are from a particular tetrahedron belonging to the \( i \)-th Voronoi cell, but for sake of clarity we will omit those details, leaving them implied.
then:
\[ f(\varphi(x)) = \| M_T(x - \mathbf{w}_i) \|_p^p; \quad \varphi(U) = \varphi(T) = T(0, \mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3). \]

Since \( \mathbf{u} = M_T(x - \mathbf{w}_i) \), the variables identifying the domain become \( \mathbf{u}_C = M_T(C_j - \mathbf{w}_i) = \mathbf{U}_j \) and \( \mathbf{u}_{\mathbf{w}_i} = M_T(\mathbf{w}_i - \mathbf{w}_i) = 0 \). We will indicate with \( T' \) the simplex \( \varphi(T) = T(0, \mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3) \) and the variable substitution formula becomes:
\[
\int_T \| M_T(x - \mathbf{w}_i) \|_p^p \det(J(M_T(x - \mathbf{w}_i))) \, dx = \int_{T'} \| \mathbf{u} \|_p^p \, d\mathbf{u}
\]

To find the Jacobian matrix we first have to decompose \( \varphi(x) \) in the three components \( \{ \varphi(x)_x, \varphi(x)_y, \varphi(x)_z \} \). Calling \( \mathbf{d} \) the distance \( (x - \mathbf{w}_i) \) and expanding \( M_T \) as in (7) we obtain:
\[
M_T \mathbf{d} = \begin{bmatrix} (\sigma_1 \mathbf{q}_1)^t \\ (\sigma_2 \mathbf{q}_2)^t \\ (\sigma_3 \mathbf{q}_3)^t \end{bmatrix} = \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix}
\]

Being \( d_x = x - w_{ix} \), each element \( d \varphi_x / dx_x \) of the Jacobian matrix it’s found in a manner similar to:
\[
\frac{d \varphi_x}{dx_x} = d (\sigma_1 q_{1x} (x - w_{ix}) + \sigma_1 q_{1y} (y - w_{iy}) + \sigma_1 q_{1z} (z - w_{iz})) = \sigma_1 q_{1x},
\]

and the Jacobian matrix of \( \varphi(x) \) is then \( M_T \).

Since the determinant of \( M_T \) is 1 we can express, as in [Levy and Liu (2010)] the energy function as:
\[
F^T_{T'} = \int_T \| M_T(x - \mathbf{w}_i) \|_p^p \, dx = \int_T \| M_T(x - \mathbf{w}_i) \|_p^p \det(M_T) \, dx = \int_{T'} \| \mathbf{u} \|_p^p \, d\mathbf{u}.
\]

Note that:
\[
\| \mathbf{u} \|_p^p = \left( \sqrt[p]{x^p_u + y^p_u + z^p_u} \right)^p \text{ with } p \text{ even}
\]
\[
x^p_u + y^p_u + z^p_u,
\]

therefore being \( \mathbf{u}^p = [x_u, y_u, z_u]^t \) we can associate with it a \( p \)-linear symmetric form using a polarization formula (see appendix B):
\[
H(u(1), u^{(2)}, \ldots, u^{(p)}) = \frac{1}{p!} \frac{\partial}{\partial \lambda_1} \cdots \frac{\partial}{\partial \lambda_p} f(\lambda_1 u^{(1)} + \ldots + \lambda_p u^{(p)})
\]
\[
= \frac{1}{p!} \frac{\partial^p}{\partial \lambda_1 \cdots \partial \lambda_p} (\lambda_1 u^{(1)} + \ldots + \lambda_p u^{(p)})^p
\]
\[
= \frac{1}{p!} \frac{\partial^p}{\partial \lambda_1 \cdots \partial \lambda_p} \left( \lambda_1 x^{(1)} + \ldots + \lambda_p x^{(p)})^p + (\lambda_1 y^{(1)} + \ldots + \lambda_p y^{(p)})^p + (\lambda_1 z^{(1)} + \ldots + \lambda_p z^{(p)})^p \right).
\]

(13)

For the sum rule we can solve the derivative for the first parenthesis and then apply the result to the other two. It’s easy to see that
\[
\frac{\partial}{\partial \lambda_1} (\lambda_1 x^{(1)} + \ldots + \lambda_p x^{(p)})^p = \begin{bmatrix} p \times (1) x^{(1)}(\lambda_1 x^{(1)} + \ldots + \lambda_p x^{(p)})^{p-1},
\end{bmatrix}
\]

and then
\[
\frac{\partial}{\partial \lambda_2} (\lambda_1 x^{(1)} + \ldots + \lambda_p x^{(p)})^{p-1} = p(p - 1)x^{(1)}(\lambda_1 x^{(1)} + \ldots + \lambda_p x^{(p)})^{p-2},
\]
and so on, until the following result is obtained:

\[
\frac{\partial^p}{\partial \lambda_1 \cdots \partial \lambda_p} (\lambda_1 x^{(1)} + \cdots + \lambda_p x^{(p)})^p = p \cdot (p - 1) \cdots (2) \cdot (1) \cdot x^{(1)} x^{(2)} \cdots x^{(p)} (\lambda_1 x^{(1)} + \cdots)^0
\]

\[
= p! x^{(1)} x^{(2)} \cdots x^{(p)}.
\]  

(14)

By putting (14) back into (13), factoring out \( p! \) to cancel the denominator and using (11) and (12), we obtain:

\[
H(u^{(1)}, u^{(2)}, \ldots, u^{(p)}) = u^{(1)} \cdots u^{(p)},
\]

which has to be \( p \)-linear and symmetric.

The symmetry of (15) is easy to see, since any changes in the order of the arguments of \( H \) will only change the order of factors inside the parenthesis, without affecting the result. On the other hand, \( p \)-linearity is proved by solving the following equation:

\[
H(u^{(1)}, \ldots, \lambda u^{(i)} + \mu u^{(i')}, \ldots, u^{(p)}).
\]

Given that:

\[
\lambda u^{(i)} + \mu u^{(i')} = \begin{bmatrix} \lambda x^{(i)} \\ \lambda y^{(i)} \\ \lambda z^{(i)} \end{bmatrix} + \begin{bmatrix} \mu x^{(i')} \\ \mu y^{(i')} \\ \mu z^{(i')} \end{bmatrix} = \begin{bmatrix} \lambda x^{(i)} + \mu x^{(i')} \\ \lambda y^{(i)} + \mu y^{(i')} \\ \lambda z^{(i)} + \mu z^{(i')} \end{bmatrix},
\]

we can solve for the first operand and then repeat the same procedure twice:

\[
\left(x^{(1)} x^{(2)} \cdots (\lambda x^{(i)} + \mu x^{(i')} \cdots \cdots x^{(p)}) = \lambda(x^{(1)} x^{(2)} \cdots x^{(i)} \cdots x^{(p)}) + \mu(x^{(1)} x^{(2)} \cdots x^{(i')} \cdots x^{(p)}),
\]

after this, it becomes easy to see that

\[
\lambda(x^{(1)} \cdots x^{(i)} \cdots x^{(p)}) + \mu(x^{(1)} \cdots x^{(i')} \cdots x^{(p)}) + \lambda(y^{(1)} \cdots y^{(i)} \cdots y^{(p)}) + \mu(y^{(1)} \cdots y^{(i')} \cdots y^{(p)})
\]

\[
+ \lambda(z^{(1)} \cdots z^{(i)} \cdots z^{(p)}) + \mu(z^{(1)} \cdots z^{(i')} \cdots z^{(p)})
\]

\[
= \lambda H(u^{(1)}, \ldots, u^{(i)}, \ldots, u^{(p)}) + \mu H(u^{(1)}, \ldots, u^{(i')}, \ldots, u^{(p)}).
\]

Finally, we can apply theorem A.1 to equation (15) to obtain:

\[
F_{L^p}^T = \int_T H(u, u, \ldots, u) du = \sum_{\alpha + \beta + \gamma = p} U_{1\alpha}^{*} U_{2\beta}^{*} U_{3\gamma}^{*}
\]

\[
= \sum_{\alpha + \beta + \gamma = p} \frac{\varphi(w_{1})^{* \alpha_{1}} \ast \varphi(C_{1})^{* \alpha_{2}} \ast \varphi(C_{2})^{* \alpha_{3}}}{(n+1)^{\frac{\alpha_{1}}{p}}}
\]

\[
2 \quad \text{Expression of } \nabla F_{L^p}^T
\]

In this section, we will obtain the expression of the overall gradient of \( F \) by combining, through the chain rule, the gradient of \( F \) on the vertices of the integration simplices together with the gradient of \( F \) on Voronoi vertices.

The usual chain rule for a function of multiple variables is:

\[
\frac{\partial f(g_1(x), g_2(x), \ldots, g_n(x))}{\partial x} = \sum_{i=1}^{n} \frac{\partial f(g_i(x))}{\partial g_i(x)} \frac{\partial g_i(x)}{\partial x}
\]

Applying the latter to \( F_{L^p}^T \) we obtain:

\[
\frac{\partial F_{L^p}^T(w, C_1, C_2, C_3)}{\partial W} = \frac{\partial F_{L^p}^T}{\partial w_i} \frac{\partial w_i}{\partial W} + \frac{\partial F_{L^p}^T}{\partial C_1} \frac{\partial C_1}{\partial W} + \frac{\partial F_{L^p}^T}{\partial C_2} \frac{\partial C_2}{\partial W} + \frac{\partial F_{L^p}^T}{\partial C_3} \frac{\partial C_3}{\partial W}
\]  

(16)
where \( \partial A/\partial B \) denotes the Jacobian matrix of \( A \) with respect to \( B \).

Since \( F^T_{L_p} \) is a scalar function and \( W \in (\mathbb{R}^n)^k \), the gradient will also belong to \( (\mathbb{R}^n)^k \).

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Note that \( \partial w_i/\partial W = \left[ \begin{array}{c} \partial w_i/\partial w_1 \\ \vdots \\ \partial w_i/\partial w_k \end{array} \right] \) and

\[
\begin{pmatrix}
\partial w_i/\partial w_j \\
\vdots \\
\partial w_i/\partial w_{jk}
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

only if \( i = j \), so that \( \partial F^T_{L_p}/\partial w_i \cdot \partial w_i = \partial F^T_{L_p}/\partial w_i \), meaning that the other elements of the \( n \cdot k \) vector will be 0.

Following this same reasoning, we see that all other addends are vectors in \( \mathbb{R}^n \) which, when multiplied each by a \( n \times (n \cdot k) \) matrix, produce vectors of dimension \( n \cdot k \).

We can also express \( F^T_{L_p} \) as a function of

\[
U_{1,2,3} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix},
\]

which is a vector in \( 3 \cdot n \) dimensions.

Putting it all together, we can expand (10) as:

\[
\frac{\partial F^T_{L_p}}{\partial W} = \frac{\partial F^T_{L_p}}{\partial U_{1,2,3}} \frac{\partial U_{1,2,3}}{\partial w_i} \frac{\partial w_i}{\partial W} + \frac{\partial F^T_{L_p}}{\partial U_{1,2,3}} \frac{\partial U_{1,2,3}}{\partial C_1} \frac{\partial C_1}{\partial W} + \frac{\partial F^T_{L_p}}{\partial U_{1,2,3}} \frac{\partial U_{1,2,3}}{\partial C_2} \frac{\partial C_2}{\partial W} + \frac{\partial F^T_{L_p}}{\partial U_{1,2,3}} \frac{\partial U_{1,2,3}}{\partial C_3} \frac{\partial C_3}{\partial W}.
\]

(17)

2.1 Derivation of \( F^T_{L_p} \) relative to the vertices of \( T \)

Starting from (17) we can now derive each component of the equation, to obtain the final result.

To solve the above derivative, Levy and Lin (2010) first define:

\[
E^T_{L_p} := \sum_{\alpha+\beta+\gamma=p} U_1^{*\alpha} * U_2^{*\beta} * U_3^{*\gamma}
\]

\[
F^T_{L_p} = \left( \frac{|T|}{n+p} \right) E^T_{L_p}
\]

so that:

\[
\frac{\partial F^T_{L_p}}{\partial W} = \frac{\partial}{\partial W} \left( \frac{|T|}{n+p} E^T_{L_p} \right) = \frac{1}{(n+p)} \left( E^T_{L_p} \partial |T| + |T| \partial E^T_{L_p} \right)
\]

The first two steps of the derivation will therefore be the two derivatives found in (18): namely \( \partial E^T_{L_p} / \partial U_{1,2,3} \) and \( \partial |T| / \partial U_{1,2,3} \).

First step

We first show the derivation of \( \partial E^T_{L_p} / \partial U_{1,2,3} \).
Recalling that
\[
\frac{\partial E_L^T}{\partial U_{1,2,3}} = \partial \left( \sum_{\alpha + \beta + \gamma = p} \left( x_1^\alpha x_2^\beta x_3^\gamma + y_1^\alpha y_2^\beta y_3^\gamma + z_1^\alpha z_2^\beta z_3^\gamma \right) \right)
\]

we can now examine the derivation step by step:

\[
\frac{\partial E_L^T}{\partial U_{1,2,3}} = \left[ \frac{\partial E_L^T}{\partial U_1}, \frac{\partial E_L^T}{\partial U_2}, \frac{\partial E_L^T}{\partial U_3} \right];
\]

that, for every single combination with \( \alpha \geq 1 \), equals to:

\[
\left[ \frac{\partial (x_1^\alpha x_2^\beta x_3^\gamma)}{\partial x_1} + \frac{\partial (y_1^\alpha y_2^\beta y_3^\gamma)}{\partial y_1} + \frac{\partial (z_1^\alpha z_2^\beta z_3^\gamma)}{\partial z_1}, \frac{\partial (x_1^\alpha x_2^\beta x_3^\gamma)}{\partial x_1} + \frac{\partial (y_1^\alpha y_2^\beta y_3^\gamma)}{\partial y_1} + \frac{\partial (z_1^\alpha z_2^\beta z_3^\gamma)}{\partial z_1}, \frac{\partial (x_1^\alpha x_2^\beta x_3^\gamma)}{\partial x_1} + \frac{\partial (y_1^\alpha y_2^\beta y_3^\gamma)}{\partial y_1} + \frac{\partial (z_1^\alpha z_2^\beta z_3^\gamma)}{\partial z_1} \right]
\]

\[
= \left[ \alpha x_1^{\alpha-1} x_2^\beta x_3^\gamma, \alpha y_1^{\alpha-1} y_2^\beta y_3^\gamma, \alpha z_1^{\alpha-1} z_2^\beta z_3^\gamma \right]
\]

And finally:

\[
\frac{\partial E_L^T}{\partial U_{1,2,3}} = \left[ \sum_{\alpha + \beta + \gamma = p; \alpha \geq 1} \alpha U_1^{\alpha-1} U_2^\beta U_3^\gamma \right]^T
\]

\[
= \left[ \sum_{\alpha + \beta + \gamma = p; \beta \geq 1} \beta U_1^{\alpha} U_2^{\beta-1} U_3^\gamma \right] + \left[ \sum_{\alpha + \beta + \gamma = p; \gamma \geq 1} \gamma U_1^{\alpha} U_2^\beta U_3^{\gamma-1} \right]
\]

**Second step**

Here we will show the derivation of \( \frac{\partial |T|}{\partial U_{1,2,3}} \).

\( T \) is a three-dimensional simplex, i.e., a tetrahedron, having one of its vertices in the origin. This means that \( U_1, U_2 \) and \( U_3 \) are all edges of the tetrahedron and therefore its volume is:

\[
|T| = 1/6 U_1 \cdot (U_2 \times U_3) = 1/6 U_2 \cdot (U_3 \times U_1) = 1/6 U_3 \cdot (U_1 \times U_2),
\]

The gradient of the volume with respect to \( U_{1,2,3} \) is:

\[
\frac{\partial |T|}{\partial U_{1,2,3}} = \frac{1}{6} \left[ [U_2 \times U_3]^t, [U_3 \times U_1]^t, [U_1 \times U_2]^t \right].
\]

To prove this, we will examine the increment \( |T|^t \) of \( |T| \) with respect to \( U_1 \), with
\( U_1^{(e)} = U_1 + \varepsilon v \). From definitions, we know that:

\[
\begin{align*}
|T|^{(e)} &= U_1^{(e)} \cdot (U_2 \times U_3) \\
&= U_1 \cdot (U_2 \times U_3) + \varepsilon v \cdot (U_2 \times U_3) \\
&= U_1 \cdot (U_2 \times U_3) + \varepsilon \frac{\partial (U_1 \cdot (U_2 \times U_3))}{\partial U_1} v + o(\varepsilon)
\end{align*}
\]

from which

\[
\varepsilon \frac{\partial (U_1 \cdot (U_2 \times U_3))}{\partial U_1} v = \varepsilon (U_2 \times U_3)^t v = \frac{\partial (U_1 \cdot (U_2 \times U_3))}{\partial U_1} = (U_2 \times U_3)^t,
\]

by iterating the same procedure on \( \partial (U_2 \cdot (U_3 \times U_1))/\partial U_2 \) and \( \partial (U_3 \cdot (U_1 \times U_2))/\partial U_3 \) we get to the result.

Intuitively the latter indicates for each vertex a variation proportional to one sixth of the rate of increase directed along the perpendicular of that side, i.e. the height.

For completeness and self-containment of this paper, we also report the demonstration of the two-dimensional derivative contained in [Levy and Liu (2010)] in appendix [C].

**Third step**

The derivatives of \( F_{L_p}^T \) with respect to the vertices of the tetrahedron are:

\[
\begin{align*}
\frac{\partial F_{L_p}^T}{\partial C_i} &= \left[ \begin{array}{c} \frac{\partial F_{L_p}^T}{\partial U_{1,2,3}} M_T; \\
\frac{\partial F_{L_p}^T}{\partial w_i} = - \frac{\partial F_{L_p}^T}{\partial C_i} \frac{\partial F_{L_p}^T}{\partial C_2} - \frac{\partial F_{L_p}^T}{\partial C_3} \end{array} \right] \\
\frac{\partial F_{L_p}^T}{\partial C_i} &= \left[ \begin{array}{c} \frac{\partial F_{L_p}^T}{\partial U_{1,2,3}} \frac{\partial U_{1,2,3}}{\partial w_i} \end{array} \right] \\
\frac{\partial F_{L_p}^T}{\partial C_i} &= \left[ \begin{array}{c} \frac{\partial F_{L_p}^T}{\partial U_{1,2,3}} \frac{\partial U_{1,2,3}}{\partial C_1} \frac{\partial F_{L_p}^T}{\partial C_1} \\
\frac{\partial F_{L_p}^T}{\partial U_{1,2,3}} \frac{\partial U_{1,2,3}}{\partial C_2} \frac{\partial F_{L_p}^T}{\partial C_2} \\
\frac{\partial F_{L_p}^T}{\partial U_{1,2,3}} \frac{\partial U_{1,2,3}}{\partial C_3} \frac{\partial F_{L_p}^T}{\partial C_3} \\
M_T \\
0 \\
0 \\
0
\end{array} \right] = \left[ \begin{array}{c} \frac{\partial F_{L_p}^T}{\partial U_1} M_T; \\
\frac{\partial F_{L_p}^T}{\partial U_2} M_T; \\
\frac{\partial F_{L_p}^T}{\partial U_3} M_T;
\end{array} \right]
\end{align*}
\]

2.2 Gradient of Voronoi vertices

Given that the simplex vertices, apart from \( w_i \), are Voronoi vertices, i.e. the intersection of \( n + 1 \) Voronoi cells each of them is the circumcenter of a Delaunay simplex of vertices \( w_i, w_j, w_k, w_l \), obtained by intersecting the three bisectors (Voronoi edges) between \( w_i \) and

\[\text{More precisely any vertex can be the intersection of at least } n + 1 \text{ vertices, but the case in which the number of vertices is more than the necessary, that we can call the degenerate case, is not discussed in [Levy and Liu (2010)], nor will be discussed here.}\]
$w_j$, $w_i$ and $w_k$, $w_i$ and $w_l$. According to Goldman (1994) the computation of the point of intersection of three planes can be condensed to:

$$
\frac{1}{n_i \cdot (n_2 \times n_3)}((p_1 \cdot n_i)(n_2 \times n_3) + (p_2 \cdot n_i)(n_3 \times n_1) + (p_3 \cdot n_i)(n_1 \times n_2)),
$$

where $p_j$ is a point on the plane $j$ and $n_j$ is the unit vector normal to the same plane.

If we define each side $L_j$ of the tetrahedron starting from $w_i$ as

$$
L_1 = w_j - w_i, \quad L_2 = w_k - w_i, \quad L_3 = w_l - w_i,
$$

we can express the points $p_j$ as follows:

$$
p_1 = \frac{w_j - w_i}{2} + w_i, \quad p_2 = \frac{w_k + w_i}{2}, \quad p_3 = \frac{w_l + w_i}{2},
$$

and the unit vectors $n_j$ can be expressed in turn as:

$$
n_1 = \frac{L_1}{|L_1|}, \quad n_2 = \frac{L_2}{|L_2|}, \quad n_3 = \frac{L_3}{|L_3|}.
$$

From (23) we can state that:

$$
n_j \times n_k = \frac{L_j \times L_k}{|L_j||L_k|} \text{ and } n_j \cdot (n_k \times n_i) = \frac{L_j \cdot (L_k \times L_i)}{|L_j||L_k||L_i|}.
$$

Eventually, substituting (22), (23) and (24) in (21), we obtain the condensed equation for each vertex $C$, defined by the Voronoi generators $w_i, w_j, w_k, w_l$:

$$
C = \frac{|L_1| |L_2| |L_3|}{L_1 \cdot (L_2 \times L_3)} \left[ \frac{w_j + w_i}{2} \cdot \frac{L_1}{|L_1|} \frac{L_2 \times L_3}{|L_2||L_3|} \right. \\
+ \left. \frac{w_k + w_i}{2} \cdot \frac{L_2}{|L_2|} \frac{L_3 \times L_1}{|L_3||L_1|} \right. \\
+ \left. \frac{w_l + w_i}{2} \cdot \frac{L_3}{|L_3|} \frac{L_1 \times L_2}{|L_1||L_2|} \right] .
$$

By solving in (25) the first addend alone, we obtain:

$$
\frac{|L_1| |L_2| |L_3|}{L_1 \cdot (L_2 \times L_3)} \left( \frac{w_j + w_i}{2} \cdot \frac{L_1}{|L_1|} \frac{L_2 \times L_3}{|L_2||L_3|} \right) = \frac{1}{L_1 \cdot (L_2 \times L_3)} \frac{(w_j + w_i)(w_j - w_i)}{2} = \frac{1}{L_1 \cdot (L_2 \times L_3)} \frac{w_j^2 - w_i^2}{2}.
$$

Then, by using the same result for the other two addends, we can write:

$$
C = \frac{1}{L_1 \cdot (L_2 \times L_3)} \left[ \frac{L_2 \times L_1}{2} \left( \frac{1}{2} w_j^2 - w_i^2 \right) \right. \\
+ \left( L_3 \times L_1 \right) \left( \frac{1}{2} w_k^2 - w_i^2 \right) \\
+ \left. \left( L_1 \times L_2 \right) \left( \frac{1}{2} w_l^2 - w_i^2 \right) \right].
$$

If we consider a matrix $A$ having $L'_1, L'_2$ and $L'_3$ as its rows, the determinant is: $\text{det} A = L_1 \cdot (L_2 \times L_3)$, and it follows from the definitions of inverse matrix and of cross product that:

$$
A^{-1} = \frac{1}{L_1 \cdot (L_2 \times L_3)} \left[ \frac{L_2 \times L_3}{|L_2 \times L_3|} \right. \\
+ \left. \frac{L_3 \times L_1}{|L_3 \times L_1|} \right].
$$
It is now easy to see that \( C \) may be found by:

\[
C = A^{-1}B, \text{ where } A = \begin{bmatrix} [w_j - w_i]^t \\ [w_k - w_i]^t \\ [w_l - w_i]^t \end{bmatrix}; \quad B = \frac{1}{2} \begin{bmatrix} w_j^2 - w_i^2 \\ w_k^2 - w_i^2 \\ w_l^2 - w_i^2 \end{bmatrix}.
\]

**Fourth step**

To calculate \( \frac{\partial C}{\partial W} \) we first need to recall some matrix derivation rules (Minka, 1997):

\[
\frac{\partial (AB)}{\partial W} = \frac{\partial (A)B + A\frac{\partial (B)}{\partial W}}{\partial W} \tag{27}
\]

\[
\frac{\partial (A^{-1})}{\partial W} = -A^{-1}\frac{\partial (A)A^{-1}}{\partial W} \tag{28}
\]

Using first (27) then (28), the expression of \( \frac{\partial C}{\partial W} \) can be expanded:

\[
\frac{\partial C}{\partial W} = \frac{\partial (A^{-1})B + A\frac{\partial (B)}{\partial W}}{\partial W} = -A^{-1}\frac{\partial (A)A^{-1}B + A^{-1}\frac{\partial (B)}{\partial W}}{\partial W} = A^{-1}\left( \frac{\partial (B)}{\partial W} - \frac{\partial (A)C}{\partial W} \right).
\]

Both \( \frac{\partial B}{\partial W} \) and \( \frac{\partial A}{\partial W} \) depend only from \( w_i, w_j, w_k \) and \( w_l \), just like the circum-center \( C \), so we can substitute \( \frac{\partial (B)}{\partial W} \) and \( \frac{\partial (A)C}{\partial W} \) with:

\[
\frac{\partial B}{\partial W} = \frac{1}{2} \begin{bmatrix} \frac{\partial w_j^2 - w_i^2}{\partial w_i} & \frac{\partial w_j^2 - w_i^2}{\partial w_j} & \frac{\partial w_j^2 - w_i^2}{\partial w_k} \\ \frac{\partial w_k^2 - w_i^2}{\partial w_i} & \frac{\partial w_k^2 - w_i^2}{\partial w_j} & \frac{\partial w_k^2 - w_i^2}{\partial w_k} \\ \frac{\partial w_l^2 - w_i^2}{\partial w_i} & \frac{\partial w_l^2 - w_i^2}{\partial w_j} & \frac{\partial w_l^2 - w_i^2}{\partial w_k} \end{bmatrix} \begin{bmatrix} -w_i^t & 0 & 0 \\ -w_j^t & w_k^t & 0 \\ -w_l^t & 0 & w_i^t \end{bmatrix} = \begin{bmatrix} -w_i^t & w_j^t & 0 \\ -w_j^t & 0 & w_k^t \\ -w_l^t & 0 & 0 \end{bmatrix};
\]

\[
\frac{\partial (A)C}{\partial W} = \begin{bmatrix} \frac{\partial [w_j - w_i]^t}{\partial w_i} & \frac{\partial [w_j - w_i]^t}{\partial w_j} & \frac{\partial [w_j - w_i]^t}{\partial w_k} \\ \frac{\partial [w_k - w_i]^t}{\partial w_i} & \frac{\partial [w_k - w_i]^t}{\partial w_j} & \frac{\partial [w_k - w_i]^t}{\partial w_k} \\ \frac{\partial [w_l - w_i]^t}{\partial w_i} & \frac{\partial [w_l - w_i]^t}{\partial w_j} & \frac{\partial [w_l - w_i]^t}{\partial w_k} \end{bmatrix} = \begin{bmatrix} -C^t & C^t & 0 \\ -C^t & 0 & C^t \\ -C^t & 0 & 0 \end{bmatrix} \tag{C}
\]

that if \( \frac{\partial [w_j - w_i]^t}{\partial w_i} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \) (that can be reasonable, since \( [w_j - w_i]^t \) is a row vector) and \( \frac{\partial A}{\partial W} \) is treated as a block matrix leads to the result:

\[
\frac{\partial (A)C}{\partial W} = \begin{bmatrix} -C^t & C^t & 0 \\ -C^t & 0 & C^t \\ -C^t & 0 & 0 \end{bmatrix},
\]

getting to the result:

\[
\frac{\partial C}{\partial W} = \begin{bmatrix} [w_j - w_i]^t \\ [w_k - w_i]^t \\ [w_l - w_i]^t \end{bmatrix}^{-1} \begin{bmatrix} [C - w_i]^t & [w_j - C]^t & 0 & 0 \\ [C - w_i]^t & 0 & [w_k - C]^t & 0 \\ [C - w_i]^t & 0 & [w_l - C]^t & 0 \end{bmatrix}, \tag{29}
\]

being the other elements of \( (\frac{\partial (B)}{\partial W} - \frac{\partial (A)C}{\partial W}) \) all zeroes.
Conclusions

The complete, expanded derivations presented here for the objective function of \( L_p \)-CVT and its gradient help highlighting the conditions of applicability of the anisotropy field, as stated in theorem 1.2. As a point of relevance, this shows the possibility of combining the mathematical framework of [Levy and Liu (2010)] with the well-known computational method of the Local Principal Component Analysis (LPCA) [Kambhatla and Leen (1997)]; in fact, the LPCA tensor can become the anisotropy term in the framework presented as long as the simple conditions stated in theorem 1.2 are fulfilled. Hopefully, the present work will be the base for further studies on the application of \( L_p \)-CVT to surface reconstruction and remeshing from noisy point clouds.

In passing, these same derivations show a minor inaccuracy in the original derivation of [Levy and Liu (2010)], with respect to surface meshing (see appendix C).

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A Integrating a \( p \)-homogeneous polynomial over an \( n \)-dimensional simplex

Considering \( n + 1 \) points, or vertices, \( x_0, x_1, \ldots, x_n \), such that the vectors \( (x_i - x_0) \) are linearly independent, the \( n \)-dimensional non-degenerate simplex \( \Delta_n \subset \mathbb{R}^n \) is the set of \( x \) such that \( x \in \Delta_n \) if and only if \( x \) is a convex combination \( \sum_0^n \lambda_i x_i \), with \( \lambda_i \geq 0 \) and \( \sum_0^n \lambda_i = 1 \).

We are interested in computing

\[
\int_{\Delta_n} q(x) \, dx,
\]

where \( \Delta_n \) is an \( n \)-dimensional simplex as described above, and \( q(x) : \mathbb{R}^n \to \mathbb{R} \) is a real \( p \)-homogeneous polynomial, i.e. \( q(\lambda x) = \lambda^p q(x) \) for all \( \lambda > 0 \), \( x \in \mathbb{R}^n \) and some integer \( p \geq 0 \).

We first need to introduce some concepts and notation. With every symmetric \( p \)-linear form \( H : (\mathbb{R}^n)^p \to \mathbb{R} \), given by:

\[
(x_1, x_2, \ldots, x_p) \mapsto H(x_1, x_2, \ldots, x_p), \quad x_1, x_2, \ldots, x_p \in \mathbb{R}^n,
\]

one may associate a \( p \)-homogeneous polynomial \( x \mapsto f(x) := H(x, x, \ldots, x) \) and conversely, using a polarization formula (see appendix B), with every \( p \)-homogeneous polynomial \( f : \mathbb{R}^n \to \mathbb{R} \) one may associate a symmetric \( p \)-linear form \( H : (\mathbb{R}^n)^p \to \mathbb{R} \), such that

\[
H(x, x, \ldots, x) = f(x).
\]

Therefore, we now consider the integration of a \( p \)-linear form \( H \) over the simplex \( \Delta_n \) [Lasserre and Avrachenkov (2004)].

\^The term degenerate indicate a simplex in which the edges starting from a given vertex aren’t linearly independent, i.e. a \( n \)-dimensional simplex that has intrinsic dimension \( d < n \).

\^Symmetric \( p \)-linear means that the polynomial is function of \( p \) variables, is invariant in respect to the order of its variables, i.e. \( H(x_1, x_2, \ldots, x_n) = H(x_2, x_1, \ldots, x_n) \), and it’s linear in each of them, i.e. \( H(x_1, \ldots, \lambda x_i, \ldots, x_n) = \lambda H(x_1, \ldots, x_i, \ldots, x_n) + \mu H(x_1, \ldots, x_i', \ldots, x_n), \forall i \leq p \).

\^Kirwan and Ryan, in [Kirwan and Ryan (1998)] state that if \( f(x) \in \mathcal{P}(\mathbb{R}^n) \), where \( \mathcal{P}(\mathbb{R}^n) \) denotes the Banach space of bounded, \( p \)-homogeneous polynomials from \( X \) into \( \mathbb{R} \), then there exists a unique bounded symmetric multi-linear mapping \( H : X^n \to \mathbb{R} \) such that \( f(x) = H(x; \ldots; x) \), \( \forall x \in X \). This result is valid, in our case, since we work in the Euclidean space, that is a banach space.
Theorem A.1 (Lasserre, Avrachenkov). Let $x_0, x_1, \ldots, x_n$ be the $n + 1$ vertices of an $n$-dimensional simplex $\Delta_n$. Then, for a symmetric $p$-linear form $H : (\mathbb{R}^n)^p \to \mathbb{R}$, one has

$$
\int_{\Delta_n} H(x, x, \ldots, x) \, dx = \frac{Vol(\Delta_n)}{(n+p)} \left[ \sum_{\alpha_i=p} H(x_0^{\alpha_0}, x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}) \right]
$$

(30)

where the notation $H(x_0^{\alpha_0}, x_1^{\alpha_1}, \ldots, x_n^{\alpha_n})$ means that $x_0$ appears $\alpha_0$ times, $x_1$ appears $\alpha_1$ times, $\ldots$, $x_n$ appears $\alpha_n$ times.

The number of possible $p$-multiset of $n + 1$ objects, that is, the $n + 1$ vertices of $\Delta_n$ as the $p$ variables of $H$, is the same of the simple $p$-combinations, i.e. without repetitions, of $(n + 1) + p - 1$ objects, that is given by the binomial coefficient $\binom{(n+1)+p-1}{p} = \binom{n+p-1}{p}$, as stated in (30).

Since every polynomial can be represented as a sum of homogeneous polynomials, (30) can be easily applied to integrate an arbitrary polynomial over a simplex.

B Polarization formula

Let $f(u)$ be a polynomial of $n$ variables $u = (u_1, u_2, \ldots, u_n)$, $f$ homogeneous of degree $p$. Let $u^{(1)}, u^{(2)}, \ldots, u^{(p)}$ be a collection of indeterminates with $u^{(i)} = (u_1^{(i)}, u_2^{(i)}, \ldots, u_n^{(i)})$. The polar form of $f$ is then a polynomial in $pn$ variables

$$
F(u^{(1)}, u^{(2)}, \ldots, u^{(p)}),
$$

which is linear in each $u^{(i)}$, symmetric among the $u^{(i)}$, and such that $F(u, u, \ldots, u) = f(u)$.

The polar form of $f$ is given by the following construction:

$$
F(u^{(1)}, u^{(2)}, \ldots, u^{(d)}) = \frac{1}{d!} \frac{\partial}{\partial \lambda_1} \ldots \frac{\partial}{\partial \lambda_d} f \left( \lambda_1 u^{(1)} + \lambda_2 u^{(2)} + \ldots + \lambda_d u^{(d)} \right).
$$

For example, if $u = (x, y)$ and $f(u) = x^2 + 3xy + 2y^2$, the polarization of $f$ is a function in $u^{(1)} = (x^{(1, y)}, y^{(1)})$ and $u^{(2)} = (x^{(2, y)}, y^{(2)})$, with $\lambda_1 u^{(1)} + \lambda_2 u^{(2)} = \lambda_1 \left[ x^{(1)} \right] + \lambda_2 \left[ x^{(2)} \right]$,

given by:

$$
F(u^{(1)}, u^{(2)}) = \frac{1}{2!} \frac{\partial}{\partial \lambda_1} \frac{\partial}{\partial \lambda_2} f(\lambda_1 u^{(1)} + \lambda_2 u^{(2)})
$$

$$
= \frac{1}{2} \frac{\partial}{\partial \lambda_1} \frac{\partial}{\partial \lambda_2} \left( \lambda_1 x^{(1)} + \lambda_2 x^{(2)} \right)^2 + 3(\lambda_1 x^{(1)} + \lambda_2 x^{(2)}) (\lambda_1 y^{(1)} + \lambda_2 y^{(2)}) + 2(\lambda_1 y^{(1)} + \lambda_2 y^{(2)})^2
$$

$$
= \frac{1}{2} \frac{\partial}{\partial \lambda_2} \left( 2x^{(1)}(\lambda_1 x^{(1)} + \lambda_2 x^{(2)}) + 3(2\lambda_1 x^{(1)} y^{(1)} + \lambda_2 x^{(2)} y^{(1)}) + \lambda_2 x^{(1)} y^{(2)} + 4y^{(1)}(\lambda_1 y^{(1)} + \lambda_2 y^{(2)}) \right)
$$

$$
= \frac{1}{2} \left( 2x^{(1)} x^{(2)} + 3x^{(2)} y^{(1)} + 3x^{(1)} y^{(2)} + 4y^{(1)} y^{(2)} \right)
$$

$$
= x^{(1, y)} x^{(2)} + \frac{3}{2} x^{(1)} y^{(2)} + \frac{3}{2} x^{(2)} y^{(1)} + 2y^{(1)} y^{(2)}
$$

C Derivation of a 2D simplex with respect to $U_{1,2,3}$

If $T$ is a surface simplex, i.e. a triangle, his area is $|T| = 1/2||N||$, with

$$
N = (U_1 - U_3) \times (U_2 - U_3) = (U_2 - U_1) \times (U_3 - U_1) = (U_3 - U_2) \times (U_1 - U_2).
$$

We have to find the gradient $\frac{\partial |T|}{\partial U_{1,2,3}}$ with respect to the vector composed by $U_1, U_2$ and $U_3$. 

12
The first three components of the resultant vector, i.e. \( \frac{\partial |T|}{\partial U_1} \) are obtained with the following method:

\[
\frac{\partial |T|}{\partial U_1} = \frac{\partial 1/2 |\mathbf{N}|}{\partial U_1} = \frac{1}{2} \frac{\partial (\mathbf{N} \cdot \mathbf{N})^{1/2}}{\partial U_1} \\
= \frac{1}{2} (\mathbf{N} \cdot \mathbf{N})^{-1/2} \frac{\partial (\mathbf{N} \cdot \mathbf{N})}{\partial U_1} \\
= \frac{1}{4} (\mathbf{N} \cdot \mathbf{N})^{1/2} \frac{\partial \mathbf{N}}{\partial U_1} \\
= \frac{1}{4} \frac{\partial \mathbf{N}}{\partial U_1} \mathbf{N}.
\]  

(31)

We can now derive \( \frac{\partial \mathbf{N}}{\partial U_1} \). We will examine the increment \( \mathbf{N}(\varepsilon) \) of \( \mathbf{N} \) with respect to \( U_1 \), with \( U_1(\varepsilon) = U_1 + \varepsilon v \). We know from definition that:

\[
\mathbf{N}(\varepsilon) = (U_1(\varepsilon) - U_3) \times (U_2 - U_3) = \mathbf{N} + \varepsilon \frac{\partial \mathbf{N}}{\partial U_1} v + o(\varepsilon),
\]  

(32)

and expanding \( \mathbf{N}(\varepsilon) \) we obtain:

\[
\mathbf{N}(\varepsilon) = (U_1 - U_3) \times (U_2 - U_3) + \varepsilon (\mathbf{N} \times (U_2 - U_3))
\]  

(33)

Any cross product between two vectors can be expressed with an antisymmetric tensor, associated to the first vector, applied to the second one, such that

\[
\mathbf{w} \times \mathbf{v} = \mathbf{Wv}, \text{ and } \mathbf{W}^t \mathbf{v} = -\mathbf{w} \times \mathbf{v} = \mathbf{v} \times \mathbf{w}, \quad \forall \mathbf{v} \in \mathbb{R}^n
\]  

(34)

therefore

\[
\left( \frac{\partial \mathbf{N}}{\partial U_1} \right) v = \mathbf{W} v = -(U_2 - U_3) \times v.
\]  

(35)

We need only the value of \( \frac{\partial \mathbf{N}}{\partial U_1} \) applied to a generic vector \( v \), to find the value of (31). Substituting (35) in (31) we obtain:

\[
\left( \frac{\partial |T|}{\partial U_1} \right) v = -\frac{1}{4|T|} [\mathbf{N} \times (U_2 - U_3)]^t v.
\]  

(36)

To obtain the entire gradient, it’s enough to obtain the equations equivalent to (36) for \( \frac{\partial}{\partial U_2} \) and \( \frac{\partial}{\partial U_3} \).

This is the result:

\[
\frac{\partial |T|}{\partial U_{1,2,3}} = -\frac{1}{4|T|} \left[ [\mathbf{N} \times (U_2 - U_3)]^t, [\mathbf{N} \times (U_3 - U_1)]^t, [\mathbf{N} \times (U_1 - U_2)]^t \right].
\]

Intuitively this result indicates for each vertex a variation proportional to half of the opposite side of the triangle (base) and with the greatest rate of increase in the orthogonal direction with respect to that side (height).
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