Non-Equilibrium Dynamics and Weakly Broken Integrability

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Motivated by dynamical experiments on cold atomic gases, we develop a quantum kinetic approach to weakly perturbed integrable models out of equilibrium. Using the exact matrix elements of the underlying integrable model we establish an analytical approach to real-time dynamics. The method addresses a broad range of timescales, from the intermediate regime of pre-thermalization to late-time thermalization. Predictions are given for the time-evolution of physical quantities, including effective temperatures and thermalization rates. The approach provides conceptual links between perturbed quantum many-body dynamics and classical Kolmogorov-Arnold-Moser (KAM) theory. In particular, we identify a family of perturbations which do not cause thermalization in the weakly perturbed regime.

Conservation laws play a ubiquitous role in constraining the dynamics of complex many-body systems. This is especially true in low-dimensional integrable systems, where their proliferation gives rise to rich phenomena. A striking example is provided by the quantum Newton’s cradle experiment [1], which shows the absence of thermalization over long timescales. The impact of conservation laws in this so-called pre-thermalization regime is directly encoded via a Generalized Gibbs Ensemble (GGE) [2–7]; each conserved quantity is associated with its own effective temperature, leading to anomalous thermalization. This has stimulated a wealth of theoretical activity, including the recent extension of hydrodynamics [8–12] to integrable systems [13–15] and its application to experiment [16, 17]. For recent reviews exploring the exotic dynamics of isolated quantum integrable systems see [18–25].

Despite recent advances in the understanding of integrable systems, real physical systems always contain perturbations. These may influence and destabilize the integrable dynamics, but their effect is hard to quantify. In the classical domain, the effect of weak perturbations is encoded in KAM theory [26], which describes the persistence of quasi-periodic orbits under small perturbations. In the quantum many-body domain, the scenario of pre-thermalization followed by slow thermalization has been widely studied in this context [27–42]; for recent reviews see [43, 44]. However, insights analogous to KAM theory have been hard to establish, and many experimentally and conceptually relevant questions remain. To what extent does quantum integrability survive in the presence of weak perturbations? How can we quantify and organize the dynamical effects of integrability destroying interactions? What are the relevant timescales?

In this paper we address these questions by developing a quantum kinetic approach to weakly perturbed integrable models out of equilibrium. We show that the dynamics of physical observables from short to long timescales can be described using the exact matrix elements of the underlying integrable model. Our findings are illustrated by numerical evaluation of the key formulae, including the time-evolution of the average densities, quasiparticle distributions, and effective temperatures. Embedding the kinetic approach into a general theory, we identify dynamical response functions which encode the timescales of thermalization. We also find a family of integrability-breaking, KAM-like perturbations, which do not lead to thermalization in the weakly coupled regime. A notable insight which emerges from our analysis is that, in one spatial dimension, thermalization and hydrodynamic diffusion are controlled by distinct families of processes, which we characterize. Our findings also provide the integrability destroying corrections to the Euler hydrodynamics of integrable systems.

Setup.— We consider the general scenario in which a spatially homogeneous one-dimensional integrable system, described by Hamiltonian $H_0$, is perturbed by an extensive integrability destroying term $V = \int dx \, v(x)$. The resulting Hamiltonian is given by $H = H_0 + \lambda V$, where $\lambda$ controls the strength of the perturbation. The Hamiltonian $H_0$ is characterized by an infinite number of mutually commuting conserved quantities $Q_i$, $i = 0, 1, 2, \ldots$, including the momentum $P = Q_1$ and the Hamiltonian $H_0 = Q_2$. In the perturbed system only two conserved quantities remain: the total energy $H$ and the total momentum $P$.

In order to explore the dynamics of the non-integrable Hamiltonian $H$ we consider a quantum quench from an initial state which is stationary under $H_0$, but which evolves under the dynamics of $H$. In light of the integrability of $H_0$ it is natural to take a GGE as the initial state, whose density matrix is given by $\rho_0 = Z^{-1}e^{-\sum_i \beta_i Q_i}$. Here $Z = \text{Tr}(e^{-\sum_i \beta_i Q_i})$ and the $\beta_i$ are the inverse effective temperatures associated with each conserved quantity $Q_i$. These are the most general states that maximize entropy with respect to all of the extensive conserved quantities of $H_0$; they therefore provide natural initial states for studying the dynamics of perturbed integrable systems.

The quench setup described above is well-suited to studying thermalization. At long times, it is expected that expectation values of local observables $\langle \Theta(x, t) \rangle = \text{Tr} [\rho_0 e^{-i H t} \Theta(x) e^{-i H t}]$ tend to the value they would take in a boosted thermal ensemble de-
scribed by $H$ and $P$. Explicitly, \[ \lim_{t \to \infty} \langle O(x,t) \rangle = \frac{1}{Z} \text{Tr} \left[ e^{-\beta s(H-\nu s P)} O(x) \right], \]
where the stationary values $\beta_s$ and $\nu_s$ are uniquely fixed by $(H)$ and $\langle P \rangle$. Thermalization is proven rigorously in various situations [45–48], and if it occurs it does so for any perturbation strength $\lambda$. From a physical perspective however, the most important questions are to what extent integrability still plays a role at finite times, and how the system reaches thermalization. For small perturbations, it could be expected that integrability strongly influences these processes, and constrains the dominant physics.

**Dynamics of Charges.**— To see the effects of the integrability-breaking term, it is instructive to examine the time-evolution of the charges $Q_i$ under the Hamiltonian $H$. To lowest order in $\lambda$, the time-evolution of the corresponding charge densities $q_i(x,t)$, can be computed within second order perturbation theory:

\[ \partial_t q_i(x,0,t) = \lambda^2 \int_{-t}^{t} ds \langle [V^0(s), Q_i] v(0) \rangle \delta, \quad (1) \]

where here and throughout we set $\hbar = k_B = 1$, and we denote the connected correlation function by $\langle \cdots \rangle^c$. Time-evolution on the left-hand side is with respect to the non-integrable Hamiltonian $H$, while time-evolution on the right-hand side is with respect to the integrable $H_0$, with $V^0(s) = e^{iH_0 s} V e^{-iH_0 s}$; see the Supplemental Material (SM).

A key feature of this perturbative approach is that it can describe both the rapid onset of pre-thermalization and the slower process of thermalization. As pre-thermalization builds up on a $\lambda$-independent timescale, the state changes abruptly but the conserved densities only receive small corrections of order $\lambda^2$, as follows from Eq. (1). As a result the pre-thermalized state is non-thermal, and is in fact close to a new GGE for the unperturbed Hamiltonian $H_0$. Afterwards, the dynamics occurs over timescales of order $1/\lambda^2$. We will refer to this as the Boltzmann regime. It is accessed by the formal $t \to \infty$ limit of Eq. (1), with $\epsilon = \lambda^2 t$ held fixed. In this limit, the unperturbed energy density is stationary, while the $t$-derivatives of other observables take finite, non-zero values, which satisfy the GGE equations of state. Proofs of these statements can be found in [40, 49]. Thus in the Boltzmann regime, the GGE continues to evolve slowly with time. The final stationary regime is expected to occur for $t \gg 1/\lambda^2$, which requires going beyond the perturbative result (1); see [31]. Nonetheless, for weakly broken integrability, the Boltzmann regime is very long in comparison with experimental timescales. Moreover, its physical properties are fully accessible using integrability as we now demonstrate.

**Form Factors.**— As the right-hand side of (1) involves time-evolution under the integrable Hamiltonian $H_0$, powerful techniques are available for its evaluation. The principal idea is that the matrix elements of the perturbing operator $v$ can be computed by means of a spectral decomposition, in terms of a suitable basis of eigenstates of $H_0$. For example, the initial GGE can be represented by a state $|\rho_0\rangle$, with $Z^{-1} \text{Tr} (\rho \Theta) = \langle \rho_0 | \Theta | \rho_0 \rangle$. Here, the quasiparticle density $\rho_0(\theta)$, as a function of the rapidity $\theta$, is fixed by the thermodynamic Bethe ansatz (TBA) [50–52]. Excited states $|\rho_1, \rho_2, \rho_3 \rangle$ involve particle and hole excitations on top of this [53–58], where $\rho$ and $\lambda$ indicate their respective sets of rapidities. These diagonalize the momentum $Q_1$, energy $Q_2$, and other conserved quantities $Q_i$, with one-particle eigenvalues given by $\kappa(\theta)$, $\varepsilon(\theta)$ and $\eta(\theta)$ respectively. Performing the spectral decomposition on Eq. (1) yields

\[ \partial_t \langle q_i(0,t) \rangle = 2 \int d\varphi \delta(\kappa) \sin \frac{\varphi}{\varepsilon} |\langle \rho_0 | \rho_1, \rho_2, \rho_3 \rangle|^2, \quad (2) \]

as shown in the SM. The integrand $d\varphi = d\rho \rho_n(\rho)$ includes the factor $\rho_n(\rho) = \prod_{\theta \in \rho} \rho_n(\theta)$. This describes the accessible ‘phase space’ given by the density of holes $\rho_n(\theta)$, and likewise for $\rho_\rho(h)$. Here $\kappa = \sum_{\rho \in \rho} \kappa(\theta) - \sum_{\gamma \in \gamma} \kappa(\gamma)$, and similarly for $\varepsilon$ and $\eta$.

The expression (2) has a simple interpretation: in accordance with [59, 60], particles and holes are in and out states of scattering processes. The change in the charge density $\langle q_i(0,t) \rangle$ is given by a weighted sum over all the momentum conserving processes, with transition rates given by the form factors squared $|\langle \rho_0 | \rho_1, \rho_2, \rho_3 \rangle|^2$ of the perturbing operator, in conformity with Fermi’s golden rule. By evaluating these matrix elements one can obtain a quantitative description of the thermalization process, from short to long timescales.

**Pre-thermalization.**— The form factor approach gives a quantitative approach to pre-thermalization which is consistent with previous results. For example, after an interaction quench, the charge densities undergo fast initial dynamics, followed by an oscillatory power-law approach to a quasi-stationary regime which persists for long times. This can be verified by applying a small $\phi^4$ perturbation to a free massive scalar field, whose form factors can be evaluated using the methods of [55]. The results are provided in the SM; similar numerical results are obtained in [61].

**Boltzmann Regime.**— After pre-thermalization, the approximate GGE continues to evolve in accordance with Eq. (1). In the Boltzmann regime the time-evolution of the state is slow, varying over long timescales of order $1/\lambda^2$. As such, the change in the state can be large, with the power-law tails describing the approach to the instantaneous GGE giving perturbatively small corrections. In this regime, the evolution is towards an (approximate) instantaneous GGE giving perturbatively small corrections. In the SM, a general H-theorem shows that (3) is consistent with thermalization.
The spectral decomposition (2) available for integrable systems allows us to recast (3) as a Boltzmann-type kinetic equation. This sums over energy and momentum conserving scattering processes with arbitrary numbers of particles. This generalizes approaches based on the kinetics of free models [16, 34, 62–69], to interacting integrable systems. Re-expressing (3) in terms of the time-dependent quasiparticle density \( \rho_p(\theta) \), which represents the time-evolving GGE (see the SM), one obtains
\[
\partial_t \rho_p(\theta) = I[\rho_p](\theta) := 
\int dp dh \, K(\theta, p) B(p \rightarrow h) [\rho_h(p) \rho_p(h) - \rho_p(p) \rho_h(h)],
\]
where
\[
B(p \rightarrow h) = 2\pi \delta(\kappa) \delta(\varepsilon) |\langle \rho_p | v | \rho_h; p, h \rangle|^2 = B(h \rightarrow p)
\]
is the matrix element for particle-hole scattering processes. In the special case of perturbations of free models, \( K(\theta, p) = \sum_{\Phi \in p} \delta(\theta - \Phi) \) and we have a generalization of the quantum Boltzmann equation to include higher-order scattering processes. If the unperturbed Hamiltonian \( H_0 \) is interacting, then \( K \) also describes the effect of indirect processes where a particle of rapidity \( \theta \) is created or destroyed in the interacting background in response to a scattering event. In this case
\[
K(\theta, p) = \sum_{\Phi \in p} K(\theta, \Phi),
\]
where
\[
K(\theta, \Phi) = \delta(\theta - \Phi) + \frac{\partial}{\partial \Phi} \left[ \frac{F(\theta, \Phi) \rho_p(\Phi)}{\rho_p(\Phi) + \rho_h(\Phi)} \right].
\]
Here, \( F(\theta, \Phi) \) is the backflow function representing the effect of adding an excitation to the interacting background; see the SM. Here we assume particle-hole symmetry, in accordance with the usual microscopic reversibility condition of the Boltzmann scattering kernel (5). We show in the SM that an arbitrary boosted thermal state is a fixed point of the time-evolution given in (4), confirming the general H-theorem presented there. We note finally that (4) is an expansion in the number of excitations, which can often be recast as a low-density expansion. This is analogous to the LeClair-Mussardo series for equilibrium expectation values [70], which is observed to converge quickly.

**Multiparticle Scattering.**— The kinetic equation (4) generically contains infinitely-many scattering processes with arbitrarily large numbers of particles \( p \rightarrow h \). In the absence of internal degrees of freedom, the \( 2 \rightarrow 2 \) scattering processes do not contribute: these preserve momenta by 1+1-dimensional kinematics, hence the term in square brackets in (4) vanishes. This is consistent with the notion that thermalization requires the non-trivial rearrangement of momenta. In generic integrable models, the higher-particle form factors are typically non-zero, thereby leading to thermalization via (4). The \( \phi^4 \) theory considered above is special, as these higher-particle form factors vanish. As such, it does not thermalize in the Boltzmann regime in 1+1 dimensions, in agreement with three-loop results for correlation functions [71, 72]. For the \( \phi^6 \) perturbation, the \( 2 \rightarrow 4 \) and \( 3 \rightarrow 3 \) processes contribute. In Fig. 1 we show the time-evolution of the rapidity distribution \( n(\theta) \) for the free scalar field theory with unit mass perturbed by \( \lambda/(\delta !) \int dx \phi^\lambda(x) \), following a quantum quench from \( \lambda = 0 \) to \( \lambda > 0 \). The initial distribution is the GGE with \( \beta_2 = \beta_6 = 0.5, \beta_4 = 0.1, \) all other \( \beta_i = 0 \). At late times \( n(\theta) \) approaches a thermal distribution as indicated by the gray solid line. Lower panel: time-evolution of the first three effective inverse temperatures for the same quench, showing a non-monotonic approach to thermalization. The large values of \( t \) reflects the standard normalization conventions for the scalar field theory, which effectively reduces the strength of the \( \phi^6 \) perturbation.
In order to expose the relevant physics, we have concentrated for simplicity on a perturbation of a free model. However, an important aspect of this work is that it applies equally well to the case of an interacting integrable model. As an example, we consider the experimentally relevant case of two Lieb-Liniger gases perturbed by a density-density coupling. We consider arbitrary interaction strengths in the low-density regime. In this case, the two degrees of freedom allow for a non-trivial $2 \to 2$ contribution in the Boltzmann regime, for further details see the SM.

**Nearly-integrable Perturbations.**— Perturbations that break integrability yet do not lead to thermalization in the Boltzmann regime can be seen as “nearly-integrable perturbations”, in analogy with the concept from KAM theory [26]. The $\phi^4$ perturbation of the free massive scalar field discussed above is such an example. We show that such perturbations exist generically. To see this, we re-write the time-evolution (3) as

$$\partial_t \langle q_i \rangle = ([v, Q_i], v),$$

where $(a, b)$ is a suitable inner product [73], defined by

$$(a, b) = \int dt dx \left\langle (1 - \mathcal{P}) [a^0(x, t)]^\dagger b(0, 0) \right\rangle^c.$$  \hspace{2cm} (9)

Here, $a^0(x, t) = e^{i H_0 t} a(x) e^{-i H_0 t}$ and $\mathcal{P}$ is the projector onto the space of charges $Q_i$; see the SM. We show in the SM that current operators $j_k$, satisfying $\partial_t q_k + \partial_x j_k = 0$, commute with the conserved charges under the inner product: $(j_k, Q_i)_a = 0$ for all $a, i$. According to (8), under a perturbation $v = j_k$, the state remains constant throughout the Boltzmann regime. Therefore current operators are nearly-integrable perturbations. This extends the notion of perturbed integrable models which preserve integrability in equilibrium [74–76, 78–80]. For example, there exist families of integrable models, $H = H_0 + V_{\lambda}$, which correspond to perturbations by current operators, $V_{\lambda} = \lambda \int dx \, j_k(x) + O(\lambda^2)$, at leading order [81]. A similar relationship holds between the sine-Gordon model [82] and the $\phi^4$ perturbation of the scalar field. The observation here is that thermalization is absent at leading order, despite these models not being integrable.

The discussion above gives a natural classification of perturbations, and an associated classification of scattering processes. Indeed, under the inner product (9), local operators form a Hilbert space $\mathcal{H}^0$ [73]. This admits an orthogonal decomposition $\mathcal{H}^0 = \mathcal{H}_N \oplus \mathcal{H}_B$, where $\mathcal{H}_N$ is the nearly-integrable subsector that commutes with $Q_i$ within $\mathcal{H}^0$, and $\mathcal{H}_B$ is the thermalizing Boltzmann subsector. In the kinetic description, operators in $\mathcal{H}_N$ only couple to $2 \to 2$ scattering processes. These, as explained above, do not lead to thermalization. It was shown in [59, 60] that such processes lead to hydrodynamic diffusion instead, as they fully determine the Onsager matrix, $\mathcal{L}_{ij} = (j_i, j_j)$ [73, 77]. Thus, there is a separation between processes leading to hydrodynamic diffusion, associated with $\mathcal{H}_N$, and those leading to thermalization, associated with $\mathcal{H}_B$.

**Thermalization and Entropy Production.**— The late time dynamics near the final, stationary state is obtained by linearizing the evolution operator [83]. In terms of the inverse effective temperatures $\beta_i$, this gives

$$\sum_j C_{ij} \partial_t \beta_j = - \sum_j \mathcal{B}_{ij} \beta_j;$$  \hspace{2cm} (10)

see the SM. Here we define the Boltzmann matrix $\mathcal{B}_{ij} = \langle [v, Q_i], [v, Q_j] \rangle$, while the static covariance matrix is $C_{ij} = \partial \langle q_j \rangle / \partial \beta_i$; both are non-negative and evaluated in the stationary state. A similar evolution equation also holds for the small deviations of the conserved densities $\delta q_i = \langle q_i \rangle - \langle q_i \rangle_s$. As $\mathcal{B}_{ij} = 0$ for either $i, j = 1$ or 2 the spectrum of $\Gamma = \mathcal{B}^{-1}$ always contains the eigenvalue 0, corresponding to the conserved modes of the Boltzmann dynamics. The rest of the spectrum controls the rate of approach to thermalization: if it extends continuously to 0 then the approach is polynomial, whereas if there is a gap of size $\gamma > 0$, it is exponential $\delta q_i \propto e^{-\gamma T}$ with $T = \lambda^2 \gamma^{-1}$ [83–85]. It is notable that the timescale $\tau$ is solely determined by the final state, with the conserved energy and momentum densities containing the only information about the initial state.

In Fig. 2, we show numerical results consistent with an exponential approach to thermalization for the $\phi^6$ perturbation. Therefore, for the $\phi^6$ perturbation, the spectrum of the Boltzmann matrix has a gap $\gamma > 0$. At high temperatures we find an increasing thermalization timescale $\tau \sim T^\alpha$ with $\alpha \approx 3/2$, corresponding to an effectively gapless regime. In contrast, at low temperatures, we observe Arrhenius behavior with $\tau \sim e^{3m/T}$, corresponding to the 3-body collisions in the $\phi^6$ theory; see SM.

The Boltzmann matrix determines the late time dynamics of all physical quantities. Notably, the production of entropy near the final stationary state takes the form

$$\partial_t s = \sum_{i,j \geq 3} \beta_i \mathcal{B}_{ij} \beta_j = ([v, \log \rho], [v, \log \rho]),$$

where $\log \rho = -\sum_i \beta_i Q_i$ is the entropy operator; see the SM. As the right-hand side in (11) is quadratic in the $\beta_i$, if there is a gap $\gamma$, the time-evolution of the entropy is also exponential, but with a rate $2\gamma$. This is twice that found in the time-evolution of the inverse temperatures and charge densities, which we confirm in Fig. 2.

Exponential decay can also be seen in correlation functions, as they are determined at large times by the conserved quantities. By projection methods, two-point functions at scaled wave numbers $k = k/\lambda^2$ in the final state behave as

$$\langle \Theta_1 \Theta_2 \rangle^\tau (\bar{k}, \bar{t}) =$$

$$- \sum_{ij} \partial_{\langle q_j \rangle} \langle \Theta_1 \rangle \exp \left[ iA_{ij} \bar{k}^2 - \Gamma \bar{t} \right]_{ij} \partial_{\bar{x}_j} \langle \Theta_2 \rangle$$

(12)

where the matrix $A_{ij} = \partial_{\langle q_j \rangle} \langle j_i \rangle$ encodes the propagation of the conserved modes, and $\Gamma$ their decay. In particular,
this gives the Lorentzian broadening of the Drude peaks associated with the broken charges, \( \int d\xi e^{i\omega \xi} \langle j_{i,jk} \rangle (k = 0, \ell) = 2 [\Lambda (\Gamma^2 + \bar{\omega}^2)^{-1} \Gamma AC]_{ik} \), see also [42]. We observe that the singularity in the complex \( \omega \)-plane that is nearest to the real line is at a distance \( \gamma \). Dynamical correlation functions in the thermal state therefore determine the rate of approach towards it. A similar situation also occurs in holographic models, where the eigenvalues of the Boltzmann matrix are analogous to quasi-normal modes, see for example [86]. As a signature of the integrability of the unperturbed model, this singularity is expected to be a branch point, because of the continuum of hydrodynamic modes parametrized by the rapidity \( \theta \).

**Hydrodynamics.**— The kinetic approach developed here is applicable beyond quenches from homogeneous states, to include integrability destroying perturbations in the hydrodynamic description of integrable models [13, 14]. In this context, the effects of integrability breaking on the diffusive scale were recently discussed in [42]. Here, we stress that the effects of weak perturbations are also manifest on the larger, Euler scale. In the Euler scaling limit \( x, t \to \infty, \lambda \to 0 \) with \( \bar{t} = \lambda^2 t \) and \( \bar{x} = \lambda^2 x \) held fixed, the entropy increase of local fluid cells occurs on Euler hydrodynamic timescales. The spectral decomposition (4) in the Boltzmann regime adds a generalised collision term \( I(\theta) \) to the fluid equations, \( \partial_t \rho_\theta(\theta) + \partial_\theta (\nu_{\text{eff}}(\theta) \rho_\theta(\theta)) = I(\theta) \), where \( \nu_{\text{eff}} \) is given in [13, 14]. This opens the door to future studies of the crossover from integrable to non-integrable hydrodynamics, including the emergence of shocks, which are absent in the former case [87–89].

**Conclusions.**— In this work we have developed a form factor approach to perturbed integrable models out of equilibrium. We have shown that one can address a broad range of timescales, including the approach to thermalization. We have provided analytical and numerical predictions for the time-evolution of physical observables, including conserved charges, effective temperatures, and rapidity distributions. We observe that the rate of thermalization for entropy is always exactly twice as large as that for conserved charges. We have also shown that there always exists a families of perturbations that do not thermalize in the weakly perturbed regime. It would be interesting to verify these predictions in experiment.

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\section{Time-Evolution of the Charge Densities}

We here derive a perturbative equation of motion for the densities of conserved charges, showing in particular that the leading term occurs in second order perturbation theory. We consider the Hamiltonian $H = H_0 + \lambda V$, composed of an integrable contribution $H_0$ and a homogeneous integrability-breaking perturbation $V = \int dx \, v(x)$. The conserved charges $Q_i = \int dx \, q_i(x)$ of the unperturbed Hamiltonian satisfy $[H_0, Q_i] = 0$. We further assume that they mutually commute, $[Q_i, Q_j] = 0$ for all $i, j$, and define $Q_1 = P$ and $Q_2 = H_0$ as in the main text. Expanding the time-evolved charge densities $q_i(x, t) = e^{iH_0 t} q_i(x) e^{-iH_0 t}$ to first order in $\lambda$, perturbation theory gives

$$q_i(x, t) = q_i^0(x, t) + i\lambda \int_0^t ds \left[ V^0(s), q_i^0(x, t) \right] + O(\lambda^2),$$

where the superscript 0 indicates time-evolution with respect to $H_0$, i.e. $q_i^0(x, t) = e^{iH_0 t} q_i(x) e^{-iH_0 t}$ and $V^0(s) = e^{iH_0 s} V e^{-iH_0 s}$. We evaluate this in the initial state with density matrix $\rho_0 = Z^{-1} e^{-\sum_i \beta_i Q_i}$.

$$[\rho_0, Q_j] = 0 \quad \forall j.$$  \hspace{1cm} (2)

This follows from the identity $[Q_i, Q_j] = 0$; taking $j = 1$ implies spatial homogeneity of the state, and $j \neq 1$ implies invariance of the state under the action of the associated charge $Q_j$. The Heisenberg equation of motion $\partial_t q_i(x, t) = i[H, q_i(x, t)]$ then gives

$$\partial_t \langle q_i(0, t) \rangle = i\lambda \langle [V, q_i^0(0, t)] \rangle - \lambda^2 \int_0^t ds \langle [V, [V^0(s), q_i^0(0, t)]] \rangle + O(\lambda^3),$$  \hspace{1cm} (3)

where spatial homogeneity allows evaluation at $x = 0$ without loss of generality. The first term on the right-hand side can be shown to vanish by repeated application of Eq. (2) for different values of $j$:

$$\langle [V, q_i^0(0, t)] \rangle = \int dx \langle [v(x), q_i^0(0, t)] \rangle = \int dx \langle [v, q_i^0(-x, t)] \rangle = \langle [v, Q_i] \rangle = 0.$$ \hspace{1cm} (4)

Similar manipulations allow the second term to be written in a more convenient form

$$\partial_t \langle q_i(0, t) \rangle = -\lambda^2 \int dx \int_0^t ds \langle [v(0), [v^0(x, s), Q_i]] \rangle$$

$$= \lambda^2 \int dx \int_{-t}^t ds \langle [v^0(x, s), Q_i] \rangle v(0) \rangle^c$$

$$= \lambda^2 \int_{-t}^t ds \langle [V^0(s), Q_i] \rangle v(0) \rangle^c,$$ \hspace{1cm} (5)

which corresponds to Eq. (1) in the main text. In the second line we have replaced the correlation function with its connected part,

$$\langle [v^0(x, s), Q_i] \rangle v(0) \rangle^c = \langle [v^0(x, s), Q_i] \rangle v(0) \rangle - \langle [v^0(x, s), Q_i] \rangle \langle v(0) \rangle,$$ \hspace{1cm} (6)

which is possible as $\langle [v^0(x, s), Q_i] \rangle \langle v(0) \rangle = 0$ by Eq. (2). This makes the convergence of the integral in $x$ immediate by the clustering properties of the state.
2 $\phi^p$ Perturbation of the Free Scalar Field

In this section we provide further details on the results presented in the main text for pre-thermalization and thermalization in a weakly-interacting bosonic field. For $\phi^p$ perturbations of the free scalar field the form factor expansion for the charge evolution truncates at finite order; for a polynomial interaction $\phi^p$ the form factors with more than $p$ particles/holes are vanishing. We consider the unperturbed Hamiltonian
\[
H_0 = \frac{1}{2} \int dx \left[ \pi^2 + (\nabla \phi)^2 + \phi^2 \right],
\]
where the normal ordering $\Theta$ is made with respect to the vacuum, and for simplicity we set the mass to 1. The Bose distribution functions associated with the GGE density matrix are
\[
n(\theta) = \frac{1}{e^{W(\theta)} - 1}, \quad \bar{n}(\theta) = 1 + n(\theta),
\]
where
\[
W(\theta) = \sum_{n=1}^{\infty} (\beta_{2n-1}\sinh(n\theta) + \beta_{2n}\cosh(n\theta)).
\]

The quasi-particle density is $\rho_{p}(\theta) = \cosh(\theta)n(\theta)/(2\pi)$, as given in the main text. The form factor expansion for correlation functions can be obtained by the methods of Ref. [1, 2, 3], by application of Wick’s theorem for bosonic commutation relations. Taking the perturbation $V = \lambda/(4!) \int dx \phi^4(x)$, only the 2 and 4 particle terms contribute to the equation of motion for the charge densities. For the numerical data presented the 4-particle terms were dominant. This yields the following evolution equation in the prethermal regime
\[
\partial_t q_i(0, t) = \frac{2}{4!} \left( \frac{\lambda}{24\pi^2} \right)^2 \int d\theta_1 d\theta_2 d\theta_3 d\theta_4 \delta(\kappa) \frac{\sin(\varepsilon t)}{\varepsilon} \\
\left[ \bar{n}_1 \bar{n}_2 \bar{n}_3 \bar{n}_4 \left( h_i(\theta_1) + h_i(\theta_2) + h_i(\theta_3) + h_i(\theta_4) \right) + 16\bar{n}_1 \bar{n}_2 \bar{n}_3 n_4 \left( h_i(\theta_1) + h_i(\theta_2) + h_i(\theta_3) - h_i(\theta_4) \right) \right.
\]
\[
+ 36\bar{n}_1 \bar{n}_2 n_3 \bar{n}_4 \left( h_i(\theta_1) + h_i(\theta_2) - h_i(\theta_3) - h_i(\theta_4) \right) + 16n_1 n_2 n_3 \bar{n}_4 \left( h_i(\theta_1) - h_i(\theta_2) - h_i(\theta_3) - h_i(\theta_4) \right) \right]
\]
\[
+ n_1 n_2 n_3 n_4 \left( -h_i(\theta_1) - h_i(\theta_2) - h_i(\theta_3) - h_i(\theta_4) \right),
\]
where $n_i = n(\theta_i)$ and $\bar{n}_i = \bar{n}(\theta_i)$. As shown in Figs. 1 and 2, pre-thermalization in the $\phi^4$ theory is seen to consist of a fast increase in the value of charge densities, followed by oscillations within a decaying envelope to the long-time pre-thermal value. Numerically, the envelope is found to decay as $t^{-\alpha}$ with $\alpha = 1.0 \pm 0.2$ for all observables and initial states studied; see Fig. 2 for an illustration. The numerical solution of Eq. (10) was obtained using the numerical integration routines in Mathematica.

In order to go beyond pre-thermalization, we turn our attention to the $\phi^6$ theory with $V = \lambda/(6!) \int dx \phi^6(x)$. As we discuss in the main text, this has a non-trivial Boltzmann regime. Here the only terms which contribute to the kinetic equation occur at the 6-particle/hole level
\[
\partial_t n(\theta) = \frac{1}{6! \cosh \theta} \left( \frac{1}{24\pi^2} \right)^2 \int d\theta_1 d\theta_2 d\theta_3 d\theta_4 d\theta_5 \delta(\kappa) \delta(\varepsilon) \\
\left[ 1200 \left( \bar{n}(\theta) \bar{n}_1 \bar{n}_2 n_3 n_4 n_5 - n(\theta) n_1 n_2 \bar{n}_3 \bar{n}_4 \bar{n}_5 \right) + 900 \left( \bar{n}(\theta) \bar{n}_1 \bar{n}_2 n_3 n_4 n_5 - n(\theta) n_1 n_2 n_3 \bar{n}_4 \bar{n}_5 \right) \right.
\]
\[
+ 450 \left( \bar{n}(\theta) \bar{n}_1 n_2 n_3 n_4 n_5 - n(\theta) n_1 \bar{n}_2 \bar{n}_3 \bar{n}_4 \bar{n}_5 \right). \]
Figure 1: Change in the particle and energy densities $q_0$ and $q_2$, respectively, for the free scalar field theory (with unit mass) perturbed by $\lambda/(4!) \int dx \phi^4(x)$, following a quantum quench from $\lambda = 0$ to $\lambda > 0$. The initial state is taken as the vacuum state (main panel), and a thermal state with $\beta = 5$ (inset). In both cases, the charge densities settle down to constant values corresponding to pre-thermalization.

Figure 2: Time-evolution of the particle and energy densities $n = q_0$ and $q_2$ after a $\lambda/(4!)\phi^4$ quench of a free scalar field with unit mass, with a thermal initial state at inverse temperature $\beta = 5$. Oscillations are observed within an envelope that is well described by a power law $t^{-1}$. 
Figure 3: (a): Exponential decay of the rate of approach to thermal equilibrium at low temperatures $\beta \gg 1$, showing the relationship $\lambda^2 \gamma \sim e^{-3\beta}$. (b): Polynomial decay of the rate of approach to thermal equilibrium at high temperatures $T \gg 1$, showing the relationship $\lambda^2 \gamma \sim T^{-\alpha}$ where $\alpha = 1.498 \pm 0.003$.

Numerical solution of this equation gives the results presented in Figs (1) and (2) in the main text. As shown in Fig 3, it is also used to find the dependence of the rate of approach to equilibrium $\gamma$ on the temperature $T$ of the final state.

In order to solve the generalized Boltzmann equation (11) numerically, we implement the energy and momentum delta functions analytically, facilitated by the change of variables $k_i = \sinh(\theta_i)$. We take a discrete set of values of $\theta$ and interpolate between $n(\theta)$ at these points to obtain the full function $n(\theta)$, as fast decay of $n(\theta)$ at large $|\theta|$ (cf. equation (9)) means that we can restrict the infinite integrals over rapidity to some suitable finite region $\Lambda$. We then solve using the forward Euler method in $t$, and evaluate the remaining triple integrals by Monte-Carlo methods. We estimate the errors at $\sim 0.1\%$ using the ratio of the standard deviation to the mean of the energy $\sigma_q/\bar{q}$ over the simulation run-time. Energy fluctuations occur as a result of the Monte-Carlo methods used, and linear interpolation between lattice points in evaluating integrals involving $n(\theta)$. Uncertainty in the fitting parameters is found to be of this order of magnitude, except for the entropy decay rate which is very sensitive to energy fluctuations (cf. Eq. (30)), and therefore the uncertainty is an order of magnitude larger.

3 Coupled Lieb-Liniger Gases

The main results of the paper apply to perturbations of interacting integrable systems as well as perturbations of free theories; the derivation is presented in Section 4 below. In order to illustrate this, let us consider a perturbed Lieb-Liniger system with the Hamiltonian

$$H = \int dx \left( -\sum_{i=a,b} \psi_i^\dagger \psi_i^2 + \sum_{i=a,b} 2c_i \psi_i^\dagger \psi_i^\dagger \psi_i \psi_i + \lambda \psi_a^\dagger \psi_b \psi_a^\dagger \psi_b \right).$$

This is a case where the system contains multiple internal degrees of freedom. This can be accommodated in the main results by allowing the rapidity to carry an additional index denoting the relevant degree of freedom.

The system (12) has two integrable points, at $\lambda = 0$ where we have a two-species Lieb-Liniger model, and at $\lambda = 4c_a = 4c_b$ when it becomes the Yang-Gaudin model. We will consider the case of a small $\lambda$ perturbation of the former integrable point. As the system has multiple particle species, $Q_i$ carries a
The spectral expansion is over product states $|\Omega\rangle = |\Omega\rangle_a \otimes |\Omega\rangle_b$. The relevant thermodynamic form factors, corresponding to the density operator in the Lieb-Liniger model, are known exactly [4].

In this model, the form factor expansion in excitation numbers can be recast as a low-density expansion, in powers of the density $D = \int d\theta \rho_p(\theta)$ of the gas. In particular, the phase space factors appearing in (4) of the main text are dominant for smaller numbers of particle-hole excitations. At low enough densities the number of excitations is typically small, by analogy with the Leclair-Mussardo formula for equilibrium expectation values [5].

We consider the lowest order contribution in the density for arbitrary Lieb-Liniger couplings $c_i$. At this order, only the two particle-hole terms remain; these do not vanish due to the presence of multiple particle species. The results are obtained by taking the zero-density limit of the density operator form factors. Eliminating the energy and momentum delta functions, we find the following kinetic equation

$$
\partial_t \rho^a_p(p) = \frac{(2\pi)^2}{16} \int \frac{dh}{|p-h|} \frac{\phi^2_a(p-h) \phi^2_b(p-h)}{(p-h)^4} \left( (p-h)^2 + c_a^2 \right) \left( (p-h)^2 + c_b^2 \right) \rho^a_p(h) \rho^b_p(p) - \rho^a_p(p) \rho^b_p(h) - \partial_t \rho^b_p(p),
$$

where the scattering phase $\phi_i(p) = 2\arctan(p/c_i)$. In this case the scattering only involves exchange of rapidities between the two gases, and indirect scattering is absent. We recall that indirect scattering processes are interpreted as processes whereby a particle of rapidity $\theta$ is created or removed by a change in the interacting background induced by scattering processes.

In Fig 4 we show the time-evolution of the distribution functions in the Boltzmann regime, where we take GGE initial conditions with $c_a = 1, c_b = 5$. The numerical solution to (13) was obtained in a similar way to (11), as outlined in section 2. We observe an exponential rate of approach to the equilibrium value, where the rate of approach for the entropy is twice that for the charges, as predicted by the theory in the main text.
4 Thermodynamic Spectral Expansion

We now provide more details on the spectral expansion leading to the scattering-like Eqs (2) and (4) in the main text. The right-hand side of equation (5) is a two-point function, allowing the spectral decomposition

$$\partial_{\lambda^{2}t} \langle q_{i}(0, t) \rangle = \int_{-\infty}^{t} ds \sum_{\Omega} \langle \rho_{p} | V^{0}(s) , Q_{i} | \Omega \rangle \langle \Omega | v(0, 0) | \rho_{p} \rangle,$$

where $\Omega$ labels a complete set of states. The state $| \rho_{p} \rangle$ can be given explicitly in free models [1, 2, 3], and using the equivalence of the microcanonical and canonical ensembles it can be interpreted in more general integrable systems as a representative Bethe state $| \rho_{p} \rangle$ for the GGE $Z^{-1} \text{Tr}(\rho \Theta) = \langle \rho_{p} | \Theta | \rho_{p} \rangle$.

The quasiparticle density function $\rho_{p}(\theta)$ is fixed by the thermodynamic Bethe ansatz (TBA) [7, 8, 9]. In particular, it is related to the state density $\rho_{s}(\theta)$ by the integral equation

$$2\pi \rho_{s}(\theta) = \partial_{\theta} p(\theta) + \int d\gamma \partial_{\theta} \phi(\theta - \gamma) \rho_{p}(\gamma),$$

where $\phi(\gamma) = -i \log S(\gamma)$ is the scattering phase, and $S(\theta_{1} - \theta_{2})$ is the two-body scattering matrix element. Particle and hole excitations on top of the state $\rho$ are denoted $| \rho_{p} ; p, h \rangle$, where $p$ (resp. $h$) are sets of particle (resp. hole) rapidities, and the associated form factors in interacting integrable models have been studied [10, 11, 12]. These states diagonalize the operators $Q = (P, H, \{ Q_{\geq 3} \})$ with one-particle eigenvalues $\eta = (\kappa(\theta), \varepsilon(\theta), \{ \eta_{\geq 3}(\theta) \})$. There is a convenient expression for these eigenvalues in terms of the backflow function $F(\gamma | \theta)$ defined by

$$2\pi F(\gamma | \theta) = \phi(\gamma - \theta) + \int d\gamma' \partial_{\gamma} \phi(\gamma - \gamma') \frac{\rho_{p}(\gamma')}{\rho_{p}(\gamma)} F(\gamma' | \theta),$$

The eigenvalues of a single excitation are related to the free particle eigenvalues $h = (p(\theta), E(\theta), \{ h_{\geq 3}(\theta) \})$ by the relation (see e.g. [13])

$$\eta(\theta) = h(\theta) - \int d\gamma F(\gamma | \theta) \frac{\rho_{p}(\gamma)}{\rho_{p}(\gamma)} \partial_{\gamma} h(\gamma).$$

In addition, the eigenvalues of a state with many particle excitations $p$ and hole excitations $h$ are additive

$$\eta(p, h) = \eta(p) - \eta(h) = \sum_{\theta \in p} \eta(\theta) - \sum_{\theta \in h} \eta(\theta).$$

The spectral expansion of (14) can now be written explicitly in terms of particle and hole excitations

$$\partial_{\lambda^{2}t} \langle q_{i}(0, t) \rangle = 2 \int d\rho dh \eta_{i}(p, h) \delta(\kappa(p, h)) \frac{\sin(\varepsilon(p, h)t)}{\varepsilon(p, h)} | \langle \rho_{p} | v_{p} | p, h \rangle |^{2},$$

with the integration measure given by

$$d\rho dh = \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{N_{p}=0}^{N} \left[ \prod_{i=1}^{N_{p}} \rho_{h}(p_{i}) dp_{i} \right] \left[ \prod_{j=1}^{N-N_{p}} \rho_{p}(h_{j}) dh_{j} \right],$$

where we have split the sum over all $N$ particles involved into the number of particles $N_{p}$ and the number of holes $N_{h} = N - N_{p}$. In addition we have defined the hole density $\rho_{h} = \rho_{s} - \rho_{p}$.

In the Boltzmann regime, obtained by taking the limit $t \to \infty$, the GGE is re-established at all times. We may use $2 \sin(\varepsilon t)/\varepsilon \to 2 \pi \delta(\varepsilon)$, as well as the relation $\langle q_{i} \rangle = \int d\theta \rho_{p}(\theta) h_{i}(\theta)$ to take the functional
derivative of equation (19) and find the kinetic equation for the distribution function, as given by Eq. (4) in the main text, with the measure
\[
dp dh = \sum_{N=0}^{\infty} \frac{1}{N!} \prod_{i=1}^{N} dp_i \prod_{j=1}^{N-N_p} dh_j.
\]
Finally, we can also expand the matrix \( B \) in Eq. (10) of the main text in form factors, giving
\[
B_{ij} = \int dp_0 \sum_{h} x_{i,j}(p,h) \eta_i(p,h) \delta(\epsilon(p,h)) \delta(\epsilon(p,h)) |\langle \rho_0|v|\rho; p, h \rangle|^2,
\]
which can be used to determine the exponential decay of correlations in the final state by Eq. (12) of the main text.

5 An H-Theorem

We now show that the general formalism, independent of any spectral expansion, predicts entropy maximization with respect to the conserved quantities at long times. For a function which decays sufficiently rapidly at infinity we have
\[
\int_{-\infty}^{\infty} ds f(s) = \lim_{T,L \to \infty} \frac{1}{2T} \int_{-T}^{T} ds' f(s-s')\tag{23}
\]
Inserting this twice into Eq. (3) in the main text, for the time and space integrals, gives the following expression
\[
\partial_t \langle q_i \rangle_{\beta(\bar{t})} = \lim_{T,L \to \infty} \frac{1}{2} \int_{-T}^{T} ds' \int_{-T}^{T} dx dx' \left[ (v(x,s)Q_j v(x',s'))_{\beta(\bar{t})} - (Q_j v(x,s) v(x',s'))_{\beta(\bar{t})} \right].\tag{24}
\]
Here, the time-evolutions are with respect to the unperturbed Hamiltonian \( H_0 \). We consider fixed \( T \) and \( L \), and denote \( W^{T,L} = \sqrt{\frac{4 T L}{\nu}} \int_{-T}^{T} dt \int_{-L}^{L} dx v(x,t) \). In the GGE state with \( \rho(\bar{t}) = Z^{-1} \exp(-\sum_i \beta_i Q_i) \), we can expand in a complete basis of common eigenstates of the charges to obtain
\[
\partial_t \langle q_i \rangle_{\beta(\bar{t})} = \lim_{T,L \to \infty} \sum_{m,n} (\rho_m(\bar{t}) - \rho_n(\bar{t})) (Q_{i,n} - Q_{i,m}) |W_{mn}^{T,L}|^2.
\]
The von-Neumann entropy density \( s = (2L)^{-1} \langle \log \rho \rangle \) has the time-derivative
\[
\partial_t s(\bar{t}) = \sum_i \beta_i(\bar{t}) \partial_t \langle q_i \rangle_{\beta(\bar{t})}.\tag{26}
\]
Using the relation \( \sum_i \beta_i Q_{i,m} = -\log \rho_m - \log Z \) and the symmetry \( |W_{mn}^{T,L}| = |W_{nm}^{T,L}| \), we obtain
\[
\partial_t s(\bar{t}) = \lim_{T,L \to \infty} \sum_{m,n} (\rho_m - \rho_n) (\log \rho_m - \log \rho_n) |W_{mn}^{T,L}|^2 \geq 0,
\]
by monotonicity of the log function.

Equation (10) in the main text also follows from the intermediate result (25) if we consider \( \rho \) near the stationary point \( \rho^0 \propto \exp(-\beta_{\text{stat}}(H_0 - \nu_{\text{stat}} P)) \). Stationarity requires that \( \rho_{mn}^0 = \rho_{nm}^0 \) for any \( m, n \) where \( \lim_{T,L \to \infty} |W_{mn}^{T,L}|^2 \) is non-vanishing. Near the stationary state we therefore have
\[
\partial_t \langle q_i \rangle_{\beta(\bar{t})} = \lim_{T,L \to \infty} \sum_{m,n} \sum_j \beta_j \rho_{mn}^0 (Q_{j,n} - Q_{j,m}) (Q_{i,n} - Q_{i,m}) |W_{mn}^{T,L}|^2.
\]
where the linearised density matrix $\rho \propto \rho^0 \left(1 - \sum_j \beta_j Q_j \right)$ due to commutativity of the charges. One can show that this equation of motion is equivalent to

$$\partial_t \langle q_i \rangle B(t) = \sum_j \langle [v, Q_j] [v, Q_j] \rangle \beta_j,$$

(29)

with the inner product on the right hand side evaluated in the stationary state. Using the definition of $\langle q_i \rangle B(t)$ and the chain rule we find that $\dot{q}_i = -\sum_j C_{ij} \beta_j$, with $C_{ij} = \langle q_i Q_j \rangle$. The static charge correlation matrix, reproducing Eq. (10) in the main text. Using (26) we also recover the first equality of Eq. (11) in the main text.

$$\partial_t s = \sum_{i,j \geq 3} \beta_i \langle [v, Q_i] [v, Q_j] \rangle \beta_j,$$

(30)

where we use the fact that $[v, Q_1] = [v, Q_2] = [h_0, Q_1] = 0$ on $\mathcal{H}'$. Therefore, $[h_0 + \lambda v, Q_i], [h_0 + \lambda v, Q_j] = \lambda^2 [v, Q_i], [v, Q_j]$. Writing the operator $\log \rho = -\sum_i \beta_i Q_i - \log Z$, we thus have that the time derivative of the entropy has the suggestive expression, in the Boltzmann regime,

$$\partial_t s = \langle [h, \log \rho] [h, \log \rho] \rangle = \langle [h, \log \rho] \rangle^2,$$

(31)

where $h = h_0 + \lambda v$ is the perturbed Hamiltonian density, in agreement with Eq. (11) in the main text.

6. The Thermal State from TBA is Stationary

In the preceding section we showed that the long-time stationary state in the Boltzmann regime maximises entropy with respect to the conserved quantities of the perturbed system. We now wish to show directly from the kinetic equation (Eq. (4) in the main text),

$$\partial_t \rho_{p}(\theta) = \int dp dh K(\theta, p) B(p \rightarrow h) \left[ \rho_{p}(p) \rho_{p}(h) - \rho_{p}(p) \rho_{h}(h) \right],$$

(32)

that the stationary state is precisely the thermal state from TBA. Consider the last term in the square brackets, where we write $\rho_{p} = \rho_{p} n$ and $\rho_{h} = \rho_{p} (1 - n)$; a similar argument holds for statistics which are not fermionic. A sufficient condition for stationarity is that

$$\prod_{p \in p} n(p) \prod_{h \in h} (1 - n(h)) - \prod_{h \in h} n(h) \prod_{p \in p} (1 - n(p)) = 0; \forall p, h.$$  

(33)

Taking the logarithm, we can ensure this condition is satisfied for all sets $p$ and $h$ if

$$\log \left( \frac{n(\theta)}{1 - n(\theta)} \right) = \beta \epsilon(\theta) + \nu \kappa(\theta),$$

(34)

as here the energy and momentum delta functions contained in $B(p \rightarrow h)$ (see Eq. (5) in the main text) ensure the condition (33) is satisfied. We thus find

$$n(\theta) = \frac{1}{1 + e^{\epsilon(\theta)}}, \quad \hat{\epsilon}(\theta) = \beta \epsilon(\theta) + \nu \kappa(\theta).$$

(35)

The TBA boosted thermal state is $n(\theta) = (1 + e^{\epsilon(\theta)})^{-1}$, with $\epsilon(\theta)$ satisfying the integral equation

$$\epsilon(\theta) = \beta (E(\theta) + \nu p(\theta)) - \int \frac{d\gamma}{2\pi} \partial_\theta \phi(\theta - \gamma) \log \left( 1 + e^{-\epsilon(\gamma)} \right).$$

(36)

Differentiating this equation and integrating by parts we find the following relation:

$$\partial_\theta \epsilon(\theta) = \beta \partial_\theta (E(\theta) + \nu p(\theta)) + \int \frac{d\gamma}{2\pi} \partial_\theta \phi(\theta - \gamma) n(\gamma) \partial_\gamma \epsilon(\gamma),$$

(37)

where $u(\theta) = \beta E(\theta) + \nu p(\theta)$. This is equivalent to relation (17) for $\hat{\epsilon}[13]$, and thus $\epsilon(\theta) = \hat{\epsilon}(\theta)$ up to a constant. Assuming that this constant vanishes, we have the desired result.
Nearly-Integrable Perturbations

As discussed in the main text, if we take a current operator \( v(x) = j_k(x) \) as the perturbation, then it is nearly integrable in the sense that

\[
\lim_{t \to \infty} \partial_t \langle q_i(0, t) \rangle = 0 \quad \forall i.
\]

(38)

To prove this we first use the definition of the projection onto conserved quantities to write

\[
\int dx \mathbb{P}[v^0(x, t), Q_i] = \int dx \sum_{j,l} \langle v^0(x, t), Q_i \rangle C_{jl}^{-1} Q_l
\]

\[
= \sum_{j,l} \langle v^0(0, t), Q_i, Q_j \rangle C_{jl}^{-1} Q_l = 0,
\]

(39)

where \( C_{jl} = \langle q_j Q_l \rangle \) is the static charge correlation matrix. This immediately shows the equality between Eqs (3) and (8) in the main text. Using the fact that the GGE state should satisfy the cluster decomposition principle at large distances, and using the continuity equation, we also have

\[
V^0(t) = -\int dx \partial_x j_0^0(x, t) \quad \text{and} \quad \partial_t E_0^k(t) = \int dx \partial_x q_0^k(x, t) =: \partial_t E_0^k(t).
\]

(40)

By the results of Ref. [14] we have that \([E_0^k(t), Q_i] = \int dx \Theta_{k,i}^0(x, t)\) for some local operator \( \Theta_{k,i} \). Hence

\[
([v, Q_i], \Theta) = \lim_{T \to \infty} \int_{-T}^{T} dt \int dx \langle [j_0^0(x, t), Q_i] \Theta(0) \rangle^c
\]

\[
= \lim_{T \to \infty} \int_{-T}^{T} dt \langle [V^0(t), Q_i] \Theta(0) \rangle^c
\]

\[
= \lim_{T \to \infty} \langle [E_0^k(T) - E_0^k(-T), Q_i] \Theta(0) \rangle^c
\]

\[
= \lim_{T \to \infty} \int dx \langle [\Theta_{k,i}(x, T) - \Theta_{k,i}(x, -T)] \Theta(0) \rangle^c
\]

\[
= 0,
\]

(41)

for every local operator \( \Theta \). In the last step we used the hydrodynamic projection principle [15], which implies that the large positive and negative time limits are equal. Setting \( \Theta = j_k \) we obtain \( \partial_t \langle q_k \rangle = 0 \quad \forall i \).

As discussed in the main text, there exists an orthogonal decomposition

\[
\mathcal{H}'' = \mathcal{H}_N \oplus \mathcal{H}_B,
\]

(42)

into a nearly-integrable component \( \mathcal{H}_N \) and a thermalizing component \( \mathcal{H}_B \). This is obtained as follows. The subspace \( \mathcal{H}_N \subset \mathcal{H}'' \) is defined by the relation written in the main text:

\[
b \in \mathcal{H}_N \quad \text{iff} \quad ([b, Q_i], a) = 0 \quad \forall \quad a \in \mathcal{H}'' , \forall i.
\]

(43)

By invariance of the state with respect to the action of \( Q_i \) (that is, the fact that the density matrix commutes with \( Q_i \)), this implies

\[
b \in \mathcal{H}_N \quad \text{iff} \quad (b, [Q_i, a]) = 0 \quad \forall \quad a \in \mathcal{H}'' , \forall i.
\]

(44)
We define $\mathcal{H}_B = \{ [Q_i, a] : a \in \mathcal{H}'', i \}$ as the union of the images of all actions $[Q_i, \cdot]$ of the $Q_i$'s on $\mathcal{H}''$. We can then see that $\mathcal{H}_B$ is indeed orthogonal to $\mathcal{H}_N$. To make this argument mathematically rigorous requires consideration of the completions of these spaces. This can be done by considering the one-parameter unitary groups generated by the actions of the $Q_i$'s. Note finally that the subspace $\mathcal{H}_N$ is referred to as the “diffusive subspace”, and denoted $\mathcal{H}_\text{diff}$, in [15].

In summary, the operators that are responsible for the diffusive effects of integrable models are also those that, seen as perturbations, do not lead to thermalization at weak coupling. The rest, corresponding to the orthogonal complement, are perturbations leading to thermalization. This is the Hilbert space expression of the separation between diffusive and thermalizing processes.

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