Linking covariant and canonical LQG II: spin foam projector

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Abstract
In a seminal paper, Kaminski \textit{et al} for the first time extended the definition of spin foam models to arbitrary boundary graphs. This is a prerequisite in order to make contact to the canonical formulation of loop quantum gravity whose Hilbert space contains all these graphs. This makes it finally possible to investigate the question whether any of the presently considered spin foam models yields a rigging map for any of the presently defined Hamiltonian constraint operators. We postulate a rigging map by summing over all abstract spin foams with arbitrary but given boundary graphs. The states induced on the boundary of these spin foams can then be identified with elements in the gauge invariant Hilbert space \(H_0\) of the canonical theory. Of course, such a sum over all spin foams is potentially divergent and requires a regularization. Such a regularization can be obtained by introducing specific cut-offs and a weight for every single foam. Such a weight could be for example derived from a generalized formal group field theory allowing for arbitrary interaction terms. Since such a derivation is, however, technical involved we forgo to present a strict derivation and assume that there exist a weight satisfying certain natural axioms, most importantly a gluing property. These axioms are motivated by the requirement that spin foam amplitudes should define a rigging map (physical inner product) induced by the Hamiltonian constraint. In the analysis of the resulting object we are able to identify an elementary spin foam transfer matrix that allows to generate any finite foam as a finite power of the transfer matrix. It transpires that the sum over spin foams, as written, does not define a projector on the physical Hilbert space. This statement is independent of the concrete spin foam model and Hamiltonian constraint. However, the transfer matrix...
potentially contains the necessary ingredient in order to construct a proper rigging map in terms of a modified transfer matrix.

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(Some figures may appear in colour only in the online journal)

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1. Motivation

To quantize a field theory one can either choose a canonical approach, quantize the Hamiltonian and solve the Schrödinger equation, or a covariant one, which rests on the path integral description going back to Feynman’s famous PhD thesis [1]. In loop quantum gravity (LQG), a background independent quantization of general relativity, the canonical formulation [2, 3] originates from a reformulation of the ADM action [4] in terms of gauge connections by Ashtekar and Barbero [5] while the covariant or spin foam model [6–8], was initiated by Reisenberger’s and Rovelli’s ‘sum over histories’ [9]. In both approaches many technical and structural difficulties arise from the constrained nature of GR deeply rooted in the diffeomorphism invariance of the theory. Particularly, the non-polynomial Hamiltonian
constraint, although a quantization has been known for a while (see [10]), is challenging and up to today the physical Hilbert space $\mathcal{H}_{\text{phys}}$ cannot be determined satisfactorily. On the other hand, spin foam models suffer from second class constraint which cannot be implemented strongly. The covariant model has matured a lot but the correct treatment of the constraints is still under debate (see e.g. [12]).

Even though both approaches differ significantly it was often emphasized in the past that they should converge to the same theory. Heuristically, the discrete time-evolution of a spin network on a spatial hypersurface, which defines a basis state in the gauge invariant kinematical Hilbert space of canonical LQG, leads to a colored 2-complex that is the main building block of spin foams. Therefore the partition functions defined by the latter can be either understood as propagator between two three-dimensional (3D) geometries or as a rigging map, a generalized projector onto $\mathcal{H}_{\text{phys}}$. This paper will especially focus on the latter train of thoughts.

The subsequent analysis will be mainly based on [23] (EPRL-model) and [15] (KKL-model). Closely related to these is the FK-approach [28]. The boundary space of the EPRL/KKL-model can be formally identified with subspaces of $\mathcal{H}_0$ which will be used here in order to define a spin foam operator $\hat{Z}[\kappa]$ for the canonical theory. Even if the operators $\hat{Z}[\kappa]$ are equipped with appropriate properties so that the sum $\sum \hat{Z}[\kappa]$ has a chance to define a projector into $\mathcal{H}_{\text{phys}}$, the object we obtain does not provide a rigging map. This conclusion is independent of the details of a spin foam model or of a Hamiltonian constraint. To prove this a method is developed to split the operator into smaller building blocks. This splitting procedure is also interesting from a purely technical point of view since it gives a better handle on the sum over all complexes $\kappa$ in the EPRL/KKL-partition function. On the positive side, the splitting property just mentioned allows to extract a spin foam transfer matrix which, if proper regularized, defines a modified transfer matrix that potentially yields a proper rigging map.

The paper is organized as follows. In section 2 the construction of spin foam models is briefly reviewed. For the reader not familiar with the covariant framework we provide a more detailed account in the appendix. Furthermore, a relation between foams and triangulations is proven.

In section 3.1 a general framework for merging both theories will be developed guided by the concept of rigging maps or group averaging methods for simpler constrained systems. On this basis, a list of properties that the operator $\hat{Z}[\kappa]$ should satisfy will be deducted. In section 3.3 a spin foam operator will be proposed that displays all the features worked out before. Section 4 gives a proof that each operator $\hat{Z}[\kappa]$ can be split into simple blocks $\hat{Z}$ based on 2-complexes which only contain a small number of internal vertices all connected to an initial spin net (see section 4.1). This result can be used to show that the proposed projector is not of the required form (section 4.2). However, $\hat{Z}$ may still contain the necessary information in order to construct a spin foam model using a modification of $\hat{Z}$ with the properties of a rigging map. We conclude by summarizing and discussing the results in section 5.

### 2. Covariant quantum gravity

In the first part of this section we will give a brief overview on spin networks and foam to introduce the notation and clarify the parameters of the model used in this manuscript. The last two section give a physical motivation for this choice.

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3 Apart from these models there exist several other approaches, for example the spinor model [29], the flux representation [30] and the hypercubic model [31].
2.1. Spin networks and spin foams

2.1.1. Spin networks. The gauge invariant Hilbert space of canonical LQG $\mathcal{H}_{\text{kin}}$, to which we will mostly refer to as kinematical space, is spanned by so-called spin network functions. A spin network (short: spin net) $s$ is a triple $(\gamma, \{j_l\}, \{\tau_n\})$ consisting of an oriented semianalytic (SA) graph $\gamma$, a labeling of the links $l$ in $\gamma$ by $\text{SU}(2)$-irreducibles $j_l \in \frac{1}{2}\mathbb{N}$ of dimension $d_{j_l}$ and an assignment of intertwiners $\tau_n \in \text{Inv}(\bigotimes_{V' \cap \gamma = n} \mathcal{H}_{j_l})$. A gauge invariant spin network function is then defined by

$$T_{\gamma,j_i}(\{g_l\}) := \prod_{l \in E(\gamma)} \sqrt{d_{j_l}} [R^{j_l}(g_l)]^{n_{l(1)}}_{m_{l(1)}} \prod_{n \in V(\gamma)} (\tau_n)^{[m_{n(1)}; m_{n(1)}, \ldots, m_{n(1)}]}_{[n_{0}; n_{0}, \ldots, n_{0}]}$$

where $E(\gamma)$ denotes the set of links in $\gamma$, $V(\gamma)$ denotes the set of nodes, $R^{j_l}(g_l)$ the representation matrices of $g_l \in \text{SU}(2)$ and $n_{l(1)}$ and $m_{l(1)}$ are magnetic indices assigned to the source $s(l)$ of the link $l$ and the target $t(l)$ respectively (compare with figure 1).

In order that $(\gamma, j_i, \tau)$ labels a linearly independent set of states we require $j_l \neq 0$ for all $l \in E(\gamma)$ and exclude two-valent nodes whose adjacent links have co-linear tangents\(^4\). Therefore the function (2.1) can be only gauge invariant if the underlying graph $\gamma$ is closed, that is, every node is contained in at least two links, since otherwise the magnetic indices are not contracted completely. Apart from that, we say that a spin net is connected if the underlying graph cannot be written as the disjoint union of two or more closed subgraphs.

The complex conjugate of a spin network $T_{\gamma,j_i}$ is obtained by reversing the orientation of all links of $\gamma$ since $\overline{R^j_{\text{inv}}(g \cdot g^{-1})} = R^j_{\text{inv}}(g^{-1} \cdot g)$ and the trace of a spin net is a map $(\gamma, j_i, \tau) \rightarrow \mathbb{C}$,

$$\text{Tr}(\gamma, j_i, \tau) := \text{Tr} \left[ \prod_n \tau_n \right]$$

defined by contracting the intertwiners.

2.1.2. Spin foams. Heuristically, the ‘time-evolution’ of a spin-network leads to a colored combinatorial 2-complex that is a collection of faces, edges and vertices\(^5\) whose cells are labeled by representation data. In order to have better control on the complexes we here restrict to convex piecewise linear (PL) ones in the sense of [21] (see appendix A.2 for a definition). The reader might be concerned that convexity is too strong if 2-complexes shall

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\(^4\) Not excluded are two-valent intertwiners whose tangents are not co-linear.

\(^5\) For convenience 1-cells are called edges and 2-cells faces. Furthermore, vertices in a graph will be mostly called ‘nodes’ and labeled by $n$ while edges in a graph will be called ‘links’ and labeled by $l$ to distinguish between graphs and 2-complexes.
describe the time-evolution of a spin-networks. However, it is always possible to approximate a semianalytic link $l$ and its ‘time-evolved’ face $l \times [0, 1]$ by a collection of convex cells which itself defines a 2-complex. Since such an approximation is somewhat arbitrary, the final model should be independent of this. Despite of this, all the following assertions remain true when convexity is dropped as long as self-intersections are excluded, which will be always assumed throughout the following. It is just convenient to keep convexity for the moment while it has to be relaxed later on.

To efficiently describe the local properties of a 2-complex we introduce the following notations:

**Notation.**

- The frontier $\partial f$ of a 2-cell $f$ is the 1-complex bounding $f$, the frontier of a link $l$ is the union of the vertices of $l$.
- The interior $\text{int} c$ of a cell $c$ is the set $c \setminus \partial c$.
- A cell $c$ is called adjacent to a different cell $c'$ if $c \cap c' \neq \emptyset$.
- The set of n-cells of $\kappa$ is denoted by $\kappa^{(n)}$.
- The vicinity $\mathcal{V}(c)$ of a cell $c$ is the set of all cells $b$ for which $c \in \partial b$.
- The total number of cells in some set $S$ is denoted by $|S|$.
- A complex $\kappa$ is called connected if for any two sub-complexes $\kappa_1, \kappa_2$ such that $\kappa = \kappa_1 \cup \kappa_2$ one can find at least one cell $c$ satisfying $\partial c \cap \kappa_1 \neq \emptyset \neq \partial c \cap \kappa_2$.

For the following purpose not all kinds of PL 2-complexes can be used but only those are of interest to which one can associate boundary graphs. This specific type will be called a foam in the sense of the following definition.

**Definition 1.**

- The interior $\kappa_{\text{int}}$ of a 2-complex $\kappa$ is the set of all faces, all edges, which are contained in more than one face, and all vertices contained in more than one internal edge.
- The boundary graph $\partial \kappa$ of a 2-complex $\kappa$ is the set of all edges (links) contained in only one face and vertices (nodes) contained in only one internal edge $e \in \kappa_{\text{int}}^{(1)}$.
- A graph $\gamma$ is said to border $\kappa$ iff there exists a one-to-one (affine) map $c : \gamma \times [0, 1] \rightarrow \kappa$ mapping each face $l \times [0, 1]$ and each edge $n \times [0, 1]$ of $\gamma \times [0, 1]$ to a unique face and a unique internal edge in $\kappa$ respectively.
- A 2-complex $\kappa$ whose boundary graph $\partial \kappa$ is the disjoint union of connected graphs $\gamma$ bordering $\kappa$ is called a foam.

Note, that the boundary graphs of a foam are not necessarily semianalytic but just 1-complexes without isolated vertices. To distinguish between semianalytic graphs in the canonical theory and the boundary graphs of a foam, the latter will be called abstract.

In the literature the boundary graph of a foam is often defined by either just the combinatorial definition (see e.g. [8]) or just by bordering graphs (see appendix of [7]). Neither of this is sufficient since for example $\partial \kappa$ is in general not a well-defined graph. Particularly, if the intersection point $n$ of two or more boundary links is contained in several internal edges then $n \notin \partial \kappa$ and consequently $\partial \kappa$ is not even a 1-complex. On the other hand, a graph bordering $\kappa$ does not have to be closed. The above definition guarantees that a) boundary graphs of foams are closed and b) that all edges adjacent to an internal vertex are itself internal (see lemma 5 and 6). This also implies that each face has at least two internal edges.

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6 A more appropriate choice would be to define the model on ball rather than PL-complexes (see section 3.2).
7 We alert the reader that by definition a graph has no faces.
Figure 2. The face orientation induces an orientation on the boundary links $l_i$ while the ordering on the internal edge $Z_{\text{int}} : \{1, 2, 3\} \rightarrow f_1, f_2, f_3$ induces the ordering on the boundary node (red) such that $l_i$ is the unique link contained in $f_i$.

Similar as for graphs one can assign an orientation to the faces and internal edges of a foam $\kappa$ in a manner independent from any specific embedding of $\kappa$. The orientation of the boundary graph is chosen such that it agrees with the induced orientation of the faces (see figure 2 and appendix A.3 for details). This constitutes an oriented foam. Since $\partial \kappa$ borders $\kappa$, internal edges $e$ intersecting the boundary graph in a connected graph $\gamma$ are either all in- or all outgoing of $\gamma$ corresponding to the embedding $\gamma \times [0, 1]$ respectively $\gamma \times [-1, 0]$. If all internal edges are outgoing of $\gamma$ it is called initial and otherwise final.

The foams can now be labeled additionally by representation data of a gauge group $G$. In covariant quantum gravity this gauge group is of course given by the Lorentz group $\text{SO}(3,1)$ or by $\text{SO}(4)$ in the Euclidean theory. Since $\text{SO}(4)$ is a compact semisimple Lie group the representation theory is comparably easy and therefore we will focus on the latter.

A spin foam $(\kappa, \{H_f\}, \{Q_e\})$ consists of an oriented foam $\kappa$ and an assignment of a Hilbert space $f \rightarrow H_f$ (irreducible representation space of $\text{SO}(4)$) to every face $f \in \kappa$. This induces a Hilbert space $H_e = \bigotimes_{f \in \kappa} H_f$ on the internal edges $e$ where the orientation of $\kappa$ determines whether $H_f$ or the contragredient representation $H_f^*$ is induced (see appendix A.3 for details). In addition to that we associate an operator $Q_e : H_{e, \text{inv}} \rightarrow H_{e, \text{inv}}$ acting on the invariant subspace $H_{e, \text{inv}}$ to each internal edge. This coloring then induces a spin net structure on the boundary of $\kappa$ (see figure 3).

Before we go on, it is important to note that the coloring does in fact not depend on the orientations of the internal edges (see [34]), it is just more convenient to keep the orientations for the later discussion.

2.1.3. Subdivisions, partition functions and gluing. For the construction of the partition function associated to a foam $\kappa$ and the later analysis subdivisions of foams play a major role.

Definition 2. If $C_1$ and $C_2$ are two complexes then $C_1$ is called a subdivision of $C_2$ iff $\overline{C_1} = C_2$ and every cell of $C_1$ is a subset of some cell of $C_2$. A subdivision is called proper if $|C_1| > |C_2|$.

A generic subdivision of a foam does not constitute a foam itself. For example, if a boundary link is split by an additional vertex then also the unique face containing the link has to be split in order that the resulting complex is again a foam in the sense of definition 1. In the following we will only consider such subdivisions which itself constitute a foam. Furthermore,
The coloring of the bulk induces a spin net on the boundary where \( \iota' \) is an intertwiner in the domain or the image of \( Q \), depending on whether the vertex carrying \( \iota' \) is the source or the target respectively of the internal edge. The intertwiner space associated to target/source of an internal edge is independent of the edge orientation and depends only on the face orientations (compare the two figures on the left).

A vertex boundary graph obtained by cutting out the vertex \( v \) along the dotted lines. The dotted lines represent the splitting edges \( e(f) \) and the bold black points the vertices \( m(e) \).

Orientation and coloring of the resulting foam should be inherited\(^8\) from the original foam. Such subdivisions will be called colored subdivisions (see appendix A.3).

**Definition 3.** Subdivide all edges \( e \) adjacent to an internal vertex \( v \in \kappa \) by a vertex \( m(e) \) in the interior of \( e \) and all faces \( f \in \mathcal{V}(v) \) by an edge \( e(f) \) with endpoints \( m(e) \) and \( m(e') \) whenever \( e, e' \in f \) and \( e, e' \in \mathcal{V}(v) \). This yields a 1-complex \( \gamma_v = \{ m(e), e(f) | e, f \in \mathcal{V}(v) \} \) called vertex boundary graph. The orientation and coloring on \( \gamma_v \) is designed such that \( (\gamma_v, j_{e(f)}, l_{m(e)}) \) is a boundary spin net of the sub-complex obtained by cutting out the wedges spanned by the half-edges \( e_{m(e)} \) with vertices \( v \) and \( m(e) \), that is, the orientation of \( e(f) \) opposes the one induced by the corresponding wedge (see figure 4).

Note, that vertex boundary graphs are always closed since every \( e \in \mathcal{V}(v) \) is contained in at least two faces. A vertex spin net therefore defines a natural contraction of the intertwiners by

\(^8\) The only freedom in choice is therefore the orientation on newly introduced internal edges of the subdivided foam.
\[ \mathcal{A}_v(\{l_e\}) = \text{Tr}(\gamma_e, j_{e(f)}, t_{m(e)}) = \text{Tr} \left[ \prod_{e \in \mathcal{V}(i)} l_e \right] \]

which constitutes the so-called vertex amplitude. Here, \( e_v \) is the half-edge of \( e \) adjacent to \( v \). Note, that all intertwiners which are not assigned to boundary nodes can be contracted in this way defining the spin foam trace

\[ \text{Tr}(\kappa, \mathcal{H}_f, Q_e) := \left[ \prod_{\mathcal{L} \in \mathcal{L}_{\kappa_0}} \sum_{l_\mathcal{L}} \langle Q_e \rangle_{l_\mathcal{L}(i)} \prod_{e \in \mathcal{E}(\mathcal{L})} \mathcal{A}_e(\{l_e\}) \right] \otimes \prod_{n} \langle e \rangle_n, \]

When, in addition, group elements \( g_\mathcal{L} \in G \) are attached to all boundary links \( l \) then one obtains the spin foam partition function\(^9\)

\[ Z[\kappa](\{g_\mathcal{L}\}) := \sum_{\{l_\mathcal{L}\}} \prod_{\mathcal{L} \in \mathcal{L}_{\kappa_0}} \langle Q_e \rangle_{l_\mathcal{L}(i)} \prod_{e \in \mathcal{E}(\mathcal{L})} \mathcal{A}_e(\{l_e\}) \left[ \text{Tr} \prod_{l_e \in (\mathcal{L}_f)_{i}^{(1)}} R_{l_e}(g) \prod_{n_e \in (\mathcal{L}_f)_{i}^{(0)}} l_{e_n} \right] \]

Notice that no claim about convergence of (2.5) is made at this point for generic \( \kappa \) which therefore may only define a ‘distributional’ linear functional on the boundary space \( \mathcal{H}_{\partial \kappa} \) spanned by spin nets based on \( \partial \kappa \). To fix one’s intuition, consider the following easy but important example.

**Definition 4.** The trivial evolution \( \kappa_0 \) is an oriented foam which has no internal vertices and whose boundary graph \( \partial \kappa \) is the disjoint union of two graphs \( \gamma_1 \) and \( \gamma_2 \) such that there exist an isomorphism \( \gamma_1 \cong \gamma \cong \gamma_2 \).

Since by definition the boundary links of \( \partial \kappa_0 \) inherit the orientation of the face in which they are contained and since for every face \( f \) there are two links \( t_f \in \gamma_1 \), \( t'_f \in \gamma_2 \) and \( l_f, l'_f \in \bar{f} \) it follows that the orientation of \( \gamma_1 \) is opposite to the one of \( \gamma_2 \). Moreover, each internal edge \( e \) is adjacent to two nodes in the boundary graph, w.l.o.g. fix \( s(e) \in \gamma_1 \) and \( t(e) \in \gamma_2 \), so that the spin net on \( \gamma_1 \) is dual to the one induced on \( \gamma_2 \). Concluding,

\[ Z[\kappa_0](\{g_\mathcal{L}\}) = \sum_{\{l_\mathcal{L}\}} \left[ \prod_{f} \frac{1}{d_{l_f}} \prod_{e \in \mathcal{E}(\mathcal{L})} \langle Q_e \rangle_{l_\mathcal{L}(i)} \right] \times \text{Tr}_{\gamma_1,i_0,i_1} \left( \{g_{\mathcal{L}}\} \right) \otimes (\text{Tr}_{\gamma_2,i_0,i_1} \left( \{g_{\mathcal{L}}\}\right))^1. \]

The partition function (2.5) is invariant if one adds or removes faces labeled by the trivial representation. Later on we will also include additional face amplitudes such that \( Z[\kappa] \) is also invariant under colored subdivisions.

Apart from colored subdivisions the gluing of foams along common closed components of their boundaries plays an important role. Suppose that the boundary graphs \( \gamma_1 \) and \( \gamma_2 \) of the foams \( \kappa_1 \) and \( \kappa_2 \) respectively are isomorphic and the orientation and coloring of the faces and internal edges touching \( \gamma_1 \) and \( \gamma_2 \) respectively are compatible then they can be glued together by identifying \( \gamma_1 \) and \( \gamma_2 \). This yields a new complex \( \kappa' \) that is a colored subdivision of the complex \( \kappa_1 \sqcup \kappa_2 \) where \( \gamma_1 = \gamma_2 = \gamma \) is removed (see figure 5). In particular, requiring compatibility of coloring and orientation amounts to stating that the spin net induced on \( \gamma_1 \) in

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\(^9\) Such a partition function can be defined in the dual connection formulation as well. See [35].
\( \kappa_1 \) is dual to that induced on \( \gamma_2 \) in \( \kappa_2 \). Then \( \mathcal{H}_{f_1} \circ f_1 = \mathcal{H}_{f_1} \equiv \mathcal{H}_{f_f} \), where \( f \in \gamma \) is contained in \( f_i \in \kappa_1 \) and \( f'_i \in \kappa_2 \) respectively, and \( Q_{e, e'} = Q_e \circ Q_{e'} \) where \( e \in \kappa, e' \in \kappa' \) such that \( e \cap e' = n \in \gamma \).

### 2.2. Triangulations and foams

One of the main ingredients of covariant LQG is the truncations of degrees of freedom by introducing a triangulation of space-time. A triangulation of a smooth compact \( n \)-manifold \( M \) is a triple \((M, \Delta, f)\) where \( \Delta \) is a (simplicial) complex and \( f: \Delta \to M \) a piecewise differential homeomorphism (see appendix A.2 for details and an extension to non-compact manifolds). In 1940 Whitehead [52] proved\(^{10}\) that any smooth manifold \( M \) has an essentially unique triangulation up to PL homeomorphisms. Moreover, the underlying polyhedron \( \Delta \) is a PL manifold which means that any point in the interior of \( \Delta \) has a neighborhood which is PL homeomorphic to an \( n \)-simplex. Thus, any \((n - 1)\)-cell in the interior of \( \Delta \) is a proper face of two \( n \)-cells and the set \( \partial \Delta \) of all \((n - 1)\)-cells contained in only one \( n \)-cell induces a proper triangulation of the boundary of \( M \) whereupon \( \partial^2 M = \emptyset \) implies that any lower dimensional \((\leq n - 2)\) cell must be contained in at least two higher dimensional cells.

Let \( \Delta := \{ A_i^{(j)} | i = 0, \ldots, n; j = 1 \ldots q_i \} \) be a (simplicial) \( n \)-complex triangulating \( \Delta \) where \( i \) labels the dimension of the cell and \( q_i \) is the number of \( i \)-cells. Let \( a_{i}^{(j)} \) denote the barycenter of \( A_i^{(j)} \). The barycenters of \( n \)-cells define the dual vertices. The \( 1 \)-cell dual \(*[A_i^{(n-1)}]\) to \( A_i^{(n-1)} = A_i^{(n)} \cap A_i^{(n+1)} \) is the union of the edge joining \( a_i^{(n)} \) and \( a_i^{(n-1)} \) and the edge joining \( a_i^{(n)} \) and \( a_i^{(n+1)} \). Inductively, the dual cell of \( A_i^{(n)} \) is defined to be the \((n-m)\)-dimensional subset of all points \( x \) for which there exist \( \lambda, \mu > 0, \lambda + \mu = 1 \), such that \( x = \lambda a_i^{(n)} + \mu b \) where \( b \) is a point in some \(*A_i^{(n+1)}\) dual to a cell \( A_i^{(n+1)} \) in the vicinity of \( A_i^{(n)} \) (see figure 6). The set of all dual cells is the dual complex \( *\Delta \) of \( \Delta \).

In general \( a_i^{(n)}, a_i^{(n-1)} \) and \( a_j^{(n)} \) are not collinear and thus dual cells are not convex but compact polyhedra.

**Lemma 1** ([22]). If \( \Delta \) is a PL \( n \)-manifold and \( A \in \Delta \) an \( m \)-cell then \(*A \) is a PL \((n-m)\)-ball (or equivalently: PL homeomorphic to an \((n-m)\)-simplex). If \( A \in \partial \Delta \) then the cell \( \sharp A \) dual to \( A \) in the subcomplex \( \partial \Delta \) is an \((n-m-1)\)-ball in the frontier of \(*A \).

\(^{10}\) Originally Whitehead proved the assertion in the \( C^1 \) category but already extended it to \( C^k \)-triangulations. To ensure uniqueness up to PL homeomorphisms and ensure that \( \Delta \) is a PL manifold the embedding map \( f \) must be sufficiently smooth, i.e. \( C^1 \) is not enough (for a counter example see [53]).
Figure 6. In two-dimensions the complex dual to the black triangulation can be constructed by joining the barycenter (blue vertices) of the triangles where the blue face is dual to the red vertex.

The object $\ast \Delta$ is generically not a cell-complex in the strict sense of definition 10. Yet, it is a ball complex\(^{11}\), that is a collection $\{B_j | j = 1, \ldots, r\}$ of $m$-balls, $m \leq n$, which obey

1. $\overline{\Delta} = \bigcup_{j=1}^{r} B_j$

2. $\overline{B}_i \cap \overline{B}_j = \emptyset$, if $i \neq j$ where $\overline{B}$ is the interior of $B$

3. $\overline{B}_j$ is a finite union of balls of lower dimension in $\ast \Delta$ and every dual $m$-ball, $m < n$, lies in the frontier of at least one $m+1$-ball.

From the third property and lemma 1 follows immediately that the subset $\ast \partial \Delta \subset \ast \Delta$, containing all $(n-1)$-balls $B_k^{(n-1)}$, which are adjacent to only one $n$-ball, and all balls in their frontier $\partial B_k^{(n-1)}$, is dual to the subcomplex $\partial \Delta$.

**Definition 5.** Let $\Delta$ be a triangulation of a compact 4-manifold then the dual 2-complex $\kappa$ is the set obtained by removing all balls of dimension greater than two from $\ast \Delta$ and additionally all 2-balls from $\ast \partial \Delta$.

Since property two and three listed above still hold every 1-ball $e$ in $\kappa$ is contained in at least one 2-ball $f$. A 1-ball $e$ is adjacent to exactly one 2-ball $f$ if and only if $e \subset \partial \Delta$ by lemma 1. As above we will call 1-balls contained in more than one 2-ball internal, otherwise it is called external. Again by lemma 1, every vertex of $\kappa$ dual to a 4-cell must be the intersection of several internal edges. By the above construction every node in the boundary is the barycenter of a 3-cell and therefore the endpoint of exactly one internal 1-ball. Besides that, the dual 1-complex of $\partial \Delta$ is closed (every node of $\partial \kappa$ must be contained in at least two 1-balls), otherwise $\partial^2 \Delta$ would not be empty, and bordering $\kappa$. This proves the first part.

**Theorem 1.** If $\kappa_\Delta$ is the 2-complex dual to a triangulation $\Delta$ of a compact 4-manifold then $\kappa_\Delta$ is combinatorially equivalent to a foam $\kappa$, i.e. there exists a bijection $g: \kappa_\Delta \rightarrow \kappa$ mapping each $n$-cell of $\kappa_\Delta$ to an $n$-cell of $\kappa$ preserving the gluing relations (if $A$ is a common face of $B$ and $C$ then $g(A)$ is a common face of $g(B)$ and $g(C)$). Moreover, $\kappa_\Delta$ is PL homeomorphic to $\kappa$.

**Proof.** To prove that $\overline{\kappa_\Delta}$ and $\overline{\kappa}$ are PL homeomorphic we construct the following subdivision $\kappa'_\Delta$ and $\kappa'$. Since dual cells are by construction the underlying polyhedra of cell-complexes PL homeomorphic to $m$-balls, we can fix a point $x$ in the interior of a dual face $f \in \kappa_\Delta$ in such a way that the straight lines connecting $x$ and any barycenter $a'_j \subset \overline{f}$ or any vertex of $f$ lies in $\overline{f}$. When splitting every face in that way we obtain a simplicial complex $\kappa'_\Delta$ which is a subdivision of $\kappa_\Delta$. On the other hand, cells in $\kappa$ are already convex so that one can choose any

11 For a proof see [21].
point in the interior of each face \( \tilde{\mathbf{f}} \in \kappa \) and each edge \( e \in \kappa \). By joining the points as above one can find a simplicial subdivision of \( \kappa \) which is combinatorially equivalent to \( \kappa'_\Delta \). Define \( h : \kappa'_\Delta \to \kappa \) by \( h(x_i) = y_i \), if \( x_i \) is a vertex of \( \kappa' \Delta \) and \( y_i \) the corresponding vertex of \( \kappa' \), and extend it linearly. This gives the desired PL homeomorphism mapping \( n \)-cells of \( \kappa'_\Delta \) to \( n \)-cells of \( \kappa' \).

2.3. Spin foams and quantum gravity

The covariant quantization of GR is based on the observation that gravity is closely related to topological BF-theories. These theories are defined on the principal \( G \)-bundle over a smooth \( D \)-dimensional manifold \( \mathcal{M} \) with connection \( A \). The basic fields are the curvature \( F[A] = dA + A \wedge A \) and a (Lie) algebra \( g \)-valued \((D-2)\)-form \( B \). Classically, the 4d BF-action

\[
S_{\text{BF}} = \int_{\mathcal{M}} \text{Tr}(B \wedge F[A])
\]

(2.7)

with gauge group \( G = \text{SO}(4) \) (or \( G = \text{SO}(3,1) \)) is equivalent to the Holst action [17] iff the \( B \)-field can be expressed in terms of tetrads \( E \) and the Hodge dual \( * \)

\[
B = *(E \wedge E) + \frac{1}{\beta} E \wedge E.
\]

(2.8)

The wedge product is taken with respect to the external indices, the trace in (2.7) contracts the internal indices and \( \beta \) is the Barbero–Immirzi parameter. The variation of (2.7) with respect to the \( B \)-field constrains the curvature to vanish and formally the path integral is given by

\[
Z_{\text{BF}}(\mathcal{M}) := \int \mathcal{D}A \int \mathcal{DB} \exp \left( i \int_{\mathcal{M}} \text{Tr}(B \wedge F) \right) = \int \mathcal{D}A \delta(F).
\]

(2.9)

To obtain a covariant model of LQG we will first discretize, then quantize \( Z_{\text{BF}} \) and finally implement the simplicity constraints (2.8).

Suppose \( \Delta \) is a triangulation of \( \mathcal{M} \) and \( \kappa \) its dual 2-complex then the discrete BF-action is obtained by smearing the \( B \)-fields on the triangles of \( \Delta \) and the curvature on the dual faces in \( \kappa \). In the quantum theory the curvature is regularized by the holonomies along the loops enclosing the faces \( f \in \kappa \) and the path integral measure \( \mathcal{D}A \) in (2.9) is replaced by the Haar measure on the gauge group, here \( \text{SO}(4) \). The final partition function is then obtained by integrating out the bulk degrees of freedom (see appendix B.1 for more details). This finally yields

\[
Z_{\text{BF}}[\kappa] = \sum_{(\rho), l} \prod_f d_{\rho_f} \prod_{\nu \in \kappa'} \mathcal{A}_{\nu}(\{t_{\nu}\}) \ T^{\text{BF}}_{\rho,\rho_f}(\{g_{e_{\nu}}\})
\]

(2.10)

where \( T^{\text{BF}} \) is a \( \text{SO}(4) \)-spin network function, \( d_{\rho_f} \) is the dimension of the \( \text{SO}(4) \)-irreducible \( \rho_f \), \( t \) are intertwiners and the vertex amplitudes \( \mathcal{A}_{\nu} \) are given by the trace of the vertex boundary spin nets (2.3). Apart from the additional face amplitude, this coincides with the partition function defined in (2.5) for an \( \text{SO}(4) \)-spin foam with trivial edge operators.

To obtain a partition function for gravity one still has to implement the simplicity constraint. In the most accepted model, the EPRL-model [23] one imposes a linearized version of the constraint in the time-gauge by a sort of master constraint approach. This essentially amounts to restricting the coloring of the foam in (2.10). In particular, the irreducibles labeling the faces are constrained to those that satisfy \( \rho = (j^+ \quad j^-) \) with \( j^\pm = \frac{1}{2}(\pm 1) j \) and the edges are labeled by \( \text{SU}(2) \)-intertwiners \( \tau^{\text{EPRL}} \) coupling additionally the \( \text{SU}(2) \)-irreducibles \( j^\pm \) and
This finally leads to the amplitudes (B.22) given in (appendix B.1.1). The coloring is, of course, only well-defined if \( |\beta \pm j| \) is a half-integer which puts additional constraints on \( \beta \) and \( j \). Yet, this problem only occurs in the Euclidean theory and can be avoided by requiring \( \beta \) to be an odd integer.

The EPRL-model has several problems. The most severe is that it is bound to simplicial triangulations that restricts the topology of the boundary graphs in the dual 2-complex. But in order to make contact with the canonical theory one needs to allow all possible graphs. This is essentially forced on us by the requirement to quantize a point separating \( * \)-subalgebra of the classical Poisson algebra (see section 3.2.1 for a detailed discussion). Apart from that, the intertwiners \( \tau_{\text{EPRL}} \) are not \( \text{SO}(4) \) invariant as one would expect.

Both problems are avoided in the KKL-approach [15]: To any arbitrary foam \( \kappa \) one assigns an amplitude of the form (2.4). Alike the EPRL-model, the faces are labeled by representations \( \rho = (j^+, j^-) \) however the edges are labeled by elements \( \xi_{\text{KKL}} \) of a specific subspace of \( \text{SO}(4) \) (or \( \text{Spin}(4) \)) intertwiners. In addition to that one introduces non-trivial edge operators to diagonalize the \( \xi_{\text{KKL}} \). This finally leads to the amplitudes (B.37) derived in (appendix B.1.2), which will be favored in the following. Nevertheless, the reader should keep in mind that it is also possible to extend the amplitude (B.22) to arbitrary foams by analogy. In fact, all constructions below work similar for (B.22).

The drawback of the KKL-approach is that it arose from a formalization of the EPRL-amplitude to arbitrary complexes. Therefore the clear geometric interpretation and the connection to BF-theory is lost a priori. On the other hand, the derivation of the EPRL-model from the classical theory is only formal as it is for example not clear how the path integral measure in (2.9) is modified by the constraints. Also the correct implementation of the non-commuting constraints and the construction of a continuum limit are still under debate.

Before we go on, let us stress once more that the motivation for considering arbitrary foams instead of only considering foams dual to simplicial triangulations mainly comes from the canonical theory and therefore is vital for the subsequent analysis. Nevertheless, it is also questionable from a purely covariant point of view whether one can restrict triangulations to let say simplicial ones instead of considering all kinds of triangulations. From the point of PL topology there exist no reason in doing so as any triangulation based on arbitrary polyhedra is PL isomorphic to a simplicial one. In addition to that, we know that up to dimension five PL manifolds are equivalent to smooth ones [60–62] so that also from the smooth point of view there exist no necessity in considering arbitrary triangulations. In fact, the discrete BF-action (B.1) is completely equivalent to the continuous form (2.7). Only the inclusion of the simplicity constraint and the approximation of the curvature by (B.2) lead to a true truncation of the degrees of freedom. But these two steps are essential for the derivation of the covariant model so that it is not clear from the outset that the theory is independent of the type of triangulation and that a restriction to simplicial triangulations is justified.

3. Spin foam projector

3.1. The general idea

Since generically zero does not lie in the point spectrum of a given family of constraints \( \{\hat{C}_I\}_{I \in I} \) one has to search weak rather than strong solutions. A weak solution \( L \in D^*_{\text{phys}} \subset \mathcal{H}_0 \) is an element in the algebraic dual of a dense domain \( D_0 \subset \mathcal{H}_0 \) for which

\[
[(\hat{C}_I)^* L](f) := L(\hat{C}_I^* f) = 0
\]

Note, even though the model can be extended to more arbitrary triangulations [19] the requirement of \( \kappa \) being dual to the triangulation still restricts the set of possible boundary graphs.
holds for all $I \in \mathcal{I}$ and $f \in \mathcal{D}_0$. Here, $\mathcal{H}_0$ is the kinematical Hilbert space of the theory in question. Moreover, $(\hat{C}_l)^*$ refers to the dual operator acting on $\mathcal{D}_0^\ast$ and $\mathcal{C}^\ast$ to the Hermitian adjoint acting on $\mathcal{H}_0$. For physical measurements and interpretation $\mathcal{D}_0^\ast$ must be equipped with a scalar product. Unfortunately, it is not possible to naively use the kinematical product $\langle \cdot | \cdot \rangle_0$ since $L$ is generically not in the topological dual. Instead, assume that $\mathcal{D}_0^\ast$ is the algebraic dual of a dense subspace $\mathcal{D}_{\text{phys}}$ of $\mathcal{H}_{\text{phys}}$ whose scalar product $\langle \cdot , \cdot \rangle_{\text{phys}}$ can be constructed by an anti-linear (rigging) map\footnote{For more details on the construction of a rigging map see e.g. [3] and references therein.}

$$\eta : \mathcal{D}_0 \rightarrow \mathcal{D}_0^\ast$$

such that

$$\langle f | f' \rangle_{\text{phys}} := \langle \eta[f] | \eta[f'] \rangle_0 := \eta[f|(f') \quad f, f' \in \mathcal{D}_0. \quad (3.3)$$

If this rigging map exists then $\mathcal{H}_{\text{phys}}$ is the completion of $\mathcal{D}_{\text{phys}} := \eta(\mathcal{D}_0) / \ker(\eta)$. For well-behaved systems $\{G_l\}$ (closed, locally compact Lie-group) a rigging map can be constructed by exponentiating the constraints

$$\eta(f|(f')) = T \int d\mu(T) \langle \exp(it'\hat{C}^\dagger f, f') \rangle_0$$

with multipliers $(t')_l \in T$ and a suitable invariant measure $\mu(T)$. Thus, a rigging map solves two problems in one stroke: it projects on the subspace of solutions and defines a scalar product.

For closed finite constraint systems a rigging map always exist. But the constraints in GR do not generate a Lie-algebra but a Lie-algebroid and it is not clear that the above procedure can be applied. Nevertheless, it is often emphasized that spin foams could provide such a rigging map even though one starts with a different action and constraint algebra and therefore with a different symplectic structure (see e.g. [12]). Ignoring these problems we want to take a rather naive point of view and regard spin foams as a computational algorithm to construct a projector onto, or at least into, the physical Hilbert space.

The spin foam partition function $Z[\kappa]$ is often interpreted as the evaluation of a two-dimensional ‘Feynman diagram’ $\kappa$ appearing in the transition amplitude

$$\int dN \langle T_{s_1} | \exp(iN\hat{H}) | T_{s_2} \rangle'' = \sum_{\kappa ; \gamma \rightarrow \gamma'} \langle T_{s_1} | Z[\kappa] | T_{s_2} \rangle$$

(3.5)

with spin nets $s_{1/2} = (j_1/2, j_2/2, t_{1/2})$. The reason for taking the adjoint spin foam amplitude $Z[\kappa]$, i.e. the amplitude $Z[\kappa^\ast]$ associated to the complex $\kappa^\ast$ obtained from $\kappa$ by reversing all internal face and edge orientations, is that it is more convenient in the later to interpret $T_{s_1}$ as the ingoing spin net. For the moment this is just a mere convention.

The sum on the right hand side of equation (3.5) presumes the existence of a tool to identify semianalytic graphs in $\mathcal{H}_0$ with PL graphs in the boundary of $\kappa^\ast$. This issue will be discussed at length in the next subsection, for the discussion below it suffice to assume that such foams exist. More precisely, we assume that the boundary spin net $T_{s_{\kappa^*, \gamma}}$ of $\kappa^\ast$ can be identified with the $\text{SU}(2)$-nets $T_{s_1} \otimes T_{s_2}'$ so that

$$\langle T_{s_1} | Z'[\kappa] | T_{s_2} \rangle = \sum_{\{j, l\}, \{e, f\}} \prod_{v \in \text{vert}} A_v \prod_{e \in \text{edge}} A_e \prod_{f \in \text{face}} A_f \langle T_{s_1}, T'_{s_1} | T_{s_2} \rangle$$

(3.6)

where vertex, edge and face amplitude, $A_v$, $A_e$ and $A_f$, depend on the model. In analogy to (3.4), one can now postulate a rigging map

$$\eta_{\gamma_0, \gamma_1} : \mathcal{H}_{\kappa, \gamma_0} \rightarrow \mathcal{H}_{\kappa, \gamma_1}$$

$$\eta_{\gamma_0, \gamma_1} : \langle T_{s_1} | \langle T_{s_2} | Z'[\kappa] | T_{s_2} \rangle \rangle.$$

(3.7)
The state $\eta[T]$ is clearly distributional and, thus, an element of the algebraic rather than the topological dual as $Z^\dagger[\kappa]$ includes an infinite sum over all labelings and the sum over all $\kappa$ with corresponding boundary graphs is infinite. Note, the only restriction that is put on the 2-complexes so far is that $\kappa$ is a foam, that means it contains only a finite number of cells and the boundary graphs are closed. If $\eta$ is a proper rigging map then it should satisfy

$$\eta[T_\gamma](\hat H T_\gamma) = \sum_{\kappa_\gamma} \sum_{\gamma_\kappa \rightarrow \gamma_\kappa} (T_\gamma \hat Z^\dagger[\kappa]|T_{\gamma_\kappa}) (T_{\gamma_\kappa} \hat H | T_\gamma) = 0 \quad (3.8)$$

for all $T_\gamma, T_{\gamma_\kappa} \in \mathcal{H}_0$. The sum over all intermediate spin nets $s_m$ including a sum over all possible graphs $\gamma_\kappa$ seems to be ill-defined since the kinematical Hilbert space of LQG is not separable. Even graphs which only differ slightly in their shape and not in their combinatorics give rise to orthogonal spin nets and thus are to be considered inequivalent. Nevertheless, only finitely many summands of (3.8) will be non-zero and this problem is avoided.

Of course, one could also include a weight $w(\kappa)$ in (3.7) as it is generated in group field theory (GFT) [24–26]. But since we take all possible 2-complexes into account, not only those dual to a simplicial triangulation, such a weight has to be generated by a formal GFT allowing for all possible interactions (see [27]). As this is technically very involved, we here just assume that $w(\kappa) = 1$ for all $\kappa$. However, it will be important for the later to note that all the following statements remain true for a non-trivial weight satisfying $w(\kappa_1 \nmid \kappa_2) = w(\kappa_1) w(\kappa_2)$. In section 4.3 this issue will be discussed in more detail.

To define $\eta$ in equation (3.7) precisely, one first has to face the problem of identifying the states induced on the boundary of $\kappa$ with states in the canonical space $\mathcal{H}_0$. If the ‘rigging map’ is build on an amplitude of the EPRL kind (see (B.22)) then the states $T^{\text{EPRL}} \in \mathcal{H}^{\text{EPRL}}$ induced on the boundary are already SU(2)-states but different from those in $\mathcal{H}_0$. If, instead, $\eta$ is defined by the KKL-amplitude (B.37) then the boundary spin network states $T^{\text{KKL}} \in \mathcal{H}^{\text{KKL}}$ are specific Spin(4) states. Fortunately, in both cases it is possible to find a projector, $P_\gamma : \mathcal{H}_\gamma^{\text{EPRL}} \rightarrow \mathcal{H}_0, \gamma$ or $P_\gamma : \mathcal{H}_\gamma^{\text{KKL}} \rightarrow \mathcal{H}_0, \gamma$, respectively, by making use of projected spin networks (see appendix B.2 for details). If the Barbero–Immirzi parameter $\beta$ is an odd integer, which will be assumed throughout the remaining analysis, then this projector is in fact injective as for this choice the intertwiners $\tau^{\text{EPRL}}, \gamma$ and $\xi^{\text{KKL}}, \gamma$ define injective maps from the space of Spin(4)-intertwiners to the space of SU(2)-intertwiners [15]. Apart from that we also postulate the following reasonable properties the rigging map should obey.

1. The map $\eta_{\gamma_1, \gamma_2}$ formally decomposes into a sum of operators $\hat Z^\dagger[\kappa] : \mathcal{H}_{0, \gamma_1} \rightarrow \mathcal{H}_{0, \gamma_2}$ whose matrix elements are proportional to the spin foam amplitude (B.22) or (B.37).
2. The operator $\hat Z^\dagger[\kappa_0]$ based on the trivial evolution (see definition 4) defines an isometry $P_\gamma$ between $\mathcal{H}_{0, \gamma}$ and $\mathcal{H}_{0, \gamma}^{\text{KKL}}$ such that $\hat Z^\dagger[\kappa_0] = P_\gamma^\dagger P_\gamma$.
3. $\hat Z^\dagger[\kappa]$ respects the equivalence relations of spin networks.
4. Splitting of internal edges and faces should leave $\hat Z^\dagger[\kappa]$ invariant.
5. Let $\kappa_1$ and $\kappa_2$ be 2-complexes such that $\kappa_1 \cap \kappa_2 = \partial \kappa_1 \cap \partial \kappa_2 = \tilde{\gamma}$ then

$$\sum_{\tilde{\gamma} \in \mathcal{H}_{0, \gamma}} (T_\tilde{\gamma} | \hat Z^\dagger[\kappa_1]|T_\tilde{\gamma}) (T_\tilde{\gamma} \hat Z^\dagger[\kappa_2]|T_\tilde{\gamma}) = (T_\tilde{\gamma} | \hat Z^\dagger[\kappa_2 \nmid \kappa_1]|T_\tilde{\gamma}) \quad (3.9)$$

where $\kappa_2 \nmid \kappa_1$ is the 2-complex obtained by gluing along the common graph $\tilde{\gamma}$ and $T_\tilde{\gamma}$, $T_\gamma$ are spin network functions living on the boundary graph of $\kappa_2 \nmid \kappa_1$.

The first point captures the details of the above argument and the second point is motivated by the heuristic interpretation of foams being two-dimensional Feynman graphs. From this point of view every internal vertex corresponds to the action of $\hat{H}$ and consequently $\kappa_0$ represents the zeroth order in $\exp(N\hat{H}) \approx 1 + \ldots$. Therefore $\langle T | \hat Z^\dagger[\kappa_0]|T \rangle$ should represent the kinematical inner product which imposes the second property.
The third requirement is necessary in order to construct a self-consistent operator. Two spin nets are equivalent if they can be obtained by the following manipulations

(a) adding new links labeled by the trivial representation
(b) creating a new node labeled by the trivial intertwiner by splitting a link.

Since every face touching \( \partial \kappa \) contributes a link in the boundary graph \( \hat{Z}^\dagger[\kappa] \) should be invariant if we add or remove a face labeled by the trivial representation. If we split link in \( \partial \kappa \) then also the adjacent face must be subdivided by a new internal edge. Therefore, the spin foam amplitude should be invariant under such splittings and also under the trivial subdivision of internal edges because it does not play a role whether the new edge \( e \) splits another internal edge or joins an internal vertex. Furthermore, the model should be independent of the way a semianalytic graph is approximated (see below).

The last condition reflects the gluing property of spin foam amplitudes

\[
Z^\dagger[\kappa_1]Z^\dagger[\kappa_2] = Z^\dagger[\kappa_2 \sharp \kappa_1]
\]

used in most models in order to fix the boundary amplitude. Furthermore, if (3.7) defines an improper projector\(^{14}\) then \( \eta \) should satisfy

\[
\eta[\eta[T]] = K \eta[T]
\]

for a constant \( K > 0 \).

### 3.2. Abstract versus embedded setting

In the last section we discussed the general idea how to combine the covariant and the canonical approach. Even though the states induced on the boundary of a spin foam are formally equivalent to spin net states on the same graph, this does not prove equivalence of both theories. Due to the structural difference of both models, it is, for example, not clear that observables agree. Also the construction of the maps (3.4) and (3.5) is only formal since the correct measure of this path integral is unknown. In this section we will argue that a strict derivation of (3.5) from BF-theory is not possible if one insists that \( \kappa \) is dual to a triangulation of space-time. Essentially, this is caused by the different topological and geometrical meaning of graphs in the canonical and covariant model and will be discussed in the first subsection. In the second part we will analyze the impact of a rigging map as postulated in (3.7) on the canonical theory focusing on the role of diffeomorphisms.

#### 3.2.1. Triangulations, foams and graphs

The first obvious obstacle when trying to combine covariant and canonical theory is that the canonical model is based on semianalytic paths instead of PL 1-cells. Nevertheless, one can always approximate a semianalytic path by PL ones. That is another important reason why we ask for invariance under trivial face splittings so that the ‘transition function’ is independent of the approximation. Of course, it is not really possible to approximate spin nets defined on semianalytic graphs by spin nets on PL graphs since the Ashtekar–Lewandowski-measure is maximally clustering in the sense that any two spin nets are orthogonal as soon as they are defined on slightly different graphs. Thus one should either modify canonical LQG to accommodate PL structures or one eventually interprets the boundary graphs of spin foam models in the semianalytic category.

Moreover, the links of a spin net in \( \mathcal{H}_0 \) can be knotted so that the ‘time-evolution’ \( \gamma \times [0, \epsilon] \) could lead to complicated self-intersections of faces. On the other hand, the Hamiltonian acts locally on the nodes and the physical impact of knotting is barely understood anyway so that we will restrict to unknotted links\(^ {15}\).

Another problem that occurs when trying to match PL and semianalytic (SA) graphs is the following: A PL cell is defined as the convex hull of its vertices and therefore completely determined by them. Yet, there are infinitely many possibilities how to glue a SA link between

\(^{14}\) Generically \( \eta \) will have no square, that is, the constant \( K \) will actually be infinite.

\(^{15}\) The knotting class of the node can be still non-trivial. See the next section for more details.
two nodes and thus several links can be glued between the same nodes. This is not possible
for PL links.

To summarize the previous argument: PL complexes are too restrictive for the purpose
of defining a rigging map but we also do not want to give up all the nice properties worked
out before. A way out of this dilemma is to use ball complexes as in section 2.2 or a more
combinatorial definition:

**Definition 6.**

- An abstract n-cell c is an n-ball whose frontier is the finite union of lower dimensional
  balls (faces).
- An abstract n-complex $C$ is a finite collection of m-balls, $m \leq n$, containing at least one
  n-cell. If $A \in C$ then also all faces of A are in $C$. If $A, B \in C$ then either $A \cap B = \emptyset$
or $A \cap B$ is a common face of A and B.

All definitions and theorems of section 2 can be immediately generalized by replacing
‘PL’ through ‘abstract’. Indeed, we only give up convexity and linearity and since balls are
path connected there exists subdivisions $C'$ of $C$ that are combinatorially equivalent to a PL
complexes (compare with theorem 1).

One might wonder why we are putting so much effort in adapting foams to graphs and
do not simply restrict the class of graphs used in the canonical theory to those which are dual
to a triangulation of the hypersurface $\Sigma$. A technical reason for this is that the Hamiltonian
constraint, as defined in [10], creates trivalent nodes that cannot be dual to a 3D polyhedron.
Obviously, this can be avoided by using a different regularization, e.g. [65], but the only known
parametrization, which leads to a non-anomalous Hamiltonian, is the original one [10].

Despite this more technical arguments, there are also severe reasons why the class of
graphs should not be restricted in the canonical model that are deeply rooted in the different
treating of geometry and topology in both theories. When quantizing the canonical theory
we start with the configuration space $\mathcal{A}$ that is the space of connections on a principal bundle
$P(\Sigma, G)$ with base manifold $\Sigma$ and gauge group $G$. This space can be embedded into the set of
homomorphisms $\text{Hom}(\mathcal{P}, G)$ from the groupoid of paths $\mathcal{P}$ on $\Sigma$ to $G$ [3]. In fact $\text{Hom}(\mathcal{P}, G)$
defines the space of generalized connections $\mathcal{A}$ which is used to construct the gauge variant
kinematical Hilbert space $H_{\text{kin}} = L^2(\mathcal{A}, \mu_{\text{AL}})$. This space is spanned by spin net functions on
all possible graphs build by gluing elements in $\mathcal{P}$, not only those ones which are dual to a triangulation.
Moreover, the holonomy flux algebra does not preserve the underlying graph
of a spin net and, therefore, also the span of spin net functions based on dual graphs is not
preserved.

On the other hand, if one only considers graphs of certain topology, say at most four-valent
ones, then the algebra of cylindrical functions over those graphs does not close on itself but
multiplying such functions with each other generates a much larger set of cylindrical functions.
One would not have quantized a point separating *subalgebra of the classical Poisson algebra.
The only circumstance where that does not happen is when only graphs are allowed that
are part of a fixed lattice. But then the built-in continuum limit of the theory is lost and
the diffeomorphism group is not allowed to act on such functions. Furthermore, the only
known anomaly free Hamiltonian operator does not preserve such functions, i.e. one cannot
implement the constraints. Therefore one cannot simply restrict the graphs in the canonical
theory.

Given all holonomies along all paths in $\Sigma$ one can reconstruct the connection. The set
$\text{Hom}(\mathcal{P}, G)$ also captures topological information since it can be related to the fundamental
group of $\Sigma$ (see e.g. [63]). Again, this information cannot be captured by a single graph $\gamma$, i.e. a finite collection of paths.

The situation changes fundamentally when $\gamma$ is dual to a non-degenerate triangulation $\Delta$ of $\Sigma$. As proven by Whitehead [52], $\Delta$ is uniquely determined up to PL homeomorphisms. Astonishingly, it can be shown that in three-dimension also every PL and every topological manifold have a unique differentiable structure up to diffeomorphisms. In other words in three-dimensions the topological (TOP), piecewise linear (PL) and smooth (DIFF) category are equivalent. The equivalence of PL and DIFF was proven independently by Smale [61], Munkres [60] and Hirsch [62] and the equivalence of TOP and DIFF by Moise [59]. A triangulation also allows to partly reconstruct a metric by defining edge length and angels at each vertex of $\Delta$.

In this sense, a graph $\gamma/\Delta$ dual to a triangulation captures much more topological and geometric information than an arbitrary graph. For example, closed graphs can be only dual to the triangulations of a closed (compact, without boundary) manifold. But to ensure gauge invariance the underlying graph of a spin net must be closed. By a theorem of Milnor [58] any compact 3D manifold $\Sigma$ can be uniquely decomposed into a finite number of prime manifolds $\Sigma_i$. A compact 3-manifold is said to be prime if it is either $S^2 \times S^1$, a non-trivial bundle over $S^1$ with fibers homeomorphic to $S^2$ (similar to the Hopf bundle) or every 2-sphere bounds a 3-ball in $\Sigma$; two prime manifolds are glued together by removing a 3-ball and identifying the newly generated boundaries. Thus any graph dual to triangulation of a compact subregion in $\Sigma$ must be either represent a prime factor of $\Sigma$, or a product thereof or must be dual to a discretized 3-ball (tetrahedron). Yet, a graph dual to a 3-ball is certainly not closed and thus the boundary graph of the associated foam would contain edges that are not embedded in $\Sigma$. This shows that any graph dual to a triangulation of a region in a spatial hypersurface and bordering a foam must be related to a prime factor. To summarize, by restricting the graphs to those dual to a triangulation one automatically encodes much more information in a spin network function than in the original set-up. Surely, this statement remains true even if one allows for all graphs dual to all possible triangulations of $\Sigma$ as they are essentially all PL homeomorphic.

Apart from that, taking the idea of the rigging map seriously, the spin foam ‘projector’ should be based on $\kappa$ which is dual to a discretization of the foliation $\mathbb{R} \times \Sigma$. However, the resulting dual foam $\kappa$ is not obviously a discrete foliation into the same discretized leaves. All of these difficulties suggest to work with arbitrary abstract foams that do not originate as the dual of an embedded discretization of $M$.

### 3.2.2. Semianalytic, piecewise analytic and abstract

In the following, we will discuss how one can realize (3.7) by using abstract complexes in the sense of definition 6 while graphs are still embedded in $\Sigma$.

Due to technical reasons, one prefers to work with semianalytic diffeomorphisms $\text{Diff}_{sa}(\Sigma)$ which are analytic except on some semianalytic submanifolds where they are of class $C^n$, $n > 0$. It was also suggested in [40] to use instead piecewise analytic diffeomorphism, i.e. functions which are almost everywhere analytic except for a finite set of points where they are continuous but not necessarily differentiable.

A diffeomorphism $\phi$ acts on spin net functions by

$$\hat{U}(\phi)T_{\gamma,jl,\iota n}(\{g_{\iota l}\}) = T_{\phi(\gamma),\phi(jl),\phi(\iota n)}(\{\phi(\iota l)\})$$

16 A method to analyze the relation between combinatorial graphs and triangulations is crystallization and leads to colored graphs as they are used in colored GFT [56]. Therefore, a comparison between our spin foam complexes and the ones used in the GFT literature (see e.g. [57]) might provide important insights. However, this goes beyond the scope of this manuscript.
leaving the labeling of links invariant, that is, \( j'_{\phi(t)} = j_t \). Of course, \( \phi \) changes the group element \( g_t \) since now the holonomy is taken along \( \phi(t) \) and can also modify the intertwiners by altering the ordering of links at \( n \).

In the subsequent discussion, two graphs are said to be PA- or SA-equivalent if there exist a PA/SA diffeomorphism \( \phi \) such that \( \phi(\gamma') = \gamma' \). In [40], the authors showed that two graphs are PA-equivalent iff one can find a one-parameter family (ambient isotopy) of homeomorphism \( h_t : \Sigma \to \Sigma, t \in [0, 1] \) with \( h_0(\gamma') = \gamma \) and \( h_1(\gamma') = \gamma' \). This kind of equivalence classes is called a singular knot. These knotting classes are countable. Consequently, the Diffpa-invariant Hilbert space \( \mathcal{H}_{\text{diff,pa}} \) must be separable.

In contrast to that, the space \( \mathcal{H}_{\text{diff,sa}} \) is non-separable. Since \( \phi \in \text{Diff}_{sa} \) is at least \( C^{(1)} \) at every point \( p \in \Sigma \) the differential \( D\phi(p) \) of \( \phi \) at \( p \) is a linear transformation in the tangent space \( T_p \Sigma \). A dilatation in \( T_p \Sigma \) only effects the parametrization of the integral curves \( c(t) \) with \( c(0) \in T_p \Sigma \) and so we may assume w.l.o.g that \( D\phi(p) \in \text{SL}(3) \) which is eight-dimensional. Now an \( n \)-tuple of lines through \( p \) in 3D is determined by \( m \geq 10 \) angles for \( n \geq 5 \) and therefore the equivalence class of an \( n \)-valent node is labeled by \( m \) \(-\)dimensional continuous parameter, so-called moduli \( \theta \). However, it can be shown that \( \mathcal{H}_{\text{diff,sa}} \) is almost the direct integral over spaces with fixed moduli \( \theta \) (see [3]).

As there exist no infinitesimal operator on \( \mathcal{H}_{\lambda} \) representing the classical diffeomorphism constraint, the Diffpa/sa invariance is imposed by a rigging map

\[
\eta_D(T_s) := \eta_{[s]_D} L_{[s]_D}
\]

\[
L_{[s]_D} := \sum_{s' \in [s]_D} \langle T_{s'}, \cdot \rangle \in D^D_0
\]

(3.11)

where, modulo technicalities [2], \([s]_D \) is the orbit of \( s = (\gamma', j, \iota) \) under diffeomorphism and the positive number \( \eta_{[s]_D} \) can be fixed such that the scalar product imposed by the rigging map (3.11) is well-defined. More in detail, \( \eta_{[s]_D} \) is equal to the product of a positive number \( \eta_{[\gamma(s)]_D} \) that depends only on the orbit of the graph \( \gamma(s) \) underlying \( s \) but so far cannot be fixed and a factor \( \eta_{[\gamma'(s)]_D} \) that is chosen such that the averaging in (3.11) respects the graph symmetries of \( s \) and the scalar product is sesquilinear.

We can proceed similarly with (3.7): In the following, two embedded spin nets belong to the same abstract equivalence class \([s]_A \) if they are embeddings of the same abstract spin net \( s_A \). Now, replace (3.11) by

\[
\eta(D) := \sum_{[r]_A \in N_A} \eta_{[r]_A} L_{[r]_A}
\]

\[
\eta_{[r]_A} := \sum_{s \in [r]_A} Z^1[k]
\]

where \( N_A \) denotes the set of equivalence classes and

\[
L_{[r]_A} = \eta_{[r]_A} \sum_{s \in [r]_A} \langle T_{s}, \cdot \rangle
\]

(3.13)

with \( \eta_{[r]_A} \) is a positive number with similar properties as \( \eta_{[s]_D} \). This definition is advantageous regarding two aspects: First it also implements diff-invariance since \([s]_D \subset [s]_A \) and second it allows us to directly work in the abstract setting. Yet, the equivalence class \([s]_A \) is huge and (3.12) does not only ‘wash out’ the embedding information but also all information about moduli or knotting classes. On the other hand, it was shown in [36] that at least in the semianalytic theory the same happens when working with embedded foams.

In fact, the motivation for introducing a rigging map of the form (3.13) originates more form the canonical than the covariant side. Since the Hamiltonian and the diffeomorphism
constraint do not commute, the Hamiltonian does not preserve the image of $\eta_D$. Therefore, it makes a difference whether the Hamiltonian is imposed before or after the diffeomorphism constraint. As suggested in [10], we here assume that $H$ is imposed first. Like many operators in the canonical setting, $H$ must be regularized such that the regularized operator $\hat{H}$ converges to $\hat{H}$ when the parameter $\epsilon$ tends to zero. This limit is taken in a weak * operator topology on $D_{\text{diff}}^* \times D$, that is $|L(\hat{H}^\epsilon f) - L(\hat{H} f)| < \delta$ for all $\epsilon < \delta(\epsilon)$ and $L \in D_{\text{diff}}^*$, $f \in D$. The limit point $\hat{H}$, which in this case can be taken as $\hat{H} = \hat{H}^0$ for an arbitrary but fixed choice $\epsilon_0$ of the regulator $\epsilon$, is an operator on the kinematical Hilbert space and not on its dual. In [10], the above limit is based on $\eta_D$ but since (3.13) also includes an averaging over spatial diffeomorphisms, $\eta_D$ can be replaced by (3.13) in the construction. We are only interested in whether its dual action annihilates the image of $\eta$. Hence, no problem appears from the diffeomorphism averaging.

### 3.3. Operator Foam

We now have all tools to construct the operator $\hat{Z}[k]$. Let $(k_A, \mathcal{H}_f, Q_e)$ be an abstract spin foam with ingoing spin net $s^i = (y^i, j^i, \iota^i_A)$ and outgoing spin net $s^f = (y^f, j^f, \iota^f_A)$. Suppose $Z[k_A]$ is the EPRL or KKL-amplitude associated to $k$ and suppose $P_{\gamma}$ is the map projecting the induced boundary states $T_{\gamma}^{\text{SF}}$, where either $T_{\gamma}^{\text{SF}} = T_{\gamma}^{\text{EPRL}}$ or $T_{\gamma}^{\text{SF}} = T_{\gamma}^{\text{KKL}}$, onto $\mathcal{H}_{0,\gamma}$ then we define $\hat{Z}[k_A]$ by

$$
\hat{Z}[k_A] : \mathcal{H}_{0,\gamma} \rightarrow \mathcal{H}_{0,\gamma},
$$

$$
\langle T_{s_f} | \hat{Z}[k_A] | T_{s_i} \rangle := \sum_{k_{\text{sf}}} \sum_{\omega \in \mathcal{A}_{\text{sf}}} \langle PT_{s_f} | T_{s_f}^{\text{SF}} | \hat{Z}[k_A] | T_{s_i}^{\text{SF}} | PT_{s_i} \rangle.
$$

(3.14)

To keep the notation simple the label $A$ will be left away in the following. By choosing appropriated edge and face amplitudes for (B.22) or (B.37) respectively the amplitude $Z[k]$ can indeed be modified such that $\hat{Z}[k]$ displays all the desired properties listed in section 3.1, i.e. it is invariant under colored subdivisions and obeys the gluing property and resolution of the identity. Moreover, it can be shown that $\hat{Z}'[\delta]$ is equal to $\hat{Z}[k^+]$. In appendix C, we explicitly prove that for the operator based on the KKL-amplitude (B.37).

**Equivalence classes.** Subdivisions and adding faces/edges labeled by the trivial representation define equivalence relations on foams/spin nets. Since they leave the amplitude/spin net function invariant one should only sum over equivalence classes in (3.7) and (3.12). If not stated otherwise it will be always assumed that a foam/graph is minimal in the following sense.

**Definition 7.** An abstract foam/graph is called minimal if it cannot be obtained from another foam/graph by subdivisions.

Note, whether an abstract foam/graph is minimal does not depend on the coloring. Given a generic foam a minimal one can be obtained by successively removing two-valent internal edges and two-valent vertices (internal as well as external). However, not all two-valent edges can be removed since it might happen that the removal of an edge generates a self-intersecting surface which is not homeomorphic to a 2-ball (see figure 7 for an example). This also shows that the minimal representatives of the equivalence classes are not unique. But since the model is independent of this choice we can safely fix a minimal representative for each equivalence class in the following. Furthermore, trivial representations will be excluded as before.
4. Does the spin foam projector provide a rigging map onto $\mathcal{H}_{\text{phys}}$?

Apart from technical issues a first test on $\eta$ is to check whether the constraints are really annihilated. By construction the gauss and diffeomorphism constraint are obviously satisfied, but the Hamiltonian constraint is not. To prove this we will first develop a method to split foams into basic building blocks. The properties of the so-defined rigging map will be discussed in the sequel.

4.1. Time ordering

The rigging map $\eta$ is naturally distinguishing between in and out-going spin nets which induces an order of the internal vertices.

**Definition 8.** Suppose $v$ is an internal vertex of an abstract minimal foam $\kappa$ with non-empty boundary graph such that there exists at least one edge $e$ joining $v$ and a node $n$ in an initial graph, then $v$ is called a vertex of first generation. Inductively a vertex of $n$th generation has at least one connection to a vertex of $(n-1)$th generation but no connections to vertices of lower generation.

If $\partial \kappa$ only contains final graphs then we proceed backwards calling internal vertices, which are connected to $\partial \kappa$ by at least one internal edge, of generation $-1$ and so forth.

If $\partial \kappa = \emptyset$ then all internal vertices are of first generation.

By definition, every internal vertex in a connected foam can be traced back along internal edges to the boundary graph and the shortest path to an initial graph, involving the least number of edges determines the generation. Suppose $\kappa$ contains a vertex $v$ which cannot be traced back to an initial part of the boundary graph, then either $\partial \kappa$ is empty or $v$ is only connected to a final graph. In the first case all vertices are of first generation while in the second case all internal vertices linked to $v$ are also detached from the initial graph. Since boundary graphs are closed and boundary nodes are only adjacent to one internal edge this is only possible if $v$ is part of a sub-foam which is completely disconnected and whose boundary graph only contains final graphs. Yet, the generation is independently defined for every completely disconnected sub-foam and therefore all internal vertices can be uniquely classified.

**Lemma 2.** Let $\mathcal{V}_n(\kappa)$ be the set of vertices of $n$th generation in $\kappa$ and suppose $e \in \kappa_{\text{int}}$ is adjacent to $v \in \mathcal{V}_n(\kappa)$ then $v' \notin e$ is either of generation $n-1, n$ or $n+1$ or $v' \in \partial \kappa$.

**Proof.** The vertex $v'$ cannot be of generation $m < n - 1$ since otherwise $v$ would be of generation lower than $n$. If $v' \notin \mathcal{V}_{n-1}$ then $e$ is either a lowermost connection of $v'$ and consequently $v' \in \mathcal{V}_{n+1}$ or $e$ is adjacent to a vertex in $\mathcal{V}_n$ or in $\partial \kappa$. Note, if $v'$ is a boundary node then it is contained in a final graph unless $n = 1$ in which case it can also be part of an initial graph. \qed
Theorem 2. Every (finite in the sense of number of cells) connected, minimal, abstract spin foam \((\kappa, \{j_f\}, \{Q_e\})\) can be split into minimal sub-foams \((\kappa', \{J'_f\}, \{Q'_e\})\) containing only vertices of \(i\)th generation with respect to the original foam such that for the colored foam holds \(\kappa = \kappa' \cdots \sharp \kappa_n\) where \(n\) is the maximal generation of \(\kappa\).

Proof. The theorem holds trivially for foams with empty boundary graph and w.l.o.g. we may assume that \(\partial \kappa\) contains at least one connected initial(final) graph.

Consider the set \(\mathcal{E}_{nk}\) of edges intersecting the boundary graph in two points. Due to lemma 5 the nodes of an edge in \(\mathcal{E}_{nk}\) must lie in different, disjoint boundary graphs. Apart from that, there exist a natural orientation of internal edges adjacent to the boundary induced by the bordering property of boundary graphs. To be consistent boundary nodes \(n_i\) in an initial graph are always mapped to \(n_i \times [0, 1]\) while nodes \(n_f\) of a final graph are mapped to \(n_f \times [-1, 0]\) (see definition 15). This implies that any edge in \(\mathcal{E}_{nk}\) must join an initial and a final graph.

If \(\kappa'\) is a foam derived from \(\kappa\) by splitting all edges in \(\mathcal{E}_{nk}\) then lemma 3 guarantees that a face \(f \in \kappa'\) is either bounded by at least two edges in \(\mathcal{E}_{1,2}(\kappa')\) or by none. Suppose \(v'_f, \ldots, v'_n\) are the vertices of first generation in \(f\) where the numbering is induced by the orientation of \(f\), i.e. no other vertex of first generation is situated between \(v'_1\) and \(v'_n\). Recall that \(f\) is path connected and homeomorphic to a 2-ball and therefore it is possible to connect \(v'_j\) and \(v'_{j+1}\) by an edge in \(f\). Even better, we can introduce such edges \(e'_{j,j+1}\) for all pairs \((v'_j, v'_{j+1})\) that are not already adjacent to the same edge in such a way that the edges \(e'_{j,j+1}\) do not intersect.

Closing the loop by joining \(v'_1\) and \(v'_n\), the face \(f\) is divided into a subface \(f'_0\) which has only vertices of first generation, \(N/2\) faces \(f'_i\) that contain exactly two edges of \(\mathcal{E}_{1,2}(\kappa')\) and at most one face \(f\) whose internal vertices are only of first generation and that intersects the initial graph in \(l_0\). Here, \(N\) is the total number of edges \(e'_1, \ldots, e'_N \in \mathcal{E}_{1,2}(\kappa')\) bounding \(f\). Since the

\[
\begin{array}{c|ccc|c|c|c}
& n & n + 1 & n + 2 & \partial \kappa \\
\hline
n - 1 & B & N.A. & N.A. & G \\
n & B & G & N.A. & G \\
n + 1 & B/G/S & S & S & S \\
\partial \kappa & B/G/S & S & S & S \\
\end{array}
\]

The set of all edges adjacent to a vertex \(v \in V_n\) and a vertex \(v'\), which is either of generation \(n + 1\) or a node in a final graph, will be denoted by \(\mathcal{E}_{n,n+1}\). Since \(\mathcal{Z}[\kappa]\) is independent of internal edge orientations, we may also assume that all edges in \(\mathcal{E}_{n,n+1}\) are oriented such that \(s(e) \in V_n\).

Lemma 3. Given a face \(f\) and an edge \(e_f \in \mathcal{E}_{n,n+1}\) in the frontier of \(f\) then there exists at least one other edge \(e'_f \in f\) that is either an element of \(\mathcal{E}_{n,n+1}\) or \(s(e'_f) \in V_m, m \leq n\) and \(t(e'_f) \in \partial \kappa\).

Proof. Since \(f\) is a closed loop the statement follows immediately (see figure 8).
Let $\kappa''$ be the complex obtained from $\kappa'$ by subdividing all faces that contain vertices of first and second generation in the above manner, then $\kappa''$ satisfies:

- $\mathcal{E}_{\partial \kappa''} = \mathcal{E}_{\partial \kappa'} = \emptyset$
- $\mathcal{E}_{1,2}(\kappa'') = \mathcal{E}_{1,2}(\kappa')$
- $\forall f \in \kappa''$ s.t. $f \cap \mathcal{E}_{1,2}(\kappa'') \neq \emptyset \ni e_f, e'_f \in \mathcal{E}_{1,2}(\kappa'')$ and $e_f, e'_f \in \tilde{f}$.

The first two statements follow directly from the fact that the newly generated edges join only vertices of first generation and the third statement is a direct consequence of the splitting procedure for single faces.

Proceed by subdividing every edge $e \in \mathcal{E}_{1,2}$ by a vertex $m(e) \in \dot{e}$ and join $m(e)$ and $m(e')$ by an edge $e(f) \in \tilde{f}$ if $e$ and $e'$ are contained in the same face $f$. Since all edges $e \in \mathcal{E}_{1,2}(\kappa'')$ are internal and therefore contained in at least two faces the set $\{m(e), e(f)\}$ give rise to a well-defined closed splitting graph $\gamma_s$ dividing $\kappa$ in two sub-foams:

- $\kappa_1$ containing only vertices of first generation whose boundary graph is the disjoint union of $\gamma_s$ and the initial graphs in $\partial \kappa$.
- $\kappa_{2 \to f}$ which joins $\gamma_s$ and the final graphs of $\kappa$.

Finally, remove all superfluous subdivisions and proceed with $\kappa_{2 \to f}$ in the same manner until no sub-foam $\kappa_i$ contains two vertices of the same generation. From lemma 2 follows immediately that the above splitting preserves the set of vertices of the same generation\(^{17}\) and thus $\kappa = \kappa_1 \sharp \cdots \sharp \kappa_n$ (see figure 9 for an example).

After the removal of all superfluous help edges and vertices the resulting blocks are again minimal. This proves the theorem. \(\square\)

The splitting procedure given in the proof is obviously unique for minimal foams whose faces are bounded by at most two edges in $\mathcal{E}_{m,m+1}$ for each $0 < m \leq n$. For this reason,

\(^{17}\) A vertex $v \in \kappa_{2,f}$ is of first generation iff it is second in $\kappa$. 

---

**Figure 9.** A foam with two first order (red) and one second order (blue) vertex. The red graphs are the initial and final spin net induced on the boundary graph. The red dashed lines in the left picture indicate a cutting net such that the new blocks on the right only contain vertices of first generation.
we will only consider such foams in the following. This restriction is not of pure technical nature but can be justified also from the space-time perspective. More precisely, it can be shown easily that a foam is of this type if it is generated by gluing together minimal foams \( \kappa_i \) with just a single internal vertex in such way that the gluing of \( \kappa_j \) to \( \kappa_1 \cdots \kappa_{j-1} \) does not change the generation of vertices in \( \kappa_1 \cdots \kappa_{j-1} \) and the generation of the vertex \( v \in \kappa_j \) with respect to the glued foam \( \kappa_1 \cdots \kappa_j \) is either equal or higher than the generation of any other vertex in \( \kappa_1 \cdots \kappa_j \), which is joined to \( v \) by an internal edge. In the discrete language this procedure corresponds to discretizing space-time `layer by layer`. Thus the above restriction can be interpreted as enforcing that a given discretization of space-time has a natural (discrete) foliation.

Due to the gluing property the operator \( \hat{Z}[\kappa] \) of a connected foam \( \kappa \) decomposes as well into sub-operators containing only vertices of the same generation
\[
\langle T_{s_i} | \hat{Z}[\kappa] | T_{s_f} \rangle = \sum_{T_{s_i} \in \mathcal{H}_{s_i}} \sum_{T_{s_f} \in \mathcal{H}_{s_f}} \langle T_{s_i} | \hat{Z}[\kappa_{i-1}] | T_{s_{i-1}} \rangle \langle T_{s_{i-1}} | \hat{Z}[\kappa_{i-2}] | T_{s_{i-2}} \rangle \cdots \langle T_{s_1} | \hat{Z}[\kappa_1] | T_{s_0} \rangle. \tag{4.1}
\]

In the next section we want to apply this to the full projector.

4.2. The time ordered projector

As above, the rigging map (3.12) can be restricted to fixed minimal representatives of the abstract equivalence classes by means of the map \( L_{s_i} \), where the weight \( \eta_{s_i} \) in (3.13) is set equal to one for simplicity. Thus \( \eta \) effectively reduces to the operator averaging \( \eta_{s_i} , \gamma, \gamma \) in (3.12). Explicitly,
\[
\eta[T_{s_i} ] (T_{s_f} ) = \sum_{\kappa \in K_{\gamma, \gamma'} (s_i, \gamma, \gamma)} \langle T_{s_i} , \hat{Z}[\kappa ] T_{s_f} \rangle \tag{4.2}
\]
where \( K_{\gamma, \gamma'} \) is the set of all abstract minimal foams with fixed initial and final boundary graph \( \gamma \) and \( \gamma' \) respectively. Similar to the Feynman graph expansion of N-point functions in ordinary QFT, ‘vacuum bubbles’, that is, interior sums over connected foams with empty boundary graph just give rise to powers of \( \eta'' [T_{s_i} ] (T_{s_f} ) \) where the superscript \( c \) indicates that only connected foams are involved. Likewise, contributions \( \kappa \) of the form \( \kappa = \kappa_i \cup \kappa_f \) with \( \partial \kappa_i = \gamma (s_i), \partial \kappa_f = \gamma (s_f) \) and \( \kappa_i \cap \kappa_f = \emptyset \) give again rise to powers of \( \eta'' [T_{s_i} ] (T_{s_f} ) \) times
\[
\eta'' [T_{s_i} ] (T_{s_f} ) \eta'' [T_{s_i} ] (T_{s_f} ) \tag{4.3}
\]
The remaining contribution comes from the set \( K_{\gamma, \gamma'} (s_i, s_f) \) of connected foams with the given boundary graphs. Now suppose that the boundary graphs decompose into several disconnected components, then for example
\[
\eta'' [T_{s_i} \otimes T_{s_2} ] (T_{s_3} \otimes T_{s_4} ) = (\eta'' [T_{s_i} ] (T_{s_3} )) (\eta'' [T_{s_2} ] (T_{s_3} )) + (\eta'' [T_{s_3} ] (T_{s_2} )) (\eta'' [T_{s_2} ] (T_{s_3} )) + (\eta'' [T_{s_3} ] (T_{s_2} )) (\eta'' [T_{s_2} ] (T_{s_3} )) \tag{4.4}
\]
where the label \( n.t. \) indicates that only foams are considered that do not split into disconnected initial and final parts as in the foregoing example.

Combining these arguments, we see that the amplitude (4.2) is known if the connected amplitude
\[
\eta'' [T_{s_i} ] (T_{s_f} ) = \sum_{\kappa \in K_{\gamma, \gamma'} (s_i, s_f)} \langle T_{s_i} , \hat{Z}[\kappa ] T_{s_f} \rangle \tag{4.5}
\]
can be computed. In fact, \( \eta \) is a rigging map for the Hamiltonian constraint \( \hat{H} (N) \) with lapse smearing function \( N \) if
\[
\eta[T_{s_i} ] (\hat{H} (N) T_{s_f} ) = 0 \quad s_i , s_f , N. \tag{4.6}
\]
In particular, equation (4.6) must also hold for \( s_i, s_f = \emptyset \). Moreover, the locality of the Hamiltonian action in combination with relation (4.3) and (4.4) imply that (4.6) is equivalent to

\[
\eta'[T_{s_i}] (\hat{H}(N) T_{s_f}) = 0 \quad \forall \, s_i, s_f, N
\]

and thus it is sufficient to consider connected foams only in the sequel.

We can now apply the splitting procedure developed in the previous section to the connected map (4.5):

\[
\eta'[T_{s_i}] (T_{s_f}) = \delta_{s_i, s_f} + \sum_{N=1}^{\infty} \sum_{\delta} \sum_{\hat{k} \in K_{\gamma(N)}^{(s_i, s_f)}} \langle T_{s_i}, \hat{Z}^1[\hat{k}_1] \cdots \hat{Z}^N[\hat{k}_N] T_{s_f} \rangle.
\]

The second sum in (4.8) extends over all ‘single-time-step’ foams \( \hat{k}_k \), \( k = 1, \ldots, N \) whose internal vertices are all of first generation and whose gluing product is contained in \( K_{\gamma(s_i), \gamma(s_f)}^{(s_i, s_f)} \), that is, consecutive foams \( \hat{k}_i \) and \( \hat{k}_{i+1} \) are glued along matching boundary graphs. The first sum runs over all possible values \( N \) of maximal generation. Given the set \( \hat{K}_{N, \gamma} \) of single time step foams with initial and final boundary graphs \( \gamma \) and \( \gamma' \) respectively, equation (4.8) can be written more explicitly as

\[
\eta'[T_{s_i}] (T_{s_f}) = \delta_{s_i, s_f} + \sum_{N=1}^{\infty} \sum_{\gamma} \cdots \sum_{\gamma' \in K_{N-1, \gamma}} \langle T_{s_i}, \hat{Z}^1[\hat{k}_1] \cdots \hat{Z}^N[\hat{k}_N] T_{s_f} \rangle
\]

with \( \gamma_0 := \gamma(s_i), \gamma_N := \gamma(s_f) \). The graphs \( \gamma_1, \ldots, \gamma_{N-1} \) belong to the aforementioned set of minimal representatives of the abstract equivalence classes on which we know how to evaluate \( Z(\hat{k}) \).

The label \( c \) on the third sum in (4.9) is to remind us that the glued product must be connected. That this is not a pure decoration can be understood form the following example: Let \( \kappa \) be a connected foam made of two tubes, one connected to the initial and the other connected to the final graph, which are joined to the sides of a donut. Then, by imposing the time-splitting it might happen that we slice the donut several times what possibly produces one-time-step foams that are not connected. Therefore, neither the graphs \( \gamma_1, \ldots, \gamma_{N-1} \) in (4.9) nor the elements in \( \hat{K}_{\gamma, \gamma'} \) can be restricted to the connected category.

Recall that the generation of a vertex is uniquely defined and therefore two components can be only joined by identifying vertices of the same generation. Concluding, since \( \hat{K}_{\gamma, \gamma'} \) is generated by cutting connected foams, all disjoined components of a single time step foam \( \hat{k} \in \hat{K}_{\gamma, \gamma'} \) must contain at least one (non-trivial) internal vertex and are bounded by non-trivial in- and outgoing graphs.

These troubles, related to the use of (4.5), cannot be avoided by working with the full rigging map (4.2). The reason for this is that the maximal generation in disconnected components of a given foam do not have to agree in general what causes ordering ambiguities when passing from (4.5) to (4.8). On the other hand, when single time step foams of the above type, bordered by non-empty final and initial graphs, are glued to a connected graph then the resulting foam will always be connected. This is true even if these building blocks consist of several disconnected components. It therefore suffices to require that either \( s_i \) or \( s_f \) is connected. Since the Hamiltonian as defined in [10] is acting locally on the nodes and can therefore only split but not glue, we prefer to restrict the domain of \( \eta[T_{s_i}] \) to the subspace \( \mathcal{H}_0 \) in which the finite linear span of connected spin nets lies dense.
If the final spin net $s_f$ is connected then the label $c$ on the sum in (4.9) can be removed and the whole expression can be simplified by introducing the spin foam transfer matrix\footnote{A similar matrix was already introduced in [75] in the context of holonomy spin foam models. However, in this work the authors only considered a very specific regular type of foams in order to identify a transfer matrix.}

\[
\hat{Z} := \sum_{\gamma',\gamma} \mathcal{P}_{\gamma'} \left[ \sum_{\hat{k} \in \hat{\gamma}_{\gamma'}} Z[\hat{k}] \right] \mathcal{P}_{\gamma}.
\]

(4.10)

Here, $\mathcal{P}_\gamma$ is the projection operator on the subspace of $\mathcal{H}_0$ consisting of the closed linear span of spin network functions over $\gamma$ (with all spins non vanishing on each link). Note that $\hat{Z}$ is still on the linear span of all spin net functions including disconnected ones.

Since $\mathcal{P}_\gamma \mathcal{P}_{\gamma'} = \delta_{\gamma,\gamma'} \mathcal{P}_{\gamma}$ the sum over single time step foams can be replaced by

\[
\mathcal{P}_\gamma \hat{Z} \mathcal{P}_{\gamma'} = \sum_{\hat{k} \in \hat{\gamma}_{\gamma'}} Z[\hat{k}]
\]

and thus (4.9) is equivalent to

\[
\eta'[\mathcal{T}_s]_c(T_{s_f}) = \delta_{s_f,s} + \sum_{N=1}^{\infty} \sum_{\gamma_1,...,\gamma_N} \langle T_{s_f}, \mathcal{P}_{\gamma(s_f)} \hat{Z}^i \mathcal{P}_{\gamma_{N-1}} \hat{Z}^i \mathcal{P}_{\gamma_{N-2}} \cdots \mathcal{P}_{\gamma_1} \hat{Z}^i \mathcal{P}_{\gamma(s)} T_s \rangle.
\]

(4.12)

Using that $T_s = \mathcal{P}_\gamma T_s$ and the fact that $\mathcal{P}_\gamma$ is the identity on $\mathcal{H}_0$ we deduce the compact formula

\[
\eta'[\mathcal{T}_s]_c(T_{s_f}) = \sum_{N=0}^{\infty} \langle T_{s_f}, (\hat{Z}^i)^N T_s \rangle \quad \forall T_s \in \mathcal{H}_0.
\]

(4.13)

The operator $\hat{Z}$ no longer refers to a given boundary graph. Therefore, dropping the requirement that $s_f$ is connected, the right hand side of (4.13) can be extended to a suitable dense subset of the whole Hilbert space $\mathcal{H}_0$ even including states that are not finite linear combinations of spin net functions (see below). One should, however, keep in mind that the equality in (4.13) only holds if $T_{s_f}$ is an element of $\mathcal{H}_0$. Nonetheless, if $\eta$ defines a rigging map then $\eta[T_{s_f}](HT_{s_f})$ must also vanish for all $T_{s_f} \in \mathcal{H}_0$.

### 4.3. Regularization and properties

To test the rigging map on the subspace $\mathcal{H}_0'$ as suggested, the formal expression (4.13) must be regularized. The strategy here is first to regularize the expression on the right hand side of (4.13) by turning the formal operator $\hat{Z}$ into a densely defined quadratic form on the form domain given by the finite linear span of spin network functions in the full Hilbert space $\mathcal{H}_0$ and afterwards restrict to $\mathcal{H}_0'$. To tame the infinite spin sums in $\hat{Z}^i[k]$ for fixed $k$, a spin cut-off $J$ has to be introduced, that is, all spins $j$ that contribute to the spin foam operator $\hat{Z}^i[k]$ are supposed to obey $j \leq J$. However, we must also impose a bound $N_f$ on the valence of the internal edges (i.e. the number of faces intersecting it) and a bound $N_e$ on the valence of internal vertices. A bound on the number of internal vertices in $\hat{k}$ is not necessary since each internal vertex of first generation must be contained in an edge of the form $n_0 \times [0,1]$ where $n_0$ is a node in an initial graph and consequently $\hat{k}$ can have at most as much internal vertices as there are nodes in the initial graph. For PL complexes this would also restrict the number of possible internal edges and faces but in the abstract category several edges can intersect at the same endpoints and faces can be glued on the same frontier. Therefore, the cut-offs $N_f$ and $N_e$ are necessary to render $\hat{K}_{\gamma',\gamma}$ finite.
Yet, this still does not turn $\hat{Z}$ into a densely defined operator as it has non vanishing matrix elements between any two spin network states (given $\gamma$ take any $\gamma'$ and let $\hat{\kappa}$ be the single time step foam such that all initial and final nodes are joined via internal edges to a single internal vertex of first generation). To cure this the elementary operators $Z[\hat{\kappa}]$ should be equipped with a weight $w(\hat{\kappa})$. This weight should be such that $w(\hat{\kappa}) := \prod_{k=1}^{N_{\kappa}} w(\hat{\kappa}_k)$ for the connected foam $\kappa := \hat{\kappa}_1 \cdots \hat{\kappa}_n$, otherwise the gluing property would be violated and the above statements would no longer be applicable. Denote the modified operator by $\hat{Z}'$ and pick the weight $w$ in such a way that

$$||\hat{Z}'_{T_s}||^2 = \sum_s |\langle T_{\gamma}, \hat{Z}'_{T_s} \rangle|^2 = \sum_{\gamma, \gamma'} \sum_{j, \ell} |\sum_{\hat{\kappa} \in \hat{K}_{\gamma, \gamma'}} w(\hat{\kappa}) \langle T_{\gamma'}, j, \ell, \hat{\kappa}, \hat{\kappa}' \rangle \rangle|^2$$

(4.14)

converges for $s = (\gamma, j, \ell)$. This is possible because firstly the set $\hat{K}_{\gamma, \gamma'}$ for given $\gamma, \gamma'$ is finite due to the bounds $N_f$ and $N_e$, secondly the sum over $j', \ell'$ for fixed $\gamma, \gamma'$ is finite due to the cut-off $J$, and thirdly, due to the restriction to the embedded representatives of abstract minimal graphs, the sum over $\gamma'$ is countable. It will therefore be sufficient to pick $w(\hat{\kappa})$ for $\hat{\kappa} \in \hat{K}_{\gamma, \gamma'}$ to be such that it suppresses the growth behavior as $\gamma, \gamma'$ become large after having performed the sum over $j', \ell', \hat{K}_{\gamma, \gamma'}$. It is likely that this growth behavior is bounded by the number

$$C(J, N_f, N_e) |E(\gamma')| + |E(\gamma')|$$

(4.15)

where $C(J, N_f, N_e)$ only depends on the cut-offs. The reason for this is that we expect polynomial growth in $J$ for every face due to the $n_j$ symbols involved in the spin foam amplitude of which there are an order of $N_fN_e(|E(\gamma')| + |E(\gamma')|)$.

Before we go one let us remark that such a cut-off in the valence of edges and faces and the cut-off in spins allows also to make contact with a regularized GFT of the type suggested in [27], which involves only a finite number of fields with a finite number of interaction terms. This opens the possibility to calculate a weight $\omega$ by GFT methods taking advantage of the numerous results on scaling behaviors of group and tensor field theories obtained so far.

In the present context it is, of course, most interesting to test whether such a weight would satisfy the mild assumptions spelled out before. However, this goes far beyond the scope of this article.

Having tamed $\hat{Z}$ like this as an operator densely defined on $\mathcal{H}_j$, which is the subspace of $\mathcal{H}_0$ defined by the spin cut-off $J$, it is still not clear that its powers are densely defined as its domain, the finite linear span of spin network functions is not preserved. Namely, the range of $\hat{Z}$ lies always in the closure of its domain. To improve on that, notice that $\hat{Z}$ is formally a symmetric operator. This follows from the reality of all amplitude factors that define the operator $\hat{Z}[\kappa]$ (at least in the Euclidian setting) and the fact that both $\kappa^*$ and $\kappa$ are elements of $\hat{K}_{\gamma, \gamma'}$. Whence (4.10) is invariant under taking adjoints and $^\dagger$ can be skipped in the following. It also follows irrespective of how $\hat{Z}$ was computed from the expression (4.13) which should define a sesqui-linear form.

Suppose $\hat{Z}$ can be extended as a self-adjoint operator on $\mathcal{H}_j$ with projection valued measure $E$. Let $\mathcal{H}_q := E([-q, q])\mathcal{H}_0$ be the closed subspace of $\mathcal{H}_0$ on which $\hat{Z}$ acts by multiplication with $\lambda \in [-q, q]$ where $0 < q < 1$. More specifically its elements are of the form

$$\psi_q := \int_{-q}^{q} dE(\lambda) \psi, \psi \in \mathcal{H}_0.$$ 

(4.16)

19 Just to give a very incomplete list [30, 67–71] and references therein.
20 We rename $\hat{Z}$ by $\hat{Z}$ again.
The operator \( A := \sum_{n=0}^{\infty} \hat{Z}^n \) acts on these vectors as
\[
A\psi_q = \int_{-q}^{q} dE(\lambda) \sum_{n=0}^{\infty} \lambda^n \psi = \int_{-q}^{q} dE(\lambda) (1 - \lambda)^{-1} \psi
\]
(4.17)

defining formally a geometric series. This implies
\[
\|A\psi_q\|^2 = \langle \psi, A^2 E([-q, q])\psi \rangle = \int d\langle \psi, E(\lambda)\psi \rangle |1 - \lambda|^{-2} \leq (1 - q)^{-2} \|\psi_q\|^2
\]
(4.18)

and whence \( A \) and any power of \( \hat{Z} \) is even bounded on \( \mathcal{H}_q \). Accordingly, on \( \mathcal{H}_q \) holds \( A = 1 + \hat{Z} \).

Let now \( \psi' = E([-q, q])\psi' \in \mathcal{H}_q \) be in the domain of the averaging map and \( \psi, \psi' \) in the domain of \( \hat{H}(N) \) for any lapse function\(^{21}\). Then, if \( \eta \) is a rigging map for \( \hat{H} \), we find
\[
0 = \eta[\psi'(\hat{H}(N)\psi_q)] = \langle \psi', A\hat{H}(N)\psi \rangle = \langle A\psi', \hat{H}(N)\psi \rangle
\]
\[
= \langle \psi', \hat{H}(N)\psi \rangle + \langle \hat{Z}\psi', A\hat{H}(N)\psi \rangle = \langle \psi', \hat{H}(N)\psi \rangle
\]
(4.19)

for all \( \psi', \psi \) in the common domain \( \mathcal{D}' \) of all \( \hat{H}(N) \) defined by the finite linear span of connected spin network functions over the allowed set of graphs. In fact, as long as \( \psi \) is connected, equation (4.19) holds also for states \( \psi' \) that are finite linear combinations of spin network functions on arbitrary graphs, including disconnected ones. Because the Hamiltonian can only split spin nets, we can therefore choose \( \psi' := \hat{H}(N)\psi \). In particular
\[
\|E([-q, q])\hat{H}(N)\psi\|^2 = 0
\]
(4.20)

has to hold for all \( 0 < q < 1 \). Thus the range of the \( \hat{H}(N) \) avoids the kernel of \( \hat{Z} \). To bring this into a familiar form, notice that it follows from the Cauchy–Schwarz identity
\[
\langle \psi', E([-q, q])\hat{H}(N)\psi \rangle = 0
\]
(4.21)

for all \( \psi, \psi' \in \mathcal{D}' \). Dividing by \( 2q \) and taking \( q \to 0 \) we conclude (in the sense of the functional calculus)
\[
\langle \psi', \delta(\hat{Z})\hat{H}(N)\psi \rangle = 0.
\]
(4.22)

We arrive at the first conclusion: if the spin foam amplitude as above defines a projector on the joint kernel of the \( \hat{H}(N) \), the range of any of the \( \hat{H}(N) \) must be orthogonal to the kernel of the spin foam Hamiltonian \( \hat{Z} \). In other words the \( \hat{H}(N) \) annihilates the kernel of \( \hat{Z} \). If true this would tell us how to construct a spin foam model given \( \hat{H}(N) \) or vice versa how to build a Hamiltonian given a spin foam model. For instance, above criterion would be satisfied if the ‘spin foam Hamiltonian’ \( \hat{Z} \) takes the form of a master constraint
\[
\hat{M} = \sum_{I,J} Z^{IJ} \hat{H}(N_I) \hat{H}(N_J)
\]
(4.23)

for a suitable choice of matrices \( Z^{IJ} \) and smearing functions \( N_I \), see \([11]\) for details where this kind of expression was considered as the source of an alternative spin foam model. In particular, such choice (\( \hat{Z} = \hat{M} \)) is bounded by zero from below and thus one can in principle make use of analytic continuation techniques in order to define the path integral rigorously (Feynman–Kac formula).

However, this identification of the kernels of \( \hat{Z} \) and \( \hat{M} \) cannot be correct. Namely, the second conclusion is the following: according to the above argumentation (4.13) should be a generalized projector on the kernel of \( \hat{Z} \). Instead of \( T_p \), \( T_q \) we pick the states \( \psi_q = E([p, q])\psi \)

\(^{21}\) \( \hat{H}(N) \) also must be projected to \( \mathcal{H}_f \).
and \( \psi'_q = E([p, q]) \psi' \) with any connected \( \psi \), any \( \psi' \) and \( 0 < p < q < 1 \). Thus

\[
\langle \psi'_q, \{ \sum_{N=0}^\infty \hat{Z}^N \} \hat{Z} \psi_q \rangle \text{ should vanish if } \hat{Z} = \hat{M} \text{. Yet, an explicit evaluation gives }
\]

\[
0 = \int_p^q d(\psi', E(\lambda) \psi) \frac{\lambda}{1 - \lambda} \tag{4.24}
\]

where the spectral measure is defined by the polarization identity. In particular for \( \psi = \psi' \)

\[
0 = \int_p^q d(\psi, E(\lambda) \psi) \frac{\lambda}{1 - \lambda} \tag{4.25}
\]

yields a contradiction unless all spectral measures \( E_\psi = \langle \psi, E(\cdot) \psi \rangle \) have no support in \( (p, q) \). Indeed, if \( \hat{Z} \) and the operators \( \hat{H}(N) \) for all choices of lapse functions really have the same kernel then the expression (4.13) is somehow incorrect and should better be replaced by the heuristic expression

\[
\eta_c[\psi'](\psi) := \langle \psi', \delta(\hat{Z}) \psi \rangle = \lim_{T \to \infty} \int_{-T}^{T} \frac{dt}{2\pi} \langle \psi', e^{it\hat{Z}} \psi \rangle. \tag{4.26}
\]

When this is formally expanded it yields again a power series in \( \hat{Z} \) as before but with different coefficients (at finite \( T \)). To make this even more obvious, suppose that by introducing the cut-offs \( J, N_f, N_e \) the operator \( \hat{Z} \) becomes bounded. By rescaling \( \hat{Z} \) by a suitable global factor, i.e. by just choosing a different weight, we may assume without loss of generality that \( ||\hat{Z}|| < 1 \).

But then

\[
\left[ \sum_n \hat{Z}^n \right] \hat{H}(N) = (1 - \hat{Z})^{-1} \hat{H}(N) = 0 \tag{4.27}
\]

is obviously a contradiction as can be seen by multiplying with the invertible operator \( 1 - \hat{Z} \) from the left. In summary, the identification of \( \hat{Z} \) with the master constraint of the \( \hat{H}(N) \) is not sustainable. Note, that the last argument can be formally also applied if one does not assume a regularization at all since (4.27) remains formally valid. Thus the conclusion does not depend on the specific regularization as long as the gluing property is not destroyed.

We conclude this subsection with some speculations about the Lorentzian case. The Lorentzian spin foam amplitudes can be glued similar to Euclidean ones. Moreover, the lifting defined in section appendix B.2 was actually first developed for Lorentzian spin foams. Yet, certain Lorentzian vertex amplitudes are not integrable \[16\] and therefore the class of foams must be restricted further. Nevertheless, the splitting defined in section 4.1 does not affect vertex amplitudes, so that this problem can be possibly ignored and thus the conclusions reached at above are not affected by switching signatures.

5. Merging covariant and canonical LQG: the current status

The extension of spin foam amplitudes based on the duals of simplicial complexes to arbitrary complexes invented in the seminal paper \[15\] offers for the first time the exciting possibility to test: (A) whether a given spin foam model defines a rigging map for a given Hamiltonian constraint, (B) whether a spin foam model has a rigging kernel at all and if yes to which Hamiltonian constraint it corresponds, or (C) how to define a spin foam model such that it defines a rigging map for a given Hamiltonian. This is not possible for spin foam models based on simplicial complexes because those necessarily have purely four-valent boundary graphs. Yet, firstly, the rigging map must act on the full LQG Hilbert space and, secondly, none of the known anomaly free versions of the Hamiltonian constraint preserves the subspace spanned by spin network states based on a purely four-valent graph.
In order to postulate such a spin foam rigging map it is necessary to sum over all possible spin foams whose boundary graphs match the initial and final graph of the would-be rigging map matrix element. For the regularization of the sum, it is unfortunately not possible to use standard GFT techniques since one usually only considers a finite number of interacting fields which dictates the valence of the dual graph. To get rid of this restriction one would have to allow all possible interaction terms in the Lagrangian based on certain invariant polynomials of arbitrarily many gauge group elements. While this is certainly possible on a formal level (see [27]) it would cause tremendous technical difficulties to handle such a theory. As this goes beyond the scope of this paper, we instead postulated a weight function that is compatible with the rules listed in subsection 3.1. These rules are established so that the sum has a chance to partially get to work by restricting on the Euclidean theory and a very specific class of foams, (see [27]) it would cause tremendous technical difficulties to handle such a theory. As this goes beyond the scope of this paper, we instead postulated a weight function that is compatible with the rules listed in subsection 3.1. These rules are established so that the sum has a chance to define a rigging map. It is of course an interesting question for further research whether such a weight can indeed be obtained from a GFT approach.

Even though the considerations in [33] have shown that the above set-up can at least partially get to work by restricting on the Euclidean theory and a very specific class of foams, the discussion in the last two sections has revealed that this is not feasible for the full model. This conclusion is totally independent of the particulars of face and vertex amplitudes and the details of the Hamiltonian constraint. The essential properties, which are needed to reach at this point, are (A) the realization of boundary states of spin foams as kinematical states of the canonical theory and (B) the gluing property of the spin foam amplitude. Let us also remark that no proof in the whole section depends on the invariance under edge or face subdivisions since they can be removed prior of the computation of $\hat{Z}$.

On the other hand, it transpires that the single time step operator $\hat{Z}$, which defines a sort of elementary transfer matrix from which any foam can be generated, is of importance. Its matrix elements between spin network states provide the ‘integral kernel’ (better: summation kernel) of the single time step spin foam evolution. The correct correspondence between $\hat{Z}$ and $\hat{M}$, the master constraint associated to all of the $\hat{H}(N)$, can currently only be speculated about which is sketched here for completeness (see [47] for similar ideas): Using a skeletonization of the interval $T$ into $n$ intervals of length $T/n$ and a Riemann sum approximation one obtains

$$2\pi \delta(\hat{M}) = \lim_{T \to \infty} \int_{-T}^{T} e^{i\hat{M}t} = \lim_{T \to \infty} \int_{0}^{T} [e^{i\hat{M}t} + e^{-i\hat{M}t}]$$

$$= \lim_{T \to \infty} \lim_{n \to \infty} \sum_{k=0}^{n} \frac{T}{n} [e^{i\hat{M}T/n} + e^{-i\hat{M}T/n}].$$

If $\hat{Z}$ would be unitary rather than symmetric then (4.13) suggests that $\hat{Z} = e^{i\hat{M}}$ for an artificial synchronization of the limits $T, n \to \infty$ by some constant $\tau = T/n$. Just that in this case the geometric series over the negative powers of $\hat{Z}$ is missing. But, as $\hat{Z}$ is symmetric, it appears that one wrongly took the geometric series of $\cos(\tau \hat{M})$ rather than the Laurent series of $e^{i\hat{M}}$ minus one. This can be interpreted as an artifact of summing over both Plebanski sectors $(\Pi \pm \pm)$ rather than keeping only the sector corresponding to the Einstein–Hilbert action\textsuperscript{22}. Both identifications for unitary and symmetric $\hat{Z}$ suppose, of course, that $\hat{Z}$ has unit norm so that the geometric series is marginally divergent as it is expected for a $\delta$-distribution. In that case it would be natural to define $\tau \hat{M} := \frac{1}{i} \ln(Z)$ (modulo $2\pi$) or $\tau(M)$ = $\arccos(Z)$ (modulo $\pi$) where the value of $\tau$ is irrelevant as $\hat{M}$ is a constraint. With this identification of $\hat{M}$ in terms of $\hat{Z}$, (4.13) should then, perhaps, be replaced by $\delta(\hat{M})$.

\textsuperscript{22} The sector $(I \pm \pm)$ is excluded by implementing the linearized constraint while the Einstein–Hilbert sector of the 4-simplex amplitude can be only specified when taking the orientation of the 4-simplex into account. See [64] for details.
A similar problem can be also observed in a very different set-up. Namely, the semiclassical limit \[55\] of the 4-simplex vertex amplitude also suffers from the appearance of multiple terms, each consisting in an exponential of the Regge-action, that resembles a series of the cosine rather than the one exponential term one would expect. In \[64\], the author showed that this can be cured for the Euclidean theory by inducing an additional constraint. It would be of course interesting to check whether such a constraint can also resolve the problems of the postulated rigging map.

Apart from that, the whole argument would break down if \( \hat{Z}[\kappa] \) cannot be split into smaller sub-foams. This can be understood as a hint that the vertex amplitude is too local in the sense that it only depends on the coloring of the adjacent faces and not on other vertices in its surrounding and therefore allows splitting. Indeed there are indications that a more non-local action is needed to reconstruct diffeomorphism invariance on a lattice \[49\]. On the other hand, also the action of the Hamiltonian constraint \[10\] is local. Whether this locality is a problem is still under debate in both approaches, the canonical as well as the covariant one.

One could, of course, also argue that the assumption on the weight in the above section is too naive and the weight possibly does not feature the same factorization properties as the non-normalized operator \( \hat{Z}[\kappa] \). For example, it might not be necessary to introduce a spin cut-off \textit{a priori} since many colorings are restricted by the compatibility conditions imposed by the labeling of the boundary spin nets. Nevertheless, there exist regions, so-called bubbles (see \[50\] and references therein), where the spins are not restricted by the boundary data. These region therefore require a separate regularization if the spins are not restricted by a general cut-off. For attempts into this direction see \[72–74\]. By splitting the foam it might happen that one cuts through such a bubble so that it is no longer present in the single time step foams. Only after gluing the blocks this structures reappear. It therefore deserves further investigations whether this bubbles can be renormalized by weights of the above form.

In addition, the equivalence class \( \kappa \sim \kappa^\prime \iff Z[\kappa] = Z[\kappa^\prime] \) is huge as many combinatorial inequivalent foams can lead to the same amplitude. For example, two regions of a foam can be swapped if the boundary spin nets induced on a closed surface around this regions are equal. To avoid over-counting it might be necessary to introduce a factor in the model determining the multiplicities of interchangeable regions. A similar factor was already advertised in order to prove a relation between summing over all foams and refining and was argued to be related to the volume of the orbit of diffeomorphism acting on a colored complex (see \[48\]). This idea is misleading here since a map, somehow related to diffeomorphism, should be at least continuous while cutting out parts of \( \kappa \) and gluing them in somewhere else does not define a continuous function. Despite, we are working with abstract complexes while a diffeomorphism is only affecting the embedding so in this sense we have already taken care of diffeomorphisms. Instead, we propose to include a purely statistical factor related to the (heuristic) expansion of the exponential in (2.9). Why could such a factor cure the problem? Suppose the constraint \( \hat{C} \) can be implemented via group averaging

\[
\int d\alpha |m\rangle\langle n| \exp(i\alpha \hat{C}) = \int d\alpha (\delta_{m,n} + \sum_N \frac{(i\alpha)^N}{N!} \sum_{m_1} \cdots \sum_{m_N} C_{m_1, m_2, \cdots, m_N} - \delta_{m,n})
\]

Expanding the exponential and inserting a resolution of unity \( \sum_m |m\rangle\langle m| \) yields

\[
\int d\alpha (\delta_{m,n} + i\alpha \sum_{m_1} C_{m_1} \left\{ \delta_{m,n} + \sum_N \frac{(i\alpha)^N}{(N+1)!} \sum_{n_1} \cdots \sum_{n_N} C_{n_1, n_2, \cdots, n_N} \right\})
\]
with $C_{mn} \equiv \langle m|\hat{C}|n\rangle$. Due to the factor $N!$ this expression cannot be written as a formal geometric series as it was done for the spin foam transfer matrix. Of course, for spin foams the situation is more complicated since matching conditions depending on the bulk structure and coloring of each foam have to be respected. The inclusion of a statistical factor $N(\kappa)$ as advertised above can only solve the problem if this factor is non-local in the sense that $N(\kappa \neq \kappa') \neq N(\kappa)N(\kappa')$ so that $N(\kappa)Z[\kappa]$ does no longer possess a gluing property.

Another hint into this direction comes from the standard GFT approach with a fixed number of interacting fields and interaction terms. In simplicial gravity or BF-theory one expects a symmetry factor $\lambda|\gamma(0)|_{\text{sym}}(\gamma)$ where $\lambda$ is the GFT coupling constant, $|\gamma(0)|$ the number of nodes in the corresponding GFT graph $\gamma$ and $\text{sym}(\gamma)$ the order of the automorphism group of $\gamma$ (see e.g. [66]). While the factor $\lambda|\gamma(0)|_{\text{sym}}(\gamma)$ clearly obeys a sort of gluing property, it will in general not be true that $\text{sym}(\gamma)$ can be written as a product of the order of automorphism groups assigned to subgraphs of $\gamma$. The situation is getting much more involved if one allows for more interaction fields and potential even if the valence of the nodes is still restricted. First, each new interaction term would introduce a new coupling constant whose interpretation in canonical LQG is rather obscure, second the combinatorics of $\gamma$ and thus the determination of $\text{sym}(\gamma)$ gets much more involved and third the derivation of the mentioned symmetry factor in GFT is so far bound to the simplicial context to the best of our knowledge. Finally, let us remark that the factorization of the proposed rigging map into spin foam transfer matrices can still be valid even if the single weight $\omega(\hat{\kappa})$ do not obey a gluing property but are such that they allow to extract a common factor for all one-time-step foams with the same boundary that than in turn satisfies a gluing property. Nevertheless, it seems very promising to study the relation of our model to GFT in more detail and rigor in future work.

On the other hand, the reason for demanding a gluing property (3.9) is deeply rooted in the ‘sum over histories’ interpretation of spin foams: two ‘histories’ glued together should yield a new ‘history’. However, it was argued earlier [42–44] that (causal) propagators do not entail a physical scalar product or projector. For instance, in [42] the author illustrated that mainly due to the absence of an extrinsic time parameter a propagator $G$ in Wheeler-deWitt Cosmology cannot define a projector since (if $G$ can be normalized) it is not idempotent23. More recently, Calcagini, Gielen and Oriti [44] analyzed different two-point functions in LQC coupled to a scalar field and found that only certain two-point functions24 $G(x',t';x,t)$ define a positive-definite physical scalar product satisfying $G^2 = G$ while all constructed causal propagators fail to either satisfy an adequate composition property or are not defining a positive-definite scalar product. Even more severe, the Feynman propagator does not even solve the constraint equation rather it defines a Green’s function. All two-point functions considered in [44] can be transformed in a ‘vertex expansion’ which closely resembles the spin foam model and was later constructed in [47]. But there are two important differences between spin foams in the full theory and two-point functions in (L)QC. For Wheeler-deWitt as well loop cosmology the properties of the propagator highly depend on the contour of integration and a super-selected sector of solutions (see [42, 44–46] and references therein) while in LQG neither the complete set of solutions nor the correct path integral measure is known. Thus, the impact of a ‘contour’ is rather obscure. Second, in the presence of a scalar field the vertex expansion in LQC is always non-local.

Obviously, the whole problematic is bypassed if $\hat{Z}$ is itself a projector. This idea is supported by the computation in [33] where it was sufficient to consider only complexes with

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23 For the renormalized projector to be idempotent the single amplitudes do not necessarily have to satisfy a gluing property. Thus there is no contradiction between demanding idempotency and violation of a gluing property.

24 So-called non-relativistic Newton–Wightman functions.

25 Except for the relativistic causal two-point function.
a single internal vertex excluding the trivial evolution. Also in [75] the authors showed that for BF-theory it is in fact possible to construct a ‘spin foam transfer matrix’ that annihilates the four-dimensional curvature. Their transfer matrix is constructed by gluing arbitrary but fixed building blocks embedded in space-time. But BF-theory is topological and therefore independent of the triangulation which is certainly not the case for quantum gravity. Therefore, it is questionable that the transfer matrix defined here could already implement the constraint. Also heuristically there is no good argument why trivial foams and larger foams with vertices of several generations should be excluded. Nevertheless, it could be enlightening to test our framework in the case of BF theory.

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Appendix A. More details on spin nets and spin foams

This appendix contains a more detailed overview on spin nets and spin foams for the reader not familiar to framework.

A.1. Spin networks

The kinematical Hilbert space $H_{\text{kin}}$ of canonical LQG is the space of complex valued, square-integrable functions $\Psi_{\gamma}(A)$ of (generalized) connections $A$ on a spatial hypersurface $\Sigma$ embedded in space-time $M$. A connection on a manifold can be reconstructed from the set of holonomies

$$h_p(A) = P\exp \left( \int_p A \right)$$  \hspace{1cm} (A.1)

along all (semianalytic) paths\(^{26}\) $p$ where $P$ denotes path ordering. Likewise, holonomies provide a map from the groupoid of paths into $SU(2)$. Instead of evaluating a holonomy along a single path one can also use finite systems of paths:

**Definition 9.** A semianalytic graph $\gamma$ embedded in $\Sigma$ is a finite set of oriented\(^{27}\) semianalytic paths (links $l$) which intersect at most in their endpoints (nodes $n$).

The space $H_{\text{kin}}$ is spanned by cylindrical functions

$$\Psi_{\gamma}(A) := \psi(h_{l_1}(A), \ldots, h_{l_n}(A))$$  \hspace{1cm} (A.2)

where $\psi$ is a function on $SU(2)^n$ and the scalar product is given by the Ashtekar–Lewandowski measure $\mu_{\text{AL}}$ which reduces to the Haar measure $\mu_H(l)$ of $SU(2)$ on every link $l \in \gamma$ (compare with [37]). More precisely, for a fixed graph $\gamma$ with $n$ links $H_{\text{kin},\gamma}$ is isomorphic to $L^2(SU(2)^n, \mu_H)$. Let $j := \{j_l\}$ be a labeling of the links by irreducible representations $H_{j_l}$ of dimension $d_{j_l} := 2j_l + 1$ and $m := \{m_l\}$ and $n := \{n_l\}$ be magnetic indices associated to the target $t(l)$ and source $s(l)$ of $l \in E(\gamma)$. Since the matrix elements of the Wigner matrices $R_{j_l}(g_l)$, $g_l \in SU(2)$, define an orthogonal basis of $H_{j_l}$ the functions

$$T_{\gamma,j,m,n}(\{g_l\}) = \prod_{l \in E(\gamma)} \sqrt{d_{j_l}} R_{j_l,m_l,n_l}(g_l).$$  \hspace{1cm} (A.3)

\(^{26}\) A path is an equivalence class of curves under reparametrization and retracing.

\(^{27}\) Taking the holonomy along a path always implies an orientation of the path.
build an orthonormal basis of \( L^2(SU(2)_n, \mu_{ij}) \). To restore gauge invariance one needs to assign an intertwiner to each node \( n \), that is a group homomorphism \( \iota : V_1 \to V_2 \). At the node \( n \) the space \( V_1 \) is formed by the tensor product of all irreducible representations \( \mathcal{H}_{j_{i_1}} \) assigned to the outgoing links \( l_i \) at \( n \) and \( V_2 \) equals the tensor product of all irreducible representations \( \mathcal{H}_{j_{l_i}'} \) assigned to the ingoing links \( l_i' \):

\[
\iota_n : \mathcal{H}_{j_{l_1}} \otimes \cdots \otimes \mathcal{H}_{j_{l_k}} \to \mathcal{H}_{j_{l_1}'} \otimes \cdots \otimes \mathcal{H}_{j_{l_r}'}.
\]

The space of all intertwiners, \( \iota_n \), constitutes a Hilbert space \( \mathcal{H}_{n, \text{inv}} \) when equipped with a scalar product \( (\cdot, \cdot) = (\tilde{\iota}_n^\dagger \iota_n) \) defined by the natural contraction of magnetic indices \( m_l, n_{l_i} \) where \( \dagger \) denotes Hermitian conjugation. Due to the compatibility conditions of recoupling theory \( \mathcal{H}_{n, \text{inv}} \) is finite dimensional. Equivalently we could define \( \iota_n \) to be an invariant tensor

\[
\iota_n : \bigotimes_{l \text{ incoming}} \mathcal{H}_{j_{l_i}'}^* \otimes \bigotimes_{l \text{ outgoing}} \mathcal{H}_{j_{l_i}} \to \mathbb{C}
\]

(A.5)

where \( \mathcal{H}_{j_{l_i}'}^* \) is the contragredient representation. Therefore, we often will identify \( \mathcal{H}_{n, \text{inv}} \) with the space of invariant tensors

\[
\text{Inv} \left( \bigotimes_{l \text{ incoming}} \mathcal{H}_{j_{l_i}'}^* \otimes \bigotimes_{l \text{ outgoing}} \mathcal{H}_{j_{l_i}} \right)
\]

(A.6)

equipped with the trace as inner product. An intertwiner depends in general on the ordering\(^{28}\) of the tensor product (A.6) which is why an ordering of the nodes \( n \) has to be introduced indicating the order of the links intersecting at \( n \). This finally yields the states (2.1) that span the Hilbert space \( \mathcal{H}_0 \). More precisely, \( \mathcal{H}_0 \) is the completion of the linear span of those functions.

### A.2. Some facts on piecewise-linear topology and triangulations

Since PL complexes are fundamental for the construction of the covariant model, they will be briefly reviewed in the sequel. More detailed introductions to piecewise-linear topology can be found for example in [21] and [22]. The exposition here mainly follows [21].

**Definition 10** ([22]).

- A compact \( n \)-cell in \( \mathbb{R}^m \), \( m \geq n \), is the convex hull of a finite set of affine independent points, called vertices, which span an \( n \)-dimensional affine subspace.
- Let \( A, B \) be compact cells and \( P \) be the hyperplane of dimension \( m \) spanned by \( B \). If \( P \cap A = B \) and \( P \cap (A \setminus B) = \emptyset \), then \( B \) is an \( m \)-face of \( A \). It is called proper if the dimension of \( B \) is strictly lower than the dimension of \( A \). The set of all proper faces of \( A \) is called the frontier \( \partial A \).
- An \( n \)-complex \( C \) is a finite union of compact \( m \)-cells, \( m \leq n \), with at least one compact \( n \)-cell such that the following two conditions hold.
  1. If \( A \in C \) then all faces of \( A \) are in \( C \).
  2. If \( A, B \in C \) then either \( A \cap B = \emptyset \) or \( A \cap B \in C \) is a common face of \( A \) and \( B \).
- The union of all cells of \( C \) is called the underlying polyhedron \( \overline{C} \).

\(^{28}\) Different orderings can be related by a change of basis in the intertwiner space.
A complex is a collection of all building blocks together with their gluing relations along common faces while the underlying polyhedron is the whole object glued together. If not necessary we do not make this explicit distinction but it should be kept in mind that these are in principle different objects. For instance \( C \) is a topological space while \( C \) itself is just a set.

A compact \( n \)-cell is homeomorphic to an \( n \)-ball and the frontier homeomorphic to an \((n - 1)\)-sphere (for a proof see e.g. [21, 22]). This can be understood by an easy example: Let \( f \) be a 2-cell with a vertex \( v \) in its interior and an edge \( e \) joining \( v \) and another vertex of \( f \) (see figure A1). If \((e, v)\) would be in the frontier of \( f \) then there would exist a straight line \( P \) with \( P \cap f = e \). But such line would divide \( f \) into two separate faces (figure on the right). Therefore the figure on the left of figure A1 is not a convex cell. On the other hand, it is also not a 2-complex since \( f \cap e = e \) is not a face of the 2-cell \( f \). This is summarized by

**Lemma 4.** Every vertex of a 2-cell \( f \in C \) is contained in exactly two 1-cells in the frontier of \( f \).

A cell-complex is called simplicial if all its cells are simplices, that is, a \( D \)-dimensional simplex is spanned by \( D + 1 \) vertices not all of which lay in a \( D - 1 \) dimensional hypersurface. Simplicial complexes are often used for triangulations. However, to be able to apply this concept also to non-compact manifolds the finiteness requirement in definition 10 has to be relaxed.

**Definition 11.** A locally finite simplicial complex \( K \subset \mathbb{R}^n \) is a collection of simplices such that

1. \( \sigma, \tau \in K \Rightarrow \sigma \cap \tau = \emptyset \) or it is a common face
2. \( \sigma \in K, \tau \) a face of \( \sigma \) then \( \tau \in K \)
3. \( \forall x \in K \exists U \in \mathbb{R}^n \) s.t. \( U \) is an open neighborhood of \( x \) meeting only finitely many simplices of \( K \).

As before \( K \) denotes the underlying polyhedron, i.e. the union of cells of \( K \). A map \( f : K \to L \) between polyhedra \( K \) and \( L \) is piecewise linear (PL) iff the graph \( \gamma(f) := \{(x, f(x))|x \in K\} \) is a polyhedron. A PL-map is simplicial if the restriction of \( f \) to any simplex \( \sigma \in K \) is linear. Note, that a simplicial map is determined completely by its values on its vertices.

A PL \( m \)-ball is PL homeomorphic to an \( m \)-simplex in \( \mathbb{R}^m \). If every point \( x \in K \) lies in the interior of a PL \( m \)-ball or \((m - 1)\)-ball then \( K \) is a PL manifold of dimension \( m \) with boundary \( \partial K \), which is the submanifold consisting of all points \( x \in K \) whose neighborhood in \( \partial K \) is homeomorphic to an \((m - 1)\)-ball.
Definition 12. Let $K$ be a locally finite (simplicial) cell complex and $M$ a smooth manifold then $f : K \to M$ is piecewise differentiable (PD) if for every point $x \in K$ one can find a closed neighborhood $U \subset K$ and a subdivision $K'_x$ of $K$ such that $U \cap K'_x$ is a finite simplicial complex and the restriction of $f$ to each simplex of $K'_x \cap U$ is smooth. The map $f$ is a PD homeomorphism if $f$ is PD, a homeomorphism and the restriction of $f$ to each simplex has an injective differential at each point.

A smooth triangulation of a smooth $n$-manifold is a triple $(M, K, f)$ where $M$ is a smooth manifold, $K$ a PL $n$-manifold and $f : K \to M$ a PD homeomorphism.

Theorem 3 (Whitehead). Every smooth $n$-manifold $M$ has a triangulation $(M, K, f)$ which is unique up to PD homeomorphism.

Originally Whitehead worked in the $C^1$-category [52] instead of smooth manifolds and PD maps. Yet in this case, $K$ is not necessarily a PL-manifold and thus the triangulation is not unique e.g. $S^5$ allows triangulation that are not PL manifolds [53]. The above theorem can be proven by showing that any map $f : K \to M$ of class $C^k$ can be approximated by a PL map. Lets assume for simplicity that $K$ is finite then for every $\epsilon, \rho > 0$ one can find a simplicial subdivision $K'$ of $K$ and a simplicial map $L_f$ defined by the values $f(x_i)$ on the vertices $x_i$ of $K'$ such that

$$\|L_f - f\| \leq \epsilon \quad \text{and} \quad \|dL_f - df\| \leq \rho$$

(A.7)
on every simplex of $K$. Furthermore, the subdivision of $K$ can be chosen fine enough such that $L_f$ is non-degenerate if $f$ is non-degenerate, i.e. the Jacobean matrix has full rank at each point of $f$.

On the other hand every PL manifold of dimension less than seven has a unique differentiable structure, thus to every PL $n$-manifold $K$ with $n < 7$, corresponds a unique triangulation $(K, f, M)$ of a smooth manifold $M$ up to diffeomorphism (see [60–62]). In dimension lower than four even every topological manifold has a unique PL and differentiable structure [59].

As shown in section 2.2 the 2-complex dual to a triangulation is a foam after definition 1. These display the following two elementary properties.

Lemma 5. Let $\kappa$ be a foam then the boundary graph is the disjoint union of closed connected graphs. A face (2-cell) $f \in \kappa$ intersects a connected graph $\gamma \subset \partial \kappa$ at most in one link $l_f$.

Proof. Suppose $\gamma \subset \partial \kappa$ is not closed then there is at least one node $n$ adjacent to one and only one link $l$ in the boundary graph. Since $\gamma$ is bordering $\kappa$, $n$ is also an endpoint of an internal edge $e_n$. But $e_n$ is contained in only one face, namely the face generated by $[0, 1] \times l$ and consequently $e_n \in \partial \kappa$.

Since whenever a connected graph $\gamma \subset \partial \kappa$ borders $\kappa$ there exists a one-to-one affine map $\gamma \times [0, 1] \to \kappa$, this implies that a face $f$ cannot intersect $\gamma$ in more than one link.

Note, lemma 5 does not exclude faces intersecting the boundary graph in several disconnected graphs $\gamma, \gamma' \in \partial \kappa, \gamma \cap \gamma' = \emptyset$.

Lemma 6. Let $v$ be an internal vertex of the foam $\kappa$ then all edges $e \in \mathcal{V}(v)$ are internal.

Proof. Suppose $e \in \mathcal{V}(v)$ is an element of $\partial \kappa$ but since $v \notin \partial \kappa$ then $\partial \kappa$ is not a graph.

Lemma 5 and lemma 6 also show that every face has at least two internal edges.
Figure A2. A 2-cell can be oriented by successively counting the edges $e_i$ in $\hat{f}$. This induces an orientation (black arrows) on $e_i$ with $s(e_i) = e_i \cap e_{i-1}$. When the face is subdivided by an edge $e$ (red) then the faces $f_1, f_2$ are oriented such that the induced orientation on $e \in \hat{f}$ is preserved. Thus $f_1$ and $f_2$ are oriented antidromic and here $f_1$ is ingoing to while $f_2$ is outgoing of the red edge.

A.3. Spin foams

To define the labeling of a foam properly one must first specify an orientation. This should be done in a way that does not depend on the embedding of the foam when using an abstract calculus.

**Definition 13.**

- The orientation of an edge $e$ determines source $s(e)$ and target $t(e)$ vertex of $e$.
- Suppose $\hat{f}$ consist of $n$ edges then define a one-to-one map $Z_f : \{1, \ldots, n\} \rightarrow \{e \mid e \in \hat{f}\}$ so that $Z_f(i) \mapsto e_i$ and $e_i \cap e_{i+1} = v_i$ is a vertex of $f$ for all $i < n$ and $e_1 \cap e_n = v_0$.
- The face orientation is the equivalence class of $Z_f$ under cyclic permutations.

Because $\hat{f}$ constitutes a closed loop (lemma 4) there exist exactly two inequivalent orientations (cyclic/anticyclic) of a 2-cell $f$. Furthermore, $Z_f$ induces an edge orientation choosing $s(e_i) = e_{i-1} \cap e_i$ and $t(e_i) = e_i \cap e_{i+1}$. This orientation is not unique if the edge is contained in the frontier of more than one face, i.e. the induced orientation of $f$ can be opposite to that of $f'$ on the common edge $e$. In this case the orientation of $f$ is antidromic to that of $f'$, otherwise it is dromic (see figure A2). Due to convexity $f$ and $f'$ intersect at most in one edge so that this definition is consistent. Even in the more general case, when faces are allowed to intersect in more than one edge but the frontiers $\hat{f}, \hat{f}'$ are still homeomorphic to $S^1$, the induced orientation on all common edges are either all opposed or all equal.

Independently from the face orientation one can still assign an edge orientation. If the induced orientation of $f$ agrees with this independent orientation then $f$ is ingoing otherwise it is called outgoing with respect to the given edge.

Besides the above, the labeling by intertwiners requires an ordering.

**Definition 14.** Let $c$ be an $n$-cell of the complex $C$ and $V^{(n+1)}(c)$ the set of all $(n+1)$ cells in the vicinity of $c$ then the bijection

$$Z_c : \{1, \ldots, m = |V^{(n+1)}(c)|\} \rightarrow V^{(n+1)}(c)$$

(A.8)

is called an ordering of $c$. Two orderings are equivalent if they only differ by cyclic permutations.

In contrast to face orientations there exist more than just two inequivalent orderings, for instance a four-valent internal edge has six inequivalent orderings.
**Definition 15.** An oriented foam is a foam $\kappa$ whose edges and faces are oriented such that all faces $f$ touching the boundary graph $\partial \kappa$ are ingoing to $v_f = f \cap \partial \kappa$. Furthermore, all internal edges $e$ carry an ordering $Z_e$ which assigns an ordering $Z_n$ on the boundary nodes $n$ by $Z_n(l_f) = Z_e(f_l)$ where $e_n$ is the unique internal edge with $n \in e_n$ and $f_l$ is the unique face containing the wedge spanned by $e_n$ and the boundary link $v_f$ (see figure 2)\textsuperscript{29}.

The orientation of a foam should be preserved under subdivisions. Suppose e.g. we split an edge $e \in \kappa$ by a vertex $v_0$, then the new edges $e_1 \cup e_2 = e$ obey $s(e_1) = s(e), t(e_1) = v_0 = s(e_2)$ and $t(e_2) = t(e)$, if $e$ is internal then $e_1, e_2$ inherit the order $Z_e$ of $e$. If $e \in \partial \kappa$ then $v_0$ is adjacent to only two boundary links and the order is unique.

Let $v, v' \in f$ be two vertices such that linking $v$ and $v'$ by an edge $e_0$ in $f$ yields two new faces $f_1 \cup f_2 = f$. The new faces inherit the orientation of $f$ so that the induced orientation on all old faces is preserved while on $e_0$ the orientations of $f_1$ and $f_2$ are antidromic. Therefore, the direction of $e_0$ can be chosen freely (see figure A2).

We now have all the tools to define the coloring of a foam more carefully. The assignment $f \rightarrow \mathcal{H}_f$ induces a space

$$e \mapsto \mathcal{H}_e := \bigotimes_{f \text{ ingoing to } e} \mathcal{H}_f \otimes \bigotimes_{f \text{ outgoing of } e} \mathcal{H}_f^\ast,$$

and its invariant subspace $\mathcal{H}_e^{\text{Inv}}$, spanned by intertwiners $\iota : \mathcal{H}_e \rightarrow \mathbb{C}$, on every internal edge $e$. Let $n_i (n_0)$ be the total number of faces ingoing to (outgoing from) the edge $e$ and $(x_{i_1}, \ldots, x_{i_0})$ be a basis of $\mathcal{H}_f$ with $d_f := \dim \mathcal{H}_f$ then $\iota_e \in \mathcal{H}_e^{\text{Inv}}$ is a tensor of rank $(n_i, n_0)$

$$\iota_e = (\iota_e)_{A_{i_1} \cdots A_{i_n}}^{A_{0_1} \cdots A_{0_n}} x_{i_1}^1 \otimes \cdots \otimes x_{i_n}^1 \otimes x_{0_1}^0 \otimes \cdots \otimes x_{0_n}^0. \quad (A.9)$$

If we now also associate an operator $Q_e : \mathcal{H}_e^{\text{Inv}} \rightarrow \mathcal{H}_e^{\text{Inv}}$ to every edge then the expansion of $Q_e$ in the basis $\{\iota_e\}$ of $\mathcal{H}_e^{\text{Inv}}$ reads

$$Q_e := (Q_e)_{\iota_{e(x)}^{\dagger}}^{\iota_{e(\gamma)}^{\dagger}} \iota_{e(\gamma)}^{\dagger} \otimes \iota_{e(x)} \quad (A.10)$$

where the dual $\iota_{e(x)}^{\dagger}$ is attached to the target and $\iota_{e(\gamma)}$ to the source of $e$.

The coloring $(\mathcal{H}_f, Q_e)$ of the bulk $\kappa_{\text{int}}$ induces a spin net structure on $\partial \kappa$: A boundary link $l_f$ contained in the unique face $f$ is labeled by $\mathcal{H}_f$ and a node $n_e \in \partial \kappa$ is labeled by $\iota_e$, if the internal edge $e$ adjacent to $n_e$ is ingoing, and by the dual intertwiner if $e$ is outgoing. By lemma 5 each boundary link $l_f \in \kappa$ is adjacent to exactly two internal edges $e, e'$ which are either both ingoing to or both outgoing of $\partial \kappa$ and therefore, if $f$ is ingoing to $e$ it is outgoing of $e'$. In both cases $\mathcal{H}_f$ is associated to $l_f$ while the dual is associated to the source (see figure 3). Thus the assignment of $\mathcal{H}_f$ and $\mathcal{H}_f^\ast$ to the edges and boundary vertices only depends on the face orientations and the model can therefore be formulated without specifying the orientation of the internal edges.

Having introduced orientations and colorings properly we can now also give a more precise definition of a colored subdivision:

**Definition 16.** A colored subdivision of a spin foam $(\kappa, \mathcal{H}_f, Q_e)$ is an oriented subdivision of $\kappa$ such that for the new colored foam $(\kappa', \mathcal{H}_f', Q_e')$ holds

1. $\mathcal{H}_f = \mathcal{H}_f' = \cdots = \mathcal{H}_f''$ if $f \in \kappa$; $f', \ldots, f'' \in \kappa'$ and $f'_0 \cup \cdots \cup f''_n = f$
2. if $e' \notin \kappa$ then $Q_{e'} = 1$ and $\iota_{e'}$ is a two-valent intertwiner
3. if $e_1', \ldots, e_n' \in \kappa'_{\text{int}}$ such that $e_1' \cup \cdots \cup e_n' = e \in \kappa_{\text{int}}$ then $Q_{e_1'} \circ \cdots \circ Q_{e_n'} = Q_e$.

\textsuperscript{29} An ordering of internal vertices is not necessary.
Appendix B. The covariant model: a brief review

B.1. Discretized BF-theory

If $\Delta$ is a simplicial triangulation of a closed manifold $\mathcal{M}$ then the vector space $C^n(\Delta)$ of formal linear combinations of $n$-cells in $\Delta$ equipped with the scalar product $(\sigma_i, \sigma_j) = \delta_{ij}$ is isometric to the space of $n$-forms with scalar product $\langle \omega, \omega' \rangle = \int \text{Tr}(\omega \wedge * \omega')$. Furthermore, there is a one-to-one correspondence between the operations $(\wedge, *, d)$ and operations in $C^n(\Delta)$ (see [41]). For example the Hodge dual acts on cells by mapping to dual cells.

Within this scheme the $B$ fields of BF-theory are smeared on $(D - 2)$-cells and $F$ on the dual faces such that we obtain the discrete action

$$S_{BF} = \sum_{e \in \Delta^{D-2}} \text{Tr} \left( \left[ \int_{\Delta^{D-2}} F \right] \left[ \int_{\Delta^{D-2}} B \right] \right).$$

Remarkably, this step is independent of the chosen triangulation due to the topological nature of BF-theory. Only after the implementation of the simplicity constraint rendering the theory local the discretization yields a truncation of local degrees of freedom.

Recall that connections of a gauge theory are naturally regularized by holonomies $h_e[A]$ along paths $e \subset \mathcal{M}$ and therefore the ‘measure $DA$’ in (2.9) can be replaced by $\prod \, d\mu_f(g_e)$. Similarly, the curvature is regularized along a loop $\alpha$ enclosing a compact 2-d-surface $f$ since in second order approximation $h_e[A] \approx 1 + F(f)$ with $F(f) = \int_f F \in G$. Thus the curvature integral in (B.1) can be replaced by

$$\int_{f \in \Delta^{(2)}} F \approx \prod_{e \in f} g_e^{e_f} = g_f$$

for $D = 4$. Here, $g_e$ are group elements attached to the edges $e$ bounding a face $f$ in the dual 2-complex$^{30}$ $\kappa$ equipped with an orientation. The order of the group elements $g_e$ in (B.2) is determined up to cyclic permutations by the orientation of the face and $e_f$ equals 1 if $f$ is ingoing and $-1$ if $f$ is outgoing of $e$. Note, that the substitution (B.2) replaces an algebra element, namely the curvature, by a group element. In the case of BF-theory this seems not to be of importance as one only considers flat connections, however, in the case of gravity this might very well effect the theory as was discussed in [32]. Despite this issue, we will here follow the usual line in spin foam models and approximate the curvature by (B.2).

Combining equation (B.2) and (B.1), (2.9) can be approximated by

$$Z_{BF}(\kappa) = \int \prod_{e \in \Delta^{(1)}} d\kappa_e \prod_{f \in \Delta^{(2)}} \delta \left( \prod_{e \in f} (g_e)^{e_f} \right).$$

The above procedure can be easily generalized to arbitrary 4-manifolds: If $\mathcal{M}$ is non-compact one has to pass over to locally finite complexes (see appendix A.2). In order to keep everything finite we will not bother about this but always assume that $\mathcal{M}$ is a compact region of space-time. In the case that $\mathcal{M}$ has a non-empty boundary the action (2.7) must be supplemented by a boundary term in order to leave the equations of motions unaltered (see e.g. [54]). Without going into too much detail, $Z_{BF}$ can be constructed as in (B.3) just that the integral is only taken over bulk-variables.

Following [19], we split each edge $e$ into half-edges $l_{i(e)}$ and $l_{i(e)}$, where $l_{i(e)}$ is adjacent to the source and $l_{i(e)}$ to the target, and reorientate the half-edges in such a way that they are

$^{30}$ For the following it is not important that $\kappa$ is a ball-complex and the reader can safely assume that $\kappa$ is a foam.
all oriented toward the splitting point. The half-edges are now labeled by group elements $g_{l_v}$ and $g_{s_v}$ obeying
\[ ge = g_{s_v} g_{l_v}^{-1}. \]  
(B.4)

After introducing these new variables, the group elements can be rearranged defining
\[ gf_v := g_{s_v}^{-1} g_{l_v} g_{l_v}^{'} \]  
(B.5)

where $l_v$ is the half-edge in the frontier of $f$ adjacent to $v$. Note, if $\epsilon_{ef} = 1$ then $l_v = l_{v(e)}$ otherwise $l_v$ is the half-edge of an edge with source $v$ (see figure B1). In these variables the discretized BF-partition function is given by
\[ Z[\kappa] = \int_{SO(4)} \left( \prod_{v \in \kappa} dg_{f_v} \right) \prod_{f \in \kappa} \delta \left( \prod_{v \in \partial f} g_{l_v} \right) \prod_{v \in \kappa} \delta \left( g_{f_v} g_{f_v}^{-1} g_{e}^{-1} \right) \prod_{v \in \kappa} \mathcal{A}_v (\{g_{f_v}\}). \]
(B.6)

Here, $\{g_{f_v}\}$ is the set of all group elements $g_{f_1}, \ldots, g_{f_n}$ assigned to the $n$ faces adjacent to $v$ and
\[ A_v (\{g_{f_v}\}) := \int \left[ \prod_{v \in \partial (v)} dg_{l_v} \right] \prod_{f \in \partial (v)} \delta \left( g_{f_v}^{-1} g_{l_v} g_{l_v}^{'} \right) \]  
(B.7)

reverses the substitution (B.4) and (B.5) in the bulk while $\delta (g_{f_v} g_{f_v}^{-1} g_{e}^{-1})$ reverses it on the boundary. By Weyl’s orthogonality formula the group convolution $\delta (g)$ can be expressed by a sum over the characters of its irreducible representations. This can be used in order to expand the vertex amplitude (B.7) in terms of spin net functions. Recall, that the Euclidean\(^{31}\) gauge group $SO(4) \simeq SU(2)_L \times SU(2)_R / \mathbb{Z}_2$ is locally defined by a left (L) and right (R) action of $SU(2)$. Because of that, irreps of $SO(4)$ are given by the tensor representations $\rho = (j^L, j^R)$ of $Spin(4) \simeq SU(2)_L \times SU(2)_R$ for which $j^L + j^R \in \mathbb{N}$. While by no means justified from the $SO(4)$ point of view, we will work with Spin(4) from the beginning in order to avoid the above limitation on spins $j^{L,R}$. The convolution $\delta (g)$ is then defined by
\[ \delta (g) = \sum_{j^L, j^R} d_{j^L} d_{j^R} \chi^{j^L} (g^{j^L}) \chi^{j^R} (g^{j^R}) \]
(B.8)

with $g = (g^{j^L}, g^{j^R})$ and $j^{L,R} \in SU(2)$.\(^{31}\)

Taking into account that every edge adjacent to an internal vertex is itself internal (lemma 6), every group element $g_{l_v}$ in equation (B.7) appears at least in two different face
distributions\textsuperscript{32}. Thus, we have to integrate over products of characters. Consider for example a vertex $v$ splitting a trivalent edge into two half-edges $h_i, h'_i$. In this case one has to compute integrals of the form

$$\mathcal{I} = \int_{\text{SU}(2)} \ldots \frac{3}{\prod_{i=1}^3} \sum_{j_L} \sum_{j_R} \left( h_i^{-1} h_i^{-1} h_i \right)$$

when evaluating (B.7) at $v$. This integral can be easily solved and yields

$$\mathcal{I} = \text{Tr}(t_i^\dagger t_i^\dagger) \text{Tr} \left[ t_i^\dagger \prod_{i=1}^3 R^h (h_i) t_i^\dagger \right]$$

with $t_i, t_i \in \text{Inv} \left( \bigotimes H_{j_L} \right)$. The second trace constitutes a (non-normalized) spin net function on the vertex graph $\gamma_v$. Using that Spin(4) functions $T_{\text{BF}}$ can be expanded in terms of SU(2) spin nets

$$T_{\gamma_v, \rho, \iota} (g_{j_f}) = T_{\gamma_v, \rho, \iota} \left( \{ g_{j_f} \} \right) \otimes T_{\gamma_v, \rho, \iota} \left( \{ g_{j_f} \} \right)$$

a vertex amplitude at $v \in \kappa_{\text{int}}$ with vertex boundary graph $\gamma_v$ is generally given by

$$A_v (\{ g_{j_f} \}) = \sum_{\{\rho_f\}, \{\iota_i\}} \prod_{f \in \gamma_v} \sqrt{\text{dim} \rho_f} \text{Tr} \left( \bigotimes_{f \in \gamma_v} \iota_i^\dagger \right) T_{\rho_f, \iota_i} \left( \{ g_{j_f} \} \right) .$$

The notation $^\dagger$ symbolizes that the intertwiners in $\text{Tr}$ are dual\textsuperscript{33} to the corresponding intertwiners in the spin net function. The sum over all labelings $\rho_f = (j_f^L, j_f^R)$ and the dimensional factor are remains of (B.8) while the summation over orthonormal intertwiners $\sum_{\iota_i}$ results from integrating products of more than three characters.

Each element $g_{j_f}$ associated to an internal vertex appears exactly twice in (B.6), once in a vertex amplitude and once in the first distribution. Thus the integration over the bulk variables $g_{j_f}$ relates the vertex amplitudes by fixing the representation associated to the faces and causes

$$Z[\kappa] = \sum_{\{\rho\}, \{\iota\}} \prod_v d_{\rho_v} \prod_{v \in \kappa_{\text{int}}} A_v (\{ t_i \}) T_{\rho_v, \iota_v} \left( \{ g_{j_f} \} \right)$$

with

$$A_v (\{ t_i \}) = \text{Tr} \left( \bigotimes_{f \in \gamma(v)} t_i \right) = \text{Tr} \left( \bigotimes_{f \in \gamma(v)} t_i^\dagger \bigotimes t_i^\dagger \right).$$

This function coincides with (2.5) where $Q_v$ is the identity except for an additional face amplitude. So far, we only quantized BF-theory and still have to impose the simplicity constraint.

\textbf{B.1.1. The EPRL-model.} To implement the simplicity constraint (2.8) in the model we need to discretize it but the non-trivial dependence on the tetrad fields is complicating the matter. Therefore, we replace (2.8) by $B = \Sigma + \frac{1}{2} \epsilon \ast \Sigma$ where $\Sigma$ is a $\mathfrak{g}$ valued two-form satisfying\textsuperscript{34}

$$\Sigma^{IJ} \wedge \Sigma^{KL} = \frac{1}{4!} \epsilon^{IJKL} \epsilon_{MNQP} \Sigma^{MN} \wedge \Sigma^{PQ} .$$

\textsuperscript{32} When restricting foams to complexes dual to a triangulation then every internal edge is adjacent to at least four faces since the smallest 3-cell in $\Delta$ is a tetrahedron.

\textsuperscript{33} For intertwiners based on 3j-symbols/Clebsch–Gordan coefficients this difference is of academic nature since they are self-dual.

\textsuperscript{34} This idea goes back to [51].
The solutions of condition (B.15) fall into five sectors:

\[ \begin{align*}
(\text{I} \pm) & \quad \Sigma = \pm E \wedge E \\
(\text{II} \pm) & \quad \Sigma = \pm * E \wedge E \\
(\text{deg}) & \quad \text{Tr}(*E \wedge E) = 0.
\end{align*} \]

Of these sectors only (II \pm) correspond to gravity with co-tetrad \( E \) and the usual Newton constant. Sector (I \pm) essentially corresponds to gravity with a rescaled Newton constant while (deg) does not correspond to a theory of gravity. Note that the resulting action in sector (II \pm) or (I \pm) is only equivalent to the Einstein–Hilbert action or the rescaled Einstein–Hilbert action respectively up to a sign generated by the sign ambiguity in the sectors and the orientation of the tetrads. An additional constraint is needed in order to restrict onto the proper Einstein–Hilbert sector (see [64]). However, in most spin foam models such a constraint is not taken into account.

As stated previously, 2-forms are naturally discretized on two-dimensional surfaces. Consider for simplicity a 4-simplex \( \sigma \) embedded in a manifold \( M \), label the vertices by \( a = 1, \ldots, 5 \) and let \( \tau_a \) be the tetrahedron not containing vertex \((a)\) and \( \Delta_{ab} \) be the triangle \( \tau_a \cap \tau_b \). Then,

\[ \Sigma_{ab}^{IJ} := \int_{\Delta_{ab}} \Sigma_{\mu
u}^{IJ} \]

(B.16)

and (B.15) is replaced by (see [23]):

1. **diagonal simplicity:** \( *\Sigma_{ab} \cdot \Sigma_{ab} = 0 \)
2. **off-diagonal simplicity:** \( *\Sigma_{ab} \cdot \Sigma_{ac} = 0 \quad \forall c \neq b, c \neq a \)
3. **dynamical simplicity.**

Furthermore, the bivectors \( \Sigma_{ab} \) are closed, \( \sum_{b \neq a} \Sigma_{ab} = 0 \), due to gauge-invariance. If \( \sigma \) is non-degenerate, meaning that the tetrahedra span 3D subspaces and can be glued such that the resulting 4-simplex \( \sigma \) spans a four-dimensional subspace, then \([B_{ab}]\) satisfy additional non-degeneracy and orientation conditions. Each non-degenerate 4-simplex determines a unique set of such bivectors and each set of bivectors satisfying the above constraints determines a 4-simplex (see [51]).

The dynamical simplicity constraint does not have to be implemented since diagonal, off-diagonal simplicity and closure already imply dynamical simplicity \(^{36}\). They can be further simplified by replacing the above quadratic constraints by the linearized expression:

\[ \forall \tau_a \in \sigma \quad \exists \gamma_a \in \mathbb{R}^4 \text{ s.t. } (N_a)_j(*\Sigma_{ab})^{IJ} = 0 \quad \forall b \neq a. \]

(B.17)

In fact, this constraint is stronger than the quadratic ones in the sense that it already excludes solutions in sector (I \pm).

If one would impose (B.16) by integration over the corresponding Lagrange multipliers, one would also need to modify the path integral measure. Instead, one chooses first to quantize discrete BF-theory and then impose (B.17) by a master constraint \( \hat{M} \). The reason for considering a master constraint lies in the fact that upon replacing the curvature by holonomies around loops \( \hat{j} \) bounding faces \( f \) in the dual complex and the \( B \) fields by invariant vector fields the former commuting constraints become non-commuting. For this reason the master constraint is also embedded weakly in the new models \([23, 28]\), that is, the BF-vertex boundary Hilbert

\(^{35}\) A 4-simplex is the complex hull of five points not all of which lie in a 3D hyperplane.

\(^{36}\) This set is stronger than the one listed above.
Plugging this back into the full partition function results in $H$ in the model when projecting the BF-Amplitude onto $\phi_{\gamma_v}$ Euclidean theory and can be avoided by requiring $\gamma_v$ to be an odd integer which puts additional constraints on $\beta$ and $\tau$. Yet, this problem only occurs in the Euclidean theory and can be avoided by requiring $\beta$ to be an odd integer.

Following the above considerations, the off-diagonal constraints are implemented weakly in the model when projecting the BF-Amplitude onto $H^{EPRL}_{\gamma_v}$:

$$A^{EPRL}_{\gamma_v}((g_{\ell_w})) = \sum_{t_j l_t} \langle T^{EPRL}_{t_j l_t} \mid A_{\gamma_v} \rangle T^{EPRL}_{t_j l_t} ((g_{\ell_w})) .$$

(B.21)

This is non-zero iff $(j^+, j^0) \equiv (j^-, j^+)$ and obviously also implements diagonal simplicity. Plugging this back into the full partition function results in

$$Z[\kappa] = \sum_{(l^\pm_{\gamma_v}) \cdot (g_{\ell_w})} \prod_{f} \left( \prod_{v \in V_{\kappa}} \left( \prod_{e_v} \sum_{i_{\ell_e}, i_{\ell_e}'} f^{\kappa}_{i_{\ell_e}, i_{\ell_e}'} A_v ([i_{\ell_e}]) \right) \right)$$

\[ \times \sum_{(l^\pm_{\gamma_v}) \cdot (g_{\ell_w})} \left( \prod_{(l^\pm_{\gamma_v}) \cdot (g_{\ell_w})} \frac{1}{\sqrt{\prod_{i_j} d_{i_j} d_{i_j}^*}} \right) T^{EPRL}_{l^\pm_{\gamma_v}} ((g_{\ell_w})) \]

(B.22)

where $f^{\kappa}_{i_{\ell_e}, i_{\ell_e}'}$ are the well known fusion coefficients [23]

$$f^{\kappa}_{i_{\ell_e}, i_{\ell_e}'} := \text{Tr}[\tau^{EPRL}_{\kappa}] t^+ i^- ] .$$

(B.23)
The above model can be extended to non-degenerate arbitrary triangulations (see [19]) by making use of Minkowski’s theorem [39] stating that a polyhedron is uniquely determined, up to inversion and translations, by its face areas and normals.

**B.1.2. The KKL-model.** The KKL-model [15] provides an extension of the EPRL-model for arbitrary triangulations. Furthermore, it uses true Spin(4) intertwiners instead of the SU(2) ones of EPRL.

Consider an arbitrary foam (section appendix A.3) whose faces are colored by irreps of Spin(4) and whose edges are labeled by operators \( Q_e \) in the induced intertwiner space. If we choose \( Q_e \) to be the identity we formally recover BF-theory (B.13), to ‘implement’ the simplicity constraint one has to restrict the coloring to EPRL data:

\[
f \mapsto \rho_f \equiv (j_f^+, j_f^-) \quad \forall f \in \kappa^{(2)}
\]

\[
e \mapsto \zeta^{\text{KKL}}(\eta_{e(c)}) \otimes \zeta^{\dagger}_{\text{KKL}}(\eta_{e(c)}^+) \quad \forall e \in \kappa_{\text{int}}^{(1)}
\]

where

\[
\zeta^{\text{KKL}} : \text{InvSU}(2) \left( \bigotimes_f \mathcal{H}_{v}^{(f)} \right) \to \text{InvSpin}(4) \left( \bigotimes_f \mathcal{H}_{v}^{(f)} \right)
\]

\[
\eta \mapsto \sum_{l^\pm} f_{l^+,l^-}^n e_{l^+,l^-} \otimes e_{l^-}
\]

maps SU(2) intertwiners \( \eta \) to Spin(4) ones. Assuming that all edges are incoming, the vertex amplitude (2.3) is given by

\[
\text{Tr} \left( \bigotimes_{e \in V(v)} \zeta_{\text{KKL}}(\eta_{e(v)}) \right) = \left[ \prod_{e \in V(v)} \sum f_{l^+,l^-}^n e_{l^+,l^-} \right] A_v(\{l^\pm\}).
\]

When each edge is labeled by an operator of the type \( Q_e = |\zeta^{\text{KKL}}(\eta)\rangle \langle \zeta^{\dagger}_{\text{KKL}}(\eta)| \) then the 

\[
Z^{\text{KKL}}[\kappa] = \sum_{\{l^\pm\}} \prod_{e \in e_{\kappa}} \left[ \prod_{n \in \kappa} \sum f_{l^+,l^-}^n e_{l^+,l^-} \right] A_v(\{l^\pm\}) T_{\text{KKL}}^{\delta_{j^+_l,j^-_l},\eta_{\kappa_n}}(\{g_{\{l\}}\})
\]

is almost the same as (B.22) but differs in the states induced on the boundary graph

\[
T_{\delta_{j^+_l,j^-_l},\eta_{\kappa_n}}^{\text{KKL}}(\{g_{\{l\}}\}) \equiv \text{Tr} \left( \prod_{n \in \delta_{j^+_l,j^-_l}} R^{l^+_n}(g_{j^+_l}^+ 1 \otimes g_{j^-_l}^-) \prod_{n \in \delta_{j^-_l,j^+_l}} \sum_{i^+_n, i^-_n} f_{i^+_n, i^-_n}^n e_{i^+_n}^+ \otimes e_{i^-_n}^- \right).
\]

In contrast to the SU(2) intertwiner \( \tau_{\text{EPRL}}(\eta) \) the intertwiner \( \zeta^{\text{KKL}}(\eta) \) is Spin(4) invariant and therefore the space \( \mathcal{H}_{\text{KKL}} \) spanned by the states (B.29) is a proper subspace of \( \mathcal{H}_{\text{BF}} \). For a visualization of the different spin nets see figure B3.

Although, (B.29) are linearly independent they are not orthogonal (see [15]) with respect to the BF-scalar product since

\[
\left( \tau_{\text{KKL}}^{\delta_{j^+_l,j^-_l},\eta_{\kappa_n}}, \tau_{\text{KKL}}^{\delta_{j^+_l,j^-_l},\eta_{\kappa_n}} \right)_{\text{BF}} \equiv \int_{\text{Spin(4)}} \prod \text{d}g_{\{l\}} T_{\delta_{j^+_l,j^-_l},\eta_{\kappa_n}}^{\text{KKL}}(\{g_{\{l\}}\}) T_{\delta_{j^+_l,j^-_l},\eta_{\kappa_n}}^{\text{KKL}}(\{g_{\{l\}}\})
\]

\[
= \prod_{l^\pm} \delta_{j^+_l,j^-_l} \prod_{n} \left[ \sum_{i^+_n, i^-_n} f_{i^+_n, i^-_n}^n e_{i^+_n}^+ \otimes e_{i^-_n}^- \right]
\]

(30)
\[ \sum_{\ell^-} f_{\ell^-} f_{\ell^-}^* \] (a) A link in \( H_{KKL} \)

\[ R_{j^-} (g^-) \]

\[ R_{j^-} (g^+) \]

\[ \tau (\eta) \]

\[ \tau (\eta') \]

\[ R_{j^+} (g^+) \]

\[ R_{j^-} (g^-) \]

Figure B3. Different graphical visualization of a link in \( H_{KKL} \) and in \( H_{EPRL} \).

and \( H_{\eta}^\eta \) is in general not even diagonal. However, the KKL-map (B.26) is injective [15] and if \( \{a\} \) is an orthonormal basis in SU(2) intertwiner space then \( \{\xi_{KKL}(a)\} \) constitutes a basis in the KKL-intertwiner space. Instead of diagonalizing this basis we can, introduce an operator \( Q \)

\[ Q : \text{Inv}_{KKL} \left( \bigotimes_e H_{\rho_e} \right) \rightarrow \text{Inv}_{KKL} \left( \bigotimes_e H_{\rho_e} \right) \] (B.31)

\[ Q[\xi_{KKL}(a)] = \sum_b Q^a_b \xi_{KKL}(b) \] (B.32)

such that

\[ \delta^a_c = \langle \xi_{KKL}(c)|Q[\xi_{KKL}(a)] \rangle_{BF} \] (B.33)

as suggested in [20]. By expanding the KKL-map \( \xi_{KKL}(a) = f_a^{a_+} a_+ \otimes a_- \) in an orthonormal basis \( \{a_+ \otimes a_-\} \) of Spin(4) intertwiners, \( Q \) can be defined equivalently by

\[ Q = \tilde{Q}^a_b (\xi_{KKL}(a))^\dagger \otimes \xi_{KKL}(b) \]

\[ = \hat{f}^a_{a_+} \hat{Q}^a_{b_+} f_{b_-}^{a_-} (a_+ \otimes a_-)^\dagger \otimes (b_+ \otimes b_-) \] (B.34)

where \( \hat{f}^a_{a_+} \hat{Q}^a_{b_+} = \delta^a_{a_+} \delta^a_{b_+} [H^{-1}]^a_{b_+} \). Thus, the natural scalar product on \( H_{KKL} \) is the product

\[ \langle T_{Y,\alpha_n,j_l}^{KKL} | T_{Y',\alpha'_n,j'_l}^{KKL} \rangle_{KKL} = \langle T_{Y,\alpha_n,j_l}^{KKL} | \prod_{n \in Y} Q_n | T_{Y',\alpha'_n,j'_l}^{KKL} \rangle_{BF} \] (B.35)

\[ = \delta_{Y,Y'} \prod_l \delta_{j_l,j'_l} \prod_n \delta_{\alpha_n,\alpha'_n} \] (B.35)

with respect to which the states \( T_{KKL} \) are orthogonal. Note, the identity operator w.r.t (B.35) is formally

\[ \left( \prod_{l \in Y} d_{j_l}^* d_{j_l} \right) I_{Y,\text{KKL}} = |T_{Y}^{KKL}\rangle_{BF} |T_{Y}^{KKL}\rangle \prod_{n \in Y} Q_n. \] (B.36)
In contrast, the EPRL-states (B.21) are already orthogonal and therefore it is possible to implement the simplicity constraint before performing the integration on the bulk variables. In the second approach the model is defined by restricting the representations of the BF-partition function (B.13). If we would have done this before integration then we would encounter additional edge amplitudes due to (B.30). Therefore, it is advisable to label each internal edge by an operator $Q_e$ so that (B.28) is replaced by

$$
\tilde{Z}_{KKL}[\kappa] = \sum_{\{ n \}} \sum_{\{ a_{(e)} \}} \prod_{e \in G_{\kappa}} \mathcal{A}_e(a_{(e)}^+, a_{(e)}^-) \prod_{v \in G_{\kappa}} \mathcal{A}_v(a_{(v)}^+) T^{BF}_{g_{e_a}, a_{(v)}^-} ([g_{f_v}]) \quad (B.37)
$$

with edge amplitude

$$
\mathcal{A}_e(a_{(e)}^+, a_{(e)}^-) := \tilde{f}_{a_{(e)}^+, a_{(e)}^-}^{\kappa} (\tilde{Q}_e)^{a_{(e)}^+} (\tilde{Q}_e)^{a_{(e)}^-}. \quad (B.38)
$$

Here, the fusion coefficients are absorbed in the edge amplitude and therefore $T^{KKL}$ had to be replaced by $T^{BF}$. Nevertheless, (B.37) still defines a distribution in $H_{KKL}$. In the following we will mainly work with the object (B.37) and just write $Z[\kappa]$ instead of $Z^{KKL}[\kappa]$ to keep the notation simple.

### B.2. Projected spin networks

The advantage of the KKL-model is the preservation of covariance (see [38]) but merging canonical and covariant approach is more complicated and will involve so-called projected spin nets [13, 14].

The difficulty is to find a map $H_0 \rightarrow H_{KKL}$ projecting the $SU(2)$ invariant functions in $H_0$ onto Spin(4) invariant functions $T^{KKL}$. To do so we first need to establish an isomorphism between $SU(2)$ and a $SU(2)$-subgroup of $Spin(4)$. Unfortunately, there exist no canonical choice of such a subgroup but one has to fix a normal $n$ left invariant by the $SU(2)$-subgroup $SU_2(n) \subset Spin(4)$.

The manifold $SU(2)$ is isomorphic to the sphere $S^3$, which is uniquely determined by the set of vectors $n \in \mathbb{R}^4$, $\|n\| = 1$, that is, we can define a bijection

$$
\omega : S^3 \rightarrow SU(2) \quad n \mapsto \omega(n) = \frac{1}{2} n_\mu \sigma^\mu \quad (B.39)
$$

where $\sigma^0 = i I_2$ and $\sigma^i$ are the Pauli matrices and construct a projection $\pi_2 : Spin(4) \rightarrow SO(4), (g_L, g_R) \mapsto E(g_L, g_R)$, such that $\omega(E(g_L, g_R) \cdot n) := g_L \omega(n) (g_R)^{-1}$. Fix $T \equiv (1, 0, 0, 0)$ then the $SU(2)$-subgroup $SU_T(2) \subset Spin(4)$ stabilizing $T$ is the set of all elements $(h, h) \in Spin(4)$. Since any normal is uniquely determined by the action of $SO(4)$ on $T$, i.e.

$$
\omega(n) = \omega(E(B_n^L, B_n^R) T) = B_n^R (B_n^L)^{-1} \quad (B.40)
$$

for some $B_n = (B_n^L, B_n^R) \in Spin(4)$, the subgroup $SU_2(n)$ is the set of all elements

$$
B_n \triangleright (h, h) = (B_n^L h (B_n^L)^{-1}, B_n^R h (B_n^R)^{-1}).
$$

Note, the projection $\pi_2$ is two-to-one because $E(g_L, g_R) = E(-g_L, -g_R)$ which is due to the fact that $Spin(4)$ is the double cover of $SO(4)$.

On the one hand, it is necessary to fix a normal in order to identify the different copies of $SU(2)$ but, on the other hand, this breaks $Spin(4)$-invariance. A way out of this dilemma is to consider spin network functions whose nodes $v$ are also labeled by normals $n_v$ that transform in the defining $SO(4)$-representation: $\Lambda \triangleright n := E(g_L, g_R) n$ for $\Lambda = (g_L, g_R) \in Spin(4)$. Let $K$ be the space of square integrable, gauge invariant functions

$$
\phi(\gamma, \{ g_L, n_v \}) = \phi(\gamma, \{ \Lambda_{x(v)} g_{(\Lambda_{x(v)}^{-1})}, \{ \Lambda \triangleright n_v \} \}) \quad (B.41)
$$
with the scalar product
\[
\langle \phi | \phi' \rangle = \delta_{\gamma,\gamma'} \left( \prod_{v} \int_{S} \! d\mathbf{n}_{v} \delta(\mathbf{n}_{v} - \mathbf{n}_{v}') \right) \sqrt{\det(\mathbf{g}_{v})} \int \left[ \prod_{l} \! dg_{l} \right] \phi(\gamma', \{ g_{l} \}, \{ n_{v} \}) \phi(\gamma, \{ g_{l} \}, \{ n_{v} \})
\]
(B.42)

Remarkably, the so-called projected spin network functions \((B.41)\) do not depend on the choice of the normal \(\mathbf{n}_{v}\). In [14], the authors have shown, using Schur orthogonality, gauge invariance \((B.41)\) and the properties of the intertwiners \((B.20)\), that \(K\) is spanned by the orthonormal functions
\[
\phi_{\gamma,\mu,\eta}(|\{ g_{l} \}, \{ n_{v} \}|) := \prod_{l \in \gamma} \sqrt{d_{\mu} d_{\eta}} \text{Tr} \left[ \prod_{l \in \gamma} R_{\frac{l}{2}}^{\frac{l}{2}} \left( (B_{n_{\mu l}}^{\frac{l}{2}})^{-1} g_{l}^{\frac{l}{2}} B_{n_{\eta l}}^{\frac{l}{2}} \right) R_{\frac{l}{2}}^{\frac{l}{2}} \left( (B_{n_{\eta l}}^{\frac{l}{2}})^{-1} g_{l}^{\frac{l}{2}} B_{n_{\mu l}}^{\frac{l}{2}} \right) \right] \times \prod_{v} \tau_{\text{EPRL}}(\eta_{v})
\]
(B.43)

Here, the coupling is not restricted in contrast to the EPRL-states where \(\eta_{v}\) couples to the highest (lowest) weight of \(\mathcal{H}_{j_{v}} \otimes \mathcal{H}_{j_{v}'}\). It is even allowed that \(\eta_{v(1)}\) and \(\eta_{v(1)}\) couple to different spins \(j_{v(1)}\), \(j_{v(1)}'\) \(\in \{ j_{v} - j_{v}', \ldots, j_{v} + j_{v}' \}\).

When fixing a time gauge, \(\mathbf{n}_{v} = \mathbf{T} \forall \gamma \in \gamma^{(0)}\), and restricting \(g_{l}\) to the subgroup \(SU(2)\) then \((B.43)\) reduce to usual \(SU(2)\)-spin network functions provided that \(j_{v(1)} = j_{v(1)}'\) and vanishes otherwise. This can be easily verified by using the equivariant property of intertwiners,
\[
[R^{h}](\gamma_{m_{1}})_{\gamma_{m_{2}}}^{\gamma_{m_{3}}} C_{j_{1},m_{1};j_{1},m_{2}}^{j_{1},m_{1};j_{1},m_{2}} = C_{j_{1},m_{1};j_{1},m_{2}}^{j_{1},m_{1};j_{1},m_{2}} [R^{h}](\gamma_{m_{1}})_{\gamma_{m_{2}}}^{\gamma_{m_{3}}},
\]
and the normalization of Clebsch–Gordan coefficients, \(C_{j_{1},m_{1};j_{1},m_{2}}^{j_{1},m_{1};j_{1},m_{2}} = \delta_{j_{1}j_{1}'} \delta_{m_{1}m_{1}'} \delta_{m_{2}m_{2}'}\). More precisely, if \(j_{v(1)} = j_{v(1)}'\)
\[
\phi_{\gamma,\mu,\eta}(|\{ g_{l} \}, \{ n_{v} \}|) = \prod_{l \in \gamma} \sqrt{d_{\mu} d_{\eta}} \text{Tr} \left[ \prod_{l \in \gamma} R_{\frac{l}{2}}^{\frac{l}{2}} (h_{l}) R_{\frac{l}{2}}^{\frac{l}{2}} (h_{l}) \prod_{v} \tau_{\text{EPRL}}(\eta_{v}) \right]
\]
(B.44)

Vice versa, kinematical states \(T \in \mathcal{H}_{0} \) to \(K\) can be lifted via the expansion of convolutions of \(SU(2)\) and spin(4) in terms of characters \(\chi\) and \(\Theta\) respectively. Explicitly,
\[
[L,T_{\gamma,\mu,\eta}](|\{ g_{l} \}, \{ n_{v} \}|) := \prod_{l} \left[ N_{l} \sum_{j_{l}} \int \text{SU}(2) \! dh_{l} dk_{l} \chi^{j_{l}}(k_{l} h_{l}) \Theta^{j_{l}}_{v} (B_{n_{\mu l}}^{-1} g_{l} B_{n_{\eta l}}, h_{l}) \right] T_{\gamma,\mu,\eta}(|\{ k_{l} \}|)
\]
(B.45)

with some normalization constant \(N_{l}\). Below, we are only interested in the case where \(j_{l}^{1}/j_{l}^{2}\) are determined by the simplicity constraint \(j_{l}^{1}/R = j^{\pm}\).

Relation between. \(K, \mathcal{H}_{\text{EPRL}}^{\mathcal{K}}\) and \(\mathcal{H}_{\text{Kkl}}^{\mathcal{K}}\) When the normals are fixed but the group elements left arbitrary then the states \((B.43)\) are obviously basis states of \(\mathcal{H}_{\text{EPRL}}^{\mathcal{K}}\). Integrating over the normals yields states in \(\mathcal{H}_{\text{Kkl}}^{\mathcal{K}}\):
\[
\int_{S} \prod_{v \in \gamma^{(0)}} \! d\mathbf{n}_{v} \phi_{\gamma,\mu,\eta}(|\{ g_{l} \}, \{ n_{v} \}|) := \int_{\text{Spin}(4)} \prod_{v \in \gamma^{(0)}} \! dB_{v} \phi_{\gamma,\mu,\eta}(|(B_{v(1)})^{-1} g_{v} B_{v(1)}|)
\]
\[
= \text{Tr} \left\{ \prod_{l \in \gamma} \sqrt{d_{\mu} d_{\eta}} \left[ R_{\frac{l}{2}}^{\frac{l}{2}} (g_{v}) R_{\frac{l}{2}}^{\frac{l}{2}} (g_{v}) \right] \prod_{v \in \gamma^{(0)}} \left[ \sum_{\frac{k_{v}^{0}}{l_{v}^{0}} \cdots \frac{k_{v}^{0}}{l_{v}^{0}}} f_{l_{v}^{0} \cdots l_{v}^{0}}^{k_{v}^{0} \cdots k_{v}^{0}} \otimes t_{v}^{0} \right] \right\}
\]
(B.46)
which shows that
\[ P_y : \mathcal{H}_{\text{inv},y} \rightarrow \mathcal{H}_{y}^{\text{K KL}} \]
\[ [P_y \psi_y](g_1) = \int \prod_v d\nu_v \left[ L \psi_y \right](g_1, \{n_v\}) \]  
(B.47)
defines an isomorphism between \( \mathcal{H}_{0,y} \) and \( \mathcal{H}_{y}^{\text{K KL}} \). For instance a normalized state in \( \mathcal{H}_0 \) (see (2.1)) is lifted to
\[ [L T_{y,j,i}](g_1, \{n_v\}) = \prod_t N_l \sqrt{d_{j_t}} \frac{d_{j_t}}{d_{j_t}'} \times \text{Tr} \left( \prod_t R_t^{j_t} ((B_{s(t)}^{pl})^{-1} g_t^s B_{s(t)}^{pr}) R_t^{\gamma_t} ((B_{s(t)}^{pl})^{-1} g_t^s B_{s(t)}^{pr}) \prod_v \tau_{\text{EPRL}}(\eta_v) \right) \]  
(B.48)and afterwards projected
\[ [P_y T_{y,j,i}](g_1) = \left( \prod_t N_l \sqrt{d_{j_t} d_{j_t}'} \right) \int \text{spin}(A) \left( \prod_v d\nu_v \right) \times \text{Tr} \left( \prod_t R_t^{j_t} ((B_{s(t)}^{pl})^{-1} g_t^s B_{s(t)}^{pr}) R_t^{\gamma_t} ((B_{s(t)}^{pl})^{-1} g_t^s B_{s(t)}^{pr}) \prod_v \tau_{\text{EPRL}}(\eta_v) \right) \]
\[ = \left( \prod_t N_l \frac{N_l}{(d_{j_t} d_{j_t}')^2} \right) \tilde{T}_{y,j,i}^{\text{K KL}}(g_1) \]  
(B.49)to an orthonormal state \( \tilde{T}_{y,j,i}^{\text{K KL}} := \left( \prod_t \sqrt{d_{j_t} d_{j_t}'} \right) T_{y,j,i}^{\text{K KL}} \) (w.r.t. (B.35)).

**Appendix C. Explicit form of \( \hat{Z}_K \)**

In the following we give an explicit expression for \( \hat{Z}_K \) defined on the KKL-amplitude (B.37) and discuss its properties.

**Definition 17.** Let \( (\kappa, \mathcal{H}_f, Q_\kappa) \) be an abstract spin foam whose faces are labeled by EPRL-triples \( (j, j^+, j^-) \) and whose edges carry an operator \( Q_\kappa : \chi^{\text{K KL}}(\mathcal{H}_{0,\text{inv}}) \rightarrow \chi^{\text{K KL}}(\mathcal{H}_{e,\text{inv}}) \) defined in (B.33). Suppose \( \delta \kappa \) is the disjoint union of an initial graph \( \gamma_i \) and final graph \( \gamma_f \) then
\[ \hat{Z}_K : \mathcal{H}_{0,\gamma_i} \rightarrow \mathcal{H}_{0,\gamma_f} \]
\[ \langle T_{\gamma_f} | \hat{Z}_K | T_{\gamma_i} \rangle_0 := \langle P_y T_{\gamma_f} | Z_K | P_y T_{\gamma_i} \rangle_{\text{K KL}}. \]  
(C.1)
Here, \( Z_K \) is the amplitude (B.37) with an additional face weight \( A_f = \tilde{d}_{j_f} d_{j_f} \) and \( P_y : \mathcal{H}_{0,y} \rightarrow \mathcal{H}^{\text{K KL}}_y \) is the isometry (B.47) with normalization constant \( N_y = (d_{j_f} d_{j_f}')^{3/2} \).

With the results of [15], it is straightforward to show that the adjoint \( \hat{Z}_K^* \) is equal to \( \hat{Z}_K \) and that the following properties hold.

---

\[ \langle T_{\gamma_f} | Z_K | T_{\gamma_i} \rangle_0 := \sum_{x, x'} \sum_{\sigma \in \mathcal{D}_{x,x'}} \left[ \langle PT_{\gamma_f} | T_{\gamma_i}^{\text{K KL}} \rangle \langle T_{\gamma_f}^{\text{K KL}} | \hat{Z}_K | T_{\gamma_i} \rangle \bigg| \langle PT_{\gamma_f} | T_{\gamma_i}^{\text{K KL}} \rangle \langle T_{\gamma_f}^{\text{K KL}} | \hat{Z}_K | T_{\gamma_i} \rangle \right]. \]
C.1.1. (a) Subdivision of edges. If $e \in \mathcal{E}_{\text{int}}$ is an internal edge which is subdivided into $e_1, e_2$ by a vertex $v_0 = t(e_1) = s(e_2)$ then

$$A_{v_0}(t^+_1, t^-_1, t^+_2, t^-_2) = (t^+_1 \otimes t^-_1) \otimes (t^+_2 \otimes t^-_2) = \delta^{t^+_1}_{e_1} \delta^{t^-_1}_{e_1} \delta^{t^+_2}_{e_2} \delta^{t^-_2}_{e_2},$$

(C.2)

and therefore

$$\sum_{i_1, i_2} \sum_{i_1, i_2} A_{v_0}(t^+_1, t^-_1, i_1, i_2, i_2) A_{v_0}(i_1, i_2, i_2, i_1, i_1) A_{v_0}(i_1, i_2, i_2, i_1, i_1)
= \sum_{i_1, i_2} \sum_{i_1, i_2} f^{i_1 i_2}_{\otimes i_1 i_2} (Q_{e_1})_{i_1 i_2} f^{i_2 i_1}_{\otimes i_1 i_2} (Q_{e_2})_{i_1 i_2} f^{i_1 i_2}_{\otimes i_1 i_2}
= f^{i_1 i_2}_{\otimes i_1 i_2} (Q_{e_1})_{i_1 i_2} f^{i_2 i_1}_{\otimes i_1 i_2} (Q_{e_2})_{i_1 i_2}
= A_{v_0}(t^+_1, t^-_1, t^+_2, t^-_2),$$

(C.3)

where $t$ and $t^+ \otimes t^-$ are normalized SU(2) respectively Spin(4) invariants assigned to $e_0$. This proves that $\hat{Z}[\kappa]$ is invariant under a subdivision of edges.

C.1.2. (b) Subdivision of faces. A colored subdivision of a face $f$ can be obtained by joining two vertices in $f$ by an edge $e_0$ lying in the interior of $f$. The sub-faces $f_1, f_2$ adjacent to $e_0$ inherit the coloring and orientation of $f$ so that the SU(2) intertwiner space attached to $e_0$ is $\text{Inv}(\mathcal{H}_{f_1} \otimes \mathcal{H}_{f_2})$. Since this space is one-dimensional the edge amplitude $A_{v_0}$ reduces to the identity. However, the edge $e_0$ also gives rise to a splitting of the vertex boundary graphs at its source $s$ and target $t$ by splitting the link $e(f) \in \gamma_{f_1}$ associated to $f$. Hence, we have to insert the unique two-valent intertwiners

$$(e_{m_1}^{m_2})_{\delta} = \frac{1}{\sqrt{d_{f_1} d_{f_2}}} (e_{m_1}^{m_2})_{\delta},$$

(C.4)

into the vertex amplitude

$$A_{f_1, f_2} = \text{Tr} \left( \cdots \left( e_{m_1}^{m_2} \cdots m_1 \cdots \right) \left( e_{m_1}^{m_2} \cdots m_1 \cdots \right) \cdots \right)
= \frac{1}{\sqrt{d_{f_1} d_{f_2}}} \text{Tr} \left( \cdots \left( e_{m_1}^{m_2} \cdots m_1 \cdots \right) \left( e_{m_1}^{m_2} \cdots m_1 \cdots \right) \cdots \right).$$

(C.5)

Here, $e_1 \in f$ and $e_2 \in f$ are the unique edges meeting at $s/f$ that also bound $f_1$ and $f_2$ respectively. Let $\kappa'$ be the complex obtained from $\kappa$ by such a subdivision then $\hat{Z}[\kappa'] = d_{f_1} d_{f_2} Z[\kappa]$ due to (C.5). To restore invariance under face splitting one needs to introduce a face amplitude $A_f$ for which

$$A_{f_1, f_2} = \frac{1}{d_{f_1} d_{f_2}} A_f,$$

(C.6)

C.1.3 (c) Gluing and resolution of the identity. Suppose $\kappa_1$ and $\kappa_2$ are foams whose boundaries $\partial \kappa_1 = \gamma_{f_1} \cup \gamma_{f_2}$ decompose each into one final $(f)$ and one initial $(i)$ graph with $\gamma_{f_1} \equiv \gamma_{f_2} \equiv \gamma$. Recall, that the states $T^{\text{KKL}}$ induced on the boundary graph are not normalized and all internal edges adjacent to a final graph are incoming thus

$$\cdots | \hat{Z}[\kappa_1]|_{T_{f_1, i_1}^{f_1, i_1}} = \sum_{f_j, i_j} \left( \prod_{l \in \gamma^{\text{fin}}} A_{f_l} \right) \prod_{n \in \gamma^{\text{int}}} f_{l_n}^{i_n} (Q_{e_n})_{i_n}^{l_n} | T_{f_1, i_1}^{\text{KKL}} \rangle_{n} | P T_{f_1, i_1}^{\text{KKL}} \rangle_{n}$$

(C.7)
where $f_l$ is the unique face containing $l \in \gamma^{(1)}$ and $e_n$ the edge adjacent to $n \in \gamma^{(0)}$. When $\kappa_1$ and $\kappa_2$ are glued (see (3.9)) along the spin net $s = (\gamma, j_i, l_n)$ this implies
\[
\sum_s \langle T_{s_2}^{f_{l_2}} | \tilde{Z}[\kappa_2] | T_s | \tilde{Z}[\kappa_1] | T_{s_1} \rangle = \prod_{i \in \gamma} \frac{A_{f_j}}{d_{f_j}^2 d_{f_{j_i}}^2} \langle T_{s_2}^{f_{l_2}} | \tilde{Z}[\kappa_2] \not\in \kappa_1 | T_{s_1} \rangle. \tag{C.8}
\]

The foam $\kappa_2 \not\in \kappa_1$ is the complex which arises when $\gamma_1$ and $\gamma_2$ are identified and then removed. More precisely, the faces $f_1^l \in \kappa_1$ and $f_2^l \in \kappa_2$ are combined to one face in $\kappa_2$ which produces an excess face amplitude. Concluding, if the face weight is fixed to $A_f = d_{f_j}^2 d_{f_{j_i}}^2$ then the amplitude is invariant under face splittings and obeys a gluing property. By an analogue computation one can also show that
\[
\langle T_{\gamma,j_i,i_n} | \tilde{Z}[\kappa_0] | T_{\gamma,j_i,i_n} \rangle = \langle (PT)_{\gamma,j_i,i_n} | (PT)_{\gamma,j_i,i_n} \rangle_\text{KKL} = \langle T_{\gamma,j_i,i_n} | T_{\gamma,j_i,i_n} \rangle_\text{0}. \tag{C.9}
\]

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