The $N_C \to \infty$ limit of QCD gives a useful approximation scheme to the physical hadronic world. A brief overview of the mesonic sector is presented. The large–$N_C$ constraints on the low-energy chiral couplings are summarized and the role of unitarity corrections is discussed. As an important illustration of the $1/N_C$ expansion techniques, the Standard Model prediction of $\varepsilon'/\varepsilon$ is reviewed.

1 Mesons at Large $N_C$

The limit of an infinite number of quark colours turns out to be a very useful starting point to understand many features of the strong interaction. The $SU(N_C)$ gauge theory simplifies considerably at $N_C \to \infty$, while keeping the most essential properties of QCD. Choosing the coupling constant $g_s$ to be of $O(1/\sqrt{N_C})$, i.e., taking the large–$N_C$ limit with $\alpha_s N_C$ fixed, there exists a systematic expansion in powers of $1/N_C$, which for $N_C = 3$ provides a good quantitative approximation scheme to the hadronic world. The combinatorics of Feynman diagrams at large $N_C$ results in simple counting rules, which characterize the $1/N_C$ expansion:

1. Dominance of planar diagrams with an arbitrary number of gluon exchanges (and a single quark loop at the edge for matrix elements of quark bilinears).

2. Non-planar diagrams are suppressed by factors of $1/N_C^2$.

3. Internal quark loops are suppressed by factors of $1/N_C$.

The summation of the leading planar diagrams is a very formidable task, which we are still unable to perform. Nevertheless, making the very plausible assumption that colour confinement persists at $N_C \to \infty$, a very successful picture of the meson world emerges.

Let us consider a generic $n$-point function of local quark bilinears $J = \bar{q} \Gamma q$:

$$\langle T (J_1 \cdots J_n) \rangle \sim O(N_C).$$

(1)

A simple diagrammatic analysis shows that at large $N_C$ the only singularities
are one-meson poles. For instance, the two-point function takes the form:

\[ \langle J(k) J(-k) \rangle = \sum_n \frac{f_n^2}{k^2 - M_n^2}. \]  

(2)

Thus:

i) \( f_n = \langle 0|J|n\rangle \sim O(\sqrt{N_C}) \) and \( M_n \sim O(1) \).

ii) There are an infinite number of meson states, since \( \langle J(k) J(-k) \rangle \) behaves logarithmically for large \( k^2 \).

iii) Mesons are free, stable and non-interacting.

At \( N_C \to \infty \), the \( n \)-point functions are given by sums of tree diagrams with free meson propagators and effective local interaction vertices among \( m \) mesons, which scale as \( V_m \sim O(N_C^{1-m/2}) \). Moreover, \( \langle 0|J|M_1 \cdots M_m \rangle \sim O(N_C^{1-m/2}) \). Each additional meson coupled to the current \( J \) or to an interaction vertex brings then a suppression factor \( 1/\sqrt{N_C} \).

Including gauge-invariant gluon operators, such as \( J_G = \text{Tr}(G_{\mu \nu} G^{\mu \nu}) \), the diagrammatic analysis can be easily extended to glue states. Since \( \langle T(J_G G_1 \cdots J_G G_m) \rangle \sim O(N_C^2) \), one derives the large–\( N_C \) counting rules \( \langle 0|J_G|G_1 \cdots G_m \rangle \sim O(N_C^{2-m}) \) and \( V[G_1, \cdots, G_m] \sim O(N_C^{2-m}) \). Thus, at \( N_C \to \infty \), glueballs are also free, stable, non-interacting and infinite in number. From the mixed correlators \( \langle T(J_1 \cdots J_n J_G G_1 \cdots J_G G_m) \rangle \sim O(N_C) \), one gets \( V[M_1, \cdots, M_p; G_1, \cdots, G_q] \sim O(N_C^{1-q-p/2}) \). Therefore, glueballs and mesons decouple at large \( N_C \), their mixing being suppressed by a factor \( 1/\sqrt{N_C} \).

Many known phenomenological features of the hadronic world are easily understood at lowest order in the \( 1/N_C \) expansion: suppression of the \( \bar{q}q \) sea (exotics), quark model spectroscopy, Zweig’s rule, light SU(3) meson nonets, narrow resonances, multiparticle decays dominated by resonant two-body final states, etc. In some cases, the large–\( N_C \) limit is in fact the only known theoretical explanation that is sufficiently general. Clearly, the expansion in powers of \( 1/N_C \) appears to be a sensible physical approximation at \( N_C = 3 \).
The large–$N_C$ limit provides a weak coupling regime to perform quantitative QCD studies. At leading order in $1/N_C$, the scattering amplitudes are given by sums of tree diagrams with physical hadrons exchanged. Crossing and unitarity imply that this sum is the tree approximation to some local effective Lagrangian. Higher-order corrections correspond to hadronic loop diagrams.

2 Chiral Symmetry

With $n_f$ massless quark flavours, the QCD Lagrangian $[\bar{q} = (\bar{u}, \bar{d}, \ldots)]$

$$\mathcal{L}_{\text{QCD}}^0 = -\frac{1}{4} G^a_{\mu\nu}G_a^{\mu\nu} + i \bar{q}_L \gamma^\mu D_\mu q_L + i \bar{q}_R \gamma^\mu D_\mu q_R$$

is invariant under global $U(n_f)_L \otimes U(n_f)_R$ transformations of the left- and right-handed quarks in flavour space: $q_{L,R} \rightarrow g_{L,R} q_{L,R}$, $g_{L,R} \in U(n_f)_L, U(n_f)_R$. Under very general assumptions it has been shown that, at $N_C \to \infty$, the symmetry group must spontaneously break down to the diagonal $U(n_f)_L + U(n_f)_R$. According to Goldstone’s theorem, $n_f^2$ pseudoscalar massless bosons appear in the theory, which for $n_f = 3$ can be identified with the $U(3)$ multiplet

$$\Phi = \begin{pmatrix} \pi^0 & \pi^- & \pi^0 \\ \eta_8 + \frac{1}{\sqrt{6}} \eta_1 & -\pi^0 & \pi^- \\ K^- & K^0 & K^+ \end{pmatrix}.$$ 

The unitary matrix

$$U(\phi) = u(\phi)^2 = \exp \left\{ i \sqrt{2} \Phi / f \right\}$$

gives a very convenient parameterization of the Goldstone fields. Under the chiral group it transforms as $U(\phi) \rightarrow g_R U(\phi) g_L^\dagger$.

The Goldstone nature of the pseudoscalar mesons implies strong constraints on their interactions, which can be most easily analyzed on the basis of an effective Lagrangian. Since there is a mass gap separating the pseudoscalar nonet from the rest of the hadronic spectrum, we can build an effective field theory (EFT) containing only the Goldstone modes. Moreover, the low-energy effective Lagrangian can be organized in terms of increasing powers of momenta (derivatives).

Let us consider an extended QCD Lagrangian, with quark couplings to external Hermitian matrix-valued fields $l_\mu, r_\mu, s, p$:

$$\mathcal{L}_{\text{QCD}} = \mathcal{L}_{\text{QCD}}^0 + \bar{q}_L \gamma^\mu l_\mu q_L + q_R \gamma^\mu r_\mu q_R - \bar{q}_L (s-ip) q_R - \bar{q}_R (s+ip) q_L.$$
The external fields can be used to incorporate the electromagnetic and semileptonic weak interactions, and the explicit breaking of chiral symmetry through the quark masses:

\[ s = \mathcal{M} + \ldots, \quad \mathcal{M} = \text{diag}(m_u, m_d, m_s). \] (6)

At lowest order in derivatives and quark masses, the most general effective Lagrangian consistent with chiral symmetry has the form:

\[
\mathcal{L}_2 = \frac{f^2}{4} \langle D_\mu U^\dagger D^\mu U + U^\dagger \chi + \chi^\dagger U \rangle, \quad \chi = 2B_0 \left( s + ip \right), \tag{7}
\]

where \( D_\mu U = \partial_\mu U - i r_\mu U \pm i U l_\mu \), \( \langle A \rangle \) denotes the flavour trace of the matrix \( A \) and \( B_0 \) is a constant, which, like \( f \), is not fixed by symmetry requirements alone. Taking functional derivatives with respect to the appropriate external fields, one finds that \( f \) equals the pion decay constant (at lowest order) \( f = f_\pi = 92.4 \text{ MeV} \), while \( B_0 \) is related to the quark condensate:

\[
B_0 = - \frac{\langle \bar{q}q \rangle}{f^2} = \frac{M^2_z}{m_u + m_d} = \frac{M^2_{B_0}}{m_s + m_d} = \frac{M^2_{B^\pm}}{m_s + m_u}. \tag{8}
\]

Formally, the chiral Lagrangian could be computed (non-perturbatively) from the QCD generating functional. The leading-order terms in \( 1/N_C \) should be of \( O(N_C^{-2-n}) \). The Lagrangian \( \mathcal{L}_2 \) obeys the correct \( N_C \) counting rules: \( f^2 \sim O(N_C), \ B_0 \sim M^2_\rho \sim U(\phi) \sim O(1) \). The \( U(\phi) \) matrix generates an expansion in powers of \( \phi/f \), giving the required \( 1/\sqrt{N_C} \) suppression for each additional meson field. Clearly, interaction vertices with \( n \) mesons scale as \( V_n \sim f^{2-n} \sim O(N_C^{1-n/2}) \). Since \( \mathcal{L}_2 \) has an overall factor of \( N_C \) and \( U \) is \( N_C \)-independent, the \( 1/N_C \) expansion is equivalent to a semiclassical expansion. Quantum corrections computed with the chiral Lagrangian will have a \( 1/N_C \) suppression for each loop.

At \( O(p^4) \), the conventional \( SU(3)_L \otimes SU(3)_R \)-invariant chiral Lagrangian is usually written as:

\[
\mathcal{L}_4 = L_1 \langle D_\mu U^\dagger D^\mu U \rangle^2 + L_2 \langle D_\mu U^\dagger D_\nu U \rangle \langle D^\mu U^\dagger D^\nu U \rangle \\
+ L_3 \langle D_\mu U^\dagger D^\mu U \rangle \langle D_\nu U^\dagger D^\nu U \rangle + L_4 \langle D_\mu U^\dagger D^\mu U \rangle \langle U^\dagger \chi + \chi^\dagger U \rangle \\
+ L_5 \langle D_\mu U^\dagger D^\mu U (U^\dagger \chi + \chi^\dagger U) \rangle + L_6 \langle U^\dagger \chi + \chi^\dagger U \rangle^2 \\
+ L_7 \langle U^\dagger \chi - \chi^\dagger U \rangle^2 + L_8 \langle \chi^\dagger U \chi^\dagger U + U^\dagger \chi U^\dagger \chi \rangle \\
- iL_9 \langle F^\mu_\nu R D_\mu U D_\nu U^\dagger \rangle + F^\mu_\nu L D_\mu U D_\nu U^\dagger + L_{10} \langle U^\dagger F^\mu_\nu R U F_{L\mu\nu} \rangle,
\]

where \( F^\mu_\nu_{L,R} \) are field-strength tensors of the \( l^\mu \) and \( r^\mu \) flavour fields.
As shown in Table 2, the phenomenologically determined values
their average order of magnitude,
Their divergent parts are renormalized by the
The fourth column shows the source used to get this information. The large–$N_C$ predictions
obtained within the single-resonance approximation are given in the last column.

Thus, at $O(p^4)$ we need ten additional coupling constants $L_i$ to determine
the low-energy behaviour of the Green functions. Terms with a single trace
are of $O(N_C)$, while those with two traces should be of $O(1)$. However, a
$3 \times 3$ matrix relation has been used to eliminate the additional structure
c $\langle D\mu U \bar{D}\nu U D^\mu U \bar{D}^\nu U \rangle$ with the result
2$\delta L_1 = \delta L_2 = -\frac{1}{3}\delta L_3 = \delta L_4 \sim O(N_C)$.
As shown in Table 2, the phenomenologically determined values of those
couplings follow the pattern suggested by the $1/N_C$ counting rules. Moreover, their average order of magnitude,
$L_i \sim f^2/(4A_\chi^2) \sim 2 \times 10^{-3}$, suggests a chiral symmetry-breaking scale $\Lambda_\chi \sim 1$ GeV.

One-loop graphs with the lowest-order Lagrangian $L_2$ contribute also at
$O(p^4)$ in the chiral expansion, but they are suppressed by a factor of $1/N_C$. Their divergent parts are renormalized by the $L_4$ couplings:

$$L_i = L_i^r(\mu) + \Gamma_i \frac{\mu^{D-4}}{32\pi^2} \left\{ \frac{2}{D-4} + \gamma_E - \log(4\pi) - 1 \right\}. \quad (10)$$

This introduces a renormalization scale dependence,

$$L_i^r(\mu_2) = L_i^r(\mu_1) + \frac{\Gamma_i}{(4\pi)^2} \log\left(\frac{\mu_1}{\mu_2}\right), \quad (11)$$

which is subleading in $1/N_C$. The phenomenological couplings given in Table 2 have been normalized at $\mu = M_\rho$.

The chiral loops generate non-polynomial contributions, with logarithms
and threshold factors as required by unitarity, which are completely predicted

| $i$ | $L_i^r(M_\rho)$ | $O(N_C)$ | Source | $\Gamma_i$ | $L_i^{N_C \to \infty}$ |
|-----|-----------------|-----------|--------|---------|---------------------|
| $2L_1 - L_2$ | $-0.6 \pm 0.6$ | $O(1)$ | $K_{e4}, \pi\pi \to \pi\pi$ | $3/16$ | $0$ |
| $L_2$ | $1.4 \pm 0.3$ | $O(N_C)$ | $K_{e4}, \pi\pi \to \pi\pi$ | $3/16$ | $1.8$ |
| $L_3$ | $-3.5 \pm 1.1$ | $O(N_C)$ | $K_{e4}, \pi\pi \to \pi\pi$ | $0$ | $-4.3$ |
| $L_4$ | $-0.3 \pm 0.5$ | $O(1)$ | Zweig rule | $1/8$ | $0$ |
| $L_5$ | $1.4 \pm 0.5$ | $O(N_C)$ | $F_K : F_\pi$ | $3/8$ | $2.1$ |
| $L_6$ | $-0.2 \pm 0.3$ | $O(1)$ | Zweig rule | $11/144$ | $0$ |
| $L_7$ | $-0.4 \pm 0.2$ | $O(1)$ | GMO, $L_3, L_8$ | $0$ | $-0.3$ |
| $L_8$ | $0.9 \pm 0.3$ | $O(N_C)$ | $M_\omega, L_5$ | $5/48$ | $0.8$ |
| $L_9$ | $6.9 \pm 0.7$ | $O(N_C)$ | $(\pi^2)_{\pi\pi}$ | $1/4$ | $7.1$ |
| $L_{10}$ | $-5.5 \pm 0.7$ | $O(N_C)$ | $\pi \to e\nu\gamma$ | $-1/4$ | $-5.4$ |

Table 1. Phenomenological values of the renormalized couplings $L_i^r(M_\rho)$ in units of $10^{-3}$. The fourth column shows the source used to get this information. The large–$N_C$ predictions obtained within the single-resonance approximation are given in the last column.
as functions of $f$ and the Goldstone masses. Although they are suppressed by a factor of $1/N_C$, the chiral logarithms can be numerically important since

$$\frac{1}{N_C} \log (A^2/M^2_\chi) \sim 4/3.$$  

### 2.1 Anomalies

Since chiral symmetry is explicitly violated by fermion anomalies at the fundamental QCD level, we need to add a functional $Z_A$ with the property that its change under chiral transformations reproduces the anomalous change of the QCD generating functional. For the non-Abelian anomalies associated with the external sources $l_\mu$ and $r_\mu$, such a functional was first constructed by Wess and Zumino, and reformulated in a nice geometrical way by Witten. It is an $O(p^4)$ effect, which is completely calculable with no free parameters. This contribution is of $O(N_C)$, because it is generated by a triangle quark loop coupled to external sources.

Much more subtle is the $U(1)_A$ gluonic anomaly which breaks the conservation of the singlet axial quark current in the chiral limit:

$$\partial_\mu (\bar{q}\gamma^\mu \gamma_5 q) = 2 n_f \omega; \quad \omega = \frac{\alpha_s}{16\pi} e^{\mu\nu\rho\sigma} G_{\mu\nu} G_{\rho\sigma}.$$  

The corresponding anomalous change of the QCD generating functional can be accounted for by adding a term $\Delta L_{\text{QCD}} = -\theta \omega$ with the appropriate chiral transformation for the so-called vacuum angle $\theta(x)$. Notice that in the large-$N_C$ limit the $U(1)_A$ anomaly is absent.

To lowest non-trivial order in $1/N_C$, the chiral symmetry breaking effect induced by the $U(1)_A$ anomaly can be taken into account in the effective low-energy theory, through the term

$$L_{U(1)_A} = -\frac{f^2}{4} \frac{a}{N_C} \left\{ \theta - \frac{i}{2} \left[ \log(\det U) - \log(\det U^\dagger) \right] \right\}^2,$$  

which breaks $U(3)_L \otimes U(3)_R$ but preserves $SU(3)_L \otimes SU(3)_R \otimes U(1)_V$.

The parameter $a$ has dimensions of mass squared and, with the factor $1/N_C$ pulled out, is booked to be of $O(1)$ in the large-$N_C$ counting rules. Its value is not fixed by symmetry requirements alone; it depends crucially on the dynamics of instantons. In the presence of the term, the $\eta_1$ field becomes massive even in the chiral limit:

$$M_{\eta_1}^2 = 3 \frac{a}{N_C} + O(M).$$  

Owing to the large mass of the $\eta_1'$, the effect of the $U(1)_A$ anomaly cannot be treated as a small perturbation. Rather, one should keep the term
together with the lowest-order Lagrangian (7). It is possible to build a consistent combined expansion in powers of momenta, quark masses and \(1/N_C\), by counting the relative magnitude of these parameters as:
\[
O(p^2) \sim O(M) \sim O(1/N_C).
\]
(15)
This expansion has been already analyzed at the next-to-leading order.

3 Resonance Chiral Theory

Let us consider a chiral-invariant Lagrangian \(\mathcal{L}(U, V, A, S, P)\), describing the couplings of resonance nonet multiplets of the type \(V(1^{--})\), \(A(1^{++})\), \(S(0^{++})\) and \(P(0^{--})\) to the Goldstone bosons.
\[
\mathcal{L}_2[V(1^{--})] = \sum_i \left\{ \frac{F_{V_i}}{2\sqrt{2}} \langle V_i^{\mu\nu} f_{+\mu\nu} \rangle + \frac{i G_{V_i}}{\sqrt{2}} \langle V_i^{\mu\nu} u_{\mu} u_{\nu} \rangle \right\},
\]
\[
\mathcal{L}_2[A(1^{++})] = \sum_i \frac{F_{A_i}}{2\sqrt{2}} \langle A_i^{\mu\nu} f_{-\mu\nu} \rangle,
\]
\[
\mathcal{L}_2[S(0^{++})] = \sum_i \left\{ c_{d_i} \langle S_i u^\mu u_\mu \rangle + c_{m_i} \langle S_i \chi^+_\mu \rangle \right\},
\]
\[
\mathcal{L}_2[P(0^{--})] = \sum_i i d_{m_i} \langle P_i \chi^- \rangle,
\]
(16)
where \(u_{\mu} \equiv i u^\dagger D_\mu U u^\dagger\), \(f_{\pm}^{\mu\nu} \equiv u F_L^{\mu\nu} u^\dagger \pm u^\dagger F_R^{\mu\nu} u\) and \(\chi_{\pm} \equiv u^\dagger \chi u^\dagger \pm u^\dagger \chi u\). The resonance couplings \(F_{V_i}, G_{V_i}, F_{A_i}, c_{d_i}, c_{m_i}\) and \(d_{m_i}\) are of \(O(\sqrt{N_C})\).

The lightest resonances have an important impact on the low-energy dynamics of the pseudoscalar bosons. Below the resonance mass scale, the singularity associated with the pole of a resonance propagator is replaced by the corresponding momentum expansion; therefore, the exchange of virtual resonances generates derivative Goldstone couplings proportional to powers of \(1/M_{V_i}^2\). At lowest order in derivatives, this gives the large-\(N_C\) predictions for the \(O(p^4)\) couplings of chiral perturbation theory (\(\chi\)PT).

\[
2 L_1 = L_2 = \sum_i \frac{C_{V_i}^2}{4 M_{V_i}^2}, \quad L_3 = \sum_i \left\{ -\frac{3 G_{V_i}^2}{4 M_{V_i}^2} + \frac{c_{d_i}^2}{2 M_{V_i}^2} \right\},
\]
\[
L_5 = \sum_i \frac{c_{d_i} c_{m_i}}{M_{S_i}^2}, \quad L_8 = \sum_i \left\{ \frac{c_{m_i}^2}{2 M_{S_i}^2} - \frac{d_{m_i}^2}{2 M_{P_i}^2} \right\},
\]
\[
L_9 = \sum_i \frac{F_{V_i} G_{V_i}}{2 M_{V_i}}, \quad L_{10} = \sum_i \left\{ \frac{F_{A_i}^2}{4 M_{A_i}^2} - \frac{F_{V_i}^2}{4 M_{V_i}^2} \right\}.
\]
(17)
All these couplings are of $O(N_C)$, in agreement with the counting made in Table 2, while for the couplings of $O(1)$ we get $2 L_1 - L_2 = L_4 = L_6 = L_7 = 0$.

Owing to the $U(1)_A$ anomaly, the $\eta_1$ field is massive and it is often integrated out from the low-energy chiral theory. In that case, the $SU(3)_L \otimes SU(3)_R$ chiral coupling $L_7$ gets a contribution from $\eta_1$ exchange:

$$L_7 = -\frac{\tilde{d}_m^2}{2 M_{\eta_1}^2}, \quad \tilde{d}_m = -\frac{f}{\sqrt{24}}.$$  

(18)

Since, $M_{\eta_1}^2 \sim O(1/NC, M)$, the coupling $L_7$ could then be considered of $O(N_C^2)$. However, the large--$N_C$ counting is no longer consistent if one takes the limit of a heavy $\eta_1$ mass ($N_C$ small) while keeping $m_s$ small.

3.1 Short-Distance Constraints

The short-distance properties of the underlying QCD dynamics impose some constraints on the low-energy EFT parameters:

1. **Vector Form Factor.** At leading order in $1/N_C$, the two-Goldstone matrix element of the vector current, is characterized by

$$F_V(t) = 1 + \sum_i F_{V_i} G_{V_i} \frac{t}{M_{V_i}^2 - t}.$$  

(19)

Since the vector form factor $F_V(t)$ should vanish at infinite momentum transfer $t$, the resonance couplings should satisfy

$$\sum_i F_{V_i} G_{V_i} = f^2.$$  

(20)

2. **Axial Form Factor.** The matrix element of the axial current between one Goldstone and one photon is parameterized by the axial form factor. From the resonance Lagrangian, one gets

$$G_A(t) = \sum_i \left\{ \frac{2 F_{V_i} G_{V_i} - F_{V_i}^2}{M_{V_i}^2} + \frac{F_{A_i}^2}{M_{A_i}^2 - t} \right\},$$  

(21)

which vanishes at $t \to \infty$ provided that

$$\sum_i \frac{2 F_{V_i} G_{V_i} - F_{V_i}^2}{M_{V_i}^2} = 0.$$  

(22)
3. \textit{Weinberg Sum Rules.} The two-point function built from a left-handed
and a right-handed vector quark current defines the correlator
\[ \Pi_{LR}(t) = \frac{f^2}{t} + \sum_i \frac{F^2_{V_i}}{M_{V_i}^2 - t} - \sum_i \frac{F^2_{A_i}}{M_{A_i}^2 - t}. \] (23)

Since gluonic interactions preserve chirality, \( \Pi_{LR}(t) \) satisfies an unsubtracted dispersion relation. Moreover, in the chiral limit it vanishes faster than \( 1/t^2 \) when \( t \to \infty \). This implies the well-known conditions:
\[ \sum_i (F^2_{V_i} - F^2_{A_i}) = f^2, \quad \sum_i (M_{V_i}^2 F^2_{V_i} - M_{A_i}^2 F^2_{A_i}) = 0. \] (24)

The second relation is correct up to very small quark-mass contributions.

4. \textit{Scalar Form Factor.} The two-pseudoscalar matrix element of the scalar
quark current contains another dynamical form factor, which for the \( K\pi \)
\[ F_{S_{K\pi}}(t) = 1 + \sum_i \frac{4 c_{m_i}}{f^2} \left\{ c_{d_i} + (c_{m_i} - c_{d_i}) \frac{M_{K}^2 - M_{A_i}^2}{M_{S_i}^2} \right\} \frac{t}{M_{S_i}^2 - t}. \] (25)

Requiring \( F_S(t) \) to vanish at \( t \to \infty \), one gets the constraints:
\[ 4 \sum_i c_{d_i} c_{m_i} = f^2, \quad \sum_i \frac{c_{m_i}}{M_{S_i}^2} (c_{m_i} - c_{d_i}) = 0. \] (26)

5. \textit{SS − PP Sum Rules.} The two-point correlation functions of two scalar
or two pseudoscalar currents would be equal if chirality was absolutely
\[ \Pi_{SS-PP}(t) = 16 B_0^2 \left\{ \sum_i \frac{c_{m_i}^2}{M_{S_i}^2 - t} - \sum_i \frac{d_{m_i}^2}{M_{P_i}^2 - t} + \frac{f^2}{8t} \right\}. \] (27)

For massless quarks, \( \Pi_{SS-PP}(t) \) vanishes as \( 1/t^2 \) when \( t \to \infty \), with a
coefficient proportional to \( \alpha_s \frac{\langle \bar{q}Tq\bar{q}Tq \rangle}{B_0^2} \). The vacuum four-quark
\[ 8 \sum_i (c_{m_i}^2 - d_{m_i}^2) = f^2, \quad \sum_i (c_{m_i}^2 M_{S_i}^2 - d_{m_i}^2 M_{P_i}^2) = \frac{3}{4} \frac{\pi \alpha_s}{f^4}. \] (28)
3.2 Single-Resonance Approximation

Let us approximate each infinite resonance sum with the contribution from the first meson nonet with the given quantum numbers. This is meaningful at low energies where the contributions from higher-mass states are suppressed by their corresponding propagators. The single-resonance approximation (SRA) corresponds to work with a low-energy EFT below the scale of the second resonance multiplets. The resulting short-distance constraints are nothing else than the matching conditions between this EFT and the underlying QCD dynamics. Thus, we are assuming that the short-distance operator product expansion provides an acceptable description at energies above 1.5 GeV.

Within the SRA, Eqs. (20), (22) and (24) determine the vector and axial-vector couplings in terms of $M_V$ and $f$:

$$F_V = 2 G_V = \sqrt{2} F_A = \sqrt{2} f, \quad M_A = \sqrt{2} M_V.$$  \hfill (29)

The scalar and pseudoscalar parameters are obtained from (26) and (28):

$$c_m = c_d = \sqrt{2} d_m = f/2, \quad M_P = \sqrt{2} M_S (1 - \delta)^{1/2}.$$  \hfill (30)

The last relation involves a small correction $\delta \approx 3 \pi \alpha_s f^2 / M_S^2 \sim 0.08 \alpha_s$, which we can neglect together with the tiny effects from light quark masses.

Inserting these predictions into Eqs. (17), one finally gets all $O(N_C P^4)$ $\chi$PT couplings, in terms of $M_V$, $M_S$ and $f$:

$$2 L_1 = L_2 = \frac{1}{4} L_0 = -\frac{1}{3} L_{10} = \frac{f^2}{8 M_V^2},$$  \hfill (31)

$$L_3 = -\frac{3 f^2}{8 M_V^2} + \frac{f^2}{8 M_S^2}, \quad L_5 = \frac{f^2}{4 M_S^2}, \quad L_8 = \frac{3 f^2}{32 M_S^2}.$$  \hfill (32)

The last column in Table 2 shows the results obtained with $M_V = 0.77$ GeV, $M_S = 1.0$ GeV and $f = 92$ MeV. Also shown is the $L_7$ prediction in [13], taking $M_{\eta} = 0.80$ GeV. The agreement with the measured values is a clear success of the large–$N_C$ approximation. It demonstrates that the lightest resonance multiplets give indeed the dominant effects at low energies.

The study of other Green functions provides further matching conditions between the hadronic and fundamental QCD descriptions. Clearly, it is not possible to satisfy all of them within the SRA. A useful generalization is the so-called Minimal Hadronic Ansatz, which consists of keeping the minimum number of resonances compatible with all known short-distance constraints for the problem at hand. Some $O(P^2)$ $\chi$PT couplings have been already analyzed in this way, by studying an appropriate set of three-point functions.
4 Unitarity Corrections

The $\chi$PT loops incorporate the unitarity field theory constraints in a perturbative way, order by order in the chiral expansion. Although subleading in the $1/N_C$ counting, these corrections may be enhanced by infrared logarithms. Their effect appears to be crucial for a correct understanding of some observables, in particular in the scalar sector, because the $S$–wave rescattering of two pseudoscalars is very strong. The combined constraints of analyticity and unitarity make possible to perform appropriate resummations of chiral logarithms, which describe the leading $1/N_C$ corrections in the resonance region.

A simple example is provided by the Omnès exponentiation of the pion form factor:

$$F_V(t) = \frac{M_V^2}{M_V^2 - t} \exp\left\{ -\frac{t}{96\pi^2 f^2} A^{(\pi)}(t) \right\}, \quad (33)$$

where $|\sigma_\pi| \equiv \sqrt{1 - 4M_\pi^2/t}$

$$A^{(\pi)}(t) = \sigma_\pi^3 \log\left( \frac{\sigma_\pi + 1}{\sigma_\pi - 1} \right) + \log\left( \frac{M_\pi^2}{\mu^2} \right) + 8 \frac{M_\pi^2}{t} - \frac{5}{3} - \delta L_9^r(\mu), \quad (34)$$

is the regularized one-loop function describing two intermediate pions (the small $KK$ loop contribution has been neglected), which arises here from an integration over the $I = J = 1 \pi\pi$ phase shift at leading order in $\chi$PT,

$$F_V(t) = Q_n(t) \exp\left\{ \frac{s^n}{\pi} \int_4M_\pi^2 \frac{dz}{z^n} \frac{\delta_1^r(z)}{z - t - i\epsilon} \right\}. \quad (35)$$

This expression is valid in the elastic region and has a polynomial ambiguity which is compensated by the subtraction function $Q_n(t)$. Only the logarithmic corrections are unambiguous. The ambiguity has been solved by matching the Omnès solution both to the $\chi$PT and large-$N_C$ (SRA) results. There remains a local indetermination at higher orders, made explicit through the constant $\delta L_9^r(\mu) \equiv 128 \pi^2 [L_9^r(\mu) - L_9^{N_C \to \infty}]$, which is next-to-leading in $1/N_C$ and does not contain any large infrared logarithm when $\mu \sim M_V$.

Equation (33) has obvious shortcomings. We have used an $O(p^2)$ approximation to the $\pi\pi$ phase shift, $\delta_1^r(t) = t \sigma_\pi^3/(96\pi f^2)$, which is a very poor (and even wrong) description at the higher end of the dispersive integration region. Nevertheless, one can always take a sufficient number of subtractions to emphasize numerically the low-energy region. Since our matching has fixed an infinite number of subtractions, this result should give a good approximation for values of $s$ not too large. Moreover, this can be phenomenologically improved with the use of the measured phase shifts.
Figure 2. Dyson-Schwinger resummation of $F_V(t)$ with effective vertices.

A more important question concerns the $\rho$ meson pole, which needs a proper treatment if one aims to describe physics around or above the resonance peak. The pole is regulated by the $\rho$ width, which vanishes at $N_C \to \infty$.

The dressed propagator can be calculated through a Dyson-Schwinger resummation constructed from effective Goldstone vertices containing both the local $\chi$PT interaction and the resonance-exchange contributions:

$$F_V(t) = \frac{M_V^2}{M_V^2 - t + \xi_\rho(t) - i M_V \Gamma_\rho(t)}, \quad (36)$$

where

$$\xi_\rho(t) - i M_V \Gamma_\rho(t) = \frac{t M_V^2}{96 \pi^2 f^2} A^{(\pi)}(t). \quad (37)$$

Thus,

$$\Gamma_\rho(t) = \theta(t - 4 M_\pi^2) \frac{t M_V}{96 \pi f^2} \sigma_\pi^3, \quad (38)$$

which at $t = M_\rho^2$ gives $\Gamma_\rho(M_\rho^2) = 144$ MeV, in reasonable agreement with the measured $\rho$ width. The intermediate $K\bar{K}$ contributions can be included through a coupled-channel resummation; the only modification is the change $A^{(\pi)}(t) \to A^{(\pi)}(t) + \frac{1}{2} A^{(K)}(t)$.

Equations (38) and (39) represent different resummations of higher-order corrections. They agree, by construction, at $O(p^4)$ in $\chi$PT and at the leading order in $1/N_C$. The result can be further improved by inserting into the Omnès exponential the phase shift predicted in (39),

$$\delta_1(t) = \arctan \left\{ \frac{M_V \Gamma_\rho(t)}{M_V^2 - t + \xi_\rho(t)} \right\} = \frac{t \sigma_\pi^3}{96 \pi f^2} + \cdots, \quad (39)$$

and imposing the appropriate matching conditions.
Similar unitarization procedures have been applied to amplitudes with $I = J = 0$, which get large corrections from infrared chiral logarithms.\(^\text{(25,34,35)}\)

5 Strangeness-Changing Non-Leptonic Weak Transitions

Since weak currents factorize at large-$\mathcal{N}_C$, a naive $1/\mathcal{N}_C$ description of $K \to \pi\pi$ would imply $A(K^0 \to \pi^0\pi^0) = 0$. In terms of isospin amplitudes, $A_0 = \sqrt{2}A_2$; \textit{i.e.}, there is no $\Delta I = 1/2$ enhancement at leading order in $1/\mathcal{N}_C$. 

| Energy Scale | Fields | Effective Theory |
|--------------|--------|-----------------|
| $M_Z$        | $W, Z, \gamma, g$ | Standard Model |
| $\lesssim m_c$ | $\gamma, g; \mu, e, \nu_i$ | $\gamma, g$; $\mu, e, \nu_i$ |
| $M_K$        | $\gamma; \mu, e, \nu_i$ | $\chi$PT |

Figure 5. Evolution from $M_Z$ to $M_K$.

A correct analysis should take into account the presence of very different mass scales. At short distances, the gluonic interactions induce large logarithmic corrections which scale as $\frac{1}{N_C} \log (M_W/\mu)$, while at large distances they generate infrared effects of the type $\frac{1}{N_C} \log (\mu/M_\pi)$. At $\mu \sim 1$ GeV, $\log (M_W/\mu) \sim 4$ and $\log (\mu/M_\pi) \sim 2$, which breaks the $1/N_C$ expansion.

The necessary summation of the short-distance logarithms is performed with the operator product expansion (OPE) and the renormalization group. After integrating out all heavy scales, one gets an effective $\Delta S = 1$ Lagrangian, defined in the three-flavour theory ($\mu < m_c$),

$$L_{\text{eff}}^{\Delta S=1} = -\frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \sum_{i=1}^{10} C_i(\mu) \tilde{Q}_i(\mu), \quad (40)$$

which is a sum of local four-fermion operators $Q_i$, constructed with the light degrees of freedom, modulated by Wilson coefficients $C_i(\mu)$ which are functions of the heavy masses. The overall renormalization scale $\mu$ separates the short- ($M > \mu$) and long- ($m < \mu$) distance contributions, which are contained in $C_i(\mu)$ and $Q_i$, respectively. The physical amplitudes are of course independent of $\mu$. The Wilson coefficients have been computed at the next-to-leading logarithmic order. All gluonic corrections of $O(\alpha_s^n t^n)$ and $O(\alpha_s^{n+1} t^n)$ are known, where $t \equiv \ln (M_1/M_2)$ refers to the logarithm of any ratio of heavy mass scales $M_1, M_2 \geq \mu$. 

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5.1 \( \chi PT \) Description

At lowest order in \( \chi PT \), the most general effective bosonic Lagrangian with the same \( SU(3)_L \otimes SU(3)_R \) transformation properties as the short-distance Lagrangian \( (40) \) contains three terms, transforming as \( (8_L, 1_R), (27_L, 1_R) \) and \( (8_L, 8_R) \), respectively. Their corresponding couplings are denoted by \( g_8, g_{27} \) and \( g_{ew} \). At tree level, the resulting \( K \to \pi \pi \) amplitudes take the form:

\[
A_0 = -\frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \sqrt{2} f_\pi \left\{ \left( g_8 + \frac{1}{9} g_{27} \right) (M_K^2 - M_\pi^2) - \frac{2}{3} f_\pi^2 e^2 g_{ew} \right\},
\]

\[
A_2 = -\frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \frac{2}{9} f_\pi \left\{ 5 g_{27} (M_K^2 - M_\pi^2) - 3 f_\pi^2 e^2 g_{ew} \right\}.
\]

(41)

The isospin amplitudes \( A_I = A_I e^{i\delta_I} \) have been computed up to next-to-leading order in \( \chi PT \). The only remaining problem is the calculation of the chiral couplings from the short-distance Lagrangian \( (40) \), which requires to perform the matching between the two EFTs. This can be easily done at \( N_C \to \infty \), because the four-quark operators factorize into currents which have well-known chiral realizations.

\[
\tilde{g}_8^\infty = \left\{ -\frac{2}{5} C_1(\mu) + \frac{3}{5} C_2(\mu) + C_4(\mu) - 16 L_5 C_6(\mu) B(\mu) \right\} f_0^{K\pi}(M_\pi^2),
\]

\[
\tilde{g}_{27}^\infty = \frac{3}{5} [C_1(\mu) + C_2(\mu)] f_0^{K\pi}(M_\pi^2),
\]

(42)

\[
e^2 [g_8 \tilde{g}_{ew}]^\infty_I = -3 \left\{ C_8(\mu) B(\mu) \xi_I + \Delta_I C(\mu) \frac{M_K^2 - M_\pi^2}{4 f_\pi^2} f_0^{K\pi}(M_\pi^2) \right\}.
\]

The effective couplings \( \tilde{g}_I^\infty \) include the local \( O(p^4) \) \( \chi PT \) corrections. They generate the factors \( \xi_0 = 1 + 4L_5 M_K^2 / f_\pi^2, \xi_2 = 1 + 4L_5 M_\pi^2 / f_\pi^2 \) and \( f_0^{K\pi}(M_\pi^2) \approx \xi_2 \), and introduce additional dependences on Wilson coefficients: \( \Delta_0 C(\mu) = [C_7 - C_9 + C_{10}(\mu)] \), \( \Delta_2 C(\mu) = -2[C_7 - C_9 - C_{10}(\mu)] \). At \( N_C \to \infty \),\n
\[
L_5^\infty = \frac{1}{2} (f_K f_\pi - f_\pi^2) / (M_K^2 - M_\pi^2) \approx 2.1 \cdot 10^{-3} \quad \text{and} \quad f_0^{K\pi}(M_\pi^2) \approx 1.02.
\]

The factorization of the operators \( Q_i \) \( i \neq 6, 8 \) does not provide any scale dependence, because their anomalous dimensions vanish at \( N_C \to \infty \). To achieve a reliable expansion in powers of \( 1 / N_C \), one needs to go to the next order, where this physics is captured. This is the reason why the study of the \( \Delta I = 1/2 \) rule has proven to be so difficult. The only anomalous dimensions which survive when \( N_C \to \infty \) are the ones corresponding to \( Q_6 \) and \( Q_8 \). These operators factorize into colour-singlet scalar and pseudoscalar currents, which are \( \mu \) dependent. This generates the factors...
B(\mu) \equiv M_K^2 / [(m_s + m_u)(\mu) f_\pi]^2 \ (m_s + m_d = m_q), which exactly cancel the 
\mu dependence of \mathcal{C}_{6,8}(\mu) at large \mathcal{N}_C. Since these two operators give the 
leading contributions to Im(g_I), the large–\mathcal{N}_C limit provides a good estimate
of the CP-violating ratio \varepsilon'/\varepsilon, while Re(g_I) gets large 1/\mathcal{N}_C corrections.\[44\]
The large–\mathcal{N}_C calculation does not produce any strong phases \delta_I. Those
phases originate in the final rescattering of the two pions and, therefore, are
generated by chiral loops which are of higher order in both the momentum and
1/\mathcal{N}_C expansions. Analyticity and unitarity require a corresponding disper-
sive effect in the moduli of the isospin amplitudes. Since the S–wave phase-shift
difference is very large, \[46\] 
\delta_0 - \delta_2 = \frac{\pi}{4} \pm 6^\circ, one should expect large
unitarity corrections.\[35\] This is confirmed by the \(O(p^4)\) \chi PT calculation, which
generates large infrared logarithms. With the normalization \[43\] 
\mathcal{A}_I^{(R)} = \mathcal{A}_I^{(R)} \times \mathcal{C}_I^{(R)}, the correction factors \mathcal{C}_I^{(R)} \equiv 1 + \Delta_L A_I^{(R)} take the values:

\begin{align*}
C_0^{(8)} &= 1.27 \pm 0.05 + 0.46 \ i , \\
C_0^{(27)} &= 2.0 \pm 0.7 + 0.46 \ i , \\
C_0^{(ew)} &= 1.27 \pm 0.05 + 0.46 \ i , \\
C_2^{(27)} &= 0.96 \pm 0.05 - 0.20 \ i , \\
C_2^{(ew)} &= 0.50 \pm 0.24 - 0.20 \ i .
\end{align*}

The quoted uncertainties correspond to changes of the \chi PT renormalization
scale between 0.6 and 1 GeV. The scale dependence is only present in t he
dispersive contributions and should cancel with the corresponding dependence
of the \(O(p^4)\) counterterms at the next-to-leading order in 1/\mathcal{N}_C.

5.2 The Standard Model Prediction for \varepsilon'/\varepsilon

The CP-violating ratio

\[ \frac{\varepsilon'}{\varepsilon} = e^{i \Phi} \frac{\omega}{\sqrt{2 |\varepsilon|}} \left[ \frac{\text{Im}(A_2)}{\text{Re}(A_2)} - \frac{\text{Im}(A_0)}{\text{Re}(A_0)} \right] , \quad \Phi \approx \delta_2 - \delta_0 + \frac{\pi}{4} \approx 0 , \]  

constitutes a fundamental test for our understanding of flavour-changing phe-
nomena. The present experimental world average,\[45\]

\[ \text{Re} \left( \frac{\varepsilon'}{\varepsilon} \right) = (17.2 \pm 1.8) \cdot 10^{-4} , \]  

provides clear evidence for the existence of direct CP violation.

The amplitudes Re(A_I), their ratio \omega = Re(A_2)/Re(A_0) \approx 1/22 and
\varepsilon are usually set to their experimentally determined values. A theoretical
calculation is then only needed for the CP-odd quantities Im(A_I), which are
dominated by the operators \(Q_6\) and \(Q_8\). To a very good approximation,\[45\]

\[ \frac{\varepsilon'}{\varepsilon} \sim \left[ B_6^{(1/2)} (1 - \Omega_{IB}) - 0.4 B_8^{(3/2)} \right] , \]  

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where the factors $B_i$ parameterize the $Q_i$ matrix elements in vacuum insertion units. The ratio

$$\Omega_{IB} = \frac{1}{\omega} \frac{\text{Im}(A_2)}{\text{Im}(A_0)} \approx 0.12 \pm 0.05$$  \hspace{1cm} (47)$$
takes into account isospin-breaking corrections, which get enhanced by $1/\omega$.

The isospin-breaking correction was originally estimated to be $\Omega_{IB} = 0.25$. Together with the usual ansatz $B_i \sim 1$, this produced a large numerical cancellation in Eq. (46) leading to unphysical low values of $\varepsilon'/\varepsilon$ around $7 \cdot 10^{-4}$. The $\chi$PT loop corrections destroy this accidental cancellation. The final result is governed by the matrix element of the gluonic penguin operator $Q_6$.

Taking into account all large logarithmic corrections at short and long distances, the Standard Model prediction for $\varepsilon'/\varepsilon$ is found to be

$$\text{Re}(\varepsilon'/\varepsilon) = (1.7 \pm 0.2^{+0.5}_{-0.3} \pm 0.5) \cdot 10^{-3} = (1.7 \pm 0.9) \cdot 10^{-3},$$  \hspace{1cm} (48)$$
in excellent agreement with the measured experimental value $45$. The first error comes from the short-distance evaluation of Wilson coefficients and the choice of low-energy matching scale $\mu$. The uncertainty coming from the strange quark mass, $(m_s + m_q)(1\text{ GeV}) = 156 \pm 25 \text{ MeV}$, is indicated by the second error. The most critical step is the matching between the short- and long-distance descriptions, which has been done at leading order in $1/N_C$. Since all ultraviolet and infrared logarithms have been resummed, our educated guess for the theoretical uncertainty associated with $1/N_C$ corrections is $\sim 30\%$ (third error).

A better determination of the strange quark mass would allow to reduce the uncertainty to the $30\%$ level. In order to get a more accurate prediction, it would be necessary to have a good analysis of next-to-leading $1/N_C$ corrections. This is a very difficult task, but progress in this direction can be expected in the next few years.

6 Summary

The large–$N_C$ limit provides a sensible approximation to the $N_C = 3$ hadronic world. Assuming confinement, the strong dynamics at $N_C \to \infty$ is given by tree diagrams with infinite sums of hadron exchanges, which correspond to the tree approximation to some local effective Lagrangian. Hadronic loops generate corrections suppressed by factors of $1/N_C$.

At very low energies the hadronic EFT describing the lightest pseudoscalar nonet is $\chi$PT, while resonance chiral theory provides the correct
framework to incorporate the massive mesonic states. The short-distance properties of QCD at large $N_C$ provide strong constraints on the chiral couplings.

The expansion in powers of $1/N_C$ turns out to be a very useful tool for quantitative non-perturbative analyses. While there is a very successful leading-order phenomenology, some important physical effects only appear at subleading topologies: the $U(1)_A$ anomaly, the anomalous dimensions of (non-penguin) four-quark operators and their associated short-distance logarithms, the infrared $\chi$PT logarithms, the resonance widths, etc. Those effects can be rigorously analyzed with appropriate tools as exemplified by the calculation of $\varepsilon'/\varepsilon$. The control of non-logarithmic corrections at the next-to-leading order in $1/N_C$ remains a challenge for future investigations.

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