Topologically massive gravity from the outside in

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Abstract
The asymptotically anti-de Sitter solutions of cosmological topologically massive gravity are analyzed for values of the mass parameter in the range $\mu \geq 1$. At non-chiral values, a new term in the Fefferman–Graham expansion is needed to capture the bulk degree of freedom. The Carlip–Deser–Waldron–Wise modes provide a basis for the pure non-Einstein solutions at all $\mu$, with nonlinear corrections appearing at higher order in the expansion.

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1. Introduction
Topologically massive gravity (TMG) [1, 2] with a negative cosmological constant appears to be the simplest theory that contains both black holes and local gravitational degrees of freedom, making it a potentially useful toy model to explore various questions in quantum gravity. However, at arbitrary values of the coupling constants, the theory appears to be unstable, containing either positive mass black holes and negative energy gravitons, or the reverse, depending on the choice of sign of the Einstein–Hilbert piece of the action. Recently, Li et al [3] proposed that with suitable boundary conditions, the local bulk degree of freedom disappears at the ‘chiral’ coupling $\mu \ell = 1$, allowing the possibility of choosing the sign of the action such that only positive mass black holes are included. With these boundary conditions, the theory becomes chiral, in that the asymptotic symmetry consists of a single copy of the Virasoro algebra [4–6], and the theory was dubbed chiral gravity [3, 7].

The claim that chiral gravity admits only non-negative energy modes has been the subject of much debate in the literature. Several non-perturbative studies found a single local propagating degree of freedom at all values of $\mu$ [5, 8, 9]; however, the boundary conditions satisfied by these modes were not investigated. At the linearized level, other authors found negative-energy modes at $\mu \ell = 1$ [10–13], but these modes either were not chiral or required different boundary conditions than those used in [3]. After some confusion in the literature, Maloney et al [7] concluded that TMG at the critical value could be divided into two theories depending on the choice of boundary conditions: chiral gravity with Brown–Henneaux [14] boundary conditions and log gravity with relaxed boundary conditions that include a logarithmic term in the asymptotic expansion of the metric. For related work
on boundary conditions, see [6, 15]. Additionally, it was shown in [7] that all stationary, axially symmetric solutions of chiral gravity are the familiar Bañados–Teitelboim–Zanelli (BTZ) black holes [16] and have non-negative energy. The authors found that the proposed counterexamples either required the relaxed boundary conditions, and thus were solutions to log gravity, or developed linearization instabilities at second order, and they speculated that all asymptotically anti-de Sitter (AdS) non-Einstein solutions at the critical point are in fact solutions to log gravity.

Recently, Compère et al [17] discovered a new class of non-Einstein solutions of chiral gravity using the Fefferman–Graham (FG) expansion [18] with Brown–Henneaux boundary conditions. They further examined a subset of the general solution that included linear perturbations from AdS3 and BTZ backgrounds. Of the solutions examined, all contained either naked singularities or closed timelike curves and, unless they can be excluded as unphysical, may render chiral gravity unstable.

This paper extends their work to non-chiral values $\mu \ell > 1$. New terms in the FG expansion are needed to capture the bulk degrees of freedom at all values of the mass parameter, and for each value of $\mu$, a similar phenomenon occurs: one of the equations of motion disappears, leaving one piece of the metric unconstrained by the equations of motion. Section 2 gives the solution to second order in this new term. The division between Einstein and non-Einstein solutions becomes explicit in this formalism: one set of terms in the expansion captures all Einstein solutions, and the second set captures the non-Einstein solutions. Section 3 examines the special case of chiral gravity, where the solution of [17] is given in light-cone coordinates. In this formalism, we see that the chiral point is just the point at which the two sets of terms overlap. Section 4 maps the Carlip–Deser–Waldron–Wise (CDWW) modes [10] onto the full solution of section 1. The CDWW modes provide a complete basis of the non-Einstein solutions only, and additional ingredients are needed to include the Einstein solutions. The solution given in section 2 agrees with CDWW to second order; however, nonlinear deviations from CDWW are found for several integral values of $\mu$ at higher order and are likely to exist for generic $\mu$. I conclude with a discussion on the significance of these solutions on the stability of chiral gravity.

2. Asymptotic solution of TMG

The equation of motion for TMG [1, 2] with a cosmological constant is

$$R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} + \Lambda g_{\mu \nu} + \frac{1}{\mu} C_{\mu \nu} = 0,$$

(1)

where $C_{\mu \nu}$ is the Cotton tensor

$$C_{\mu \nu} = \epsilon_{\mu}{}^{a \beta} \nabla_a \left( R_{\beta \nu} - \frac{1}{4} R g_{\beta \nu} \right).$$

(2)

In the discussion below, we work in units where $\Lambda = -1$. By the Bianchi identity, the Cotton tensor is symmetric, traceless and covariantly conserved. Taking the trace of (1), we find that the solution is a spacetime of constant scalar curvature

$$R = -6,$$

(3)

and the equation becomes

$$R_{\mu \nu} + 2 g_{\mu \nu} + \frac{1}{\mu} C_{\mu \nu} = 0.$$  

(4)

Pure Einstein solutions are those for which $R_{\mu \nu} = -2 g_{\mu \nu}$, and it is apparent from (2) that the Cotton tensor is identically zero for all Einstein metrics. Thus, TMG contains all of the
ordinary Einstein solutions plus the massive propagating modes for which the Cotton tensor is non-zero.

In particular, TMG admits an asymptotically AdS solution which can be written in Gaussian normal coordinates as
\[ ds^2 = dr^2 + g_{ij} dx^i dx^j, \]
where
\[ g_{ij}^{(0)} = \lim_{r \to \infty} e^{-2r} g_{ij}(x, r) \]
is the metric on the boundary. Here, we take Greek indices to run over all coordinates, and Latin indices run over the non-radial coordinates. Due to a theorem by Fefferman and Graham [18], the two-dimensional metric \( g_{ij} \) can always be expanded near the boundary in powers of \( e^r \):
\[ g_{ij} = e^{2r} \left( g_{ij}^{(0)} + e^{-2r} g_{ij}^{(2)} + e^{-4r} g_{ij}^{(4)} + \cdots \right) \]
For three-dimensional Einstein gravity, the expansion terminates at \( g_{ij}^{(4)} \) (all higher order terms are zero), and the first three terms are sufficient to capture all AdS_{3} and BTZ solutions to pure Einstein gravity [19].

As noted in [20], the form of the expansion depends on the bulk theory, and other theories may exhibit different asymptotic behavior. Because all solutions of Einstein gravity are also solutions of TMG, we need at least the terms in (7) in the expansion for TMG. However, these terms alone do not capture the propagating degree of freedom of TMG at \( \mu \neq 1 \), the series (7) still terminates at \( g_{ij}^{(4)} \), and the Cotton tensor vanishes to all orders. Thus to include all Einstein and non-Einstein solutions, at all values of the mass parameter \( \mu > 1 \), the expansion is
\[ g_{ij} = e^{2r} \left( g_{ij}^{(0)} + e^{-2r} g_{ij}^{(2)} + e^{-4r} g_{ij}^{(4)} \right) \quad \text{Einstein} \]
\[ + e^{2r} \left( e^{-(\mu+1)r} g_{ij}^{(\mu+1)} + e^{-(\mu+3)r} g_{ij}^{(\mu+3)} + \cdots \right) \quad \text{non-Einstein}. \]
In this form, it becomes apparent that the odd integral values of the mass parameter are just the values at which the two expansions overlap.

Now the procedure is to plug the expansion into the equations of motion (4), collect terms of the same power of \( e^r \) and set the coefficients equal to zero. Thus, we can solve for the higher order terms algebraically in terms of the lower order terms. The boundary metric \( g_{ij}^{(0)} \) may be left as a free field; however, the process is simplified by the use of a constant boundary metric and light-cone coordinates \( x^+ = \frac{1}{\sqrt{2}}(t + \phi) \) and \( x^- = \frac{1}{\sqrt{2}}(t - \phi) \). Consistent with the conventions of [17], I set \( g_{ij}^{(0)} = -1 \) and \( \sqrt{-g}e^{-r} = 1 \). The solutions for \( g_{ij}^{(2)} \) and \( g_{ij}^{(4)} \) at non-critical values are
\[ g_{++}^{(2)} = L(x^+) \quad \text{and} \quad g_{--}^{(2)} = L(x^-) \]
\[ g_{+-}^{(4)} = -\frac{1}{2} L(x^+) \tilde{L}(x^-). \]
As noted previously, the solutions encompassed in these terms contain only the ordinary Einstein solutions [19], including the BTZ black hole [16]. For \( \mu > 1 \), the equations set \( g_{ij}^{(\mu+1)} \)

1 Note that I propose this new series as an ansatz. This differs from the view of [20] in which the correct asymptotic expansion should be derived from the bulk theory via the AdS/CFT correspondence. However, this should not affect the outcome—the solutions found using this expansion still satisfy the equations of motion.
and $g_{-+}^{(\mu+1)}$ to zero, but the equations for $g_{++}^{(\mu+1)}$ disappear, and this component is unconstrained by the theory. The solution is

$$
g_{++}^{(\mu+1)} = F(x^+, x^-)$$

$$
g_{++}^{(\mu+3)} = \frac{1}{2\mu + 6} \partial_\mu \partial_- F$$

$$
g_{+-}^{(\mu+3)} = -\frac{(\mu + 1)^2 - 2}{2\mu(\mu + 3)} \bar{L} F - \frac{1}{2\mu(\mu + 3)} \partial^2 F$$

$$
g_{++}^{(\mu+5)} = \ldots .$$

(11)

The calculation for generic $\mu$ was performed by hand. Additionally, a Maple script was written to solve the equations of motion order by order at integral values of $\mu$, and results match (11) for $\mu = 1, 2$ and 3. A quick check reveals that the asymptotically conserved Virasoro charges do not depend on the function $F(x^+, x^-)$, see e.g. the derivation of conserved charges in [6]. Additionally, since the new set of terms appears at higher order (than $g_{(2)}$) in the asymptotic expansion, i.e. deeper in the interior, the function $F$ does not appear in the holographic stress tensor [12, 20]. For the full solution (10) and (11), we find a non-zero Cotton tensor (and therefore non-Einstein solutions) only when $F(x^+, x^-) \neq 0$. Thus, the function $F$ contains the massive, propagating modes unique to TMG.

3. Chiral gravity

Two new features appear at the critical point $\mu = 1$. First, the asymptotic expansion acquires a logarithmic term [20]

$$
g_{ij} = e^{2r} \left( g_{ij}^{(0)} + r e^{-2r} g_{ij}^{(1)} + e^{-2r} g_{ij}^{(2)} + \ldots \right).$$

(12)

As discovered by Grumiller and Johansson [12], the logarithmic term contains a new branch of solutions only accessible at $\mu = 1$. At $\mu \neq 1$ the equations of motion force $g_{(1)} = 0$, and this term is not present in the general expansion (9). Chiral gravity [3, 7] is the subset of the theory at $\mu = 1$ with the logarithmic mode turned off. Here, the functions $F(x^+, x^-)$ and $L(x^-)$ overlap, and the general solution in light-cone coordinates to third order ($g_{(6)}$) is

$$
g_{++}^{(2)} = F(x^+, x^-)$$

$$
g_{--}^{(2)} = \bar{L}(x^-)$$

$$
g_{++}^{(4)} = \frac{1}{2} \partial_\mu \partial_- F$$

$$
g_{+-}^{(4)} = -\frac{1}{4} \bar{L} F - \frac{1}{8} \partial_-^2 F$$

$$
g_{++}^{(6)} = \frac{1}{6} F \partial_\mu^2 F + \frac{1}{360} \partial_\mu^3 \partial_-^2 F - \frac{1}{960} (\partial_- F)^2$$

$$
g_{+-}^{(6)} = -\frac{1}{16} \bar{L} \partial_\mu \partial_- F - \frac{1}{960} \partial_\mu \partial_-^3 F$$

$$
g_{--}^{(6)} = \frac{1}{4} \bar{L} \partial_-^2 F + \frac{1}{120} (\partial_- \bar{L})(\partial_- F) + \frac{1}{384} \partial_-^4 F.$$  

(13)

In agreement with [17], we find that Einstein solutions are the subset for which the function $F$ depends only on $x^+$. When $\partial_- F = 0$, the Cotton tensor vanishes to all orders, and the expansion (12) terminates at $g_{(4)}$. Note that the requirement for vanishing Cotton tensor at the chiral value is more stringent than that at non-chiral values, for which $F = 0 \iff C_{\mu\nu} = 0$. 

4
4. Revisiting CDWW

Previously, Carlip et al \cite{10, 11} found a complete set of solutions—the CDWW modes—to the linearized equations of motion with asymptotically AdS$_3$ boundary conditions at all values of $\mu$. These solutions share some important features with (11), namely

- the solutions are invariant under $\mu \to -\mu$ and a chirality flip ($x^+ \leftrightarrow x^-$) and
- they exhibit a $\mu$-dependent asymptotic behavior, with different fall-off conditions for each component of the metric.

Given the similarities, it is natural to ask if (11) contains the CDWW modes. However, CDWW solved for the linearized Einstein tensor, and these modes must first be converted into perturbations of the metric in Gaussian normal coordinates before a direct comparison can be made. This can always be done in three dimensions, since a perturbation of the Einstein tensor uniquely determines a perturbation in the metric. The solutions for each component of the linearized Einstein tensor are the Bessel functions given in equation (A.28) of \cite{11}, for example

$$H_{++} = \frac{\omega^2}{\omega^2} \exp[i(\omega_+ x^+ + \omega_- x^-)] z J_{\mu-2}(\omega z) + \text{h.c.}$$

is shown here for comparison. In this expression, $z$ is the radial coordinate related to the $r$ of the previous section by $z = e^{-r}$; $\omega_+$ and $\omega_-$ are eigenvalues of the $SL(2, \mathbb{R})$ generators $i\partial_+$ and $i\partial_-$, and $\omega^2 = -2\omega_+ \omega_-$. Equation (5.5) of \cite{11} relates the Einstein tensor to the metric perturbations via differential equations such as

$$H_{++} = -\frac{1}{2} z \partial_z (z \partial_z + 2) g_{++}.$$  

The final step is to expand the modes (14) in powers of $z$ and solve (15) for the metric perturbations order by order. The CDWW modes as metric perturbations are

$$g_{++} = \frac{\omega^2 \omega^{-3} \exp[i(\omega_+ x^+ + \omega_- x^-)]}{2^{\mu-3}(\mu + 1) \Gamma(\mu)} \left[ z \mu^{-1} + \frac{\omega^2 z \mu^4}{2^2(\mu + 3)} + \frac{\omega^2 \omega^{-3} z \mu^4}{2^2(\mu + 5)(\mu + 3) \mu} \right]$$

$$g_{+-} = \frac{\omega^2 \mu^4 \exp[i(\omega_+ x^+ + \omega_- x^-)]}{2^\mu(\mu + 3) \Gamma(\mu + 2)} \left[ z \mu^4 + \frac{\omega^2}{2^2(\mu + 5) z \mu^4} \right]$$

$$g_{--} = \frac{-\omega^2 \omega^{-3} \exp[i(\omega_+ x^+ + \omega_- x^-)]}{2^\mu(\mu + 5) \Gamma(\mu + 4)} z^{\mu+3}.$$  

With the identification

$$F(x^+, x^-) = -\frac{\omega^2 \omega^{-3}}{(\mu + 1)2^{\mu-3} \Gamma(\mu)} \exp[i(\omega_+ x^+ + \omega_- x^-)]$$

the CDWW modes can be written as

$$g_{++} = F(x^+, x^-) z^{\mu-1} + \frac{\omega^2 z^{\mu+1}}{2(\mu + 3)} \partial_+ F + \frac{\omega^2 z^{\mu+3}}{8(\mu + 5)(\mu + 3) \mu} \partial_+^3 F$$

$$g_{+-} = -\frac{\omega^2 z^{\mu+1}}{2\mu(\mu + 3)} \partial_+^2 F - \frac{\omega^2 z^{\mu+3}}{4(\mu + 5)(\mu + 3) \mu} \partial_+ \partial_3 F$$

$$g_{--} = \frac{\omega^2 z^{\mu+3}}{4(\mu + 5)(\mu + 3) \mu(\mu + 2) \mu} \partial_+ \partial_3^2 F.$$  

Comparison with the asymptotic solutions (11) and (13) reveals that CDWW correctly gives all numerical coefficients for pure derivatives of $F(x^+, x^-)$. However, the full solution is more than just a background Einstein solution plus the CDWW modes. It also contains interactions
between the background and CDWW, e.g. the piece proportional to \( \bar{\ell} F \) in (11). Additionally, nonlinear deviations from CDWW appear at higher orders. In the chiral solution (13), for example, CDWW fails to include the nonlinear terms \( F \partial^2 F - F - \frac{1}{160} (\partial - F)^2 \) in the \( g_+^{(6)} \) component.

While the \( (\mu + 5) \) and higher order solution at generic \( \mu \) have not been found, solutions at integral values of \( \mu \) have been explored using Maple, and several patterns emerge. For all \( \mu \) examined, the first nonlinear deviation from CDWW appears in the \( g_+^{(6)} \) component at order \( 2\mu + 4 \) and contains terms proportional to \( F \partial^2 F - F - \frac{1}{160} (\partial - F)^2 \). In the case of odd \( \mu \), these deviations only appear at higher orders in the expansion (9). In the case of even \( \mu \), this nonlinearity turns on a new series of terms in the expansion (9), although these new terms do not represent new degrees of freedom. Note that, even at these higher orders, CDWW still gives the correct numerical coefficients of the pure derivative terms. I expect these features to hold for generic \( \mu \). Table 1 shows the first departure from CDWW for \( \mu = 1, 2 \) and 3.

The CDWW modes at \( \mu = 1 \) were originally proposed as a counterexample to the positivity theorem—while the modes blow up in the interior, Carlip et al [10] created finite-energy superpositions of the modes with negative energy. However, the original CDWW modes (16), as well as several other proposed counterexamples to positivity, were shown to develop a linearization instability at second order [7, 21]. For example, the Giribet–Kleban–Porrati modes [13] at second order require the relaxed logarithmic boundary conditions and thus are not a linear approximation to an exact solution of chiral gravity. This is not the case with CDWW—the nonlinear completion of CDWW, including a background Einstein metric, interaction terms, and the nonlinear terms of table 1, is a solution of chiral gravity satisfying strict Brown–Henneaux boundary conditions. This extended CDWW should be reconsidered as a potential candidate for violating the positivity theorem. The next step, left for the interested reader, is to construct finite-energy superpositions of this extended CDWW.

5. Discussion

It remains an open question whether physical non-Einstein solutions of chiral gravity exist. The technique used here—working from the boundary in and solving the equations of motion order by order—offers an alternative approach to the perturbative techniques used in most of the papers on the subject. Using this approach, I have constructed the general solution of TMG with strict Brown–Henneaux boundary conditions at all values of the mass parameter \( \mu \geq 1 \). The solutions at each \( \mu \) share the same basic structure and can be written as the sum of an Einstein metric, a purely non-Einstein metric and interactions between the two. The non-Einstein solution is characterized by a single function \( F \) which can be expanded on to the CDWW modes of [10], with nonlinear corrections to CDWW appearing at higher order. In particular, the general solution at the critical value \( \mu = 1 \) contains these extended CDWW modes, and these modes do not require the relaxed boundary conditions of log gravity. Since chiral gravity shares these features with non-chiral TMG, which is generally thought to be unstable, this result raises questions about the classical stability of chiral gravity. However,
the task of constructing physically significant non-Einstein solutions to chiral TMG remains incomplete. Of the solutions (13) examined in [17] all contained either naked singularities or naked closed timelike curves which violate causality, rendering the solutions unphysical. However, they considered only a subset of functions $F$ with finite FG expansion, and some superposition of the extended CDWW modes offers an intriguing possibility.

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Appendix. FG expansion of the equations of motion

This section contains the equations of motion expanded to sixth order using the standard FG expansion (7). The full covariant equations are fairly tedious and will not be included (see [23] for the covariant fourth-order equations). Instead, the calculations are greatly simplified in the light-cone gauge using the boundary metric and Levi-Civita conventions listed in section 2. The equations are symmetric under $\mu \to -\mu$ and $x^+ \to x^-$. Without loss of generality, we restrict attention to positive $\mu$. The second-order equations are [23]

$$\text{Tr}(g^{-1}(g_{(2)}) = -\frac{1}{2} R(g_{(0)})$$

$$\left(1 - \frac{1}{\mu}\right) \partial_- g^{(2)}_{++} = 0$$

$$\left(1 + \frac{1}{\mu}\right) \partial_+ g^{(2)}_{--} = 0. \quad (A.1)$$

For all positive $\mu$, $g^{(2)}_{--} = \bar{L}(x^-)$; however, the equation for $g^{(2)}_{++}$ disappears at $\mu = 1$, leaving this component of the metric unconstrained by the equations of motion. At $\mu \neq 1$, the solution reduces to the Einstein solution $g^{(2)}_{++} = L(x^+)$. At second order, the $\{r, r\}$ and $\{x^i, x^j\}$ ($i, j = 1, 2$) equations are identically zero, leaving only the $\{r, x^i\}$ equations. At higher order, all six equations of motion are present. However, the $\{r, x^i\}$ equations are derivatives of the $\{x^i, x^j\}$ equations and contain no new information. Hence, we need only solve the $\{r, r\}$ and $\{x^i, x^j\}$ equations algebraically in terms of lower orders. The fourth-order equations are

$$g^{(4)}_{++} = -\frac{1}{4} g^{(2)}_{++} g^{(2)}_{--} - \frac{1}{8\mu} \partial^2 g^{(2)}_{++}$$

$$\left(1 - \frac{3}{\mu}\right) g^{(4)}_{++} = -\frac{1}{4\mu} \partial_+ \partial_- g^{(2)}_{++}$$

$$\left(1 + \frac{3}{\mu}\right) g^{(4)}_{--} = 0. \quad (A.2)$$

These equations possess their own critical point $\mu = 3$. When $\mu \neq 3$, the equations completely determine the components of $g_{(4)}$ in terms of the $g_{(2)}$, and the solution is either the chiral solution (13) when $\mu = 1$ or the Einstein solution (10) for $\mu \neq 1, 3$. However, at $\mu = 3$, one of the equations disappears, leaving $g^{(4)}_{++}$ unconstrained, and the full solution depends on the three functions $g^{(2)}_{++} = L(x^+)$, $g^{(2)}_{--} = \bar{L}(x^-)$ and $g^{(4)}_{++} = F(x^+, x^-)$.
The same structure is repeated at propagating modes for non-odd integral values of the mass parameter. Instead, the new set of \( \bar{g}_\mu^6 = \frac{1}{4} \partial_\mu \partial_{\bar{\mu}} F(x^+, x^-) \) does not equal odd integral values, \( \mu = 3 \). The table shows the non-chiral solutions to sixth order. Here, we use \( g_\mu^{(2)} = L(x^+) \) and \( g_\mu^{(2)} = \bar{L}(x^-) \). In all cases, \( g_\mu^{(4)} = \bar{g}_\mu^{(4)} = 0 \). Together with (13), these are the complete set of solutions with the un-modified FG expansion (7) obeying Brown–Henneaux boundary conditions.

| \( \mu = 3 \) | \( \mu = 5 \) | \( \mu \neq 1, 3, 5 \) |
|---|---|---|
| \( g_\mu^{(4)} \) | \( F(x^+, x^-) \) | 0 |
| \( g_\mu^{(4)} \) | \( -\frac{1}{4} L(x^+) \bar{L}(x^-) \) | \( -\frac{1}{4} L(x^+) \bar{L}(x^-) \) |
| \( g_\mu^{(6)} \) | \( \frac{1}{2} \partial_\mu \partial_{\bar{\mu}} F \) | \( F(x^+, x^-) \) |
| \( g_\mu^{(6)} \) | \( \frac{1}{2} \partial \bar{\mu} \partial_{\mu} \) | 0 |
| \( g_\mu^{(6)} \) | \( \partial \bar{\mu} g^{(2)} \) | 0 |
| \( g_\mu^{(6)} \) | \( \partial \bar{\mu} g^{(4)} \) | 0 |

This feature is repeated in the sixth-order equations:

\[
\begin{align*}
\left( 1 - \frac{5}{\mu} \right) g_\mu^{(6)} &= \left( 1 - \frac{6}{\mu} \right) \left[ -\frac{2}{3} g_\mu^{(2)} g_{-}^{(2)} - \frac{1}{6} g_{-}^{(2)} \right] \\
&+ \frac{1}{24} \left( 1 - \frac{3}{\mu} \right) g_\mu^{(2)} \partial_{\bar{\mu}} g^{(2)} + \frac{1}{6} \partial_{\bar{\mu}} \partial_{\mu} g^{(4)} + \frac{1}{6} \partial_{\bar{\mu}} \partial_{\mu} g^{(4)}
\end{align*}
\]

The same structure is repeated at \( \mu \neq 5 \), \( g_\mu^{(6)} \) is determined entirely in terms of the lower order terms. At the critical value \( \mu = 5 \), the \( g_\mu^{(6)} \) component is unconstrained. The equations are easily solved in each of the cases \( \mu = 3, 5 \) and \( \mu \neq 1, 3, 5 \) and displayed in table A.1.

When \( \mu \) does not equal odd integral values, \( g_\mu^{(6)} \) is identically zero, and the full solution (with the un-modified FG expansion (7)) is equivalent to the Einstein solution. I have expanded the equations out to tenth order using Maple and have confirmed that \( g_\mu^{(8)} = g_\mu^{(10)} = 0 \). Additionally, the Cotton tensor \( C_{\mu\nu} = 0 \) to tenth order.

The solutions at \( \mu = 3, 5 \) share several important features. First, the full solution is characterized by three functions \( g_\mu^{(2)} = L(x^+) \), \( g_\mu^{(2)} = \bar{L}(x^-) \) and the unconstrained term \( g_\mu^{(4)} = F(x^+, x^-) \) in the case \( \mu = 3 \) or \( g_\mu^{(4)} = F(x^+, x^-) \) in the case \( \mu = 5 \). The first non-zero component of the Cotton tensor is

\[
C_{\mu\nu} = \begin{cases} 
\frac{1}{4} \partial_\mu \partial_{\bar{\nu}} \partial_{\bar{\nu}} F e^{-2\nu} & \mu = 1 \\
12 F e^{-2\nu} & \mu = 3 \\
60 F e^{-4\nu} & \mu = 5 
\end{cases}
\]

Thus at \( \mu = 3, 5 \), the requirement for a non-Einstein solution is simply that \( F \neq 0 \). In contrast, the solution at the chiral point is characterized by only two functions \( F(x^+, x^-) \) and \( \bar{L}(x^-) \), and non-Einstein solutions require the more stringent requirement that \( \partial_{\bar{\nu}} F \neq 0 \).

The solutions above indicate that the un-modified FG expansion (7) does not capture the propagating modes for non-odd integral values of the mass parameter. Instead, the new set of
terms (9) is required. To see this, we can add the term $e^{(2-m)}g_{ij}^{(m)}$ to the FG expansion and ask what constraints the equations of motion place on $g_{(m)}$. The equations force the $- -$ and $+ -$ components to zero, but the $++$ equation is

$$
\left(1 - \frac{m-1}{\mu}\right) g_{++}^{(m)} = 0.
$$

(A.5)

Similar to (A.2) and (A.3), this equation disappears at the critical value $\mu = m-1$, leaving $g_{++}^{(m)}$ unconstrained. If we call $g_{++}^{(m)} = F(x^+, x^-)$, then the first non-zero component of the Cotton tensor is $C_{++} = \left(\frac{1}{2}m^3 - \frac{3}{2}m^2 + m\right) Fe^{(2-m)x}$.

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