Bianchi -I Cosmology with Magnetic Field in Lyra Geometry.

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Abstract

A new class of Bianchi -I cosmological model with magnetic field within the framework of Lyra geometry has been presented. Exact solutions to the field equations for the model are obtained. The physical and kinematical behaviors of the model have been discussed.

1. INTRODUCTION

The aim of cosmology is to determine the large scale structure of the Universe. The present day observations indicate that the Universe at large scale is homogeneous and isotropy. It is well known that in general theory of relativity spatially homogeneous space times belong to either to the Bianchi classifications or to Kantowski-Sachs class and interpreted as cosmological model [1].

In last few decades there has been considerable interest in alternative theory of gravitation. The most important among them being scalar tensor theories proposed by Lyra [2] and Brans-Dicke [2]. Lyra proposed a modification of Riemannian geometry by introducing a gauge function into the structure less manifold that bears a close resemblance to Weyl’s geometry. In general relativity Einstein succeeded in geometrising gravitation by identifying the metric tensor with gravitational potentials. In scalar tensor theory of Brans-Dicke on the other hand, the scalar field remains alien to the geometry. Lyra’s geometry is more in keeping with the spirit of Einstein’s principle of geometrisation since both the scalar and tensor fields have more or less intrinsic geometrical significance. In consecutive investigations Sen [3] and Sen and Dunn [4] proposed a new scalar tensor theory of gravitation and constructed an analog of the Einstein field equation based on Lyra’s geometry which in normal gauge may be written as

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\[ R_{ab} - \frac{1}{2}g_{ab}R + \frac{3}{2}\phi_a\phi_b - \frac{3}{4}g_{ab}\phi_c\phi^c = -8\pi GT_{ab} \] (1)

Where \( \phi_a \) is the displacement vector and other symbols have their usual meaning as in Riemannian geometry.

Halford[Aust.J.Phys. 23, 833, (1970) ] has pointed out that the constant displacement field \( \phi_a \) in Lyra’s geometry play the role of cosmological constant \( \Lambda \) in the normal general relativistic treatment. According to Halford the present theory predicts the same effects within observational limits, as far as the classical solar system tests are concerned, as well as tests based on the linearised form of field equations.

Subsequent investigations were done by several authors in scalar tensor theory and cosmology within the frame work of Lyra geometry [5].

It is well known that the magnetic field has a significant role at the cosmological scale and is present in galactic and intergalactic spaces . The occurrence of magnetic fields on galactic scales and their importance for a variety of Astrophysical phenomena has been pointed out by several authors[6] . Melvin[M.A.Melvin (1975),Ann.N.Y.Acad.Sci. 262,253] in his cosmological solution for dust and electromagnetic field argues that the presence of magnetic field is not as unrealistic as it appears to be , because for a large part of the history of evolution matter was highly ionized and matter and field were smoothly coupled . Latter during cooling as a result of expansion of ions combined to form neutral matter. The cosmological models with magnetic field have been discussed by a number of authors in general relativity . But as far as our knowledge there has not been any work in literature where Lyra’s geometry has been considered to study cosmological model with magnetic field . So it is interesting to study cosmological model with magnetic field within the frame work of Lyra geometry. In the present work , we consider Bianchi -I cosmological model with magnetic field based on Lyra geometry.

2. The Basic Equations

We consider an axially symmetric Bianchi -I metric as

\[ ds^2 = -dt^2 + e^{2\alpha}dx^2 + e^{2\beta}(dy^2 + dz^2) \] (2)

where \( \alpha = \alpha(t) \) and \( \beta = \beta(t) \).

Now the energy momentum tensor for the cosmological model with a magnetic field along the x - direction is given by

\[ T^a_b = T^a_b(c) + T^a_b(m) \text{ with} \]

\[ T_{ab}(c) = (p + \rho)U_aU_b + pg_{ab}; U_iU^i = -1 \] (3)

Where \( U_i \) is the four velocity , p is the pressure and \( \rho \) is mass energy density and
\[ T_{ab}(m) = \frac{1}{4\pi} [F_{a}^{c} F_{b}^{c} - \frac{1}{4} F_{c}^{cd} \delta_{ab}^{b}] \] (4)

In the above, \( T_{ab}(c) \) is the stress energy tensor for a perfect fluid and \( F_{ab} \) is the electromagnetic field tensor.

Further, since the magnetic field is being assumed in the \( x \)-direction, \( T_{23} \) is the only non-zero component of the electromagnetic field tensor.

Maxwell equation \( F_{[ab,d]} = 0 \) and \( [F^{ab} \sqrt{-g} ]_{a} = 0 \), lead to the result

\[ T_{23} = A \] (5)

\( A \) being a constant quantity. So, the components of the stress energy tensor for the electromagnetic field are:

\[ E^{0}_0 = E^{1}_1 = -E^{2}_2 = -E^{3}_3 = -\frac{A^{2}}{8\pi} e^{-4\beta} \] (6)

The time-like displacement vector \( \phi_i \) in (1) is given by

\[ \phi_i = (\beta_0(t), 0, 0, 0) \] (7)

Now choosing units such that \( 8\pi G = 1 \), the field equation (1) becomes with equations (2), (3) and (6):

\[ 2\alpha'\beta' + (\beta')^2 = \rho + \frac{3}{4} \beta_0^2 + \frac{A^{2}}{8\pi} e^{-4\beta} \] (8)

\[ 2\beta'' + 3(\beta')^2 = -p - \frac{3}{4} \beta_0^2 + \frac{A^{2}}{8\pi} e^{-4\beta} \] (9)

\[ \alpha'' + (\alpha')^2 + \beta'' + (\beta')^2 + \alpha'\beta' = -p - \frac{3}{4} \beta_0^2 - \frac{A^{2}}{8\pi} e^{-4\beta} \] (10)

\([' \text{ denotes the differentiation w.r.t. 't'}]\]

We assume the equation of state as

\[ p = m\rho [0 \leq m \leq 1] \] (11)

The proper volume, expansion scalar \( \theta \) and shear scalar \( \sigma^2 \) are respectively given by

\[ R^3 = e^{(\alpha+2\beta)} \] (12)

\[ \theta = \alpha' + 2\beta' \] (13)

\[ \sigma^2 = (\alpha')^2 + 2(\beta')^2 - \frac{1}{3} \theta^2 \] (14)

3. Solutions:
Since the number of unknown parameters appearing in the model exceeds the number of field equations by one, we require one more equation to find the exact solutions. This is achieved by assuming the following relation between the metric coefficients

$$\alpha = a\beta$$  \hspace{1cm} (15)

Where 'a' is an arbitrary constant.

From the field equations (9) and (10), by using eq.(15), we get

$$\beta'' + (a + 2)(\beta')^2 = \frac{A^2}{4(a - 1)\pi}e^{-4\beta}$$  \hspace{1cm} (16)

The first integral of eq.(16) gives:

$$e^{2(a+2)\beta}(\beta')^2 = -\frac{A^2}{8a(a - 1)\pi}e^{2a\beta} + D$$  \hspace{1cm} (17)

(Where D is an integration constant)

The first integral can be written in the integral form:

$$\int \frac{e^{2\beta}d\beta}{\sqrt{De^{-2a\beta} - \frac{A^2}{4(a-1)\pi}}} = \pm(t - t_0)$$  \hspace{1cm} (18)

(\(t_0\) is another integration constant)

We shall solve the above integral for three different cases: \(a = -1, -2\) and \(-\frac{1}{2}\).

For other cases the solutions will not be obtained in closed form of \(\beta\).

**Case - I : a = -1 :**

Here we get an expression of \(\beta\) as

$$e^{2\beta} = \frac{D^2(t - t_0)^2 + \frac{A^2}{16\pi}}{D}$$  \hspace{1cm} (19)

The other parameters are obtained as

$$R^3 = \sqrt{\frac{D^2(t - t_0)^2 + \frac{A^2}{16\pi}}{D}}$$  \hspace{1cm} (20)

$$\theta = \sqrt{\frac{D^2}{D^2(t - t_0)^2 + \frac{A^2}{16\pi}} - \frac{A^2D^2}{16\pi} - \frac{1}{D^2(t - t_0)^2 + \frac{A^2}{16\pi}}}$$  \hspace{1cm} (21)
\[
\sigma^2 = \frac{8}{3} \left[ \frac{D^2}{D^2(t-t_0)^2 + \frac{A^2}{16\pi}} - \frac{A^2D^2}{16\pi D^2(t-t_0)^2 + \frac{A^2}{16\pi}} \right] 
\]

\[
\rho = p = 0
\]

\[
3\beta^2_0 = \frac{mD^2/(1-m)}{D^2(t-t_0)^2 + \frac{A^2}{16\pi}} - \frac{(2-m)A^2D^2/16\pi(1-m)}{[D^2(t-t_0)^2 + \frac{A^2}{16\pi}]^2}
\]

Behavior of the model:

In this case the solutions degenerate into singularity-free vacuum solution based on Lyra geometry. The space time of the class of solution is to represent an expanding Universe for \( t > 0 \).

The Kinematical variables \( \theta \) and \( \sigma^2 \) will be vanished as \( t \to \infty \). We also see that the ratio \( \sigma \) to \( \theta \) is constant. This implies that there is no possibility that the model may got isotropized in some latter time i.e. it remain anisotropic for all time.

In this model particle horizon exists because

\[
\int_{t_0}^{t} \frac{D}{\sqrt{D^2(t-t_0)^2 + \frac{A^2}{16\pi}}} dt' 
\]

is a convergent integral.

The gauge function decreases with the evolution of the model and at \( t \to \infty, \beta \to 0 \). So the concept of Lyra geometry will not linger for infinite time.

**Case - II : a = -2 :**

For this case integral (18) can at once be evaluated to yield

\[
e^{2\beta} = \frac{A}{\sqrt{48\pi D}} \cosh 2\sqrt{D(t-t_0)}
\]

Here we get an expression of \( \rho \) as

\[
\rho = -\frac{4D}{(1-m) \cosh^2 2\sqrt{D(t-t_0)}}
\]

Since \( m \leq 1 \), the model violets the positivity energy condition. So we reject this case.
Case - III : $a = -\frac{1}{2}$ :

We can now find a solution of the integral (18) as

$$e^\beta = \frac{1}{D} [u^2 + \frac{A^4}{36\pi^2 u^2} - \frac{A^2}{16\pi}] \quad (27)$$

Where

$$u = \sqrt{\frac{3}{2} D^2 t + \sqrt{\frac{9}{4} D^4 t^2 + \frac{A^6}{54\pi^3}}} \quad (28)$$

[ Here we neglect the integration constant $t_0$ . ]

In view of (27) one can easily find out the other parameters as

$$R^3 = \sqrt[3]{\frac{1}{D} [u^2 + \frac{A^4}{36\pi^2 u^2} - \frac{A^2}{16\pi}]} \quad (29)$$

$$\theta = \frac{3}{2} \sqrt{De^{-3\beta} - \frac{A^2}{6\pi} e^{-4\beta}} \quad (30)$$

$$\sigma^2 = \frac{23}{12} [De^{-3\beta} - \frac{A^2}{6\pi} e^{-4\beta}] \quad (31)$$

$$\rho = \frac{A^2 D^2}{12p(1-m)} [u^2 + \frac{A^4}{36\pi^2 u^2} - \frac{A^2}{16\pi}]^2 \quad (32)$$

$$\frac{3}{4} \beta_0^2 = \frac{A^2 (3m-1)}{24p(1-m)} e^{-4\beta} - \frac{D}{(1-m)} e^{-3\beta} \quad (33)$$

**Behavior of the model :**

In this case we get the solution of matter field Universe . Here we note that our space time is singularity free . It has a particle horizon because

$$\int_{t_0}^t \sqrt{\frac{D}{u^2 + \frac{A^4}{36\pi^2 u^2} - \frac{A^2}{16\pi}}} dt'$$

is a convergent integral .

The solutions represent an expanding model of the Universe . At $t \to \infty$ the expansion
ceases. The gauge function decays during its evolution.

4. Concluding Remarks:

In this work, we have discussed Bianchi-I cosmological model in Lyra geometry considering a source free magnetic field, the reasons for which has been discussed in the introduction. We have obtained three sets of solutions with a special choice of metric coefficients, viz., $\alpha = a\beta$. For a particular value of $a$, say, $a = -1$, we observe that our model represents a singularity free empty Universe. In the absence of the magnetic field, our model reduces to the model as obtained by Singh and Singh (the model has a initial singularity)[J.Math.Phys. 32, 2956, (1991)]. But the nature of the solution are changed due to the presence of the magnetic field (here the model is singularity free). The other choice of $a$, say, $a = -\frac{1}{2}$, we get a solution of the matter filled Universe.

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