On the generalized dimensions of multifractal eigenstates

J A Méndez-Bermúdez, A Alcazar-López and Imre Varga

1 Instituto de Física, Benemérita Universidad Autónoma de Puebla, Apartado Postal J-48, Puebla 72570, Mexico
2 Elméleti Fizika Tanszék, Fizikai Intézet, Budapesti Műszaki és Gazdaságtudományi Egyetem, H-1521 Budapest, Hungary

E-mail: jmendezb@ifuap.buap.mx

Received 12 August 2014
Accepted for publication 30 September 2014
Published 10 November 2014

Abstract. Recently, based on heuristic arguments, it was conjectured that an intimate relation exists between any multifractal dimensions, $D_q$ and $D_{q'}$, of the eigenstates of critical random matrix ensembles: $D_{q'} \approx qD_q[q' + (q - q')D_q]^{-1}$, $1 \leq q, q' \leq 2$. Here, we verify this relation by extensive numerical calculations on critical random matrix ensembles and extend its applicability to $q < 1/2$, but also to deterministic models producing multifractal eigenstates and to generic multifractal structures. We also demonstrate, for the scattering version of the power-law banded random matrix model at criticality, that the scaling exponents $\sigma_q$ of the inverse moments of Wigner delay times, $\langle \tau_W^{-q} \rangle \propto N^{-\sigma_q}$, where $N$ is the linear size of the system, are related to the level compressibility $\chi$ as $\sigma_q \approx q(1 - \chi)[1 + q\chi]^{-1}$ for a limited range of $q$, thus providing a way to probe level correlations by means of scattering experiments.

Keywords: finite-size scaling, disordered systems (theory)

ArXiv ePrint: 1303.5665
1. Introduction

Multifractality [1,2] appears to be an essential feature of electronic states in disordered systems right at the Anderson transition, which has been demonstrated in several experiments [3]. The detailed nature of such complex structures has received renewed interest and recently many interesting results have emerged [4–7].

The multifractal dimensions characterizing these states have been obtained mainly using numerical simulations. Exact, analytical estimates are available only perturbatively, therefore heuristic relations should help a more in-depth understanding of the complexity of these states. In a recent paper [8] we have already presented such relations, especially in the case of various random matrix ensembles, so here we wish to extend those results to different observables and to a number of models where either analytical or numerical results are available. We will also demonstrate the robustness of the relations first presented in [8] giving suggestions for possible experimental tests.
As mentioned above, the spatial fluctuations of the eigenstates are captured in a way whereby the mean generalized inverse participation numbers scale with system size

\[ \langle \sum_{i=1}^{N} |\Psi_i|^2q \rangle \sim N^{-(q-1)D_q}, \tag{1} \]

where \( \langle \cdots \rangle \) is the average over some states within an eigenvalue window and over random realizations of the matrix. In the regime of strong localization the scaling is trivial since these moments are essentially independent of system size, resulting in \( D_q \to 0 \) for all \( q \). In the regime of weak disorder, however, the states appear to be perturbed extended Bloch-states, whose moments all scale with the embedding dimension, i.e. \( D_q \to d \), for all \( q \). Right at the critical point and even in its vicinity, the \( D_q \) dimensions result in being a nonlinear function of the parameter \( q \).

Some of these generalized dimensions have an immediate meaning, for instance, as \( q \to 1 \) we arrive at the scaling of the information entropy of the eigenstates as

\[ \langle -\sum_{i=1}^{N} |\Psi_i|^2 \ln |\Psi_i|^2 \rangle \sim D_1 \ln N. \tag{2} \]

A further well-known and widely-used dimension is called the correlation dimension \( D_2 \), which is extracted from the inverse participation number from equation (1) using \( q = 2 \). This exponent shows up in the power-law scaling of the density–density correlation function in the energy domain as \[ \langle |\Psi_\mu|^2 |\Psi_\nu|^2 \rangle \propto \left| E_0 E_\mu - E_\nu \right|^{1-D_2/d}, \tag{3} \]
as well as in the auto-correlation in space as \[ \langle |\Psi_k|^2 |\Psi_l|^2 \rangle \propto \left( \frac{N}{|r_k - r_l|^d} \right)^{d-D_2}, \tag{4} \]

where \( r_k \) and \( r_l \) denote the position of sites \( k \) and \( l \), respectively. In equation (3) the energy scale \( E_0 \) is of the order of the bandwidth \([9]\).

At the disorder driven Anderson transition not only the eigenstates but also the spectra show unusual behavior. The characterization of the fluctuations of the spectrum can be done in many ways \([11]\). An often employed quantity, the level compressibility \( \chi \), in connection with the energy asymptotic limit of the level number variance, is related to medium and long-range spectral correlations. Its definition is given as

\[ \Sigma^{(2)}(E) = \left( \langle n(E)^2 \rangle - \langle n(E) \rangle^2 \right) \sim \chi E, \tag{5} \]

using \( n(E) \) as the number of states in an interval of length \( E \) if \( E \gg 1 \), i.e. much beyond the mean level spacing. In a metal the spectral fluctuations can be mor-e-or-less described by standard random matrix theory, and therefore the compressibility vanishes, \( \chi \to 0 \), while in the extreme of strong disorder, i.e. when the spectrum is uncorrelated, Poisson statistics yield \( \chi = 1 \), giving \( \chi \) a character of order parameter. Right at the transition multifractality shows up in statistical fluctuations that are intermediate between these two extremes, i.e. \( 0 < \chi < 1 \).

An even more interesting result is that the quantities which describe the statistical fluctuations of the spectra and those for the eigenstates may be related as first pointed
out a long time ago in [12]. For the critical three-dimensional Anderson transition and the two-dimensional quantum-Hall transition it was shown earlier that:

$$2\chi + \frac{D_2}{d} = 1.$$  \hspace{1cm} \text{(6)}

This relation should obviously hold approximately only because the range of the correlation dimension and that of the level compressibility are limited as $0 \leq D_2/d \leq 1$ and $0 \leq \chi \leq 1$, respectively, leaving the validity of (6) to the limit of weak-multifractality; i.e. if $D_2 \to d$ then $\chi \to 0$.

More recently, Bogomolny and Giraud [4] showed that for systems at criticality in $d$-dimensions $D_1$ and $\chi$ can be related in a very simple way:

$$\chi + \frac{D_1}{d} = 1.$$  \hspace{1cm} \text{(7)}

Furthermore, using as evidence various critical random matrix ensembles, in [4] it was shown that:

$$\frac{D_q}{d} = \begin{cases} \frac{\Gamma(q-1/2)}{\sqrt{\pi} \Gamma(q)} (1-\chi), & 1-\chi \ll 1 \\ 1-q\chi, & \chi \ll 1 \end{cases}.$$  \hspace{1cm} \text{(8)}

The latter relation seemed to be valid for any type of multifractality.

Looking at equations (6) and (7) it seems evident that there may exist further relations between generalized dimensions and the level compressibility. Moreover, one may conjecture that the generalized dimensions of the multifractal spectrum must also be intimately linked together.

In [8] we presented evidence that a series of relations between various generalized dimensions, $D_q$ and $D_{q'}$, and the level compressibility $\chi$ does indeed exist, allowing for a generalization that contains equation (7) exactly and equation (6) within the appropriate limit. In order to prove that, numerical simulations of various critical random matrix ensembles were used. In the present work we wish to give a series of details that were not included in our earlier work. Moreover, by the use of our relations between generalized dimensions we state a clear link between the spectral and scattering properties of disordered systems at the metal-insulator transition. Finally, by exploring both, a deterministic model having a self-similar potential that produces multifractal eigenstates and multifractal objects expressly constructed by the use of iteration algorithms, we also show that our results are not restricted to random matrix models.

2. Model and heuristic relations

In [4] equations (7) and (8) were shown to be correct numerically for the Power-Law Banded Random Matrix (PBRM) model [2, 13, 14] at criticality. Below we will make use of this model to briefly present the heuristic relations already published in our earlier work. Furthermore, we expand the applicability of our approach to an extended range of multifractal dimensions as well as to scattering quantities.

The PBRM model describes one-dimensional (1d) samples of length $N$ with random long-range hoppings. This model is represented by $N \times N$ real symmetric ($\beta = 1$) or complex hermitian ($\beta = 2$) matrices whose elements are statistically independent random
variables drawn from a normal distribution with zero mean and a variance given by 
\( \langle |H_{nm}|^2 \rangle = \beta^{-1} \) and 
\[
\langle |H_{nm}|^2 \rangle = \frac{1}{2(1 + [\sin (\pi |m - n|/N)/(\pi b/N)]^{2b})},
\]
where \( b \) and \( \mu \) are parameters. In equation (9) the PBRM model is in its periodic version; i.e. the 1d sample is in a ring geometry. Theoretical considerations \([2, 13–15]\) and detailed numerical investigations \([2, 16, 17]\) have verified that the PBRM model undergoes a transition at \( \mu = 1 \) from localized states for \( \mu > 1 \) to delocalized states for \( \mu < 1 \). This transition shows key features of the disorder driven Anderson metal-insulator transition \([2]\), including the multifractality of eigenfunctions and non-trivial spectral statistics. Thus, the PBRM model possesses a line of critical points \( b \in (0, \infty) \) in the case of \( \mu = 1 \). In the following we will focus on the PBRM model at criticality, \( \mu = 1 \).

By tuning the parameter \( b \), from \( b \ll 1 \) to \( b \gg 1 \), the states cross over from the nature of strong-multifractality \( D_q \to 0 \), which corresponds to localized-like or insulator-like states, to weak-multifractality \( D_q \to 1 \) showing rather extended, i.e. metallic-like states. Meanwhile, at the true Anderson transition in \( d = 3 \) or at the integer quantum-Hall transition in \( d = 2 \), the states belong to the weakly multifractal regime, i.e. \( d - D_2 \ll d \), the PBRM model allows for an investigation without such a limitation. The evolution of the generalized dimensions as a function of the parameter \( b \) therefore represent this behavior, i.e. \( D_q \to 1 \) for \( b \gg 1 \) and within the limit of \( b \ll 1 \) the multifractal dimensions vanish as \( D_q \sim b^{[2, 14]} \).

The multifractal dimensions, especially \( D_1 \) and \( D_2 \) and their dependence on the parameter \( b \) in the case of the PBRM model, have been at the focus of several of our works \([18, 19]\). In those works it was demonstrated that simple phenomenological relations can be identified: \( D_1 \approx [1 + (\alpha_1 b)^{-1}]^{-1} \) and \( D_2 \approx [1 + (\alpha_2 b)^{-1}]^{-1} \) where \( \alpha_{1,2} \) are fitting constants. These continuous functions are trivial interpolations between the limiting cases of low-\( b \) and large-\( b \), taking half of the harmonic mean of the two cases. In \([8]\) we generalized and proposed the following heuristic expression for an extended range of the parameter \( q \): 
\[
D_q \approx \left[ 1 + (\alpha_q b)^{-1} \right]^{-1}, (10)
\]
as a global fit for the multifractal dimensions \( D_q \) of the PBRM model for both symmetry classes, \( \beta = 1 \) and \( \beta = 2 \). In figure 1 we show fits of equation (10) to numerically obtained \( D_q \)'s as a function of \( b \) for some values of \( q \) for the PBRM model with \( \beta = 2 \) (the case \( \beta = 1 \) has already been reported in \([8]\)) and in figure 2(a) we plot the values of \( \alpha_q \) extracted from the fittings; here, for comparison purposes, we report both cases: \( \beta = 1 \) and \( \beta = 2 \). We observe that equation (10) fits reasonably well the numerical \( D_q \) for \( q > 1/2 \). It is important to stress that equation (10) reproduces well the \( b \)-dependencies predicted analytically \([2]\) for the limits \( b \ll 1 \) and \( b \gg 1 \).

The multifractal dimensions of figure 1 were extracted from the linear fit of the logarithm of the inverse mean eigenfunction participation numbers versus the logarithm of \( N \), see equation (1). \( D_1 \) was extracted from the linear fit of the mean eigenfunction entropy versus the logarithm of \( N \), see equation (2). We used system sizes of \( N = 2^n \) with \( 8 \leq n \leq 13 \). The average was performed over \( 2^{n-3} \) eigenvectors with eigenvalues around the band center of \( 2^{16-n} \) realizations of the random matrices.

We have also verified that 
\[
\chi \approx (1 + \alpha_x b)^{-1}, (11)
\]
doi:10.1088/1742-5468/2014/11/P11012
reproduces qualitatively well the $b$-dependencies predicted analytically [2,20] in the small- and large-$b$ limits:

$$
\chi = \begin{cases} 
  1 - 4b, & \beta = 1 \\
  1 - \pi \sqrt{2b} + \frac{4}{3} (2 - \sqrt{3}) \pi^2 b^2, & \beta = 2 \\
  \frac{1}{2 \beta \pi b} & b \gg 1
\end{cases} \quad (12)
$$

Moreover, the parameter $\alpha_\chi$ must be equal to $\alpha_1$ for equations (10) and (11) to fulfill relation (7).

As a consequence of equating $b$ in equations (10) and (11) we get a direct relation

$$
\chi \approx (1 - D_q) \left[ 1 + (\gamma_q - 1) D_q \right]^{-1} \quad (13)
$$

with $\gamma_q = \alpha_1 / \alpha_q$. We observed that $\gamma_q \approx q$ in the range $0.8 < q < 2.5$, as plotted in figure 2(b), so in this range of $q$ values we can write simplified relations between $\chi$ and $D_q$:

$$
\chi \approx \frac{1 - D_q}{1 + (q-1) D_q} \quad \text{and} \quad D_q \approx \frac{1 - \chi}{1 + (q-1) \chi} \quad (14)
$$

The expression for $D_q$ in equation (14) reproduces equation (8) exactly for $q = 1$ and $q = 2$ and approximately for $1 < q < 2.5$. Moreover, equation (14) combined with equation (7) allows us to express any $D_q$ in terms of, for example, $D_1$:

$$
D_q \approx D_1 \left[ q + (1 - q)D_1 \right]^{-1} \quad (15)
$$

We also noticed that by equating $\chi$ for different $D_q$'s from equation (14) we were able to get recursive relations between them:

$$
q' D_{q'} (1 - D_{q'})^{-1} = q D_q (1 - D_q)^{-1} \quad (16)
$$

which in case of taking $q' = q + 1$ leads to

$$
D_{q+1} = q D_q (1 + q - D_q)^{-1} \quad (17)
$$

Notice that all these relations can be expressed using the fact that the ratio $q D_q / (1 - D_q)$ is independent of $q$. These expressions also provide a relation between the correlation

doi:10.1088/1742-5468/2014/11/P11012 6
dimension and the information dimension or between the correlation dimension and the compressibility of the spectrum:
\[ D_2 = D_1 (2 - D_1)^{-1} = (1 - \chi)(1 + \chi)^{-1}. \] (18)
It is relevant to add that in the weak multifractal regime, i.e. when \( \chi \to 1 \), equation (18) reproduces the relation given in equation (6) with \( d = 1 \), reported in [12].

### 2.1. The case \( q < 1/2 \)

For \( q < 1/2 \), equation (10) cannot be directly applied. However, the regime \( q < 1/2 \) could also be explored within our approach by the combination of equation (10) and the symmetry relation [21]
\[ \Delta_q = \Delta_{1-q} \quad \text{with} \quad \Delta_q = D_q(q - 1) - d(q - 1), \] (19)
implying that it is possible to link the multifractal dimensions with indexes \( q < 1/2 \) to those with \( q > 1/2 \). Therefore, we get
\[ D_q \approx \frac{1 - 2q}{1 - q} + \frac{q}{1 - q} \left( \frac{\alpha_{1-q} b}{1 + \alpha_{1-q} b} \right), \] (20)
for \( q < 1/2 \); that is, once we know the coefficients \( \alpha_q \) for \( q > 1/2 \) we can use them to get \( D_q \) for \( q < 1/2 \). Moreover, by the use of equation (17) it is possible to write \( D_q \), for \( q < 1/2 \), as a function of any specific \( D_q \) with \( q > 1/2 \). For example
\[ D_q \approx \frac{1 - 2q}{1 - q} + \frac{q}{1 - q} \left( \frac{D_1}{1 + q(D_1 - 1)} \right), \] (21)

---

**Figure 2.** (a) \( \alpha_q \) and (b) \( \gamma_q = \alpha_1/\alpha_q \) as a function of \( q \) for the PBRM model at criticality with \( \beta = 1 \) and \( \beta = 2 \). The red dashed line in (b) equal to \( q \) is plotted to guide the eye. The error bars in (a) are the rms error of the fittings.
provides $D_q$ for $q < 1/2$ in terms of the information dimension. Moreover, we can write down relations between $\chi$ and $D_q$ with $q < 1/2$:

$$\chi \approx \frac{1 - D_q}{q(2 - D_q)} \quad \text{and} \quad D_q \approx \frac{1 - 2q\chi}{1 - q\chi}. \quad (22)$$

Finally, it is important to stress that equations (20)–(21) get the form

$$D_q = (1 - 2q)(1 - q)^{-1} \quad (23)$$

in the limit $b \to 0$, which has been derived analytically in [6, 21–23].

2.2. Wigner delay times

As mentioned in the Introduction, the modeling and analysis of multifractal states in disordered systems at the Anderson-transition has been a subject of intensive research activity for many decades [1, 2]. Moreover, since the properties of the closed system, i.e. the fractality of the eigenstates, strongly influence the scattering and transport properties of the corresponding open system, the interest has also been extended to critical scattering systems. In particular, much attention has been focused on the probability distribution functions of the resonance widths and Wigner delay times [19, 21, 24–31], as well as the transmission or dimensionless conductance [32–48].

Among many relevant results, here we want to focus on those related to Wigner delay times $\tau_W$ and recall that

(i) for disordered systems at criticality the inverse moments of Wigner delay times $\langle \tau_W^{-q} \rangle$ scale as [21, 24, 25]

$$\langle \tau_W^{-q} \rangle \propto N^{-\sigma_q} \quad \text{where} \quad \sigma_q \equiv qD_{q+1}; \quad (24)$$

(ii) for the PBRM model at criticality the typical values of the Wigner delay times scale as [19]

$$\tau_{W}^{\text{typ}} \propto N^{\sigma_\tau} \quad \text{where} \quad \sigma_\tau = D_1, \quad (25)$$

with $\tau_{W}^{\text{typ}} \equiv \exp (\ln \tau_W)$.

Note that equations (24) and (25) provide a way to probe the properties of a critical system (i.e. the fractality of its eigenstates) by means of scattering experiments. Moreover, we can also relate spectral properties to scattering properties by:

(i) combining equations (14) and (24):

$$\sigma_q \approx \frac{q(1 - \chi)}{1 + q\chi} \quad \text{and} \quad \chi \approx \frac{q - \sigma_q}{q(\sigma_q + 1)}; \quad (26)$$

(ii) combining equations (18) and (25):

$$\sigma_\tau = 1 - \chi. \quad (27)$$
Also, we can express any $\sigma_q$ as a function of, say, $\sigma_1$:
\[
\sigma_q = 2q\sigma_1[(1 + q) + (1 - q)\sigma_1]^{-1}.
\]

Finally, note that we can obtain recursive relations for $\sigma_q$'s in analogy to equation (16):
\[
(q' + 1)\sigma_{q'}(q' - \sigma_{q'})^{-1} = (q + 1)\sigma_q(q - \sigma_q)^{-1},
\]
or
\[
\sigma_{q'} = q'(q + 1)\sigma_q[q(q' + 1) + (q - q')\sigma_q]^{-1},
\]
which leads to
\[
\sigma_{q+1} = (q + 1)^2\sigma_q[q(q + 2) - \sigma_q]^{-1},
\]
when $q' = q + 1$.

3. Numerical results for the PBRM model

In this section we present a numerical justification of our analytical relations derived above using the PBRM model.

3.1. Multifractal exponents

In figure 3 we plot $(1 - D_q)[1 + (q - 1)D_q]^{-1}$ as a function of $b$ for several values of $q$ for the PBRM model at criticality with $\beta = 1$ and $\beta = 2$, and observe good correspondence with the analytical prediction for $\chi$; that is, we verify the validity of equation (14). For completeness, in figure 3(a) we also include independent, numerically obtained values of $\chi$ (taken from [49]). In the insets of figure 3 we plot $qD_q(1 - D_q)^{-1}$ as a function of $b$, see equation (16), which for the PBRM model acquires the simple form
\[
qD_q(1 - D_q)^{-1} \approx \alpha_1 b.
\]
The fact that all curves $qD_q(1 - D_q)^{-1}$ versus $b$ fall one on top of the other makes evident its independence of $q$.

In figure 4 we present $D_q$ as a function of $q$ for the PBRM model with $\beta = 1$ and $\beta = 2$ for some values of $b$. As black and red dashed lines we also include equations (15), for $q > 1/2$, and (21), for $q < 1/2$, respectively. In both equations we used the values of $D_1$ we obtained numerically. We observe very good correspondence between the numerical data and equations (21) and (15) mainly for $-4 < q < 0.2$ and $0.8 < q < 4$, respectively.

3.2. Wigner delay times

We obtain Wigner delay times $\tau_W$ by turning the isolated system, represented by the PBRM model, into a scattering one by attaching one semi-infinite single-channel lead using perfect coupling. Since we are dealing here with the periodic version of the PBRM model, all sites are bulk sites and the place at which we attach the lead is irrelevant. To compute $\tau_W$ we use the effective Hamiltonian approach described in [19,24]. For statistical processing a large number of disorder realizations is used. Each disorder realization gives one value of $\tau_W$. We used $N = 50, 100, 200, 400$, and $800$ getting $10^6, 10^6, 10^5, 10^5$, and $10^4$ values of $\tau_W$, respectively. Then, the exponents $\sigma_q[\sigma_{\tau}]$ were extracted from the
On the generalized dimensions of multifractal eigenstates

Figure 3. $(1 - D_q)[1 + (q-1)D_q]^{-1}$ (see equation (14)) as a function of $b$ for the PBRM model at criticality with (a) $\beta = 1$ and (b) $\beta = 2$. The red dashed lines are the analytical predictions for $\chi$ given by equation (12). The blue symbols in (a) are independent numerically obtained values of $\chi$ (taken from [49]). Insets: $qD_q(1 - D_q)^{-1}$ as a function of $b$; see equation (32). The red dashed line equal to $\alpha_1 b$ is plotted to guide the eye. (a) $\alpha_1 = 4$ and (b) $\alpha_1 = 9.4$ were used.

linear fit of the logarithm of the averaged inverse moments of Wigner delay times (typical Wigner delay times) versus the logarithm of $N$, see equation (24) (equation (25)). We are concentrating here on the PBRM model with $\beta = 1$ only.

We start by noticing that if we combine equations (10) and (24) we get a heuristic expression for $\sigma_q$ as a function of $b$:

$$\sigma_q \approx \frac{q}{1 + (\alpha_{q+1}b)^{-1}}. \quad (33)$$

In figure 5 we compare equation (33) to the numerically obtained $\sigma_q$ as a function of $b$ for some values of $q$. We observe that equation (33) fits reasonably well the numerical $\sigma_q$ for $q \geq 0.1$. In figure 5 we also include the independent, numerically obtained values of $D_q$, which further verifies the validity of relation (24) [21,24,25].

In figure 6 we plot $(q - \sigma_q)[q(\sigma_q + 1)]^{-1}$ as a function of $b$ for the PBRM model at criticality; that is, we verify the validity of equation (26). We also plot the analytical prediction for $\chi$ given in equation (12) and observe good correspondence with the numerical data. In the inset we plot $\sigma_q/(q - \sigma_q)$ as a function of $b$, see equation (29).
which for the PBRM model acquires the simple form
\[ \sigma_q(q - \sigma_q)^{-1} \approx \alpha_{q+1} b. \] (34)

Finally, in figure 7 we show \( \sigma_\tau \) as a function of \( b \) for the PBRM model at criticality with \( \beta = 1 \). To test the validity of equation (27) we compare \( \sigma_\tau \) with the numerically obtained \( D_1 \) and with the theoretical prediction for \( 1 - \chi \). We again observe good correspondence.

4. Other critical ensembles

Remember that relations (14)–(18) were obtained from the combination of equations (10) and (11). That is, relations (14)–(18) are expected to work in particular for the PBRM model at criticality. However, equations (14) reproduce equations (7) and (8), which were shown to be valid for the PBRM model, but also for other critical ensembles [4]. Therefore, the question arises to what extent relations (14)–(18) are valid for critical ensembles different to the PBRM model. So, in the following we verify the validity of equations (14)–(18) for other well-known critical ensembles.

doi:10.1088/1742-5468/2014/11/P11012
4.1. Calogero–Moser ensembles

The Calogero–Moser (CM) $N$-particle systems yield three ensembles of $N \times N$ Hermitian matrices of the form [5,6]

\[ H_{mn} = p_m \delta_{mn} + ig(1 - \delta_{mn})V(m - n), \tag{35} \]

where $p_m$ are independent Gaussian random variables with zero mean and unit variance, $g$ is a free parameter which drives the multifractality of the eigenstates, and $V(m - n)$ is
one of the three following functions:

\[
\frac{1}{m-n}, \quad \frac{1}{N \sinh[(m-n)/N]}, \quad \frac{1}{N \sin[(m-n)/N]}
\]

These ensembles were denoted as [5, 6] CMR, CMH, and CMT, respectively.

In figures 8 and 9 we plot \(D_q\) as a function of \(q\) for the CMR and CMT ensembles, respectively, for several values of \(g\). To have an independent verification of our predictions, the data reported in these figures were taken from [6]. We compare the numerical data with our equations for \(D_q\) with \(q < 1/2\), equation (21), and \(q > 1/2\), equation (15); using as input, the values of \(D_1\) obtained by the interpolation of the curves \(D_q\) versus \(q\). We observe for both ensembles that our predictions reproduce reasonably well the numerical data.

4.2. The Ruijsenaars–Schneider ensemble and intermediate quantum maps

The Ruijsenaars–Schneider ensemble (RSE) proposed in [50] is defined as matrices of the form

\[
H_{mn} = \exp(i\Phi_m) \frac{1 - \exp(2\pi ig)}{N[1 - \exp(2\pi i(m - n + g)/N)]},
\]

where \(1 \leq m \leq n\), \(\Phi_m\) are independent random phases distributed between 0 and \(2\pi\), and \(g\) is a free parameter independent on \(N\).

Now, in figure 10 we present \(D_q\) as a function of \(q\) for the RSE for several values of \(g\). The data were also taken from [6]. As for the CM ensembles, here we observe that our predictions reproduce reasonably well the numerical data for both \(D_q\) with \(q < 1/2\) and \(q > 1/2\). The values of \(D_1\) we used as input in equations (21) and (15) were obtained by the interpolation of the curves \(D_q\) versus \(q\).
We used \( D_1 = 0.911, 0.771, 0.565, 0.226 \) and 0.0216.

Also, in [8] we tested some of our predictions for a variant of the RSE, introduced in [51], with the name of the intermediate quantum maps (IQM) model; see also [52].
On the generalized dimensions of multifractal eigenstates

Figure 10. $D_q$ as a function of $q$ for the RSE for several values of $g$. The data was taken from [6]. Black [Red] dashed lines are equation (15) (equation (21)). We used $D_1 = 0.99, 0.914, 0.75, 0.512, 0.193$ and 0.022.

In this model the parameter $g$ of the RSE is replaced by $cN/g$ with $cN = \pm 1 \mod g$, $g$ being the parameter of the IQM model. For the IQM model we substituted $\chi \approx 1/g$ or $D_1 \approx 1 - 1/g$, analytical expressions reported in [51], into equation (14) or (15), respectively, to get

$$D_q \approx (g - 1) (g + q - 1)^{-1}. \quad (39)$$

Here we just want to add that by the use of equations (37) and (38) for the RSE, and equation (39) for the IQM model, we can demonstrate the independence of $qD_q(1 - D_q)^{-1}$ on $q$ (already shown for the PBRM model in figure 3). In fact, by substituting the above-mentioned expressions into equation (16) we get

$$qD_q(1 - D_q)^{-1} \approx (g - 1)^{-2} - 1, \quad (40)$$

and

$$qD_q(1 - D_q)^{-1} \approx k^2(g - k)^{-2} - 1, \quad (41)$$

for $0 < g < 1$ and $|g - k| \ll 1$ with $k \geq 2$, respectively, for the RSE; and

$$qD_q(1 - D_q)^{-1} \approx g - 1, \quad (42)$$

for the IQM model. Then, in figure 11 we plot $qD_q(1 - D_q)^{-1}$ for the RSE and the IQM model for several values of $q$. We also include the equations given above in red dashed lines. We observe a rather good correspondence between the numerical data and equations (40)–(42).

4.3. Higher dimensional models

The generalization of equations (14)–(18) to higher dimensional systems ($d > 1$) can be done if $D_q$ is replaced by $D_q/d$ in equations (14)–(18). Then, below we explore the applicability of our results to the quantum Hall transition in $d = 2$ and the Anderson transition in $d = 3$. 

doi:10.1088/1742-5468/2014/11/P11012
Figure 11. $qD_q(1-D_q)^{-1}$ as a function of $g$ for (a) the RSE and (b) IQM model for several values of $q$. Red lines are (a) equations (40)--(41) and (b) equation (42). The multifractal dimensions $D_q$ reported in this figure were computed using the same matrix sizes and ensemble realizations as for the PBRM model.

In figure 12(a) we plot $D_q$ as a function of $q$ for the quantum Hall transition (QHT). The data for $D_q$ were taken from [53]. We also include the prediction for $D_q$ given by equation (15) (where $D_q$ has been replaced by $D_q/2$) using $D_1 = 1.7405 \pm 0.0002$ [53]. We observe that the prediction of equation (15) is a reasonably good approximation for $D_q$ in the interval $0 < q < 1.2$.

In figure 12(b) we plot $D_q$ and $D_q^{\text{typ}}$ as a function of $q$, for the 3d Anderson model at criticality, together with the prediction for $D_q$ given by equation (15) (where $D_q$ has been replaced by $D_q/3$) using $D_1 = 1.93 \pm 0.01$ [54]. The data for $D_q$ and $D_q^{\text{typ}}$ were taken from [55]. The multifractal dimensions $D_q^{\text{typ}}$ were extracted from the scaling of the typical participation numbers $I_q^{\text{typ}} \equiv \exp \langle \ln I_q \rangle$ with the system size $N$, from the relation

$$I_q^{\text{typ}} \sim N^{-(q-1)D_q^{\text{typ}}}.$$  

(43)

We observe that the prediction of equation (15) is reasonably good for $D_q^{\text{typ}}$ with $0 < q < 4$. In contrast, equation (15) do not reproduce the numerical $D_q$ when $q > 1$. We have also substituted $D_1 = 1.97 \pm 0.002$ (obtained from the interpolation of the $D_q$ data from [55]) into equation (15) but the resulting $D_q$ curve is very similar to that with $D_1 = 1.93 \pm 0.01$, so we do not show it in figure 12(b).
On the generalized dimensions of multifractal eigenstates

Figure 12. (a) $D_q$ as a function of $q$ for the QHT. The red dashed line is the prediction for $D_q$ given by equation (15) using $D_1 = 1.7405 \pm 0.0002$ [53]. The numerical data for $D_q$ was taken from [53]. (b) $D_q$ and $D_{typ}$ as a function of $q$ for the 3d Anderson model at criticality. The red dashed line is the prediction for $D_q$ given by equation (15) using $D_1 = 1.93 \pm 0.01$ [54]. The numerical data for $D_q$ and $D_{typ}$ was taken from [55].

5. Applicability to deterministic models

In the previous section we verified that relations (14)–(18) are valid for critical random matrix ensembles in 1d, and also, to some extent, to higher dimensional models at criticality. The common feature in the systems used above is the presence of multifractal eigenstates. However, note that not only disordered models produce them. It is well known that deterministic models having self-similar potentials also possess multifractal eigenstates, see for example [56, 57]. Moreover, multifractal objects can be expressly constructed by the use of iteration algorithms. As examples we can mention the Cantor set and the set produced by the baker's map.

Below, we test the applicability of our expressions relating multifractal exponents now to the multifractal eigenstates of a tight-binding model having a self-similar potential and to multifractal sets produced by iteration algorithms.

doi:10.1088/1742-5468/2014/11/P11012
5.1. Off-diagonal one dimensional Fibonacci lattice

According to Fujiwara et al [56] the multifractal spectrum of a one-dimensional Fibonacci sequence can be represented by the inflation rule $T_{n+1} = T_n T_{n-1}$, where $T_1 = A$ and $T_2 = AB$, so $T_3 = ABA$ and so on. In this case the Schrödinger equation

$$t_{j+1} \psi_{j+1} + t_{j-1} \psi_{j-1} = E \psi_j$$

has a multifractal solution at the bandcenter, $E = 0$. Then, by defining the parameter $g = t_{AB}/t_{AA}$ the generalized dimensions of the eigenstates take the form [56]

$$D_q = (3 \ln \sigma)^{-1} (q - 1)^{-1} \ln \left[ \lambda^q (g^2) / \lambda (g^{2q}) \right]$$

where $\sigma = (\sqrt{5} + 1) / 2$ is the golden mean and $\lambda(x) = (2x)^{-(x+1)^2 + \sqrt{(x+1)^4 + 4x^2}}$. Hence, the information dimension and the correlation dimension read as

$$D_1 = (3 \ln \sigma)^{-1} \left[ \ln \lambda(g^2) - g^2 \ln g^2 \lambda(g^2) / \lambda(g^2) \right]$$

and

$$D_2 = (3 \ln \sigma)^{-1} \left[ 2 \ln \lambda(g^2) - \ln \lambda(g^4) \right]$$

respectively.

Then, in figure 13 we plot $D_q$, computed from equation (44), as a function of $q$ for the Fibonacci lattice for several values of $g$. The dashed line is the prediction for $D_q$ given by equation (15) with $D_1$ calculated from equation (45). Again, as for the eigenstates of disordered models, here we observe that equation (15) reproduces rather well the multifractal dimensions of the eigenstates of the Fibonacci lattice, mainly in the range of $1 < q < 4$.

5.2. The 2-measure, 1-scale Cantor set or binomial branching process

In order to generate a multifractal distribution we use a Cantor set with two measures [58]. Take the $[0, 1]$ interval and divide it into two parts, which in our case can cover the whole interval, and even we can choose them to be equal. So let us partition the unit interval into two equal halves. Now let us introduce another measure: Let us associate the probability $g$ with one of the intervals and $(1 - g)$ with the other one. Then let us do the same
procedure with the two subintervals. In this way we start with the unit interval with probability measure 1 in the 0th approximation, then the 1st iteration gives two half intervals with probabilities $g$ and $1-g$. After the 2nd iteration we get four intervals, with length one-fourth each, but with measures $g^2$, $g(1-g)$, $(1-g)g$, and $(1-g)^2$. Proceeding further iteratively the distribution will be a multifractal. Such a distribution may also be achieved using a binomial branching process [59].

Since the distribution can be obtained in a recursive way, the generalized entropies of these distributions can be traced back to the 1st iteration, so we can write down $D_q$ immediately, see [59, 60] for details. Therefore

$$D_q = \ln [g^q + (1-g)^q] [(1-q) \ln 2]^{-1}$$ (47)

and

$$D_1 = -[g \ln g + (1-g) \ln(1-g)] (\ln 2)^{-1},$$ (48)

where the factor $\ln 2$ comes from the fact that the unit interval has been divided into two pieces of length $1/2$, or in other words the branching is always two-fold.

In figure 14 we plot $D_q$, computed from equation (47), as a function of $q$ for the Cantor set for several values of $g$. The dashed line is the prediction for $D_q$ given by equation (15) with $D_1$ calculated from equation (48). Again, as for the Fibonacci lattice, we observe that equation (15) reproduces rather well the multifractal dimensions of the the Cantor set, mainly in the range of $1 < q < 4$.

5.3. Generalized baker’s map

The generalized baker’s map is defined as a transformation of the unit square $[0, 1] \times [0, 1]$ with the following rules: [61] We first divide the unit square into two pieces, $y < g$ and $y > g$; $g = (0, 1)$. We then compress the two pieces in the horizontal direction by different factors, $\lambda_a$ for the piece in $y < g$, and $\lambda_b$ for the piece in $y > g$; where $\lambda_a + \lambda_b \leq 1$. Then we vertically stretch the lower piece by a factor $1/g$ and the upper piece by a factor $1/(1-g)$, so that both are of unit length. We then take the upper piece and place it back in the unit square with its right vertical edge coincident with the right vertical edge of the unit square. Thus, the generalized baker’s map is a mapping of the unit square into two
stripes within the square, one in $0 \leq x \leq \lambda_a$, and another one in $1 - \lambda_b \leq x \leq 1$. Applying the map a second time, maps the two stripes into four stripes. Application of the map $n$ more times results in more stripes of narrower width, where the widths approach zero as $n$ approaches infinity. In fact, the intersection of the attractor with a horizontal line is a Cantor set. In the particular case of $\lambda = \lambda_a = \lambda_b$ we have [61]

$$D_q = 1 + \frac{1}{q-1} \frac{\ln(g^q + (1-g)^q)}{\ln(\lambda)}. \quad (49)$$

Finally, in figure 15 we plot $D_q$, computed from equation (49), as a function of $q$ for the generalized baker’s map for several values of $g$ and $\lambda = 0.5$. The dashed line is the prediction for $D_q$ given by equation (15) with $D_1$ extracted from the interpolation of the data $D_q$ versus $q$.

Note that this multifractal is embedded in 2d, so we make the substitution $D_q \rightarrow D_q/2$ in equation (15). We observe that equation (15) reproduces well the multifractal dimensions of the generalized baker’s map mainly when $g \rightarrow 1/2$. When $g \rightarrow 0$ we observe good correspondence in the range $1/2 < q < 1$, only.

6. Conclusions

In this paper we propose heuristic relations on one hand between the generalized multifractal dimensions, $D_q$ and $D'_q$, for a relatively wide range of the parameter $q$, and on the other hand between these dimensions and the level compressibility $\chi$. As a result we find a general framework embracing an earlier result [12] and a recent one [4]. Our proposed relations have been corroborated by numerical simulations on: Various random matrix ensembles, a deterministic model having a self-similar potential whose eigenstates have multifractal properties, and multifractal objects expressly constructed by the use of iteration algorithms. Of course the analytical relations and the numerical simulations set limitations on the validity for a certain range of the parameter $q$. Therefore, our results are obviously approximate. Hence they call for further theoretical as well as numerical investigations.

doi:10.1088/1742-5468/2014/11/P11012
Moreover, since our relations between the generalized dimensions and the level compressibility allowed us to state a clear link between the spectral and scattering properties of disordered systems at the metal-insulator transition, it may be interesting to explore the consequences of our results on the quantities characterizing the dynamical properties of critical random matrix ensembles; which have been the focus of very recent investigations [62–64].

We believe that our results may find applications in several recently studied models characterized by multifractal eigenstates such as deterministic self-similar potentials [56, 57], quantum spin chains [65], Dirac fermions in the presence of random magnetic fields [66], disordered graphene [67], and other critical random matrix ensembles [4,68].

Acknowledgments

The authors are greatly indebted to V Kravtsov for useful discussions. This work was partially supported by VIEP-BUAP (Grant MEBJ-EXC14-1), PIFCA (Grant BUAP-CA-169), the Alexander von Humboldt Foundation, and the Hungarian Research Fund (OTKA) grant K108676.

References

[1] Janssen M 1998 Phys. Rep. 295 1
Janssen M 1994 Int. J. Mod. Phys. B 8 943
[2] Evers F and Mirli n A D 2008 Rev. Mod. Phys. 80 1355
[3] Richardella A, Roushan P, Mack S, Zhou B, Huse D A, Awschalom D D and Yazdani A 2010 Science 327 665
Faez S, Strybulevych A, Page J H, Lagendijk A and van Tiggelen B A 2009 Phys. Rev. Lett. 103 155703
Hashimoto K, Sohrmann C, Wiebe J, Inaoka T, Meier F, Hirayama Y, Römer R A, Wiesendanger R and Morgenstern M 2008 Phys. Rev. Lett. 101 256802
[4] Bogomolny E and Giraud O 2011 Phys. Rev. Lett. 106 044101
[5] Bogomolny E and Giraud O 2011 Phys. Rev. E 84 036212
[6] Bogomolny E and Giraud O 2012 Phys. Rev. E 85 046208
[7] Rushkin I, Ossipov A and Fyodorov Y V 2011 J. Stat. Mech. L03001
Méndez-Bermúdez J A, Akazary-Lopez A and Varga I 2012 Europhys. Lett. 98 37006
[8] Cuevas E and Kravtsov V E 2007 Phys. Rev. B 76 235119
[9] Wegner F 1980 Z. Phys. B 36 209
[10] Mehta M 1991 Random Matrices (Boston: Academic)
[11] Mehta M 1991 Random Matrices (Boston: Academic)
[12] Chalker J T, Kravtsov V E and Lerner I V 1996 JETP Lett. 64 386
Klesse R and Metzler M 1997 Phys. Rev. Lett. 79 721
[13] Mirlin A D, Fyodorov Y V, Dittes F-M, Quezada J and Seligman T H 1996 Phys. Rev. E 54 3221
[14] Mirlin A D 2000 Phys. Rep. 326 259
[15] Kravtsov V E and Muttalib K A 1997 Phys. Rev. Lett. 79 1913
Kravtsov V E and Tsvelik A M 2000 Phys. Rev. B 62 9888
[16] Cuevas E, Ortuno M, Gasparian V and Perez-Garrido A 2001 Phys. Rev. Lett. 88 016401
[17] Varga I and Braun D 2000 Phys. Rev. B 61 R11859
Varga I 2002 Phys. Rev. B 66 094201
[18] Méndez-Bermúdez J A, Kottos T and Cohen D 2006 Phys. Rev. E 73 036204
[19] Méndez-Bermúdez J A and Varga I 2006 Phys. Rev. B 74 125114
[20] Kravtsov V E, Yevtushenko O M and Cuevas E 2006 J. Phys. A: Math. Gen. 39 2021
Kravtsov V E, Yevtushenko O M and Cuevas E 2011 J. Phys. A: Math. Theor. 44 189501

doi:10.1088/1742-5468/2014/11/P11012
On the generalized dimensions of multifractal eigenstates

Mirlin A D, Fyodorov Y V, Mildenberger A and Evers F 2006 Phys. Rev. Lett. 97 046803
Montluc C and Garel T 2010 J. Stat. Mech. P09015
Kravtsov V E 2013 private communication
Méndez-Bermúdez J A and Kottos T 2005 Phys. Rev. B 72 064108
Ossipov A and Fyodorov Y V 2005 Phys. Rev. B 71 125133
Ossipov A, Kottos T and Geisel T 2003 Europhys. Lett. 62 719
Fyodorov Y V 2003 JETP Lett. 78 250
Texier C and Combet A 1999 Phys. Rev. Lett. 82 4220
Ossipov A, Kottos T and Geisel T 2000 Phys. Rev. B 61 11411
Kottos T and Weiss M 2002 Phys. Rev. Lett. 89 056401
Monthus C and Garel T 2010 J. Stat. Mech. P09015
Weiss M, Méndez-Bermúdez J A and Kottos T 2006 Phys. Rev. B 73 045103
Shapiro B 1990 Phys. Rev. Lett. 65 1510
Markoš P 1994 Europhys. Lett. 26 431
Markoš P 1999 Phys. Rev. Lett. 83 588
Slevin K and Ohtsuki T 1997 Phys. Rev. Lett. 78 4083
Slevin K, Ohtsuki T and Kawarabayashi T 2000 Phys. Rev. Lett. 86 3594
Wang X, Li Q and Soukoulis C M K 1998 Phys. Rev. B 58 3576
Rühländer M and Soukoulis C M 2001 Physica B 296 32
Rühländer M, Markoš P and Soukoulis C M 2001 Phys. Rev. B 64 172202
Rühländer M, Markoš P and Soukoulis C M 2001 Phys. Rev. B 64 212202
Travènec I and Markoš P 2002 Phys. Rev. B 65 113109
Schweitzer L and Markoš P 2005 Phys. Rev. Lett. 95 256805
Schweitzer L and Markoš P 2006 J. Phys. A: Math. Gen. 39 3221
Jansen M, Metzler M and Zirnbauer M R 1999 Phys. Rev. B 59 15836
Senouci K and Zekri N 2002 Phys. Rev. B 66 212201
Monthus C and Garel T 2009 Phys. Rev. B 79 205120
Monthus C and Garel T 2009 J. Stat. Mech. P07033
Méndez-Bermúdez J A, Gopar V A and Varga I 2010 Phys. Rev. B 82 125106
Martínez-Mendoza A J, Méndez-Bermúdez J A and Varga I 2010 AIP. Conf. Proc. 1319 41
Slevin K 2014 private communication
Bogomolny E, Giraud O and Schmit C 2009 Phys. Rev. Lett. 103 054103
Martin J, Giraud O and Georgeot B 2008 Phys. Rev. E 77 035201
Martin J, Garcia-Mata I, Giraud O and Georgeot B 2010 Phys. Rev. E 82 046206
Evers F, Mildenberger A and Mirlin A D 2008 Phys. Rev. Lett. 101 116803
Rodríguez A, Vasquez L J, Slevin K and Römer R A 2010 Phys. Rev. Lett. 105 046403
Vasquez L J 2010 High precision multifractal analysis in the 3D Anderson model of localization PhD Thesis
The University of Warwick
Fujitara T, Kohmoto M and Tokihito T 1989 Phys. Rev. B 40 7413
Woloszyn M and Spisak B J 2012 Eur. Phys. J. B 85 10
Mandelbrot B 1977 Fractals: Form, Chance and Dimension (San Francisco: W H Freedman)
Halsey T C, Jensen M H, Kadanoff L P, Procaccia I and Shraiman B 1986 Phys. Rev. A 33 1141
Miki H and Honjo H 2013 J. Phys. Soc. Japan 82 034002
Varga I, Pipek J, Janssen M and Pracz K 1996 Europhys. Lett. 36 437
Ott E 2002 Chaos in Dynamical Systems (Cambridge: Cambridge University Press)
Kravtsov V E, Ossipov A and Yevtushenko O M 2011 J. Phys. A: Math. Theor. 44 305003
Kravtsov V E, Yevtushenko O M, Snajberk P and Cuevas E 2012 Phys. Rev. E 86 021136
Garcia-Mata I, Martin J, Giraud O and Georgeot B 2012 Phys. Rev. E 86 056215
Atas Y Y and Bogomolny E 2012 Phys. Rev. E 86 021104
Atas Y Y and Bogomolny E 2014 Phil. Trans. R. Soc. A 372 20120520
Chamon C C, Murdy C and Wen X 1996 Phys. Rev. Lett. 77 4194
Chen X, Hsu B, Hughes T L and Fradkin E 2012 Phys. Rev. B 86 134201
Klefftogiannis I and Evangelou S N 2013 arXiv:1304.5968
Barrios-Vargas J E and Naumis G G 2014 2D Mater. 1 011009
Fyodorov Y V, Ossipov A and Rodríguez A 2009 J. Stat. Mech. L12001

doi:10.1088/1742-5468/2014/11/P11012