Study on caustic formation in Dirac-Born-Infeld type scalar field systems

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Abstract. Formation of caustics in the Dirac-Born-Infeld type scalar field systems has been investigated for two generic of potentials, viz., exponentially decreasing and inverse power law potentials. The study reveals that in the case of exponentially decreasing rolling massive potential, there are multi-valued regions and regions of likely to be caustics in the field configuration. The formation of caustics is inevitable for the inverse power law potentials under consideration in Minkowski spacetime, whereas caustics do not form in this case in the Friedmann-Robertson-Walker Universe.

1. Introduction

The widely accepted idea of the cause of accelerated expansion of the present Universe [1] is the existence of a mysterious form of energy, dubbed as dark energy. The dark energy exerts a large negative pressure for the present accelerated expansion and is the dominant fraction of the energy content of our Universe. This idea believes on the validity of the general theory of relativity (GTR) over all energy scales, however, there are many alternative ideas as well, which see the phenomenon of accelerated expansion from different perspectives (e.g., see [2]). As a viable physical explanation of the observed late time cosmic acceleration, a variety of scalar field models have been proposed as a source of dark energy over the years, which are basically motivated by the fundamental theories of high energy physics [2]. The Dirac-Born-Infeld (DBI) scalar field model, referred as the tachyon field, is the string theoretical model, in which, at early time when the field remain sub-dominant and frozen, the expansion dynamics is governed by the background fluid, and as the background energy density becomes comparable to the field energy density with the evolution of time, the field begins to evolve and subsequently overtakes the background to become the dominant component of the Universe [3]. Although the models, based upon the DBI scalar, have exciting features of the cosmological dynamics, they may lead to the formation of caustics, where the second, and the higher order derivatives of the field become singular. Caustic formation is an undesirable consequence in the field theoretical models in cosmology, which can be used to investigate the fundamental shortcoming of the field theory for a specific potential.

In this paper, we have investigated the issue of caustic formation in the DBI scalar field systems for inverse power law potentials, and also for the exponentially decreasing potential, by
extending the analysis given in [4], where it is shown that caustics inevitably form in tachyon system with potentials decaying faster than $\phi^{-2}$ at infinity [5]. In Section 2, we have discussed briefly about the DBI type general scalar field equation. Numerical results on the formation of caustics are presented in the Section 3. We have summarized our main results in the last Section.

2. The Dirac-Born-Infeld type scalar field system

The DBI type action for a scalar field (tachyon field) $\phi$ is given by [3, 4]:

$$S = -\int d^4x \sqrt{-g} \, V(\phi) \sqrt{1 + \partial_\mu \phi \partial^\mu \phi},$$

(1)

where $V(\phi)$ is the potential of the field $\phi$. The field equation derived from the action in Eq. (1) is:

$$\nabla_\mu \nabla^\mu \phi = \frac{\nabla_\mu \nabla_\nu \phi}{1 + \nabla_\alpha \phi \nabla^\alpha \phi} \nabla^\mu \phi \nabla^\nu \phi - \frac{V_\phi(\phi)}{V(\phi)} = 0,$$

(2)

where the covariant derivative of the field $\phi$ with respect to the metric $g_{\mu\nu}$ is denoted by $\nabla_\mu$. In the 1+1-dimensional Minkowski space, the Eq. (2) takes the form:

$$\ddot{\phi} = (1 + \phi'^2)^{-1} \left[ (1 - \phi'^2 + \phi'^2) \left( \phi'' - \frac{V_\phi(\phi)}{V(\phi)} \right) + 2\phi' \phi' \phi'' - \phi'^2 \phi'' \right],$$

(3)

where the dot and prime over $\phi$ indicate the time and space derivatives of the field $\phi$ respectively.

Since, in this paper, we are interested only on the formation of caustics in the tachyon field, we have only considered the inhomogeneous tachyon field where $\phi', \phi'' > 0$, however, they should be sufficiently less than unity for a realistic scalar field in cosmology [4]. Under this consideration, the Eq. (3) can be written as:

$$\ddot{\phi} = (1 - \phi'^2 + \phi'^2) \left( \phi'' - \frac{V_\phi(\phi)}{V(\phi)} \right) = P(\phi)Q(\phi),$$

(4)

where

$$P(\phi) = 1 - \phi'^2 + \phi'^2, \quad Q(\phi) = \phi'' - \frac{V_\phi(\phi)}{V(\phi)}.$$  

(5)

The equation of $P(\phi)$ contains both the time and space derivatives of the field, and therefore, this parameter can be used as an indicator of the pattern of evolution of the field governed by a particular field potential. For example, if the field $\phi(x, t)$ rapidly approaches the configuration $P(\phi) = 0$, it then indicates that the time and space evolution of the field is such that it rapidly approaches $\phi'^2 - \phi'^2 = 1$. Similarly, if $P(\phi) \approx 1$, it implies that, $\phi'^2 \approx \phi'^2$, i.e., the type of evolution of the field $\phi(x, t)$ with respect to space and time are nearly equal. The extensive numerical solutions of the Eq. (4) for the scalar field with the two different classes of field potentials of our interest clearly showed the above behaviour of $P(\phi)$ [5]. Thus, the Eq. (4) has two attractors, one for $P(\phi) = 0$ and the other for $P(\phi) = 1$, depending on the field potential. Corresponding to the above two attractors, we have obtained the parametric solutions for the field $\phi(x, t)$ [5]. This analysis can be easily enhanced to the more realistic case of expanding, spatially flat, Friedmann-Robertson-Walker (FRW) universe, assuming a small inhomogeneous field perturbation for a specific field potential [5].
3. Caustics in DBI system

We have studied the possibility of the formation of caustics in the DBI techyon system with the aforesaid two generic classes of potentials, and the numerical results are presented.

The exponentially decreasing rolling massive scalar field potential is given by:

\[ V(\phi) = V_0 e^{-\frac{1}{2}M^2\phi^2}, \quad (6) \]

where \( V_0 \) and \( M \) are constants. The inhomogeneous scalar field given in Eq. (4) for this potential can be written as:

\[ \ddot{\phi} = \left( 1 - \dot{\phi}^2 + \phi^2 \right) \left( \phi'' + M^2\phi \right). \quad (7) \]

We have solved the Eq. (7) for different initial values of the field \( \phi \) and for different values of the parameters \( M, \phi' \) and \( \phi'' \). We have taken the optimum values of all these parameters from our extensive numerical simulation to use the solution of the Eq. (7) for \( \phi(x, t) \), to study the patterns of variations of \( P(\phi) \) and \( Q(\phi) \) with respect to time [5]. From this study, we have considered the solution of \( P(\phi) \) with \( M = 1 \) and \( \phi_0 = 0 \), and the initial field configuration as \( \phi_i(q) = exp(-q^2) \) [4], to obtain the characteristic curves, which are shown in the left panel of the Figure 1. From these characteristic curves, it is observed that there are likely to be caustics as well as multi-valued regions in the field profile, which are independent of the initial conditions of the field [5].

![Figure 1](image)

**Figure 1.** The particle trajectories in \( 1 + 1 \) Minkowski space obtained for potentials given in Eqs. (6) and (8) respectively.

Next, we have considered the inverse power law potential of the form, given by:

\[ V(\phi) = V_0 \phi^{-n}, \quad 0 < n < 2. \quad (8) \]

The late time accelerated expansion of the universe corresponds to \( 0 \leq n \leq 2 \), and accordingly we have restricted the value of the exponent \( n \) within this range for our work. The inhomogeneous scalar field given in Eq. (4), in this case takes the form:

\[ \ddot{\phi} = \left( 1 - \dot{\phi}^2 + \phi^2 \right) \left( \phi'' + \frac{n}{\phi} \right). \quad (9) \]

We have followed the same numerical procedure as in the previous case for this potential also. The plots of the characteristic curves for this potential are shown in the right panel of the Figure 1, for \( n = 1.0 \) with \( \phi_0 = 1.0 \) [5]. This panel of the figure clearly shows the formation of caustics and multi-valued regions for the inverse power law potential.

As expansion works against caustics formation, we have expected not to encounter caustics in the case of exponentially decreasing potential, but even caustics are more distinctly form for
the expanding universe in this case, which is clear from the left panel of the Figure 2. In the case of the inverse power law potential, there is no caustics in the field configuration as seen from in the right panel of the Figure 2. Since, in the case of this potential, the dark energy behaves as a late time attractor, and the characteristic curves, obtained by taking the form of $a(t) \sim t^{2/3(1+w)}$, incorporating all time dark energy effect, are very similar to the characteristic curves of 1 + 1-dimension at late time [5].

4. Conclusion
The caustic formation in mass free space is an important issue for the viability of the various scalar field models in the context of the dark energy paradigm. We have examined the formation of caustic in the tachyon system for two generic classes of potential, viz, massive rolling scalar potentials and inverse power law potentials. Our study is supported by analytical estimation as well as the extensive numerical simulations [5]. We have observed that caustics are formed in the tachyon field with exponentially decreasing massive scalar potential for both 1 + 1-dimensional Minkowski space, and flat FRW expanding universe. The caustic formation is more definite in the case of the expanding universe than the case of 1 + 1-dimensional Minkowski space. However, the expanding universe dilutes the effect of caustics as clear from the figures, but cannot render the situation caustic free, which is generally true for a tachyon potential that decays faster than $1/\Phi^2$ at infinity. For inverse power law potentials, caustics are formed with the multi-valued regions in the Minkowski spacetime. It is interesting to observe that no caustics are formed in the field of the flat FRW expanding universe for the inverse power law potentials as the effect of expansion can compete with the tendency of caustic formation. Dark energy as a late time attractor of the dynamics in this case gives rise to cosmic repulsion, allowing one to avoid caustics and multi-valued regions in the field profile. This behaviour of the field with inverse power law potentials agrees very well with the result obtained by assuming dark energy dominance at all times. Hence, the inverse power law is found suitable to explain the late time cosmic acceleration [5].

References
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