THE LERCH ZETA FUNCTION II. ANALYTIC CONTINUATION

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Abstract. This is the second of a series of four papers that study algebraic and analytic structures associated with the Lerch zeta function. The Lerch zeta function
\[ \zeta(s, a, c) := \sum_{n=0}^{\infty} \frac{e^{2\pi i n a}}{(n+c)^s} \]
was introduced by Lipschitz in 1857, and is named after Lerch, who showed in 1887 that it satisfied a functional equation. Here we analytically continue \( \zeta(s, a, c) \) as a function of three complex variables. We show that it is well-defined as a multivalued function on the manifold \( \mathcal{M} := \{(s, a, c) \in \mathbb{C} \times (\mathbb{C} \setminus \mathbb{Z}) \times (\mathbb{C} \setminus \mathbb{Z})\} \), and that this analytic continuation becomes single-valued on the maximal abelian cover of \( \mathcal{M} \). We compute the monodromy functions describing the multivalued nature of this function on \( \mathcal{M} \), and determine various of its properties.

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1. INTRODUCTION

The Lerch zeta function is defined by the Dirichlet series
\[ \zeta(s, a, c) := \sum_{n=0}^{\infty} \frac{e^{2\pi i n a}}{(n+c)^s}. \] (1.1)

It already appeared in a paper of Lipschitz [17] in 1857, but is named after Lerch [15], who showed in 1887 that it satisfies a functional equation. This functional equation, called Lerch’s transformation formula, states that
\[ \zeta(1-s, a, c) = (2\pi)^{-s} \Gamma(s) \left\{ e^{\frac{\pi i c}{2}} e^{-2\pi i c} \zeta(s, -c, a) + e^{-\frac{\pi i c}{2}} e^{2\pi i c (1-a)} \zeta(s, c, 1-a) \right\}, \] (1.2)
and is valid for complex $a$ with $\Im(a) \geq 0$ and real $c$ with $0 < c < 1$, cf. Erdelyi [5, p. 29]. (For $\Im(a) > 0$ and real $c > 0$, the Dirichlet series (1.1) is an entire function of $s$, because its coefficients are rapidly decreasing.) In part I we derived two symmetrized four term functional equations, as given by Weil [26], which are valid for real $a$ and $c$, and for all $s \in \mathbb{C}$. These apply to the two functions

$$L^\pm(s, a, c) := \zeta(s, a, c) \pm e^{-2\pi i a} \zeta(s, 1 - a, 1 - c),$$

see Theorem 2.1 below. Special cases of the Lerch zeta function include $a = 0$ which gives the Hurwitz zeta function

$$\zeta(s, c) = \sum_{n=0}^{\infty} \frac{1}{(n + c)^s},$$

named after Hurwitz [8], and the case $c = 1$, which gives

$$e^{-2\pi i a} F(s, a) = e^{-2\pi i a} \left( \sum_{n=1}^{\infty} \frac{e^{2\pi i a}}{n^s} \right),$$

where $F(s, a)$ is the periodic zeta function, cf. Apostol [2, p. 257]. The Riemann zeta function occurs when $a = 0$ and $c = 1$. These four term functional equations, stated in Theorem 2.1 below, can be used to derive the known functional equations of the Hurwitz and periodic zeta functions.

This paper is the second in a series of four papers that study algebraic and analytic structures associated with the Lerch zeta function. Our object here is to obtain the analytic continuation of the Lerch zeta function as a function of three complex variables. Our main result analytically continues the function to an essentially maximal domain of holomorphy, viewing it as a multivalued function of three variables. Then we determine properties of the extension to this domain, noting particularly the differential-difference equations and differential equations which these functions satisfy, described below and in more detail in §2.

First, we analytically continue the Lerch zeta function to a single-valued function on the universal cover $\tilde{\mathcal{M}}$ of the non-simply connected manifold

$$\mathcal{M} := \{(s, a, c) \in \mathbb{C} \times (\mathbb{C} \setminus \mathbb{Z}) \times (\mathbb{C} \setminus \mathbb{Z})\}. \quad (1.5)$$

The resulting function may alternatively be viewed as a multivalued function on $\mathcal{M}$, and we compute the monodromy functions describing the multivaluedness. We show that the resulting function becomes single valued on the maximal abelian cover $\tilde{\mathcal{M}}^{ab}$ of $\mathcal{M}$; see §2. The manifold $\mathcal{M}$ proves convenient to study because this is the largest manifold on which the two functions $L^\pm(s, a, c)$ appearing in the functional equations are well-defined as multivalued functions. That is, the manifold $\mathcal{M}$ has the symmetry that the map $\phi_R : \mathcal{M} \rightarrow \mathcal{M}$ defined by

$$\phi_R(s, a, c) = (1 - s, 1 - c, a)$$

defines an automorphism of order 4 which respects the complex structure on $\mathcal{M}$. This map lifts to an automorphism of the universal cover $\tilde{\mathcal{M}}$ which leaves invariant a “base point region”, the fundamental polycylinder defined in §2. We show in §8 that some of
the punctures in the $c$-plane of $\mathcal{M}$ are removable singularities, so that there is a further extension of this analytic continuation to a multi-valued function on the universal cover $\tilde{\mathcal{M}}^\#$ of the extended manifold

$$\mathcal{M}^\# := \{(s, a, c) \in \mathbb{C} \times (\mathbb{C} \setminus \mathbb{Z}) \times (\mathbb{C} \setminus \mathbb{Z}_{\leq 0})\}. \quad (1.6)$$

This manifold is obtained from $\mathcal{M}$ by filling in the points where $c = n \geq 1$ is a positive integer. However $\mathcal{M}^\#$ is not preserved by the automorphism $\phi_R$, and the four-term functional equations cannot be extended to be defined at all points of its universal cover $\tilde{\mathcal{M}}^\#$. The universal cover $\tilde{\mathcal{M}}^\#$ appears to be a maximal domain of holomorphy for the analytic continuation in three variables. It is certainly maximal except for the possible addition of lower-dimensional strata, of real codimension at least two, see §9 for further remarks.

Second, we observe that the function $\zeta(s, a, c)$ satisfies two differential-difference equations, namely

$$\left(\frac{1}{2\pi i} \frac{\partial}{\partial a} + c\right) \zeta(s, a, c) = \zeta(s - 1, a, c), \quad (1.7)$$

and

$$\frac{\partial}{\partial c} \zeta(s, a, c) = -s \zeta(s + 1, a, c). \quad (1.8)$$

This has the important consequence that it satisfies a linear partial differential equation namely

$$\left(\frac{1}{2\pi i} \frac{\partial}{\partial a} \frac{\partial}{\partial c} + c \frac{\partial}{\partial c}\right) \zeta(s, a, c) = -s \zeta(s, a, c). \quad (1.9)$$

These three operators all act equivariantly with respect to the covering map from the universal cover $\tilde{\mathcal{M}} \to \mathcal{M}$. It follows that all the monodromy functions satisfy these differential-difference equations and the linear partial differential equation (1.9). Here the differential operators $\frac{\partial}{\partial a}, \frac{\partial}{\partial c}$ are viewed as acting on a complex domain, in the sense of Hille [7].

Third, we study properties of the monodromy functions. We obtain the analytic continuation of the functional equations given in part I, and show that these imply a system of linear dependencies among the monodromy functions. We also study the vector space $\mathcal{V}_s$ spanned by monodromy functions, with the parameter $s$ held fixed. In this case there occur further linear dependencies among monodromy functions, for integer values of $s$. We note that at non-positive integers $s = -m \leq 0$ all monodromy functions for $\zeta(s, a, c)$ vanish identically. It follows that the value of the function $\zeta(s, a, c)$ is then well-defined on the manifold $\mathcal{M}$ rather than being defined only on a covering manifold $\tilde{\mathcal{M}}$. This gives a property characterizing the these values of $s$ as “special values”.

A logical continuation of this work is to study the change of variable $z := e^{2\pi i a}$, which gives the Lerch transcendent $\Phi(s, z, c)$, given by

$$\Phi(s, z, c) := \sum_{n=0}^{\infty} \frac{z^n}{(n + c)^s}. \quad (1.10)$$

This is done in part III, where we analytically continue the function $\Phi(s, z, c)$ to a (nearly) maximal domain of holomorphy, and determine its monodromy functions. The
Lerch transcendent is closely related to the polylogarithm \( L_{i,m}(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^m} \) under the specialization \( c = 1 \) and \( s = m \) is a positive integer. We use this to obtain results on generalized polylogarithms and their monodromy.

There has been very extensive prior work on analytic continuation of the Lerch zeta function, some of it given in terms of the Lerch transcendent, described further in part III. In particular there are many individual results on analytic continuations in various subsets of the variables, including continuations on various “singular strata” above, for example that of the Hurwitz zeta function. Most relevant to this work, in 2000 Kanemitsu, Katsurada and Yoshimoto [9, Theorem 1 and Theorem 4] obtained an analytic continuation of the Lerch zeta function and Lerch transcendent in three complex variables, to a single-valued function on various large (non-maximal) simply-connected domains in \( \mathbb{C}^3 \). In §2 we compare methods; here our objective is to go further and determine completely the multivalued nature of the analytic continuation. We recently discovered that in 1906 E. W. Barnes [3] discussed analytic continuation of the Lerch transcendent \( \Phi(s, z, c) \), and noted features of its multivalued nature. His work appears to give another approach to analytic continuation of these functions in three variables.

For general treatments of the Lerch zeta function we refer to the books of Laurenˇ cikas and Garunkštis [14], Srivastava and Choi [24, Chap. 2] and Kanemitsu and Terada [10, Chaps. 3-5].

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2. Summary of results

The main results of this paper concern the multivalued analytic continuation of the Lerch zeta function, summarized in Theorems 2.1-2.3 below. A more detailed description of the multivaluedness is given in theorems in §4.

To effect the analytic continuation we use four ingredients: (i) the series representation \((1.1)\), (ii) the integral representation \((2.2)\) below, (iii) the functional equations, both the four-term symmetric functional equations of part I, and the three-term asymmetric functional equation (Lerch transformation formula), analytically continued to a suitable domains, and (iv) the difference-differential equation \((1.7)\). In comparison, the analytic continuation of Kanemitsu, Katsurada and Yoshimoto [9] made use of (i), (ii) above for the Lerch transcendent; they also used the Lerch transformation formula and various other expansions not considered here.

The series representation \((1.1)\) defines \( \zeta(s, a, c) \) as a single-valued analytic function of three variables on the polycylinder

\[
\mathcal{U} := \{ s : s \in \mathbb{C} \} \times \{ a : \Im(a) > 0 \} \times \{ c : \Re(c) > 0 \} \subset \mathbb{C}^3.
\]
We also use the following integral representation of the Lerch zeta function:

\[ \zeta(s, a, c) := \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}e^{-ct}}{1-e^{2\pi ia}e^{-t}} dt. \]  

This defines a single-valued analytic function on the polycylinder

\[ \mathcal{U} := \{ s : \Re(s) > 0 \} \times \{ a : \Im(a) > 0 \} \times \{ c : \Re(c) > 0 \}, \]  

which is smaller than the region \( (2.1) \). However this integral can be used to give an analytic continuation to a region in the \( a \)-variable allowing negative imaginary part, if the real part of \( a \) is suitably restricted. Erdelyi [5, p. 27] notes that in terms of the variable \( z = e^{2\pi ia} \) the integral above converges on the region \( z \in \mathbb{C} \setminus \mathbb{R} \geq 1 \). In particular the right side of \( (2.2) \) defines \( \zeta(s, a, c) \) as a single-valued analytic function on the fundamental polycylinder

\[ \Omega := \{ s : 0 < \Re(s) < 1 \} \times \{ a : 0 < \Re(a) < 1 \} \times \{ c : 0 < \Re(c) < 1 \}. \]  

The Lerch zeta function satisfies a two four-term functional equations, in the real variables \( 0 < a < 1, 0 < c < 1 \) given in part I [11, Theorem 2.1]. These functional equations were obtained by Weil [26, p. 57]. Our first result, Theorem 2.1 below, gives analytically continued versions of these functional equations valid on the fundamental polycylinder, and permits a further analytic continuation in the \( s \)-variable to a single-valued function on the larger region

\[ \tilde{\Omega} := \{ s : s \in \mathbb{C} \} \times \{ a : 0 < \Re(a) < 1 \} \times \{ c : 0 < \Re(c) < 1 \}, \]  

which we call the extended fundamental polycylinder. We recall the symmetrized Lerch zeta functions

\[ L^\pm(s, a, c) := \zeta(s, a, c) \pm e^{2\pi i a} \zeta(s, 1-a, 1-c), \]  

and we let \( \hat{L}^\pm(s, a, c) \) denote the same functions times an appropriate archimedean factor specified in the following result.

**Theorem 2.1.** (Lerch Functional Equations on Polycylinders).

1. On the fundamental polycylinder \( \Omega \), which requires \( 0 < \Re(s) < 1 \), the function

\[ \hat{L}^+(s, a, c) := \pi^{-\frac{3}{2}} \Gamma\left(\frac{s}{2}\right) (\zeta(s, a, c) + e^{-2\pi ia} \zeta(s, 1-a, 1-c)) \]  

is holomorphic in all variables and satisfies the functional equation

\[ \hat{L}^+(s, a, c) = e^{-2\pi ic} \hat{L}^+(1-s, 1-c, a). \]  

In addition the function

\[ L^-(s, a, c) := \pi^{-\frac{s+1}{2}} \Gamma\left(\frac{s+1}{2}\right) (\zeta(s, a, c) - e^{-2\pi ia} \zeta(s, 1-a, 1-c)) \]  

is holomorphic in all variables and satisfies the functional equation

\[ \hat{L}^-(s, a, c) = ie^{-2\pi ic} \hat{L}^-(1-s, 1-c, a). \]  

2. The function \( \zeta(s, a, c) \) analytically continues to a holomorphic function of three variables on the extended fundamental polycylinder \( \tilde{\Omega} \), which allows \( s \in \mathbb{C} \). Both functions \( L^\pm(s, a, c) \) analytically continue to \( \tilde{\Omega} \) as holomorphic functions of three variables and the functional equations \( (2.7) \) and \( (2.10) \) hold on \( \tilde{\Omega} \).
Theorem 2.1 is proved in §3. The three-term functional equation of the Lerch zeta function (Lerch’s transformation formula) also holds on $\tilde{\Omega}$ and is derived as Corollary 3.1.

The extended fundamental polycylinder is a large convex domain in $\mathbb{C}^3$ on which the Lerch zeta function is single-valued. The Dirichlet series (1.1) representation defines another convex domain (2.1) on which it is single-valued. The union of these two domains is simply connected but is not convex.

The transformation $(s,a,c) \mapsto (1-s,1-c,a)$ is an automorphism of the extended fundamental polycylinder of period four. Iterating it yields

$$\tilde{L}^\pm(s,a,c) = (-1)^k e^{-2\pi i a} \tilde{L}^\pm(s,1-a,1-c), \text{ for } k = 0,1 \quad (2.11)$$

and

$$\tilde{L}^\pm(s,a,c) = (-i)^k e^{-2\pi i ac + 2\pi ic} \tilde{L}^\pm(1-s,c,1-a), \text{ for } k = 0,1. \quad (2.12)$$

Here the notation $L^+$ corresponds to $k = 0$ and $L^-$ corresponds to $k = 1$, i.e. $\pm = (-1)^k$.

Our next two results concern the multivalued analytic continuation of the Lerch zeta function. Let $\mathcal{M} = \mathbb{C} \times (\mathbb{C} \setminus \mathbb{Z}) \times (\mathbb{C} \setminus \mathbb{Z})$, and let $\tilde{\mathcal{M}}$ denote its universal cover, identified with homotopy classes of curves starting from the base point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. This basepoint is in the extended fundamental polycylinder, which is canonically embedded as a subset of the universal cover $\tilde{\mathcal{M}}$. We let $Z([\gamma])$ denote the analytic continuation of $\zeta(s,a,c)$ along a path $\gamma : [0,1] \to \mathcal{M}$ in $(s,a,c)$-space which has base point $\gamma(0) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, and whose other endpoint is at $\gamma(1) = (s,a,c)$. We let $[\gamma]$ denote the homotopy class of the path $\gamma$ with fixed endpoints on the manifold $\mathcal{M}$. We will establish that the value $Z([\gamma])$ depends only on this homotopy class. We view $Z([\gamma])$ as an analytic function element when we allow the endpoint $\gamma(1)$ to vary, and in this case we write $Z(s,a,c,[\gamma])$ to explicitly indicate the dependence on the endpoint. The function, $Z(s,a,c,[\gamma])$ represents a branch of the Lerch zeta function lying above $(s,a,c)$ (Here we are considering a covering manifold of a 3-dimensional complex manifold, so these branches are locally 3-dimensional complex manifolds.) In §4 we prove the following result.

**Theorem 2.2.** (Lerch Analytic Continuation) *The Lerch zeta function $\zeta(s,a,c)$ analytically continues to a single-valued holomorphic function $Z(s,a,c,[\gamma])$ on the universal cover $\tilde{\mathcal{M}}$ of the manifold

$$\mathcal{M} = \{(s,a,c) \in \mathbb{C} \times (\mathbb{C} \setminus \mathbb{Z}) \times (\mathbb{C} \setminus \mathbb{Z})\},$$

in which $[\gamma] \in \pi_1(\tilde{\mathcal{M}}, x_0)$ for $x_0 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $\pi([\gamma]) = (s,a,c)$ denotes the image of the covering map $\pi : \tilde{\mathcal{M}} \to \mathcal{M}$. Furthermore the function $Z(s,a,c,[\gamma])$ is single-valued on the maximal abelian covering manifold $\tilde{\mathcal{M}}^{ab}$ of $\tilde{\mathcal{M}}$.*

Given a multivalued function $F$ on $\mathcal{M}$ and two loops $\tau_1$ and $\tau_2$ in $\mathcal{M}$ based at the point $x_0 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, we define the monodromy function $M(F, [\tau_1], [\tau_2])$ to be the multivalued function whose value on a path $\gamma$ having $\gamma(0) = x_0$ is given by

$$M(F, [\tau_1], [\tau_2])([\gamma]) := F([\tau_2\gamma]) - F([\tau_1\gamma]). \quad (2.13)$$
The name “monodromy function” reflects that the standard definition of monodromy is the value of this function $M(F, [\tau_1], [\tau_2])$ at the base point $x_0$.

To determine all monodromy functions it suffices to compute them in the special case that $[\tau_1]$ is the identity, in which case we use the simplified notation $M_{[\tau_2]}(F) := M(F, [\tau_2], [Id])$. In §4 we derive explicit formulas for the monodromy functions of the function $Z$, for a set of generators of $\pi_1(M, x_0)$, see Theorem 4.5. We show that the monodromy functions vanish identically on the derived subgroup (commutator subgroup) $\pi_1(M, x_0)$, for a set of generators of $\pi_1(M, x_0)$, see Theorem 4.6. This implies that $Z(s, a, c, [\gamma])$ defines a single-valued function on the maximal abelian covering manifold $\tilde{M}^{ab}$ of $M$, as stated above.

The monodromy computations reveal that all monodromy functions corresponding to loops around the integer points $c = n \geq 1$ vanish. These points have removable singularities, and we obtain the following extended analytic continuation, whose proof is deferred to §8.

**Theorem 2.3.** (Lerch Extended Analytic Continuation) The Lerch zeta function $\zeta(s, a, c)$ analytically continues to a single-valued holomorphic function $Z(s, a, c, [\gamma])$ on the universal cover $\tilde{M}^\#$ of the manifold

\[ \tilde{M}^\# := \{(s, a, c) \in \mathbb{C} \times (\mathbb{C} \setminus \mathbb{Z}) \times (\mathbb{C} \setminus \mathbb{Z}_{\leq 0}) : \}, \]

in which $[\gamma] \in \pi_1(\tilde{M}^\#, x_0)$ for $x_0 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $\pi^\#([\gamma]) = (s, a, c)$ denotes the image of the covering map $\pi^\# : \tilde{M}^\# \to M^\#$. Furthermore the function $Z(s, a, c, [\gamma])$ is single-valued on the maximal abelian covering manifold $(\tilde{M}^\#)^{ab}$ of $M^\#$.

In §5-§7 we determine various properties of the Lerch monodromy functions, associated to the manifold $M$. First, in §5 we observe that the Lerch zeta function satisfies the two differential-difference equations (1.7) and (1.8), under analytic continuation. From this we deduce that it also satisfies a linear partial differential equation (2.12) in the $(s, a, c)$-variables. The differential-difference equations and the partial differential equation are equivariant under the action of $\pi_1(M, x_0)$, so we deduce that the monodromy functions satisfy the same difference-differential equations and partial differential equation (Theorem 5.1). Second, in §6 we specify linear dependencies among the Lerch monodromy functions that follow from the functional equations (Theorem 6.1). Third, in §7 we study the vector space $\mathcal{V}_s$ generated by the Lerch monodromy functions when the variable $s$ is fixed. Here each monodromy function $M_{[\tau]}([\gamma])$ corresponds to some element of $\mathcal{V}$. Note that the associated monodromy representation describing the multivaluedness is an induced action

\[ \rho : \pi_1(M, x_0) \to GL(\mathcal{V}_s), \]

where $\mathcal{V}_s$ is a (generally infinite-dimensional) complex vector space spanned by the germs of function elements of $Z$ above $x_0$. We obtain a family of representations $\rho_s : \pi_1(M, x_0) \to GL(\mathcal{V}_s)$. The size of the vector space $\mathcal{V}_s$ can change radically as $s$ varies, it degenerates on certain lower dimensional strata. In particular we observe that
for $s \in \mathbb{Z}$ there are extra linear dependencies, not implied by the functional equations, satisfied by the monodromy functions. We show that the monodromy functions vanish identically when $s = -m \leq 0$ is a nonpositive integer, so that $\zeta(-m, a, c)$ is a single-valued function of $(a, c) \in (\mathbb{C} \setminus \mathbb{Z}) \times (\mathbb{C} \setminus \mathbb{Z})$, cf. Theorem 7.2 We explicitly determine a basis of the vector space $V_s$ for all complex $s$. For $s$ not a nonpositive integer, the vector space $V_s$ is infinite-dimensional.

In section §8, we give the analytic continuation of $\zeta(s, a, c)$ to the extended manifold $M^\#$, and prove Theorem 2.3.

**Notation.** The hat notation $\hat{L}$ always denotes a “completed” function multiplied by an appropriate archimedean Euler factor. In the following $\Re(s)$ and $\Im(s)$ denote the real and imaginary parts of a complex variable $s$, respectively.

### 3. Functional equations

As a preliminary step, we recall the two functional equations derived in part I, originally due to A. Weil [26], valid for real $(a, c)$ with $0 < a < 1, 0 < c < 1$. We prove Theorem 2.1 analytically continuing them to the fundamental polycylinder $\Omega$, noting that $\Omega$ is invariant under the symmetries of the functional equations.

For all real $a$ and for real $c$ with $0 < c < 1$,

$$L^+(s, a, c) := \zeta(s, a, c) + e^{-2\pi ia} \zeta(s, 1-a, 1-c) = \sum_{n=-\infty}^{\infty} e^{2\pi i a |n + c|^s} .$$

(3.1)

We also set

$$L^-(s, a, c) = \zeta(s, a, c) - e^{-2\pi ia} \zeta(s, 1-a, 1-c) = \sum_{n=-\infty}^{\infty} e^{2\pi i a \text{sgn}(n + c)|n + c|^{-s}} ,$$

(3.2)

in which

$$\text{sgn}(x) = \begin{cases} 
-1 & \text{if } x < 0 , \\
0 & \text{if } x = 0 , \\
1 & \text{if } x > 0 .
\end{cases}$$

(3.3)

If $a \notin \mathbb{Z}$ these series converge absolutely and define analytic functions of $s$ for $\Re(s) > 1$; they converge conditionally (summing from $-N$ to $N$ and letting $N \to \infty$) and define analytic functions of $s$ for $\Re(s) > 0$. If $a \in \mathbb{Z}$ they converge absolutely and define analytic functions of $s$ for $\Re(s) > 1$.

**Lemma 3.1.** (Lerch Functional Equations) For real $a, c$ with $0 < a < 1$ and $0 < c < 1$, and with $\pm = (-1)^k$ the functions

$$\hat{L}^\pm(s, a, c) := \pi^{-\frac{s+k}{2}} \Gamma\left(\frac{s+k}{2}\right) L^\pm(s, a, c) \quad \text{for } k = 0, 1 ,$$

(3.4)

analytically continue to $s \in \mathbb{C}$ as holomorphic functions of $s$, and satisfy the functional equation

$$\hat{L}^\pm(s, a, c) = i^k e^{-2\pi i ac} \hat{L}^\pm(1-s, 1-c, a) \quad \text{for } k = 0, 1 .$$

(3.5)
Proof. This follows from part I, see [11] Theorem 1.1. This result was originally obtained by Weil [26] eqn. (15) on page 57. Weil’s notation \((a, x, y, s)\) corresponds to our notation \((k, c, -a, 2s)\). His functions \(S_k(x, y, s)\) defined by [26] eqn. (10) on p. 56 are related to our functions by \(S_0(x, y, s) = L^+(2s, -y, x)\) and \(S_1(x, y, s) = L^-(2s - 1, -y, x)\). Some terms involve \(-a\) where we have \(1 - a\) above, but Weil’s functions are defined to be periodic (mod 1) in the \(a\)-variable. □

The proof in part I of the functional equations above apply more generally to all real \(a\), including the cases \(a = 0\) and 1 with \(0 < c \leq 1\). That is, one can define \(\hat{L}^\pm(s, a, c)\) for \(0 < c \leq 1\), with \(a = 0\) or 1, meromorphically continue it to \(s \in \mathbb{C}\), and obtain an appropriate functional equation. In these cases \(\hat{L}^+(s, a, c)\) has a simple pole at \(s = 1\) and no other singularities while \(\hat{L}^-(s, a, c)\) is holomorphic for \(s \in \mathbb{C}\). The case \(\hat{L}^+(s, 0, 1)\) corresponds to the Riemann zeta function, and \(\hat{L}^+(s, 0, c)\) for \(0 < c < 1\) to the Hurwitz zeta function. Part I observed that an extension of \(\hat{L}^\pm(s, a, c)\) to \((a, c) \in \mathbb{R} \times \mathbb{R}\) existed that preserved the functional equation and gave meromorphic functions in \(s\) for each fixed \((a, c)\), at the cost that this function is discontinuous in \(a\) and \(c\) at integer values, for certain ranges of \(s\). This provides evidence that there cannot exist a general analytic continuation in all three complex variables that includes the points \(a \in \mathbb{Z}\).

Proof of Theorem 2.7. (i). The functional equations consist of four terms expressed in terms of the Lerch zeta function by (2.7) and (2.9), and each of the four terms is a holomorphic function of three complex variables in the fundamental polycylinder

\[
\Omega = \{ s : 0 < \Re(s) < 1 \} \times \{ a : 0 < \Re(a) < 1 \} \times \{ c : 0 < \Re(c) < 1 \}.
\]

These functional equations then hold by analytic continuation in the three variables in the entire domain \(\Omega\). Indeed treat \(s\) and \(c\) as fixed, with \(0 < c < 1\) and arbitrary \(\{ s : 0 < \Re(s) < 1 \}\), and vary \(a\). Since the relation holds for the interval \(0 < a < 1\), it analytically continues to all values of \(a\) in the region \(\{ a : 0 < \Re(a) < 1 \}\). Once this is done, we now can choose an arbitrary point \((s, a)\) in \(\{ s : 0 < \Re(s) < 1 \} \times \{ a : 0 < \Re(a) < 1 \}\) and vary \(c\). Since it holds for real \(c\), \(0 < c < 1\), it analytically continues to \(\{ c : 0 < \Re(c) < 1 \}\) and \(\Omega\) is covered.

(ii). The integral (2.2) defines \(\zeta(s, a, c)\) as an analytic function of three variables in the domain

\[
\Omega^+ := \{ s : \Re(s) > 0 \} \times \{ a : 0 < \Re(a) < 1 \} \times \{ c : 0 < \Re(c) < 1 \}.
\]  

(3.6)

We can suitably combine the functional equations for \(\hat{L}^+(s, a, c)\) and \(\hat{L}^-(s, a, c)\) to eliminate one of the four zeta function terms that occur, to recover the Lerch transformation formula in the slightly modified form

\[
\zeta(1 - s, a, c) := (2\pi)^{-s}\Gamma(s) \left[ e^{\frac{i\pi s - 2\pi i c}{2}} \zeta(1 - c, a) + e^{-\frac{i\pi s - 2\pi i c}{2}} \zeta(s, c, 1 - a) \right].
\]  

(3.7)

This formula is valid for \(a, c\) real with \(0 < a < 1\), \(0 < c < 1\) and for \(s\) in the strip \(\{ s : 0 < \Re(s) < 1 \}\), and is proved in Theorem 5.4 of part I. (The identities \(\Gamma(\frac{s}{2})\Gamma(\frac{s + 1}{2}) = \sqrt{2\pi} 2^{\frac{s}{2} - s}\Gamma(s)\) and \(\Gamma(s)\Gamma(1 - s) = \frac{\pi}{\sin \pi s}\) are used in the derivation of (3.7).) For \(\Re(s) > 1\)
both terms on the right side are jointly analytic functions of all three variables in the region $\Omega^+$, and taking this as a definition of the left side effects the analytic continuation of $\zeta(s, a, c)$ to $\Re(s) < 0$ in all three variables. The functional equations of Theorem 2.1 are then inherited by analytic continuation in the $s$-variable, when the other two variables are held constant. The formula (3.7) expresses $\zeta(1 - s, a, c)$ as a holomorphic function for $\Re(s) > 0$, because all terms on the right are holomorphic there. Thus $\zeta(s, a, c)$ is holomorphic in the region

$$\Omega^- := \{ s : \Re(s) < 1 \} \times \{ a : 0 < \Re(a) < 1 \} \times \{ c : 0 < \Re(c) < 1 \}$$

which with $\Omega^+$ covers the extended fundamental polycylinder $\tilde{\Omega} = \{ s : s \in \mathbb{C} \} \times \{ a : 0 < \Re(a) < 1 \} \times \{ c : 0 < \Re(c) < 1 \}$. It remains to establish the holomorphicity in $s$ of the functions $L^\pm(s, a, c)$, for $s \in \mathbb{C}$. Since $\zeta(s, a, c)$ is holomorphic in all variables, we must establish there are no poles produced by their Gamma function factors at negative integers. This now follows from the functional equations taking $s$ to $1 - s$, which equate them to values at positive integers where the holomorphy follows from that of $\zeta(s, a, c)$ and $\zeta(s, 1 - a, 1 - c)$. \hfill \qed

The proof above also establishes the following variant of the Lerch transformation formula, valid on the extended fundamental polycylinder (2.5).

**Theorem 3.2.** (Extended Lerch Transformation Formula) For all $(s, a, c)$ in the extended fundamental polycylinder $\tilde{\Omega}$ there holds

$$\zeta(1 - s, a, c) = (2\pi)^{-s} \Gamma(s) \left[ e^{\pi i s/2} e^{-2\pi i ac} \zeta(s, 1 - c, a) + e^{-\pi i s/2} e^{-2\pi i(a - 1)} \zeta(s, c, 1 - a) \right].$$

(3.8)

**Proof.** This holds for $(s, a, c) \in \tilde{\Omega}$ using (3.7). \hfill \qed

The extended Lerch transformation formula above differs from the original Lerch transformation formula (1.2) in having the term $\zeta(s, 1 - c, a)$ in place of $\zeta(s, -c, a)$. This formula still agrees with (1.2) because the function $\zeta(s, a, c)$ in Lerch’s original paper is periodic (mod one) in the $a$-variable for $\Im(a) > 0$, and then by extension for parts of the real axis, so that

$$\zeta(s, -c, a) = \zeta(s, 1 - c, a)$$

(3.9)

holds in a suitable domain. The extended Lerch transformation formula (3.8) keeps all Lerch zeta function terms inside the extended fundamental polycylinder. Under analytic continuation the equality (3.9) is preserved only if the correct branches of the function are chosen on both sides, cf. Theorem 6.1.

4. **Analytic continuation and monodromy functions**

In this section we analytically continue $\zeta(s, a, c)$ to a multivalued function on the manifold

$$\mathcal{M} := \mathbb{C} \times (\mathbb{C} \setminus \mathbb{Z}) \times (\mathbb{C} \setminus \mathbb{Z}),$$

(4.1)
which is coordinatized with the three complex variables \((s, a, c)\), and prove Theorem 4.2.

We fix the base point \(x_0 := (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})\) in the fundamental polycylinder \(\Omega\) and consider analytic continuation of the Lerch zeta function along a path \(\gamma: [0, 1] \to \mathcal{M}\) emanating from the base point \(\gamma(0) = x_0\) to an endpoint \(\gamma(1) = (s, a, c) \in \mathcal{M}\). The analytic continuation will depend only on the homotopy class \([\gamma]\) of the path (where homotopies hold endpoints fixed).

**Definition 4.1.** The multi-valued function element \(Z(\gamma)\) denotes the analytic continuation of the Lerch zeta function \(\zeta(s, a, c)\) along the path \(\gamma\), starting from the base point \((s, a, c) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})\). We shall sometimes denote this by \(Z(s, a, c; \gamma)\), where \((s, a, c)\) are the coordinate values at the endpoint of \(\gamma\), and view it as a single-valued holomorphic function in a small open neighborhood of the endpoint \((s, a, c)\).

Below we will use the notations \(Z([\gamma])\) and \(Z([s, a, c, [\gamma]])\), anticipating the dependence of function elements only on the homotopy class of the path; this latter property is only established in the proof of Theorem 4.5 below.

We may alternatively regard \(Z([\gamma])\) as a holomorphic function on the universal covering manifold \(\hat{\mathcal{M}}\) of \(\mathcal{M}\). The universal cover \(\hat{\mathcal{M}}\) is the collection of homotopy classes \([\gamma]\) of paths from \(x_0\), and the projection \(\pi: \hat{\mathcal{M}} \to \mathcal{M}\) is the endpoint

\[
\pi([\gamma]) := \gamma(1) .
\] (4.2)

The lifted base point in the universal cover is \(\hat{x}_0 := [x_0] \in \hat{\mathcal{M}}\), and we let \(\hat{x} = [\gamma]\) denote a general point in \(\hat{\mathcal{M}}\). Since \(\hat{\mathcal{M}}\) is a covering space of \(\mathcal{M}\), it inherits its complex-analytic structure. Thus if we can analytically continue \(\zeta(s, a, c)\) along a path \(\gamma\), it will automatically be holomorphic in a neighborhood of its endpoint. When we want to view \(Z([\gamma])\) as a holomorphic function element in a neighborhood of \([\gamma]\), we write \(Z(s, a, c, [\gamma])\). As the extended fundamental polycylinder \(\hat{\Omega}\) is simply connected, we shall identify a point \(x \in \hat{\Omega}\) with a path \(\gamma\) from \(x_0\) to \(x\) which remains entirely in \(\hat{\Omega}\). In this way we embed \(\hat{\Omega}\) in \(\hat{\mathcal{M}}\). Given one path \(\gamma\) from \(x_0\) to \(x\), we may describe all homotopy classes of paths from \(x_0\) to \(x\) by \([\tau\gamma]\), where \([\tau]\) runs through all classes of loops in the fundamental group \(\pi_1(\mathcal{M}, x_0)\) of \(\mathcal{M}\) with base point \(x_0\).

We now specify a set \(\mathcal{G}\) of generators of the homotopy group \(\pi_1(\mathcal{M}, x_0)\). In view of (4.3), the \(\pi_1(\mathcal{M}, x_0)\) is the product of the homotopy groups of the three manifolds \(\mathcal{C}, \mathcal{X}_a = \mathcal{C} \setminus \mathbb{Z}\) and \(\mathcal{Y}_c = \mathcal{C} \setminus \mathbb{Z}\). The homotopy group \(\pi_1(\mathcal{C}, \frac{1}{2})\) is trivial. The group \(\pi_1(\mathcal{X}_a, \frac{1}{2})\) is a free group on countably many generators \(\{[X_n] : n \in \mathbb{Z}\}\), in which \(X_n\) is a small counterclockwise-oriented loop around the point \(a = n \in \mathbb{Z}\) in the \(a\)-plane, which is initially reached by a path from the base point \(a = \frac{1}{2}\) which lies in the upper half-plane \(\{a : \Im(a) > 0\}\) and which returns to \(\frac{1}{2}\) along the same path. The group \(\pi_1(\mathcal{Y}_c, \frac{1}{2})\) is defined similarly, with generators \(\{[Y_n] : n \in \mathbb{Z}\}\), where \([Y_n]\) traverses a small counterclockwise loop around the point \(c = n\) in the \(c\)-plane, reached along a path in \(\{c : \Im(c) > 0\}\). These give the set of generators

\[
\mathcal{G} := \{[X_n] : n \in \mathbb{Z}\} \cup \{[Y_n] : n \in \mathbb{Z}\}
\] (4.3)

of \(\pi_1(\mathcal{M}, x_0)\).
We determine explicit formulas for the multivalued function elements $Z([\gamma])$ in terms of the action of the generating set $G$, given in Theorem 4.5 below. We first outline the proof approach. We begin by analytically continuing $\zeta(s,a,c)$ to holomorphic functions on a collection $\{U_j\}$ of open, simply connected subdomains $U_j$ of $\mathcal{M}$, each containing $\Omega$ such that their union covers $\mathcal{M}$. Hence we have extended $\zeta$ to a holomorphic function $Z$ on some open subset $\mathcal{W}$ of $\tilde{\mathcal{M}}$ whose projection to $\mathcal{M}$ is the whole space. Next we observe that for each point $x$ in the intersection of two subdomains, we have obtained two values $Z([\gamma_1])$ and $Z([\gamma_2])$, following two paths $\gamma_1$, $\gamma_2$ from $x_0$ to $x$ in each subdomain. We call the difference $Z([\gamma_2]) - Z([\gamma_1])$ the monodromy of $Z$ at $[\gamma_1]$ with respect to the loop $[\tau] \in \pi_1(\mathcal{M},x_0)$, where $\tau = \gamma_2\gamma_1^{-1}$ is the composition of the path $\gamma_2$ followed $\gamma_1$ traced backwards. We view this as a function of the path $[\gamma_1]$, called the monodromy function $M_{[\tau]}(Z)([\gamma_1])$, and formalize this in the following definitions.

**Definition 4.2.** Let $f : \tilde{\mathcal{M}} \to \mathbb{C}$ be a continuous function on the universal cover $\tilde{\mathcal{M}}$ of a manifold $\mathcal{M}$ and let $[\tau] \in \pi_1(\mathcal{M},x_0)$ be a homotopy class. The $[\tau]$-translated function $Q_{[\tau]}(f) : \tilde{\mathcal{M}} \to \mathbb{C}$ is defined by

$$Q_{[\tau]}(f)([\gamma]) := f([\tau\gamma]) ,$$

where $\gamma$ is a path with basepoint $\gamma(0) = x_0$ and $\tau\gamma$ is the composed path.

Each element $[\tau]$ of $\pi_1(\mathcal{M},x_0)$ corresponds to a fixed-point-free homeomorphism $\phi_{[\tau]}$ of $(\tilde{\mathcal{M}},\tilde{x}_0)$ which commutes with the projection $\pi : \tilde{\mathcal{M}} \to \mathcal{M}$ (see Hatcher [6, Chap. 1.3]). Then we have, in terms of $\phi_{[\tau]}$,

$$Q_{[\tau]}(f)(\tilde{x}) = f(\phi_{[\tau]}(\tilde{x})) .$$

Each $Q_{[\tau]}$ acts as a linear operator on the vector space $C^0(\tilde{\mathcal{M}})$ of continuous complex-valued functions on $\tilde{\mathcal{M}}$, defining an representation of the group $\pi_1(\mathcal{M},x_0)^{\text{opp}}$ having the opposite multiplication, i.e.

$$Q_{[\tau_2]}Q_{[\tau_1]} = Q_{[\tau_1\tau_2]} .$$

We have $Q_{[\tau^{-1}]} = Q_{[\tau]}^{-1}$ and the operators $\hat{Q}_{[\tau]} := Q([\tau])^{-1}$ give a linear representation of $\pi_1(\mathcal{M},x_0)$ on the vector space $C^0(\tilde{\mathcal{M}})$.

**Definition 4.3.** Let $f : \tilde{\mathcal{M}} \to \mathbb{C}$ be a continuous function and let $[\tau] \in \pi_1(\mathcal{M},x_0)$ be a homotopy class. The monodromy function $M_{[\tau]}(f) : \tilde{\mathcal{M}} \to \mathbb{C}$ of $f$ at $[\tau]$ is defined by

$$M_{[\tau]}(f) := (Q_{[\tau]} - I)(f) .$$

That is, for all paths $\gamma : [0,1] \to \mathcal{M}$ with base point $\gamma(0) = x_0$,

$$M_{[\tau]}(f)([\gamma]) := f([\tau\gamma]) - f([\gamma]) .$$

Monodromy functions obey the following relation.

**Lemma 4.4.** For a single-valued function $f(\tilde{x})$ on $\tilde{\mathcal{M}}$ and any $[\tau_1]$, $[\tau_2] \in \pi_1(\mathcal{M},x_0)$, we have

$$M_{[\tau_1\tau_2]}(f) = M_{[\tau_1]}(f) + M_{[\tau_2]}(f) + M_{[\tau_2]}(M_{[\tau_1]}(f)) .$$
Proof. We have
\[ M_{[r_1,r_2]}(f)([\gamma]) = f([r_1 \tau_2 \gamma]) - f([\gamma]), \]
and
\[ M_{[r_1]}(f)([\gamma]) + M_{[r_2]}(f)([\gamma]) = f([r_1 \gamma]) - f([\gamma]) + f([r_2 \gamma]) - f([\gamma]), \]
while
\[ M_{[r_2]}(M_{[r_1]}(f))(\gamma) = M_{[r_1]}(f)([r_2 \gamma]) - M_{[r_1]}(f)(\gamma) \]
\[ = (f([r_1 r_2 \gamma]) - f([r_2 \gamma])) - (f([r_1 \gamma]) - f([\gamma])). \]
Combining these formulae yields (4.9). \qed

To continue outlining the proof, we obtain a partially defined monodromy function
\[ M_{[\tau]}(Z)([\gamma]) \]
defined only for certain paths \( \gamma \) lying in one subdomain \( U_j \) and ending in the overlapped area. However the monodromy functions \( M_{[\tau]}(Z) \), each defined on an open subset of \( \tilde{\mathcal{M}} \), are “simpler” functions than \( Z \), and can themselves be directly analytically continued to \( \tilde{\mathcal{M}} \) without knowledge of \( Z \). We are then able to analytically continue \( Z \) along all loops \( [\tau] \in \pi_1(\mathcal{M}, x_0) \), by writing \( [\tau] \) as a word in the generating set \( \mathcal{G} \), and repeatedly applying Lemma 4.4 to compute \( [\tau] \) to compute \( M_{[\tau]}(Z) \) and finally obtain
\[ Z([\tau]) := M_{[\tau]}(Z)([\gamma]) + Z([\gamma]), \]
for \([\gamma] \) contained in \( U_j \), and \( [\tau] \in \pi_1(\mathcal{M}, x_0) \). Since the sets \( \{U_j\} \) cover \( \mathcal{M} \), this defines a single-valued holomorphic function \( Z \) on \( \tilde{\mathcal{M}} \), which agrees with \( \zeta(s,a,c) \) on the fundamental polycylinder. The resulting function \( Z \) is the desired function by uniqueness of analytic continuation along paths, using Cauchy’s theorem and the fact that \( \mathcal{M} \) is simply connected.

The following result establishes analytic continuation and explicitly gives monodromy functions associated to the generating set \( \mathcal{G} \). In this result, the principal branch of the function \( \log z \) is the branch of \( \log z \) on \( \mathbb{C}^* \) which is real on the positive real axis, with the branch defined by making a cut along the negative imaginary axis, so that on the negative real axis one has \( \log(-x) := \log x + \pi i \), where \( x \in \mathbb{R}_{>0} \).

**Theorem 4.5.** (Lerch monodromy functions) The Lerch zeta function \( \zeta(s,a,c) \) analytically continues to a single-valued holomorphic function \( Z \) on the universal cover \( \tilde{\mathcal{M}} \) of \( \mathcal{M} \). The monodromy functions of \( Z \) on the generators of \( \pi_1(\mathcal{M}, x_0) \) are given as follows.

(i-a) For \( n \in \mathbb{Z} \), on the extended fundamental polycylinder \( \tilde{\Omega} \) given by (2.5) we have
\[ M_{[X_n]}(Z)(s,a,c) = -\frac{(2\pi i)^{s} e^{2\pi i a}}{\Gamma(s)}(a-n)^{s-1} e^{-2\pi i c(a-n)}, \]
where \((a-n)^{s-1} := e^{(s-1)\log(a-n)}\) using the principal branch of the logarithm. This function analytically continues to a single-valued function on \( \tilde{\mathcal{M}} \).

(i-b) For \( n \in \mathbb{Z} \), on any path \( \gamma \) in \( \mathcal{M} \) from \( x_0 \) to an endpoint \( x_1 \) lying in the multiply connected region
\[ \mathcal{M}_s := \{s\} \times (\mathbb{C} \setminus \mathbb{Z}) \times (\mathbb{C} \setminus \mathbb{Z}) \]
we have
\[ M_{[X_n]^{-1}}(Z)([\gamma]) = -e^{-2\pi i s} M_{[X_n]}(Z)([\gamma]) \],
and
\[ M_{[X_n]}(Z)([\gamma]) = \frac{e^{\pm 2\pi isk} - 1}{e^{\pm 2\pi is} - 1} M_{[X_n]}(Z)([\gamma]) \quad \text{for} \quad k \geq 1. \quad (4.14) \]

(ii-a) For \( n \in \mathbb{Z} \), on the extended fundamental polycylinder \( \tilde{\Omega} \) there holds
\[ M_{[Y_n]}(Z)(s, a, c, [\gamma]) = \begin{cases} 0 & \text{for } n \geq 1, \\ e^{-2\pi is}(c - n)^{-s} & \text{for } n \leq 0, \end{cases} \quad (4.15) \]

where \( (c - n)^{-s} := e^{-s\log(c - n)} \) using the principal branch of the logarithm. This function analytically continues to a single-valued function on \( \widetilde{\mathcal{M}} \).

(ii-b) For \( n \in \mathbb{Z} \), on any path \( \gamma \) in \( \mathcal{M} \) from \( x_0 \) to an endpoint \( x_1 \) lying in \( M_s \) we have we have
\[ M_{[Y_n]}^{-1}(Z)([\gamma]) = -e^{2\pi is} M_{[Y_n]}(Z)([\gamma]), \quad (4.16) \]

and
\[ M_{[X_n]}^{\pm k}(Z)([\gamma]) = \frac{e^{\pm 2\pi isk} - 1}{e^{\pm 2\pi is} - 1} M_{[X_n]}^{\pm 1}(Z)([\gamma]) \quad \text{for} \quad k \geq 1. \quad (4.17) \]

Remark. With our convention on the principal branch of the logarithm, the formula (4.11) has the more explicit form
\[ M_{[X_n]}(Z)(s, a, c) = \begin{cases} -(2\pi)^s a \frac{\Gamma(s)}{\Gamma(s)} e^{\pi is(s-1)}(n - a)^{s-1} e^{-2\pi ic(a-n)} & \text{if } n \geq 1, \\ -(2\pi)^s a \frac{\Gamma(s)}{\Gamma(s)} (a - n)^{s-1} e^{-2\pi ic(a-n)} & \text{if } n \leq 0. \end{cases} \]

Proof. Recall that the extended fundamental polycylinder \( \tilde{\Omega} \) is embedded as a subset of \( \tilde{\mathcal{M}} \) with base point \((\frac{1}{2}, \frac{1}{2}, \frac{1}{2})\) and that on this region the function \( Z \) agrees with \( \zeta(s, a, c) \).

We start with the integral representation of the Lerch zeta function
\[ \zeta(s, a, c) := \frac{1}{\Gamma(s)} \int_0^\infty \frac{e^{-ct}}{1 - e^{2\pi ia}e^{-t}} t^{s-1} dt. \quad (4.18) \]

Let \( \mathcal{A}_L \) denote the \( a \)-plane cut along all the half-lines \( M_k = \{ k - it : 0 \leq t < \infty \} \) for \( k \in \mathbb{Z} \), so that
\[ \mathcal{A}_L := \mathbb{C} \setminus \{ M_k : k \in \mathbb{Z} \}. \quad (4.19) \]

This cut region is pictured in Figure 1.

The integral (4.18) defines a single-valued function on the simply-connected domain
\[ \mathcal{U}_1 := \{ s : \Re(s) > 0 \} \times \{ a : a \in \mathcal{A}_L \} \times \{ c : \Re(c) > 0 \}. \quad (4.20) \]

Furthermore we may obtain a similar result for a domain of altered shape by considering instead
\[ \zeta(s, a, c) := \frac{1}{\Gamma(s)} \int_{L_u,\epsilon} \frac{e^{-ct}}{1 - e^{2\pi ia}e^{-t}} t^{s-1} dt, \quad (4.21) \]

in which \( L_{u,\epsilon} \) is a deformed path from 0 to \( \infty \) in the \( t \)-plane on the real axis which makes a clockwise detour around the point \( u \in \mathbb{R}_{>0} \) through a circle of radius \( \epsilon \), with \( 0 < \epsilon < \min(u, 1/2) \), as shown in Figure 2.

The integral (4.21) defines a single-valued holomorphic function on the simply-connected domain
\[ \mathcal{U}_1(u, \epsilon) := \{ s : \Re(s) > 0 \} \times \mathcal{A}_{L_{u,\epsilon}} \times \{ c : \Re(c) > 0 \} \]
in which \( A_{L_u,\epsilon} \) denotes the \( a \)-plane cut along all the deformed half-lines
\[
L_k^-(u, \epsilon) := \left\{ a = k - \frac{i}{2\pi} t : t \in L_{u,\epsilon} \right\} \quad \text{for} \quad k \in \mathbb{Z},
\]
so that
\[
A_{L_u,\epsilon} := \mathbb{C} \setminus \{ L_k^-(u, \epsilon) : k \in \mathbb{Z} \}.
\] (4.22)

As \( u \) and \( \epsilon \) vary over \( 0 < u < \infty \) and \( 0 < \epsilon < \min(u, 1/2) \), the domains \( U_1(u, \epsilon) \) cover the entire domain
\[
U_1^\infty := \{ s : \Re(s) > 0 \} \times \{ a : a \in \mathbb{C} \setminus \mathbb{Z} \} \times \{ c : \Re(c) > 0 \}.
\] (4.23)

Moreover, for \( u > 0 \) and \( 0 < \epsilon < \min(u, 1/2) \), the open half-disk
\[
D_{u,\epsilon}(n) := \left\{ a : \left| a - \left( n - \frac{iu}{2\pi} \right) \right| < \epsilon \quad \text{and} \quad \Re(a) > n \right\}
\] (4.24)
lies in the intersection of the domains \( A_{L_u,\epsilon} \) and \( A_L \), but in homotopically different components, see Figure 3. We use this fact to compute \( M_{[X_n]}(Z) \), where \( Z \) denotes the analytic continuation of the function \( \zeta(s, a, c) \) from the base point \( x_0 \). We can find
two paths $\gamma_1, \gamma_2$ with base point $x_0 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ which hold $c = s = 1/2$ fixed and vary $a$, such that $\gamma_1(1) = \gamma_2(1) = (\frac{1}{2}, a, \frac{1}{2})$ with $a \in D_{u, \epsilon}(n)$, while $\gamma_1([0, 1]) \subseteq A_L$ and $\gamma_2([0, 1]) \subseteq A_{L_u, \epsilon}$. Then $\gamma_2 \gamma_1^{-1}$ is homotopic to $X_n$ in $M$. Hence

$$Z([\gamma_2]) - Z([\gamma_1]) = Z([\gamma_2 \gamma_1^{-1}]) - Z([\gamma_1]) = M_{[X_n]}(Z)([\gamma_1]) .$$

The contours $\gamma_1$ and $\gamma_2$ are pictured in Figure 3.

![Figure 3: Contours for $M_{[X_n]}(Z)$.](image)

Now we have

$$Z([\gamma_2]) = \frac{1}{\Gamma(s)} \int_{L_u, \epsilon} e^{-ct} \frac{e^{-ct}}{1 - e^{2\pi i a e^{-t}}} t^{s-1} dt$$

$$= \frac{1}{\Gamma(s)} \int_{L_u, \epsilon} e^{-ct} \frac{1}{1 - e^{2\pi i a e^{-t}}} t^{s-1} dt + \frac{1}{\Gamma(s)} \int_{C_u, \epsilon} e^{-ct} \frac{e^{-ct}}{1 - e^{2\pi i a e^{-t}}} t^{s-1} dt ,$$

in which $C_{u, \epsilon}$ is a closed clockwise oriented semicircular contour centered at $u$ with radius $\epsilon$ together with the negatively oriented line segment $[u - \epsilon, u + \epsilon]$. The residue theorem gives

$$Z([\gamma_2]) = Z([\gamma_1]) - \frac{2\pi i}{\Gamma(s)} Res_{t=2\pi i(a-n)} \left( \frac{e^{-ct}}{1 - e^{2\pi i a e^{-t}}} t^{s-1} \right) ,$$

in which $a = (n - \frac{m}{2\pi}) + (x + iy)$ with $x > 0$ and $x^2 + y^2 < \epsilon^2$. We deduce that

$$M_{[X_n]}(Z)([\gamma_1]) = -\frac{(2\pi i)^s}{\Gamma(s)} (a - n)^{s-1} e^{-2\pi i c (a-n)} \text{ for } n \in \mathbb{Z}$$

holds near the endpoint $(s, a, c)$ of $\gamma_1$. Here

$$(2\pi i)^s := \exp(s \log 2\pi i)) = (2\pi)^s e^{\frac{a}{2}} ,$$

where $\log 2\pi i := \log 2\pi + i\pi$ is the value on the principal branch of $\log s$, and

$(a - n)^{s-1} := \exp((s-1) \log(a-n))$ with $\log(a-n)$ determined by the principal branch of the logarithm, observing that $\Re(a - n) > 0$ at the endpoint of $\gamma_1$, and $a - n$ never crosses
the negative imaginary axis when tracing the path $\gamma_1$ backwards. The function \emph{defined} by the right side of (4.27) on $A_L$ analytically continues to a holomorphic function on the universal cover $\widetilde{M}$, and its only branch points over $M$ are at $\mathbb{C} \times \{a = n\} \times (\mathbb{C} \setminus \mathbb{Z})$. In particular when this function is analytically continued inside $A_L$ back to the fundamental polycylinder along $[\gamma_1]^{-1}$, we obtain the functions given in (4.11).

Next, we use the series representation of the Lerch zeta function

$$\zeta(s, a, c) = \sum_{n=0}^{\infty} e^{2\pi i n a} (n + c)^{-s},$$

which defines the principal branch of $\zeta(s, a, c)$ as a holomorphic function on the simply-connected domain

$$\mathcal{U} = \{s : s \in \mathbb{C}\} \times \{a : \Im(a) > 0\} \times \{c : \Re(c) > 0\}.$$  

(4.29)

For integers $n \geq 0$, the function

$$F_n(s, a, c) := e^{2\pi i n a} (n + c)^{-s} = e^{2\pi i n a} e^{-s \ln(n + c)}$$

on $\mathcal{U}$ analytically continues to a multi-valued function on $M$ whose branch locus is $\mathbb{C} \times (\mathbb{C} \setminus \mathbb{Z}) \times \{c = -n\}$. Thus all the monodromy functions $M_{\gamma_l}^M(F_n) = 0 \ (l \in \mathbb{Z})$, while

$$M_{\gamma_l}^M(F_n) = \begin{cases} 
(e^{-2\pi i s} - 1)F_n & \text{if } l = -n, \\
0 & \text{if } l \neq -n,
\end{cases}$$

(4.31)

holds in the submanifold $\{s\} \times (\mathbb{C} \setminus \mathbb{Z}) \times (\mathbb{C} \setminus \mathbb{Z})$. By separating the terms $\sum_{n=0}^{N} e^{2\pi i n a} (n + c)^{-s}$, we may rewrite (4.28) as

$$\zeta(s, a, c) = \sum_{n=0}^{N} e^{2\pi i n a} (n + c)^{-s} + \sum_{n=N+1}^{\infty} e^{2\pi i n a} (n + c)^{-s}$$

$$= \sum_{n=0}^{N} F_n(s, a, c) + e^{2\pi i (N+1) a} \zeta(s, a, c + N + 1).$$

(4.32)

We have already shown that the last term $\zeta(s, a, c + N + 1)$ can be holomorphically continued to simply connected regions covering the domain

$$\mathcal{U}_1^\infty(N) := \{s : \Re(s) > 0\} \times \{a : a \in \mathbb{C} \setminus \mathbb{Z}\} \times \{c : \Re(c) > -N - 1, c \notin \mathbb{Z}_{\leq 0}\}.$$  

and the same holds for each of the $F_n$. Therefore the right side of (4.32) defines an analytic continuation for the left side $\zeta(s, a, c)$. Letting $N \to \infty$, we conclude that $\zeta(s, a, c)$ can be holomorphically continued to simply connected regions covering the domain

$$\mathcal{M}^+ := \{s : \Re(s) > 0\} \times \{a : a \in \mathbb{C} \setminus \mathbb{Z}\} \times \{c : c \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}\}.$$  

(4.33)

Finally we use the extended Lerch transformation formula given in Theorem 3.2 to extend $\zeta(s, a, c)$ to a holomorphic function on simply connected regions covering the domain

$$\mathcal{M}^- := \{s : \Re(s) < 1\} \times \{a : a \in \mathbb{C} \setminus \mathbb{Z}\} \times \{c : c \in \mathbb{C} \setminus \mathbb{Z}\}.$$  

The resulting continuation is holomorphic since $\Gamma(s)$ is holomorphic in the half-plane $\Re(s) > 0$. 
We next compute the monodromy functions of $Z$ along $[Y_n]$, e.g. $M_{[Y_n]}(Z)$. A loop in $\mathcal{M}$ based at $x_0$ around $c = 1, 2, 3, \ldots$ that remains in the positive $c$-plane and along which $a = s = \frac{1}{2}$ are held fixed is contractible in $\mathcal{M}^+$, hence we find that

$$M_{[Y_n]}(Z) \equiv 0 \quad \text{for} \quad n = 1, 2, 3, \ldots$$  \quad (4.34)

In view of (4.32), (4.31) and (4.34), for integer $n \leq 0$, the monodromy $M_{[Y_n]}(Z)$ arises from the monodromy of $F_{-n}$. In other words, on the extended fundamental polycylinder $\tilde{\Omega}$,

$$M_{[Y_n]}(Z)(s, a, c) = (e^{-2\pi is} - 1)e^{-2\pi is(a - n)^{-s}}$$ for $n = 0, -1, -2, \ldots$

as given in (4.15).

To complete the proof of Theorem 4.5 we verify the remaining parts of assertions (i)--(ii). We have already verified the formulas (4.11) and (4.15). Now Lemma 4.4 applied to $\tau_2 = \tau_1^{-1}$ gives

$$0 = M_{[d]}(Z) = M_{[\tau_1]}(Z) + M_{[\tau_1]}^{-1}(Z) + M_{[\tau_1]}^{-1}(M_{[\tau_1]}(Z)) \quad \text{.} \quad (4.35)$$

Choosing $[\tau_1] = [X_n]$, we evaluate the last term on the right side using (4.27), to obtain

$$M_{[X_n]}^{-1}(M_{[X_n]}(Z)) = M_{[X_n]}^{-1}(\frac{(2\pi i)^s}{\Gamma(s)}(a - n)^{-1}e^{-2\pi is(a - n)}) = (e^{-2\pi is} - 1)M_{[X_n]}(Z).$$

We deduce from (4.35) that for a path $\gamma$ with $\gamma(0) = x_0$ and other endpoint in $\mathcal{M}_s$ that

$$M_{[X_n]}^{-1}(Z)([\gamma]) = -e^{-2\pi is}M_{[X_n]}(Z)([\gamma]) \quad \text{.}$$

To verify the formulas for $M_{[X_n]}^{\pm k}(Z)$ we also use Lemma 4.4. We proceed by induction on $k$. Treating first $k \geq 1$, the base case $k = 1$ holds trivially and using the induction hypothesis we obtain

$$M_{[X_n]}^{k+1}(Z) = M_{[X_n]}^k(Z) + M_{[X_n]}(Z) + M_{[X_n]}(M_{[X_n]}^k(Z)).$$

$$= \frac{e^{2\pi is} - 1}{e^{2\pi is} - 1}M_{[X_n]}(Z) + M_{[X_n]}(Z) + (e^{2\pi is} - 1)\left(\frac{e^{2\pi is} - 1}{e^{2\pi is} - 1}M_{[X_n]}(Z)\right)$$

$$= \frac{e^{2\pi is(k+1)} - 1}{e^{2\pi is} - 1}M_{[X_n]}(Z),$$

completing the induction step. The proof for $k \leq -1$ is similar. Finally we deduce the formulas for $M_{[Y_n]}^{-1}(Z)$ and $M_{[Y_n]}^{\pm k}$ by analogous arguments. \hfill \Box

The next result determines all monodromy functions for $Z$ and shows that $Z$ is single-valued on the maximal abelian cover $\tilde{\mathcal{M}}^{ab}$ of $\mathcal{M}$.

**Theorem 4.6.** Suppose that $[\tau] \in \pi_1(\mathcal{M}, x_0)$ satisfies

$$[\tau] = [S_1]^{\epsilon_1}[S_2]^{\epsilon_2} \cdots [S_m]^{\epsilon_m},$$

with each $[S_i]$ in the set of generators $\mathcal{G}$ of $\pi_1(\mathcal{M}, x_0)$, in which each $\epsilon_i$ equals one of $\pm 1$. Then the monodromy function $M_{[\tau]}(Z)$ for the Lerch zeta function on the universal cover $\tilde{\mathcal{M}}$ satisfies

$$M_{[\tau]}(Z) = \sum_{[S] \in \mathcal{G}} M_{[S]^{k}(S)}(Z) \quad \text{,} \quad (4.37)$$
with
\[ k(S) := \sum_{\{i : S_i = S\}} \epsilon_i , \quad \text{for each} \quad S \in \mathcal{G} . \quad (4.38) \]

In particular, \( M_\tau(Z) \) vanishes identically for \( \tau \) in the commutator subgroup
\[ \Gamma := (\pi_1(M, x_0))' = [\pi_1(M, x_0) : \pi_1(M, x_0)] \quad (4.39) \]
of \( \pi_1(M, x_0) \). Thus \( Z \) is a single-valued function on the quotient manifold \( \tilde{M}^{ab} := \Gamma \backslash \tilde{M} \) with \( \Gamma \) acting as a group of homeomorphisms of \( \tilde{M} \).

**Remark.** The manifold \( \tilde{M}^{ab} \) is the maximal unramified abelian covering manifold of \( M \) with projection map \( \pi_{ab} : \tilde{M}^{ab} \to M \) induced from \( \pi : \tilde{M} \to M \). We let \( \pi_D : \tilde{M} \to \tilde{M}^{ab} \) denote the projection from the universal cover to \( \tilde{M}^{ab} \).

**Proof.** Recall that \( \mathcal{G} = \{[X_n] : n \in \mathbb{Z}\} \cup \{[Y_n] : n \in \mathbb{Z}\} \). We first note that for any two distinct generators \([S_1], [S_2] \in \mathcal{G}\) the Lerch monodromy functions are independent in the sense that
\[ M_{[S_1]^{k_1}}(M_{[S_2]^{k_2}}(Z)) = 0 . \quad (4.40) \]
This follows by a direct computation from Theorem 4.5 because \( M_{[S_2]^{k_2}}(Z) \) has non-trivial monodromy only around the generator \([S_2]\) in \( \mathcal{G} \) and no other generator. Now Lemma 4.4 gives for \( [S_1] \neq [S_2] \) that
\[ M_{[S_1]^{k_1}}(M_{[S_2]^{k_2}}(Z)) = M_{[S_1]^{k_1}}(Z) + M_{[S_2]^{k_2}}(Z) . \quad (4.41) \]
We prove Theorem 4.6 by induction on \( m \). The base case \( m = 1 \) holds because both sides of (4.37) are then identical; (4.41) shows that the case \( m = 2 \) also holds. For the induction step, assume it holds for \( m - 1 \geq 1 \). By Lemma 4.4 we have
\[ M_\tau(Z) = M_{[S_1]^{s_1} \ldots [S_m]^{s_m}}(Z) = M_{[S_1]^{s_1} \ldots [S_{m-1}]^{s_{m-1}}}(Z) + M_{[S_m]^{s_m}}(Z) + M_{[S_m]^{s_m}}(M_{[S_1]^{s_1} \ldots [S_{m-1}]^{s_{m-1}}}(Z)). \]
The induction hypothesis gives
\[ M_{[S_1]^{s_2} \ldots [S_{m-1}]^{s_{m-1}}}(Z) = \sum_{S \in \mathcal{G}} M_{[S]^{k'(S)}}(Z) \quad (4.42) \]
in which \( k'(S) = k(S) \) if \( S \neq S_m \) and \( k'(S) = k(S) - \epsilon_m \) if \( S = S_m \). Applying this formula gives
\[ M_{[S_m]^{s_m}}(M_{[S_1]^{s_1} \ldots [S_{m-1}]^{s_{m-1}}}(Z)) = \sum_{S \in \mathcal{G}} M_{[S_m]^{s_m}}(M_{[S]^{k'(S)}}(Z)) \]
\[ = M_{[S_m]^{s_m}}(M_{[S_m]^{k'(S_m)}}(Z)) \]
\[ = M_{[S_m]^{s_m+k'(S_m)}}(Z) - M_{[S_m]^{s_m}}(Z) - M_{[S_m]^{k'(S_m)}}(Z), \]
where we used (4.40) to remove all terms except $S = S_m$ and applied Lemma 4.4 to obtain the last expression. Thus

$$M_{[\tau]}(Z) = \left( \sum_{S \in G} M_{[S]}(Z) \right) + M_{[S_m]}(Z) + M_{[S_m]^{k'}(S_m)}(Z) - M_{[S_m]^{k}(S_m)}(Z)$$

which completes the induction step, and (4.37) follows.

The commutator subgroup of a free group on a set of generators $G$ is well known to be the set of words $[\tau] = [S_1]^{k_1} \cdots [S_m]^{k_m}$ in the generators for which all exponents $k(S)$ satisfy

$$k(S) := \sum_{\{i: S_i = S\}} k_i = 0 \text{ for all } S \in G.$$

Now (4.37) gives

$$M_{[\tau]}(Z) \equiv 0 \text{ for all } [\tau] \in \pi_1(M, x_0)^{\prime}.$$

The conclusion about $\tilde{M}_{ab}$ is immediate.

Proof of Theorem 2.2. This is an immediate consequence of Theorem 4.5 and Theorem 4.6.

Remark. We have derived formulas for $M$ as the base manifold, rather than the extended manifold $M^\#$, because the largest manifold on which the analytically continued functions $L^\pm(s, a, c)$ appearing in the functional equation remain well-defined (on a suitable covering manifold). In addition the manifold $M$ possesses a global automorphism of order 4 corresponding to the functional equation: $\phi_R : M \to M$, given by $\phi_R(s, a, c) = (1 - s, 1 - c, a)$, which does not extend to $M^\#$.

5. Differential-difference operators and Lerch monodromy functions

In the next three sections we derive properties of the Lerch zeta function and its monodromy functions $M_{[\tau]}(Z)$ as a function of the variables $(s, a, c) \in M$. Here we show that they satisfy two differential-difference equations, and a linear partial differential equation in the operators $\frac{\partial}{\partial a}$ and $\frac{\partial}{\partial c}$, on the universal cover $\tilde{M}$.

The differential-difference equations involve the partial differential operators $\frac{\partial}{\partial a}$ and $\frac{\partial}{\partial c}$ viewed on $M$, and extending to the universal cover $\tilde{M}$. We define the raising operator

$$D^+_L := \frac{\partial}{\partial c},$$

and the lowering operator

$$D^-_L := \frac{1}{2\pi i} \frac{\partial}{\partial a} + c.$$
We define the *Lerch differential operator* as

\[ D_L := D_L^+ D_L^- = \left( \frac{1}{2\pi i} \frac{\partial}{\partial a} + c \right) \frac{\partial}{\partial c} \cdot \]  

(5.3)

The differential-difference equations satisfied by the Lerch zeta function in the region where the series expansion (1.1) converges uniformly are

\[ D_L^+ \zeta(s, a, c) = -s \zeta(s + 1, a, c), \]  

(5.4)

and

\[ D_L^- \zeta(s, a, c) = \zeta(s - 1, a, c). \]  

(5.5)

Combining (5.2) and (5.4) shows that the Lerch zeta function satisfies the linear differential equation

\[ D_L \zeta(s, a, c) = -s \zeta(s, a, c), \]  

a fact previously observed by Okubo [22, p. 1057].

Theorem 5.1 below gives the analytic continuation of these equations. It uses the fact that the operators \( \frac{\partial}{\partial a} \) and \( \frac{\partial}{\partial c} \) lift to partial differential operators on \( \widehat{\mathcal{M}} \) which are *equivariant* with respect to the group \( G_M \) of diffeomorphisms of \( \widehat{\mathcal{M}} \) that commute with the covering map \( \pi : \widehat{\mathcal{M}} \to \mathcal{M} \). Here \( G_M \simeq \pi_1(\mathcal{M}, x_0) \simeq \{ Q[\tau] : [\tau] \in \pi_1(\mathcal{M}; x_0) \} \).

**Theorem 5.1. (Lerch Differential-Difference Operators)**

1. The analytic continuation \( Z(s, a, c, [\gamma]) \) of the Lerch zeta function on the universal cover \( \widehat{\mathcal{M}} \) satisfies the two differential-difference equations

\[ \left( \frac{1}{2\pi i} \frac{\partial}{\partial a} + c \right) Z(s, a, c, [\gamma]) = Z(s - 1, a, c, [\gamma_-]) \]  

(5.6)

and

\[ \frac{\partial}{\partial c} Z(s, a, c, [\gamma]) = -s Z(s + 1, a, c, [\gamma_+]), \]  

(5.7)

in which \([\gamma_+]\) (resp. \([\gamma_-]\)) denote paths which first traverse \( \gamma \) and then traverse a path from the endpoint of \( \gamma \) that changes the \( s \)-variable only, moving from \( s \) to \( s + 1 \) (resp. \( s - 1 \)).

2. The analytic continuation \( Z(s, a, c, [\gamma]) \) on \( \widehat{\mathcal{M}} \) satisfies the linear partial differential equation

\[ D_L Z(s, a, c, [\gamma]) = -s Z(s, a, c, [\gamma]), \]  

(5.8)

where \( D_L := \left( \frac{1}{2\pi i} \frac{\partial}{\partial a} \frac{\partial}{\partial c} + c \frac{\partial}{\partial c} \right) \).

3. Each Lerch monodromy function \( M[\tau](Z)(s, a, c, [\gamma]) \) for each \([\tau] \in \pi_1(\mathcal{M}, x_0)\) satisfies the two differential-difference equations and the differential equation above on \( \widehat{\mathcal{M}} \).

**Proof.** (1) The Lerch zeta function \( \zeta(s, a, c) \) is defined by the Dirichlet series (1.1) in the simply-connected region

\[ U = \{ s : s \in \mathbb{C} \} \times \{ a : \Re(a) > 0 \} \times \{ c : \Re(c) > 0 \}. \]  

(5.9)

In this region, \( \zeta(s, a, c) \) satisfies the differential-difference equation

\[ \left( \frac{1}{2\pi i} \frac{\partial}{\partial a} + c \right) \zeta(s, a, c) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (2\pi in) e^{2\pi ina} (n + c)^{-s} + \sum_{n=0}^{\infty} ce^{2\pi ina} (n + c)^{-s} \]  

\[ = \zeta(s - 1, a, c). \]  

(5.10)
It also satisfies the differential-difference equation

\[
\frac{\partial}{\partial c} \zeta(s, a, c) = -s \sum_{n=0}^{\infty} e^{2\pi ina} (n + c)^{-s-1}
\]

\[
= -s\zeta(s + 1, a, c) .
\]

(5.11)

These equations carry over to \(\tilde{M}\) by analytic continuation using paths based at \(x_0\).

Note that from any point \(x = (s, a, c, [\gamma]) \in \tilde{M}\), there is a unique single-valued analytic continuation possible in the \(s\)-variable, in any surface holding the \((a, c)\)-variables fixed.

The difference operators \(s \to s \pm 1\) are interpreted to represent motions made in the \(s\)-variable on such a surface. The paths \([\gamma_-]\) represent the path \([\gamma]\) extended by a path in the \(s\)-variable made in this surface, starting from the endpoint of \(\gamma\), moving from \(s\) to \(s \pm 1\).

(2) Combining the differential-difference equations (5.10) and (5.11) on the simply connected domain \(U\) gives

\[
\left( \frac{1}{2\pi i} \frac{\partial}{\partial a} + c \right) \frac{\partial}{\partial c} \zeta(s, a, c) = -s\zeta(s, a, c)
\]

valid on \(U\). This shows that (5.8) holds for \(Z\) in the fundamental polycylinder intersected with \(\Im(a) > 0\). It now holds by analytic continuation for all \(Z([\gamma])\) for all paths \(\gamma\) based at \(x_0\).

(3) By definition, we have

\[
M_{[\tau]}(Z)([\gamma]) := Z(s, a, c, [\tau\gamma]) - Z(s, a, c, [\gamma])
\]

Now we may apply the two differential-difference operators (and their composition) \(D^\pm_L\) to both sides of this equation. By (1), (2) both terms on the right side satisfy these equations, hence \(M_{[\tau]}([\gamma]) = M_{[\tau]}(Z)(s, a, c, [\gamma])\) satisfies (5.6), (5.7), (5.8) in place of \(Z(s, a, c, [\gamma])\).

Remarks. (1) The relation \(D_L f = -sf\) can be verified directly for the monodromy formulae in Theorem 4.5 and provides a consistency check on their correctness.

(2) The partial differential equation (5.8) is a local condition in a neighborhood of a point \((s, a, c, [\gamma]) \in \tilde{M}\) and is preserved under analytic continuation. In contrast, the functional equation for the Lerch zeta function involves difference operators, which are not local. Under analytic continuation equality is preserved in the functional equation only if the proper branches of the function are chosen, cf. Theorem 6.1 below. For example the relation \(\zeta(s, a, c) = \zeta(s, a + 1, c)\) holds on the domain (5.9), but under analytic continuation the right and left sides of this equality may be on different sheets of the universal cover \(\tilde{M}\).

(3) All four terms in the functional equation in Theorem 2.1 separately satisfy the partial differential equation of Theorem 5.1. That is, (5.8) holds for \(f(s, a, c)\) equal to any of \(\zeta(s, a, c), e^{-2\pi ia}\zeta(s, 1-a, 1-c), e^{-2\pi iac}\zeta(1-s, 1-c, a)\) and \(e^{-2\pi iac+2\pi ic}\zeta(1-s, c, 1-a)\).
6. Functional equations and monodromy functions

There are linear relations between monodromy functions implied by the functional equations. The functional equation in Theorem 6.1 analytically continues to the universal cover \( \tilde{M} \) of \( M \), provided each of the four terms continues along a separate path from the base point \( x_0 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \). Given a path \( \gamma(t) = \{(s(t), a(t), c(t)) : 0 \leq t \leq 1\} \) in \( M \) starting at \( x_0 \), we define the path \( \theta(\gamma) \) by

\[
\theta(\gamma)(t) := (1 - s(t), 1 - c(t), a(t)), \quad 0 \leq t \leq 1.
\]

In particular \( \theta \) acts on loops in \( M \) based at \( x_0 \). This map on loops is well-defined on homotopy classes and induces a homomorphism

\[
\theta : \pi_1(M, x_0) \to \pi_1(M, x_0),
\]

which is an automorphism of order 4.

**Theorem 6.1.** (1) The map \( \theta : \pi_1(M, x_0) \to \pi_1(M, x_0) \) is an automorphism of order 4 satisfying

\[
\theta([X_n]) = [Y_n] \quad \text{for} \quad n \in \mathbb{Z},
\]

\[
\theta([Y_n]) = [X_{n-1}] \quad \text{for} \quad n \in \mathbb{Z}.
\]

(2) For each \( [\tau] \in \pi_1(M, x_0) \) the Lerch zeta monodromy functions \( M_{[\tau]}(Z) \) satisfy the linear relations

\[
\pi^{-\frac{s}{2}} \Gamma \left(\frac{s}{2}\right) \left\{ \frac{e^{-2\pi i a} M_{[\tau]}(Z)}{\Gamma(s)} \right\} + e^{-2\pi i a} M_{[\tau]}(Z) \right\}
\]

\[
e^{2\pi i c} M_{[\tau]}(Z) \right\}
\]

and

\[
\frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \left\{ \frac{e^{-2\pi i a} M_{[\tau]}(Z)}{\Gamma(s)} \right\} + e^{-2\pi i a} M_{[\tau]}(Z) \right\}
\]

\[
e^{2\pi i c} M_{[\tau]}(Z) \right\}
\]

in which \( \gamma \) is any path with \( \gamma(0) = x_0 \).

**Proof.** The action (6.3) of the automorphism \( \theta \) on generators of \( \pi_1(M, x_0) \) is easily computed from the definition. The relations (6.4) and (6.5) now follow from analytically continuing the functional equations (2.8) and (2.10) for \( L^\pm(s, a, c) \) along a path (6.1). □

**Remark.** Theorem 6.1 provides another consistency check on the monodromy functions. As an example, taking \( [\tau] = [X_0] \) and using Theorem 4.5 to compute monodromy functions one finds that (6.6) asserts that

\[
e^{-2\pi i a} \pi^{-\frac{1-s}{2}} \Gamma \left(\frac{1-s}{2}\right) \left\{ e^{2\pi i s} - 1 \right\} a^{-1} (1-a) + 0.
\]

This can be verified directly using suitable gamma function identities.
7. Lerch monodromy vector spaces

In this section we determine properties of the restricted monodromy functions obtained when the parameter $s$ is viewed as fixed. The values $s \in \mathbb{Z}$ are “special values,” where the vector space spanned by the monodromy functions simplifies.

For fixed $s \in \mathbb{C}$ and $[\tau] \in \pi_1(\mathcal{M}, \mathbf{x}_0)$, denote by $Q^s_{[\tau]}(Z)$ (resp. $M^s_{[\tau]}(Z)$) the restriction of the functions $Q_\tau(Z)$ (resp. $M_{[\tau]}(Z)$) to the manifold

$$\tilde{M}_s := \pi^{-1}(\{s\} \times (\mathbb{C} \setminus \mathbb{Z}) \times (\mathbb{C} \setminus \mathbb{Z}))$$

which is a path-connected analytic submanifold of the universal cover $\tilde{M}$ of $\mathcal{M}$.

**Definition 7.1.** The Lerch monodromy space $\mathcal{V}_s$ at $s \in \mathbb{C}$ is the vector space generated by all germs of function elements of $Z$, which is generated by $Z^s$ and by all $\{Q^s_{[\tau]}(Z) : [\tau] \in \pi_1(\mathcal{M}, \mathbf{x}_0)\}$, where $Q^s_{[\tau]}(Z)$ is the function $Q_{[\tau]}(Z)$ restricted to $s \in \mathbb{C}$ as above. This vector space consists of all finite $\mathbb{C}$-linear combinations of the specified generators $Q^s_{[\tau]}(Z)$.

It is immediate that this vector space is spanned by the function $Z^s = Q^s_{[id]}(Z)$ together with all the functions $M^s_{[\tau]}(Z)$ regarded as functions on $\tilde{M}_s$, i.e.

$$\mathcal{V}_s := \mathbb{C}[Z] + \left( \sum_{[\tau] \in \pi_1(\mathcal{M}, \mathbf{x}_0)} \mathbb{C}[M^s_{[\tau]}(Z)] \right),$$

using the fact that $M^s_{[\tau]}(Z) = Q^s_{[\tau]}(Z) - Z^s$.

**Theorem 7.2.** (Lerch Monodromy Spaces) The Lerch monodromy space $\mathcal{V}_s$ depends on the parameter $s \in \mathbb{C}$ as follows.

(i) If $s \in \mathbb{Z}$ with $s = -m \leq 0$, then all Lerch monodromy functions vanish identically, i.e.

$$M^{-m}_{[\tau]}(Z) = 0 \quad \text{for all} \quad [\tau] \in \pi_1(\mathcal{M}, \mathbf{x}_0).$$

Thus $\mathcal{V}_{-m} = \mathbb{C}[Z^{-m}]$ is one-dimensional.

(ii) If $s \in \mathbb{Z}$ with $s = m \geq 1$, then $\mathcal{V}_m$ is an infinite-dimensional vector space, and has a basis consisting of the function $Z^m$ together with $\{M^m_{[X_n]}(Z) : n \in \mathbb{Z}\}$.

(iii) If $s \notin \mathbb{Z}$, then $\mathcal{V}_s$ is an infinite-dimensional vector space, and has as a basis the function $Z^s$ together with the functions $\{M^s_{[X_n]}(Z) : n \in \mathbb{Z}\} \cup \{M^s_{[Y_n]}(Z) : n \in \mathbb{Z}_{\leq 0}\}$.

**Proof.** (i) Let $s = -m \in \mathbb{Z}_{\leq 0}$. Theorem 4.5 shows that the monodromy $M_{[\tau]}^s(Z)$ vanishes identically when $[\tau] = [S]^k$ for any generator $S \in \mathcal{G}$. Indeed, this holds for $[\tau] = [X_n]^k$ because $\overline{\tau}_{(s)} = 0$ when $s \in \mathbb{Z}_{\leq 0}$ and it holds for $[\tau] = [Y_n]^k$ because $e^{-2\pi i s} - 1 = 0$ for $s \in \mathbb{Z}$. Now Theorem 4.6 shows that all monodromy functions $M^{-m}_{[\tau]}(Z)$ vanish identically. In this case (7.2) shows that $\mathcal{V}_{-m} = \mathbb{C}[Z^m]$ is one-dimensional.

(ii) For $s = m \in \mathbb{Z}_{> 0}$ the monodromy functions $M^s_{[X_n]}(Z)$ vanish identically as in part (i). However Theorem 4.5 shows that for $n \in \mathbb{Z}$ the monodromy function

$$M^s_{[X_n]}(Z) = c_n(a - n)^{s-1} e^{-2\pi i (a-n)c},$$

vanishes identically as in part (i).
in which \( c_n \) is a nonzero constant (which itself depends on \( s \)). Furthermore all \( M_{[X_n]}(Z) \) have exactly the same form, so are linearly dependent on \( M_{[X_n]}(Z) \). Theorem 3.6 now shows that \( \{M_{[X_n]}(Z) : n \in \mathbb{Z} \} \) together with \( Z^m \) form a spanning set for the vector space \( \mathcal{V}_m \). It is a basis because any finite subset is linearly independent, by considering them as functions of \( (a, c) \) in a small disk around \((s, \frac{1}{2}, \frac{1}{2})\) with \( s \) held constant. Thus \( \mathcal{V}_m \) is infinite-dimensional.

(iii) By a similar argument to (ii), the functions in

\[
\mathcal{B} := \{M_{[X_n]}(Z) : n \in \mathbb{Z}\} \cup \{M_{[Y_n]}(Z) : n \in \mathbb{Z}_{\leq 0}\} \cup \{Z^s\}
\]

comprise a spanning set of the vector space \( \mathcal{V}_s \). The functions \( M_{[Y_n]}(Z) \) have the form \( c_ne^{-2\pi in(a - n)^s} \) for nonzero \( c_n \) (which depends on \( s \)). The \( \mathbb{C} \)-linear independence of any finite subset of the functions in \( \mathcal{B} \) can be established by considering them in a small disk in the \((a, c)\)-plane around \((s, \frac{1}{2}, \frac{1}{2})\), holding \( s \) constant. Thus \( \mathcal{V}_s \) is infinite-dimensional.

We can associate to the vector space \( \mathcal{V}_s \) an induced monodromy representation

\[
\rho_s : \pi_1(\mathcal{M}, x_0) \to \text{End}(\mathcal{V}_s),
\]

defined for \( [\sigma] \in \pi_1(\mathcal{M}, x_0) \) by

\[
\rho_s([\sigma])(Q_{[\tau]}^s(Z)) := Q_{[\sigma^{-1}\tau]}^s(Z).
\]

Now one has

\[
\rho_s([\sigma_1])\rho_s([\sigma_2]) = \rho_s([\sigma_1\sigma_2]),
\]

and we note that

\[
\rho_s([\sigma])(M_{[\tau]}^s(Z)) = \rho_s([\sigma]) \left( Q_{[\tau]}^s(Z) - Q_{[\sigma^{-1}\tau]}^s(Z) \right)
= Q_{[\sigma^{-1}\tau]}^s(Z) - Q_{[\sigma^{-1}]^s(Z)}
= M_{[\sigma^{-1}\tau]}^s(Z) - M_{[\sigma^{-1}]^s(Z)}.
\]

This representation has a kernel that depends on \( s \), since there are relations among \( M_{[S]}^s(Z) \) for a fixed generator \([S]\) as \( d \in \mathbb{Z} \) varies. For \( s \not\in \mathbb{Z} \) the quotient group

\[
G_s := \pi(\mathcal{M}, x_0) / \ker(\rho_s)
\]

can be identified with the free abelian group \( \mathbb{Z}[\tilde{\mathcal{G}}] \) on the generating set \( \tilde{\mathcal{G}} = \{[X_n] : n \in \mathbb{Z}\} \cup \{[Y_n] : n \leq 0\} \), and the quotient representation \( \tilde{\rho}_s : \mathbb{Z}[\tilde{\mathcal{G}}] \to \text{End}(\mathcal{V}_s) \) studied as \( s \) varies.

Remarks. (1) The multi-valued nature of the Lerch zeta function encodes information about the interaction of the additive and multiplicative structures of \( \mathbb{Z} \). The vanishing of all monodromy functions when \( s = -m \) is a non-positive integer is another aspect of the viewpoint that these points are “special values” that contain “universal” information.

(2) In part III [12, Sect. 5] we will show that for such \( s = -m \), taking “special values” \( L^\pm(-m, a, c) \) for \( n \in \mathbb{Z}_{>0} \) at rational values of \( a \) and \( c \) with \( 0 < a, c \leq 1 \) yields data sufficient to construct by \( p \)-adic interpolation all \( p \)-adic \( L \)-functions associated to \( \mathbb{Q} \).
8. Extended analytic continuation

In this section we complete the analytic continuation of the Lerch zeta function \( \zeta(s, a, c) \) to a multivalued function \( Z(s, a, c, [\gamma]) \) defined over the manifold

\[
\mathcal{M}^\# := \{(s, a, c) \in \mathbb{C} \times (\mathbb{C} \setminus \mathbb{Z}) \times (\mathbb{C} \setminus \mathbb{Z}_{\leq 0})\},
\]

obtained from \( \mathcal{M} \) by filling in all points \( \{c : c = n \geq 1\} \). The vanishing of the Lerch monodromy functions

\[
M_{[\gamma_n]}(Z) = 0 \quad \text{for} \quad n = 1, 2, 3, \ldots
\]

(as given in Theorem 4.5) implies that the function \( Z(s, a, c, [\gamma]) \) is single-valued in a (“punctured”) open neighborhood of any point \( (s, a, c) \in \mathcal{M} \), with \( c = n \) for integer \( n \geq 1 \). However a proof is required that the singularity at \( c = n \) can be removed.

**Proof of Theorem 2.3.** The proof parallels the analytic continuation to \( \mathcal{M} \), so we omit many details. We must show that for each \( (s, a) \in \mathbb{C} \times (\mathbb{C} \setminus \mathbb{Z}) \) and for each \([\gamma]\) with endpoint \( x_1 \) lying over \((s, a, n) \in \mathcal{M}^\# \), there is a local open neighborhood in three complex variables where the function \( Z(s, a, n, [\gamma]) \) is holomorphic. This reduces to showing holomorphicity in three variables for \( \zeta(s, a, c) \) on the principal sheet, plus holomorphicity of each possible monodromy function \( M_{[\tau]}(s, a, c) \) on the principal sheet. The form of the monodromy functions in Theorem 4.5 and Theorem 4.6 indicates that all of them are holomorphic in an open three-dimensional neighborhood of any point with \( c = n \), so it remains to check the case of \( \zeta(s, a, c) \) itself.

First, the earlier proof showed that Dirichlet series expansion (1.1) gives an analytic continuation of \( \zeta(s, a, c) \) to the region

\[
U = \{s : s \in \mathbb{C}\} \times \{a : \Im(a) > 0\} \times \{c : \Re(c) > 0\}.
\]

Second, the integral representation

\[
\zeta(s, a, c) := \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}e^{-ct}}{1 - e^{2\pi ia}e^{-t}} dt.
\]

defines a single-valued analytic extension to the region

\[
W_1 = \{s : \Re(s) > 0\} \times \{a : 0 < \Re(a) < 1\} \times \{c : \Re(c) > 0\}.
\]

Third, the periodicity of the function \( \zeta(s, a, c) \) under \( a \mapsto a + 1 \) in the region \( U \), when compared to \( W_1 \), means that we can use this periodicity to extend the continuation in \( W_1 \) to the lower half-plane in strips of width one, to the region

\[
W_2 = \{s : \Re(s) > 0\} \times \{a : a \in A_L\} \times \{c : \Re(c) > 0\}
\]

in which

\[
A_L := \mathbb{C} \setminus \{M_k : k \in \mathbb{Z}\}
\]

is pictured in Figure 1. Fourth, using the monodromy functions \( M_{[X_n]}(Z) \) to circle around all integer points in the \( a \)-plane, we can now analytically continue this function to a multi-valued function over the manifold

\[
W_3 = \{s : \Re(s) > 0\} \times \{a : a \in \mathbb{C} \setminus \mathbb{Z}\} \times \{c : \Re(c) > 0\},
\]

which will be single-valued on the maximal abelian cover of this manifold.
Fifth, we use the differential-difference equation

$$
\zeta(s - 1, a, c) = \left( \frac{1}{2\pi i} \frac{\partial}{\partial a} + c \right) (\zeta(s, a, c)) \tag{8.6}
$$

to analytically continue from $s$ to $s - 1$ in the $s$-variable, from the region $\mathcal{W}_3$. Here we use Theorem 5.1 and the fact that the differential-difference equation continues to hold at points where $c = n \geq 1$, in the region $\mathcal{W}_3$, by analytic continuation. We now have obtained analytic continuation as a multi-valued function over the manifold

$$
\mathcal{W}_4 = \{ s : s \in \mathbb{C} \} \times \{ a : a \in \mathbb{C} \setminus \mathbb{Z} \} \times \{ c : \Re(c) > 0 \},
$$

which is single-valued over the maximal abelian cover. (Note that we cannot use the functional equation to make the continuation in $s$, at points with $c = n \geq 1$.)

Sixth, we extend the analytic continuation to the rest of the $c$-plane, omitting non-positive integers, using Theorem 2.2 and the fact that this function agrees with $Z(s, a, c, [\gamma])$ in the resulting region. Since all resulting monodromy functions are holomorphic at points where $c = n \geq 1$, we obtain a multi-valued function defined over the manifold

$$
\mathcal{M}^\# = \{ (s, a, c) \in \mathbb{C} \times (\mathbb{C} \setminus \mathbb{Z}) \times (\mathbb{C} \setminus \mathbb{Z}_{\leq 0}) \}.
$$

Since no monodromy is added at points $c = n$, the fact that this function is single-valued on the maximal abelian cover $\tilde{\mathcal{M}}^\#$ of $\mathcal{M}^\#$ follows from Theorem 2.2.

Remarks. (1) Applying the transformation $\phi_R(s, a, c) = (1 - s, 1 - c, a)$ defined on $\mathbb{C}^3$, we deduce that the function $R(\zeta)(s, a, c) := e^{2\pi i ac} \zeta(s, 1 - c, a)$ implicitly appearing in the extended Lerch transformation formula (3.5) has a multi-valued analytic continuation defined over the extended manifold

$$
\phi_R(\mathcal{M}^\#) = \mathbb{C} \times (\mathbb{C} \setminus \mathbb{Z}_{\geq 1}) \times (\mathbb{C} \setminus \mathbb{Z}). \tag{8.7}
$$

Similar results hold for $R_k^\#(\zeta)$ for $k = 2, 3$, giving analytic continuations above the manifolds $\phi_R(\mathcal{M}^\#)$.

(2) The proof above used the differential-difference operator (5.2) to analytically continue from $\Re(s) > 0$ to $s \in \mathbb{C}$. An alternative approach to analytically continue in the $s$-variable is to use a different contour integral representation of the Lerch zeta function, as done by Barnes [3]. That integral uses the same right hand side in (8.3), with the integral taken over a keyhole contour on the complex plane cut along the positive real axis, that initially comes in from $+\infty$ along the (lower) positive real axis, circles clockwise around $s = 0$ on a small circle and goes back to $+\infty$ along the (upper) real axis. This integral equals $(e^{2\pi is} - 1)\zeta(s, a, c)$.

9. Concluding remarks

The analytic continuation of the Lerch zeta function given in this paper is to an essentially maximal domain of holomorphy in three complex variables. It is also possible to define analytic continuations of the Lerch zeta function in fewer variables which make sense on various “singular strata”, i.e. regions $(s, a, c)$ in which either $a$ or $c$, or both, take integer values, corresponding to the punctures in the manifold $\mathcal{M}^\#$. These functions
can sometimes be defined as continuous limits from “non-singular” ranges of $(s, a, c)$, as discussed in part I, and are then analytic in fewer variables on such ranges. For example, the Hurwitz zeta function corresponds to fixing $a = 0$ and taking $0 < c < 1$, and we view it as belonging to a “singular stratum” of real codimension two (i.e. this stratum is a 2-dimensional complex manifold, where we vary the other two variables.) As noted in part I, it can be obtained as a limiting value of the Lerch zeta function on $M$, provided that $\Re(s) > 1$. It is well-known that the Hurwitz zeta function can be meromorphically continued in the $s$-variable, and has a simple pole at $s = 1$. Our viewpoint here is that the polar singularity is a signal that this function belongs to a “singular stratum.”

One can also show by the methods of this paper that it possesses a multi-valued analytic continuation in the $(s, c)$-variables, having additional singularities at non-positive integer values of $c$. The Riemann zeta function is associated to various integer points in both the $a$-variable and $c$-variable. For the point $a = 1, c = 1$ it lies on a singular stratum of real codimension 2, and arises as a continuous limiting value of the Lerch zeta function on $M$, when $\Re(s) > 1$. According to part I, it is also (in some sense) associated to the point $a = 0, c = 0$, which lies on a singular stratum of real codimension 4, but there the Lerch zeta function has no continuous limiting value for any $s \in \mathbb{C}$. We note that specialization of the Lerch functional equation in Theorem 2.1 at $(a, c) = (1, 1)$ necessarily requires that data at $(a, c) = (0, 0)$ be included. It seems an interesting topic to further identify and classify all the “singular strata” and to understand how the functions (or collection of functions) on each singular stratum arise by “degeneration” from the multivalued function above.

Finally we have observed that the strata corresponding to positive integer $c = n \geq 1$ are in fact “nonsingular,” as indicated by Theorem 2.3. The truly “singular strata” can be recognized by the appearance of discontinuities, already illustrated in part I [11, Theorem 2.3].

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