On the Interpretation of the Balance Function

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Abstract. We construct a simple toy model and explicitly demonstrate that the Balance Function (BF) can become negative for some values of the rapidity separation and hence can not have any probabilistic interpretation. In particular, the BF can not be interpreted as the probability density for the balancing charges to occur separated by the given rapidity interval.

1 Introduction

In experiments, the netcharge fluctuations usually are studied \cite{1, 2} by calculating the quantity $\nu_{\text{dyn}}$, defined as:

$$
\nu_{\text{dyn}}(\delta \eta) \equiv \frac{\langle n_+(n_+ - 1) \rangle}{\langle n_+ \rangle^2} + \frac{\langle n_-(n_- - 1) \rangle}{\langle n_- \rangle^2} - 2 \frac{\langle n_+ n_- \rangle}{\langle n_+ \rangle \langle n_- \rangle},
$$

where $n^+$ and $n^-$ is a number of positive and negative particles observed in the pseudorapidity interval $\delta \eta$. In some cases there is more convenient to modify the normalization of this quantity introducing \cite{3}:

$$
\nu_s(\delta \eta) \equiv - \frac{\langle n_+ \rangle + \langle n_- \rangle}{4} \nu_{\text{dyn}}(\delta \eta).
$$

This variable is closely connected with the so-called Balance Function (BF) \cite{4}, usually defined as

$$
B(\eta_1, \eta_2) = \frac{1}{2} \left[ \frac{\rho_+(\eta_1, \eta_2)}{\rho_+(\eta_1)} + \frac{\rho_-(\eta_1, \eta_2)}{\rho_-(\eta_1)} - \frac{\rho_{++}(\eta_1, \eta_2)}{\rho_+(\eta_1)} - \frac{\rho_{--}(\eta_1, \eta_2)}{\rho_-(\eta_1)} \right],
$$

where $\rho_+(\eta)$, $\rho_-(\eta_1, \eta_2)$ and so on are the inclusive and double inclusive pseudorapidity distributions of corresponding charged particles (on the correspondence with other possible alternative definitions of the BF see e.g. \cite{5}).

In the most simple way the connection between the $\nu_s(\delta \eta)$ and the $B(\eta_1, \eta_2)$ can be established in mid-rapidity region at LHC energies, where the translation invariance in rapidity is valid. In this case the single inclusive distributions are constant: $\rho_+(\eta) = \langle n_+ \rangle / \delta \eta$, $\rho_-(\eta) = \langle n_- \rangle / \delta \eta$ and the double inclusive distributions depend only on the differences of their arguments: $\rho_{+-}(\eta_1, \eta_2) = \rho_{+-}(\eta_1 - \eta_2)$ and so on. Hence the BF also will depend only on the $\eta_1 - \eta_2 \equiv \Delta \eta$.

The charge symmetry is also well satisfied in this case:

$$
\langle n_+ \rangle = \langle n_- \rangle, \quad \omega_n = \omega_{-n}, \quad n_n \equiv D_{n_+}/\langle n_+ \rangle, \quad D_{n_+} \equiv \langle n_+^2 \rangle - \langle n_+ \rangle^2.
$$

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Then the expressions (2) for $\nu_s(\delta \eta)$ and (3) for $B(\eta_1,\eta_2)$ reduce to
\[
\nu_s(\delta \eta) = \frac{(n_+ n_-) - (n_+ (n_+ - 1))}{\langle n_+ \rangle} = 1 + \frac{(n_+ n_-) - (n_+^2)}{\langle n_+ \rangle}
\] (5)

and
\[
B(\eta_1 - \eta_2) = \frac{\rho_{+-}(\eta_1 - \eta_2) - \rho_{++}(\eta_1 - \eta_2)}{\rho^0_+}.
\] (6)

Then by the direct integration of (6) we get
\[
\nu_s(\delta \eta) = \frac{1}{\delta \eta} \int_{\delta \eta} d\eta_1 \int_{\delta \eta} d\eta_2 B(\eta_1 - \eta_2),
\] (7)

where we have taken into account the normalization conditions (13) and (14) (see the next Section).

Since by the definition (3) the BF is symmetric: $B(\Delta \eta) = B(-\Delta \eta)$, the integral (7) can be written as follows (see e.g. the Appendix A in the paper [6]):
\[
\nu_s(\delta \eta) = \frac{1}{\delta \eta} \int_{\delta \eta} d\eta_1 \int_{\delta \eta} d\eta_2 B(\eta_1 - \eta_2) = \frac{1}{\delta \eta} \int_{-\delta \eta/2}^{\delta \eta/2} d\eta_1 \int_{-\delta \eta/2}^{\delta \eta/2} d\eta_2 B(\eta_1 - \eta_2)
\]
\[
= \frac{1}{\delta \eta} \int_{-\delta \eta}^{\delta \eta} d(\delta \eta) B(\Delta \eta) t_{\delta \eta}(\Delta \eta) = \frac{2}{\delta \eta} \int_{0}^{\delta \eta} d(\delta \eta) B(\Delta \eta) (\delta \eta - \Delta \eta),
\] (8)

where $t_{\delta \eta}(\Delta \eta)$ is the usual phase space triangular weight function:
\[
t_{\delta \eta}(\Delta \eta) = \theta(-\Delta \eta)(\delta \eta + \Delta \eta) + \theta(\Delta \eta)(\delta \eta - y)\theta(\delta \eta - |\Delta \eta|) \geq 0
\] (9)

(see the Fig. A.1 in the paper [6]).

In paper [4] the authors state that "The BF would represent the probability that the balancing charges were separated by $\Delta \eta$ (in our formalism we include a division by $\Delta \eta$ to express $B(\Delta \eta)$ as a density)." Nevertheless in the Introduction Section of the paper [2] it is mentioned that the value of $\nu_{d\eta}(\delta \eta)$ can be both negative and positive: "A negative value of $\nu_{d\eta}$ signifies the dominant contribution from correlations between pairs of opposite charges. On the other hand, a positive value indicates the significance of the same charge pair correlations."

By formula (2) this means that in some cases the $\nu_s(\delta \eta)$ can take negative values. Then by formula (8) we see that in this case the BF must also be negative at least at some values of $\Delta \eta$ to ensure the negative value of the integral (5), as the triangular weight function (9) is positive: $t_{\delta \eta}(\Delta \eta) \geq 0$. But if the $\nu_s(\delta \eta)$ and the BF $B(\Delta \eta)$ can take negative values they can not have any probabilistic interpretation, in particular that mentioned in paper [4].

In present short note we explicitly confirm this fact by direct calculations for very simple toy model.

Note also in the conclusion of this section that the $\nu_s(\delta \eta)$ is simply connected with the variable $\Sigma(n_+, n_-)$,
\[
\nu_s(\delta \eta) = 1 - \Sigma(n_+, n_-),
\] (10)
denoted in [7] as $\Sigma(n^+_F, n^-_F)$.

### 2 Model independent definitions and relations

We start with the definitions of inclusive and double inclusive pseudorapidity distributions of charged particles:
\[
\rho_\pm(\eta) \equiv \frac{dN_{ch}^\pm}{d\eta}, \quad \rho_{+-}(\eta_1, \eta_2) \equiv \frac{d^2N_{ch}^{+-}}{d\eta_1 d\eta_2}, \quad \rho_{++}(\eta_1, \eta_2) \equiv \frac{d^2N_{ch}^{++}}{d\eta_1 d\eta_2}, \quad \rho_{+-}(\eta_1, \eta_2) \equiv \frac{d^2N_{ch}^{-+}}{d\eta_1 d\eta_2},
\] (11)
Then we define the two-particle correlation functions by a standard way, \cite{1}:

\begin{equation}
\int_{\delta \eta} d\eta \, \rho_+ (\eta) = \langle n_+ \rangle ,
\end{equation}

\begin{equation}
\int_{\delta \eta} d\eta_1 \int_{\delta \eta_2} d\eta_2 \, \rho_+ (\eta_1, \eta_2) = \langle n_+ (n_+ - 1) \rangle .
\end{equation}

\begin{equation}
\int_{\delta \eta} d\eta_1 \int_{\delta \eta_2} d\eta_2 \, \rho_+ (\eta_1, \eta_2) = \langle n_+ n_- \rangle .
\end{equation}

Then we define the two-particle correlation functions by a standard way, \cite{1}:

\begin{equation}
C_{++}(\eta_1, \eta_2) \equiv \frac{\rho_+ (\eta_1, \eta_2)}{\rho_+ (\eta_1) \rho_+ (\eta_2)} - 1 , \quad C_{+-}(\eta_1, \eta_2) \equiv \frac{\rho_-(\eta_1, \eta_2)}{\rho_+ (\eta_1) \rho_- (\eta_2)} - 1 .
\end{equation}

In mid-rapidity region at LHC energies, when the translation invariance in rapidity and the charge symmetry, mentioned above, take place, these formulae can be simplified, using that

\begin{equation}
\rho_+ (\eta) = \rho_- (\eta) = \rho_0 = \text{const} = \langle n_+ \rangle / \delta \eta ,
\end{equation}

\begin{equation}
\rho_+ (\eta_1, \eta_2) = \rho_+ (\eta_1 - \eta_2) , \quad \rho_- (\eta_1, \eta_2) = \rho_+ (\eta_1 - \eta_2)
\end{equation}

and hence

\begin{equation}
C_{++}(\eta_1, \eta_2) = C_{++}(\eta_1 - \eta_2) , \quad C_{+-}(\eta_1, \eta_2) = C_{+-}(\eta_1 - \eta_2) .
\end{equation}

Then by (12)-(17) we have

\begin{equation}
\rho_+^0 \rho_+ \int_{\delta \eta} d\eta_1 \int_{\delta \eta_2} d\eta_2 \, C_{++}(\eta_1 - \eta_2) = \langle n_+ (n_+ - 1) \rangle - \langle n_+ \rangle^2 .
\end{equation}

\begin{equation}
\rho_+^0 \rho_- \int_{\delta \eta} d\eta_1 \int_{\delta \eta_2} d\eta_2 \, C_{+-}(\eta_1 - \eta_2) = \langle n_+ n_- \rangle - \langle n_+ \rangle \langle n_- \rangle .
\end{equation}

Using now definition (5) we express the \( \nu_s (\delta \eta) \) through the correlation functions \( C_{+-} \) and \( C_{++} \) by the model independent way:

\begin{equation}
\nu_s (\delta \eta) = \frac{\rho_0}{\delta \eta} \int_{\delta \eta} d\eta_1 \int_{\delta \eta_2} d\eta_2 \left[ C_{++}(\eta_1 - \eta_2) - C_{++}(\eta_1 - \eta_2) \right] .
\end{equation}

Simultaneously from formula (6) for the BF we have

\begin{equation}
B(\eta_1 - \eta_2) = \rho_0 \cdot \left[ C_{+-}(\eta_1 - \eta_2) - C_{++}(\eta_1 - \eta_2) \right] .
\end{equation}

3 The models with independent identical sources

In models with independent identical sources the following formula \cite{6} for \( C(\eta_1, \eta_2) \) takes place (see a simple proof in Appendix A):

\begin{equation}
C(\eta_1, \eta_2) = \frac{\Lambda (\eta_1, \eta_2) + \omega_{N}}{\langle N \rangle} ,
\end{equation}

where \( N \) is a number of sources, which fluctuates event by event around some mean value, \( \langle N \rangle \), with some scaled variance, \( \omega_{N} = D_{N} / \langle N \rangle \).
The $\Lambda(\eta_1, \eta_2)$ is the two-particle correlation function characterizing a single source. It is defined similarly to $C(\eta_1, \eta_2)$, but taking into account only particles produced by a given source:

$$\Lambda_{++}(\eta_1, \eta_2) = \frac{\lambda_{++}(\eta_1, \eta_2)}{\lambda_{+}(\eta_1)\lambda_{+}(\eta_2)} - 1, \quad \Lambda_{+-}(\eta_1, \eta_2) = \frac{\lambda_{+-}(\eta_1, \eta_2)}{\lambda_{+}(\eta_1)\lambda_{-}(\eta_2)} - 1,$$

where

$$\lambda_{+}(\eta) = \frac{dN_{ch}^+}{d\eta}, \quad \lambda_{+}(\eta_1, \eta_2) = \frac{d^2N_{ch}^{++}}{d\eta_1 d\eta_2}, \quad \lambda_{-}(\eta_1, \eta_2) = \frac{d^2N_{ch}^{-+}}{d\eta_1 d\eta_2},$$

are inclusive and double inclusive pseudorapidity distributions of charged particles produced by a given source. They are normalized as follows:

$$\int d\eta \lambda_{+}(\eta) = \langle \mu_+ \rangle,$$

$$\int d\eta_1 \int d\eta_2 \lambda_{++}(\eta_1, \eta_2) = \langle \mu_+(\mu_+ - 1) \rangle.$$

$$\int d\eta_1 \int d\eta_2 \lambda_{+-}(\eta_1, \eta_2) = \langle \mu_+ \mu_- \rangle.$$

In mid-rapidity region at LHC energies, when the translation invariance in rapidity and the charge symmetry take place, these formulae can again be simplified, using that

$$\lambda_{+}(\eta) = \lambda_{-}(\eta) = \lambda_0 = \lambda_0 = \text{const} = \frac{\langle \mu_+ \rangle}{\delta\eta} = \frac{\langle n_+ \rangle}{\langle N \rangle} = \frac{\rho_0}{\langle N \rangle},$$

$$\lambda_{+}(\eta_1, \eta_2) = \lambda_{+}(\eta_1 - \eta_2), \quad \lambda_{-}(\eta_1, \eta_2) = \lambda_{-}(\eta_1 - \eta_2)$$

and hence

$$\Lambda_{++}(\eta_1, \eta_2) = \Lambda_{++}(\eta_1 - \eta_2), \quad \Lambda_{+-}(\eta_1, \eta_2) = \Lambda_{+-}(\eta_1 - \eta_2).$$

Then

$$\Lambda_{++}(\eta_1 - \eta_2) = \frac{\lambda_{++}(\eta_1 - \eta_2)}{\lambda_0^0 \lambda_0^0} - 1, \quad \Lambda_{+-}(\eta_1 - \eta_2) = \frac{\lambda_{+-}(\eta_1 - \eta_2)}{\lambda_0^0 \lambda_0^0} - 1.$$

Substituting now the general connection (22) into formula (20) we finally express the $\nu_s(\delta\eta)$ through the correlation functions $\Lambda_{+-}$ and $\Lambda_{++}$ of a single source:

$$\nu_s(\delta\eta) = \frac{\lambda_0^0}{\lambda_0^0} \int d\eta_1 \int d\eta_2 \left[ \frac{\Lambda_{+-}(\eta_1 - \eta_2) - \Lambda_{++}(\eta_1 - \eta_2)}{\Lambda_{++}(\eta_1 - \eta_2) - \lambda_{++}(\eta_1 - \eta_2)} \right].$$

Note that a dependence on $\langle N \rangle$ and $\omega_N = D_N/\langle N \rangle$ is canceled what proves the strongly intensive behavior of this variable in the case with identical sources.

We see this also from the fact that formula (31) coincides with the definition (5) when replacing all engaged quantities by the corresponding ones for one source. That also can be written as

$$\nu_s(\delta\eta) = \frac{\langle n_+ n_- \rangle - \langle n_+ (n_+ - 1) \rangle}{\langle n_+ \rangle} = \frac{\langle \mu_+ \mu_- \rangle - \langle \mu_+ (\mu_+ - 1) \rangle}{\langle \mu_+ \rangle}$$

in any model with identical courses.
As mentioned in the Introduction the $\nu_s(\delta \eta)$ is simply connected with the balance function
$B(\eta_1 - \eta_2)$. In any model with the identical independent sources in the central region, where
the translation invariance in rapidity and the charge symmetry take place, we have (see e.g.
Section 5 of the paper [5]):

$$B(\eta_1 - \eta_2) = \lambda^0_+ \cdot \left[ \Lambda_{+-}(\eta_1 - \eta_2) - \Lambda_{-+}(\eta_1 - \eta_2) \right]. \quad (33)$$

One can immediately obtain this formula substituting (22) into (21) and taking into account
the relation (28).

Comparing formulas (31) and (33) we see that the general relation (7):

$$\nu_s(\delta \eta) = \frac{1}{\delta \eta} \int_{\delta \eta} d\eta \int_{\delta \eta} d\eta' B(\eta_1 - \eta_2),$$

of course, is true in this particular case.

4 Toy model with production of correlated charge pairs by a source

Let us consider at first the very simple model, when each source always produces only one
plus-minus pair, with plus and minus particles being uniformly distributed in some wide
interval $(-Y/2, Y/2)$, $Y \gg 1$.

In this simple model

$$\lambda^0_+ = \frac{1}{Y}, \quad \lambda_{+-}(\eta_1 - \eta_2) = 0, \quad \lambda_{-+}(\eta_1 - \eta_2) = \frac{1}{Y^2}. \quad (34)$$

To test these formulae we can use the normalization conditions (25)-(27) in the whole accept-
tance $Y$:

$$\int_Y d\eta \lambda_+(\eta) = 1, \quad (35)$$

$$\int_Y d\eta_1 \int_Y d\eta_2 \lambda_{+-}(\eta_1 - \eta_2) = 0. \quad (36)$$

$$\int_Y d\eta_1 \int_Y d\eta_2 \lambda_{-+}(\eta_1 - \eta_2) = 1. \quad (37)$$

Then by (30) we have

$$\Lambda_{+-}(\eta_1 - \eta_2) = -1, \quad \Lambda_{-+}(\eta_1 - \eta_2) = 0. \quad (38)$$

As expected we see no correlation between plus and minus particles produced from the same
source, $\Lambda_{+-}(\eta_1 - \eta_2) = 0$, and a strong anticorrelation between plus and plus particles from
one source, $\Lambda_{++}(\eta_1 - \eta_2) = -1$, because the only one plus particle, produced from a source,
can’t be simultaneously at both $\eta_1$ and $\eta_2$ pseudorapidities.

Substituting all this now in formula (31) we find

$$\nu_s(\delta \eta) = \frac{1}{Y \delta \eta} \delta \eta^2 \left[ 0 - (-1) \right] = \frac{\delta \eta}{Y}. \quad (39)$$

The interpretation of the $\nu_s(\delta \eta) = \frac{\delta \eta}{Y}$ as the probability to find the negatively charged particle
in the rapidity interval $\delta \eta$ under condition that we have already the positively charged particle
in this interval looks very suspicious. Since, as we can see from formulae (38) and (39),
this result arises not due to correlation between plus and minus particles but due to a strong
anticorrelation between plus and plus particles in this simple model.
To verify these suspicions let us consider a little bit more sophisticated model, when each source always produces two plus-minus pairs, with two plus and two minus particles being uniformly distributed in some wide interval \((-Y/2, Y/2), Y \gg 1\).

In this version of the model

\[
\lambda_0^+ = \frac{2}{Y}, \quad \lambda_{++}(\eta_1 - \eta_2) = \frac{2}{Y^2}, \quad \lambda_{+-}(\eta_1 - \eta_2) = \frac{4}{Y^2}.
\]

(40)

Again we can test these formulae using the normalization conditions (25)-(27) in the whole acceptance \(Y\):

\[
\int_Y d\eta \lambda_\pm(\eta) = 2, \quad \int_Y d\eta_1 \int_Y d\eta_2 \lambda_{++}(\eta_1 - \eta_2) = 2, \quad \int_Y d\eta_1 \int_Y d\eta_2 \lambda_{+-}(\eta_1 - \eta_2) = 4.
\]

(41) (42) (43)

Then by (30) we have

\[
\Lambda_{++}(\eta_1 - \eta_2) = -\frac{1}{2}, \quad \Lambda_{+-}(\eta_1 - \eta_2) = 0.
\]

(44)

As expected again we see no correlation between plus and minus particles produced from the same source, \(\Lambda_{+-}(\eta_1 - \eta_2) = 0\), and attenuation of the anticorrelation between plus and plus particles from one source, \(\Lambda_{++}(\eta_1 - \eta_2) = -\frac{1}{2}\), because now two plus particles are produced from a source and \(\Lambda_{++}(\eta_1 - \eta_2) = \frac{2}{Y^2} > 0\).

Substituting all this in formula (31) we find that again

\[
\nu_s(\delta\eta) = \frac{2}{Y \delta\eta} \delta\eta^2 \left[0 - \left(-\frac{1}{2}\right)\right] = \frac{\delta\eta}{Y}.
\]

(45)

It is easy to prove that in model, when each source always produces \(k\) plus-minus pairs, with \(k\) plus and \(k\) minus particles being uniformly distributed in some wide interval \((-Y/2, Y/2), Y \gg 1\), we'll have

\[
\nu_s(\delta\eta) = \frac{k}{Y \delta\eta} \delta\eta^2 \left[0 - \left(-\frac{1}{k}\right)\right] = \frac{\delta\eta}{Y}.
\]

(46)

The interpretation of the \(\nu_s(\delta\eta) = \frac{\delta\eta}{Y}\) as the probability to find the negatively charged particle in the rapidity interval \(\delta\eta\) under condition that we have already the positively charged particle in this interval still holds, since in each event we have equal number of plus and minus particles uniformly distributed in some wide interval \((-Y/2, Y/2), Y \gg 1\), as in the initial version of the model with one charge pair production by a source. Nevertheless it looks strange since it based not on correlations between plus and minus particles but on anticorrelations between plus and plus particles in this simple model.

Note that in this case by formula (33) the BF ("the probability density") is equal to \(1/Y\):

\[
B(\Delta\eta) = \frac{k}{Y} \left[0 - \left(-\frac{1}{k}\right)\right] = \frac{1}{Y}.
\]

(47)

what after integration over rapidity interval \(\delta\eta\) by (8) again leads to the formula (46).
5 Toy model with production of correlated charge pairs by a source

As we can see in previous section this result, \( \nu_s(\delta \eta) = \frac{\delta \eta}{Y} \), arises due to plus-plus anticorrelation, \( \Lambda_{++}(\eta_1 - \eta_2) = -\frac{1}{Y} \), in the version of the model with production of \( k \) independent plus-minus pairs by each source. After multiplying by \( \lambda_0^+ = \frac{1}{T} \) and the integration we just have this result.

So, in present section we are trying to introduce some additional plus-plus correlation, formulating a more complex version of the model.

5.1 Strong correlation between identical charges from a source

Let us consider at first the model in which each source always produces two plus-minus pairs, so that the rapidities of both positive particles coincide and the same is true for both minus particles (the maximally strong correlation between identical charges), whereas the rapidities of the plus pair and the minus pair themselves are uniformly distributed in some wide interval \( (-Y/2, Y/2) \), \( Y \gg 1 \).

In this version of the model
\[
\lambda_0^+ = \frac{2}{Y}, \quad \lambda_{++}(\eta_1 - \eta_2) = \frac{2}{Y} \delta(\eta_1 - \eta_2), \quad \lambda_{+-}(\eta_1 - \eta_2) = \frac{4}{Y^2}.
\]

Again we can test these formulae using the normalization conditions (25)-(27) in the whole acceptance \( Y \):
\[
\int_Y d\eta \lambda_{\pm}(\eta) = 2, \quad \int_Y d\eta_1 \int_Y d\eta_2 \lambda_{++}(\eta_1 - \eta_2) = 2, \quad \int_Y d\eta_1 \int_Y d\eta_2 \lambda_{+-}(\eta_1 - \eta_2) = 4.
\]

Then by (30) we have
\[
\Lambda_{++}(\eta_1 - \eta_2) = \frac{2}{Y} \delta(\eta_1 - \eta_2) - 1, \quad \Lambda_{+-}(\eta_1 - \eta_2) = 0.
\]

As expected we see again no correlation between plus and minus particles produced from the same source, but we see now strong additional \( \frac{4}{Y^2} \delta(\eta_1 - \eta_2) \) correlation between positive particles from one source.

Substituting all this in formula (31) we find
\[
\nu_s(\delta \eta) = \frac{2}{Y} \frac{\delta \eta}{\delta \eta} \left[ 0 \cdot \delta \eta^2 - \left( \frac{Y}{2} \cdot \delta \eta - 1 \cdot \delta \eta^2 \right) \right] = \frac{2\delta \eta}{Y} - 1.
\]

Before to make any conclusions we verify this important result using the simple formula (32):
\[
\nu_s(\delta \eta) = \frac{\langle n_+ n_- \rangle - \langle n_+ (n_+ - 1) \rangle}{\langle n_+ \rangle} = \frac{\langle \mu_+ \mu_- \rangle - \langle \mu_+ (\mu_+ - 1) \rangle}{\langle \mu_+ \rangle}
\]

In this version of the model
\[
\langle \mu_+ \rangle = \sum_{\mu_+ \geq 1} P(\mu_+) \mu_+ = P(1) \cdot 1 + P(2) \cdot 2 = 0 \cdot 1 + \frac{\delta \eta}{Y} \cdot 2 = 2 \frac{\delta \eta}{Y},
\]

\[
\langle \mu_- \rangle = \sum_{\mu_- \geq 1} P(\mu_-) \mu_- = P(1) \cdot 1 + P(2) \cdot 2 = 0 \cdot 1 + \frac{\delta \eta}{Y} \cdot 2 = 2 \frac{\delta \eta}{Y},
\]

\[
\langle \mu_+ \mu_- \rangle = \sum_{\mu_+ \geq 1} \sum_{\mu_- \geq 1} P(\mu_+) P(\mu_-) \mu_+ \mu_- = P(1)^2 \cdot 1 \cdot 1 + P(1) P(2) \cdot 1 \cdot 2 + P(2)^2 \cdot 2 \cdot 2 = 0 \cdot 1 + \frac{\delta \eta}{Y} \cdot 2 + \frac{\delta \eta}{Y} \cdot 4 = 2 \frac{\delta \eta}{Y} + 2 \frac{\delta \eta}{Y} = \frac{4 \delta \eta}{Y},
\]

\[
\langle \mu_+ (\mu_+ - 1) \rangle = \sum_{\mu_+ \geq 1} \frac{1}{\mu_+} P(\mu_+) \mu_+ (\mu_+ - 1) = P(1)^2 \cdot 1 \cdot 0 + P(1) P(2) \cdot 1 \cdot 1 + P(2)^2 \cdot 2 \cdot 1 = 0 \cdot 0 + \frac{\delta \eta}{Y} \cdot 1 + \frac{\delta \eta}{Y} \cdot 2 = \frac{3 \delta \eta}{Y},
\]

\[
\langle \mu_+ \rangle = \frac{2 \delta \eta}{Y} - 1.
\]
\[ \langle \mu_+ \mu_- \rangle = \sum_{\mu_+ \geq 1 \mu_- \geq 1} P(\mu_+, \mu_-) \mu_+ \mu_- = P(1, 1) \cdot 1 + P(1, 2) \cdot 2 + P(2, 1) \cdot 2 + P(2, 2) \cdot 4 \]  

\[ = 0 \cdot 1 + 0 \cdot 2 + 0 \cdot 2 + \frac{\delta \eta \, \delta \eta}{Y} \cdot 4 = 4 \left( \frac{\delta \eta}{Y} \right)^2 , \]

\[ \langle \mu_+ (\mu_+ - 1) \rangle = \sum_{\mu_+ \geq 2} P(\mu_+) \mu_+ (\mu_+ - 1) = P(2) \cdot 2 = \frac{\delta \eta}{Y} \cdot 2 = 2 \frac{\delta \eta}{Y} . \]  

Then by formula (52) we find

\[ \nu_s (\delta \eta) = \frac{\langle \mu_+ \mu_- \rangle - \langle \mu_+ (\mu_+ - 1) \rangle}{\langle \mu_+ \rangle} = \frac{4 \left( \frac{\delta \eta}{Y} \right)^2 - 2 \frac{\delta \eta}{Y}}{2} = \frac{2 \delta \eta - 1}{Y} , \]  

that coincides with (53).

So, concluding we see that although for the whole interval at \( \delta \eta = Y \) we have \( \nu_s (Y) = 1 \), as expected, nevertheless the value of the \( \nu_s (\delta \eta) \) at \( \delta \eta < Y/2 \) becomes negative and hence can not have any probabilistic interpretation.

Note that by formula (33) the BF in this case is as follows

\[ B(\Delta \eta) = - \delta (\Delta \eta) + \frac{2}{Y} , \]  

what after integration over rapidity interval \( \delta \eta \) by (8) again leads to the formula (57).

### 5.2 Gentle correlation between identical charges from a source

From the model construction it is clear that if we’ll use instead of the \( \delta \)-function any enough narrow distribution normalized by unity, we’ll arrive to the same conclusion. Really, let us use in this subsection instead of the \( \delta \)-function the step distribution normalized to unity and spread over interval from \(-a\) to \(a\) (\(a > 0\)):

\[ \delta (\Delta \eta) \rightarrow h_a (\Delta \eta) \equiv \frac{1}{2a} \theta (a - |\Delta \eta|) , \]  

In this case for the version of the model, described in the previous Subsection 5.1, we have

\[ \lambda_+^0 = \frac{2}{Y} , \quad \lambda_{++}(\eta_1 - \eta_2) = \frac{2}{Y - a/2} h_a (\eta_1 - \eta_2) , \quad \lambda_{+-}(\eta_1 - \eta_2) = \frac{4}{Y^2} . \]  

Using the formulae (8) we can check that the factor \(2/(Y - a/2)\) ensures the correct normalization condition (26) for the \( \lambda_{++}(\eta_1 - \eta_2)\):

\[ \int_Y d\eta \, \lambda_+ (\eta) = 2 , \]  

\[ \int_Y d\eta_1 \int_Y d\eta_2 \, \lambda_{++}(\eta_1 - \eta_2) = \]  

\[ = \int_{-Y/2}^{Y/2} d\eta_1 \int_{-Y/2}^{Y/2} d\eta_2 \, \lambda_{++}(\eta_1 - \eta_2) = \frac{2}{Y - a/2} \int_Y d(\Delta \eta) \, h_a (\Delta \eta) \, t_Y (\Delta \eta) = 2 , \]

\[ \int_Y d\eta_1 \int_Y d\eta_2 \, \lambda_{+-}(\eta_1 - \eta_2) = 4 . \]  

So, we see that although for the whole interval at \( \delta \eta = Y \) we have \( \nu_s (Y) = 1 \), as expected, nevertheless the value of the \( \nu_s (\delta \eta) \) at \( \delta \eta < Y/2 \) becomes negative and hence can not have any probabilistic interpretation.
Then by (30) we have
\[ \Lambda_{++}(\eta_1 - \eta_2) = \frac{Y^2}{2Y-a} h_a(\eta_1 - \eta_2) - 1, \quad \Lambda_{+-}(\eta_1 - \eta_2) = 0. \] (64)

By the formula (33) we find now the BF:
\[ B(\Delta \eta) = -\frac{Y}{Y-a/2} h_a(\Delta \eta) + \frac{2}{Y}. \] (65)

For \(|\Delta \eta| < a\) by (59) we have
\[ B(\Delta \eta) = -\frac{Y}{(2Y-a)a} + \frac{2}{Y}. \] (66)

It is easy to check that at \(|\Delta \eta| < a < (1 - 1/\sqrt{2}) Y \approx 0.29 Y\) the BF is negative, \(B(\Delta \eta) < 0\), and can’t be interpreted as a probability density.

We can now calculate \(\nu_s(\delta \eta)\) by the integration of the expression (65) over rapidity interval \(\delta \eta\) using the formulae (7) and (8):
\[ \nu_s(\delta \eta) = \frac{1}{\delta \eta} \int_{-\delta \eta}^{\delta \eta} d(\Delta \eta) B(\Delta \eta) t_{\delta \eta}(\Delta \eta) = \int_{-\delta \eta}^{\delta \eta} d(\Delta \eta) \frac{2\delta \eta - 2\delta \eta/a}{2 - a/Y} \theta(a - |\Delta \eta|) t_{\delta \eta}(\Delta \eta). \] (67)

Then we find
\[ \nu_s(\delta \eta) = \frac{2\delta \eta}{Y} - \frac{2 - a/\delta \eta}{2 - a/Y} \quad \text{at} \quad \delta \eta > a, \] (68)

and
\[ \nu_s(\delta \eta) = \frac{2\delta \eta}{Y} - \frac{\delta \eta/a}{2 - a/Y} = \left(\frac{2}{Y} - \frac{Y}{(2Y-a)a}\right) \delta \eta \quad \text{at} \quad \delta \eta < a. \] (69)

From formula (68) we see that at \(a \to 0\) the \(\nu_s(\delta \eta)\) go to the result (57), obtained in previous Subsection 5.1.

By formula (69) we see that at
\[ \delta \eta < a < (1 - 1/\sqrt{2}) Y \approx 0.29 Y \] (70)

(the same condition as the condition obtained from formula (66) for the BF) the \(\nu_s(\delta \eta)\) is negative, \(\nu_s(\delta \eta) < 0\), and can’t be interpreted as a probability. Note that occurs for rather wide correlation function \(\lambda_{++}(\eta_1 - \eta_2)\) (60), with \(a\) compared to \(Y\), as follows from condition (70).

In conclusion of this subsection we perform one more check of the obtained formulae. Clear that this model with \(a = Y\) corresponds to the absence of the correlation between the same charge particles from a source. Hence, in this case we have a source always emitting two pairs of uncorrelated plus-minus particles. This version of the model was already considered in Section 4 (the case with \(k = 2\)).

Really, if in formula (60) we put \(a = Y\), then the formula for \(\lambda_{++}(\eta_1 - \eta_2)\), (60), reduces to (40):
\[ \lambda_{++}(\eta_1 - \eta_2) = \frac{2}{Y - a/2} h_a(\eta_1 - \eta_2) \to \frac{2}{Y^2}. \]
The formula for $\Lambda_{++}(\eta_1 - \eta_2)$, (64), reduces to (44):

$$\Lambda_{++}(\eta_1 - \eta_2) = \frac{Y^2}{2Y - a} h_a(\eta_1 - \eta_2) - 1 \to -\frac{1}{2}.$$ 

The formula for $B(\Delta \eta)$, (65), reduces to (47):

$$B(\Delta \eta) = -\frac{Y}{Y - a/2} h_a(\Delta \eta) + \frac{2}{Y} \to \frac{1}{Y}.$$

The formula for $\nu_s(\delta \eta)$, (69), reduces to (45):

$$\nu_s(\delta \eta) = \left(\frac{2}{Y} - \frac{Y}{(2Y - a) a}\right) \delta \eta \to \frac{\delta \eta}{Y}.$$ 

So, we see that the model considered in this Subsection 5.2 with gentle correlation between identical charges, given by the function (see formulas (59) and (60)):

$$\lambda_{++}(\Delta \eta) = \frac{2}{Y - a/2} h_a(\Delta \eta) = \frac{1}{(2Y - a) a} \theta(a - |\Delta \eta|), \quad (71)$$

with arbitrary value of the correlation width parameter $a$, $0 < a \leq Y$, on the one hand at $a \to 0$ go to the model with strong correlation between identical charges considered in the previous Subsection 5.1 and on the other hand at $a = Y$ go to the model with uncorrelated charge pairs production by a source considered in the Section 4.

We see also that the negative values of the BF $B(\Delta \eta)$ at $\Delta \eta < a$ and the $\nu_s(\delta \eta)$ at $\delta \eta < a$ already occur when we introduce the rather weak correlation between same charge particles with the value of $a$ compared to $Y$, namely at $a < (1 - 1/\sqrt{2}) Y \approx 0.29 Y$ as follows from condition (70).

6 Conclusion

In this short note by constructing of a simple toy model we explicitly demonstrate that the values of the $\nu_s(\delta \eta)$ and hence the $\nu_{d_{\text{yon}}}(\delta \eta)$,

$$\nu_s(\delta \eta) \equiv -\frac{\langle n \rangle}{4} \nu_{d_{\text{yon}}}(\delta \eta),$$

can be both negative and positive, so it can not have any probabilistic interpretation, as e.g. the probability that the balancing charges occur in the same rapidity interval $\delta \eta$.

By relation

$$\nu_s(\delta \eta) = \frac{1}{\delta \eta} \int_{\delta \eta} d\eta_1 \int_{\delta \eta} d\eta_2 B(\eta_1 - \eta_2),$$

it follows that in this case the BF must also be negative at least for some values of $\Delta \eta = \eta_1 - \eta_2$ to ensure the negative value of the integral. We also check it explicitly calculating the BF in our toy model.

But if the BF $B(\Delta \eta)$ can take negative values it can not have any probabilistic interpretation in general case. In particular, the BF can not be interpreted as the probability density for the balancing charges to occur separated by the rapidity interval $\Delta \eta$, as it was formulated in the paper [4].
Appendix. A proof of the formula (22)

For a class of events with fixed number of sources $N$, following the paper [1], we have

\[ \rho^{(N)}(\eta) = N\lambda(\eta), \]  
(72)

\[ \rho^{(N)}_2(\eta_1, \eta_2) = N\lambda_2(\eta_1, \eta_2) + N(N-1)\lambda(\eta_1)\lambda(\eta_2). \]  
(73)

Then averaging over events with different number of sources $N$ we find

\[ \rho(\eta) = \sum_N P(N)\rho^{(N)}(\eta) = \sum_N P(N)N\lambda(\eta) = \langle N \rangle \lambda(\eta), \]  
(74)

\[ \rho_2(\eta_1, \eta_2) = \sum_N P(N)\rho^{(N)}_2(\eta_1, \eta_2) = \langle N \rangle \lambda_2(\eta_1, \eta_2) + \langle N(N-1) \rangle \lambda(\eta_1)\lambda(\eta_2). \]  
(75)

Using now definitions (15) and (23) we have

\[ C(\eta_1, \eta_2) = \frac{\rho_2(\eta_1, \eta_2)}{\rho(\eta_1)\rho(\eta_2)} - 1 = \frac{\langle N \rangle \lambda_2(\eta_1, \eta_2) - \lambda(\eta_1)\lambda(\eta_2)}{\langle N \rangle \lambda(\eta_1)\lambda(\eta_2)} + \frac{\langle N^2 \rangle - \langle N \rangle^2}{\langle N \rangle^2} - 1 = \frac{\Lambda(\eta_1, \eta_2)}{\langle N \rangle} + \frac{\langle N^2 \rangle - \langle N \rangle^2}{\langle N \rangle^2} = \frac{\Lambda(\eta_1, \eta_2) + \omega_N}{\langle N \rangle}. \]  
(76)

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