A note on tridiagonal matrices with zero main diagonal

Alexander Dyachenko*  Mikhail Tyaglov†
Email: diachenko@sfedu.ru  Email: tyaglov@mail.ru

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Abstract

We find the spectrum of an arbitrary irreducible complex tridiagonal matrix with two-periodic main diagonal provided that the spectrum of the matrix with the same sub- and superdiagonals and zero main diagonal is known. Our result substantially generalises some of the recent results on the Sylvester-Kac matrix and its certain main principal submatrix.

1 Introduction

The present note is intended to simplify and generalise some recent findings on certain modifications of the so-called Sylvester-Kac matrix

$$K_N := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ N & 0 & 2 & \cdots & 0 & 0 & 0 \\ 0 & N-1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & N-1 & 0 \\ 0 & 0 & 0 & \cdots & 2 & 0 & N \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$ 

This matrix seemingly appeared for the first time in a work of J.J. Sylvester [10] and later was studied by a number of mathematicians in XIX century, see [9] for references. In XX century, Mark Kac rediscovered this matrix in his famous work on the Brownian motion [6]. A good survey on the Sylvester-Kac matrix was presented by O. Taussky and J. Todd in [11]. References to some recent results on the Sylvester-Kac matrix are given, e.g., in [2]. Here we avoid reviewing this topic: instead, we show that some recent results around Sylvester-Kac-like matrices can be extended to an interesting property of a much larger class of tridiagonal matrices.

In the paper [7], the author added to $K_N$ the main diagonal where the entries with even and odd indices have the same absolute value but opposite signs, and found the determinant of that matrix. The authors of [8] calculate the determinant of a matrix obtained from $K_N$ by adding a non-zero two-periodic main diagonal. Note that a shorter proof of the result of [8] was given in [3]. In Section 4 we show that the results of [3, 8] immediately follow from the result of [7].

In the paper [4], the authors deal with a certain submatrix of the Sylvester-Kac matrix $K_N$ (considered much earlier by A. Caley [1]). Among other results, they found the determinant and the eigenvalues of this submatrix with an added two-periodic main diagonal.

In this paper, we generalise the aforementioned results of the works [3, 4, 7, 8] to the class of arbitrary (irreducible) complex tridiagonal matrices with zero main diagonal. In Section 5 we show that the findings...
of [7] can be easily extended (we give an elementary independent proof) to such a class of matrices. This result allows us to generalise [8] and (partially) [4]. Namely, we determine the eigenvalues and the determinant of a tridiagonal matrix with two-periodic main diagonal in terms of the eigenvalues of the same matrix but with zero main diagonal, see Section 4. Note that at the same time, the matrix considered in Section 3 is a particular case of the matrix from Section 4.

Disclaimer. We claim that this version of the manuscript is preliminary, for reference only, and to be substantially extended rather soon.

2 Tridiagonal matrices with zero main diagonal

Consider an $n \times n$ complex irreducible tridiagonal matrix whose main diagonal only contains zero entries:

$$J_n = \begin{pmatrix} 0 & c_1 & 0 & \cdots & 0 & 0 \\ a_1 & 0 & c_2 & \cdots & 0 & 0 \\ 0 & a_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & c_{n-1} \\ 0 & 0 & 0 & \cdots & a_{n-1} & 0 \end{pmatrix}, \quad a_k, c_k \in \mathbb{C} \setminus \{0\}. \quad (2.1)$$

Since the main diagonal of $J_n$ is zero, the spectrum of $J_n$ is symmetric w.r.t. zero.

**Theorem 2.1.** The spectrum of the matrix $J_n$ has the form

$$\sigma(J_{2l}) = \{ \pm \lambda_1, \pm \lambda_2, \ldots, \pm \lambda_l \} \quad \text{and} \quad \sigma(J_{2l+1}) = \{ 0, \pm \lambda_1, \pm \lambda_2, \ldots, \pm \lambda_l \}. \quad (2.2)$$

Here the numbers $\lambda_i$ are not necessary distinct or non-zero.

**Proof.** Let $P_k(z), k = 1, \ldots, n,$ be the characteristic polynomial of the $k^{th}$ leading principal submatrix of $J_n$. Then the following three-term recurrence relations hold

$$P_{k+1}(z) = z P_k(z) - a_k c_k P_{k-1}(z), \quad k = 0, 1, \ldots, n - 1, \quad (2.3)$$

with $P_{-1}(z) \equiv 0, P_0(z) \equiv 1$. From (2.3) it follows that the polynomials $P_k(z)$ do not depend on $a_k$ and $c_k$ separately – only on the product $a_k c_k$. Therefore, the matrices $J_n$ and $-J_n$ have the same eigenvalues, and hence the spectrum of $J_n$ is symmetric w.r.t. 0. In particular, if $n$ is odd, the matrix $J_n$ is singular. \qed

3 Tridiagonal matrices with alternating signs main diagonal

Consider the matrix

$$A_n = J_n + x E_n, \quad (3.1)$$

where the matrix $E_n = \{ e_{ij} \}_{i,j=1}^n$ is defined as follows

$$e_{ij} = \begin{cases} (-1)^{i-j} & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

and $J_n$ is as in (2.1). The following fact holds.

**Theorem 3.1.** The spectrum of the matrix $A_n$ defined in (3.1) has the form

$$\sigma(A_{2l}) = \{ \pm \sqrt{\lambda_1^2 + x^2}, \pm \sqrt{\lambda_2^2 + x^2}, \ldots, \pm \sqrt{\lambda_l^2 + x^2} \}, \quad (3.2)$$

and

$$\sigma(A_{2l+1}) = \{ x, \pm \sqrt{\lambda_1^2 + x^2}, \pm \sqrt{\lambda_2^2 + x^2}, \ldots, \pm \sqrt{\lambda_l^2 + x^2} \}, \quad (3.3)$$

where $\{ \lambda_i \}_{i=1}^l$ belong to the spectrum of $J_n$, see (2.2).

\footnote{It is clear that the Sylvester-Kac matrix and the matrix considered in [4] belong to this class, since they are irreducible tridiagonal and have zero main diagonal.}
Indeed, the square of the matrix $A_n$ has the form
\[
A_n^2 = (J_n + xE_n)^2 = J_n^2 + x J_n E_n + x E_n J_n + x^2 I_n = J_n^2 + x^2 I_n,
\]
where $I_n$ is the $n \times n$ identity matrix. Here we made use of the easily verifiable identity $J_n E_n + E_n J_n = 0$.

It is clear now that the spectrum of the matrix $A_n^2$ has the form
\[
\sigma(A_n^2) = \{\lambda_i^2 + x^2, \lambda_i^2 + x^2, \ldots, \lambda_i^2 + x^2\} \quad \text{and} \quad \sigma(A_{2l+1}) = \{x^2, \lambda_1^2 + x^2, \lambda_2^2 + x^2, \ldots, \lambda_l^2 + x^2\}.
\]

For the rest of the proof we assume that all $\lambda_j$ are nonzero and distinct, the general case then follows by continuity. Let $\vec{v}_{2j}$ and $\vec{v}_{2j+1}$ be the eigenvectors of $J_n$, corresponding to the eigenvalues $\lambda_j$ and $-\lambda_j$, respectively. Then
\[
A_n^2 \vec{v}_{2j} = (\lambda_j^2 + x^2) \vec{v}_{2j} \quad \text{and} \quad A_n^2 \vec{v}_{2j+1} = (\lambda_j^2 + x^2) \vec{v}_{2j+1}.
\]

In other words, each point of the spectrum of $A_n^2$ is of even geometric multiplicity – only excluding $x^2$ when $n = 2l + 1$ (cf. [7] Chapters VII-VIII). The eigenspaces of $A_n$ are, in turn, one-dimensional (that is, $A_n$ is non-derogatory), and hence for each $j$ both $\sqrt{\lambda_j^2 + x^2}$ and $-\sqrt{\lambda_j^2 + x^2}$ are necessarily the eigenvalues of $A_n$. Consequently, the spectrum of $A_{2l}$ satisfies (4.2). To get (4.3), it is enough to additionally show that $x \in \sigma(A_{2l+1})$. This inclusion holds since $\det(zI - A_{2l+1})$ is divisible by $z - x$, which can be proved by induction.

As a corollary of this theorem one gets the following formulæ generalising the result of [7]
\[
\det A_{2l} = (-1)^l \prod_{k=1}^l (x^2 + \lambda_k^2), \quad \det A_{2l+1} = (-1)^l x \prod_{k=1}^l (x^2 + \lambda_k^2).
\]

### 4 Tridiagonal matrices with two-periodic main diagonal

Consider now the matrix
\[
B_n = \begin{pmatrix}
  b_1 & c_1 & 0 & \ldots & 0 & 0 \\
  a_1 & b_2 & c_2 & \ldots & 0 & 0 \\
  0 & a_2 & b_3 & \ldots & 0 & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & 0 & \ldots & b_{n-1} & c_{n-1} \\
  0 & 0 & 0 & \ldots & a_{n-1} & b_n
\end{pmatrix}, \quad a_k, c_k \in \mathbb{C} \setminus \{0\}, \quad b_k = \begin{cases} x & \text{if } k \text{ is odd}, \\
 y & \text{if } k \text{ is even}. \end{cases}
\]

It is easy to see that
\[
B_n = J_n + \frac{x - y}{2} E_n + \frac{x + y}{2} I_n,
\]
where $J_n$ is defined in (2.1), so from (3.2)–(3.3) we obtain
\[
\sigma(B_{2l}) = \left\{ \frac{x + y}{2} \pm \frac{1}{2} \sqrt{4\lambda_i^2 + (x - y)^2}, \frac{x + y}{2} \pm \frac{1}{2} \sqrt{4\lambda_i^2 + (x - y)^2}, \ldots, \frac{x + y}{2} \pm \frac{1}{2} \sqrt{4\lambda_i^2 + (x - y)^2} \right\},
\]
and
\[
\sigma(B_{2l+1}) = \left\{ x, \frac{x + y}{2} \pm \frac{1}{2} \sqrt{4\lambda_i^2 + (x - y)^2}, \frac{x + y}{2} \pm \frac{1}{2} \sqrt{4\lambda_i^2 + (x - y)^2}, \ldots, \frac{x + y}{2} \pm \frac{1}{2} \sqrt{4\lambda_i^2 + (x - y)^2} \right\}.
\]

These formulæ can be obtained from (3.2)–(3.3) by replacing $x$ with $\frac{x - y}{2}$ and then by adding $\frac{x + y}{2}$ to all the eigenvalues.

From (4.1)–(4.2), one can easily obtain that the determinant of $B_n$ has the form
\[
\det B_{2l} = \prod_{k=1}^l (xy - \lambda_k^2), \quad \det B_{2l+1} = x \prod_{k=1}^l (xy - \lambda_k^2).
\]

On letting $J_n$ to be the Sylvester-Kac matrix or its main principal submatrix, the formulæ (4.1)–(4.3) generalise the results of the works [5, 6], as well as an analogous transition in [7].
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