Wilson Loops in string duals of Walking and Flavored Systems.

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We consider the VEV of Wilson loop operators by studying the behavior of string probes in solutions of Type IIB string theory generated by $N_c$ D5 branes wrapped on an $S^2$ internal manifold. In particular, we focus on solutions to the background equations that are dual to field theories with a walking gauge coupling as well as for flavored systems. We present in detail our walking solution and emphasize various general aspects of the procedure to study Wilson loops using string duals. We discuss the special features that the strings show when probing the region associated with the walking of the field theory coupling.

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References
I. INTRODUCTION

In this paper we want to study the behavior of non-local operators of gauge theories, making use of the gauge-string correspondence. We are in particular interested in a specific class of supergravity solutions that are closely related to what goes under the name of walking in the field theory language. In the introduction we summarize the basic notions and ideas that will feature prominently in the paper: the treatment of Wilson loops in the gauge-string correspondence, the concepts of confinement and screening in gauge theories and the meaning of walking dynamics (with a view on its role within dynamical electro-weak symmetry breaking).

It must be stressed that we do not know the precise nature of the field-theory dual of some of the examples we are going to consider in the body of the paper. The Wilson loop studied here is a very important quantity, that may help identifying such dual theory.

A. AdS/CFT and Wilson loops

According to general ideas of holography and more concretely to the Maldacena conjecture [1], a quantum conformal field theory in dimension $d$ is equivalent to a quantum theory in $AdS_{d+1}$ space. In general, the idea is that local operators in the CFT couple to fields in the $AdS$ side, in such a way that correlators of conformal fields are related to amplitudes in the quantum theory in $AdS$, as explained in [2]. For instance, since the CFT-side of the equivalence must contain the energy-momentum tensor $T_{μν}$ among its operators, there must be a field on the $AdS$-side that couples to it. Since this field is the graviton, then the theory on $AdS_{d+1}$ space must be a quantum theory of gravity.

One can use the correspondence to study non-local operators on the CFT-side. In particular, if the field theory is a pure Yang-Mills theory, an example of such operator is the Wilson loop [3]. These objects couple to extended objects, excited in the $AdS$ side of the correspondence. The Wilson loops (path ordered exponentials of the holonomy of the gauge field along a curve $C$) are one of the most interesting observables of such a gauge theory,

$$ W(C) \equiv \frac{1}{N_c} Tr P e^{i \oint_{C} A_μ dx^μ}. \tag{1} $$

The loop itself and products of them provide a basis of gluonic gauge invariant operators.

The Wilson loop along a curve $C$ is computed in the dual string theory by calculating the action of a string bounded by $C$ at the boundary of the $AdS$ space. More concretely$^1$,

$$ \langle W(C) \rangle = \int_{∂F(C)} DF e^{-S[F]} \tag{2} $$

where $F$ denotes all the fields of the string theory and $∂F$ their boundary values. A good approximation to this path integral is by steepest descent. The Wilson loop is then related to the area of the minimal surface bounded by the curve $C$, spanned by classical string configurations (with Nambu-Goto action $S_{NG}$) that explore the bulk of $AdS$.

All of this was first proposed ten years ago in [4]. In the meantime, this proposal motivated lots of developments, see [5] for beautiful and influential papers on this line. See also [6] for a review. In particular, the ideas and techniques of the original $AdS/CFT$ correspondence have been assumed to generalize to a large class of systems, and have been used to relate field theories that are not conformal (and hence more closely related to phenomenological applications) with backgrounds that are not $AdS$, extending the framework to what is more appropriately referred to as gauge-string duality.

B. Confinement and Screening

In its original definition [3], the Wilson loop computes the phase factor associated to a closed trajectory for a very massive quark in the fundamental representation (it can also be generalized to other representations). The quark-antiquark (static) potential can be read from the VEV of the Wilson loop. Choosing a rectangular loop of sides

$^1$ The fundamental string is actually dual to the generalized Wilson loop, containing a term that couples the coordinates of the internal space with the scalars of the brane. This is due to the fact that the string ending on the brane is source of the electric field, generating $A_μ$ but also of the scalars on the brane, from which is pulling. See [4] for a clear discussion of this.
\( L_{QQ}, T \), in the first approximation for large times \( T \to \infty \),

\[
\langle W(C) \rangle \sim e^{-E_{QQ}T},
\]

where \( E_{QQ} \) is the quark-pair energy. In the limit of large \( 't \) Hooft coupling, the steepest descent approximation mentioned above yields the identification

\[
\langle W(C) \rangle \sim e^{-E_{QQ}T} \sim e^{-S_{NG}}.
\]

The description of the Wilson loop in QCD in terms of a string partition function is not new. Well before its modern formulation, the ideas behind gauge/string correspondence have been used to show that the potential of a quark-antiquark pair separated by a distance \( L_{QQ} \) gets a correction of the form \( \frac{c}{L_{QQ}} \) due to quantum fluctuations of the Nambu-Goto action [7].

The definition of confinement we will adopt is the following. Consider a \( SU(N_c) \) gauge theory with matter fields in generic representations. We decouple (make infinitely massive) all fields with non-zero N-ality, and then introduce a single particle-antiparticle pair of non-dynamical fields with non-zero N-ality as a test probe for the system (effectively, the pair dynamics is quenched). We then compute the work needed to separate the particle-antiparticle test pair up to a distance \( L_{QQ} \). If the work approaches \( E_{QQ} \approx \sigma L_{QQ} \) for large separations, the theory is confining\(^2\). The quantity \( \sigma \) is a representation dependent constant, the string tension.

C. Walking technicolor

Walking technicolor [8] is a framework within which the phenomenological difficulties of dynamical electro-weak symmetry breaking might find a very natural and elegant dynamical solution, thanks to the fact that, in contrast to QCD-like theories, the guidelines provided by naive dimensional analysis are violated. This is because of large anomalous dimensions controlling the dynamics over a large regime of energies. The word \textit{walking} refers to the fact that the new dynamics is strongly coupled over a large range of energies, where its fundamental coupling exhibits a \( \beta \)-function which is anomalously small in respect to the coupling itself. A behavior of this type is expected in theories which flow onto strongly-coupled IR fixed-points, and it is reasonable to assume that it persists also when such IR fixed points are only approximate, though in this case a degree of ambiguity as to the meaning of \textit{approximate} invites some caution.

Besides being affected by the usual calculability limitations due to the strong coupling (as in QCD and in QCD-like technicolor), the walking dynamics itself makes this framework very hard to work with. New, non-perturbative instrument are needed in order to understand the (effective) field theory properties of a walking theory. Very recent years saw a lot of progress towards a better understanding of walking dynamics both from the lattice [9] and thanks to ideas borrowed from the gauge-string duality. See for instance [10] for a list of references focused on the precision electro-weak parameters.

In this paper, we will use the word \textit{walking} to refer to backgrounds in which there exists a interval in the \( \rho \) radial direction over which the geometry is determined by some coupling that shows an evolution with the radial coordinate that is anomalously slow. This agrees with the standard definition of walking, where the beta-function of the fundamental gauge-coupling is, over some range of energy, anomalously small in comparison with what expected from the strength of the actual coupling itself. Not all couplings show this behavior: this also agrees with standard definitions, in the sense that relevant operators must be present in order to drive the RG flow away from a possible IR fixed point, and scale invariance is present only in the sense that in the walking region the relevant couplings are so small that their effect can be neglected. However, our definition is significantly less restrictive that the usual one: we do not require the presence of an actual fixed point in the flow, and correspondingly we do not have an approximately AdS background, nor can we recover an AdS background by dialing some parameter to some special value.

D. General idea: string-theory as a laboratory for walking dynamics.

Motivated by the difficulties described in the previous subsection, in ref. [11] a more general program is proposed, based upon gauge/string correspondence in order to go beyond the low-energy effective field theory description. The

\(^2\) Another equivalent way of defining confinement is by computing the VEV of the Polyakov loop, that if vanishing indicates a confining theory. Also, the perimeter law of a \('t\) Hooft loop indicates confinement.
The proposal is to study the dynamics of theories that yield walking behavior, but that are not necessarily related to the electro-weak symmetry. In short, one would like to use string theory as a laboratory in which to study the general properties of walking dynamics by itself, in isolation from its complicated realization within an explicit model of dynamical electro-weak symmetry breaking. In ref. [11], it is shown that in the context of Type IIB string theory on a background generated by a stack of $N_c$ D5 branes, there exists a very large class of solutions to the background equations for which a suitably defined gauge coupling exhibits the basic properties of a putative walking theory. The running of the gauge coupling flattens over a large range of intermediate energies, but restarts at low energies, until the space ends into a (good) singularity in the deep IR, so that no exact IR fixed point exists. The fact that this is not a walking technicolor theory (there is no electro-weak symmetry in the set-up, and hence no mass generation in the usual sense), together with the large-$N_c$ expansion, yields the advantage that we avoid the complications due to mixing of weakly-coupled and strongly-coupled properties of the theory. For instance, the spectrum of the spin-0 sector of the theory can be studied, and has been studied [12], yielding remarkable surprises.

In this paper we take a further step in this direction, by studying the behavior of the Wilson loop in backgrounds of this class. As we will explain, we can use the techniques developed in the context of gauge-string duality, by studying the background with a probe string. In particular, we will study Wilson loops in the dual walking QFT. For technical reasons that will be explained in the body of the paper, in order for this program to be carried out we will also need to generalize further the class of backgrounds in [11]. These new solutions have been already introduced in [12]. We explain here in deeper detail how to generate them, characterize them, and relate them to the literature.

E. Outline

The paper is organized as follows: we set up notation and introduce a set of important ideas in section II revising the bibliography and adding important new ingredients and derivations. Then we apply these to well-established examples of gauge-string duality in section III providing a simple and compact set of exercises that are intended to yield some guidance in the following sections, in which the dynamics is far from well-understood. Section IV presents our new walking solution and a study of the dynamics of the Wilson loop as a function of the length of the walking region. Section V studies the results derived in section II when applied to background that encode the dynamics of fundamental fields. We then conclude in section VI.

II. GENERAL THEORY

In this section we present general results for Wilson loops, computed using the ideas of [4]. Some of the results here have been derived long ago (see for example [14]), but our approach will be different and some new and useful points will be specially emphasized.

We study the action for a string in a background of the generic form

$$ds^2 = -g_{tt}dt^2 + g_{xx}d\vec{x}^2 + g_{\rho\rho}d\rho^2 + g_{ij}d\theta^i d\theta^j.$$  

We assume that the functions $(g_{tt}, g_{xx}, g_{\rho\rho})$ depend only on the radial coordinate $\rho$. By contrast, $g_{ij}$ for the internal (typically compact) space can also depend on other coordinates. Whatever are the internal coordinates, they will play no role in what follows. This is because we will choose a configuration for a probe string that is not excited on the $\theta^i$ directions, hence in what follows, we will ignore the internal space.\footnote{Strings or other objects that extend in the internal space filling part of it can be treated as an effective string, analogous to the one we are studying. If these objects are allowed to vibrate in the internal space, then a generalization of the present treatment should be done.}

A. Equations of motion

The configuration we choose is,

$$t = \tau, \quad x = x(\sigma), \quad \rho = \rho(\sigma).$$

and compute the Nambu-Goto action

$$S = \frac{1}{2\pi\alpha'} \int_{[0,T]} d\tau \int_{[0,2\pi]} d\sigma \sqrt{-\det G_{\alpha\beta}}.$$
The induced metric on the 2-d world-volume is 
\[ G_{\alpha\beta} = g_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu, \]
where
\[ G_{\tau\tau} = -g_{tt}, \quad G_{\sigma\sigma} = g_{xx}(dx/d\sigma)^2 + g_{\rho\rho}(d\rho/d\sigma)^2. \] (8)

Defining for convenience \( f(\rho)^2 = g_{tt}g_{xx}, \) \( g(\rho)^2 = g_{tt}g_{\rho\rho}, \) the Nambu-Goto action is
\[ S = \frac{T}{2\pi\alpha'} \int_0^{2\pi} d\sigma \sqrt{f^2 x'(\sigma)^2 + g^2 \rho'(\sigma)^2} = \frac{T}{2\pi\alpha'} \int_0^{2\pi} d\sigma L. \] (9)

Notice that we consider the situation in which the string does not couple to a NS B-field.

We first compute the Euler-Lagrange equations from Eq. (7) and then we specify them for the ansatz in Eq. (6).

We get that for the \((t, x, \rho)\) coordinates the eqs. of motion read, respectively,
\[
\begin{align*}
\partial_\tau \left[ \frac{1}{L} (f^2 x'^2 + g^2 \rho'^2) \right] & = 0, \\
\partial_\sigma \left[ \frac{1}{L} f^2 x' \right] & = 0, \\
\partial_\sigma \left[ \frac{1}{L} g^2 \rho' \right] & = \frac{1}{L} (x'^2 f f' + \rho'^2 g g'),
\end{align*}
\] (10)

where \( X' = \frac{dX(\sigma)}{d\sigma} \) for any function \( X. \)

The first equation in (10) is solved because we assume a background metric independent of time (we consider the system at the equilibrium). The second equation in (10) is solved if the quantity inside brackets is a constant (that we call \( C \)), which implies that,
\[ \frac{d\rho}{d\sigma} = \pm \frac{dx}{d\sigma} V_{eff}(\rho) \leftrightarrow \frac{d\rho}{dx} = \pm V_{eff}(\rho). \] (11)

One can check that the third equation in (10) is solved if Eq. (11) is imposed. Then, we need to work with just one equation. Defining
\[ V_{eff}(\rho) = \frac{f(\rho)}{C g(\rho)} \sqrt{f^2(\rho) - C^2}, \] (12)
we write it as
\[
\begin{align*}
\frac{d\rho}{d\sigma} = \pm \frac{dx}{d\sigma} V_{eff}(\rho) \leftrightarrow \frac{d\rho}{dx} = \pm V_{eff}(\rho).
\end{align*}
\] (13)

Another way to arrive to the last version of Eq. (13) is to consider a restricted ansatz for the string configuration \([t = \tau, \quad x = \sigma, \quad \rho = \rho(\sigma)]\) and use the conserved Hamiltonian derived from Eq. (9) to get an expression for \( \rho(\sigma) \) that is precisely Eq. (13).

The kind of solution we are interested in can be depicted as follows: a string that hangs from infinite radial position at \( x = 0 \) and drops down towards smaller \( \rho \) as \( x \) increases. Once it arrives at the smallest \( \rho \) compatible with the solution, namely \( \rho_0 \), it starts growing in the radial direction up to infinite \( \rho \) where \( x = L_{QQ}/2 \), see Fig. (1).4 This means that in the two distinct regions \( x < L_{QQ}/2 \) and \( x > L_{QQ}/2 \) the equations of motion will differ only in a sign
\[
\begin{align*}
x < \frac{L_{QQ}}{2} & \quad \frac{dp}{dx} = -V_{eff}(\rho) \\
x > \frac{L_{QQ}}{2} & \quad \frac{dp}{dx} = V_{eff}(\rho)
\end{align*}
\] (14)

We can now formally integrate the equations of motion
\[ x(\rho) = \begin{cases} 
L_{QQ} - \int_{\rho}^{\rho_0} \frac{dp}{V_{eff}(\tau)} & x < \frac{L_{QQ}}{2} \\
\int_{\rho_0}^{\rho} \frac{dp}{V_{eff}(\tau)} & x > \frac{L_{QQ}}{2}
\end{cases} \] (15)

---

4 For future reference we refer to the lowest end of the radial coordinate as \( \hat{\rho}_0 \).
or more compactly

$$\left| x(\rho) - \frac{L_{QQ}}{2} \right| = \int_{\rho_0}^{\rho} \frac{dr}{V_{\text{eff}}(r)}.$$  \hspace{1cm} (16)

In what follows we will use only one of the solutions in Eq. (14) unless explicitly noted.

1. **Boundary conditions**

We need to specify the boundary conditions for the string in Eq. (6). This is an open string, vibrating in the bulk of a closed string background. Following the ideas of [4], we add a D-brane at a very large radial distance where the open string will end. This string will then satisfy a Dirichlet boundary condition at \( \rho \to \infty \). This means that for large values of the radial coordinate \( dx/d\sigma \) must vanish. The only way of satisfying the equation of motion in Eq. (11) for \( \rho \to \infty \), given that the left hand side has to be non vanishing, is to have a divergent \( V_{\text{eff}}(\rho) \):

$$\lim_{\rho \to \infty} V_{\text{eff}}(\rho) = \infty.$$  \hspace{1cm} (17)

This implies that there are restrictions on the asymptotic behavior of the background functions \([f(\rho), g(\rho)]\) in order for the string proposed in Eq. (6) to exist. We will come back to this in the following sections. Before proceeding, a brief digression is needed. When studying Eqs. (13)-(17) the reader may find unsatisfactory that the restriction we have proposed above, namely

$$\left. \frac{d\rho}{dx} \right|_{\rho \to \infty} = V_{\text{eff}} \big|_{\rho \to \infty} \to \infty$$  \hspace{1cm} (18)

looks dependent of our choice of the radial coordinate. This is actually not the case, because we could rewrite this restriction in a more covariant form in the following way. We define a couple of vectors \(^5\) in the \( x, \rho \) directions,

\[ \vec{v}^x = \frac{dx}{d\sigma} \partial_x, \quad \vec{v}^\rho = \frac{d\rho}{d\sigma} \partial_\rho, \]  \hspace{1cm} (19)

\(^5\) We thank Johannes Schmude for the discussions that lead to this.
compute the ratio $\mu$ of their norms and impose that this is divergent on the surface $D$ on which the string has to satisfy the Dirichlet condition:

$$\mu = \frac{g_{\rho\rho} \left( \frac{d\rho}{d\sigma} \right)^2}{g_{xx} \left( \frac{dx}{d\sigma} \right)^2} \bigg|_D \to \infty. \quad (20)$$

Combining this with Eq. (18), and evaluating at the boundary

$$\mu = \frac{g_{\rho\rho} V_{eff}^2}{g_{xx}} \bigg|_D = \frac{f^2(D) - C^2}{C^2} \to \infty. \quad (21)$$

This last expression is free of coordinate ambiguities, as it comes from operating with invariants (norms of vectors). It is however easier to work within a specific choice of coordinates, which we will do in the body of the paper. As we will see explicitly, this occurs in the examples we will study below.

2. Turning points

Once Eq. (17) is satisfied the string will move to smaller values of the radial coordinate down to a turning point $\rho_0$ where the quantity $\frac{d\rho}{dx}(\rho_0) = 0$, i.e. $V_{eff} = 0$. In principle, there could be points where $V_{eff}$ vanishes because of either isolated zeros of $f(\rho)$, or diverging point of $g(\rho)$. However, we will not consider this kind of inversion points, since we are interested in solutions of the equations of motion that allow the string to probe the entire radial direction. Hence the turning point can be placed in any possible $\rho_0$, with $\hat{\rho}_0 < \rho_0 < \infty$ (where $\hat{\rho}_0$ is the end of the space). Thus we will restrict ourselves to forms of $V_{eff}$ where the inversion point is given by imposing $C = f(\rho_0)$. Furthermore, in the next section we will also impose that the envelop of the string is convex in a neighbourhood of $\rho_0$, hence insuring that gauge theory quantities like the separation between the pair of quarks and its energy will be continuous functions of $\rho_0$. It is then clear from Eq. (13) that $V_{eff}(\rho)$ controls not only the boundary condition at infinity, but also the possibility for the string to turn around and come back to the brane at infinity.

B. Energy and Separation of the $Q\bar{Q}$ pair.

We now follow the standard treatment for Wilson loops summarized in [6]. If our probe-string hangs from infinity, turns around at a point $\rho_0$ as described above and goes back to the $D$-brane at infinity, we can then compute gauge theory quantities, like the separation between the two ends of the string, which can be thought of as the separation between a quark-antiquark pair living on the $D$-brane and coupled to the end-points of the string. And we can compute the Energy of the pair of quarks, that we associate with the length of the string (computed along its path in the bulk). Both of these quantities will be functions of the turning point $\rho_0$.

The standard expressions that we will use can be derived easily. Indeed, for the $Q\bar{Q}$ separation we only need to compute $\int dx$. To calculate the energy of the $Q\bar{Q}$ pair, we compute the action of the string and subtract the action of two ‘rods’ that would fall from infinity to the end of the space $^6$. The results are [6],

$$L_{QQ}(\rho_0) = 2 f(\rho_0) \int_{\rho_0}^{\infty} \frac{g(z)}{f(z)} \frac{dz}{\sqrt{f^2(z) - f^2(\rho_0)}},$$

$$E_{QQ}(\rho_0) = f(\rho_0) L_{QQ}(\rho_0) + 2 \int_{\rho_0}^{\infty} \frac{g(z)}{f(z)} \sqrt{f^2(z) - f^2(\rho_0)} dz - 2 \int_{\hat{\rho}_0}^{\infty} g(z) dz. \quad (22)$$

As discussed above, the constant $C$ defined around Eq. (11) must be taken to be $C = f(\rho_0)$. Using Eq. (12) we can rewrite

$$L_{QQ}(\rho_0) = 2 \int_{\rho_0}^{\infty} \frac{dz}{V_{eff}(z)} \quad (23)$$

$^6$ Notice that these ‘rods’ need not be strings, it is just a way of renormalizing the infinite mass of the quarks.
As discussed in section [II.A.1] we must impose that for large values of the radial coordinate \( r_{\text{eff}} \) diverges. This, however, is not enough to ensure that the integral above receives a finite contribution from the upper end of the integral, we then have to require that for large radial coordinate \( r_{\text{eff}} \) diverges at least as

\[
V_{\text{eff}} \sim r^{-\beta},
\]

with \( \beta > 1 \). Then, the quantity \( L_{QQ} \) can be finite or infinite depending on the IR \( (r \to r_0 \equiv \hat{r}_0) \) asymptotics of the background, meaning that we consider the turning point at the end of the space at the lower end of the integral. If, expanding around the turning point \( r_0 \), we have

\[
V_{\text{eff}} \sim (r - r_0)^\gamma,
\]

then the separation of the quark pair is infinite for \( \gamma \leq 1 \) and finite for \( \gamma < 1 \). We will see examples of both behaviors in the following sections.

Another reasonable condition that we could impose is that \( r_0 \) is the first zero of \( V_{\text{eff}} \) for which the string has positive convexity (a minimum). This can be easily expressed as a condition on the background functions. In fact, let us assume that near \( r_0 \) the effective potential in Eq. (12) behaves as \( V_{\text{eff}} = \kappa (r - r_0)^\gamma \). Then, in a neighbourhood of \( r_0 \), we have

\[
\frac{dr}{dx} = \pm V_{\text{eff}} = \pm (\kappa (r - r_0)^\gamma) + O((r - r_0)^{\gamma + 1})
\]

\[
\frac{d^2r}{dx^2} = \frac{d}{dx} \left( \frac{dr}{dx} \right) = \pm \frac{d^2V_{\text{eff}}(r)}{dr^2} \frac{dr}{dx} = V_{\text{eff}}(r) \frac{dV_{\text{eff}}(r)}{dr} = \kappa^2 (r - r_0)^{2\gamma - 1} + O((r - r_0)^{2\gamma}),
\]

by iteration and induction we obtain (we choose only one of the branches of Eq. (14)),

\[
\frac{d^n r}{dx^n} = (V_{\text{eff}}(r) \frac{d}{dr})^n r = \kappa^n \Pi_{j=0}^{n-1} [j(\gamma - 1) + 1] (r - r_0)^{(\gamma - 1)n + 1} + O((r - r_0)^{(\gamma - 1)n + 2}).
\]

In order to have a minimum we impose that the first non vanishing derivative at \( r = r_0 \) is an even derivative (its value has to be positive). So, for even \( n \) we need (this result will not depend on the choice of branches above),

\[
\gamma = 1 - \frac{1}{n},
\]

from this it follows that \( \gamma \) cannot be bigger than one. This means that the only possibility of obtaining a string that can be stretched up to infinite distance will come from the case \( \gamma = 1 \) (contrast this with the result of [14]). The solutions with infinite length will have all even derivatives vanishing at \( r_0 \). Had we considered non-integer corrections to the leading term in Eq. (20), so that near \( r_0 \) the function

\[
V_{\text{eff}} = \kappa_1 (r - r_0)^\gamma + \kappa_2 (r - r_0)^{\gamma + \epsilon}
\]

for very small values of \( \epsilon \) we would have had a formula like the one in Eq. (27) where the exponent is the same, but the coefficient changes. For large values of \( \epsilon \), we have, to leading order, the same result as in Eq. (27). The reasoning given above applies.

These constraints ensure that there exists a unique trajectory for the string as function of \( r_0 \). It should be also noticed that the profile of the string is not analytic function and in particular in \( r_0 \) it turns out to be a \( C^2 \) function.

1. Some exact results.

Next, let us derive an expression of the energy \( E_{QQ} \) as function of the inter-quark separation \( L_{QQ} \). For this purpose, it is useful to introduce the function \( K[x] = \sqrt{x^{-1}} \). The expression for \( L_{QQ}(r_0) \) in Eq. (22) reads

\[
L_{QQ}(r_0) = \lim_{r_1 \to \infty} 2 \int_{r_0}^{r_1} \frac{g(z)}{f(z)} K \left[ \frac{f(z)}{f(r_0)} \right] dz.
\]

Some of these expressions have been derived in past collaborations with Angel Paredes.
Performing the derivative
\[
\frac{dL_{QQ}}{d\rho_0} = -2 \frac{g(\rho_0)}{f(\rho_0)} K[1] + \lim_{\rho_1 \to \infty} \frac{\rho_1 g(z)}{f(z)} \partial_{\rho_0} K \left[ \frac{f(z)}{f(\rho_0)} \right]
\]
(31)
Now, we use the following identity,
\[
\partial_{\rho_0} K \left[ \frac{f(z)}{f(\rho_0)} \right] = -\partial_z K \left[ \frac{f(z)}{f(\rho_0)} \right] \left( \frac{f(z)}{f(\rho_0)} \right) \frac{f'(\rho_0)}{f'(z)}
\]
(32)
integrating by parts and after some algebra, we get
\[
\frac{dL_{QQ}}{d\rho_0} = -2 \lim_{\rho_1 \to \infty} \partial_{\rho_0} \log f(\rho_0) \left\{ \frac{g(\rho_1)}{f'(\rho_1)} K \left[ \frac{f(\rho_1)}{f(\rho_0)} \right] - \int_{\rho_0}^{\rho_1} dz \ K \left[ \frac{f(z)}{f(\rho_0)} \right] \partial_z \left( \frac{g(z)}{f(z)} \right) \right\},
\]
(33)
or
\[
\frac{dL_{QQ}}{d\rho_0} = 2 \partial_{\rho_0} \log f(\rho_0) \lim_{\rho_1 \to \infty} \left\{ \int_{\rho_0}^{\rho_1} dz \ K \left[ \frac{f(z)}{f(\rho_0)} \right] \partial_z \left( \frac{g(z)}{f(z)} \right) - \frac{f(\rho_1)}{f'(\rho_1)V_{eff}(\rho_1)} \right\}.
\]
(34)
Similarly, we compute the derivative for the energy function finding that
\[
\frac{dE_{QQ}}{d\rho_0} = f(\rho_0) \frac{dL_{QQ}}{d\rho_0},
\]
and we get
\[
\frac{dE_{QQ}}{dL_{QQ}} = f(\rho_0).
\]
(36)
Notice that (after integration) Eq. (36) is an exact expression for $E_{QQ}$ in terms of $L_{QQ}$, the non triviality residing in the function $\rho_0(L_{QQ})$, this expression was also derived in [13]. Also, Eq. (36) yields the force between the quark pair. It should be stressed that we are assuming that we will be able to invert $\rho_0$ as function of $L_{QQ}$. Whenever this cannot be done, the solution would be valid only in the regions of monotonicity of $L_{QQ}(\rho_0)$.

Let us comment briefly about the existence of cusps in our strings. If near the lower point of the string (typically situated in the end of the geometry) the quantity $V_{eff}(\rho)$ diverges, then there will be a cusp in the shape of the string. This can be easily understood using Eq. (13) and an expansion of $V_{eff}(\rho)$ of the form discussed in Eq. (16).

If $V_{eff}(\rho) \propto (\rho - \rho_0)^{\gamma}$ with $\gamma < 0$, one can integrate the equation
\[
\rho - \rho_0 \propto \left[ (1 - \gamma) x - \frac{L_{QQ}}{2} \right]^{\frac{1}{1-\gamma}},
\]
(37)
which implies that the shape of the string $\rho(x)$ is not analytic at $\frac{L_{QQ}}{2}$.

C. Leading and subleading behaviors. Inversion points

Let us focus on the case that, near the end of the space (in the IR), $V_{eff} \sim (\rho - \rho_0)$ (the string stretches up to infinite length $L_{QQ} (\rho_0) \to \infty$). The subleading corrections to the Energy of the quark pair [14] read
\[
E_{QQ} = f(\rho_0)L_{QQ} + \kappa + O(\epsilon^{\alpha L_{QQ}}) (\log L_{QQ})^b.
\]
(38)
This formula can be obtained as an expansion around $\rho_0 \to \rho_0$, or equivalently expanding Eq. (36) around $L_{QQ} \to \infty$. Notice that no power-law corrections or Luscher-terms appear in these corrections: they would appear if $\gamma > 1$ in Eq. (28), something that we ruled out. Contrast this with the result of [14]. Because Eq. (25) and the discussion around it rely only on the generic properties of $V_{eff}$, such power-law corrections can only descend from higher-derivative corrections to Eq. (4), which cannot be repackaged into the form of Eq. (13).

In the following sections, we will discuss these subleading behaviors in various examples and precisely find the coefficients that apply to each particular case. Before proceeding, a few comments are due in order to avoid ambiguities. The partition function of the string is studied by using two different expansions, in $\alpha'$ and $g_s$. First, the Nambu-Goto action is characterized by the string tension, and one can think of expanding any physical quantity (correlation
function) in powers of $\alpha'$. This procedure is effectively equivalent to quantizing the (particle-like) excitations of the string, and in this sense $\alpha'$ corrections can be associated with loops of the dynamics of the string modes. One can also rephrase the results in terms of a classical effective theory, in which the $\alpha'$ corrections are encoded in higher derivative corrections to the Nambu-Goto classical action. By doing so, one can associate the resulting $E_{\text{QQ}}$ to the expectation value of a Wilson loop in the dual theory, and hence interpret the generalization of Eq. (38) in terms of the actual quantum corrections to the quark-antiquark potential at large distance, for a confining string, as done for example in \cite{15}. As explained above, this procedure yields power-law corrections to the leading-order result $E_{\text{QQ}} \propto L_{\text{QQ}}$. This is not what we are doing in this paper. All the results we obtain are at the leading-order in $\alpha'$, the calculations being completely classical and based on the Nambu-Goto action. In particular, this explains why we do not find a Luscher term.

We are going to truncate at the leading-order also the expansion in $g_s$, which controls processes where the string breaks. This is also done in \cite{15}, where the quantum corrections that are computed are the ones in $\alpha'$, but the whole analysis treats the string itself as effectively free. Not only quantum corrections related to the $g_s$ expansion, but the whole many-body nature of string-theory is neglected in this way. This is a very important limitation: strings that can stretch up to infinite separation for the quark pair can be treated in this way, while theories in which fragmentation and hadronization take place via string-breaking are accessible to our approach only up to a scale smaller than the scale of breaking itself (real world QCD, if it admits a string dual description, should fall into this class in which $\alpha'$ corrections must be included and actually dominate the dynamics from the scale of breaking and below).

We will see in the following examples in which Eq. (25) holds with $\gamma < 1$. This will yield a finite value for the maximal displacement in the Minkowski directions between the quark-antiquark pair. Studying the large-$L_{\text{QQ}}$ limit would necessarily require the inclusion both of quantum effects in the $\alpha'$ expansion, but also the effect of string breaking, which might be decoupled in the large quark-mass limit.

Another important comment is the following: if the function $L_{\text{QQ}}(\rho_0)$ is not monotonic, then it is not invertible, and $\rho_0(L_{\text{QQ}})$ is multivalued. If this is the case, it will happen that for some given $L_{\text{QQ}}$ we can find that also $E_{\text{QQ}}(L_{\text{QQ}})$ is multivalued. Among all the possible values of $E_{\text{QQ}}$ for a given $L_{\text{QQ}}$ (all of which satisfy the equations of motion) we will refer to the lowest one as the stable solution, while the others will represent either excited or unstable states. The Van der Waals liquid-gas system provides a nice realization of these excited and unstable states, and we report its treatment in Appendix \cite{B}.

The existence of inversion points (or extrema of $L_{\text{QQ}}(\rho_0)$) implies that the first derivative $\frac{dL_{\text{QQ}}}{d\rho_0}$ vanishes. Using the expression in Eq. (34),

$$\lim_{\rho_1 \to \infty} \left\{ \int_{\rho_0}^{\rho_1} dz \ K \left[ \frac{f(z)}{f'(|\rho_0|)} \right] \frac{\partial_z \left[ g(z) \right]}{f'(z)} - \frac{f(\rho_1)}{f'(\rho_1)|V_{eff}(\rho_1)|} \right\} = 0. \quad (39)$$

If it happens that $\frac{f(\rho_1)}{f'(|\rho_0|)|V_{eff}(\rho_1)|} \to 0$ for large values of $\rho_1$, then a necessary condition for the existence of turning points is that the integral in Eq. (34) vanishes, or more simply that the integrand changes sign. Notice, nevertheless, that this is not a criterion of practical application except for some particular easy backgrounds as we will see in section \cite{V}.

To close this section let us stress that given the constraints we have specified along this section, the shape of the string that we are proposing as solution is the only allowed shape, since there can be only one minimum of $\rho(x)$.

\section{III. SOME WELL-UNDERSTOOD EXAMPLES}

In this section we illustrate the points of the previous section in a set of very well known examples. All of them are string-theory backgrounds that are simple enough that the dynamics of the probe string can be treated analytically. We will write the 10-dimensional background in string frame.

\subsection{A. The case of $\text{AdS}_5 \times S^5$}

This is certainly the main example, where the conjecture was originally proposed \cite{1} and the ideas for Wilson loops described in the introduction first developed \cite{4}. After a rescaling of the radial coordinate, the metric reads,

$$ds^2 = \alpha' \left[ \frac{\rho^2}{R^2} dx_{1,3}^2 + \frac{R^2}{\rho^2} d\rho^2 + R^2 d\Omega_5^2 \right], \quad (40)$$
where \( R^2 = \sqrt{4\pi g_s N_c} \) is dimensionless, \( \rho \) has dimensions of inverse-length, and the constant \( \alpha' \) in front of the metric ensures that when we compute the functions \( f, g \) defined below Eq. (8) and plug into Eq. (7) the factors of \( \alpha' \) will cancel (this will happen in the examples discussed below also). Hence we have,

\[
\begin{align*}
\left\{ 
\begin{array}{l}
g(\rho)^2 = \alpha'^2 \\
f(\rho)^2 = \alpha'^2 \frac{\rho^4}{R^2} \\
C^2 = f(\rho_0)^2 = \alpha'^2 \frac{\rho_0^3}{R^2}
\end{array}
\right. \\
V_{\text{eff}} = \frac{\rho^2}{\rho_0^2 R^2} \sqrt{\rho^4 - \rho_0^4}
\end{align*}
\]  

(41)

One can check that the functions of the background respect all the constraints we imposed in section II regarding the boundary conditions and convexity. We can exactly integrate \( L_{QQ}(\rho_0) \)

\[
L_{QQ}(\rho_0) = 2 \int_{\rho_0}^{\infty} d\rho \frac{R^2 \rho_0^2}{\rho^2} = (2\pi)^{\frac{3}{2}} \frac{R^2}{\rho_0 \Gamma(\frac{1}{2})^2}
\]

(42)

Since it is possible to invert the relation we can then write \( \rho_0(L_{QQ}) \), and using Eq. (46)

\[
dE_{QQ} \ dn_{QQ} = \frac{(2\pi)^3 R^2}{L_{QQ}^2 \Gamma(\frac{1}{2})^2} \Rightarrow E_{QQ}(L_{QQ}) = -\frac{(2\pi)^3 R^2}{L_{QQ}^2 \Gamma(\frac{1}{2})^2}
\]

(43)

in agreement with the result of [4]. Of course, the separation for the quark pair diverges for \( \rho_0 \to 0 \) (the end of the space). Also notice that in this case, the expression of \( L_{QQ}(\rho_0) \) is invertible. There are no corrections to Eq. (43).

A few words of comment might be useful to a reader who is not familiar with this set-up. The Yang-Mills coupling

\[
\alpha \equiv \frac{g^2}{4\pi}
\]

\[\equiv \frac{1}{2Y M^2(2\pi)^2}
\]

is the coordinate on the circle,

\[\equiv \frac{1}{4\pi g_Y \sqrt{\alpha'} N_c}
\]

\[\equiv \frac{1}{\rho_0 \sqrt{\alpha' \Lambda^3}}
\]

is dimensionless, so that \( \rho \) has dimensions of inverse-length, as in the previous subsection, and so does \( \Lambda \). The relevant functions are,

\[
\left\{ 
\begin{array}{l}
f(\rho)^2 = \alpha'^2 \frac{\rho^4}{R^2} \\
g(\rho) = \alpha'^2 \frac{\rho^3}{R^2 \Lambda} \\
C^2 = f(\rho_0)^2 = \alpha'^2 \frac{\rho_0^3}{R^2}
\end{array}
\right. \\
V_{\text{eff}} = \sqrt{\frac{(\rho^3 - \Lambda^3)(\rho^3 - \rho_0^3)}{(\rho_0 R)^3}}
\]

(45)

hence,

\[
L_{QQ} = 2(\rho_0 R)^{3/2} \int_{\rho_0}^{\infty} d\rho \frac{1}{\sqrt{(\rho^3 - \rho_0^3)(\rho^3 - \Lambda^3)}}
\]

(46)

One can explicitly perform the integral above, but the result, in terms of special functions, is not very illuminating.

To get a better handle on the underlying dynamics, we consider the case in which \( \rho_0 = \Lambda \). In this case, we get

\[
L_{QQ}(\rho_0) = -\frac{2\sqrt{R^3 \rho_0^5}}{6\rho_0^4} \sqrt{\frac{12}{\pi}} \arctan \left( \frac{2\rho + \rho_0}{\sqrt{3}\rho_0} \right) + \log \left( \frac{\rho^2 + \rho \rho_0 + \rho_0^2}{(\rho - \rho_0)^2} \right)
\]

(47)

that we can see diverges logarithmically for \( \rho = \rho_0 \). So, the string here again, has infinite length as is expected in a dual of a confining field theory. If \( \rho_0 > \Lambda \) then the separation of the pair is computed from the integral of Eq. (46) and turns out to be finite.

B. Witten-Sakai-Sugimoto Model

This model, based on D4 branes wrapped on a circle with SUSY breaking periodicity conditions [16] received a great deal of attention thanks to the observation by Sakai and Sugimoto [17] that the introduction of \( N_f \ll N_c \) flavor D8 branes as probes allows to construct a model where a peculiar realization of chiral symmetry is spontaneously broken. The metric reads

\[
\frac{ds^2}{\alpha'} = \left( \frac{\rho}{\bar{\rho}} \right)^{3/2} \left[ dx_{1,3}^2 + \hat{F}(\rho) dx_4^2 \right] + \left( \frac{R}{\rho} \right)^{3/2} \left[ \frac{d\rho^2}{\hat{F}(\rho)} + \rho^2 d\Omega_4^2 \right]
\]

(44)

where \( x_4 \) is the coordinate on the circle, \( R^3 = \pi g_s \sqrt{\alpha'} N_c \), and \( \hat{F}(\rho) = \frac{\rho^3}{\rho^3 - \Lambda^3} \). Notice that in this case the gauge coupling \( g_Y^2 M^2 = 2\pi g_Y^2 \sqrt{\alpha'} \) is dimensional, so that \( \rho \) has dimensions of inverse-length, as in the previous subsection, and so does \( \Lambda \). The relevant functions are,

\[
\left\{ 
\begin{array}{l}
f'(\rho) = \alpha'^2 \frac{\rho^3}{R^2} \\
g'(\rho) = \alpha'^2 \frac{\rho^2}{R^2 \Lambda} \\
C^2 = f(\rho_0)^2 = \alpha'^2 \frac{\rho_0^3}{R^2}
\end{array}
\right. \\
V_{\text{eff}} = \sqrt{\frac{(\rho^3 - \Lambda^3)(\rho^3 - \rho_0^3)}{(\rho_0 R)^3}}
\]

(45)

hence,

\[
L_{QQ} = 2(\rho_0 R)^{3/2} \int_{\rho_0}^{\infty} d\rho \frac{1}{\sqrt{(\rho^3 - \rho_0^3)(\rho^3 - \Lambda^3)}}
\]

(46)

One can explicitly perform the integral above, but the result, in terms of special functions, is not very illuminating. To get a better handle on the underlying dynamics, we consider the case in which \( \rho_0 = \Lambda \). In this case, we get

\[
L_{QQ}(\rho_0) = -\frac{2\sqrt{R^3 \rho_0^5}}{6\rho_0^4} \sqrt{\frac{12}{\pi}} \arctan \left( \frac{2\rho + \rho_0}{\sqrt{3}\rho_0} \right) + \log \left( \frac{\rho^2 + \rho \rho_0 + \rho_0^2}{(\rho - \rho_0)^2} \right)
\]

(47)

that we can see diverges logarithmically for \( \rho = \rho_0 \). So, the string here again, has infinite length as is expected in a dual of a confining field theory. If \( \rho_0 > \Lambda \) then the separation of the pair is computed from the integral of Eq. (46) and turns out to be finite.
On the other hand, it is possible to iteratively invert the relation between $L_{QQ}$ and $\rho_0$ for $\rho_0 \sim \Lambda$ and hence for $L_{QQ} \rightarrow \infty$

$$\rho_0(L_{QQ}) = \Lambda + 4\sqrt{3}e^{-\frac{\pi}{2\sqrt{3}}} \frac{3L_{QQ}\sqrt{\Lambda}}{4\Lambda^2} \Lambda + \ldots$$

(48)

and from Eq. (48) we get for the Energy of the pair $E_{QQ}$ in terms of the separation $L_{QQ}$,

$$E_{QQ}(L_{QQ}) \bigg|_{L_{QQ} \rightarrow \infty} = L_{QQ} \left( \frac{\Lambda}{R} \right)^{\frac{3}{2}} - O(\varepsilon^{2/3}) \frac{3L_{QQ}\sqrt{\Lambda}}{4\Lambda^2} \Lambda) + \ldots$$

(49)

C. D5 branes on $S^2$

In this example the system consists of $N_c$ D5 branes that wrap a two cycle inside the resolved conifold preserving four supercharges ($\mathcal{N} = 1$ in four dimensions) \[18\]. After a geometrical transition takes place, a background dual to this field theory contains a metric, dilaton and RR three form. In this case $g^2_{YM} \equiv (2\pi)^3 g_s \alpha'$. We quote only the part of the metric relevant to this computation (after a rescaling of the Minkowski coordinates is done)

$$ds^2 = \frac{1}{g_s \alpha' N_c} e^{\phi} \left[ dx^2_{1,3} + d\rho^2 + ds^2_{int} \right].$$

(50)

The relevant functions read,

$$g(\rho) = f(\rho) = e^{\phi} g_s \alpha' N_c$$

$$e^{2\phi} = \frac{2^{2/3}}{\sqrt{\rho \coth(2\rho) - e^{2\phi(\rho)}}} \left( \frac{\rho^2}{\sinh(2\rho)} \right)^{-\frac{1}{3}}$$

$$V_{eff} = e^{-\phi(\rho_0)} \sqrt{e^{2\phi(\rho)} - e^{2\phi(\rho_0)}}$$

(51)

The integral defining the separation of the quark pair cannot be evaluated explicitly, but we can check that the upper limit of the integral gives a finite contribution (it goes as $\int_0^\infty \rho^{2} e^{\rho} d\rho$), while from the lower extremum of the string reaches the end of the space ($\rho_0 \rightarrow 0$) we get

$$L_{QQ} \sim \int_{\rho_0=0}^{\rho_0=\infty} \frac{d\rho}{\sqrt{e^{2\phi(\rho)} + \ldots}} \sim \lim_{\rho_0 \rightarrow 0} \log(\rho_0) \rightarrow \infty.$$

(52)

indicating that (like in the Witten-Sakai-Sugimoto example discussed above) strings that reach the end of the space correspond to an infinite separation between the quark pair, which in turn indicates the absence of screening (expected from the QFT that contains only fields with zero N-ality).

We compute the subleading terms in the Energy of the pair in terms of the separation to obtain,

$$E_{QQ} = e^{\phi(0)} L_{QQ} + O(e^{-2\Lambda L_{QQ}})$$

(53)

as above a clear sign of a confining dual QFT.

D. Klebanov-Strassler Model

Certainly, the Klebanov-Strassler model is the cleanest example for a dual to a four-dimensional field theory that confines in the IR and approaches a conformal point in the UV (modulo important subtleties) \[19\]. Here again, we will quote only the relevant part of the metric,

$$ds^2 = h(\rho)^{-1/2} dx^2_{1,3} + h(\rho)^{1/2} e^{A/3} \frac{d\rho^2}{6K(\rho)^2} + ds^2_{int}$$

(54)

with

$$K^3(\rho) = \frac{\sinh 2\rho - 2\rho}{2\sinh^3(\rho)}, \quad h(\rho) = \frac{2^{2/3}(g_s \alpha' N_c)^2 e^{-8/3}}{\sinh^2 x} \int_0^\infty \frac{x \coth x - 1}{\sinh^2 x} (\sinh 2x - 2x)^{1/3}. $$

(55)
The functions \( f, g, V_{eff} \) read,
\[
f^2 = h(\rho)^{-1}, \quad g^2 = \frac{\epsilon^{4/3}}{6K(\rho)^2}, \quad V_{eff} = \frac{\sqrt{6h(\rho_0)K(\rho)}}{\sqrt{h(\rho)^{-1}} - h(\rho_0)^{-1}} \tag{56}
\]
In this example again, using the asymptotics for the various functions, it can be checked that when the string approaches the end of the space \( \rho_0 = 0 \) the separation of the quark pair diverges logarithmically, as in the two previous examples. Also, the energy of the pair reads
\[
E_{QQ} = f(0)L_{QQ} + O(e^{-\frac{2\epsilon^{2/3}}{h(0)}L_{QQ}}) \tag{57}
\]
with \( a_0 = \frac{h(0)}{(g, c_\alpha, N_\epsilon, -\epsilon^3)^2} = 1.1398 \). As above, it is clear that the model shows confining behavior.

IV. WALKING SOLUTIONS IN THE D5 SYSTEM, UNFLAVORED.

This section is devoted to the specific case of a class of solutions to the D5 system that exhibit walking behavior in the IR, in the sense that a suitably defined gauge coupling becomes almost constant in a finite range of energies \([11]\). We remind the reader about the set-up, based on the geometry produced by stacking on top of each other \( N_c \) D5-branes that wrap on a \( S^2 \) inside a CY3-fold and then taking the strongly-coupled limit of the gauge theory on this stack (that leaves us in the supergravity approximation for this system). We introduce the class of solutions of interest and apply the formalism of the previous sections to these solutions. As we will see, these solutions do confine, in the sense defined earlier on, but also show a remarkable behavior in the walking energy region, leading to a phenomenology reminiscent of a first order phase transition. As a result, the leading-order, long-distance behavior of the quark-antiquark potential is linear, but the coefficient is very different from what computed in section III C.

A. General set-up.

We start by recalling the basic definitions that yield the general class of backgrounds obtained from the D5 system, which includes \([18]\) as a very special case. We start from the action of type-IIB truncated to include only gravity, dilaton and the RR 3-form \( F \):
\[
S_{11B} = \frac{1}{G_{110}} \int d^{10}x \sqrt{-g} \left[ R - \frac{1}{2} (\partial \phi)^2 - \frac{\epsilon^2}{12} F_3^2 \right]. \tag{58}
\]
We define the \( SU(2) \) left-invariant one forms as,
\[
\tilde{\omega}_1 = \cos \psi d\tilde{\theta} + \sin \psi \sin \tilde{\theta} d\tilde{\varphi}, \quad \tilde{\omega}_2 = -\sin \psi d\tilde{\theta} + \cos \psi \sin \tilde{\theta} d\tilde{\varphi}, \quad \tilde{\omega}_3 = d\psi + \cos \tilde{\theta} d\tilde{\varphi}. \tag{59}
\]
and write an ansatz for the solution \([22]\) assuming that the functions appearing in the background depend only the radial coordinate \( \rho \), but not on \( x \) nor the 5 angles \( \theta, \tilde{\theta}, \phi, \tilde{\phi}, \psi \) (in string frame):
\[
ds^2 = \alpha' g_s e^{\phi(\rho)} \left[ \frac{d\tilde{\omega}_1^2}{\alpha' g_s} + e^{2k(\rho)} d\tilde{\theta}^2 + e^{2h(\rho)} (d\theta^2 + \sin^2 \theta d\varphi^2) + \right.
\]
\[
+ \left. \frac{e^{2g(\rho)}}{4} \left( (\tilde{\omega}_1 + a(\rho) d\tilde{\theta})^2 + (\tilde{\omega}_2 - a(\rho) \sin \theta d\varphi)^2 \right) + \frac{e^{2k(\rho)}}{4} (\tilde{\omega}_3 + \cos \theta d\varphi)^2 \right],
\]
\[
F_3 = \frac{N_c}{4} \left[ -(\tilde{\omega}_1 + b(\rho) d\tilde{\theta}) \wedge (\tilde{\omega}_2 - b(\rho) \sin \theta d\varphi) \wedge (\tilde{\omega}_3 + \cos \theta d\varphi) + 
\]
\[
b' d\rho \wedge (-d\theta \wedge \tilde{\omega}_1 + \sin \theta d\varphi \wedge \tilde{\omega}_2) + (1 - b(\rho)^2) \sin \theta d\theta \wedge d\varphi \wedge \tilde{\omega}_3 \right]. \tag{60}
\]

The system of BPS equations can be rearranged in a convenient form, by rewriting the functions of the background in terms of a set of functions \( P(\rho), Q(\rho), Y(\rho), \tau(\rho), \sigma(\rho) \) as \([21]\)
\[
4e^{2h} = \frac{P^2 - Q^2}{P \cosh \tau - Q}, \quad e^{2g} = P \cosh \tau - Q, \quad e^{2k} = 4Y, \quad a = \frac{P \sinh \tau}{P \cosh \tau - Q}, \quad N_c b = \sigma. \tag{61}
\]
Using these new variables, one can manipulate the BPS equations to obtain a single decoupled second order equation for $P(\rho)$, while all other functions are simply obtained from $P(\rho)$ as follows:

\[
Q(\rho) = (Q_0 + N_c) \cosh \tau + N_c(2\rho \cosh \tau - 1),
\]
\[
\sinh \tau(\rho) = \frac{1}{\sinh(2\rho - 2\hat{\rho}_0)}, \quad \cosh \tau(\rho) = \coth(2\rho - 2\hat{\rho}_0),
\]
\[
Y(\rho) = \frac{P'}{8};
\]
\[
e^{4\phi} = \frac{e^{4\phi_0} \cosh(2\hat{\rho}_0)^2}{(P^2 - Q^2)Y \sinh^2 \tau},
\]
\[
\sigma = \tanh \tau(Q + N_c) = \frac{(2N_c\rho + Q_0 + N_c)}{\sinh(2\rho - 2\hat{\rho}_0)}.
\]

(62)

The second order equation mentioned above reads,

\[
P'' + P' \left( \frac{P' + Q'}{P - Q} + \frac{P' - Q'}{P + Q} - 4 \coth(2\rho - 2\hat{\rho}_0) \right) = 0.
\]

(63)

In the following we will fix the integration constant $Q_0 = -N_c$, so that no singularity appears in the function $Q(\rho)$. We also choose $\hat{\rho}_0 = 0$ for notational convenience, together with $\alpha'q_s = 1$. For our purposes, it is also convenient to fix $8e^{4\phi_0} = 1$. With all of this, the functions we need for the probe string are:

\[
f^2(\rho) = e^{2\phi} = \sqrt{\frac{\sinh^2(2\rho)}{(P^2 - Q^2)P'}},
\]

(64)

\[
g^2(\rho) = \frac{1}{2} P' f^2(\rho),
\]

(65)

\[
V_{eff}^2(\rho) = \frac{2}{C^2P'} \left( \sqrt{\frac{\sinh^2(2\rho)}{(P^2 - Q^2)P'} - C^2} \right).
\]

(66)

**B. Walking solutions.**

The simplest and better understood solution of the system described in the previous subsection is defined by

\[
\hat{P} = 2N_c\rho.
\]

(67)

It belongs to class I, and it has already been presented in section III C, where is shown the study of the behaviour of the Wilson loop\(^8\).

For this background it is possible to show that

\[
f^2(0) = \frac{1}{N_c \sqrt{2N_c}},
\]

(68)

and that for $C^2 = f^2(0)$, expanding around $\rho \sim 0$

\[
V_{eff}^2(\rho) = \frac{8\rho^2}{9N_c} + \cdots.
\]

(69)

In particular, on the basis of what we already discussed around Eq. (25) this means that $L_{QQ}$ and $E_{QQ}$ **diverge** for $C^2 \to f^2(0)$.

In \([11]\), it was observed that there exists a class of well-behaved solutions for which a suitably defined (four-dimensional) gauge coupling exhibits a **walking** regime, meaning that for a long range in the radial coordinate $\rho_{IR} <

---

\(^8\) see Appendix A for the definition of class I and II and for more details about the solutions to the equation for $P$.
\( \rho < \rho_* \) the running becomes very slow, and the gauge coupling effectively is constant. These solutions depend on two free parameters \( c \) and \( \alpha \), and can be obtained recursively, by assuming \( c \) large and expanding in powers of \( N_c/c \), so that

\[
P(\rho) = \sum_{n=0}^{\infty} c^{1-n} P_{1-n}.
\]  

(70)

with

\[
P_1(\rho) \equiv (\cos^3 \alpha + \sin^3 \alpha (\sinh(4\rho) - 4\rho))^{1/3}.
\]  

(71)

In particular, for \( c/N_c \to +\infty \), the solution is very well approximated by \( P \approx c P_1 \).

The solutions of the form in Eq. (70) belong to class II in the language introduced in Appendix A. This type of solution is not suited for the present study, because of the exponential behavior of \( c/N \).

In particular, for \( \alpha = 0 \) would imply that the solution is not a free parameter \( c \).

Indeed, there are no asymptotic (in the UV) solutions that behave as \( e^{c \rho} \). Allowing for \( c_2 \neq 0 \) would imply that the solution is not a deformation of \( \tilde{P} \), but rather that the expansion in Eq. (73) breaks down, and the solution (if regular) falls in class II. The procedure of allowing for a small component \( c_1 \neq 0 \) can be interpreted as the insertion of a small, relevant deformation, which does not affect the UV-asymptotics, but has very important physical effects in the IR.

We have now obtained an important result: at least up to large values of \( \rho \), there exists a class of solutions that approach asymptotically the \( \tilde{P} \) solution. We cannot prove that such solutions are well behaved all the way to \( \rho \to 0 \). However, we can use this result in setting up the boundary conditions (at large-\( \rho \)) and numerically solve Eq. (63).
towards the IR. By inspection, these solutions are precisely the ones we were looking for. They start deviating significantly from $\hat{P}$ below some $\rho_\ast > 0$, below which $P$ is approximately constant. We plot in Fig. 2 two such solutions, with $\rho_\ast \simeq 4$ and $\rho_\ast \simeq 9$, together with the $\hat{P}$ solution for the same value of $N_c$. We also plot in Fig. 3 the functions appearing in metric $(e^{2g}, e^{2k}, e^{2k}, \phi)$ for the same solutions. Notice the behavior of $e^{2g}$ for $\rho \to 0$, but also the fact that the dilaton $\phi$ is finite for $\rho \to 0$.

A short digression is due at this point. The reader should be aware that in the presented case we are likely working with a singular background, as the function $e^{2g}$, a warp factor inside the internal space, is divergent at $\rho = 0$. The problem of singularities in gauge-string duality has a long history. Surely it is better to work with non-singular backgrounds, but it is possible to obtain interesting information also from singular manifolds. The procedure, developed along the years, to decide whether one is dealing with a “good singularity” or not, consists in analyzing all the physical observables and showing that no track of the singularity can be found in them. One specific suggestion is to look at the behavior of the $g_{tt}$ component of the metric (in the present case, the dilaton) and make sure that it is finite. Many are in literature the examples of such a class of backgrounds. To corroborate the idea that our background falls in this class, we decide to investigate not only the value of the Wilson loop but also two invariants of the metric, namely the Ricci scalar and a suitable contraction of the Ricci tensor. As can be seen in Fig. 4 no trace of singularities can be found in these two observables for our background, in support of the idea that the backgrounds we are considering are indeed acceptable.

![Figure 2](image-url)

**Fig. 2:** The numerical solutions for $P(\rho)/N_c$ used in the analysis as an example, for $N_c = 100$. The three solutions correspond to the $\hat{P}$ case with $\rho_\ast = 0$, and to two new numerical solutions with, respectively, $\rho_\ast \simeq 4$ and $\rho_\ast \simeq 9$. The numerical solutions can be plotted up to $\rho \simeq 150$, but in the following we will truncate them at $\rho_1 = 30$.

### C. Probes: numerical study.

We set-up the configuration of the string by assuming that its extremes are attached to the brane at $\rho = \rho_1 \gg 1$, and treat this as a UV cut-off. In the numerical study and in the resulting plots, we used $\rho_1 = 30$. The string is stretched in the Minkowski direction $x = x(\rho)$, with $x(\rho_1) = 0$ for convenience. We vary the integration constant $C^2 > f^2(0)$. For each choice of $C$ we define $\rho_0$ as $V_{eff}^2(\rho_0) = 0$. In this way, the coordinates of the string are $(x(\rho), \rho)$, where

$$x(\rho) = \int_\rho^{\rho_1} \frac{dr}{V_{eff}(r)}.$$  

(79)

The Minkowski distance between the the end-points of the string is hence $L_{QQ} = 2x(\rho_0)$. For the energy, because in the numerical study we do not remove the UV cut-off, rather that Eq. (22), we use the unsubtracted action (setting
FIG. 3: The functions ($e^{2g}$, $e^{2h}$, $e^{2k}$, $\phi$) appearing in the metric for the same solutions as in Fig. 2, computed rescaling $P \to P/N_c$ and $Q \to Q/N_c$.

FIG. 4: The (10-dimensional) Ricci scalar $R$ and the scalar $R_2 \equiv R_{MN}R_{PQ}g^{MP}g^{NQ}$, plotted as a function of the radial coordinate $\rho$, for several numerical solutions in the class discussed in the body of the paper. Each curve corresponds to a different value of $\rho_\ast$. Notice that both scalars are finite in the $\rho \to 0$ limit.

$$T/(2\pi\alpha') = 1$$ evaluated up to the cut-off:

$$E_{QQ} = 2 \int_{\rho_0}^{\rho_1} dr \sqrt{\frac{f^2(r)g^2(r)}{f^2(r) - C^2}}. \quad (80)$$

The numerical results obtained for the three different solutions $P$ are shown in Fig. 5 and Fig. 6.

Let us first focus our attention on the $\hat{P}$ case of Eq. (67). The deeper the string probes the radial coordinate (smaller values of $\rho_0$), the longer the separation $L_{QQ}$ between the end-points on the UV brane, in agreement with our criteria and results of sections II and III.

Two regimes can be identified: as long as $\rho_0 > \rho_{IR}$, then $L_{QQ}$ varies very little with $\rho_0$, while for small $\rho_0$, further reductions of $\rho_0$ imply much longer $L_{QQ}$. The scale $\rho_{IR} \sim O(1)$ is the scale in which the function $Q$ changes from linear to approximately quadratic in $\rho$, and is also the scale below which the gaugino condensate is appearing (the function $b(\rho)$ in the background is non-zero). This result is better visible in the upper panel of Fig. 5. The dependence of $L_{QQ}$ on $\rho_0$ is monotonic, but shows two very different behaviors for $\rho_0 < \rho_{IR}$ and $\rho_0 > \rho_{IR}$, respectively. The transition between the two is completely smooth.
FIG. 5: Upper panel, the radial coordinate $\rho_0$ of the middle point of the string as a function of $L_{QQ}$. Middle and lower panel, the energy $E_{QQ}$ as a function of the quark-antiquark separation $E_{QQ}(L_{QQ})$. The three solutions in Fig. 2 are used, with the same color-coding.
The physical meaning of this behavior is well illustrated by studying the total energy $E_{QQ}$ of the classical configurations, as a function of $L_{QQ}$ and of $\rho_0$ (see the middle and lower panel of Fig. [5]). One sees that for small $L_{QQ}$, the energy grows very fast with $L_{QQ}$, until a critical value beyond which the dependence becomes linear. We have already studied analytically this behavior, which can be interpreted in terms of the linear behavior of the quark-antiquark potential obtained from the Wilson loop in agreement with the discussion of sections [II] and [III]. The energy is also a monotonic function of $\rho_0$.

By comparing with the solutions that walk in the IR, one sees that a very different behavior appears. Starting from the upper panel in Fig. 5 one sees that as long as $\rho_0 > \rho_*$, the dependence of $L_{QQ}$ from $\rho_0$ reproduces the $\bar{P}$ case. Beginning from such large $\rho_0$, we start pulling the string down to smaller values of $\rho_0$, and follow the classical evolution. Provided we do this adiabatically, we can describe the motion of the string as the set of classical equilibrium solutions we obtained in the previous section. Going to smaller $\rho_0$, $L_{QQ}$ increases, and nothing special happens until the tip of the string touches $\rho_0 \simeq \rho_*$. At this point $L_{QQ} = L_{\text{max}}$. From here on, the string can keep probing smaller values of $\rho_0$ only at the price of becoming shorter in the Minkowski direction (smaller $L_{QQ}$). Another change happens when $L_{QQ} = L_{\text{min}}$, at which point the tip of the string entered the bottom section of the space, $\rho_0 < \rho_{1R}$. From here on, further reducing $\rho_0$ requires larger values of $L_{QQ}$. Asymptotically for $\rho_0 \to 0$, the separation between the end-points of the string is diverging, $L_{QQ} \to \infty$.

Even more interesting is the behavior of the energy (lower panel of Fig. 5): for very short $L_{QQ}$, and again for very large-$L_{QQ}$, it is just a monotonic function, but for a range $L_{\text{min}} < L_{QQ} < L_{\text{max}}$ there are three different configurations allowed by the classical equations for the string we are studying 9. One of the three solutions (smoothly connected to the small-$L_{QQ}$ configurations) is just the Coulombic potential already seen with $\bar{P}$. The highest energy one is an unstable configuration, with much higher energy. The third solution (smoothly connected to the unique solution with $L_{QQ} > L_{\text{max}}$) reproduces the linear potential typical of confinement. Notice (from the lower panel of Fig. 4) that the solution at large-$L_{QQ}$ is linear, but has a slope much larger than what seen in the $\bar{P}$ case. This co-existence of several disjoint classical solutions is expected in systems leading to phase transitions (see Appendix B). The instability/metastability/instability of the solutions can be illustrated by comparing Fig. 5 with Fig. 10 in Appendix [B]. This is just an analogy, and one should not push it too far. However, identifying the pressure $P$, volume $V$ and Gibbs free energy $G$ as $L_{QQ} \leftrightarrow P$, $\rho_0 \leftrightarrow V$ and $E_{QQ} \leftrightarrow G$, one sees obvious similarities. In particular, there is a critical distance $L_{\text{min}} < L_c < L_{\text{max}}$ at which the minimum of $E_{QQ}$ is not differentiable.

In order to better understand and characterize the solutions we find, it is useful to look more in details at the shape of the string configurations, focusing in particular on the middle panel in Fig. 4 in which we plot the string configuration that solves the equations of motion for various values of $\rho_0$, on the background with $\rho_* \simeq 4$. Consider those strings that penetrate below $\rho_*$. Besides having a shorter $L_{QQ}$, and higher $E_{QQ}$ than those for which $\rho_0 > \rho_*$, this strings show another interesting feature. They start developing a non trivial structure around their middle point, that becomes progressively more curved the further the string falls at small $\rho$. Ultimately, this degenerates into a cusp-like configuration, which disappears once $\rho_0$ approaches the end of the space. Notice that, as a result, the three different solutions for $L_{\text{min}} < L_{QQ} < L_{\text{max}}$ have three very different geometric configurations. One (stable or metastable) configuration is completely featureless, and practically indistinguishable from the solutions in the background generated by $\bar{P}$. The second (stable or metastable) configuration shows a funnel-like structure below $\rho_*$, and then the string lies very close to the end of the space. The third (unstable) solution presents a highly curved configuration around its middle point 10. All of this seems to be consistent with the fact that in the region $\rho_{1R} < \rho < \rho_*$, the background has a higher curvature, and as a result the classical configurations prefer to lie either in the far-UV or deep-IR. It must be noted here that the Ricci curvature is indeed bigger in this region, but that it converges to a constant for $\rho \to 0$.

D. Comments on this section.

We conclude this section by summarizing here three important lessons we learned.

First, we are able to derive the linear $E_{QQ}(L_{QQ})$ expected from confinement. This linear behavior emerges for $L_{QQ} \gg L_c$, $L_c$ being the critical distance between the end-points of the string, of the order of the distance $L_{QQ}$.

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9 Notice that the qualitative behavior of the solutions with $\rho_* \simeq 4$ and $\rho_* \simeq 9$ are identical. However, we were not able to follow numerically the solution with larger $\rho_*$, and hence the plots show only two such solution. The existence of the third is assured by the fact that this background must yield a confining potential.

10 This should not be interpreted literally as a cusp. The classical solutions we found are always continuous and differentiable, and this structure disappears once the string approaches the end of the space.
FIG. 6: The strings in \((x, \rho)\)-plane, obtained with various choices of \(C^2 > f^2(0)\). Top to bottom, the three numerical solutions corresponding to increasing values of \(\rho_*\).
computed for the string the tip of which reaches $\rho_{IR}$. This fact provides a physical meaning for the scale $\rho_{IR}$, which clearly shows in the (scheme-dependent) background functions and in the gauge coupling 11.

In particular, we can conclude that the walking solution we looked at is dual to a confining theory, but a very different one from the one of the non-walking solution $\hat{P}$. This is signaled by the fact that the leading order behavior of the walking solution is related to the value of $V_{eff}$ in $\rho_0 \gtrsim \rho_s$ while the non-walking is related to the zero of $V_{eff}$ in $\rho_0 \gtrsim 0$, and hence the presence of this two different solutions has to be related to the existence of two different scales in the background.

A limitation of this formalism is that, as we explained at length, it does not allow to study the sub-leading corrections to the linear behavior, which would require a treatment in which $\alpha'$-suppressed quantum corrections are included. It would be very interesting to calculate these corrections, which might provide useful information as to the nature of the dual theory.

The second important lesson we learn is related to the second scale $\rho_s$ appearing in solutions with walking behavior. Again, this scale appears in $P$ and as a consequence in all the functions in the background, including the (scheme-dependent) gauge coupling. The main point is that, provided $\rho_s > \rho_{IR}$, $\rho_s$ has a very important physical meaning: it separates the small-$L_{QQ}$ regime, where the background and all physical quantities are the same as in the original $\hat{P}$ solution, from the large-$L_{QQ}$ regime, in which the dual theory is completely different from the dual to the $\hat{P}$ solution. The scale $\rho_s$ is not just a scheme-dependent fluke effect: its value is somehow related to the coefficient of the linear leading behavior of the quark-antiquark potential.

The third lesson we learn is that in the region $\rho_{IR} < \rho < \rho_s$, the dynamics being more strongly coupled than elsewhere, classical solutions are unstable. The string configurations that are stable are those that either do not reach $\rho_s$, or those that go through this region only in order to reach the region near the end of the space, where the string can lie up to indefinitely large separations (in the Minkowski directions). This is a very interesting result, that might be related with what found in [12], where it is shown that for large values of $\rho_s$ the spectrum of scalar glueballs on these backgrounds splits into a set of towers of heavy states and some light state, separated by a large gap. This is going to be studied elsewhere [24].

V. WILSON LOOP IN A FIELD THEORY WITH FLAVORS

In this section we consider the effects of fundamental (flavor) degrees of freedom on the Wilson loop, using backgrounds that encode the dynamics of fields charged under a flavor group. As discussed in the Introduction, we expect that confinement does not take place, instead the theory will screen. We will observe the existence of a maximal length, requiring the string to break. As we explained, we are not including $g_s$ effects hence the pair-creation will not be accessible to the description we are giving here. In 20, these effects are taken into account by constructing the screened solution explicitly.

The construction of string backgrounds where the effects of flavors is included was considered in a large variety of models, see [25]. We will concentrate on the models developed in [20]. We follow the treatment and notation of [21], as done in the previous sections. Other authors have studied effects similar to the ones described in this section by using different string models, see [27]-[31].

At this point, let us stress that, what we will do in the following is to use as probe a closed string in a classical theory with no $g_s$ correction. Such an observable cannot describe the physics of the broken $QQ$ pair in the QFT sector; it will have a good overlap with the state describing the $\hat{Q}\hat{Q}$ pair only for small separation distance among the quarks. If the distance among the quark is increased another configuration, that will mimic the breaking of the pair, will become more energetically favorable. On the other hand if we decide to measure only the Wilson loop we will see no signal of such a breaking. Conversely if a cusps appears in the profile of the string used to measure the wilson loop, this signal has to be read as the loss of validity of the approximation imposed on our theory. While this signal cannot be interpreted as corresponding physical quantity (the distance at which the cusp appears has nothing to do with the scale of breaking of the $QQ$ pair), the observation can give a suggestion about the range of validity of our approximation. The results of this section should be understood as the application of the formalism of section [11] to the case of flavored background, with the proviso of the validity of the Nambu-Goto approximation and the comment made above about the screening phenomena. Aside form this, the numerical solution Fig. 7 below is new material included here.

In order to find the background solutions, we will need to solve a differential equation that is a generalization of

11 A change of scheme in the string picture corresponds to a redefinition of the radial coordinate.
the one discussed in the previous section, Eq. (63):

\[
P'' + (P' + N_f)\left[\frac{P' + Q' + 2N_f}{P - Q} + \frac{P' - Q' + 2N_f}{P + Q} - 4 \coth(2\rho)\right] = 0, \\
Q(\rho) = \frac{2N_c - N_f}{2}(2\rho \coth(2\rho) - 1).
\] (81)

The relation to the functions that appear explicitly in the background is given in Eqs. (3.18) and (3.19) of [21]. There are various known solutions to Eq. (81). We focus on the so called type-I solutions that are known only as a series expansion. For small values of the radial coordinate (\(\rho \to 0\)), the expansion is given in Eq. (4.24) of [21], while for large values of the radial coordinate (\(\rho \to \infty\)) is given in Eqs. (4.9)-(4.11) of [21]:

\[
P(\rho \sim 0) = P_0 - N_f \rho + \frac{4}{3} e^3 P_0^2 \rho^3 + \cdots, \\
P(\rho \sim \infty) = Q + N_c(1 + \frac{N_f}{4Q} + \frac{N_f(N_f - 2N_c)}{8Q^2} + \cdots). 
\] (82)

In the \(N_f \to 0\) limit this can be matched with the expansion Eq. (72). Below, we find it useful to have the IR (\(\rho \to 0\)) asymptotics for the functions

\[
e^{4\phi} \sim \frac{8e^{4\phi_0}}{e^3 P_0^4} \left[1 + \frac{4N_f}{P_0^2} \rho + \frac{10N_f^2}{P_0^4} \rho^2\right] + \ldots, \quad e^{2k} \sim 2e^3 \left[\frac{P_0^2 \rho^2 - 2P_0 N_f \rho^3}{P_0^4}\right] + \ldots
\] (83)

and in the UV (\(\rho \to \infty\)),

\[
e^{4\phi} \sim e^{4\phi_0} \frac{1}{\rho}, \quad e^{2k} \sim 1.
\] (84)

We plot in Fig. 7 a set of numerical solutions, which reproduce the asymptotic behaviors in Eq. (82), for several different values of \(N_f/N_c\). The existence of these solutions smoothly joining the asymptotic behaviors has been assumed in [21]. Here we are explicitly showing that this assumption is correct. Numerically, we learn that (provided \(P_0\) is not too small) the function \(P\) is convex, as shown in Fig. 7. Let us move to the study of string probes in these backgrounds.
In the light of the discussion of section II A, we form the combinations

\[ f^2 = g_{tt}g_{xx} = e^{2\phi(\rho)} \frac{\sqrt{8e^{2\phi_0}\sinh(2\rho)}}{(P^2 - Q^2)(P' + N_f)}, \quad C^2 = e^{2\phi(\rho_0)} \]

\[ g^2 = g_{tt}g_{\rho\rho} = e^{2\phi(\rho)+2k(\rho)} \frac{\sqrt{2e^{2\phi_0}\sinh(2\rho)\sqrt{P'} + N_f}}{\sqrt{(P^2 - Q^2)}}, \]

\[ V_{eff} = \frac{1}{e^{k(\rho)}} \sqrt{e^{2\phi(\rho)} - C^2} = \frac{1}{e^{k(\rho)}} \sqrt{e^{2\phi(\rho) - 2\phi(\rho_0)} - 1}. \]  

(85)

We will choose for convenience the integration constant \( \phi_0 \) such that \( 8e^{4\phi_0} = e^3 P_0^4 \), hence we will have \( \phi(\rho = 0) = 0 \).

Following the discussion below Eq. (23) and using Eq. (84), the contribution to \( L_{QQ} \) of the integral for large values of the radial coordinate goes like

\[ L_{QQ}(\rho_0 \to \infty) \sim \int_{\rho_0}^{\infty} d\rho \frac{1}{e^{2\rho}} \]

(86)

which is finite. Similarly, using Eq. (83), the lower-end of the integral contributes with

\[ L_{QQ}(\rho_0 \to 0) \sim \int_{\rho_0}^{0} d\rho \sqrt{\rho}, \]

(87)

which is itself finite. In this case we can see that in Eq. (25) the quantity \( \gamma = -\frac{1}{2} \), according to the discussion around Eq. (25) implies a finite \( L_{QQ} \). Contrast this with the examples studied in section III and V.

The fact that near \( \rho = 0 \) the quantity \( V_{eff} \sim \rho^{-\frac{1}{2}} \) implies, using Eq. (13), that near the IR there must be a cusp-like behavior. A plot of the shape of the string makes this clear, see Fig. 8 where we string probing a background with \( N_f/N_c = 1.2 \). In contrast with the example studied in section IV these backgrounds have a divergent Ricci scalar at \( \rho = 0 \), in agreement with the fact that only for \( \rho_0 = 0 \) the string presents a cusp. This is the only example presented in this paper with a true cusp in the string profile.
A. The case \( N_f = 2N_c \).

Finally, we turn our attention to the very peculiar, limiting case with \( N_f = 2N_c \). The solution is described in Eq. (4.22) of the first paper in \[26\]. It corresponds to

\[
Q = 4N_c \frac{(2 - \xi)}{\xi(4 - \xi)}, \quad P = N_c + \sqrt{N_c^2 + Q^2},
\]

and the functions \( f(\rho), g(\rho) \) read in this case

\[
f^2 = g^2 = e^{2\rho_0 + 2 \rho}
\]

where now the radial coordinate is defined in the interval \(-\infty < \rho < \infty\). The explicit computation of Eqs. (22), now for the lower end of the integral \( \rho_0 \to -\infty \), gives

\[
L_{QQ} = g_s \alpha' N_c \pi, \quad E_{QQ} = 0
\]

We illustrate in Fig. 9 the behavior of the string on this background. Notice how the length of the string is fixed \( L_{QQ} = \pi \), in units where \( \alpha' g_s N_c = 1 \), irrespectively of how deep into the radial direction the string probes the background. Another interesting fact is that the quantity of Eq. (89) vanishes identically in this background, meaning that \( dL_{QQ}/d\rho_0 = 0 \) for any \( \rho_0 \), implying that it is always possible to find a solution with any given \( \rho_0 \) and with the same \( L_{QQ} \), as Fig. 9 clearly indicates. The field theory dual to this background is quite peculiar. On the one hand, the numerology \( N_f = 2N_c \) brings to mind the situation in which the theory becomes conformal. Nevertheless, we should not forget that this is a background constructed with wrapped branes, so, clearly a scale is introduced (the scale set by the inverse size of the cycle wrapped). It may be the case that once the singularity is resolved one finds in the deep IR an \( AdS_5 \) space. But the theory should remind that in the UV conformality is broken, hence developing the scale \( \Lambda^{-2} = \alpha' g_s N_c \), that is typical of six dimensional theories. Notice that the only allowed separation, according to the calculations above is precisely \( \pi \) in units of this scale and the energy is zero. These results do not need to be generic for all solutions with \( N_f = 2N_c \), but for this particular one discussed above. Note also that in this case, even when the background is singular in the IR, the string does not become cuspy, as the case studied in the previous section.

\begin{center}
\includegraphics[width=0.8\textwidth]{fig9}
\end{center}

FIG. 9: The string hanging on the special background in Eq. (88).

VI. SUMMARY AND CONCLUSIONS

Let us summarize what we learnt in this paper and emphasize the reasons that motivated our study. In section II we recovered many well-known results about holographic Wilson loops, but at the same time introduced a certain
amount of formalism and some new results that allow us to make a systematic study of many qualitative features of string probes dual to Wilson loops in field theory. For example, we introduced the quantity $V_{\text{eff}}$ in Eq. (12) which dictates the behavior of the string near the boundary of the space ($\rho \to \infty$) and tells us if the Dirichlet boundary condition is attained. At the same time, $V_{\text{eff}}$ near the IR tells us about the possibility (or not) of stretching the probe string indefinitely. Assuming certain characteristic power-law behavior of the function $V_{\text{eff}}$ near the IR, we were able to derive a condition on the power ensuring the presence of a minimum (or turning point). In that same section, we derived an exact expression of the relation between the energy and the separation of the quark pair. We also found an integral expression giving a necessary condition for the existence of turning points in the string probing the bulk and the condition for the existence of cusps.

In section III we applied the formalism described above to some well-understood examples and made some comments about the absence of Luscher terms in the $E_{QQ} - L_{QQ}$ relation.

The material in section IV was derived with the following motivation: in the paper [11] a new background was proposed to be dual to a QFT with a walking regime (for a particular coupling). Nevertheless, aside from an argument based on symmetries given in that paper, it could have been the case that this walking region is just a fluke of the coordinates choice that dissapears under a simple diffeomorphism (in the dual field theory, gauge couplings and beta functions are scheme dependent). To clarify if there is any real physical effect in the walking region we computed Wilson loops in the (putative) walking field theory. In order to do this, we needed to find a new walking solution that allows the string to satisfy the boundary conditions discussed in section II. The new solution was found numerically as a perturbation of the solution in [18]. We gave an interpretation of this new background in the dual field theory as the appearance of a VEV for a quasi-marginal operator. When studying the dynamics of strings in this new background we observed that there are various physical effects associated with the presence and length of the walking regime: the value of the Ricci scalar, the separation between the quark pair, the relation between the energy and the separation of the pair, etc. In conclusion, the walking regime has physically observable effects, hence it can not erased by a change of coordinates (conversely, it is not an effect of a choice of scheme in the field theory). Recent experience seems to suggest that whenever we deal with a system with two independent scales, the phenomenology of the Wilson loop will be similar to what we described in section IV.

Finally we moved to the study of the Wilson loops in field theories with flavors. Here we benefitted from some work done in the past, where the asymptotic behavior of the background functions was given. We constructed numerically the full solution in terms of a convenient formalization of the problem described in [21]. We applied the material of section II to this case and checked that the probe string was behaving as expected (screening).

We believe that this paper clarifies a considerable number of new and interesting points. Certainly, it pushes forward the idea of using string theory methods to study models of walking dynamics. This may become important at the moment of modelling the mechanism electroweak symmetry breaking. On the phenomenological side we provide a set of tools that can be applied to other backgrounds and may be even used in bottom-up approaches to the dual of QCD.

It would be nice to find new models of walking dynamics and apply the ideas in this paper to study and compare features. It would also be interesting to apply the formalism developed here to some of the examples of backgrounds dual to field theories with flavors, for which certain aspects of Wilson loops were studied [27] - [31].
A: UV asymptotic solutions.

In this appendix we present a short digression about the asymptotic behavior of the solutions to Eq. (63), in order to make contact with a possible field theory interpretation of the results that have been presented in the section [UV].

Generically, one can expand all the functions appearing in the background, and associate the integration constants appearing in this expansion with the insertion in the UV of the dual field theory of operators with scaling behavior determined by the $\rho$-dependence of the corresponding term in the expansion. This is reminiscent of what is done in the case of backgrounds that are asymptotically AdS, though in the present case the procedure is less solid, and the results should be taken with caution.

We can start from the function $Q$.

$$Q \sim N_c \left(2\rho - 1 + O(e^{-4\rho})\right), \quad (A1)$$

where we dropped factors as powers of $\rho$ in the exponential corrections. We will do so in rest of this analysis, the logic of which will not be affected.

If one thinks that the two behaviors correspond to the insertion of an operator of dimension $d$ and of its $6 - d$ dimensional coupling in the (six-dimensional) theory living on the $D5$ branes, this means that this expression should be written as

$$Q \sim z^d + z^{6-d}, \quad (A2)$$

with $z$ proportional to a length scale. By comparison with the expansion of $Q$, one is lead to the identification $\rho = \frac{1}{3} \log z$. This result agrees with what was done by comparing an appropriately defined beta-function to the NSVZ beta function in [22]. One can interpret the two terms in this expansion as the deformation of the theory by the insertion of a marginally relevant operator of dimension $d \sim 6 - \epsilon$, with coupling of dimension $6 - d \sim \epsilon$. Notice that the coefficient of the $O(e^{-4\rho})$ correction is very small, and has a sizable effect only at very small values of $\rho$. This is not due to the fact that we chose a particular value of the integration constant $Q_0$, in order to avoid the arising of a pathology in the IR. Allowing for $Q_0$ would yield the same expansion, and the modification of the coefficients would be such that only for $\rho < \rho_{IR}$ would the sub-leading correction have an effect. The scale $\rho_{IR} \sim O(1)$ has an important role in the present study.

Another important quantity in the background is

$$b(\rho) = \frac{2\rho}{\sinh(2\rho)} \sim O(e^{-2\rho}), \quad (A3)$$

which by means of the previous identification is equivalent to $b \sim z^3$. This can be interpreted as the VEV of a dimension-3 operator (customarily identified with the gaugino condensate in the dimensionally-reduced 4-dimensional low-energy theory). This is a relevant deformation. Because this is a more relevant operator than the VEV mentioned above, it always dominates in the IR over the sub-leading correction to $Q$, which can be ignored. Notice also that $b(\rho)$ is practically vanishing for $\rho > \rho_{IR}$. This provides an intuitive explanation for this scale: it is the scale at which the gaugino condensate emerges dynamically. It also suggests that the expansion of $Q$, rather than identifying an independent operator, might be interpreted in terms of the insertion of an operator loosely corresponding to the square of the gaugino condensate.

There exist two different classes of UV-asymptotic solutions for $P$ [21]:

$$P \sim 2N_c\rho + O(e^{-4\rho}) \quad (\text{class I}), \quad (A4)$$

$$P \sim O(e^{4/3\rho}) + O(e^{-4/3\rho}) + O(e^{-8/3\rho}) \quad (\text{class II}). \quad (A5)$$

In class I, there is only one integration constant, in the $O(e^{-4\rho})$ term. Notice that $P$ has the same leading and sub-leading components as $Q$. However, the sub-leading correction depends on a free parameter: the corresponding VEV can be enhanced in such a way that there be a range of $\rho$ over which the dynamics is dominated by this deformation, over the deformation present in $b(\rho)$, while the latter will become important only at very small values of $\rho$.

Solutions in the class II are rather different. The independent coefficients appear in the $O(e^{4/3\rho})$ and $O(e^{-8/3\rho})$ terms, and the former cannot be dialed to zero independently of the latter (see [21] for details). Using the same identification between $\rho$ and $z$, the leading order component of $P$ scales as $z^{-2}$, and can be interpreted as the insertion of a dimension-8 operator in the six-dimensional theory. It is somewhat natural to think that it is related to a gauge coupling is six-dimensions. The presence of sub-leading corrections that scale as $z^2$, and $z^4$ suggest that the gravity field $P$ should not be interpreted as a simple operator in the underlying dual dynamics, and that some caution should be used. One might want to interpret the $z^4$ as the insertion of the VEV of a four dimensional operator. But it could as well arise from the combination of the coupling scaling as $z^{-2}$ and the same VEV of marginal operator scaling as $z^6$ that is present in class I. All of this should not be taken too literally, but provides some guidance in what we do in the body of the paper, when constructing the class of walking solutions we are interested in.
B: Van der Waals gas.

Here we summarize some aspect of first order phase transitions that plays an important conceptual role in the body of the paper. By way of example, we remind the reader about the classical treatment of the van der Waals gas, in terms of its pressure $P$, temperature $T$ and volume $V$ of $N$ moles of particles, by means of the equation of state

$$ P = \frac{NRT}{V - bN} - \frac{N^2 a}{V^2}, \tag{B1} $$

where $R, b, a$ are constants.

In Fig. 10 we plot one isotherm. The condition for stability of the equilibrium $(\partial^2 F / \partial V^2)_T = - (\partial P / \partial V)_T > 0$ is not satisfied in some region. This implies that a phase transition is taking place.

![FIG. 10: The pressure $P$ as a function of the volume $V$ (left panel) and the Gibbs free energy $G$ as a function of the pressure $P$ (right panel) for the same isotherm curve.](image)

In order to understand what the physical trajectory followed by the system at equilibrium is, we plot in Fig. 10 the Gibbs free energy $G = G(T, P) = G(P)$ for the same isotherm. From this plot, one sees that the system evolves on the path $ABCOQ$, where $C = N$, in such a way that for every choice of $P$ the Gibbs free energy $G$ is at its minimum. The evolution is smooth along $ABC$ (gas phase: $|\partial P / \partial V|$ is small), but at $C = N$ the free energy is not differentiable, signaling that a first-order phase transition is taking place. In the $(P, V)$ plane the system runs along the horizontal line (constant $P$) joining $C$ and $N$. This explains the Maxwell rule introducing a curve of constant pressure that separates two regions of equal areas above and below it, delimited by the original isotherm. Afterwards, the evolution follows smoothly the curve $NOQ$ (liquid phase: $|\partial P / \partial V|$ is large).

While the trajectory $ABCNOQ$ follows the stable equilibrium configurations, it is possible to have the system evolving along the path $CDE$ or $LMN$. Both these paths represent metastable configurations, because $(\partial P / \partial V)_T < 0$. Indeed, these metastable states can be realized in laboratory experiments. For example, in a bubble chamber or in supercooled water, the metastability is exploited as a detector device, because small perturbations induced by passing-by charged particles are sufficient to drive the system out of the state and into the stable minimum. The evolution along the path $EFHIL$ is completely unstable and not realized physically (it is a local maximum, as clear from the right panel of Fig. 10 and by the fact that $(\partial P / \partial V)_T > 0$).

The analogy with the examples in the main body of the paper is apparent. Notice for instance that the pressure $P$ in this system as a function of the volume $V$ behaves as a non monotonic function, hence there are inversion points in the curve (the points $E, L$ in figure 10), in the same sense in which we discuss the presence of inversion points in the body of the paper.
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