Graded identities of block-triangular matrices *

Diogo Diniz Pereira da Silva e Silva †

Unidade Acadêmica de Matemática e Estatística
Universidade Federal de Campina Grande
Campina Grande, PB, Brazil

Thiago Castilho de Mello ‡

Instituto de Ciência e Tecnologia
Universidade Federal de São Paulo
São José dos Campos, SP, Brazil

April 17, 2015

Abstract

Let $F$ be an infinite field and $UT(d_1, \ldots, d_n)$ be the algebra of upper block-triangular matrices over $F$. In this paper we describe a basis for the $G$-graded polynomial identities of $UT(d_1, \ldots, d_n)$, with an elementary grading induced by an $n$-tuple of elements of a group $G$ such that the neutral component corresponds to the diagonal of $UT(d_1, \ldots, d_n)$. In particular, we prove that the monomial identities of such algebra follows from the ones of degree up to $2n - 1$. Our results generalize for infinite fields of arbitrary characteristic, previous results in the literature which were obtained for fields of characteristic zero and for particular $G$-gradings.

1 Introduction

Let $F$ be an infinite field and $UT(d_1, \ldots, d_n)$ the algebra of upper block triangular matrices. It is the subalgebra of the matrix algebra $M_{d_1+\ldots+d_n}(F)$

* partially supported by Fapesp grant No. 2014/10352-4, CNPq grant No. 461820/2014-5, CNPq grant No. 480139/2012-1 and CAPES grant No. 99999.001558/2014-05
† diogo@dme.ufcg.edu.br
‡ tcmello@unifesp.br
consisting of the matrices

\[
\begin{pmatrix}
A_{11} & A_{12} & \cdots & A_{1n} \\
0 & A_{22} & \cdots & A_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_{nn}
\end{pmatrix},
\]

where \(A_{ij}\) is a block of size \(d_i \times d_j\). In this paper we study the graded polynomial identities of upper block triangular matrix algebras \(UT(d_1, \ldots, d_n)\) over an infinite field \(F\). These algebras appear in the classification of minimal varieties (see [22]) and are generalizations of the matrix algebras (when \(n = 1\)) and the algebra \(UT_n(F)\) of upper triangular matrices (when \(d_1 = \cdots = d_n = 1\)).

One of the main problems in the theory of PI-algebras is the (generalized) Specht problem about the existence, for a given class of algebras, of finite basis for the \(T\)-ideals of identities. This problem for the ordinary identities of associative algebras over a field of characteristic zero was solved by Kemer (see [24], [25]). In the case of associative algebras graded by a finite group it was solved by I. Sviridova [32] in the case of abelian groups and by E. Aljadeff and A. Kanel-Belov [1] in the general case. Over fields of positive characteristic however the situations is different and ideals of identities without finite basis exist (see for example [9], [23], [31]). The basis for the graded identities of \(UT(d_1, \ldots, d_n)\) in our main result (Theorem 3.7) is finite, provided that \(G\) is finite.

The algebras of block triangular matrices admit gradings by any group \(G\) in which the elementary matrices are homogeneous. These are called elementary gradings (or good gradings, see [8]). The algebras \(UT_n(F)\) admit elementary gradings only (see [33]). Over an algebraically closed field of characteristic 0 every grading on \(M_n(F)\) by a finite group is obtained by a certain tensor product construction from an elementary grading and a fine grading (see [7]). If moreover the group is abelian an analogous result holds for the algebra \(UT(d_1, \ldots, d_n)\) (see [34]).

Explicit basis for the identities are known for a few algebras only and for the algebras \(UT(d_1, \ldots, d_n)\) (over an infinite field) the only known basis for the ordinary identities are for the algebras \(M_2(F)\) (see [30], [20], [26]) and \(UT_n(F)\) (see [28]). In general ideal of identities of \(UT(d_1, \ldots, d_n)\) is the product of the ideals of identities of the matrix algebras \(M_{d_i}\) (see [21]). An analogous property for the graded identities of block triangular matrix algebras was studied in [12]. Elementary gradings on \(UT_n(F)\) and the corresponding graded identities were studied in [16] and in particular it was
proved that elementary gradings can be distinguished by their graded identities. An analogous result for $UT(d_1, \ldots, d_n)$ with an elementary grading by an abelian group was obtained in [17].

When $\text{char} F = 0$ a complete description of the $\mathbb{Z}_2$-graded identities of $M_2(F)$ (and other PI-algebras) was given in [11]. Analogous basis for the identities of $M_n(F)$ with elementary $\mathbb{Z}$ and $\mathbb{Z}_n$ gradings were determined by Vasilovsky in [35], [36]. These results were also established for infinite fields (see [27], [4], [5]) and analogous results were obtained for related algebras (see [11],[13],[14]). Graded identities of $M_n(F)$ were studied more generally in [6] and in particular a basis for the graded identities of $M_n(F)$ with certain elementary gradings was determined. The elementary $G$-gradings considered are the ones induced by an $n$-tuple of pairwise different elements of $G$. The result considering an infinite field was obtained in [18]. In this case the basis is analogous to the one obtained by Vasilovsky and some monomial identities may be necessary. Recall that a $G$-grading on an algebra $A$ is called nondegenerate if the ideal of graded identities of $A$ contains no monomials. These types of gradings were studied in [2], [3]. Vasilovsky proved that $\mathbb{Z}_n$-grading on $M_n(F)$ is nondegenerate and that for the $\mathbb{Z}$-gradings one needs to consider the monomial identities of degree 1 corresponding to the homogeneous components of dimension 0.

In this paper we prove that the basis given in [6] holds for the algebras $UT(d_1, \ldots, d_n)$ over an infinite field $F$. Moreover we prove that it is only necessary to include the monomial identities of degree up to $2n - 1$ in the basis. The ideas used are similar to those in [6] and [4], [5]. We remark that in [10] a similar result was proved for $\mathbb{Z}_n$-graded identities.

2 Preliminaries

In this paper we consider associative algebras over an infinite field $F$ and vector spaces are also considered over $F$.

2.1 Graded algebras and graded polynomial identities

Let $A$ be an algebra and $G$ a group. A $G$-grading on $A$ is a vector space decomposition $A = \oplus_{g \in G} A_g$ compatible with the multiplication of the algebra in the sense that the inclusions

$$A_g A_h \subseteq A_{gh}$$

hold for any $g$ and $h$ in $G$. A nonzero element $a$ in $\cup_{g \in G} A_g$ is called a homogeneous element. Clearly to every homogeneous element $a$ corresponds
an element $g$ in $G$ such that $a \in A_g$. We say that this $g$ is the degree of $a$ in the given $G$-grading. The set $\{g \in G | A_g \neq 0\}$ is the support of the grading and is denoted by $\text{supp} A$.

A subspace $V$ of $A$ is a homogeneous subspace if $V = \oplus_{g \in G} (V \cap A_g)$. A subalgebra $B$ is a homogeneous subalgebra if it is homogeneous as a subspace and in this case $B = \oplus_{g \in G} B_g$, where $B_g = B \cap A_g$, is a $G$-grading on $B$. The $G$-grading on a homogeneous subalgebra $B$ of $A$ is assumed to be this one.

Let $X = \bigcup X_g$ be a disjoint union of a family of countable sets $X_g = \{x_g^{(1)}, x_g^{(2)}, \ldots \}$ and $F \langle X \rangle$ be the free associative algebra. A polynomial $f(x_g^{(1)}, \ldots, x_g^{(n)})$ is a graded polynomial identity for $A = \oplus_{g \in G} A_g$ if we have $f(a_g^{(1)}, \ldots, a_g^{(n)}) = 0$ for any $a_g^{(1)} \in A_g, \ldots, a_g^{(n)} \in A_g$. The set $T_G(A)$ of all graded polynomial identities of $A$ is an ideal of $F \langle X \rangle$ invariant under all graded endomorphisms of this algebra, i.e. it is a $T_G$-ideal. If $S$ is a set of polynomials in $F \langle X \rangle$ the intersection $U$ of all $T_G$-ideals containing $S$ is a $T_G$-ideal. In this case, we say that $S$ is a basis for $U$. Two sets are equivalent if they generate the same $T_G$-ideal. Since the field $F$ is infinite it is well known that every polynomial $f$ in $F \langle X \rangle$ is equivalent to a finite collection of multihomogeneous identities. Hence we may reduce our considerations to multihomogeneous polynomials.

### 2.2 Elementary gradings on block-triangular matrices

Let $(g_1, \ldots, g_m)$ be an $m$-tuple of elements of $G$ and $A = M_m(F)$ be the full matrix algebra of order $m$. If we set $A_g$ to be the subspace spanned by the elementary matrices $e_{ij}$ such that $g_i^{-1} g_j = g$ then we have

$$A = \oplus_{g \in G} A_g$$

and this decomposition is a $G$-grading. Let $B$ be a subalgebra of $A$ generated by elementary matrices. Then $B$ is a homogeneous subalgebra. In particular $UT(d_1, \ldots, d_n)$ is a homogeneous subalgebra of $M_m(F)$, where $d_1 + \cdots + d_n = m$. We say that the $G$-grading on $UT(d_1, \ldots, d_m)$ (and more generally on $B$) is the elementary grading induced by $(g_1, \ldots, g_n)$.

Let $e$ denote the unit of the group $G$ and consider $B = UT(d_1, \ldots, d_m)$ with the elementary grading induced by $(g_1, \ldots, g_n)$. The elementary matrices $e_{ii}$ have degree $e$ and therefore the dimension of the component $B_e$ is $\geq n$. We have $\text{dim} B_e = n$ if and only if the elements in the $n$-tuple inducing the grading are pairwise distinct. Equivalently the polynomial $x_e^{(1)} x_e^{(2)} - x_e^{(2)} x_e^{(1)}$ is a graded identity for $B$. 
2.3 Generic Graded Algebras

Let \( g = (g_1, \ldots, g_n) \) be a \( n \)-tuple of pairwise distinct elements of \( G \). Denote by \( A = \bigoplus_{g \in G} A_g \) the algebra \( M_n(F) \) with the elementary grading induced by \( g \). Let \( B \) be a subalgebra of \( A \) with basis \( \{e_{i_1j_1}, \ldots, e_{i_lj_l}\} \) as a vector space. Denote by \( G_0 \) (resp. \( G_0^A \)) the support of the grading on \( B \) (resp. \( A \)).

Let \( g \) be an element in the support \( G_0 \) of the grading of \( B \). Denote by \( D^g \) the set of indexes \( i \in \{i_1, \ldots, i_l\} \) such that for some \( j \in \{j_1, \ldots, j_l\} \) the matrix unit \( e_{ij} \) has degree \( g \). Recall that the \( n \)-tuple \( g \) consists of pairwise distinct elements of \( G \). This implies that for each \( i \in D^g \) there exists exactly one index in \( \{j_1, \ldots, j_l\} \), denoted by \( \hat{g}(i) \), such that \( e_{i\hat{g}(i)} \in B_g \). Thus we obtain a function \( \hat{g} : D_g \to \{j_1, \ldots, j_l\} \) for each \( g \in G_0 \). With this notation \( \{e_{i\hat{g}(i)} \mid i \in D_g\} \) is a basis for \( B_g \).

Denote by \( \Omega \) the algebra of polynomials in commuting variables

\[
\Omega = F[x_{ij}^{(k)} \mid i, j = 1, 2, \ldots, n; k = 1, 2, \ldots].
\]

The algebra \( M_n(\Omega) \) has a natural \( G \)-grading where the homogeneous component of degree \( g \) is the subspace generated by the matrices \( m_{ij}e_{ij} \), where \( e_{ij} \in A_g \) and \( m_{ij} \) is a monomial in \( \Omega \).

Definition 2.1 For each \( g \in G_0 \) and each natural number \( k \) the element

\[
\xi_g^{(k)} = \sum_{i \in D_g} x_{i\hat{g}(i)}^{(k)} e_{i\hat{g}(i)}
\]

of \( M_n(\Omega) \) is called a graded generic element. The algebra \( G(B) \) generated by the \( \xi_g^{(k)} \), \( g \in G_0 \), \( k = 1, 2, \ldots \) is called the algebra of graded generic elements of \( B \).

The algebra \( G(B) \) is a homogeneous subalgebra of \( M_n(\Omega) \) and is a graded algebra with the inherited grading. If \( B = A \) the above construction yields the graded algebra \( G(A) \) of generic elements of \( A \). The generic element in \( G(A) \) corresponding to \( g \in G_0^A \) and \( k \) will be denoted by \( \xi_g^{(k,A)} \). The following result is well known.

Theorem 2.2 Let \( F \) be an infinite field. The algebra \( G(B) \) is isomorphic as a graded algebra to the relatively free \( G \)-graded algebra \( F(\langle X \rangle) / \text{Id}_G(B) \).

Proof. The homomorphism \( \Theta : F(\langle X \rangle) \to G(B) \) induced by mapping \( x_g^{(i)} \mapsto \xi_g^{(i)} \) is clearly onto. Moreover as in the case of the generic matrix algebra (see [22, Theorem 1.4.4]) we have \( \ker \Theta = T_G(B) \) and the result follows. \( \square \)
3 The main result

Given \( g_1, g_2, \ldots, g_p \in G_0 \) we consider the composition \( \nu = \tilde{g}_p \cdots \tilde{g}_1 \) of the corresponding functions. This may not be well defined and we will prove in the next lemma that in this case the monomial \( x_{\tilde{g}_1}^{(1)} \cdots x_{\tilde{g}_p}^{(p)} \) is a graded identity for \( B \). Otherwise its domain \( D_\nu = D_{\tilde{g}_p} \cdots \tilde{g}_1 \) is the set of \( i \in \{i_1, \ldots, i_l\} \) for which the image \( \tilde{g}_p(i) \cdots (\tilde{g}_1(i)) \) is well defined. In this case \( \{e_{i\nu(i)}| i \in D_\nu\} \) is a basis for the subspace spanned by \( B_{\tilde{g}_1} \cdots B_{\tilde{g}_p} \).

Lemma 3.1 Let \( h_1, h_2, \ldots, h_p \) be elements in \( G_0 \). If \( D_{\tilde{h}_p \cdots \tilde{h}_1} = \emptyset \) then \( \xi_{h_1} \xi_{h_2} \cdots \xi_{h_p} = 0 \). Moreover if the set \( D_{\tilde{h}_p \cdots \tilde{h}_1} \) is nonempty then the \( i \)-th line of the matrix \( \{\xi_{h_1} \xi_{h_2} \cdots \xi_{h_p}\} \) is nonzero if and only if \( i \in D_{\tilde{h}_p \cdots \tilde{h}_1} \). In this case if \( j = \tilde{h}_p \cdots \tilde{h}_1(i) \), the only nonzero entry in the \( i \)-th line is a monomial of \( \Omega \) in the \( j \)-th column.

Proof. The proof is by induction on the length \( p \) of the product. The result for \( p = 1 \) follows directly from Definition 2.1. Hence we consider \( p > 1 \) and assume the result for products of length \( p - 1 \). Let us consider first the case \( D_{\tilde{h}_p \cdots \tilde{h}_1} \neq \emptyset \). In this case \( D_{\tilde{h}_p \cdots \tilde{h}_1} \neq \emptyset \) and we denote \( \nu = \tilde{h}_{p-1} \cdots \tilde{h}_1 \).

The induction hypothesis implies that there exists monomials \( m_i \), where \( i \in D_{\tilde{h}_p \cdots \tilde{h}_1} \), such that

\[
\xi_{h_1}^{(i_1)} \xi_{h_2}^{(i_2)} \cdots \xi_{h_p}^{(i_p)} = \left( \sum_{i \in D_{\tilde{h}_{p-1} \cdots \tilde{h}_1}} m_i e_{i\nu(i)} \right) \left( \sum_{j \in D_{\tilde{h}_p}} \xi_{j \tilde{h}_p(j)}^{(i)} e_{j \tilde{h}_p(j)} \right). \tag{1}
\]

Note that \( e_{i\nu(i)} e_{j \tilde{h}_p(j)} \neq 0 \) for some \( j \) if and only if \( i \in D_{\tilde{h}_{p-1} \cdots \tilde{h}_1} \) and in this case the product equals \( e_{i \tilde{h}_p(j)} \). Hence we obtain

\[
\xi_{h_1}^{(i_1)} \xi_{h_2}^{(i_2)} \cdots \xi_{h_p}^{(i_p)} = \sum_{i \in D_{\tilde{h}_{p-1} \cdots \tilde{h}_1}} (m_i \xi_{\nu(i) \tilde{h}_p(\nu(i))}^{(i_p)} e_{i \tilde{h}_p(\nu(i))}),
\]

and the result follows. Now assume that \( D_{\tilde{h}_p \cdots \tilde{h}_1} = \emptyset \). If \( D_{\tilde{h}_{p-1} \cdots \tilde{h}_1} = \emptyset \) then by the induction hypothesis \( \xi_{h_1}^{(i_1)} \xi_{h_2}^{(i_2)} \cdots \xi_{h_{p-1}}^{(i_{p-1})} = 0 \) and the result holds. Moreover if \( D_{\tilde{h}_{p-1} \cdots \tilde{h}_1} \neq \emptyset \) then we may write the product \( \xi_{h_1}^{(i_1)} \xi_{h_2}^{(i_2)} \cdots \xi_{h_p}^{(i_p)} \) as in (1). Since \( D_{\tilde{h}_p \cdots \tilde{h}_1} = \emptyset \) every product \( e_{i\nu(i)} e_{j \tilde{h}_p(j)} \) equals zero and therefore \( \xi_{h_1}^{(i_1)} \xi_{h_2}^{(i_2)} \cdots \xi_{h_p}^{(i_p)} = 0 \). 

\[\square\]
Corollary 3.2 If a monomial $x_{h_1}^{(i_1)} \ldots x_{h_p}^{(i_p)}$ in $F(X)$ is a graded identity for $B$ then it is a consequence of a monomial in $T_G(B)$ of length at most $2n-1$.

Proof. The result follows if we prove that every monomial in $T_G(B)$ of length $p > 2n-1$ is a consequence of a monomial of length at most $p-1$. Let $m = x_{h_1}^{(i_1)} \ldots x_{h_p}^{(i_p)}$ be a monomial identity for $B$. Clearly we may assume that $h_i \in G_0$, $i = 1, 2, \ldots, p$. If $D_{h_r h_{r-1} \ldots h_1} = \emptyset$ for some $r < p$ then Lema 3.1 implies that $\xi^{(i_1)}_{h_1} \ldots \xi^{(i_r)}_{h_r} = 0$. Hence $x_{h_1}^{(i_1)} \ldots x_{h_p}^{(i_p)}$ is an identity for $B$ and $m$ is a consequence of this monomial. Thus we assume that $D_r = D_{h_r h_{r-1} \ldots h_1}$ is nonempty for $r < p$ and denote by $I_r$ the image of the composition $\hat{h_r} h_{r-1} \ldots \hat{h_1}$. Notice that

$$D_1 \supseteq D_2 \supseteq \cdots \supseteq D_{p-1} \supseteq D_p = \emptyset. \quad (2)$$

Assume that there exists $r$ such that $D_r = D_{r+1} = D_{r+2}$. The equality $D_r = D_{r+2}$ implies that $I_r \subseteq D_{h_{r+2}}$. Clearly $D_{h_{r+2}} \subseteq D_{\hat{h}}$, where $h = h_{r+1} h_{r+2}$. Therefore $I_r \subseteq D_{\hat{h}}$ and we conclude that $D_{h_{r+1} h_{r-1} \ldots h_1} = D_{\hat{h}}$. Since $D_r = D_{r+2}$ this implies that the compositions $\hat{h} h_{r+1} \ldots \hat{h_1}$ and $h_{r+2} h_{r+1} \ldots h_1$ have the same domain. Moreover the equality in $G$, $h_1 h_2 \cdots h_r h = h_1 h_2 \cdots h_{r+1} h_{r+2}$ implies that for every $i \in D_{r+2}$, $\hat{h} h_r \ldots \hat{h_1}(i) = h_{r+2} h_{r+1} h_r \ldots h_1(i)$. Hence $\hat{h} h_r \ldots \hat{h_1} = h_{r+2} h_{r+1} h_r \ldots h_1$ and therefore we have $D_{h_{r+2} h_{r+1} \ldots h_1} = D_p = \emptyset$. It follows from Lemma 3.1 that the monomial $m' = x_{h_1}^{(i_1)} \ldots x_{h_r}^{(i_r)} x_{h_{r+1}}^{(i_{r+1})} x_{h_{r+2}}^{(i_{r+2})} \ldots x_{h_p}^{(i_p)}$, where $i_{r+1} \notin \{i_1, \ldots, i_r\}$, is an identity for $B$. Clearly $m$ is a consequence of $m'$. It remains only to verify that if $p > 2n-1$ there exists $r$ such that $D_r = D_{r+1} = D_{r+2}$. First notice that if $|D_1| = n$ then $\{i_1, \ldots, i_r\} = \{1, 2, \ldots, n\} = \{1, \ldots, j_1\}$ and $\hat{h}$ is a bijection in this set. Therefore $D_p = \emptyset$ implies that $D_{h_{j_2} h_{j_1} \ldots h_2} = \emptyset$. By Lemma 3.1 the monomial $x_{h_2}^{(i_2)} x_{h_3}^{(i_3)} \ldots x_{h_p}^{(i_p)}$ is an identity and clearly $m$ is a consequence of it. Therefore we may assume now that $|D_1| \leq n - 1$. In this case there are at most $n - 1$ proper inclusions in (2) and if $p > 2n-1$ there are two consecutive equalities, i.e., there exists $r$ such that $D_r = D_{r+1} = D_{r+2}$. \hfill \Box

We consider the following graded polynomials:

$$x_e^{(1)} x_e^{(2)} - x_e^{(2)} x_e^{(1)} \text{, if } e \in G_0 \quad (3)$$

$$x_g^{(1)} x_{g^{-1}}^{(2)} - x_g^{(3)} x_{g^{-1}}^{(2)} x_g^{(1)} \text{, if } e \neq g \text{ and } B_g \neq 0, \quad (4)$$

$$x_g^{(1)} \text{, if } B_g = 0. \quad (5)$$

7
Lemma 3.3 The algebra $B$ with the elementary grading induced by an $n$-tuple $g = (g_1, \ldots, g_n)$ of pairwise distinct elements of $G$ satisfies the graded polynomial identities (3) – (5).

Proof. Clearly the polynomials in (5) are identities for $B$. Since the elements in $g = (g_1, \ldots, g_n)$ are pairwise different if $e \in G_0$ the graded generic matrices $ξ^{(i)}_g$ are diagonal. Hence we have the graded identity (3). Since (4) is multilinear in order to verify that it is a graded identity substitute $x^{(1)}_g, x^{(3)}_g$ by $e_{ij}, e_{kl} \in B_g$ respectively and $x^{(2)}_{g^{-1}}$ by $e_{rs} \in B_{g^{-1}}$. If $(e_{ij}e_{rs}e_{kl}) \neq 0$ then $j = r$ and $s = k$. Moreover $e_{is}$ and $e_{rl}$ are in $A_e$ and therefore $i = s$ and $r = l$. Hence in this case $e_{ij} = e_{kl}$ and the result of the substitution is zero. Analogously if $(e_{kl}e_{rs}e_{ij}) \neq 0$ the result is zero. The remaining case to consider is $(e_{ij}e_{rs}e_{kl}) = 0 = (e_{kl}e_{rs}e_{ij})$ and the result is also 0. □

Proposition 3.4 [18, Lemma 4.6] Let $U_A$ denote the $T_G$-ideal generated by the identities (3) – (5) satisfied by the matrix algebra $A$ and let $ξ^{(i,A)}_g$, $g \in G_0^A$, $i = 1, 2, \ldots$ denote the generic elements in $G(A)$. If the monomials $m(x^{(1)}_{h_1}, \ldots, x^{(p)}_{h_p})$ and $n(x^{(1)}_{h_1}, \ldots, x^{(p)}_{h_p})$ in $F\langle X \rangle$ are such that the matrices $n(ξ^{(1,A)}_{h_1}, \ldots, ξ^{(p,A)}_{h_p})$ and $m(ξ^{(1,A)}_{h_1}, \ldots, ξ^{(p,A)}_{h_p})$ have the same position the same non-zero entry then

$$m(x^{(1)}_{h_1}, \ldots, x^{(p)}_{h_p}) \equiv n(x^{(1)}_{h_1}, \ldots, x^{(p)}_{h_p}) \mod U_A.$$ 

Next we generalize this proposition to the case of a subalgebra $B$ of $A = M_n(F)$ generated by elementary matrices. Note that the algebra $G(B)$ is a homomorphic image of the algebra $G(A)$ by Theorem 2.2. The homomorphism constructed in the following remark will be useful.

Remark 3.5 We construct a homomorphism from $G(A)$ to $G(B)$ as follows: the map $x^{(k)}_{ij} \mapsto χ_{ij}x^{k}_{ij}$ where $χ_{ij} = 1$ if $e_{ij} \in B_g$ and $χ_{ij} = 0$ if $e_{ij} \notin B_g$ induces an endomorphism $θ$ of $Ω$ extending this map. Hence $Θ : M_n(Ω) \to M_n(Ω)$ given by $Θ(∑p_{ij}e_{ij}) = ∑θ(p_{ij})e_{ij}$ is an endomorphism of $M_n(Ω)$. From the definition of $θ$ if follows that $Θ(ξ^{(k,A)}_{ij}) = ξ^{(k)}_{ij}$ and therefore $Θ(G(A)) = G(B)$. The restriction to $G(A)$ gives the desired homomorphism (also denoted by $Θ$).

Corollary 3.6 Let $B$ be a subalgebra of $M_n(F)$ generated by elementary matrices with the induced $G$-grading and $U_0$ be the $T_G$-ideal generated by the identities (3) – (5) satisfied by the graded algebra $B$. If $m(x^{(1)}_{h_1}, \ldots, x^{(p)}_{h_p})$
and $n(x_1^{(1)}, \ldots, x_{hp}^{(p)})$ are two monomials in $F \langle X \rangle$ such that the matrices
$m(\xi_1^{(1)}, \ldots, \xi_{hp}^{(p)})$ and $n(\xi_1^{(1)}, \ldots, \xi_{hp}^{(p)})$ have in the same position the same
nonzero entry then
\[ m(x_1^{(1)}, \ldots, x_{hp}^{(p)}) \equiv n(x_1^{(1)}, \ldots, x_{hp}^{(p)}) \pmod{U_0}. \]

Proof. Let $\tilde{m}(x_1^{(1)}, \ldots, x_{gn}^{(n)})$ be a monomial in $F \langle X \rangle$. Let $\Theta$ be the homomorphism constructed in the previous remark. We have
\[ \Theta(\tilde{m}(\xi_1^{(i_1,A)} \cdots \xi_{gn}^{(i_n,A)})) = \tilde{m}(\xi_1^{(i_1)} \cdots \xi_{gn}^{(i_n)}). \] (6)

It follows from Lemma 3.1 that the entries of $\tilde{m}(\xi_1^{(i_1,A)} \cdots \xi_{gn}^{(i_n,A)})$ are monomials in $\Omega$. Note that if $p$ is a monomial in $\Omega$ then $\theta(p)$ is either 0 or $p$. Hence (6) implies that the nonzero entries of $\tilde{m}(\xi_1^{(i_1)} \cdots \xi_{gn}^{(i_n)})$ equal the corresponding entries of $\tilde{m}(\xi_1^{(i_1,A)} \cdots \xi_{gn}^{(i_n,A)})$. Thus the matrices $m(\xi_1^{(i_1,A)} \cdots \xi_{hp}^{(p,A)})$ and $n(\xi_1^{(i_1,A)} \cdots \xi_{hp}^{(p,A)})$ have in the same position the same nonzero entry. Therefore Proposition 3.4 implies that $m \equiv n \pmod{U_A}$. The result is then a consequence of the inclusion $U_A \subseteq U_0$. To prove this we verify that every generator of $U_A$ is in $U_0$ and this follows from the inclusion $G_0 \subset G^A_0$. \hfill $\square$

**Theorem 3.7** Let $G$ be a group and let $g = (g_1, \ldots, g_n) \in G^n$ induce an elementary $G$-grading of $M_n(F)$, where the elements $g_1, \ldots, g_n$ are pairwise
different. If $B$ is a subalgebra of $M_n(F)$ generated by elementary matrices $e_{ij}$
then a basis of the graded polynomial identities of $B$ consists of (3) – (5) and
a finite number of identities of the form $x_{h1} \cdots x_{hp}$, where $2 \leq p \leq 2n - 1$.

Proof. Let $U$ be the $T_G$-ideal of $F \langle X \rangle$ generated by the polynomials (3) – (5)
together with the monomial identities $x_1^{(1)} \cdots x_{hp}^{(p)}$ of $B$ with $2 \leq p \leq 2n - 1$.
It follows from Lemma 3.3 that $U \subseteq T_G(B)$. Hence to prove the theorem it is
enough to show that every multihomogeneous $G$-graded identity of $B$ lies in $U$.
Assume, on the contrary, that $f$ is a multihomogeneous graded identity
that does not lie in $U$. We write $f \equiv \sum_{i=1}^{k} \alpha_i m_i \pmod{U}$, where the $\alpha_i$ are
non-zero scalars and $m_i$ are monomials in $F \langle X \rangle$. We may assume that the
number $k$ of nonzero coefficients is minimal. If $k = 1$ then $m_1$ is an identity
for $B$ and Corollary 3.2 implies that it lies in $U$ which is a contradiction. We
now consider $k > 1$. Denote by $\overline{m_i}$ the matrix in $G(B)$ that is the result of
substituting every variable $x_g^{(j)}$ in $m_i$ for the corresponding generic matrix
$\xi_g^{(i)}$. By the minimality of $k$ the monomials $m_i$ are not identities for $B$ and
in particular \( \overline{m_1} \) has a nonzero entry. Moreover we have

\[-\alpha_1 \overline{m_1} = \sum_{i=2}^{k} \alpha_i \overline{m_i}.\]

It follows from Lemma 3.1 that the nonzero entries of the matrices \( \overline{m_i} \) are monomials in \( \Omega \). Therefore there exists a \( j > 1 \) such that \( \overline{m_j} \) and \( \overline{m_1} \) have in the same position the same nonzero entry. Thus Corollary 3.6 implies that \( m_1 \equiv m_j \) modulo \( U \). Hence \( f \equiv (\alpha_1 + \alpha_j)m_1 + \sum_{i \neq j} m_i \) modulo \( U \). This last polynomial is an identity for \( B \) that does not lie in \( U \) with fewer nonzero coefficients than \( f \) and this is a contradiction. □

Recall that a \( G \)-grading on an algebra \( A \) is called nondegenerate if for every integer \( r \) and any tuple \( (g_1, \ldots, g_r) \in G^r \) the monomial \( x_e^{(1)} \cdots x_e^{(r)} \) is not a graded identity for \( A \) (see [2, Observation 2.2]). A stronger condition is that \( A g A h = A gh \) for every \( g, h \in G \) and in this case the grading is called strong. The \( \mathbb{Z}^n \)-grading considered by Vasilovsky in [36] is strong and in particular nondegenerate and the basis determined consists of (1) and (2).

In the next corollary we consider elementary \( G \)-gradings on \( M_n(F) \) that are closely related to this grading.

**Corollary 3.8** Let \( G \) be a finite group with unit \( e \) and let \( M_n(F) \) be endowed with an elementary grading such that \( x_e^{(1)} x_e^{(2)} - x_e^{(2)} x_e^{(1)} \) is a graded polynomial identity. If the \( G \)-grading is nondegenerate then a basis for the graded identities of \( M_n(F) \) consists of the polynomials (3) and (4). Moreover in this case the grading is strong and \( G \) is a group of order \( n \).

**Proof.** Let \( g = (g_1, \ldots, g_n) \in G^n \) be a tuple inducing the elementary grading. If \( g_i = g_j \) for some \( i \neq j \) then the elementary matrices \( e_{ij} \) and \( e_{ji} \) have degree \( e \) and \( e_{ij} e_{ji} - e_{ji} e_{ij} \neq 0 \). Hence \( x_e^{(1)} x_e^{(2)} - x_e^{(2)} x_e^{(1)} \) is not a graded identity and this is a contradiction. Thus the elements in the tuple \( g \) are pairwise different. Since the grading is nondegenerate it follows from Theorem 3.7 that the polynomials (3) and (4) are a basis for the graded identities. Now we prove the last assertion. Note that since \( g \) consists of pairwise different elements we have \( |G| \geq n \). We claim that if \( |G| > n \) then there exists \( g_1, \ldots, g_n \in G \) such that \( x^{(1)}_{g_1} \cdots x^{(n)}_{g_n} \) is a graded identity. Clearly it follows from this claim that \( |G| = n \). We construct the sequence as follows: since \( |G| > n \) we let \( g_1 \in G \) such that none of the elementary matrices \( e_{11}, e_{12}, \ldots, e_{1n} \) have degree \( g_1 \). Clearly the first line of \( \xi^{(1)}_{g_1} \) is zero. Then we choose \( g_2 \) such that the second line of \( \xi^{(1)}_{g_1 g_2} \) is zero. Inductively we
choose $g_i$ such that the $i$-th line of $\xi_g^1$ is zero, where $g = g_1 \cdots g_i$. Note that the first line of $\xi_{g_1}^{(1)} \xi_{g_2}^{(2)}$ is zero since the first line of $\xi_{g_1}^{(1)}$ is zero. Moreover the second line of $\xi_{g_1}^{(1)} \xi_{g_2}^{(2)}$ is also zero because the second line of $\xi_{g_1g_2}^{(1)}$ is zero. It follows by induction that the first $i$ lines of $\xi_{g_1}^{(1)} \cdots \xi_{g_i}^{(i)}$ are zero. Hence $\xi_{g_1}^{(1)} \cdots \xi_{g_n}^{(n)} = 0$ and it follows from Lemma 3.1 that $x_{g_1}^{(1)} \cdots x_{g_n}^{(n)}$ is a graded identity for $M_n(F)$. Now we prove that the grading is strong. Let $g \in G$. Note that for each $i$ the elementary matrices $e_{i1}, \ldots, e_{in}$ have pairwise different degrees and since $|G| = n$ the sequence of degrees is just a reordering of the elements of $G$. Thus there exists $j$ such that $e_{ij}$ has degree $g$. Hence we obtain $A_g A_h = A_{gh}$ for any $g, h \in G$. 

Remark 3.9 The proof of the last assertion in the previous lemma is based on the proof of Lemma 3.3 in [2]. In this lemma a characterization of non-degenerate gradings on finite dimensional $G$-simple algebras is given.

Corollary 3.10 Let $G$ be a group. If $UT(d_1, \ldots, d_n)$ has an elementary grading such that the polynomials (3) and (4) are a basis for the graded polynomial identities of this graded algebra then $n = 1$, i.e., $UT(d_1, \ldots, d_n) = M_{d_1}(F)$.

Proof. If $n > 1$ then we apply the previous lemma to each block $A_{ii}$ to obtain a monomial that is a graded identity for $M_{d_i}$. The product of copies of these monomials in disjoint sets of variables is a monomial $m$ such that the result of any substitution lies in the jacobson radical $J$ of $UT(d_1, \ldots, d_n)$. Since $J$ is a nilpotent ideal, say $J^k = 0$, the product of $k$ copies of $m$ in disjoint sets of variables is a monomial identity. 

4 Matrices over the Grassmann algebra

We now turn our attention to matrices over the Grassmann algebra.

In this section we suppose $F$ is a field of characteristic zero and we denote by $E$ the Grassmann algebra of an infinite dimensional vector space over $F$ with its natural $\mathbb{Z}_2$-grading $E = E_0 \oplus E_1$ induced by the length of its monomials. For more information concerning the Grassmann algebra, see [19].

We use the results of the previous sections and results of [15] to find a basis for the $G \times \mathbb{Z}_2$-graded polynomial identities of $UT(d_1, \ldots, d_n; E)$: the algebra of block-triangular matrices over the Grassmann algebra, which is isomorphic to the tensor product $UT(d_1, \ldots, d_n) \otimes E$, and more generally of
the algebra $B \otimes E$, where $B$ is a $G$-graded subalgebra of $M_n(F)$ generated by elementary matrices with an elementary grading induced by an $n$-tuple $(g_1, \ldots, g_n)$ of pairwise distinct elements of $G$.

If $B$ is a $G$-graded algebra the algebra $B \otimes E$ has a natural $G \times \mathbb{Z}_2$-grading induced by the gradings of $B$ and of $E$. In such grading, the homogeneous component of degree $(g, \delta)$ is $(B \otimes E)_{(g, \delta)} = B_g \otimes E_\delta$.

In order to work with the $G \times \mathbb{Z}_2$-graded identities of $B \otimes E$, we now consider the free associative algebra $F \langle Z \rangle$, with $Z = X' \cup Y'$, where $X' = \bigcup X_g'$ is the set of graded variables with $G \times \mathbb{Z}_2$-degree $(g,0)$ and $Y' = \bigcup Y_g'$ is the set of graded variables with $G \times \mathbb{Z}_2$-degree $(g,1)$. We denote elements of $X_g'$ and $Y_g'$ respectively by $x_g^{(i)}$ and $y_g^{(i)}$, for $i \in \mathbb{N}$ and $g \in G$. From now on, the variables labeled as $z_g^{(i)}$ may be $x_g^{(i)}$ or $y_g^{(i)}$.

Recall that in [15] the authors define a map $\zeta_J$, for $J \subseteq \mathbb{N}$, which maps multilinear identities of $B$ into identities of $B \otimes E$. Such map is defined as follows.

First, we observe that $F\langle Z \rangle$ is both a $\mathbb{Z}_2$-graded algebra and $G$-graded algebra. Concerning the $\mathbb{Z}_2$-grading of $F\langle Z \rangle$, one defines the map $\zeta$ as follows. If $m$ is a multilinear monomial let $i_1 < \cdots < i_k$ be the indices with odd $\mathbb{Z}_2$-degree occurring in $m$. Then for some $\sigma \in \text{Sym}\{i_1, \ldots, i_k\}$, we write

$$m = m_0 y_{g_i \sigma(i_1)} m_1 \cdots y_{g_k \sigma(i_k)} m_{k+1}$$

where $m_0, \ldots, m_{k+1}$ are monomials on even variables only. Then we define

$$\zeta(m) = (-1)^\sigma m$$

**Definition 4.1** Let $J \subseteq \mathbb{N}$. We define $\varphi_J : F \langle X \rangle \to F \langle Z \rangle$ to be the unique $G$-homomorphism of algebras defined by

$$\varphi_J(x_g^{(i)}) = \begin{cases} x_g^{(i)} & \text{if } i \notin J \\ y_g^{(i)} & \text{if } i \in J \end{cases}$$

Also for a multilinear monomial $m$ we define $\zeta_J(m) = \zeta(\varphi_J(m))$

The map $\zeta_J$ extends by linearity to the space of all multilinear polynomials in $F\langle X \rangle$ and for each multilinear polynomial in $F\langle X \rangle$, $\zeta_J(f)$ is also a multilinear polynomial in $F\langle Z \rangle$.

We now recall Theorem 11 of [15].

**Theorem 4.2** Let $A$ be a $G$-graded algebra and $\mathcal{E} \subset F\langle X \rangle G$ be a system of multilinear generators for $T_G(A)$. Then the set

$$\{\zeta_J(f) \mid f \in \mathcal{E}, J \subseteq \mathbb{N}\}$$
is system of multilinear generators of $T_{G \times \mathbb{Z}_2}(A \otimes E)$

Since the basis of the graded polynomial identities of $UT(d_1, \ldots, d_m)$, described in Theorem 3.7 contains polynomials in at most $2n - 1$ variables, it is enough to consider $J \subset \{1, \ldots, 2n - 1\}$.

**Lemma 4.3** Applying the map $\zeta_J$ to the polynomial $x_e^{(1)} x_e^{(2)} - x_e^{(2)} x_e^{(1)}$ we obtain up to endomorphisms of $F \langle X \rangle$ the following polynomials

\[
x_e^{(1)} z_e^{(2)} - z_e^{(2)} x_e^{(1)}
\]

Applying the map $\zeta_J$ (for $J \subseteq \{1, 2, 3\}$) to the polynomial $x_g^{(1)} x_{g^{-1}}^{(2)} x_g^{(3)} - x_g^{(3)} x_{g^{-1}}^{(2)} x_g^{(1)}$ we obtain up to endomorphisms of $F \langle X \rangle$ the polynomials

\[
x_g^{(1)} y_{g^{-1}}^{(2)} z_g^{(3)} - z_g^{(3)} y_{g^{-1}}^{(2)} x_g^{(1)}
\]

Finally, if $m = x_{g_1}^{(1)} \cdots x_{g_p}^{(p)}$ is a $G$-graded monomial identity of the algebra $B$, generated by elementary matrices, for some $1 \leq p \leq 2n - 1$ then up to some endomorphism of $F \langle X \rangle$, $\zeta_J(m) = z_{g_1}^{(1)} \cdots z_{g_p}^{(p)}$.

**Proof.** The proof consist of several applications of the map $\zeta_J$ for $J \subseteq \mathbb{N}$. For $f_1 = x_e^{(1)} x_e^{(2)} - x_e^{(2)} x_e^{(1)}$, it is enough to consider $J \subseteq \{1, 2\}$. So consider $J = \{1\}$ and $J = \{2\}$. Then $\zeta_{\{1\}}(f_1) = y_e^{(1)} x_e^{(2)} - x_e^{(2)} y_e^{(1)}$ and $\zeta_{\{2\}}(f_1) = -(y_e^{(2)} x_e^{(1)} - x_e^{(1)} y_e^{(2)})$, and the latter is the image of the former, by the endomorphism of $F \langle Z \rangle$, which permutes the indexes 1 and 2 of the variables and multiply the result by $-1$. For this reason, up to an endomorphism of $F \langle Z \rangle$, the image of $f_1$, for $|J| \leq 1$ is $x_e^{(1)} z_e^{(2)} - z_e^{(2)} x_e^{(1)}$. If $J = \{1, 2\}$, one obtains $\zeta_J = y_e^{(1)} y_e^{(2)} + y_e^{(2)} y_e^{(1)}$.

Similarly, one obtains the images by $\zeta_J$ of the polynomials

\[
f_2 = x_g^{(1)} x_{g^{-1}}^{(2)} x_g^{(3)} - x_g^{(3)} x_{g^{-1}}^{(2)} x_g^{(1)}
\]

By applying the above lemma and theorem we obtain:
Corollary 4.4 Let $F$ be a field of characteristic zero, $G$ be a group and let $g = (g_1, \ldots, g_n) \in G^n$ induce an elementary $G$-grading of $M_n(F)$, where the elements $g_1, \ldots, g_n$ are pairwise different. If $B$ is a subalgebra of $M_n(F)$ generated by matrix units $e_{ij}$, then a basis of the graded polynomial identities of the algebra $B \otimes E$ consists of the polynomials $(10) - (16)$ and a finite number of identities of the form $z_{g_1}^{(1)} \cdots z_{g_p}^{(p)}$, with $2 \leq p \leq 2n - 1$, for each $g_1, \ldots, g_p \in G_0$ such that $x_{g_1}^{(1)} \cdots x_{g_p}^{(p)}$ is a graded identity of $B$.

Remark 4.5 It interesting to observe that in characteristic $p$ case the map $\zeta_J$ also maps multilinear identities of $B$ into multilinear identities of $B \otimes E$. But such identities may not be enough to generate all $G \times \mathbb{Z}_2$-graded identities of $B \otimes E$, since in positive characteristic, the identities may not be generated by the multilinear ones.

As an example one can consider the field $F$, as the algebraic closure of the prime field $\mathbb{Z}_p$, graded by the trivial group. The ideal of identities of $F$ are generated by the polynomial $[x_1, x_2]$, but the algebra $F \otimes E$, which is isomorphic to $E$, satisfies the $\mathbb{Z}_2$-graded identity

$$St_p(y_1, \ldots, y_p) = \sum_{\sigma \in S_p} (-1)^{\sigma} y_{\sigma(1)} \cdots y_{\sigma(p)}$$

which is not in the $T_{\mathbb{Z}_2}$-ideal generated by the image of $[x_1, x_2]$ by $\zeta_J$.

Problems involving relations between identities in positive characteristic and in characteristic zero are quite difficult. See for example [29, Problem e), p. 185].

Acknowledgements

This work was completed while the first author was a postdoctoral fellow at Memorial University of Newfoundland. He would like to thank prof. Yuri Bahturin for useful discussions on this subject.

References

[1] E. Aljadeff, A. Kanel-Belov, Representability and Specht problem for $G$-graded algebras, Adv. Math., 225 (5), (2010), 2391–2428.

[2] E. Aljadeff, D. Ofir On group gradings on PI-algebras, J. Algebra 428 (2015), 403–424.

[3] E. Aljadeff, D. Ofir On regular $G$-gradings, Trans.; Amer. Math. Soc. (2015), in press.
[4] S. S. Azevedo, *Graded identities for the matrix algebra of order n over an infinite field*, Comm. Algebra 30 (2002), no. 12, 5849–5860.

[5] S. S. Azevedo, *A basis for \(\mathbb{Z}\)-graded identities of matrices over infinite fields*, Serdica Math. J. 29 (2003), no. 2, 149–158.

[6] Yu. Bahturin, V. Drensky, *Graded polynomial identities of matrices*, Linear Algebra and its Applications, 357 (2002), 15–34.

[7] Yu. Bahturin, M. V. Zaicev *Group gradings on matrix algebras*, Canad. Math. Bull. 45 (2002), no. 4, 499–508.

[8] M Bărăscu, S. Dăscălescu *Good gradings on upper block triangular matrix algebras*, Communications in Algebra, 41 (2013), 4290–4298.

[9] A. Ya. Belov, *On non-Specht varieties* (Russian), Fundam. Prikl. Mat. 5 (1999), No. 1, 47–66.

[10] L. Centrone, T. C. de Mello, *On \(\mathbb{Z}_n\)-graded identities of block-triangular matrices*, Linear and Multilinear Algebra 63 (2015), no. 2, 302–313.

[11] O. M. Di Vincenzo, *On the graded identities of \(M_{1,1}(E)\)*, Israel J. Math. 80 (1992), no.3, 323–335.

[12] O. M. Di Vincenzo, R. La Scala, *Block-triangular matrix algebras and factorable ideals of graded polynomial identities*.

[13] O. M. Di Vincenzo, V. Nardozza, *Graded polynomial identities of verbally prime algebras*, Journal of Algebra and its Applications, Vol. 6, No. 3 (2007), 385–401.

[14] O. M. Di Vincenzo, V. Nardozza, *\(\mathbb{Z}_{k+1}\times\mathbb{Z}_2\) identities for \(M_{k,l}\otimes E\)*, Rend. Sem. Mat. Univ. Padova, Vol. 108 (2002)

[15] O. M. Di Vincenzo, V. Nardozza, *Graded polynomial identities for tensor products by the Grassmann Algebra*, Comm. Algebra, Vol. 31 (2003), no. 3, 1453–1474.

[16] O. M. Di Vincenzo, P. Koshlukov, A. Valenti *Gradings on the algebra of upper triangular matrices and their graded identities*, Journal of Algebra 275 (2004) 550–566.

[17] O. M. Di Vincenzo, E. Spinelli *Graded polynomial identities on upper block triangular matrix algebras*, Journal of Algebra 415 (2014), 50–64.
[18] D. Diniz, *On the graded identities for elementary gradings in matrix algebras over infinite fields*, Linear Algebra and its Applications 439 (2013), 1530–1537.

[19] V. Drensky, *Free algebras and PI-algebras: Graduate Course in Algebra*, Springer, Singapore, (1999).

[20] V. Drensky, *A minimal basis for a second-order matrix algebra over a field of characteristic 0*, Algebra i Logika 20, no.3 (1980), 282–290 [in Russian]; Algebra and Logic 20, no. 3 (1981), 188–194 [Engl. transl.].

[21] A. Giambruno, M. Zaicev, *Minimal varieties of algebras of exponential growth*, Adv. Math. 174 (2003), no. 2, 310–323.

[22] A. Giambruno, M. Zaicev, *Polynomial Identities and Asymptotic Methods*, Mathematical Surveys and Monographs, Volume 122 (2005).

[23] A. V. Grishin, *Examples of T-spaces and T-ideals over a field of characteristic 2 without the finite basis property* (Russian, English summary), Fundam. Prikl. Mat. 5 (1999), No. 1, 101–118.

[24] A. Kemer, *Solution of the problem as to whether associative algebras have a finite basis of identities*, Dokl. Akad. Nauk SSSR 298 (1988), 273–277; translation in Soviet Math. Dokl. 37 (1988), 60–64.

[25] A. Kemer, *Ideals of Identities of Associative Algebras*, Translations of Monographs, vol. 87, Amer. Math. Soc., Providence, RI, 1991.

[26] P. Koshlukov, *Basis of the identities of the matrix algebra of order two over a field of characteristic p ≠ 2*, J. Algebra 241 (2001), 410–434.

[27] P. Koshlukov, S. S. Azevedo, *Graded identities for T-prime algebras over fields of positive characteristic*, Israel J. Math. 128 (2002), 157–176.

[28] Yu. N. Maltsev, *A basis for the identities of the algebra of upper triangular matrices* (Russian), Algebra i Logika 10 (1971), 393–400. Translation: Algebra and Logic 12 (1973), 242–247.

[29] C. Procesi, *Rings with Polynomial Identities*, Dekker, New York, 1973.

[30] Yu. P. Razmyslov, *Finite basing of the identities of a matrix algebra of second order over a field of characteristic zero*, Algebra i Logika 12, no. 1 (1973), 83–113 [in Russian]; Algebra and Logic 12 (1973), 47–63 [Engl. transl.].
[31] V. V. Shchigolev, *Examples of infinitely based T-ideals* (Russian, English summary), Fundam. Prikl. Mat. 5 (1999), No. 1, 307–312.

[32] I. Sviridova, *Identities of pi-algebras graded by a finite abelian group*, Comm. Algebra 39 (2011), no. 9, 3462–3490.

[33] A. Valenti, M. V. Zaicev, *Group gradings on upper triangular matrices* Arch. Math. 89 (2007), 33–40.

[34] A. Valenti, M. V. Zaicev, *Abelian gradings on upper block triangular matrices* Canad. Math. Bull. 55 (2012), no. 1, 208–213.

[35] S. Yu. Vasilovsky *Z-graded polynomial identities of the full matrix algebra* Comm. Algebra 26 (1998), no. 2, 601–612.

[36] S. Yu. Vasilovsky *Zn-graded polynomial identities of the full matrix algebra of order n* Proc. Amer. Math. Soc. 127 (1999), no. 12, 3517–3524.