The Scalar Einstein-aether theory

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We consider an Einstein-aether type Lorentz-violating theory of gravity in which the aether vector field $V_\mu$ is represented as the gradient of a scalar field $\phi$, $V_\mu = \nabla_\mu \phi$. A self interacting potential for the scalar aether field is considered, as well as the possibility of a coupling between the hydrodynamic matter flux and the aether field, with the imposition of the timelike nature of the aether vector. The gravitational field equations and the equation of motion of the scalar field are derived by varying the action with respect to the metric and $\phi$. In the absence of matter flux and scalar field coupling the effective energy-momentum tensor of the scalar aether is conserved. The matter flux-aether coupling generates an extra force acting on massive test particles and consequently the motion becomes non-geodesic. The Newtonian limit of the theory is investigated and the generalized Poisson equation for weak gravitational fields is obtained. The cosmological implications of the theory are also considered and it is shown that in the framework of the Scalar Einstein-aether theory both decelerating and accelerating cosmological models can be constructed.

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I. INTRODUCTION

In 1955 G. Szekeres\textsuperscript{[1]} proposed an extension of general relativity in which the cosmic time is introduced as a new field variable. The proposed formalism also allowed for a rigorous definition of the concept of aether. In proposing such an extension of general relativity Szekeres was motivated by the contradiction between the principle of equivalence, that all frames of reference are equivalent, and that there are no physical processes which would distinguish one particular frame from the other, and the Weyl postulate, fundamental in cosmology, which requires the existence of an absolute time. In order to give a rigorous definition of the aether, Szekeres introduced a scalar field variable $\phi$, called the cosmic time and postulated an interaction between fields associated with the metric and the $\phi$ fields. The introduction of the cosmic time field $\phi$ allows for the definition of the aether, which “is a state of motion determined uniquely by the gradient of $\tau$ at every point of the space time continuum”\textsuperscript{[1]}. With the help of the gradient $Q_\mu$ of the cosmic time, $Q_\mu = \nabla_\mu \phi = \partial \phi / \partial x^\mu$, Szekeres constructed the action of the gravitational field as

$$S = \int \left\{ R + \left[ \frac{1}{2} \beta (\nabla^\nu Q^\mu \nabla_\nu Q_\mu) - \frac{\gamma}{\phi^2} \right] + L_m \right\} \sqrt{-g} d^4x,$$

(1)

where $R$ is the curvature scalar, $L_m$ the matter action, and $\beta$ and $\gamma$ are constants. The term $-\gamma / \phi^2$ in the action describes the effect of the cosmological constant on the gravitational dynamics. The gravitational field equations corresponding to action (13) were obtained in [1] and their physical implications (cosmological models, spherically symmetric vacuum solution, planetary orbits, gravitational energy and gravitational waves) were investigated in detail. An extension of the aether model was considered in [2], where a more general Lagrangian of the form $L = R + \gamma_1 S_1 + \gamma_2 S_2$ was constructed with $\gamma_1, \gamma_2$ being constants and $S_1, S_2$ are scalar densities formed from the cosmic time field $\phi$ and the metric tensor components, representing the energy density of the cosmic time and the interaction of the $\phi$ and $g_{\mu\nu}$ fields, respectively. The authors adopt $S_1 \sim \phi^{-2}$, and consider two choices for $S_2$, $S_2 = (C_\mu^\nu C_\nu^\mu)^2$ and $S_2 = (C_\mu^\nu C_\nu^\mu)$, respectively, where $C_\mu^\nu = \nabla_\mu Q_\nu$. The physical implications of this model were also analyzed in detail. The gravitational motion in the presence of aether drift was considered in [3]. It was shown that the path of a free particle is geodesic provided that its absolute velocity is small compared to the velocity of light. Hence the aether
drift does not have a direct effect on the motion of particles in slow motion. However, it influences and modifies the external gravitational field of a massive body and the associated geodesics. In the case of the field generated by a spherically symmetric object, due to the action of the aether, some asymmetry is present which makes it possible to detect and measure the aether drift vector $\partial \phi / \partial x^\mu$ experimentally. The effect is small, of the same order of magnitude as that of the general relativistic corrections to the Newtonian theory, with the velocity of aether drift in the Solar System being of the order of 100 km/s.

Recently T. Jacobson has proposed a Lorentz-violating theory of gravity with an “aether” vector field $V_\mu$, determining a preferred rest frame at each space-time point; the so-called Einstein-aether (EA) gravity theory [4]. More precisely, $V_\mu$ breaks local boost invariance, while rotational symmetry in a preferred frame is preserved [5]. The most general action for the pure EA theory is given by [6]

$$ S_{ae} = \frac{1}{2k^2} \int d^4x \sqrt{-g} \left[ R + K_{\lambda\sigma}^{\mu\nu} \nabla_\mu V^\lambda \nabla_\nu V^\sigma + \lambda (V^\mu V^\mu + 1) \right] + S_m, $$

(2)

where the Lagrange multiplier $\lambda$ constrains the vector field $V^\mu$ being timelike. The tensor $K_{\lambda\sigma}^{\mu\nu}$ is given by

$$ K_{\lambda\sigma}^{\mu\nu} = c_0 g^{\mu\nu} g_{\lambda\sigma} + c_1 \delta_\mu^\nu \delta_\lambda^\sigma + c_2 \delta_\mu^\lambda \delta_\nu^\sigma + c_3 V^\nu V^\nu g_{\lambda\sigma}, $$

(3)

where $c_i$, $i = 0, 1, 2, 3$ are the dimensionless free parameters of the EA theory. The action given by Eq. (2) extends the standard Einstein-Hilbert action for the metric with the addition of a kinetic term for the aether, containing four dimensionless coefficients $c_i$, $i = 0, 3$ which couple the aether to the metric through the covariant derivatives and a non-dynamical Lagrange multiplier $\lambda$.

One of the other representations of a Lorentz violating extension of general relativity was proposed by P. Horava [7]. The Horava-Lifshitz (HL) theory was written as an attempt to build a UV completion of general relativity by adding higher order spatial derivatives to the theory without adding higher order time derivatives. This results in modification of the graviton propagator in such a way that the theory becomes power counting renormalizable. Horava assumed a preferred space-like foliation of space-time which can be described by a scalar field and the lapse function $N$ which breaks local boost invariance, while rotational symmetry in a preferred frame is preserved [5]. However, it was shown that this scalar mode for gravity causes problems such as instability and strong coupling at low energies [8]. This problem can be avoided if one adds all the possible terms which respect the symmetry of the theory to the action [9]. One can then obtain a theory where its strong coupling scale is pushed to sufficiently high energies. The theory can be considered as an effective field theory which we denoted by BPS theory.

The HL and the EA theories are both modifications of gravity which break the Lorentz symmetry. This suggests that these theories may be related to each other. In fact, in the limit where higher than second order derivative terms of the HL theory can be ignored (which corresponds to the IR limit of the theory), one obtains the EA theory with an additional constraint that the aether vector should be hypersurface orthogonal [9, 10]. Moreover, because all spherically symmetric solutions are hypersurface orthogonal, one can expect that all spatial derivatives of $N$ vanish. Also many different terms become identical up to a total derivative which makes the calculations tractable. However, it was shown that this scalar mode for gravity causes problems such as instability and strong coupling at low energies [8]. This problem can be avoided if one adds all the possible terms which respect the symmetry of the theory to the action [9]. One can then obtain a theory where its strong coupling scale is pushed to sufficiently high energies. The theory can be considered as an effective field theory which we denoted by BPS theory.

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An interesting alternative dark matter model was introduced by Milgrom [14] in which Newton’s second law is modified for very small accelerations, commonly known as MOND. A relativistic version of MOND was proposed by Bekenstein [15] where gravity is mediated by three fields, a tensor field with an associated metric compatible connection, a timelike one form field, and a scalar field respectively. However, in [16] it was shown that Bekenstein’s theory can be reformulated as a Vector-Tensor theory akin to EA theory with non-canonical kinetic terms. The total TeVeS action can be entirely written in the matter frame, and is given by

$$ S_{TeVeS} = \frac{1}{16\pi G} \int \left[ R + \tilde{K}^{abmn} \nabla_a A_m \nabla_b A_n + \frac{V(\mu)}{A^2} \right] \sqrt{-g} d^4x + S_m \left[ g^{ab} \right], $$

(4)
where $\mu$ is a non-dynamical field.

The physical and cosmological implications of the EA type theories have been intensively investigated. Time independent spherically symmetric solutions of this theory were studied in [17] and a three-parameter family of solutions was found. Asymptotic flatness restricts the solutions to a two parameter class and requiring the aether to be aligned with the timelike Killing field further restricts them to one parameter, the total mass. The static aether solutions are given analytically up to the solution of a transcendental equation. Black Holes in EA theory were studied in [18]. To be causally isolated, a black hole interior must trap matter fields as well as all aether and metric modes. The theory possesses spin-0, spin-1, and spin-2 modes whose speeds depend on the four coupling coefficients. The gravitational spectrum of black holes in the EA theory was considered in [19], while numerical simulations of the gravitational collapse, neutron star structure, strong field effects and generic properties of black holes were analyzed in [20]. Post-Newtonian approximations, solar system and galactic and extra-galactic tests of the theory were obtained and discussed in [21]. By coupling a scalar field to the timelike vector in [22] it was shown via a tunneling approach that the universal horizon radiates as a black body at a fixed temperature even if the scalar field equations also violate local Lorentz invariance. A comprehensive study of the cosmological effects of the EA theory was performed in [23] and observational data were used to constrain it. In conjunction with the previously determined consistency and experimental constraints, it was found that an EA universe can fit observational data over a wide range of its parameter space but requires a specific re-scaling of the other cosmological densities. Another interesting application of aether theory in cosmology has been discussed in [24] where the authors proposed a new class of theories where energy always flows along timelike geodesics, mimicking dark energy.

The primordial perturbations generated during a stage of single-field inflation were analyzed in [25]. Quantum fluctuations of the inflaton and aether fields would seed long wavelength adiabatic and isocurvature scalar perturbations, as well as transverse vector perturbations. Scalar and vector perturbations may leave significant imprints on the cosmic microwave background. The primordial spectra and their contributions to temperature anisotropies were obtained and some of the phenomenological constraints that follow from observations were formulated. The linear perturbation equations were constructed in a covariant formalism and the CMB B-mode polarization, using the CAMB code, was modified so as to incorporate the effects of the aether vector field [26]. Several families of accelerating universe solutions to an EA gravity theory were derived in [27]. These solutions provide possible descriptions of inflationary behavior in the early universe and late-time cosmological acceleration. By taking a special form of the Lagrangian density of the aether field it was shown in [28] that the EA theory may represent an alternative to the standard dark energy model. A dynamical systems analysis to investigate the future behavior of EA cosmological models with a scalar field coupling to the expansion of the aether and a non-interacting perfect fluid was performed in [29]. The stability of the equilibrium solutions were analyzed and the results were compared to the standard inflationary cosmological solutions and previously studied cosmological EA models. A class of spatially anisotropic cosmological models in EA theory with a scalar field in which the self-interacting potential depends on the timelike aether vector field through the expansion and shear scalars was investigated in [30]. The cosmological evolution of EA models with a power-law like potential, using the method of dynamical systems, was studied in [31]. In the absence of matter, there are two attractors which correspond to an inflationary universe in the early epoch, or a de Sitter universe at late times. In the case where matter is present, if there is no interaction between dark energy and matter, there are only two de Sitter attractors and no scaling attractor exists. The consequences of Lorentz violation during slow-roll inflation were analyzed in [32]. If the scale of Lorentz violation is sufficiently small compared to the Planck mass and the strength of the scalar-aether coupling is suitably large, then the spin-0 and spin-1 perturbations grow exponentially and spoil the inflationary background. The effects of such a coupling on the Cosmic Microwave Background (CMB) are too small to be visible to current or near-future CMB experiments.

It is the goal of the present paper to consider a scalar formulation of the EA theory where the aether four-vector $V_\mu$ can be represented as the gradient of a scalar function, so that $V_\mu = \nabla_\mu \phi$. Therefore we consider a gravitational theory where the standard Einstein-Hilbert action is modified by considering aether kinetic terms which are coupled to the metric via the coupling coefficients given by Eq. (3). Moreover, we impose the timelike constraint on the gradient of the aether scalar field via a Lagrange multiplier. A self-interacting potential for the scalar field is then included in the action. In addition, we consider the possibility of a coupling/interaction between the matter flux and the gradient of the scalar field. In this way the matter in motion would feel the effects of the aether which can directly influence the dynamics of massive test particles. The gravitational field equations corresponding to the scalar EA action are obtained by varying the metric as well as the scalar field. In the absence of matter current-aether field coupling the covariant divergence of the effective energy-momentum tensor corresponding to the aether field is zero. The matter flux-aether field coupling induces non-conservation of the energy-momentum tensor. The corresponding particle motion is non-geodesic and the equation of motion of a massive test particle is obtained for this case. The Newtonian limit of the theory and the generalized Poisson equation is then derived. We briefly consider the cosmological implications of the theory and show that, depending on the values of coupling constants $c_i$, $i = 0, 1, 2, 3$, a large number of cosmological solutions, both accelerating and decelerating, can be obtained.
The present paper is organized as follows. The gravitational field equations of the scalar EA theory are presented in Section II. The equation of motion of massive test particles, the Newtonian limit of the theory and the generalized Poisson equation are obtained in Section III. The cosmological implications of the theory are considered in Section IV. We discuss and conclude our results in Section V. The computation of the divergence of the energy-momentum tensor of the theory is presented in detail in the Appendix.

II. GRAVITATIONAL FIELD EQUATIONS OF THE SCALAR EA THEORY

The vector EA theory is defined by the action given by Eq. (2). In the following we consider that the aether vector field $V_\mu$ can be represented as the gradient of a scalar function, that is

$$V_\mu = \nabla_\mu \phi.$$  \hspace{1cm} (5)

After such substitution, it turns out that the terms with coefficients $c_0$ and $c_1$ becomes equal. We then propose the scalar EA (SEA) gravitational theory action as

$$S_{SEA} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[R + c_1 \nabla_\mu \nabla_\nu \phi \nabla^\mu \nabla_\nu \phi + c_2 (\Box \phi)^2 + c_3 \nabla^\mu \phi \nabla_\mu \phi \nabla^\sigma \nabla_\sigma \phi + c_4 \rho u^\sigma \nabla_\sigma \phi \right. \\
\left. - V(\phi) + \lambda (\nabla_\mu \phi \nabla^\mu \phi + \epsilon) \right] + S_m,$$  \hspace{1cm} (6)

where $\epsilon = \pm 1,$ and $c_4$ is a constant. For $\epsilon = 1$ we assume that, analogous to the vector EA case, the scalar function $\phi$ is normalized via $\nabla_\mu \phi \nabla^\mu \phi = -1.$ The term $\rho u^\sigma \nabla_\sigma \phi$ represents a possible interaction between the matter hydrodynamic flux $j^\sigma = \rho u^\sigma$ and the aether vector. We have also added to the action a self-interacting scalar field potential $V(\phi).$

One can also write the above action in a compact way as

$$S_{SEA} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[R + (\nabla_\mu \nabla_\nu \phi) K^{\mu\nu,\rho\sigma} (\nabla_\rho \nabla_\sigma \phi) + c_4 \rho u^\sigma \nabla_\sigma \phi - V(\phi) + \lambda (\nabla_\mu \phi \nabla^\mu \phi + \epsilon) \right] + S_m,$$  \hspace{1cm} (7)

where we have defined

$$K^{\mu\nu,\rho\sigma} = c_1 g^{\mu\rho} g^{\nu\sigma} + c_2 g^{\mu\nu} g^{\rho\sigma} + c_3 g^{\nu\sigma} \nabla^\mu \phi \nabla^\rho \phi.$$  \hspace{1cm} (8)

Varying action (6) with respect to the Lagrange multiplier $\lambda$ we immediately find that

$$\nabla_\mu \phi \nabla^\mu \phi = -\epsilon,$$  \hspace{1cm} (9)

which introduces a preferred direction for the space-time. Varying with respect to the metric leads to the equation of motion

$$G_{\mu\nu} + K_{\mu\nu} + \frac{1}{2} V(\phi) g_{\mu\nu} + \lambda \nabla_\mu \phi \nabla_\nu \phi = \kappa^2 T_{\mu\nu},$$  \hspace{1cm} (10)

where we have used equation (9) to simplify the above equation and we have defined

$$K_{\mu\nu} = c_1 \left( \nabla^\lambda (\nabla_\lambda \phi \nabla_\mu \phi \nabla_\nu \phi) - 2 \Box (\nabla_\mu \phi \nabla_\nu \phi) - \frac{1}{2} g_{\mu\nu} \nabla_\alpha \nabla_\beta \phi \nabla^\alpha \nabla^\beta \phi \right) \\
+ c_2 \left( g_{\mu\nu} \nabla^\lambda \phi \nabla_\lambda \Box \phi + \frac{1}{2} g_{\mu\nu} (\Box \phi)^2 - 2 \nabla_\lambda (\phi \nabla_\nu \phi) \Box \phi \right) + \frac{1}{2} c_4 T_{\mu\nu} u^\alpha \nabla_\alpha \phi.$$  \hspace{1cm} (11)

The equation of motion for the aether scalar becomes

$$c_1 \nabla_\nu \Box \nabla^\nu \phi + c_2 \Box^2 \phi - \frac{1}{2} c_4 \nabla_\mu (\rho u^\mu) - \frac{1}{2} \frac{dV}{d\phi} - \nabla_\mu (\lambda \nabla^\mu \phi) = 0.$$  \hspace{1cm} (12)

One should note that the $c_3$ term does not contribute to the equations of motion. This is because we have used the constraint equation (9) and its derivative

$$\nabla_\mu \nabla^\alpha \nabla^\mu \phi = 0.$$  \hspace{1cm} (13)
We also note that in the special case of $V(\phi) = 0$, the scalar equation becomes a total derivative $\nabla_\mu J^\nu = 0$, with

$$J^\nu = c_1 \Box \nabla^\nu \phi + c_2 \nabla^\nu \Box \phi - \frac{1}{2} c_4 \rho u^\nu - \lambda \nabla^\nu \phi,$$

which gives $\sqrt{-g} J^\nu = \text{const}$. Assuming that the constant being zero, and multiplying the whole equation by $\nabla_\nu \phi$, one can obtain the Lagrange multiplier $\lambda$ as

$$\epsilon \lambda = \frac{1}{2} c_4 \rho u^\nu \nabla_\nu \phi - c_1 \nabla^\nu \Box \nabla_\nu \phi - c_2 \nabla_\nu \phi \nabla^\nu \Box \phi.$$

In the following we assume that the matter content of the Universe is represented by a perfect fluid with energy density $\rho$ and thermodynamic pressure $p$, with energy-momentum tensor given by

$$T^\mu_\nu = (\rho + p) u^\mu u_\nu + p \delta^\mu_\nu,$$

where $u^\mu$ is the velocity four-vector of the matter. Using the equations of motion, one can prove that the terms proportional to $c_1, c_2$ and $c_3$ do not contribute to the covariant derivative of the ordinary matter energy-momentum tensor. One can then obtain (see the Appendix for details) the covariant divergence of the matter energy-momentum tensor as

$$\nabla^\mu T^\mu_\nu = \frac{c_4 \left[ T^\mu_\nu \nabla^\mu (u^\alpha \nabla_\alpha \phi) - \nabla_\alpha (\rho u^\alpha) \nabla^\nu \phi \right]}{2\kappa^2 - c_4 u^\beta \nabla^\beta \phi}.$$

If $c_4 = 0$, that is, we neglect the possible coupling between the matter flux $j^\alpha$ and the aether vector, the matter energy-momentum is conserved, $\nabla^\mu T^\mu_\nu = 0$.

### III. EQUATION OF MOTION OF A MASSIVE TEST PARTICLE AND THE NEWTONIAN LIMIT

In this Section we obtain the equation of motion for a massive test particle moving in a SEA universe. The Newtonian limit of the theory is considered and the generalized Poisson equation for the gravitational weak field is derived.

#### A. The equation of motion of massive test particles

Taking the divergence of Eq. (16) and defining the projection operator $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ one obtains

$$\nabla_\mu T^\mu_\nu = h^{\mu\nu} \nabla_\mu p + u^\nu u_\mu \nabla^\mu \rho + (\rho + p) \left( u^\nu \nabla_\mu u^\mu + u^\mu \nabla_\mu u^\nu \right).$$

Multiplying the equation above by $h^\nu_\mu$ we have

$$h^\lambda_\mu \nabla_\mu T^\mu_\nu = (\rho + p) \nabla_\mu \nabla^\mu \phi + h^{\nu\lambda} \nabla_\nu p,$$

where we have used the relation $u_\mu \nabla_\nu u^\nu = 0$. Noting that

$$u^\mu \nabla_\mu u^\lambda = \frac{d^2 x^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu} u^\mu u^\nu,$$

and using Eq. (17) we obtain the equation of motion for a massive test particle as

$$\frac{d^2 x^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu} u^\mu u^\nu = f^\lambda,$$

where

$$f^\lambda = \frac{h^{\lambda\nu}}{\rho + p} \left( c_4 \left[ \nabla_\nu (\rho u^\alpha \nabla_\alpha \phi) - \nabla_\nu \phi \nabla_\alpha (\rho u^\alpha) \right] - 2\kappa^2 \nabla_\nu \phi \right).$$

$f^\lambda$ is the extra force which leads to non-geodesic motion for a massive test particle in the SEA universe. We note that if $c_4$ vanishes, the above equation reduces to the standard geodesic equation for a perfect fluid. We also note that the extra force is perpendicular to the particle four-velocity, $f^\nu u_\nu = 0$. 
B. The Newtonian Limit of the SEA theory

In order to obtain the Newtonian limit of the theory we first show that the equation of motion Eq. (21) can also be obtained by a variational principle. We assume that the extra force given by Eq. (22) can be written formally as

\[ f^\lambda = (g^\mu^\nu + u^\nu u^\lambda) \nabla_\nu \ln \sqrt{Q}, \]  

(23)

where \( Q \) is a dimensionless quantity. We note that when \( Q \) tends to unity we recover the standard geodesic equation of general relativity. Now, in order to obtain the form of \( Q \) in the Newtonian limit of SEA theory we assume that the density of the physical system is small and one may ignore the pressure \( p \ll \rho \). In this case, using Eqs. (22) and (23), one has

\[ \nabla_\nu \ln \sqrt{Q} = \frac{1}{\rho} \frac{c_4}{c_2} \phi \nabla_\nu \phi \frac{(\rho u^\alpha)}{\rho} - \frac{2\kappa}{2} \nabla_\nu \phi. \]

(24)

In the Newtonian limit the function \( \phi \) depends only on \( r \) and the velocity of the particle satisfies \( u^\mu = \delta_0^\mu / \sqrt{g_0} \). One can then easily show that \( u^\beta \nabla_\beta \phi \approx 0 \). We can also assume that \( \phi = \phi(\rho) \) and

\[ \nabla_\alpha (\rho u^\alpha) = Z(\rho). \]

(25)

Expanding Eq. (24) about the background density \( \rho_0 \), one has

\[ \nabla_\nu \ln \sqrt{Q} = -\frac{c_4}{2\kappa} \phi'(\rho_0) \left[ \frac{Z(\rho_0)}{\rho} + \frac{Z'(\rho_0)}{\rho} \right] \nabla_\nu \rho. \]

(26)

The first term of the above equation is proportional to \( \nabla_\nu \ln \rho \), which can be further simplified to

\[ \nabla_\nu \ln \rho = \nabla_\nu \ln(\rho + \delta \rho) \approx \frac{1}{\rho_0} \nabla_\nu \delta \rho, \]

(27)

where \( \delta \rho = \rho - \rho_0 \). One may then obtain the following expression for \( \sqrt{Q} \)

\[ \sqrt{Q} \approx 1 - \frac{\alpha}{\rho_0} \delta \rho, \]

(28)

where we have defined

\[ \alpha = \frac{c_4}{2\kappa} \phi'(\rho_0) \left[ \frac{Z(\rho_0)}{\rho} + \frac{Z'(\rho_0)}{\rho} \right]. \]

(29)

We note that one may obtain the equation of motion Eq. (21) by starting with the formal definition of the extra-force given in Eq. (23), and varying the modified particle motion action \[ 33 \]

\[ S_p = \int \sqrt{Q} \sqrt{g_{\mu\nu} u^\mu u^\nu} ds. \]

(30)

One can see that in the case \( \sqrt{Q} \rightarrow 1 \), the standard geodesic equation is obtained. In the Newtonian limit the standard line element for a dust particle can be written as

\[ \sqrt{g_{\mu\nu} u^\mu u^\nu} ds \approx \left( 1 + \psi - \frac{\vec{v}^2}{2} \right) dt, \]

(31)

where \( \psi \) is the Newtonian potential and \( \vec{v} \) is the 3D velocity of the particle. Using Eqs. (28) and (31) one can write the action (30) as

\[ S_p = \int \left( 1 + \psi - \frac{\vec{v}^2}{2} - \frac{\alpha}{\rho_0} \delta \rho \right) dt. \]

(32)

Varying the above action gives us

\[ \ddot{\vec{a}} = -\ddot{\vec{\nabla}} \psi + \ddot{\vec{a}}_E, \]

(33)

where the first term is the Newtonian acceleration \( \ddot{\vec{a}}_N \) and the second term is the extra acceleration having the form

\[ \ddot{\vec{a}}_E(\rho) = \frac{\alpha}{\rho_0} \ddot{\vec{\nabla}} \rho. \]

(34)

The extra acceleration depends on the gradient of the matter density. The above theory shows that in the region of space-time where the matter density is (approximately) constant, the extra acceleration becomes zero or is negligibly small.
C. The generalized Poisson equation

Taking the trace of the gravitational field equation, Eq. (10), and assuming that $V(\phi) = 0$, one has

$$ R = (c_1 + 2c_2)(\Box \phi)^2 + (c_1 + 3c_2)\nabla^\nu \phi \nabla_\nu \Box \phi + c_1 \nabla^\nu \phi \nabla_\nu \phi - c_4 u^\alpha \nabla_\alpha \phi + \kappa^2 \rho. \tag{35} $$

One may easily obtain the solution

$$ g_{\mu\nu} = \eta_{\mu\nu}, \quad \phi_0 = 2t + x + y + z, \quad \lambda = 0, \tag{36} $$
as a background solution of the theory. One should note that, because of the constraint equation (9), the aether scalar should depend on time. So one cannot obtain a fully static solution as a background solution.

Assuming that the metric takes the form

$$ ds^2 = -(1 - 2A(x))dt^2 + (1 - 2\psi(x))(dx^2 + dy^2 + dz^2), \tag{37} $$

and

$$ \phi = \phi_0 + \varphi(x), \tag{38} $$
one can see that the constraint equation (9) can be satisfied, to first order if

$$ \psi = \frac{4}{3}A - \frac{1}{3} \sum_{i=1}^{3} \partial_i \varphi. \tag{39} $$

One can then obtain, up to first order

$$ R = \frac{22}{3} \nabla^2 A - \frac{4}{3} \sum \nabla^2 \partial_i \varphi. \tag{40} $$

The generalized Poisson equation for the SEA theory now reads

$$ \nabla^2 A = \frac{2}{11} \sum \nabla^2 \partial_i \varphi + \frac{3}{22}(c_1 + 3c_2)\nabla^\nu \phi_0 \nabla_\nu \Box \varphi + \frac{3}{22}c_1 \nabla^\nu \Box \nabla_\nu \varphi + \frac{3}{22}(\kappa^2 - 2c_4)\rho. \tag{41} $$

IV. COSMOLOGICAL SOLUTIONS

In this Section we will study the cosmological implications of the SEA theory. We will restrict our study to homogeneous and isotropic cosmological models, with the line element given by the flat Friedmann-Robertson-Walker metric

$$ ds^2 = -dt^2 + a(t)^2 dx^2. \tag{42} $$

We also assume that the scalar aether field is homogeneous and therefore has the form $\phi = \phi(t)$. We also assume that the matter content of the universe has a perfect fluid form

$$ T^\mu_\nu = \text{diag}[-\rho(t), p(t), p(t), p(t)], \tag{43} $$

and the velocity of the particle is $u^\alpha = [1, 0, 0, 0]$. In this case the constraint equation Eq. (9) can be solved for $\phi$ to give

$$ \phi = t + \alpha_1, \tag{44} $$

where $\alpha_1$ is an integration constant and we have assumed that $\epsilon = 1$ so that the aether vector becomes a time-like vector field. With these assumptions, equation (44) becomes obvious since the FRW metric already has a time-like preferred direction $\partial/\partial t$ and the aether vector should be identical to that direction up to a shift. The metric and scalar field equations can then be obtained from (10) and (12) as

$$ 3[3(c_1 + c_2) - 2]H^2 - 6c_2 \dot{H} + (2\kappa^2 - c_4)\rho - 2\lambda = 0, \tag{45} $$

$$ (c_1 + 3c_2 - 2)(3H^2 + 2\dot{H}) - (2\kappa^2 - c_4)\rho = 0, \tag{46} $$

and

$$ c_2 \ddot{H} - (2c_1 - 3c_2)H\dot{H} - 3c_1 H^3 + \frac{1}{3} \dot{\lambda} - \frac{1}{6} c_4 \dot{\rho} + \frac{1}{2}(2\lambda - c_4 \rho)H = 0, \tag{47} $$

where we have assumed that $V(\phi) = 0$. The conservation equation (17) reduces to

$$ 2(c_1 - \kappa^2)(\dot{\rho} + 3H\rho) + 3(c_4 - 2\kappa^2)H\rho = 0. \tag{48} $$
A. Vacuum solutions

In the case of zero energy-momentum tensor $T_{\mu\nu} = 0$ one can easily obtain the dust-like solution $a = a_0 t^{2/3}$, $a_0 = \text{constant}$, with the Lagrange multiplier

$$\lambda = \frac{2(3c_1 + 6c_2 - 2)}{3t^2}. \quad (49)$$

Now, let us consider the case

$$c_2 = \frac{2 - c_1}{3}, \quad (50)$$

which simplifies the equations. In this case one obtains a self-accelerating solution $a = a_0 \exp (H_0 t)$, $H_0 = \text{constant}$ with

$$\lambda = 3c_1 H_0^2. \quad (51)$$

There is also a power-law solution $a = a_0 t^n$ with the Lagrange multiplier of the form

$$\lambda = \frac{n [2 + c_1 (3n - 1)]}{t^2}. \quad (52)$$

B. The case $c_4 = 0$

In this case, the energy-momentum tensor becomes conserved due to equation (17). One may then obtain the matter dominated solution $a = a_0 t^{2/3}$, $a_0 = \text{constant}$

$$p = 0, \quad \rho = \frac{\rho_0}{t^2}, \quad \lambda = \frac{\lambda_0}{t^2}, \quad (53)$$

with

$$\lambda_0 = 2c_1 + 4c_2 - \frac{2}{3} + \rho_0 \kappa^2. \quad (54)$$

One can also obtain a radiation solution $a = a_0 t^{1/2}$ with

$$p = \frac{\rho_0}{3t^2}, \quad \rho = \frac{\rho_0}{t^{3/2}}, \quad \lambda = \frac{3(2c_2 + c_1)}{4t^2} + \frac{\kappa^2 \rho_1}{t^{3/2}}, \quad (55)$$

where

$$\rho_0 = \frac{3(2 - c_1 - 3c_2)}{8\kappa^2}. \quad (56)$$

and $\rho_1$ is an integration constant.

C. The case $c_4 \neq 0$

In this case one has the matter-dominated solution $a = a_0 t^{2/3}$, $a_0 = \text{constant}$. Putting $p = 0$ in (17) one may prove that the energy-density should behave as

$$\rho = \frac{\rho_0}{t^2}, \quad (57)$$

as in the case where one has energy-momentum conservation. In this case we obtain

$$\lambda = \frac{(6\kappa^2 - 3c_4) \rho_0 - 8 + 24c_2 + 12c_1}{6t^2}. \quad (58)$$
We have also a self-accelerating solution \( a = a_0 e^{H_0 t} \), \( H_0 = \text{constant} \), with

\[
\begin{align*}
p &= \frac{3H_0^2 (2 - c_1 - 3c_2)}{c_4 - 2\kappa^2}, \\
\rho &= \frac{3H_0^2 (c_1 + 3c_2 - 2)}{2(c_4 - \kappa^2)} + \rho_0 e^{-3H_0 t}, \\
\lambda &= \frac{1}{2} (2\kappa^2 - c_4) \rho + \frac{3}{2} H_0^2 \left[ 3(c_1 + c_2) - 2 \right].
\end{align*}
\]

(59)

The above equation shows that the energy-density has a cosmological constant part. In order to get rid of the cosmological constant term one should again impose the condition (50) which would make the solution a self-accelerating one with an energy-momentum tensor behaving as dust.

V. DISCUSSION AND FINAL REMARKS

In this paper we have considered a scalar version of the vector EA type theories where the aether vector field is represented by the gradient of a scalar function. In this way the basic physical characteristics of the aether can be described in terms of a single scalar function \( \phi \), whose coupling to the metric is accomplished via its gradient. A self-interacting potential of the scalar aether field can also be added to the theory as well as a possible coupling between the hydrodynamic flow of the matter, described by the flux \( j^\sigma \), and the aether scalar. In the presence of such a coupling the energy-momentum tensor of the matter is not conserved and an extra force is generated. If no such coupling exists, the matter energy-momentum tensor is conserved since the covariant divergence of the effective energy-momentum tensor of the scalar aether is identically zero. In the presence of the matter flow-aether coupling, in the weak field limit, the total acceleration of a massive test particle is \( \ddot{a} = \ddot{a}_N + \ddot{a}_E \). As shown in [34], the Newtonian acceleration can be expressed as \( \ddot{a}_N \approx (|\ddot{a}|/2|\ddot{a}_E|) \ddot{a} \), or, equivalently, \( a = \sqrt{2a_E a_N} \), a relation which is very similar to the acceleration equation introduced in the MOND approach to dark matter [14]. Since \( a_N = GM/r^2 \), where \( M \) is the mass of the central body, it follows that \( a \approx \sqrt{a_E GM/r} = v_{tg}^2 /r \), where \( v_{tg} \) is the rotational velocity of a massive test particle under the influence of a central force. Therefore, it follows that \( v_{tg}^2 \rightarrow v_{\infty}^2 = \sqrt{a_E GM} = \text{constant} \), pointing to the presence of an extra force due to the coupling between hydrodynamic motion and aether which may explain the constancy of the galactic rotation curves, usually attributed to the presence of dark matter.

We have also investigated the cosmological implications of the theory by studying the cosmological evolution of a flat, homogeneous and isotropic Universe. Depending on the values of the coupling constants \( c_i \), \( i = 0, 1, 2, 3 \), several classes of cosmological models can be constructed. For simplicity, in our analysis we have neglected the possible physical effects of the scalar field self-interacting potential \( V \). In particular, a de Sitter type cosmological expansion is possible in the presence of a hydrodynamic flow-scalar field interaction. In this scenario, in the long time limit, the matter energy-density tends to of a constant value. Power law solutions can be obtained for dust and radiation filled Universes as well.

In conclusion, we have proposed a scalar version of the EA type theories. The scalar version of the EA theory was also considered in [9] as an attempt to build a healthy extension of HL gravity in IR (the BPS theory). However the present model differs from the above theory by an additional condition which fixes the relation between the scalar field and the aether vector [13]. We have also obtained the basic field equations, and we have briefly explored the basic cosmological implications of the theory. A more detailed analysis of the cosmological behavior of the model, as well as of the astrophysical implications of the theory will be presented in a separate publication.
Appendix A: Proof of the energy-momentum conservation

In this Section we will derive equation Eq. (17) in detail. After taking the covariant divergence of Eq. (10), and using Eq. (12) to eliminate $dV/d\phi$, we obtain

$$\kappa^2 \nabla^\mu T_{\mu \nu} = c_1 (\nabla^\mu S_{1 \mu \nu} + \nabla_{\alpha} \nabla_{\mu} \nabla^\alpha \phi \nabla_{\nu} \phi)$$

$$+ c_2 (\nabla^\mu S_{2 \mu \nu} + \Box^2 \phi \nabla_{\nu} \phi) + c_3 [\nabla^\mu S_{3 \mu \nu}$$

$$+ \nabla_{\alpha} (\nabla^\mu \phi \nabla^\alpha \phi \nabla_{\mu} \nabla_{\nu} \phi + \Box \phi \nabla^\rho \phi \nabla^\nu \phi \nabla^\alpha \phi) \nabla_{\nu} \phi]$$

$$+ c_4 [\nabla^\mu S_{4 \mu \nu} - \frac{1}{2} \nabla_{\alpha} (\rho u^\alpha) \nabla_{\nu} \phi]$$

$$+ \nabla^\mu [\lambda \nabla_{\lambda} \phi \nabla_{\phi} \phi - \frac{1}{2} \lambda g_{\mu \nu} (\nabla_{\rho} \phi \nabla_{\mu} \phi + \epsilon)]$$

$$- \nabla_{\mu} (\lambda \nabla^\mu \phi) \nabla_{\nu} \phi,$$

(A1)

where we have defined

$$S_{1 \mu \nu} = \Box \phi \nabla_{\mu} \phi - 2 \nabla_{(\mu} \phi \nabla_{\nu) \phi + \nabla^\lambda \phi \nabla_{\lambda} \nabla_{\mu} \phi - \frac{1}{2} g_{\mu \nu} \nabla_{\lambda} \phi \nabla_{\lambda} \phi$$

(A2)

$$S_{2 \mu \nu} = g_{\mu \nu} \nabla^\lambda \phi \nabla_{\lambda} \Box \phi + \frac{1}{2} g_{\mu \nu} (\Box \phi)^2 - 2 \nabla_{(\mu} \phi \nabla_{\nu)} \Box \phi$$

(A3)

$$S_{3 \mu \nu} = \nabla^\alpha \phi \nabla^\beta \phi \nabla_{\mu} \phi \nabla_{\nu} \phi - \nabla_{\alpha} \nabla_{\beta} \phi \nabla^\alpha \nabla^\beta \phi \nabla_{\mu} \phi \nabla_{\nu} \phi$$

$$- \frac{1}{2} g_{\mu \nu} \nabla^\alpha \phi \nabla^\beta \phi \nabla_{\alpha} \phi \nabla_{\beta} \phi$$

(A4)

$$S_{4 \mu \nu} = \frac{1}{2} T_{\mu \nu} u^\alpha \nabla_{\alpha} \phi.$$

(A5)

Using equations (9) and (13), one can easily prove that the last two lines of equation (A1) is zero. Expanding the term which corresponds to the constant $c_1$ in Eq. (A1) results in

$$2 \nabla^\lambda \phi \nabla_{\mu [\lambda} \nabla_{\nu \phi} + 2 \nabla^\alpha \nabla^\beta \phi \nabla_{\mu [\beta \nabla_{\nu \phi]} + 2 \nabla_{\mu \phi} \nabla_{\nu \phi} \nabla_{\mu [\alpha \nabla_{\beta \phi]}} \nabla_{\alpha \phi},$$

(A6)

which is identically zero due to the relations

$$2 \nabla_{[\mu} \nabla_{\nu]} T_{\rho \sigma} = T_{\alpha \rho \sigma \mu} + T_{\rho \alpha} R_{\sigma \nu \mu},$$

(A7a)

$$2 \nabla_{[\mu} \nabla_{\nu]} A_{\rho} = A_{\alpha} R_{\rho \nu \mu \alpha}.$$  

(A7b)

The term corresponding to constant $c_2$ vanishes by the substitution of the tensor $S_{2 \mu \nu}$. The term corresponding to the constant $c_3$ can be written as

$$2 \nabla_{\nu} \phi \nabla_{\mu} \phi \nabla_{\sigma} \nabla_{\alpha} \phi \nabla_{\alpha \sigma \phi} + 2 \nabla_{\nu} \phi \nabla_{\beta} \phi \nabla_{\beta} \phi \nabla_{\sigma \alpha \phi}$$

$$+ 2 \nabla_{\nu} \phi \nabla_{\beta} \phi \nabla_{\phi} \nabla_{\alpha \sigma} \phi + 2 \nabla_{\nu} \phi \nabla_{\beta} \phi \nabla_{\alpha} \nabla_{\mu} \phi \nabla_{\mu \phi} \nabla_{\nu \phi},$$

(A8)

which is zero due to identities (A7). Equation (A1) then reduces to

$$\kappa^2 \nabla^\mu T_{\mu \nu} = c_4 \left[ \frac{1}{2} \nabla_{\mu} (T_{\mu \nu} u^\alpha \nabla_{\alpha} \phi) - \frac{1}{2} \nabla_{\alpha} (\rho u^\alpha) \nabla_{\nu} \phi \right],$$

(A9)

which after expanding and isolating the covariant divergence of the energy-momentum tensor reduces to equation (17) in the main text.

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