COINCIDENCE OF SCHUR MULTIPLIERS OF THE DRURY-ARVESON SPACE

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Abstract. In a purely multi-variable setting (i.e., the issues discussed in this note are not interesting in the single variable operator theory setting), we show that the coincidence of two operator valued Schur class multipliers of a certain kind on the Drury-Arveson space is characterized by the fact that the associated colligations (or a variant, obtained canonically) are 'unitarily coincident' in a sense to be made precise in this article.

1. Introduction

Given a Hilbert space $\mathcal{U}$ (all Hilbert spaces in this note are over the complex field and are separable), $H^2_d(\mathcal{U})$ denotes the reproducing kernel Hilbert space corresponding to the kernel

$$k(z, w) = \frac{I_{\mathcal{U}}}{1 - \langle z, w \rangle}; \quad (z, w) \in \mathbb{B}_d \times \mathbb{B}_d.$$ 

It consists of $\mathcal{U}$ valued holomorphic functions on the Euclidean unit ball $\mathbb{B}_d$ in $\mathbb{C}^d$.

If $\mathcal{Y}$ is another Hilbert space, then a multiplier $\varphi$ is a $\mathcal{B}(\mathcal{U}, \mathcal{Y})$ valued holomorphic function on $\mathbb{B}_d$ such that $\varphi f \in H^2_d(\mathcal{Y})$ for all $f \in H^2_d(\mathcal{U})$. Here $\varphi f$ denotes the pointwise product. It is a well known consequence of the closed graph theorem that the linear map $f \mapsto \varphi f$ is a bounded operator $M_{\varphi}$ from $H^2_d(\mathcal{U})$ into $H^2_d(\mathcal{Y})$. The multipliers form a Banach space with the norm

$$\|\varphi\| := \|M_{\varphi}\|.$$ 

The closed unit ball of this Banach space is called the Schur class $S_d(\mathcal{U}, \mathcal{Y})$. The following theorem to be found in [1] and [2] describes the Schur class.

Theorem 1.1. Let $\varphi$ be a $\mathcal{B}(\mathcal{U}, \mathcal{Y})$ valued function defined on $\mathbb{B}_d$. Then the following are equivalent:

1. $\varphi \in S_d(\mathcal{U}, \mathcal{Y})$.
2. The kernel $k_{\varphi} : \mathbb{B}_d \times \mathbb{B}_d \to \mathcal{B}(\mathcal{Y})$ given by

$$k_{\varphi}(z, w) = \frac{I_{\mathcal{Y}} - \varphi(z)\varphi(w)^*}{1 - \langle z, w \rangle}$$

is positive semi-definite.
(3) There exists an auxiliary Hilbert space $X$ and a unitary connecting operator (or colligation) $U$ of the form:

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A_1 & B_1 \\ & \ddots & \ddots \\ & A_d & B_d \\ C & D \end{pmatrix} : \left( \begin{array}{c} X \\ U \end{array} \right) \rightarrow \left( \begin{array}{c} X^d \\ Y \end{array} \right)$$

such that $\varphi(z)$ can be realized as $\varphi(z) = D + C(I_X - ZA)^{-1}ZB$, where $Z$ denotes the row tuple $(z_1 I_X, \ldots, z_d I_X)$ for $z \in \mathbb{B}_d$.

(4) There exists an auxiliary Hilbert space $X$ and a contractive colligation operator $U$ such that $\varphi$ can be realized in the same form as above.

A realization, as in (3) above, of a Schur multiplier is commutative if the operators $A_i$ commute among themselves. Not all Schur multipliers admit commutative realizations. See Proposition 3.3 of [5].

**Definition 1.2.** The colligation operators

$$U_1 = \begin{pmatrix} A_1 \\ C_1 \end{pmatrix} : \left( \begin{array}{c} K_1 \\ \mathcal{U} \end{array} \right) \rightarrow \left( \begin{array}{c} K_1^d \\ \mathcal{Y} \end{array} \right)$$

and

$$U_2 = \begin{pmatrix} A_2 \\ C_2 \end{pmatrix} : \left( \begin{array}{c} K_2 \\ \mathcal{V} \end{array} \right) \rightarrow \left( \begin{array}{c} K_2^d \\ \mathcal{W} \end{array} \right)$$

are said to be ‘unitarily coincident’ if there exist unitary operators

$$\Lambda : K_1 \rightarrow K_2, \quad \Omega : \mathcal{U} \rightarrow \mathcal{V}, \quad \text{and} \quad \Omega_2 : \mathcal{Y} \rightarrow \mathcal{W}$$

satisfying

$$\begin{pmatrix} A^d & 0 \\ 0 & \Omega_2 \end{pmatrix} \begin{pmatrix} A_1 \\ C_1 \end{pmatrix} = \begin{pmatrix} A_2 \\ C_2 \end{pmatrix} \begin{pmatrix} \Lambda & 0 \\ 0 & \Omega_1 \end{pmatrix}.$$
See [5] for more on such $\varphi$.

In single variable theory, the de Branges-Rovnyak space is always invariant under the backward shift operator and satisfies the one variable analog of the difference quotient inequality. To consider Schur functions satisfying the multivariable inequality is a natural extension of the one variable theory.

**Definition 1.3.** If a realization $(A \ C \ B \ D)$ of a Schur class function $\varphi$ is such that $A_j = M_{k\varphi}^* \mid_{H(k\varphi)}$ for $j = 1, \ldots, d$, $Cf = f(0)$ for $f \in H(k\varphi)$ and $D = \varphi(0)$ then it is called a functional model realization.

Let us note that the Schur functions admitting functional model realizations are precisely the ones which are considered in this paper as per the assumptions before, [5].

The difference quotient inequality is then equivalently expressed as

$$
\sum_{j=1}^{d} A_j^* A_j + C^* C \leq I_{H(k\varphi)}.
$$

A functional model realization of a Schur multiplier is not unique.

2. Preliminaries and notations

2.1. Coincidence. We define the notion of coincidence of two Schur multipliers.

**Definition 2.1.** A $\varphi \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ is said to **coincide** with $\psi \in \mathcal{S}_d(\mathcal{V}, \mathcal{W})$, if there exist unitary operators $\alpha : \mathcal{U} \rightarrow \mathcal{V}$ and $\beta : \mathcal{Y} \rightarrow \mathcal{W}$ such that $\beta \varphi(z) = \psi(z) \alpha$ holds for all $z \in \mathbb{B}^d$.

In this section, $\varphi \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ and $\psi \in \mathcal{S}_d(\mathcal{V}, \mathcal{W})$ are two such functions which coincide.

**Lemma 2.2.** There is a unique linear isomorphism $\Gamma : H(k\varphi) \rightarrow H(k\psi)$ such that

$$
\Gamma(\sum_{t=1}^{m} k_\varphi(\cdot, w_t) y_t) = \sum_{t=1}^{m} k_\psi(\cdot, w_t) \beta y_t.
$$

Moreover, if we identify the Hilbert spaces $H^2_d(\mathcal{Y})$ and $H^2_d \otimes \mathcal{Y}$ (and similarly for $H^2_d(\mathcal{W})$), then $\Gamma = (I_{H^2_d} \otimes \beta) \mid_{H(k\varphi)}$.

Proof : Uniqueness is clear because of density of the vectors $\sum_{t=1}^{m} k_\varphi(\cdot, w_t) y_t$.

For existence we need to show that

$$
\left\| \sum_{t=1}^{m} k_\psi(\cdot, w_t) \beta y_t \right\|_{H(k\varphi)}^2 = \left\| \sum_{t=1}^{m} k_\varphi(\cdot, w_t) y_t \right\|_{H(k\varphi)}^2,
$$

which easily follows from coincidence. The rest is straightforward computation.

To prove that $\Gamma = (I_{H^2_d} \otimes \beta) \mid_{H(k\varphi)}$, note that, as a vector

$$
k_\varphi(\cdot, w) = \frac{I_{\mathcal{Y}} - \varphi(\cdot) \varphi(w)^*}{1 - \langle \cdot, w \rangle} y \in H^2_d(\mathcal{Y}).
$$
Thus,

\[ (I_{H^2_d} \otimes \beta) \, k_\psi(\cdot, w) y \in H^2_2(W). \]

Let \( h \in W \) and \( k(\cdot, w')h \) denote an elementary tensor in \( H^2_2(W) \). Then we have,

\[
\langle (I_{H^2_d} \otimes \beta)k_\psi(\cdot, w)y, k(\cdot, w')h \rangle_{H^2_2(W)} = \langle k_\psi(\cdot, w)y, (I_{H^2_d} \otimes \beta^*)k(\cdot, w')h \rangle_{H^2_2(Y)} \\
= \langle k_\psi(\cdot, w)y, k(\cdot, w')\beta^*y \rangle_{H^2_2(Y)} \\
= \langle k_\psi(w', w)y, \beta^*y \rangle_Y \\
= \langle \beta k_\psi(w', w)y, h \rangle_Y \\
= \langle k_\psi(w', w)\beta y, h \rangle_Y \\
= \langle k_\psi(\cdot, w)\beta y, k(\cdot, w')h \rangle_{H^2_2(W)}. 
\]

The second equality from the end in the above follows from the coincidence of the two Schur multipliers. This shows that \( \Gamma = (I_{H^2_d} \otimes \beta)|_{H(k_\psi)} \) on a dense subspace of \( H(k_\psi) \). Finally, by contractive inclusion of \( H(k_\psi) \) in \( H^2_2(W) \) one has the required result. \( \square \)

2.2. The two de Branges - Rovnyak spaces. Now the following intertwining relation is immediate:

\[ \Gamma(M^*_x \otimes I_Y)|_{H(k_\psi)} = (M^*_x \otimes I_W)|_{H(k_\psi)} \Gamma. \]

From the above, it is easy to see that if one of the Schur multipliers is such that its corresponding de Branges-Rovnyak space is backward shift invariant, the elements of which satisfy the difference quotient inequality then the same is also true for the other Schur multiplier which is coincident to it.

Consider the Hilbert space \( H(k_\psi)^d \). Let \( W^*k_\psi(\cdot, w)y := \begin{pmatrix} \overline{w_1}k_\psi(\cdot, w)y \\ \vdots \\ \overline{w_d}k_\psi(\cdot, w)y \end{pmatrix} \in H(k_\psi)^d \) and let \( \mathcal{D}_\psi = \overline{\text{span}} \{ W^*k_\psi(\cdot, w)y : w \in \mathbb{B}_d, y \in Y \} \). Similarly we define \( \mathcal{D}_\psi \). On \( H(k_\psi)^d \), let

\[
\hat{\Gamma} = \begin{pmatrix} \Gamma & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Gamma \end{pmatrix} \quad \text{and} \quad A^\psi = \begin{pmatrix} A^\psi_1 \\ \vdots \\ A^\psi_d \end{pmatrix}.
\]

By the action of the unitary \( \Gamma \), one has \( \hat{\Gamma}(\mathcal{D}_\psi) = \mathcal{D_\psi} \) and thus \( \hat{\Gamma}(\mathcal{D}_\psi^+) = \mathcal{D}_\psi^+ \). Let \( R^\varphi : \mathcal{D}_\varphi \to \mathcal{U} \) be given by \( W^*k_\varphi(\cdot, w)y \mapsto (\varphi(w)^*-\varphi(0)^*)y \). Next, we define some operators considered in Section 2 of [4] and follow the notations therein closely for our convenience. In what follows, \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) is a functional model realization as in Definition 3.

- \( T^\psi_{11} : \mathcal{D}_\psi \to H(k_\psi) \) is given by \( T^\psi_{11} = A^*|_{\mathcal{D}_\psi} \), where \( A^* = (A^*_1, \ldots, A^*_d) \).
- \( T^\psi_{12} : \mathcal{D}_\psi \oplus Y \to H(k_\psi) \) is given by \( T^\psi_{12} = \begin{pmatrix} A^*|_{\mathcal{D}_\psi} & C^* \end{pmatrix} \).
- \( T^\psi_{22} : \mathcal{D}_\varphi \oplus Y \to \mathcal{U} \) is given by \( T^\psi_{22} = \begin{pmatrix} R^\varphi & \varphi(0)^* \end{pmatrix} \).
Note that \( \text{ran} T^\varphi_{12} \subset \mathcal{U}^{0\perp}_{\varphi} \). We know that any functional model realization of \( \varphi \) is the adjoint of the colligation written in terms of the above operators:

\[
\begin{pmatrix}
T^\varphi_{11} & T^\varphi_{12} \\
X & T^\varphi_{22}
\end{pmatrix}
\begin{pmatrix}
D^\perp & D^\varphi \\
\mathcal{Y} & \mathcal{Y}
\end{pmatrix}
\rightarrow
\begin{pmatrix}
H(k_\varphi) \\
\mathcal{U}
\end{pmatrix}
\]

where \( H(k_\varphi) = D^\perp \oplus D^\varphi \) and \( X : D^\varphi \rightarrow \mathcal{U} \) is the non-unique component of the operator \( B^\varphi = \left( X \ R^\varphi \right)^* : \mathcal{U} \rightarrow H(k_\varphi) \) appearing in the colligation. See Section 2 of [4] for more details. Similarly, one may consider the analogous operators \( T^\varphi_{ij} \) and \( T^\psi_{ij} \) for \( i, j = 1, 2 \). They are easy consequences of the intertwining property of \( \Gamma \) and coincidence of the Schur multipliers.

\[
\Gamma \ T^\varphi_{11} = T^\varphi_{11} \ \tilde{\Gamma} |_{D^\varphi} \ , \ \Gamma \ T^\varphi_{12} = T^\psi_{12} \left( \begin{array}{c|c}
\Gamma |_{D^\varphi} & 0 \\
0 & \beta \\
\end{array} \right) \ , \ \alpha \ T^\psi_{22} = T^\psi_{22} \left( \begin{array}{c|c}
\Gamma |_{D^\varphi} & 0 \\
0 & \beta \\
\end{array} \right).
\]

Define

1. \( G^\varphi_1 : \text{ran} \ (I_{H(k_\varphi)} - T^\varphi_{12} T^\varphi_{12})^\frac{1}{2} \rightarrow D^\perp \)
2. \( (I_{H(k_\varphi)} - T^\varphi_{12} T^\varphi_{12})^\frac{1}{2} f \mapsto T^\varphi_{11} f, \ f \in H(k_\varphi) \).

and

3. \( G^\psi_2 : \text{ran} \ (I_{D^\varphi \oplus \mathcal{Y}} - T^\varphi_{12} T^\varphi_{12})^\frac{1}{2} \rightarrow \mathcal{U} \)
4. \( (I_{D^\varphi \oplus \mathcal{Y}} - T^\varphi_{12} T^\varphi_{12})^\frac{1}{2} l \mapsto T^\varphi_{22} l, \ l \in D^\varphi \oplus \mathcal{Y} \).

Note that \( \text{ran} G^\varphi_2 = \text{ran} T^\varphi_{22} \subset \mathcal{U}^{0\perp}_{\varphi} \). Similarly one has the operators \( G^\varphi_1 \) and \( G^\psi_2 \).

**Lemma 2.3.** There are intertwining relations of \( G^\varphi_j \) and \( G^\psi_j \) for \( j = 1, 2 \) as given below:

\[
\left( \Gamma |_{D^\varphi} \right) G^\varphi_1 = G^\psi_1 \Gamma \ \text{and} \ \alpha \ G^\varphi_2 = G^\psi_2 \left( \begin{array}{c|c}
\Gamma |_{D^\varphi} & 0 \\
0 & \beta \\
\end{array} \right).
\]

**Proof:** The proof follows from the following relations

\[
\bullet \ \Gamma \left( I_{H(k_\varphi)} - T^\varphi_{12} T^\varphi_{12} \right) = \left( I_{H(k_\varphi)} - T^\psi_{12} T^\psi_{12} \right) \Gamma
\]

\[
\bullet \left( \begin{array}{c|c}
\Gamma |_{D^\varphi} & 0 \\
0 & \beta \\
\end{array} \right) \left( I_{D^\varphi \oplus \mathcal{Y}} - T^\varphi_{12} T^\varphi_{12} \right) = \left( I_{D^\varphi \oplus \mathcal{W}} - T^\psi_{12} T^\psi_{12} \right) \left( \begin{array}{c|c}
\Gamma |_{D^\varphi} & 0 \\
0 & \beta \\
\end{array} \right)
\]

\( \square \)

**2.3. Some new colligations.** Let \( \mathcal{U}^0 = \{ u \in \mathcal{U} \mid \varphi(z)u \equiv 0 \} \). \( \mathcal{V}^0_\psi \) is similarly defined. Note that by virtue of coincidence one has \( \alpha(\mathcal{U}^0_\varphi) = \mathcal{V}^0_\psi \) and therefore \( \alpha(\mathcal{U}^{0\perp}_\varphi) = \mathcal{V}^{0\perp}_\psi \).

Let \( \varphi \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y}) \) be such that it admits coisometric functional model realizations. Let \( U^\varphi = \begin{pmatrix} A^\varphi & B^\varphi \\ C^\varphi \end{pmatrix} \) be a coisometric functional model colligation of
Consider the projection map \( J : \mathcal{N} \to \mathcal{U}^0 \) as a partial isometry. Its adjoint \( J^* : \mathcal{U}^0 \to \mathcal{N} \) is the inclusion map. The \( B(\mathcal{N}, \mathcal{Y}) \)-valued function \( \varphi_J(z) := \varphi(z)J \), on \( \mathcal{B}_d \) and the operator valued kernel

\[
k_{\varphi_J}(z, w) = \frac{I_\mathcal{Y} - \varphi_J(z)\varphi_J(w)^*}{1 - \langle z, w \rangle}
\]

on \( \mathcal{B}_d \times \mathcal{B}_d \) satisfy

\[
k_{\varphi_J}(z, w) = \frac{I_\mathcal{Y} - \varphi_J(z)\varphi_J(w)^*}{1 - \langle z, w \rangle} = \frac{I_\mathcal{Y} - \varphi(z)JJ^*\varphi(w)^*}{1 - \langle z, w \rangle} = \frac{I_\mathcal{Y} - \varphi(z)\varphi(w)^*}{1 - \langle z, w \rangle} = k_{\varphi}(z, w).
\]

Thus \( \varphi_J \in \mathcal{S}_d(\mathcal{N}, \mathcal{Y}) \). As a consequence we have \( H(k_{\varphi_J}) = H(k_{\varphi}) \) and therefore \( \varphi_J \) also admits functional model realizations. Also note that

\[
\mathcal{N}^0_{\varphi_J} = \mathcal{U}^0_{\varphi_J} \oplus \text{ran} (I_{\mathcal{D}_{\varphi}^\perp} - G_1^\varphi G_1^{\varphi*})^{1/2}.
\]

The following are easy consequences of the definition of \( J \):

\[
T_{11}^{\varphi}; T_{12}^{\varphi}; T_{22}^{\varphi}; J^*T_{22}^{\varphi}; G_1^{\varphi*} = G_1^\varphi; G_2^{\varphi*} = J^*G_2^\varphi.
\]

Let \( U^\varphi \) be a fixed functional model realization of \( \varphi \). We know that a functional model realization is always weakly coisometric. Then from Theorem 2.7 of [4] we know that the realization has an explicit description and the adjoint of the non-unique operator \( B^\varphi \) is given by

\[
\left( -G_2^{\varphi*}T_{12}^{\varphi*} + \xi^\varphi(I_{\mathcal{D}_{\varphi}^\perp} - G_1^\varphi G_1^{\varphi*})^{1/2} \right) ; \mathcal{D}_{\varphi}^\perp \oplus \mathcal{D}_{\varphi} \to \mathcal{U},
\]

for some contraction \( \xi^\varphi : \text{ran} (I_{\mathcal{D}_{\varphi}^\perp} - G_1^\varphi G_1^{\varphi*})^{1/2} \to \mathcal{U}^0 \). Let \( \xi_{iso}^\varphi \) denote the isometric operator defined by

\[
\xi_{iso}^\varphi : \text{ran} (I_{\mathcal{D}_{\varphi}^\perp} - G_1^\varphi G_1^{\varphi*})^{1/2} \to \mathcal{U}^0 \oplus \text{ran} (I_{\mathcal{D}_{\varphi}^\perp} - G_1^\varphi G_1^{\varphi*})^{1/2}
\]
where, 

\[ h \mapsto \xi \varphi h \oplus \Delta \varphi h. \]

By using the sufficiency criterion of Theorem 2.7 (2) of [4], we construct the following coisometric functional model realization of \( \varphi_j \):

\[
U_{\varphi_j} = \begin{pmatrix}
A_{\varphi_j}^* & C_{\varphi_j}^* \\
B_{\varphi_j}^* & J^* \varphi(0)^* 
\end{pmatrix} : \begin{pmatrix}
H(k_\varphi) & \varnothing \\
\varnothing & N
\end{pmatrix} \to \begin{pmatrix}
H(k_\varphi) & 0 \\
0 & N
\end{pmatrix},
\]

where,

\[
B_{\varphi_j}^* = \begin{bmatrix}
- \xi \varphi G_2^* T_{12} G_1^{\varphi_j} + \xi \varphi I_{D_2} - G_2^* G_1^{\varphi_j} \frac{\partial}{\partial \varphi}
\end{bmatrix} J^* R^n : D_\varphi^\perp \oplus D_\varphi \to N.
\]

3. The main result and its proof

The main result of this note is the following theorem.

**Theorem 3.1.** Let \( \varphi \in S_d(\mathcal{U}, \mathcal{Y}) \) and \( \psi \in S_d(\mathcal{V}, \mathcal{W}) \) be such that both admit functional model realizations. Then the following are equivalent:

1. \( \varphi \) coincides with \( \psi \).
2. \( \dim \mathcal{U}_{\varphi} = \dim \mathcal{V}_{\psi} \) and for any two arbitrary choices of functional model colligations \( \mathcal{U}_{\varphi} \) and \( \mathcal{V}_{\psi} \) of \( \varphi \) and \( \psi \) respectively, there exists an auxiliary Hilbert space \( \mathcal{H} \) such that either the canonically obtained realization \( U_{\mathcal{H}} \varphi \) is unitarily coincident to \( U_{\mathcal{H}} \psi \) or \( U_{\mathcal{H}} \psi \) is unitarily coincident to \( U_{\mathcal{H}} \varphi \). Here \( U_{\mathcal{H}} \psi \) is defined as before.

As usual, one of the implications is easier than the other, and that is (2) \( \Rightarrow \) (1). Let condition (2) hold. Without loss of generality let us assume that \( U_{\mathcal{H}} \varphi \) is unitarily coincident to \( U_{\mathcal{H}} \psi \). Then their corresponding transfer functions coincide. Thus we have that \( \begin{bmatrix} \varphi_j & 0 \end{bmatrix} \) coincides with \( \psi_j \). As a consequence, there exist unitary operators \( \beta : \mathcal{Y} \to \mathcal{W} \) and \( \alpha : \mathcal{N} \oplus \mathcal{H} \to \mathcal{N} \) such that \( \alpha(U_{\mathcal{H}}^{0\perp}) = \mathcal{V}_{\psi}^{0\perp} \), satisfying the coincidence identity. Since \( \dim \mathcal{U}_{\varphi} = \dim \mathcal{V}_{\psi} \), choose any unitary \( \tau : \mathcal{U}_{\varphi} \to \mathcal{V}_{\psi} \) and consider the unitary operator

\[
\alpha = \begin{pmatrix}
\hat{\alpha} |_{\mathcal{U}_{\varphi}^{0\perp}} & 0 \\
0 & \tau
\end{pmatrix} : \mathcal{U}_{\varphi}^{0\perp} \oplus \mathcal{U}_{\varphi} \to \mathcal{V}.
\]

Finally, the coincidence of \( \varphi \) and \( \psi \) follow from the easy computation below. Let \( h \in \mathcal{U} \) and \( z \in \mathbb{H} \), then

\[
\psi(z) \alpha h = \psi(z) \alpha(P_{\mathcal{U}_{\varphi}^{0\perp}} h \oplus P_{\mathcal{U}_{\varphi}} h)
= \psi(z) (\hat{\alpha} |_{\mathcal{U}_{\varphi}^{0\perp}} P_{\mathcal{U}_{\varphi}^{0\perp}} h \oplus \tau P_{\mathcal{U}_{\varphi}} h)
= \psi(z) (\hat{\alpha} |_{\mathcal{U}_{\varphi}^{0\perp}} P_{\mathcal{U}_{\varphi}^{0\perp}} h)
= \psi(z) J^* \hat{\alpha} (P_{\mathcal{U}_{\varphi}^{0\perp}} h \oplus 0)
= \psi_j(z) \hat{\alpha} (P_{\mathcal{U}_{\varphi}^{0\perp}} h \oplus 0)
= \beta \varphi_j(z) (P_{\mathcal{U}_{\varphi}^{0\perp}} h)
= \beta \varphi(z) (P_{\mathcal{U}_{\varphi}^{0\perp}} h) = \beta \varphi(z) h.
\]

That proves (2) \( \Rightarrow \) (1). The converse proof will follow from the following theorem about coincidence of Schur multipliers admitting coisometric functional model
realizations. When the multipliers admit coisometric functional model realizations, the partial isometry $J$ does not figure in the characterization.

**Theorem 3.2.** Let $\varphi \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ and $\psi \in \mathcal{S}_d(\mathcal{V}, \mathcal{W})$ be such that both admit coisometric functional model realizations. Then the following are equivalent:

1. $\varphi$ coincides with $\psi$.
2. $\dim \mathcal{U}_\varphi^0 = \dim \mathcal{V}_\psi^0$ and for any two arbitrary choices of coisometric functional model realizations say $U^\varphi$ and $U^\psi$ of $\varphi$ and $\psi$ respectively, there exists an auxiliary Hilbert space $\mathcal{M}$ such that either $U^\varphi_\mathcal{M}$ is unitarily coincident to $U^\psi$ or $U^\psi_\mathcal{M}$ is unitarily coincident to $U^\varphi$.

Proof: (1) $\Rightarrow$ (2). Let $\varphi$ and $\psi$ be coincident. Then there exist unitaries say, $\alpha : \mathcal{U} \to \mathcal{V}$ and $\beta : \mathcal{Y} \to \mathcal{W}$, such that
\[
\beta \varphi(z) = \psi(z) \alpha
\]
holds for all $z \in \mathbb{B}_d$.

Let $U^\varphi$ and $U^\psi$ be any two coisometric functional model realizations of $\varphi$ and $\psi$ respectively. By Theorem 2.7 of [4], the colligation $U^\varphi$ can be explicitly described as:
\[
U^\varphi = \begin{pmatrix} A^\varphi & B^\varphi \\ C^\varphi & \varphi(0) \end{pmatrix} : \begin{pmatrix} H(k_\varphi) \\ \mathcal{U} \end{pmatrix} \to \begin{pmatrix} H(k_\varphi)^d \\ \mathcal{Y} \end{pmatrix}
\]
where the non-unique operator $B^{\varphi*} : H(k_\varphi)^d \to \mathcal{U}$ is given by
\[
\begin{pmatrix} -G_2^\varphi T_2^\varphi I_{G_1^\varphi}^2 - I_{D_D^\varphi} - G_1^\varphi I_{G_1^\varphi}^2 \end{pmatrix} : D_D^\varphi \to \mathcal{U},
\]
for some isometry $\zeta^\varphi : \text{ran} \left( I_{D_D^\varphi} - G_1^\varphi G_1^\varphi \right)^{\frac{1}{2}} \to \mathcal{U}_\varphi^0$. The non-uniqueness of $B^\varphi$ comes from dependence of the parameter $\zeta^\varphi$.

Similarly, one has a description of the chosen colligation $U^\psi$ in terms of the operators $G_2^\psi, T_2^\psi, G_1^\psi$ and some isometric operator $\zeta^\psi : \text{ran} \left( I_{D_D^\psi} - G_1^\psi G_1^\psi \right)^{\frac{1}{2}} \to \mathcal{V}_\psi^0$.

Due to coincidence of the Schur multipliers one has
\[
\Gamma^d |_{D_D^\psi} \left( I_{D_D^\psi} - G_1^\psi G_1^\varphi \right)^{\frac{1}{2}} = \left( I_{D_D^\psi} - G_1^\psi G_1^\varphi \right)^{\frac{1}{2}} \Gamma^d |_{D_D^\varphi}.
\]

Thus,
\[
\Gamma^d |_{D_D^\psi} \left( \text{ran} \left( I_{D_D^\psi} - G_1^\psi G_1^\varphi \right)^{\frac{1}{2}} \right) = \text{ran} \left( I_{D_D^\psi} - G_1^\psi G_1^\varphi \right)^{\frac{1}{2}}.
\]

If
\[
\dim \left( \mathcal{U}_\psi^0 \ominus \text{ran} \zeta^\psi \right) \leq \dim \left( \mathcal{V}_\psi^0 \ominus \text{ran} \zeta^\psi \right),
\]
then choose a Hilbert space $\mathcal{M}$ such that
\[
\dim \left[ \mathcal{M} \oplus \left( \mathcal{U}_\psi^0 \ominus \text{ran} \zeta^\psi \right) \right] = \dim \left( \mathcal{V}_\psi^0 \ominus \text{ran} \zeta^\psi \right).
\]

By virtue of the fact that $\zeta^{(1)}$ is an isometry (and hence a unitary onto its range) choose a unitary $\delta : \text{ran} \zeta^\varphi \to \text{ran} \zeta^\psi$ and $\delta' : \mathcal{M} \oplus \left( \mathcal{U}_\psi^0 \ominus \text{ran} \zeta^\psi \right) \to \mathcal{V}_\psi^0 \ominus \text{ran} \zeta^\psi$ such that $\delta \oplus \delta' : \mathcal{U}_\psi^0 \oplus \mathcal{M} \to \mathcal{V}_\psi^0$ is unitary and the following diagram commutes.
As \( \alpha(U^0_\psi) = V^0_\psi \), we consider the unitary operator \( \tilde{\alpha} : U \oplus M \to V \) given by
\[
\tilde{\alpha} = \left( \begin{array}{cc}
\alpha & 0 \\
0 & \delta \oplus \delta'
\end{array} \right) : U \oplus M \to V.
\]

By definition of \( \tilde{\alpha} \) we have
\[
\beta \varphi_M(z) = \psi(z) \tilde{\alpha}
\]
for all \( z \in \mathcal{B}_d \).

Consider the operator \( \left[ \begin{array}{cc}
B^\varphi & 0 \\
0 & 0
\end{array} \right] : \mathcal{H}(k^d) \to U \oplus M \) acting as:
\[
h \mapsto B^{\varphi^*} h \oplus 0.
\]

For \( h = h_1 \oplus h_2 \) where \( h_1 \in \mathcal{D}_\varphi \) and \( h_2 \in \mathcal{D}_\psi^\perp \) one has
\[
\left[ \begin{array}{cc}
B^\varphi & 0 \\
0 & 0
\end{array} \right]^* h = (-G^\varphi G^\psi T G^\psi T G^\varphi h_2 + R^\varphi h_1) \oplus (\zeta^\varphi (I_{\mathcal{D}_\psi^\perp} - G^\psi G^\psi^* \frac{1}{2} h_2 + 0) \).
\]
The first term in brackets belongs to \( U^0_\varphi \) and the second term belongs to \( U^0_\psi + M \).

Taking into account the fact (due to coincidence of \( \varphi \) and \( \psi \)) that
\[
-\alpha G^\varphi G^\psi T G^\psi T G^\varphi = -G^\psi T G^\psi T G^\varphi \Gamma^d|_{\mathcal{D}_\varphi^\perp}
\]
and from the construction of the unitary \( \tilde{\alpha} \) it is easy to see that
\[
\tilde{\alpha} \left[ \begin{array}{cc}
B^\varphi & 0 \\
0 & 0
\end{array} \right]^* = B^{\psi^*} \Gamma^d.
\]

These relations lead to the fact that
\[
\left( \begin{array}{cc}
\Gamma'^d & 0 \\
0 & \beta
\end{array} \right) U^\varphi_M = U^\psi \left( \begin{array}{cc}
\Gamma & 0 \\
0 & \tilde{\alpha}
\end{array} \right)
\]
and our claim is proved. If, in (5), the inequality holds the other way, then the same argument with \( \varphi \) and \( \psi \) interchanged, would give us the unitary coincidence of \( U^\psi_M \) and \( U^\varphi \).

(2) \( \Rightarrow \) (1). Without loss of generality, let us assume that there exists a Hilbert space \( \mathcal{M} \) such that \( U^\varphi_M \) is unitarily coincident to \( U^\psi \). Then we know that the corresponding transfer functions coincide. But the transfer function of \( U^\psi_M \) is \( \varphi_M \).

Let \( \hat{\alpha} : U \oplus M \to V \) and \( \beta : \mathcal{Y} \to \mathcal{W} \) be unitaries such that
\[
\beta \varphi_M(z) = \psi(z) \hat{\alpha}
\]
holds for all \( z \in \mathbb{B}_d \).

From above we conclude that \( \hat{\alpha}(U^\perp) = V^\perp \), since \((U \oplus M)^0 = U^0 \oplus M\). Also, by virtue of the assumption \( \dim U^0 = \dim V^0 \) choose a unitary \( \tau : U^0 \to V^0 \) and consider the unitary operator

\[
\alpha = \begin{pmatrix} \hat{\alpha}_{U^0} & 0 \\ 0 & \tau \end{pmatrix} : \mathcal{U} \to \mathcal{V}.
\]

Now it is easy to see that \( \beta \varphi(z) = \psi(z) \alpha \) holds for all \( z \in \mathbb{B}_d \) and our claim is proved.

Now we complete the proof of the main theorem. Let \( \varphi \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y}) \) and \( \psi \in \mathcal{S}_d(\mathcal{V}, \mathcal{W}) \) and let \( \varphi \) coincide with \( \psi \). Let \( \alpha \) and \( \beta \) the unitary operators as before. Also let \( \mathcal{N}' \) and \( J' \) be the Hilbert space and the partial isometry counterparts of \( \mathcal{N} \) and \( J \) respectively for the Schur multiplier \( \psi \). Then it is easy to see that the operator

\[
\alpha \oplus \Gamma^d|_{D^\perp} \qquad \text{is a unitary operator from } \mathcal{N} \text{ onto } \mathcal{N}'.
\]

Considering the above unitary operator it is easy to see that \( \varphi_J \) and \( \psi_J \) coincides. Indeed, for \( m \in \mathcal{N} \) and \( z \in \mathbb{B}_d \),

\[
\psi_J(z)(\alpha \oplus \Gamma^d|_{D^\perp})m = \psi(z)J'(\alpha \oplus \Gamma^d|_{D^\perp})m = \psi(z)\alpha(P_{U^0} m) = \beta \varphi(z)(P_{U^0} m) = \beta \varphi(z)Jm = \beta \varphi_J(z)m.
\]

Now to complete the proof \( (1) \Rightarrow (2) \) of the main theorem, if \( (1) \) holds then \( \varphi_J \) coincides with \( \psi_J \). Considering their coisometric functional model realizations \( U^\varphi_J \) and \( U^\psi_J \) obtained canonically from an arbitrary but fixed choice \( U^\varphi \) and \( U^\psi \) of functional model realizations of \( \varphi \) and \( \psi \), an application of Theorem 3.2 proves our claim. \( \square \)

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