INTERMEDIATE ASSOUAD-LIKE DIMENSIONS FOR MEASURES

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Abstract. The upper and lower Assouad dimensions of a metric space are local variants of the box dimensions of the space and provide quantitative information about the ‘thickest’ and ‘thinnest’ parts of the set. Less extreme versions of these dimensions for sets have been introduced, including the upper and lower quasi-Assouad dimensions, θ-Assouad spectrum, and Φ-dimensions.
In this paper, we study the analogue of the upper and lower Φ-dimensions for measures. We give general properties of such dimensions, as well as more specific results for self-similar measures satisfying various separation properties and discrete measures.

1. Introduction

1.1. Background. The upper and lower Assouad dimensions of a metric space are local variants of the box dimensions of the space and provide quantitative information about the ‘thickest’ and ‘thinnest’ parts of the set. The analogous upper and lower Assouad dimensions for measures, denoted \( \dim A \mu \) and \( \dim L \mu \) respectively, were introduced by Käenmäki et al in [24, 25] and by Fraser and Howroyd in [9], where they were originally called upper and lower regularity dimensions respectively. In recent years, a number of less extreme versions of these dimensions for sets have been introduced, including the (upper and lower) quasi-Assouad dimensions, [3, 29], the θ-Assouad spectrum, [12], and the (most general) Φ-dimensions, [16]. These dimensions can all be different and provide more refined information about the geometry of the set.

One reason for the interest in the upper Assouad dimension of a measure is that it is finite if and only if the measure is doubling, meaning there is some constant \( c > 0 \) such that \( \mu(B(z,r)) \geq c\mu(B(z,2r)) \) for all \( z \in \text{supp} \mu \). However, as many interesting measures, such as self-similar measures that do not satisfy the open set condition, often fail to be doubling, less extreme dimensional notions for measures may also provide more insightful information. Hence the motivation for studying the more moderate (upper and lower) quasi-Assouad dimensions of measures, [21, 22] and the θ-Assouad spectrum for measures, [10,11]. In this paper, we will introduce and study the analogue of the upper and lower Φ-dimensions for measures.

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To explain how these dimensions are defined, we recall that for the upper Assouad dimension of the measure \( \mu \), we determine the infimal \( \alpha \) such that
\[
\mu(B(z,R))/\mu(B(z,r)) \leq C \left( \frac{R}{r} \right)^\alpha
\]
for all \( z \in \text{supp} \mu \) and \( r < R \). The lower Assouad dimension is similar, asking for the supremal \( \beta \) such that \( \mu(B(z,R))/\mu(B(z,r)) \geq C(R/r)^\beta \) for all \( z \in \text{supp} \mu \) and \( r < R \). As is the case for the quasi-Assouad dimensions and the \( \theta \)-Assouad dimension, the upper and lower \( \Phi \)-dimensions, denoted \( \dim_{\Phi} \mu \) and \( \dim_{\Phi} \mu \), are computed by further restricting the choice of \( r \), requiring that \( r \leq R^{1+\Phi(R)} \). The \( (\text{quasi-}) \) Assouad dimensions and \( \theta \)-Assouad spectrum are all special cases of \( \Phi \)-dimensions. For example, the Assouad dimensions are the special case of \( \Phi = 0 \).

We refer the reader to Definition 2.6 for the precise definitions of all these notions.

1.2. Overview of the paper. In Section 2 we establish basic properties of these dimensions. For example, we show that
\[
\dim_{L} \mu \leq \dim_{\Phi} \mu \leq \sup_{z} \dim_{\text{loc}} \mu(z) \leq \sup \dim_{\text{loc}} \mu(z) \leq \dim_{\Phi} \mu \leq \dim_{A} \mu
\]
where \( \dim_{\text{loc}} \mu(z) \) and \( \dim_{\text{loc}} \mu(z) \) are the upper and lower local dimensions of \( \mu \) at \( z \in \text{supp} \mu \). If the function \( \Phi(x) \to 0 \) as \( x \to 0 \), then the \( \Phi \)-dimensions lie between the quasi-Assouad and Assouad dimensions of the measure. We show that \( \dim_{\Phi} \mu \geq \dim_{\Phi} \text{supp} \mu \) and \( \dim_{\Phi} \mu \leq \dim_{\Phi} \text{supp} \mu \) if \( \mu \) is a doubling measure. Examples are given to see that all these inequalities can be strict.

It is clear that if \( \Phi(x) \geq \Psi(x) \) for all \( x > 0 \), then \( \dim_{\Phi} \mu \leq \dim_{\Psi} \mu \), and conversely for the lower dimensions. In Proposition 2.14 we prove, more specifically, that if there exists \( \lambda < 1 \) such that \( \Phi(x) \geq \lambda \Psi(x) \) for all \( x \), then
\[
\dim_{\Phi} \mu \geq \lambda \dim_{\Psi} \mu \text{ and } \dim_{\Psi} \mu \leq \dim_{\Phi} \mu + (1 - \lambda) \dim_{A} \mu.
\]
It follows that if \( \lim_{x \to 0} \Phi(x)/\Psi(x) \to 1 \), then the upper \( \Phi \) and \( \Psi \)-dimensions coincide, as do the lower \( \Phi \) and \( \Psi \)-dimensions if \( \mu \) is doubling. Moreover, if \( \Phi \) and \( \Psi \) are both constant functions, then \( \dim_{\Phi} \mu \) and \( \dim_{\Psi} \mu \) are simultaneously finite for any measure \( \mu \).

In Theorem 2.21 bounds are given for the \( \Phi \)-dimensions in terms of the exponents \( s,t \) satisfying \( cr^t \leq \mu(B(z,r)) \leq Cr^s \) for all \( z \in \text{supp} \mu \) and all \( r \). Indeed, if \( \Phi(x) \to \infty \) as \( x \to 0 \), then \( \dim_{\Phi} \mu \) is the supremum of such \( t \) and \( \dim_{\Phi} \mu \) is the supremum of such \( s \) if, in addition, \( \mu \) is doubling. This improves upon results in [10].

In [11], it is asked if an absolutely continuous measure with positive lower Assouad dimension has its density function in \( L^p \) for some \( p > 1 \). In Proposition 2.31 we answer this in the negative.

In Section 3 we study the dimensional properties of self-similar measures. In contrast to the case for general measures, and even for self-similar measures satisfying the open set condition, in Theorem 3.2 we see that if \( \mu \) is a self-similar measure satisfying the strong separation condition, then
\[
\sup_{z \in \text{supp} \mu} \dim_{\text{loc}} \mu(z) = \dim_{\Phi} \mu \text{ and } \dim_{\Phi} \mu = \inf_{z \in \text{supp} \mu} \dim_{\text{loc}} \mu(z).
\]
In Theorem 3.3 we characterize the finiteness of \( \dim_{\Phi} \mu \), in terms of a doubling-like property, for any self-similar measure \( \mu \) on \( \mathbb{R} \) whose support is an interval and which satisfies the weak separation condition. Consequently, any equicontinuous,
self-similar measure $\mu$ with interval support and satisfying the weak separation condition has the property that $\dim_{\Phi} \mu < \infty$ for all non-zero, constant functions $\Phi$. If, in addition, the probabilities associated with the left-most and right-most contractions in the underlying IFS are minimal, then we even have $\dim_{\Phi} \mu < \infty$ for any $\Phi$ satisfying $\Phi(x)/\Psi(x) \to \infty$ as $x \to 0$ for $\Psi(x) = \log |\log x|/|\log x|$. In particular, $\dim_{\Phi_A} \mu < \infty$ for such measures $\mu$. We give an example to show that the function $\Psi(x)$ is sharp. We also prove that $\dim_{\Phi_A} \mu = \infty$ for any biased Bernoulli convolution with contraction factor the inverse of the golden mean, thus the extra assumption on the probabilities is also a necessary condition.

In [9], Fraser and Howroyd compute the upper Assouad dimension of discrete measures of the form $\mu = \sum_{n=1}^{\infty} p_n \delta_{a_n}$. for summable $p_n$ either of the form $n^{-\lambda}$ for $\beta^{-n}$, and for $a_n$ of similar form. Of course, the support of such a measure is $\{a_n\}_{n=1}^{\infty} \cup \{0\}$. These measures are the focus of Section 4. We extend the results of [9] to all the upper $\Phi$-dimensions and allow $\mu\{0\} > 0$. The relationship between $a_n$, $p_n$ and the $\Phi$-dimension proves to be somewhat intricate, often depending on the limiting behaviour of $\Phi(x)$ as it relates to $a_n$ and $p_n$.

2. Basic Properties of the $\Phi$-Dimensions

2.1. Definitions.

**Definition 2.1.** A map $\Phi : (0, 1) \to \mathbb{R}^+$ is called a dimension function if $x^{1+\Phi(x)}$ decreases to 0 as $x$ decreases to 0. We will write $D$ for the space of all dimension functions.

Special examples of dimension functions include the constant functions $\Phi(x) = \delta \geq 0$ and the functions $\Phi(x) = 1/|\log x|$ or $|\log x|$. It is useful to observe that as $x^{1+\Phi(x)} \leq x$, $x^{1+\Phi(x)} \to 0$ as $x \to 0$ for any $\Phi \in D$.

**Notation 2.2.** Given a bounded metric space $X$, we denote the open ball centred at $x \in X$ with radius $R$ by $B(x, R)$. The notation $N_r(X)$ will mean the least number of balls of radius $r$ that cover $X$. We write $\text{diam}E$ for the diameter of $E \subseteq X$.

**Notation 2.3.** When we write $f \sim g$ we mean there are constants $a, b > 0$ such that $af(x) \leq g(x) \leq bf(x)$ for all $x$ in the domain of the functions $f, g$. When we write $f \preceq g$ we mean there is a constant $c$ such that $f(x) \leq cg(x)$ for all $x$.

**Definition 2.4** (García, Hare, Mendivil [16]). Let $\Phi$ be a dimension function. The **upper and lower $\Phi$-dimensions** of $E \subseteq X$ are given by

$$\dim_{\Phi} E = \inf \left\{ \alpha : (\exists C_1, C_2 > 0)(\forall 0 < r \leq R^{1+\Phi(R)} \leq R < C_1) \quad N_r(B(z, R) \cap E) \leq C_2 \left(\frac{R}{s}\right)^\alpha \forall z \in E \right\}$$

and

$$\dim_{\Phi} E = \sup \left\{ \alpha : (\exists C_1, C_2 > 0)(\forall 0 < r \leq R^{1+\Phi(R)} \leq R < C_1) \quad N_r(B(z, R) \cap E) \leq C_2 \left(\frac{R}{s}\right)^\alpha \forall z \in E \right\}.$$ 

**Remark 2.5.**

(i) The **upper Assouad** and **lower Assouad dimensions** of $E$, [2],[25], and denoted $\dim_A E$ and $\dim_L E$, are the special cases of the upper and lower $\Phi$-dimensions with $\Phi = 0$.

(ii) If we let $\Phi_{\theta}(x) = 1/\theta - 1$ for all $x$, then the upper and lower $\Phi_{\theta}$-dimensions are (slight modifications) of the **upper and lower $\theta$-Assouad spectrum** introduced in [12].
The upper and lower quasi-Assouad dimensions, denoted \( \dim_{qA} E \) and \( \dim_{qL} E \) and introduced in [3, 29], can be defined as the limit as \( \theta \to 1 \) of the upper and lower \( \Phi \)-dimensions, respectively.

A metric space \( X \) has finite upper Assouad dimension if and only if it is doubling, meaning there is a constant \( M \) such that any ball in \( X \) of radius \( R \) can be covered by at most \( M \) balls of radius \( R/2 \), [23]. The space \( X \) has positive lower Assouad dimension if and only if it is uniformly perfect, meaning there is a constant \( c > 0 \) so that \( B(z,r) \setminus B(z,cr) \neq \emptyset \) whenever \( z \in X \) and \( R \) is at most the diameter of \( X \), [25].

By a measure we will always mean a Borel probability measure on the metric space \( X \). The analogues of the Assouad dimensions for measures (also known as the upper and lower regularity dimensions), the quasi-Assouad dimensions and the \( \theta \)-Assouad spectrum for measures have been extensively studied, c.f., [9–11, 21, 22, 24, 25]. Motivated by these notions, we introduce the larger class of upper and lower \( \Phi \)-dimensions for measures.

**Definition 2.6.** Let \( \Phi \) be a dimension function and let \( \mu \) be a measure on the metric space \( X \).

The upper and lower \( \Phi \)-dimensions of \( \mu \) are given by

\[
\overline{\dim}_{\Phi} \mu = \inf \left\{ \alpha : \left( \exists C_1, C_2 > 0 \right) \left( \forall 0 < r < R^{1+\Phi(R)} \leq R \leq C_1 \right) \frac{\mu(B(x,R))}{\mu(B(x,r))} \leq C_2 \left( \frac{R}{r} \right)^\alpha \ \forall x \in \text{supp} \mu \right\}
\]

and

\[
\underline{\dim}_{\Phi} \mu = \sup \left\{ \alpha : \left( \exists C_1, C_2 > 0 \right) \left( \forall 0 < r < R^{1+\Phi(R)} \leq R \leq C_1 \right) \frac{\mu(B(x,R))}{\mu(B(x,r))} \geq C_2 \left( \frac{R}{r} \right)^\alpha \ \forall x \in \text{supp} \mu \right\}.
\]

**Remark 2.7.**

(i) The upper and lower Assouad dimensions of \( \mu \), introduced by Käenmäki et al in [24, 25] and Fraser and Howroyd in [9], and denoted \( \dim_A \mu \) and \( \dim_L \mu \) respectively, are the upper and lower \( \Phi \)-dimensions with \( \Phi \) the constant function 0.

(ii) If we let \( \Phi_\theta = 1/\theta - 1 \), then \( \overline{\dim}_{\Phi_\theta} \mu \) and \( \underline{\dim}_{\Phi_\theta} \mu \) are the upper and lower \( \theta \)-Assouad spectrum. The upper and lower quasi-Assouad dimensions of \( \mu \) are given by

\[
\dim_{qA} \mu = \lim_{\theta \to 1} \overline{\dim}_{\Phi_\theta} \mu, \quad \dim_{qL} \mu = \lim_{\theta \to 1} \underline{\dim}_{\Phi_\theta} \mu.
\]

See [21, 22].

To be precise, the \( \theta \)-Assouad spectrum of a set \( E \), as introduced in [12], only required consideration of \( r = R^{1/\theta} \). However, it was shown in [8] that if we denote this dimension by \( \overline{\dim}^{\theta} E \), then \( \overline{\dim}^{\theta} E = \sup_{\psi \leq \theta} \overline{\dim}^{\psi} E \). The analogous statements were proved for the lower \( \theta \)-Assouad spectrum of sets in [4] and for the upper and lower \( \theta \)-Assouad spectrum of measures in [22].

We note that the same proof as given for sets in [16] Prop. 2.15] shows that given any measure \( \mu \), there are dimension functions \( \Phi, \Psi \) such that \( \overline{\dim}_{\Phi} \mu = \dim_{qA} \mu \) and \( \underline{\dim}_{\Psi} \mu = \dim_{qL} \mu \).

**Remark 2.8.**
(i) It is known (see [9]) that a measure has finite upper Assouad dimension if and only if it is **doubling**, meaning there is a constant $C$ such that

$$
\mu(B(z, 2R)) \leq C \mu(B(z, R)) \text{ for all } R \leq \text{diam} X, \ z \in \text{supp} \mu.
$$

(ii) If a measure has an atom, then all of its lower $\Phi$-dimensions are 0. More generally, a measure $\mu$ has positive lower Assouad dimension if and only if $\mu$ is **uniformly perfect** (see [22,24]) which means there are positive constants $a, b$ such that

$$
\mu(B(z, R)) \geq b \mu(B(z, aR)) \text{ for all } z \in \text{supp} \mu \text{ and } R \leq \text{diam} X \text{ or, equivalently, there are constants } c > 1 \text{ and } a > 0 \text{ such that}
$$

$$
\mu(B(z, R)) \geq c \mu(B(z, aR)) \text{ for all } R \leq \text{diam} X, \ z \in \text{supp} \mu.
$$

2.2. **Preliminary results.** Here are some easy facts about these dimensions.

**Proposition 2.9.**

(i) For all dimension functions $\Phi$,

$$(2.1) \quad \dim_{\text{L}} \mu \leq \dim_{\Phi} \mu \leq \dim_{A} \mu$$

and

$$(2.2) \quad \overline{\dim}_{\Phi} \mu \geq \overline{\dim}_{\Phi} \text{supp} \mu.$$ 

If $\mu$ is doubling, then

$$(2.3) \quad \dim_{\Phi} \mu \leq \dim_{A} \text{supp} \mu.$$ 

(ii) If $\Phi(x) \leq \Psi(x)$ for all $x > 0$, then

$$(2.4) \quad \overline{\dim}_{\Phi} \mu \leq \overline{\dim}_{\Psi} \mu \text{ and } \underline{\dim}_{\Phi} \mu \leq \underline{\dim}_{\Psi} \mu.$$ 

(iii) If $\Phi(x) \to 0$ as $x \to 0$, then

$$(2.5) \quad \overline{\dim}_{\Phi} \mu \leq \dim_{\text{L}} \mu \text{ and } \underline{\dim}_{A} \mu \leq \underline{\dim}_{\Phi} \mu.$$ 

(iv) If there exists $x_0 > 0$ such that $\Phi(x) \leq C/|\log x|$ for $0 < x \leq x_0$, then

$$
\underline{\dim}_{\Phi} \mu = \dim_{A} \mu \text{ and } \overline{\dim}_{\Phi} \mu = \dim_{\text{L}} \mu.
$$

**Proof.** The first statement in (i) is obvious. For the second, we remark that it was shown [9,21] that $\dim_{\Phi} \text{supp} \mu$ is dominated by both $\dim_{A} \mu$ and $\dim_{\text{L}} \mu$. The same arguments work here for the upper $\Phi$-dimensions. For the lower $\Phi$-dimensions the arguments are similar to those found in [22,24] for the special cases of the (quasi-) lower Assouad dimensions.

To prove the claims of (iv), it is enough to study $\mu(B(z, R))/\mu(B(z, r))$ for $R \geq r \geq R^{1+\Phi(R)} \geq e^{-C} R$. For such $r$, we have

$$
1 \leq \frac{\mu(B(z, R))}{\mu(B(z, r))} \leq \frac{\mu(B(z, R))}{\mu(B(z, R^{1+\Phi(R)})�}.
$$

Since $R/r \sim 1$, it follows that the Assouad and $\Phi$-dimensions coincide.

Statements (ii) and (iii) follow easily from the definitions. \qed

**Remark 2.10.** The inequalities of (2.1)-(2.5) can all be strict. In [21], examples are given to show that $\dim_{(q)A} \mu > \dim_{(q)A} \text{supp} \mu$. Similar examples can be constructed for the upper and lower $\Phi$-dimensions to see the strictness in (2.2) and (2.3). In [16, Theorem 3.5], formulas are given for the upper and lower $\Phi$-dimensions of central Cantor sets. Using these formulas many examples are given there to illustrate the strictness of the analogues of the inequalities in (2.1), (2.4) and (2.5) when the
measure $\mu$ is replaced by the Cantor set $E$. However, if $\mu$ is the uniform Cantor measure on the Cantor set $E$, then the upper or lower $\Phi$-dimension of $\mu$ coincides with that of $E$.

In fact, it is shown in [16] that given any $0 < \alpha < \beta < 1$, there is a central Cantor set $E \subseteq [0, 1]$ with $\{\dim_{\Phi} E : \Phi \in D, \lim_{x \to 0} \Phi(x) = 0\} = [\alpha, \beta] = [\dim_{\Phi} A E, \dim_{\Phi} A]$. Taking $\mu$ to be the uniform Cantor measure on this Cantor set gives the same support is defined as $\dim_{\Phi} x \Phi$. This can be decomposed into 3 subintervals of level $n$, namely $[3a/3^n + 1, (3a + 1)/3^{n+1}]$, $[(3a + 1)/3^n + 1, (3a + 2)/3^{n+1}]$ and $[(3a + 2)/3^n + 1, 3(a + 1)/3^{n+1}]$. We will call these the left child, the middle child and the right child, respectively, of the

We will construct probability measures with support in $[0, 1]$ with that of $E$, namely $[\dim_{\Phi} A E, \dim_{\Phi} A]$. This can be decomposed into 3 subintervals of level $n+1$, namely $[3a/3^n + 1, (3a + 1)/3^{n+1}]$, $[(3a + 1)/3^n + 1, (3a + 2)/3^{n+1}]$ and $[(3a + 2)/3^n + 1, 3(a + 1)/3^{n+1}]$. We will call these the left child, the middle child and the right child, respectively, of the

We recall the definition of the local dimension of a measure.

**Definition 2.11.** The **lower local dimension** of a measure $\mu$ at a point $z$ in its support is defined as

$$\dim_{\text{loc}}^\mu(z) = \liminf_{r \to 0} \frac{\log \mu(B(z, r))}{\log r}.$$  

By replacing $\liminf$ with $\limsup$ we obtain the **upper local dimension** and if the lower and upper local dimensions are equal, then we call the quantity the **local dimension** of $\mu$ at $z$.

**Proposition 2.12.** For any dimension function $\Phi$,

$$\dim_{\Phi} \mu \leq \inf_{z \in \text{supp} \mu} \dim_{\text{loc}}^\mu(z) \leq \dim_{\Phi} \mu \leq \sup_{z \in \text{supp} \mu} \dim_{\text{loc}}^\mu(z) \leq \dim_{\Phi} \mu.$$  

**Proof.** The middle inequalities around $\dim_{\Phi} \mu$ are standard, see [5, ch. 10]. The inequality $\dim_{\text{loc}} \dim_{\text{loc}}^\mu(z) \leq \dim_{\Phi} \mu$ obviously holds if $d = \dim_{\Phi} \mu = \infty$, so assume $d < \infty$. Fix $\varepsilon > 0$ and choose $C_1$ such that for all $r \leq R^{1+\Phi(r)} \leq R \leq C_1$ we have

$$\frac{\mu(B(z, R))}{\mu(B(z, r))} \leq \left(\frac{R}{r}\right)^{d+\varepsilon}.$$  

Taking logarithms and dividing by $-\log r$, we see that

$$\frac{\log(\mu(B(z, R)))}{-\log r} + \frac{\log(\mu(B(z, r)))}{\log r} \leq (d + \varepsilon) \frac{\log R}{-\log r} + d + \varepsilon.$$  

Keeping $R$ fixed and letting $r \to 0$ gives

$$\lim_{r \to 0} \frac{\log(\mu(B(z, r)))}{\log r} \leq d + \varepsilon.$$  

That proves $\sup_{z} \dim_{\text{loc}}^\mu(z) \leq d$. The argument for the lower $\Phi$-dimension is similar. \qed

Here is an example illustrating strictness in (2.6).

**Example 2.13** (Examples where $\dim_{\Phi} \mu > \sup_{z} \dim_{\text{loc}}^\mu(z)$ or $\dim_{\Phi} \mu < \inf_{z} \dim_{\text{loc}}^\mu(z)$). We will construct probability measures with support in $[0, 1]$ by specifying the measure of each of the triadic subintervals of $[0, 1]$.

Consider a level $n$, triadic subinterval, $[a/3^n, (a+1)/3^n]$ for integers $a \in \{0, 1, \ldots, 3^n-1\}$. This can be decomposed into 3 subintervals of level $n+1$, namely $[3a/3^n + 1, (3a + 1)/3^{n+1}]$, $[(3a + 1)/3^n + 1, (3a + 2)/3^{n+1}]$ and $[(3a + 2)/3^n + 1, 3(a + 1)/3^{n+1}]$. We will call these the left child, the middle child and the right child, respectively, of the
original parent interval. We will define the measures by proscribing the ratio of the measure of a child with the measure of the parent.

We begin by choosing an increasing sequence \( \{n_j\} \), with \( n_{j+1} \gg n_j \). Let \( M_{n_j} := [3^{-n_j}, 2 \cdot 3^{-n_j}] \). Inductively, let \( M_{n_j+k+1} \) be the middle child of \( M_{n_j+k} \) for \( k = 0, 1, \ldots, n_j \) and let \( L_{n_j+k+1} \) (resp. \( R_{n_j+k+1} \) ) be the left (resp. right) child of \( M_{n_j+k} \).

Given a sequence \( \{p_j\}_{j=1}^{\infty} \) with \( 0 \leq p_j \leq 1 \), we define the measure \( \mu(p_j) = \mu \) by the rule that the ratio of the \( \mu \)-measure of the middle child to the measure of its parent is \( p_j \) for \( k = 0, \ldots, n_j \) and the ratio of the \( \mu \)-measure of the left or right child to the measure of its parent is \( \frac{1-p_j}{2} \). Thus, for \( k = 0, 1, \ldots, n_j \)

\[
\mu(M_{n_j+k+1}) = p_j \mu(M_{n_j+k})
\]

and

\[
\mu(L_{n_j+k+1}) = \mu(R_{n_j+k+1}) = \left( \frac{1-p_j}{2} \right) \mu(M_{n_j+k}).
\]

For all other children of all other parents we set the ratio of the measure of the child to the parent to be \( 1/3 \).

One can see that for any nested sequence of triadic intervals the ratio of the measure of a parent to a child is \( 1/3 \), except possibly a finite number of times. This gives us that

\[
\dim_{\text{loc}} \mu(z) = 1 \text{ for all } z \in [0,1].
\]

Put \( \Phi(x) = 1 \). Let \( x_j = \frac{1}{3} 3^{-n_j} + 1 \) be the midpoint of the triadic subinterval \( M_{n_j} = [3^{-n_j}, 2 \cdot 3^{-n_j}] \). Put \( R_j = 3^{-n_j}/2 \) and \( r_j = 3^{-2n_j}/6 \). We note that \( \Phi(R_j) = 1 \), thus \( R_j^{1+\Phi(R_j)} = 3^{-2n_j}/4 = 3r_j/2 \). As \( B(x_j, r_j) \) is a triadic interval at level \( 2n_j+1 \), and it and all its ancestors back to level \( n_j + 1 \) are middle children, we see that

\[
\frac{\mu(B(x_j, R_j))}{\mu(B(x_j, r_j))} = \left( \frac{1}{p_j} \right)^{n_j+1} = (3^{n_j+1})^{-\log_3 p_j}.
\]

Since \( R_j/r_j = 3^{n_j+1} \), it follows that

\[
\dim_{\Phi} \mu \geq \max(1, -\log_3(\lim \sup_p p_j)).
\]

In a similar fashion, we have that

\[
\dim_{\Phi} \mu \leq \min(1, -\log_3(\lim \inf_p p_j)).
\]

In fact, we have equality in both cases, as similar reasoning shows. We leave these details to the reader.

Here are some explicit examples.

- If \( \lim \inf_p p_j = 1/4 \) then
  \[1 = \sup_{z \in \text{supp} \mu} \dim_{\text{loc}} \mu(z) < \dim_{\Phi} \mu = \log 4/\log 3.\]

- If \( \lim \inf_p p_j = 0 \) then
  \[1 = \sup_{z \in \text{supp} \mu} \dim_{\text{loc}} \mu(z) < \dim_{\Phi} \mu = \infty.\]

- If \( \lim \sup_p p_j = 1/2 \) then
  \[
  \log 2/\log 3 = \dim_{\Phi} \mu < \inf_{z \in \text{supp} \mu} \dim_{\text{loc}} \mu(z) = 1
  \]
Proposition 2.14. Let \( \Phi, \Psi \in \mathcal{D} \). Suppose there are constants \( 0 < \lambda < 1 \) and \( x_0 > 0 \) such that
\[
\Phi(x) \geq \lambda \Psi(x) \text{ for all } 0 < x \leq x_0.
\]
Then for any \( \varepsilon > 0 \) and small enough \( R \),
\[
0 = \dim_{\Phi} \mu < \inf_{z \in \text{supp} \mu} \dim_{\Phi, \mu}(z) = 1
\]

2.3. Comparing \( \Phi \)-Dimensions. As commented earlier, it is immediate from the definition that if \( \Phi \geq \Psi \), then \( \dim_{\Phi} \mu \leq \dim_{\Psi} \mu \) and conversely for the lower dimensions. If we know more about the relative sizes of \( \Phi \) and \( \Psi \), more can be said about the corresponding dimensions.

**Proposition 2.14.** Let \( \Phi, \Psi \in \mathcal{D} \). Suppose there are constants \( 0 < \lambda < 1 \) and \( x_0 > 0 \) such that
\[
\Phi(x) \geq \lambda \Psi(x) \text{ for all } 0 < x \leq x_0.
\]
Then for any measure \( \mu \) on \( \mathcal{E} \) we have
\[(i) \int \dim_{\Phi} \mu \geq \lambda \int \dim_{\Psi} \mu \text{ and } \dim_{\Phi} \mu \leq \dim_{\Psi} \mu + (1 - \lambda) \dim_{\mathcal{A}} \mu ;
(ii) \int \dim_{\Phi} \mu \leq \lambda \int \dim_{\Psi} \mu \text{ and } \dim_{\Phi} \mu \leq \dim_{\Psi} \mu + (1 - \lambda) \dim_{\mathcal{A}} \mu .
\]

**Proof.** (i) We begin with the upper dimensions. We can assume \( \dim_{\Phi} \mu > 0 \) for otherwise there is nothing to prove. Choose positive real numbers \( \alpha \) and \( \beta \) such that \( \dim_{\Phi} \mu = \sup \alpha_n \) and \( \dim_{\Psi} \mu = \inf \beta_n \). Suppose \( x_n \in \text{supp} \mu, R_n \to 0 \) and \( R_n \leq R_n^{1+\Psi(R_n)} \).

Hence
\[
\mu(B(x_n, R_n)) = \mu(B(x_n, r_n)) \geq \left( \frac{R_n}{r_n} \right)^{\alpha_n} \geq R_n^{\Psi(R_n)^{\alpha_n}}
\]

and this implies that \( \dim_{\Phi} \mu \leq \lambda \dim_{\Psi} \mu .
\]

Now we consider the lower dimensions. We can assume \( \dim_{\Phi} \mu < \infty \) and \( \dim_{\mathcal{A}} \mu < \infty \). Suppose \( x_n \in \text{supp} \mu, R_n \to 0 \) and \( R_n \leq R_n^{1+\Phi(R_n)} \) are chosen such that
\[
\mu(B(x_n, R_n)) = \mu(B(x_n, r_n)) \leq \left( \frac{R_n}{r_n} \right)^{\alpha_n}
\]

where \( \alpha_n \) is \( \dim_{\Phi} \mu \), and again assume that for all but finitely many \( n \) we have
\[
R_n^{1+\Phi(R_n)} \leq R_n^{1+\Phi(R_n)}.
\]

Then, for any \( \varepsilon > 0 \) and small enough \( R_n \), we have
\[
\frac{\mu(B(x_n, R_n))}{\mu(B(x_n, R_n^{1+\Phi(R_n)}))} = \frac{\mu(B(x_n, R_n))}{\mu(B(x_n, R_n^{1+\Phi(R_n)}))} \cdot \frac{\mu(B(x_n, R_n^{1+\Phi(R_n)}))}{\mu(B(x_n, R_n^{1+\Phi(R_n)}))} \leq \frac{C(R_n^{\varepsilon})}{R_n^{(1 - \lambda)\Psi(R_n)(\dim_{\mathcal{A}} \mu + \varepsilon)}} \\
\leq CR_n^{-(1 - \lambda)\Psi(R_n)(\dim_{\mathcal{A}} \mu + \varepsilon)} \\
\leq CR_n^{-(1 - \lambda)\Psi(R_n)(\dim_{\mathcal{A}} \mu + \varepsilon)}.
\]
and this obviously implies $\dim_\Phi \mu \leq \dim_\Psi \mu + (1 - \lambda) \dim_A \mu$.

\[ \text{[ii]} \] The proof for sets is essentially the same. \qedhere

We have the following corollaries as an immediate consequence.

\textbf{Corollary 2.15.}

\begin{itemize}
  \item[(i)] If $\Phi(x)/\Psi(x) \to 1$ as $x \to 0$, then $\overline{\dim}_\Phi \mu = \overline{\dim}_\Psi \mu$. The same statement holds for the lower dimensions if, in addition, $\mu$ is doubling.
  \item[(ii)] If $\Phi(x) \to 0$ as $x \to 0$, then $\overline{\dim}_\Phi \mu = \overline{\dim}_\Psi \mu$ where $\Phi_\theta$ is the constant function $\theta$.
  \item[(iii)] If $\Phi \sim \Psi$, then $\overline{\dim}_\Phi \mu < \infty$ if and only if $\overline{\dim}_\Psi \mu < \infty$. In particular, if $\Phi$ and $\Psi$ are positive constant functions, then $\overline{\dim}_\Phi \mu < \infty$ if and only if $\overline{\dim}_\Psi \mu < \infty$.
\end{itemize}

\textbf{Remark 2.16.} It would be interesting to know if the assumption of a doubling measure is necessary for the second statement of \[ \text{[ii]} \].

If $\Phi(x)/\Psi(x)$ does not tend to 1, we do not, in general, have equality of the dimensions as the next result illustrates.

\textbf{Proposition 2.17.} Suppose $\Phi, \Psi$ are dimension functions decreasing to 0 as $x \to 0$ with $\Psi(x) |\log x| \to \infty$ as $x \to 0$. Assume there is some constant $\eta > 0$ such that $\Phi(x) \geq 1 + \eta \Psi(x)$ for all $x$ small. Then there is a measure $\mu$ such that $\overline{\dim}_\Phi \mu < \overline{\dim}_\Psi \mu$.

\textbf{Proof.} This is essentially a consequence of \[ \text{[16]} \text{ Theorem 3.8] where the analogous result was shown for sets. Indeed, it is shown that under these assumptions, there is a central Cantor set $E$ with $\overline{\dim}_\Phi E < \overline{\dim}_\Psi E$. If we choose $\mu$ to be the uniform measure on this Cantor set, then $\overline{\dim}_\Phi \mu = \overline{\dim}_\Psi E$, while $\overline{\dim}_\Psi \mu = \overline{\dim}_\Psi E$. \qedhere

\textbf{Remark 2.18.} We recall that the condition $\limsup_{x \to 0} \Psi(x) |\log x| < \infty$ implies that the $\Psi$-dimension coincides with the Assouad dimension (Proposition 2.14[iv]), hence the necessity of the hypothesis $\Psi(x) |\log x| \to \infty$. Later in the paper (Cor. 2.22[iv]), we will prove that if $\Phi, \Psi \to \infty$, then $\overline{\dim}_\Phi \mu = \overline{\dim}_\Psi \mu$ for all measures $\mu$, regardless of the comparative sizes of $\Phi, \Psi$.

What might be thought of as the analogue of Cor. 2.15[iii] for the lower dimension (namely, that $\overline{\dim}_\Phi \mu > 0$ if and only if $\overline{\dim}_\Psi \mu > 0$ when $\Phi \sim \Psi$) need not be true, even when the Assouad dimension of the measure is finite, as the next example illustrates.

\textbf{Example 2.19 (An example of a doubling measure and dimension functions $\Phi \sim \Psi$, with $\overline{\dim}_\Phi \mu = 0$, but $\overline{\dim}_\Psi \mu > 0$).} We will let $\Phi, \Psi$ be the constant functions 1, 2 respectively. Choose a sequence of integers $\{n_j\}$ with $n_{j+1} \geq 9n_j$ and take as $\mu$ the corresponding measure given in Example 2.13 with $p_j = 1$ for all $j$. Thus $\overline{\dim}_\Phi \mu = 0$.

To see that $\mu$ is a doubling measure, consider any $x \in \text{supp} \mu$ and the balls $B(x, 3^{-n})$ and $B(x, 3^{-(n+1)})$. The smaller of these balls contains a triadic interval $I$ at level $n+1$ which contains $x$. (If there is a choice for $I$, choose one of positive $\mu$-measure.) The parent, $J$, of $I$ is a triadic interval of level $n$ and has the property that $\mu(J)/3 \leq \mu(I) \leq \mu(J)$. The adjacent triadic intervals at level $n$, say $J^-$ and $J^+$, have measure either equal to that of $\mu(J)$ or 0.
We have that \( I \subseteq B(x, 3^{-n-1}) \subseteq B(x, 3^{-n}) \subseteq J^- \cup J \cup J^+ \). This gives that
\[
\frac{\mu(B(x, 3^{-n}))}{\mu(B(x, 3^{-(n+1)}))} \leq \frac{\mu(J^- \cup J \cup J^+)}{\mu(I)} \leq 9.
\]
Thus \( \mu \) is doubling and hence has finite Assouad dimension.

We will now show that \( \dim_{\Phi} \mu \geq 1/4 > 0 \). Let \( x \in \text{supp} \mu \) and \( r \leq R^{1+\Psi(R)} = R^3 \).
Pick \( n \) maximal and \( N \) minimal such that
\[
B(x, r) \subseteq I_n \subseteq I_N \subseteq B(x, R),
\]
where \( I_n \) and \( I_N \) are triadic intervals of length \( 3^{-n} \) and \( 3^{-N} \) respectively. Choose a sequence of triadic intervals \( I_k \), of level \( k \), containing \( x \), so that \( I_n \leq I_{n+1} \subseteq I_k \subseteq I_N \) for each \( k = N, \ldots, n-1 \).

We remark that as \( n \sim 3N \) and \( n_{j+1} \geq 9n_j \), there can be at most one choice of \( j \) with \( \{n_j, \ldots, 2n_j\} \cap \{N, \ldots, n\} \) non-empty. By the construction of \( \mu \), for \( k \in \{N, \ldots, n\} \), either \( \mu(I_k) = \mu(I_{k-1})/3 \) (the measure of the child is 1/3rd of that of the parent) or \( \mu(I_k) = \mu(I_{k-1}) \) (the measure of the child equals that of the parent), with equality only on levels \( k \) where \( n_j \leq k \leq 2n_j \) for this (unique) choice of \( n_j \). Hence, for all \( N \leq k < n_j \) and all \( 2n_j < k \leq n \) we have \( \mu(I_k) = \mu(I_{k-1})/3 \). One can check that at least \( 1/4 \) of these children will have measure equal to \( 1/3 \) the measure of their parent and this gives that
\[
\frac{\mu(B(x, R))}{\mu(B(x, r))} \geq \frac{\mu(I_N)}{\mu(I_n)} \geq \left( \frac{1}{3} \right)^{(N-n)/4}.
\]
As \( R/r \sim 3^{n-N} \), it follows that \( \dim_{\Phi} \mu \geq 1/4 \).

We remark that it is possible to have \( \dim_{\Phi} \mu < \infty \) for all non-zero, constant dimension functions and yet \( \dim_{\Phi} \mu = \infty \). In Proposition [3.9] we will prove that the biased Bernoulli convolution with contraction factor the inverse of the golden mean has this property, as does the measure in the next example.

**Example 2.20** (A measure \( \mu \) having \( \dim_{\Phi} \mu = \infty \), but \( \overline{\dim}_{\Phi} \mu < \infty \) for all \( \Phi = \theta > 0 \)). We define the measure \( \mu \) on the diadic subintervals of \([0,1]\) by specifying that the ratio of the measure of the left child of a diadic subinterval to that of its parent is \( 2/3 \), while the ratio of the measure of the right child to the parent is \( 1/3 \).

Let \( r = 2^{-(n+[\theta n]+2)} \), \( R = 2^{-n} + 2^{-(n+[\theta n]+2)} \) and take \( x \) to be the midpoint of the diadic interval of level \( n + [\theta n] + 1 \) that has \( 1/2 \) as its right end point. Then \( B(x, r) \) is this diadic interval, while \( B(x, R) \) contains the diadic interval of level \( n \) to the right of \( 1/2 \) and is contained in the union of this level \( n \) diadic interval and the level \( n-1 \) diadic interval immediately to the left of \( 1/2 \). Thus \( R/r \sim 2^{\theta n} \) and
\[
\frac{\mu(B(x, R))}{\mu(B(x, r))} \sim \frac{(2/3)^n}{(1/3)^{n+\theta n}}.
\]
Hence
\[
\overline{\dim}_{\Phi} \mu \sim \frac{\log 2 + \theta \log 3}{\theta \log 2} = 1 + \frac{\log 3}{\log 2}.
\]
This tends to infinity as \( \theta \to 0 \), thus \( \dim_{\Phi} \mu = \infty \).
2.4. Regularity-like properties. Recall that a measure is called \( s \)-regular if there is a constant \( c > 0 \) such that for all \( x \in \text{supp} \mu \) and \( r \leq \text{diam}(\text{supp} \mu) \) we have \( c^{-1}r^s \leq \mu(B(x, r)) \leq cr^s \). Clearly, all the \( \Phi \)-dimensions agree for regular measures, c.f., [24,25], but the converse is not true, as seen in [22, Example 2.7].

Following [10], we will define the **upper Minkowski dimension** of a compactly supported measure \( \mu \) to be
\[
\dim M \mu = \inf \{ t : \exists B > 0 \text{ so that } \inf_{z \in \text{supp} \mu} \mu(B(z, r)) \geq Br^t \forall r \leq \text{diam}(\text{supp} \mu) \}
\]
and the **Frostman dimension** of \( \mu \) to be
\[
\dim F \mu = \sup \{ s : \exists A > 0 \text{ so that } \sup_{z \in \text{supp} \mu} \mu(B(z, r)) \leq Ar^s \forall r \leq \text{diam}(\text{supp} \mu) \}.
\]

Note that \( \sup \{ \dim M \mu \} \leq \dim M \mu \) and \( \inf \{ \dim M \mu \} \geq \dim M \mu \).

In [10], Fraser and Käenmäki show that for the constant function \( \Phi = 1/\theta - 1 \), \( \dim F \mu \leq \dim M \mu \) and \( \dim M \mu \leq \dim M \mu \leq (\dim M \mu)/(1 - \theta) \). Here is an extension of this result.

**Theorem 2.21.** Let \( \mu \) be a measure with compact support and suppose \( \Phi \in \mathcal{D} \). Put \( L = \lim \sup_{x \to 0} \Phi(x)^{-1} \). Then
\[
\dim F \mu \geq \dim M \mu \geq \dim F \mu - L(\dim M \mu - \dim F \mu)
\]
and
\[
\dim M \mu \leq \dim M \mu \leq \dim M \mu + L(\dim M \mu - \dim F \mu).
\]

**Proof of Theorem 2.21.** We first observe that for any fixed \( \rho > 0 \),
\[
\inf \{ \mu(B(z, \rho)) : z \in \text{supp} \mu \} > 0.
\]
This is an elementary compactness argument. Assume not. Then for some \( \rho > 0 \) there exists a sequence \( z_n \in \text{supp} \mu \) such that \( z_n \to z_0 \in \text{supp} \mu \) and \( \mu(B(z_n, \rho)) \to 0 \). Choose \( N \) such that for all \( n \geq N \) we have \( \|z_n - z_0\| \leq \rho/2 \). Then \( B(z_0, \rho/2) \subseteq B(z_n, \rho) \) and hence \( \mu(B(z_0, \rho/2)) \leq \mu(B(z_n, \rho)) \). This implies that \( \mu(B(z_0, \rho/2)) = 0 \), a contradiction to \( z_0 \) being in the support of \( \mu \).

Let \( D = \dim M \mu \) and \( d = \dim M \mu \). We will first prove the left side inequalities. Of course, the second is obvious if \( D = \infty \), so assume otherwise. Fix \( 0 < \varepsilon < 1 \) and choose \( \rho \) such that for all \( r \leq R^{1+\Phi(R)} \leq R \leq \rho \) and \( z \in \text{supp} \mu \),
\[
C_1 \left( \frac{R}{r} \right)^{d-\varepsilon} \leq \frac{\mu(B(z, R))}{\mu(B(z, r))} \leq C_2 \left( \frac{R}{r} \right)^{D+\varepsilon}.
\]
Assume \( r \leq \rho^{1+\Phi(\rho)} \). For some constant \( C_\rho > 0 \) we have
\[
\frac{C_\rho}{\mu(B(z, r))} \leq \mu(B(z, \rho))/\mu(B(z, r)) \leq C_2 \left( \frac{\rho}{r} \right)^{D+\varepsilon} = C_2 \rho^{D+\varepsilon} r^{D+\varepsilon}.
\]
Consequently, for a suitable constant \( B \),
\[
\mu(B(z, r)) \geq C_\rho C_2^{-1} \rho^{-(D+\varepsilon)} r^{D+\varepsilon} = Br^{D+\varepsilon}.
\]
As this is true for all \( \varepsilon > 0 \), we deduce that \( \dim M \mu \leq D \).

We can similarly conclude that \( \dim F \mu \geq d \) since
\[
\frac{1}{\mu(B(z, r))} \geq \mu(B(z, \rho))/\mu(B(z, r)) \geq C_1 \left( \frac{\rho}{r} \right)^{d-\varepsilon} = C_1 \rho^{d-\varepsilon} r^{D+\varepsilon}.
\]

Now we prove the right side inequalities. For notational ease, put \( a = \dim F \mu \) and \( b = \dim M \mu \). There is no loss of generality in assuming \( b < \infty \). Take \( \varepsilon > 0 \).
For any \( q \) and \( z \in \text{supp} \mu \) we have \( \mu(B(z, q)) \leq Aq^{a-\varepsilon} \) and \( \mu(B(z, q)) \geq Bq^{b+\varepsilon} \) for positive constants \( A, B \) depending on \( \varepsilon \).

Suppose \( r = R^{1+\Psi(R)} \) with \( \Psi(R) \geq \Phi(R) \). Then for \( C = B/A \) we have
\[
\frac{\mu(B(z, R))}{\mu(B(z, r))} \geq \frac{B}{A} \left( \frac{R^{b+\varepsilon}}{r^{a-\varepsilon}} \right) = CR^{b+\varepsilon-(a-\varepsilon)(1+\Psi(R))} = C \left( \frac{R}{r} \right)^{a-\varepsilon-(b-a+2\varepsilon)/\Psi(R)} \geq C \left( \frac{R}{r} \right)^{a-\varepsilon-(b-a+2\varepsilon)/\Psi(R)}.
\]
As this holds for all \( \varepsilon > 0 \), it follows that
\[
d \geq \lim_{R \to 0} \inf_{R \to 0} (a - (b - a)/\Phi(R)) = a - L(b - a).
\]
The argument for \( D \) is similar. \( \square \)

The following corollaries are immediate.

**Corollary 2.22.**
\begin{enumerate}
\item If \( \Phi(x) \to \infty \) as \( x \to 0 \), then \( \overline{\dim}_\Phi \mu = \overline{\dim}_M \mu \). If, in addition, \( \overline{\dim}_M \mu < \infty \), then \( \overline{\dim}_\Phi \mu = \overline{\dim}_F \mu \).
\item If \( \Phi_1, \Phi_2 \to \infty \), then \( \overline{\dim}_{\Phi_1} \mu = \overline{\dim}_{\Phi_2} \mu \). If \( \overline{\dim}_M \mu < \infty \), then also \( \overline{\dim}_{\Phi_1} \mu = \overline{\dim}_{\Phi_2} \mu \).
\end{enumerate}

**Remark 2.23.**
\begin{enumerate}
\item It was shown in [16, Prop. 2.8] that if \( \Phi(x) \to \infty \) as \( x \to 0 \), then \( \overline{\dim}_\Phi E \) is the upper box (or Minkowski) dimension of \( E \), while \( \overline{\dim}_\Phi E \) is the lower box dimension if, in addition, \( \underline{\dim}_\Phi E > 0 \).
\item We do not know if the assumption that \( \overline{\dim}_M \mu < \infty \) is necessary.
\end{enumerate}

**Corollary 2.24.** Let \( \Psi_\theta = 1/\theta - 1 \) for \( \theta \in (0, 1) \). Then
\[
\overline{\dim}_M \mu \leq \overline{\dim}_{\Psi_\theta} \mu \leq \overline{\dim}_M \mu - \frac{\theta \dim_F \mu}{1 - \theta}
\]
and
\[
\dim_F \mu - \frac{\theta \overline{\dim}_M \mu}{1 - \theta} \leq \overline{\dim}_{\Psi_\theta} \mu \leq \dim_F \mu.
\]
Furthermore, \( \lim_{\theta \to 0} \overline{\dim}_{\Psi_\theta} \mu = \overline{\dim}_M \mu \) and if \( \overline{\dim}_M \mu < \infty \), then \( \lim_{\theta \to 0} \overline{\dim}_{\Psi_\theta} \mu = \dim_F \mu \).

Recall that in Proposition 2.12 it was shown that \( \overline{\dim}_\Phi \mu \geq \sup_{z \in \text{supp} \mu} \dim_{\text{loc}} \mu(z) \). Another consequence of the theorem is that we can show it is possible to have \( \overline{\dim}_\Phi \mu = \infty \) for all \( \Phi \in \mathcal{D} \) and yet \( \dim_{\text{loc}} \mu(z) \leq 1 \) for all \( z \in \text{supp} \mu \).

**Example 2.25** (A measure \( \mu \) with \( \dim_{\text{loc}} \mu(z) = 1 \) for all \( z \), but \( \overline{\dim}_M \mu = \infty \)). We can achieve this with a slight modification of the strategy of Example 2.13. Instead of assigning special weights \( p_j \) on levels \( n_j + k \) for \( k = 0, ..., n_j \), do this on levels \( n_j + k \) for \( k = 0, ..., n_j^2 \) and choose \( n_j + 1 \geq n_j^2 \). Let \( \Phi(x) = \lceil \log_2 x \rceil \to \infty \) as \( x \to 0 \). By choosing \( p_j \) with \( \lim \inf_{j} p_j = 1 \), we can construct \( \mu \) with the property that \( \dim_{\text{loc}} \mu(z) = 1 \) for all \( z \in \text{supp} \mu \), but \( \overline{\dim}_\Phi \mu = \infty \). Since \( \Phi(x) \to \infty \) as \( x \to 0 \), we have \( \overline{\dim}_M \mu = \infty \) from Corollary 2.22 and therefore \( \overline{\dim}_\Psi \mu = \infty \) for all dimension functions \( \Psi \).
Similarly, by taking \( p_1 \) with \( \limsup p_1 = 0 \), we can construct a measure \( \nu \) with \( \dim_{F} \nu = \dim_{F} \phi = 0 \) for all dimension functions \( \phi \), while \( \dim_{loc} \nu(z) = 1 \) for all \( z \).

2.5. Smoothness Properties. In [11], Fraser and Troscheit show that if \( \mu \) is a uniformly perfect, absolutely continuous measure supported on \([0,1]\), with monotonic density function \( f \), then \( f \in L^p(\mathbb{R}) \) for some \( p > 1 \). They asked if this was true without the monotonicity assumption. Here we will show that the answer to this question is no.

In this subsection (only), we will think of \([0,1]\) both as a subset of \( \mathbb{R} \) and as the group \( \mathbb{T} \) under addition mod 1. When we consider balls in the latter sense, we will use the notation \( B_T \).

When we say a measure on \([0,1]\) is symmetric, we will mean that \( \mu(E) = \mu(1-E) \) for all Borel sets \( E \subseteq [0,1] \subseteq \mathbb{R} \). (Of course, if we view \([0,1]\) as \( \mathbb{T} \), then \( 1-E = -E \).

Lemma 2.26. Let \( \mu \) be a measure supported on \([0,1] \subseteq \mathbb{R} \) that is symmetric.

(i) If there are constants \( a, c > 0 \) such that \( \mu(B(z,R)) \geq c \mu(B(z,aR)) \) for all \( z \in \text{supp} \mu \) and \( R \leq 1 \), then
\[
\mu(B_T(z,R)) \geq \frac{c}{2} \mu(B_T(z,aR)) \text{ for all } z \in \text{supp} \mu \text{ and } R \leq 1.
\]

(ii) Similarly, if there are constants \( a, c > 0 \) such that \( \mu(B_T(z,R)) \geq c \mu(B_T(z,aR)) \) for all \( z \in \text{supp} \mu \) and \( R \leq 1 \), then
\[
\mu(B(z,R)) \geq \frac{c}{2} \mu(B(z,aR)) \text{ for all } z \in \text{supp} \mu \text{ and } R \leq 1.
\]

Proof. First, note that \( \mu(B(z,R)) = \mu(B(z,R) \cap [0,1]) \leq \mu(B_T(z,R)) \) for all \( z \in [0,1] \) and \( R \leq 1 \). If \( B(z,R) \cap [0,1] = B(z,R) \), then \( \mu(B(z,R)) = \mu(B_T(z,R)) \).

But 1 – \( z + R - 1 \) \( \geq \) \( z - R \), hence \( \mu(B_T(z,R)) \leq 2 \mu(B(z,R)) \). The argument if \( z - R < 0 \) is similar. Consequently, we also have \( \mu(B(z,R)) \geq \frac{1}{2} \mu(B_T(z,R)) \).

Both parts follow easily from these observations. \( \square \)

Lemma 2.27. Let \( \mu \) be the uniform Cantor measure on the classical middle-third Cantor set \( C \).

(i) For every \( z \in C \) and \( R \leq 1 \) we have
\[
\mu(B(z,R)) \geq 8 \mu(B(z,R/3^4)).
\]

(ii) For every \( z \in [0,1] \) and \( R \leq 1 \) we have
\[
\mu(B(z,R)) \geq 8 \mu(B(z,R/3^5)).
\]

Remark 2.28. We emphasize that in [11] the bound holds for all \( z \in C \) while in [ii] it must hold for all \( z \in [0,1] \).

Proof. First, suppose \( R = 3^{-N} \) and let \( z \in C \). Consider the Cantor intervals \( I_j \) of levels \( j = N, N+3 \) that contain \( z \). Note that \( B(z,3^{-N}) \) contains \( \text{int} I_N \) and since the gaps adjacent to \( I_{N+3} \) have length at least \( 3^{-(N+3)} \), \( B(z,3^{-(N+3)}) \cap C \subseteq I_{N+3} \). Consequently,
\[
\mu(B(z,3^{-N})) \geq 2^{-N} \text{ and } \mu(B(z,3^{-(N+3)})) \leq 2^{-(N+3)}.
\]

Hence \( \mu(B(z,3^{-N})) \geq 8 \mu(B(z,3^{-(N+3)})) \).
Now suppose $0 < R \leq 1$ and the integer $N$ is chosen with $3^{-(N+1)} < R \leq 3^{-N}$.
Suppose $z \in C$. Then
\[ \mu(B(z, R)) \geq \mu(B(z, 3^{-(N+1)})) \geq 8\mu(B(z, 3^{-(N+4)})) \geq 8\mu(B(z, R/3^5)). \]

(ii) If $z \in C$ there is nothing to prove, so assume otherwise. If $B(z, R/3^5) \cap C$ is empty there is, again, nothing to prove. So assume otherwise. Then $z$ belongs to one of the gaps in the construction of the Cantor set and the distance to the nearest endpoint of that gap, $w$, is at most $R/3^5$. Hence $B(z, R/3^5) \subseteq B(w, 2R/3^5)$ and $B(w, 2R/3) \subseteq B(z, R)$. As $w \in C$, we can apply part (i) to deduce that
\[ \mu(B(z, R/3^5)) \leq \mu(B(w, 2R/3^5)) \leq \frac{1}{8}\mu(B(w, 2R/3)) \leq \frac{1}{8}\mu(B(z, R)). \]

Suppose $\mu, \nu$ are measures supported on $[0, 1]$. When we write $\mu \ast \nu$ we will mean the convolution taken over $\mathbb{T}$. Thus $\mu \ast \nu$ is another measure supported on $[0, 1]$, which we can either think of as a measure on $\mathbb{T}$ or on $\mathbb{R}$.

**Lemma 2.29.** Suppose $\mu, \nu$ are measures on $[0, 1]$. If there are constants $a, c > 0$ such that $\mu(B_T(z, R)) \geq c\mu(B_T(z, aR))$ for all $z \in [0, 1]$ and $R \leq 1$, then
\[ \mu \ast \nu(B_T(z, R)) \geq c\mu \ast \nu(B_T(z, aR)) \quad \text{for all } z \in [0, 1] \text{ and } R \leq 1. \]

**Proof.** Let $z \in [0, 1]$ and $R \leq 1$. With addition being understood mod 1, we have
\[
\begin{align*}
\mu \ast \nu(B_T(z, R)) &= \int \int 1_{B_T(z, R)}(x + y)d\mu(x)d\nu(y) \\
&= \int \mu(B_T(z - y, R))d\nu(y) \\
&\geq c\mu(B_T(z - y, aR))d\nu(y) = c\mu \ast \nu(B_T(z, aR)).
\end{align*}
\]

Since the Cantor measure $\mu$ is symmetric, combining these lemmas gives the following useful fact.

**Corollary 2.30.** If $\mu$ is the uniform Cantor measure on the classical Cantor set and $\nu$ is any symmetric measure on $[0, 1]$, then
\[ \mu \ast \nu(B(z, R)) \geq 2\mu \ast \nu(B(z, R/3^5)) \quad \text{for all } z \in [0, 1] \text{ and } R \leq 1. \]

**Proof.** Lemma 2.27 gives that $\mu(B(z, R)) \geq 8\mu(B(z, R/3^5))$ for all $z \in [0, 1]$ and $R \leq 1$. From Lemma 2.29, $\mu(B_T(z, R)) \geq 4\mu(B_T(z, R/3^5))$ and then Lemma 2.29 implies
\[ \mu \ast \nu(B_T(z, R)) \geq 4\mu \ast \nu(B_T(z, R/3^5)) \quad \text{for all } z \in [0, 1] \text{ and } R \leq 1. \]

To complete the argument, call upon Lemma 2.29 in the negative.

**Proposition 2.31.** There is an absolutely continuous measure $\nu$ with density function $f$ having the property that $\dim_L \nu > 0$, but $f \notin L^p$ for any $p > 1$. 
Proof. We will give an explicit example. Let $K_n$ denote the $n$'th Fejer kernel on $T = [0,1]$, 
\[ K_n(x) = \sum_{j=-n}^{n} \left( 1 - \frac{|j|}{n} \right) e^{ijx}, \]
and inductively define integers $N_m \in \{3^k\}_{k=1}^{\infty}$ with $N_1 = 3$ and $N_{m+1} > 3^{2m}N_m$. Put 
\[ g(x) = \sum_{m=1}^{\infty} m^{-2}K_{32^m}(N_mx). \]
Note that $g \in L^1$ since $\|K_n\|_1 = 1$, and $g$ is symmetric. Let $\mu$ denote the uniform Cantor measure on the classical Cantor set and let $\nu = g * \mu$ (where the convolution is on $T$).

Since $g \in L^1$, the measure $\nu$ is absolutely continuous (whether viewed as a measure on $T$ or $\mathbb{R}$). By Corollary 2.30 $\nu$ is uniformly perfect and hence has positive lower Assouad dimension, as explained in Remark 2.31.

We will check that its density function, $f$, does not belong to $L^p(T)$ for any $p > 1$ by verifying that the Fourier transform $(\hat{f}(n))_{n=-\infty}^{\infty} \notin \ell^q$ for any $q < \infty$. An appeal to the Hausdorff-Young inequality will then imply $f \notin L^p(T)$ for any $p > 1$. Since a function supported on $[0,1]$ belongs to $L^p(\mathbb{R})$ if and only if it belongs to $L^p(T)$, this will complete the argument.

It is immediate from the definitions that 
\[ \hat{g}(n) = \frac{1}{m^2} \left( 1 - \frac{|n|}{32^m} \right) \text{ if } n \in \{ \pm 1, \ldots, \pm 32^m \} \cdot N_m \text{ for some } m \in \mathbb{N} \]
and that $\hat{g}(n) = 0$ if $n \notin \bigcup_m \{ \pm 1, \ldots, \pm 32^m \} \cdot N_m$. Thus $\hat{g}(n) \geq 2/(3m^2)$ on 
\[ \{ \pm 1, \ldots, \pm 32^{m-1} \} \cdot N_m. \]

It is well known that $|\hat{\mu}(3^k)| = |\hat{\mu}(3)| \neq 0$ for all $k \geq 1$. Thus 
\[ |\hat{g}(n)\hat{\mu}(n)| = |\hat{f}(n)| \geq \frac{2}{3m^2} |\hat{\mu}(3)| \]
for each $n \in S_m = \{ \{ \pm 1, \ldots, \pm 32^{m-1} \} \cdot N_m \} \cap \{3^k\}_{k=1}^{\infty}$. Since $|S_m| = 2(2^m - 1)$ and the choice of the integers $N_m$ ensures that the sets $S_m$ are disjoint, we have 
\[ \sum_{n \in \mathbb{Z}} |\hat{f}(n)|^q \geq \sum_{m=1}^{\infty} \left( \frac{2}{3m^2} |\hat{\mu}(3)| \right)^q 2(2^m - 1) = \infty \]
for each $q < \infty$. Thus $(\hat{f}(n))_{n=-\infty}^{\infty} \notin \ell^q$ for any $q < \infty$ and that completes the proof.

\[
\square 
\]

3. $\Phi$-Dimensions of Self-similar measures

3.1. Self-similar measures and separation properties. In this section, our focus will be on self-similar measures that satisfy various separation conditions. We begin with useful notation.

Consider the iterated function system (IFS), where the maps $S_j : X \to X$ are similarities with contraction factors $r_j$ for $j = 0, \ldots, m$ and $m \geq 1$. Assume, also, that we are given probabilities $\{p_j\}_{j=0}^{m}$, meaning $p_j > 0$ and $\sum_{j=0}^{m} p_j = 1$. By the self-similar measure $\mu$ associated with the IFS $\{S_j\}_{j=0}^{m}$ and the probabilities
This measure will have as its support \( K \), the unique, non-empty, compact set \( K \) satisfying \( K = \bigcup_{j=0}^{m} S_j(K) \), known as the self-similar set associated with the IFS.

Let \( \Sigma \) be the set of all finite words on the alphabet \( \{0, 1, \ldots, m\} \). Given \( w \in \Sigma \), say \( w = (j_1, \ldots, j_n) \), let \( w^\tau = (j_1, \ldots, j_{n-1}) \), \( S_w = S_{j_1} \circ \cdots \circ S_{j_n} \),

\[
r_w = \prod_{i=1}^{n} r_{j_i} \quad \text{and} \quad p_w = \prod_{i=1}^{n} p_{j_i}.
\]

Note that \( r_w \) is the contraction factor of \( S_w \). Let

\[
r_{\min} = \min |r_j| > 0
\]

and put

\[
\Lambda_n = \{ w \in \Sigma : |r_w| \leq r_{\min}^n \text{ and } |r_w^-| > r_{\min}^n \}.
\]

If the IFS consists of equicontractive similarities (all \( r_j = r_{\min} \in (0, 1) \)), then \( \Lambda_n \) consists of the words \( w \) of length \( n \). More generally, there exist \( a, b > 0 \) such that \( w \in \Lambda_n \) implies \( an \leq |w| \leq bn \). Note that for each \( n \),

\[
K = \bigcup_{\sigma \in \Lambda_n} S_\sigma(K).
\]

IFS satisfying the following definitions have been much studied.

**Definition 3.1.** The IFS \( \{S_j\}_{j=0}^{m} \), and any associated self-similar measure, are said to satisfy:

(i) The strong separation condition (SSC) if the sets \( S_j(K) \) are disjoint for \( j = 0, \ldots, m \);

(ii) The open set condition (OSC) if there is a bounded, non-empty, open set \( U \) such that \( S_j(U) \subseteq U \) for each \( j \) and the sets \( S_j(U) \) are disjoint;

(iii) The weak separation condition (WSC) if there is some \( x_0 \in \mathbb{R} \) and integer \( M \) such that for any \( n \in \mathbb{N} \) and finite word \( \sigma \), any closed ball of radius \( r_{\min}^n \) contains no more than \( M \) distinct points of the form \( S_\sigma(S_r(x_0)) \) for \( \sigma \in \Lambda_n \).

The definition we have given of the WSC is a restricted case of the original definition due to Lau and Ngai, [27]. Many equivalent properties can be found in [31].

It is well known that

\[
\text{SSC} \subseteq \text{OSC} \subseteq \text{WSC}
\]

and that both these inclusions are proper. For example, the IFS with the two similarities \( S_0(x) = x/2, S_1(x) = x/2 + 1/2 \) on \( \mathbb{R} \) satisfies the OSC, but not the SSC. The IFS \( S_\rho = \{S_0(x) = px, S_1(x) = px + 1 - \rho\} \) where \( \rho \) is the inverse of a Pisot number and the IFS \( S_d = \{S_j(x) = x/d + (d-1)jx/(dn) : j = 0, 1, \ldots, m\} \) where \( 2 \leq d \leq m \) are integers, satisfy the WSC but not the OSC. In the case of the IFS \( S_\rho \), any associated self-similar measure is known as a Bernoulli convolution and is said to be biased if \( p_0 \neq p_1 \). In the case of the IFS \( S_d \), for a suitable choice
of probabilities, the self-similar measure is the m-fold convolution of the uniform Cantor measure on the Cantor set with contraction factor 1/d.

3.2. Self-similar measures satisfying the strong separation condition. It was shown in [4] that self-similar sets arising from an IFS that satisfies the open set condition have equal upper and lower Assouad dimensions (and hence also all Φ-dimensions). This is not true for self-similar measures. For instance, the measure of Example 2.20 is the self-similar measure arising from the IFS with $S_1(x) = x/2$, $S_1(x) = x/2 + 1/2$ and probabilities $2/3, 1/3$. This IFS satisfies the open set condition and yet we have $\dim_{q,A,\mu} = \infty$, while $\dim_{q,\mu} < \infty$ for all non-zero constant functions $\Phi$.

However, we cannot produce such an example with a self-similar measure that satisfies the strong separation property, as our next result shows.

**Theorem 3.2.** Assume $\mu$ is a self-similar measure that satisfies the strong separation condition. For any dimension function $\Phi$ we have

$$\dim_{\Phi,\mu} = \min\{\dim_{\mu}(z) : z \in \text{supp}\mu\}$$

and

$$\overline{\dim}_{\Phi,\mu} = \max\{\dim_{\mu}(z) : z \in \text{supp}\mu\}.$$ 

**Proof.** Assume the measure $\mu$ arises from the IFS $\{S_j\}_{j=0}^n$ that satisfies the SSC, with probabilities $\{p_j\}$, and that $K$ is the associated self-similar set. It is well known (see [5, ch. 11]) that if the contraction factor of $S_j$ is $r_j$, then

$$\{\dim_{\mu}(z) : z \in \text{supp}\mu\} = \left[\min_j \frac{\log p_j}{\log r_j}, \max_j \frac{\log p_j}{\log r_j}\right] := [\theta, \Theta].$$

Of course, this means $r_j^\Theta \leq p_j \leq r_j^\theta$ for all $j$.

As the upper and lower $\Phi$-dimensions are bounded (below and above, respectively) by the maximum and minimum local dimensions (Proposition 2.12), it will be enough to show that there are constants $C_0, C_1 > 0$ such that for all $x \in \text{supp}\mu$, $R \leq \text{diam}(\text{supp}\mu)$ and $0 < r \leq R^{1+\Phi(R)}$, we have

$$C_0 \left(\frac{R}{r}\right)^\theta \leq \frac{\mu(B(x, R))}{\mu(B(x, r))} \leq C_1 \left(\frac{R}{r}\right)^\Theta$$

to see that $\overline{\dim}_{\Phi,\mu} = \Theta$ and $\underline{\dim}_{\Phi,\mu} = \theta$.

Fix such $x$, $R$ and $r \leq R^{1+\Phi(R)}$ and choose integers $n, m$ so that $r_{\min}^n \leq R \leq r_{\min}^m$ and $r_{\min}^m \leq r \leq r_{\min}^{m-1}$. Obtain $w \in \Lambda_n$ and $w\sigma \in \Lambda_m$ such that $x \in S_{w\sigma}(K)$. Then

$$|r_w| \leq r_{\min}^n \leq R < r_{\min}^{n-2}|r_w|$$

and

$$|r_{w\sigma}| \leq r_{\min}^m \leq r < r_{\min}^{m-2}|r_{w\sigma}|,$$

so

$$\frac{R}{r} \geq \frac{|r_w|}{r_{\min}^2|r_{w\sigma}|} = \frac{r_{\min}^2}{|r_w|}.$$

Since $S_{w}(K) \subseteq B(x, R)$ and $S_{w\sigma}(K) \subseteq B(x, r)$ we have

$$\mu(B(x, R)) \geq p_{w\sigma} \text{ and } \mu(B(x, r)) \geq p_w.$$ 

Because the IFS satisfies the strong separation condition, there is some $\varepsilon > 0$ such that $d(S_i(K), S_j(K)) \geq \varepsilon$ for all $i \neq j$. Consequently, for any word $\tau$ and $i \neq j$, $d(S_{\tau_i}(K), S_{\tau_j}(K)) \geq \varepsilon |r_{\tau}|$. 

Choose an integer $L$ such that $\varepsilon r_{\min}^{-(L-1)} > 2$. Let $W$ be the set of words $v \in \Lambda_n$ such that $S_v(K) \cap B(x, R) \neq \emptyset$, so $B(x, R) \cap K \subseteq \bigcup_{v \in W} S_v(K)$. We claim that the words $v \in W$ must have a common ancestor $\tau \in \Lambda_{n-L}$. If not, there would be a pair $v, v' \in W$ with different ancestors at level $n - L$. But, then,
\[
d(S_v(K), S_{v'}(K)) \geq \varepsilon r_{\min}^{n-L},
\]
which exceeds the diameter of $B(x, R)$, and this is impossible. Thus $B(x, R) \cap K \subseteq S_\tau(K)$ where $\tau$ is the common ancestor. Moreover, as $w \in W$, $p_\tau \leq p_w (\min p_i)^{-L}$, so
\[
\mu(B(x, R)) \leq \mu(S_\tau(K)) = p_\tau \leq p_w c_1
\]
for $c_1 = (\min p_i)^{-L}$.

These facts, together with the definition of $\Theta$, implies
\[
\frac{\mu(B(x, R))}{\mu(B(x, r))} \leq \frac{p_w c_1}{p_w \sigma} \leq \frac{c_1}{p_\sigma} \leq c_1 \left(\frac{1}{|r_\sigma|}\right)^\Theta \leq C_1 \left(\frac{R}{r}\right)^\Theta
\]
for a suitable choice of $C_1$.

As a similar upper bound can be found for $\mu(B(x, r))$, the lower bound follows in the same manner. \(\square\)

3.3. **Self-similar measures satisfying the weak separation condition.** In this subsection we will assume the measure $\mu$ arises from an IFS $\{S_j\}_{j=0}^m$ of similarities $S_j(x) = r_j x + d_j$ on $\mathbb{R}$ that satisfies the WSC. We will also assume that the self-similar set (and support of the measure) $K = [0, 1]$. We continue to use the notation of the previous subsection.

It was proven in \[19\] that such measures have the property that there is some $a > 0$ such that

\[
|S_\sigma(w) - S_\tau(z)| \geq ar_{\min}^n
\]
whenever $\sigma, \tau \in \Lambda_n, w, z \in \{0, 1\}$ and $S_\sigma(w) \neq S_\tau(z)$. This property is very helpful in studying the dimensional properties of $\mu$.

It is convenient to introduce further notation. For each $n \in \mathbb{N}$, let $h_1, \ldots, h_{s_n}$ denote the set of elements of $\{S_\sigma(0), S_\sigma(1) : \sigma \in \Lambda_n\}$, listed in increasing order. The intervals, $[h_j, h_{j+1}]$, are called the **net intervals of level** $n$. In what follows $\Delta_n$ will always denote a net interval of level $n$ and $\Delta_n(x)$ will be a level $n$ net interval containing $x$ (noting that there could be two choices if $x$ is a boundary point $h_i$.) We write $\ell(I)$ for the length of the interval $I$. From (3.1) it follows that

\[
a_{\min}^n \leq \ell(\Delta_n) \leq r_{\min}^n.
\]

Put
\[
P_n(\Delta_n) = \sum_{w \in \Lambda_n} p_w, \quad \text{if } S_w[0,1] \supseteq \Delta_n.
\]

Let
\[
p = \min p_j^M
\]
where $M$ is the maximum length of any word $w$ such that there exists an integer $m$ and word $\sigma \in \Lambda_{m-1}$ with $\sigma w \in \Lambda_m$.

The definitions ensure that if $\Delta_n \subseteq \Delta_{n-1}$, then

\[
P_{n-1}(\Delta_{n-1}) \geq P_n(\Delta_n) \geq pP_{n-1}(\Delta_{n-1}).
\]
Furthermore, as $\ell(S_\sigma[0,1]) \leq r_{\min}^n$ whenever $\sigma \in \Lambda_n$, we have
\begin{equation}
\mu(B(x, r_{\min}^n)) \geq P_n(\Delta_n(x)) \geq \mu(\Delta_n(x)).
\end{equation}

It was shown in [21, Cor. 4.6] that these measures $\mu$ satisfy $\dim_{\text{qc}} \mu < \infty$ if and only if $\mu$ has the doubling-like property that for every $\varepsilon > 0$ there is a constant $C$ such that
\begin{equation}
\mu(B(x, R)) \leq CR^{-2\varepsilon} \mu(B(x, R/2))
\end{equation}
for all $x \in \text{supp}\mu$ and $0 < R \leq 1$. Motivated by this, we introduce the following definition of $\Phi$-doubling.

Recall that a function $\Phi$ is said to be \textbf{doubling} if there is a constant $c > 0$ such that
\begin{equation}
\Phi(x) \leq c\Phi(x/2)
\end{equation}
whenever $x > 0$. Doubling dimension functions include $\Phi = \delta$, $\Phi(x) = 1/|\log x|$ and $\Phi(x) = \log \log x/|\log x|$.

\textbf{Definition 3.3.} \textit{We will say the measure $\mu$ on $X$ is $\Phi$-doubling if there are constants $C \geq 1$, $\gamma > 0$ such that
\begin{equation}
\mu(B(x, R)) \leq CR^{-\gamma\Phi(R)} \mu(B(x, R/2))
\end{equation}
for all $x \in \text{supp}\mu$ and $0 < R \leq 1$.}

Notice that if $\Phi = 0$, this is the usual definition of a doubling measure.

Given $n \in \mathbb{N}$, let
\begin{equation}
\phi(n) = n\Phi(r_{\min}^n) \geq 0.
\end{equation}
It is easy to check that if $\Phi$ is a doubling function, then $\mu$ is $\Phi$-doubling if and only if there is a (possibly different) constant $C \geq 1$ such that
\begin{equation}
\mu(B(x, r_{\min}^n)) \leq C^{1 + \phi(n)} \mu(B(x, r_{\min}^{n+1}))
\end{equation}
for all $x \in \text{supp}\mu$ and $n \in \mathbb{N}$. Note that a repeated application of (3.4) shows that for each positive integer $k$, there is a constant $C_k \geq 1$ such that
\begin{equation}
\mu(B(x, r_{\min}^n)) \leq C_k^{1 + \phi(n)} \mu(B(x, r_{\min}^{n+k})).
\end{equation}

The property of being $\Phi$-doubling can be described in terms of the measure of net intervals.

\textbf{Lemma 3.4.} \textit{Assume $\mu$ is a self-similar measure that satisfies the WSC and has support $[0,1]$. Then $\mu$ is $\Phi$-doubling if and only if there is a constant $C_0 \geq 1$ such that
\begin{equation}
\mu(\Delta_n) \geq C_0^{-1 + \phi(n)} \mu(\Delta_n^*)
\end{equation}
whenever $\Delta_n^*$ is a level $n$ net interval adjacent to the level $n$ net interval $\Delta_n$.}

\textit{Proof.} Fix $a > 0$ so that $\ell(\Delta_n) \geq ar_{\min}^n$ for all net intervals $\Delta_n$ of level $n$ and all $n \in \mathbb{N}$.

Suppose $\mu$ is $\Phi$-doubling. Let $\Delta_n$ be any level $n$ net interval and $\Delta_n^*$ be an adjacent net interval. Let $x$ denote the midpoint of $\Delta_n$.

The doubling assumption ensures that for a suitable constant $C \geq 1$,
\begin{align*}
\mu(\Delta_n) &= \mu(B(x_{\Delta_n}, \ell(\Delta_n)/2)) \geq \mu(B(x_{\Delta_n}, ar_{\min}^n/2)) \geq C^{-1 + \phi(n)} \mu(B(x_{\Delta_n}, 2r_{\min}^n)) \geq C^{-1 + \phi(n)} \mu(\Delta_n^*),
\end{align*}
where the last inequality holds because $B(x_{\Delta_n}, 2r_{\min}^n) \supseteq \Delta_n^*$. 

Conversely, assume there exists a constant $C_0 \geq 1$ such that for all $n$, $\mu(\Delta_n) \geq C_0^{-1(1+\phi(n))} \mu(\Delta^*_{n})$. Fix $x \in [0,1]$ and suppose $x \in \Delta_n$. (If $x$ is a boundary point of a net interval, choose either net interval.) Let $\Delta^{(1)}_n$ be the level $n$ net interval immediately to its right, and more generally, let $\Delta^{(j)}_n$ be the net interval of level $n$ immediately to the right of $\Delta^{(j-1)}_n$ (should it exist). By repeated application of Proposition 3.5,

$$\mu(B(x, r^n_{\min})) \geq \mu(\Delta_n) \geq C_0^{-1(1+\phi(n))} \mu(\Delta^{(1)}_n) \geq C_0^{-k(1+\phi(n))} \mu(\Delta^{(k)}_n).$$

Choose $k$ so that $[x, x + 2r^n_{\min}] \cap [0,1] \subseteq \bigcup_{j=0}^{k} \Delta^{(j)}_n$; notice $k \leq 1 + 2/a$. For the constant $C_1 = C_0^k$, we have

$$\mu([x, x + 2r^n_{\min}]) \leq \mu\left(\bigcup_{j=0}^{k} \Delta^{(j)}_n\right) \leq kC_1^{1+\phi(n)} \mu(\Delta_n) \leq kC_1^{1+\phi(n)} \mu(B(x, r^n_{\min})).$$

We similarly bound $\mu([x - 2r^n_{\min}, x])$ and hence deduce that

$$\mu(B(x, r^n_{\min})) \geq \frac{1}{2k} C_1^{-1(1+\phi(n))} \mu(B(x, 2r^n_{\min})).$$

This suffices to prove that $\mu$ is $\Phi$-doubling. \[\square\]

We now characterize $\Phi$-doubling in terms of the upper $\Phi$-dimensions.

**Proposition 3.5.** Assume $\mu$ is a self-similar measure on $\mathbb{R}$ that satisfies the weak separation condition and has support $[0,1]$. Suppose that $\Phi$ is an increasing, doubling, dimension function. Then $\dim_{\Phi} \mu < \infty$ if and only if $\mu$ is $\Phi$-doubling.

**Proof.** Fix $a > 0$ so that $\ell(\Delta_n) \geq ar^n_{\min}$ for all level $n$ net intervals $\Delta_n$.

First, suppose that $d = \dim_{\Phi} \mu < \infty$ and fix $\varepsilon > 0$. By the definition of the upper $\Phi$-dimension, there is a constant $C = C(\varepsilon)$ such that for any suitably large integer $n$ we have

$$\mu(B(x, 2r^n_{\min})) / \mu(B(x, ar^n_{\min} + \phi(n)/2)) \leq Cr^{-\phi(n)(d+\varepsilon)}$$

for all $x \in [0,1]$. (3.6)

Consider any level $n$ net interval of level $\Delta_n$, with midpoint $x$. Then

$$B(x, ar^n_{\min} + \phi(n)/2) \cap [0,1] \subseteq \Delta_n,$$

while $B(x, 2r^n_{\min})$ contains both $\Delta_n$ and the two adjacent level $n$ net intervals. Let $\Delta^*_n$ denote either adjacent interval. Then (3.6) gives

$$\mu(\Delta_n) \geq \mu(B(x, ar^n_{\min} + \phi(n)/2)) \geq C^{-1}r_{\min}^{-\phi(n)(d+\varepsilon)} \mu(B(x, 2r^n_{\min})) \geq C^{-1}r_{\min}^{-\phi(n)(d+\varepsilon)} \mu(\Delta_n) \geq C_1^{-1(1+\phi(n))(d+\varepsilon)} \mu(\Delta^*_n),$$

for $C_1 = \max(r_{\min}^{-1}, C)^{1/(d+\varepsilon)} \geq 1$. By Lemma 3.3, $\mu$ is $\Phi$-doubling.

Conversely, assume $\mu$ is $\Phi$-doubling. Fix $x \in [0,1]$ and $N \in \mathbb{N}$. Let $\Delta_N$ denote the level $N$ net interval containing $x$ (taking either, if there is a choice) and let $\Delta^R_N$ and $\Delta^L_N$ denote the two adjacent, level $N$ net intervals to the right and left respectively.

According to the Lemma, the $\Phi$-doubling condition implies

$$\mu(\Delta^R_N) \leq C_0^{1+\phi(N)} \mu(\Delta_N)$$

and

$$\mu(\Delta^L_N) \leq C_0^{1+\phi(N)} \mu(\Delta_N).$$

(3.4)
and similarly for $\mu(\Delta^L_N)$. Since $B(x, r^N_{\min} a) \cap [0, 1] \subseteq \Delta_N \cup \Delta^R_N \cup \Delta^L_N$, (3.3) implies

$$\mu(B(x, r^N_{\min} a)) \leq \mu(\Delta_N \cup \Delta^R_N \cup \Delta^L_N) \leq 3C_0^{1+\phi(N)} \mu(\Delta_N) \leq 3C_0^{1+\phi(N)} P_N(\Delta_N).$$

Choose any integer

$$n \geq N(1 + \Phi(r^{N+1}_{\min} a)).$$

Let $\Delta_n \subseteq \Delta_N$ be the net interval of level $n$ containing $x$. From (3.2) and (3.3) we see that

$$\frac{\mu(B(x, r^N_{\min} a))}{\mu(B(x, r^n_{\min}))} \leq 3C_0^{1+\phi(N)} P_N(\Delta_N(x)) \leq 3C_0^{1+\phi(N)} p^{-(n-N)}.$$

The doubling assumption of $\Phi$ ensures there is some $\beta > 0$ (independent of $N$) such that

$$(3.7) \quad \Phi(r^{N+1}_{\min} a) \geq \beta \Phi(r^N_{\min}),$$

so

$$n - N = N \Phi(r^{N+1}_{\min} a) \geq \beta \phi(N).$$

Taking $s, t \geq 0$ such that $C_0 = r_{\min}^{-s}$ and $p = r_{\min}^{t}$, we have

$$(3.8) \quad \frac{\mu(B(x, r^N_{\min} a))}{\mu(B(x, r^n_{\min}))} \leq 3r_{\min}^{-s(1+\phi(N))} r_{\min}^{-t(n-N)} \leq 3r_{\min}^{-s-t+s/\beta}(n-N) \leq C \left( \frac{r^N_{\min}}{r^n_{\min}} \right)^\alpha$$

for $\alpha \geq t + s/\beta$ and another constant $C \geq 1$. That proves $\dim_\Phi \mu \leq \alpha < \infty$. □

Remark 3.6. It would be interesting to know if this result holds for all measures.

The IFS $\{r x, r x + 1 - r\}$, where $r$ is the inverse of a Pisot number (such as the golden mean), and the IFS $\{x/d + (d-1)x/(dm) \}_{j=0}^m$, for integers $2 \leq d \leq m$, are examples of IFS that do not satisfy the OSC, but satisfy a separation property stronger than the WSC known as finite type. This notion was introduced by Ngai and Wang in [39]. For equicontractive IFS of similarities on $\mathbb{R}$ it can be defined as follows.

Definition 3.7. Let $S = \{S_j\}_{j=0}^m$ be an equicontractive IFS of similarities on $\mathbb{R}$ with contraction factor $0 < r_{\min} < 1$. The IFS, or any associated self-similar measure, is said to be of finite type if there is a finite set $F \subseteq \mathbb{R}$ such that if $v, w$ are words on $\{0, 1, ..., m\}$ of length $n$, and $c$ is the diameter of the self-similar set, then either

$$|S_v(0) - S_w(0)| > cr_n^{1/n} \text{ or } r_{\min}^{-n} |S_v(0) - S_w(0)| \in F.$$

An IFS that is of finite type satisfies the WSC. Conversely, it is proven in [19] that any equicontractive, self-similar measure that satisfies the WSC and has support $[0, 1]$ is of finite type. Any equicontractive IFS that satisfies the OSC with the open set being $(0, 1)$ is also of finite type.

It is known that an IFS of finite type has the property that there are only finitely many values for $\ell(\Delta_n)r_{\min}^{-n}$, over all level $n$ net intervals and all $n$.

Corollary 3.8. Suppose $\mu$ is any equicontractive, self-similar, finite type measure with support $[0, 1]$. Then $\dim_\Phi \mu < \infty$ for any dimension function $\Phi = \delta > 0$.

Proof. Choose $a > 0$ such that $ar_n^{1/n} \leq \ell(\Delta_n) \leq r_{\min}^{n}$ for all level $n$ net intervals $\Delta_n$ and fix an integer $k$ such that $r_{\min}^{k} \leq a/2.$
Let $\Delta_n$ be any level $n$ net interval and suppose $x$ is its midpoint. Choose a word $\omega$ of length $n + k$ so that $x \in S_{\omega}[0, 1] \subseteq \Delta_n$. Thus

$$\mu(\Delta_n) \geq \mu(S_{\omega}[0, 1]) \geq (\min p_j)^{n+k}.$$ 

It is known that for any finite type measure there is a constant $A$ such that $\mu(\Delta_n) \leq A^n$, [17]. Since $\phi(n) = n\delta$ when $\Phi = \delta$, it easily follows from this that (3.5) is satisfied for such $\Phi$. Hence $\mu$ is $\Phi$-doubling and therefore the upper $\Phi$-dimension is finite for all non-zero constant functions $\Phi$. □

The measure $\mu$ studied in Example [2,20] is of finite type and has support $[0, 1]$. As we saw in that example, $\dim_\Phi \mu < \infty$ for all $\Phi = \delta \neq 0$, but $\dim_{qA} \mu = \infty$, showing the sharpness of the corollary. The biased Bernoulli convolutions discussed next are another class of such examples.

**Proposition 3.9.** Let $\mu$ be the biased Bernoulli convolution arising from the IFS

$\{\rho x, \rho x + 1 - \rho\}$ with probabilities $p, 1 - p$, where $p > 1/2$ and $\rho$ is the inverse of the golden mean. Then $\mu$ is an equicontractive, self-similar measure of finite type with support $[0, 1]$, but $\dim_{qA} \mu = \infty$.

**Proof.** It is well known that this IFS is of finite type, c.f. [6]. As explained there, the net intervals of level $n$ can all be labelled by $n+1$-tuples, $(1, \gamma_1, \ldots, \gamma_n)$, where $\gamma_i \in \{2, \ldots, 7\}$ (and the allowed choices for $\gamma_{i+1}$ depend on $\gamma_i$) and

$$\rho^{n+3} \leq \ell(\Delta_n) \leq \rho^n.$$ 

Two adjacent net intervals of level four are $\Delta_0 = (1, 3, 5, 6, 3)$ and $\Delta_1 = (1, 3, 5, 7, 5)$ which lies immediately to its right. The net interval

$$\Delta_{0}^{(k)} := (1, 3, 5, 6, 3, (5, 7)^k, 5)$$

is the right-most descendent of $\Delta_0$ at level $5+2k$, and adjacent to it is the left-most descendent of $\Delta_1$ at the same level,

$$\Delta_{1}^{(k)} := (1, 3, 5, 7, 5, (3, 5)^k, 3).$$

From the calculations of [IFS, Section 4] (in the notation used there $c_1 = 3$, $c_2 = 5$ and $c_1 = 7$), it follows that $\mu(\Delta_{0}^{(k)}) \sim \|T_0^k\|$ and $\mu(\Delta_{1}^{(k)}) \sim \|T_1^k\|$ where

$$T_0 = \begin{bmatrix} p(1-p) & p(1-p) \\ 0 & (1-p)^2 \end{bmatrix} \quad \text{and} \quad T_1 = \begin{bmatrix} p^2 & 0 \\ (1-p)^2 & p(1-p) \end{bmatrix},$$

and the matrix norm $\|T\| = \sum_{i,j} |T_{ij}|$ when $T = (T_{ij})$.

An induction argument shows that

$$(T_0)^{2^k} = \begin{bmatrix} p(1-p)^{2^k} & A_k \\ 0 & (1-p)^{2k+1} \end{bmatrix}, \quad (T_1)^{2^k} = \begin{bmatrix} p^{2^{k+1}} & 0 \\ B_k & (p(1-p))^{2^k} \end{bmatrix}.$$
with
\[ A_k = p(1-p) \prod_{i=0}^{k-1} ((p(1-p))^{2^i} + (1-p)^{2^{i+1}}) \]
\[ = p(1-p) \prod_{i=0}^{k-1} (p(1-p))^{2^i} \prod_{i=0}^{k-1} \left( 1 + \left( \frac{1-p}{p} \right)^{2^i} \right) \]
\[ = (p(1-p))^{2^k} \prod_{i=0}^{k-1} \left( 1 + \left( \frac{1-p}{p} \right)^{2^i} \right) \]
and
\[ B_k = (1-p)^2 \prod_{i=0}^{k-1} ((1-p)^{2^{i+1}} + (p(1-p))^{2^i}) \]
\[ = (1-p)^2 2^{k+1-2} \prod_{i=0}^{k-1} \left( 1 + \left( \frac{1-p}{p} \right)^{2^i} \right). \]

Since \( 1 - p < p, \prod_{i=0}^{k-1} \left( 1 + ((1-p)/p)^{2^i} \right) \) converges to a constant \( 0 < c < \infty \).

Hence there are positive constants \( A, B \) such that for large enough \( k \)
\[ \left\| T_0^{2^k} \right\| = (p(1-p))^{2^k} + A_k + (1-p)^{2^{k+1}} \]
\[ \leq (p(1-p))^{2^k} (1 + 2c + ((1-p)/p)^{2^k}) \]
\[ \leq A(p(1-p))^{2^k} \]
and similarly
\[ \left\| T_1^{2^k} \right\| \geq B p^{2^{k+1}}. \]

Let \( x_k \) be the midpoint of \( \Delta_0^{(2^k)} \) and \( R_k = 2 \rho^{5+2^{k+1}}. \) Then \( R_k \geq \ell(\Delta_0^{(2^k)}) + \ell(\Delta_1^{(2^k)}), \) so \( B(x_k, R_k) \supseteq \Delta_0^{(2^k)} \cup \Delta_1^{(2^k)} \) and therefore
\[ \mu(B(x_k, R_k)) \geq \mu(\Delta_1^{(2^k)}) \sim \left\| T_1^{2^k} \right\| \geq B p^{2^{k+1}}. \]

Put \( r_k = R_k^{1+\delta} \) for fixed \( \delta > 0. \) If \( k \) is sufficiently large, then
\[ r_k \leq \rho^{5+2^{k+3}}/2 \leq \ell(\Delta_0^{(2^k)})/2 \]
and therefore \( B(x_k, r_k) \subseteq \Delta_0^{(2^k)}. \) It follows that
\[ \mu(B(x_k, r_k)) \leq \mu(\Delta_0^{(2^k)}) \sim \left\| T_0^{2^k} \right\| \leq A(p(1-p))^{2^k}. \]

Consequently,
\[ \frac{\mu(B(x_k, R_k))}{\mu(B(x_k, r_k))} \geq \frac{B p^{2^{k+1}}}{A(p(1-p))^{2^k}} = \frac{B}{A} \left( \frac{p}{1-p} \right)^{2^k}, \]
while \( R_k/r_k = R_k^{-\delta} = 2^{-\delta} \rho^{-\delta(5+2^{k+1})}. \) Thus
\[ \dim_{\Phi_2} \mu \geq \frac{\log(p/(1-p))}{2\delta \, |\log \rho|} \]
and therefore
\[ \dim_{\Phi_2} A \mu = \lim_{\delta \to 0} \dim_{\Phi_2} \mu = \infty. \]
An equicontractive self-similar measure of finite type is called regular if the probabilities associated with the left and right-most contractions are equal and minimal. One example is an $m$-fold convolution of a uniform Cantor measure on a Cantor set with contraction factor $1/d$ for $d \in \mathbb{N}$. Another is a uniform (but not biased) Bernoulli convolution with contraction factor the inverse of a Pisot number.

**Corollary 3.10.** Suppose $\mu$ is an equicontractive, self-similar, regular, finite type measure. Then $\dim_\Phi \mu < \infty$ whenever $\Phi(x) \geq \log|\log x|/|\log x|$ for all $x \leq 1$. In particular, $\dim_{qA} \mu < \infty$ for such measures $\mu$.

**Proof.** For such measures $\mu$, it is known that $\mu(\Delta_n) \geq Cn\mu(\Delta_n^*)$. [17], thus $\mu$ is $\Phi$-doubling for such $\Phi$. $\square$

The measures studied in Example 2.20 and Proposition 3.9 illustrate the necessity of the hypothesis of regularity. The following example shows the sharpness of the function $\log|\log x|/|\log x|$.

**Example 3.11** (An equicontractive, self-similar measure of finite type that has full support, is regular and has $\dim_\Phi \mu = \infty$ for all $\Phi(x) \ll \log|\log x|/|\log x|$). Consider the IFS, $S_j(x) = x/3 + d_j$ with $d_0 = 0$, $a = 1/3$, $b = 2/3$ and probabilities $p_j = 1/4$ for all $j$. Let $\mu$ be the associated self-similar measure. This example was studied in [24, Ex. 5.11]. The two net intervals of level $n$ with endpoint 1/2 have length $3^{-n}/2$. The $\mu$-measure of the right interval is at most $c_14^{-n}$, while the measure of the left is at least $c_2n4^{-n}$ for some $c_1, c_2 > 0$. Take $x_n$ the midpoint of the right interval, $R_n = 3^{-n}/2$ and $r_n = R_{n+1}^{\Phi(R_n)} \leq 3^{-n}/4$. Hence there exist constants $C, \alpha < \infty$ such that

$$\frac{c_2}{c_1}n \leq \frac{\mu(B(x, R_n))}{\mu(B(x, r_n))} \leq C \left(\frac{R_n}{r_n}\right)^\alpha$$

only if

$$\Phi(R_n) = \frac{\log n}{\log|\log R_n|} \geq \frac{\log|\log R_n|}{|\log R_n|}.$$

4. **Discrete Measures**

Let $\{a_n\}_{n=1}^\infty$ be a decreasing sequence tending to 0 and $\{p_n\}_{n=0}^\infty$ a set of probabilities, $p_n \geq 0$, such that $0 < \sum_{n=0}^\infty p_n < \infty$. We define a discrete measure $\mu$ with support $E := \{a_n\}_{n=1}^\infty \cup \{0\}$, by

$$\mu = \sum_k p_k \delta_{a_k} + p_0 \delta_0.$$

Thus $\mu(F) = \sum_{n:a_n \in F} p_n$ for any Borel set $F \subseteq \mathbb{R}\setminus\{0\}$ and $\mu\{0\} = p_0$. If we normalize $\mu$, then it is a probability measure and normalizing does not change $\Phi$-dimensions.

It was shown in [14] and [15] that if the sequence of gaps $\{a_n - a_{n+1}\}_{n=1}^\infty$ is also decreasing (such as when $a_n = \beta^{-n}$ or $n^{-\lambda}$ for $\beta > 1$ or $\lambda > 0$), then both the upper Assouad and quasi-Assouad dimensions of $E$ are either 0 or 1, although not necessarily the same value for the same set $E$. In [15, Example 2.18], it was shown that this need not be true for upper $\Phi$-dimensions, even for dimension functions $\Phi$.
with upper Φ-dimensions lying between the upper quasi-Assouad and Assouad dimensions. Thus it is natural to ask about the Φ-dimensions for measures supported on such sets.

As these measures have atoms, their lower Φ-dimensions are always zero, so it is only the upper Φ-dimensions that are unknown. In [9], Fraser and Howroyd determined \( \dim A \mu \) for such measures \( \mu \) when \( p_0 = 0 \) and either all \( p_n \) are equal to \( n^{-\lambda} \) or all are equal to \( \beta^{-n} \) for \( n \in \mathbb{N} \), and likewise for \( a_n \) (although with possibly different values for \( \lambda \) or \( \beta \)). Here, we will continue to focus on these choices for \( p_n \) and \( a_n \), for \( n \in \mathbb{N} \).

To state our results, it is convenient to let

\[
L = L_\Phi = \limsup_{x \to 0} \Phi(x)^{-1} \quad \text{and} \quad \Psi(x) = \frac{\log |\log x|}{|\log x|}.
\]

For \( \beta > 1 \) and \( \lambda > 0 \), put

\[
(4.1) \quad s = \frac{\beta - 1}{\lambda} \quad \text{and} \quad t = \frac{\beta}{\lambda + 1}.
\]

Note that \( s \leq t \) if and only if \( t \leq 1 \).

**Theorem 4.1.** Assume \( \mu = p_0 \delta_0 + \sum p_n \delta_{a_n} \) and suppose \( \Phi \) is any dimension function.

(i) “Polynomial-polynomial”: Suppose that for all \( n \in \mathbb{N} \), \( p_n = n^{-\beta} \) and \( a_n = n^{-\lambda} \) for \( \beta > 1 \) and \( \lambda > 0 \). If \( p_0 = 0 \), then \( \dim_{\Phi} \mu = \left\{ \begin{array}{ll}
\max(1, s) & \text{if } L \geq \lambda \\
\max(t + L(t - s), s) & \text{if } L \leq \lambda
\end{array} \right. \),

while if \( p_0 \neq 0 \), then \( \dim_{\Phi} \mu = \left\{ \begin{array}{ll}
sL + \max(1, s) & \text{if } L \geq \lambda \\
(1 + L) \max(s, t) & \text{if } L \leq \lambda
\end{array} \right. \).

(ii) “Exponential-exponential”: Suppose that for all \( n \in \mathbb{N} \), \( p_n = \beta^{-n} \) and \( a_n = \lambda^{-n} \) for \( \beta, \lambda > 1 \). Then \( \dim_{\Phi} \mu = \left\{ \begin{array}{ll}
(1 + L) \frac{\log \beta}{\log \lambda} & \text{if } p_0 \neq 0 \\
\frac{\log \lambda}{\log \lambda} & \text{if } p_0 = 0
\end{array} \right. \).

(iii) “Mixed rates”: (Exponential-polynomial) Suppose that for all \( n \in \mathbb{N} \), \( p_n = \beta^{-n} \) and \( a_n = n^{-\lambda} \) for \( \beta > 1 \) and \( \lambda > 0 \). Then \( \dim_{\Phi} \mu = \infty \).

(iv) “Mixed rates”: (Polynomial-exponential) Suppose that for all \( n \in \mathbb{N} \), \( p_n = n^{-\beta} \) and \( a_n = \lambda^{-n} \) for \( \beta, \lambda > 1 \). Then \( \dim_{\Phi} \mu = \left\{ \begin{array}{ll}
\lim_{x \to 0} \beta^{\Psi(x)/\Phi(x)} & \text{if } p_0 \neq 0 \\
\lim_{x \to 0} \frac{\Psi(x)}{\Phi(x)} & \text{if } p_0 = 0
\end{array} \right. \).

Before beginning the proof, we will list some immediate corollaries.

**Corollary 4.2.**

(i) If \( p_0 \neq 0 \), then \( \dim A \mu = \infty \) (in all cases). If \( p_0 = 0 \), then \( \dim A \mu = \infty \) in the mixed rates cases, \( \dim A \mu = \max(1, s) \) in the polynomial-polynomial case and \( \dim A \mu = \log \beta/\log \lambda \) in the exponential-exponential case.
Lemma 4.4. Let $\dim_{qA} \mu = 0$ (regardless of the choice of $p_0$).

(ii) The upper quasi-Assouad dimension coincides with the upper Assouad dimension except in the polynomial-exponential case when $\dim_{qA} \mu = 0$.

(iii) If $E = 0 \cup \{\lambda^{-n}\}_{n=1}^\infty$ and $\Phi(x)/\Psi(x) \to \infty$ as $x \to 0$, then $\overline{\dim}_\Phi E = 0$.

Proof. To compute the upper Assouad dimension, just note that when $\Phi = 0$, then $L_\Phi = \infty$ (so $L \geq \lambda$) and $\lim_{x \to 0} \frac{\Phi(x)}{\Psi(x)} = \infty$. To compute the upper quasi-Assouad dimension, let $\Phi_\delta$ be the constant function $\delta > 0$, observe that $L_{\Phi_\delta} \to \infty$ as $\delta \to 0$ and use the fact that $\dim_{qA} \mu = \lim_{\delta \to 0} \overline{\dim}_{\Phi_\delta} \mu$.

Finally, if $\Phi(x)/\Psi(x) \to \infty$, then, taking $\mu$ as in the mixed rate case, $0 = \overline{\dim}_\Phi \mu \geq \overline{\dim}_\Psi \mu = \dim_\Phi E$. \(\square\)

We will give the details of the proof of the theorem in the polynomial-polynomial case. The other cases require essentially no new ideas and are less complicated because of the good properties of geometric series and the fact that exponentials overwhelm polynomials in the asymptotic sense.

We begin with two elementary lemmas.

Lemma 4.3. Under the assumptions and notation of Theorem 4.1, in the polynomial-polynomial case

$$\mu(B(a_k, R)) \sim \begin{cases} 
\max(R^s, p_0) & \text{if } R > a_k \\
(a_k + R)^s - (a_k - R)^s & \text{if } a_k - a_{k+1} < R \leq a_k \\
\frac{a_k}{\beta/\lambda} & \text{if } R \leq a_k - a_{k+1}
\end{cases}$$

and $\mu(B(0, R)) \sim \max(R^s, p_0)$.

Proof. These observations follow from the fact that

$$\mu(B(a_k, R)) \sim \sum_{n: a_k \in (a_k - R, a_k + R)} n^{-\beta} \quad \text{if } R > a_k \text{ and } p_0 \neq 0 \quad \text{otherwise}.$$ 

When $R \leq a_k - a_{k+1}$, then $\mu(B(a_k, R)) = \mu\{a_k\} = k^{-\beta} = \frac{a_k^{\beta/\lambda}}{a_k}$, as claimed.

When $a_k - a_{k+1} < R \leq a_k$, then choose integers $N \geq k + 1$ and $M$ such that $a_{N+1} < a_k - R \leq a_N$ and $a_M \leq a_k + R < a_{M-1}$. (Put $N = \infty$ if $R = a_k$.) We have

$$\mu(B(a_k, R)) = \sum_{j=M}^{N} j^{-\beta} \sim M^{-\beta+1} - N^{-\beta+1} \sim (a_k + R)^s - (a_k - R)^s.$$

The reasoning is similar if $R > a_k$. \(\square\)

Lemma 4.4. Let $s > 0$. There are constants $c_1, c_2 > 0$, depending only on $s$, such that

$$c_1 a^{s-1} x \leq (a + x)^s - (a - x)^s \leq c_2 a^{s-1} x$$

whenever $0 \leq x \leq a$.

Proof. This is clear if $a/2 \leq x \leq a$ and otherwise follows quickly from the Mean value theorem. \(\square\)

Proof of Theorem 4.1. We remind the reader that we are proving the polynomial-polynomial case. Throughout the proof we will use the notation

$$X(k, r, R) = \frac{\mu(B(a_k, R))}{\mu(B(a_k, r))} \quad \text{and} \quad X(0, r, R) = \frac{\mu(B(0, R))}{\mu(B(0, r))}.$$
**Step 1:** We will first assume that $x^\Phi(x) \to 0$ as $x \to 0$.

**Upper bound on $\dim_{\Phi}\mu$:** As the arguments are often quite similar, we will handle the cases $p_0 = 0$ and $p_0 > 0$ concurrently.

Since $X(0, r, R) \sim 1$ if $p_0 \neq 0$ and comparable to $(R/r)^s$ if $p_0 = 0$, we easily see that $X(0, r, R) \lesssim (R/r)^\alpha$ for any $\alpha > 0$ when $p_0 \neq 0$, and any $\alpha > s$ if $p_0 = 0$. Hence we now focus on balls centred at $a_k$, $k \in \mathbb{N}$.

As it often arises, we will set

$$b_k = a_k - a_{k+1} \sim a_k^{(\lambda+1)/\lambda}.$$  

If $R \leq b_k$, then also $r \leq b_k$ and $X(k, r, R) \sim 1$, so any $\alpha > 0$ suffices.

Thus, there remain two cases to study: $R > a_k$ and $b_k < R \leq a_k$.

**Case 1:** $R > a_k$.

(i) Suppose $r > a_k$. If $p_0 \neq 0$, then $X(k, r, R) \sim 1 \lesssim (R/r)^\alpha$ for any $\alpha > 0$. If $p_0 = 0$, then $X(k, r, R) \sim (R/r)^s$.

(ii) Next, suppose $r \in (b_k, a_k]$. Then, in addition to the inequality $r \leq R^{1+\Phi(R)}$, we also have

$$a_k^{(\lambda+1)/\lambda} \lesssim r \leq a_k < R.$$  

From the lemmas we know

$$X(k, r, R) \sim \max(p_0, R^s) \frac{\max(p_0, R^s)}{(a_k + r)^s - (a_k - r)^s} \sim \frac{\max(p_0, R^s)}{a_k^{s-1} r}.$$  

If $s \geq 1$, then $X(k, r, R) \lesssim \max(p_0, R^s) r^{-s}$. When $p_0 = 0$, $\alpha > s$ is clearly sufficient to have $X(k, r, R) \lesssim (R/r)^\alpha$. If $p_0 \neq 0$, it will be enough for $\alpha$ to satisfy $p_0 r^{-s} \lesssim (R/r)^\alpha$ for small $R$ and all $r = R^{1+\Phi(R)}$ with $\Psi(R) \geq \Phi(R)$. Equivalently, we want to satisfy

$$1 \lesssim R^{s(1+\Psi(R)) - \alpha\Psi(R)},$$

for small $R$, hence $\alpha > s(L + 1)$ is enough.

When $s < 1$ (equivalently, $s < t$), then $a_k^{-1} r \geq r \left(\min(R, r^{-\lambda/(\lambda+1)})\right)^{s-1}$. If this minimum is $R$, (which can occur only if $\Phi(R) \leq 1/\lambda$), then

$$X(k, r, R) \lesssim \frac{\max(p_0, R^s)}{R^{s-1} r} = \begin{cases} p_0 R^{1-s} r^{-1} & \text{if } p_0 \neq 0 \\ R^{-1} r^{-1} & \text{if } p_0 = 0 \end{cases}.$$  

Again, putting $r = R^{1+\Phi(R)}$ with $\Psi \geq \Phi$, it is easy to check that the requirement $X(k, r, R) \lesssim (R/r)^\alpha$ is satisfied with $\alpha > 1$ when $p_0 = 0$ and with $\alpha > 1 + sL$ when $p_0 \neq 0$.

If, instead, $\min(R, r^{-\lambda/(\lambda+1)}) = r^{\lambda/(\lambda+1)}$, then we have $r = R^{1+\Phi(R)}$ with $\Psi(R) \geq \max(\Phi(R), 1/\lambda)$. Moreover $a_k^{-1} r \gtrsim r^t$, thus $X(k, r, R) \lesssim \max(p_0, R^s) r^{-t}$. If $p_0 = 0$, it will be enough to have $R^s r^{-t} \lesssim (R/r)^\alpha$, and this happens if

$$\alpha > t + (t-s)/\Psi(R).$$

If $L > \lambda$, then $\Phi(R) < 1/\lambda$ for small enough $R$, so $1/\Psi(R) \leq \lambda$. Thus $\alpha > t + (t-s)\lambda = 1$ suffices. Similarly, $\alpha > t + L(t-s)$ is sufficient when $L \leq \lambda$. If $p_0 \neq 0$, we will want $p_0 r^{-t} \lesssim (R/r)^\alpha$ and this is satisfied by any $\alpha > \beta$ if $L > \lambda$, and for any $\alpha > t(1+L)$ when $L \leq \lambda$. 
Here is a summary of the choices of $\alpha$ for which $X(k, r, R) \leq c(R/r)^\alpha$

in case 1(ii): If $p_0 = 0$, then it is sufficient to have

$$\alpha > \begin{cases} 
  s & \text{if } s \geq 1 \\
  1 & \text{if } s < 1 \text{ and } L \geq \lambda \\
  t + L(t - s) & \text{if } s < 1 \text{ and } L < \lambda 
\end{cases}.$$  

If $p_0 \neq 0$, then we can take

$$\alpha > \begin{cases} 
  s(L + 1) & \text{if } s \geq 1 \\
  \max(1 + sL, \beta) & \text{if } s < 1 \text{ and } L \geq \lambda \\
  t(1 + L) & \text{if } s < 1 \text{ and } L < \lambda 
\end{cases}.$$  

(iii) Otherwise, $r \leq b_k \sim a_k^{1+1/\lambda}$, say $r = a_k^{1+\Psi(R)}$ where $\Psi(R) \geq \max(\Phi(R), 1/\lambda)$.

In this case

$$X(k, r, R) \lesssim \max(p_0, R^s) a_k^{-\beta/\lambda} \lesssim \max(p_0, R^t) r^{-t}. $$

If $p_0 = 0$, it suffices to have $\alpha > t + (t - s)/\Psi(R)$. If $s \geq 1$, (equivalently, $s \geq t$) this inequality is satisfied with any $\alpha > t$, while if $s < 1$ the reasoning is similar to the arguments in case (ii). Likewise, the reasoning when $p_0 \neq 0$ is similar to case (ii).

To summarize: If $p_0 = 0$, it is enough to have

$$\alpha > \begin{cases} 
  t & \text{if } s \geq 1 \\
  1 & \text{if } s < 1 \text{ and } L \geq \lambda \\
  t + L(t - s) & \text{if } s < 1 \text{ and } L < \lambda 
\end{cases}.$$  

while if $p_0 \neq 0$, then we can take

$$\alpha > \begin{cases} 
  \beta & \text{if } L \geq \lambda \\
  t(1 + L) & \text{if } L < \lambda 
\end{cases}.$$  

Case 2: $b_k < R \leq a_k$. Here the calculations are the same regardless of the choice of $p_0$.

(i) Suppose $r \leq b_k$, say $r = a_k^{1+\Psi(R)}$ where $\Psi(R) \geq \Phi(R)$. Here, $\mu(B(a_k, r)) \sim a_k^{\beta/\lambda}$, so as $a_k^{-1} \lesssim \min(R^{-1}, r^{-\lambda/(\lambda+1)})$, $X(k, r, R) \sim a_k^{s-1-\beta/\lambda} R \lesssim R \min(r^{-1}, R^{-(1+1/\lambda)}).$

By consideration of the two possible choices for the minimum, it can be checked that $\alpha > \min(1, L/\lambda)$ will suffice.

(ii) Otherwise, $b_k < r < R \leq a_k$ (a choice which can only occur if $L \geq \lambda$), and then it is easy to see that $\alpha > 1$ is sufficient, so again $\alpha > \min(1, L/\lambda)$ will work.

It is a tedious exercise to check that these constraints on $\alpha$ imply that the values specified in the Proposition are upper bounds on $\dim_\Phi \mu$.

Lower bound on $\dim_\Phi \mu$: We turn now to proving $\dim_\Phi \mu$ is as large as claimed.

First, suppose $p_0 = 0$. In this case,

$$\dim_{\text{loc}} \mu(0) = \lim_n \frac{\log \mu(B(0, n^{-\lambda}))}{\log n^{-\lambda}} \sim \frac{\beta - 1}{\lambda} = s,$$

so it is certainly true that $\overline{\dim}_\Phi \mu \geq s$ for all choices of $\Phi$.  

Essentially the same arguments as in [12], show that if $\Phi$ is the constant function $1/\theta - 1$, then

$$\underline{\dim}_\Phi E = \min \left(1, \frac{1}{(1 + \lambda)(1 - \theta)} \right).$$

As $\underline{\dim}_\Phi \mu \geq \underline{\dim}_\Phi E$, it follows that if $L \geq \lambda$, then, also, $\underline{\dim}_\Phi \mu \geq 1$. Hence if $L \geq \lambda$, then $\underline{\dim}_\Phi \mu \geq \max(1, s)$.

Next, suppose $0 \leq L < \lambda$ and $t > s$ (for otherwise, $\max(t + L(t - s), s) = s$).

Choose $R_j \to 0$ such that $\Phi(R_j) \to 1/L$ and put $r_j = R_j^{1+\Phi(R_j)}$. Choose $k = k_j$ such that $a_k - a_{k+1} \geq r > a_{k+1} - a_{k+2}$. Since $\Phi(R_j) > 1/\lambda$ for large $j$, one can check that $R_j > a_k$ and hence

$$X(k, r_j, R_j) \sim \frac{R_j^s}{a_k^\beta} \geq R_j^{s-\beta s(1+\Phi(R_j))},$$

while $(R_j/r_j)^\alpha \sim R_j^{-\alpha \Phi(R_j)}$. Thus in order to satisfy $X(k, r_j, R_j) \leq (R_j/r_j)^\alpha$ for all $j$, we require

$$1 \leq R_j^{-\Phi(R_j)(\alpha-(t+(t-s)/\Phi(R_j)))}.$$

Since $R_j^{-\Phi(R_j)} \to \infty$ and $\Phi(R_j) \to 1/L$, we see that $\alpha \geq t + L(t - s)$ is a necessary condition.

Now assume $p_0 \neq 0$ and first suppose $0 \leq L < \lambda$. With the same choice of $r_j, R_j$ and $a_k$ as above, we have $X(k, r_j, R_j) \sim a_k^{-\beta/\lambda}$. It follows that we require $\alpha \geq t(1 + L)$.

If, instead, we pick $k = k_j$ such that $a_k \leq R_j^{1+\Phi(R_j)} < a_{k-1}$ and put $r_j = a_k$, then $r_j \leq R_j^{1+\Phi(R_j)}$. With these choices for $k, r_j, R_j$, we have

$$X(k, r_j, R_j) \sim p_0 a_k^{-s} \sim R_j^{-s(1+\Phi(R_j))},$$

and one can deduce that $\alpha \geq s(L+1)$ is a necessary condition to satisfy $X(k, r_j, R_j) \leq (R_j/r_j)^\alpha$.

Lastly, assume $L \geq \lambda$. Put $r_j = R_j^{1+\Phi(R_j)}$ and choose $k = k_j$ such that $a_k < R_j \leq a_{k-1}$. For large $j$, $r_j \geq a_k - a_{k+1} \sim a_k^{1/\lambda}$. Since $r_j/R_j = R_j^{\Phi(R_j)} \to 0$, we can assume $r_j < a_k$. Hence $X(k, r_j, R_j) \sim p_0 R_j^{-s+\Phi(R_j)}$ and it follows that $\alpha \geq 1 + sL$ is required.

This completes the proof that $\underline{\dim}_\Phi \mu$ is as claimed in the statement of the Theorem for the polynomial-polyomial case when $x^{\Phi(x)} \to 0$.

**Step 2:** We now consider the case that $\delta = \limsup_{x \to 0} x^{\Phi(x)} > 0$ or, equivalently, $\limsup_{x \to 0} \Phi(x) \log x < \infty$.

For $p_0 > 0$, choose $R_j \to 0$ such that $R_j^{\Phi(R_j)} \to \delta > 0$. Pick $k = k_j$ such that $a_k < R_j \leq a_{k-1}$ and let $r_j = \min(R_j^{1+\Phi(R_j)}, a_k)$, so that $R_j/r_j \sim 1$. If $N_j$ is chosen such that $a_{N_j+1} \leq a_k + r_j \leq a_{N_j}$, then $\mu(B(a_k, r_j)) \leq \sum_{i=N_j}^{\infty} p_i \to 0$ as $j \to \infty$. But $\mu(B(a_k, R_j)) \geq p_0$. Consequently, $X(k, r_j, R_j) \to \infty$ as $j \to \infty$ and forces $\underline{\dim}_\Phi \mu = \infty$ for all such $\Phi$.

So, suppose $p_0 = 0$ and define

$$\Psi_0(x) = \frac{\sqrt{\log \log x}}{\log x} \text{ and } \Phi_0(x) = \max(\Psi_0(x), \Phi(x)).$$
Then $\Phi_0 \in \mathcal{D}$ and as $\Phi_0(x) \geq \Psi_0(x)$ for all $x$, $x^{\Phi_0(x)} \to 0$. Furthermore, $\Phi(x) \leq \Phi_0(x)$, hence $\dim_{\Phi_0} \mu \leq \dim_{\Phi} \mu \leq \dim_{A} \mu$. Since $\Phi_0 \leq \Phi + \Psi_0$, one can verify that $L_{\Phi_0} = \infty$ and thus by the first part of the theorem, $\dim_{\Phi_0} \mu = \max(1, s)$. It was shown in [9] that $\dim_{A} \mu = \max(1, s)$ and hence also $\dim_{\Phi} \mu = \max(1, s)$, as claimed in the statement of the Theorem for the polynomial-polynomial case. \[\square\]

Remark 4.5. The choice of $\Psi_0$ was made to ensure that the arguments for the other choices of $p_n$ and $a_n$ are virtually the same in the final steps of the proof.

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