Parametric Representation for the Multisoliton Solution
of the Camassa-Holm Equation

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The parametric representation is given to the multisoliton solution of the Camassa-Holm equation. It has a simple structure expressed in terms of determinants. The proof of the solution is carried out by an elementary theory of determinants. The large time asymptotic of the solution is derived with the formula for the phase shift. The latter reveals a new feature when compared with the one for the typical soliton solutions. The peakon limit of the phase shift is also considered, showing that it reproduces the known result.

KEYWORDS: Camassa-Holm equation, soliton, peakon, parametric representation

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§1. Introduction

In this paper, we report some new results associated with the multisoliton solution of the Camassa-Holm (CH) equation

$$u_t + 2\kappa^2 u_x - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}. \quad (1)$$

Here, $u = u(x, t)$, $\kappa$ is a positive parameter and the subscripts $t$ and $x$ appended to $u$ denote partial differentiation. Originally, this equation has been found in a purely mathematical search of recursion operators connected with the integrable partial differential equations. Recently, equation (1) has attracted considerable interest since it has been derived as a model equation for shallow-water waves. In addition, the equation has been shown to be completely integrable. With this as a turning point, a large number of works have been devoted to studying the mathematical structure of the equation. A recent paper describes a short history and relevant references concerning the CH and related equation. Almost all the works have been focused on the case $\kappa = 0$ for which the CH equation exhibits peakon solutions which are represented by piecewise analytic functions and whose dynamics are now well understood. However, when $\kappa \neq 0$, several new features appear in solutions. In particular, solutions recover analytic nature, but they are expressed by a parametric form like $u = u(y, t), x = x(y, t)$ where $y$ is a new coordinate variable. A difficult technical problem is to find the inverse mapping $x = x(y, t)$. To date, this problem is resolved only for particular cases. In this respect, we remark that an approach based on the inverse scattering transform method (IST) provides an explicit form of the inverse mapping in terms of Wronskian determinants. Nevertheless, the general $N$-soliton formula is not available yet.

The main purpose in this paper is to present a complete description of the general $N$-soliton solution in the form of parametric representation. We show that the solution exhibits a simple structure expressed in terms of two determinants. In §2, we shortly summarize the derivation of a system of equation (associated CH equation) equivalent to
equation (1). In §3, we describe the main result in which a parametric representation is given to the $N$-soliton solution of the CH equation. The proof of the solution is carried out by an elementary theory of determinants. In §4, we explore the asymptotic form of the $N$-soliton solution for large time and show that the interaction process of solitons reveals a new feature when compared with the typical one for the Korteweg-de Vries equation. In §5, we summarize the results and discuss some open problems related to the CH equation.

§2. Associated CH equation

It has been demonstrated that the CH equation can be recast into a more tractable form by means of an appropriate coordinate transformation. Here, we give a short summary of the results. Introducing the new variable $r$ in accordance with the relation

$$r^2 = u - u_{xx} + \kappa^2,$$

the CH equation (1) can be put into the form

$$r_t + (ur)_x = 0,$$

where the boundary condition for $r$ is $r(\pm \infty, t) = \kappa$. Then, we define the coordinate transformation $(x, t) \rightarrow (y, t')$ by

$$dy = rdr - urdt, \quad dt' = dt.$$

In the following analysis, we use the time variable $t$ in place of $t'$ by virtue of the second relation in (4). Transforming (3) by means of (4), it becomes

$$r_t + r^2 u_y = 0,$$

and $u$ is expressed in terms of $r$ as

$$u = r^2 - r(\ln r)_y - \kappa^2.$$
We term the system of equations (5) and (6) the associated CH equation. If we substitute (6) into (5), we obtain an alternative but more convenient form in the following analysis, which is

\[ Q_t = r_y, \]  

where

\[ Q = \frac{1}{2} \frac{r_{yy}}{r} - \frac{1}{4} \left( \frac{r_y}{r} \right)^2 + \frac{1}{4} \left( \frac{1}{r^2} - \frac{1}{\kappa^2} \right). \]  

By eliminating the variable \( r \) from (7) and (8), we can see that \( Q \) evolves according to the following nonlinear wave equation

\[ Q_t + 2\kappa^3 Q_y + 4\kappa^2 Q Q_t + 2\kappa^2 Q_y \partial_y^{-1} Q_t - \kappa^2 Q_{yyy} = 0, \]

where \( \partial_y^{-1} = -\int_y^\infty dy \) is an integral operator. An important observation is that equation (9) can be identified with a model equation for shallow-water waves. This fact enables us to obtain the \( N \)-soliton solution of (9) in the \((y, t)\) coordinate system. To complete the solution, however, one must revert to the original \((x, t)\) coordinate system via the inverse mapping

\[ x_y = \frac{1}{r(y, t)}, x_t = u(y, t). \]  

The most difficult ingredient in the analysis is to integrate (10) for the \( N \)-soliton solution.

§3. Parametric representation of the \( N \)-soliton solution

Now, the main result in this paper is summarized as follows: The \( N \)-soliton solution of the CH equation (1) can be written in a parametric representation

\[ u(y, t) = \left( \ln \frac{f_2}{f_1} \right)_t, \]

\[ x = \frac{y}{\kappa} + \ln \frac{f_2}{f_1} + d. \]

Here, \( f_1 = f_1(y, t) \) and \( f_2 = f_2(y, t) \) have the determinantal expressions

\[ f_1 = |G|, f_2 = |H|, \]
with the \( N \times N \) matrices \( G \) and \( H \) whose elements are given respectively by

\[
G = (g_{ij}), g_{ij} = \frac{p_i / q_i}{p_i - q_i} e^{\xi_i} \delta_{ij} + \frac{1}{p_i - q_j}, (i, j = 1, 2, ..., N),
\]

\[
H = (h_{ij}), h_{ij} = \frac{q_i / p_i}{p_i - q_i} e^{\xi_i} \delta_{ij} + \frac{1}{p_i - q_j}, (i, j = 1, 2, ..., N),
\]

where \( \delta_{ij} \) is Kronecker’s delta and the parameter \( d \) in (12) is an integration constant. By virtue of the parametrization (17), the \( N \)-soliton solution is characterized completely by the \( 2N \) parameters \( k_i \) and \( \xi_{i0} \), \( (i = 1, 2, ..., N) \). In terms of the parameters \( k_i \), the phase variable \( \xi_i \) of the \( i \)th soliton may be written in the form

\[
\xi_i = k_i \left( y - \frac{2\kappa^3}{1 - \kappa^2 k_i^2} t - \xi_{i0} \right), (i = 1, 2, ..., N),
\]

where we have put \( \xi_{i0} = -k_i y_{i0} \).

Let us now outline the proof of (11) and (12) in which two bilinear identities (23) and (38) below will play an essential role. First, we write the \( N \)-soliton solution of equation (9) in a determinantal form

\[
Q = -2(\ln f)_{yy}, f = |F|,
\]

where \( F \) is an \( N \times N \) matrix with elements

\[
F = (f_{ij}), f_{ij} = \frac{e^{\xi_i}}{p_i - q_i} \delta_{ij} + \frac{1}{p_i - q_j}, (i, j = 1, 2, ..., N).
\]

Substituting (19) into (7) and integrating the resultant equation by \( y \) under the boundary condition \( r \to \kappa, |y| \to \infty \), we obtain

\[
r = \kappa - 2(\ln f)_{ty}.
\]
A crucial observation in the present analysis is that \( r \) can be represented in terms of \( f, f_1 \) and \( f_2 \) as
\[
  r = \kappa \frac{f_1 f_2}{f^2}.
\]
(22)

It follows from (21) and (22) that
\[
  \kappa f_1 f_2 = \kappa f^2 - 2(f f_{ty} - f_t f_y).
\]
(23)

This is a bilinear identity among determinants. We note that analogous relations have been studied in the direct proof of the multiperiodic solutions of the Benjamin-Ono and nonlocal nonlinear Schrödinger equations while employing an elementary theory of determinants.\(^{10}\)

For later convenience, we first introduce some notations as well as formulas for determinants and then describe the main result. Matrices and cofactors associated with any \( N \times N \) matrix \( A = (a_{ij}) \) are defined as follows:

\[
  A(a_i; b_i) = \begin{pmatrix}
    a_{11} & \ldots & a_{1N} & b_1 \\
    \vdots & \ddots & \vdots & \vdots \\
    a_{N1} & \ldots & a_{NN} & b_N \\
    a_1 & \ldots & a_N & 0
  \end{pmatrix},
\]
(24)

\[
  A(a_i, b_i; c_i, d_i) = \begin{pmatrix}
    a_{11} & \ldots & a_{1N} & c_1 & d_1 \\
    \vdots & \ddots & \vdots & \vdots & \vdots \\
    a_{N1} & \ldots & a_{NN} & c_N & d_N \\
    a_1 & \ldots & a_N & 0 & 0 \\
    b_1 & \ldots & b_N & 0 & 0
  \end{pmatrix},
\]
(25)

\[
  A_{ij} = \frac{\partial |A|}{\partial a_{ij}}, \quad A_{ij,kl} = \frac{\partial^2 |A|}{\partial a_{ik} \partial a_{jl}}.
\]
(26)

Here, \( A_{ij} \) and \( A_{ij,kl} \) are the first and second cofactors, respectively. The following formulas are used frequently in the present analysis:

\[
  |A(a_i, b_i; c_i, d_i)| = \left| A \right| z - \sum_{i,j=1}^{N} A_{ij} x_i y_j,
\]
(27)

\[
  |A(a_i, b_i; c_i, d_i)||A| = |A(a_i; c_i)||A(b_i; d_i)| - |A(a_i; d_i)||A(b_i; c_i)|,
\]
(28)
\[ \sum_{i,j=1}^{N} (f_i + g_j) a_{ij} A_{ij} = \sum_{i=1}^{N} (f_i + g_i) |A|, \]  
(29)

\[ \sum_{r,s=1}^{N} (f_r + g_s) a_{rs} A_{is} A_{rj} = (f_i + g_j) A_{ij} |A|. \]  
(30)

Formula (28) is Jacobi’s identity and formulas (29) and (30) follow from the expansion formulas for determinants, \[ \sum_{k=1}^{N} a_{ik} A_{jk} = \delta_{ij} |A|; \sum_{k=1}^{N} a_{ki} A_{kj} = \delta_{ij} |A|. \]

Let us now prove (23). Using the rule for the differentiation of determinant and formula (29) with \( A = F, f_i = p_i \) and \( g_i = -q_i \), we obtain

\[ |F|_y = - \sum_{i,j=1}^{N} F_{ij} + \sum_{i=1}^{N} (p_i - q_i) |F|. \]  
(31)

Similarly, we find from (29) with \( A = F, f_i = p_i^{-1} \) and \( g_i = -q_i^{-1} \) that

\[ |F|_t = \frac{1}{2} \sum_{i,j=1}^{N} \frac{F_{ij}}{p_i q_j} + \frac{1}{2} \sum_{i=1}^{N} \left( \frac{1}{p_i} - \frac{1}{q_i} \right) |F|. \]  
(32)

If we use formula (30) with \( A = F, f_r = p_r \) and \( g_s = -q_s \) as well as Jacobi’s identity of the form \( |F| F_{il} F_{ij} - F_{ij} F_{il} \), we can derive the formula for the \( y \) derivative of \( F_{ij} \)

\[ F_{ij,y} = \left\{ - \sum_{l,s=1}^{N} F_{ls} + \sum_{l=1}^{N} (p_l - q_l) |F| \right\} \frac{F_{ij}}{|F|} + \sum_{l,s=1}^{N} \frac{F_{is} F_{lj}}{|F|} - (p_i - q_j) F_{ij}. \]  
(33)

Differentiating (32) by \( y \) and inserting (31) and (32), we obtain the expression for \( |F|_{ty} \).

With (31) and (32), this result is substituted in the right-hand side of (23) to obtain the relation

\[ \kappa f^2 - 2(ff_{ty} - f_{ty}) = \kappa |F|^2 - \kappa \sum_{i,j=1}^{N} \frac{F_{ij}}{p_i} \sum_{l,k=1}^{N} \frac{F_{ik}}{q_k} + \kappa |F| \sum_{i,j=1}^{N} \frac{p_i - q_j}{p_i q_j} F_{ij}. \]  
(34)

Further simplification is possible by applying (27) to (34). This gives

\[ \kappa f^2 - 2(ff_{ty} - f_{ty}) = \kappa (|F| + |F(1;p_i^{-1})|(|F| - |F(q_i^{-1};1)|)). \]  
(35)
On the other hand, using (27) together with the basic formulas for determinants, we can show that

\[ f_1 = |G| = \prod_{j=1}^{N} (p_j/q_j)(|F| + |F(1,p_i^{-1})|), \]  

(36)

\[ f_2 = |H| = \prod_{j=1}^{N} (q_j/p_j)(|F| - |F(q_i^{-1};1)|). \]  

(37)

The identity (23) follows immediately from (35), (36) and (37).

The identity below also plays an important role:

\[ f_2^2 - f_1f_2 = \kappa(f_1f_2, y) - f_1,yf_2), \]  

(38)

which we shall now show. First, it follows from (36) and (37) that

\[ f_2^2 - f_1f_2 = |F|(|F(q_i^{-1};1)| - |F(1,p_i^{-1})|) + |F(1,p_i^{-1})||F(q_i^{-1};1)|. \]  

(39)

Formulas similar to (31) now take the form

\[ |G|_y = \prod_{j=1}^{N} (p_j/q_j)|F(q_i;p_i^{-1})| + \sum_{i=1}^{N} (p_i - q_i)|G|, \]  

(40)

\[ |H|_y = \prod_{j=1}^{N} (q_j/p_j)|F(q_i^{-1};p_i)| + \sum_{i=1}^{N} (p_i - q_i)|H|. \]  

(41)

Then, we calculate the quantity \( P \equiv f_2^2 - f_1f_2 + \kappa(f_1f_2, y) - f_1,yf_2) \). Substitution of (36), (37), (40) and (41) into \( P \) yields

\[ P = |F|(|F(q_i^{-1};1)| - \kappa|F(q_i^{-1};p_i)| - |F(1,p_i^{-1})| + \kappa|F(q_i;p_i^{-1})|) \]

\[ +|F(1,p_i^{-1})||F(q_i^{-1};1)| - \kappa|F(q_i;p_i^{-1})||F(q_i^{-1};1)| - \kappa|F(q_i^{-1};p_i)||F(1,p_i^{-1})|. \]  

(42)

Let \( P_1 \) be the sum of terms involving \( |F| \) and \( P_2 \) be the rest. Owing to the basic formula for determinant, \( \alpha|F(a_i;b_i)| + \beta|F(a_i;c_i)| = |F(a_i;\alpha b_i + \beta c_i)| \) and the relation \( 1 - \kappa p_i = \kappa q_i \) which follows directly from (17), \( P_1 \) reduces to

\[ P_1 = \kappa|F|(|F(q_i^{-1};q_i)| - |F(p_i^{-1};p_i)|). \]  

(43)
Using Jacobi’s identity and (17), we can show that

\[ P_2 = \kappa |F||F(p_i, 1; p_i^{-1}, q_i^{-1})|. \]  

(44)

After a few manipulations, we finally arrive at the relation

\[ P = P_1 + P_2 = \frac{\kappa |F|}{\prod_{i=1}^{N} p_i q_i} (|\tilde{F}(q_i; p_i)| - |\tilde{F}(p_i; q_i)|), \]  

(45)

where \( \tilde{F} = (\tilde{f}_{ij}) \) is an \( N \times N \) matrix with elements

\[ \tilde{f}_{ij} = p_i (q_i f_{ij} + 1), (i, j = 1, 2, ..., N). \]  

(46)

Thanks to (17) and (20), we see that \( \tilde{F} \) is a symmetric matrix and hence \( |\tilde{F}(q_i; p_i)| = |\tilde{F}(p_i; q_i)| \), implying that \( P = 0 \). Thus, we complete the proof of (38).

The relations (22) and (38) immediately lead to our main result. Indeed, it follows from (22) and (38) that

\[ \frac{1}{r} - \frac{1}{\kappa} = \left( \ln \frac{f_2}{f_1} \right)_y. \]  

(47)

We rewrite equation (5) in the form

\[ \left( \frac{1}{r} - \frac{1}{\kappa} \right)_t = u_y, \]  

(48)

and substitute (47) into (48). Integrating the resultant equation in \( y \) under the boundary condition \( u \to 0, |y| \to \infty \), we obtain the expression (11) for \( u(y, t) \). In view of (47), a system of equations (10) which determine the inverse mapping can be integrated. This gives rise to (12). Note that the parameter \( d \) is independent of \( t \) as confirmed by (48) and the second equation in (10). The expressions (11) and (12) give a complete description of the parametric representation for the \( N \)-soliton solution of the CH equation. In the process of constructing soliton solutions, we have noticed that it is better to use the determinants \( f_1 \) and \( f_2 \) in place of the determinant \( f \). It should be pointed out, however, that \( f_1 \) and \( f_2 \) exhibit the same functional form as \( f \) except phase factors. To see this, let \( f = \)
$f(\xi_1, ..., \xi_N)$. Then, it follows from (14), (15) and (20) that $f_1 = f(\xi_1 + \ln(p_1/q_1), ..., \xi_N + \ln(p_N/q_N))$ and $f_2 = f(\xi_1 + \ln(q_1/p_1), ..., \xi_N + \ln(q_N/p_N))$.

§4. Asymptotic behavior of the $N$-soliton solution

A new feature of the $N$-soliton solution appears in the interaction process of solitons. Here, we address on this problem. The procedure for investigating the asymptotic behavior of the solution can now be performed straightforwardly. The core part of the calculation is to evaluate $f_1$ and $f_2$ by utilizing the formula for the Cauchy determinant

$$d(m, m + 1, ..., n) \equiv \left| \begin{array}{cccc}
(p_i - q_i) & (p_i - q_j) & \cdots & (p_i - q_n) \\
(p_i - q_1) & (p_i - q_j) & \cdots & (p_i - q_n) \\
\vdots & \vdots & \ddots & \vdots \\
(p_i - q_1) & (p_i - q_j) & \cdots & (p_i - q_n) \\
\end{array} \right| = \prod_{m \leq i < j \leq n} \frac{(p_i - p_j)(q_i - q_j)}{(p_i - q_j)(q_i - p_j)}. \quad (49)$$

To this end, we order the magnitude of the velocity of each soliton in the $(x, t)$ coordinate system as $c_1 > c_2 > ... > c_N$ where

$$c_i = \frac{2\kappa^2}{1 - k^2k_i^2}, \quad (i = 1, 2, ..., N). \quad (50)$$

We take the limit $t \to -\infty$ with the phase variable $\xi_i$ of the $i$th soliton being fixed. Since then other phase variables behave like $\xi_1, \xi_2, ..., \xi_{i-1} \to +\infty, \xi_{i+1}, \xi_{i+2}, ..., \xi_N \to -\infty,$ $f_1$ has the leading-order asymptotic of the form

$$f_1 \sim \exp \left[ \sum_{j=1}^{i-1} \xi_i \right] \left( \prod_{l=1}^{i-1} \frac{p_l}{q_l} \right) \left( \prod_{s=1}^{N} \frac{1}{p_s - q_s} \right) \times \left( \frac{p_i}{q_i} e^{\xi_i} d(i + 1, i + 2, ..., N) + d(i, i + 1, ..., N) \right). \quad (51)$$

By invoking (49) and (17), we obtain

$$\frac{d(i + 1, i + 2, ..., N)}{d(i, i + 1, ..., N)} = \prod_{j=i+1}^{N} \left( \frac{k_i + k_j}{k_i - k_j} \right)^2. \quad (52)$$

Substitution of (52) into (51) gives

$$f_1 \sim \exp \left[ \sum_{j=1}^{i-1} \xi_j \right] \left( \prod_{l=1}^{i-1} \frac{p_l}{q_l} \right) \left( \prod_{s=1}^{N} \frac{1}{p_s - q_s} \right) d(i, i + 1, ..., N) \left( \frac{p_i}{q_i} e^{\xi_i - \gamma_i^(-)} + 1 \right), \quad (53)$$
where

\[ \gamma_i^{(-)} = \sum_{j=i+1}^{N} \ln \left( \frac{k_i - k_j}{k_i + k_j} \right)^2, \quad (i = 1, 2, ..., N). \]  

(54)

Similarly, in the limit of \( t \to -\infty \), \( f_2 \) has the asymptotic form

\[ f_2 \sim \exp \left[ \sum_{j=1}^{i-1} \xi_j \right] \left( \prod_{l=1}^{i-1} \frac{q_l}{p_l} \right) \left( \prod_{s=1}^{N} \frac{1}{p_s - q_s} \right) d(i, i + 1, ..., N) \left( \frac{q_i}{p_i} e^{\xi_i - \gamma_i^{(-)}} + 1 \right). \]  

(55)

It turns out from (11), (53) and (55) that \( u \) is represented by a superposition of \( N \) solitons

\[ u \sim \sum_{i=1}^{N} u_i (\xi_i - \gamma_i^{(-)}), \]  

(56)

where \( u_i \) is a one-soliton solution given by\(^{3,5,6}\)

\[ u_i(\xi_i) = \frac{2\kappa \tilde{c}_i k_i^2}{1 + \kappa^2 k_i^2 + (1 - \kappa^2 k_i^2) \cosh \xi_i} \left( \tilde{c}_i = \kappa c_i \right). \]  

(57)

In the same limit, the mapping relation (12) becomes

\[ x - c_i t - x_{i0} \sim \frac{\xi_i}{\kappa k_i} + \ln \left( \frac{1 - \kappa k_i}{1 + \kappa k_i} e^{\xi_i - \gamma_i^{(+)}} + \frac{1 + \kappa k_i}{1 + \kappa k_i} e^{\xi_i - \gamma_i^{(-)}} + 1 - \kappa k_i \right) \]

\[ - \sum_{j=1}^{i-1} \ln \left( \frac{1 + \kappa k_j}{1 - \kappa k_j} \right)^2 - \ln \left( \frac{1 + \kappa k_i}{1 - \kappa k_i} \right) + d, \quad (i = 1, 2, ..., N), \]  

(58)

where \( x_{i0} = y_{i0}/\kappa \). In the limit of \( t \to +\infty \), on the other hand, the expressions corresponding to (56), (54) and (58) are given respectively by

\[ u \sim \sum_{i=1}^{N} u_i (\xi_i - \gamma_i^{(+)}) \]  

(59)

\[ \gamma_i^{(+)} = \sum_{j=1}^{i-1} \ln \left( \frac{k_i - k_j}{k_i + k_j} \right)^2, \quad (i = 1, 2, ..., N), \]  

(60)

\[ x - c_i t - x_{i0} \sim \frac{\xi_i}{\kappa k_i} + \ln \left( \frac{1 - \kappa k_i}{1 + \kappa k_i} e^{\xi_i - \gamma_i^{(+)}} + \frac{1 + \kappa k_i}{1 + \kappa k_i} e^{\xi_i - \gamma_i^{(+)} + 1 - \kappa k_i} \right) \]

\[ - \sum_{j=i+1}^{N} \ln \left( \frac{1 + \kappa k_j}{1 - \kappa k_j} \right)^2 - \ln \left( \frac{1 + \kappa k_i}{1 - \kappa k_i} \right) + d, \quad (i = 1, 2, ..., N). \]  

(61)
Let $\Delta_i$ be the phase shift of the $i$th soliton in the $(x, t)$ coordinate system. This quantity can be evaluated simply with an appropriate use of (58) and (61). The result is

$$\Delta_i = \frac{1}{\kappa k_i}(\gamma_i^{(+)} - \gamma_i^{(-)}) + \sum_{j=1}^{i-1} \ln \left( \frac{1 + \kappa k_j}{1 - \kappa k_j} \right)^2 - \sum_{j=i+1}^{N} \ln \left( \frac{1 + \kappa k_j}{1 - \kappa k_j} \right)^2, (i = 1, 2, .., N). \quad (62)$$

The phase shift consists of two contributions. The first term on the right-hand side of (62) comes from $u(y, t)$ and the rest terms from $x(y, t)$. Note that the first term is the same as the phase shift arising from the interaction of $N$ solitons for the KdV equation. However, due to the mapping (12), additional terms appear as indicated by (62). The case $N = 2$ in (62) recovers a formula already derived by the IST.\(^{11}\)

It is also an interesting problem to investigate the characteristics of the $N$-soliton solution in the limit of $\kappa \to 0$. Here, we are concerned only with the limiting form of the phase shift. We find that the appropriate limiting procedure can be performed if one puts $\kappa k_i = 1 - \epsilon_i$ and takes the limit $\epsilon_i \to 0$ while keeping $c_i (i = 1, 2, .., N)$. At the same time, the conditions $\epsilon_i c_i = \epsilon_j c_j (i, j = 1, 2, .., N)$ must be imposed by virtue of the relation between $c_i$ and $k_i$ (see (50)). The formula (62) then reduces to

$$\Delta_i = -\sum_{j=1}^{i-1} \ln \left( \frac{c_i}{c_i - c_j} \right)^2 + \sum_{j=i+1}^{N} \ln \left( \frac{c_i}{c_i - c_j} \right)^2, (i = 1, 2, .., N). \quad (63)$$

In the special case of $N = 2$, (63) coincides with a formula presented in ref. 4).

\section*{§5. Discussion}

We have obtained a simple parametric representation for the $N$-soliton solution of the CH equation. The solution is expressed compactly by the two determinants $f_1$ and $f_2$. This finding is of essential importance. If one calculates the solution $u$ from (16) and (21) in terms of a single variable $f$, then resulting expression becomes a complicated fashion and it would not be of practical use in investigating the structure of the solution. Introduction of the variables $f_1$ and $f_2$ also leads to an analytical form (12) for the inverse mapping. The explicit form of the $N$-soliton solution makes it possible to construct other
class of solutions. For instance, the rational soliton solutions may be obtained from it by taking appropriate long wave limits \( k_i \to 0 (i = 1, 2, ..., N) \).\(^{11}\)

Recently, the CH equation has been generalized to a two-dimensional version by applying an asymptotic expansion method to a system of water-wave equations.\(^{12}\) It is an interesting problem to investigating its integrability. The method developed in this paper may be useful in constructing multisoliton solutions, if they exist. Furthermore, the Degasperis-Procesi (DP) equation is a current interest in soliton theory.\(^{13,14}\) Although the DP equation has a form similar to the CH equation, its mathematical structure is quite different from that of the CH equation.\(^{14}\) Quite recently, we have succeeded in obtaining the multisoliton solution of the DP equation by means of a reduction procedure for the multisoliton solution of the Kadomtsev-Petviashvili equation.\(^{15}\) The solution can be written in a parametric form analogous to the corresponding solution of the CH equation. However, the simple expressions like (11) and (12) are not at hand yet for the general \( N \)-soliton solution. This problem is currently being investigated.

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