Nonparametric signal detection with small values of type I and type II error probabilities

Mikhail Ermakov

October 2020

Institute of Problems of Mechanical Engineering RAS, Bolshoy pr., 61, VO, 1991178 St. Petersburg and St. Petersburg State University, Universitetsky pr., 28, Petrodvorets, 198504 St. Petersburg, RUSSIA

AMS subject classification: 62F03, 62G10, 62G2

keywords: Neymann test, consistency, signal detection, goodness-of-fit tests

Abstract

We consider problem of signal detection in Gaussian white noise. Test statistics are linear combinations of squares of estimators of Fourier coefficients or $L_2$-norms of kernel estimators. We point out necessary and sufficient conditions when nonparametric sets of alternatives have a given rate of exponential decay for type II error probabilities.

1 Introduction

In hypothesis testing and confidence estimation we usually work with events having small error probabilities. In paper we explore probabilities of such events arising in problems of signal detection in Gaussian white noise. For tests of Neymann type and for tests generated $L_2$-norms of kernel estimators, we provide comprehensive description of nonparametric sets of alternatives having given rates of convergence to zero of type II error probabilities.

If problem of nonparametric hypothesis testing is the problem of testing hypothesis on adequacy of statistical model choice, the alternatives can be arbitrary. The only natural requirement is that functions from set of alternatives does not belong to hypothesis. Thus we need to explore consistency of nonparametric sets of alternatives without any assumptions. We should not be limited by assumptions of their parametric structure or ownership to some class of smooth functions [3, 11]. Such an approach has been developed in [2, 6, 7] and this line is continued in this paper.

In papers [6, 7] we established necessary and sufficient conditions of uniform consistency of nonparametric sets of alternatives for wide spread nonparametric

\footnote{Research has been supported RFFI Grant 20-01-00273.}
tests: Kolmogorov, Kramer-von Mises, chi-squared with number of cells increasing with growing of sample size, Neymann, Bickel-Rosenblatt. The results were provided for nonparametric sets of alternatives assigned both in terms of distribution function and density.

The paper provides a comprehensive description of uniformly consistent nonparametric sets of alternatives having a given rate of convergence of type II error probabilities to zero as noise power tends to zero. Such a description is provided, both in terms of the distance method (asymptotics of distance between hypothesis and alternatives) and in terms of rate of convergence to zero $L_2$-norms of signals belonging to sets of alternatives. As mentioned we consider tests of Neymann type [13] and tests generated $L_2$-norms of kernel estimators [1, 8].

Let we observe random process $Y_\varepsilon(t), t \in [0, 1]$, defined stochastic differential equation

$$dY_\varepsilon(t) = S(t) \, dt + \varepsilon \, dw(t), \quad \varepsilon > 0,$$

(1.1)

where $S \in L_2(0, 1)$ and $dw(t)$ - Gaussian white noise.

We verify hypothesis

$$\mathbb{H}_0 : S(t) = 0, \quad t \in [0, 1],$$

(1.2)

versus simple alternatives

$$\mathbb{H}_\varepsilon : S(t) = S_\varepsilon(t), \quad t \in [0, 1], \quad \varepsilon > 0.$$  

(1.3)

It is clear that the asymptotic of type II error probabilities for arbitrary family of sets of alternatives $\Psi_\varepsilon \subset L_2(0, 1), \varepsilon > 0$, is unambiguously described such a setup. We can always extract from the sets $\Psi_\varepsilon, \varepsilon > 0$, a family of simple alternatives $S_\varepsilon, \varepsilon > 0$, having worst order of asymptotic of type II error probabilities and to explore this asymptotic.

Paper is organized as follows. In section 2 we provide definitions of uniform consistency in zones of large and moderate deviation probabilities. In section 3 the asymptotics of type I and type II error probabilities in large and moderate deviations zones are pointed out if test statistics are linear combinations of squared estimates of Fourier coefficients of signal. In section 4 we introduce the notion of maxisets for zones of large deviation probabilities. Maxisets for test statistics are the largest convex, center-symmetric sets such that, if, from such a maxiset, we delete $L_2$-balls, having centers at zero and having a given rates of convergence of radii to zero as the noise power tends to zero, we get uniformly consistent family of sets of alternatives in large deviation sense. In section 5 we establish that Besov bodies $B^s_{2,\infty}(P_0), P_0 > 0$ are maxisets. Parameter $s$ depends on rates of convergence $L_2$-norms $\|S_\varepsilon\|$ to zero. We show that any family of simple alternatives $S_\varepsilon, \varepsilon > 0$, having given rates of convergence of $L_2$-norms $\|S_\varepsilon\|$ to zero, is consistent in terms of large deviation probabilities, if and only if, each function $S_\varepsilon$ admits representation as sum of two functions: $S_1\varepsilon \in B^s_{2,\infty}(P_0)$ with the same rates of convergence of $\|S_1\varepsilon\|$ to zero as $\|S_\varepsilon\|$ and orthogonal function $S_2\varepsilon$. The functions $S_1\varepsilon$ are smoother than $S_2\varepsilon$ and
functions \( S_{2\varepsilon} \) are more rapidly oscillating. In section 6 we establish similar results if test statistics are \( L^2 \)-norms of kernel estimators of signal. In section 7 proof of Theorems is provided.

We use letters \( c \) and \( C \) as a generic notation for positive constants. Denote \( 1_{\{A\}} \) the indicator of an event \( A \). Denote \([a]\) whole part of real number \( a \). For any two sequences of positive real numbers \( a_n \) and \( b_n \), \( a_n = O(b_n) \) and \( a_n \asymp b_n \) imply respectively \( a_n < C b_n \) and \( c a_n \leq b_n \leq C a_n \) for all \( n \) and \( a_n = o(b_n) \), \( a_n \ll b_n \) implies \( a_n/b_n \to 0 \) as \( n \to \infty \). We also use notation \( a_n = \Omega(b_n) \) which means \( b_n \leq C a_n \) for all \( n \).

Define function
\[
\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\{-t^2/2\} \, dt, \quad x \in \mathbb{R}^1,
\]
of standard normal distribution.

Let \( \phi_j, 1 \leq j < \infty \), be orthonormal system of functions onto \( L^2(0, 1) \). For each \( P_0 > 0 \) define set
\[
\mathbb{B}_{2\infty}^s(P_0) = \left\{ S : S = \sum_{j=1}^{\infty} \theta_j \phi_j, \sup_{\lambda>0} \sum_{|j|>\lambda} \theta_j^2 \leq P_0, \theta_j \in \mathbb{R}^1 \right\}. \tag{1.4}
\]

With some conditions on the basis \( \phi_j \), \( 1 \leq j < \infty \), functional space
\[
\mathbb{B}_{2\infty}^s = \left\{ S : S = \sum_{j=1}^{\infty} \theta_j \phi_j, \sup_{\lambda>0} \sum_{|j|>\lambda} \theta_j^2 < \infty, \theta_j \in \mathbb{R}^1 \right\}
\]
is Besov space \( \mathbb{B}_{2\infty}^s \) (see [10]). In particular, this holds if \( \phi_j \), \( 1 \leq j < \infty \) is trigonometric basis.

If \( \phi_j(t) = \exp\{2\pi i j x\}, x \in (0, 1), j = 0, \pm 1, \ldots \), denote
\[
\mathbb{B}_{2\infty}^s(P_0) = \left\{ S : f = \sum_{j=-\infty}^{\infty} \theta_j \phi_j, \sup_{\lambda>0} \sum_{|j|>\lambda} |\theta_j|^2 \leq P_0 \right\}.
\]
Since functions \( \phi_j \) are complex here, then \( \theta_j \) are complex numbers as well, and \( \theta_j = \theta_{-j} \) for all \( -\infty < j < \infty \).

Denote \( \mathbb{B}_{2\infty}^s \) – Banach space generated by balls \( \mathbb{B}_{2\infty}^s(P_0) \), \( P_0 > 0 \).

2 Consistency in large and moderate deviation zone

For any test \( L_\varepsilon \), \( \varepsilon > 0 \), denote \( \alpha(L_\varepsilon) = E_0(L_\varepsilon) \) – its type I error probability and \( \beta(L_\varepsilon, S) = E_S(1 - L_\varepsilon) \) – its type II error probability for alternative \( S \in L_2(0, 1) \).

Let \( r_\varepsilon \to \infty \) as \( \varepsilon \to 0 \). If we have
\[
|\log \beta(L_\varepsilon, S_\varepsilon)| = \Omega(r_\varepsilon^2) \tag{2.1}
\]
for all tests \( L_\varepsilon \), \( \alpha(L_\varepsilon) = \alpha_\varepsilon \), such that \( r_\varepsilon^{-2} \log \alpha(L_\varepsilon) \to 0 \) as \( \varepsilon \to 0 \), then we say that family of alternatives \( S_\varepsilon, \varepsilon > 0 \), is \( r_\varepsilon^2 \)-consistent from large deviation viewpoint (\( r_\varepsilon - LD \)-consistent).

Note that any \( r_\varepsilon^2 - LD \)-consistent family of alternatives \( S_\varepsilon \) is consistent, while the reverse is not obligatory. If (2.1) does not hold, we say that family of alternatives is \( r_\varepsilon^2 - LD \) inconsistent.

3 Asymptotic of type I and type II error probabilities of quadratic test statistics

Using orthonormal system of functions \( \phi_j \), \( 1 \leq j < \infty \), we can rewrite stochastic differential equation (1.1) in terms of sequence model (see [10])

\[
y_{\varepsilon j} = \theta_j + \varepsilon \xi_j, \quad 1 \leq j < \infty,
\]

where

\[
y_{\varepsilon j} = \int_0^1 \phi_j dY_\varepsilon(t), \quad \xi_j = \int_0^1 \phi_j dw(t) \quad \text{and} \quad \theta_j = \int_0^1 S \phi_j dt.
\]

Denote \( y_\varepsilon = \{y_{\varepsilon j}\}_{j=1}^\infty \) and \( \theta = \{\theta_j\}_{j=1}^\infty \).

For alternatives \( S_\varepsilon \) we put \( \theta_{\varepsilon j} = \int_0^1 S_\varepsilon \phi_j dt \).

We can consider \( \theta \) as a vector in Hilbert space \( \mathbb{H} \) with the norm \( \| \theta \| = \left( \sum_{j=1}^\infty \theta_j^2 \right)^{1/2} \). In what follows, we shall implement the same notation of norm \( \| \cdot \| \) for the space \( L_2 \) and for \( \mathbb{H} \). The sense of this notation will be always clear from context.

We explore the test statistics of Neyman type tests

\[
T_\varepsilon(Y_\varepsilon) = \varepsilon^{-2} \sum_{j=1}^\infty \kappa_{\varepsilon j}^2 y_{\varepsilon j}^2 - \rho_\varepsilon^2,
\]

where \( \rho_\varepsilon^2 = \sum_{j=1}^\infty \kappa_{\varepsilon j}^2 \).

Suppose that coefficients \( \kappa_{\varepsilon j}^2 \) satisfy the following assumptions.

A1. For any \( \varepsilon > 0 \) sequence \( \kappa_{\varepsilon j}^2 \) is decreasing.

A2. There holds

\[
C_1 < A_\varepsilon = \varepsilon^{-4} \sum_{j=1}^\infty \kappa_{\varepsilon j}^4 < C_2.
\]

Denote \( \kappa_{\varepsilon}^2 = \kappa_{\varepsilon k_\varepsilon}^2 \), where \( k_\varepsilon = \sup \{ k : \sum_{j<k} \kappa_{\varepsilon j}^2 \leq \frac{1}{2} \rho_\varepsilon^2 \} \).

A3. There are \( C_1 \) and \( \lambda > 1 \), such that for any \( \delta > 0 \) and for each \( \varepsilon > 0 \) and all \( \kappa_{\varepsilon(\delta)} = [(1 + \delta)k_\varepsilon] \) we have \( \kappa_{\varepsilon,k_{\varepsilon(\delta)}}^2 < C_1 (1 + \delta)^{-1}\lambda \kappa_{\varepsilon}^2 \).

A4. We have \( \kappa_{\varepsilon 1}^2 \gtrsim \kappa_{\varepsilon}^2 \) as \( \varepsilon \to 0 \). For any \( c > 1 \) there is \( C \) such that \( \kappa_{\varepsilon,[ck_\varepsilon]}^2 \gtrsim C \kappa_{\varepsilon}^2 \) for all \( \varepsilon > 0 \).
Define test 

\[ L_\varepsilon = 1_{(A_\varepsilon^{-1/2}Y_\varepsilon > x_\alpha)}. \]

Denote \( D_\varepsilon(S_\varepsilon) = \varepsilon^{-4} \sum_{j=1}^{\infty} \chi^2_j \theta^2_j = \varepsilon^{-4} (T_\varepsilon(S_\varepsilon) + \varepsilon^2 \rho^2_\varepsilon). \) Denote 

\[ B_\varepsilon(S_\varepsilon) = \frac{D_\varepsilon(S_\varepsilon)}{(2A_\varepsilon)^{1/2}}. \]

**Theorem 3.1.** Assume A1 - A4.

Let \( x_{\alpha_\varepsilon} \to \infty \) and let \( x_{\alpha_\varepsilon} = o(k_\varepsilon^{1/6}) \) as \( \varepsilon \to 0. \) Then we have 

\[ \alpha(L_\varepsilon) = (1 - \Phi(x_{\alpha_\varepsilon}))(1 + o(1)). \] (3.3)

Let \( \alpha_\varepsilon \to 0, \ x_{\alpha_\varepsilon} = O(B_\varepsilon(S_\varepsilon)) \) and let 

\[ D_\varepsilon(S_\varepsilon) \to \infty, \quad D_\varepsilon(S_\varepsilon) = o(k_\varepsilon^{1/6}), \quad B_\varepsilon(S_\varepsilon) - x_{\alpha_\varepsilon} = \Omega(B_\varepsilon(S_\varepsilon)) \] (3.4)

as \( \varepsilon \to 0. \) Then we have 

\[ \beta(L_\varepsilon, S_\varepsilon) = \Phi(x_{\alpha_\varepsilon} - B_\varepsilon(S_\varepsilon))(1 + o(1)). \] (3.5)

Let \( x_{\alpha_\varepsilon} \to \infty \) and let \( x_{\alpha_\varepsilon} = o(k_\varepsilon^{1/2}) \) as \( \varepsilon \to 0. \) Then we have 

\[ \sqrt{2} |\log \alpha(L_\varepsilon)| = x_{\alpha_\varepsilon}(1 + o(1)), \quad \alpha_\varepsilon \to 0 \] (3.6)

as \( \varepsilon \to 0. \) Let \( \alpha_\varepsilon \to 0, \ x_{\alpha_\varepsilon} = O(B_\varepsilon(S_\varepsilon)) \) and let 

\[ D_\varepsilon(S_\varepsilon) \to \infty, \quad D_\varepsilon(S_\varepsilon) = o(k_\varepsilon^{1/2}), \quad B_\varepsilon(S_\varepsilon) - x_{\alpha_\varepsilon} = \Omega(B_\varepsilon(S_\varepsilon)) \] (3.7)

as \( \varepsilon \to 0. \) Then we have 

\[ \sqrt{2} |\log \beta(L_\varepsilon, S_\varepsilon)| = (B_\varepsilon(S_\varepsilon) - x_{\alpha_\varepsilon})(1 + o(1)), \] (3.8)

Let \( \alpha_\varepsilon \to 0, \ x_{\alpha_\varepsilon} = O(B_\varepsilon(S_\varepsilon)) \) and let

\[ r_\varepsilon \to \infty, \quad \frac{B_\varepsilon(S_\varepsilon)}{r_\varepsilon} \to \infty, \quad B_\varepsilon(S_\varepsilon) - x_{\alpha_\varepsilon} = \Omega(B_\varepsilon(S_\varepsilon)) \]

as \( \varepsilon \to 0. \) Then we have \( r_\varepsilon^{-2} |\log \beta(L_\varepsilon, S_\varepsilon)| \to \infty \) as \( \varepsilon \to 0. \)

Denote 

\[ r_\varepsilon^2 = \varepsilon^4 D_\varepsilon(S_\varepsilon) = \sum_{j=1}^{\infty} \chi^2_j \theta^2_j = T_\varepsilon(S_\varepsilon) + \varepsilon^2 \rho^2_\varepsilon. \]
Theorem 3.2. Assume $A1 - A4$.
Let $x_{\alpha_r} \to \infty$ and let $k_\varepsilon = \Omega(x_{\alpha_r}^2)$ as $\varepsilon \to 0$. Then we have
\[
\log P_0(T_\varepsilon(Y_\varepsilon) > x_{\alpha_r}) \leq -\frac{1}{2} \kappa_{\varepsilon_1}^{-2} x_{\alpha_r}(1 + o(1)). \tag{3.9}
\]
Let $k_\varepsilon = \Omega(D_\varepsilon^2(S_\varepsilon))$ and $\varepsilon(\tau_\varepsilon - x_{\varepsilon_1}^{1/2}) \to \infty$ as $\varepsilon \to 0$ additionally. Then we have
\[
\log P_{S_\varepsilon}(T_\varepsilon(Y_\varepsilon) < x_{\alpha_r}) \leq -\frac{1}{2} \kappa_{\varepsilon_1}^{-2}(\tau_\varepsilon - x_{\varepsilon_1}^{1/2})^2(1 + o(1)). \tag{3.10}
\]
The equality in (3.10) is attained for $S_\varepsilon = \{\theta_{\varepsilon_1}\}_{j=1}^\infty$, where $\theta_{\varepsilon_1} = \varepsilon_{\varepsilon_1}^{-1} \tau_\varepsilon$ and $\theta_{\varepsilon_j} = 0$ for $j > 1$.

In Theorems 3.1 and 6.2 coefficients $\kappa_{\varepsilon_j}^2$ have different normalization in comparison with coefficients $\kappa_{\varepsilon_j}^2$ in [4]. At the same time Theorems 3.1 and 3.2 are obtained by simple modification of proofs of Lemmas 2 and 3 in [4]. To use notation $\kappa_{\varepsilon_j}^2$ instead of $\kappa_{\varepsilon_j}^2$ in the proof of Lemmas 2 and 3 in [4], we should put
\[
\kappa_{\varepsilon_j}^2 = \frac{\sum_{j=1}^{\infty} \kappa_{\varepsilon_j}^2 \theta_{\varepsilon_j} \kappa_{\varepsilon_{j+1}}^2}{\sum_{j=1}^{\infty} \kappa_{\varepsilon_j}^2}, \quad 1 \leq j < \infty, \tag{3.11}
\]
Normalization 3.2 was implemented in Theorems on asymptotic normality of test statistics $T_\varepsilon(Y_\varepsilon)$ in [6, 8].

4 Maxisets in large deviation zone

Maxiset definition in the zone of large deviation probabilities is akin to the definition in [4] introduced for exploration of uniform consistency in problems of nonparametric hypothesis testing. The only difference is replacement of notion of consistency with notion of $\varepsilon^{-2\omega} = LD$ consistency, $0 < \omega \leq 1$. Just like in [6] we explore maxisets for alternatives $S_\varepsilon$, approaching with hypothesis in $L_2$-norm with the rate of convergence $\varepsilon^{2r}$, $0 < r < 1/2$, as $\varepsilon \to 0$.

Let $\Xi, \Xi \subset L_2(0, 1)$, be Banach space with the norm $\|\cdot\|_\Xi$. Denote $U = \{f : \|f\|_\Xi \leq 1, S \in \Xi\}$ ball in $\Xi$. In what follows, we suppose that set $U$ is compact in $L_2(0, 1)$ (see [4, 8]).

Define subspaces $\Pi_k, 1 \leq k < \infty$, using induction.

Denote $d_1 = \max\{\|S\|, S \in U\}$ and define function $e_1 \in U$ such that $\|e_1\| = d_1$. Denote $\Pi_1 \subset L_2(0, 1)$ linear subspace generated by function $e_1$.

For $i = 2, 3, \ldots$ denote $d_i = \max\{\rho(S, \Pi_{i-1}), S \in U\}$, where $\rho(S, \Pi_{i-1}) = \min\{\|S - g\|, g \in \Pi_{i-1}\}$. Define function $e_i, e_i \in U$, such that $\rho(e_i, \Pi_{i-1}) = d_i$. Denote $\Pi_i$ linear space generated by functions $e_1, \ldots, e_i$.

For any function $S \in L_2(0, 1)$ denote $S_{\Pi_i}$ projection of function $S$ onto subspace $\Pi_i$ and put $S_i = S - S_{\Pi_i}$.

Set $U$ is called maxiset and functional space $\Xi$ is called maxispace, if the following two statements hold:
**5 Necessary and sufficient conditions of \( \varepsilon^{-2\omega} - LD \) - consistency of families of simple alternatives \( S_\varepsilon \) converging to hypothesis in \( L_2 \)**

We explore necessary and sufficient conditions of \( \varepsilon^{-2\omega} - LD \)-consistency of families of simple alternatives \( S_\varepsilon, \varepsilon > 0 \), such that \( \|S_\varepsilon\| \asymp \varepsilon^{2r} \) as \( \varepsilon \to 0 \). Here \( 0 < r < 1/2 \) and \( 0 < 2\omega < 1 - 2r \). Thus \( r_\varepsilon \asymp \varepsilon^{-\omega} \) as \( \varepsilon \to 0 \). We put \( k_\varepsilon \asymp \varepsilon^{-4+8r+4\omega} \).

For this setup, A1-A4 implies

\[
\kappa_\varepsilon^2 \asymp \varepsilon^{4-4r-4\omega}, \quad \bar{A}_\varepsilon \asymp \varepsilon^{-4} \sum_{j=1}^{\infty} \kappa_{ej}^2 \asymp \varepsilon^{-4\omega} \tag{5.1}
\]

as \( \varepsilon \to 0 \).

Theorems of section and subsequent proofs are akin to the statements and the proofs of Theorems 4.1 and 4.4-4.10 in [6] with the above orders \( \kappa_\varepsilon^2, \bar{A}_\varepsilon \) and \( k_\varepsilon \). For the proofs it suffices to substitute the above orders in the proofs of corresponding Theorems in [6]. This will be demonstrated in the proof of the property \( ii. \) of maxiset in Theorem 5.2. All other proofs will be omitted.

### 5.1 Analytical form of necessary and sufficient conditions of \( \varepsilon^{-2\omega} - LD \)-consistency

Results are provided in terms of Fourier coefficients of functions \( S = S_\varepsilon = \sum_{j=1}^{\infty} \theta_{ej} \phi_j \).

**Theorem 5.1.** Assume A1-A4. Family of alternatives \( S_\varepsilon, \|S_\varepsilon\| \asymp \varepsilon^{2r}, \) is \( \varepsilon^{-2\omega} - LD \)-consistent, if and only if, there are \( c_1, c_2 \) and \( \varepsilon_0 > 0 \) such that there holds

\[
\sum_{|j| < c_2 k_\varepsilon} |\theta_{ej}|^2 > c_1 \varepsilon^{4r} \tag{5.2}
\]

for all \( \varepsilon < \varepsilon_0 \).

Versions of Theorem 5.1 and Theorem 5.6 given below, hold also for test statistics based on \( L_2 \)-norm of kernel estimators. In this setup index \( j \) have positive and negative values and coefficients \( \theta_{ej} \) may be complex numbers. By this reason, we write \( |j| \) instead of \( j \) and \( |\theta_{ej}| \) instead of \( \theta_{ej} \) in (5.2) and (5.6).
5.2 Maxisets. Qualitative structure of consistent families of alternatives.

Denote $s = \frac{F}{2-4r-2r}$. Then $r = \frac{(2-2\omega)s}{1+4r}$.

**Theorem 5.2.** Assume A1-A4. Then the balls $B_{2\infty}^\omega (P_0)$, $P_0 > 0$, are maxisets for test statistics $T_\omega (Y_\omega)$.

**Theorem 5.3.** Assume A1-A4. Then family of alternatives $S_{\varepsilon}$, $\|S_{\varepsilon}\| \asymp \varepsilon^{2r}$, is $\varepsilon^{-2\omega} - \text{LD}$ consistent, if and only if, there is maxiset $B_{2\infty}^\omega (P_0)$, $P_0 > 0$, and family of functions $S_{1\varepsilon} \in B_{2\infty}^\omega (P_0)$, $\|S_{1\varepsilon}\| \asymp \varepsilon^{2r}$, such that $S_{1\varepsilon}$ is orthogonal $S_{\varepsilon} - S_{1\varepsilon}$, that is, we have

$$\|S_{\varepsilon}\|^2 = \|S_{1\varepsilon}\|^2 + \|S_{\varepsilon} - S_{1\varepsilon}\|^2. \quad (5.3)$$

**Theorem 5.4.** Assume A1-A4. Then, for any $\delta > 0$, for any $\varepsilon^{-2\omega} - \text{LD}$ consistent family of alternatives $S_{\varepsilon}$, $\|S_{\varepsilon}\| \asymp \varepsilon^{2r}$, there is maxiset $B_{2\infty}^\omega (P_0)$, $P_0 > 0$, and family of functions $S_{1\varepsilon}$, $\|S_{1\varepsilon}\| \asymp \varepsilon^{2r}$, $S_{1\varepsilon} \in B_{2\infty}^\omega (P_0)$, such that there hold:

$S_{1\varepsilon}$ is orthogonal $S_{\varepsilon} - S_{1\varepsilon}$,

for any tests $L_{\varepsilon}$, satisfying (3.7), there is $\varepsilon_0 = \varepsilon_0 (\delta) > 0$, such that, for $\varepsilon < \varepsilon_0$, there hold

$$|\log \beta (L_{\varepsilon}, S_{\varepsilon}) - \log \beta (L_{\varepsilon}, S_{1\varepsilon})| \leq \delta |\log \beta (L_{\varepsilon}, S_{\varepsilon})| \quad (5.4)$$

and

$$|\log \beta (L_{\varepsilon}, S_{\varepsilon}) - \log \beta (L_{\varepsilon}, S_{1\varepsilon})| \leq \delta |\log \beta (L_{\varepsilon}, S_{\varepsilon})|. \quad (5.5)$$

5.3 Interaction of $\varepsilon^{-2\omega} - \text{LD}$ consistent and $\varepsilon^{-2\omega} - \text{LD}$ inconsistent alternatives. Purely $\varepsilon^{-2\omega} - \text{LD}$ consistent families of alternatives

We say that $\varepsilon^{-2\omega} - \text{LD}$ consistent family of alternatives $S_{\varepsilon}$, $\|S_{\varepsilon}\| \asymp \varepsilon^{2r}$, is purely $\varepsilon^{-2\omega} - \text{LD}$ consistent, if there is no inconsistent subsequence of alternatives $S_{1\varepsilon}$, $\varepsilon_i \to 0$ as $i \to \infty$, such that $S_{1\varepsilon}$, is orthogonal $S_{\varepsilon} - S_{1\varepsilon}$, and $\|S_{1\varepsilon}\| > c_1 \varepsilon^{2r}$.

**Theorem 5.5.** Assume A1-A4. Let family of alternatives $S_{\varepsilon}$, $\|S_{\varepsilon}\| \asymp \varepsilon^{2r}$, be $\varepsilon^{-2\omega} - \text{LD}$ consistent. Then for any $\varepsilon^{-2\omega} - \text{LD}$ inconsistent family of alternatives $S_{1\varepsilon}$, $\|S_{1\varepsilon}\| \asymp \varepsilon^{2r}$, there holds

$$\lim_{\varepsilon \to 0} \frac{\log \beta (L_{\varepsilon}, S_{\varepsilon}) - \log \beta (L_{\varepsilon}, S_{\varepsilon} + S_{1\varepsilon})}{\log \beta (L_{\varepsilon}, S_{\varepsilon})} = 0.$$  

**Theorem 5.6.** Assume A1-A4. Family of alternatives $S_{\varepsilon}$, $\|S_{\varepsilon}\| \asymp \varepsilon^{2r}$, is purely $\varepsilon^{-2\omega} - \text{LD}$ consistent, if and only if, for any $\delta > 0$, there is such a constant $C_1 = C_1 (\delta)$, that there holds

$$\sum_{|j| > C_1 \varepsilon} |\theta_{ij}|^2 \leq \delta \varepsilon^{4r} \quad (5.6)$$
for all \( \varepsilon < \varepsilon_0(\delta) \).

**Theorem 5.7.** Assume A1-A4. Then family of alternatives \( S_\varepsilon, \|S_\varepsilon\| \sim \varepsilon^{2r} \), is purely \( \varepsilon^{-2\omega} - LD \)-consistent, if and only if, for any \( \delta > 0 \) there is maxiset \( \mathcal{B}_2^\infty(P_0) \) and family of functions \( S_{1\varepsilon} \in \mathcal{B}_2^\infty(P_0) \) such that \( \|S_\varepsilon - S_{1\varepsilon}\| \leq \delta \varepsilon^{2r} \) for all \( \varepsilon < \varepsilon_0(\delta) \).

**Theorem 5.8.** Assume A1-A4. Then family of alternatives \( S_\varepsilon, \|S_\varepsilon\| \sim \varepsilon^{2r} \), is purely \( \varepsilon^{-2\omega} - LD \)-consistent, if and only if, for any \( \varepsilon^{-2\omega} - LD \)-inconsistent sequence of alternatives \( S_{1\varepsilon}, \|S_{1\varepsilon}\| \sim \varepsilon_i^{2r}, \varepsilon_i \to 0 \) as \( i \to \infty \), there holds

\[
\|S_{\varepsilon} + S_{1\varepsilon}\|^2 = \|S_{\varepsilon}\|^2 + \|S_{1\varepsilon}\|^2 + o(\varepsilon^{4r}).
\]  

**Remark 5.1.** Let \( \varepsilon_{2j}^2 > 0 \) for \( j \leq l_\varepsilon \), and let \( \varepsilon_{2j}^2 = 0 \) for \( j > l_\varepsilon \), where \( l_\varepsilon \sim \varepsilon^{-4+4r+4\omega} \) as \( \varepsilon \to 0 \). Analysis of proof of Theorems show that Theorems 3.1, 3.2 and 5.1-5.8 remains valid for this setup, if assumption A4 we replace with

A5. For any \( c, 0 < c < 1 \), there is \( c_1 \) such that \( \varepsilon_{2j}^2 \leq c_1 \varepsilon_{2j}^2 \) for all \( \varepsilon > 0 \).

In all corresponding proofs we put \( \varepsilon_{2j}^2 = \varepsilon_{2j}^2 \) and \( k_\varepsilon = l_\varepsilon \). Theorems 5.6 is correct with the following modifications. It suffices suppose that \( C_1(\varepsilon) < 1 \). Proof of corresponding versions of Theorems 3.1, 3.2 and 5.1-5.8 is similar and is omitted.

### 6 Tests based on kernel estimators

We explore the same problem of signal detection in Gaussian white noise \([1,2], [1,3]\). Suppose additionally, that signal \( S_\varepsilon \) belong to \( L_2^{per}(\mathbb{R}^1) \) the set of 1-periodic functions such that \( S_\varepsilon(t) \in L_2(0,1), t \in [0,1) \). This allows to extend our model on real line \( \mathbb{R}^1 \) putting \( w(t+j) = w(t) \) for all integer \( j \) and \( t \in [0,1) \) and to write the forthcoming integrals over all real line (see \([3]\)).

Define kernel estimators

\[
\hat{S}_\varepsilon(t) = \frac{1}{h_\varepsilon} \int_{-\infty}^{\infty} K\left(\frac{t-u}{h_\varepsilon}\right) dY_\varepsilon(u), \quad t \in (0,1),
\]

where \( h_\varepsilon > 0, h_\varepsilon \to 0 \) as \( \varepsilon \to 0 \).

The kernel \( K \) is bounded function such that the support of \( K \) is contained in \([-1/2,1/2], K(t) = K(-t) \) for \( t \in \mathbb{R}^1 \) and \( \int_{-\infty}^{\infty} K(t) \, dt = 1 \).

Denote \( K_h(t) = \frac{1}{h} K\left(\frac{t}{h}\right), t \in \mathbb{R}^1 \) and \( h > 0 \).

Define test statistics

\[
T_{1\varepsilon}(Y_\varepsilon) = T_{1ch_\varepsilon}(Y_\varepsilon) = \varepsilon^{-2}h_\varepsilon^{1/2} \gamma_1^{-1}(\|\hat{S}_\varepsilon\|^2 - \varepsilon^2 h_\varepsilon^{-1} \|K\|^2),
\]

where

\[
\gamma^2 = 2 \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} K(t-s)K(s)\,ds \right)^2 \, dt.
\]
Statistics \( \|\tilde{S}_\varepsilon\|^2 - \varepsilon^2 h_\varepsilon^{-1}\|K\|^2 \) is estimator of functional value

\[
T_\varepsilon(S_\varepsilon) = \int_0^1 \left( \frac{1}{h_\varepsilon} \int K \left( \frac{t-s}{h_\varepsilon} \right) S_\varepsilon(s) \, ds \right)^2 \, dt.
\]

Denote \( L_\varepsilon = 1_{\{T_\varepsilon(Y_\varepsilon) > x_\alpha_\varepsilon\}} \), where \( x_\alpha_\varepsilon \) is defined by equation \( \alpha(L_\varepsilon) = \alpha_\varepsilon \).

Problem admits interpretation in the framework of sequence model. Let we observe realization of random process \( Y_\varepsilon(t), t \in [0, 1] \).

For \(-\infty < j < \infty\), denote

\[
\hat{K}(jh) = \int_{-1}^1 \exp\{2\pi i j t\} K_h(t) \, dt, \quad h > 0,
\]

\[
y_{\varepsilon j} = \int_0^1 \exp\{2\pi i j t\} dY_\varepsilon(t), \quad \xi_j = \int_0^1 \exp\{2\pi i j t\} \, dw(t),
\]

\[
\theta_{\varepsilon j} = \int_0^1 \exp\{2\pi i j t\} S_\varepsilon(t) \, dt.
\]

In this notation we can define estimator in the following form

\[
\hat{\theta}_{\varepsilon j} = \hat{K}(jh) y_{\varepsilon j} = \hat{K}(jh) \theta_{\varepsilon j} + \varepsilon \hat{K}(jh) \xi_{\varepsilon j}, \quad -\infty < j < \infty. \tag{6.2}
\]

Then test statistics \( T_\varepsilon \) admits the following representation:

\[
T_\varepsilon(Y_\varepsilon) = \varepsilon^{-2} h_\varepsilon^{1/2} \gamma^{-1} \left( \sum_{j=-\infty}^{\infty} |\hat{\theta}_{\varepsilon j}|^2 - \varepsilon^2 \sum_{j=-\infty}^{\infty} |\hat{K}(jh)|^2 \right). \tag{6.3}
\]

If we put \( |\hat{K}(jh)|^2 = \kappa_{\varepsilon j}^2 \), we get, that test statistics \( T_\varepsilon(Y_\varepsilon) \) in this section and sections 3 and 5 are almost indistinguishable. Results similar to section 3 were obtained in [5]. However for obtaining results similar to section 5 one needs their interpretation in terms of Fourier coefficients.

Main difference of setup of section 6 is the presence of heterogeneous Gaussian white noise in the model.

**Theorem 6.1.** Let \( \|S_\varepsilon\| \asymp \varepsilon^{2r} \) and \( h_\varepsilon \asymp \varepsilon^{4-8r-4\omega}, 0 < 2\omega < 1 - 2r \). Then Theorems 5.1–5.8 are valid for this setup with the only difference that \( \tilde{B}^2_{2\infty}(P_0) \) is replaced with \( \tilde{B}^2_{2\infty}(P_0) \).

Proof of Theorems is based on the following version of Theorem 3.1 (see Theorems 2.1 and 2.2, [3]).

**Theorem 6.2.** Let \( h_\varepsilon \to 0 \) as \( \varepsilon \to 0 \). Let \( \alpha_\varepsilon \equiv \alpha(K_\varepsilon) = o(1) \).

If \( 1 \ll \sqrt{2 \log \alpha_\varepsilon} \ll h_\varepsilon^{-1/6} \),

then \( x_\alpha_\varepsilon \) is defined equation \( \alpha_\varepsilon = \Phi(x_\alpha_\varepsilon)(1 + o(1)) \) as \( \varepsilon \to 0 \).

If \( \varepsilon^{-2} h_\varepsilon^{1/2} T_\varepsilon(S_\varepsilon) - \sqrt{2 \log \alpha_\varepsilon} > c\varepsilon^{-2} h_\varepsilon^{1/2} T_\varepsilon(S_\varepsilon) \) \( (6.4) \)
and 
\[ \varepsilon^2 h^{-1/2}_\varepsilon << T_\varepsilon(S_\varepsilon) << \varepsilon^2 h^{-2/3}_\varepsilon = o(1), \]
as \( \varepsilon \to 0 \), then the following relation holds
\[ \beta(L_\varepsilon, S_\varepsilon) = \Phi(x_{\alpha_\varepsilon} - \gamma^{-1} \varepsilon^{-2} h^{1/2}_\varepsilon T_\varepsilon(S_\varepsilon))(1 + o(1)). \]

(6.5)

If 
\[ 1 << \sqrt{2 |\log \alpha_\varepsilon|} << h^{-1/2}_\varepsilon, \]
then 
\[ x_{\alpha_\varepsilon} = \sqrt{2 |\log \alpha_\varepsilon|}(1 + o(1)). \]

(6.4) holds then we have
\[ 2 \log \beta(L_\varepsilon, S_\varepsilon) = -(x_{\alpha_\varepsilon} - \gamma^{-1} \varepsilon^{-2} h^{1/2}_\varepsilon T_\varepsilon(S_\varepsilon))^2(1 + o(1)). \]

(6.6)

If 
\[ \varepsilon^2 x_{\alpha_\varepsilon}/h^{1/2}_\varepsilon \to 0 \] as \( \varepsilon \to 0 \), then
\[ \lim_{\varepsilon \to 0} (\log \alpha_\varepsilon)^{-1} \log \beta(L_\varepsilon, S_\varepsilon) = \infty. \]

Note that
\[ T_\varepsilon(S_\varepsilon) = \sum_{j=-\infty}^{\infty} |\hat{K}(j \varepsilon)|^2 |\theta_j|^2. \]

(6.7)

7 Proof of Theorems

7.1 Proof of Theorem 3.2

Denote \( z^2 = \kappa^2 \varepsilon x_{\alpha_\varepsilon} \). Implementing Chebyshev inequality, we have
\[
\begin{align*}
P_s(T_\varepsilon - \tau^2_\varepsilon < z^2 - \tau^2_\varepsilon) &\leq \exp \{t(z^2 - \tau^2_\varepsilon)\} E_0 \left[ \exp \left\{ -t \sum_{j=1}^{\infty} \kappa_j^2(y_j - \varepsilon^2) + 2t \sum_{j=1}^{\infty} \kappa_j^2 y_j \theta_j \right\} \right] d y \\
&= \exp \{t(z^2 - \tau^2_\varepsilon)\} \\
&\times \lim_{m \to \infty} (2\pi \varepsilon^2)^{-m/2} \int \left[ \exp \left\{ -t \sum_{j=1}^{m} \kappa_j^2 y_j^2 \varepsilon + \frac{1}{2} \sum_{j=1}^{m} \left( 1 + 2t \kappa_j^2 \varepsilon^2 \right) y_j^2 \right\} \right] d y \\
&+ t \sum_{j=1}^{m} \kappa_j^2 y_j^2 + 2t \sum_{j=1}^{m} \kappa_j^2 y_j \theta_j \pm \sum_{j=1}^{m} 2t^2 \kappa_j^4 \theta_j^2 \varepsilon^2 \\
&= \exp \{t(z^2 - \tau^2_\varepsilon)\} \exp \left\{ \frac{1}{2} \sum_{j=1}^{\infty} \log(1 + 2t \kappa_j^2 \varepsilon^2) + \sum_{j=1}^{\infty} \frac{2t^2 \kappa_j^4 \theta_j^2 \varepsilon^2}{1 + 2t \kappa_j^2 \varepsilon^2} \right\}.
\end{align*}
\]
Note that
\[ \sum_{j=1}^{\infty} \log(1 + 2t\kappa_j^2\varepsilon^2) \asymp k_\varepsilon \ll \varepsilon^{-2}\tau^2_\varepsilon. \]

It is easy to see that, if \( \sum_{j=1}^{\infty} \kappa_j^2\theta_j^2 = \text{const} \), then supremum on \( \theta_j \) of formula under the sign of the second exponent is reached when \( \kappa_j^2\theta_j^2 = \tau^2_\varepsilon \).

Thus the problem is reduced to maximization on \( t \)
\[ \exp\left\{ t(z^2 - \tau^2_\varepsilon) + \frac{2t^2\kappa^2\tau^2_\varepsilon\varepsilon^2}{1 + 2t\kappa^2\varepsilon^2} \right\}. \] (7.2)

By straightforward calculations, we get that maximum on \( t \) is attained if
\[ 2t = \varepsilon^{-2}\kappa^{-2}_\varepsilon(\tau_\varepsilon z^{-1} - 1). \] (7.3)

Substituting (7.3) to (7.2), we get right hand-side of (3.10).

### 7.2 Proof of Theorem 3.1

Test statistics \( T_\varepsilon \) are sums of independent random variables. Therefore the standard reasoning of proof of Cramer Theorem are implemented [13, 15]. This reasoning has been realized in proof of Lemma 4 in [4]. If hypothesis is valid, then proof of Theorem 3.1 completely coincides with proof of Lemma 4 in [4]. If alternative is valid, the difference is only in the evaluation of the residual terms arising in the proof of Lemma 4 in [4].

Such differences will arise first of all in the evaluation of the residual term in (3.51) in [4].

If (3.7) holds, using \( \kappa_\varepsilon^2 \asymp \varepsilon^2 k_\varepsilon^{-1/2} \), we have
\[ \varepsilon^{-6} \sum_{j=1}^{\infty} \kappa^4_{\varepsilon j} \theta^2_{\varepsilon j} \asymp \varepsilon^{-6} \kappa^2_\varepsilon \sum_{j=1}^{\infty} \kappa^2_{\varepsilon j} \theta^2_{\varepsilon j} \asymp k_\varepsilon^{-1/2} D_\varepsilon(S_\varepsilon), \] (7.4)
or, for \( \kappa_{\varepsilon j} \) notation, we have
\[ \varepsilon^{-6} \sum_{j=1}^{\infty} \kappa^4_{\varepsilon j} \theta^2_{\varepsilon j} \asymp \varepsilon^{-6} \kappa^2_\varepsilon \sum_{j=1}^{\infty} \kappa^2_{\varepsilon j} \theta^2_{\varepsilon j} = o\left( \varepsilon^{-4} \kappa^2_\varepsilon \sum_{j=1}^{\infty} \kappa^4_{\varepsilon j} \right). \] (7.5)

Using (7.5), we can replace remainder terms in (3.53), (3.55)–(3.57) in [4] by the factor \((1 + o(1))\). This allows to replace (3.43) (see Lemma 4 in [4]) with (3.8).

If (3.7) holds, then we have
\[ (B_\varepsilon(S_\varepsilon) - x_{a_\varepsilon}) \varepsilon^{-6} \sum_{j=1}^{\infty} \kappa^4_{\varepsilon j} \theta^2_{\varepsilon j} \asymp \varepsilon^{-2} \kappa^2_\varepsilon \left( \varepsilon^{-4} \sum_{j=1}^{\infty} \kappa^2_{\varepsilon j} \theta^2_{\varepsilon j} \right)^3 \]
\[ \asymp k^1_\varepsilon \left( \varepsilon^{-4} \sum_{j=1}^{\infty} \kappa^2_{\varepsilon j} \theta^2_{\varepsilon j} \right)^3 = o(1), \] (7.6)
or, for $\kappa_{\epsilon j}$ notation, we have

\[
\varepsilon^{-6} \sum_{j=1}^{\infty} \kappa_{\epsilon j}^4 \theta_{\epsilon j}^2 \succ \varepsilon^{-6} \kappa_{\epsilon j}^2 \sum_{j=1}^{\infty} \kappa_{\epsilon j}^4 \theta_{\epsilon j}^2 \\
\succeq \varepsilon^{-6} \kappa_{\epsilon j}^2 \sum_{j=1}^{\infty} \kappa_{\epsilon j}^4 \theta_{\epsilon j}^2 \succ \varepsilon^{-6} k_{\varepsilon}^6 = o(1). \tag{7.7}
\]

Using (7.7), we can remain all estimates of the residual terms in the proof of Lemma 4 in [4] and get the validity of these estimates for proof of (3.5).

Proof of (3.3) and (3.6) for type I error probabilities is based on similar estimates. It suffices to put $x_{\alpha} = \varepsilon^{-4} \sum_{j=1}^{\infty} \kappa_{\epsilon j}^2 \theta_{\epsilon j}^2$ in above mentioned estimates.

### 7.3 Proof of Theorem 5.2

We verify only ii. in definition of maxiset. Proof of i. is akin to [6] and is omitted.

Suppose the contrary. Then $S = \sum_{j=1}^{\infty} \tau_j \phi_j \notin \bar{B}_{s\infty}$. This implies that there is sequence $m_l$, $m_l \rightarrow \infty$ as $l \rightarrow \infty$, such that

\[
m_{2s}^{2l} \sum_{j=m_l}^{\infty} \tau_j^2 = C_l, \tag{7.8}
\]

where $C_l \rightarrow \infty$ as $l \rightarrow \infty$.

Define sequence $\eta_l = \{\eta_{lj}\}_{j=1}^{\infty}$ such that $\eta_{lj} = 0$ if $j < m_l$ and $\eta_{lj} = \tau_j$, if $j \geq m_l$.

We put $\hat{S}_l = \sum_{j=1}^{\infty} \eta_{lj} \phi_j$.

For alternatives $\hat{S}_l$ we define sequence $n_l$ such that

\[
||\eta_l||^2 \asymp \varepsilon_l^{4r} = n_l^{-2r} \asymp m_l^{-2s} C_l. \tag{7.9}
\]

Then

\[
n_l \asymp C_l^{-1/(2r)} m_l^{s/r} \asymp C_l^{-1/(2r)} m_l^{-1/(s-2r)}. \tag{7.10}
\]

Hence we have

\[
m_l \asymp C_l^{(1-2r-\omega)/r} n_l^{-2-4r-2\omega}. \tag{7.11}
\]

By A4, (7.11) implies

\[
\kappa_{2n_l m_l}^2 = o(\kappa_{n_l}^2). \tag{7.12}
\]

Using (5.1), A2 and (7.12), we get

\[
A_{\epsilon_l}(\eta_l) = n_l^2 \sum_{j=1}^{\infty} \kappa_{\epsilon_l j}^2 \eta_{lj}^2 \leq n_l^2 \kappa_{\epsilon_l m_l}^2 \sum_{j=m_l}^{\infty} \theta_{n_l j}^2 \\
\asymp n_l^{-2-4r} \kappa_{\epsilon_l m_l}^2 \asymp \varepsilon_l^{-4\omega} \kappa_{\epsilon_l m_l}^2 k_{\epsilon_l}^2 = o(\varepsilon_l^{-4\omega}). \tag{7.13}
\]

By Theorem 3.1 (7.13) implies $\varepsilon^{-2\omega} - LD$ inconsistency of sequence of alternatives $\hat{S}_l$. 

7.4 Proof of version of Theorem 5.2 for tests based on kernel estimators

We verify only ii. Suppose the contrary. Let \( S = \sum_{j=-\infty}^{\infty} \tau_j \phi_j \notin \gamma U \) for all \( \gamma > 0 \). Then there is sequence \( m_l, m_l \to \infty \) as \( l \to \infty \), such that

\[
m_l^{2s} \sum_{|j| \geq m_l} |\tau_j|^2 = C_l,
\]

where \( C_l \to \infty \) as \( l \to \infty \).

It is easy to see (see [6]) that we can define sequence \( m_l \) such that

\[
\sum_{m_l \leq |j| \leq 2m_l} |\tau_j|^2 \asymp \varepsilon_l^{4r} = n_l^{-2r} > \delta C_l m_l^{-2s},
\]

where \( \delta, 0 < \delta < 1/2, \) does not depend on \( l \).

Define sequence \( \eta_l = \{ \eta_j \}_{j=-\infty}^{\infty} \) such that \( \eta_j = \tau_j, |j| \geq m_l, \) and \( \eta_j = 0 \) otherwise.

Denote \( \tilde{S}_l(x) = \sum_{j=-\infty}^{\infty} \eta_j \exp\{2\pi i j x\} \).

For alternatives \( \tilde{S}_l(x) \) we define sequence \( n_l \) such that \( \| \tilde{S}_l(x) \| \asymp n_l^{-r} \).

Then

\[
n_l \asymp C_l^{-1/(2r)} m_l^{s/r}.
\]

We have \( |\hat{K}(\omega)| \leq \hat{K}(0) = 1 \) for all \( \omega \in R^1 \) and \( |\hat{K}(\omega)| > c > 0 \) for all \( |\omega| < b \).

Therefore, if we put \( h_l = h_{n_l} = 2^{-1} b^{-1} m_l^{-1} \), then, by (7.15), there is \( C > 0 \) such that for all \( h > 0 \), we have

\[
T_{\varepsilon_l}(\tilde{S}_l, h_l) = \sum_{|j|<m_l} |\hat{K}(jh_l) \eta_j|^2 \asymp \sum_{j=m_l}^{\infty} |\hat{K}(jh_l) \eta_j|^2 = C T_{\varepsilon_l}(\tilde{S}_l, h_l).
\]

Therefore, we can put \( h = h_l \) in further reasoning.

Using (7.15), we get

\[
T_{\varepsilon_l}(\tilde{S}_l) = \sum_{|j|>m_l} |\hat{K}(jh_l) \eta_j)|^2 \asymp \sum_{j=m_l}^{2m_l} |\eta_j|^2 \asymp n_l^{-2r}.
\]

If we put in (7.14), (7.11), \( k_l = [h_{\varepsilon_l}^{-1}] \) and \( m_l = k_l \), then we get

\[
h_{\varepsilon_l}^{-1/2} \asymp C_l^{(2r-1)(1-2r)}/n_l^{2r-1+\omega}.
\]

Using (7.16) and (7.17), we get

\[
\varepsilon_l^{-2} T_{\varepsilon_l}(\tilde{S}_l) h_{\varepsilon_l}^{1/2} \asymp C_l^{-(1-2r)/2}.
\]

By Theorem 6.2 this implies inconsistency of sequence of alternatives \( \tilde{S}_l \).
References

[1] P.J. Bickel, M. Rosenblatt, On some global measures of deviation of density function estimates.— Ann. Statist. 1 (1973), 1071-1095.

[2] L. Comminges, A. Dalalyan Minimax testing of a composite null hypothesis defined via a quadratic functional in the model of regression.— Electronic Journal of Statistics, 7 (2013), 146–190.

[3] M.S. Ermakov, Minimax detection of a signal in a Gaussian white noise— Theory Probab. Appl., 35 667-679 (1990).

[4] M.S. Ermakov, Testing nonparametric hypothesis for small type I and type II error probabilities.— Problems of Information Transmission. 44 (2008), 54-73.

[5] M.S. Ermakov, Nonparametric signal detection with small type I and type II error probabilities. Statistical Inference for Stochastic Processes 14 (2011), 1-19.

[6] M.S. Ermakov, On uniform consistency of nonparametric tests. I.—J Math Sci 258 (2021), 802-837.

[7] M.S. Ermakov, On uniform consistency of nonparametric tests. II. arXiv.org 2004.07039 (2022), 1-21.

[8] J. Horowitz, V. Spokoiny, An adaptive, rate-optimal test of a e parametric mean-regression model against a nonparametric alternative.— Econometrica, 69 (2001), 599-631.

[9] I.A. Ibragimov, R.Z. Has’minskii, (1977) On the estimation of an infinite-dimensional parameter in Gaussian white noise.— Soviet Mathematics. Doklady, 18 (1977), 1307-1309.

[10] I.A. Ibragimov, R.Z. Has’minskii, Statistical estimation: Asymptotic theory, Springer, N.Y. (1981).

[11] Yu.I. Ingster, I.A. Suslina, Nonparametric goodness-of-fit testing under gaussian models. Lecture Notes in Statistics 169 Springer: N.Y. (2002).

[12] G. Kerkyacharian, D. Picard, Density estimation by kernel and wavelets methods: optimality of Besov spaces.— Statist. Probab. Lett. 18 (1993), 327 - 336.

[13] J. Neyman, Smooth test for goodness of fit. — Skand. Aktuarietidsskr., 1–2 (1937), 149–199.

[14] L. V. Osipov, On probabilities of large deviations for sums of independent random variables.— Theory Probab. Appl., 17 (1973), 309–331.
[15] V. V. Petrov, *On the probabilities of large deviations for sums of independent random variables.*— Theory Probab. Appl., **10** (1965), 287–298.

[16] V. Rivoirard, *Maxisets for linear procedures.*— Statist. Probab. Lett. **67** (2004) 267-275.