Critical Percolation of Free Product of Groups

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Abstract

In this article we study percolation on the Cayley graph of a free product of groups.

The critical probability $p_c$ of a free product $G_1 * G_2 * \cdots * G_n$ of groups is found as a solution of an equation involving only the expected subcritical cluster size of factor groups $G_1, G_2, \ldots, G_n$. For finite groups these equations are polynomial and can be explicitly written down. The expected subcritical cluster size of the free product is also found in terms of the subcritical cluster sizes of the factors. In particular, we prove that $p_c$ for the Cayley graph of the modular group $\text{PSL}_2(\mathbb{Z})$ (with the standard generators) is the unique root of the polynomial $2p^5 - 6p^4 + 2p^3 + 4p^2 - 1$ in the interval $(0,1)$.

In the case when groups $G_i$ can be “well approximated” by a sequence of quotient groups, we show that the critical probabilities of the free product of these approximations converge to the critical probability of $G_1 * G_2 * \cdots * G_n$ and the speed of convergence is exponential. Thus for residually finite groups, for example, one can restrict oneself to the case when each free factor is finite.

We show that the critical point, introduced by Schonmann, $p_{\exp}$ of the free product is just the minimum of $p_{\exp}$ for the factors.

1 Introduction

1.1 Percolation

We will use the notation $\mathcal{G} = (V, E)$ for a graph with the vertex set $V$ and the edge set $E$. A graph $\mathcal{G}$ is said to be locally finite if each vertex has finitely many neighbors, and transitive if for any two vertices $u, v$ in $V$ there is an automorphism of $\mathcal{G}$ mapping $u$ to $v$.

An edge of the graph is called a bond. A Bernoulli bond percolation on $\mathcal{G}$ is a product probability measure $P_p$ on the space $\Omega = \{0, 1\}^E$, the subsets of the edge set $E$. For any realization $\omega \in \Omega$, the bond $e \in E$ is said open if $\omega(e) = 1$ and closed otherwise. For $0 \leq p \leq 1$ the product measure is defined via $P_p(\omega(e) = 1) = p$ for all $e \in E$. Thus each bond is open with probability $p$ independently of all other bonds. We write $E_p$ for the expected value with respect to $P_p$.

For any realization $\omega$, open edges form a random subgraph of $\mathcal{G}$. An (open) cluster is a connected component of such subgraph $\omega$. An open cluster containing the origin is denoted by $C$ and the number of vertices in $C$ by $|C|$. Percolation function is defined to be the probability that the origin is contained in an infinite cluster, i.e. $\theta(p) = P_p(|C| = \infty)$.

Depending on the parameter $p$, every subgraph $\omega$ has either no infinite cluster, or infinitely many infinite clusters (non-uniqueness phase), or only one infinite cluster (uniqueness phase) $P_p$-almost surely. Häggström and Peres [14] have shown that for transitive graphs there are two phase transition values of $p$: $p_c$ and $p_u$, such that for $0 \leq p < p_c$ all clusters are finite, non-uniqueness phase occurs for $p_c < p < p_u$ and if $p_u < p \leq 1$ there is unique infinite cluster $P_p$-a.s. The critical probability is then equivalently defined by

$$p_c = \inf\{p : \theta(p) > 0\}.$$
There is another critical value of $p$ which can be defined based on the probability of open path between two vertices.

$$p_{\text{exp}} = \sup \{ p : \exists c, \gamma > 0 \forall x, y \in V P_p(x \leftrightarrow y) \leq Ce^{-\gamma \text{dist}(x,y)} \}$$

This critical point lies between $p_c$ and $p_u$ as pointed out by Schonmann [21].

As $p$ approaches $p_c$, the behavior of the percolation function and mean cluster size is studied. Assume $\theta(p)$ is continuous at $p_c$ and that

$$\theta(p) \approx (p - p_c)^{\beta} \quad \text{as } p \searrow p_c,$$

$$E_p(|C|) \approx (p_c - p)^{\gamma} \quad \text{as } p \nearrow p_c.$$

Then we say that $\beta$ and $\gamma$ are critical exponents.

The Cayley graph of a group $G$ with respect to the finite set of generators $S$ is the graph $G$ with vertices $V = G$ and $\{ g, h \} \in E$ iff $g^{-1}h \in S$ (with the appropriate multiplicity). This graph is always locally finite and transitive.

Percolation characteristics of a Cayley graph ($p_c, \beta, \gamma$, etc.) of the group are important invariants of the Cayley graph and the group related to the spectral radius, $l_2$-Betti numbers, the Cheeger constant, amenability, etc.

For example if a group is amenable then the non-uniqueness phase is empty, i.e. $p_c = p_u$. On the other hand Pak and Smirnova-Nagnibeda [19] showed that if $G$ is non-amenable then there is a generating set $S$ of $G$ such that the percolation on a Cayley graph of $G$ with respect to $S$ has nontrivial non-uniqueness phase. The general problem whether this is true for all Cayley graphs of non-amenable groups is still open.

Recall that the Cheeger constant of a graph $G(V, E)$ is defined by

$$h(G) = \inf_{K} \frac{\partial K}{|K|},$$

where $K$ is a finite subset of $V$ and $\partial K$, the boundary of $K$, contains all edges in $E$ with exactly one endpoint in $K$.

There are several general inequalities involving $p_c$. The critical probability $p_c$ of a quotient graph does not exceed the $p_c$ of the original graph (see Campanino [5]). In particular, the $p_c$ of a Cayley graph of any factor group of $G$ is at most the $p_c$ of a Cayley graph of $G$ itself (with respect to the corresponding generating sets).

It is easy to show (using the expected cluster size) that

$$p_c \geq \frac{1}{2|S| - 1}, \quad (1)$$

where the equality holds for free groups (i.e. when the Cayley graph is a tree). On the other hand Benjamini and Schramm [4] proved that

$$p_c \leq \frac{1}{h(G) + 1}, \quad (2)$$

and again the equality holds for free groups.

Gaboriau [9] related harmonic Dirichlet functions on a graph to those on the infinite clusters in the uniqueness phase. He also proved that the first $\ell^2$-Betti number of a group does not exceed $\frac{1}{2}(p_u - p_c)$.

Note also that random subgraphs of the Cayley graph are crucial in the study of generic properties of a group and the average case complexity of the word problem [13].

Probabilistic properties of Cayley graphs (say, properties of the random walks) have been extensively studied [24], but properties of percolation initiated by Benjamini and Schramm [4] have not been studied as much mostly because it is usually difficult to find the explicit values of the percolation characteristics even for relatively simple graphs.
Explicit values of \( p_c \) are known only for some special cases. For example, for lattices in \( \mathbb{R}^2 \) the value of \( p_c \) is obtained using dual graphs (for \( \mathbb{R}^d \) with \( d \geq 3 \) the values of \( p_c \) are not known). For square lattice, Kesten [16] proved \( p_c = 1/2 \), for triangular lattice \( p_c = 2\sin(\pi/18) \), and for hexagonal lattice \( p_c = 1 - 2\sin(\pi/18) \) (see Grimmett [10]). Ziff and Scullard [25] recently found \( p_c \) for a larger class of lattices in \( \mathbb{R}^2 \) (they considered graphs which can be decomposed onto certain self-dual arrangement). Grimmett and Newmann [11] studied the percolation on a direct product of regular tree with \( \mathbb{Z} \), they discuss how \( p_c \) and \( p_u \) changes with the degree of the tree. Lyons studied percolation on arbitrary trees [17].

Note that many of the graphs from the previous paragraph are Cayley graphs of groups: the square lattice in \( \mathbb{R}^2 \), the triangular lattice in \( \mathbb{R}^2 \), some trees, and the direct product of a tree and \( \mathbb{Z} \).

The critical exponents are known for some lattices in \( \mathbb{R}^2 \), for \( \mathbb{R}^d \), \( d \geq 19 \), trees and more generally Cayley graphs with infinitely many ends. For last three examples \( \beta = 1, \gamma = -1 \).

In this article we will focus on the Cayley graphs of free products of groups \( G_1 \ast G_2 \ast \cdots \ast G_n \). Some probabilistic properties of such graphs have been studied before. For example Mairesse and Mathéus [18] considered the transient nearest-neighbor random walk on the Cayley graph of free product of finite groups. We obtain, among other results, explicit formulas for \( p_c \) (as solutions of some equations), for the Schonmann’s critical point \( p_{\text{exp}} \), and for the right derivative of the percolation function at the critical point (that gives more information than the critical exponent). In the case of a free product of finite groups (say, \( \text{PSL}(2, \mathbb{Z}) \)), \( p_c \) is obtained as a root of an explicitly written polynomial (in particular, \( p_c \) is an algebraic number).

### 1.2 Free products: critical probability

The Cayley graph of a free product of groups has a tree-graded structure [6]: it is a union of subgraphs \( M_j, j \in \mathbb{N} \), each \( M_j \) is a copy of the Cayley graph of one of the group \( G_i \), different \( M_j \) and \( M_k \) intersect by at most one point, and every simple loop in the Cayley graph is in one of the \( M_i \). Note that the results of this paper can be generalized to arbitrary transitive locally finite tree-graded graphs.

A non-trivial free product of groups (except \( C_2 \ast C_2 \)) has infinitely many ends. This implies \( p_u = 1 \) (as noticed for example by Lyons [17]). But the critical probabilities \( p_c \) and \( p_{\text{exp}} \) were not known even in the case of the modular group \( \text{PSL}(2, \mathbb{Z}) \) which is the free product of cyclic groups of orders 2 and 3.

We start with the following result giving an equation for \( p_c \) in the case of free product of two groups.

**Theorem 1.** Let \( G_1 = \langle S_1 \rangle \), \( G_2 = \langle S_2 \rangle \) be two finitely generated groups. Consider the Cayley graph of the free product \( G_1 \ast G_2 \) with respect to the generating set \( S_1 \cup S_2 \).

Then the critical probability \( 0 < p_c \leq 1 \) is the unique solution of the following equation

\[
(\chi_1(p) - 1)(\chi_2(p) - 1) = 1,
\]

where \( \chi_i(p) = E_p(|C|_{G_i}) \) denotes the expected size of the cluster containing the origin in the Cayley graph of group \( G_i \) with respect to the generating set \( S_i \).

The critical probability \( p_c \) of the free product is 1 if and only if \( |G_1| = |G_2| = 2 \) (in that case the group \( G_1 \ast G_2 \) is virtually cyclic).

The next theorem gives the expected size of a cluster for subcritical \( p \) in the Cayley graph of a free product of two groups.

**Theorem 2.** Let \( G_1 \ast G_2 \) be as in Theorem 1. Then for \( p < p_c \), the mean cluster size satisfies

\[
E_p(|C|_{G_1 \ast G_2}) = \frac{\chi_1(p)\chi_2(p)}{\chi_1(p) + \chi_2(p) - \chi_1(p)\chi_2(p)}.
\]
Note that by Theorem 1, the denominator in formula (4) is equal to 1 if \( p = 0 \) and is decreasing to 0 as \( p \to p_c(G_1 \ast G_2) \).

The next two corollaries generalize the above theorems to the free product of an arbitrary number of groups. They follow by induction.

**Corollary 3.** Let \( \mathcal{G} \) be a Cayley graph of the free product of \( n \) non-trivial finitely generated groups \( G_i, i = 1, \ldots, n \), with respect to the set of generators \( \bigcup_{i=1}^n S_i \), where \( S_i \) is a generating set of \( G_i \).

The expected size of the cluster at the origin is equal to

\[
E_p(|C|_{G_1 \ast \cdots \ast G_n}) = \frac{\prod_{i=1}^n \chi_i(p)}{\sum_{j=1}^n \prod_{i=1, i \neq j}^n \chi_i(p) - (n-1) \prod_{i=1}^n \chi_i(p)}
\]

for \( p < p_c \) and it is infinity for \( p \geq p_c \).

**Corollary 4.** Let \( \mathcal{G} \) be a Cayley graph of the free product of \( n \) non-trivial finitely generated groups \( G_i, i = 1, \ldots, n \), with respect to the set of generators \( \bigcup_{i=1}^n S_i \), where \( S_i \) is a generating set of \( G_i \). Then the critical probability \( 0 < p_c \leq 1 \) of \( \mathcal{G} \) is the unique solution of the following equation

\[
\sum_{j=1}^n \prod_{i=1, i \neq j}^n \chi_i(p) - (n-1) \prod_{i=1}^n \chi_i(p) = 0,
\]

where \( \chi_i(p) = E_p(|C|_{G_i}) \) denotes the expected size of the component containing origin in the Cayley graph of \( G_i \) with respect to the set of generators \( S_i \).

We give two proofs of Theorem 1. The first proof is direct, and the second one uses the theory of branching processes [20]. Theorem 2 and Corollaries 3, 4 also can be obtained as applications of the theory of branching processes.

The next theorem shows, in particular, that in the case of free products of residually finite groups, the critical probability can be obtained as a limit of a fast converging sequence of algebraic numbers (critical probabilities of free products of finite groups).

Suppose that a finitely generated group \( G = \langle S \rangle \) has surjective homomorphisms \( \phi_i: G \to F_i \) such that \( \phi_i \) is injective on a ball of radius \( i \) in the Cayley graph of \( G \) (i.e. on the set of all products of elements from \( S \cup S^{-1} \) of length at most \( i \)). Then we shall say that \( G \) is well approximated by \( F_i \). By the Cayley graph of \( F_i \) we shall always mean the Cayley graph with respect to \( \phi_i(S) \).

**Theorem 5.** Suppose that each of the (non-trivial) finitely generated group \( G_i = \langle S_i \rangle \) is well approximated by groups \( H_i^1, i = 1, 2 \). Then

\[
p_c(G_1 \ast G_2) = \lim_{j \to \infty} p_c(H_i^1 \ast H_i^2).
\]

More precisely there exist \( C, \gamma > 0 \) such that

\[
0 \leq p_c(H_i^1 \ast H_i^2) - p_c(G_1 \ast G_2) \leq Ce^{-\gamma j}.
\]

A similar result holds (by induction) for a free product of any finite number of groups.

Finally note that although inequalities (1) and (2) become equalities for the free group (with free generators) [4], already for the free products of finite cyclic groups both inequalities become strict (see Proposition 13). Thus these inequalities only give rough estimates for \( p_c \).

The next proposition gives the Schonmann’s critical point for free products.

**Proposition 6.** Consider the free product \( G_1 \ast \cdots \ast G_n \) of \( n \) finitely generated groups. Then

\[
p_{\exp}(G_1 \ast \cdots \ast G_n) = \min_{1 \leq i \leq n} \{ p_{\exp}(G_i) \},
\]

where for finite \( G_i \) we define \( p_{\exp}(G_i) = 1 \).
In particular for the free product of finite groups $p_{\exp}$ is equal to one.

**Corollary 7.** Consider the free product $G_1 \ast \cdots \ast G_n$ of $n$ nontrivial finitely generated groups, which is not virtually Z. Then

$$p_c(G_1 \ast \cdots \ast G_n) < \min_{1 \leq i \leq n} \{p_c(G_i)\} \leq \min_{1 \leq i \leq n} \{p_{\exp}(G_i)\} = p_{\exp}(G_1 \ast \cdots \ast G_n) \leq p_u(G_1 \ast \cdots \ast G_n) = 1$$

The first inequality will be shown to be strict for any free product, which is not virtually Z. If it is virtually cyclic, i.e. $C_2 \ast C_2$, all mentioned critical values are equal to one.

### 1.3 Free products: critical exponents

Häggström and Peres [14] proved that the function $\theta(p)$ is continuous for $p > p_c$ for any Cayley graph. A free product of nontrivial groups (which is not virtually cyclic) is non-amenable, so the percolation dies at $p_c$ (for the proof see [3]) and therefore the percolation function $\theta(p)$ is continuous also at $p_c$. This allows us to consider the critical exponents.

We introduced two critical exponents $\beta$ and $\gamma$, but there are several others describing the behavior near the critical point $p_c$. Physicists believe that the numerical values of critical exponents depend only on the underlying space and not on the structure of the particular lattice.

Values of critical exponents are well known for trees ($\beta = 1, \gamma = -1$), so called mean-field values. Hara and Slade [15] proved that the critical exponents take their mean-field values in $\mathbb{Z}^d$ for $d > 19$ by verifying the triangle condition introduced by Aizenman and Newman [2]. Schonmann [22] proved that the critical exponents take their mean-field values for all non-amenable planar graphs with one end, and for unimodular graphs with infinitely many ends. The later case covers Cayley graphs of free products.

The equation for percolation function found in the proof of Theorem 1 allows us to evaluate the right derivative of the percolation function using implicit differentiation. One can also compute the left derivative of the expected cluster size function of a free product at $p_c$. The formulas for these derivatives immediately imply Schonmann’s result that the values of critical exponents of a free product are mean-field. In particular the formula for the derivative of the cluster size of a free product is the following. It involves the derivative of cluster sizes in the factor groups:

$$\frac{d}{d-\rho}(\mathbb{E}_{p_c}(\mathcal{C}|G_1 \ast G_2))^{-1} = \left. \frac{d}{d-\rho} \chi_1(p) \right|_{p=p_c} (1 - \chi_2(p_c)) + \left. \frac{d}{d-\rho} \chi_2(p) \right|_{p=p_c} (1 - \chi_1(p_c)) \chi_1(p_c) \chi_2(p_c)$$

Schonmann [21] also proved the mean-field criticality for highly non-amenable graphs, i.e. such that $h(G) > D(G)/\sqrt{2}$, where $D(G)$ is the maximal degree of vertices in $G$ (in case of a Cayley graph it is just the degree of the origin $deg(o)$). The question for general non-amenable graph remains open. Define the spectral radius of $G$ by

$$R(G) = \limsup_{n \to \infty} (\# \text{ of closed walks of length } n \text{ at } o)^{1/n}.$$ 

Schonmann used in his argument that if $p_c < 1/R(G)$ then the triangle condition is satisfied and thus the critical exponents take their mean-field values. For any non-amenable group, Pak and Smirnova-Nagnibeda construct a finite generating set with the property $p_c R(G) < 1$. Combining these two results we get the following statement.

**Proposition 8.** Every finitely generated non-amenable group has a finite generating set such that the Cayley graph with respect to this generating set has mean-field valued critical exponents.

Sapir conjectured that the mean-field criticality should be true for a class of hyperbolic groups which includes free products (see Conjecture 1 below). Recall that a group $G$ is called Gromov-hyperbolic if
for some $\delta > 0$, in every geodesic triangle of the Cayley graph, one side is in the $\delta$-neighborhood of the union of two other sides. A group is called elementary if it contains a cyclic subgroup of finite index. For example, free products of finite groups are hyperbolic and if the groups are of order $> 2$, the free products are not elementary.

A well known conjecture in percolation theory claims that the critical exponents of all lattices in $\mathbb{R}^2$ are the same ($\beta = \frac{5}{36}$, $\gamma = \frac{-43}{18}$) as proved for triangular lattices by Smirnov and Werner [23]). The motivation for this conjecture is that for every lattice $L$ in $\mathbb{R}^2$ the Gromov-Hausdorff limit of rescaled copies of $L$, $L/2$, $L/3$, ... is isometric to $\mathbb{R}^2$ with the $L_1$-metric, i.e. any two lattices in $\mathbb{R}^2$ have the same asymptotic cones ([12]). Sapir conjectured that this is true in general: if two groups have isometric asymptotic cones then their critical exponents should coincide. In particular, since all asymptotic cones of all non-elementary Gromov-hyperbolic groups are isometric (they are isometric to the universal $\mathbb{R}$-tree of degree continuum by a result of Dyubina and Polterovich [7]), the following statement should follow:

**Conjecture 1** (M. Sapir). *If the asymptotic cones of Cayley graphs of two groups $G, G'$ are isometric, then their critical exponents are the same.*

In particular every Cayley graph of a non-elementary hyperbolic group has mean-field valued critical exponents.

Note that Benjamini and Schramm [4] asked the question whether all Cayley graphs of non-amenable groups have mean-field criticality. The positive answer to this question would of course imply the second part of Conjecture 1.

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## 2 Critical probability $p_c$

### 2.1 A recursive expression for percolation function

First let us describe the structure of a Cayley graph $G$ of the free product of two groups. Every vertex $v$ is contained in exactly two basic subgraphs induced by vertices: the basic subgraph of the first type, $G_1$, is an induced subgraph of $G$ consisting of vertices obtained from $v$ by multiplying on the right by elements of $G_1$, and the basic subgraph of type two, $G_2$, is defined similarly for $G_2$. Each of the basic subgraphs of type one (type two) is isomorphic to the Cayley graph of $G_1$ (resp. $G_2$). Every vertex of the Cayley graph of $G$ is the (unique) common vertex of a subgraph of type 1 and a subgraph of type 2. In the case of finite cyclic groups with single generator, $G_i$'s are cycles.

Since the Cayley graph of a finitely generated group is locally finite, the origin is in an infinite cluster if and only if there exists an infinite open simple path starting from the origin. Note that if $G_1$ and $G_2$ are finite, any infinite simple path in the Cayley graph of $G$ has to intersect infinitely many basic subgraphs and if it leaves a subgraph, it can never return to it. In the general case, any simple infinite path starting at the vertex $v$ is of one of the following two types: type one has first edge in the basic subgraph $G_1$ (and uses no edges of the basic subgraph $G_2$ which contains $v$); type two starts with an edge from basic subgraph $G_2$.

Let $A$ be the probability that there is an infinite path of type one starting at the origin $o$ and let $B$ be defined similarly for the type two paths. These two events are independent. Moreover we can recursively express $A$ using $B$ and vice versa as follows.

$$1 - A = \sum_{\omega \text{ subgraph of } G_1} P_p(\omega)(1 - B)^{|C|_{G_1}},$$

where $P_p(\omega)$ is the probability of $\omega = (V(\omega), E(\omega))$ to be the open subgraph of $G_1 = (V(G_1), E(G_1))$. The left side is the probability that there is no infinite path (starting at the origin) of type one. The
right side is obtained by conditioning on at which vertex the infinite path could leave \( G_1 \). Specifically for a fixed realization \( \omega \) we compute probability that there is no infinite path of type two starting at a vertex of the connected component of the origin in \( G_1 \). Probability of the existence of an infinite path of type two starting at any specific vertex is equal to \( B \) because Cayley graphs are homogeneous (and so it does not matter which vertex we take as the origin). Therefore the right side is equal to the probability that there is no simple infinite path starting at the origin and leaving \( G_1 \) at any vertex (different from the origin) connected to the origin.

Note that for finite groups

\[ P_p(\omega) = p^{\vert E(\omega)\vert}(1-p)^{\vert E(G_1)\vert - \vert E(\omega)\vert}, \]

and \( \vert C\vert_{G_1} \) is the number of vertices in the connected component of \( \omega \) containing \( o \) in \( G_1 \).

For infinite \( G_1 \) the summation should be replaced by integration, or we shall use short expression using the expectation in the probability space restricted to \( G_1 \):  
\[ 1 - A = E_p(1 - B)^{\vert C\vert_{G_1} - 1}. \]

One can modify the equation (7) by summing only over all connected subgraphs of \( G_i \) containing \( o \):

\[ 1 - A = \sum_{K \text{ finite connected subgraph of } G_1 \text{ containing } o} P'_p(K)(1 - B)^{\vert C\vert_{G_1} - 1}. \]

Here \( P'_p(K) = p^{\vert E(K)\vert}(1 - B)^{\vert \partial K\vert} \) where \( \vert \partial K\vert \) denotes the size of the (external) boundary of \( K \) in \( G_1 \), i.e. the number of edges of \( G_1 \) not in \( K \) with at least one end vertex (maybe both) in \( K \).

Similar equalities hold for \( G_2 \). Let us now formally rewrite the summation to be over the size of the connected component containing \( o \) of a random subgraph \( \omega \) of \( G_i \). Denote by \( Q_i(n) \) the probability that this component is of size \( n \), i.e. \( Q_i(n) = P_p(\vert C\vert_{G_i} = n) \). Define a recurrent walk through function \( g_i, \ i = 1, 2 \) for \( 0 \leq p, t \leq 1 \) by:

\[ g_i(p, t) = 1 - \sum_{n=1}^{\vert G_i\vert} (1 - t)^{n-1} Q_i(n) \]  \hspace{1cm} (8)

Notice that \( g_i \) is very close to the moment generating function of the cluster size in \( G_i \) (see for example [8]). This will become handy when we take the derivatives at \( t = 0 \) in Proposition 10.

If \( G_i \) is an infinite group then we sum over infinitely many (non-negative) values, but the sum is always bounded by 1 from above (since \( \sum_{n=1}^{\infty} Q_i(n) \leq 1 \)). On the other hand \( g_i \) is always bigger than the probability of having an infinite cluster at the origin just in \( G_i \) (i.e. \( \theta_{G_i}(p) \)).

In the case when \( G_i \) is a cyclic group of finite order \( m \), and so \( G_i \) is a finite cycle, connected subgraphs containing \( o \) are just arcs. Probability of a specific arc of length \( n < m \) is \( (1 - p)^2 p^{n-1} \) and the number of those containing origin is just \( n \). Therefore

\[ Q_i(n) = n(1-p)^2 p^{n-1} \text{ for } n < m, \]
\[ Q_i(m) = (m(1-p) + p)p^{m-1}. \]

It simplifies the summation as follows.

\[ g_i(p, t) = 1 - \sum_{j=1}^{m-1} (j(1-p)^2(p(1-t))^{j-1}) + (m(1-p) + p)(p(1-t))^{m-1} \]  \hspace{1cm} (9)

The percolation function is given by

\[ \theta = A + B - AB, \]
where
\[
A = g_1(p, B), \\
B = g_2(p, g_1(p, B)).
\] (10)

It remains to determine for which values of $p$ the last equation in (10) has a positive solution.

### 2.2 The critical probability

We start with describing general properties of the walk-through function $g_i$. In particular we will show that the function $g_0(B) = g_2(p, g_1(p, B)) - B$ is concave and equal to 0 for $B = 0$. This allows us to decide whether the equation (10) has a positive solution based on the derivative of $g_p(B)$ at zero.

We will use the following result of Aizenman and Barsky [1]. In fact they proved it for $\mathbb{Z}^d$ but their argument works for any transitive graph as noticed by Lyons and Peres [17].

**Lemma 9** (Aizenman, Barsky). If $p < p_c$ then the mean cluster size is finite in any transitive graph.

**Proposition 10** (Properties of $g_i$). Let $g_i, i = 1, 2$ be defined as above for arbitrary finitely generated groups $G_i$ (with some choice of generators). Assuming $0 < t < 1, 0 < p < 1$, we have:

1. $0 \leq g_i(p, t) \leq 1$.
2. $\frac{\partial^k g_i(p, t)}{\partial t^k} \bigg|_{t=0} = (-1)^{k+1} E_p[|C|_{G_i} - 1]|C|_{G_i} - 2)\ldots(|C|_{G_i} - k)]$ for $k = 1, 2, \ldots$ and $p < p_c(G_i)$.
3. $g_2(p, g_1(p, t)) - t$ is concave in $t$ for $p < \min(p_c(G_1), p_c(G_2))$.

**Proof.** Using the definition of $g_i$ given by (8) we may evaluate it for $t = 0$ and $t = 1$. Recall that $Q_i(n)$ is the probability that the component containing origin in $G_i$ has size $n$,

\[
g_i(p, 0) = 1 - \sum_{n=1}^{[G_i]} Q_i(n) = Q_i(\infty) \geq 0
\]

\[
g_i(p, 1) = 1 - Q_i(1) = 1 - (1 - p)^{D(G_i)} \leq 1
\]

where $D(G_i)$ is the degree of the origin (and any other vertex) in the graph $G_i$. Now let us take the formal derivative.

\[
\frac{\partial g_i(p, t)}{\partial t} = \sum_{n=1}^{[G_i]} (n - 1)(1 - t)^{n-2}Q_i(n) \geq 0
\]

\[
\frac{\partial g_i(p, t)}{\partial t} \leq \frac{\partial g_i(p, t)}{\partial t} \bigg|_{t=0} = \sum_{n=1}^{[G_i]} (n - 1)Q_i(n) = \chi_i(p) - 1,
\] (11)

Note that $\chi_i(p) = [G_i]$ and for $p < p_c(G_i)$, $\chi_i(p) < \infty$ by Lemma 9. This in particular implies the continuity and differentiability of $g_i$ and $\chi_i(p)$ for $p < p_c$.

From above we can conclude that $0 \leq g_i \leq 1$ as expected from the fact that for particular values of $t$, $g_i$ gives the probability of an infinite path in a part of the Cayley graph as defined above.

Now we compute the $n$-th derivative (term by term) and evaluate it at $t = 0$.

\[
\frac{\partial^k g_i(p, t)}{\partial t^k} = (-1)^{k+1} \sum_{n=k+1}^{[G_i]} (n - 1)(n - 2)\ldots(n - k)(1 - t)^{n-k-1}Q_i(n)
\] (12)
If \( p < p_c(G_i) \) then \( Q_i(\infty) = 0 \) and we have (by the definition of the expectation)

\[
\left. \frac{\partial^k g_i(p, t)}{\partial t^k} \right|_{t=0} = (-1)^{k+1} E_p[|C|G_i - 1]|C|G_i - 2| \ldots |C|G_i - k] \]

This proves part ii.

Using the above formula for the derivative of \( g_i \) we see that the odd numbered derivatives are positive and the even numbered derivatives are negative. Thus in particular \( g_i \) is increasing and concave in \( t \) for all \( p \).

As soon as \( |G_i| > 2 \), \( g_i \) is strictly concave and \( \chi_i(p) \) is increasing. Since the composition of two concave functions is concave, \( g_2(p, g_1(p, t)) - t \) is concave in \( t \) for \( p < \min(p_c(G_1), p_c(G_2)) \), and if \((|G_1| - 1)(|G_2| - 1) > 1 \) then it is strictly concave.

Now we are ready to show for which \( p \) the percolation function is positive, that is to find \( p_c \) of free product of two groups.

**Proof of Theorem 1.** Clearly \( p_c(G_1 \ast G_2) \leq \min(p_c(G_1), p_c(G_2)) \) so we need to decide for which \( p < \min(p_c(G_1), p_c(G_2)) \) the equation \( g_2(p, g_1(p, t)) - t = 0 \) has a positive solution.

In what follows we always assume \( p < \min(p_c(G_1), p_c(G_2)) \) and set \( q_p(t) = g_2(p, g_1(p, t)) - t. \)

Since \( g_1(p, 0) = Q_i(\infty) = 0 \) we have \( q_p(0) = g_2(p, g_1(p, 0)) = 0 \). On the other hand \( g_2(p, g_1(p, 1)) \leq 1 \) and thus \( q_p(1) \leq 0 \). By Proposition 10 iii. \( q_p(t) \) is concave in \( t \). Therefore there is at most one change in the monotonicity of \( q_p(t) \) on the unit interval, in particular the function is either decreasing all the time or there is \( t_0 \) such that \( q_p(t) \) is increasing the interval \((0, t_0)\) and decreasing on \((t_0, 1)\), see Picture 1. Thus the equation \( q_p(t) = 0 \) has a positive solution \((0 \text{ is always a solution})\) if and only if the function \( q_p(t) \) has positive derivative at \( t = 0 \).

Now using Proposition 10 ii. we have:

\[
\left. \frac{q_p(t)}{\partial t} \right|_{t=0} = \left. \frac{\partial g_2(p, g_1(p, t)) - t}{\partial t} \right|_{t=0} = \left. \frac{\partial g_2(p, t)}{\partial t} \right|_{t=g_1(p,0)=0} \left. \frac{\partial g_1(p, t)}{\partial t} \right|_{t=0} - 1 = (\chi_2(p) - 1)(\chi_1(p) - 1) - 1
\]
where $\chi_i(p)$ is the expected size of the component of $G_i$ containing the origin. It is an increasing function of $p$.

Therefore the value of derivative of $\varphi_p(t)$ at $t = 0$ is increasing function of $p$. For $p = 0$ it equals $-1$ and for $p_1 = \min(p_c(G_1), p_c(G_2))$ we have

$$\frac{\varphi_{p_1}(t)}{\partial t} \bigg|_{t=0} = (|G_1| - 1)(|G_2| - 1) - 1 \geq 0$$

Therefore there exists unique $0 < p_0 \leq \min(p_c(G_1), p_c(G_2))$ such that

$$\frac{\varphi_{p_0}(t)}{\partial t} \bigg|_{t=0} = 0.$$

or equivalently

$$(\chi_1(p_0) - 1)(\chi_2(p_0) - 1) = 1.$$  

The case $C_2 \ast C_2$ (the group is virtually $\mathbb{Z}$) is the only one when $p_0 = 1$ and its percolation function $\theta(p) = 0$ for $0 \leq p < 1$, and $\theta(1) = 1$. In all other cases we conclude that the equation $\varphi_p(t) = 0$ has unique nonzero solutions if and only if $p > p_0$. So $p_0$ is the critical probability $p_c(G_1 \ast G_2)$ of the Cayley graph of free product $G_1 \ast G_2$. Note that if $(|G_1| - 1)(|G_2| - 1) \geq 1$ (i.e. both groups are nontrivial and one of them has more than 2 elements) then $p_c(G_1 \ast G_2) < \min p_c(G_1)$.

This finishes the proof of Theorem 1.

2.3 The expected cluster size

The expected cluster size function can be expressed recurrently in a similar way as the percolation function $\theta(p)$. In this section, we consider the free product of arbitrary number of groups.

**Proposition 11.** Let $G_i = \langle S_i \rangle$ be a finitely generated group, $i = 1 \ldots n$. Denote by $\chi_i(p)$ the expected size of a cluster at the origin in the Cayley graph of $G_i$ with respect to the generating set $S_i$. Then the expected size of the cluster at the origin in a Cayley graph of the free product $G_1 \ast \cdots \ast G_n$ with respect to the generating set $S_1 \cup \cdots \cup S_n$ satisfies

$$E_p(|C|_{G_1 \ast \cdots \ast G_n}) = \frac{\prod_{i=1}^n \chi_i(p)}{\sum_{j=1}^n \prod_{i=1, i \neq j}^n \chi_i(p) - (n - 1) \prod_{i=1}^n \chi_i(p)}$$  

for $p < p_c(G_1 \ast \cdots \ast G_n)$.

**Proof.** First assume $n=2$. If we remove the origin from the Cayley graph it splits into two parts: part $P_1$ consists of vertices $v$ such that any simple path from the origin to $v$ first visits a vertex corresponding to an element of $G_1$ (in fact $S_1$), part $P_2$ is defined similarly for $G_2$. Denote $|C|_{P_1} = |C \cap P_1|_{G_1 \ast G_2}$, size of the cluster containing the origin intersected with the part $P_1$. Now we can write

$$E_p(|C|_{G_1 \ast G_2}) = E_p(|C|_{P_1}) + E_p(|C|_{P_2}) + 1,$$

where the constant 1 represents the origin. One can represent $|C|_{P_1}$ as a random sum of random variables:

$$|C|_{P_1} = \sum_{o \neq x \in C \cap G_1} Y_x,$$

where each $Y_x$ has the same distribution as $|C|_{P_2} + 1$. Using Wald’s identity (see for example [8]) we have:

$$E_p(|C|_{P_1}) = (\chi_1(p) - 1)(E_p(|C|_{P_2}) + 1)$$

$$= \chi_1(p) - 1 + (\chi_1(p) - 1)(\chi_2(p) - 1)(E_p(|C|_{P_1}) + 1)$$

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A similar equality holds for $E_p(|C|_{p_2})$. Combining into one equation and solving for $E_p(|C|_{G_1 \ast G_2})$ we obtain:

$$E_p(|C|_{G_1 \ast G_2}) = \chi_1(p) + \chi_2(p) - 1 + (\chi_1(p) - 1)(\chi_2(p) - 1)(E_p(|C|_{G_1 \ast G_2}) + 1)$$

$$= \frac{\chi_1(p)\chi_2(p)}{\chi_1(p) + \chi_2(p) - \chi_1(p)\chi_2(p)}$$

(14)

Note that the formula in the denominator coincides with the one in the equation (3) after rearrangement. Therefore it is positive for $p < p_c$ and equal to 0 at $p_c$. Thus the expected cluster size is given by formula (14) for $p < p_c$ and it tends to infinity as $p$ approaches $p_c$ (the expected size of the cluster is equal to infinity for $p \geq p_c$).

Suppose now that the result holds for any free product of at most $n$ groups. Consider $G_1 \ast \cdots \ast G_{n+1}$. We can view it as a free product of two groups $G_1 \ast \cdots \ast G_n$ and $G_{n+1}$. By induction we know the expected size of the component in both of those groups and we can apply formula (14). Therefore

$$E_p(|C|_{G_1 \ast \cdots \ast G_{n+1}}) = \frac{E_p(|C|_{G_1 \ast \cdots \ast G_n})\chi_{n+1}(p)}{E_p(|C|_{G_1 \ast \cdots \ast G_n}) + \chi_{n+1}(p) - E_p(|C|_{G_1 \ast \cdots \ast G_n})\chi_{n+1}(p)}$$

$$= \prod_{i=1}^{n} \chi_i(p) \chi_{n+1}(p) \delta^{-1},$$

where

$$\delta = \prod_{i=1}^{n} \chi_i(p) + \chi_{n+1}(p) \left( \sum_{j=1}^{n} \prod_{i=1, i \neq j}^{n} \chi_i(p) - (n-1) \prod_{i=1}^{n} \chi_i(p) \right) - \chi_{n+1}(p) \prod_{i=1}^{n} \chi_i(p)$$

$$= \sum_{j=1}^{n+1} \prod_{i=1, i \neq j}^{n+1} \chi_i(p) - n \prod_{i=1}^{n+1} \chi_i(p).$$

For $p < p_c$ all expressions are finite and the denominator is non-zero (follows from the discussion of the free product of two groups).

\[\square\]

Corollary 4 about $p_c$ for $G_1 \ast \cdots \ast G_n$ now follows from Proposition 11. Let us look at the formula for the expected cluster size more closely. The numerator in formula (13) is finite, positive and increasing for $p < \min\{p_c(G_i)\}$. The denominator is equal to 1 for $p = 0$ and is decreasing in $p$ for $p < p_c$ since each $\chi_i(p)$ is increasing in $p$ and

$$\frac{\partial}{\partial \chi_k(p)} \left( \sum_{j=1}^{n} \prod_{i=1, i \neq j}^{n} \chi_i(p) - (n-1) \prod_{i=1}^{n} \chi_i(p) \right) = \sum_{j=1, j \neq k}^{n} (1 - \chi_j(p)) \prod_{i=1, i \neq j, k}^{n} \chi_i(p)$$

$$< 0.$$

As $p$ approaches $\min\{p_c(G_i)\}$ the expression in the denominator becomes negative, and thus there exists unique $p$ such that

$$\sum_{j=1}^{n} \prod_{i=1, i \neq j}^{n} \chi_i(p) - (n-1) \prod_{i=1}^{n} \chi_i(p) = 0.$$

The solution is the critical probability $p_c$ which proves Corollary 4.
2.4 Comparison with branching processes

The expected cluster size can be viewed as a population size of a branching process in the following way. Consider the free product of two groups \( G_1 \ast G_2 \). Let us define a branching process. The origin is the only element of generation zero. The first generation consists of those vertices of the basic subgraph \( G_1 \setminus \{ o \} \) that are connected to the origin by open paths, the \( 2i \)-th (resp. \( 2i+1 \)-st) generation contains vertices, which are connected to some vertex of the previous generation by a nontrivial open path included in a subgraph of type \( G_2 \) (type \( G_1 \), resp.). If we consider only even generations, then we obtain a simple Galton-Watson branching process. Again using the Wald’s identity (see for example theorem 2A on page 201 in [20]) we obtain that the expected size of one generation is precisely \((\chi_1(p) - 1)(\chi_2(p) - 1)\). By the Basic theorem of branching processes (see for example theorem 2A on page 201 in [20]), such a branching process terminates if the expected generation size is at most one, otherwise the population size grows to the infinity. Therefore the critical value of \( p \) occurs for \((\chi_1(p) - 1)(\chi_2(p) - 1) = 1\) as claimed by Theorem 1. The result can be generalized to the free product of arbitrary number of groups or to a general tree-graded vertex-transitive graph.

3 The critical point \( p_{\text{exp}} \)

Next we would like to find the \( p_{\text{exp}} \) of the free product. That is to decide for which \( p \) the connectivity function decays exponentially with the distance.

Proof of Proposition 6. Clearly \( p_{\text{exp}}(G_1 \ast \cdots \ast G_n) \leq \min\{p_{\text{exp}}(G_i)\} \). For the converse inequality suppose that \( p < \min\{p_{\text{exp}}(G_i)\} \). Then there exist \( C_i, \gamma_i > 0 \), for \( 1 \leq i \leq n \) such that

\[
P_p(x \leftrightarrow y \text{ in } G_i) \leq C_i e^{-\gamma_i \text{dist}(x,y)}.
\]

Clearly if \( G_i \) is finite this estimate can be done for any \( p \). Define \( C = \max\{C_i\}, \gamma = \min\{\gamma_i\} \). Any (simple) path from \( x \) to \( y \) can be divided into finite number of pieces, where each piece lies in one special subgraph \( G_i \) and two consequent subgraphs share exactly one vertex. Thus there is a finite set of cut points \( z_1, \ldots, z_k \), the same for all paths connecting \( x \) and \( y \). Let \( z_0 = x \) and \( z_{k+1} = y \), we have

\[
P_p(x \leftrightarrow y) = \prod_{i=0}^{k} P_p(z_i \leftrightarrow z_{i+1}),
\]

\[
\text{dist}(x,y) = \sum_{i=0}^{k} \text{dist}(z_i, z_{i+1}).
\]

Now there exists \( K > 0 \) and \( 0 < \alpha \leq \gamma \) such that if \( \text{dist}(z_i,z_{i+1}) > K \) then

\[
P_p(z_i \leftrightarrow z_{i+1}) \leq C e^{-\gamma \text{dist}(z_i,z_{i+1})} \leq e^{-\alpha \text{dist}(z_i,z_{i+1})}.
\]

Considering only the state of edges adjacent to \( z_i \) we obtain following rough estimate:

\[
P_p(z_i \leftrightarrow z_{i+1}) \leq 1 - (1 - p)^{\deg(o)}.
\]

The right side is strictly less than one thus there exists \( 0 < \beta \leq \alpha \) such that

\[
1 - (1 - p)^{\deg(o)} \leq e^{-\beta K}.
\]

Therefore if \( \text{dist}(z_i,z_{i+1}) \leq K \) then

\[
P_p(z_i \leftrightarrow z_{i+1}) \leq e^{-\beta K} \leq e^{-\beta \text{dist}(z_i,z_{i+1})},
\]
and for $\text{dist}(z_i, z_{i+1}) > K$ we have 
\[ P_p(z_i \leftrightarrow z_{i+1}) \leq e^{-\alpha \text{dist}(z_i, z_{i+1})} \leq e^{-\beta \text{dist}(z_i, z_{i+1})}. \]
Combining the above two estimates we obtain 
\[ P_p(x \leftrightarrow y) = e^{-\beta \text{dist}(x,y)}. \]
Therefore the connectivity function has an exponential decay at $p$ and we have proved 
\[ p_{\exp}(G_1 * \cdots * G_n) \geq \min\{p_{\exp}(G_i)\}. \]

4 Approximation results

Proof of Theorem 5. Recall that we consider a sequence of factor groups $H_i^j = G_i/N_i^j$ and want to show that $p_e(G_1 * G_2) = \lim_{j \to \infty} p_e(H_i^j * H_j^i)$. Let $\phi$ be the factor map $G_i \to H_i^j$. Consider a Cayley graph of $G_i$ with respect to generating set $S$ and a Cayley graph of $H_i^j$ with respect to generating set $\phi(S)$. Then any path in $H_i^j$ from origin $o$ to a vertex $x$ can be lifted to a unique path in $G_i$ from the origin $o$ to a vertex $y$, s.t. $\phi(y) = x$. On the other hand the image under $\phi$ of a simple path does not have to be simple. (A similar argument was used by Campanino [5].) Thus
\[
E_p(|C|_{H_i^j}) = \sum_{x \in H_i^j} P_p(o \leftrightarrow x)
\]
\[
= \sum_{x \in H_i^j} P_p(\text{at least one path } o \leftrightarrow x \text{ is open})
\]
\[
\leq \sum_{x \in H_i^j} \sum_{y : \phi(y) = x} P_p(\text{at least one path } o \leftrightarrow y \text{ is open})
\]
\[
= \sum_{y \in G_i} P_p(o \leftrightarrow y)
\]
\[
= \chi_i(p)
\]
From Theorem 1 we know that $p_e$ is a solution of equation (3). Therefore $p_e(G_1 * G_2) \leq p_e(H_i^j * H_j^i)$ for all $j$ (that also follows from [5]).

Now assume that $p$ is such that $\chi_i(p) < \infty$, $i = 1, 2$. Then by Schonmann [21] there exist $C, \gamma > 0$ such that $P_p(|C|_{G_i} \geq n) \leq Ce^{-\gamma n}$ and
\[
0 \leq \chi_i(p) - E_p(|C|_{H_i^j}) \leq 2\sum_{k=j}^{\infty} kP_p(|C|_{G_i} = k)
\]
\[
\leq 2Cje^{\gamma j} + \sum_{k=j+1}^{\infty} e^{-\gamma k} \leq \frac{4je^{\gamma j}}{1 - e^{-\gamma}} \leq C'e^{-\gamma'j},
\]
where $C' > 0$ (later also $C'' > 0$). Then
\[
0 \leq (\chi_1(p) - 1)(\chi_2(p) - 1) - (E_p(|C|_{H_1^j}) - 1)(E_p(|C|_{H_2^j}) - 1) =
\]
\[
= (\chi_1(p) - 1)(\chi_2(p) - E_p(|C|_{H_2^j})) + (\chi_1(p) - E_p(|C|_{H_1^j}))(E_p(|C|_{H_2^j}) - 1) \leq
\]
\[
\leq (\chi_1(p) - 1)(\chi_2(p) - E_p(|C|_{H_2^j}))(\chi_1(p) - 1) \leq
\]
\[
\leq C'' e^{-\gamma'j}
\]

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Since \( p_c(G_1 * G_2) \leq \min\{p_c(G_i)\} \) we can take the derivative of their cluster sizes and the following derivative

\[
\frac{d}{dp}(\chi_1(p) - 1)(\chi_2(p) - 1) \bigg|_{p=p_c(G_1 * G_2)}
\]

is positive. There exist \( \epsilon \) and \( \delta > 0 \) such that the derivative is bigger than \( \delta \) on the interval \( [p_c(G_1 * G_2), p_c(G_1 * G_2) + \epsilon] \). Let \( p_0 := p_c(G_1 * G_2) + C''\delta^{-1}e^{-\gamma j} \) then for \( j \) large enough \( p_0 \in [p_c(G_1 * G_2), p_c(G_1 * G_2) + \epsilon] \) and

\[
(E_{p_c(G_1 * G_2)}(|C|_{H_1^n}) - 1)(E_{p_c(G_1 * G_2)}(|C|_{H_2^n}) - 1) \leq (E_{p_0(G_1 * G_2)}(|C|_{G_1}) - 1)(E_{p_0(G_1 * G_2)}(|C|_{G_2}) - 1) = 1

(E_{p_0}(|C|_{H_1^n}) - 1)(E_{p_0}(|C|_{H_2^n}) - 1) \geq (E_{p_0}(|C|_{G_1}) - 1)(E_{p_0}(|C|_{G_2}) - 1) - C''e^{-\gamma j}

\geq 1 + \delta(p_c(G_1 * G_2) - p_0) - C''e^{-\gamma j} \geq 1
\]

Therefore \( p_c(H_1^j * H_2^j) \in [p_c(G_1 * G_2), p_0] \) and \( 0 \leq p_c(H_1^j * H_2^j) - p_c(G_1 * G_2) \leq C''\delta^{-1}e^{-\gamma j} \rightarrow 0 \). Thus \( p_c(G_1 * G_2) = \lim_{j \to \infty} p_c(H_1^j * H_2^j) \) which completes the proof. \( \square \)

5 Examples

In Section 2.1, we have presented the expression (9) of the walk through function \( g_t \) for finite cyclic groups. Another relatively easy case is a free group. In order to find \( p_c \) of the free product where one factor is a cyclic group or a free group, we need to know the expected cluster size.

Proposition 12. The expected cluster size in the Cayley graph of the cyclic group \( C_m \) or the free group \( F_n \) with respect to a standard set of generators is given by the following formula (in the case of the free group the formula holds for \( p < p_c = \frac{\gamma \delta}{2m-1} \))

\[
\begin{align*}
E_p(|C|_{C_m}) &= \frac{1 + p - p^m(m+1) - p^{m+1}(m-1)}{1 - p}, \\
E_p(|C|_{F_n}) &= \frac{1 + p}{1 - (2m-1)p}.
\end{align*}
\]

Proof. The expression for the cluster size in a cyclic group can be obtained by taking the derivative of \( g_t \) given by (9) and evaluating it at \( t = 0 \).

\[
\begin{align*}
E_p(|C|_{C_m}) &= 1 + \sum_{j=1}^{m-1} j(j - 1)(1 - p)^2p^{j-1} + (m(1 - p) + p)(m-1)p^{m-1} \\
&= 1 + \frac{(2^2p^2 - 2p + 1) - m(3p^2 + 4p - 1) + 2p^2p^{m-1} - 2p}{p - 1} + (m(1 - p) + p)(m-1)p^{m-1} \\
&= \frac{1 + p - p^m(m^m - p^{m+1}) - p^m - p^{m+1}}{1 - p}.
\end{align*}
\]

In order to evaluate the mean cluster size in free group the following observation will be useful. The size of the cluster can be viewed as a sum of indicators that a given vertex is connected to the origin. And the expectation of a sum is a sum of expectations, therefore:

\[
E_p(|C|_G) = \sum_{x \in G} P_p(x \text{ is connected to the origin } o).
\]

In the tree, the probability that \( x \) is connected to \( o \) is equal to \( p^d \), where \( d \) is the distance between \( x \) and \( o \). The number of vertices at a given distance \( d \) is equal to \( 2u(2n - 1)^{d-1} \). We plug it into the
Proposition 13. Consider the free product of two finite cyclic groups $C_m * C_n$, such that $(m - 1)(n - 1) > 1$ with the natural generators. Then the inequalities (1), (2) are strict, that is

$$\frac{1}{3} < p_c < \frac{1}{h(C_m * C_n) + 1}.$$ 

Proof. The first inequality follows from the observation that $E_p(|C|_{n}) > E_p(|C|_{m})$ for all $m$ and $p > 0$. For the second inequality we will show that

$$h(C_n * C_m) \leq 2 - \max \left( \frac{2m}{n(m - 1)}, \frac{2n}{m(n - 1)} \right) < 1/p_c - 1. \quad (16)$$

Consider a set $S_1$ consisting of a cycle of $C_n$ at the origin in the Cayley graph of $C_n * C_m$. We construct $S_k$ inductively by including the whole cycle $C_n$ at every point of the boundary of $S_{k-1}$ and then adding the next generation of cycles $C_n$ to the $S_{k-1}$.
Then
\[
|\partial S_k| = 2n
\]
\[
|S_k| = n
\]
\[
|\partial S_{k+1}| = (n-1)(m-1)|\partial S_k|
\]
\[
= 2n((n-1)(m-1))^k
\]
\[
|S_{k+1}| = |S_k| + |\partial S_k|(m-1)n
\]
\[
= n + n^2(m-1) \sum_{i=0}^{k-1} ((n-1)(m-1))^i
\]
\[
= n + n^2(m-1)\frac{((n-1)(m-1))^{k-1} - 1}{(n-1)(m-1) - 1}
\]
\[
\frac{|\partial S_{k+1}|}{|S_{k+1}|} = \frac{2n((n-1)(m-1))^k(mn-m-n)}{n(mn-m-n) + n^2(m-1)((n-1)(m-1))^{k-1}}
\]
\[
\xrightarrow{k \to \infty} \frac{2(mn-m-n)}{n(m-1)}
\]
\[
h(C_n \ast C_m) \leq 2 - \frac{2m}{n(m-1)}
\]

Assume that \(n \leq m\) and set
\[
p_1 = \frac{1}{3 - \frac{2}{n}} \leq \frac{1}{3 - \frac{2m}{n(m-1)}} \leq \frac{1}{h(C_n \ast C_m) + 1}.
\] (17)

It is enough to prove \(p_c < p_1\). By Theorem 1, it means we need to show that
\[
(E_{p_1}(|C|_{C_m}) - 1)(E_{p_1}(|C|_{C_n}) - 1) > 1.
\]

We have:
\[
(E_{p_1}(|C|_{C_n}) - 1) = \frac{2p_1 + p_1^{n+1}(n-1) - p_1^n(n+1)}{1 - p_1}
\]
\[
= 1 + \frac{1 - (n^2 + n - 1)\left(\frac{n}{3n-2}\right)^n}{n-1}
\]

Now for \(n \geq 3\):
\[
(n^2 + n - 1)\left(\frac{1}{3} + \frac{2}{3n-2}\right)^n \leq (n^2 + n - 1)\left(\frac{3}{7}\right)^n < 1.
\]

Thus if \(3 \leq n \leq m\) we have
\[
(E_{p_1}(|C|_{C_m}) - 1)(E_{p_1}(|C|_{C_n}) - 1) \geq (E_{\frac{m}{m-1}}(|C|_{C_m}) - 1)(E_{\frac{n}{n-1}}(|C|_{C_n}) - 1) > 1.
\]

It remains to consider the case \(n = 2\). If \(m = 3\) we will use the stronger estimate in (17) \(p_2 := \frac{1}{3 - \frac{1}{n(m-1)}} = \frac{3}{2}\); otherwise we use again \(p_1\), now equal to \(\frac{1}{2}\). By plugging in we obtain
\[
(E_{p_2}(|C|_{C_2}) - 1)(E_{p_2}(|C|_{C_3}) - 1) = 1.4
\]
\[
(E_{p_2}(|C|_{C_2}) - 1)(E_{p_1}(|C|_{C_m}) - 1) \geq (E_{p_1}(|C|_{C_2}) - 1)(E_{p_1}(|C|_{C_3}) - 1) = 1.2
\]

Therefore in all cases we obtained that \(p_c < 1/(h(C_m \ast C_n) + 1)\), as required. \(\Box\)
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