ON UNIVERSAL EQUIVALENCE OF PARTIALLY COMMUTATIVE METABELIAN LIE ALGEBRAS

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In this paper, we consider partially commutative metabelian Lie algebras whose defining graphs are cycles. We show that such algebras are universally equivalent iff the corresponding cycles have the same length. Moreover, we give an example showing that the class of partially commutative metabelian Lie algebras such that their defining graphs are trees is not separable by universal theory in the class of all partially commutative metabelian Lie algebras.

Key Words: Lie algebra; Metabelian; Partially commutative; Universal equivalence; Universal theory.

2010 Mathematics Subject Classification: 17B99.

1. INTRODUCTION

Let $G = \langle X, E \rangle$ be an undirected graph without loops with the finite set of vertices $X = \{x_1, \ldots, x_n\}$ and the set of edges $E (E \subseteq X \times X)$. We denote the elements of $E$ by $\{x, y\}$.

Suppose that $R$ is an associative commutative ring $R$ with a unit. A partially commutative Lie algebra over $R$ is an $R$-algebra $\mathcal{L}_R(X; G)$ with the set of generators $X$ and the set of defining relations of the form

$$[x_i, x_j] = 0,$$

where $\{x_i, x_j\} \in E$.

Henceforth, Lie product of $g$ and $h$ is denoted by $[g, h]$. $G$ is called the defining graph of the corresponding algebra.

One can also define partially commutative Lie algebras in some variety of Lie algebras. In this case, a partially commutative Lie algebra in a variety is a Lie $R$-algebra defined by a set of generators, defining relations, and the set of identities holding in this variety. In this paper, we consider partially commutative Lie algebras in the variety of metabelian Lie algebras.

So, the definition of partially commutative Lie algebras is analogous to ones of other partially commutative structures such as groups, monoids, etc. (see [4]).

Received May 25, 2013; Communicated by V. A. Artamonov.
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Partially commutative groups are studied very heavily nowadays (see [2, 5–
7, 12–16]). In some papers (for example, in [6, 7, 15]), universal theories of partially
commutative metabelian groups were studied.

Partially commutative algebras (both associative and Lie algebras) were
studied less. Although there are some results obtained for other partially
commutative structures (see [1, 3, 8–11]). In [11], for instance, partially commutative
Lie algebras whose defining graphs are trees were considered, and the universal
theories of these algebras were studied.

In this paper, we continue studying universal theories of partially commutative
Lie algebras.

In Section 2, preliminary definitions and results are given.

In Section 3, the partially commutative Lie algebras whose graphs are cycles
are considered. It is shown that two such algebras are universally equivalent iff the
cycles have the same length.

In Section 4, some graph transformation is defined. It is shown that if a
partially commutative metabelian Lie algebra is obtained from another one by
applying this transformation, then both algebras have the same universal theory.
The paper finishes by giving an example of two universally equivalent algebras such
that the defining graph of the first one is a tree, while the defining graph of the
second one is not.

2. PRELIMINARIES

Let $R$ be an integral domain, and let $G = \langle X; E \rangle$ be an undirected graph with
the set of vertices $X$ and the set of edges $E$. By $M(X; G)$, denote the partially
commutative metabelian Lie algebra with the set of generators $X$ and the defining
graph $G$, i.e., the Lie $R$-algebra $M_R(X)/I$, where $M_R(X)$ is the free metabelian Lie
$R$-algebra with the set of generators $X$, and $I$ is the ideal of $M_R(X)$ generated by the
set of relations $\{[x_i, x_j] = 0 | x_i, x_j \in X, \text{ such that } \{x_i, x_j\} \in E\}$.

If $\{x, y\} \in E$, then we write $x \leftrightarrow y$. Similarly, suppose that $X' \subseteq X$ and $x \in X$
is adjacent to all vertices in $X'$. Then we write $x \leftrightarrow X'$. These pieces of notation are
going to be used as global ones. Namely, they mean that a vertex is adjacent to
another vertex (or to all vertices in a subset of $X$) in $G$, i.e., in the defining graph of the
initial partially commutative metabelian Lie algebra $M(X; G)$.

In particular, since $G$ has no loops, $x \leftrightarrow X'$ implies $x \notin X'$.

In this paper, we assume $X = \{x_0, x_1, \ldots, x_{n-1}\}$. We also suppose that $R$
is an integral domain containing the ring of integers $\mathbb{Z}$ as a subring. We denote Lie
monomials as bracketed lowercase Latin letters (for example, $[u]$) to keep the same
notation as in some other papers.

**Definition 2.1.** Let $[u]$ be a Lie monomial in the set of generators $X$. The
multidegree of $[u]$ is the vector $\delta = (\delta_0, \delta_1, \ldots, \delta_{n-1})$, where $\delta_i$ is the number of
occurrences of $x_i$ in $[u]$.

Let us denote by $\text{mdeg}([u])$ the multidegree of $[u]$ and by $\text{mdeg}_i(u)$ the number
of $x_i$ in $[u]$, i.e., the $i$th coordinate of $\text{mdeg}([u])$.

The glued multidegree of $[u]$ is the vector $\text{mdeg}([u]) = (\delta_0, \delta_1, \ldots, \delta_{n-2} + \delta_{n-1})$,
where $\delta_i = \text{mdeg}_i([u])$ as above.
Let us introduce some pieces of notation associated with graphs. Let \( H \) be an arbitrary undirected graph. By \( V(H) \) and \( E(H) \) denote the set of the vertices and the set of the edges of this graph, respectively. Next, let \( V' \subseteq V(H) \). By \( H[V'] \), denote the subgraph of \( H \), generated by the set \( V' \).

Let \([u]\) be a Lie monomial not equal to zero in \( M(X; G) \). By \( X[u] \), denote the set \( \{x_i \in X \mid \text{mdeg}_i([u]) \neq 0\} \). Correspondingly, \( G[u] \) is a subgraph of \( G \) generated by \( X[u] \). Likewise, if \( v \) is an associative monomial, then denote by \( X_v \) the set of generators occurring in \( v \). We also denote by \( G_v \) the graph \( G(X_v) \).

In [11], the explicit description of a basis of a Lie algebra \( M(X; G) \) was found. Suppose that the set \( X \) is linearly ordered. Then the basis of \( M(X; G) \) is the set of monomials of the form \([u] = \cdots [x_i, x_{i_1}], \ldots, x_{i_k}] \) having the following properties:

1. \( x_{i_2} < x_{i_1} \);
2. \( x_{i_k} \leq x_{i_{k-1}} \leq \cdots \leq x_{i_1} \);
3. \( x_{i_k} \) and \( x_{i_{k-1}} \) are in different connected components of \( G[u] \);
4. Let \( H \) be the connected component of \( G[u] \) containing \( x_{i_1} \), then \( x_{i_1} \) is the greatest element among the vertices in \( V(H) \).

Such monomials are called basis monomials. Note, that the set of basis monomials depends on an order of the set \( X \). However, whatever ordering is used the set of all basis monomials of \( M(X; G) \) forms a basis of this algebra.

**Definition 2.2.** If all monomials of a Lie polynomial \( g \) have the same multidegree \( \vec{\delta} \), then we call such polynomial homogeneous and write \( \text{mdeg}(g) = \vec{\delta} \), where \( \vec{\delta} \) is the multidegree of a monomial in \( g \).

Since the set of identities and the set of defining relations of a partially commutative metabelian Lie algebra are homogeneous, the following statement holds. If \( 0 = \sum g_i \) in \( M(X; G) \), where all \( g_i \) are homogeneous Lie polynomials of pairwise distinct multidegrees, then for any \( i \) we have \( g_i = 0 \) in this algebra. In particular, if \( g \) is homogeneous, then for any order of \( X \) all basis monomials in \( g \) have the same multidegree.

Let \( g \) be an element such that its decomposition to the linear combination of basis monomials is a homogeneous polynomial. Then for any other order of \( X \) the decomposition of \( g \) to the linear combination of basis monomials is also homogeneous. Moreover, the multidegrees of the corresponding polynomials are same for all orders. So we can define homogeneous elements in an obvious way. Indeed, since all identities and defining relations of a partially commutative metabelian Lie \( R \)-algebra are homogeneous, any transformation of a nonzero Lie monomial gives a homogeneous linear combination of the same multidegree. So, we can also define the multidegree of a homogeneous element of \( M(X; G) \) as follows. Let \( g \) be a homogeneous element. Consider a decomposition of this element to a linear combination of basis monomials (with respect to any order on \( X \)). Then by definition, put \( \text{mdeg}(g) = \text{mdeg}([u]) \), where \([u]\) is an arbitrary basis Lie monomial contained in this decomposition.
Let \([u] = [\ldots [x_{i_1}, x_{i_2}], \ldots, x_{i_k}]\). It is easy to show (see, for example, [11]) that for any permutation of \(x_{i_1}, \ldots, x_{i_k}\), we obtain a monomial equal to \([u]\) in \(M'(X)\), and consequently these monomials are also equal in \(M(X; G)\). That means
\[
[\ldots [[x_{i_1}, x_{i_2}], x_{i_3}], \ldots, x_{i_k}] = [\ldots [[x_{i_1}, x_{i_2}], x_{\sigma(i_3)}], \ldots, x_{\sigma(i_k)}],
\]
where \(\sigma\) is a permutation of \(\{i_1, i_2, \ldots, i_k\}\).

Let \(R[X]\) be the set of all commutative associative polynomials over \(R\). It follows from the last paragraph that the derived subalgebra \(M'(X; G)\) of \(M(X; G)\) is an \(R[X]\)-module with respect to the adjoint representation. Denote by \(u.f\) the element of \(M'(X; G)\) obtained by acting the element \(f \in R[X]\) on \([u] \in M'(X; G)\).

Namely, let us define \(u.f\) inductively:

1. \(u.y = [u, y]\) for any \(y \in X\);
2. Let \(f = y_1y_2\ldots y_m\) for \(m \geq 2\), and let \(f_0 = y_1y_2\ldots y_{m-1}\); then \(u.f = (u.f_0).y_m\);
3. Finally, if \(f = g + s\), where \(s\) is a commutative associative monomial, then \(u.f = u.g + u.s\).

**Definition 2.3.** Let \(g\) be an arbitrary element of the algebra \(M(X; G)\), and let \(C(g)\) be the centralizer of \(g\). The set
\[
C(g) = C(g) \cap M'(X; G)
\]
is called the derived centralizer of \(g\).

For the derived centralizers of the generators of \(M(X; G)\), the following theorem holds (see [11]).

**Theorem 2.4.** Let \(M(X; G)\) be a metabelian partially commutative Lie \(R\)-algebra, where \(X = \{x_0, x_1, \ldots, x_{n-1}\}\). Then for any elements \(x_{i_1}, x_{i_2}, \ldots, x_{i_m}\) and for any \(x_{i_1}, x_{i_2}, \ldots, x_{i_m} \in R \setminus \{0\}\), we have
\[
C\left(\sum_{j=1}^{m} x_{i_j}x_{i_j}\right) = \bigcap_{j=1}^{m} C(x_{i_j})
\]

Finally, let us remind some terminology related universal theories of algebraic systems. Let \(\Phi\) be a formula having no free variables and including elements of an algebra \(A\). Then by definition, put \(A \models \Phi\) if \(\Phi\) holds on \(A\).

**Definition 2.5.** An \(\exists\)-sentence is a formula of the form
\[
\exists w_1 \ldots w_m \Phi(w_1, \ldots, w_m),
\]
having no free variables. Here \(\Phi(w_1, \ldots, w_m)\) is a formula of predicate calculus in the corresponding algebraic system such that this formula does not contain quantifiers.
Lemma 3.1. Let all vertices be connected consecutively.

a) If properties hold:
whose defining graph is the cycle of the length $n$

b) Let $|i - j| > 1$ and $x, y \in R \setminus \{0\}$. Then the derived centralizer $\mathcal{C}(x_i + x_j)$ consists of all linear combinations of nonzero Lie monomials $[u_i]$ such that $X_{[u_i]} = X \setminus \{x_i, x_j\}$.

Moreover, any element of $\mathcal{C}(x_i + x_j)$ can be represented in the form $f = [x_{i-1}, x_{i+1}]g$, where $g$ is an associative polynomial.

c) If $m \geq 3$ and $x_1, \ldots, x_m \in R \setminus \{0\}$, then $\mathcal{C}(\sum_{j=1}^{m} x_j x_j) = 0$.

Proof. It follows from Theorem 2.4 that for any $x_j \in R \setminus \{0\}$ ($i = 1, 2, \ldots, m$) the equation $\mathcal{C}(\sum_{j=1}^{m} x_j x_j) = \bigcap_{j=1}^{m} \mathcal{C}(x_j)$ holds in any algebra $M(X; G)$.

By construction of basis monomials, we can see that, if $[u] \in \mathcal{C}(x_i) \setminus \{0\}$ for a Lie monomial $[u]$, then $\text{mdeg}([u]) = 0$.

a) By [11], we only have to prove the assertion for nonzero Lie monomials. Let $[u] \in \mathcal{C}(x_i + x_{i+1})$ be a Lie monomial such that $[u] \neq 0$ in $M(C_n; X)$. In this
case, \( G_u \) has at least two chains. Therefore, there exists \( j \in \mathbb{Z}_n \) such that \( x_j \notin X_u \), and there is a representation \([u] = [x_i, x_j], w\), where \( w \) is an associative monomial and \( x_i, x_j \) are generators from different chains. Besides, since \([u]x_i = [u]x_{i+1} = 0\) these generators should be in the same chain in the graphs \( G(X_u \cup \{x_i\}) \) and \( G(X_u \cup \{x_{i+1}\})\), but this is impossible. Indeed, \( x_i \) and \( x_{i+1} \) are adjacent in \( C_n \).

Therefore, we have to add both vertices to \( G_u \) to get one chain in the obtained graph instead of two ones in \( G_u \).

b) Let \( g \in \langle x_i + \beta x_j \rangle \), where \(|i - j| > 1\). As in a), it is sufficient to prove the statement for nonzero Lie monomials. Let us suppose that \([u] \neq 0 \) in \( M(X; C_n) \).

We can write \([u] = [x_i, x_j], w\) for some associative monomial \( w \). Let \( T_1, T_2, \ldots, T_p \) be the chains of \( G(X_u) \) \( (p \geq 2 \) because \([u] \neq 0 \) in \( M(X; C_n) \)). We can assume without loss of generality that \( x_i \) is in \( T_1 \), and \( x_j \) is in \( T_2 \). Since \([u] \in \langle x_i \rangle \), we have \([u]x_i = 0\).

Consequently, the vertices of the chains \( T_1 \) and \( T_2 \) are in the same connected component of \( G(X_u \{x_i\})\). So, \( x_i \) connects the subgraph \( T_1 \) with the subgraph \( T_2 \). Namely, in the graph \( C_n \), the vertex \( x_i \) is adjacent to some vertex of \( T_2 \) the vertex \( x_j \) is adjacent to some vertex of \( T_2 \) as well. Using the same arguments for \( x_j \), we obtain that in \( C_n \), the vertex \( x_j \) is adjacent to a vertex in \( T_1 \), and \( x_i \) is adjacent to a vertex in \( T_2 \) as well. This is possible only if there are only two chains in \( G_u \) and one endpoint of each chain is adjacent to \( x_i \) while the other endpoint of each chain is adjacent to \( x_j \). Consequently, \( X_u = X \{x_i, x_j\} \).

There are only two connected components in \( G_u \) and the vertices \( x_{i-1} \) and \( x_{i+1} \) are in the different connected components. Therefore, one of these vertices is in the same connected component with \( x_i \) and the other one is in the same connected component with \( x_j \).

In [11], it was shown that, if \([u_1] \) is a nonzero Lie monomial and \([u_2] \) is obtained from \([u_1] \) by switching two generators from the same connected component of \( G_{[u_1]} \), then \([u_1] = [u_2] \) in \( M(X; G) \).

Thus, we can switch \( x_{i-1} \) with the vertices in \( \{x_i, x_j\} \) lying in the same connected component as \( x_{i-1} \) and \( x_{i+1} \) with another vertex of this set. Applying the anticommutativity identity to the obtained monomial (if it is necessary), we obtain a monomial in the form \( [x_{i+1}, x_{i-1}], v \), which is equal to \([u] \) in \( M(X; G) \).

c) It is sufficient to show that \( \langle x_i + \beta x_j + \gamma x_k \rangle = 0 \). If at least two among these three vertices are adjacent in \( C_n \), then the proof follows from a).

Suppose that \(|i - j| > 1\), \(|j - k| > 1\), and \(|i - k| > 1\). Let \( T_1, T_2, \ldots, T_p \) be chains of \( G_u \). Obviously, \( p \geq 3 \) in this case. All arguments are similar to ones in b).

Let \([u] = [x_i, x_j], w\), where \( w \) is an associative monomial. We can assume without loss of generality that \( x_i \) is in a chain \( T_1 \), and \( x_j \) is in a chain \( T_2 \). Since \([u]x_i = 0\), we get \( x_i \) adjacent to some vertex in \( T_1 \), and \( x_j \) is adjacent to some vertex in \( T_2 \) as well. That means that \( x_i \) connects \( T_1 \) and \( T_2 \) in a connected component in \( G(X_u \{x_i\})\). Analogously, since \([u]x_j = 0\), we obtain \( x_j \) is adjacent to some vertex in \( T_1 \) and \( x_j \) is adjacent to some vertex in \( T_2 \) as well. But in this case, \( T_1 \) and \( T_2 \) together should contain all vertices in \( X \{x_i, x_j\} \). This is impossible because there are at least tree chains. So the proof is completed. \( \square \)

Let \( m \geq 4 \). Consider the formula

\[
\Phi(m) = \exists z_0, z_1, \ldots, z_{m-1} \Theta(z_0, z_1, \ldots, z_{m-1}),
\]

(2)
where

\[
\Theta(z_0, z_1, \ldots, z_{m-1}) = \left( \bigwedge_{i \in \mathbb{Z}_m} [z_i, z_{i+1}] = 0 \land \bigwedge_{i, j \in \mathbb{Z}_m, |i-j| > 1} [z_i, z_j] \neq 0 \right)
\land \bigwedge_{i, j \in \mathbb{Z}_m, |i-j| \neq 1} ([z_i, z_{i+2}], z_j) \neq 0.
\]

It is easy to see that this formula holds in \(M(X, C_m)\). Indeed, let us put \(g_i\) to be equal \(x_i\) for any \(i \in \mathbb{Z}_m\). Obviously, \([x_i, x_j] = 0\) iff \(|i - j| \leq 1\). We also can see that \([[x_i, x_{i+2}], x_j] = 0\) iff the vertices \(x_i\) and \(x_{i+2}\) are in the same connected component of \(G((x_i, x_{i+2}, x_j))\). But this is true only in the case

\[|j - i| = |j - (i + 2)| = 1. \quad (3)\]

Note that if \(m \geq 5\) then for any \(k, l \in \mathbb{Z}_m\) such that \(|k - l| = 2\) there is a unique \(t\) such that \(|k - t| = |l - t| = 1\). So, if \(k - t = 1\), then we may assume that \(l = i\), then \(k = i + 2\) and \(t = i + 1 = j\). Otherwise, we suppose that \(k = i\), then \(l = i + 2\) and again \(t = i + 1 = j\).

However, if \(m = 4\), then the assertion from the last paragraph is not true. Indeed, let \(k, l \in \mathbb{Z}_4\) be such that \(|k - l| = 2\). Then for either \(t \in \mathbb{Z}_4\) such that \(t \neq k\) and \(t \neq l\), we have \(|k - t| = |l - t| = 1\). As in the last paragraph, we can put \(k = i\) if \(l - t = 1\), and \(l = i\) in the other case.

Let us prove some conditions the elements \(g_0, g_1, \ldots, g_{m-1} \in M(X; C_n)\) should satisfy to \(M(X; C_n) \models \Theta(g_0, g_1, \ldots, g_{m-1})\) true. But before doing that, let us note that any \(g_i\) can be written as

\[g_i = \hat{g}_i + g_i',\]

where \(g_i' \in M'(X; C_n)\) and

\[\hat{g}_i = \sum_{j=1}^{r_i} x_{i,j} a_{i,j}; \quad (4)\]

moreover, for any fixed \(i\) all indices \(k_{i,l}\) are distinct and \(z_{i,k_i} \in R \setminus \{0\}\). So, \(\hat{g}_i\) is the linear part of \(g_i\). Thus, we have

\[[g_i, g_j] = [\hat{g}_i + g_i', \hat{g}_j + g_j']
\quad = [\hat{g}_i, \hat{g}_j] + [\hat{g}_i, g_j'] + [g_i', \hat{g}_j] + [g_i', g_j']
\quad = [\hat{g}_i, \hat{g}_j] - [g_j', g_i] + [g_i', \hat{g}_j]. \quad (5)\]

Besides, we need the following formula:

\[[[g_i, g_j], g_k] = [[[g_i, g_j], \hat{g}_k] + [g_i, g_j'] g_k']
\quad = [[[g_i, g_j], \hat{g}_k] + [[g_i, g_j], g_k']
\quad = [[[g_i, g_j], \hat{g}_k]. \quad (6)\]
Lemma 3.2. Let \( m \geq 5 \), and let \( g_0, g_1, \ldots, g_{m-1} \) be the elements of \( M(X; C_n) \) (\( n \geq 3 \)) such that \( M(X; C_n) \models \Theta(g_0, g_1, \ldots, g_{m-1}) \). Then for all \( i \in \mathbb{Z}_m \) we have \( \hat{g}_i \neq 0 \) in \( M(X; C_n) \).

**Proof.** Let \( \hat{g}_i = 0 \) for some \( i \). Since \( m \geq 5 \), we have \(|(i + 3) - i| > 1 \). Obviously, \([g_{i+3}, g_i] = 0 \) because \( M(X, C_n) \) is metabelian and \([g_{i+3}, g_i], g_i \in M'(X; C_n)\). We get a contradiction to (2). Therefore, for any \( i \in \mathbb{Z}_m \), we obtain \( \hat{g}_i \neq 0 \).

Lemma 3.3. Let \( m \geq 5 \), and let \( g_0, g_1, \ldots, g_{m-1} \) be the elements of \( M(X; C_n) \) (\( n \geq 4 \)) such that \( M(X; C_n) \models \Theta(g_0, g_1, \ldots, g_{m-1}) \). Then for all \( i \in \mathbb{Z}_m \) we have \( r_i \leq 2 \), where \( r_i \) are taken in (4). Moreover, if \( r_i = 2 \) for some \( i \in \mathbb{Z}_m \), then \( |k_{i,1} - k_{i,2}| > 1 \), where \( k_{i,1} \) and \( k_{i,2} \) are also taken in (4).

**Proof.** Let \( \hat{g}_i \) have at least three nonzero summands for some \( i \); i.e., let \( r_i \geq 3 \). Since \([g_{i+1}, g_{i-1}], g_i \in M'(X; C_n), (6) \) implies

\[
0 = [(g_{i+1}, g_{i-1}), \hat{g}_i] = \left[ g_{i+1}, g_{i-1} \right] \sum_{l=1}^{r_i} z_i x_{k,1}.
\]

Hence, \([g_{i+1}, g_{i-1}] \in \mathcal{C}(\sum_{l=1}^{r_i} z_i x_{k,1}) \). By Lemma 3.1 c), \([g_{i+1}, g_{i-1}] = 0 \). This contradicts (2) because for \( m \geq 5 \) we have \(|(i + 1) - (i - 1)| > 1 \).

Let \( \hat{g}_i = z_{i,1} x_j + z_{i,2} x_{j+1} \) for some \( j \in \mathbb{Z}_m \) and for \( z_{i,1}, z_{i,2} \in R\{0\} \) (i.e., in the case of \( r_i = 2 \) and \( |k_{i,1} - k_{i,2}| = 1 \)). Then by Lemma 3.1 a), \([g_{i+1}, g_{i-1}] = 0 \), and we also obtain a contradiction.

Lemma 3.4. Let \( m \geq 5 \), and let \( g_0, g_1, \ldots, g_{m-1} \) be the elements of \( M(X; C_n) \) (\( n \geq 4 \)) such that \( M(X; C_n) \models \Theta(g_0, g_1, \ldots, g_{m-1}) \). If \( r_{i-1} = r_{i+1} = 1 \) for some \( i \in \mathbb{Z}_m \), then \( r_i = 1 \) where \( r_i, r_{i-1}, r_{i+1} \) are taken in (4).

**Proof.** Suppose that \( r_{i-1} = r_{i+1} = 1 \) and \( r_i = 2 \) for some \( i \in \mathbb{Z}_m \). So, let \( \hat{g}_i = ax_j + bx_k \) (where \( |j - k| > 1 \)), \( \hat{g}_{i-1} = \gamma x_j \), and \( \hat{g}_{i+1} = \delta x_i \).

By (2), we have \([g_{i-1}, g_i] = 0 \). Therefore, by (5) and homogeneity of identities and relations in a metabelian partially commutative Lie algebra we obtain \([\hat{g}_{i-1}, \hat{g}_i] = 0 \) in \( M(X; C_n) \). That is \( \sum_{j} ax_j + bx_k \gamma x_j + bx_k \delta x_i = 0 \). Consequently, \( |j - s| < 1 \) and \( |k - s| < 1 \). There are two cases possible: either \( s \) is equal to one of the elements \( j, k \) or it is distinct from both of them.

1. We can assume without loss of generality that \( j = s \). Since \( r_i = 2 \), we have \( j \neq k \). Therefore, \( |j - k| = 1 \). But this contradicts Lemma 3.3.

2. Let \( |j - s| = |k - s| = 1 \). We can assume without loss of generality that \( j = s - 1 \) and \( k = s + 1 \).

Consider \([g_i, g_{i-1}] \). By the same argument as above, \( |j - t| = |k - t| = 1 \). We again get two cases.

2.1. Let \( t = s \). By (2), \([g_{i-2}, g_i], g_{i-1} = 0 \). Consequently, by (6), we get \([g_{i-2}, g_i], \hat{g}_{i-1} = 0 \). That means \([g_{i-2}, g_i] \in \mathcal{C}(x_j) \). Now, by (6), we
obtain \([\hat{g}_{i-2}, g_i], g_{i+1}] = [[\hat{g}_{i-2}, g_i], \hat{g}_{i+1}] = 0\) (remind that \(\hat{g}_{i+1} = \delta x_i\)). We get a contradiction to \(2\) because \((i - 2) - (i + 1) \neq 1\) in \(\mathbb{Z}_m\) for \(m \geq 5\).

2-2. Let \(t \neq s\). This is possible only if \(n = 4\). Renumbering generators if it is necessary, we can suppose that \(\hat{g}_t = \alpha x_1 + \beta x_3\), \(\hat{g}_{t-1} = \gamma x_0\), \(\hat{g}_{i+1} = \delta x_2\).

Consider \(g_{i+2}\). By \(5\) and homogeneity of identities and relations in a metabelian partially commutative Lie algebra, \([\hat{g}_{i+1}, \hat{g}_{i+2}] = 0\). Remember that \([x_p, x_q] = 0\) iff \(|p - q| \leq 1\). Consequently, \(\hat{g}_{i+2} = \zeta x_1 + \eta x_2 + \theta x_3\), where \(\zeta, \eta, \theta \in R\). Moreover, by Lemma 3.3, either \(\eta = 0\) or \(\zeta = \theta = 0\).

2-2-1. If \(\eta = 0\), then \(g_{i+2} = \zeta x_1 + \theta x_3\). By \(6\), we have \([g_{i-1}, g_{i+1}], \hat{g}_{i}] = 0\). Hence, \([g_{i-1}, g_{i+1}] \in \mathcal{C}(\hat{g}_{i}) = \mathcal{C}(x_1) \cap \mathcal{C}(x_3)\). But in this case \([g_{i-1}, g_{i+1}] \in \mathcal{C}(\hat{g}_{i+2})\) even if neither \(\zeta\) nor \(\theta\) is equal to 0. So, \([g_{i-1}, g_{i+1}], \hat{g}_{i+2}] = 0\). Since \(m \geq 5\), we have \(|(i - 1) - (i + 2)| \neq 1\) in \(\mathbb{Z}_m\), and we get a contradiction to \(2\).

2-2-2. If \(\zeta = \theta = 0\), then Lemma 3.2 implies \(\eta \neq 0\). We have \(g_{i+2} = \eta x_2\).

By \(6\), we obtain \([g_i, g_{i+2}], \hat{g}_{i}] = 0\). Therefore, \([g_i, g_{i+2}] \in \mathcal{C}(\hat{g}_{i+2}) = \mathcal{C}(x_2)\). But in this case \([g_i, g_{i+2}] \in \mathcal{C}(\hat{g}_{i+2})\). So, by \(6\), we obtain \([g_i, g_{i+2}] \in \mathcal{C}(\hat{g}_{i+2})\). Again we get a contradiction to \(2\).

\(\square\)

Lemma 3.5. Let \(m \geq 5\), and let \(g_0, g_1, \ldots, g_{m-1}\) be the elements of \(M(X; C_n)\) \((n \geq 4)\) such that \(M(X; C_n) \supset \Theta(g_0, g_1, \ldots, g_{m-1})\). Then \(r_i = 1\) for all \(i \in \mathbb{Z}_m\).

Proof. By Lemma 3.4 we are left to show that the case of \(r_i = r_{i+1} = 2\) is impossible.

Suppose that \(\hat{g}_t = \alpha x_1 + \beta x_3\) and \(\hat{g}_{i+1} = \gamma x_1 + \delta x_3\), where \(\alpha, \beta, \gamma, \delta \neq 0\) and \(|j - k| > 1, |s - t| > 1\). Then \(5\), \(2\), and homogeneity of identities and relations of a partially commutative metabelian Lie algebra imply

\[0 = [\hat{g}_t, \hat{g}_{i+1}] = \alpha \gamma [x_j, x_i] + \alpha \delta [x_j, x_i] + \beta \gamma [x_k, x_i] + \beta \delta [x_k, x_i].\]  
(7)

Indeed, only these summands in \(5\) are the products of two generators. Consider several cases.

1. Let \(j, k, s, t\ be distinct. Then \(7\) and homogeneity of identities and relations of partially commutative metabelian Lie algebra imply

\[[x_j, x_i] = [x_j, x_i] = [x_k, x_i] = [x_k, x_i] = 0.\]  
(8)

By the first two parts of \(8\), \(|j - s| = |j - t| = 1\). Then \(x_j, x_i, x_s\ are connected successfully in \(C_n\), namely \(x_i\ is adjacent to both other vertices. For the same reason, we obtain \(|k - s| = |k - t| = 1\) (considering two last parts of \(8\)). Therefore, \(x_k\ is also adjacent to both \(x_i\, and \(x_s\). This is possible only in the case of \(n = 4\).

We can assume without loss of generality that \(\hat{g}_t = \alpha x_0 + \beta x_2\), \(\hat{g}_{i+1} = \gamma x_1 + \delta x_3\).

Consider \(g_{i+2}\). Let \(\hat{g}_{i+2} = \zeta x_0 + \eta x_1 + \theta x_2 + \kappa x_3\). By \(6\), \([g_{i+1}, g_{i+2}], \hat{g}_{i+2}] = 0\) if \(\zeta = \theta = 0\ or \eta = \kappa = 0\).

1-1. Let \(\zeta = \theta = 0\). Then \(\hat{g}_{i+2} = \eta x_1 + \kappa x_3\). By \(6\), \([g_i, g_{i+2}], \hat{g}_{i+2}] = 0\), therefore \([g_i, g_{i+2}] \in \mathcal{C}(\hat{g}_{i+2}) = \mathcal{C}(x_1) \cap \mathcal{C}(x_3)\). But in this case, \([g_i, g_{i+2}] \in \mathcal{C}(\hat{g}_{i+2})\)
and it does not matter whether neither $\eta$ nor $\kappa$ is equal to 0. Hence, by (6), we obtain $[[g_r, g_{r+1}], g_{r+2}] = 0$, but this contradicts to (2).

1.2. Let $\eta = \kappa = 0$. Then $\hat{g}_{i+2} = \zeta x_0 + \theta x_2$. By analogy with Case 1-1, (6) implies $[[g_{i-1}, g_{i+1}], \hat{g}_i] = 0$. Therefore, $[g_{i-1}, g_{i+1}] \in \mathcal{C}(\hat{g}_i) = \mathcal{C}(x_0) \cap \mathcal{C}(x_2)$. But then $[g_{i-1}, g_{i+1}] \in \mathcal{C}(\hat{g}_{i+2})$, and it does not matter whether neither $\zeta$ nor $\theta$ is equal to 0. Thus, by (6), we obtain $[[g_{i-1}, g_{i+1}], g_{i+2}] = 0$. But this contradicts (2) because if $m \geq 5$ then $(i - 1) - (i + 2) \neq 1$.

2. Let $\{j, k\} \cap \{s, t\}$ have one element. Without loss of generality, we can suppose that $j = s$, $k \neq t$. In this case, (7) implies $z\theta[x_i, x_j] + \beta_i [x_i, x_j] + \beta \delta[x_i, x_s] = 0$. By homogeneity of identities and relations of a partially commutative metabelian Lie algebra, $[x_i, x_j] = [x_s, x_t] = [x_i, x_s] = 0$. Since $j \neq k$ and $j \neq t$, we have $|j - k| = |j - t| = 1$. As $k \neq t$, we obtain $|k - t| = 2$. But then $[x_i, x_s] \neq 0$. It is a contradiction. Consequently, this case is impossible.

3. Finally, let $\{j, k\} = \{s, t\}$. We can assume without loss of generality that $s = j$ and $t = k$. So, $g_{s+1} = x_i + \delta x_1$. By (6) we obtain $[[g_{s-1}, g_{s+1}], \hat{g}_i] = 0$. Thus, $[g_{s-1}, g_{s+1}] \in \mathcal{C}(\hat{g}_i) = \mathcal{C}(x_i) \cap \mathcal{C}(x_s)$. Then $[g_{s-1}, g_{s+1}] \in \mathcal{C}(\hat{g}_{s+1})$. Consequently, $[g_{s-1}, g_{s+1}] \in \mathcal{C}(\hat{g}_{s+1})$. So, by (6) we obtain $[[g_{s-1}, g_{s+1}], g_{s+2}] = 0$, but this contradicts (2).

Taking into account Lemma 3.4, we have considered all cases and have seen that none of them is appropriate. This completes the proof. □

Theorem 3.6. Let R be an integral domain containing $\mathbb{Z}$ as a subring, and let $X = \{x_1, x_2, \ldots, x_m\}$, $Y = \{y_1, y_2, \ldots, y_m\}$. The partially commutative metabelian Lie algebras $M(X; C_n)$ and $M(Y; C_m)$ are universally equivalent iff $m = n$.

Proof. The converse is obvious. Indeed, if $m = n$, then $M(X; C_n)$ and $M(Y; C_m)$ are isomorphic.

Let us prove the direct statement. We can assume without loss of generality that $m > n$. If $n = 3$, then the statement is trivial. Indeed, $M(X; C_3)$ is an abelian Lie algebra. Therefore, the following formula holds in this algebra:

$$\Psi = \forall g, h[g, h] = 0.$$ 

But $\Psi$ does not hold in $M(X; C_n)$ for $m \geq 4$. As a counterexample, we can put $g = x_1$, $h = x_2$. Since $m > n \geq 3$, we have $m \geq 4$, and consequently, $|1 - 3| > 1$. Therefore, $[x_1, x_2] \neq 0$ in $M(X; C_n)$.

Suppose that $m > n \geq 4$. Let $g_0, g_1, \ldots, g_{m-1}$ make true (2) in $M(X; C_n)$. By Lemmas 3.2–3.5, for all $i \in \mathbb{Z}_m$ we have $\hat{g}_i = \alpha_i x_{j_i}$, where $\alpha_i \neq 0$.

Easy to see that $|j_i - j_{i+1}| \leq 1$ for any $i \in \mathbb{Z}_m$. Indeed, (5) and homogeneity of identities and relations in a partially commutative metabelian Lie algebra imply $[\hat{g}_i, \hat{g}_j] = \alpha_{j_i} \alpha_{j_{i+1}} [x_{j_i}, x_{j_{i+1}}] = 0$, and the relation between $j_i$ and $j_{i+1}$ is immediate.

Let us show that there exist $i, k \in \mathbb{Z}_m$ such that $j_i = j_k$ and therefore $x_{j_i} = x_{j_k}$.

If there exists $i \in \mathbb{Z}_m$ such that $x_{j_k} = x_{j_{k+1}}$, then there is nothing to prove. Suppose that $|j_i - j_{i+1}| = 1$ for all $i \in \mathbb{Z}_m$. Chose $l_0$, such that $j_{l_0}$ is a minimal among all $j_i$'s. If $j_{l_0} \neq 0$, then $j_{l_0+1} = j_{l_0-1} = j_{l_0} + 1$. 


Let $j_0 = 0$. If $j_{i+1} = j_{i-1} = 1$ or $j_{i+1} = j_{i-1} = n - 1$, then there is nothing to prove. Thus, we can suppose that $j_{i+1} \neq j_{i-1}$. Then we may assume without loss of generality that $j_{i+1} = 1$, $j_{i-1} = n - 1$. Since $j_i$ are distinct, we obtain

$$j_0 < j_{i+1} \cdots < j_{i+m-1} = j_{i-1}. \quad (9)$$

Indeed, $i_0 + m - 1 = i_0 - 1$ in $\mathbb{Z}_m$, and consequently, $j_{i+1} - j_i = 1$ for all $i \in \mathbb{Z}_m$. But $j_{i-1} = m - 1 > n - 1$. This contradiction shows that the set of inequalities (9) is impossible. Therefore, for some $i, k \in \mathbb{Z}_m$ we have $j_i = j_k$. Then, by (6), we have $[(g_{i-1}, g_{i+1}), \tilde{g}_k] = 0$, and consequently, $[g_{i-1}, g_{i+1}] \in \mathcal{C}(\tilde{g}_k) = \mathcal{C}(x_i)$. This implies $[g_{i-1}, g_{i+1}] \in \mathcal{C}(\tilde{g}_k)$, so $[[g_{i-1}, g_{i+1}], \tilde{g}_k] = 0$. But since $m \geq 5$ and $k \neq i$, at least one of the values $|k - (i - 1)|$ and $|k - (i + 1)|$ is not equal to 1. This contradiction to (2) completes that proof.

4. TO THE QUESTION ON ALGEBRAS WITH THE SAME UNIVERSAL THEORIES

Let $M(X; G)$ be a partially commutative metabelian Lie algebra and $G(X, E)$ its defining graph. For a vertex $x \in X$, let us put $x^⊥ = \{y \mid d(x, y) \leq 1\}$, where $d(x, y)$ is the distance between $x$ and $y$. Namely, it is the least number $l$ such that there exists a path that connects $x$ and $y$ and goes through $l$ edges. Let us introduce a binary relation $\sim_⊥$ on $X$. By definition, put $x \sim_⊥ y$ iff $x_⊥ = y_⊥$. Evidently, this is an equivalence relation.

If $x \sim_⊥ y$ and $x \neq y$, then $x \leftrightarrow y$. Therefore, $x$ and $y$ are in the same connected component of any subgraph of $G$ containing both these vertices.

Let us prove a couple of statements about $\sim_⊥$.

**Lemma 4.1.** Let $\tilde{X}_x^⊥$ be the equivalence class of $x$ with respect to $\sim_⊥$. Then the subgraph $G(\tilde{X}_x^⊥)$ of $G$ is a complete graph. Moreover, for $y \in X \setminus \tilde{X}_x^⊥$ we have $y \leftrightarrow x$ iff $y \leftrightarrow X_x^⊥$.

**Proof.** Let $z \in \tilde{X}_x^⊥$. Then by definition $x^⊥ = z^⊥$. Since clearly $x \in x^⊥$, we have $x \leftrightarrow z$.

Since $\sim_⊥$ is an equivalence relation, we can repeat the arguments above for any vertex in $\tilde{X}_x^⊥$ instead of $x$. So, each vertex in $X_x^⊥$ is adjacent to all other vertices of this set.

Now, let $y \in X \setminus \tilde{X}_x^⊥$. Then $y \leftrightarrow x$ iff $y \in x_⊥$. For any vertex in $z \in \tilde{X}_x^⊥$ we have $z^⊥ = x^⊥$. So, we obtain $y \leftrightarrow z$. Consequently, all vertices in $\tilde{X}_x^⊥$ are adjacent to the same vertices in $X \setminus \tilde{X}_x^⊥$. Thus, $\leftrightarrow \tilde{X}_x^⊥$.

**Lemma 4.2.** Let $G = \langle X; E \rangle$ be a graph, $Y \subseteq X$, and $H = \langle Y; F \rangle$ be a subgraph of $G$ generated by a set $Y$. Let also $x, y \in Y$ be such that $x \sim_⊥ y$ in $G$. Finally, let $H' = H(Y \setminus \{y\})$. Then the number of connected components in $H$ and $H'$ is same. Moreover, two vertices not equal to $y$ are in the same connected component of $H'$ iff they are in the same component of $H$. 


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Proof. Let $z_1$ and $z_2$ be in the same connected component of $H$. Then there exists a path connecting $z_1$ and $z_2$

\[(y_0 = z_1, y_1, \ldots, y_{s-1}, y_s = z_2).\] (10)

If no vertex of this path is equal to $y$, then there is nothing to prove. Assume the converse, Namely, suppose that $y_l = y$ for some $l \in \{1, 2, \ldots, s - 1\}$. Then $y_{l+1} \leftrightarrow y$ and $y_{l+1} \leftrightarrow y$. However, $y \sim x$. Therefore, by Lemma 4.1, $y_{l+1} \leftrightarrow x$ and $y_{l+1} \leftrightarrow x$. So, we can change $y$ by $x$ in the path (10). We get another path connecting $z_1$ and $z_2$. If this path still goes through $y$, let us repeat the procedure described above. Finally, we can obtain a path not going through $y$. Consequently, $z_1$ and $z_2$ are in the same connected component of $H'$.

The converse is obvious because any path in $H'$ is also a path in $H$.

In particular, the number of connected components $H$ and $H'$ is same. This concludes the proof. \bbox

Let $G$ be a graph such that $\sim_1$ is not identical (diagonal) on it, and let $M(X; G)$ be a partially commutative metabelian Lie algebra with the defining graph $G$. Suppose that the set $X'_{\sim_1} \cup \{1\}$ contains more than one element. Denote by $X'$ the set $X \setminus \{x_{n-1}\}$ and by $G'$ the graph $G(X')$. Without loss of generality, it can be assumed that $x_{n-1} \sim x_{n-2}$. From this point on, we consider an order on $X$ for that $x_{n-1}$ and $x_{n-2}$ are two least vertices and $x_{n-2} < x_{n-1}$. The order of other vertices does not matter so we can fix any order having the indicated property. Respectively, we consider the order on $X'$ that is obtained from the order on $X$ by removing $x_{n-1}$.

For any $\lambda \in R \setminus \{0\}$, let us define the map $\varphi_\lambda : X \rightarrow M(X'; G')$ as follows:

\[\varphi_\lambda(x_i) = \begin{cases} x_i, & \text{if } i \neq n - 1; \\ \lambda x_{n-2} & \text{if } i = n - 1. \end{cases} \] (11)

This map can be extended up to a homomorphism from $M(X; G)$ to $M(X'; G')$ uniquely.

Indeed, each homomorphism keeps addition and multiplication. Therefore, we can represent the image of any element as a linear combination of the elements that are the products of generators of $M(X; G)$. Therefore, this homomorphism unique provided it exists.

Let us show that the extension of $\varphi_\lambda$ to the entire $R$-algebra Lie $M(X; G)$ is really a homomorphism (let us denote this extension also by $\varphi_\lambda$). It suffices to check that the extension keeps all identities and relations of the metabelian partially commutative Lie algebra $M(X; G)$. All identities hold in $M(X'; G')$ because $M(X'; G')$ is also a metabelian Lie algebra. Let $[x_i, x_j] = 0$ in $M(X; G)$. It means that $x_i \leftrightarrow x_j$. If $i, j \neq n - 1$, then $\varphi_\lambda([x_i, x_j]) = [\varphi_\lambda(x_i), \varphi_\lambda(x_j)] = [x_i, x_j] = 0$ in $M(X'; G')$. If $[x_i, x_{n-1}] = 0$ for $i \neq n - 1$, then $x_i$ is adjacent to $x_{n-1}$; therefore, $x_i$ is also adjacent to $x_{n-2}$ and so $x_i, x_{n-2} = 0$. We obtain $\varphi_\lambda([x_i, x_{n-1}]) = [\varphi_\lambda(x_i), \varphi_\lambda(x_{n-1})] = \lambda [x_i, x_{n-2}] = 0$.

By $\text{mdeg}([w])$ we also denote the multidegree of a homogeneous element $[w]$ of the algebra $M(X'; G')$. Since it is clear what algebra we are talking about, there is no ambiguity.
Let us set \( \lambda \) and consider \( \varphi_j \). Suppose that \([u]\) and \([v]\) are nonzero Lie monomials of \( M(X; G) \) such that \( \varphi_j([u]) \) and \( \varphi_j([v]) \) are not equal to zero in \( M(X'; G') \). Clearly, \( \text{mdeg}(\varphi([u])) = \text{mdeg}(\varphi([v])) \) iff \( \text{mdeg}([u]) = \text{mdeg}([v]) \). Indeed, easy to see that \( \text{mdeg}(\varphi([u])) = \text{mdeg}([u]) \) and \( \text{mdeg}(\varphi([v])) = \text{mdeg}([v]) \).

Consider an element \( g \in M(X; G) \setminus \{0\} \). There exists the decomposition

\[
g = \sum_{\delta, x_i} g_{\delta, x_i},
\]

where \( g_{\delta, x_i} \) is a nonempty linear combination of basis monomials starting with \( x_i \), appearing with nonzero coefficients, and having the glued multidegree \( \delta \). It is easy to see how to obtain this decomposition. First of all, let us represent \( g \) as a linear combination of basis monomials. Then, for each \( g_{\delta, x_i} \) we need to choose the summands starting with the required generator and having the required glued multidegree. Since basis monomials are linearly independent, all \( g_{\delta, x_i} \) that consist of at least one basis monomial with a non-zero coefficient are not equal to zero in \( M(X; G) \).

Let \( \tilde{\delta}_0 = (e_0, e_1, \ldots, e_{n-2}) \) be a glued multidegree. If representation (12) of \( g \) contains \( g_{\tilde{\delta}_0, x_i} \), then we can write

\[
g_{\tilde{\delta}_0, x_i} = \sum_{j=0}^{e_{n-2}} x_j [u_{i,j}],
\]

where \([u_{i,j}]\) is the monomial of \( M(X; G) \) defined as follows. Its multidegree is

\[
\text{mdeg}([u_{i,j}]) = (e_0, e_1, \ldots, e_{n-3}, e_{n-2} - j, j),
\]

it starts with \( x_i \), and it is a basis monomial in the case of \( x_i \neq 0 \). Let us also notice that by definition of \( g_{\tilde{\delta}_0, x_i} \) there exists \( j \) such that \( x_j \neq 0 \).

**Lemma 4.3.** If \([u_{i,j}]\) is a basis monomial for some \( j \), then \([u_{i,0}]\) is also a basis monomial.

**Proof.** If \( j = 0 \), then there is nothing to prove.

Let \( j = e_{n-2} \). Since \( x_{n-2} \sim_{e_{n-2}} x_{n-1} \), the graphs \( G(X_{[u_{n-2}, x_{n-2}]} \) and \( G(X_{[u_{n-1}]} \) are clearly isomorphic and the map taking \( x_{n-1} \) to \( x_{n-2} \), and \( x_i \) to \( x_i \) for all other \( i \) is an isomorphism. Moreover, \( x_{n-1} \) is the least generator, and therefore, \( x_{n-2} \) is on the second place in \([u_{i,j}]\). Thus, the described isomorphism of graphs takes \( x_{n-1} \) to \( x_{n-2} \) and \( x_{n-2} \) is the least generator in \( X_{[u_{i,0}]} \). Therefore, \( x_{n-2} \) is on the second place in \( \varphi_j([u_{i,j}]) \). So, if \([u_{i,e_{n-2}}]\) is a basis monomial, then \( x_i \) and \( x_{n-1} \) are in different connected components of \( G(X_{[u_{i,e_{n-2}}]} \). Consequently, the images of these vertices are also in different connected components of \( G(X_{[u_{i,0}]} \).

Finally, suppose that \( j \neq 0, e_{n-2} \). By Lemma 4.2, it is easy to see that \( x_i \) and \( x_{n-1} \) are in different connected components, i.e., \( x_i \) and \( x_{n-2} \) are also in different connected components.

By Lemma 4.2, for all graphs \( G(X_{[u_{i,j}]} \), the connected components not containing \( x_{n-2} \) and \( x_{n-1} \) are same. Therefore, if \( x_i \) is the largest vertex in its
connected component of $G(X_{[u,0]})$ it is also the largest vertex in the corresponding component of $G(X_{[u,0]})$. □

**Corollary 4.4.**

1. $\varphi_\lambda(g_{\tilde{\delta},x_i}) = 0$ iff $\sum_{j=0}^{\infty} g_{\tilde{\lambda},x_j} = 0$.
2. If $\varphi_\lambda(g_{\tilde{\delta},x_i}) \neq 0$ then this is a multiple of some basis monomial.

**Proof.** Note that

$$\varphi_\lambda([u_{i,j}]) = \lambda^j[u_{i,0}].$$

We may write this because $X_{[u,0]} \subseteq X'$ and so $[u_{i,0}]$ can be considered as an element of $M(X'; G')$.

Now, one can easily obtain both assertions from Lemma 4.3. □

**Lemma 4.5.** Let $g$ be a nonzero element of $M(X; G)$, and let $\varphi_\lambda$ a homomorphism defined by (11) for some $\lambda \in R \setminus \{0\}$. Then $\varphi_\lambda(g) = 0$ in $M(X'; G')$ iff $\varphi_\lambda(g_{\tilde{\delta},x_i}) = 0$ for all components $g_{\tilde{\delta},x_i}$ of decomposition (12).

**Proof.** Let $g$ be a nonzero element of $M(X; G)$ such that $\varphi_\lambda(g) = 0$. By (12), we obtain $\sum_{\tilde{\delta},x_i} \varphi_\lambda(g_{\tilde{\delta},x_i}) = \varphi_\lambda(g) = 0$. Suppose that $\varphi_\lambda(g_{\tilde{\delta},x_i}) \neq 0$ for some glued multidegree $\tilde{\delta} = (\varepsilon_0, \ldots, \varepsilon_{n-2})$ and some generator $x_i$. Show that the following expression is not equal to zero in $M(X'; G')$:

$$\varphi_\lambda \left( \sum_i g_{\tilde{\delta}_i,x_i} \right) = \sum_i \varphi_\lambda(g_{\tilde{\delta}_i,x_i}).$$

Indeed, by (15) and Corollary 4.4, we obtain $\sum_{\tilde{\delta}_i} \varphi_\lambda(g_{\tilde{\delta}_i,x_i}) = \sum_{\tilde{\delta}_i} \beta_i [u_{i,0}]$, where $\beta_i$ are some elements in $R$. Since $\beta_i [u_{i,0}] = \varphi_\lambda(g_{\tilde{\delta}_i,x_i})$, some $\beta_i$ are different. Therefore, $\sum_i \varphi_\lambda(g_{\tilde{\delta}_i,x_i}) \neq 0$ in $M(X'; G')$. But this is impossible. Indeed, by homogeneity of identities and relations a partially commutative metabelian Lie algebra, if a Lie polynomial is equal to zero in $M(X'; G')$, then all summands of the decomposition of this polynomial as the sum of homogeneous elements should also be equal to zero in $M(X'; G')$. We are left to notice that $\varphi_\lambda(\sum_{\tilde{\delta}_i} g_{\tilde{\delta}_i,x_i})$ is just such summand. □

**Lemma 4.6.** Let $g \in M(X; G) \setminus \{0\}$. Then there exists $\lambda_0 \in \mathbb{Z}^+$ such that $\varphi_\lambda(g) \neq 0$ for any $\lambda \geq \lambda_0$.

**Proof.** It follows from Lemma 4.5 that $\varphi_\lambda([g]) \neq 0$ in $M(X'; G')$ iff $\varphi_\lambda(g_{\tilde{\delta},x_i}) = 0$ for all $g_{\tilde{\delta},x_i}$ appearing in the decomposition (12). Therefore, it suffices to prove the assertion of the lemma in the case of $g = g_{\tilde{\delta},x_i}$ that is not equal to zero in $M(X; G)$, where $x_i$ is a generator and $\tilde{\delta} = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{n-2})$ is a glued multidegree.
Let \( g = \sum_{j=0}^{e_{n-2}} x_j [u_{i,j}] \) where \( u_{i,j} \) is a basis monomial such that it starts with \( x_i \) and \( \text{mdeg}([u_{i,j}]) = (e_0, e_1, \ldots, e_{n-3}, e_{n-2} - j, j) \). By (14), we have \( \varphi(g) = (\sum_{j=0}^{e_{n-2}} x_j \lambda^j)[u_{0,0}] \). Consider the polynomial \( p(\lambda) = \sum_{j=0}^{e_{n-2}} x_j \lambda^j \). Since \( R \) is an integral domain, this polynomial has at most \( e_{n-2} \) positive integer roots. Thus, \( \lambda_0 \) can be chosen by one greater than the largest positive integer root of \( p(\lambda) \). If \( p(\lambda) \) has no positive integer root, we can take, for example, \( \lambda_0 = 1 \).

**Theorem 4.7.** Let \( X \) be a finite set, \( G \) a graph. Suppose that there exists \( x \in X \) such that the \( \sim_\perp \)-equivalence class \( \overline{X}_\perp \) has more than one element. Finally, let \( X' = X \setminus \{x\} \) and \( G' = G(X') \). Then the algebras \( M(X; G) \) and \( M(X'; G') \) are universally equivalent.

**Proof.** By Theorem 2.8, it suffices to show that for each finite submodel in \( M(X; G) \) there exists an isomorphic submodel in \( M(X'; G') \) and vice versa.

The converse is obviously true because \( M(X'; G') \) is a subalgebra of \( M(X; G) \). Let us prove the direct statement.

Let \( x = x_{n-1} \) and \( x_{n-2} \sim_\perp x_{n-1} \). Let also \( \Gamma = \{g_1, \ldots, g_m\} \) be a finite set of the elements of \( M(X; G) \). Extend \( \Gamma \) adding the elements \( g_i - g_j, g_i + g_j - g_k, [g_i, g_j] - g_k \) for all \( i, j, k = 1, 2, \ldots, m \), and denote by \( \Gamma' \) the obtained set. It is sufficient to show that there exists \( \lambda \) such that the kernel of \( \varphi : M(X; G) \to M(X'; G') \) is disjoint with \( \Gamma \). If it is the case, then the images of the elements in \( \Gamma \) are distinct. Moreover, if \( g_i \neq g_j + g_k \) or \( g_i \neq [g_j, g_k] \), then the images of \( g_i \) and \( g_j + g_k \) (images of \( g_i \) and \( [g_j, g_k] \) respectively) are not equal either.

By Lemma 4.6, for any nonzero \( g \in \Gamma' \), there exists \( \lambda_0(g) \) such that for any \( \lambda \geq \lambda_0(g) \) the following inequality holds: \( \varphi_\lambda(g) \neq 0 \). Let \( \lambda_0 \) be maximal among \( \lambda_0(g) \) for all \( g \in \Gamma' \). Then for any \( \lambda \geq \lambda_0 \) and any \( g \in \Gamma' \), we obtain \( \varphi_\lambda(g) \neq 0 \).

So, the universal theories of \( M(X; G) \) and \( M(X'; G') \) coincide. \( \Box \)

Let \( G = \langle X; E \rangle \) be a graph. Suppose that there exists a \( \sim_\perp \)-equivalence class containing at least two vertices. Let \( x \) be a vertex of such class, \( X' = X \setminus \{x\} \), and \( G' = G(X') \). Then by Theorem 4.7, the universal theories of \( M(X; G) \) and \( M(X'; G') \) coincide. Moreover, it is easy to see that for any equivalence relation if we remove an element from any equivalence class, then all other elements of this class still remain in the same equivalence class and other equivalence classes do not change.

So, if an obtained graph still contains a \( \sim_\perp \)-equivalence class with at least two vertices, then we can repeat the procedure described above. By Theorem 4.7, we again get a partially commutative metabelian Lie algebra that is universally equivalent to the initial one and so on.

Let \( G \) be any graph. We can remove all but one vertices from each \( \sim_\perp \)-equivalence class of this graph. The universal theories of the initial and final graphs coincide. The obtained graph is called the compaction of \( G \). Let us denote it by \( \overline{G} \).

Figure 1, there is an example of two graphs with the same universal theories such that one of them is a tree while the other one is not.

Finally, let us note that the converse of Theorem 4.7 is not true. Namely, even if the algebras \( M(X; G) \) and \( M(Y; H) \) are universally equivalent it does not mean that we can obtain \( H \) from \( G \) adding and removing the vertices to \( \sim_\perp \)-equivalence classes. Indeed, it is easy to see that all compactions of a graph are isomorphic. On the other hand, a compaction of any tree with more than two vertices is this tree
itself. But in [11], it was shown that if defining graphs of two partially commutative metabelian Lie algebras are trees then these algebras can be universally equivalent even if their defining graphs are not isomorphic.

**FUNDING**

The author is supported by the Russian Foundation for Basic Research (Grant 12-01-00084) and by the Ministry of Science of Russian Federation (state assignment No. 214/138, project 1052).

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