Newton polytope of good symmetric functions

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Abstract

We introduce a general class of symmetric functions that has saturated Newton polytope and their Newton polytope has integer decomposition property. The class covers numerous previously studied symmetric functions.

2020 Mathematics Subject Classification. 52B20, 05E05.
Keywords and phrases. Newton polytope, Symmetric functions.

1 Introduction

In combinatorics, if a convex polytope equals the convex hull of its integer points, we say that it is a lattice polytope. Studying lattice polytopes is important because of their connections in many other domains. For instance, in mathematical optimization, if a system of linear inequalities defines a polytope, then we can use linear programming to solve integer programming problems for this system (see [Bar17]). In algebraic geometry, lattice polytopes are used to study projective toric varieties (see [CLS11], [Ful16]). In particular, there is an important class of lattice polytopes, which are the convex hull of exponent vectors of a polynomial, say Newton polytope. The Newton polytope is a central object in tropical geometry (see [KKE21]), and they are used to characterizing Grobner bases (see [Stu96]).

Lattice polytopes are studied by Ehrhart polynomials (see [Eug62]). Important properties of Ehrhart polynomials such as unimodality and log-concavity are related to the integer decomposition property (IDP) of the lattice polytope (see [OH06,BR07,SVL13]). In [BGH+21], the authors studied the Newton polytope of inflated symmetric Grothendieck polynomials. The integer decomposition property is a consequence of the saturated property (SNP). The result is a generalization of Schur functions, symmetric Grothendieck polynomials in [EY17].

In this paper, we introduce a general class of symmetric functions that has SNP and with Newton polytope has IDP (see Theorem 4.2 and Corollary 4.3). Our class covers symmetric functions in [EY17,MTY19,BGH+21,MMS22]: Schur functions, Stembridge’s symmetric polynomials, Stanley’s symmetric polynomials, Reutenauer’s symmetric polynomials, cycle index polynomials, Schur $P$-polynomials and Schur $Q$-polynomials, generic $(q,t)$-evaluation of a symmetric Macdonald polynomials, classical resultants, inflated Grothendieck polynomials, chromatic symmetric functions co-bipartite graphs, Dyck paths, $(3+1)$-free posets. It also covers other symmetric functions, for instance, dual Grothendieck polynomials in [LP07].

Acknowledgments: This paper was written while the second author visited Vietnam Institute for advanced study in Mathematics (VIASM). He would like to thank VIASM for...
very kind support and hospitality. The third author is supported by UAlbany Research Foundation.

2 Newton polytope

A polytope $\mathcal{P}$ in $\mathbb{R}^m$ is the convex hull $Conv(v_1, \ldots, v_k)$ of finite many points $v_1, \ldots, v_k \in \mathbb{R}^m$. The vertex set of $\mathcal{P}$ is the minimal set $V$ in $\mathbb{R}^m$ such that $\mathcal{P} = Conv(V)$. Algebraically, a point $v \in \mathcal{P}$ is a vertex if, $v = tw + (1-t)u$ for some $w, u \in \mathcal{P}$, $t \in (0,1)$ implies $w = u = v$. We say that $\mathcal{P}$ is a lattice polytope if $V$ is a subset of $\mathbb{Z}^m$.

**Example 2.1.** The convex hull $\mathcal{P}$ of twelve points in $\mathbb{R}^3$ below is a lattice polytope.

$$
(3,1,0), (3,0,1), (1,0,3), (0,1,3), (0,3,1), (1,3,0),
(2,2,0), (2,0,2), (0,2,2),
(2,1,1), (1,1,2), (1,2,1).
$$

The permutations of $(3,1,0)$ are vertices of the polytope $\mathcal{P}$. In the picture below, $\mathcal{P}$ is the blue hexagon.

Let $\mathcal{P}$ be a lattice polytope. For a positive integer $t$, let $t\mathcal{P} = \{tv \mid v \in \mathcal{P}\}$. We say that $\mathcal{P}$ has integer decomposition property (IDP) if, for any positive integer $t$ and $p \in t\mathcal{P} \cap \mathbb{Z}^m$, there are $t$ points $v_1, \ldots, v_t \in \mathcal{P} \cap \mathbb{Z}^m$ such that $p = v_1 + \cdots + v_t$.

**Example 2.2.** Let $\mathcal{P}$ be the lattice polytope in Example 2.1. Then $3\mathcal{P}$ is the convex hull of six points

$$
(9,3,0), (9,0,3), (3,0,9), (0,3,9), (0,9,3), (3,9,0).
$$

We see that $(9,2,1) \in 3\mathcal{P} \cap \mathbb{Z}^3$ and is the sum of three points in $\mathcal{P} \cap \mathbb{Z}^3$.

$$(9,2,1) = (3,1,0) + (3,1,0) + (3,0,1).$$

Let $f(x) = \sum_{\alpha \in \mathbb{Z}^m_{\geq 0}} c_\alpha x^\alpha \in \mathbb{C}[x_1, \ldots, x_m]$. The support of $f$ is defined by

$$Supp(f) = \{\alpha \in \mathbb{Z}^m_{\geq 0} \mid c_\alpha \neq 0\}.$$ 

The Newton polytope of $f$ is defined by

$$Newton(f) = Conv(Supp(f)).$$

We say that $f$ has saturated Newton polytope (SNP) if $\alpha \in Newton(f) \cap \mathbb{Z}^m$, then $\alpha \in Supp(f)$. 


Example 2.3. Let \( f(x_1, x_2, x_3) \) be the polynomial

\[
x^{(3,1,0)} + x^{(3,0,1)} + x^{(1,0,3)} + x^{(0,1,3)} + x^{(0,3,1)} + x^{(1,3,0)} \\
+ x^{(2,2,0)} + x^{(2,0,2)} + x^{(0,2,2)} \\
+ 2x^{(2,1,1)} + 2x^{(1,1,2)} + 2x^{(1,2,1)}.
\]

The set \( \text{Supp}(f) \) contains twelve points in Example 2.1. Then \( \text{Newton}(f) \) is the polytope \( \mathcal{P} \) in Example 2.1. Since \( \text{Newton}(f) \cap \mathbb{Z}^3 = \text{Supp}(f) \), \( f \) has SNP.

3 Schur functions

A partition with at most \( m \) parts is a sequence of weakly decreasing nonnegative integers \( \lambda = (\lambda_1, \ldots, \lambda_m) \). The size of partition \( \lambda \) is defined by \( |\lambda| = \sum_{i=1}^{m} \lambda_i \). Each partition \( \lambda \) is presented by a Young diagram \( Y(\lambda) \) that is a collection of boxes such that the leftmost boxes of each row are in a column, and the numbers of boxes from top row to bottom row are \( \lambda_1, \lambda_2, \ldots \), respectively. A semistandard Young tableau of shape \( \lambda \) with entries from \( \{1, \ldots, m\} \) is a filling of the Young diagram \( Y(\lambda) \) by the ordered alphabet \( \{1 < \cdots < m\} \) such that the entries in each column are strictly increasing from top to bottom, and the entries in each row are weakly increasing from left to right. A Young tableau \( T \) is said to have content \( \alpha = (\alpha_1, \alpha_2, \ldots) \) if \( \alpha_i \) is the number of entries \( i \) in the tableau \( T \). We write

\[ x^T = x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots. \]

For each partition \( \lambda \) with at most \( m \) parts, the Schur function \( s_\lambda(x_1, \ldots, x_m) \) is defined as the sum of \( x^T \), where \( T \) runs over the semistandard Young tableaux of shape \( \lambda \) with filling from \( \{1, \ldots, m\} \).

Example 3.1. Vector \((3,1,0)\) is a partition. The Young diagram of \((3,1,0)\) is

\[
\begin{array}{ccc}
\& \& \\
\& \& \\
\& \& \\end{array}
\]

The following filling is a semistandard tableau of shape \((3,1,0)\) and content \((1,2,1)\).

\[
\begin{array}{ccc}
1 & 2 & 3 \\
\& 2 \\
\end{array}
\]

Schur function \( s_{(3,1,0)}(x_1, x_2, x_3) \) is the polynomial \( f \) in Example 2.3.

4 Good symmetric functions

Let \( \alpha \) and \( \beta \) be partitions with at most \( m \) parts. We say \( \beta \) is bigger than \( \alpha \) and write \( \beta \geq \alpha \) if and only if \( \beta_i \geq \alpha_i \) for all \( i \). If \( \alpha, \beta \) are partitions of the same size, we say \( \beta \) dominates \( \alpha \) and write \( \beta \succeq \alpha \) if \( \sum_{i=1}^{j} \beta_i \geq \sum_{i=1}^{j} \alpha_i \) for all \( j \geq 1 \).

Example 4.1. \((3,1,0) < (3,3,3)\) and \((3,2,0) \succeq (3,1,1)\).
Let $F(x_1, \ldots, x_m)$ be a linear combination of Schur functions associated to partitions with at most $m$ parts. We can collect Schur functions appearing in $F$ associated with partitions of the same size to a bracket. We say that $F$ is **good** if it satisfies the following conditions:

(a) The support of each bracket equals the union of supports of its Schur elements.

(b) Suppose that there are $l + 1$ brackets in condition (a). In each bracket, we pick up the $\triangleright$-maximum partition if exists. Then $\triangleright$-maximum partitions have form

$$\alpha = \lambda^0 < \cdots < \lambda^l = \beta,$$

where $\alpha \leq \beta$ are fixed partitions and for each $i > 0$, $\lambda^i$ is obtained from $\lambda^{i-1}$ by adding a box in the northmost row of $\lambda^{i-1}$ such that the addition gives a Young diagram, $\alpha < \lambda^i \leq \beta$.

**Theorem 4.2.** Let $F$ be a good linear combination of Schur functions. Then $F$ has SNP and $\mathrm{Newton}(F)$ has IDP.

**Corollary 4.3.** Let $F$ be a linear combination of Schur functions such that it satisfies condition (b). If any two Schur functions in the same bracket of $F$ have the same sign, then $F$ is good. In particular, $F$ has SNP and $\mathrm{Newton}(F)$ has IDP.

**Proof.** The sign condition is a particular case of condition (a). \hfill $\Box$

**Example 4.4.** Let $F(x_1, x_2, x_3)$ be

$$s_{(3,1,0)} - (3s_{(3,2,0)} + 6s_{(3,1,1)}) + (3s_{3,3,0} + 18s_{(3,2,1)}) - (18s_{(3,3,1)} + 4s_{(3,2,2)}) + 44s_{(3,3,2)} - 55s_{(3,3,3)}.$$

Schur functions in the same bracket have the same sign. The $\triangleright$-maximum partitions $\lambda^i$ for $i = 0, \ldots, 5$ chosen from brackets have form

$$\alpha = (3,1,0) < (3,2,0) < (3,3,0) < (3,3,1) < (3,3,2) < (3,3,3) = \beta.$$

$F$ is a good symmetric function.

We need the following facts to prove Theorem 4.2.

**Proposition 4.5.** ([Rad52, Proposition 2.5]) Let $\alpha, \beta$ be partitions of the same size. Then, $\mathrm{Newton}(s_\alpha) \subseteq \mathrm{Newton}(s_\beta)$ if and only if $\alpha \triangleright \beta$.

**Lemma 4.6.** ([EY17, Theorem 0.1]) Let $\alpha$ be a partition with at most $m$ parts. Then $s_\alpha$ has SNP with $\mathrm{Newton}$ polytope being the convex hull of the $S_m$-orbit of $\alpha$.

**Proof of Theorem 4.2.** We first prove that $F$ has SNP.

1. Let $F = \sum_\mu C_\mu s_\mu$ with $C_\mu \neq 0$. By condition (a) of $F$, we have

$$\mathrm{Supp}(F) = \bigcup_\mu \mathrm{Supp}(s_\mu).$$

Then

$$\mathrm{Newton}(F) = \mathrm{Conv}(\bigcup_\mu \mathrm{Supp}(s_\mu)).$$

We need to prove that $\mathrm{Newton}(F)$ has IDP. By Lemma 4.6, $s_\alpha$ has SNP with Newton polytope being the convex hull of the $S_m$-orbit of $\alpha$. Therefore, $F$ has SNP.

The support of each bracket equals the union of supports of its Schur elements.

Then

$$\mathrm{Newton}(F) = \mathrm{Conv}(\bigcup_\mu \mathrm{Supp}(s_\mu)).$$

By Proposition 4.5, $\mathrm{Newton}(s_\alpha) \subseteq \mathrm{Newton}(s_\beta)$ if and only if $\alpha \triangleright \beta$.

**Proof of Corollary 4.3.** We first prove that $F$ has SNP.

1. Let $F = \sum_\mu C_\mu s_\mu$ with $C_\mu \neq 0$. By condition (a) of $F$, we have

$$\mathrm{Supp}(F) = \bigcup_\mu \mathrm{Supp}(s_\mu).$$

Then

$$\mathrm{Newton}(F) = \mathrm{Conv}(\bigcup_\mu \mathrm{Supp}(s_\mu)).$$

The support of each bracket equals the union of supports of its Schur elements.

Then

$$\mathrm{Newton}(F) = \mathrm{Conv}(\bigcup_\mu \mathrm{Supp}(s_\mu)).$$

By Proposition 4.5, $\mathrm{Newton}(s_\alpha) \subseteq \mathrm{Newton}(s_\beta)$ if and only if $\alpha \triangleright \beta$. \hfill $\Box$
Let $\alpha = \lambda^0 < \lambda^1 < \cdots < \lambda^l = \beta$ be the $\triangleright$-maximum partitions in condition (b) of $F$. By Lemma 4.6, the right-hand side of (2) is

$$\bigcup_{\mu} \text{Supp}(s_{\mu}) = \bigcup_{i=0}^l \text{Supp}(s_{\lambda^i}).$$

(4)

Therefore, by (2), (4),

$$\text{Supp}(F) = \bigcup_{i=0}^l \text{Supp}(s_{\lambda^i}).$$

(5)

By Proposition 4.5,

$$\text{Conv}(\bigcup_{\mu} \text{Supp}(s_{\mu})) = \text{Newton}(s_{\mu}) \subseteq \text{Newton}(s_{\lambda^i}) = \text{Conv}(\text{Supp}(s_{\lambda^i}))$$

for some $i$. It implies that the right-hand side of (3) is

$$\text{Conv}(\bigcup_{\mu} \text{Supp}(s_{\mu})) = \text{Conv}(\bigcup_{i=0}^l \text{Newton}(s_{\lambda^i})).$$

(6)

Hence by (3), (6), we have

$$\text{Newton}(F) = \text{Conv}(\bigcup_{i=0}^l \text{Newton}(s_{\lambda^i})).$$

(7)

2. Let $p$ be a point in $\text{Newton}(F) \cap \mathbb{Z}^m$. By (7), $p$ has form $p = \sum_{i=0}^l c_i v^i$ for some $v^i \in \text{Newton}(s_{\lambda^i})$, and some $c_i \geq 0$, $\sum_{i=1}^l c_i = 1$. We see that $v^i$ is not a partition in general. However, if we denote the sum of its coordinates by $|v^i|$, then $|v^i| = |\lambda^i|$. Then $|p| = \sum_{i=0}^l c_i |\lambda^i|$ is between $|\lambda^0|$ and $|\lambda^l|$, because of (1). Thus $|p| = |\lambda^j|$ for some $j \in [0, l]$, because $\lambda^i$ is obtained from $\lambda^{i-1}$ by adding a box. Let $p'$ be the partition obtained from $p$ by rearranging coordinates of $p$ in descending order. We have $p' \leq \beta^j$ by the same arguments of [EY17, Claim C]. By Lemma 4.6, Proposition 4.5, $p$ is a point in

$$\text{Newton}(s_{p'}) \cap \mathbb{Z}^m \subseteq \text{Newton}(s_{\lambda^j}) \cap \mathbb{Z}^m = \text{Supp}(s_{\lambda^j}) \subseteq \text{Supp}(F).$$

(8)

Therefore we conclude that $F$ has SNP.

Now we show that $\text{Newton}(F)$ has IDP.

1. We have proven that $F$ has SNP. Then by (5), Lemma 4.6, we have

$$\text{Newton}(F) \cap \mathbb{Z}^m = \text{Supp}(F) = \bigcup_{i=0}^l \text{Supp}(s_{\lambda^i}) = \bigcup_{i=0}^l \text{Newton}(s_{\lambda^i} \cap \mathbb{Z}^m).$$

(9)

2. Suppose that $\alpha = (\alpha_1, \ldots, \alpha_m)$ and $\beta = (\beta_1, \ldots, \beta_m)$. For $i = 1, \ldots, m - 1$, set $\lambda^i = (\beta_1, \ldots, \beta_i, \alpha_{i+1}, \ldots, \alpha_m)$. Set $\lambda^0 = \alpha$, $\lambda^m = \beta$. Then $\alpha = \lambda^0 < \cdots < \lambda^m = \beta$ is a subchain of (1). We have

$$\text{Newton}(F) = \text{Conv}(\bigcup_{i=0}^m \text{Newton}(s_{\lambda^i})).$$

(10)

Indeed, $\text{Newton}(F)$ is the convex hull of its vertex set. We can get (10) from (7) by showing that a partition $\lambda^j$ not of form $\lambda^i$ is not a vertex of $\text{Newton}(F)$. It is trivial because $\lambda^j = \frac{1}{2}(\lambda^{j-1} + \lambda^{j+1})$.  



5
3. For a positive integer $t$, we construct a chain of form (1)

$$ta = \Lambda^0 < \cdots < \Lambda^L = t\beta. \tag{11}$$

Set $F_t = \sum_{i=0}^{L} s_{\Lambda^i}$. Then $F_t$ is a good linear combination of Schur functions and $\Lambda^{(i)} = t\lambda^{(i)}$ for each $i = 0, \ldots, m$. By (10), we have

$$\text{Newt}(F_t) = \text{Conv}(\bigcup_{i=0}^{m} \text{Newt}(s_{\Lambda^{(i)}})) = t\text{Conv}(\bigcup_{i=0}^{m} \text{Newt}(s_{\Lambda^{(i)}}))$$

$$= t\text{Newt}(F). \tag{12}$$

4. Let $p$ a point in $t\text{Newt}(F) \cap \mathbb{Z}^m$. By (12), $p$ is a point in $\text{Newt}(F_t) \cap \mathbb{Z}$. Since $F_t$ has SNP, by (9), it is a point in $\text{Newt}(s_{\Lambda^i}) \cap \mathbb{Z}$ for some $\Lambda^i$ in (11). Hence, $p$ is the content of some semistandard tableau $T$ of shape $\Lambda^i$ with filling from $\{1, \ldots, m\}$. For $j = 1, \ldots, t$, let $T_j$ be the semistandard tableau obtained by taking $j'$-th column of $T$ for $j' \equiv j \mod t$. Let $\theta(j)$ be the shape of tableau $T_j$. Let $v_j$ be the content of tableau $T_j$. Then $p = v_1 + \cdots + v_t$. We also have $\alpha \leq \theta(j) \leq \beta$. So there is a unique partition $\lambda^k$ in chain (1) such that $\theta(j) \preceq \lambda^k$. Then by Proposition 4.5, $v_j$ is a point in $\text{Newt}(s_{\theta(j)}) \cap \mathbb{Z}^m \subseteq \text{Newt}(s_{\lambda^k}) \cap \mathbb{Z}^m$.

So by (9), $v_j$ is a point of $\text{Newt}(F) \cap \mathbb{Z}^m$. Therefore we conclude that $\text{Newt}(F)$ has IDP.

\[\square\]

**Remark 4.7.** In general, we do not have equality (2). Instead, we know

$$\text{Supp}(F) \subseteq \bigcup_{\mu} \text{Supp}(s_{\mu}).$$

For example,

$$\text{Supp}(s_{(3,1,0)} - s_{(2,2,0)}) \subseteq \text{Supp}(s_{(3,1,0)}) \cup \text{Supp}(s_{(2,2,0)}).$$

**Example 4.8.** We look at Example 4.4. The subchain $\lambda^{(i)}$ for $i = 0, \ldots, 3$ in the proof of Theorem 4.2 is

$$\alpha = (3,1,0) = (3,1,0) < (3,3,0) < (3,3,3) = \beta.$$ 

In this case, $\lambda^{(0)} = \lambda^{(1)}$. The vertex set of $\text{Newt}(F)$ is the union of $S_3$-orbits of partitions $(3,1,0), (3,3,0), (3,3,3)$.

5 Applications

Theorem 4.2 covers the following cases. Known results are:

- SNP and IDP of inflated symmetric Grothendieck polynomials $G_{h,\lambda}$ (see [EY17, Theorem 0.1], [BGH+21, Proposition 21, Theorem 27]). Indeed, condition (a) comes from the validity of sign condition in Corollary 4.3. The partitions $\alpha, \beta$ in condition (b) are $\lambda$, the $\geq$-maximum of the set $A(h, \lambda)$ in [BGH+21, Lemma 18], respectively.
• SNP and IDP of the following symmetric functions: Stembridge’s symmetric polynomials, Stanley’s symmetric polynomials, Reutenauer’s symmetric polynomials, cycle index polynomials, Schur $P$-polynomials and Schur $Q$-polynomials, generic $(q,t)$-evaluation of a symmetric Macdonald polynomials, classical resultants (see [MTY19, Theorem 2.20, Theorem 2.30, Theorem 2.32, Theorem 2.28, Theorem 3.1, Proposition 3.5, Proposition 3.6, Theorem 5.8]). The SNP of these polynomials are inferred by [MTY19, Propostion 2.5], that is a particular case of Corollary 4.3.

• SNP and IDP of chromatic symmetric functions of co-bipartite graphs, Dyck paths, $(3+1)$-free posets (see [MMS22, Proposition 3.1, Theorem 4.1, Theorem 5.7]). Indeed, the condition (a) come from [MMS22, Proposition 2.9]. The partitions $\alpha, \beta$ in condition (b) are the same.

Unknown results are:

• SNP and IDP of dual Grothendieck polynomials $g_\lambda$ (see [LP07]). Indeed, by [LP07, Theorem 9.8], condition (a) comes from the validity of sign condition in Corollary 4.3. The partitions $\alpha, \beta$ in condition (b) are $(\lambda_1), \lambda$, respectively.

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