Log-decay $F$-isocrystals on higher dimensional varieties

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Abstract

Let $k$ be a perfect field of positive characteristic and let $X$ be a smooth irreducible quasi-compact scheme over $k$. The Drinfeld-Kedlaya theorem states that for an irreducible $F$-isocrystal on $X$, the gap between consecutive generic slopes is bounded by one. In this note we provide a new proof of this theorem. Our proof utilizes the theory of $F$-isocrystals with $r$-log decay. We first show that a rank one $F$-isocrystal with $r$-log decay is overconvergent if $r < 1$. Next, we establish a connection between slope gaps and the rate of log-decay of the slope filtration. The Drinfeld-Kedlaya theorem then follows from a simple patching argument.

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1 Introduction

1.1 Motivation

Let $k$ be a perfect field of positive characteristic and let $X$ be a smooth irreducible quasi-compact scheme over $k$. When studying motives over $X$, one typically studies their $\ell$-adic realization for some $\ell \neq p$. These are lisse $\ell$-adic sheaves on $X$, which correspond to continuous $\ell$-adic representations of $\pi_1^\et(X)$. While lisse $\ell$-adic sheaves are sufficient for studying the $\ell$-adic and archimedean properties of a motive, thus far they have been insufficient for studying $p$-adic questions. For example, for a smooth proper fibration $f : Y \to X$, we know that the Frobenius eigenvalues of $R^i_f \eta_\ell(Q_k)$ at a closed point $x \in X$ has $\ell$-adic valuation zero, but there does not exist such a sweeping general statement about the $p$-adic valuations. In general, the $p$-adic valuations will change as $x$ varies. It is therefore natural to ask how the $p$-adic valuations behave as one varies $x$ (i.e. how does the $p$-adic Newton polygon of the characteristic polynomial of the Frobenius vary). By considering the $p$-adic realization of a motive, which are $F$-isocrystals, there are several beautiful statements about the variation of these Newton polygons. The first general result along these lines is due to Grothendieck, and says that the Newton polygon of a generic point lies below the Newton polygon of any specialization. Another significant result is the de Jong-Oort purity theorem, which tells us these Newton polygons are constant on an open subscheme and jump on a closed subscheme of codimension one. More recently, we have the Drinfeld-Kedlaya theorem (see [4]). This theorem states that for an irreducible $F$-isocrystal, the gaps between slopes of the generic Newton polygon are bounded by one. The purpose of this article is to provide a new proof of this theorem using $F$-isocrystals with logarithmic decay. Along the way, we describe an interesting connection between the slope filtration and the log-decay condition.

1.2 Statement of the main result and proof outline

Let $M$ be either a convergent $F$-isocrystal or an overconvergent $F$-isocrystal on $X$ whose rank is $n$ (see [3]. For any point $x \in X$, we may associate to $M|_x$ rational numbers $a_1^x(M), \ldots, a_n^x(M) \in \Q$,
The proof of Drinfeld and Kedlaya in [4] can be summarized as follows: first they prove that if

$$|a^{i+1}_\eta(M) - a^i(M)| \leq 1,$$

for each $i$.

In the first step of the proof of Theorem 1.1 we study a rank one isocrystals $M$ on $\mathbb{G}_{m,k}^n \times \mathbb{A}_k^m$ with $r$-log decay. In Theorem 1.2 we prove that if $r < 1$ then a tensor power $M \otimes p^k$ has a log-connection. When $M$ has a compatible Frobenius structure, we find that a higher tensor power extends to $\mathbb{A}_k^{n+m}$. This implies that the representation of $\pi^\text{et}(\mathbb{G}_{m,k}^n \times \mathbb{A}_k^m)$ corresponding to $M$ is potentially unramified at the coordinate planes, and thus $M$ is overconvergent by a result of Kedlaya (see [12, Theorem 2.3.7]). It would be interesting to know if all isocrystals on $\mathbb{G}_{m,k}^n \times \mathbb{A}_k^m$ with $r$-log-decay are overconvergent when $r < 1$.

For the second step of the proof, we consider an overconvergent $F$-isocrystal $M$ on $\mathbb{G}_{m,k}^n \times \mathbb{A}_k^m$ whose Newton polygon remains constant for each point $x \in \mathbb{G}_{m,k}^n \times \mathbb{A}_k^m$. Thus $M$ obtains a slope filtration in the category of convergent $F$-isocrystals. That is, if $a_x^i(M) < a_x^{i+1}(M)$, there exists a convergent sub-$F$-isocrystal $M_i$ of $M$, such that $a_x^j(M_i) = a_x^j(M)$ for all $j \leq i$ and all $x \in \mathbb{G}_{m,k}^n \times \mathbb{A}_k^m$. In Proposition 5.5 we prove that $M_i$ has $r_i$-log-decay, where $r_i = a_x^{i+1}(M) - a_x^i(M)$. Combining this with the results of the previous paragraph, we see that $M_i$ is overconvergent if $a_x^{i+1} - a_x^i(M) > 1$.

The final step involves a geometric patching argument. We first consider a generically étale alteration $Y \to X$ with compactification $\overline{Y}$ such that $\overline{Y} - Y$ is a normal crossing divisor. Using the results of [9] we find finite étale maps locally on $Y$ onto $\mathbb{G}_{m,k}^n \times \mathbb{A}_k^m$ this allows us to use the ideas of the previous paragraphs.

1.3 Relationship with previous approaches

The proof of Drinfeld and Kedlaya in [4] can be summarized as follows: first they prove that if $U \subset X$ is a dense open subscheme, the restriction functor from convergent $F$-isocrystals on $X$ to $F$-isocrystals on $U$ is fully faithful. This builds upon several other difficult fully faithful results due to Kedlaya and Shiho (see [8], [11, Theorem 5.1], and [15]). Let $M$ be an $F$-isocrystal on $X$. Let $U \subset X$ be the locus of points where $a_y^i(M) = a_x^i(M)$. When we restrict $M$ to $U$, we obtain a slope filtration. In particular, $M|_U$ corresponds to an element of $\text{Ext}^1(M_1, M_2)$, where the slopes of $M_1$ are less than those of $M_2$. In [4], they show that $\text{Ext}^1(M_1, M_2)$ is trivial when the smallest slope of $M_2$ is greater than one plus the largest slope of $M_1$. Therefore a gap larger than one in the slopes means $M|_U = M_1 \oplus M_2$. This decomposition provides idempotent morphisms from $M|_U \to M|_U$, which extend to $M$ by the aforementioned fully faithfulness result.

Our proof can be viewed as orthogonal to the Drinfeld-Kedlaya approach in two facets. First, instead of restricting to the constant locus of the Newton polygon, we prove that the Newton polygon remains constant along all of $X$. This allows us to completely bypass any sort of fully faithfulness result when $M$ is convergent and only use the fully faithfulness of the overconvergent $F$-isocrystals to convergent $F$-isocrystals functor when $M$ is overconvergent (see [8]). Second is the proof that $\text{Ext}^1(M_1, M_2)$ is trivial, which can be traced back to Kedlaya’s thesis (see [7] Theorem...
The underlying idea is that when the gaps between the slopes are larger than one, the connection is preserved at the corresponding step of the descending slope filtration due to de Jong (see [2]). The descending slope filtration only exists over some purely inseparable pro-cover, but using the connection it is possible to descend part of the filtration to the original base. This is in contrast to our proof, where we show that the pertinent step of the ascending slope filtration descends using the notion of $r$-logarithmic decay.

It is also worth mentioning that Drinfeld and Kedlaya assume that $M$ is indecomposable. This is decidedly stronger than Theorem 1.1, where we assume $M$ is irreducible. However, for the applications in [4] and the other applications we are aware of (e.g. [14]), Theorem 1.1 is sufficient. Of course, one could apply the Ext^1 result in [4] together with Theorem 1.1 to obtain this more general result.

Finally, let us mention previous work of the author, where we proved Theorem 1.1 for curves over a finite field (see [13, Corollary 7.4]). In this work, Theorem 1.1 was a corollary of a difficult monodromy theorem for rank one convergent $F$-isocrystals and an analysis of the slope filtration. In particular, using a monodromy theorem ([13, Corollary 4.16]) and class field theory, we showed that for $r < 1$, a rank one $F$-isocrystal with $r$-log-decay is overconvergent. This is the same as Theorem 4.4. However, we maintain that the present approach is preferable and necessary. First, the proof of Theorem 4.4 in this article relies on a study of the underlying differential equation. This elementary approach completely bypasses the technical monodromy theorem used in [13]. It is also amenable to the higher dimensional situation, where ramification theory is much more technical. Second, in this article we may take our ground field to be any perfect field $k$. Lastly, in this paper we deal with varieties of arbitrary dimension. Although one could use the Lefschetz theorem for $F$-isocrystals, due to Abe and Esnault (see [1]), to obtain results on higher dimensional varieties, this only works for finite fields. It also has the downside of being less direct of than the proof presented here.

### 1.4 A remark on logarithmic decay $F$-isocrystals

The notion of $F$-isocrystals with a log-decay Frobenius structure was introduced by Dwork-Sperber and plays a prominent role in Wan’s work on unit-root $L$-functions (see [5], [16], and [17]). The log-decay condition for Frobenius structures arise naturally in the study of unit-root $F$-isocrystals. However, they only study $F$-isocrystals over $B_{dM,k}$. In [13], the author studied $F$-isocrystals with log-decay in both the Frobenius structure and the differential equation over curves. We studied the rate of log-decay of the slope filtration and the monodromy properties of $F$-isocrystals with log-decay. In the present article, we utilize the log-decay notion for a general variety $X$ in a somewhat ad-hoc manner. We find an alteration $Y \to X$, whose compactification $\overline{Y}$ is smooth and $\overline{Y} - Y$ is a normal crossing divisor $D$. We then we cover $\overline{Y}$ with open subschemes that admit an etale map onto $\mathbb{A}^n$, that take $D$ to coordinate planes. This lets us use an explicit definition of $r$-log-decay for $F$-isocrystals on $B_{dM,k}$. Although sufficient for our applications, it is not clear if the property of having $r$-log-decay is intrinsic to an $F$-isocrystal on $X$. What if we choose a different alteration $Y$ or find different etale maps onto $\mathbb{A}^n$? It would be interesting to either find an intrinsic definition of logarithmic decay or to prove that the ad-hoc notion used in this article is intrinsic.
2 Rings of functions on polyannuli

Let \( K \) be \( W(k)[\frac{1}{p}] \) and let \( \sigma \) the lift of the Frobenius morphism on \( k \). Let \( n > 0 \) and let \( m \geq 0 \). Consider indeterminates \( T_1, \ldots, T_{n+m} \). We then define

\[
\mathcal{A} = K \langle T_1^\pm, \ldots, T_n^\pm, T_{n+1}, \ldots, T_{n+m} \rangle
\]

\[
\mathcal{A}_{(i)} = K \langle T_1^\pm, \ldots, T_i^\pm, T_{i+1}^\pm, \ldots, T_n^\pm, T_{n+1}, \ldots, T_{n+m} \rangle
\]

We may extend \( \sigma \) to \( \mathcal{A} \) by having \( \sigma \) send \( T_i \) to \( T_i^p \). We let \( |.|_1 \) be the Gauss norm on \( \mathcal{A} \). Let \( x(T) \in \mathcal{A} \). For each \( i = 1, \ldots, n \) we may write

\[
x(T) = \sum_{d \in \mathbb{Z}} a_d T_i^d,
\]

with \( a_d \in \mathcal{A}_{(i)} \) and \( |a_d|_1 \to 0 \) as \( |d| \to \infty \). We refer to this as the \( T_i \)-adic expansion of \( x(T) \). Using this expansion, we define some truncations of \( x(T) \):

\[
W_{<}^{(i)}(x(T)) = \sum_{d < -1} a_d T_i^d
\]

\[
W_{\geq}^{(i)}(x(T)) = \sum_{d \geq 0} a_d T_i^d
\]

Note that \( W_{<}^{(i)}(W_{\geq}^{(j)}(x(T))) = W_{\geq}^{(j)}(W_{<}^{(i)}(x(T))) \). Next, we define the \( T_i \)-adic \( j \)-th partial valuation as

\[
w_j^{(i)}(x(T)) = \min_{v(a_d) \leq j} \{ d \}.
\]

Using these partial valuations, we define the ring of overconvergent functions and the ring of \( r \)-log-decay functions:

\[
\mathcal{A}^{\dagger} = \left\{ x(T) \in \mathcal{A} \mid \text{there exists } c > 0 \text{ such that for } i = 1, \ldots, n \right. \\
\text{ and } j \gg 0, \text{ we have } w_j^{(i)}(x(T)) \geq -cj \right\}
\]

\[
\mathcal{A}^{r} = \left\{ x(T) \in \mathcal{A} \mid \text{there exists } c > 0 \text{ such that for } i = 1, \ldots, n \right. \\
\text{ and } j \gg 0, \text{ we have } w_j^{(i)}(x(T)) \geq -cp^rj \right\}
\]

Note that \( \sigma \) restricts to endomorphisms of \( \mathcal{A}^{\dagger} \) and \( \mathcal{A}^{r} \). For \( r < 1 \) and \( x \in \mathcal{A}^{r} \), the \( T_i \)-adic primitive of \( W_{<}^{(i)}(x) \) converges to an element of \( \mathcal{A}^{r+1} \):

\[
\int W_{<}^{(i)}(x(T))dT_i = \sum_{d < -1} a_d \frac{T_i^{d+1}}{d}.
\]

We define the \( T_i \)-adic residue to be

\[
\text{Res}_{i}(x(T)) = a_{-1}.
\]
3 \( F \)-isocrystals

Let \( X \) be a smooth irreducible quasi-compact scheme over \( k \). We will freely use the notion of convergent and overconvergent \( F \)-isocrystals. For a high level overview, we recommend [10]. We let \( F \) \( - \) \( \text{Isoc}^\dagger(X) \) denote the category of overconvergent \( F \)-isocrystals on \( X \) (see [10] for precise definitions) and we \( F \) \( - \) \( \text{Isoc}(X) \) denote the category of convergent \( F \)-isocrystals on \( X \). For a dense open immersion \( U \subset X \) we let \( F \) \( - \) \( \text{Isoc}(U, X) \) denote the category of \( F \)-isocrystals on \( U \) overconvergent along \( X - U \). For any finite extension \( E \) of \( \mathbb{Q}_p \) we let \( F \) \( - \) \( \text{Isoc}(X) \otimes E \) (resp \( F \) \( - \) \( \text{Isoc}(U, X) \otimes E \)) denote the category whose objects are objects in \( F \) \( - \) \( \text{Isoc}(X) \) (resp \( F \) \( - \) \( \text{Isoc}(U, X) \)) with a \( \mathbb{Q}_p \)-linear action of \( E \). Given an open subscheme \( V \subset U \) and \( W \subset X \), such that \( V \subset W \) is an open immersion, there is a natural restriction functor \( F \) \( - \) \( \text{Isoc}(U, X) \to F \) \( - \) \( \text{Isoc}(V, W) \). If \( M \) is an object of \( F \) \( - \) \( \text{Isoc}(U, X) \), we refer to the image of \( M \) in \( F \) \( - \) \( \text{Isoc}(V, W) \) as the restriction of \( M \) to the pair \((V, W)\).

Now let \( M \) be an object of \( F \) \( - \) \( \text{Isoc}(X) \). For any \( x \in X \) we define \( b^i_x(M) = a^1_x(M) + \cdots + a^i_x(M) \) and let \( NP_x(M) \) be the lower convex hull of the vertices \((i, b^i_x(M))\). If \((i, y)\) is a vertex of \( NP_x(M) \) for all \( x \in X \), then by a theorem of Katz there exists a rank \( i \) subobject \( M_i \) of \( M \) in \( F \) \( - \) \( \text{Isoc}(X) \) such that \( a^j_x(M_i) = a^j_x(M) \) for all \( j \leq i \).

3.1 \( F \)-isocrystals on \( \mathbb{G}_{m, k} \) as \((\sigma^f, \nabla)\)-modules over \( \mathcal{A}, \mathcal{A}^r \), and \( \mathcal{A}^\dagger \)

When \( U = \mathbb{G}_{m, k} \times_k \mathbb{A}^m_k \) and \( X = \mathbb{A}^{n+m}_k \), we may view objects of \( F \) \( - \) \( \text{Isoc}(U) \) and \( F \) \( - \) \( \text{Isoc}(U, X) \) as differential equations over the rings introduced in \( \S 2 \) with a compatible Frobenius structure. Let \( R \) be either \( \mathcal{A}^\dagger \), \( \mathcal{A}' \), or \( \mathcal{A} \).

**Definition 3.1.** A \( \sigma^f \)-module is a locally free \( R \)-module \( M \) equipped with a \( \sigma^f \)-semilinear endomorphism \( \varphi : M \to M \) whose linearization is an isomorphism. More precisely, we have \( \varphi(am) = \sigma^f(a)\varphi(m) \) for \( a \in R \) and \( \varphi : R \otimes_{\sigma^f} M \to M \) is an isomorphism.

**Definition 3.2.** Let \( \Omega_R \) be the module of differentials of \( R \) over \( K \). Let \( \delta_T : R \to \Omega_R \) to be the exterior derivative. A \( \nabla \)-module over \( R \) is a locally free \( R \)-module \( M \) equipped with a connection. That is, \( M \) comes with a \( K \)-linear map \( \nabla : M \to \Omega_R \) satisfying the Liebnitz rule: \( \nabla(am) = \delta_T(a)m + a\nabla(m) \).

**Definition 3.3.** By abuse of notation, define \( \sigma^f : \Omega_R \to \Omega_R \) be the map induced by pulling back the differential along \( \sigma^f \). A \((\sigma^f, \nabla)\)-module is an \( R \)-module \( M \) that is both a \( \sigma^f \)-module and a \( \nabla \)-module with the following compatibility condition:

\[
\begin{array}{ccc}
M \xrightarrow{\nabla} & M \otimes \Omega_R & \xrightarrow{\sigma^f} \\
\downarrow{\sigma^f} & \downarrow{\sigma^f \otimes \sigma^f} & \\
M \xrightarrow{\nabla} & M \otimes \Omega_R.
\end{array}
\]

We denote the category of \((\sigma^f, \nabla)\)-modules over \( R \) by \( \Phi^{f, \nabla}_R \). We obtain functors \( \Phi^{f, \nabla}_{\mathcal{A}^\dagger} \to \Phi^{f, \nabla}_{\mathcal{A}'} \) and \( \Phi^{f, \nabla}_{\mathcal{A}'} \to \Phi^{f, \nabla}_{\mathcal{A}^r} \) by base changing to \( \mathcal{A} \). We say that an object \( M \) of \( \Phi^{f, \nabla}_{\mathcal{A}^r} \) is overconvergent (resp. has \( r \)-log-decay) if it lies in the essential image of \( \Phi^{f, \nabla}_{\mathcal{A}^r} \to \Phi^{f, \nabla}_{\mathcal{A}^\dagger} \) (resp. \( \Phi^{f, \nabla}_{\mathcal{A}'} \to \Phi^{f, \nabla}_{\mathcal{A}^r} \)). There are equivalences of categories

\[
\begin{align*}
\Phi^{f, \nabla}_{\mathcal{A}^r} & \leftrightarrow F - \text{Isoc}^r(\mathbb{G}_{m, k} \times_k \mathbb{A}^m_k) \otimes \mathbb{Q}_p^f, \\
\Phi^{f, \nabla}_{\mathcal{A}^\dagger} & \leftrightarrow F - \text{Isoc}^\dagger(\mathbb{G}_{m, k} \times_k \mathbb{A}^m_k, \mathbb{A}^{n+m}_k) \otimes \mathbb{Q}_p^f.
\end{align*}
\]
There are natural functors $\epsilon_f : M_{A,\nabla} \rightarrow M_{fA,\nabla}$ and $\epsilon_f^* : M_{fA,\nabla} \rightarrow M_{A,\nabla}$, which are obtained by iterating the Frobenius map $f$ times.

4 Connections on polyannuli with $r$-log decay for $r < 1$

In this section we study rank one $\nabla$-modules with small rates of logarithmic decay.

**Lemma 4.1.** Let $M$ be an integrable $\nabla$-module over $A$ or $A^r$ with rank one. Let $e$ be a basis of $M$ and write

$$\nabla(e) = f_1(T) dT_1 + \cdots + f_{n+m}(T) dT_{n+m}.$$  

We have $\text{Res}_i(f_i) \in K$ for $i = 1, \ldots, n$. Furthermore, if $W^{(i)}_<(f_i) = 0$ then for each $j$ we have $W^{(i)}_<(f_j) = 0$.

**Proof.** Since $\nabla$ is integrable we know $\partial_i f_j = \partial_j f_i$. The lemma follows immediately.

**Proposition 4.2.** Let $M$ be a rank one integrable $\nabla$-module over $A^r$ for some $r < 1$. Then there exists $m$ such that $M \otimes p^m$ has regular singularities.

**Proof.** Let $e$ be a basis of $M$ and let $c_{1,1}, \ldots, c_{1,n+m} \in A^r$ with $\nabla(e) = \sum c_{1,i} dT_i \otimes e$. Since $r < 1$ we know that

$$h_1 = \int W^{(1)}_<(c_{1,1}) dT_1$$

converges in $A$. Thus for $\tau$ sufficiently large we may consider the basis $e_1 = \exp(p^\tau h_1)e^{\otimes p}$ of $M^{\otimes p^\tau} \otimes A$. We have

$$\nabla^{\otimes p^\tau}(e_1) = \sum_{i=1}^d c_{2,i} dT_i \otimes e_1$$

$$c_{2,i} = \partial_i h_1 + p^\tau c_{1,i}.$$  

By our definition of $h_1$ we know that $W^{(1)}<(c_{2,1}) = 0$, so by Lemma 4.1 we have $W^{(1)}<(c_{2,i})$ is zero for each $i$. From Lemma 4.1 we know that $c_{2,2} = W^{(1)}_(c_{2,2})$. As $W^{(1)}_<(\partial_i h_1) = 0$ this gives

$$W^{(2)}<(c_{2,2}) = W^{(2)}_(W^{(1)}_(c_{2,2}))$$

$$= p^\tau W^{(2)}_(W^{(1)}(c_{1,2})),$$

which is contained in $A^r$. In particular

$$h_2 = \int W^{(2)}_(c_{2,2}) dT_2$$

converges in $A$. After increasing $\tau$, we may consider the basis $e_2 = \exp(h_2)e_1$ of $M^{\otimes p^\tau} \otimes A$. We have

$$\nabla^{\otimes p^\tau}(e_2) = \sum_{i=1}^d c_{3,i} dT_i \otimes e_2$$

$$c_{3,i} = \partial_i(h_1 + h_2) + p^\tau c_{1,i}.$$
Note that $W^{(j)}_{<}(c_{3,i}) = 0$ for $j < 3$ and all $i$ from Lemma 4.1. In particular $W^{(1)}_{\geq}(W^{(2)}_{\geq}(c_{3,3})) = c_{3,3}$. Since the truncation operators commute with each other and $W^{(j)}_{\geq}(\partial h_{j}) = 0$ for $j = 1, 2$, we find as above that

$$W^{(3)}_{<}(c_{3,3}) = p^{r}W^{(3)}_{\geq}(W^{(1)}_{\geq}(W^{(2)}_{\geq}(c_{1,3}))),$$

which is contained in $\mathcal{A}^{r}$. This allows us to define $h_{3}$. The proposition follows from repeating this process.

\begin{proposition}
Let $M$ be a $(\sigma^{f}, \nabla)$-module over $\mathcal{A}^{r}$ for some $r < 1$. Then a tensor power of $M$ extends to $K\langle T_{1}, \ldots, T_{n+m} \rangle$.
\end{proposition}

\begin{proof}
By Proposition 4.2 we may assume that $M$ has regular singularities. This means $M = e\mathcal{A}^{r}$ and

$$\nabla(e) = f_{1}dT_{1} + \ldots + f_{n+m}dT_{n+m},$$

where $f_{i} = \frac{q}{r} + g_{i}$ with $c_{i} \in K$ and $g_{i} \in T_{i}K\langle T_{1}, \ldots, T_{n+m} \rangle$. Let $\alpha \in \mathcal{A}^{r}$ satisfy $\sigma^{f}(e) = ae$. The compatibility between $\sigma^{f}$ and $\nabla$ imply

$$T_{i}\frac{\partial a}{\alpha} = qf_{i}^{\sigma^{f}} - f_{i}. \quad (1)$$

This gives $\text{Res}(\frac{\partial a}{\alpha}) = qc_{i}^{\sigma^{f}} - c_{i}$. This residue is an integer $n_{i}$, so we have $c_{i} = \frac{n_{i}}{q-1}$. It follows that $M^{\otimes(q-1)}$ has a solution, and thus $\nabla$ extends to $K\langle T_{1}, \ldots, T_{n+m} \rangle$. The compatibility between $\sigma^{f}$ and $\nabla$ implies that the Frobenius also extends to $K\langle T_{1}, \ldots, T_{n+m} \rangle$.
\end{proof}

\begin{theorem}
Let $M$ be a rank one object of $\mathbb{F} - \text{Isoc}(\mathbb{G}_{m}^{n} \times \mathbb{A}_{k}^{n+m})$, so that we may regard $M$ as $(\sigma, \nabla)$-module over $\mathcal{A}$. If the connection descends to $\mathcal{A}^{r}$ for some $r < 1$ then $M$ is overconvergent.
\end{theorem}

\begin{proof}
We may assume that $M$ is unit-root, and therefore corresponds to a $p$-adic character $\rho : \pi_{1}(\mathbb{G}_{m}^{n} \times \mathbb{A}_{k}^{n}) \to E^{\times}$, where $E$ is a finite extension of $\mathbb{Q}_{p}$. By Proposition 4.3 we $M^{\otimes r}$ extends to an $F$-isocrystal on $\mathbb{A}_{k}^{n+m}$, meaning that $\rho^{\otimes r}$ extends to a representation of $\pi_{1}(\mathbb{A}_{k}^{n+m})$. This implies $\rho$ is potentially unramified as in [12] Definition 2.3.6. By [12] Theorem 2.3.7 we know that $M$ is overconvergent along the divisor $T_{1} \ldots T_{n} = 0$ in $\mathbb{A}_{k}^{n+m}$.
\end{proof}

5 Slope filtrations and log-decay

Let $M$ be a free $(\sigma, \nabla)$-module over $\mathcal{A}^{d}$ of rank $d$. We will assume that the Newton polygon of $x^{*}M$ remains constant as we vary over all points $x : \text{Spec}(k_{0}) \to \mathbb{G}_{m,k}^{n} \times \mathbb{A}^{n}$, that the slopes are non-negative, and that the slope zero occurs exactly once. In particular, when we look at the image of $M$ has a rank one subobject $M^{u-r}$ in the category $\mathbf{M}^{\Phi_{\mathcal{A}}}$. The main result of this section is:

\begin{theorem}
Let $s$ be the smallest nonzero slope of $M$ and let $r = \frac{1}{s}$. There exists $f$ such that $\epsilon_{f}(M^{u-r})$ has $r$-log-decay.
\end{theorem}
We first introduce some auxiliary subrings of $A^r$ and $A^\dagger$. We define

$$A^{r,c} = \left\{ x(T) \in A \ \middle| \ \begin{array}{ll} w_k^{(i)}(x(T)) \geq -c p^{\sigma f k} & \text{for } k > 0, \\
\text{and } w_0^{(i)}(x(T)) \geq 0 & \text{for } i = 1, \ldots, d \end{array} \right\},$$

$$A^{\dagger,r,c} = A^{r,c} \cap A^{\dagger}.$$

Note that $A^{r,c}$ is $p$-adically complete, unlike $A^r$. When $R$ is any of these rings, we let $O_R$ denote the ring of elements in $R$ whose Gauss norm is less than or equal to one. Now for $f$ large enough, there exists $\omega \in K$ with $v_{p,f}(\omega) = s$. The following Lemma follows from the definition of $A^{r,c}$.

**Lemma 5.2.** We have the following:

1. For $x \in A^{r,c}$ we have $x^{\sigma f} \in A^{r,p,c}$ and $\omega x \in A^{r,p^{-1}c}$.

2. Let $x \in A$ with $w_0(x) \geq 0$. If $\omega x \in A^{r,c}$ then $x \in A^{r,p^{-1}c}$.

By our assumptions on the slope of $M$, there exists a basis of $M$ whose Frobenius is given by a matrix

$$A = \begin{pmatrix} A_{1,1} & \omega A_{1,2} \\
\omega A_{2,1} & \omega A_{2,2} \end{pmatrix},$$

where $A_{i,j}$ are matrices with entries in $O_{A^\dagger}$ and $A_{1,1} \in O_{A^\dagger}^\times$. The connection is given by the differential matrices

$$C_1 dT_1 + \cdots + C_{n+m} dT_{n+m},$$

where the $C_i$ are $d \times d$ matrices with entries in $O_{A^\dagger}$. For $a \in A$ we let $K_{u,v}(a)$ be the matrix with $a$ in the $(u,v)$-entry and zero elsewhere. We let $L_{u,v}(a) = 1_d + K_{u,v}(a)$, where $1_d$ is the $d \times d$ identity matrix.

**Lemma 5.3.** After a change of variables, we may assume the following holds for all $i$:

1. $w_0^{(i)}(A_{1,1}^{-1}) \geq 0$
2. $w_0^{(i)}(A_{1,2}) \geq 0$ and $w_0^{(i)}(A_{2,2}) \geq 0$.

**Proof.** To prove the first claim, it is enough to prove $w_0^{(i)}(A_{1,1}) \leq 0$. Let $C_i = \begin{pmatrix} T_i^k & 0 \\
0 & 1_{d-1} \end{pmatrix}$. For $k$ large enough, we see that $C_i A_i^{-\sigma f}$ satisfies the desired property. For the second claim we set $D_i = \begin{pmatrix} 1 & 0 \\
k^{-1} T_i & 1_{d-1} \end{pmatrix}$. When $k$ is large enough, we find that $D_i A_i D_i^{-\sigma f}$ satisfies the second property without changing the $(1,1)$-entry. \hfill \Box

By Lemma 5.3 we know that for $c$ sufficiently large the entries of $A_{1,1}^{-1}, \omega A_{2,1}, \omega A_{1,2}$ and $\omega A_{2,2}$ are contained in $O_{A^{\dagger,r,c}}$.

**Lemma 5.4.** Let $x, y \in O_{A^\dagger}$. Assume that $x^{-1}, \omega y \in O_{A^{\dagger,r,c}}$ and $w_0(y) \geq 0$. Then

1. We have $x^{-\sigma f} \omega y \in O_{A^{\dagger,r,c}}$. 

2. If $\omega | y$ then $(x + y)^{-1} \in O_{\mathcal{A}^{t, r, c}}$.

Proof. We know that $y \in O_{\mathcal{A}^{t, r, p, \ell, c}}$, which means $x^{-\sigma f} y \in O_{\mathcal{A}^{t, r, p, \ell, c}}$. This implies $x^{-\sigma f} \omega y \in O_{\mathcal{A}^{t, r, c}}$. Since $O_{\mathcal{A}^{t, r, c}}$ we know the geometric series $(1 + x^{-1} y)^{-1}$ is contained in $O_{\mathcal{A}^{t, r, c}}$. As $1 + x^{-1} y \in O_{\mathcal{A}^{t}}$, we know that $(1 + x^{-1} y)^{-1} \in O_{\mathcal{A}^{t, r, c}}$. Thus $(x + y)^{-1} = x^{-1}(1 + x^{-1} y)^{-1}$ is contained in $O_{\mathcal{A}^{t, r, c}}$. 

Proposition 5.5. There exists $N = \begin{pmatrix} 1 & 0 \\ N_{2,1} & 1_{d-1} \end{pmatrix}$ where $N_{2,1}$ has entries in $O_{\mathcal{A}^{t, r, c}}$ and $NAN^{-\sigma f}$ is of the form $A' = \begin{pmatrix} A_{1,1}' & \omega A_{1,2} \\ 0 & A_{2,2}' \end{pmatrix}$.

Proof. We will show inductively that there exists $N_k = \begin{pmatrix} 1 & 0 \\ N_{2,1,k} & 1_{d-1} \end{pmatrix}$ such that:

1. $A_k = N_k AN_k^{-\sigma f}$ is of the form $\begin{pmatrix} A_{1,1,k} & \omega A_{1,2} \\ \omega^k A_{2,1,k} & \omega A_{2,2,k} \end{pmatrix}$

2. The entries of $A_{1,1,k}^{-1}, \omega A_{1,2,k}, \omega A_{2,2,k},$ and $\omega^k A_{2,1,k}$ are contained in $O_{\mathcal{A}^{t, r, c}}$

3. We have $w_0^{(i)}(A_{1,1,k}) \geq 0$, $w_0^{(i)}(A_{1,2,k}) \geq 0$ and $w_0^{(i)}(A_{2,2,k}) \geq 0$.

4. For all $k$ we have $N_k \equiv N_{k-1} \mod \omega^k$.

The result will follow by taking the limit of the $N_k$ as $k \to \infty$. When $k = 1$ this follows from Lemma 5.3. Now let $k > 1$ and assume $N_k$ exists. We define $N_k = \begin{pmatrix} 1 & 0 \\ -A_{1,1,k}^{-1} & 0 \end{pmatrix}$, and set $A_{k+1} = N_k A_k N_k^{-\sigma f}$. It is immediate that 1, 3, and 4 are satisfied. We can verify 2 using Lemma 5.4.

Proof of Theorem 5.1. Let $N$ and $A'$ be as in Proposition 5.5. After changing basis by $N$, the connection is given by the matrix of 1-forms:

$$D_1 dT_1 + \cdots + D_{n+m} dT_{n+m},$$

where $D_i = \partial_i M + C_i M$. In particular, the $D_i$ has entries in $O_{\mathcal{A}^{t}}$. Compatibility between the connection and Frobenius give the relation:

$$\partial_i A' + D_i A' = q A'^{\sigma f} D_i.$$

Write $D_i = \begin{pmatrix} R_i & S_i \\ U_i & V_i \end{pmatrix}$, where $V$ is a $(d-1) \times (d-1)$-matrix. Continuing with the notation from Proposition 5.5 and considering the lower left corner, we obtain $U_i A'_{1,1} = q A'_{2,2} U_i^{\sigma f}$. 

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From here it is clear that \( U_i = 0 \). It follows that the \( R_i \) and \( A_{i,1}' \) describe the unit-root sub-\( F \)-isocrystal \( M^{u-r} \) of \( M \). As the connection and Frobenius structure are defined over \( \mathcal{O}_A' \), we see that \( M^{u-r} \) has \( r \)-log-decay.

\[ \square \]

6 Bounded slope theorem

**Proposition 6.1.** Let \( M \) be a rank \( d \) object of \( F - \text{Isoc}(\mathbb{G}_{m,k}^n \times \mathbb{A}_k^m, \mathbb{A}_k^{n+m}) \). Assume that the Newton polygon of \( M \) remains constant on \( x \in \mathbb{G}_{m,k}^n \times \mathbb{A}_k^m \). Let \( \eta \) be the generic point of \( \mathbb{G}_{m,k}^n \times \mathbb{A}_k^m \). We assume that \( a^{i+1}_\eta(M) - a^i_\eta(M) > 0 \) and let \( M_i \) denote the sub-object of \( M \) in \( F - \text{Isoc}(\mathbb{G}_{m,k}^n \times \mathbb{A}_k^m) \) with \( a^j_\eta(M_i) = a^j_\eta(M) \) for \( j \leq i \). If \( a^{i+1}_\eta(M) - a^i_\eta(M) > 1 \), then \( M_i \) is overconvergent along \( \mathbb{A}_k^{n+m} \).

**Proof.** Assume \( a^{i+1}_\eta(M) - a^i_\eta(M) > 1 \). Let \( r = \frac{1}{a^{i+1}_\eta(M) - a^i_\eta(M)} \). Since that \( M_i \) is overconvergent if and only if \( \det(M_i) \) is overconvergent, we may replace \( M \) with \( \Lambda^\text{rank}(M_i)M \) twisted so that the smallest slope is 0 and prove that the unit-root sub-\( F \)-isocrystal \( M^{u-r} \) is overconvergent. By Theorem 5.1, we know that \( \epsilon_f(M^{u-r}) \) has \( r \)-log-decay for some \( f > 0 \). As \( r < 1 \), we know from Theorem 4.4 that \( \epsilon_f(M^{u-r}) \) is overconvergent along \( \mathbb{A}_k^{n+m} \). The functor \( \epsilon_f \) for the corresponding Galois representations corresponds to the composition \( \rho : \pi_1(\mathbb{G}_{m,k}^n \times \mathbb{A}_k^m) \to GL_n(\mathbb{Q}_p) \to GL_n(\mathbb{Q}_p^f) \).

It follows that \( M^{u-r} \) has finite monodromy, and therefore is overconvergent.

\[ \square \]

**Lemma 6.2.** Let \( A_0 \) and \( B_0 \) be smooth \( k \)-algebras. Let \( f_0 : A_0 \to B_0 \) be a finite étale morphism. Let \( B \) (resp \( A \)) be a \( p \)-adically complete \( W(k) \)-algebra with \( B \otimes_{W(k)} k = B_0 \) (resp \( A \otimes_{W(k)} k = A_0 \)) and let \( f : A \to B \) be a lifting of \( f_0 \). If \( B \) are flat then \( f \) is finite étale.

**Proof.** Since \( f_0 \) is finite étale, there exists \( g_0(x_0) \in A_0[x_0] \) of degree \( d \) such that \( B_0 = A_0[x_0]/g_0(x_0) \) and \( g'_0(x_0) \) is a unit in \( B_0 \). Let \( x \in B \) be a lift of \( x_0 \). We claim that \( B \) is isomorphic to \( M = A \oplus xA \oplus x^{d-1}A \) as an \( A \)-module via the natural map \( M \to B \). It suffices to show \( \theta_n : M \otimes \mathbb{Z}/p^n\mathbb{Z} \to B \otimes \mathbb{Z}/p^n\mathbb{Z} \) is an isomorphism for all \( n \). When \( n = 1 \) this is true because \( f_0 \) is finite étale. Assume that \( \theta_n \) is an isomorphism. Let \( y \in B \otimes \mathbb{Z}/p^{n+1}\mathbb{Z} \). We can find \( x \in M \otimes \mathbb{Z}/p^n\mathbb{Z} \) such that \( \theta_n(x) - y \in p^n(B \otimes \mathbb{Z}/p^{n+1}) \). Since \( \theta_0 \) is an isomorphism we can find \( z \in M \) such that \( p^n\theta_n(z) = \theta_n(x) - y \). This proves surjectivity. The injectivity of \( \theta_{n+1} \) follows from the flatness assumption. This shows that \( B \) is a finite \( A \)-algebra. Furthermore, there exists \( g(T) \in A[T] \) of degree \( d \) such that \( B = A[T]/g(T) \) and \( x \) corresponds to \( T \). Clearly \( g \) reduces to \( g_0 \) modulo \( p \), so we know \( g'(x) \neq 0 \) is a unit in \( B \).

\[ \square \]

**Theorem 6.3.** (Drinfeld-Kedlaya) Let \( k \) be perfect field of characteristic \( p \). Let \( X \) be a smooth irreducible quasi-compact scheme over \( k \). Let \( M \) be an irreducible object of \( F - \text{Isoc}(X) \) or \( F - \text{Isoc}^\dagger(X) \). Then for each \( i \)

\[
|a^{i+1}_\eta(M) - a^i_\eta(M)| \leq 1.
\]

**Proof.** We first take \( M \) to be an object of \( F - \text{Isoc}(X) \). Let \( \eta \in X \) be a generic point. Assume that \( a^{i+1}_\eta(M) - a^i_\eta(M) > 1 \). We will show that for every closed point \( x \in X \), we have \( b^i_\eta(M = b^i_\eta(M), \) which will imply \( M \) is not irreducible. By replacing \( M \) with a twist of \( \Lambda^iM \), we may assume that
\(b^1_\eta(M) = 0\) and \(b^2_\eta(M) > 1\). The de Jong-Oort purity theorem (see [3]) tells us that the locus in \(X\) where \(b^2_k > 0\) is a closed subscheme \(D \subset X\) of codimension 1. Let \(x_0 \in D\). Let \(i : D \hookrightarrow X\) be a smooth curve containing \(x_0\). We further assume that the set theoretic intersection of \(D\) and \(C\) is equal to \(\{x_0\}\) and let \(U = C - \{x_0\}\). As \(U \cap D\) is empty, there exists a rank one unit-root convergent sub-\(F\)-isocrystal \(M^{u-r}\) contained in \(M_U\). After shrinking \(C\), we may find a morphism \(f : C - \{x_0\} \to \mathbb{G}_m\) that is finite étale of degree \(d\). Consider \(N = f_*M_U\), which is overconvergent and the subcrystal \(N^{u-r} = f_*M^{u-r}\). By Proposition 6.1 we know that \(N^{u-r}\) is overconvergent, which implies \(M^{u-r}\) is overconvergent. Using [12, Theorem 2.3.7] we see that \((M^{u-r})^{\otimes n}\) extends to all of \(C\) for \(n \gg 0\). Thus the smallest slope of \(M^{\otimes n}\) is zero, which means \(b^1_{x_0} = 0\).

Next, let \(M\) be an object of \(\mathbf{F-ISoc}^1(X)\) and assume that \(a^0_{i+1}(M) - a^0_i(M) > 1\). By the previous paragraph, we know that there is a convergent sub-\(F\)-isocrystal \(M_i \subset M\). We claim that \(M_i\) is overconvergent. As in the previous paragraph, we may assume \(b^0_\eta(M) - b^1_\eta(M) > 1\) and prove that \(M_1\) is overconvergent. After twisting, we can assume \(b^1_\eta(M) = 0\). Let \(\pi: Y \rightarrow X\) be a generically étale morphism such that \(Y\) has a smooth compatification \(\overline{Y}\) and \(E = \overline{Y} - Y\) is a normal crossing divisor. Let \(N = f^*M\) and \(N_i = f^*M_i\). For \(x \in E\), we let \(U \subset \overline{Y}\) be an affine neighborhood of \(x\) and let \(V = U - (U \cap E)\). Let \(U\) be a smooth lifting of \(U\) over \(W(k)\) and let \(U\) be the rigid fiber of \(\mathcal{U}\). Then we may regard \(N\) as a locally free sheaf \(\mathcal{N}\) on a strict neighborhood \(\mathcal{V}\) of the tube \(|V|\) in \(\mathcal{U}\) with a connection and a compatible Frobenius structure. After shrinking \(U\), we may assume that \(\mathcal{N}\) is a free \(\mathcal{O}_{\mathcal{V}}\)-module. By [9, Theorem 2] there exists a finite étale morphism \(\pi_0: U \rightarrow \mathbb{A}^{m+n}\) such that \(\pi(U \cap E)\) is the union of \(m\) coordinate hyperplanes. Now consider \(\pi_*(N)\), which is an object of \(\mathbf{F-ISoc}^1(\mathbb{G}_{m,k} \times \mathbb{A}^m_k, \mathbb{A}^{m+n}_k)\) and \(\pi_*(N_i)\), which is the unit-root sub-\(F\)-isocrystal of \(\pi_*(N)\). We may lift \(\pi_0\) to a map \(\pi: \mathcal{U} \rightarrow \text{Spec}(W(k)[T_1, \ldots, T_{n+m}])\) and by Lemma 6.2 we see that \(\pi\) is finite étale. In particular, the map \(\pi^{rig}\) is finite étale and thus \((\pi^{rig})_*(\mathcal{N})\) is free. In particular, we may regard \(\pi_*(N)\) as a free \((\sigma, \nabla)\)-module over \(\mathcal{A}^1\). \((\pi^{rig})_*(\mathcal{N})\). By Proposition 6.1 we know that \(\pi_*(N_1)\) is overconvergent, which in turn means \(N_1\) is overconvergent when restricted to \((V, U)\). Therefore \(N_1\) is overconvergent.

Now let \(W\) be a dense open subset of \(X\) and \(Z \subset Y\) such that \(f: Z \rightarrow W\) is a finite étale morphism of degree \(d\). Note that \(f_*N_1\) is isomorphic to \((M^{u-r})_W\) and \(f_*N_1\) is overconvergent, which means that \(M_1|_W\) is overconvergent. It follows that \(M_1\) is overconvergent.

\[\square\]

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