Abstract

The article contains a detailed description of the connection between finite depth inclusions of II$_1$-subfactors and finite C$^*$-tensor categories (i.e. C$^*$-tensor categories with dimension function for which the number of equivalence classes of irreducible objects is finite). The (N, N)-bimodules belonging to a II$_1$-subfactor N $\subset$ M with finite Jones index form a C$^*$-tensor category with dimension function. Conversely, taking an object of a finite C$^*$-tensor category C we construct a subfactor A $\subset$ R of the hyperfinite II$_1$-factor R with finite index and finite depth. For this subfactor we compute the standard invariant and show that the C$^*$-tensor category of the corresponding (A, A)-bimodules is equivalent to a subcategory of C. We illustrate the results for the C$^*$-tensor category of the unitary finite dimensional corepresentations of a finite dimensional Hopf-*-algebra.

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Introduction

The theory of subfactors was established by V.F.R. Jones in his famous paper [8]. The goal of this paper is to sketch the connection between C$^*$-tensor categories and subfactors of type II$_1$-factors. We will use J.E. Roberts’ theory of dimension for C$^*$-tensor categories (see [22], [23], and [14]), but our approach to subfactors starting from C$^*$-tensor categories is different from that in [14], in contrast to [14] we restrict ourselves to factors of type II$_1$. 
Some parts of this article seem to be known to some experts (A. Wassermann for example), but the author could only find the expositions of J.E. Roberts and R. Longo. This article is the abridged version of a part of a paper (see [24]) accepted as a Habilitationsschrift by the Technische Universität München.

$C^*$-tensor categories with dimension function (and some other properties) are called compact $C^*$-tensor categories in this paper, one may regard them as a concept to deal with more general symmetries than those described by (compact) groups. Hopf algebras, in particular quantum groups, may also be regarded as concepts describing generalized symmetries. As one expects, there is a close connection between the concepts of $C^*$-tensor categories and Hopf algebras.

Compact $C^*$-tensor categories are based on the following idea: Consider the unitary finite dimensional representations of a compact group $G$. For these representations one has intertwining operators. One may introduce subrepresentations, finite direct sums and tensor products of representations. There is a conjugate representation for every representation. One can produce a category with an additional structure from these ingredients. The objects are representations, the morphisms are intertwining operators. The space of morphisms is endowed with an additional structure, which is related to $C^*$-algebras. Moreover, the tensor product gives a product operation on the objects. A further property of this category is the existence of conjugates. Compact $C^*$-tensor categories behave in a similar way as these categories. While in the $C^*$-tensor category of a compact group the tensor product operation is commutative, the product need not be commutative for general compact $C^*$-tensor categories. J.E. Roberts introduced a dimension for the objects of compact $C^*$-tensor categories which in contrast to the group case need not be a natural number in general.

S. Doplicher and J.E. Roberts showed the following (see [4]): Every compact $C^*$-tensor category, for which the tensor product operation is commutative in a certain strict way, is equivalent to the $C^*$-tensor category of a suitable compact group.

In the theory of subfactors compact $C^*$-tensor categories appear in a natural way. A. Ocneanu had the idea to develop the theory of subfactors by considering bimodules. For a pair $N \subset M$ of $\text{II}_1$-factors with finite index, he considered the $(N,N)$-, $(N,M)$-, $(M,N)$- and $(M,M)$-bimodules contained in $L^2(M_k)$ for some $k \in \mathbb{N} \cup \{0\}$; here $M_k$ denotes the $\text{II}_1$-factor after a $k+1$-fold application of the basic construction. The occurring $(N,N)$-bimodules form a compact $C^*$-tensor category, where the product operation for the bimodules is the $N$-tensor product $\otimes_N$.

An important subject of this paper is a method for the construction of subfactors of the hyperfinite $\text{II}_1$-factor from a given $C^*$-tensor category. Compact $C^*$-tensor categories, for which the number of the equivalence classes of the irreducible objects is finite, are called finite $C^*$-tensor categories. We
construct a finite depth subfactor of the hyperfinite II$_1$-factor for a given object of a finite $C^*$-tensor category. The method is very general. Important special cases of this construction are certain subfactors associated with finite groups as well as the subfactors due to H. Wenzl. In particular, an explicit construction of the $C^*$-tensor categories associated with H. Wenzl’s subfactors is given.

Let us now give a more detailed outline of this paper:

In Section 1 $C^*$-tensor categories and especially compact $C^*$-tensor categories are introduced and basic facts about these structures are presented.

Section 2 contains the proof that the $(N, N)$-bimodules associated with a subfactor $N \subset M$ of a II$_1$-factor $M$ with finite index form a compact $C^*$-tensor category. Moreover we study the subfactors $N \subset N \rtimes H$, where $N$ is a II$_1$-factor, $H$ is a finite dimensional Hopf-$*$-algebra acting on $N$ by an outer action and $N \rtimes H$ is the corresponding crossed product. We show that the $C^*$-tensor category of the $(N, N)$-bimodules is equivalent to the $C^*$-tensor category of the unitary corepresentations of $H^{}^{\text{cop}}$, where $H^{}^{\text{cop}}$ is the Hopf-$*$-algebra emerging from $H$ by reversing the comultiplication.

In Section 3 we deal with the construction of finite depth subfactors of the hyperfinite II$_1$-factor from a given object of a finite $C^*$-tensor category $C$. For that purpose we use H. Wenzl’s technics which were developed in Chapter 1 of [29] to investigate subfactors generated by a ladder of commuting squares. The standard invariant for these subfactors is computed. This Section also contains a variant of this construction, which imitates Wenzl’s construction of subfactors in [29] and [30]. Furthermore we illustrate our construction in case $C$ is the $C^*$-tensor category of the unitary representations of a finite group or more generally the category of the unitary corepresentations of a finite dimensional Hopf-$*$-algebra.

In Section 4 we show that the $C^*$-tensor category of the $(A, A)$-bimodules associated with the subfactor $A \subset B$ constructed in Section 3 is equivalent to a subcategory of the $C^*$-tensor category $C$, from which we started to construct the subfactor. The proof involves a lot of computations in the category $C$ and in the II$_1$-factors of the Jones tower for $A \subset B$.

In this paper we do not discuss concrete examples of compact $C^*$-tensor categories with objects having a non-integer dimension. We intend to deal with $C^*$-tensor categories associated with H. Wenzl’s subfactors ([29] and [30]) in a forthcoming paper. Another future project is to generalize the constructions of this paper by using 2-$C^*$-tensor categories instead of $C^*$-tensor categories and including the $(N, M)$-, $(M, N)$- and $(M, M)$-bimodules.
1 $C^*$-tensor categories

Mainly following J.E. Roberts, we introduce $C^*$-tensor categories; in contrast to him we don’t restrict ourselves to strict $C^*$-tensor categories. As in [15] we assume that the objects of a category form a set. We fix a universe and call a set small if it is an element of this universe (see [15], Section I.6). We suppose that all sets appearing in this paper are small. Henceforth we will omit all details in this context.

Let $C$ be a category. $\text{Ob }C$ denotes the set of objects of $C$. The set of morphisms ($=$ arrows) with source $\rho \in \text{Ob }C$ and target $\sigma \in \text{Ob }C$ is denoted by $(\rho, \sigma)$ and the identity morphism of $\rho$ by $1_\rho$.

**Definition 1.1.** (i) A category $C$ is called a $C^*$-category, if the following hold:

(a) The space $(\rho, \sigma)$ is a complex Banach space for all objects $\rho, \sigma \in \text{Ob }C$.

(b) The composition of morphisms gives a bilinear map $(S, R) \mapsto S \circ R$ such that $\|S \circ R\| \leq \|S\| \cdot \|R\|$.

(c) There is an antilinear involutive contravariant functor $*: C \longrightarrow C$ such that $\sigma = \sigma^*$ for $\sigma \in \text{Ob }C$ and $\|R^* \circ R\| = \|R\|^2$ for $R \in (\rho, \sigma)$.

(d) $\dim (\rho, \rho) \geq 1$ for every object $\rho \in \text{Ob }C$.

In particular, $(\rho, \rho)$ is a $C^*$-algebra with unit $1_\rho$ for every object $\rho$ of a $C^*$-category.

(ii) Let $C$ be a $C^*$-category and $\rho, \sigma \in \text{Ob }C$. A morphism $U \in (\rho, \sigma)$ is called a linear isometry, if $U^* \circ U = 1_\rho$. A linear isometry satisfying $U \circ U^* = 1_\sigma$ is called unitary.

(iii) Let $C$ be a $C^*$-category endowed with the following product structure:

(a) For $\rho, \sigma \in \text{Ob }C$ there is a product object $\rho \sigma$, and for morphisms $T \in (\rho, \sigma)$ and $T' \in (\rho', \sigma')$ there is a morphism $T \times T' \in (\rho \rho', \sigma \sigma')$. The mapping $(T, T') \mapsto T \times T'$ is bilinear, and we have $(T \times T')^* = T^* \times (T')^*$, $1_\rho \times 1_\sigma = 1_{\rho \sigma}$ for $\rho, \sigma \in \text{Ob }C$ as well as the interchange law

$$S \times S' \circ T \times T' = (S \circ T) \times (S' \circ T')$$

whenever the left side is defined. (To save brackets we evaluate $\times$ before $\circ$.)

(b) For all objects $\rho, \sigma$ and $\tau$ of $C$ there is a unitary operator $a(\rho, \sigma, \tau) \in (\rho(\sigma \tau), (\rho \sigma) \tau)$ such that the pentagonal diagram
commutes for objects $\phi, \rho, \sigma$ and $\tau$. Furthermore $a(\rho, \sigma, \tau)$ is natural in $\rho, \sigma, \tau$.

(c) There is a distinguished object $\iota$ in $\mathcal{C}$ (the unit object). For each object $\rho$ in $\mathcal{C}$ there are unitary operators $l_\rho \in (\iota \rho, \rho)$ and $r_\rho \in (\rho \iota, \rho)$ satisfying the following properties:

They are natural in $\rho$, it holds $l_\iota = r_\iota$, and the diagram

\[
\begin{array}{ccc}
\rho(\iota \sigma) & \xrightarrow{a(\rho, \iota, \sigma)} & (\rho \iota) \sigma \\
1_\rho \times l_\sigma & \downarrow & r_\rho \times 1_\sigma \\
\rho \sigma & & \\
\end{array}
\]

commutes for all $\rho, \sigma \in \text{Ob } \mathcal{C}$.

Then $(\mathcal{C}, \times, a, l, r)$ (or simply $\mathcal{C}$) is said to be a $C^*$-tensor category.

(iv) The $C^*$-tensor category $\mathcal{C}$ is called strict if $\rho(\sigma \tau) = (\rho \sigma) \tau$, $a(\rho, \sigma, \tau) = 1_{\rho(\sigma \tau)}$, $\rho \iota = \iota \rho = \rho$ and $l_\rho, r_\rho = 1_\rho$ for all objects $\rho, \sigma, \tau$ of $\mathcal{C}$. We also write $\rho \sigma \tau$ instead of $\rho(\sigma \tau)$ for objects $\rho, \sigma, \tau$ of a strict $C^*$-tensor category.

(v) A $C^*$-tensor category $\mathcal{D}$ is called a (full) $C^*$-tensor subcategory of the $C^*$-tensor category $\mathcal{C}$ if $\text{Ob } \mathcal{D} \subset \text{Ob } \mathcal{C}$, if the morphism spaces $(\rho, \sigma)$ of $\mathcal{D}$ coincide with those of $\mathcal{C}$ for $\rho, \sigma \in \text{Ob } \mathcal{D}$, and if one obtains the remaining structure of $\mathcal{D}$ by restricting the $C^*$-tensor structure of $\mathcal{C}$ onto $\text{Ob } \mathcal{D}$.

**Definition 1.2.** (i) Two objects $\rho$ and $\sigma$ of a $C^*$-category $\mathcal{C}$ are called equivalent if and only if there is a unitary operator $U \in (\rho, \sigma)$. An object $\rho$ is called irreducible if and only if $(\rho, \rho) = C1_\rho$. Clearly ‘equivalent’ defines an equivalence relation on $\text{Ob } \mathcal{C}$. $[\rho]$ denotes the equivalence class of the object $\rho$, $[\mathcal{C}]$ the set of equivalence classes of $\text{Ob } \mathcal{C}$ and $[[\mathcal{C}]]$ the set of equivalence classes of irreducible objects of $\mathcal{C}$.
(ii) Let $\rho$ be an object of $\mathcal{C}$ and $E$ a projection in $(\rho, \rho)$. (In this paper projections are assumed to be orthogonal.) An object $\sigma$ of $\mathcal{C}$ is called a subobject of $\rho$ corresponding to $E$ if there is a linear isometry $V \in (\sigma, \rho)$ such that $V \circ V^* = E$. The notation $\sigma \leq \rho$ means that $\sigma$ is a subobject of $\rho$.

(iii) An object $\tau$ of $\mathcal{C}$ is called the direct sum $\bigoplus_{i=1}^n \rho_i$ of the objects $\rho_i$ if there are linear isometries $V_i \in (\rho_i, \tau)$ for $i = 1, \ldots, n$ such that $\sum_{i=1}^n V_i \circ V_i^* = 1_\tau$.

(iv) The $C^*$-category $\mathcal{C}$ is said to have subobjects (or to be closed under subobjects) if there is a subobject of $\rho$ corresponding to $E$ for every object $\rho$ of $\mathcal{C}$ and for every projection $E \in (\rho, \rho)$. $\mathcal{C}$ is said to have (finite) direct sums if a direct sum $\rho \oplus \sigma$ exists for all objects $\rho, \sigma \in \text{Ob}\, \mathcal{C}$.

1.3 Remarks:

(1) A subobject $\sigma$ of an object $\rho \in \text{Ob}\, \mathcal{C}$ corresponding to a projection $E \in (\rho, \rho)$ is unique up to equivalence. If a linear isometry $V \in (\sigma, \rho)$ is chosen as in Definition 1.2 (ii) then

$$\alpha : (\sigma, \sigma) \longrightarrow E(\rho, \rho) E, \quad T \mapsto V \circ T \circ V^*,$$

defines a surjective isomorphism from the $C^*$-algebra $(\sigma, \sigma)$ onto the $C^*$-algebra $E(\rho, \rho) E$.

(2) Two projections $E$ and $F$ in $(\rho, \rho)$ belong to the same subobject $\sigma$ if and only if they are equivalent in $(\rho, \rho)$ (i.e. there is a partial isometry $U \in (\rho, \rho)$ such that $U^* \circ U = E$ and $U \circ U^* = F$.)

(3) Let us assume that a direct sum $\tau = \bigoplus_{i=1}^n \rho_i$ of objects $\rho_i \in \text{Ob}\, \mathcal{C}$ is defined as in Definition 1.2 (iii). Let $\tau'$ be another direct sum of the objects $\rho_i$ $(i = 1, \ldots, n)$ and let $V_i' \in (\rho_i, \tau')$ be the corresponding linear isometries. Then there is a unitary operator $U \in (\tau, \tau')$ such that

$$U \circ V_i \circ V_i^* \circ U^* = V_i' \circ (V_i')^* \quad (i = 1, \ldots, n),$$

in particular a direct sum $\bigoplus_{i=1}^n \rho_i$ is unique up to equivalence.

(4) Let $\tau$ be a direct sum $\rho_1 \oplus \rho_2$, and let $E := V_1 \circ V_1^*$ and $F := V_2 \circ V_2^*$. There is a unique surjective linear isometric map $\iota = \iota_{\rho_1, \rho_2}$ from $(\rho_1, \rho_2)$ onto

$$F(\rho_1, \rho_1) E := \{ x \in (\tau, \tau) : F \circ x \circ E = x \} \subset (\tau, \tau)$$

given by $\iota(T) = V_2 \circ T \circ V_1^*$ for $T \in (\rho_1, \rho_2)$. We obtain $\iota_{\rho_1, \rho_2}(T^*) = \iota_{\rho_2, \rho_1}(T^*)$. 

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(5) Let \( \mathcal{C} \) be a \( C^* \)-tensor category. If \( \rho_1 \) and \( \rho_2 \) are objects of \( \mathcal{C} \) and if \( \sigma_i \) is a subobject of \( \rho_i \) for \( i = 1, 2 \) then \( \rho_1 \rho_2 \) is a subobject of \( \sigma_1 \sigma_2 \).

The proof of the Remarks is just an easy exercise.

**Definition 1.4.** (i) Let \( \mathcal{C} \) and \( \mathcal{D} \) be \( C^* \)-categories and \( F : \mathcal{C} \rightarrow \mathcal{D} \) a covariant functor. \( F \) is called a \( C^* \)-functor if

\[
T \in (\rho, \sigma) \mapsto F(T) \in (F(\rho), F(\sigma))
\]

is linear for all objects \( \rho, \sigma \in \text{Ob} \mathcal{C} \) and \( F(T^*) = F(T)^* \) holds for \( T \in (\rho, \sigma) \).

(ii) Let \( \mathcal{C} \) and \( \mathcal{D} \) be \( C^* \)-tensor categories, \( \iota_\mathcal{C} \) and \( \iota_\mathcal{D} \) their unit objects and \( F : \mathcal{C} \rightarrow \mathcal{D} \) a \( C^* \)-functor. For objects \( \rho, \sigma \in \text{Ob} \mathcal{C} \) let \( U_{\rho\sigma} \) be a unitary operator of \((F(\rho)F(\sigma), F(\rho \sigma))\) and \( J \) a unitary operator of \((F(\iota_\mathcal{C}), \iota_\mathcal{D})\) such that the following properties are satisfied:

(a) \( U_{\rho\sigma} \) is natural in \( \rho \) and \( \sigma \).

(b) For \( \rho, \sigma, \tau \in \text{Ob} \mathcal{C} \) the diagram

\[
\begin{array}{ccc}
1_{F(\rho)} \times U_{\sigma\tau} & F(\rho)F(\sigma\tau) & U_{\rho,\sigma\tau} \rightarrow F(\rho(\sigma\tau)) \\
\downarrow & \downarrow & \downarrow \\
F(\rho)(F(\sigma)F(\tau)) & (F(\rho)F(\sigma))F(\tau) & F(\rho\sigma)F(\tau)
\end{array}
\]

commutes.

(c) The diagram

\[
\begin{array}{ccc}
F(\iota_\mathcal{C})F(\rho) & \xrightarrow{U_{\iota_\mathcal{C}\rho}} F(\iota_\mathcal{C}\rho) & U_{\iota_\mathcal{D}\rho} \rightarrow F(l_\rho) \\
J \times 1_{F(\rho)} & \downarrow & \downarrow \\
\iota_\mathcal{D}F(\rho) & l_{F(\rho)} & F(\rho)
\end{array}
\]

commutes for every object \( \rho \) in \( \mathcal{C} \) just as the corresponding diagram for \( r_\rho \) and \( r_{F(\rho)} \).

Then \((F, (U_{\rho\sigma})_{\rho,\sigma}, J)\) is called a \( C^* \)-tensor functor. Usually we just write \( F \) for the \( C^* \)-tensor functor.
(iii) A $C^*$-tensor functor $F : C \rightarrow D$ is called a ($C^*$-tensor) equivalence of the $C^*$-tensor categories $C$ and $D$ if $F$ is full and faithful and if for each object $\tau$ of $D$ there is an object $\rho$ of $C$ such that $F(\rho)$ is equivalent to $\tau$.

(iv) Let $C$ and $D$ be strict $C^*$-tensor categories. A covariant functor $F : C \rightarrow D$ is called a strict $C^*$-tensor functor if $F(\rho_\sigma) = F(\rho)F(\sigma)$ for $\rho, \sigma \in \text{Ob } C$, $F(\iota_C) = \iota_D$ and if $(F, (1_{F(\rho_\sigma)})_{\rho,\sigma}, 1_{\iota_D})$ is a $C^*$-tensor functor.

1.5 Remarks:

(1) Let $F : C \rightarrow D$ be a $C^*$-functor. We obtain $\|F(T)\| \leq \|T\|$ for all morphisms $T$ of $C$. If $F$ is faithful, then $\|F(T)\| = \|T\|$ for all morphisms $T$ of $C$. (Use the corresponding results for homomorphisms of $C^*$-algebras.)

(2) Let $C$ be a $C^*$-tensor category. Remark (1) implies $\|1_{\rho} \times T\| \leq \|T\|$ and $\|T \times 1_{\rho}\| \leq \|T\|$ for any object $\rho$ of $C$ and any morphism $T$ of $C$. Hence $\|S \times T\| \leq \|S\| \cdot \|T\|$ for all morphisms $S$ and $T$ of $C$.

An equivalence $F : C \rightarrow D$ induces a bijection $[F]$ from the set $[C]$ onto the set $[D]$. Equivalences satisfy the expected properties:

Proposition 1.6. (i) Let $C, D$ and $E$ be $C^*$-tensor categories. If $(F, (U_{\rho_\sigma})_{\rho,\sigma}, J)$ is an equivalence from $C$ to $D$ and $(G, (V_{\phi_\psi})_{\phi,\psi}, K)$ is an equivalence from $D$ to $E$, then $(G \circ F, (G(U_{\rho_\sigma}) \circ V_{F(\rho_\phi)})_{\rho,\phi}, K \circ G(J))$ is an equivalence from $C$ to $E$.

(ii) If $F : C \rightarrow D$ is an equivalence of the $C^*$-tensor categories $C$ and $D$ then there is a $C^*$-tensor equivalence $G : D \rightarrow C$ such that $[G]$ is the inverse of $[F]$.

Part (i) is easy to verify. Since the author failed to find a suitable reference for (ii), a proof of (ii) is presented in the Appendix.

Proposition 1.7. Every $C^*$-tensor category is equivalent to a strict $C^*$-tensor category.

First this result was proved in [10] for tensor categories without $C^*$-structure. (A more accessible reference is [9], Theorem XI.5.3.) The transfer to $C^*$-tensor categories is obvious. The Proposition allows us to work only with strict $C^*$-tensor categories. This will abbreviate many computations, as we don’t need the operators $a(\rho,\sigma,\tau), l_\rho$ and $r_\rho$. 

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Definition 1.8.  (i) Let $\rho$ be an object of the $C^*$-tensor category $\mathcal{C}$. An object $\bar{\rho}$ is said to be conjugate to $\rho$ if there are $R \in \langle \iota, \rho \rho \rangle$ and $\bar{R} \in \langle \iota, \rho \bar{\rho} \rangle$ such that

$$l_{\rho} \circ \bar{R}^* \times 1_{\rho} \circ a(\rho, \bar{\rho}, \rho) \circ 1_{\rho} \times R \circ r_{\rho}^* = 1_{\rho} \quad \text{and} \quad (1)$$
$$l_{\bar{\rho}} \circ R^* \times 1_{\bar{\rho}} \circ a(\bar{\rho}, \rho, \bar{\rho}) \circ 1_{\bar{\rho}} \times \bar{R} \circ r_{\bar{\rho}}^* = 1_{\bar{\rho}} \quad (2)$$

hold. $(R, \bar{R})$ is called a pair of conjugation operators for $\rho$ and $\bar{\rho}$. If the $C^*$-tensor category $\mathcal{C}$ is strict, $(1)$ and $(2)$ mean

$$R^* \times 1_{\rho} \circ a(\rho, \bar{\rho}, \rho) \circ 1_{\rho} \times r_{\rho}^* = 1_{\rho} \quad \text{and} \quad (3)$$
$$R^* \times 1_{\bar{\rho}} \circ a(\bar{\rho}, \rho, \bar{\rho}) \circ 1_{\bar{\rho}} \times r_{\bar{\rho}}^* = 1_{\bar{\rho}} \quad (4)$$

From now on, if $\rho$ is an object of a $C^*$-tensor category $\mathcal{C}$, $\bar{\rho}$ denotes an object of $\mathcal{C}$ conjugate to $\rho$.

(ii) A strict $C^*$-tensor category is called a regular $C^*$-tensor category if it has subobjects and finite direct sums and $(\iota, \iota) = \mathcal{C}_{1_\iota}$ is satisfied for the unit object $\iota$.

(iii) A regular $C^*$-tensor category $\mathcal{C}$ is called a compact $C^*$-tensor category if every object $\rho$ of $\mathcal{C}$ possesses a conjugate.

(iv) A compact $C^*$-tensor category $\mathcal{C}$ is called a finite $C^*$-tensor category if $\dim(C)$ is finite.

Lemma 1.9. Let $\mathcal{C}$ and $\mathcal{D}$ be $C^*$-tensor categories, let $(F, (U_{\rho\sigma})_{\rho,\sigma}, J)$ be a $C^*$-tensor functor from $\mathcal{C}$ to $\mathcal{D}$, and let $\rho$ be an object of $\mathcal{C}$ having a conjugate $\bar{\rho}$. Then $F(\bar{\rho})$ is conjugate to $F(\rho)$. If $(R, \bar{R})$ is a pair of conjugation operators for $\rho$ and $\bar{\rho}$ then

$$(S, \bar{S}) = (U_{\rho\bar{\rho}}^* \circ F(R) \circ J^*, U_{\rho\bar{\rho}}^* \circ F(\bar{R}) \circ J^*)$$

is a pair of conjugation operators for $F(\rho)$ and $F(\bar{\rho})$.

Proof: We verify Equation $(1)$ for the pair $(S, \bar{S})$ and omit the similar proof of $(2)$. $(1)$ for $(R, \bar{R})$ implies

$$F(l_{\rho}) \circ F(\bar{R})^* \times 1_{\rho} \circ F(a(\rho, \bar{\rho}, \rho)) \circ F(1_{\rho} \times R) \circ F(r_{\rho}^*) = 1_{F(\rho)}. \quad (5)$$

Using Definition 1.4 (ii), we replace

$$F(l_{\rho}) \quad \text{by} \quad l_{F(\rho)} \circ J \times 1_{F(\rho)} \circ U_{\iota\rho}^*;$$
$$F(\bar{R})^* \times 1_{\rho} \quad \text{by} \quad U_{\iota\rho}^* \circ F(\bar{R})^* \times 1_{F(\rho)} \circ U_{\rho\bar{\rho}, \rho}^*;$$
$$F(a(\rho, \bar{\rho}, \rho)) \quad \text{by} \quad U_{\rho\bar{\rho}, \rho} \circ U_{\rho\bar{\rho}} \times 1_{F(\rho)} \circ a(F(\rho), F(\bar{\rho}), F(\rho)) \circ$$
$$\circ 1_{F(\rho)} \times U_{\rho\bar{\rho}}^* \circ U_{\rho, \rho\bar{\rho}}^*;$$

and so on.
In this way Equation (5) yields
\[ l_{F(\rho)} \circ J \times 1_{F(\rho)} \circ F(\overline{R}) \times 1_{F(\rho)} \circ U_{\rho \bar{\rho}} \times 1_{F(\rho)} \circ a(F(\rho), F(\bar{\rho}), F(\rho)) \circ 1_{F(\rho)} \times U_{\rho} \times 1_{F(\rho)} \times F(\overline{R}) \circ 1_{F(\rho)} \times J^* \circ r_{F(\rho)} = 1_{F(\rho)}, \]
and Equation (1) has been shown for \( S \) and \( \overline{S} \).

Definition 1.8 (ii) is due to J.E. Roberts for strict \( C^* \)-tensor categories (see [23]). He introduced a dimension for objects possessing a conjugate in [23]. In the following we will give a brief summary (without proofs) of his theory.

In the following let \( C \) be a regular \( C^* \)-tensor category and let \( \rho \) be an object of \( C \). We start with the Frobenius reciprocity (see Lemma 2.1 in [14]):

Lemma 1.10. Assume that \( \rho \) has a conjugate \( \bar{\rho} \), let \((R, \overline{R})\) be a pair of conjugation operators for \( \rho \) and \( \bar{\rho} \) and let \( \sigma \) and \( \tau \) be objects of \( C \).

\[ S \mapsto 1_{\bar{\rho}} \times S \circ R \times 1_\sigma \text{ is a bijective linear map from } (\rho \sigma, \tau) \text{ onto } (\sigma, \bar{\rho} \tau). \text{ The inverse is given by } S' \mapsto \overline{R}^* \times 1_\tau \circ 1_{\bar{\rho}} \times S'. \]

Similarly \( T \in (\sigma \rho, \tau) \mapsto T \times 1_{\bar{\rho}} \circ 1_\sigma \times \overline{R} \in (\sigma, \tau \bar{\rho}) \) and \( T' \in (\sigma, \tau \bar{\rho}) \mapsto 1_\tau \times R^* \circ T' \times 1_\rho \in (\sigma \rho, \tau) \) are inverse linear maps.

We note the following results:

Proposition 1.11. (i) Let \( \bar{\rho} \) be conjugate to \( \rho \). An object \( \tau \) of \( C \) is conjugate to \( \rho \) if and only if \( \bar{\rho} \) and \( \tau \) are equivalent.

(ii) If \( \rho \) is irreducible and \( \bar{\rho} \) conjugate to \( \rho \), \( \bar{\rho} \) is irreducible, too.

(iii) If \( \rho \) has a conjugate the \( C^* \)-algebra \( (\rho, \rho) \) is finite dimensional, in particular \( \rho \) is a finite direct sum of irreducible objects.

(iv) Let \( \sigma \) and \( \rho \) be irreducible objects of \( C \). \( \rho \) is conjugate to \( \sigma \) if and only if there are operators \( R \in (\iota, \sigma \rho) \) and \( \overline{R} \in (\iota, \rho \sigma) \) such that \( \overline{R}^* \times 1_\rho \circ 1_\rho \times R \neq 0 \).

R. Longo and J.E. Roberts proved Proposition 1.11 in [14] ((i), (ii) after Lemma 2.1, (iii) in Lemma 3.2 and (iv) in Lemma 2.2).

1.12 Remarks:

(1) Each minimal projections of \( (\rho, \rho) \) corresponds to an irreducible subobject \( \sigma \) of \( \rho \). Two minimal projections \( E \) and \( F \) of \( (\rho, \rho) \) correspond to the same subobject \( \sigma \) of \( \rho \), if and only if they are equivalent in \( (\rho, \rho) \). If \( A \) is a simple direct summand of \( (\rho, \rho) \) and if \( \phi \) is an irreducible subobject of \( \rho \) belonging to a minimal projection \( E \) of \( A \), then we attach the equivalence class \([\phi] \in [\mathbb{C}] \) to \( A \). In this way we get a bijective correspondence between the equivalence classes of the irreducible objects of \( C \) contained in \( \rho \) and the simple direct summands of \( (\rho, \rho) \).
(2) Let \( \rho \) be an object of \( \mathcal{C} \) having a conjugate. \( 1_\rho \) has a decomposition
\[ 1_\rho = \sum_{i=1}^{n} P_i \]
into minimal projections of \((\rho, \bar{\rho})\). For \( i = 1, \ldots, n \) let \( p_i \) denote an irreducible object corresponding to \( P_i \) \((i = 1, \ldots, n)\). Then \( \rho \) is a direct sum \( \bigoplus_{i=1}^{n} p_i \) of these objects.

If \( 1_\rho = \sum_{j=1}^{m} Q_j \) is another decomposition of \( 1_\rho \) into minimal projections, then \( m = n \) and there exists a permutation \( \pi \) of \( \{1, \ldots, n\} \) such that \( P_i \) is equivalent to \( Q_{\pi(i)} \) for \( i = 1, \ldots, n \). Hence the decomposition of \( \rho \) into irreducible objects is unique in the following meaning: If \( \rho = \bigoplus_{i=1}^{n} p_i \) and \( \rho = \bigoplus_{j=1}^{m} \rho'_j \) are two decompositions of \( \rho \) into irreducible objects, then \( n = m \) and there is a permutation \( \pi \) of \( \{1, \ldots, n\} \) such that \( p_i \) and \( \rho'_{\pi(i)} \) are equivalent for \( i = 1, \ldots, n \).

### 1.13 The statistical dimension

We assume that \( \rho \) has a conjugate \( \bar{\rho} \). For the moment let us suppose that \( \rho \) is irreducible. Let \( (S, \overline{S}) \) be a pair of conjugation operators for \( \rho \) and \( \bar{\rho} \). In general the operators \( S^* \circ S \) and \( \overline{S^*} \circ \overline{S} \) are not equal. But there is a complex number \( \alpha \in \mathbb{C} \setminus \{0\} \) such that \( R = \alpha S \) and \( \bar{R} = \frac{1}{\alpha} \overline{S} \) are a pair of standard conjugation operators, i.e., that

\[ R^* \circ R = \overline{R^*} \circ \overline{R} \]

holds. By identifying \((\iota, \iota) = \mathbb{C}1_\iota \) with \( \mathbb{C} \), we may regard \( R^* \circ R \) as a positive number, which is called the (statistical) dimension \( d(\rho) \) of \( \rho \). The statistical dimension \( d(\rho) \) does not depend on the choice of the conjugate \( \bar{\rho} \) and the standard conjugation operators \( R \) and \( \bar{R} \) (see below). We point out that the dimension is not a natural number in general.

We are going to define standard conjugation operators and the dimension, if \( \rho \) is necessarily irreducible. There are irreducible objects \( \sigma_1, \ldots, \sigma_n \) and linear isometries \( W_i \in (\sigma_i, \rho) \), \( \overline{W_i} \in (\overline{\sigma_i}, \bar{\rho}) \) \((i = 1, \ldots, n)\) satisfying

\[ 1_\rho = \sum_{i=1}^{n} W_i \circ W_i^* \quad \text{and} \quad 1_{\bar{\rho}} = \sum_{i=1}^{n} \overline{W_i} \circ \overline{W_i^*}. \]

Moreover, we have standard conjugation operators \( R_i \in (\iota, \overline{\sigma_i}) \) and \( \overline{R_i} \in (\iota, \sigma_i \overline{\iota}) \) for \( \iota = 1, \ldots, n \). Put \( R := \sum_{i=1}^{n} W_i \times W_i \circ R_i \) and \( \bar{R} := \sum_{i=1}^{n} \overline{W_i} \times \overline{W_i} \circ \overline{R_i} \). \((R, \bar{R})\) is a pair of conjugation operators for \( \rho \) and \( \bar{\rho} \). Pairs of conjugation operators are called standard, when they are obtained in this way. We get \( R^* \circ R = \overline{R^*} \circ \overline{R} \), and this number is called the statistical dimension \( d(\rho) \) of \( \rho \). \( d(\rho) \) is independent of the choice of the standard pair \((R, \bar{R})\) according to the following Lemma (see Lemma 2 in [23]):

**Lemma 1.14.** Let \( \bar{\rho} \) be any conjugate of \( \rho \) and let \( T \in (\iota, \bar{\rho} \rho) \) be given. There is an operator \( T \in (\iota, \bar{\rho} \rho) \) such that \((T, \bar{T})\) is a standard pair of conjugation operators, if and only if there is a unitary operator \( U \in (\bar{\rho}, \bar{\rho}) \) such that \((U \times 1_\rho) \circ R = T \).
The conjugate object and the statistical dimension have the following properties:

**Theorem 1.15.** Let \( \rho \) and \( \sigma \) be objects of \( \mathcal{C} \) having conjugates \( \bar{\rho} \) and \( \bar{\sigma} \).

(i) \( \rho \oplus \sigma \) is conjugate to \( \bar{\rho} \oplus \bar{\sigma} \) and \( d(\rho \oplus \sigma) = d(\rho) + d(\sigma) \).

(ii) \( \rho \) is also conjugate to \( \bar{\rho} \) and \( d(\rho) = d(\bar{\rho}) \) holds.

(iii) Every subobject of \( \rho \) has a conjugate.

(iv) The product \( \bar{\sigma} \bar{\rho} \) is conjugate to \( \rho \sigma \). If \( (R_\rho, \bar{R}_\rho) \) is a standard pair of conjugation operators for \( \rho \) and \( \bar{\rho} \) and \( (R_\sigma, \bar{R}_\sigma) \) for \( \sigma \) and \( \bar{\sigma} \), then

\[
(R_{\rho\sigma}, \bar{R}_{\rho\sigma}) = (1_\sigma \times R_\rho \times 1_\sigma \circ R_\sigma, 1_\rho \times \bar{R}_\sigma \times 1_\rho \circ \bar{R}_\rho)
\]

is a standard pair of conjugation operators for \( \rho \sigma \) and \( \bar{\sigma} \bar{\rho} \). The equation \( d(\rho \sigma) = d(\rho) \cdot d(\sigma) \) is satisfied.

Part (i) and (ii) are almost obvious. Part (iii) follows from Theorem 2.4 and Part (iv) from Lemma 3.6 and Corollary 3.10 (a) in [14].

**1.16 The \( C^* \)-tensor category of the unitary corepresentations of a finite dimensional Hopf-\( * \)-algebra**

The most natural example for a compact \( C^* \)-tensor category is the category \( \mathcal{U}_G \) of the unitary finite dimensional representations of a compact group \( G \). In [14] R. Longo and J.E. Roberts deal with the more general case of the unitary finite dimensional corepresentations of a compact matrix pseudogroup in the sense of S. Woronowicz ([31]). In this article we only need the special case of a finite dimensional Hopf-\( * \)-algebra.

We assume that the reader is familiar with Hopf-\( * \)-algebras and mention [26], [9], [10], and [31] as basic references. Let us introduce the finite \( C^* \)-tensor category \( \mathcal{U}_H \) of the unitary corepresentations of the finite dimensional Hopf-\( * \)-algebra \( H \). \( \Delta : H \rightarrow H \otimes H \) denotes the comultiplication of \( H \), \( \epsilon : H \rightarrow \mathbb{C} \) the counit of \( H \) and \( S : H \rightarrow H \) the antipode of \( H \). The objects of \( \mathcal{U}_H \) are the unitary corepresentations \( \sigma : V_\sigma \rightarrow V_\sigma \otimes H \) of \( H \) on a (small) finite dimensional Hilbert space \( V_\sigma \). We recall that a unitary corepresentation \( \sigma : V_\sigma \rightarrow V_\sigma \otimes H \) is a linear map satisfying

\[
(\sigma \otimes id_H) \circ \sigma = (id_{V_\sigma} \otimes \Delta) \circ \sigma \quad \text{and} \quad (id_{V_\sigma} \otimes \epsilon) \circ \sigma = id_{V_\sigma}
\]

as well as

\[
\sum_{(v), (w)} \langle v^{(1)}, w^{(1)} \rangle w^{(2)*} v^{(2)} = \langle v, w \rangle 1 \quad \text{for} \ v, w \in V_\sigma \quad (7)
\]
where $\sigma(v) = \sum_v v^{(1)} \otimes v^{(2)}$ (Sweedler notation for corepresentations). If $\rho : V_\rho \rightarrow V_\rho \otimes H$ and $\sigma : V_\sigma \rightarrow V_\sigma \otimes H$ are objects of $U_H \, (\rho, \sigma)$ is the Banach space of the intertwining operators $T : V_\rho \rightarrow V_\sigma$ of the corepresentations $\rho$ and $\sigma$ (i.e. $T$ satisfies $(T \otimes \text{id}_H) \circ \rho = \sigma \circ T$.) The composition $\circ$ is the composition of maps and $T^* \in (\sigma, \rho)$ is the adjoint operator of $T \in (\rho, \sigma)$. The tensor product $\rho \sigma$ is the tensor product $\rho \otimes \sigma$ of the corepresentations $\rho$ and $\sigma$ defined by

$$\rho \otimes \sigma : V_\rho \otimes V_\sigma \rightarrow V_\rho \otimes V_\sigma \otimes H, \ (\rho \otimes \sigma)(v \otimes w) = \sum_{(v), (w)} v^{(1)} \otimes w^{(1)} \otimes v^{(2)} w^{(2)},$$

and $T \times T'$ is the tensor product $T \otimes T'$ of linear operators for morphisms $T \in (\rho, \sigma)$ and $T' \in (\rho', \sigma')$. The unit object is the identity corepresentation $\iota : \mathbb{C} \rightarrow \mathbb{C} \otimes H, \ i(\lambda) = \lambda \otimes 1$. The associativity constraint $a(\rho, \sigma, \tau)$ is the canonical unitary operator from $V_\rho \otimes (V_\sigma \otimes V_\tau)$ onto $(V_\rho \otimes V_\sigma) \otimes V_\tau$, and $l_\rho$ (resp. $r_\rho$) is the canonical unitary operator from $\mathbb{C} \otimes V_\rho$ (resp. $V_\rho \otimes \mathbb{C}$) onto $V_\rho$. Using the usual identifications of these Hilbert spaces, we may regard $U_H$ as a strict $C^*$-tensor category.

The conjugate object for an object $\sigma \in \text{Ob} \, U_H$ is the contragredient corepresentation $\bar{\sigma}$ of $\sigma$. Let $\mathcal{B} = (v_1, \ldots, v_n)$ be an orthonormal base of $V_\sigma$ and $(a_{ij})_{i,j=1}^n$ be the matrix coefficients of $\sigma$ with respect to $\mathcal{B}$, i.e. $\sigma(v_j) = \sum_{i=1}^n v_i \otimes a_{ij}$ for $i = 1, \ldots, n$. Then $\bar{\sigma}$ is given on the dual Hilbert space $V_\sigma$ by $\bar{\sigma}(\overline{v_j}) = \sum_{i=1}^n \overline{v_i} \otimes S(a_{ji})$, where $(\overline{v_1}, \ldots, \overline{v_n})$ is the base dual to $\mathcal{B}$. $\bar{\sigma}$ is a unitary corepresentation of $H$ with respect to the inner product on $V_\sigma$. (According to [10], a corepresentation $\sigma$ fulfills the unitarity property (7) if and only if

$$S(a_{ij}) = a_{ji}^* \quad \text{for} \ i, j = 1, \ldots, n.$$  

Using this characterization of the unitarity and applying $S^2 = \text{id}_H$, one concludes that $\bar{\sigma}$ is unitary.)

Observe that $\sigma$ is unitary if and only if the matrix coefficients satisfy

$$\sum_{j=1}^n a_{ij} a_{kj}^* = \delta_{ik} \mathbbm{1} \quad \text{for} \ i, k = 1, \ldots, n$$

(see [10]). Hence

$$((\sigma \otimes \bar{\sigma}) \sum_{j=1}^n v_j \otimes \overline{v_j}) = \sum_{i,k=1}^n v_i \otimes \overline{v_k} \otimes \sum_{j=1}^n a_{ij} \cdot a_{kj}^* = \sum_{i=1}^n v_i \otimes \overline{v_i} \otimes \mathbbm{1}.$$ 

Thus the restriction of $\sigma \otimes \bar{\sigma}$ onto $\mathbb{C} \sum_{j=1}^n v_j \otimes \overline{v_j}$ is equivalent to the identity corepresentation $\iota$ and

$$R : \mathbb{C} \rightarrow V_\sigma \otimes \overline{V_\sigma}, \ \lambda \mapsto \lambda \sum_{j=1}^n v_j \otimes \overline{v_j}.$$
intertwines the corepresentations \( \iota \) and \( \sigma \otimes \bar{\sigma} \). Similarly,

\[
R : \mathbb{C} \rightarrow V_\sigma \otimes V_\sigma, \lambda \mapsto \lambda \sum_{j=1}^{n} v_j \otimes v_j,
\]

is an intertwining operator for the corepresentations \( \iota \) and \( \bar{\sigma} \otimes \sigma \). Now we get

\[
(\bar{R}^* \otimes 1_\sigma \circ 1_\sigma \otimes R) w = (\bar{R}^* \otimes 1_\sigma) w \otimes \sum_{j=1}^{n} v_j \otimes v_j =
\sum_{j=1}^{n} \langle w \otimes \bar{v}_j, \sum_{i=1}^{n} v_i \otimes \bar{v}_i \rangle v_j = \sum_{j=1}^{n} \langle w, v_j \rangle v_j = w
\]

for \( w \in V_\sigma \) and conclude \( \bar{R}^* \otimes 1_\sigma \circ 1_\sigma \otimes R = 1_\sigma \). The same argument yields \( R^* \otimes 1_\sigma \circ 1_\sigma \otimes R = 1_\sigma \). As \( R^* \circ R = n = R^* \circ \bar{R} \), we have \( d(\sigma) = n = \dim V_\sigma \), if \( \sigma \) is irreducible. By applying Theorem 1.15 (i) we obtain \( d(\sigma) = \dim V_\sigma \) for every object \( \sigma \in \text{Ob} \mathcal{U}_H \). We point out that the equation \( d(\sigma) = \dim V_\sigma \) is not valid for general compact quantum groups, as \( \bar{\sigma} \) need not be unitary with respect to the dual inner product on \( V_\sigma \). (Observe that \( S^2 = id_H \) is not satisfied in general.)

\section{The \( C^* \)-tensor category of the \((N, N)\)-bimodules for subfactors \( N \subset M \)}

First let us fix some notations concerning subfactors and the basic construction. In order to avoid subtleties we will assume that every von Neumann algebra appearing in this paper acts on a separable Hilbert space.

\subsection{The basic construction and the Jones’ tower}

Let \( A \subset B \) be an inclusion of two finite von Neumann algebras \( A \) and \( B \) with the same unit, and let \( \text{tr} \) be a faithful normal normalized trace of \( B \). \( L^2(B) \) is defined as the completion of \( B \) with respect to the inner product \( (x, y) \mapsto \text{tr}(xy^*) \). We denote an element \( x \) of \( B \) by \( \overline{x} \) if \( x \) is regarded as an element of \( L^2(B) \). The Hilbert space \( L^2(A) \) (defined by the trace \( \text{tr} | A \)) is a closed subspace of \( L^2(B) \). \( B \) acts normally and faithfully on \( L^2(B) \) by left multiplication. So we may consider \( B \) as a von Neumann algebra on \( L^2(B) \).

Let \( e \) denote the orthogonal projection from \( L^2(B) \) onto \( L^2(A) \). The von Neumann algebra \( B_1 \) generated by \( B \) and \( e \) is called the basic construction for \( A \subset B \) (and for the trace \( \text{tr} \)). \( e \) is called the Jones projection. \( e \) maps \( B \) onto \( A \), the restriction

\[
E = e|B : B \rightarrow A
\]
of $e$ is a normal faithful conditional expectation from $B$ onto $A$, in particular we have $E(a_1 b a_2) = a_1 E(b) a_2$ for $a_1, a_2 \in A$ and $b \in B$ and $E(b) \geq 0$ for $b \geq 0$. For $b \in B$, $E(b)$ is the unique element of $A$ satisfying

$$tr(E(b) a) = tr(b a) \quad \text{for every } a \in A. \quad (8)$$

$E$ is called the conditional expectation from $B$ to $A$ corresponding to the trace $tr$.

Let $B^{op}$ be the von Neumann algebra opposite to $B$. ($B^{op}$ is equal to $B$ as a complex vector space, the multiplication law is reversed, that means $b^{op} c := c \cdot b$ for $b, c \in B$, and the involution $*$ is the same as in $B$.) We have a normal representation $\rho : B^{op} \to L(L^2(B))$ of $B^{op}$ given by $\rho(b) \overline{x} = \overline{xb}$ for $b, x \in B$ (multiplication from right). The basic construction satisfies the relation

$$B_1 = \rho(A^{op})'.$ \quad (9)$$

Let $J_B$ be the involutive antiunitary operator on $L^2(B)$ given by $J_B \overline{x} = \overline{x^*}$ for $x \in B$. Often we abbreviate $J_B$ to $J$. We get $\rho(b) = J b^* J$ for $b \in B$. From equation (9) we conclude

$$B_1 = JA'J. \quad (10)$$

For a $\Pi_1$-factor $L$, let $tr_L$ denote the unique normalized trace of $L$. Now let $N \subset M$ be an inclusion of $\Pi_1$-factors (with the same unit). The basic construction $M_1$ (with respect to $tr_M$) is a factor of type $\Pi_1$ if and only if the Jones index $\beta := [M : N] < \infty$. From now on let us assume that $\beta$ is finite. In this case $M_1$ is the $\mathbb{C}$-vector space generated by $m_1em_2$, $m_1, m_2 \in M$ (see [3], Theorem 3.6.4). The subfactor $M \subset M_1$ is called the subfactor dual to $N \subset M$. We obtain $[M_1 : M] = [M : N] < \infty$, so we are able to repeat the basic construction infinitely many times and get the so called Jones tower

$$N = M_{-1} \subset M = M_0 \subset M_1 \subset M_2 \subset M_3 \subset \ldots ,$$

where $M_{k+1}$ is the basic construction for the subfactor $M_{k-1} \subset M_k$. For $k \in \mathbb{N} \cup \{0\}$ let $e_k \in M_{k+1}$ denote the orthogonal projection from $L^2(M_k)$ onto $L^2(M_{k-1})$ and $E_k : M_k \to M_{k-1}$ the corresponding conditional expectation. We have the following relations:

$$e_k e_{k \pm 1} e_k = \beta^{-1} e_k \quad \text{and} \quad (11)$$

$$e_k e_l = e_l e_k \quad \text{for } |k - l| \geq 2. \quad (12)$$

Moreover we have

$$e_k x e_k = E_k(x) e_k \quad \text{for } x \in M_k \quad \text{and} \quad (13)$$

$$E_k(e_{k-1}) = \beta^{-1} 1. \quad (14)$$

The trace $tr_{M_{k+1}}$ on $M_{k+1}$ satisfies the Markov property

$$\beta tr_{M_{k+1}}(xe_k) = tr_{M_{k+1}}(x) \quad \text{for } x \in M_k. \quad (15)$$
2.2 $C^*$-tensor categories and bimodules

Let $N$ be a II$_1$-factor. An easy access to $(N, N)$-bimodules may be found in [23] (see also [2], [3], [12] and [13]). We assume the notation from [23] and recall that an $(N, N)$-bimodule $N\mathcal{H}_N$ is called regular if the intersection of the left and of the right bounded elements of $\mathcal{H}$ is dense in $\mathcal{H}$. For every $(N, N)$-bimodule $N\mathcal{H}_N$, there is the conjugate bimodule $\overline{N\mathcal{H}_N}$, where $\mathcal{H}$ is equal to $\mathcal{H}$ as a real vector space, but the inner product and the scalar multiplication are conjugate. The left action $\lambda(n) = \rho(n^*)$ and $\overline{\rho}(n) = \lambda(n^*)$ for $n \in N$.

We introduce the $C^*$-tensor category $\mathcal{B}_N$ of all (small) regular $(N, N)$-bimodules. The objects are the (small) regular $(N, N)$-bimodules. For two objects $\rho = \mathcal{H}_N$ and $\sigma = \mathcal{K}_N$ of $\mathcal{B}_N$, the $(N, N)$-linear continuous operators from $\mathcal{H}$ into $\mathcal{K}$ (endowed with the operator norm) form the complex Banach space $(\rho, \sigma)$. Endowed with the usual $*$-operation $*: (\rho, \sigma) \rightarrow (\sigma, \rho)$, $\mathcal{B}_N$ is a $C^*$-category with subobjects and finite direct sums.

The product structure is given by the tensor product $\otimes_N$. We define $T \times T' := T \otimes_N T'$ for $T \in (\rho, \sigma)$ and $T' \in (\rho', \sigma')$. The unit object $\iota$ is $\mathcal{N}L^2(N)_N$. The associativity constraint $a(\rho, \sigma, \tau)$ as well as the maps $l_\rho$ and $r_\rho$ are defined as in [23], Section 1.

Let $N \subset M$ be an inclusion of type II$_1$-factors such that $[M : N] < \infty$. Let $\mathcal{B}_{N \subset M}$ denote the full $C^*$-tensor subcategory of $\mathcal{B}_N$ whose object set consists of all (small) $(N, N)$-bimodules which are equivalent to a finite direct sum of $(N, N)$-bimodules contained in $\mathcal{N}L^2(M_k)_N$ for some $k \in \mathbb{N} \cup \{0\}$. Observe $\mathcal{L}^2(M_{k-1}) \otimes_N \mathcal{L}^2(M_{l-1}) \cong \mathcal{L}^2(M_{k+l-1})$ (for example see [23]), hence the $\otimes_N$-tensor product of two objects of $\mathcal{B}_{N \subset M}$ is an object of $\mathcal{B}_{N \subset M}$.

From now on we will treat the $C^*$-tensor category $\mathcal{B}_{N \subset M}$, as if it were strict. (This is allowed by Proposition [17]) The following result is probably known to some experts, but the author could not find any reference.

**Proposition 2.3.** The $C^*$-tensor category $\mathcal{B}_{N \subset M}$ is compact. It is finite if and only if the subfactor $N \subset M$ has finite depth.

If $\mathcal{H}$ is an $(N, N)$-bimodule of $\mathcal{B}_{N \subset M}$ the conjugate bimodule $\overline{\mathcal{H}}$ is conjugate to $\mathcal{H}$ in the sense of Definition [7,8]. We have

$$d(\mathcal{H}) = \sqrt{c(\lambda(N), \mathcal{H}) \cdot c(\rho(N), \mathcal{H})}$$

(16)

for any irreducible $(N, N)$-bimodule $N\mathcal{H}_N \in \text{Ob} \mathcal{B}_{N \subset M}$ (where $c(\lambda(N), \mathcal{H})$ denotes the coupling constant of $\lambda(N)$ in $\mathcal{H}$).

2.4 Remarks:

(1) For any bimodule $\mathcal{H} \in \mathcal{B}_{N \subset M}$, $d(\mathcal{H})$ is equal to the minimal dimension of $\mathcal{H}$ which is the square root of the minimal index $[\rho_{\mathcal{H}}(N) : \lambda_{\mathcal{H}}(N)]_{\text{min}}$ for the subfactor $\lambda_{\mathcal{H}}(N) \subset \rho_{\mathcal{H}}(N)'$. (We will not define the minimal index here, see [3], [7] or [11] for the definition).
For the so-called extremal subfactors \( N \subset M \) (see \[21\]), we have

\[ d(\mathcal{H}) = c(\lambda(N), \mathcal{H}) = c(\rho(N), \mathcal{H}) \]

for every \( \mathcal{H} \in \text{Ob} \mathcal{B}_{N \subset M} \). We point out that finite depth subfactors and irreducible subfactors are extremal.

If \( N \neq M \) every object of \( \mathcal{B}_{N \subset M} \) is indeed equivalent to a sub-bimodule of \( L^2(M_k) \) for a suitable \( k \in \mathbb{N} \).

The Remarks (1) and (2) are shown at the end of the proof of the Proposition, where we make free use of S. Popa’s definitions and results in \[21\]. The easy proof of (3) is left to the reader.

**Proof of the Proposition:** It suffices to show the assertions for a sub-bimodule \( \mathcal{H} \) of \( _N L^2(M)_M \), as it is possible to carry out the same proof for \( M_k (k \in \mathbb{N}) \) instead of \( M \).

Let \( \mathcal{H} \) be \( pL^2(M) \) where \( p \) is an orthogonal projection of \( N' \cap M_1 \). \( M \) possesses a finite base as a right \( N \)-module, as it was shown in \[18\], Proposition 1.3. A version of this result may be found in \[5\], Theorem 3.6.4, we will modify the proof there in order to get a right \( N \)-module base also for \( pM \) (\( pL^2(M) \)).

As in the proof in \[5\] there are finitely many partial isometries \( w_1, \ldots, w_m \in M_1 \setminus \{0\} \) such that

\[ w_j^* w_j \leq e_0 \]  

(17)

for \( j = 1, \ldots, m \) and \( \sum_{j=1}^m w_j w_j^* = p \). Similarly we find partial isometries \( w_{m+1}, \ldots, w_n \in M_1 \setminus \{0\} \) such that (17) holds for \( j = m+1, \ldots, n \) and \( \sum_{j=m+1}^n w_j w_j^* = 1 - p \). For \( j = 1, \ldots, n \) there is a unique \( v_j \in M \) such that \( w_j = v_j e_0 \). As in \[5\] we see that \( (v_i : i = 1, 2, \ldots, n) \) fulfils the properties of a Pimsner-Popa basis:

(a) \( E_0(v_i^* v_j) = 0 \) for \( i \neq j \).

(b) \( f_i := E_0(v_i^* v_i) \) is a projection satisfying \( v_i f_i = v_i \).

(c) Every \( x \in M \) has a unique expansion \( x = \sum_{j=1}^n v_j y_j \) with \( y_j \in N \). In fact, we have \( y_j = E_0(v_j^* x) \).

Furthermore, we obviously get the following relations:

(d) \( \sum_{j=1}^m v_j e_0 v_j^* = p \) and

(e) \( p.\overline{x} = \sum_{j=1}^m v_j e_0 v_j^* \overline{x} = \sum_{j=1}^m v_j E_0(v_j^* x) \) for \( x = \sum_{j=1}^n v_j y_j \in M \).

From (e) we conclude that

(f) \( pL^2(M) \cap M \) is dense in \( pL^2(M) \).
The conjugate bimodule $\overline{H}$ is isomorphic to $p^{op}L^2(M)$, where $p^{op} = JpJ$ and $J$ is the antiunitary operator $J_M$. We get $p^{op}L^2(M) = [\overline{x} : x \in pL^2(M) \cap M]$ and identify $\overline{H}$ and $p^{op}L^2(M)$.

Using the canonical isomorphism between $L^2(M_1)$ and $L^2(M) \otimes_N L^2(M)$, we regard $K := pL^2(M) \otimes_N p^{op}L^2(M)$ as an $(N,N)$-sub-bimodule of $L^2(M_1)$. $\overline{p} \in L^2(M_1)$ belongs to $K$, as $\overline{p} = \beta^{-1/2} \sum_{j=1}^m v_j \otimes_N \overline{v}_j$ by Property (d) and $\overline{v}_j \in pL^2(M)$ for $j = 1, \ldots, m$ according to Property (e). Since $p$ and $N$ commute,

$$\overline{R} : L^2(N) \longrightarrow K = pL^2(M) \otimes_N p^{op}L^2(M), \; \overline{m} \longmapsto \beta^{1/2} \overline{p}_0 \in K \subset L^2(M_1),$$

defines a continuous $(N,N)$-linear operator. For $x \in K \cap M_1$ we get

$$(1 - p)x = (1 - p)x \in (1 - p)(pL^2(M) \otimes_N L^2(M)) = (1 - p)pL^2(M) \otimes_N L^2(M) = \{0\}$$

(by applying Theorem 2.2 and Corollary 2.3 in [25]). Hence $x = px$. Moreover we have

$$\langle \overline{R}^* \overline{x}, \overline{m} \rangle = \langle \overline{x}, \overline{R} \overline{m} \rangle = \beta^{1/2} \text{tr}_{M_1}(x(pn)^*) = \beta^{1/2} \text{tr}_{M_1}(pxn^*) = \beta^{1/2} \text{tr}_{M_1}(E_0(E_1(x)) n^*)$$

for $n \in N$, as $E_0 \circ E_1 : M_1 \longrightarrow N$ is the conditional expectation onto $N$ associated with the unique normalized trace $\text{tr}_{M_1}$ of $M_1$. Hence $\overline{R}^* \overline{x} = \beta^{1/2} E_0(E_1(x))$. Let $R$ be defined as $\overline{R}$ where $p$ is replaced by $p^{op}$. We compute

$$(R^* \times 1_{\overline{H}}) \circ (1_{\overline{H}} \times \overline{R}) \overline{m} = \beta^{1/2} (R^* \times 1_{\overline{H}}) \overline{n} \otimes_N \overline{p}_0 = \beta^{3/2} (R^* \times 1_{\overline{H}}) \sum_{j=1}^m m e_0 v_j e_0 \overline{v}_j = \beta^{1/2} \sum_{j=1}^m R^* \sum_{j=1}^m \overline{m} e_0 v_j \otimes_N \overline{v}_j$$

$$= \sum_{j=1}^m \overline{E_0(E_1(m e_0 v_j))} v_j = \beta \sum_{j=1}^m \overline{E_0(v_j m^* s_j)} = J(p.\overline{m}) = p^{op}.\overline{m} = \overline{m}$$

for every $m \in p^{op}L^2(M) \cap M$. (In line 2 we used the canonical isomorphisms from $L^2(M_2)$ onto $L^2(M) \otimes_N L^2(M_1)$ and $L^2(M_1) \otimes_N L^2(M)$, see [25] or consider the proof of Theorem 4.1.) By applying (f) with $p^{op}$ instead of $p$ and reversing the parts of $R$ and $\overline{R}$, we see that $R$ and $\overline{R}$ are conjugation operators for $\overline{H}$. For $\overline{T} \in N$ we have

$$(\overline{R}^* \circ \overline{R}) \overline{T} = \beta E_0(E_1(p)) = \beta \text{tr}_{M_1}(E_0(E_1(p))) \overline{T} = \beta \text{tr}_{M_1}(p) \overline{T} = c(\lambda(N), L^2(M)) \cdot \text{tr}_{M_1}(p) \overline{T} = c(\lambda(N), pL^2(M)) \overline{T}.$$
We get the observation that the dimension \(d\) and the coupling constant are additive for direct sums (see \([12]\) for the minimal dimension), we get Remark (1) for any bimodule of \(\text{Ob} \ B_{N\subset M}\).

If \(N \subset M\) is extremal \(N \subset M_{2k-1}\) is extremal for every \(k \in \mathbb{N}\) by 1.2.5 (iii) and (iv) in \([21]\). This implies \(\text{tr}_{M_{2k-1}}(p) = \text{tr}_{M_{2k-1}}(p^{op})\) for every minimal projection \(p \in \mathcal{N}' \cap M_{2k-1}\) by 1.2.5 (i) in \([21]\). We obtain

\[
\begin{align*}
c(\rho(N), pL^2(M_{k-1})) &= \text{tr}_{M_{2k-1}}(p) \cdot c(\rho(N), L^2(M_{k-1})) = \\
&= \text{tr}_{M_{2k-1}}(p) \cdot \min[M : N]^k = \text{tr}_{M_{2k-1}}(p^{op}) \cdot \min[M : N]^k = \\
&= c(\rho(N), p^{op}L^2(M_{k-1})) = c(\lambda(N), pL^2(M_{k-1})),
\end{align*}
\]

and Remark (2) follows for every irreducible bimodule of \(\text{Ob} \ B_{N\subset M}\). Now the observation that the dimension \(d\) and the coupling constant are additive for direct sums completes the proof of Remark (2).

We consider the example \(N \subset N \rtimes H\), where \(H\) is a finite dimensional Hopf-*-algebra acting on \(N\) by an outer action. Observe that every irreducible subfactor \(N \subset M\) with finite index and depth 2 is isomorphic to a subfactor of that kind. (This result was announced by A. Ocneanu, for complete proofs see \([1]\) or \([27]\).) First we recall some definitions and results for the reader's convenience (see \([28]\) for example).

## 2.5 Hopf-*-algebras and their actions on \(\text{II}_1\)-factors

(1) Let \(N\) be a unital *-algebra and \(H\) be a finite dimensional Hopf-*-algebra. A bilinear map

\[
H \times N \longrightarrow N, \ (a, x) \longmapsto \alpha(a)x,
\]

is called an action of \(H\) on \(N\), if
(a) \( \alpha \) is a nondegenerate representation of the algebra \( H \) on the linear space \( N \),

(b) \( \alpha(a)(xy) = \sum_{(a)} \alpha(a^{(1)})x \cdot \alpha(a^{(2)})y \) for \( a \in H, \ x, y \in N \), where \( \Delta(a) = \sum_{(a)} a^{(1)} \otimes a^{(2)} \) (Sweedler notation),

(c) \( \alpha(a)1 = \epsilon(a)1 \) for \( a \in H \) and

(d) \( (\alpha(a)x)^* = \alpha(S(a)^*)x^* \) for \( a \in H \) and \( x \in N \).

(We use the convention \( \alpha(a)x \cdot y := (\alpha(a)x)y \).)

(2) The algebra \( NH := \{ x \in N : \alpha(a)x = \epsilon(a)x \text{ for } a \in H \} \)

is called the fixed point algebra under the action \( \alpha \).

(3) The crossed product \( N \rtimes_{\alpha} H = N \rtimes H \) is a unital *-algebra that as a linear space coincides with the algebraic tensor product \( N \otimes H \). The multiplication is given by \( (x \otimes a) \cdot (y \otimes b) = \sum_{(a)} x \cdot \alpha(a^{(1)})y \otimes a^{(2)}b \) and the *-operation by \( (x \otimes a)^* = \sum_{(a)} \alpha(a^{(1)*})x^* \otimes a^{(2)*} \).

(4) There is a unique faithful normalized trace \( \tau \) (see [10]) of \( H \) called the Haar trace such that \( (\tau \otimes id_H)(\Delta(a)) = \tau(a)1 = (id_H \otimes \tau)(\Delta(a)) \)

for any \( a \in H \). If \( p \) is a minimal central projection of \( H \) such that \( pH \cong L(\mathbb{C}^d) \), then \( \tau(f) = d/ \dim H \) for every minimal projection \( f \) of \( pH \).

One easily concludes that the comultiplication \( \Delta : H \rightarrow H \otimes H \) is a unitary corepresentation with respect to the inner product \( (a, b) \mapsto \tau(ab^*) \).

(5) If \( N \) is a factor of type \( \Pi_1 \) and \( \alpha \) an action of the finite dimensional Hopf-*-algebra \( H \) on \( N \), then \( NH \) as well as \( N \rtimes H \) are von Neumann algebras (see [28]). \( N \) is embedded into \( N \rtimes H \) by \( x \mapsto x \otimes 1 \). \( tr_N \otimes \tau \) is a normalized finite trace of the von Neumann algebra \( N \rtimes H \) (see [28], Proposition 2.7).

(6) An action \( \alpha \) of \( H \) on a factor \( N \) of type \( \Pi_1 \) is called outer if \( (NH)' \cap N = \mathbb{C}1 \)

and if \( \alpha \) is a faithful representation of \( H \). If \( \alpha \) is an outer action \( N \rtimes H \) is a \( \Pi_1 \)-factor. Furthermore, we have

\[ N' \cap (N \rtimes H) = \mathbb{C}1 \]  \hspace{1cm} (19)

(see [28], Proposition 3.2). According to (5) we may identify \( L^2(N \rtimes H) \) with \( L^2(N) \otimes H \) where \( H \) is endowed with the inner product given by the Haar trace \( \tau \).
(7) Let $H$ be a finite dimensional Hopf-\$-algebra. If we use the reversed comultiplication

$$\Delta^{\text{cop}} : H \rightarrow H \otimes H, \quad a \mapsto \sum_{(a)} a^{(2)} \otimes a^{(1)},$$

instead of $\Delta$ on $H$ and do not change the remaining structure on $H$, $H$ is a Hopf-\$-algebra again (see [9], Corollary III.3.5). We denote this Hopf-\$-algebra by $H^{\text{cop}}$. (For general Hopf-algebras the antipode $S$ has to be replaced by $S^{-1}$, but $S = S^{-1}$ is satisfied for finite dimensional Hopf-\$-algebras.)

Similarly, we are able to introduce another Hopf-\$-algebra structure on $H$ by using the reversed multiplication $a \circ b := ba$ (and leaving the remaining structure unchanged). This Hopf-\$-algebra is called $H^{\text{op}}$.

The connection between finite dimensional Hopf-\$-algebras and subfactors has also been studied by R. Longo (see [13]), whose approach is based on index theory for infinite factors, in particular on sector theory.

**Proposition 2.6.** Let $H$ be a finite dimensional Hopf-\$-algebra and let $N$ be a factor of type $\text{II}_1$ endowed with an outer action $\alpha$ of $H$.

(i) Let $\sigma : V_{\sigma} \rightarrow V_{\sigma} \otimes H^{\text{cop}}$ be a (small) unitary finite dimensional corepresentation of $H^{\text{cop}}$. On the Hilbert space $H_{\sigma} = L^2(N) \otimes V_{\sigma}$ there are a left action $\lambda$ of $N$ determined by $\lambda(m) \overline{n} \otimes v = \overline{mn} \otimes v$ and a right action $\rho$ of $N$ determined by

$$\rho(m) \overline{n} \otimes v = \sum_{(v)} n \cdot \alpha(v^{(2)})m \otimes v^{(1)}$$

($m, n \in N$, $v \in V_{\sigma}$). Endowed with these actions $H_\sigma$ is an $(N,N)$-bimodule belonging to the objects of the $C^*$-tensor category $\mathcal{B}_{N \subset N \rtimes H}$.

(ii) The following data determine an equivalence $(F, (U_{\rho \sigma})_{\rho, \sigma}, J)$ of the $C^*$-tensor categories $\mathcal{U}_{H^{\text{cop}}}$ and $\mathcal{B}_{N \subset N \rtimes H}$:

$F : \mathcal{U}_{H^{\text{cop}}} \rightarrow \mathcal{B}_{N \subset N \rtimes H}$ is the $C^*$-tensor functor given by $F(\sigma) = H_{\sigma}$,

$$F(T) = id_{L^2(N)} \otimes T \text{ for } T \in (\rho, \sigma) \text{ and } \rho, \sigma \in \text{Ob } \mathcal{U}_{H^{\text{cop}}},$$

$U_{\rho \sigma} : H_\rho \otimes_N H_\sigma \rightarrow H_{\rho \sigma}$ is defined by

$$U_{\rho \sigma} (\overline{n_1} \otimes v) \otimes_N (\overline{n_2} \otimes w) = \sum_{(v)} n_1 \cdot \alpha(v^{(2)})n_2 \otimes v^{(1)} \otimes w \quad (20)$$

for $n_1, n_2 \in N$, $v \in V_{\rho}$ and $w \in V_{\sigma}$,

$J : H_\epsilon = L^2(N) \otimes \mathbb{C} \rightarrow L^2(N)$ is defined by $J(\overline{\pi} \otimes \gamma) = \gamma \overline{\pi}$ for $\gamma \in \mathbb{C}$ and $n \in N$.

**Remark:** Let $G$ be a finite group and $H$ the Hopf-\$-algebra $\text{Fun}(G)$ of all complex valued functions on $G$. $N \rtimes H$ coincides with the usual crossed
product $N \rtimes G$, where $G$ acts on $N$ by an outer action in the usual meaning. The Hopf-$\ast$-algebras $\text{Fun}(G)$ and $\text{Fun}(G)^{\text{cop}}$ are isomorphic (where the isomorphism is given by $f \mapsto \hat{f}, \hat{f}(g) = f(g^{-1})$). Hence the $C\ast$-tensor category $\mathcal{B}_{N \rtimes N \rtimes G}$ is equivalent to the $C\ast$-tensor category $\mathcal{U}_G$ of all unitary finite dimensional representations of $G$.

**Proof:** (i) Obviously $\lambda$ defines a left action of $N$. For $m \in N$, $\rho(m)$ is well defined on the algebraic tensor product $N \otimes V_\sigma$. It is easy to see that $\rho(m)$ is continuous on $N \otimes V_\sigma$ with respect to the Hilbert space norm. Hence $\rho(m)$ has a unique extension to a continuous linear operator on $L^2(N) \otimes V_\sigma$. We easily get $\rho(1) = 1$. We show

$$\rho(m_1 m_2) = \rho(m_2) \rho(m_1) \quad \text{for all } m_1, m_2 \in N$$

(21)

and

$$\rho(m^*) = \rho(m)^* \quad \text{for } m \in N;$$

(22)

which implies that $\rho$ is a right action of $N$ on $H_\sigma$. Obviously $\rho$ and $\lambda$ commute such that $H_\sigma$ is actually an $(N, N)$-bimodule. For $H_\sigma \in \text{Ob} \, \mathcal{B}_{N \rtimes N \rtimes H}$ see Part (ii) (b) of the proof.

The computation

$$\rho(m_2) \rho(m_1) \overline{n} \otimes \varepsilon = \sum_{(\varepsilon)} \sum_{(v)} n \cdot \alpha(v(2)) m_1 \cdot \alpha(v(1)) m_2 \otimes \varepsilon^{(1)(1)} =$$

$$\sum_{(\varepsilon)} \sum_{(v)} n \cdot \alpha(v(2)) m_1 \cdot \alpha(v(1)) m_2 \otimes \varepsilon^{(1)} = \quad (*)$$

$$\sum_{(v)} n \cdot \alpha(v(2)) (m_1 m_2) \otimes \varepsilon^{(1)} = \quad \rho(m_1 m_2) \overline{n} \otimes \varepsilon$$

($n \in N$ and $v \in V_\sigma$) shows Equation (21). (Observe that the Sweedler notation for $H^{\text{cop}}$ is used in the computation, but in (*) the comultiplication of $H$ is needed.)

For $n_1, n_2 \in N$ and $v, w \in V_\sigma$ we get

$$\langle \rho(m^*) \overline{n_1} \otimes v, \overline{n_2} \otimes w \rangle = \sum_{(v)} \text{tr}_N \left( n_1 \cdot \alpha(v(2)) m^* \cdot n_2^* \right) \langle v^{(1)}, w \rangle$$

(23)

as well as

$$\langle \overline{n_1} \otimes v, \rho(m) \overline{n_2} \otimes w \rangle = \sum_{(w)} \text{tr}_N \left( n_1 \cdot (n_2 \cdot \alpha(w(2)) m)^* \right) \langle v, w^{(1)} \rangle$$

$$= \sum_{(w)} \text{tr}_N \left( n_2^* \cdot n_1 \cdot \alpha(S(w(2)^*)) m^* \right) \langle v, w^{(1)} \rangle.$$  

(24)

Since $\sigma$ is unitary, we know

$$\sum_{(v)} \langle v^{(1)}, S(v^{(2)}) \rangle = \sum_{(w)} \langle v, w^{(1)} \rangle \, w^{(2)*} \quad \text{for } v, w \in V_\sigma.$$
(see [10], (1.42)). An application of the antipode $S$ yields

$$
\sum_{(v)} \langle v^{(1)}, w^{(2)} \rangle = \sum_{(w)} \langle v, w^{(1)} \rangle S(w^{(2)*})
$$

(observe $S^2 = id_H$), and the right sides of (24) and (23) coincide. So Equation (22) has been shown.

(ii) (a) Let $\sigma$ and $\pi$ be unitary corepresentations of $H^\text{cop}$. We will prove that a continuous linear operator $S$ from $\mathcal{H}_\sigma$ into $\mathcal{H}_\pi$ is $(N, N)$-linear if and only if there is an operator $T \in (\sigma, \pi)$ such that $S = \mathbf{1}_{L^2(N)} \otimes T$.

Obviously, $(1 \otimes T) \lambda_\sigma(m) = \lambda_\pi(m) (1 \otimes T)$ for every $T \in (\sigma, \pi)$ and $m \in N$, where $\lambda_\sigma$ denotes the left action of $N$ on $\mathcal{H}_\sigma$. The corresponding equation for the right action is shown by the following computation:

$$
\rho_\sigma(m) (1 \otimes T) \pi \otimes v = \sum_{(Tv)} n \cdot \alpha((Tv)^{(2)}) m \otimes (Tv)^{(1)} = \sum_{(v)} n \cdot \alpha(v^{(2)}) m \otimes Tv^{(1)} = (1 \otimes T) \rho_\sigma(m) \pi \otimes v \quad (n \in N, v \in V_\sigma).
$$

Now let $S$ be any continuous $(N, N)$-linear operator in $\mathcal{H}_\sigma$. Let $\lambda_0$ (resp. $\rho_0$) denote the canonical left action (resp. right action) of $N$ on $L^2(N)$. The following computation shows $\rho(m) = \rho_0(m) \otimes 1$ for every $m \in N^H$:

$$
\rho(m) \pi \otimes v = \sum_{(v)} (n \cdot \epsilon(v^{(2)}) m \otimes v^{(1)}) = \pi m \otimes v
$$

for every $n \in N$ and $v \in V_\sigma$. Hence

$$
S \in \left( (\lambda_0(N) \cup \rho_0(N^H)) \otimes \mathbb{C} \mathbf{1} \right)' = \left( \lambda_0(N)' \cap \rho_0(N^H)' \right) \otimes \mathbb{L}(V_\sigma) = \left( \rho_0(N) \cap \rho_0(N^H)' \right) \otimes \mathbb{L}(V_\sigma) = \mathbb{C} \mathbf{1} \otimes \mathbb{L}(V_\sigma).
$$

There is an operator $T \in \mathbb{L}(V_\sigma)$ such that $S = \mathbf{1} \otimes T$. $S \rho(m) \mathbf{T} \otimes v = \rho(m) S \mathbf{T} \otimes v$ for $m \in N$ and $v \in V_\sigma$ implies

$$
\sum_{(v)} \alpha(v^{(2)}) m \otimes Tv^{(1)} = \sum_{(Tv)} \alpha((Tv)^{(2)}) m \otimes (Tv)^{(1)}.
$$

Since $\alpha$ is a faithful action, we get

$$
\sum_{(v)} Tv^{(1)} \otimes v^{(2)} = \sum_{(Tv)} (Tv)^{(1)} \otimes (Tv)^{(2)}
$$

for every $v \in V_\sigma$. It follows $T \in (\sigma, \sigma)$.

At last we consider a continuous $(N, N)$-linear operator $S$ from $\mathcal{H}_\sigma$ into $\mathcal{H}_\pi$. $S$ may be regarded as an $(N, N)$-linear operator in $\mathcal{H}_\sigma \oplus \mathcal{H}_\pi = \mathcal{H}_{\sigma \oplus \pi}$.
and we are able to apply the considerations from above (with $\sigma \oplus \pi$ instead of $\sigma$) and find that there is an operator $T \in (\sigma, \pi) \subset (\sigma \oplus \pi, \sigma \oplus \pi)$ such that $S = 1 \otimes T$ (observe Remark 1.3 (4)).

(b) $\Delta^{\text{cop}}$ is a unitary corepresentation of $H^{\text{cop}}$ with respect to the Haar trace $\tau$ of $H$. Obviously the the $(N, N)$-bimodule $\mathcal{H}_{\Delta^{\text{cop}}}$ is equal to the $(N, N)$-bimodule $L^2(N \rtimes H) (=L^2(N) \otimes H$ as a Hilbert space). From we conclude that every irreducible corepresentation $\sigma$ of $H^{\text{cop}}$ is contained in $\Delta^{\text{cop}}$ and that $\mathcal{H}_\sigma \in \text{Ob } \mathcal{B}_{N \subset N \times H}$. Hence $F$ is a full and faithful $C^*$-functor. Let $\mathcal{H}$ be an irreducible object of $\mathcal{B}_{N \subset N \times H}$. As $N \subset N \times H$ is a subfactor of depth 2, $\mathcal{H}$ is equivalent to a sub-bimodule of $N L^2(N \rtimes H)_N = \mathcal{H}_{\Delta^{\text{cop}}}$. Therefore there is an irreducible corepresentation $\sigma$ of $H^{\text{cop}}$ such that $\mathcal{H}$ is equivalent to $\mathcal{H}_\sigma$.

(c) Let $\rho$ and $\sigma$ be unitary corepresentations of $H^{\text{cop}}$. It is easy to see that $\xi \otimes_N (\overline{\eta} \otimes w) \longmapsto (\xi \otimes_N \overline{\eta}) \otimes w$ ($\xi \in \mathcal{H}_\rho$, $n \in N$, $w \in V_\sigma$) defines a unitary operator $W_{\rho\sigma}$ from $\mathcal{H}_\rho \otimes_N \mathcal{H}_\sigma = \mathcal{H}_\rho \otimes_N (L^2(N) \otimes V_\sigma)$ onto $(\mathcal{H}_\rho \otimes_N L^2(N)) \otimes V_\sigma$.

Now we easily conclude that $U_{\rho\sigma} := (r_{\mathcal{H}_\rho} \otimes \text{id}_{V_\sigma}) \circ W_{\rho\sigma}$ is the unique continuous linear operator satisfying $(20)$, where $r_{\mathcal{H}_\rho}$ is the canonical unitary operator from $\mathcal{H}_\rho \otimes_N L^2(N)$ onto $\mathcal{H}_\rho$. Obviously $U_{\rho\sigma}$ is unitary and left $N$-linear.

We get
\[
U_{\rho\sigma} \circ \rho(m) \ (\overline{\pi} \otimes v) \otimes_N (\overline{1} \otimes w) = U_{\rho\sigma} \sum_{(w)} (\overline{\pi} \otimes v) \otimes_N \left( \alpha(w^{(2)})m \otimes w^{(1)} \right) = \\
\sum_{(v), (w)} n \cdot \alpha(v^{(2)}w^{(2)})m \otimes v^{(1)} \otimes w^{(1)} = \\
\rho(m) \overline{\pi} \otimes v \otimes w = \rho(m) \circ U_{\rho\sigma} (\overline{m} \otimes v) \otimes_N (\overline{1} \otimes w)
\]

for all $m, n \in N$, $v \in V_\rho$, $w \in V_\sigma$. Since
\[
\text{span } \{(\overline{\pi} \otimes v) \otimes_N (\overline{1} \otimes w) : v, w \in V_\rho, n \in N\}
\]
is dense in $\mathcal{H}_\rho \otimes_N \mathcal{H}_\sigma$, we have verified that $U_{\rho\sigma}$ is right $N$-linear.

Routine arguments show that the relations (a), (b) and (c) of Definition 1.4 (ii) are satisfied. (The computation for (b) is easier, if one uses the density argument from above.) Hence $F$ is a $C^*$-tensor equivalence. ■

3 Subfactors defined by $C^*$-tensor categories

Let a finite $C^*$-tensor category $\mathcal{C}$ be given. The goal of this Section is to construct a subfactor belonging to an object $\sigma$ of $\mathcal{C}$ and to discuss the properties of this subfactor.

For an object $\rho$ of $\mathcal{C}$ $(R_\rho, \overline{R}_\rho)$ denotes a standard pair of conjugation operators for $\rho$ and $\overline{\rho}$.
3.1 Some observations

Let \( \rho, \sigma \) and \( \tau \) be objects of \( \mathcal{C} \).

(1) There is a distinguished normalized faithful trace \( \text{tr}_\rho \) of the finite dimensional \( \mathcal{C}^* \)-algebra \((\rho, \rho)\). \( \text{tr}_\rho \) is uniquely determined by its values on minimal projections. Let \( E \) be a minimal projection and \( \pi \) a subobject of \( \rho \) corresponding to \( E \). Then

\[
\text{tr}_\rho(E) := \frac{d(\pi)}{d(\rho)}.
\]  

(26)

Conversely this relation determines a faithful trace, as equivalent minimal projections have the same positive number.

Obviously the trace \( \text{tr}_\rho \) is faithful. Theorem 1.15 (i) shows that \( \text{tr}_\rho \) is normalized. If \( A, B \in (\rho, \sigma) \) then

\[
d(\rho) \text{tr}_\rho(B^*A) = d(\sigma) \text{tr}_\sigma(AB^*).
\]  

(27)

(Observe that \( d(\rho) \text{tr}_\rho \) is the trace introduced in \([\text{14}]\) after Lemma 3.7; thus \((27)\) is a consequence of Lemma 3.7 in \([\text{14}]\).)

(2) The \( \ast \)-algebras \((\rho, \rho)\) and \((\sigma, \sigma)\) are embedded into the \( \ast \)-algebra \((\rho \sigma, \rho \sigma)\) by

\[
1 \rho \times \bar{R}_\sigma \circ (T \times 1 \sigma) \times 1 \rho \circ 1 \rho \times \bar{R}_\sigma = (\bar{R}_\sigma \circ \bar{R}_\sigma) \cdot T \quad \text{and} \\
R^*_\rho \times 1 \sigma \circ 1 \rho \times (1 \rho \times T) \circ R_\rho \times 1 \sigma = (R^*_\rho \circ R_\rho) \cdot T
\]

we conclude that the embeddings are injective. R. Longo and J.E. Roberts proved (see Corallary 3.10 in \([\text{14}]\))

\[
d(\rho) \text{tr}_\rho(S_1T^*_1) \cdot d(\sigma) \text{tr}_\sigma(S_2T^*_2) = d(\rho \sigma) \text{tr}_{\rho \sigma}((S_1 \times S_2)(T_1 \times T_2)^*)
\]

for all \( S_1, T_1 \in (\rho, \rho), S_2, T_2 \in (\sigma, \sigma) \). Using \( d(\rho \sigma) = d(\rho)d(\sigma) \) we get

\[
\text{tr}_{\rho \sigma} | (\rho, \rho) = \text{tr}_\rho \quad \text{and} \quad \text{tr}_{\rho \sigma} | (\sigma, \sigma) = \text{tr}_\sigma.
\]  

(28)

(3) The linear maps

\[
\Phi^\rho_\sigma : (\rho \sigma, \rho \sigma) \longrightarrow (\sigma, \sigma), \quad X \longmapsto \frac{1}{d(\rho)} R^*_\rho \times 1 \sigma \circ 1 \rho \circ X \circ R_\rho \times 1 \sigma,
\]

and

\[
\Psi^\rho_\sigma : (\sigma \rho, \sigma \rho) \longrightarrow (\sigma, \sigma), \quad X \longmapsto \frac{1}{d(\rho)} 1 \sigma \times R^*_\rho \circ X \times 1 \rho \circ 1 \sigma \times \bar{R}_\rho,
\]

are conditional expectations. (In \([\text{14}]\) R. Longo and J.E. Roberts used the more general concept of left and right inverses, which we do not need here.) \( \Phi^\rho_\sigma \) and \( \Psi^\rho_\sigma \) do not depend on the choice of \( R_\rho \) and \( \bar{R}_\rho \) (see Lemma 3.3 in
R. Longo and J.E. Roberts showed that \( \Phi_\rho^\omega : (\rho, \rho) \rightarrow (\omega, \rho) = \mathbb{C} \) and \( \Psi_\rho^\omega : (\rho, \rho) \rightarrow (\rho, \omega) = \mathbb{C} \) are just \( \text{tr}_\rho \) (see Lemma 3.3 in [14]). We have

\[
\Phi_\tau^\rho \circ \Phi_\sigma^\omega = \Phi_{\tau \sigma}^{\rho \omega} \quad \text{as well as} \quad \Psi_\tau^\rho \circ \Psi_\sigma^\omega = \Psi_{\tau \sigma}^{\rho \omega}.
\]

Using Equation (3) we get

\[
\Phi_{\tau \sigma}^{\rho \omega}(X) = \frac{1}{d(\rho \sigma)} \left( R_{\sigma}^* \circ 1_{\omega} \times R_{\rho}^* \times 1_{\sigma} \right) \times 1_\tau \circ 1_{\omega} \times X \circ \left( 1_{\omega} \times R_{\rho} \times 1_{\sigma} \circ R_{\sigma} \right) \times 1_\tau = \\
\frac{1}{d(\rho) d(\sigma)} R_{\sigma}^* \times 1_\omega \circ 1_{\omega} \times \left( R_{\rho}^* \times 1_{\sigma} \circ 1_{\omega} \times R_{\rho} \times 1_{\sigma} \right) \circ R_{\sigma} \times 1_\tau = \\
= \Phi_{\tau \sigma}^\rho(\Phi_{\sigma \tau}^\rho(X))
\]

for any \( X \in (\rho \sigma \tau, \rho \sigma \tau) \). The same method also shows the second relation. We note

\[
\text{tr}_\sigma \circ \Phi_\sigma^\rho = \text{tr}_{\rho \sigma} \quad \text{and} \quad \text{tr}_\sigma \circ \Psi_\sigma^\rho = \text{tr}_{\sigma \rho}.
\]

as special cases. Equation (29) means that \( \Phi_\sigma^\rho \) (resp. \( \Psi_\sigma^\rho \)) is a conditional expectation corresponding to the trace \( \text{tr}_{\rho \sigma} \) (resp. \( \text{tr}_{\sigma \rho} \)).

(4)

\[
(\rho \sigma, \rho \sigma) \subset (\rho \sigma \tau, \rho \sigma \tau) \cup (\sigma \tau, \sigma \tau)
\]

(30)

is a commuting square of finite dimensional von Neumann algebras (in the meaning of [5], Section 4.2) with respect to the trace \( \text{tr}_{\rho \sigma \tau} \), as \( \Phi_{\sigma \tau}^\rho(X \times 1_\tau) = \Phi_{\tau \sigma}^\rho(X) \times 1_\tau \) holds for any \( X \in (\rho \sigma, \rho \sigma) \).

(5) The Bratteli diagram for the inclusion \( (\rho, \rho) \subset (\rho \sigma, \rho \sigma) \) is described as follows:

The vertices in the lower line (resp. upper line) are in one to one correspondence to the simple direct summands of \( (\rho, \rho) \) (resp. \( (\rho \sigma, \rho \sigma) \)) and are labelled by the equivalence classes \([\phi]\) (resp. \([\psi]\)) of the irreducible subobjects \( \phi \) (resp. \( \psi \)) of \( \rho \) (resp. \( \rho \sigma \)). The number of edges between the vertices belonging to \([\phi]\) and to \([\psi]\) is the dimension of \( \psi \phi \sigma \), i.e. the number of times that \( \psi \) is contained in \( \phi \sigma \). We leave the easy proof to the reader.

One can deal with the inclusion \( (\rho, \rho) \subset (\sigma \rho, \sigma \rho) \) in the same way, one merely has to replace \( \rho \sigma \) by \( \sigma \rho \) and \( \phi \sigma \) by \( \sigma \phi \).

Let \([\mathcal{C}, \rho] := \{[\phi] \in [\mathcal{C}] : \phi \leq \rho\}\). Obviously the Bratteli diagrams for \( (\rho, \rho) \subset (\rho \sigma, \rho \sigma) \) and \( (\tau, \tau) \subset (\tau \sigma, \tau \sigma) \) are the same if \([\mathcal{C}, \rho] = [\mathcal{C}, \tau] \).
Lemma 3.2. Let us consider the inclusion

$$A := (\rho, \rho) \subset B := (\rho\sigma, \rho\sigma) \subset C := (\rho\sigma\bar{\sigma}, \rho\sigma\bar{\sigma})$$

of finite dimensional \*$\*$-algebras. Moreover, let $B(A, B)$ be the basic construction for $A \subset B$ and for the trace $tr_{\rho\sigma}$, let $e$ be the Jones projection of $B(A, B)$, and let

$$f := \frac{1}{d(\sigma)}1_{\rho} \times \left( \bar{R}_\sigma \circ \bar{R}_\sigma^* \right) \in C.$$  

$D := \text{span } B f B$ is a (two-sided) ideal of $C$, and there is an isomorphism $\alpha$ from the basic construction $B(A, B)$ onto $D$ such that $\alpha \vert B = \text{id}_B$ and $\alpha(e) = f$. If every irreducible subobject of $\rho\sigma\bar{\sigma}$ is a subobject of $\rho$ then $D = C$.

The trace $tr_{\rho\sigma}$ satisfies the Markov relation

$$tr_{\rho\sigma} \left( (X \times 1_\sigma) \circ f \right) = \frac{1}{d(\sigma)^2} tr_{\rho\sigma}(X)$$

for $X \in (\rho\sigma, \rho\sigma)$.

The Lemma is also true for $A = (\rho, \rho)$, $B = (\sigma\rho, \sigma\rho)$, $C = (\bar{\sigma}\sigma\rho, \bar{\sigma}\sigma\rho)$, and $f = \frac{1}{d(\sigma)}(R_\sigma \circ R_\sigma^*) \times 1_{\rho}$. The Markov relation for this case is

$$tr_{\bar{\sigma}\sigma\rho} \left( (1_\sigma \times X) \circ f \right) = \frac{1}{d(\sigma)^2} tr_{\sigma\rho}(X).$$

Proof: We will only prove the first case.

$$x \in (\rho, \rho) \mapsto xf = x \times \frac{1}{d(\sigma)}(\bar{R}_\sigma \circ \bar{R}_\sigma^*) \in (\rho\sigma\bar{\sigma}, \rho\sigma\bar{\sigma})$$

is an injective homomorphism. For $b \in B$ we get $f b f = \Psi_{\rho}(b)f$, as the following computation using the interchange law shows:

$$\Psi_{\rho}(b)f = \frac{1}{d(\sigma)^2} (1_{\rho} \times \bar{R}_\sigma^* \circ b \times 1_\sigma \circ 1_{\rho} \times \bar{R}_\sigma) \times 1_{\rho} \sigma \circ 1_{\rho} \times (\bar{R}_\sigma \circ \bar{R}_\sigma^*) =$$

$$\frac{1}{d(\sigma)^2} 1_{\rho} \times \bar{R}_\sigma \circ (1_{\rho} \times \bar{R}_\sigma^* \circ b \times 1_\sigma \circ 1_{\rho} \times \bar{R}_\sigma) \circ 1_{\rho} \times \bar{R}_\sigma^* = f b f.$$

Now Proposition 2.6.9 in [3] implies that there is an isomorphism $\alpha$ having the desired properties, and that $D$ is an ideal of the \*$\*$-algebra $\langle B, f \rangle$ generated by $B$ and $f$. We have to show that $D$ is an ideal of $C$.

The subobject of $\sigma\bar{\sigma}$ corresponding to the projection $f_0 := \frac{1}{d(\sigma)} \bar{R}_\sigma \circ \bar{R}_\sigma^* \in (\sigma\bar{\sigma}, \sigma\bar{\sigma})$ is the unit object $\iota$. In particular every irreducible subobject of $\rho\iota = \rho$ is also a subobject of $\rho\sigma\bar{\sigma}$. Let $I$ be the ideal of $C$ consisting of those simple direct summands of $C$ for which the associated irreducible subobjects of $\rho\sigma\bar{\sigma}$ are also subobjects of $\rho$. We will show $I = D$.
Let \( \{q_1, q_2, \ldots, q_n\} \) be a maximal set of pairwise non equivalent minimal projections of \( A \) and let \( \{\pi_1, \ldots, \pi_n\} \) be the associated irreducible objects of \( \mathcal{C} \). The projection \( q_j f = q_j \times f_0 \) (regarded as an element of \( C \)) corresponds to the same irreducible object \( \pi_j \) of \( \mathcal{C} \) as the projection \( q_j \) (regarded as an element of \( A \)), in particular \( q_j f \) is a minimal projection of \( C \). Let \( I_j \) (resp. \( D_j \)) be the simple direct summand of \( I \) (resp. \( D \)) containing \( q_j f \). Every minimal projection \( p \) of \( D_j \) is equivalent to \( q_j f \) in \( C \), too, and belongs to the direct summand \( I_j \) of \( I \). So \( D_j \) is contained in \( I_j \) for \( j = 1, \ldots, n \).

As \( D = \bigoplus_{j=1}^n D_j \), \( D \) is a subalgebra of \( I \). On the other hand the inclusion matrices for \( B \subset D \) and \( B \subset I \) are the same.

Let \( B = \bigoplus_{k=1}^m B_k \) be the decomposition of \( B \) into a direct sum of minimal simple ideals and let \( \tau_k \) be the irreducible object of \( C \) belonging to \( B_k \). The vertices corresponding to \( B_k \) and \( I_j \) are connected by \( \dim (\pi_j, \tau_k \sigma) \) edges. Since \( D \) is the basic construction for \( A \subset B_k \), \( B_k \) and \( D_j \) are connected by \( \dim (\pi_k, \tau_j \sigma) \) edges (see [3], Proposition 2.4.1 (b)). The Frobenius reciprocity law (see Lemma [14]) yields \( \dim (\tau_k, \pi_j \sigma) = \dim (\pi_j, \tau_k \sigma) \).

So \( \dim D = \dim I \), and \( D = I \) follows. It remains to verify the Markov relation:

\[
\text{tr}_{\rho \sigma} \left((X \times 1_\sigma) \cdot \frac{d(\sigma)}{1} (1_\rho \times (\bar{R}_\sigma \circ \bar{R}_\sigma^*))) = \right) \quad \text{(by Equation (27))}
\]

\[
\frac{d(\sigma)}{1^3} \text{tr}_\rho (1_\rho \times \bar{R}_\sigma^* \circ X \times 1_\sigma \circ 1_\rho \times \bar{R}_\sigma) = \frac{d(\sigma)}{1^2} \text{tr}_\rho (\bar{\Phi}_\rho^* (X)) = \frac{d(\sigma)}{1^2} \text{tr}_\rho (X) \quad \text{for } X \in (\rho \sigma, \rho \sigma). \]

\[\blacksquare\]

### 3.3 The subfactors

For an object \( \sigma \) of \( \mathcal{C} \) we introduce the following notation:

\[
\sigma(0) = \nu, \sigma(1) = \sigma, \sigma(2) = \sigma \sigma, \sigma(3) = \sigma \sigma \sigma, \sigma(4) = \sigma \sigma \sigma \sigma, \ldots \quad \text{and} \quad \bar{\sigma}(0) = \nu, \bar{\sigma}(1) = \bar{\sigma}, \bar{\sigma}(2) = \bar{\sigma} \bar{\sigma}, \bar{\sigma}(3) = \bar{\sigma} \bar{\sigma} \bar{\sigma}, \bar{\sigma}(4) = \bar{\sigma} \bar{\sigma} \bar{\sigma} \bar{\sigma}, \ldots
\]

We will consider the tower

\[
\begin{align*}
(\sigma, \bar{\sigma}) & \subset (\bar{\sigma}, \bar{\sigma}) \subset (\bar{\sigma}(2), \sigma(2)) \subset (\bar{\sigma}(3), \sigma(3)) \subset \ldots \\
(\nu, \nu) & \subset (\bar{\sigma}, \bar{\sigma}) \subset (\bar{\sigma}(2), \sigma(2)) \subset (\bar{\sigma}(3), \sigma(3)) \subset \ldots.
\end{align*}
\]

of finite dimensional \( * \)-algebras. In the following we abbreviate \( (\sigma(n - 1), \bar{\sigma}(n - 1)) \) to \( A^n \) and \( (\bar{\sigma}(n - 1), \sigma(n - 1)) \) to \( B^n \). There is a trace \( \text{tr} \) of the \( * \)-algebra \( B^\infty := \bigcup_{n=1}^\infty B^n \) given by \( \text{tr} | B^n = \text{tr}_{\sigma(n - 1)} \).

Observation 3.1 (1) and (2) show that \( \text{tr} \) is well defined and faithful and that \( \text{tr} | A^n = \text{tr}_{\sigma(n - 1)} \).

If \( d(\sigma) > 1 \), the subsequent considerations will show that the tower (32) (endowed with the trace \( \text{tr} \)) fulfils the periodicity assumptions used in H.
Wenzl’s subfactor construction (Theorem 1.5 of [29]). In particular this implies the following: Let \( \pi \) be the GNS representation of the state \( \text{tr} \) of the \( \ast \)-algebra \( B^\infty \) and let \( A = \pi(A^\infty)'' \) (with \( A^\infty = \bigcup_{n=1}^\infty A^n \)) and \( B = \pi(B^\infty)'' \). Then \( A \) and \( B \) are \( \text{II}_1 \)-factors.

According to Observation 3.1 (4), the squares appearing in the tower \((32)\) commute. Observe that every subobject of \( \sigma(n-1) \) is also a subobject of \( \sigma(n+1) \). As the C*-tensor category \( \mathcal{C} \) is finite, Observation 3.1 (5) implies that the Bratteli diagram for \( A^{n-1} \subset A^n \) is the same as that for \( A^{n+1} \subset A^{n+2} \) if \( n \) is sufficiently large. The same holds for \( (B^n)_{n\in\mathbb{N}} \). Using Observation 3.1 (5) again, we see that the inclusion matrices for \( A^n \subset B^n \) and \( A^{n+2} \subset B^{n+2} \) coincide for a sufficiently large \( n \).

By induction, we see that the Bratteli diagram for \( A^n \subset A^{n+1} \) is connected. (That means: For any two vertices of the graph there is a sequence of edges connecting these vertices.) The case \( n = 0 \) is obvious. The step \( n-1 \to n \) is a conclusion from the proof of Lemma 3.2. The Bratteli diagram for \( A^n \subset A^{n+1} \) is the mirror image of the Bratteli diagram for \( A^{n-1} \subset A^n \), where new vertices for \( A^{n+1} \) are added and connected with some vertices of \( A^n \) by new edges. Now one easily concludes that the inclusion matrix for \( A^n \subset A^{n+2} \) is primitive for every \( n \in \mathbb{N} \).

We have to exclude the case \( d(\sigma) = 1 \). If \( d(\sigma) > 1 \) we get \( B^n \subsetneq B^{n+2} \) for every \( n \in \mathbb{N} \).

For \( m \in \mathbb{N} \) we consider the tower

\[
B^1_m = \left( \sigma(m+1), \sigma(m+1) \right) \subset \cdots \subset B^3_m = \left( \overline{\sigma(2)} \sigma(m+1), \overline{\sigma(2)} \sigma(m+1) \right) \subset \cdots \subset B^{m+1}_m = \left( \overline{\sigma(3)} \sigma(m+1), \overline{\sigma(3)} \sigma(m+1) \right) \cdots .
\]

Let \( B_m \) be the unique \( \text{II}_1 \)-factor containing \( B^\infty_m = \bigcup_{n=1}^\infty B^n_m \) as an ultra-weakly dense subalgebra. \( B_{m-1} \) is canonically embedded into \( B_m \).

**Theorem 3.4.** Let \( d(\sigma) > 1 \).

(i) \( [B : A] = d(\sigma)^2 \).

(ii) There is an isomorphism \( j \) from the basic construction \( B(A,B) \) for \( A \subset B \) onto the \( \text{II}_1 \)-factor \( B_1 \) such that \( j \mid B = \text{id}_B \) and \( j(e_0) = f_0 \), where \( e_0 \) is the Jones projection in \( B(A,B) \) and \( f_0 = \frac{1}{d(\sigma)} R_{\sigma} \circ R_{\sigma}^* \in B^1_1 = (\sigma \sigma, \sigma \sigma) \subset B_1 \).

(iii) The Jones tower for \( A \subset B \) may be identified with

\[
B_{-1} = A \subset B_0 = B \subset B_1 \subset B_2 \subset B_3 \subset \ldots .
\]
The Jones projection \( f_m \in B_{m+1} \) for \( B_{m-1} \subset B_m \) is
\[
f_m = \frac{1}{d(\sigma)} 1_{\sigma(m)} \times \left( R_\sigma \circ R_\sigma^* \right) \in B_{m+1}^1 ~ \text{for} ~ m ~ \text{even and}
\]
\[
f_m = \frac{1}{d(\sigma)} 1_{\sigma(m)} \times \left( R_\sigma \circ R_\sigma^* \right) \in B_{m+1}^1 ~ \text{for} ~ m ~ \text{odd}.
\]

(iv) The subfactor \( A \subset B \) has finite depth. The standard invariant
\[
\mathcal{C}_1 = A' \cap A \subset A' \cap B \subset A' \cap B_1 \subset A' \cap B_2 \subset \ldots
\]
\[
\mathcal{C}_1 = B' \cap B \subset B' \cap B_1 \subset B' \cap B_2 \subset \ldots
\]
of \( A \subset B \) is equal to
\[
(\iota, \iota) \subset (\sigma, \sigma) \subset (\sigma \bar{\sigma}, \sigma \bar{\sigma}) \subset (\sigma \bar{\sigma} \sigma, \sigma \bar{\sigma} \sigma) \subset \ldots
\]
\[
(\iota, \iota) \subset (\bar{\sigma}, \bar{\sigma}) \subset (\bar{\sigma} \sigma, \bar{\sigma} \sigma) \subset (\bar{\sigma} \sigma \bar{\sigma}, \bar{\sigma} \sigma \bar{\sigma}) \subset \ldots.
\]
Especially the subfactor \( A \subset B \) is irreducible (i.e. \( A' \cap B = \mathcal{C}_1 \)) if and only if \( \sigma \) is an irreducible object.

One easily shows the following

3.5 Remarks:

(1) If one chooses another conjugate \( \tilde{\sigma} \) of \( \sigma \) instead of \( \bar{\sigma} \) the tower \((32)\)
is isomorphic to the tower formed with \( \tilde{\sigma} \) instead of \( \sigma \), as an easy consideration shows. Hence the subfactor does not depend on the choice of the conjugate \( \tilde{\sigma} \).

(2) If \( \mathcal{C} \) and \( \mathcal{D} \) are finite \( \mathcal{C}^* \)-tensor categories and \( F \) is an equivalence of the \( \mathcal{C}^* \)-tensor categories \( \mathcal{C} \) and \( \mathcal{D} \) then the subfactor constructed with the object \( \sigma \) of \( \mathcal{C} \) is isomorphic to the subfactor constructed with the object \( F(\sigma) \) of \( \mathcal{D} \).

Proof of Theorem 3.4: (i) We consider the inclusion
\[
A^{2n+1} = \left( \sigma(2n), \bar{\sigma}(2n) \right) \subset B^{2n+1} = \left( \sigma(2n) \sigma, \bar{\sigma}(2n) \sigma \right)
\]
for a sufficiently large \( n \). The object \( \sigma(2n) \sigma \bar{\sigma} = (\sigma \bar{\sigma})^n \) contains the same irreducible subobjects as \( \sigma(2n) \), hence \( B^{2n+1}_\sigma \) is the basic construction for \( A^{2n+1} \subset B^{2n+1}_\sigma \) according to Lemma 3.2 and \( \text{tr}_{\sigma(2n) \sigma} \) is a faithful Markov trace of modulus \( d(\sigma)^2 \) for this inclusion. Theorem 1.5 in [23] (along with Proposition 2.7.2 in [5]) yields \([B : A] = d(\sigma)^2\).
(ii) By using [18], Corollary 1.8 we will show that \( A \subset B \) is the basic construction downwards: We get \([B_1 : B] = d(\sigma)^2\) as in the proof for \( A \subset B \). Lemma 3.2 implies
\[
d(\sigma)^2 \text{tr}_{B_1}(xf_0) = \text{tr}_B(x)
\]
for every \( x \in B \). Therefore
\[
\frac{1}{[B_1 : B]} \text{tr}_B(x) = \text{tr}_{B_1}(xf_0) = \text{tr}_B(E_B(xf_0)) = \text{tr}_B(xE_B(f_0))
\]
for every \( x \in B \) and \( E_B(f_0) = [B_1 : B]^{-1}1 \). Hence there is an isomorphism \( \iota \) from the basic construction \( B(P, B) \) for the subfactor \( P := \{ f_0 \}' \cap B_1 \subset B \) onto the \( \text{II}_1 \)-factor \( B_1 \) such that \( \iota \mid B = id_B \) and \( \iota^{-1}(f_0) \) is the Jones projection. Obviously \( A \) is contained in \( P \) and \([B : P] = [B_1 : B] = [B : A] < \infty \). Hence \( A = P \), and we get the assertion.

(iii) follows by applying the argument from (ii) repeatedly.

(iv) Obviously, for every \( m \in \mathbb{N} \) \( B^1_m = (\sigma(m + 1), \sigma(m + 1)) \) is contained in \( A' \cap B_m \). Let \( n \) be a sufficiently large odd number, and let \( q \) be a minimal projection of \( A^n = (\sigma(n - 1), \sigma(n - 1)) \) which corresponds to the identity object \( \iota \) of \( \mathcal{C} \). The subobject of \( \sigma(n - 1)\sigma(m + 1) \) corresponding to the projection \( p = q \times 1_{\sigma(m+1)} \) is \( \iota \cdot \sigma(m + 1) = \sigma(m + 1) \), hence
\[
p((A^n)' \cap B^a_m) = (pA^a p)' \cap pB^a_m p = (\mathbb{C} p)' \cap pB^a_m p = pB^a_m p
\]
is isomorphic to \((\iota \sigma(m + 1), \iota \sigma(m + 1)) \) (compare Remark 1.3 (1) and (5)). Theorem 1.6 in [25] implies \( \dim A' \cap B_m \leq \dim (\sigma(m + 1), \sigma(m + 1)) = \dim B^1_m \). So \( B^1_m = A' \cap B_m \) has been shown. The same method yields
\[
B' \cap B_m = (\iota \sigma(m - 1), \iota \sigma(m - 1)) \subset B^1_m
\]
for \( m \geq 1 \).

Since the number of direct summands in the sequence \( (A' \cap B_m)_m \) is bounded, the inclusion \( A \subset B \) has finite depth. \( \blacksquare \)

3.6 Example

We regard the subfactor \( A \subset B \) if \( \mathcal{C} \) is the finite \( C^* \)-tensor category \( \mathcal{B}_{N \subset M} \) for a subfactor \( N \subset M \) of finite depth and if \( \sigma \) is the bimodule \( N L^2(M)_N \).

For an \((N, N)\)-bimodule \( \mathcal{H} \), \( \mathcal{L}_{N,N}(\mathcal{H}) \) denotes the von Neumann algebra of all operators of \( \mathcal{L}(\mathcal{H}) \) commuting with \( \rho(N) \) and \( \mathcal{L}_{N,N}(\mathcal{H}) \) the von Neumann algebra of all operators commuting with \( \rho(N) \) and \( \lambda(N) \).

We have \( \sigma = \bar{\sigma} \) and \( (\sigma^k, \sigma^k) = \mathcal{L}_{N,N}(L^2(M)^{\otimes^N}) \). Corollary 2.3 in [25] tells us that for every \( n > 1 \) there is an isomorphism \( J_n : M_{2n-1} \rightarrow \ldots \)
\( \mathcal{L}_{-,N}(L^2(M)^{\otimes n_i}) \) such that
\[
J_n(m) = J_k(m) \otimes_{N_i} \text{id}_{L^2(M)^{\otimes n_k}} \quad \text{for } m \in M_{2k-1},
\]
\[
J_n(N' \cap M_{2k-1}) = \mathcal{L}_{N,N}(L^2(M)^{\otimes k}) \otimes_{N_i} \text{Cid}_{L^2(M)^{\otimes n_k}}, \quad \text{and}
\]
\[
J_n(M'_1 \cap M_{2k-1}) = \text{Cid}_{L^2(M)} \otimes_{N_i} \mathcal{L}_{N,N}(L^2(M)^{\otimes n_k-1}) \otimes_{N_i} \text{Cid}_{L^2(M)^{\otimes n_k}}
\]
hold for \( 1 \leq k \leq n \). Hence the standard invariant of \( A \subset B \) is isomorphic to
\[
\mathbb{C}1 = N' \cap N \subset N' \cap M_1 \subset N' \cap M_3 \subset N' \cap M_5 \subset \ldots,
\]
\[
\mathbb{C}1 = M'_1 \cap M_1 \subset M'_1 \cap M_3 \subset M'_1 \cap M_5 \subset \ldots.
\]
But this tower of finite dimensional von Neumann algebras is also isomorphic to the standard invariant of the subfactor \( N \subset M_1 \): From \([19]\) (compare also \([25]\)) we conclude that there is an isomorphism \( J_2 \) from \( M_3 \) onto the basic construction \( B(N,M_1) \) such that the restriction of \( J_2 \) onto \( M_1 \) is the identity. The same argument shows that we may identify \( M_5 \) with the basic construction \( B(M_1,M_3) \) and so on. Hence the Jones tower for \( N \subset M_1 \) is isomorphic to
\[
N \subset M_1 \subset M_3 \subset M_5 \subset \ldots.
\]
Now let us assume that \( M \) is the hyperfinite \( \Pi_1 \)-factor. \([M_1:M] < \infty\) implies that \( M_1 \) is also isomorphic to the hyperfinite \( \Pi_1 \)-factor (see Lemma 2.1.18 in \([8]\)). Since \( A \subset B \) and \( N \subset M_1 \) have the same standard invariant and are of finite depth, they are isomorphic (according to Popa’s result \([20]\)).

We modify the preceding construction of the subfactors by always using \( \sigma \) instead of \( \sigma \) and \( \bar{\sigma} \) alternately. For example we obtain Wenzl’s Hecke algebra subfactors by this method, if we take a \( C^* \)-tensor category \( \mathcal{C} \) which, roughly speaking, consists of finite-dimensional representations of the quantum group \( U_q sl_k \), where \( q \) is a root of unity. (This seems to be known, although the author does not know any detailed reference. The author intends to publish a detailed approach elsewhere.)

Here the factors \( A \) and \( B \) are given by the tower
\[
B^1 = (\sigma, \sigma) \subset B^2 = (\sigma^2, \sigma^2) \subset B^3 = (\sigma^3, \sigma^3) \subset B^4 = (\sigma^4, \sigma^4) \subset \ldots
\]
\[
A^1 = (\iota, \iota) \subset A^2 = (\sigma, \sigma) \subset A^3 = (\sigma^2, \sigma^2) \subset A^4 = (\sigma^3, \sigma^3) \subset \ldots,
\]
where the embedding of \( A^n \) into \( B^n \) is given by \( T \mapsto T \times 1_{\sigma} \) and the embedding of \( A^n \) (resp. \( B^n \)) into \( A^{n+1} \) (resp. \( B^{n+1} \)) by \( T \mapsto 1_{\sigma} \times T \).

Additionally to the assumption \( \nu(d(\sigma)) > 1 \), we require the following condition for the object \( \sigma \):

**Assumption 3.7.** There is a \( \nu \in \mathbb{N} \setminus \{1\} \) such that \( \bar{\sigma} \) is contained in \( \sigma^{\nu-1} \) (and consequently \( \iota \leq \sigma^\nu \) holds).
Remark: One easily shows that $\sigma$ fulfils the condition if for every irreducible subobject $\tau$ of $\sigma$ there exists a $\nu_\tau \in \mathbb{N}$ such that $\iota \leq \tau^{\nu_\tau}$.

We get a similar result as before:

**Theorem 3.8.** Let $\sigma$ be an object of $\mathcal{C}$ satisfying Assumption 3.7. The sequences $(A^n)_{n \in \mathbb{N}}$ and $(B^n)_{n \in \mathbb{N}}$ satisfy the assumptions of Wenzl’s Theorem 1.5 in [29]. The subfactor $A \subset B$ defined by this tower has the following properties:

(i) $[B : A] = d(\sigma)^2$.

(ii) The tower

$$
B_m^1 = (\sigma(m + 1), \sigma(m + 1)) \subset \\
B_m^2 = (\sigma(\sigma(m + 1), \sigma(m + 1)) \subset \\
B_m^3 = (\sigma^2(\sigma(m + 1), \sigma^2 \sigma(m + 1)) \subset \\
B_m^4 = (\sigma^3(\sigma(m + 1), \sigma^3 \sigma(m + 1)) \subset \ldots
$$

defines a unique $\text{II}_1$-factor $B_m$. The Jones tower for $A \subset B$ is isomorphic to

$$B_{-1} = A \subset B_0 = B \subset B_1 \subset B_2 \subset B_3 \subset \ldots,$$

where the Jones projection $f_m \in B_{m+1}$ is given by

$$
f_m = \frac{1}{d(\sigma)} \mathbf{1}_{\sigma(m)} \times \left( \bar{R}_\sigma \circ \bar{R}_\sigma^* \right) \in B_{m+1}^1 \text{ for } m \text{ even and }

f_m = \frac{1}{d(\sigma)} \mathbf{1}_{\sigma(m)} \times \left( R_\sigma \circ R_\sigma^* \right) \in B_{m+1}^1 \text{ for } m \text{ odd}.
$$

(iii) The standard invariant for $A \subset B$ is

$$(\iota, \iota) \subset (\sigma, \sigma) \subset (\sigma \bar{\sigma}, \sigma \bar{\sigma}) \subset (\sigma \bar{\sigma} \sigma, \sigma \bar{\sigma} \sigma) \subset \ldots
\subset (\overline{\sigma}, \overline{\sigma}) \subset (\overline{\sigma} \overline{\sigma}, \overline{\sigma} \overline{\sigma}) \subset \ldots.
$$

Hence the standard invariant of this subfactor is the same as the standard invariant of the subfactor constructed with the same object $\sigma$ in Theorem 3.4. It follows from the main result of [20] that the two subfactors are isomorphic.

**Proof:** The proof is similar to that of the preceding results.

(1) If $\rho$ is an object of $\mathcal{C}$, we have

$$\rho \leq \sigma^{2^2} \rho \leq \sigma^{3^2} \rho \leq \ldots$$

with the consequence that

$$[[\mathcal{C}, \rho]] \subset [[\mathcal{C}, \sigma^{2^2} \rho]] \subset [[\mathcal{C}, \sigma^{3^2} \rho]] \subset \ldots.$$
Since \( C \) is finite, there is a \( k_{\rho} \in \mathbb{N} \) such that the tower \((34)\) becomes stationary from \([C, \sigma^{k_{\rho}}\rho]\) on. By putting \( \rho = \sigma^n\sigma(m + 1) \) and applying Observation \(3.1\) \((5)\), we obtain that the towers

\[
B_m^1 \subset B_m^2 \subset B_m^3 \subset B_m^4 \subset \ldots
\]

\((m \in \{-1, 0\} \cup \mathbb{N})\) are periodic with period \( \nu \).

\(2\) We have \( \bar{\sigma}\sigma \leq \sigma^{\nu} \) as well as \( \sigma\bar{\sigma} \leq \sigma^{\nu} \) and obtain

\[
\sigma^n \leq \sigma^n\sigma(2m) \leq \sigma^n\sigma^{\nu m}
\]

for \( m, n \in \mathbb{N} \cup \{0\} \). From Part (1) we conclude \([\bar{C}, \sigma^{n+\nu m}] = [\bar{C}, \sigma^n]\), if \( n \) is sufficiently large. Relation \((35)\) implies \([\bar{C}, \sigma^n]\) equals \([\bar{C}, \sigma^n\sigma(2m)]\). Therefore the numbers of the direct summands of \( B_{2m-1}^{n+1} \) and \( B_{2m+1}^{n+1} \) agree for a sufficiently large \( n \). Lemma \(3.2\) implies that \( B_{2m}^{n+1} \) is the basic construction for \( B_{2m-1}^{n+1} \). If \([\bar{C}, \sigma^n\sigma(2m)] = [\bar{C}, \sigma^n\sigma(2m + 2)]\) then \([\bar{C}, \sigma^n\sigma(2m)\bar{\sigma}] = [\bar{C}, \sigma^n\sigma(2m + 2)\bar{\sigma}]\). By applying Lemma \(3.2\) again, we obtain that \( B_{2m+2}^{n+1} \) is the basic construction for \( B_{2m}^{n+1} \)

We will verify in Step (3) and (4) that the inclusion matrix for \( B_m^{n+\nu} \) is primitive if \( n \) is sufficiently large. After having finished, we are able to proceed as in the proof of Theorem \(3.4\), and the proof of the Theorem is completed.

\(3\) Let \( s \) be a multiple of \( \nu \) such that \([C, \sigma^s] \supset [C, \sigma^{\nu k}]\) for every \( k \in \mathbb{N} \) and let \( \pi \) and \( \rho \) be irreducible objects of \( C \) such that \([\bar{\pi}], [\bar{\rho}] \in [C, \sigma^s]\). Applying the Frobenius reciprocity law twice, we get

\[
0 < \dim(\rho, \sigma^s) = \dim(\iota, \sigma^s\bar{\rho}) = \dim(\sigma^s, \bar{\rho}) = \dim(\bar{\sigma}^s, \bar{\rho}).
\]

Since \( \bar{\sigma}^s \) is a subobject of \( \sigma^{(\nu-1)s} \), \( \bar{\rho} \) is a subobject of \( \sigma^{(\nu-1)s} \). So \([\bar{\rho}] \in [C, \sigma^s]\) and \(0 < \dim(\rho, \sigma^s) = \dim(\iota, \sigma^s\rho)\). Hence \( \iota \leq \sigma^s\rho \) and \( \pi \leq \sigma^s\iota \) yields \( \pi \leq \sigma^s\rho \).

\(4\) From \( d(\sigma) > 1 \) we conclude \( B_m^n \neq B_{m+s}^{n+s} \). We intend to see that the matrix \( D \) describing the inclusion

\[
B_m^n = (\sigma^{-1}\sigma(m + 1), \sigma^{n-1}\sigma(m + 1)) \subset B_m^{n+s} = (\sigma^{n+\nu-1}\sigma(m + 1), \sigma^{n+\nu-1}\sigma(m + 1))
\]

is primitive for \( n > s \) and \( m \in \mathbb{N} \cup \{-1, 0\} \). \( D \) is quadratic for \( n > s \), as \([C, \sigma^{-1}\sigma(m + 1)] = [C, \sigma^{n+\nu-1}\sigma(m + 1)]\).

Let \( \rho \) and \( \pi \) be irreducible subobjects of \( \sigma^{-1}\sigma(m + 1) \). Since \( \bar{\sigma} \leq \sigma^{-1} \), there is an \( r \in \mathbb{N} \) such that \( \rho \) and \( \pi \) are subobjects of \( \sigma^{s+r} \). \( (\pi, \sigma^{s+r}) \neq 0 \) implies \( (\bar{\sigma}^r\pi, \sigma^s) \neq 0 \). Hence there is an irreducible subobject \( \bar{\pi} \) of \( \sigma^s \pi \) such that \( (\bar{\pi}, \sigma^s) \neq 0 \). We also have an irreducible subobject \( \bar{\rho} \) of \( \bar{\sigma}^r \rho \) such that \( (\bar{\rho}, \sigma^s) \neq 0 \). Part (3) implies \( (\bar{\pi}, \sigma^{2s}\bar{\rho}) \neq 0 \) with the consequence
that \((\sigma^r \pi, \sigma^2 \pi \rho) \neq 0\) and \((\pi, \sigma^r \rho \sigma^r \bar{\rho}) \neq 0\). Using \(\bar{\sigma} \leq \sigma^{n-1}\) we get \((\pi, \sigma^{2s+\nu \rho}) \neq 0\). Since \(\sigma^s\) has the same irreducible subobjects as \(\sigma^{2s+\nu \rho}\), \(\pi\) is a subobject of \(\sigma^s \rho\).

The matrix \(D^{s/\nu}\) describes the inclusion \(B_m^a \subset B_m^{n+s}\). The simple direct summands of \(B_m^a\) and \(B_m^{n+s}\) are in one to one correspondence with the irreducible subobjects of \(\sigma^{n-1}\sigma(m+1)\) (compare Observation 3.1 (5)) and the coefficient of \(D^{s/\nu}\) associated with the irreducible subobjects \(\rho\) of \(\sigma^{n-1}\sigma(m+1)\) and \(\pi\) of \(\sigma^{n+s-1}\sigma(m+1)\) is the number of times that \(\pi\) is contained in \(\sigma^s \rho\). According to the considerations from above, this number is always greater than 0. So \(D\) is primitive.

We illustrate the construction of Section 3.3 in case \(\mathcal{C}\) is the \(C^*\)-tensor category \(\mathcal{U}_H\) of the finite dimensional unitary corepresentations of a finite dimensional Hopf-\(*\)-algebra \(H\). Let \(\sigma : V \rightarrow V \otimes H\) be an object of \(\mathcal{U}_H\) and \(A \subset B\) the associated subfactor. We introduce the abbreviations

\[
V^1 := \mathbb{C}, \ V^2 := \overline{V}, \ V^3 := V \otimes \overline{V}, \ V^4 := \overline{V} \otimes V \otimes \overline{V}, \text{ and so on,}
\]

and consider the tower

\[
\mathbb{L}(V^1) \subset \mathbb{L}(V^2) \subset \mathbb{L}(V^3) \subset \ldots
\]

of finite dimensional \(*\)-algebras. Let \(N\) be the unique \(\text{II}_1\)-factor containing \(N^\infty := \bigcup_{n=1}^\infty \mathbb{L}(V^n)\) as an ultra-strongly dense \(*\)-subalgebra.

Since the \(*\)-algebra \(A^n\) is contained in \(\mathbb{L}(V^n)\) for every \(n \in \mathbb{N}\), \(A\) may be regarded as a subfactor of \(N\). We will describe \(A\) as a fixed point algebra \(N^K\) under an action of the finite dimensional Hopf-\(*\)-algebra \(K := (H^o)^{\text{cop}}\), where \(H^o := \{f : H \rightarrow \mathbb{C} : f \text{ linear}\}\) denotes the Hopf-\(*\)-algebra dual to \(H\). (Details of the definition of \(H^o\) can be found in [10].)

There is a one to one correspondence between the finite dimensional unitary corepresentations of \(H\) and the finite dimensional non-degenerate \(*\)-representations of \(H^o\). If \(\rho : V_\rho \rightarrow V_\rho \otimes H\) is a unitary corepresentation of \(H\), the associated \(*\)-representation \(\rho^o : H^o \rightarrow \mathbb{L}(V_\rho)\) is defined by \(\rho^o(f)w = (id_{V_\rho} \otimes f)(\rho(w))\) (compare [10]). \(\rho^o\) is also a \(*\)-representation of \(K\), as \(H^o\) and \(K\) are the same \(*\)-algebras. We write \(\rho^c\) for \(\rho^o\), whenever we regard \(\rho^o\) as a \(*\)-representation of \(K\).

**Lemma 3.9.** (i) Let \(\mu\) be a nondegenerate \(*\)-representation of the Hopf-\(*\)-algebra \(K\) on the Hilbert space \(V_\mu\). Then an action \(\alpha_\mu\) of \(K\) on \(\mathbb{L}(V_\mu)\) is defined by

\[
\alpha_\mu(a)x := \sum_{(a)} \mu(a^{(1)}) x \mu(S(a^{(2)})
\]

\((\Delta(a) = \sum_{(a)} a^{(1)} \otimes a^{(2)}, \text{ Sweedler notation for } K)\) for every \(a \in K\) and \(x \in \mathbb{L}(V_\mu)\). The relation

\[
\mathbb{L}(V_\mu)^K = \mu(K)^{\prime}
\]

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holds.
In particular, the fixed point algebra \( L(V^\rho)^K \) under the action \( \alpha_{\rho^c} \) of \( K \) is equal to \( (\rho, \rho) \).

(ii) \( \alpha_{\rho^c} \) is unitarily equivalent to the representation \( (\overline{\rho} \otimes \rho)^o \) of \( H^o \), if we regard \( \alpha_{\rho^c} \) as a representation of \( H^o \) and endow \( L(W) \) with the inner product given by the normalized trace \( tr \) of \( L(W) \). In particular, the representation \( \alpha_{\rho^c} \) respects the \( \ast \)-operation.

Proof: (i) \( \alpha_\mu \) is an action of \( K \) by Example 2.5 (2) in [17]. The proof of \( \mu(K)' \subset L(V_\mu)^K \) is easy. For \( x \in L(V_\mu)^K \) and \( a \in K \) we get

\[
x \cdot \mu(a) = \sum (a) \epsilon(a^{(1)}) x \cdot \mu(a^{(2)}) = \\
\sum (a), (a^{(1)}) \mu(a^{(1)}) \cdot x \cdot \mu(S(a^{(1)(2)})) \cdot \mu(a^{(2)}) = \\
\sum (a), (a^{(2)}) \mu(a^{(1)}) \cdot x \cdot \mu(S(a^{(2)(1)})) \cdot \mu(a^{(2)(2)}) = \\
\sum (a) \mu(a^{(1)}) \cdot x \cdot \mu\left( \sum (a^{(2)}) S(a^{(2)(1)}) \cdot a^{(2)(2)} \right) = \\
\sum (a) \mu(a^{(1)}) \cdot x \cdot \mu(\epsilon(a^{(2)}) 1) = \\
\sum (a) \mu\left( \epsilon(a^{(2)}) a^{(1)} \right) \cdot x = \mu(a) \cdot x.
\]

Therefore \( x \) belongs to \( \mu(K)' \).

(ii) Let \( (a_{ij})_{i,j} \) be the matrix coefficients of \( \rho \) with respect to an orthonormal base \( B := (w_1, \ldots, w_s) \) of \( V_\rho \). We get

\[
\rho^o(f) w_j = \sum_{i=1}^s f(a_{ij}) w_i \quad \text{and} \quad \overline{\rho^o}(f) \overline{w_j} = \sum_{i=1}^s f(S(a_{ji})) \overline{w_i}. \quad (37)
\]

Let \( \epsilon_{ij} \in L(W) (i, j = 1, \ldots, s) \) be defined by \( \epsilon_{ij} w_k = \delta_{j,k} w_i \). The unitary operator \( U : L(V_\rho) \rightarrow V_\rho \otimes V_\rho \) determined by \( \sqrt{s} \epsilon_{ij} \rightarrow \overline{w_j} \otimes w_i \) intertwines
the representations $\alpha_{\rho^c}$ and $(\bar{\rho} \otimes \rho)^o$ of $H^o$:

$$\sqrt{s} U \alpha_{\rho^c}(f) \epsilon_{ij} =$$

$$\sqrt{s} U \sum_{(f)} \rho^o(f^{(2)}) \cdot \epsilon_{ij} \cdot \rho^o(S^o(f^{(1)})) = \quad \text{(by using (37))}$$

$$\sqrt{s} U \sum_{k,l=1}^s f^{(2)}(a_{kl}) S^o(f^{(1)})(a_{ij}) \epsilon_{kl} =$$

$$\sum_{k,l=1}^s S^o(f^{(1)})(a_{ij}) \overline{w}_l \otimes f^{(2)}(a_{kj}) w_k =$$

$$\sum_{l=1}^s f^{(1)}(S(a_{jl})) \overline{w}_l \otimes \sum_{k=1}^s f^{(2)}(a_{kl}) w_k = \quad \text{(by using (37))}$$

$$(\bar{\rho} \otimes \rho)^o(f) \overline{w}_j \otimes w_i = \quad (\bar{\rho} \otimes \rho)^o(f) U \sqrt{s} \epsilon_{ij}$$

for $f \in H^o$ and $i,j = 1, \ldots, s$. (For the first ‘=’ observe that the Sweedler notation uses the comultiplication of $H^o$ and not of $K$.)

**Lemma 3.10.** There is an action $\alpha_{\infty}$ of $K$ on $N_{\infty}$ given by

$$\alpha_{\infty}(f) x = \alpha_{\rho^c(\tau)^c}(f) x \quad \text{for } x \in L(V^{n+1}). \quad (38)$$

**Proof:** It is necessary to show that Definition (38) is compatible with the embedding in the tower (36). (To this purpose we need $K = (H^o)^{cop}$ instead of $H^o$.) Let $\rho : V_\rho \rightarrow V_\rho \otimes H$ and $\tau : V_\tau \rightarrow V_\tau \otimes H$ be unitary corepresentations of $H$. We prove

$$\alpha_{(\rho \otimes \tau)^c}(f)(1_{V_\rho} \otimes x) = 1_{V_\rho} \otimes \alpha_{\tau^c}(f)x$$

for $f \in K$ and $x \in L(V_\tau)$. An easy computation shows

$$(\rho \otimes \tau)^c(f) = \sum_{(f)} \rho^c(f^{(2)}) \otimes \tau^c(f^{(1)}) \quad \text{for } f \in K$$

(where the Sweedler notation for $K$ is used). From now on $S$ denotes the
antipode of \( K \). We obtain

\[
\alpha(\rho \otimes \tau)(f) (1 \otimes x) = \\
\sum_{(f)} \left( \sum_{(f^{(1)})} \rho^c(f^{(1)(2)}) \otimes \tau^c(f^{(1)(1)}) \right) \cdot (1 \otimes x) \\
\left( \sum_{(S(f^{(2)}))} \rho^c((S(f^{(2)}))^{(2)}) \otimes \tau^c((S(f^{(2)}))^{(1)}) \right) = \\
\sum_{(f)} \sum_{(f^{(1)})} \sum_{(f^{(2)})} \rho^c(f^{(1)(2)}) \cdot 1 \cdot \rho^c(S(f^{(2)(1)})) \otimes \tau^c(f^{(1)(1)}) \cdot x \cdot \tau^c(S(f^{(2)(2)})) = \\
\sum_{(f)} \sum_{(f^{(1)})} \sum_{(f^{(1)(2)})} \rho^c(f^{(1)(2)}) \cdot \rho^c(S(f^{(1)(2)})) \otimes \tau^c(f^{(1)(1)}) \cdot x \cdot \tau^c(S(f^{(2)})) = \\
\sum_{(f)} \sum_{(f^{(1)})} \rho^c(\epsilon(f^{(1)(2)})) 1 \otimes \tau^c(f^{(1)(1)}) \cdot x \cdot \tau^c(S(f^{(2)})) = \\
1 \otimes \sum_{(f^{(1)})} \tau^c\left( \sum_{(f^{(1)})} \epsilon(f^{(1)(2)}) f^{(1)(1)} \right) \cdot x \cdot \tau^c(S(f^{(2)})) = \\
1 \otimes \alpha_\tau(f)x.
\]

(For the second ' \( = \) ' observe \( \Delta(S(a)) = \sum_{(a)} S(a^{(2)}) \otimes S(a^{(1)}) \).) \( \blacksquare \)

We extend \( \alpha_\infty \) to an action \( \alpha \) of \( K \) on \( N \): by Lemma 3.9 (ii), we know that \( \alpha_{\sigma(n)} \) is a \( \ast \)-representation of the finite dimensional \( C^* \)-algebra \( K \) on the Hilbert space \( L(V^n) \) for every \( n \in \mathbb{N} \). Hence \( ||\alpha_{\sigma(n)}(f)|| \leq ||f|| \) for \( f \in K \), where \( ||f|| \) denotes the \( C^* \)-norm of \( f \). We conclude that the linear operator \( \alpha_\infty(f) : N^\infty \to N^\infty \) is continuous and satisfies the relation \( ||\alpha_\infty(f)|| \leq ||f|| \), if we regard \( N^\infty \) as a subspace of the Hilbert space \( L^2(N) \). Hence \( \alpha_\infty(f) \) has a unique extension to a continuous linear operator \( u(f) : L^2(N) \to L^2(N) \). It is straightforward to verify that \( f \in K \to u(f) \) is a nondegenerate \( \ast \)-representation of \( K \) on \( L^2(N) \).

Now

\[
\alpha(f)x = \sum_{(f)} u(f^{(1)})x u(S(f^{(2)})) \quad (f \in K)
\]

defines an action \( \alpha \) of \( K \) on \( L(L^2(N)) \) (see Lemma 3.9 (i)). For \( n \in N_\infty \subset N \subset L(L^2(N)) \) we have \( \alpha(f)n = \alpha_\infty(f)n \), as the computa-
\[
\lambda(\alpha(f)n) \overline{m} = \sum_{(f)} \alpha_\infty(f^{(1)}) \left( n \cdot \alpha_\infty(S(f^{(2)}))m \right) = \\
\sum_{(f)} \alpha_\infty(f^{(1)(1)}) n \cdot \alpha_\infty(f^{(1)(2)} S(f^{(2)}))m = \\
\sum_{(f)} \alpha_\infty(f^{(1)}) n \cdot \alpha_\infty(\epsilon(f^{(2)}) \mathbf{1})m = \\
\alpha_\infty(\sum_{(f)} \epsilon(f^{(2)}) f^{(1)}) n \cdot m = \lambda(\alpha_\infty(f)) \overline{m}
\]

for \( m \in N^\infty \) shows. Since \( x \in L(L^2(N)) \mapsto \alpha(f)x \in L(L^2(N)) \) is strongly continuous and \( \alpha(f)x \in N \) for \( x \in N^\infty \), we find that \( \alpha(f) \) maps \( N \) into \( N \) for every \( f \in K \). So the restriction of \( \alpha \) to an action of \( K \) on \( N \) extends \( \alpha_\infty \). We denote this restriction by \( \alpha \) again.

**Lemma 3.11.** The fixed point algebra \( N^K \) under the action \( \alpha \) is equal to \( A \).

**Proof:** There is a unique minimal central projection \( e \) of \( K \) such that \( fe = \epsilon(f)e \) for every \( f \) of \( K \) (see [17], Theorem 2.2 (4)). (If \( K \) is the group algebra \( C[G] \) for a finite group \( G \), then \( e := 1/|G| \sum_{g \in G} g \).) By Proposition 2.12 in [17],

\[
N^K = \{ \alpha(e)x : x \in N \}.
\]

From Lemma 3.11 (i) we conclude \( A^\infty = N^\infty \cap N^K \). For \( x \in N \) there is a sequence \( (x_i)_{i \in \mathbb{N}} \subset N^\infty \) strongly converging to \( x \) (in \( L^2(N) \)). Then the sequence \( (\alpha(e)x_i)_i \) converges strongly to \( \alpha(e)x \), hence \( \alpha(e)x \in A \) and \( N^K \subset A \). The other inclusion is obvious. 

**Lemma 3.12.** \( (N^K)' \cap N = C1 \).

**Proof:** It is not difficult to see that the towers \( (A^n)_n \) and \( (L(V^n))_n \) describing the subfactor \( N^K \subset N \) fulfil the periodicity assumptions of [29], Theorem 1.5. Note that

\[
E_n : L(V^n) \rightarrow A^n, \ x \mapsto \alpha(e)x,
\]

is the conditional expectation from \( L(V^n) \) onto the fixed point algebra \( L(V^n)^K = A^n \) corresponding to the unique normalized trace on \( L(V^n) \) (use Proposition 2.12 in [17] and observe Lemma 3.11 (ii)).

We will apply Theorem 1.6 in [29] in order to estimate the dimension of \( (N^K)' \cap N \). For every \( n \in \mathbb{N} \) there is a minimal projection \( p \) in \( A^{2n+1} \) such
that the restriction \(\sigma(2n)|pV^{2n+1}\) of the corepresentation \(\sigma(2n)\) is equivalent to the identity corepresentation. So \(\dim pV^{2n+1} = 1\), and

\[
\dim (N^K)' \cap N \leq \dim p\left((A^{2n+1})' \cap L(V^{2n+1})\right) = 1
\]

follows. \(\blacksquare\)

Similarly, it is possible to write \(B\) as a fixed point algebra under an action of \(K\):

\[N \otimes L(V)\) is the II \(_1\)-factor containing \[
\bigcup_{n=1}^{\infty} L(V^n \otimes V) = \bigcup_{n=1}^{\infty} L(V^n) \otimes L(V)
\]
as an ultra-strongly dense \(*\)-subalgebra. As before we introduce an action \(\beta\) of \(K\) on \(N \otimes L(V)\) such that

\[
\beta(f)x = \alpha_{\sigma(n) \otimes \sigma}(f)x \quad \text{for} \quad x \in L(V^n \otimes V).
\]

We get \((N \otimes L(V))^K = B\) as well as \(((N \otimes L(V))^K)' \cap N \otimes L(V) = \mathbb{C}1\). Hence the subfactor \(A \subset B\) may be written as an inclusion of fixed point algebras

\[N^K \subset (N \otimes L(V))^K, \quad (40)\]

where the embedding of \(N\) into \(N \otimes L(V)\) is given by \(n \mapsto n \otimes 1\).

The restriction of the action \(\beta\) onto \(N\) is in general not equal to \(\alpha\), as the proof of Lemma 3.11 does not work for \(x \otimes id_{V_\rho} (x \in L(V_\rho))\) instead of \(id_{V_\rho} \otimes x (x \in L(V_\rho))\) except in the group case. If \(H\) is equal to the commutative Hopf-\(\ast\)-algebra \(\text{Fun}(G) := \{f : G \to \mathbb{C}\}\) for a finite group \(G\), \(\sigma\) may be considered as a unitary finite dimensional representation of \(G\) and the action \(\beta\) of \(G\) is equal to

\[\alpha \otimes \text{Ad} \sigma : G \to \text{Aut} N \otimes L(V),
\]

\[(\alpha \otimes \text{Ad} \sigma)(g)(n \otimes x) = \alpha(g)n \otimes \sigma(g)x\sigma(g)^{-1} \quad \text{for} \quad g \in G, \ n \in N, \ x \in L(V).
\]

By applying Lemma 3.9 (ii), one obtains

**Lemma 3.13.** The action \(\alpha\) (resp. \(\beta\)) is an outer action of \(K\) on \(N\) (resp. \(N \otimes L(V)\)) if there is an \(n \in \mathbb{N}\) such that \((\sigma \sigma)^n\) (resp. \((\bar{\sigma} \sigma)^n\)) contains every irreducible unitary corepresentation of \(K\).

In particular, the exposition from above describes a method for constructing an outer action of a finite dimensional Hopf-\(\ast\)-algebra \(K\) on the hyperfinite II \(_1\)-factor, if we use a unitary corepresentation \(\sigma\) of \(H = (K^{\text{cop}})\) as in Lemma 3.13. (For example, choose the comultiplication \(\Delta\) of \(H\) as the corepresentation \(\sigma\).)
4 The $C^*$-tensor category associated with the constructed subfactors

In this Section we determine the $C^*$-tensor category of the $(A, A)$-bimodules for the subfactors $A \subset B$ from Section 3. We assume the notation used there.

Let $\rho$ be an object of the finite $C^*$-tensor category $\mathcal{C}$. $\mathcal{C}_\rho$ denotes the following full $C^*$-tensor subcategory of $\mathcal{C}$:

the objects of $\mathcal{C}_\rho$ form the smallest subset $\mathcal{O}_\rho$ of $\text{Ob} \mathcal{C}$ with the following properties:

(a) $\iota, \rho \in \mathcal{O}_\rho$.

(b) If $\tau \in \mathcal{O}_\rho$ then every object equivalent to $\tau$ and every subobject of $\tau$ belongs to $\mathcal{O}_\rho$.

(c) If $\tau, \phi \in \mathcal{O}_\rho$ then $\tau \in \mathcal{O}_\rho$ and $\tau \phi \in \mathcal{O}_\rho$.

(d) Finite direct sums of objects of $\mathcal{O}_\rho$ are objects of $\mathcal{O}_\rho$.

Obviously, $\mathcal{C}_\rho$ is a finite $C^*$-tensor category.

Theorem 4.1. Let $A \subset B$ be the subfactor from Section 3.3 or Theorem 3.8. There is an equivalence $G : \mathcal{C}_{\sigma\bar{\sigma}} \rightarrow \mathcal{B}_{A \subset B}$ of the $C^*$-tensor categories $\mathcal{C}_{\sigma\bar{\sigma}}$ and $\mathcal{B}_{A \subset B}$ such that $A L^2(B)_A$ is equivalent to $G(\sigma\bar{\sigma})$.

Corollary 4.2. For every finite $C^*$-tensor category $\mathcal{C}$ there is a subfactor $A \subset B$ of the hyperfinite factor $B$ with finite index such that the $C^*$-tensor category $\mathcal{B}_{A \subset B}$ is equivalent to $\mathcal{C}$.

Proof: In Theorem 1.1 take an object $\sigma$, which contains every irreducible object of $\mathcal{C}$ as a subobject. □

Corollary 4.3. Let $N \subset M$ and $P \subset Q$ be two inclusions of II$_1$-factors with finite index and finite depth. If the standard invariants of $N \subset M$ and $P \subset Q$ are isomorphic, the $C^*$-tensor categories $\mathcal{B}_{N \subset M}$ and $\mathcal{B}_{P \subset Q}$ are equivalent.

Proof: Assume that the standard invariants of $N \subset M$ and $P \subset Q$ are isomorphic. Let $A \subset B$ (resp. $C \subset D$) be the subfactor from Section 3.3 with $\mathcal{C} = \mathcal{B}_{N \subset M}$ (resp. $\mathcal{C} = \mathcal{B}_{P \subset Q}$) and $\sigma =_N L^2(M)_N$ (resp. $\rho L^2(Q)_P$).

According to Example 3.6, the standard invariants of $N \subset M_1$ and $A \subset B$ are isomorphic as well as the the standard invariants of $P \subset Q_1$ and $C \subset D$. Moreover, the standard invariant of $N \subset M_1$ is isomorphic to the standard invariant of $P \subset Q_1$. $B$ and $D$ are hyperfinite II$_1$-factors, so Popa’s classification [20] of finite depth subfactors implies that the subfactors $A \subset B$ and $C \subset D$ are isomorphic. Theorem 4.1 shows that the $C^*$-tensor category $\mathcal{B}_{A \subset B} = \mathcal{B}_{C \subset D}$ is equivalent to $\mathcal{B}_{N \subset M}$ as well as to $\mathcal{B}_{P \subset Q}$. □

Intending to work with a modified version of the $C^*$-tensor category $\mathcal{C}_{\sigma\bar{\sigma}}$ in the proof of Theorem 4.1 we carry out this modification in the following observations.
4.4 Observations

(1) Let $\mathcal{D}$ be a strict $C^*$-tensor category and let $\mathcal{D}^P$ denote the following $C^*$-tensor category: The objects of $\mathcal{D}^P$ are pairs $(\rho, P)$, where $\rho$ is an object of $\mathcal{D}$ and $P$ a projection of $(\rho, \rho) \setminus \{0\}$. The space of morphisms is

$$((\rho, P), (\psi, Q)) := \{ T \in (\rho, \psi) : QT P = T \}$$

for any objects $(\rho, P)$ and $(\psi, Q)$ of $\mathcal{D}^P$. Endowed with the Banach space structure, the composition law, and the $*$-operation from $\mathcal{D}$, $\mathcal{D}^P$ is a $C^*$-category with subobjects, as one easily sees.

Let the product of objects be given by $(\rho, P) \cdot (\psi, Q) = (\rho \psi, P \times Q)$. The product $S_1 \times S_2$ of morphisms $S_i \in ((\rho_i, P_i), (\psi_i, Q_i), i = 1, 2$, is defined as in $\mathcal{D}$. Obviously $S_1 \times S_2$ belongs to $((\rho_1 \rho_2, P_1 \times P_2), (\psi_1 \psi_2, Q_1 \times Q_2))$. The unit object in $\mathcal{D}^P$ is $(\iota, 1\iota)$. Endowed with this product structure, $\mathcal{D}^P$ is a strict $C^*$-tensor category.

Let $F : \mathcal{D} \rightarrow \mathcal{D}^P$ be the functor given by $F(\rho) = (\rho, 1\rho)$ for $\rho \in \text{Ob } \mathcal{D}$ and $F(T) = T \in ((\rho, 1\rho), (\psi, 1\psi))$ for $T \in (\rho, \psi)$. Obviously $F$ is a full and faithful strict $C^*$-tensor functor. It is a $C^*$-tensor equivalence if $\mathcal{D}$ has subobjects. (Assume that $(\rho, P)$ is an object of $\mathcal{D}^P$ and $\phi$ is a subobject of $\rho$ corresponding to $P$. Then $F(\phi)$ is equivalent to $(\rho, P)$.)

(2) Let $\mathcal{D}$ be a strict $C^*$-tensor category and let $S$ be a subset of $\text{Ob } \mathcal{D}$ satisfying the following properties:

(a) Each equivalence class of $\text{Ob } \mathcal{D}$ contains an element of $S$, and the unit object $\iota$ belongs to $S$.

(b) $\rho, \phi \in S$ implies $\rho \phi \in S$.

We get a modified $C^*$-tensor category $\mathcal{D}_S$ if we reduce the object set of $\mathcal{D}$ to the set $\text{Ob } \mathcal{D}_S := S$ and take the remaining structure of $\mathcal{D}$. Obviously the functor $G : \mathcal{D}_S \rightarrow \mathcal{D}$ defined by

$$G(\phi) = \phi \text{ for } \phi \in S \quad \text{and} \quad G(T) = T \text{ for every morphism } T \in \mathcal{D}_S$$

is a strict $C^*$-tensor equivalence.

(3) We will apply Observation (2) to the $C^*$-tensor category $\mathcal{D} = (C_{\sigma\sigma})^P$ if $d(\sigma) \ne 1$. Let

$$S = \{((\sigma\bar{\sigma})^m, P) : m \in \mathbb{N} \cup \{0\}, P \text{ a projection } \ne 0 \text{ of } ((\sigma\bar{\sigma})^m, (\sigma\bar{\sigma})^m)\}.$$

It is not difficult to see that $S$ satisfies the Properties (a) and (b) from Observation (2). Therefore we are able to introduce the $C^*$-tensor category $\mathcal{D}_{\sigma\sigma} := ((C_{\sigma\sigma})^P)_S$, which we use in the proof of the Theorem. By using the Observations (1) and (2) and Proposition 1.6 (i) and (ii), we see that there
is an equivalence \( H : \mathcal{C}_{\sigma \bar{\sigma}} \to \mathcal{D}_{\sigma \bar{\sigma}} \) of the \( C^* \)-tensor categories \( \mathcal{C}_{\sigma \bar{\sigma}} \) and \( \mathcal{D}_{\sigma \bar{\sigma}} \) such that \( H(\sigma \bar{\sigma}) \) is equivalent to \( (\sigma \bar{\sigma}, 1_{\sigma \bar{\sigma}}) \).

**Proof of the Theorem:** (1) We only deal with the case of Theorem 3.4, as the proofs do not differ. We fix a standard pair \((R_{\phi}, \bar{R}_{\phi})\) of conjugation operators for every object \( \phi \) of \( \mathcal{C} \). For all objects \( \rho, \phi, \psi \) of \( \mathcal{C} \) we introduce a map \( f(\rho, \phi, \psi) \) from \((\phi \bar{\phi}, \psi \bar{\psi})\) into the set \( \mathbb{L}((\rho \phi, \rho \phi), (\rho \psi, \rho \psi)) \) of all linear maps from \((\rho \bar{\phi}, \rho \bar{\phi})\) into \((\rho \bar{\psi}, \rho \bar{\psi})\):

For \( K \in (\phi \bar{\phi}, \psi \bar{\psi}) \) let

\[
f(\rho, \phi, \psi)(K) : (\rho \phi, \rho \phi) \to (\rho \psi, \rho \psi),
\]

\[
S \mapsto \sqrt{\frac{d(\phi)}{d(\psi)}} 1_{\rho \phi} \times R^*_\phi \circ 1_{\rho} \times K \times 1_{\psi} \circ 1_{\phi \psi} \circ 1_{\rho} \times \bar{R}_{\phi} \times 1_{\psi}.
\]

We will often abbreviate \( f(\rho, \phi, \psi)(K) \) to \( f(\rho)(K) \). We obtain the following properties:

(a) \( f(\rho, \phi, \psi)(K) \) is \((D, D)\)-linear for every \( K \in (\phi \bar{\phi}, \psi \bar{\psi}) \) where \( D := (\rho, \bar{\rho}) \).

(b) \( K \in (\phi \bar{\phi}, \psi \bar{\psi}) \mapsto f(\rho)(K) \) is linear and injective.

(c) \( f(\rho)(L \circ K) = f(\rho)(L) \circ f(\rho)(K) \) for every \( K \in (\phi \bar{\phi}, \psi \bar{\psi}) \) and \( L \in (\psi \bar{\psi}, \tau \bar{\tau}) \) (\( \phi, \psi, \tau \in \text{Ob} \mathcal{C} \)).

(d) \( f(\rho)(K)^* = f(\rho)(K^*) \) for every \( K \in (\phi \bar{\phi}, \psi \bar{\psi}) \), where the linear space \((\rho \bar{\phi}, \rho \bar{\phi})\) (resp. \((\rho \psi, \rho \psi)\)) is endowed with the inner product \( \langle S, T \rangle = \text{tr}_{\rho \phi}(ST^*) \) (resp. \( \text{tr}_{\rho \phi}(ST^*) \)).

(e) \( f(\rho, \phi, \psi)(1_{\phi}) = id_{(\rho \phi, \rho \phi)} \).

(f) \( f(\rho)(K)(1_{\rho_1} \times S) = 1_{\rho_1} \times f(\rho_2)(K)(S) \) for \( \rho = \rho_1 \rho_2, \ K \in (\phi \bar{\phi}, \psi \bar{\psi}), \) and \( S \in (\rho_2 \phi, \rho_2 \bar{\phi}) \).

The Properties (a) and (f) are obvious. Property (e) is an easy consequence of the defining relations for the conjugation operator. We show the remaining relations:

ad (b): The linearity is obvious. We prove \( f(\rho)(K) \neq 0 \) for \( K \neq 0 \). First we consider the case that \( \phi = \psi \) and \( K \neq 0 \) is positive. The computation

\[
\text{tr}_{\rho \phi \bar{\phi}} \left( 1_{\rho \phi} \times R_{\phi} \circ f(\rho)(K)(1_{\rho \bar{\phi}}) \circ 1_{\rho} \times \bar{R}_{\phi} \times 1_{\phi} \right) =
\]

\[
\text{tr}_{\rho \phi \bar{\phi}} \left( 1_{\rho \phi} \times (R_{\phi} \circ \bar{R}_{\phi}^*) \circ 1_{\rho} \times K \times 1_{\phi} \circ 1_{\rho} \times (\bar{R}_{\phi} \circ \bar{R}_{\phi}^*) \times 1_{\phi} \right) = \text{by Equation (B1)}
\]

\[
d(\phi)^{-1} \text{tr}_{\rho \phi \bar{\phi}} \left( 1_{\rho} \times (K \circ 1_{\rho} \times (\bar{R}_{\phi} \circ \bar{R}_{\phi}^*)) \right) = \text{by Equation (C7)}
\]

\[
d(\phi)^{-2} \text{tr}_{\rho}(1_{\rho} \times \bar{R}_{\phi} \circ 1_{\rho} \times K \circ 1_{\rho} \times \bar{R}_{\phi}) =
\]

\[
d(\phi)^{-2} \text{tr}_{\rho}(\Psi^\phi_{\rho}(K)) = d(\phi)^{-2} \text{tr}_{\rho \phi}(K) > 0
\]
yields \( f(\rho)(K) \neq 0 \). Now let \( \phi, \psi \) and \( K \neq 0 \) be arbitrary. We get \( f(\rho)(K^*)f(\rho)(K) = f(\rho)(K^* \circ K) \neq 0 \) by applying Property (c).

ad (c): For \( S \in (\rho \phi, \rho \phi) \) we have

\[
\begin{align*}
f(\rho)(L) & \left( f(\rho)(K)(S) \right) = \\
& \sqrt{\frac{d(\tau)}{d(\phi)}} \ 1_{\rho \tau} \times R^*_\tau \circ 1_{\rho} \times L \times 1_{\tau} \circ 1_{\rho \psi} \times R^*_\psi \times 1_{\psi \tau} \circ 1_{\rho} \times K \times 1_{\psi \psi \tau} \circ \\
& \circ S \times 1_{\phi \psi \psi \tau} \circ 1_{\rho} \times R_\phi \times 1_{\psi \psi \tau} \circ 1_{\rho} \times R_\psi \times 1_{\tau}.
\end{align*}
\]

The interchange law implies

\[
1_{\rho} \times R_\phi \times 1_{\psi \psi \tau} \circ 1_{\rho} \times R_\psi \times 1_{\tau} = 1_{\rho \phi \psi \psi} \times R_\psi \times 1_{\tau} \circ 1_{\rho} \times R_\psi \times 1_{\tau}.
\]

By applying this equation and shifting \( R_\psi \) to the left, we obtain

\[
\begin{align*}
f(\rho)(L) & \left( f(\rho)(K)(S) \right) = \\
& \sqrt{\frac{d(\tau)}{d(\phi)}} \ 1_{\rho \tau} \times R^*_\tau \circ 1_{\rho} \times L \times 1_{\tau} \circ 1_{\rho \psi} \times R^*_\psi \times 1_{\psi \tau} \circ 1_{\rho \psi \psi} \times R_\psi \times 1_{\tau} \circ \\
& \circ 1_{\rho} \times K \times 1_{\tau} \circ S \times 1_{\phi \psi \tau} \circ 1_{\rho} \times R_\phi \times 1_{\tau} = \\
& f(\rho)(L \circ K)(S).
\end{align*}
\]

By doing so we used the conjugation relation (41) for \( R_\psi \) and \( R_\psi \).

ad (d): We have to verify

\[
\text{tr}_{\rho \psi} \left( f(\rho)(K)(S) \right) T^* = \text{tr}_{\rho \phi} \left( S \ f(\rho)(K^*)(T)^* \right)
\] (41)

for \( S \in (\rho \phi, \rho \phi) \) and \( T \in (\rho \psi, \rho \psi) \).

Applying the conjugation relation (42) for \( R_\psi \) and \( R_\psi \), we get

\[
T = 1_{\rho \psi} \times R^*_\psi \circ T \times 1_{\psi \psi} \circ 1_{\rho} \times R_\psi \times 1_{\psi}
\]
and conclude

\[
\begin{align*}
\text{tr}_{\rho \psi} \left( f(\rho)(K)(S) \right) T^* = \\
\sqrt{\frac{d(\psi)}{d(\phi)}} \ \text{tr}_{\rho \psi} \left( 1_{\rho \psi} \times R^*_\psi \circ 1_{\rho} \times K \times 1_{\psi} \circ S \times 1_{\phi \psi} \circ 1_{\rho} \times R_\phi \times 1_{\psi} \circ \\
\circ 1_{\rho} \times R^*_\psi \times 1_{\psi} \circ T^* \times 1_{\psi} \circ 1_{\rho \psi} \times R_\psi \circ \\
\circ 1_{\rho} \times R_\phi \times 1_{\psi} \circ 1_{\rho} \times R^*_\phi \times 1_{\psi} \circ T^* \times 1_{\phi \psi} \right) =
\text{by Equation (27)}
\end{align*}
\]

\[
d(\phi)^{-1/2} \ d(\psi)^{5/2} \ \text{tr}_{\rho \psi \psi} \left( 1_{\rho \psi} \times R_\psi \circ R^*_\psi \circ 1_{\rho} \times K \times 1_{\psi} \circ S \times 1_{\phi \psi} \circ \\
\circ 1_{\rho} \times R_\phi \times 1_{\psi} \circ 1_{\rho} \times R^*_\phi \times 1_{\psi} \circ T^* \times 1_{\phi \psi} \right) =
\text{by Equation (22)}
\]

\[
d(\phi)^{-1/2} \ d(\psi)^{3/2} \ \text{tr}_{\rho \psi \psi} \left( 1_{\rho} \times K \circ S \times 1_{\phi \psi} \circ 1_{\rho} \times R_\phi \circ 1_{\rho} \times R^*_\phi \times T^* \times 1_{\psi} \right).
\]
The same calculation for
\[ \text{tr}_{\rho \phi} \left( S f(\rho)(K^*)(T) \right) = \text{tr}_{\rho \phi} \left( f(\rho)(K^*)(T) \right) \]
yields
\[
\text{tr}_{\rho \phi} \left( S f(\rho)(K^*)(T) \right) = d(\psi)^{-1/2} d(\phi)^{3/2} \text{tr}_{\rho \phi}(1_\rho \times K^* \circ T \times 1_\psi \circ 1_\rho \times R_\psi \circ 1_\rho \times R_{\phi}^* \circ S^* \times 1_{\phi})
\]
\[
= d(\psi)^{-1/2} d(\phi)^{3/2} \text{tr}_{\rho \phi}(S \times 1_\phi \circ 1_\rho \times R_\phi \circ 1_\rho \times R_{\phi}^* \circ T^* \times 1_\psi \circ 1_\rho \times K).
\]
So by applying Equation (27) we find that the results of both computations coincide, and that Equation (11) has been established.

(2) We will use the maps \( f(\rho) \) from Part (1) of the proof in order to define an \((A, A)\)-linear map \( F_m : L^2(B_{m-1}) \rightarrow L^2(B_{l-1}) \) for \( K \in \left( \sigma(m) \bar{\sigma}(m), \sigma(l) \bar{\sigma}(l) \right) = \left( (\sigma \bar{\sigma})^m, (\bar{\sigma} \sigma)^l \right), m, l \in \mathbb{N} \cup \{0\} \).

First we define a linear map \( F_m : L^2(B_{m-1}) \rightarrow L^2(B_{m-1}) \) into \( B_{l-1} \subset L^2(B_{l-1}) \) by
\[
F(K) S = f(\sigma(m), \sigma(l)) (S) \in B_{l-1} \subset L^2(B_{l-1})
\]
for \( S \in B_{m-1}^n \). In order to do that, we choose a standard pair \((R_{\sigma(m)}, \bar{R}_{\sigma(m)})\) of conjugation operators for \( \sigma(m) \) and \( \bar{\sigma}(m) \), \( m \in \mathbb{N} \cup \{0\} \), in the following way: we fix a standard pair \((R_\sigma, \bar{R}_\sigma)\) for \( \sigma \) and \( \bar{\sigma} \) and put \( R_{\sigma(0)} := \bar{R}_{\sigma(0)} := 1 \), \( R_{\sigma(0)} := \bar{R}_\sigma \) and \( R_{\sigma(0)} := \bar{R}_\sigma \). For \( m > 1 \) we define \((R_{\sigma(m)}, \bar{R}_{\sigma(m)})\) by several times applying Equation (3), in which we use \((R_\sigma, \bar{R}_\sigma)\) as a standard pair of conjugation operators for \( \bar{\sigma} \) and \( \sigma \).

By the Property (f) from Part (1), \( F(K) \) is well defined. According to the rules in (1),
\[
L \in ((\sigma \bar{\sigma})^m, (\sigma \bar{\sigma})^m) \rightarrow f(\sigma(n), \sigma(m), \sigma(m)) (L) \in (\sigma(n) \sigma(m), \bar{\sigma}(n) \sigma(m))
\]
is a representation of the finite dimensional \( C^\ast \)-algebra \((\sigma \bar{\sigma})^m, (\sigma \bar{\sigma})^m\). It follows \( \| f(\sigma(n)) (L) \| \leq \| L \| \) for every \( L \in ((\sigma \bar{\sigma})^m, (\sigma \bar{\sigma})^m) \). So
\[
\| f(\sigma(n)) (K) \|^2 = \| f(\sigma(n)) (K^*) f(\sigma(n)) (K) \| = \| f(\sigma(n)) (K^* K) \| \leq \| K^* K \| = \| K \|^2
\]
and \( F(K) \) is continuous on \( B_{m-1}^\infty \). Property (a) for \( f(\rho) \) in (1) shows that \( F(K) \) is \((A^\infty, A^\infty)\)-linear. As \( B_{m-1}^\infty \) is dense in \( L^2(B_{m-1}) \), \( F(K) \) has a unique extension to a continuous \((A, A)\)-linear map from \( L^2(B_{m-1}) \) onto \( L^2(B_{l-1}) \). We denote this extension by \( F(K) \) again.
In order to note the properties of the assignment $K \mapsto F(K)$ we have to introduce unitary $(A, A)$-linear operators

$$W_{m,l} : L^2(B_{m-1}) \otimes_A L^2(B_{l-1}) \longrightarrow L^2(B_{m+l-1}), \ l, m \in \mathbb{N} \cup \{0\}.$$ 

According to [25], there is a unique $(A, A)$-linear unitary operator $U_k : L^2(B) \otimes_A \mathbb{C}^k \longrightarrow L^2(B_{k-1})$ for $k \in \mathbb{N}$ such that

$$U_k(x_1 \otimes_A \cdots \otimes_A x_k) =$$

$$= d(\sigma)^{(k-1)/2} x_1 f_0 x_2 f_1, x_3 f_2, x_4 \cdots x_{k-1} f_{k-2} x_k$$

for $x_1, \ldots, x_k \in B$. The abbreviation $f_{r,s}, \ r, s \in \mathbb{N} \cup \{0\}$, is defined by

$$f_{r,s} := \begin{cases} f_r \cdot f_{r+1} \cdot \ldots \cdot f_{s-1} \cdot f_s & \text{for } r < s, \\ f_r & \text{for } r = s, \\ f_r \cdot f_{r-1} \cdot \ldots \cdot f_{s+1} \cdot f_s & \text{for } r > s. \end{cases}$$

Now we put

$$W_{m,l} := \begin{cases} U_{m+l} \circ (U_m^{-1} \otimes_A U_l^{-1}) & \text{for } m, l \in \mathbb{N}, \\ r_{L^2(B_{m-1})} & \text{for } l = 0, \\ l_{L^2(B_{l-1})} & \text{for } m = 0, \end{cases}$$

where $l_{L^2(B_{l-1})}$ (resp. $r_{L^2(B_{l-1})}$) is the canonical unitary operator from $L^2(B_{l-1}) \otimes_A L^2(A)$ (resp. $L^2(A) \otimes_A L^2(B_{l-1})$) onto $L^2(B_{l-1})$. Especially we have

$$W_{m,l} \cdot x_1 f_0 x_2 \cdots f_{m-2} x_m \otimes_A y_1 f_0 y_2 \cdots f_{l-2} y_l =$$

$$d(\sigma)^{ml} x_1 f_0 \cdots f_{m-2} x_m f_{m-1,0} y_1 f_{m,0} \cdots f_{m+l-2} y_l =$$

$$d(\sigma)^{ml} x_1 f_{0,m+l-2} \cdots x_m f_{l-1,0} y_1 f_{l,0} \cdots f_{0} y_l$$

for $x_1, \ldots, x_m, y_1, \ldots, y_l \in B$ and $m, l \in \mathbb{N}$. The maps

$$F = F_{m,l} : \left( (\sigma \bar{\sigma})^m, (\sigma \bar{\sigma})^l \right) \longrightarrow \mathcal{L}_{A,A} \left( L^2(B_{m-1}), L^2(B_{l-1}) \right),$$

$$K \mapsto F(K),$$

satisfy the following relations:

(a) The map $F_{m,l}$ is linear and bijective for $m, l \in \mathbb{N} \cup \{0\}$.

(b) $F(L \circ K) = F(L) \circ F(K)$ for every $K \in ((\sigma \bar{\sigma})^m, (\sigma \bar{\sigma})^l)$ and $L \in (\sigma \bar{\sigma})^k, (\bar{\sigma} \sigma)^k) (m, l, k \in \mathbb{N} \cup \{0\}).$

(c) $F(K)^* = F(K^*)$ for every $K \in ((\sigma \bar{\sigma})^m, (\sigma \bar{\sigma})^l)$ (m, l $\in \mathbb{N} \cup \{0\}$).

(d) $F(1_{(\sigma \bar{\sigma})^m}) = id_{L^2(B_{m-1})}$ for $m \in \mathbb{N} \cup \{0\}$. 

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(e) \( F(K \times L) = W_{l,r} \circ (F(K) \otimes_A F(L)) \circ W^*_{m,s} \) for every \( K \in ((\sigma \bar{\sigma})^m, (\sigma \bar{\sigma})^l) \) and \( L \in ((\sigma \bar{\sigma})^s, (\sigma \bar{\sigma})^r) \) \((m, l, r, s \in \mathbb{N} \cup \{0\})\).

Property (e) and the surjectivity of the map \( F_{m,l} \) will be shown in Step (3) of the proof. The other properties are obvious or immediate consequences of the Properties (a) - (f) from Step (1).

Let \( \mathcal{D}_{\sigma \bar{\sigma}} \) be the finite \( C^* \)-tensor category introduced in Observation 4.4. Using Property (b) we find that there is a functor \( \mathcal{G} : \mathcal{D}_{\sigma \bar{\sigma}} \rightarrow \mathcal{B}_{ABC} \) such that

\[
\mathcal{G}((\sigma \bar{\sigma})^m, P)) = F(P) L^2(B_{m-1}) \quad \text{and} \quad \mathcal{G}(K) = F(K) \quad \text{for } K \in \left( (\sigma \bar{\sigma})^m, ((\sigma \bar{\sigma})^l, Q) \right),
\]

where \( \left( (\sigma \bar{\sigma})^m, P), ((\sigma \bar{\sigma})^l, Q) \right) \) is regarded as a subspace of the morphism space \( ((\sigma \bar{\sigma})^m, (\sigma \bar{\sigma})^l) \) in the category \( \mathcal{C} \). For objects \( \phi = ((\sigma \bar{\sigma})^m, P) \) and \( \psi = ((\sigma \bar{\sigma})^l, Q) \) in \( \mathcal{D}_{\sigma \bar{\sigma}} \), let \( U_{\phi \psi} \) be the restriction of \( W_{m,l} \) to a unitary operator from \( F(P) L^2(B_{m-1}) \otimes_A F(Q) L^2(B_{l-1}) \) onto \( F(P \times Q) L^2(B_{m+l-1}) \). (Observe that \( F(P \times Q) = W_{m,l} \circ (F(P) \otimes_A F(Q)) \circ W^*_{m,l} \) holds according to Property (e).)

Now we conclude that \( (\mathcal{G}, (U_{\phi \psi}, \phi, \psi, id_{L^2(A)})) \) is a \( C^* \)-tensor equivalence. \( \mathcal{G} \) is a \( C^* \)-functor according to the Properties (a) and (c). Definition 4.4 (ii) is satisfied, Part (a) follows from Property (e), Part (c) is obvious, and Part (b) is an easy consequence of the fact that

\[
W_{m+l,k} \circ (W_{m,l} \otimes_A id_{L^2(B_{k-1})}) = W_{m,k+l} \circ (id_{L^2(B_{m-1})} \otimes_A W_{l,k})
\]

is satisfied for \( k, l, m \in \mathbb{N} \cup \{0\} \).

From Property (a) we conclude that Definition 4.4 (iii) is satisfied. Using the results from Observation 4.4 and Proposition 4.6 we are able to replace \( \mathcal{D}_{\sigma \bar{\sigma}} \) by the equivalent \( C^* \)-tensor category \( \mathcal{C}_{\sigma \bar{\sigma}} \) and get the assertion of the Theorem.

(3) We will prove Property (e) and the surjectivity in Property (a) from Step (2). In order to facilitate some computations we develop a special notation: We fix an object \( \rho \) of \( \mathcal{C} \) and define the operators

\[
R_{[\rho]}^{[m]} := \begin{cases} 
1_{\rho \sigma(l)} \times \bar{R}_\sigma \times 1_{\sigma(l-1)} & \text{if } l \text{ is even}, \\
1_{\rho \sigma(l)} \times R_\sigma \times 1_{\sigma(m-l-1)} & \text{if } l \text{ is odd and } m \geq l + 1, \\
1_{\rho \sigma(l)} \times R_\sigma & \text{if } l \text{ is odd and } m = l 
\end{cases}
\]

(belonging to \( (\rho \sigma(m), \rho \sigma(m + 2)) \)) for \( m \geq l \geq 0 \),

\[
R_{[\rho]}^{[m]} := \begin{cases} 
1_{\rho \sigma(l)} \times \bar{R}_\sigma^* \times 1_{\sigma(l-2)} & \text{if } l \text{ is even}, \\
1_{\rho \sigma(l)} \times R_\sigma^* \times 1_{\sigma(m-l-3)} & \text{if } l \text{ is odd and } m \geq l + 3, \\
1_{\rho \sigma(l)} \times R_\sigma^* & \text{if } l \text{ is odd and } m = l + 2
\end{cases}
\]
(belonging to \((\rho \sigma(m), \rho \sigma(m - 2))\)) for \(m \geq l + 2\) and \(l \in \mathbb{N} \cup \{0\},\)

\[
S^{[m]} := \begin{cases} 
S \times 1_{\sigma(m-l)} & \text{if } l \text{ is even}, \\
S \times 1_{\delta\sigma(m-l-1)} & \text{if } l \text{ is odd and } m \geq l + 1, \\
S & \text{if } l \text{ is odd and } m = l
\end{cases}
\]

for any operator \(S \in (\rho \sigma(l), \rho \sigma(l))\) and \(m \geq l \geq 0\) and

\[
K^{[m]}_{[\rho]l} := 1_{\rho\sigma(l)} \times K \times 1_{\sigma(m-l-2r)} \in (\rho \sigma(m), \rho \sigma(m - r + s))
\]

for \(K \in ((\sigma\bar{\sigma})^r, (\sigma\bar{\sigma})^s), l\) even and \(m \geq l + 2r\). The indices put in brackets \([\ ]\) are usually omitted if the meaning is clear from the context. For instance we have

\[
1_{\rho} \times \bar{R}_{\sigma(m)} = R_{m-1}^{2m-2} \circ \ldots \circ R_1^2 \circ R_0^0 = R_{m-1} \ldots R_1 R_0^0
\]

(we often omit the sign \(\circ\)) as well as

\[
f(\rho)(K)(S) = d(\sigma)^{-\frac{l}{2}} R_i^{l+2} R_1^{l+4} \ldots R_2^{l+2m} S_{l+2m} R_{m-1}^{l+2m-2} \ldots R_1^{l+2} R_0^0
\]

for \(K \in ((\sigma\bar{\sigma})^m, (\sigma\bar{\sigma})^l)\) and \(S \in (\rho \sigma(m), \rho \sigma(m))\) \((m, l \in \mathbb{N})\).

(a) We prove Property (e) from Step (2) for \(m = l = r = s = 1\). For \(n \in \mathbb{N} \cup \{0\}, \rho = \sigma(n)\) and \(S, T \in (\rho \sigma, \rho \sigma) = B^{n+1}\) we compute

\[
f(\rho)(K \times L)(S f_0 T) = \frac{1}{d(\sigma)} R_i^4 R_3^6 R_6^5 K_0^6 S^6 R_0^4 R_0^6 T^6 R_1^4 R_0^2 = \frac{1}{d(\sigma)} R_i^4 R_3^2 R_2^1 R_2^1 K_0^1 S R_0^1 R_1^1 T R_0^2 = \frac{1}{d(\sigma)} R_i^4 R_3^2 R_2^1 R_2^1 K_0^1 S R_2^1 R_0^1 T R_0^2 = \frac{1}{d(\sigma)} R_i^4 R_3^2 R_2^1 R_2^1 K_0^1 S R_0^1 R_0^1 T R_0^2 = \frac{1}{d(\sigma)} R_i^4 R_3^2 R_2^1 R_2^1 K_0^1 S R_0^1 R_0^1 T R_0^2 = (\text{with } C := f(\rho)(K)(S))
\]

\[
\frac{1}{d(\sigma)} R_i^4 R_3^2 R_2^1 L R_0^1 T R_0^2 = \frac{1}{d(\sigma)} R_i^4 R_3^2 R_2^1 L R_0^1 T R_0^2 = \frac{1}{d(\sigma)} R_i^4 R_3^2 L R_0^1 T R_0^2 = \frac{1}{d(\sigma)} C R_0^1 L R_0^1 T R_0^2 = (\text{with } D := f(\rho)(L)(T))
\]

\[
\frac{1}{d(\sigma)} C R_0^1 R_0^1 T R_0^2 = f(\rho)(K)(S) f_0 f(\rho)(L)(T).
\]

During the computation we used the rule

\[
R_i^{m+2} \circ R_i^m = 1_{\rho\sigma(m)}
\]

(42)
several times. (The case '-' follows from the defining relations (3) and (4) of the conjugation operators, for the case '+' we have to apply $\ast$ to the Equations (3) and (4)). Moreover, the interchange law from Definition 1.1 (ii) (a) was used very often. We present a detailed proof of the equation $R_2^1 R_0^2 = R_0^4 R_0^2$ (which has been used in line 5 of the preceding computation) as an example:

$$R_2^1 R_0^2 = \mathbf{1}_\rho \times \mathbf{1}_{\sigma \rho} \times \bar{R}_\sigma \circ \mathbf{1}_\rho \times \bar{R}_\sigma \times \mathbf{1}_\epsilon =$$

$$\mathbf{1}_\rho \times \bar{R}_\sigma \times \bar{R}_\sigma = \mathbf{1}_\rho \times \bar{R}_\sigma \times \mathbf{1}_{\sigma \rho} \circ \mathbf{1}_\rho \times \mathbf{1}_\epsilon \times \bar{R}_\sigma = R_0^4 R_0^2.$$

We get

$$F(K \times L) W_{1,1} \overline{S} \otimes_A \overline{T} = d(\overline{\sigma(n)}) (K \times L)(Sf_0 T) =$$

$$d(\sigma(n)) (K)(S) f_0 f(\overline{\sigma(n)}) (L)(T) = W_{1,1} F(K)(S) \otimes_A F(L)(T)$$

for $S, T \in B^{n+1} \subset L^2(B) (n \in \mathbb{N})$, and (e) is established for $m = l = s = r = 1$.

(b) If we replace $\sigma$ by $\sigma(m)$ for $m > 0$ in (a), we obtain

$$f(\rho)(K \times L)(Sg_m T) = f(\rho)(K)(S) g_m f(\rho)(L)(T) \quad (43)$$

for $K, L \in (\sigma \overline{\sigma})^m, (\sigma \overline{\sigma})^m)$, $S, T \in (\rho \sigma(m), \rho \sigma(m))$ ($\rho = \overline{\sigma(n)}$) and

$$g_m := \frac{1}{d(\sigma(m))} \mathbf{1}_\rho \times (\bar{R}_{\sigma(m)} \circ \bar{R}_{\sigma(m)^2}) \in (\rho(\sigma \overline{\sigma})^m, \rho(\sigma \overline{\sigma})^m) = B_{2m-1}^{n+1} \subset B_{2m-1}.$$

We get Property (e) from Step (2) for $m = l = s = r \in \mathbb{N}$ as before if we are able to prove

$$W_{m,m} \overline{S} \otimes_A \overline{T} = d(\overline{\sigma(m)}) \overline{S} \overline{g}_m \overline{T} \quad \text{for} \ S, T \in B_{m-1}. \quad (44)$$

First we will show

$$g_m = d(\sigma)^{(m-1)m} f_{m-1,0} \cdot f_{m,1} \cdot \ldots \cdot f_{2m-2,m-1}. \quad (45)$$

We have

$$f_{m+k-1,k} = \frac{1}{d(\sigma)^m} R_{m+k-1}^{2m-2} R_{m+k-1}^* R_{m+k-2} \circ \ldots \circ R_{k+1}^* R_k R_k^* R_{2m}^* =$$

$$\frac{1}{d(\sigma)^m} R_{m+k-1}^{2m-2} R_k^* R_{2m}^* \quad (46)$$

By induction on $k$ we will prove

$$R_{m-1}^{2m-2} \circ \ldots \circ R_{m-k}^* R_{m-k-1}^* R_0^* R_1^* \circ \ldots \circ R_k^* R_{2m}^* =$$

$$R_{m-1}^{2m-2} R_0^* R_m R_1^* R_{m+1}^* R_2^* \circ \ldots \circ R_{m+k-1}^* R_k^* \quad (47)$$
for $k = 0, \ldots, m - 1$. The case $k = m - 1$ shows Equation (45), as an application of Equation (46) yields.

The case $k = 0$ in Equation (47) is obvious, the following computation shows the implication $k \to k + 1$, $(k < m - 1)$:

\[
R^{2m-2}_{m-1} R_0^* \circ \cdots \circ R_{m+k-1} R_k^* R_{m+k} R_{k+1}^{2m} = \quad \text{(by induction hypothesis)}
\]
\[
R^{2m-2}_{m-1} \circ \cdots \circ R_{m-k} R_{m-k-1} R_0^* R_1^* \circ \cdots \circ R_k^* R_{m+k} R_{k+1}^{2m} =
\]
\[
R^{2m-2}_{m-1} \circ \cdots \circ R_{m-k-1} R_{m-k-2} R_0^* \circ \cdots \circ R_k^* R_{k+1}^{2m}.
\]

It suffices to verify Equation (43) for

\[
T = y_1 f_0 y_2 \cdots y_{m-1} f_{m-2,0} y_m,
\]

where $y_1, y_2, \ldots, y_m \in B$. The following computation deals with this case:

\[
W_{m,m} \overline{S} \otimes_A \overline{T} =
\]
\[
d(\sigma)^m S f_{m-1,0} y_1 f_{m,0} y_2 \cdots y_{m-1} f_{m-2,0} y_m =
\]
\[
d(\sigma)^m S f_{m-1,0} f_{m,1,0} y_2 f_{m+1,0} y_3 \cdots y_{m-1} f_{m-2,0} y_m =
\]
\[
\ldots =
\]
\[
d(\sigma)^m S f_{m-1,0} f_{m,1} \cdots f_{m-2,0} f_{m-1,0} y_2 f_{m+1,0} y_3 \cdots y_{m-1} f_{m-2,0} y_m =
\]
\[
d(\sigma)^m S g_m T.
\]

(c) For $S \in B_{m-1}$ we prove

\[
F_{0,1}(\overline{R}_\sigma) \overline{S} = d(\sigma)^{1/2} \overline{S} \in L^2(B) \quad \text{for } m = 0, \quad (48)
\]
\[
F_{m,m+1}(1_{(\sigma)^m} \times \overline{R}_\sigma) \overline{S} = d(\sigma)^m \overline{S f_{m-1,0}} \in L^2(B_m) \quad \text{for } m \geq 1 \quad (49)
\]

and

\[
F_{m,m+1}(\overline{R}_\sigma \times 1_{(\sigma)^m}) \overline{S} = d(\sigma)^m \overline{f_{0,m-1} S} \in L^2(B_m) \quad \text{for } m \geq 1. \quad (50)
\]

The first equation is a consequence of

\[
f(\overline{\sigma(n)})(\overline{R}_\sigma)(S) = d(\sigma)^{1/2} R_1^* R_0^* S^1 = d(\sigma)^{1/2} S^1
\]

for $S \in (\overline{\sigma(n)}, \overline{\sigma(n)}) = A^{n+1}$ $(n \in \mathbb{N} \cup \{0\})$. Intending to derive Equation (49) we state

\[
f(\rho)(1_{(\sigma)^m} \times \overline{R}_\sigma)(S) =
\]
\[
d(\sigma)^{1/2} R_{m+1}^{m+3} \circ \cdots \circ R_{2m}^{3m+1} R_{2m+1}^{3m+3} R_{2m+2}^{3m+1} S^{3m+1} R_{m-1}^{3m+1} \circ \cdots \circ R_0^{m+1}
\]

for $\rho = \overline{\sigma(n)}$ and $S \in B_{m-1}^{n+1}$. Using $R_{2m+1}^{3m+3} R_{2m+2}^{3m+1} = 1_{\rho \sigma(3m+1)}$ and shifting $R_{3m+1}^{2m}$ to the right, we get

\[
f(\rho)(1_{(\sigma)^m} \times \overline{R}_\sigma)(S) =
\]
\[
d(\sigma)^{1/2} R_{m+1}^{m+3} \circ \cdots \circ R_{2m-1}^{3m-1} S^{3m+1} R_{m-1}^{3m+1} \circ \cdots \circ R_1^{m+1} R_0^{m-1} R_0^{m+1}.
\]

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Next we shift $R_{2m-1}^{3m+1}$ to the right and so on such that
\[
f(\rho)(1_{(\sigma\sigma)^m} \times \tilde{R}_{\sigma})(S) = \]
\[
d(\sigma)^{1/2} R_{m+1}^{m+3} \circ \ldots \circ R_{2m-2}^{m} S R_{m-1} \circ \ldots \circ R_2 R_1 R_0 R_0^{m+1} = \]
\[
\ldots = \ldots = \]
\[
d(\sigma)^{1/2} S^{m+1} R_{m-1}^* R_0^* R_0^{m+1} = d(\sigma)^{m+1/2} S f_{m-1,0}
\]
follows. So Equation (49) has been shown.

We use the following computation in order to prove Equation (50):
\[
f(\rho)(\tilde{R}_{\sigma} \times 1_{(\sigma\sigma)^m})(S) = d(\sigma)^{1/2} R_{m+1}^{m+3} \circ \ldots \circ R_{2m-2}^{m} S R_{m-1} \circ \ldots \circ R_2 R_1 R_0 R_0^{m+1} = \]
\[
d(\sigma)^{1/2} R_{m-1}^* R_0^* \ldots \circ R_{2m-3}^* S R_{m-1} \circ \ldots \circ R_2 R_1 R_1 R_0 R_0^{m+1} = \]
\[
\ldots = \ldots = \]
\[
d(\sigma)^{1/2} R_{m-1}^* S R_{m-1}^* \ldots \circ R_2 R_1 R_1 R_1 R_0^{m+1} = \]
\[
d(\sigma)^{1/2} R_{m-1}^* S R_{m-1}^* \ldots \circ R_2 R_1 R_1 R_0^{m+1} = \]
\[
d(\sigma)^{m+1/2} f_{0,m-1} S.
\]
In line 2 we shifted $R_{0}^{3m+1}$ to the left, in line 3 - 5 the computation is similar to that of Equation (49), in line 7 we used Equation (42) several times.

(d) Now we are able to verify Property (e) in Step (2) for any arbitrary $m, l, s, r \in \mathbb{N} \cup \{0\}$. Let $M$ be a natural number satisfying $M \geq \max\{2, m, l, s, r\}$. For an integer $\nu > 0$ let $\tilde{R}_{\sigma}^\times \nu$ denote the $\nu$-fold product $\tilde{R}_{\sigma} \times \ldots \times \tilde{R}_{\sigma}$ and let $\tilde{R}_{\sigma}^\times 0 := 1_r$.

\[
K := (\tilde{R}_{\sigma}^\times (M-l) \times 1_{(\sigma\sigma)^m}) \circ K \circ (\tilde{R}_{\sigma}^\times (M-m) \times 1_{(\sigma\sigma)^m})^* \quad \text{and} \quad
\]
\[
\tilde{L} := (1_{(\sigma\sigma)^m} \times \tilde{R}_{\sigma}^\times (M-r)) \circ L \circ (1_{(\sigma\sigma)^m} \times \tilde{R}_{\sigma}^\times (M-s))^* \]

are operators of $((\sigma\sigma)^M, (\sigma\sigma)^M)$ such that the relation
\[
F(\tilde{K} \times \tilde{L}) = W_{M,M} \circ (F(\tilde{K}) \otimes_A F(\tilde{L})) \circ W_{M,M}^\ast
\]
is satisfied. Using $\tilde{R}_{\sigma}^\ast \circ \tilde{R}_{\sigma} = d(\sigma)1_r$ we get
\[
K = \frac{1}{d(\sigma)_{2M-m-l}} \left(\tilde{R}_{\sigma}^\times (M-l) \times 1_{(\sigma\sigma)^m}\right)^* \circ \tilde{K} \circ \left(\tilde{R}_{\sigma}^\times (M-m) \times 1_{(\sigma\sigma)^m}\right)
\]
and
\[
L = \frac{1}{d(\sigma)_{2M-s-r}} \left(1_{(\sigma\sigma)^m} \times \tilde{R}_{\sigma}^\times (M-r)\right)^* \circ \tilde{L} \circ \left(1_{(\sigma\sigma)^m} \times \tilde{R}_{\sigma}^\times (M-s)\right).
\]

We intend to show
\[
F(\tilde{R}_{\sigma}^\times (M-m) \times 1_{(\sigma\sigma)^m+s}) = W_{M,s} \circ \left(F(\tilde{R}_{\sigma}^\times (M-m) \times 1_{(\sigma\sigma)^m}) \otimes_A id_{L^2(B_{s-1})}\right) \circ W_{m,s}^\ast.
\]

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The cases \( s = 0 \) and \( M = m \) are obvious, the cases \( s \geq 1 \) and \( 0 < m < M \) follow from the following computation for \( T \in B_{s-1} \) and 
\[ S = x_1f_{0,m-2}x_2 \cdots x_{m-1}f_0x_m \in B_{m-1} \quad (x_1, \ldots, x_m \in B) \]:
\[ F(R^{(M-m)}_\sigma \times 1_{(\sigma;\tau)}^{m+s}) \circ W_{m,s} \otimes_A T = \]
\[ d(\sigma)^{Ms + s^2} \cdot f_0, M+s-2 \cdot f_0, M+s-1 x_1 f_0, M+s-2 x_2 \cdots x_{m-1} f_0 s x_m f_0, s-1 T = \]
\[ d(\sigma)^{s^2} W_{m,s} F(R^{(M-m)}_\sigma \times 1_{(\sigma;\tau)}^m) S \otimes_A T. \]

The case \( s \geq 1 \) and \( m = 0 < M \) requires a separate computation which we leave to the reader. If we make use of Formula (\ref{eq12}) instead of Formula (\ref{eq14}) a computation similar as above yields
\[ F(1_{(\sigma;\tau)}^{M+s} \times R^{(M-s)}_\sigma) = \]
\[ W_{M,M} \circ (id_{L^2(B_{M-1})} \otimes_A F(1_{(\sigma;\tau)}^s \times R^{(M-s)}_\sigma)) \circ W_{M,s}^* \quad (53) \]

By applying the Relations (\ref{eq12}), (\ref{eq13}), (\ref{eq14}), (\ref{eq11}) and (\ref{eq53}) we get
\[ F(K \times L) = \]
\[ \frac{1}{d(\sigma)^{4M-(m+l+s+r)}} F(R^{(M-l)}_\sigma \times 1_{(\sigma;\tau)}^{l+r})^* \circ F(1_{(\sigma;\tau)}^{M+r} \times R^{(M-r)}_\sigma)^* \]
\[ F(\tilde{K} \times \tilde{L}) \circ F(1_{(\sigma;\tau)}^{M+s} \times \tilde{R}^{(M-s)}_\sigma) \circ F(\tilde{R}^{(M-m)}_\sigma \times 1_{(\sigma;\tau)}^{m+s}) = \]
\[ \frac{1}{d(\sigma)^{4M-(m+l+s+r)}} W_{l,r} \circ (F(\tilde{R}^{(M-l)}_\sigma \times 1_{(\sigma;\tau)}^r)^* \otimes_A id_{L^2(B_{-1})}) \]
\[ (id_{L^2(B_{M-1})} \otimes_A F(1_{(\sigma;\tau)}^r \times \tilde{R}^{(M-r)}_\sigma)^*) \]
\[ (F(\tilde{K}) \otimes_A F(\tilde{L})) \circ (id_{L^2(B_{M-1})} \otimes_A F(1_{(\sigma;\tau)}^s \times \tilde{R}^{(M-s)}_\sigma)) \circ \]
\[ (F(\tilde{R}^{(M-m)}_\sigma \times 1_{(\sigma;\tau)}^m) \otimes_A id_{L^2(B_{-1})}) \circ W_{m,s}^* \quad W_{l,r} \circ (F(K) \otimes_A F(L)) \circ W_{m,s}^*. \]

(e) It remains to show that the linear maps
\[ F_{m,l} : ((\sigma;\tau)^m, (\sigma;\tau)^l) \rightarrow \mathcal{L}_{A,A}(L^2(B_{m-1}), L^2(B_{l-1})) \]
are surjective for \( m, l \in \mathbb{N} \cup \{0\} \). It is well known that there is a canonical linear isomorphism from \( A' \cap B_{2m-1} \) onto \( \mathcal{L}_{A,A}(L^2(B_{m-1}), L^2(B_{m-1})) \) for \( m \in \mathbb{N} \) (see \cite{28} for example). Furthermore, Theorem \ref{thm3.4} (iv) shows \( A' \cap B_{2m-1} \cong ((\sigma;\tau)^m, (\sigma;\tau)^m) \) for \( m \in \mathbb{N} \cup \{0\} \), and \( F_{m,m} \) is surjective.

We regard the case \( l < m \); consider an operator
\[ X \in \mathcal{L}_{A,A}(L^2(B_{m-1}), L^2(B_{l-1})). \] Since \( F_{m,m} \) is surjective, there is an operator \( K \in ((\sigma;\tau)^m, (\sigma;\tau)^m) \) such that
\[ F_{m,m}(K) = F_{l,m}(1_{(\sigma;\tau)}^l \times \tilde{R}^{(m-l)}_\sigma) \circ X. \]
Using $\bar{R}_\sigma \ast \bar{R}_\sigma = d(\sigma)1$, we get
\[
F_{m,l}(\frac{1}{d(\sigma)^{m-l}}(1_{(\sigma\bar{\sigma})^l} \times \bar{R}_\sigma^{(m-l)}(\sigma)) \circ K) = \\
F_{l,l}(\frac{1}{d(\sigma)^{m-l}}(1_{(\sigma\bar{\sigma})^l} \times (\bar{R}_\sigma^{(m-l)}(\sigma)) \circ X = X.
\]

At last we reduce the case $l > m$ to the case $l < m$ by applying the $\ast$-operation. 

\section{Finite dimensional Hopf-$\ast$-algebras as examples}

We consider the case that $\mathcal{C}$ is the $C^\ast$-tensor category $\mathcal{U}_H$, where $H$ is a finite dimensional Hopf-$\ast$-algebra. For the sake of simplicity we assume $\mathcal{C}_{\sigma\bar{\sigma}} = \mathcal{C}$. That means that there is an $n \in \mathbb{N}$ such that every irreducible corepresentation of $H$ is contained in $(\sigma\bar{\sigma})^n$.

Theorem 1.1 tells us that $\mathcal{B}_{A \subset B}$ is equivalent to $\mathcal{U}_H$.

On the other hand, the considerations in Section 3 imply that $A \subset B$ is equal to an inclusion $N^K \subset (N \otimes L(V))^K$ of fixed point algebras, where the action $\alpha$ of $K := (H^\alpha)^{\text{cop}}$ on $N$ is outer. There is an outer action of the dual Hopf-$\ast$-algebra $K^\alpha$ on $M := N^K$ such that the subfactor $N^K \subset N$ is isomorphic to $M \subset M \rtimes K^\alpha$. One easily sees that $\mathcal{B}_{A \subset B} = \mathcal{B}_{N^K \subset N \otimes L(V)} = \mathcal{B}_{N^K \subset N}$. Hence Proposition 2.3 yields that $\mathcal{B}_{N^K \subset N}$ is equivalent to $\mathcal{U}_{(K^\alpha)^{\text{cop}}}$. Clearly,
\[
(K^\alpha)^{\text{cop}} = ((H^\alpha)^{\text{cop}})^{\text{cop}} = ((H^\alpha)^{\text{op}})^{\text{cop}} = H^{\text{op} \text{cop}}
\]
holds. The Hopf-$\ast$-algebras $H$ and $H^{\text{op} \text{cop}}$ are isomorphic, as an application of the antipode $S$ of $H$ shows. So we conclude that $\mathcal{B}_{A \subset B}$ is equivalent to a full $C^\ast$-tensor subcategory of $\mathcal{U}_H$, and Theorem 1.1 has been confirmed for this special case.

\section{Appendix: Proof of Proposition 1.5 (ii)}

We will proceed similarly as for the equivalences of arbitrary categories (for example see [3], Proposition XI.1.5), but more details have to be verified.

Let an equivalence $(F,(U_{\rho\sigma}),J)$ of the $C^\ast$-tensor categories $\mathcal{C}$ and $\mathcal{D}$ be given. For each object $\phi \in \text{Ob } \mathcal{D}$ we choose an object $G(\phi)$ of $\mathcal{C}$ such that $F(G(\phi))$ is equivalent to $\phi$ and a unitary operator $W_{\phi} \in (\phi, F(G(\phi)))$. Especially we put
\[
G(\iota_\mathcal{D}) := \iota_\mathcal{C} \quad \text{and} \quad W_{\iota_\mathcal{D}} := J^* \in (\iota_\mathcal{D}, F(\iota_\mathcal{C})). \quad (56)
\]
The linear map $F_{\rho,\sigma} : T \in (\rho, \sigma) \mapsto F(T) \in (F(\rho), F(\sigma))$ is invertible for all objects $\rho$ and $\sigma$. For $S \in (\phi, \psi)$ ($\phi, \psi \in \text{Ob } \mathcal{D}$) we define
\[
G(S) := F_{G(\phi), G(\psi)}^{-1}(W_\phi \circ S \circ W_\psi^*) \in (G(\phi), G(\psi)). \quad (57)
\]
One easily checks that a full and faithful $C^*$-functor $G : \mathcal{D} \to \mathcal{C}$ is defined in this way.

We note that Definition (57) implies

$$F(G(S)) \circ W_\phi = W_\psi \circ S \quad \text{for } S \in (\phi, \psi). \quad (58)$$

In order to define a $C^*$-tensor equivalence we introduce unitary operators $V_{\phi\psi} : G(\phi) G(\psi) \to G(\phi \psi)$ by putting

$$V_{\phi\psi} := F^{-1}_{G(\phi) G(\psi), G(\phi \psi)} (W_{\phi\psi} \circ W_\phi^* \times W_\psi^* \circ U_{G(\phi), G(\psi)})$$

for all objects $\phi$ and $\psi$ of $\mathcal{D}$. We will prove that $(G, (V_{\phi\psi})_{\phi, \psi}, 1_C)$ is a $C^*$-tensor equivalence with $[G] = [F]^{-1}$. For this purpose we have to verify (a), (b) and (c) in Definition 1.4 (ii), the remaining assumptions are obviously satisfied.

(a) We consider the following diagram for objects $\phi, \phi', \psi$ and $\psi'$ of $\mathcal{D}$ and morphisms $R \in (\phi, \phi')$ and $S \in (\psi, \psi')$:

$$\begin{array}{cccc}
F(G(\phi) G(\psi)) & \to & F(G(R) \times G(S)) & \to & F(G(\phi') G(\psi')) \\
U^*_{G(\phi), G(\psi)} & \downarrow & (1) & \downarrow & U^*_{G(\phi'), G(\psi')} \\
F(G(\phi)) F(G(\psi)) & \to & F(G(R)) \times F(G(S)) & \to & F(G(\phi')) F(G(\psi')) \\
W^*_\phi \times W^*_\psi & \downarrow & (2) & \downarrow & W^*_\phi' \times W^*_\psi' \\
\phi \psi & \downarrow & R \times S & \downarrow & \phi' \psi' \\
W_{\phi\psi} & \downarrow & (3) & \downarrow & W_{\phi'\psi'} \\
F(G(\phi \psi)) & \to & F(G(R \times S)) & \to & F(G(\phi' \psi')) \\
\end{array}$$

Definition 1.4 (ii) (a) implies that diagram (1) commutes. Furthermore the diagrams (2) and (3) commute according to Relation (58). Hence the exterior diagram is commuting, and the application of $F^{-1}$ shows

$$G(R \times S) \circ V_{\phi \psi} = V_{\phi' \psi'} \circ G(R) \times G(S).$$

(b) We have to check

$$G(a(\phi, \psi, \eta)) \circ V_{\phi, \psi, \eta} \circ 1_{G(\phi)} \times V_{\psi, \eta} = V_{\phi \psi, \eta} \circ V_{\phi \psi} \times 1_{G(\eta)} \circ a(G(\phi), G(\psi), G(\eta)) \quad (59)$$
for all objects \( \phi, \psi, \eta \in \text{Ob } D \). We know

\[
F(1_{G(\phi)} \times V_{\psi \eta}) = U_{G(\phi),G(\psi \eta)} \circ 1_{F(G(\phi))} \times F(V_{\psi \eta}) \circ U^*_{G(\phi),G(\psi \eta)}.
\] (60)

By applying \( F \) on both sides of Equation (59) and using Equation (60) as well as the corresponding relation for \( F(V_{\psi \phi} \times 1_{G(\eta)}) \), we find that (59) is equivalent to the equation

\[
F(G(a(\phi, \psi, \eta))) \circ W_{\phi(\psi \eta)} \circ W^*_{\phi} \times W^*_{\psi} \circ U^*_{G(\phi),G(\psi \eta)} \circ 1_{F(G(\phi))} \times W_{\psi \eta} \circ 1_{F(G(\phi))} \times U^*_{G(\phi),G(\psi \eta)} \circ W_{\psi \eta} \circ U^*_{G(\phi),G(\psi \eta)} \circ 1_{F(G(\phi))} \times W_{\psi \eta} = W_{\phi(\psi \eta)} \circ W^*_{\phi} \times W^*_{\psi} \circ U^*_{G(\phi),G(\psi \eta)} \circ W_{\phi \psi} \times 1_{F(G(\eta))} \circ (W^*_{\phi} \times W^*_{\psi}) \times 1_{F(G(\eta))} \circ a(F(G(\phi)), F(G(\psi)), F(G(\eta))).
\] (61)

(58) implies

\[
F(G(a(\phi, \psi, \eta))) = W_{\phi(\psi \eta)} \circ a(\phi, \psi, \eta) \circ W^*_{\phi(\psi \eta)},
\]
moreover

\[
F(a(G(\phi), G(\psi), G(\eta))) \circ U_{G(\phi),G(\psi \eta)} \circ 1_{F(G(\phi))} \times U_{G(\psi),G(\eta)} =
U_{G(\phi),G(\psi),G(\eta)} \circ U_{G(\phi),G(\psi \eta)} \circ 1_{F(G(\eta))} \circ a(F(G(\phi)), F(G(\psi)), F(G(\eta))).
\]

Inserting these relations into (61) we see that Equation (59) is equivalent to

\[
a(\phi, \psi, \eta) \circ W^*_{\phi} \times W^*_{\psi} \circ 1_{F(G(\phi))} \times W_{\psi \eta} \circ 1_{F(G(\phi))} \times (W^*_{\psi} \times W^*_{\eta}) =
W^*_{\phi \psi} \times W^*_{\eta} \circ W_{\phi \psi} \circ 1_{F(G(\eta))} \circ (W^*_{\phi} \times W^*_{\psi}) \times 1_{F(G(\eta))} \circ a(F(G(\phi)), F(G(\psi)), F(G(\eta)))
\]

and to

\[
a(\phi, \psi, \eta) \circ W^*_{\phi} \times (W^*_{\psi} \times W^*_{\eta}) =
(W^*_{\phi} \times W^*_{\psi}) \times W^*_{\eta} \circ a(F(G(\phi)), F(G(\psi)), F(G(\eta))).
\]

The last relation is satisfied, because \( a(\phi, \psi, \eta) \) is natural in \( \phi, \psi \) and \( \eta \).

(c) By Definition (54) we have to establish the equation

\[
l_{G(\phi)} = G(l_{\phi}) \circ V_{cD,\phi}
\] (62)

for every object \( \phi \in \text{Ob } D \). Equation (62) is equivalent to

\[
F(l_{G(\phi)}) = F(G(l_{\phi})) \circ W_{cD,\phi} \circ J \times W^*_{\phi} \circ U^*_{cD,G(\phi)}.
\] (63)

From Definition 14 (ii) (c) we conclude

\[
F(l_{G(\phi)}) \circ U_{cD,G(\phi)} = l_{F(G(\phi))} \circ J \times 1_{F(G(\phi))}.
\]
By inserting this relation into (63) we obtain that Equation (63) is equivalent to
\[ l_{F(G(\phi))} \circ J \times 1_{F(G(\phi))} = F(G(l_\phi)) \circ W_{\iota_\mathcal{D} \phi} \circ J \times W_\phi^*. \] (64)

Since \( l_\phi \) is natural in \( \phi \),
\[ l_{F(G(\phi))} \circ 1_{\iota_\mathcal{D}} \times W_\phi = W_\phi \circ l_\phi \]
holds, and Equation (58) yields
\[ W_\phi \circ l_\phi = F(G(l_\phi)) \circ W_{\iota_\mathcal{D} \phi}. \]

By applying those two equations we see that Equation (64) is satisfied. ■

References

[1] M.-Cl. David, Paragroupe d’Adrian Ocneanu et Algebre de Kac, Pac. J. Math 172 (1996), 331 - 363.

[2] Y. Denizeau, J.F. Havet, Correspondances d’indice fini I: Indice d’un vecteur, J. Operator Theory 32 (1994), 111 - 156.

[3] Y. Denizeau, J.F. Havet, Correspondances d’indice fini II: Indice d’un correspondance, to appear in: Proceedings of OT15 conference.

[4] S. Doplicher, J.E. Roberts, A new duality theory for compact groups, Invent. Math. 98 (1989), 157 - 218.

[5] F. Goodman, P. de la Harpe, V.F.R. Jones, Coxeter graphs and towers of algebras, Springer, 1989.

[6] J.F. Havet, Espérance conditionelle minimale, J. Operator Theory 24 (1990), 33 - 55.

[7] F. Hiai, Minimizing indices of conditional expectations onto a subfactor, Publ. Res. Inst. Math. Sci. 27 (1988), 673 - 678.

[8] V.F.R. Jones, Index for subfactors, Invent. Math. 72 (1983), 1 - 25.

[9] Chr. Kassel, Quantum groups, Springer, New York, Heidelberg, Berlin, 1995.

[10] T.H. Koornwinder, General compact quantum groups, a tutorial, appears as part of Compact quantum groups and q-special functions, in: M. Picardello (ed.), Representations of Lie groups and Quantum groups, Congman, Harlow.

[11] R. Longo, Index of subfactors and statistics of quantum fields, I, Comm. Math. Phys. 126 (1989), 217 - 247.
[12] R. Longo, Minimal index and braided subfactors, J. Funct. Anal. 109 (1992), 98 - 112.

[13] R. Longo, A duality for Hopf algebras and for subfactors I, Comm. Math. Phys. 159 (1994), 133 - 150.

[14] R. Longo, J.E. Roberts, A theory of dimension, preprint, Università di Roma Tor Vergata, 1995.

[15] S. Mac Lane, Categories for the working mathematician, Springer, New York, Heidelberg and Berlin, 1971.

[16] S. Mac Lane, Natural associativity and commutativity, Rice Univ. Studies 49 (1963), 28 - 46.

[17] C. Peligrad, W. Szymański, Saturated actions of finite dimensional Hopf \( \ast \)-algebras on \( C^\ast \)-algebras, Math. Scand. 75 (1994), 217 - 239.

[18] M. Pimsner, S. Popa, Entropy and index for subfactors, Ann. Sci. École Norm. Sup. 19 (1986), 57 - 106.

[19] M. Pimsner, S. Popa, Iterating the basic construction, Trans. Amer. Math. Soc. 310 (1988), 127 - 133.

[20] S. Popa, Classification of subfactors: the reduction to commuting squares, Invent. Math. 101 (1990), 19 - 43.

[21] S. Popa, Classification of amenable subfactors of type II, Acta Math. 172 (1994), 352 - 445.

[22] J.E. Roberts, Lectures given at the Graduiertenkolleg of the Universität München, 1994.

[23] J.E. Roberts, The statistical dimension, conjugation and the Jones index, preprint.

[24] R. Schafitlzel, \( C^\ast \)-tensor categories in the theory of subfactors of \( \text{II}_1 \)-factors, Habilitationsschrift, Technische Universität München, 1996.

[25] R. Schafitlzel, A note on bimodules and \( \text{II}_1 \)-subfactors, preprint, 1996.

[26] M.E. Sweedler, Hopf algebras, W.A. Benjamin, Inc., New York, 1969.

[27] W. Szymański, Finite index subfactors and Hopf algebra crossed products, Proc. Amer. Math. Soc. 120 (1994), 519 - 528.

[28] W. Szymański, Actions of Hopf-\( \ast \)-algebras on \( C^\ast \)-algebras and von Neumann algebras, preprint.

[29] H. Wenzl, Hecke algebras of type \( A_n \) and subfactors, Invent. Math. 92 (1988), 349 - 383.
[30] H. Wenzl, *Quantum groups and subfactors of Lie type B, C, and D*, Comm. Math. Phys. 133 (1990), 383 - 433.

[31] S.L. Woronowicz, *Compact matrix pseudogroups*, Comm. Math. Phys. 111 (1987), 613 - 665.

[32] S. Yamagami, *A note on Ocneanu’s approach to Jones’ index theory*, Internat. J. Math. 4 (1993), 859 - 871.

[33] S. Yamagami, *Modular theory for bimodules*, preprint.