Solving fractional nonlinear partial differential equations by the modified Kudryashov method

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Abstract. There are more and more methods for transforming nonlinear partial differential equations into ordinary differential equations by using the traveling wave transform. In this paper, the modified Kudryashov method is used to use the new traveling wave transform, and the exact solution of the space-time fractional equal-width equation is obtained by means of symbolic computation. Search for solution of the equal-width equation becomes more direct and simple. It is also suitable for solving a large number of similar fractional-order nonlinear partial differential equations, the method provides a new idea for solving fractional-order nonlinear partial differential equations.

1. Introduction

Due to the fractional integral theory is only carried out in theory, its significance for real life is not obvious. With the extensive combination of this theory and practice, it is more and more used to deal with the problems in the rheology, material, mechanical systems, biological mathematics and other application fields. In particular, the fractional partial differential equations which were abstracted from practical problems have become the research focus of mathematics workers. It becomes very important to obtain the exact solution of the fractional partial differential equation in order to better understand and explain these practical problems. In recent years, there are many relatively methods for obtain the exact solutions of fractional partial differential equations. For example, the first integral method[1], \( (G'/G) \)-expansion method[2], the exp-function method[3], the Jacobi elliptic function expansion method[4], the homogenous balance method [5], the exponential rational function method[6], and others. In this paper, we mainly use the modified Kudryashov method [7] to solve the exact solution of fractional partial differential equation under the definition of modified Riemann-Liouville [8]. So as to prove the correctness and practicability of this method, we will apply it to the space-time fractional equal-width equation [9]. In the past few decades, many researchers have given different definitions of fractional derivatives, the most famous of which is the form of Riemann-Liouville fractional calculus [10]. It was proposed by Riemann. In this paper, the calculation is carried out under the more perfect definition of Riemann-Liouville derivative which was proposed by Jumarie.

The modified order Riemann-Liouville derivative is defined as follows:
The relevant properties that need to be used in the calculation are given as follow:

When \( 0 < \beta < 1 \), we have

\[
D_\beta^\gamma t = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\beta)} t^{\gamma-\beta}, \quad \gamma > 0 ,
\]

(2)

\[
D_\beta^\gamma (f(t)g(t)) = g(t)D_\beta^\gamma f(t) + f(t)D_\beta^\gamma g(t)
\]

(3)

\[
D_\beta^\gamma f[g(t)] = f'[g(t)]D_\beta^\gamma g(t) = D_\beta^\gamma f[g(t)]g'[t]^{\gamma}
\]

(4)

For given nonlinear FPDES

\[
P(u, u_x, u_{xx}, D_\beta^\gamma u, D_\beta^\gamma u, \ldots)
\]

(5)

Where \( D_\beta^\gamma u \) and \( D_\beta^\gamma u \) are the modified Riemann-Liouville derivative.

The main steps of the modified Kudryashov method are as follows:

Step 1 We introduce the following transformation:

\[
u(x,t) = u(\xi), \quad \xi = \frac{\alpha_1 x^\beta}{\Gamma(\beta+1)} + \frac{\alpha_2 t^\beta}{\Gamma(\beta+1)} + \xi_0 \]

(6)

Where \( \alpha_1 \) and \( \alpha_2 \) are arbitrary constants. By setting the transformation (1.6) into Eq. (1.5), we obtain

\[
G = (u, u_\xi, u_{\xi\xi}, u_{\xi\xi\xi}, \ldots)
\]

(7)

Step 2 We suppose that the exact solutions of the eq. (7) can be written as follows

\[
u(\xi) = \sum_{i=0}^N a_i Q^i(\xi)
\]

(8)

Where \( Q = \frac{1}{1 \pm a^\xi} \) and the function \( Q \) satisfies the equation

\[
Q_\xi = \ln a(Q^2 - Q)
\]

(9)

Step 3 According to the above content, we assume that the solution of Eq. (1.7) can be expressed in the form

\[
u(\xi) = a_0 Q^N + \ldots
\]

(10)

Above \( N \) is a nature number. It can be determined by the homogeneous balance principle.

Step 4 Substituting the Eq. (10) into the Eq. (7), a set of overdetermined equations can be obtained by making the coefficient of \( Q^N \) to zero. The exact solution of the Eq. (5) can be obtained by solving the equations with Mathematica.
2. Application of the modified Kudryashov method

Classical KDV equation:
\[ u_t + uu_x + u_{xxx} = 0 \]

It simulates the motion of linear one-dimensional water wave with time.

After that, Morrison [11] proposed one-dimensional partial differential equations:
\[ u_t + uu_x + \mu u_{xxx} = 0 \]

Which plays a major role in the study of nonlinear dispersive waves since it describes a broad class of physical phenomena such as shallow water waves and ion acoustic plasma waves.

Because of the generalization of fractional order theory, we mainly study the space-time fractional Equal-Width equation
\[ D_t^\alpha \Phi(x,t) + \mu_t D_x^\beta \Phi^2(x,t) - \mu_x D_x^{3\beta} \Phi(x,t) = 0 \]

Eq. (11) can be transformed into ordinary differential equation by means of complex transformation and integral.
\[ \alpha_2 \Omega + 2\mu_1 \alpha_1 \Omega^2 + \mu_2 \alpha_1^2 \Omega^2 = 0 \]

Balancing \( \Omega^2 \) and \( \Omega' \) in Eq(12), we compute
\[ N = 2 \]

so the form of the solution can be suppose as follows:
\[ \Omega(\xi) = a_0 + a_1 Q + a_2 Q^2 \]

Substituting Eq. (13) into Eq. (12) and equating the coefficients of \( Q^0 \) to zero, we obtain
\[ Q^0: \alpha_2 a_0 + 2a_1 a_2 Q^2 = 0 \]
\[ Q^1: 2a_0 (\ln a) a_1 + \mu_1 a_0 + \alpha_2 a_1 + 4 \mu_1 a_2 a_1 = 0 \]
\[ Q^2: -3a_1 (\ln a)^2 a_2 + 4a_1 (\ln a)^2 a_2 + 2\mu_1 a_2 a_1 + 4 \mu_1 a_2 a_2 = 0 \]
\[ Q^3: 2a_0 (\ln a)^3 a_2 - 10a_1 (\ln a)^2 a_2 + 4 \mu_1 a_2 a_2 = 0 \]
\[ Q^4: 2a_0 (\ln a)^4 a_2 + 4a_1 (\ln a)^4 a_2 + 2\mu_1 a_2 a_2 = 0 \]

Solving the resulting system of algebraic equations we obtain four group solutions.

Case 1:
\[ a_0 = 0, \quad a_1 = -\frac{3i a_2 \ln a \sqrt{\mu_2}}{\mu_1}, \quad a_2 = \frac{3i a_2 \ln a \sqrt{\mu_2}}{\mu_1}, \quad a_1 = -\frac{i}{\ln a \sqrt{\mu_2}} \]

Inserting Eq. (14) into Eq. (13), the following solution of Eq. (11) is obtained:
\[ u_1(x,t) = \frac{-3i\alpha_2 \ln \sqrt{\mu_2}}{\mu_1} \cdot \frac{1}{1 + 1 + a} + \frac{3i\alpha_2 \ln \sqrt{\mu_2}}{\mu_1} \cdot \left( \frac{1}{1 + a} \right)^2 \]

\[ u_2(x,t) = \frac{-3i\alpha_2 \ln \sqrt{\mu_2}}{\mu_1} \cdot \frac{1}{1 - a} + \frac{3i\alpha_2 \ln \sqrt{\mu_2}}{\mu_1} \cdot \left( \frac{1}{1 - a} \right)^2 \]

**Figure 1.** 3D Plot of \( u_1 \) by choosing \( a = 10, \ a_2 = \mu_1 = \frac{\zeta}{\alpha} = \mu_2 = \beta = 1 \)

Case 2:

\[ a_0 = 0, \ a_1 = \frac{3i\alpha_2 \ln \sqrt{\mu_2}}{\mu_1}, \ a_2 = -\frac{3i\alpha_2 \ln \sqrt{\mu_2}}{\mu_1}, \ a_1 = \frac{i}{\ln \sqrt{\mu_2}} \]

Substituting Eq. (15) into Eq. (13), we obtain the following solution of Eq. (11):

\[ u_3(x,t) = \frac{3i\alpha_2 \ln \sqrt{\mu_2}}{\mu_1} \cdot \frac{1}{1 + 1 + a} + \frac{3i\alpha_2 \ln \sqrt{\mu_2}}{\mu_1} \cdot \left( \frac{1}{1 + a} \right)^2 \]

\[ u_4(x,t) = \frac{3i\alpha_2 \ln \sqrt{\mu_2}}{\mu_1} \cdot \frac{1}{1 - a} + \frac{3i\alpha_2 \ln \sqrt{\mu_2}}{\mu_1} \cdot \left( \frac{1}{1 - a} \right)^2 \]
Figure 2. 3D plot of $u_3$ by choosing $a = e$, $\alpha_2 = \mu_1 = \xi_0 = \beta = 1$, $\mu_2 = -1$

Case 3:

$$a_0 = -\frac{\alpha_2 \ln a \sqrt{\mu_2}}{2 \mu_1}, \quad a_1 = \frac{3 \alpha_2 \ln a \sqrt{\mu_2}}{\mu_1}, \quad a_2 = -\frac{3 \alpha_2 \ln a \sqrt{\mu_2}}{\mu_1}, \quad \alpha_1 = \frac{1}{\ln a \sqrt{\mu_2}} \quad (16)$$

Using Eq. (16) into Eq. (13), the following solution of Eq. (11) is obtained:

$$u(x, t) = -\frac{\alpha_2 \ln a \sqrt{\mu_2}}{2 \mu_1} + \frac{3 \alpha_2 \ln a \sqrt{\mu_2}}{\mu_1} \cdot \frac{1}{1 + a} - \frac{3 \alpha_2 \ln a \sqrt{\mu_2}}{\mu_1} \cdot \left( \frac{1}{1 + a} - \frac{1}{1 - a} \right)$$

Figure 3. 3D Plot of $u_5$ by choosing $a = e$, $\alpha_2 = 0.5$, $\mu_1 = \xi_0 = \mu_2 = \beta = 1$

Case 4:

$$a_0 = \frac{\alpha_2 \ln a \sqrt{\mu_2}}{2 \mu_1}, \quad a_1 = -\frac{3 \alpha_2 \ln a \sqrt{\mu_2}}{\mu_1}, \quad a_2 = \frac{3 \alpha_2 \ln a \sqrt{\mu_2}}{\mu_1}, \quad \alpha_1 = -\frac{1}{\ln a \sqrt{\mu_2}} \quad (17)$$
Inserting Eq. (17) into Eq. (13), we obtain the following solution of Eq. (11):

\[
\begin{align*}
    u_1(x,t) &= \frac{\alpha_2 \ln a \sqrt{\mu_2}}{2\mu_1} - \frac{3\alpha_2 \ln a \sqrt{\mu_2}}{\mu_1} \frac{1}{1 + \alpha \left( \frac{1}{1 + \alpha} \right)^{1/(1+\beta)}} + \frac{3\alpha_2 \ln a \sqrt{\mu_2}}{\mu_1} \left( \frac{1}{1 - \alpha \left( \frac{1}{1 - \alpha} \right)^{1/(1+\beta)}} \right)^2 \\
    u_2(x,t) &= \frac{\alpha_2 \ln a \sqrt{\mu_2}}{2\mu_1} - \frac{3\alpha_2 \ln a \sqrt{\mu_2}}{\mu_1} \frac{1}{1 - \alpha \left( \frac{1}{1 - \alpha} \right)^{1/(1+\beta)}} + \frac{3\alpha_2 \ln a \sqrt{\mu_2}}{\mu_1} \left( \frac{1}{1 + \alpha \left( \frac{1}{1 + \alpha} \right)^{1/(1+\beta)}} \right)^2
\end{align*}
\]

Figure 4. 3D Plot, color plot and 2D Plot by choosing \( a = e, \ \alpha_2 = \mu_1 = \xi_0 = \mu_2 = \beta = 1, \ t_0 = 0 \)

Four kinds of solutions of the equation are obtained, the first three of which are soliton solutions commonly mentioned by us. Case 1 obtains dark soliton solutions because of the wave fall to the bottom. In case 2 and 3, the bright soliton solutions are obtained. Both the dark soliton solution and the bright soliton solution can not change the shape of the solitary wave when it propagates. That is to say, the shape in the graph can be propagated separately. In case 4, the wave is a strange wave which is a kind of nonlinear local wave with double limitations in time and space and the peak of the wave is very high from the evolution diagram of the wave.

3. Conclusion
We successfully obtain the exact solutions of the space-time fractional equal-width equation by using the modified Kudryashov method and draw the corresponding 3D snapshots for the solutions with Mathematica software. It shows the applicability of the modified Kudryashov method to obtain the exact solutions of fractional partial differential equations. This method is more concise and direct. Therefore, the modified Kudryashov method plays a very important role in the application of nonlinear fractional partial differential equations.
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