Crossing of the \( w = -1 \) barrier in two-fluid viscous modified gravity

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Abstract

Singularities in the dark energy late universe are discussed, under the assumption that the Lagrangian contains the Einstein term \( R \) plus a modified gravity term of the form \( R^\alpha \), where \( \alpha \) is a constant. It is found, similarly as in the case of pure Einstein gravity [I. Brevik and O. Gorbunova, Gen. Rel. Grav. 37, 2039 (2005)], that the fluid can pass from the quintessence region \( (w > -1) \) into the phantom region \( (w < -1) \) as a consequence of a bulk viscosity varying with time. It becomes necessary now, however, to allow for a two-fluid model, since the viscosities for the two components vary differently with time. No scalar fields are needed for the description of the passage through the phantom barrier.

KEY WORDS: Dark energy, viscous cosmology, modified gravity, Big Rip

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1 Introduction

The possibility of crossing the \( w = -1 \) barrier in dark energy physics is a topic that has attracted a great deal of interest. It is

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usually assumed that the equation of state for the cosmic fluid can be written in the form

\[ p = w \rho \equiv (\gamma - 1) \rho. \]  

(1)

Here the equation of state parameter \( w = -1 \) or \( p = -\rho \) corresponds to a "vacuum fluid", with bizarre thermodynamic properties such as possibly negative entropies \[1\]. Cosmological observations such as SNe Ia \[2\], WMAP \[3\], SDSS \[4\] and the Chandra X-ray Observatory \[5\] indicate that the present universe is accelerating. Moreover, based upon the observed data it has been conjectured that \( w \) is a varying function of time. For instance, as discussed in Ref. \[6\], \( w \) might have been around 0 at redshift \( z \sim 1 \) and may be around -1.2 today. (Cf. also the very recent analysis of observational constraints on the dark energy \[7\].) Perhaps \( w \) even an oscillating function in time.

In view of these circumstances the analysis of a possible crossing of the phantom barrier \( w = -1 \), from the quintessence region \((-1 < w < -1/3)\) into the phantom region \( w < -1 \), is obviously of physical interest. It ought to be noted that both quintessence and phantom fluids lead to the inequality \( \rho + 3p \leq 0 \), thus breaking the strong energy condition.

The occurrence of a phantom fluid leads inevitably to a singularity in the future, called the Big Rip \[8, 9, 10, 11\]. Often, the crossing of the phantom barrier is described in terms of one or more scalar fields, besides the gravitational field. For instance, as shown recently in Ref. \[12\], one can consider inflation, dark energy and dark matter under the same standard: phantom-nonphantom transitions may appear such that the universe could have been effectively quantum-like both at early times and will behave similarly at late times. Cf. also the related papers \[13, 19\]. Here, only one scalar field is considered. In several papers - cf., for instance, Refs. \[14, 15, 16, 17, 18\] - cosmological models with two scalar fields (or fluids) are considered. In Ref. \[6\], arguments based upon a stability analysis are given why phantom barrier crossing cannot be effectuated in terms of one scalar field alone.

One interesting alternative to a scalar field theory is to allow for some sort of modification of the standard Einstein theory.
One kind of approach is to introduce higher order derivatives of scalar fields in the Lagrangian [19, 20]. The form of modified gravity that we shall be concerned with here is to introduce a term $R^\alpha$ in the gravitational action, where $\alpha$ is a constant. A recent introduction to modified gravity of this kind (including also Gauss-Bonnet gravity) is given by Nojiri and Odintsov [21]. Some other papers dealing with modified gravity from different viewpoints are Refs. [22, 23, 24, 25, 26, 27].

The body of literature in this field of research is large, and the above list of references is not intended to be exhaustive. We turn now, however, to the main theme of the present paper, which is to introduce a bulk viscosity $\zeta$ in the cosmic fluid. Bulk viscosity is compatible with a universe that is spatially isotropic. We shall take $\zeta$ to be dependent on time. In the paper [28] dealing with Einstein’s gravity, we showed how the ansatz of letting $\zeta$ be proportional to the scalar expansion $\theta = 3H$ can drive the fluid into the phantom region even if it starts from the quintessence region in the non-viscous case. In Ref. [29] we considered a more general situation with the action

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} (f_0 R^\alpha + L_m),$$

where $\kappa^2 = 8\pi G$, $f_0$ and $\alpha$ are constants, and $L_m$ is the matter Lagrangian. This is the kind of action recently analyzed by Abdalla et al. [30]. Based upon a natural assumption for the time variation of the scale factor, we verified that the field equations are satisfied for general $\alpha$, and we investigated the ansatz of letting $\zeta$ be proportional to $\theta^{2\alpha-1}$. This kind of modified gravity was further considered in Ref. [31]. We found the same kind of behavior also in this case: the fluid can in principle make a viscosity-generated passage from the quintessence region into the phantom region. There are some obvious advantages of this kind of theory: the theory is mathematically simple, it makes use of standard concepts from fluid mechanics only (in addition to gravity), and there is no need of introducing a scalar field.

The approach of Ref. [31] is however physically incomplete, in the following sense: it involves the gravitational correction term $R^\alpha$ only, in the Lagrangian. It would be more appropriate
to include the Einstein term $R$ also, so that the action takes the form
\[ S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left( R + f_0 R^\alpha + L_m \right). \] (3)

This form of action is the starting point for the investigation in the present paper. As we shall see, the fluid possesses essentially the same property as before; it can slide through the phantom barrier into the phantom region as a consequence of viscosity. But there is one essential difference: the theory now requires a \textit{two-component} model of the fluid. The two components belonging to the Einstein gravity and to the modified gravity vary differently with time. It becomes natural to speculate if this necessity of dividing the cosmic fluid into two components is not after all another demonstration of the difficulties with a one-component model that have been found previously in other investigations, within a conventional scalar field approach.

The present series of works is of course not the first time that the viscosity concept has been introduced in cosmology. Misner was probably the first to introduce viscosity in the cosmic fluid in his study of how initial anisotropies in the early universe become relaxed [32]. Another early paper is that of Padmanabhan and Chitre [33]. An extensive review of the development up to 1990 is given by Grøn [34]. We have ourselves dealt earlier with viscous entropy production in the early universe [35], and with viscous fluids on the Randall-Sundrum branes [36, 37]. Ren and Meng recently discussed a cosmological model with a dark viscous fluid described by an effective equation of state, and compared with SNe data [38]. Cataldo et al. considered viscous dark energy and phantom evolution within the framework of the Eckart theory of relativistic irreversible thermodynamics [39]. Kofinas et al. considered the crossing of the $w = -1$ barrier from a brane-bulk energy exchange scenario containing an induced gravity curvature correction term [40]. As discussed by Nojiri and Odintsov [41] and by Capozziello et al. [7], a dark fluid with a time dependent bulk viscosity can be considered as a fluid with an inhomogeneous equation of state.

In the next section we present the field equations, and the conservation equation for energy, and derive herefrom two ex-
pressions, Eqs. (16) and (17), for the coefficient $B$ defined in Eq. (12). A positive value of $B$ leads to a Big Rip. Consistency of the formalism requires that the coefficients $\tau_E$ and $\tau_\alpha$ introduced in Eqs. (14) and (15) are related to each other. Examples with positive values of $\alpha$ are discussed in Sect. 3. The case of negative $\alpha$, discussed in Sect. 4, is exceptional, since the physical quantities associated with the modified fluid component does not go to infinity at Big Rip, but rather to zero. Still, the Hubble factor, as well as the physical quantities associated with the Einstein component, diverge.

2 General formalism

We assume the spatially flat FRW metric

$$ds^2 = -dt^2 + a^2(t)dx^2,$$

(4)

put the cosmological constant $\Lambda$ equal to zero, and adopt the energy-momentum tensor of the viscous fluid in the standard form

$$T_{\mu\nu} = \rho U_\mu U_\nu + \bar{p} h_{\mu\nu},$$

(5)

where $h_{\mu\nu} = g_{\mu\nu} + U_\mu U_\nu$ is the projection tensor and $\bar{p} = p - \zeta \theta$ the effective pressure. In comoving coordinates, the four-velocity of the fluid is $U^0 = 1, U^i = 0$. From variation of the action (3) we obtain the equations of motion

$$-\frac{1}{2} g_{\mu\nu} R(1 + f_0 R^{a-1}) + R_{\mu\nu} (1 + \alpha f_0 R^{a-1})$$

$$-\alpha f_0 \nabla_\mu \nabla_\nu R^{a-1} + \alpha f_0 g_{\mu\nu} \nabla^2 R^{a-1} = \kappa^2 T_{\mu\nu},$$

(6)

where $T_{\mu\nu}$ corresponds to the term $L_m$ in the Lagrangian.

Of main interest is the (00)-component of this equation. Using that

$$R_{00} = -3\ddot{a}/a, \quad R = 6(H + 2H^2),$$

(7)

as well as $T_{00} = \rho$, we obtain

$$3H^2 + \frac{1}{2} f_0 R^a - 3\alpha f_0 (\dot{H} + H^2) R^{a-1}.$$
\begin{align*}
+3\alpha(\alpha - 1)f_0HR^{\alpha-2}\dot{R} &= \kappa^2 \rho. \quad (8)
\end{align*}

An important property of Eq. (6) is that the covariant divergence of the LHS is equal to zero \[41\]. Thus

\[\nabla^\nu T_{\mu\nu} = 0, \quad (9)\]

just as in the case of Einstein’s gravity. Energy-momentum conservation is a consequence of the field equations. Contracting the expression \[5\] for the energy-momentum tensor with \(U^\mu\) we obtain the energy conservation equation

\[\dot{\rho} + (\rho + p)3H = 9\zeta H^2. \quad (10)\]

We now differentiate the expression \[8\] with respect to \(t\), and insert \(\dot{\rho}\) from Eq. \[10\]. After some calculation we then obtain

\[6\dot{H} + 9\gamma H^2 + \frac{3}{2}\gamma f_0 R^{\alpha} - 3\alpha f_0 [(3\gamma - 2)\dot{H} + 3\gamma H^2] R^{\alpha-1}\]

\[+3\alpha(\alpha - 1)f_0[(3\gamma - 1)H\dot{R} + \dot{R}] R^{\alpha-2} + 3\alpha(\alpha - 1)(\alpha - 2)f_0 \dot{R}^2 R^{\alpha-3} = 9\kappa^2 \zeta H. \quad (11)\]

Recalling that \(R = 6(\dot{H} + 2H^2)\), we see that this is a nonlinear differential equation for \(\dot{H}(t)\). We shall seek for solutions that are compatible with the following basic ansatz for the Hubble parameter:

\[H = \frac{H_0}{X}, \quad \text{where} \quad X \equiv 1 - BH_0t. \quad (12)\]

Here \(B\) is a nondimensional quantity whose value is dependent on \(\alpha\). We take the initial time to be \(t_0 = 0\), and give a subscript zero to quantities referring to this instant. If Big Rip shall occur \((H \to \infty)\), \(B\) has to be positive.

From the mathematical structure of Eq. \[11\] it is clear that we cannot eliminate the time dependent terms in the governing equation for \(B\) simply by letting \(\zeta\) be proportional to a power of the scalar expansion \(\theta\). This feature contrasts that found in
earlier papers \cite{28} and \cite{31}. It becomes now necessary to allow for a two-fluid model. We shall write $\zeta = \zeta(t)$ as a sum of a term $\zeta_E$ referring to Einstein gravity and a term $\zeta_\alpha$ referring to modified gravity:

$$\zeta = \zeta_E + \zeta_\alpha; \quad (13)$$

where

$$\zeta_E = \tau_E \theta = 3\tau_E H, \quad (14)$$

$$\zeta_\alpha = \tau_\alpha \theta^{2\alpha - 1} = \tau_\alpha (3H)^{2\alpha - 1}, \quad (15)$$

$\tau_E$ and $\tau_\alpha$ being constants. Then Eq. (11) can be satisfied for the Einstein part and the modified part separately. We get in this way two different algebraic expressions determining the constant $B$:

$$B = -\frac{3}{2} \gamma + \frac{9}{2} \kappa^2 \tau_E, \quad (16)$$

$$(B + 2)^{\alpha - 1}\left\{9(2 - \alpha)\gamma + 3[\alpha + 3\gamma + \alpha(2\alpha - 3)(3\gamma - 1)]B + 6\alpha(\alpha - 1)(2\alpha - 1)B^2\right\} = \frac{18\kappa^2}{f_0} \left(\frac{3}{2}\right)^\alpha \tau_\alpha; \quad (17)$$

the time-dependent terms drop out. If $f_0 = 0$ (Einstein gravity only), Eq. (16) is in accordance with Ref. \cite{28}. If the Einstein term is absent, Eq. (17) agrees with Ref. \cite{31}.

One important property follows at once from the compatibility of Eqs. (16) and (17): there must be a relationship between $\tau_E$ and $\tau_\alpha$. This relationship $\tau_\alpha = \tau_\alpha(\tau_E)$ depends on the values of $\alpha$ and $f_0$, as well as on the fluid parameter $\gamma = w + 1$. There seems to be no direct physical reason behind this dependence; it results simply from the mathematical consistency of the model.

In the next section we shall turn to considering specific models. We note, however, the general expressions for $B$ that follow directly from the energy conservation equation (10): as this equation has to be fulfilled for the two fluid components separately,

$$\dot{\rho}_E + (\rho_E + p_E)3H = 9\zeta_E H^2, \quad (18)$$

$$\dot{\rho}_\alpha + (\rho_\alpha + p_\alpha)3H = 9\zeta_\alpha H^2, \quad (19)$$
we get
\[ B = -\frac{3\gamma}{2} + \frac{27\tau_E}{2} \frac{H_0^2}{\rho_{0E}}, \quad (20) \]
\[ B = -\frac{3\gamma}{2\alpha} + \frac{3\tau_\alpha (3H_0)^{2\alpha}}{2\alpha \rho_{0\alpha}}. \quad (21) \]

We here made use of the time dependent relations
\[ \zeta_\alpha = \tau_\alpha \left( \frac{3H_0}{X} \right)^{2\alpha-1}, \quad \rho_E = \frac{\rho_{0E}}{X^2}, \quad \rho_\alpha = \frac{\rho_{0\alpha}}{X^{2\alpha}}. \quad (22) \]

In order to calculate \( B \) from the expressions (20) or (21), at least one of the initial densities \( \rho_{0E} \) or \( \rho_{0\alpha} \) have to be known.

Equation (20) agrees with Eq. (16) in view of the first Friedmann equation \( 3H_0^2 = \kappa^2 \rho_{0E} \) at \( t = 0 \).

3 Positive values of \( \alpha \): Some examples

3.1 Small deviations from Einstein’s gravity

It is natural, for exemplification, to start with a modified gravity system where the exponent \( \alpha \) in the Lagrangian (3) is small. We shall take \( \alpha \) to be positive, and also assume that \( f_0 \) is positive and small:
\[ \alpha \ll 1, \quad f_0 \ll 1. \quad (23) \]

As approximatively \( R^\alpha = 1 + \alpha \ln R \), we see that terms containing \( f_0 \alpha \) are of second order and thus negligible. The product \( f_0 R^\alpha \) can thus simply be replaced with \( f_0 R \), and the action (3) reduces to
\[ S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} [(1 + f_0)R + L_m]. \quad (24) \]

From this equation, or directly from Eq. (3), it follows that the Ricci tensor can be taken to be the same as in Einstein’s gravity. The energy-momentum tensor becomes however rescaled with
the factor \(1/(1 + f_0)\) (or \(1 - f_0\)). The field equations \((6)\) reduce to
\[
-\frac{1}{2}g_{\mu\nu}R + R_{\mu\nu} = \frac{\kappa^2 T_{\mu\nu}}{1 + f_0}.
\] (25)

The first Friedmann equation (corresponding to \(\mu = \nu = 0\)) now becomes
\[
3H^2 = \frac{\kappa^2}{1 + f_0} \rho.
\] (26)

As for the determination of the constant \(B\) we obtain for the Einstein component the same equation \((16)\) as before. For the \(\alpha\)-component \((17)\), when combined with \((16)\), we obtain the simple relation
\[
\tau_\alpha = f_0 \tau_E,
\] (27)
valid for all values of \(\gamma\). Note the dimensions: \([f_0] = \text{cm}^{2(\alpha - 1)}, \,[\tau_E] = \text{cm}^{-2}, \,[\tau_\alpha] = \text{cm}^{2(\alpha - 2)}\). (For clarity we write \(\tau_\alpha\) instead of \(\tau_1\).)

Under the present conditions, \(\tau_\alpha \ll \tau_E\).

Already from Eq. \((16)\) it is clear that \(B\) can be greater than zero, thus leading to a Big Rip, if \(\tau_E\) is big enough. Let us summarize the time dependencies of the physical quantities:
\[
\zeta_E = \frac{3\tau_E H_0}{X}, \quad \rho_E = \frac{\rho_{0E}}{X^2}, \quad (28)
\]
\[
\zeta_\alpha = \frac{3\tau_E f_0 H_0}{X}, \quad \rho_\alpha = \frac{\rho_{0\alpha}}{X^2}, \quad (29)
\]
with \(X = 1 - BH_0 t\) as before. Moreover, \(\rho = \rho_E + \rho_\alpha\) for all \(t\).

### 3.2 The case \(\alpha = 1/2\)

This case, corresponding to
\[
S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left( R + f_0 \sqrt{R} + L_m \right), \quad (30)
\]
is mathematically simplifying, and can moreover be of physical interest. Equation \((17)\) yields the following quadratic equation for \(B\):
\[
B^2 + 6(\gamma - 4T^2)B + 9\gamma^2 - 48T^2 = 0, \quad (31)
\]
with
\[ T = \frac{\kappa^2}{f_0} \tau_\alpha. \] (32)

The product of the two roots for \( B \) is equal to \( 9\gamma^2 - 48T^2 \). If we put \( \gamma = 0 \), which is the most interesting case, we thus see that the product is always negative and there is one positive and one negative root. The positive root necessarily leads to a Big Rip. When \( \gamma = 0 \) we get from Eqs. (31) and (32), when taking Eq. (16) into account,

\[ \tau_\alpha = \frac{9f_0\tau_E}{4\sqrt{12 + 27\kappa^2\tau_E}}. \] (33)

Still, \( \tau_\alpha \) is of order \( f_0\tau_E \), what is physically reasonable.

In this case \( \zeta_E \) and \( \rho_E \) vary with time as in Eq. (28), whereas

\[ \zeta_\alpha = \tau_\alpha = \text{const}, \quad \rho_\alpha = \frac{\rho_0}{X}. \] (34)

The bulk viscosity corresponding to the \( \alpha \)-fluid component is thus a constant. The density \( \rho_\alpha \) decreases more slowly with time than does \( \rho_E \propto X^{-2} \).

### 3.3 The case \( \alpha = 2 \)

This case is quadratic modified gravity, in its simplest form. From Eq. (17) we get the following cubic equation for \( B \):

\[ B^3 + \left( 2 + \frac{3}{4}\gamma \right) B^2 + \frac{3}{2}\gamma B - \frac{9}{8}T = 0. \] (35)

Let us put \( \gamma = 0 \). If we draw a curve representing the expression \((B^3 + 2B^2 - 9T/8)\) versus \( B \), we see that it has a local maximum at \( B = -4/3 \) and a local negative minimum at \( B = 0 \). There is thus one single positive root of the equation, for all positive \( T \). This root leads in turn to a viscosity-generated Big Rip. A more detailed discussion of this case is given in Sect. 4.2 in [31]. We only note here the time dependencies of the \( \alpha \)-fluid component:

\[ \zeta_\alpha = \tau_\alpha \left( \frac{3H_0}{X} \right)^3, \quad \rho_\alpha = \frac{\rho_0}{X^4}. \] (36)
4 Negative values of $\alpha : \alpha = -1$

If $\alpha$ takes negative values, the situation becomes qualitatively different in the following way: Positive values of $B$ will always lead to a Big Rip singularity for the Hubble parameter, as $H = H_0/X \to \infty$ at a finite value of $t$. Similarly, both $\zeta_E \propto X^{-1}$ and $\rho_E \propto X^{-2}$ will diverge. However, both $\zeta_\alpha \propto X^{-(2\alpha-1)}$ and $\rho_\alpha \propto X^{-2\alpha}$ go to zero at the Big Rip. The Einstein fluid component and the $\alpha$-fluid component thus behave quite differently.

Let us consider $\alpha = -1$ as the most typical example:

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left( R + \frac{f_0}{R} + L_m \right). \tag{37}$$

From Eq. (17) we get the following equation for $B$:

$$B^2 + \frac{13 - 9\gamma + 8T}{2} B - \frac{19\gamma - 16T}{4} = 0, \tag{38}$$

$T$ being being given by Eq. (32) as before.

Consider first the vacuum fluid case $\gamma = 0$. The product $B_1B_2$ of the two roots is positive, whereas the sum $(B_1 + B_2)$ of them is negative. There is no positive root, thus no Big Rip, in this case. The same is true for $\gamma < 0$. The case of general $\gamma$ is analyzed similarly by observing that $B_1B_2$ has the same sign as $(16T - 9\gamma)$, whereas $(B_1 + B_2)$ has the same sign as $(9\gamma - 3 - 8T)$. On physical grounds, we expect that $T \ll 1$. It follows that for $\gamma < 16T/9$ there are two negative roots, whereas for $\gamma > 16T/9$ there is one positive and one negative root. In the latter case there is thus a Big Rip possible, in the special sense explained above. Still, there is no need of a scalar field.

5 Summary. Remarks on Big Rip classification

Summary. The main purpose of this paper has been to investigate whether the passage through the phantom barrier, from the quintessence region ($w > -1$) into the phantom region ($w < -1$)
can be described as a viscosity-generated phenomenon in the
general case when the action is as in Eq. (3). As shown ear-
lier [28] such a mechanism works well in the case of Einstein’s
gravity, if the bulk viscosity $\zeta \propto \theta$ where $\theta$ is the scalar expan-
sion. Also, in the more general case of pure modified gravity
(Lagrangian of the form $R^\alpha$), the same mechanism was found
to work if $\zeta \propto \theta^{2\alpha-1}$ [29, 31], what is a natural generalization of
our basic ansatz.

The case where the action is as in Eq. (3) is more physical
than those considered in [29, 31] since the Einstein component
and the modified component are now combined into one La-
grangian. It turns out that the system shows the same kind
of behavior as previously: there exists in principle a viscosity-
driven passage through the barrier $w = -1$. It becomes however
necessary to introduce a two-fluid model, since the bulk viscosi-
ties for the Einstein component and the modified component
vary differently with time. Also, physical quantities such as the
density vary differently for the two fluid components.

Perhaps is this kind of behavior yet another indication of
the necessity of introducing a two-component fluid model in the
late universe, as emphasized, for instance, by Vikman within
the framework of the scalar field picture [6].

It ought to be mentioned again that the present model
requires the two coefficients $\tau_E$ and $\tau_\alpha$ to be related. This re-
quirement follows from Eqs. (16) and (17). There appears to
be no direct physical reason why this should be so; rather, the
condition is a consequence of the mathematical consistency of
the formalism.

**On the Big Rip classification.** There are actually several
variants of the Big Rip phenomenon, and it is of interest to trace
out to what category the singularities encountered in the present
paper belong. In Refs. [8, 9, 10] two different types of Big Rip
were discussed. In Ref. [11] a classification of the known types
of Big Rip was made and two new types were discovered, so that
there are according to this four types in all. Let $t = t_s$ be the
instant when Big Rip occurs. From Ref. [11] we reproduce the
following classification:
• Type I: For $t \to t_s$, $a \to \infty$, $\rho \to \infty$ and $|p| \to \infty$
• Type II: For $t \to t_s$, $a \to a_s$, $\rho \to \rho_s$ and $|p| \to \infty$
• Type III: For $t \to t_s$, $a \to a_s$, $\rho \to \infty$ and $|p| \to \infty$
• Type IV: For $t \to t_s$, $a \to a_s$, $\rho \to 0$, $|p| \to \infty$ and higher derivatives of $H$ diverge.

Here $t_s, a_s$ and $\rho_s$ are constants with $a_s \neq 0$. Type I is the case one usually associates with the Big Rip concept, emerging when $w < -1$.

In the present case, when $H$ is given as in Eq. (12), the scale factor varies with time as

$$a(t) = \frac{1}{(1 - BH_0 t)^{1/B}}.$$  \hspace{1cm} (39)

Thus, if $B > 0$, $a \to \infty$ when $t \to t_s = 1/(BH_0)$. It means that only Type I is an actual option in our case. If the parameter $\alpha > 0$ in the action (3), we have a Type I Big Rip both for the Einstein fluid component and the $\alpha$-component. If $\alpha < 0$, the Einstein component still belongs to the Type I category, but the $\alpha$-component belongs to a new type, different from the ones listed above, since $\zeta_\alpha \to 0$, $\rho_\alpha \to 0$, $p_\alpha = w\rho_\alpha \to 0$.

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References

[1] Brevik, I., Nojiri, S., Odintsov, S. D. and Vanzo, L.: Phys. Rev. D 70, 043520 (2004).

[2] Riess, A. G. et al.: Astrophys. J. 607, 665 (2004); Knop, R. A. et al.: Astrophys. J. 598, 102 (2003); Riess, A. G. et al.: Astron. J. 116, 1009 (1998); Perlmutter, S. et al.: Astrophys. J. 517, 565 (1999).

[3] Bennett, C. L. et al.: Astrophys. J. Suppl. 148, 1 (2003); Spergel, D. N. et al.: Astrophys. J. Suppl. 148, 175 (2003).

[4] Tegmark, M. et al.: Phys. Rev. 69, 103501 (2004); Tegmark, M. et al.: Astrophys. J. 606, 702 (2004); Seljak, U. et al.: Phys. Rev. D 71, 103515 (2005).

[5] Allen, S. W., Schmidt, H., Fabian, A. C. and van Speybroeck, L.: Mon. Not. R. Astron. Soc. 353, 457 (2004).

[6] Vikman, A.: Phys. Rev. D 71, 023515 (2005).

[7] Capozziello, S., Cardone, V. F., Elizalde, E., Nojiri, S. and Odintsov, S. D.: Preprint astro-ph/0508350v3.

[8] Caldwell, R. R., Kamionkowski, M. and Weinberg, N. N.: Phys. Rev. Lett. 91, 071301 (2003).

[9] McInnes, B.: J. High Energy Phys. 0208, 029 (2002).

[10] Barrow, J. D.: Class. Quant. Grav. 21, L79 (2004).

[11] Nojiri, S., Odintsov, S. D. and Tsujikawa, S.: Phys. Rev. D 71, 063004 (2005).

[12] Capozziello, S., Nojiri, S. and Odintsov, S. D.: Phys. Lett. B 632, 597 (2006).

[13] Li, M., Feng, B. and Zhang, X.: J. Cosmology and Astroparticle Phys. 12, 002 (2005).

[14] Nojiri, S. and Odintsov, S. D.: Phys. Rev. D 72, 023003 (2005).
[15] Wei, H. and Cai, R. G.: Phys. Lett. B 634, 9 (2006).

[16] Feng, B., Wang, X. and Zhang, X.: Phys. Lett. B 607, 35 (2005).

[17] Zhao, G.-B., Xia, J.-O., Li, M., Feng, B. and Zhang, X.: Phys. Rev. D 72, 123515 (2005).

[18] Elizalde, E., Nojiri, S., Odintsov, S. D. and Wang, P.: Phys. Rev. D 71, 103504 (2005).

[19] Li, M., Feng, B. and Zhang, X.: J. Cosmology and Astropart. Phys. 12, 002 (2005).

[20] Anisimov, A., Babichev, E. and Vikman, A.: J. Cosmology and Astropart. Phys. 06, 006 (2005).

[21] Nojiri, S. and Odintsov, S. D.: Preprint hep-th/0601213.

[22] Gognola, G., Elizalde, E., Nojiri, S., Odintsov, S. D. and Zerbini, S.: Preprint hep-th/0601008. J. Cosmology and Astropart. Phys. 0502, 010 (2005).

[23] Nojiri, S. and Odintsov, S. D.: Phys. Rev. D 68, 123512 (2003); Gen. Rel. Grav. 36, 1765 (2004); Phys. Lett. B 631, 1 (2005).

[24] Carroll, S. M., Duvvuri, V., Trodden, M. and Turner, M. S.: Phys. Rev. D 70, 043528 (2004).

[25] Nojiri, S., Odintsov, S. D. and Sasaki, M.: Phys. Rev. D 71, 123509 (2005); Carter, B. M. N. and Neupane, I. P.: Preprints hep-th/0512262, hep-th/0510109.

[26] Barrow, J. D.: Phys. Lett. B 180, 335 (1987); Phys. Lett. B 183, 285 (1987); Nucl. Phys. B 310, 743 (1988).

[27] Clifton, T. and Barrow, J. D.: Phys. Rev. D 72, 103005 (2005); Phys. Rev. D 72, 123003 (2005); Class. Quant. Grav. 23, L1 (2006).
[28] Brevik, I. and Gorbunova, O.: Gen. Rel. Grav. 37, 2039 (2005).

[29] Brevik, I., Gorbunova, O. and Shaido, Y. A.: Int. J. Mod. Phys. D 14, 1899 (2005).

[30] Abdalla, M. C. B., Nojiri, S. and Odintsov, S.: Class. Quant. Grav. 22, L35 (2005).

[31] Brevik, I.: Int. J. Mod. Phys. D, in press [gr-qc/0601100].

[32] Misner, C. W.: Astrophys. J. 151, 431 (1968).

[33] Padmanabhan, T. and Chitre, S. M.: Phys. Lett. A 120, 433 (1987).

[34] Grøn, Ø.: Astrophys. Space Sci. 173, 191 (1990).

[35] Brevik, I. and Heen, L. T.: Astrophys. Space Sci. 219, 99 (1994).

[36] Brevik, I. and Hallanger, A.: Phys. Rev. D 69, 024009 (2004).

[37] Brevik, I., Børven, J. M. and Ng, S.: Gen. Rel. Grav., in press [gr-qc/0512026].

[38] Ren, J. and Meng, X. H.: Phys. Lett. B 633, 1 (2006); Preprint astro-ph/0602462.

[39] Cataldo, M., Cruz, N. and Lepe, S.: Phys. Lett. B 619, 5 (2005).

[40] Kofinas, G., Panotopoulos, G. and Tomaras, N.: Preprint hep-th/0510207.

[41] Koivisto, T.: Preprint gr-qc/0505128.