Wigner’s inequalities in quantum field theory

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We present a relativistic generalization of the Wigner inequality for the scalar and pseudoscalar particles decaying to two spin particles (fermions and photons.) We consider Wigner’s inequality with the full spin anticorrelation (with the non-relativistic analog) as well as the case with the full spin correlation. The possibility for relativistic testing of Bohr’s complementarity principle is shown.

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I. INTRODUCTION

Since the inception of quantum mechanics in the first quarter of the 20-th century, disputes abound around two closely related issues:

1) Is the probabilistic nature of quantum theory predictions and the confirmation by experimental measurements is a reflection of the objective laws of the microcosm, or is the indeterminism is a consequence of our ignorance of some “subtle interactions” among microparticles that would provide theoretical predictions and experimental measurements like in deterministic classical mechanics. For example, in addition to the well-known measurable properties of elementary particles like mass, charge, spin, lepton and baryon numbers, color, weak isospine, etc., particles may have properties, which in principle, cannot be measured with macroscopic analyzers. This lack of information about the values of these variables makes the predictions of quantum mechanics probabilistic. This concept is known as the theory of hidden variables of quantum mechanics.

2) The the particle parameters described by noncommuting operators are elements of a physical reality simultaneously and independently of the act of measurement or are the particle parameters fundamentally inseparable from the design and capabilities of a particle detector as postulated in the Bohr principle of complementarity.

These issues are essential not only for non-relativistic quantum mechanics (NQM) under which they were intensely debated (a comprehensive review can be found in [1]), but also for quantum field theory (QFT). In the framework of QFT this topic was highlighted in a few papers (e.g. [2] – [5]). More complete bibliography may be found in these works.

The experimental answer to the second of the above issues may be given by Bell’s inequalities. They were introduced for the first time by J.S.Bell in 1964 – 1966 [6] and then modified by Clauser, Horne, Shimony and Holt in 1969 [7]. In Bell’s original work three dichotomic variables $A$, $B$ and $C$ were introduced. These variables simultaneously were elements of a physical reality due to the existence of a set of a hidden variables $\lambda$. The expected values of these dichotomic variables satisfy the following inequality:

$$|\langle AB \rangle - \langle AC \rangle| \leq 1 + \langle BC \rangle.$$  \hspace{1cm} (1.1)

From (1.1) follows the Clauser-Horne-Shimony-Holt (CHSH) inequality for four dichotomic variables with spectre $\pm 1$:

$$|\langle AB \rangle + \langle AC \rangle + \langle DB \rangle - \langle DC \rangle| \leq 2.$$  \hspace{1cm} (1.2)

Dichotomic variables $A$, $B$, $C$ and $D$ may be naturally implemented in the form of a spin $1/2$ projection on any non-parallel directions $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$. However from the experimentalist point of view it is more convenient to use photon polarization and “flavour–CP” quantum numbers of neutral $K^-$ and $B^-$-mesons. For example, there is a recent paper of the Belle collaboration of the precise test of Bell’s inequalities in neutral $B^-$-mesons [8].

It is widely believed that for the derivation of Bell’s inequalities (1.1) and (1.2), the existence of local, context–depending hidden variables $\lambda$ is required. Thus, the violation of Bell’s inequalities is often considered as a disproof of the existence of a wide class of hidden variables. This view comes from a classical work [6]. However, this view is wrong.

1 having a spectre of only two values, in this case $\pm 1$

2 i.e. depending not only on the particle state but on the state of an analyzer
It was shown in the paper [4] that for the derivation of (1.1) and (1.2) it is enough for only the nonnegative joint probabilities \( W(A, B, C) \) and \( W(A, B, C, D) \) to exist. These probabilities should satisfy Kolmogorov’s probability axioms. The existence of such probabilities is a mathematical reflection of the following statement: \( (A, B, C, D) \) parameters of a given quantum system are simultaneously elements of physical reality. In addition we can assume that the existence of the nonnegative joint probabilities is provided by the hidden variables. Again we emphasize the fact that this assumption is not necessary for the derivation of formulae (1.1) and (1.2).

From these considerations it follows that Bell’s inequalities open the possibility for a direct experimental test of the Bohr principle of complementarity. Actually, the violation of (1.1) and (1.2) shows that observables \( A, B, C \) and \( D \) described in NQM and QFT by noncommuting operators do not exist simultaneously as elements of physical reality. The choice of these observables is determined not only by a macroscopic system, but also by an analyzer used. This is clearly in favor of the principle of complementarity.

Currently there are no suggestions for how to test the principle of complementarity in the relativistic area for particles with non-zero masses. It is not natural to try to perform such tests on current colliders in the reactions or decays of elementary particles. For these tests, proper relativistic generalization of Bell’s inequalities should be introduced for particular processes. Usually elementary particles are used for such tests as in the decay \( \eta_c \to \Lambda \Lambda \) (see [10]). The overview of the main ideas for testing Bell’s inequalities in HEP may be found in [11].

In the another set of publications [2] – [5], Bell’s inequalities are studied in the framework of a formal algebraic quantum field theory (AQFT). In this approach the value of a maximum possible violation of (1.2) in QFT [2] was found. Also it was shown that the correlation between the entangled particles remains even after the local measurement of one of the particles [4]. This fact though is quite obvious because the signal propagation speed in QFT is limited by the speed of light. However, in AQFT there is no particular suggestion for testing these predictions.

In this paper we attempt to write a relativistic generalization of Bell’s inequalities for specific decays of elementary particles. It turns out that the most natural way for relativistic generalization of Bell’s inequalities is Wigner form [12], which is not dependent on the normalization of states and allows a direct test of Bohr principle of complimentarity in the relativistic region. The spin projections of photons and relativistic fermions to various directions were chosen as the observables with noncommuting operators. Note that it is convenient to use a relativistic generalization of spin 1/2 from the work [13].

The paper is organized as follows: in Section [II] a few variants of Bell’s inequalities were obtained. These are suggested for testing of the principle of complimentarity in QFT. In Sections [III] and [IV] these inequalities are applied to decays of a scalar and pseudoscalar particle into a fermion-antifermion pair. Bell’s inequalities for the decays into two photons in final state are presented in Section [V].

Some definitions and calculations can be found in the Appendices.

II. BELL’S INEQUALITIES IN WIGNER FORM

Bell’s inequalities in forms (1.1) or (1.2) are not suitable enough for relativistic generalization. First, wave functions are used for the derivation, so it cannot be used in QFT. Second, the operators, corresponding to \( A, B, C, D \) values (usually the particles spin operators), should be generalized themselves in a relativistic case. It is desirable to find a variant of Bell’s inequalities without the above difficulties.

Such variant was proposed by E. Wigner [12] in 1970. In this section we will show that Bell’s inequalities in Wigner form could be written in two different forms. The first form corresponds to the decay of a pseudoscalar particle into fermion-antifermion pair or into two photons. It coincides with [12] for non-relativistic QM. The second form corresponds to the decay of a scalar particle into a fermion and antifermion or into a photon pair. This variant is usually not considered in NQM due to some natural obstacles.

A. Bell’s inequalities for two-body decays of pseudoscalar particle

Let us consider the decay of a resting particle with mass \( M \) to a fermion-antifermion pair, where we label the momentum of the antifermion as \( \vec{k}_1 \), the momentum of the fermion as \( \vec{k}_2 \) and their masses as \( m_1 \) and \( m_2 \) accordingly. Then \( \vec{k}_1 = -\vec{k}_2 \) and \( M > m_1 + m_2 \). If the decay is induced by the strong or electromagnetic interaction, the flavours of fermions are conserved \( (m_1 = m_2 = m) \) as well as \( P \)-parity \( (P_{ff} = (-1)^{L_{ff} + 1} = -1) \). The full momentum of the system is conserved, so \( J_{ff} = 0 \). Then, the orbital momentum and the spin of the fermion-antifermion pair is \( L_{ff} = S_{ff} = 0 \). This leads to the full anticorrelation of the fermion “2” spin and the antifermion “1” spin projections on any direction determined by the vector \( \vec{a} \):

\[
s_{a}^{(2)} = -s_{a}^{(1)}
\]  

(2.1)
Next suppose that the fermion and antifermion spin projections on three non parallel directions $\vec{a}, \vec{b}$ and $\vec{c}$ to be the elements of the physics reality at the same time. In the appendix A.1 it is shown that such an assumption leads to the following inequality:

$$w\left(s_a^{(2)} = +\frac{1}{2}, s_b^{(1)} = +\frac{1}{2}\right) \leq w\left(s_a^{(2)} = +\frac{1}{2}, s_c^{(1)} = +\frac{1}{2}\right) + w\left(s_c^{(2)} = +\frac{1}{2}, s_b^{(1)} = +\frac{1}{2}\right). \tag{2.2}$$

for the probabilities for fermion and antifermion to have spin projection of $+1/2$ (at the same time) on any two of three directions $\vec{a}, \vec{b}$ and $\vec{c}$. Since only the decay probabilities appear in (2.2) it is equally applicable in QFT and NQM. Let all the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ to lie on the same ($xz$)–plane. In non-relativistic QM this leads to the transformation of (2.2) into the following inequality (see [12]):

$$\sin^2\frac{\theta_{ab}}{2} \leq \sin^2\frac{\theta_{ac}}{2} + \sin^2\frac{\theta_{bc}}{2} \tag{2.3}$$

where $\theta_{\alpha\beta} = \theta_{\alpha} - \theta_{\beta}$, $\{\alpha, \beta\} = \{a, b, c\}$. The inequality (2.3) is violated when vectors $\vec{a}$ and $\vec{b}$ form an angle less than $\pi$, and vector $\vec{c}$ bisects this angle. The evidence of the violation of (2.3) is a direct experimental confirmation of the Bohr principle of complementarity. It is possible to write an inequality analogous to (2.2) for the neutral resting particle decays into two pions, for example $\pi^0 \rightarrow 2\gamma$. In this case the system of two photons with negative $P$–parity has the full momentum of zero, as well as orbital momentum and spin, i.e. $J_{\gamma\gamma} = L_{\gamma\gamma} = S_{\gamma\gamma} = 0$. This implies that for a photon with linear polarisation, there is a full anticorrelation of polarizations by any direction $\vec{a}$, which is perpendicular to the direction of the photon propagation. If we label the polarisation of photon with $\lambda_{\alpha}^{(1,2)}$ and the corresponding states with indices “1” and “2”, then the analog of (2.2) for photons will be:

$$w\left(\lambda_a^{(1)} = 1, \lambda_b^{(2)} = 1\right) \leq w\left(\lambda_a^{(1)} = 1, \lambda_c^{(2)} = 1\right) + w\left(\lambda_c^{(1)} = 1, \lambda_b^{(2)} = 1\right). \tag{2.4}$$

**B. Bell’s inequalities for two-body decay of scalar particle**

Consider the decay of a scalar particle with mass $M$ (in its rest frame) to a fermion-antifermion pair. In the case of strong or electromagnetic decay, due to the conservation of full momentum and $P$–parity, the orbital momentum and spin of the fermion-antifermion pair equals 1 $L_{\bar{f}f} = S_{\bar{f}f} = 1$. The fermions are in $P$–wave and have a full spin correlation for any direction $\vec{a}$:

$$s_a^{(2)} = s_a^{(1)}. \tag{2.5}$$

Again, like in subsection [1.1A] suppose that spin projections of fermion and antifermion on any three non-parallel directions $\vec{a}, \vec{b}$ and $\vec{c}$ are simultaneously elements of a physical reality. Then we can obtain the following equation:

$$w\left(s_a^{(2)} = +\frac{1}{2}, s_b^{(1)} = -\frac{1}{2}\right) \leq w\left(s_a^{(2)} = +\frac{1}{2}, s_c^{(1)} = -\frac{1}{2}\right) + w\left(s_c^{(2)} = +\frac{1}{2}, s_b^{(1)} = -\frac{1}{2}\right). \tag{2.6}$$

The inequality (2.6) is not considered in NQM – in this case the fermions are not in a singlet but in a triplet spin state, i.e. in non-relativistic QM the inequalities (2.2) and (2.6) are essentially different. In the framework of QFT the probabilities in both inequalities are calculated uniformly.

In the case of the decay of a scalar particle into two photons, e.g. $H^0 \rightarrow \gamma\gamma$, in the particle’s rest frame, there appears a two-photon state with $J_{\gamma\gamma} = L_{\gamma\gamma} = S_{\gamma\gamma} = 2$, fully correlated by linear polarization in any direction orthogonal to photons impulses. Then the inequality analog of (2.2) for this case will be:

$$w\left(\lambda_a^{(1)} = 1, \lambda_b^{(2)} = 2\right) \leq w\left(\lambda_a^{(1)} = 1, \lambda_c^{(2)} = 2\right) + w\left(\lambda_c^{(1)} = 1, \lambda_b^{(2)} = 2\right). \tag{2.7}$$
III. BELL’S INEQUALITIES IN QFT FOR THE DECAY OF A PSEUDOSCALAR PARTICLE INTO TWO FERMIONS

In the case of conserved $P$–parity the decay of a pseudoscalar particle to a fermion-antifermion pair can be described using an effective Hamiltonian:

$$\mathcal{H}^{PS}(x) = g \varphi(x) \left( f(x) \gamma^5 f(x) \right)_N,$$  \hspace{1cm} (3.1)

where $g$ is the effective coupling constant, $\varphi(x)$ is the field of the pseudoscalar particle, and $f(x)$ is the fermionic field. In the decays described by (3.1) the masses of fermion and antifermion should be equal. In all equations below we will use the index “1” for antifermion and the index “2” for fermion (and for masses too). The unitary normal vector $\mathbf{b}$ coincides with the fermion propagation direction, i.e. $\vec{k}_2 = \left| \vec{k}_2 \right| \hat{n}$. Let us sequentially consider three possible cases.

### A. The decay of a resting scalar particle

Let the pseudoscalar be at rest at the origin of the coordinate system and place a spin state analyzers at infinity to measure spin projections in a planes parallel to $(xz)$ plane. If the spin projections of the fermion on the $\vec{a}$ direction and of the antifermion on the $\vec{b}$ direction are equal to $+1/2$, the decay amplitude can be written as follows:

$$A \left( s_a^{(2)} = + \frac{1}{2}, s_b^{(1)} = + \frac{1}{2} \right) = - g \bar{u}(\vec{k}_2, s_a^{(2)}) = +1/2, \bar{a} \right) \gamma^5 v(\vec{k}_1, s_b^{(1)}) = +1/2, \bar{b} \right) =
$$

$$= g \sqrt{\frac{\varepsilon_2 + m_2}{\varepsilon_1 + m_1}} (M + m_1 - m_2) \chi_4^{\dagger}(\bar{a}) \chi_{-}(\bar{b}).$$

This amplitude does not depend on angular variables. Then, taking into account the first equation of (B6), the probability can be written in the following way:

$$w \left( s_a^{(2)} = + \frac{1}{2}, s_b^{(1)} = + \frac{1}{2} \right) = f(M, m_1, m_2, \theta, \phi) \sin^2 \frac{\theta_{ab}}{2}. \hspace{1cm} (3.2)$$

If the direction of fermion propagation is not taken into account, then

$$f(M, m_1, m_2, \theta, \phi) = \frac{g^2}{16 \pi} \frac{M^2 - (m_1 - m_2)^2}{M^3} \chi^{1/2}(M^2, m_1^2, m_2^2), \hspace{1cm} (3.3)$$

where $\chi(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc$ is the triangular function, defining the dependency of the probability on the phase space.

From (3.2) and (3.3) it follows that in the framework of QFT the Bell inequality (2.2) reduces to (2.3), which was obtained in the non-relativistic approach.

If take into account the fermion propagation direction, i.e. select only the fermions with fixed values of $\hat{\theta}$ and $\bar{\phi}$, then the equation (3.3) should be modified as

$$f(M, m_1, m_2, \hat{\theta}, \bar{\phi}) = \frac{g^2}{16} \frac{M^2 - (m_1 - m_2)^2}{M^3} \chi^{1/2}(M^2, m_1^2, m_2^2) \frac{\sin \hat{\theta}}{(2 \pi)^2}. \hspace{1cm} (3.4)$$

Equation (3.4) reflects the fact that if both fermion and antifermion propagate along the $z$-axis (i.e. $\sin \hat{\theta} = 0$), they wouldn’t be registered and inequality (2.2) becomes meaningless. We emphasize the fact that in the non-relativistic approach this obvious deduction cannot be made.

### B. Taking into account the finite distance from the analyzers

Next we look at how the inequality (2.3) could evolve if both of the spin analyzers are parallel to the $(xz)$ plane and cross the $y$-axis at a distance $\pm L/2$ from the coordinates origin. This case may be better described if instead of the Hamiltonian (3.1) we use

$$\tilde{H}^{PS}(x) = g \varphi(x) \left( f(x) \gamma^5 f(x) \right)_N \theta \left( \frac{L}{2} - y \right) \theta \left( \frac{L}{2} + y \right), \hspace{1cm} (3.5)$$
where \( \theta(L/2 \pm y) \) is the Heaviside function\(^3\).

Let \( k_1 = (k_{1x}, k_{1y}, k_{1z}) \), \( k_2 = (k_{2x}, k_{2y}, k_{2z}) \) and let the decaying pseudoscalar meson still be at rest. Then, the Hamiltonian (3.5) leads us to the following expression for the two-body phase space:

\[
d\Phi_2 = (2\pi)^3 \delta(k_{1x} + k_{2x}) \delta(k_{1z} + k_{2z}) \delta(M - \varepsilon_1 - \varepsilon_2) L 
\left( \frac{\sin(L(k_{1y} + k_{2y})/2)}{L(k_{1y} + k_{2y})/2} \right)^2 \frac{d\vec{k}_1}{(2\pi)^3 2\varepsilon_1} \frac{d\vec{k}_2}{(2\pi)^3 2\varepsilon_2}.
\]

Using the first two \( \delta \)-functions it is easy to get rid of the integration over \( dk_{1x} \) and \( dk_{1z} \). This leads to: \( k_{2x} = -k_{1x} = k_x \) and \( k_{2z} = -k_{1z} = k_z \). In order to integrate over \( d\vec{k}_2 \) in the momentum space it is convenient to switch to cylindrical coordinates \( k_T \) and \( \gamma \) as follows: \( k_x = k_T \cos \gamma \) and \( k_z = k_T \sin \gamma \). Then

\[
d\vec{k}_2 = \frac{1}{2} d k_T^2 \, d\gamma \, d k_{2y}.
\]

After integrating over \( d\vec{k}_2^2 \), we use the remaining \( \delta \)-function the symmetric variables of integration \( k_y = k_{1y} + k_{2y} \) and \( \Delta_y = k_{1y} - k_{2y} \), we get the following for the two-body phase space:

\[
d\Phi_2 = \frac{1}{2^6 \pi^3} \frac{L}{M} \left( \frac{\sin(L k_{y}/2)}{L k_y/2} \right)^2 d\gamma \, d k_y \, d\Delta_y.
\] (3.6)

Removing all of the \( \delta \)-functions in the phase space yields \( k_T, \varepsilon_1 \) and \( \varepsilon_2 \) equal to:

\[
k_T = \frac{1}{2M} \lambda^{1/2} \left( M^2, (k_y + \Delta_y)^2/4 + m^2, (k_y - \Delta_y)^2/4 + m_y^2 \right);
\]

\[
\varepsilon_1 = \frac{1}{2M} \left( M^2 + m_y^2 - m_x^2 + k_y \Delta_y \right);
\]

\[
\varepsilon_2 = \frac{1}{2M} \left( M^2 - m_y^2 + m_x^2 - k_y \Delta_y \right).
\] (3.7)

In cylindrical coordinates the scalar product and vector product of the momentums \( \vec{k}_1 \) and \( \vec{k}_2 \) can be written as follows:

\[
(\vec{k}_2 \, \vec{k}_1) = \frac{1}{4} \left( k_y^2 - \Delta_y^2 \right) - k_T^2;
\]

\[
\vec{k}_2 \times \vec{k}_1 = k_y k_T (-\sin \gamma, 0, \cos \gamma).
\] (3.8)

Using (3.8) and the second equation from (130) one can calculate the value of the convolution:

\[
\epsilon^{ijk} k_i^1 k_i^2 \, w_{a+}^{i} = \left( \vec{w}_{++}, \vec{k}_2 \times \vec{k}_1 \right) = - k_y k_T \sin \left( \gamma + \frac{\chi^1}{2} \right).
\] (3.9)

When the projections of the fermion spin on the direction of \( \vec{a} \) and the antifermion spin on the direction of \( \vec{b} \) both are equal to \(+1/2\), the decay amplitude can be written as follows:

\[
\tilde{A} \left( s^{(2)} = + \frac{1}{2}, s^{(1)} = + \frac{1}{2} \right) = g \left[ \sqrt{\varepsilon_1 + m_1} \sqrt{\varepsilon_2 + m_2} \chi^\dagger_+ (\vec{a}) \chi^- (\vec{b}) - \right.
\]

\[
- \sqrt{\varepsilon_1 - m_1} \sqrt{\varepsilon_2 - m_2} \chi^\dagger_+ (\vec{a}) \left( \frac{\vec{\sigma} \, \vec{k}_2}{k_2} \right) \chi_- (\vec{b}) \bigg| \frac{\vec{k}_1}{k_1} \bigg| \chi^\dagger_- (\vec{b}) \right].
\] (3.10)

Considering (3.8), (3.9), the expansion for \( \sigma^i \sigma^j \) and the identity \( |\vec{k}_{1,2}| = \sqrt{\varepsilon_{1,2} - m_{1,2}^2} \), the following expression for the amplitude can be derived:

\[
\tilde{A} \left( s^{(2)} = + \frac{1}{2}, s^{(1)} = + \frac{1}{2} \right) = \frac{g M \left[ \tilde{A}^{(1)}_{++} \sin \frac{\chi^1}{2} + i \frac{k_y}{k_T} \tilde{A}^{(2)}_{++} \sin \left( \gamma + \frac{\chi^1}{2} \right) \right]}{\sqrt{\varepsilon_1 + m_1} / M \sqrt{\varepsilon_2 + m_2} / M}.
\]

\(^3\) Note that \( x \) in (3.3) means 4-vector \( x^\mu = (t, x, y, z) \), while \( y \) is a 3-rd component of this 4-vector.
where the dimensionless real functions

\[
\tilde{A}^{(1)}_{++} = \frac{\varepsilon_1 + m_1}{M} \frac{\varepsilon_2 + m_2}{M} + \frac{1}{4 M^2} (4 k_y^2 + \Delta_y^2 - k_y^2);
\]
\[
\tilde{A}^{(2)}_{++} = \frac{k_T}{M}.
\]

(3.11)

both do not depend on the polar angle \( \gamma \). The amplitude therefore can easily be squared and integrated over \( d\gamma \). Then, considering (3.6) for the differential probability of the decay of the resting pseudoscalar particle:

\[
dw \left( s_a^{(2)} = + \frac{1}{2}, s_b^{(1)} = + \frac{1}{2} \right) = \frac{g^2}{2 \pi^2} \frac{M^2 L}{(\varepsilon_1 + m_1)(\varepsilon_2 + m_2)} \left( \frac{\sin(L k_y/2)}{L k_y/2} \right)^2 \times \]
\[
\times \left[ 2 \left| \tilde{A}^{(1)}_{++} \right|^2 \sin^2 \frac{\theta_{ab}}{2} + \frac{k_y^2}{M^2} \left| \tilde{A}^{(2)}_{++} \right|^2 \right] dk_y d\Delta_y.
\]

(3.12)

From (3.12) it follows that when spin analyzers are located at a finite distance, a constant adjustment appears in (2.3). However this adjustment is quite unimportant. Actually, in (3.12) the main integral contribution resides in the area \( k_y \in [-1/L, +1/L] \). We suppose \( L \) to be macroscopic – hence \( 1/L \ll M \), i.e.

\[
\frac{k_y^2}{M^2} \sim \frac{1}{(M L)^2} \ll 1.
\]

For example if \( L \sim 2 \text{ cm} \), and \( M \sim 1 \text{ GeV} \), then \( 1/(M L)^2 \sim 10^{-28} \), which is below the current available experimental precision by many orders of magnitude. Thus if the distance between the spin analyzers is macroscopic, then this adjustment is unimportant. The integration over \( d\Delta_y \) doesn’t affect this conclusion, because the amplitudes \( \tilde{A}^{(1,2)}_{++} \) are smooth functions of variable \( \Delta_y \) (this fact follows from (3.11)).

Later in this article we always suppose the analyzers to reside at infinity.

C. The adjustment due to the non-antiparallelity of the fermion and antifermion momenta

In the previous subsection we showed that due to the non-conservation of the momentum projection on the \( y \)-axis in the rest frame of a pseudoscalar particle, the angle between the vectors \( \vec{k}_2 \) and \( \vec{k}_1 \) had a small deviation from \( \pi \). Due to the violation of antiparallelity of the two vectors in the Bell inequality (2.3) a quadratic correction appeared by the small parameter \( k_y/M \).

The antiparallelity of the vectors \( \vec{k}_2 \) and \( \vec{k}_1 \) can be caused by the emission of a soft photon from one of the fermions. The energy \( \omega \) of the soft photon can be below the detection threshold and a lot less than energies \( \varepsilon_1 \) and \( \varepsilon_2 \) (these are of the order of \( M/2 \)). It is well known from standard QED that in the first approximation by \( \omega/\varepsilon_{1,2} \) the amplitude of the emission of a soft photon can be factorized by the amplitude for the no-emission process and by a factor corresponding to the emission of a soft photon. Therefore in the case of soft photon emission the possible corrections for (2.3) may only appear starting in the second order by the small parameter \( \omega/\varepsilon_{1,2} \sim \omega/M \).

Let us prove the general statement. Let \( \vec{k}_1 = |\vec{k}_1| \vec{n}_1 \) and \( \vec{k}_2 = |\vec{k}_2| \vec{n} \) and the conservation law

\[
|\vec{k}_1| \vec{n}_1 + |\vec{k}_2| \vec{n} = |\vec{p}| \vec{\ell},
\]

(3.13)

where the vector \( \vec{\ell} \) in not parallel to the vectors \( \vec{n}_1 \) and \( \vec{n} \). Additionally let \( E = \varepsilon_1 + \varepsilon_2 \) and

\[
\frac{|\vec{p}|}{M} \ll 1.
\]

Then

\[
w \left( s_a^{(2)} = + \frac{1}{2}, s_b^{(1)} = + \frac{1}{2} \right) = g^2 f_0(M, m_1, m_2) \sin^2 \frac{\theta_{ab}}{2} + O \left( \frac{|\vec{p}|^2}{M^2} \right).
\]

(3.14)
Actually, from (3.10) and (3.13) it follows that if vectors \( \vec{n}_1 \) and \( \vec{n} \) are not antiparallel, then the amplitude can be written down in the form:\(^4\)

\[
A \left( s_a^{(2)} = + \frac{1}{2}, s_b^{(1)} = + \frac{1}{2} \right) = g \sqrt{\frac{\varepsilon_2 + m_2}{\varepsilon_1 + m_1}} \left[ \left( E + m_1 - m_2 - \sqrt{\varepsilon_2 - m_2 \varepsilon_2 + m_2} |\vec{p}| \ell^i n^i \right) \sin \frac{\theta_{ab}}{2} - i \sqrt{\frac{\varepsilon_2 - m_2}{\varepsilon_2 + m_2}} |\vec{p}| \epsilon^{ijk} n^i \ell^j w_{++}^k \right].
\]  

(3.15)

The series expansion by the small parameter \(|\vec{p}|/M\) gives:

\[
E = E^{(0)} + E^{(1)} \left( \frac{|\vec{p}|}{M} \ell^i n^i \right) + O \left( \frac{|\vec{p}|^2}{M^2} \right); \\
\varepsilon_{1,2} = \varepsilon_{1,2}^{(0)} + \varepsilon_{1,2}^{(1)} \left( \frac{|\vec{p}|}{M} \ell^i n^i \right) + O \left( \frac{|\vec{p}|^2}{M^2} \right); \\
d\Phi_n = \frac{d\Omega}{4\pi} \left( \Phi_2^{(0)} + \Phi_2^{(1)} \left( \frac{|\vec{p}|}{M} \ell^i n^i \right) + O \left( \frac{|\vec{p}|^2}{M^2} \right) \right) d\Phi_{n-2},
\]

where the phase space \( d\Phi_n \) includes the integration over the variables different from the angle variables of the fermion, and considers a possible emission of \( n - 2 \) soft photons, whilst \( d\Omega = d\cos \theta d\phi \) selects the integration over the fermion propagation direction. The explicit form of the coefficients of the expansion in (3.10) depends on the source of the momentum \( \vec{p} \). For example, if that momentum results from the Brownian motion of the decaying pseudoscalar particle, then \( E \approx M \left( 1 + \frac{1}{2} \frac{|\vec{p}|^2}{M^2} \right) \). Hence \( E^{(0)} = M, E^{(1)} = 0 \).

In accordance with (3.15) and (3.10) for the decay probability it can be written:

\[
w \left( s_a^{(2)} = + \frac{1}{2}, s_b^{(1)} = + \frac{1}{2} \right) = g^2 \int d\Omega_{n-1} \int \frac{d\Omega}{4\pi} \left( \alpha^{(0)} \sin^2 \frac{\theta_{ab}}{2} + \alpha^{(1)}_1 \frac{|\vec{p}|}{M} \ell^i n^i + \alpha^{(1)}_2 \frac{|\vec{p}|}{M} \epsilon^{ijk} n^i \ell^j Im \left( w_{++}^k \right) + O \left( \frac{|\vec{p}|^2}{M^2} \right) \right).
\]

(3.17)

Given that

\[
\int \frac{d\Omega}{4\pi} = 1 \quad \int \frac{d\Omega}{4\pi} n^i = \langle n^i \rangle_{\Omega} = 0,
\]

then (3.14) immediately follows from (3.17).

Note the statement (3.14) is quite general. It’s true for the Brownian motion of a decaying particle, the uncertainties from the composition of the initial state, the interaction between the final state fermions (e.g. via the Coulomb force), or the interaction with a weak external field. In the beginning of the current subsection two cases complying with (3.14) were presented: the emission of soft photons and a phase space limit of the decay. The equality (3.14) may be usefull not only for the Bell inequalities. It can be easily adapted to the various tasks in quantum teleportation and quantum measurements.

**IV. THE BELL INEQUALITIES IN QFT FOR THE DECAY OF A SCALAR PARTICLE INTO TWO FERMIONS**

The effective Hamiltonian of the decay of a scalar particle to a fermion-antifermion pair can be written in exactly the same way as (3.1):

\[
\mathcal{H}^{(S)}(x) = g \varphi(x) \left( \bar{f}(x) f(x) \right) N,
\]

(4.1)

where \( \varphi(x) \) is a field of the scalar particle. Like in section III, the index “1” will always be for antifermion, and index “2” for fermion. The unitary vector \( \vec{n} \) is set to the direction of the fermion propagation, while the vector \( \vec{n}_1 \) is set to the antifermion direction.

We will only consider the case when the scalar particle is resting and again spin analyzers are placed at infinity in the planes parallel to the \((xz)\)-plane. Let us consider two cases:

---

\(^4\) Later we assume \( \epsilon^{123} = \epsilon_{123} = +1 \).
A. Decay of a scalar particle when the direction of the fermion propagation is not fixed

If the projection of the fermion spin on \( \vec{a} \) direction is equal to +1/2 and the projection of the antifermion spin on \( \vec{b} \) direction is equal to −1/2, then in accordance with (4.1), the decay amplitude has the following form:

\[
A \left( s_a^{(2)} = + \frac{1}{2}, s_b^{(1)} = - \frac{1}{2} \right) = -g \vec{u}(\vec{k}_2, s_a^{(2)}) = +1/2, \vec{u}) \ e(\vec{k}_1, s_b^{(1)} = -1/2, \vec{b}) = g \sqrt{\frac{\varepsilon_2 - m_2}{\varepsilon_1 + m_1}} (M + m_1 + m_2) \chi_+^\dagger(\vec{a}) (\vec{a}) \chi_+(\vec{b}).
\]

In the last expression we took into account that in the rest frame of the scalar particle the unitary vectors \( \vec{n}_1 \) and \( \vec{n} \) are related as:

\[
\vec{n}_1 = -\frac{|\vec{k}_2|}{|\vec{k}_1|} \vec{n}.
\]

Since the amplitude does not depend on the angular variables, the expression for the probability has the following form:

\[
w \left( s_a^{(2)} = + \frac{1}{2}, s_b^{(1)} = - \frac{1}{2} \right) = \int \frac{|A(s_a^{(2)} = +1/2, s_b^{(1)} = -1/2)|^2}{2 M} d\Phi_2,
\]

If the momentum direction of the fermion is not fixed, then the integration should be done over the whole space angle. Assuming the integration

\[
\int \frac{d\Omega}{4 \pi} \left| \chi_+^\dagger(\vec{a}) (\vec{a}) \chi_+(\vec{b}) \right|^2 = \frac{1}{3} \left| \chi_+^\dagger(\vec{a}) \vec{a} \chi_+(\vec{b}) \right|^2
\]

and the last of the identities (B6), we get for the decay probability:

\[
w \left( s_a^{(2)} = + \frac{1}{2}, s_b^{(1)} = - \frac{1}{2} \right) = \frac{g^2}{48 \pi} \frac{M^2 - (m_1 + m_2)^2}{M^3} \lambda^{1/2}(M^2, m_1^2, m_2^2) \left( 1 + \sin^2 \frac{\theta_{ab}}{2} \right).
\]

From (4.2) it follows that the Bell inequalities (2.4) are reduced to the following trigonometric inequality:

\[
\sin^2 \frac{\theta_{ab}}{2} \leq 1 + \sin^2 \frac{\theta_{ac}}{2} + \sin^2 \frac{\theta_{bc}}{2},
\]

Unlike (2.3), the inequality (4.3) is always true. Hence (4.3) is unusable to test the principle of complementarity.

B. Decay of the scalar particle in the case when the direction of fermion propagation is fixed

Now we show that it is still possible to test the Bell inequalities for the decays of scalar particles. Let us consider a fermion propagating along some chosen direction defined by angles \( \theta \) and \( \phi \). In this case the angular part of the probability \( w \left( s_a^{(2)} = + \frac{1}{2}, s_b^{(1)} = - \frac{1}{2} \right) \) has the form

\[
\sin \theta \left| \chi_+^\dagger(\vec{a}) (\vec{a}) \chi_+(\vec{b}) \right|^2 = \sin \theta \left( \sin^2 \theta \cos^2 \phi \sin^2 \frac{\kappa_{ab}}{2} + \cos^2 \theta \cos^2 \frac{\kappa_{ab}}{2} + \frac{1}{2} \sin(2\theta) \cos \phi \sin \kappa_{ab} + \sin^2 \theta \sin^2 \phi \sin^2 \frac{\theta_{ab}}{2} \right).
\]

If we require maximal violation of Bell inequalities, we should choose \( \theta = \pi/2 \) in (4.4). Then:

\[
w \left( s_a^{(2)} = + \frac{1}{2}, s_b^{(1)} = - \frac{1}{2} \right) \sim \cos^2 \phi \sin^2 \frac{\kappa_{ab}}{2} + \sin^2 \phi \sin^2 \frac{\theta_{ab}}{2}.
\]
With $\phi = \pi/2$ (i.e. in the case when the fermion is propagating along the $y$–axis orthogonal to the direction of the spin projections) the substitution of (4.5) into (2.6) leads to the trigonometric inequality (2.3). If one chooses $\phi = 0$, the following new trigonometric inequality can be obtained:

$$
\sin^2 \frac{K_{ab}}{2} \leq \sin^2 \frac{K_{ac}}{2} + \sin^2 \frac{K_{bc}}{2},
$$

(4.6)

which is violated when $\theta_a = \pi - \alpha$, $\theta_b = \alpha$, $\theta_c = 3\alpha$ and $\pi/2 > \alpha > \pi/4$.

The linear combination of the inequalities (2.3) and (4.6) with weights $\sin^2 \phi$ and $\cos^2 \phi$ respectively is harder to violate than each of the two equations individually. The conditions of the violation are different for (2.3) and for (4.6).

For the decay of the scalar particle to a fermion-antifermion pair small deviations from the antiparallel state of the vectors $\vec{n}_1$ and $\vec{n}$ are quadratic by a small parameter only in the case of integration over the $d\Omega$. This leads to a never-violated inequality (4.3). For any chosen direction these deviations are proportional to the first power of $|\vec{p}|$ ($\vec{\ell} \vec{n}$).

V. BELL’S INEQUALITIES FOR THE DECAYS OF SCALAR AND PSEUDOSCALAR PARTICLES INTO TWO PHOTONS

Consider the decay of a spin-less particles into two photons in the final state. Below we will consider a Higgs boson as a scalar and a $\pi^0$ meson as a pseudoscalar.

As long as $P$–parity is conserved, the amplitude of $H^0 \rightarrow \gamma \gamma$ decay has the following form:

$$
A^{H^0 \rightarrow \gamma \gamma} = F_H \epsilon^\mu (\lambda^{(1)}) \epsilon^*_\nu (\lambda^{(2)}),
$$

where $\vec{k}_2 = \omega \vec{n} = \omega (0, 1, 0)$. The 4-vectors of the photon polarization in the direction $\vec{a}$, orthogonal to $\vec{n}$, are defined as follows:

$$
\epsilon^\mu (\lambda^{(1,2)}_a) = (0, \sin \theta_a, 0, \cos \theta_a); \quad \epsilon^\mu (\lambda^{(1,2)}_b) = (0, \cos \theta_a, 0, -\sin \theta_a).
$$

In the case when one photon has a polarization “1” in the direction $\vec{a}$ and a polarization “2” in the direction orthogonal to $\vec{b}$, the amplitude can be written as follows:

$$
A^{H^0 \rightarrow \gamma \gamma} (\lambda^{(1)}_a = 1, \lambda^{(2)}_b = 2) = \epsilon^\mu (\lambda^{(1)}_a) \epsilon^*_\nu (\lambda^{(2)}_b) = -F_H \sin \theta_{ab}.
$$

Hence, Bell’s inequality (2.7) reduces to the following trigonometric inequality:

$$
\sin^2 \theta_{ab} \leq \sin^2 \theta_{ac} + \sin^2 \theta_{bc},
$$

(5.1)

which is violated when the vectors $\vec{a}$ and $\vec{b}$ form an acute angle, while $\vec{c}$ is its bisector. The inequality (5.1) is analogous to the trigonometric inequality (2.3).

At the Large Hadron Collider, where the Higgs bosons may be born, the plane of $\ell^+ \ell^-$–pair should be used as a spin projector. The invariant mass of a lepton pair should be small in order to effectively exclude the contribution from a virtual photon. Effects of the exchange interaction are negligible, because both of lepton pairs are largely separated in phase space in the rest frame of $H^0$.

Let’s now consider a decay $\pi^0 \rightarrow \gamma \gamma$. The amplitude of that decay has the form:

$$
A^{\pi^0 \rightarrow \gamma \gamma} = F_\pi \epsilon_{\mu \nu \alpha \beta} \epsilon^\alpha (\lambda^{(1)}) \epsilon^\beta (\lambda^{(2)}) k^\mu_1 k^\beta_2.
$$

Given that in this problem $k^\mu_1 = \omega (1, 0, -1, 0)$, $k^\mu_2 = \omega (1, 0, 1, 0)$ and $\epsilon_{0123} = -\epsilon^{0123} = +1$, we have:

$$
A^{\pi^0 \rightarrow \gamma \gamma} (\lambda^{(1)}_a = 1, \lambda^{(2)}_b = 1) = -2 \omega^2 F_\pi \sin \theta_{ab}.
$$

Substitution of the amplitude into inequality (2.4) again leads to the inequality (5.1).

Thus, Bell’s inequalities in Wigner form for the decay of the scalar (2.7) and pseudoscalar (2.4) particle into two photons lead to the same trigonometric inequality (5.1), which can be used for the experimental test of the Bohr principle of complementarity as in inequality (2.3).
If we do not require $P$-parity conservation, then the decay amplitude of a pseudoscalar or scalar meson $P^0$ with mass $M_P$ into a photon pair has the following form:

$$A^{P^0 \rightarrow \gamma\gamma} = e^{*\mu}(\lambda^{(1)}) e^{*\nu}(\lambda^{(2)}) \left[ A \varepsilon_{\mu\nu\alpha\beta} k_1^{\alpha} k_2^{\beta} - i B \left( k_2^{\mu} k_1^{\nu} - g_{\mu\nu} M_P^2 \right) \right],$$

For this amplitude the inequality (2.4) is transformed into a trigonometric inequality:

$$\left( |A|^2 - |B|^2 \right) \sin^2 \theta_{ab} \leq |B|^2 + \left( |A|^2 - |B|^2 \right) \left( \sin^2 \theta_{ac} + \sin^2 \theta_{bc} \right). \quad (5.2)$$

The inequality (5.2) can be violated only when $|A| \geq \sqrt{2}|B|$. The violation reaches a maximum when $|B| = 0$. In this case, inequality (5.2) is transformed to (5.1) obtained above. Also, the amplitude $A^{P^0 \rightarrow \gamma\gamma}$ transforms inequality (2.7) into

$$\left( |B|^2 - |A|^2 \right) \sin^2 \theta_{ab} \leq |A|^2 + \left( |B|^2 - |A|^2 \right) \left( \sin^2 \theta_{ac} + \sin^2 \theta_{bc} \right), \quad (5.3)$$

Transposing $A$ and $B$ we again have the inequality (5.2).

The inequalities (5.2) and (5.3) are not affected if one of the final particles is a vector meson, instead of a photon. For example, both inequalities can be written for the decay $B^0_d \rightarrow K^{*0}\gamma$. However, in this case $|A| = |B|$. Therefore, the formulae (5.2) and (5.3) become trivial statements: $|B| \geq 0$ and $|A| \geq 0$ respectively. The same trivialization of Bell’s inequalities occurs in rare radiative decays $B^0_s \rightarrow \gamma\gamma$. In both cases the cause of the triviality stems from these decays going through loop diagrams, which are reduced to the effective tensor operator $\bar{s} \sigma^{\mu\nu}(1 + \gamma^5)b$. If the contributions from tensor and pseudotensor quark currents are essentially different, then the inequalities (5.2) and (5.3) may be violated. This is possible for example in LR-models.
VI. CONCLUSION

The following conclusions are drawn for this paper.

1. It is shown that the relativistic generalization of Bell’s inequalities in Wigner form for the decay of a resting pseudoscalar particle into a fermion-antifermion pair reproduces the non-relativistic result (2.3) for the decay of a singlet state into the two states with spin 1/2.

2. We proved that the corrections due to small deviations from the exact antiparallel state of a fermion and antifermion are quadratic by the small parameter $|\vec{k}_1 + \vec{k}_2|/M$.

3. For the case of scalar particle decay into a fermion-antifermion pair, we obtained a new type of Bell’s inequalities in Wigner form (2.6) based on a full correlation of spin projections of fermion and antifermion on any direction.

4. It is shown that the inequality (2.6) may lead to the trigonometric inequality (2.3) as well as to a new trigonometric inequality (4.6) depending on the selection of fermions by the propagation direction relative to spin analyzers.

5. The decay of a scalar and pseudoscalar particle into two photons leads to a new trigonometric inequality (5.1). This inequality can be experimentally tested at current colliders.

6. If $P$-parity is not conserved the inequality (5.1) may be generalized to inequalities (5.2) and (5.3). The inequalities (5.2) and (5.3) are not affected if one of the final particles is a vector meson, instead of a photon.

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Appendix A: A derivation of Bell’s inequalities in Wigner form

In this appendix we give a detailed derivation of Bell’s inequalities in Wigner form for decays of pseudoscalar and scalar particles.

1. Derivation of Bell inequalities for a two-body decay of a pseudoscalar particle

Consider a decay of a pseudoscalar particle into two fermions. Let us make a key assumption for the derivation. The spin projections of fermion and antifermion on three non-parallel directions $\vec{a}$, $\vec{b}$, $\vec{c}$ are simultaneously elements of a physical reality. Then we can speak of nonnegative number of fermion-antifermion pairs, with $s_a^{(2)} = +1/2$, $s_b^{(1)} = +1/2$ and $s_c^{(1)} = +1/2$. Denote the number of such pairs as $N(s_a^{(2)} = +1/2, s_b^{(1)} = +1/2, s_c^{(1)} = +1/2)$. Now it is easy to obtain the number of pairs with spin projections only on two directions:

$$N\left(s_a^{(2)} = \frac{1}{2}, s_b^{(1)} = \frac{1}{2}, s_c^{(1)} = \frac{1}{2}\right) = N\left(s_a^{(2)} = \frac{1}{2}, s_b^{(1)} = \frac{1}{2}, s_c^{(1)} = \frac{1}{2}\right) +$$

$$+ N\left(s_a^{(2)} = \frac{1}{2}, s_b^{(1)} = \frac{1}{2}, s_c^{(1)} = \frac{1}{2}\right). \quad (A1)$$

In analogy

$$N\left(s_a^{(2)} = \frac{1}{2}, s_c^{(1)} = \frac{1}{2}\right) = N\left(s_a^{(2)} = \frac{1}{2}, s_b^{(1)} = \frac{1}{2}, s_c^{(1)} = \frac{1}{2}\right) +$$

$$+ N\left(s_a^{(2)} = \frac{1}{2}, s_b^{(1)} = \frac{1}{2}, s_c^{(1)} = \frac{1}{2}\right). \quad (A2)$$
And finally
\[ N \left( s_c^{(2)} = +\frac{1}{2}, s_b^{(1)} = +\frac{1}{2} \right) = N \left( s_a^{(2)} = +\frac{1}{2}, s_c^{(2)} = +\frac{1}{2}, s_b^{(1)} = +\frac{1}{2} \right) + \]
\[ + N \left( s_a^{(2)} = -\frac{1}{2}, s_c^{(2)} = +\frac{1}{2}, s_b^{(1)} = +\frac{1}{2} \right) \]
or, using the anticorrelation condition (2.1) for the direction \( \vec{c} \),
\[ N \left( s_c^{(2)} = +\frac{1}{2}, s_b^{(1)} = +\frac{1}{2} \right) = N \left( s_a^{(2)} = +\frac{1}{2}, s_b^{(1)} = -\frac{1}{2} \right) + \]
\[ + N \left( s_a^{(2)} = -\frac{1}{2}, s_c^{(1)} = +\frac{1}{2}, s_b^{(1)} = +\frac{1}{2} \right). \] (A3)

In equalities (A1) – (A3) each term in the right part is nonnegative. Hence we can write a kind of “triangle inequality”:
\[ N \left( s_a^{(2)} = +\frac{1}{2}, s_b^{(1)} = +\frac{1}{2} \right) \leq N \left( s_a^{(2)} = +\frac{1}{2}, s_c^{(1)} = -\frac{1}{2} \right) + \]
\[ + N \left( s_a^{(2)} = -\frac{1}{2}, s_b^{(1)} = +\frac{1}{2} \right). \] (A4)

Since the number of fermion-antifermion pairs is inversely proportional to the decay probability, we immediately have the inequality (2.2).

The basic inequality (A4) may be rewritten in a few equivalent forms. For example if \( \vec{b} \) and \( \vec{c} \) are changed to opposite directions, then using the condition (2.1) one can obtain the inequality
\[ N \left( s_a^{(2)} = +\frac{1}{2}, s_b^{(1)} = -\frac{1}{2} \right) \leq N \left( s_a^{(2)} = +\frac{1}{2}, s_c^{(1)} = +\frac{1}{2} \right) + \]
\[ + N \left( s_b^{(2)} = +\frac{1}{2}, s_c^{(1)} = +\frac{1}{2} \right). \]

If we change only the direction \( \vec{c} \) to the opposite, then the following inequality appears:
\[ N \left( s_a^{(2)} = +\frac{1}{2}, s_b^{(1)} = +\frac{1}{2} \right) \leq N \left( s_a^{(2)} = +\frac{1}{2}, s_c^{(1)} = -\frac{1}{2} \right) + \]
\[ + N \left( s_b^{(2)} = -\frac{1}{2}, s_b^{(1)} = +\frac{1}{2} \right). \]

It can be weakened by rewriting it in the following form:
\[ N \left( s_a^{(2)} = +\frac{1}{2}, s_b^{(1)} = +\frac{1}{2} \right) + N \left( s_a^{(2)} = +\frac{1}{2}, s_c^{(1)} = +\frac{1}{2} \right) + \]
\[ + N \left( s_b^{(2)} = +\frac{1}{2}, s_b^{(1)} = +\frac{1}{2} \right) \leq N_{tot}. \]

If the vectors \( \vec{a} \), \( \vec{b} \) and \( \vec{c} \) lie in the same plane, then in the framework of NQM, the last inequality is reduced to the following trigonometric inequality:
\[ \sin^2 \frac{\theta_{ab}}{2} + \sin^2 \frac{\theta_{ac}}{2} + \sin^2 \frac{\theta_{bc}}{2} \leq 2, \]
which is violated when the angle between the vectors \( \vec{a} \) and \( \vec{b} \) is close to \( \pi \), and the direction \( \vec{c} \) bisects this angle. This condition is more strict than the condition of violation of the inequality (2.3).

Since all the variants of Bell’s inequalities in Wigner form above either equivalent to the inequality (A4) or weaker, we will consider only the relativistic generalization of (A4).
2. Derivation of Bell’s inequalities for the decay of a scalar particle into two fermions

Consider the case of a resting scalar particle of mass \( M \), that decays into a fermion and an antifermion. Like in Appendix A1 we assume that the spin projections of the fermion and antifermion on three non-parallel directions set by unitary vectors \( \vec{a}, \vec{b} \) and \( \vec{c} \), are simultaneously elements of a physical reality. Then:

\[
N \left( s^{(2)}_a = + \frac{1}{2}, s^{(1)}_b = - \frac{1}{2} \right) = N \left( s^{(2)}_a = + \frac{1}{2}, s^{(1)}_b = - \frac{1}{2}, s^{(1)}_c = + \frac{1}{2} \right) + \\
+ N \left( s^{(2)}_a = + \frac{1}{2}, s^{(1)}_b = - \frac{1}{2}, s^{(1)}_c = - \frac{1}{2} \right)
\]  

(A5)

In analogy

\[
N \left( s^{(2)}_a = + \frac{1}{2}, s^{(1)}_c = - \frac{1}{2} \right) = N \left( s^{(2)}_a = + \frac{1}{2}, s^{(1)}_c = + \frac{1}{2}, s^{(1)}_b = - \frac{1}{2} \right) + \\
+ N \left( s^{(2)}_a = + \frac{1}{2}, s^{(1)}_c = - \frac{1}{2}, s^{(1)}_b = + \frac{1}{2} \right)
\]  

(A6)

And finally

\[
N \left( s^{(2)}_c = + \frac{1}{2}, s^{(1)}_b = - \frac{1}{2} \right) = N \left( s^{(2)}_c = + \frac{1}{2}, s^{(1)}_b = - \frac{1}{2}, s^{(1)}_a = + \frac{1}{2} \right) + \\
+ N \left( s^{(2)}_c = + \frac{1}{2}, s^{(1)}_b = - \frac{1}{2}, s^{(1)}_a = - \frac{1}{2} \right)
\]  

(A7)

or, considering the correlations of spin projections on the direction \( \vec{c} \):

\[
N \left( s^{(2)}_c = + \frac{1}{2}, s^{(1)}_b = - \frac{1}{2} \right) = N \left( s^{(2)}_c = + \frac{1}{2}, s^{(1)}_b = + \frac{1}{2}, s^{(1)}_a = - \frac{1}{2} \right) + \\
+ N \left( s^{(2)}_c = + \frac{1}{2}, s^{(1)}_b = - \frac{1}{2}, s^{(1)}_a = + \frac{1}{2} \right)
\]  

(A8)

From equalities (A5) – (A7) follows the inequality:

\[
N \left( s^{(2)}_a = + \frac{1}{2}, s^{(1)}_b = - \frac{1}{2} \right) \leq N \left( s^{(2)}_a = + \frac{1}{2}, s^{(1)}_c = - \frac{1}{2} \right) + \\
+ N \left( s^{(2)}_c = + \frac{1}{2}, s^{(1)}_b = - \frac{1}{2} \right)
\]  

(A9)

and its probabilistic analog — the inequality (2.6). Like formula (A3), the inequality (A8) can be transformed to other equivalent inequalities by switching the directions of \( \vec{b} \) and \( \vec{c} \).

Appendix B: A relativistic spin 1/2 operator and solutions for a free Dirac equality

Let the free Dirac particle of mass \( m \) propagate in the lab coordinate system over the direction defined by a unitary vector

\[
\vec{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta),
\]  

(B1)

where \( \theta \in [0, \pi], \phi \in [0, 2\pi] \). In this coordinate system the particle has energy \( \varepsilon_p \) and momentum \( \vec{p} = |\vec{p}| \vec{n} \).

The solution of the free Dirac equation in the standard representation for the particle has the form:

\[
u(\vec{p}, s_a, \vec{a}) = \begin{pmatrix} \frac{\sqrt{\varepsilon_p + m}}{\sqrt{\varepsilon_p - m}} (\vec{a} \vec{n}) & \chi_{s_a}(\vec{a}) \end{pmatrix},
\]  

(B2)

and for the antiparticle has the form:

\[
v(\vec{p}, s_a, \vec{a}) = \begin{pmatrix} \frac{\sqrt{\varepsilon_p - m}}{\sqrt{\varepsilon_p + m}} (\vec{a} \vec{n}) & \xi_{-s_a}(\vec{a}) \end{pmatrix},
\]  

(B3)
where \( s_a = \pm 1/2 \) is a spin projection on a unitary vector direction

\[
\vec{a} = (\sin \theta_a \cos \phi_a, \sin \theta_a \sin \phi_a, \cos \theta_a) .
\]

Two-component spinors \( \chi_{s_a}(\vec{a}) \) and \( \xi_{-s_a}(\vec{a}) \) obey the normalization conditions \( \chi_{s_a}(\vec{a})\dagger \chi_{s_a}(\vec{a}) = \delta_{s_a s'_a} \) and \( \xi_{-s_a}(\vec{a}) = -2 s_a \chi_{-s_a}(\vec{a}) \).

The solution (12) must be an eigenfunction of a projection operator

\[
(\vec{a} \vec{O}) u(\vec{p}, s_a, \vec{a}) = 2 s_a u(\vec{p}, s_a, \vec{a}),
\]

corresponding to eigenvalues \( 2s_a = \pm 1 \). The operator \( \vec{O} \) is a relativistic generalization of a spin 1/2 operator for a free particle and can be written as (13):

\[
\vec{O} = -\gamma^5 \vec{\gamma} + \gamma^5 \frac{\vec{p}}{\bar{\epsilon}_p} + \frac{\bar{\epsilon}_p (\bar{\gamma}_5 \vec{p})}{\bar{\epsilon}_p (\bar{\epsilon}_p + m)},
\]

where the matrix \( \gamma^5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3 \). Components of the operator (13) satisfy the standard commutation relations for doubled components of non-relativistic spin 1/2 operator \( (\gamma^{123} = +1) \)

\[
[O^i, O^j] = 2 i \epsilon^{ijk} O^k .
\]

This allows one to test the Bohr principle of complimentarity in QFT in analogy with NQM.

From the explicit form of the operator \( \vec{O} \) follows formulae for two-component spinors:

\[
\begin{align*}
\chi_{s_a} = +1/2(\vec{a}) & \equiv \chi_+(\vec{a}) = \begin{pmatrix} \cos \frac{\theta_a}{2} e^{-i\phi_a/2} \\ \sin \frac{\theta_a}{2} e^{i\phi_a/2} \end{pmatrix}, \\
\chi_{s_a} = -1/2(\vec{a}) & \equiv \chi_-(\vec{a}) = \begin{pmatrix} -\sin \frac{\theta_a}{2} e^{-i\phi_a/2} \\ \cos \frac{\theta_a}{2} e^{i\phi_a/2} \end{pmatrix} .
\end{align*}
\]

Hence, when \( \phi_a = \phi_b = 0 \)

\[
\begin{align*}
\chi^\dagger_+(\vec{a}) \chi_-(\vec{b}) &= \sin \frac{\theta_{ab}}{2}; \\
\vec{w}_{++} &= \chi^\dagger_+(\vec{a}) \vec{\sigma} \chi_-(\vec{b}) = \begin{pmatrix} \cos \frac{\kappa_{ab}}{2}, -i \cos \frac{\theta_{ab}}{2}, -\sin \frac{\kappa_{ab}}{2} \end{pmatrix}; \\
\vec{w}_{+-} &= \chi^\dagger_+(\vec{a}) \vec{\sigma} \chi_+(\vec{b}) = \begin{pmatrix} \sin \frac{\kappa_{ab}}{2}, i \sin \frac{\theta_{ab}}{2}, \cos \frac{\kappa_{ab}}{2} \end{pmatrix},
\end{align*}
\]

where \( \theta_{\alpha \beta} = \theta_{\alpha} - \theta_{\beta}, \kappa_{\alpha \beta} = \theta_{\alpha} + \theta_{\beta} \) and \( \{\alpha, \beta\} = \{a, b, c\} \).

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We present a relativistic generalization of the Wigner’s inequality for the scalar and pseudoscalar particles decaying to two particles with spin (fermions and photons.) We consider Wigner’s inequality with the full spin anticorrelation (with the non-relativistic analog) as well as the case with the full spin correlation. The latter case may be obtained by a special choice of the plane of measurement of the spin projections on the direction of propagation of fermions. The possibility for relativistic testing of Bohr’s complementarity principle is shown.

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I. INTRODUCTION

Since the inception of quantum mechanics in the first quarter of the 20-th century, disputes abound around two closely related issues:

1) Is the probabilistic nature of predictions of quantum theory and the confirmation by experimental measurements a reflection of the objective laws of the microcosm, or is the indeterminism a consequence of our ignorance of some “subtle interactions” among microparticles that would provide theoretical predictions and experimental measurements like in the case of deterministic classical mechanics. For example, in addition to the well-known measurable properties of elementary particles like mass, charge, spin, lepton and baryon numbers, color, weak isospine, etc., particles may have properties, which in principle, cannot be measured with macroscopic analyzers. This lack of information about the values of these variables makes the predictions of quantum mechanics probabilistic. This concept is known as the theory of hidden variables of quantum mechanics.

2) Are the particle parameters described by noncommuting operators elements of a physical reality simultaneously and independently of the act of measurement or are they fundamentally inseparable from the design and capabilities of a particle detector as it is postulated by the Bohr’s complementarity principle.

These issues are essential not only for non-relativistic quantum mechanics (NQM) under which they were intensely debated (a comprehensive review may be found in [1], [2]), but also for quantum field theory (QFT). In the framework of QFT this topic was highlighted in a few papers (e.g. [3] – [6]). More complete bibliography may be found in these works.

The experimental answer to the second of the above issues may be given by Bell’s inequalities. They were introduced for the first time by J.S.Bell in 1964 – 1966 [7] and then modified by Clauser, Horne, Shimony and Holt in 1969 [8]. In Bell’s original work three dichotomic variables \( A, B, \) and \( C \) were introduced. These variables were elements of the physical reality simultaneously due to the existence of some set of hidden variables \( \lambda \). The expected values of these dichotomic variables satisfy the following inequality:

\[
|\langle AB \rangle - \langle AC \rangle| \leq 1 + \langle BC \rangle.
\]  

(1.1)

From (1.1) follows the Clauser-Horne-Shimony-Holt (CHSH) inequality for four dichotomic variables with spectre \( \pm 1 \):

\[
|\langle AB \rangle + \langle AC \rangle + \langle DB \rangle - \langle DC \rangle| \leq 2.
\]  

(1.2)

The dichotomic variables \( A, B, C \) and \( D \) may be naturally implemented in the form of spin \( 1/2 \) projection on any non-parallel directions \( \vec{a}, \vec{b}, \vec{c} \) and \( \vec{d} \). However from the experimentalist point of view it is more feasible to use photon polarization and “flavour–CP” quantum numbers of neutral \( K \)– and \( B \)–mesons. For example, there is a recent paper of the Belle collaboration of the precise test of Bell’s inequalities in neutral \( B \)–mesons [9].

It is widely believed that for the derivation of Bell’s inequalities (1.1) and (1.2), the existence of local, context–depending\(^2\) hidden variables \( \lambda \) is required. Thus, the violation of Bell’s inequalities is often considered as a disproof of

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1 having a spectre of only two values, in this case \( \pm 1 \)

2 i.e. depending not only on the particle state but on the state of an analyzer
the existence of a wide class of hidden variables. This view comes from classical work\textsuperscript{7}. However this view is wrong. It was shown in the paper\textsuperscript{10} that for the derivation of (1.1) and (1.2) it is enough for only the nonnegative joint probabilities $W(A, B, C)$ and $W(A, B, C, D)$ to exist. These probabilities should satisfy Kolmogorov’s probability axioms. The existence of such probabilities is a mathematical reflection of the following statement: $(A, B, C, D)$ parameters of a given quantum system are simultaneously elements of physical reality. In addition we can assume that the existence of the nonnegative joint probabilities is provided by the hidden variables. Again we emphasize the fact that this assumption is not necessary for the derivation of formulae (1.1) and (1.2).

From these considerations it follows that Bell’s inequalities open the possibility for a direct experimental test of the Bohr’s complementarity principle. Actually, the violation of (1.1) and (1.2) shows that observables $A$, $B$, $C$ and $D$ described in NQM and QFT by noncommuting operators do not exist simultaneously as elements of physical reality\textsuperscript{3}. The choice of these observables is determined not only by a macroscopic system, but also by an analyzer used. This is clearly in favor of the principle of complementarity.

Currently there are no suggestions for how to test the principle of complementarity in the relativistic area for particles with non-zero masses. It is natural to try to perform such tests on current colliders in the reactions or decays of elementary particles. For these tests, proper relativistic generalization of Bell’s inequalities should be introduced for particular processes. Usually elementary particles are used for such tests as in the decay $\eta_c \to \Lambda\bar{\Lambda}$ (see\textsuperscript{11}). The overview of the main ideas for testing Bell’s inequalities in HEP may be found in\textsuperscript{12}.

In the another set of publication\textsuperscript{3} – \textsuperscript{6}, Bell’s inequalities are studied in the framework of a formal algebraic quantum field theory (AQFT). In this approach the value of a maximum possible violation of (1.2) in QFT\textsuperscript{4} was found. Also it was shown that the correlation between the entangled particles remains even after the local measurement of one of the particles\textsuperscript{3}. This fact though is quite obvious because the signal propagation speed in QFT is limited by the speed of light. However, in AQFT there is no particular suggestion for testing of these predictions.

In this paper we attempt to write a relativistic generalization of Bell’s inequalities for specific decays of elementary particles. It turns out that the most natural way for relativistic generalization of Bell’s inequalities is Wigner form\textsuperscript{13}, which is not dependent on the normalization of states and allows a direct test of Bohr’s complementarity principle in the relativistic region. The spin projections of photons and relativistic fermions to various directions were chosen as the observables with noncommuting operators. Note that it is convenient to use a relativistic generalization of spin 1/2 from the work\textsuperscript{14}.

The paper is organized as follows: in Section II a few variants of Bell’s inequalities were obtained. These are suggested for testing of the principle of complementarity in QFT. In Sections III and IV these inequalities are applied to decays of a scalar and pseudoscalar particle into a fermion-antifermion pair. Bell’s inequalities for the decays into two photons in final state are presented in Section V.

Some definitions and calculations can be found in the Appendices.

II. BELL’S INEQUALITIES IN WIGNER FORM

Bell’s inequalities in forms (1.1) or (1.2) are not suitable enough for relativistic generalization. First, wave functions are used for the derivation, so it cannot be used in QFT. Second, the operators, corresponding to $A$, $B$, $C$, $D$ values (usually the particles spin operators), should be generalized themselves in a relativistic case. It is desirable to find a variant of Bell’s inequalities without the above difficulties.

Such variant was proposed by E.Wigner\textsuperscript{13} in 1970. In this section we will show that Bell’s inequalities in Wigner form may be written in two different forms. The first form corresponds to the decay of a pseudoscalar particle into fermion-antifermion pair or into two photons. It coincides with\textsuperscript{13} for non-relativistic QM. The second form corresponds to the decay of a scalar particle into a fermion and antifermion or into a photon pair. This variant is usually not considered in NQM due to some natural obstacles.

A. Bell’s inequalities for two-body decays of pseudoscalar particle

Let us consider the decay of a resting particle with mass $M$ to a fermion-antifermion pair, where we label the momentum of the antifermion as $k_1^\ast$, the momentum of the fermion as $k_2^\ast$ and their masses as $m_1$ and $m_2$ accordingly. Then $k_1^\ast = -k_2^\ast$ and $M > m_1 + m_2$. If the decay is induced by the strong or electromagnetic interaction, the flavours

\textsuperscript{3} In NQM the violation of the Bell’s inequalities may also be caused by the non-locality of the theory itself. Such alternative interpretation is excluded in QFT, which is local by derivation.
of fermions are conserved \((m_1 = m_2 = m)\) as well as \(P\)-parity \((P_{f \bar{f}} = (-1)^{L_{f \bar{f}} + 1} = -1)\). The full momentum of the system is conserved, so \(J_{f \bar{f}} = 0\). Then, the orbital momentum and the spin of the fermion-antifermion pair is \(L_{f \bar{f}} = S_{f \bar{f}} = 0\). This leads to the full anticorrelation of the fermion “2” spin and the antifermion “1” spin projections on any direction determined by the vector \(\vec{a}\):

\[
S^{(2)}_a = -S^{(1)}_a
\]  

(2.1)

Next suppose that the fermion and antifermion spin projections on three non parallel directions \(\vec{a}, \vec{b}\) and \(\vec{c}\) are the elements of the physical reality at the same time. In the appendix A.1 it is shown that such an assumption leads to the following inequality:

\[
w \left( S^{(2)}_a = \frac{1}{2}, S^{(1)}_a = \frac{1}{2} \right) \leq w \left( S^{(2)}_a = \frac{1}{2}, S^{(1)}_a = \frac{1}{2} \right) + w \left( S^{(2)}_c = \frac{1}{2}, S^{(1)}_b = \frac{1}{2} \right)
\]  

(2.2)

for the probabilities for fermion and antifermion to have spin projection of +1/2 (at the same time) on any two of three directions \(\vec{a}, \vec{b}\) and \(\vec{c}\). Since only the decay probabilities appear in (2.2) it is equally applicable in QFT and NQM. Let all the vectors \(\vec{a}, \vec{b}\) and \(\vec{c}\) to lie on the same \(XZ\)-plane. In non-relativistic QM this leads to the transformation of (2.2) into the following inequality (see [13]):

\[
\sin^2 \frac{\theta_{ab}}{2} \leq \sin^2 \frac{\theta_{ac}}{2} + \sin^2 \frac{\theta_{bc}}{2}
\]  

(2.3)

where \(\theta_{ab} = \theta_{\alpha} - \theta_{\beta}\) \(\{\alpha, \beta\} = \{a, b, c\}\). The inequality (2.3) is violated when vectors \(\vec{a}\) and \(\vec{b}\) form an angle less than \(\pi\), and vector \(\vec{c}\) bisects this angle. The evidence of the violation of (2.3) is a direct experimental confirmation of the Bohr’s complementarity principle. It is possible to write an inequality analogous to (2.2) for the neutral resting particle decays into two pions, for example \(\pi^0 \rightarrow 2\gamma\). In this case the system of two photons with negative \(P\)-parity has the full momentum of zero, as well as orbital momentum and spin, i.e. \(J_{\gamma \gamma} = L_{\gamma \gamma} = S_{\gamma \gamma} = 0\). This implies that for a photon with linear polarisation, there is a full anticorrelation of polarizations by any direction \(\vec{a}\), which is perpendicular to the direction of the photon propagation. If we label the polarisation of photon with \(\lambda^{(1,2)}_a\) and the corresponding states with indices “1” and “2”, then the analog of (2.2) for photons will look like:

\[
w \left( \lambda^{(1)}_a = 1, \lambda^{(2)}_b = 1 \right) \leq w \left( \lambda^{(1)}_a = 1, \lambda^{(2)}_c = 1 \right) + w \left( \lambda^{(1)}_c = 1, \lambda^{(2)}_b = 1 \right)
\]  

(2.4)

B. The condition of the full spin correlation in the decays of a pseudoscalar particles and a new expression for Bell’s inequalities

Let us consider the decay of a resting scalar particle with mass \(M\) into a fermion-antifermion pair. In the case of strong or electromagnetic decay \(P\)-parity is conserved, leading to the condition \(L_{f \bar{f}} = S_{f \bar{f}} = 1\) for \(J_{f \bar{f}} = 0\). It is easy to see that in this case if the projection of the full spin of the \(f \bar{f}\)-pair on the direction \(\vec{a}\) is equal \(S^{a}_{f \bar{f}} = \pm 1\), then the full correlation exists between the spin projections of the fermions on this direction:

\[
S^{(2)}_a = S^{(1)}_a.
\]  

(2.5)

In the case of \(S^{a}_{f \bar{f}} = 0\), the full anticorrelation exists instead, like in the case of the decay of a pseudoscalar particle. With the arbitrary orientation of the plane of measurement the both cases may take place simultaneously – the correlation and the anticorrelation, and the experimental possibility of the testing of the Bell’s inequalities will suffer. However it is possible to choose such relative position of the measurement plane and the propagation direction of fermions that the contribution from the anticorrelation will be insignificant. In such experimental configuration it becomes possible to test the Bell’s inequalities with full correlation.

Let \(\theta\) and \(\phi\) be zenith and azimuth angle of the vector \(\vec{n}\) accordingly, and \(\bar{\theta}\) be the angle between the \(\vec{a}\) and \(\vec{n}\) vectors. The angle \(\theta_a\) defines the position of the vector \(\vec{a}\) in the \(XZ\) plane. The \(\bar{\theta}\)-dependence of the amplitude of the decay of pseudoscalar meson \(S\) into a fermion-antifermion pair is:

\[
A(S \rightarrow f \bar{f}) \sim \left| \langle S_{f \bar{f}} = 1, S^{a}_{f \bar{f}} = 0 \mid H \mid S_S = 0, S^{a}_S = 0 \rangle \right| \cos \bar{\theta} + \\
+ \left| \langle S_{f \bar{f}} = 1, S^{a}_{f \bar{f}} = +1 \mid H \mid S_S = 0, S^{a}_S = 0 \rangle \right| - \\
- e^{2i\phi} \left| \langle S_{f \bar{f}} = 1, S^{a}_{f \bar{f}} = -1 \mid H \mid S_S = 0, S^{a}_S = 0 \rangle \right| \sin \bar{\theta}.
\]
One can see that the anticorrelations are gone in case of angle $\tilde{\theta} = \pi/2$. Then using the cosine theorem:

$$0 = \cos \theta \cos \theta_a + \sin \theta \sin \theta_a \cos \phi.$$ 

When testing the Bell’s inequalities in Wigner’s form for the full spin correlation it is necessary to choose such values of the angles $\theta$ and $\phi$ that the cosine theorem hold for the angle $\theta_a \in [0, \pi]$. This is possible when

$$\theta = \phi = \tilde{\theta} = \pi/2,$$

i.e. in the case when the fermions propagate in the direction of the axis $Y$, perpendicular to the polarization measurement plane $XZ$. Later we always will assume such experimental configuration when talking about the decays of the scalar particles. Again, like in the subsection II A suppose that spin projections of fermion and antifermion on $XZ$-measurement plane. One can see that the anticorrelations are gone in case of angle $\tilde{\theta}$ of the angles $\theta$.

Considering the probabilities for the correlation and the anticorrelation above $\theta = \phi = \pi/2$ it is possible to write the inequality analogous to the inequality (2.7) for the case of the full anticorrelation:

$$w\left(\lambda_{a}^{(1)} = 1, \, \lambda_{b}^{(2)} = 2\right) \leq w\left(\lambda_{a}^{(1)} = 1, \, \lambda_{c}^{(2)} = 2\right) + w\left(\lambda_{c}^{(1)} = 1, \, \lambda_{b}^{(2)} = 2\right).$$

(2.8)

III. BELL’S INEQUALITIES IN QFT FOR THE DECAY OF A PSEUDOSCALAR PARTICLE INTO TWO FERMIONS

In the case of conserved $P$–parity the decay of a pseudoscalar particle to a fermion-antifermion pair can be described using an effective Hamiltonian:

$$\mathcal{H}^{(PS)}(x) = g \varphi(x) \left(\bar{f}(x) \gamma^5 f(x)\right)_{N},$$

(3.1)

where $g$ is the effective coupling constant, $\varphi(x)$ is the field of the pseudoscalar particle, and $f(x)$ is the fermionic field. In the decays described by (3.1) the masses of fermion and antifermion should be equal. In the all equations below we will use the index “1” for antifermion and the index “2” for fermion (same for their masses). The unitary normal vector (3.1) coincides with the fermion propagation direction, i.e. $\vec{k}_2 = |\vec{k}_2| \hat{n}$. Let us sequentially consider three possible cases.

A. The decay of a resting pseudoscalar particle

Let the pseudoscalar be at rest at the origin of the coordinate system and place a spin state analyzers at infinity to measure spin projections in a planes parallel to $XZ$-plane. If the spin projections of the fermion on the $\vec{a}$ direction
and of the antifermion on the $\vec{b}$ direction are equal to $+1/2$, the decay amplitude can be written as follows:

$$A \left( s_a^{(2)} = \frac{1}{2}, s_b^{(1)} = \frac{1}{2} \right) = -g \bar{u}(\vec{k}_2, s^{(2)}_a = +1/2, \vec{a}) \gamma^5 v(\vec{k}_1, s^{(1)}_b = +1/2, \vec{b}) =$$

$$= g \sqrt{\epsilon_2 + m^2_{2}} \frac{1}{\epsilon_1 + m_1} (M + m_1 - m_2) \chi^+_a(\vec{a}) \chi_-^b(\vec{b}).$$

This amplitude does not depend on angular variables. Then, taking into account the first equation of (3.1), the probability can be written in the following way:

$$w \left( s_a^{(2)} = \frac{1}{2}, s_b^{(1)} = \frac{1}{2} \right) = f(M, m_1, m_2, \theta, \phi) \sin^2 \frac{\theta_{ab}}{2}. \quad (3.2)$$

If the direction of fermion propagation is not taken into account, then

$$f(M, m_1, m_2, \theta, \phi) = \frac{g^2}{16 \pi} \frac{M^2 - (m_1 - m_2)^2}{M^3} \lambda^{1/2}(M^2, m_1^2, m_2^2), \quad (3.3)$$

where $\lambda(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc$ is the triangular function, defining the dependency of the probability on the phase space.

From (3.2) and (3.3), it follows that in the framework of QFT the Bell inequality (22) reduces to (23), which was obtained in the non-relativistic approach.

If take into account the fermion propagation direction, i.e. to select only the fermions with fixed values of $\tilde{\theta}$ and $\tilde{\phi}$, then the equation (3.3) should be modified as:

$$f(M, m_1, m_2, \tilde{\theta}, \tilde{\phi}) = \frac{g^2}{16} \frac{M^2 - (m_1 - m_2)^2}{M^3} \lambda^{1/2}(M^2, m_1^2, m_2^2) \sin \tilde{\theta} \frac{2}{(2 \pi)^2}. \quad (3.4)$$

Equation (3.4) reflects the fact that if both fermion and antifermion propagate along the $z$-axis (i.e. $\sin \tilde{\theta} = 0$), they wouldn’t be registered and inequality (23) becomes meaningless. We emphasize the fact that in the non-relativistic approach this obvious deduction cannot be made.

### B. The adjustment due to the non-antiparallelity of the fermion and antifermion momenta

In the previous subsection we showed that due to the non-conservation of the momentum projection on the $y$-axis in the rest frame of a pseudoscalar particle, the angle between the vectors $\vec{k}_2$ and $\vec{k}_1$ had a small deviation from $\pi$. Due to the violation of antiparallelity of the two vectors in the Bell inequality (23) a quadratic correction appeared by the small parameter $\epsilon_2/M$.

The antiparallelity of the vectors $\vec{k}_2$ and $\vec{k}_1$ can be caused by the emission of a soft photon from one of the fermions. The energy $\omega$ of the soft photon can be below the detection threshold and a lot less than energies $\epsilon_1$ and $\epsilon_2$ (these are of the order of $M/2$). It is well known from standard QED that in the first approximation by $\omega/\epsilon_{1,2}$ the amplitude of the emission of a soft photon can be factorized by the amplitude for the no-emission process and by a factor corresponding to the emission of a soft photon. Therefore in the case of soft photon emission the possible correctons for (23) may only appear starting in the second order by the small parameter $\omega/\epsilon_{1,2} \sim \omega/M$.

Let us prove the general statement. Let $\vec{k}_1 = |\vec{k}_1| \vec{n}_1$ and $\vec{k}_2 = |\vec{k}_2| \vec{n}_2$ and the conservation law

$$|\vec{k}_1| \vec{n}_1 + |\vec{k}_2| \vec{n}_2 = |\vec{p}| \vec{\ell}, \quad (3.5)$$

where the vector $\vec{\ell}$ is not parallel to the vectors $\vec{n}_1$ and $\vec{n}$. Additionally let $E = \epsilon_1 + \epsilon_2$ and

$$\frac{|\vec{p}|}{M} \ll 1.$$ 

Then

$$w \left( s_a^{(2)} = +\frac{1}{2}, s_b^{(1)} = +\frac{1}{2} \right) = g^2 f_0(M, m_1, m_2) \sin^2 \frac{\theta_{ab}}{2} + O \left( \frac{|\vec{p}|^2}{M^2} \right). \quad (3.6)$$
Actually, with zero angular momentum there is a spherical symmetry. Small deviations from such symmetry may only be quadratic by any direction, which immediately leads to the formula \(3.6\). However we also prove this fact strictly.

When the projections of the fermion spin on the direction of \(\vec{a}\) and the antifermion spin on the direction of \(\vec{b}\) both are equal to \(+1/2\), the decay amplitude can be written as follows:

\[
\tilde{A}
(s^{(2)}_a + \frac{1}{2}, s^{(1)}_b + \frac{1}{2}) = g \left[ \sqrt{\varepsilon_1 + m_1} \sqrt{\varepsilon_2 + m_2} \chi^\dagger (\vec{a}) \chi (\vec{b}) - \sqrt{\varepsilon_1 - m_1} \sqrt{\varepsilon_2 - m_2} \chi^\dagger (\vec{a}) \right] \left( \frac{\tilde{\sigma}_{k_2}}{|k_2|} \frac{\tilde{\sigma}_{k_1}}{|k_1|} \chi (\vec{b}) \right).
\]

(3.7)

From (3.7) and (3.5) it follows that if vectors \(\vec{n}_1\) and \(\vec{n}\) are not antiparallel, then the amplitude can be written down in the form:

\[
A
(s^{(2)}_a + \frac{1}{2}, s^{(1)}_b + \frac{1}{2}) = g \sqrt{\frac{\varepsilon_2 + m_2}{\varepsilon_1 + m_1}} \left[ (E + m_1 - m_2 - \sqrt{\varepsilon_2 - m_2} |\vec{p}| \ell^i n^i) \sin \frac{\theta_{ab}}{2} - i \sqrt{\frac{\varepsilon_2 - m_2}{\varepsilon_2 + m_2}} |\vec{p}| \epsilon^{ijk} n^i \ell^j w^k_{++} \right].
\]

The series expansion by the small parameter \(|\vec{p}|/M\) gives:

\[
E = E^{(0)} + E^{(1)} \frac{|\vec{p}|}{M} \ell^i n^i + O \left( \frac{|\vec{p}|^2}{M^2} \right);
\]

\[
\varepsilon_{1,2} = \varepsilon_{1,2}^{(0)} + \varepsilon_{1,2}^{(1)} \frac{|\vec{p}|}{M} \ell^i n^i + O \left( \frac{|\vec{p}|^2}{M^2} \right);
\]

\[
d\Phi_n = \frac{d\Omega}{4\pi} \left( \Phi_2^{(0)} + \Phi_2^{(1)} \frac{|\vec{p}|}{M} \ell^i n^i + O \left( \frac{|\vec{p}|^2}{M^2} \right) \right) d\Phi_{n-2},
\]

where the phase space \(d\Phi_n\) includes the integration over the variables different from the angle variables of the fermion, and considers a possible emission of \((n-2)\) soft photons, whilst \(d\Omega = d\cos \theta d\phi\) selects the integration over the fermion propagation direction. The explicit form of the coefficients of the expansion in (3.9) depends on the source of the momentum \(\vec{p}\). For example, if that momentum results from the Brownian motion of the decaying pseudoscalar particle, then \(E \approx M \left( 1 + \frac{|\vec{p}|^2}{M^2} \right)\). Hence \(E^{(0)} = M, E^{(1)} = 0\).

In accordance with (3.8) and (3.9) for the decay probability it can be written:

\[
\int \frac{d\Omega}{4\pi} = 1 \quad \int \frac{d\Omega}{4\pi} n^i = \langle n^i \rangle_{\Omega} = 0,
\]

then (3.6) immediately follows from (3.10).

Note the statement (3.6) is quite general. It’s true for the Brownian motion of a decaying particle, the uncertainties from the composition of the initial state, the interaction between the final state fermions (e.g. via the Coulomb force), or the interaction with a weak external field. In the beginning of the current subsection two cases complying with (3.6) were presented: the emission of soft photons and a phase space limit of the decay. The equality (3.6) may be useful not only for the Bell inequalities. It can be easily adapted to the various tasks in quantum teleportation and quantum measurements.

---

4 Later we assume \(\epsilon^{123} = \epsilon_{123} = +1\).
As an example let us consider the influence of two parallel spin analyzers situated in the XZ–plane and crossing the Y–axis at the distance ±L/2 from the coordinates origin. Then the uncertainty of \( k_y \sim 1/L \)

\[
\frac{k_y^2}{M^2} \sim \frac{1}{(M L)^2} \ll 1.
\]

For example if \( L \sim 2 \text{ cm} \), and \( M \sim 1 \text{ GeV} \), then \( 1/(M L)^2 \sim 10^{-28} \), which is below the current available experimental precision by many orders of magnitude. Thus if the distance between the spin analyzers is macroscopic, then this adjustment is unimportant. Later in this article we always suppose the analyzers to reside at infinity.

IV. THE BELL INEQUALITIES IN QFT FOR THE DECAY OF A SCALAR PARTICLE INTO TWO FERMIONS

The effective Hamiltonian of the decay of a scalar particle to a fermion-antifermion pair can be written in exactly the same way as (3.1):

\[
\mathcal{H}^{(S)}(x) = g \varphi(x) \left( \bar{f}(x) f(x) \right)_N,
\]

where \( \varphi(x) \) is a field of the scalar particle. Like in section III, the index “1” will always be for antifermion, and index “2” for fermion. The unitary vector \( \vec{n} \) is set to the direction of the fermion propagation, while the vector \( \vec{n}_1 \) is set to the antifermion direction.

We will only consider the case when the scalar particle is resting and again spin analyzers are placed at infinity in the planes parallel to the XZ–plane.

A. Decay of the scalar particle in the case when the direction of fermion propagation is fixed

As it was shown in the section II B the spin correlations are only possible when \( \theta = \phi = \pi/2 \). That’s why we will consider the decay of a scalar particle into fermion-antifermion pair for fixed direction.

Let us consider a fermion propagating along some chosen direction defined by angles \( \theta \) and \( \phi \). In this case the angular part of the probability \( w \left( s_a^{(2)} = + \frac{1}{2}, \ s_b^{(1)} = - \frac{1}{2} \right) \) has the form

\[
\sin \theta \left| \chi_{+}(\vec{a}) \left( \bar{\sigma} \vec{n} \right) \chi_{+}(\vec{b}) \right|^2 = \sin \theta \left( \sin^2 \theta \cos^2 \phi \sin^2 \frac{k_{ab}}{2} + \cos^2 \theta \cos^2 \frac{k_{ab}}{2} + \frac{1}{2} \sin(2\theta) \cos \phi \sin k_{ab} + \sin^2 \theta \sin^2 \phi \sin^2 \frac{\theta_{ab}}{2} \right).
\]

According to the section II B \( \theta = \phi = \pi/2 \) in (4.2). Then:

\[
w \left( s_a^{(2)} = + \frac{1}{2}, \ s_b^{(1)} = - \frac{1}{2} \right) \sim \sin^2 \frac{\theta_{ab}}{2},
\]

and we get the Bell’s inequalities in the form (2.3) as for the case of a pseudoscalar particle decay. However in this case the direction is fixed, and much higher experimental statistics is required for testing of the inequalities. For the decay of the scalar particle to a fermion-antifermion pair small deviations from the antiparallel state of the vectors \( \vec{n}_1 \) and \( \vec{n} \) are proportional to the first power of \( |\vec{p}| (\vec{\ell} \vec{n}) \).

V. BELL’S INEQUALITIES FOR THE DECAYS OF SCALAR AND PSEUDOSCALAR PARTICLES INTO TWO PHOTONS

Consider a decay of a particle without spin into two photons in the final state. Below we will use Higgs boson as a scalar and a \( \pi^0 \) meson as a pseudoscalar.

As long as \( P \)–parity is conserved, the amplitude of \( H^0 \to \gamma \gamma \) decay has the following form:

\[
A^{H^0 \to \gamma \gamma} = F_H \epsilon^\mu(\lambda(1)) \epsilon^*_\mu(\lambda(2)),
\]
where $k_2 = \omega \vec{n} = \omega (0, 1, 0)$. The 4-vectors of the photon polarization in the direction $\vec{a}$, orthogonal to $\vec{n}$, are defined as follows:

\[e^\mu (\lambda_\alpha^{(1,2)} = 1) = (0, \sin \theta_\alpha, 0, \cos \theta_\alpha); \quad e^\mu (\lambda_\beta^{(1,2)} = 2) = (0, \cos \theta_\alpha, 0, - \sin \theta_\alpha).\]

In the case when the first photon has polarization “1” in the direction $\vec{a}$ and the second photon has polarization “2” in the direction orthogonal to $\vec{b}$, the amplitude can be written as follows:

\[A^{H^0 \rightarrow \gamma \gamma} (\lambda_\alpha^{(1)} = 1, \lambda_\beta^{(2)} = 2) = F_H e^\mu (\lambda_\alpha^{(1)} = 1) e^\nu (\lambda_\beta^{(2)} = 2) = - F_H \sin \theta_{ab}.\]

In the case when the first photon has polarization “1” in the direction $\vec{a}$ and the second photon has polarization “1” in the direction $\vec{b}$, the amplitude can be written as follows:

\[A^{H^0 \rightarrow \gamma \gamma} (\lambda_\alpha^{(1)} = 1, \lambda_\beta^{(2)} = 1) = F_H e^\mu (\lambda_\alpha^{(1)} = 1) e^\nu (\lambda_\beta^{(2)} = 1) = F_H \cos \theta_{ab}.\]

Hence, Bell’s inequality (2.9) reduces to the following trigonometric inequality:

\[\sin^2 \theta_{ab} \leq \frac{1}{2} + \sin^2 \theta_{ac} + \sin^2 \theta_{bc}, \quad (5.1)\]

It is a new version of Bell’s inequality, which in principle may be violated. However, the range of the angles where it is violated is very narrow, and it is written with the condition (2.6).

At the Large Hadron Collider, where the Higgs bosons may be born, the following decay is possible: $H \rightarrow \gamma^* \gamma^* \rightarrow (\ell^+ \ell^-)(\ell^+ \ell^-)$. In this case, each of $(\ell^+ \ell^-)$ planes may be used as a spin projector for each photon. The invariant mass of a lepton pair should be small in order to effectively exclude the contribution from a highly virtual photon (such photons have additional longitudinal polarisation). Effects of the exchange interaction are negligible, because both of lepton pairs are largely separated in phase space in the rest frame of $H^0$.

Let’s now consider a decay $\pi^0 \rightarrow \gamma \gamma$. The amplitude of that decay has the form:

\[A^{\pi^0 \rightarrow \gamma \gamma} = F_\pi \epsilon_{\mu \alpha \beta} e^\nu (\lambda^{(1)}) e^\kappa (\lambda^{(2)}) k_1^\alpha k_2^\beta.\]

Given that in this problem $k_1^\mu = \omega (1, 0, -1, 0)$, $k_2^\mu = \omega (1, 0, 1, 0)$ and $\epsilon_{0123} = - \epsilon_{0123} = +1$, we have:

\[A^{\pi^0 \rightarrow \gamma \gamma} (\lambda_\alpha^{(1)} = 1, \lambda_\beta^{(2)} = 1) = - 2 \omega F_\pi \sin \theta_{ab}.\]

Substitution of the amplitude into inequality (2.4) again leads to the inequality

\[\sin^2 \theta_{ab} \leq \sin^2 \theta_{ac} + \sin^2 \theta_{bc}, \quad (5.2)\]

which is violated when the vectors $\vec{a}$ and $\vec{b}$ form an acute angle, while $\vec{c}$ is its bisector. The inequality (5.2) is analogous to the trigonometric inequality (2.3).

Thus, Bell’s inequalities in Wigner form for the decay of the scalar (2.9) and pseudoscalar (2.4) particle into two photons lead to two trigonometric inequalities (5.1) and (5.2), which can be used for the experimental test of the Bohr’s complementarity principle as well as inequality (2.3).

If we do not require $P$-parity conservation, then the decay amplitude of a pseudoscalar or scalar meson $P^0$ with mass $M_P$ into a photon pair has the following form:

\[A^{P^0 \rightarrow \gamma \gamma} = e^\nu (\lambda^{(1)}) e^\mu (\lambda^{(2)}) \left[ A \epsilon_{\mu \alpha \beta} k_1^\alpha k_2^\beta - i B \left( k_2 \gamma_\mu k_1 - g_{\mu \nu} \frac{M_P^2}{2} \right) \right],\]

For this amplitude the inequality (2.4) is transformed into a trigonometric inequality:

\[|A|^2 - |B|^2 \sin^2 \theta_{ab} \leq |B|^2 + (|A|^2 - |B|^2) \left( \sin^2 \theta_{ac} + \sin^2 \theta_{bc} \right), \quad (5.3)\]

The inequality (5.3) can be violated only when $|A| \geq \sqrt{2} |B|$. The violation reaches a maximum when $|B| = 0$. In this case, inequality (5.3) is transformed to (5.2) obtained above.

The inequality (5.3) is not affected if one of the final particles is a vector meson, instead of a photon. For example, the inequality may be written for the decay $B^0 \rightarrow K^{*0} \gamma$. However, in this case $|A| = |B|$. Therefore, the formula (5.3) becomes a trivial statement: $|B| \geq 0$. The same trivialization of Bell’s inequalities occurs in rare radiative decays $B^0 \rightarrow \ell \gamma$. In this case the cause of the triviality stems from these decays going through loop diagrams, which are reduced to the effective tensor operator $s \sigma^{\mu \nu} (1 + \gamma^5) b$. If the contributions from tensor and pseudotensor quark currents are essentially different, then the inequality (5.3) may be violated. This is possible for example in LR-models.
VI. CONCLUSION

The following conclusions are drawn for this paper.

1. It is shown that the relativistic generalization of Bell’s inequalities in Wigner form for the decay of a resting pseudoscalar particle into a fermion-antifermion pair reproduces the non-relativistic result (2.3) for the decay of a singlet state into the two states with spin 1/2.

2. We proved that the corrections due to small deviations from the exact antiparallel state of a fermion and antifermion are quadratic by the small parameter $|\vec{k}_1 + \vec{k}_2|/M$.

3. For the case of a scalar particle decay into a fermion-antifermion pair, we obtained a new type of Bell’s inequalities in Wigner form (2.7) for the full spin correlations using the special experimental configuration (2.6). It is shown that the inequality (2.7) may lead to the trigonometric inequality (2.3), same as for pseudoscalar particle.

4. The decay of a scalar and pseudoscalar particle into two photons leads to a new trigonometric inequalities (5.1) and (5.2) accordingly, assuming that $P$-parity is conserved. These inequalities may be experimentally tested at current colliders.

5. If $P$-parity is not conserved the inequality (5.2) may be generalized to inequality (5.3). The inequality (5.3) is not affected if one of the final particles is a vector meson, instead of a photon.

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Appendix A: A derivation of Bell’s inequalities in Wigner form

In this appendix we give a detailed derivation of Bell’s inequalities in Wigner form for decays of pseudoscalar and scalar particles.

1. Derivation of Bell inequalities for a two-body decay of a pseudoscalar particle

Consider a case of a full anticorrelation of spin projections. It may appear for example for the decay of a pseudoscalar particle into fermion-antifermion pair (see section II A). Let us make a key assumption for the derivation. The spin projections of fermion and antifermion on three non-parallel directions $\vec{a}$, $\vec{b}$ and $\vec{c}$ are simultaneously elements of a physical reality. Then we can speak of nonnegative number of fermion-antifermion pairs, with $s_a^{(2)} = +1/2$, $s_b^{(1)} = +1/2$ and $s_c^{(1)} = +1/2$. Denote the number of such pairs as $N(s_a^{(2)} = +1/2, s_b^{(1)} = +1/2, s_c^{(1)} = +1/2)$. Now it is easy to obtain the number of pairs with spin projections only on two directions:

$$N\left(s_a^{(2)} = +1/2, s_b^{(1)} = +1/2\right) = N\left(s_a^{(2)} = +1/2, s_b^{(1)} = +1/2, s_c^{(1)} = +1/2\right) + N\left(s_a^{(2)} = +1/2, s_b^{(1)} = +1/2, s_c^{(1)} = -1/2\right). \quad (A1)$$

In analogy

$$N\left(s_a^{(2)} = +1/2, s_c^{(1)} = +1/2\right) = N\left(s_a^{(2)} = +1/2, s_b^{(1)} = +1/2, s_c^{(1)} = +1/2\right) + N\left(s_a^{(2)} = +1/2, s_b^{(1)} = -1/2, s_c^{(1)} = +1/2\right). \quad (A2)$$
And finally

\[ N \left( s_c^{(2)} = + \frac{1}{2}, s_b^{(1)} = + \frac{1}{2} \right) = N \left( s_a^{(2)} = + \frac{1}{2}, s_c^{(2)} = + \frac{1}{2}, s_b^{(1)} = + \frac{1}{2} \right) + 
\]
\[ + N \left( s_a^{(2)} = - \frac{1}{2}, s_c^{(2)} = + \frac{1}{2}, s_b^{(1)} = + \frac{1}{2} \right) \]

or, using the anticorrelation condition (2.1) for the direction \( \vec{c} \),

\[ N \left( s_c^{(2)} = + \frac{1}{2}, s_b^{(1)} = + \frac{1}{2} \right) = N \left( s_a^{(2)} = + \frac{1}{2}, s_b^{(1)} = + \frac{1}{2}, s_c^{(1)} = - \frac{1}{2} \right) + 
\]
\[ + N \left( s_a^{(2)} = - \frac{1}{2}, s_b^{(1)} = + \frac{1}{2}, s_c^{(1)} = - \frac{1}{2} \right). \]  \hspace{1cm} (A3)

In equalities (A1) – (A3) each term in the right part is nonnegative. Hence we can write a kind of “triangle inequality”:

\[ N \left( s_a^{(2)} = + \frac{1}{2}, s_b^{(1)} = + \frac{1}{2} \right) \leq N \left( s_a^{(2)} = + \frac{1}{2}, s_c^{(1)} = + \frac{1}{2} \right) + 
\]
\[ + N \left( s_b^{(2)} = + \frac{1}{2}, s_c^{(1)} = + \frac{1}{2} \right). \]  \hspace{1cm} (A4)

Since the number of fermion-antifermion pairs is inversely proportional to the decay probability, we immediately have the inequality (2.2).

The basic inequality (A4) may be rewritten in a few equivalent forms. For example if \( \vec{b} \) and \( \vec{c} \) are changed to opposite directions, then using the condition (2.1) one can obtain the inequality

\[ N \left( s_a^{(2)} = + \frac{1}{2}, s_b^{(1)} = - \frac{1}{2} \right) \leq N \left( s_a^{(2)} = + \frac{1}{2}, s_c^{(1)} = - \frac{1}{2} \right) + 
\]
\[ + N \left( s_b^{(2)} = + \frac{1}{2}, s_c^{(1)} = + \frac{1}{2} \right). \]

If we change only the direction \( \vec{c} \) to the opposite, then the following inequality appears:

\[ N \left( s_a^{(2)} = + \frac{1}{2}, s_b^{(1)} = + \frac{1}{2} \right) \leq N \left( s_a^{(2)} = + \frac{1}{2}, s_c^{(1)} = - \frac{1}{2} \right) + 
\]
\[ + N \left( s_b^{(2)} = - \frac{1}{2}, s_c^{(1)} = + \frac{1}{2} \right), \]

It can be weakened by rewriting it in the following form:

\[ N \left( s_a^{(2)} = + \frac{1}{2}, s_b^{(1)} = + \frac{1}{2} \right) + N \left( s_a^{(2)} = + \frac{1}{2}, s_c^{(1)} = + \frac{1}{2} \right) + 
\]
\[ + N \left( s_b^{(2)} = + \frac{1}{2}, s_c^{(1)} = + \frac{1}{2} \right) \leq N_{tot}. \]

If the vectors \( \vec{a}, \vec{b} \) and \( \vec{c} \) lie in the same plane, then in the framework of NQM, the last inequality is reduced to the following trigonometric inequality:

\[ \sin^2 \frac{\theta_{ab}}{2} + \sin^2 \frac{\theta_{ac}}{2} + \sin^2 \frac{\theta_{bc}}{2} \leq 2, \]

which is violated when the angle between the vectors \( \vec{a} \) and \( \vec{b} \) is close to \( \pi \), and the direction \( \vec{c} \) bisects this angle. This condition is more strict than the condition of violation of the inequality (2.3).

Since all the variants of Bell’s inequalities in Wigner form above either equivalent to the inequality (A4) or weaker, we will consider only the relativistic generalization of (A4).
Consider the case of a full spin projections correlation. The pure full correlation doesn’t appear in the decays. Usually there are contributions with correlation and with anticorrelation as well. In this case there are much less possibilities for the violation of Bell’s inequalities in QFT than in the cases of the pure full correlation or of the pure full anticorrelation. However, like it was shown in the section 11 with use the experimental configuration 270, QFT predicts the full spin correlation for example for the case of the decay of a scalar particle. Like in Appendix A1 we assume that the spin projections of the fermion and antifermion on three non-parallel directions set by unitary vectors \( \vec{a}, \vec{b} \) and \( \vec{c} \), are simultaneously elements of a physical reality. Then:

\[
N \left( s_a^{(2)} = + \frac{1}{2}, s_b^{(1)} = - \frac{1}{2} \right) = N \left( s_a^{(2)} = + \frac{1}{2}, s_b^{(1)} = - \frac{1}{2}, s_c^{(1)} = + \frac{1}{2} \right) + N \left( s_a^{(2)} = + \frac{1}{2}, s_b^{(1)} = - \frac{1}{2}, s_c^{(1)} = - \frac{1}{2} \right). \tag{A5}
\]

In analogy

\[
N \left( s_a^{(2)} = + \frac{1}{2}, s_c^{(1)} = - \frac{1}{2} \right) = N \left( s_a^{(2)} = + \frac{1}{2}, s_b^{(1)} = + \frac{1}{2}, s_c^{(1)} = - \frac{1}{2} \right) + N \left( s_a^{(2)} = + \frac{1}{2}, s_b^{(1)} = + \frac{1}{2}, s_c^{(1)} = + \frac{1}{2} \right). \tag{A6}
\]

And finally

\[
N \left( s_c^{(2)} = + \frac{1}{2}, s_b^{(1)} = - \frac{1}{2} \right) = N \left( s_a^{(2)} = + \frac{1}{2}, s_b^{(1)} = - \frac{1}{2}, s_c^{(1)} = + \frac{1}{2} \right) + N \left( s_a^{(2)} = - \frac{1}{2}, s_b^{(1)} = + \frac{1}{2}, s_c^{(1)} = - \frac{1}{2} \right). \tag{A7}
\]

or, considering the correlations of spin projections on the direction \( \vec{c} \):

\[
N \left( s_c^{(2)} = + \frac{1}{2}, s_b^{(1)} = - \frac{1}{2} \right) = N \left( s_a^{(2)} = + \frac{1}{2}, s_b^{(1)} = - \frac{1}{2}, s_c^{(1)} = + \frac{1}{2} \right) + N \left( s_a^{(2)} = + \frac{1}{2}, s_b^{(1)} = - \frac{1}{2}, s_c^{(1)} = + \frac{1}{2} \right). \tag{A8}
\]

From equalities (A5) – (A7) follows the inequality:

\[
N \left( s_a^{(2)} = + \frac{1}{2}, s_b^{(1)} = - \frac{1}{2} \right) \leq N \left( s_a^{(2)} = + \frac{1}{2}, s_c^{(1)} = - \frac{1}{2} \right) + N \left( s_c^{(2)} = + \frac{1}{2}, s_b^{(1)} = - \frac{1}{2} \right). \tag{A8}
\]

and its probabilistic analog — the inequality (2.7). Like formula (A4), the inequality (A8) can be transformed to other equivalent inequalities by switching the directions of \( \vec{b} \) and \( \vec{c} \).

### Appendix B: A relativistic spin 1/2 operator and solutions for a free Dirac equality

Let the free Dirac particle of mass \( m \) propagate in the lab coordinate system over the direction defined by a unitary vector

\[
\vec{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta), \tag{B1}
\]

where \( \theta \in [0, \pi), \phi \in [0, 2\pi) \). In this coordinate system the particle has energy \( \varepsilon_p \) and momentum \( \vec{p} = |\vec{p}| \vec{n} \).

The solution of the free Dirac equation in the standard representation for the particle has the form:

\[
u(\vec{p}, s_a, \vec{a}) = \left( \frac{\sqrt{\varepsilon_p + m}}{\sqrt{\varepsilon_p - m}} (\vec{\sigma} \vec{n}) \chi_{sa}(\vec{a}) \right), \tag{B2}
\]
and for the antiparticle has the form:

\[ v(\vec{p}, s_a, \vec{a}) = \left( \frac{\sqrt{\varepsilon_p - m}}{\sqrt{\varepsilon_p + m}} (\sigma \vec{n}) \xi_{-s_a}(\vec{a}) \right), \]  

(B3)

where \( s_a = \pm 1/2 \) is a spin projection on a unitary vector direction

\[ \vec{a} = (\sin \theta_a \cos \phi_a, \sin \theta_a \sin \phi_a, \cos \theta_a). \]

Two-component spinors \( \chi_{s_a}(\vec{a}) \) and \( \xi_{-s_a}(\vec{a}) \) obey the normalization conditions \( \chi_{s_a}(\vec{a})^\dagger \chi_{s'_a}(\vec{a}) = \delta_{s_a s'_a} \) and \( \xi_{-s_a}(\vec{a}) = -2 s_a \chi_{-s_a}(\vec{a}) \).

The solution (B2) must be an eigenfunction of a projection operator

\[ (\vec{a} \vec{O}) u(\vec{p}, s_a, \vec{a}) = 2 s_a u(\vec{p}, s_a, \vec{a}), \]  

(B4)

corresponding to eigenvalues \( 2 s_a = \pm 1 \). The operator \( \vec{O} \) is a relativistic generalization of a spin 1/2 operator for a free particle and can be written as [14]:

\[ \vec{O} = -\gamma^5 \gamma^+ + \gamma^5 \frac{\vec{p}}{\varepsilon_p} + \frac{\vec{p} \gamma_5 (\gamma^+ \gamma^0 \gamma^5 \gamma^-, \vec{p})}{\varepsilon_p (\varepsilon_p + m)} \]  

(B5)

where the matrix \( \gamma^5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3 \). Components of the operator (B5) satisfy the standard commutation relations for doubled components of non-relativistic spin 1/2 operator \( (\epsilon^{123} = +1) \)

\[ [O^i, O^j] = 2 i e^{ijk} \vec{O}^k. \]

This allows one to test the Bohr’s complementarity principle in QFT in analogy with NQM.

From the explicit form of the operator \( \vec{O} \) follows formulae for two-component spinors:

\[ \chi_{s_a} = +1/2(\vec{a}) \equiv \chi_+(\vec{a}) = \left( \cos \frac{\theta_a}{2} e^{-i \phi_a / 2}, \sin \frac{\theta_a}{2} e^{i \phi_a / 2} \right), \]
\[ \chi_{s_a} = -1/2(\vec{a}) \equiv \chi_-(\vec{a}) = \left( -\sin \frac{\theta_a}{2} e^{-i \phi_a / 2}, \cos \frac{\theta_a}{2} e^{i \phi_a / 2} \right). \]

Hence, when \( \phi_a = \phi_b = 0 \)

\[ \chi_+^\dagger(\vec{a}) \chi_-(\vec{b}) = \sin \frac{\theta_{ab}}{2}, \]
\[ \vec{w}_{++} = \chi_+^\dagger(\vec{a}) \vec{\sigma} \chi_-(\vec{b}) = \left( \cos \frac{\kappa_{ab}}{2}, -i \cos \frac{\theta_{ab}}{2}, -\sin \frac{\kappa_{ab}}{2} \right); \]
\[ \vec{w}_{+-} = \chi_+^\dagger(\vec{a}) \vec{\sigma} \chi_+(\vec{b}) = \left( \sin \frac{\kappa_{ab}}{2}, i \sin \frac{\theta_{ab}}{2}, \cos \frac{\kappa_{ab}}{2} \right); \]  

(B6)

where \( \theta_{\alpha \beta} = \theta_{\alpha} - \theta_{\beta} \), \( \kappa_{\alpha \beta} = \theta_{\alpha} + \theta_{\beta} \) and \( \{\alpha, \beta\} = \{a, b, c\} \).

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