Channel Coding and Lossy Source Coding
Using a Constrained Random Number Generator

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Abstract
Stochastic encoders for channel coding and lossy source coding are introduced with a rate close to the fundamental limits, where the only restriction is that the channel input alphabet and the reproduction alphabet of the lossy source code are finite. Random numbers, which satisfy a condition specified by a function and its value, are used to construct stochastic encoders. The proof of the theorems is based on the hash property of an ensemble of functions, where the results are extended to general channels/sources and alternative formulas are introduced for channel capacity and the rate-distortion region. Since an ensemble of sparse matrices has a hash property, we can construct a code by using sparse matrices, where the sum-product algorithm can be used for encoding and decoding by assuming that channels/sources are memoryless.

Index Terms
Shannon theory, channel coding, lossy source coding, information spectrum methods, LDPC codes, sum-product algorithm

I. INTRODUCTION
The aim of this paper is to introduce a channel code and a lossy source code for general channels/sources including additive Gaussian, Markov, and non-stationary channels/sources. The only assumption is that the input alphabet for channel coding and the reproduction alphabet for lossy source coding are finite. We prove that the fundamental limits called the channel capacity and the boundary of the rate-distortion region are achievable with the

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proposed codes. We introduce stochastic encoders for constructing the codes and we can easily modify these encoders to make them deterministic. Let $\mathcal{X}^n$ be the cartesian power of a set $\mathcal{X}$, and $x$ denotes an element of $\mathcal{X}^n$. To construct stochastic encoders, we use a sequence of random numbers subject to a distribution $\tilde{\mu}$ on $\mathcal{X}^n$ defined as

$$
\tilde{\mu}(x) = \begin{cases}
\frac{\mu(x)}{\mu((x : Ax=c))} & \text{if } Ax = c \\
0 & \text{if } Ax \neq c
\end{cases}
$$

for a given probability distribution $\mu$ on $\mathcal{X}^n$, a function $A : \mathcal{X}^n \rightarrow \{Ax : x \in \mathcal{X}^n\}$, and $c \in \{Ax : x \in \mathcal{X}^n\}$. Let us call a generator for this type of random number a constrained random number generator.

One contribution of this paper is to extend the results of [21] to general channels/sources. In [21], the direct part of the channel coding theorem and the lossy source coding theorem for a discrete stationary memoryless channel/source are shown based on the hash property of an ensemble of functions, which is an extension of random bin coding [4], the set of all linear functions [6], and the two-universal class of hash functions [9]. In this paper, alternative general formulas for the channel capacity and rate-distortion region are introduced and the achievability of the proposed codes is proved based on a stronger version of hash property introduced in [22][23][24][25]. Since an ensemble of sparse matrices has a hash property, we can construct codes by using sparse matrices.

Another contribution of this paper is that we introduce a practical algorithm for the proposed code for a (non-stationary) memoryless (asymmetric) channel/source. We introduce an practical algorithm for a constrained random generator by using a sparse matrix and a sum-product algorithm [1][17], where we assume that a channel/source is (non-stationary) memoryless. There are many ways to construct channel codes [3][11][18] and lossy source codes [12][26][19][32] by using sparse matrices. These approaches assume that a channel/source is stationary memoryless and symmetric, or a quantization map [10, Section 6.2] is used for an asymmetric channel/source. On the other hand, the only requirement for the proposed code is that the input alphabet for channel coding and the reproduction alphabet for lossy source coding are finite.

It should be noted that a similar idea has appeared in [28][33], where they introduced random bin coding (privacy amplification) and Slepian-Wolf decoding\(^1\) (information reconciliation) for the construction of codes, and their proofs are based on the fact that the output

\(^1\)It should be noted that the idea of using Slepian-Wolf decoding has already been mentioned in [20][21].
statistics of random binning are uniformly distributed. Furthermore, the encoding functions seem to be impractical. This paper describes the explicit practical construction of encoding functions and theorems are proved simply and rigorously based on the technique reported in [24], where it is proved that we can use sparse matrices for the construction of codes.

This paper is organized as follows. Section II reviews formulas for the channel capacity and the rate-distortion region based on the information spectrum method introduced in [13][14][31]. Alternative formulas for the channel capacity and the rate-distortion region are also introduced. Section III describes the notion of a hash property, which is stronger than that introduced in [21]. Several lemmas are introduced that will be used in the proof of the theorems. Section IV deals with the construction of a channel code and Section V describes the construction of a lossy code. Section VI describes an algorithm for a constrained random number generator by using a sum-product algorithm. The conversion from stochastic encoders into deterministic encoders is discussed in this section. Theorems and lemmas are proved in Section VII. Some lemmas are shown in Appendix.

II. FORMAL DESCRIPTION OF PROBLEMS AND GENERAL FORMULAS FOR CHANNEL CAPACITY AND RATE DISTORTION REGION

This section provides a formal description of the problems and reviews formulas for the channel capacity and the rate distortion region. All the results in this paper are presented by using the information spectrum method introduced in [13][14][31], where the consistency and stationarity of channels/sources are not assumed. It should be noted that all the results reported in this paper can be applied to stationary ergodic channels/sources and stationary memoryless channels/sources.

Throughout this paper, we denote the probability of an event by \( P(\cdot) \) and denote the probability distribution of a random variable \( U \) by \( \mu_U \).

We call a sequence \( U \equiv \{U^n\}_{n=1}^{\infty} \) of random variables a general source, where \( U^n \in \mathcal{U}^n \). For a general source \( U \), we define the spectral sup-entropy rate \( \overline{H}(U) \) and the spectral inf-entropy rate \( \underline{H}(U) \) as

\[
\overline{H}(U) \equiv \inf \left\{ \theta : \lim_{n \to \infty} P \left( \frac{1}{n} \log \frac{1}{\mu_{U^n}(U^n)} > \theta \right) = 0 \right\},
\]

\[
\underline{H}(U) \equiv \sup \left\{ \theta : \lim_{n \to \infty} P \left( \frac{1}{n} \log \frac{1}{\mu_{U^n}(U^n)} < \theta \right) = 0 \right\}.
\]
It is known that both \( \bar{H}(U) \) and \( H(U) \) are equal to the entropy rate of \( U \) when \( U \) is stationary ergodic, that is,
\[
\bar{H}(U) = H(U) = \lim_{n \to \infty} \frac{H(U^n)}{n},
\]
where \( H(U^n) \) is the entropy of \( U^n \). When \( U \) is stationary memoryless, both \( \bar{H}(U) \) and \( H(U) \) are equal to the entropy \( H(U^1) \).

For a pair \( (U, V) = \{(U^n, V^n)\}_{n=1}^{\infty} \) of general sources, we define the spectral conditional sup-entropy rate \( \bar{H}(U|V) \), the spectral conditional inf-entropy rate \( H(U|V) \), the spectral sup-mutual information rate \( \overline{I}(U; V) \), and the spectral inf-mutual information rate \( I(U; V) \) as
\[
\begin{align*}
\bar{H}(U|V) & \equiv \inf \left\{ \theta : \lim_{n \to \infty} P \left( \frac{1}{n} \log \frac{1}{\mu_{U^n|V^n}(U^n|V^n)} > \theta \right) = 0 \right\}, \\
H(U|V) & \equiv \sup \left\{ \theta : \lim_{n \to \infty} P \left( \frac{1}{n} \log \frac{1}{\mu_{U^n|V^n}(U^n|V^n)} < \theta \right) = 0 \right\}, \\
\overline{I}(U; V) & \equiv \inf \left\{ \theta : \lim_{n \to \infty} P \left( \frac{1}{n} \log \frac{\mu_{U^nV^n}(U^n, V^n)}{\mu_{U^n}(U^n)\mu_{V^n}(V^n)} > \theta \right) = 0 \right\}, \\
I(U; V) & \equiv \sup \left\{ \theta : \lim_{n \to \infty} P \left( \frac{1}{n} \log \frac{\mu_{U^nV^n}(U^n, V^n)}{\mu_{U^n}(U^n)\mu_{V^n}(V^n)} < \theta \right) = 0 \right\},
\end{align*}
\]
where \( \mu_{U^nV^n} \) is the joint probability distribution corresponding to \( (U^n, V^n) \). It is known that both \( \bar{H}(U|V) \) and \( H(U|V) \) are equal to the conditional entropy rate of \( U \) given \( V \), and both \( \overline{I}(U; V) \) and \( I(U; V) \) are equal to the mutual information rate between \( U \) and \( V \), when \( (U, V) \) is stationary ergodic, that is,
\[
\begin{align*}
\bar{H}(U|V) & = H(U|V) = \lim_{n \to \infty} \frac{H(U^n|V^n)}{n}, \\
\overline{I}(U; V) & = I(U; V) = \lim_{n \to \infty} \frac{I(U^n; V^n)}{n},
\end{align*}
\]
where \( H(U^n|V^n) \) is the conditional entropy of \( U^n \) given \( V^n \) and \( I(U^n; V^n) \) is the mutual information between \( U^n \) and \( V^n \). When \( (U, V) \) is stationary memoryless, both \( \bar{H}(U|V) \) and \( H(U|V) \) are equal to the conditional entropy \( H(U^1|V^1) \) and both \( \overline{I}(U; V) \) and \( I(U; V) \) are equal to the mutual information \( I(U^1; V^1) \).

### A. Channel Capacity

In the following, we introduce the definition of the channel capacity for a general channel. Let \( X^n \) and \( Y^n \) be the alphabets of a channel input \( X^n \) and a channel output \( Y^n \), respectively.
A sequence \( W \equiv \{p_{y^n|x^n}\}_{n=1}^\infty \) of conditional probability distributions is called a general channel.

**Definition 1:** For a general channel \( W \), we call a rate \( R \) achievable if for all \( \delta > 0 \) and all sufficiently large \( n \) there is a pair consisting of an encoder \( \varphi_n : \mathcal{M}_n \to \mathcal{X}^n \) and a decoder \( \psi_n : \mathcal{Y}^n \to \mathcal{M}_n \) such that

\[
\frac{1}{n} \log |\mathcal{M}_n| \geq R
\]

\[
P(\psi_n(Y^n) \neq M_n) \leq \delta,
\]

where \([1/n] \log |\mathcal{M}_n|\) represents the rate of the code, \( M_n \) is a random variable of the message corresponding to the uniform distribution on \( \mathcal{M}_n \) and the joint distribution \( \mu_{M_nY^n} \) is given as

\[
\mu_{M_nY^n}(m, y) \equiv \frac{\mu_{Y^n|X^n}(y|\varphi_n(m))}{|\mathcal{M}_n|}.
\]

The channel capacity \( C(W) \) is defined by the supremum of the achievable rate.

For a general channel \( W \), the channel capacity \( C(W) \) is derived in [31] as

\[
C(W) = \sup_X I(X; Y), \tag{1}
\]

where the supremum is taken over all general sources \( X = \{X^n\}_{n=1}^\infty \) and the joint distribution \( \mu_{X^nY^n} \) is given as

\[
\mu_{X^nY^n}(x, y) \equiv \mu_{Y^n|X^n}(y|x)\mu_{X^n}(x). \tag{2}
\]

We introduce the following lemma, which will be proved in Section VII-A. It should be noted that this capacity formula is a straightforward generalization of that obtained by Shannon [30].

**Lemma 1:** For a general channel \( W \),

\[
C(W) = \sup_X \left[ H(X) - \overline{H}(X|Y) \right], \tag{3}
\]

where the supremum is taken over all general sources \( X \) and the joint distribution of \( (X, Y) \) is given as (2).

**Remark 1:** When \( W \) is stationary ergodic, it is sufficient that the supremum on the right hand side of (1) and (3) is taken over all stationary ergodic sources. When \( W \) is stationary memoryless, it is sufficient that the supremum on the right hand side of (1) and (3) is taken over all stationary memoryless sources. For these reasons, the lemma is trivial in these cases.
In this paper, we construct a channel code whose rate is close to the channel capacity given by (3). Constructed code is given by a pair consisting of a stochastic encoder $\Phi_n : \mathcal{M}_n \to \mathcal{X}^n$ and a decoder $\psi_n : \mathcal{Y}^n \to \mathcal{M}_n$, where $\mathcal{M}_n$ is a set of messages.

**Remark 2:** It should be noted that the capacity formulas (1) and (3) are satisfied when a stochastic encoder is allowed. In fact, by considering the average over stochastic encoders and using the random coding argument, we can construct a deterministic encoder from a stochastic encoder. Thus the rate of the stochastic encoder should be upper bounded by the channel capacity. On the other hand, the channel capacity is achievable with a stochastic encoder because a deterministic encoder is one type of stochastic encoder.

### B. Rate-Distortion Region

In the following, we introduce the achievable rate-distortion region for a general source. Let $\mathcal{Y}^n$ be a source alphabet and $\mathcal{X}^n$ be a reproduction alphabet\(^2\). Let $d_n : \mathcal{X}^n \times \mathcal{Y}^n \to [0, \infty)$ be a distortion function.

**Definition 2 ([14, Def. 5.3.1]):** We call a pair $(R, D)$ consisting of a rate $R$ and a distortion $D$ **achievable** if for all $\delta > 0$ and all sufficiently large $n$ there is a pair consisting of an encoder $\varphi_n : \mathcal{Y}^n \to \mathcal{M}_n$ and a decoder $\psi_n : \mathcal{M}_n \to \mathcal{Y}^n$ such that

\[
\frac{1}{n} \log |\mathcal{M}_n| \leq R \tag{4}
\]

\[
P(d_n(\psi_n(\varphi_n(Y^n)), Y^n) > D) \leq \delta. \tag{5}
\]

The **achievable rate-distortion region** $\mathcal{R}(\mathcal{Y})$ is defined by the set of all achievable pairs $(R, D)$.

**Remark 3:** It should be noted that the factor $1/n$ appears in [14, Def. 5.3.1]. This difference is not essential because we can replace $d_n$ by $[1/n]d_n$ throughout this paper.

For a pair $(\mathcal{X}, \mathcal{Y})$ of general sources, let $\overline{\mathcal{D}}(\mathcal{X}, \mathcal{Y})$ be defined as

\[
\overline{\mathcal{D}}(\mathcal{X}, \mathcal{Y}) \equiv \inf \left\{ \theta : \lim_{n \to \infty} P(d_n(X^n, Y^n) > \theta) = 0 \right\}.
\]

\(^2\)It should be noted that the roles of $\mathcal{X}^n$ and $\mathcal{Y}^n$ are the reverse of those in the conventional definition of the rate-distortion theory.
For a general source $Y$, the rate-distortion region $\mathcal{R}(Y)$ is derived in [27][14, Theorem 5.4.1] as

$$
\mathcal{R}(Y) = \bigcup_{W} \left\{ (R, D) : \frac{\mathcal{T}(X; Y)}{D(X; Y)} \leq R \right\},
$$

where the union is taken over all general channels $W \equiv \{\mu_{X^n|Y^n}\}_{n=1}^{\infty}$ and the joint distribution $\mu_{X^nY^n}$ is given as

$$
\mu_{X^nY^n}(x, y) \equiv \mu_{X^n|Y^n}(x|y)\mu_{Y^n}(y).
$$

We introduce the following lemma, which is proved in Section VII-B.

**Lemma 2:** For a general source $Y$,

$$
\mathcal{R}(Y) = \bigcup_{W} \left\{ (R, D) : \frac{\mathcal{H}(X) - \mathcal{H}(X|Y)}{D(X; Y)} \leq R \right\},
$$

where the union is taken over all channels $W$ and the joint distribution of $(X, Y)$ is given as (7).

**Remark 4:** When $X$ is stationary ergodic, it is sufficient that the union on the right hand side of (6) and (8) is taken over all stationary ergodic channels. When $X$ is stationary memoryless, it is sufficient that the union on the right hand side of (6) and (8) is taken over all stationary memoryless channels. For these reasons, the lemma is trivial in these cases.

In this paper, we construct a fixed-rate lossy source code, where $(R, D)$ is close to the boundary of the region given by the right hand side of (8). Constructed code is given by a pair consisting of a stochastic encoder $\Phi_n : \mathcal{X}_n \to \mathcal{M}_n$ and a decoder $\psi_n : \mathcal{M}_n \to \mathcal{Y}^n$, where $\mathcal{M}_n$ is a set of codewords.

**Remark 5:** Similarly to Remark 2, formulas (6) and (8) are satisfied when a stochastic encoder is allowed. In fact, by considering the average over the stochastic encoders and using the random coding argument, we can construct a deterministic encoder from a stochastic encoder without any loss of encoding rate. Thus the rate-distortion pair of the stochastic encoder should be in the rate-distortion region. On the other hand, the rate-distortion region is achievable with a stochastic encoder because a deterministic encoder is one type of stochastic encoder.

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3The rate-distortion function, which is the infimum of $R$ such that $(R, D)$ is achievable for a given $D$, is derived in [27][14, Theorem 5.4.1].
Remark 6: It should also be noted that we have similar results that are obtained in this paper by assuming
\[ d_{\text{max}} \equiv \max_{n,x,y} d_n(y,x) < \infty, \]
where (5) is replaced by the average distortion criterion
\[ E_Y^n [d_n(\psi_n(\varphi_n(Y^n)), Y^n)] \leq D + \delta. \]

III. \((\alpha, \beta)\)-HASH PROPERTY

In this section, we introduce the hash property\(^4\) introduced in [22][23][24][25] and its implications.

Throughout this paper, we use the following definitions and notations. The set \(\mathcal{U} \setminus \mathcal{V} \equiv \mathcal{U} \cap \mathcal{V}^c\) denotes the set difference. Let \(A\mathbf{u}\) denote a value taken by a function \(A : \mathcal{U}^n \rightarrow \overline{\mathcal{U}}\) at \(\mathbf{u} \in \mathcal{U}^n\), where \(\mathcal{U}^n\) is the domain of \(A\) and \(\overline{\mathcal{U}}\) is the region of \(A\). It should be noted that \(A\) may be nonlinear. When \(A\) is a linear function expressed by an \(l \times n\) matrix, we assume that \(\mathcal{U} \equiv \text{GF}(q)\) is a finite field and the range of functions is \(\mathcal{U}^l\). For a set \(A\) of functions, let \(\text{Im}A\) and \(\text{Im}_A\) be defined as
\[ \text{Im}A \equiv \{ A\mathbf{u} : \mathbf{u} \in \mathcal{U}^n \} \]
\[ \text{Im}_A \equiv \bigcup_{A \in A} \text{Im}A. \]
We define a set \(C_A(c)\) and \(C_{AB}(c, m)\) as
\[ C_A(c) \equiv \{ \mathbf{u} : A\mathbf{u} = c \} \]
\[ C_{AB}(c, m) \equiv \{ \mathbf{u} : A\mathbf{u} = c, A\mathbf{u} = m \}. \]

In the context of linear codes, \(C_A(c)\) is called a coset determined by \(c\). The random variables of a function \(A\) and a vector \(c \in \text{Im}A\) are denoted by the sans serif letters \(A\) and \(c\), respectively. It should be noted that the random variable of a \(n\)-dimensional vector \(\mathbf{u} \in \mathcal{U}^n\) is denoted by the Roman letter \(U^n\) that does not represent a function, which is the way it has been used so far.

\(^4\)In [22][23][24][25], it is called the ‘strong hash property.’ Throughout this paper, we call it simply the ‘hash property.’
A. Formal Definition and Basic Properties

Here, we introduce the hash property for an ensemble of functions. It requires stronger conditions than those introduced in [21].

Definition 3: Let $\mathcal{A} \equiv \{\mathcal{A}_n\}_{n=1}^{\infty}$ be a sequence of sets such that $\mathcal{A}_n$ is a set of functions $A : U^n \to \text{Im} \mathcal{A}_n$. For a probability distribution $p_{\mathcal{A},n}$ on $\mathcal{A}_n$, we call a sequence $(\mathcal{A}, p_{\mathcal{A}}) \equiv \{(\mathcal{A}_n, p_{\mathcal{A},n})\}_{n=1}^{\infty}$ an ensemble. Then, $(\mathcal{A}, p_{\mathcal{A}})$ has an $(\alpha_\mathcal{A}, \beta_\mathcal{A})$-hash property if there are two sequences $\alpha_\mathcal{A} \equiv \{\alpha_\mathcal{A}(n)\}_{n=1}^{\infty}$ and $\beta_\mathcal{A} \equiv \{\beta_\mathcal{A}(n)\}_{n=1}^{\infty}$, depending on $\{p_{\mathcal{A},n}\}_{n=1}^{\infty}$, such that

$$\lim_{n \to \infty} \alpha_\mathcal{A}(n) = 1 \quad \text{(H1)}$$

$$\lim_{n \to \infty} \beta_\mathcal{A}(n) = 0 \quad \text{(H2)}$$

and

$$\sum_{u' \in U^n \setminus \{u\} : p_{\mathcal{A},n}(\{A : Au = Au'\}) > \frac{\alpha_\mathcal{A}(n)}{|\text{Im} \mathcal{A}_n|}} p_{\mathcal{A},n}(\{A : Au = Au'\}) \leq \beta_\mathcal{A}(n) \quad \text{(H3)}$$

for any $n$ and $u \in U^n$. Throughout this paper, we omit the dependence of $\mathcal{A}$, $p_{\mathcal{A}}$, $\alpha_\mathcal{A}$ and $\beta_\mathcal{A}$ on $n$.

Remark 7: In [21][23], an ensemble is required to satisfy the condition

$$\lim_{n \to \infty} \frac{1}{n} \log \frac{|U_n|}{|\text{Im} \mathcal{A}_n|} = 0,$$

where $U_n$ is the range of functions. This condition is omitted because it is unnecessary for the results reported in this paper.

Let us remark on the condition (H3). This condition requires the sum of the collision probabilities $p_{\mathcal{A}}(\{A : Au = Au'\})$, which is greater than $\alpha_\mathcal{A}/|\text{Im} \mathcal{A}|$, to be bounded by $\beta_\mathcal{A}$, where the sum is taken over all $u'$ except $u$. An intuitive interpretation of (H3) will be provided in Section III-B by using an ensemble of sparse matrices. It should be noted that this condition implies

$$\sum_{\substack{u \in T \cap U^n \setminus \{u'\} : p_{\mathcal{A},n}(\{A : Au = Au'\}) > \frac{\alpha_\mathcal{A}(n)}{|\text{Im} \mathcal{A}_n|}}} p_{\mathcal{A}}(\{A : Au = Au'\}) \leq |T \cap T'| + \frac{|T||T'|\alpha_\mathcal{A}}{|\text{Im} \mathcal{A}|} + \min\{|T|, |T'|\} \beta_\mathcal{A} \quad \text{(H3') for any $T, T' \subset U^n$, which is introduced in [21]. A stronger condition (H3) is required for Lemmas 3 and 5, which appear later. The proof of (H3') is given in Appendix B for the completeness of this paper.}

It should be noted that when $\mathcal{A}$ is a two-universal class of hash functions [9] and $p_{\mathcal{A}}$ is the uniform distribution on $\mathcal{A}$, then $(\mathcal{A}, p_{\mathcal{A}})$ has a $(1, 0)$-hash property, where 1 and 0 denote...
the constant sequences of 1 and 0, respectively. Random bin coding [4] and the set of all linear functions [6] are examples of the two-universal class of hash functions. An ensemble of sparse matrices satisfying a hash property is introduced in Section III-B.

We have the following lemma, where it is unnecessary to assume the linearity of functions assumed in [21]. The proof is given in Appendix C for the completeness of this paper.

**Lemma 3 ([22, Lemma 4]):** Let \((\mathcal{A}, p_A)\) and \((\mathcal{B}, p_B)\) be ensembles satisfying an \((\alpha_A, \beta_A)\)-hash property and an \((\alpha_B, \beta_B)\)-hash property, respectively. Let \(A \in \mathcal{A}\) (resp. \(B \in \mathcal{B}\)) be a set of functions \(A : \mathcal{U}^n \rightarrow \text{Im}(A)\) (resp. \(B : \mathcal{U}^n \rightarrow \text{Im}(B)\)). Let \((A, B) \in \mathcal{A} \times \mathcal{B}\) be a function defined as

\[(A, B)u \equiv (A u, B u) \quad \text{for each } u \in \mathcal{U}^n.\]

Let \(p_{AB}\) be a joint distribution on \(\mathcal{A} \times \mathcal{B}\) defined as

\[p_{AB}(A, B) \equiv p_A(A)p_B(B) \quad \text{for each } (A, B) \in \mathcal{A} \times \mathcal{B}.\]

Then the ensemble \((\mathcal{A} \times \mathcal{B}, p_{AB})\) has an \((\alpha_{AB}, \beta_{AB})\)-hash property, where \((\alpha_{AB}, \beta_{AB})\) is defined as

\[
\alpha_{AB} \equiv \alpha_A \alpha_B \\
\beta_{AB} \equiv \beta_A + \beta_B.
\]

The following lemma is related to the **collision-resistance property**, that is, if the number of bins is greater than the number of items then there is an assignment such that every bin contains at most one item. The proof is given in Appendix D for the completeness of this paper.

**Lemma 4 ([21, Lemma 1]):** If \((\mathcal{A}, p_A)\) satisfies (H3'), then

\[p_A (\{A : [\mathcal{G} \setminus \{u\}] \cap C_A(A u) \neq \emptyset\}) \leq \frac{|\mathcal{G}| \alpha_A}{|\text{Im}(A)|} + \beta_A\]

for all \(\mathcal{G} \subset \mathcal{U}^n\) and \(u \in \mathcal{U}^n\).

We show the collision-resistance property from Lemma 4. Let \(\mu_U\) be the probability distribution on \(\mathcal{G} \subset \mathcal{U}^n\). We have

\[
E_{\mathcal{A}} [\mu_U (\{u : [\mathcal{G} \setminus \{u\}] \cap C_A(A u) \neq \emptyset\})] \leq \sum_{u \in \mathcal{G}} \mu_U(u)p_A (\{A : [\mathcal{G} \setminus \{u\}] \cap C_A(A u) \neq \emptyset\})
\]

\[\leq \sum_{u \in \mathcal{G}} \mu_U(u) \left[ \frac{|\mathcal{G}| \alpha_A}{|\text{Im}(A)|} + \beta_A \right]
\]

\[\leq \frac{|\mathcal{G}| \alpha_A}{|\text{Im}(A)|} + \beta_A. \quad (9)
\]
By assuming that $|G|/|\text{Im}A|$ vanishes as $n \to \infty$, we have the fact that there is a function $A$ such that
\[
\mu_U (\{ u : [G \setminus \{ u \}] \cap C_A(Au) \neq \emptyset \}) < \delta
\]
for any $\delta > 0$ and sufficiently large $n$. Since the relation $[G \setminus \{ u \}] \cap C_A(Au) \neq \emptyset$ corresponds to an event where there is $u' \in G$ such that $u$ and $u'$ are different members of the same bin (they have the same codeword determined by $A$), we have the fact that the members of $G$ are located in different bins (the members of $G$ can be decoded correctly) with probability close to one.

The following lemma is related to the balanced coloring property, which is analogous to [2, Lemma 3.1][7, Lemma 17.3]. This lemma implies that there is a function $A$ such that $T$ is almost equally partitioned by $A$ with respect to a measure $Q$. We use this property instead of the saturation property [21], that is, if the number of bins is greater than the number of items there is an assignment such that every bin contains at least one item. The proof is given in Appendix E for the completeness of this paper.

**Lemma 5 ([24, Lemma 4]):** If $(A, p_A)$ satisfies (H3), then
\[
E_A \left[ \sum_c \left| \frac{Q(T \cap C_A(c))}{Q(T)} - \frac{1}{|\text{Im}A|} \right| \right] \leq \sqrt{\alpha_A - 1 + \frac{[\beta_A + 1]|\text{Im}A| \max_{u \in T} Q(u)}{Q(T)}} \tag{10}
\]
for any function $Q : U^n \to [0, \infty)$ and $T \subset U^n$, where
\[
Q(T) \equiv \sum_{u \in T} Q(u).
\]

**Remark 8:** In [2, Lemma 3.1] and [7, Lemma 17.3], the absolute value on the left hand side of (10) is upper-bounded by $\varepsilon/|\text{Im}A|$ for all $c \in \text{Im}A$ and $Q \in Q$ provided that $\varepsilon^2 > 3|\text{Im}A| \log(2|\text{Im}A||Q|) \max_{u \in T} Q(u)$, where $Q$ is a finite set of probability distributions.

We show the balanced coloring property. From Lemma 5, we have the fact that there is a function $A$ such that
\[
\sum_c \left| \frac{Q(T \cap C_A(c))}{Q(T)} - \frac{1}{|\text{Im}A|} \right| \leq \sqrt{\alpha_A - 1 + \frac{[\beta_A + 1]|\text{Im}A| \max_{u \in T} Q(u)}{Q(T)}}.
\]
By assuming that $Q(T) \leq 1$ and $|\text{Im}A| \max_{u \in T} Q(u)$ vanishes as $n \to \infty$, we have
\[
\left| \frac{Q(T \cap C_A(c))}{|\text{Im}A|} - \frac{Q(T)}{|\text{Im}A|} \right| \leq \sum_c \left| \frac{Q(T \cap C_A(c)) - Q(T)}{|\text{Im}A|} \right|
\]

\(^5\)See also [8, Remark on Lemma B.1].
\[
Q(T) \sum_c \left| \frac{Q(T \cap C_A(c))}{Q(T)} - \frac{1}{|\text{Im} A|} \right|
\leq \sqrt{\alpha_A - 1 + [\beta_A + 1]} |\text{Im} A| \max_{u \in T} Q(u)
\leq \delta
\] 
(11)

for all \( c \in \text{Im} A, \delta > 0, \) and sufficiently large \( n. \) Since \( \{T \cap C_A(c)\}_{c \in \text{Im} A} \) is a partition of \( T, \) we have the fact that the set \( T \) is almost equally partitioned with respect to a measure \( Q, \) where \( c \) represents the color of a set \( T \cap C_A(c). \)

B. Hash Property for Ensembles of Matrices

In the following, we discuss the hash property for an ensemble of matrices.

In the last section we discussed that the uniform distribution on the set of all linear functions has a strong \((1, 0)\)-hash property because it is a universal class of hash functions.

In the following, we introduce another ensemble of matrices.

First, we introduce the average spectrum of an ensemble of matrices given in [3]. Let \( U \) be a finite field and \( A \) be a set of linear functions \( A : U^n \rightarrow U^l. \) It should be noted again that \( A \) can be expressed by an \( l \times n \) matrix.

Let \( t(u) \) be the type\(^6\) of \( u \in U^n, \) which is characterized by the empirical probability distribution of the sequence \( u. \) Let \( H \) be a set of all types of length \( n \) except \( t(0), \) where \( 0 \) is the zero vector. For a probability distribution \( p_A \) on a set of \( l \times n \) matrices and a type \( t, \) let \( S(p_A, t) \) be defined as

\[
S(p_A, t) \equiv \sum_{A \in A} p_A(A)|\{u \in U^n : Au = 0, t(u) = t\}|
\]

which is called the expected number of codewords that have type \( t \) in the context of linear codes. For a given \( \widehat{H}_A \subset H, \) we define \( \alpha_A(n) \) and \( \beta_A(n) \) as

\[
\alpha_A(n) \equiv \frac{|\text{Im} A|}{|U|^l} \max_{t \in \widehat{H}_A} S(p_A, t)
\]

\( \beta_A(n) \equiv \sum_{t \in H \setminus \widehat{H}_A} S(p_A, t), \) 
(12)

(13)

where \( p_A \) denotes the uniform distribution on the set of all \( l \times n \) matrices.

The following lemma provides a sufficient condition for an ensemble of matrices to satisfy a strong hash property. The proof is given in Appendix F for the completeness of this paper.

\(^6\)In [21], it is called a histogram that is characterized by the number of occurrences of each symbol in the sequence \( u. \) The type and the histogram are essentially the same when \( n \) is fixed.
Lemma 6 ([22, Theorem 1]): Let \((\mathcal{A}, p_A)\) be an ensemble of matrices and assume that 
\[ p_A \{ A : Au = 0 \} \] 
depends on \(u\) only through the type \(t(u)\). If \((\alpha_A, \beta_A)\), defined by (12) and (13), satisfies (H1) and (H2), then \((\mathcal{A}, p_A)\) has a strong \((\alpha_A, \beta_A)\)-hash property.

Next, we introduce the ensemble of \(q\)-ary sparse matrices introduced in [21], which is the \(q\)-ary extension of the ensemble proposed in [18]. Let \(U \equiv \text{GF}(q)\) and \(l \equiv nR\) when \(0 < R < 1\) is given, where \(q\) is a prime number or a power of a prime number. We generate an \(l \times n\) matrix \(A\) with the following procedure, where at most \(\tau\) random nonzero elements are introduced in every row.

1) Start from an all-zero matrix.
2) For each \(i \in \{1, \ldots, n\}\), repeat the following procedure \(\tau\) times:
   a) Choose \((j, a) \in \{1, \ldots, l\} \times [\text{GF}(q) \setminus \{0\}]\) uniformly at random.
   b) Add \(a\) to the \((j, i)\)-element of \(A\).

Assume that \(\tau = O(\log n)\) is even and let \((\mathcal{A}, p_A)\) be an ensemble corresponding to the above procedure. Let \(\mathcal{H}_A \subset \mathcal{H}\) be a set of types satisfying the requirement that the weight (the number of occurrences of non-zero elements) is large enough. Let \((\alpha_A, \beta_A)\) be defined by (12) and (13). Then \(\alpha_A\) measures the difference between the ensemble \((\mathcal{A}, p_A)\) and the ensemble of all \(l \times n\) matrices with respect to the high-weight part of the average spectrum, and \(\beta_A\) provides the upper bound of the probability that the code \({u \in U^n : Au = 0}\) has low-weight codewords. It is proved in [21, Theorem 2] that \((\alpha_A, \beta_A)\) satisfy (H1) and (H2) if we adopt an appropriate \(\mathcal{H}_A\). Then, from Lemma 6, we have the fact that this ensemble has a strong \((\alpha_A, \beta_A)\)-hash property. It should be noted that the convergence speed of \((\alpha_A, \beta_A)\) depends on how fast \(\tau\) grows in relation to the block length. The analysis of \((\alpha_A, \beta_A)\) is given in the proof of [21, Theorem 2].

IV. CONSTRUCTION OF CHANNEL CODE

This section introduces a channel code. The idea for the construction is drawn from [20][21][24]. It should be noted that we assume that the channel input alphabet \(X^n\) is a finite set but allow the channel output alphabet \(Y^n\) to be an arbitrary (infinite, continuous) set.

For given \(r > 0\) and \(R > 0\), let \((\mathcal{A}, p_A)\) and \((\mathcal{B}, p_B)\) be ensembles of functions \(A : X^n \rightarrow \text{Im} A\) and \(B : X^n \rightarrow \text{Im} B\) satisfying

\[ r = \frac{1}{n} \log |\text{Im} A| \]

\(^7\)It should be noted that \((j, i)\)-element of the matrix is not overwritten by \(a\) when the same \(j\) is chosen again.
Encoder
\[
\begin{align*}
c & \xrightarrow{m} \tilde{X}_{AB}^n \\
& \xrightarrow{x}
\end{align*}
\]

Decoder
\[
\begin{align*}
y & \xrightarrow{c} x \\
& \xrightarrow{B} m
\end{align*}
\]

Fig. 1. Construction of Channel Code

\[
R = \frac{1}{n} \log |\text{Im } B|,
\]

respectively, where we define \( M_n \equiv \text{Im } B \) and \( R \) represents the rate of the code. We fix functions \( A \in \mathcal{A} \), \( B \in \mathcal{B} \), and a vector \( c \in \text{Im } A \) so that they are available for constructing an encoder and a decoder.

We use a constraint random number generator to construct an encoder. Let \( \tilde{X}^n \equiv \tilde{X}_{AB}^n(c, m) \) be a random variable corresponding to the distribution

\[
\mu_{\tilde{X}_n}(x) \equiv \begin{cases} 
\frac{\mu_{X^n}(x)}{\mu_{X^n}(C_{AB}(c, m))}, & \text{if } x \in C_{AB}(c, m), \\
0, & \text{if } x \notin C_{AB}(c, m),
\end{cases}
\]

(14)

where \( \mu_{X^n} \) is the probability distribution of the channel input random variable \( X^n \).

We define the stochastic encoder \( \Phi_n : \text{Im } B \rightarrow \mathcal{X}^n \) and the decoder \( \psi_n : \mathcal{Y}^n \rightarrow \text{Im } B \) as

\[
\Phi_n(m) \equiv \tilde{X}_{AB}^n(c, m) \quad (15)
\]

\[
\psi_n(y) \equiv Bx_A(c|y), \quad (16)
\]

where we declare an encoding error when \( \mu_{X^n}(C_{AB}(c, m)) = 0 \) and \( x_A \) is defined as

\[
x_A(c|y) \equiv \arg \max_{x' \in C_A(c)} \mu_{X^n|Y^n}(x'|y).
\]

(17)

The flow of vectors is illustrated in Fig. 1.

Remark 9: It should be noted that (15) is different from the encoder defined in [21] whereas (16) is the same. In [21], the encoder is defined based on typical sets, where \( x \in \mathcal{T}_{X|Y} \).

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is satisfied when $x \in T_{X,\gamma}$ and $y \in T_{Y|X,\gamma}(x)$. We changed the definition of the encoder because a general channel may not satisfy this property.

The error probability $\text{Error}(A, B, c)$ is given by

$$\text{Error}(A, B, c) \equiv \sum_{m : \mu_{X^n}(C_{AB}(c, m)) = 0} \frac{1}{|M_n|} + \sum_{m : \mu_{X^n}(C_{AB}(c, m)) > 0, x \in C_{AB}(c, m), y = \psi_m(y)} \mu_{X^n}(x) \mu_{Y^n}(y|x) \mu_{X^n}(x) \mu_{Y^n}(C_{AB}(c, m)).$$  \hspace{1cm} (18)

We have the following theorem, where the proof is given in Section VII-C.

**Theorem 1:** If $r, R > 0$ satisfy

$$r > H(X|Y)$$ \hspace{1cm} (19)

$$r + R < H(X),$$ \hspace{1cm} (20)

then for any $\delta > 0$ and all sufficiently large $n$ there are functions $A \in \mathcal{A}$, $B \in \mathcal{B}$, and a vector $c \in \text{Im}A$ such that

$$\text{Error}(A, B, c) \leq \delta.$$  \hspace{1cm} (21)

The channel capacity is achievable with the proposed code by letting $X$ be a source that attains the supremum on the right hand side of (3).

**Remark 10:** From (18) and (21), we have the fact that $C_{AB}(c, m) \neq \emptyset$ with probability close to 1 by letting $\delta \to 0$ because

$$\sum_{m : C_{AB}(c, m) = \emptyset} \frac{1}{|M_n|} \leq \sum_{m : \mu_{X^n}(C_{AB}(c, m)) = 0} \frac{1}{|M_n|} \leq \text{Error}(A, B, c) \leq \delta.$$ \hspace{1cm} (22)

Furthermore, we can find $c \in \text{Im}A \subset \text{Im}A$ satisfying (21) because $\text{Error}(A, B, c) = 1$ when $c \in \text{Im}A \setminus \text{Im}A$.

Next, we consider a special case for the proposed code, which provides an interpretation of the conventional linear codes [3][10]. It should be noted that a constrained random number generator is unnecessary.

Let us assume that $\mu_{X^n}$ is the uniform distribution on $X^n$ and $(\mathcal{A}, p_A)$ is an ensemble of matrices $A : X^n \to X^l$ satisfying

$$r = \frac{1}{n} \log |\text{Im}A|.$$
when \( 0 < r < \log |\mathcal{X}| \) is given. We fix a matrix \( A \in \mathcal{A} \) and a vector \( c \in \text{Im} A \subset \text{Im} A \) so that they are available for constructing an encoder and a decoder.

Let \( \mathcal{M}_n \) be a set of all messages that is a linear space satisfying \( |\mathcal{M}_n| = |C_A(c)| \) for all \( c \in \text{Im} A \). Since \( A \) is a linear function, there is a bijective linear function \( G : \mathcal{M}_n \rightarrow C_A(0) \), which is known as a generator matrix. The rate \( R \) of the code is given as

\[
R \equiv \frac{1}{n} \log |\mathcal{M}_n|.
\]

Since for a given \( c \in \text{Im} A \) there is \( x_c \) such that \( Ax_c = c \), then we have the fact that \( A[Gm + x_c] = c \) for all \( m \in \mathcal{M}_n \). Since \( G \) is a linear function, there is a linear function \( B : \mathcal{X}^n \rightarrow \mathcal{M}_n \) such that \( BGm = m \) for all \( m \in \mathcal{M}_n \). We define a deterministic encoder \( \varphi_n : \mathcal{M}_n \rightarrow C_A(c) \) and a decoder \( \psi_n : \mathcal{Y}^n \rightarrow \mathcal{M}_n \) as

\[
\varphi_n(m) \equiv Gm + x_c,
\]

\[
\psi_n(y) \equiv B[x_A(c|y) - x_c]
\]

where \( x_A \) is defined as (17). The error probability \( \text{Error}(A, c) \) is given by

\[
\text{Error}(A, c) \equiv \sum_{x \in C_A(c), y} \frac{\mu_{\mathcal{Y}^n|\mathcal{X}^n}(y|x)}{|C_A(c)|}.
\]

We have the following corollary, which is shown in Section VII-D.

**Corollary 2:** If \( r \) satisfies

\[
\overline{H}(X|Y) < r < \log |\mathcal{X}|,
\]  

(23)

then for any \( \delta > 0 \) and all sufficiently large \( n \) there are a matrix \( A \in \mathcal{A} \) and a vector \( c \in \text{Im} A \) such that

\[
R \geq \log |\mathcal{X}| - r
\]  

(24)

\[
\text{Error}(A, c) \leq \delta
\]  

(25)

When the supremum on the right hand side of (3) is achieved by \( X \) corresponding to the uniform distribution, the channel capacity

\[
C(W) = \log |\mathcal{X}| - \overline{H}(X|Y)
\]  

(26)

is achievable with the proposed code by letting \( r \rightarrow \overline{H}(X|Y) \). Assuming that \( \mathcal{X} = \mathcal{Y} = \mathcal{Z} \) is a finite field, the capacity

\[
C(W) = \log |\mathcal{X}| - \overline{H}(Z)
\]  

(27)
for a channel with additive noise $Z = \{Y^n - X^n\}_{n=1}^{\infty}$ is achievable with the proposed code by letting $r \to \mathbb{P}(Z)$.

Remark 11: In [5, Theorem 7.2.1], the capacity of a discrete stationary memoryless weakly symmetric channel is given as

$$C(W) = \log |\mathcal{Y}| - H(\text{row of the transition matrix}),$$

which is another expression of (26). It should be noted that the formula (26) is valid for a weakly symmetric channel and is well-defined as long as $|\mathcal{X}|$ is finite. It should also be noted that the capacity (26) for a symmetric output channel (e.g. an additive Gaussian noise channel) is achieved by $X$ corresponding to the uniform distribution. For a channel with additive noise $Z$, the channel capacity (27) is derived in [31][14, Example 3.2.1] when $\mathcal{X} = \mathcal{Y} = \mathcal{Z} = \{0, 1\}$. Formula (27) is an extension to a general finite alphabet.

V. CONSTRUCTION OF LOSSY SOURCE CODE

This section introduces a lossy source code. The idea for the construction is drawn from [20][21][24]. It should be noted that we assume that a reproduction alphabet $\mathcal{X}^n$ is finite set but a source alphabet $\mathcal{Y}^n$ is allowed to be arbitrary (infinite, continuous) set.

For given $r > 0$ and $R > 0$, let $(A, p_A)$ and $(B, p_B)$ be ensembles of functions $A : \mathcal{X}^n \to \text{Im}A$ and $B : \mathcal{X}^n \to \text{Im}B$ satisfying

$$r = \frac{1}{n} \log |\text{Im}A|$$
$$R = \frac{1}{n} \log |\text{Im}B|,$$

respectively. We fix functions $A \in A$, $B \in B$, and a vector $c \in \text{Im}A$ so that they are available for constructing an encoder and a decoder.

Let $\mu_{X^n}$ be defined as

$$\mu_{X^n}(x) \equiv \sum_{y} \mu_{X^n|Y^n}(x|y)\mu_{Y^n}(y),$$

where $\mu_{Y^n}$ is the probability distribution of a source $Y^n$ and we assume that the conditional probability distribution $\mu_{X^n|Y^n}$ is given. We use a constrained random number generator to construct an encoder. Let $\tilde{X}^n \equiv \tilde{X}^n_A(c|y)$ be a random variable corresponding to the distribution

$$\mu_{\tilde{X}^n|Y^n}(x|y) \equiv \begin{cases} \frac{\mu_{X^n|Y^n}(x|y)}{\mu_{X^n|Y^n}(c_A(c)|y)}, & \text{if } x \in C_A(c), \\ 0, & \text{if } x \notin C_A(c). \end{cases}$$ (28)
We define the stochastic encoder $\Phi_n : Y^n \rightarrow \text{Im} B$ and the decoder $\psi_n : \text{Im} B \rightarrow X^n$ as

\begin{align*}
\Phi_n(y) &\equiv B\overline{X}_A^n(c|y) \quad (29) \\
\psi_n(m) &\equiv x_{AB}(c, m), \quad (30)
\end{align*}

where we declare an encoding error when $\mu_{X^n|Y^n}(C_A(c)|y) = 0$ and $x_{AB}$ is defined as

\begin{equation}
x_{AB}(c, m) \equiv \arg \max_{x' \in C_{AB}(c, m)} \mu_{X^n}(x'). \quad (31)
\end{equation}

The flow of vectors is illustrated in Fig. 2.

The error probability $\text{Error}(A, B, c, D)$ is given as

\begin{equation}
\text{Error}(A, B, c, D) \equiv P\left(d_n(\psi_n(\Phi_n(Y^n)), Y^n) > D\right), \quad (32)
\end{equation}

where we define $d_n(\psi_n(\Phi_n(y)), y) \equiv \infty$ when $\mu_{X^n|Y^n}(C_A(c)|y) = 0$. We have the following theorem, where the proof is given in Section VII-E.

**Theorem 3:** If $r, R > 0$ satisfy

\begin{align*}
r &< H(X|Y) \quad (33) \\
r + R &> H(X), \quad (34)
\end{align*}

then for any $\delta > 0$ and all sufficiently large $n$ there are functions $A \in A$, $B \in B$, and a vector $c \in \text{Im} A$ such that

\begin{equation}
\text{Error}(A, B, c, D) \leq P\left(d_n(X^n, Y^n) > D\right) + \delta. \quad (35)
\end{equation}
By assuming that \( \{ \mu_{X^n|Y^n} \}_{n=1}^{\infty} \) satisfies
\[
\overline{D}(X, Y) < D,
\]
we have the fact that
\[
\lim_{n \to \infty} P \left( d_n(X^n, Y^n) > D \right) = 0
\]
from the definition of \( \overline{D}(X, Y) \). Then, by letting \( n \to \infty, \delta \to 0, \) and \( r \to H(X|Y) \), we have the fact that for any \((R, D)\) close to the boundary of \( \mathcal{R}(Y) \) there is a sequence of proposed codes such that
\[
\lim_{n \to \infty} \text{Error}(A, B, c, D) = 0
\]
\[\text{Remark 12:}\] We can find \( c \in \text{Im}A \subset \text{Im}A \) satisfying (35) because Error\((A, B, c) = 1\) when \( C_A(c) = \emptyset \).

Next, we consider a special case of the proposed code, which provides an interpretation of the conventional code introduced in [19][12]. It should be noted that a constrained random number generator is unnecessary.

Let us assume that \( \mu_{X^n} \) is the uniform distribution on \( X^n \) and \((A, p_A)\) is an ensemble of matrices \( A : X^n \to X^l \) satisfying
\[
r = \frac{1}{n} \log |\text{Im}A|
\]
when \( r > 0 \) is given. We fix a matrix \( A \in A \) and a vector \( c \in \text{Im}A \subset \text{Im}A \) so that they are available for constructing an encoder and a decoder.

Since \( C_A(0) \) is a linear space, there is a surjective linear function \( B : X^n \to C_A(0) \). We use the encoder defined by (29). The rate \( R \) of the code is given as
\[
R \equiv \frac{1}{n} \log |C_A(0)|.
\]
Furthermore, since \( B \) is surjective, there is a bijective linear function \( x'_{AB} : \text{Im}A \times C_A(0) \to X^n \) such that \( x'_{AB}(Ax, Bx) = x \) for all \( x \). We replace the function \( x_{AB} \) by \( x'_{AB} \) in the definition of the decoder (30). Let Error\((A, c, D)\) be the error probability given as
\[
\text{Error}(A, c, D) \equiv P \left( d_n(\psi_n(\Phi_n(Y^n)), Y^n) > D \right).
\]
We have the following corollary, which is shown in Section VII-F.

\[\text{Corollary 4:}\] If \( r \) satisfies
\[
r < H(X|Y),
\]
(36)
then for any $\delta > 0$ and all sufficiently large $n$ there are a matrix $A \in \mathcal{A}$ and a vector $c \in \text{Im}A$ such that

$$R \leq \log |\mathcal{X}| - r + \delta$$

(37)

$$\text{Error}(A, c, D) \leq P(d(X^n, Y^n) > D) + \delta.$$  

(38)

When the boundary of the right hand side of (8) is attained with a general channel $W \equiv \{\mu_{X^n|Y^n}\}_{n=1}^{\infty}$ such that $\mu_{X^n}$ is uniform, for any $(R, D)$ close to the boundary of $\mathcal{R}(\mathcal{Y})$ there is a sequence of proposed codes such that

$$\lim_{n \to \infty} \text{Error}(A, c, D) = 0.$$  

VI. CONSTRAINED RANDOM NUMBER GENERATION BY USING SUM-PRODUCT ALGORITHM

In this section, we introduce an algorithm for generating random numbers subject to the distributions (14) and (28) by assuming that $\mu_{X^n}$ and $\mu_{X^n|Y^n}$ are memoryless, that is, they are given by

$$\mu_{X^n}(x) = \prod_{i=1}^{n} \mu_{X_i}(x_i)$$

(39)

$$\mu_{X^n|Y^n}(x|y) = \prod_{i=1}^{n} \mu_{X_i|Y_i}(x_i|y_i)$$

for each $x \equiv (x_1, \ldots, x_n)$ and $y \equiv (y_1, \ldots, y_n)$, respectively. In the following, we construct a random number generator subject to the distribution $\mu_{\tilde{X}_n}$ defined by

$$\mu_{\tilde{X}_n}(x) \equiv \begin{cases} \frac{\mu_{X^n}(x)}{\mu_{X^n}(C_A(c))} & \text{if } x \in C_A(c) \\ 0 & \text{if } x \notin C_A(c) \end{cases}$$

(40)

for a $\mu_{X^n}$ given by (39), $A$, and $c \in \text{Im}A$. It should be noted that (14) can be reduced to (40) by considering a function $(A, B) : \mathcal{X}^n \to \text{Im}A \times \text{Im}B$ defined as $(A, B)x \equiv (Ax, Bx)$, and (28) can also be reduced to (40) by letting $\mu_{X^n|Y^n}(\cdot|y_i)$ for a given $y$.

Let us assume that there is a family $\{Z_j\}_{j \in J}$ of sets such that $\text{Im}A \subset \times_{j \in J} Z_j$. For a set of local functions $\{f_j : \mathcal{X}^{|S_j|} \to Z_j\}_{j \in J}$, the sum-product algorithm [1][17] calculates a real-valued global function $g$ on $\mathcal{X}$ defined as

$$g(x_i) \equiv \frac{\sum_{x \setminus \{x_i\}} \prod_{j \in J} f_j(x_{S_j})}{\sum_x \prod_{j \in J} f_j(x_{S_j})}$$

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approximately, where the summation $\sum x_{\setminus \{x_i\}}$ is taken over all $x \in \mathcal{X}^n$ except for the variable $x_i$ and the function $f_j$ depends only on the set of variables $x_{S_i} \equiv \{x_j\}_{j \in S_i}$. It should be noted that the algorithm calculates the global function exactly when the corresponding factor graph has no loop. Let $\pi_{x_i \to f_j}(x_i)$ and $\sigma_{f_j \to x_i}(x_i)$ be messages defined as
\[
\pi_{x_i \to f_j}(x_i) \equiv \prod_{j' \in \mathcal{J} \setminus \{j\}; \sigma_{x_i \to f_j}(x_i) i \in S_j} \sigma_{f_j \to x_i}(x_i)
\]
\[
\sigma_{f_j \to x_i}(x_i) \equiv \frac{\text{\sum}_{x_{S_j \setminus \{i\}}} f_j(x_{S_i}) \prod_{i' \in S_j \setminus \{i\}} \pi_{x_{i' \to f_j}}(x_{i'})}{\text{\sum}_{x_{S_j}} f_j(x_{S_i}) \prod_{i' \in S_j \setminus \{i\}} \pi_{x_{i' \to f_j}}(x_{i'})},
\]
where the summation $\sum_{S}$ is taken over all $\{x_i\}_{i \in S}, \pi_{x_i \to f_j}(x_i) \equiv 1$ when there is no $j' \in \mathcal{J} \setminus \{j\}$ such that $i \in S_{j'}$ and $\sigma_{f_j \to x_i}(x_i) \equiv f_j(x_i)/\sum_{x_j} f_j(x_i)$ when $S_j = \{i\}$. The sum-product algorithm is performed by repeating the above operations for every message $\sigma_{f_j \to x_i}(x_i)$ and $\pi_{x_i \to f_j}(x_i)$ satisfying $i \in S_j$ and finally calculating the approximation of the global function as
\[
g(x_i) \approx \prod_{j \in \mathcal{J}; \iota \in S_j} \sigma_{f_j \to x_i}(x_i),
\]
where we assign initial values to $\pi_{x_i \to f_j}(x_i)$ and $\sigma_{f_j \to x_i}(x_i)$ when they appear on the right hand side of the above operations and are undefined.

In the following, we introduce an algorithm for constrained random number generation. For each $i \in \{1, \ldots, l\}$, let $\mathbf{a}_i : \mathcal{X}^{|S_i|} \to \mathbb{Z}$ be a function such that
\[
A x = (\mathbf{a}_1(x_{S_1}), \mathbf{a}_2(x_{S_2}), \ldots, \mathbf{a}_l(x_{S_l})),
\]
where the $i$-th component $\mathbf{a}_i$ of $A$ depends only on the set of variables $x_{S_i} \equiv \{x_j\}_{j \in S_i}$. For example, when $A \equiv (a_{i,j})$ is an $l \times n$ sparse matrix with a maximum row weight $w$, we have $\mathcal{X} = \mathbb{Z}$, the set $S_i$ defined as
\[
S_i \equiv \{j \in \{1, \ldots, n\} : a_{i,j} \neq 0\}
\]
satisfies $|S_i| \leq w$ for all $i \in \{1, \ldots, l\}$, and $\mathbf{a}_i(x_{S_i})$ is defined as the inner product $\mathbf{a}_i \cdot x$ of vectors $\mathbf{a}_i$ and $x$. Let $x^i \equiv (x_i, \ldots, x_j)$, where $x^j$ is a null string if $i > j$. Let $c \equiv (c_1, \ldots, c_l) \in \mathbb{Z}^l$. Let $\chi(\cdot)$ be defined as
\[
\chi(S) \equiv \begin{cases} 
1, & \text{if the statement } S \text{ is true} \\
0, & \text{if the statement } S \text{ is false.}
\end{cases}
\]

**Constrained Random Number Generation Algorithm:**

**Step 1** Let $k = 1$. 
Step 2 Calculate the conditional probability distribution \( P_{\tilde{X}_k|\tilde{X}_1^k} (x_k|x_1^{k-1}) \) defined as

\[
P_{\tilde{X}_k|\tilde{X}_1^k} (x_k|x_1^{k-1}) = \frac{\sum_{j=k}^{\infty} \prod_{j=k}^{\infty} \mu_{X_j}(x_j) \prod_{i=1}^{l} \chi(a_i(x_{S_i}) = c_i)}{\sum_{j=k}^{\infty} \prod_{j=k}^{\infty} \mu_{X_j}(x_j) \prod_{i=1}^{l} \chi(a_i(x_{S_i}) = c_i)}.
\] (42)

It should be noted that the sum-product algorithm can be employed to obtain (42), where \( \{\mu_{X_j}\}_{j=k}^{\infty} \) and \( \{\chi(a_i(x_{S_i}) = c_i)\}_{i=1}^{l} \) are local functions and we substitute the generated sequence \( x_1^{k-1} \) for (42). If \( \chi(a_i(x_{S_i}) = c_i) \) is a constant after the substitution of \( x_1^{k-1} \), we can recode the constant in preparation for the future.

Step 3 Generate and recode a random number \( x_k \) corresponding to the distribution \( P_{\tilde{X}_k|\tilde{X}_1^k} \).

Step 4 If \( k = n \), output \( x \equiv x_1^n \) and terminate.

Step 5 If for the generated sequence \( x_1^n \) there is a unique \( x_{k+1}^n \) such that \( x \equiv x_1^n \in C_A(x) \), obtain the unique vector \( x_{k+1}^n \), output \( x \), and terminate.

Step 6 Let \( k \leftarrow k + 1 \) and go to Step 2.

Remark 13: We can omit Step 5 if it is hard to execute.

Remark 14: When \( A \) is a linear function with rank \( l' \), by checking whether \( k = l' \) or not at Step 5, we can easily determine whether or not for a given \( x_1^k \) there is a unique \( x_{k+1}^n \) such that \( x_1^n \in C_A(x) \). We can obtain the unique \( x_{k+1}^n \) from \( x_1^{l'} \) by using a linear operation.

Remark 15: It should be noted that the memoryless condition on \( X^n \) is not essential for the description of the algorithm. The algorithm is well-defined when we use the formula

\[
\mu_{X^n}(x) = \prod_{i=1}^{n} \mu_{X_i|X_1^{i-1}}(x_i|x_1^{i-1})
\]

and replace \( \mu_{X_i}(x_i) \) by \( \mu_{X_i|X_1^{i-1}}(x_i|x_1^{i-1}) \) for \( i \geq 2 \) in (42). However, the sum-product algorithm may not find a good approximation in general because the corresponding factor graph may have many loops.

We have the following theorem, which is shown in Section VII-G.

Theorem 5: Assume that (42) is computed exactly. Then the proposed algorithm generates \( x \equiv x_1^n \) subject to the probability distribution given by (40).

In the following, we consider a situation where we can use a real number \( \omega \) subject to the uniform distribution on \([0, 1)\). We modify the proposed algorithm, where the basic idea comes from the interval algorithm introduced in [15] and is analogous to the arithmetic coding [29]. It should be noted that only Steps 1, 3 are modified.

Interval Constrained Random Number Generation Algorithm:

Step 1 Let \( k = 1 \) and \( [\underline{a}_1, \overline{a}_1) \equiv [0, 1) \).
Step 2 Calculate the conditional probability distribution $p_{\tilde{X}_k | \tilde{X}_{k-1}}$ defined by (42).

Step 3 Partition the interval $[\theta_{k-1}, \theta_k)$ into sub-intervals that are labeled corresponding to the elements in $\mathcal{X}$, where the sub-interval width is subject to the ratio $p_{\tilde{X}_k | \tilde{X}_{k-1}}(x_k | x_{k-1}^k)$.

Let $[\theta_k, \theta_k')$ be a sub-interval that contains $\omega$, that is, $\omega \in [\theta_k, \theta_k')$ is satisfied for a given $\omega$. Let $x_k$ be a label that corresponds to the sub-interval $[\theta_k, \theta_k')$ and record it.

Step 4 If $k = n$, output $x \equiv x_n$ and terminate.

Step 5 If for the generated sequence $x_1^n$ there is a unique $x_{n+1}$ such that $x \equiv x_1^n \in C_A(x)$, obtain the unique vector $x_{n+1}$, output $x$, and terminate.

Step 6 Let $k \leftarrow k + 1$ and go to Step 2.

From Theorem 5, we have the fact that the probability of selecting $\omega \in [0, 1)$ is equal to the width of the sub-interval $[\theta_{k'}, \theta_{k'})$, which is equal to the probability $\mu_{\tilde{X}_n}(x)$ of a generated sequence $x$, where $k'$ is the value of $k$ when the algorithm is terminated.

It should be noted that we can construct a deterministic code from a stochastic code by fixing a random number $\omega \in [0, 1)$. In fact, by using the random coding argument, we can show that there is a random number $\omega \in [0, 1)$ such that the error probability is sufficiently small. This is because, from Theorems 1 and 3, the average error probability with respect to the random variable corresponding to a random number on $[0, 1)$ is sufficiently small.

Remark 16: Instead of a real number $\omega$, we can use a binary random sequence $\omega_1, \omega_2, \ldots$ subject to the uniform distribution on $\{0, 1\}$ by letting $\omega \equiv 0.\omega_1\omega_2\cdots \in [0, 1)$, which is the binary expansion of a real number. Since we can estimate $\mu_{\tilde{X}_n}(C_A(c)) = 1/|\text{Im} A|$ approximately and the average entropy of $\tilde{X}_n(c)$ is given as

$$E_c \left[ H(\tilde{X}_n(c)) \right] = \sum_c \frac{1}{|\text{Im} A|} \sum_{x \in C_A(c)} \frac{\mu_{\tilde{X}_n}(x)}{\mu_{\tilde{X}_n}(C_A(c))} \log \frac{\mu_{\tilde{X}_n}(C_A(c))}{\mu_{\tilde{X}_n}(x)}$$

$$= H(X^n) - \log |\text{Im} A|, \quad (43)$$

the required length of the binary sequence can be estimated approximately as at least $H(X^n) - \log |\text{Im} A|$.

VII. PROOFS OF THEOREMS

A. Proof of Lemma 1

Since

$$I(X; Y) \geq H(X) - H(Y | X)$$
for any \((X, Y)\), we have

\[
C(W) = \sup_{X} I(X; Y) \\
\geq \sup_{X} \left[ H(X) - H(X|Y) \right].
\]

(44)

In the following, we prove that

\[
C(W) \leq \sup_{X} \left[ H(X) - H(X|Y) \right],
\]

(45)

which completes the proof of the lemma.

From the definition of \(C(W)\), we have the fact that for any \(\delta > 0\) and sufficiently large \(n\) there is a pair consisting an encoder \(\varphi_n : \mathcal{M}_n \to \mathcal{X}^n\) and a decoder \(\psi_n : \mathcal{Y}^n \to \mathcal{M}_n\) such that

\[
\liminf_{n \to \infty} \frac{1}{n} \log |\mathcal{M}_n| \geq C(W) - \delta
\]

(46)

\[
\lim_{n \to \infty} P(\psi_n(Y^n) \neq \mathcal{M}_n) = 0.
\]

(47)

We can assume\(^8\) that \(\mathcal{M}_n \subset \mathcal{X}^n\) without loss of generality. Since the distribution \(\mu_{\mathcal{M}_n}\) of \(\mathcal{M}_n\) is uniform on \(\mathcal{M}_n\), we have the fact that

\[
\frac{1}{n} \log \frac{1}{\mu_{\mathcal{M}_n}(x)} = \frac{1}{n} \log |\mathcal{M}_n| \\
\geq \liminf_{n \to \infty} \frac{1}{n} \log |\mathcal{M}_n| - \delta
\]

(48)

for all \(x \in \mathcal{M}_n\), \(\delta > 0\), and sufficiently large \(n\). Since

\[
\frac{1}{n} \log \frac{1}{\mu_{\mathcal{M}_n}(x)} = \infty
\]

for every \(x \notin \mathcal{M}_n\), we have the fact that

\[
\frac{1}{n} \log \frac{1}{\mu_{\mathcal{M}_n}(x)} \geq \liminf_{n \to \infty} \frac{1}{n} \log |\mathcal{M}_n| - \delta
\]

for every \(x \in \mathcal{X}^n\), \(\delta > 0\) and sufficiently large \(n\). This implies that

\[
\lim_{n \to \infty} P\left(\frac{1}{n} \log \frac{1}{\mu_{\mathcal{M}_n}(M_n)} < \liminf_{n \to \infty} \frac{1}{n} \log |\mathcal{M}_n| - \delta\right) = 0.
\]

(49)

\(^8\)This assumption is used merely so that \(\mathcal{M} \equiv \{M_n\}_{n=1}^\infty\) is a general source satisfying \(M_n \in \mathcal{X}^n\). It should be noted that \(\mathcal{M}_n\) and \(\{\varphi_n(m) : m \in \mathcal{M}_n\}\) are different subsets of \(\mathcal{X}^n\) in general. We could define a channel code by a subset \(\mathcal{M}_n\) of \(\mathcal{X}^n\) as defined in \([14][31]\) instead of introducing an encoder \(\varphi_n\). We introduce an encoder \(\varphi_n\) to consider a stochastic encoder.
Let $\mathcal{M} \equiv \{M_n\}_{n=1}^{\infty}$ be a general source. Then we have

$$\liminf_{n \to \infty} \frac{1}{n} \log |M_n| - \delta \leq H(\mathcal{M})$$

from (49) and the definition of $H(\mathcal{M})$. We have

$$C(W) \leq \liminf_{n \to \infty} \frac{1}{n} \log |M_n| + \delta \leq H(\mathcal{M}) + 2\delta$$

where the first inequality comes from (46), the second inequality comes from (50), and the equality comes from (47) and Lemma 7. We have (45) by letting $\delta \to 0$.  

B. Proof of Lemma 2

Since

$$\mathcal{T}(X;Y) \leq \mathcal{H}(X) - H(X|Y)$$

for any $(X, Y)$, we have

$$\mathcal{R}(Y) = \bigcup_{w} \left\{ (R, D) : \begin{array}{l} \mathcal{T}(X;Y) \leq R \\ \mathcal{H}(X;Y) \leq D \end{array} \right\} \supset \bigcup_{w} \left\{ (R, D) : \begin{array}{l} \mathcal{H}(X) - H(X|Y) \leq R \\ \mathcal{H}(X;Y) \leq D \end{array} \right\}.$$

In the following, we prove that

$$\mathcal{R}(Y) \subset \bigcup_{w} \left\{ (R, D) : \begin{array}{l} \mathcal{H}(X) - H(X|Y) \leq R \\ \mathcal{H}(X;Y) \leq D \end{array} \right\},$$

which completes the proof of the lemma.

Assume that $(R, D) \in \mathcal{R}(Y)$. From (6), we have the fact that for all $\delta > 0$ and all sufficiently large $n$, there is a pair consisting an encoder $\varphi_n$ and a decoder $\psi_n$ satisfying (4) and (5). Let $\hat{X}^n \equiv \psi_n(\varphi_n(Y^n)) \in \mathcal{X}^n$. Then we have

$$P \left( \frac{1}{n} \log \frac{1}{\mu_{\hat{X}^n}(\hat{X}^n)} > R + \varepsilon \right) \leq P \left( \frac{1}{n} \log \frac{1}{\mu_{\hat{X}^n}(\hat{X}^n)} \geq \frac{1}{n} \log |M_n| + \varepsilon \right) \leq 2^{-n\varepsilon}$$

(53)
for any $\varepsilon > 0$, where the first inequality comes from (4), and the second inequality comes from [14, Lemma 2.6.2] and the fact that the cardinality of the domain of $\hat{X}^n$ is at most $|M_n|$. By letting $n \to \infty$, we have the fact that a general source $\hat{X} \equiv \{\psi_n(\varphi_n(Y^n))\}_{n=1}^{\infty}$ satisfies

$$\overline{H}(\hat{X}) - H(\hat{X} | Y) \leq \overline{H}(\hat{X}) \leq R + \varepsilon. \quad (54)$$

By letting $\varepsilon \to 0$, we have

$$\overline{H}(\hat{X}) - H(\hat{X} | Y) \leq R. \quad (55)$$

On the other hand, we have

$$\lim_{n \to \infty} P \left( d_n(\hat{X}^n, Y^n) > D \right) = 0$$

from (5) by letting $n \to \infty$ and $\delta \to 0$. This implies that

$$\overline{D}(\hat{X}, Y) \leq D. \quad (56)$$

Then we have

$$(R, D) \in \bigcup_{W} \left\{ (R, D) : \overline{H}(\hat{X}) - H(\hat{X} | Y) \leq R \right\}.$$

which implies (52).

C. Proof of Theorem 1

We omit dependence on $n$ of $X$ and $Y$ when they appear in the subscript of $\mu$.

From (19) and (20), we have the fact that there is $\varepsilon > 0$ satisfying

$$r > \overline{H}(X | Y) + \varepsilon \quad (57)$$

$$r + R < H(X) - \varepsilon. \quad (58)$$

Let $\mathcal{T}_X \subset X^n$ and $\mathcal{T}_{X|Y} \subset X^n \times Y^n$ be defined as

$$\mathcal{T}_X \equiv \left\{ x : \frac{1}{n} \log \frac{1}{\mu_X(x)} \geq H(X) - \varepsilon \right\} \quad (59)$$

$$\mathcal{T}_{X|Y} \equiv \left\{ (x, y) : \frac{1}{n} \log \frac{1}{\mu_{X|Y}(x|y)} \leq \overline{H}(X|Y) + \varepsilon \right\}. \quad (60)$$

Assume that $(x, y) \in \mathcal{T}_{X|Y}$ and $x_A(Ax|y) \neq x$. Then we have the fact that there is $x' \in C_A(Ax)$ such that $x' \neq x$ and

$$\mu_{X|Y}(x'|y) \geq \mu_{X|Y}(x|y) \geq 2^{-n[H(X|Y) + \varepsilon]}.$$
This implies that \([\mathcal{T}_{X|Y}(y) \setminus \{x\}] \cap C_A(Ax) \neq \emptyset\), where \(\mathcal{T}_{X|Y}(y)\) is defined as

\[
\mathcal{T}_{X|Y}(y) \equiv \{x : (x, y) \in \mathcal{T}_{X|Y}\}.
\]

We have

\[
E_A \left[ \chi(x_A(Ax|y) \neq x) \right] \leq p_A \left( \{A : [\mathcal{T}_{X|Y}(y) \setminus \{x\}] \cap C_A(Ax) \neq \emptyset\} \right)
\leq \frac{|\mathcal{T}_{X|Y}(y)| \alpha_A}{|\text{Im}A|} + \beta_A
\leq 2^{-n[r - \mathcal{T}(X|Y) - \varepsilon]} \alpha_A + \beta_A
\]

(61)

for all \((x, y) \in \mathcal{T}_{X|Y}\), where \(\chi(\cdot)\) is defined by (41), the second inequality comes from Lemma 4, and the third inequality comes from the fact that \(|\mathcal{T}_{X|Y}(y)| \leq 2^{n[\mathcal{T}(X|Y) + \varepsilon]}\). We have the fact that

\[
E_A \left[ \sum_{x,y} \mu_{XY}(x, y) \chi(x_A(Ax|y) \neq x) \right]
= \sum_{(x,y) \in \mathcal{T}_{X|Y}} \mu_{XY}(x, y) E_A \left[ \chi(x_A(Ax|y) \neq x) \right] + \sum_{(x,y) \notin \mathcal{T}_{X|Y}} \mu_{XY}(x, y) E_A \left[ \chi(x_A(Ax|y) \neq x) \right]
\leq 2^{-n[r - \mathcal{T}(X|Y) - \varepsilon]} \alpha_A + \beta_A + \mu_{XY}([\mathcal{T}_{X|Y}]^c),
\]

(62)

where the last inequality comes from (61). We also have the fact that

\[
E_{AB} \left[ \sum_{c,m} \mu_X(C_{AB}(c, m)) - \frac{1}{|\text{Im}A||\text{Im}B|} \right]
\leq E_{AB} \left[ \sum_{c,m} \mu_X(C_{AB}(c, m) \cap \mathcal{T}_X) - \frac{\mu_X(\mathcal{T}_X)}{|\text{Im}A||\text{Im}B|} \right]
+ E_{AB} \left[ \sum_{c,m} \left[ \mu_X(C_{AB}(c, m) \cap [\mathcal{T}_X]^c) + \frac{\mu_X([\mathcal{T}_X]^c)}{|\text{Im}A||\text{Im}B|} \right] \right]
= \mu_X(\mathcal{T}_X) E_{AB} \left[ \sum_{c,m} \frac{\mu_X(C_{AB}(c, m) \cap \mathcal{T}_X)}{\mu_X(\mathcal{T}_X)} - \frac{1}{|\text{Im}A||\text{Im}B|} \right] + 2 \mu_X([\mathcal{T}_X]^c)
\leq \mu_X(\mathcal{T}_X) \sqrt{\frac{\alpha_A - 1 + [\beta_A + 1]|\text{Im}A||\text{Im}B| \max_{x \in \mathcal{T}_X} \mu_X(x)}{\mu_X(\mathcal{T}_X)}} + 2 \mu_X([\mathcal{T}_X]^c)
\leq \sqrt{\alpha_A - 1 + [\beta_A + 1]2^{-n[H(X) - r - R - \varepsilon]}} + 2 \mu_X([\mathcal{T}_X]^c),
\]

(63)
where the second inequality comes from Lemma 5. Then we have

\[
E_{ABc}[\text{Error}(A, B, c)]
\]

\[
= E_{AB} \left[ \sum_{c,m: \mu_X(c_{AB}(c,m)) > 0} \frac{1}{|\text{Im}A||\text{Im}B|} \mu_X(c_{AB}(c,m)) + \sum_{c,m: \mu_X(c_{AB}(c,m)) = 0} \frac{\mu_{XY}(x, y)}{|\text{Im}A||\text{Im}B|} \mu_X(c_{AB}(c,m)) \right]
\]

\[
= E_{AB} \left[ \sum_{c,m: \mu_X(c_{AB}(c,m)) = 0} \frac{1}{|\text{Im}A||\text{Im}B|} \mu_X(c_{AB}(c,m)) \right]
\]

\[
+ \sum_{c,m: \mu_X(c_{AB}(c,m)) > 0} \sum_{x: \mu_X(c_{AB}(c,m)) > 0} \sum_{x: \mu_X(c_{AB}(c,m)) > 0} \mu_X(c_{AB}(c,m)) \sum_{x: \mu_X(c_{AB}(c,m)) > 0} \sum_{x: \mu_X(c_{AB}(c,m)) > 0} \mu_X(c_{AB}(c,m)) \mu_{XY}(x, y)
\]

\[
\leq E_A \left[ \sum_{x,y} \mu_{XY}(x, y) \chi(x_{A}(Ax|y) = x) \right] + E_{AB} \left[ \sum_{c,m} \mu_X(c_{AB}(c,m)) - \frac{1}{|\text{Im}A||\text{Im}B|} \right]
\]

\[
\leq 2^{-n[r - \overline{H}(X|Y)] - \epsilon} \alpha_A + \beta_A + \mu_{XY}([\overline{I}_X|Y]) + \sqrt{\alpha_{AB} - 1 + \beta_{AB} + 1]2^{-n[\overline{H}(X|Y)] - r - \epsilon}} + 2\mu_X([\overline{I}_X]),
\]

(64)

where c is a random variable corresponding to the uniform distribution on ImA, the first inequality comes from the fact that

\[
\sum_{c,m: \mu_X(c_{AB}(c,m)) > 0} \mu_{XY}(x, y) \left[ \frac{1}{|\text{Im}A||\text{Im}B|} \mu_X(c_{AB}(c,m)) - 1 \right]
\]

\[
\leq \sum_{c,m: \mu_X(c_{AB}(c,m)) > 0} \left[ \frac{1}{|\text{Im}A||\text{Im}B|} \mu_X(c_{AB}(c,m)) - 1 \right] \mu_X(c_{AB}(c,m))
\]

\[
= \sum_{c,m: \mu_X(c_{AB}(c,m)) > 0} \left[ \mu_X(c_{AB}(c,m)) - \frac{1}{|\text{Im}A||\text{Im}B|} \right]
\]

\[
= \sum_{c,m} \left[ \mu_X(c_{AB}(c,m)) - \frac{1}{|\text{Im}A||\text{Im}B|} \right] - \sum_{c,m: \mu_X(c_{AB}(c,m)) = 0} \frac{1}{|\text{Im}A||\text{Im}B|},
\]

(65)
and the second inequality comes from (62), (63). From (57), (58), (64) and the fact that \(\alpha_A \to 1, \beta_A \to 0, \alpha_{AB} \to 1, \beta_{AB} \to 0, \mu_X([T_X]^c) \to 0, \mu_{XY}([T_{XY}]^c) \to 0\) as \(n \to \infty\), we have the fact that there are functions \(A \in \mathcal{A}, B \in \mathcal{B}\), and a vector \(c \in \text{Im}A\) satisfying (21).

\[\blacksquare\]

D. Proof of Corollary 2

Inequality (24) is shown as

\[R \equiv \frac{1}{n} \log |\mathcal{M}_n| = \frac{1}{n} \log \frac{|X^n|}{|\text{Im}A|} \geq \log |X| - r,
\]

where the inequality comes from the definition of \(r\) and the fact that \(\text{Im}A \subset \text{Im}A\).

Since \(\mu_X^n\) is uniform and for given \(c \in \text{Im}A\) and \(m \in \mathcal{M}_n\) there is a unique \(x \in C_{AB}(c, m)\), we have the fact that

\[\frac{1}{|C_A(c)|} = \frac{\mu_X^n(x)}{|\mathcal{M}_n| \mu_X^n(C_{AB}(c, m))}\]

for all \(m\). Then we have

\[E_{Ac}[\text{Error}(A, c)] \leq 2^{-n[r-r(\overline{H}(X|Y)-\epsilon)]} \alpha_A + \beta_A + \mu_{XY}([T_{X|Y}]^c) + \sqrt{\alpha_{AB} - 1 + (\beta_{AB} + 1)2^{-n[H(X)-r-R-\epsilon]} + 2\mu_X([T_X]^c)},\]

from (64). From (23), (66), and the fact that \(\alpha_A \to 1, \beta_A \to 0, \mu_{XY}([T_{X|Y}]^c) \to 0, \mu_X([T_X]^c) \to 0\) as \(n \to \infty\), we have the fact that for any \(\delta > 0\) and sufficiently large \(n\) there are functions \(A \in \mathcal{A}\), and a vector \(c \in \text{Im}A\) satisfying (25) for all \(\delta > 0\) and sufficiently large \(n\).

Now, we prove (27) following the proof presented in [31][14, Example 3.2.1]. Assume that \(\mu_{Y^n|X^n}\) is a channel with additive noise \(Z = \{Y^n - X^n\}_{n=1}^\infty\). Since the channel \(\mu_{Y^n|X^n}\) is weakly symmetric (see [5, p.190]), then the reverse channel \(\mu_{X^n|Y^n}\) is also weakly symmetric when the channel input distribution \(\mu_X^n\) is uniform. This implies that \(\overline{H}(X|Y)\) does not depend on \(Y\) and

\[\overline{H}(X|Y) = \overline{H}(X|0) = \overline{H}(-Z) = \overline{H}(Z).
\]

We have

\[I(X;Y) \leq \overline{H}(X) - \overline{H}(X|Y)\]
\[ \leq \log |\mathcal{X}| - H(X|Y) \]
\[ \leq \log |\mathcal{X}| - \overline{H}(Z). \]  

(67)

This implies that \( \log |\mathcal{X}| - \overline{H}(Z) \geq C(W) \). On the other hand, the supremum on the right hand side of (3) is achieved by assuming that \( \mu_{X^n} \) is the uniform distribution on \( \mathcal{X}^n \). This implies that \( \log |\mathcal{X}| - \overline{H}(Z) \) is the capacity of this channel. ■

**E. Proof of Theorem 3**

We omit the dependence on \( n \) of \( X \) and \( Y \) when they appear in the subscript of \( \mu \).

From (33) and (34), we have the fact that there is \( \varepsilon > 0 \) satisfying
\[ r < H(X|Y) - \varepsilon \]  
\[ r + R > \overline{H}(X) + \varepsilon. \]  

(68)  

(69)

Let \( \mathcal{T}_X \subset \mathcal{X}^n \) and \( \mathcal{T}_{X|Y} \subset \mathcal{X}^n \times \mathcal{Y}^n \) be defined as
\[ \mathcal{T}_X \equiv \left\{ x : \frac{1}{n} \log \frac{1}{\mu_X(x)} \leq \overline{H}(X) + \varepsilon \right\} \]
\[ \mathcal{T}_{X|Y} \equiv \left\{ (x, y) : \frac{1}{n} \log \frac{1}{\mu_{X|Y}(x|y)} \geq \overline{H}(X|Y) - \varepsilon \right\}. \]

Assume that \( x \in \mathcal{T}_X \) and \( x_{AB}(Ax, Bx) \neq x \). Then we have the fact that there is \( x' \in C_{AB}(Ax, Bx) \) such that \( x' \neq x \) and
\[ \mu_X(x') \geq \mu_X(x) \geq 2^{-n[\overline{H}(X)+\varepsilon]}. \]

This implies that \( [\mathcal{T}_X \setminus \{x\}] \cap C_{AB}(Ax, Bx) \neq \emptyset \). Then we have
\[ E_{AB} [\chi(x_A(Ax, Bx) \neq x)] \leq p_{AB} \left( \{ (A, B) : [\mathcal{T}_X \setminus \{x\}] \cap C_{AB}(Ax, Bx) \neq \emptyset \} \right) \]
\[ \leq \frac{|\mathcal{T}_X|}{|\text{Im} A|} + \beta_{AB} \]
\[ \leq 2^{-n[r-H(X)+\varepsilon]} \alpha_{AB} + \beta_{AB}, \]  

(70)

where \( \chi(\cdot) \) is defined by (41), the second inequality comes from Lemma 4, and the last inequality comes from the fact that \( |\mathcal{T}_X| \leq 2^n[r-H(X)+\varepsilon] \). We have the fact that
\[ E_{AB} \left[ \sum_x \mu_X(x) \chi(x_{AB}(Ax, Bx) \neq x) \right] \]
\[ = \sum_{x \in \mathcal{T}_X} \mu_X(x) E_{AB} [\chi(x_{AB}(Ax, Bx) \neq x)] + \sum_{x \notin \mathcal{T}_X} \mu_X(x) E_{AB} [\chi(x_{AB}(Ax, Bx) \neq x)] \]
where the second inequality comes from (70). We also have the fact that

$$E_A \left[ \sum_{c, y} \mu_Y(y) \left| \mu_{X|Y}(C_A(c)|y) - \frac{1}{|\text{Im}A|} \right| \right]$$

$$\leq E_A \left[ \sum_{c, y} \mu_{X|Y}(T_{X|Y}(y)|y) \mu_Y(y) \left| \frac{\mu_{X|Y}(C_A(c) \cap T_{X|Y}(y)|y)}{\mu_{X|Y}(T_{X|Y}(y)|y)} - \frac{1}{|\text{Im}A|} \right| \right]$$

$$+ E_A \left[ \sum_{c, y} \mu_{X|Y}(C_A(c) \cap [T_{X|Y}(y)]^c|y) \mu_Y(y) \right] + E_A \left[ \sum_{c, y} \frac{\mu_{X|Y}([T_{X|Y}(y)]^c|y) \mu_Y(y)}{|\text{Im}A|} \right]$$

$$= \sum_{y} \mu_{X|Y}(T_{X|Y}(y)|y) \mu_Y(y) \left[ \sum_{c} \frac{\mu_{X|Y}(C_A(c) \cap T_{X|Y}(y)|y)}{\mu_{X|Y}(T_{X|Y}(y)|y)} - \frac{1}{|\text{Im}A|} \right]$$

$$+ 2 \mu_{XY}([T_{X|Y}]^c)$$

$$\leq \sum_{y} \mu_{X|Y}(T_{X|Y}(y)|y) \mu_Y(y) \sqrt{\frac{\alpha_{AB} - 1 + [\beta_{AB} + 1]|\text{Im}A| \max_{x \in T_X} \mu_X(x)}{\mu_{X|Y}(T_{X|Y}(y)|y)} + 2 \mu_X([T_{X|Y}]^c)}$$

$$\leq \sqrt{\alpha_{A} - 1 + [\beta_{A} + 1]2^{-n[\mu_{X|Y} - r/\epsilon]} + 2 \mu_{X|Y}([T_{X|Y}]^c)},$$

where the second inequality comes from Lemma 5. Then we have

$$E_{A|BC}[\text{Error}(A, B, c, D)]$$

$$\leq E_{A|BC} \left[ \sum_{y} \mu_Y(y) + \sum_{x \in C_A(c)} \frac{\mu_{X|Y}(x|y) \mu_Y(y)}{\mu_{X|Y}(C_A(c)|y)} \right]$$

$$= E_A \left[ \sum_{c, y} \frac{\mu_Y(y)}{|\text{Im}A|} \right]$$

$$+ \sum_{c, x, y: x \in C_A(c), \mu_{X|Y}(C_A(c)|y) > 0, d_n(x, y) > D \ or \ x_{AB}(Bx) \neq x} \mu_{XY}(x, y) \left[ 1 + \frac{1}{|\text{Im}A| \mu_{X|Y}(C_A(c)|y)} - 1 \right]$$
inequality comes from the fact that and the third inequality comes from (71), (72). From (68), (69), (73) and the fact that there are functions \( \beta \). Proof of Corollary 4

\[
\leq P(d_n(X^n, Y^n) > D) + E_{AB} \left[ \sum_x \mu_X(x) \chi(x_{AB}(Ax, Bx) \neq x) \right]
\]

\[
+ E_A \left[ \sum_{c \in Y} \mu_Y(y) \left| \mu_{X|Y}(C_A(c)|y) - \frac{1}{|\text{Im}A|} \right| \right]
\]

\[
\leq P(d_n(X^n, Y^n) > D) + 2^{-n[R - \overline{R}]} \alpha_{AB} + \beta_{AB} + \mu_X([\overline{T}_X]^c)
\]

\[
+ \sqrt{\alpha_A - 1 + [\beta_A + 1]2^{-n[\overline{H}(X|Y) - r - \epsilon]} + 2\mu_{XY}([\overline{T}_X|Y]^c)},
\]

where \( c \) is a random variable corresponding to the uniform distribution on \( \text{Im}A \), the second inequality comes from the fact that

\[
\sum_{c \in Y, \mu_{X|Y}(C_A(c)|y) > 0} \mu_{X|Y}(C_A(c)|y) \left| \frac{1}{|\text{Im}A|} \right| - 1
\]

\[
= \sum_{c \in Y, \mu_{X|Y}(C_A(c)|y) > 0} \mu_Y(y) \left| \mu_X(C_A(c)|y) - \frac{1}{|\text{Im}A|} \right| - \sum_{c \in Y, \mu_{X|Y}(C_A(c)|y) = 0} \mu_Y(y)
\]

and the third inequality comes from (71), (72). From (68), (69), (73) and the fact that \( \alpha_A \rightarrow 1, \beta_A \rightarrow 0, \alpha_{AB} \rightarrow 1, \beta_{AB} \rightarrow 0, \mu_X([\overline{T}_X]^c) \rightarrow 0, \mu_{XY}([\overline{T}_X|Y]^c) \rightarrow 0 \) as \( n \rightarrow \infty \), we have the fact that there are functions \( A \in \mathcal{A}, B \in \mathcal{B} \), and a vector \( c \in \text{Im}A \) satisfying (35).

\[\Box\]

F. Proof of Corollary 4

Since \( x'_{AB}(c, Bx) = x \) is satisfied for all \( x \), we can substitute

\[
\chi(x_{AB}(c, Bx) \neq x) = 0
\]

in the derivation of (73) and obtain

\[
E_{Ac} \left[ \text{Error}(A, c, D) \right]
\]

\[
\leq P(d(X^n, Y^n) > D) + \sqrt{\alpha_A - 1 + [\beta_A + 1]2^{-n[\overline{H}(X|Y) - r - \epsilon]} + 2\mu_{XY}([\overline{T}_X|Y]^c)}. \quad (75)
\]

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On the other hand, from Lemma 5, we have
\[
E_{A_c} \left[ \left| \frac{|\text{Im} A||C_A(c)|}{|X^n|} - 1 \right| \right] = E_A \left[ \sum_c \left| \frac{|C_A(c)|}{|X^n|} - \frac{1}{|\text{Im} A|} \right| \right] \\
\leq \sqrt{\alpha_A - 1 + \frac{[\beta_A + 1]|\text{Im} A|}{|X^n|}} \\
= \sqrt{\alpha_A - 1 + [\beta_A + 1]2^{-n[\log |X| - r]}}.
\] (76)

By using the Markov inequality, (36), (75), and the fact that \( \alpha_A \to 1, \beta_A \to 0, \mu_{XY}([\mathcal{I}_{X|Y}]^c) \to 0 \) as \( n \to \infty \), we have the fact that for any \( \delta > 0 \) and sufficiently large \( n \) there are functions \( A \in \mathcal{A} \), and a vector \( c \in \text{Im} A \) satisfying (38) and
\[
\left| \frac{|\text{Im} A||C_A(c)|}{|X^n|} - 1 \right| < 1
\] (77)
for sufficiently large \( n \). Then we have the fact that \( c \in \text{Im} A \subset \text{Im} A \) because the left hand side of (77) is equal to 1 when \( c \in \text{Im} A \setminus \text{Im} A \). From (77) and the fact that \( A \) is a linear function, we have
\[
\frac{|\text{Im} A||C_A(0)|}{|X^n|} = \frac{|\text{Im} A||C_A(c)|}{|X^n|} < 2
\] (78)
and
\[
R = \frac{1}{n} \log |C_A(0)| \\
\leq \frac{1}{n} \log \frac{2|X^n|}{|\text{Im} A|} \\
\leq \log |X| - r + \delta,
\] (79)
for all \( \delta > 0 \) and sufficiently large \( n \).

G. Proof of Theorem 5

Let \( g'_0 \) and \( g'_k : \mathcal{X} \to \mathcal{Z}_k \) be defined as
\[
g'_0(x) = \sum_{x} \prod_{j=1}^{n} \mu_{X_j}(x_j) \prod_{i=1}^{l} \chi(a_i(x_{S_i}) = c_i)
\]
\[
g'_k(x_k) = \mu_{X_k}(x_k) \sum_{x_{k+1}^{n}} \prod_{j=k+1}^{n} \mu_{X_j}(x_j) \prod_{i=1}^{l} \chi(a_i(x_{S_i}) = c_i)
\]
for a given \( c = (c_1, \ldots, c_l) \in \mathcal{Z}^l \). Then we have
\[
P_{X_k|X_1^{k-1}}(x_k|x_1^{k-1}) = \frac{\sum_{x_{k+1}^{n}} \prod_{j=k}^{n} \mu_{X_j}(x_j) \prod_{i=1}^{l} \chi(a_i(x_{S_i}) = c_i)}{\sum_{x_{k+1}^{n}} \prod_{j=k}^{n} \mu_{X_j}(x_j) \prod_{i=1}^{l} \chi(a_i(x_{S_i}) = c_i)}
\]
\[ g'_k(x_k) = \frac{g'_k(x_k)}{\sum_{x_k} g'_k(x_k)}. \]  
(82)

If the algorithm terminates with \( k = n \) at Step 4, we have

\[ g'_n(x_n) = p_{X_n}(x_n). \]  
(83)

On the other hand, if the algorithm terminates with \( k = k' \) at Step 5, we have

\[ g'_{k'}(x_{k'}) = p_{X_{k'}}(x_{k'}) \sum_{x_{k'+1}^n} \prod_{j=k'+1}^n p_{X_j}(x_j) \prod_{i=1}^l \chi(a_i(x_{S_i}) = c_i) \]

\[ = \prod_{j=k'}^n p_{X_j}(x_j), \]  
(84)

where the second equality comes from the fact that for a given \( x_{k'}^n \) there is a unique \( x_{k'+1}^n \) such that \( x_1^n \in C_A(x) \). Since (83) is a special case of (84) with \( k' = n \), we assume that the algorithm terminates at \( k = k' \) in the following.

Since

\[ \sum_{x_1} g'_1(x_1) = \sum_{x_1} \mu_{X_1}(x_1) \sum_{x_2^n} \prod_{j=2}^n \mu_{X_j}(x_j) \prod_{i=1}^l \chi(a_i(x_{S_i}) = c_i) \]

\[ = g'_0, \]  
(85)

and

\[ \sum_{x_k} g'_k(x_k) = \sum_{x_k} p_{X_k}(x_k) \sum_{x_{k+1}^n} \prod_{j=k+1}^n \mu_{X_j}(x_j) \prod_{i=1}^l \chi(a_i(x_{S_i}) = c_i) \]

\[ = \sum_{x_k^n} \prod_{j=k}^n \mu_{X_j}(x_j) \prod_{i=1}^l \chi(a_i(x_{S_i}) = c_i) \]

\[ = \frac{g_{k-1}(x_{k-1})}{\mu_{X_{k-1}}(x_{k-1})}, \]  
(86)

for \( k \geq 2 \), we have the fact that (40) is rephrased as

\[ \mu_{\tilde{X}^n}(x) = \frac{\prod_{i=1}^l \mu_{X_i}(x_i) \prod_{i=1}^l \chi(a_i(x_{S_i}) = c_i)}{\sum_x \prod_{j=1}^l \mu_{X_j}(x_j) \prod_{i=1}^l \chi(a_i(x_{S_i}) = c_i)} \]

\[ = \frac{g'_{k'}(x_{k'})}{g'_0} \prod_{j=1}^{k'-1} \mu_{X_j}(x_j) \]

\[ = \frac{g'_1(x_1)}{g'_0} \prod_{k=2}^{k'} \frac{\mu_{X_{k-1}}(x_k) g'_k(x_k)}{g'_{k-1}(x_{k-1})} \]

\[ = \prod_{k=1}^{k'} \frac{\mu_{X_{k-1}}(x_k) g'_k(x_k)}{g'_{k-1}(x_{k-1})}, \]  
(87)
\[
\begin{align*}
&= \prod_{k=1}^{k'} \frac{g'_k(x_k)}{\sum_{x_k} g'_k(x_k)} \\
&= \prod_{k=1}^{k'} p_{\bar{X}_k|\bar{X}_1^{k-1}}(x_k|x_1^{k-1}),
\end{align*}
\]  
where the first equality comes from (40), the second equality comes from (80), (84), we denote \( g'_0(x_0) \equiv g'_0 \) in the fourth equality, the fifth equality comes from (85), (86), and the last equality comes from (82).

Since the algorithm generates a sequence \( x \equiv x^n_1 \) subject to \( \prod_{k=1}^{k'} p_{\bar{X}_k|\bar{X}_1^{k-1}}(x_k|x_1^{k-1}) \), the proposed algorithm generates \( x \) subject to the probability distribution given by (40). 

\[ \blacksquare \]

**APPENDIX**

We prove the lemmas used in the proofs of the theorems. Some proofs are presented for the completeness of this paper.

A. **Lemma Analogous to Fano Inequality**

We prove the following lemma which is analogous to the Fano inequality. It should be noted that a stronger version of this lemma has been proved in [16, Lemma 4].

**Lemma 7:** Let \((U, V) \equiv \{(U^n, V^n)\}_{n=1}^{\infty}\) be a pair consisting of two sequences of random variables. If there is \( \{\psi_n\}_{n=1}^{\infty} \) such that

\[
\lim_{n \to \infty} P(\psi_n(V^n) \neq U^n) = 0,
\]

then

\[
\Pi(U|V) = 0.
\]

**Proof:** For \( \gamma > 0 \), let

\[
G \equiv \left\{ (u, v) : \frac{1}{n} \log \frac{1}{\mu_{U^nV^n}(u|v)} \geq \gamma \right\}
\]

\[
\mathcal{S} \equiv \{(u, v) : \psi_n(v) = u\}.
\]

Then we have

\[
\mu_{U^nV^n}(G) = \mu_{U^nV^n}(G \cap \mathcal{S}) + \mu_{U^nV^n}(G \cap \mathcal{S}^c)
\]

\[
= \mu_{U^nV^n}(G \cap \mathcal{S}^c) + \sum_{(u, v) \in G \cap \mathcal{S}} \mu_{U^nV^n}(u, v)
\]

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\[
\begin{align*}
&= \mu_{U^nV^n}(G \cap S^c) + \sum_{v} \mu_{V^n}(v) \sum_{\psi_n(v) = u} \mu_{U^n|V^n}(u|v) \\
&\leq \mu_{U^nV^n}(G \cap S^c) + \sum_{v} \mu_{V^n}(v) \sum_{u: \psi_n(v) = u} 2^{-n\gamma} \\
&\leq P(\psi_n(V^n) \neq U^n) + 2^{-n\gamma},
\end{align*}
\]  
where the first inequality comes from the definition of \( G \) and the last inequality comes from the fact that for all \( v \) there is a unique \( u \) satisfying \( \psi_n(v) = u \). From this inequality and (88), we have
\[
\lim_{n \to \infty} P\left(\frac{1}{n} \log \frac{1}{\mu_{U^n|V^n}(U^n|V^n)} \geq \gamma \right) = 0.
\]
Then we have
\[
0 \leq \overline{H}(U|V) \leq \gamma
\]
from the definition of \( \overline{H}(U|V) \). We have (89) by letting \( \gamma \to 0 \).

\section*{B. Proof of (H3')}

If an ensemble satisfies (H3), then we have
\[
\sum_{u \in T} p_A(\{A : Au = Au'\}) = \sum_{u \in T \cap T'} p_A(\{A : Au = Au'\})
\]
\[
+ \sum_{u \in T} \sum_{u' \in T \setminus \{u\}} p_A(\{A : Au = Au'\})\frac{\alpha_A}{|\text{Im} A|} + \sum_{u \in T} \beta_A
\]
\[
\leq |T \cap T'| + \sum_{u \in T} \sum_{u' \in T \setminus \{u\}} p_A(\{A : Au = Au'\})\frac{\alpha_A}{|\text{Im} A|} + \sum_{u \in T} \beta_A
\]
\[
\leq |T \cap T'| + \frac{|T||T'|\alpha_A}{|\text{Im} A|} + \beta_A
\]
\[
\leq |T \cap T'| + \frac{|T||T'|\alpha_A}{|\text{Im} A|} + \min\{|T|, |T'|\}\beta_A
\]
for any \( T \) and \( T' \) satisfying \( |T| \leq |T'| \).

\[\tag{91}\]

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C. Proof of Lemma 3

Let

\[ p_{A,u,u'} \equiv p_A(\{A : Au = Au'\}) \]
\[ p_{B,u,u'} \equiv p_B(\{B : Bu = Bu'\}) \]
\[ p_{AB,u,u'} \equiv p_{AB}(\{(A,B) : (A,B)u = (A,B)u'\}) \]

Then we have

\[ \sum_{u' \in \mathcal{U}^{n} \setminus \{u\}; p_{AB,u,u'} > \frac{\alpha_{AB}}{\| \text{Im}A \| \| \text{Im}B \|}} p_{AB,u,u'} \]
\[ \leq \sum_{u' \in \mathcal{U}^{n} \setminus \{u\}; p_{A,u,u'p_{B,u,u'}} > \frac{\alpha_{AB}}{\| \text{Im}A \| \| \text{Im}B \|}} p_{A,u,u'p_{B,u,u'}} \]
\[ = \sum_{u' \in \mathcal{U}^{n} \setminus \{u\}; p_{A,u,u'} > \frac{\alpha_{AB}}{\| \text{Im}A \|}} p_{A,u,u'} + \sum_{u' \in \mathcal{U}^{n} \setminus \{u\}; p_{B,u,u'} > \frac{\beta_{AB}}{\| \text{Im}B \|}} p_{B,u,u'} \]
\[ \leq \sum_{u' \in \mathcal{U}^{n} \setminus \{u\}; p_{A,u,u'} > \frac{\alpha_{AB}}{\| \text{Im}A \|}} p_{A,u,u'} + \sum_{u' \in \mathcal{U}^{n} \setminus \{u\}; p_{B,u,u'} > \frac{\beta_{AB}}{\| \text{Im}B \|}} p_{B,u,u'} \]
\[ = \beta_A + \beta_B \]
\[ = \beta_{AB} \tag{92} \]

where the first inequality comes from the fact that \( \text{Im}A \times \mathcal{B} \subset \text{Im}A \times \text{Im}B \) and \( A, B \) are mutually independent, and the last inequality comes from the fact that \( p_{A,u,u'} \leq 1 \), \( p_{B,u,u'} \leq 1 \). Since \((\alpha_{AB}, \beta_{AB})\) satisfies (H1) and (H2), we have the fact that \((\mathcal{A} \times \mathcal{B}, p_{AB})\) has an \((\alpha_{AB}, \beta_{AB})\)-hash property.

D. Proof of Lemma 4:

We have

\[ p_A(\{A : \mathcal{G} \setminus \{u\} \cap C_A(Au) \neq \emptyset\}) \]
\[ \leq \sum_{u' \in \mathcal{U}^{n} \setminus \{u\}} p_A(\{A : Au = Au'\}) \]
\[ \leq |\{u\} \cap |G \setminus \{u\}| |G| + \frac{|\{u\}| |G \setminus \{u\}| \alpha_{A}}{|\text{Im} \cdot A|} + \min\{|\{u\}|, |G \setminus \{u\}|\} \beta_{A} \]
\[ \leq \frac{|G| \alpha_{A}}{|\text{Im} \cdot A|} + \beta_{A}, \]  
where the second inequality comes from (H3') by letting \( T \equiv \{u\} \) and \( T' \equiv G \setminus \{u\} \). \( \blacksquare \)

E. Proof of Lemma 5

Let \( p_{A,u,u'} \) be defined as

\[ p_{A,u,u'} \equiv p_{A} (\{A : Au = Au'\}) \]

Then we have

\[
E_{A} \left[ \left( \sum_{u' \in T} Q(u) \chi(Au = c) \right)^2 \right] 
\]
\[
= E_{A} \left[ \sum_{u \in T} Q(u) \sum_{u' \in T} Q(u') \chi(Au = Au') E_{c} [\chi(Au' = c)] \right] 
\]
\[
= \frac{1}{|\text{Im} \cdot A|} \sum_{u \in T} Q(u) \left[ \sum_{u' \in T \setminus \{u\}} Q(u') p_{A,u,u'} + \sum_{u' \in T \setminus \{u\}} Q(u') p_{A,u,u'} + Q(u) \right] 
\]
\[
\leq \frac{1}{|\text{Im} \cdot A|} \sum_{u \in T} Q(u) \left[ \sum_{u' \in T \setminus \{u\}} Q(u') \frac{\alpha_{A}}{|\text{Im} \cdot A|} + \sum_{u' \in T \setminus \{u\}} p_{A,u,u'} + 1 \right] \max_{u' \in T} Q(u) \]
\[
\leq \frac{Q(T)^2 \alpha_{A}}{|\text{Im} \cdot A|^2} + \frac{Q(T) [\beta_{A} + 1] \max_{u \in T} Q(u)}{|\text{Im} \cdot A|}, \]  
where \( \chi(\cdot) \) is defined by (41), the second equality comes from the fact that the uniqueness of the value \( Au' \) implies

\[ E_{c} [\chi(Au' = c)] = \frac{1}{|\text{Im} \cdot A|} \sum_{c} \chi(Au' = c) \]
\[ = \frac{1}{|\text{Im} \cdot A|} \]

for any \( A \in A \) and \( u' \in \mathcal{U}^n \) when the distribution of \( c \) is uniform on \( \text{Im} \cdot A \). Then the lemma is shown as

\[ E_{A} \left[ \sum_{c} \frac{Q(T \cap C_{A}(c))}{Q(T)} - \frac{1}{|\text{Im} \cdot A|} \right] = E_{A} \left[ \sum_{c} \frac{1}{|\text{Im} \cdot A|} \frac{Q(T \cap C_{A}(c))}{Q(T)} - 1 \right] \]
\[ E_{A_c} \left[ \sqrt{\frac{Q(T \cap C_A(c)) |\text{Im}A|}{Q(T)} - 1} \right]^2 \]
\[ \leq \sqrt{E_{A_c} \left[ \frac{Q(T \cap C_A(c)) |\text{Im}A|}{Q(T)} - 1 \right]^2} \]
\[ = \frac{|\text{Im}A|^2}{Q(T)^2} E_{A_c} \left[ \left( \sum_{u \in T} Q(u) \chi(Au = c) \right)^2 \right] - 1 \]
\[ \leq \alpha_A - 1 + \frac{[\beta_A + 1]|\text{Im}A| \max_{u \in T} Q(u)}{Q(T)}, \quad (96) \]

where the third equality comes from the fact that \( \{C_A(c)\}_{c \in \text{Im}A} \) is a partition of \( U^n \) and the last inequality comes from (94).

\[ F. \text{ Proof of Lemma 6} \]

For a type \( t \), let \( C_t \) be defined as

\[ C_t \equiv \{ u \in U^n : t(u) = t \}. \]

We assume that \( p_A(\{A : Au = 0\}) \) depends on \( u \) only through the type \( t(u) \). For a given \( u \in C_t \), we define

\[ p_{A,t} \equiv p_A(\{A : Au = 0\}). \]

We use the following lemma, which is proved for the completeness of the paper.

\textit{Lemma 8 ([21, Lemma 9]):} Let \( (\alpha_A, \beta_A) \) be defined by (12) and (13). Then

\[ \alpha_A = |\text{Im}A| \max_{t \in \mathcal{H}_A} p_{A,t} \quad (97) \]
\[ \beta_A = \sum_{t \in \mathcal{H} \setminus \mathcal{H}_A} |C_t| p_{A,t} \quad (98) \]

where \( \mathcal{H} \) is a set of all types of length \( n \) except for the type of the zero vector.

\textit{Proof:} We have

\[ S(p_A,t) = \sum_A p_A(A) \sum_{u \in C_t : Au = 0} 1 \]
\[ = \sum_{u \in C_t} \sum_{A : Au = 0} p_A(A) \]
\[ = |C_t| p_{A,t}. \quad (99) \]
Similarly, we have

\[ S(p_{\mathcal{K}, t}) = |C_t|p_{\mathcal{K}, t} \]
\[ = |C_t||U|^{-l}, \quad (100) \]

where the last equality comes from the fact that

\[ p_{\mathcal{K}, t} = \frac{|U|^{n-1}l}{|U|^n} \]
\[ = |U|^{-l} \quad (101) \]

because we can find \(|U|^{n-1}l\) matrices \(\overline{A}\) to satisfy \(\overline{A}u = 0\) for a given \(u \in C_t\). The lemma can be shown immediately from (12), (13), (99), and (100).

Now we prove Lemma 6. It is enough to show (H3) because (H1), (H2) are satisfied from the assumption of the lemma. Since function \(A\) is linear, we have

\[ p_A(\{A : Au = Au'\}) = p_A(\{A : A[u - u'] = 0\}) \]
\[ = p_{A, t(u - u')} \quad (102) \]

Then, for \(u \neq u'\) satisfying \(t(u - u') \in \widehat{H}_A\), we have

\[ p_A(\{A : Au = Au'\}) = p_{A, t(u - u')} \leq \max_{t \in \widehat{H}_A} p_{A, t} \]
\[ = \frac{\alpha_A}{|\text{Im}A|}, \quad (103) \]

where the last inequality comes from (97). Then we have the fact that \(p_A(\{A : Au = Au'\}) > \alpha_A/|\text{Im}A|\) implies \(t(u - u') \notin \widehat{H}_A\). Finally, we have

\[ \sum_{u' \in U \setminus \{u\}: \quad \frac{p_A(\{A : Au = Au'\})}{p_A(\{A: Au = Au'\}) > \frac{\alpha_A}{|\text{Im}A|}}} \leq \sum_{u' \in U \setminus \{u\}: \quad t(u - u') \in H \setminus \widehat{H}_A} p_{A, t(u - u')} \]
\[ \leq \sum_{t \in H \setminus \widehat{H}_A} \sum_{u' \in U \setminus \{u\}: \quad t(u - u') = t} p_{A, t} \]
\[ \leq \sum_{t \in H \setminus \widehat{H}_A} |C_t|p_{A, t} \]
\[ = \beta_A, \quad (104) \]

where the equality comes from (98).
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