Klein and Conformal Superspaces, 
Split Algebras and Spinor Orbits

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Abstract

We discuss $\mathcal{N} = 1$ Klein and Klein-Conformal superspaces in $D = (2, 2)$ space-time dimensions, realizing them in terms of their functor of points over the split composition algebra $\mathbb{C}_s$. We exploit the observation that certain split forms of orthogonal groups can be realized in terms of matrix groups over split composition algebras; this leads to a natural interpretation of the sections of the spinor bundle in the critical split dimensions $D = 4, 6$ and $10$ as $\mathbb{C}_s^2$, $\mathbb{H}_s^2$ and $\mathbb{O}_s^2$, respectively. Within this approach, we also analyze the non-trivial spinor orbit stratification that is relevant in our construction since it affects the Klein-Conformal superspace structure.
Contents

1 Introduction 2

2 Split Algebras 7

3 Quadratic Jordan Algebras over Split Algebras 8

4 Spinors
   4.1 Pure Spinors ................................................................. 10
   4.2 Classification ................................................................. 11
   4.3 Spinors and Space-Time Signature: Lorentz versus Klein .......... 12

5 Vectors and Spinors of the Klein group Spin(2, 2)
   5.1 Spinor Orbits and Representatives ..................................... 14

6 Klein-Conformal group and Spin(3, 3)
   6.1 A Further Group Isomorphism ........................................... 18

7 Klein and Klein-Conformal $\mathcal{N} = 1$ Superspaces 20

A Symplectic Realization of Spin(3, 3) 23

B Supergeometry 24
1 Introduction

Supersymmetry (Susy) is a deep and elegant symmetry relating half-integer spin fields (fermions, constituents of matter) to integer-spin fields (bosons, giving rise to interactions). Such a symmetry was originally formulated, as a global symmetry of fields, back in the early 70’s in former Soviet Union by physicists Gol’fand and Likhtman [1], Volkov and Akulov [2], and independently in Europe by Wess and Zumino [3].

A major advance in the formulation of supersymmetric theories in space-time, which then allowed for the construction of manifestly invariant interactions, was due to Salam and Strathdee, who were the first to introduce the concept of superfield [4, 5]. In fact, depending on the number s and t of spacelike resp. timelike dimensions, space-time Susy recasts bosonic and fermionic fields into multiplet structures, each providing a certain representation of such an underlying symmetry. Within the simplest formulation of Susy, in which a unique fermionic generator exists besides the bosonic ones, fields defined in a space $M^{s,t} \cong \mathbb{R}^{s,t}$ (which in the case $s = 3$ and $t = 1$ yields the usual Minkowski space-time) are assembled into a unique object, named superfield, defined into the so-called $\mathcal{N} = 1$, $(s + t)$-dimensional superspace $M^{s,t|1}$, which is characterized by the presence of an anti-commuting Grassmannian coordinate besides the usual commuting bosonic coordinates of $M^{s,t}$.

Such developments eventually led to major advances in Quantum Field Theory, constituting the foundational pillars on which consistent candidates for a unified theory encompassing Quantum Gravity and the Standard Model of particle interactions were constructed. In combination with local gauge invariance, global Susy allowed for the formulation of Supersymmetric Yang-Mills Theories (SYM’s) [6]. In such a framework, Susy gives rise to remarkable cancellations between bosons and fermions in their quantum corrections, thus allowing for a study of SYM’s beyond perturbation theory. This generally provides a framework for a possible solution of the hierarchy problem, for the search of natural candidates for dark matter, as well as for addressing the conceptual issue of the dark energy.

In presence of general diffeomorphisms covariance, Susy becomes a local symmetry. In 1976, Ferrara, Freedman, Van Niewenhuizen [7] and Deser and Zumino [8] succeeded in formulating Susy as a local symmetry and coupling it to General Relativity. This resulted into the first formulation of supergravity, providing a low-energy effective description of more fundamental theories such as superstrings and M-theory, and playing a crucial role in supersymmetry breaking, an essential ingredient of all realistic elaborations beyond the Standard Model.

Also in its world-sheet formulation, Susy is one of the main tools for the construction of the most promising frameworks - the aforementioned superstring theory and M-theory - in which Quantum Theory and General Relativity may be reconciled and consistently formulated (cfr. e.g. [9, 10]). Quite recently, local Susy also proved to be a surprisingly successful tool in the investigation of the properties and dynamics of black holes, the endpoints of gravitational collapse, in which an horizon surface acts as a cosmic censor for the possible formation of a space-time singularity.

Susy had a major impact in Mathematics, as well (cfr. [11] for an excellent introduction). It gave rise also to a vast, deep and flourishing arena of mathematical investigation, inspiring generations of mathematicians to change their approach to geometry, both from the differential and algebraic point of view. In such frameworks, the symmetries of superspaces are naturally described by superalgebras and supergroups, the super-generalizations of the usual concept of algebras and groups.

Nowadays, superseding the more traditional sheaf theoretic approach, supergroups and superspaces are investigated by exploiting the elegant machinery of the functor of points, originally introduced by Grothendieck in algebraic geometry (see e.g. [12, 13]). Remarkably, such a deeply abstract point of view, formalized and developed by Shvarts [14] and Voronov [15], shares surprising similarities with the physicists’ approach in the aforementioned early times of Susy, in which points in the superspace were understood by exploiting Grassmann algebras, which are nothing but superalgebras over a superspace consisting of a point [16]. The subsequent work of Manin [17, 18] applied the powerful abstract machinery of the functor of points to the theory of superspaces and superschemes; ultimately, this led
to the development of the theory of superflags and super-Grassmannians.

However, sharing the same approach as in [19] and essentially relying on [20] and [21], we would like to point out that in the present investigation we will strive to leave abstract subtleties pertaining to the formal machinery of functor of points on the background, though employing its descriptive power while dealing with $T$-points of a supergroup or with a superspace.

An intriguing aspect of Susy is its deep relation to the four normed division algebras $A = \mathbb{R}$ (real numbers), $\mathbb{C}$ (complex numbers), $\mathbb{H}$ (quaternions, or Hamilton numbers), $\mathbb{O}$ (octonions, or Cayley numbers), especially involving super-twistors [23, 24, 25, 26]. In fact, non-Abelian YM theories are supersymmetric (thus giving rise to SYM’s) only if the space-time dimension is $D = 3, 4, 6$ or $10$ (and the same is true for the Green-Schwarz superstring), named critical dimension. In this context, the consistent formulation of Susy relies on the vanishing of a certain trilinear expression relying on the existence of $A$, whose real dimension is respectively given by $D - 2$ [27, 28, 29, 30].

Motivated by attempts at explaining the remarkable fact that (super)gravity scattering amplitudes can be obtained from those of (S)YM theories (cfr. e.g. [31]), in [32] Duff and collaborators exploited normed division algebras $A$’s in order to obtain the massless spectrum and the multiplet structure of supergravity theories in various dimensions by tensoring SYM multiplets (also cfr. subsequent developments in [33, 34]). The core of their main argument relies on the observation that the entries of second row of the order-2 split magic square $L_2(A_s, \mathbb{B})$ [35, 36, 37]

\begin{align*}
\begin{array}{cccc}
\mathbb{R} & \mathbb{C} & \mathbb{H} & \mathbb{O} \\
\mathbb{C} & \mathfrak{so}(2,1) & \mathfrak{so}(3,1) & \mathfrak{so}(5,1) & \mathfrak{so}(9,1)
\end{array}
\end{align*}

(1.1)
can be naturally represented as $\mathfrak{sl}(2, A)$, then yielding the isomorphisms of Lie algebras (cfr. [38], as well as [39] [42] and Refs. therein)

\[ \mathfrak{sl}(2, A) \cong \mathfrak{so}(q + 1, 1), \]

(1.2)
where

\[ q := \dim_{\mathbb{R}} A = 1, 2, 4, 8 \text{ for } A = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}, \text{ respectively}, \]

(1.3)
and $\mathfrak{so}(q + 1, 1)$ is the Lie algebra of the Lorentz group in $D = q + 2$ dimensions. Analogously, the third line of $L_2(A_s, \mathbb{B})$, i.e.

\begin{align*}
\begin{array}{cccc}
\mathbb{R} & \mathbb{C} & \mathbb{H} & \mathbb{O} \\
\mathbb{H} & \mathfrak{so}(3,2) & \mathfrak{so}(4,2) & \mathfrak{so}(6,2) & \mathfrak{so}(10,2)
\end{array}
\end{align*}

(1.4)
can be reinterpreted by noting the following Lie algebraic isomorphism [37]

\[ \tilde{\mathfrak{sp}}(4, A) \cong \mathfrak{so}(q + 2, 2), \]

(1.5)
with $\mathfrak{so}(q + 2, 2)$ standing for the conformal Lie algebra in $D = q + 2$, and $\tilde{\mathfrak{sp}}(4, A)$ denoting the Barton-Sudbery symplectic algebra, in which the matrix transposition is replaced by the Hermitian conjugation, differently from the usual definition of symplectic algebras [10, 37]. The Lie algebraic isomorphisms (1.2)-(1.5) have been recently extended to the Lie group level (considering the spin covering of the Lorentz and conformal groups, namely Spin$(q + 1, 1)$ resp. Spin$(q + 2, 2)$), by explicit constructions worked out in a series of paper [41, 42, 43, 44] by Dray, Manogue and collaborators. In particular, in [41] a Lie group version of the aforementioned order-2 split magic square $L_2(A_s, \mathbb{B})$ was constructed and studied.

Conformal symmetry also plays a crucial role in Physics and in Mathematics. While it is usually associated to massless particles, it also characterizes, possibly as an approximated symmetry, a number of physical systems in certain regimes of their dynamics.
Conformal symmetry also provides the foundation of an important branch of geometry, named conformal geometry, in which equivalence classes of metrics are exploited for a manifest, locally Weyl-invariant formulation of the equations governing the evolution of physical systems. In fact, conformal geometry enjoys a natural and remarkably elegant formulation as curved Cartan geometry, and essentially relies on the so-called Weyl-covariant differential calculus, also known as tractor calculus. This is the conformal-covariant generalization of the ordinary differential calculus; it was originally constructed in [35] (cfr. also [46, 47] for more physicists’ minded treatments, and [48] for an application to the AdS/CFT correspondence) and subsequently generalized to all parabolic geometries in [49].

Minkowski $D$-dimensional space-time $\mathbb{M}^{D-1,1}$ (or the aforementioned generalizations $\mathbb{M}^{s,D-s}$ thereof) cannot support a linear implementation of conformal symmetry, and a compactification procedure, which amounts to adding suitable points at infinity, is needed. This framework has been formalized and developed by Fefferman and Graham in [50], especially for curved manifolds. A simple instance of flat conformal geometry is provided by the Dirac cone construction, in which the $D$-dimensional compactified Minkowski space $\mathbb{M}^{D-1,1}$ is obtained as a particular section of the space of light like rays in the so-called conformal space $\mathbb{M}^{D,2}$.

In [51], the compactified 3-dimensional Minkowski space $\mathbb{M}^{3,1}$ was constructed, along with its $\mathcal{N} = 1$ supersymmetric extension $\mathbb{M}^{2,1,1}$, in terms of a Lagrangian manifold over the twistor space $\mathbb{R}^{6}$, by exploiting the Lie group isomorphism $\text{Spin}(3, 2) \cong \text{Sp}(4, \mathbb{R})$. Taking inspiration from the isomorphisms (1.2)-(1.5) and also relying on [11], in [19] a symplectic characterization of the 4-dimensional (compactified and real) Minkowski space $\mathbb{M}^{3,1}$ and $\mathcal{N} = 1$ Poincaré superspace $\mathbb{M}^{3,1,1}$ was given, exploiting the Lie group isomorphism $\text{Spin}(4, 2) \cong \text{Sp}(4, \mathbb{C})$. Therein, it was also argued the possibility to extend the approach also to the other critical dimensions $D = 6$ and 10, thus providing a uniform and elegant description of $\mathcal{N} = 1$ Poincaré superspaces $\mathbb{M}^{q+1,1,1}$ in critical dimensions $D = q + 2$ in terms of the four normed division algebras $\mathbb{A}$’s.

In the present paper, we shall be interested in space-time signatures characterized by the same number of spacelike and timelike dimensions: $s = t$. The corresponding signature is usually named Kleinian (or also ultrahyperbolic). Usually, Susy, SYM’s and supergravity theories in such a signature are investigated by focussing on suitably Wick-rotated versions of the corresponding theories in Lorentz signature (cfr. e.g. [52], and Refs. therein). However, also other, more exotic, possibilities can be considered, such as compactifications of the so-called $M'$-theory or $M^*$-theory (see e.g. [53, 54, 55]). Geometries in Kleinian signature currently remains a vast and yet unexplored realm, displaying a rich mathematical structure, whose little knowledge is essentially based on a few studies scattered in literature (cfr. e.g. [56, 57, 58, 59, 60, 62]).

Although considering Kleinian signature might seem at first a purely mathematical divertissement, important motivations are actually provided by Physics. The computation and the study of symmetries of scattering amplitudes in SYM’s and in supergravity highlighted the relevance of Kleinian signature, especially in 4 dimensions; indeed, in [61] Ooguri and Vafa showed that $D = 4$ is the critical dimension of the $\mathcal{N} = 2$ superstring, whose bosonic part is given by a self-dual metric of signature $s = t = 2$. It is also worth pointing out here that 4-dimensional Kleinian signature is essentially related to twistors [62], thus providing a powerful computational tool in the investigation of scattering amplitudes [63].

The present paper is then devoted to the study of the 4-dimensional Klein space $\mathbb{M}^{2,2}$, viewed inside the related Klein-conformal space $\mathbb{K}$ as well as of their supersymmetric extensions, namely the Klein $\mathcal{N} = 1$ superspace $\mathbb{M}^{2,2,1}$ and the corresponding Klein-conformal $\mathcal{N} = 1$ superspace. By recalling the split counterparts of the division algebras, namely $\mathbb{A}_s = \mathbb{C}_s$ (split complex numbers), $\mathbb{H}_s$ (split quaternions) and $\mathbb{O}_s$ (split octonions), we rely on the observation that the entries of second row of the order-2 doubly-split magic square $\mathcal{E}_2(\mathbb{A}_s, \mathbb{H}_s) = [55, 60, 67]$

\footnote{Technically this is called the big cell inside the Klein-conformal (super)space.}
\[
\begin{array}{cccc}
\mathbb{R} & \mathbb{C}_s & \mathbb{H}_s & \mathbb{O}_s \\
\mathbb{C}_s & \text{so}(2,1) & \text{so}(2,2) & \text{so}(3,3) & \text{so}(5,5) \\
\end{array}
\]  

(1.6)

can be naturally represented as \(\mathfrak{sl}(2,\mathbb{A}_s)\), then yielding the isomorphisms of Lie algebras (cfr. e.g. [64], and Refs. therein)

\[
\mathfrak{sl}(2,\mathbb{A}_s) \cong \text{so}(q/2+1, q/2+1),
\]  

(1.7)

where \(q\) is here defined as

\[
q \equiv \dim_{\mathbb{R}} \mathbb{A}_s = 2, 4, 8 \text{ for } \mathbb{A}_s = \mathbb{C}_s, \mathbb{H}_s, \mathbb{O}_s, \text{ respectively},
\]  

(1.8)

and \(\text{so}(q/2+1, q/2+1)\) is the Lie algebra of the Klein group in \(D = q + 2\). It is then natural to think, in analogy with the non split case, that the third line of \(\mathcal{L}_2 (\mathbb{A}_s, \mathbb{B}_s)\), i.e.

\[
\begin{array}{cccc}
\mathbb{R} & \mathbb{C}_s & \mathbb{H}_s & \mathbb{O}_s \\
\mathbb{H}_s & \text{so}(3,2) & \text{so}(3,3) & \text{so}(4,4) & \text{so}(6,6) \\
\end{array}
\]  

(1.9)

can be reinterpreted by means of the following Lie algebraic isomorphism

\[
\tilde{\mathfrak{sp}}(4, \mathbb{A}_s) \cong \text{so}(q/2+2, q/2+2),
\]  

(1.10)

and \(\text{so}(q/2+2, q/2+2)\) is the Lie algebra of the Klein-conformal group in \(D = q + 2\). More in detail, in this paper we give an explicit proof and take advantage of the Lie group isomorphism \(\text{Spin}(2,2) \cong \text{SL}(2, \mathbb{H}_s)\) and \(\text{Spin}(3,3) \cong \text{Sp}(4, \mathbb{C}_s)\), by constructions similar to the ones made in [41] and [19]. While in our treatment the construction and the Lie group isomorphisms analogues of (1.7) and (1.10) are explicitly worked out in those cases, nothing seemingly prevents us from putting forward the conjecture that our approach equally works well in the other critical dimensions with ultrahyperbolic signature, i.e. in \(D = (3,3)\) and in \(D = (5,5)\).

We will point out that the Klein-conformal space in \(D = 4, 6 \) or 10 dimensions may be respectively regarded as a certain Lagrangian manifold over the three aforementioned normed split algebras \(\mathbb{A}_s\)'s. In fact, the inner motivation of the present analysis also relies on the belief that a deeper understanding of the relation between Susy and split normed algebras \(\mathbb{A}_s\)'s from a supergeometric point of view could provide interesting insights on the classical and quantum properties of SYM's and supergravity theories in critical dimensions with ultrahyperbolic signature.

Our approach to \(\mathcal{M}^{2,2}\) and its \(\mathcal{N} = 1\) super-extensions will follow closely the one of [19], which in turn developed a procedure exploited in [66, 21], in which the complex 4-dimensional Minkowski (super)space was realized inside a complex flag (super)manifold, with the conformal group \(\text{SL}(4, \mathbb{C})\) acts naturally. It is here worth remarking that this is a more physics-oriented approach, in which superspaces come along with the supergroups describing their supersymmetries; this is to be contrasted to the approach e.g. of [18], in which super-Grassmannians and superflags are essentially conceived as complex entities and constructed by themselves. It should also be recalled that in [66, 21] real forms of four-dimensional Minkowski and conformal (super)spaces were introduced through suitable involutions, compatible with the natural (supersymmetric) action of the Poincaré and conformal \(\mathcal{N} = 1\), \(D = (3,1)\) supergroups.

In the present study, by essentially adapting the treatment of [19] to Kleinian signature and thus leaving the complex structure and superflags on the background, we will find a much richer mathematical structure with respect to the Minkowski case studied in [19] itself. Such a deep difference can ultimately be traced back to the fact that the action of the Klein and Klein-conformal group on its

\footnote{While the generalization to the \(D = (3,3)\) is straightforward, the case \(D = (5,5)\) may be plagued by further issues, which actually arise also in the Lorentzian case \(D = (9,1)\), due to the known problem of constructing the superconformal algebra in \(D > 6\) [64]. We aim at tackling this problem in a future project.}
irreducible spinor representation, that can be identified with $\mathbb{C}^2_s$ and $\mathbb{H}^2_s$, is not transitive, and the corresponding spinor space gets then stratified into orbits, defined by suitable invariant constraints. Remarkably, this has deep consequences in the construction of the Klein (super)space, since one must from the beginning choose a particular pair of orbit representative: in this paper, we focus only on one particular choice of pair of spinors, called generic. We point out that such a phenomenon of spinor stratification is absent in Lorentzian signature, in which case the whole spinor representation space - apart from its origin - consists of a unique orbit of the (spin covering of the) Lorentz group $\text{Spin}(q + 1, 1)$. This uniquely determines the construction of Minkowski and conformal superspaces. We will determine the isotropy groups (also named stabilizers) of the spinor orbits, as well as the constraints which define them. Relying on the theory of Clifford algebras, spinor algebras and their representations, we will highlight the relevance of the interplay between split algebras and the dimensions and reality properties of spinors of space-time symmetries in Kleinian signature, which in turn are ultimately based on the representability of the relevant spinor representation spaces as 2-dimensional vector spaces over $\mathbb{A}_s$ [67] (also cf. [68], and Refs. therein).

It is also worth anticipating here that the symmetry of the order-2 doubly-split magic square $\mathcal{L}_2(\mathbb{A}_s, \mathbb{B}_s)$ (as opposed to the order-2 split magic square $\mathcal{L}_2(\mathbb{A}_s, \mathbb{B})$, which is not symmetric) - promoted to the Lie group level by relying on the work of Dray, Manogue and collaborators [41] - will play an important role in our treatment. Indeed, the Klein-conformal group $\text{Spin}(3, 3)$ in 4 dimensions, besides occurring in the entry $\mathcal{L}_2(\mathbb{H}_s, \mathbb{C}_s)$ and thus being characterized as $\text{Spin}(3, 3) \cong \text{Sp}(4, \mathbb{C}_s)$, also appears in the entry $\mathcal{L}_2(\mathbb{C}_s, \mathbb{H}_s)$, and as such it enjoys the isomorphism $\text{Spin}(3, 3) \cong \text{SL}(2, \mathbb{H}_s)$, as well. In other words, $\text{Spin}(3, 3)$ can be regarded as the Klein-conformal group in $D = (2, 2)$, namely as $\text{Spin}(q/2 + 2, q/2 + 2)$ with $q = 2$, or as the Klein group in $D = (3, 3)$, namely as $\text{Spin}(q/2 + 1, q/2 + 1)$ with $q = 4$. Since the spinor stratification of $\text{Spin}(q/2 + 1, q/2 + 1)$ over $\mathbb{A}_s^2$ is known, this latter observation immediately allows for the knowledge of the spinor stratification of the twistor space $\mathbb{C}_s^4 \cong \mathbb{H}_s^2$ relevant for the explicit construction of the Klein space $M^{2,2}$ as a suitable section of the $D = (3, 3)$ Klein-conformal space. In our treatment, we will present an explicit derivation of the aforementioned Lie group isomorphisms, as well as of the above geometric construction.

We conclude by briefly mentioning the possible implications of our analysis for the fascinating task of space-time quantization, on which many approaches have been pursued and many research venues have been explored in literature. E.g., in [69] the quantum deformation of the complex (chiral) Minkowski and conformal superspaces was investigated by exploiting the formal machinery of flag varieties developed in [72] [73]. The more direct approach which stems from the present study is essentially the one developed in [19]; it exhibits an intrinsic elegance based on split algebras $\mathbb{A}_s$’s, and it may pave the way to the intriguing task to construct a quantum deformation of both real Klein and Klein-conformal $\mathcal{N} = 1$ superspaces.

The plan of the paper is as follows

In Section 2 we introduce split composition algebras $\mathbb{A}_s$, setting the notation used in the present work, while in Section 3 we discuss the construction of quadratic Jordan algebras over $\mathbb{A}_s$.

Section 4 reports on the classification of the spinor bundles in critical dimensions, stressing out the differences between Lorentz and Kleinian signature.

In Section 5, we focus our attention on the $D = (2, 2)$ case, which is related to the split complex algebra $\mathbb{C}_s$, by realizing explicitly the action of the Klein group on vectors, $2 \times 2$ Hermitian matrices over $\mathbb{C}_s$, and spinors, identified with vectors in $\mathbb{C}_s^2$; in particular, we compute the orbit stratification of spinors, and derive corresponding representatives.

In Section 6, we then extend our analysis to the conformal case, and discuss the symplectic realization of $\text{Spin}(3, 3)$, whose proof can be found in the Appendix A.

Finally, Section 7 deals with the $D = (2, 2)$ construction of the $\mathcal{N} = 1$ Klein superspace viewed inside the Klein-conformal $\mathcal{N} = 1$ superspace. In the Appendix B, we also give a short introduction
to the basic Supergeometry ingredients needed for a better understanding of this last Section.

2 Split Algebras

Addressing the reader to extended treatments given e.g. in [74] and [75] (also cfr. App. A of [76], and Refs. therein), we present here some basic definitions on the split algebras $C_s$ and $H_s$, useful for the subsequent treatment.

For each of the composition, normed division algebras $C$ (complex numbers), $H$ (Hamilton numbers, or quaternions) and $O$ (Cayley numbers, or octonions), one can respectively construct, by suitably adapting the Cayley-Dickson procedure, the corresponding split (composition) algebras $C_s$ (split complex numbers), $H_s$ (split quaternions) and $O_s$ (split octonions); these are characterized by the fact that some of the imaginary units square to 1 instead of $-1$.

More in detail, one starts constructing the split complex numbers $C_s$, also named hyperbolic numbers, as

$$C_s := \{ \alpha + j \beta \mid j^2 = 1, \alpha, \beta \in \mathbb{R} \}; \quad (2.1)$$

this algebra is equipped with a natural conjugation

$$a = \alpha + j \beta \rightarrow \alpha - j \beta =: \overline{a}, \quad (2.2)$$

which is used in order to define the norm

$$|a|^2 := a \overline{a} = \alpha^2 - \beta^2. \quad (2.3)$$

Not all elements in $C_s$ are invertible; in fact, it holds that

$$\frac{1}{a} = \frac{\overline{a}}{|a|^2}. \quad (2.4)$$

therefore, an element of $C_s$ with vanishing norm, i.e. $a = \alpha \pm j \alpha$, is non-invertible. Then, we denote by $C^*_s$ the invertible elements of $C_s$:

$$C^*_s := \{ \alpha + j \beta \mid \alpha \neq \pm \beta \}. \quad (2.5)$$

Every (non-zero) non-invertible element must be of the form $aE$ or $a\overline{E}$, with $E := 1 + j$ and $\alpha \in \mathbb{R}$. Moreover, it is here worth noting the following useful relations:

$$E^2 = 2E, \quad \overline{E}^2 = 2\overline{E}; \quad (2.6)$$

$$E \overline{E} = 0; \quad (2.7)$$

$$aE = (\alpha + \beta)E, \quad \forall a = \alpha + j \beta \in C_s. \quad (2.8)$$

Moreover, we observe that every element $a = \alpha + j \beta$ can be uniquely decomposed according to the following

$$a = \alpha_+ E + \alpha_- \overline{E}, \quad \alpha_\pm := \frac{1}{2}(\alpha \pm \beta) \quad (2.9)$$

It should also be remarked that a non-invertible element is always a zero divisor, due to (2.7).

By the iterating the Cayley-Dickson procedure, we then proceed constructing the split quaternions

$$H_s := \{ a + kc \mid k^2 = -1, a, c \in C_s \}, \quad (2.10)$$

which, as their divisional counterparts $H$, are non-commutative. Explicitly, any element $h \in H_s$ can be written as

$$h = (\alpha + j \beta) + k(\gamma + j \delta) = \alpha + j \beta + k \gamma + (kj) \delta,$$
where $h_R$ and $h_I$ respectively denote the real and imaginary part of the split quaternion $h$. Moreover, $j$, $k$ and $kj$ are three “imaginary” units, whose multiplication rules are summarized in the following table:

|   | $k$  | $kj$ | $j$ |
|---|------|------|-----|
| $k$ | $-1$ | $-j$ | $kj$ |
| $kj$ | $j$  | $1$  | $k$  |
| $j$  | $-kj$| $-k$ | $1$  |

(2.11)

In $\mathbb{H}_s$, the conjugation is defined as

$$h = h_R + kh_I \longrightarrow \overline{h}_R - kh_I =: h^*,$$

(2.12)

or explicitly:

$$h = \alpha + j\beta + k\gamma + (kj)\delta \longrightarrow \alpha - j\beta - k\gamma - (kj)\delta =: h^*.$$  

(2.13)

The norm of a split quaternion then reads

$$|h|^2 := hh^* = \alpha^2 + \gamma^2 - \beta^2 - \delta^2.$$  

(2.14)

It is straightforward to check that the invertible split quaternions $\mathbb{H}_s^*$ are given by

$$\mathbb{H}_s^* := \{ \alpha + j\beta + k(\gamma + j\delta) \mid \alpha^2 + \gamma^2 \neq \beta^2 + \delta^2 \}.$$  

(2.15)

Due to the aforementioned non-commutativity, one should properly discuss left and right invertibility; nevertheless, it can be proved that left and right inverse coincide.

It is also worth pointing out that one can construct the following isomorphism between $\mathbb{H}_s$ and the space of $2 \times 2$ matrices with $\mathbb{C}_s$-valued entries

$$N : = \{ M \in \mathbb{M}_2(\mathbb{C}_s) \mid \overline{M}\epsilon = \epsilon M \},$$

(2.16)

$$\epsilon : = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

(2.17)

by means of the map

$$Z : \mathbb{H}_s \rightarrow N,$$

$$h \mapsto \begin{pmatrix} h_R & h_I \\ -h_I & h_R \end{pmatrix}.$$  

(2.18)

When considering matrices with $\mathbb{H}_s$-valued entries, one can apply the map $Z$ entry-wise.

Finally, split octonions $\mathbb{O}_s$ are obtained from $\mathbb{H}_s$ by further iterating the Cayley-Dickson procedure:

$$\mathbb{O}_s := \{ h + lf \mid l^2 = -1 \text{, } h, f \in \mathbb{H}_s \}.$$  

(2.19)

We will not further deal with the algebra $\mathbb{O}_s$, since this not relevant for the present investigation (for a very recent excellent account, we address to the monograph [77]).

For convenience in the subsequent treatment, it is here worth recalling the definition of two symmetries which can be associated to split algebras: the norm-preserving symmetry and the triality symmetry.

As it can be seen from (2.3) and (2.14), the squared norm of a split algebra element is given by the symmetric bilinear form $\eta_{ab} = \eta^{ab}$ with signature $(q/2, q/2)$, and $a, b = 1, \ldots, q$, with $q$ defined in (1.8) being the real dimension of the split algebra. This is in fact the canonical inner product on the Klein space $\mathbb{M}^{q/2, q/2} \cong \mathbb{R}^{q/2, q/2}$, which is preserved by $\text{SO}(q/2, q/2) =: \text{SO}(\mathbb{A}_s)$ (whose Lie algebra we denote by $\mathfrak{so}(q/2, q/2) =: \mathfrak{so}(\mathbb{A}_s)$). Thus, $\text{SO}(\mathbb{A}_s)$ is named as the norm-preserving group of $\mathbb{A}_s$ itself.

8
Then, let us consider the following Lie algebra \( \text{tri}(\mathbb{A}_s) \):

\[
\text{tri}(\mathbb{A}_s) := \{ (A, B, C) \mid A(x, y) = B(x)y + xC(y), \ A, B, C \in \mathfrak{so}(q/2, q/2), \ x, y \in \mathbb{A}_s \}. \tag{2.20}
\]

This algebra, appearing explicitly in the magic square formula of Barton and Sudbery \cite{37, 79} (see also e.g. \cite{37}), is named as the \textit{triality symmetry algebra} of \( \mathbb{A}_s \), and the corresponding Lie group \( \text{Tri}(\mathbb{A}_s) \) is referred to as the \textit{triality group} of \( \mathbb{A}_s \) itself.

In general, it holds that \( \text{SO}(\mathbb{A}_s) \) is a (not necessarily proper) subgroup of \( \text{Tri}(\mathbb{A}_s) \), and thus one can define the following (symmetric) cosets:\(^3\) (for further elucidation, see e.g. \cite{79, 80, 81, 82}, and Refs. therein):

\[
\tilde{A}_q := \text{Tri}(\mathbb{A}_s) \sim \text{SO}(\mathbb{A}_s) \cong \begin{cases} 
q = 2 : \text{SO}(1, 1), \\
q = 4 : \text{Sp}(2, \mathbb{R}) \\
q = 8 : \text{Id},
\end{cases}
\tag{2.21}
\]

whose relevance will be exploited further below. For completeness, and later convenience, we also report the analogue result for the four normed division algebras \cite{22} \( \mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \) (for which \( q = 1, 2, 4, 8 \), respectively):

\[
\mathcal{A}_q := \text{Tri}(\mathbb{A}) \sim \text{SO}(\mathbb{A}) \cong \begin{cases} 
q = 1 : \text{Id}, \\
q = 2 : \text{U}(1), \\
q = 4 : \text{USp}(2) \\
q = 8 : \text{Id}.
\end{cases}
\tag{2.22}
\]

### 3 Quadratic Jordan Algebras over Split Algebras

Referring to thorough treatments given e.g. in \cite{83, 84} for references and details, we shall here give a brief account of quadratic Jordan algebras.

A \textit{Jordan algebra} over a field \( \mathbb{F} \) (which we shall henceforth assume to be \( \mathbb{R} \), unless otherwise specified) is an algebra \( J \) with a symmetric product\(^4\)

\[
X \circ Y = Y \circ X \in J, \ \forall X, Y \in J \tag{3.1}
\]

which satisfies the \textit{Jordan identity}

\[
X \circ (Y \circ X^2) = (X \circ Y) \circ X^2, \tag{3.2}
\]

where \( X^2 := X \circ X \). Therefore, a Jordan algebra is commutative and generally non-associative.

Given a Jordan algebra \( J \), one can define a \textit{norm} \( \mathbb{N} : J \to \mathbb{R} \) over it, satisfying the composition property \cite{55}

\[
\mathbb{N}[2X \circ (Y \circ X) - (X \circ Y) \circ Y] = \mathbb{N}^2(X)\mathbb{N}(Y). \tag{3.3}
\]

The \textit{degree} \( p \), of the norm form as well as of \( J \), is defined by \( \mathbb{N}(\lambda X) = \lambda^p \mathbb{N}(X) \), where \( \lambda \in \mathbb{R} \). A \textit{Euclidean Jordan algebra} is a Jordan algebra for which the condition \( X \circ X + Y \circ Y = 0 \) implies that \( X = Y = 0 \) for all \( X, Y \in J \); they are sometimes called \textit{compact} Jordan algebras, since their automorphism groups are compact.

In the present investigation, we are interested in a particular class of simple, \textit{quadratic} Euclidean Jordan algebras (degree \( p = 2 \); the algebras of such a class \cite{56} are denoted by \( J^C_2, J^H_2, J^O_2 \), and they are generated by Hermitian \((2 \times 2)\)-matrices over the split composition algebras \( \mathbb{A}_s = \mathbb{C}_s, \mathbb{H}_s, \mathbb{O}_s \), respectively :

\[
J = \begin{pmatrix} \alpha & Z \\ \bar{Z} & \beta \end{pmatrix} \in J^A_s, \tag{3.4}
\]

\(^3\)\textit{Id} denotes the group identity element throughout.
where $\alpha, \beta \in \mathbb{R}$ and $Z \in \mathbb{A}_s$, and the bar stands for the conjugation pertaining to the algebra under consideration; moreover, the Jordan product $\circ$ is realized as (one half) the matrix anticommutator.

The set of linear invertible transformations leaving the quadratic norm of $J^{k_s}_2$ invariant is the so-called reduced structure group $\text{Str}_0\left(J^{k_s}_2\right)$ of $J^{k_s}_2$ itself, and it holds that (recall (1.8))

$$\text{Str}_0\left(J^{k_s}_2\right) = \text{Spin}(q/2 + 1, q/2 + 1).$$

In other words, the reduced structure group of $J^{k_s}_2$ is the Klein group $\text{Spin}(q/2 + 1, q/2 + 1)$ in $D = q + 2$.

### 4 Spinors

In this Section, we provide some basic definitions and results on spinors, useful for the subsequent treatment; for further details and elucidation, we address the reader e.g. to [87, 88, 89], and Refs. therein.

We will henceforth assume $D = s + t$ even (in view of the specific case we will be interested in below, namely $D = 4$ and $s = t = 2$).

Let us start and consider the properties of (irreducible) spinor representations of the spin covering group $\text{Spin}(s,t)$ of pseudo-orthogonal groups $\text{SO}(s,t)$. For more details, cfr. e.g. [90, 55], and Refs. therein. Let $V$ be a real vector space of dimension $D = s + t$, with basis $\{e_a\} (a = 1, \ldots, D)$ and signature $(s,t) : V \cong \mathbb{R}^{s,t}$. Then, $V$ admits a non-degenerate symmetric bilinear form $\eta$ with signature $(s,t)$, which in the basis $\{e_a\}$ is given by the metric

$$\eta_{ab} = \eta^{ab} = \begin{pmatrix} +, \ldots, +, & -, \ldots, - \end{pmatrix}_{s \atop t}. \quad (4.1)$$

The group $\text{Spin}(V)$ is defined as the unique double-covering of the identity-connected component of $\text{SO}(s,t)$. A spinor representation of $\text{Spin}(V)^\mathbb{C}$ is an irreducible complex representation whose highest weights are the fundamental weights corresponding - within usual convention - to the right extreme nodes in the Dynkin diagram.

A spinor representation of $\text{Spin}(V)$ over the reals $\mathbb{R}$ (which we will be interested in) is an irreducible representation over $\mathbb{R}$, whose complexification is a direct sum of spin representations. Two parameters, namely the signature $\rho := s - t \mod(8)$ and the dimension $D = s + t \mod(8)$, classify the properties of the spinor representation (cfr. e.g. [90], and Refs. therein).

When $s = t$ (and thus $\rho = 0$), the real space $V \cong \mathbb{R}^{s,s}$ is named Klein space, its signature $(s,t) = (s,s)$ Kleinian (or hyperbolic), and the corresponding spin group $\text{Spin}(s,s)$ is named Klein group.

#### 4.1 Pure Spinors

The Clifford algebra $\mathcal{C}(s,t)$ associated to $V$ is generated by the $s + t$ Dirac gamma matrices $\Gamma^a$’s obeying

$$\left\{\Gamma^a, \Gamma^b\right\} = 2\eta^{ab}, \quad (4.2)$$

Note that $D = q + 2$ corresponds to the critical space-time dimensions of superstring theory. In fact, there is a deep relationship between supersymmetry and division algebras; cfr. e.g. [23, 24, 29, 30], and Refs. therein.

Note that in general $\mathcal{C}(s,t)$ is not isomorphic to $\mathcal{C}(t,s)$, even if $\text{Spin}(s,t) \cong \text{Spin}(t,s)$ (and thus $\text{SO}(s,t) \cong \text{SO}(t,s)$); cfr. e.g. [90, 91].
where \( I \) denotes the identity matrix. By \( \psi \) we denote a \( 2^{(s+t)/2} \)-dimensional spinor, namely a vector of the \( 2^{(s+t)/2} \)-dimensional representation space \( S \) of \( \mathcal{C}(s,t) \); for \( z \in V \), \( \psi \) is defined by the Cartan equation \[ (4.3) \]

\[ z_a \Gamma^a \psi = 0, \]

yielding the existence of a totally null plane of dimension \( d \leq (s+t)/2 \), denoted by \( T_d(\psi) \). In \( D = s+t \) even dimensions (as we are assuming throughout; cf. the start of the present Section), \( \psi \) does not provide an irreducible representation for \( \text{Spin}(s,t) \).

A "volume element" in the Clifford algebra \( \mathcal{C}(s,t) \) can be defined by introducing the gamma matrix \( \Gamma_{s+t+1} := \Gamma_1 \Gamma_2 \ldots \Gamma_{s+t} \), which anticommutes with all \( \Gamma_a \)'s; it can be used to construct an invariant projector \( P_\pm \) and we denote by \( \psi \pm \) the chiral (or Weyl) spinors, namely the \( 2^{(s+t)/2-1} \)-dimensional spinors defined by

\[ \psi \pm := P_\pm \psi, \]

implying the corresponding chiral Cartan–Weyl equations to read

\[ z_a \Gamma^a P_\pm \psi = 0. \]

Equ. \((4.3)\) define a \( d \)-dimensional totally null plane \( T_d(\psi^\pm) \), and each of the chiral spinors \( \psi \pm \) provides an irreducible representation for \( \text{Spin}(s,t) \). The existence of chiral spinors determines the splitting of the \( \mathcal{C}(s,t) \)-representation space \( S \) (with generic element \( \psi \)) into the direct sum of two \( \text{Spin}(s,t) \)-representation spaces \( S^\pm \) (with generic elements \( \psi \pm \)):

\[ S = S^+ \oplus S^-. \]

For \( d = (s+t)/2 \), i.e. for the maximal dimension of \( T_d(\psi^\pm) \), the corresponding Weyl spinor \( \psi \pm \) is named pure, and \( T_{(s+t)/2}(\psi^\pm) \cong \pm \psi^\pm \). Cartan himself stressed out the importance of this equivalence, which indeed establishes the crucial link between spinor geometry and projective Euclidean geometry. Actually, Cartan named such spinors simple, and the nowadays customary naming pure is due to Chevalley.

It should be remarked that the dimension of \( T_{(s+t)/2}(\psi^\pm) \) increases linearly with \( (s+t)/2 \), while that of the pure \( \psi^\pm \)'s increases as \( 2^{(s+t)/2-1} \); consequently, for high \( (s+t)/2 \)'s, pure spinors will be given by the solutions of suitable (quadratic) constraining relations, named pure spinor constraints, which allow to separate (in a \( \text{Spin}(V) \)-invariant way) the space of pure spinors from the space of "impure" ones. In fact, all spinors are pure for \( (s+t)/2 = 1, 2, 3 \) (i.e. in \( D = 2, 4, 6 \) dimensions), while for \( (s+t)/2 = 4, 5, 6, 7, \ldots \) (i.e. in \( D = 8, 10, 12, 14, \ldots \) dimensions) pure spinors are subject to 1, 10, 66, 364, \ldots constraints, respectively; in general, in \( D = s+t \) dimensions there are \( \binom{s+t}{(s+t)/2-4} \) pure spinor constraints.

For instance, in \( D = s+t = 10 \) dimensions, there are 10 pure spinor constraints, given by

\[ \psi \Gamma^a \psi = 0, \forall a = 1, \ldots, 10, \]

which are especially relevant for the formulation of the pure spinor formalism of superstrings (see e.g. for an introduction).

### 4.2 Classification

The problem of classifying spinors is usually formulated in subsequent steps as: (i) determining the structure of the spinor orbits \( \mathcal{O} \)'s under the action of the Spin group; (ii) computing the isotropy (stabilizer) group \( \mathcal{H} \subset \text{Spin} \) of each orbit \( \mathcal{O} \); and (iii) determining the algebra of invariants of the spinor representation space \( S \).
The orbit \( \mathcal{O}_\psi \) of a well-defined spinor representative \( \psi \) under the Spin group is a coset manifold, whose structure is determined by the isotropy group \( \mathcal{H}_\psi \) of \( \psi \):

\[
\mathcal{O}_\psi \simeq \frac{\text{Spin}}{\mathcal{H}_\psi};
\]

in general, the embedding of \( \mathcal{H}_\psi \) into Spin is not maximal nor symmetric; thus, the coset \( \mathcal{O}_\psi \) is usually non-symmetric.

Classification of spinors was first studied by Chevalley [92], who considered the orbit of pure spinors. He found that, in general, the orbit of pure spinors is the orbit of least dimension (or, equivalently, the stabilizer of pure spinors is the largest one among all spinor stabilizers). Chevalley’s analysis classifies spinors in all dimensions up to \( D = s + t = 6 \); as mentioned above, in these cases all spinors are pure.

Igusa has then classified spinors in dimensions up to \( D = s + t = 12 \) [95]. For each spinor orbit, he provided a well-defined representative, as well as the stabilizer of the orbit itself. Using similar techniques, full classifications of spinors have been worked out in more than 12 dimensions by Kac and Vinberg [96], Popov [97], Zhu [98], Antonyan and Elashvili [99], but very little is known beyond 16 dimensions. A nice summary of the spinor classification programme has been recently accounted in [100] (for what concerns pure spinors, see also e.g. [101]).

Spinors in critical dimensions \( D = s + t = q + 2 = 3, 4, 6, 10 \) have also been studied by Bryant [56, 102], whose approach exploited the connection between spinors and the four normed division algebras \( \mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \). As a physical application, such results have been recently applied to the gauging of \( \mathcal{N} = (1,0) \) magic [103] chiral supergravities in \( D = 6 \) (Lorentzian : \( s = 5, t = 1 \)) space-time dimensions in [104].

### 4.3 Spinors and Space-Time Signature: Lorentz versus Klein

Before treating in some detail the irreducible spinor representations of the Klein group \( \text{Spin}(2,2) \) in Sec. 5 (which will then be instrumental for the introduction of the Klein and conformal \( D = (2,2) \) \( \mathcal{N} = 1 \) superspaces in Sec. 7), we now briefly recall the crucial differences between Lorentzian and Klein spinors in critical dimensions \( D = q + 2 \) (for \( q = 2, 4, 8 \), especially for what concerns the representability in terms of division and split algebras, respectively. In the specific case of \( D = 4 \), this reasoning will also highlight the important differences between the approach exploited in the present investigation and the one considered in [19] (note that we will anticipate some results, which will then be obtained and discussed in the treatment of subsequent Sections).

As far as notation is concerned, by \( M_p(\mathbb{R}) \) (resp. \( M_p(\mathbb{C}) \)) we will denote the algebra of \( p \times p \) matrices with entries in \( \mathbb{R} \) (resp. \( \mathbb{C} \)) (consistently with (2.16)). Instead, \( M_p(\mathbb{H}) \) will denote the set of \( p \times p \) complex matrices satisfying the quaternionic condition

\[
\overline{M} = -\Omega M \Omega,
\]

where the bar denotes conjugation in \( \mathbb{C} \), and \( \Omega \) is the symplectic metric (for \( p = 2, \Omega = \epsilon (2.17) \)). If \( \Omega \) is non-degenerate, (4.9) implies \( p \) to be even, and \( M \) can be written as a \( p/2 \times p/2 \) matrix whose entries are quaternionic. It should also be stressed that we will be considering the Clifford algebras as real algebras throughout (cfr. e.g. Tables 1 and 2 of [90]).

- \( D = 10 \) \((\leftrightarrow q = 8, \text{ thus corresponding to } \mathbb{O}_8 \text{ or } \mathbb{O})\). Let us first consider the **Klein case** : \( D = (5,5) \), namely \( s = t = 5 \), and thus \( \rho = 0 \). The Clifford algebra \( \mathcal{C}(5,5) \), as a real algebra, is isomorphic to real \( 32 \times 32 \) matrices:

\[
\mathcal{C}(5,5) \cong M_{32}(\mathbb{R}),
\]

with \( \text{dim}_R \mathcal{C}(5,5) = 32^2 = 2^{10} \). The spinor representation space \( S \) of \( \mathcal{C}(5,5) \) is real, with real dimension \( 2^5 = 32 \), and it splits into chiral spinor representation spaces \( S^\pm \) as given by (4.10).
Each of $S^\pm$ is real, with real dimension $2^4 = 16$ : namely, it is a Majorana-Weyl spinor representation space. After \[67\] (also cfr. \[68\], and Refs. therein), a Majorana-Weyl spinor $\psi^\pm$ of Spin$(5, 5) \cong$ SL$(2, \mathbb{O})$ can be represented by a vector in $\mathbb{O}^2$ (from (2.21), recall that $\mathcal{A}_8 \cong \text{Id}$):

\[
\begin{align*}
\psi^+ & \cong \mathbb{O}^2 \cong \left\{ \begin{array}{l}
16 \\
16'
\end{array} \right\} \text{ of Spin}(5, 5).
\end{align*}
\]

(4.11)

Let us then consider the Lorentz case : $D = (9, 1)$, namely $s = 9$, $t = 1$, and thus $\rho = 8 = 0 \mod(8)$. Since $\rho$ and $D$ are the same as the Klein case previously considered, the spinor properties coincide. Indeed, the Clifford algebra $\mathcal{C}(9, 1)$, as a real algebra, is isomorphic to real $32 \times 32$ matrices :

\[
\mathcal{C}(9, 1) \cong M_{32}(\mathbb{R}),
\]

(4.12)

with dim$_\mathbb{R}\mathcal{C}(9, 1) = 32^2 = 2^{10}$, and the spinor representation space $S$ of $\mathcal{C}(9, 1)$ is real, with real dimension $2^5 = 32$. Each of the chiral spinor representation spaces $S^\pm$ is Majorana-Weyl, with real dimension $2^4 = 16$. Once again, after \[67\] (also cfr. \[68\], and Refs. therein), a Majorana-Weyl spinor $\psi^\pm$ of Spin$(9, 1) \cong$ SL$(2, \mathbb{O})$ can be represented by a vector in $\mathbb{O}^2$ (from (2.22), recall that $\mathcal{A}_8 \cong \text{Id}$):

\[
\begin{align*}
\psi^+ & \cong \mathbb{O}^2 \cong \left\{ \begin{array}{l}
16 \\
16'
\end{array} \right\} \text{ of Spin}(9, 1).
\end{align*}
\]

(4.13)

- $D = 6$ ($\leftrightarrow q = 4$, thus corresponding to $\mathbb{H}_s$ or $\mathbb{H}$). Let us first consider the Klein case : $D = (3, 3)$, namely $s = t = 3$, and thus $\rho = 0$. The Clifford algebra $\mathcal{C}(3, 3)$, as a real algebra, is isomorphic to real $8 \times 8$ matrices :

\[
\mathcal{C}(3, 3) \cong M_8(\mathbb{R}),
\]

(4.14)

with dim$_\mathbb{R}\mathcal{C}(3, 3) = 8^2 = 2^6$. The spinor representation space $S$ of $\mathcal{C}(3, 3)$ is real, with real dimension $2^3 = 8$, and it splits into chiral spinor representation spaces $S^\pm$, which are also real and with real dimension $2^2 = 4$ : namely, they are Majorana-Weyl 4-dimensional spinor representation spaces. Therefore, a generic element $\psi = \psi^+ + \psi^- \in S$, namely a non-chiral spinor of Spin$(3, 3) \cong$ SL$(4, \mathbb{R}) \cong$ SL$(2, \mathbb{H}_s) \cong$ Sp$(4, \mathbb{C})$ (cfr. (4.15) below), can be represented by a vector in $\mathbb{H}_s^2$ (from (2.22), recall that $\mathcal{A}_4 \neq \text{Id}$) :

\[
\psi = \psi^+ + \psi^- \cong \mathbb{H}_s^2 \cong (4, 2) \text{ of Spin}(3, 3) \times \mathcal{A}_4,
\]

(4.15)

where $\mathcal{A}_4 \cong$ SL$(2, \mathbb{R}) \cong$ Sp$(2, \mathbb{R})$ has been recalled from (2.21). Note that the presence of a non-trivial $\mathcal{A}_4 \neq \text{Id}$ (2.21) is crucial for the consistency of the spinor properties with the representability in terms of split algebras. Let us then consider the Lorentz case : $D = (5, 1)$, namely $s = 5$, $t = 1$, and thus $\rho = 4$. The Clifford algebra $\mathcal{C}(5, 1)$, as a real algebra, is isomorphic to quaternionic $4 \times 4$ matrices (in the sense specified above) :

\[
\mathcal{C}(5, 1) \cong M_4(\mathbb{H}),
\]

(4.16)

with dim$_\mathbb{R}\mathcal{C}(5, 1) = 4^2 = 2^4$. Thus, the spinor representation space $S$ of $\mathcal{C}(5, 1)$ is quaternionic, with complex dimension $2^3 = 8$. Each of the chiral spinor representation spaces $S^\pm$ is quaternionic, with complex dimension $2^2 = 4$. After \[67\] (also cfr. \[68\], and Refs. therein), a quaternionic (also named symplectic-Majorana-Weyl) spinor $\psi^\pm$ of Spin$(5, 1) \cong$ SU$^+(4) \cong$ SL$(2, \mathbb{H})$ can be represented by a vector in $\mathbb{H}_s^2$ :

\[
\psi^+ \cong \mathbb{H}_s^2 \cong \left\{ \begin{array}{l}
(4, 2) \\
(\overline{4}, 2)
\end{array} \right\} \text{ of Spin}(5, 1) \times \mathcal{A}_4,
\]

(4.17)
where \( \mathcal{A}_4 \cong \text{SU}(2) \cong \text{USp}(2) \) has been recalled from (2.22). Again, let us point out that the presence of a non-trivial \( \mathcal{A}_q \neq 1 \) (2.22) is crucial for the consistency of the spinor properties with the representability in terms of division algebras.\(^8\) Note that in (1.17) the bar denotes the conjugation in \( \mathbb{C} \).

- \( D = 4 \) (\( \leftrightarrow q = 2 \), thus corresponding to \( \mathbb{C}_s \) or \( \mathbb{C} \)). Let us first consider the **Klein case**: \( D = (2, 2) \), namely \( s = t = 2 \), and thus \( \rho = 0 \); this will be the case considered in detail in the next Sections. The Clifford algebra \( \mathcal{C}(2, 2) \), as a real algebra, is isomorphic to real \( 4 \times 4 \) matrices:

\[
\mathcal{C}(2, 2) \cong M_4(\mathbb{R}),
\]

with \( \text{dim}_{\mathbb{R}} \mathcal{C}(3, 3) = 4^2 = 2^4 \). The spinor representation space \( S \) of \( \mathcal{C}(3, 3) \) is real, with real dimension \( 2^2 = 4 \), and it splits into chiral spinor representation spaces \( S^\pm \), which are also real and with real dimension 2: namely, they are **Majorana-Weyl** 2-dimensional spinor representation spaces. Thus, a generic element \( \psi = \psi^+ + \psi^- \in S \), namely a **non-chiral** spinor of Spin(2, 2) \( \cong \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) \equiv \text{SL}(2, \mathbb{C}_s) \) (cfr. (5.10) below), can be represented by a vector in \( \mathbb{C}^2_s \):

\[
\psi = \psi^+ + \psi^- \cong \mathbb{C}^2_s \cong (2, 1)_+ + (1, 2)_- \quad \text{of Spin}(2, 2) \times \mathcal{A}_2, \tag{4.19}
\]

where the “+” and “−” subscripts denote weights with respect to \( \mathcal{A}_2 \cong \text{SO}(1, 1) \) (cfr. (2.21)). Again, we observe that the presence of a non-trivial \( \mathcal{A}_q \neq 1 \) (2.21) is crucial for the consistency of the spinor properties with the representability in terms of split algebras. Also, note the non-simple nature of Spin(2, 2) \( \cong \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) \) yields the spinor split \( \psi = (2, 1)_+ + (1, 2)_- \), as well as the chirality interpretation of \( \mathcal{A}_2 \) itself (see below). Let us then consider the **Lorentz case**: \( D = (3, 1) \), namely \( s = 3, \ t = 1 \), and thus \( \rho = 2 \). The Clifford algebra \( \mathcal{C}(3, 1) \), as a real algebra, is isomorphic to real \( 4 \times 4 \) matrices:

\[
\mathcal{C}(3, 1) \cong M_4(\mathbb{R}), \tag{4.20}
\]

with \( \text{dim}_{\mathbb{R}} \mathcal{C}(5, 1) = 4^2 = 2^4 \). The spinor representation space \( S \) of \( \mathcal{C}(3, 1) \) is real, with real dimension \( 2^2 = 4 \). Each of the chiral spinor representation spaces \( S^\pm \) is complex, with complex dimension 2. Therefore, a **chiral complex** spinor \( \psi^+ \) (or \( \psi^- \)) of Spin(3, 1) \( \cong \text{SL}(2, \mathbb{C}) \) can be represented by a vector in \( \mathbb{C}^2 \):

\[
\psi^+ \cong \mathbb{C}^2 \cong \mathbb{C}^2 \cong (2, 1)_+ + (1, 2)_- \quad \text{of Spin}(3, 1) \times \mathcal{A}_2; \tag{4.21}
\]

\[
\psi^- \cong \overline{\psi^+} \cong \mathbb{C}^2 \cong (2, 1)_- + (1, 2)_+ \quad \text{of Spin}(3, 1) \times \mathcal{A}_2; \tag{4.22}
\]

where the “+” and “−” subscripts here denote charges with respect to \( \mathcal{A}_2 \cong U(1) \) (cfr. (2.22)). Again, we stress that the presence of a non-trivial \( \mathcal{A}_q \neq 1 \) (2.22) is crucial for the consistency of the spinor properties with the representability in terms of division algebras. The comparison between (1.19) and (1.21)-(1.22) explains the necessary differences between the approach exploited in the present investigation and the one considered in [19]. Note that in (1.22) the bar in \( \mathbb{R}^2 \) denotes the conjugation in \( \mathbb{C} \), whereas the bar in \( \overline{\psi^+} \) denotes the spinor conjugation, which in turn - because of the representability \( \psi^+ \cong \mathbb{C}^2 \) - is *induced* by the conjugation in \( \mathbb{C} \) itself.

## 5 Vectors and Spinors of the Klein group Spin(2, 2)

We are now going to consider in some detail the irreducible spinor representations of the *Klein group* Spin(2, 2), namely of Spin(\( V \)), where \( V \) is the *Klein space* \( \mathbb{M}^{2,2} \cong \mathbb{R}^{2,2} \). As mentioned above, this

---

\(^8\)Concerning physical applications, the relevance of \( \mathcal{A}_q \) (2.22) as a part of the \( U \)-duality symmetry of \( \mathcal{N} = (1, 0) \) chiral *magic* supergravity theories in \( D = (5, 1) \) dimensions has been recently exploited in [101] (cfr. Table 2 and Sec. 3.2 therein).
latter is a 4-dimensional real vector space with Kleinian signature, \( i.e. \) with \( s = 2 \) spacelike dimensions and \( t = 2 \) timelike dimensions (thus, having \( \rho = 0 \)).

As reported in Sec. 4.3, the theory of spinor algebras (see \( e.g. \) [90]) yields that the non-chiral spinor representation \( \psi \) is real, of dimension \( 2^{D/2} = 4 \). This provides an irreducible representation of the Clifford algebra \( C(2,2) \) \(^{(1.18)}\); however, since \( D = 4 \) is even, such a representation \( \psi \) is not irreducible under \( \text{Spin}(2,2) \), and the corresponding representation space \( S \) splits into two \( \text{Spin}(2,2) \)-irreducible Majorana-Weyl spinor subspaces\(^9\), as given by \((1.6)\), each of real dimension 2. Thus, one can reconsider \((1.19)\), writing

\[
\psi_{(2,2)} = (2,1)_+ \oplus (1,2)_- \cong \begin{pmatrix} a \\ c \end{pmatrix}, \quad a, c \in \mathbb{C}_s.
\]

As noted below \((1.19)\), subscripts “+” and “-” in \((5.1)\) denote weights with respect to \( \tilde{A}_2 \cong \text{SO}(1,1) \) \(^{(2.2)}\); on the other hand, they also represent the chirality, since \( \psi^+ \) and \( \psi^- \) are Majorana-Weyl spinors of real dimension 2 with opposite chirality. Thus, in \( D = 4 \) Kleinian dimensions \( A_2 \cong \text{SO}(1,1) \), commuting with the Klein group \( \text{Spin}(2,2) \), can actually be identified the chirality operator in \( \mathbb{C}_s^2 \).

Summarizing, \( \text{Spin}(2,2) \times \text{SO}(1,1) \), has the following three representations of (real) dimension 4:

1. The (non-chiral) spinor representation \( \psi \) \((5.1)\).

2. Its conjugate spinor representation

\[
\bar{\psi}_{(2,2)} = (2,1)_- \oplus (1,2)_+ \cong \begin{pmatrix} \bar{a} \\ \bar{c} \end{pmatrix},
\]

where it is immediate to realize that, by virtue of the representability of \( \psi \) as \( \mathbb{C}_s^2 \), the conjugation in \( S \) is induced by the conjugation\(^10\) \((2.2)\) in \( \mathbb{C}_s \).

3. The vector \( x := (2,2)_0 \), which (differently from the spinor representations at points 1 and 2 above) descends to an irreducible representation of \( \text{SO}(2,2) \times \text{SO}(1,1) \). Consistently, it is given by the tensor product of the Majorana-Weyl spinors \( \psi^+ \) and \( \psi^- \) (or of their conjugate; \( cfr. \) \( e.g. \) Table 3 of [90], with \( D = 4 \) and \( k = 1 \)):

\[
x_{(2,2)_0} := \psi^+ \otimes \psi^- = \psi^+_+ \otimes \psi^-_-.
\]

\((5.3)\)

\( x \) \((5.3)\) can be consistently represented as an element of \( J^2_{\mathbb{C}_s} \), as follows. In the standard basis of \( M^{2,2} \), \( x^a = (x^1, \ldots, x^4) \) \((a = 1, \ldots, 4 = s + t)\); then, its components can be rearranged as entries of the following \( 2 \times 2 \) Hermitian matrix (recall \((3.3)\) and \((6.8)\)):

\[
\mathcal{X} := \begin{pmatrix} x^+ & \bar{a} \\ a & x^- \end{pmatrix} \in J^2_{\mathbb{C}_s},
\]

\((5.4)\)

where \( a := x^3 + jx^2 \in \mathbb{C}_s \), and \( \mathbb{R} \ni x^+ := x^1 \pm x^4 \), and the bar denotes the conjugation in \( \mathbb{C}_s \) (see \((2.2)\)). The so-called trace reversal \( \tilde{\mathcal{X}} \) of \( \mathcal{X} \) is defined as follows:

\[
\tilde{\mathcal{X}} := - \begin{pmatrix} x^- & -\bar{a} \\ -a & x^+ \end{pmatrix} \in J^2_{\mathbb{C}_s}.
\]

\((5.5)\)

Then, by recalling \((2.3)\), we observe that

\(^9\)In this case, the chiral projectors on \( S^2 \) are real, as well.

\(^{10}\)After the remarks below \((1.21)-(1.22)\), the same holds in \( D = (3,1) \), as a consequence of the representability in terms of \( \mathbb{C}^2 \).
\[ \det \mathcal{X} = (x^1)^2 - (x^4)^2 - |z|^2 = (x^1)^2 + (x^2)^2 - (x^3)^2 = \eta_{ab} x^a x^b, \quad (5.6) \]
or equivalently

\[ \mathcal{X} \tilde{\mathcal{X}} = \tilde{\mathcal{X}} \mathcal{X} = : -\eta_{ab} x^a x^b \mathbb{I}, \quad (5.7) \]
where the metric \( \eta_{ab} = \eta^{ab} \) is given by (4.11) with \( s = t = 2 \), and \( \mathbb{I} \) denotes the \( 2 \times 2 \) identity matrix. In other words, recalling (4.5), one can conclude that the squared norm \( |x|^2 \) of \( x \) (as a vector in \( \mathbf{M}^{2 \times 2} \)) is given by the quadratic norm of \( x \) as an element (5.4) of \( J^2 \) itself:

\[ |x|^2 = x^a x^b \eta_{ab} = \det \mathcal{X} = N(x). \quad (5.8) \]

Let us now consider the following transformations:

\[ J^2 \rightarrow J^2, \quad \mathcal{X} \rightarrow \lambda^\dagger \mathcal{X} \lambda = \mathcal{X}', \quad \lambda \in M_2(C_s), \quad (5.9) \]
where \( \dagger \) stands for transposition times conjugation \( (2.2) \) in the underlying split algebra \( C_s \). **Klein transformations** are defined as those transformations (5.9) in which \( \lambda \in \text{SL}(2, C_s) \); it is then immediate to realize that such transformations induce orthogonal transformations in \( \mathbf{M}^{2 \times 2} \), since they do preserve the determinant of \( \mathcal{X} \), and thus \( |x|^2 \). In particular, \( \text{SL}(2, C_s) \cong \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) \) doubly covers \( \text{SO}(2, 2) \), and it is then possible to identify it (or, more precisely, its identity-connected component) with the Spin group \( \text{Spin}(2, 2) \), which we anticipated above to be named **Klein group** in 4 dimensions. In other words, \( \text{SL}(2, C_s) \) acts naturally on \( J^2 \) as the spin covering of \( \text{SO}(2, 2) \). Thus, the following group isomorphisms hold:

\[ \text{Spin}(2, 2) \cong \text{SL}(2, C_s) \cong \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}). \quad (5.10) \]

As we have mentioned above, in signature (2, 2) spinors are Majorana, and they are identified with vectors in \( C^2_s \). We identify them with the vector representation of \( \text{SL}(2, C_s) \), i.e. \( C^2_s \). It is here instructive to observe that, as an \( \text{SL}(2, C_s) \)-module, \( C^2_s \) is not irreducible. This can be realized by decomposing every vector in \( C^2_s \) according to (2.3) as

\[ \begin{pmatrix} a \\ c \\ \psi \end{pmatrix}_{\psi} = \begin{pmatrix} \alpha_+ \\ \gamma_+ \\ \psi_\tau \end{pmatrix}_{\psi_\tau} + \begin{pmatrix} \alpha_- \\ \gamma_- \\ \psi_\tau \end{pmatrix}_{\psi_\tau}, \quad \alpha_\pm, \gamma_\pm \in \mathbb{R}; \quad (5.11) \]

analogously, any element of \( M_2(C_s) \) can be split as follows:

\[ \begin{pmatrix} a & b \\ c & d \end{pmatrix}_M = \begin{pmatrix} \alpha_+ & \beta_+ \\ \gamma_+ & \delta_+ \end{pmatrix}_M E + \begin{pmatrix} \alpha_- & \beta_- \\ \gamma_- & \delta_- \end{pmatrix}_M E. \quad (5.12) \]

Consider now a matrix \( M = M_E E + M_\tau E \) \( \in \text{SL}(2, C_s) \); then, \( 2M_E \in \text{SL}(2, \mathbb{R}) \) and \( 2M_\tau \in \text{SL}(2, \mathbb{R}) \) and every \( \text{SL}(2, C_s) \)-module \( \psi \in C^2_s \) splits into two irreducible submodules on which \( M \) acts by an \( \text{SL}(2, \mathbb{R}) \) matrix. To see this, we observe that \( \det M = 2 \det M_E + 2 \det M_\tau E \), from which one obtains that the unitarity of \( M \) implies \( \det M_E = \det M_\tau = \frac{1}{4} \), and thus \( 2M_E \) and \( 2M_\tau \) are \( \text{SL}(2, \mathbb{R}) \)-matrices. Then, the action on any spinors splits as

\[ M \psi = (2M_E) \psi_E + (2M_\tau) \psi_\tau. \quad (5.13) \]

Consistent with (5.11), we thus identify \( \psi_E \) and \( \psi_\tau \) with the Majorana-Weyl spinors \( \psi^+ \) resp. \( \psi^- \) of opposite chirality.
5.1 Spinor Orbits and Representatives

Let us now discuss how the linear action of Spin(2,2) (×SO(1,1)) on the spinor ψ = (2,1)+ ⊕ (1,2)− (or, equivalently, on its conjugate \( \overline{\psi} \)) determines the stratification of the corresponding spinor representation space \( S \) into orbits. The crucial outcome of our analysis (in agreement with literature; cfr. e.g. \([56, 102]\), and Refs. therein) is that Spin(2, 2) \( \cong \text{SL}(2, \mathbb{C}) \) (cfr. (5.10)) does not act transitively on \( \mathbb{C}^2_s \).

We start by noting that the orbit of \( e_1 := (1,0)^t \in \mathbb{C}^2_s \) contains all elements of the form \((a, c)^t\) with \(a\) and/or \(c\) invertible; in fact:

\[
\begin{align*}
\begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} a \\ c \end{pmatrix}, \\
\begin{pmatrix} a & -c^{-1} \\ c & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} a \\ c \end{pmatrix}.
\end{align*}
\]

Thus, we are henceforth going to deal only with elements \((a, c)^t \in \mathbb{C}^2_s\) with both \(a\) and \(c\) non-invertible.

By recalling the remark below (2.5) and Eq. (2.8), this amounts to consider both \(a\) and \(c\) either zero or \(u \in \mathbb{E}\) or \(u' \in \mathbb{E}'\), with \(u, u' \in \mathbb{C}^x_s\) and \(\mathbb{E} := 1 + j\).

We notice that

\[
\begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} u\mathbb{E} \\ u'\mathbb{E} \end{pmatrix} = \begin{pmatrix} \mathbb{E} \\ uu'\mathbb{E} \end{pmatrix};
\]

(5.16)

furthermore, \((\mathbb{E}, v\mathbb{E})^t\) lies in the orbit of \((\mathbb{E}, \mathbb{E})^t\), because

\[
\begin{pmatrix} 1 & 0 \\ 1 - v & 1 \end{pmatrix} \begin{pmatrix} \mathbb{E} \\ v\mathbb{E} \end{pmatrix} = \begin{pmatrix} \mathbb{E} \\ \mathbb{E} \end{pmatrix}.
\]

(5.17)

Therefore, up to conjugation in \(S \cong \mathbb{C}^2_s\) (induced by the conjugation in \(\mathbb{C}_s\), and mapping the spinor \(\psi\) into its conjugate \(\overline{\psi}\)), in \(S\) one needs to consider (besides \((0, 0)^t\) and \((1, 0)^t\)) only the following elements:

\[
1: \begin{pmatrix} \mathbb{E} \\ \mathbb{E} \end{pmatrix}, \quad 2: \begin{pmatrix} \mathbb{E} \\ 0 \end{pmatrix}, \quad 3: \begin{pmatrix} 0 \\ \mathbb{E} \end{pmatrix}, \quad 4: \begin{pmatrix} 0 \\ \mathbb{E} \end{pmatrix};
\]

(5.18)

in other words, one can disregard the multiplication by invertible split complex numbers (as well as the conjugation in \(\mathbb{C}_s\)) when dealing with the stratification of the spinor representation space \(S\).

By definition of group orbit, in order to establish the stratification structure of \(\mathbb{C}^2_s\) under the action of the Klein group in four dimensions, we have to determine which elements in \(\mathbb{C}^2_s\) are connected through the action of an element \(g \in \text{SL}(2, \mathbb{C}_s)\). Let us then analyze the elements listed in (5.18):

1. This element belongs to the orbit of \((\mathbb{E}, \mathbb{E})^t\), because:

\[
\begin{pmatrix} (1 + \mathbb{E})/2 \\ (1 - \mathbb{E})/2 \end{pmatrix} \begin{pmatrix} (1 - \mathbb{E})/2 \\ (1 + \mathbb{E})/2 \end{pmatrix} \begin{pmatrix} \mathbb{E} \\ \mathbb{E} \end{pmatrix} = \begin{pmatrix} 0 \\ 2\mathbb{E} \end{pmatrix}.
\]

(5.19)

2. A similar argument also shows that \((\mathbb{E}, 0)^t\) is in the orbit of \((\mathbb{E}, \mathbb{E})^t\).

3. Quite surprisingly, the element \((\mathbb{E}, \mathbb{E})^t\) can be proved to lie in the orbit of \((1, 0)^t\), because:

\[
\begin{pmatrix} \mathbb{E} & -1/2 \\ \mathbb{E} & 1/2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbb{E} \\ \mathbb{E} \end{pmatrix}.
\]

(5.20)

\[\text{The upperscript "t" denotes transposition.}\]
4. There exists no transformation of $SL(2, \mathbb{C}_s)$ connecting $(\mathcal{E}, \mathcal{E})^t$ to $(1, 0)^t$. In fact, if this were the case, one would have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathcal{E} \\ \mathcal{E} \end{pmatrix},$$

(5.21)

hence $a = c = \mathcal{E}$. This cannot be, since otherwise the determinant $ad - bc = \mathcal{E}(d - b)$ would be a zero divisor.

From this analysis, it follows that the orbits of $\mathbb{C}_s^2$ under the action of the Klein group $SL(2, \mathbb{C}_s)$ (up to conjugation in $\mathbb{C}_s^2$, equivalent to conjugation in $S$) are characterized by one of the following three well-defined representatives:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \mathcal{E} \\ \mathcal{E} \end{pmatrix}.$$  

(5.22)

Thus, besides the trivial orbit (given by the origin $(0, 0)^t$ of $\mathbb{C}_s^2$), $(1, 0)^t$ and $(\mathcal{E}, 0)^t$ (or equivalently $(\mathcal{E}, 0)^t$) are well-defined representatives of the orbit stratification. In particular, the representative $(1, 0)^t$ is stabilized by any matrix of $SL(2, \mathbb{C}_s)$ of the form

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \sim \mathbb{C}_s \sim \mathbb{R}^2,$$

(5.23)

while the stabilizer of $(\mathcal{E}, 0)^t$ reads

$$M_{\mathbb{F}_2} \oplus \begin{pmatrix} 1 \\ \beta_+ \\ 0 \\ \frac{1}{\beta_+} \end{pmatrix} \mathcal{E} \cong SL(2, \mathbb{R}) \ltimes \mathbb{R}.$$  

(5.24)

Summarizing, we obtained

$$O_{(1, 0)^t} \cong \frac{\text{Spin}(2, 2)}{\mathbb{R}^2}, \quad \text{dim}_{\mathbb{R}} = 4; \quad (5.25)$$

$$O_{(\mathcal{E}, 0)^t} \cong \frac{\text{Spin}(2, 2)}{\text{SL}(2, \mathbb{R}) \ltimes \mathbb{R}}, \quad \text{dim}_{\mathbb{R}} = 2. \quad (5.26)$$

Since $\text{dim}_{\mathbb{R}} O_{(1, 0)^t} = \text{dim}_{\mathbb{R}} S = 4$, such an orbit can be regarded as the generic one; consequently, $O_{(\mathcal{E}, 0)^t}$ is the non-generic spinor orbit.

### 6 Klein-Conformal group and $\text{Spin}(3, 3)$

We are now going to consider the Klein-ambient space $M^{3, 3} \cong \mathbb{R}^{3, 3}$ and the corresponding Klein group in 6 dimensions, namely $\text{Spin}(3, 3)$. In this case, $s = t = 3$, and thus again $\rho = 0$. In turn, $\text{Spin}(3, 3)$ can also be regarded as the conformal group of $M^{2, 2} \cong \mathbb{R}^{2, 2}$ itself.

In complete analogy with the treatment of $\text{Spin}(2, 2)$ given above, one can identify a vector $x^A = (x^1, \cdots, x^6) \quad (A = 1, \ldots, 6)$ in $M^{3, 3}$ with an element of the quadratic simple Jordan algebra $J_{2, \mathbb{H}_s}$ over split quaternions $\mathbb{H}_s$, by rearranging the vector components as entries of the $2 \times 2$ Hermitian matrix

$$\mathcal{V} = \begin{pmatrix} \hat{x}^+ & z^* \\ z & \hat{x}^- \end{pmatrix} \in J_{2, \mathbb{H}_s},$$  

(6.1)

where $z := x^5 + jx^1 + kx^4 + (kj)x^2 \in \mathbb{H}_s$, $\mathbb{R} \ni \hat{x}_\pm := x^3 \pm x^6$, and the star denoting the conjugation in $\mathbb{H}_s$ (cfr. (2.12)). By recalling the definition (2.14), the quadratic form associated to the metric

\[\text{This observation will also give rise to the chain of isomorphisms} \quad (6.14) \quad (\text{holding both at Lie algebra and at Lie group level). It can be traced back to the symmetry of the doubly-split Magic Square of order 2} \quad (11) \quad (37) \quad (11).\]
\(\eta_{AB} = \eta^{AB}\) of signature \((3, 3)\) (given by \((\text{I.1})\) with \(s = t = 3\)) of \(M^{3,3}\) is then obtained by computing\(^{13}\)

\[
\det \mathcal{V} = (x^3)^2 - (x^6)^2 - |z|^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 - (x^4)^2 - (x^5)^2 - (x^6)^2 := \eta_{AB} x^A x^B, \quad (6.2)
\]

or equivalently

\[
\mathcal{V} \tilde{\mathcal{V}} = \tilde{\mathcal{V}} \mathcal{V} = - \eta_{AB} x^A x^B \mathbb{I}, \quad (6.3)
\]

where \(\tilde{\mathcal{V}}\) is the trace reversal of \(\mathcal{V}\). In other words, recalling \((\text{I.3})\), one can conclude that the squared norm \(|x|^2\) of \(x\) (as a vector in \(M^{3,3}\)) is given by the quadratic norm of \(x\) as an element \((\text{6.1})\) of \(J_{2}^{\mathbb{H}_{s}}\) itself:

\[
|x|^2 = x^A x^A \eta_{AB} = \det \mathcal{V} = N (x). \quad (6.4)
\]

Let us now consider the following transformations:

\[
J_{2}^{\mathbb{H}_{s}} \rightarrow J_{2}^{\mathbb{H}_{s}},
\]

\[
\mathcal{V} \rightarrow \lambda \mathcal{V} \lambda =: \mathcal{V}', \quad \lambda \in M_2(\mathbb{H}_{s}),
\]

where \(\dagger\) stands for transposition times conjugation \((\text{2.12})\) in the underlying split algebra \(\mathbb{H}_{s}\). \(\textit{Klein-conformal transformations}\) are defined as those transformations \((\text{6.3})\) in which \(\lambda \in \text{SL}(2, \mathbb{H}_{s})\), where the special linear group is defined as (recall \((\text{2.18})\))

\[
\text{SL}(2, \mathbb{H}_{s}) := \{ M \in M_2(\mathbb{H}_{s}) \mid \text{det} (Z(M)) = 1 \}; \quad (6.6)
\]

or equivalently (recall \((\text{2.17})\))

\[
\text{SL}(2, \mathbb{H}_{s}) := \left\{ M \in \text{SL}(4, \mathbb{C}) \mid \begin{array}{cc} \epsilon & 0 \\ 0 & \epsilon \end{array} \right\} M \right\}. \quad (6.7)
\]

It is then immediate to realize that such transformations induce orthogonal transformations in \(M^{3,3}\) (and correspondingly \textit{conformal} transformations in \(M^{2,2}\)), since they do preserve \(\det \mathcal{V}\) and thus \(|x|^2\). In particular, \(\text{SL}(2, \mathbb{H}_{s})\) doubly covers \(\text{SO}(3, 3)\), and it is then possible to identify it (or, more precisely, its identity-connected component) with the \textit{Spin} group \(\text{Spin}(3, 3)\). In other words, \(\text{SL}(2, \mathbb{H}_{s})\) acts naturally on \(J_{2}^{\mathbb{H}_{s}}\) as the spin covering of \(\text{SO}(3, 3)\). This establishes the group isomorphism

\[
\text{Spin}(3, 3) \cong \text{SL}(2, \mathbb{H}_{s}). \quad (6.8)
\]

### 6.1 A Further Group Isomorphism

For the subsequent treatment, we find convenient to present also another isomorphism involving \(\text{Spin}(3, 3)\), namely\(^{14}\)

\[
\text{Spin}(3, 3) \cong \widetilde{\text{Sp}(4, \mathbb{C})}. \quad (6.9)
\]

In order to prove it, we start from the \(4 \times 4\) matrix given by \((\text{A.4})\) in the App. \(\text{A}\) which we report below for convenience’s sake :

\[
\mathcal{X} := \begin{pmatrix}
\hat{x}_+ \epsilon & \hat{x} \epsilon \\
-\hat{x} \epsilon & -\hat{x}_- \epsilon
\end{pmatrix}. \quad (6.10)
\]

Then, one can compute that

\[
\mathcal{X}^\dagger \Omega \mathcal{X} = \eta_{AB} x^A x^B \Omega. \quad (6.11)
\]

\(^{13}\)It should be here remarked that the determinant of \(2 \times 2\) Hermitian matrices with \(\mathbb{H}\)- or \(\mathbb{H}_{s}\)-valued entries is well defined \([53]\).

\(^{14}\)The tilde in \(\widetilde{\text{Sp}(4, \mathbb{C})}\) denotes the peculiar definition \((\text{6.13})\) - after \([40, 37]\) - of the symplectic group by the matrix Hermitian-conjugate (and not by the matrix transpose, as usually done).
where † stands for transposition times conjugation 2.17 in \( \mathbb{C}_s \), and \( \Omega \) here denotes for the 4 × 4 symplectic metric (recall 2.17)

\[
\Omega = \begin{pmatrix}
0 & I \\
-I & 0
\end{pmatrix} = \mathbb{I} \otimes \epsilon.
\] (6.12)

Therefore, one can define the symplectic group à la Barton and Sudbery [40, 37]:

\[
\tilde{\text{Sp}}(4, \mathbb{C}_s) := \{ M \in \text{SL}(4, \mathbb{C}_s) \mid M^\dagger \Omega M = \Omega \},
\] (6.13)

and any transformation of the form

\[
X \rightarrow X = \lambda X \lambda^t, \quad \text{with} \quad \lambda \in \tilde{\text{Sp}}(4, \mathbb{C}_s),
\] (6.14)

preserves the (squared) norm \( |x|^2 \) in \( M^{3,3} \) and induces a Klein-conformal transformation in \( M^{2,2} \), thus providing an alternative realization of \( \text{Spin}(3, 3) \), and then inducing the isomorphism (6.9) .

We thus obtain the following chain of group isomorphisms:

\[
\text{Spin} (3, 3) \cong \text{SL}(2, \mathbb{H}_s) \cong \tilde{\text{Sp}}(4, \mathbb{C}_s).
\] (6.15)

We have already discussed in Section 4.3 that spinors of \( \text{Spin} (3, 3) \) can be interpreted as vectors of \( \mathbb{H}^2_s \cong \mathbb{C}_s^4 \) on which, in analogy with the \( \text{Spin} (2, 2) \) case, \( \text{SL}(2, \mathbb{H}_s) \) does not act transitively; this determines the stratification into orbits. Even if the stratification reveals to be more evident using the special linear group over split quaternions, in the next section we will use instead the symplectic group over split complexes to realize the Klein superspace as the space of 2|0 totally isotropic subspaces in \( \mathbb{C}_s^4 \), i.e. the Lagrangian superspace.

We will focus on the case in which the representative super plane is given by a pair of generic vectors of \( \mathbb{C}_s^4 \) whose even part is a generic spinor. One has of course the possibility to choose other isotropic subspaces as representative given by other combinations (namely, non generic-generic or non generic-non generic) of spinors. Since the action of the Klein group stratifies the spinor space, we expect to obtain different and intriguing constructions. We leave a detailed analysis for a future project, while in this paper we focus on the generic-generic case for spinor representatives.

7 Klein and Klein-Conformal \( \mathcal{N} = 1 \) Superspaces

We can now proceed to construct the \( \mathcal{N} = 1 \) Klein-conformal and Klein superspaces in \( D = (2, 2) \). Supermanifolds, and in particular the \( \mathcal{N} = 1 \) Minkowski and conformal superspaces in \( D = (3, 1) \), have been studied intensively in the past years. A thorough account of such a broad field of investigation lies well beyond the scope of this paper; we here confine ourselves to addressing the interested reader to [66], and Refs. therein, for an exhaustive bibliography.

In order to construct the \( \mathcal{N} = 1 \) Klein-conformal and Klein superspaces in \( D = (2, 2) \), we will exploit a procedure which is very similar to the one of [19]; however, some extra attention should be paid in the definition of the functor of points of a \( \mathbb{C}_s \)-group. We could give our definitions in full generality, but for clarity’s sake we do prefer to adapt them to our specific framework. We have provided App. [13] for the basic facts of supergeometry and supergroups; for more details on the technicalities involved, we address the reader e.g. to Ch. 10 of [20].

The \( A \)-points of the general linear supergroup over \( \mathbb{C}_s \) are given by (see App. [13] (3.6):

\[
\text{GL}(m|n)(A) = \left\{ \begin{pmatrix}
a & \alpha \\
\beta & b
\end{pmatrix} \right\} = \text{Hom}_{\text{salg}}(\mathbb{C}_s|\text{GL}(m|n)), A),
\] (7.1)

where \( a, b, \alpha, \beta \) are matrices with entries in \( A \) (roman and greek lowercase letters denote even resp. odd entries throughout), and \( a \) and \( b \) are invertible.
If we regard GL(m|n) as a real supergroup, we can define its $A$-points (where here $A$ is a real superalgebra):

$$(GL(m|n)(A))_R = \text{Hom}_{(salg)}(\mathbb{C}_s[[x_{ij}, \xi_{kl}]][\det(x_{ij})^{-1}_{1\leq i,j \leq m}, \det(x_{ij})^{-1}_{m+1\leq i,j \leq m+n}], A \otimes \mathbb{C}_s)$$

(7.2)

(see App. B (B.10)).

We now define, in complete analogy to [19], the symplectic orthogonal supergroup $\tilde{SpO}(4|1)$ as the (real) subsupergroup of GL(m|n)$_R$ given as:

$$\tilde{SpO}(4|1)(A) = \{ \Lambda \in (GL(4|1))_R(A) \mid \Lambda^\dagger \Lambda = J \}, \text{ with } J = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (\Omega \ 0 \ 0).$$

where $\Lambda^\dagger := \overline{\Lambda}^t$ (with $t$ here denoting the supertranspose) and the conjugation is consistently understood in $\mathbb{C}_s$, as detailed in the treatment above. If

$$\Lambda = \begin{pmatrix} B & \alpha \\ \beta & u \end{pmatrix}, \quad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \beta = (\beta_1, \beta_2), \quad \alpha = (\alpha_1, \alpha_2)^t,$$

with $\beta_i, \alpha_i \in A^2 \ (i = 1, 2)$, from the condition

$$\Lambda^\dagger J \Lambda = J,$$

one obtains the following set of equations:

$$\begin{cases}
B^\dagger J B + \beta^\dagger \beta = J; \\
B^\dagger J \alpha + \beta^\dagger u = 0; \\
-a^\dagger J B + u^\dagger \beta = 0; \\
-a^\dagger J \alpha + u^\dagger u = 1.
\end{cases} \quad \iff \begin{cases}
a^\dagger c^\dagger a + b^\dagger \beta_1 = 0; \\
a^\dagger d^\dagger b + \beta_2^\dagger \beta_2 = 1; \\
b^\dagger c^\dagger d^\dagger a + \beta_1^\dagger \beta_1 = 0; \\
b^\dagger c^\dagger d^\dagger b + \beta_2^\dagger \beta_2 = 0; \\
b^\dagger d^\dagger a_1 + a^\dagger a_2 + \beta_1^\dagger u = 0; \\
b^\dagger d^\dagger a_1 + b^\dagger a_2 + \beta_2^\dagger u = 0; \\
b^\dagger a_1 - a_1^\dagger a_2 + u^\dagger u = 1.
\end{cases} \quad (7.5)$$

We now consider the (real) supermanifold $\mathcal{L}$ of 2|0 totally isotropic subspaces in $\mathbb{C}_s^{4|1}$. Let us take $\{e_1, e_2, e_3, e_4, e\}$ the canonical basis for $\mathbb{C}_s^{4|1}$. We define $\mathcal{L}$ as the orbit of the super subspace $\text{span}_{\mathbb{C}_s}\{e_1, e_2\}$ under the natural action of the real supergroup $\tilde{SpO}(4|1)$. This is a supermanifold, and if $A$ is a local $\mathbb{C}_s$-superalgebra, one obtains

$$\mathcal{L}(A) = \left\{ \begin{pmatrix} a \\ c \\ \beta_1 \end{pmatrix} \mid a^\dagger c^\dagger a + \beta_1^\dagger \beta_1 = 0 \right\} / \text{GL}_2(A).$$

(7.6)

It should be here stressed that $A$ needs to be taken local in order to express in an easier way the action of $\tilde{SpO}(4|1)$ on $\mathcal{L}$; we address the reader to Chs. 2 and 4 of [66] for a detailed treatment of this technical point.

**Remark.** The real supergroup $\tilde{SpO}(4|1)$ does not act transitively on the superspace $\mathbb{C}_s^{4|1}$; in the standard (i.e., non-super) case, we have mentioned such a feature in the previous section and in Sec. 5.1 for the Klein case. However, this fact will not influence our treatment, since we realize the Klein $\mathcal{N} = 1$ superspace as an open inside the $\tilde{SpO}(4|1)$-orbit $\mathcal{L}$ of $\text{span}_{\mathbb{C}_s}\{e_1, e_2\}$, i.e. of the generic-generic spinor case.
We consider the open subset of $\mathcal{L}$ consisting of those subspaces corresponding to a invertible. We call it $\mathbf{M}^{2,2|1}$ : it will be our model for the $D = (2,2) \, N = 1$ Klein superspace, while $\mathcal{L}$ is topologically the compactification of $\mathbf{M}^{2,2|1}$, and it is the $D = (2,2) \, N = 1$ Klein-conformal superspace. By multiplying by a suitable element of $\text{GL}_2(A)$ we have:

$$
\mathbf{M}^{2,2|1}(A) = \left\{ \begin{pmatrix} \mathbb{I} \\ \mathcal{Y} \\ \zeta \end{pmatrix} \left| \mathcal{Y}^\dagger = \mathcal{Y} + \zeta^\dagger \zeta \right. \right\} .
$$

(7.7)

Here $A$ is a commutative superalgebra, not necessarily local as before.

Notice that $\mathcal{Y} = ca^{-1}$, $\zeta = \beta_1 a^{-1}$ with respect to the expression in (7.6). Hence, the equation is obtained immediately from (7.6) by setting $a = 1$. This is precisely the condition found in [21] and in [19]. Furthermore, we remark that the relation:

$$
\mathcal{Y}^\dagger = \mathcal{Y} + \zeta^\dagger \zeta
$$

for $\zeta = 0$ reduces to the condition of $\mathcal{Y}$ to be Hermitian (in the context of $\mathbb{C}_s$). A comparison with (5.4) shows that this is precisely the condition for an element in $M_2(\mathbb{C}_s)$ to belong to $\mathbf{M}^{2,2}$. Thus, the $\mathbb{C}_s$ points of the supermanifold $\mathbf{M}^{2,2|1}$ coincide with the Klein space $\mathbf{M}^{2,2}$ discussed above, and this justifies the use of our super-terminology.

We now proceed to examine the Klein-Poincaré supergroup, acting on $\mathbf{M}^{2,2|1}$. We start by noticing that the supergroup functor

$$
\tilde{\text{sKP}}(A) := \left\{ \begin{pmatrix} L & 0 & 0 \\ M & R & R\phi \\ d\chi & 0 & d \end{pmatrix} \right\} \subset \text{SpO}(4|1)(A)
$$

(7.8)

leaves $\mathbf{M}^{2,2|1}$ invariant ($A$ as usual is a commutative superalgebra). This subgroup is representable (see App. [22] for the definition of representable supergroup functor). In fact the real superalgebra representing it is obtained as a quotient of $\mathbb{R}[\tilde{\text{SpO}}(4|1)]$, namely setting to zero those generators corresponding to the positions where we have zeros for the $A$-points in (7.8). Notice that its reduced group (see App. [22] (17.9)) is the Klein-Poincaré group itself.

We then define $\text{sKP}$ as the Klein-Poincaré supergroup. Its $A$-points are given by (7.8). Applying the equations in (7.5) to $\text{sKP}(A)$, one obtains

$$
R = (L^\dagger)^{-1}, \quad \phi = \chi^\dagger, \quad ML^{-1} = (ML^{-1})^\dagger + (L^\dagger)^{-1} \chi^\dagger \chi L^{-1},
$$

(7.9)

yielding

$$
\tilde{\text{sKP}}(A) = \left\{ \begin{pmatrix} L \\ M \quad (L^\dagger)^{-1} \\ d\chi \end{pmatrix} \right\} \left( \begin{pmatrix} 0 & 0 \\ (L^\dagger)^{-1} \chi^\dagger \\ d \end{pmatrix} \right) .
$$

(7.10)

Then, the action on $\mathbf{M}^{2,2|1}$ (7.7) can be readily computed to yield :

$$
\tilde{\text{sKP}} \times \mathbf{M}^{2,2|1} \rightarrow \mathbf{M}^{2,2|1}
$$

$$
\begin{pmatrix} L \\ M \\ d\chi \end{pmatrix} \begin{pmatrix} \mathbb{I} \\ \mathcal{Y} \\ \zeta \end{pmatrix} \mapsto \begin{pmatrix} \mathbb{I} \\ ML^{-1} + (L^\dagger)^{-1} \mathcal{Y}L^{-1} + (\chi L^{-1})^\dagger \zeta L^{-1} \\ d\chi L^{-1} + d\zeta L^{-1} \end{pmatrix} .
$$

(7.11)

We end this Section with an important observation that relates our construction of the Klein-Poincaré
supergroup with our previous treatment of spinors of Spin(2, 2). The Klein-Poincaré supergroup contains as its closed subgroup the \textit{Klein supergroup}, whose functor of points is given by:

\[
\tilde{\mathcal{K}}(A) = \begin{pmatrix}
(R^t)^{-1} & 0 & 0 \\
0 & R & R\phi \\
\phi^\dagger & 0 & d
\end{pmatrix}.
\] (7.12)

As for its counterpart in Lorentz signature, \(\tilde{\mathcal{K}}\) is obtained from \(\tilde{\mathcal{K}}P\) by removing the inhomogeneous translational part given by \(M\) (note we here use the variables \(R = (L^{-1})^t, \phi = \chi^t\)). The corresponding Lie superalgebra reads

\[
\text{Lie}(\tilde{\mathcal{K}}) = \begin{pmatrix}
-r^t & 0 & 0 \\
0 & r & r\varphi \\
D\varphi^t & 0 & D
\end{pmatrix}.
\] (7.13)

If \(\mathfrak{g}_0\) and \(\mathfrak{g}_1\) respectively denote the even and odd part of \(\mathfrak{g} := \text{Lie}(\tilde{\mathcal{K}})\), there is a natural action of \(\mathfrak{g}_0\) on \(\mathfrak{g}_1\). Indeed, in this framework, it holds that \(\mathfrak{g}_0 = \mathfrak{g}_0^t \oplus \mathfrak{c}_s\), where \(\mathfrak{g}_0^t\) is the Lie algebra of the spin group \(\text{SL}_2(\mathbb{C}) \cong \text{Spin}(2, 2)\) \((\text{cfr. (5.11)})\), and \(\mathfrak{c}_s\) corresponds to dilatations. As one can readily check, the action of \(\mathfrak{g}_0\) on the odd part \(r\varphi\) (that is \(\mathfrak{g}_1\)) is precisely the spinor representation \(\mathbb{C}_2^s\) studied in previous Sections.

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\section*{A Symplectic Realization of Spin(3, 3)}

Consider the canonical basis \(\{e_\mu\}\) for \(\mathbb{C}_4^s\), and \(\{e^\mu\}\) its dual basis \((\mu = 1, ..., 4)\). A natural inner product \(\langle \bullet, \bullet \rangle\) in \(\Lambda^2\mathbb{C}_4^s\) can be defined as follows

\[
\langle \bullet, \bullet \rangle : \Lambda^2\mathbb{C}_s^4 \otimes \Lambda^2\mathbb{C}_s^4 \rightarrow \mathbb{C}_s, \\
x \wedge y, z \wedge w \quad \mapsto \quad (x \wedge y \wedge z \wedge w)(e^1 \wedge e^2 \wedge e^3 \wedge e^4).
\] (A.1)

Note that \(\tilde{\text{Sp}}(4, \mathbb{C}_s)\) \((6.13)\) acts in an obvious way on \(\Lambda^2\mathbb{C}_4^s\) preserving the inner product \(\langle \bullet, \bullet \rangle\).

We are now going to determine a real 6-dimensional subspace of \(\Lambda^2\mathbb{C}_s^4\), which is stable under \(\tilde{\text{Sp}}(4, \mathbb{C}_s)\) and on which \(\langle \bullet, \bullet \rangle\) takes real values. To this aim, let us define the \textit{symplectic} inner product

\[
\langle \bullet, \bullet \rangle_{\Omega} : \mathbb{C}_s^4 \otimes \mathbb{C}_s^4 \rightarrow \mathbb{C}_s, \\
x, y \quad \mapsto \quad y^\dagger \Omega x,
\] (A.2)

where \(\Omega\) is given by \((6.12)\).

Then, one can use \(\langle \bullet, \bullet \rangle\) and \(\langle \bullet, \bullet \rangle_{\Omega}\) in order to construct the isomorphisms \(\phi : \Lambda^2\mathbb{C}_s^4 \congto (\Lambda^2\mathbb{C}_s^4)^*\) and \(\varphi : \mathbb{C}_s^4 \congto (\mathbb{C}_s^4)^*\). It is then possible to naturally identify \((\Lambda^2\mathbb{C}_s^4)^* \cong \Lambda^2(\mathbb{C}_s^4)^4\), and use it to construct the \(\tilde{\text{Sp}}(4, \mathbb{C}_s)\)-invariant isomorphism of \(\Lambda^2\mathbb{C}_s^4\) into itself as \(\Phi := \varphi^{-1} \otimes \varphi^{-1} \cdot \phi\). This identifies a subspace of \(\Lambda^2\mathbb{C}_s^4\) on which \(\Phi\) acts as the identity operator. A convenient basis of such a
subspace reads as follows:

\[
E_1 = \frac{1}{\sqrt{4}} e_1 \wedge e_4 - \frac{1}{\sqrt{2}} e_2 \wedge e_3; \quad E_4 = \frac{1}{\sqrt{4}} e_1 \wedge e_4 + \frac{1}{\sqrt{2}} e_2 \wedge e_3;
E_2 = j \frac{1}{\sqrt{2}} e_1 \wedge e_3 + j \frac{1}{\sqrt{2}} e_2 \wedge e_4; \quad E_5 = \frac{1}{\sqrt{2}} e_2 \wedge e_4 - \frac{1}{\sqrt{2}} e_1 \wedge e_3;
E_3 = \frac{1}{\sqrt{2}} e_1 \wedge e_2 - \frac{1}{\sqrt{2}} e_3 \wedge e_4; \quad E_6 = \frac{1}{\sqrt{2}} e_1 \wedge e_2 + \frac{1}{\sqrt{2}} e_3 \wedge e_4,
\]

(A.3)

and it can be checked that within the such a subspace the inner product \(< \bullet, \bullet >\) takes real values, and has signature \((3, 3)\).

Therefore, any vector in this subspace can be represented as antisymmetric \(4 \times 4\) matrix of the form

\[
X := \begin{pmatrix}
0 & x_3 + x_6 & -x_5 + jx_2 & x_1 + x_4 \\
-x_3 - x_6 & 0 & x_4 - x_1 & x_5 + jx_2 \\
x_5 - jx_2 & -x_4 + x_1 & 0 & x_6 - x_3 \\
-x_1 - x_4 & -x_5 - jx_2 & x_3 - x_6 & 0
\end{pmatrix}
= \begin{pmatrix}
\hat{x}_+ \varepsilon & \mathcal{X} \varepsilon \\
-\hat{\mathcal{X}} \varepsilon & -\hat{x}_- \varepsilon
\end{pmatrix}
= \epsilon \otimes \begin{pmatrix}
\hat{x}_+ & \mathcal{X} \\
-\mathcal{X} & -\hat{x}_-
\end{pmatrix},
\]

(A.4)

where in the last step definition (2.17) has been recalled, \(\hat{x}_\pm := x^5 \pm x^6 \in \mathbb{R}\), and \(\mathcal{X}, \mathcal{X} \in J_2^C\) (cfr. definitions (5.4)- (5.5)).

**B Supergeometry**

In this appendix we recall few well known facts about superalgebras and more in general supergeometry. We refer the reader to [20] and the references within for more details.

Let \(k\) be a commutative algebra. For our purposes, it is enough to consider the cases of \(k = \mathbb{R}, \mathbb{C}, \mathbb{C}_s\).

A super vector space is a \(\mathbb{Z}/2\mathbb{Z}\)-graded vector space \(V = V_0 \oplus V_1\); the elements of \(V_0\) are called even and elements of \(V_1\) are called odd. Notice that a parity of a vector \(v\), denoted by \(p(v)\), is not defined in general, but, since any element may be expressed as the sum of homogeneous ones, it suffices to consider only homogeneous vectors in all of the statements relying on linearity. The super dimension of a super vector space \(V\) is the pair \((p, q)\), where \(\text{dim}(V_0) = p\) and \(\text{dim}(V_1) = q\) as ordinary vector spaces. When the dimension of \(V\) is \(p|q\), we can find a basis \(\{e_1, \ldots, e_p\} \) of \(V_0\) and a basis \(\{\varepsilon_1, \ldots, \varepsilon_q\}\) of \(V_1\) so that

\[
V = \text{span}\{e_1, \ldots, e_p, \varepsilon_1, \ldots, \varepsilon_q\}.
\]

(B.1)

For us, the most relevant example is \(\mathbb{C}_s^{4|1} = \text{span}\{e_1, \ldots, e_4, \varepsilon_1 =: \varepsilon\}\) (when there is just one odd basis element we omit the numbering).

A superalgebra over \(k\) is a super vector space \(A\) together with a multiplication preserving parity. \(A\) is commutative if

\[
xy = (-1)^{p(x)p(y)}yx
\]

(B.2)

The prototype of a commutative superalgebra is the polynomial superalgebra, generated by the even indeterminates \(t_1, \ldots, t_m\), which commute, and the odd ones \(\theta_1, \ldots, \theta_n\), which anticommute: \(\theta_i \theta_j = -\theta_j \theta_i\), hence \(\theta_i^2 = 0\). We denote such superalgebra with \(k[t_1, \ldots, t_m, \theta_1, \ldots, \theta_n]\). The reader may safely think of such superalgebra when we make our statements regarding commutative superalgebras.

If \(A\) is the polynomial superalgebra, we have:

\[
A_0 = \left\{ f_0 + \sum_{r \text{ even}} f_I t_I | I = \{i_1 < \ldots < i_r\} \right\}, \quad A_1 = \left\{ \sum_{s \text{ odd}} f_J \theta_J | J = \{j_1 < \ldots < j_s\} \right\}.
\]

(B.3)
where we are using the multi-index notation and \( f_I, f_J \in k[t_1, \ldots, t_n] \) the ordinary polynomial algebra in the commuting variables \( t_1, \ldots, t_n \).

Let \( V \) be a vector space and \( A \) a commutative superalgebra. We define:

\[
V(A) = (A_0 \otimes V_0) \oplus (A_1 \otimes V_1).
\]  

If \( V = k^{p|q} \), we most immediately have

\[
V(A) = \{(a_1, \ldots, a_p, \alpha_1, \ldots, \alpha_q) \mid a_i \in A_0, \alpha_j \in A_1\}
\]  

We define the \textit{A-points} of the \textit{general linear supergroup} \( GL(p|q)(A) \), as the parity preserving linear maps from \( V(A) \) to itself. An easy calculation shows that:

\[
GL(p|q)(A) = \left\{ \begin{pmatrix} a_{ij} & a_{il} \\ a_{kj} & a_{kl} \end{pmatrix} \mid a_{ij}, a_{kl} \in A_0, \quad a_{il}, a_{kj} \in A_1 \right\}
\]  

where \( 1 \leq i, j \leq p, p+1 \leq k, l \leq p+q \) and \( \det(a_{ij}), \det(a_{kl}) \) are invertible. This is an ordinary group with the matrix multiplication. The super nature of this geometric object lies into the anticommuting entries of its odd part, namely the \( \alpha_{rs} \)'s.

We can identify \( GL(p|q)(A) \) with the group of superalgebra morphisms from the superalgebra

\[
k[GL(p|q)] := k[x_{ij}, \xi_{kl}][\det(x_{ij})^{-1}_{1 \leq i,j \leq p}, \det(x_{ij})^{-1}_{p+1 \leq i,j \leq p+q}]
\]  

to the superalgebra \( A \). Let us see this identification through an example (the general case is a straightforward modification of it). Consider

\[
GL(1|1)(A) = \left\{ \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix} \mid a, b \in A_0, \quad \alpha, \beta \in A_1, \quad a, b \text{ invertible} \right\}
\]  

and \( k[GL(1|1)] = k[x, y, \xi, \eta][x^{-1}, y^{-1}] \). A morphism \( \phi : k[x, y, \xi, \eta][x^{-1}, y^{-1}] \rightarrow A \) is determined by the images of the generators, namely \( \phi(x) = a, \phi(y) = b, \phi(\xi) = \alpha, \phi(\eta) = \beta \), where \( a, b \) are invertible in \( A_0 \) and \( \alpha, \beta \in A_1 \). The identification of \( \phi \) with a matrix in the form \( (B.8) \) is then immediate.

The identification between \( GL(p|q)(A) \) and the set of morphisms of superalgebras as above, denoted by \( \text{Hom}(salg)(k[GL(p|q)], A) \), allows us to say that the general linear supergroup is \textit{represented} by the superalgebra \( k[GL(p|q)] \). The information contained in \( GL(p|q)(A) \) for all \( A \) is effectively contained in the superalgebra \( k[GL(p|q)] \). More appropriately, we call \textit{general linear supergroup over} \( k \) and we denote it by \( GL(p|q) \), the functor that associates to a given commutative superalgebra \( A \) the group \( GL(p|q)(A) \). The reader does not need to be familiar with the theory of categories, but should be aware that a supergroup functor \( G \) is a way of giving, for any commutative superalgebra \( A \), a group, denoted by \( G(A) \), that behaves nicely when we change \( A \) (namely, if we have a morphism \( A \rightarrow B \), this morphism should naturally induce another morphism \( G(A) \rightarrow G(B) \)). Furthermore, to fully deserve the name of supergroup, the functor \( G \) must be \textit{representable}, that is, there is a superalgebra \( k[G] \), playing the role of \( k[GL(p|q)] \), so that we can identify \( G(A) \), the \( A \)-points of the supergroup functor, with the morphisms \( k[G] \rightarrow A \). However, for the present work, we shall not be interested in these subtleties: all of the supergroup functors we consider in this paper are indeed representable.

The \textit{reduced group} associated to a supergroup is the ordinary group that we obtain by taking \( A = k \).

For example, for \( GL(p|q) \):

\[
GL(p|q)(k) = \left\{ \begin{pmatrix} a_{ij} & 0 \\ 0 & a_{kl} \end{pmatrix} \right\} = GL(p) \times GL(q)
\]  

because the only value in a field \( k \) that the nilpotent variables \( \alpha_{rs} \) can take is zero.
At this point we need to make a step forward in this theory and look at the differences in the choice of $k$. So to mark the difference between the different $k$’s, we speak of $k$-supergroups or we say that a supergroup is defined over $k$. For the purpose of the present paper, we need to consider $\mathbb{C}_s$-supergroups, that we want to view as supergroups over $\mathbb{R}$. Let us look at an example and consider the supergroup GL(1|1) over $\mathbb{C}_s$; again, the general case is not conceptually different. The superalgebra representing the supergroup is $\mathbb{C}_s[z, w, \zeta, \eta][z^{-1}, w^{-1}]$ (see (B.8)). This superalgebra will give us the $A$-points of GL(1|1), when $A$ is a $\mathbb{C}_s$-superalgebra, while now we want to determine the $A$-points of GL(1|1) as a real supergroup, that is when $A$ is a real superalgebra. We then define the $A$-points of the $\mathbb{C}_s$-supergroup GL(1|1), viewed as $\mathbb{R}$-supergroup, the $A \otimes \mathbb{C}_s$ points of GL(1|1):

$$GL(1|1)_\mathbb{R}(A) = GL(1|1)(A \otimes \mathbb{C}_s) = \text{Hom}_{\text{salg}}(\mathbb{C}_s[z, w, \zeta, \eta][z^{-1}, w^{-1}], A \otimes \mathbb{C}_s)$$

where the tensor product is over $\mathbb{R}$. In fact, a morphism $\psi : \mathbb{C}_s[z, w, \zeta, \eta][z^{-1}, w^{-1}] \rightarrow A \otimes \mathbb{C}_s$ is specified once we know $\psi(z), \psi(w), \psi(\zeta), \psi(\eta)$. Let us look at $\psi(z) = a \otimes 1 + b \otimes j$. The image of $z$ is effectively recovered by the pair $(a, b)$ with $a, b \in A_0$. So we see that a complex indeterminate $z$ is associated with two real indeterminates. The images of the 4 $\mathbb{C}_s$-generators $z, w, \psi, \zeta$ give 8 elements of the real algebra $A$, as one expects (in analogy to what we expect for ordinary vector spaces or algebras: the complex coordinates double their number, when viewed as real). For the $\mathbb{C}_s$ general linear supergroup:

$$(GL(p|q)(A))_\mathbb{R} = \text{Hom}_{\text{salg}}(\mathbb{C}_s[[x_{ij}, \xi_{kl}],[\det(x_{ij})]_{1 \leq i,j \leq p}, [\det(x_{ij})]_{p+1 \leq i,j \leq p+q}], A \otimes \mathbb{C}_s)$$

(B.10)

The definition for a generic $\mathbb{C}_s$-supergroup is:

$$G_\mathbb{R} = \text{Hom}_{\text{salg}}(\mathbb{C}_s[G], A \otimes \mathbb{C}_s)$$

(B.11)

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