A REMARK ON LIOUVILLE PROPERTY OF STRONGLY TRANSITIVE ACTIONS

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Abstract. Liouville property of actions of discrete groups can be reformulated in terms of existence co-Følner sets. Since every action of amenable group is Liouville, the property can be served as an approach for proving non-amenability. The verification of this property is conceptually different than finding a non-amenable action. There are many groups that are defined by strongly transitive actions. In some cases amenability of such groups is an open problem. We define $n$-Liouville property of action to be Liouville property of point-wise action of the group on the sets of cardinality $n$. We reformulate $n$-Liouville property in terms of additive combinatorics and prove it for $n=1, 2$. The case $n \geq 3$ remains open.

1. Introduction

Let $G$ be a countable discrete group acting on a set $X$. A measure $\mu$ on $G$ is non-degenerate if $supp(\mu)$ generate $G$ as semigroup. Define transition probabilities from $x$ to $y$ in $X$ induce by the measure $\mu$ by

$$p_\mu(x, y) = \sum_{g \in G, gx = y} \mu(g).$$

A function $f : X \to \mathbb{R}$ is $\mu$-harmonic if

$$f(x) = \sum_{y \in X} f(y)p_\mu(x, y).$$

The action of $G$ on $X$ is called $\mu$-Liouville if all bounded $\mu$-harmonic functions are constant. In particular, this definition implies that $\mu$-Liouville actions are transitive. We call an action Liouville if there is a non-degenerate measure $\mu$ on $G$ which makes it $\mu$-Liouville.

One of the main motivations to study Liouville actions is its relation to amenability. A renowned result of Kaimanovich and Vershik, [8], states that the group is not amenable if and only if the action of the group on itself is not Liouville. However, if a transitive action of $G$ on a set is not Liouville then the action of $G$ in itself is not Liouville either. This follows from a simple fact that if $f : X \to \mathbb{R}$ is $\mu$-Liouville then for every $x \in X$ the function $f_x : G \to \mathbb{R}$ defined by $f_x = f(gx)$ is $\mu$-Liouville. Thus, having a bounded non-constant $\mu$-harmonic function on $X$ one obtains a bounded non-constant $\mu$-harmonic on $G$. 

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This approach to non-amenability have been recently used by Kaimanovich, [7], as a suggested attempt to show non-amenability of Thompson group $F$. In particular, Kaimanovich showed that for every finitely supported non-degenerate measure $\mu$ on Thompson group $F$ the action of the group on dyadic rationals is not $\mu$-Liouville. In [6], Zheng and the author showed that this action is Liouville. An action of $G$ on $X$ is called strongly transitive if for every finite subsets $E$ and $F$ of $X$ of the same cardinality there is $g$ in $G$ such that $g(E) = F$. The main property of the Thompson group $F$ that was used in [6] is strong transitivity of order preserving action on a totally ordered set and the existence of an element in the group with infinite orbit.

This paper aims to study Liouville property of actions on the sets of a fixed cardinality. An action of a group $G$ on a set $X$ is called $n$-Liouville if there is a non-degenerate measure $\mu$ on the subsets of $X$ of cardinality $n$ such that every bounded $\mu$-harmonic function is constant. One of the main ingredients will be the following theorem/definition of Liouville action.

Let $\mu$ be a finitely additive measure on $G$ and $x \in X$, denote by $(m \cdot x)$ the finitely additive measure on $X$ given by $(m \cdot x)(A) = m(\{g \in G : gx \in A\})$.

**Theorem 1.** Let $G$ act transitively on a set $X$, then the following are equivalent:

1. the action is Liouville, i.e., there exists a non-degenerate symmetric measure $\mu$ on $G$ such that $X$ is $\mu$-Liouville;
2. there exists a finitely additive measure $m$ on $G$ such that $m \cdot x = m \cdot y$ for any $x, y$ on $X$.
3. for every $\varepsilon > 0$ and for every finite $F \subseteq X$ there exists a finite set $E \subset G$ such that we have
   $$|Ex \Delta Ey| \leq \varepsilon|E|$$ for every $x, y \in F$.

One should compare the last condition with the definition of amenable action, i.e., the existence of Følner sets:

For every $\varepsilon > 0$ and for every finite $E \subseteq G$ there exists a finite set $F \subset X$ such that we have
$$|gF \Delta hF| \leq \varepsilon|F|$$ for every $g, h \in E$.

We use the condition (3) to show that the action of Thompson group $F$ on the dyadic rationals is 1- and 2-Liouville. However, the case of $n$-Liouville property for $n \geq 3$ is not feasible for us at the current moment. It is tempting for us to conjecture that Thompson group $F$ does not have 3-Liouville property. A negative answer to the following seemingly easy question implies that Thompson $F$ does not have 3-Liouville property.
action, thus it is not amenable:

Is it true that for every \( \varepsilon > 0 \) there is a set \( V \subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N} \) such that

\[
|\{(p_1(x), p_2(x)) : x \in V\} \cap \{(p_2(x), p_3(x)) : x \in V\}| 
\]

\[
\geq (1 - \varepsilon)|V|
\]

where \( p_i \) is a projection on the corresponding coordinate. Here we assume possibility of \( p_i(x) = p_i(y) \) for distinct \( x \) and \( y \), and consider \( p_i(x) \) as a set with multiplicity.

The question above is a open. The following condition which implies the question above is equivalent to 3-Liouville property.

For every \( n \geq 3 \) and for every \( \varepsilon > 0 \) there is a finite set \( V \) in \( \mathbb{N} \times \mathbb{N} \times \ldots \times \mathbb{N} \) (product taken \( n \) times), such that

\[
|\bigcap_{1 \leq k \leq n; 1 \leq i \leq k; k+1 \leq m \leq n} \{(p_i(x) + p_{i+1}(x) + \ldots + p_k(x), p_{k+1}(x) + \ldots + p_m(x)) : x \in V\}| 
\]

\[
\geq (1 - \varepsilon)|V|
\]

A reformulation of \( n \)-Liouville property is very similar to the one of 3-Liouville. Instead of intersecting vectors in \( \mathbb{N} \times \mathbb{N} \) one considers vectors of dimension \( n - 1 \). We will specify it in the later section.

2. Følner type sets and Liouville actions

Let \( \mu \) be a finitely additive measure on \( G \) and \( x \in X \), denote by \( (m \cdot x) \) the finitely additive measure on \( X \) given by \( (m \cdot x)(A) = m(\{g \in G : gx \in A\}) \).

Theorem 2. Let \( G \) act transitively on a set \( X \), then the following are equivalent:

1. the action is Liouville, i.e., there exists a non-degenerate symmetric measure \( \mu \) on \( G \) such that \( X \) is \( \mu \)-Liouville;

2. there exists a finitely additive measure \( m \) on \( G \) such that \( m \cdot x = m \cdot y \) for any \( x, y \) on \( X \);

3. for every \( \varepsilon > 0 \) and for every finite \( F \subseteq X \) there exists a finite set \( E \subseteq G \) such that we have

\[
|Ex \Delta Ey| \leq \varepsilon|E| \text{ for every } x, y \in F.
\]

Proof. Condition (1) implies (2): let \( \mu \) be a measure such that the action on \( X \) is \( \mu \)-Liouville, then convolution powers of approximate a mean on \( G \) such that (2) is satisfied. To show the converse, given such a mean \( m \) on \( G \), let \( v_n \) be a sequence of
finite support probability measure that approximate $m$, then one can build a measure using the Kaimanovich-Vershik method (or as in [6]) which is Liouville.

The equivalence of (3) and (2) follows from the same considerations as the proof of equivalence of existence of Følner sets and approximately invariant $l_1$-functions (Reiter's condition), see [2] for example. □

3. 1- and 2-Liouville actions

The 1-Liouville property was shown in [6]. In this section we will consider 2—Liouville property, the case of 1-Liouville will be also clear from our considerations.

An action of $G$ on $X$ is called strongly transitive if for every finite subsets $E$ and $F$ of $X$ of the same cardinality there is $g$ in $G$ such that $g(E) = F$. Denote by $\mathcal{P}_n(X)$ the set of all subsets of $X$ of cardinality $n$.

We will consider the group $F_{\mathbb{R}}$ of all piece-wise linear homeomorphisms of $\mathbb{R}$ with slopes as powers of 2 and breaking points of the first derivative in dyadic rationals. Thompson group $F$ is the subgroup of $F_{\mathbb{R}}$ made of the homeomorphisms that are identity outside of the interval $[0,1]$. It contains many other copies of Thompson group $F$, for example subgroups $G_n$ that are identity outside of $[-2^n,2^n]$. The group $F_{\mathbb{R}}$ itself is an abelian extension of the union of $G_n$. Thus, Thompson group $F$ is amenable if and only if $F_{\mathbb{R}}$ is amenable. The consideration of $F_{\mathbb{R}}$ allows scaling of dyadic rationals to natural numbers.

**Theorem 3.** The action of Thompson group $F$ and the group $F_{\mathbb{R}}$ on dyadic rationals is 2-Liouville.

**Proof.** Denote by $X$ the set of dyadic rationals for the case of $F_{\mathbb{R}}$ and dyadic rationals intersected with $[0,1]$ for Thompson group $F$. Let us firstly consider the group $F_{\mathbb{R}}$ and a set $V$ of all subsets of cardinality 2. For every $\varepsilon > 0$ we have to arrange a finite subset $E \subset F_{\mathbb{R}}$ such that

$$|Ex \Delta Ey| \leq \varepsilon |E|$$

for every $x, y \in V$. (1)

By multiplying the set $E$ on the right by an element of the form $f(x) = 2^i x$ with $i$ large enough, we can assume that all elements of $V$ are the subsets of natural numbers. For $x = \{x_1, x_2\} \in V$ define $\hat{x} = |x_2 - x_1|$, and for a subset $Q$ of $\mathcal{P}_2(X)$ denote $\hat{Q} = \{\hat{x} : x \in Q\}$ and consider this set with multiplicities. In particular, the condition (1) obviously implies

$$|\hat{E}x \Delta \hat{E}y| \leq \varepsilon |E|$$

for every $x, y \in V$. (2)

We claim that the existence of sets that satisfy the condition (2) implies the existence of the sets that satisfy the condition (1). Fix $\varepsilon > 0$ and let $E$ be a set that satisfy (2). Obviously, $Ex$ and $Ey$ might not even intersect. However, we can blur $Ex$ and $Ey$ the way that the intersection will be large. To do that define $g(x) = x + 1$. Then the set $E' = \{g, g^2, \ldots, g^n\} \cdot E$ will satisfy

$$|E'x \Delta E'y| \leq \varepsilon' |E'|$$

for every $x, y \in V$. (3)
where \( \varepsilon' \) tends to 0, when \( \varepsilon \) tends to 0. Indeed, it is easy to see that if \( x \) and \( y \) are two elements of \( P_2(X) \) with \( \tilde{x} = \tilde{y} \), then
\[
a_n = |\{g, g^2, \ldots, g^n\} x \cap \{g, g^2, \ldots, g^n\} y|/n \rightarrow 1
\]
when \( n \) goes to infinity. However,
\[
|E'x \cap E'y| \geq (1 - \varepsilon)a_n |E| = (1 - \varepsilon)a_n |E'| \text{ for every } x, y \in V,
\]
which implies the claim.

In order to show the existence of sets that satisfy the condition (2) let us consider one of the easiest and most illustrative cases from which the most general case will be clear. Consider the case where \( V \) is the set of all subsets of cardinality 2 which are supported on 4 points of natural numbers \( n_1, n_2, n_3 \) and \( n_4 \). We claim that if for every \( \varepsilon > 0 \) there exists a set finite set \( W \subset N \times N \times N \), considered with multiplicities, such that
\[
|p_1(W) \cap p_2(W) \cap p_3(W) \cap \{p_1(w) + p_2(w) : w \in W\} \cap \{p_2(w) + p_3(w) : w \in W\} \cap \ldots \cap \{p_1(w) + p_2(w) + p_3(w) : w \in W\}| \geq (1 - \varepsilon)|W|,
\]
then there is a finite subset \( E \) of the group that satisfies the condition (2) Here \( p_i \) is the projection of \( W \) onto the \( i \)-th coordinate (the sets \( p_i(W) \) are considered with multiplicities). Indeed, fix natural numbers \( r_1, r_2, r_3, \) and let \( A \) be a set of natural numbers which is almost invariant by multiplication by \( r_i, r_i + r_j \) and \( r_1 + r_2 + r_3 \) for all \( i \neq j \), then the set \( W = \{(r_1a, r_2a, r_3a) : a \in A\} \) satisfies the condition.

Now the set \( E' \) is constructed as follows. To each element \( (w_1, w_2, w_3) \) in \( W \) we associate an element of the group that sends \( n_1 \) to 0, \( n_2 \) to \( w_1 \), \( n_3 \) to \( w_1 + w_2 \) and \( n_4 \) to \( w_1 + w_2 + w_3 \). This can be arranged because the group acts strongly transitively on \( X \). The set \( E' \) is chosen to be the set of all elements in the group associated to the elements of \( W \). It is clear that the intersection of projections above guarantees the large intersection of the sets \( E'x \) and \( E'y \). For example, if \( x = \{0, 1\} \) and \( y = \{1, 2\} \), then the fact that the intersection of \( p_1(W) \) and \( p_2(W) \) is large guarantees that the intersection of \( E'x \) and \( E'y \) is large.

The case when the elements of \( V \) are supported on larger sets, say on a set of cardinality \( n \), follows from existence for every \( \varepsilon \) of a set \( W \subset N \times \ldots \times N \) (where the product is taken \( n - 1 \) times) which satisfy
\[
|p_1(W) \cap \ldots \cap p_{n-1}(W) \cap \bigcap_{1 \leq i \leq n-2} \{p_i(w) + p_{i+1}(w) : w \in W\} \cap \ldots \cap \{p_1(w) + p_2(w) + \ldots p_n(w) : w \in W\}| \geq (1 - \varepsilon)|W|,
\]
Such sets exist and can be chosen of the form $W = \{(r_1a, r_2a, \ldots, r_{n-1}a) : a \in A\}$, where $A$ is a set of natural numbers invariant under multiplication of corresponding sums of $r_i$.

The proof of the above theorem generalizes in obvious way to the following theorem.

**Theorem 4.** Let a discrete group $G$ act strongly transitive on a totally ordered set $X$ in order preserving way. Assume that there is a sequence of elements $g_n$ in $G$ which admits arbitrarily large orbits. Then the action is Liouville.

It is not known to us whether there exists an amenable group satisfying the theorem above. If it exists using technique of [6] we can deduce that the action of any group (in particular, Thompson group $F$) that satisfy the theorem is $n$-Liouville for any $n$. Note that there are examples of amenable groups that act strongly transitive on sets and admit elements of infinite orbits, for example those coming from topological full groups [3], [4], [4].

**4. $n$-Liouville actions: open problems**

Let us firstly consider the case $n = 3$ and reformulation of 3-Liouville property. The following theorem is a straightforward adaptation of the first part of the proof of Theorem 3.

**Theorem 5.** The action of Thompson group $F$ or the group $F\mathbb{R}$ on dyadic rationals is 3-Liouville if and only if for every $n$ and $\varepsilon > 0$ there exists a set $W \subset \mathbb{N} \times \ldots \times \mathbb{N}$ ($n$-fold product) such that

$$\left| \bigcap_{1 \leq k \leq n; 1 \leq i \leq k; k+1 \leq m \leq n} \left\{(p_1(x) + p_{i+1}(x) + \ldots + p_k(x), p_{k+1}(x) + \ldots + p_m(x)) : x \in W\right\} \right| \geq (1 - \varepsilon)|W|.$$

We are tempted to conjecture that these sets do not exist. The later would imply non-amenability of Thompson group $F$. Even the following seeming easy subproblem is currently out of our rich.

**Problem 6.** Is it true that for every $\varepsilon > 0$ there is a finite subset $W$ of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$, such that

$$|\{(p_1(x), p_2(x)) : x \in V\} \cap \{(p_2(x), p_3(x)) : x \in V\} \cap \{(p_1(x) + p_2(x), p_3(x)) : x \in V\} \cap \{(p_1(x), p_2(x) + p_3(x)) : x \in V\}| \geq (1 - \varepsilon)|V|$$

The problem above is equivalent to the existence of sets from Theorem 2 (3) when $V$ is chosen to be all subsets of cardinality 3 supported on 4 points.
References

[1] Brin, M., The chameleon groups of Richards J. Thompson: automorphisms and dynamics. Publications Mathmatiques de l’IHES 84 (1996): 5-33.

[2] Juschenko, K., Amenability of discrete groups by examples. Current version available at http://www.math.northwestern.edu/~juschenk/book.html.

[3] Juschenko, K., Monod, N., Cantor systems, piecewise translations and simple amenable groups. Annals of Math, (2013) 775-787

[4] Juschenko, K., Nekrashevych, V., de la Salle, M., Extensions of amenable groups by recurrent groupoids. Inventiones mathematicae 206, no. 3 (2016): 837-867.

[5] Juschenko, K., Matte Bon, N., Monod, N., de la Salle, M., Extensive amenability and an application to interval exchanges. Ergodic Theory and Dynamical Systems (2016): 1-25.

[6] Juschenko, K. and Zheng, T., 2016. Infinitely supported Liouville measures of Schreier graphs. arXiv preprint arXiv:1608.03554.

[7] Kaimanovich, V., Thompson’s group F is not Liouville, Preprint, arXiv:1602.02971

[8] Kaimanovich, V. A., Vershik, A. M., Random walks on discrete groups: boundary and entropy, The Annals of Probability, (1983) 457-490.

[9] Lyons R., Peres Y., Probability on Trees and Networks, Cambridge University Press, 2016.

[10] Mishchenko, P., Boundary of the action of Thompson group F on dyadic numbers, Preprint, arXiv:1512.03083

[11] Savchuk, D., Schreier graphs of actions of Thompson’s group F on the unit interval and on the Cantor set, Geometriae Dedicata, V.175 (2015), 355-372

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