Multiscale Loop Vertex Expansion for Cumulants. Part I: The $T^4_3$ Model

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Abstract

We construct cumulants up to an finite order of a tensor field theory perturbed by a quartic term, the $T^4_3$ model. The method we use is the multi-scale loop vertex expansion. We prove analyticity and Borel summability of the ordinary coupling constant $g$.

keywords Cumulants; Constructive Field Theory

Mathematics Subject Classification 81T08

1 Introduction

For a general exposition of constructive field theory see [1, 2, 3], and for a general view on random tensors see [4]. The loop vertex expansion (LVE) was introduced in 2007 as a new tool in constructive field theory in order to deal with matrix fields [5]. The essential ingredients of LVE are the Hubbard-Stratonovich intermediate field representation [6, 7], the replica method [8] and the BKAR formula [9, 10].

A main feature of the LVE is that it is written in terms of trees which are exponentially bounded. It means that the outcome of the LVE is convergent whereas the usual perturbative expansion diverges. For a review of the LVE, we suggest [11]; for the actual mechanism of replacing Feynman graphs, which are not exponentially bounded, by trees, see [12]. For the LVE applied to cumulants when renormalisation is absent, see [13, 14].
In order to deal with quantum fields when renormalisation is present, we aimed at combining the LVE with the multi-scale approach that is developed in [15, 16] and synthesized in the book [3].

We have already performed the initial steps on this road. In [17] we presented a simple combinatorial model with renormalization. It is a model of conjugate vector fields with a quartic interaction and a particular propagator. The main result of [17] is that this divergence is renormalized by using a Wick-ordered interaction. Then the MLVE has been successfully applied to various general super-renormalisable fields of increasing complexity [18, 19, 20]. The articles [17, 18, 19, 20] are restricted to the partition function and its logarithm (the free energy) and does not include cumulants (connected Schwinger functions in the terminology of field theorists).

We choose the $T^4_3$ model investing in [19] as a initial benchmark for studying cumulants in quantum field theory when renormalisation is present, because this model is part of the "tensor track" [21, 22, 23, 24, 25, 26, 27, 28], and is the simplest interacting tensor field theory which has a power counting almost similar to ordinary $\phi^4_2$, the initial constructive model. But remark that our cumulants depend on $J$ and $\bar{J}$. They cannot be defined constructively but only perturbatively. One finds no notion of scalar cumulants or topological expansion for cumulants in our article, all things that are defined in [13]. In the papers [13] and [14], the authors takes advantage of the Weingarten calculus because it’s relevant, the emphasis being on the large $N$ expansion. But here, our view is different. Since we break the $U(N)$ invariance, it’s the constructive aspect that we are after. Therefore we are satisfied with a theorem like Theorem 2 where the arguments of our sources are bounded by $K$.

Recently the subject of cumulants in probability from a combinatorial point of view has attracted increasing interest, see [29, 30]. We think that this MLVE with cumulants (MLVEC) is interesting to that point of view. We think also it can be extended for example to various groups such as the $O(N)$ and $Sp(N)$ groups. We think in addition that it can be used in $T^4_5$ [31, 32] and in the formalism of the Borel-Ecalle of trans-series [33, 34].

Acknowledgments We thank warmly the anonymous referee for pointing out many important corrections to the initial version of this paper. We also acknowledge the support of the CEA-LIST through the chair “Artificial Intelligence and Complexity”.

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2 The model and our main result

Before introducing the $T^4_3$ model, let us adopt the following notation:

- We always define $\lambda := \sqrt{2g}$ throughout this paper, so that $g = \frac{\lambda^2}{2}$.
- We adopt the convention $\sum_\varnothing = 0$, $\prod_\varnothing = 1$.
- We write $i := \sqrt{-1}$. We also denote by $\mathbb{I}_n$ the identity matrix on $\mathbb{R}^n$ or $\mathbb{C}^n$ and by $\mathbf{1}_n$ the $n \times n$ matrix with all entries equal to 1, depending on context, i.e. whereas the underlying field is $\mathbb{R}$ or $\mathbb{C}$.
- We write $a \lesssim b$ if there is a constant $K > 0$ such that $a \leq Kb$.

From now on we will switch to an integral notation, more adapted to the MLVE, and more reminiscent of the functional integration in quantum field theory. Our model is the $T^4_3$ model described in [19]. The authors of [19] consider conjugate rank-3 tensor fields $T_n$, $\bar{T}_n$ with

$$d\mu_C(T, \bar{T}) = \left( \prod_{n, \bar{n} \in \mathbb{Z}^3} \frac{dT_n dT_{\bar{n}}}{2i\pi} \right) |\text{Det } C|^{-1} e^{-\sum_n T_n C_{n, \bar{n}}^{-1} T_{\bar{n}}},$$

(1)

$$n = \{n_1, n_2, n_3\} \in \mathbb{Z}^3, \quad \bar{n} = \{\bar{n}_1, \bar{n}_2, \bar{n}_3\} \in \mathbb{Z}^3.$$

(2)

where the bare propagator $C$ has unit mass:

$$C_{n, \bar{n}} = \delta_{n, \bar{n}} C(n), \quad C(n) \equiv \frac{1}{n_1^2 + n_2^2 + n_3^2 + 1}. \quad \text{(3)}$$

The bare partition function is then

$$Z_0(g) = \int e^{-\frac{g}{4} \sum_c V^c(T, \bar{T})} d\mu_C(T, \bar{T}) = \int e^{-\lambda^2 \sum_c V^c(T, \bar{T})} d\mu_C(T, \bar{T})$$

(4)

where $g = \frac{\lambda^2}{2}$ is the coupling constant and

$$V^c(T, \bar{T}) = \sum_{n, \bar{n}, p, \bar{p}} \left( T_n \bar{T}_{\bar{n}} \prod_{c' \neq c} \delta_{n_{c' \bar{n}_{c'}}} \delta_{p_{c' \bar{p}_{c'}}} \right) \delta_{n_{c' \bar{n}_{c'}}} \delta_{p_{c' \bar{p}_{c'}}} \left( T_p \bar{T}_{\bar{p}} \prod_{c' \neq c} \delta_{p_{c' \bar{p}_{c'}}} \right)$$

(5)

are the three quartic interaction terms of random tensors at rank three. See Figure 1.

1By Fourier transform these fields can be considered also as ordinary scalar fields $T(\theta_1, \theta_2, \theta_3)$ and $\bar{T}(\bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3)$ on the three torus $T_3 = U(1)^3$. [31]
The bare amplitude for $M$ is the sum of three amplitudes with color $c$, each of which is a non-trivial function of the single incoming momentum $n_c$

$$A(M) = \sum_c A(M_c), \quad A(M_c)(n_c) = -g \sum_{p \in [-N,N]^3} \frac{\delta(p_c - n_c)}{p^2 + 1}. \quad (6)$$

The sum over $p$ diverges logarithmically as $N \to \infty$. In fact the perturbative amplitudes of the $T_3^4$ model are finite in the $N \to \infty$ limit, except for a mild divergence of self-loops which yield a logarithmically divergent sum. This divergence is itself renormalized by using a Wick-ordered interaction.

Let us make this point a bit more precise. The authors of [19] use two cutoffs; first of all the ”cubic cutoff”, with $p \in [-N, N]^3$; then another cutoff, roughly similar to the first, because the cubic cutoff is not very well adapted to the rotation invariant $n^2$ term in the propagator, nor very convenient for multi-slice analysis as in [17]. The second cutoff is still sharp in the momentum space $\ell_2(\mathbb{Z})^3$, but not longer factorize over colors. In the traditional way of constructive quantum field theory [15, 16], it means they fix an integer $M > 1$ as ratio of a geometric progression $M^j$ and defined this second cutoff as follows:

$$1_{\leq 1} = 1_1 = 1_{1+n_1^2+n_2^2+n_3^2 \leq M^2}, \quad (7)$$

$$1_{\leq j} = 1_{1+n_1^2+n_2^2+n_3^2 \leq M^j}, \quad \text{for } j \geq 2, \quad (8)$$

$$1_j = 1_{\leq j} - 1_{\leq j-1}, \quad \text{for } j \geq 2. \quad (9)$$

Then they define the ultraviolet cutoff as a maximal slice index $j_{\text{max}}$ so that the previous $N$ roughly corresponds to $M^{j_{\text{max}}}$.

The intermediate field representation (known in quantum field theory as the Hubbard-Stratonovich transformation) decomposes the quartic interaction using intermediate real scalar fields.
Definition 1. Let us define the operator $\vec{\sigma}$ as in [19];

$$\vec{\sigma} = \sigma_1 \otimes I_2 \otimes I_3 + I_1 \otimes \sigma_2 \otimes I_3 + I_1 \otimes I_2 \otimes \sigma^3.$$  \hspace{1cm} (10)

The renormalized partition function with ultraviolet cutoff is [19]

$$Z(g, j_{\max}) = \int d\nu(\vec{\sigma}) e^{-\text{Tr} \log \left[ 1 - i\lambda C^{1/2} \vec{\sigma} C^{1/2} \right]} + i\lambda \sum_c \sum_{n_c} A(n_c) \sigma^c_{n_c} + D + E,$$ \hspace{1cm} (11)

with

$$A(n_c) = \sum_{p \in \mathbb{Z}^3} \frac{n_c^2}{(n_c^2 + p^2 + 1)(p^2 + 1)} \leq O(1) \log(1 + |n_c|),$$ \hspace{1cm} (12)

$$D \equiv \sum_c \delta V_1^c + \delta V_2^c + \delta V_{bm}^c = \frac{\lambda^2}{2} \sum_c \sum_{n_c} A^2(n_c),$$ \hspace{1cm} (13)

$$E \equiv \sum_c \delta V_2^c = \frac{\lambda^2}{2} \int d\nu(\vec{\sigma}) \text{Tr}(C\vec{\sigma})^2.$$ \hspace{1cm} (14)

Remark that this renormalized partition function of the $T_4^3$ model (contrary to the $\phi_4^2$ case) remains positive for $g$ or $\lambda$ real positive.

It is proved in [19] that

Theorem 1. Fix $\rho > 0$ small enough. The series for $\log Z(g, j_{\max})$ is absolutely and uniformly in $j_{\max}$ convergent for $g$ in the small open cardioid domain $\text{Card}_\rho$ defined by $|g| < \rho \cos[(\text{Arg} g)/2]$. Its ultraviolet limit $\log Z(g) = \lim_{j_{\max} \to \infty} \log Z(g, j_{\max})$ is therefore well-defined and analytic in that cardioid domain; furthermore it is the Borel sum of its perturbative series in powers of $g$.

Remark 1. That domain, where considered in $g$, is a cardioid, see Figure 2.

Furthermore it is proved in [19] that the first order term in $g$ cancels exactly in this $\sigma$ representation of $\log Z(g, j_{\max})$. Therefore as a corollary the following statement on $Z(g, j_{\max})$ is correct:

Corollary 1.

$$Z(g, j_{\max}) = 1 + O_1(g) = 1 + O_1\left(\frac{\lambda^2}{2}\right),$$ \hspace{1cm} (15)

$$O_1(g) = O_1\left(\frac{\lambda^2}{2}\right) \lesssim |\lambda^2| \leq K_1 |\lambda^2|,$$ \hspace{1cm} (16)

where we recall that $a \lesssim b$ means that there is a constant $K_1 > 0$ such that $a \leq K_1 b$. 

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Remark 2. It is natural to exchange $g$ or $\lambda$ depending on context because the intermediate field representation is written according to $\lambda$.

Enough with [19]. The renormalized partition function with tensor sources $\bar{J}, J$ writes:

$$ Z(g, j_{\text{max}}, J) = e^{i\lambda P_{c} P_{nc} A(n_{c})} e^{D + E}. $$

(18)

Definition 2. For all $k \geq 1$, one defines the cumulant of order $k$ by the following relation:

$$ K_{k}^{j_{\text{max}}} (g, \{a, \bar{a}\}) := \left[ \frac{\partial^{2k}}{\partial J_{a_{1}} \cdots \partial J_{\bar{a}_{k}} \partial J_{a_{1}} \cdots \partial J_{\bar{a}_{k}}} \log Z(g, j_{\text{max}}, J) \right]_{\{J\} = 0}. $$

(19)

Let us fix $k_{\text{max}}$. Our main result extend those of [19] to the cumulants of order $k$ with $k \leq k_{\text{max}}$.

Theorem 2. Let $1 \leq k \leq k_{\text{max}}$ and $\|(a, \bar{a})\|$ bounded by $K$. Let $g \in \mathbb{C}$, $g = |g|e^{i\gamma}$ be in the domain

$$ |g| < \rho \cos[\gamma/2], \quad \rho = \frac{1}{5}. $$

(20)
Then the series \(19\) is absolutely and uniformly in \(j_{\max}\) convergent in that domain. Its ultraviolet limit

\[
\lim_{j_{\max} \to \infty} \mathcal{R}^k_{j_{\max}}(g, \{a, \bar{a}\}) = \mathcal{R}^k(g, \{a, \bar{a}\})
\]

is therefore well-defined and analytic in that domain; furthermore it is the Borel sum of its perturbative series in powers of \(g\).

The proof of this theorem is given in Section \(3\).

**Remark 3.** The norm used in Theorem \(2\) is the Hermitian norm

\[
\|(a, \bar{a})\| = \sqrt[3]{\sum_{j=1}^{k} \|a_j\|}.
\]

**3 Proof of Theorem \(2\)**

Let us now come to the heart of the proof. We define \(n\) as in (2) and \(C(n, \bar{n})\) and \(C(n)\) as in (3). Hence

\[
C = C(n), \quad C(n) = \frac{1}{n} = \frac{1}{n_1^2 + n_2^2 + n_3^2 + 1}.
\]

**Definition 3.** We define

\[
R = [\mathbb{I} - i\lambda C^{1/2}\bar{\sigma}C^{1/2}]^{-1},
\]

\[
O_2(\lambda^2) = [\mathbb{I} - i\lambda C^{1/2}\bar{\sigma}C^{1/2}]^{-1} - e^{i\lambda C^{1/2}\bar{\sigma}C^{1/2}}.
\]

Hence

\[
\|O_2(\lambda^2)\| \lesssim \lambda^2 \leq K_2 \lambda^2.
\]

**Proposition 1.**

\[
CR = C[\mathbb{I} - i\lambda C^{1/2}\bar{\sigma}C^{1/2}]^{-1} = \frac{\delta_{n, \bar{n}} e^{i\lambda C^{1/2}\bar{\sigma}C^{1/2}}}{n} + O_3(\lambda^2)
\]

with

\[
\|O_3(\lambda^2)\| \leq \|\frac{\delta_{n, \bar{n}}}{n} [O_1(\frac{\lambda^2}{2}) + O_2(\lambda^2)]\| \leq \|\frac{\delta_{n, \bar{n}}}{n} \|\| (K_1 + K_2) \lambda^2 \| \cdot (28)
\]
Proof. Let us define $O_1(\lambda^2)$ as in (15)-(16) and $O_2(\lambda^2)$ as in (25)-(26). Then

$\begin{align*}
[\mathbb{I} - i\lambda C^{1/2}\sigma C^{1/2}]^{-1} &= [\mathbb{I} + i\lambda C^{1/2}\sigma C^{1/2}] + O_4(\lambda^2), \\
e^{i\lambda C^{1/2}\sigma C^{1/2}} &= [\mathbb{I} + i\lambda C^{1/2}\sigma C^{1/2}] + O_5(\lambda^2).
\end{align*}$

Therefore

$\begin{align*}
\frac{\delta_{n,\bar{n}}}[1 + \lambda C^{1/2}\sigma C^{1/2}] &= \frac{\delta_{n,\bar{n}}}{n} e^{i\lambda C^{1/2}\sigma C^{1/2}} + O_3(\lambda^2), \\
\frac{\delta_{n,\bar{n}}}{n} &\leq \frac{\delta_{n,\bar{n}}}{n} \|O_4(\lambda^2) - O_5(\lambda^2)\|, \\
\|O_3(\lambda^2)\| &\leq \|\frac{\delta_{n,\bar{n}}}{n}\|(K_1 + K_2)\lambda^2\|.
\end{align*}$

Definition 4. We define

$\begin{align*}
3^k_{j_{\max}}(g) &:= \int d\nu(\sigma) \left[ \frac{\delta_{n,\bar{n}} e^{i\lambda C^{1/2}\sigma C^{1/2}}}{n^k} \right] e^{-\text{Tr} \log [\mathbb{I} - i\lambda C^{1/2}\sigma C^{1/2}]}, \\
z_0 &:= 3_{j_{\max}}^0(g) - 1 = Z(g, j_{\max}) - 1, \\
z_k &:= k! 3^k_{j_{\max}}(g) \text{ for } k \geq 1, \\
3^k_{j_{\max}}(J, \bar{J}) &:= \frac{(J_aJ_{\bar{a}})^k}{[k!]^2}, \\
3_k(a, \bar{a}) &:= \frac{1}{k!} \left\{ \frac{\partial^{2k}}{\partial J_a \partial \bar{J}_{\bar{a}} \partial J_{a_1} \partial \bar{J}_{\bar{a}_1} \cdots \partial J_{a_k} \partial \bar{J}_{\bar{a}_k}} \left[ 3^k_{j_{\max}}(J, \bar{J}) \right] \right\}_{\{J\}=0}.
\end{align*}$

It is clear that $3^0_{j_{\max}}(J, \bar{J}) = 3_0(a, \bar{a}) = 1$. Hence by appealing to Proposition 1 and Definition 4 we can write:

$\begin{align*}
\log Z(g, j_{\max}, J) &= \log \left[ \sum_{k=0}^{k_{\max}} 3^k_{j_{\max}}(g) 3^k_{j_{\max}}(J, \bar{J}) \right] \\
&= \log \left[ 1 + z_0 + \sum_{k=1}^{k_{\max}} z_k 3_k(a, \bar{a}) \right].
\end{align*}$
Proposition 2. Suppose \( \|(a, \bar{a})\| \) is bounded by \( K \). Then
\[
\| z_k(a, \bar{a}) \| \leq K. \tag{41}
\]

Proof. Using [4], we write for \( k \geq 1 \):
\[
\left\| \frac{\partial^{2k}}{\partial J_{a_1} \ldots \partial J_{a_k} \partial \bar{J}_{a_1} \ldots \partial \bar{J}_{a_k}} \left[ k!^3 \right] J=0 \right\| = \frac{1}{k!} \sum_{\pi \in \Theta_k} \prod_{(a_i, \bar{a}_j) \in \pi} \delta_{a_i, \bar{a}_j} \tag{42}
\]
\[
\leq \prod_{(a_i, \bar{a}_j) \in \{1, \ldots, k\}} \delta_{a_i, \bar{a}_j} \tag{43}
\]
\[
\leq \prod_{(a_i, \bar{a}_j) \in \{1, \ldots, k_{\text{max}}\}} \delta_{a_i, \bar{a}_j} \tag{44}
\]
\[
\leq K. \tag{45}
\]

Therefore taken into account Proposition 1, (40) is expressed only in \( z_k = Z_{k, \text{max}}(g) \) for any \( 0 \leq k \leq k_{\text{max}} \). In short we can write
\[
\log \left[ 1 + z_0 + \sum_{k=1}^{k_{\text{max}}} z_k \delta_k(a, \bar{a}) \right] \leq \log \left[ 1 + z_0 + K \sum_{k=1}^{k_{\text{max}}} z_k \right]. \tag{46}
\]

Proposition 3. Suppose \( \| z_0 \| \leq \frac{1}{2} \) and, for \( 1 \leq k \leq k_{\text{max}} \)
\[
\| z_k \| \leq \frac{1}{K} \left[ \frac{1}{4} \right]^k. \tag{47}
\]

Recalling Definition 2-3-4 we can write:
\[
\| R_{\text{max}}^k (g, \{a, \bar{a}\}) \| \leq \frac{1}{5}. \tag{48}
\]

Proof. Recalling the analyticity of the power series of the log and (40)
\[
\| R_{\text{max}}^k (g, \{a, \bar{a}\}) \| = \| \log \left[ 1 + z_0 + \sum_{k=1}^{k_{\text{max}}} z_k \delta_k(a, \bar{a}) \right] \| \tag{49}
\]
\[
\leq \| \log \left[ 1 + z_0 + K \sum_{k=1}^{k_{\text{max}}} z_k \right] \| \tag{50}
\]
\[
\leq \sum_{m=1}^{\infty} \frac{1}{m} \left[ \frac{1}{2} - \sum_{k=1}^{k_{\text{max}}} \frac{1}{4k} \right]^m \tag{51}
\]
\[
\leq \sum_{m=1}^{\infty} \frac{1}{m} \left[ \frac{1}{2} - \frac{1}{3} \right]^m = \sum_{m=1}^{\infty} \frac{1}{m} \left[ \frac{1}{6} \right]^m \leq \frac{1}{5}. \tag{52}
\]
Proposition 4.  

- Provided $K_1 |\lambda^2| \leq \frac{1}{2}$ is true, $\|z_0\| \leq \frac{1}{2}$.

- Provided, for $1 \leq k \leq k_{\text{max}}$,

$$\|(K_1 + K_2)\lambda^2\| = [k_{\text{max}}!] \frac{1}{K} \left[\frac{1}{4}\right]^k \tag{53}$$

is true, $\|z_k\| \leq \frac{1}{K} \left[\frac{1}{4}\right]^k$ is true.

Proof. By Corollary 1

$$\|z_0\| \leq \|Z(g, j_{\text{max}}) - 1\| \leq K_1 |\lambda^2| \leq \frac{1}{2}. \tag{54}$$

Then we turn to the second part.

$$1 \leq j_{\text{max}} = 1 + n_1^2 + n_2^2 + n_3^2 \leq M^{2j_{\text{max}}}, \tag{55}$$

so

$$1 \leq 1 + n_1^2 + n_2^2 + n_3^2 \leq M^{2j_{\text{max}}}, \tag{56}$$

therefore for $k \geq 1$:

$$\frac{1}{[M^{2j_{\text{max}}}]^k} \leq \frac{1}{[1 + n_1^2 + n_2^2 + n_3^2]^k} \leq 1, \tag{57}$$

and $\delta_{n, \bar{n}} = 1$, hence

$$\frac{1}{[M^{2j_{\text{max}}}]^k} \leq \|\delta_{n, \bar{n}}\| \leq 1.$$  

By Definition 4 and Proposition 1 if $\|(K_1 + K_2)\lambda^2\| \leq [k_{\text{max}}!] \frac{1}{K} \left[\frac{1}{4}\right]^k$, so provided

$$\lambda^2 = 2g \leq [k_{\text{max}}!] \frac{1}{(K_1 + K_2)} \frac{1}{K} \left[\frac{1}{4}\right]^k, \tag{58}$$

then

$$\|z_k\| \leq [k_{\text{max}}!] \int d\nu(\bar{\sigma}) \left\|\frac{\delta_{n, \bar{n}}}{n^k}\right\| \left\|e^{i\lambda C^{1/2} \partial C^{1/2}}\right\| \left\|e^{-\text{Tr} \log[I - i\lambda C^{1/2} \partial C^{1/2}]}\right\| \tag{59}$$

$$\leq \frac{1}{K} \left[\frac{1}{4}\right]^k. \tag{60}$$

Let us come finally to the proof of Theorem 2.
Proof. For the series (19), hence for the analyticity uniformly with respect to \( j_{\text{max}} \) of the cumulants \( R_{j_{\text{max}}}^k (g, \{ a, \bar{a} \}) \), we rely on Propositions 1-3 and on a statement that a convergent series depending uniformly on a parameter \( j_{\text{max}} \in [K, \infty[ \) is uniformly analytic with respect to that parameter.

For analyticity part of the ultraviolet limit of the cumulants

\[
\lim_{j_{\text{max}} \to \infty} R_{j_{\text{max}}}^k (g, \{ a, \bar{a} \}) = R^k (g, \{ a, \bar{a} \})
\]

we rely on the following statement: if a analytic function \( R_{j_{\text{max}}} \) depend uniformly on a parameter \( j_{\text{max}} \in [K, \infty[ \), it is analytic with respect to the limit \( j_{\text{max}} \to \infty \) of that parameter.

For the Borel summability in powers of \( g \) of \( R^k (g, \{ a, \bar{a} \}) \) we rely on the following argument: for all \( 1 \leq k \leq k_{\text{max}} \) and for all \( \{ a, \bar{a} \} \in B(K) \), there exists two constants \( K_3 \) and \( \sigma \) independent of \( k_{\text{max}} \) and \( \{ a, \bar{a} \} \) such that the Taylor rest term of order \( r \) of the cumulants \( R^k (g, \{ a, \bar{a} \}) \), denoted by \( R_{\{ a, \bar{a} \}, r}^k (g) \) obeys the following bound:

\[
| R_{\{ a, \bar{a} \}, r}^k (g) | \leq K_3 \sigma^r r! |g|^r.
\]

This bound prove that the cumulants verify the hypotheses of the Nevanlinna-Sokal theorem.

So the Theorem 2 is proved. \( \square \)

4 Appendix A

No paper about the LVE should forget an appendix about the BKAR formula [9, 10] since this formula is crucial to the LVE. Any quantity \( F \) in quantum field theory which is an integral over a Gaussian complex measure can be combinatorially represented as a sum over the set \( \mathcal{F} \) of oriented forests. For readers who want to look further into BKAR formula and oriented forests, ordered or not, see [12, 35]. We recalled the BKAR formula in the notations of [35].

Lemma 1 (BKAR formula). The Taylor BKAR formula for oriented forests...
\( F(\mathcal{M}) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mathcal{F} \in \mathfrak{F}_n} \int dw_{\mathcal{F}} \partial_{\mathcal{F}} \int d\mu_{C(x_{ij}^\mathcal{F})}(\mathcal{M}) F_n(\mathcal{M}) \bigg|_{x_{ij}=x_{ij}^\mathcal{F}(w)}, \) \quad (63)

where \( \int dw_{\mathcal{F}} := \prod_{(i,j) \in \mathcal{F}} \int_0^1 dw_{ij}, \quad \partial_{\mathcal{F}} := \prod_{(i,j) \in \mathcal{F}} \frac{\partial}{\partial x_{ij}}, \) \quad (64)

\[ x_{ij}^\mathcal{F}(w) := \begin{cases} \inf_{(k,l) \in \mathcal{P} \overset{\mathcal{F}}{\leftrightarrow} i,j} w_{kl} & \text{if } \mathcal{P} \overset{\mathcal{F}}{\leftrightarrow} i,j \text{ exists,} \\ 0 & \text{if } \mathcal{P} \overset{\mathcal{F}}{\leftrightarrow} i,j \text{ does not exist.} \end{cases} \] \quad (65)

In this formula \( w_{ij} \) is the weakening parameter of the edge \((i, j)\) of the forest, and \( \mathcal{P} \overset{\mathcal{F}}{\leftrightarrow} i,j \) is the unique path in \( \mathcal{F} \) joining \( i \) and \( j \) when it exists.

**Proof.** See [9, 10, 12, 35]. Oriented forests simply distinguish edges \((i, j)\) and \((j, i)\), so we have edges with arrows. It allows to distinguish between operators \( \frac{\partial}{\partial \mathcal{M}_i^\dagger} \frac{\partial}{\partial \mathcal{M}_j} \) and \( \frac{\partial}{\partial \mathcal{M}_j^\dagger} \frac{\partial}{\partial \mathcal{M}_i} \). Remember that a main property of the forest formula is that the symmetric \( n \times n \) matrix \( C\{x_{ij}^\mathcal{F}\} = x_{ij}^\mathcal{F}(w) + x_{ji}^\mathcal{F}(w) \) is positive for any value of \( w_{kl} \), hence the Gaussian measure \( d\mu_{C(x_{ij}^\mathcal{F})}(\mathcal{M}) \) is well-defined. \( \square \)

## 5 Appendix B

We recall the Nevanlinna-Sokal theorem [36, 37, 2]. Here we follow the notations of [4], Appendix C, with the following important modification: \( N \), who is in [4], Appendix C, a simple integer, is replaced by \( N \in [K, \infty[ \times B(K) \) where \( B(K) \in \mathbb{C}^{2k_{\max}} \) is simply the ball of radius \( K \).

**Theorem 3.** A function \( f(\lambda, N) \) with \( \lambda \in \mathbb{C} \) and \( N \in [K, \infty[ \times B(K) \) is said to be Borel summable in \( \lambda \) uniformly in \( N \) if:

- \( f(\lambda, N) \) is analytic in a disk \( \Re(\lambda^{-1}) > (2R)^{-1} \) with \( R \in \mathbb{R}_+ \) independent of \( N \).

- \( f(\lambda, N) \) admits a Taylor expansion at the origin with uniform bound on the Taylor remainder:

\[
f(\lambda, N) = \sum_{k=0}^{r-1} f_{N,k} \lambda^k + R_{N,r}(\lambda), \quad |R_{N,r}(\lambda)| \leq K \sigma^r r! |\lambda|^r, \quad (66)
\]

\(^2\)For Borel-LeRoy modifications, see [38].
for some constants $K$ and $\sigma$ independent of $\mathcal{N}$.

If $f(\lambda, \mathcal{N})$ is Borel summable in $\lambda$ uniformly in $\mathcal{N}$ then:

$$B(t, \mathcal{N}) = \sum_{k=0}^{\infty} \frac{1}{k!} f_{\mathcal{N},k} t^k,$$

is an analytic function for $|t| < \sigma^{-1}$ that admits an analytic continuation in the strip $\{z| |\Im z| < \sigma^{-1}\}$ such that $|B(t, \mathcal{N})| \leq B e^{t/R}$ for some constant $B$ independent of $\mathcal{N}$ and $f(\lambda, \mathcal{N})$ is given by the absolutely convergent integral:

$$f(\lambda, \mathcal{N}) = \frac{1}{\lambda} \int_{0}^{\infty} dt B(t, \mathcal{N}) e^{-\frac{t}{\lambda}}.$$  

In other words, the Taylor expansion of $f(\lambda, \mathcal{N})$ at the origin is Borel summable, and $f(\lambda, \mathcal{N})$ is its Borel sum.

For applying the Nevanlinna-Sokal theorem to the results of this paper, simply put $\lambda = \sqrt{2g}$, $2R = \rho^{k_{\max}}$, $\mathcal{N} = \{j_{\max}, \{a, \bar{a}\}\} \in [K, \infty] \times \mathcal{B}(K)$.

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