An inequality for the number of vertices with an interval spectrum in edge labelings of regular graphs

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Abstract

We consider undirected simple finite graphs. The sets of vertices and edges of a graph \(G\) are denoted by \(V(G)\) and \(E(G)\), respectively. For a graph \(G\), we denote by \(\delta(G)\) and \(\eta(G)\) the least degree of a vertex of \(G\) and the number of connected components of \(G\), respectively. For a graph \(G\) and an arbitrary subset \(V_0 \subseteq V(G)\) \(G[V_0]\) denotes the subgraph of the graph \(G\) induced by the subset \(V_0\) of its vertices. An arbitrary nonempty finite subset of consecutive integers is called an interval. A function \(\varphi : E(G) \rightarrow \{1, 2, \ldots, |E(G)|\}\) is called an edge labeling of the graph \(G\), if for arbitrary different edges \(e' \in E(G)\) and \(e'' \in E(G)\), the inequality \(\varphi(e') \neq \varphi(e'')\) holds. If \(G\) is a graph, \(x\) is its arbitrary vertex, and \(\varphi\) is its arbitrary edge labeling, then the set \(S_G(x, \varphi) \equiv \{\varphi(e)/e \in E(G), e\text{ is incident with }x\}\) is called a spectrum of the vertex \(x\) of the graph \(G\) at its edge labeling \(\varphi\). If \(G\) is a graph and \(\varphi\) is its arbitrary edge labeling, then \(V_{int}(G, \varphi) \equiv \{x \in V(G)/ S_G(x, \varphi)\text{ is an interval}\}\). For an arbitrary \(r\)-regular graph \(G\) with \(r \geq 2\) and its arbitrary edge labeling \(\varphi\), the inequality

\[
|V_{int}(G, \varphi)| \leq \left\lfloor \frac{3 \cdot |V(G)| - 2 \cdot \eta(G[V_{int}(G, \varphi)])}{4} \right\rfloor.
\]

is proved.

Keywords: edge labeling, interval spectrum, regular graph, cubic graph.
Math. Classification: 05C15, 05C78

We consider undirected simple finite graphs. The sets of vertices and edges of a graph \(G\) are denoted by \(V(G)\) and \(E(G)\), respectively. For a graph \(G\), we denote by \(\delta(G)\) the least degree of a vertex of \(G\). For any graph \(G\) we define a parameter \(\eta(G)\) by the following way: if \(G\) is empty then \(\eta(G) \equiv 0\), otherwise \(\eta(G)\) is equal to the number of connected components of \(G\). If \(G\) is a graph, \(x \in V(G), y \in V(G)\), then \(\rho_G(x, y)\) denotes the distance between the vertices \(x\) and \(y\) in \(G\). If \(G\) is a graph, \(x \in V(G)\), and \(V_0 \subseteq V(G)\), then \(\rho_G(x, V_0)\) denotes the distance in the graph \(G\) between its vertex \(x\) and the subset \(V_0\) of its vertices. For a graph \(G\) and an arbitrary subset \(V_0 \subseteq V(G)\) \(G[V_0]\) denotes the subgraph of the graph \(G\) induced by the subset \(V_0\) of its vertices.
For any graph $G$ and its arbitrary subgraph $H$, let us define the subgraph $Surr[HinG]$ of the graph $G$ as follows:

$$V(Surr[HinG]) \equiv \{ x \in V(G) \mid \rho_G(x, V(H)) \leq 1 \},$$

$$E(Surr[HinG]) \equiv E(H) \cup \{ (x, y) \in E(G) \mid x \in V(Surr[HinG]) \setminus V(H), y \in V(H) \}.$$

An arbitrary nonempty finite subset of consecutive integers is called an interval. A function $\phi : E(G) \to \{1, 2, \ldots, |E(G)|\}$ is called an edge labeling of the graph $G$, if for arbitrary different edges $e' \in E(G)$ and $e'' \in E(G)$, the inequality $\varphi(e') \neq \varphi(e'')$ holds. For a graph $G$, the set of all its edge labelings is denoted by $\tau(G)$.

If $G$ is a graph, $x \in V(G), \varphi \in \tau(G)$, then the set $S_G(x, \varphi) \equiv \{ \varphi(e)/e \in E(G), e \text{ is incident with } x \}$ is called a spectrum of the vertex $x$ of the graph $G$ at its edge labeling $\varphi.$ If $G$ is a graph, $\varphi \in \tau(G)$, then $V_{int}(G, \varphi) \equiv \{ x \in V(G) \mid S_G(x, \varphi) \text{ is an interval} \}.$ The terms and concepts which are not defined can be found in [1].

An upper bound for the cardinality of the set $V_{int}(G, \varphi)$ is obtained in that cases when $G$ is a regular graph with $\delta(G) \geq 2$ and $\varphi \in \tau(G)$.

First we recall the following

**Proposition 1.** [2] Let $G$ be a graph with $\delta(G) \geq 2$. Let $\varphi \in \tau(G)$ and $V_{int}(G, \varphi) \neq \emptyset$. Then $G[V_{int}(G, \varphi)]$ is a forest, each connected component of which is a simple path.

**Theorem 1.** If $G$ is a $r$-regular graph, $r \geq 2$, $\varphi \in \tau(G)$, then

$$|V_{int}(G, \varphi)| \leq \left[ \frac{r \cdot |V(G)| - 2 \cdot \eta(G[V_{int}(G, \varphi)])}{2 \cdot (r - 1)} \right].$$

**Proof.** Let $\eta(G[V_{int}(G, \varphi)]) = k$.

**Case 1.** $V_{int}(G, \varphi) = \emptyset$.

In this case the required inequality is the following evident one:

$$0 \leq \left[ \frac{r \cdot |V(G)|}{2 \cdot (r - 1)} \right].$$

**Case 2.** $V_{int}(G, \varphi) \neq \emptyset$.

In this case $k \geq 1$. Since $\delta(G) = r \geq 2$, then, by the proposition [1] $G[V_{int}(G, \varphi)]$ is a forest with $k$ connected components, each of which is a simple path.

Let $P_1, \ldots, P_k$ be all connected components of the forest $G[V_{int}(G, \varphi)]$.

It is not difficult to see that for $\forall i, 1 \leq i \leq k$, the equality $|E(Surr[P_i \cap G])| = (r - 1) \cdot |V(P_i)| + 1$ holds.

Let us also note that (if $k \geq 2$) for arbitrary integers $i'$ and $i''$ satisfying the inequality $1 \leq i' < i'' \leq k$, the relation $E(Surr[P_{i'} \cap G]) \cap E(Surr[P_{i''} \cap G]) = \emptyset$ holds.

Taking into account the evident relation $(\bigcup_{i=1}^{k} E(Surr[P_i \cap G])) \subseteq E(G)$, we obtain

$$|E(G)| = \frac{r \cdot |V(G)|}{2} \geq \left| \bigcup_{i=1}^{k} E(Surr[P_i \cap G]) \right| = \sum_{i=1}^{k} |E(Surr[P_i \cap G])| =$$

$$= \sum_{i=1}^{k} ((r - 1) \cdot |V(P_i)| + 1) = k + (r - 1) \cdot \sum_{i=1}^{k} |V(P_i)| = k + (r - 1) \cdot |V_{int}(G, \varphi)|,$$
\[
|V_{\text{int}}(G, \varphi)| \leq \frac{1}{r-1} \cdot \left( \frac{r \cdot |V(G)|}{2} - k \right) = \frac{r \cdot |V(G)| - 2k}{2 \cdot (r-1)}.
\]

Consequently,
\[
|V_{\text{int}}(G, \varphi)| \leq \left\lfloor \frac{r \cdot |V(G)| - 2k}{2 \cdot (r-1)} \right\rfloor.
\]

The Theorem is proved.

Corollary 1. If \( G \) is a \( r \)-regular graph, \( r \geq 2 \), \( \varphi \in \tau(G) \), then
\[
|V_{\text{int}}(G, \varphi)| \leq \left\lfloor \frac{r \cdot |V(G)| - 2}{2 \cdot (r-1)} \right\rfloor.
\]

Corollary 2. If \( G \) is a cubic graph, \( \varphi \in \tau(G) \), then
\[
|V_{\text{int}}(G, \varphi)| \leq \left\lfloor \frac{3 \cdot |V(G)| - 2 \cdot \eta(G[V_{\text{int}}(G, \varphi)])}{4} \right\rfloor.
\]

Corollary 3. If \( G \) is a cubic graph, \( \varphi \in \tau(G) \), then
\[
|V_{\text{int}}(G, \varphi)| \leq \left\lfloor \frac{3 \cdot |V(G)| - 2}{4} \right\rfloor.
\]

References

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