Principles of Discrete Time Mechanics: V. The Quantisation of Maxwell’s Equations

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Abstract

Principles of discrete time mechanics are applied to the quantisation of Maxwell’s equations. Following an analysis of temporal node and link variables, we review the classical discrete time equations in the Coulomb and Lorentz gauges and conclude that electro-magneto duality does not occur in pure discrete time electromagnetism. We discuss the role of boundary conditions in our mechanics and how temporal discretisation should influence very early universe dynamics. Quantisation of the Maxwell potentials is approached via the discrete time Schwinger action principle and the Faddeev-Popov path integral. We demonstrate complete agreement in the case of the Coulomb gauge, obtaining the vacuum functional and the discrete time field commutators in that gauge. Finally, we use the Faddeev-Popov method to construct the discrete time analogues of the photon propagator in the Landau and Feynman gauges, which casts light on the break with relativity and possible discrete time analogues of the metric tensor.

Throughout this paper the acronym CT refers to continuous time, whereas DT refers to discrete time. Readers familiar with the principles and methodology discussed in the earlier papers of this series may skip the introduction, but are advised that the general notation has been improved and is discussed in section 2.

1 Introduction

The development of quantum field theory in the second quarter of the twentieth century was accompanied by speculation about the microscopic nature of time and space, motivated by the spectacle of quantised fields dynamically evolving over a classical space-time in no way different to the bland Riemannian space-time continuum used by Einstein in general relativity. Various problems in quantum field theory such as the divergences in the renormalisation programme and ambiguities in operator products were believed to be associated in some way with the microscopic description of space-time, but relatively little was done to investigate the issue in any depth. Although there were occasional attempts to investigate alternative mathematical descriptions of space-time, such as the notable work of Snyder [1, 2], it was
more usual to circumvent difficulties with ad hoc procedures such as point splitting of operator products with no modification of the underlying space-time, or by the use of space-time lattices, which were always assumed to be an approximation to the continuum. It was only with the advent of the space-time foam approach to quantum gravity and the reinterpretation of superstring theory in the nineteen eighties that it became generally acceptable to talk about the microscopic nature of space-time as more than likely quite different to the continuum normally assumed in field theory. It is in this context that our work should be seen.

In the current series of paper on $DT$ mechanics [3-6] we investigate the consequences of taking literally the hypothesis that time is discrete on an incredibly small scale. Our original motivation is discussed in the first paper of this series [3]. Compared with quantum gravity and superstring theory, ours is a very modest and limited step. However, it turns out to have enormous consequences, principally conceptual, altering virtually all aspects of the laws of mechanics and how we view space-time. Along the way a number of sacred cows have to be sacrificed. For instance, without continuity with respect to time there is no differentiation with respect to time. Therefore we are forced into the construction of a mechanics without velocities. This immediately raises the question of what replaces Lagrangians, which are normally functions of dynamical variables and their temporal derivatives. Since we have no velocities in our theory, we cannot construct conjugate momenta in the traditional way, as these are defined as derivatives of Lagrangians with respect to velocities. So we appear not to have a phase space, and consequently we do not have a Hamiltonian formulation or Poisson brackets in the normal sense of the word. This goes hand in hand with the lack of continuous translations in time and with the absence of a generator of such transformations. This raises questions about quantisation, but we have shown in earlier papers of this series that these can answered. The price we pay for this is that we end up no longer doing exactly what we were doing before, amounting to a paradigm shift in the language of Thomas Kuhn.

We regard it as an important principle that we do not simply modify $CT$ equations of motion by replacing temporal derivatives with ad hoc differences. Although that works in some situations in Newtonian mechanics, it becomes more subtle in the presence of gauge invariance. Our approach is to start from the beginning, rewriting the action integral as an action sum and developing the consequences rigorously from there. In this series of papers we have not confined our interest to particular models which happen to be amenable to temporal discretisation. The microscopic nature of space and time will affect all dynamical processes. Our interest has been in the subject of $DT$ mechanics as a whole, in both its classical and quantised forms, applied to point and field systems.

An important consequence of temporal discretisation is that it rejects the notion that space and time form a four dimensional continuum. This notion has been successfully exploited in the special and general theories of relativity throughout this century and is a cornerstone of the theories of quantum gravity and superstring theory (which paradoxically eventually undermine this very idea). We appear to have taken a step back towards the separation of absolute space and time in Newtonian mechanics. Discretising time also raises questions any relativist would ask, which are: in which inertial frame is time discrete and what dictates this choice? Our discrete time mechanics is not Lorentz covariant, and we have to address this issue as well as others.

We believe that there are satisfactory answers to these particular questions which accord
with modern cosmology. Consider a CT Freidmann-Robertson-Walker space-time, i.e., one for which CT co-ordinates can be chosen so that the Riemannian pseudo-metric distance rule takes the standard form

\[ ds^2 = dt^2 - R(t)^2 d\sigma^2. \]  

Here \( R(t) \) is a function of the co-moving time \( t \) only and \( d\sigma \) is the distance element of some spherical, flat, or hyperbolic 3-space. Given that the gross space-time structure of the expanding universe is well represented by such a choice of co-ordinates and by such a distance rule, consider now the cosmic microwave background radiation (CMB) field discovered by Penzias and Wilson in 1964 [7]. It has been pointed out by a number of authors [8, 9], that, contrary to the principles of relativity, this radiation field can be used to define a local absolute inertial frame \( F_P \) at each point \( P \) in space-time. Such a frame is unique up to Euclidean transformations such as rotations. For an observer at \( P \) instantaneously at rest relative to \( F_P \), the CMB radiation field will appear isotropic to a very high degree. That this is physically meaningful is borne out by the empirical observation that the earth appears to be moving at a speed of about \( 500 - 600 \) km/sec relative the local \( F_P \) frame, a phenomenon called the dipole effect [10]. It is somewhat ironic that long after the Michelson-Morley experiment failed to detect an aether carrying radiation (and thereby supporting the principle of special relativity), Penzias and Wilson discovered a plenum consisting entirely of radiation which may be used to define absolute local inertial frames.

Our thinking is as follows. Suppose we take seriously the hypothesis that time is really discrete. Our fundamental criterion is that any dynamics based on this idea should not make predictions at odds with scientifically determined (i.e. empirical) facts. We shall call this the empirical principle. It has been part of our programme to determine where if anywhere temporal discretisation actually clashes with this principle. So far, we have not been able to rule \( DT \) mechanics out on this basis.

Always mindful of this principle, we should now carefully sift out and identify those additional concepts and ideas which, despite our traditional training and inclination, are really no more than contemporary belief structures which happen to be compatible with the empirical principle. We shall call such ideas idealisations. We should feel free to dispense with idealisations if absolutely necessary, provided we do not clash with the empirical principle. It is particularly important to identify idealisations which are currently popular because of their elegance and mathematical content; these are usually the hardest to remove, because mathematical elegance is frequently taken as a principle in physics.

A good example of such an idealisation is the Poincaré group. This particular structure cannot occur in our theory because we do not have the freedom to make continuous displacements in time; this property emerges only in the continuous time limit. What replaces the Poincaré group in \( DT \) mechanics is still under investigation. What can be said however is that any \( DT \) analogue will almost certainly be more complicated mathematically. The absence of the Poincaré group however does not unduly worry us here and should not be used as a criticism, because our mechanics can still satisfy the empirical principle for the following reason. In our \( DT \) mechanics there is a fundamental interval of time \( T \) and generally, we can show agreement with \( CT \) theory certainly at the \( O(T^0) \) level. Moreover, disagreement invariably occurs at the \( O(T^2) \) level. Since we imagine \( T \) is of the order of the Planck time \( T_P \equiv \sqrt{\hbar G/c^5} \simeq 5 \times 10^{-44} \) sec, our theory should be good for the current
level of experimental accuracy. As yet we have no explanation of the value of $T$, but then we cannot explain the value of $c$, $\hbar$, $G$ or the electric charge either.

Continuing this line of thought, we argue that temporal discreteness would have influenced both the dynamical origin of the universe and its subsequent evolution. The Planck epoch is a term used to denote the interval from the origin of time to $T_P$. By the end of that interval, it is generally believed that gravity had decoupled from the other interactions and space started to expand in a pre-inflationary context. It is during the Planck epoch that $CT$ field theories are generally considered to be either invalid or seriously incomplete, and it is possible that some version of $DT$ mechanics (though not necessarily the one we are considering here) is the appropriate theory to describe very early universe dynamics. If true, then a generally covariant description in the fashion of general relativity might not be at all appropriate during this epoch, with the best description perhaps involving some preferred frame of reference.

It may be the case that after the Planck time and before false vacuum inflation, different regions of the Universe had different local temporal discretisation frames, randomly distributed, analogous to pre-inflationary early universe monopoles or ferromagnetic domains in a solid. If so, we would argue that in much the same way as the monopole problem was removed by inflation, only one actual temporal discretisation frame would survive inflation into our local universe. All the others would be beyond the event horizon.

The discretisation frame holding in our visible universe would most likely be linked to the dynamical processes involved with the phase transition from the false vacuum to radiation and matter, and we would expect it to leave a signature. This signature would be the existence of the co-moving frame used in the FRW metric (1), and subsequently, the isotropy frame of the $CMB$. We propose that the discrete time parameter discussed in this paper be identified as the discretised version of the local co-moving time (assumed to be the coordinate time in our local $F_P$) in our neighbourhood. We are assuming here that the $CMB$ isotropy frame and the cosmic matter rest frame coincide locally [9].

Now over laboratory scales associated with particle scattering experiments, we could ignore the local aspect of this discretisation and regard a global discrete time frame as a very good approximation, in much the same way that special relativity is a good approximation to general relativity in the laboratory. Hence we end up with the equivalent of a unique discretisation of Minkowski space-time in our neighbourhood.

A $DT$ analogue of general relativity awaits investigation; it will require abandoning the principle of general covariance on a microscopic level, with a generally covariant description emerging only in the $CT$ limit. It is possible that our fundamental interval $T$ is itself determined dynamically by the local matter densities and by the dynamics. There is a precedent for this idea. In 1983 Lee published a study of discrete time field theory [11], where his interval of time $T_n$ varied and was a dynamical variable coupled to matter. In our earlier work and in this paper, time is discrete but otherwise is as passive as time in $CT$ relativistic field theory. We work with a fixed $T$ throughout.

In this paper we develop further the discrete time Maxwell’s equations first discussed in [3, 4]. In the next section we review our notation, which has been somewhat overhauled and compactified compared to earlier papers in the series, but with no change in content. Following that, we review the $DT$ mechanics formalism, extending the discussion to include the dynamical variables associated with temporal links. Our view of discrete time mechanics
has evolved from our original picture of dynamical variables changing over successive instants of time (nodes) only to one where there are dynamical variables on the nodes and on the intervals or links between the nodes. A gauge theory such as Maxwell’s equations (and undoubtedly gravitation) involves a dynamical interplay between node variables and link variables.

Then we review the DT Maxwell’s equations in the Lorentz and Coulomb gauges. Working in the former gauge presents a particularly interesting challenge in DT mechanics because it is a relic of the Lorentz symmetry which DT mechanics undermines. At first sight it appears less natural than the Coulomb gauge defined in the local absolute rest frame in which time is discretised. We recall that the use of Lorentz gauges and attempts to maintain manifest Lorentz covariance in CT quantum electrodynamics leads to problems with state vectors, requiring the Gupta-Bleuler formulation or equivalent technology.

We then consider the quantisation of the free Maxwell fields. Using the DT Schwinger action principle which was successfully used previously for the DT quantisation of the scalar and Dirac fields, we find the vacuum functional and the free field commutators in the Coulomb gauge. Although we do not have a Hamiltonian formulation in our theory, we do have gauge symmetry, and we may use the Faddeev-Popov approach to the quantisation of gauge fields in our theory. We find that the Faddeev-Popov vacuum functional for the DT free electromagnetic theory in the DT Coulomb gauge coincides precisely with the vacuum functional obtained by solving the DT Schwinger functional differential equations in the same gauge.

Finally, and this was found to be the hardest task, we construct the DT analogues of the photon propagator in the Landau and Feynman gauges, in preparation for future applications to QED. We find that there is an interesting breakdown of what happens in CT theory, with DT analogues of the metric tensor making appearances. These differ in interesting ways, depending on whether we are in space time or in momentum space, and indicate that a reformulation of general relativity via DT mechanics will be quite instructive.

2 Notation and conventions

In addition to the natural unit system where \( c = \hbar = 1 \), we use the following conventions throughout this paper. Given \( T \) is the fundamental time step in DT mechanics then an event in our space-time has natural coordinates \( x \equiv (n, x) \), where \( n \) is an integer. The CT limit

\[
T \to 0, \; n \to \infty, \; nT = t \quad (t \text{ fixed})
\]

then corresponds to the CT coordinate time \( t \). Our coordinates are natural in the following sense. We work in an inertial frame of reference such that the CMB dipole anisotropy is absent locally. In this frame we fix cartesian spatial axes with coordinates \( x \equiv (x^1, x^2, x^3) \) such that the distant stars do not appear to rotate relative to such axes. Finally we choose an instant of discrete time as our temporal origin of coordinates and count fundamental intervals of \( T \) forwards and backwards, giving us the integer coordinate \( n \) referred to above. Our discussion in this paper does not include any cosmological aspects such as Hubble expansion, other than using the CMB to determine our frame of reference. We shall be concerned with local coordinates relevant to particle physics in the present epoch.
We denote integration over space-time by the sum/integral symbol
\[
\int \equiv T \sum_{n=-\infty}^{\infty} \int d^3x.
\] (3)

Given a variable \( f_x \equiv f_n(x) \) indexed by a discrete temporal index \( n \) and by a continuous
spatial index \( x \) we define the DT fourier transform \( \tilde{f}_p \) of \( f_x \) by
\[
\tilde{f}_p \equiv \tilde{f}(z, p) \equiv \int f_x e^{p \cdot x},
\] (4)

where
\[
e_{px} \equiv z^n e^{-ip \cdot x}
\] (5)
and \( z \) is complex and non-zero. We shall assume that such a transform exists and defines a
function analytic in some annular region in the complex-\( z \) plane centred on the origin and
including the unit circle. This of course puts restrictions on the sequence defined by \( f_n \).
Although \( p \) is the direct equivalent of the spatial components of the momentum space four-
vector of \( CT \) mechanics, the analogue of the time component \( p^0 \) of the momentum space
four vector is not \( z \). We shall be interested in taking \( z \) on the unit circle, and then it is the
principal argument of \( z \) which is related to \( p^0 \). We shall use the symbol \( p \) to denote the set
\( p \equiv (z, p) \).

If the DT fourier transform exists and has a Laurent expansion in some annular region
centred on the origin in the complex \( z \)-plane and containing the unit circle, then we can
construct the inverse transform. This is given by
\[
f_x = \oint_p \tilde{e}_{px} \tilde{f}_p,
\] (6)

where
\[
\oint_p \equiv \frac{1}{2\pi i T} \oint \frac{dz}{z} \int \frac{d^3p}{(2\pi)^3}
\] (7)
with the \( z \)-integral being over the unit circle in the complex plane taken in the anticlockwise
sense and
\[
\tilde{e}_{px} \equiv z^{-n} e^{ip \cdot x}.
\] (8)

We could attempt to define our dynamics in the \((z, p)\) transform space, restricting our
dynamical variables to be functions in the complex \( z \)-plane for which the inverse transform
exists. No new dynamical content should emerge from this approach, but it may be
mathematically more secure. However, we will normally assume that coming first from the
space-time direction is valid. That after all is the conventional way to define field theories.

The analogues of the identity operators (delta functions) in our formalism are defined as
follows. If \( \delta_n \) is the \( DT \) Kronecker delta, satisfying
\[
\delta_n = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0 \end{cases}
\] (9)
where \( n \) is an integer, then we define the four dimensional \( DT \) Dirac delta \( \delta_x \) by
\[
\delta_{x-y} \equiv \frac{\delta_{n-m}}{T} \delta^3(x - y), \quad T > 0, \quad (10)
\]
where \( y \equiv (m, y) \). Then
\[
\oint p \delta_{x-y} = f_y, \quad (11)
\]
An integral representation of \( \delta_{x-y} \) is
\[
\delta_{x-y} = \oint p \delta_{x-y} = \oint p e^{p_x} \bar{e}_{py}, \quad (12)
\]
The corresponding operator \( \tilde{\delta}_{p-q} \) in \( DT \) fourier transform space satisfies the relation
\[
\oint p \tilde{\delta}_{p-q} = \tilde{f}_q, \quad q \equiv (u, q), \quad (13)
\]
where \( u \) is complex. A \( DT \) sum/integral representation of \( \tilde{\delta}_{p-q} \) is given by
\[
\tilde{\delta}_{p-q} = \oint x \bar{e}_{px} e_{qx}, \quad (14)
\]
The classical step operator \( U_n \) acting on any temporally indexed variable \( f_n \) is defined by
\[
U_n f_n \equiv f_{n+1}, \quad U_n^{-1} f_n \equiv f_{n-1}, \quad (15)
\]
with powers of \( U \) defined in the obvious way, viz
\[
U_n^a f_n = f_{n+a}, \quad (16)
\]
where \( a \) is an integer. We note that
\[
U_n f_m = f_m, \quad m \neq n. \quad (17)
\]
Throughout our mechanics we shall deal with operators and equations defined via linear combinations of the step operators. An operator of the form
\[
P(U_n) \equiv c_0 U_n^a + c_1 U_n^{a+1} + \ldots + c_r U_n^{a+r} \quad (18)
\]
will be called an \( r^{th} \) order operator. Then an equation involving an \( r^{th} \) order operator will be called \( r^{th} \) order. Important first order operators are the forwards and backwards differences defined by
\[
\Delta^+_n \equiv U_n - 1, \quad \Delta^-_n \equiv 1 - U_n^{-1} \quad (19)
\]
respectively, and from them we define the second order symmetric difference
\[
\Delta_n \equiv \frac{1}{2} (\Delta^+_n + \Delta^-_n) = \frac{1}{2} (U_n - U_n^{-1}). \quad (20)
\]
First temporal derivatives are invariably replaced by one of three possible operators, defined by
\[
D^+_n \equiv \frac{\Delta^+_n}{T}, \quad D^-_n \equiv \frac{\Delta^-_n}{T}, \quad D^n \equiv \frac{\Delta^n}{T},
\]
whereas the second derivative is invariably replaced by the operator
\[
D^2_n \equiv D^+_n D^-_n = D^-_n D^+_n = \frac{U_n - 2 + U^{-1}_n}{T^2}.
\]

It is a feature of our DT mechanics that the formalism will tell us which of the above difference operators we need to use in a given context. For example, our investigations into the DT Schrödinger equation \([4]\) and the Dirac equation \([5]\) show that in these equations we have to replace \(\partial/\partial t\) by the second order symmetric operator \(D^2_n\). This has important consequences as far as the solutions of the equations are concerned, because the second order symmetric difference leads to a DT equation of motion which acts like a second order CT equation of motion, rather than a first order equation of motion, and this generates the oscillon solutions discussed in \([4]\) and \([5]\). Fortunately, we found that in the second quantised theory, these oscillons correspond to states with unphysical norm, and so are not observable asymptotically as ordinary particles. Their role as virtual particles in QED will be considered in the next paper of this series.

In our mechanics, *time reversal* amounts to the interchange \(U_n \leftrightarrow U^{-1}_n\). We shall encounter various difference operators which have the symmetry property that they are invariant to time reversal. We shall refer to such operators as \(T\) symmetric. They are important and useful to us, and in DT fourier transform space they are real functions of \(z\) provided \(z\) is on the unit circle (we note that \(z^* = z^{-1}\) holds only on the unit circle). With this definition, we see \(D^2_n\) is a second order \(T\)-symmetric operator. Another important second order \(T\)-symmetric operator which occurs throughout our mechanics is given by
\[
S_n \equiv \frac{1}{6} \left( U_n + 4 + U^{-1}_n \right).
\]
The factor of \(1/6\) and the \(4\) can readily be understood in terms of our virtual path procedure, discussed below. In the CT limit \([2]\), if it exists, these operators can be replaced by
\[
D^+_n \rightarrow \partial_t, \quad D^-_n \rightarrow \partial_t, \quad D^n \rightarrow \partial_t, \quad D^2_n \rightarrow \partial_t^2, \quad S_n \rightarrow 1.
\]

There is no trace of \(S_n\) in CT theory, but it plagues DT mechanics, occurring in unpredictable places and introducing temporal nonlocality in unexpected places. It makes the rewriting of CT mechanics into a DT form far from easy, particularly in the case of Maxwell’s equations. This nonlocality also enters at the level of the metric tensor, and suggests that an attempt to formulate general relativity into a DT framework will require thinking of the metric tensor as a non-local in time operator (and hence as a more dynamical object), rather than as a set of local functions forming the components of a rather bland second rank tensor. Another important operator is the DT d’Alembertian, which turns out to be the second order \(T\)-symmetric difference-differential operator
\[
\Box_x \equiv D^2_n - S_n \nabla^2_x,
\]
which appeared in our studies of the DT Klein Gordon equation [4, 5] and the DT Dirac equation [6]. This operator is also important in our formulation of Maxwell’s equations.

We note the useful result

\[ \oint_x f_x \overrightarrow{P}(U_n)g_x = \oint_x f_x \overrightarrow{P}(U_n^{-1})g_x. \]  

(27)

From this we see that for T-symmetric operators, we may write

\[ \oint_x f_x \overrightarrow{P}(U_n)g_x = \oint_x f_x \overrightarrow{P}(U_n)g_x. \]  

(28)

The DT fourier transform gives the following useful result:

\[ \oint_x e^{px} P(U_n) f_x = P(z^{-1}) \tilde{f}_p, \quad z \neq 0. \]  

(29)

We may use the above to find the DT Feynman propagator \( \Delta_{Fx} \), which satisfies the equation [4]

\[ (\Box_x + m^2 S_n) \Delta_{Fx} = -\delta_x. \]  

(30)

Taking the DT fourier transform of this equation we find

\[ (p^2 - m^2 S_z) \tilde{\Delta}_{Fp} = 1, \]  

(31)

where we define

\[ p^2 \equiv -D^2 z - S_z p \cdot p, \]  

(32)

with

\[ D^2 z \equiv \frac{z - 2 + z^{-1}}{T^2}, \quad S_z \equiv \frac{z + 4 + z^{-1}}{6}. \]  

(33)

A solution of interest in particle theory is

\[ \Delta_{Fx} = \oint_p \tilde{e}_{px} \frac{1}{p^2 - m^2 S_z + i\epsilon}, \]  

(34)

where we choose a DT analogue of the Feynman +i\epsilon prescription. The singularity structure in the complex z-plane of the integrand in the above is particularly interesting. First, we can prove that the equation

\[ p^2 - m^2 S_z + i\epsilon = 0 \]  

(35)

has no solution on the unit circle in the complex z-plane for any value of the linear momentum \( p \). To prove this, write \( z = \exp(iz) \). Then (35) becomes

\[ \frac{2 (\cos \theta - 1)}{T^2} - \frac{(\cos \theta + 2)}{3} (p \cdot p + m^2) + i\epsilon = 0, \]  

(36)

which has no solution for real \( \theta \) if \( \epsilon > 0 \). This result is important because it means we have a fully closed contour of integration over the unit circle in the complex z-plane, requiring no principal value discussion.
Next, we note that (35) is a quadratic in \( z \), with roots \( z_1, z_2 \) satisfying the relation

\[ z_1 z_2 = 1. \]  

(37)

Hence we deduce that the denominator in (34) contributes one simple pole inside the unit circle and one outside. By looking carefully at the location of these poles as the spatial momentum \( p \) varies, we see that two distinct patterns of behaviour emerge [3]. For momentum in the elliptic regime, corresponding to momenta bounded by \( T|p| < \sqrt{12} \), the simple pole inside the unit circle is just inside and gives the equivalent of a trigonometric solution when we use the calculus of residues. For momentum in the hyperbolic regime, on the other hand, given by \( T|p| > \sqrt{12} \), the simple pole interior to the unit circle starts to move towards \( z = 0 \), giving a damped exponential solution when we use the calculus of residues.

Finally, we note that if we had taken a slightly different prescription, viz

\[ \Delta_{Fx} \equiv \frac{1}{\oint p \bar{e}_p e^{p} - (m^2 - i\epsilon) Sz} \]  

(38)

it is not hard to see that our conclusions would be exactly the same.

3 Link and node variables in DT mechanics

The discretisation of time requires us to change the way we think about dynamical variables. Given a lattice structure to time, we can identify two distinct geometrical components. These are the node sites, which correspond to events at times \( nT \), where \( n \) is an integer, and the links between these nodes. For a one dimensional lattice, links and nodes are mathematically dual, but in our theory this mathematical duality does not carry over into the physics. Some of our dynamical variables are defined at nodes whereas others are defined at sites, and generally this occurs in such a way that interchanging links and nodes is not a symmetry of the theory. In particular, matter fields corresponding to massive particles are defined on nodes only. We have found that electric fields occur as link variables whereas magnetic fields are node variables, so that electro-magnetic duality does not appear to occur here. This seems a natural way to explain the absence of Maxwellian magnetic monopoles.

We show now that the dynamical rules for link variables are similar to but not identical to those for node variables. First, let a generic node field dynamical variable be denoted by \( A_\alpha^n (x) \), where \( n \) is the time, \( x \) is the spatial position, and \( \alpha \) is some extra label such as a vector, spinorial or group index. Likewise, denote a typical link variable by \( \phi_\beta^n (x) \). As discussed in [3, 4] we base our dynamics on the system function \( F^n \equiv \int dxF^n \), where the system function density \( F^n \) has the form

\[ F^n \equiv \mathcal{F}(A_n, A_{n+1}, \phi_n, \nabla A_n, \nabla A_{n+1}, \nabla \phi_n), \]  

(39)

where we have suppressed the spatial coordinates and field labels. This is the discrete time analogue of a Lagrangian of the standard form

\[ L \equiv \int dx \mathcal{L}(\varphi, \nabla \varphi, \dot{\varphi}) \]  

(40)
and is equally generic, in that all of the system functions we need to use are of this form. We note that link and node variables are treated differently right at this point, in that the system function is first order in the node variables but zeroth order in the link variables. This emphasises further that as far as dynamics is concerned, link and node variables are not dual.

In DT mechanics the action integral becomes an action sum. The action sum from initial time $MT$ to final time $NT > MT$ is given by

$$A_{NM}^N \equiv T \sum_{n=M}^{N-1} F^n$$

and the equations of motion are obtained by Cadzow’s action principle \[12\]

$$\delta A_{NM} \over c = 0,$$

for suitable variations of the fields. Here and elsewhere we shall use the symbol $\equiv$ to denote an equality holding by virtue of the equations of motion. For an arbitrary variation:

$$A_\alpha^n \to A_\alpha^n + \delta A_\alpha^M, \quad M \leq n \leq N$$

$$\phi^\beta_n \to \phi^\beta_n + \delta \phi^\beta_n, \quad M \leq n < N$$

we find

$$\delta A_{NM} = T \int d^3x \left\{ \delta A_\alpha^M (x) \frac{\delta}{\delta A_\alpha^M (x)} F^M + \sum_{n=M+1}^{N-1} \delta A_\alpha^n (x) \frac{\delta}{\delta A_\alpha^n (x)} F^n + \delta A_\alpha^N (x) \frac{\delta}{\delta A_\alpha^N (x)} F^{N-1} + \sum_{n=M+1}^{N-1} \delta \phi^\alpha_n (x) \frac{\delta}{\delta \phi^\alpha_n (x)} F^n \right\},$$

where we use the summation convention for the field labels $\alpha$ only. For fixed end-point, but otherwise arbitrary variations, viz.

$$\delta A_\alpha^M (x) = \delta A_\alpha^N (x) = 0$$

we find the functional derivative equations of motion

$$\frac{\delta}{\delta A_\alpha^n (x)} \{ F^n + F^{n-1} \} \over c = 0, \quad M < n < N$$

$$\frac{\delta}{\delta \phi^\alpha_n (x)} F^n = 0, \quad M \leq n < n, \quad M < n < N$$

which reduce to

$$\frac{\partial}{\partial A_\alpha^n (x)} \{ F^n + F^{n-1} \} \over c = \nabla \cdot \frac{\partial}{\partial \nabla A_\alpha^n (x)} \{ F^n + F^{n-1} \}, \quad M < n < N$$

$$\frac{\partial}{\partial \phi^\beta_n (x)} F^n = \nabla \cdot \frac{\partial}{\partial \nabla \phi^\beta_n (x)} F^n, \quad M \leq n < N.$$
We see here once more the essential difference between the node field dynamics and that of the link fields. The former involve genuine second order difference equations of motion (48) corresponding to second order temporal derivatives in CT theory. For the links however, equations (49) are at most first order difference equations, and such equations are analogous to constraint equations in CT mechanics rather than equations of motion. All of this is intimately tied in with the boundary conditions required to solve these equations. CT theories with constraints and gauge symmetries will have DT analogues where both sort of equations occur. The electromagnetic equations discussed in this paper give a basic example of these ideas.

4 Classical electromagnetism

4.1 The charge free Maxwell equations

Our discrete time formulation of the charge free Maxwell’s equations starts with the CT electromagnetic potentials \( A^\mu \equiv (\phi, A) \) which are used to construct the physical electric and magnetic fields \( E \) and \( B \). A clear distinction has to be made here between the nature of the electric scalar potential \( \phi \) and the magnetic vector potential \( A \) in DT mechanics. The former is associated with the temporal interval or link connecting times \( t_n \equiv nT \) and \( t_{n+1} \), whereas the latter is associated with nodes, or times \( t_n \) themselves. This distinction also manifests itself in the difference between the physical electric field \( E \) and the magnetic field \( B \), which are likewise associated with temporal links and nodes respectively. The electric scalar potential associated with the link connecting time \( t_n \) and \( t_{n+1} \) at spatial position \( x \) will be denoted by the symbol \( \phi_n(x) \), rather than by (say) \( \phi_{n+\frac{1}{2}}(x) \). Although our notation suggests a bias towards \( t_n \) at the expense of \( t_{n+1} \), this is not really the case.

One of the principles we have applied to DT mechanics is minimality. By this we mean our belief that if we are really working at an irreducibly fundamental level, then the dynamics should be as basic and uncomplicated as possible. Now as discussed in \([3]\) a system function is more like a Hamilton’s principle function than a Lagrangian, being a function of the dynamical degrees of freedom at the end points of a fundamental interval of time \( T \). In principle, we cannot probe below this level in DT mechanics. We could simply postulate a system function, but a better approach would be to take some CT theory such as Maxwell’s electromagnetism and construct a suitable system function directly from its CT Lagrangian. If we did not use the CT Lagrangian as a guide, then we could not hope to guess an appropriate system function.

Our approach uses the auxiliary concept of virtual path. This is discussed fully in \([3, 4]\). The basic idea is to replace the CT fields by functions of the node and link variables smeared in some way over a given interval \([nT, (n+1)T] \). From the point of view of CT mechanics, this introduces a degree of non-locality in time, but this is inevitable. For some fields, such as the neutral scalar field, the function is a linear interpolation, but in the case of matter fields involving gauge invariance, such as Dirac fields, the virtual paths are highly non-linear. Fortunately, the virtual paths for the Maxwell potential fields are straightforward. The virtual path \( \phi_{n\lambda} \) for the electric potential (a link variable) is defined by

\[
\phi_{n\lambda}(x) \equiv \phi_n(x), \quad 0 \leq \lambda \leq 1,
\]  

(50)
where we use the variable $\lambda$ to interpolate between the two ends of a given link. Given that the magnetic vector potential is a node variable, its associated virtual path $A_{n\lambda}$ is defined by

$$A_{n\lambda}(x) \equiv \lambda A_{n+1}(x) + \bar{\lambda} A_n(x), \quad 0 \leq \lambda \leq 1,$$

(51)

where $A_n$, $A_{n+1}$ are the dynamically meaningful field values at the ends of a given link and $\bar{\lambda} \equiv 1 - \lambda$.

The above virtual paths are defined over the interval $[nT, (n+1)T]$ For a set of successive time intervals, we observe that virtual paths for node variables are continuous in time, whereas virtual paths for link variables are not. This is tied in with the nature of their respective dynamics. Note that no physical meaning can be attributed to the virtual path in a genuine $DT$ theory, and as we have stressed before [3], these paths are used only as a tool in obtaining a system function with built in properties such as gauge invariance. Beyond that, they have no significance.

The $DT$ version of Maxwell’s equations comes with a $DT$ version of gauge invariance. A local $DT$ gauge transformation involves a gauge function $\chi$ associated with nodes rather than links. A gauge function value at time $n$ and position $x$ will be denoted by $\chi_n(x)$ and is assumed differentiable with respect to $x$. The virtual path for a gauge function is given by

$$\chi_{n\lambda}(x) \equiv \lambda \chi_{n+1}(x) + \bar{\lambda} \chi_n(x), \quad 0 \leq \lambda \leq 1.$$

(52)

In $CT$ electromagnetism a local gauge transformation is defined by the replacements

$$A^\mu \to A'^\mu = A^\mu + \partial^\mu \chi.$$

(53)

In our theory this becomes

$$\phi_n \to \phi'_n = \phi_n + D^+ n \chi_n, \quad (54)$$

$$A_n \to A'_n = A_n - \nabla \chi_n. \quad (55)$$

When we talk about local gauge transformations, we shall mean these last two equations.

Turning now to the physical fields, we define the gauge invariant electric and magnetic fields via the potentials:

$$E_n \equiv -\nabla \phi_n - D^+ n A_n, \quad B_n \equiv \nabla \times A_n.$$  

(56)

These satisfy the homogeneous $DT$ Maxwell equations

$$\nabla \cdot B_n = 0, \quad \nabla \times E_n + D^+ n B_n = 0.$$  

(57)

To construct a gauge invariant system function, we recall that the $CT$ Lagrange density for electromagnetism is given in terms of the Faraday tensor $F_{\mu \nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$, viz

$$\mathcal{L} = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu}$$

$$= -\frac{1}{2} \partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2} \partial_\nu A_\mu \partial^\mu A^\nu$$

$$= \frac{1}{2} (\dot{A}^i + \partial_i \phi)(\dot{A}^i + \partial_i \phi) + \frac{1}{2} (\partial_i \dot{A}^i \partial_j A^j - \partial_i A^i \partial_j A^j).$$  

(58)
The gauge invariant system function density for the charge free system is obtained by replacing the fields in the above \( CT \) Lagrange density by their virtual path forms and integrating with respect to \( \lambda \) over the interval \([0, 1]\);

\[
\mathcal{F}^n \equiv \langle L(A_{n\lambda}, \phi_{n\lambda}) \rangle = \langle \frac{1}{2}(T^{-1}\partial_{\lambda}A_{n\lambda}^i + \partial_{\lambda}\phi_{n\lambda})(T^{-1}\partial_{\lambda}A_{n\lambda}^i + \partial_{\lambda}\phi_{n\lambda}) + \frac{1}{2} (\partial_{\lambda}A_{n\lambda}^i \partial_{\lambda}A_{n\lambda}^i - \partial_{\lambda}A_{n\lambda}^i \partial_{\lambda}A_{n\lambda}^i) \rangle
\]

\[
= \frac{1}{2} (D_n^+ A_n + \nabla\phi_n) \cdot (D_n^+ A_n + \nabla\phi_n) + \frac{1}{2} ((\partial_{\lambda}A_{n\lambda}^i \partial_{\lambda}A_{n\lambda}^i - \partial_{\lambda}A_{n\lambda}^i \partial_{\lambda}A_{n\lambda}^i)),
\]

where the angular brackets denote an integral over \( \lambda \), i.e.

\[
\langle f_\lambda \rangle \equiv \int_0^1 f(\lambda) \, d\lambda.
\]

Here we use the virtual path replacement \( \partial_t \rightarrow T^{-1}\partial_{\lambda} \). For convenience we have not multiplied the system function by a factor \( T \) as was done in earlier papers of this series, so that it now has the physical dimensions of a Lagrangian rather than an action.

The equations of motion for the magnetic potential (a node variable) are given by

\[
\frac{\partial}{\partial A_i} \{ \mathcal{F}^n + \mathcal{F}^{n-1} \} = \frac{1}{c} \frac{\partial}{\partial_j} \frac{\partial}{\partial A_i} \{ \mathcal{F}^n + \mathcal{F}^{n-1} \},
\]

which reduce to

\[
\Box_n A_n + \nabla\Lambda_n = 0,
\]

where

\[
\Lambda_n \equiv D_n^- \phi_n + \nabla \cdot S_n A_n
\]

is the \( DT \) Lorentz function. For the scalar potential (a link variable), the equation of motion is

\[
\frac{\partial \mathcal{F}^n}{\partial \phi_n} = \nabla \cdot \frac{\partial \mathcal{F}^n}{\partial \nabla \phi_n},
\]

which reduces to

\[
D_n^+ \nabla \cdot A_n + \nabla^2 \phi_n = 0.
\]

This implies the equation

\[
\Box_n \phi_n - D_n^+ \Lambda_n = 0.
\]

Equations (62–66) are \( DT \) gauge invariant, as can be readily verified. The reason is that our system function is gauge invariant, and this guarantees that the equations of motion are gauge invariant.

### 4.2 The \( DT \) Lorentz gauge

In the \( DT \) Lorentz gauge we set

\[
\Lambda_n = 0,
\]

\[sans-serif\]
and then the potentials satisfy the massless DT Klein-Gordon equations

\[
\square_n A_n = 0, \quad \square_n \phi_n = 0.
\]  

(68)  

(69)

It is always possible to work in this gauge, as the following argument shows. Suppose that we started off with a configuration of fields for which the DT Lorentz function was non-zero, i.e.

\[
\Lambda_n \equiv D_n - n \phi_n + S_n \nabla \cdot A_n \neq 0
\]

(70)

Now consider the gauge transformation

\[
A'_n = A_n - \nabla \chi_n, \quad \phi'_n = \phi_n + D_n \chi_n
\]

(71)

where the gauge function \( \chi_n \) satisfies the DT inhomogeneous Klein-Gordon equation

\[
\square_n \chi_n = -\Lambda_n.
\]

(72)

Then we find

\[
\Lambda'_n \equiv D_n - n \phi'_n + S_n \nabla \cdot A'_n = 0,
\]

(73)

as required. We note that using our experience with the DT Klein-Gordon propagators in earlier papers, we may write down a particular solution to (72) in the form

\[
\chi_x = \int_y \Delta_{Fx-y} A_y,
\]

(74)

where \( \Delta_{Fx} \) is the DT scalar massless Feynman propagator satisfying the equation

\[
\square_x \Delta_{Fx} = -\delta_x.
\]

(75)

It is noteworthy that the Lorentz condition

\[
\Lambda_n \equiv D_n - n \phi_n + S_n \nabla \cdot A_n = 0
\]

(76)

is second order in time, that is, relates field values on three successive nodes and on the two links between them. This will interact in some way with the equations of motion, which are also second order, and so we can expect trouble. We shall see that the Lorentz condition and its generalisation demands special attention when we come to work out the electromagnetic propagator in the DT Feynman gauge. In particular, we will have to consider the existence of an inverse operator, \( S_n^{-1} \), which is highly non-local in time.

### 4.3 The physical fields

Turning to the physical (gauge invariant) electric and magnetic fields, we may write the system function density (59) in the form

\[
\mathcal{F}^n = \frac{1}{2} E_n^2 - \frac{1}{6} (B_{n+1} \cdot B_{n+1} + B_{n+1} \cdot B_n + B_n \cdot B_n),
\]

(77)
which differs from the more familiar form

\[ \mathcal{F}_n^* = \frac{1}{2} \mathbf{E}_n^2 - \frac{1}{2} \mathbf{B}_n \cdot \mathbf{S}_n \mathbf{B}_n \]  

(78)

by a total temporal difference and so gives the same equations of motion. Applying Cadzow’s equation to (77) we find

\[ \nabla \cdot \mathbf{E}_n = 0, \quad D_n^{-} \mathbf{E}_n = \nabla \times \mathbf{S}_n \mathbf{B}_n. \]  

(79)

Using (57) and (79) we find the physical electromagnetic fields satisfy the DT massless Klein-Gordon equations

\[ \Box_n \mathbf{E}_n = 0, \quad \Box_n \mathbf{B}_n = 0. \]  

(80)

We may readily construct the analogues of the conserved total linear momentum and angular momentum using the method described in Appendix B. For example, we find the DT analogue of the Poynting vector is

\[ \mathbf{P}_n \equiv \int d^3 \mathbf{x} \left\{ \mathbf{E}_n \times \mathbf{B}_n + \frac{1}{6} \mathbf{T} \mathbf{B}_n \nabla \mathbf{B}_n \right\} = \mathbf{P}_{n+1}. \]  

(81)

The expression for the total angular momentum is left as a exercise.

### 4.4 Maxwell’s equations in the presence of charges

In the presence of electric charges the system function density (77) is replaced by

\[ \mathcal{F}_n^* [\mathbf{j}] = \mathcal{F}_n^* - \phi_n \rho_n + \frac{1}{2} \mathbf{A}_{n+1} \cdot \mathbf{j}_{n+1} + \frac{1}{2} \mathbf{A}_n \cdot \mathbf{j}_n \]  

(82)

where \( \rho_n (\mathbf{x}) \) and \( \mathbf{j}_n (\mathbf{x}) \) are the discrete time charge density and charge current respectively. The homogeneous equations (57) remain unaltered but now the equations of motion become

\[ \nabla \cdot \mathbf{E}_n = \rho_n, \quad \nabla \times \mathbf{S}_n \mathbf{B}_n - D_n^{-} \mathbf{E}_n = \mathbf{j}_n. \]  

(83)

These equations are consistent provided the equation of continuity

\[ D_n^{-} \rho_n + \nabla \cdot \mathbf{j}_n = 0 \]  

(84)

for electric charge holds. The dynamical equations of motion (83) may be written in the form

\[ -\nabla^2 \phi_n - D_n^{+} \nabla \cdot \mathbf{A}_n = \rho_n, \]  

(85)

\[ \Box_n \mathbf{A}_n + \nabla \Lambda_n = \mathbf{j}_n. \]  

(86)

Equation (85) may also be rewritten in the form

\[ \Box_n \phi_n - D_n^{+} \Lambda_n = S_n \rho_n, \]  

(87)
In the Lorentz gauge \((86)\) and \((87)\) become

\[
\Box_x \phi_x = S_n \rho_x, \quad (88)
\]

\[
\Box_x A_x = j_x. \quad (89)
\]

We note here the appearance of the non-local \(T\)-symmetric operator \(S_n\).

Quantisation is more convenient in the Coulomb gauge, where we set

\[
\nabla \cdot A_n = 0. \quad (90)
\]

Then the equations of motion \((83)\) and \((84)\) become

\[
\nabla^2 \phi_n = -\rho_n, \quad (91)
\]

\[
\Box_n A_n + \nabla D_n^- \phi_n = j_n. \quad (92)
\]

We see once again that the scalar potential \(\phi_n\) cannot be regarded as a dynamical field in the same way as the components of the vector potential are. Equation \((91)\) is zeroth order in time whereas \((92)\) as a full second order dynamical equation. This is the \(DT\) analogue of the situation in \(CT\) electromagnetism, where a direct application of Dirac’s constraint analysis shows that the momentum conjugate to the scalar potential vanishes. This has important consequences when we develop our quantisation via the Schwinger action principle, discussed next.

### 4.5 comment

It comes as a surprise to see that charge density as formulated in our mechanics turns out to be a link variable. A naive guess would have us take it to be a node variable, on the grounds that electric charge is carried by matter fields, which in our theory are node variables. This is another example where the unravelling of dynamics from a \(CT\) to a \(DT\) framework forces us to re-evaluate our understanding of the various components of dynamics.

For example, consider the preparation of a state containing charged particles. In view of the above comment about charge being a link variable, we see that it must be insufficient in some way to think of such a state as being completely defined or specified at a given instant or node of time only. If we wanted to measure the total charge of the system, for instance, we would have to consider the fields on the link to which this node is attached, and the fields on the node at the other end of this link as well.

This raises another interesting thought; the link pointing forwards in time from a given node is different to the link pointing backwards in time. Therefore, the meaning of what constitutes a state in \(DT\) mechanics must depend on whether it is regarded as an initial state or as a final state.

It is such examples which lead to the conclusion that in \(DT\) cosmology, time could not be considered to have a beginning at a point only; it would be necessary to say something about the first link as well. If indeed it is correct to think of our fundamental interval \(T\) as equivalent to the Planck scale \(T_P\), then the origin of the universe in \(DT\) mechanics would require us to regard the so-called \(Planck\) epoch as just the first link. Then it would be wrong
in this context to imagine any form of dynamical evolution process occurring during that epoch. This is in direct contrast with fundamental theories such as quantum gravity and superstring theory based on continuous time, where presumably, there is scope for a great deal of dynamical interaction during the Planck epoch.

Carrying on this line of thought, we would have to accept that dynamical fields on nodes such as the vector potentials $\mathbf{A}$ could only start dynamical evolution after the Planck epoch, since they satisfy a second order equation of motion. On the other hand, link variables such as the scalar potential $\phi$ would not have such a restriction. The essential point here is that $\text{DT}$ mechanics alters our perception of boundary conditions. Fortunately, our discussion in this paper involves the present epoch, and we may assume time runs from remote past to remote future without any qualms about boundary conditions at the origin of time.

5 Quantisation in the Coulomb gauge

Quantisation in $\text{DT}$ field theory is readily tackled via the $\text{DT}$ Schwinger action principle, which we shall now state and use to determine the $\text{DT}$ electromagnetic field commutators and vacuum functional in the Coulomb gauge. It is a merit of Schwinger’s approach and also of Feynman’s path integral approach to quantisation that the emphasis is on the physically useful amplitudes of the theory, rather than on the operators themselves, such as occurs in the canonical quantisation process. The process of imposing naive canonical commutators between dynamical variables and their conjugate momenta can be expected to fail in $\text{DT}$ mechanics for a number of reasons: we do not have a Hamiltonian framework in our mechanics; we do not have a constraint theory in the fashion of Dirac for gauge field dynamics; and the construction of conjugate momenta is straightforward only in the case of systems which are normal $[3]$. Fortunately, none of these reasons prevent us from quantising Maxwell’s equations.

5.1 The $\text{DT}$ Schwinger action principle

When we use the $\text{DT}$ Schwinger action principle, we work in the Heisenberg picture and consider matrix elements between states associated with different times. These correspond to preparation and observation, that is, initial and final states. If $|\Psi, M\rangle$ is the state we have prepared at time $MT$, and $|\Phi, N\rangle$ is a state we are asking questions about at time $NT > MT$, then the infinitesimal change $\delta \langle \Phi, N|\Psi, M\rangle$ in the transition amplitude $\langle \Phi, N|\Psi, M\rangle$ due to infinitesimal changes in the external sources is defined by

$$\delta \langle \Phi, N|\Psi, M\rangle = i \langle \Phi, N|\delta A^{NM}|\Psi, M\rangle, \quad N > M$$

(93)

where $\delta A^{NM}$ is the infinitesimal change in the action sum operator

$$A^{NM} \equiv T \sum_{n=M}^{N-1} \int d^3x F^n.$$  

(94)

In the case of the electromagnetic field, we take the free field system function density $[7]$ and introduce arbitrary infinitesimal sources $\rho_n, j_n$ in the manner of Schwinger $[13]$. Since
these sources are arbitrary, we must take care to ensure that the charges which couple to the electromagnetic fields satisfy the equation of continuity \((84)\). Following Schwinger and anticipating the use of the Coulomb gauge, the system function density in the presence of the sources is given by

\[
F^n[j] = F^n - \phi_n \rho_n^c + \frac{1}{2} A_{n+1} \cdot j_{n+1}^c + \frac{1}{2} A_n \cdot j_n^c,
\]

where

\[
\rho_n^c(x) \equiv \rho_n(x),
\]

\[
j_n^c(x) \equiv j_n(x) + \nabla_x \int_y G_{Cx-y} \left[ D_{n}^{-} \rho_{y} + \nabla_y j_{y} \right]
\]

are the conserved charge densities constructed out of the independent external densities \(\rho_n\) and \(j_n\), and \(G_C\) is the Coulomb Green’s function which satisfies the equation

\[
\nabla_x^2 G_{C} = -\delta_x.
\]

This has particular solution

\[
G_{C} = \frac{\delta_n}{4\pi|x|T}.
\]

We note the conserved charge densities satisfy the DT equation of continuity

\[
D_n^+ \rho_n^c(x) + \nabla_x \cdot j_n^c(x) = 0,
\]

regardless of the values of the independent densities. The above coupling ensures that we are free to vary \(\rho_n\) and \(j_n\) arbitrarily whilst still ensuring that the electromagnetic fields are coupled to conserved charges.

Functional differentiation is defined via

\[
\frac{\delta}{\delta \rho_n(x)} \rho_m(y) = \delta_{x-y} \equiv \frac{\delta_{n-m}}{T} \delta^3(x-y)
\]

and similarly for the currents. Then with the external sources coupled as in \((95)\) we find

\[
\frac{i\delta}{\delta \rho_n(x)} \langle \Phi, n + 1 | \Psi, n \rangle_j = \langle \Phi, n + 1 | \phi_n(x) | \Psi, n \rangle_j
\]

\[
\frac{-i\delta}{\delta j_n(x)} \langle \Phi, n | \Psi, n - 1 \rangle_j = \frac{1}{2} \langle \Phi, n | A_n^i(x) | \Psi, n - 1 \rangle_j
\]

\[
\frac{-i\delta}{\delta j_n(x)} \langle \Phi, n + 1 | \Psi, n \rangle_j = \frac{1}{2} \langle \Phi, n + 1 | A_n^i(x) | \Psi, n \rangle_j,
\]

using the Coulomb gauge (or transversality) condition

\[
\langle \Phi | \nabla \cdot A_n(x) | \Psi \rangle = 0
\]

for all states \(\Psi, \Phi\).
If now we assume the existence of \( \text{in} \) and \( \text{out} \) vacua, and if \( Z[j] \equiv \langle 0_{\text{out}} | 0_{\text{in}} \rangle_j \) denotes the vacuum functional in the presence of the external sources, then direct application of the \( DT \) Schwinger action principle gives the functional derivatives

\[
\frac{i\delta}{\delta \rho_n(x)} Z[j] = \langle 0_{\text{out}} | \phi_n(x) | 0_{\text{in}} \rangle_j, \quad (106)
\]

\[
-\frac{i\delta}{\delta j_n^i(x)} Z[j] = \langle 0_{\text{out}} | A_n^i(x) | 0_{\text{in}} \rangle_j. \quad (107)
\]

In the Coulomb gauge the quantum analogues of Cadzow’s equations are

\[
\nabla^2 \phi_n(x) = -\rho_n^c(x), \quad (108)
\]

\[
\square_n \langle 0_{\text{out}} | A_n(x) | 0_{\text{in}} \rangle_j + \nabla_x D_n^- \phi_n(x) Z[j] = j_n^c(x) Z[j], \quad (109)
\]

taking the scalar potential in the Coulomb gauge to be a c-number. From these equations we deduce

\[
\phi_x = \oint_y G_{Cx-y} \rho_y \quad (110)
\]

and

\[
\square_x \langle 0_{\text{out}} | A_x^i | 0_{\text{in}} \rangle_j = j_x^c i Z[j] + \partial_i^x \partial_j^y \oint_y G_{Cx-y} j_y^c Z[j]. \quad (111)
\]

Note that we have switched notation, as we shall do frequently, using the symbols \( x, y \) to denote \( (n, x) \) and \( (m, y) \) respectively.

The second functional derivative of this last equation gives the \( DT \) time-ordered product

\[
\langle 0 | \tilde{T} A_x^i A_y^j | 0 \rangle = i \Delta_{Fx-y} \delta_{ij} + \oint_z \Delta_{Fx-z} \partial_i^z \partial_j^y G_{Cx-y} \quad (112)
\]

in the absence of the sources. This is equivalent to

\[
\langle 0 | \tilde{T} A_n^i(x) A_m^j(y) | 0 \rangle = i \int \frac{d^3p}{(2\pi)^3} \left[ \delta_{ij} - \frac{p_i^j p^j}{p^2} \right] \tilde{\Delta}_{F}^{n-m}(p) e^{-ip \cdot (x-y)} \quad (113)
\]

where \( \tilde{\Delta}_{F}^{n}(p) \) is the fourier transform

\[
\tilde{\Delta}_{F}^{n}(p) = \int d^3x \Delta_n^F(x) e^{-ip \cdot x} \quad (114)
\]

of the temporally indexed Greens’ function, which satisfies the \( DT \) massless field equation

\[
\square_n \Delta_n^F(x) = -\frac{\delta_n^3}{T} \delta^3(x). \quad (115)
\]

In the absence of external sources we may choose the radiation gauge, defined by

\[
\phi_n(x) = 0, \quad \langle \Phi | \nabla \cdot A_n(x) | \Psi \rangle = 0, \quad \forall \Psi, \Phi. \quad (116)
\]
Using the above time ordered products (113), we can extract the following unequal-time commutators:

$$\langle 0 | \left[ \tilde{A}_{n+1}^i(p), \tilde{A}_{n}^j(q) \right] | 0 \rangle = -i \Gamma_p \left[ \delta_{ij} - \frac{p^i p^j}{p^2} (2\pi)^3 \delta^3(p - q) \right]$$ (117)

$$\langle 0 | \left[ \tilde{A}_{n+1}^{i+}(p), \tilde{A}_{n}^{j}(q) \right] | 0 \rangle = -i \Gamma_p \left[ \delta_{ij} - \frac{p^i p^j}{p^2} (2\pi)^3 \delta^3(p - q) \right]$$ (118)

where

$$\Gamma_p = \frac{6T}{6 + T^2 p^2}$$ (119)

and

$$\tilde{A}_{n}^i(p) \equiv \int d^3x e^{-ip \cdot x} A_{n}^i(x),$$ (120)

$$\tilde{A}_{n}^{i+}(p) \equiv \int d^3x e^{ip \cdot x} A_{n}^i(x).$$ (121)

Hence we obtain the result

$$\langle 0 | \left[ A_{n+1}^i(x), A_{n}^j(y) \right] | 0 \rangle = -i \int \frac{d^3p}{(2\pi)^3} e^{ip \cdot (x-y)} \left[ \delta_{ij} - \frac{p^i p^j}{p^2} \right] \frac{6T}{6 + T^2 p^2}$$ (122)

which is precisely the same as for the scalar field discussed in [5] apart from the modified Kronecker delta, necessary to preserve the transversality condition (105).

As a final step, we may suppose that the commutators of the fields are c-numbers, in the language of Dirac, and then we arrive at the operator commutator statement

$$\left[ A_{n+1}^i(x), A_{n}^j(y) \right] = -i \int \frac{d^3p}{(2\pi)^3} e^{ip \cdot (x-y)} \left[ \delta_{ij} - \frac{p^i p^j}{p^2} \right] \frac{6T}{6 + T^2 p^2}$$ (123)

which amounts to our DT quantisation prescription. We may use (123) and the operator equation of motion

$$\Box_n A_n(x) = 0,$$ (124)

to deduce the equal time commutators

$$\left[ \tilde{A}_{n}^i(p), \tilde{A}_{n}^{i+}(q) \right] = 0,$$ (125)

which is equivalent to

$$\left[ A_{n}^i(x), A_{n}^{i+}(y) \right] = 0.$$ (126)

Photon creation and annihilation operators are defined by

$$a_{n}(p, \lambda) \equiv \frac{i}{\Gamma_p} \int d^3x e^{i\theta p \cdot x} \epsilon(p, \lambda) \cdot \left[ A_{n+1}(x) - e^{i\theta p} A_{n}(x) \right]$$ (127)

$$a_{n}^{+}(p, \lambda) \equiv \frac{-i}{\Gamma_p} \int d^3x e^{-i\theta p \cdot x} \epsilon(p, \lambda) \cdot \left[ A_{n+1}(x) - e^{-i\theta p} A_{n}(x) \right]$$ (128)
where
\[ \cos \theta_p = \frac{6 - 2T^2 p^2}{6 + T^2 p^2} \]  
(129)
and
\[ \epsilon(p, \lambda) \cdot \epsilon(p, \lambda') = \delta_{\lambda \lambda'}, \quad \epsilon(p, \lambda) \cdot p = 0. \]  
(130)
In these definitions we have taken the momentum \( p \) to be in the elliptic regime. From our work in previous papers with the \( DT \) harmonic oscillator, we know that momentum in the hyperbolic region \( |p| T > \sqrt{12} \) would give wave-functions which either decay to zero or diverge in time. The implications are that, like scalar and Dirac particles discussed in earlier papers of this series, there is a natural cutoff in our mechanics in the photon spectrum. This may have important repercussions in discussions involving for example the black body spectrum.

A direct application of the commutation rules (123 - 126) gives
\[
\left[ a_n(p, \lambda), a_n^+(q, \lambda') \right] = 2|p| \sqrt{1 - \frac{T^2 p^2}{12}} \delta_{\lambda \lambda'} (2\pi)^3 \delta^3(p - q),
\]  
(131)
\[
\left[ a_n(p, \lambda), a_n(q, \lambda') \right] = 0,
\]  
(132)
which shows explicitly that we will obtain a spectrum of polarised photon states, but only up to the parabolic barrier \(|p| T < \sqrt{12}\), as discussed above. We note that an expansion in powers of \( T \) gives
\[
\left[ a_n(p, \lambda), a_n^+(q, \lambda') \right] = 2|p| \delta_{\lambda \lambda'} (2\pi)^3 \delta^3(p - q) + O(T^2),
\]  
(133)
which supports the results discussed in earlier papers of this series that Lorentz symmetry emerges very rapidly from our mechanics if, as we imagine, \( T \) is of the order of the Planck time or less.

5.2 Comments

i) Fixing the value of \( \theta \) to equal \( \theta_p \) in (129) is the \( DT \) equivalent to a mass-shell constraint in \( CT \) field theory;

ii) The cutoff in the high momentum photon particle spectrum occurs for a dynamical reason. Consider the following toy model; suppose \( \psi_n \) is a complex valued variable evolving according to the \( DT \) harmonic oscillator equation [3]
\[
\psi_{n+1} = \frac{2\eta \psi_n - \psi_{n-1}}{c},
\]  
(134)
where \( \eta \) is real. Now take another complex sequence \((z_n)\) satisfying the same equation. Then we can readily prove that the constructions
\[
a_n \equiv i \left( z_n^* \psi_{n+1} - z_{n+1}^* \psi_n \right)
\]
\[
a_n^* \equiv -i \left( z_n \psi_{n+1}^* - z_{n+1} \psi_n^* \right)
\]  
(135)
are invariants of the motion, viz
\[
a_{n,c} = a_{n-1,c}, \quad a_{n,c}^* = a_{n-1,c}^*,
\]  
(136)
regardless of the value of $\eta$. In a field theory, the equivalent of $\eta$ is determined by the momentum $p$.

Now we ask where the cutoff in comes from. The answer is found in the pattern of possible behaviour found in the $DT$ harmonic oscillator. We showed in [3] that bounded motion occurs when $\eta$ is restricted to the range $-1 < \eta < 1$, corresponding to the trigonometric solutions of the $CT$ harmonic oscillator. We call this regime the elliptic region. For $\eta = \pm 1$ we have the parabolic regime, and for $|\eta| > 1$ we have the hyperbolic regime, where the solutions either diverge or collapse to zero in the limit of infinite time. It is this which creates the cutoff. If we attempted to create particle states with momentum in the hyperbolic regime, we would find physically unacceptable behaviour occurring in matrix elements after a sufficiently long time. Such states would not be stationary in the conventional sense. An analogous phenomenon occurs in $CT$ quantum wave mechanics, where we reject solutions to wave equations on the basis that they have unacceptable behaviour at large spatial distances.

6 The vacuum functional

A fundamental problem in quantum gauge field theory is the construction of the vacuum functional $Z[j]$. In this section we shall first solve the functional differential equations obtained via the $DT$ Schwinger action principle to obtain the pure electromagnetic vacuum functional in the presence of external sources in the Coulomb gauge. A functional integral approach then becomes of interest as an independent means of calculating the same quantity and confirming that the $DT$ Schwinger action principle is sound.

6.1 The $DT$ Schwinger vacuum functional

In the Coulomb gauge we take $\phi_n$ to be a classical field and the $A_n$ to be quantum fields, satisfying the $DT$ Schwinger action principle relations

$$\phi_x Z[j] = \frac{i\delta}{\delta\rho_x} Z[j], \quad \langle A^i_x \rangle_j = \frac{-i\delta}{\delta j^i_x} Z[j]$$

(137)

If now we take the equations of motion

$$\phi_x = \oint_y G_{C} G_{C-y}\rho_y$$

(138)

$$\Box_x \langle 0^\text{out}| A^i_x |0^\text{in}\rangle_j = \left( j^c_i + \partial^c_i \partial^c_j \oint_y \Delta F_{x-y} j^c_j \right) Z[j]$$

(139)

and write the vacuum functional as the product

$$Z[j] = Z[\rho] Z[j]$$

(140)

then we may readily integrate the resulting functional differential equations to find the $DT$ Schwinger action functional

$$Z[j] \sim \exp \left\{ -\frac{1}{2} \oint_x \oint_y \left[ \rho_x G_{C-x-y} \rho_y + j^i_x \Delta F_{x-y} j^i_y - \oint_z \nabla \cdot j_x \Delta F_{x-y} G_{C-y} \nabla \cdot j_x \right] \right\}.$$

(141)
### 6.2 The \( DT \) Faddeev-Popov vacuum functional

It is a remarkable feature of quantum gauge field dynamics that we may use two completely different routes to determine the \( CT \) vacuum functional \( Z[j] \) expressed as a path integral in \( CT \) field theory. Superficially these seem quite different. One way is to work out the details of the physical phase space dynamics via Dirac’s constraint mechanics and then define \( Z[j] \) in terms of the non-redundant or physical degrees of freedom. The other route is via the Faddeev-Popov gauge symmetry approach. These apparently unrelated approaches give the same result.

The former approach involves the Hamiltonian \[14, 15\] and therefore has no analogue in \( DT \) mechanics as far as we understand at this time (because we do not have a Hamiltonian approach in our mechanics, there being no generator of continuous translations in discrete time).

Fortunately, the Faddeev-Popov method approach uses gauge symmetry arguments to achieve the same end, and there is nothing in those arguments which prevents us from employing exactly the same method in \( DT \) gauge theory. By following the standard arguments \[14, 15\] we arrive at the expression

\[
Z[j] \sim \prod_n \int [dA_n] [d\phi_n] \Delta_g [A,\phi] \delta[g(A,\phi)] \exp \{iS[j]\} \tag{142}
\]

where \( g \) is the gauge fixing function. Here we have taken the gauge fixing function to reside on the nodes. This is consistent with the Coulomb gauge discussed next and with the \( DT \) analogues of the Landau and Feynman gauges discussed after that.

If we choose the Coulomb gauge then

\[
g_n = \nabla \cdot A_n, \tag{143}
\]

the \( \Delta_g \) factor is independent of the fields, and the action integral reduces to

\[
S[j] = \sum_x \left\{ -\frac{1}{2} \nabla^2 \phi - \frac{1}{2} A \cdot \Box A - \rho \phi + j \cdot A \right\}. \tag{144}
\]

Then using the functional integral result

\[
\int [d\psi] \exp \left\{ i \int dx \psi \mathcal{M} \psi + i \int dx j \psi \right\} \sim \sqrt{\det M^{-1}} \exp \left\{ -\frac{1}{2} i \int dx j M^{-1} j \right\} \tag{145}
\]

we recover the Schwinger vacuum functional \[144\] exactly. This confirms the consistency of our approach.

### 7 More general gauges

The Faddeev-Popov approach allows us to consider more general gauges. We now follow the standard gauge fixing approach to construct the \( DT \) analogues of the Landau and Feynman gauge propagators. First we note that the vacuum functional \[142\] is independent of the gauge fixing function, so we can choose

\[
g_x \equiv \Lambda_x - \omega_x, \tag{146}
\]
where \( \omega_x \equiv \omega_n(x) \) is an arbitrary function on DT space-time and \( \Lambda \) is the DT Lorentz function (33). Now we can functionally integrate over \( \omega \), weighting the integrand by a suitable weighting function. The essential step here is to choose the weighting function to be the exponential

\[
\exp \left( \frac{-i}{2\alpha} \int_x \omega_x N(U_n) \omega_x \right)
\]

(147)

where \( N(U_n) \) is some \( T \)-symmetric operator to be determined and \( \alpha \) is a constant. After integrating out \( \omega \) via the functional delta function, we can take the vacuum functional to be given by

\[
Z[j] \sim \int [dA] [d\phi] \exp \{ iS_0 + iS_G + iS_j \},
\]

(148)

where \( S_0, S_G \) and \( S_j \) are the free electromagnetic action sum, the gauge fixing term and the source term respectively. We now consider these three terms separately. The most efficient approach is to work with the four component object

\[
A_\mu^0 \equiv (\phi_n, A_n),
\]

(149)

which looks like a Lorentz four-vector, but it should be stressed that this is a matter of convenience only. For one thing, \( A_0^0 \equiv \phi_n \) is a link variable whereas the other components are node variables.

Suppressing the space-time indices \((n, x)\), we find

\[
S_0 \equiv \oint \left\{ -\frac{1}{2} A^i \Box A^i + \frac{1}{2} \partial_i A^i S \partial_j A^j - \frac{1}{2} \phi \nabla^2 \phi + \partial_i \phi D^+ A^i \right\}
\]

\[
= \frac{1}{2} \oint A^\mu R_{\mu\nu} A^\nu
\]

(150)

where the operator

\[
R_{\mu\nu} \equiv \left[ \begin{array}{cc} -\nabla^2 & -D^+ \partial_i \\ -D^- \partial_i & -\Box \delta_{ij} - S \partial_i \partial_j \end{array} \right].
\]

(151)

written in \((1, 3) \times (1, 3)\) block form acts to the right.

The gauge fixing term is given by

\[
S_G \equiv -\frac{1}{2\alpha} \oint (D^- \phi + S \partial_i A^i) N (D^- \phi + S \partial_i A^i)
\]

\[
= +\frac{1}{2} \alpha^{-1} \oint A^\mu S_{\mu\nu} A^\nu
\]

(152)

where the operator

\[
S_{\mu\nu} = \left[ \begin{array}{cc} ND^2 & ND^+ S \partial_i \\ NSD^- \partial_i & NS^2 \partial_i \partial_j \end{array} \right]
\]

(153)

acts to the right. The current term is just

\[
S_j \equiv - \oint j_\mu A^\mu,
\]

(154)
where \( j_\mu \equiv (\rho, -j) \). Hence

\[
Z [j] \sim \int [dA] \exp \left\{ \frac{1}{2} i \oint A^\mu M_{\mu\nu} A^\nu - i \oint j_\mu A^\mu \right\}
\]

(155)

where

\[
M_{\mu\nu} \equiv R_{\mu\nu} + \alpha^{-1} S_{\mu\nu} = \begin{bmatrix}
-\nabla^2 + \alpha^{-1} ND^2 & (\alpha^{-1} NS - 1) D^+ \partial_i \\
(\alpha^{-1} NS - 1) D^- \partial_i & -\Box \delta_{ij} + (\alpha^{-1} NS^2 - S) \partial_i \partial_j
\end{bmatrix}
\]

(156)

Assuming \( M_{\mu\nu} \) has an inverse, we may integrate to find

\[
Z [j] \sim \exp \left\{ \frac{i}{2} j_\mu G^{\mu\nu} j_\nu \right\}
\]

(157)

where

\[
M_{\mu\nu} G^{\nu\lambda} = -\delta^\lambda_\mu.
\]

(158)

The calculation of \( G^{\nu\lambda} \) goes as follows. We note (158) is equivalent to

\[
\oint_{x-y} \nabla_x \delta_{x-y} G^{\nu\lambda}_{y-z} = -\delta^\lambda_\mu \delta_{x-z},
\]

(159)

i.e

\[
\oint_{x} e_{px} \nabla_{x} G^{\nu\lambda}_{x} = -\delta^\lambda_\mu.
\]

(160)

Assuming \( G^{\nu\lambda}_x \) has a DT fourier transform \( \tilde{G}^{\nu\lambda}_q \), we may write

\[
\oint_{x} e_{px} \nabla_{x} \tilde{G}^{\nu\lambda}_q = -\delta^\lambda_\mu.
\]

(161)

Now \( \tilde{e}_{qx} \) is an eigenfunction of the operator \( \nabla_{x} \), so we may write

\[
\nabla_{x} \tilde{e}_{qx} = \tilde{M}_{\mu\nu q} \tilde{e}_{qx}
\]

(162)

where

\[
\tilde{M}_{\mu\nu q} \equiv \begin{bmatrix}
q^2 + \alpha^{-1} NuD^2 u & i (1 - \alpha^{-1} NuSu) D^- uq^i \\
i (1 - \alpha^{-1} NuSu) D^+ uq^i & q^2 \delta_{ij} - (\alpha^{-1} NuS^2 u - Su) q^i q^j
\end{bmatrix}
\]

(163)

in block form, where \( q \equiv (u, q) \). Hence we find

\[
\tilde{M}_{\mu\nu p} \tilde{G}^{\nu\lambda}_p = -\delta^\lambda_\mu.
\]

(164)

Now if we write

\[
\tilde{M}_{\mu p} \equiv \begin{bmatrix}
a & b^p T \\
b^p & c + dpp^T
\end{bmatrix}
\]

(165)
then a solution is
\[
A = -\frac{c + dp^2}{ac + p^2\Delta}, \quad B = \frac{b^*}{ac + p^2\Delta}, \quad C = \frac{1}{c}, \quad D = \frac{\Delta}{c[ac + p^2\Delta]}
\]
(166)
where \(\Delta \equiv ad - b^*b\). Now from (163) we read off
\[
a = p^2 + \alpha^{-1}NzD^2z, \quad b = i(1 - \alpha^{-1}NzSz)D^{-1}z
\]
\[
c = p^2, \quad d = Sz - \alpha^{-1}NzS^2z
\]
(167)
and now we may readily compute the components of \(\tilde{G}^{\nu\lambda}_{\mu}\).

The solution is complicated in the case of arbitrary \(N\). We may now make a careful choice which reduces the complexity and brings us closer to the CT results. The particular choice which works here is
\[
Nz \equiv \frac{1}{Sz},
\]
(168)
which means that \(N(U_n)\) is the operator inverse of \(S(U_n)\).

Some concern may be expressed at this choice, as it involves step operators of all orders and may be undefined for some sequences. However, we can allay fears by noting that in the DT fourier transform space, the reciprocal of \(Sz \equiv (z + 4 + 1/z)/6\) exists, provided we are on the unit circle in the complex \(z\) plane. We may use this to define the inverse operator \(N(U_n) = S^{-1}(U_n)\) by its action in DT fourier transform space, viz
\[
S^{-1}(U_n)f_n = \oint_{sz} \tilde{f}_z z^n S^z
\]
(169)
We note that \(Sz\) has two zeros, located at \(z = -2 + \sqrt{3}\) and \(z = -2 - \sqrt{3}\). The first is inside the unit circle and contributes an additional simple pole away from the origin. Hence the above integral should be well defined and readily evaluated using the calculus of residues.

With the above choice for \(Nz\), the DT Landau gauge propagator is obtained by setting \(\alpha = 0\) and then we find
\[
\tilde{G}^{\mu\nu}_L(p) = \frac{1}{p^2p^2} \begin{bmatrix}
-S^2zp^2 & iSzD^-zp^T \\
iSzD^+zp & -p^2\delta_{ij} - Szp^ip^j
\end{bmatrix}
\]
(170)
in block form. This tends to the correct CT Landau gauge propagator
\[
\tilde{G}^{\mu\nu}_L(p) = \frac{1}{p^2} \left( \eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right)
\]
\[
= \frac{1}{p^2p^2} \begin{bmatrix}
-p^2 & -p^0p^T \\
-p^0p & -p^2\delta_{ij} - p^ip^j
\end{bmatrix}
\]
(171)
in the CT limit
\[
T \to 0, \quad Sz \to 1, \quad -iD^+z \to p^0, \quad -iD^-z \to p^0.
\]
(172)
Here we have used the reparametrisation
\[ z \equiv e^{i\theta} \] (173)
where
\[ \theta = Tp^0 + O(T^3), \] (174)
as discussed in [5].

The DT Feynman gauge propagator is obtained by setting \( \alpha = 1 \) and choosing \( Nz = (Sz)^{-1} \) gives
\[ \tilde{G}_F^{\mu\nu}(p) = \frac{1}{p^2} \left[ \begin{array}{cc} Sz & 0^T \\ 0 & -\delta_{ij} \end{array} \right], \] (175)
which tends to the CT Feynman gauge propagator
\[ \tilde{G}_F^{\mu\nu}(p) = \frac{\eta^{\mu\nu}}{p^2} = \frac{1}{p^2} \left[ \begin{array}{cc} 1 & 0^T \\ 0 & -\delta_{ij} \end{array} \right] \] (176)
in the CT limit, as expected.

Two comments on this last result are in order.

1. Our theory has not assumed any metric structure to space-time. We note the appearance of a factor \( Sz \) in the \( 0-0 \) component in (175). The precise relationship of our DT space-time and any DT analogue of the CT metric tensor in special and general relativity awaits further investigation;

2. In the DT Feynman gauge we find the first functional derivatives with respect to the external sources of the vacuum functional gives
\[ \langle 0_{\text{out}}|\phi_x|0_{\text{in}}\rangle_j = \frac{i\delta}{\delta \rho_x} Z[j] = S_n \rho_n Z[j], \] (177)
which is consistent with the classical equation (88).

8 Concluding remarks

The stage is now set for the next paper in this series, which will be to develop the DT Feynman rules for QED and investigate the renormalization programme. One of the principal motivations of this series was the possibility that the introduction of a new scale \( T \) would allow a more convincing approach to the elimination of divergences than other formulations. Whether this works remains to be seen. We note that one of the early and successful regularisation methods, the Pauli-Villiars technique, employs the introduction of a very large mass scale. In some sense, this is the other side of the coin to our approach, which is to consider very small time scales. Either way, there is a suggestion or hint that something intimately involved with cosmological scales, either extremely large or extremely small, is at the core of the problems with field theory.
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A A first order formulation

In the first paper of this series [3] we interpreted Cadzow’s equations of motion (46) as an expression of momentum conservation at the node sites. The momentum in that context is clearly a node quantity. We shall call such a momentum a node momentum. In this section we consider the possibility of introducing link variables related to node variables via Legendre transformations and these we shall call link momenta. The node and link momentum concepts are different in DT mechanics, but it turns out that in the CT limit $T \to 0$ they coincide. It is part of the problem of quantisation in DT mechanics that it is not clear a priori whether node or link momenta should be used in setting up canonical quantisation algebras such as the Weyl-Heisenberg algebra. In the previous papers of this series we avoided this question by using the DT analogue of the Schwinger action principle to obtain quantum commutation relations consistent with our DT mechanics.

The reason we should consider link momenta is because there is more than one way to arrive at the Euler-Lagrange equations for configuration space variables in CT mechanics. The usual way is called the second order formulation, wherein we start with the Lagrangian $L \equiv L(q, \dot{q}, t)$ and use Hamilton’s principle directly to get the Euler-Lagrange equations

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) = \frac{\partial L}{\partial q},$$

(178)

Another route is via the so-called first order formulation, which is related to the Hamiltonian phase space approach, but pretends to live in an extended configuration space. We define the conjugate momenta

$$p \equiv \frac{\partial L}{\partial \dot{q}}.$$

(179)

construct the Hamiltonian $H(p, q, t)$, and then define the first order Lagrangian

$$\tilde{L} \equiv p \cdot \dot{q} - H.$$

(180)

Now we apply Hamilton’s action principle to $\tilde{L}$ and get the extended configuration space equations

$$\frac{d}{dt} \left( \frac{\partial \tilde{L}}{\partial \dot{q}} \right) - \frac{\partial \tilde{L}}{\partial q} = 0,$$

(181)

$$\frac{\partial \tilde{L}}{\partial p} = 0.$$

(182)

These equations are disguised versions of Hamilton’s equations.
\[ \dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = \frac{\partial H}{\partial q}. \]  

(183)

in phase space and are equivalent to the CT Euler-Lagrange equations obtained from the second order formulation. This approach to configuration space mechanics is the basis of the ADM formulation of quantum gravity [16]. We note that equations (181) and (182) are remarkably similar to the two sorts of DT equations of motion (46 and 47) for node and link variables respectively. This leads us to expect the DT analogue of phase space momentum to be a link variable, rather than a node variable.

When it comes to gauge theories, the construction of link momentum for matter fields is altered in a subtle way related to the virtual paths used to maintain gauge invariance, but none of this subtlety is seen or required in the CT limit.

For completeness, we show that we may rewrite the free electromagnetic equations in a first order formulation. We introduce the gauge invariant link variable \( \pi_n \) and define the system function

\[ \mathcal{F}_1^n \equiv \pi_n \cdot D_n^+ A_n - \frac{1}{2} \pi_n \cdot \pi_n + \pi_n \cdot \nabla \phi_n - \frac{1}{4} \langle B_n \nabla \cdot B_n \rangle. \]  

(184)

Then the equations of motion are

\[ D_n^- \pi_n + S_n \nabla \times B_n = 0, \]  

(185)

\[ \nabla \cdot \pi_n = 0, \]  

(186)

\[ \pi_n = D_n^+ A_n + \nabla \phi_n, \]  

(187)

which are equivalent to the second order equations derived above. By inspection, we see that \( \pi_n = -E_n \), which was to be expected from an analysis of the CT first order formulation.

### B Invariants of the motion

Consider a symmetry of the system function, i.e. a transformation of the node and link variables

\[ A_n^\alpha \rightarrow A_n^\alpha + \delta A_n^\alpha, \quad \phi_n^\alpha \rightarrow \phi_n^\alpha + \delta \phi_n^\alpha \]  

(188)

such that the system function is left unchanged. Then

\[ \delta F^n = \int d^3x \left\{ \frac{\delta F^n}{\delta A_n^\alpha(x)} \frac{\delta}{\delta A_n^\alpha(x)} + \frac{\delta F^n}{\delta A_{n+1}^\alpha(x)} \frac{\delta}{\delta A_{n+1}^\alpha(x)} \right. \]

\[ + \sum_{n=M}^{N-1} \delta \phi_n(x) \frac{\delta}{\delta \phi_n(x)} F^n \right\} = 0 \]  

(189)

Using the DT equations of motion (46, 47) we find the invariant of the motion

\[ C_n \equiv T \int d^3x \delta A_n^\alpha(x) \left( \frac{\delta}{\delta A_n^\alpha(x)} F^n + \nabla \cdot \left( \frac{\partial F^n}{\partial A_n^\alpha(x)} \right) \right) = C_n^{n+1}, \]  

(190)
which we refer to as a Maeda-Noether invariant \cite{17}. Although variations of the link variables are involved in the overall variation (and indeed are essential), partial derivatives with respect to link variables do not occur in the explicit construction of the above invariant. This underlines the fact that link variables are not equivalent to node variables in a dynamical sense.
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