Large Deviations Principles for Langevin Equations in Random Environment and Applications∗

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Abstract

In contrast to the study of Langevin equations in a homogeneous environment in the literature, the study on Langevin equations in randomly-varying environments is relatively scarce. Almost all the existing works require random environments to have a specific formulation that is independent of the systems. This paper aims to consider large deviations principles (LDPs) of Langevin equations involving a random environment that is a process taking value in a measurable space and that is allowed to interact with the systems, without specified formulation on the random environment. Examples and applications to statistical physics are provided. Our formulation of the random environment presents the main challenges and requires new approaches. Our approach stems from the intuition of the Smoluchowski-Kramers approximation. The techniques developed in this paper focus on the relation between the solutions of the second-order equations and the associate first-order equations. We obtain the desired LDPs by showing a family of processes enjoy the exponential tightness and local LDPs with an appropriate rate function.

Keywords. Langevin equations, statistical physics, large deviations principle, Smoluchowski-Kramers approximation

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Running Title. LDPs of Langevin equation in Random Environment

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1 Introduction

Langevin equations are used to describe the motion of particles in a fluid due to collisions with the molecules of the fluid; see e.g., [18], which have been studied intensively in both mathematics and physics communities. Take for instance, small particles with strong damping [4], which are formulated as

$$\dot{x}_\varepsilon(t) = b(x_\varepsilon(t)) - \frac{\lambda}{\varepsilon} \dot{x}_\varepsilon(t) + \sqrt{\varepsilon \sigma} \dot{B}(t).$$

By letting $X_\varepsilon^t = x_\varepsilon(t/\varepsilon)$, $w(t) = \sqrt{\varepsilon}B(t/\varepsilon)$, we obtain the so-called chemical Langevin equation

$$\varepsilon^2 \dot{X}_\varepsilon^t = b(X_\varepsilon^t) - \lambda \dot{X}_\varepsilon^t + \sqrt{\varepsilon \sigma} \dot{w}(t).$$

Likewise, the motion of a small particle with mass $\mu$ in the force field $b(x) + \sqrt{\varepsilon} \sigma \dot{w}$ with a friction proportional to the velocity and the friction coefficient $\lambda$ is described by the following equation due to the Newton law

$$\mu \ddot{x}_\varepsilon^t = b(x_\varepsilon^t) - \lambda \dot{x}_\varepsilon^t + \sqrt{\varepsilon \sigma} \dot{w}(t).$$

When $\mu = \varepsilon^2$, this equation becomes (1.1). Much effort is devoted to the study of equation (1.1) and its applications; see e.g., [3] [5] [6] [11] [23] and references therein.

While a time-homogeneous environment is usually used with the force field $b$ not depending on any other random process, we consider a randomly-varying environment in this work. We consider

$$b(x) \sim b(t, x, \xi), \quad \lambda \sim \lambda_\varepsilon(t, x), \quad \sigma \sim \sigma_\varepsilon(t, x),$$

where $\xi$ indicates the random environment, which may or may not interact with the system. As a consequence, equation (1.1) becomes

$$\begin{align*}
\varepsilon^2 \dot{X}_\varepsilon^t = b(t, X_\varepsilon^t, \xi_{t/\varepsilon}) - \lambda_\varepsilon(t, X_\varepsilon^t)X_\varepsilon^t + \sqrt{\varepsilon \sigma_\varepsilon(t, X_\varepsilon^t)} \dot{w}(t), \\
X_0^\varepsilon = x_0 \in \mathbb{R}^d, \quad \dot{X}_0^\varepsilon = x_1 \in \mathbb{R}^d,
\end{align*}$$

(1.2)

where $w(t)$ is an $m$-dimensional standard Brownian motion and $\dot{w}(t)$ is its formal derivative, $\xi_{t/\varepsilon}$ is a random process, which may or may not depend on $X_\varepsilon^t$, $w(t)$ and which takes value in a measurable space $\mathcal{M}$ describing how the status of environment changes randomly in time and state. The fast scale $\xi_{t/\varepsilon}$ is, in fact, obtained after rescaling $X_\varepsilon^t = x_\varepsilon(t/\varepsilon)$.

Natural and important questions in mathematical physics and statistical mechanics include: What is the asymptotic behavior of $\{(X_\varepsilon^t)\}_{\varepsilon > 0}$? Can we obtain an averaging principle for $\xi$? Can such a second-order system be approximated by the corresponding overdamped system (the Smoluchowski-Kramers approximation)? What is the tail probability of the convergence? We aim to address these questions by obtaining a large deviations principle (LDP for short) for $\{X_\varepsilon^t\}_{\varepsilon > 0}$ in the space of continuous functions. Large deviations principles play an important role in equilibrium and non-equilibrium statistical mechanics, multi-fractals, and thermodynamic formulation of chaotic systems; see [8] [9] [26] and references therein.

In this paper, we first establish a LDP of (1.2). The result will then be specified under different settings of $\xi_{t/\varepsilon}$ such as diffusion processes, jump processes, and a Markov switching environment. Applications to mathematical physics and statistical mechanics are then treated. The classical Smoluchowski-Kramers approximation is dealt with in the presence of another random process interacting with the system.

Let us assume that for each fixed time $t$ and fixed state $X_\varepsilon^t = x$, as $\varepsilon \to 0$, $\xi_{t/\varepsilon}$ (which may depend on both $t$ and $X_\varepsilon^t$) has an invariant measure denoted by $\pi_t,x$. Intuitively, as $\varepsilon \to 0$, the
behavior of equation (1.2) takes 3 phases. Phase 1, letting $\varepsilon^2 \to 0$, (1.2) behaves as an overdamped Langevin equation

$$\dot{X}_t^\varepsilon = \frac{\varepsilon \sigma(t, X_t^\varepsilon)}{\lambda_\varepsilon(t, X_t^\varepsilon)} w(t).$$

Phase 2, letting $\varepsilon \to 0$, the ergodicity of $\xi_{t/\varepsilon}$ leads to the approximation

$$\dot{X}_t^\varepsilon = \frac{\overline{b}(t, X_t^\varepsilon)}{\lambda_0(t, X_t^\varepsilon)} + \sqrt{\varepsilon} \sigma(t, X_t^\varepsilon) \dot{w}(t),$$

where $\overline{b}(t, x) := \int_{\mathcal{M}} b(t, x, z) \pi_{t,x}(dz)$. Phase 3, letting $\sqrt{\varepsilon} \to 0$, the small diffusion presents less influence and the system tends to be concentrated on the averaged system

$$\overline{X}_t = \frac{\overline{b}(t, \overline{X}_t)}{\lambda_0(t, \overline{X}_t)},$$

where $\lambda_0$ is a limit (as $\varepsilon \to 0$) of sequence of functions $\{\lambda_\varepsilon\}$. Not only our work provides a rigorous analysis for these intuitions, but also show that the tail probability of the convergence is exponentially small under appropriate conditions.

**Related works.** In mathematical physics and statistical mechanics, Langevin equations [18], and stochastic acceleration [16, 17] among others, were studied in [5, 11] for the Smoluchowski-Kramers approximation, [4, 6] for the LDPs and MDPs (moderate deviations principles) of Langevin equations in the absence of the random environment, and [23] for LDPs of Langevin equations under the random environment given by a Markov switching process taking values in a finite set with a fast jump rate. However, the studies of the subject involving random environment is still scarce. Moreover, almost all of the existing works requires the random fields be independent of the system and/or have specific formulation.

Because much attention has been devoted to the study of large deviations principles (LDPs) for families of stochastic processes given by first-order stochastic differential equations (SDEs), LDPs for the first-order SDEs have been relatively well understood. Consider the following SDE

$$dY^\varepsilon(t) = b(t, Y^\varepsilon(t), \xi_{t/\varepsilon}) dt + \sqrt{\varepsilon} \sigma(t, Y^\varepsilon(t), \xi_{t/\varepsilon}) dw(t),$$

where $w(t)$ is a standard Brownian motion and $\xi_{t/\varepsilon}$ is a random process that may or may not depend on $w(t)$ and $Y^\varepsilon(t)$. When $\xi_{t/\varepsilon}$ is deterministic almost surely, the study of large deviations is an extension of the Freidlin-Wentzell theory [12]. When $\xi_{t/\varepsilon}$ is a random process independent of $B(t)$ and $Y^\varepsilon(t)$, the LDPs of $\{Y^\varepsilon\}_{\varepsilon > 0}$ has been addressed in [20] for $\xi_{t/\varepsilon}$ being a fast diffusion process having coefficients independent of $Y^\varepsilon(t)$ and driven by another Brownian motion independent of $w(t)$, in [14] for $\xi_{t/\varepsilon}$ being an exponentially ergodic process taking values in a general measurable space, and in [15] for $\xi_{t/\varepsilon}$ being a Markovian switching process taking values in a finite set. From another angle, much effort has been devoted to the study of the coupled system (i.e., $\xi_{t/\varepsilon}$ depending on $Y^\varepsilon(t)$ and $w(t)$). Perhaps one of the natural expressions is to assume $\xi_{t/\varepsilon}$ to be a solution of a fast-varying stochastic differential equation in the setting of fast-slow SDEs. Such cases have been studied in [27, 28] for some coupled systems in which some coefficients do not depend on both slow and fast processes, and in [24] for fully-coupled systems with all coefficients depending on both slow and fast processes and with the driving noises being correlated. Moreover, fully-coupled systems in which $\xi_{t/\varepsilon}$ being a jump process taking values in a finite set was considered in [3]. However, in contrast to the works on the first-order equations, the study on the second-order equations is still scarce.
Our contributions. In this work, we provide a new approach compared with [4, 6, 23] as well as the works for first-order equations, and generalize the results in [4, 5, 11, 23] in statistical physics applications. We study Langevin equations in a random environment that is not assumed to have a specific form and allowed to interact with the system. The formulations on state space and the process $\xi_t$ are main challenges and require careful handling and new approaches. Without specific structure of $\xi_t$, we could not have a representation formula as in [1, 2] for the solution processes. Thus the weak convergence approach of [4] and the results in [23] are no longer applicable because we do not assume any specific structure of $\xi_t$ and do not assume the noise moves much faster. By establishing the LDP for a Langevin dynamics in random fields, we provide some insight into the statistical inference for the motions of a net of small particles, which is shown to be equivalent to homogeneous environment obtained by averaging. This fact plays an important role in practice because, typically, the heterogeneity is often much difficult to analyze and simulate than the homogeneity. We also generalize the principle of least action for the environment with the presence of the heterogeneity. From a technical point of view, it is the first work consider the large deviations of a second-order stochastic differential equations in random environment, without specific formulation for the random environment.

Our method and approach. Our techniques and method rely on the relation between the solutions of the second-order equations and the associate first-order equations. Our approach stems from the intuition of the Smoluchowski-Kramers approximation. Our proof of the main results is based on the property that if a family of processes enjoys the exponential tightness and a local LDP with an appropriate rate function, then it satisfies the LDP with the same rate function. One of the difficulties stems from handling the diffusion part of the solution of (1.2) with a term $\frac{1}{\varepsilon^2} \int_0^t H_\varepsilon(s)ds$, where

$$H_\varepsilon(t) := \sqrt{\varepsilon} e^{-A_\varepsilon(t)} \int_0^t e^{A_\varepsilon(s)} \sigma_\varepsilon(s, X_\varepsilon^s)dw(s), \quad A_\varepsilon(t) := \frac{1}{\varepsilon^2} \int_0^t \lambda_\varepsilon(r, X_\varepsilon^r)dr.$$ 

The large factor $\frac{1}{\varepsilon}$ in $\frac{1}{\varepsilon^2} \int_0^t H_\varepsilon(s)ds$ requires detailed estimates for $H_\varepsilon(t)$. But we cannot move $e^{-A_\varepsilon(t)}$ inside the stochastic integral in Itô’s sense. But, we do need this random variable $(e^{-A_\varepsilon(t)})$ to balance the large factor $e^{A_\varepsilon(s)}$ inside the stochastic integral.

To prove the exponential tightness, we use an extended Puhalskii’s criteria [19, Theorem 3.1] and [10, Remark 4.2]. The challenge in this part is to estimate $H_\varepsilon(t)$ with high probability, which cannot be handled in Itô’s sense or by martingale estimates. By using regularity of the solution to interpret $H_\varepsilon(t)$ in pathwise sense and its suitable decomposition, we are able to obtain desired properties needed for the exponential tightness. Under the assumption on the local LDP of the family of solutions of associated to the first-order equations (obtained by taking the intuition of the Smoluchowski-Kramers approximation), the family of processes $\{X^\varepsilon\}_{\varepsilon > 0}$ satisfies a local LDP. Here, we need to connect the solutions of the second-order and the associate first-order equations. Although we do not expect they are exponentially equivalent, we expect that they are exponentially equivalent in a “local sense”. The challenge here is that we need a term, which leverages the decay (as $\varepsilon \to 0$) of the distance between solutions of the second-order and the first-order equations. By looking at the behavior around a fix function $\varphi$ from auxiliary frozen systems, we are able to replace $H_\varepsilon$ by a stochastic process, in which we can move the random variable from outside to inside the stochastic integral. It is also noted that even when these types of stochastic integrals can be understood in Itô’s sense, they are no longer martingales. However, by using techniques borrowed from handling stochastic convolution, used in stochastic partial differential equations, we can obtain the desired estimates.

Outline of the paper. The rest of paper is arranged as follows. Section 2 formulates the problem
and states our main results. Section 3 is devoted to some specific $\xi$'s, applications, and discussions. The proofs of main results are given in Section 4. To avoid the interruption, proofs of some technical results needed in the proof of main results are postponed to an appendix.

## 2 Formulation and Main Results

We use $|\cdot|$ to denote the Euclid norm for vectors or matrices, $\langle \cdot, \cdot \rangle$ the inner product, and $C([0,1], \mathbb{R}^d)$ the space of continuous functions on $[0,1]$ endowed with the sup-norm $\| \cdot \|$. Denote by $\nabla_t$ and $\nabla_x$ the partial derivatives with respect to the variables $t$ and $x$, respectively. We work with $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, a complete filtered probability space with the filtration satisfying the usual condition. Let $w(t)$ be an $m$-dimensional standard Brownian motion and $\xi(t)$ be a random process that may or may not depend on $w(t)$ and that take values in a measurable space $\mathcal{M}$. We use the letter $C$ with or without subscripts to represent a generic positive constant, whose values may change for different usage. The letters $\hat{C}$ and $\tilde{C}$ with or without subscripts are constants to be specified later. The constants $C$, $\hat{C}$, and $\tilde{C}$ are independent of $\varepsilon$. We begin with the following definition; see e.g., [9].

**Definition 2.1.** A family of stochastic processes $\{Y^\varepsilon\}_{\varepsilon \geq 0}$ in $C([0,1], \mathbb{R}^d)$ is said to enjoy the large deviations principle (LDP) with a rate function $I$ if the following conditions are satisfied:

- $I : C([0,1], \mathbb{R}^d) \rightarrow [0, \infty]$ is inf-compact, that is, the level sets $\{I(f) \leq L\}$ are compact in $C([0,1], \mathbb{R}^d)$ for any $L > 0$.
- For any open subset $G$ of $C([0,1], \mathbb{R}^d)$,
  $$\lim_{\varepsilon \rightarrow 0} \inf \varepsilon \log \mathbb{P}(Y^\varepsilon \in G) \geq -I(G) := -\inf_{f \in G} I(f).$$
- For any closed subset $F$ of $C([0,1], \mathbb{R}^d)$,
  $$\limsup_{\varepsilon \rightarrow 0} \varepsilon \log \mathbb{P}(Y^\varepsilon \in F) \leq -I(F) := -\inf_{f \in F} I(f).$$

Our main goal of this paper is to establish a LDP for the family of processes $\{X^\varepsilon = (X^\varepsilon_t)_{t \in [0,1]}\}_{\varepsilon \geq 0}$, which are solutions of the second-order stochastic differential equations (SDEs) with random environment given by

$$\begin{cases} 
\varepsilon^2 \ddot{X}_t^\varepsilon = b(t, X_t^\varepsilon, \xi_t/\varepsilon) - \lambda_\varepsilon(t, X_t^\varepsilon) \dot{X}_t^\varepsilon + \sqrt{\varepsilon} \sigma_\varepsilon(t, X_t^\varepsilon) \dot{w}(t), \\
X_0^\varepsilon = x_0 \in \mathbb{R}^d, \quad \dot{X}_0^\varepsilon = x_1 \in \mathbb{R}^d.
\end{cases}$$

(2.1)

With $X^\varepsilon$ denoting the solution of (2.1), the pair $(X^\varepsilon, p^\varepsilon)$ is the solution of the following system of first-order SDEs

$$\begin{cases} 
\dot{X}_t^\varepsilon = p_t^\varepsilon, \quad X_0^\varepsilon = x_0 \in \mathbb{R}^d, \\
\varepsilon^2 \dot{p}_t^\varepsilon = b(t, X_t^\varepsilon, \xi_t/\varepsilon) - \lambda_\varepsilon(t, X_t^\varepsilon) p_t^\varepsilon + \sqrt{\varepsilon} \sigma_\varepsilon(t, X_t^\varepsilon) \dot{w}(t), \quad p_0^\varepsilon = x_1 \in \mathbb{R}^d.
\end{cases}$$

(2.2)

For simplicity, we assume $x_0, x_1$ to be non-random and fixed. More general cases can be handled similarly; see Remark 3. To proceed, we make the following assumptions on the coefficients of (2.1), almost all of them are similar to that used in the literature; see e.g., [4, 6, 23], and the assumption on the local LDPs for the corresponding first-order equations.
Assumption 2.1. Suppose that

- \( b(\cdot, \cdot, \cdot) : \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{M} \to \mathbb{R}^d \) is measurable and that there exists a constant \( C > 0 \) satisfying
  \[
  |b(t, x, \xi) - b(t, y, \xi)| \leq C |x - y|, \quad \text{for all } t \geq 0, x, y \in \mathbb{R}^d, \xi \in \mathcal{M};
  \]

- for each \( \varepsilon > 0 \), \( \sigma_\varepsilon(t, x) : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d \times m} \) is continuously differentiable functions with respect to \( t \) and \( x \) and satisfies
  \[
  \limsup_{\varepsilon \to 0} \sup_{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d} ||\sigma_\varepsilon(t, x)|| + ||\varepsilon^{-1} \nabla_t \sigma_\varepsilon(t, x)|| < \infty; \quad \sup_{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d} ||\nabla_x \sigma_\varepsilon(t, x)|| \leq C \varepsilon^2;
  \]

- for each \( \varepsilon > 0 \), the mapping \( \lambda_\varepsilon(\cdot, \cdot) : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R} \) twice continuously differentiable functions and satisfies
  \[
  \limsup_{\varepsilon \to 0} \sup_{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d} |\lambda_\varepsilon(t, x)| + |\nabla_t \lambda_\varepsilon(t, x)| < \infty; \quad \sup_{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d} |\nabla_x \lambda_\varepsilon(t, x)| + |\nabla_{xx} \lambda_\varepsilon(t, x)| \leq C \varepsilon^2;
  \]

\[
\kappa_0 := \liminf_{\varepsilon \to 0} \inf_{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d} \lambda(t, x) > 0.
\]

Assumption 2.2. [Assumptions on the first-order equation]. Assume that the family \( \{q_\varepsilon = (q_\varepsilon^t)_{t \in [0,1]}\}_{\varepsilon > 0} \) of solutions of the following stochastic differential equation

\[
\begin{align*}
\dot{q}_t^\varepsilon &= \frac{b(t, q_t^\varepsilon, t/\varepsilon)}{\lambda_\varepsilon(t, q_t^\varepsilon)} + \sqrt{\varepsilon} \sigma_\varepsilon(t, q_t^\varepsilon) \dot{w}(t), \\
q_0^\varepsilon &= x_0 \in \mathbb{R}^d,
\end{align*}
\]

satisfies the local LDP (see Definition 4.2) in \( \mathcal{C}([0,1], \mathbb{R}^d) \) with a rate function \( \hat{I}(\cdot) \), and \( \hat{I}(\varphi) = \infty \) if \( \varphi \) is not absolutely continuous.

Remark 1. Equation (2.3) is obtained by using the Smoluchowski-Kramers approximation. Intuitively, when \( \varepsilon \to 0 \), the terms being higher-order of \( \varepsilon \) in the second-order equation (2.1) converge to 0 much faster and then stay around 0 for a long time (compared with other terms). Therefore, roughly, as \( \varepsilon \to 0 \), equation (2.1) is approximated by

\[
0 = b(t, X_t^\varepsilon, t/\varepsilon) - \lambda_\varepsilon(t, X_t^\varepsilon) \dot{X}_t^\varepsilon + \sqrt{\varepsilon} \sigma_\varepsilon(t, X_t^\varepsilon) \dot{w}(t).
\]

As a result, we obtain (2.3).

Remark 2. As we mentioned briefly in the introduction, it is well-known that Assumption 2.2 on the local LDPs of the first-order SDEs (2.3) is not restrictive. The LDPs for the first-order SDEs in random environment are well-established with explicit rate function in the literature under different formulations of different random processes \( \xi_t \); see some examples of our results in Section 3.1.

Our main result is the LDP for the family of solutions of (2.1). This is given in the following theorem.

Theorem 2.1. Under Assumptions 2.1 and 2.2 the family of solutions \( \{X_\varepsilon\}_{\varepsilon > 0} \) of (2.1) satisfies the LDP in \( \mathcal{C}([0,1], \mathbb{R}^d) \) with rate function \( \hat{I}(\cdot) \) in Assumption 2.2.
**Specification:** Not only Theorem 2.1 gives us a limit as $\varepsilon \to 0$, but also tail probability estimate of the convergence. Let us assume that $\tilde{I}(\varphi) = 0$ has unique solution $\varphi^*$. [If we assume a specific form of $\xi_{t/\varepsilon}$, we can obtain explicit formula for $\tilde{I}$. In such a case, it can be verified that $\tilde{I}$ verifies the condition assumed in this remark; see Section 3.1.] Let $B(\varphi^*)$ be an arbitrary neighborhood of $\varphi^*$ and $B^c(\varphi)$ be its completion in $C([0,1],\mathbb{R}^d)$. We have $\tilde{I}(B^c(\varphi^*)) > 0$. Otherwise, if $\tilde{I}(B^c(\varphi)) = 0$, there exists $\{\varphi_k\}_{k=1}^\infty \subset B^c(\varphi^*)$ such that $\lim_{k \to \infty} \tilde{I}(\varphi_k) = 0$. Due to $\tilde{I}$ is good rate function, there exists a convergent subsequence (still denoted by $\varphi_k$) and with limit $\varphi^{**} \in B^c(\varphi^*)$. Since $\tilde{I}$ is lower semi-continuous, $0 \leq \tilde{I}(\varphi^{**}) = \tilde{I}(\lim_{k \to \infty} \varphi_k) \leq \liminf_{k \to \infty} \tilde{I}(\varphi_k) = 0$. It leads to $\tilde{I}(\varphi^{**}) = 0$, which is a contradiction. That means $\tilde{I}(B^c(\varphi^*)) > 0$. Thus, from the LDP of $\{X_t^\varepsilon\}_{\varepsilon > 0}$, the probability $\mathbb{P}(X_t^\varepsilon \in B^c(\varphi^*)) \approx \exp\{-\frac{\tilde{I}(B^c(\varphi^*))}{\varepsilon}\}$ tends to 0 exponentially fast.

**Remark 3.** It will be seen in the proof of Theorem 2.1 that the LDP of $\{X_t^\varepsilon\}_{\varepsilon > 0}$ still holds if the initial value $x_1$ depending on $\varepsilon$ (i.e., $x_1 = x_1^\varepsilon$) satisfies $\varepsilon x_1^\varepsilon$ being bounded as $\varepsilon \to 0$. For example, we may replace the initial condition $x_1$ by $x_1^\varepsilon = x_1/\varepsilon$, which occurs in some applications to physics after scaling the time. In a more general setting, we can also allow both $x_0, x_1$ to be random and depending on $\varepsilon$ as well. To be more specific, we can replace the initial values $x_0, x_1$ by $x_0^\varepsilon, x_1^\varepsilon$ and assume that $\limsup_{\varepsilon \to 0} \varepsilon |x_1^\varepsilon| < \infty$ a.s. and $\{x_0^\varepsilon\}_{\varepsilon > 0}$ obeys the LDP in $\mathbb{R}^d$ with the rate function $I_0$ so that the rate function for (2.3) becomes $I_0 + \tilde{I}$ in the sense that $(I_0 + \tilde{I})(\varphi) = I_0(\varphi_0) + \tilde{I}(\varphi)$ (see [24] Section 9). Then Theorem 2.1 still holds with the rate function $I_0 + \tilde{I}$.

**Remark 4.** The results presented in Theorems 2.1 as well as others in Section 3.1 can be extended to the space $C([0,T],\mathbb{R}^d)$ of continuous functions on $[0,T]$ endowed with the sup-norm topology for any $T > 0$. As a consequence, these LDPs still hold in $C([0,\infty),\mathbb{R}^d)$, the space of continuous function on $[0,\infty)$ endowed with the local supremum topology defined by the metric

$$
\sum_{n=1}^\infty \frac{1}{2^n} \left(1 \wedge \sup_{t \leq n} |\varphi_t - \psi_t|\right), \quad \forall \varphi, \psi \in C([0,\infty),\mathbb{R}^d).
$$

This fact follows from the Dawson-Gärtner theorem; see [9, Theorem 4.6.1], which states that it is sufficient to check the LDPs in $C([0,T],\mathbb{R}^d)$ for any $T$ in the uniform metric.

### 3 Specifications, Examples, and Discussions

In this section, we first provide several specifications of the process $\xi_{t/\varepsilon}$. Then we consider some examples in statistical mechanics.

#### 3.1 Special Cases of $\xi_{t/\varepsilon}$: Diffusion, Jump, and Switching Processes

**Diffusion processes.** We assume the noise process $\xi_t$ is given by a diffusion and then $\xi_{t/\varepsilon}$ is a fast diffusion in $\mathcal{M} = \mathbb{R}^l$. Thus, we consider the second-order system involving fast and slow processes

$$
\begin{align*}
\varepsilon^2 \ddot{X}_t^\varepsilon &= b(t, X_t^\varepsilon, Y_t^\varepsilon) - \lambda_0(t, X_t^\varepsilon) \dot{X}_t^\varepsilon + \sqrt{\varepsilon} \sigma_\varepsilon(t, X_t^\varepsilon) \dot{w}(t), \\
Y_t^\varepsilon &= \frac{1}{\varepsilon} F(t, X_t^\varepsilon, Y_t^\varepsilon) + \frac{1}{\sqrt{\varepsilon}} G(t, X_t^\varepsilon, Y_t^\varepsilon) \dot{w}(t), \\
X_0^\varepsilon &= x_0 \in \mathbb{R}^d, \quad \dot{X}_0^\varepsilon = x_1 \in \mathbb{R}^d, \quad Y_0^\varepsilon = y_0 \in \mathbb{R}^l,
\end{align*}
$$

(3.1)

where $Y_t^\varepsilon \in \mathbb{R}^l$, $F(t, x, y) : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^l \to \mathbb{R}^l$, $G(t, x, y) : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^l \to \mathbb{R}^{l \times n}$ are measurable functions, and $\dot{w}(t)$ is an $n$-dimensional standard Brownian motions. Moreover, we allow $\dot{w}(t)$ to be correlated with $w(t)$ and denote its correlation matrix by $\Sigma$ (i.e., $\Sigma$ is a $m \times n$ matrix and its $(i, j)$-th entry is the correlation of the $i$-th component of $w(t)$ and the $j$-th component of $\dot{w}(t)$).
Together with Assumption 2.1 for the coefficient \( b, \lambda, \sigma \), we make the following assumptions for \( F, G \) (see [24]), which is much milder than that of \( b, \lambda, \sigma \). The smoothness conditions on \( b, \lambda, \sigma \) have two main purposes. First, the conditions enable us to treat the stochastic integral, which cannot be estimated as in the case of the first-order SDEs. Second, the conditions are needed to establish a (local sense) exponential equivalence of the solutions of the second-order SDEs and its associated first-order SDEs.

**Assumption 3.1.** The functions \( G(t, x, y) \) (as well as \( G(t, x, y)[G(t, x, y)]^\top \) ) are bounded locally in \((t, x)\) and globally in \(y\) and are continuous in \((x, y)\). The function \( F(t, x, y) \) is measurable and locally bounded in \((t, x, y)\) and is Lipschitz continuous in \(y\) and continuous locally uniformly in \((t, x)\). The functions \( F(t, x, y) \) and \( G(t, x, y)[G(t, x, y)]^\top \) are continuous in \(x\) locally uniformly in \(t\) and uniformly in \(y\). \( G(t, x, y)[G(t, x, y)]^\top \) is of class \( C^1 \) in \(y\), with the first partial derivatives being bounded and Lipschitz continuous in \(y\) locally uniformly in \((t, x)\), and \( \text{div}_y G(t, x, y)[G(t, x, y)]^\top \) is continuous in \((x, y)\). Moreover, for any \(t, N > 0\),

\[
\limsup_{|y| \to \infty} \sup_{s \in [0,t]} \frac{[F(s, x, y)]^\top y}{|y|^2} < 0.
\]

Let \( \lambda_0(t, x), \sigma_0(t, x) \) is the limit of \( \lambda_\varepsilon(t, x), \sigma_\varepsilon(t, x) \) in the sense of that

\[
\limsup_{\varepsilon \to 0} \sup_{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d} (|\lambda_\varepsilon(t, x) - \lambda(t, x)| + |\sigma_\varepsilon(t, x) - \sigma_\varepsilon(t, x)|) = 0.
\]

**Assumption 3.2.** The matrix \( G(t, x, y)[G(t, x, y)]^\top \) is positive definite uniformly in \(y\) and locally uniformly in \((t, x)\). Either \( \sigma_0(t, x) = 0 \) for all \((t, x) \in \mathbb{R}_+ \times \mathbb{R}^d\) or the matrix

\[
G(t, x, y)[G(t, x, y)]^\top - \sigma_0(t, x)\Sigma[G(t, x, y)]^\top \left( G(t, x, y)[G(t, x, y)]^\top \right)^{-1} G(t, x, y)\Sigma^\top \sigma_0(t, x) \Sigma^\top
\]

is positive definite uniformly in \(y\) and locally uniformly in \((t, x)\).

Applying Theorem 2.1 and [24 Corollary 2.1], we have the following result.

**Theorem 3.1.** Assume assumptions 2.1, 3.1, and 3.2 hold. The family of processes \( \{X^\varepsilon\}_{\varepsilon > 0} \) satisfies the LDP in \( \mathcal{C}([0,1], \mathbb{R}^d) \) with the rate function \( I \) given as follows. If \( \varphi \) is absolutely continuous and \( \varphi_0 = x_0 \), then

\[
I(\varphi) = \int_0^1 \sup_{\beta \in \mathbb{R}^d} \left( \beta^\top \varphi_\varepsilon - \sup_{m \in \mathcal{P}(\mathbb{R}^d)} \left( \beta^\top \int_{\mathbb{R}^d} \frac{b(s, \varphi_\varepsilon, y)m(y)}{\lambda_0(s, y)} dy \right) \right) - \frac{1}{2} \beta^\top \left( \int_{\mathbb{R}^d} \frac{\sigma_0(s, y)\sigma_0(s, y)^\top}{\lambda_0(s, y)} m(y) dy \right) \beta
\]

\[
- \sup_{h \in \mathcal{C}_0^1(\mathbb{R}^d)} \int_{\mathbb{R}^d} \left( \nabla h(y)^\top \left( \frac{1}{2} \text{div}_y G(s, \varphi_\varepsilon, y)[G(s, \varphi_\varepsilon, y)]^\top m(y) \right) \right)
\]

\[
- \int_{\mathbb{R}^d} \left( F(s, \varphi_\varepsilon, y)m(y) - G(s, \varphi_\varepsilon, y)\Sigma^\top \sigma_0(s, y) \Sigma^\top \beta m(y) \right)
\]

\[
- \int_{\mathbb{R}^d} \left( \frac{1}{2} \nabla h(y)^\top G(s, \varphi_\varepsilon, y)[G(s, \varphi_\varepsilon, y)]^\top [\nabla h(y)] dy \right) ds.
\]

Otherwise, \( I(\varphi) = \infty \). In the above, \( \mathcal{P}(\mathbb{R}^d) \) is the space of probability density functions \( m(s) \) in \( \mathbb{R}^d \) such that \( m \in W^{1,1}(\mathbb{R}^d) \) and \( \sqrt{m} \in W^{1,2}(\mathbb{R}^d) \) with \( W^{1,2}(\mathbb{R}^d) \) being Sobolev (and local Sobolev) spaces with appropriate (indicated) exponents in \( \mathbb{R}^d \), \( \mathcal{C}_0^1(\mathbb{R}^d) \) is the space of continuously differentiable functions with compact support in \( \mathbb{R}^d \).
Jump processes. Here, we assume $\xi_t$ is a jump process taking finite values, which depends on the slow process as well. To be more precise, assume $\mathcal{M} = \{1, \ldots, |\mathcal{M}|\}$ is a finite set. Similar to [3], the evolution of the jump fast component is constructed through a jump intensity function $c(x, y) = c_y(x) : \mathbb{R}^d \times \mathcal{M} \to [0, \infty)$ and a transition probability function $r(x, y, y') = r_{yy'}(x) : \mathbb{R}^d \times \mathcal{M} \times \mathcal{M} \to [0, 1]$ as follows.

Assumption 3.3. According to [3], we make following assumption for the jump process and construct the rate function $F$ is an

\[
F = \int_{\mathbb{R}^d \times \mathcal{M} \times \mathcal{M}} r_{yy'}(x) \, d\mu(y, y') + \int_{\mathbb{R}^d \times \mathcal{M}} c_y(x) \, d\mu(y).
\]

Assume that for all $(x, y, y') \in \mathbb{R}^d \times \mathcal{M} \times \mathcal{M}$, $y \neq y'$, and $T = \{(y, y') \in \mathcal{M} \times \mathcal{M} : r_{yy'}(x) > 0 \text{ for some } x \in \mathbb{R}^d\}$. For $(i, j) \in T$, let $N_{ij}$ be a Poisson random measure on $[0; \xi] \times [0, T] \times \mathbb{R}_+$ with intensity measure $\mu_\xi \otimes \mu_T \otimes \mu_\infty$, where $\mu_T$ and $\mu_\infty$ denote the Lebesgue measures on $[0, T]$ and $\mathbb{R}_+$, respectively such that for $t \in [0, T]$,

\[
\overline{N}_{ij}(A \times [0, t] \times B) - t \mu_\xi(A) \mu_\infty(B)
\]

is an $\mathcal{F}_t$-martingale for all $A \in \mathcal{B}[0, \xi]$ and $B \in \mathcal{B}(\mathbb{R}_+)$ with $\mu_\infty(B) < 1$. Then, we define

\[
N_{ij}^{\varepsilon^{-1}}(dr \times dt) = \overline{N}_{ij}(dr \times dt \times [0, \varepsilon^{-1}])
\]

a Poisson random measure on $[0, \xi] \times [0, T]$ with intensity measure $\varepsilon^{-1} \mu_\xi \otimes \mu_T$. The processes $(N_{ij}^{\varepsilon^{-1}})_{(i, j) \in T}$ are taken to be mutually independent. We assume that for $0 \leq s \leq t \leq T$,

\[
\{w(t) - w(s) ; N_{ij}^{\varepsilon^{-1}}(A \times (s; t] \times B) : A \in \mathcal{B}[0, \xi], B \in \mathcal{B}(\mathbb{R}_+), (i, j) \in T\}
\]

is independent of $\mathcal{F}_s$. Now, we consider the following system

\[
\begin{align*}
\varepsilon^2 \dot{X}^\varepsilon_i &= b(X^\varepsilon_i, Y^\varepsilon_i) - \lambda X^\varepsilon_i + \sqrt{\varepsilon} \sigma \dot{w}(t), \\
dY^\varepsilon_i &= \sum_{(i, j) \in T} \int_{r \in [0, \xi]} (j - i) \mathbb{1}_{\{Y^\varepsilon(r-) = i\}} \mathbb{1}_{E_{ij}(Y^\varepsilon)^n}(r) N_{ij}^{\varepsilon^{-1}}(dr \times dt), \\
X^\varepsilon_i &= x_0 \in \mathbb{R}^d, \quad X^\varepsilon_i = x_1 \in \mathbb{R}^d, \quad Y^\varepsilon_i = y_0 \in \mathcal{M}.
\end{align*}
\]

According to [3], we make following assumption for the jump process and construct the rate function as follows.

Assumption 3.3. Function $c$ is bounded and there exists finite constant $C > 0$ such that for all $y, y' \in \mathcal{M}$ and $x_1, x_2 \in \mathbb{R}^d$,

\[
|c_y(x_1) - c_y(x_2)| + |r_{yy'}(x_1) - r_{yy'}(x_2)| \leq C|x_1 - x_2|.
\]

Moreover,

\[
\inf_{x \in \mathbb{R}^d} \min_{y, z \in \mathcal{M}} \sum_{n=1}^{\{|\mathcal{M}|\}} r^{n}_{yz}(x) > 0, \quad \inf_{x \in \mathbb{R}^d} \min_{y \in \mathcal{M}} c_y(x) > 0, \quad \inf_{x \in \mathbb{R}^d, (y, y') \in T} \min_{x \in \mathbb{R}^d, (y, y') \in T} r_{yy'}(x) > 0.
\]

For $\psi = (\psi(j))_{j \in \mathcal{M}}$, with $\psi_j : [0, \xi] \to \mathbb{R}_+$ being a measurable map for every $j$, define

\[
\Phi^\psi_{ij}(x) = \left\{ \begin{array}{ll}
\int_{E_{ij}(x)} \psi_j(z) \mu_\xi(d(z), & \text{if } i \neq j, \\
- \sum_{y: y \neq j} \Phi^\psi_{ij}(x), & \text{if } i = j,
\end{array} \right.
\]

and

\[
\mathcal{R} = \{ v = (v_{ij})_{(i, j) \in T}, v_{ij} : [0, 1] \times [0, \xi] \to \mathbb{R}_+ \text{ is measurable for all } (i, j) \in T \}.
\]
For $\varphi \in C([0, 1], \mathbb{R}^d)$, let $\mathcal{V}(\varphi)$ be the collection of all
\[
(\varphi = (u, v, \pi) \in \mathcal{M}([0, 1] : \mathbb{R}^m)^{|\mathcal{M}|} \times \mathcal{R} \times \mathcal{M}([0, 1] : \mathcal{P}(\mathcal{M})),
\]
[where $\mathcal{M}([0, 1] : \mathcal{P}(\mathcal{M}))$, $\mathcal{M}([0, T] : \mathbb{R}^d)$ denote the space of measurable maps from $[0, 1]$ to $\mathcal{P}(\mathcal{M})$ and from $[0, 1]$ to $\mathbb{R}^d$, respectively, with $\mathcal{P}(\mathcal{M})$ being the space of probability measures on $\mathcal{M}$ equipped with the topology of weak convergence], such that $\int_0^1 \|u_i(s)\|^2 \pi_i(s)ds < \infty$ for each $i \in \mathcal{M}$, and
\[
\varphi_s = x_0 + \sum_{j \in \mathcal{M}} \int_0^t \frac{b(\varphi_s, j)}{\lambda} \pi_j(s)ds + \sum_{j \in \mathcal{M}} \frac{\sigma u_j(s) \pi_j(s)}{\lambda} ds,
\]
and
\[
\sum_{j \in \mathcal{M}} \pi_j(s) \Phi_{ji}^{v_j(s, \cdot)}(\varphi_s) = 0, \text{a.e. } s \in [0, 1], \forall i \in \mathcal{M}.
\]
Combining Theorem 2.1 and [3] yields the following result.

**Theorem 3.2.** Let assumptions 2.1 and 3.3 hold. Then the family of processes $\{X^\varepsilon \}_{\varepsilon > 0}$ satisfies the LDP in $\mathcal{C}([0, 1], \mathbb{R}^d)$ with the rate function $I$ given by
\[
I(\varphi) = \inf_{(u, v, \pi) \in \mathcal{V}(\varphi)} \left\{ \sum_{i \in \mathcal{M}} \frac{1}{2} \int_0^1 \|u_i(s)\|^2 \pi_i(s)ds + \sum_{(i, j) \in \mathcal{T}} \int_{[0, \lambda \times [0, 1]} \ell(u_{ij}(s, z)) \pi_i(s) \mu_z(dz)ds \right\}, \tag{3.3}
\]
where $\ell(x) = x \ln x - x + 1$.

Note that another representation for the rate function $I$ can be found in [3, (2.18)].

**Markov chains.** We consider $\xi_{t/\varepsilon} = \alpha_{t/\varepsilon}$ to be a Markov switching process independent of the Brownian motion $w(t)$ taking value in a finite state space $\mathcal{M}$ such that $\alpha_{t/\varepsilon}$ has generator $Q(t)/\varepsilon$ with $Q(t) \in \mathbb{R}^{|\mathcal{M}| \times |\mathcal{M}|}$ being a generator of a continuous-time, irreducible Markov chain. Consider the following system
\[
\begin{cases}
\varepsilon^2 \dot{X}_{t/\varepsilon} = b(t, X_{t/\varepsilon}, \alpha_{t/\varepsilon}) - \lambda(t, X_{t/\varepsilon}) X_{t/\varepsilon} + \sqrt{\varepsilon} \sigma(t, X_{t/\varepsilon}) \dot{w}(t), \\
X_0 = x_0 \in \mathbb{R}^d, \quad X_{t/\varepsilon} = x_1 \in \mathbb{R}^d.
\end{cases}
\tag{3.4}
\]
As stated in Theorem 2.1, the family of solutions of (3.4) satisfies the LDP as well. The rate function can be established as follows (see e.g., [15, Theorem 4.3])
\[
I(\varphi) = \left\{ \begin{array}{ll}
\int_0^1 L(\varphi_s, \dot{\varphi}_s, s)ds & \text{if } \varphi \text{ is absolutely continuous}, \\
\infty & \text{otherwise}.
\end{array} \right.
\]
In the above, $L(x, \gamma, s)$ is the Fenchel-Legendre transform of the $H$-functional, i.e.,
\[
L(x, \gamma, s) := \sup_{\beta \in \mathbb{R}^d} [(\gamma, \beta) - H(x, \beta, s)]
\]

\footnote{Another possible approach is to specialize the rate function from [3] after formulating the Markovian switching in sense of jump process}
and \( H(x, \beta, s) \) is the function such that (see e.g., \[15\] Lemma 4.1 or \[28\] Theorem 1) for the proof of its existence and properties

\[
\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{E}_t \exp \left\{ \frac{1}{\varepsilon} \int_0^T \left( \frac{[\beta_\varepsilon]^T b(s, \varphi_s, \beta_\varepsilon)}{\lambda_\varepsilon(s, \varphi_s)} + \frac{[\sigma_\varepsilon(s, \varphi_s)]^T \beta_\varepsilon}{2\lambda_\varepsilon^2(s, \varphi_s)} \right) ds \right\} = \int_0^T H(\varphi_s, \beta_s, s) ds
\]

for any step functions \( \varphi_s \) and \( \beta_s \) in \( \mathbb{R}^d \) and \( \mathbb{E}_t \) indicates the expectation with respect to the initial value \( \alpha^\varepsilon(0) = i \).

### 3.2 Statistical Mechanics Examples

In this section, we consider some examples in statistical mechanics of small particles in a random environment, which generalizes that of \[11, 6, 23\]. Let us assume that \( \xi_t \) is an ergodic process, which may depend on \( X_t^\varepsilon \), and assume for each fixed \( t \) and fixed state \( X_t^\varepsilon = x \), \( \xi_{t/\varepsilon} \), as \( \varepsilon \to 0 \), has invariant measure denoted by \( \pi_{t,x} \).

**Smoluchowski-Kramers approximation.** Consider an overdamped approximation

\[
\dot{q}_t^\varepsilon = \frac{b(t, q_t^\varepsilon, \xi_{t/\varepsilon})}{\lambda_\varepsilon(t, q_t^\varepsilon)} + \sqrt{\varepsilon \frac{\sigma_\varepsilon(t, q_t^\varepsilon)}{\lambda_\varepsilon(t, q_t^\varepsilon)}} \dot{w}(t). \tag{3.5}
\]

In homogeneous environment, it is well-known that the Langevin equation can be simplified to the overdamped approximation, which is also commonly referred to as a Smoluchowski-Kramers approximation \[22\]. Our result in Proposition 4.4, one of main steps in the proof of our main result, in Section 4.4 is a generalization of this classical result to the case of presence of another interacting random process.

**Principle of Least Action.** As seen in Section 3.1, under certain conditions, the family of solutions \( \{q_t^\varepsilon\}_{\varepsilon>0} \) of overdamped approximations (3.3) satisfies the LDP with the rate function denoted by \( \tilde{I}(\cdot) \), which can be given explicitly as in Section 3.1 depending on the formulation of \( \xi_{t/\varepsilon} \).

Applying our results, the solution \( \{X^\varepsilon\}_{\varepsilon>0} \) of (2.1) satisfies the LDP with the same rate function \( \tilde{I} \). Theorem 2.1 shows that

\[
- \inf_{\varphi \in \mathcal{B}^o} \tilde{I}(\varphi) \leq \liminf_{\varepsilon \to 0} \varepsilon \log \mathbb{P}\{X^\varepsilon \in B\} \leq \limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}\{X^\varepsilon \in B\} \leq - \inf_{\varphi \in \mathcal{B}} \tilde{I}(\varphi), \tag{3.6}
\]

where \( B^o \) and \( \mathcal{B} \) denote the interior and closure of \( B \) in \( \mathcal{C}([0,1], \mathbb{R}^d) \).

If we let \( P_\varepsilon[X] \) be the probability density functional or law over different trajectories \( \{X^\varepsilon\}_{\varepsilon>0} \) in a given time interval \([0,1] \). Then a LDP for the random paths \( \{X^\varepsilon\}_{\varepsilon>0} \) indicates that

\[
P_\varepsilon[\varphi] \sim e^{-\tilde{I}(\varphi)/\varepsilon}, \quad \varepsilon \to 0.
\]

We denote by \( \varphi^* \), the solution to

\[
\dot{\varphi}^*_t = \frac{\overline{b}(t, \varphi^*_t)}{\lambda_0(t, \varphi^*_t)},
\]

where

\[
\overline{b}(t, x) := \int_\mathcal{M} b(t, x, z) \pi_{t,x}(dz), \tag{3.7}
\]

and \( \lambda_0 \) is the limit of functions \( \{\lambda_\varepsilon\}_{\varepsilon \to 0} \). It is easy to check that \( I(\varphi^*) = 0 \); see some explicit formulas of \( \tilde{I} \) in Section 3.1. Hence, the random path of particles in time-inhomogeneous environment contributes around of the path of \( \varphi^* \) as \( \varepsilon \to 0 \) with exponential tail, i.e., the probability of the path
of $X^\varepsilon$ far from that of $\varphi^*$ is exponentially small. In addition, because of the formula of $b(\cdot, \cdot)$, one sees that the “equilibrium” $\varphi^*$ is obtained by considering the particle in a new environment, which averages the time-inhomogeneous environment. So, in the random environment changing in time and state, we can know clearly the statistical physics of small particles if we know the invariant measure $\pi_{t,x}$ of $\xi_{t/\varepsilon}$ describing how the external force fluctuates with each fixed time $t$ and state $x$.

We can view $\hat{I}$ as the action of the system, i.e.,

$$\hat{I}[\varphi] = \int L(\varphi_s, \dot{\varphi}_s) ds.$$ 

In above, $L(\cdot, \cdot)$ is the Legendre transform of the Hamiltonian $H(\cdot, \cdot)$; the details of which can found in [13] and references therein. The classical principle of least action states that the actual path taken by $\varphi^*$ is an extremum of $\hat{I}$. Under this observation, our result generalizes the principle of least action as follows. The LDP indicates that

$$P\{X^\varepsilon \in B\} \sim e^{-\frac{1}{\varepsilon} \inf_{\varphi \in B} \hat{I}(\varphi)},$$

The probability is determined by the path that minimizes the rate function. This corresponds to minimizing the action in order to find the path that is taken by the system, where the integral $\int_0^T L(\varphi, \dot{\varphi}, s) ds$ is considered as the Action. Our results show that the principle of least action still holds in random environment modeled for small particles. Moreover, the statistical inference for the system in inhomogeneous models can be obtained by averaging the time-inhomogeneous factors (in the sense of equation (3.7)). This fact plays an important role in practice because, typically, the heterogeneity is often much difficult to analyze and simulate than homogeneity.

4 Proof of Main Results

In this section, we give the proof of Theorem 2.1. We begin with some basic definitions and preliminaries of large deviations theory; for further details, we refer the reader to [8, 9, 19].

**Definition 4.1.** A family of stochastic processes $\{Y^\varepsilon\}_{\varepsilon > 0}$ is said to be exponentially tight in the space $C([0, 1], \mathbb{R}^d)$, if there exists an increasing sequence of compact subsets $\{K_L\}_{L \geq 1}$ of $C([0, 1], \mathbb{R}^d)$ such that

$$\lim_{L \to \infty} \limsup_{\varepsilon \to 0} \varepsilon \log P(Y^\varepsilon \notin K_L) = -\infty.$$ 

**Definition 4.2.** A family of stochastic processes $\{Y^\varepsilon\}_{\varepsilon > 0}$ is said to satisfy the local LDP in $C([0, 1], \mathbb{R}^d)$ with rate function $\hat{I}$, if for any $\varphi \in C([0, 1], \mathbb{R}^d)$,

$$\lim_{\delta \to 0} \limsup_{\varepsilon \to 0} \varepsilon \log P(Y^\varepsilon \in B(\varphi, \delta)) = \lim_{\delta \to 0} \liminf_{\varepsilon \to 0} \varepsilon \log P(Y^\varepsilon \in B(\varphi, \delta)) = -\hat{I}(\varphi),$$

where $B(\varphi, \delta)$ is the ball centered at $\varphi$ with radius $\delta$ in $C([0, 1], \mathbb{R}^d)$.

The following is a well-known result in large deviations theory; see e.g., [8, 9, 19].

**Proposition 4.1.** The exponential tightness and the local LDP for a family $\{Y^\varepsilon\}_{\varepsilon > 0}$ in $C([0, 1], \mathbb{R}^d)$ with local rate function $\hat{I}$ imply the full LDP in $C([0, 1], \mathbb{R}^d)$ for this family with rate function $\hat{I}$.
4.1 A Road Map

To help the reading, we provide a road map of our approach. To establish the family of processes \( \{X^\varepsilon\}_{\varepsilon > 0} \) satisfying the LDP, we prove that it enjoys the exponential tightness and the local LDP thanks to Proposition 4.1.

**Representation of the solution.** First, we carefully examine the solutions. Since \( X^\varepsilon_t \) is the solution of a second-order differential equation, we need to solve the equation for its derivative first by using the variation of parameter formula. We rewrite equation (2.1) as (2.2) and solve (2.2) to find \( p^\varepsilon_t \). Then, we can the formula for \( X^\varepsilon_t \) as

\[
X^\varepsilon_t = x_0 + x_1 \int_0^t e^{-A^\varepsilon(s)} ds + \frac{1}{\varepsilon^2} \int_0^t \int_0^s e^{-A^\varepsilon(s,r)} b(r, X^\varepsilon_r, \xi_{r/\varepsilon}) dr ds + \frac{1}{\varepsilon^2} \int_0^t H^\varepsilon(s) ds,
\]

where for any \( 0 \leq s < t \leq 1, \varepsilon > 0 \),

\[
A^\varepsilon(t, s) := \frac{1}{\varepsilon^2} \int_s^t \lambda^\varepsilon(r, X^\varepsilon_r) dr, \quad A^\varepsilon(t) = A^\varepsilon(t, 0),
\]

\[
H^\varepsilon(t) := \sqrt{\varepsilon e^{-A^\varepsilon(t)}} \int_0^t e^{A^\varepsilon(s)} \sigma^\varepsilon(s, X^\varepsilon_s) dw(s).
\]

The large factor \( \frac{1}{\varepsilon^2} \) in \( \frac{1}{\varepsilon^2} \int_0^t H^\varepsilon(s) ds \) is a main challenge for us to obtain the desired estimates directly. Therefore, we use an integration by parts formula to overcome the difficulty; see Section 4.2.

**Exponential tightness.** It suffices to prove \( \{X^\varepsilon\}_{\varepsilon > 0} \) satisfying the (extended) Puhalskii’s criteria (see [19, Theorem 3.1] and [10, Remark 4.2]), which are

\[
\lim_{L \to \infty} \limsup_{\varepsilon \to 0} \varepsilon \log P\left( \|X^\varepsilon\| > L \right) = -\infty, \quad (4.1)
\]

\[
\lim_{\delta \to 0} \limsup_{\varepsilon \to 0} \sup_{s \in [0,1]} \varepsilon \log P\left( \sup_{t \leq s + \delta} |X^\varepsilon_t - X^\varepsilon_s| > \ell \right) = -\infty, \quad \forall \ell > 0. \quad (4.2)
\]

To prove (4.1), one has to carefully estimate the term \( H^\varepsilon(t) \) and prove that

\[
\lim_{L \to \infty} \limsup_{\varepsilon \to 0} \varepsilon \log P\left( \|H^\varepsilon\| > L \right) = -\infty, \quad (4.3)
\]

The difficulty is that we cannot move the non-adapted variable \( e^{-A^\varepsilon(t)} \) into the stochastic integral in Itô’s sense and use estimates for martingales. On the other hand, we need the term \( (e^{-A^\varepsilon(s)}) \) to balance the large term \( e^{A^\varepsilon(s)} \) in the stochastic integral. In this case, we need to use the regularity of the coefficient to estimate the stochastic integral in path wise sense (see e.g., [4]). We bound the stochastic process \( H^\varepsilon(t) \) by a function of \( \sqrt{\varepsilon} \|w\| \) and use the LDP for Brownian motions; see the details in Section 4.3.1.

For the “exponential equi-continuity condition” (4.2), the subtlety is due to the “non-adaptedness” mentioned above and the “non-integral form” of \( H^\varepsilon(t) \), which require more delicate analysis to estimate its (uniform) changes in small time. We need to decompose \( H^\varepsilon(t) \) as

\[
H^\varepsilon(t) = \sqrt{\varepsilon e^{-M^\varepsilon(t)}} h^\varepsilon(t),
\]

where \( M^\varepsilon(t) := M^\varepsilon(t, 0) \) with \( M^\varepsilon(t, s) := A^\varepsilon(t, s) - \overline{A}^\varepsilon(t, s) \), and

\[
\overline{A}^\varepsilon(t, s) := \frac{1}{\varepsilon^2} \int_s^t \lambda^\varepsilon(r, 0) dr, \quad \overline{A}^\varepsilon(t) = \overline{A}^\varepsilon(t, 0), \quad h^\varepsilon(t) := e^{-\overline{A}^\varepsilon(t)} \int_0^t e^{A^\varepsilon(r)} \sigma^\varepsilon(r, X^\varepsilon_r) dw(r).
\]
This decomposition separates $H_\varepsilon(t)$ into two parts. One of them is $e^{-M_\varepsilon(t)}$, whose changes in time can be estimated by using the regularity of $\lambda$ and the boundedness established in (1.1). The other is $h_\varepsilon(t)$, in which we can move $e^{-\mathcal{L}_\varepsilon(t)}$ into the stochastic integral in Itô’s sense. Although the integrand now is adapted, such integral is a stochastic convolution but it is not a martingale. Therefore, we cannot directly estimate the tail probability $\mathbb{P}(\sup_t |h_\varepsilon(t)| > L)$ as in the martingale cases. By keeping $t$ frozen, $h_\varepsilon(t)$ can be viewed as an element of a sequence of martingales. Then the exponential inequality for martingale helps us to estimate the change in small time intervals of $h_\varepsilon(t)$. Therefore, by a technical Lemma 4.1 we are able to obtain the desired estimates. The details of this part are in Section 4.3.2.

**Local LDPs.** Using the intuition of the Smoluchowski-Kramers approximation, we give some local estimates (similar to “exponential equivalence property”, but in the local sense) for $X_\varepsilon^t$ and $q_\varepsilon^t$, the solutions of the associated first-order equation. The calculations of this part are relatively complex, whose details are in Section 4.4. Our intuitions and ideas are as follows.

Given an absolutely continuous function $\varphi \in \mathcal{C}([0, 1], \mathbb{R}^d)$ and a neighborhood $B(\varphi, \theta)$ (the ball with center $\varphi$ and radius $\theta$ in $\mathcal{C}([0, 1], \mathbb{R}^d)$), we prove that there is a neighborhood $B(\varphi, \overline{\varphi})$ of $\varphi$ such that the difference between the probabilities of $X_\varepsilon \in B(\varphi, \theta)$ and the that of $q_\varepsilon \in B(\varphi, \overline{\varphi})$ is exponentially small. As was seen, the distance of $X_\varepsilon$ and $q_\varepsilon$ depends on $H_\varepsilon$. Since we cannot move the non-adapted random variable inside the stochastic integral in the Itô sense, estimate (1.3) cannot be improved. We need a term to leverage the decay (as $\varepsilon \to 0$) of the distance between $X_\varepsilon$ and $q_\varepsilon$ and get an “exponentially equivalent” property in the local sense. By looking at the behavior of the families $\{X_\varepsilon\}$ and $\{q_\varepsilon\}$ around a fixed (absolutely continuous) function $\varphi$, we are able to give estimates for

$$\left| \mathbb{P} \left( \| q_\varepsilon - \varphi \| < \theta \right) - \mathbb{P} \left( \| X_\varepsilon - \varphi \| < \overline{\theta} \right) \right|$$

depending on $\| H_\varepsilon \varepsilon \|$, where

$$H_\varepsilon^\varphi(t) = \sqrt{\varepsilon} e^{-A_\varepsilon^\varphi(t)} \int_0^t e^{A_\varepsilon^\varphi(s)} \sigma_\varepsilon(r, \varphi_r) dw(r),$$

and

$$A_\varepsilon^\varphi(t, s) = \frac{1}{\varepsilon^2} \int_s^t \lambda_\varepsilon(r, \varphi_r) dr; \quad A_\varepsilon^\varphi(t) = A_\varepsilon^\varphi(t, 0).$$

In contrast to $H_\varepsilon(t)$, we can write

$$H_\varepsilon^\varphi(t) = \sqrt{\varepsilon} \int_0^t e^{-A_\varepsilon^\varphi(t) + A_\varepsilon^\varphi(s)} \sigma_\varepsilon(r, \varphi_r) dw(r),$$

and understand it in Itô’s sense due to the independence of $\varphi_t$ and $w(t)$. Again, it is noted that $H_\varepsilon^\varphi(t)$ is not a martingale with respect to $t \in [0, 1]$. However, by estimating its change in small time intervals, we can obtain the following (Lemma 4.2)

$$\mathbb{P} \left( \sup_{t \in [0, 1]} |H_\varepsilon^\varphi(t)| > \ell \right) \leq \overline{M}_1 \exp \left\{ -\frac{\overline{M}_2 \ell^2}{\varepsilon^2} \right\}, \forall \ell > 0,$$

for some constants $\overline{M}_1, \overline{M}_2$, independent of $\varepsilon$ and $\ell$. This fact allows us to neglect $H_\varepsilon^\varphi$ asymptotically.

The terms left by decomposition process when estimating $\mathbb{P}(\| X_\varepsilon - \varphi \| < \overline{\theta})$ are either controlled by $\varepsilon$, or $\mathbb{P}(\| H_\varepsilon^\varphi \| > \ell)$, or $\mathbb{P}(\| q_\varepsilon - \varphi \| < \theta)$. As a consequence, we can study the behavior of $X_\varepsilon$ around $\varphi$ using that of $q_\varepsilon$. Then, the local LDP of the family of the solutions of the first-order equations allows us to obtain the local LDP for $\{X_\varepsilon\}_{\varepsilon > 0}$; see details in Section 4.4.
4.2 Representation Formula for Solutions

Under Assumption 2.1 equation 2.2 admits a unique solution \((X^\varepsilon, p^\varepsilon) \in \mathcal{C}([0, 1], \mathbb{R}^d)\). Therefore, equation 2.1 has a unique solution \(X^\varepsilon \in \mathcal{C}([0, 1], \mathbb{R}^d)\). From equation 2.2, by a variation of parameter formula, we obtain

\[
p_t^\varepsilon = x_1 e^{-A_\varepsilon(t)} + \frac{1}{\varepsilon^2} \int_0^t e^{-A_\varepsilon(t,s)} b(s, X_s^\varepsilon, \xi_{s/\varepsilon}) ds + \frac{1}{\varepsilon^2} H_\varepsilon(t), \tag{4.4}
\]

where for any \(0 \leq s \leq t \leq 1, \varepsilon > 0,\)

\[
A_\varepsilon(t, s) := \frac{1}{\varepsilon^2} \int_s^t \lambda_\varepsilon(r, X_r^\varepsilon) dr, \quad A_\varepsilon(t) = A_\varepsilon(t, 0), \quad H_\varepsilon(t) := \sqrt{\varepsilon} e^{-A_\varepsilon(t)} \int_0^t e^{A_\varepsilon(s)} \sigma_\varepsilon(s, X_s^\varepsilon) dw(s).
\]

Therefore, we obtain the formula for \(X_t^\varepsilon\) as follows

\[
X_t^\varepsilon = x_0 + x_1 \int_0^t e^{-A_\varepsilon(s)} ds + \frac{1}{\varepsilon^2} \int_0^t \int_0^s e^{-A_\varepsilon(t,r)} b(r, X_r^\varepsilon, \xi_{r/\varepsilon}) dr ds + \frac{1}{\varepsilon^2} \int_0^t H_\varepsilon(s) ds. \tag{4.5}
\]

Using an integration by parts formula, we have

\[
X_t^\varepsilon = x_0 + \int_0^t \frac{b(s, X_s^\varepsilon, \xi_{s/\varepsilon})}{\lambda_\varepsilon(s, X_s^\varepsilon)} ds + \sqrt{\varepsilon} \int_0^t \frac{\sigma_\varepsilon(s, X_s^\varepsilon)}{\lambda_\varepsilon(s, X_s^\varepsilon)} dw(s) + R_\varepsilon(t), \tag{4.6}
\]

where

\[
R_\varepsilon(t) := x_1 \int_0^t e^{-A_\varepsilon(s)} ds - \frac{1}{\lambda_\varepsilon(t, X_t^\varepsilon)} \int_0^t e^{-A_\varepsilon(t,s)} b(s, X_s^\varepsilon, \xi_{s/\varepsilon}) ds \\
- \int_0^t \frac{1}{\lambda^2_\varepsilon(s, X_s^\varepsilon)} \left( \int_0^s e^{-A_\varepsilon(s,r)} b(r, X_r^\varepsilon, \xi_{r/\varepsilon}) dr \right) \left( \nabla_s \lambda_\varepsilon(s, X_s^\varepsilon) + \langle \nabla_X \lambda_\varepsilon(s, X_s^\varepsilon), p_s^\varepsilon \rangle \right) ds \\
- \frac{1}{\lambda_\varepsilon(t, X_t^\varepsilon)} H_\varepsilon(t) - \int_0^t \frac{1}{\lambda^2_\varepsilon(s, X_s^\varepsilon)} H_\varepsilon(s) \left( \nabla_s \lambda(s, \varepsilon^2 X_s^\varepsilon) + \langle \nabla_X \lambda(s, X_s^\varepsilon), p_s^\varepsilon \rangle \right) ds \\
=: \sum_{i=1}^5 R_\varepsilon^{(i)}(t). \tag{4.7}
\]

4.3 Exponential Tightness of \(\{X^\varepsilon\}_{\varepsilon > 0}\)

In this section, we investigate the exponential tightness of \(\{X^\varepsilon\}_{\varepsilon > 0}\) in \(\mathcal{C}([0, 1], \mathbb{R}^d)\) by proving 4.1 and 4.2. In what follows, whenever the estimates involve random variables, they should be understood in the sense of with probability 1 if it is not specified otherwise, which is our convention henceforth.

4.3.1 Proof of 4.1

We begin with the following Proposition, whose proof is postponed to the Appendix.
Proposition 4.2. There is a finite constant $\hat{C}_0$, independent of $\varepsilon$ such that
\[ \|H_\varepsilon\| \leq \hat{C}_0 \varepsilon \|w\|. \] (4.8)
As a result, there is a finite constant $\hat{C}_1$, independent of $\varepsilon$ such that
\[ \|X_\varepsilon\| \leq \hat{C}_1 \Gamma(\hat{C}_1 \varepsilon \|w\|), \]
where
\[ \Gamma(v) := (1 + ve^v + v^2 e^{2v})(1 + v)e^{1+ve^v}, \quad v \geq 0. \]

Proof of (4.1). We have from Proposition 4.2 that
\[ \|X_\varepsilon\| \leq \hat{C}_1 \Gamma(\hat{C}_1 \varepsilon \|w\|), \] (4.9)
for some finite constant $\hat{C}_1$ independent of $\varepsilon$ and $\Gamma(\cdot)$ as in Proposition 4.2. On the other hand, by the LDP for the Brownian motion $w(t)$, we have
\[ \lim_{L \to \infty} \limsup_{\varepsilon \to 0} \varepsilon \log P\{\sqrt{\varepsilon} \varepsilon \|w\| \geq L\} = -\infty. \] (4.10)
Combining (4.9) and (4.10), we obtain
\[ \lim_{L \to \infty} \limsup_{\varepsilon \to 0} \varepsilon \log P\{\|X_\varepsilon\| \geq L\} \leq \lim_{L \to \infty} \limsup_{\varepsilon \to 0} \varepsilon \log P\{\sqrt{\varepsilon} \varepsilon \|w\| \geq M(L, \hat{C}_1)\} = -\infty, \]
where $M(L, \hat{C}_1)$ is a constant depending on $L$ and $\hat{C}_1$ that tends to $\infty$ as $L \to \infty$. Therefore, the proof is complete.

4.3.2 Proof of (4.2)

Proposition 4.3. For any $\ell > 0$, we have
\[ \lim_{\delta \to 0} \limsup_{\varepsilon \to 0} \sup_{s \in [0,1]} \varepsilon \log P\left( \sup_{s \leq t \leq s+\delta} |H_\varepsilon(t) - H_\varepsilon(s)| > \ell \right) = -\infty. \]

Proof. Let $\ell > 0$ be fixed. Let $\delta \in (0,1)$ be fixed but otherwise arbitrary and $\varepsilon \in (0,1)$. In what follows, we mainly work with $s, t \in [0,1]$, $0 \leq t - s \leq \delta$. Denote
\[ \overline{A}_\varepsilon(t, s) := \frac{1}{\varepsilon^2} \int_s^t \lambda_\varepsilon(r, 0)dr, \quad \overline{A}_\varepsilon(t) := \overline{A}_\varepsilon(t, 0), \quad M_\varepsilon(t) := A_\varepsilon(t, s) - A_\varepsilon(t, 0), \quad M_\varepsilon(t) := M_\varepsilon(t, 0). \]
Using property of $\lambda$, it is easily seen that
\[ |M_\varepsilon(t, s)| \leq \kappa_1 \int_s^t |X_\varepsilon^r|dr, \] (4.11)
where $\kappa_1$ is some finite constant, independent of $\varepsilon, t, s$. By definition of $H_\varepsilon(t)$ in Section 4.2, we rewrite
\[ H_\varepsilon(t) = \sqrt{\varepsilon} e^{-M_\varepsilon(t)} h_\varepsilon(t), \quad h_\varepsilon(t) := e^{-\overline{A}_\varepsilon(t)} \int_0^t e^{A_\varepsilon(r)} \sigma_\varepsilon(r, X_\varepsilon^r)dw(r), \] (4.12)
and denote
\[ h_\varepsilon(t, s) := h_\varepsilon(t) - h_\varepsilon(s) \]
\[ = \int_s^t e^{-A_\varepsilon(t,r)+M_\varepsilon(r)} \sigma_\varepsilon(r, X_r^\varepsilon) dw(r) - (1 - e^{-A_\varepsilon(t,s)}) \int_0^s e^{-A_\varepsilon(s,r)+M_\varepsilon(r)} \sigma_\varepsilon(r, X_r^\varepsilon) dw(r). \] (4.13)

In the above, we can move \( A_\varepsilon(t) \) into the Itô’s integral because of independence of \( A_\varepsilon(t) \) and \( w(t) \). Moreover, it is noted that \( h_\varepsilon(t) \) (resp. \( h_\varepsilon(t, s) \)) is \( \mathbb{R}^d \)-valued, and we will denote by \( h_\varepsilon^{(i)}(t) \) (resp. \( h_\varepsilon^{(i)}(t, s) \)) their \( i \)-th component, \( i = 1, \ldots, d \). Using (4.12), we have following decomposition
\[ H_\varepsilon(t) - H_\varepsilon(s) = (\sqrt{\varepsilon} e^{-M_\varepsilon(t)} h_\varepsilon(t) - \sqrt{\varepsilon} e^{-M_\varepsilon(t)} h_\varepsilon(s)) + (\sqrt{\varepsilon} e^{-M_\varepsilon(t)} h_\varepsilon(s) - \sqrt{\varepsilon} e^{-M_\varepsilon(s)} h_\varepsilon(s)) \]
\[ =: K_\varepsilon^{(1)}(t, s) + K_\varepsilon^{(2)}(t, s). \]

Next we proceed to estimate \( K_\varepsilon^{(1)} \) and \( K_\varepsilon^{(2)} \). Although \( -A_\varepsilon(t, r) \) and \( M_\varepsilon(r) \) are adapted with respect to filtration generated by the Brownian motion, \( \{h_\varepsilon^{(i)}(t, s)\}_{t \geq s} \) is not a martingale with respect to \( t \). However, if we frozen \( t \), the sequence
\[ \int_s^{t'} e^{-A_\varepsilon(t,r)+M_\varepsilon(r)} \sigma_\varepsilon^{(i)}(r, X_r^\varepsilon) dw(r) \]
is a martingale with respect to \( t' \in [s, t] \) and has the quadratic variation,
\[ \int_s^{t'} e^{-2A_\varepsilon(t,r)+2M_\varepsilon(r)}|\sigma_\varepsilon^{(i)}(r, X_r^\varepsilon)|^2 dr, \]
where \( \sigma_\varepsilon^{(i)} \) is \( i \)-th row of \( \sigma_\varepsilon \). Because of (4.11), we have
\[ \int_s^t e^{-2A_\varepsilon(t,r)+2M_\varepsilon(r)}|\sigma_\varepsilon^{(i)}(r, X_r^\varepsilon)|^2 dr \leq C e^{2\kappa_1\|X^\varepsilon\|}\|t - s\|. \]

Similarly, using the fact \((1 - e^{-u})^2 \leq u, \forall u > 0, \int_0^s e^{-A_\varepsilon(s,r)} dr \leq \int_0^s \frac{-\kappa_0(s-r)}{\varepsilon^2} dr \leq \varepsilon^2 \|X^\varepsilon\|\|t - s\|\), one has
\[ (1 - e^{-A_\varepsilon(t,s)}) \int_0^s e^{-A_\varepsilon(s,r)+M_\varepsilon(r)} \sigma_\varepsilon^{(i)}(r, X_r^\varepsilon) dw(r) \]
is an element of sequence of martingale with quadratic variation bounded by \( C e^{2\kappa_1\|X^\varepsilon\|\|t - s\|}. \)

Therefore, by applying exponential inequality for martingale (see e.g., [21] Theorem 7.4, p. 44)) for the above two stochastic integrals, we have from (4.13) that for all \( s \leq s_1 \leq s_2 \leq t \leq s + \delta \),
\[ \mathbb{P} \left\{ |h_\varepsilon^{(i)}(s_2, s_1)| \geq \frac{\delta^{1/8}|s_2 - s_1|^{1/8}}{d\sqrt{\varepsilon}} + \frac{1}{Cd\sqrt{\varepsilon}\delta^{1/4}|s_2 - s_1|^{3/8} e^{2\kappa_1\|X^\varepsilon\|}|s_2 - s_1|} \right\} \]
\[ \leq 4 \exp \left\{ \frac{-1}{2d^2 C \varepsilon \delta^{1/4}|s_2 - s_1|^{1/4}} \right\}. \]

Therefore, one has that for all \( s \leq s_1 \leq s_2 \leq t \leq s + \delta \),
\[ \mathbb{P} \left\{ |h_\varepsilon^{(i)}(s_2, s_1)| \geq \left( \frac{\delta^{1/8}}{d\sqrt{\varepsilon}} + \frac{e^{2\kappa_1\|X^\varepsilon\|\delta^{1/4}}}{d\sqrt{\varepsilon}} \right) |s_2 - s_1|^{1/8} \right\} \leq 4 \exp \left\{ \frac{-1}{2d^2 C \varepsilon \delta^{1/4}|s_2 - s_1|^{1/4}} \right\}. \] (4.14)

To proceed, we have following lemma.
Lemma 4.1. Assume that $Y(t)$ is a continuous stochastic process, that $L$ is a random variable, and that there are constants $\alpha_1$ and $\alpha_2 > 0$ such that

$$\mathbb{P}(|Y(s_2) - Y(s_1)| \geq L|s_2 - s_1|^{1/8}) \leq \alpha_1 \exp \left\{ -\frac{\alpha_2}{|s_2 - s_1|^{1/4}} \right\}, \forall s_2, s_1 \in [0, 1]. \quad (4.15)$$

There are constants $C_1, C_2, \text{ and } C_3 > 0$ such that

$$\mathbb{P}(\sup_{t \in [0,1]} |Y(t)| \geq C_3 L) \leq C_1 \alpha_1 \exp \{ -C_2 \alpha_2 \}.$$

Now, we obtain from (4.14) and Lemma 4.1 that

$$\mathbb{P} \left\{ \sup_{t \in [s,s+\delta]} |h_{\epsilon}(t)| \geq \frac{\hat{C}_4 \delta^{1/8}}{d} \mathbb{E}^{1/\delta^{1/4}} + \frac{\hat{C}_4 e^{2\kappa_1} \|X^\epsilon\| \delta^{1/4}}{d} \right\} \leq \hat{C}_2 \exp \left\{ -\frac{\tilde{C}_3}{\epsilon \delta^{1/8}} \right\}, \quad i = 1, \ldots, d,$$

for some positive constants $\hat{C}_2, \hat{C}_3, \tilde{C}_2$, independent of $\epsilon, t, s, \delta$, which implies that

$$\mathbb{P} \left\{ \sup_{t \in [s,s+\delta]} \left| K_{\epsilon}(t, s) \right| \geq \frac{\hat{C}_4 \delta^{1/8} e^{\kappa_1 \|X^\epsilon\|}}{d} \mathbb{E}^{1/\delta^{1/4}} + \frac{\tilde{C}_3 e^{3\kappa_1} \|X^\epsilon\| \delta^{1/4}}{d} \right\} \leq d \hat{C}_2 \exp \left\{ -\frac{\tilde{C}_3}{\epsilon \delta^{1/8}} \right\}. \quad (4.16)$$

Combining (4.16), the fact that $|K_{\epsilon}(t, s)| \leq \sqrt{\epsilon} e^{-M(t)} |h_{\epsilon}(t, s)|$, and (4.11), one has that for all $s, t \in [0, 1]$, $0 \leq t - s \leq \delta$,

$$\mathbb{P} \left\{ \sup_{t \in [s,s+\delta]} \left| K_{\epsilon}(t, s) \right| \geq \hat{C}_4 \delta^{1/8} e^{\kappa_1 \|X^\epsilon\|} + \frac{\hat{C}_4 e^{3\kappa_1} \|X^\epsilon\| \delta^{1/4}}{d} \right\} \leq d \hat{C}_2 \exp \left\{ -\frac{\tilde{C}_3}{\epsilon \delta^{1/8}} \right\}. \quad (4.17)$$

From (4.17) and the logarithm equivalence principle [9, Lemma 1.2.15], we have the following estimates

$$\limsup_{\epsilon \to 0} \sup_{s \in [0,1]} \epsilon \log \mathbb{P} \left( \sup_{s \leq t \leq s + \delta} \left| K_{\epsilon}(t, s) \right| > \ell \right)$$

$$\leq \limsup_{\epsilon \to 0} \sup_{s \in [0,1]} \epsilon \log \left( \mathbb{P} \left( \sup_{s \leq t \leq s + \delta} \left| K_{\epsilon}(t, s) \right| \geq \hat{C}_4 \delta^{1/8} e^{\kappa_1 \|X^\epsilon\|} + \frac{\hat{C}_4 e^{3\kappa_1} \|X^\epsilon\| \delta^{1/4}}{d} \right) \right)$$

$$+ \mathbb{P} \left( \frac{\hat{C}_4 \delta^{1/8} e^{\kappa_1 \|X^\epsilon\|}}{d} \geq \frac{\ell}{2} \right) + \mathbb{P} \left( \frac{\tilde{C}_3 e^{3\kappa_1} \|X^\epsilon\| \delta^{1/4}}{d} \geq \frac{\ell}{2} \right)$$

$$= \limsup_{\epsilon \to 0} \epsilon \log \left( \hat{C}_2 \exp \left\{ -\frac{\tilde{C}_3}{\epsilon \delta^{1/8}} \right\} \right) \vee \mathbb{P} \left( \hat{C}_4 \delta^{1/8} e^{\kappa_1 \|X^\epsilon\|} \geq \frac{\ell}{2} \right) \vee \mathbb{P} \left( \tilde{C}_3 e^{3\kappa_1} \|X^\epsilon\| \delta^{1/4} \geq \frac{\ell}{2} \right). \quad (4.18)$$

Now, letting $\delta \to 0$, it is seen that

$$\limsup_{\delta \to 0} \sup_{\epsilon \to 0} \epsilon \log \frac{d \hat{C}_2 \exp \left\{ -\frac{\tilde{C}_3}{\epsilon \delta^{1/8}} \right\}}{d} = -\infty,$$

and from (4.1) that

$$\limsup_{\delta \to 0} \sup_{\epsilon \to 0} \epsilon \log \mathbb{P} \left( \hat{C}_4 \delta^{1/8} e^{\kappa_1 \|X^\epsilon\|} \geq \frac{\ell}{2} \right)$$

$$= \limsup_{\delta \to 0} \sup_{\epsilon \to 0} \epsilon \log \mathbb{P} \left( \hat{C}_4 e^{3\kappa_1} \|X^\epsilon\| \delta^{1/4} \geq \frac{\ell}{2} \right) \quad (4.20)$$

$$= -\infty.$$
Combining (4.18), (4.19), and (4.20) leads to that
\[
\lim_{\delta \to 0} \limsup_{\varepsilon \to 0} \sup_{s \in [0,1]} \varepsilon \log \mathbb{P}\left( \sup_{s \leq t \leq s + \delta} \left| K_\varepsilon^{(1)}(t, s) \right| > \ell \right) = -\infty. \tag{4.21}
\]

Next, we prove similar results for \( K_\varepsilon^{(2)}(t, s) \). The mean value theorem and (4.11) imply that
\[
\left| e^{-M_\varepsilon(t)} - e^{-M_\varepsilon(s)} \right| \leq \kappa_1 \left\| X_\varepsilon^{\kappa_1} \right\| \left\| X_\varepsilon^{\kappa_1} \right\| |t - s|.
\]
Therefore, for all \( s, t \in [0,1] \), \( s \leq t \leq s + \delta \)
\[
\sup_{t \in [s, s + \delta]} \left| K_\varepsilon^{(2)}(t, s) \right| \leq \kappa_1 \sqrt{\varepsilon} \left\| X_\varepsilon^{\kappa_1} \right\| \left\| X_\varepsilon^{\kappa_1} \right\| |h_\varepsilon(s)|. \tag{4.22}
\]

Since
\[
\int_0^s e^{-2\pi \varepsilon (r, \varepsilon^2 X_\varepsilon^r)} \| \sigma^1 (r, \varepsilon^2 X_\varepsilon^r) \|^2 dr \leq C e^{2\kappa_1 \left\| X_\varepsilon^{\kappa_1} \right\|} \int_0^s e^{-2\pi \varepsilon (r, \varepsilon^2 X_\varepsilon^r)} dr \leq \widehat{C}_\varepsilon e^{2\kappa_1 \left\| X_\varepsilon^{\kappa_1} \right\|}, i = 1, \ldots, d,
\]
where \( \widehat{C}_\varepsilon \) is a finite constant depending only on \( \kappa_0 \) and function \( \sigma \); by exponential inequality for martingale (see e.g., [21, Theorem 7.4, p. 44]) again and similar process of getting (4.16), we have
\[
\mathbb{P}\left\{ \left| h_\varepsilon(s) \right| \geq \frac{1}{\sqrt{\varepsilon}} + \frac{1}{C_\varepsilon^2 \varepsilon} \right\} \leq 2d \exp \left\{ -\frac{1}{d \varepsilon} \cdot \frac{2}{d \widehat{C}_\varepsilon \varepsilon^2} \right\} \leq 2d \exp \left\{ -\frac{2}{C_\varepsilon d^2 \varepsilon^3} \right\}. \tag{4.23}
\]
Combining (4.22) and (4.23) enables us to obtain that
\[
\mathbb{P}\left\{ \sup_{t \in [s, s + \delta]} \left| K_\varepsilon^{(2)}(t, s) \right| \geq \kappa_1 \left\| X_\varepsilon^{\kappa_1} \right\| \left\| X_\varepsilon^{\kappa_1} \right\| + \kappa_1 \left\| X_\varepsilon^{\kappa_1} \right\| \left\| X_\varepsilon^{\kappa_1} \right\| |h_\varepsilon(s)| \right\} \leq 2d \exp \left\{ -\frac{2}{C_\varepsilon d^2 \varepsilon^3} \right\}. \tag{4.24}
\]
Therefore, by a similar argument for getting (4.21), we obtain
\[
\lim_{\delta \to 0} \limsup_{\varepsilon \to 0} \sup_{s \in [0,1]} \varepsilon \log \mathbb{P}\left( \sup_{s \leq t \leq s + \delta} \left| K_\varepsilon^{(2)}(t, s) \right| > \ell \right) = -\infty. \tag{4.21}
\]
As a consequence, we have
\[
\lim_{\delta \to 0} \limsup_{\varepsilon \to 0} \sup_{s \in [0,1]} \varepsilon \log \mathbb{P}\left( \sup_{s \leq t \leq s + \delta} |H_\varepsilon(t) - H_\varepsilon(s)| > \ell \right) = -\infty.
\]

**Proof of (4.22).** Let \( \ell > 0 \) be fixed. Let \( \delta \in (0,1) \) be fixed but otherwise arbitrary and \( \varepsilon \in (0,1) \).

In what follows, we mainly work with \( s, t \in [0,1], 0 \leq t - s \leq \delta \). Because of (4.6), we have
\[
|X_t^\varepsilon - X_s^\varepsilon| \leq \int_s^t \frac{b(r, X_r^\varepsilon, \xi_{r/\varepsilon})}{\lambda_\varepsilon(r, X_r^\varepsilon)} dr + \sqrt{\varepsilon} \int_s^t \frac{\sigma(r, X_r^\varepsilon)}{\lambda_\varepsilon(r, X_r^\varepsilon)} dw(r) + |R_\varepsilon(t) - R_\varepsilon(s)|. \tag{4.25}
\]
Since
\[
\int_s^t \frac{b(r, X_r^\varepsilon, \xi_{r/\varepsilon})}{\lambda_\varepsilon(r, X_r^\varepsilon)} dr \leq C(1 + \left\| X_\varepsilon^{\kappa_1} \right\|) |t - s|,
\]
it is easily seen from (3.1) that
\[
\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \sup_{s \in [0, 1]} \varepsilon \log P \left( \sup_{s \leq t \leq s + \delta} \left| \int_s^t \frac{b(r, X^\varepsilon_r, \xi_r)}{\lambda^\varepsilon (r, X^\varepsilon_r)} \, dr \right| > \ell \right) = -\infty.  \tag{4.26}
\]
By our assumptions on \( \lambda \) and \( \sigma \), the coefficient of the diffusion \( \sqrt{\varepsilon} \int_s^t \frac{\sigma_\varepsilon(r, X^\varepsilon_r)}{\lambda^\varepsilon (r, X^\varepsilon_r)} \, dw(r) \) is uniformly bounded. Therefore, the Bernstein inequality \cite{25} pp. 153-154 yields
\[
\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \sup_{s \in [0, 1]} \varepsilon \log P \left( \sup_{s \leq t \leq s + \delta} \sqrt{\varepsilon} \int_s^t \frac{\sigma_\varepsilon(r, X^\varepsilon_r)}{\lambda^\varepsilon (r, X^\varepsilon_r)} \, dw(r) > \ell \right) = -\infty.  \tag{4.27}
\]
Next, we consider the term \( |R^\varepsilon(t) - R^\varepsilon(s)| \). We have
\[
|R^\varepsilon(t) - R^\varepsilon(s)| \leq \sum_{i=1}^5 |R^{(1)}_\varepsilon(t) - R^{(1)}_\varepsilon(s)|.  \tag{4.28}
\]
First, it is clear that
\[
\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \sup_{s \in [0, 1]} \varepsilon \log P \left( \sup_{s \leq t \leq s + \delta} \left| R^{(1)}_\varepsilon(t) - R^{(1)}_\varepsilon(s) \right| > \ell \right) = -\infty.  \tag{4.29}
\]
Second, we have
\[
|R^{(2)}_\varepsilon(t) - R^{(2)}_\varepsilon(s)| \leq \frac{1}{\lambda^\varepsilon (t, X^\varepsilon_t)} \left| \int_s^t e^{-A^\varepsilon(t, r)} b(r, X^\varepsilon_r, \xi_r) \, dr \right|
+ \frac{|\lambda^\varepsilon(t, X^\varepsilon_t) - \lambda^\varepsilon(s, X^\varepsilon_s)|}{\lambda^\varepsilon(t, X^\varepsilon_t) \lambda^\varepsilon(s, X^\varepsilon_s)} \left| \int_0^s e^{-A^\varepsilon(t, r)} b(r, X^\varepsilon_r, \xi_r) \, dr \right|
\leq C(1 + \|X^\varepsilon\|) |t - s| + C(1 + \|X^\varepsilon\|)(|t - s| + \varepsilon^2\|X^\varepsilon\|),
\]
and then, (4.1) gives us that
\[
\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \sup_{s \in [0, 1]} \varepsilon \log P \left( \sup_{s \leq t \leq s + \delta} \left| R^{(2)}_\varepsilon(t) - R^{(2)}_\varepsilon(s) \right| > \ell \right) = -\infty.  \tag{4.30}
\]
Third, one has from definition of \( R^{(3)}_\varepsilon \), property of \( \lambda^\varepsilon(\cdot, \cdot) \) and (A.3) that
\[
|R^{(3)}_\varepsilon(t) - R^{(3)}_\varepsilon(s)|
= \left| \int_s^t \frac{1}{\lambda^\varepsilon(r, X^\varepsilon_r)} \left( \int_0^r e^{-A^\varepsilon(r, r')} b(r', X^\varepsilon_{r'}, \xi_{r'}) \, dr' \right) \left( \nabla_s \lambda^\varepsilon(r, X^\varepsilon_r) + \left( \nabla X \lambda^\varepsilon(r, X^\varepsilon_r), p^\varepsilon_r \right) \right) \, dr \right|
\leq C(1 + \|X^\varepsilon\|)(1 + \|H^\varepsilon\|)|t - s|,
\]
which combined with Proposition 4.2 and the LDP of \( \sqrt{\varepsilon}\|w\| \) implies that
\[
\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \sup_{s \in [0, 1]} \varepsilon \log P \left( \sup_{s \leq t \leq s + \delta} \left| R^{(3)}_\varepsilon(t) - R^{(3)}_\varepsilon(s) \right| > \ell \right) = -\infty.  \tag{4.31}
\]
Next, we have that
\[
|R^{(4)}_\varepsilon(t) - R^{(4)}_\varepsilon(s)| \leq \frac{1}{\lambda^\varepsilon(t, X^\varepsilon_t)} |H^\varepsilon(t) - H^\varepsilon(s)| + \frac{|\lambda^\varepsilon(t, X^\varepsilon_t) - \lambda^\varepsilon(s, X^\varepsilon_s)|}{\lambda^\varepsilon(t, X^\varepsilon_t) \lambda^\varepsilon(s, X^\varepsilon_s)} |H^\varepsilon(s)|
\leq C|H^\varepsilon(t) - H^\varepsilon(s)| + C\|H^\varepsilon\|(|t - s| + \varepsilon^2\|X^\varepsilon\|).
\]
Hence, using Proposition 4.3 to take care of the term $|H_\varepsilon(t) - H_\varepsilon(s)|$, and using (4.11) and Proposition 4.2 to take care of the term $\|H_\varepsilon\|(\|t - s\| + \varepsilon^2\|X_\varepsilon\|)$, we can obtain
\[
\lim_{\delta \to 0} \lim \sup_{\varepsilon \to 0} \varepsilon \log P \left( \sup_{s \in [0,1]} \left| R_\varepsilon^{(4)}(t) - R_\varepsilon^{(4)}(s) \right| > \ell \right) = -\infty. \quad (4.32)
\]

Finally, $R_\varepsilon^{(5)}$ is handled similarly as that of $R_\varepsilon^{(3)}$. Since
\[
|R_\varepsilon^{(5)}(t) - R_\varepsilon^{(5)}(s)| \leq C\|H_\varepsilon\|(1 + \|H_\varepsilon\|)|t - s|,
\]
using Proposition 4.2 and the LDP of $\sqrt{\varepsilon\|w\|}$, one has
\[
\lim_{\delta \to 0} \lim \sup_{\varepsilon \to 0} \varepsilon \log P \left( \sup_{s \in [0,1]} \left| R_\varepsilon^{(5)}(t) - R_\varepsilon^{(5)}(s) \right| > \ell \right) = -\infty. \quad (4.33)
\]

We obtain from (4.28)-(4.33) that
\[
\lim_{\delta \to 0} \lim \sup_{\varepsilon \to 0} \varepsilon \log P \left( \sup_{s \in [0,1]} \left| R_\varepsilon(t) - R_\varepsilon(s) \right| > \ell \right) = -\infty. \quad (4.34)
\]
Combining (4.25), (4.26), (4.27), and (4.34), we get (4.2).

\[\square\]

4.4 Local LDP of $\{X_\varepsilon\}_{\varepsilon > 0}$

We begin this section with the following Proposition, which provides a kind of “exponential equivalence property” of $X_\varepsilon$ and $q^\varepsilon$ in the “local sense”.

**Proposition 4.4.** For any $\theta > 0$, $N > 0$, and $\varphi \in C([0,1],\mathbb{R}^d)$ that is absolutely continuous, there exist $\overline{\theta}_1, \overline{\theta}_2 > 0$, independent of $\varepsilon$ and $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$,
\[
P \{ \|q^\varepsilon - \varphi\| < \theta \} \geq P \{ \|X_\varepsilon - \varphi\| < \overline{\theta}_1 \} - \exp \left\{ -\frac{N}{\varepsilon} \right\}, \quad (4.35)
\]
\[
P \{ \|X_\varepsilon - \varphi\| < \theta \} \geq P \{ \|q^\varepsilon - \varphi\| < \overline{\theta}_2 \} - \exp \left\{ -\frac{N}{\varepsilon} \right\}. \quad (4.36)
\]
Moreover, $\overline{\theta}_1$ and $\overline{\theta}_2$ are to be specified later (see (4.69)).

**Proof.** For each $\varphi(\cdot) \in C([0,1],\mathbb{R}^d)$ that is absolutely continuous, we denote by $X_\varepsilon^{\varepsilon,\varphi}$ the solution of
\[
\begin{align*}
\varepsilon^2 \ddot{X}_t^{\varepsilon,\varphi} &= b(t, \varphi_t, \xi_{t/\varepsilon}) - \lambda_\varepsilon(t, \varphi_t) \dot{X}_t^{\varepsilon,\varphi} + \sqrt{\varepsilon} \sigma_\varepsilon(t, \varphi_t) \dot{w}(t), \\
X_0^{\varepsilon,\varphi} &= x_0 \in \mathbb{R}^d; \quad X_t^{\varepsilon,\varphi}(0) = x_1 \in \mathbb{R}^d,
\end{align*}
\]
and by $q_\varepsilon^{\varepsilon,\varphi}$ the solution of
\[
\begin{align*}
\dot{q}_0^{\varepsilon,\varphi} &= \frac{b(t, \varphi_t, \xi_{t/\varepsilon})}{\lambda_\varepsilon(t, \varphi_t)} + \sqrt{\varepsilon} \frac{\sigma_\varepsilon(t, \varphi_t)}{\lambda_\varepsilon(t, \varphi_t)} \dot{w}(t), \\
q_0^{\varepsilon,\varphi} &= x_0 \in \mathbb{R}^d.
\end{align*}
\]
Recall that $C$ is a generic positive constant whose value may change for different appearances. The constant $C$ may depend on initial values $x_0, x_1$ and coefficients $b, \lambda, \sigma$, but is independent of $\varepsilon$ and $\varphi$. Note that the time variable $t$ is always assumed to be in $[0,1]$. Now, it is readily seen that

$$|X_t^\varepsilon - \varphi_t| \leq |X_t^\varepsilon - X_t^{\varepsilon,\varphi}| + |X_t^{\varepsilon,\varphi} - q_t^{\varepsilon,\varphi}| + |q_t^{\varepsilon,\varphi} - q_t^\varepsilon| + |q_t^\varepsilon - \varphi_t|. \quad (4.39)$$

**Step 1: Estimate of $|X_t^\varepsilon - X_t^{\varepsilon,\varphi}|$.** The following decomposition will be used often in the proof

$$u(t)v(t) - u(s)v(s) = u(t)(v(t) - v(s)) + v(s)(u(t) - u(s)).$$

As in Section 4.2 we have

$$X_t^{\varepsilon,\varphi} = x_0 + x_1 \int_0^t e^{-A(s,\varphi)}ds + \frac{1}{\varepsilon^2} \int_0^t \int_0^s e^{-A(s,r)}b(r, \varphi_r, \xi_{r/\varepsilon})dr ds + \frac{1}{\varepsilon^2} \int_0^t H^{\varepsilon}(s)ds, \quad (4.40)$$

where

$$A^\varepsilon(t,s) = \frac{1}{\varepsilon^2} \int_s^t \lambda_\varepsilon(r, \varphi_r)dr, \quad A^\varepsilon(t) = A^\varepsilon(t,0),$$

$$H^{\varepsilon}(t) = \sqrt{\varepsilon} e^{-A^\varepsilon(t)} \int_0^t e^{A^\varepsilon(r)} \sigma_\varepsilon(r, \varphi_r)dw(r).$$

Thus, using integration by parts, we have

$$X_t^{\varepsilon,\varphi} = x_0 + \int_0^t \frac{b(s, \varphi_s, \xi_{s/\varepsilon})}{\lambda_\varepsilon(s, \varphi_s)} ds + \frac{\sqrt{\varepsilon}}{\lambda_\varepsilon(s, \varphi_s)} dw(s) + R^\varepsilon(t), \quad (4.41)$$

where

$$R^\varepsilon(t) := x_1 \int_0^t e^{-A^\varepsilon(s)} b(s, \varphi_s, \xi_{s/\varepsilon}) ds - \frac{1}{\lambda_\varepsilon(t, \varphi_t)} \int_0^t e^{-A^\varepsilon(t,s)} b(s, \varphi_s, \xi_{s/\varepsilon}) ds$$

$$- \int_0^t \frac{1}{\lambda^2_\varepsilon(s, \varphi_s)} \left( \int_0^s e^{-A^\varepsilon(s,r)} b(r, \varphi_r, \xi_{r/\varepsilon}) dr \right) (\nabla_s \lambda_\varepsilon(s, \varphi_s) + (\nabla X \lambda_\varepsilon(s, \varphi_s), \varphi_s)) ds$$

$$- \frac{1}{\lambda_\varepsilon(t, \varphi_t)} H^{\varepsilon}(t) - \int_0^t \frac{1}{\lambda^2_\varepsilon(s, \varphi_s)} H^{\varepsilon}(s) \left( \nabla_s \lambda_\varepsilon(s, \varphi_s) + (\nabla X \lambda_\varepsilon(s, \varphi_s), \varphi_s) \right) ds$$

$$= \sum_{i=1}^5 R^{\varepsilon,(i)}(t).$$

Therefore, we obtain from (4.3), (4.40), and an integration by parts formula (applied to the stochastic integral only) that

$$|X_t^\varepsilon - X_t^{\varepsilon,\varphi}| \leq \frac{1}{\varepsilon^2} \int_0^t \int_0^s \left( e^{-A(s,r)} b(r, X^\varepsilon_r, \xi_{r/\varepsilon}) - e^{-A^\varepsilon(s,r)} b(r, \varphi_r, \xi_{r/\varepsilon}) \right) dr ds$$

$$+ |D_t^\varepsilon(t)| + \left| R^{(1)}_k(t) - R^{(1)}_k(t) \right| + \left| R^{(4)}_k(t) - R^{(4)}_k(t) \right| + \left| R^{(5)}_k(t) - R^{(5)}_k(t) \right|,$$

where

$$D_t^\varepsilon(t) := \sqrt{\varepsilon} \int_0^t \left( \frac{\sigma_\varepsilon(s, X^\varepsilon_s)}{\lambda_\varepsilon(s, X^\varepsilon_s)} - \frac{\sigma_\varepsilon(s, \varphi_s)}{\lambda_\varepsilon(s, \varphi_s)} \right) dw(s).$$

From the fact $A_\varepsilon(s) A^\varepsilon_\varepsilon(s) \geq \frac{\kappa_\varepsilon^2}{\varepsilon^4}$, and the property of $\lambda$, we can obtain that

$$\left| e^{-A(s)} - e^{-A^\varepsilon(s)} \right| \leq C e^{-\frac{\kappa_\varepsilon^2}{\varepsilon^4}} \frac{1}{\varepsilon^2} \int_0^s |X^\varepsilon_r - \varphi_r| dr = C e^{-\frac{\kappa_\varepsilon^2}{\varepsilon^4}} \int_0^s |X^\varepsilon_r - \varphi_r| dr. \quad (4.43)$$
Therefore, we obtain from the Lipschitz property of the coefficients and (4.43) that
\[
\begin{align*}
\int_0^s \left| e^{-A_\varepsilon(t,s)} b(r, X_\varepsilon^t, \xi_{r/\varepsilon}) - e^{-A_\varepsilon^e(t,s)} b(r, \varphi_r, \xi_{r/\varepsilon}) \right| \, dr \\
\leq \int_0^s \left| e^{-A_\varepsilon(t,s)} b(r, X_\varepsilon^t, \xi_{r/\varepsilon}) - b(r, \varphi_r, \xi_{r/\varepsilon}) \right| \, dr + \int_0^s \left| b(r, \varphi_r, \xi_{r/\varepsilon}) \right| \, dr \\
\leq C(1 + \|\varphi\|)^2 \sup_{0 \leq r \leq s} |X_\varepsilon^r - \varphi_r|.
\end{align*}
\]
(4.44)

A consequence of (4.43) is that
\[
\left| R_{\varepsilon}^{(1)}(t) - R_{\varepsilon}^{e,(1)}(t) \right| \leq C \varepsilon^2 \sup_{s \in [0,t]} |X_\varepsilon^s - \varphi_s|,
\]
(4.45)

and by (4.44)
\[
\left| \frac{1}{\varepsilon^2} \int_0^t \int_0^s \left( e^{-A_\varepsilon(t,s)} b(r, X_\varepsilon^t, \xi_{r/\varepsilon}) - e^{-A_\varepsilon^e(t,s)} b(r, \varphi_r, \xi_{r/\varepsilon}) \right) \, dr \, ds \right| \leq C(1 + \|\varphi\|) \int_0^t \sup_{0 \leq r \leq s} |X_\varepsilon^r - \varphi_r| \, ds.
\]
(4.46)

It is well-known that in a compact interval, an absolutely continuous function \( \varphi \) is also of bounded variation. Moreover, \( \varphi \) is differentiable almost everywhere and if we denote the derivative by \( \dot{\varphi} \), then \( \dot{\varphi} \) is integrable. Hence, the argument in the proof of Proposition 1.2 enables us to conclude that (A.1)-(A.3) are valid for the function \( \varphi(\cdot) \). As a result, by using (4.4) for \( e^{A_\varepsilon(s)} \sigma_\varepsilon(s, X_\varepsilon^s) \) and \( w(s) \), and \( e^{A_\varepsilon^e(s)} \sigma_\varepsilon(s, \varphi_s) \) and \( w(s) \) to estimate pathwise stochastic integrals
\[
\int_0^t e^{A_\varepsilon(s)} \sigma_\varepsilon(s, X_\varepsilon^s) dw(s) \quad \text{and} \quad \int_0^t e^{A_\varepsilon^e(s)} \sigma_\varepsilon(s, \varphi_s) dw(s),
\]
we obtain
\[
H_{\varepsilon}(t) - H_{\varepsilon}^e(t) \\
= \sqrt{\varepsilon} \left( \sigma_\varepsilon(t, X_\varepsilon^t) - \sigma_\varepsilon(t, \varphi_t) \right) w(t) \\
- \sqrt{\varepsilon} \int_0^t w(s) \left( e^{-A_\varepsilon(t,s)} - e^{-A_\varepsilon^e(t,s)} \right) \left[ \frac{\lambda_\varepsilon(s, X_\varepsilon^s)}{\varepsilon^2} \sigma_\varepsilon(s, X_\varepsilon^s) + \nabla_\varepsilon \sigma_\varepsilon(s, X_\varepsilon^s) + \nabla_\varepsilon \nabla_\varepsilon \sigma_\varepsilon(s, X_\varepsilon^s) p_\varepsilon \right] \, ds \\
+ \sqrt{\varepsilon} \int_0^t w(s) e^{-A_\varepsilon^e(t,s)} \left[ \frac{\lambda_\varepsilon(s, \varphi_s)}{\varepsilon^2} \sigma_\varepsilon(s, \varphi_s) - \frac{\lambda_\varepsilon(s, X_\varepsilon^s)}{\varepsilon^2} \sigma_\varepsilon(s, \varepsilon^2 X_\varepsilon^s) \right] \, ds \\
+ \sqrt{\varepsilon} \int_0^t w(s) e^{-A_\varepsilon^e(t,s)} \left[ \frac{\lambda_\varepsilon(s, \varphi_s)}{\varepsilon^2} \sigma_\varepsilon(s, \varphi_s) - \frac{\lambda_\varepsilon(s, X_\varepsilon^s)}{\varepsilon^2} \sigma_\varepsilon(s, \varepsilon^2 X_\varepsilon^s) \right] \, ds \\
+ \nabla_\varepsilon \sigma_\varepsilon(s, \varphi_s) + \nabla_\varepsilon \sigma_\varepsilon(s, \varphi_s) \dot{\varphi}_s - \nabla_\varepsilon \sigma_\varepsilon(s, \varepsilon^2 X_\varepsilon^s) - \nabla_\varepsilon \sigma_\varepsilon(s, \varepsilon^2 X_\varepsilon^s) p_\varepsilon \right] \, ds \\
= : \sum_{i=1}^4 B_{\varepsilon}^{\varphi,y}(t).
\]
(4.47)

Note that
\[
|B_{\varepsilon}^{\varphi,y}(t)| \leq C \sqrt{\varepsilon} \varepsilon^2 \|w\| |X_\varepsilon^t - \varphi_t|.
\]
(4.48)
Using (A.8) and (4.43), we have

\[ |B_{\varepsilon}^{\varphi,(2)}(t)| \leq C\sqrt{\varepsilon} \|w\| \left( \frac{1}{\varepsilon^2} + 1 + \|H_\varepsilon\| \right) \int_0^t \left| e^{-A_\varepsilon(t,s)} - e^{-A_\varepsilon(t,s)} \right| ds \]

\[ \leq C\sqrt{\varepsilon} \|w\| \left( \frac{1}{\varepsilon^2} + \|H_\varepsilon\| \right) \sup_{s \in [0,t]} |X_\varepsilon^s - \varphi_s| \int_0^t e^{-\kappa t(s-t)} (t-s) ds. \quad (4.49) \]

A change of variable leads to

\[ \int_0^t \exp \left\{ -\kappa \frac{s}{\varepsilon^2} \right\} \cdot \sqrt{\varepsilon} ds = \varepsilon^2 \int_0^{\sqrt{\varepsilon}} e^{-\kappa r^2} r dr \leq C\varepsilon^2. \quad (4.50) \]

Combining (4.49) and (4.50) implies that

\[ |B_{\varepsilon}^{\varphi,(2)}(t)| \leq C\sqrt{\varepsilon} \|w\| \left( \varepsilon^2 + \varepsilon^4 \|H_\varepsilon\| \right) \sup_{s \in [0,t]} |X_\varepsilon^s - \varphi_s|. \quad (4.51) \]

Next, it is readily seen that

\[ |B_{\varepsilon}^{\varphi,(3)}(t)| \leq C\sqrt{\varepsilon} \|w\| \int_0^t |X_\varepsilon^s - \varphi_s| ds. \quad (4.52) \]

On the other hand, we have

\[ B_{\varepsilon}^{\varphi,(4)}(t) = \sqrt{\varepsilon} e^{-A_\varepsilon(t)} \int_0^t w(s) de^{A_\varepsilon(s)} (\sigma_{\varepsilon}(s, \varphi_s) - \sigma_{\varepsilon}(s, X_\varepsilon^s)) \]

\[ = \sqrt{\varepsilon} w(t) (\sigma_{\varepsilon}(t, \varphi_t) - \sigma_{\varepsilon}(t, X_\varepsilon^t)) - \sqrt{\varepsilon} \int_0^t e^{-A_\varepsilon(t,s)} (\sigma_{\varepsilon}(s, \varphi_s) - \sigma_{\varepsilon}(s, X_\varepsilon^s)) dw(s). \quad (4.53) \]

Thus, we obtain from (4.53) that

\[ |B_{\varepsilon}^{\varphi,(4)}(t)| \leq C\sqrt{\varepsilon} \varepsilon^2 \|w\| |X_\varepsilon^t - \varphi_t| + |D_{\varepsilon}^2(t)|, \quad (4.54) \]

where

\[ D_{\varepsilon}^2(t) := \sqrt{\varepsilon} \int_0^t e^{-A_\varepsilon(t,s)} (\sigma_{\varepsilon}(s, \varphi_s) - \sigma_{\varepsilon}(s, X_\varepsilon^s)) dw(s). \]

Combining (4.48), (4.51), (4.52), and (4.54) implies that

\[ |H_\varepsilon(t) - H_{\varepsilon}^\varphi(t)| \leq C\sqrt{\varepsilon} \varepsilon^2 \|w\| \left( 1 + \varepsilon^2 \|H_\varepsilon\| \right) \sup_{s \in [0,t]} |X_\varepsilon^s - \varphi_s| \]

\[ + C\varepsilon \sqrt{\varepsilon} \|w\| \int_0^t |X_\varepsilon^s - \varphi_s| ds + C|D_{\varepsilon}^2(t)|. \quad (4.55) \]

Therefore, a standard calculation allows us to obtain that

\[ \left| R_{\varepsilon}^{(4)}(t) - R_{\varepsilon}^{\varphi,(4)}(t) \right| \leq \left| \left( \frac{1}{\lambda_{\varepsilon}(t, X_\varepsilon^t)} - \frac{1}{\lambda_{\varepsilon}(t, \varphi_t)} \right) H_{\varepsilon}^\varphi(t) + \frac{1}{\lambda_{\varepsilon}(t, X_\varepsilon^t)} (H_\varepsilon(t) - H_{\varepsilon}^\varphi(t)) \right| \]

\[ \leq C\varepsilon^2 \|H_\varepsilon\| |X_\varepsilon^t - \varphi_t| + C\sqrt{\varepsilon} \varepsilon^2 \|w\| \left( 1 + \varepsilon^2 \|H_\varepsilon\| \right) \sup_{s \in [0,t]} |X_\varepsilon^s - \varphi_s| \]

\[ + C\varepsilon \sqrt{\varepsilon} \|w\| \int_0^t |X_\varepsilon^s - \varphi_s| ds + C|D_{\varepsilon}^2(t)|. \quad (4.56) \]
Next, we have
\[
R^{(5)}_\varepsilon(t) - R^{(5)}_\varepsilon(t) = \int_0^t \left( \frac{H_\varepsilon(s) \nabla_\lambda \lambda(s, X^\varepsilon_s)}{\lambda^2(s, X^\varepsilon_s)} - \frac{H_\varepsilon(s) \nabla_\lambda \lambda(s, \varphi_s)}{\lambda^2(s, \varphi_s)} \right) ds + \int_0^t \left( \frac{H_\varepsilon(s) \nabla_X \lambda(s, X^\varepsilon_s)}{\lambda^2(s, X^\varepsilon_s)} - \frac{H_\varepsilon(s) \nabla_X \lambda(s, \varphi_s)}{\lambda^2(s, \varphi_s)} \right) ds
\]
\[= B^{(5)}_\varepsilon(t) + B^{(6)}_\varepsilon(t).\]

It can be seen that
\[
|B^{(5)}_\varepsilon(t)| \leq C \int_0^t |H_\varepsilon(s) - H_\varepsilon(s)| ds + C \varepsilon^2 \|H_\varepsilon\| \int_0^t |X^\varepsilon_s - \varphi_s| ds.
\]

On the other hand, using (A.8), we get
\[
|B^{(6)}_\varepsilon(t)| \leq C(1 + \|H_\varepsilon\|) \int_0^t |H_\varepsilon(s) - H_\varepsilon(s)| ds + C \left[ 1 + \varepsilon^2 \int_0^1 |\dot{\varphi}_s| ds + \|H_\varepsilon\| \right] \|H_\varepsilon\|.
\]

These equations imply
\[
\begin{aligned}
|R^{(5)}_\varepsilon(t) - R^{(5)}_\varepsilon(t)| &\leq C(1 + \|H_\varepsilon\|) \int_0^t |H_\varepsilon(s) - H_\varepsilon(s)| ds + C \varepsilon^2 \|H_\varepsilon\| \int_0^t |X^\varepsilon_s - \varphi_s| ds \\
&+ C \left[ 1 + \varepsilon^2 \int_0^1 |\dot{\varphi}_s| ds + \|H_\varepsilon\| \right] \|H_\varepsilon\|.
\end{aligned}
\]

Hence, by combining (4.42), (4.45), (4.46), (4.56), (4.57), and (4.55), we obtain
\[
|X^\varepsilon_t - X^{\varepsilon,\varphi}_t| \leq C \varepsilon^2 (1 + \sqrt{\varepsilon} \|w\|) (1 + \|H_\varepsilon\|)^2 (1 + \|H_\varepsilon\|) \sup_{s \in [0,t]} |X^\varepsilon_s - \varphi_s| \\
+ C(1 + \|\varphi\|) \left[ (1 + \sqrt{\varepsilon} \|w\|) (1 + \|H_\varepsilon\|) + \varepsilon^2 \|H_\varepsilon\| \right] \int_0^t \sup_{r \in [0,s]} |X^\varepsilon_r - \varphi_r| ds
\]
\[+ C \left[ 1 + \varepsilon^2 \int_0^1 |\dot{\varphi}_s| ds + \|H_\varepsilon\| \right] \|H_\varepsilon\| + C \sup_{r \in [0,1]} (|D^1 T(r)| + |D^2 T(r)|).
\]

**Step 2: Estimate of** $|X^{\varepsilon,\varphi}_t - q^{\varepsilon,\varphi}_t|$. In views of (4.37) and (4.38), we have
\[
\varepsilon^2 \dot{X}^{\varepsilon,\varphi}_t = -\lambda_\varepsilon(t, \varphi_t)(\dot{X}^{\varepsilon,\varphi}_t - q^{\varepsilon,\varphi}_t).
\]

Therefore,
\[
|X^{\varepsilon,\varphi}_t - q^{\varepsilon,\varphi}_t| = \varepsilon^2 \left| \int_0^t \frac{\dot{X}^{\varepsilon,\varphi}_s}{\lambda_\varepsilon(s, \varphi_s)} ds \right| \leq \varepsilon^2 \kappa_0 \left| \int_0^t \dot{X}^{\varepsilon,\varphi}_s ds \right| = \varepsilon^2 \kappa_0 |p^{\varepsilon,\varphi}_t - x_1|,
\]
where $p^{\varepsilon,\varphi}_t$ is derivative of $X^{\varepsilon,\varphi}_t$ defined similarly to $p^\varepsilon_t$. As a consequence of (4.59) and (A.8), we have
\[
|q^{\varepsilon,\varphi}_t - X^{\varepsilon,\varphi}_t| \leq C(\varepsilon^2 + \|H_\varepsilon\|).
\]

**Step 3: Estimate of** $|q^{\varepsilon,\varphi}_t - q^\varepsilon_t|$. Note that (4.37) and (4.38) imply
\[
\begin{align*}
q^{\varepsilon,\varphi}_t - q^\varepsilon_t &= \left( \frac{b(t, \varphi_t, \xi^{t/\varepsilon}_t)}{\lambda_\varepsilon(t, \varphi_t)} - \frac{b(t, q^\varepsilon_t, \xi^{t/\varepsilon}_t)}{\lambda_\varepsilon(t, q^\varepsilon_t)} \right) + \sqrt{\varepsilon} \left( \frac{\sigma_\varepsilon(t, \varphi_t)}{\lambda_\varepsilon(t, \varphi_t)} - \frac{\sigma_\varepsilon(t, q^\varepsilon_t)}{\lambda_\varepsilon(t, q^\varepsilon_t)} \right) w(t), \\
q^{\varepsilon,\varphi}_0 &= q^\varepsilon_0 = x_0.
\end{align*}
\]
Lemma 4.3.

\[ |q_t^\varepsilon - q_t^\varepsilon| \leq C(\varepsilon^2 \| \varphi \| + 1) \| q^\varepsilon - \varphi \| + |D_3^\varepsilon(t)|, \quad (4.61) \]

where

\[ D_3^\varepsilon(t) := \sqrt{\varepsilon} \int_0^t \left( \frac{\sigma(s, \varepsilon^2 \varphi_s)}{\lambda(s, \varepsilon^2 \varphi_s)} - \frac{\sigma(s, 0)}{\lambda(s, 0)} \right) dw(s). \]

**Step 4: Estimates of \( \| X^\varepsilon - \varphi \| \).** Applying (4.58), (4.60), and (4.61) to (4.39), we have

\[
\sup_{r \in [0,t]} |X_r^\varepsilon - \varphi_r| \leq \tilde{C}\varepsilon^2(1 + \| \varphi \| + \| \varphi \|^2) + \tilde{C}(\varepsilon^2 \| \varphi \| + 1) \| q^\varepsilon - \varphi \| + \tilde{C}[1 + \varepsilon^2 \int_0^1 |\varphi_s| ds + \| H_{\varepsilon} \|] \| H_{\varepsilon} \| + \tilde{C}(1 + \| \varphi \|)[(1 + \varepsilon^2 \| w \|)(1 + \| H_{\varepsilon} \|)^2(1 + \| H_{\varepsilon} \|)] \sup_{s \in [0,t]} |X_s^\varepsilon - \varphi_s| + \tilde{C} \int_0^t \sup_{r \in [0,s]} |X_r^\varepsilon - \varphi_r| ds + \tilde{C} \sup_{r \in [0,1]} (|D_1^\varepsilon(r)| + |D_2^\varepsilon(r)| + |D_3^\varepsilon(r)|),
\]

for a positive finite constant \( \tilde{C} \), independent of \( \varepsilon \) and \( \varphi \).

**Final Step.** To proceed, we need a couple of lemmas. To avoid interruption, the proofs of these lemmas are relegated to the appendix.

**Lemma 4.2.** There are constants \( \overline{M}_1 \) and \( \overline{M}_2 \) independent of \( \ell \) and \( \varepsilon \) such that

\[
P\left\{ \sup_{t \in [0,1]} |H_{\varepsilon}^1(t)| > \ell \right\} \leq \overline{M}_1 \exp \left\{ -\frac{\overline{M}_2 \varepsilon^2}{\varepsilon^2} \right\}, \quad \text{for all } t, s \in [0,1], \ 0 < \varepsilon < 1, \ \ell > 0. \quad (4.63)\]

**Lemma 4.3.** There is a constant \( \overline{M}_3 \) independent of \( \varepsilon \) such that

\[
P\left\{ \sup_{t \in [0,1]} (|D_1^\varepsilon(t)| + |D_2^\varepsilon(t)| + |D_3^\varepsilon(t)|) \geq \varepsilon + \overline{M}_3 \varepsilon^2(\| X^\varepsilon \| + \| \varphi \|)^2 \right\} \leq \exp \left\{ -\frac{\varepsilon^2}{\varepsilon^2} \right\}.
\]

With the two lemmas at hand, we proceed to complete the proof of the proposition. Now, let \( \theta, N > 0 \) be arbitrary and fixed. By the LDP for the Brownian motion \( w(\cdot) \) and Proposition 4.2 there exists a constant \( L = L(N) > 0 \) and \( \varepsilon_1 = \varepsilon_1(L) \in (0,1) \) such that

\[
P(\Omega_1^1) \leq \exp \left\{ -\frac{3N}{\varepsilon} \right\}, \quad \Omega_1^1 := \{ \varepsilon \| w \| + \| H_{\varepsilon} \| + \| X_{\varepsilon} \| > L \} \text{ for all } \varepsilon < \varepsilon_1. \quad (4.64)\]

In view of Lemma 4.3 there is an \( \varepsilon_2 = \varepsilon_2(N, \overline{M}_3) \in (0,1) \) satisfying

\[
P(\Omega_2^2) \leq \exp \left\{ -\frac{3N}{\varepsilon} \right\}, \quad \Omega_2^2 := \{ \| D_1^\varepsilon \| + \| D_2^\varepsilon \| + \| D_3^\varepsilon \| > \varepsilon + \overline{M}_3 \varepsilon^2(\| X^\varepsilon \| + \| \varphi \|)^2 \} \text{ for all } \varepsilon < \varepsilon_2. \quad (4.65)\]

There is a small \( \ell = \ell(N, \theta, \varphi) \in (0,1) \) satisfying

\[
\tilde{C}\ell \left[ 1 + \frac{1}{\varepsilon} \int_0^1 |\varphi_s| ds + L \right] e^{2\tilde{C}(1+\| \varphi \|)(1+L)^2+1} \leq \frac{\theta}{8}.
\]

(4.66)

By Lemma 4.2 there is an \( \varepsilon_3 = \varepsilon_3(\ell, \overline{M}_1, \overline{M}_2, N) \in (0,1) \) such that

\[
P(\Omega_3^3) \leq \exp \left\{ -\frac{3N}{\varepsilon} \right\}, \quad \Omega_3^3 := \{ \| H_{\varepsilon} \| > \ell \} \text{ for all } \varepsilon < \varepsilon_3. \quad (4.67)\]
There is an \( \varepsilon_4 = \varepsilon_4(N, \theta, \varphi) \in (0, 1) \) such that for all \( \varepsilon < \varepsilon_4 \)
\[
\bar{C}_\varepsilon^2 (1 + L)^3 < \frac{1}{4}, \quad \bar{C}_\varepsilon (2 + \|\varphi\| + \|\varphi\|^2) e^{2 \bar{C}_\varepsilon (1 + \|\varphi\|)((1 + L)^2 + 1)} \leq \frac{\theta}{8} \quad \text{and} \quad \bar{C}_\varepsilon [1 + M_3 (L + \|\varphi\|)^2] e^{2 \bar{C}_\varepsilon (1 + \|\varphi\|)((1 + L)^2 + 1)} \leq \frac{\theta}{8}. \tag{4.68}
\]
Let
\[
\bar{\theta}_1 = \frac{\theta}{8 \bar{C}_\varepsilon (\varepsilon^2 \|\varphi\| + 1) \exp \{2 \bar{C}_\varepsilon (1 + \|\varphi\|)((1 + L)^2 + 1) \}}, \tag{4.69}
\]
and
\[
\Omega^0_\varepsilon = \{ \|q^\varepsilon - \varphi\| < \bar{\theta}_1 \} \setminus (\bigcup_{i=1}^3 \Omega^i_\varepsilon).
\]
Then it is clear that \( \bar{\theta}_1 \) is independent of \( \varepsilon \). Moreover, note that
\[
\mathbb{P} \left( \bigcup_{i=1}^3 \Omega^i_\varepsilon \right) \leq 3 \exp \left\{ -\frac{3N}{\varepsilon} \right\} \leq \exp \left\{ -\frac{N}{\varepsilon} \right\}
\]
for all \( \varepsilon < \varepsilon_5 \) for some \( \varepsilon_5 = \varepsilon_5(N) \in (0, 1) \).

Now, for any \( \varepsilon < \varepsilon_0 := \min \{ \varepsilon_i : i = 1, \ldots, 5 \} \) and \( \omega \in \Omega^0_\varepsilon \), we have from (4.64), (4.65), and (4.67) that
\[
\sup_{s \in [0, t]} |X^\varepsilon_s - \varphi_s| \leq 2 \bar{C}_\varepsilon (2 + \|\varphi\| + \|\varphi\|^2) + 2 \bar{C}_\varepsilon \left( \varepsilon^2 \|\varphi\| + 1 \right) \bar{\theta}_1 + \bar{C}_\varepsilon \left[ 1 + \int_0^1 |\dot{\varphi}_s| ds + L \right] \\
+ 2 \bar{C}_\varepsilon [1 + M_3 (L + \|\varphi\|)^2] + 2 \bar{C}_\varepsilon (1 + \|\varphi\|) (1 + L)^2 + 1] \int_0^t \sup_{r \in [0, s]} |X^\varepsilon_r - \varphi_r| ds. \tag{4.70}
\]
Applying Gronwall’s inequality to (4.70) and then using (4.66), (4.68), and (4.69), we obtain that
\( \|X^\varepsilon - \varphi\| < \theta \), for any \( \varepsilon < \varepsilon_0 \) and \( \omega \in \Omega^0_\varepsilon \). Therefore, one has that for any \( \varepsilon < \varepsilon_0 \),
\[
\mathbb{P} \{ \|q^\varepsilon - \varphi\| < \theta \} \geq \mathbb{P} \{ \|X^\varepsilon - \varphi\| < \bar{\theta}_1 \} - \exp \left\{ -\frac{N}{\varepsilon} \right\}. \]

By noting that
\[
|q_t^\varepsilon - \varphi_t| \leq |X_t^\varepsilon - \varphi_t| + |X_t^\varepsilon - X_t^{\varepsilon, \varphi}| + |X_t^{\varepsilon, \varphi} - q_t^{\varepsilon, \varphi}| + |q_t^{\varepsilon, \varphi} - q_t^\varepsilon|,
\]
(4.36) can be obtained. Therefore, Proposition 4.4 is proved. \( \square \)

Applying Assumption 2.2 and Proposition 4.4 enables us to obtain the following theorem.

**Theorem 4.1.** The sequence \( \{X^\varepsilon\}_{\varepsilon > 0} \) satisfies the local LDP. That is, for any \( \varphi \in \mathcal{C}([0, 1], \mathbb{R}^d) \), one has
\[
\lim_{\varepsilon \to 0} \lim_{\theta \to 0} \sup_{\varepsilon} \varepsilon \log \mathbb{P} (X^\varepsilon \in B(\varphi, \theta))
= \lim_{\theta \to 0} \inf_{\varepsilon} \varepsilon \log \mathbb{P} (X^\varepsilon \in B(\varphi, \theta))
= -\tilde{I}(\varphi),
\]
where \( B(\varphi, \varepsilon) \) is the ball centered at \( \varphi \) with radius \( \varepsilon \).
Proof. We divide the proof by treating the lower bounds and upper bounds.

**Lower bound of local LDPs.** We first prove

\[
\lim_{\theta \to 0} \liminf_{\varepsilon \to 0} \varepsilon \log P (X^\varepsilon \in B(\varphi, \theta)) \geq -\tilde{I} (\varphi).
\]

Since it is trivial if \( \tilde{I} (\varphi) = \infty \), we assume that \( \tilde{I} (\varphi) < \infty \) and \( \varphi \) is absolutely continuous. For any \( r > 0 \), since \( \{q^\varepsilon\}_{\varepsilon > 0} \) satisfies the local LDP with rate function \( \tilde{I} \), there is a \( \theta_1 \) such that

\[
\liminf_{\varepsilon \to 0} \varepsilon \log P (q^\varepsilon \in B(\varphi, \theta_1)) \geq -\tilde{I} (\varphi) + 2r.
\]

Let \( N_1 \) be sufficiently large such that

\[
\exp \left\{ -\frac{\tilde{I} (\varphi) + 2r}{\varepsilon} \right\} - \exp \left\{ -\frac{N_1}{\varepsilon} \right\} \geq \exp \left\{ -\frac{\tilde{I} (\varphi) + r}{\varepsilon} \right\}.
\]

By Proposition 4.3, there are \( \theta_1 \) and \( \varepsilon_0 \) such that for any \( \varepsilon < \varepsilon_0 \)

\[
P \{ \| X^\varepsilon - \varphi \| < \theta_1 \} \geq P \{ \| q^\varepsilon - \varphi \| < \theta_1 \} - \exp \left\{ -\frac{N_1}{\varepsilon} \right\}.
\]

As a consequence, one concludes that for any \( r > 0 \), there is a \( \overline{\theta}_1 \) satisfying

\[
\liminf_{\varepsilon \to 0} \varepsilon \log P (X^\varepsilon \in B(\varphi, \overline{\theta}_1)) \geq \liminf_{\varepsilon \to 0} \varepsilon \log \left( P (q^\varepsilon \in B(\varphi, \theta_1)) - \exp \left\{ -\frac{N_1}{\varepsilon} \right\} \right)
\]

\[
\geq -\tilde{I} (\varphi) + r.
\]

Therefore, we obtain the lower bound for local LDPs.

**Upper bound of local LDPs.** It is easily seen that if \( \varphi \) is absolutely continuous, a similar argument to the process of obtaining lower bound of local LDP yields that

\[
\lim_{\theta \to 0} \limsup_{\varepsilon \to 0} \varepsilon \log P (X^\varepsilon \in B(\varphi, \theta)) \leq -\tilde{I} (\varphi).
\]

Now, we consider \( \varphi \in \mathcal{C}([0, 1], \mathbb{R}^d) \), which is not absolutely continuous and \( I (\varphi) = \infty \). We aim to prove that

\[
\lim_{\theta \to 0} \limsup_{\varepsilon \to 0} \varepsilon \log P (X^\varepsilon \in B(\varphi, \theta)) = -\infty.
\]

For any \( R > 0 \), since \( \{q^\varepsilon\}_{\varepsilon > 0} \) satisfies the local LDP with rate function \( \tilde{I} \) and \( \tilde{I} (\varphi) = \infty \), there is a \( \theta_2 \in (0, 1) \) such that

\[
\limsup_{\varepsilon \to 0} \varepsilon \log P (q^\varepsilon \in B(\varphi, \theta_2)) \leq -R.
\]

Let \( N_2 > R \). By Proposition 4.4 and Proposition 4.6, there is a \( \overline{\theta}_2 \in (0, \theta_2 / 2) \) such that for any \( \phi \in B(\varphi, 1) \), \( \phi \) is absolutely continuous and there is \( \varepsilon_0 = \varepsilon_0 (\phi) \) satisfying

\[
P \{ \| X^\varepsilon - \phi \| < 2\overline{\theta}_2 \} \leq P \{ \| q^\varepsilon - \phi \| < \frac{\theta_2}{2} \} + \exp \left\{ -\frac{N_2}{\varepsilon} \right\}, \ \forall \varepsilon < \varepsilon_0 (\phi).
\]
Let $\phi \in B(\varphi, \theta_2)$ be an absolutely continuous function (such $\phi$ does always exist due to denseness of absolutely continuous functions). As a consequence, we have

$$\limsup_{\varepsilon \to 0} \varepsilon \log P(X^\varepsilon \in B(\varphi, \theta_2)) \leq \limsup_{\varepsilon \to 0} \varepsilon \log P(X^\varepsilon \in B(\phi, \theta_2))$$

$$\leq \limsup_{\varepsilon \to 0} \varepsilon \log \left( P(q^\varepsilon \in B(\phi, \theta_2)) + \exp \left\{ -\frac{N_2}{\varepsilon} \right\} \right)$$

$$\leq \limsup_{\varepsilon \to 0} \varepsilon \log \left( P(q^\varepsilon \in B(\varphi, \theta_2)) + \exp \left\{ -\frac{N_2}{\varepsilon} \right\} \right)$$

$$= \max \left\{ \limsup_{\varepsilon \to 0} \varepsilon \log (P(q^\varepsilon \in B(\varphi, \theta_2))), -N_2 \right\}$$

$$\leq -R.$$

Therefore, if $\phi$ is not absolutely continuous, then

$$\lim_{\theta \to 0} \limsup_{\varepsilon \to 0} \varepsilon \log P(X^\varepsilon \in B(\varphi, \theta)) = -\infty.$$ 

So, the proof is complete.

\[\square\]

A Proofs of Technical Results

Proof of Proposition 4.2. If $f \in C^1([0, t])$ and $g \in C([0, t])$, then the Stieltjes integral

$$\int_0^t f(s)dg(s), \quad t \geq 0$$

is well defined and the following integration by parts formula holds

$$\int_{t_1}^{t_2} f(s)dg(s) = f(t_2)g(t_2) - f(t_1)g(t_1) - \int_{t_1}^{t_2} g(s)f'(s)ds, \quad 0 \leq t_1 < t_2 \leq t.$$ 

(A.1)

In addition, if $g(0) = 0$, as a consequence of (A.1),

$$\int_0^t f(s)dg(s) = g(t)f(0) + \int_0^t (g(t) - g(s))f'(s)ds, \quad t \geq 0.$$ 

(A.2)

Thus, we can apply the integration by parts formula (A.2) for $f(s) = e^{A\varepsilon(s)}\sigma(s, \varepsilon^2X^\varepsilon_s)$ and $g(s) = w(s)$ to get

$$\int_0^t e^{A\varepsilon(s)}\sigma_\varepsilon(s, X^\varepsilon_s)dw(s) = \sigma_\varepsilon(0, x_0)w(t)$$

$$+ \int_0^t e^{A\varepsilon(s)} \left[ \frac{\lambda_\varepsilon(s, X^\varepsilon_s)}{\varepsilon^2} \sigma_\varepsilon(s, X^\varepsilon_s) + \nabla_x \sigma_\varepsilon(s, X^\varepsilon_s) + \nabla_x \sigma_\varepsilon(s, X^\varepsilon_s)p^\varepsilon_s \right] (w(t) - w(s)) ds.$$ 

(A.3)

Therefore, by multiplying $\sqrt{\varepsilon}e^{-A\varepsilon(t)}$ to both sides of (A.3), taking the norm on both sides of the equation, using boundedness assumptions on $\lambda_\varepsilon$ and $\sigma_\varepsilon$, and carrying out the detailed calculations, we obtain

$$|H_\varepsilon(t)| \leq C\sqrt{\varepsilon} \|w\| \left( 1 + \frac{1}{\varepsilon^2} \int_0^t e^{-A\varepsilon(t,s)}ds + \varepsilon^2 \int_0^t e^{-A\varepsilon(t,s)}|p^\varepsilon_s|ds \right).$$ 

(A.4)

29
Combining (A.4) and the fact \( A_\varepsilon(t, s) \geq \frac{\kappa_0(t - s)}{\varepsilon^2} \geq 0 \) implies that

\[
|H_\varepsilon(t)| \leq C\sqrt{\varepsilon} \|w\| \left( 1 + \varepsilon^2 \int_0^t |p_\varepsilon| ds \right), \quad \forall t \in [0, 1]. \tag{A.5}
\]

Now, we are ready to estimate \(|p_\varepsilon(t)|\). Thanks to (4.4) and the fact \( A_\varepsilon(t, s) \geq \frac{\kappa_0(t - s)}{\varepsilon^2} \) again, we have

\[
|p_\varepsilon| \leq |x_1| e^{-A_\varepsilon(t)} + C \int_0^t e^{-\frac{\kappa_0(t-s)}{\varepsilon^2}} (1 + |X_\varepsilon|) ds + \frac{1}{\varepsilon^2} |H_\varepsilon(t)|. \tag{A.6}
\]

Because of (4.5) and Young’s inequality, one has

\[
|X_\varepsilon| \leq C + C \int_0^t (1 + |X_\varepsilon|) ds + \frac{1}{\varepsilon^2} \int_0^t |H_\varepsilon(s)| ds
\leq C \left( 1 + \frac{1}{\varepsilon^2} \int_0^t |H_\varepsilon(s)| ds \right) + C \int_0^t |X_\varepsilon| ds.
\]

Then applying Gronwall’s inequality leads to

\[
|X_\varepsilon| \leq C \left( 1 + \frac{1}{\varepsilon^2} \int_0^t |H_\varepsilon(s)| ds \right). \tag{A.7}
\]

Hence, we obtain from (A.6) and (A.7) that

\[
|p_\varepsilon(t)| \leq C \left( 1 + \frac{1}{\varepsilon^2} \sup_{s \in [0, t]} |H_\varepsilon(s)| \right). \tag{A.8}
\]

Applying (A.8) to (A.5) yields that

\[
\sup_{s \in [0, t]} |H_\varepsilon(s)| \leq C\sqrt{\varepsilon} \|w\| \left( 1 + \int_0^t \sup_{r \in [0, s]} |H_\varepsilon(r)| ds \right), \quad \forall t \in [0, 1].
\]

Thus, Gronwall’s inequality implies that

\[
\|H_\varepsilon\| \leq \tilde{C}_0 \sqrt{\varepsilon} \|w\| e^{\tilde{C}_0 \sqrt{\varepsilon} \|w\|},
\]

for some finite constant \( \tilde{C}_0 \), independent of \( \varepsilon \).

Next, we proceed to estimate \( X_\varepsilon \) by using the representation (4.6). It is easily seen that for all \( t \in [0, 1] \),

\[
|R_\varepsilon^{(1)}(t)| \leq C, \quad |R_\varepsilon^{(2)}(t)| \leq C \left( 1 + \int_0^t \sup_{r \in [0, s]} |X_\varepsilon| ds \right), \quad |R_\varepsilon^{(4)}(t)| \leq C|H_\varepsilon(t)|. \tag{A.9}
\]

By assumptions on bounded derivative of \( \lambda \) and (A.8), one can obtain that

\[
|R_\varepsilon^{(3)}(t)| \leq C(1 + \|H_\varepsilon\|) \int_0^t \sup_{r \in [0, s]} |X_\varepsilon| ds. \tag{A.10}
\]

Similarly, one has

\[
|R_\varepsilon^{(5)}(t)| \leq C\|H_\varepsilon\|(1 + \|H_\varepsilon\|). \tag{A.11}
\]
Therefore, we have that

\[ |R_\varepsilon(t)| \leq C(1 + \|H_\varepsilon\| + \|H_\varepsilon\|^2) + C(1 + \|H_\varepsilon\|) \int_0^t \sup_{r \in [0,s]} |X_r^\varepsilon| ds. \]  

(A.12)

Moreover, a similar process of getting (A.3) helps us to estimate

\[ \sqrt{\varepsilon} \int_0^t \frac{\sigma(s, \varepsilon^2 X_s^\varepsilon)}{\lambda(s, \varepsilon^2 X_s^\varepsilon)} dw(s), \]

which together with (A.8) implies that

\[ \left| \sqrt{\varepsilon} \int_0^t \frac{\sigma(s, \varepsilon^2 X_s^\varepsilon)}{\lambda(s, \varepsilon^2 X_s^\varepsilon)} dw(s) \right| \leq C \sqrt{\varepsilon} \|w\|(1 + \|H_\varepsilon\|). \]  

(A.13)

It follows from (4.16), (A.12), and (A.13) that

\[ \sup_{s \in [0,t]} |X_s^\varepsilon| \leq C(1 + \|H_\varepsilon\| + \|H_\varepsilon\|^2)(1 + \sqrt{\varepsilon} \|w\|) + C(1 + \|H_\varepsilon\|) \int_0^t \sup_{r \in [0,s]} |X_r^\varepsilon| ds. \]  

(A.14)

Therefore, it follows (A.14) and the Gronwall inequality that

\[ \|X^\varepsilon\| \leq C(1 + \|H_\varepsilon\| + \|H_\varepsilon\|^2)(1 + \sqrt{\varepsilon} \|w\|) e^{C(1 + \|H_\varepsilon\|)} \leq \tilde{C}_1 \Gamma(\tilde{C}_1 \sqrt{\varepsilon} \|w\|), \]  

(A.15)

where \( \tilde{C}_1 \) is a positive finite constant, independent of \( \varepsilon \) and

\[ \Gamma(v) := (1 + ve^v + v^2 e^{2v})(1 + v)e^{1+v} - v. \]

The proof is complete.

**Proof of Lemma 4.1.** The proof is similar to [11, Proof of Theorem 4.2]. Define the grid \( G_n = \{ \frac{i}{2^n} : 0 \leq i \leq 2^n \} \). Two points \( u = \frac{i}{2^n}, v = \frac{j}{2^n} \in G_n \) are said to be nearest neighbors if \( |i - j| \leq 1 \). Then for any \( u \in G_n \), there exists a path \( 0 = q_0, v_1, \ldots, v_N = u \) of points in \( G_n \) such that each pair \( v_{i-1} \) and \( v_i \) are nearest neighbors in some grid \( G_m, m \leq n \), and at most one of such pairs consists of points, which are nearest neighbors in any given grid \( G_m \). Indeed, we can write \( u = 0.k_1 \ldots k_N \) in the binary (base 2) expansion and let \( v_m = 0.k_1 k_2 \ldots k_m \). Next, let \( D(n) \) be the event that for all nearest neighbors \( u, v \in G_n \), we have \( |Y(u) - Y(v)| \leq 2L^{-0.125n} \). From (4.15), for each pair of nearest neighbors \( u, v \in G_n \), we have

\[ \mathbb{P}(|Y(u) - Y(v)| > 2L^{-0.125n}) \leq \alpha_1 \exp\{ -\alpha_2 2^{0.25n} \}. \]

Because there are \( 2^n \) nearest neighbors in \( G_n \), one gets

\[ \mathbb{P}(D(n)^c) \leq 2^n \alpha_1 \exp\{ -\alpha_2 2^{0.25n} \} \leq C_1 \alpha_1 \exp\{ -C_2 2^{0.25n} \}, \]

for some positive constants \( C_1, C_2 \), independent of \( n \). Hence, let \( D = \cap_{n=0}^{\infty} D(n) \) and summing the previous estimates over \( n \), we have

\[ \mathbb{P}(D^c) \leq C_1 \alpha_1 \exp\{ -C_2 \alpha_2 \}, \]

where \( C_1, C_2 \) may be different than before. Moreover, in the event \( D \) one has that for any \( u \in \cup_{n=0}^{\infty} G_n \), there is a path \( 0 = v_1; v_2; \ldots; v_N = u \) with \( v_{i-1}, v_i \) are nearest neighbors in some \( G_n \) and then,

\[ |Y(u)| \leq \sum_{n=1}^{N} |Y(v_{n-1}) - Y(v_n)| \leq \sum_{n=1}^{\infty} L2^{-0.125n} \leq C_3 L. \]

Therefore, we conclude the proof of Lemma 4.1.
Proofs of Lemmas 4.2 and 4.3

A standard calculation shows that

$$H_\varepsilon^\varphi(t) - H_\varepsilon^\varphi(s) = \sqrt{\varepsilon} e^{-A^\varphi_\varepsilon(t)} \int_0^t e^{A^\varphi_\varepsilon(r)} \sigma_\varepsilon(r, \varphi_r) dw(r) - \sqrt{\varepsilon} e^{-A^\varphi_\varepsilon(s)} \int_0^s e^{A^\varphi_\varepsilon(r)} \sigma_\varepsilon(r, \varphi_r) dw(r)
$$

$$= \sqrt{\varepsilon} \int_s^t e^{-A^\varphi_\varepsilon(t,r)} \sigma_\varepsilon(r, \varphi_r) dw(r) - \sqrt{\varepsilon}(1 - e^{-A^\varphi_\varepsilon(t,s)}) \int_0^s e^{-A^\varphi_\varepsilon(s,r)} \sigma_\varepsilon(r, \varphi_r) dw(r).$$

(A.16)

As used often in this paper, the first stochastic integral is an element of a sequence of martingales with quadratic deviation bounded by $C\varepsilon^2(1 - e^{-\frac{\varepsilon}{C\varepsilon^2}})$, then using the fact $1 - e^{-u} \leq \sqrt{u}$, $\forall u > 0$ that is bounded by $C\varepsilon^2 \sqrt{|t - s|}$. Similarly, by using the fact $(1 - e^{-u})^2 \leq \sqrt{u}$, $\forall u > 0$, the second stochastic integral is an element of a sequence of martingales with quadratic deviation bounded by the $C\varepsilon^2 \sqrt{|t - s|}$. Therefore, an application of exponential martingale inequality [21, Theorem 7.4, p. 44] allows us to obtain that $\forall t, s \in [0, 1]$,

$$P \{ |H_\varepsilon^\varphi(t) - H_\varepsilon^\varphi(s)| > \ell \} \leq \exp \left\{ - \frac{C\ell^2}{\varepsilon^2 |t - s|^2} \right\},$$

(A.17)

where $C$ is some finite constant, independent of $\ell, \varepsilon$. With this property, the technique and argument to obtain (4.63) is similar to that of Lemma 4.1. Similarly, the proof of Lemma 4.3 is obtained by using exponential martingale inequality [21, Theorem 7.4, p. 44].
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