Towards an Algebraic Theory of Analogical Reasoning in Logic Programming

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Abstract. Analogy-making is an essential part of human intelligence and creativity. This paper proposes an algebraic model of analogical reasoning in logic programming based on the syntactic composition and decomposition of programs. The main idea is to define analogy in terms of modularity and to derive abstract forms of concrete programs from a ‘known’ source domain which can then be instantiated in an ‘unknown’ target domain to obtain analogous programs. To this end, we introduce algebraic operations for program modularity and illustrate, by giving numerous examples, that programs have nice decompositions. Interestingly, our work suggests a close relationship between modularity, generalization, and analogy which we believe should be explored further in the future. In a broader sense, this paper is a first step towards an algebraic (and mainly syntactic) theory of analogical reasoning in logic-based knowledge representation and reasoning systems, with potential applications to fundamental AI-problems like computational learning and creativity.

Keywords: Artificial General Intelligence · Knowledge Transfer · Computational Learning.

1 Introduction

Analogy-making is an essential part of human intelligence and creativity, with applications to such diverse tasks as commonsense reasoning, problem solving, and story telling (cf. (Hofstadter, 2001; Gust, Krumnack, Kühnberger, & Schwering, 2008)).

This paper proposes an algebraic model of analogical reasoning in logic programming based on the syntactic composition and decomposition of programs. The main idea is to define analogy in terms of modularity and to derive abstract forms of concrete programs from a ‘known’ source domain which can then be instantiated in a ‘novel’ target domain to obtain analogous programs. To this end, we introduce algebraic operations for program modularity and illustrate, by giving numerous examples, that programs have nice decompositions.

In general, ‘creative’ analogies contain some form of unexpectedness, in the sense that objects (possibly from different domains) are put together in a novel and often unforeseen way. Consider the following running example.
Example 1. Imagine two domains, one consisting of numerals of the form 0, s(0), s(s(0)), ..., formally representing the natural numbers, and the other made up of lists [ ], [a], [a, [b]], and so on. We know from basic arithmetic what it means to add two numbers (i.e., numerals), e.g., \(\text{plus}(s(0), s(0), s(s(0)))\) expresses the fact that one plus one equals two. Now suppose we want to transfer the concept of addition to the list domain. We can then ask—by analogy—the following question: What does it mean to ‘add’ two lists? There are (at least) two possible answers: One is to say that a list \(\ell\), say, is the ‘addition’ of two lists \(\ell_1\) and \(\ell_2\), say, if the number of elements in \(\ell\) equals the sum of the number of elements in \(\ell_1\) and \(\ell_2\). In this case, we do not take the inner structure of lists into account and we directly transfer the concept of number to lists, e.g., in analogy to \(\text{plus}(s(0), s(0), s(s(0)))\), the fact \(\text{plus}([a], [b], [c, c])\) is perfectly derivable under this interpretation. However, if we do respect the inner structure of lists, we obtain a second interpretation in which \(\ell\) is the ‘addition’ of \(\ell_1\) and \(\ell_2\) if \(\ell\) is the list \(\ell_1\) appended by \(\ell_2\). In this case, we have specialized the analogy to the concrete domain at hand. This simple example shows the inexact nature of analogies.

We can now ask a further question: Given that we ‘know’ how to add numbers, can we use this knowledge to answer questions in the list domain? It is evident that in the first case, where we only cared about the number of elements in a list, we can simulate ‘addition’ of lists via addition of numerals. The more interesting case is the second one, that is, given that multiple lists have the same length and thus map to the same numeral, can we still ‘add’ (i.e., append) lists via the (seemingly) simpler addition of numerals? Surprisingly, the answer is again ‘yes’, and we will return to this question, in a more formal way, in Example 4 and 8 (in which we also discuss the important role of (un)typing for analogical reasoning).

The rest of the paper is structured as follows. The next section is preliminary and recalls the syntax and proof theory of an untyped version of higher-order Horn logic programming. Section 3 introduces two algebraic operations for program modularity, namely ‘composition’ and ‘concatenation’, on which the notions of analogy and logic program forms will be based. The following Sections 4 and 5 constitute the main part of the paper. In Section 4 we define the notion of analogy and show how analogies can be used to transfer knowledge across different domains by formally and precisely discussing Example 1. After that we introduce in Section 5 the notion of logic program form and show that forms subsume analogies. We then sketch, by giving detailed examples, a procedure for analogical reasoning via decomposition, abstraction, and specialization. Finally, we conclude the paper with a discussion and some ideas for future research.

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1 One can distinguish between different elements of a list.
2 Note that we still have the property that the length of the list \(\ell\) is the sum of the lengths of \(\ell_1\) and \(\ell_2\).
3 What we mean here is a simple mapping between lists and numerals which makes no use of any complicated form of encoding.
For excellent surveys on computational models of analogical reasoning we refer the reader to (Hall, 1989; Prade & Richard, 2014a, 2014b).

2 Preliminaries

We recall the syntax and proof theory of higher-order Horn logic programs where we restrict ourselves to the \textit{untyped} case.

2.1 Syntax

Let $\Sigma$ be an unranked nonempty alphabet whose elements are called \textit{symbols}. Terms are formed as usual from variables $x, y, z, \ldots$ and symbols from $\Sigma$, where constants are treated as 0-ary symbols. A term is \textit{ground} if it contains no variables. The set of ground terms is called the \textit{Herbrand universe} of $\Sigma$ and is denoted by $HU_\Sigma$.

An (\textit{untyped higher-order Horn logic}) program over $\Sigma$ (or $\Sigma$-program) is a set of rules of the form

$$t_0 \leftarrow t_1, \ldots, t_k, \quad k \geq 0,$$

(1)

where $t_0, \ldots, t_k$ are terms. A program is called \textit{ground} if each of its rule is ground.

It will be convenient to define, for a rule $r$ of the form (1),

$$H(r) = \{t_0\} \text{ (head)}$$

and

$$B(r) = \{t_1, \ldots, t_k\} \text{ (body)}$$

extended to programs by $H(P) = \bigcup_{r \in P} H(r)$ and $B(P) = \bigcup_{r \in P} B(r)$. A \textit{fact} is a rule with empty body, whereas a \textit{proper rule} is a rule which is not a fact. We denote the facts (resp., proper rules) of $P$ by $P^\emptyset$ (resp., $P^\bullet$). We will occasionally make some of the variables occurring in a program $P$ explicit by writing, e.g., $P(x, y)$ for the program containing the variables $x, y$.

Substitutions and unifiers are defined as usual and we denote the most general unifier of two terms $s, t$ by $mgu(s, t)$. Substitutions operate on expressions as usual and we denote the application of a substitution $\theta$ on an expression $E$ by $E[\theta]$.

For a symbol $q \in \Sigma$ and a rule $r$ of the form (1), we define the \textit{wrapping} of $r$ via $q$ by

$$q(r) = q(t_0) \leftarrow q(t_1), \ldots, q(t_k),$$

extended to programs by $q(P) = \{q(r) \mid r \in P\}$. The \textit{reverse} of a program $P$ is obtained by reversing all the arrows in $P$, that is,

$$\overrightarrow{P} = \{t \leftarrow H(r) \mid r \in P : t \in B(r)\}.$$

A multiset $S$ of rules is a \textit{submultiset} of $P$, denoted $S \subseteq P$, if $S$ consists only of rules from $P$. 
2.2 SLD-Resolution

Logic programs compute via a restricted form of resolution, called SLD-resolution, as follows.

For simplicity, we consider here only the ground case. Let $Q$ be a ground query $t_1, \ldots, t_k$, $k \geq 1$, and suppose that for some $i$, $1 \leq i \leq k$, and $m \geq 0$, $r = t_i \leftarrow t'_1, \ldots, t'_m$ is a rule from $\text{gnd}(P)$. Then $Q'$ given by

$$\leftarrow t_1, \ldots, t_{i-1}, t'_1, \ldots, t'_{m}, t_{i+1}, \ldots, t_k$$

is called a resolvent of $Q$ and $r$, and $t_i$ is called the selected term of $Q$. By iterating this process we obtain a sequence of resolvents which is called an SLD-derivation. A derivation can be finite or infinite. If its last query is empty then we speak of an SLD-refutation of the original query $Q$. In this case we have derived an SLD-proof of $t_1, \ldots, t_k$. A failed SLD-derivation is a finite derivation which is not a refutation. In case $t$ is a ground term with an SLD-proof from $P$, we say that $t$ is an SLD-consequence of $P$ and write $P \vdash t$.

Example 2. In this paper, we will be interested in three basic data structures: numerals, lists, and (binary) trees. The corresponding type programs are constructed as follows. Let $\text{Nat} = \{0/0, s/1\}$, $\text{List} = \{\text{nil}/0, \text{cons}/2\}$, and $\text{Tree} = \{0/0, f/3\}$ be alphabets. As is customary in logic programming, we abbreviate lists by writing, e.g., $[a, b]$ instead of $\text{cons}(a, \text{cons}(b, \text{nil}))$ and so on. Then, we define the $\text{Nat}$-, $\text{List}$-, and $\text{Tree}$-type programs $\omega(x), \lambda(u, x), \tau(u, x, y)$, respectively, by

$$0 \leftarrow \quad [ ] \leftarrow \quad 0 \leftarrow$$

$$s(x) \leftarrow x, \quad [u | x] \leftarrow x, \quad f(u, x, y) \leftarrow x, y.$$

3 Modularity

In the rest of the paper, $P$ and $R$ will denote logic programs over some common alphabet $\Sigma$ if not specified otherwise.

This section introduces two (syntactic) composition operations on which we will base our notions of analogy and logic program forms.

Definition 1. We define the composition of $P$ and $R$ by

$$P \circ R = \{H(r\theta) \leftarrow B(S\theta) \mid r \in P, S \subseteq R : H(S\theta) = B(r\theta), \theta = \text{mgu}(B(r), H(S))\}.$$
In the sequel, we simply write $PR$ if the composition operation is understood. We denote the neutral element with respect to composition by $1$.

Roughly, we obtain the composition of $P$ and $R$ by resolving all body atoms in $P$ with the ‘matching’ rule heads of $R$. Composition is thus related to SLD-resolution in the sense that we can answer queries across different domains via decomposition (cf. Example 3).

Example 3. Reconsider the program $\omega$ of Example 2 generating the natural numbers. By composing the only proper rule in $\omega$ with itself, we obtain

$$(\omega \bullet \omega)^2 = \{s(x) \leftarrow x\} \circ \{s(x) \leftarrow x\} = \{s(s(x)) \leftarrow x\}.$$  

Note that this program, together with the single fact in $\omega$, generates the even numbers. Let us therefore define the program

$$\varepsilon = \omega^0 \cup (\omega \bullet \omega)^2 = \begin{cases} 0 \leftarrow & \\ s(s(x)) \leftarrow x & \end{cases}.$$  

(2)

We will come back to this example in (Section 5) when studying logic program forms.

In some cases, a program is the ‘concatenation’ of two other programs on an atomic level. A typical example is the program

$$\text{Length} = \begin{cases} \text{length}([\ ]), 0 \leftarrow & \\ \text{length}([u \mid x], s(y)) \leftarrow & \end{cases}$$

which is, roughly, the ‘concatenation’ of the List-type program $\lambda(u, x)$ in the first argument and the Nat-type program $\omega(y)$ in the second argument wrapped by the symbol $\text{length}$ (cf. Example 2). This is formalized as follows.

Definition 2. We define the concatenation of $P$ and $R$ inductively as follows:

1. For symbols $p, q$ and sequences of terms $s, t$,

$$p(s) \cdot q(t) = \begin{cases} p(s, t) & \text{if } p = q, \\ \text{undefined} & \text{otherwise}. \end{cases}$$

2. For rules $r, r'$,

$$r \cdot r' = (H(r) \cdot H(r')) \leftarrow \{t \cdot t' | t \in B(r), t' \in B(r')\}.$$  

3. Finally, the concatenation of $P$ and $R$ is given by

$$P \cdot R = \{r \cdot r' | r \in P, r' \in R\}.$$  

We can now formally decompose the program $\text{Length}$ from above with respect to concatenation by

$$\text{Length} = \text{length}(\lambda(u, x)) \cdot \text{length}(\omega(y)).$$  

We will return to decompositions of this form in Section 5.
4 Analogy

In this section, we formalize the concept of two programs, possibly from different domains, being syntactically analogous (or similar).

**Definition 3.** We say that \( P \) can be (one-step) simulated by \( R \), in symbols \( P \preceq R \), if there are programs \( X, Y \) over \( \Sigma_P \cup \Sigma_R \) such that \( P = XRY \). In this case, we call the programs \( X, Y \) the left and right transfer programs, respectively. If \( P \preceq R \) and \( R \preceq P \), we say that \( P \) and \( R \) are analogous (or similar) and write \( P \approx R \).

In other words, \( P \) can be simulated by \( R \) if there is a decomposition of \( P \) (over the joint alphabet \( \Sigma_P \cup \Sigma_R \)) in which \( R \) occurs as a factor, that is, if \( P \) can be algebraically constructed from \( R \). The intuition is that the transfer programs \( X, Y \) in the decomposition of \( P \) function as ‘bridges’ for transferring information between \( P \) and \( R \), as is illustrated by the following example.

**Example 4.** To return to our introductory Example 1, consider the well-known program

\[
\text{Plus} = \begin{cases}
\text{plus}(0, y, y) \leftarrow \\
\text{plus}(s(x), y, s(z)) \leftarrow \\
\text{plus}(x, y, z)
\end{cases},
\]

implementing the addition of positive numbers (represented as numerals). We can now ask—by analogy—what it means to ‘add’ two lists. As we have informally discussed in Example 1 there are at least two possible answers. The first is to say that a list \( \ell \), say, is the ‘addition’ of the lists \( \ell_1, \ell_2 \), say, if the number of elements of \( \ell \) equals the sum of the number of elements in \( \ell_1 \) and \( \ell_2 \). This solution is represented by the program

\[
\text{Plus}_{\text{List}} = \begin{cases}
\text{plus}([ ], [ ], [ ]) \leftarrow \\
\text{plus}([ ], [u | y], [v | z]) \leftarrow \\
\text{plus}([ ], y, z) \\
\text{plus}([u | x], y, [v | z]) \leftarrow \\
\text{plus}(x, y, z)
\end{cases}.
\]

Note that the program \( \text{Plus}_{\text{List}} \) does not take the inner structure of lists into account; e.g., we can derive

\[
\text{Plus}_{\text{List}} \vdash \text{plus}([a], [a], [b, [b]]).
\]

The program \( \text{Plus}_{\text{List}} \) is arguably the ‘freest’ translation of \( \text{Plus} \) into the list domain in the sense that for any rule \( r \) consisting only of shared symbols (e.g., \( \text{plus} \)), we have

\[
\text{Plus}_{\text{List}} \vdash r \iff \text{Plus} \vdash r.
\]
For example, the rule
\[ r = plus(y, x, z) \leftarrow plus(x, y, z), \]
expressing commutativity of addition, is derivable from both programs. However, if we do take the inner structure of lists into account, we obtain a second solution where \( \ell \) is the ‘addition’ of \( \ell_1 \) and \( \ell_2 \) if \( \ell \) is \( \ell_1 \) appended by \( \ell_2 \). This case is represented by the program
\[
\text{Append} = \begin{cases} 
\text{plus}([ ], y, y) \leftarrow \\
\text{plus}([u | x], y, [v | z]) \leftarrow \\
\text{plus}(x, y, z)
\end{cases}
\]
In a sense, we can say that Append is a specialization of Plus_{List} to the concrete list domain. As a consequence, Append does not satisfy (3).

We now want to show that all of the three programs in this example are analogous according to Definition 3. We first show that Plus and Append are analogous. For this, we define the transfer program \( X_1 \) by
\[
X_1 = \begin{cases} 
\text{plus}(0, y, y) \leftarrow \\
\text{plus}([ ], y, y) \\
\text{plus}(s(x), y, s(z)) \leftarrow \\
\text{plus}([u | x], y, [v | z])
\end{cases}
\]
and compute
\[
Plus = X_1 \circ \text{Append} \quad \text{and} \quad \text{Append} = X_1 \circ Plus.
\]
We now show that Plus and Plus_{List} are analogous by defining the transfer program
\[
X_2 = \begin{cases} 
\text{plus}(0, y, y) \leftarrow \\
\text{plus}([ ], [ ], [ ]) \\
\text{plus}(s(x), y, s(z)) \leftarrow \\
\text{plus}([ ], [u | y], [u | z]) \\
\text{plus}(s(x), y, s(z)) \leftarrow \\
\text{plus}([u | x], y, [u | z])
\end{cases}
\]
and by computing
\[
Plus = X_1 \circ Plus_{List} \quad \text{and} \quad Plus_{List} = X_2 \circ Plus.
\]
This shows
\[
Plus \approx Plus_{List} \approx Append.
\]
Finally, we want to mention that there are many other (list) programs analogous to Plus (with different ‘bridges’, of course), which shows the inexact nature of analogy (see the discussion in Section 7).
The following SLD-derivation demonstrates how we can use the transfer program $X_1$ from above to append two lists via the seemingly simpler program for the addition of numerals:

\[
\begin{align*}
&\leftarrow \text{plus}([a], [b, c], [a, b, c]) \\
&\text{X}_1 \leftarrow \text{plus}(s([ ]), [b, c], s([b, c])) \\
&\text{Plus} \leftarrow \text{plus}(0, [b, c], [b, c]) \\
&\text{Plus} \leftarrow \square.
\end{align*}
\]

This shows, via a translation to numerals,

\[\text{Append} \vdash \text{plus}([a], [b, c], [a, b, c]).\]

Interestingly, the SLD-derivation above contains the ‘entangled’ terms $s([ ])$ and $s([a, b])$ which are neither numerals nor lists, and we believe that such ‘dual’ syntactic objects—which under the conventional doctrine of programming are not well-typed—are characteristic for analogical reasoning across different domains and deserve special attention (this is discussed in Section 7).

The programs in Example 4 are syntactically almost identical. We now give an example in which the programs differ more substantially.

**Example 5.** In (Tausend & Bell, 1991) the authors derive a mapping between the program *Append* from Example 4 above and the program

\[
\begin{align*}
\text{Member} = \left\{ 
\begin{array}{l}
\text{member}(u, [u \mid x]) \leftarrow \\
\text{member}(u, [v \mid x]) \leftarrow \\
\text{member}(u, x)
\end{array}
\right\},
\end{align*}
\]

which computes list membership. We can now ask—by analogy to (Tausend & Bell, 1991)—whether *Member* can be simulated by *Append* (and vice versa) according to our definition. Define the transfer programs

\[
\begin{align*}
W = \left\{ 
\begin{array}{l}
\text{member}(u, [u \mid x]) \leftarrow \\
\text{member}(u, [v \mid x]) \leftarrow \text{plus}([v \mid x], u, [v \mid x])
\end{array}
\right\}
\end{align*}
\]

and

\[Z = \{ \text{plus}(x, y, z) \leftarrow \text{member}(y, x) \}.\]

It is not hard to compute

\[\text{Member} = W \circ \text{Append} \circ Z,\]
which shows that membership can indeed be simulated by the program for appending lists, i.e.,

\[ \text{Member} \preceq \text{Append}. \]  

Interestingly enough, the transfer programs \( W, Z \), which in combination permute the first two arguments of \( \text{Append} \) and ‘forget’ about the last one, resemble the mapping computed in (Tausend & Bell, 1991) by other means. The converse of \( (4) \) fails, roughly, since \( \text{Member} \) does not contain enough ‘syntactic structure’ to simulate \( \text{Append} \). The intuitive reason is that \( \text{member} \) is a binary predicate, whereas \( \text{plus} \) is ternary.

The following simple example shows that analogy and (logical) equivalence are ‘orthogonal’ concepts.

**Example 6.** The empty program is equivalent to the propositional program \( a \leftarrow \neg a \) consisting of a single rule. Since we cannot obtain the rule \( a \leftarrow a \) from the empty program via composition, equivalence does not imply analogy. For the other direction, a simple computation shows that the programs \( P = \{ a, b \leftarrow a \} \) and \( R = \{ a, b \leftarrow a, b \} \) are analogous; however, \( P \) and \( R \) are not SLD-equivalent.

## 5 Logic Program Forms

In this section, we study the process of deriving abstract forms of concrete programs from a ‘known’ source domain which can then be instantiated in a ‘novel’ target domain to obtain analogous programs.

We are now ready to introduce the main notion of the paper.

**Definition 4.** A (logic program) form is any well-formed expression containing programs, program variables, and the composition operations introduced above. Formally, a form is defined by the grammar

\[ \varphi ::= \varphi \cup \varphi \mid \varphi \circ \varphi \mid \varphi \cdot \varphi \mid \varphi^\theta \mid \varphi[\theta] \mid q(\varphi) \mid P \mid X \]

where \( \theta \) is a substitution, \( q \in \Sigma \) is a symbol, \( P \) is a program, and \( X \) is a program variable.

We will usually write forms in boldface to distinguish them from ordinary programs.

Logic program forms induce transformations of programs in the obvious way by replacing program variables with concrete programs. A particularly interesting form, derived from the definition of simulation, is given, for any program \( P \), by

\[ \text{Sim}(P)(X, y) = X \circ P \circ y. \]
Of course, we can now reformulate 3 by saying that $P$ is simulated by $R$ if, and only if, there are some transfer programs $X, Y$ such that $P = \text{Sim}(R)(X, Y)$. For instance, we can reformulate the program $\text{Append}$ from Example 4 as

\[ \text{Append} = \text{Sim}(\text{Plus})(X_1, 1). \]

This shows that program forms subsume the concepts of simulation and analogy. Forms are strictly ‘more expressive’ than analogies in the sense that two programs may share some abstract property and still not be analogous according to 3. More precisely, for any form $\varphi$ and programs $P, R$ we can say that $\varphi(P)$ and $\varphi(R)$ share the abstract property expressed by $\varphi$; however, in general, we cannot expect $\varphi(P)$ and $\varphi(R)$ to be analogous. This shows that program forms can capture, in a functorial way, abstract similarities which are ‘invisible’ under the lens of analogy.

**Example 7.** In Example 3 we have constructed the program $\varepsilon$, representing the even numbers, from $\omega$ by inheriting its fact and by iterating its proper rule once. By replacing $\omega$ in (2) with a program variable $X$, we arrive at the form

\[ \text{Even}(X) = X^0 \cup (X^1)^2. \quad (5) \]

We can now instantiate this form with arbitrary programs to transfer the concept of being ‘even’ to other domains. For example, consider the program

\[ Reverse = \begin{cases} \text{reverse}([], []) \leftarrow \\ \text{reverse}([u \mid x], y) \leftarrow \text{reverse}(x, z), \\ \text{plus}(z, [u], y) \end{cases} \cup \text{Append} \]

for reversing lists. By instantiating the form (5) with $Reverse$, we obtain the program $\text{Even}(Reverse)$ given by

\[ \begin{align*} 
\text{reverse}([], []) & \leftarrow \\
\text{reverse}([u_1, u_2 \mid x], [u_3 \mid y]) & \leftarrow \\
\text{reverse}(x, z), \\
\text{plus}(z, [u_2], [u_3 \mid w]), \\
\text{plus}(w, [u_1], y) \\
\text{plus}([], y, y) & \leftarrow \\
\text{plus}([u_1, u_2 \mid x], y, [u_1, u_2 \mid z]) & \leftarrow \\
\text{plus}(x, y, z). 
\end{align*} \]

One can verify that $\text{Even}(Reverse)$ is a program for reversing lists of even length. Similarly, if $Sort$ is a program for sorting lists, then $\text{Even}(Sort)$ is a program for sorting ‘even’ lists and so on.

\[ ^4 \text{Recall from Example 4 that } Append \text{ uses the predicate (i.e., symbol) plus instead of append.} \]
We now come back to Example 4 from the perspective of logic program forms.

**Example 8.** Reconsider the program \( P_{\text{Plus}} \) of Example 4 given by

\[
P_{\text{Plus}} = \begin{cases} 
\text{plus}(0, y, y) \leftarrow \\
\text{plus}(s(x), y, s(z)) \leftarrow \\
\text{plus}(x, y, z)
\end{cases}.
\]

We now want to derive a form \( \text{Plus} \) from \( P_{\text{Plus}} \) which abstractly represents addition. Our first observation is that \( P_{\text{Plus}} \) is, essentially, the concatenation of \( \omega \) in the first and last parameter together with a middle part. Formally, we can deconcatenate \( P_{\text{Plus}} \) by

\[
P_{\text{Plus}} = \text{plus}(\omega(x)) \cdot \left\{ \begin{cases} 
\text{plus}(y, y) \leftarrow \\
\text{plus}(y) \leftarrow \\
\text{plus} \leftarrow \\
\text{plus}(\omega(z)) \cdot \end{cases} \right\}.
\]

By generalizing the two instances of \( \omega \) in the above expression, we immediately obtain the form \( \text{Plus}(X, Y) \) given by

\[
\text{plus}(X) \cdot \left\{ \begin{cases} 
\text{plus}(y, y) \leftarrow \\
\text{plus}(y) \leftarrow \\
\text{plus} \leftarrow \\
\text{plus}(Y) \cdot \end{cases} \right\}.
\]

We can now instantiate this form to obtain analogous programs for addition. For instance, we can derive the program \( \text{Append} \) of Example 4 by instantiating the form with the List-type program \( \lambda \) of Example 2 as follows:

\[
\text{Append} = \text{Plus}(\lambda(u, x), \lambda(u, z)).
\]

As a further example, we want to define the ‘addition’ of (binary) trees by instantiating the form \( \text{Plus} \) with the Tree-type program \( \tau \). Note that we now have multiple choices:

1. As in the case of lists, we can choose to ‘bind’ the variables \( u \) and \( v \).
2. Additionally, since \( \tau(u, x, y) \) contains two bound variables (i.e., \( x, y \)), we have four more possibilities.

Let us first consider the program

\[
\text{Plus}(\tau(u, x, x), \tau(u, z, z))
\]

given by

\[
\begin{align*}
\text{plus}(0, y, y) & \leftarrow \\
\text{plus}(f(u, x, x), y, f(u, z, z)) & \leftarrow \\
\text{plus}(x, y, z).
\end{align*}
\]

This program ‘appends’ the tree in the second argument to each leaf of the \textit{symmetric} tree in the first argument.
Note that all of the above programs are syntactically almost identical, e.g., we can transform Plus into Append via a simple rewriting of terms. The next program shows that we can derive programs from Plus which syntactically differ more substantially from the above programs. Concretely, the program

\[
\text{Plus}(\tau(u, x_1, x_2), \tau(u, z_1, z_2))
\]

given by

\[
\begin{align*}
\text{plus}(0, y, y) & \leftarrow \\
\text{plus}(f(u, x_1, x_2), y, f(u, z_1, z_2)) & \leftarrow \\
& \quad \text{plus}(x_1, y, z_1), \\
& \quad \text{plus}(x_2, y, z_1), \\
& \quad \text{plus}(x_1, y, z_2), \\
& \quad \text{plus}(x_2, y, z_2).
\end{align*}
\]

is SLD-equivalent to program (6). However, in some situations, this more complicated representation is beneficial. For example, we can now remove the second and third body term to obtain the more compact program

\[
\begin{align*}
\text{plus}(0, y, y) & \leftarrow \\
\text{plus}(f(u, x_1, x_2), y, f(u, z_1, z_2)) & \leftarrow \\
& \quad \text{plus}(x_1, y, z_1), \\
& \quad \text{plus}(x_2, y, z_2).
\end{align*}
\]

This program, in analogy to program (6), ‘appends’ the tree in the second argument to each leaf of the not necessarily symmetric tree in the first argument and thus generalizes (6).

The above examples motivate the following Procedure A for the transfer of knowledge from a ‘known’ source domain \(S\) to a ‘novel’ target domain \(T\) via logic program forms:

1. Decompose the source program \(S \in S\) into factors \(S_1, \ldots, S_n \in S, n \geq 1\).
2. Replace the programs \(S_1, \ldots, S_n\) in the decomposition of \(S\) by program variables \(X_1, \ldots, X_n\) to obtain the form \(S(X_1, \ldots, X_n)\).
3. Instantiate the form \(S\) with ‘known’ programs \(P_1, \ldots, P_n \in T\) from the target domain to obtain the target program \(P = S(P_1, \ldots, P_n) \in T\).
4. If necessary, specialize the program \(P\) —given some further information on the specification of the intended program—to the concrete domain at hand.

Of course, each step of Procedure A is, in its current form, vague and we leave its formalization as a starting point for future research.
6 Related Work

Arguably, the most prominent (symbolic) model of analogical reasoning to date is Gentner’s *Structure-Mapping Theory* (or SMT) (Gentner, 1983), first implemented in (Falkenhainer, Forbus, & Gentner, 1989). Our approach shares with Gentner’s SMT its symbolic and syntactic nature. However, while in SMT mappings are constructed with respect to meta-logical considerations—for instance, Gentner’s *systematicity principle* prefers connected knowledge over independent facts—in our framework mappings are realized as (transfer) programs satisfying mathematically well-defined properties.

Some work has been done on analogical reasoning in the setting of logic programming. Most notably are the early works of (Haraguchi, 1986; Haraguchi & Arikawa, 1986, 1987), which implement Winston’s principle (cf. (Winston, 1982)):

Assume that the premises \( t_1, \ldots, t_k \) logically imply \( t_0 \) in the source domain and that the analogous premises \( t_1', \ldots, t_k' \) hold in the target domain. Then we can conclude \( t_0' \) in the target domain which is analogous to \( t_0 \).

In practice, the main difficulty for implementing this principle is the detection of two syntactic objects being analogous. In our framework, the analogical mapping between ‘similar’ logical atoms is implicitly established in the computation of transfer programs, thus respecting the syntactic structure of the encapsulating programs.

Finally, we briefly want to mention the works of (Davies & Russell, 1987), (Costantini, Lanzarone, & Sbarbaro, 1995), and (Tausend & Bell, 1991), dealing with analogical reasoning in logic programming, and the works of (Qiu, 1994; Iwayama, Satoh, & Arima, 1992) which study the nonmonotonic aspects of analogies.

7 Discussion and Future Work

In this paper, we proposed an algebraic model of analogical reasoning in logic programming. More precisely, we introduced algebraic operations for program modularity and defined the notions of analogy and logic program forms in terms of syntactic decompositions of programs.

In general, ‘creative’ analogies contain some form of *unexpectedness*, in the sense that objects (possibly from different domains) are put together in a *novel* and often *unforeseen* way. For instance, in Example 4 we have—*directly*—used the (seemingly) simpler program for addition of numbers in the list domain for appending lists, where ‘entangled’ terms of the form \( s([ ]) \) and \( s([a,b]) \), which are neither numerals nor lists, occurred. We believe that ‘dual’ syntactic objects of this form—which under the conventional doctrine of programming are not well-typed (see, for example, the remarks in (Apt, 1997, §5.8.2))—are characteristic for ‘creative’ analogies across different domains and deserve special attention.
More precisely, if we think of (logic) programs as ‘tools’ for solving problems, then using those tools ‘creatively’ often requires, as in the example mentioned above, a tight coupling between seemingly incompatible objects. However, if those objects are statically typed by the programmer, then this might prevent the creation and exploitation of useful analogical ‘bridges’ for knowledge transfer.

In a broader sense, this paper is a first step towards an algebraic (and mainly syntactic) theory of analogical reasoning in logic-based knowledge representation and reasoning systems, with potential applications to fundamental AI-problems like computational learning and creativity.

7.1 Outlook

A major task is to formalize each step of Procedure A (cf. Section 5):

1. What are the considerations when systematically computing decompositions of $S$ and under which conditions do (unique) ‘prime’ decompositions exist?
2. Which factors in the decomposition of $S$ should be replaced by program variables? (For instance, in Example 8 when computing the form $\text{Plus}(x, y)$ we have not replaced the middle factor dealing with the variable $y$ by a program variable).
3. How can we systematically choose the right programs $P_1, \ldots, P_n$ in the target domain to obtain the intended target program $P$? (For instance, in Example 8 it does not make much sense to substitute the program $\text{Plus}$ into the form $\text{Plus}$).
4. How can we systematically incorporate additional information for the specification of the target program (e.g., in the form of positive and negative examples) into our framework?

For practical concerns, efficient implementations of Procedure A are of interest. From a theoretical point of view, a mathematical study of (the relationship between) modularity, analogy, and logic program forms is desirable.

Analogical proportions are statements of the form ‘$a$ is to $b$ as $c$ is to $d$’ and are another form of analogical reasoning. It is interesting to incorporate analogical proportions into our framework by using the algebraic operations introduced in this paper.

Another interesting line of research is to recast the ideas in this paper to nonmonotonic logic programming under the answer set semantics (cf. (Brewka, Eiter, & Truszczynski, 2011)). This task is non-trivial due to negation as failure occurring in rule bodies (and heads), and we want to mention that some work has already been done in this direction (cf. (Qiu, 1994; Iwayama et al., 1992; Prade & Richard, 2011)).

Finally, motivated by the discussion above, we believe that it is interesting to investigate the role of (polymorphic) (un)typing in computational models of analogy-making for computational learning and creativity.
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