Boundary term, extended Witten identities and positivity of energy

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In terms of two-spinors a chiral formulation of general relativity with the Ashtekar Lagrangian and its Hamiltonian formalism in which the basic dynamic variables are the dyad spinors are presented. The extended Witten identities are derived. A new expression of the Hamiltonian boundary term is obtained. Using this expression and the extended Witten identities the proof of the positive energy theorem is extended to a case including momentum and angular momentum.

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I. INTRODUCTION

It is generally agreed that gravitational energy exists, but because of the equivalence principle it cannot be localized [1]. A suitable expression which could provide a physical reasonable description of the energy-momentum density for gravitating system has lone been sought. All candidates had several shortcoming. In particular they violated a fundamental theoretical requirement—that gravitational energy should be positive—as well as requirements concerning localization and reference frame independence.

During the last 30 years one of the greatest achievements in classical general relativity is certainly the proof of the positivity of the total gravitational energy, both at spatial and null infinity [2-4]. It is precisely positivity that makes this notion not only important (because of its theoretical significance), but a useful tool as well in the everyday practice of working relativists. The successes in the proof of the positivity of energy inspired several relativists to search for expressions of the gravitational energy-momentum at the quasilocal level, i.e. to associate these physical quantities with closed spacelike 2-surfaces \( S \) (see, for instance, Ref. [5] and the references therein).

Quasilocal energy is important for its several potential applications. It can be used to define binding energy of stars. In numerical relativity it can tell us how to cut off the data on a noncompact region to a compact region where the quasilocal mass of the compact region approximates the ADM mass of the noncompact region. When we evolve the data according to Einstein equations, we need local control of energy to see how the space changes. The positivity of quasilocal energy is essential for such an investigation. This could be regarded as a generalization of the energy method in the theory of a nonlinear hyperbolic system.

There are several contenders for a good definition of quasilocal energy. The two that interest us are Brown and York’s ‘canonical quasilocal energy’[6], and the various definitions based on the integral over \( S \) of the Witten-Nester two-form (the two-form used in Witten’s proof of the positive energy theorem [3]). The approach of Brown and York starts with standard Hamilton-Jacobi theory and thus is somewhat different from more traditional approaches based on Noether techniques. Brown and York have determined what geometric entity play the role of quasilocal energy in general relativity. The canonical quasilocal energy is supposed to be the energy of the gravitational and matter fields contained in a spatial volume \( \Sigma \), whose boundary two-surface is \( S = \partial \Sigma \). This approach gives the correct quasilocal energy surface density and allows one the freedom to assign the reference point of the quasilocal energy.

The approach based on the Witten-Nester two-form uses spinorial methods, and the different definitions are distinguished by the choice of supplementary equation the \( S \)-spinors are supposed to satisfy, for example the Sen-Witten equation [3], the Dougan-Mason equation [7], or the Ludvigsen-Vickers equation [8]. Since Witten’s positive energy proof can be understood in terms of the Hamiltonian, the Hamiltonian density associated with this proof provides a locally positive localization—and thus has real promise as a truly physical energy-momentum density for gravitational fields. To fulfill this promise certain features need further consideration: an outstanding one concerns the role of—and even the need for—the spinor field, which seemed rather mysterious. Although there are very beautiful argument for some nice important results, it is not clear as to how much the Witten argument really captures the correct physics. Moreover, all the spinor formulations cannot give angular momentum and the center-of-mass moment [9].

The two approaches are conceptually distinct, even though in some circumstances one might expect their numerical value to coincide. Therefore, it is of interest to establish a connection between the Brown-York quasilocal energy and the Witten-Nester form. Indeed, such a connection supports the use of the Witten-Nerster form in the quasilocal context, and, furthermore, provides one with a non-spinor vantage point from which the various supplementary spinor equations can be examined. The general relationship between the Brown-York quasilocal energy and the common "spinorial definition of quasilocal energy" constructed from the Witten-Nester two-form has been established by Lau
Then the scalar curvature $R$ splits into two parts.

In this paper we start with the self-dual Lagrangian following Lau’s route and then use Witten-Ashtekar-Horowitz’s method [11] to prove the positive energy theorem. To this end we have to derive a set of extended Witten identities and then apply them to the positivity proof. In the discussion on positivity proof and energy-momentum quasilocalization the boundary term has important roles. The value of the Hamiltonian "on shell" is determined by the Hamiltonian boundary term. The "natural" Hamiltonian boundary term inherited from the Lagrangian can—and should—be adjusted. The general expression for boundary term depends on the choice of variables, a displacement vector field (e.g. translation for energy momentum and rotation for angular momentum), a reference configuration and boundary conditions. Being different from most of the works on positivity of energy, in this paper by choosing a boundary term with a nonvanishing shift we will extend the proof to involve momentum and even angular momentum.

The organization of this paper is as follows. In Sec. II, a review of some results of Lau is given in a rather different appearance. The Lagrangian of general relativity is written in terms of two-spinors and then is decomposed into two independent parts, the self-dual and the anti-self-dual part. The self-dual part is identified with a version of the Ashtekar Lagrangian [12] given by Goldberg [13] and then can be taken as the gravitational Lagrangian. Sec. III is the Hamiltonian formulation in which the dyad spinors are chosen as the basic dynamic variables while the Ashtekar connections appear in the conjugate momenta. In Sec. IV the Hamiltonian boundary term is rewritten as a suitable form for the application of the extended Witten identities. The extended Witten identities are derived in Sec. V and are used to calculate the boundary term and prove the positivity of the energy in Sec. VI. Finally, Sec. VII is devoted to some conclusions and remarks.

II. THE SELF-DUAL LAGRANGIAN OF GR

We start by rewriting some results of Lau [10] in a rather different appearance in order to adapt to our purpose. Using the spin connection one-form $ω_{a}^{KL}$ the curvature two-form $R_{abKL}$ of the spacetime $M$ can be written as [14]

$$R_{abKL} = \nabla_{a}^{\Gamma} ω_{bKL} - \nabla_{b}^{\Gamma} ω_{aKL} - ω_{a}^{M} ω_{bKL} + ω_{b}^{M} ω_{aKL},$$

where $\nabla_{a}^{\Gamma}$ is the covariant derivative defined by the affine connection $Γ_{a}^{bc}$, for example, the covariant derivative of the cotetrad one-form $e^{J}_{b}$ is

$$\nabla_{a}^{\Gamma} e^{J}_{b} = e^{J}_{b} e_{a}^{\Gamma} = \partial_{a} e_{b}^{J} - Γ_{ab}^{c} e^{J}_{c}.$$

Then the scalar curvature $R$ can be written as

$$R = 2\nabla_{a}^{\Gamma} (e^{K}_{a} e^{Lb} ω_{bKL}) + e^{K}_{a} e^{Lb} ω_{a}^{L} ω_{bKL} - e^{K}_{a} e^{Lb} ω_{a}^{L} ω_{bKL},$$

and the gravitational Lagrangian $L_{G}$ consists of the Moller Lagrangian [15] and a divergence term:

$$L_{G} = \sqrt{-g} R$$

$$= \sqrt{-g} (e^{K}_{a} e^{Lb} ω_{a}^{L} ω_{bKL} - e^{K}_{a} e^{Lb} ω_{a}^{L} ω_{bKL}) + 2\partial_{a} (\sqrt{-g} e^{K}_{a} e^{Lb} ω_{bKL}).$$

Passing on to two-spinor expression [16], the spin connection reads

$$ω_{A′B′CC′} = ω_{A′B′C′} \epsilon_{B′C′} + ω_{A′B′C′} \epsilon_{B′C′}$$

$$= ω_{A′B′CC′} + ω_{A′B′CC′},$$

where

$$ω_{A′B′CC′} = ω_{A′B′C′} \epsilon_{B′C′},$$

and

$$ω_{A′B′CC′} = ω_{A′B′C′} \epsilon_{B′C′},$$

is the self-dual and the anti-self-dual part of the connection $ω_{A′B′CC′}$, respectively, and then the scalar curvature $R$ splits into two parts.
\[ R = R^+ + R^-, \]  
where the self-dual and the anti-self-dual part of \( R \) is, respectively
\[ R^+ = 2\nabla_a \left( e^{KK'a} e^{LL'b} \omega_{bKL} \epsilon_{L'K'} \right) + \omega^K_{\, I} K \omega^{LK'}_{\, L} - \omega^{KK'I}_{\, I} L \omega^{L'K'}_{\, K'} \] \tag{7}  
\[ R^- = 2\nabla_a \left( e^{KK'a} e^{LL'b} \omega_{bKL} \epsilon_{L'K'} \right) + \omega^K_{\, I} K \omega^{LK'}_{\, L} - \omega^{KK'I}_{\, I} L \omega^{L'K'}_{\, K'}. \] \tag{8}  
Consequently, the gravitational Lagrangian \( L_G = \sqrt{-g} R \) splits into two parts
\[ L_G = L^+_G + L^-_G. \]  
The self-dual part \( L^+_G \) consists of the self-dual Moller Lagrangian [13,15] and a divergence term:
\[ L^+_G = \sigma \left( \omega^{KK'I}_{\, I} K \omega^{LK'}_{\, L} - \omega^{KK'I}_{\, I} L \omega^{L'K'}_{\, K'} \right) + \partial_a \left( 2\sigma e^{KK'a} e^{LL'b} \omega_{bKL} \epsilon_{L'K'} \right) \] \tag{9}  
where
\[ \sigma = \sqrt{-g} \] \tag{10}  
is the determinant of the SL(2,C) soldering form \( \sigma_{\mu AA'} \) on the spacetime manifold \( M \). Since \( L^+_G \) and \( L^-_G \) depend on \( \omega_{AA'B} C \) and \( \omega_{AA'B} C' \) respectively, and are independent of each other, we can choose the self-dual part \( L^+_G \) as the Lagrangian of the gravitational field, which is identified with a version of the Ashtekar Lagrangian [12] given by Goldberg [13].

### III. THE HAMILTONIAN FORMALISM OF THE SELF-DUAL GRAVITY

To put the theory in Hamiltonian form, we will assume that the spacetime \( M \) is topologically \( \Sigma \times \mathbb{R} \) for some spacelike submanifold \( \Sigma \) and assume that there exists a time function \( t \) with nowhere vanishing gradient \( (dt)_a \) such that each \( t = \text{const.} \) surface \( \Sigma_t \) is diffeomorphic to \( \Sigma \). \( t^a \) will denote the time flow vector field \( (t^a (dt)_a = 1) \), while \( n^a \) will denote the future-pointing timelike vector field \( (n^a n_a = -1) \) normal to the \( t = \text{const.} \) surfaces. Then the intrinsic metric \( g_{ab} \) of \( \Sigma \), the lapse \( N \), and the shift \( N^a \) are induced by
\[ g_{ab} = q_{ab} - n_a n_b, \]  
and
\[ t^a = N n^a + N^a, \]  
respectively. In the spinor notation [16], the normal vector \( n^{AA'} \) defines an isomorphism from the space of primed spinors to the space of unprimed spinors:
\[ \alpha^{+A} = i\sqrt{2} n^{AA'} \alpha_{A'}, \]  
\[ \omega_{ABCD} = i\sqrt{2} n^{A'} A' \omega_{AA'CD}, \]  
\[ n^{AB} = i \sqrt{2} n^{BA'} n^{A'} = i \sqrt{2} \epsilon^{AB}. \] \tag{11}  
In this notation
\[ g_{ab} = q_{ab} - n_a n_b, \]  
reads
\[ g^{ABCD} = q^{ABCD} - n^{AB} n^{CD}, \]  
or
\[ \epsilon^{AC} \epsilon^{BD} = -\epsilon^{A(C} \epsilon^{D)B} + \frac{1}{2} \epsilon^{AB} \epsilon^{CD}. \] 

In terms of the dyad spinors
\[ \zeta_0^C = \sigma^C, \zeta_1^C = \iota^C, \]
the self-dual connection can be expressed as
\[ \omega_{ABC}^D = \zeta^b \nabla_{AB} \zeta^b D, \]

Using these results we can decompose the quantities appearing in \( \mathcal{L}_G^+ \) into the tangential and the normal parts to the surface \( \Sigma_t \), for example, for a four-vector \( V^{AB} \) we have
\[ V^{AB} = q^{AB}_{CD} V^{CD} - n^{AB} n_{CD} V^{CD} = V^{(AB)} + V^{[AB]}, \]

where \( V^{(AB)} \) and \( V^{[AB]} \) is the tangential and the normal part of \( V^{AB} \) to the surface \( \Sigma_t \), separately. Then the Lagrangian \( \mathcal{L}_G^+ \) can be rewritten as
\[ \mathcal{L}_G^+ = 4\sqrt{2} i \sigma \nabla^{(AB)} \zeta^a B \zeta_a A - N \sigma [\nabla_{(AB)} \zeta^a a \nabla^{(CD)} \zeta^a C - \nabla_{(AB)} \zeta^a C \nabla^{(BC)} \zeta^a A] + \sqrt{2} i \sigma N^{CD} \zeta^a C \nabla^{(AB)} \zeta^a B \zeta_a B \]
\[ - \partial_{(AB)} \left( 2 N \sigma \zeta^{AB} \nabla^{(CB)} \zeta_a C - \sqrt{2} i \sigma N^{CD} \zeta^{AB} \nabla^{(CD)} \zeta_a B \right), \]

where
\[ \dot{\zeta}_{aA} = \iota^{CD} \nabla^{a} C \zeta_{aA}, \]

and
\[ \sigma = \frac{4 \sigma}{N} = \frac{\sqrt{-g}}{N}. \]

The dyad spinors \( \zeta_{aA} \) are chosen as the basic dynamic variables and then the canonical momenta conjugate to \( \zeta_{aA} \) are
\[ \hat{p}^a A = \frac{\partial \mathcal{L}_G^+}{\partial \dot{\zeta}_{aA}} = 4\sqrt{2} i \sigma \nabla^{(AB)} \zeta^a B. \]

The gravitational Hamiltonian density can be computed
\[ \mathcal{H}_G = \hat{p}^a A \dot{\zeta}_{aA} - \mathcal{L}_G^+ \]
\[ = N \sigma [\nabla_{(AB)} \zeta^a a \nabla^{(CD)} \zeta^a C - \nabla_{(AB)} \zeta^a C \nabla^{(BC)} \zeta^a A] + \sqrt{2} i \sigma N^{CD} \zeta^a C \nabla^{(AB)} \zeta^a B \zeta_a B \]
\[ + \partial_{(AB)} \left( 2 N \sigma \zeta^{AB} \nabla^{(CB)} \zeta_a C - \sqrt{2} i \sigma N^{CD} \zeta^{AB} \nabla^{(CD)} \zeta_a B \right). \]

Using
\[ \nabla^{(AB)} \zeta^a B = \frac{i}{4\sqrt{2} \sigma} \hat{p}^a A, \]
we obtain
\[ \mathcal{H}_G = N \left[ \frac{1}{32\sigma} \vec{p}_{aA} \tilde{\sigma}^a - \sigma \nabla^\Gamma (AB) \zeta_c \tilde{\sigma}^c \nabla^\Gamma (BC) \zeta_a \right] \]
\[ - \frac{1}{4} N^{CD} \left[ 3 \left( \nabla^\Gamma (CD) \zeta_a A \right) \tilde{p}^a A + \sigma \zeta_a A \tilde{p}^a A \nabla^\Gamma (CD) \right] \]
\[ + \partial (AB) \left( \frac{\sqrt{2} i}{4} N \zeta_a A \tilde{p}^a A - \sqrt{2} i \sigma N^{CD} \zeta_a A \nabla^\Gamma (CD) \zeta_a B \right) \]

(20)

i. e.

\[ \mathcal{H}_G = NH + N^{AB} \mathcal{H}_{AB} + \partial (AB) \tilde{B}^{AB}, \]

(21)

where the Hamiltonian constraint

\[ \mathcal{H} = \frac{1}{32\sigma} \vec{p}_{aA} \tilde{\sigma}^a - \sigma \nabla^\Gamma (AB) \zeta_c \tilde{\sigma}^c \nabla^\Gamma (BC) \zeta_a, \]

(22)

the momentum constraint

\[ \mathcal{H}_{AB} = - \frac{3}{4} \tilde{p}^a C \nabla^\Gamma (AB) \zeta_a C - \frac{1}{4} \sigma \zeta_a C \nabla^\Gamma (AB) \left( \frac{1}{\sigma} \tilde{p}^a C \right), \]

(23)

and the boundary term

\[ \tilde{B}^{AB} = 2 N \sigma \zeta^{AB} \gamma^\Gamma (CB) \zeta_c C - \sqrt{2} i \sigma N^{CD} \zeta^A a \nabla^\Gamma (CD) \zeta_a B \]
\[ = \frac{\sqrt{2} i}{4} N \zeta_a A \tilde{p}^a B - \sqrt{2} i \sigma N^{CD} \zeta^A a \nabla^\Gamma (CD) \zeta_a B, \]

(24)

separately. Being different from Ashtekar’s formulation, here the dyad spinors \( \zeta_a A \) are chosen as the basic dynamic variables and the conjugate momenta \( \tilde{p}^a A \) are related to Ashtekar’s variables \( \omega (AB) \) \[12,13\] by

\[ \omega (AC) B = \frac{i}{4\sqrt{2} \sigma} \zeta_B a \tilde{p}_{aA}. \]

IV. BOUNDARY TERMS

Being different from most of the works on positivity of energy in which the shift \( N^{AB} \) is chosen to vanish, we consider a more general boundary term \( \tilde{B}^{AB} \) which allows a particular choice of \( N^{AB} \). Choosing the triad

\[ \sigma_1^{AB} = \frac{i}{\sqrt{2}} \left( o^A o^B - \ell^A \ell^B \right) = \frac{1}{\sqrt{2}} \left( m^a + m^a \right), \]
\[ \sigma_2^{AB} = \frac{1}{\sqrt{2}} \left( o^A o^B + \ell^A \ell^B \right) = - \frac{i}{\sqrt{2}} \left( m^a - m^a \right), \]
\[ \sigma_3^{AB} = \frac{i}{\sqrt{2}} \left( -o^A o^B - \ell^A \ell^B \right) = \frac{1}{\sqrt{2}} \left( p^a - n^a \right), \]

(25)

on \( \Sigma \) then the normal unit vector to the boundary \( \partial \Sigma \) is

\[ v^{AB} = \sigma_3^{AB} = - \frac{i}{\sqrt{2}} \left( o^A \ell^B + \ell^A o^B \right). \]

We chose

\[ N = - \chi^2, \]
\[ N^{AB} = \chi^2 \left( m^{AB} + m^{AB} + v^{AB} \right) \]
\[ = \frac{i}{\sqrt{2}} \chi^2 \left( o^A o^B + \ell^A \ell^B - o^A \ell^B \right), \]

(27)
and compute the boundary integral

\[ \oint \tilde{B}^{AB} dS_{AB} = \oint \tilde{B}^{AB} v_{AB} dS \]

\[ \quad = -\sqrt{2} \int \sigma \chi^2 \left( m^b m_a \nabla b m^a - m^b m_a \nabla b n^a \right) dS \]

\[ \quad - \frac{\sqrt{2}}{2} \int \sigma \chi^2 \left( m^b - m^b - \frac{3}{2} m^b + \frac{1}{2} n^b \right) \left( n_a \nabla b m^a - m_a \nabla b m^a \right) dS \]

\[ = - \int \left[ \sigma \chi^2 k \right. \left. - \frac{\sqrt{2}}{2} \sigma \chi^2 \left( m^b - m^b - \frac{3}{2} m^b + \frac{1}{2} n^b \right) \left( n_a \nabla b m^a - m_a \nabla b m^a \right) \right] dS \]

(29)

where \( k \) is the trace of the extrinsic curvature of \( \partial \Sigma \). The first term of the integral is just the "unreferenced" Brown-York quasilocal energy when \( \chi^2 = 1 \), and the second term comes from the particular choice (27) of \( N^{AB} \) and then includes the contribution of momentum and angular momentum on which our interest is concentrated.

For further calculation and discussion we rewritten the expression of \( \tilde{B}^{AB} \) as follows.

Since

\[ \partial_{(AB)} \left( \sqrt{2} i \sigma N^{CD} \omega_{CD}^{AB} \right) \]

\[ = \sqrt{2} i \sigma D_{(AB)}^{(2)} (N^{CD} \omega_{CD}^{AB}) \]

\[ = \sqrt{2} i \sigma \epsilon^{AE} \epsilon^{BF} \epsilon^{CG} \epsilon^{DH} D_{(EF)}^{(3)} (N_{GH} \omega_{CDAB}) \]

\[ = \sqrt{2} i \sigma (\epsilon^{DB} \epsilon^{FH} + \epsilon^{FD} \epsilon^{BH}) \left( \epsilon^{CA} \epsilon^{EG} + \epsilon^{EC} \epsilon^{AG} \right) D_{(EF)}^{(3)} (N_{GH} \omega_{CDAB}) \]

\[ = \sqrt{2} i \sigma D_{(AB)}^{(3)} (N^{AB} \omega_{CD}^{CD}) - \sqrt{2} i \sigma D_{(AB)}^{(3)} (N^{AD} \omega_{BC}^{BD}) \]

\[ - \sqrt{2} i \sigma D_{(AB)}^{(3)} (N^{CA} \omega_{BD}^{CD}) + \sqrt{2} i \sigma D_{(AB)}^{(3)} (N^{CD} \omega_{AB}^{AB}) \]

\[ = \sqrt{2} i \sigma \partial_{(AB)} \left( \sigma N^{AB} \zeta^{CB} \nabla^{(CD)} \zeta_{bD} + \sigma N^{CD} \zeta^{CB} \nabla^{(AB)} \zeta_{bD} \right) \]

\( \tilde{B}^{AB} \) can be written as

\[ \tilde{B}^{AB} = 2N \sigma \zeta^{AB} \nabla^{(CB)} \zeta_{aC} - \sqrt{2} i \sigma N^{AB} \zeta^{CB} \nabla^{(CD)} \zeta_{bD} \]

\[ - \sqrt{2} i \sigma N^{CD} \zeta^{CB} \nabla^{(AB)} \zeta_{aD}. \]

(30)

Introducing spinors \( \lambda^A \) and \( \lambda^+A \) by

\[ o^A = \frac{1}{\chi} \lambda^A, \quad i^A = \frac{1}{\chi} \lambda^+A \]

(31)

and noting

\[ \nabla_{AA} \lambda^C = \partial_{AA} \lambda^C, \]

(32)

we can compute

\[ \tilde{B}^{AB} = 2N \sigma \frac{1}{\chi^2} \left( \lambda^A \delta^{(CB)} \lambda^\dagger_C - \lambda^+A \delta^{(CB)} \lambda^A_C \right) + 2N \sigma \frac{1}{\chi^2} \left( \lambda^+A \lambda_C - \lambda^A \lambda^+_C \right) \delta^{(CB)} \chi \]

\[ - \sqrt{2} i \sigma N^{AB} \frac{1}{\chi^2} \left( \lambda^C \partial_{(CD)} \lambda^+D - \lambda^+C \partial_{(CD)} \lambda^D \right) \]

\[ - \sqrt{2} i \sigma N^{CD} \frac{1}{\chi^2} \left( \lambda^C \partial^{(AB)} \lambda^+_D - \lambda^+_C \partial^{(AB)} \lambda_D \right). \]

Noting

\[ N = -\chi^2 = -\lambda_A \lambda^+_A, \]

(33)

and
\[ N^{AB} = -m^{AB} - \overline{m}^{AB} - v^{AB} \]
\[ = \frac{i}{\sqrt{2}} \left( \lambda^A \lambda^B + \lambda^{+A} \lambda^{+B} + \lambda^{(A} \lambda^{+B)} \right). \]  

(34)

we obtain

\[ \tilde{B}^{AB} = -\frac{\sigma}{\lambda^2} \left( \lambda^A \lambda^B + \lambda^{+A} \lambda^{+B} + \lambda^{(A} \lambda^{+B)} \right) \left( \lambda^C \partial_{(CD)} \lambda^{+D} - \lambda^{+C} \partial_{(CD)} \lambda^D \right) \]
\[ -2\sigma \left( \lambda^A \partial^{(CB)} \lambda^C - \lambda^{+A} \partial^{(CB)} \lambda_C \right) - \sigma \left( \lambda^{+A} \lambda_C - \lambda^A \lambda^+_C \right) \partial^{(CB)} \ln \lambda^2 \]
\[ + \sigma \left( \lambda^{+D} \partial^{(AB)} \lambda^D + \lambda^D \partial^{(AB)} \lambda^+_D + \frac{1}{2} \lambda^D \partial^{(AB)} \lambda^+_D + \frac{1}{2} \lambda^{+D} \partial^{(AB)} \lambda^+_D \right). \]  

(35)

V. THE EXTENDED WITTEN IDENTITY

In order to deal with the boundary term \( \tilde{B}^{AB} \) we follow Witten [3], Ashtekar and Horowitz [11] and derive some useful identities. In a two-spinor formulation a Dirac spinor \( \varepsilon \) and Dirac matrices \( \gamma^I \) can be expressed

\[ \varepsilon = \left( \begin{array}{c} \chi^A \\ \sigma \end{array} \right), \gamma^I = \sqrt{2} \left( \begin{array}{cc} 0 & \sigma I_{A'} \\ \sigma I_{A'} & 0 \end{array} \right), \]

respectively, and then the Witten identity [3]

\[ -\nabla^2 \varepsilon = (i \nabla)^2 \varepsilon = -\sum_i \nabla_i \nabla_i \varepsilon + 4\pi G (T_{00} + \sum_j T_{0j} \gamma^0 \gamma^j) \varepsilon. \]

(36)

becomes two two-spinor identities

\[ -2\nabla^{AC} \nabla_{CB} \lambda^B = -\nabla^{BC} \nabla_{BC} \lambda^A + 4\pi G (T_{00} \lambda^A + \sqrt{2} T_{0B} \lambda^B), \]
\[ -2\nabla_{A'C'} \nabla^{C'B'} \sigma_{B'} = -\nabla^{B'C'} \nabla_{B'C'} \sigma_{A'} + 4\pi G (T_{00} \sigma_{A'} + \sqrt{2} T_{0B} \sigma_{B'}), \]

(37)

where \( \nabla_{AB} \) and \( \nabla_{A'B'} \) denote the covariant derivatives defined by the spin connections \( \omega_{ABC}^{D} \) and \( \sigma_{A'B'C'D'} \), respectively. Multiplying the first identity with \( \lambda^{+A} \) leads to

\[ 2\lambda^{+A} \nabla_{(AC)} \nabla^{(CB)} \lambda_B = \lambda^{+A} \nabla_{(BC)} \nabla^{(BC)} \lambda_A - 4\pi G (T_{00} \lambda^{+A} \lambda_A + \sqrt{2} T_{0AB} \lambda^{+A} \lambda^B). \]

Since

\[ \lambda^{+A} \nabla_{(BC)} \nabla^{(BC)} \lambda_A = \nabla_{(BC)} \left( \lambda^{+A} \nabla^{(BC)} \lambda_A \right) - \left( \nabla_{(BC)} \lambda^{+A} \right) \nabla^{(BC)} \lambda_A \]
\[ = D_{BC} \left( \lambda^{+A} \nabla^{(BC)} \lambda_A \right) - \left( \nabla_{(BC)} \lambda^{+A} \right) \nabla^{(BC)} \lambda_A, \]

then we have

\[ \sigma D_{BC} \left( \lambda^{+A} \nabla^{(BC)} \lambda_A \right) = 2\sigma \lambda^{+A} \nabla_{(AC)} \nabla^{(CB)} \lambda_B + \sigma \left( \nabla_{(BC)} \lambda^A \right) ^{+} \nabla^{(BC)} \lambda_A \]
\[ + 4\pi G \sigma (T_{00} \lambda^{+A} \lambda_A + \sqrt{2} T_{0AB} \lambda^{+A} \lambda^B). \]

Integrating it leads to

\[ \int_{S_{(AB)}} \sigma \lambda^{+A} \nabla^{(BC)} \lambda_A dS_{(AB)} = \int_{\Sigma} \sigma D_{BC} \left( \lambda^{+A} \nabla^{(BC)} \lambda_A \right) dv \]
\[ = 2 \int_{\Sigma} \sigma \lambda^{+A} \nabla_{(AC)} \nabla^{(CB)} \lambda_B dv + \int_{\Sigma} \sigma \left( \nabla_{(BC)} \lambda^A \right) ^{+} \nabla^{(BC)} \lambda_A dv \]
\[ + 4\pi G \int_{\Sigma} \sigma (T_{00} \lambda^{+A} \lambda_A + \sqrt{2} T_{0AB} \lambda^{+A} \lambda^B) dv. \]

(38)
Multiplying the first identity of (37) with $\lambda^A$ leads to
\[
D_{BC} \left( \lambda^A \nabla^{(BC)} \lambda_A \right) = 2 \lambda^A \nabla_{(AC)} \nabla^{(CB)} \lambda_B + 4\sqrt{2} \pi G T_{0AB} \lambda^A \lambda^B.
\]
Integrating yields
\[
\oint \sigma \lambda^A \nabla^{(BC)} \lambda_A dS_{(AB)} = 2 \int_{\Sigma} \sigma D_{BC} \left( \lambda^A \nabla^{(BC)} \lambda_A \right) dV
= 2 \int_{\Sigma} \sigma \lambda^A \nabla_{(AC)} \nabla^{(CB)} \lambda_B dV + 4\sqrt{2} \pi G \int_{\Sigma} \sigma T_{0AB} \lambda^A \lambda^B dV. \tag{39}
\]
By the similar way, from the second identity of (37) we obtain
\[
\oint \sigma \lambda^A \nabla_a \lambda^+_A dS^a = 2 \int_{\Sigma} \sigma \lambda^A \nabla_{(A'C')} \nabla^{(C'B')} \lambda_{B'} dV + 4\sqrt{2} \pi G \int_{\Sigma} \sigma \lambda^A \nabla_{(BC)} \nabla^{(BC)} \lambda_A dV
- i\sqrt{2} \oint \sigma \lambda^B K_{CD} \lambda^+_A dS^{(BC)} \tag{40}
\]
and
\[
\oint \sigma \lambda^A \nabla_a \lambda^+_A dS^a
= 2 \int_{\Sigma} \sigma \lambda^A \nabla_{(A'C')} \nabla^{(C'B')} \lambda_{B'} dV + \int_{\Sigma} \nabla_{(BC)} \lambda^A \left( \nabla^{(BC)} \lambda_A \right)^+ dV
+ 4\pi G \int_{\Sigma} \sigma (T_{00} \lambda^A \lambda_{A'} + \sqrt{2} \lambda^A \nabla_{(A'C')} \lambda_{B'} dV + i\sqrt{2} \oint \sigma \lambda^B K_{CD} \lambda^+_A dS^{(CD)}
- i\sqrt{2} \int_{\Sigma} \sigma K_{CDA} \lambda^B \left( \lambda^A \nabla^{(CD)} \lambda_A^+ - \lambda_{B'} \nabla^{(CD)} \lambda_A \right) dV + 2 \int_{\Sigma} \sigma K_{CDA} \lambda^B \lambda^+_E dV. \tag{41}
\]
(38), (39), (40), and (41) can be called the extended Witten identities.

VI. THE CALCULATION OF THE BOUNDARY INTEGRAL

Now we are ready to calculate the boundary integral. Since
\[
\zeta_{0A} = a_A = \frac{1}{\chi} \lambda_A, \zeta_{1A} = \iota_A = \frac{1}{\chi} \lambda^+_A,
\]
then
\[
\nabla_{AA'} \lambda^C = \partial_{AA'} \lambda^C,
\]
and
\[
\omega_{AA'B}^C = \zeta_{B}^b \nabla_{AA'} \zeta_6^C
= \frac{1}{\chi^2} \left( \lambda_B \partial_{AA'} \lambda^C - \lambda^+_B \partial_{AA'} \lambda^C \right) + \frac{1}{\chi^3} \partial_{AA'} \chi \left( \lambda_B \lambda^C - \lambda_B \lambda^+ C \right).
\]
and
\[
\nabla_{AA'} \lambda_B : = \partial_{AA'} \lambda_B - \omega_{AA'B}^C \lambda_C
= \partial_{AA'} \lambda_B - \frac{1}{\chi^2} \left( \lambda_B \partial_{AA'} \lambda^+ C - \lambda^+_B \partial_{AA'} \lambda^C \right) \lambda_C + \frac{1}{\chi} \partial_{AA'} \chi \lambda_B.
\]
Using this result we obtain
\[
\lambda^+ B \nabla_{AA'} \lambda_B = 2 \lambda^+ B \partial_{AA'} \lambda_B - \chi \partial_{AA'} \chi,
\]
and
\[ \lambda^B \nabla_{AA'} \lambda_B = \lambda^B \partial_{AA'} \lambda_B - \partial_{AA'} \lambda^C \lambda_C = 2 \lambda^B \partial_{AA'} \lambda_B. \]

By the same way we find that
\[ \nabla_{AA'} \lambda_B^+ = \partial_{AA'} \lambda_B^+ - \omega_{AA'} B^C \lambda_C^+ = \partial_{AA'} \lambda_B^+ - \frac{1}{\chi} \left( \lambda_B \partial_{AA'} \lambda^+ - \lambda^+_B \partial_{AA'} \lambda^C \right) \lambda_C^+ + \frac{1}{\chi} \partial_{AA'} \chi \lambda_B^+, \]
which leads to
\[ \lambda^B \nabla_{AA'} \lambda_B^+ = 2 \lambda^B \partial_{AA'} \lambda_B^+ + \chi \partial_{AA'} \chi, \]
and
\[ \lambda^+ B \nabla_{AA'} \lambda_B^+ = 2 \lambda^+ B \partial_{AA'} \lambda_B^+. \]

Then we have
\[ \lambda^+ B \partial_{AA'} \lambda_B + \lambda^B \partial_{AA'} \lambda_B^+ = \frac{1}{2} \left( \lambda^+ B \nabla_{AA'} \lambda_B + \lambda^B \nabla_{AA'} \lambda_B^+ \right), \]
\[ \lambda^B \partial_{AA'} \lambda_B = \frac{1}{2} \lambda^B \nabla_{AA'} \lambda_B, \]
\[ \lambda^+ B \partial_{AA'} \lambda_B^+ = \frac{1}{2} \lambda^+ B \nabla_{AA'} \lambda_B^+, \]
and
\[ \lambda^+ D \partial^{(AB)} \lambda_D + \lambda^D \partial^{(AB)} \lambda_D^+ = \frac{1}{2} \left( \lambda^+ D \nabla^{(AB)} \lambda_D + \lambda^D \nabla^{(AB)} \lambda_D^+ \right), \]
\[ \lambda^D \partial^{(AB)} \lambda_D = \frac{1}{2} \lambda^D \nabla^{(AB)} \lambda_D, \]
\[ \lambda^+ D \partial^{(AB)} \lambda_D^+ = \frac{1}{2} \lambda^+ D \nabla^{(AB)} \lambda_D^+. \]

Using these results and (35) we obtain
\[ \oint B^{AB} dS_{(AB)} = \oint \frac{\sigma}{\chi^2} \left( \lambda^A \lambda^B + \lambda^+ A \lambda^+ B + \lambda^A \lambda^B \right) \left( \lambda^C \partial_{(CD)} \lambda^+ D + \lambda^+ C \partial_{(CD)} \lambda^D \right) dS_{(AB)} \]
\[ - \frac{1}{2} \oint \sigma \left( \lambda^A \partial^{(CB)} \lambda_C^+ - \lambda^+ A \partial^{(CB)} \lambda_C \right) dS_{(AB)} \]
\[ - \frac{1}{2} \oint \sigma \left[ \lambda^+ A \lambda_C - \lambda^+ A \lambda_C^+ \right] \partial^{(CB)} \chi + \frac{\chi}{2} \partial^{(AB)} \chi \right] dS_{(AB)} \]
\[ + \frac{1}{2} \oint \sigma \left( \lambda^+ D \nabla^{(AB)} \lambda_D^+ + \lambda^D \nabla^{(AB)} \lambda_D + \lambda^+ D \nabla^{(AB)} \lambda_D \right) dS_{(AB)} \]

From the extended Witten identities (38), (39), and (40) one finds
\[ \frac{1}{2} \oint \sigma \left( \lambda^+ D \nabla^{(AB)} \lambda_D^+ + \lambda^D \nabla^{(AB)} \lambda_D + \lambda^+ D \nabla^{(AB)} \lambda_D \right) dS_{(AB)} \]
\[ = \int \sigma \lambda^A \nabla^{(AB)} \lambda_B dV + \int \sigma \lambda^A \nabla^{(AB)} \lambda_B dV + \int \sigma \lambda^A \nabla^{(AB)} \lambda_B dV + \int \sigma \lambda^A \nabla^{(AB)} \lambda_B dV \]
\[ + \frac{1}{2} \int \sigma \left( \nabla^{(BC)} \lambda_A^+ \nabla^{(BC)} \lambda_A dV + 2 \pi G \int \sigma (T_{00}) \lambda^+ A \lambda_A + \sqrt{2} T_{0AB} \lambda^+ A \lambda_B \right) dV \]
\[ - \frac{i \sqrt{2}}{2} \oint \sigma \lambda^B K^{CDA} \Lambda \lambda_A^+ dS_{(CD)}, \]
and then the boundary integral becomes
\[ \int B^{AB} dS_{(BC)} \]
\[ = - \int \frac{\sigma}{\chi^2} \left( \lambda^A \lambda^B + \lambda^+ \lambda^+ + \lambda^{(A} \lambda^{B)} \right) \left( \lambda^C \partial_{(CD)} \lambda^+ - \lambda^+ \partial_{(CD)} \lambda^D \right) dS_{(AB)} \]
\[ - 2 \int \sigma \left( \lambda^A \beta^{(CB)} \lambda^+ - \lambda^+ \beta^{(CB)} \lambda^C \right) dS_{(BC)} \]
\[ - \int \sigma \left[ \frac{2}{\chi} \left( \lambda^+ \lambda_C - \lambda^A \lambda_C^+ \right) \beta^{(CB)} \chi + \frac{1}{2} \chi \beta^{(AB)} \chi \right] dS_{(BC)} \]
\[ + \int \Sigma \sigma \chi \left( B_{(C)B} \right)^+ \left( B_{(C')}B' \right) \lambda_A dV + \frac{1}{2} \int \Sigma \sigma \left( \nabla_{(BC)} \lambda^A \right)^+ \left( \nabla_{(BC')} \lambda_A \right) dV + 2 \pi G \int \Sigma \sigma T_{00} \lambda^+ \lambda_A + \sqrt{2} T_{0AB} \lambda^+ \lambda^B dV \]
\[ - \frac{i \sqrt{2}}{2} \oint \sigma \lambda^+ B^{CD} A^+ \lambda_A dS_{(CD)}. \] (42)

When the Witten equations
\[ \nabla^{(CB)} \lambda_B = 0, \nabla^{(C'B')} \lambda_{B'} = 0, \]
are satisfied we have
\[ \int B^{AB} dS_{(AB)} \]
\[ = - \int \frac{\sigma}{\chi^2} \left( \lambda^A \lambda^B + \lambda^+ \lambda^+ + \lambda^{(A} \lambda^{B)} \right) \left( \lambda^C \partial_{(CD)} \lambda^+ - \lambda^+ \partial_{(CD)} \lambda^D \right) dS_{(AB)} \]
\[ - 2 \int \sigma \left( \lambda^A \beta^{(CB)} \lambda^+ - \lambda^+ \beta^{(CB)} \lambda^C \right) dS_{(AB)} \]
\[ - \int \sigma \left[ \frac{2}{\chi} \left( \lambda^+ \lambda_C - \lambda^A \lambda_C^+ \right) \beta^{(CB)} \chi + \frac{1}{2} \chi \beta^{(AB)} \chi \right] dS_{(AB)} \]
\[ + \frac{1}{2} \int \Sigma \sigma \left( \nabla_{(BC)} \lambda^A \right)^+ \left( \nabla_{(BC')} \lambda_A \right) dV + 2 \pi G \int \Sigma \sigma T_{00} \lambda^+ \lambda_A + \sqrt{2} T_{0AB} \lambda^+ \lambda^B dV \]
\[ - \frac{i \sqrt{2}}{2} \oint \sigma \lambda^+ B^{CD} A^+ \lambda_A dS_{(CD)}. \] (43)

Noting
\[ dS_{(AB)} = v_{(AB)} dS, \]
and
\[ v^{AB} = - \frac{i}{\sqrt{2}} \left( \sigma^{A} \nu^{B} + \nu^{A} \sigma^{B} \right), \]
one finds
\[ - \int \frac{\sigma}{\chi^2} \left( \lambda^A \lambda^B + \lambda^+ \lambda^+ + \lambda^{(A} \lambda^{B)} \right) \left( \lambda^C \partial_{(CD)} \lambda^+ - \lambda^+ \partial_{(CD)} \lambda^D \right) dS_{(AB)} \]
\[ = - \frac{i}{\sqrt{2}} \oint \sigma \left( \lambda^C \partial_{(CD)} \lambda^+ - \lambda^+ \partial_{(CD)} \lambda^D \right) dS. \]
And according to the definition of the extrinsic curvature of \( \Sigma \)
\[ K_{CDAB} = - \nabla_{(CD)} \sigma_{0AB} \]
\[ = - \frac{i}{\sqrt{2}} \nabla_{(CD)} \left( - \sigma^A \nu^B + \nu^A \sigma^B \right), \]
we have
\[ \lambda^+ B K_{CDAB} \lambda^+ = 0. \]

Finally, the boundary integral becomes

\[
\oint \tilde{B}^{AB} dS_{(AB)} = -\frac{i}{\sqrt{2}} \oint \sigma \left( \lambda^C \partial_{(CD)} \lambda^+ - \lambda^+ \partial_{(CD)} \lambda^- \right) dS \\
- 2 \oint \sigma \left( \lambda^A \partial^{(CB)} \lambda^+_C - \lambda^+ \partial^{(CB)} \lambda^-_C \right) v_{(AB)} dS \\
- \oint \sigma \left[ \frac{1}{\sqrt{2}} (\lambda^+ \lambda^-_C - \lambda^- \lambda^+_C) \partial^{(CB)} \lambda + \frac{1}{2} \lambda \partial^{(AB)} \lambda \right] v_{(AB)} dS \\
+ \frac{1}{2} \int_{\Sigma} \sigma (\nabla_{(BC)} \lambda^A)^+ \nabla^{(BC)} \lambda_A dV \\
+ 2\pi G \int_{\Sigma} \sigma (T_{00} \lambda^+ \lambda_A + \sqrt{2} T_{0AB} \lambda^+ \lambda^B) dV. \tag{44}
\]

This is manifestly non-negative if the gravitational source satisfies the dominant energy condition and then the positivity of the quasilocal energy can be proved easily. Starting from the choice (34) of the shift vector \( N^{AB} \), the prove can be extended to include momentum and angular momentum.

**VII. CONCLUSIONS AND REMARKS**

Using a spinor method the Einstein-Hilbert Lagrangian is decomposed into the self-dual and the anti-self-dual part. The self-dual part is identified with a version of the Ashtekar Lagrangian given by Goldberg. Starting from this Lagrangian and choosing the dyad spinors as the basic dynamical variables a Hamiltonian formulation is developed. A set of extended Witten identities is derived. Choosing a particular shift vector \( N^{AB} \) and applying the extended Witten identities the proof of the positive energy theorem is extended to a case including momentum and angular momentum.

In both the approach of Brown-York and the approach of Witten-Nester the boundary term has important roles. The "natural" Hamiltonian boundary term inherited from the Lagrangian can—and should—be adjusted. The choice of the Hamiltonian boundary term is the key point in the discussion of this paper. Being different from most of the works on positivity of energy in which the shift \( N^{AB} \) is chosen to vanish, we consider a more general boundary term \( \tilde{B}_{AB} \) which includes the ‘unreferred’ Brown-York quasilocal energy and allows a particular choice of \( N^{AB} \).

However, we do not deal with the issue of reference point since it has been shown [10] that when the Witten-Nester expression is evaluated on solution spinors to the Sen-Witten equation (obeying appropriate boundary conditions), an implicit reference point for the energy is set. The further investigation of the boundary term in some particular cases will be given in a forthcoming paper elsewhere.

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